Alpha condensates and nonlocalized cluster structures

Yasuro Funaki

RIKEN, Nishina Center, Wako 351-0198, Japan
E-mail: funaki@riken.jp

Abstract. We discuss a container structure for non-gaslike cluster states, in which single Tohsaki-Horiuchi-Schuck-Röpke (THSR) wave functions are shown to be almost 100 % equivalent to the full solutions of the corresponding RGM/GCM equations, for the inversion doublet band states in \(^{20}\)Ne, \(\alpha\)-linear-chain states, and \(\alpha + \alpha + \Lambda\) cluster states in \(^{9}\)Be. The recognition of the fact that the THSR wave function describes well not only gaslike cluster states but also non-gaslike cluster states is a recent remarkable development of nuclear cluster physics. This fact tells us that the cluster structure is composed of cluster-mean-field motion under the constraint of inter-cluster Pauli repulsion, in which we call the cluster-mean-field potential the container. We demonstrate that the evolution of the cluster structure of a nucleus is governed by the size parameter of the cluster-mean-field potential (container), for \(^{16}\)O nucleus.

1. Introduction
The gaslike cluster structure is a novel type of nuclear clustering. The structure was first suggested in the study of \(^{12}\)C, in which the Hoyle state (the second \(0^+\) state of \(^{12}\)C at 7.65 MeV) was found to have a large root-mean-square (r.m.s.) radius and a loosely coupled \(S\)-wave component dominantly with respect to the relative \(\alpha-\alpha\) motions. The microscopic cluster model calculations such as the resonating group method (RGM), generator coordinate method (GCM), and orthogonality condition model (OCM) reproduced almost of all experimental data available of the Hoyle state and revealed its structure [1]. After that, about 20 years later, the Hoyle state was reinvestigated by using the \(\alpha\)-condensation-like wave function, which we now refer to as the Tohsaki-Horiuchi-Schuck-Röpke (THSR) wave function [2], focusing on an analogy to the \(\alpha\) condensation in nuclear matter [3]. After it was found in 2003 that the fully-microscopic wave functions of the 3\(\alpha\) RGM and GCM obtained long time ago are almost 100 % equivalent to single THSR wave functions, we have had a new understanding of the Hoyle state that is the 3\(\alpha\) condensate, where the 3\(\alpha\) particles loosely couple with each other like a gas and occupy the lowest energy \(S\)-orbit of mean-field-like single-\(\alpha\) potential [4]. This equivalence relation of the RGM/GCM wave function to single THSR wave function is also found in the case of \(^{8}\)Be ground state [5].

In contrast, non-gaslike cluster states have been recognized as having localized structures of clusters, which are obviously quite different from the \(\alpha\) condensate states mentioned above. A typical example is the inversion doublet band states of \(\alpha + ^{16}\)O in \(^{20}\)Ne [6], which have parity-violating intrinsic deformation. Another example is a linear-chain-state of \(\alpha\) particles, where all \(\alpha\) clusters are aligned on a line with a rigid-body configuration [7]. \(^{9}\)\(\Lambda\)Be nucleus is also such...
an example, where the additional \( \Lambda \) particle shrinks the core \(^8\)Be and compact and non-gaslike \(^2\) cluster structure seems to be realized. Thus, the concept of the localized clustering is the important basis to understand these structure states. The very recent calculations, however, upset this common sense by revealing that all those structure states can also be represented by the single configurations of the THSR wave function with nearly 100% accuracy \([8, 9, 10, 11, 12]\]. The THSR wave function has a container structure that the clusters are nonlocalized and move around in a whole nuclear volume (container) characterized by a size parameter “\( B \)”. We will report in this contribution that the container structure is realized not only for the gaslike cluster states but also for the non-gaslike cluster states, by showing that the THSR-type wave functions describe well non-gaslike cluster states mentioned above.

2. THSR description of inversion-doublet bands in \(^{20}\)Ne

In this section, we show that the \(^{16}\)O + \( \alpha \) Brink-GCM wave functions of the states belonging to the inversion-doublet bands of \(^{20}\)Ne have almost 100% square overlaps with single THSR wave functions. The THSR wave function was introduced in order to describe cluster-gaslike states, especially \( \alpha \)-condensate-like states.

\[
\Psi_{\text{THSR}}^{J} = C_{J} P_{M0}^{J} \int dR \exp \left\{ -\frac{4(R_{x}^{2} + R_{y}^{2})}{5\beta_{z}^{2}} - \frac{4R_{z}^{2}}{5\beta_{z}^{2}} \right\} \psi_{^{16}\text{O} + \alpha}^{B} \left( \frac{4}{5} R_{x} - \frac{1}{5} R_{z} \right),
\]

(1)

\[
= C_{J}^{'} \exp \left(-\frac{10}{b^{2}} \mathbf{X}_{G}^{2} \right) P_{M0}^{J} A \left[ \exp \left\{ -\frac{8(r_{x}^{2} + r_{y}^{2})}{5B_{z}^{2}} - \frac{8r_{z}^{2}}{5B_{z}^{2}} \right\} \phi(^{16}\text{O})\phi(\alpha) \right],
\]

(2)

where \( B_{k}^{2} = b^{2} + 2\beta_{k}^{2} (k = x, y, z), r = \mathbf{X}_{\alpha} - \mathbf{X}_{0}, \) and \( \mathbf{X}_{G} \) is total c.o.m. coordinate. \( \Psi_{^{16}\text{O} + \alpha}^{B} (\frac{4}{5} R_{x} - \frac{1}{5} R_{z}) \) is the \( \alpha + ^{16}\text{O} \) Brink wave function and \( C_{J} \) and \( C_{J}^{'} \) are normalization constants. This THSR wave function \( \Psi_{J}^{\text{THSR}} \) has positive parity and therefore can have only even angular momentum \( J \). The negative-parity THSR wave function with odd \( J \) is constructed by the following procedure

\[
\phi_{\text{THSR}, J}^{-} = \lim_{S_{x} \to 0} C_{J}^{-} P_{M0}^{J} \frac{1 - P_{\pi}}{2} A \left[ \exp \left\{ -\frac{8(r_{x}^{2} + r_{y}^{2})}{5B_{z}^{2}} - \frac{8(r_{z} - S_{z})^{2}}{5B_{z}^{2}} \right\} \phi(^{16}\text{O})\phi(\alpha) \right],
\]

(3)
where $C_J$\(^{(-)}\) is normalization constant and \((1 - P_x)/2\) is the projection operator onto negative parity. If $B_x = B_z = B$, $\Phi^{\text{THSR.}(J^-)}$ has the following form

$$\Phi^{\text{THSR.}(J^-)} = D_J^{(-)} A \left[ r^J Y_{JM}(\hat{r}) \exp \left\{ \frac{-8r^2}{5B^2} \right\} \phi(\alpha) \right]. \quad (4)$$

In Fig. 1, we show the calculated energy spectrum in comparison with the experimental data. The squared overlap values of the single THSR wave function with the $\alpha + ^{16}\text{O}$ Brink-GCM wave function are shown in parentheses. As mentioned above, we can see that both wave functions are almost 100% equivalent to each other. The inversion-doublet rotational bands in $^{20}\text{Ne}$ consist of the plus-parity band which is the ground-state band and the minus-parity band which is the $K^\pi = 0^-$ band upon the $1^-$ state at $E_x = 5.80$ MeV. The minus-parity band levels are all above the $^{16}\text{O} + \alpha$ threshold and have large $\alpha$-decay widths whose reduced widths are comparable with the Wigner limit value. In general the inversion-doublet rotational bands are generated from a parity-violating intrinsic deformed state. As the parity-violating intrinsic state of the inversion-doublet bands, spatially-localized cluster structure of $^{16}\text{O} + \alpha$ is assigned [6]. Therefore the existence of the inversion-doublet rotational bands has been regarded as a convincing evidence of the spatial localization of clusters. On the other hand, the THSR wave function describes the nonlocalized motion of clusters. The almost 100% equivalence of the Brink-GCM wave functions and single THSR wave functions for the $^{20}\text{Ne}$ inversion-doublet band states is thus really striking.

**Figure 2.** Nucleon-density distribution of the $2\alpha$ prolate THSR wave function with $(\beta_x, \beta_y, \beta_z) = (1.78, 1.78, 8.75 \ \text{fm})$ [10].

**Figure 3.** Nucleon-density distribution of the $^{16}\text{O} + \alpha$ hybrid-Brink-THSR wave function with $S_z = 0.6 $ fm and $(\beta_x, \beta_y, \beta_z) = (0.9, 0.9, 2.5 \ \text{fm})$ [10].

The problem of localized clustering vs nonlocalized clustering was already noticed in 2002 in the THSR study of $\alpha$-$\alpha$ clustering in $^8\text{Be}$. Ref. [5] reported that the Brink-GCM wave function of the ground state of $^8\text{Be}$ is 100% equivalent to single THSR wave function. This equivalence looks like questioning the well-known dumbbell picture for the intrinsic shape of $^8\text{Be}$. However it was shown recently [10] that the nucleon-density distribution of the intrinsic THSR wave function of $2\alpha$ ground state displays clearly localized clustering of $\alpha$-$\alpha$ as seen in Fig. 2. The reason why the THSR wave function which adopts the relative-motion wave function expressing the nonlocalized clustering yields the nucleon-density distribution showing the localized clustering is attributed to the inter-cluster Pauli repulsion which makes two clusters stay apart from each other avoiding
close coming to each other. Figure 3 shows the nucleon intrinsic density distribution of the $^{16}\text{O} + \alpha$ hybrid-Brink-THSR wave function Eq. (3), which is defined as the one-body density of the intrinsic wave function,

$$\mathbf{\phi^{hyb-B-THSR}} = A \exp \left\{ - \frac{8(r_x^2 + r_y^2)}{5B_x^2} - \frac{8(r_z - S_z)^2}{5B_z^2} \right\} \mathbf{\phi^{(16}\text{O})\phi(\alpha)}, \tag{5}$$

where $S_z$ is set to be 0.6 fm and $(\beta_x, \beta_y, \beta_z) = (0.9, 0.9, 2.5 \text{ fm})$ [10]. We observe in this figure that, in spite of the small value of $S_z = 0.6$ fm, the inter-cluster distance between $^{16}\text{O}$ and $\alpha$ is about 3.6 fm. Clearly the large inter-cluster distance of about 3.6 fm cannot be attributed to the small value of $S_z = 0.6$ fm, but should be attributed to the effective spatial localization of $^{16}\text{O}$ and $\alpha$ clusters in the prolate THSR wave function with $(\beta_x, \beta_y, \beta_z) = (0.9, 0.9, 2.5 \text{ fm})$ with $S_z = 0$, which is nothing but the intrinsic THSR wave function giving the minimum energy for the $0^+$ state.

3. THSR description of $3\alpha$ and $4\alpha$ linear-chain-states

If we follow the conclusion of the study of the inversion doublet bands of $^{20}\text{Ne}$, the basic feature of the cluster dynamics is nonlocalized clustering and the localization of clusters is due to the inter-cluster Pauli repulsion. A confirmation of this conclusion was given in Ref. [11] by the investigation of the linear-chain structure of $\alpha$ clusters which is a typical and well-known example of localized cluster structure. It was reported that the Brink-GCM wave functions of the $3\alpha$ and $4\alpha$ linear-chain states have about 98 % and 94 % square overlaps with $3\alpha$ and $4\alpha$ single THSR wave functions, respectively. These large overlap values are to be compared with the maximum square overlap values with $3\alpha$ and $4\alpha$ single Brink wave functions which are 78 % and 48 %, respectively. These results urge us to regard that the Brink-GCM wave functions of the $3\alpha$ and $4\alpha$ linear-chain states have one-dimensional $\alpha$-condensate-like character rather than the conventional picture of localized $\alpha$ clusters on a line. However, just like the case of $^{16}\text{O} + \alpha$ inversion doublet, the nucleon-density distribution of the intrinsic THSR wave functions of linear-chain states show the localized $\alpha$ clusters because of inter-$\alpha$ Pauli repulsion, which is displayed in Fig. 4.

4. THSR description of $^{9}\text{Be}$

The above-mentioned THSR descriptions of localized clustering states is not only for ordinary nuclei but also possible for hypernuclei. For example, the so-called Hyper-THSR wave function
describing the $2\alpha + \Lambda$ hypernucleus with good angular momentum can be introduced as
follows [12]:

$$
\Psi^H_J(\beta) = \hat{P}_{MK}^J \Phi^H_{2\alpha-\Lambda}^{THSR}(\beta), \quad \Phi^H_{2\alpha-\Lambda}^{THSR}(\beta) = \Phi^{THSR}_{2\alpha}(\beta) \sum_{\kappa} f_{\kappa}(\beta, \kappa) \varphi_{\Lambda}(\kappa),
$$

(6)

where the $\Lambda$ particle simply couples to the $2\alpha$ core nucleus in an $S$ wave. Its radial part is
expanded in terms of Gaussian basis functions, $\varphi_{\Lambda}(\kappa) = (\pi/2\kappa)^{-3/4} \exp(-\kappa r_{2\alpha-\Lambda}^2)$ with
respect to the width parameter $\kappa$, where $r_{2\alpha-\Lambda} = r_{\Lambda} - (X_1 + X_2)/2$. Here we also assume an axially-
symmetric deformation in Eq. (6), $\beta_x = \beta_y = 0 \equiv \beta_{\perp} \neq \beta_z$. The coefficients of the expansion
$f_{\kappa}(\beta_{\perp}, \beta_z, \kappa)$ are then determined by solving the following Hill-Wheeler-type equation of motion,

$$
\sum_{\kappa'} \langle \hat{P}_{MK}^J(\beta_{\perp}, \beta_z) \varphi_{\Lambda}(\kappa') | \hat{H} - E(\beta_{\perp}, \beta_z) | \hat{P}_{MK}^J(\beta_{\perp}, \beta_z) \varphi_{\Lambda}(\kappa') \rangle f_{\kappa}(\beta_{\perp}, \beta_z, \kappa') = 0.
$$

(7)

This Hyper-THSR wave function is characterized only by the parameters $\beta_{\perp}$ and $\beta_z$, which
correspond to a spatial extension of the whole nucleus. It is reported that the $\Lambda$ particle invokes
spatial core shrinkage in many hypernuclei. It should be mentioned that such core shrinkage
effect is expected to be taken into account very naturally by this parametrization of $\beta$, in general
hypernuclei, since it specifies the dilatation of the whole nucleus. In the present work, only the
$S$-wave component of the $\Lambda$ particle is considered, for simplicity.

In order to compare the single Hyper-THSR wave function with the Brink-GCM wave
function, we calculate the following squared overlap:

$$
O_J(\beta_{\perp}, \beta_z) = \frac{|\langle \Psi^H_J(\beta_{\perp}, \beta_z) | \tilde{\Psi}^B_J \rangle|^2}{\langle \Psi^H_J(\beta_{\perp}, \beta_z) | \Psi^H_J(\beta_{\perp}, \beta_z) \rangle \langle \Psi^B_J | \Psi^B_J \rangle},
$$

(8)

with

$$
\tilde{\Psi}^B_J = \sum_{R,S} f^{(J,\lambda=0)}(R, S) |u_J(R), \psi_{\lambda=0}(S) \rangle J.
$$

(9)

The above wave function $\tilde{\Psi}^B_J$ is the Brink-GCM wave function projected onto the model space
with the angular-momentum channel $(L, \lambda) = (J, 0)$.

The squared overlap between the single THSR wave functions and the Brink-GCM wave
functions defined in Eq. (9) are 99.5 %, 99.4 %, and 97.7 % for the $0^+$, $2^+$, and $4^+$ states,
respectively. The calculated r.m.s. radii are 2.31 fm, 2.29 fm, and 2.20 fm, for the $0^+$, $2^+$, and
$4^+$ states, respectively. These values are all much smaller than the calculated r.m.s. value for
$^8$Be, 2.8 fm, and therefore in $^9$Be very localized $2\alpha + \Lambda$ cluster structure is realized, due to the
shrinkage effect of the $\Lambda$ particle. Thus this equivalence result means that THSR wave function
works very well for localized cluster structure states in hypernuclei.

5. Evolution of cluster structures

We here demonstrate for the $4\alpha$ system that the evolution of cluster states can be expressed
by dilatation of a container. We first obtain the eigenstates of $^{16}$O, $\Phi^{(4\alpha\lambda=0)}_{4\alpha}$, by solving the Hill-
Wheeler equation by taking $\beta$ as generator coordinates with a constraint of $\beta_0 \equiv \beta_{\perp} = \beta_z$.
These wave functions are expressed as the superposed ones of the THSR wave function with
different $\beta_0$ values. It is interesting to see how these superposed wave functions are expressed by
a single component THSR wave function $\Phi^{THSR,(4\alpha\lambda=0)}_{4\alpha}(\beta_0)$, since this point is concerned with the
container picture, in which the $\beta$ parameter characterizes a shape of the whole nucleus. We have
reported in Ref. [13] that the $(0^+_2)_{THSR}$, $(0^+_3)_{THSR}$, and $(0^+_4)_{THSR}$ states, i.e. $\lambda = 2, 3, 4$ for the
wave functions, $\Phi_{4\alpha,\lambda}^{J=0}$, respectively, in the THSR ansatz correspond to the $(0^+_2)_{\text{OCM}}$, $(0^+_3)_{\text{OCM}}$, and $(0^+_4)_{\text{OCM}}$ states in the OCM calculation [14], which have the structures of $^{12}\text{C}(0^+)$ and $\alpha$ in an $S$-wave, of $^{12}\text{C}(0^+)$ and $\alpha$ in a higher nodal $S$-wave, and of the $4\alpha$ condensate, respectively.

We first construct orthonormal wave functions from the THSR wave functions as follows:

$$\tilde{\Phi}_\lambda(\beta_0) = N_\lambda P_{\lambda-1} \Phi_{4\alpha,\lambda}^{\text{THSR}(J=0)}(\beta_0), \quad (\lambda = 1, \cdots, 4),$$

where $N_\lambda$ is the normalization constant and $P_{\lambda-1}$ are projection operators defined by $P_0 = 1$ and for $\lambda \geq 2$ by

$$P_{\lambda-1} = 1 - \sum_{k=1}^{\lambda-1} |\Phi_{4\alpha,k}^{J=0}\rangle \langle \Phi_{4\alpha,k}^{J=0}|, \quad (\lambda = 2, 3, 4).$$

The wave function $\tilde{\Phi}_\lambda(\beta_0)$ depends on the parameter $\beta_0$ and is orthogonal to the wave functions $\Phi_{4\alpha,\lambda}^{J=0}$ of the $(0^+_\lambda)_{\text{THSR}}$ states with $\lambda' < \lambda$. We then calculate the following squared overlap amplitudes between the wave functions $\tilde{\Phi}_\lambda(\beta_0)$ and $\Phi_{4\alpha,\lambda}^{J=0}$,

$$\Theta_\lambda(\beta_0) = \left| \left\langle \Phi_{4\alpha,\lambda}^{J=0} \right| \tilde{\Phi}_\lambda(\beta_0) \right|^2, \quad (\lambda = 1, \cdots, 4).$$

Table 1. The largest values of the squared overlaps in Fig. 5 are shown for the four THSR states, $(0^+_\lambda)_{\text{THSR}} (\lambda = 1, \cdots, 4)$, together with the corresponding $\beta_0$ values. Figure taken from Ref. [13].

| $\lambda$ | $\Theta_\lambda(\beta_0)$ | $\beta_0$ (fm) |
|-----------|----------------|----------------|
| 1         | 0.98           | 1.2            |
| 2         | 0.98           | 2.5            |
| 3         | 0.98           | 4.0            |
| 4         | 0.96           | 6.5            |

In Fig. 5 we show $\Theta_\lambda(\beta_0)$ of Eq. (12) as a function of $\beta_0$. For any wave functions $\Phi_{4\alpha,\lambda}^{J=0}$ ($\lambda = 1, \cdots, 4$), $\Theta_\lambda(\beta_0)$ ($\lambda = 1, \cdots, 4$) takes values close to 100 % at optimal values of $\beta_0$.

We also show in Table 1 the largest squared overlaps and the corresponding $\beta_0$ values for the four states. We can see that the optimal $\beta_0$ values increase from the $(0^+_2)_{\text{THSR}}$ to the $(0^+_4)_{\text{THSR}}$ states. The r.m.s. radii also increase with the increase of $\beta_0$. This means that the
clustering excitations, which are usually described by the eigenfunctions of the Hill-Wheeler equation, are very accurately characterized by the variation of the $\beta_0$ value, i.e. dilatation of the container, from the most compact $(0^+_1)_{\text{THSR}}$ state to the most dilute gaslike $(0^+_4)_{\text{THSR}}$ state with the 4\(\alpha\) condensate structure, through the intermediate density $(0^+_2)_{\text{THSR}}$ states with the $^{12}\text{C} + \alpha$ cluster structures. This means that the evolution of the cluster excitations, which are given as a result of solving the Hill-Wheeler equation, can be described by dilatation of the container, from a smaller value of $\beta_0$ to a larger one. This also gives a support of the correctness of the container picture. The more sophisticated description of $^{12}\text{C} + \alpha$ configuration on the container picture is to use an extended version of the THSR wave function in Ref. [15].

6. Conclusion

We demonstrated that the single THSR wave functions is almost 100 % equivalent to the full solutions of the RGM/GCM equations for the inversion doublet band states in $^{20}\text{Ne}$, $\alpha$-linear-chain states, and $\alpha + \alpha + \Lambda$ cluster states in $^{9}\text{Be}$. The recognition of the fact that the THSR wave function describes well not only cluster-gaslike states but also non-gaslike cluster states is a recent remarkable development of nuclear cluster physics. This fact tells us that the cluster structure is composed of cluster-mean-field motion under the constraint of inter-cluster Pauli repulsion. The single-cluster wave function is the Gaussian wave packet around the c.o.m. coordinate which is the lowest orbit of the cluster-mean-field potential that we call the container. The evolution of the cluster structure of a nucleus is governed by the size parameter of the cluster-mean-field potential (container). We demonstrated that the $^{16}\text{O}$ nucleus, starting from the compact shell-model structure for the ground state, develops the cluster structure, from the $\alpha + ^{12}\text{C}$ structure to the $4\alpha$ structure, with the increase of the size parameter $\beta$.

References

[1] Fujiwara Y, Horiuchi H, Ikeda K, Kamimura M, Katô K, Suzuki Y, and Uegaki E, Prog. Theor. Phys. Supple. 68 (1980) Chapt.2, p.29
[2] Tohsaki A, Horiuchi H, Schuck P, and Röpke G, Phys. Rev. Lett. 2001 87 192501
[3] Röpke G, Schenell A, Schuck P, and Nozières P, 1998 Phys. Rev. Lett. 80 3177
[4] Funaki Y, Tohsaki A, Horiuchi H, Schuck P, and Röpke G, 2003 Phys. Rev. C 67 051306(R)
[5] Funaki Y, Horiuchi H, Tohsaki A, Schuck P, and Röpke G, 2002 Prog. Theor. Phys. 108 297
[6] Horiuchi H and Ikeda K, 1968 Prog. Theor. Phys. 40 277
[7] Morinaga H, 1966 Phys. Lett. 21 78
[8] Zhou B, Funaki Y, Horiuchi H, Ren Z, Röpke G, Schuck P, Tohsaki A, Xu C, and Yamada T, 2012 Phys. Rev. C 86 014301
[9] Zhou B, Funaki Y, Horiuchi H, Ren Z, Röpke G, Schuck P, Tohsaki A, Xu C, and Yamada T, 2013 Phys. Rev. Lett. 110 262501
[10] Zhou B, Funaki Y, Horiuchi H, Ren Z, Röpke G, Schuck P, Tohsaki A, Xu C, and Yamada T, 2014 Phys. Rev. C 89 034319
[11] Suhara T, Funaki Y, Zhou B, Horiuchi H, and Tohsaki A, 2014 Phys. Rev. Lett. 112 062501
[12] Funaki Y, Yamada T, Hiyama E, Zhou B, and Ikeda K, arXiv: 1405.6067, to appear in 2014 Prog. Theor. Ext. Phys.
[13] Funaki Y, Yamada T, Tohsaki A, Horiuchi H, Röpke G, and Schuck P, 2010 Phys. Rev. C 82 024312
[14] Funaki Y, Yamada T, Horiuchi H, Röpke G, Schuck P, and, Tohsaki A, 2008 Phys. Rev. Lett. 101 082502
[15] Funaki Y, arXiv: 1408.5855