ONSAGER’S CONJECTURE ALMOST EVERYWHERE IN TIME

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ABSTRACT. In recent works by Isett [16], and later by Buckmaster, De Lellis, Isett and Székelyhidi Jr. [2], iterative schemes were presented for constructing solutions belonging to the Hölder class $C^{1/5-\varepsilon}$ of the 3D incompressible Euler equations which do not conserve the total kinetic energy. The cited work is partially motivated by a conjecture of Lars Onsager in 1949 relating to the existence of $C^{1/3-\varepsilon}$ solutions to the Euler equations which dissipate energy. In this note we show how the later scheme can be adapted in order to prove the existence of non-trivial Hölder continuous solutions which for almost every time belong to the critical Onsager Hölder regularity $C^{1/3-\varepsilon}$ and have compact temporal support.

0. Introduction

In what follows $\mathbb{T}^3$ denotes the 3-dimensional torus, i.e. $\mathbb{T}^3 = S^1 \times S^1 \times S^1$. Formally, we say $(v,p)$ solves the incompressible Euler equations if

\[
\begin{cases}
\partial_t v + \text{div} \ v \otimes v + \nabla p = 0 \\
\text{div} \ v = 0
\end{cases}
\]  

Suppose $v$ is such a solution, then we define its kinetic energy, as

\[
E(t) := \frac{1}{2} \int_{\mathbb{T}^3} |v(x,t)|^2 \, dx.
\]

A simple calculation applying integration by parts yields that for any classical solution of (1) the kinetic energy is in fact conserved in time. This formal calculation does not however hold for distributional solutions to Euler (cf. [20, 21, 3, 7, 22, 9]).

In fact in the context of 3-dimensional turbulence, flows dissipating energy in time have long been considered. A key postulate of Kolmogorov’s K41 theory [17] is that for homogeneous, isotropic turbulence, the dissipation rate is non-vanishing in the inviscid limit. In particular, defining the structure functions for homogeneous, isotropic turbulence

\[
S_p(\ell) := \left\langle \left( (v(x + \ell) - v(x)) \cdot \frac{\ell}{\ell} \right)^p \right\rangle,
\]
where \( \ell \) denotes a spatial vector of length \( \ell \), Kolmogorov’s famous four-fifths law can be stated as

\[
S_3(\ell) = -\frac{4}{5} \varepsilon_d \ell,
\]

where here \( \varepsilon_d \) denotes the mean energy dissipation per unit mass. More generally, Kolmogorov’s scaling laws can be stated as

\[
S_p(\ell) = C_p \varepsilon_d^{p/3} \ell^{p/3},
\]

for any positive integer \( p \).

A well known consequence of the above scaling laws is the Kolmogorov spectrum, which postulates a scaling relation on the ‘energy spectrum’ of a turbulent flow (cf. [15, 13]). It was this observation that provided motivation for Onsager to conjecture in his famous note [19] on statistical hydrodynamics, the following dichotomy:

(a) Any weak solution \( v \) belonging to the Hölder space \( C^{\theta} \) for \( \theta > \frac{1}{3} \) conserves the energy.

(b) For any \( \theta < \frac{1}{3} \) there exist weak solutions \( v \in C^{\theta} \) which do not conserve the energy.

Part (a) of this conjecture has since been resolved: it was first considered by Eyink in [12] following Onsager’s original calculations and later proven by Constantin, E and Titi in [5]. Subsequently, this later result was strengthened by showing that under weakened assumptions on \( v \) (in terms of Besov spaces) kinetic energy is conserved [11, 4].

Part (b) remains an open conjecture and is the subject of this note. The first constructions of non-conservative Hölder-continuous \( (C^{1/10 - \varepsilon}) \) weak solutions appeared in work of De Lellis and Székelyhidi Jr. [8], which itself was based on their earlier seminal work [10] where continuous weak solutions were constructed. Furthermore, it was shown in the mentioned work that such solutions can be constructed obeying any prescribed smooth non-vanishing energy profile. In recent work [16], P. Isett introduced a number of new ideas in order to construct non-trivial \( 1/5 - \varepsilon \) Hölder-continuous weak solutions with compact temporal support. This construction was later improved by Buckmaster, De Lellis and Székelyhidi Jr. [2], following more closely the earlier work [10, 8], in order construct \( 1/5 - \varepsilon \) Hölder-continuous weak solution obeying a given energy profile.

In this note we give a proof of the following theorem.

**Theorem 0.1.** There exists is a non-trivial continuous vector field \( v \in C^{1/5 - \varepsilon}(T^3 \times (-1, 1), \mathbb{R}^3) \) with compact support in time and a continuous scalar field \( p \in C^{2/5 - 2\varepsilon}(T^3 \times (-1, 1)) \) with the following properties:

(i) The pair \((v, p)\) solves the incompressible Euler equations (1) in the sense of distributions.
(ii) There exists a set \( \Omega \subset (-1,1) \) of Hausdorff dimension strictly less than 1 such that if \( t \notin \Omega \) then \( v(\cdot,t) \) is Hölder \( C^{1/3-\varepsilon} \) continuous and \( p \) is Hölder \( C^{2/3-2\varepsilon} \) continuous.

Relation to intermittency. The theory of intermittency is born of an effort to explain the experimental and numerical evidence (e.g. [1]) of measurable discrepancies from the scaling laws (3) (cf. [14]). In this direction, Mandelbrot conjectured [18] that at the inviscid limit, turbulence concentrates (in space) on a fractal set of Hausdorff dimension strictly less than 3.

It is interesting to note that the solutions constructed in order to prove the above theorem have a fractal structure in time: namely, the set of times for which \( v \) is not Hölder \( C^{1/3-\varepsilon} \) continuous is contained in a Cantor-like set with Hausdorff dimension strictly less than 1. Since the phenomena observed does not relate to the structure functions from which intermittency was originally postulated it is clearly far-fetched to label such a phenomena as intermittency. Nevertheless, it is the opinion of the author that the parallels to the notion of intermittency remain of interest.

0.1. Euler-Reynolds system and the convex integration scheme. In order to prove Theorem 0.1 we construct an iteration scheme in the style of [2], which is itself based on the schemes presented in [10, 8]. At each step \( q \in \mathbb{N} \) we construct a triple \((v_q, p_q, \hat{R}_q)\) solving the Euler-Reynolds system (see Definition 2.1 in [10]):

\[
\begin{align*}
\partial_t v_q + \text{div} (v_q \otimes v_q) + \nabla p_q &= \text{div} \hat{R}_q \\
\text{div} v_q &= 0.
\end{align*}
\]

The initial triple \((v_0, p_0, \hat{R}_0)\) will be non-trivial with compact support in time; all triples thereafter will be defined inductively as perturbations of the proceeding triples. The perturbation

\[ w_q := v_q - v_{q-1}, \]

will be composed of weakly interacting perturbed Beltrami flows (see Section 1) oscillating at frequency \( \lambda_q \), defined in such a way to correct for the previous Reynolds error \( \hat{R}_{q-1} \).

\[ ^1 \text{More precisely, the Hausdorff dimension } d \text{ is such that } 1 - d > C\varepsilon^2 \text{ for some positive constant } C. \]
In order to ensure convergence of the sequence $\nu_q$ to a continuous weak $C^{1/5-\varepsilon}$ solution of Euler, we will require the following estimates to be satisfied

\begin{equation}
\|w_q\|_0 + \frac{1}{\lambda_q} \|\partial_t w_q\|_0 + \frac{1}{\lambda_q^2} \|w_q\|_1 \leq \lambda_q^{-1/5+\varepsilon_0} \quad (5)
\end{equation}

\begin{equation}
\|p_q - p_{q-1}\|_0 + \frac{1}{\lambda_q} \|\partial_t (p_q - p_{q-1})\|_0 + \frac{1}{\lambda_q^2} \|p_q - p_{q-1}\|_2 \leq \lambda_q^{-2/5+2\varepsilon_0} \quad (6)
\end{equation}

\begin{equation}
\left\|\hat{R}_q\right\|_0 + \frac{1}{\lambda_q} \left\|\hat{R}_q\right\|_1 \leq \lambda_q^{-2/5+2\varepsilon_0} \quad (7)
\end{equation}

for some $\varepsilon_0 > 0$ strictly smaller than $\varepsilon$. Here and throughout the article, $\|\cdot\|_\beta$ for $\beta = m + \kappa$, $\beta \in \mathbb{N}$ and $\kappa \in [0,1)$ will denote the usual spatial Hölder $C^{m,\kappa}$ norm. As a minor point of deviation from [2], we keep track of second order spatial derivative estimates of $p_q - p_{q-1}$, whereas in [2] first order estimates – which in the present work are implicit by interpolation – were sufficient. These second order estimates will be used in order to obtain slightly improved bounds on the Reynolds stress (see Section 5).

It is perhaps worth noting that aside from the second order estimate on $p_q - p_{q-1}$, up to a constant multiple, the above estimates are consistent with the estimates given in [2].

In order to ensure that our sequence convergences to a non-trivial solution, we will impose the addition requirement that

\begin{equation}
\sum_{q=1}^{\infty} \|w_q\|_0 < \frac{1}{2} \|\nu_0\|_0, \quad (8)
\end{equation}

for times $t \in [-1/8, 1/8]$.

The principle new idea of this work is that in addition to the estimates given above, we will keep track of sharper, time localized estimates. As a consequence of these sharper estimates, it can be shown that for any given time $t \in (-1, 1)$ outside a prescribed set $\Omega$ of Hausdorff dimension strictly less than 1, there exists a $N = N(t)$ such that

\begin{equation}
\|w_q\|_0 + \frac{1}{\lambda_q} \|\partial_t w_q\|_0 + \frac{1}{\lambda_q^2} \|w_q\|_1 \leq \lambda_q^{-1/3+\varepsilon_0} \quad (9)
\end{equation}

\begin{equation}
\|p_q - p_{q-1}\|_0 + \frac{1}{\lambda_q} \|\partial_t (p_q - p_{q-1})\|_0 + \frac{1}{\lambda_q^2} \|p_q - p_{q-1}\|_2 \leq \lambda_q^{-2/3+2\varepsilon_0} \quad (10)
\end{equation}

\begin{equation}
\left\|\hat{R}_q\right\|_0 + \frac{1}{\lambda_q} \left\|\hat{R}_q\right\|_1 \leq \lambda_q^{-2/3+2\varepsilon_0} \quad (11)
\end{equation}

for every $q \geq N$.  

\textsuperscript{2}In [2] the estimates corresponding to (5)-(7) are written in terms of a sequence of parameters $\delta_q$ which in the context of the present paper are defined to be $\delta_q := \lambda_q^{-2/5+2\varepsilon_0}$ (cf. Section 3 and Section 6).

\textsuperscript{3}Here and throughout the paper we suppress the dependence on the time variable $t$.  

0.2. The main iteration proposition and the proof of Theorem 0.1

Proposition 0.2. For every small \( \varepsilon_0 > 0 \), there exists an \( \alpha > 1 \), \( d < 1 \) and a sequence of parameters \( \lambda_0, \lambda_1, \ldots \) satisfying \( 1/2\lambda_0^{\alpha^n} < \lambda_q < 2\lambda_0^{\alpha^n} \) such that the following holds. A sequence of triples \((v_q, p_q, R_q)\) can be constructed with temporal support confined to \([-1/2, 1/2]\] solving (4) and satisfying the estimates (5-8). Moreover, for any \( \delta > 0 \), there exists an integer \( M \) such that if \( \Xi_M \) denotes the set of times \( t \) such that there exists a \( q \geq M \) satisfying either

\[
\|w_q\|_0 + \frac{1}{\lambda_q}\|w_q\|_1 > \lambda_q^{-1/3+\varepsilon_0}, \text{ or }
\]

\[
\|p_q - p_{q-1}\|_0 + \frac{1}{\lambda_q}\|p_q - p_{q-1}\|_1 > \lambda_q^{-2/3+2\varepsilon_0},
\]

then there exists a cover of \( \Xi_M \) consisting of a sequence of balls of radius \( r_i \) such that

\[
\sum_{i} r_i^d < \delta.
\]

Proof of Theorem 0.1. Fix \( \varepsilon_0 = \varepsilon/2 \) and let \((v_q, p_q, R_q)\) be a sequence as in Proposition 0.2. It follows then easily that \((v_q, p_q)\) converge uniformly to a pair of continuous functions \((v, p)\) satisfying \((1)\), having compact temporal support. Moreover, by interpolating the inequalities \((5)\) and \((6)\) we obtain that \(v_q\) converges in \(C^{1/5-\varepsilon}\) and \(p_q\) in \(C^{2/5-2\varepsilon}\).

In order to prove (ii) we first fix \( \delta > 0 \) and let \( M \) and \( \Xi_M \) be as in Proposition 0.2. Hence by assumption if \( t \notin \Xi_M \)

\[
\|w_q\|_0 + \frac{1}{\lambda_q}\|w_q\|_1 \leq \lambda_q^{-1/3+\varepsilon_0}
\]

\[
\|p_q - p_{q-1}\|_0 + \frac{1}{\lambda_q}\|p_q - p_{q-1}\|_1 \leq \lambda_q^{-2/3+2\varepsilon_0},
\]

for all \( q \geq M \). Thus interpolating the inequalities above we obtain that \(v - v_M\) is bounded in \(C^{1/5-\varepsilon}\) and \(p - p_M\) in \(C^{2/5-2\varepsilon}\). By \((5)\) and \((6)\), the pair \((v_M, p_M)\) are \(C^1\) bounded and thus it follows that \(v\) and \(p\) are bounded in \(C^{1/3-\varepsilon}\) and \(C^{2/3-2\varepsilon}\) respectively. Letting \( \delta \) tend to zero we obtain our claim. \(\square\)

0.3. Plan of the paper. After recalling in Section 1 some preliminary notation from the paper [10], in Section 2 we give the precise definition of the sequence of triples \((v_q, p_q, R_q)\). In Section 3 we list a number of inequalities that we will require on the various parameters of our scheme. The Sections 4 and 5 will focus on estimating, respectively, \(w_{q+1} = v_{q+1} - v_q\) and \(\dot{R}_{q+1}\). These estimates are then collected in Section 6 where Proposition 0.2 will be finally proved. Throughout the entire article we will rely heavily on the arguments of [2] – in some sense the scheme presented here is a simple variant of that given in [2] – as such the present paper is intentionally structured in a similar manner to [2] in order to aide comparison.
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1. Preliminaries

Throughout this paper we denote the $3 \times 3$ identity matrix by $\text{Id}$. In this section we state a number of results found in [10] which are fundamental to the present scheme as well its predecessors [10, 8, 2].

1.1. Geometric preliminaries. The following two results will form the cornerstone in which to construct the highly oscillating flows required by our scheme.

**Proposition 1.1** (Beltrami flows). Let $\bar{\lambda} \geq 1$ and let $A_k \in \mathbb{R}^3$ be such that

$$A_k \cdot k = 0, \ |A_k| = \frac{1}{\sqrt{2}}, \ A_{-k} = A_k$$

for $k \in \mathbb{Z}^3$ with $|k| = \bar{\lambda}$. Furthermore, let

$$B_k = A_k + i \frac{k}{|k|} \times A_k \in \mathbb{C}^3.$$ 

For any choice of $a_k \in \mathbb{C}$ with $a_k = a_{-k}$ the vector field

$$W(\xi) = \sum_{|k| = \bar{\lambda}} a_k B_k e^{i k \cdot \xi}$$

is real-valued, divergence-free and satisfies

$$\text{div} (W \otimes W) = \nabla |W|^2$$

(16)

Furthermore

$$\langle W \otimes W \rangle = \int_{T^3} W \otimes W \, d\xi = \frac{1}{2} \sum_{|k| = \bar{\lambda}} |a_k|^2 \left( \text{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|} \right).$$

(17)

**Lemma 1.2** (Geometric Lemma). For every $N \in \mathbb{N}$ we can choose $r_0 > 0$ and $\bar{\lambda} > 1$ with the following property. There exist pairwise disjoint subsets

$$A_j \subset \{ k \in \mathbb{Z}^3 : |k| = \bar{\lambda} \} \quad j \in \{1, \ldots, N\}$$

and smooth positive functions

$$\gamma_{(j)}(k) \in C^\infty (B_{r_0}(\text{Id})) \quad j \in \{1, \ldots, N\}, \ k \in A_j,$$

such that

\[^4\]Here $B_{r_0}(\text{Id})$ denotes the ball around $\text{Id}$ of radius $r_0$ under the usual matrix operator norm $|A| := \max_{|v| = 1} |Av|.$
Proposition 1.5. Following Schauder estimates (Proposition G.1 (ii), Appendix G of [2]) and commutator estimates (Proposition H.1 Appendix H of [2]), we have the identity

\[ R = \frac{1}{2} \sum_{k \in \Lambda_j} \left( \gamma_{k}^{(j)}(R) \right)^2 \left( \text{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|} \right) \quad \forall R \in B_{r_0}(\text{Id}). \quad (18) \]

1.2. The operator \( \mathcal{R} \). The following operator will be used in order to deal with the Reynolds Stresses arising from our iteration scheme.

**Definition 1.3.** Let \( v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3) \) be a smooth vector field. We then define \( \mathcal{R}v \) to be the matrix-valued periodic function

\[ \mathcal{R}v := \frac{1}{4} (\nabla P u + (\nabla P u)^T) + \frac{3}{4} (\nabla u + (\nabla u)^T) - \frac{1}{2} (\text{div} u) \text{Id}, \]

where \( u \in C^\infty(\mathbb{T}^3, \mathbb{R}^3) \) is the solution of

\[ \Delta u = v - \int_{\mathbb{T}^3} v \text{ in } \mathbb{T}^3 \]

with \( \int_{\mathbb{T}^3} u = 0 \) and \( P \) is the Leray projection onto divergence-free fields with zero average.

**Lemma 1.4** (\( \mathcal{R} = \text{div}^{-1} \)). For any \( v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3) \) we have

(a) \( \mathcal{R}v(x) \) is a symmetric trace-free matrix for each \( x \in \mathbb{T}^3 \);
(b) \( \text{div} \mathcal{R}v = v - \int_{\mathbb{T}^3} v \).

1.3. Schauder and commutator estimates on \( \mathcal{R} \). We recall the following Schauder estimates (Proposition G.1 (ii), Appendix G of [2]) and commutator estimates (Proposition H.1 Appendix H of [2]).

**Proposition 1.5.** Let \( k \in \mathbb{Z}^3 \setminus \{0\} \) be fixed. For a smooth vector field \( a \in C^\infty(\mathbb{T}^3, \mathbb{R}^3) \) let \( F(x) := a(x)e^{i\lambda_k \cdot x} \). Then we have

\[ \| \mathcal{R}(F) \|_0 \leq \frac{C}{\lambda^{1-\alpha}} \| a \|_0 + \frac{C}{\lambda^{m-\alpha}} |a|_m + \frac{C}{\lambda^m} |a|_{m+\alpha}, \quad (19) \]

for \( m = 0, 1, 2, \ldots \) and \( \alpha \in (0, 1) \), where \( C = C(\alpha, m) \).

**Proposition 1.6.** Let \( k \in \mathbb{Z}^3 \setminus \{0\} \) be fixed. For any smooth vector field \( a \in C^\infty(\mathbb{T}^3, \mathbb{R}^3) \) and any smooth function \( b \), if we set \( F(x) := a(x)e^{i\lambda_k \cdot x} \), we then have

\[ \| b, \mathcal{R}(F) \|_0 \leq C \lambda^{\alpha-2} \| a \|_0 \| b \|_1 + C \lambda^{\alpha-m} (\| a \|_{m-1+\alpha} \| b \|_{1+\alpha} + \| a \|_\alpha \| b \|_{m+\alpha}) \quad (20) \]

for \( m = 0, 1, 2, \ldots \) and \( \alpha \in (0, 1) \), where \( C = C(\alpha, m) \).

2. The construction of the triples \((v_q, p_q, \tilde{R}_q)\)

2.1. The initial triple \((v_0, p_0, \tilde{R}_0)\). Let \( \chi_0 \) be a smooth non-negative function, compactly supported on the interval \([-1/4, 1/4] \), bounded above by 1
and identically equal to 1 on \([-1/8, 1/8]\). We now set our initial velocity to be the divergence-free vector field

\[
v_0(t, x) := \frac{1}{2} \lambda_0^{\frac{3}{2} + \varepsilon_0} \chi_0(t) (\cos(\lambda_0 x_3), \sin(\lambda_0 x_3), 0),
\]

where here we use the notation \(x = (x_1, x_2, x_3)\). The initial pressure \(p_0\) is then defined to be identically zero. Finally if we set

\[
\dot{R}_0 = \frac{1}{2} \lambda_0^{\frac{3}{2} + \varepsilon_0} \chi_0'(t) \begin{pmatrix}
0 & 0 & \sin(\lambda_0 x_3) \\
0 & 0 & -\cos(\lambda_0 x_3) \\
\sin(\lambda_0 x_3) & -\cos(\lambda_0 x_3) & 0
\end{pmatrix},
\]

we obtain

\[
\partial_t v_0 + \text{div} (v_0 \otimes v_0) + \nabla p_0 = \text{div} \dot{R}_0.
\]

Hence the triple \((v_0, p_0, \dot{R}_0)\) is a solution to the Euler-Reynolds system (4).

Furthermore, it follows immediately that

\[
\left\| \dot{R}_0 \right\|_0 + \frac{1}{\lambda_0} \left\| \dot{R}_0 \right\|_1 \leq C \lambda_0^{-\frac{1}{2} + \varepsilon_0}.
\]

Thus if \(\lambda_0\) is sufficiently large we obtain (5-7) for \(q = 0\).

**Remark 1.** The choice of initial triple \((v_0, p_0, \dot{R}_0)\) is not of any great importance: any choice satisfying the conditions set out in Section 0.2 and is such that \(|v_0| \approx \lambda_0^{-1/5 + \varepsilon_0}\) for times \(t \in [-1/8, 1/8]\) should suffice.

**2.2. The inductive step.** The procedure of constructing \((v_{q+1}, p_{q+1}, \dot{R}_{q+1})\) in terms of \((v_q, p_q, \dot{R}_q)\) follows in the same spirit as that of the scheme outlined in [2] with a few minor modifications in order to satisfy the specific requirements of Proposition 0.2.

We will assume that \(\lambda_0\) is chosen large enough such that

\[
\sum_{j < q} \lambda_j^{2/3} \leq \lambda_q^{2/3} \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda_j^{-1/5 + \varepsilon_0} \leq \frac{\lambda_0^{-1/5 + \varepsilon_0}}{4} \leq \frac{1}{8}.
\]

(21)

Notice as a direct consequence (9) follows from (5) and the definition of \(v_0\).

We fix a symmetric non-negative convolution kernel \(\psi\) with support confined to \([-1, 1]\).

With a slight abuse of notation, we will use \((v, p, \dot{R})\) for \((v_q, p_q, \dot{R}_q)\) and \((v_1, p_1, \dot{R}_1)\) for \((v_{q+1}, p_{q+1}, \dot{R}_{q+1})\).

As was done in [2], we discretize time into intervals of size \(\mu^{-1}\) for some large parameter \(\mu\) to be chosen later.

The choice of cut-off functions \(\chi = \chi^{(q+1)}\) used in this article will differ slightly to that described in [2]. Specifically, we define \(\chi\) to be a smooth function such that for a small parameter \(\varepsilon_1 > 0\) (to be chosen later) \(\chi\) satisfies the following conditions:

- The support of \(\chi\) is contained in \(-\frac{1}{2} - \frac{\lambda_1^{-\varepsilon_1}}{4}, \frac{1}{2} + \frac{\lambda_1^{-\varepsilon_1}}{4}\).
In the range \((-\frac{1}{2} + \frac{\lambda_{q+1}^{-\varepsilon_1}}{4}, \frac{1}{2} - \frac{\lambda_{q+1}^{-\varepsilon_1}}{4})\) we have \(\chi \equiv 1\).

The sequence \(\{\chi^2(x-l)\}_{l \in \mathbb{Z}}\) forms a partition of unity of \(\mathbb{R}\), i.e.

\[
\sum_{l \in \mathbb{Z}} \chi^2(x-l) = 1.
\]

For \(N \geq 0\) we have the estimates

\[
|\partial^N_x \chi| \leq C \lambda_{q+1}^{N \varepsilon_1},
\]

where the constant \(C\) depends only on \(N\) – in particular it is independent of \(q\).

In [2], \(\chi\) was simply chosen to be a \(C^\infty_c(-\frac{3}{4}, \frac{3}{4})\) function, independent of the iteration \(q\), satisfying the third condition. Having defined \(\chi\), we adopt the notation \(\chi(t) := \chi(\mu t - l)\). The fundamental difference to choice of \(\chi\) in [2] is the extra factor \(\lambda_{q+1}^{-\varepsilon_1}\) appearing in the definition. A consequence of this modification is that the Lebesgue measure of the set

\[
\bigcap_{q=1}^{\infty} \bigcup_{q'=q}^{\infty} \bigcup_{l} \text{support}(\chi_{q',l})
\]

is zero. We will see this will provide us with a key ingredient in order to prove a.e. in time \(C^{1/3-\varepsilon}\) convergence of the sequence \(v_q\).

For each \(l\) define the amplitude function

\[
\rho_l = 2r_0^{-1} \left\| R(\cdot, l \mu^{-1}) \right\|_0.
\]

The function \(\rho_l\) will play a similar role to the \(\rho_l\) found in [2]: the comparatively simpler definition above reflects the fact that we are only interested in correcting for the Reynolds error and are not attempting to construct a solution to Euler with a prescribed energy as was done in [2]. In particular, up to a constant multiple, the amplitude function \(\rho_l\) defined here provides a lower bound for the amplitude defined in [2], and moreover is potentially significantly smaller.

Keeping in mind the new choices of \(\rho_l\) and \(\chi_l\), the construction of \((v_1, p_1, \hat{R}_1)\) proceeds in exactly the same manner as that described in [2], with the minor exception that the mollification parameter \(\ell\) will be chosen explicitly to be

\[
\ell = \lambda_{q+1}^{-1+\varepsilon_1}.
\]

In particular assuming \(\frac{\alpha-1}{2} > \varepsilon_1\) and \(\lambda_0\) is chosen sufficiently large, we have

\[
\frac{1}{\lambda_q} \leq \ell \leq \frac{1}{\lambda_{q+1}}.
\]

For comparison, the choice of \(\ell\) taken in [2] was \(\ell := \delta_{q+1}^{-1/3} \delta_q^{-1/4} \lambda_q^{-1/4} \lambda_{q+1}^{-3/4}\). The parameter \(\varepsilon_1\) may be taken arbitrarily small, and consequently, the choice of \(\ell\) taken here will be significantly smaller than that taken in [2].
For completeness we recall the remaining steps required to construct the triple \((v_1, p_1, R_1)\).

Having set

\[
R_l(x) := \rho_l \text{Id} - \tilde{R}(x, l\mu^{-1})
\]

and \(v_\ell = v \ast \psi_\ell\), we define \(R_{\ell,l}\) to be the unique solution to the following transport equation

\[
\begin{cases}
\partial_t R_{\ell,l} + v_\ell \cdot \nabla R_{\ell,l} = 0 \\
R_{\ell,l}(x, l\mu^{-1}) = R_l \ast \psi_\ell.
\end{cases}
\]

For every integer \(l \in [-\mu, \mu]\), we let \(\Phi_l : \mathbb{R}^3 \times (-1, 1) \to \mathbb{R}^3\) be the solution of

\[
\begin{cases}
\partial_t \Phi_l + v_\ell \cdot \nabla \Phi_l = 0 \\
\Phi_l(x, l\mu^{-1}) = x.
\end{cases}
\]

Applying Lemma 1.2 with \(N = 2\), we denote by \(\Lambda^e\) and \(\Lambda^o\) the corresponding families of frequencies in \(\mathbb{Z}^3\) and set \(\Lambda := \Lambda^o + \Lambda^e\). For each \(k \in \Lambda\) and each \(l \in \mathbb{Z} \cap [0, \mu]\) we then define

\[
a_{kl}(x, t) := \sqrt{\rho_l \gamma_k} \left( \frac{R_{\ell,l}(x, t)}{\rho_l} \right),
\]

\[
w_{kl}(x, t) := a_{kl}(x, t) B_k e^{i\lambda_{q+1}k \cdot \Phi_l(x, t)}.
\]

The perturbation \(w = v_1 - v\) is then defined as the sum of a “principal part” and a “corrector”. The “principal part” being the map

\[
w_\alpha(x, t) := \sum_{l \text{ odd}, k \in \Lambda^o} \chi_l(t) w_{kl}(x, t) + \sum_{l \text{ even}, k \in \Lambda^e} \chi_l(t) w_{kl}(x, t).
\]

The “corrector” \(w_c\) is then defined in such a way that the sum \(w = w_\alpha + w_c\) is divergence free:

\[
w_c = \sum_{kl} \chi_l \left( \frac{i}{\lambda_{q+1}} \nabla a_{kl} - a_{kl}(D\Phi_l - \text{Id})k \right) \times \frac{k \times B_k}{|k|^2} e^{i\lambda_{q+1}k \cdot \Phi_l}.
\]

The new pressure is defined as

\[
p_1 = p - \frac{|w_\alpha|^2}{2} - \frac{1}{3} |w_c|^2 - \frac{2}{3} \langle w_\alpha, w_c \rangle - \frac{2}{3} \langle v - v_\ell, w \rangle.
\]
and finally we set $\hat{R}_1 = R^0 + R^1 + R^2 + R^3 + R^4 + R^5$, where

\begin{align*}
R^0 &= \mathcal{R} (\partial_t w + v_\ell \cdot \nabla w + w \cdot \nabla v_\ell) \\
R^1 &= \mathcal{R} \text{div} \left( w_o \otimes w_o - \sum_i \chi_i^2 R_{i,l} - \frac{|w_o|^2}{2} \text{Id} \right) \\
R^2 &= w_o \otimes w_c + w_c \otimes w_o + w_c \otimes w_c - \frac{|w_o|^2 + 2|w_o, w_o|}{3} \text{Id} \\
R^3 &= w \otimes (v - v_\ell) + (v - v_\ell) \otimes w - \frac{2(w - v \cdot v_\ell)}{3} \text{Id} \\
R^4 &= \dot{R} - \dot{R} \ast \psi_\ell \\
R^5 &= \sum_i \chi_i^2 (\dot{R}_{i,l} + \dot{R} \ast \psi_\ell).
\end{align*}

2.3. Compact support in time. By construction it follows that if for each integer $j$ the triple $(v_j, p_j, \dot{R}_j)$ is supported in the time interval $[T, T']$ then $(v_{j+1}, p_{j+1}, \dot{R}_{j+1})$ is supported in the time interval $[T - \mu_j^{-1}, T' + \mu_j^{-1}]$. Therefore since $(v_0, p_0, \dot{R}_0)$ is supported in the time interval $[-1/4, 1/4]$, it follows by induction that if we assume

$$
\mu_j \geq 2^{j+2}
$$

then triple $(v_q, p_q, \dot{R}_q)$ is supported in the time interval

$$
[-1/4 - \sum_{j=1}^q \mu_j^{-1}, 1/4 + \sum_{j=1}^q \mu_j^{-1}] \subset [-1/2, 1/2].
$$

3. Ordering of parameters

In order to better aid comparison to arguments of [2], we introduce a sequence of strictly decreasing parameters $\delta_q < 1$. In Section 3 we will provide an explicit definition of $\delta_q$, but for now we restrict ourselves to specifying a number of inequalities that $\delta_q$ will need to satisfy. Analogously to [2] we will assume the following estimates

\begin{align*}
\frac{1}{\lambda_q} \|v_q\|_1 &\leq \delta_q^{1/2} \\
\frac{1}{\lambda_q} \|p_q\|_1 + \frac{1}{\lambda_q^2} \|p_q\|_2 &\leq \delta_q \\
\left\|\dot{R}_q\right\|_0 + \frac{1}{\lambda_q} \left\|\dot{R}_q\right\|_1 &\leq \frac{1}{C_0} \delta_{q+1} \\
\left\|\partial_t + v \cdot \nabla \dot{R}_q\right\|_0 &\leq \delta_{q+1} \delta_q^{1/2} \lambda_q,
\end{align*}

where $C_0 > 1$ is a large number to be specified in the next section.
Furthermore, we will assume in addition that the following parameter inequalities are satisfied

\[
\sum_{j<q} \delta_j \lambda_j \leq \delta_q \lambda_q, \quad \frac{\delta_q^{1/2} \lambda_q \ell}{\delta_{q+1}^{1/2}} \leq 1, \quad \frac{\delta_q^{1/2} \lambda_q}{\mu} \leq \lambda_{q+1}^{-\varepsilon_1} \quad \text{and} \quad \frac{1}{\lambda_{q+1}} \leq \frac{\delta_{q+1}^{1/2}}{\mu}.
\] (35)

The sequence \(\delta_q\) will be applied in the context of proving \(1/5 - \varepsilon\) convergence of the velocities \(v_q\); however note that unlike the case in [2], the sequence does not appear explicitly in the definition of the triples \((v_q, p_q, \hat{R}_q)\).

In order to prove a.e. time \(1/3 - \varepsilon\) convergence, we will require localized estimates (in time). To this aim, we fix a time \(t_0 \in (-1, 1)\) and set \(l_{q+1}\) to be the unique integer such that \(\mu t_0 \in \left[-\frac{1}{2} + l_{q+1}, \frac{1}{2} + l_{q+1}\right)\). We now introduce a new sequence of strictly decreasing parameters \(\delta_{q,t_0}\) such that for a given time \(t\) satisfying \(|\mu t - l_{q+1}| \leq 1\) we have the following estimates

\[
\frac{1}{\lambda_q} \|v_q\|_1 \leq \delta_{q,t_0}^{1/2} \quad \text{(36)}
\]
\[
\frac{1}{\lambda_q} \|p_q\|_1 + \frac{1}{\lambda_q^2} \|p_q\|_2 \leq \delta_{q,t_0} \quad \text{(37)}
\]
\[
\left\|\hat{R}_q\right\|_0 + \frac{1}{\lambda_q} \left\|\hat{R}_q\right\|_1 \leq \frac{1}{C_0} \delta_{q+1,t_0} \quad \text{(38)}
\]
\[
\left\|\partial_t + v \cdot \nabla \hat{R}_q\right\|_0 \leq \delta_{q+1,t_0} \delta_{q,t_0}^{1/2} \lambda_q. \quad \text{(39)}
\]

Analogously to (35) we assume the following inequalities are satisfied

\[
\sum_{j<q} \delta_{j,t_0} \lambda_j \leq \delta_{q,t_0} \lambda_q, \quad \frac{\delta_{q,t_0}^{1/2} \lambda_q \ell}{\delta_{q+1,t_0}^{1/2}} \leq 1, \quad \text{and} \quad \frac{\delta_{q,t_0}^{1/2} \lambda_q}{\mu} \leq \lambda_{q+1}^{-\varepsilon_1}. \quad \text{(40)}
\]

The last inequality being a trivial consequence of (35) and the inequality \(\delta_{q,t_0} \leq \delta_q\). Observe that we do not assume a condition akin to the last inequality of (35). This remark is worth keeping in mind as we will apply the arguments of [2] extensively, where such a condition was present. Luckily, this condition is only really required at one specific point in the paper: the estimation of

\[
\left\|\partial_t \hat{R}_1 + v_1 \cdot \nabla \hat{R}_1\right\|_0,
\]

for which on a subset of time we will present sharper estimates. This condition was also used in a few isolated cases in [2] in order to simplify a number of terms arising from estimates, however this was primarily done for aesthetic reasons.
4. Estimates on the perturbation

In order to bound the perturbation, we apply nearly identical arguments used in Section 3 of [2].

We recall the following notation from [2]

\[ \phi_{kl}(x, t) := e^{i\lambda_{q+1}k \cdot [\Phi_l(x, t) - x]}, \]

\[ L_{kl} := a_{kl} B_k + \left( \frac{i}{\lambda_{q+1}} \nabla a_{kl} - a_{kl}(D\Phi_l - \text{Id}) \right) \frac{k \times B_k}{|k|^2}. \]

The perturbation \( w \) can then be written as

\[ w = \sum_{kl} \chi_l L_{kl} \phi_{kl} e^{i\lambda_{q+1}k \cdot x} = \sum_{kl} \chi_l L_{kl} e^{i\lambda_{q+1}k \cdot \Phi_l}. \]

For reference we note that as a consequence of (5), and (21) we have

\[ \|v_q\|_0 \leq 1. \] (41)

We also recall that as a consequence of simple convolution inequalities together with the inequalities (36) we have for a fixed \( t_0 \), \( N \geq 1 \) and times \( t \) satisfying \( |\mu_{q+1} t - l_{q+1}| < 1 \)

\[ \|v_q\|_N \leq \delta_{q,t_0}^{1/2} \lambda_q \ell^{-N+1}. \] (42)

With this notation we now present a minor variant of Lemma 3.1 from [2].

**Lemma 4.1.** Fix a time \( t_0 \in (-1, 1) \) and let \( l_{q+1} \) be as before, i.e. the unique integer such that \( t_0 \in [-1/2 + l_{q+1}, 1/2 + l_{q+1}) \). Assuming the series of inequalities listed in Section 3 hold then we have the following estimates. For \( t \) such that \( |\mu t - l_{q+1}| \leq 1 \) and \( l \in \{l_{q+1} - 1, l_{q+1}, l_{q+1} + 1\} \) we have

\[ \|D\Phi_l\|_0 \leq C \] (43)

\[ \|D\Phi_l - \text{Id}\|_0 \leq C \delta_{q,t_0}^{1/2} \lambda_q \mu \] (44)

\[ \|D\Phi_l\|_N \leq C \delta_{q,t_0}^{1/2} \lambda_q \mu \ell^{-N}, \quad N \geq 1 \] (45)

Moreover,

\[ \|a_{kl}\| + \|L_{kl}\| \leq C \delta_{q+1,t_0}^{1/2} \] (46)

\[ \|a_{kl}\|_N \leq C \delta_{q+1,t_0}^{1/2} \lambda_q \ell^{-N}, \quad N \geq 1 \] (47)

\[ \|L_{kl}\|_N \leq C \delta_{q+1,t_0}^{1/2} \ell^{-N}, \quad N \geq 1 \] (48)

\[ \|\phi_{kl}\|_N \leq C \lambda_{q+1} \delta_{q,t_0}^{1/2} \lambda_q \mu \ell^{N-1} + C \left( \delta_{q,t_0}^{1/2} \lambda_q \lambda_{q+1} \right)^N \mu \ell^{-N}, \quad N \geq 1. \] (49)
Consequently, for any $N \geq 0$

$$
\| w_c \|_N \leq C_0^{\delta^{1/2}} \left( \frac{\lambda_q}{\lambda_{q+1}} + \frac{\delta^{1/2}}{q_{t0}} \lambda_q \right) \lambda^N_{q+1} \tag{50}
$$

$$
\leq C_0^{\delta^{1/2}} \frac{\delta^{1/2}}{\mu} \lambda^N_{q+1}, \tag{51}
$$

$$
\| w_0 \|_N \leq C_0^{\delta^{1/2}} \lambda^N_{q+1} \tag{52}
$$

$$
\leq C_0^{\delta^{1/2}} \lambda^N_{q+1}. \tag{53}
$$

The constants appearing in the above estimates depend only on $N$ and the constant $C_0$ given in (33) and (38). In particular for a fixed $N$, the constants appearing in (35) - (38) and (50) - (53) can be made arbitrarily small by taking $C_0$ to be sufficiently large. Furthermore, the weaker estimates (51) and (53) hold uniformly in time.

The proof of the above lemma follows from essentially exactly the same arguments to those given in the proof of Lemma 3.1 from [2] – making use of our new sequence of parameters $\delta_{q,t0}$. The only minor point of departure from [2] is the appearance of the term $\frac{\lambda^N}{\lambda_{q+1}}$ in (50). This is related to the fact that we do not have a parameter ordering akin to the last inequality of (35) for the parameters $\delta_{q,t0}$. Nevertheless, since $\delta_{q,t0} \leq \delta_q$, the estimate is sharper than the corresponding estimate of [2] and hence we obtain (51). From the definition of $w_c$ we have

$$
\| w_c \|_N \leq C \sum_{kl} \chi_l \left( \frac{1}{\lambda^N_{q+1}} \| a_{kl} \|_{N+1} + \| a_{kl} \|_0 \| D\Phi_l - \text{Id} \|_N + \| a_{kl} \|_N \| D\Phi_l - \text{Id} \|_0 \right) + C \| w_c \|_0 \sum_l \chi_l \left( \lambda^N_{q+1} \| D\Phi_l \|_{N} + \lambda_{q+1} \| D\Phi_l \|_{N+1} \right).
$$

Hence applying (43)-(47) and applying the inequalities from Section 3 we obtain (50).

**Corollary 4.2.** Under the assumptions of Lemma 4.1 we have

$$
\lambda^{-1}_{q+1} \| v_1 \|_1 + \| w \|_0 \leq \delta^{3/2}_{q+1,t0} \tag{54}
$$

$$
\lambda^{-2}_{q+1} \| p_1 \|_2 + \lambda^{-1}_{q+1} \| p_1 \|_1 + \| p_1 - p \|_0 \leq \delta_{q+1,t0}. \tag{55}
$$
Lemma 4.3. Under the assumptions of Lemma 4.1 we have
\[ t \]
Consequently for \( t \) in the range \(|t\mu - l_{q+1}| \leq 1/2(1 - \lambda_{q+1}^{-1})\) we have
\[ D_t w_c \parallel_N \leq C\delta_{q,t_0}^\lambda q \lambda q N^{t_0} \] (61)
\[ D_t w_o \parallel_N \equiv 0. \] (62)
Moreover we have the following estimates which are valid uniformly in time
\[ D_t w_c \parallel_N \leq C\delta_{q,t_0}^\lambda q \lambda q N^{t_0+\epsilon_1} \] (63)
\[ D_t w_o \parallel_N \leq C\delta_{q,t_0}^\lambda q \lambda q N^{t_0+\epsilon_1}. \] (64)
Again, we note that the constants \( C \) depend only on our choice of \( C_0 \); in particular, the constants appearing in (58) - (61) can be made arbitrarily small by taking \( C_0 \) sufficiently large.

Proof. First note that (58), (63) and (64) follow by exactly the same arguments as those given in Lemma 3.2 of [2] - making use of our new sequence of parameters \( \delta_{q,t_0} \). However in contrast [2], time derivatives falling on \( \chi_l \) for some \( l \) pick up an additional factor of \( \lambda_{q+1}^{\epsilon_1} \), which explains this additional factor appearing in (63) and (64).

To prove (54) and (55), in addition to using our new parameters \( \delta_{q,t_0} \), we will take advantage of our second order inductive estimates on the pressure in order to obtain sharper estimates than those found in [2].
Consider (56), we note that by the arguments of [2] we obtain that
\[ \|D_t v\|_N \leq \|\nabla p * \psi\|_N + \|\text{div} \tilde{R} * \psi\|_N + C \lambda_q^2 \ell^{1-N} \delta_{q,t_0} \]
Then from the inductive estimates (37) of the pressure \(p\), the estimate (38) on the Reynolds stress \(\tilde{R}\), together with standard convolution estimates, we obtain (56). From (23) and since \(\delta_{q+1,t_0} \leq \delta_{q,t_0}\) we obtain (57).

We now consider the estimate (59).
\[
D_t^2 L_{kl} = \left( - \frac{i}{\lambda_{q+1}} (D_t Dv_\ell)^T \nabla a_{kl} + \frac{i}{\lambda_{q+1}} Dv_\ell^T Dv_\ell^T \nabla a_{kl} + \right.
\left. - a_{kl} D\Phi_1 Dv_\ell Dv_\ell k + a_{kl} D\Phi_1 D_t Dv_\ell k \right) \times \frac{k \times B_k}{|k|^2}.
\]
Note that \(D_t Dv_\ell = D D_t v_\ell - Dv_\ell Dv_\ell\), so that
\[ \|D_t Dv_\ell\|_N \leq \|D_t v_\ell\|_{N+1} + C \|D v_\ell\|_N \|D v_\ell\|_0 \leq C(\delta_{q,t_0} \lambda_q^2 \ell^{-N} + \delta_{q+1,t_0} \lambda_q \ell^{-N-1}) (1 + \lambda_q \ell) \leq C \delta_{q,t_0} \lambda_q^2 \ell^{-N} + C \delta_{q+1,t_0} \lambda_q \ell^{-N-1}. \]
Hence utilizing the estimates in Lemma 4.1 we obtain
\[ \|D_t^2 L_{kl}\|_N \leq C \delta_{q+1,t_0} \lambda_q \ell^{-N} \left( \delta_{q,t_0} \lambda_q + \frac{\delta_{q+1,t_0}}{\ell} \right) \left( 1 + \frac{\lambda_q}{\lambda_{q+1}} + \frac{\delta_{q+1,t_0}}{\mu} \right) \leq C \delta_{q+1,t_0} \lambda_q \ell^{-N} (\delta_{q,t_0} \lambda_q + \delta_{q+1,t_0} \ell^{-1}). \]
Thus we obtain (59). The estimate (61) follows also as a consequence of (23), Lemma 4.1 and (58).

**Remark 2.** While (56) and (59) are the analogues of the corresponding estimates in [2], (56) and (59) are sharper and are derived taking into account the bounds on the second derivatives of the pressure (37).

## 5. Estimates on the Reynolds stress

In this section we describe the estimates on Reynolds stress, which follow by applying the arguments of Section 5 of [2] to the present scheme. The main result is the following proposition, which is a sharper, time localized version of Proposition 5.1 of [2], providing estimates for a subset of the times in the complement of the regions where the cut-off functions overlap.

**Proposition 5.1.** Fix \(t\) in the range \(|\mu - l_{q+1}| < \frac{1}{2}(1 - \lambda_{q+1}^{-\varepsilon_1})\). There is a constant \(C\) such that, if \(\delta_{q,t_0}, \delta_{q+1,t_0}\) and \(\mu\), satisfy (40), then we have
\[ \|\tilde{R}^0\|_0 + \frac{1}{\lambda_{q+1}} \|\tilde{R}^{\theta}\|_1 + \frac{1}{\delta_{q+1,t_0}^2 \lambda_{q+1}} \|D_t \tilde{R}^0\|_0 \leq C \delta_{q+1,t_0}^{1/2} \delta_{q,t_0}^{1/2} \lambda_q \ell \] (65)
Thus

\[ \| R^1 \|_0 + \frac{1}{\lambda_{q+1}} \| R^1 \|_1 + \frac{1}{\delta_{q+1,t_0} \lambda_{q+1}} \| D_t R^1 \|_0 \leq C \frac{\delta_{q+1,t_0}^{3/2} \lambda_q \lambda_{q+1}^{\epsilon_1}}{\mu} + C \delta_{q+1,t_0}^{1/2} \delta_{q,t_0}^{1/2} \lambda_q \ell \]  

(66)

\[ \| R^2 \|_0 + \frac{1}{\lambda_{q+1}} \| R^2 \|_1 + \frac{1}{\delta_{q+1,t_0} \lambda_{q+1}} \| D_t R^2 \|_0 \leq C \frac{\delta_{q+1,t_0}^{3/2} \lambda_q \lambda_{q+1}^{\epsilon_1}}{\mu} + C \delta_{q+1,t_0}^{1/2} \delta_{q,t_0}^{1/2} \lambda_q \ell \]  

(67)

\[ \| R^3 \|_0 + \frac{1}{\lambda_{q+1}} \| R^3 \|_1 + \frac{1}{\delta_{q+1,t_0} \lambda_{q+1}} \| D_t R^3 \|_0 \leq C \frac{\delta_{q+1,t_0}^{3/2} \lambda_q \lambda_{q+1}^{\epsilon_1}}{\mu} + C \delta_{q+1,t_0}^{1/2} \delta_{q,t_0}^{1/2} \lambda_q \ell \]  

(68)

\[ \| R^4 \|_0 + \frac{1}{\lambda_{q+1}} \| R^4 \|_1 + \frac{1}{\delta_{q+1,t_0} \lambda_{q+1}} \| D_t R^4 \|_0 \leq C \frac{\delta_{q+1,t_0}^{3/2} \lambda_q \lambda_{q+1}^{\epsilon_1}}{\mu} + C \delta_{q+1,t_0}^{1/2} \delta_{q,t_0}^{1/2} \lambda_q \ell \]  

(69)

\[ \| R^5 \|_0 + \frac{1}{\lambda_{q+1}} \| R^5 \|_1 + \frac{1}{\delta_{q+1,t_0} \lambda_{q+1}} \| D_t R^5 \|_0 \leq C \frac{\delta_{q+1,t_0}^{3/2} \lambda_q \lambda_{q+1}^{\epsilon_1}}{\mu} + C \delta_{q+1,t_0}^{1/2} \delta_{q,t_0}^{1/2} \lambda_q \ell \]  

(70)

Thus

\[ \| \hat{R}_1 \|_0 + \frac{1}{\lambda_{q+1}} \| \hat{R}_1 \|_1 + \frac{1}{\delta_{q+1,t_0} \lambda_q} \| D_t \hat{R}_1 \|_0 \leq C \left( \frac{\delta_{q+1,t_0}^{3/2} \lambda_q \lambda_{q+1}^{\epsilon_1}}{\mu} + \delta_{q+1,t_0}^{1/2} \delta_{q,t_0}^{1/2} \lambda_q \ell \right), \]  

(71)

\[ \| \partial_t \hat{R}_1 + v_1 \cdot \nabla \hat{R}_1 \|_0 \leq C \delta_{q+1,t_0}^{1/2} \lambda_{q+1} \left( \frac{\delta_{q+1,t_0}^{1/2} \lambda_q \lambda_{q+1}^{\epsilon_1}}{\mu} + \delta_{q+1,t_0}^{1/2} \delta_{q,t_0}^{1/2} \lambda_q \ell \right). \]  

(72)

The arguments will be a minor variation to those found in Proposition 5.1 of [2], the key differences being:

(A) Since for all \( l \) we have \( \chi'_l \) is identically zero for times \( t \) in the range \( |t \mu - l_{q+1}| < \frac{1}{2}(1 - \lambda_{q+1}^{\epsilon_1}) \), no positive powers of \( \mu \) will appear as a consequence of differentiating in time.

(B) As previously mentioned in Section 3 in contrast to the case in [2] we do not have an estimate of the type

\[ \frac{1}{\lambda_{q+1}} \leq \frac{\delta_{q+1,t_0}^{1/2}}{\mu} \]  

(73)

at our disposal.
(C) In many of the material derivative estimates in \cite{2} the estimate \( \delta_{q}^{1/2} \lambda_{q} \leq \mu \) was used in order to simplify terms: we will avoid employing such an estimate, although in its place we will sometimes use the estimate \( \delta_{q_{0}0}^{1/2} \lambda_{q_{0}} \leq \delta_{q_{0}1}^{1/2} \lambda_{q_{0}1} \).

(D) In \cite{2} a new constant \( \epsilon > 0 \) was introduced in order to state the analogous estimates: in order to minimize the number of small constants, we simply use \( \epsilon \) and apply the identity \( \ell = \lambda_{q_{1}}^{1} - \lambda_{q_{1}}^{1} \) in order to reduce the number of terms in the estimates.

(E) No term of the type

\[
\delta_{q+1,0}^{1/2} \lambda_{q+1} \mu \ell ,
\]

appears in the estimate \((65)\) and within the brackets of the right hand sides of \((71)\) and \((72)\). This is related to the fact that in \cite{2} the authors did not keep track of second derivatives of the pressure (see Remark \(2)\).

Proof. Keeping in mind the observations (A), (B), (C) and (D) above, the proof of \((67)-(70)\) follows by applying nearly identical arguments to that found in Proposition 5.1 of \cite{2}. Indeed the estimates on \(R^{2}, R^{3}, R^{4}\) and \(R^{5}\), depend on the \(C^{0}\) of \(w_{o}, w_{c}, v, \hat{R}, D_{t}w_{o}, D_{t}w_{c}, D_{t}v_{\ell}, D_{t}\hat{R}\) and the \(C^{1}\) norm of \(w_{o}, w_{c}, v, p, \hat{R}\). For bounding these quantities we use the estimates \((36)-(39)\), together with the estimates from Lemmas \(4.1\) and \(4.3\), which are analogous to the corresponding estimates ones in \cite{2}. The estimate \((71)\) easily follows as a consequence of \((65)-(70)\), and \((72)\) follows from \((71)\) together with the observation

\[
\| \partial_{t} \hat{R}_{1} + v_{1} \cdot \nabla \hat{R}_{1} \|_{0} \leq \| D_{t} \hat{R}_{1} \|_{0} + (\| v - v_{\ell} \|_{0} + \| w \|_{0}) \| \hat{R}_{1} \|_{1} .
\]

Therefore we will restrict ourselves to proving the estimates \((65)\) and \((66)\). For reasons of brevity, in what follows we adopt the abuse of notation \(l_{q} = l_{q+1}\).

Estimates on \(R^{0}\). Recall from \cite{2} that, by the definition of \(R^{0}\) given by \cite{2}, taking into account Propositions \(1.5\) and \(1.6\) and applying the decomposition

\[
D_{t}R^{0} = ([D_{t}, R] + RD_{t})(\partial_{t}w + v_{\ell} \cdot \nabla w + w \cdot \nabla v_{\ell})
\]

\[
= ([v_{\ell} \cdot \nabla, R] + RD_{t})(\partial_{t}w + v_{\ell} \cdot \nabla w + w \cdot \nabla v_{\ell})
\]

we need to bound the terms \(\Omega_{kl}\) where

\[
\partial_{t}w + v_{\ell} \cdot \nabla w + w \cdot \nabla v_{\ell} = \sum_{kl} \Omega_{kl} e^{i\lambda_{q+1} k \cdot x} ,
\]

that is

\[
\Omega_{kl} := (\chi'_{l} L_{kl} + \chi_{l} D_{l} L_{kl} + \chi_{l} L_{kl} \cdot \nabla v_{\ell}) \phi_{kl} .
\]

5 Such a term imposes strong restrictions on the choice of \(\ell\) to ensure convergence and is in part the reason for the complicated choice of \(\ell\) taken in \cite{2}.
and the terms $\Omega'_{kl}$ where
\[
D_t (\partial_t w + v_\ell \cdot \nabla w + w \cdot \nabla v_\ell) := \sum_k \Omega'_{kl} e^{i\lambda_{q+1}^kx},
\]
that is
\[
\Omega'_{kl} := \left( \partial_t^2 + \chi_l L_{kl} + 2\partial_t \chi_l D_t L_{kl} + \chi_l D_t^2 L_{kl} + \partial_t \chi_l L_{kl} \cdot \nabla v_\ell + \chi_l D_t L_{kl} \cdot \nabla v_\ell + \chi_l L_{kl} \cdot \nabla D_t v_\ell - \chi_l L_{kl} \cdot \nabla v_\ell \cdot \nabla v_\ell \right) \phi_{kl}.
\]
Precisely, applying Propositions 1.5 with $\alpha = \varepsilon_1$ we obtain
\[
\| R^0 \|_0 \leq C \sum_{kl} \left( \ell \| \Omega'_{kl} \|_0 + \lambda_{q+1}^{1-N} \ell \| \Omega_{kl} \|_N + \lambda_{q+1}^{-N} \| \Omega'_{kl} \|_{N+\varepsilon_1} \right),
\]
and by Propositions 1.5 and 1.6 and the decomposition (75) we obtain
\[
\| D_t R^0 \|_0 \leq C \sum_{kl} \left( \ell \| \Omega'_k \|_0 + \lambda_{q+1}^{1-N} \ell \| \Omega'_{kl} \|_N + \lambda_{q+1}^{-N} \| \Omega'_{kl} \|_{N+\varepsilon_1} \right) + \ell \lambda_{q+1}^{-1} \| v_\ell \|_1 \| \Omega_{kl} \|_1 + \lambda_{q+1}^{-N} \ell \left( \| \Omega_{kl} \|_{N+\varepsilon_1} \| v_\ell \|_{1+\varepsilon_1} + \| \Omega_{kl} \|_{1+\varepsilon_1} \| v_\ell \|_{N+\varepsilon_1} \right).
\]
Observe that since we assumed $|t\mu - l_1| < 1/2(1 - \lambda_{q+1}^{-\varepsilon_1})$ we have that $\Omega_{kl}, \Omega'_{kl} \equiv 0$ for all $l \neq l_1$. Moreover, since on the given temporal range $\chi_{l_1} \equiv 1$ and $\chi'_l \equiv 0$, we have
\[
\Omega_{kl_1} := \left( D_t L_{kl_1} + L_{kl_1} \cdot \nabla v_\ell \right) \phi_{kl_1},
\]
and
\[
\Omega'_{kl_1} := \left( D_t^2 L_{kl_1} + D_t L_{kl_1} \cdot \nabla v_\ell + L_{kl_1} \cdot \nabla D_t v_\ell - L_{kl_1} \cdot \nabla v_\ell \cdot \nabla v_\ell \right) \phi_{kl_1}.
\]
Applying Lemmas 4.1, Lemma 13, 12 and 10 we obtain
\[
\| \Omega_{kl_1} \|_N \leq C \delta_{q+1,t_0}^{1/2} \delta_{q,t_0}^{1/2} \lambda_q \ell^{-N}.
\]
Similarly we obtain
\[
\| \Omega'_{kl_1} \|_N \leq C \delta_{q+1,t_0}^{1/2} \lambda_q \ell^{-N} \left( \delta_{q,t_0} \lambda_q + \delta_{q+1,t_0} \ell^{-1} \right)
\]
\[
\leq C \delta_{q+1,t_0}^{1/2} \delta_{q,t_0}^{1/2} \lambda_q \lambda_{q+1} \ell^{-N}.
\]
**Remark 3.** Note we implicitly used the estimates (54) and (59) which rely on second order inductive estimates on the pressure (see Remark 2). In [21], only first order estimates of the pressure were assumed, resulting in an additional error term of the type (74).
Hence choosing $N$ large enough such that $N \varepsilon_1 > 3$, then combining (78)-(82) we obtain (85).

**Estimates on $R^1$.** Recall that a key ingredient to the estimation of $R^1$ involves estimating

$$f_{klk'} := \chi_{l} \chi_{l'} a_{kl} a_{k'l'} \phi_{kl} \phi_{k'l'}$$

and

$$D_t \left( \nabla f_{klk'} e^{i\lambda q_1 (k+k') \cdot x} e^{-i\lambda q_1 (k+k') \cdot x} = \Omega''_{klk'} \right).$$

More precisely, using Proposition 1.5 it was shown in [2] that

$$\| R^1 \|_0 \leq C \sum_{(k,l),(k',l')} \left( \ell \| f_{klk'} \|_1 + \lambda_{q+1}^{-1} \ell \| f_{klk'} \|_{N+1} + \lambda_{q+1}^{-N} \| f_{klk'} \|_{N+1+\varepsilon_1} \right)$$

$$\| R^1 \|_1 \leq C \lambda_{q+1} \sum_{(k,l),(k',l')} \left( \ell \| f_{klk'} \|_1 + \lambda_{q+1}^{-1} \ell \| f_{klk'} \|_{N+1} + \lambda_{q+1}^{-N} \| f_{klk'} \|_{N+1+\varepsilon_1} \right)$$

$$+ C \sum_{(k,l),(k',l')} \left( \ell \| f_{klk'} \|_2 + \lambda_{q+1}^{-1} \ell \| f_{klk'} \|_{N+2} + \lambda_{q+1}^{-N} \| f_{klk'} \|_{N+2+\varepsilon_1} \right)$$

and using Proposition 1.5 and (1.6) together with the identity $D_t R = [v_t \cdot \nabla, R] + RD_t$ that

$$\| D_t R^1 \|_0 \leq C \sum_{(k,l),(k',l')} \left( \ell \| \Omega''_{klk'} \|_1 + \lambda_{q+1}^{-1} \ell \| \Omega''_{klk'} \|_{N+1} + \lambda_{q+1}^{-N} \| \Omega''_{klk'} \|_{N+1+\varepsilon_1} \right)$$

$$+ \ell \lambda_{q+1}^{-1} \| f_{klk'} \|_1 \| v_t \|_1$$

$$+ \lambda_{q+1}^{-1} \ell \left( \| f_{klk'} \|_{N+1+\varepsilon_1} \| v_t \|_{1+\varepsilon_1} + \| f_{klk'} \|_{2+\varepsilon_1} \| v_t \|_{N+\varepsilon_1} \right).$$

Again as a consequence of our assumption $|t \mu - l_1| < \varepsilon_1$ we have that if either $l \neq l_1$ or $l' \neq l_1$ then $f_{klk'} \equiv 0$ and $\Omega''_{klk'} \equiv 0$. Moreover we have

$$\Omega''_{klk'} := - (a_{kl} Dv^T_k \nabla a_{kl} + a_{kl} Dv^T_{l} \nabla a_{kl} ) \phi_{kl} \phi_{k'l}$$

$$- \lambda_{q+1} a_{kl} a_{k'l} (D\Phi_x Dv^T_k + D\Phi_x Dv^T_{k'} ) \phi_{kl} \phi_{k'l}.$$

Estimating $f_{klk'}$ and $\Omega''_{klk'}$ we have from Lemma 4.1 and Lemma 4.3 for $N \geq 1$

$$\| f_{klk'} \|_N \leq C \delta_{q+1,t_0} \ell_1^{-1-N} \left( \lambda_q + \frac{\delta_{q,t_0}^2 \lambda_q \lambda_{q+1}}{\mu} \right),$$

(83)
and
\[
\begin{align*}
\left\| \Omega_{k,l}^{m} \right\|_{0} & \leq C \delta_{q+1}^{1/2} \delta_{0}^{1/2} \lambda_{q} \left( \lambda_{q} + \lambda_{q+1} \right) \\
\left\| \Omega_{k,l}^{m} \right\|_{N} & \leq C \delta_{q+1}^{1/2} \delta_{0}^{1/2} \lambda_{q} \lambda_{q+1} N,
\end{align*}
\]
for \( N \geq 1. \)

Combining the above estimates and again selecting \( N \) such that \( N \epsilon_{1} > 3 \) we obtain (86).

We now state uniform estimates for the new Reynolds stress. Taking advantage of some of the additional observations used previously to prove Proposition 5.1 by applying nearly identical arguments to that of Proposition 5.1 of [2] we obtain the following Proposition.

**Proposition 5.2.** There is a constant \( C \) such that, if \( \delta_{q}, \delta_{q+1} \) and \( \mu \), satisfy (85), then we have
\[
\begin{align*}
\left\| \check{R}_{1} \right\|_{0} & + \frac{1}{\lambda_{q+1}^{1/2}} \left\| \check{R}_{1} \right\|_{1} + \frac{1}{\mu} \left\| D_{t} \check{R}_{1} \right\|_{0} \leq C \left( \delta_{q+1}^{1/2} \mu \lambda_{q} \lambda_{q+1} + \frac{\delta_{q+1}^{1/2} \delta_{q} \lambda_{q+1} \lambda_{q+1}}{\mu} \right), \\
\left\| \partial_{t} \check{R}_{1} + v_{1} \cdot \nabla \check{R}_{1} \right\|_{0} & \leq C \delta_{q+1}^{1/2} \lambda_{q} \left( \delta_{q+1}^{1/2} \mu \lambda_{q+1}^{2} + \frac{\delta_{q+1}^{1/2} \delta_{q} \lambda_{q+1}^{2}}{\mu} \right).
\end{align*}
\]

In contrast to [2], the extra factors of \( \lambda_{q+1}^{1/2} \) appearing in (86) and (87) are due to the fact that in the present scheme derivatives falling on \( \chi_{l} \) pick up an extra factor of \( \lambda_{q+1}^{1/2} \). A second point of difference to [2] is that unlike [2], no terms of the form
\[
\delta_{q+1}^{1/2} \delta_{q}^{1/2} \lambda_{q} \ell + \frac{\delta_{q+1}^{1/2} \delta_{q} \lambda_{q}}{\lambda_{q+1}^{1/2} \ell \mu}
\]
appear within the brackets of the right hand sides of (86) and (87). The absence of the first term in (88) can be easily explained by the fact that by (85) we have
\[
\delta_{q+1}^{1/2} \delta_{q}^{1/2} \lambda_{q} \ell \leq \frac{\delta_{q+1}^{1/2} \delta_{q} \lambda_{q} \lambda_{q+1}^{1/2}}{\mu}.
\]
The absence of the second term in (88) follows by the same reasoning as the absence of an analogous term in Proposition 5.1 (see the comment (D) after the statement of Proposition 5.1 and Remarks 2 and 3).

### 6. Choice of the parameters and conclusion of the proof

We begin by noting that we have not imposed any upper bounds on the choice of \( \lambda_{0} \) and thus we are free to choose \( \lambda_{0} \) to be as large as need be: in what follows we will use this fact multiple times without further comment.
1/5 – \varepsilon convergence. We now make the following parameter choices
\[
\alpha := 1 + \varepsilon_0, \quad \lambda_q = \lfloor \lambda_0^q \rfloor, \\
\varepsilon_1 := \frac{\varepsilon_0^2}{18}, \quad \delta_q := \lambda_q^{-2/5 + 2\varepsilon_0}, \\
\mu := \delta_q^{1/4} \delta_q^{1/4} \lambda_q^{1/2} \lambda_{q+1}^{1/2},
\]
here \(|a|\) denotes the largest integer smaller than \(a\). It is worth noting that with the above choices, our definition of \(\mu\) agrees with the definition given in [2].

Having made the above choices it is clear the inequalities (30) and (35) are satisfied. Moreover assuming (31)–(34), it follows as a consequence of Corollary 4.2 that (31) and (32) are satisfied with \(q\) replaced by \(q + 1\). In order to show (33) and (34) with \(q\) replaced with \(q + 1\) we note that with our choices of parameters we obtain from (86) and (87) that
\[
\left\| \tilde{R}_{q+1} \right\|_0 + \frac{1}{\lambda_{q+1}} \left\| \tilde{R}_{q+1} \right\|_1 \leq C \delta_{q+1}^{1/2} \mu \lambda_{q+1}^{\varepsilon_1}
\]
and
\[
\frac{1}{\delta_{q+1}^{1/2} \lambda_{q+1}} \left\| (\partial_t + v_{q+1} \cdot \nabla) \tilde{R}_{q+1} \right\|_0 \leq C \delta_{q+1}^{1/2} \mu \lambda_{q+1}^{2\varepsilon_1}.
\]

Hence since
\[
\delta_{q+1}^{1/2} \mu \lambda_{q+1}^{2\varepsilon_1} \leq C \lambda_{q-2/5 + \frac{6\varepsilon_0}{5} + \frac{5\varepsilon_0^2}{3} + \frac{\varepsilon_0^3}{6}}, \leq C \delta_{q+2} \lambda_{q-2}^{-\varepsilon_0^2}
\]
we obtain both (33), and (34) with \(q\) replaced by \(q + 1\). Since the inequalities (31)–(34) hold for \(q = 0\), we obtain by induction that the inequalities hold for \(q \in \mathbb{N}\). The inequalities (51)–(77) with \(q\) replaced by \(q + 1\) then follow as a consequence of Corollary 4.2 together with the estimates on \(v, p, \tilde{R}, w, w_o\), and \(w_c\). In particular, one may derive time derivative estimates on \(w\) and \(p_1 - p\) from the simple decomposition \(\partial_t = D_t - v_t \cdot \nabla\) and the estimates
\[
\left\| \partial_t w \right\|_0 \leq \left\| \partial_t w_o \right\|_0 + \left\| \partial_t w_c \right\|_0
\]
and
\[
\left\| \partial_t \left( p_{q+1} - p_q \right) \right\|_0 \leq (\left\| w_c \right\|_0 + \left\| w_o \right\|_0)(\left\| \partial_t w_c \right\|_0 + \left\| \partial_t w_o \right\|_0) + 2\left\| w \right\|_0 \left\| \partial_t v \right\|_0 + \ell \left\| v_1 \right\| \left\| \partial_t w \right\|_0
\]
and
\[
\left\| \partial_t \left( p_{q+1} - p_q \right) \right\|_0 \leq (\left\| w_c \right\|_0 + \left\| w_o \right\|_0)(\left\| D_t w_o \right\|_0 + \left\| D_t w_c \right\|_0 + \left\| w_o \right\|_1 + \left\| w_c \right\|_1) + 2\left\| w \right\|_0 \left( \left\| \partial_t v + v \cdot \nabla v \right\|_0 + \left\| v_1 \right\|_1 + \ell \left\| v_1 \right\| \left\| \partial_t w \right\|_0
\]
\[
\leq (\left\| w_c \right\|_0 + \left\| w_o \right\|_0)(\left\| D_t w_o \right\|_0 + \left\| D_t w_c \right\|_0 + \left\| w_o \right\|_1 + \left\| w_c \right\|_1) + 2\left\| w \right\|_0 \left( \left\| p_1 \right\| + \left\| \tilde{R} \right\|_1 + \left\| v_1 \right\|_1 + \ell \left\| v_1 \right\| \left\| \partial_t w \right\|_0
\]
The required estimates then follow as a consequence from (31)–(34), (51), (53), (57), (59) and (61).
1/3 − ε convergence. Let us define \( U^{(q)} \) to be the set

\[
U^{(q)} = \bigcup_{l \in [-\mu_q, \mu_q]} [\mu_q^{-1}(l + 1/2 - \lambda_q^{-\varepsilon_1}), \mu_q^{-1}(l + 1/2 + \lambda_q^{-\varepsilon_1})],
\]

i.e. a union of \( \sim 2\mu_q \) balls of radius \( \lambda_q^{-\varepsilon_1}\mu_q^{-1} \) and define

\[
V^{(q)} = \bigcap_{q' = q}^\infty U^{(q')}.
\]

Observe that \( V^{(q)} \) can be covered by a sequence of balls of radius \( r_i \) such that

\[
\sum r_i^d \leq 3 \sum_{q' = q}^\infty \lambda_q^{-d\varepsilon_1}\mu_{q'}^{-1-d}.
\]

Thus assuming

\[
d > \frac{(1 + \alpha)(-\frac{1}{5} + \varepsilon_0 + 1)}{(1 + \alpha)(-\frac{1}{5} + \varepsilon_0 + 1) + 2\alpha\varepsilon_1},
\]

it follows that the right hand side of (89) converges to zero as \( q \) tends to infinity.

From this point on we assume \( d < 1 \) is fixed, satisfying (90) – which we note is possible due to the fact the right hand side of (90) is strictly less than 1.

For any time \( t_0 \in \bigcap_N V^{(N)} \) we simply set \( \delta_{q,t_0} = \delta_q \) for all \( q \).

Now suppose \( t_0 \notin V^{(N)} \) for some integer \( N \), furthermore assume \( N \) to be the smallest such integer. We now make the following parameter choices

\[
\delta_{q+1,t_0} := \begin{cases} 
\lambda_q^{-2/5 + 2\varepsilon_0} & \text{if } q \leq N \\
\max \left( \lambda_q^{-2/5 + 2\varepsilon_0}, \lambda_{q+1}^{-2/5 + 2\varepsilon_0} \right) & \text{if } q > N
\end{cases}
\]

It follows that

\[
\frac{\delta_{q,t_0}\lambda_q}{\delta_{q+1,t_0}\lambda_{q+1}} \geq \lambda_q^{\varepsilon_0/3},
\]

from which we obtain (91) assuming \( \varepsilon_0 \) is sufficiently small. Applying Corollary 4.3 and Proposition 5.1 iteratively we see that (36)-(39) hold for all \( q \geq N \). In particular, in order to show (38) for \( q \) replaced by \( q + 1 \) we note that by Proposition 5.1 we have for all times \( t \) satisfying \( |t\mu_q - l_q| < 1/2(1 - \lambda_q^{-\varepsilon_1}) \)

\[
\|\tilde{R}_1\| + \frac{1}{\lambda_{q+1}}\|\tilde{R}_1\| \leq C\delta_{q+1,t_0}^{1/2}\delta_{q,t_0}^{1/2}\lambda_q\ell + C\delta_{q+1,t_0}^{1/2}\lambda_q\lambda_{q+1}^{\varepsilon_1}.
\]

(91)
Notice that if $|\mu_{q+2}^1 t - l_{q+2}| \leq 1$ then

$$|t \mu_{q+1} - l_{q+1}| \leq \frac{\mu_{q+1}}{\mu_{q+2}} |\mu_{q+2}^1 t - l_{q+2}| + \left| \frac{\mu_{q+1}^1 t_{q+2} - l_{q+1}}{\mu_{q+2}} \right|$$

$$\leq \frac{\mu_{q+1}}{\mu_{q+2}} + \frac{\mu_{q+1}^1}{\mu_{q+2}} \left| l_{q+2} - t_0 \right| + \left| \mu_{q+1}^1 t_0 - l_{q+1} \right|$$

$$< \frac{2\mu_{q+1}}{\mu_{q+2}} + \left| \mu_{q+1}^1 t_0 - l_{q+1} \right|$$

$$< 2\lambda_q^{-\eps_q/4} + 1/2 - \lambda_q^{-\eps_q/1}$$

$$< 1/2(1 - \lambda_q^{-\eps_q/1}).$$

Thus (91) holds for times $t$ in the range $|\mu_{q+2}^1 t - l_{q+2}| < 1$.

Taking logarithms of $I$ and $II$ we obtain

$$\ln I \leq \left( 1 + \frac{\eps_0}{2} \right) \ln \delta_{q,t_0} + \left( \frac{\eps_0^2}{18} + \frac{\eps_0^3}{18} - \eps_0 \right) \ln \lambda_q + C \quad (92)$$

and

$$\ln II \leq \left( \frac{3}{2} + \eps_0 \right) \ln \delta_{q,t_0} + \left( \frac{1}{5} - \frac{7\eps_0}{5} - \frac{4\eps_0^2}{9} + O(\eps_0^3) \right) \ln \lambda_q + C. \quad (93)$$

Note by definition we have

$$\ln \delta_{q+2,t} \geq (1 + \eps_0)^2 \ln \delta_{q,t_0} - \left( \frac{2\eps_0^2}{9} + O(\eps_0^3) \right) \ln \lambda_q. \quad (94)$$

Thus since $\delta_{q,t_0} \geq \lambda_q^{-2/3 + 2\eps_0}$, combining (92) and (94) we obtain

$$\ln \left( \frac{I}{\delta_{q+2,t}} \right) \leq \left( \frac{-3\eps_0}{2} - \eps_0^2 \right) \ln \delta_{q,t_0} + \left( \frac{5\eps_0^2}{18} - \eps_0 + O(\eps_0^3) \right) \ln \lambda_q + C$$

$$\leq \left( \frac{-\eps_0^2}{4} + O(\eps_0^3) \right) \ln \lambda_q + C. \quad (95)$$

Similarly, since $\delta_{q,t_0} \leq \lambda_q^{-2/3 + 2\eps_0}$, combining (93) and (94) we obtain

$$\ln \left( \frac{II}{\delta_{q+2,t}} \right) \leq \left( \frac{1}{2} - \eps_0 - \eps_0^2 \right) \ln \delta_{q,t_0} + \left( \frac{1}{5} - \frac{7\eps_0}{5} - \frac{2\eps_0^2}{9} + O(\eps_0^3) \right) \ln \lambda_q + C$$

$$\leq \left( -\eps_0^2 + O(\eps_0^3) \right) \ln \lambda_q + C. \quad (96)$$

Hence assuming $\eps_0$ is sufficiently small, from (95) and (96) we obtain (38) for $q$ replaced by $q+1$.

Observe also that there exists an $N'$ such that for all $q \geq N + N'$ we have

$$\delta_{q,t_0} = \lambda_q^{-2/3 + 2\eps_0},$$

and hence the inequality (12) is never satisfied for $q \geq N + N'$. Thus

$$\Xi^{N+N'} \subset V^N.$$
In particular $N'$ can be chosen universally, independent of $N$. Fixing $\delta > 0$ and choosing $N$ such that $V^N$ can be covered by a sequence of balls of radius $r_i$ satisfying

$$\sum r_i^d < \delta,$$

we obtain that if we set $M = N + N'$ then \([13] \) is satisfied which concludes the proof of Proposition \([0.2] \).

**Remark 4.** For the sake of completeness we note that analogously to the estimates \([5]-[7] \), the estimates \([9]-[11] \) follow as a consequence of Lemma \([4.1] \), Lemma \([4.3] \) and Proposition \([5.1] \) – here the set $\Omega$ can be taken explicitly to be

$$\Omega := \bigcap_{q=1}^{\infty} V^{(q)}.$$

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