Axiomatic TQFT, Axiomatic DQFT, and Exotic 4-Manifolds

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Abstract

In this article we prove that any unitary, axiomatic topological quantum field theory in four-dimensions can not detect changes in the smooth structure of $M$, a simply connected, closed (compact without boundary), oriented smooth manifold. However, as Donaldson-Witten theory (a topological quantum field theory but not an axiomatic one) is able to detect changes in the smooth structure of such an $M$, this seemingly leads to a contradiction. This seeming contradiction is resolved by introducing a new set of axioms for a “differential quantum field theory”, which in truth only slightly modify the naturality and functoriality axioms of a topological quantum field theory, such that these new axioms allow for a theory to detect changes in smooth structure.
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1 Introduction

The fountainhead of modern topological quantum field theory can be traced back to the work of Witten [24]. There he proved that the Morse inequalities [16] can be obtained through the use of a certain supersymmetric version of quantum mechanics. The next major milestone in the study of topological quantum field theory was also authored by Witten [25]. There he showed that the Donaldson polynomials [7] can be interpreted as observables of a certain four-dimensional quantum field theory. Subsequently, Witten [26] authored *Quantum Field Theory and the Jones Polynomial*, for which, in large part, he received the Fields Medal. There he proved that the Jones polynomial [11] can be interpreted as an observable of a certain three-dimensional quantum field theory. It was after the publication of this paper that the flood gates opened and the volume of papers dealing with topological quantum field theory began to greatly increase.

With this increased volume of work on topological quantum field theory, many mathematicians started to become more interested in the subject. However, due to the methods used, in particular the mathematically ill-defined path-integral [2], many of the results were valid to a physicist’s level of rigour, but not to a mathematician’s. This soon changed when Atiyah [3], motivated by Witten [26] and Segal [19], axiomatized the foundations of topological quantum field theory. This axiomatization made it possible for mathematicians to obtain rigorous results.

However, Atiyah’s axiomatization is based on experiences from topological quantum field theories in three or fewer dimensions. The axiomatization is rarely used in four or more dimensions. Hence, there are a dearth of results using axiomatic topological quantum field theory in four or more dimensions, and the axioms themselves may contain hidden “biases” that “favor” three or fewer dimensions. In particular, application of the axiomatization to four dimensions [22] tends to yield theories that are “trivial” in that they can not detect changes in smooth structure.

In this article we will take a first step in to higher dimensions and examine axiomatic topological quantum field theory in four dimensions. We prove, formalizing the difficulties expressed by Thurston [22], that in four dimensions any unitary, axiomatic topological quantum field theory can not detect changes in the smooth structure of $M$, a simply connected, closed (compact without boundary), oriented smooth four-manifold. This moti-

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1Thurston’s mathoverflow answer [22] and subsequent discussion were the original motivation for this article.
vates us to slightly modify the axioms of a topological quantum field theory so that it is possible for an axiomatic topological quantum field theory to detect changes in the smooth structure of such an $M$. Thus, these modified axioms could more accurately be dubbed axioms of a differential quantum field theory.

2 Axiomatic TQFT

In his ground–breaking work Witten [25] introduced an “informal” definition of a topological quantum field theory, a quantum field theory on a smooth manifold $M$ that is independent of the metric placed on $M$. Atiyah [3], motivated by Witten’s informal definition and Segal’s [19] axiomatization of two-dimensional conformal field theory, then axiomatized topological quantum field theory. Over the years several authors have explored and refined Atiyah’s axiomatization, see [18] and [23], resulting in the current formulation [4], which we describe below.

2.1 Axiomatic TQFT

An $(n+1)$-dimensional topological quantum field theory, from now on abbreviated TQFT, over a field $F$ assigns to every closed, oriented $n$-dimensional smooth manifold $X$ a finite dimensional vector space $\mathcal{H}(X)$ over $F$ and assigns to every $(n+1)$-dimensional cobordism $W$ from $X_-$ to $X_+$ an $F$ linear map,

$$Z(W, X_-, X_+) : \mathcal{H}(X_-) \to \mathcal{H}(X_+).$$

(1)

Recall that given two closed, oriented $n$-dimensional smooth manifolds $X_\pm$ a cobordism from $X_-$ to $X_+$ is a compact, oriented $(n+1)$-dimensional smooth manifold $W$ such that $\partial W = X_- \sqcup X_+$, where $\partial W$ is the boundary of $W$ and $\sqcup$ denotes disjoint union. The assignments $\mathcal{H}(X)$ and $Z(W, X_-, X_+)$ must satisfy the following axioms.

2.1.1 Naturality

**Axiom 2.1** (Naturality). Any orientation–preserving diffeomorphism of closed, oriented $n$-dimensional smooth manifolds $f : X \to X'$ induces an isomorphism $f^* : \mathcal{H}(X) \to \mathcal{H}(X')$. For an orientation–preserving diffeomorphism $g$ from the cobordism $(W, X_-, X_+)$ to the cobordism $(W', X'_-, X'_+)$, the

\[ f^* (Z(W', X'_-, X'_+)) = Z(W, X_-, X_) \]

\[ g^* (Z(W, X_-, X_+)) = Z(W', X'_-, X'_+) \]

Note, we use $f$ to denote the orientation–preserving diffeomorphism and the isomorphism. Context should prevent any confusion in this regard.
following diagram is commutative.

\[
\begin{array}{ccc}
\mathcal{H}(X_-) & \xrightarrow{g_{X_-}} & \mathcal{H}(X'_-) \\
\downarrow Z(W) & & \downarrow Z(W') \\
\mathcal{H}(X_+) & \xrightarrow{g_{X_+}} & \mathcal{H}(X'_+) \\
\end{array}
\]

Note, $Z(W)$ is shorthand for $Z(W, X_-, X_+)$ and $Z(W')$ is shorthand for $Z(W', X'_-, X'_+)$. 

2.1.2 Functoriality

**Axiom 2.2** (Functoriality). If a cobordism $(W, X_-, X_+)$ is obtained by gluing two cobordisms $(M, X_-, X)$ and $(M', X', X_+)$ using an orientation-preserving diffeomorphism $f : X \to X'$, then the following diagram is commutative.

\[
\begin{array}{ccc}
\mathcal{H}(X_-) & \xrightarrow{Z(M)} & \mathcal{H}(X_+) \\
\downarrow Z(W) & & \downarrow Z(M') \\
\mathcal{H}(X) & \xrightarrow{f} & \mathcal{H}(X') \\
\end{array}
\]

2.1.3 Normalization

**Axiom 2.3** (Normalization). For any closed, oriented $n$-dimensional smooth manifold $X$, the $F$ linear map

\[ Z(X \times [0,1]) : \mathcal{H}(X) \to \mathcal{H}(X) \]

is the identity.

2.1.4 Multiplicativity

**Axiom 2.4** (Multiplicativity). There are functorial isomorphisms

\[ \mathcal{H}(X \amalg Y) \to \mathcal{H}(X) \otimes \mathcal{H}(Y) \]

and

\[ \mathcal{H}() \to F \]

---

3 The formal definition of gluing is given in Chapter VI Section 5 of Kosinski [14].
such that the diagrams

\[
\begin{array}{c}
\mathcal{H}((X_1 \amalg X_2) \amalg X_3) \longrightarrow (\mathcal{H}(X_1) \otimes \mathcal{H}(X_2)) \otimes \mathcal{H}(X_3) \\
\downarrow \downarrow \downarrow \\
\mathcal{H}(X_1 \amalg (X_2 \amalg X_3)) \longrightarrow \mathcal{H}(X_1) \otimes (\mathcal{H}(X_2) \otimes \mathcal{H}(X_3))
\end{array}
\]

and

\[
\begin{array}{c}
\mathcal{H}(X \amalg \emptyset) \longrightarrow \mathcal{H}(X) \otimes \mathbb{F} \\
\downarrow \downarrow \\
\mathcal{H}(X) \longrightarrow \mathcal{H}(X)
\end{array}
\]

commute. Note, the vertical maps are induced by the obvious diffeomorphisms and the standard vector space isomorphisms.

2.1.5 Symmetry

**Axiom 2.5 (Symmetry).** The isomorphism

\[
\mathcal{H}(X \amalg Y) \longrightarrow \mathcal{H}(Y \amalg X)
\]

induced by the obvious diffeomorphism corresponds to the standard isomorphism of vector spaces

\[
\mathcal{H}(X) \otimes \mathcal{H}(Y) \longrightarrow \mathcal{H}(Y) \otimes \mathcal{H}(X).
\]

2.2 Remarks

Before continuing on with the remainder of this article, there are a few points of note that easily follow from the above axioms and that we will have need of later.

First, an axiomatic TQFT defines invariants for closed, oriented \((n+1)\)-dimensional smooth manifolds. In more detail, a closed, oriented \((n+1)\)-dimensional smooth manifold \(W\) can be thought of as a cobordism from \(\emptyset\) to \(\emptyset\). Thus, \(Z(W) \in Hom_{\mathbb{F}}(\mathbb{F}, \mathbb{F}) = \mathbb{F}\), and \(Z(W) \in \mathbb{F}\) is simply a numerical invariant of \(W\).

Second, any compact, oriented \((n+1)\)-dimensional smooth manifold \(W\) with boundary can be thought of as a cobordism from \(\emptyset\) to \(\partial W\). Thus, \(Z(W) \in Hom_{\mathbb{F}}(\mathbb{F}, \mathcal{H}(\partial W)) = \mathcal{H}(\partial W)\). So, \(Z(W)\) in this case is simply a vector in \(\mathcal{H}(\partial W)\). This vector \(Z(W)\) is called the vacuum vector of \(W\) and we will find it of great use in what follows.
Finally, for a closed, oriented \( n \)-dimensional smooth manifold \( X \) the manifold \( X \times [0, 1] \) can be considered as a cobordism from \( \overline{X} \sqcup X \) to \( \emptyset \), where \( \overline{X} \) is \( X \) with its orientation reversed. Hence, \( Z(X \times [0, 1]) \) can be viewed as an \( \mathbb{F} \) linear map
\[
Z(X \times [0, 1]) : \mathcal{H}(\overline{X}) \otimes \mathcal{H}(X) \to \mathbb{F}.
\]
This gives a functorial isomorphism of \( \mathcal{H}(\overline{X}) = \mathcal{H}(X)^* = \text{Hom}_\mathbb{F}(\mathcal{H}(X), \mathbb{F}) \).

Thus, if a closed, oriented \((n+1)\)-dimensional smooth manifold \( W \) is obtained by gluing \( M \) to \( M' \), where \( \partial M = \partial M' \), then Axiom 2.2, the functoriality axiom, implies \( Z(W) = \langle Z(M')|Z(M) \rangle \in \mathbb{F} \), where \( Z(M) \) and \( Z(M') \) are viewed as vacuum vectors and \( \langle Z(M')|Z(M) \rangle \) is defined as the value of \( Z(M') \in \mathcal{H}(\partial M)^* \) acting on \( Z(M) \in \mathcal{H}(\partial M) \).

### 2.3 Unitarity

An additional axiom that is sometimes used in conjunction with the above set of standard axioms is that of unitarity.

**Axiom 2.6** (Unitarity). For any compact, oriented \((n+1)\)-dimensional smooth manifold \( W \) with non-zero \( Z(W) \in \mathcal{H}(\partial W) \) the element \( Z(W) = \langle Z(M')|Z(M) \rangle \in \mathbb{F} \) is not zero.

Unitarity is sometimes, but not always, taken as an axiom of TQFT. However, all “physical” theories, for example the standard model [17] and general relativity [10], are unitary. Thus, we will assume that any axiomatic TQFT that we deal with obeys the unitarity axiom.

### 3 Akbulut Corks and Exotic Four-Manifolds

The wellspring of many an idea related to exotic four-manifolds can be traced back to the work of Akbulut [1]. In this foundational work Akbulut found that for a certain smooth four-manifold \( M \) one can make an exotic copy \( M' \) of \( M \), a manifold homeomorphic but not diffeomorphic to \( M \), by cutting out and regluing \( A_C \), a certain four-dimensional smooth submanifold of \( M \), by an involution of its boundary \( \partial A_C \). This smooth four-manifold \( A_C \) later became known as an Akbulut cork.

This means of generating exotic four-manifolds was later generalized in a preprint of Curtis and Hsiang. The proofs in this preprint were then simplified and extended through the work of Curtis, Freedman, Hsiang, and Strong [6], Matveyev [15], Bizaca, and Kirby [13].
In this section, to place these developments in the proper context, we will review the theorems that built up to the discovery of Akbulut corks, Smale’s h-cobordism theorem \[20\] and Freedman’s h-cobordism theorem \[8\], as well as reviewing the theorems presented in the above series of papers. These theorems will be presented without proofs. The interested reader can refer to original works and/or to Chapter 9 of Gompf and Stipsicz \[21\] where most of this material is covered.

### 3.1 Smale’s h-Cobordism Theorem

Classification of four-dimensional smooth manifolds up to diffeomorphism can best be understood, strangely enough, by looking first at the classification of smooth manifolds up to diffeomorphism in greater than four dimensions. Looking at the results in higher dimensions serves to put the results in four dimensions in to the proper context.

The key result used to classify manifolds up to diffeomorphism in greater than four dimensions is Smale’s h-cobordism theorem \[20\]. This theorem establishes a criteria through which one can determine if two simply connected, closed, oriented smooth \(n\)-manifolds, where \(n > 4\), are diffeomorphic. It is this theorem which we will now review.

However, before presenting Smale’s h-cobordism theorem, we must introduce a definition \[21\]. Two simply connected smooth manifolds \(X_-\) and \(X_+\) are h-cobordant if there exists a cobordism \(W\) from \(X_-\) to \(X_+\) such that the inclusions \(i_\pm : X_\pm \hookrightarrow W\) are homotopy equivalences. Given this definition we can now state Smale’s h-cobordism theorem \[21\].

**Theorem 3.1** (Smale’s h-Cobordism Theorem). *If \(W\) is an h-cobordism between the \(n\)-dimensional smooth manifolds \(X_-\) and \(X_+\), where \(n > 4\), then \(W\) is diffeomorphic to \(X_- \times [0, 1]\). In particular \(X_-\) is diffeomorphic to \(X_+\).*

With this one can see that if two \(n\)-dimensional smooth manifolds are h-cobordant and \(n > 4\), then these two manifolds are diffeomorphic. In practice this often simplifies the process of determining if two manifolds are diffeomorphic, as proving two manifolds are h-cobordant is often easier than directly proving they are diffeomorphic.

This theorem can be used to classify smooth manifolds up to diffeomorphism in more than four dimensions. However, as we will see, this result fails to be true in four dimensions, where a strictly “weaker” result holds. This “weaker” result is the subject of Freedman’s h-cobordism theorem to which we now turn.
3.2 Freedman’s h-Cobordism Theorem

One may hope that the techniques used to prove Smale’s h-cobordism theorem could be generalized to accommodate the case $n = 4$. However, this is not possible\(^4\). The best one can do in four dimensions is Freedman’s h-cobordism theorem [21].

**Theorem 3.2** (Freedman’s h-Cobordism Theorem). *If $W$ is an h-cobordism between the four-dimensional smooth manifolds $X_-$ and $X_+$, then $W$ is homeomorphic to $X_- \times [0, 1]$. In particular $X_-$ is homeomorphic to $X_+$.*

Thus, if two four-dimensional smooth manifolds are h-cobordant, then these two manifolds are homeomorphic. In four-dimensions this result can not be improved upon. In other words, there exist four-dimensional smooth manifolds $X_-$ and $X_+$ that are h-cobordant and not diffeomorphic [21]. As they are h-cobordant, Freedman’s h-cobordism theorem implies they are homeomorphic. But, as they are not diffeomorphic, $X_+$ is an exotic version of $X_-$, a manifold homeomorphic but not diffeomorphic to $X_-$. In fact, the original results of Akbulut [1] provide such a pair.

As it is a result we will require later, we pause here to note that one can strengthen Freedman’s h-cobordism theorem in the following manner [21].

**Theorem 3.3** (Strengthened Freedman’s h-Cobordism Theorem). *Two simply connected, closed, oriented, four-dimensional smooth manifolds $X_-$ and $X_+$ are homeomorphic if and only if they are h-cobordant.*

3.3 Akbulut Corks

The results of Akbulut [1], along with Smale’s and Freedman’s h-cobordism theorems, lead one to conjecture that it might be possible to “excise” a submanifold $A$ from $W$, a five-dimensional h-cobordism from $X_-$ to $X_+$, such that the remainder $W - \text{int}(A)$ is diffeomorphic to $(X_- - \text{int}(A)) \times [0, 1]$. Thus, all of the “strangeness” that occurs in four dimensions would be contained in $A$, and $W - \text{int}(A)$ would be “trivial”. This conjecture, and in fact much more, is true, as was found by Curtis, Freedman, Hsiang, and Strong [6], Matveyev [15], Bižaca, and Kirby [13].

The formal summary of the flurry of work contained in the above articles is given by the following theorem [13].

\(^4\)The main problem is that “Whitney’s Trick”, which works in more than four dimensions, fails in four dimensions [21].
Theorem 3.4 (Précis of Akbulut Corks). If $W$ is a five-dimensional $h$-cobordism between two smooth four-manifolds $X_-$ and $X_+$, then there exists a five-dimensional $h$-cobordism $A \subset W$ from the smooth four-manifold $A_- \subset X_-$ to the smooth four-manifold $A_+ \subset X_+$ with the following properties:

1. $A_-$, and hence $A$ and $A_+$, is contractible.
2. $W - \text{int}(A)$ is diffeomorphic to $(X_- - \text{int}(A_-)) \times [0,1]$.
3. $W - A$, and hence $X_- - A$ and $X_+ - A$, is simply connected.
4. $A$ is diffeomorphic to $D^5$, the standard five-dimensional disk with boundary.
5. $A_- \times [0,1]$ and $A_+ \times [0,1]$ are diffeomorphic to $D^5$.
6. $A_-$ is diffeomorphic to $A_+$ by a diffeomorphism which, when restricted to $\partial A_- = \partial A_+$, is an involution.

The manifolds $A_\pm$ identified above are Akbulut corks and are a generalization of the manifolds first discovered by Akbulut [1] in his foundational work.

Given $W$, $X_\pm$, and $A_\pm$ as appear in the previous theorem, one can easily prove the following results. As a result of (2), $X_- - \text{int}(A_-)$ is diffeomorphic to $X_+ - \text{int}(A_+)$. The definitions of $X_\pm$ and $A_\pm$ imply $X_\pm = (X_\pm - \text{int}(A_\pm)) \cup \text{id}(A_\pm)$. Thus, as a result of (6), $X_- = (X_+ - \text{int}(A_+)) \cup \text{id}(A_-)$ and $X_+ = (X_- - \text{int}(A_-)) \cup I A_-$, where $I$ is the involution of $\partial A_-$ from (6) and all equivalences are up to diffeomorphism.

Now, assume one has a simply connected, closed, oriented, smooth four-manifold $M$ along with $M'$, a manifold homeomorphic but not diffeomorphic to $M$. (In other words, $M'$ is an exotic version of $M$.) As a result of the strengthened version of Freedman’s $h$-cobordism theorem, $M$ is $h$-cobordant to $M'$. Thus, as a result of the argument in the previous paragraph, there exists an Akbulut cork $A_C \subset M$ such that $M = (M - \text{int}(A_C)) \cup \text{id} A_C$ and $M' = (M - \text{int}(A_C)) \cup I A_C$, where $I$ is the involution of $\partial A_C$ given in (6).

4 Axiomatic TQFT and Exotic 4-Manifolds

This section will be dedicated to proving our main theorem.

Theorem 4.1. In four-dimensions any unitary, axiomatic topological quantum field theory can not detect changes in the smooth structure of $M$, a
simply connected, closed (compact without boundary), oriented smooth four-manifold.

Proof. Assume there exists a smooth manifold $M'$ homeomorphic but not diffeomorphic to $M$, in other words $M'$ has a different smooth structure than $M$. We will prove that $Z(M) = Z(M')$ for any unitary, axiomatic topological quantum field theory.

As $M$ and $M'$ are homeomorphic, Theorem 3.3, the strengthened Freedman’s h-cobordism theorem, implies that there exists an h-cobordism $W$ from $M$ to $M'$.

As there exists an h-cobordism $W$ from $M$ to $M'$, Theorem 3.4 implies that there exists an Akbulut cork $A_C \subset M$ such that

$$M = (M - \text{int}(A_C)) \cup_{id} A_C$$

and

$$M' = (M - \text{int}(A_C)) \cup_I A_C,$$

where $I$ is the involution of $\partial A_C$ given in part (6) of Theorem 3.4.

As $M = (M - \text{int}(A_C)) \cup_{id} A_C$, the results of Section 2.2 imply the equality

$$Z(M) = \langle Z(M - \text{int}(A_C)) | Z(A_C) \rangle$$

Similarly, as $M' = (M - \text{int}(A_C)) \cup_I A_C$, the results of Section 2.2 along with Axiom 2.2, the functoriality axiom, imply

$$Z(M') = \langle Z(M - \text{int}(A_C)) | I(Z(A_C)) \rangle,$$

where $I$ is the isomorphism of $\mathcal{H}(\partial A_C)$ induced by the involution $I$ of $\partial A_C$. Thus, to prove $Z(M) = Z(M')$ we only need to prove $Z(A_C) = I(Z(A_C))$, or, equivalently, we need to prove $Z(A_C) - I(Z(A_C)) = 0$.

If $Z(A_C) - I(Z(A_C)) = 0$, then we are done. So, we can thus safely assume that $Z(A_C) - I(Z(A_C)) \neq 0$. Hence, Axiom 2.6, the unitarity axiom, implies that if the product $\langle Z(A_C) - I(Z(A_C)) | Z(A_C) - I(Z(A_C)) \rangle = 0$, then $Z(A_C) - I(Z(A_C)) = 0$. So, if we can prove that $\langle Z(A_C) - I(Z(A_C)) | Z(A_C) - I(Z(A_C)) \rangle = 0$, we are done.
\[
\langle Z(A_C) - I(Z(A_C)) | Z(A_C) - I(Z(A_C)) \rangle 
= \langle Z(A_C) | Z(A_C) \rangle - \langle Z(A_C) | I(Z(A_C)) \rangle - \langle I(Z(A_C)) | Z(A_C) \rangle + \langle I(Z(A_C)) | I(Z(A_C)) \rangle 
= Z(A_C \cup id A_C) - Z(A_C \cup I A_C) - Z(A_C \cup I A_C) + Z(A_C \cup id A_C) 
= 2(Z(A_C \cup id A_C) - Z(A_C \cup I A_C)),
\]
where in the second to last line we have used the fact that \( I \) is an involution and thus \( I^2 = id \). As a result of the previous computation, we find that our desired conclusion follows if we can prove \( Z(A_C \cup id A_C) - Z(A_C \cup I A_C) = 0 \).

Now, part (5) of Theorem 3.4, précis of Akbulut corks, implies \( A_C \times [0,1] \) is diffeomorphic to \( D^5 \), the standard five-dimensional disk with boundary. As \( \partial(A_C \times [0,1]) = A_C \cup id A_C \), this implies \( A_C \cup id A_C = S^4 \), where \( S^4 \) is the standard four-dimensional sphere. Thus, \( Z(A_C \cup id A_C) = Z(S^4) \).

Part (4) of Theorem 3.4, précis of Akbulut corks, implies \( A \), of Theorem 3.4, is diffeomorphic to \( D^5 \). As \( \partial A = A_C \cup I A_C \) in our case, this implies \( A_C \cup I A_C = S^4 \). Thus, \( Z(A_C \cup I A_C) = Z(S^4) \).

Collecting the results of the last two paragraphs,
\[
Z(A_C \cup id A_C) - Z(A_C \cup I A_C) = Z(S^4) - Z(S^4) = 0.
\]
So, tracing all our previous steps, we have proven \( Z(M) = Z(M') \).  

\[\square\]

5 Remarks

The results of Theorem 4.1 seem, somehow, unsatisfying. It is well known that Donaldson-Witten theory [25] is a TQFT, in Witten’s informal sense, that is able to detect changes in the smooth structure of \( M \), a simply connected, closed, oriented smooth four-manifold. So, it comes as somewhat of a surprise that any unitary, axiomatic TQFT can not detect changes in the smooth structure of such an \( M \). It feels as if axiomatic TQFT is lacking something that is present in Donaldson-Witten theory, and indeed this is the case. However, the modifications that one must make to axiomatic TQFT in order to allow it to detect changes in smooth structure are relatively easy to spot upon thinking a bit about what is happening in the scenario above.

Axiomatic TQFT in four-dimensions is, rather unsurprisingly, a four-dimensional theory. So, in particular, all of its symmetries should arise

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from symmetries that appear naturally in four-dimensions. For example, Axiom 2.1, the naturality axiom, implies that any axiomatic TQFT in four-dimensions is invariant with respect to four-dimensional diffeomorphisms. This makes sense. This is a purely four-dimensional symmetry that arises in a purely four-dimensional theory. However, by contrast, Axiom 2.1 also states that any orientation–preserving diffeomorphism of closed, oriented, three-dimensional smooth manifolds \( f : X \to X' \) induces an isomorphism \( f : \mathcal{H}(X) \to \mathcal{H}(X') \). At first glance this seems harmless, but, in fact, it is not.

The involution \( I \) of \( \partial A_C \) from part (6) of Theorem 3.4, précis of Akbulut corks, is a diffeomorphism of \( \partial A_C \) that does not arise from a diffeomorphism of \( A_C \). In other words one can not continue \( I \) over \( A_C \) as a diffeomorphism. The best one can do is to continue \( I \) over \( A_C \) as a homeomorphism. So, the assertion in Axiom 2.1 that \( I \) gives rise to an isomorphism \( I : \mathcal{H}(\partial A_C) \to \mathcal{H}(I(\partial A_C)) \) is asserting that there exists a symmetry in the four-dimensional theory that has no natural origin in four-dimensions, as there exists no four-dimensional diffeomorphism of \( A_C \) that when restricted to \( \partial A_C \) yields \( I \). In other words, it is, without any “physical” justification, enlarging the symmetry group of the theory. In point of fact, it is just this enlarged symmetry group that we are seeing in Theorem 4.1.

The modifications that one must make to the TQFT axioms such that they allow for detection of changes in smooth structure are rather straightforward. To wit, one must limit the set of orientation-preserving diffeomorphisms that give rise to isomorphisms of \( \mathcal{H}(X) \). More specifically, if \( X \) is a closed, oriented \( n \)-dimensional smooth submanifold of a compact, oriented \( (n+1) \)-dimensional smooth manifold \( W \), then any orientation-preserving diffeomorphism \( f \) of \( X \) that arises as a restriction of an orientation-preserving diffeomorphism of \( W \) induces an isomorphism \( f : \mathcal{H}(X) \to \mathcal{H}(f(X)) \). If \( f' \) is an orientation-preserving diffeomorphism of \( X \) that does not arise in such a manner, then its action on \( \mathcal{H}(X) \) is undefined. If we call an orientation-preserving diffeomorphism \( f \) that arises in such a manner a restricted orientation-preserving diffeomorphism, then the naturality and functoriality TQFT axioms must be modified in the following manner so as to allow for detection of changes in smooth structure.

\[5\]The easiest way to see this is to note that if one could continue \( I \) over \( A_C \) as a diffeomorphism, then one could prove Smale’s h-cobordism in four-dimensions, a result known to be false.

\[6\]One immediately sees that if one uses these new axioms, the proof of Theorem 4.1 fails.
5.1 Naturality

**Axiom 5.1** (Naturality). Any orientation-preserving diffeomorphism $f$ of $X$, a closed, oriented $n$-dimensional smooth submanifold of $W$ a compact, oriented $(n+1)$-dimensional smooth manifold, that arises as a restriction of an orientation-preserving diffeomorphism of $W$ induces an isomorphism $f : \mathcal{H}(X) \rightarrow \mathcal{H}(f(X))$. For an orientation-preserving diffeomorphism $g$ from the cobordism $(W, X_-, X_+)$ to the cobordism $(W', X'_-, X'_+)$, the following diagram is commutative.

\[
\begin{array}{ccc}
\mathcal{H}(X_-) & \xrightarrow{g|_{X_-}} & \mathcal{H}(X'_-)
\\
Z(W) \downarrow & & \downarrow Z(W')
\\
\mathcal{H}(X_+) & \xrightarrow{g|_{X_+}} & \mathcal{H}(X'_+)
\end{array}
\]

Note, $Z(W)$ is shorthand for $Z(W, X_-, X_+)$ and $Z(W')$ is shorthand for $Z(W', X'_-, X'_+)$. 

5.2 Functoriality

**Axiom 5.2** (Functoriality). If a cobordism $(W, X_-, X_+)$ is obtained by gluing two cobordisms $(M, X_-, X)$ and $(M', X', X_+)$ using an orientation-preserving diffeomorphism $f$ where $f : X \rightarrow X'$ and $f$ can be viewed as the restriction of an orientation-preserving diffeomorphism of $W$, then following diagram is commutative.

\[
\begin{array}{ccc}
\mathcal{H}(X_-) & \xrightarrow{Z(W)} & \mathcal{H}(X_+)
\\
Z(M) \downarrow & & \downarrow Z(M')
\\
\mathcal{H}(X) & \xrightarrow{f} & \mathcal{H}(X')
\end{array}
\]

6 Conclusion

6.1 Remarks

The standard formulation of axiomatic TQFT [4] is sufficient for many situations in fewer than four dimensions. However, in four-dimensions the standard axiomatic formulation requires some small modifications if it is to detect changes in smooth structure. These modifications are required as there exist orientation-preserving diffeomorphisms of $\partial A_C$ that do not extend to orientation-preserving diffeomorphisms of the smooth four-manifold.
$A_C$. (In fewer than four-dimensions such diffeomorphisms do not exist.) If these small modifications are made, one obtains a set of axioms that allow for the detection of changes in the smooth structure of a four-manifold.

6.2 Axiomatic DQFT

We call the construct resulting from the modified axioms *axiomatic differential quantum field theory*. In summary its axioms are as follows.

6.2.1 Naturality

**Axiom 6.1** (Naturality). *Any orientation-preserving diffeomorphism* $f$ of $X$, a closed, oriented $n$-dimensional smooth submanifold of $W$ a compact, oriented $(n+1)$-dimensional smooth manifold, that arises as a restriction of an orientation-preserving diffeomorphism of $W$ induces an isomorphism $f \colon \mathcal{H}(X) \to \mathcal{H}(f(X))$. For an orientation-preserving diffeomorphism $g$ from the cobordism $(W, X_-, X_+)$ to the cobordism $(W', X'_-, X'_+)$, the following diagram is commutative.

$$
\begin{array}{ccc}
\mathcal{H}(X_-) & \xrightarrow{g|_{X_-}} & \mathcal{H}(X'_-)
\\
\downarrow Z(W) & & \downarrow Z(W')
\\
\mathcal{H}(X_+) & \xrightarrow{g|_{X_+}} & \mathcal{H}(X'_+)
\end{array}
$$

Note, $Z(W)$ is shorthand for $Z(W, X_-, X_+)$ and $Z(W')$ is shorthand for $Z(W', X'_-, X'_+)$.  

6.2.2 Functoriality

**Axiom 6.2** (Functoriality). *If a cobordism* $(W, X_-, X_+)$ *is obtained by gluing two cobordisms* $(M, X_-, X)$ and $(M', X', X_+)$ *using an orientation-preserving diffeomorphism* $f$ *where* $f : X \to X'$ *and* $f$ *can be viewed as the restriction of an orientation-preserving diffeomorphism* $W_*$ then following

---

7 If they existed in fewer than four-dimensions, then there would exist exotic manifolds in three or fewer dimensions. There exist no such manifolds.

8 Note, $\mathcal{H}(X)$ may also have to be infinite dimensional in four-dimensions.
6.2.3 Normalization

**Axiom 6.3 (Normalization).** For any closed, oriented \( n \)-dimensional smooth manifold \( X \), the \( F \) linear map

\[
Z(X \times [0,1]) : \mathcal{H}(X) \to \mathcal{H}(X)
\]

is the identity.

6.2.4 Multiplicativity

**Axiom 6.4 (Multiplicativity).** There are functorial isomorphisms

\[
\mathcal{H}(X \amalg Y) \to \mathcal{H}(X) \otimes \mathcal{H}(Y)
\]

and

\[
\mathcal{H}(\emptyset) \to F
\]

such that the diagrams

\[
\begin{align*}
\mathcal{H}((X_1 \amalg X_2) \amalg X_3) & \to (\mathcal{H}(X_1) \otimes \mathcal{H}(X_2)) \otimes \mathcal{H}(X_3) \\
\mathcal{H}(X_1 \amalg (X_2 \amalg X_3)) & \to \mathcal{H}(X_1) \otimes (\mathcal{H}(X_2) \otimes \mathcal{H}(X_3))
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{H}(X \amalg \emptyset) & \to \mathcal{H}(X) \otimes F \\
\mathcal{H}(X) & \to \mathcal{id} \to \mathcal{H}(X)
\end{align*}
\]

commute. Note, the vertical maps are induced by the obvious diffeomorphisms and the standard vector space isomorphisms.
6.2.5 Symmetry

Axiom 6.5 (Symmetry). The isomorphism

\[ \mathcal{H}(X \amalg Y) \rightarrow \mathcal{H}(Y \amalg X) \]

induced by the obvious diffeomorphism corresponds to the standard isomorphism of vector spaces

\[ \mathcal{H}(X) \otimes \mathcal{H}(Y) \rightarrow \mathcal{H}(Y) \otimes \mathcal{H}(X). \]

7 Afterward

Upon distributing this preprint, it has come to the author’s attention that a proof of a result similar to Theorem 4.1 was given as Theorem 4.1 of Freedman et al. [9]. In addition, the author was informed of a research program with a focus similar to that of this preprint. This research program was launched by Freedman, Kitaev, Nayak, Slingerland, Walker, and Wang in [9], continued by Kreck and Teichner in [12], and furthered by Calegari, Freedman, and Walker in [5].
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