On the entangled ergodic theorem

TANJA EISNER AND DÁVID KUNSZENTI-KOVÁCS

Abstract. We study the convergence of the so-called entangled ergodic averages

\[ \frac{1}{N^k} \sum_{n_1, \ldots, n_k = 1}^{N} T_{m-1}^{n_{\alpha(m)}} A_{m-1}^{n_{\alpha(m-1)}} A_{m-2} \ldots A_1^{n_{\alpha(1)}} , \]

where \( k \leq m \) and \( \alpha : \{1, \ldots, m\} \to \{1, \ldots, k\} \) is a surjective map. We show that, on general Banach spaces and without any restriction on the partition \( \alpha \), the above averages converge strongly as \( N \to \infty \) under some quite weak compactness assumptions on the operators \( T_j \) and \( A_j \). A formula for the limit based on the spectral analysis of the operators \( T_j \) and the continuous version of the result are presented as well.

Mathematics Subject Classification (2010): 47A35 (primary); 37A30 (secondary).

1. Introduction

The classical mean ergodic theorem has inspired many mathematicians and led to several generalisations and extensions. We mention Berend, Lin, Rosenblatt, Tempelman [3] for modulated and subsequential ergodic theorems and e.g. Kra [13] for an overview on multiple ergodic theorems as well as for the history of the subjects and further references.

In this note we study a further extension of the mean ergodic theorem, namely the so-called entangled ergodic theorem. Let \( \alpha : \{1, \ldots, m\} \to \{1, \ldots, k\} \) be a surjective map for some positive integers \( k \leq m \), and take \( T_1, \ldots, T_m \) and \( A_1, \ldots, A_{m-1} \) to be linear operators on a Banach space \( X \). We investigate the convergence of the entangled Cesáro means

\[ \frac{1}{N^k} \sum_{n_1, \ldots, n_k = 1}^{N} T_{m-1}^{n_{\alpha(m)}} A_{m-1}^{n_{\alpha(m-1)}} A_{m-2} \ldots A_1^{n_{\alpha(1)}} . \] (1.1)

This type of ergodic theorems was introduced by Accardi, Hashimoto, Obata [1] motivated by quantum stochastics and was then studied by Liebscher [15] and

Received December 16, 2010; accepted in revised form June 4, 2011.