Euclidean and chemical distances in ellipses percolation

Marcelo Hilário\textsuperscript{1} and Daniel Ungaretti\textsuperscript{2}

\textsuperscript{1,2}Universidade Federal de Minas Gerais
\textsuperscript{2}Universidade de São Paulo

March 18, 2021

Abstract

The ellipses model is a continuum percolation process in which ellipses with random orientation and eccentricity are placed in the plane according to a Poisson point process. A parameter $\alpha$ controls the tail distribution of the major axis’ distribution and we focus on the regime $\alpha \in (1, 2)$ for which there exists a unique infinite cluster of ellipses and this cluster fulfills the so called highway property. We prove that the distance within this infinite cluster behaves asymptotically like the (unrestricted) Euclidean distance in the plane. We also show that the chemical distance between points $x$ and $y$ behaves roughly as $c \log \log |x - y|$.

1 Introduction

In this paper we study both the chemical and Euclidean distances in the ellipses model introduced in [16]. It is a Boolean percolation in the plane with defects given by random ellipses centered at points given by a Poisson point process with intensity
$u > 0$. Given the position of the centers, the eccentricities and orientations of the ellipses are independent. The minor axes always have length one and they make uniformly distributed angles with the horizontal direction. The lengths of the major axes are drawn independently from a heavy-tailed distribution $\rho$ supported on $[1, \infty)$ that satisfies $\rho[r, \infty) = cr^{-\alpha}$ for $r \geq 1$. Therefore, while the parameter $u$ controls the amount of ellipses appearing in the picture, the parameter $\alpha$ controls how eccentric they are.

In [16], phase transition and connectivity properties for the ellipses model were studied as functions of these two parameters. Here we will focus on $\alpha \in (1, 2)$, the regime in which, for any choice of $u > 0$, there exists a unique infinite cluster of ellipses that, in addition, satisfies what we refer to as the highway property. Roughly, it means that after scaling the probability of connecting two regions using a single ellipse becomes close to one.

Let $\mathcal{D}(x, y)$ denote the minimum length of a polygonal path from $x$ to $y$ which lies entirely inside the set covered by the ellipses. We call $\mathcal{D}(\cdot, \cdot)$ the Euclidean distance restricted to the set of ellipses or sometimes the internal distance. Also, for any two points $x$ and $y$ in the infinite cluster of ellipses, denote by $D(x, y)$ the chemical distance between them, i.e. the minimum number of ellipses that a continuous path from $x$ to $y$ contained entirely inside the cluster of ellipses has to intersect. The Euclidean distance in the plane, sometimes called the unrestricted Euclidean distance, is denoted by $|x - y|$.

Let us now state our main results

**Theorem 1** (Euclidean distance). Consider the ellipses model with parameters $u > 0$ and $\alpha \in (1, 2)$. For $x, y \in \mathbb{R}^2$ and $\delta > 0$,

$$\lim_{|x-y| \to \infty} \mathbb{P}\left(1 \leq \frac{\mathcal{D}(x, y)}{|x-y|} \leq 1 + \left| x - y \right|^{\frac{2-4}{2+\delta}} \mid x \leftrightarrow y \right) = 1.$$  \hspace{1cm} (1)

**Theorem 2** (Chemical distance). Consider the ellipses model with parameters $u > 0$ and $\alpha \in (1, 2)$. For $x, y \in \mathbb{R}^2$ and $\delta > 0$,

$$\lim_{|x-y| \to \infty} \mathbb{P}\left(\frac{1 - \delta}{\log(\frac{2}{\alpha} - 1)} \leq \frac{D(x, y)}{\log \log |x-y|} \leq \frac{2 + \delta}{\log(\frac{2}{\alpha})} \mid x \leftrightarrow y \right) = 1.$$  \hspace{1cm} (2)
To understand geometric properties of infinite clusters is a problem of major interest in percolation theory. Models for which the chemical distance was studied include Bernoulli percolation and first-passage percolation [1, 11, 12]; random interlacements [6]; random walk loop soup [7]; and Gaussian free field [8, 9]. General conditions for a percolation model on $\mathbb{Z}^d$ to have a unique infinite cluster in which Euclidean and chemical distances are comparable are provided in [9]. Theorems 1 and 2 show that ellipses model does not fit into the conditions of [9]. This is due to the presence of long ellipses. A similar behavior can be observed in Poisson cylinder model [18] and long-range percolation [2, 15], as we discuss next.

1.1 Comparing with long-range models

Ellipses model is closely related to other two percolation models that allow for arbitrarily long connections: Poisson cylinders model on $\mathbb{R}^d$ and long-range percolation on $\mathbb{Z}^d$. In principle one could try to leverage these relations in order to obtain estimates for the distances in ellipses model, and indeed some of our results are obtained this way.

We emphasize that the highway property is shared by these three models, with the immediate adaptations that connection of far away regions is accomplished using a single cylinder or a single open edge for Poisson cylinders and long-range model, respectively. The highway property is the main tool to ensure that, in all three models, the distance inside the infinite cluster is asymptotically equivalent to the unrestricted Euclidean distance in the plane.

However, the behavior of the chemical distance differs completely in each of these models. Before elaborating on these differences, we give a quick introduction to Poisson cylinder model and long-range percolation.

Poisson Cylinders. Poisson cylinders model consists of a random collection of bi-infinite cylinders of radius one whose axes are given by a Poisson point process on the space of all the lines (i.e. affine one-dimensional subspaces) in $\mathbb{R}^d$ with $d \geq 3$, see [18] for details. Distances within clusters of cylinders were studied in [5] and [14]. In [5] the authors prove that almost surely any two cylinders are linked by
a sequence composed of at most $d - 2$ other intersecting cylinders, implying that the chemical distance is bounded. For the Euclidean distance on the other hand, in [14] the authors prove a shape theorem showing that if $x, y \in \mathbb{R}^d$ are points in the infinite cluster then the internal distance between $x$ and $y$ is asymptotically $|x - y| + O(|x - y|^{1/2 + \varepsilon})$, for any $\varepsilon > 0$.

One straightforward connection between Poisson cylinders and ellipses model is to study the intersection of the random cylinders with any given 2-dimensional plane. As shown in [19] this intersection is a collection of ellipses whose law is an instance of the ellipses model with $\alpha = 2$ when $d = 3$ and, with $\alpha > 2$ in higher dimensions. Thus, this natural coupling between ellipses model and Poisson cylinder model is not helpful to draw conclusions when $\alpha$ ranges in $(1, 2)$.

**Long-range percolation.** Fix $\beta, s > 0$ and consider the bond percolation model, known as long-range percolation, in which for each $x \neq y \in \mathbb{Z}^d$ an open edge connects $x$ and $y$ with probability

$$p_{xy} = 1 - \exp\left(-\beta|x - y|^{-s}\right).$$

Different expressions for $p_{xy}$ may be considered but it is usually assumed that it decays roughly as $\beta|x - y|^{-s + o(1)}$ for some positive $\beta$ and $s$.

Let us now explain how long-range percolation and ellipses model relate to each other. Notice that both models have one parameter that controls the density ($u$ and $\beta$, respectively) and another that controls the distribution of long connections ($\alpha$ and $s$, respectively). Essentially, a discretization of ellipses model leads to a long-range percolation with parameters satisfying the following relations

$$s = 2 + \alpha \quad \text{and} \quad u = \frac{\pi \beta}{\alpha 2^\alpha}. \quad (3)$$

The coupling is given as follows. Take $B := [-1/2, 1/2)^2$ and for $x \in \mathbb{R}^2$ write $B_x = x + B$ so that $(B_x)_{x \in \mathbb{Z}^2}$ forms a tiling of $\mathbb{R}^2$. For a realization of the ellipses model with parameters $u$ and $\alpha$ associate with every ellipse the two extremities of its major axis. Now embed $\mathbb{Z}^2$ in $\mathbb{R}^2$ in the natural way and define two sites $x \neq y \in \mathbb{Z}^2$ to be $\xi$-connected and write $x \sim_\xi y$ if there is an ellipse whose major axis has
one extremity in $B_x$ and the other in $B_y$. Inserting open edges between pairs of \( \xi \)-connected sites leads to a long-range percolation model whose parameters \( s \) and \( \beta \) satisfy (3), as we show in Section 2.

We will explore this coupling to translate results about long-range percolation to results about ellipses model. However, there are some key points that must be dealt with when comparing connectivity in these models using the coupling described above.

A first issue is that, in some situations, connectivity is favored in ellipses model. In fact, when two long ellipses cross each other it may occur that the resulting open edges in the long-range model belong to different components. This suggests that connectivity properties in these two models may differ. Indeed in [16, Theorem 1.2] it is shown that in ellipses model with \( \alpha \in (1, 2) \) the covered set percolates for any intensity \( u > 0 \). The corresponding long-range percolation (with \( s \in (3, 4) \) by (3)) does not percolate for sufficiently small \( \beta \), since when \( \sum_{z \in \mathbb{Z}^2} p_{0z} < 1 \) the open cluster of the origin is dominated by a subcritical Galton-Watson tree.

A second issue affects connectivity in the opposite direction. Notice that having \( x \sim \xi y \sim \xi z \) does not ensure that, in the underlying ellipses model, the corresponding ellipses overlap since it may occur that two ellipses intersect the box $B_y$ without touching each other, see Figure 1.

![Figure 1: Possible problems when coupling ellipses model and long-range. On the left, ellipses that do not intersect lead to a single connected component of $\xi$-edges; on the right, ellipses that do intersect lead to disjoint components of $\xi$-edges.](image)

We now present the results about long-range percolation that we use. We refer the reader to [2, Section 1.3] for a summary on the chemical distance for different regimes of \( s \). We will be mainly interested in the case \( d = 2 \) and \( s \in (3, 4) \), which corresponds to ellipses model with \( \alpha \in (1, 2) \). Results for any \( d \geq 2 \) and \( s \in (d, 2d) \) are discussed in papers [2–4].
Our estimate for Euclidean distance in Theorem 1 builds on a construction from [2] that relies on the above mentioned highway property. Let us exemplify this property for the long-range model with \( d \geq 2 \) and \( s \in (d, 2d) \). Denote \(|x - y| = N\) and for \( \gamma \in (s/2d, 1)\) consider \( B = \mathbb{Z}^d \cap [-N\gamma/2, N\gamma/2]^d\). Then, the probability of the event \( \{ B_x \leftrightarrow B_y \} \) that there is an open edge connecting a site in \( B_x = x + B \) to another site in \( B_y = y + B \) can be estimated using

\[
\mathbb{P}(\{ B_x \leftrightarrow_1 B_y \}^c) = \prod_{x' \in B_x, y' \in B_y} (1 - p_{x'y'}) = \exp \left[ -\frac{\beta |B_x||B_y|}{(N + O(N^\gamma))^s} \right] \sim e^{-\beta N^{2d\gamma - s}} \quad (4)
\]

as \( N \to \infty \). The estimate in (4) is in the core of the hierarchical construction from [2] which leads to the main result therein: the chemical distance between two points \( x, y \) on the infinite cluster behaves asymptotically as

\[
D(x, y) = (\log |x - y|)^{\Delta + o(1)} \quad \text{as } |x - y| \to \infty,
\]

where \( \Delta = \frac{\log 2}{\log(2d/s)} \).

This same argument shows that \( D(x, y) \sim |x - y| \), although not mentioned in [2]. We present (a simplified version of) their hierarchical construction in Section 3, and use it as a fundamental tool for controlling the Euclidean distance traversed by a path in ellipses model.

As we have seen above, in all three models the Euclidean distance restricted to the covered set and the unrestricted distance are asymptotically the same. The chemical distance can be seen as an alternative measure of connectedness for these models and through this lens they behave very differently, presenting different orders of magnitude. For Poisson cylinders the chemical distance is bounded by a constant, for ellipses model it grows as \( \log \log |x - y| \), and for the long-range model it grows as \( (\log |x - y|)^\Delta \). We will see that this discrepancy between ellipses model and long-range percolation may be explained as a consequence of the first issue above.
1.2 Idea of proofs

We need to obtain lower and upper bounds on the distance between points $x$ and $y$ that belong to the same cluster of ellipses.

Our estimates for the lower bounds are simpler to obtain. The lower bound for the internal distance appearing in (1) is simply the unrestricted distance $|x - y|$ in the plane, and although obvious, we do not have any improvement for it. The lower bound for the chemical distance appearing in (2), follows from an elementary induction argument. One could try to improve the bounds using the BK inequality like in the lower bound in [3] but our argument seems to provide the correct order of magnitude in a simpler way.

The proofs for the upper bounds appearing in both Theorems 1 and 2 follow similar strategies. The first step is to show that, with high probability, there exists a set of few overlapping ellipses that allows us to traverse from a local region containing $x$ to another local region containing $y$ without deviating too much. The second step consists of connecting locally the points $x$ and $y$ to this structure. This is the content of a deterministic construction in Lemma 7.

Let us discuss some details of each proof. The proof of Theorem 1 is based on a coupling of ellipses model and site-bond long-range percolation model on a renormalized lattice. The probability of an edge being open will be given by the coupling with long-range percolation described above. Only a subset of the underlying Poisson point process defining the ellipses model is used for this coupling. The remaining (independent) part is used for defining a site percolation model on a lattice of renormalized sites that correspond to boxes in the original lattice. Roughly, a site is considered open (or good) if the corresponding box is good meaning that the cluster of ellipses near this box is sufficiently well-connected. This definition is based on an idea from [1].

On the event that $x \leftrightarrow y$ in ellipses model, the bond percolation part and the site percolation part are then combined to create a short path connecting $x$ to $y$. This is done in two steps:

Hierarchical construction. This is essentially the construction from [2] based on
the highway property (4). When $|x-y| = N$ is large, with very high probability, there is an open edge connecting small neighborhoods around $x$ and $y$. This idea can be iterated to build what we call a hierarchy, see Definition 1 and Figure 2. In words, a hierarchy is a collection of long edges (or highways) that essentially connects $x$ to $y$, leaving only some gaps that are much shorter than the highways.

**Gluing procedure.** Given that we have found a hierarchy the original problem is then replaced by the problem of building connections across the remaining gaps. For that, we use the site percolation part of our coupling. The definition of good boxes will ensure that neighboring good boxes have intersecting clusters of ellipses. Moreover, the renormalization scheme is performed so that the probability of a box being good is highly supercritical. Therefore, even when a gap that we want to cross has some bad boxes around it, we can still contour these bad boxes by paying a low price in terms of distance and probability. This is accomplished through a large deviation bound on the size of bad clusters, see Section 3.2.

After these two steps are completed, we have with high probability a path of ellipses connecting $x$ and $y$ whose length is well-controlled. This establishes the upper bound in Theorem 1.

The reader who is familiar with the hierarchical construction of [2] and the renormalization procedure of [1] may notice that, in our proof of Theorem 1, we define events that are much simpler than the ones appearing in the original constructions. This is possible due to the existence of long overlapping ellipses that overlap, a phenomenon with no counterpart in long-range or Bernoulli percolation.

The proof for the upper bound for the chemical distance in Theorem 2 does not rely on the same coupling with long-range percolation as in Theorem 1 since this coupling does not explore the possibility of using long ellipses to its full potential. Instead, our argument involves choosing a rapidly increasing sequence of rectangles and studying the event that they are crossed in the hardest direction by a single ellipse. By a Borel-Cantelli argument, this construction provides 'enhanced highways' that cross large distances more efficiently.
Remarks on the notation. Throughout the paper we use $c, C$ to denote generic positive constants that can change from line to line. Numbered constants $c_0, c_1, c_2$, are kept fixed. Also, our asymptotic notation uses

- both $f = o(g)$ and $f \ll g$ to denote $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$;
- $f = O(g)$ to denote $|f| \leq C|g|$ for some constant $C$;
- $f = \Theta(g)$ to denote $c|g| \leq |f| \leq C|g|$;
- $f \sim g$ to denote $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$.

2 Couplings, highways and hierarchies

In this Section we collect some results from the literature that will be used in the proofs of Theorems 1 and 2.

Ellipses model. Ellipses model is defined via a Poisson point process (PPP) on $\mathbb{R}^2 \times \mathbb{R}^+ \times (-\pi/2, \pi/2]$ with intensity measure

$$u \cdot dz \otimes \alpha R^{-(1+\alpha)} dR \otimes \frac{1}{\pi} dV,$$

(5)

For each point $(z, R, V)$ in the PPP, place an ellipse centered at $z$ whose minor axis has length 1 and whose major axis has length $R$ and forms an angle $V$ with respect to the horizontal direction. The multiplicative parameter $u > 0$ controls the density of ellipse whereas the exponent $\alpha > 0$ controls the tail of major axis’ distribution. We refer the reader to [16] for an account on the phase transition for percolation on the covered set with respect to parameters $u$ and $\alpha$.

Define the event $LR_1(l; k)$ that an ellipse crosses the box $[0, l] \times [0, kl]$ from left to right. The next lemma uncovers the range of parameters in which ellipses model presents the highway property:

Lemma 1 (Proposition 5.1 of [16]). Let $\alpha > 1$. There is a constant $c_0 = c_0(\alpha) > 0$ such that for every $k, l > 0$ with $lk > 2$

$$1 - e^{-c_0^{-1}u(k\wedge k^{-\alpha})^{2-\alpha}} \leq \mathbb{P}(LR_1(l; k)) \leq 1 - e^{-cu(k^{2-\alpha}v^k)^{2-\alpha}}.$$

(6)
Therefore, when \( \alpha \in (1, 2) \) and \( k \) is fixed, we have \( \mathbb{P}(LR_1(l;k)) \to 1 \) as \( l \to \infty \), showing that the highway property holds in this range of \( \alpha \).

A second useful estimate is a similar bound for the probability that there is an ellipse that traverses an annulus. Let \( B(l) \) denote the Euclidean ball of radius \( l \) centered at the origin in \( \mathbb{R}^2 \) and denote its boundary by \( \partial B(l) \). For two disjoint regions \( A_1 \) and \( A_2 \) write \( A_1 \leftrightarrow_1 A_2 \) if there is one ellipse that intersects both \( A_1 \) and \( A_2 \). We have:

**Lemma 2.** Let \( \alpha > 1 \). There is a constant \( c_i = c_i(\alpha) > 0 \) such that for every \( l_1, l_2 \) with \( l_2 - l_1 \geq 2 \) and \( l_1 \geq 1 \) we have

\[
\mathbb{P}(B(l_1) \leftrightarrow_1 \partial B(l_2)) \leq 1 - \exp\left[-uc_i l_1 \cdot (l_2 - l_1)^{1-\alpha}\right].
\]

(7)

**Proof.** See [16, Lemma 6.1]. The estimate for \( \mu(\Gamma_{12}) \) implies (7). \( \Box \)

**Coupling long-range with continuous model.** There is a canonical coupling mentioned in [15] and used in [4] between long-range percolation model and a Poisson Point Process \( \xi \) on \( \mathbb{R}^d \times \mathbb{R}^d \) with intensity measure

\[
\mu_{\beta,s} := \frac{\beta}{|x-y|^s} \, dx \, dy.
\]

(8)

We may interpret each point \( (x, y) \in \xi \) as giving rise to a segment connecting \( x \) and \( y \). This coupling is useful to make the renormalization scaling more transparent. In fact, for \( a > 0 \) if \( \xi' := \{(ax, ay); (x, y) \in \xi\} \) then the intensities of \( \xi \) and \( \xi' \) are related by

\[
\mu_{\beta}(dx' \, dy') = \frac{\beta}{|x'-y'|^s} \, dx' \, dy' = \frac{\beta}{a^s|x-y|^s} \, a^d \, dx \, a^d \, dy = \mu_{\beta a^{2s-d}}(dx \, dy).
\]

(9)

This scaling property is behind the highway property in case \( s \in (d, 2d) \), since the intensity appearing on the right-hand side tends to infinity as \( a \) grows. Also notice that when \( s = 2d \) the model is scale-invariant and there is no hope that a similar property is satisfied in that case.

For disjoint regions \( A_1 \) and \( A_2 \) we write \( A_1 \sim_\xi A_2 \) and say that \( A_1 \) and \( A_2 \) are connected if there is \( (\hat{x}, \hat{y}) \in (A_1 \times A_2) \cap \xi \). Take \( B := [-1/2, 1/2]^d \) and for \( x \in \mathbb{Z}^d \)}
consider $B_x = x + B$. We say that two sites $x \neq y \in \mathbb{Z}^d$ are $\xi$-connected if $B_x \sim_\xi B_y$ and denote this event by $x \sim_\xi y$.

Lemma 3 below yields estimates on the probability of connecting two distant boxes and shows that this coupling indeed produces a long-range percolation model.

**Lemma 3** (Connecting boxes). Let $B(l) = [-l/2, l/2]^d$ and $z \in \mathbb{R}^d$ and $B_z(l) = z + B(l)$. We have that $\mathbb{P}(B(l) \sim_\xi B_z(l)) = 1$ if and only if $B(l) \cap B_z(l) \neq \emptyset$. Moreover, we have

$$\mathbb{P}(B(l) \sim_\xi B_z(l)) \sim \beta l^{2d} |z|^{-s} \text{ as } z \to \infty. \quad (10)$$

**Proof.** We begin by noticing that for $x \in B(l)$ and $y \in B_z(l)$ we have that

$$|x - y| \geq |-z + x + (z - y)|_\infty \geq |z|_\infty - l.$$  

Thus, when $|z|_\infty > l$ we can write

$$\mathbb{P}(B(l) \sim_\xi B_z(l)) = 1 - \exp \left[ -\beta \int_{B(l) \times B_z(l)} |x - y|^{-s} \, dx \, dy \right]$$

$$\leq 1 - \exp \left[ -\beta (|z|_\infty - l)^{-s} \cdot l^{2d} \right] < 1$$

and as $|z| \to \infty$ we have $|x - y|^{-s} = |z|^{-s} (1 + O(|z|^{-1}))$, implying

$$\mathbb{P}(B(l) \sim_\xi B_z(l)) = 1 - \exp \left[ -|z|^{-s} (\beta + O(|z|^{-1})) \cdot l^{2d} \right] \sim \beta l^{2d} |z|^{-s}$$

with implied constants depending on $d$, $s$ and $l$. Also, when $|z|_\infty = l$ one can verify that $\mathbb{P}(B(l) \sim_\xi B_z(l)) = 1$. The fact that boxes $B(l)$ and $B_z(l)$ must share at least a corner will imply the integral diverges for $s \in (d, 2d)$.

Also, if we restrict our intensity measure to only allow for segments whose lengths are larger than some fixed value, say $\mu_{\beta, s} := \beta |x - y|^{-s} 1_{|x - y| > \kappa} \, dx \, dy$ we get a model in which nearest neighbors are no longer connected with probability 1, but that has the same behavior on long edges.

**Change of variables and ellipses model.** Now, let us restrict ourselves to the
case \( d = 2 \). Here we use a change of variables to verify that the PPP's with intensity measures (8) and (5) may be viewed as reparametrizations of each other.

Instead of parametrizing a line segment in \( \mathbb{R}^2 \) specifying its endpoints \( x \) and \( y \), we can use its middle point \( z = (z_1, z_2) \), its radius \( R \) and the angle if forms with a given direction, \( V \). This change of variables is given by \( \Psi : \mathbb{R}^2 \times \mathbb{R}^+ \times \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right) \rightarrow \mathbb{R}^4 \) such that

\[
\Psi(z_1, z_2, R, V) = (z_1 + R \cos V, z_2 + R \sin V, z_1 - R \cos V, z_2 - R \sin V).
\]

It is straightforward to check that the Jacobian matrix \( J \) satisfies

\[
J = \begin{bmatrix}
1 & 0 & \cos V & -R \sin V \\
0 & 1 & \sin V & R \cos V \\
1 & 0 & -\cos V & R \sin V \\
0 & 1 & -\cos V & -R \cos V
\end{bmatrix}
\]

and \( \det J = 4R \).

Therefore, for any measurable \( A \subset \mathbb{R}^4 \)

\[
\int_A \frac{\beta}{|x - y|^s} \, dx \, dy = \int_{\Psi^{-1}(A)} \frac{\beta}{(2R)^s} \cdot (4R) \, dz \, dR \, dV = \int_{\Psi^{-1}(A)} \frac{4\beta}{2^s} \cdot R^{1-s} \, dz \, dR \, dV.
\]

The usual parametrization of ellipses percolation is based on measure

\[
u \, dz \otimes \alpha R^{-(1+\alpha)} \, dR \otimes \frac{dV}{\pi}.
\]

Comparing these measures, we obtain that we can relate parameters \( \beta, s \) used in the endpoint parametrization with the \( u, \alpha \) parametrization of ellipses model, which leads to the relations in (3):

\[
s = 2 + \alpha \quad \text{and} \quad u = \frac{\pi \beta}{\alpha 2^{\alpha}}.
\]

Using relation (3) we see that Lemma 3 can also be used to estimate connection probabilities on ellipses model.
Hierarchical construction. Consider long-range percolation on $\mathbb{Z}^d$ with parameters $\beta$ and $s$. The highway property in (4) ensures that, for fixed $\gamma \in (s/2d, 1)$ and $|x - y| =: N$ large, there is an open edge connecting points in neighborhoods of size $N^\gamma$ around $x$ and $y$ with high probability. This idea can be iterated to build what we call a hierarchy, see Definition 1 and Figure 2. When a hierarchy exists the problem of finding a path from $x$ to $y$ can be replaced by finding paths between well-separated pairs of points that are however much closer than the original pair $(x, y)$. This construction, introduced by [2], is reproduced below.

We use $\sigma \in \{0, 1\}^k$ to encode the leaves of a binary tree of depth $k$, by considering that $\varnothing$ is the root vertex, and 0 and 1 denote the left and right children of $\varnothing$, respectively. We append digits to the right of a word $\sigma \in \{0, 1\}^k$ in order to create longer words, e.g., $\sigma 1 \in \{0, 1\}^{k+1}$ is the word that encodes the right child of $\sigma$.

**Definition 1 (Hierarchy).** For $n \geq 1$ and $x, y \in \mathbb{Z}^d$ we say that a collection

$$\mathcal{H}_n(x, y) = \{(z_\sigma); \sigma \in \{0, 1\}^k, 1 \leq k \leq n, z_\sigma \in \mathbb{Z}^d\}$$

(11)

is a hierarchy of depth $n$ if

1. $z_0 = x$ and $z_1 = y$.
2. For all $0 \leq k \leq n - 2$ and all $\sigma \in \{0, 1\}^k$, $z_{\sigma 00} = z_{\sigma 0}$ and $z_{\sigma 11} = z_{\sigma 1}$.
3. For all $0 \leq k \leq n - 2$ and all $\sigma \in \{0, 1\}^k$ with $z_{\sigma 01} \neq z_{\sigma 10}$ the edge $(z_{\sigma 01}, z_{\sigma 10})$ is open.
4. Each edge $(z_{\sigma 01}, z_{\sigma 10})$ as in 3. appears exactly once in $\mathcal{H}_k(x, y)$.

Note that the definition of a hierarchy does not take into account the distances between the points $z_\sigma$.

It will be useful to think of hierarchies as being constructed successively. In view of the computation in (4), in the first step we may try to link a pair of sites $z_{01}$ and $z_{10}$ that belong to neighborhoods of size roughly $N^\gamma$ around $z_0$ and $z_1$, respectively (recall that we are assuming $\gamma \in (s/2d, 1)$ as in the paragraph above (4)). Having succeeded to do so in the first $k$ steps, for each $\sigma \in \{0, 1\}^k$ we try to link $z_{\sigma 01}$ and $z_{\sigma 10}$ belonging to neighborhoods of size roughly $N^{\gamma k}$ around $z_{\sigma 0}$ and $z_{\sigma 1}$ respectively. Ideally, when we reach depth $n$ we will be left with $2^n - 1$ gaps which are pairs of
Figure 2: Hierarchy $H_n(x, y)$ provides a collection of highways connecting all pairs $(z_{\sigma 01}, z_{\sigma 10})$ with $\sigma \in \{0, 1\}^{n-2}$. To ensure $x$ is connected to $y$ it suffices to connect the remaining $2^{n-1}$ gaps, that are either of the form $(z_{\sigma 00}, z_{\sigma 01})$ or $(z_{\sigma 10}, z_{\sigma 11})$.

sites of type $(z_{\sigma 00}, z_{\sigma 01})$ or $(z_{\sigma 10}, z_{\sigma 11})$ with sites in each pair at a distance of order $N^\gamma$. The reader may consult Figure 2 for an illustration of this iterative procedure. Note that, by the discrete nature of the long-range model the procedure cannot be iterated indefinitely.

The above discussion motivates the definition of the event $B_n(x, y)$ that there is a hierarchy $H_n(x, y)$ of depth $n$ satisfying that, for all $0 \leq k \leq n-2$ and all $\sigma \in \{0, 1\}^k$

$$|z_{\sigma 01} - z_{\sigma 00}|_\infty \quad \text{and} \quad |z_{\sigma 10} - z_{\sigma 11}|_\infty \quad \text{belong to} \quad \left[\frac{1}{2} N_{k+1}, N_{k+1}\right], \quad (12)$$

where $N_k := N^\gamma$. The following lemma is a simplified version of [2, Lemmas 4.2 and 4.3] and provides appropriate choices of parameters so that the above idealized picture is achieved with high probability.

**Lemma 4 (Hierarchy).** Fix $\varepsilon > 0$ and $\gamma \in \left(\frac{s}{2d}, 1\right)$. For $x, y \in \mathbb{Z}^d$ and $N := |x - y|$, let $n \in \mathbb{N}$ be the greatest positive integer such that

$$n \log(1/\gamma) \leq \log(2) N - \varepsilon \log(3) N. \quad (13)$$

There is $N'(\varepsilon, \gamma, d)$ and $b = b(d) \in (0, 1)$ such that if $N \geq N'$ then for any hierarchy $H_n(x, y)$ satisfying (12) we have

$$\forall 0 \leq k \leq n - 2, \quad \text{and} \quad \sigma \in \{0, 1\}^k, \quad |z_{\sigma 01} - z_{\sigma 10}| \in [b N_k, b^{-1} N_k]. \quad (14)$$
Moreover, there is a positive constant $c = c(\beta, d, s)$ such that
\[
\mathbb{P}(B_n(x, y)^c) \leq 2^{n-1} \cdot e^{-cN_n^{2(2d-\gamma-s)}} \leq \exp[-c e^{(2d-\gamma-s)(\log(2) N)^c}].
\] (15)

Remark 1. Let $\Delta' := \frac{\log 2}{\log(1/\gamma)}$. The definition of $n$ yields:
\[2^n \leq (\log N)^{\Delta'} \quad \text{and} \quad \exp[\varepsilon (\log(2) N)] \leq N_n \leq e^{(1/\gamma)(\log(2) N)^c}.\] (16)

In fact, by the definition of $n$ and $N_n$ we can write
\[N_n := N^n \gamma = \exp[\exp[\log(2) N + n \log \gamma]],\]
and also $\varepsilon \log(3) N \leq \log(2) N + n \log \gamma \leq \varepsilon \log(3) N + \log(1/\gamma)$.

Therefore,
\[\exp[\varepsilon \log(3) N] \leq N_n \leq e^{\exp[\varepsilon \log(3) N + \log(1/\gamma)]} = e^{(1/\gamma)(\log(2) N)^c}.\]

The other inequality in (16) follows from $n \log(1/\gamma) \leq \log(2) N$.

Proof of Lemma 4. By (16), we have that scales $N_k$ are well-separated meaning that, for $0 \leq k \leq n - 1$,
\[
\frac{N_k}{N_{k+1}} = N_k^{1-\gamma} \geq N_n^{1-\gamma} \geq e^{(1-\gamma)(\log(2) N)^c}.
\] (17)

Let $0 \leq k \leq n - 2$ and $\sigma \in \{0, 1\}^k$. By (12) we have that
\[|z_{\sigma_0} - z_{\sigma_1}|_{\infty} = \Theta(N_k) \quad \text{and} \quad |z_{\sigma_0} - z_{\sigma_01}|_{\infty}, |z_{\sigma_1} - z_{\sigma_10}|_{\infty} = \Theta(N_{k+1})\]
and thus the triangular inequality and (17) imply
\[|z_{\sigma_01} - z_{\sigma_10}|_{\infty} = \Theta(N_k) + 2\Theta(N_{k+1}) = \Theta(N_k),\]
and (14) follows.

To obtain (15) we partition $B_n^c$ according to the first depth $k$ at which we fail to
find highways. For that value of $k$ the event $\mathcal{B}_{k-1}$ occurs and there is a hierarchy $\mathcal{H}_{k-1}$ satisfying (12). Let us fix a gap in $\mathcal{H}_{k-1}$, which is either the form $(z_{\sigma 00}, z_{\sigma 01})$ or $(z_{\sigma 00}, z_{\sigma 01})$ with $\sigma \in \{0, 1\}^{k-3}$. For the corresponding pair of neighborhoods, say

$$\{z': \frac{1}{2}N_{k-1} \leq |z_{\sigma 00} - z'|_\infty \leq N_{k-1}\} \text{ and } \{z': \frac{1}{2}N_{k-1} \leq |z_{\sigma 01} - z'|_\infty \leq N_{k-1}\},$$

none of the edges between these neighborhoods must be open. By (12) these neighborhoods are centered at sites whose distance belongs to $[\frac{1}{2}N_{k-2}, N_{k-2}]$. Moreover, each neighborhood has $cN_{k-1}^d$ vertices. A straightforward adaptation of the argument leading to (4) applied to scale $N_{k-2}$ shows that the probability of not finding an open edge linking a fixed pair of neighborhoods is bounded above by

$$\exp[-\beta cN_{k-1}^{2d}N_{k-2}^{\gamma-s}] \leq \exp[-cN_n^{2d\gamma-s}],$$

where $c = c(\beta, d, s) > 0$. Since there are $2^{k-2}$ pairs of neighborhoods we have:

$$\mathbb{P}(\mathcal{B}_n^c) = \sum_{k=2}^{n} \mathbb{P}(\mathcal{B}_{k-1} \cap \mathcal{B}_k^c) \leq \sum_{k=2}^{n} 2^{k-2} e^{-cN_n^{2d\gamma-s}} \leq 2^{n-1} e^{-cN_n^{2d\gamma-s}}.$$

The last bound in (15) follows from (16).

\[\Box\]

### 3 Bounding the Euclidean distance

Using the hierarchical construction from Biskup [2] we obtain a collection of highways that provides the main contribution for finding open paths between two distant sites. However, the remaining gaps must still be connected in order to find an open path between the original two points that actually uses these highways. In [2] this is accomplished by requiring that the vertices $z_{\sigma}$ in $\mathcal{H}_n(x,y)$ belong to sufficiently large but local clusters. For our model we take a similar but different strategy.

The idea is to make a hybrid approach, considering a renormalized lattice to define a site-bond percolation model. The bond percolation part will be coupled to a long-range percolation model. Independently of this bond percolation part we define a site percolation that will be used to glue together these highways, using an idea from
We begin describing the renormalization procedure. Partition $\mathbb{Z}^2$ into a collection of boxes $(B_x; x \in \mathbb{Z}^2)$ where $B_x = B_x(K) := Kx + [-K/2, K/2]^2$. The exact choice of $K$ will depend on the parameters $u, \alpha$ of the model and is deferred till Lemma 5. Each box $B_x$ is assigned an enlarged box and a core, defined respectively as

$$B'_x := Kx + [-\frac{9K}{10}, \frac{9K}{10}]^2 \quad \text{and} \quad B''_x := Kx + [-\frac{K}{10}, \frac{K}{10}]^2.$$

We say that the box of the origin $B_o$ is good if the event

$$\left\{ \left[ \frac{5K}{10}, \frac{7K}{10} \right] \times \left[ -\frac{9K}{10}, -\frac{7K}{10} \right] \sim \xi \left[ \frac{5K}{10}, \frac{7K}{10} \right] \times \left[ \frac{7K}{10}, \frac{9K}{10} \right] \right\}$$

occurs, as well as the three similar events resulting from (18) by rotations by $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$ around the origin, see Figure 3. The events $\{B_x \text{ is good}\}$ are defined analogously; in words, a box $B_x$ is good if it is enclosed by a well-positioned circuit of overlapping ellipses contained in its enlarged box, see Figure 3. We also say that a site $x$ is good if its respective renormalized box, $B_x$, is good. If $B_x$ is good, we denote by $O_x$ a circuit of ellipses that realizes such event, chosen according to some predetermined rule.

Our site-bond percolation model is defined via the following configurations:

**Site percolation:** $$(\omega_x) := \left( 1 \{B_x \text{ is good}\}; x \in \mathbb{Z}^2 \right);$$

**Bond percolation:** $$(\omega_{xy}) := \left( 1 \{B''_x \sim \xi B''_y\}; x, y \in \mathbb{Z}^2, x \neq y \right).$$

It follows from our construction that $(\omega_x)$ and $(\omega_{xy})$ are independent processes since they are defined in terms of the PPP $\xi$ restricted to disjoint regions of $\mathbb{R}^4$. For the same reason, $(\omega_x)$ is an independent (Bernoulli) site percolation process on $\mathbb{Z}^2$.

Our definition of a good box is close to the definition of good boxes used in [1]. Essentially, it ensures that in a cluster of good boxes one is able to move from one

Antal and Pisztora [1].
Figure 3: A renormalized box $B_x$ is good if there is a carefully positioned surrounding circuit of ellipses $O_x$ contained in its enlarged box $B'_x$; we also highlight its core $B''_x$. On the right, we emphasize that *-neighboring good boxes must have their respective outer circuits interlaced. Outer circuits and cores of a same box are shown in matching colors to help visualization.

box to a neighboring one remaining inside the covered set. This holds not only when moving along the coordinate directions (to a box that shares a side) but also when moving diagonally (to a box that shares a single vertex). We briefly discuss this notion of connectivity now introducing notation that is very similar to that of [1].

**On *-connected sets.** For $x, y \in \mathbb{Z}^d$ define

$$x \sim y \text{ if } |x - y| = 1 \quad \text{and} \quad x \sim^* y \text{ if } |x - y|_{\infty} = 1.$$ 

Given a configuration $\omega \in \{0, 1\}^{\mathbb{Z}^d}$, we say that a site $x \in \mathbb{Z}^d$ is good if $\omega_x = 1$. Otherwise $x$ is said bad. Denote by $C^*_x$ the bad cluster (with respect to $\sim^*$) containing $x$. We use the convention that $C^*_x = \emptyset$ if $x$ is good. For a finite subset $\Lambda \subset \mathbb{Z}^d$ define its outer and inner boundaries by

$$\partial^o \Lambda := \{x \in \Lambda^c; \exists y \in \Lambda, x \sim y\} \quad \text{and} \quad \partial^i \Lambda := \{x \in \Lambda; \exists y \in \Lambda^c, x \sim y\},$$

respectively. We use the convention that $\partial^o C^*_x = \{x\}$ when $x$ is good.

For $\Lambda$ finite, its complementary set $\Lambda^c$ contains a finite number of connected com-
ponents $\Lambda_1, \ldots, \Lambda_k$. Exactly one of them, say $\Lambda_1$ is infinite; the other ones, if any, are called holes. When holes exist, we define $\hat{\Lambda} := \Lambda \cup \Lambda_2 \cup \ldots \cup \Lambda_k$ which may be regarded as the result of filling all holes in $\Lambda$. We also define the external outer boundary and the external inner boundary of $\Lambda$ respectively as

$$
\partial_e^{o} \Lambda := \partial^{o} \hat{\Lambda} \quad \text{and} \quad \partial_e^{i} \Lambda := \partial^{i} \hat{\Lambda}.
$$

An important topological fact is given by [1, Statement (3.35)]:

for any finite $\Lambda$ that is $\ast$-connected we have that

$$
\partial_e^{o} \Lambda \quad \text{and} \quad \partial_e^{i} \Lambda \quad \text{are both} \quad \ast\text{-connected.} \quad (19)
$$

This is important because whenever we find a region composed of bad sites, we can contour that bad region using its exterior boundary of good sites.

**On the renormalized model.** Recall that the PPP $\xi$ can be parametrized using either $(s, \beta)$ or $(u, \alpha)$, by (3).

**Lemma 5.** The following properties hold for the renormalized model:

**P1.** Fix $p \in (0, 1)$ and $\beta_0 > 0$. There is $K(u, \alpha, p, \beta_0)$ large enough such that $(\omega_x)$ dominates independent site percolation with parameter $p$ and $(\omega_{xy})$ dominates long-range percolation with $p_{xy} = 1 - e^{-\beta_0 |x-y|-s}$.

**P2.** If $W$ is a $\ast$-connected set of good sites then all the surrounding circuits $O_x$, $x \in W$ are contained in the same connected component of ellipses.

**P3.** If $C^*_x$ is finite, then $\partial_e^{o} C^*_x$ is a $\ast$-connected set of good sites.

**Proof.** Property **P2** is a straightforward geometric consequence of the definition (see Figure 3) and Property **P3** follows from (19).

We now prove Property **P1**. Denote by $A_0$ the event in (18) and by $A_i$, for $1 \leq i \leq 3$ the three similar events resulting from (18) by rotations by $\frac{\pi}{2}$, $\pi$ and $\frac{3\pi}{2}$ around the origin, respectively. Since $\xi$ is invariant with respect to translations and rotations, any $A_i$ has probability

$$
\mathbb{P}(A_i) = \mathbb{P}\left((-K, K)^2 \sim_{\xi} \left(\frac{K}{M}, 0\right) + [-K, K]^2\right) \geq 1 - e^{-\beta \left(\max_{x,y} |x-y|\right)^{-s}(4\pi)^4}
$$
where the maximum runs over all points \( x \) and \( y \) in the first and second boxes, respectively. The maximum is a constant multiple of \( K \), so we can write
\[
P(A_i) \geq 1 - e^{-cK^{4-s}} \to 1 \quad \text{as} \quad K \to \infty
\]
for some constant \( c = c(\beta, s) > 0 \), since we are assuming \( s \in (3, 4) \). Then, FKG inequality implies \( P(B_n \text{ is good}) \) also tends to 1.

For the probability of an edge being open, we notice that \( B''_x \sim_x B''_y \) is a scaling by \( K/5 \) of event
\[
\left\{ \left[ -\frac{1}{2}, \frac{1}{2} \right]^2 \sim_x 5(y - x) + \left[ -\frac{1}{2}, \frac{1}{2} \right]^2 \right\}.
\] (20)

By (9) we can relate the probability of \( B''_x \sim_x B''_y \) with that of the event in (20) under a rescaled long-range model whose intensity can be made as high as we want by increasing \( K \). This completes the proof of Property \( \textbf{P1} \).

\[\Box\]

### 3.2 Gluing highways

Given two fixed sites \( x, y \in \mathbb{R}^2 \), Lemma 4 roughly states that, for the long-range model in the renormalized lattice, hierarchies exist with very high probability. On the event \( x \leftrightarrow y \) we want to use the highway structure entailed by one of these hierarchies in order to find a path that connects \( x \) to \( y \) efficiently.

For \( z \in \mathbb{R}^2 \), let \( a(z) \in \mathbb{Z}^2 \) be the unique site in \( \mathbb{Z}^2 \) such that \( z = Ka(z) + [-K/2, K/2]^2 \). The distance between the original points \( x \) and \( y \) and the distance between their respective counterparts \( a(x) \) and \( a(y) \) in the renormalized lattice can be compared as
\[
|x - y| = |(x - Ka(x)) - (y - Ka(y)) + K(a(x) - a(y))| \\
= K \cdot |a(x) - a(y)| + KO(1).
\] (21)

Here the \( L_2 \)-norm could be replaced by any other norm on \( \mathbb{R}^2 \). Lemma 4 implies that \( \mathcal{B}_n(a(x), a(y)) \) has probability close to 1 (provided that \( N \) is large and \( n \) satisfies (13)). Conditional on \( \mathcal{B}_n \), we can find a collection of sites \( \{z_\sigma; \sigma \in \{0, 1\}^n\} \) together
with the endowed highway structure connecting some of them. If there is more than one choice, just pick one of them according to a predetermined rule.

We still have to ensure that all the remaining gaps, that is, all the $2^{n-1}$ edges of type $(z_{\sigma 00}, z_{\sigma 01})$ or $(z_{\sigma 10}, z_{\sigma 11})$ for $\sigma \in \{0,1\}^{n-2}$ are connected with high probability. This can be done with the aid of an argument from [1].

We can specify each gap uniquely as $(z_{\sigma 0}, z_{\sigma 1})$, with $\sigma \in \{0,1\}^{n-1}$. Writing $\tilde{N} := |a(x) - a(y)|$ and $\tilde{N}_k = N^{\gamma_k}$, on the event $B_n(a(x), a(y))$ we have

$$|z_{\sigma 1} - z_{\sigma 0}|_{\infty} \in \left[\frac{1}{2} \tilde{N}_{n-1}, \tilde{N}_{n-1}\right] \text{ for every } \sigma \in \{0,1\}^{n-1},$$

(22)

see (12).

Moreover, (14) guarantees that the highways of the form $(z_{\sigma 01}, z_{\sigma 10})$ with $\sigma \in \{0,1\}^k$ and $0 \leq k \leq n - 2$ have length $|z_{\sigma 01} - z_{\sigma 10}| = \Theta(\tilde{N}_k)$. From the point of view of our original ellipses model, a highway connecting $z_{\sigma 01}$ and $z_{\sigma 10}$ represents an actual ellipse $E_\sigma$ that realizes the event $B_{z_{\sigma 10}}'' \sim \xi B_{z_{\sigma 01}}''$. Thus,

$$\text{diam}(E_\sigma) = K \cdot \left(|z_{\sigma 10} - z_{\sigma 01}| + O(1)\right)$$

(23)

and consequently the

$$(\text{number of renormalized boxes intersected by } E_\sigma) \in \left[c_2 \tilde{N}_k, c_2^{-1} \tilde{N}_k\right]$$

(24)

for a positive constant $c_2$ that will remain fixed from now on. Also, the site percolation process $(\omega_x)$ is independent of the collection $(z_\sigma; \sigma \in \{0,1\}^n)$.

For each gap $(z_{\sigma 0}, z_{\sigma 1})$, with $\sigma \in \{0,1\}^{n-1}$ write $m_\sigma := |z_{\sigma 0} - z_{\sigma 1}|_1$ and fix a deterministic path (according to a predetermined rule) of $m_\sigma + 1$ neighboring sites that realize this distance, meaning

$$z_{\sigma 0} = z_{\sigma 0}^{(\sigma)}, z_{\sigma 1}^{(\sigma)}, \ldots, z_{m_\sigma}^{(\sigma)} = z_{\sigma 1} \text{ with } z_j^{(\sigma)} \sim z_{j+1}^{(\sigma)} \text{ for } 0 \leq j < m_\sigma.$$

Recall that $C_z^\ast$ denotes the $\ast$-connected cluster of bad sites containing $z$ and define $C^\ast(z) := C_z^\ast \cup \partial^\ast C_z^\ast$. We look at the random subset of $\mathbb{R}^2$ composed by the boxes
associated to the bad clusters of the sites along the path \((z^{(\sigma)})\):

\[
W_\sigma := \bigcup_{j=0}^{m_\sigma} \left( \bigcup_{z \in C^*_{(z^{(\sigma)})}} B'_z \right).
\]

(25)

Denoting by \(#W_\sigma\) the number of sites in the renormalized lattice that one needs to explore to find \(W_\sigma\), we have:

**Lemma 6.** For every \(a > 0\) there exists \(c_3 = c_3(a, p) > 0\) such that

\[
P\left( B_n \cap \left( \bigcup_{\sigma \in \{0,1\}^n} \{ #W_\sigma \geq aN_{n-2} \} \right) \right) \leq 2^{n-1} \cdot e^{-c_3N_{n-2}} \ll 1.
\]

(26)

**Proof.** We have \(#\bar{C}^*_z\) = 1 when \(z\) is good. Otherwise, since each site of \(C^*_z\) has at most 8 neighbors

\[
#W_\sigma \leq \sum_{j=0}^{m_\sigma} #\bar{C}^*(z_j^{(\sigma)}) \leq 1 + m_\sigma + 8 \sum_{j=0}^{m_\sigma} #C^*(z_j^{(\sigma)}).
\]

Also, using an argument from [1] based on a previous construction in Fontes and Newman [10] (see the proof of Theorem 4), if \((\bar{C}^*_z)_{z \in \mathbb{Z}^2}\) is a collection of independent random subsets of \(\mathbb{Z}^2\) with \(\bar{C}^*_z \overset{d}{=} C^*_o\), then \((\bar{C}^*_z)\) dominates stochastically \((C^*_z)\).

Defining \(Y_z := #\bar{C}^*_z\), we have that \((Y_z, z \in \mathbb{Z}^2)\) are i.i.d. random variables with the same distribution as \(#C^*_o\). We have for \(Y_j := Y_{z_j^{(\sigma)}}\) that

\[
#W_\sigma \leq 1 + m_\sigma + 8 \sum_{j=0}^{m_\sigma} #\bar{C}^*(z_j^{(\sigma)}) \overset{d}{=} 1 + m_\sigma + 8 \sum_{j=0}^{m_\sigma} Y_j.
\]

Notice that \(C^*_o\) is a \(*\)-cluster of bad sites in a Bernoulli site percolation of parameter \(p\) and by Lemma 5 we can start the construction with \(p\) sufficiently close to 1 so that the probability of a site being bad, \(1 - p\), is subcritical, and then choose \(K(u, \alpha, p, \beta_0)\) accordingly. Exponential decay of cluster size (see e.g. [13, Theorem (6.75)]) yields \(\psi(p) > 0\) such that \(h(p) := E[e^{\psi(p)Y}] < \infty\). Hence, for any fixed \(\sigma \in \{0, 1\}^{n-1}\), an
application of Markov’s Inequality yields

\[ \mathbb{P}(\mathcal{B}_n \cap \{\#W_{\sigma} \geq a\tilde{N}_{n-2}\}) \leq \mathbb{P}\left(\sum_{j=0}^{m_{\sigma}} Y_j \geq \frac{a}{8}\tilde{N}_{n-2} + O(m_{\sigma})\right) \]

\[ \leq \exp\left[-\frac{a\psi(p)}{8}\tilde{N}_{n-2} + O(m_{\sigma})\right] \cdot \mathbb{E}\left[e^{\psi(p)\sum_{j=0}^{m_{\sigma}} Y_j}\right] \]

\[ = \exp\left[-\frac{a\psi(p)}{8}\tilde{N}_{n-2} + O(m_{\sigma}) + \log h(p) \cdot m_{\sigma}\right]. \]

Finally, by (22) we can write

\[ m_{\sigma} = |z_{\sigma 0} - z_{\sigma 1}|_1 = \Theta(\tilde{N}_{n-1}) \ll \tilde{N}_{n-2}. \]

The estimate on (26) follows from a union bound and the bounds obtained in (16). □

Recall the definition of \( c_2 \) in (24) and take \( a = \frac{1}{3}c_2 \) at Lemma 6. Define

\[ \mathcal{W}_n := \bigcap_{\sigma \in \{0,1\}^{n-1}} \{\#W_{\sigma} \leq \frac{c_2}{3}\tilde{N}_{n-2}\}. \quad (27) \]

We have \( \mathbb{P}(\mathcal{B}_n \cap \mathcal{W}_n^c) \ll 1 \), meaning that with high probability every \( W_{\sigma} \) is too small to contain any highway. Fix some \( \sigma \in \{0,1\}^{n-1} \). By P2 and P3 on Lemma 5 we can use the external inner boundary \( \partial_i^e W_{\sigma} \), a *-connected set of good boxes, to glue together the highways that arrive at \( z_{\sigma 0} \) and \( z_{\sigma 1} \), the procedure is illustrated in Figure 4.

Suppose that we know that \( \{x \leftrightarrow y\} \cap \mathcal{B}_n(a(x), a(y)) \cap \mathcal{W}_n \) has occurred and fix a path \( \mathcal{P} \) connecting \( x \) to \( y \). Although \( \mathcal{P} \) can be arbitrarily long, after the gluing process we can build a path \( \mathcal{P}' \) from \( x \) to \( y \) whose length is controlled. Let \( 0, 1 \in \{0,1\}^{n-1} \) be the all zeroes and all ones words, respectively.

**Definition 2.** Path \( \mathcal{P}' \) is defined as follows:

1. Follow \( \mathcal{P} \) from \( x \) till it hits the first outer circuit \( O_z \) of a good box \( B_z \), with \( z \in \partial_i^e W_{\widehat{0}} \).
2. When \( \mathcal{P}' \) first gets to \( \partial_i^e W_{\sigma} \) with \( \sigma \neq \widehat{1} \), use outer circuits to move towards the next highway.
Figure 4: Region $W_σ$ (light gray) explores bad boxes (dark gray) on a deterministic path of boxes (thick lines) and $\partial^i W_σ$ is made of good boxes. When $\#W_σ$ is small, the highways arriving at $B_{zσ0}$ and $B_{zσ1}$ can be connected through $\partial^i W_σ$.

3. When $P'$ arrives at a highway, move in a straight line till intersecting the next $\partial^i W_σ$.
4. When $P'$ gets to $\partial^i W_1$, use outer circuits to move to the last point of $P$ that intersects a circuit $O_z$ in $\partial^i W_1$ and then use $P$ to move to $y$.

We have good estimates for the length of path $P'$ when moving on highways or when using outer circuits of some $W_σ$. However, some parts of $P'$ could be wiggly (when following along $P$) and that could possibly add a considerable amount to the total length. The next lemma allows us to improve the estimate on the length of a path inside the covered set $E$ when we move inside a bounded region.

**Lemma 7 (Distance on small scales).** Let $W \subset \mathbb{R}^2$ be a bounded connected set and let $x \in W$. If $x \leftrightarrow \partial W$ then

$$D(x, \partial W) \leq \frac{2}{π} \text{Vol}(W) + 1,$$  \hspace{1cm} (28)

and consequently

$$d(x, \partial W) \leq \left(\frac{2}{π} \text{Vol}(W) + 1\right) \cdot \text{diam}(W).$$  \hspace{1cm} (29)
Proof. Denote by $\{e_i; 1 \leq i \leq m\}$ the set of all ellipses that intersect $W$, which is almost surely finite since $W$ is bounded. Since $x \leftrightarrow \partial W$ there is some point $y \in \partial W$ that can be reached from $x$ by a path contained in $\mathcal{E}$. Take a path $\mathcal{P}$ that connects $x$ and $y$ without self-intersections. For any fixed ellipse $e$ used by $\mathcal{P}$, if $\mathcal{P} \cap e$ is not a straight line we can reduce the length of $\mathcal{P}$ by connecting its first and last visit to that ellipse directly. This modified path may intersect $\partial W$ before reaching $y$, but in this case we simply replace $y$ by the first point of $\partial W$ that was reached. Thus, we can restrict ourselves to polygonal paths.

Let $f : [0,1] \rightarrow \mathbb{R}^2$ be a continuous and injective parametrization of $\mathcal{P}$ with $f(0) = x$ and $f(1) = y$, and define $I_j = f^{-1}(e_j)$. By the properties of $\mathcal{P}$, we know that each $I_j$ is a closed interval and that $[0,1] = \bigcup_{j=1}^m I_j$. We build a minimal set of ellipses that covers $\mathcal{P}$ by doing a greedy exploration. We can assume that $x \in e_1$ and define $i_1 := 1$. Then, inductively define $i_{j+1}$ as the index of an ellipse that intersects $e_{i_j}$ and with rightmost point of $I_{i_{j+1}}$ closest to 1. Since we have a finite collection, the process ends on some index $i_n$; relabeling if necessary, we can consider $i_j = j$ for $1 \leq j \leq n$.

By construction we have that $\mathcal{P} \subset \bigcup_{j=1}^n e_j$ and each $e_j$ only intersects $e_{j-1}$ and $e_{j+1}$. Finally, by the same reasoning as in the beginning of the proof we can assume that $e_n$ is the first ellipse to intersect $\partial W$. We can bound the size of $n$ by using the fact that $\{e_i; i \text{ odd}, 1 \leq i \leq n-1\}$ and $\{e_i; i \text{ even}, 1 \leq i \leq n-1\}$ are disjoint collections inside $W$. Since each $e_i$ contains a ball of radius 1, we have that

$$n - 1 \leq 2 \cdot \frac{\text{Vol}(W)}{\pi},$$

and we proved (28). The bound on (29) follows by using that on each $e_i$ the length of $\mathcal{P}$ is bounded by $\text{diam}(W)$. \qed

### 3.3 Proof of Theorem 1

We now have all the ingredients to bound the Euclidean distance between distant points inside a same cluster of ellipses.
Proof of Theorem 1. We can assume \( \delta \in (0, 1 - \frac{2+\alpha}{4}) \) and define \( \gamma := \frac{2+\alpha}{4} + \delta \). We analyze the probability of \( \mathcal{D}(x, y) \) being large by decomposing this event with respect to \( B_n(a(x), a(y)) \cap W_n \), obtaining

\[
\mathbb{P}(x \leftrightarrow y, \mathcal{D}(x, y) > N + N^{\frac{2+\alpha}{4} + \delta}) = \mathbb{P}(\{x \leftrightarrow y\} \cap B_n \cap W_n, \mathcal{D}(x, y) > N + N^\gamma) + o(1),
\]

since both \( \mathbb{P}(B_n^c) \) and \( \mathbb{P}(B_n \cap W_n^c) \) tend to zero with \( N \) by Lemmas 4 and 6, respectively. On event \( \{x \leftrightarrow y\} \cap B_n \cap W_n \) there is a path \( P \) between \( x \) and \( y \) and we use \( P \) to build a path \( P' \) as in Definition 2.

Using Lemma 7 we replace the parts of \( P' \) that use \( P \) on steps 1. and 4. by a path satisfying the bound on (29). Actually, we do not lose much by applying (29) at every \( W_\sigma \), since

\[
\text{Vol}(W_\sigma) \leq \left(\frac{n}{5}K\right)^2 \cdot \#W_\sigma \quad \text{and} \quad \text{diam}(W_\sigma) \leq \left(\frac{n}{5}K\right)^\gamma \cdot \#W_\sigma.
\]

By (16) we have that \( \#W_\sigma \leq \frac{1}{3}c_2\tilde{N}_{n-2} \leq \frac{1}{3}c_2 \cdot e^{(1/\gamma)^3(\log^2\tilde{N})^\epsilon} \), implying

\[
\text{Vol}(W_\sigma) \cdot \text{diam}(W_\sigma) \leq c \cdot K^3 \cdot e^{2/\gamma^3(\log^2\tilde{N})^\epsilon}.
\]

The length of \( P' \) can be estimated by

\[
l(P') \leq c \sum_\sigma \text{Vol}(W_\sigma) \cdot \text{diam}(W_\sigma) + \sum_{\sigma'} \text{diam}(E_{\sigma'})
\]

where the index in the second sum runs over all highways. Since there are \( 2^{n-1} \) gaps, the bounds on (16) imply

\[
\sum_\sigma \text{Vol}(W_\sigma) \cdot \text{diam}(W_\sigma) \leq cK^3 \cdot (\log \tilde{N})^\Delta \cdot e^{2/\gamma^3(\log^2\tilde{N})^\epsilon}.
\]

For the second sum, notice that in the long-range model, for each \( 0 \leq k \leq n - 2 \)
there are $2^k$ highways of size about $\tilde{N}_k$. We can therefore write

$$\sum_{\sigma'} \text{diam}(E_{\sigma'}) = \text{diam}(E_{\sigma}) + \sum_{k=1}^{n-2} 2^k \cdot K \Theta(\tilde{N}_k) \tag{23}$$

$$\overset{(16)}{=} K \cdot (|z_{10} - z_{01}| + O(1)) + K2^{n-1}O(\tilde{N}_1)$$

$$\overset{(21)}{=} K \cdot (\tilde{N} + \Theta(\tilde{N}_1)) + KO((\log \tilde{N})^{\Delta'}\tilde{N}_1)$$

implying the bound $l(P') \leq N + O((\log N)^{\Delta'}N^\gamma)$. Since $\gamma = \frac{2+\alpha}{4} + \delta$ and $\delta$ can be taken arbitrarily small, we conclude the proof of (1). \hfill \Box

### 4 Bounding chemical distance

Now we turn to investigating the chemical distance. The same construction as in the proof of Theorem 1 also provides an upper bound for the chemical distance between $x$ and $y$, since it implies

$$D(x, y) \leq \sum_{\sigma} \text{Vol}(W_{\sigma}) + \sum_{\sigma'} 1 \leq cK^2e^{(1/\gamma)^3(\log(2)\tilde{N})e} + 2^{n-1} \leq (\log |x - y|)^{\Delta'} + o(1). \tag{30}$$

However, we can actually achieve a better bound. In fact, although our collection of highways provides a structure of long ellipses that links far away points efficiently in terms of their Euclidean distance, it may be conceivable that the optimal strategy to minimize the chemical distance might differ. An improvement to the bound in (30) is given in the next result:

**Proposition 1.** For any $\delta > 0$ it holds that

$$\lim_{|x - y| \to \infty} \mathbb{P}\left(D(x, y) \leq \frac{2 + \delta}{\log(2/\alpha)} \cdot \log \log |x - y| \bigg| x \leftrightarrow y \right) = 1. \tag{31}$$

**Proof.** The main construction for this bound is a way of moving faster than through highways, see Figure 5. This construction has no counterpart on discrete long-range
model, since it leverages on the property that two ellipses that cross in their middle section are connected.

We consider a sequence \((l_n; n \geq 0)\) of increasing lengths which is defined recursively by \(l_n = l_{n-1}^{2/\alpha}(\log l_{n-1})^{-1}\). The value of \(l_0\) is fixed later. Consider also the following collection of boxes

\[ B_n = \begin{cases} [0, l_n] \times [0, l_{n-1}] & \text{if } n \text{ odd} \\ [0, l_{n-1}] \times [0, l_n] & \text{if } n \text{ even} \end{cases} \]

and define event \(A_n\) in which box \(B_n\) is crossed in its longest direction by one ellipse. By Lemma 1 and our choice of sequence \((l_n)\) we have

\[
\sum_{n \geq 1} \mathbb{P}(A_n^c) \leq \sum_{n \geq 1} \exp[-uc_0^{-1} l_n^{-\alpha} l_{n-1}^2] = \sum_{n \geq 1} \exp[-uc_0^{-1} (\log l_{n-1})^\alpha]
\]

Now, we check that the series above converges by estimating the growth rate of sequence \((l_n)\). Notice that \(l_n \leq l_{n-1}^{2/\alpha} \leq l_0^{2/\alpha}\) and also that this upper bound implies

\[
l_n = l_{n-1}^{2/\alpha} \cdot (\log l_{n-1})^{-1} \geq l_{n-2}^{2/\alpha} \cdot (2/\alpha)^{-n-1} (\log l_0)^{-1}
\]

\[
\geq l_0^{(2/\alpha)^n} (2/\alpha)^{-\sum_{j=1}^n (2/\alpha)^{j-1}} (\log l_0)^{-\sum_{j=1}^n (2/\alpha)^{j-1}}.
\]

Computing the sums on the last line one obtains as \(n \to \infty\) that

\[
\sum_{j=1}^n (n - j) (2/\alpha)^{j-1} = (2/\alpha)^n \left( \frac{2/\alpha}{((2/\alpha) - 1)^2} + o(1) \right),
\]

\[
\sum_{j=1}^n (2/\alpha)^{j-1} = (2/\alpha)^n \left( \frac{1}{(2/\alpha) - 1} + o(1) \right),
\]

which leads to

\[
l_n \geq \exp \left\{ \left( \frac{2}{\alpha} \right)^n \left\{ \log l_0 - \frac{(2/\alpha) \log(2/\alpha)}{((2/\alpha) - 1)^2} - \frac{\log \log l_0}{(2/\alpha) - 1} + o(1) \right\} \right\}
\]

and thus for some large \(l_0 = l_0(\alpha)\) the coefficient in curly brackets can be estimated.
from below by $2 + o(1)$, which implies
\[
\sum_{n \geq 1} \mathbb{P}(A_n^c) \leq \sum_{n \geq 1} \exp[-uc((2/\alpha)^{\alpha})^{n-1}] < \infty
\]
since $(2/\alpha)^{\alpha} > 1$. This means we can make $\mathbb{P}(\cap_{n \geq n_0} A_n)$ arbitrarily close to one by taking $n_0$ sufficiently large. Notice that on this event we move faster than when using highways, since we can get from $B_{n_0}$ to distance $l_n$ using only $n - n_0$ ellipses.

Besides faster highways, we also build a useful collection of circuits. Let $U_n^0$ be the event in which box $[-2l_n, 2l_n] \times [l_n, 2l_n]$ is crossed in its longest direction with one ellipse and let $U_n^j$ be the analogous events obtained by rotating this box counterclockwise by $j \cdot \pi/2$, $j = 1, 2, 3$. Defining events $C_n := \cap_{j=1}^4 U_n^j$, we have by Lemma 1
\[
\sum_{n \geq 1} \mathbb{P}(C_n^c) \leq \sum_{n \geq 1} 4\mathbb{P}(\cap_{j=1}^4 U_n^j) \leq 4 \sum_{n \geq 1} \exp[-uc \cdot l_n^{2-\alpha}] < \infty.
\]

Moreover, on $C_n$ we have a circuit made of four ellipses, whose supporting lines form a convex quadrilateral $Q_n$ that surrounds $[-l_n, l_n]^2$ but stays inside $[-2l_{n_0}, 2l_{n_0}]^2$.

Now we are ready to prove (31). Without loss of generality, we can assume $y$ is the origin. Fix any $\varepsilon > 0$ and choose $n_0$ sufficiently large such that
\[
\mathbb{P}(\cap_{n \geq n_0} A_n) \geq 1 - \varepsilon \quad \text{and} \quad \mathbb{P}(\cap_{n \geq n_0} C_n) \geq 1 - \varepsilon.
\]

Let us also define $A_n(x)$ and $C_n(x)$ as the events analogous to $A_n$ and $C_n$ but considering that $x$ is the origin. Thus, if we define event
\[
V := \left( \bigcap_{n \geq n_0} A_n \right) \cap \left( \bigcap_{n \geq n_0} C_n \right) \cap \left( \bigcap_{n \geq n_0} A_n(x) \right) \cap \left( \bigcap_{n \geq n_0} C_n(x) \right),
\]
we can write
\[
|\mathbb{P}(o \leftrightarrow x) - \mathbb{P}(\{o \leftrightarrow x\} \cap V)| \leq \mathbb{P}(V^c) \leq 4\varepsilon.
\]

On event $\{o \leftrightarrow x\} \cap V$ we have some path $\mathcal{P}$ of ellipses connecting $o$ to $x$. For $|x| > 2l_{n_0}$ path $\mathcal{P}$ intersects quadrilaterals $Q_{n_0}$ and $Q_{n_0}(x)$.

Let us define $n_1 = n_1(x)$ as the first index $k$ such that $x + B_{n_0} \subset [-l_k, l_k]^2$. We have
that
\[ l_{n_1} \geq |x| + l_{n_0} > l_{n_1-1}, \quad \text{implying} \quad n_1 = \frac{\log \log |x|}{\log(2/\alpha)} + O(1). \]

Finally, notice that event $\cap_{n \geq n_0} A_n$ ensures $Q_{n_0}$ is connected to $Q_{n_1}$ by a path $P$ of at most $n_1$ ellipses. We can also find a path $P(x)$ of at most $n_1$ ellipses connecting $Q_{n_0}(x)$ to $Q_{n_1}$, when we consider event $\cap_{n \geq n_0} A_n(x)$. Thus, we can bound the chemical distance of $o$ and $x$ by the number of ellipses in the following path $P'$:

(i) Move from $o$ to $Q_{n_0}$ using the minimal number of ellipses and then follow circuit $Q_{n_0}$ till meeting $P \cap Q_{n_0}$.

(ii) Move from $P \cap Q_{n_0}$ to $P \cap Q_{n_1}$ and then follow circuit $Q_{n_1}$ till you meet $P(x) \cap Q_{n_1}$. Move from $P(x) \cap Q_{n_1}$ to $P(x) \cap Q_{n_0}(x)$.

(iii) Follow circuit $Q_{n_0}(x)$ till you meet a point of $Q_{n_0}(x)$ connected to $x$ by a path inside $Q_{n_0}(x)$ that uses a minimal number of ellipses.

By Lemma 7 we can bound the number of ellipses used in each of the steps (i) and (iii) by $\frac{2}{\pi}(2l_{n_0})^2 + 1$. Moreover, moving between points in a same quadrilateral takes at most 4 ellipses. This leads to
\[
D(o, x) \leq 2 \cdot \left( \frac{2}{\pi} \cdot 4l_{n_0}^2 + 1 \right) + 3 \cdot 4 + 2n_1 \leq \frac{2}{\log(2/\alpha)} \log \log |x| + O(1).
\]

Taking the limit as $|x|$ tends to infinity, one obtains for arbitrary $\varepsilon > 0$ that
\[
\lim_{|x| \to \infty} \mathbb{P}(x \leftrightarrow o, \ D(x, o) > \frac{2 + \delta}{\log(2/\alpha)} \cdot \log \log |x|) \leq 4\varepsilon.
\]
\[ \square \]

4.1 Lower bound for chemical distance

The argument from [3], due originally to Trapman [17], cannot provide us a lower bound for the chemical distance since we already have an upper bound for $D(x, y)$ of order $\log \log |x - y|$. It is possible to employ a similar strategy based on BK inequality [20, 21], but here we are able to use a more elementary approach.
Figure 5: Construction of short path $P'$, depicted by a zigzag line. On the right, we show event $\cap_{n \geq n_0} A_n$ in which we have an ‘improved highway’. On the left, the improved highways $P$ and $P(x)$ are connected to quadrilaterals $Q_{n_0}, Q_{n_0}(x)$ and $Q_{n_1}(x)$ to form $P'$.

For $0 < l_1 < l_2$, we make a slight abuse of notation by denoting the chemical distance between sets $B(l_1)$ and $\partial B(l_2)$ by $D(l_1, l_2)$. Instead of working with $D(o, x)$ directly, we investigate the quantity $D(1, |x|)$.

**Proposition 2.** Let $\gamma := \frac{\alpha - 1}{2}$ and define $C := \max\{15, u2^{\alpha - 1}c_1\}$, where $c_1$ is given by Lemma 2. For every $n \geq 0$ we have

$$\mathbb{P}(D(1, |x|) \leq 2^n) \leq C^n |x|^{-2\gamma 2^n}, \text{ for } |x| > 2^{\gamma 1 - n - 2^n}. \quad (32)$$

**Proof.** The proof is by induction. By Lemma 2 we have for $|x| > 2$ that

$$\mathbb{P}(D(1, |x|) = 1) \leq uc_1(|x| - 1)^{1-\alpha} \leq uc_1 2^{\alpha - 1} |x|^{1-\alpha} \leq C|x|^{-2\gamma},$$

and we proved case $n = 0$. For the induction step, we notice that for $n \geq 0$

$$\mathbb{P}(D(1, |x|) \leq 2^{n+1}) \leq \mathbb{P}(D(1, l) \leq 2^n) + \mathbb{P}(D(l, |x|) \leq 2^n) \leq \mathbb{P}(D(1, l) \leq 2^n) + (7l) \cdot \mathbb{P}(D(1, |x| - l) \leq 2^n),$$

31
where in the last inequality we used union bound and the fact that any \( \partial B(l) \) with \( l > 2 \) can be covered by at most \( 7l \) balls of radius 1 and some of these balls must be connected to \( \partial B(|x|) \). If the induction hypothesis can be applied, the choice

\[
l = \exp\left[\left(\log |x|\right) \cdot \frac{2\gamma^{2^n}}{1 + 2\gamma^{2^n}}\right]
\]

ensures that the two terms in the sum above are approximately the same size, since it makes

\[
l^{-2\gamma^{2^n}} = l \cdot |x|^{-2\gamma^{2^n}} = \exp\left[\left(\log |x|\right) \cdot \frac{-4\gamma^{2n+1}}{1 + 2\gamma^{2n}}\right].
\]

Let us denote \( b_n = 2\gamma^{1-n-2^n} \) and suppose \( |x| > b_{n+1} \). Then, it is easy to check that our choice of \( l \) satisfies \( l > b_n \) if and only if

\[
\exp\left[\left(\log 2\right)\gamma^{-n-2^n+1} \cdot \frac{2\gamma^{2^n}}{1 + 2\gamma^{2^n}}\right] \geq \exp\left[\left(\log 2\right)\gamma^{1-n-2^n}\right].
\]

Inequality (33) is equivalent to

\[
\gamma^{-1-2^n} = \gamma^{-n-2^n+1-2^n} \geq 1 + \frac{1}{2\gamma^{2^n}}, \quad \text{or} \quad 1 \geq \gamma^{1+2^n} + \gamma^n.
\]

Since \( \gamma = \frac{a-1}{2} \in (0, 1/2) \) and \( n \geq 0 \), we have that (33) is satisfied. Analogously, we must check that \( |x| - l \geq b_n \). This can be done by noticing

\[
\frac{l}{|x|} = |x|^{-2\gamma^{2^n}} < \exp\left[-\left(\log 2\right)\frac{\gamma^{-n-2^n+1}}{1 + 2\gamma^{2^n}}\right] \leq 2^{-\gamma^{2^n}} \leq 2^{-2},
\]

which implies that \( |x| - l \geq \frac{3}{4} |x| \geq \frac{3}{4} b_n > b_{n-1} \). Thus, we are allowed to use the induction hypothesis. Using the bound

\[
(1 - l/|x|)^{-2\gamma^{2^n}} \leq (1 - 1/4)^{-2\gamma^{2^n}} \leq (4/3)^{2\gamma} < 2,
\]

32
we can write
\[
\mathbb{P}(D(1, |x|) \leq 2^{n+1}) \leq C^n l^{-2\gamma^2 n} + 7l \cdot C^n (|x| - l)^{-2\gamma^2 n} \leq (1 + 14) \cdot C^n \cdot l^{-2\gamma^2 n} \\
\leq C^{n+1} \exp \left( \frac{\log |x|}{1 + 2\gamma^2 n} \right).
\]

Finally, we can estimate
\[
\frac{\mathbb{P}(D(1, |x|) \leq 2^{n+1})}{C^{n+1} |x|^{-2\gamma^2 n+1}} \leq \exp \left( \frac{(\log |x|) \cdot 2\gamma^2 n+1}{1 + 2\gamma^2 n} \right) \leq 1,
\]
since \(1 - \frac{2}{1+2\gamma^2 n} \in (-1, 0)\). \(\square\)

From Proposition 2, we can conclude the following

**Corollary 1.** Let \(0 < \delta < \frac{1}{\log(1/\gamma)}\), where \(\gamma := \frac{a-1}{2}\). Then we have
\[
\lim_{|x| \to \infty} \mathbb{P}(D(1, |x|) \leq \delta \log \log |x|) = 0. \tag{34}
\]

**Proof.** Let us choose
\[
n = \lfloor (\log 2)^{-1} \cdot (\log \delta + \log \log \log |x|) \rfloor,
\]
which makes \(2^n \in \left[ \frac{\delta}{2} \log \log |x|, \delta \log \log |x| \right]\). Notice that \(|x| > 2^{\gamma^{1-n-2n}}\) for large \(|x|\), since \(1/\delta > \log(1/\gamma)\) implies
\[
\log \log 2^{\gamma^{1-n-2n}} = \log \log 2 + (2^n + n - 1) \log(1/\gamma)
\leq \frac{2^n}{\delta} \leq \log \log |x|.
\]

Hence, by Proposition 2 we have
\[
\mathbb{P}(D(1, |x|) \leq 2^n) \leq C^n |x|^{-2\gamma^2 n}
\leq \exp \left[ \log C \cdot n - 2 \log |x| \cdot \gamma \log \log |x| \right]
\leq \exp \left[ \frac{\log C}{\log 2} \cdot \log \log \log |x| + O(1) - 2(\log |x|)^{1-\delta \log(1/\gamma)} \right].
\]

33
which tends to zero since $\delta \log(1/\gamma) < 1$. 

References

[1] Peter Antal and Agoston Pisztora. On the chemical distance for supercritical bernoulli percolation. *The Annals of Probability*, pages 1036–1048, 1996.

[2] Marek Biskup. On the scaling of the chemical distance in long-range percolation models. *The Annals of Probability*, 32(4):2938–2977, 2004.

[3] Marek Biskup. Graph diameter in long-range percolation. *Random Structures & Algorithms*, 39(2):210–227, 2011.

[4] Marek Biskup and Jeffrey Lin. Sharp asymptotic for the chemical distance in long-range percolation. *Random Structures & Algorithms*, 55(3):560–583, 2019.

[5] Erik I Broman and Johan Tykesson. Connectedness of poisson cylinders in euclidean space. In *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, volume 52, pages 102–126. Institut Henri Poincaré, 2016.

[6] Jiří Černý, Serguei Popov, et al. On the internal distance in the interlacement set. *Electronic Journal of Probability*, 17, 2012.

[7] Yinshan Chang. Supercritical loop percolation on $\mathbb{Z}^d$ for $d \geq 3$. *Stochastic Processes and their Applications*, 127(10):3159–3186, 2017.

[8] Jian Ding and Li Li. Chemical distances for percolation of planar gaussian free fields and critical random walk loop soups. *Communications in Mathematical Physics*, 360(2):523–553, 2018.

[9] Alexander Drewitz, Balázs Ráth, and Artëm Sapozhnikov. On chemical distances and shape theorems in percolation models with long-range correlations. *Journal of Mathematical Physics*, 55(8):083307, 2014.

[10] Luiz Fontes and Charles M Newman. First passage percolation for random colorings of $\mathbb{Z}^d$. *The Annals of Applied Probability*, 3(3):746–762, 1993.
[11] Olivier Garet and Régine Marchand. Asymptotic shape for the chemical distance and first-passage percolation on the infinite bernoulli cluster. *ESAIM: Probability and Statistics*, 8:169–199, 2004.

[12] Olivier Garet, Régine Marchand, et al. Large deviations for the chemical distance in supercritical bernoulli percolation. *The Annals of Probability*, 35(3):833–866, 2007.

[13] Geoffrey Grimmett. *Percolation*. Springer, 1999.

[14] Marcelo Hilario, Xinyi Li, and Petr Panov. Shape theorem and surface fluctuation for poisson cylinders. *Electronic Journal of Probability*, 24, 2019.

[15] Charles M Newman and Lawrence S Schulman. One dimensional $1/|j - i|^s$ percolation models: The existence of a transition for $s \leq 2$. *Communications in Mathematical Physics*, 104(4):547–571, 1986.

[16] Augusto Teixeira and Daniel Ungaretti. Ellipses percolation. *Journal of Statistical Physics*, 168(2):369–393, 2017.

[17] Pieter Trapman. The growth of the infinite long-range percolation cluster. *The Annals of Probability*, 38(4):1583–1608, 2010.

[18] Johan Tykesson and David Windisch. Percolation in the vacant set of poisson cylinders. *Probability theory and related fields*, 154(1-2):165–191, 2012.

[19] D Ungaretti. *Planar continuum percolation: heavy tails and scale invariance*. PhD thesis, IMPA, August 2017.

[20] Jacob van den Berg. A note on disjoint-occurrence inequalities for marked poisson point processes. *Journal of applied probability*, pages 420–426, 1996.

[21] Jacob van den Berg and Harry Kesten. Inequalities with applications to percolation and reliability. *Journal of applied probability*, pages 556–569, 1985.