Torsion theories and TTF theories in Birkhoff subcategories of simplicial groups

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Abstract

There is a lattice of torsion theories in simplicial groups such that the torsion/torsion-free categories are given by simplicial groups with truncated Moore complex below/above a certain degree. We study the restriction of these torsion theories to certain subcategories of simplicial groups. First, when restricted to the category of simplicial groups with trivial above Moore complex, we establish connections between the torsion categories and, the categories of abelian groups and central extensions in groups. As a second case, we study some aspects of torsion in the category Ashley’s reduced crossed complexes. We prove this category is semi-abelian and that does present weaker versions of TTF theories.

1 Introduction

Semi-abelian categories [15] extend the validity of classical homological algebras to non-abelian categories such as groups, Lie algebras, etc. Torsion theories, originally introduced for abelian categories, have been studied in semi-abelian categories (and other more general non-abelian settings) in [4] and [7].

The category \(\text{Grpd}(\mathcal{X})\) of internal groupoids in \(\mathcal{X}\), which is semi-abelian when \(\mathcal{X}\) is so, exhibits two examples of torsion theories given by the pairs:

\[(\text{Ab}(\mathcal{X}), \text{Eq}(\mathcal{X})) \quad \text{and} \quad (\text{ConnGrpd}(\mathcal{X}), \text{Dis}(\mathcal{X}))\]

where \(\text{Ab}(\mathcal{X})\), \(\text{Eq}(\mathcal{X})\), \(\text{ConnGrpd}(\mathcal{X})\) and \(\text{Dis}(\mathcal{X})\) are, respectively, the subcategories of \(\text{Grpd}(\mathcal{X})\) of internal abelian groups, internal equivalence relations, connected groupoids and discrete internal groupoids ([4], [11]).

For the case of \(\mathcal{X} = \text{Grp}\) the category of groups, the category of internal groupoids is equivalent to the category \(\mathcal{X}\text{Mod}\) of Whitehead’s crossed modules, thus the torsion theories of internal groupoids correspond to the torsion theories in crossed modules: (abelian groups, inclusion of normal monomorphisms) and (central extension in groups, groups as discrete crossed modules).

In [19], the examples of torsion theories of internal groupoids in groups have been expanded to a linearly ordered lattice \(\mu(\text{Grp})\) of torsion theories in the category \(\text{Simp}(\text{Grp})\) of simplicial groups. In such a way that the categories \(\text{Dis}(\text{Grp})\), \(\text{Eq}(\text{Grp})\) and \(\text{Grpd}(\text{Grp})\) are torsion-free subcategories of \(\text{Simp}(\text{Grp})\). On the other hand, the categories \(\text{Ab}(\text{Grp})\) and \(\text{ConnGrpd}(\text{Grp})\) are not torsion subcategories of \(\text{Simp}(\text{Grp})\). In this work we will show that when we restrict the torsion theories of \(\mu(\text{Grp})\) to the subcategory \(\mathcal{M}_{n\geq}\) of simplicial groups with truncated Moore complex above \(n\). First, we obtain that the corresponding lattice \(\mu(\mathcal{M}_{n\geq})\) has a minimal element where the torsion category is equivalent to \(\text{Ab}\) the category of abelian groups. Second, that the following (the second smallest) torsion category can be characterized using central extensions in groups, in a similar way as the torsion category \(\text{ConnGrpd}(\text{Grp})\) of \(\text{Grpd}(\text{Grp})\) is equivalent to surjective crossed modules.

The subcategory \(\mathcal{M}_{n\geq}\) is a Birkhoff subcategory of \(\text{Simp}(\text{Grp})\), this means a regular epireflective subcategory closed under subobjects and quotients in \(\text{Simp}(\text{Grp})\). Birkhoff subcategories of a semi-abelian category are again semi-abelian. As a second case study, we prove that the subcategory...
of Dakin’s $T$-group complexes is also a Birkhoff subcategory of $\text{Simp}(\text{Grp})$, and hence, it is semi-abelian. In order to study torsion theories in $T$-group complexes, it will be easier to work to the equivalent category $\text{Crs}(\text{Grp})$ of Ashley’s reduced crossed complexes. Our interest in reduced crossed complexes lies in their ‘similar behaviour’ as chain complexes. In particular, torsion theories in $\text{Crs}(\text{Grp})$ present examples of TTF theories in a weak sense. Used mainly in categories of modules over rings ([16]), a torsion torsion-free theory (or TTF theory for short) in an abelian category $X$ is a triplet of full subcategories $(C, T, F)$ such that $(C, T)$ and $(T, F)$ in $X$ are torsion theories in $X$.

In section 2, we recall basic facts of torsion theories in semi-abelian categories, as well as, the definition of the lattice $\mu(\text{Grp})$ of torsion theories in $\text{Simp}(\text{Grp})$. Section 3 and 4 studies torsion theories in the subcategories $\mathcal{M}_{n\geq}$ and $\text{Crs}(\text{Grp})$, respectively. Section 5, studies how reduced crossed complexes present examples of TTF theories in a weak sense. To this end we introduce, on one hand, CTF-theories, subcategories that are torsion-free and mono-coreflective. On the other hand, $\mathcal{E}$-torsion theories, torsion theories relative to a particular class of objects $\mathcal{E}$ of $\text{X}$.

### 2 Torsion theories in simplicial groups with truncated Moore complex

A category $X$ is called semi-abelian [15] if it is pointed with binary coproducts, Barr exact and Bourn protomodular.

**Definition 2.1.** A torsion theory in a semi-abelian category $X$ is a pair $(\mathcal{T}, \mathcal{F})$ of full and replete subcategories of $X$ such that:

**TT1** A morphism $f : T \to F$ with $T$ in $\mathcal{T}$ and $F$ in $\mathcal{F}$ is a zero morphism.

**TT2** For any object $X$ in $X$ there is a short exact sequence:

$$0 \longrightarrow T_X \xrightarrow{\epsilon_X} X \xrightarrow{\eta_X} F_X \longrightarrow 0$$

with $T_X$ in $\mathcal{T}$ and $F_X$ in $\mathcal{F}$.

In a torsion theory $(\mathcal{T}, \mathcal{F})$, $\mathcal{T}$ is called the torsion category and its objects are called torsion objects, in a similar way, $\mathcal{F}$ is the torsion-free category. Any subcategory is called torsion (torsion-free resp.) if it is the torsion (torsion-free resp.) category of a torsion theory. The torsion category $\mathcal{T}$ is a normal mono-coreflective subcategory of $X$, i.e., the inclusion $J : \mathcal{T} \to X$ has a right adjoint $T : X \to \mathcal{T}$ and each component of the counit $\epsilon_X : JT(X) \to X$ is a normal monomorphism (a kernel of some arrow in $X$). Similarly, $\mathcal{F}$ is a normal epi-reflective subcategory of $X$, i.e., the inclusion $I$ has a left adjoint $F$ and each component $\eta_X : X \to IF(X)$ is a normal epimorphism:

$$\mathcal{T} \xlongleftarrow{\perp} X \xlongleftarrow{\perp} \mathcal{F}.$$  \hspace{1cm} (2)

Then, it is easy to observe that $\mathcal{T}$ is closed under colimits in $X$ (those that exist in $X$) and $\mathcal{F}$ is closed under limits in $X$. Both $\mathcal{T}$ and $\mathcal{F}$ are closed under extensions in $X$, this means that given a short exact sequence in $X$:

$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$$

with $A$ and $B$ in $\mathcal{T}$ (resp. in $\mathcal{F}$) then $X$ is in $\mathcal{T}$ (resp. in $\mathcal{F}$). A torsion theory $(\mathcal{T}, \mathcal{F})$ is called hereditary if $\mathcal{T}$ is closed under subobjects in $X$, i.e., given a monomorphism $m : M \to T$ with $T$ in $\mathcal{T}$ then $M$ is also torsion. Conversely, a torsion theory is called cohereditary if $\mathcal{F}$ is closed under quotients in $X$, i.e., given a normal epimorphism $q : F \to Q$ with $F$ in $\mathcal{F}$ then $Q$ is also in $\mathcal{F}$. It is useful to recall the following result.
Proposition 2.2. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory of a semi-abelian category $\mathcal{X}$, if $(\mathcal{T}, \mathcal{F})$ is hereditary then $\mathcal{T}$ is a semi-abelian category. Accordingly, if $(\mathcal{T}, \mathcal{F})$ is hereditary then $\mathcal{F}$ is a semi-abelian category.

Given two torsion theories $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{S}, \mathcal{G})$ in $\mathcal{X}$, we have that $\mathcal{T} \subseteq \mathcal{S}$ if and only if $\mathcal{G} \subseteq \mathcal{F}$. This allows to introduce a partial order in the class of torsion theories in a category $\mathcal{X}$ by $(\mathcal{T}, \mathcal{F}) \leq (\mathcal{S}, \mathcal{G})$ if $\mathcal{T} \subseteq \mathcal{S}$. There is a bottom element and a top element given by the trivial torsion theories $0 := (0, \mathcal{X})$ and $\mathcal{X} := (\mathcal{X}, 0)$.

The simplicial category $\Delta$ has, as objects, finite ordinals $[n] = \{0, 1, \ldots, n\}$ and as morphisms monotone functions. A simplicial group is a functor $X : \Delta^{op} \rightarrow \text{Grp}$, we will denote $\text{Simp}($Grp$)$ for the category of simplicial groups. A simplicial group $X$ can be equivalently defined by the following data: a family of objects $\{X_n\}_{n \in \mathbb{N}}$ in $\mathcal{X}$, the face morphisms $d_i : X_n \rightarrow X_{n-1}$ and the degeneracies morphisms $s_i : X_n \rightarrow X_{n+1}$:

$$
X = \ldots X_n \xrightarrow{\begin{array}{c} \overleftarrow{d_1} \\ \vdots \\ \overleftarrow{d_n} \end{array}} \ldots X_1 \xrightarrow{\begin{array}{c} \overleftarrow{s_1} \\ \vdots \\ \overleftarrow{s_0} \end{array}} X_0
$$

satisfying the simplicial identities:

$$
d_i d_j = d_{j-1} d_i \quad \text{if} \quad i < j
$$

$$
s_i s_j = s_{j+1} s_i \quad \text{if} \quad i \leq j
$$

$$
d_i s_j = \begin{cases} 
  s_{j-1} d_i & \text{if} \quad i < j \\
  1 & \text{if} \quad i = j \text{ or } i = j + 1 \\
  s_j d_i & \text{if} \quad i > j + 1.
\end{cases}
$$

Since $\text{Simp}($Grp$)$ is a functor category with codomain a semi-abelian category, it is semi-abelian.

2.3. An $n$-truncated simplicial group $X$ is a simplicial group where the objects $X_i$, the face morphisms $d_i$ and degeneracies morphisms $s_i$ are defined up to level $n$ and they satisfy the simplicial identities (those that make sense). Let $\text{Simp}_n($Grp$)$ be the category of $n$-truncated simplicial groups then there is a truncation functor:

$$
\text{tr}_n : \text{Simp}($Grp$) \longrightarrow \text{Simp}_n($Grp$)
$$

which simply forgets everything above degree $n$. For all $n$, the functor $\text{tr}_n$ has a left adjoint $\text{sk}_n$ called the $n$-skeleton, and a right adjoint $\text{cosk}_n$ named the $n$-coskeleton: $\text{sk}_n \dashv \text{tr}_n \dashv \text{cosk}_n$. We will write $\text{Sk}_n = \text{sk}_n \text{tr}_n$ and $\text{Cosk}_n = \text{cosk}_n \text{tr}_n$.

In addition, for $n = 0$, the functor $\text{sk}_0$ has a left adjoint $\pi_0$ where $\pi_0(X) = \text{coeq}(d_0, d_1)$ is the coequalizer of the face morphisms $d_0, d_1 : X_1 \rightarrow X_0$. Moreover, the adjunctions $\pi_0 \dashv \text{sk}_0 \dashv \text{tr}_0 \dashv \text{cosk}_n$ correspond to the functors $\pi_0 \dashv \text{Dis} \dashv (\_)_0 \dashv \text{Ind}$. For a group $G$, $\text{Dis}(G)$ is the discrete simplicial group of $G$

$$
\text{Dis}(G) = \ldots \xrightarrow{1} G \xrightarrow{1} G \xrightarrow{1} G \xleftarrow{1} G
$$

and $\text{Ind}(G)$ is the indiscrete simplicial group of $G$

$$
\text{Ind}(G) = \ldots \xrightarrow{p_0} G^4 \xrightarrow{p_0} G^3 \xrightarrow{p_0} G^2 \xrightarrow{p_0} G
$$

where $G^n$ is the $n$-fold product, $(X)_0 = X_0$ and the morphisms $p_i$ are induced by product projections.
A chain complex $M$ is a family of morphisms $\{\delta_n: M_n \to M_{n-1}\}_{n \in \mathbb{N}}$ such that $\delta_n \delta_{n+1} = 0$ for all $n$. A chain complex $M$ is proper if for each differential $\delta_n$ the monomorphism $m_n$ of the normal epi/mono factorization $(e_n, m_n)$ of $\delta_n$ is a normal monomorphism:

$$\cdots \to M_{n+1} \xrightarrow{\delta_{n+1}} M_n \xrightarrow{\delta_n} M_{n-1} \xrightarrow{\delta_{n-1}} \cdots$$

The category of chain complexes and the subcategory of proper chain complexes in groups will be denoted as $\text{chn}(\text{Grp})$ and $\text{pch}(\text{Grp})$.

**Definition 2.4.** The Moore normalization functor $N: \text{Simp}(\text{Grp}) \to \text{chn}(\text{Grp})$ is defined as follows. Let $X$ be a simplicial group then $N(X)$ is the group chain complex

$$\cdots \to N(X)_n \xrightarrow{\delta_n} N(X)_{n-1} \to \cdots \to N(X)_1 \xrightarrow{\delta_1} N(X)_0$$

such that $N(X)_0 = X_0$ and

$$N(X)_n = \bigcap_{i=0}^{n-1} \ker(d_i: X_n \to X_{n-1})$$

and differentials $\delta_n = d_n \circ \cap \ker(d_i): N(X)_n \to N(X)_{n-1}$ for $n \geq 1$. The Moore chain complex $N(X)$ of a simplicial group is proper. It is known that the functor $N$ preserves finite limits and normal epimorphisms, and also it is conservative.

We will write $\mathcal{M}_{n\geq}$ for the subcategory of simplicial groups with trivial Moore complex above degree $n$. Similarly, $\mathcal{M}_{\geq n}$ is the subcategory of simplicial groups with trivial Moore complex below degree $n$.

In [19] it is proved that $\mathcal{M}_{n\geq}$ is a torsion-free subcategory of $\text{Simp}(\text{Grp})$, and respectively, $\mathcal{M}_{\geq n}$ is a torsion subcategory for all $n$. Indeed, the corresponding torsion theories form a linearly ordered lattice $\mu(\text{Grp})$ in $\text{Simp}(\text{Grp})$. We recall how these are defined and useful properties.

**Theorem 2.5.** [19] There is a linear order lattice $\mu(\text{Grp})$ of torsion theories in $\text{Simp}(\text{Grp})$:

$$0 \leq \cdots \leq \mu_{n+1} \leq \mu_{n+1} \leq \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_2 \leq \mu_1 \leq \mu_0 \leq \mu_{0\geq} \leq \text{Simp}(\text{Grp})$$

where

1. The torsion theory $\mu_{n\geq}$ is given by the pair $(\text{Ker}(\text{Cot}_n), \mathcal{M}_{n\geq})$, where $\text{Cot}_n : \text{Simp}(\text{Grp}) \to \mathcal{M}_{n\geq}$ (introduced in [21]) is the left adjoint of the inclusion $i : \mathcal{M}_{n\geq} \to \text{Simp}(\text{Grp})$ and $\text{Ker}(\text{Cot}_n)$ is the full subcategory of simplicial groups $X$ such that $\text{Cot}_n(X) = 0$.
2. The torsion theory $\mu_{\geq n}$ is given by the pair $(\mathcal{M}_{\geq n}, \mathcal{F}_{tr_n})$, where $\mathcal{F}_{tr_n}$ is the subcategory of simplicial groups $X$ with $\eta_X$ monic where $\eta$ is the unit of the adjunction $\text{tr}_n \dashv \text{cosk}_n$.
3. Let $X$ be a simplicial group and $M$ its Moore complex. For $X$ the associated short exact sequence of $\mu_{n\geq}$ under normalization is the short exact sequence in chain complexes (written
4. Let $X$ be a simplicial group and $M$ its Moore complex. For $X$ the associated short exact sequence of $\mu \geq n$ under normalization is the short exact sequence in chain complexes (written vertically):

$$
\cdots \to M_{n+2} \xrightarrow{\delta_{n+2}} M_{n+1} \xrightarrow{\delta_{n+1}} \delta_{n+1}(X_{n+1}) \to 0 \to \cdots
$$

As shown form diagram (3) and (4) the coreflector of $\mathcal{M}_{\geq n}$ and respectively the reflector of $\mathcal{M}_{n \geq}$ behave, at the level of Moore complexes, as the truncation functors introduced by L. Illusie in [14]. As a consequence, torsion/torsion-free objects can be characterized by their Moore complex as follows.

**Corollary 2.6.** [19] Let $X$ be a simplicial group with Moore complex $M$, then

1. $X$ belongs in $\text{Ker}(\text{Cot}_n)$ if and only if $M$ is of the form

$$
\cdots \to M_{n+2} \xrightarrow{\delta_{n+2}} M_{n+1} \xrightarrow{\delta_{n+1}} M_n \xrightarrow{\delta_n} 0 \to 0 \to \cdots
$$

with $\delta_n$ a normal epimorphism.

2. $X$ belongs in $\mathcal{F}_{tr_n}$ if and only if $M$ is of the form

$$
\cdots \to 0 \xrightarrow{\delta_{n+1}} M_{n+1} \xrightarrow{\delta_n} M_n \xrightarrow{\delta_{n-1}} M_{n-1} \xrightarrow{\delta_{n-2}} M_{n-2} \to \cdots
$$

with $\delta_{n+1}$ a normal monomorphism.

3 **Torsion subcategories of $\mathcal{M}_{n \geq}$**

The lattice $\mu(\text{Grp})$

$$
0 \leq \cdots \leq \mu_{\geq 2} \leq \mu_{1 \geq} \leq \mu_{\geq 1} \leq \mu_{0 \geq} \leq \text{Simp(Grp)}.
$$

extends the torsion theories in internal groupoids $\text{Grpd(Grp)}$

$$
0 = (0, \text{Grpd(Grp)}) \leq (\text{Ab, Eq(Grp)}) \leq (\text{ConnGrpd(Grp), Dis(Grp)}) \leq \text{Grpd(Grp)}
$$
in the sense that the torsion-free categories of the first 3 largest non-trivial torsion theories, \( \mu_{1 \geq} \leq \mu_{2 \geq} \leq \mu_{0 \geq} \), in \( \text{Simp}(\text{Grp}) \) are the torsion-free categories \( \text{Grpd}(\text{Grp}), \text{Eq}(\text{Grp}) \) and \( \text{Dis}(\text{Grp}) \), respectively) of \( \text{Grpd}(\text{Grp}) \).

On the other hand, the torsion categories of \( \text{Simp}(\text{Grp}) \) are not as easily described as in the case of \( \text{Grpd}(\text{Grp}) \). However, as in the case \( \text{Grpd}(\text{Grp}) = M_{1 \geq} \), some similarities arise when we restrict the lattice \( \mu(\text{Grp}) \) to subcategories of the form \( M_{n \geq} \). To this end, we recall how simplicial complexes with operations following the work of Loday in [17], D. Conduché in [8] and, P. Carrasco and A. M. Cegarra in [6].

### 3.1 The case of \( M_{1 \geq} \) and crossed modules

For completeness sake we quickly recall how torsion theories in internal groupoids correspond to torsion theories in Whitehead’s crossed modules.

We will write group actions acting on the left \( b(a) \) and each group \( G \) is consider to act on itself by conjugation as \( g'(g) = gg'g^{-1} \).

**Definition 3.1.** A crossed module (in groups) is a morphism of groups \( \delta : A \to B \) with a groups action \( B \to \text{Aut}(A) \) such that:

1. \( \delta(b(a)) = b\delta(a)b^{-1} \) (\( \delta \) is equivariant).
2. \( \delta(a)a' = aa'a^{-1} \) (Peiffer identity).

If \( \delta \) only satisfy axiom 1 then it is called a precrossed module. We will denote the category of crossed modules and of precrossed modules as \( \mathcal{X}\text{Mod} \) and \( \mathcal{P}\mathcal{X}\text{Mod} \), respectively.

From [17], the following categories are equivalent:

1. The category \( M_{1 \geq} \) of simplicial groups with trivial Moore complex above degree 1.
2. The category \( \text{Grpd}(\text{Grp}) \) of internal groupoids in groups.
3. The category \( \mathcal{X}\text{Mod} \) of crossed modules in groups.
4. The category of \( \text{Cat-1-groups} \).

Indeed, given a simplicial group \( X \) with Moore complex \( M = N(X) \), the differential \( \delta_1 : M_1 \to M_0 \) is a precrossed module with the action given by conjugation with \( s_0 \). Furthermore, \( \delta_1 \) is a crossed module if and only if \( M_i = 0 \) for \( i > 1 \).

We will write \( \mu(M_{1 \geq}) \) for the lattice given by the torsion theories of \( \mu(\text{Grp}) \) restricted to \( M_{1 \geq} \), i.e., we consider the torsion theory \( (\mathcal{T} \cap M_{1 \geq}, \mathcal{F} \cap M_{1 \geq}) \) for each element \( (\mathcal{T}, \mathcal{F}) \) of \( \mu(\text{Grp}) \). We will write \( \mu_{n \geq} \) and \( \mu_{\geq n} \) for the corresponding restrictions of \( \mu_{n \geq} \) and \( \mu_{\geq n} \).

**Proposition 3.2.** The lattice \( \mu(M_{1 \geq}) \) is given by

\[
0 \leq \mu_{n \geq} \leq \mu_{0 \geq} \leq \mu_{\geq n} \leq M_{1 \geq}.
\]
Recall that $\mu(\mathsf{Grp})$ is given by

$$
\begin{array}{c}
\text{Simp}(\mathsf{Grp}) = \\
\mu_0 \geq K_{\mathsf{Cot}_0} \quad \text{Simp} (\mathsf{Grp}) = \\
\mu \geq 1 \geq M \quad \text{Simp} (\mathsf{Grp}) = \\
M_1 \geq K_{\mathsf{Cot}_1} \quad \text{Simp} (\mathsf{Grp}) = \\
M_2 \geq K_{\mathsf{Cot}_2} \quad \text{Simp} (\mathsf{Grp}) = \\
\ldots \quad \ldots \quad \ldots \quad \ldots
\end{array}
$$

(5)

Since $M_{1\geq}$ is itself a torsion-free subcategory comprised in the lattice $\mu(\mathsf{Grp})$, the restriction of each torsion theory below $\mu_{1\geq}$ is the trivial torsion theory $(0, M_{1\geq})$ in $M_{1\geq}$.

It is easy to observe that the torsion-free categories $E_{\mathsf{Grp}}$ and $D_{\mathsf{Grp}}$ in $\text{Simp}(\mathsf{Grp})$ are also torsion-free categories of $M_{1\geq}$. We can conclude the following.

**Proposition 3.3.** The lattice $\mu(M_{1\geq})$ corresponds to

$$0 \leq (\text{Ab}(\mathsf{Grp}), E_{\mathsf{Grp}}) \leq (\text{ConnGrpd}(\mathsf{Grp}), D_{\mathsf{Grp}}) \leq M_{1\geq}.$$ 

Moreover, under the Moore normalization this lattice corresponds to the lattice of torsion theories in $X\text{Mod}$

$$0 \leq (\text{Ab}, \text{Mono}) \leq (C_{\text{ext}}, D_{\text{dis}}) \leq X\text{Mod},$$

where

1. $\text{Ab}$ is the category of crossed modules of the form $A \to 0$ for an abelian group $A$.
2. $\text{Mono}$ is the category of crossed modules given by the inclusion of a normal subgroup $i : N \to G$.
3. $\text{Cext}$ is the category of crossed modules given by central extensions, epimorphisms $p : G \to Q$ with a $\ker(p) \leq Z(G)$.
4. $\text{Dis}$ is the category of crossed modules given of the form $0 \to G$ for a group $G$.

**Proof.** In a torsion theory the torsion and the torsion-free category determine each other uniquely, so if $E_{\mathsf{Grp}}(\mathsf{Grp})$ and $D_{\mathsf{Grp}}(\mathsf{Grp})$ are torsion-free categories of $\mu_{0\geq}$ and $\mu_{1\geq}$ the torsion categories must correspond accordingly to $\text{Ab}(\mathsf{Grp})$ and $\text{ConnGrpd}(\mathsf{Grp})$.

The second statement is well-known, for instance the equivalences are mentioned in [3], [11] and [20].
3.2 The case of $\mathcal{M}_{n\geq}$

Following the previous case, we will write $\mu(\mathcal{M}_{n\geq})$ for the restriction of $\mu(\text{Grp})$ to $\mathcal{M}_{n\geq}$ and $\mu'_{i\geq}$, $\mu'_{i\geq}$ for the restrictions of the torsion theories $\mu_{i\geq}$, $\mu_{i\geq}$.

Similar to Proposition 3.2, also from the diagram (4) we have the following result.

**Proposition 3.4.** The lattice $\mu(\mathcal{M}_{n\geq})$ is given by

$$0 \leq \mu'_{i\geq} \leq \mu'_{i-1\geq} \leq \mu'_{i-2\geq} \cdots \leq \mu'_{1\geq} \leq \mu'_{0\geq} \leq \mathcal{M}_{n\geq}.$$ 

Just as in $\mathcal{M}_{1\geq}$ the subcategories $\text{Dis}(\text{Grp})$, $\text{Eq}(\text{Grp})$ and $\text{Grpd}(\text{Grp})$ are torsion-free subcategories of $\mathcal{M}_{n\geq}$. In order to characterise the torsion categories with recall some categories introduced by D. Conduché.

**Definition 3.5.** A group chain complex

$$L \xrightarrow{\delta_2} M \xrightarrow{\delta_1} N$$

is called a 2-crossed module if $N$ acts on $L$ and $M$ and the differentials $\delta_2, \delta_1$ are equivariant ($N$ acts over itself with conjugation), and there is a mapping

$$\{ , \} : M \times M \longrightarrow L$$

satisfying:

1. $\delta_2\{m_0, m_1\} = m_0 m_1 m_0^{-1} \delta_1(m_0)(m_1^{-1});$
2. $\delta_2(l_0)\delta_2(l_1) = [l_0, l_1];$
3. $\delta_2(l, m)\{m, \delta_2(l)\} = l \delta_1(m)l^{-1};$
4. $\{m_0, m_1m_2\} = \{m_0, m_1\}\{m_0, m_2\}\{\delta_2(m_0, m_2)^{-1}, \delta_1(m_0)m_1\};$
5. $\{m_0 m_1, m_2\} = \{m_0, m_1m_2m_1^{-1}\} \delta_1(m_0)\{m_1, m_2\};$
6. $\{m_0, m_1\} = \{n_0, n_1\}.$

The map $\{ , \}$ is called the Peiffer lifting.

A morphism of 2-crossed modules is a morphism of chain complexes that preserves the group actions and the Peiffer lifting. The category of 2-crossed complexes will be denoted as $\mathcal{X}_{\text{Mod}}$.

In a 2-crossed module the Peiffer lifting defines an action of $M$ over $L$ as $m(l) = l\{\delta_2(l)^{-1}, m\}$, so $\delta_2$ is indeed a crossed module. On the other hand, $\delta_1$ is only a precrossed module. A crossed module is a 2-crossed module by setting $L = 0$. If a 2-crossed module has $N = 0$ then the equations get simplify, thus we obtain a reduced 2-crossed module as follows.

**Definition 3.6.** A reduced 2-crossed module is a group morphism $\delta : L \rightarrow M$ with a map $\{ , \} : M \times M \rightarrow L$ satisfying:

1. $\delta\{m_0, m_1\} = [m_0, m_1],$
2. $\{\delta(l_0), \delta(l_1)\} = [l_0, l_1],$
3. $\{\delta(l), m\}\{m, \delta(l)\} = 1,$
4. $\{m_0, m_1m_2\} = \{m_0, m_1\}\{m_0, m_2\}\{m_2, m_0, m_1\},$
5. $\{m_0 m_1, m_2\} = \{m_0, m_1m_2m_1^{-1}\}\{m_1, m_2\}.$
The category of reduced 2-crossed modules will be denoted as $R_2\times Mod$.

**Definition 3.7.** [S] A stable crossed module is a group morphism $\delta : L \to M$ with a map $\{ , \} : M \times M \to L$ satisfying:

1. $\delta\{m_0, m_1\} = [m_0, m_1]$,
2. $\{\delta(l_0), \delta(l_1)\} = [l_0, l_1]$,
3. $\{m_1, m_0\} = \{m_0, m_1\}^{-1}$,
4. $\{m_0m_1, m_2\} = \{m_0m_1m_0^{-1}, m_0m_2m_0^{-1}\}\{m_0, m_2\}$.

We will denote $St\times Mod$ for the category of stable crossed modules.

The underlying morphism $\delta : L \to M$ of reduced 2-crossed module or of a stable crossed module is in fact a crossed module. Similar to the case of crossed modules/ internal groupoids, these categories characterise simplicial groups via the normalization functor.

**Theorem 3.8.** [S]

1. The category $\mathcal{M}_{2\geq}$ of simplicial groups with trivial Moore complex above degree 2 is equivalent to the category $\times Mod$ of 2-crossed modules.
2. The category $\mathcal{M}_{1,2}$ of simplicial groups with trivial Moore complex except at degrees 1,2 is equivalent to the category $R_2\times Mod$ of reduced 2-crossed modules.
3. The category $\mathcal{M}_{p,p+1}$ of simplicial groups with trivial Moore complex except at degrees $p, p + 1$ for $p \geq 2$ is equivalent to the category $St_2\times Mod$ of stable crossed modules.

The bottom torsion categories of $\mathcal{M}_{n\geq}$ can be characterized in a similar way as the torsion categories of internal groupoids.

**Theorem 3.9.** For $n = 2$, consider the lattice of torsion theories $\mu(\mathcal{M}_{2\geq})$:

$$0 \leq \mu_{2\geq} \leq \mu_{1\geq} \leq \mu_{1\geq} \leq \mu\{\geq\} \leq \mathcal{M}_{2\geq}$$

For the bottom torsion categories we have the equivalences:

1. the torsion category $\mathcal{M}_{2\geq} \cap M_{2\geq} = \mathcal{M}_{1,2}$ of $\mu_{2\geq}$ is equivalent to the category $R_2\times Mod$ of reduced 2-crossed modules;
2. the torsion category $Ker(Cot_1) \cap M_{2\geq}$ of $\mu_{1\geq}$ is equivalent to the category $R_2\times Mod \cap Cext$ of reduced 2-crossed modules $\delta : L \to M$ with $\delta$ a central extension;
3. the torsion category $\mathcal{M}_{2\geq} \cap M_{2\geq}$ of $\mu\{\geq\}$ is equivalent to the category of 2-crossed modules of the form $A \to 0 \to 0$ and hence, it is also equivalent to the category $Ab$ of abelian groups.

**Proof.** 1) It follows immediately from the fact that $\mathcal{M}_{2\geq} \cap M_{2\geq}$ is equivalent by definition to the category $\mathcal{M}_{1,2}$ of simplicial groups with trivial Moore complex except at degrees 1,2. and Theorem 3.8.

2) A simplicial group $X$ belongs to $Ker(Cot_1) \cap M_{2\geq}$ if and only if its Moore complex is trivial except $\delta_2 : M_2 \to M_1$ which is, in addition, surjective (form Corollary 2.6). This happen if and only if $\delta_2$ is a central extension, since a 2-reduced crossed module is in particular a crossed module.

3) The category $\mathcal{M}_{2\geq} \cap M_{2\geq}$ is equivalent to the category of simplicial groups with trivial except at degree 2, so it corresponds to a 2-crossed module of the form $L \to 0 \to 0$. Since, in a 2-crossed module the morphism $\delta_2$ is a crossed module then $L$ must be an abelian group. □
Theorem 3.10. For $n > 2$, in $\mu(M_{n\geq})$ consider the bottom torsion theories:

$$0 \leq \mu'_{\geq n} \leq \mu'_{n-1} \geq \mu'_{\geq n-1} \ldots$$

Then for the torsion categories we have the equivalences:

1. the torsion category $M_{\geq n-1} \cap M_{n\geq} = M_{n,n-1}$ of $\mu'_{\geq n-1}$ is equivalent to the category $St\times Mod$ of stable crossed modules;

2. the torsion category $\text{Ker}(\text{Cot}_{n-1}) \cap M_{n\geq}$ of $\mu'_{n-1}$ is equivalent to the category $St\times Mod \cap Cext$ of stable crossed modules $\delta : L \to M$ with $\delta$ a central extension;

3. the torsion category $M_{\geq n} \cap M_{n\geq}$ of $\mu'_{\geq n}$ is equivalent to the category $Ab$ of abelian groups.

Proof. It follows from 3) of Theorem 3.8, the proof is similar to Theorem 3.9.

Corollary 3.11. A simplicial group $X$ belong to the torsion category $\text{Ker}(\text{Cot}_{n-1}) \cap M_{n\geq}$ of $M_{n\geq}$ if and only if its Moore complex $M$ is a central extension in groups.

Corollary 3.12. The categories $\times Mod$ of $2$-crossed modules, $R\times Mod$ of reduced $2$-crossed modules and $St\times Mod$ of stable crossed modules are semi-abelian.

Proof. From Proposition 2.2 the categories $M_{n\geq}$ are semi-abelian. In particular, for $n = 2$ the category of $2$-crossed modules is semi-abelian. Similarly, $R\times Mod$ and $St\times Mod$ are semi-abelian from Theorems 3.9, 3.10 and Proposition 2.2.

3.3 The case of $M_{\geq n}$

A dual behaviour can be noticed when we work with the torsion categories $M_{\geq n}$, which are also semi-abelian (again form Proposition 2.2). When considering $\mu(M_{\geq n})$, the restriction of $\mu(Grp)$ to $M_{\geq n}$, we first obtain that the torsion theories above $\mu_{n\geq}$ are trivialized and second, the upper torsion-free categories are equivalent to abelian groups and the different kinds of crossed modules.

Proposition 3.13. The lattice $\mu(M_{\geq n})$ is given by:

$$0 \leq \ldots \leq \mu'_{n+1} \geq \mu'_{\geq n+1} \leq \mu'_{n} \geq \ldots \leq M_{\geq n}$$

where $\mu'_{\geq n}$, $\mu'_{\geq n}$ are the restriction of the torsion theories of $\mu(Grp)$.

Theorem 3.14. For the category $M_{\geq 1}$ we have

1. the torsion-free category of $\mu'_{\geq}$ given by $M_{1\geq} \cap M_{\geq 1}$ is equivalent to the category of abelian groups;

2. the torsion-free category of $\mu'_{\geq}$ given by $M_{2\geq} \cap M_{\geq 1}$ is equivalent to the category $R\times Mod$ of Reduced $2$-crossed modules.

For the category $M_{\geq n}$ and $n \leq 2$ we have

1. the torsion-free category of $\mu'_{n\geq}$ given by $M_{n\geq} \cap M_{\geq n}$ is equivalent to the category of abelian groups;

2. the torsion-free category of $\mu'_{n+1\geq}$ given by $M_{n+1\geq} \cap M_{\geq n}$ is equivalent to the category $St\times Mod$ of stable crossed modules.

Proof. The proof is similar to the case of torsion categories in $M_{n\geq}$.
4 Torsion theories in reduced crossed complexes

Introduced by M. K. Dakin [9], a T-complex is a Kan simplicial object that admits a canonical filler for horns, for example internal groupoids have this property. Following the work of N. Ashley in [1] and the observations in [6] a group T-complex (a T-complex in simplicial groups) can be defined as

\[ X \text{ with } M_n \cap D_n = 0 \]

where \( M \) is the Moore complex of \( X \) and \( D \) is the graded subgroup of \( X \) generated by the degenerated elements of \( X \).

**Definition 4.1.** [1] A reduced crossed complex \( M \), or a crossed complex in groups, is a proper chain complex

\[ M = \ldots \rightarrow M_n \xrightarrow{\delta_n} M_{n-1} \rightarrow \ldots \rightarrow M_2 \xrightarrow{\delta_2} M_1 \xrightarrow{\delta_1} M_0 \]

where

1. \( M_n \) is abelian for \( n \geq 2 \);
2. \( M_0 \) acts on \( M_n \) for \( n \geq 1 \) and the restriction to \( \delta_1(M_1) \) acts trivially on \( M_n \) for \( n \geq 2 \);
3. \( \delta_n \) preserves the action of \( M_0 \) and \( \delta_1 : M_1 \rightarrow M_0 \) is a crossed module.

A morphism of reduced crossed complexes is a chain complex morphism that preserves all actions. We will write \( \text{Crs}(\text{Grp}) \) for the category of reduced crossed complexes.

It is proved in [1] that the category of group T-complexes is equivalent to the category of reduced crossed complexes via the Moore normalization. And in [10] it is shown that the category of reduced crossed complexes is an epi-reflective subcategory of simplicial groups.

**Corollary 4.2.** The category of Dakin’s group T-complexes is semi-abelian. Hence, the category \( \text{Crs}(\text{Grp}) \) of reduced crossed complexes is also semi-abelian.

**Proof.** From [10], we have that group T-complexes is a normal epi-reflective subcategory of \( \text{Simp}(\text{Grp}) \). In fact, it is a Birkhoff subcategory so it is semi-abelian. Indeed, we just need to prove that it is closed under regular epimorphism in \( \text{Simp}(\text{Grp}) \). So, let \( X \) be a group T-complex, a simplicial group with \( M^n_X \cap D^n_X = 0 \) and \( f : X \rightarrow Y \) a regular epimorphism where \( M^n_X, M^n_Y \) are the Moore subobjects and \( D^n_X, D^n_Y \) are the graded subgroups generated by degenerated elements of \( X \) and \( Y \) respectively.

Since the Moore normalization preserves regular epimorphisms, then the restriction \( f_M : M^n_X \rightarrow M^n_Y \) is also a regular epimorphism. Now the restriction \( f_D : D^n_X \rightarrow D^n_Y \) is surjective, since we have a commutative diagram:

\[ \begin{array}{ccc}
X_n & \xrightarrow{s_1} & X_m \\
\downarrow & & \downarrow f \\
Y_n & \xleftarrow{s_1} & Y_m
\end{array} \]

so for a degenerate element \( s_1(y_m) \) in \( Y_n \) (\( s_1 \) a composition of degenerate morphisms), there is \( x_m \) in \( X_M \) such that \( s_1f(x_m) = f s_1(x_m) \). Finally, since both \( f_M \) and \( f_D \) are regular epimorphisms the morphism:

\[ f_M \times f_D : M^n_X \times_{X_n} D^n_X \rightarrow M^n_Y \times_{Y_n} D^n_Y \]

is also a regular epimorphism, so if \( M^n_X \times_{X_n} D^n_X = M^n_X \cap D^n_X = 0 \), then so is \( M^n_Y \cap D^n_Y = 0 \). □

Our interest in reduced crossed complexes lies in their similar behaviour to chain complexes, thus torsion theories of simplicial groups can be easily studied when restricted to subcategory of \( \text{Crs}(\text{Grp}) \). To this end, we recall some properties of reduced crossed complexes, we refer the reader to [5] for the details and proofs.
4.3. An \( n \)-reduced crossed complex is an \( n \)-truncated chain complex \( M \):

\[
M = M_n \xrightarrow{\delta_n} M_{n-1} \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_2} M_2 \xrightarrow{\delta_1} M_1 \xrightarrow{\delta_1} M_0
\]
satisfying all the axioms of a reduced crossed complex (those that make sense), thus we have a category \( \text{Crs}(\text{Grp})_{\geq} \) of \( n \)-truncated crossed complexes. Clearly, a 1-truncated reduced crossed complex is nothing but a crossed module, thus \( \text{Crs}(\text{Grp})_{\geq} = \text{XMod} \).

Moreover, we have the functors for all \( n \in \mathbb{N} \):

- the truncation functor \( \text{tr}_n : \text{Crs}(\text{Grp}) \to \text{Crs}(\text{Grp})_{\geq} \)
  \[
  \text{tr}_n(M) = M_n \xrightarrow{\delta_n} M_{n-1} \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_2} M_2 \xrightarrow{\delta_1} M_1 \xrightarrow{\delta_1} M_0
  \]
- the skeleton functor (or natural embedding) \( \text{sk}_n : \text{Crs}(\text{Grp})_{\geq} \to \text{Crs}(\text{Grp}) \)
  \[
  \text{sk}_n(M) = \ldots \to 0 \to 0 \xrightarrow{\delta_n} M_n \xrightarrow{\delta_2} M_{n-1} \xrightarrow{\delta_2} \cdots ;
  \]
- the coskeleton functor \( \text{cosk}_n : \text{Crs}(\text{Grp})_{\geq} \to \text{Crs}(\text{Grp}) \)
  \[
  \text{cosk}_n(M) = \ldots \to 0 \to \ker(\delta_n) \xrightarrow{\delta_2} M_n \xrightarrow{\delta_2} M_{n-1} \xrightarrow{\delta_2} \cdots ;
  \]
- the cotruncation \( \text{cot}_n : \text{Crs}(\text{Grp}) \to \text{Crs}(\text{Grp})_{\geq} \)
  \[
  \text{cot}_n(M) = M_n/\delta_{n+1}(M_{n+1}) \xrightarrow{\delta_2} M_{n-1} \xrightarrow{\delta_2} M_{n-2} \xrightarrow{\delta_2} \cdots .
  \]

We will write \( \text{Sk}_n = \text{sk}_n\text{tr}_n, \text{Cosk}_n = \text{cosk}_n\text{tr}_n \) and \( \text{Cot}_n = \text{sk}_n\text{cot}_n \). These functors give a string of adjunctions

\[
\text{Crs}(\text{Grp})_{\geq} \xrightarrow{\text{sk}_n} \text{Crs}(\text{Grp}) \xleftarrow{\text{tr}_n} \text{Crs}(\text{Grp})_{\geq}
\]

It is worth mentioning that the adjunctions of simplicial groups \( \text{sk}_n \dashv \text{tr}_n \dashv \text{cosk}_n \) restricted to crossed complexes correspond (under Moore normalization) to \( \text{sk}_n \dashv \text{tr}_n \dashv \text{cosk}_n \) and Porter’s cotruncation \( \text{Cot}_n \) corresponds to \( \text{Cot}_n \). In addition, unlike the general case of simplicial groups, the adjunction \( \text{cot}_n \dashv \text{sk}_n \) for reduced crossed complexes.

On the other hand, it will be useful to write \( \text{Crs}(\text{Grp})_{\geq n} \) for the subcategory of reduced crossed complexes with \( M_i = 0 \) for \( n > i \). For all \( n \geq 1 \), \( \text{Crs}(\text{Grp})_{\geq n} \) equivalent to the category of \( \text{ch}(\text{Ab})_{\geq n} \) chain complexes of abelian groups. Moreover, we can easily define the dual functors:

- \( \text{tr}'_n : \text{Crs}(\text{Grp}) \to \text{Crs}(\text{Grp})_{\geq n} \);
- \( \text{sk}'_n : \text{Crs}(\text{Grp})_{\geq n} \to \text{Crs}(\text{Grp}) \);
- \( \text{cot}'_n : \text{Crs}(\text{Grp}) \to \text{Crs}(\text{Grp})_{\geq n} \).

However, only the adjunction \( \text{sk}'_n \dashv \text{cot}'_n \) holds.

**Proposition 4.4.** Let \( \mu(\text{Crs}(\text{Grp})) \) be the lattice given by the restriction of torsion theories in \( \mu(\text{Grp}) \) to \( \text{Crs}(\text{Grp}) \):

\[
\mu(\text{Crs}(\text{Grp})) = \cdots \leq \mu'_{2 \geq} \leq \mu'_{1 \geq} \leq \mu'_{2 \geq} \leq \mu'_{0 \geq} \leq \text{Crs}(\text{Grp}).
\]

Then, the torsion theories \( \mu'_{n \geq} \) and \( \mu'_{2 \geq} \) can be expressed with the functors \( \text{cot}_n \dashv \text{sk}_n \dashv \text{tr}_n \dashv \text{cosk}_n \):

\[
\mu'_{n \geq} = (\text{Ker}(\text{cot}_n), \text{Crs}(\text{Grp})_{n \geq}), \quad \mu'_{2 \geq} = (\text{Crs}(\text{Grp})_{\geq n}, \mathcal{F}_{\text{tr}_n}).
\]
Proof. It follows from the fact that the equivalence between group $T$-complexes and reduced crossed complexes is given by the Moore normalization, and also from the short exact sequences in Theorem 2.5 and the previous observations.

**Theorem 4.5.** For $n > 1$, let $\mu(Crs(Grp)_{\geq n})$ be the lattice given by restriction of $\mu(Grp)$ to $Crs(Grp)_{\geq n}$:

$$
\mu(Crs(Grp)_{\geq n}) = 0 \leq \mu'_{\geq n} \leq \mu'_{n-1} \leq \cdots \leq \mu'_{2} \leq \mu'_{1} \leq \mu_{0,2} \leq Crs(Grp)_{\geq n}.
$$

Then for the torsion categories we have the equivalences:

1. $Ker(\cot_{n-1}) \cap Crs(Grp)_{\geq n}$, the torsion category of $\mu'_{n-1}$, is equivalent to the category $CExt(Ab)$ of central extension in abelian groups, i.e., surjective morphisms.

2. $Crs(Grp)_{\geq n} \cap Crs(Grp)_{\geq n}$, the torsion category of $\mu'_{\geq n}$, is equivalent to the category $Ab$ of abelian groups.

**Proof.** The proof is similar to Theorem 3.9.

5 **Torsion torsion-free theories**

**Definition 5.1.** A torsion torsion-free theory in $X$, or TTF theory, is a triplet $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ of full subcategories of $X$ such that $(\mathcal{C}, \mathcal{T})$ and $(\mathcal{T}, \mathcal{F})$ are torsion theories in $X$. A subcategory $\mathcal{T}$ of $X$ is called a torsion torsion-free category (or a TTF category) if there are subcategories $\mathcal{C}$ and $\mathcal{F}$ such that $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ is a TTF theory.

TTF theories were introduced in [16] with applications mainly to categories of modules over rings, in particular they are used to study when any object of a category is a joint of torsion objects for different torsion theories. TTF theories have been studied in different non-abelian settings, for example in triangulated categories in [3]. In our context of simplicial groups, we present examples of TTF theories in a weak sense.

5.1 **TTF theories in chain complexes**

Let $X$ be a semi-abelian category and $chn(X)$ and $pch(X)$ the categories of chain complexes and proper chain complexes, respectively. We will write $chn(X)_{\geq n}$ and $chn(X)_{\geq n}$ for the category of chain complexes bounded above/below $n$.

We have the adjunctions:

$$
\cot_n \dashv \sk_n \dashv \tr_n \dashv \cosk_n : \begin{cases}
\mathcal{C}(X) \\
\mathcal{C}(X)_{\geq n}
\end{cases}
$$

as well as their duals:

$$
\cosk'_n \dashv \tr'_n \dashv \sk'_n \dashv \cot'_n : \begin{cases}
\mathcal{C}(X)_{\geq n} \\
\mathcal{C}(X)
\end{cases}
$$

defined similarly as in 4.3. These adjunction can be restricted to adjunctions between $pch(X)$ and $pch(X)_{\geq n}$ (or $pch(X)_{\geq n}$ respectively).

Similar to simplicial groups, torsion theories are given by the cotruncation functors $\cot$ and $\cot'$ as follows.
Theorem 5.2. [19] Let \( \mathcal{X} \) be a semi-abelian category, then we have:

1. in \( \text{chn} \( \mathcal{X} \) \) we have a torsion theory \( (\text{chn} \( \mathcal{X} \) \geq n, \mathcal{F}_{\text{tr}_{n-1}}) \):

\[
\begin{array}{ccc}
\text{sk}'_n & \downarrow & \text{ch}(\mathcal{X}) \geq n \\
\circlearrowleft & & \circlearrowright \\
\cot'_n & & \text{F}_{\text{tr}_{n-1}}
\end{array}
\]

2. the previous torsion theory is restricted to a torsion theory in \( \text{pch} \( \mathcal{X} \) \) as \( (\text{pch} \( \mathcal{X} \) \geq n, \mathcal{M}_n) \):

\[
\begin{array}{ccc}
\text{sk}'_n & \downarrow & \text{pch}(\mathcal{X}) \geq n \\
\circlearrowleft & & \circlearrowright \\
\cot'_n & & \mathcal{M}_n
\end{array}
\]

3. the category \( \text{chn} \( \mathcal{X} \) \geq n \) is a normal epi-reflective subcategory of \( \text{chn} \( \mathcal{X} \) \), but it is not a torsion-free subcategory.

4. in \( \text{pch} \( \mathcal{X} \) \) we have a torsion theory \( (\mathcal{E}_n, \text{pch}(\mathcal{X})_{n-1} \geq) \):

\[
\begin{array}{ccc}
\mathcal{E}_n & \downarrow & \text{pch}(\mathcal{X}) \geq n \\
\circlearrowleft & & \circlearrowright \\
\text{sk}_{n-1} & & \text{cot}_{n-1}
\end{array}
\]

where

- \( \mathcal{F}_{\text{tr}_{n-1}} \) is the subcategory of chain complexes such that their component of the unit of \( \text{tr}_{n-1} \dashv \text{cosk}_{n-1} \) is monic.
- \( \mathcal{M}_n \) is the subcategory of proper chain complexes such that \( M_i = 0 \) for \( i > n \) and the differential \( \delta_n \) is monic.
- \( \mathcal{E}_n \) is the subcategory of proper chain complexes such that \( M_i = 0 \) for \( n-1 > i \) and the differential \( \delta_n \) is epic.

The short exact sequences of the torsion theories in 2) and 3) are given, for a proper chain complex \( M \), as in diagrams (3) and (4) in Theorem 2.5 respectively.

In addition to this torsion theories, we can add the following.

Theorem 5.3. Let \( \mathcal{X} \) be a semi-abelian category. For each \( n \in \mathbb{Z} \) the pair \( (\text{chn}(\mathcal{X})_{n-1} \geq, \text{chn}(\mathcal{X}) \geq n) \) is a hereditary cohereditary torsion theory in \( \text{chn} \( \mathcal{X} \) \). Moreover, the reflector and coreflector are given by \( \text{sk}_{n-1} \dashv \text{tr}_{n-1} \) and \( \text{tr}'_n \dashv \text{sk}'_n \):

\[
\begin{array}{ccc}
\text{sk}_{n-1} & \downarrow & \text{tr}_{n-1} \circlearrowright \\
\text{ch}(\mathcal{X})_{n-1} \geq & & \text{ch}(\mathcal{X}) \geq n \\
\circlearrowleft & & \circlearrowright \\
\text{tr}'_n & \downarrow & \text{sk}'_n
\end{array}
\]
Proof. For \( X \in \text{ch}(\mathcal{X})_{n-1} \geq \) and \( Y \in \text{ch}(\mathcal{X})_{n} \geq n \) it is clear that a morphism \( \text{sk}_{n-1}(X) \rightarrow \text{sk}'_{n}(Y) \) must be trivial:

\[
\text{sk}_{n-1}(X) = \quad \ldots \rightarrow 0 \rightarrow X_{n-1} \rightarrow X_{n-2} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\text{sk}'_{n}(Y) = \quad \ldots \rightarrow Y_{n+1} \rightarrow Y_{n} \rightarrow 0 \rightarrow 0 
\]

Since limits and colimits are computed component-wise in \( \text{ch}(\mathcal{X}) \), the short exact sequence of the torsion theory for a chain complex \( X \) in \( \text{ch}(\mathcal{X}) \) is given by:

\[
\ldots \rightarrow 0 \rightarrow 0 \rightarrow X_{n-1} \rightarrow X_{n-2} \rightarrow \ldots \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\ldots \rightarrow X_{n+1} \rightarrow X_{n} \rightarrow X_{n-1} \rightarrow X_{n-2} \rightarrow \ldots \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\ldots \rightarrow X_{n+1} \rightarrow X_{n} \rightarrow 0 \rightarrow 0 \rightarrow \ldots 
\]

\[\Box\]

Corollary 5.4. Let \( \mathcal{X} \) be semi-abelian category. For each \( n \in \mathbb{Z} \) the triplets of full subcategories

\[(\text{ch}(\mathcal{X})_{n-1} \geq, \text{ch}(\mathcal{X})_{n} \geq, \mathcal{F}_{\text{tr}_{n-1}})\]

are TTF theories in \( \text{ch}(\mathcal{X}) \). Moreover, by restriction these determine the TTF theories in \( \text{pch}(\mathcal{X}) \)

\[(\text{pch}(\mathcal{X})_{n-1} \geq, \text{pch}(\mathcal{X})_{n} \geq, \mathcal{M}_{\mathcal{N}}_{n})\]

Similarly, the triplet of subcategories in \( \text{ch}(\mathcal{X}) \)

\[(\text{Ker}(\text{cot}_{n-1} \geq), \text{ch}(\mathcal{X})_{n-1} \geq, \text{ch}(\mathcal{X})_{n} \geq)\]

determines the TTF theories in \( \text{pch}(\mathcal{X}) \)

\[(\mathcal{E}\mathcal{P}_{n}, \text{pch}(\mathcal{X})_{n-1} \geq, \text{pch}(\mathcal{X})_{n} \geq)\].

5.2 Weak TTF theories in chain complexes with operations

Through this work we have mention the similarities between torsion theories in simplicial groups and in chain complexes, this happens since they are both defined with similar set of adjunctions \((\text{sk}_{n} \dashv \text{tr}_{n} \dashv \text{cosk}_{n} \text{ and } \text{cot}_{n} \dashv \text{sk}_{n} \dashv \text{tr}_{n} \dashv \text{cosk}_{n})\). However, the TTF theories in chain complexes cannot be easily adapted to the simplicial case, not even in the initial example of internal groupoids. For instance, in \( \text{Grpd}(\text{Grp}) \) the subcategory of discrete groupoids \( \text{Dis}(\text{Grp}) \cong \mathcal{M}_{\mathcal{O}_{>}} \) is a torsion-free subcategory (with reflector \( \pi_{0} \dashv \text{Dis} \) and also mono-coreflective (with the adjunction \( \text{Dis} \dashv \text{tr}_{0} \)) but not normal mono-coreflective, so it is not a torsion subcategory. In general, the subcategories \( \mathcal{M}_{\mathcal{N}}_{n} \) are only torsion-free.

Two different kinds of torsion theories are introduced.

Definition 5.5. For a class of objects \( \mathcal{E} \) of \( \mathcal{X} \), a pair \((\mathcal{T}, \mathcal{F})\) of full subcategories of \( \mathcal{X} \) will be called a \( \mathcal{E} \)-torsion theory or a torsion theory relative to the class \( \mathcal{E} \) if:

TT1 for all \( X \in \mathcal{T} \) and \( Y \in \mathcal{F} \), every morphism \( f : X \rightarrow Y \) is zero;

TT2’ for every object \( X \in \mathcal{E} \) exists a short exact sequence

\[
0 \rightarrow T_{X} \xrightarrow{t_{X}} X \xrightarrow{f_{X}} F_{X} \rightarrow 0
\]

with \( T_{X} \in \mathcal{T} \) and \( F_{X} \in \mathcal{F} \).
As a first example of a $\mathcal{E}$-torsion theory we have the pair $(\text{Ker}(\cot_n), ch(X)_{n\geq})$ in $chn(X)$ as in Theorem 5.2.

**Lemma 5.6.** In $ch(X)$ the category of chain complexes the pair

$$(\text{Ker}(\cot_n), ch(X)_{n\geq})$$

and $\mathcal{E}$ the class of proper chain complexes $pch(X)$ is a $\mathcal{E}$-torsion theory in $ch(X)$.

**Proof.** The objects in $\text{Ker}(\cot_n)$ are the chain complexes $X$ such that $X_i = 0$ for $n > i$ and the differential $\delta_{n+1}$ has a trivial cokernel. Thus, to verify TT1 it suffices to notice that given a commutative diagram:

$$
\begin{array}{ccc}
X_{n+1} & \xrightarrow{\delta_{n+1}} & X_n \\
\downarrow{f_{n+1}} & & \downarrow{f_n} \\
0 & \xrightarrow{} & Y_n
\end{array}
$$

with $\delta_{n+1}$ a morphism with trivial cokernel then the morphism $f$ must be trivial. TT2' holds since it has been established that the restriction of the pair $(\text{Ker}(\cot_n), ch(X)_{n\geq})$ to proper chain gives a torsion theory $(\mathcal{E}P_n, pch(X)_{n\geq})$ in $pch(X)$. In addition, for a proper chain complex $M$ the short exact sequence is given by diagram (3) in Theorem 2.5. 

Any normal epireflective subcategory yield a $\mathcal{E}$-torsion theory. Let $F \dashv I : X \to A$ be a normal epireflective subcategory of $X$ with unit $\eta$. Following [H], we consider two subcategories of $X$:

$$T_F = \{ T \mid T \cong \ker(\eta_X) \text{ for some } X \}$$

and

$$Ker(F) = \{ X \mid F(X) = 0 \}.$$

Clearly, we have $Ker(F) \subseteq T_F$. The pair $(T_F, F)$ satisfies axiom TT2 of a torsion theory while the pair $(Ker(F), F)$ satisfy axiom TT1. Indeed, if $Ker(F) = T_F$ we have a torsion theory. Clearly, in the relative case we have the following:

**Lemma 5.7.** Let $F \dashv I : X \to A$ be a normal epireflective subcategory of $X$ with unit $\eta$. If $\mathcal{E} = \{ X \mid F(\ker(\eta_X)) = 0 \}$ then the pair $(Ker(F), F)$ is a $\mathcal{E}$-torsion theory.

**Example 5.8.** The category of $Ab$ of abelian groups is a normal epireflective subcategory of $Grp$ (in fact, it is a Birkhoff subcategory) where the reflector is the abelianization functor $ab(G) = G/G'$ where $G'$ is the commutator subgroup. Hence, if $Pr$ is the category of perfect groups, groups $G$ such that $G' = G$, then the pair $(Pr, Ab)$ is a $\mathcal{E}$-torsion theory with respect the class $\mathcal{E}$ of groups such that $(G')' = G'$ or equivalently, groups with a perfect commutator.

We introduce our main definition.

**Definition 5.9.** Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in $\mathcal{X}$, if $\mathcal{F}$ is a mono-coreflective category of $\mathcal{X}$ we will call $(\mathcal{T}, \mathcal{F})$ a $\text{CTF theory}$. This means, that the embedding $I$ of $\mathcal{F}$ into $\mathcal{X}$ has both a right and left adjoint, $F \dashv I \dashv C$:

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{J} & \mathcal{X} \\
\downarrow{T} & \text{ } & \downarrow{I} \\
\mathcal{F} & \xrightarrow{C} & \mathcal{X}
\end{array}
$$
Clearly, in a TTF theory \((\mathcal{C}, \mathcal{T}, \mathcal{F})\) the pair \((\mathcal{C}, \mathcal{T})\) is a CTF-theory. In [7], it is proved that a normal mono-coreflective subcategory closed under extension is in fact a torsion category, so a CTF theory with \(\mathcal{F}\) a normal-mono-coreflective subcategory is a TTF theory.

The subcategory of discrete crossed modules \(\text{Dis}\) in \(\mathbb{X}Mod\) behaves almost as a torsion torsion-free subcategory, it presents an example of a CTF theory as well as a relative \(\mathcal{E}\)-torsion theory. To this end, recall that a monomorphism in \(\mathbb{X}Mod\) is given by an injective crossed module morphism \(f = (f_1, f_0):\)

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow f_1 & & \downarrow f_0 \\
X & \longrightarrow & Y.
\end{array}
\]

In addition, \(f\) is a normal subcrossed module if and only if \(A, B\) are normal subgroups of \(X, Y\), and the conditions \(y(a) \in A\) and \(b(x)x^{-1} \in X\) hold for all \(a \in A, b \in B, y \in Y\) and \(x \in X\).

**Proposition 5.10.** In \(\mathbb{X}Mod\) consider the triplet of subcategories:

\((\text{CExt}, \text{Dis}, \text{Ab})\)

then:

1. the pair \((\text{CExt}, \text{Dis})\) is a CTF theory in \(\mathbb{X}Mod\);
2. the pair \((\text{Dis}, \text{Ab})\) is an \(\mathcal{E}\)-torsion theory where \(\mathcal{E}\) is the class of crossed modules \(\delta: A \rightarrow B\) where the action \(B \rightarrow \text{Aut}(A)\) is trivial.

**Proof.**

1) The discrete functor \(D\) has right adjoint \((\_)_0\) where the component of the counit for a crossed module \(\delta: A \rightarrow B\) is given by horizontal arrows in the diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow & & \downarrow \delta \\
B & \longrightarrow & B \\
\end{array}
\]

which is a monomorphism since the pair \((0, 1)\) are injective morphisms.

2) It is clear that the pair \((\text{Dis}, \text{Ab})\) satisfies TT1 since in a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow f_1 & & \downarrow f_0 \\
G & \longrightarrow & 0
\end{array}
\]

the morphism \(f = (f_1, f_0)\) is zero. For TT2', recall that the unit Diagram (6) is a normal monomorphism in \(\mathbb{X}Mod\) if and only if \(b(a)a^{-1} = 0\), i.e., the action of \(B\) over \(A\) is trivial. From the Peiffer identity \(\delta(a)(a') = aa'a^{-1}\), a crossed module with trivial action also has \(A\) as an abelian group then we have the short exact sequence in \(\mathbb{X}Mod\):

\[
\begin{array}{ccc}
0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow \delta & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & B & \longrightarrow & B & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}
\]
Remark 5.11. The pair of subcategories $\text{Dis}$ and $\text{Ab}$ of $\text{XMod}$ present another example of an $\mathcal{E}$-torsion theory in $\text{XMod}$. This time as $(\text{Ab}, \text{Dis})$ and $\mathcal{E}$ as the subcategory $\text{Mod}$ of modules of groups.

For a module we mean a pair $(A, G)$ such that $G$ is a group and $A$ is an abelian groups with a group action $G \to \text{Aut}(A)$. Then a module $(A, G)$ is a crossed modules as $\delta = 0 : A \to G$. In fact, the subcategory $\text{Mod}$ is a Birkhoff subcategory of $\text{XMod}$. The associated short exact sequence of the $\mathcal{E}$-torsion theory for a module $(A, G)$ is

$$
0 \longrightarrow A \longrightarrow A \longrightarrow 0 \longrightarrow 0 \\
0 \longrightarrow 0 \longrightarrow G \longrightarrow G \longrightarrow 0.
$$

The category of reduced crossed complexes present a similar example as Proposition 5.10.

5.12. In $\text{Crs}(\text{Grp})$ for each $n \geq 0$ consider the full subcategory $\text{Crs}(\text{Grp})_{\geq n}$ of reduced crossed complexes $M$ who are trivial in degrees below $n$:

$$M = \ldots \longrightarrow M_{n+1} \longrightarrow M_n \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots ;$$

for all $n > 0$ the category $\text{Crs}(\text{Grp})_{\geq n}$ is equivalent to the category $\text{ch}(\text{Ab})_{\geq n} \cong \text{ch}(\text{Ab})$ of chain complexes in abelian groups.

Thus, for $n \geq 2$ we have a functor $\text{tr}'_n : \text{Crs}(\text{Grp}) \to \text{Crs}(\text{Grp})_{\geq n}$ defined for a crossed complex $M$ by

$$\text{tr}'_n(M) = \ldots \longrightarrow M_{n+1} \longrightarrow M_n \longrightarrow 0 \longrightarrow \ldots .$$

The natural chain complex morphism $f : M \to \text{tr}'_n(M)$:

$$
\ldots \longrightarrow M_{n+1} \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \ldots \longrightarrow M_1 \longrightarrow M_0 \\
\downarrow 1 \downarrow 1 \downarrow 0 \downarrow 0 \downarrow 0 \\
\ldots \longrightarrow M_{n+1} \longrightarrow M_n \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \longrightarrow 0
$$

is a morphism in $\text{Crs}(\text{Grp})$ if and only if all the actions $M_0 \to \text{Aut}(M_i)$ are trivial for $i \geq n$. Indeed, $M$ should satisfy $m_0m_n = f_n(m_0m_n) = f_0(m_0) = m_0$ and $m_n = m_n$ for all $m_0 \in M_0$ and $m_n \in M_n$. In particular, this condition holds if $\delta_1 : M_1 \to M_0$ is a central extension, since in a crossed complex the restrictions of the actions $\delta_1(M_1) \to \text{Aut}(M_0)$ are trivial.

Proposition 5.13. For $n \geq 2$, consider the triplet of subcategories:

$$(\text{Ker}(\text{cot}_{n-1}), \text{Crs}(\text{Grp})_{n-1\geq}, \text{Crs}(\text{Grp})_{\geq n}) .$$

in $\text{Crs}(\text{Grp})$. Then:

1. the pair $(\text{Ker}(\text{cot}_{n-1}), \text{Crs}(\text{Grp})_{n-1\geq})$ is CTF theory, i.e., the subcategory $\text{Crs}(\text{Grp})_{n-1\geq}$ is mono-coreflective;

2. the pair $(\text{Crs}(\text{Grp})_{n-1\geq}, \text{Crs}(\text{Grp})_{\geq n})$ is an $\mathcal{E}$-torsion theory where $\mathcal{E}$ is the class of crossed complexes $M$ with all actions $M_0 \to \text{Aut}(M_i)$ trivial for $i \geq n$;

3. if $M$ is a crossed complex with $\delta_1 : M_1 \to M_0$ a crossed module central extension then $M$ belongs to $\mathcal{E}$.

In particular, for $n = 2$ this holds for the triplet:

$$(\text{Ker}(\text{cot}_1), \text{XMod}, \text{chn}(\text{Ab})_{\geq 2}) .$$

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Proof. 1) From Proposition 4.4. \( \mu_{n-1} = (\text{Ker}(\cot_{n-1}), \text{Crs}(\text{Grp})_{n-1}) \) is a torsion theory. It suffices to notice that the counit of \( \text{sk}_{n-1} \dashv \text{tr}_{n-1} \) given by

\[
\begin{array}{ccccccc}
\ldots & \rightarrow & 0 & \rightarrow & M_{n-1} & \rightarrow & M_{n-2} & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \rightarrow & M_n & \rightarrow & M_{n-1} & \rightarrow & M_{n-2} & \rightarrow & \ldots
\end{array}
\]

is monic since each component is an injective morphism.

2) It is clear that the pair \( (\text{Crs}(\text{Grp})_{n-1}, \text{Crs}(\text{Grp})_{\geq n}) \) satisfies TT1 of the definition of a \( \mathcal{E} \)-torsion theory. Now, let \( M \) be a crossed complex with trivial actions \( M_0 \rightarrow \text{Aut}(M_i) \) and consider the morphisms \( \text{tr}_{n-1}(M) \rightarrow M \rightarrow \text{tr}'_n(M) \) in \( \text{Crs}(\text{Grp}) \):

\[
\begin{array}{ccccccc}
\ldots & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & M_{n-1} & \rightarrow & M_{n-2} & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \rightarrow & M_{n+1} & \rightarrow & M_n & \rightarrow & M_{n-1} & \rightarrow & M_{n-2} & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \rightarrow & M_{n+1} & \rightarrow & M_n & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \ldots
\end{array}
\]

recall that the morphism \( M \rightarrow \text{tr}'_n(M) \) is indeed a morphism in \( \text{Crs}(\text{Grp}) \) since the actions are trivial. It is a short exact sequence in \( \text{Crs}(\text{Grp}) \) since it is a short exact sequence as chain complexes and the forgetful functor is conservative.

3) It follows from the definition of crossed complex that if \( \delta_1 \) is surjective the actions \( \delta_1(M_i) = M_0 \rightarrow \text{Aut}(M_i) \) are trivial.

\[\square\]

5.3 A semi-abelian splitting CTF theory

In an abelian category a torsion theory \( (T, F) \) is called splitting if the torsion subobject \( t(X) \) of \( X \) is a direct summand. In a semi-abelian category we will call a torsion theory \( (T, F) \) splitting if for every object \( X \) the associated exact sequence splits:

\[
0 \rightarrow t(X) \rightarrow X \xleftarrow{\pi} X/t(X) \rightarrow 0
\]

In \( R\text{Mod} \) the category of modules over the ring \( R \), a central idempotent element of \( R \) induces a splitting torsion theory \( (T, F) \) (also called centrally splitting), and even yields a TTF theory \( (F, T, F) \). Connections of splitting torsion theories and TTF theories are studied in [16].

Example 5.14. Let \( K\text{Hopf}_{\text{coc}} \) the category of cocommutative Hopf algebras over the field \( K \) of characteristic 0. In [12], the category \( K\text{Hopf}_{\text{coc}} \) is proved that it is a semi-abelian category and have a torsion theory \( (K\text{Lie}, \text{Grp}) \) where the \( K \)-Lie algebras are consider as the primitive Hopf algebras and \( \text{Grp} \) is consider as the category of group Hopf algebras. Indeed, the associated short exact sequence is given by the Cartier-Gabriel-Moore-Milnor-Kostant theorem: for every \( K \)-algebra \( H \) there is a split short exact sequence

\[
0 \rightarrow U(L_H) \rightarrow H \xleftarrow{\pi} K[G_H] \rightarrow 0
\]

where \( K[G_H] \) is the group algebra of the group-like elements \( G_H \) of \( H \) and \( U(L_H) \) is the enveloping algebra of the primitive elements \( L_H \) of \( H \).

In [13], it is proved that the functor \( \mathcal{G} : K\text{Hopf}_{\text{coc}} \rightarrow \text{Grp} \) that takes the group-like elements and \( K[\_] : \text{Grp} \rightarrow K\text{Hopf}_{\text{coc}} \) yield the adjunctions \( \mathcal{G} \dashv K[\_] \) and \( K[\_] \dashv \mathcal{G} \). So finally, \( (K\text{Lie}, \text{Grp}) \) is a splitting CTF theory.
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