The direct scattering problem for the perturbed $\text{Gr}(1,2)^{\geq 0}$
Kadomtsev-Petviashvili solitons

Derchyi Wu
Institute of Mathematics, Academia Sinica, Taipei, Taiwan
e-mail: mawudc@gate.sinica.edu.tw

August 20, 2019

Abstract

Regular Kadomtsev-Petviashvili (KP) solitons have been investigated and classified successfully by the Grassmannian. We provide rigorous analysis for the direct scattering problem of perturbed $\text{Gr}(1,2)^{\geq 0}$ KP solitons.

1 Introduction

If the amplitude is small and the wave length is large of a quasi-two dimensional water wave, then the dynamics can be approximated by the Kadomtsev-Petviashvili II (KPII) equation

$$(-4u_{x_3} + u_{x_1 x_1 x_1} + 6uu_{x_1})_{x_1} + 3u_{x_2 x_2} = 0,$$
$$u = u(x_1, x_2, x_3) \in \mathbb{R}, \quad x_1, x_2 \in \mathbb{R}, x_3 \geq 0.$$ (1.1)

Interesting features of the water wave can be reproduced by the KPII line solitons which have been discovered in 1970’s [16], [17], [10]. Precisely, a regular KPII line soliton can be constructed by

$$u(x) = u(x_1, x_2, x_3) = 2\partial^2_{x_1} \ln \tau(x),$$ (1.2)

where the $\tau$-function is given by the Wronskian determinant

$$\tau(x) = \text{Wr}(f_1, f_2, \cdots, f_N),$$
$$f_i(x) = \sum_{j=1}^{N} a_{ij} E_j(x), \quad 1 \leq i \leq N, \quad N < M,$$
$$E_j(x) = e^{\theta_j}, \quad \theta_j = \kappa_j x_1 + \kappa_j^2 x_2 + \kappa_j^3 x_3, \quad \kappa_1 < \cdots < \kappa_M,$$
$$A = (a_{ij}) \in \text{Gr}(N, M)^{\geq 0} \subset \text{Gr}(N, M).$$ (1.3)

the Grassmannian $\text{Gr}(N, M)$ denotes the set of $N$-dimensional subspaces in $\mathbb{R}^M$, and $\text{Gr}(N, M)^{\geq 0}$ is the subset of elements whose maximal minors are all non-negative [5], [8]. Since 2000’s, there has been important progress in studying properties and classification theory of these KPII line solitons.
Throughout this report, (1.2) defined by (1.3), are called \( \text{Gr(N,M}_\geq 0 \) KP solitons for simplicity.

The well-posedness problem of the KPII equation (1.1) with initial data \( u_c(x,y) \) where \( u_c(x-ct,y) \) is a KP solution has been solved by Molinet-Saut-Tzvetkov [13]. Their result shows that the deviation of the KPII solution from the initial data could evolve exponentially. Taking

\[
N = 1, \quad M = 2, \quad \kappa_1 = -\kappa_2, \quad A = (1,1)
\]

which are the simplest KPII 1-line solitons produced by the KdV 1-soliton solutions, Mizumachi establishes excellent \( L^2 \)-orbital stability and \( L^2 \)-instability theories for \( \text{Gr}(1,2)_\geq 0 \) KP solitons [11].

An important alternative approach to study the stability problem of \( \text{Gr(N,M}_\geq 0 \) KP solitons is the inverse scattering theory (IST) based on the Lax pair

\[
\begin{align*}
(-\partial_{x_2} + \partial_{x_1}^2 + u)\Psi(x,\lambda) &= 0, \\
(-\partial_{x_3} + \partial_{x_1}^3 + \frac{3}{2}u\partial_{x_1} + \frac{3}{4}u_{x_1} + \frac{3}{4}\partial_{x_1}^{-1}u_{x_2} - \lambda^3)\Psi(x,\lambda) &= 0
\end{align*}
\]

of the KPII equation. Indeed, the IST is to establish a bijective maps between the Lax equation

\[
(-\partial_{x_2} + \partial_{x_1}^2 + u)\Psi(x,\lambda) = 0,
\]

(defined by the KP solution) and a Cauchy integral equation (defined by the scattering data of the Lax equation). Substantial and important works on algebraic characterization and formal IST have been studied by Boiti-Pempenelli-Pogrebkov-Prinari [2], [3], [4], [5], Villarroel-Ablowitz [19]. In particular, the most remarkable characteristic, discontinuities for the Green function and eigenfunction of the Lax equation (1.6) were discovered by Boiti, Pempenelli, Pogrebkov, and Prinari (cf [2], [3], [5]). But a rigorous IST for perturbed \( \text{Gr(N,M}_\geq 0 \) KP solitons is still open.

Under the assumption (1.4), based on a KdV theory [14], [1], rigorous analysis for the direct scattering theory of perturbed \( \text{Gr}(1,2)_\geq 0 \) KP solitons has been carried out in [21]. To generalize the theory to arbitrary perturbed \( \text{Gr}(N,M)_\geq 0 \) KP solitons, the Sato (or \( \tau \)-function) approach [5] is not avoidable. The goal of this report is to adopt the Sato approach to provide a rigorous theory of the direct problem for general perturbed \( \text{Gr}(1,2)_\geq 0 \) solitons which consists of all KPII 1-line solitons with oblique directions and phase shifts. More precisely, using the convention \( x = (x_1,x_2,0) \), \( k = (k_1,k_2) \), \( k_j \geq 0 \), \( \partial_x = \partial_{x_1}^{k_1}\partial_{x_2}^{k_2} \), \( |x| = |x_1| + |x_2| \), \( |k| = k_1 + k_2 \), our results are stated
as: for
\[ u(x) = u_0(x) + v_0(x), \]
\[ u_0(x) = \frac{(\kappa_1 - \kappa_2)^2}{2} \text{sech}^2 \frac{\theta_1(x) - \theta_2(x) - \ln a}{2}, \]
\[ v_0(x) \in \mathbb{R}, \]
\[ \partial_x^k v_0 \in L^1 \cap L^\infty, |k| \leq 4, \]
\[ |v_0|_{L^1 \cap L^\infty} \ll 1, \]

i. We prove the existence of the eigenfunction of the Lax equation (1.6) by establishing uniform estimates of the Green function;

ii. We justify a Cauchy integral equation for the eigenfunction by deriving estimates of the spectral transform.

The contents of the paper are as follows. In Section 2, for a Lax equation (1.6) defined by a perturbed Gr(1,2)≥0 KP soliton, we introduce a proper boundary data and the Green function using the Sato theory. Then we provide algebraic and analytic characterization, including a uniform estimate, of the Green function.

In Section 3, we prove the existence and study the \( \overline{\partial} \)-scattering data of the eigenfunction, define the forward scattering transform \( T \), and derive estimates for the spectral map \( C T m \) where \( C \) is the Cauchy integral operator. Finally, in Section 4, we justify the initial eigenfunction satisfies a singular Cauchy integral equation and show that the singular Cauchy equation reduces to a Gr(1,2)≥0 KP soliton if the continuous scattering data is 0.

Acknowledgments. We feel indebted to A. Pogrebkov and Y. Kodama for introducing the Sato theory of the KP hierarchy. We would like to pay respects to the pioneer IST theory done by Boiti, Pempinelli, Pogrebkov, Prinari. This research project was partially supported by NSC 107-2115-M-001-002-.

2 The Green function

Setting \( x_3 = 0, N = 1, M = 2, \kappa_1 < \kappa_2, A = (1, a), a > 0 \) in (1.2) and (1.3), we obtain
\[ \tau(x) = \det \text{Wr}(f_1) = e^{\theta_1} + e^{\theta_2 + \ln a} = 2e^{\theta_1 + \theta_2 + \ln a} \cosh \frac{\theta_1 - \theta_2 - \ln a}{2} \]
and the Gr(1,2)≥0 KP soliton
\[ u_0(x) = \frac{(\kappa_1 - \kappa_2)^2}{2} \text{sech}^2 \frac{\theta_1(x) - \theta_2(x) - \ln a}{2}. \]  (2.1)

For the Lax equation (1.6), defined by the perturbed Gr(1,2)≥0 KP soliton
\[ u(x) = u_0(x) + v_0(x), \]  (2.2)
we impose the boundary value data

\[ \Psi(x, \lambda) \rightarrow \varphi(x, \lambda), \ x \rightarrow \infty, \quad (2.3) \]

where

\[ \varphi(x, \lambda) = e^{\lambda x_1 + \lambda^2 x_2} e^{\frac{\theta_1(x_1 + \frac{1}{\lambda} x_2 - \frac{1}{2\lambda^2} \cdot \cdot \cdot)}{\tau(x)}} + a e^{\theta_2(x_1 + \frac{1}{\lambda} x_2 - \frac{1}{2\lambda^2} \cdot \cdot \cdot)} \]

\[ = e^{\lambda x_1 + \lambda^2 x_2} e^{\frac{\theta_1(x_1 + \frac{1}{\lambda} x_2 - \frac{1}{2\lambda^2} \cdot \cdot \cdot)}{\tau(x)}} \]

\[ \equiv e^{\lambda x_1 + \lambda^2 x_2} \chi(x, \lambda) \equiv e^{\lambda x_1 + \lambda^2 x_2} \xi(x, \lambda) \]

is the Sato eigenfunction and \( \chi(x, \lambda) \) is the Sato normalized eigenfunction [5, (2.12)], [6, Theorem 6.3.8., (6.3.13)], [8, Proposition 2.2, (2.21)] satisfying

\[ L\varphi(x, \lambda) \equiv (-\partial_{x_2} + \partial_{x_1}^2 + u_0(x)) \varphi(x, \lambda) = 0, \]

\[ L\chi(x, \lambda) \equiv (-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1} + u_0(x)) \chi(x, \lambda) = 0. \]

(2.5)

If we renormalize the eigenfunction \( \Psi(x, \lambda) = e^{\lambda x_1 + \lambda^2 x_2} m(x, \lambda) \), then the boundary value problem (2.3) turns into

\[ Lm(x, \lambda) = -v_0(x) m(x, \lambda), \]

\[ m(x, \lambda) \rightarrow \chi(x, \lambda), \ x \rightarrow \infty. \]

(2.6)

Define the Green functions \( G(x, x', \lambda) \) and \( G(x, x', \lambda) \)

\[ L\mathcal{G}(x, x', \lambda) = \delta(x - x'), \quad L\mathcal{G}(x, x', \lambda) = \delta(x - x'), \]

\[ \mathcal{G}(x, x', \lambda) = e^{\lambda(x_1 - x'_1) + \lambda^2(x_2 - x'_2)} G(x, x', \lambda). \]

(2.7)

In the following, we explain one approach of Boiti et al [5] to derive the explicit formula of the Green functions. To this aim, we introduce the Sato adjoint eigenfunction

\[ \psi(x, \lambda) \]

\[ = e^{-(\lambda x_1 + \lambda^2 x_2)} \tau(x + \frac{1}{\lambda}) \]

\[ \equiv e^{-(\lambda x_1 + \lambda^2 x_2)} \frac{\theta_1(x_1 + \frac{1}{\lambda} x_2 + \frac{1}{2\lambda^2} \cdot \cdot \cdot)}{\tau(x)} + a e^{\theta_2(x_1 + \frac{1}{\lambda} x_2 + \frac{1}{2\lambda^2} \cdot \cdot \cdot)} \]

\[ = e^{-(\lambda x_1 + \lambda^2 x_2)} \frac{\theta_1(x_1 + \frac{1}{\lambda} x_2 + \frac{1}{2\lambda^2} \cdot \cdot \cdot)}{\tau(x)} \]

\[ \equiv e^{-(\lambda x_1 + \lambda^2 x_2)} \xi(x, \lambda), \]

(2.8)
and \( \xi(x, \lambda) \), the Sato normalized adjoint eigenfunction \([5 \ (2.12)], \ [6] \) Theorem 6.3.8., \((6.3.13)\), satisfying

\[
\begin{align*}
\mathcal{L}^1 \psi(x, \lambda) &\equiv (\partial_{x_1} + \partial_{x_1}^2 + u_0(x)) \psi(x, \lambda) = 0, \\
\mathcal{L}^1 \xi(x, \lambda) &\equiv (\partial_{x_1} + \partial_{x_1}^2 - 2\lambda \partial_{x_1} + u_0(x)) \xi(x, \lambda) = 0.
\end{align*}
\]  

(2.9)

Note that for \( \forall x \in \mathbb{R}^2 \) fixed, \( \chi(x, \cdot) \) is a rational function normalized at \( \infty \) and with a simple pole at 0; and \( \xi(x, \cdot) \) is a rational function normalized at \( \infty \) with simple poles at \( \kappa_1, \kappa_2 \), and vanishes at 0. Let

\[
\begin{align*}
\psi_j(x) &= \varphi(x, \kappa_j) = e^{\kappa_j x_1 + \kappa_j^2 x_2} \chi_j(x), \\
\psi_j(x) &= \text{res}_{\lambda = \kappa_j} \psi(x, \lambda) = e^{-(\kappa_j x_1 + \kappa_j^2 x_2)} \xi_j(x).
\end{align*}
\]

(2.10)

Lemma 2.1.

\[
\sum_{j=1}^{2} \varphi_j(x) \psi_j(x') = 0.
\]

Proof. Using (2.4) and (2.8), one obtains

\[
\begin{align*}
\varphi_1(x) \psi_1(x') &= \frac{\kappa_1 - \kappa_2}{\kappa_1} e^{\kappa_1 x_1 + \kappa_2 x_2} \psi_1(x') \\
\varphi_2(x) \psi_2(x') &= \frac{\kappa_2}{\kappa_2} e^{\kappa_2 x_1 + \kappa_2 x_2} \psi_2(x').
\end{align*}
\]

\[
\begin{align*}
\varphi_1(x) \psi_1(x') &= \frac{\kappa_1 - \kappa_2}{\kappa_1} e^{\kappa_1 x_1 + \kappa_2 x_2} \psi_1(x') \\
\varphi_2(x) \psi_2(x') &= \frac{\kappa_2}{\kappa_2} e^{\kappa_2 x_1 + \kappa_2 x_2} \psi_2(x').
\end{align*}
\]

\[\Box\]

Lemma 2.2. \([5 \ Eq. \ 3.1] \) Let \( \theta \) be the Heaviside function. Then

\[
\begin{align*}
\mathcal{G}(x, x', \lambda) &= \mathcal{G}_c(x, x', \lambda) + \mathcal{G}_d(x, x', \lambda), \\
\mathcal{G}_c &= -\frac{sgn(x_2-x_2')}{2\pi} \int_{\mathbb{R}} \theta((s^2 - \lambda^2)^2) (x_2 - x_2') \varphi(x, \lambda_R + is) \psi(x', \lambda_R + is) ds, \\
\mathcal{G}_d &= -\frac{\theta(x_2 - x_2)}{2\pi} \{\theta(\lambda_R - \kappa_1) \varphi_1(x) \psi_1(x') + \theta(\lambda_R - \kappa_2) \varphi_2(x) \psi_2(x')\}.
\end{align*}
\]

(2.11)

Proof. We follow the proof of \([4] \). Namely, the Green function \( \mathcal{G} \) will be constructed via an orthogonality relation of \( \varphi(x, \lambda) \psi(x', \lambda) \) superposed with an appropriate cutoff function.

To establish an orthogonality relation, note \( \varphi(x, \lambda + i\lambda') \psi(x', \lambda + i\lambda') = e^{[\lambda+i\lambda')(x_1-x_1')+[\lambda+i\lambda')^2(x_2-x_2')]}(x, \lambda + i\lambda') \xi(x', \lambda + i\lambda') \). Hence applying the Fourier inversion theorem, introducing a new variable \( \lambda + i\lambda' = \lambda_R + is \), using the residue theorem, and Lemma 2.1 one has

\[
\begin{align*}
\frac{\delta(x_2-x_2')}{2\pi} \int \varphi(x, \lambda + i\lambda') \psi(x', \lambda + i\lambda') d\lambda' &= \frac{\delta(x_2-x_2')}{2\pi} \int e^{(\lambda_R + is)(x_1-x_1')} \chi(x, \lambda_R + is) \xi(x', \lambda_R + is) ds
\end{align*}
\]
Proposition 2.1. The Green function $G(x, x', \lambda)$ satisfies the analytic constraint, for $\forall x_2 - x'_2 \neq 0$, $\lambda \neq \kappa_1, \kappa_2$,

$$|G(x, x', \lambda)| \leq C(1 + \frac{1}{\sqrt{|x_2 - x'_2|}}),$$

and for any Schwartz function $f$,

$$\lim_{|x| \to \infty} G(x, x', \lambda) \ast f(x') \to 0,$$

$G \ast f(x, \lambda) \equiv \int G(x, x', \lambda)f(x')dx'$.

(2.14)

(2.15)
Proof. From (2.7) and Lemma 2.2 one needs to show uniform estimates of $G_d(x, x', \lambda)$ and $G_c(x, x', \lambda)$ which can be written as

$$G_d(x, x', \lambda) = -e^{\lambda(x'_1-x_1)+\lambda^2(x'_2-x_2)} \sum_{j=1}^{\infty} \theta(x'_2-x_2) \theta(\lambda R - \kappa_j) \varphi_j(x) \psi_j(x').$$

and

$$G_c(x, x', \lambda) = -\frac{1}{2\pi} \int_{\mathbb{R}} ds \, \text{sgn}(x_2-x'_2) \theta((s^2 - \lambda^2_f)(x_2-x'_2)) \chi(x, \lambda R + is) \times \xi(x', \lambda R + is) e^{\lambda - (\lambda R + is)(x'_1-x_1) + [\lambda^2 - (\lambda R + is)^2](x'_2-x_2)}$$

$$= -\frac{e^{i\lambda(x'_1-x_1)+2\lambda R(x'_2-x_2)}}{2\pi} \int_{\mathbb{R}} ds \, \text{sgn}(x_2-x'_2) \theta((s^2 - \lambda^2_f)(x_2-x'_2))$$

$$\times e^{(s^2-\lambda^2_f)(x'_2-x_2)-i[s(x'_1-x_1)+2\lambda R(x'_2-x_2)]} \chi(x, \lambda R + is) \xi(x', \lambda R + is),$$

(2.17)

**Step 1 (Estimates for $G_d$):** From Lemma 2.1, (2.4), (2.8), (2.10), (2.16), the dominated convergence theorem, and the Riemann-Lebesgue lemma, one has

$$|G_d(x, x', \lambda)| = | -e^{\lambda(x'_1-x_1)+\lambda^2(x'_2-x_2)} \theta(x'_2-x_2) \theta(\kappa_2 - \lambda R) \theta(\lambda R - \kappa_1) \varphi_1(x) \psi_1(x') |$$

$$= | -\frac{\theta(x'_2-x_2) \theta(\kappa_2 - \lambda R) \theta(\lambda R - \kappa_1) \varphi_1(x) \psi_1(x')}{(1 + e^{\theta_2(x'_2-x_2)} + e^{\theta_2(x'_2-x_2)})} |$$

$$\leq C \left| \frac{\theta(x'_2-x_2) \theta(\kappa_2 - \lambda R) \theta(\lambda R - \kappa_1) \varphi_1(x) \psi_1(x')}{(1 + e^{\theta_2(x'_2-x_2)} + e^{\theta_2(x'_2-x_2)})} \right|$$

$$\leq C,$$

(2.18)

and

$$G_d(x, x', \lambda) \to \begin{cases} 0, & \lambda \to \kappa_j^+, \\ +\theta(x'_2-x_2) \chi_2(x) \xi_2(x'), & \lambda \to \kappa_j^+, \\ -\theta(x'_2-x_2) \chi_1(x) \xi_1(x'), & \lambda \to \kappa_j^+, \\ 0, & \lambda \to \kappa_j^-, \\ \kappa_j^+ = \kappa_j + 0^+ e^{i\alpha}, & \kappa_j^+ = \kappa_j + 0^+ e^{i(\pi + \alpha)}, & 0 \leq \alpha \leq \pi, \\ \lim_{|x| \to \infty} G_d(x, x', \lambda) \ast f(x') = 0, & \text{for } \lambda \neq \kappa_j \text{ fixed,} \\ \text{and any Schwartz function } f. 
\end{cases}$$

(2.19)

**Step 2 (A decomposition for $G_c$):** From (2.4), (2.8), (2.17),

$$G_c(x, x', \lambda) = -\frac{e^{i\lambda(x'_1-x_1)+2\lambda R(x'_2-x_2)}}{2\pi} \int_{\mathbb{R}} ds \, \text{sgn}(x_2-x'_2) \theta((s^2 - \lambda^2_f)(x_2-x'_2))$$

$$\times (1 - \frac{\kappa_1 - \kappa_2}{1 + e^{[-(\kappa_1 - \kappa_2)] x_1 + (\kappa_1^2 - \kappa_2^2) x_2 - i\alpha} - \kappa_2}$$

$$\times (1 + \frac{\kappa_1 - \kappa_2}{1 + e^{[-(\kappa_1 - \kappa_2)] x_1 + (\kappa_1^2 - \kappa_2^2) x_2 - i\alpha}}).$$

7
So if \( \lambda \in D_{\kappa_1}^c \cap D_{\kappa_2}^c \),

\[
|G_c(x, x', \lambda)| \leq C \theta(x_2 - x'_2) (\int_{-\infty}^{-|\lambda|} + \int_{|\lambda|}^{\infty}) e^{(s^2 - \lambda_1^2)}(x_2' - x_2) ds + C \theta(x'_2 - x_2) \int_{|\lambda|}^{\infty} e^{(s^2 - \lambda_1^2)}(x_2' - x_2) ds,
\]

and one can either look for estimates for special functions \( \text{erfc}(z) \) and Dawson’s integral or a direct estimate (see Step 1 in [21]) to derive

\[
|G_c(x, x', \lambda)| \leq C(1 + \frac{1}{\sqrt{|x_2 - x'_2|}}), \quad \lambda \in D_{\kappa_1}^c \cap D_{\kappa_2}^c.
\] (2.20)

Furthermore, the dominated convergence theorem and Riemann-Lebesgue lemma imply

\[
\lim_{|x| \to \infty} G_c(x, x', \lambda) \ast f(x') = 0, \quad \text{for } \lambda \neq \kappa_j \text{ fixed, and any Schwartz function } f.
\] (2.21)

Hence it remains to show the estimates for \( \lambda \in D_{\kappa_j}^c \).

For \( \lambda \in D_{\kappa_j}^c \), decompose

\[
G_c(x, x', \lambda) = -\frac{\sigma(\lambda, \lambda_1 - \lambda_2) + \lambda R(\lambda_1 - \lambda_2) + \lambda R(\lambda_1 - \lambda_2)}{2\pi} (I_j + II_j + III_j + IV_j),
\] (2.22)

with

\[
I_j = \int_{-\kappa}^{\kappa} \text{sgn}(x_2 - x'_2) \theta((s^2 - \lambda_1^2)(x_2 - x'_2)) \chi(x, \lambda R + is) \xi(x', \lambda R + is) \times \left[ e^{is} \frac{s}{\lambda R + is - \kappa} \right] ds,
\]

\[
II_j = \int_{-\kappa}^{\kappa} \text{sgn}(x_2 - x'_2) \theta((s^2 - \lambda_1^2)(x_2 - x'_2)) \chi(x, \lambda R + is) \xi(x', \lambda R + is) \times \left[ e^{is} \frac{s}{\lambda R + is - \kappa} \right] ds,
\]

\[
III_j = \int_{-\kappa}^{\kappa} \text{sgn}(x_2 - x'_2) \theta((s^2 - \lambda_1^2)(x_2 - x'_2)) \chi(x, \lambda R + is) \xi(x', \lambda R + is) \times e^{is} \left[ \frac{s}{\lambda R + is - \kappa} \right] ds,
\]

\[
IV_j = \left( \int_{-\kappa}^{\kappa} \text{sgn}(x_2 - x'_2) \theta((s^2 - \lambda_1^2)(x_2 - x'_2)) \chi(x, \lambda R + is) \xi(x', \lambda R + is) e^{is} \left[ \frac{s}{\lambda R + is - \kappa} \right] ds \right).
\] (2.23)

By the same method as that for (2.20), we have

\[
|II_j| < C, \quad |IV_j| \leq C(1 + \frac{1}{\sqrt{|x_2 - x'_2|}}), \quad \lambda \in D_{\kappa_j}^c.
\] (2.24)

**Step 3 (Estimates for III_j):** Since

\[
III_j = \theta(x_2 - x'_2) \left( \int_{-\kappa}^{\kappa} \frac{\chi(x, \lambda R + is) \xi(x')}{\lambda R + is - \kappa} ds - \theta(x'_2 - x_2) \int_{-\kappa}^{\kappa} \frac{\chi(x, \lambda R + is) \xi(x')}{\lambda R + is - \kappa} ds \right)
\]

\[
= -i\chi_j(x) \xi_j(x') \left( \theta(x_2 - x'_2) \int_{-\kappa}^{\kappa} \frac{1}{s - i(\lambda R - \kappa_j)} ds - \theta(x'_2 - x_2) \int_{-\kappa}^{\kappa} \frac{1}{s - i(\lambda R - \kappa_j)} ds \right).
\]
By logarithmic function,
\[
\begin{align*}
\lim_{\lambda \to \kappa_j} \int_{-\kappa}^{\kappa} \frac{1}{s-i(\lambda R-\kappa_j)} \, ds &= i\pi (2\theta(\lambda R - \kappa_j) - 1), \\
\int_{-|\lambda_j|}^{|\lambda_j|} \frac{1}{s-i(\lambda R-\kappa_j)} \, ds &= 2\pi i \theta(\lambda R - \kappa_j) - 1 + 2i \cot^{-1} \frac{\lambda_R-\kappa_j}{|\lambda_j|}, \quad \lambda \in D_{\kappa_j}^\infty.
\end{align*}
\]  
(2.25)

Here
\[
\cot^{-1} \frac{\lambda_R-\kappa_j}{|\lambda_j|} = \begin{cases} 
\alpha, & 0 < \alpha \leq \pi, \quad \lambda \in D_{\kappa_j}^\infty, \\
2\pi - \alpha, & \pi \leq \alpha < 2\pi, \quad \lambda \in D_{\kappa_j}^\infty,
\end{cases}
\]  
(2.26)

As a result,
\[
|III_j| \leq C, \quad \lambda \in D_{\kappa_j}^\infty,
\]  
(2.27)

and
\[
-2 \int_{-|\lambda_j|}^{|\lambda_j|} \frac{1}{s-i(\lambda R-\kappa_j)} \, ds,
\]  
(2.28)

**Step 4 (Estimates for I\(_j\))**: We follow the same method as that in [21] to derive estimates for I\(_j\). Setting \(y_1 = x_1 - x'_1\), \(y_2 = x_2 - x'_2\), estimates for I\(_j\) are reduced to
\[
\begin{align*}
I_{jn}^j(y_1, y_2, \lambda) &= -\theta(-y_2) \int_{-|\lambda_j|}^{|\lambda_j|} \frac{e^{(\lambda_R-s)^2 y_2 + i\alpha(y_1 + 2\lambda R y_2)} - 1}{s-i(\lambda R-\kappa_j)} \, ds, \\
I_{jn}^j(y_1, y_2, \lambda) &= \theta(y_2) \int_{-|\lambda_j|}^{|\lambda_j|} \frac{e^{(\lambda_R-s)^2 y_2 + i\alpha(y_1 + 2\lambda R y_2)} - 1}{s-i(\lambda R-\kappa_j)} \, ds.
\end{align*}
\]  
(2.29)

In this step, we study I\(_{jn}^j\) by considering cases
\[
\begin{align*}
(1a) & \quad (\lambda_R - \kappa_j)(y_1 + 2\lambda R y_2) \geq 0, \quad |\lambda_j| - |\lambda_R - \kappa_j| \leq \frac{1}{2} |\lambda_j|, \\
(1b) & \quad (\lambda_R - \kappa_j)(y_1 + 2\lambda R y_2) \geq 0, \quad |\lambda_j| - |\lambda_R - \kappa_j| \geq \frac{1}{2} |\lambda_j|, \\
(1c) & \quad (\lambda_R - \kappa_j)(y_1 + 2\lambda R y_2) < 0.
\end{align*}
\]  
(2.30)

In Case (1a), |\(\lambda_R - \kappa_j| \geq \frac{|\lambda_j|}{2}. So
\[
|I_{jn}^j| \leq \theta(-y_2) \int_{-|\lambda_j|}^{|\lambda_j|} \frac{e^{(\lambda_R-s)^2 y_2 + i\alpha(y_1 + 2\lambda R y_2)} - 1}{s-i(\lambda R-\kappa_j)} \, ds
\leq C \int_{-|\lambda_j|}^{|\lambda_j|} \frac{1}{|\lambda_j|} \, ds \leq C.
\]  
(2.31)

In Case (1b) or (1c), we deform the real interval –|\(\lambda_j| \leq s \leq |\lambda_j| to the semicircle \(\Gamma\), defined by
\[
\Gamma \equiv \{ s = se^{i\beta} \in \mathbb{C} : s = |\lambda_j|, \ (y_1 + 2\lambda R y_2) \sin \beta > 0 \} \subset \Omega, \\
\Omega \equiv \{ s = se^{i\beta} \in \mathbb{C} : 0 \leq s \leq |\lambda_j|, \ (y_1 + 2\lambda R y_2) \sin \beta > 0 \},
\]  
9
We consider the following cases,

\[ e^{(\lambda_1^2-s^2)y_2 + is(y_1+2\lambda_Ry_2)} \] is uniformly bounded on the half disk \( \Omega \),

\[ i(\lambda_R - \kappa_j) \in \Omega \] in Case (1b).

Besides, (1b) or (1c) implies

\[ \text{Distance}\{\Gamma, i(\lambda_R - \kappa_j)\} \geq |\lambda_j| \frac{1}{2}. \]

Therefore,

\[ |I_j^{in}| = |\theta(-y_2) \int_{-|\lambda_j|}^{\lambda_j} \frac{e^{(\lambda_1^2-s^2)y_2 + is(y_1+2\lambda_Ry_2)}}{s-i(\lambda_R - \kappa_j)} \frac{1}{|\lambda_j|} ds| \]
\[ \leq |\theta(-y_2) \int_{\Gamma} \frac{e^{(\lambda_1^2-s^2)y_2 + is(y_1+2\lambda_Ry_2)}}{s-i(\lambda_R - \kappa_j)} \frac{1}{|\lambda_j|} ds| \]
\[ + C \delta((\lambda_R - \kappa_j)(y_1 + 2\lambda_Ry_2)) \theta(-y_2) \delta(\lambda_j) - |\lambda_R - \kappa_j| \] (2.32)
\[ \times e^{(\lambda_1^2+(\lambda_R-\kappa_j)^2)y_2-(\lambda_R-\kappa_j)(y_1+2\lambda_Ry_2)} + C \]
\[ \leq 2 |\int_{\Gamma} \frac{C}{|\lambda_j|} ds| + C \leq C. \]

Hence estimates for \( I_j^{in} \) follows from (2.31) and (2.32).

**Step 5 (Estimates for \( I_j \) (continued)):** We consider the following cases,

\[(2a) \quad |\lambda_j| \sqrt{|y_2|} \geq 1, \]
\[(2b) \quad |\lambda_j| \sqrt{|y_2|} \leq 1. \]

In case of (2a), let \( \xi = \frac{\omega}{|\lambda_j|}. \)

\[ |I_j^{out}| = C |\theta(y_2)| \left\{ (\int_{-\frac{1}{|\lambda_j|}}^{\frac{1}{|\lambda_j|}} \frac{e^{\lambda_1^2\xi^2+(\lambda_1^2+y_1^2)}}{s-i(\lambda_R - \kappa_j)} \frac{1}{|\lambda_j|} d\xi \right\} \]
\[ - (\int_{-\kappa}^{\kappa} \frac{1}{|\lambda_j|} ds) \right\} | \]
\[ \leq C (\int_{-\frac{1}{|\lambda_j|}}^{\frac{1}{|\lambda_j|}} e^{-\xi^2} d\xi + C \]
\[ = Ce^{\sqrt{e^{-1} - e^{-\kappa^2}}} + C \leq C. \]

In case (2b), let \( s = \frac{\omega}{\sqrt{y_2}}. \)

\[ |I_j^{out}| = C |\theta(y_2)| \left( \int_{-\frac{1}{|\lambda_j|}}^{\frac{1}{|\lambda_j|}} e^{(\lambda_1^2+y_1^2)} \frac{1}{\omega-i(\lambda_R - \kappa_j)\sqrt{y_2}} d\omega \right) \]
\[ + \theta(y_2) \left( \int_{-\frac{1}{|\lambda_j|}}^{\frac{1}{|\lambda_j|}} e^{\lambda_1^2\xi^2+y_1^2} \frac{1}{\omega-i(\lambda_R - \kappa_j)\sqrt{y_2}} d\omega \right) \]
\[ - (\int_{-\kappa}^{\kappa} \frac{1}{|\lambda_j|} ds) \right\} | \]
\[ \leq C(A_1 + A_2 + A_3), \]
where

\[
A_1 = |\theta(y_2)(\int_{-1}^{1} e^{i\omega y_1 + 2\lambda R y_2 - \omega^2} - e^{i\omega y_1 + 2\lambda R y_2} - e^{-i\omega y_1 + 2\lambda R y_2}) d\omega | + (\int_{-1}^{1} e^{i\omega y_1 + 2\lambda R y_2} - e^{-i\omega y_1 + 2\lambda R y_2} - e^{-i\omega y_1 + 2\lambda R y_2}) d\omega | \leq A_{11} + A_{12},
\]

\[
A_2 = |\theta(y_2)(\int_{-1}^{1} e^{i\omega y_1 + 2\lambda R y_2} - e^{i\omega y_1 + 2\lambda R y_2} - e^{i\omega y_1 + 2\lambda R y_2}) d\omega | \leq C.
\]

Applying the mean value theorem to \(A_{11}\),

\[
A_{11} = \left| \theta(y_2)(\int_{-1}^{1} e^{i\omega y_1 + 2\lambda R y_2} - e^{i\omega y_1 + 2\lambda R y_2} - e^{i\omega y_1 + 2\lambda R y_2}) d\omega | \leq \theta(y_2)(\int_{-1}^{1} e^{i\omega y_1 + 2\lambda R y_2} - e^{i\omega y_1 + 2\lambda R y_2} - e^{i\omega y_1 + 2\lambda R y_2}) d\omega | \leq C,
\]

and the argument of (1a), (1b), (1c) in Step 4 to

\[
A_{12} = |(\int_{-1}^{1} e^{i\omega y_1 + 2\lambda R y_2} - e^{i\omega y_1 + 2\lambda R y_2} - e^{i\omega y_1 + 2\lambda R y_2}) d\omega |,
\]

one can derive uniform boundedness for \(A_{12}\). Therefore, we have justify \(|I_{j}^{\text{out}}| \leq C\) in case (2b).

Combining Step 1 to Step 5, we prove (2.14).

\[\square\]

**Lemma 2.3.** The Green function \(G\), defined by (2.7), satisfies the algebraic constraint

\[
G(x, x', \lambda) = \overline{G(x, x', \lambda)}.
\]
Moreover, there exist $\mathcal{G}_j$, $\omega_j$, such that

$$
G(x, x', \lambda) = \begin{cases} 
\mathcal{G}_1(x, x') + \frac{1}{2}\chi_1(x)\xi_1(x') \cot^{-1} \frac{\lambda \cdot \xi_1}{|\lambda|} + \omega_1(x, x', \lambda), & \lambda \in D_{\kappa_1}^x, \\
\mathcal{G}_2(x, x') - \frac{1}{2}\chi_2(x)\xi_2(x') \cot^{-1} \frac{\kappa_2 - \xi_2}{|\lambda|} + \omega_2(x, x', \lambda), & \lambda \in D_{\kappa_2}^x,
\end{cases}
$$

(2.37)

with $\cot^{-1} \frac{\lambda \cdot \xi_1}{|\lambda|}$, $\cot^{-1} \frac{\kappa_2 - \xi_2}{|\lambda|}$ defined by (2.26),

$$
|\mathcal{G}_j|_{L^\infty(D_{\kappa_j})}, \quad |\omega_j|_{L^\infty(D_{\kappa_j})} \leq C(1 + \frac{1}{\sqrt{|x_2 - x_2'|}}),
$$

(2.38)

and the symmetry

$$
\mathcal{G}_2(x, x')e^{\kappa_2(x_1 - x_1') + \kappa_2^2(x_2 - x_2')} = e^{\kappa_1(x_1 - x_1') + \kappa_1^2(x_2 - x_2')}\mathcal{G}_1(x, x'),
$$

$$
G(x, x', \kappa_2 + 0^+ e^{i\alpha})e^{\kappa_2(x_1 - x_1') + \kappa_2^2(x_2 - x_2')} = e^{\kappa_1(x_1 - x_1') + \kappa_1^2(x_2 - x_2')}G(x, x', \kappa_1 + 0^+ e^{i(\pi + \alpha)}).
$$

(2.39)

Proof. Step 1 (Proof for (2.36)) : First of all, applying (2.24), (2.25), (2.10), Lemma 2.2 and by a change of variables $s \mapsto -s$, one can prove the algebraic constraint (2.36).

Step 2 (Proof for (2.37)) : For fixed $x, x'$, asymptotic (2.37) can be obtained via the dominated convergence theorem, (2.23), (2.19), (2.28), estimates of (2b) in Step 5 of Proposition 2.1 and definition (2.26). Moreover, the error estimate (2.38) follows from a similar argument (more elaborating) as that for deriving (2.11). We omit the details for simplicity and refer [21, Lemma 3.1] for a similar detailed proof.

Step 3 (Proof of (2.39)) : From (2.26), it suffices to establish

$$
\mathcal{G}(x, x', \kappa_2 + 0^+ e^{i\alpha}) = \mathcal{G}(x, x', \kappa_1 + 0^+ e^{i(\pi + \alpha)}).
$$

(2.40)

We now exploit the approach in [19, Proposition 9 (i)] to prove (2.40). For fixed $x \neq 0$, $0 < \alpha \leq 2\pi$, let

$$
\lambda_2 = \lambda_2, R + i\lambda_2, I = \kappa_2 + 0^+ e^{i\alpha},
$$

$$
\lambda_1 = \lambda_1, R + i\lambda_1, I = \kappa_1 + 0^+ e^{i(\pi + \alpha)}.
$$

(2.41)

Then from Lemma 2.1 and Lemma 2.2 immediately, one has

(i) \quad for $0 < \alpha < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \alpha < 2\pi$,
$$
\mathcal{G}_d(x, x', \lambda_2) = \mathcal{G}_d(x, x', \lambda_1) = 0;
$$

(ii) \quad for $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$,
$$
\mathcal{G}_d(x, x', \lambda_2) = \mathcal{G}_d(x, x', \lambda_1) = -\theta(x_2 - x_2')\varphi_1(x)\psi_1(x').
$$

(2.42)
On the other hand,

\[( -2\pi )G_c(x, x', \lambda_2) = \theta(x_2 - x_2') \int_R \varphi(x, is + \lambda_2 R) \psi(x', is + \lambda_2 R) ds \]

\[- \int_{|\lambda_2 |}^{|\lambda_2 |} \varphi(x, is + \lambda_2 R) \psi(x', is + \lambda_2 R) ds ; \]

\[( -2\pi )G_c(x, x', \lambda_1) = \theta(x_2 - x_2') \int_R \varphi(x, is + \lambda_1 R) \psi(x', is + \lambda_1 R) ds \]

\[- \int_{|\lambda_1 |}^{|\lambda_1 |} \varphi(x, is + \lambda_1 R) \psi(x', is + \lambda_1 R) ds . \]

\[(2.43) \]

Deforming the contour, applying the residue theorem and Lemma 2.11,

\[\theta(x_2 - x_2') \int_R \varphi(x, is + \lambda_2 R) \psi(x', is + \lambda_2 R) ds \]

\[= \theta(x_2 - x_2') \int_R \varphi(x, is + \lambda_2 R) \psi(x', is + \lambda_2 R) ds \]

\[+ (-2\pi) \theta(x_2 - x_2') \int_R \varphi(x, is + \lambda_2 R) \psi(x', is + \lambda_2 R) ds \]

\[= \theta(x_2 - x_2') \int_R \varphi(x, is + \lambda_2 R) \psi(x', is + \lambda_2 R) ds \]

\[= \theta(x_2 - x_2') \int_R \varphi(x, is + \lambda_1 R) \psi(x', is + \lambda_1 R) ds . \]

\[(2.44) \]

On the other hand, the residue theorem, Lemma 2.11, and the dominated convergence theorem imply

\[\int_{|\lambda_2 |}^{|\lambda_2 |} \varphi(x, is + \lambda_2 R) \psi(x', is + \lambda_2 R) ds \]

\[= + (-i) \varphi_1 (x) \psi_1 (x') \int_{|\lambda_2 |}^{|\lambda_2 |} \frac{1}{s-i(\lambda_2 R - \kappa_2)} ds \]

\[= + (-i) \varphi_1 (x) \psi_1 (x') \int_{|\lambda_1 |}^{|\lambda_1 |} \frac{1}{s-i(\lambda_1 R - \kappa_1)} ds \]

\[= \int_{|\lambda_1 |}^{|\lambda_1 |} \varphi(x, is + \lambda_1 R) \psi(x', is + \lambda_1 R) ds . \]

\[(2.45) \]

Consequently, (2.40) follows from (2.41) - (2.45). \[\square \]

**Lemma 2.4.** \[\square \] For \( \lambda_1 \neq 0 \),

\[\partial_{\lambda'} G(x, x', \lambda) = - \frac{sgn(\lambda_1)}{2\pi i} e^{(\lambda_1 - \lambda)(x_1 - x'_1) + (\lambda_1 - \lambda)(x_2 - x'_2)} \chi(x, \lambda) \xi(x', \lambda) \]

**Proof.** Using the change of variables \( \lambda + i\lambda' = \lambda_1 + is \) in (2.11),

\[\partial_{\lambda'} G_c(x, x', \lambda) = \int_{-2\pi}^{2\pi} \frac{\partial_{\lambda'} G_c(x, x', \lambda)}{\lambda_1} \]

\[= \int_{-2\pi}^{2\pi} \left[ \int_{-\infty}^{-2\lambda_1} + \int_{-\infty}^{0} \right] \chi(x, \lambda + i\lambda') \xi(x', \lambda + i\lambda') \]

\[\times e^{[(\lambda_1 + i\lambda')\lambda](x_1 - x'_1) + [(\lambda_1 + i\lambda')^2 - \lambda^2](x_2 - x'_2) d\lambda'} \]

\[\times \int_{2\pi}^{2\lambda_1} \chi(x, \lambda + i\lambda') \xi(x', \lambda + i\lambda') \]

\[\times e^{[(\lambda_1 + i\lambda')\lambda](x_1 - x'_1) + [(\lambda_1 + i\lambda')^2 - \lambda^2](x_2 - x'_2) d\lambda'} , \quad \text{if } \lambda_1 > 0 , \]

\[\int_{-2\pi}^{2\pi} \left[ \int_{-\infty}^{-2\lambda_1} + \int_{-\infty}^{0} \right] \chi(x, \lambda + i\lambda') \xi(x', \lambda + i\lambda') \]

\[\times e^{[(\lambda_1 + i\lambda')\lambda](x_1 - x'_1) + [(\lambda_1 + i\lambda')^2 - \lambda^2](x_2 - x'_2) d\lambda'} , \quad \text{if } \lambda_1 < 0 . \]
\[ \begin{align*}
&= -\frac{\text{sgn}(\lambda_1)}{2\pi} \delta_{\lambda_1 = -2\lambda} e^{((\lambda+i\lambda') - \lambda)(x_1-x') + [(\lambda+i\lambda')^2 - \lambda^2](x_2-x')^2} \\
&\times \chi(x, \lambda + i\lambda') \xi(x', \lambda + i\lambda') \\
&- \frac{1}{2\pi} \int_{\mathbb{R}} e^{((\lambda+i\lambda') - \lambda)(x_1-x') + [(\lambda+i\lambda')^2 - \lambda^2](x_2-x')^2} \Xi(x_2 - x_2', \lambda, \lambda + \lambda') \\
&\times \partial_x^2 \chi(x, \lambda + i\lambda') \xi(x', \lambda + i\lambda') d\lambda'
\end{align*} \]

Here
\[ \Xi(x_2, \lambda, s) = \theta(x_2) \chi_{\{s, -|\lambda_1|\}}(\lambda_2, \lambda_1, \lambda_1) - \theta(-x_2) \chi_{\{s, |\lambda_1|\}}(\lambda_2, \lambda_1, \lambda_1), \]
\[ \chi_A(s) \text{ denotes the characteristic function of the set } A. \] (2.46)

Using \[ \frac{1}{\pi} \partial_\lambda \left( \frac{1}{\lambda - a} \right) = \delta_{\lambda = a} \delta_{\lambda_1 = a_1}, \]
one has
\[ \partial_\lambda^2 \chi(x, \lambda + i\lambda') \xi(x', \lambda + i\lambda') = \pi \chi_1(x) \xi_1(x') \delta_{\lambda_1 = -\lambda_1} + \pi \chi_2(x) \xi_2(x') \delta_{\lambda_1 = \kappa_2} \delta_{\lambda_1 = -\lambda_1}. \]

So
\[ \begin{align*}
\partial_\lambda^2 G_c(x, x', \lambda) &= -\frac{\text{sgn}(\lambda_1)}{2\pi} e^{(\lambda - \lambda_1)(x_1-x_1') + [\lambda^2 - \lambda_1^2](x_2-x_2')^2} \chi(x, \lambda) \xi(x', \lambda) \\
&+ \frac{1}{\pi} \theta(x_2' - x_2) e^{-i\lambda_1(x_1-x_1') + (2i\kappa_1 \lambda_1 + \lambda_1^2)(x_2-x_2')^2} \chi_1(x) \xi_1(x') \delta_{\lambda_1 = \kappa_1} \\
&+ \frac{1}{\pi} \theta(x_2' - x_2) e^{-i\lambda_1(x_1-x_1') + (2i\kappa_2 \lambda_1 + \lambda_1^2)(x_2-x_2')^2} \chi_2(x) \xi_2(x') \delta_{\lambda_1 = \kappa_2} \\
&= -\frac{\text{sgn}(\lambda_1)}{2\pi} e^{(\lambda - \lambda_1)(x_1-x_1') + [\lambda^2 - \lambda_1^2](x_2-x_2')^2} \chi(x, \lambda) \xi(x', \lambda) \\
&+ \frac{1}{\pi} \theta(x_2' - x_2) e^{-\lambda(x_1-x_1') - \lambda^2(x_2-x_2')^2} \varphi_1(x) \psi_1(x') \delta_{\lambda_1 = \kappa_1} \\
&+ \frac{1}{\pi} \theta(x_2' - x_2) e^{-\lambda(x_1-x_1') - \lambda^2(x_2-x_2')^2} \varphi_2(x) \psi_2(x') \delta_{\lambda_1 = \kappa_2}. \quad (2.47)
\end{align*} \]

Besides, a direct computation yields
\[ \begin{align*}
\partial_\lambda^2 G_d(x, x', \lambda) &= -\delta_{\lambda_1} [e^{-\lambda(x_1-x_1') - \lambda^2(x_2-x_2')} \theta(x_2' - x_2) \\
&\times \{ \theta(\lambda_R - \kappa_1) \varphi_1(x) \psi_1(x') + \theta(\lambda_R - \kappa_2) \varphi_2(x) \psi_2(x') \}] \\
&= -\frac{1}{2} e^{-\lambda(x_1-x_1') - \lambda^2(x_2-x_2')} \theta(x_2' - x_2) \varphi_1(x) \psi_1(x') \delta_{\lambda_1 = \kappa_1} \\
&+ \frac{1}{2} e^{-\lambda(x_1-x_1') - \lambda^2(x_2-x_2')} \theta(x_2' - x_2) \varphi_2(x) \psi_2(x') \delta_{\lambda_1 = \kappa_2}. \quad (2.48)
\end{align*} \]

Combining (2.47) and (2.48), we prove the lemma. \qed

3 The eigenfunction and spectral transformation

Based on the characterization of the Green function \( G \), we can provide the \( \partial \) data of \( m \) in Theorem 14 and 2.
Theorem 1. If \( \partial^k v_0 \in L^1 \cap L^\infty, \ |k| \leq 2, \ |v_0|_{L^1 \cap L^\infty} \ll 1, v_0(x) \in \mathbb{R}, \) then for fixed \( \lambda \in \mathbb{C} \setminus \{0, \kappa_1, \kappa_2\}, \) there is a unique solution \( m(x, \lambda) \) to the spectral equation

\[
Lm(x, \lambda) = -v_0(x)m(x, \lambda),
\]

\[
\lim_{|x| \to \infty} (m(x, \lambda) - \chi(x, \lambda)) = 0,
\]

(3.1)

where the spectral operator \( L \) and \( \chi \) are defined by (2.4) and (2.7).

Moreover, \( m(x, \lambda) = m(x, \lambda) \), and for fixed \( x \in \mathbb{R}^2 \), \( m(x, \lambda) \) satisfies

\[
|(1 - E_0)m(x, \lambda)| \leq C|v_0|_{L^1 \cap L^\infty},
\]

(3.2)

\[
m(x, \lambda) = \frac{m_{\kappa_1, 0}(x)}{1 - \gamma \cot^{-1} \frac{\lambda - \kappa_1}{|\lambda|}}, \quad m_{\kappa_1, 0}(x, \lambda) = \frac{\Theta_1(x)}{1 - \gamma \cot^{-1} \frac{\lambda - \kappa_1}{|\lambda|}},
\]

(3.3)

\[
m_{\kappa_1, 0}(x, \lambda) = \frac{\Theta_1(x)}{1 - \gamma \cot^{-1} \frac{\lambda - \kappa_1}{|\lambda|}}.
\]

(3.4)

\[
\Theta_1(x) = (1 + \Theta_1 * v_0)^{-1} \chi_1 \in \mathbb{R},
\]

\[
\gamma = -\frac{1}{2} \int \xi_1(x)v_0(x) \Theta_1(x)dx \in \mathbb{R}.
\]

(3.5)

Proof. Step 1 (Proof of (3.1)-(3.3)): Applying Proposition 2.1 and the assumption \( \partial^k v_0 \in L^1 \cap L^\infty, \ 0 \leq |k| \leq 2, \ |v_0|_{L^1 \cap L^\infty} \ll 1, \) for \( \lambda \neq 0, \) one can prove the unique solvability of the integral equation

\[
m(x, \lambda) = \chi(x, \lambda) - G * v_0 m(x, \lambda), \quad m(x, \lambda) \in L^\infty.
\]

(3.6)

where the * operator is defined by (2.15). Besides, from (2.7) and Lemma 2.2, the unique solvability of (3.1) is equivalent to that of (3.6).

Applying (3.6) and Proposition 2.1

\[
|(1 - E_0)m(x, \lambda)| = |(1 + G * v_0)^{-1}(1 - E_0)\chi| \leq C|v_0|_{L^1 \cap L^\infty}.
\]

So (3.2) is justified. Similarly, for \( \lambda \in D_0, \) using (2.4), (3.6), and Proposition 2.1

\[
m = (1 + G * v_0)^{-1}(\frac{\lambda_0}{\lambda} + [\chi - \frac{\lambda_0}{\lambda}]).
\]
One has
\[
m_{res} = (1 + G_0^* \ast v_0)^{-1} \chi_0
\]
\[
m_0, r = (1 + G \ast v_0)^{-1} \left[ \chi - \frac{\Theta}{\lambda} \right] + (1 + G \ast v_0)^{-1} \frac{G \ast G_0^*}{\lambda} \ast v_0 (1 + G_0^* \ast v_0)^{-1} \chi_0.
\]
(3.7)

So (3.3) follows.

Step 2 (Proof of (3.3)-(3.5)) : For \( \lambda = \lambda_R + i \lambda_I \in D_{\kappa_j}, j = 1, 2 \), applying (2.14), (3.6), and defining
\[
\varphi_j(x, x', \alpha) = \begin{cases} 1 + \left[ \mathcal{G}_1 + \frac{1}{\pi} \chi_1(x) \xi_1(x') \cot^{-1} \frac{\lambda_R - \kappa_1}{|\lambda_I|} \right] \ast v_0, \\ 1 + \left[ \mathcal{G}_2 - \frac{1}{\pi} \chi_2(x) \xi_2(x') \cot^{-1} \frac{\kappa_2 - \lambda_R}{|\lambda_I|} \right] \ast v_0, \end{cases}
\]
(3.8)
\[
\Theta_j(x) = \left[ 1 + \mathcal{G}_j(x, x') \ast v_0(x') \right]^{-1} \chi_j(x'),
\]
\[
\gamma_1 = -\frac{1}{\pi} \int \xi_1(x') v_0(x) \Theta_1(x) dx,
\]
\[
\gamma_2 = \frac{1}{\pi} \int \xi_2(x') v_0(x) \Theta_2(x) dx,
\]

one has
\[
m(x, \lambda) = (1 + \varphi_j^{-1} \omega_j \ast v_0)^{-1} \varphi_j^{-1} \chi(x, \lambda),
\]
and
\[
\begin{align*}
\uparrow m_{\kappa_{1,0}}(x, \lambda) &= \varphi_1^{-1} \chi_1(x, \lambda) \\
&= (1 + \left[ \mathcal{G}_1 + \frac{1}{\pi} \chi_1(x) \xi_1(x') \cot^{-1} \frac{\lambda_R - \kappa_1}{|\lambda_I|} \right] \ast v_0)^{-1} \chi_1 \\
&= \left[ 1 + \mathcal{G}_1 \ast v_0 \right]^{-1} \chi_1 \\
&\quad + \left[ 1 + \mathcal{G}_1 \ast v_0 \right]^{-1} \frac{1}{\pi} \chi_1(x) \xi_1(x') \cot^{-1} \frac{\lambda_R - \kappa_1}{|\lambda_I|} \ast v_0 \left[ 1 + \mathcal{G}_1 \ast v_0 \right]^{-1} \chi_1 \\
&\quad + \left[ 1 + \mathcal{G}_1 \ast v_0 \right]^{-1} \frac{1}{\pi} \chi_1(x) \xi_1(x') \cot^{-1} \frac{\lambda_R - \kappa_1}{|\lambda_I|} \ast v_0 \left[ 1 + \mathcal{G}_1 \ast v_0 \right]^{-1} \chi_1 + \cdots \\
&= \Theta_1(x) + \gamma_1 \cot^{-1} \frac{\lambda_R - \kappa_1}{|\lambda_I|} \Theta_1 + \left( \gamma_1 \cot^{-1} \frac{\lambda_R - \kappa_1}{|\lambda_I|} \right)^2 \Theta_1 + \cdots \\
&= \frac{\Theta_1(x)}{1 - \gamma_1 \cot^{-1} \frac{\lambda_R - \kappa_1}{|\lambda_I|}}.
\end{align*}
\]
\[
\begin{align*}
\uparrow m_{\kappa_{2,0}}(x, \lambda) &= \varphi_2^{-1} \chi_2(x, \lambda) \\
&= (1 + \left[ \mathcal{G}_2 + \frac{1}{\pi} \chi_2(x) \xi_2(x') \cot^{-1} \frac{\kappa_2 - \lambda_R}{|\lambda_I|} \right] \ast v_0)^{-1} \chi_2 \\
&= \left[ 1 + \mathcal{G}_2 \ast v_0 \right]^{-1} \chi_2 \\
&\quad + \left[ 1 + \mathcal{G}_2 \ast v_0 \right]^{-1} \frac{1}{\pi} \chi_2(x) \xi_2(x') \cot^{-1} \frac{\kappa_2 - \lambda_R}{|\lambda_I|} \ast v_0 \left[ 1 + \mathcal{G}_2 \ast v_0 \right]^{-1} \chi_2 \\
&\quad + \left[ 1 + \mathcal{G}_2 \ast v_0 \right]^{-1} \frac{1}{\pi} \chi_2(x) \xi_2(x') \cot^{-1} \frac{\kappa_2 - \lambda_R}{|\lambda_I|} \ast v_0 \left[ 1 + \mathcal{G}_2 \ast v_0 \right]^{-1} \chi_2 + \cdots \\
&= \Theta_2 + \gamma_2 \cot^{-1} \frac{\kappa_2 - \lambda_R}{|\lambda_I|} \Theta_2 + \left( \gamma_2 \cot^{-1} \frac{\kappa_2 - \lambda_R}{|\lambda_I|} \right)^2 \Theta_2 + \cdots \\
&= \frac{\Theta_2(x)}{1 - \gamma_2 \cot^{-1} \frac{\kappa_2 - \lambda_R}{|\lambda_I|}}.
\end{align*}
\]
To investigate the symmetries between $\Theta_j$ and $\gamma_j$, we combining (2.39) with (2.7), (2.14), and

$$
\chi_2(x) = - \frac{\kappa_1}{\kappa_2} \chi_1(x) e^{(\kappa_1-\kappa_2)x_1 + (\kappa_1^2-\kappa_2^2)x_2 - \ln a},
$$

$$
\xi_2(x) = \frac{\kappa_2}{\kappa_1} \xi_1(x) e^{-(\kappa_1-\kappa_2)x_1 - (\kappa_1^2-\kappa_2^2)x_2 + \ln a},
$$

we obtain

$$
\Theta_2(x) = - \frac{\kappa_1}{\kappa_2} e^{(\kappa_1-\kappa_2)x_1 + (\kappa_1^2-\kappa_2^2)x_2 - \ln a} \Theta_1(x),
$$

$$
\gamma_1 = \gamma_2 = \gamma
$$

which, combining with (2.26), prove (3.4).

\[\square\]

**Theorem 2.** Suppose $\partial^k_x v_0 \in L^1 \cap L^\infty$, $|k| \leq 2$, $|v_0|_{L^1 \cap L^\infty} \ll 1$, and $v_0(x) \in \mathbb{R}$. Then

$$
\partial^k_x m(x, \lambda) = s_c(\lambda) e^{(\lambda-\lambda)x_1 + (\lambda^2-\lambda^2)x_2} m(x, \lambda) , \quad \lambda \neq 0,
$$

with

$$
s_c(\lambda) = \frac{\text{sgn}(\lambda)}{2\pi i} \int e^{-[\lambda-\lambda]x_1 + (\lambda^2-\lambda^2)x_2} \xi(x, \lambda) v_0(x) m(x, \lambda) dx
$$

$$
\equiv \frac{\text{sgn}(\lambda)}{2\pi i} \xi v_0 m(\lambda, \lambda).
$$

Moreover, if $\partial^k_x v_0 \in L^1 \cap L^\infty$, $|k| \leq 2$, then

$$
|(1 - E_{D_{\kappa_1}} \cup D_{\kappa_2}) s_c|_{L^2((\lambda_1, \lambda_2) \cap L^\infty) \leq C \sum_{|k| \leq 2} |\partial^k_x v_0|_{L^1 \cap L^\infty},
$$

and if $(1 + |x|) \partial^k_x v_0 \in L^1 \cap L^\infty$, $|k| \leq 2$, then

$$
s_c(\lambda) = \begin{cases} 
+ \frac{\text{sgn}(\lambda)}{\lambda-\lambda} & + \frac{\text{sgn}(\lambda)}{\lambda-\lambda} + \text{sgn}(\lambda) h_1(\lambda), \quad \lambda \in D^\kappa_1, \\
+ \frac{\text{sgn}(\lambda)}{\lambda-\lambda} & + \frac{\text{sgn}(\lambda)}{\lambda-\lambda} + \text{sgn}(\lambda) h_2(\lambda), \quad \lambda \in D^\kappa_2, \\
\text{sgn}(\lambda) h_0(\lambda), & \lambda \in D^\kappa_0,
\end{cases}
$$

where $E_{\mu, \alpha}, D^\kappa_\alpha$, $\cot^{-1} \frac{\kappa_2-\lambda}{|\lambda|}$, $\cot^{-1} \frac{\kappa_2-\lambda}{|\lambda|}$, $\gamma$ are defined by Definitions 1, (2.26), and (3.5). Moreover,

$$
|\gamma|_{L^\infty} \leq |v_0|_{L^1}, \quad \sum_{0 \leq |\mu| \leq 1} \partial^k_{\lambda_1} \partial^k_{\lambda_2} h_k|_{L^\infty} \leq C |(1 + |x|) v_0|_{L^1 \cap L^\infty},
$$

$$
\sum_{0 \leq |\mu| \leq 1} \partial^k_{\lambda_1} \partial^k_{\lambda_2} h_k(\lambda) = h_k(\lambda).
$$
Proof. Step 1 (Proof of (3.10)): Denote \( \rho(x, \lambda) = e^{i(\lambda - \lambda)x_1 + (\lambda^2 - \lambda^2)x_2} \). Note \( \rho(x, \lambda) \) is annihilated by the heat operator \( p_\lambda(D) \equiv -\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1} \). So \( p_\lambda(D)f = e^{i(\lambda - \lambda)x_1 + (\lambda^2 - \lambda^2)x_2} p_\lambda(D)e^{-(\lambda - \lambda)x_1 + (\lambda^2 - \lambda^2)x_2}f \) which yields
\[
G_\lambda \rho(x, \lambda) = \rho(x, \lambda) G_\lambda.
\] (3.15)
Therefore, for \( \lambda_I \neq 0 \), by Lemma 2.1 (3.6), (3.11), and (3.15),
\[
\begin{align*}
\partial_\lambda & m(x, \lambda) \\
& = \partial_\lambda \left[ (1 + G_\lambda \ast v_0)^{-1} \chi \right] \\
& = -(1 + G_\lambda \ast v_0)^{-1} (\partial_\lambda G_\lambda \ast v_0) m(x, \lambda) \\
& = -(1 + G_\lambda \ast v_0)^{-1} sgn(\lambda_I) \rho(x, \lambda) \chi(x, \lambda) \ast v_0 m \\
& = s_c(\lambda)(1 + G_\lambda \ast v_0)^{-1} \rho(x, \lambda) \chi(x, \lambda) \\
& = s_c(\lambda) \rho(x, \lambda)(1 + G_\lambda \ast v_0)^{-1} \chi(x, \lambda) \\
& = s_c(\lambda)e^{i(\lambda - \lambda)x_1 + (\lambda^2 - \lambda^2)x_2} m(x, \lambda).
\end{align*}
\]
Step 2 (Proof of (3.13), (3.14)): From (2.8), (3.4), (3.5), and (3.11),
\[
\begin{align*}
s_c(\lambda) &= sgn(\lambda_I) \int \int e^{-(\lambda - \lambda)x_1 + (\lambda^2 - \lambda^2)x_2} \xi(x, \lambda) v_0(x) m(x, \lambda) dx \\
& \quad \left( \begin{array}{ll}
\frac{\text{sgn}(\lambda_I)}{\lambda^2} \int \xi_1(x) v_0(x) \Theta_1(x) 1 - \gamma \cot^{-1} \frac{\lambda}{|\lambda_I|} \ dx \\
+ \frac{\text{sgn}(\lambda_I)}{\lambda - \lambda_1} \int \xi_2(x) v_0(x) \Theta_2(x) 1 - \gamma \cot^{-1} \frac{\lambda}{|\lambda_I|} \ dx \\
+ \frac{\text{sgn}(\lambda_I)}{\lambda - \lambda_2} \int \xi(x, 0) v_0(x) m_{res}(x) dx \\
\end{array} \right) \text{sgn}(\lambda_I) h_0(\lambda), \\
& \quad \begin{cases}
\lambda \in D_0^\frown \\
\lambda \in D_{\eta_1}^\frown \\
\lambda \in D_{\eta_2}^\frown
\end{cases}
\end{align*}
\]
which is (3.13) and estimates for (3.14) can be derived directly.

Step 3 (Proof of (3.12)): Using a similar argument as in Step 2, one can prove \( |E_0 s_c|_{L^\infty} \leq C \) as well. Moreover, from (3.11), the Fourier theory, and Theorem \( \Box \)
\[
\begin{align*}
& \int \int |(1 - E_{D_{\eta_1} \cup D_{\eta_2}}(\lambda)) s_c(\lambda)|^2 |\lambda_I| d\lambda \wedge d\lambda \\
& \leq C \int \int \sum_{k,j \leq 2} |\partial_{x_1}^k \partial_{x_2}^j v_0|^2 |\lambda_I| d\lambda \wedge d\lambda \\
& \leq C \sum_{k,j \leq 2} |\partial_{x_1}^k \partial_{x_2}^j v_0|^2 |\lambda_I| d\lambda_L \wedge L^\infty.
\end{align*}
\]
Based on the characterization of the eigenfunction \( m \), we define the eigenfunction space \( W \equiv W_x \) and the spectral transformation \( T \) in Definition [2] and [3].

**Definition 2.** The eigenfunction space \( W \equiv W_x \) is the set of functions

1. \( \phi(x, \lambda) = \phi(x, \lambda) \);
2. \( (1 - E_0)\phi(x, \lambda) \in L^\infty \);
3. \( \phi(x, \lambda) = \phi_{res}(x) + \phi_{0,r}(x, \lambda), \lambda \in D_0^\times \),
   \[ \phi_{res}(x), \lambda \phi_{0,r}(x, \lambda), \frac{\phi_{0,r}(x, \lambda)}{1 + |x|} \in L^\infty(D_0); \]
4. \( \phi(x, \lambda_2) = s_d e^{(\kappa_1 - \kappa_2)x_1 + (\kappa_2^2 - \kappa_1^2)x_2} \phi(x, \lambda_1), \)
   \( \lambda_2 = \kappa_2 + 0^+ e^{i\alpha}, \lambda_1 = \kappa_1 + 0^+ e^{i(x + \alpha)}, s_d = -\frac{\kappa_1}{\kappa_2} e^{-\ln a}, \)
   \[ \phi(x, \lambda), \frac{\phi(x, \lambda)}{1 + |x|} \in L^\infty(D_{\kappa_2}). \]

**Definition 3.** Define \( \{0; \kappa_1, \kappa_2, s_d, s_c(\lambda)\} \) as the set of scattering data, where 0, location of the simple pole, \( \kappa_j \), location of discontinuities, and \( s_d \equiv -\frac{\kappa_1}{\kappa_2} e^{-\ln a} \), the norming constant, are the *discrete scattering data*; and \( s_c(\lambda) \), the *continuous scattering data*, is defined by (3.11). Denote \( T \) as the forward scattering transform by

\[ T(\phi)(x, \lambda) = s_c(\lambda) e^{(\lambda - \lambda_2)x_1 + (\lambda - \lambda_2)^2 x_2} \phi(x, \lambda). \]  (3.16)

**Definition 4.** Let \( C \) be the Cauchy integral operator defined by

\[ C(\phi)(x, \lambda) = C_\lambda(\phi) = -\frac{1}{2\pi i} \int \frac{\phi(x, \zeta)}{\zeta - \lambda} d\zeta \wedge d\zeta. \]  (3.17)

We now provide estimates on the spectral transform of \( m \) in order to formulate a Cauchy integral equation in Section [4].

**Theorem 3.** Suppose \( \partial_x^k v_0 \in L^1 \cap L^\infty, |k| \leq 2, |v_0|_{L^1 \cap L^\infty} \ll 1 \). Then

\[ |CTm|_{L^\infty} \leq C(1 + |x|) \sum_{|k| \leq 2} |\partial_x^k v_0|_{L^1 \cap L^\infty}, \]

\[ CTm(x, \lambda) \to 0, \text{ as } |\lambda| \to \infty, \lambda_I \neq 0. \]

**Proof.** Step 1 \((z \in \{\kappa_1, \kappa_2\})\): From (3.10), applying Stokes’ theorem,

\[ -\frac{1}{2\pi i} \int_{D_{\kappa_1} \cup D_{\kappa_2} \cup D_{\lambda, x}} \frac{\phi(x, \zeta)}{\zeta - \lambda} m(x, \zeta) d\zeta \wedge d\zeta = -\frac{1}{2\pi i} \int_{D_{\kappa_1} \cup D_{\kappa_2} \cup D_{\lambda, x}} \frac{\partial m(x, \zeta)}{\zeta - \lambda} d\zeta \wedge d\zeta \]

\[ = -\frac{1}{2\pi i} \int_{D_{\kappa_1} \cup D_{\kappa_2} \cup D_{\lambda, x}} \frac{m(x, \zeta)}{\zeta - \lambda} d\zeta \]

\[ = -\frac{1}{2\pi i} \int_{D_{\kappa_1} \cup D_{\kappa_2} \cup D_{\lambda, x}} \frac{m(x, \zeta)}{\zeta - \lambda} d\zeta + \frac{1}{2\pi i} \int_{\partial D_{\kappa_1}} \frac{m(x, \zeta)}{\zeta - \lambda} d\zeta + \frac{1}{2\pi i} \int_{\partial D_{\kappa_2}} \frac{m(x, \zeta)}{\zeta - \lambda} d\zeta \]

(3.18)
where \( k = \frac{1}{2} \min \{ |\kappa_1|, |\kappa_2|, \kappa_2 - \kappa_1 \} \) is defined by Definition 1. Note, by \( \lambda \neq \kappa_j \), (3.14), and (3.13),

\[
\begin{align*}
- \frac{1}{2\pi i} \int_{D_{\lambda,\epsilon}} s_c(\zeta) e^{\zeta - \xi_1 + (\zeta - \zeta)^2} x m(x, \lambda) d\zeta \wedge d\xi & \to 0, \\
- \frac{1}{2\pi i} \int_{D_{\lambda,\epsilon}} s_c(\zeta) e^{\zeta - \xi_1 + (\zeta - \zeta)^2} x m(x, \lambda) d\zeta \wedge d\zeta & \to 0, \\
\frac{1}{2\pi i} \int_{\partial D_{\lambda,\epsilon}} m(x, \lambda) d\zeta & \to 0, \\
\frac{1}{2\pi i} \int_{\partial D_{\lambda,\epsilon}} m(x, \lambda) d\zeta & \to m(x, \lambda), \text{ as } \epsilon \to 0.
\end{align*}
\]

Therefore

\[
|CTE_m|_{L^\infty} = \left| \left| - \frac{1}{2\pi i} \int \frac{E_{x, s_c(\zeta)} e^{\zeta - \xi_1 + (\zeta - \zeta)^2} x m(x, \lambda)}{\zeta - \lambda} d\zeta \wedge d\xi \right| \right|_{L^\infty}
\]

\[
= |m(x, \lambda)| - \frac{1}{2\pi i} \int_{|\zeta - \lambda| = \kappa} \frac{m(x, \xi)}{\zeta - \lambda} d\zeta \left| L^\infty \right|
\]

\[
\leq C |v_0|_{L^1 \cap L^\infty}.
\]

**Step 2 (Near 0):** From (3.10),

\[
\partial_x \left( m(x, \lambda) - \frac{m_0(\lambda)}{\lambda} \right) = s_c(\lambda) e^{(\lambda - \lambda)x_1 + (\lambda^2 - \lambda^2)x_2 m(x, \lambda)}, \lambda \in D_0 / \mathbb{R}.
\]

Applying Stokes’ theorem,

\[
\begin{align*}
- \frac{1}{2\pi i} \int_{D_{\lambda,\epsilon}} s_c(\zeta) e^{\zeta - \xi_1 + (\zeta - \zeta)^2} x m(x, \lambda) d\zeta \wedge d\xi & \to 0, \\
- \frac{1}{2\pi i} \int_{D_{\lambda,\epsilon}} s_c(\zeta) e^{\zeta - \xi_1 + (\zeta - \zeta)^2} x m(x, \lambda) d\zeta \wedge d\zeta & \to 0, \\
\frac{1}{2\pi i} \int_{\partial D_{\lambda,\epsilon}} m(x, \lambda) d\xi & \to 0, \\
\frac{1}{2\pi i} \int_{\partial D_{\lambda,\epsilon}} m_0(x, \lambda) d\xi & \to m_0(x, \lambda), \text{ as } \epsilon \to 0.
\end{align*}
\]

Note, for \( x \in \mathbb{R}^2, \lambda \neq 0 \) fixed, (3.13), and (3.13),

\[
\begin{align*}
- \frac{1}{2\pi i} \int_{D_{\lambda,\epsilon}} s_c(\zeta) e^{\zeta - \xi_1 + (\zeta - \zeta)^2} x m(x, \lambda) d\zeta \wedge d\xi & \to 0, \\
- \frac{1}{2\pi i} \int_{D_{\lambda,\epsilon}} s_c(\zeta) e^{\zeta - \xi_1 + (\zeta - \zeta)^2} x m(x, \lambda) d\zeta \wedge d\zeta & \to 0, \\
\frac{1}{2\pi i} \int_{\partial D_{\lambda,\epsilon}} m_0(x, \lambda) d\xi & \to 0, \\
\frac{1}{2\pi i} \int_{\partial D_{\lambda,\epsilon}} m_0(x, \lambda) d\zeta & \to m_0(x, \lambda), \text{ as } \epsilon \to 0.
\end{align*}
\]
Therefore

\[
|CTE_0m|_{L^\infty} = | - \frac{1}{2\pi i} \int \frac{E_0s_c(\zeta)e^{i\zeta x_1}(\zeta^2 - \zeta^2)x_2 m(x,\zeta)}{\zeta - \lambda} d\zeta|_{L^\infty} \\
= |m_0, r(x, \lambda) - \frac{1}{2\pi i} \int_{|\zeta| = \kappa} \frac{m_0, r(x, \zeta)}{\zeta - \lambda} d\zeta|_{L^\infty} \\
\leq C(1 + |x|) |v_0|_{L^1 L^\infty},
\]

(3.22)

Step 3 (Near \(\infty\)): The proof can be applied to \(\forall \phi \in W\). Via a change of variables

\[
2\pi i \xi = \zeta - \bar{\zeta}, \quad 2\pi i \eta = \bar{\zeta} - \zeta^2, \\
\zeta = -i \pi \xi + \frac{\eta}{2\pi}, \quad d\bar{\zeta} \wedge d\zeta = \frac{i}{|\zeta|} d\xi d\eta,
\]

(3.23)

and from (3.12), [20, Lemma 2.II]

\[
p_{\lambda}(\xi, \eta) = (2\pi \xi)^2 - 4\pi i \xi \lambda + 2\pi i \eta, \quad \Omega_{\lambda} = \{ (\xi, \eta) \in \mathbb{R}^2 : |p_{\lambda}(\xi, \eta)| < 1 \}, \\
\left| \frac{1}{p_{\lambda}} \right|_{L^1(\Omega_{\lambda}, d\xi d\eta)} \leq \frac{C}{(1 + |\lambda|^2)^{1/4}} \leq \frac{1}{p_{\lambda}} \left| \frac{1}{L^2(\Omega_{\lambda}, d\xi d\eta)} \right| \\
\leq \frac{C}{(1 + |\lambda|^2)^{1/4}}
\]

(3.24)

along with (3.22) and (3.12), we obtain

\[
|C[1 - E_{D_0 \cup D_{\kappa_1} \cup D_{\kappa_2}}]T \phi| \\
\leq C \left| \int \frac{[1 - E_{D_0 \cup D_{\kappa_1} \cup D_{\kappa_2}}(\zeta)] s_c(\zeta)e^{i(\zeta^2 - \zeta^2)x_2} \phi} {\zeta - \lambda} d\zeta \wedge d\zeta} \right| \\
\leq C \left| \frac{[1 - E_0]}{L^\infty} \int \frac{[1 - E_{D_0 \cup D_{\kappa_1} \cup D_{\kappa_2}}(\zeta)] s_c(\zeta)} {(2\pi)^2 - 4\pi i \xi \lambda + 2\pi i \eta} d\xi d\eta \right| \\
\leq C \left| \frac{[1 - E_0]}{L^\infty} \{ [1 - E_{D_0} (\zeta)] s_c(\zeta) \} L^2(d\xi d\eta) \left| \frac{1}{p_{\lambda}} \right| L^2(\Omega_{\lambda}, d\xi d\eta) \right| \\
\leq C |\phi|_{L^\infty} \{ [1 - E_{D_0} (\zeta)] s_c(\zeta) \} L^\infty(d\xi d\eta) \left| \frac{1}{p_{\lambda}} \right| L^1(\Omega_{\lambda}, d\xi d\eta) \}
\]

(3.25)

and tends to 0 as \(|\lambda| \to \infty, \lambda I \neq 0\).

We make several remarks about Theorem 3 since it is necessary for the justification of a Cauchy integral equation for \(m\).

- Due to (3.12), there is a missing direction in the \(\lambda\)-plane (the real axis) for \(s_c(\lambda)\) to decay no matter how smooth the initial data \(v_0(x)\) is. Therefore, boundedness of \(m(x, \lambda)\) on \(\lambda \in D_0^c \cap D_{\kappa_1}^c \cap D_{\kappa_2}^c\) is vital in deriving uniform estimates there.
Boiti-Pempinelli-Pogrebkov’s eigenfunction [5], defined by
\[ \lambda m(x, \lambda), \]
which is not bounded near \( \infty \), cannot be admissible.

In [21], via a KdV approach, the boundary condition, i.e., the Sato eigenfunction \( \varphi(x, \lambda) \), is replaced by the KdV eigenfunction \( \psi_-(x, \lambda) \)

\[ \psi_-(x, \lambda) = (1 + \frac{2i\kappa}{\lambda-i\kappa} \frac{1}{1+e^{2ixi\kappa}})e^{(-i\lambda)x} + (\lambda-i\lambda)^2 \]

So the eigenfunction for the boundary value problem of the Lax equation is
\[ \frac{\lambda}{\lambda-i\kappa} \tilde{m}(x, \lambda), \quad \tilde{m} \in L^\infty, \]
whileas \( i\kappa \) is also a singularity of the corresponding \( \tilde{s}_c \). Therefore, a standard Cauchy Theorem (shown in the proof of (3.19)) will collapse at \( D_{i\kappa, \epsilon}, \partial D_{i\kappa, \epsilon} \).

We then introduced a regularization at \( i\kappa \)
\[ (\lambda - i\kappa) \frac{\lambda}{\lambda-i\kappa} \tilde{m}(x, \lambda) \]  (3.26)

to remedy the problem near \( i\kappa \). But (3.26) becomes unbounded at \( \infty \).

Finally, we introduce an extra pole \( 2i\kappa \) to tame the singularities at \( \infty \) and obtain
\[ \frac{\lambda-i\kappa}{\lambda-2i\kappa} \frac{\lambda}{\lambda-i\kappa} \tilde{m}(x, \lambda) \]  (3.27)

The eigenfunction (3.27) is admissible.

4 The Cauchy integral equation

Theorem 4. If
\[ u_0(x) = \frac{(\kappa_1-\kappa_2)^2}{2} \text{sech}^2 \frac{\theta_1-\theta_2-\ln a}{2}, \]
\[ \partial_t^k v_0 \in L^1 \cap L^\infty, \quad |k| \leq 4, \]
\[ |v_0|_{L^1 \cap L^\infty} \ll 1, \quad v_0(x) \in \mathbb{R}, \]
then the eigenfunction \( m \) derived from Theorem 4 satisfies
\[ m(x, \lambda) \in W \]  (4.1)
and the Cauchy integral equation
\[ m(x, \lambda) = 1 + \frac{m_{\text{res}}(x)}{\lambda} + CTm, \quad \forall \lambda \neq 0, \]  (4.2)
where \( W \) is defined by Definition 2.
Proof. Theorem 1 implies, for $x$ fixed,
\begin{equation}
m(x, \lambda) - \frac{m_{res}(x)}{\lambda} \in L^\infty, \tag{4.3}
\end{equation}
\begin{equation}
E_{0,n}Tm(x, \lambda) \in L^1(d\vec{\lambda} \wedge d\lambda), \tag{4.4}
\end{equation}
for $\forall n > 0$. Here $E_{z,a}$ is defined by Definition 1. Exploiting (4.4) and applying [13, §1, Theorem 1.13, Theorem 1.14], one derives
\begin{equation}
\partial_x CE_{0,n}Tm(x, \lambda) = E_{0,n}Tm(x, \lambda) \in L^1(d\vec{\lambda} \wedge d\lambda). \tag{4.5}
\end{equation}

Therefore, together with Theorem 2, Theorem 3,
\begin{equation}
\partial_x \left[ m(x, \lambda) - \frac{m_{res}(x)}{\lambda} - CTm(x, \lambda) \right] = 0. \tag{4.6}
\end{equation}

For $\forall x$ fixed, applying Theorem 3 (4.3), (4.6), and Liouville’s theorem, one concludes
\begin{equation}
m(x, \lambda) = g(x) + \frac{m_{res}(x)}{\lambda} + CTm(x, \lambda). \tag{4.7}
\end{equation}

Equation (3.1) and a direct computation yield:
\begin{equation}
\begin{aligned}
u(x)m(x, \lambda) & = (\partial_{x_2} - \partial_{x_1}^2 - 2\lambda \partial_{x_2}) m(x, \lambda) \\
& = (\partial_{x_2} - \partial_{x_1}^2 - 2\lambda \partial_{x_1}) \left[ g(x) + \frac{m_{res}(x)}{\lambda} \right] \\
& + (\partial_{x_2} - \partial_{x_1}^2 - 2\lambda \partial_{x_1}) CTm.
\end{aligned} \tag{4.8}
\end{equation}

Note that
\begin{align*}
\partial_{x_1} CTm &= C[(\vec{\lambda} - \lambda)Tm + T(\partial_{x_1} m)], \\
\partial_{x_1}^2 CTm &= C[(\vec{\lambda} - \lambda)^2Tm + 2(\vec{\lambda} - \lambda)T(\partial_{x_1} m) + T(\partial_{x_1}^2 m)], \\
\partial_{x_2} CTm &= C[(\vec{\lambda}^2 - \lambda^2)Tm + T(\partial_{x_2} m)].
\end{align*}

Applying the Fourier transform theory, if $v_0(x)$ has 4 derivatives in $L^1 \cap L^\infty$, then
\begin{align*}
1 - E_{Dx_1 \cup Dx_2}(\lambda)(\vec{\lambda} - \lambda)s_c(\lambda), \\
1 - E_{Dx_1 \cup Dx_2}(\lambda)(\vec{\lambda} - \lambda)^2s_c(\lambda), \\
1 - E_{Dx_1 \cup Dx_2}(\lambda)(\vec{\lambda}^2 - \lambda^2)s_c(\lambda),
\end{align*}
are all bounded in $L^\infty \cap L^2(\lambda_I|d\vec{\lambda} \wedge d\lambda)$. Therefore, if $\partial_x^k v_0 \in L^1 \cap L^\infty$, $0 \leq |k| \leq 4$, one can adapt the proof of Step 1 - Step 3 in Theorem 3 and derive, as $|\lambda| \to \infty$, $\lambda_I \neq 0$,
\begin{equation}
(\partial_{x_2} - \partial_{x_1}^2 - 2\lambda \partial_{x_1}) CTm \to o(|\lambda|).
\end{equation}
So comparing growth in (4.8), we conclude (4.7) turns into
\[
m(x, \lambda) - 1 = g(x_2) - 1 + \frac{m_{res}(x)}{\lambda} + C T m(x, \lambda).
\] (4.9)

Fix \(x_2\), and let \(\epsilon\) be given. Let \(x_1 \gg 1, |\lambda| \gg 1, \lambda_I \neq 0\), one has
\[
|\frac{m_{res}(x)}{\lambda} + C T m(x, \lambda)| \leq \frac{\epsilon}{2}
\]
by Theorem [3] and
\[
|m(x, \lambda) - 1| \leq \frac{\epsilon}{2}
\]
by the boundary property (3.1). So we justify \(g \equiv 1\) and establish (4.2).

Theorem 4 implies that the residue \(m_{res}(x)\) at the simple pole \(\lambda = 0\) and \(m(x, \kappa_j + 0^+ e^{i\alpha})\) at \(\kappa_j\) satisfy the linear system
\[
\begin{align*}
\frac{m_{res}(x)}{\kappa_1} &= -1 + m(x, \kappa_1 + 0^+ e^{i\alpha}) - C_{\kappa_1+0^+ e^{i\alpha}} T m, \\
\frac{m_{res}(x)}{\kappa_2} &= -1 + m(x, \kappa_2 + 0^+ e^{i\alpha}) - C_{\kappa_2+0^+ e^{i\alpha}} T m, \\
m(x, \kappa_2 + 0^+ e^{i\alpha}) &= s_d e^{(\kappa_1-\kappa_2)x_1 + (\kappa_1^2-\kappa_2^2)x_2} m(x, \kappa_1 + 0^+ e^{i\alpha})
\end{align*}
\] (4.10)
for \(\forall 0 < \alpha < 2\pi\), with \(T, s_d\), and \(C\) defined by Definition [3].

**Example 4.1.** If \(s_c \equiv 0\), \(s_d = -\frac{s_1}{s_2} e^{-\text{ln} a}\). So (4.2) and (4.10) reduce to
\[
\begin{align*}
m(x, \lambda) &= 1 + \frac{m_{res}(x)}{\lambda}, \\
m(x, \kappa_2) &= s_d e^{(\kappa_1-\kappa_2)x_1 + (\kappa_1^2-\kappa_2^2)x_2} m(x, \kappa_1)
\end{align*}
\]
which yield
\[
\begin{align*}
\frac{m_{res}(x)}{\kappa_1} &= -1 + m(x, \kappa_1), & (4.11) \\
\frac{m_{res}(x)}{\kappa_2} &= -1 + m(x, \kappa_2), & (4.12) \\
m(x, \kappa_2) &= s_d e^{(\kappa_1-\kappa_2)x_1 + (\kappa_1^2-\kappa_2^2)x_2} m(x, \kappa_1) & (4.13)
\end{align*}
\]

Namely,
\[
\begin{align*}
m(x, \kappa_1) &= +\frac{\kappa_1-\kappa_2}{\kappa_1} \frac{1}{1+e^{(\kappa_1-\kappa_2)x_1+(\kappa_1^2-\kappa_2^2)x_2-\text{ln} a}}, \\
m(x, \kappa_2) &= -\frac{\kappa_1-\kappa_2}{\kappa_2} \frac{1}{1+e^{-(\kappa_1-\kappa_2)x_1+(\kappa_1^2-\kappa_2^2)x_2-\text{ln} a)},
\end{align*}
\]
and
\[
\begin{align*}
m(x, \lambda) &= 1 - \frac{\kappa_1}{\lambda} (e^{-[(\kappa_1-\kappa_2)x_1+(\kappa_1^2-\kappa_2^2)x_2-\text{ln} a]} + \frac{\kappa_2}{1+e^{(\kappa_1-\kappa_2)x_1+(\kappa_1^2-\kappa_2^2)x_2-\text{ln} a})} \\
&= \chi(x, \lambda).
\end{align*}
\]
References

[1] M.J. Ablowitz, P.A. Clarkson: Solitons, nonlinear evolution equations and inverse scattering. London Mathematical Society lecture note series, 149 (1991), Cambridge University Press.

[2] M. Boiti, F. Pempinelli, A. Pogrebkov, B. Prinari: Inverse scattering transform for the perturbed 1-soliton potential of the heat equation. Phys. Lett. A 285 (2001), no. 5-6, 307-311.

[3] M. Boiti, F. Pempinelli, A. K. Pogrebkov, B. Prinari: Inverse scattering theory of the heat equation for a perturbed one-soliton potential. J. Math. Phys. 43 (2002), no. 2, 1044-1062.

[4] M. Boiti, F. Pempinelli, A. K. Pogrebkov: Green’s function of heat operator with pure soliton potential. ArXiv:1201.0152v1, 1-10, [nlin.SI] 30 Dec 2011.

[5] M. Boiti, F. Pempinelli, A. K. Pogrebkov: IST of KPII equation for perturbed multisoliton solutions. Topology, geometry, integrable systems, and mathematical physics, 49-73, Amer. Math. Soc. Transl. Ser. 2, 234, Amer. Math. Soc., Providence, RI, 2014.

[6] L. A. Dickey: Soliton equations and Hamiltonian systems. Advanced Series in Mathematical Physics, 12, (1991). World Scientific Publishing Co., Inc., River Edge, NJ.

[7] P. Griffiths, J. Harris: Principles of algebraic geometry, Pure and applied mathematics (1978) A Wiley-Interscience publication, John Wiley & Sons.

[8] Y. Kodama: KP solitons and the Grassmannians. Combinatorics and geometry of two-dimensional wave patterns. SpringerBriefs in Mathematical Physics, 22, (2017). Springer, Singapore.

[9] Y. Kodama: Solitons in two-dimensional shallow water. SIAM.

[10] V. B. Matveev: Darboux transformation and explicit solutions of the Kadomtcev-Petviashvily equation, depending on functional parameters. Lett. Math. Phys. 3 (1979), no. 3, 213216.

[11] T. Mizumachi: Stability of line solitons for the KP-II equation in $\mathbb{R}^2$, Mem. Amer. Math. Soc., 238 (2015), no. 1125, vii+95 pp.
[12] V. B. Matveev, M. A. Salle: Darboux transformations and solitons. *Springer series in nonlinear dynamics* (1991), Berlin ; New York : Springer-Verlag.

[13] L. Molinet, J. Saut, N. Tzvetkov: Global well-posedness for the KP-II equation on the background of a non-localized solution. *Ann. Inst. H. Poincare Anal. Non Lineaire* 28 (2011), no. 5, 653-676.

[14] S. Novikov, S. V. Manakov, L. P. Pitaevski, V. E. Zakharov: Theory of solitons. The inverse scattering method. *Contemporary Soviet Mathematics*, (1984), Consultants Bureau [Plenum], New York and London.

[15] S. G. Krantz: Function theory of several complex variables, *Pure and applied mathematics* (1982) A Wiley-Interscience publication, John Wiley & Sons.

[16] J. Satsuma: N-soliton solution of the two-dimensional Korteweg-deVries equation. *J. Phys. Soc. Japan* 40 (1976), no. 1, 286290.

[17] J. Satsuma: A Wronskian Representation of N-Soliton Solutions of Nonlinear Evolution Equations. *J. Phys. Soc. Japan* 46 (1979), no. 1, 359-360.

[18] I. N. Vekua, *Generalized analytic functions*, 1962, Pergamon Press, London-Paris-Frankfurt; Addison-Wesley Publishing Co., Inc., Reading, Mass.

[19] J. Villarroel, M. J. Ablowitz: On the initial value problem for the KPII equation with data that do not decay along a line. *Nonlinearity* 17 (2004), no. 5, 1843-1866.

[20] M. V. Wickerhauser: Inverse scattering for the heat operator and evolutions in $2 + 1$ variables. *Comm. Math. Phys.* 108 (1987), no. 1, 67-89.

[21] D. C. Wu: The direct problem for the perturbed Kadomtsev - Petviashvili II one line solitons. *arXiv:1807.01420* (2018) 1-33.