Reflection theorems for number rings
generalizing the Ohno-Nakagawa identity

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Abstract

The Ohno-Nakagawa (O-N) reflection theorem is an unexpectedly simple identity relating the number of \(GL_2\(\mathbb{Z}\)-classes of binary cubic forms (equivalently, cubic rings) of two different discriminants \(D, -27D\); it generalizes cubic reciprocity and the Scholz reflection theorem. In this paper, we present a new approach to this theorem using Fourier analysis on the adelic cohomology \(H^1(\mathbb{A}_K, M)\) of a finite Galois module, modeled after the celebrated Fourier analysis on \(\mathbb{A}_K\) used in Tate’s thesis. This method reduces reflection theorems of O-N type to local identities. We establish reflection theorems of O-N type for cubic forms and rings over arbitrary number fields, and also for quadratic forms counting by a peculiar invariant \(a(b^2 - 4ac)\). We also find relations for the number of forms over \(\mathbb{Z}[1/N]\) and for forms of highly non-squarefree discriminant (discriminant reduction).

In a sequel to this paper, we will deal with reflection theorems for quartic rings, \(2 \times 3 \times 3\) symmetric boxes, and binary quartic forms. In these cases the local step is much more involved.

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1 Introduction

In 1932, using the then-new machinery of class field theory, Scholz [44] proved that the class groups of the quadratic fields \(\mathbb{Q}(\sqrt{D})\) and \(\mathbb{Q}(\sqrt{-3D})\), whose discriminants are in the ratio \(-3\), have 3-ranks differing by at most 1. This is a remarkable early example of a reflection theorem. A related theorem due to Leopoldt [29] relates different components of the \(p\)-torsion of the class group of a number field containing \(\mu_p\) when decomposed under the Galois group of that field. Applications of such reflection theorems are far-ranging: for instance, Ellenberg and Venkatesh [20] use reflection theorems of Scholz type to prove upper bounds on \(\ell\)-torsion in class groups of number fields, while Mihăilescu [32] uses Leopoldt’s generalization to simplify a step of his monumental proof of the Catalan conjecture. Through the years, numerous reflection principles for different generalizations of ideal class groups have come into print. A very general reflection theorem for Arakelov class groups is due to Gras [23].

In 1997, a beautiful and unexpected generalization was conjectured by Ohno [41] on the basis of numerical data and proved by Nakagawa [35], for which reason we will call it the Ohno-Nakagawa (O-N) theorem. It relates the numbers of binary cubic forms, or equivalently cubic rings, of two discriminants which are in the ratio \(-27 = -3^3\):

**Theorem 1.1** (Ohno–Nakagawa). For a nonzero integer \(D\), let \(h(D)\) be the number of \(GL_2(\mathbb{Z})\)-orbits of binary cubic forms

\[ f(x, y) = ax^3 + bx^2y + cy^2 + dy^3 \]

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of discriminant $D$, each orbit weighted by the reciprocal of its number of symmetries (i.e. stabilizer in $\text{GL}_2(\mathbb{Z})$). Let $h_3(D)$ be the number of such orbits $f(x,y)$ such that the middle two coefficients $b,c$ are multiples of 3, weighted in the same way.

Then for every nonzero integer $D$, we have the exact identity

$$h_3(-27D) = \begin{cases} 3h(D), & D > 0 \\ h(D), & D < 0. \end{cases} \tag{1.1}$$

By the well-known index-form parametrization (see Theorem \ref{index-form} below), $h(D)$ also counts the cubic rings of discriminant $D$ over $\mathbb{Z}$, weighted by the reciprocal of the order of the automorphism group. It turns out that $h_3(D)$ counts those rings $C$ for which $3 | \text{tr}_C/\mathbb{Z} \xi$ for every $\xi \in C$ (see Proposition \ref{index-form}). When $D$ is a fundamental discriminant, we get back Scholz’s reflection theorem, as Nakagawa points out (35, Remark 0.9).

The O-N theorem was quite unexpected, because $\text{GL}_2(\mathbb{Z})$-orbits of binary cubics have been tabulated since Eisenstein without unearthing any striking patterns. Even the exact normalizations $h(D)$, $h_3(D)$ had been in use for over two decades: the functional equation for the Shintani zeta functions \cite{shintani} relates the generating function of one to the other. Thus, O-N allows the functional equation to be diagonalized into a self-reflective form \cite{taniguchi}. The Shintani zeta functions encapsulate the distribution of discriminants of cubic rings and fields, and their simple poles at 1 and 5/6 reflect the first- and second-order terms in counting such discriminants \cite{bhp} \cite{taniguchi}.

1.1 Main results of this paper

In this paper, we prove a generalization of O-N to binary cubic forms over $\mathcal{O}_K$, for all number fields $K$, verifying and extending a conjecture of Dioses \cite{dioses} Conjecture 1.1:

**Theorem 1.2** (O-N for traced binary cubic forms). Let

$$\mathcal{V}_{a,t}(\mathcal{O}_K) := \{ f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 : a \in a, b \in t, c \in ta^{-1}, d \in a^{-2} \},$$

a representation of

$$\mathcal{G}_a := \text{SL}(\mathcal{O}_K \oplus a).$$

For nonzero $D \in t^3a^{-2}$, define the class number

$$h_{a,t}(D) = \sum_{\Phi \in \mathcal{G}_a \backslash \mathcal{V}_{a,t}(\mathcal{O}_K)} \frac{1}{|\text{Stab}_{\mathcal{G}_a} \Phi|} \text{disc} \Phi = D$$

Then we have the global reflection theorem

$$h_{a,t}(D) = \frac{3\# \{ v | \infty \in (K_\gamma)^2 \}}{N_{K/Q}(t)} \cdot h_{a^3,t^{-3}l^{-1}}(-27D). \tag{1.2}$$

The theorem can also be interpreted as an equality between the numbers of cubic rings over $\mathcal{O}_K$ of constrained discriminant, Steinitz class, and trace ideal (Theorem \ref{O-N}). As a corollary, we derive an extra functional equation for the corresponding Shintani zeta functions (Corollary \ref{functional-equation}). We also prove a reflection theorem in the function-field setting, for covers of a curve (Theorem \ref{reflection}.

Although $h(D)$ are not orders of any class groups, we call them class numbers to generalize Gauss’s notion of class number, which counts $\text{SL}_2\mathbb{Z}$-equivalence classes of binary quadratic forms with fixed discriminant. When $D$ is squarefree in the appropriate sense, we get back versions of Scholz’s reflection theorem for number fields, which are well known (see Ellenberg–Venkatesh \cite{ellenberg-venkatesh}, Lemma 3.3). As a corollary to our methods, we prove some discriminant-reducing identities (Section \ref{identities}), useful tools in arithmetic statistics (see Bhargava, Taniguchi, and Thorne \cite{bhargava-taniguchi-thorne}) that eliminate troublesome discriminant divisibility conditions.

We also find a new reflection theorem counting binary quadratic forms, not by discriminant, but by an unusual invariant: the product $a(b^2 - 4ac)$ of the discriminant and the leading coefficient, which is not invariant under $\text{GL}_2$, but under a suitable solvable subgroup:

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Theorem 1.3 (“Quadratic O-N”). Let $K$ be a number field of class number 1. For $\tau$ a divisor of 2 and $I \neq 0$, let $V_r(I)(\mathcal{O}_K)$ be the set of binary quadratic forms $f = ax^2 + bxy + cy^2$ over $\mathcal{O}_K$ with $\tau b$ and

$$I(f) = a(b^2 - 4ac) = I,$$

a set with an action of the algebraic group

$$\Gamma(\mathcal{O}_K) = \left\{ \begin{bmatrix} u & t \\ 0 & u^{-2} \end{bmatrix} : u \in \mathcal{O}_K^\times, t \in \mathcal{O}_K \right\}.$$

Then for any $I, \tau \in \mathcal{O}_K$ with $\tau \mid 2$,

$$\sum_{f \in \Gamma(\mathcal{O}_K) \setminus V_r(I)(\mathcal{O}_K)} \frac{1}{|\text{Stab}_{\Gamma(\mathcal{O}_K)}f|} = \frac{|N_{K/Q}(\tau)|}{2^{r_2(K)}} \sum_{f \in \Gamma(\mathcal{O}_K) \setminus V_{2r-1}(4\tau - 4I)(\mathcal{O}_K)} \frac{1}{|\text{Stab}_{\Gamma(\mathcal{O}_K)}f|},$$

where $r_2(K)$ is the number of complex places of $K$.

This can be interpreted as a relation between two-variable Shintani zeta functions. Over $\mathbb{Z}$, the statement becomes delightfully elementary:

Theorem 1.4 (“Quadratic O-N”). If $n$ is a nonzero integer, let $q(n)$ be the number of integer quadratic polynomials $f(x) = ax^2 + bx + c$ with

$$I(f) = a(b^2 - 4ac) = n,$$

up to the transformation $x \mapsto x + t$ ($t \in \mathbb{Z}$). Let $q_2(n)$, $q^+(n)$, and $q^+_2(n)$, respectively, be the number of these $f$ such that $2|b$ (for $q_2$), such that the roots of $f$ are real (for $q^+$), or which satisfy both conditions (for $q^+_2$). Then for all nonzero integers $n$,

$$q^+_2(4n) = q(n)$$

$$q_2(4n) = 2q^+(n).$$

The methods of this paper also apply to a reflection theorem conjectured by Nakagawa [30] for pairs of ternary quadratic forms, which parametrize quartic rings. Here the combinatorics are formidable (see [34]) and this reflection theorem will be discussed in a sequel to this paper.

1.2 Methods

Several proofs of O-N are now in print [35, 31, 10, 22], all of which consist of two main steps:

- A “global” step that uses global class field theory to understand cubic fields, equivalently $GL_2(\mathbb{Q})$-orbits of cubic forms;
- A “local” step to count the rings in each cubic field, equivalently the $GL_2(\mathbb{Z})$-orbits in each $GL_2(\mathbb{Q})$-orbit, and put the result in a usable form.

In this paper, the distinction between these steps will be formalized and clarified.

For the global step, we use Fourier analysis on adelic cohomology, inspired by the famous use of Fourier analysis on the adeles in Tate’s thesis [50]. In a paper related to this one, the author and Alberts [1] use harmonic analysis on adelic cohomology to get asymptotics for the number of Galois cohomology classes over a global field, counting by discriminant-like invariants. Our method is applicable on a quite general class of representations of algebraic groups, which we call composed varieties. The term “composed” refers to the presence of a composition law on the orbits, which corresponds to the addition law of a Galois cohomology group $H^1(K, M)$. Our guiding example is the variety of binary cubic forms of discriminant $D$ with its action of $\Gamma = SL_2$, where rational orbits correspond to $H^1(K, M)$ for an appropriate module $M$ of order 3.

For the local step, it is necessary to compute or biject orbits on the appropriate $p$-adic varieties.
1.3 Outline of the paper

In Sections 2 and 3 we lay out preliminary matter pertaining to Galois cohomology and to rings over a Dedekind domain, respectively. In Section 4 we present the notion of composed varieties, on which we perform the novel technique of Fourier analysis of the local and global Tate pairings to get our main local-to-global reflection engine (Theorems 4.14 and 4.15). The remainder of the paper will consist of applications of this engine.

In Section 5 we prove quadratic O-N, Theorems 1.3 and 1.4. In Section 6, we prove cubic O-N, Theorem 1.2 and its corollaries. In Section 7, we show some more general reflection theorems of O-N type, including discriminant-reduction identities, that can be proved using the same tools.

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1.5 Notation

The following conventions will be observed in the remainder of the paper.

We denote by \( \mathbb{N} \) the set of nonnegative integers.

If \( P \) is a statement, then

\[
1_P = \begin{cases} 
1 & \text{if } P \text{ is true} \\
0 & \text{if } P \text{ is false}.
\end{cases}
\]

If \( S \) is a set, then \( 1_S \) denotes the characteristic function \( 1_S(x) = 1_{x \in S} \).

An algebra will always be commutative and of finite rank over a field, while a ring or order will be a finite-rank, torsion-free ring over a Dedekind domain, containing 1. An order need not be a domain.

If \( \mathcal{O}_K \) is a Dedekind domain, \( K \) is its field of fractions, \( V \) is an \( n \)-dimensional vector space over \( K \) and \( A, B \subseteq V \) are two full-rank lattices, we denote by the index \( [A : B] \) the unique fractional ideal \( \mathfrak{c} \) such that

\[
\Lambda^n A = \mathfrak{c}\Lambda^n B
\]
as lattices in the top exterior power \( \Lambda^n V \). Note that if \( A \supseteq B \) and

\[
A/B \cong \mathcal{O}_K/\mathfrak{c}_1 \oplus \cdots \oplus \mathcal{O}_K/\mathfrak{c}_r
\]
as \( \mathcal{O}_K \)-modules, then

\[
[A : B] = \mathfrak{c}_1 \mathfrak{c}_2 \cdots \mathfrak{c}_r.
\]

If \( L \) is a \( K \)-algebra, \( \mathfrak{a} \subseteq L \) is an order, and \( a \subseteq L \) is a fractional ideal, the index \( [\mathcal{O} : a] \) is called the norm of \( a \) and will be denoted by \( N_{\mathcal{O}}(a) \) or, when the context is clear, by \( N(a) \). The norm is multiplicative for \( \mathcal{O} \)-ideals that are invertible (i.e. locally principal), but not for general \( \mathcal{O} \)-ideals.

If \( K \) is a field, we denote by \( \overline{K} \) the separable closure of \( K \).

If \( a, b \in L \) are elements of a local or global field, a separable closure thereof, or a finite product of the preceding, we write \( a|b \) to mean that \( b = ca \) for some \( c \) in the appropriate ring of integers \( \mathcal{O}_L \). If \( ab \) and \( b|a \), we say that \( a \) and \( b \) are associates and write \( a \sim b \). Note that \( a \) and \( b \) may be zero-divisors.

If \( S \) is a finite set, we let Sym(S) denote the set of permutations of \( S \); thus \( S_n = \text{Sym}(\{1, \ldots, n\}) \).

If \( |S| = |T| \), and if \( g \in \text{Sym}(S) \), \( h \in \text{Sym}(T) \) are elements, we say that \( g \) and \( h \) are conjugate if there is a bijection between \( S \) and \( T \) under which they correspond. Likewise when we say that two subgroups \( G \subseteq \text{Sym}(S) \), \( H \subseteq \text{Sym}(T) \) are conjugate.

We will use the semicolon to separate the coordinates of an element of a product of rings. For instance, in \( \mathbb{Z} \times \mathbb{Z} \), the nontrivial idempotents are \((1; 0)\) and \((0; 1)\).

If \( n \) is a positive integer, then \( \zeta_n \) denotes a primitive \( n \)th root of unity in \( \overline{\mathbb{Q}} \), while \( \tilde{\zeta}_n \) denotes the \( n \)th root of unity

\[
\tilde{\zeta}_n = (1; \zeta_n; \zeta_n^2; \ldots; \zeta_n^{n-1}) \in \overline{\mathbb{Q}}^n.
\]
Throughout the proofs of the local reflection theorems, we will fix a local field \( K \), its valuation \( v = v_K \) (which has value group \( \mathbb{Z} \)), its residue field \( k_K \) of order \( q \), and a uniformizer \( \pi = \pi_K \). Often the choice of uniformizer is immaterial, but not always (for example, a tamely ramified cubic extension will be described without loss of generality as \( R = K[\sqrt[3]{\pi}] \)). We put

\[
\epsilon = \begin{cases} v_K(2) & \text{in the quadratic case} \\ v_K(3) & \text{in the cubic case} 
\end{cases}
\]

Note that \( \epsilon \) is either the absolute ramification index of \( K \) or 0. We let \( \mathfrak{m}_K \) denote the maximal ideal of \( K \), and likewise let \( \mathfrak{m}_K \) be the maximal ideal of the ring \( \mathcal{O}_K \) of algebraic integers over \( K \); note that \( \mathfrak{m}_K \) is not finitely generated. We also allow \( v = v_K \) to be applied to elements of \( K \), the valuation being scaled so that its restriction to \( K \) has value group \( \mathbb{Z} \). We use the absolute value bars \([\cdot]\) for the corresponding metric (i.e. multiplicative valuation), whose normalization will be left undetermined.

If \( R/\mathcal{O}_K \) is a ring, we denote by \( R^{N=1} \) the subgroup of units of norm 1 down to \( K \). Implicitly, the group operation is multiplication, so that \( R^{N=1}[n] \), for instance, denotes the \( n \)th roots of unity of norm 1 (of which there may be more than \( n \) if \( R \) is not a domain).

## 2 Étale algebras and their Galois groups

If \( K \) is a field, an \emph{étale algebra} over \( K \) is a finite-dimensional separable commutative algebra over \( K \), or equivalently, a finite product of finite separable extension fields of \( K \). A treatment of étale algebras is found in Milne ([33], chapter 8): here we summarize this theory and prove a few auxiliary results that will be of use.

An étale algebra \( L \) of rank \( n \) admits exactly \( n \) maps \( t_1, \ldots, t_n \) (of \( K \)-algebras) to a fixed separable closure \( \bar{K} \) of \( K \). We call these the \emph{coordinates} of \( L \); the set of them will be called \( \text{Coord}(L/K) \) or simply \( \text{Coord}(L) \). Together, the coordinates define an embedding of \( L \) into \( \bar{K}^n \), which we call the \emph{Minkowski embedding} because it subsumes as a special case the embedding of a degree-\( n \) number field into \( \mathbb{C}^n \), which plays a major role in algebraic number theory, as in Delone-Faddeev [18].

For any element \( \gamma \) of the absolute Galois group \( G_K \), the composition \( \gamma \circ t_i \) with any coordinate is also a coordinate \( t_j \), so we get a homomorphism \( \phi = \phi_L : G_K \to \text{Sym}(\text{Coord}(L)) \) such that

\[
\gamma(t(x)) = (\phi_L \gamma)(x)
\]

for all \( x \in L, t \in \text{Coord}(L) \). This gives a functor from étale \( K \)-algebras to \( G_K \)-sets (sets with a \( G_K \)-action), which is denoted \( F \) in Milne’s terminology. A functor going the other way, which Milne calls \( A \), takes \( \phi : G_K \to \mathcal{S}_n \) to

\[
L = \{ (x_1, \ldots, x_n) \in \bar{K}^n \mid \gamma(x_i) = x_{\phi_{\gamma}(i)} \forall \gamma \in G_K, \forall i \}
\]

(2.1)

**Proposition 2.1** ([33], Theorem 7.29). The functors \( F \) and \( A \) establish a bijection between

- étale extensions \( L/K \) of degree \( n \), up to isomorphism, and
- \( G_K \)-sets of size \( n \) up to isomorphism; that is to say, homomorphisms \( \phi : G_K \to \mathcal{S}_n \), up to conjugation in \( \mathcal{S}_n \).

Moreover, the bijection respects base change, in the following way:

**Proposition 2.2.** Let \( K_1/K \) be a field extension, not necessarily algebraic, and let \( L/K \) be an étale extension of degree \( n \). Then \( L_1 = L \otimes_K K_1 \) is étale over \( K_1 \), and the associated Galois representations \( \phi_{L/K}, \phi_{L_1/K_1} \) are related by the commutative diagram

\[
\begin{array}{ccc}
G_K & \xrightarrow{\phi_{L/K}} & G_K \\
\downarrow{\phi_{L_1/K_1}} & \Downarrow{\phi_{L/K}} & \downarrow{\phi_{L_1/K_1}} \\
\text{Sym}(\text{Coord}_{K_1}(L_1)) & \sim & \text{Sym}(\text{Coord}_K(L)) \\
\end{array}
\]

(2.2)
objects are in bijection: $M$ (Proposition 2.7) explicit meaning in terms of field extensions of not yet been written down fully, a gap that we fill in here. We begin by describing Galois modules.

The Galois group holds the answers to various natural questions about an étale algebra. The next proposition is given without proof, since it follows immediately from the functorial character of the correspondence in Proposition 2.1.

**Proposition 2.3.** The subextensions $L' \subseteq L$ of an étale extension $L/K$, correspond to the equivalence relations $\sim$ on $\text{Coord}(L)$ stable under permutation by $G(L/K)$, under the bijection $\sim \mapsto L' = \{ x \in L : \iota(x) = \iota'(x) \text{ whenever } \iota \sim \iota' \}$.

Important for us will be two notions pertaining to the Galois group.

**Definition 2.4.** Let $G \subseteq S_n$ be a subgroup. A $G$-extension of $K$ is a degree-$n$ étale algebra $L$ with a choice of subgroup $G' \subseteq \text{Sym}(\text{Coord}(L))$ that is conjugate to $G$ and contains $G(L/K)$, plus a conjugacy class of isomorphisms $G' \cong G$: the conjugacy being in $G$, not in $S_n$. The added data is called a $G$-structure on $L$.

**Proposition 2.5.** $G$-extensions $L/K$ up to isomorphism are in bijection with homomorphisms $\phi : G_K \rightarrow G$, up to conjugation in $G$.

**Proof.** Immediate from Proposition 2.1

If $G$ embeds into $\text{Sym}(G)$ by the Cayley embedding, a $G$-extension $L/K$ has a group of automorphisms preserving the $G$-structure that is isomorphic to $G$. Such an $L$ will be called a $G$-torsor: it is a suitable analogue of a Galois extension in the realm of étale algebras. Note that every quadratic extension is a $C_2$-torsor.

This notion of $G$-extension makes it easy to define resolvent algebras in great generality:

**Definition 2.6.** Let $G \subseteq S_n$, $H \subseteq S_m$ be subgroups and $\rho : G \rightarrow H$ be a homomorphism. Then for every $G$-extension $L/K$, the corresponding $\phi_L : G_K \rightarrow G$ may be composed with $\rho$ to yield a map $\phi_R : G_K \rightarrow H$, which defines an étale extension $R/K$ of degree $m$. This $R$ is called the resolvent of $L$ under the map $\rho$.

For example, since there is a surjective map $\rho_{4,3} : S_4 \rightarrow S_3$, every quartic étale algebra $L/K$ has a cubic resolvent $R$, which appears in Bhargava [20]. Likewise, the sign map can be viewed as a homomorphism $\text{sgn} : S_n \rightarrow S_2$, attaching to every étale algebra $L$ a quadratic resolvent $T$. If $L = K[\theta]/f(\theta)$ is generated by a polynomial $f$, and if $\text{char} \, K \neq 2$, then it is not hard to see that $T = K[\sqrt{\text{disc} f}]$ where $\text{disc} f$ is the polynomial discriminant. Note that $T$ still exists even if $\text{char} \, K = 2$. We have that $T \cong K \times K$ is split if and only if the Galois group $G(L/K)$ is contained in the alternating group $A_n$.

If $\rho : G \rightarrow H$ is invertible, then it defines a bijection between $G$-extensions and $H$-extensions. Two extensions related by such a correspondence will be called mutual resolvents.

## 2.1 A fresh look at Galois cohomology

Galois cohomology is one of the basic tools in the development of class field theory. It is usually presented in a highly abstract fashion, but certain Galois cohomology groups, specifically $H^1(K, M)$ for finite $M$, have explicit meaning in terms of field extensions of $M$. It seems that this interpretation is well known but has not yet been written down fully, a gap that we fill in here. We begin by describing Galois modules.

**Proposition 2.7 (a description of Galois modules).** Let $M$ be a finite abelian group, and let $K$ be a field. Let $M^\times$ denote the subset of elements of $M$ of maximal order $m$, the exponent of $M$. The following objects are in bijection:

(a) Galois module structures on $M$ over $K$, that is, continuous homomorphisms $\phi : G_K \rightarrow \text{Aut} \, M$;

(b) $(\text{Aut} \, M)$-torsors $T/K$;
(c) \((\text{Aut} \ M)\)-extensions \(L_0/K\), where \(\text{Aut} \ M \hookrightarrow \text{Sym} \ M\) in the natural way;

(d) \((\text{Aut} \ M)\)-extensions \(L^-/K\), where \(\text{Aut} \ M \hookrightarrow \text{Sym} \ M^-\) in the natural way.

**Proof.** For item (d), to make sense, we need that \(M^-\) generates \(M\); this follows easily from the classification of finite abelian groups.

The bijections are immediate from Proposition 2.5. ■

We will denote \(M\) with its Galois-module structure coming from these bijections by \(M_\phi\), \(M_T\), or \(M_{L_0}\). Note that \(T\), \(L_0\), and \(L^-\) are mutual resolvents.

**Example 2.8.** For example (and we will return to this case frequently), if we let \(M = C_3\) be the smallest group with nontrivial automorphism group: \(\text{Aut} \ M \cong C_2\). Then the Galois module structures on \(M\) are in natural bijection with \(C_2\)-torsors over \(K\), that is, quadratic étale extensions \(T/K\). If \(\text{char} \ K \neq 2\), these can be parametrized by Kummer theory as \(T = K[\sqrt{D}]\), \(D \in K^\times / (K^\times)^2\). The value \(D = 1\) corresponds to the split algebra \(T = K \times K\) and to the module \(M\) with trivial action. We have an isomorphism

\[ M_T \cong \{0, \sqrt{D}, -\sqrt{D}\} \]

Note that the zeroth cohomology group \(H^0(K, M)\) has a ready parametrization:

**Proposition 2.9.** Let \(M = M_{L_0}\) be a Galois module. The elements of \(H^0(K, M)\) are in bijection with the degree-1 field factors of \(L_0\).

**Proof.** Proposition 2.7 establishes an isomorphism of \(G_K\)-sets between the coordinates of \(L_0\) and the points of \(M\). A degree-1 field factor corresponds to an orbit of \(G_K\) on \(\text{Coord}(L_0)\) of size 1, which corresponds exactly to a fixed point of \(G_K\) on \(M\). ■

Deeper and more useful is a description of \(H^1\). For an abelian group \(M\), let \(\text{GA}(M) = M \times \text{Aut} \ M\) be the semidirect product under the natural action of \(\text{Aut} \ M\) on \(M\). We can describe \(\text{GA}(M)\) more explicitly as the group of affine-linear transformations of \(M\); that is, maps

\[ a_g,t(x) = gx + t, \quad g \in \text{Aut} \ M, t \in M \]

composed of an automorphism and a translation, the group operation being composition. In particular, we have an embedding

\[ \text{GA}(M) \hookrightarrow \text{Sym}(M). \]

**Proposition 2.10 (a description of \(H^1\)).** Let \(M = M_\phi = M_{L_0}\) be a Galois module.

(a) \(Z^1(K, M)\) is in natural bijection with the set of continuous homomorphisms \(\psi : G_K \rightarrow \text{GA}(M)\) such that the following triangle commutes:

\[
\begin{array}{ccc}
G_K & \xrightarrow{\psi} & \text{GA}(M) \\
& \downarrow{\phi} & \downarrow{\psi} \\
& \text{Aut} \ M & \\
\end{array}
\]

(b) \(H^1(K, M)\) is in natural bijection with the set of such \(\psi : G_K \rightarrow \text{GA}(M)\) up to conjugation by \(M \subseteq \text{GA}(M)\).

(c) \(H^1(K, M)\) is also in natural bijection with the set of \(\text{GA}(M)\)-extensions \(L/K\) (with respect to the embedding \(\text{GA}(M) \hookrightarrow \text{Sym}(M)\)) equipped with an isomorphism from their resolvent \((\text{Aut} \ M)\)-torsor to \(T\).
Proof. By the standard construction of group cohomology, $Z^1$ is the group of continuous crossed homomorphisms

$$Z^1(K, M) = \{ \sigma : G_K \to M \mid \sigma(\gamma \delta) = \sigma(\gamma) + \phi(\gamma)\sigma(\delta) \}.$$ 

Send each $\sigma$ to the map

$$\psi : G_K \to \text{GA}(M)$$

$$\gamma \mapsto a_{\phi(\gamma), \sigma(\gamma)}.$$ 

It is easy to see that the conditions for $\psi$ to be a homomorphism are exactly those for $\sigma$ to be a crossed homomorphism, establishing [a]. For [b] we observe that adding a coboundary $\sigma_a(\gamma) = \gamma(a) - a$ to a crossed homomorphism $\sigma$ is equivalent to post-conjugating the associated map $\psi : G_K \to \text{GA}(M)$ by $a$. As to [c], a $\text{GA}(M)$-extension carries the same information as a map $\psi$ up to conjugation by the whole of $\text{GA}(M)$. Specifying the isomorphism from the resolvent $(\text{Aut } M)$-torsor to $T$ means that the map $\pi \circ \psi = \phi : G_K \to \text{Aut}(M)$ is known exactly, not just up to conjugation. Hence $\psi$ is known up to conjugation by $M$. ■

Remark 2.11. The zero cohomology class corresponds to the extension $L_0$, with its structure given by the embedding Aut $M \hookrightarrow \text{GA}(M)$. This can be seen to be the unique cohomology class whose corresponding $\text{GA}(X)$-extension has a field factor of degree 1.

If $K$ is a local field, a cohomology class $\alpha \in H^1(K, M)$ is called unramified if it is represented by a cocycle $\alpha : \text{Gal}(K/K) \to M$ that factors through the unramified Galois group $\text{Gal}(K^{ur}/K)$. The subgroup of unramified coclasses is denoted by $H^1_{ur}(K, M)$. If $M$ itself is unramified (and we will never have to think about unramified cohomology in any other case), this is equivalent to the associated étale algebra $L$ being unramified.

2.1.1 The Tate dual

As usual, if $M$ is a Galois module and the exponent $m$ of $M$ is not divisible by char $K$, then

$$M' = \text{Hom}(M, \mu_m)$$

is also a Galois module, called the Tate dual of $M$. The modules $M$ and $M'$ have the same order and are isomorphic as abstract groups, though not canonically; as Galois modules, they are frequently not isomorphic at all.

Example 2.12. If $M = M_{K[\sqrt{D}]}$ is one of the order-3 modules studied in Example 2.8, then

$$M' = M_{K[\sqrt{-3D}]}.$$ 

This explains the $D \to -3D$ pattern in the Scholz reflection theorem and its generalizations, including cubic Ohno-Nakagawa.

Example 2.13. A module $M$ of underlying group $C_2 \times C_2$ is always self-dual, regardless of what Galois-module structure is placed on it. This can be proved by noting that $M$ has a unique alternating bilinear form.

Particularly notable for us are the cases when $\text{GA}(M)$ is the full symmetric group $\text{Sym}(M)$, for then every étale algebra $L/K$ of degree $|M|$ has a (unique) $\text{GA}(M)$-affine structure. It is easy to see that there are only four such cases:

- $M = \{1\}$, $\text{GA}(M) \cong S_1$
- $M = \mathbb{Z}/2\mathbb{Z}$, $\text{GA}(M) \cong S_2$
- $M = \mathbb{Z}/3\mathbb{Z}$, $\text{GA}(M) \cong C_3 \rtimes C_2 \cong S_3$
- $M = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\text{GA}(M) \cong (C_2 \times C_2) \rtimes S_3 \cong S_4$. 

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For degree exceeding 4, not every étale algebra arises from Galois cohomology, a restriction that plays out in the existing literature on reflection theorems. For instance, Cohen, Rubinstein-Salzedo, and Thorne \[11\] prove a reflection theorem in which one side counts $D_p$-dihedral fields of prime degree $p \geq 3$. From our perspective, these correspond to cohomology classes of an $M = C_p$ whose Galois action is by $\pm 1$. The Tate dual of such an $M$ can have Galois action by the full $(\mathbb{Z}/p\mathbb{Z})^\times$, and indeed they count extensions of Galois group $\text{GA}(C_p)$ on the other side of the reflection theorem.

2.2 Extensions of Kummer theory to explicitize Galois cohomology

If $\text{char } K \neq 2$, we have the familiar Kummer-theory parametrization

$$H^1(K, \mathbb{Z}/2\mathbb{Z}) \cong K^\times / (K^\times)^2.$$ 

The cohomology of modules with underlying group $C_3$ can also be described explicitly.

**Proposition 2.14.** Let $K$ be a field with $\text{char } K \neq 3$, and let $M$ be a $K$-Galois module with underlying group $C_3$ (char $K \neq 3$). Let $T'$ be the quadratic étale algebra corresponding to the Tate dual $M'$ under the bijection of Proposition \[2.10\]. Then we have a group isomorphism

$$H^1(K, M) \cong T'^N / (T'^N = 1)^3$$

in which $\alpha \in T'^N / (T'^N = 1)^3$ corresponds to the cubic extension $L/K$ generated by the image of the $K$-linear map

$$\kappa : K \to \overline{K}^3$$

$$\xi \mapsto (\text{tr}_{K^2/K} \xi \omega \sqrt{\delta}) \in (K^2)^N = 1[3],$$

where $\sqrt{\delta} \in \overline{K}^2$ is chosen to have norm 1, and $\omega$ ranges through the set

$$(K^2)^N = 1[3] = \{(1; 1), (\zeta_3; \zeta_3^2); (\zeta_3^2; \zeta_3)\}$$

of cube roots of 1 in $\overline{K}^2$ of norm 1. Indeed

$$L = K + \kappa(T').$$

**Proof.** Since elements of $H^1(K, M)$ correspond, by Proposition \[2.10\] to cubic étale algebras, the parametrization follows easily from the classical cubic formula. See Knus and Tignol \[28\, Proposition 5.13\] for a treatment of the closely related quartic case ($M \cong C_2 \times C_2$).

**Remark 2.15.** Large parts of our work for modules of order 3 apply to all cyclic modules of prime order. However, for the sake of brevity, we omit some of these generalizations.

Assume now that $K$ is a local field. Our next step will be to understand the (local) Tate pairing, which is given by a cup product

$$\langle , \rangle_T : H^1(K, M) \times H^1(K, M') \to H^2(K, \mu_m) \cong \mu_m.$$ 

In many cases this pairing can be described explicitly. For instance, if $M \cong \mathbb{Z}/m\mathbb{Z}$ has trivial $G_K$-action, then $M' \cong \mu_m$, and we have a Tate pairing

$$\langle , \rangle_T : H^1(K, \mathbb{Z}/m\mathbb{Z}) \times H^1(K, \mu_m) \to \mu_m.$$ 

Now $H^1(K, \mathbb{Z}/m\mathbb{Z}) \cong \text{Hom}(K, \mathbb{Z}/m\mathbb{Z})$ parametrizes $\mathbb{Z}/m\mathbb{Z}$-torsors $L$, while by Kummer theory, $H^1(K, \mu_m) \cong K^\times / (K^\times)^m$. The Tate pairing in this case is none other than the Artin symbol $\phi_L : K^\times \to \text{Gal}(L/K) / \mu_m$ whose kernel is the norm group $N_{L/K}(L^\times)$ (see Neukirch \[37\, Prop. 7.2.13\]). If, in addition, $\mu_m \subseteq K$, then $H^1(K, \mathbb{Z}/m\mathbb{Z})$ is also isomorphic to $K^\times / (K^\times)^m$, and the Tate pairing is an alternating pairing

$$\langle , \rangle : K^\times / (K^\times)^m \times K^\times / (K^\times)^m \to \mu_m.$$
classically called the Hilbert symbol (or Hilbert pairing). It is defined in terms of the Artin symbol by
\[ \langle a, b \rangle = \phi_K(\sqrt{b})(a). \]  
(2.4)

In particular, \(\langle a, b \rangle = 1\) if and only if \(a\) is the norm of an element of \(K[\sqrt{b}]\). This can also be described in terms of the splitting of an appropriate Severi-Brauer variety; for instance, if \(m = 2\), we have \(\langle a, b \rangle = 1\) exactly when the conic
\[ ax^2 + by^2 = z^2 \]
has a \(K\)-rational point. See also Serre ([16], §§XIV.1–2). (All identifications between pairings here are up to sign; the signs are not consistent in the literature and are irrelevant for the present paper.) Pleasantly, for the types of \(M\) of concern to us, the Tate pairing can be expressed simply in terms of the Hilbert pairing.

We extend the Hilbert pairing to étale algebras in the obvious way: if \(L = K_1 \times \cdots \times K_s\), then
\[ \langle (a_1; \ldots; a_s), (b_1; \ldots; b_s) \rangle_L := \langle a_1, b_1 \rangle_{K_1} \cdots \langle a_s, b_s \rangle_{K_s}. \]

Note that if \(a\) is a norm from \(L[\sqrt{b}]\) to \(L\), then \(\langle a, b \rangle_L = 1\), but the converse no longer holds. We then have the following:

**Proposition 2.16 (The Tate pairing for Galois modules of order 3).** Let \(K\) be a local field. For \(M, T'\) as in Proposition [2.14], let \(M'\) be the Tate dual of \(M\), and let \(T\) be the corresponding étale algebra, corresponding to the \(G_K\)-set \(T'\) of elements of maximal order in \(M\), just as \(T'\) corresponds to \(M'\). The Tate pairing
\[ (\bullet, \bullet) : H^1(K, M) \times H^1(K, M') \to H^2(K, \mu_m) \cong C_m \]
is then the restriction of the Hilbert pairing on \(E := T[\mu_3]\), which naturally contains both \(T\) and \(T'\).

**Proof.** Note that \(E\) is an extension that splits both \(M\) and \(M'\). Since \([E : K] = 4\) is coprime to \([M] = 3\), restriction and corestriction embed \(H^1(K, M)\) and \(H^1(K, M')\) as direct summands of \(H^1(E, \mathbb{Z}/3\mathbb{Z})\). This allows us to reduce to the case where both \(M\) and \(M'\) are trivial, and then the conclusion follows from the definition of the Hilbert pairing. \[ \blacksquare \]

We can describe even more explicitly the group
\[ H^1(K, M) \cong T'^N = 1 / (T'^N = 1)^3. \]
(2.5)

If \(\alpha \in T'^N = 1 / (T'^N = 1)^3\), we normalize \(\alpha\) by multiplying by cubes until \(|\alpha - 1|\) is minimal. Using the division algorithm in \(\mathbb{Z}\), we let \(\ell = \ell(\alpha)\) and \(h = h(\alpha)\) be the integers such that
\[ 2v_K(\alpha - 1) = 3\ell + h, \quad 1 \leq h \leq 3. \]

We call \(\ell\) the level, and \(h\) the offset, of the element \(\alpha\), or equivalently of the associated coclass \(\sigma\) or \(GA(C_3)\)-extension \(L\). The level and offset have the following pleasant properties:

**Proposition 2.17 (levels and offsets).** Let \(M\) be a Galois module with underlying group \(C_3\) over a local field \(K\) with \(\text{char} \ K \neq 3\).

(a) The discriminant of the cubic algebra \(L\) corresponding to an \(\alpha \in T'^N = 1 / (T'^N = 1)^3\), normalized to be closest to 1, is given by
\[ v_K(\text{Disc}(L/K)) = \max\{0, 3\ell + 2 - 2v_K(\alpha - 1)\} \]
\[ = \max\{0, 3\ell + 2 - 3\ell - h\} \]  
(2.6)

(b) The level \(\ell\) of a coclass determines its offset \(h\) uniquely in the following way:

(i) If \(0 \leq \ell \leq e\), then \(h \in \{1, 2\}\) is determined by
\[ h \equiv \ell + b \mod 2, \]
where \(b\) is given by
\[ b = v_K(\text{disc} T) = \begin{cases} 0 & \text{if } M \text{ is unramified} \\ 1 & \text{if } M \text{ is ramified.} \end{cases} \]

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(ii) If $\ell = -1$, then $h = 3$.

e) For all $i$, $-1 \leq i \leq e$, the level space

$$
\mathcal{L}_i = \mathcal{L}_i(M) = \{ \sigma \in H^1(K, M) : \ell(\sigma) \geq i \}
$$

consisting of coclasses of level at least $i$ is a subgroup of $H^1(K, M)$.

(d) $\mathcal{L}_e = H^1_{ur}(K, M)$, and we define $\mathcal{L}_{e+1} = \{0\}$.

e) For $0 \leq i \leq e$,

$$|\mathcal{L}_i| = q^{-i}|H^0(K, M)|.$$

(f) $\mathcal{L}_{-1}$ is the whole of $H^1(K, M)$, and

$$|H^1(K, M)| = q^e|H^0(K, M)| \cdot |H^0(K, M')|.$$

(g) For $-1 \leq i \leq e+1$, with respect to the Tate pairing between $H^1(K, M)$ and $H^1(K, M')$,

$$\mathcal{L}_i(M)^\perp = \mathcal{L}_{e-i}(M').$$

Proof. Parts (a)–(f) follow easily from results in Del Corso–Dvornicich [12], namely Proposition 7, Corollary 2, and Lemmas 5–7. Part (g) upon reducing the Tate pairing to a Hilbert symbol, follows from Nguyen-Quang-Do [38, Proposition 3.4.1.1].

The following corollary will be used repeatedly in our applications:

Corollary 2.18. For $0 \leq i \leq e$, the characteristic function $L_i$ of the level space $\mathcal{L}_i$ has Fourier transform given by

$$\widehat{L}_i = q^{-i}L_{e-i}.$$ (2.7)

where $q = |k_K|$.

Proof. Immediate from Proposition 2.17, parts (e) and (g).

If $K$ is a tame local field, that is, $\text{char } k_K \neq 3$, the structure of $H^1(K, M)$ is well known. We put

$$e = 0, \quad \mathcal{L}_{-1} = \{0\}, \quad \mathcal{L}_0 = H^1_{ur}(K, M), \quad \mathcal{L}_1 = H^1(K, M)$$

and observe that Proposition 2.17, parts (d)–(f) and Corollary 2.18 still hold.

The wild function field case $K = \mathbb{F}_q((t))$ admits a similar treatment, but now the number of levels is infinite. We do not address this case here.

3 Rings over a Dedekind domain

3.1 Discriminants

For this work, we will need a robust notion of a discriminant of an order $O$ in an étale algebra $L$ over the field of fractions $K$ of a Dedekind domain $O_K$, char $K \neq 2$. The usual discriminant ideal is unsuitable because it drops information about the sign of the discriminant, which figures prominently in reflection theorems of O-N type. The extensions of O-N in Dioses [19] and Cohen–Rubinstein-Salzedo–Thorne [11] each employ an ad-hoc notion of discriminant that incorporates the splitting data of an order at the infinite primes. Here we explain the variant that we will use.

Since the $2n$-fold multilinear form

$$\tau : O^{2n} \to O_K$$

$$(\xi_1, \xi_2, \ldots, \xi_n, \eta_1, \eta_2, \ldots, \eta_n) \mapsto \det[\text{tr} \xi_i \eta_j]_{i,j=1}^n.$$ (3.1)
is alternating in the \( \xi \)'s and also in the \( \eta \)'s, it can be viewed as a bilinear form on the rank-1 lattice \( \Lambda^n \mathcal{O} \). Identifying \( \Lambda^n \mathcal{O} \) with a (fractional) ideal \( \mathcal{I} \) of \( \mathcal{O}_K \) (the Steinitz class of \( \mathcal{O} \)), we can write

\[
\tau(\xi) = D\xi^2
\]

for some nonzero \( D \in \mathfrak{c}^{-2} \). Had we rescaled the identification \( \Lambda^n \mathcal{O} \to \mathcal{I} \) by \( \lambda \in \mathcal{K}^\times \), \( D \) would be multiplied by \( \lambda^2 \). We call the pair \( (\mathfrak{c}, D) \), up to the equivalence \( (\mathfrak{c}, D) \sim (\lambda \mathfrak{c}, \lambda^{-2}D) \), the **discriminant** of \( \mathcal{O} \) and denote it by \( \text{Disc} \).

The class of \( D \) in \( \mathcal{K}^\times/(\mathcal{K}^\times)^2 \) depends only on \( L \). The following proposition is not difficult:

**Proposition 3.1.** If \( L \) is an étale algebra over \( K \) of discriminant \( (\mathfrak{c}, D) \), then \( T = \mathcal{K}[\sqrt{D}] \) is the discriminant torsor of \( L \); that is, the diagram of Galois structure maps

\[
\begin{array}{ccc}
G_K & \xrightarrow{\phi_L} & \mathcal{S}_n \\
\downarrow \phi_T & & \downarrow \text{sgn} \\
\mathcal{S}_2 & & 
\end{array}
\]

commutes.

There is notable integral structure on \( (\mathfrak{c}, D) \) as well.

**Lemma 3.2 (Stickelberger’s theorem over Dedekind domains).** If \( (\mathfrak{c}, D) \) is the discriminant of an order \( \mathcal{O} \), then \( D \equiv t^2 \mod 4\mathfrak{c}^{-2} \) for some \( t \in \mathfrak{c}^{-1} \).

**Remark 3.3.** When \( \mathcal{O}_K = \mathbb{Z} \), Lemma 3.2 states that the discriminant of an order is congruent to 0 or 1 mod 4: a nontrivial and classical theorem due to Stickelberger. Our proof is a generalization of the most familiar one for Stickelberger’s theorem, due to Schur [45].

**Proof.** Since \( D \in \mathfrak{c}^{-2} \), the conclusion can be checked locally at each prime dividing 2 in \( \mathcal{O}_K \). We can thus assume that \( \mathcal{O}_K \) is a DVR and in particular that \( \mathfrak{c} = (1) \). Now there is a simple tensor \( \xi_1 \land \cdots \land \xi_n \) that corresponds to the element \( 1 \in \mathfrak{c} \). By definition,

\[
\sqrt{D} = \tau_0(\xi_1, \ldots, \xi_n) = \det[\kappa_h(\xi_i)]_{i,h} = \sum_{\pi \in \mathcal{S}_n} (\text{sgn}(\sigma) \prod_i \kappa_{\pi(i)}(\xi_i)) = \rho - \bar{\rho}, \tag{3.2}
\]

where

\[
\rho = \sum_{\pi \in \mathcal{A}_n} \prod_i \kappa_{\pi(i)}(\xi_i)
\]

lies in \( \mathcal{T}_2 \) by symmetry and \( \bar{\rho} \) is its conjugate. By construction, \( \rho \) is integral over \( \mathcal{O}_K \), that is to say, \( \rho + \bar{\rho} \) and \( \rho \bar{\rho} \) lie in \( \mathcal{O}_K \). Now

\[
D = (\rho - \bar{\rho})^2 = (\rho + \bar{\rho})^2 - 4\rho \bar{\rho}
\]

is the sum of a square and a multiple of 4 in \( \mathcal{O}_K \). \( \square \)

We can now state the notion of discriminant as we would like to use it.

**Definition 3.4.** A **discriminant** over \( \mathcal{O}_K \) is an equivalence class of pairs \( (\mathfrak{c}, D) \), with \( D \in \mathfrak{c}^{-2} \) and \( D \equiv t^2 \mod 4\mathfrak{c}^{2} \) for some \( t \in \mathfrak{c}^{-1} \), up to the equivalence relation

\[
(\mathfrak{c}, D) \sim (\lambda \mathfrak{c}, \lambda^{-2}D).
\]

We have thus shown that if \( \mathcal{O} \) is an order, then \( \text{Disc} \mathcal{O} \) is a discriminant in the above sense. The discriminant recovers the discriminant ideal via \( \mathfrak{d} = \text{Disc} \mathcal{O} \).

We will often denote a discriminant by a single letter, such as \( D \). When elements or ideals of \( \mathcal{O}_K \) appear in discriminants, they are to be understood as follows:

\[
D \ (D \in \mathcal{O}_K) \quad \text{means} \quad ((1), D) \quad \text{(3.3)}
\]

\[
\mathfrak{c}^2 \ (\mathfrak{c} \subseteq K) \quad \text{means} \quad (\mathfrak{c}, 1). \quad \text{(3.4)}
\]
The seemingly counterintuitive convention \(3.4\) is motivated by the fact that, if \(c = (c)\) is principal, then \((c, 1)\) is the same discriminant as \(((1), c^2)\).

With these remarks in place, the reader should not have difficulty reading and proving the following important relation:

**Proposition 3.5.** If \(\mathcal{O} \supseteq \mathcal{O}'\) are two orders in an étale algebra \(L\), then
\[
\text{Disc } \mathcal{O}' = [\mathcal{O} : \mathcal{O}']^2 \cdot \text{Disc } \mathcal{O}.
\]

### 3.2 Quadratic rings

The discriminant can also be viewed as a *quadratic resolvent ring* for any order of finite rank, because discriminants are in bijection with quadratic rings:

**Proposition 3.6 (the parametrization of quadratic rings).** Let \(\mathcal{O}_K\) be a Dedekind domain of characteristic not \(2\). For every discriminant \(D\), there is a unique quadratic ring \(\mathcal{O}_D\) having discriminant \(D\).

**Proof.** Note first that the theorem is true when \(\mathcal{O}_K = K\) is a field: by Kummer theory, quadratic étale algebras over \(K\) are parametrized by \(K^\times/(K^\times)^2\), as are discriminants; and it is a simple matter to check that \(\text{Disc } K[\sqrt{D}] = D\). We proceed to the general case.

For existence, let \(D = (c, D)\) be given. By definition, \(D\) is congruent to a square \(t^2 \mod 4c^2\), \(t \in c^{-1}\). Consider the lattice
\[
\mathcal{O} = \mathcal{O}_K \oplus c\xi, \quad \xi = \frac{t + \sqrt{D}}{2} \in L = K[\sqrt{D}].
\]
To prove that \(\mathcal{O}\) is an order in \(L\), it is enough to verify that \((c\xi)(d\xi) \in \mathcal{O}\) for any \(c, d \in c\), and this follows from the computation
\[
\xi^2 = \xi(t - \bar{\xi}) = t\xi - \left(\frac{t^2 - D}{4}\right)
\]
and the conditions \(t \in c^{-1}, t^2 - D \in 4c^{-2}\).

Now suppose that \(\mathcal{O}_1\) and \(\mathcal{O}_2\) are two orders with the same discriminant \(D = (c, D)\). Their enclosing \(K\)-algebras \(L_1, L_2\) have the same discriminant \(D\) over \(K\), and hence we can identify \(L_1 = L_2 = L\). Now project each \(\mathcal{O}_i\) along \(\pi : L \rightarrow L/K\) is an \(\mathcal{O}_K\)-lattice \(c_i\) in \(L/K\), which is a one-dimensional \(K\)-vector space: indeed, we naturally have \(L/K \cong \Lambda^2L\), and upon computation, we find that \(\text{Disc } \mathcal{O}_i = (c_i, D)\). Consequently \(c_1 = c_2 = c\). Now, for each \(\beta \in c\), the fiber \(\pi^{-1}(c) \cap \mathcal{O}_i\) is of the form \(\beta_i + \mathcal{O}_K\) for some \(\beta_i\). The element \(\beta_1 - \beta_2\) is integral over \(\mathcal{O}_K\) and lies in \(K\), hence in \(\mathcal{O}_K\). Thus \(\mathcal{O}_1 = \mathcal{O}_2\).

**Remark 3.7.** Using this perspective of quadratic resolvent ring, the notion of discriminant extends to characteristic \(2\). We omit the details.

### 3.3 Cubic rings

Cubic and quartic rings have parametrizations, known as *higher composition laws*, linking them to certain forms over \(\mathcal{O}_K\) and also to ideals in resolvent rings. The study of higher composition laws was inaugurated by Bhargava in his celebrated series of papers \([3, 4, 5, 6]\), although the gist of the parametrization of cubic rings goes back to work of F.W. Levi \([30]\). Later work by Deligne and by Wood \([52, 54]\) has extended much of Bhargava’s work from \(\mathbb{Z}\) to an arbitrary base scheme. In a previous paper \([39]\), the author explained how a representative sample of these higher composition laws extend to the case when the base ring \(A\) is a Dedekind domain. In the present work, we will need a few more; fortunately, there are no added difficulties, and we will briefly run through the statements and the methods of proof.

**Theorem 3.8 (the parametrization of cubic rings).** Let \(A\) be a Dedekind domain with field of fractions \(K\), \(\text{char } A \neq 3\).

1. **Cubic rings** \(\mathcal{O}\) over \(A\), up to isomorphism, are in bijection with cubic maps
\[
\Phi : M \rightarrow \Lambda^2 M
\]
between a two-dimensional $A$-lattice $M$ and its own Steinitz class, up to isomorphism, in the obvious sense of a commutative square

\[
\begin{array}{cccc}
M_1 & \sim & M_2 \\
\downarrow \Phi_1 & & \downarrow \Phi_2 \\
\Lambda^2 M_1 & \sim & \det i & \Lambda^2 M_2.
\end{array}
\]

The bijection sends a ring $O$ to the index form $\Phi : O/A \to \Lambda^2(O/A)$ given by

\[x \mapsto x \wedge x^2.\]  

(b) If $O$ is nondegenerate, that is, the corresponding cubic $K$-algebra $L = K \otimes_A O$ is étale, then the map $\Phi$ is the restriction, under the Minkowski embedding, of the index form of $\tilde{K}^3$, which is

\[\Phi : \tilde{K}^3/\tilde{K} \to \tilde{K}^2/\tilde{K} \quad (x;y;z) \mapsto ((x-y)(y-z)(z-x),0).\] (3.5)

(c) Conversely, let $L$ be a cubic étale algebra over $K$. If $\tilde{O} \subseteq L/K$ is a lattice such that $\Phi$ sends $\tilde{O}$ into $\Lambda^2 \tilde{O}$, then there is a unique cubic ring $O \subseteq L$ such that, under the natural identifications, $O/A = \tilde{O}$.

Proof. (a) The proof is quite elementary, involving merely solving for the coefficients of the unknown multiplication table of $O$. The case where $A$ is a PID is due to Gross ([24], Section 2): the cubic ring having index form $f(x \xi + y \eta) = ax^3 + bx^2y + cxy^2 + dy^3$

has multiplication table

\[\xi \eta = -ad, \xi^2 = -ac + b\xi - a\eta, \eta^2 = -bd + d\xi - c\eta.\] (3.6)

For the general Dedekind case, see my [39], Theorem 7.1. It is also subsumed by Deligne’s work over an arbitrary base scheme; see Wood [52] and the references therein.

(b) This follows from the fact that the index form respects base change. The index form of $\tilde{K}^3/\tilde{K}$ is a Vandermonde determinant that can easily be written in the stated form.

(c) We have an integral cubic map $\Phi|_O : \tilde{O} \to \Lambda^2 \tilde{O}$, which is the index form $\Phi_O$ of a unique cubic ring $O$ over $O_K$. But over $K$, $\Phi_O$ is isomorphic to the index form of $L$. Since $L$ (as a cubic ring over $K$) is determined by its index form, we obtain an identification $O \otimes_{O_K} K \cong L$ for which $O/A$, the projection of $O$ onto $L/K$, coincides with $\tilde{O}$. The uniqueness of $O$ holds by considerations like those in Proposition 3.6, as $O$ must lie in the integral closure $O_L$ of $K$ in $L$.

In this paper we only deal with nondegenerate rings, that is, those of nonzero discriminant, or equivalently, those that lie in an étale $K$-algebra. Consequently, all index forms $\Phi$ that we will see are restrictions of (3.5).

When cubic algebras are parametrized Kummer-theoretically, the resolvent map becomes very explicit and simple:

**Proposition 3.9 (explicit Kummer theory for cubic algebras).** Let $R$ be a quadratic étale algebra over $K$ (char $K \neq 3$), and let

\[L = K + \kappa(R)\]

be the cubic algebra of resolvent $R' = R \otimes K[\mu_3]$ (the Tate dual of $R$) corresponding to an element $\delta \in K^{N=1}$ in Proposition 2.12, where

\[\kappa(\xi) = \left(\text{tr}_{\tilde{K}^2/K} \xi \omega^3 \delta\right)_{\omega \in \tilde{K}^2}^{N=1} \in \tilde{K}^3\]
so $\kappa$ maps $R$ bijectively onto the traceless plane in $L$. Then the index form of $L$ is given explicitly by

$$\Phi : L/K \to \Lambda^2(L/K)$$

$$\kappa(\xi) \mapsto 3\sqrt{-3}\xi^3 \land 1,$$

where we identify

$$\Lambda^2 L/K \cong \Lambda^3 L \cong \Lambda^2 R' \cong R'/K \cong \sqrt{-3} \cdot R/K$$

using the fact that $R'$ is the discriminant resolvent of $L$.

**Proof.** Direct calculation, after reducing to the case $K = K$, $\delta = 1$, $L = K \times K \times K$. □

**Proposition 3.10 (self-balanced ideals in the cubic case).** Let $\mathcal{O}_K$ be a Dedekind domain, $\text{char } K \neq 3$, and let $R$ be a quadratic étale extension. A self-balanced triple in $R$ is a triple $(B, I, \delta)$ consisting of a quadratic order $B \subseteq R$, a fractional ideal $I$ of $B$, and a scalar $\delta \in (KB)\kappa$ satisfying the conditions

$$\delta I^3 \subseteq B, \quad N(I) = (t) \text{ is principal, and } \quad N(\delta)t^3 = 1,$$

(a) Fix $B$ and $\delta \in R^\times$ with $N(\delta)$ a cube $t^{-3}$. Then the mapping

$$I \mapsto \mathcal{O} = \mathcal{O}_K + \kappa(I)$$

defines a bijection between

- self-balanced triples of the form $(B, I, \delta)$, and
- subrings $\mathcal{O} \subseteq L$ of the cubic algebra $L = K + \kappa(R)$ corresponding to the Kummer element $\delta$, such that $\mathcal{O}$ is 3-traced, that is, $\text{tr}(\xi) \in 3\mathcal{O}_K$ for every $\xi \in L$.

(b) Under this bijection, we have the discriminant relation

$$\text{disc } C = -27 \text{disc } B.$$  

**Proof.** The mapping $\kappa$ defines a bijection between lattices $I \subseteq R$ and $\kappa(I) \subseteq L/K$. The difficult part is showing that $I$ fits into a self-balanced triple $(B, I, \delta)$ if and only if $\kappa(I)$ is the projection of a 3-traced order $\mathcal{O}$. Note that if $(B, I, \delta)$ exists, it is unique, as the requirement $[B : I] = (t)$ pins down $B$.

Rather than establish this equivalence directly, we will show that both conditions are equivalent to the symmetric trilinear form

$$\beta : I \times I \times I \to \Lambda^2 R$$

$$(\alpha_1, \alpha_2, \alpha_3) \mapsto \delta \alpha_1\alpha_2\alpha_3$$

taking values in $t^{-1}\Lambda^2 I$.

In the case of self-balanced ideals, this was done over $\mathbb{Z}$ by Bhargava [3] Theorem 3]. Over a Dedekind domain, it follows from the parametrization of balanced triples of ideals over $B$ [39] Theorem 5.3], after specializing to the case that all three ideals are identified with one ideal $I$. It also follows from the corresponding results over an arbitrary base in Wood [54] Theorem 1.4].

In the case of rings, we compute by Proposition 3.9 that $\beta$ is the trilinear form attached to the index form of $\kappa(I)$. By Theorem 3.2(c) the diagonal restriction $\beta(\alpha, \alpha, \alpha)$ takes values in $t^{-1}\Lambda^2(I)$ if and only if $\kappa(I)$ lifts to a ring $\mathcal{O}$. We wish to prove that $\beta$ itself takes values in $t^{-1}\Lambda^2(I)$ if and only if $\mathcal{O}$ is 3-traced. Note that both conditions are local at the primes dividing 2 and 3, so we may assume that $\mathcal{O}_K$ is a DVR. With respect to a basis $(\xi, \eta)$ of $I$ and a generator of $t^{-1}\Lambda^2(I)$, the index form of $\mathcal{O}$ has the form

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3, \quad a, \ldots, d \in \mathcal{O}_K.$$
If this is the diagonal restriction of $\beta$, then $\beta$ itself can be represented as a 3-dimensional matrix

\[
\begin{array}{ccc}
b/3 & b/3 & c/3 \\
a & & \\
c/3 & d & \\
b/3 & & c/3 \\
\end{array}
\]

which is integral exactly when $b, c \in 3O_K$. Since the trace ideal of $O$ is generated by $\text{tr}(1) = 3$, $\text{tr}(\xi) = -b$, $\text{tr}(\eta) = c$ (by reference to the multiplication table (3.6)), this is also the condition for $O$ to be 3-traced, establishing the equivalence.

The discriminant relation (3.10) follows easily from the definition of $\kappa$. ■

4 Composed varieties

It has long been noted that orbits of certain algebraic group actions on varieties over a field $K$ parametrize rings of low rank over $K$, which can also be identified with the cohomology of small Galois modules over $K$. The aim of this section is to explain all this in a level of generality suitable for our applications.

Taniguchi and Thorne (see [48]) have used Fourier analysis on the space of binary cubic forms over $F_q$ to get the functional equation for the Shintani zeta function of forms satisfying local conditions at primes. Despite the similarities, our work is essentially independent from theirs. Our work is much closer to that of Frei, Loughran, and Newton [21], which uses harmonic analysis on the idele class group to study the Hasse norm principle.

**Definition 4.1.** Let $K$ be a field and $\bar{K}$ its separable closure. A composed variety over $K$ is a quasi-projective variety $V$ over $K$ with an action of a quasi-projective algebraic group $\Gamma$ over $K$ such that:

(a) $V$ has a $K$-rational point $x_0$;

(b) the $\bar{K}$-points of $V$ consist of just one orbit $\Gamma(\bar{K})x_0$;

(c) the point stabilizer $M = \text{Stab}_{\Gamma(\bar{K})} x_0$ is a finite abelian subgroup.

The term composed is derived from Gauss composition of binary quadratic forms and the “higher composition laws” of the work of Bhargava and others, from which we derive many of our examples.

**Proposition 4.2.**

(a) Once a base orbit $\Gamma(K)x_0$ is fixed, there is a natural injection

$$\psi: \Gamma(K)\backslash V(K) \hookrightarrow H^1(K, M)$$

by which the orbits $\Gamma(K)\backslash V(K)$ parametrize some subset of the Galois cohomology group $H^1(K, M)$.

(b) The $\Gamma(K)$-stabilizer of every $x \in V(K)$ is canonically isomorphic to $H^0(K, M)$.

**Proof.** (a) Let $x \in V(K)$ be given. Since there is only one $\Gamma(\bar{K})$-orbit, we can find $\gamma \in \Gamma(\bar{K})$ such that $\gamma(x_0) = x$. For any $g \in \text{Gal}(\bar{K}/K)$, $g(\gamma)$ also takes $x_0$ to $x$ and so differs from $\gamma$ by right-multiplication by an element in $\text{Stab}_{\Gamma(\bar{K})} x_0 = M$. Define a cocycle $\sigma_x : \text{Gal}(\bar{K}/K) \to M$ by

$$\sigma_x(g) = g(\gamma) \cdot \gamma^{-1}.$$ 

It is routine to verify that
Proposition 4.3. The parametrizations \( \psi_{x_0}, \psi_{x_1} : \Gamma(K) \backslash V(K) \to H^1(K, M) \) corresponding to two basepoints \( x_0, x_1 \in V(K) \) differ only by translation:

\[
\psi_{x_1}(x) = \psi_{x_0}(x) - \psi_{x_0}(x_1),
\]

under the isomorphism between the stabilizers \( M \) established in the previous proposition.

Proof. Routine calculation.

While \( \psi \) is always injective, it need not be surjective, as we will see by examples in the following section.

Definition 4.4.

(a) A composed variety is \textit{full} if \( \psi \) is surjective, that is, it includes a \( \Gamma(K) \)-orbit for every cohomology class in \( H^1(K, M) \).

(b) If \( K \) is a global field, a composed variety is \textit{Hasse} if for every \( \alpha \in H^1(K, M) \), if the localization \( \alpha_v \in H^1(K_v, M) \) at each place \( v \) lies in the image of the local parametrization

\[
\psi_v : V(K_v) \backslash \Gamma(K_v) \to H^1(K_v, M),
\]

then \( \alpha \) also lies in the image of the global parametrization \( \psi \).
4.1 Examples

Composed varieties are plentiful. In this section, \( K \) is any field not of one of finitely many bad characteristics for which the exposition does not make sense.

**Example 4.5.** The group \( \Gamma = \mathbb{G}_m \) can act on the variety \( V = \mathbb{A}^1 \setminus \{0\} \), the punctured affine line, by

\[
\lambda(x) = \lambda^n \cdot x.
\]

There is a unique \( \overline{K} \)-orbit. The point stabilizer is \( \mu_n \), and the parametrization corresponding to this composed variety (choosing basepoint \( x_0 = 1 \)) is none other than the Kummer map

\[
K^\times/(K^\times)^n \rightarrow H^1(K, \mu_n).
\]

That \( V \) is full follows from Hilbert’s Theorem 90.

**Example 4.6.** Let \( V \) be the variety of binary cubic forms \( f \) over \( K \) with fixed discriminant \( D_0 \). This has an algebraic action of \( \text{SL}_2 \), which is transitive over \( \overline{K} \) (essentially because \( \text{PSL}_2 \) carries any three points of \( \mathbb{P}^1 \) to any other three), and there is a ready-at-hand basepoint

\[
f_0(X, Y) = X^2 Y - \frac{D}{4} Y^3.
\]

The point stabilizer \( M \) is isomorphic to \( \mathbb{Z}/3\mathbb{Z} \), but twisted by the character of \( K(\sqrt{D}) \); that is, \( M \cong \{0, \sqrt{D}, -\sqrt{D}\} \) as sets with Galois action. Coupled with the appropriate higher composition law (Theorem \( \text{K.N} \)), this recovers the parametrization of cubic \( \epsilon \)tal algebras with fixed quadratic resolvent by \( \text{H}^1(K, M) \) in Proposition \( \text{2.7} \). To see that it is the same parametrization, note that a \( \gamma \in \Gamma(\overline{K}/K) \) that takes \( f_0 \) to \( f \) is determined by where it sends the rational root \( [1 : 0] \) of \( f_0 \), so the three \( \gamma \)'s are permuted by \( \text{Gal}(\overline{K}/K) \) just like the three roots of \( f \). In particular, \( V \) is full.

**Remark 4.7.** Note that in this framework, \( \text{SL}_2 \) has priority over \( \text{GL}_2 \). When we consider \( \text{GL}_2 \)-orbits of binary cubic forms, the point stabilizer is \( S_3 \) (or even larger for the untwisted action: see \( \text{6.1} \) for a discussion of the twist), corresponding to the six automorphisms of \( K \times K \times K \). In the statement of a theorem like O-N (Theorem \( \text{1.1} \)), one can freely replace \( \text{GL}_2 \) by \( \text{SL}_2 \), or indeed any finite-index subgroup, since the class numbers \( h(D) \), \( h_3(D) \) simply scale by the index.

**Example 4.8.** Continuing with the sequence of known ring parametrizations, we might study the variety \( V \) of pairs of ternary quadratic forms with fixed discriminant \( D_0 \). This has one orbit over \( \overline{K} \) under the action of the group \( \Gamma = \text{SL}_2 \times \text{SL}_3 \); unfortunately, the point stabilizer is isomorphic to the alternating group \( A_4 \), which is not abelian.

So we narrow the group, which widens the ring of invariants and requires us to take a smaller \( V \). We let \( \Gamma = \text{SL}_3 \) alone act on pairs \( (A, B) \) of ternary quadratic forms, which preserves the resolvent

\[
g(X, Y) = 4 \text{det}(AX + BY),
\]

a binary cubic form. We let \( V \) be the variety of \( (A, B) \) for which \( g = g_0 \) is a fixed separable polynomial. These parametrize quartic \( \epsilon \)tal algebras \( L \) over \( K \) whose cubic resolvent \( R \) is fixed. There is a natural base orbit \( (A_0, B_0) \) whose associated \( L \cong K \times R \) has a linear factor. The point stabilizer \( M \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), with the three non-identity elements permuted by \( \text{Gal}(\overline{K}/K) \) in the same manner as the three roots of \( g_0 \). We have reconstructed the parametrization of quartic \( \epsilon \)tal algebras with fixed cubic resolvent by \( \text{H}^1(K, M) \) in Proposition \( \text{2.7} \). In particular, \( V \) is full.

**Example 4.9.** Alternatively, we can consider the space \( V \) of binary quartic forms whose invariants \( I = I_0 \), \( J = J_0 \) are fixed. The orbits of this space have been found useful for parametrizing 2-Selmer elements of the elliptic curve \( E : y^2 = x^3 + xI + J \), because the point stabilizer is \( M \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) with the Galois-module structure \( E[2] \). This space \( V \) embeds into the space of the preceding example via a map which we call the Wood embedding after its prominent role in Wood’s work \( \text{[53]} \):

\[
f \mapsto (A, B)
\]

\[
ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 \mapsto \begin{pmatrix}
1/2 & 1/2 & 1/2
\end{pmatrix}
\begin{pmatrix}
a & b/2 & c/3
b/2 & c/3 & d/2
c/3 & d/2 & e
\end{pmatrix}.
\]
In general, $V$ is not full. For instance, over $K = \mathbb{R}$, if $E$ has full 2-torsion, there are only three kinds of binary quartics over $\mathbb{R}$ with positive discriminant (positive definite, negative definite, and those with four real roots) which cover three of the four elements in $H^1(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$. Two of these three (positive definite, four real roots) form the subgroup of elements whose corresponding $E$-torsor $z^2 = f(x,y)$ is soluble at $\infty$: these are the ones we retain when studying $\text{Sel} E$. The fourth element of $H^1(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ yields étale algebras whose $(A,B)$ has
\[
A = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 1 \end{bmatrix},
\]
a conic with no real points. However, over global fields, it is possible to show that $V$ is Hasse, using the Hasse-Minkowski theorem for conics.

**Remark 4.10.** Because of the extreme flexibility afforded by general varieties, it is reasonable to suppose that any finite $K$-Galois module $M$ appears as the point stabilizer of some full composed variety over $K$. In fact, there are only finitely many orbits of $G$ acting on it, equipped with an identification of the generic fiber with $(V,\Gamma)$. Then $G(C)$, the group of globally regular elements of $G$, embeds into $\Gamma(K)$, and the $\Gamma(K)$-orbits on $V(K)$ decompose into $G(C)$-orbits.

### 4.2 Integral models; localization of orbit counts

Let $K$ be a global field, that is, either a number field or the function field of a curve over a finite field. Let $\mathcal{C}$ be the corresponding scheme of integers: that is, in the number-field case, $\mathcal{C} = \text{Spec} \mathcal{O}_K$ is affine, and in the function-field case, $\mathcal{C}$ is the appropriate projective curve. If $v$ is a place of $K$, we denote by $K_v$ the completion and $\mathcal{O}_v$ its ring of integers (if $v$ is archimedean, we set $\mathcal{O}_v = \mathcal{O}$.)

Let $(V, \Gamma)$ be a composed variety, and let $(\mathcal{V}, G)$ be an integral model, that is, a pair of a flat separated scheme and a flat algebraic group over $\mathcal{C}$ acting on it, equipped with an identification of the generic fiber with $(V,\Gamma)$. Then $G(C)$, the group of globally regular elements of $G$, embeds into $\Gamma(K)$, and the $\Gamma(K)$-orbits on $V(K)$ decompose into $G(C)$-orbits.

**Example 4.11.** If $K$ is a number field, the composed variety $(V, \Gamma)$ of binary cubic forms over $\mathcal{O}_K$ of discriminant $D_0 \in \mathcal{O}_K$, studied in Example 4.6, has an integral model consisting of $G = SL_2$, considered as an algebraic group over $\mathcal{O}_K$, and the variety $V$ of binary cubic forms over $\mathcal{O}_K$ of discriminant $D_0$. Alternatively, the variety $V'$ of binary cubic $1331$-forms over $\mathcal{O}_K$ of discriminant $D_0$ furnishes another integral model. If $K$ has a nontrivial class group, there are even more integral models that will show up in the statement of Theorem 6.3.

The preceding example is indicative of the level of generality in which we will work. However, we write the definitions and results of this section abstractly, to enable wide applicability.

**Lemma 4.12 (localization of global class numbers).** Let $(\mathcal{V}, G)$ be an integral model for a composed variety $(V, \Gamma)$. For each place $v$, let
\[
w_v : G(\mathcal{O}_v) \setminus V(\mathcal{O}_v) \to \mathbb{C}
\]
be a function on the local orbits, which we call a local weighting. Suppose that:

(i) $(V, \Gamma)$ is Hasse.

(ii) $G$ has class number one (in another terminology, satisfies strong approximation), that is, the natural localization embedding

\[G(C) \setminus \Gamma(K) \to \bigoplus_v G(\mathcal{O}_v) \setminus \Gamma(K_v)\]

is surjective.

(iii) For each place $v$, there are only finitely many orbits of $G(\mathcal{O}_v)$ on $V(\mathcal{O}_v)$. This ensures that the weighted local orbit counter
\[
g_v : H^1(K_v, M) \to \mathbb{C}, \quad \alpha \mapsto \sum_{\gamma \in G(\mathcal{O}_v) \setminus V(\mathcal{O}_v)} w_v(\gamma x_\alpha)
\]

is surjective.
takes finite values. (Here \( x, \) is a representative of the \( \Gamma(K_v) \)-orbit corresponding to \( \alpha \). If there is no such orbit because \( V \) is not full, we take \( g_{v,w_v}(\alpha) = 0 \).)

(iv) For almost all \( v \), \( G(\mathcal{O}_v) \backslash V(\mathcal{O}_v) \) consists of at most one orbit in each \( \Gamma(K_v) \)-orbit, and \( w_v = 1 \) identically.

Then the global integral points \( V(C) \) consist of finitely many \( G(C) \)-orbits, and the global weighted orbit count can be expressed in terms of the \( g_{v,w_v} \) by

\[
h_{w_v} := \sum_{G(C)x \in G(C) \backslash V(C)} \frac{\prod_v w_v(x)}{|\text{Stab}_{G(C)} x|} = \frac{1}{|H^0(K,M)|} \sum_{\alpha \in H^1(K,M)} \prod_v g_{v,w_v}(\alpha). \tag{4.1}
\]

**Proof.** Grouping the \( G(C) \)-orbits into \( \Gamma(K) \)-orbits, it suffices to prove that for all \( \alpha \in H^1(K,M) \),

\[
\sum_{G(C)x \in \Gamma(K)x_{\alpha}} \frac{\prod_v w_v(x)}{|\text{Stab}_{G(C)} x|} = \frac{1}{|H^0(K,M)|} \prod_v g_{v,w_v}(\alpha). \tag{4.2}
\]

If there is no \( x_{\alpha} \), the left-hand side is zero by definition, and at least one of the \( g_{v,w_v}(\alpha) \) is also zero since \( V \) is Hasse, rendering (4.2) trivial. So we fix an \( x_{\alpha} \). The right-hand side of (4.2), which is finite by hypothesis (iv) since \( \alpha \) is unramified almost everywhere, can be written as

\[
\frac{1}{|H^0(K,M)|} \sum_{G(C)\gamma \in G(C)\backslash V(C)} \prod_v w_v(\gamma_v x_{\alpha}),
\]

the sum being over systems of \( \gamma_v \in \Gamma(K_v) \) such that \( \gamma_v x_{\alpha} \) is \( \mathcal{O}_v \)-integral. Since \( G \) has class number one, each such system glues uniquely to a global orbit \( G(C)\gamma, \gamma \in \Gamma(K) \), for which \( \gamma x_{\alpha} \) is \( \mathcal{O}_v \)-integral for all \( v \), that is, \( C \)-integral. Thus the right-hand side of (4.2) is now transformed to

\[
\frac{1}{|H^0(K,M)|} \sum_{G(C)\gamma \in G(C)\backslash V(C)} \prod_v w_v(\gamma_v x_{\alpha}).
\]

Now each \( \gamma \) corresponds to a term of the left-hand side of (4.2) under the map

\[
G(C) \backslash \Gamma(K) \rightarrow G(C) \backslash V(K)
\]

\[
G(C)\gamma \mapsto G(C)\gamma x_{\alpha}.
\]

The fiber of each \( G(C)x \) has size

\[
|\text{Stab}_{\Gamma(K)} x : \text{Stab}_{G(C)} x| = \frac{|H^0(K,M)|}{|\text{Stab}_{G(C)} x|}.
\]

So we match up one term of the left-hand side, having value

\[
\prod_v w_v(x)/|\text{Stab}_{G(C)} x|,
\]

with \( |H^0(K,M)|/|\text{Stab}_{G(C)} x| \)-many elements on the right-hand side. Multiplying by the outlying factor \( 1/|H^0(K,M)| \), this completes the proof. \( \blacksquare \)

### 4.3 Fourier analysis of the local and global Tate pairings

We now introduce the main innovative technique of this paper: Fourier analysis of local and global Tate duality. In structure we are indebted to Tate’s celebrated thesis [50], in which he

1. constructs a perfect pairing on the additive group of a local field \( K \), taking values in the unit circle \( \mathbb{C}^{N=1} \), and thus furnishing a notion of Fourier transform for \( \mathbb{C} \)-valued \( L^1 \) functions on \( K \);
2. derives thereby a pairing and Fourier transform on the adele group $\mathbb{A}_K$ of a global field $K$;

3. proves that the discrete subgroup $K \subseteq \mathbb{A}_K$ is a self-dual lattice and that the Poisson summation formula

$$
\sum_{x \in K} f(x) = \sum_{x \in K} \hat{f}(x)
$$

holds for all $f$ satisfying reasonable integrability conditions.

In this paper, we work not with the additive group $K$ but with a Galois cohomology group $H^1(K, M)$. The needed theoretical result is Poitou-Tate duality, a nine-term exact sequence of which the middle three terms are of main interest to us:

$$(\text{finite kernel}) \to H^1(K, M) \to \bigoplus_v H^1(K_v, M) \to H^1(K, M')^\vee \to (\text{finite cokernel}).$$

(While the finite kernel and cokernel are often nonzero, in all the examples that will be of interest to us, they will vanish: but this will not affect our proofs.) This can be interpreted as saying that $H^1(K, M)$ and $H^1(K, M')$ (where $M' = \text{Hom}(M, \mu)$ is the Tate dual) map to dual lattices in the respective adelic cohomology groups

$$H^1(\mathbb{A}_K, M) = \bigoplus_v H^1(K_v, M) \quad \text{and} \quad H^1(\mathbb{A}_K, M') = \bigoplus_v H^1(K_v, M'),$$

which are mutually dual under the product of the local Tate pairings

$$\langle \{\alpha_v\}, \{\beta_v\} \rangle = \prod_v \langle \alpha_v, \beta_v \rangle \in \mu.$$

Here, for $K$ a local field, the local Tate pairing is given by the cup product

$$(\bullet, \bullet) : H^1(K, M) \times H^1(K, M') \to H^2(K, \mu) \cong \mu.$$

It is well known that this pairing is perfect. (The Brauer group $H^2(K, \mu)$ is usually described as being $\mathbb{Q}/\mathbb{Z}$ but, having no need for a Galois action on it, we identify it with $\mu$ to avoid the need to write an exponential in the Fourier transform.) Now, for any sufficiently nice function $f : H^1(\mathbb{A}_K, M) \to \mathbb{C}$ (locally constant and compactly supported is more than enough), we have Poisson summation

$$\sum_{\alpha \in H^1(K, M)} f(\alpha) = c_M \sum_{\beta \in H^1(K, M')} \hat{f}(\beta)$$

for some constant $c_M$ which we think of as the covolume of $H^1(K, M)$ as a lattice in the adelic cohomology. (In fact, by examining the preceding term in the Poitou-Tate sequence, $H^1(K, M)$ need not inject into $H^1(\mathbb{A}_K, M)$, but maps in with finite kernel; but this subtlety can be absorbed into the constant $c_M$.)

We apply Poisson summation to the local orbit counters $g_i$ defined in the preceding subsection and get a very general reflection theorem.

**Definition 4.13.** Let $K$ be a local field. Let $(V^{(1)}, \Gamma^{(1)})$ and $(V^{(2)}, \Gamma^{(2)})$ be a pair of composed varieties over $K$ whose associated point stabilizers $M^{(1)}$, $M^{(2)}$ are Tate duals of one another, and let $(V^{(i)}, G^{(i)})$ be an integral model of $(V^{(i)}, \Gamma^{(i)})$. Two weightings on orbits

$$w^{(i)} : G(\mathcal{O}_K) \setminus V(\mathcal{O}_K) \to \mathbb{C}$$

are called (mutually) dual with duality constant $c \in \mathbb{Q}$ if their local orbit counters $g_{w^{(i)}}$ are mutual Fourier transforms:

$$g^{(2)} = c \cdot \hat{g}^{(1)}.$$  \hfill (4.4)

where the Fourier transform is scaled by

$$\hat{f}(\beta) = \frac{1}{H^0(K, M)} \sum_{\alpha \in H^1(K, M)} f(\alpha).$$

An equation of the form (4.4) is called a local reflection theorem. If the constant weightings $w^{(1)} = 1$ are mutually dual, we say that the two integral models $(V^{(i)}, G^{(i)})$ are naturally dual.
Theorem 4.14 (local-to-global reflection engine). Let $K$ be a number field. Let $(V^{(1)},\Gamma^{(1)})$ and $(V^{(2)},\Gamma^{(2)})$ be a pair of composed varieties over $K$ whose associated point stabilizers $M^{(1)}$, $M^{(2)}$ are Tate duals of one another. Let $(V^{(i)},\mathcal{G}^{(i)})$ be an integral model for each $(V^{(i)},\Gamma^{(i)})$, and let

$$w_v^{(i)} : \mathcal{G}^{(i)}(\mathcal{O}_v) \setminus V^{(i)}(\mathcal{O}_v) \to \mathbb{C}$$

be a local weighting on each integral model. Suppose that each integral model and local weighting satisfies the hypotheses of Lemma 4.12 and suppose that at each place $v$, the two integral models are dual with some duality constant $c_v \in \mathbb{Q}$. Then the weighted global class numbers are in a simple ratio:

$$\hat{h}\{w_v^{(2)}\} = \prod_v c_v \cdot \hat{h}\{w_v^{(1)}\}.$$

Proof. By Lemma 4.12,

$$\hat{h}\{w_v^{(1)}\} = \frac{1}{|H^0(K, M^{(i)})|} \sum_{\alpha \in H^1(K, M^{(i)})} \prod_v g_{v,w_v^{(i)}}(\alpha).$$

At almost all $v$, each $g_{v,w_v^{(i)}}$ is supported on the unramified cohomology, and must be constant there because otherwise its Fourier transform would not be supported on the unramified cohomology. However, $g_{v,w_v^{(i)}}$ cannot be identically 0 because of the existence of a global basepoint. So for such $v$,

$$g_{v,w_v^{(i)}} = 1_{H^0_w(K, M^{(i)})} \quad \text{and} \quad c_v = 1.$$

In particular, the product $\prod_v g_{v,w_v^{(i)}}$ is a locally constant, compactly supported function on $H^1(K, A_K)$, which is more than enough for Poisson summation to be valid.

Since the pairing between the adelic cohomology groups $H^1(A_K, M^{(i)})$ is made by multiplying the local Tate pairings, a product of local factors has a Fourier transform with a corresponding product expansion:

$$\prod_v \hat{g}_{v,w_v^{(i)}} = \prod_v \hat{g}_{v,w_v^{(1)}} = \prod_v c_v \cdot \prod_v \hat{g}_{v,w_v^{(2)}}.$$

We then apply Poisson summation to get a formula for the ratio of the global weighted class numbers:

$$\hat{h}\{w_v^{(2)}\} = \frac{|H^0(K, M^{(1)})|}{|H^0(K, M^{(2)})|} \cdot \frac{c_{M^{(1)}}}{c_{M^{(2)}}} \cdot \prod_v c_v \cdot \hat{h}\{w_v^{(1)}\}.$$

This gives the desired identity, except for determining the scalar $c_M$, which depends only on the Galois module $M = M^{(1)}$. This can be ascertained by applying Poisson summation to just one function $f : H^1(A_K, M) \to \mathbb{C}$ for which either side is nonzero. The easiest such $f$ to think of is the characteristic function of a compact open box

$$X = \prod_v X_v,$$

with $X_v = H^1(K, M)$ for almost all $v$. Such a specification is often called a Selmer system, and the sum

$$\sum_{\alpha \in H^1(K, M)} 1_{\alpha \in X_v \forall v}$$

is the order of the Selmer group $\text{Sel}(X)$ of global cohomology classes obeying the specified local conditions. Poisson summation becomes a formula for the ratio $|\text{Sel}(X)|/|\text{Sel}(X^1)|$ as a product of local factors, commonly known as the Greenberg-Wiles formula. By appealing to any of the known proofs of the Greenberg-Wiles formula (see Darmon, Diamond, and Taylor [16, Theorem 2.19] or Jorza [27, Theorem 3.11]), we pin down the value

$$c_M = \frac{|H^0(K, M')|}{|H^0(K, M)|}$$

which arises from a global Euler-characteristic computation. $\blacksquare$
At certain points in this paper, it will be to our advantage to consider multiple integral models at once. The following theorem has sufficient generality.

**Theorem 4.15 (local-to-global reflection engine: general version).** Let $K$ be a global field. Let $(V^{(1)}, \Gamma^{(1)})$ and $(V^{(2)}, \Gamma^{(2)})$ be a pair of composed varieties over $K$ whose associated point stabilizers $M^{(1)}$, $M^{(2)}$ are Tate duals of one another. For each place $v$ of $K$, let

$$\left\{ (V_{jv}^{(i)}, \mathcal{G}^{(i)}_{jv}) : jv \in J^{(i)}_v \right\}$$

be a family of integral models for each $(V^{(i)})$ indexed by some finite set $J^{(i)}_v$, and let

$$w^{(i)}_{jv} : \mathcal{G}^{(i)}(\mathcal{O}_v) \backslash V^{(i)}(\mathcal{O}_v) \rightarrow \mathbb{C}$$

be a weighting on the orbits of each integral model. Similarly to Lemma 4.12 and Theorem 4.14, assume that

1. $(V, \Gamma)$ is Hasse.
2. For each combination of indices $j = (jv)_v$, $jv \in J^{(1)}_v$, the local integral models $(V_{jv}^{(i)}, \mathcal{G}^{(i)}_{jv})$ glue together to form a global integral model $(V_{jv}^{(1)}, \mathcal{G}^{(1)}_{jv})$. (Since the integral models are equipped with embeddings $\mathcal{G}^{(1)}_{jv} \rightarrow V^{(1)}_v$, the gluing is seen to be unique; and its existence will be obvious in all the examples we consider.)
3. Each such $\mathcal{G}^{(1)}_{jv}$ has class number one.
4. For each $jv$, there are only finitely many orbits of $\mathcal{G}^{(1)}_{jv}$ on $V^{(1)}_v$, ensuring that the local orbit counter $g_{jv, w_{jv}}$ takes finite values.
5. For almost every $v$, the index set $J_v = \{ jv \}$ has just one element, with the corresponding integral model $V^{(1)}_v$ consisting of at most one orbit in each $\Gamma(K_v)$-orbit, and $w^{(i)}_{jv} = 1$ identically.
6. At each $v$, we have a local reflection theorem

$$\sum_{jv \in J^{(1)}_v} g_{jv, w_{jv}} = \sum_{jv \in J^{(2)}_v} g_{jv, w_{jv}}.$$  

Then the class numbers of the global integral models $(V_{jv}^{(i)}, \mathcal{G}^{(i)}_{jv})$ with respect to the weightings $w^{(i)}_{jv} = \prod_v w^{(i)}_{jv}$ satisfy global reflection:

$$\sum_{j \in \prod_v J^{(i)}_v} h \left( V^{(i)}_j, w^{(i)}_j \right) = \sum_{j \in \prod_v J^{(2)}_v} h \left( V^{(2)}_j, w^{(2)}_j \right).$$

**Proof:** Except for complexities of notation, the proof closely follows the preceding one. The first five hypotheses ensure that each global integral model $(V_{jv}^{(i)}, \mathcal{G}^{(i)}_{jv})$ satisfies the hypotheses of Lemma 4.12 so its class number is representable as a sum over the lattice of global points in adelic cohomology:

$$h \left( V^{(i)}_j, w^{(i)}_j \right) = \frac{1}{|H^0(K, M^{(i)})|} \sum_{\alpha \in H^1(K, M^{(i)})} \prod_v g_{jv, w_{jv}}(\alpha).$$

When we sum over all $j$, the contributions of each $\alpha$ factorize to give

$$\sum_{j \in \prod_v J^{(i)}_v} h \left( V^{(i)}_j, w^{(i)}_j \right) = \frac{1}{|H^0(K, M^{(i)})|} \sum_{\alpha \in H^1(K, M^{(i)})} \prod_v \sum_{jv \in J^{(i)}_v} g_{jv, w_{jv}}(\alpha).$$

But by the assumed local reflection identity, we have

$$\left( \prod_v \sum_{jv \in J^{(i)}_v} g_{jv, w_{jv}} \right) \overset{\wedge}{=} \prod_v \left( \sum_{jv \in J^{(i)}_v} g_{jv, w_{jv}} \right) \overset{\wedge}{=} \prod_v \sum_{jv \in J^{(2)}_v} g_{jv, w_{jv}}.$$  

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So we get the desired identity from Poisson summation. The scale factor was determined in proving the previous theorem.

**Remark 4.16.** Unlike in the previous theorem, we have not included duality constants $c_v$, but the same effect can be obtained by taking the appropriate constant for the weighting $w^{(i)}_j$.

### 4.4 Examples

As one might guess, there are many pairs of composed varieties whose point stabilizers $M^{(1)}$, $M^{(2)}$ are Tate duals; and, given any integral models, it is usually possible to concoct weights $w^{(i)}$ that are mutually dual, thereby getting reflection theorems from Theorem 4.14. More noteworthy is when a pair of integral models are naturally dual at all finite places. Even more significant is if a group $G$ acts on a large variety $\Lambda$, leaving certain functions $I$ on $\Lambda$ invariant, such that *every* level set of $I$ is an integral model for a composed variety with natural duality. This is the case for O-N.

We have found three families of naturally dual composed varieties of this sort:

| $\Gamma$ | $\Lambda$ | $I$ | Parametrizes | $M$ |
|---|---|---|---|---|
| $\left\{ \begin{bmatrix} \lambda & t \\ 0 & \lambda^{-2} \end{bmatrix} \right\} \subset \text{GL}_2$ | Quadratic forms, $\text{Sym}^2(2)$ | $a(b^2 - 4ac)$ | ? | $C_2$ |
| $\text{SL}_2$ | Cubic forms, $\text{Sym}^3(2)$ | Discriminant | Cubic rings / 3-torsion in quadratic rings | $C_3$ |
| $\text{SL}_3$ | Pairs of ternary quadratic forms, $\text{Sym}^2(3) \oplus 2$ | Cubic resolvent | 2-torsion in cubic rings | $C_2 \times C_2$ |

These three representations will be considered in detail in Section 5 and in a sequel to this paper, respectively. In each case, there is a local reflection that pairs two integral models of $\mathcal{V}$ over $\mathcal{O}_K$ which look alike over $K$.

**Remark 4.17.** In the latter two cases, the integral models are *dual* under an identification of $\mathcal{V}$ with its dual $\mathcal{V}^*$ (which are isomorphic, up to an outer automorphism of $\Gamma = \text{SL}_3$ in the last case). But in the quadratic case, $\mathcal{V}^*$ decomposes into $K$-orbits according to a different invariant $J = a/\Delta^2$, and the integral orbit counts are infinite, so the alignment with duals in the classical sense must be considered at least partly coincidental.

Closely related to the quartic rings example is the action of $\text{SL}_2$ on binary quartic forms $\text{Sym}^4(2)$. Here, the orbits are parametrized by a subset of a cohomology group $H^1(K, M) \cong C_2 \times C_2$ (as a group) cut out by a quadratic relation. Nevertheless, we will state some interesting reflection identities for these spaces in a sequel to this paper.

More generally, we can consider the space $\Lambda$ of pairs $(A, B)$ of $n$-ary quadratic forms, on which $\Gamma = \text{SL}_n$ acts preserving a binary $n$-ic resolvent

$$I = \det(Ax - By).$$

Although we do not consider it in this paper, preliminary investigations suggest that its integral models are naturally dual to one another for $n$ odd, yielding a corresponding global reflection theorem. This composed variety figures prominently in the study of Selmer elements of hyperelliptic curves [7].

On the other hand, the following families of composed varieties do not admit natural duality:

- The action of $\mathbb{G}_m$ on the punctured affine line by multiplication by $n$th powers. The orbits do parametrize $H^1(K, \mu_m) = \mathbb{K}^\times/(\mathbb{K}^\times)^n$. But over a local or global field, there are infinitely many integral orbits in each rational orbit.

- The action of $\text{SO}_2$ (the group of rotations preserving the quadratic form $x^2 + xy + y^2$) on binary cubic forms of the shape

$$f(x, y) = ax^3 + bx^2y + (-3a + b)xy^2 + ay^3$$

which is symmetric under the threefold shift $x \mapsto y$, $y \mapsto -x - y$. This representation is used by Bhargava and Shnidman [9] to parametrize cyclic cubic rings, that is, those with an automorphism of
order 3. The failure of natural duality is not hard to see. Within the representation over $\mathbb{Z}_p$ for $p \equiv 1 \mod 3$, take the composed variety where the discriminant is $p^2$. The cohomology group $H^1(\mathbb{Q}_p, \mathbb{Z}/3\mathbb{Z})$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, and a function $f$ on it may be written as a matrix

$$
\begin{array}{c|ccc}
  f(0) & f(\alpha) & f(2\alpha) \\
  f(\beta) & f(\alpha + \beta) & f(2\alpha + \beta) \\
  f(2\beta) & f(\alpha + 2\beta) & f(2\alpha + 2\beta)
\end{array}
$$

in which the zero-element and the unramified cohomology $\langle \alpha \rangle$ are marked off by dividers.

The six ramified cohomology elements each have one integral orbit, corresponding to the maximal order; the three unramified cohomology elements—the zero element for $\mathbb{Q}_p^3$, and the other two for the degree-3 unramified field extension in its two orientations—all have no integral orbits, because the three orders

$$\{(x_1, x_2, x_3) \in \mathbb{Z}_p^3 : x_i \equiv x_j \mod p\}$$

are all asymmetric under the threefold automorphism of $\mathbb{Z}_p^3$. So we get a local orbit counter

$$
\begin{array}{c|ccc}
  0 & 0 & 0 \\
  1 & 1 & 1 \\
  1 & 1 & 1
\end{array}
$$

whose Fourier transform

$$
\begin{array}{c|ccc}
  2 & -1 & -1 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{array}
$$

has mixed signs and thus cannot be the local orbit counter of any composed variety. Similar obstructions to natural duality appear in many of the composed varieties parametrizing rings with automorphisms found by Gundlach [25].

5 Quadratic forms by superdiscriminant

We begin with the simplest Galois module $M \cong \mathbb{Z}/2\mathbb{Z}$.

There are many full composed varieties whose point stabilizer is of order 2, and the one we take is, to say the least, one of the more unexpected. The group $\text{GL}_2$ acts on the space $V = \operatorname{Sym}^2(2) = \{ax^2 + bxy + cy^2 : a, b, c \in \mathbb{G}_a\}$

of binary quadratic forms in the natural way. Let $\Gamma$ be the algebraic subgroup, defined over $\mathbb{Z}$, of elements of a peculiar form:

$$\left\{ \begin{bmatrix} u & t \\ 0 & u^{-2} \end{bmatrix} : u \in \mathbb{G}_m, t \in \mathbb{G}_a \right\}.$$

Abstractly, this group is a certain semidirect product of $\mathbb{G}_a$ by $\mathbb{G}_m$. As is not too hard to verify, the restriction of $\Lambda$ to $\Gamma$ has a single polynomial invariant, the superdiscriminant

$$I := aD = a(b^2 - 4ac).$$

Because $\Gamma$ fixes the point $[x : y] = [1 : 0]$, there is no harm in writing forms in $V$ inhomogeneously as $f(x) = ax^2 + bx + c$, as was done in stating Theorem 1.4

Remark 5.1. The group $\Gamma$ is not reductive, that is, does not fit into the classical Dynkin-diagram parametrization for Lie groups. Non-reductive groups are decidedly in the minority within the whole context of using orbits to parametrize arithmetic objects, but they have occurred before: Altuğ, Shankar, Varma, and Wilson [2] count $D_4$-fields using orbits of pairs of ternary quadratic forms under a certain nonreductive subgroup of $\text{GL}_2 \times \text{SL}_3$. 

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Then the variety
\[ V(I) = \{ f \in V : I(f) = I \} \]
is full composed. We take the basepoint
\[ f_0 = Ix^2 + \frac{1}{4I} \]
of discriminant 1. Then the rational orbits are parametrized by \( D = b^2 - 4ac \in K^\times/(K^\times)^2 \) consistent with the parametrization of their splitting fields via Kummer theory.

Now we introduce integral models. Suppose \( \mathcal{O}_K \subseteq K \) is a PID with field of fractions \( K \). If \( \tau \in \mathcal{O}_K \) divides 2, then
\[ V_\tau = \{ ax^2 + bx + c : a, c \in \mathcal{O}_K, \tau \mid b \} \]
is a \( \text{GL}_2(\mathcal{O}_K) \)-invariant lattice in \( V \). For any \( I \in \mathcal{O}_K \), we can take \( (V, \mathcal{G}) = (V_\tau(I), \Gamma(\mathcal{O}_K)) \) as an integral model for \( V(I) \). For it to have any integral points, we must have \( \tau^2 \mid I \).

Our first local reflection theorem says that each of these integral models has a natural dual.

**Theorem 5.2 ("Local Quadratic O-N")**. Let \( K \) be a non-archimedean local field, \( \text{char} \, K \neq 2 \). For \( I, \tau \in \mathcal{O}_K \) elements dividing 2, the integral models
\[ V_\tau(I) \text{ and } V_{2^{e-1}}(4\tau^{-4}I) \]
are naturally dual with scale factor the absolute norm \( N(\tau) = |\mathcal{O}_K/\tau\mathcal{O}_K| \). In other words, the local orbit counters are related by
\[ \hat{g}_{\tau}(I) = N(\tau) \cdot \hat{g}_{2^{e-1}(4\tau^{-4}t)} \cdot \tau \]

**Proof.** We prove this result by explicitly computing the local orbit counter \( g_{\tau}(I) \), which sends each \( [D] \in K^\times/(K^\times)^2 \cong H^1(K, \mathbb{Z}/2\mathbb{Z}) \) to the number of cosets \( [\gamma] \in \Gamma(\mathcal{O}_K) \backslash \Gamma(K) \) such that \( \gamma v_0 \in V_\tau(I)\mathcal{O}_K \), where \( v_0 \) is an arbitrary vector in \( V(K) \) with \( I(v_0) = I \) and \( D(v_0) = D \). Let \( t = v(\tau) \) and \( e = v(2) \); we have \( e > 0 \) exactly when \( K \) is 2-adic, and
\[ 0 \leq t \leq e. \]

A coset \( [\gamma] \) is specified by two pieces of information. First is the valuation \( v(u) \) of the diagonal elements; this is equivalent to specifying \( v(D) \) and \( v(a) \), where, as is natural we set
\[ \gamma v_0 = ax^2 + bxy + cy^2 \text{ and } D = b^2 - 4ac. \]

Second, we specify \( t \) modulo the appropriate integral sublattice. If (as we may assume) \( v(u) = 0 \), then \( t \) is defined modulo 1, which is the same as specifying \( b \) modulo 2a. So the problem of computing \( g_{\tau}(I) \) devolves onto computing how many \( b \in \tau\mathcal{O}_K \), up to translation by \( 2a \), yield an integral value for
\[ c = \frac{b^2 - D}{4a}; \]
that is, we must solve the quadratic congruence
\[ b^2 \equiv D \mod 4a. \]

The answer, in general, depends on how close \( D \) is to being a square in \( K \). So we will express our answer in terms of the level spaces introduced in Proposition 2.17. Here the level of a coclass \( [\alpha] \), \( \alpha \in K^\times \), is defined in terms of the discriminant of \( K[\sqrt{\alpha}] \), which, by Proposition 2.11, can be computed from the minimal distance \( |\alpha - 1| \), over all rescalings of \( \alpha \) by squares. The level spaces thus correspond to the natural filtration of \( K^\times/(K^\times)^2 \) by neighborhoods of 1:
\[ L_i = \begin{cases} K^\times/(K^\times)^2, & i = -1 \\ \{ [\alpha] \in \mathcal{O}_K^\times/(\mathcal{O}_K^\times)^2 : \alpha \in 1 + \pi^{2e-2i}\mathcal{O}_K \}, & 0 \leq i \leq e \\ \{ 1 \}, & i = e + 1. \end{cases} \]

Let \( L_i \) be the characteristic function of \( L_i \). By Corollary 2.18, the Fourier transform of each \( L_i \) is a scalar multiple of \( L_{e-i} \).
group the solutions into $S$ families. In particular, group homomorphism is called the thickness $\tilde{1}$ family supported just on the trivial class and the cokernel of this homomorphism has size $\left\lceil \frac{m}{2} \right\rceil$. The basic idea, which will be a recurring one, is to group the solutions into families that have a constant number of solutions over some subset $S \subseteq K^\times/(K^\times)^2$. The subset $S$ will be called the support of the family, and the number of solutions for each $D \in S$ will be called the thickness of the family.

The condition $b \in (\tau)$ is equivalent to $v(D) \geq 2t$. If $v(D) \geq v(4a)$, then (5.2) simplifies to $4a|b^2$, that is,

$$v(b) \geq \left\lceil e + \frac{1}{2}v(a) \right\rceil.$$ 

Since we are counting values of $b$ modulo $2a$, the number of solutions is simply

$$q^{(e+v(a)) - \left\lfloor e - \frac{1}{2}v(a) \right\rfloor} = q^{\left\lfloor \frac{1}{2}v(a) \right\rfloor}.$$ 

We get a family with this thickness, supported on either $\mathcal{L}_0$ or $\mathcal{L}_{-1} \setminus \mathcal{L}_0$ according as $v(D)$ is even or odd.

If $v(D) < v(4a)$, then $b^2$ must be actually able to cancel at least the leading term of $D$ to get any solutions. In particular, $v(D)$ must be even. Let $\tilde{D} = D/\pi^{v(D)}$, and let $\tilde{b} = b/\pi^{\frac{1}{2}v(D)}$, so $\tilde{b}$ must be a unit satisfying

$$\tilde{b}^2 \equiv \tilde{D} \mod \frac{4a}{D}. \tag{5.3}$$

Let $m = v(4a/D)$. If $m \geq 2e + 1$, a unit is a square modulo $\pi^m$ only if it is a square outright, so we get a family supported just on the trivial class $1 \in K^\times/(K^\times)^2$. Otherwise, we have $1 \leq m \leq 2e$, and the support is $L_{[m/2]}$. The corresponding thicknesses are easy to compute. The $\tilde{b}$ satisfying (5.3) form a fiber of the group homomorphism

$$\phi = \ast^2 : \left( \mathcal{O}_K/\pi^{v(2a)} \right)^\times \to \left( \mathcal{O}_K/\pi^{v(4a)} \right)^\times,$$

and the cokernel of this homomorphism has size $[\mathcal{L}_0 : \mathcal{L}_i]$, so the thickness is

$$|\ker \phi| = [\mathcal{L}_0 : \mathcal{L}_i] \cdot \frac{\left| \left( \mathcal{O}_K/\pi^{v(2a)} \right)^\times \right|}{\left| \left( \mathcal{O}_K/\pi^{v(4a)} \right)^\times \right|} = [\mathcal{L}_0 : \mathcal{L}_i] \cdot \frac{(1 - \frac{1}{q}) \cdot q^{v(2a)} - \frac{1}{2}v(D)}{(1 - \frac{1}{q}) \cdot q^{v(4a)} - v(D)} = [\mathcal{L}_0 : \mathcal{L}_i] \cdot q^v(D - e) \cdot \begin{cases} q^{v(D) - e + \left\lceil m/2 \right\rceil} = q^{v(a)/2}, & 1 \leq m \leq 2e \\ 2q^{v(D)}, & m \geq 2e + 1. \end{cases}$$

By way of illustration, we tabulate the contributions to $g_{\tau,t}$ when $e = 2$ in Table 1. By pairing $L_i$ with $L_{2-i}$, it is already easy to check many examples of Theorem 5.2. We have shown the subdivision of the table into three zones given by the inequalities:

- **Zone I:** $v(D) \geq v(4a)$

| $\downarrow v(a); v(D) \to$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------------------------|---|---|---|---|---|---|---|---|---|
| 0 | $L_2$ | $L_1$ | $L_0$ | $L_1 - L_0$ | $L_0$ | $L_1 - L_0$ | $L_0$ | $L_0$ | $L_0$ |
| 1 | $L_2$ | $L_1$ | $L_0$ | $L_0$ | $L_0$ | $L_1 - L_0$ | $L_0$ | $L_0$ | $L_0$ |
| 2 | $L_3$ | $qL_2$ | $qL_1$ | $qL_0$ | $q(L_1 - L_0)$ | $qL_0$ | $qL_0$ | $qL_0$ | $qL_0$ |
| 3 | $L_3$ | $qL_3$ | $qL_1$ | $qL_0$ | $q(L_1 - L_0)$ | $qL_0$ | $qL_0$ | $qL_0$ | $qL_0$ |
| 4 | $L_3$ | $qL_3$ | $q^2L_2$ | $qL_1$ | $q^2L_0$ | $q^2L_0$ | $q^2L_0$ | $q^2L_0$ | $q^2L_0$ |

Table 1: Families of quadratic forms with $e = 2$
Figure 1: Three zones used in the local counting of quadratic forms by superdiscriminant

- Zone II: \( v(a) < v(D) \leq v(4a) \)
- Zone III: \( v(a) > v(D) \).

In general, the shapes of these zones, together with the needed condition \( v(D) \geq 2t \), will look as in Figure 1. The feature to be noted is that, under the transformation \( t \mapsto e - t \), the shape of Zone II is flipped about a diagonal line and Zones I and III are interchanged. This will be the basis for our proof of Theorem 5.2; but of a computer, it is adaptable to proving other reflection theorems.

Reflection computationally, by means of a generating function. We present the second method; with the aid of a computer, it is adaptable to proving other reflection theorems.

Let

\[
F(Z) = \sum_{n \geq 0} g_{\tau, n} Z^n,
\]

a formal power series whose coefficients are functions of \( D \in K^\times/(K^\times)^2 \). We write \( F = F_1 + F_{II} + F_{III} \), where \( F_X \) is the contribution coming from Zone \( X \) in the preceding analysis.

Writing \( i = v(a) \) and \( d = v(D) \), we proceed to compute

\[
F_1 = \sum_{t \geq 0} \sum_{d \geq 2t + 1} \left\{ \begin{array}{ll} q^{i/2} L_0 Z^{t+d}, & d \text{ even} \\
q^{i/2} (L_{-1} - L_0) Z^{t+d}, & d \text{ odd} \end{array} \right.
\]

\[
= Z^{2e} \sum_{t \geq 0} \sum_{d \geq t + 1} \left\{ \begin{array}{ll} q^{i/2} L_0 Z^{t+d}, & d \text{ even} \\
q^{i/2} (L_{-1} - L_0) Z^{t+d}, & d \text{ odd} \end{array} \right.
\]

Splitting \( i = 2t f + t_p \), where \( 0 \leq t_p \leq 1 \), and likewise \( d = 2d f + d_p \), we get

\[
F_1 = Z^{2e} \sum_{t_p = 0}^1 \sum_{t_f \geq 0} \left( \sum_{d_f \geq t_f} q^{i_f} (-1)^d L_0 Z^{d+2t_f+t_p} + \sum_{d_f \geq t_f} q^{i_f} L_{-1} Z^{2d_f+2t_f+t_p+1} \right)
\]

\[
= Z^{2e} \sum_{t_p = 0}^1 \sum_{t_f \geq 0} \left( \frac{q^{i_f} (-1)^t L_0 Z^{2t_f+t_p+1}}{1 + Z} + \frac{q^{i_f} Z^{t_f+t_p+1} L_{-1}}{1 - Z^2} \right)
\]

\[
= Z^{2e} \sum_{t_p = 0}^1 \left( \frac{(-1)^t Z^{2t_p} L_0}{(1 + Z)(1 - qZ^2)} + \frac{Z^{t_p+1} L_{-1}}{(1 - Z^2)(1 - qZ^4)} \right)
\]

\[
= Z^{2e} \left( \frac{(1 - Z^2)}{(1 + Z)(1 - qZ^2)} L_0 + \frac{Z(1 + Z)}{(1 - Z^2)(1 - qZ^4)} L_{-1} \right)
\]

\[
= Z^{2e} \frac{(1 - Z)}{1 - qZ^4} L_0 + \frac{Z^{2e+1}}{(1 - Z)(1 - qZ^4)} L_{-1},
\]

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For Zone II, which appears only when $e > 0$, the most sensible way to evaluate the sum

$$F_{II} = \sum_{i \geq 0} \sum_{i \leq d < d + 2e, d \geq 2t} q^{i/2} \left[ Z^{i+\left(\frac{i}{2}\right)} - \frac{4}{Z} L_{e+\left(\frac{i}{2}\right)} - \frac{4}{Z} Z^{i+d} \right]$$

is to group terms with the same level $L_j$. We have $j = e + \left\lfloor \frac{i}{2} \right\rfloor - \frac{d}{Z}$, so the values of $i$ and $j$ determine $d$. The condition $d \geq 2t$ reduces to $i \geq 2(j + t - \ell)$; the other condition $i \leq d < i + 2e$ is automatically satisfied if $1 \leq j \leq e - 1$, while if $j = 0$ or $j = e$, we must have $i$ odd or $i$ even respectively. For $1 \leq j \leq e - 1$, the $L_j$-piece of $F_{II}$ is therefore

$$\sum_{i \geq \max\{0, 2(j + t - \ell)\}} q^{[i/2]} L_j Z^{i+2(e+\left\lfloor \frac{i}{2}\right\rfloor - 2j)}$$

$$= \sum_{i_p=0}^{1} \sum_{i \geq \max\{0, j + t - \ell\}} q^{i \ell} Z^{i \ell + i_p + 2e - 2j} L_j$$

$$= \left( \sum_{i_p=0}^{1} Z^{i_p} \right) \left( \sum_{i \geq \max\{0, j + t - \ell\}} (qZ^4)^{i \ell} \right) Z^{2e-2j} L_j$$

$$= (1 + Z) (qZ^4)^{\max\{0, j + t - \ell\}} \cdot Z^{2e-2j} L_j.$$

For $j = 0$ and $j = e$, since $i_p$ can only take one of its two values, the initial factor $1 + Z$ is to be replaced by $Z$ and $1$ respectively. Finally, Zone III presents no particular difficulties:

$$F_{III} = \sum_{d \geq 2t} \sum_{d \text{ even}} 2q^{d/2} L_{e+1} Z^{i+d}$$

$$= 2 \sum_{d \geq 2t} q^{d/2} \cdot \frac{Z^{2d+1}}{1 - Z} \cdot L_{e+1}$$

$$= \frac{2q^e Z^{4t+1}}{(1 - Z)(1 - qZ^4)} L_{e+1}.$$

Summing up, we get for $e \geq 1$ (the case $e = 0$ can be handled similarly)

$$F = F_1 + F_{II} + F_{III}$$

$$= \frac{Z^{2e+1}}{(1 - Z)(1 - qZ^4)} L_{e+1} + \frac{Z^{2e}(1 - Z)}{1 - qZ^4} L_{e+1} + \frac{Z^{2e+1}}{1 - qZ^4} L_{e+1} + \sum_{1 \leq j \leq e-1} \frac{(1 + Z) (qZ^4)^{\max\{0, j + t - \ell\}}}{1 - qZ^4} \cdot Z^{2e-2j} L_j$$

$$+ \frac{(qZ^4)^t}{1 - qZ^4} L_{e+1} + \frac{2q^e Z^{4t+1}}{(1 - Z)(1 - qZ^4)} L_{e+1}$$

$$= \frac{Z^{2e+1}}{(1 - Z)(1 - qZ^4)} L_{e+1} + \frac{Z^{2e}}{1 - qZ^4} L_{e+1} + \sum_{1 \leq j \leq e-1} \frac{(1 + Z) (qZ^4)^{\max\{0, j + t - \ell\}}}{1 - qZ^4} \cdot Z^{2e-2j} L_j$$

$$+ \frac{(qZ^4)^t}{1 - qZ^4} L_{e+1} + \frac{2Z (qZ^4)^t}{(1 - Z)(1 - qZ^4)} L_{e+1}.$$

Now the evident symmetry between the coefficients of $L_j$ and $L_{e-j}$, when the transformation $t \mapsto e - t$ is made, establishes the theorem.

Inserting this into the local-global reflection engine produces global reflection theorems, as announced in the introduction.
Proof of Theorem 1.3. We verify the hypotheses of Lemma 4.12 on the two integral models $V_\tau(I)$ and $V_{2\tau-1}(4\tau^{-4}I)$ defined in the theorem statement.

(i) $V(I)$ is Hasse because it is full, as previously noted.

(ii) To check that $\Gamma$ has class number 1, it suffices to check the factors $\mathbb{G}_m$ and $\mathbb{G}_a$ of which $\Gamma$ is a semidirect product. The former of these has the same class number as $K$, explaining the restriction in the theorem statement.

(iii) The finiteness of the local orbit counter follows from the formulas for it computed in the previous theorem.

(iv) Finally, at almost all places, we plug in $e = t = 0$ to get $F = L_0$, establishing the needed convergence.

Now we plug in the local reflection itself, which we obtain from the preceding lemma. We keep track of the constants $c_v$ accrued:

- If $v \nmid 2\infty$, the integral models are naturally dual with constant $c_v = 1$.
- If $v | 2$, the integral models are naturally dual with constant $c_v = [\mathcal{O}_v : \tau\mathcal{O}_v]$. Multiplying over all $v | 2$ and using that $\tau | 2$ yields a factor
  \[ \prod_{v | 2} c_v = [\mathcal{O}_K : \tau\mathcal{O}_K] = |N_{K/\mathbb{Q}}\tau|. \]
- If $v$ is real, the integral models are no longer naturally dual at $v$. We place the non-natural weighting
  \[ w^{(2)} = 1_0 \]
  that picks out $\alpha \in H^1(K, \mathbb{Z}/2\mathbb{Z})$ that vanish at $v$, that is, forms with positive discriminant at $v$. This is the Fourier transform of $w^{(1)} = 1$, so $c_v = 1$.
- Finally, if $v$ is complex, then the integral models are certainly naturally dual at $v$, because $|H^1| = 1$. However, the scaling of the Fourier transform by $1/|H^0(\mathbb{C}, M)| = 1/2$ requires that we take $c_v = 1/2$.

Multiplying these constants gives the constant claimed in Theorem 1.3. ■

Proof of Theorem 1.4. We specialize further to the case $K = \mathbb{Q}$. We replace $\Gamma(\mathbb{Z})$ by its index-2 subgroup, the group $\mathbb{Z}$ of translations. This merely doubles all orbit counts, and it acts freely on quadratics with nonzero discriminant, so we can suppress all mention of stabilizers for the clean statement given in Theorem 1.4. ■

Remark 5.3. The condition that $\mathcal{O}_K$ be a PID in Theorem 1.3 can be dropped, but then $\Gamma$ no longer has class number 1, and each side of the theorem becomes a sum of orbit counts on $\text{Cl}(\mathcal{O}_K)$-many global integral models that locally look alike. We do not spell out the details here. We wonder whether such a method works in general to circumvent the class-number-1 hypothesis in Theorem 4.14.

It is not hard to compute all quadratics $f$ of a fixed superdiscriminant $I$. The leading coefficient $a$ must be a divisor of $I$ (possibly negative), and there are only finitely many of these. Then, by replacing $x$ by $x + t$ where $t$ is an integer nearest to $-b/(2a)$, we can assume that $b$ lies in the window $-|a| < b \leq |a|$. We can try each of the integer values in this window, checking whether

\[ c = \frac{ab^2 - I}{4a^2} \]

comes out to an integer.
Example 5.4. There are five quadratics of superdiscriminant 15:

| \( f(x) \) | \( q \) | \( q^+ \) | \( q_2 \) | \( q_2^+ \) |
|-----------------|--------|--------|--------|--------|
| \(-x^2 + x - 4\) | ✓ | ✓ | ✓ | ✓ |
| \(15x^2 + x\) | ✓ | ✓ | ✓ | ✓ |
| \(15x^2 - x\) | ✓ | ✓ | ✓ | ✓ |
| \(15x^2 + 11x + 2\) | ✓ | ✓ | ✓ | ✓ |
| \(15x^2 - 11x + 2\) | ✓ | ✓ | ✓ | ✓ |

One might think we left out \(-x^2 - x - 4\), but it is equivalent to another quadratic on the list:

\[-x^2 - x - 4 = -(x + 1)^2 + (x + 1) - 4.\]

So we get the totals

\( q(15) = 5 \) and \( q^+(15) = 4. \)

There are 18 quadratics of superdiscriminant 60:

| \( f(x) \) | \( q \) | \( q^+ \) | \( q_2 \) | \( q_2^+ \) |
|-----------------|--------|--------|--------|--------|
| \(x^2 - 15\) | ✓ | ✓ | ✓ | ✓ |
| \(-x^2 - 15\) | ✓ | ✓ | ✓ | ✓ |
| \(-3x^2 + 2x - 2\) | ✓ | ✓ | ✓ | ✓ |
| \(-3x^2 - 2x - 2\) | ✓ | ✓ | ✓ | ✓ |
| \(-4x^2 + x - 1\) | ✓ | ✓ | ✓ | ✓ |
| \(-4x^2 - x - 1\) | ✓ | ✓ | ✓ | ✓ |
| \(15x^2 + 2x\) | ✓ | ✓ | ✓ | ✓ |
| \(15x^2 - 2x\) | ✓ | ✓ | ✓ | ✓ |
| \(15x^2 + 8x + 1\) | ✓ | ✓ | ✓ | ✓ |
| \(15x^2 - 8x + 1\) | ✓ | ✓ | ✓ | ✓ |
| \(60x^2 + x\) | ✓ | ✓ | ✓ | ✓ |
| \(60x^2 - x\) | ✓ | ✓ | ✓ | ✓ |
| \(60x^2 + 31x + 4\) | ✓ | ✓ | ✓ | ✓ |
| \(60x^2 - 31x + 4\) | ✓ | ✓ | ✓ | ✓ |
| \(60x^2 + 41x + 7\) | ✓ | ✓ | ✓ | ✓ |
| \(60x^2 - 41x + 7\) | ✓ | ✓ | ✓ | ✓ |
| \(60x^2 + 49x + 10\) | ✓ | ✓ | ✓ | ✓ |
| \(60x^2 - 49x + 10\) | ✓ | ✓ | ✓ | ✓ |

Counting carefully, we get

\( q(60) = 18, \quad q_2(60) = 8, \quad q^+(60) = 13, \quad q_2^+(60) = 5. \)

The equalities

\( q_2^+(60) = 5 = q(15) \) and \( q_2(60) = 8 = 2 \cdot 4 = 2q^+(15) \)

are instances of Theorem 1.4. From the same theorem, we derive, without computation, that

\( q_2^+(240) = q(60) = 18 \) and \( q_2(240) = 2q^+(60) = 26. \)

Example 5.5. More generally, looking at \( n = p_1p_3 \), where \( p_1 \equiv 1 \pmod{4} \) and \( p_3 \equiv 3 \pmod{4} \) are primes, the counts involve certain Legendre symbols. For instance, the combination \( a = p_3, \ b^2 - 4ac = p_1 \) is feasible if and only if the congruence

\( b^2 \equiv p_1 \mod{4p_3} \)

has a solution, which happens exactly when \( \left( \frac{p_1}{p_3} \right) = 1 \). Examining all factorizations of \( I \) and \( 4I \), we find that

\( q^+(p_1p_3) = 5 + \left( \frac{p_1}{p_3} \right) \) and \( q_2(4p_1p_3) = 10 + 2 \left( \frac{p_3}{p_1} \right). \)
Thus our reflection theorem recovers the quadratic reciprocity law

\[
\left( \frac{p_1}{p_3} \right) = \left( \frac{p_3}{p_1} \right).
\]

We wonder: does there exist a proof of Theorem 1.4 using no tools more advanced than quadratic reciprocity?

6 Cubic Ohno-Nakagawa

6.1 Statements of results

The space \( V(K) \) of binary cubic forms over a local or global field \( K \) can have many integral models. Let \( V_{O_K} \) be the lattice of binary cubic forms with trivial Steinitz class; these can be written as

\[
V_{O_K} = \{ ax^3 + bx^2y + cxy^2 + dy^3 : a, b, c, d \in O_K \}.
\]

In this subsection we abbreviate the form \( ax^3 + bx^2y + cxy^2 + dy^3 \) to \( (a,b,c,d) \). A theorem of Osborne classifies all lattices \( L \subseteq V_{O_K} \) that are \( \text{GL}_2(O_K) \)-invariant and primitive, in the sense that \( p^{-1}L \not\subseteq V(O_K) \) for all finite primes \( p \) of \( O_K \) (any lattice can be scaled by a unique fractional ideal to become integral):

**Theorem 6.1** (Osborne [53], Theorem 2.2). A primitive \( \text{GL}_2(O_K) \)-invariant lattice in \( V(O_K) \) is determined by any combination of the primitive \( \text{GL}_2(O_{K,p}) \)-invariant lattices in the completions \( V(O_{K,p}) \), which are:

(a) If \( p \mid 3 \), the lattices \( \Lambda_{p,i} = \{(a,b,c,d) : b \equiv c \equiv 0 \text{ mod } p^i \} \), for \( 0 \leq i \leq v_p(3) \);

(b) If \( p \mid 2 \) and \( N_{K/Q}(p) = 2 \), the five lattices

\[
\begin{align*}
\Lambda_{p,1} &= V(O_{K,p}), \\
\Lambda_{p,2} &= \{(a,b,c,d) \in V(O_{K,p}) : a + b + d \equiv a + c + d \equiv 0 \text{ mod } p \}, \\
\Lambda_{p,3} &= \{(a,b,c,d) \in V(O_{K,p}) : a + b + c \equiv b + c + d \equiv 0 \text{ mod } p \}, \\
\Lambda_{p,4} &= \{(a,b,c,d) \in V(O_{K,p}) : b + c \equiv 0 \text{ mod } p \}, \\
\Lambda_{p,5} &= \{(a,b,c,d) \in V(O_{K,p}) : a \equiv d \equiv b + c \text{ mod } p \},
\end{align*}
\]

(c) For all other \( p \), the maximal lattice \( V(O_{K,p}) \) only.

From the perspective of algebraic geometry, if \( p \mid 2 \), the latter four lattices are not true integral models, because they lose their \( \text{GL}_2 \)-invariance as soon as we extend scalars so that the residue field has more than 2 elements. By contrast, if \( p \mid 3 \), the \( \text{GL}_2 \)-invariance of the space \( L_p \), can be established purely formally. This integral model, which we will call the space of \( p^i \)-traced forms, will be the subject of our main reflection theorem in this part.

Although Osborne deals only with the case of \( V(O_K) \), his method generalizes easily to the lattice

\[
V(O_K, a) = \{ ax^3 + bx^2y + cxy^2 + dy^3 : a \in a, b \in O_K, c \in a^{-1}, d \in a^{-2} \}
\]

that pops up when considering the maps

\[
\Phi : M \rightarrow \Lambda^2 M
\]

that appear in the higher composition law Theorem [53] Here the relevant action of

\[
\Gamma(O_K, a) = \text{Aut}_{O_K}(O_K \oplus a) = \{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \text{GL}_2(K) : a_{ij} \in a^{j-i} \}
\]

is nontrivial on both \( M \) and \( \Lambda^2 M \), thus affecting \( V(O_K, a) \) via a twisted action

\[
\left( \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \Phi \right)(x,y) = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\Phi(a_{11}x + a_{21}y, a_{12}x + a_{22}y).
\]
(Compare Cox [13], p. 142 and Wood [31], Theorem 1.2.) The twist by the determinant does not affect invariance of lattices (and thus is immaterial for Osborne’s theorem) but renders the action faithful, while otherwise scalar matrices that are cube roots of unity would act trivially. We sidestep this issue entirely by restricting the action to the group $SL_2$, which preserves the discriminant $D \in \mathfrak{a}^{-2}$ of the form. The corresponding ring has discriminant $(a, D)$.

For instance, over $K = \mathbb{Z}$ there are ten primitive invariant lattices, comprising five types at 2 and two types at 3. The O-N-like reflection theorems relating all the types at 2 were computed by Ohno and Taniguchi [42]; over a number field, they become yet more convoluted and will not be considered here. On the other hand, the behavior at 3 is robust. We begin by making some definitions needed to track the behavior of cubic forms and rings at primes dividing 3.

If $\mathcal{O}$ is a ring of finite rank over a Dedekind domain $\mathcal{O}_K$, define its trace ideal $\text{tr}(\mathcal{O})$ to be the image of the trace map $\text{tr}_{\mathcal{O}/\mathcal{O}_K} : \mathcal{O} \rightarrow \mathcal{O}_K$. Note that $\text{tr}(\mathcal{O})$ is an ideal of $\mathcal{O}_K$ and, since $1 \in \mathcal{O}$ has trace $n = \deg(\mathcal{O}/\mathcal{O}_K)$, it is a divisor of the ideal $(n)$. In particular, if $\mathcal{O}_K$ is a DVR, this notion is uninteresting unless $\mathcal{O}_K$ has residue characteristic dividing $n$. Let $t$ be an ideal of $\mathcal{O}_K$ dividing $(n)$. We say that the ring $\mathcal{O}$ is $t$-traced if $\text{tr}(\mathcal{O}) \subseteq t$.

By Theorem 3.8, we can parametrize cubic orders $\mathcal{O}$ by their Steinitz class $a$ and index form

$$\Phi(x \xi + y \eta) = (ax^3 + bx^2y + cxy^2 + dy^3)(\xi \wedge \eta)$$

relative to a decomposition $\mathcal{O} = \mathcal{O}_K \oplus \mathcal{O}_K \xi \oplus \mathcal{O}_K \eta$, where $a \in \mathfrak{a}$, $b \in \mathcal{O}_K$, $c \in \mathfrak{a}^{-1}$, and $d \in \mathfrak{a}^{-2}$. Then a short computation using the multiplication table from Theorem 3.8 shows that, if $(1, \xi, \eta)$ is a normal basis, then $\text{tr}(\xi) = -b$ and $\text{tr}(\eta) = c$, so $\text{tr}(\mathcal{O}) = (3, 6a)$. Thus the based $t$-traced rings over $\mathcal{O}_K$ are parametrized by the rank-$4$ lattice of cubic forms

$$V_{a, t}(\mathcal{O}_K) := \{ax^3 + bx^2y + cxy^2 + dy^3 : a \in \mathfrak{a}, b \in t, c \in \mathfrak{a}^{-1}, d \in \mathfrak{a}^{-2}\},$$

on which $GL(\mathcal{O} \oplus \mathfrak{a})$ acts by the twisted action (6.1). For instance, if $\mathcal{O}_K = \mathbb{Z}$, $a = (1)$, and $t = (3)$, this is the lattice of integer-matrix cubic forms considered in the introduction. Our goal in this section is to prove a generalization for all number fields $K$ and spaces $V(\mathcal{O}_K, a, t)$.

**Theorem 6.2 (“Local cubic O-N’’).** Let $K$ be a nonarchimedean local field, char $K \neq 3$. Let $V(D)$ be the composed variety of binary cubic forms of discriminant $D$, under the action of the group $\Gamma = SL_2$. If $\alpha \in K^\times$ and $\tau \mid 3$ in $\mathcal{O}_K$, let $V_{\alpha, \tau}(D)$ be the integral model of $V(\mathbb{Q}(D))$ consisting of forms of the shape

$$f(x, y) = a\alpha x^3 + b\tau x^2y + c\alpha^{-1}\tau xy^2 + d\alpha^{-2}y^3,$$

together with its natural action of $G_{\alpha} = SL(\mathcal{O}_K \oplus \alpha \mathcal{O}_K)$. Then the integral models

$$(V_{1, \tau}(D), SL_2 \mathcal{O}_K) \quad \text{and} \quad (V_{1, 3\tau^{-1}(-2\tau^{-6}D)}, SL_2 \mathcal{O}_K),$$

and consequently

$$(V_{\alpha, \tau}(D), G_{\alpha}) \quad \text{and} \quad (V_{\alpha^{-3}, 3\tau^{-1}(-2\tau^{-6}D), G_{\alpha\tau^{-3}}})$$

are naturally dual with duality constant $N_{\mathbb{Q}/\mathbb{Q}}(\tau) = |\mathcal{O}_K/\tau \mathcal{O}_K|$.

The two formulations are easily seen to be equivalent. The first one is the one we will prove, but the second one has the needed form of a local reflection theorem to apply at each place to get Theorem 1.2.

**Proof of Theorem 6.2** Use Theorem 6.2 at each finite place. At the infinite places, the two integral models are necessarily naturally dual because $H^1(\mathbb{R}, M_D) = 0$; but the duality constant depends on $H^0$, which depends on the sign of $D$ at each real place, as desired.

Observe that taking $K = \mathbb{Q}$, $a = 1$, $t = 1$ recovers Ohno-Nakagawa (Theorem 1.1). We can rewrite Theorem 6.2 in terms of cubic rings:

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Theorem 6.3 (O-N for traced cubic rings). Let $K$ be a number field, and let $a, t$ be ideals of $O_K$ with $t | 3$. Define $h_{a,t}(D)$ to be the number of $t$-traced cubic rings over $O_K$ with Steinitz class $a$ and discriminant $(a, D)$, each ring weighted by the reciprocal of its number of automorphisms:

$$h_{a,t}(D) = \sum_{\text{Disc } O = (a, D) \text{ t-traced}} \frac{1}{|\text{Aut}_K O|}.$$ 

Then we have the global reflection theorem

$$h_{a,t}(D) = 3\# \{v| \infty \in D(K_v)^2\} \frac{1}{N_{K/Q}(t)} \cdot h_{a t^{-3}, 3t^{-1}}(-27D). \tag{6.4}$$

We can also rewrite our results in terms of Shintani zeta functions.

Definition 6.4. Let $K$ be a number field. If $O/O_K$ is a cubic ring of nonzero discriminant, the signature $\sigma(O)$ of $O$ is the Kummer element $\alpha \in (K \otimes_{\mathbb{Q}} \mathbb{R})^\times /((K \otimes_{\mathbb{Q}} \mathbb{R})^\times)^2$ corresponding to the quadratic resolvent of $O$. That is, it takes the value $\alpha_v = +1$ or $-1$ at each real place $v$ of $K$ according as $O_v \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{C}$, and $\alpha_v = 1$ at each complex place.

Definition 6.5. Given a number field $K$, a signature $\sigma \in (K \otimes_{\mathbb{Q}} \mathbb{R})^\times /((K \otimes_{\mathbb{Q}} \mathbb{R})^\times)^2$, an ideal class $[a] \in \text{Cl}(K)$, and an ideal $t | 3$, we define the Shintani zeta function

$$\xi_{K, \sigma, [a], t}(s) = \sum_{O} \frac{1}{|\text{Aut}_K(O)|} N_{K/Q}(\text{disc}_KL)^{-s}$$

where the sum ranges over all cubic orders $O$ over $O_K$ having signature $\sigma$, Steinitz class $a$, and trace ideal contained in $t$. We also define the Shintani zeta function with unrestricted Steinitz class

$$\xi_{K, \sigma, t}(s) = \sum_{[a] \in \text{Cl}(K)} \xi_{K, \sigma, [a], t}(s).$$

Remark 6.6. We follow the (confusing) tradition of denoting Shintani zeta functions by the Greek letter xi.

Remark 6.7. By Minkowski’s theorem on the finite count of number fields with bounded degree and discriminant, each term $n^{-s}$ has a finite coefficient, so the Shintani zeta function at least makes sense as a formal Dirichlet series. It generalizes the Shintani zeta functions for rings over $\mathbb{Z}$ mentioned in the introduction. Datskovsky and Wright [17] study an adelic version of the Shintani zeta function; they show that $\xi_{K, \sigma, (1)}$ and $\xi_{K, \sigma, (3)}$ are entire meromorphic with at most simple poles at $s = 1$ and $s = 5/6$, satisfying an explicit functional equation. We surmise that the same method will prove the same for $\xi_{K, \sigma, [a], t}$. However, we do not consider the analytic properties here.

Then we have the following corollary, which generalizes Conjecture 1.1 of Dioses [19].

Corollary 6.8 (the extra functional equation for Shintani zeta functions). Let $K$ be a number field, $\sigma \in (K \otimes_{\mathbb{Q}} \mathbb{R})^\times /((K \otimes_{\mathbb{Q}} \mathbb{R})^\times)^2$ a signature, $[a] \in \text{Cl}(K)$ an ideal class, and $t | 3$ an ideal. Then the Shintani zeta function $\xi_{K, \sigma, [a], t}(s)$ satisfies an extra functional equation

$$\xi_{K, \sigma, [a], t}(s) = \frac{3\# \{v| \infty \in \sigma_v = 1 + 3[K:Q]|s\}}{N_{K/Q}(t)^{1+6s}} \xi_{K, -\sigma, [at^{-3}], 3t^{-1}}(s), \tag{6.5}$$

Hence, summing over all $a$,

$$\xi_{K, \sigma, t}(s) = \frac{3\# \{v| \infty \in \sigma_v = 1 + 3[K:Q]|s\}}{N_{K/Q}(t)^{1+6s}} \xi_{K, -\sigma, 3t^{-1}}(s), \tag{6.6}$$

Proof. Fix $a$ and $t$. Sum Theorem 6.3 over all $D \in t^4a^{-2}$ of signature $\sigma$, weighting each $D$ by $N_{K/Q}(Da^2)^{-s}$.
the norm of the discriminant of the associated cubic rings. Then the left-hand side of the summed equality matches that of (6.5). The right-hand side involves rings with discriminant ideal $27Da^2t^{-6}$, so a compensatory factor of

$$\frac{N_{K/Q}(Da^2)^{-s}}{N_{K/Q}(27Da^2t^{-6})^{-s}} = \frac{3^{[K:Q]s}}{N_{K/Q}(t)^{6s}}$$

must be added to the right-hand side to pull out the desired Shintani zeta function. \qed

We will not let this section end without an example of the function-field case, which works just as easily:

**Theorem 6.9.** Let $q \equiv 5 \mod 6$ be a prime power. Let $\mathcal{C}$ be a complete curve over $\mathbb{F}_q$, and $D$ a divisor on $\mathcal{C}$. Let $B \rightarrow \mathcal{C}$ be a double cover (possibly singular) with branch divisor $D$, and let $L = K(\mathcal{C})[\sqrt{d}]$ be its function field. Let $L' = K(\mathcal{C})[\sqrt{-3d}]$ be the $\mathbb{F}_q$-twist of $L$. Then among the triple covers $\mathcal{A} \rightarrow \mathcal{C}$ with branch divisor $D$, the quadratic resolvent algebras $L$ and $L'$ occur equally often, when each cover $\mathcal{A}$ is weighted by the reciprocal of its number of symmetries preserving $\mathcal{C}$.

**Proof.** By a remark of Deligne fleshed out by Wood ([52], Theorem 2.1), triple covers $\mathcal{A} \rightarrow \mathcal{C}$ are parametrized by rank-2 vector bundles $E$ equipped with a cubic form $\phi \in (\text{Sym}^3 E \otimes \Lambda^2 E)(\mathcal{C})$. For $\mathcal{A}$ to have branch divisor $D$, the bundle $E$ must have fixed (first) Chern class (the analogue of the Steinitz class in the function-field setting), namely $[D]$, so $E$ can be viewed as fixed; and $\phi$, which is locally a binary cubic form, must have discriminant divisor $D$.

Consider the integral composed variety $V_{D,d}$ of cubic forms on $E$ with discriminant $d$, with the action of the sheaf $\mathcal{G}_{D,d}$ of determinant-1 automorphisms of $E$ (which locally looks like $\text{SL}_2$). We check that $(V_{D,d}, \mathcal{G}_{D,d})$ and $(V_{D,-3d}, \mathcal{G}_{D,-3d})$ are locally dual integral models (the local reflection follows from the tame case of Theorem 6.2). Hence, by Theorem 1.13 we get the desired global reflection. \qed

**Remark 6.10.** In characteristic 2, there is an analogous theorem, but the $\mathbb{F}_q$-twist $L'$ is slightly harder to describe explicitly; we omit the details. If $q \equiv 1 \mod 3$, then since $-3$ is a square, global reflection is trivial if cast in this manner; however, function-field analogues of the discriminant-reducing formulas of Section 7.2 are possible. In characteristic 3, things are quite different and will not be discussed in this paper.

**Example 6.11.** On the curve $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}_q}$, where $q \equiv 5 \mod 6$, consider the divisor

$$D = 2(\infty) + 2(0).$$

The following triple covers have branch divisor $D$:

- The union of three copies of $\mathcal{C}$, two of them glued together by simple nodes at $\infty$ and 0. It has a twofold symmetry and quadratic resolvent $K \times K$.
- The union of three copies of $\mathcal{C}$, one pair glued by a simple node at $\infty$ and a different pair glued by a simple node at 0. It is asymmetric and has quadratic resolvent $K \times K$.
- The union of $\mathcal{C}$ and the arithmetically irreducible, geometrically reducible double cover made by gluing together the two points at infinity on the conic $\mathcal{Q} = \{[X : Y : Z] \in \mathbb{P}^2_{\mathbb{F}_q} : X^2 + 3Y^2 = 0\}$, mapping to $\mathcal{C}$ via $[X : Y : Z] \mapsto [X : Z]$. This cover has a twofold symmetry $y \mapsto -y$ and has quadratic resolvent $K[\sqrt{-3}]$.
- $\mathbb{P}^1$ itself, mapping to $\mathcal{C}$ by $t \mapsto t^3$. This is the only irreducible cover of the lot; it is asymmetric (there are no cube roots of 1 in $K$) and has quadratic resolvent $K[\sqrt{-3}]$.

Overall, the weighted number of covers of branch divisor $D$ is $3/2$ for each resolvent, in accord with the theorem.

We now prove Theorem 6.24 first in the tame case where char $k_K \neq 3$, and then in the wild case where $K$ is 3-adic.
6.2 The tame case

Proof of the tame case of Theorem 6.2. Fix $D \in \mathcal{O}_K$. Let $T = K[\sqrt{D}]$ be the corresponding quadratic algebra, and $T' = K[\sqrt{-3D}]$. For brevity we will write $H^i(T)$ for the cohomology $H^i(K, M_T)$ of the corresponding order-3 Galois module, and $H^i(T')$ likewise.

Denote by $f(\sigma)$, for $\sigma \in H^1(T)$, the number of orders of discriminant $D$ in the corresponding cubic algebra $L_\sigma$; and likewise, denote by $f'(\tau)$, for $\tau \in H^1(T')$, the number of orders of discriminant $-3D$ in $L_\tau$. Our task is to prove that $f' = \hat{f}$. We note that if $-3$ is a square in $\mathcal{O}_K^\times$, then $T = T'$ and $f = f'$.

Note that $f$ is even: $\sigma$ and $-\sigma$ are parametrized by the same cubic algebra with opposite orientations of its resolvent. The Fourier transform of an even, rational-valued function on a 3-torsion group $H^1(T)$ is again even and rational-valued. So far, so good.

Our method will be first to prove the duality at 0: that is, that

$$f''(0) = \hat{f}(0) \quad (6.7)$$
$$f(0) = \hat{f}'(0). \quad (6.8)$$

Let us explain how this implies that $f' = \hat{f}$. We compute $|H^1(T)|$ using the self-orthogonality of unramified cohomology:

$$|H^1(T)| = |H^1(T)^{un}| \cdot |H^1(T')^{un}| = |H^0(T)| \cdot |H^0(T')|.$$ 

So there are basically three cases:

(a) If neither $D$ nor $-3D$ is a square in $K_v$, then $H^1(T) \cong H^1(T') \cong 0$, and (6.7) trivially implies that $f' = \hat{f}$.

(b) If one of $D$, $-3D$ is a square, then $H^1(T)$ and $H^1(T')$ are one-dimensional $\mathbb{F}_3$-vector spaces. The space of even functions on each is 2-dimensional, and

$$g \mapsto (g(0), \hat{g}(0))$$
$$g' \mapsto (\hat{g}'(0), g'(0))$$

are corresponding systems of coordinates on them. Consequently, the two equations (6.7) and (6.8) together imply that $\hat{f} = f'$.

(c) Finally, if $D$ and $-3D$ are both squares, then $f$ is a function on the two-dimensional $\mathbb{F}_3$-space $H^1(T) \cong H^1(T')$ which we would like to prove self-dual. Note that $H^1(T)$ has four subspaces $W_1, \ldots, W_4$ of dimension 1. Consider the following basis for the five-dimensional space of even functions on $H^1(T)$:

$$f_1 = 1w_1, \ldots, f_4 = 1w_4, f_5 = 10.$$ 

Note that $f_1, \ldots, f_4$ are self-dual (the Tate pairing is alternating, so any one-dimensional subspace is isotropic), while $f_5$ is not: indeed $\hat{f}_5(0) \neq f_5(0)$. Thus if (6.7) holds, then $f$ is a linear combination of $f_1, \ldots, f_4$ only and hence $\hat{f} = f$.

We have now reduced the theorem to a pair of identities, (6.7) and (6.8). By symmetry, it suffices to prove (6.8), which may be written

$$f(0) = \hat{f}'(0) = \frac{1}{|H^0(T)|} \sum_{\tau \in H^1(T')} f'(\tau). \quad (6.9)$$

The proof is clean and bijective.

The sum on the right-hand side of (6.9) counts all cubic orders of discriminant $-3D$. Any cubic order $C$ of discriminant $-3D$ can be assigned an ideal in $T$ as follows. Let $L$ be the fraction algebra of $C$. By Proposition 2.14 we have the description

$$L = K + \{\xi \sqrt[3]{\delta} + \xi^2 \sqrt[3]{\delta} \mid \xi \in T'\}$$

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for some $\delta \in T'^{N=1}/(T'^{N=1})^3$; and so, since 3 is invertible in $\mathcal{O}_K$,
\[
C = \mathcal{O}_K + \{\xi \sqrt[3]{\delta} + \xi \sqrt[3]{\delta}^2 \mid \xi \in \mathfrak{c}\}
\]
(6.10)
for some lattice $\mathfrak{c}$ in $T'$. Now by Proposition 3.10, we get that $(\mathcal{O}_D, \mathfrak{c}, \delta)$ is a self-balanced ideal, that is,
\[
\delta \mathfrak{c}^3 \subseteq \mathcal{O}_D, \quad N(\mathfrak{c}) = (t) \text{ is principal, and } N(\delta)t^3 = 1,
\]
(6.11)
Now $\mathfrak{c}$ need not be invertible in $\mathcal{O}_D$; but let $\mathcal{O} = \text{End} \mathfrak{c}$. Then $\mathfrak{c}$ is an invertible $\mathcal{O}$-ideal (here we use that $\mathcal{O}$ is quadratic; we are essentially using that $\mathcal{O}$ is Gorenstein over $\mathcal{O}_K$). Note that $\mathcal{O} = \mathcal{O}_{D/\pi^{2i}}$ for some positive integer $i$. Then form the shadow
\[
\mathfrak{a} = \frac{\delta \mathfrak{c}^3}{\pi^i}.
\]
Since the norm is multiplicative on invertible ideals, the properties of $\mathfrak{c}$ in (6.11) can be recast as properties of $\mathfrak{a}$:
\[
\mathfrak{a} \subseteq \text{End} \mathfrak{a} \quad \text{and} \quad N_{\mathcal{O}_D}(\mathfrak{a}) = 1.
\]
(6.12)
By sending
\[
\mathfrak{a} \mapsto C_\mathfrak{a} = \mathcal{O}_K + 0 \times \mathfrak{a} \subseteq K \times T,
\]
this $\mathfrak{a}$ parametrizes one of the rings counted by $f(0)$. (The first part of (6.12) ensures that $C_\mathfrak{a}$ is closed under multiplication, and the second part ensures that $\text{disc} C_\mathfrak{a} = D$.)

It remains to show that each shadow $\mathfrak{a}$ comes from exactly $|H^0(T')|$-many $C$, equivalently $(\mathfrak{c}, \delta)$, corresponding to each shadow $\mathfrak{a}$, up to multiplying $\delta$ by a cube and modifying $\mathfrak{c}$ by the appropriate equivalence relation
\[
\mathfrak{c} \mapsto \lambda \mathfrak{c}, \quad \delta \mapsto \lambda^{-3}\delta \quad (\lambda \in T'^\times).
\]
(6.13)
Let $\mathfrak{a} = \alpha \mathcal{O}$ be the given shadow, where $\mathcal{O} = \mathcal{O}_{D/\pi^{2i}} = \text{End} \mathfrak{a}$. Without loss of generality, we may scale $\alpha$ by an element of $\mathcal{O}_K^\times$ so that $N(\pi^i \alpha)$ is a cube. Clearly $\mathfrak{c}$ must be invertible with regard to $\mathcal{O}$. The possible $(\mathfrak{c}, \delta)$ may be found by fixing $\mathfrak{c} = \mathcal{O}$ and taking $\delta = \pi^i \alpha \varepsilon$ where $\varepsilon \in \mathcal{O}^\times$ has norm 1. Now in the equivalence relation (6.13), the multipliers $\lambda$ preserving $\mathfrak{c} = \mathcal{O}$ are $\lambda \in \mathcal{O}^\times$, so we must consider $\varepsilon$ up to $(\mathcal{O}^\times)^3$. Since $N : \mathcal{O}^\times/(\mathcal{O}^\times)^3 \to \mathcal{O}_K^\times/(\mathcal{O}_K^\times)^3$ is surjective, the number of distinct $\varepsilon$, which is the number of algebras $L_\delta$ in which we find rings, is
\[
\frac{|\mathcal{O}^\times/(\mathcal{O}^\times)^3|}{|\mathcal{O}_K^\times/(\mathcal{O}_K^\times)^3|}.
\]
However, for each $\delta$, the contribution of $\mathfrak{a}$ to $f(\delta)$ includes every $\mathfrak{c} \subseteq T$ such that $\delta \mathfrak{c}^3 = \delta \mathcal{O} = \pi^{-i} \mathfrak{a}$. The number of such $\mathfrak{c}$ is
\[
\frac{|\mathcal{O}_T^\times[3]|}{|\mathcal{O}^\times[3]|},
\]
and the total number of cubic orders we seek is the product
\[
\frac{|\mathcal{O}^\times/(\mathcal{O}^\times)^3|}{|\mathcal{O}_K^\times/(\mathcal{O}_K^\times)^3|} \cdot \frac{|\mathcal{O}_T^\times[3]|}{|\mathcal{O}^\times[3]|}.
\]
(6.14)
To maneuver this into the required form, first note that
\[
\frac{|\mathcal{O}^\times/(\mathcal{O}^\times)^3|}{|\mathcal{O}^\times[3]|} = \frac{|\mathcal{O}_T^\times/(\mathcal{O}_T^\times)^3|}{|\mathcal{O}_T^\times[3]|},
\]
by the Snake Lemma, since $\mathcal{O}_T^\times/(\mathcal{O}^\times)^3$ is finite; so (6.14) takes the form
\[
\frac{|\mathcal{O}_T^\times/(\mathcal{O}_T^\times)^3|}{|\mathcal{O}_K^\times/(\mathcal{O}_K^\times)^3|},
\]

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We will now group all cubic rings whose resolvent torsor is $T/K$ of Lemma 6.12. Let equally to both sides of the theorem. The remainder of this section will be spent in carrying this out.

The proof of Theorem 6.2 will then consist in exhibiting an involution between the families of support

Note that $L_i \in \mathcal{L}_{c-i}$.

The support and thickness of each family are contained in étale algebras $L$ belonging to some level space $\mathcal{L}_i$; this $\mathcal{L}_i$ is called the support $\text{supp}(\mathcal{F})$ of the family.

Each $L \in \mathcal{L}_i$ has the same number of orders in the family; this number is called the thickness $\text{th}(\mathcal{F})$ of the family.

The proof of Theorem 6.2 will then consist in exhibiting an involution between the families of support $\mathcal{L}_i$ and $\mathcal{L}_{c-i}$ which affects their thicknesses, discriminants, and trace ideals in such a manner that they contribute equally to both sides of the theorem. The remainder of this section will be spent in carrying this out.

**Lemma 6.12.** Let $T/K$ be a quadratic étale algebra and let $n_T = v_K(\text{disc } T) \in \{0, 1\}$.

Then the cubic orders whose resolvent torsor is $T$ can be partitioned into families $\mathcal{F}_{n,k}$ indexed by the pairs of integers $(n, k)$ satisfying the conditions

$$0 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor, \quad n \equiv n_T \mod 2,$$

with the following properties:

1. The rings in $\mathcal{F}_{n,k}$ have discriminant ideal $(\pi^n)$ and trace ideal $(\pi^{\min(k,c)})$.

2. The support and thickness of each $\mathcal{F}_{n,k}$ depend on which of three zones the pair $(n, k)$ belongs to, as follows:

   | Zone | $(n, k)$ | $\text{supp}(\mathcal{F}_{n,k})$ | $\text{th}(\mathcal{F}_{n,k})$ |
   |------|---------|----------------|----------------|
   | $I$  | $0 \leq k < \left\lfloor \frac{n}{6} \right\rfloor$ | $\mathcal{L}_{\max}$ | $|H^0(T)|$ |
   | $II$ | $\frac{n}{6} \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor - \frac{n}{6} + \frac{e}{2}$ | $\mathcal{L}_{e-2k+\left\lceil \frac{e}{2} \right\rceil}$ | $q^{\left\lceil \frac{e}{2} \right\rceil - k}$ |
   | $III$ | $\left\lfloor \frac{n}{3} \right\rfloor - \frac{n}{6} + \frac{e}{2} < k \leq \left\lfloor \frac{n}{3} \right\rfloor$ | $\mathcal{L}_{\min}$ | $q^{\left\lceil \frac{e}{2} \right\rceil - k}$ |

(6.16)
The shapes of these zones follow are like those in quadratic O-N, as shown in Figure 2.

Remark 6.13. Zone I, which is supported on \( L_{\text{max}} = \{0\} \), consists precisely of those rings whose structure uses in an essential way that \( L \) is split, that is, has more than one field factor. Zone II has the approximate shape of a band of constant width,

\[ \frac{n}{6} \leq k \leq \frac{n}{6} + \frac{e}{2} \]

but the dependency on the value of \( n \) modulo 3 attests to a waviness of the boundary between zones II and III that cannot be avoided.

Our first step is to compute \( \text{tr}(O_L) \) for the maximal orders of all cubic extensions \( L \). The answer is delightfully simple.

Proposition 6.14. Let \( L/K \) be a cubic étale algebra over a 3-adic field having level \( \ell \) and offset \( h \). The trace ideal of the maximal order of \( L \) is given by

\[ v(\text{tr}(O_L)) = \min\{e, e - \ell(L)\} \]

where \( \ell(L) \in \{-1, 0, \ldots, e\} \) is the level. Moreover, if \( L \) is totally ramified,

- If \( h = 2 \), \( \text{tr}(O_L) \) is generated by the trace of a uniformizer.
- If \( h = 1 \), \( \text{tr}(O_L) \) is generated by the trace of an element of valuation 2.

Proof. If \( L \) is split or unramified, we have \( \ell(L) = e \) and \( \text{tr}(O_L) = (1) \), so we may assume \( L/K \) is a totally ramified field extension. Upon relating the level and offset to the discriminant, the result follows from the relation

\[ \min_{\xi \in L^*} v_L \left( \frac{\text{tr} \xi}{\xi} \right) = v_K(\text{disc } L) - [L : K] + 1 \]

due to Hyodo \[26, \text{equation 1–4} \].

Proof of Lemma 6.12. We will now enumerate all the orders in every cubic \( K \)-algebra \( L \) and arrange them into families. Let \( L \) have level \( e - k_0 \), offset \( h \), and discriminant ideal \((\pi^{n_0})\), so \( \text{tr}(O_L) = (3, \pi^{k_0}) \) and, by Proposition 2.17(a),

\[ n_0 = 3k_0 + 2 - h. \]

Pick a basis \([1, \xi_0, \eta_0]\) for \( O_L \) so that \( \eta_0 \) is traceless and \( \text{tr}(\xi_0) \) is a generator for the trace ideal \( \text{tr}(O_K) \). By Proposition 6.14, if \( L \) is totally ramified and \( h \in \{1, 2\} \), we can assume that \( v_L(\xi_0) = 3 - h \), which implies that

\[ \Phi_{O_L}(\xi_0) \sim \pi^{2-h} \] (6.17)

because a uniformizer generates the whole of \( O_L \) while an element of \( L \)-valuation 2 generates the unique subring of index \( m_K \). In the cases where \( L \) is not totally ramified, it is easy to arrange for (6.17) to hold as well.

Any order \( C \subseteq L \) then has a unique basis of the form

\[ [1, \xi = \pi^t \xi_0 + uw_0, \eta = \pi^t \eta_0] \] (6.18)
Such a $C$ has discriminant valuation
\[ n = v(\text{disc } C) = n_0 + 2i + 2j \]
and trace ideal
\[ (3, \text{tr}(\pi^j \xi_0)) = \pi^{\min\{e, k_0 + 1\}}. \]
Let $k = k_0 + i$. With one exception, namely when $L$ is a URE (which case we will handle later), we will place such a ring $C$ into the family $\mathcal{F}_{n,k}$. We must now compute the sizes of the families we have thus constructed.

Whether or not a lattice $C$ with a basis \(6.15\) is actually a ring is determined by the integrality of its index form. The following lemma reduces the number of coefficients to be checked from four to two.

**Lemma 6.15.** Let $\xi, \eta \in \mathcal{O}_L$ be integral elements of a nondegenerate cubic algebra $L$ over the field of fractions $K$ of a Dedekind domain $\mathcal{O}_K$ such that the sublattice $C = \mathcal{O}_K \langle 1, \xi, \eta \rangle$ is of full rank. If the outer coefficients
\[ \Phi_C(\xi) = \frac{\Phi_{\mathcal{O}_L}(\xi)}{\pi^{[\mathcal{O}_L : C]}} \quad \text{and} \quad \Phi_C(\eta) = \frac{\Phi_{\mathcal{O}_L}(\eta)}{[\mathcal{O}_L : C]} \]
of the index form of $C$ are integral, then the entire index form of $C$ is integral and $C$ is a ring.

**Proof.** If the whole index form of $C$ is integral, then there is a ring $C' = \langle 1, \xi', \eta' \rangle$ with the same index form (with respect to its basis) as $C$. Using the identity of index forms for $C$ over $K$, we can embed $C'$ into $L$ with $\xi' = \xi + u, \eta' = \eta + v$ for some $u, v \in K$. But since $\xi', \xi, \eta'$ are integers and $\mathcal{O}_K$ is integrally closed, we have $u, v \in \mathcal{O}_K$ so $C = C'$ is a ring.

So it suffices to prove that the index form is integral. This is a local statement, so we may assume that $\mathcal{O}_K$ is a DVR. Passing to a finite extension, we may assume that $L \cong K \times K \times K$ is totally split. Let
\[ \xi = (u_1; u_2; u_3) \quad \text{and} \quad \eta = (v_1; v_2; v_3). \]
Then
\[ D = [\mathcal{O}_L : C] = \det \begin{bmatrix} 1 & 1 & 1 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \]
We are given that the outer coefficients
\[ a = \frac{1}{D}(u_1 - u_2)(u_2 - u_3)(u_3 - u_1) \quad \text{and} \quad d = \frac{1}{D}(v_1 - v_2)(v_2 - v_3)(v_3 - v_1) \]
of the index form of $C$ are integral. We wish to prove that the same applies to the two middle coefficients. By symmetry, we can consider just the $x^2y$-coefficient
\[ b = \frac{1}{D} \left[ \sum_{i=1}^{3} (u_i - u_{i+1})(u_{i+1} - u_{i+2})(v_{i+2} - v_i) \right]. \]
(Here, and for the rest of the proof, indices are modulo 3.) It is easy to verify that
\[ b = -u_1 + 2u_2 - u_3 + 3D(u_1 - u_2)(u_2 - u_3)(v_3 - v_1). \]
So it is enough to show that
\[ r_1 = \frac{1}{D}(u_1 - u_2)(u_2 - u_3)(v_3 - v_1) \]
is integral, or more generally any of the three
\[ r_i = \frac{1}{D}(u_i - u_{i+1})(u_{i+1} - u_{i+2})(v_{i+2} - v_i). \]
But we see that
\[ a^2 d = r_1 r_2 r_3. \]
Since $a$ and $d$ have nonnegative valuation, the three $r_i$ cannot all have negative valuation, completing the proof.

\[ \square \]
Remark 6.16. The hypothesis that $L$ be nondegenerate is likely nonessential.

We can now resume the proof of Lemma 6.12. Assume that $L$ is not a URE, so $h \in \{1, 2\}$. Let the index form of $O_L$ be

$$\Phi_{O_L}(x\xi_0 + y\eta_0) = ax^3 + bx^2y + cxy^2 + dy^3.$$ 

Of the coefficients of the index form of $C$, we focus on the outer coefficients,

$$\Phi_C(\xi) = \frac{an^{3i} + b\pi^j u + c\pi^i u^2 + du^3}{\pi^{i+j}}$$

and

$$\Phi_C(\eta) = du^{2j-i}.$$ 

The latter coefficient is the simpler one, depending only on $i$ and $j$. Due to the tracelessness of $\eta_0$, we have $3|c$. By (6.17), we have $v_K(d) = 2 - h$. Thus the condition $\Phi_C(\eta) \in \mathcal{O}_K$ comes out to $2j - i + 2 - h \geq 0$, which simplifies to $n \geq 3k$.

(Incidentally, when $k \leq e$, the relation $n \geq 3k$ expresses an important relation between the trace ideal of a ring and its discriminant, generalizing the observation that an integer-matrix cubic form has discriminant divisible by 27.)

There thus remains the $\xi$-condition $\Phi_C(\xi) \in \mathcal{O}_K$, which informally states that $\xi$ is a root of $\Phi_{O_L}$ modulo $\pi^{i+j}$. Now $\Phi_{O_L}$ is a homogeneous binary form, and it is natural to consider its roots on the projective lines $\mathbb{P}^1(\mathcal{O}_K/\pi^m)$; the factorization over the field $\mathcal{O}_K/\pi^m$, for instance, gives the splitting type $\sigma(\mathcal{O}_L)$. However, in our situation there is a distinguished point on this projective line, at least for $m < e - k_0$, namely the traceless point $\eta_0$; and the line is thereby subdivided into an affine line and a portion at infinity. The point at infinity modulo $m$ is a root of $\Phi_{O_L}$ if and only if $h < 2$. This is the motivation for the calculations to be undertaken now.

Suppose first that we are in zone I, that is, $n > 6k$, which translates into $j > 2i + 2 - h$. Then the $\Phi_C(\xi)$ condition

$$\pi^{j-2i} \mid a + bu^{2i} + cu^2\pi^{-2i} + du^3\pi^{-3i}$$

is clearly dominated by a non-integral last term unless $u$ is of the form $\pi^i u'$, in which case it simplifies to

$$\pi^{j-2i} \mid a + bu' + cu^2 + du^3.$$ 

In other words, $\xi_0 + u'\eta_0$ must be a root of $\Phi_{O_L}$ modulo $m^{j-2i}$. The condition $j - 2i > 2 - h$ rules out any contribution from a multiple root modulo $m$, which never lifts to mod $m^2$ (or else $\mathcal{O}_L$ would be nonmaximal). So the only roots that contribute are the simple roots occurring if $L$ has splitting type 111, 12, or $1^21$. There are $|H^0(T)|$ simple roots, and none of them are traceless (to be explicit, they are at $(1;0)$ for each decomposition $L \cong K \times T$ into a linear and a quadratic algebra). By Hensel’s lemma, each simple root has a unique lift to any modulus. So the solutions $u'$ form a union of $|H^0(T)|$ congruence classes modulo $m^{j-2i}$. Since $u' = u/\pi^i$ is defined modulo $m^{j-2i}$, there are

$$|H^0(T)| \cdot q^i = |H^0(T)| \cdot q^k$$

rings for each pair $(i, j)$. This completes the construction of the families $\mathcal{F}_{n,k}$ in zone I.

Now suppose that $(n,k)$ is in zone II or III, still assuming that $L$ is not a URE: we have

$$3k \leq n \leq 6k$$

or, and the $(i, j)$ coordinates,

$$i \leq 2j + 2 - h \quad \text{and} \quad j \leq 2i + \frac{n_0}{2} - 2 + h. \quad (6.19)$$

We claim that there are rings for this pair $(i, j)$ if and only if $L \in \mathcal{L}_{e-2k+\lfloor \frac{n}{3}\rfloor}$, which may be also written as

$$k_0 \leq 2k - \frac{n-2}{3} \quad \text{or in $(i, j)$ coordinates as} \quad j \leq 2i - 1 + h. \quad (6.20)$$

Assume first that $k_0 > 0$, that is, $L$ has splitting type $1^3$. Then the fact that $\xi$ is a generator of the trace ideal imply that $v_L(\xi) = h$, and hence that $v_K(a) = h - 1$. Now it is easy to show that

$$v(\Phi_{O_L}(\xi)) = v(a\pi^{3i} + bu\pi^{2i} + cu^2\pi^i + du^3) = \min\{3i + h - 1, 3v(u) + 2 - h\}$$


because the sum is dominated by its first term if \( \pi^{i+h-1}|u \) and by its last term otherwise. So a necessary condition for there to be rings is that \( i + j \leq 3i + h - 1 \), which is equivalent to (6.20). If this condition holds, then the \( \xi \)-condition simply becomes

\[
v(u) \geq \frac{i + j - 2 + h}{3}
\]

and we get

\[
q^{-\left\lfloor \frac{i+j-2+h}{3} \right\rfloor} = q^{\left\lfloor \frac{2i+j-h-i}{3} \right\rfloor} = q^{\left\lfloor \frac{\pi}{3} - k \right\rfloor}
\]
solutions.

If \( L \) has one of the other splitting types, then our task is simplified by the facts that \( k_0 = 0 \) and \( n_0 = 2 - h \in \{0, 1\} \). Note that the second inequality of (6.19) implies (6.20), so we are only trying to prove that there are solutions in this case. If (6.21) does not hold, then the \( dv^3 \) term dominates in \( \Phi_{\xi, L} (\xi) \) and we do not get a solution. If (6.21) holds, we leave it to the reader to check the inequalities that imply (even without knowing anything about \( a, b, \) and \( c \)) that \( \pi^{i+j} \) divides each term of \( \Phi_{\xi, L} (\xi) \). So we get the same number of solutions as in the preceding case.

Lastly, we must address the exceptional case that \( h = 3 \), that is, \( L = K[\sqrt[3]{\eta}] \) is a URE. We take \( \xi = \sqrt[3]{\eta} \) and \( \eta = (\sqrt[3]{\eta})^2 \), which are both traceless; and we have the explicit index form

\[
\Phi_{\xi, L}(x \xi + y \eta) = x^3 - \pi y^3.
\]

This resembles the index form for a ramified \( L \) with \( h = 1 \), and analogously to that case, we compute that rings appear only for the pairs \((i, j)\) with

\[
i \leq 2j + 1 \quad \text{and} \quad j \leq 2i,
\]
each such \((i, j)\) yielding \( q^{\left\lfloor \frac{2i+j-1}{3} \right\rfloor} \) solutions. We must place these rings into the families \( \mathcal{F}_{n,k} \) of zone III so that the discriminant valuations and thicknesses match up. There is a unique choice:

\[
n = n_0 + 2i + 2j = 3e + 2i + 2j + 2
\]
\[
k = \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{2j+1-i}{3} \right\rfloor.
\]

We leave it to the reader that this establishes a bijection between the pairs \((i, j)\) in the region (6.22) with the pairs \((n, k)\) in zone III. In this way, each family in zone III is supported on the whole of \( \mathcal{L}_1 \). The discriminant and thickness are correct by construction, completing the proof.

A by-product of the foregoing is a formula for the number of traced orders in a given cubic algebra:

**Theorem 6.17 (the traced subring zeta function).** Let \( L \) be a cubic algebra over a 3-adic field \( K \) with discriminant disc \( L = m^d_K \). Let \( d \) and \( t \) be integers satisfying the necessary restrictions

\[
d \geq d_0, \quad d \equiv d_0 \mod 2, \quad 0 \leq t \leq e_K.
\]

Then the number \( g(L, d, t) \) of \( m^d_K \)-traced orders of discriminant \( m^d_K \) in \( L \) is a linear combination of the three functions

\[
g^3(d_0, d, t) = \frac{q^r - 1}{q - 1}, \quad r = \begin{cases} 0, & d < 3t \\
\left\lfloor \frac{d}{3} \right\rfloor - t + 1, & 3t \leq d \leq 6t - d_0 \\
\left\lfloor \frac{d-d_0}{6} \right\rfloor + 1, & d \geq 6t - d_0
\end{cases}
\]
\[
g^3(d_0 = 0, d, t) = \frac{q^r - 1}{q - 1}, \quad r = \begin{cases} 0, & d < 3t \\
\left\lfloor \frac{d}{3} \right\rfloor - t + 1, & 3t \leq d \leq 6t \\
\frac{d}{3} - 2 \left\lfloor \frac{d}{6} \right\rfloor + 1, & d \geq 6t
\end{cases}
\]
\[
g^{3\text{SR}}(d_0 = 0 \ or \ 1, d, t) = \frac{q^r - q^t}{q - 1}, \quad r = \begin{cases} t, & d < 6t \\
\left\lfloor \frac{d}{6} \right\rfloor - t + 1, & d \geq 6t
\end{cases}
\]

in a manner dependent on the splitting type of \( L \):
\[ \sigma(L) = 1^3: g = g^{13} \]
\[ \sigma(L) = 1^21: g = g^{13} + g^{SR} \]
\[ \sigma(L) = 3: g = g^3 \]
\[ \sigma(L) = 12: g = g^3 + g^{SR} \]
\[ \sigma(L) = 111: g = g^3 + 3g^{SR}. \]

Remark 6.18. When \( t = 0 \) (and here we need no longer assume that \( \text{char} k_F = 3 \)), we recover the formulas for orders in a cubic field computed by Datskovsky and Wright and written more explicitly by Nakagawa and the author.

Proof of Theorem 6.2. Once Lemma 6.12 is proved, we can prove Theorem 6.2 quite simply by sending the family \( \mathcal{F} = \mathcal{F}_{n,k} \) to \( \mathcal{F}' = \mathcal{F}'_{n',k'} \), where
\[ n' = n + 3e - 6t \]
\[ k' = \left\lfloor \frac{n}{3} \right\rfloor - k + e - t. \]

If the original \( \mathcal{F} \) satisfied the bounds \( t \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor \), then it is easy to see that \( t' \leq k' \leq \left\lfloor \frac{n}{3} \right\rfloor \) where \( t' = e - t \), and likewise \( n' \equiv n_T \mod 2 \) implies \( n' \equiv n_T \). Thus \( \mathcal{F}' \) is a family of rings of resolvent torsor \( T' \) whose trace ideal is contained in \((\pi^{e-t})\). It is not hard to see that \( \mathcal{F}' \) lies in zone III, II, or I according as \( \mathcal{F} \) lies in zone I, II, or III. We leave it to the reader to check the needed identities
\[ \text{supp}(\mathcal{F}') = \text{supp}(\mathcal{F})^\perp \]
\[ \text{th}(\mathcal{F}') = \frac{|\text{supp}(\mathcal{F})|}{|H^0(T)|} \cdot \text{th}(\mathcal{F}). \]

Remark 6.19. The method of the above proof can also be adapted to the tame case.

7 Non-natural weightings

Now that we know that the integral models \( V_t, V_{q t^{-1}} \) of binary cubic forms are naturally dual, we can further look for duals for non-natural weightings. This has applications to counting cubic rings satisfying local conditions. We restrict ourselves to primes not dividing 3co.

For simplicity we work over \( \mathbb{Z} \), though the techniques extend. Denote by \( M_D \) the group \( \mathbb{Z}/3\mathbb{Z} \) with Galois action given by the quadratic character corresponding to \( \mathbb{Q}(\sqrt{D}) \). Denote by \( V(R) \) the space of binary cubic forms over a ring \( R \).

As in Section 4, if \( W : V(O_{h_0}) \to \mathbb{C} \) is a locally constant weighting invariant under \( \text{GL}_2(O_{h_0}) \), we denote by \( h(D, W) \) the number of \( \text{GL}_2 \mathbb{Z} \)-classes of binary cubic forms over \( \mathbb{Z} \) of discriminant \( D \), each form \( \Phi \) weighted by
\[ W(\Phi) \]
\[ \frac{1}{|\text{Stab}_{\text{GL}_2 \mathbb{Z}}(\Phi)|} \]

If \( W = \prod_p W_p \) is a product of local weightings, then our local-to-global reflection engine (Theorems 4.13 and 4.15) produces identities relating different \( h(D, W) \), if we can find a dual for each \( W_p \).
7.1 Local weightings given by splitting types

Let $\sigma \in \{111, 12, 3, 1^{2}1, 1^{3}, 0\}$ be one of the six splitting types a binary cubic form can have at a prime. Let

$$T(\sigma) = T_p(\sigma) : V(\mathbb{Z}_p) \to \{0, 1\}$$

be the selector that takes the value 1 on binary cubic forms of splitting type $\sigma$. Then the associated weighted local orbit counter

$$g_D (T_p(\sigma)) : H^1(\mathbb{Q}_p, M_D) \to \mathbb{N}$$

attaches to each cubic algebra $L$ of discriminant $K(\sqrt{D})$ its number of orders of discriminant $D$ and splitting type $\sigma$.

There is another construction of interest to us. If $a \in \mathbb{Q}$, then the varieties

$$V_{\mathbb{Z}}(D = D_0) \quad \text{and} \quad V_{\mathbb{Z}}(D = a^2D_0)$$

(7.1)

do not in general look alike. However, their base-changes to $\mathbb{Q}$ are isomorphic, being related by any $g \in \text{GL}_2(\mathbb{Q})$ of determinant $a$. Hence the two varieties (7.1) can be viewed as two integral models for $V_{\mathbb{Q}}(D = D_0)$. Coupled with Theorem 4.15, this viewpoint is very flexible. We denote by $Z_p$ the transformation that applies

$$\begin{bmatrix} 1/p \\ 1 \end{bmatrix}$$

to the vectors of an integral model of $V_{\mathbb{Q}}$ and conjugates $G$ accordingly. Observe that

$$g_D (T_p(\sigma)Z_p^n) = g_{Dp^{-2n}}(T_p(\sigma)),$$

and similarly for global class numbers. It is not hard to see that $h(D, W)$ is meaningful for any $W$ in the $\mathbb{Q}$-algebra generated by the $T_p(\sigma)$’s and the $Z_p$’s for all $p$.

Over $\mathbb{Z}_p$, we still have $Z_p$, and we sometimes omit the subscript, as every $Z_\ell$ with $\ell \neq p$ has no effect on the integral model. We define $Z = Z_\pi$ for integral models over a general local field similarly.

**Lemma 7.1.** Let $K$ be a local field, $\text{char} \ k_K \neq 3$, and let $D \in \mathcal{O}_K \setminus \{0\}$. Then the weightings

$$T(1^3)Z \quad \text{and} \quad 2 \cdot T(111) - T(3)$$

are dual with duality constant 1; that is, the associated local orbit counters satisfy

$$g_{-3\pi^2D}(1^3) = 2g_D(111) - g_D(3).$$

(7.2)

**Proof.** The right-hand side of (7.2) can be written as

$$|H^0(M_D)| \cdot 1_0 - 1_{H^1_\text{ur}},$$

so it suffices to show that the left-hand side is the Fourier transform of this, namely

$$1 - 1_{H^1_\text{ur}} = 1_{H^1_\text{ram}}.$$

Look at binary cubic forms $f(x, y)$ of splitting type $1^3$ and discriminant $-3\pi^2D$. Changing coordinates, we can assume

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \equiv x^3 \mod \pi.$$

Then note that $\text{disc} \ f \equiv -27a^2d^2 \equiv -27d^2 \mod \pi^3$, so the only way that $f$ can have discriminant $-3\pi^2D$ is if $\pi^2 \nmid d$. Then $f$ is an Eisenstein polynomial, the index form of a maximal order in a totally ramified extension $L$. Hence the weighting counting such $f$ is $1_{H^1_\text{ram}}$, as desired. 

Plugging this, together with the natural duality of Theorem 6.2 at the other primes, into Theorem 4.14 yields results such as the following:

**Theorem 7.2.** Let $p \in \mathbb{Z}$ be a prime, $p \neq 3$. For all integers $D$ such that $p \nmid D$,

$$\frac{1}{c_\infty}h_3(-27p^2D, T_p(1^3)) = 2h(D, T_p(111)) - h(D, T_p(3))$$

(7.3)

$$c_\infty h(p^2D, T_p(1^3)) = 2h_3(-27D, T_p(111)) - h_3(-27D, T_p(3))$$

(7.4)

where $c_\infty = 3$ for $D > 0$, $c_\infty = 1$ for $D < 0$. 

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7.2 Discriminant reduction

This can be used to improve a step that often occurs in arithmetic statistics, namely the production of discriminant-reducing identities that express the number of forms with certain non-squarefree discriminant in terms of lower discriminants.

For $N$ a positive integer, let $h(D, R_N)$ be the number of classes of binary cubic forms of discriminant $D$, each weighted not only by the reciprocal of its number of automorphisms but also by its number of roots in $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. Equivalently, consider the natural congruence subgroup

$$\text{GL}^0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(\mathbb{Z}) : b \equiv 0 \mod N \right\},$$

and let $h(D, R_N)$ be the number of $\text{GL}^0(N)$-orbits of cubic $111N$-forms (integral forms with a marked root) of discriminant $D$ over $\mathbb{Z}$, each weighted by the reciprocal of its stabilizer in $\text{GL}^0(N)$. (If $N > 1$, it is easy to prove that these stabilizers are trivial.) If $3 \nmid N$, denote by $h_3(D, R_p)$ the analogous weighted count of $133N$-forms. For a prime $N = p$, we have

$$h(D, R_p) = 3h(D, T_p(111)) + h(D, T_p(12)) + 2h(D, T_p(1^21)) + h(D, T_p(111)) + (p + 1)h(D, T_p(0)).$$

This enables us to state succinctly the following theorem.

**Theorem 7.3 (discriminant reduction).** Let $p \neq 3$ be a prime, and $D$ an integer divisible by $p^2$. Then

$$h(D) = h\left(\frac{D}{p^2}, R_p\right) + h\left(\frac{D}{p^3}, R_p\right) + \frac{1}{c_\infty} \left(2h_3\left(\frac{-27D}{p^2}, T_p(111)\right) - h_3\left(\frac{-27D}{p^2}, T_p(3)\right)\right),$$

where $c_{\infty} = 3$ for $D > 0$, $c_{\infty} = 1$ for $D < 0$.

**Proof.** The cubic rings $C$ counted by the left-hand side can be divided into maximal and nonmaximal at $p$. If $C$ is maximal at $p$, then $C \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is the ring of integers of a totally tamely ramified cubic extension of $\mathbb{Q}_p$ and $p^2 \mid D$. By Theorem 7.2, such rings are counted by the last term.

If $C$ is nonmaximal at $p$, then $C$ sits with $p$-power index inside an overring $C'$. By considering $C' + pC$, we can take an inclusion $C \subset C'$ of one of the following forms:

- $C$ has index $p$ in a $C' = C_1$ of discriminant $D/p^2$. Here the index form of $C_1$ must have a marked root modulo $p$ so that the transformation

  $$\Phi_{C_1}(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \mapsto \Phi_C(x, y) = p^2ax^3 + pbx^2y + cxy^2 + \frac{d}{p}y^3$$

  keeps the form integral. This accounts for the term $h(D/p^2, R_p)$.

- $C$ has index $p^2$ in a $C' = C_2$ of discriminant $D/p^4$ with $C_2/C \cong (\mathbb{Z}/p\mathbb{Z})^2$. This requires that the index form of $C$ have content divisible by $p$; we have

  $$\Phi_C = \frac{1}{p}C_2.$$  

  This accounts for the term $h(D/p^4)$.

Observe that $C_2$ is unique if it exists. A choice of $C_1$ corresponds to a choice of multiple root of $\Phi_C$, which is unique if $\Phi_C$ is nonzero modulo $p$. Thus, the only chance of overcounting occurs when $C$ admits both a $C_2$ and one or more $C_1$’s. The $C_1$’s are all the subrings of index $p$ in $C_2$ and thus correspond to the roots of $\Phi_{C_2}$ modulo $p$. So we subtract 1 (more precisely, $1/|\text{Aut } C_2|$) for each root of a form $\Phi_{C_2}$ counted in the term $h(D/p^4)$. That is, we subtract $h(D/p^4, R_p)$, yielding the claimed total. \qed

More generally, we can reduce at multiple primes at once. Let $T_p^{\max} : V(\mathcal{O}_{A_0}) \to \mathbb{Z}$ be the selector for rings maximal at $p$. 

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Theorem 7.4 (discriminant reduction). Let $q = p_1 \cdots p_r$ be a squarefree integer, $3 \nmid q$. If $D$ is a nonzero integer divisible by $q^2$, then for any $t < q$, 

$$h(D) = \sum_{\substack{q=q_1q_2q_3 \quad q_1 \leq t \quad q_1 \leq t \quad q_1 > t}} h \left( \frac{D}{q_2q_3}, \prod_{p|q_1} T_p^{\text{max}} \prod_{p|q_2 \land p|q_3} R_p \prod_{p|q_3} (1 - R_p) \right) +$$

$$+ \frac{1}{c_\infty} \sum_{\substack{q=q_1q_2q_3 \quad q_1 > t \quad q_1 \leq t \quad q_1 > t}} h_3 \left( \frac{-27D}{q_1^2q_3}, \prod_{p|q_1} 1_p^{2 \parallel D} (R_p - 1) \prod_{p|q_3} 1_p^{2 \parallel D} (1 - R_p) \right)$$

and

$$h_3(-27D) = \sum_{\substack{q=q_1q_2q_3 \quad q_1 \leq t \quad q_1 \leq t \quad q_1 > t}} h_3 \left( \frac{-27D}{q_2q_3}, \prod_{p|q_1} T_p^{\text{max}} \prod_{p|q_2} R_p \prod_{p|q_3} (1 - R_p) \right) +$$

$$+ c_\infty \sum_{\substack{q=q_1q_2q_3 \quad q_1 > t \quad q_1 \leq t \quad q_1 > t}} h_3 \left( \frac{D}{q_1q_3}, \prod_{p|q_1} 1_p^{2 \parallel D} (R_p - 1) \prod_{p|q_3} 1_p^{2 \parallel D} (1 - R_p) \right).$$

Remark 7.5. If we take $t = \sqrt{q}$, we find that all discriminants appearing are at most $-27D/q$.

Proof. Since a ring of discriminant $D$ is maximal or nonmaximal at each of the primes dividing $q$, we have

$$h(D) = \sum_{q_1q_2q_3=q} h \left( D, \prod_{p|q_1} T_p^{\text{max}} (1^3) \prod_{p|q_2} T_p^{\text{nonmax}} \right),$$

where $T_p^{\text{max}} (1^3) = T_p^{\text{max}} \cdot T_p(1^3)$ and $T_p^{\text{nonmax}} = 1 - T_p^{\text{max}}$ (as is natural). We transform each term in one of two ways, depending on whether $q_1 \leq t$.

If $q_1 \leq t$, we simply replace each $T_p^{\text{nonmax}}$ by $Z_p R_p + Z_p^2 (1 - R_p)$ by the method of the preceding theorem, which respects local conditions at other primes. We get a sum

$$h \left( D, \prod_{p|q_1} T_p^{\text{max}} (1^3) \prod_{p|q_2} T_p^{\text{nonmax}} \right)$$

$$= h \left( D, \prod_{p|q_1} T_p^{\text{max}} (1^3) \prod_{p|q_2} (Z_p R_p + Z_p^2 (1 - R_p)) \right)$$

$$= \sum_{q_1q_2q_3=q} h \left( D, \prod_{p|q_1} T_p^{\text{max}} \prod_{p|q_2} R_p \prod_{p|q_3} (1 - R_p) \right).$$

If $q_1 > t$, we reflect. A dual of $T_p^{\text{max}}$, when restricted to discriminants $D$ that are divisible by $p^2$, is $1_p^{2 \parallel D} Z_p (R_p - 1)$ by Lemma 7.1. Hence a dual of $T_p^{\text{nonmax}} = 1 - T_p^{\text{max}}$ on the same discriminants is $1 + 1_p^{2 \parallel D} Z_p (1 - R_p)$. Applying the reflection theorem,

$$h \left( D, \prod_{p|q_1} T_p^{\text{max}} (1^3) \prod_{p|q_2} T_p^{\text{nonmax}} \right)$$

$$= \frac{1}{c_\infty} h_3 \left( -27D, \prod_{p|q_1} 1_p^{2 \parallel D} Z_p (R_p - 1) \prod_{p|q_2} (1 + 1_p^{2 \parallel D} Z_p (1 - R_p)) \right)$$

$$= \frac{1}{c_\infty} \sum_{q_1q_2q_3} h_3 \left( \frac{-27D}{q_1^2q_3}, \prod_{p|q_1} 1_p^{2 \parallel D} (R_p - 1) \prod_{p|q_3} 1_p^{2 \parallel D} (1 - R_p) \right).$$

Summing over $q_1$ yields the first identity. The second is proved in the same way.
7.3 Binary cubic forms over $\mathbb{Z}[1/N]$

For $N$ a squarefree integer, it is natural to ask what happens if we invert finitely many primes and count binary cubic forms of discriminant $D \neq 0$ over $\mathbb{Z}[1/N]$, up to the action of the relevant group $\text{SL}_2(\mathbb{Z}[1/N])$. There are still only finitely many for each degree, owing to Hermite’s theorem on the finiteness of the number of number fields with prescribed degree and set of ramified primes. However, Cremona’s reduction theory \cite{14} for binary forms over $\mathbb{Z}$ does not carry over so easily to this setting. Hence, it is valuable to have a reflection theorem to tell us what the number of forms will be.

Note that $D$ matters only up to multiplication by the squares in $\mathbb{Z}[1/N]^{\times}$; hence we can restrict our attention to $D \in \mathbb{Z}$ that are fundamental at each prime $p \mid N$. (If $p \neq 2$, this means that $p^2 \mid D$. If $p = 2$, this means that $D = 1 \mod 4$ or $D = 8, 12 \mod 16$. However, we allow $D$ to be non-fundamental at primes not dividing $N$.)

We do not have O-N for forms over $\mathbb{Z}[1/N]$ in the same formulation as over $\mathbb{Z}$. Instead, we have something better: a reflection theorem relating forms over $\mathbb{Z}[1/N]$ to forms over $\mathbb{Z}$ with certain splitting conditions, which are much easier to count.

**Definition 7.6.** Let $N > 1$ be a squarefree integer. Denote by $R_N^\chi$ the local weighting that weights a binary cubic form by its number of simple roots modulo $N$, in other words,

$$R_N^\chi = \prod_{p \mid N} \left(3T_p(111) + T_p(12) + T_p(1^21)\right)$$

Thus $h(D, R_N^\chi)$ is half the number of integer binary cubic forms

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$

such that $N \mid d$ and $\gcd(c, N) = 1$, up to the action of the familiar congruence subgroup

$$\Gamma^0(N) = \left\{ \begin{bmatrix} a & N b \\ c & d \end{bmatrix} \in \text{SL}_2 \right\}.$$  

**Remark 7.7.** The group $\Gamma^0(N)$ acts freely on nondegenerate forms for $N \geq 2$, so we have elided the mention of stabilizers. The factor of 2 appears because of transitioning from $\text{GL}_2$ to $\text{SL}_2$.

**Theorem 7.8.** Let $N$ be a squarefree integer.

For $0 \neq D \in \mathbb{Z}[1/N]$, let $h_{\mathbb{Z}[1/N]}(D)$ be the number of $\text{SL}_2(\mathbb{Z}[1/N])$-orbits of integral binary cubic forms over $\mathbb{Z}[1/N]$, each weighted by the reciprocal of its stabilizer in $\text{SL}_2(\mathbb{Z}[1/N])$. If $3 \mid N$, define $h_{3,\mathbb{Z}[1/N]}(D)$ to be the same count, counting only 1331-forms (that is, forms whose middle two coefficients belong to the ideal $3\mathbb{Z}[1/N] \subset \mathbb{Z}[1/N]$).

Now let $D \in \mathbb{Z}$ be a discriminant. As usual, let

$$c_{D, \infty} = \begin{cases} 3 & D > 0 \\ 1 & D < 0. \end{cases}$$

Then:

(a) If $3 \mid N$, then

$$2h_{3,\mathbb{Z}[1/N]}(-27D) = c_{D, \infty} \cdot h(D, R_N^\chi).$$

(b) If $3 \nmid N$ and $27 \mid D$, then

$$2h_{\mathbb{Z}[1/N]}\left(\frac{D}{27}\right) = \frac{c_{D, \infty}}{3} \cdot h_3(D, R_N^\chi)$$

(c) If $3 \mid N$, then

$$2h_{\mathbb{Z}[1/N]}(-3D) = c_{D, \infty} \cdot h(D, R_N^\chi).$$

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We take the same composed varieties \( (V^{(i)}, \Gamma^{(i)}) \) of binary cubic forms as before. However, we carefully take new integral models \( (V^{(i)}, G^{(i)}) \).

On the right-hand side, we take the scheme \( V^{(1)} \) of binary cubic \( 111N \)-forms (for \([a]\) and \([c]\)) or \( 133N \)-forms (for \([b]\)) of discriminant \( D \). (For \([c]\) the distinction disappears.) This \( V^{(1)} \) does not admit an algebraic action of \( SL_2 \), but it does admit an algebraic action of the congruence subgroup

\[
\Gamma^0(N) = \left\{ \begin{bmatrix} a & Nb \\ c & d \end{bmatrix} \in SL_2 \right\}.
\]

We take \( G^{(1)} = \Gamma^0(N) \), viewed as a group scheme over \( \mathbb{Z} \). We also impose the \( G^{(1)} \)-invariant weighting \( w_p^{(1)} = 1 \) at the primes \( p \mid N \) to impose the condition that the third coefficient \( c \) be coprime to \( N \). At \( p \nmid N \), we use natural weighting (i.e. \( w_p = 1 \)).

On the left-hand side, we take the scheme \( V^{(2)} \) of binary cubic forms over \( \mathbb{Z}[1/N] \), either \( 1331 \)-forms of discriminant \( -27D \) (for \([a]\)), \( 1111 \)-forms of discriminant \( -D/27 \) (for \([b]\)), or \( 1111 \)-forms of discriminant \( -3D \) (for \([c]\)), using in all cases the natural action of \( G^{(2)} = SL_2 \) over \( \mathbb{Z}[1/N] \) and natural weighting. Note that if \( 3 \mid N \), then the discriminants \( -3D \) and \( -27D \) are interchangeable.

It is evident that the global class numbers of these integral models match the quantities studied in the theorem. The factors of 2 arise from switching between \( GL_2 \) and \( SL_2 \). The checking of most of the conditions of Theorem \( 11.14 \) is routine, so we content ourselves with checking the local duality.

When \( p \nmid N \), the integral model is isomorphic at \( p \) to that used for O-N, so we already have the needed duality with an appropriate duality constant \( c_{D,p} \). This includes the infinite prime, at which the duality constant \( c_{D,\infty} \) tracks the sign of \( D \) as in O-N. Otherwise, \( c_{D,p} = 1 \), except that in \([b]\) there appears a factor of \( c_{D,3} = 1/3 \).

When \( p \mid N \), the computation of the local class numbers is not difficult:

- As to \( V^{(1)} \), we look for forms \( f \) of discriminant \( D \) with a marked simple root modulo \( p \). Note first that \( f \) must have a root in \( \mathbb{P}^1(\mathbb{Q}_p) \) by Hensel’s lemma, so \( g_{w_p^{(1)}}^{(1)} \) is supported on the zero cohomology class. We claim that

\[
g_{w_p^{(1)}}^{(1)} = \left| H^0(\mathbb{Q}_p, M^{(1)}) \right| \cdot 1_0. \tag{7.5}
\]

This amounts to counting the orders \( C \) in the split algebra \( L = K \times K\sqrt{D} \) with a marked root of the index form. When \( D \) is fundamental, the desired count is evidently \( |H^0(\mathbb{Q}_p, M^{(1)})| \), the number of roots of the index form of \( C = \mathcal{O}_L \). When we turn to non-maximal \( C \), since the discriminant is divisible by \( p \), there must be a multiple root as well as a simple root. So, after a suitable change of basis, the index form must be congruent to \( x^2y \) modulo \( p \).

\[
\phi_C(x, y) = pax^3 + bxy^2 + pcy^3 + pdy^3 \equiv x^2y \mod p.
\]

If \( p \nmid d \), we find that \( C \) is maximal. So there is a unique covering of index \( p \), a ring \( C' \) whose index form is

\[
\phi_{C',p} \cdot \phi_C \left( x, \frac{y}{p} \right) = p^2ax^3 + bxy^2 + cxy^2 + \frac{d}{p}y^3.
\]

This \( C' \) again has a marked simple root mod \( p \) at \([1 : 0]\), and the passage from \( C \) to \( C' \) is found to be bijective, explaining why \( g_{w_p^{(1)}}^{(1)} \) is unchanged upon multiplying \( D \) by \( p^2 \).

One subtle case is worthy of mention, although it does not affect the proof: If \( D = 1 \), there are three roots of \( f(x, y) = xy(x + y) \) to mark, so \( g^{(1)}(0) = 3 \). Note that although these give equivalent \( 1111 \)-forms, the definition of the local orbit counter demands that we count each coset in \( \Gamma^0(\mathbb{Z}_p) \big/ SL_2(\mathbb{Q}_p) \) separately, and the automorphisms of \( f \) do not lie in \( \Gamma^0(\mathbb{Z}_p) \), and so we get \( g^{(1)}(0) = 3 \cdot 1_0 = |H^0(\mathbb{Q}_p, M^{(1)})| \cdot 1_0 \).

- As to \( V^{(2)} \), since the completion of \( \mathbb{Z}[1/N] \) at \( p \) is \( \mathbb{Q}_p \), the local orbit counter counts cosets in \( SL_2(\mathbb{Q}_p) \big/ SL_2(\mathbb{Q}_p) \) that keep a certain form \( f \) “integral” over \( \mathbb{Q}_p \) (a vacuous condition). There is obviously only one such coset, regardless of the cohomology class of \( f \), so \( g^{(2)} : H^1(\mathbb{Q}_p, M) \rightarrow \mathbb{N} \) is identically 1.
Thus $V^{(1)}$ and $V^{(2)}$ are dual with duality constant 1.

Example 7.9. Take $N = 15$ and $D = 1$. The unique form $f$ of discriminant 1 has $3 \cdot 3 = 9$ roots modulo $N$, all simple, but because of the sixfold symmetry of $f$, we get $h(D, R_N^*) = 3/2$. Part $[c]$ of the theorem then assures us that the number of $\text{SL}_2(\mathbb{Z}[1/15])$-classes of forms over $\mathbb{Z}[1/15]$ of discriminant $-3$ is $3 \cdot 3/2 = 9$, when forms are weighted by stabilizer. In fact there are nine, none having any stabilizer:

$$3x^3 - \frac{1}{3^2} y^3, \quad x^3 \pm \frac{1}{3^2} y^3, \quad 5x^3 \pm \frac{1}{3 \cdot 5} y^3, \quad \frac{5}{3} x^3 \pm \frac{1}{5} y^3, \quad 5 \cdot 3x^3 \pm \frac{1}{5 \cdot 3^2} y^3.$$

References

[1] Brandon Alberts and Evan O’Dorney. Harmonic analysis and statistics of the first Galois cohomology group. Res. Math. Sci., 8(50), 2021.

[2] Salim Ali Altuğ, Arul Shankar, Ila Varma, and Kevin H. Wilson. The number of quartic $D_4$-fields ordered by conductor. Preprint (2017), available at arxiv.org/abs/1704.01729.

[3] Manjul Bhargava. Higher composition laws. I. A new view on Gauss composition, and quadratic generalizations. Ann. of Math. (2), 159(1):217–250, 2004.

[4] Manjul Bhargava. Higher composition laws. II. On cubic analogues of Gauss composition. Ann. of Math. (2), 159(2):865–886, 2004.

[5] Manjul Bhargava. Higher composition laws. III. The parametrization of quartic rings. Ann. of Math. (2), 159(3):1329–1360, 2004.

[6] Manjul Bhargava. Higher composition laws. IV. The parametrization of quintic rings. Ann. of Math. (2), 167(1):53–94, 2008.

[7] Manjul Bhargava. Most hyperelliptic curves over $\mathbb{Q}$ have no rational points, 2013. Preprint, available at arxiv.org/abs/1308.0395.

[8] Manjul Bhargava, Arul Shankar, and Jacob Tsimerman. On the Davenport-Heilbronn theorems and second order terms. Invent. Math., 193(2):439–499, 2013.

[9] Manjul Bhargava and Ariel Shnidman. On the number of cubic orders of bounded discriminant having automorphism group $C_3$, and related problems. Algebra Number Theory, 8(1):53–88, 2014.

[10] Manjul Bhargava, Takashi Taniguchi, and Frank Thorne. Improved error estimates for the davenport-heilbronn theorems, 2021. Preprint, available at arxiv.org/abs/2107.12819.

[11] Henri Cohen, Simon Rubinstein-Salzedo, and Frank Thorne. Identities for field extensions generalizing the Ohno-Nakagawa relations. Compos. Math., 151(11):2059–2075, 2015.

[12] Ilaria Del Corso and Roberto Dvornicich. The compositum of wild extensions of local fields of prime degree. Monatsh. Math., 150(4):271–288, 2007.

[13] David A. Cox. Primes of the form $x^2 + ny^2$: Fermat, class field theory, and complex multiplication. Pure and Applied Mathematics (Hoboken). John Wiley & Sons, Inc., Hoboken, NJ, second edition, 2013.

[14] J. E. Cremona. Reduction of binary cubic and quartic forms. LMS J. Comput. Math., 2:64–94, 1999. Available (with corrections) at https://homepages.warwick.ac.uk/staff/J.E.Cremona/papers/r34jcm.pdf.

[15] J. E. Cremona. Corrigendum: “Reduction of binary cubic and quartic forms” [LMS J. Comput. Math. 2 (1999), 64–94]. LMS J. Comput. Math., 4:73, 2001.

[16] Henri Darmon, Fred Diamond, and Richard Taylor. Fermat’s last theorem. In Current Developments in Mathematics, 1994.
[17] Boris Datskovsky and David J. Wright. The adelic zeta function associated to the space of binary cubic forms. II. Local theory. *J. reine angew. Math.*, 367:27–75, 1986.

[18] B. N. Delone and D. K. Faddeev. *The theory of irrationalities of the third degree*. Translations of Mathematical Monographs, Vol. 10. American Mathematical Society, Providence, R.I., 1964.

[19] Jorge Dioses. *Generalizing the theorem of Nakagawa on binary cubic forms to number fields*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–Oklahoma State University. Available at https://www.proquest.com/docview/1080790215.

[20] Jordan S. Ellenberg and Akshay Venkatesh. Reflection principles and bounds for class group torsion. *Int. Math. Res. Not.*, 2007. Article rnm002, 18 pp. https://doi.org/10.1093/imrn/rnm002.

[21] Christopher Frei, Daniel Loughran, and Rachel Newton. The Hasse norm principle for abelian extensions. *Amer. J. Math.*, 140(6):1639–1685, 2018.

[22] Xia Gao. On the Ohno-Nakagawa theorem. *J. Number Theory*, 189:186–210, 2018.

[23] Georges Gras. Théorèmes de réflexion. *J. Théor. Nombres Bordeaux*, 10(2):399–499, 1998.

[24] Benedict H. Gross and Mark W. Lucianovic. On cubic rings and quaternion rings. *J. Number Theory*, 129(6):1468–1478, 2009.

[25] Fabian Gundlach. *Parametrizing Extensions with Fixed Galois Group*. ProQuest LLC, Ann Arbor, MI, 2019. Thesis (Ph.D.)–Princeton University. Available at https://fabiangundlach.org/phd-thesis.pdf.

[26] Osamu Hyodo. Wild ramification in the imperfect residue field case. In *Galois representations and arithmetic algebraic geometry (Kyoto, 1985/Tokyo, 1986)*, volume 12 of *Adv. Stud. Pure Math.*, pages 287–314. North-Holland, Amsterdam, 1987.

[27] Andrei Jorza. Math 160c Spring 2013 Caltech. Applications of global class field theory. Course notes. Available at https://www3.nd.edu/~ajorza/courses/m160c-s2013/overview/m160c-s2013.pdf.

[28] Max-Albert Knus and Jean-Pierre Tignol. Quartic exercises. *Int. J. Math. Math. Sci.*, 2003. Article ID 284672, 61 pages. http://dx.doi.org/10.1155/S0161171203203458.

[29] H. W. Leopoldt. Über Einheitengruppe und Klassenzahl reeller abelscher Zahlkörper. *Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Nat.*, 1953(2):48 pp., 1953.

[30] Friedrich Wilhelm Levi. Kubische Zahlkörper und binäre kubische Formenklassen [Cubic number fields and cubic form classes]. *Leipz. Ber.*, 66:26–37, 1914.

[31] Monica Marinescu. A dual description of integral binary cubic forms and the Ohno-Nakagawa identities. 2015. Senior thesis, Princeton University.

[32] Preda Mihăilescu. Reflection, Bernoulli numbers and the proof of Catalan’s conjecture. In *European Congress of Mathematics*, pages 325–340. Eur. Math. Soc., Zürich, 2005.

[33] James S. Milne. Fields and Galois theory (v4.30), 2012. Available at www.jmilne.org/math/.

[34] Jin Nakagawa. Orders of a quartic field. *Mem. Amer. Math. Soc.*, 122(583):viii+75, 1996.

[35] Jin Nakagawa. On the relations among the class numbers of binary cubic forms. *Invent. Math.*, 134(1):101–138, 1998.

[36] Jin Nakagawa. A conjecture on the zeta functions of pairs of ternary quadratic forms. *Amer. J. Math.*, 143(2):335–410, 2021.

[37] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2000.
[38] Thong Nguyen-Quang-Do. Filtration de $K^*/K^{*p}$ et ramification sauvage. *Acta Arith.*, 30(4):323–340, 1976.

[39] Evan M. O’Dorney. Rings of small rank over a Dedekind domain and their ideals. *Res. Math. Sci.*, 3(8), 2016.

[40] Evan M. O’Dorney. On a remarkable identity in class numbers of cubic rings. *Journal of Number Theory*, 176:302–332, 2017.

[41] Yasuo Ohno. A conjecture on coincidence among the zeta functions associated with the space of binary cubic forms. *Amer. J. Math.*, 119(5):1083–1094, 1997.

[42] Yasuo Ohno and Takashi Taniguchi. Relations among Dirichlet series whose coefficients are class numbers of binary cubic forms II. *Math. Res. Lett.*, 21(2):363–378, 2014.

[43] Charles A. Osborne. $GL_2(K)-$invariant lattices in the space of binary cubic forms with coefficients in the number field $K$. *Proc. Amer. Math. Soc.*, 142(7):2313–2325, 2014.

[44] Arnold Scholz. Über die Beziehung der Klassenzahlen quadratischer Körper zueinander. *J. reine angew. Math.*, 166:201–203, 1932.

[45] I. Schur. Elementarer Beweis eines Satzes von L. Stickelberger. *Math. Zeits.*, 29:464–465, 1929.

[46] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.

[47] Takuro Shintani. On Dirichlet series whose coefficients are class numbers of integral binary cubic forms. *J. Math. Soc. Japan*, 24:132–188, 1972.

[48] Takashi Taniguchi and Frank Thorne. Orbital exponential sums for prehomogeneous vector spaces. Preprint (2016), available at [https://arxiv.org/abs/1607.07827](https://arxiv.org/abs/1607.07827).

[49] Takashi Taniguchi and Frank Thorne. Secondary terms in counting functions for cubic fields. *Duke Math. J.*, 162(13):2451–2508, 2013.

[50] John T. Tate. Fourier analysis in number fields, and Hecke’s zeta-functions. In J. W. S. Cassels and A. Fröhlich, editors, *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 305–347. Thompson, Washington, D.C., 1967.

[51] Melanie Matchett Wood. Gauss composition over an arbitrary base. *Adv. Math.*, 226(2):1756–1771, 2011.

[52] Melanie Matchett Wood. Parametrizing quartic algebras over an arbitrary base. *Algebra Number Theory*, 5(8):1069–1094, 2011.

[53] Melanie Matchett Wood. Quartic rings associated to binary quartic forms. *Int. Math. Res. Not.*, 2012(6):1300–1320, 2012.

[54] Melanie Matchett Wood. Parametrization of ideal classes in rings associated to binary forms. *J. reine angew. Math.*, 689:169–199, 2014.