Moving Beyond Chi-Squared in Nuclei and Neutron Stars

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Abstract. Interpreting data and understanding the accompanying theoretical models often requires a statistical analysis. Frequently, this analysis takes the form of an minimization of a function, $\chi^2$, which quantifies the extent to which a model can reproduce a set of experimental data. Similar procedures have proven successful at providing us basic knowledge of the nucleon-nucleon interaction from experimental measurements of nuclear masses, radii, and other low-energy observables. For neutron stars, masses and radii are extremely difficult to measure and models are currently severely underconstrained. For this reason, Bayesian analysis has become useful in analyzing neutron star data and obtaining constraints on the mass-radius curve and the equation of state of dense matter. In this work, I briefly review both types of analysis and suggest ways in which future fits to low-energy nuclear data might be improved by lessons learned from the Bayesian analysis of neutron star data. A few constraints on the nuclear symmetry energy are also described, and a possible method for improving current constraints is proposed.

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1. Chi-squared, nuclear masses, and charge radii

The prototypical problem to be discussed is the fitting of a model, i.e. an energy density functional, to nuclear masses and charge radii. A review the basic formalism highlights the various assumptions which are often made. Using a set of $N_d$ data points, $D_i$, and a model with $N_p$ parameters, $p_j$, one defines a function, $\chi^2$, by

$$\chi^2 = \sum_i^{N_d} \left[ \frac{D_i - P_i(p)}{\sigma_{\text{exp},i}} \right]^2,$$

(1)
where $P_i$ is the model prediction for data point with index $i$ and $\sigma_{\text{exp},i}$ is the experimental uncertainty of the $i$-th data point. Minimizing $\chi^2$ over the $N_p$ dimensional parameter space gives the best fit to the data. Alternatively, one maximizes a likelihood function

$$L = \exp \left( -\chi^2 / 2 \right),$$

which is equal to a product of Gaussians for each data point. The formulation into a likelihood function makes it clear that an implicit assumption of an independent and Gaussian distribution for each data point has already been made.‡ If this assumption is correct and the deviations of the data from the model predictions follow a Gaussian distribution, then the value of $\chi^2$ follows the chi-squared distribution and a “good fit” is one where $\chi^2 \approx N_d - N_p$. Note that, in an underconstrained system, $N_d - N_p < 0$ and this test is not useful.

If the model is a linear function of the parameters, then the model serves as a linear transformation of the multivariate Gaussian determined by the data and therefore the likelihood function is also Gaussian. Even with nonlinear models, the likelihood function is often approximately Gaussian because of the central limit theorem, in which case it can be parameterized by a best fit vector $\mu$ and a covariance matrix, $\Sigma$:

$$L(p) \approx A \exp \left[ -\frac{1}{2} (p - \mu)^T \Sigma^{-1} (p - \mu) \right],$$

with a normalization factor $A$ fixed so that Eq. 2 holds.§

The correlation matrix $C$ can be obtained from $\Sigma$ by $C_{ij} = \Sigma_{ij} / \sqrt{\Sigma_{ii} \Sigma_{jj}}$. Modern fitting codes often obtain the covariance matrix from the Jacobian of $\chi^2$. The correlation matrix, as obtained this way, is a correct representation of the correlations between model parameters only if the likelihood function is approximately given by the form in Eq. 3. It has become commonplace to report correlation matrices without verifying that this approximation is actually valid.

In the context of nuclear masses, there is evidence that the likelihood functions implied by typical models are indeed more complicated than the simple form above. Explicit calculation shows that the likelihood function simple Bethe-Weisacker mass formula is multi-modal because of the pairing contribution. Skyrme models [4], energy density functionals which describe nuclear masses and charge radii, are linear in parameters $t_0, \ldots, t_3$, but not linear in the parameter, $\alpha$, which characterizes density-dependent two-nucleon force. This non-linearity could mean that likelihood is not

‡ The assumption of independence is violated in nuclear mass data; the construction of the atomic mass evaluation is done in such a way that neighboring nuclear masses are non-trivially correlated [1, 2]. Future atomic mass evaluations should make sure that these correlations are widely accessible. In any case, these correlations may not be important here because the experimental uncertainties are much smaller than the systematic uncertainties. For an example of fitting correlated data in the context of hadron mass spectroscopy, see Ref. [3].

§ Note that the likelihood function is not normalized with $A = \left[ (2\pi)^{N_p} |\Sigma| \right]^{-1/2}$ so that its integral is unity. The likelihood function, unlike the prior distribution, is not a probability density function. This detail is important below in the discussion of Bayes factors.
fully Gaussian, though this has not been fully studied. Some evidence for non-Gaussian behavior is present in the posterior probability distributions for the Skyrme-like parameters in Fig. 1 of Ref. [5] which are fit to a large set of data comprising mostly of nuclear masses and charge radii. Covariant mean-field models may also not generate Gaussian likelihoods because of the non-linearity (which is required to reproduce saturation) in the mean-field equation for the scalar-isoscalar meson [6].

If the likelihood function was a multivariate Gaussian, one would expect the minimization of $\chi^2$ to be numerically trivial (so long as the likelihood was evaluated with sufficient numerical accuracy). For example, the Broyden-Fletcher-Goldfarb-Shanno method is guaranteed to converge if the objective function has a quadratic expansion near the extremum. The result that the fit of the Skyrme-like parameter $C_\rho \nabla J_1$ depends slightly on the initial guess in Ref. [7] (in a fit similar to that in Ref. [5] described earlier) means that, in contrast, this minimization is not trivial. This is not surprising as the posterior distribution of this same parameter in Fig. 1 in Ref. [5] appears multi-modal.

It is currently not possible to generate energy density functionals which describe heavier nuclei with systematic uncertainties which are not much larger than the $\sigma_{\text{exp},i}$. Thus, it is common to redefine $\chi^2$ as

$$\chi^2 = \sum_i \frac{(D_i - P_i)^2}{\sigma_{\text{exp},i}^2 + \sigma_{\text{sys},i}^2},$$

where $\sigma_{\text{sys},i}$ is the systematic uncertainty for each data point. Often $\sigma_{\text{exp},i}$ is small enough that it can be ignored. There are now $N_p + N_d$ parameters: the $N_p$ model parameters and a parameter $\sigma_{\text{sys},i}$ for each data point. Equivalently, one can rewrite each $\sigma_{\text{sys},i}^2$ as a weight, $w_i$, to be fixed by some recipe. There are several ways to proceed. It is possible to perform a Bayesian analysis with all $N_p + N_d$ parameters, but this is computationally difficult. The alternative is to reduce the number of parameters. A frequent choice is to employ a fixed uncertainty for each type of data point, e.g. $\sigma_{\text{mass}}$ for nuclear masses and $\sigma_{\text{radius}}$ for charge radii. A similar choice is manifest in the literature through the presence of fitting protocols: to choose not to fit data point $j$ is equivalent to assuming that the systematic uncertainty for the model to predict the $j$-th data point, $\sigma_j$, is so large that the corresponding term in $\chi^2$ is negligible. Early fits (see review in e.g. Ref. [8]) used only doubly-magic nuclei, on the basis that open-shell systems contained correlations which were unlikely to be correctly computed by Hartree-Fock calculations. Ref. [9] discusses this issue and suggests fixing the denominators by separately normalizing $\chi^2$ for each type of data (their Eq. 4). A slightly different procedure is suggested below. Finally, computing several systematic uncertainties using iteratively-reweighted least squares may be useful. Studies of how relative variations in the $\chi^2$ weights might affect the interpretation of the data, in the context of low-energy nuclear data, are in their infancy.

I have found empirically that a Bayesian analysis in this form requires prior distributions for $\sigma_{\text{sys},i}$ that fall off as $\exp(-\sigma^2)$ to to ensure the trivial solution with $\sigma \to \infty$ does not dominate the results. More work on this is in progress.
An implicit assumption lurking in this discussion is the assumption that the systematic uncertainties in the denominators are themselves uncorrelated. If this assumption fails, then one must reformulate the $\chi^2$ function to take into account the uncertainties. This problem has been found particularly relevant for fitting parton distribution functions (see in particular Appendix B of Ref. [10]). If correlations between parameters and correlations between the systematic uncertainties are not important, and if the $\chi^2$ function is of the form given above, one can estimate parameter uncertainties by varying the parameters, one at a time, until $\chi^2(p) = N_d - N_p + 1$. However, it seems likely that the systematic uncertainties of masses and radii are indeed correlated. Models with no three-nucleon forces often predict saturation at a higher density and with a larger binding energy than found in laboratory nuclei [11], and these models naturally lead both to smaller masses and smaller radii.

In the context of fits to nuclear masses and charge radii, the model parameters are almost always correlated. If the likelihood function is nearly Gaussian, then one can determine the parameter uncertainty including these correlations by marginalizing over the multivariate Gaussian (which can be done using the covariance matrix and does not require any integration).

2. Bayesian analysis for neutron star masses and radii

For neutron stars, the problem is how to determine the mass-radius ($M - R$) curve from a small sample of poorly constraining observations. (A review is available in Ref. [12], only some of which is particularly relevant to this work and reproduced here.) The statistical analysis is complicated for two reasons. The first is that the currently available data has large statistical (and large systematic) uncertainties. There are effectively more parameters than constraints. Second, the mass-radius curve need not be a function in the mathematical sense, it may fail either or both of the horizontal or vertical line tests. (Given the central energy density, $\varepsilon_c$, the relations $M(\varepsilon_c)$ and $R(\varepsilon_c)$ are both well-defined functions.) An example of how this happens is Ref. [13], where a phase transition induces a new branch in the $M - R$ curve.

Bayesian analysis allows one to solve both of these issues, at the expense of introducing an unknown function, a prior probability distribution function. Also, the two-dimensional nature of the neutron star problem requires that $N_d$ parameters for the neutron star masses, $M_i$, are also added. Some researchers have been slow to employ Bayesian inference because of the ambiguity in determining prior distributions, but Ref. [14] has shown that the prior is not a separate entity but a part of the model which can be checked and verified by comparing to data.

In the case of nuclear mass fitting described above, the likelihood function was a product of $N_d$ Gaussian probability distributions for each data point. The neutron star case is handled similarly (c.f. Eq. 31 in Ref. [15]): the likelihood function is a product over a data set for each neutron star. Each data set is of the form $D(M, R)$, a two-dimensional probability distribution for the mass and radius of a neutron star. These
probability distributions have complicated shapes, and the likelihood is not Gaussian. As discussed above, the traditional correlation matrix thus does not contain all of the information on correlations present in the problem.

As a demonstration, the correlation matrix is represented in Fig. 1, for the analysis of neutron star data with the HLPS + Model C parameterization from Ref. [16] based on Ref. [17] and Model C from Ref. [18]. There are 17 model parameters, 7 from the EOS and 10 neutron star masses. The 7 EOS parameters are given in Table 1. The four panels each represent the correlation matrices inferred from different subsets of the Markov chain Monte Carlo simulation of the likelihood function. The upper-left panel was computing using only points where the likelihood was within a factor of two of the maximum value at the best fit. The upper-right panel used points where the likelihood was a factor of 8 within the maximum. The lower-right panel used all the Monte Carlo points. The strong correlation between parameters 2 and 3 is related to the correlation between S and L as obtained in Ref. [17]. The strong anti-correlation between pressure parameters is also expected. The pressure cannot be too small over a large range in density or the maximum mass will like below the observed value. The pressure cannot also be too large over a large range in density or the radii will be too large to reproduce the data. Qualitatively, the pressure tends to be either small at moderate densities and large at high densities, or vice versa. Finally, it is clear that many of the correlations near the best fit (upper-left panel) are not representative of the full data set (lower-right panel).

This result is further demonstrated in the left panel of Fig. 2, where the correlation coefficient between the high density pressure parameter, $P_4$ and the mass of the neutron star in the M13 globular cluster is shown as a function of the range of likelihoods considered, $L_{\text{min}}/L_{\text{max}}$. The two parameters appear anti-correlated near the best fit. In reality, they are not correlated. This is expected, as the neutron star in M13 is typically lower in mass, and thus less sensitive to the pressure at higher densities. The right panel of Fig. 2 shows that the Gaussian approximation to the likelihood also gives a different range for the radius of a $1.4 M_\odot$ neutron star implying slightly larger radii.

| Parameter index | Description |
|-----------------|-------------|
| 1               | Compressibility |
| 2               | Energy per baryon of neutron matter |
| 3               | Derivative of energy per baryon of neutron matter |
| 4               | $P_1 \equiv P(\varepsilon = 2 \text{ fm}^{-4}) - P(n_B = 0.16 \text{ fm}^{-3})$ |
| 5               | $P_2 \equiv P(\varepsilon = 3 \text{ fm}^{-4}) - P(\varepsilon = 2 \text{ fm}^{-4})$ |
| 6               | $P_3 \equiv P(\varepsilon = 5 \text{ fm}^{-4}) - P(\varepsilon = 3 \text{ fm}^{-4})$ |
| 7               | $P_4 \equiv P(\varepsilon = 7 \text{ fm}^{-4}) - P(\varepsilon = 5 \text{ fm}^{-4})$ |
| 8-17            | Individual neutron star masses |

Table 1. The list of parameters for neutron star EOS parameterization described in the text and corresponding to the correlation matrices displayed in Fig. 1.
In Bayesian inference, marginalization is often used to determine model parameters. The posterior probability distribution for the $i$-th parameter is determined from

$$P(p_i) \propto \int dp_1 dp_2 \ldots dp_{i-1} dp_{i+1} \ldots dp_{N_p} dM_1 \ldots dM_{N_M} \mathcal{L}(p, M) \mathcal{P}(p, M)$$

(5)

where $\mathcal{L}$ is the likelihood and $\mathcal{P}$ is the prior distribution. This integral is typically evaluated using a Monte Carlo method. One may directly parameterize the $M - R$ curve and compare it to the data or parameterize the EOS of dense matter, $P(\varepsilon)$ and then use the TOV equations to compute the $M - R$ curve at each point in the Monte Carlo integration. Each parameterization (given a fixed data set) is a different model.
A particular advantage of this formalism, as applied in Ref. [15, 19, 18, 12, 20, 16], is that one need not assume any correlations between the properties of matter near the saturation density and matter in the inner core of the neutron star. Matter in the core may be best described, for example, by quark degrees of freedom and have little similarity with neutrons and protons at the saturation density.

Models can be compared with each other by computing the Bayes factor. The Bayes factor of Model A with respect to Model is the ratio:

\[ B_{AB} \equiv E_A / E_B \]

where \( E \) is the evidence:

\[ E = \int dp \, dM \, L(p, M) \mathcal{P}(p, M) \]  

Note that the two models need not have the same number of parameters. The Bayes factor is equivalent to betting odds: \( B_{AB} = 10 \) implies that model A is ten times more likely than model B.

Several different models can be compared with Bayes factors, a strategy employed in Ref. [12]. This work used Bayes factors and an analytical model of the neutron star atmosphere to show that different values of the “hydrogen column density” were favored. The hydrogen column density is a parameter which characterizes the extent to which X-rays are absorbed between the neutron star and the detector. Ref. [12] also found a model where some neutron stars have Helium atmospheres was favored. Recent work in Ref. [21], including more observational data, has confirmed a lower hydrogen column density for the neutron star in the \( \omega \) Cen globular cluster and a Helium atmosphere for
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3. Predictions, the nuclear symmetry energy, and the neutron skin thickness of lead

Given any function $F(p)$ of the model parameters, one can use the best fit to compute the predicted value, $F(p = \mu)$. To compute the uncertainty in the prediction, one can compute a probability distribution

$$P(F) = \int dp L(p) \delta[F(p) - F]$$

(7)

This integral is easier if $L$ can be accurately described by a multivariate Gaussian, because one can directly sample the likelihood function using the Cholesky decomposition of the covariance matrix. When the Gaussian approximation is inadequate, as in the neutron star problem discussed above, and the likelihood function must be indirectly sampled, e.g. through Markov chain Monte Carlo. It is sometimes efficient to use the Gaussian approximation as a proposal distribution in a Metropolis-Hastings step.

A quantity of central interest in low-energy nuclear physics is the nuclear symmetry energy, the energy cost to create an isospin asymmetry. Given the energy per baryon of neutron matter as a function of the baryon density, $E_{\text{neut}}(n_B)$, and the energy per baryon of nuclear matter $E_{\text{nuc}}(n_B)$, the symmetry energy can be defined as the difference $S(n_B) \equiv E_{\text{neut}}(n_B) - E_{\text{nuc}}(n_B)$. Of particular interest is the value of the symmetry energy at the nuclear saturation density, $S \equiv S(n_B = n_0)$, and its derivative, $L \equiv 3n_0S'(n_B = n_0)$.

Arguably, two of the best ways to obtain constraints on the nuclear symmetry energy are from nuclear data and neutron star observations described in sections 1 and 2 above. Nuclear masses offer a strong constraint on a linear combination of $S$ and $L$, but do not determine the two quantities separately [22]. Neutron star radius measurements and measurements of the neutron skin thickness of lead both strongly constrain $L$ and are weakly correlated with $S$ [23].

In part motivated by the connection between the neutron skin thickness in lead, $\delta R$, and the parameter $L$, several recent experiments have measured the skin thickness. The PREX experiment, which uses parity violating electron scattering, found $\delta R = 0.33^{+0.06}_{-0.18}$ fm [24]. This weak probe of nuclear structure should have smaller systematic uncertainties than those from strongly interacting probes. A more precise (but more model-dependent) result of $\delta R = 0.15^{+0.04}_{-0.06}$ fm was recently obtained from photoproduction of pions at the MAMI electron beam facility in Ref. [25]. Using probability distributions determined by neutron star radius measurements, Ref. [18] predicted the neutron skin thickness would be less than 0.20 fm, consistent with both the recent PREX and MAMI measurements. In this work, fits employed four separate models (named A, B, C, and D) and different interpretations of the neutron star data.

\footnote{Because this section is more general, the explicit reference to the neutron star masses, $M$, is removed.}
in order to attempt to assess the systematic uncertainty (see Fig. 4 of Ref. [18]). It remains to be seen if the prediction of the neutron skin thickness of lead from neutron star observations in Ref. [18] will prove true with more accurate experiments, such as PREX II.

In general, uncontrolled systematic uncertainties are particularly pernicious for predicted values: there is no method which guarantees that reality lies between any computed confidence interval which is obtained from the data. In the context of energy density functionals, there is a long history of using several nearly equivalent models of the same data in order to estimate the systematic uncertainties, as done in the neutron star analysis above, as earlier done in Refs. [26, 27, 28] and as discussed in Ref. [9].†

Studies of inter-model dependence are important, even when one is not focused on predicting a particular observable, because they help diagnose cases where a model may fit the data accurately with an incorrect physical mechanism.

4. Completeness versus accuracy

The purpose of making a model is two-fold: the first is to provide insight regarding the physical mechanisms which underlie the observed data, and second is to make predictions. These two purposes are sometimes at odds: the ability of a model to make predictions must often strike a balance between attempting to describe the most complete set of currently available data, and describing a small set of data with high accuracy. This qualitative picture is behind the practice (in Bayesian inference) of using different prior distributions for parameter estimation than those used for model comparison.

Much of the recent work surrounding the description of nuclear structure observables has focused on generating an energy density functional. The Kohn-Sham theorem suggests that, if the correct energy density functional was found, one could accurately describe all nuclei at the level of single reference Hartree-Fock-Bogoliubov (see e.g. Ref. [29]). In many works, the purpose here is mainly that of completeness: one describes as large of a data set as is possible in order to obtain the best energy density functional. Part of the promise of chiral effective theory is that we might not have to sacrifice completeness for accuracy (see review in Ref. [30]), and progress is being made in this direction in lighter nuclei. On the other hand, very accurate descriptions of neutron-rich calcium isotopes have been obtained with coupled cluster methods [31] with interactions which might otherwise have difficulty describing nuclear matter at the saturation density [32].

In the context of nuclear masses, this tension between completeness and accuracy is demonstrated in Fig. 3, where two mass models from Refs. [33, 34] are used to predict the mass of \(^{137}\text{Sn}\). In the left panel, the full experimental mass data set from

† These studies can be viewed as an inexpensive and rough way of performing a hierarchical Bayesian analysis. In particular, they are only representative of the systematic uncertainty so long as they faithfully represent the space of reasonable models.
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Figure 3. A demonstration of the competition between completeness and accuracy. Solid lines use the model from Ref. [33] and dashed lines use the mass model from Ref. [34]. Left panel: the predicted binding energy of $^{137}$Sn when fitting all measured nuclear masses. Right panel: the binding energy of $^{137}$Sn when fitting only Sn, In, and Sb isotopes.

Ref. [35, 36]. The mass models predict rather small statistical uncertainties, but the systematic uncertainties are at least 3 MeV. In the right panel, the models are fit to a more limited data set in the Sn region, and while the statistical uncertainties are nearly unchanged the systematic uncertainties appear smaller. Of course, two models are insufficient to make any conclusive statements about the magnitude of systematic uncertainties, but this plot illustrates the basic point.

Constraints on the nuclear symmetry energy will be improved by understanding this tension between completeness and accuracy. It might be possible to obtain models which more accurately describe the symmetry energy by fitting to only carefully selected nuclear data and avoiding complications of attempting to describe nuclei which are currently not well-described by Hartree-Fock-Bogoliubov. In more detail, the best constraints on the symmetry energy will come from an fitting several models to a data set which is specifically optimized to generate the smallest combination of (i) the uncertainty in the fit and (ii) the systematic uncertainty implied by the variation between models. This optimization of this data set should include varying the individual weights, $w_i$, in the $\chi^2$ function to ensure the highest possible accuracy for the symmetry energy. In addition to nuclear masses and charge radii described above, giant resonance data will likely be helpful. When the systematic uncertainties of neutron star mass and radius observations are sufficiently small, they will also provide a powerful constraint.
5. Discussion

The analysis of neutron star data has a low computational requirement; a single solution of the TOV equations is much faster than the accurate determination of several nuclear masses and charge radii from an energy density functional. This low computational requirement has enabled the use of Bayesian analysis to attempt to understand currently available neutron star data. A similar statistical analysis for the matching of energy density functionals to low-energy nuclear data is not yet possible. Nevertheless, the analogy permits recommendations on future directions for fits to energy density functionals.

- Likelihood functions should be published and made widely accessible.* If the likelihood function is sufficiently accurate for predictions, then publishing the best fit and the covariance matrix is sufficient. Otherwise, the likelihood function can either be tabulated or represented by a Markov chain of several points. In the neutron star case, this latter procedure was used in Ref. [16]; a list of points sampled from the likelihood function was provided for each model. This list can be directly used by other authors to provide predictions based on the same astronomical data. Also, the differences between the likelihood functions between models can be used to obtain an estimate on the systematic uncertainty as described above.

- The classical $\chi^2$ procedure can be applied in cases where it is inappropriate, clouding our ability to properly interpret the data. Important methods to alleviate this issue include examining the residuals from the best fit model [9], and systematically refitting with part of the data set removed (as e.g. done in the neutron star context in Ref. [37]). In addition, one critical issue is the potential for correlations between our systematic uncertainties. More work needs to be done to understand how these correlations might potentially affect results. These correlations can be explicitly modeled using the method described in Ref. [10].

- A method for obtaining modern constraints on the nuclear symmetry energy is proposed in the previous section. Generalizing this method, the ability of a model to make predictions should drive the data set which is included in $\chi^2$ fits and the associated weights that are used. This can only be effectively done when the systematic uncertainties of the model are assessed, either by a reasonable physical argument or a study of the variation among several models. The exception to this is when a more complete description of the data is likely to provide physical insight. The ability to make more accurate predictions will be useful, in particular, to those communities (like neutron star astronomers) who need nuclear data for their models.

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References

[1] Audi G, Davies W G and Lee-Whiting G E 1986 Nucl. Instrum. Methods Phys. Res. A 249 443 URL http://dx.doi.org/10.1016/0168-9002(86)90700-X
[2] Audi G, Wapstra A H and Thibault C 2003 Nucl. Phys. A 729 337 URL http://dx.doi.org/10.1016/j.nuclphysa.2003.11.003
[3] Michael C 1994 Phys. Rev. D 49 2616–2619 URL http://dx.doi.org/10.1103/PhysRevD.49.2616
[4] Skyrme T H R 1959 Nucl. Phys. 9 615 URL http://dx.doi.org/10.1016/0029-5582(58)90345-6
[5] Schunck N, McDonnell J D, Higdon D, Sarich J and Wild S 2014 arXiv:1406.4374 URL http://arxiv.org/abs/1406.4374
[6] Roca-Maza X, Paar N and Colò G 2014 arXiv:1406.1885 URL http://arxiv.org/abs/1406.1885
[7] Wild S M, Sarich J and Schunck N 2014 arXiv:1406.5464 URL http://arxiv.org/abs/1406.5464
[8] Lunney D, Pearson J M and Thibault C 2003 Rev. Mod. Phys. 75 1021 URL http://dx.doi.org/10.1103/RevModPhys.75.1021
[9] Dobaczewski J, Nazarewicz W and Reinhard P G 2014 J. Phys. G. 41 074001 URL http://dx.doi.org/10.1088/0954-3899/41/7/074001
[10] Stump D, Pumpkin J, Brock R, Casey D, Huston J, Kalk J, Lai H L and Tung W K 2001 Phys. Rev. D 65 014012 URL http://dx.doi.org/10.1103/PhysRevD.65.014012
[11] Coester F, Cohen S, Day B and Vincent C M 1970 Phys. Rev. C 1 769–776 URL http://dx.doi.org/10.1103/PhysRevC.1.769
[12] Lattimer J M and Steiner A W 2014 Eur. Phys. J. A 50 40 URL http://dx.doi.org/10.1140/epja/i2014-14040-y
[13] Alford M G, Han S and Prakash M 2013 Phys. Rev. D 88 083013 URL http://dx.doi.org/10.1103/PhysRevD.88.083013
[14] Gelman A and Shalizi C R 2012 Brit. Jour. Math. Stat. Psych. 66 8 URL http://dx.doi.org/10.1111/j.2044-8317.2011.02037.x
[15] Steiner A W, Lattimer J M and Brown E F 2010 Astrophys. J. 722 33 URL http://dx.doi.org/10.1088/0004-637X/722/1/33
[16] Steiner A W, Gandolfi S, Fattoyev F J and Newton W G 2014 arXiv:1403.7546 URL http://arxiv.org/abs/1403.7546
[17] Hebeler K, Lattimer J M, Pethick C J and Schwenk A 2013 Astrophys. J. 773 11 URL http://dx.doi.org/10.1088/0004-637X/773/1/11
[18] Steiner A W, Lattimer J M and Brown E F 2013 Astrophys. J. Lett. 765 5 URL http://dx.doi.org/10.1088/2041-8205/765/1/L5
[19] Steiner A W and Gandolfi S 2012 Phys. Rev. Lett. 108 081102 URL http://dx.doi.org/10.1103/PhysRevLett.108.081102
[20] Lattimer J M and Steiner A W 2014 Astrophys. J. 784 123 URL http://dx.doi.org/10.1088/0004-637X/784/2/123
[21] Heinke C O, Cohn H N, Lugger P M, Webb N A, Ho W C G, Anderson J, Campana S, Bogdanov S, Haggard D, Cool A M and Grindlay J E 2014 arXiv:1406.1497 URL http://arxiv.org/abs/1406.1497
[22] Farine M, Pearson J M and Rouben B 1978 Nucl. Phys. A 304 317 URL 10.1016/0375-9474(78)90241-5
Moving Beyond Chi-Squared in Nuclei and Neutron Stars

[23] Horowitz C J and Piekarewicz J 2001 *Phys. Rev. Lett.* **86** 5647–5650 URL http://dx.doi.org/10.1103/PhysRevLett.86.5647

[24] Abrahamyan S et al (PREX collaboration) 2012 *Phys. Rev. Lett.* **108** 112502 URL http://dx.doi.org/10.1103/PhysRevLett.108.112502

[25] Tarbert C M et al (Crystal Ball at MAMI and A2 Collaboration) 2014 *Phys. Rev. Lett.* **112** 242502 URL http://dx.doi.org/10.1103/PhysRevLett.112.242502

[26] Furnstahl R J 2002 *Nucl. Phys. A* **706** 85 URL http://dx.doi.org/10.1016/S0375-9474(02)00867-9

[27] Stone J R, Miller J C, Koncewicz R, Stevenson P D and Strayer M R 2003 *Phys. Rev. C* **68** 034324 URL http://dx.doi.org/10.1103/PhysRevC.68.034324

[28] Steiner A W, Prakash M, Lattimer J M and Ellis P J 2005 *Phys. Rep.* **411** 325 URL http://dx.doi.org/10.1016/j.physrep.2005.02.004

[29] Kortelainen M, McDonnell J, Nazarewicz W, Reinhard P G, Sarich J, Schunck N, Stoitsov M V and Wild S M 2012 *Phys. Rev. C* **85**(2) 024304 URL http://dx.doi.org/10.1103/PhysRevC.85.024304

[30] Epelbaum E, Hammer H W and Meißner U G 2009 *Rev. Mod. Phys.* **81** 1773–1825 URL http://dx.doi.org/10.1103/RevModPhys.81.1773

[31] Hagen G, Hjorth-Jensen M, Jansen G R, Machleidt R and Papenbrock T 2012 *Phys. Rev. Lett.* **109** 032502 URL http://dx.doi.org/10.1103/PhysRevLett.109.032502

[32] Hagen G, Papenbrock T, Hjorth-Jensen M and Dean D J 2013 *arXiv:1312.7872* URL http://arxiv.org/abs/1312.7872

[33] Dieperink A E L and Van Isacker P 2009 *Eur. Phys. J. A* **42** 269 URL http://dx.doi.org/10.1140/epja/i2009-10869-3

[34] Steiner A W 2012 *Phys. Rev. C* **85** 055804 URL http://dx.doi.org/10.1103/PhysRevC.85.055804

[35] Audi G, Wang M, Wapstra A H, Kondev F G, MacCormick M, Xu X and Pfeiffer B 2012 *Chin. Phys. C* **36** 1287 URL http://dx.doi.org/10.1088/1674-1137/36/12/002

[36] Wang M, Audi G, Wapstra A H, Kondev F G, MacCormick M, Xu X and Pfeiffer B 2012 *Chin. Phys. C* **36** 1603 URL http://dx.doi.org/10.1088/1674-1137/36/12/003

[37] Guillot S, Servillat M, Webb N A and Rutledge R E 2013 *Astrophys. J.* **772** 7 URL http://dx.doi.org/10.1088/0004-637X/772/1/7