Causal structures in the four dimensional Euclidean space

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It is shown that in the 4d Euclidean space there are two causal structures defined by the temporal field. One of them is well-known Minkowski spacetime. In this case the gravitational potential (the positive definite Riemann metric) and temporal field satisfy the Einstein equations with trivial energy-momentum tensor. However, in the case of the second causal structure the gravitational potential and temporal field should be connected with some nontrivial energy-momentum tensor. We consider the simplest case with energy-momentum tensor of the real scalar field and derive exact solution of the field equations. It can be viewed as the ground to consider the second causal structure on the equal footing with the Minkowski spacetime, i.e., as an object interesting from the physical point of view, especially in the framework of the field theory.

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I. INTRODUCTION

Einstein [1] set up the problem of including gravity into the framework of the Faraday concept of field and formulated the principles of gravitational physics (General Theory of Relativity). However, it turned out that all important consequences of the theory were not generally covariant. The reason for the appearance of noncovariant results in the theory with the principle of general covariance as a cornerstone of all building is still unclear. The problems turned out to be so serious that one begins to gradually consider them as verifications of the exceptional properties of the gravitational field such as universality, nonlocality and nonscreening. A variety of examples of the discrepancy between the first principles of General Relativity and the results that follow from it can be found, for example, in [2, 3, 4, 5, 6]. This includes all kinds of the so-called pseudo-energy tensor of the gravitational field, the background metrics introduced in one form or another and alike.
In [7], one of the authors (I.B.P.) has given a simple solution of this actual and paradoxical problem in question. The essence of new approach is as follows. Einstein placed in the foundation of General Relativity the principle of equivalence and tightly connected with it the principle of general covariance [1],[3]. The principle of general covariance has an intimate physical meaning, which in particular implies that coordinates (that one uses for description of physical phenomena) have neither physical nor geometric significance. They are simply parameters. From this follows an evident and exact requirement to formulate general covariant theory of the gravitational field without using the well-known representations about space and time. Only in the case when one is able to analyze how time and space emerge in the framework of the general covariance principles, it is possible to provide answers to such a key problem as essence of time and space. A principal result of the analysis underlies in the fact that time is not an additional dimension but a field [7]. The field concept of time discloses the reason behind those difficulties that exist in General Relativity and removing them it uncovers General Relativity as the self-consistent physical theory in which energy is as fundamental as in quantum mechanics.

The notion of smooth manifold is a geometric basis of General Relativity because this structure does not distinguish intrinsically between different coordinate systems (the principle of general covariance is naturally included into this notion). A manifold is not given apriori (as it is usually presupposed) but is defined by the physical system. It was shown [7], that to this end one needs to put in correspondence to the gravitational field a positive definite Riemann metrics, i.e., the field that was first introduced by Riemann in the framework of geometrical concepts [3, 6]. A smooth four dimensional manifold that corresponds to a physical system is called a physical manifold.

In this paper we consider a four-dimensional Euclidean space as physical manifold and hence gravitational potential is known. By virtue of it the main equation for the temporal field can be exactly solved. We consider two exact solutions to this equation. One of them is invariant with respect to the translation along the stream of time and other one is invariant with respect to the rotations. In the Sect. II we demonstrate that the first solution defines the causal structure, which is known as the Minkowski spacetime. In Sect. III the second causal structure, defined by the second solution, is considered. In Sect. IV we consider a system of gravitational, temporal and real scalar fields and give exact solutions to the corresponding equations. This enables us to find important physical quantities such as energy density, energy flow vector of the gravitational and scalar fields, respectively, and total energy of the system in question. On this ground one can consider the second causal structure (like the Minkowski spacetime) as an object interesting from the physical point of view.

II. USUAL CAUSAL STRUCTURE

The vector space \( \mathbb{R}^4 \) is the set of 4-tuples of real numbers. The symbol \( \vec{u} \) denotes the vector in \( \mathbb{R}^4 \) with components \((u^1, u^2, u^3, u^4)\); \( \vec{u} \cdot \vec{v} \) denotes the usual scalar product \( \vec{u} \cdot \vec{v} := u^1 v^1 + u^2 v^2 + u^3 v^3 + u^4 v^4 \). The distance function be

\[
d(\vec{u}, \vec{v}) = \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})}, \quad d(\vec{u}, 0) = |\vec{u}|.
\]

In accordance with [7], the causal structure on \( \mathbb{R}^4 \) is defined by the metric \( g_{ij} = \delta_{ij} \) and the scalar field \( f(u^1, u^2, u^3, u^4) \) (field of time or temporal field), which is the solution to the
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equation

\[ \left( \frac{\partial f}{\partial u^1} \right)^2 + \left( \frac{\partial f}{\partial u^2} \right)^2 + \left( \frac{\partial f}{\partial u^3} \right)^2 + \left( \frac{\partial f}{\partial u^4} \right)^2 = 1. \]  \hspace{1cm} (2.1)

Space cross-section of the \( R^4 \) is defined by the field of time. For a given real number \( t \), it is given by the equation

\[ f(u^1, u^2, u^3, u^4) = t. \]

It is evident that the equation (2.1) has the solution

\[ f(u^1, u^2, u^3, u^4) = \rightarrow a \cdot \rightarrow u, \]

where \( \rightarrow a = (a^1, a^2, a^3, a^4) \) is a unit constant vector

\[ \rightarrow a \cdot \rightarrow a = 1. \]

The stream of time is defined in the case in question as a congruence of lines of time on the physical manifold \( R^4 \). Analytically, the lines of time are defined as a solution of the autonomous system of differential equations

\[ \frac{du^i}{dt} = g^{ij} \frac{\partial f}{\partial u^j} = t^i = a^i. \]

The general solution is a straight line that goes through the fixed point \( \rightarrow u_0 \):

\[ \rightarrow u (t) = \rightarrow a (t - t_0) + \rightarrow u_0. \]  \hspace{1cm} (2.2)

The causal structure in question defines the interval as follows. Let

\[ \rightarrow u_s = \rightarrow u - 2 \rightarrow a (\rightarrow a \cdot \rightarrow u) \]

be the vector symmetrical \( \rightarrow u \) with respect to the vector \( \rightarrow a \). Then in the coordinates \( u^1, u^2, u^3, u^4 \) the interval can be presented as follows:

\[ s^2 = \rightarrow u \cdot \rightarrow u_s = \rightarrow u \cdot \rightarrow u - 2(\rightarrow a \cdot \rightarrow u)^2 = -| \rightarrow u | \cos 2\theta, \]

where \( \theta \) is an angle between \( \rightarrow a \) and \( \rightarrow u \). To be sure that \( s \) is really well-known interval, let us introduce the system of coordinates compatible with causal structure \([7]\). To this end suppose that all initial data in (2.2) belong to the space-section

\[ \rightarrow a \cdot \rightarrow u_0 = t_0, \]  \hspace{1cm} (2.3)

(the space \( R^3 \) orthogonal to the vector \( \rightarrow a \)).

Let us consider the natural system of four orthogonal unit vectors

\[ \rightarrow E_0 = (a^1, a^2, a^3, a^4), \quad \rightarrow E_1 = (-a^4, -a^3, a^2, a^1), \]

\[ \rightarrow E_2 = (a^3, -a^4, -a^1, a^2), \quad \rightarrow E_3 = (-a^2, a^1, -a^4, a^3), \]

\( \rightarrow E_0 = \rightarrow a \). Now the general solution to equation (2.3) has the form

\[ \rightarrow u_0 = t_0 \rightarrow E_0 + x \rightarrow E_1 + y \rightarrow E_2 + z \rightarrow E_3. \]

Substituting this representation into formula (2.3) we obtain that the congruence of the lines of time can be written in the following form:

\[ \rightarrow u = t \rightarrow E_0 + x \rightarrow E_1 + y \rightarrow E_2 + z \rightarrow E_3. \]  \hspace{1cm} (2.4)
We can consider (2.4) as the coordinate transformation (with unit Jacobean). It is easy to see that in the coordinates \(t, x, y, z\), the interval takes the form:

\[
s^2 = x^2 + y^2 + z^2 - t^2.
\]

Let us consider one more example. If we put

\[
\hat{P}_0 = \vec{E}_0 \cdot \vec{\nabla}, \quad \hat{P}_1 = \vec{E}_1 \cdot \vec{\nabla}, \quad \hat{P}_2 = \vec{E}_2 \cdot \vec{\nabla}, \quad \hat{P}_3 = \vec{E}_3 \cdot \vec{\nabla},
\]

where

\[
\vec{\nabla} = \left( \frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^3}, \frac{\partial}{\partial u^4} \right),
\]

then the Dirac equation in the coordinates \(u^1, u^2, u^3, u^4\) reads

\[
i\gamma^\mu \hat{P}_\mu \psi = \frac{mc}{\hbar} \psi.
\]

The Dirac equation in the coordinates \(t, x, y, z\) has an ordinary form

\[
i(\gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z})\psi = \frac{mc}{\hbar} \psi.
\]

One can work in either the coordinates \(u^1, u^2, u^3, u^4\) or in the coordinates \(t, x, y, z\), but in the first case the physical results should not depend on the choice of the constant vector \(\vec{a}\) (a direction of time flux).

**III. THE NEW CAUSAL STRUCTURE**

As it is shown, the considered causal structure on \(\mathbb{R}^4\) is the Minkowski space-time. Now we shall show that there is a new causal structure on \(\mathbb{R}^4\) that is invariant with respect to the rotations. Indeed, equation (2.4) has a remarkable solution

\[
f(u^1, u^2, u^3, u^4) = \sqrt{u \cdot \vec{u}} = \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2}.
\]

Thus, in the first case the space sections of the \(\mathbb{R}^4\) are \(\mathbb{R}^3\) (\(\vec{a} \cdot \vec{u} = t\)) and in the second one the space sections are 3D spheres \(S^3\) \(|\vec{u}| = \sqrt{u \cdot \vec{u}} = \tau\). In this case, the lines of time are the solutions of the system of equations

\[
\frac{du^i}{d\tau} = \frac{\partial f}{\partial u^i} = \frac{u^i}{f} = \frac{u^i}{\sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2}}.
\]

It can be shown that the general solution of this system is

\[
u^i(\tau) = u^i_0 \frac{\tau}{\tau_0},
\]

during which the initial data belong to the space-section

\[
\vec{u}_0 \cdot \vec{u}_0 = \tau_0^2.
\]
We see that the second causal structure is the congruence of rays coming from initial point \((0, 0, 0, 0)\) of \(R^4\) (origin). Thus, the coordinate space \(R^4\) can be fibered on \(R^3\) (the stream of time is the bundle of parallel straight lines) or \(S^3\) (the stream of time is the bundle of the rays coming from one point).

The action for the point particle defined by the second causal structure has the form

\[
S = -mc \int_{p}^{q} \sqrt{1 - \tau^2 \omega^2} d\tau,
\]

where \(\omega = dl/d\tau\) and \(dl\) is the element of arc on the unit 3D sphere. Really, \(\vec{du} \cdot \vec{du} = d\tau^2 + \tau^2 d\ell^2\), and \(\vec{u} \cdot \vec{du} = \tau d\tau\). Let us introduce the system of coordinates compatible with causal structure. Rewrite the relation (3.3) in the parametric form

\[
\begin{align*}
    u^1 &= \tau_0 \sin \alpha \sin \vartheta \cos \phi, \\
    u^2 &= \tau_0 \sin \alpha \sin \vartheta \sin \phi, \\
    u^3 &= \tau_0 \sin \alpha \cos \vartheta, \\
    u^4 &= \tau_0 \cos \alpha,
\end{align*}
\]

where \((0 \leq \alpha, \vartheta \leq \pi, 0 \leq \phi \leq 2\pi)\). Inserting this representation into (3.3) we come to the conclusion that the congruence of the lines of time can be written as follows:

\[
\begin{align*}
    u^1 &= \tau \sin \alpha \sin \vartheta \cos \phi, \\
    u^2 &= \tau \sin \alpha \sin \vartheta \sin \phi, \\
    u^3 &= \tau \sin \alpha \cos \vartheta, \\
    u^4 &= \tau \cos \alpha.
\end{align*}
\]

We can consider the transformation (3.4) as the coordinate transformation. In the coordinate \((\tau, \alpha, \vartheta, \phi)\) we have the metric

\[
ds^2 = d\tau^2 + \tau^2 \left[d\alpha^2 + \sin^2 \alpha \left(d\vartheta^2 + \sin^2 \vartheta d\phi^2\right)\right],
\]

with

\[
\sqrt{g} = \tau^3 \sin^2 \alpha \sin \vartheta
\]

and gradient of temporal field has the form \(t^i = (0, 0, 0, 1)\).

**IV. EXACT SOLUTION OF THE FIELD EQUATIONS**

In this section we study the self-consistent system of the gravitational, temporal and real scalar field equations for the reason to follow. The geometrical equations of the gravitational field read

\[
G_{ij} + T_{ij} = \varepsilon t_i t_j, \quad \varepsilon = \varepsilon_m + \varepsilon_g,
\]

where \(\varepsilon_m\) and \(\varepsilon_g\) are the energy densities of the matter and gravitational field, respectively and given by

\[
\begin{align*}
    \varepsilon_m &= T_{ij} t^i t^j, \\
    \varepsilon_g &= G_{ij} t^i t^j.
\end{align*}
\]
Here
\[ G_{ij} = \tilde{R}_{ij} - \frac{1}{2} \tilde{g}_{ij} \tilde{R} \]
is the Einstein tensor defined by the subsidiary metric
\[ \tilde{g}_{ij} = g_{ij} - 2 t_i t_j, \quad \tilde{g}^{ij} = g^{ij} - 2 t^i t^j. \]

Note that quantities like \( R \) and \( R_{ij} \) are related to the metric \( g_{ij} \) while those with tilde, i.e., \( \tilde{R} \) and \( \tilde{R}_{ij} \) are generated from the subsidiary metric \( \tilde{g}_{ij} \); \( t^i \) is the gradient of temporal field defined as
\[ t^i = g^{ij} \frac{\partial f}{\partial u^i}. \]

It is easy to see that the first causal structure defined by the \( g_{ij} = \delta_{ij} \) and \( f = a_i u^i \) is the solution to the Eq. (4.1) with \( T_{ij} = 0 \) and in this case \( \varepsilon_g = 0 \). But for the second causal structure defined by the \( g_{ij} = \delta_{ij} \) and \( f = \sqrt{u^i u^i} \) is not the solution to the Eq. (4.1) with \( T_{ij} = 0 \). We see that second causal structure will be physical field under the condition that the energy-momentum tensor is not trivial. We shall give exact solution of this problem for the case with energy-momentum tensor of real scalar field.

The scalar field Lagrangian we choose in the form:
\[ \mathcal{L}_\varphi = \frac{1}{2} \tilde{g}^{ij} \partial_i \varphi \partial_j \varphi + V(\varphi). \] (4.3)

Here \( V(\varphi) \) is some unknown potential energy that should be fixed as a result of the solution of Eq. (4.1). Taking into account that
\[ \tilde{R} = \frac{12}{f^2}, \quad \tilde{R}_{ij} = \frac{4}{f^2} (g_{ij} - t_i t_j), \] (4.4)
for the Einstein tensor one finds
\[ G_{ij} = -\frac{2}{f^2} g_{ij} + \frac{8}{f^2} t^i t^j. \] (4.5)

Now in account of the fact that in our case both \( R \) and \( R_{ij} \) are trivial, from (4.2b) we find
\[ \varepsilon_g = 6/f^2. \] (4.6)

In accordance with (4.5) and (4.3) we consider the case when \( \varphi = \varphi(f) \) with \( f = f(u) = \sqrt{u_i u^i} = \sqrt{u_1 u^1 + u_2 u^2 + u_3 u^3 + u_4 u^4} \). On account of the energy-momentum tensor \( T_{ij} \)
\[ T_{ij} = \partial_i \varphi \partial_j \varphi - \tilde{g}^{ij} \mathcal{L}_\varphi, \] (4.7)
we find the energy density of the scalar field as
\[ \varepsilon_\varphi = T_{ij} t^i t^j = \frac{1}{2} \varphi'^2 + V(\varphi), \] (4.8)
whereas, for the energy flow vector of the scalar field we find
\[ \Pi_i = \varepsilon_\varphi t_i - T_{ij} t^j = 0. \] (4.9)
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In view of what has been told, from (4.1) we find
\[
\frac{1}{2} \varphi'^2 - V(\varphi) - \frac{2}{f^2} = 0. \tag{4.10}
\]

On the other hand, the equation of motion
\[
\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j \varphi) - \frac{dV(\varphi)}{d\varphi} = 0, \tag{4.11}
\]
gives
\[
\varphi'' + \frac{3}{f} \varphi' + \frac{dV(\varphi)}{d\varphi} = 0. \tag{4.12}
\]
Excluding \( \varphi \) from (4.10) and (4.12) we find the second order differential equation for \( \varphi \):
\[
2\varphi' \varphi'' + \frac{3\varphi'^2}{f} + \frac{4}{f^3} = 0. \tag{4.13}
\]
The Eq. (4.13) allows the following first integral:
\[
\varphi'^2 = 4L \frac{f^3}{f^3} - \frac{4}{f^2}. \tag{4.14}
\]
Here \( L \) is the constant of integration. From Eq. (4.14) it follows that the physical manifold is not \( \mathbb{R}^4 \) but a ball \( f^2 \leq L^2 \). For \( \varphi \) finally we find
\[
\varphi = \mp 4 \left[ \sqrt{\frac{L}{f}} - 1 + \arcsin \sqrt{\frac{f}{L}} \right] \pm 2\pi, \tag{4.15}
\]
where the integration constant \( C = 2\pi \) is defined from the condition \( \varphi(L) = 0 \). The corresponding potential now can be found from (4.10):
\[
V(\varphi) = 2L \frac{f^3}{f^3} - \frac{4}{f^2}. \tag{4.16}
\]
Using the results obtained we find the energy density of the scalar, temporal and gravitational field system. It gives
\[
\varepsilon = \varepsilon_{\varphi} + \varepsilon_g = \frac{\varphi'^2}{2} + V(\varphi) + \frac{6}{f^2} = \frac{2L}{f^3} - \frac{2}{f^2} + \frac{2L}{f^3} - \frac{4}{f^2} + \frac{6}{f^2} = \frac{4L}{f^3}. \tag{4.17}
\]
The law of energy conservation reads
\[
\nabla_i (\varepsilon t^i) = 0.
\]
In our case
\[
\nabla_i = \frac{\partial}{\partial u^i}, \quad \varepsilon = \frac{4L}{f^3}, \quad t^i = \frac{u^i}{f}
\]
and hence
\[
\nabla_i (\varepsilon t^i) = 4L \frac{\partial}{\partial u^i} (\frac{u^i}{f_4}) = 4L (\frac{4}{f^4} - \frac{4}{f^3} u^i \frac{\partial}{\partial u^i} f) = 0.
\]
In the system of coordinates compatible with causal structure we should have

\[ \frac{\partial}{\partial \tau} (\sqrt{g} \varepsilon) = 0 \]

and this is the case because from (3.6) and (4.17) it follows that

\[ \sqrt{g} \varepsilon = \tau^3 \sin^2 \alpha \sin \vartheta \cdot \frac{4L}{\tau^3} = 4L \sin^2 \alpha \sin \vartheta. \]

If we integrate with respect to the variable \( \alpha, \vartheta, \phi \) we find for the total energy of the system

\[ E = 8\pi^2 L. \]  

(4.18)

We see that the solution obtained leads to the reasonable result. It can be shown also that the energy flow vector of the gravitational field is trivial.

Thus, we found exact solution of the corresponding equations. With this we found all important physical quantities: the energy density, an energy flow vector of the gravitational and scalar field and total energy of the system in question.

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