Hopf Algebra Symmetry and String Theory

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We investigate the Hopf algebra structure in string worldsheet theory and give a unified formulation of the quantization of string and the space-time symmetry. We reformulate the path integral quantization of string as a Drinfeld twist at the worldsheet level. The coboundary relation shows that the Drinfeld twist defines a module algebra which is equivalent to operators with normal ordering. Upon applying the twist, the space-time diffeomorphism is deformed into a twisted Hopf algebra, while the Poincaré symmetry is unchanged. This suggests a characterization of the symmetry: unbroken symmetries are twist invariant Hopf subalgebras, while broken symmetries are realized as twisted ones. We provide arguments that relate this twisted Hopf algebra to symmetries in path integral quantization.

§1. Introduction

String theory is a promising candidate for realizing the unified theory of quantum gravity and field theory for elementary particles. There is evidence to support the hypothesis that closed string theory contains general relativity. However, it is formulated only as a perturbation theory around a specific background, and therefore the concepts in classical general relativity such as general covariance are not manifest.

As an example, consider the worldsheet theory of strings in the flat Minkowski space as a target. The quantization of the theory should be Poincaré covariant, and thus we find that there are massless spin 2 graviton states in the spectrum. The massless spectrum requires, in general, the existence of space-time gauge symmetry and diffeomorphisms, even though it is not a manifest symmetry of the worldsheet theory. Moreover, scattering amplitudes among gravitons and other massless excitations are reproduced by the (super)gravity at low energy (see for example Ref.1 and references therein). It is believed that this worldsheet theory is merely an expansion of the full string theory (in which the diffeomorphism is manifest) around a specific vacuum, and the condensation of gravitons would describe another background.

From these general considerations, it is evident that there is a close connection among the quantization on the worldsheet, the Poincaré covariance and the space-time gauge symmetry as well as the general covariance, but they are linked in a far
from direct way. In this paper, we propose a framework to describe this connection by studying the Hopf algebra structure of string worldsheet theory.

The use of Hopf algebras in this paper is motivated by the use of the twisted Hopf algebra in the development of noncommutative geometry\(^2\)–\(^4\) and also by the recent progress in understanding the global or local symmetry on the Moyal-Weyl noncommutative space.\(^5\)–\(^8\),\(^10\) In Refs.7), 8) and 10) it is proposed that the explicit breaking of the Poincaré symmetry is remedied by considering it not as a group but as a Hopf algebra. The key idea is that the Moyal-Weyl \(*\)-product is considered as a twisted product equipped with a Drinfeld twist of the Hopf algebra for the Poincaré-Lie algebra. In other words, both the noncommutativity and the modification of the symmetry are controlled by a single twist. It is then generalized to the twisted version of the diffeomorphism on the Moyal-Weyl space.\(^8\) In string theory with a background \(B\)-field, the effective theory of \(D\)-branes is described by a gauge theory on the same Moyal-Weyl space.\(^12\) Therefore, it is expected that there is a corresponding twisted Hopf algebra structure in string theory.

However, in this paper we do not focus on the case with a non-zero \(B\) field background; the purpose of this paper is to formulate a framework applicable to more general situations. We will see that the twisted Hopf algebra has a similar structure even in a background with a vanishing \(B\) field. From the viewpoint of the Hopf algebra structure presented here, both the quantization and the space-time symmetry are controlled by a single twist. Of course, we can include the nontrivial \(B\)-field background in the formulation developed in this paper, and we will report on the case of a \(B\)-field background in a separate paper.\(^13\)

In this paper, we study a Hopf algebra structure in string worldsheet theory in the Minkowski background and its covariant quantization as an example, but in a form that enables one to apply it to more general cases. We use the functional description of strings and define a Hopf algebra that consists of functional diffeomorphism variations as well as of worldsheet variations, and we also define its module algebra of classical functionals. We then reformulate the path integral (functional integral) quantization of strings in terms of the twisted Hopf algebra for functionals. This formulation leads to our proposal that each choice of twist defines a quantization scheme, which is a general concept not limited to our example. By the fact that the twisted Hopf algebra is isomorphic to the original one, but is accompanied by a normal ordering, we clarify its relation with the operator formulation. Although this quantization is carried out by the twist in the Hopf subalgebra of worldsheet variations, the space-time diffeomorphism is also deformed to the twisted Hopf algebra. It turns out that the Poincaré-Lie algebra remains unaltered under the twist and is therefore regarded as a true symmetry, while a full diffeomorphism is broken but it is maintained as a twisted symmetry.

The paper is organized as follows: In §2, we first present formulae written in standard string theory textbooks, which will be reformulated in Hopf algebra language throughout the paper. Then, we define a Hopf algebra structure within classical string theory: a Hopf algebra consists of functional variations including diffeomorphisms and a corresponding module algebra of classical functionals. In §3, we reformulate the known path integral quantization of strings as a twist of the Hopf algebra
and the module algebra defined above. The isomorphism between the twisted Hopf algebra and the normal ordered algebra is also studied to relate it to the operator formulation of strings. In §4, we focus on the space-time symmetry in this twisted Hopf algebra, and how the twisting deforms a classical diffeomorphism while keeping the Poincaré-Lie algebra invariant. To relate the notion of the symmetry in the ordinary path integral, we rewrite Hopf algebra identities in the form of Ward-like identities among correlation functions. Section 5 is devoted to a discussion and conclusion. We summarize the basic facts about Hopf algebras and their twisting in Appendix A, and about Hopf algebra cohomology in Appendix C. Appendices B and D are devoted to technical proofs.

§2. Hopf algebra in string theory

In this section, we first define the notation and remind the reader of some formulae in string theory that we will consider in this paper. Then, we give a definition of the Hopf algebra and its module algebra of functionals, which appears in the classical worldsheet theory of strings.

2.1. Preliminaries

We consider bosonic closed strings as well as open strings and take a space-filling D-brane for simplicity. We start with the $\sigma$ model of the bosonic string with flat $d$-dimensional Minkowski space as the target. The action in the conformal gauge is

$$S_0[X] = \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2z \eta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu,$$

where the worldsheet $\Sigma$ can be any Riemann surface with boundaries, and typically, we take it to be the complex plane (upper half plane) for a closed string (open string, respectively). $z^a = (z, \bar{z})$ are the complex coordinates on the worldsheet. The flat metric in the target space $\mathbb{R}^d$ is represented by $\eta_{\mu\nu}^i$. We frequently use the metric $\eta_{i}^{\mu\nu}$ to raise and lower the indices. The worldsheet field $X^\mu(z^a) = X^\mu(z, \bar{z})$ is often abbreviated to $X^\mu(z)$ unless stated otherwise.

We will consider correlation functions of the form

$$\langle V_1(z_1) \cdots V_n(z_n) \rangle_0$$

and their properties under space-time transformations. Here, the vacuum expectation value (VEV) $\langle \cdots \rangle_0$ is defined by the path integral for the worldsheet field $X^\mu(z, \bar{z})$ with the action $S_0$ defined in (2.1):

$$\langle \mathcal{O} \rangle_0 = \frac{\int \mathcal{D}X_0 e^{-S_0}}{\int \mathcal{D}X e^{-S_0}},$$

and $V_i(z_i)$ denotes a vertex operator inserted at $z_i$ on the worldsheet. The quantization we discuss in this paper is the above path integral average over the field $X$ under the fixed topology, since it is sufficient for describing our ideas. Therefore, for instance, the $(b,c)$-ghost part of the full correlation function is omitted. We also do
not perform the integration over moduli parameters since we are only considering correlation functions on the fixed worldsheet in this paper.

In the operator formulation, a local vertex operator $V(z)$ is well-defined by taking a product of field operators $X^\mu(z, \bar{z})$, their derivatives $\partial X^\mu(z)$, $\bar{\partial} X^\mu(\bar{z})$, and the higher derivatives by applying the oscillator normal ordering to avoid the divergences appearing in the operator product. On the other hand, in the path integral there are also divergences at the coincidence point, and these divergences are regularized either by removing the self-contraction by hand, or equivalently, by subtracting them via the formula$^1$

$$:F[X] := \mathcal{N}_0 F[X], \quad (2.4)$$

where

$$\mathcal{N}_0 = \exp\left\{ -\frac{1}{2} \int d^2 z \int d^2 w \ G_0^{\mu\nu}(z, w) \frac{\delta}{\delta X^\mu(z)} \frac{\delta}{\delta X^\nu(w)} \right\}, \quad \text{(2.5)}$$

which is called the conformal normal ordering. Here, $G_0^{\mu\nu}(z, w)$ is the free propagator on the worldsheet defined through

$$\langle X^\mu(z) X^\nu(w) \rangle_0 = G_0^{\mu\nu}(z, w) \quad \text{(2.6)}$$

and its function form depends on the worldsheet topology. For instance, on the complex plane, it is

$$G_0^{\mu\nu}(z, w) := \eta^{\mu\nu} G_0(z, w) = -\frac{\alpha'}{2} \eta^{\mu\nu} \ln |z - w|^2. \quad \text{(2.7)}$$

In this case, the conformal normal ordering coincides with the oscillator normal ordering, but the subtraction using (2.7) works in general and we refer to (2.4) as the normal ordering in the following. Note that $:F[X] :$ itself is a power series expansion with divergent coefficients, and therefore it should be understood only together with the path integral.

The local vertex operators may be located either in the bulk or on the boundary of the worldsheet. The product of two local vertex operators in any correlation function is a time-ordered product in the operator formalism, which has the natural correspondence with the path integral formulation. It is rewritten by the normal ordered (but bi-local) vertex operators using Wick’s theorem:

$$:F[X] : :G[X]: = \exp\left\{ \int d^2 z \int d^2 w \ \eta^{\mu\nu} G_0(z, w) \frac{\delta_F}{\delta X^\mu(z)} \frac{\delta_G}{\delta X^\nu(w)} \right\} F[X] G[X] :,$$

where the derivative $\delta_F(\delta_G)$ acts on $F(G)$. This formula is again valid only inside the path integral VEV. In addition, by using the Taylor expansion around $w$ with respect to $(z - w)$, the r.h.s. coincides with the usual operator product expansion. As we shall see, the above formulae (2.5) and (2.8) always require careful application at the functional level. One of the purposes of this paper is to give a simple algebraic characterization of these formulae as Hopf algebra actions. In this formulation, not the operators but only the functional calculi are used, and there are no complications when the formal divergent expansions appear in the normal ordering formula.
After formulating the VEV in functional language, we shall discuss the symmetry of the correlation functions (2.2). If the action $S_0$ and the measure are invariant under the variation of the worldsheet fields $X^\mu(z) \to X^\mu(z) + \delta X^\mu(z)$, this defines a symmetry in the quantum theory, and we obtain a Ward identity associated with such a variation:

$$0 = \sum_{i=1}^n \langle V_1(z_1) \cdots \delta V_i(z_i) \cdots V_n(z_n) \rangle_0.$$  \hspace{1cm} (2.9)

Here the infinitesimal transformation of any local vertex operator $V(z) =: F[X]: (z)$ is given by the commutation relation with the symmetry generator, and it has the same form as the classical transformation. It is written in terms of first-order functional derivatives as

$$\delta V(x) = - : \int d^2w \, \delta X^\mu(w) \frac{\delta F[X]}{\delta X^\mu(w)} :.$$  \hspace{1cm} (2.10)

In our case, the unbroken space-time symmetry consists of Poincaré transformations generated by

$$P^\mu = -i \int d^2 z \, \eta^{\mu\lambda} \frac{\delta}{\delta X^\lambda(z)},$$

$$L^{\mu\nu} = -i \int d^2 z \, X^{[\mu}(z) \eta^{\nu]\lambda} \frac{\delta}{\delta X^\lambda(z)},$$  \hspace{1cm} (2.11)

where $P^\mu$ are the generators of the translation and $L^{\mu\nu}$ are the Lorentz generators. Note the position of the normal ordering operation in (2.10). To obtain the quantum transformation law and a similar identity in the case that a variation is not a symmetry, we again need to take care of the ordering and the divergences. We see in §4 that the transformation law of broken symmetries should be twisted in the Hopf algebra sense.

### 2.2. Hopf algebra for classical functional variations

Before discussing the quantized theory of strings, we consider a Hopf algebra structure and the related module algebra structure at the classical level, which underlies the quantization of the string worldsheet theory. We will use functionals and functional derivatives as our main tools. Actually, this structure does not depend on the action $S_0$, nor on the conformal symmetry, and thus it is background independent.

**Classical functionals as module algebra**

Classically, the string variable $X^\mu(z)$ ($\mu = 0, \cdots, d-1$) is a set of classical functions defining the embedding map $X$ of a worldsheet $\Sigma$ into a target space $\mathbb{R}^d$:

$$X : \Sigma \ni z \mapsto X(z) = \left( X^0(z, \bar{z}), \cdots, X^{d-1}(z, \bar{z}) \right) \in \mathbb{R}^d.$$  \hspace{1cm} (2.12)

Any function on the space-time $\mathbb{R}^d$ is mapped to the worldsheet function via the pull-back $X^* : C^\infty(\mathbb{R}^d) \to C^\infty(\Sigma)$ as $f \mapsto (X^*f)(z) = f[X(z)]$. The pull-back of a 1-form $\omega \in \Omega^1(\mathbb{R}^d)$ is also defined by $X^*(\omega_\mu dX^\mu) = \omega_\mu(X(z)) \partial_\mu X^\mu(z) dz_\alpha \in \Omega^1(\Sigma)$. 


They can be extended to any tensor field on the space-time. Therefore, any field on the space-time (D-brane) is realized as a worldsheet field. For example, a scalar (tachyon) field and a gauge field give

\[ (X^* \phi)(z) = \phi[X(z)] = \int d^4k \phi(k)e^{ikX(z)}, \]
\[ (X^* A_\mu)_{\alpha}(z) = A_\mu[X(z)]\partial_\alpha X^\mu(z). \]  

(2.13)

A complex valued functional \( I[X] \) of \( X \) is defined on the space of embeddings as a \( \mathbb{C} \)-linear map \( I : \text{Map}(\Sigma, \mathbb{R}^d) \to \mathbb{C} \). It is typically given by the integrated form over the world sheet \( \Sigma \) as

\[ I[X] = \int d^2z \rho(z)F[X(z)]. \]  

(2.14)

where \( F[X(z)] \) is a component of a pull-back tensor field such as (2.13) and \( \rho(z) \) is a weight function (distribution). The action functional \( S_0[X] \) in (2.1) is a simple example. Note that a pull-back function \( F[X(z)] \) defines a functional when we fix \( z \) at some point \( z_i \in \Sigma \). Thus, we also consider a functional with an additional label \( z_i \) by choosing the delta function as the weight function \( \rho(z) \),

\[ F[X](z_i) = \int d^2z \delta^{(2)}(z - z_i)F[X(z)], \]  

(2.15)

which we call a local functional at \( z_i \). We also write it simply as \( F[X(z_i)] \) when this does not cause confusion. The types of functionals given by (2.14) and (2.15) correspond to an integrated vertex operator and a local vertex operator after quantization, respectively.

Now let \( \mathcal{A} \) be the space of complex valued functionals comprising of the embedding \( X^\mu(z) \) and its worldsheet derivatives \( \partial_\alpha X^\mu(z) \) described above. We define the multiplication of two functionals as \( I_1I_2[X] = I_1[X]I_2[X] \), where the r.h.s. is multiplication in \( \mathbb{C} \). This leads to the multiplication of two local functionals as \( FG[X](z_1, z_2) = F[X(z_1)]G[X(z_2)] \). In order that this product is an element of \( \mathcal{A} \), bilocal functionals at \( (z_1, z_2) \) should be included in \( \mathcal{A} \). By including all multi-local functionals with countable labels, \( \mathcal{A} \) forms an algebra over \( \mathbb{C} \). We denote this product as a map \( m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \):

\[ m(F \otimes G) = FG. \]  

(2.16)

Note that the product is commutative and associative.

**Hopf algebra of functional vector fields**

Next, let us define a Hopf algebra acting on the classical functionals \( \mathcal{A} \). Consider an infinitesimal variation of the embedding function \( X^\mu(z) \to X^\mu(z) + \xi^\mu[X(z)] \), which is a diffeomorphism from the viewpoint of the target space. Then, the change of a functional is generated by a first order functional derivative of the form

\[ \xi = \int d^2w \xi^\mu[w]\frac{\delta}{\delta X^\mu[w]}, \]  

(2.17)
where the functional derivative is defined by
\[ \frac{\delta}{\delta X^\mu(z)} X^\nu(w) := \delta^\mu_\nu \delta^{(2)}(z - w). \] (2.18)

The object \( \xi \) in (2.17) is a functional version of the vector field acting on \( \mathcal{A} \) and it is a derivation of the algebra \( \mathcal{A} \). For a local functional \( F[X] \) its action (Lie derivative along \( \xi \)) is written as \( \xi \triangleright F[X] = (\xi^\mu \partial_\mu F)[X] \). It is related to the variation of the functional under the diffeomorphism as \( \delta F[X] = -\xi \triangleright F[X], \) *1*

The object \( \xi \) can be extended to the following expression by including world sheet variations:
\[ \xi = \int d^2 w \xi^\mu(w) \frac{\delta}{\delta X^\mu(w)}, \] (2.19)
where \( \xi^\mu(w) \) is a weight function (distribution) on the worldsheet of the following two classes.

i) \( \xi^\mu(w) \) is a pull-back of a target space function \( \xi^\mu(w) = \xi^\mu[X(w)] \). It corresponds to a target space vector field as defined above.

ii) \( \xi^\mu(w) \) is a function of \( w \) but is independent of \( X(w) \) and its derivatives. It corresponds to a change of the embedding \( X^\mu(z) \to X^\mu(z) + \xi^\mu(z) \) and is used to derive the equation of motion. We also admit functions such as \( \xi^\mu(w, z_1, \cdots) \) with some additional labels \( z_1, \cdots \). The functional derivative itself is an example of this, i.e., by setting \( \xi^\mu(w, z) = \delta^\mu_\nu \delta(w - z) \) in eq. (2.19).

Such a mixture of space-time vector fields and worldsheet variations becomes important in the following sections. Note that in this paper we do not consider another class with \( \xi^\mu(w) = e^a(w) \partial_a X^\mu(w) \), corresponding to an infinitesimal coordinate transformation \( w^a \to w^a + e^a(w) \) on the worldsheet.

We denote the space of all such vector fields \( \xi \) (2.19) as \( \mathfrak{X} \) and, in particular, the \( \xi \) in class ii) as \( \mathfrak{C} \). We write its action on \( \mathcal{A} \) as \( \xi \triangleright F \). By successive transformations \( \xi \triangleright (\eta \triangleright F) \), we see that functional vector fields form a Lie algebra with the Lie bracket
\[ [\xi, \eta] = \int d^2 w \left( \xi^\mu \frac{\delta \eta^\nu}{\delta X^\mu} - \eta^\mu \frac{\delta \xi^\nu}{\delta X^\mu} \right)(w) \frac{\delta}{\delta X^\nu(w)} \] (2.20)

We can then define the universal enveloping algebra \( \mathcal{H} = U(\mathfrak{X}) \) of \( \mathfrak{X} \) over \( \mathbb{C} \), which has a natural cocommutative Hopf algebra structure \( (U(\mathfrak{X}); \mu, \iota, \Delta, \epsilon, S) \) **2**. The defining maps given on elements \( \xi, \eta \in \mathfrak{X} \) are
\begin{align*}
\mu(\xi \otimes \eta) &= \xi \cdot \eta, \quad \iota(k) = k \cdot 1, \\
\Delta(1) &= 1 \otimes 1, \quad \Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi, \\
\epsilon(1) &= 1, \quad \epsilon(\xi) = 0, \\
S(1) &= 1, \quad S(\xi) = -\xi, \quad (2.21)
\end{align*}

*1* Note that \( F[X] \) is a scalar functional so that \( F'[X'] = F[X] \). As usual, the variation of \( F[X] \) is defined by the difference at the same “point” \( X \) and it is written by \((-1)\) times the Lie derivative along \( \xi \) as \( \delta F[X] = F'[X] - F[X] = -\langle \xi^\mu \partial_\mu F \rangle[X] \).

**2** This is a generalization of the Hopf algebra of vector field discussed in Ref.8). For a similar approach see also Ref.9).
where $k \in \mathbb{C}$. The coproduct $\Delta(\xi)$ implies that $\xi$ is a primitive element, which follows from the Leibniz rule of functional derivatives. The product $\mu$ is defined by successive transformations of $\eta$ and $\xi$ and it is also denoted by $\xi : \eta$. It gives higher order functional derivatives thus that the vector space $U(\mathfrak{X})$ consists of elements of the form

$$h = \int d^2 z_1 \cdots \int d^2 z_k \xi_1^{\lambda_1}(z_1) \frac{\delta}{\delta X^{\lambda_1}(z_1)} \cdots \xi_k^{\lambda_k}(z_k) \frac{\delta}{\delta X^{\lambda_k}(z_k)} .$$

(2.22)

As usual, the maps are uniquely extended to any such element of $U(\mathfrak{X})$ by the algebra (anti-) homomorphism.

The algebra $\mathcal{A}$ of functionals is now considered to be an $\mathcal{H}$-module algebra. The action of the element $h \in \mathcal{H}$ on $F \in \mathcal{A}$ is denoted by $h \triangleright F$ as above. The action on the product of two elements in $\mathcal{A}$ is defined by

$$h \triangleright m(F \otimes G) = m(\Delta(h) \triangleright (F \otimes G)) ,$$

(2.23)

which represents the covariance of the module algebra $\mathcal{A}$ under diffeomorphisms or worldsheet variations.

In particular, the Poincaré transformations are generated by (2.11). It is easy to see that they satisfy the standard commutation relation for the Poincaré-Lie algebra, $\mathcal{P} = \mathbb{R}^{d \times \mathfrak{so}(1, d-1)}$, and that $\mathcal{P}$ is a Lie subalgebra of $\mathfrak{X}$. As a result, their universal envelope $U(\mathcal{P})$ is also a Hopf subalgebra of $\mathcal{H} = U(\mathfrak{X})$.

Another Hopf subalgebra $U(\mathfrak{C}) \subset \mathcal{H}$ is that generated by worldsheet variations. Such variations form an abelian Lie subalgebra $\mathfrak{C}$, and thus the algebra $U(\mathfrak{C})$ is a commutative and cocommutative Hopf algebra.

§3. Quantization as a twist on the worldsheet

So far we have only dealt with classical functionals of $X^\mu(z)$. In string theory, the field $X^\mu(z)$ must be quantized. The quantization can be achieved using the functional integral over all possible embedding functions $\{X^\mu(z)\}$ weighted with a Gaussian-type functional $e^{-S_0[\{X\}]_0}$ as given in eq. (2.3). Since $S_0$ is quadratic in $X$, the VEV of a functional $\langle I[\{X\}] \rangle_0$ is completely determined by the Wick contraction. We show that the same VEV is reproduced simply by a twist of a Hopf algebra. This leads to our proposal that a Hopf algebra twist is a quantization. By using cohomological results, we clarify the relation between the twist and the normal ordering, which gives a more rigorous characterization of the path integral VEV.

3.1. Wick contraction as a Hopf algebra action

We first take a heuristic approach to rewriting the path integral VEVs in terms of the Hopf algebra $\mathcal{H}$ introduced in §2.2. To this end, we consider the following two maps $\mathcal{N}_0^{-1} : \mathcal{A} \to \mathcal{A}$ and $\tau : \mathcal{A} \to \mathfrak{C}$ where

$$\mathcal{N}_0^{-1} = \exp \left\{ \frac{1}{2} \int d^2 z \int d^2 w G^{\mu\nu}_0(z, w) \frac{\delta}{\delta X^\mu(z)} \frac{\delta}{\delta X^\nu(w)} \right\} ,$$

(3.1)

$$\tau(I[\{X\}]_0) = I[\{X\}]_0 \bigg|_{X=0} .$$

(3.2)
Here $N_0^{-1}$ is an element of $\mathcal{H}$ and this Hopf algebra action gives the contraction with respect to the free propagator (2.6), while $\tau$ extracts the scalar terms independent of $X$ in the functional. Note that if $I[X]$ contains a local functional $F[X](z)$, $\tau(I[X])$ also depends on the label $z$ in general, i.e., it is a complex function of $z$.

The path integral average (2.3) of a functional $I[X] \in \mathcal{A}$ can be written as a composition of these maps as $\tau \circ N_0^{-1} \triangleright : \mathcal{A} \to \mathbb{C}$:

$$\langle I[X] \rangle_0 = \tau(N_0^{-1} \triangleright I[X]) . \quad (3.3)$$

This is simply a rewriting of the formula derived in the standard path integral argument:

1) We consider the generating functional $Z[J]$ by temporarily introducing the source $J_\mu(z)$ for $X_\mu(z)$, where the VEV of $I[X]$ is given as $I[\frac{\delta}{\delta J}]Z[J]|_{J=0}$. Then, upon removing $J_\mu(z)$ from the expression by replacing it with the functional derivative of $X_\mu$, we obtain (3.3). Therefore, we also call (3.3) the VEV as in the case of the path integral.

However, for any functional $I[X]$ corresponding to a composite operator of $X$, the above map suffers from divergences originating from self-contractions. To remove these divergences, each functional inserted into the VEV is considered to be a normal ordered functional. Let $I[X]$ be a single local functional $F[X](z)$. The normal ordering is also given by a Hopf algebra action:

$$:F[X]: (z) = N_0 \triangleright F[X](z) , \quad (3.4)$$

where the subtraction $N_0 \in \mathcal{H}$ is given in eq. (2.5), which is the inverse of the contraction $N_0^{-1}$ (3.1) in $\mathcal{H}$. Then, its VEV is

$$\langle :F[X]: (z) \rangle_0 = \tau(F[X](z)) . \quad (3.5)$$

In particular, for any normal ordered local functional without a scalar term, its VEV is zero. It is known that this (conformal) normal ordering coincides with the oscillator normal ordering for $\Sigma = \mathbb{C}$. In that case, (3.5) corresponds to the characterization of the oscillator vacuum.

If the functional $I[X]$ is a multi-local functional at $z_1, z_2, \ldots$ given by a product of these normal ordered functionals, the path integral formula is given exactly by eq. (3.3) which leads to the multi-variable functions of $z_1, z_2, \ldots$. In particular, let us consider the VEV of the product of two local functionals $:F[X]: (z)$ and $:G[X]: (w)$, given by

$$\sigma(z, w) = \langle :F[X]: (z) :G[X]: (w) \rangle_0 . \quad (3.6)$$

Using the above introduced maps, we can rewrite the correlation function as a sequence of maps as

$$\sigma(z, w) = \tau \circ N_0^{-1} \triangleright m [(N_0 \otimes N_0) \triangleright (F[X] \otimes G[X])] = \tau \circ m [\Delta(N_0^{-1})(N_0 \otimes N_0) \triangleright (F[X] \otimes G[X])] , \quad (3.7)$$

where in the second line we used the covariance (2.23) of a Hopf algebra action on the product. This coproduct $\Delta(N_0^{-1}) \in \mathcal{H} \otimes \mathcal{H}$ shows that Wick contractions act separately on both $F$ and $G$ as self-contractions, and also as intercontractions.
between $F$ and $G$, but, because of the $(N_0 \otimes N_0)$ factor, only the latter is effective. Thus, the net contraction is characterized by an element of $H \otimes H$,

$$\mathcal{F}_0^{-1} = \Delta(N_0^{-1})(N_0 \otimes N_0).$$  \tag{3.8}$$

We show that the inverse of this operator defined by

$$\mathcal{F}_0 := \exp \left\{ - \int d^2z \int d^2w \frac{\delta \delta X^\mu}{\delta X^\mu(z) \otimes \delta X^\nu(w)} \right\}.$$ \tag{3.9}$$

satisfies (3.8). For this, we write $F_0 := \exp(F_0)$ in (3.9) and $N_0 := \exp(N_0)$ in (2.5).

Using the explicit form of $N_0$, the coproduct of $N_0$ is given by

$$\Delta(N_0) = N_0 \otimes 1 + 1 \otimes N_0 + F_0.$$ \tag{3.10}$$

from the standard Leibniz rule of the functional derivative. Here we have used the fact that $F_0$ is symmetric under the exchange of tensor factors, owing to the property of the Green function: $G_0^{\mu\nu}(z,w) = G_0^{\nu\mu}(w,z)$. Then, the relation (3.10) leads to

$$\Delta(N_0) = e^{\Delta(N_0)} = e^{N_0 \otimes 1 + 1 \otimes N_0 + F_0}$$

$$= (N_0 \otimes 1) (1 \otimes N_0) \mathcal{F}_0 = (N_0 \otimes N_0) \mathcal{F}_0.$$ \tag{3.11}$$

Therefore, $\mathcal{F}_0$ (and $\mathcal{F}_0^{-1}$) is written in terms of $N_0$ as follows:

$$\mathcal{F}_0 = \partial N_0^{-1} = (N_0^{-1} \otimes N_0^{-1}) \Delta(N_0),$$

$$\mathcal{F}_0^{-1} = \partial N_0 = \Delta(N_0^{-1})(N_0 \otimes N_0).$$ \tag{3.12}$$

As a result, we can write the correlation function $\sigma(z,w)$ as

$$\sigma(z,w) \equiv \langle :F[X]:(z) :G[X]:(w)\rangle_0 = \tau \circ m \left[ \mathcal{F}_0^{-1} \triangleright (F[X] \otimes G[X]) \right].$$ \tag{3.13}$$

This formula is algebraically well-defined, where the subtraction of the divergence is already taken into account. It contains only the divergences expected from the operator product of the two local operators.

Formula (3.13) is a typical form of a twisted product triggered by a twist of a Hopf algebra (Drinfeld twist). The main observation here is that the Wick contraction is a Hopf algebra action of an element $\mathcal{F}_0^{-1} \in \mathcal{H} \otimes \mathcal{H}$. If $\mathcal{F}_0$ is a twist element, the product inside the path integral is given by the twisted product $m_{\mathcal{F}_0} = m \circ \mathcal{F}_0^{-1}$. This is indeed the case as we will see below.

### 3.2. Quantization as a Hopf algebra twist

The above discussion motivates us to regard the quantization of strings as a Hopf algebra twist. In this subsection, we give a simple quantization procedure for defining the VEV on this basis, which coincides with the path integral counterpart for our example. For a general theory of the Hopf algebra twist, see Ref.2) (see also Appendix A).

Let $\mathcal{H} = U(\mathfrak{g})$ be the Hopf algebra of functional vector fields and let $\mathcal{A}$ be the algebra of classical functionals, which is an $\mathcal{H}$-module algebra with product $m$. 

Suppose that there is a twist element (counital 2-cocycle) $F_0 \in \mathcal{H} \otimes \mathcal{H}$, that is, it is invertible, counital with $(\text{id} \otimes \epsilon)F_0 = 1$ and satisfies the 2-cocycle condition
\[(F_0 \otimes \text{id})(\Delta \otimes \text{id})F_0 = (\text{id} \otimes F_0)(\text{id} \otimes \Delta)F_0 . \tag{3.14}\]

It is easy to show that our $F_0 \in \mathcal{H} \otimes \mathcal{H}$ (3.9) satisfies all these conditions (see Appendix B).

Given a twist element $F_0$, the twisted Hopf algebra $\mathcal{H}_F_0$ can be defined by the same algebra and the counit as $\mathcal{H}$, but with a twisted coproduct and antipode
\[\Delta_{F_0}(h) = F_0 \Delta(h) F_0^{-1}, \quad S_{F_0}(h) = US(h)U^{-1} \tag{3.15}\]
for all $h \in \mathcal{H}$, where $U = \mu(\text{id} \otimes S)F_0$. Correspondingly, a $\mathcal{H}$-module algebra $\mathcal{A}$ is twisted to the $\mathcal{H}_{F_0}$-module algebra $\mathcal{A}_{F_0}$. It is identical to $\mathcal{A}$ as a vector space but is accompanied by the twisted product
\[m_{F_0}(F \otimes G) = m \circ F_0^{-1} \triangleright (F \otimes G) . \tag{3.16}\]

This twisted product is associative owing to the cocycle condition (3.14). We also denote it as $F \ast_{F_0} G$ using a more familiar notation, i.e., the star product. Note that $\mathcal{H}_{F_0}$ is still cocommutative for our twist element $F_0$ (3.9), and thus the twisted product remains commutative.

We define the VEV for the twisted module algebra $\mathcal{A}_{F_0}$ simply as the map $\tau : \mathcal{A}_{F_0} \rightarrow \mathbb{C}$ introduced in (3.2). For any element $I[X] \in \mathcal{A}_{F_0}$ the map gives
\[\tau (I[X]) . \tag{3.17}\]

If $I[X]$ is a product of two elements in $\mathcal{A}_{F_0}$, using the above notation, their correlation function $\sigma(z, w)$ follows from (3.17) as
\[\sigma(z, w) = \tau(F[X(z)] \ast_{F_0} G[X(w)]), \tag{3.18}\]
which coincides with the path integral version of $\sigma(z, w)$ in (3.13) for $F_0$ in (3.9). Because the cocycle condition guarantees the associativity of the twisted product, the correlation function of $n$ local functionals is similarly
\[\sigma(z_1, ..., z_n) = \tau(F_1[X(z_1)] \ast_{F_0} F_2[X(z_2)] \cdots \ast_{F_0} F_n[X(z_n)]), \tag{3.19}\]
which again coincides with the path integral. Therefore, for the twist element $F_0$ in (3.9), this process of twisting is identical with the path integral. We emphasize that the process does not depend on the action $S_0$ but only on the twist element $F_0$. Moreover, the twist element in our example $F_0$ in (3.9) is only accompanied by the Hopf subalgebra of the worldsheet variations. Indeed, $F_0 \in U(\mathcal{C}) \otimes U(\mathcal{C})$ and it is determined by the free propagator $G_{0}^{\mu \nu}(z, w)$. It is then easy to generalize our twist element to more general twist elements of the same form as (3.9) but with different Green functions $G_{0}^{\mu \nu}(z, w)$. They correspond to different worldsheet theories with quadratic actions $S_0$. Our proposal is that given a Hopf algebra $\mathcal{H}$ and a module algebra $\mathcal{A}$ defined in terms of classical functionals as in the previous
section, then for any twist element, the resulting twisted Hopf and module algebras give a quantization on the worldsheet. A different choice of the twist element gives a different quantization scheme. We will come back to this point in the next section from the viewpoint of the space-time symmetry.

It is instructive at this stage to compare the twisted product $\star_{F_0}$ in this paper and the star product in deformation quantization\(^{14}\) in quantum mechanics, because they share the same property.\(^{4}\) Both theories can be described by classical variables even after the quantization. The latter is generalized to field theories\(^{15}\) and also to string theory.\(^{16}\) We will discuss this point in a separate paper and do not develop it further here, but a few remarks about this issue are in order.

In deformation quantization, a classical Poisson algebra of observables on the phase space is deformed by replacing its commutative product with a star product. It is accompanied by a (formal) deformation parameter $\hbar$ such that in the limit $\hbar \to 0$ the undeformed algebra is recovered. A basic example is the phase space $\mathbb{R}^{2n}$ equipped with a symplectic structure $\omega$, where the algebra $C^\infty(\mathbb{R}^{2n})$ of complex functions is extended to $C^\infty(\mathbb{R}^{2n})[[\hbar]]$, a formal power series in $\hbar$, and the product is twisted by $e^{-\frac{i}{2}\hbar \sum \omega^{ij} \partial_i \otimes \partial_j}$, where $\omega^{ij}$ is the inverse of a symplectic matrix. This algebra with the star product corresponds to the operator formulation in quantum mechanics in the Schrödinger picture, where the information on the time evolution is contained in the wave function. This correspondence is essentially the same for deformation quantization in field theories.

Comparing this with the twist $F_0$ (3.9), formally, the propagator $G^\mu_0(z, w)$ plays the role of $\hbar \omega^{ij}$. However, we should keep in mind the following differences: to be explicit, we take the worldsheet $\Sigma = \mathbb{C}$ for simplicity. In this case, the propagator $G^\mu_0(z, w)$ is given by (2.7). First, a twist can also be accompanied by a deformation parameter. In our case, it is $\alpha'$, because the loop expansion parameter in front of the action is $\alpha'$ (we fix $\hbar = 1$). The Hopf algebra $\mathcal{H}$ and the module algebra $\mathcal{A}$ are considered to be already extended to include $\alpha'$ by dimensional reasoning. Therefore, the generic elements of $\mathcal{A}_{F_0}$ can contain $\alpha'$ and the twisted product gives a power series in $\alpha'$ relative to these. Second, our twisted product depends on the dynamical evolution (it is free) on the worldsheet, which is more alike to the Heisenberg picture in quantum mechanics. This explains the missing factor of $\ln |z - w|$ in the deformation quantization, and therefore the remaining $\eta^{\mu\nu}$ correspond to $\omega^{ij}$. From the viewpoint of the target space-time, it would be better to regard a deformation parameter as $\alpha' \times$ (worldsheet distributions). For a given target space $\mathbb{R}^d$ and a (choice of) background metric $\eta_{\mu\nu}$, we have a twisted product defined by $F_0$. Third, there is an important difference between our twist and that of the deformation quantization. In the latter the factor in the exponential is antisymmetric under the interchange of the partial derivatives, while $F_0$ is symmetric in this sense. This implies that the twist $F_0$ is formally trivial, in contrast to the deformation quantization.

---

\(^{4}\) In the case that the phase space is a Poisson-Lie group, this deformation is equivalent to the Hopf algebra twist of the universal enveloping algebra of the dual Lie algebra $U_h(\mathfrak{g})$.\(^{11}\)
3.3. Normal Ordering

Here we clarify the meaning of the formula for the VEV in the path integral (3.3) and in the r.h.s. of (3.13). Recall that our twist $\mathcal{F}_0$ can be written in terms of $N_0$ as (3.12). This is also the case for any twist element $\mathcal{F}_0 \in U(C) \otimes U(C)$ of the form (3.9). From the viewpoint of the Hopf algebra cohomology, this means that the twist element $\mathcal{F}_0$ is a coboundary and thus it is trivial. See Appendix C. (There, we set $H_{\chi} = H_{\chi} = 1 \otimes 1$, $H_{\psi} = H_{\mathcal{F}_0}$, $\gamma = N_0^{-1} \in \mathcal{H}$). Then, there is an isomorphism between the Hopf algebras $\hat{\mathcal{H}}$ and $\mathcal{H}_{\mathcal{F}_0}$ (module algebras $\hat{\mathcal{A}}$ and $\mathcal{A}_{\mathcal{F}_0}$ respectively) summarized as

$$\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\text{twist}} & \mathcal{H}_{\mathcal{F}_0} \\
\n & \sim & \\
\mathcal{A} & \xrightarrow{\text{twist}} & \mathcal{A}_{\mathcal{F}_0}
\end{array} \quad \text{(3.20)}$$

In the diagram, the left row is a classical pair ($\mathcal{H}, \mathcal{A}$), and the middle and right rows are their quantum counterparts. Here the map $\mathcal{H}_{\mathcal{F}_0} \sim \mathcal{H}$ is given by the inner automorphism $h \mapsto N_0 h N_0^{-1} \equiv \tilde{h}$, and the map $\mathcal{A}_{\mathcal{F}_0} \sim \mathcal{A}$ is given by $F \mapsto N_0 \triangleright F \equiv F'$. We call $\mathcal{H}_{\mathcal{F}_0}$ (or $\mathcal{A}_{\mathcal{F}_0}$) the twisted Hopf algebra (module algebra) while $\mathcal{H}$ (or $\mathcal{A}$) is called the normal ordered Hopf algebra (module algebra). The reason why we distinguish between the classical ($\mathcal{H}, \mathcal{A}$) and the normal ordered ($\hat{\mathcal{H}}, \hat{\mathcal{A}}$) pairs (they are formally the same) is explained below.

To understand the physical meaning of this diagram, let us focus on the module algebras (we will discuss the Hopf algebra action in the next section). Since a functional $F \in \mathcal{A}_{\mathcal{F}_0}$ is mapped to $N_0 \triangleright F \equiv F'$, the elements in $\hat{\mathcal{A}}$ are normal ordered functionals. The VEV (3.17) for $\mathcal{A}_{\mathcal{F}_0}$ implies that we should identify (3.3) as the definition of the VEV for $\hat{\mathcal{A}}$, i.e., the map $\tau \circ N_0^{-1} : \hat{\mathcal{A}} \rightarrow \mathbb{C}$. The product in $\mathcal{A}_{\mathcal{F}_0}$ is mapped to that in $\hat{\mathcal{A}}$:

$$N_0 \triangleright m \circ \mathcal{F}_0^{-1} \triangleright (F \otimes G) = m \circ (N_0 \otimes N_0) \triangleright (F \otimes G) \quad \text{,} \; \text{(3.21)}$$

which is a direct consequence of the coboundary relation (3.12). An equivalent but more familiar expression $: (F *_{\mathcal{F}_0} G) :: := F : G ::$ is simply (2.8), the time ordered product of the vertex operators, which is again equivalent to (3.13) in the path integral average

$$\langle :F[X] : (z) : G[X] : (w) : \rangle_0 = \langle :F[X](z) *_{\mathcal{F}_0} G[X](w) : \rangle_0 \quad \text{.} \; \text{(3.22)}$$

From these considerations, all the quantities and operations in the path integral average, such as in the l.h.s. of (3.13) should be understood as the objects in the normal ordered Hopf algebra and module algebra. The isomorphism implies that, formally, the quantization is performed either by a twist $\mathcal{F}_0$ or by changing the element determined by $N_0$ in the path integral. The latter corresponds to the operator formulation.

However, there are some differences between the twist quantization and the path integral in the following sense. Note that the twist from $\mathcal{A}$ to $\mathcal{A}_{\mathcal{F}_0}$ changes the product but it does not change the elements. Therefore, a classical functional
$F$ does not suffer from a quantum correction ($\alpha'$-correction) under the twist. On the other hand, the map $\mathcal{A} \rightarrow \hat{\mathcal{A}} : F \mapsto \mathcal{F}$ changes the elements while it does not change the operations. Because $\mathcal{N}_0$ contains $\alpha'$, the normal ordered functional $\mathcal{F}$ is necessarily a power series in $\alpha'$ (relative to $F$) and each term in the series is always divergent because of the propagator at the coincident point. Therefore, $\mathcal{F}$ should be distinguished from the classical functional $F$. Nevertheless, this does not mean that the normal ordered module algebra $\hat{\mathcal{A}}$ is ill-defined; rather one should think of it as an artifact of the description, which is based on the classical functional. In fact, in the path integral, the normal ordered functionals give finite results but the classical functional is divergent. There is a similar argument in the deformation quantization approach to field theories: only the normal ordered operator corresponding to this divergent functional is well-defined within the canonical quantization, while a Weyl ordered operator corresponding to a classical functional has a divergence due to the infinite zero point energy. In this sense, if we adopt the description based on the normal ordering of operators, $\hat{\mathcal{A}}$ is the natural object and is well-defined in the path integral average.

However, if we consider a different choice of the background in string theory, there is a significant difference between twisted and normal ordered descriptions. The latter is highly background-dependent, because both, the element $\mathcal{F} \in \hat{\mathcal{A}}$ and the VEV $\tau \circ \mathcal{N}_0^{-1}$ contain $\mathcal{N}_0$. As seen in the example of the propagator in (2.7), $\mathcal{N}_0$ depends on the background metric $\eta_{\mu\nu}$. This corresponds in the operator formulation to the property that a mode expansion of the string variable $X^{\mu}(z)$ and the oscillator vacuum are background-dependent. Therefore, the description of the quantization that makes $\mathcal{A}$ well-defined is only applicable to that background and we need another mode expansion for another background. On the other hand, elements in twisted Hopf and module algebras are not altered, thus they have a background-independent meaning. All the effects are controlled by only the single twist element $\mathcal{F}_0$; thus, the background dependence is clear. In this respect, we can claim that the quantization as a Hopf algebra twist is a more general concept than the ordinary treatments. One of the advantages of this viewpoint will become clearer when we consider the space-time symmetry in the next section.

We finish this subsection with a remark: A Hopf algebra structure underlying the Wick contraction and the normal ordered product has been already considered in the literature. In their approach, the algebra with a normal ordered product was an untwisted Hopf algebra (symmetric algebra) and, by twisting with the propagator (Laplace pairing) the twisted module algebra became an algebra with a time ordered product. One difference between Ref.18) and our treatment is that the approach in the former is based on the mode expansion. The approach in Ref.18) may be related to ours but we do not discuss the details here in this paper.

§4. Space-time symmetry

In the previous section, we formulated the quantization as a twist of a Hopf algebra. The VEV of a product of local vertex operators was formulated as a twisted product $\star_{\mathcal{F}_0}$ of functionals in the module algebra and the map $\tau$. The twist of
the module algebra was a consequence of the twist of the Hopf algebra acting on the classical local functionals. Here we focus on the twisted Hopf algebra itself, in particular on its relation with the space-time symmetry. After discussing the general structure of the twisted Hopf algebra, we see how the diffeomorphism is realized in a fixed background. We also give identities among correlation functions, such as the Ward identity.

4.1. Twisted Hopf algebra and its action

Here, we continue to describe the process of twisting discussed in §3.2. We start with describing the effect of the twist \( \mathcal{H} \rightarrow \mathcal{H}_{\mathcal{F}_0} \) acting on the module algebra, then we discuss the (formal) isomorphism \( \mathcal{H}_{\mathcal{F}_0} \simeq \hat{\mathcal{H}} \) in (3.20).

An action of an element \( h \in \mathcal{H} \) on a classical functional \( h \triangleright I[X] \in \mathcal{A} \) represents a variation under a classical transformation (diffeomorphism or worldsheet variation). The twist element \( \mathcal{F}_0 \) causes a twisting of the Hopf algebra \( \mathcal{H} \rightarrow \mathcal{H}_{\mathcal{F}_0} \), and the consistency of the action (i.e., covariance) requires that the twisted functional algebra \( \mathcal{A}_{\mathcal{F}_0} \) is again an \( \mathcal{H}_{\mathcal{F}_0} \)-module algebra. Since each element in \( \mathcal{H}_{\mathcal{F}_0} \) as well as in \( \mathcal{A}_{\mathcal{F}_0} \) is the same as the corresponding classical element, the variation of the local functional has the same representation \( h \triangleright F[X] \) as the classical transformation. However, since the coproduct is deformed into \( \Delta_{\mathcal{F}_0}(h) = \mathcal{F}_0 \Delta(h) \mathcal{F}_0^{-1} \), the action is not the same as the classical transformation when \( I[X] \) is a product of several local functionals. The covariance of the twisted action on the twisted product (3.16) of two functionals in \( \mathcal{A}_{\mathcal{F}_0} \) is guaranteed by the covariance of the original module algebra (2.23) as

\[
\begin{align*}
    h \triangleright m_{\mathcal{F}_0}(F \otimes G) &= h \triangleright m \circ \mathcal{F}_0^{-1} \triangleright (F \otimes G) \\
    &= m \circ \Delta(h) \mathcal{F}_0^{-1} \triangleright (F \otimes G) \\
    &= m \circ \mathcal{F}_0^{-1} \Delta_{\mathcal{F}_0}(h) \triangleright (F \otimes G) \\
    &= m_{\mathcal{F}_0} \Delta_{\mathcal{F}_0}(h) \triangleright (F \otimes G) .
\end{align*}
\]

(4.1)

In this way the Hopf algebra and the module algebra are twisted in a consistent manner.

From the viewpoint of quantization, the twisted module algebra \( \mathcal{A}_{\mathcal{F}_0} \) together with the map \( \tau : \mathcal{A}_{\mathcal{F}_0} \rightarrow \mathbb{C} \) defines a VEV in a quantization of the string worldsheet theory. Then, the twisted Hopf algebra \( \mathcal{H}_{\mathcal{F}_0} \) should be regarded as a set of quantum symmetry transformations, which is consistent with the quantized (twisted) product. In other words, classical space-time symmetries should also be twisted under the twist quantization. The corresponding variation inside the VEV \( \tau(I[X]) \) is given by

\[
\tau(h \triangleright I[X])
\]

(4.2)

and this appears in the various relations involving the symmetry transformation.

Next recall the (formal) isomorphism \( \mathcal{H}_{\mathcal{F}_0} \simeq \hat{\mathcal{H}} \) in (3.20) (see also Appendix C). Under the isomorphism map, \( F \overset{\sim}{\rightarrow} :F: = N_0 \triangleright F \) of module algebras, the action of \( h \in \mathcal{H}_{\mathcal{F}_0} \) on \( \mathcal{A}_{\mathcal{F}_0} \) is mapped to the action of \( \tilde{h} = N_0 h N_0^{-1} \in \hat{\mathcal{H}} \) on \( \hat{\mathcal{A}} \) as

\[
h \triangleright F \overset{\sim}{\rightarrow} N_0 \triangleright (h \triangleright F) = \tilde{h} \triangleright :F: .
\]

(4.3)
Correspondingly, the action on the product is \( \hat{h} \triangleright (F \ast_{\mathcal{F}_0} G) \sim \& \hat{h} \triangleright (F :: G:) \). The covariance of the \( \hat{H} \)-action on \( \hat{A} \) can be proven by applying \( \mathcal{N}_0 \) to both sides of (4.1):

\[
\tilde{h} \triangleright (F :: G:) = \mathcal{N}_0 h \triangleright (F \ast_{\mathcal{F}_0} G) \\
= m \circ \Delta(\mathcal{N}_0 h)\mathcal{F}_0^{-1} \triangleright (F \otimes G) \\
= m \circ \Delta(\hat{h})(\mathcal{N}_0 \otimes \mathcal{N}_0) \triangleright (F \otimes G). \quad (4.4)
\]

As argued in §3, some elements in the normal ordered algebra contain the formal divergent series in the functional language, and thus the above equation has only a meaning under the path integral. For example, for a single local insertion, (4.2) leads to

\[
\langle \tilde{h} \triangleright :F[X]: (z) \rangle_0 = \tau(h \triangleright F[X(z)]), \quad (4.5)
\]

and for a product of local functionals

\[
\langle \tilde{h} \triangleright (F[X] : (z) :G[X] : (w)) \rangle_0 = \tau(h \triangleright m(\mathcal{F}_0^{-1} \triangleright (F[X] \otimes G[X]))) \quad (4.6)
\]

In this way it is always possible to convert the action of the twisted Hopf algebra into that of the normal ordered Hopf algebra. However, we will see below that the structure of the diffeomorphism is far simpler written in terms of the twisted Hopf algebra than in terms of the normal ordered Hopf algebra. Related to this, another way to give a well-defined meaning to \( F : \) is to replace it by the normal-ordered operator. In this case, the action \( \tilde{h} \triangleright \) should also be replaced with an operation involving operators and it becomes strongly background-dependent owing to \( \mathcal{N}_0 \) being included in the definition of \( \tilde{h} \).

As we have seen in §3, the quantization itself is performed within a worldsheet twist, namely, \( \mathcal{F}_0 \) in (3.9) is a twist element of a Hopf subalgebra \( U(\mathfrak{X}) \). However, this twist affects the whole Hopf algebra \( \mathcal{H} = U(\mathfrak{X}) \). This can be understood as follows: Any classical vector field \( \xi \) is originally primitive, \( \Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi \), that is, it obeys the Leibniz rule. After twisting, \( \xi \) in \( \mathcal{H}_{\mathcal{F}_0} \) acts on a single functional \( F \) in the same way, \( \xi \triangleright F \), as in the classical case, but it is not, in general, primitive now, since the coproduct is twisted \( \Delta_{\mathcal{F}_0}(\xi) = \mathcal{F}_0 \Delta(\xi) \mathcal{F}_0^{-1} \). This occurs when \( \mathcal{F}_0 \) does not commute with \( \Delta(\xi) \).\(^4\) Therefore, as a consequence of the twist quantization, the diffeomorphism of space-time, in general, cannot be separately considered but it should be twisted as well.

In this respect, the universal enveloping algebra \( U(\mathcal{P}) \subset \mathcal{H} \) for the Poincaré Lie algebra \( \mathcal{P} \) is a special case, since even after the twisting, it is identical with the original \( U(\mathcal{P}) \). This can be seen by proving that the twist does not alter the coproduct \( \Delta_{\mathcal{F}_0}(u) = \Delta(u) \) as well as the antipode \( S_{\mathcal{F}_0}(u) = S(u) \) for \( \forall u \in U(\mathcal{P}) \). See Appendix D for the proof. Therefore, \( U(\mathcal{P}) \) is also a Hopf subalgebra of \( H_{\mathcal{F}_0} \).

With our choice of the twist element \( \mathcal{F}_0 \) (3.9), we argued that the twist quantization coincides with the ordinary quantization, in which the Poincaré covariance is assumed to be at the quantum level. This suggests that, in general, the twist-invariant Hopf subalgebra corresponds to the unbroken symmetry, while the full

\(^4\) By the isomorphism, the same property holds for the normal ordered Hopf algebra \( \hat{\mathcal{N}} \). An element \( \xi = \mathcal{N}_0 \xi \mathcal{N}_0^{-1} \) is not primitive unless \( [\mathcal{N}_0, \xi] = 0 \) owing to the factor \( \Delta(\mathcal{N}_0) \).
diffeomorphism should be twisted under the quantization of the chosen twist element. Below we elaborate on the physical meaning of the twisted Hopf algebra from the viewpoint of background-(in)dependence. In particular, in the following subsections, we discuss the meaning of the twisted diffeomorphism in this context and the characterization of the broken/unbroken symmetries together with the relation with the various identities among the correlation functions.

**Identities in path integrals**

Before we discuss the various identities related to the symmetries in the twisted Hopf algebra, we recall the ordinary path integral relations for the symmetry transformations. In the path integral, the identities are obtained using the fact that any change of variables gives the same result. In particular, under the constant shift $X^\mu(z) \mapsto X^\mu(z) + \varepsilon^\mu$, where $\varepsilon^\mu$ is a constant, it gives the identity

$$0 = \int \mathcal{D}X \frac{\delta}{\delta X^\mu(z)} (e^{-S_0} \mathcal{O}) .$$

(4.7)

More generally, consider an arbitrary infinitesimal change of variables $X' ^\mu(z) = X^\mu(z) + \xi^\mu(z)$. Then,

$$0 = \int \mathcal{D}X' e^{-S_0[X']} \mathcal{O}[X'] - \int \mathcal{D}X e^{-S_0[X]} \mathcal{O}[X]$$

$$= \int \mathcal{D}X e^{-S_0[X]} \{J \mathcal{O} - (\xi \triangleright S_0) \mathcal{O} + \xi \triangleright \mathcal{O} \} ,$$

(4.8)

where the first term is the Jacobian obtained from the variation of the measure $\mathcal{D}X' := \mathcal{D}X(1 + J)$, and the second term is the variation of the action, $S_0[X'] = S_0[X] + \xi \triangleright S_0[X]$.

Here we have used Hopf algebra notation for the action of the functional derivative.

We can derive various identities from (4.8) as follows:

(i) If $\xi$ generates a worldsheet variation independent of $X$ (i.e., $\xi \in \mathcal{C}$), then the measure is manifestly invariant, $\mathcal{D}X' = \mathcal{D}X$, and (4.8) reduces to

$$0 = \int \mathcal{D}X e^{-S_0} \{-(\xi \triangleright S_0) \mathcal{O} + \xi \triangleright \mathcal{O} \} ,$$

(4.9)

which is used to derive the Schwinger-Dyson equation.

(ii) The case $\xi$ generates a space-time symmetry: If $\xi = \xi[\mathcal{X}(z)]$ is a target space vector field, but the measure and the action are invariant under $\xi$, then only the third term remains, and we obtain the Ward identity

$$0 = \int \mathcal{D}X e^{-S_0} \{\xi \triangleright \mathcal{O} \} ,$$

(4.10)

where the transformation acts only on the insertions $\mathcal{O}$.

(iii) Eq. (4.8) is also used to derive Noether’s theorem in the path integral language (see, for example, Ref.1)). Under the same assumption as stated in (ii), but

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*It is related to the variation $\delta_\xi S_0 = -\xi \triangleright S_0$. See also §2.2.
extending the variation to \( X' = X + \rho(z)\xi \) by an arbitrary distribution \( \rho(z) \) on the worldsheet, the measure is still invariant. However, the variation of the action is written as \( \int J^a \partial_a \rho(z) \), where \( J^a \) is the Noether current, leading to the identity

\[
0 = \int \mathcal{D}X e^{-S_0} \rho(z) \left\{ - \left( \int dS_a J^a \right) \mathcal{O} + \xi \triangleright \mathcal{O} \right\}. \tag{4.11}
\]

Note that the insertion of the Noether current (charge) is written as the classical variation of the operator insertion. Thus, the above equation corresponds to the operator identity of the symmetry transformation at the quantum level, \([Q, \mathcal{O}] = \delta \mathcal{O}\), where in the l.h.s., \( Q \) is the generator of the transformation and the r.h.s. is the classical variation of the operator \( \mathcal{O} \).

(iv) If the vector field does not preserve the action \( S_0 \), it is not a classical symmetry but \((4.8)\) still represents a broken Ward identity that incorporates the change of the action \( S_0 \) (and the measure).

In the following we derive the same type of identities in the Hopf algebra language, where the path integral VEV is replaced by the algebraic operation \( \tau(I[X]) \).

Before we start the derivations, we point out the following: The variation in the integrand in \((4.8)\) is the sum of that of the action \( S_0 \) and of each insertion. In particular, the action \( \xi \triangleright \mathcal{O} \) on the multiple insertions satisfies the Leibniz rule. It states that formally the vector field \( \xi \) should be an element of the classical Hopf algebra \( \mathcal{H} \).\footnote{\( \xi \sim 0 \) and \( \mathcal{O} \) are abbreviations, that are more rigorously defined by the \( t \to 0 \) limit of the action of the flow \( u(t) = \exp (t\xi) \) generated by \( \xi \).} However, at the same time, it is implicitly assumed that each insertion is understood to be normal ordered one in the path integral method. Therefore, we should be careful when this normal ordering is applied. In other words, we need to understand the change of variables in the Hopf algebra language to characterize the path integral identities correctly.

4.2. Twisted Hopf algebra as a symmetry

We clarify the relation between the Hopf algebra action of the twisted \( \mathcal{H}_{\mathcal{F}_0} \) (normal ordered \( \mathcal{H} \)), the Hopf algebra described in the previous subsection and the change of variables representing a symmetry transformation. Although they are identical classically, the relation is not trivial after the quantization.

Let \( \xi \in \mathfrak{X} \) be a vector field and let \( u = e^\xi \in \mathcal{H} = U(\mathfrak{X}) \) be an element of the classical Hopf algebra. It is a group like element, \( \Delta(u) = u \otimes u \) and it acts on both \( \mathcal{A} \) and \( \mathcal{H} \). Its action on the variable \( X \) defines \( X'_{\mu} = u \triangleright X^\mu \) as a new variable. Then its action on any functional, \( u \triangleright I[X] = I[X'] \), is considered to be the transformation law caused by the change of variables\footnote{Although the VEV itself belongs to the quantum theory, the variation is classical, and the transformed classical functional is integrated giving a new VEV.}.

Of course, \( I[X'] \) is also an element of the classical functionals \( \mathcal{A} \). Since \( u \) is group like, the transformation law for the product of functionals is the product of each functional. In particular, the classical diffeomorphism is given by \( u = e^\xi \) with \( \xi[X] \) being the pull-back of a space-time
vector field.

Note that the adjoint action of $\xi$ on $h \in H$ is defined by the Lie bracket $[\xi, h]$. Then the action of $u$ on the functional derivative gives

\[
\frac{\delta}{\delta X'(\mu)(z)} := u \triangleright \frac{\delta}{\delta X^\mu(z)}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \xi^n \triangleright \frac{\delta}{\delta X^\mu(z)}
\]

\[
= e^{\xi} \frac{\delta}{\delta X^\mu(z)} e^{-\xi}
\]

\[
= u \frac{\delta}{\delta X^\mu(z)} u^{-1}, \quad (4.12)
\]

which is a finite version of the chain rule under the change of variables. The functional derivative is transformed such that the relation

\[
\frac{\delta}{\delta X'(\mu)(z)} \triangleright X'(\nu)(w) = u \frac{\delta}{\delta X^\nu(z)} u^{-1} \triangleright X'(\nu)(w) = \delta_{\mu}^{(2)} \delta(z - w) \quad (4.13)
\]

is maintained after the change of variables. $\frac{\delta}{\delta X}(\nu)$ can be used to construct $X$ just as the original functional derivative do. The transformed functional derivatives are primitive since

\[
\Delta \left( \frac{\delta}{\delta X'\mu} \right) = (u \otimes u) \Delta \left( \frac{\delta}{\delta X^\mu} \right) (u^{-1} \otimes u^{-1}) = \frac{\delta}{\delta X'\mu} \otimes 1 + 1 \otimes \frac{\delta}{\delta X^\mu}. \quad (4.14)
\]

Their commutator vanishes,

\[
\left[ \frac{\delta}{\delta X'(\mu)} \cdot \frac{\delta}{\delta X'(\nu)} \right] = u \left[ \frac{\delta}{\delta X^\mu(z)} \cdot \frac{\delta}{\delta X^\nu(w)} \right] u^{-1} = 0. \quad (4.15)
\]

It is straightforward to show that for the element $h \in U(\mathfrak{c})$ we have $h' = u \triangleright h = uhu^{-1}$, which is equivalent to replacing all the functional derivatives in $h$ with $\frac{\delta}{\delta X'(\mu)}$.

For example, $N'_{0} = uN_{0}u^{-1}$. For an arbitrary element $h \in H$, $u$ also acts on the coefficient function.

Let $F_{0}$ be a fixed twist element in $U(\mathfrak{c}) \otimes U(\mathfrak{c})$. The action of $u$ on $F_{0}$ is written as

\[
F'_{0} = u \triangleright F_{0}
\]

\[
= \Delta(u)F_{0}\Delta(u^{-1})
\]

\[
= (u \otimes u)F_{0}\Delta(u^{-1}). \quad (4.16)
\]

This means that $u$ is a coboundary (this is always the case for a group like element) in a cohomological sense and $F'_{0}$ is a new twist element equivalent to $F_{0}$ (see Appendix C). It is also easy to show directly that $F'_{0}$ satisfies the cocycle condition by using the expression

\[
F'_{0} = \exp \left\{ - \int d^2 z \int d^2 w G_{0}^{\mu}(z, w) \frac{\delta}{\delta X'(\mu)(z)} \otimes \frac{\delta}{\delta X'(\nu)(w)} \right\}, \quad (4.17)
\]
since the proof depends only on the properties of the functional derivatives $\frac{\delta}{\delta X^\mu}$ in (4.14) and (4.15) (see Appendix B).

Now consider the action of $u$ on the twisted module algebra $A_{F_0}$. For any functional $I[X] \in A_{F_0}$, its transformation law is the same as that of $A$, $I[X'] = u \triangleright I[X]$. This is also true for the star product of two local functionals $I[X] = F[X] *_{F_0} G[X]$ when it is considered as a functional of $X$ after the star product is performed. Because of (4.1), this action of $u$ is nothing but the twisted Hopf algebra action as

$$u \triangleright (F[X] *_{F_0} G[X]) = m \circ \Delta(u) F_0^{-1} \triangleright (F \otimes G)$$

$$= m \circ F_0^{-1} \Delta_{F_0}(u) \triangleright (F \otimes G).$$

(4.18)

However, the same action can also be written using (4.16) as

$$u \triangleright (F[X] *_{F_0} G[X]) = m \circ \Delta(u) F_0^{-1} \triangleright (F \otimes G)$$

$$= m \circ F_0^{-1}(u \otimes u) \triangleright (F \otimes G)$$

$$= F[X'] *_{F_0} G[X'].$$

(4.19)

The r.h.s. is equivalent to the replacement of each $X$ with $X'$ as well as each $\frac{\delta}{\delta X^\mu}$ with $\frac{\delta}{\delta X'^\mu}$ before the star product is performed. This gives a good understanding of the twisted diffeomorphism.\(^3\) Comparing these two expressions we see that the action of $u$ as the twisted Hopf algebra (4.18) is simply the classical diffeomorphism in which the twist element has also been transformed (4.19). Here the change of the twist element itself is converted to the change of functionals through the twisted coproduct $\Delta_{F_0}$ while keeping the twist element invariant.

Moreover, (4.19) is seen as a product in the new twisted module algebra $A_{F_0'}$ twisted by $F_0'$. From (4.16), the new twisted algebra is isomorphic to the original one. Denoting this isomorphism as $\rho : A_{F_0} \to A_{F_0'}$, given by $\rho(F) = u \triangleright F$, then (4.19) implies that $\rho(F *_{F_0} G) = \rho(F) *_{F_0'} \rho(G)$. This gives a new viewpoint for the twist, that is, a change of background under $\rho$ relates two twists $A \to A_{F_0}$ and $A \to A_{F_0'}$ with each other. It is equivalent to regard the new twist element (4.17) as

$$F_0' := \exp \left\{ - \int d^2 z \int d^2 w \sum_{\mu \nu} G^{\mu \nu}_0(z, w) \frac{\delta}{\delta X^\mu(z)} \otimes \frac{\delta}{\delta X^\nu(w)} \right\},$$

$$G^{\mu \nu}_0(z, w) = G^{\mu \nu}(z, w) - \partial_\rho \xi^\mu(z) G^{\nu \rho}_0(z, w) - G^{\mu \rho}_0(z, w) \partial_\rho \xi^\nu(w),$$

(4.20)

where the change of the propagator coincides with the transformation of our fixed background metric $\eta^{\mu \nu}$ under the diffeomorphism $u^{-1} = e^{-\xi[X]}$. From this viewpoint, an element $u = e^{\xi[X]}$ of a space-time diffeomorphism that keeps the twist element invariant, $F_0' = u \triangleright F_0 = F_0$, is a symmetry (isometry) in the ordinary sense.

\(^3\) There is essentially the same argument in the context of noncommutative gravity in Ref.19). In fact, (4.19) can be rewritten in terms of the variation by using the relation mentioned in the previous footnote. Then we obtain $\delta_X(F *_{F_0} G)(X) = (\delta_X F *_{F_0} G)(X) + (F *_{F_0} \delta_X G)(X) + (F *_{F_0} \delta_X G)(X)$, which is the formula in Ref.19). Here, the product in the third term is defined by inserting $\delta_X F_0^{-1} = - F_0^{-1} \partial_\xi = - [\Delta_\xi, F_0^{-1}]$. 

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From the quantization viewpoint, we should fix a twist element $F_0$ to quantize the theory. Accompanied by this element, the classical Hopf algebra becomes a twisted Hopf algebra $H_{F_0}$ acting as a quantized transformation. Since a particular metric is chosen by fixing the twist, a full diffeomorphism is not manifest from the symmetry viewpoint. However, the above argument shows that the twisted diffeomorphism is a remnant of the classical diffeomorphism. This is the essence of our proposal, that the twist governs the quantization and the space-time symmetry in a consistent way. A fixed twist element determines the quantization scheme as well as the background metric, but the diffeomorphism is retained as a twisted Hopf algebra. This is a good starting point for discussing the general covariance by extending the above argument.

**Vacuum expectation value**

We now consider the effect on the VEV (3.17) under the change of variables $X \rightarrow X'$, as in the path integral argument (4.8). For a functional in $A_{F_0}$, it was shown above that $I[X'] = u \triangleright I[X]$. Next, $\tau$ is replaced with $\tau'$, which sets $X' = 0$ in the functional $I[X']$. One can show that $\tau'$ is written as an operation on $X$ as

$$\tau' = \tau \circ u^{-1} \triangleright .$$

Combining these contributions, we have the identity

$$\tau'(I[X']) = \tau(u^{-1} \triangleright (u \triangleright I[X])) = \tau(I[X]),$$

which implies that the change of variables keeps the VEV invariant.

Equation (4.22) as well as (4.27) below are desirable properties of the VEV corresponding to the path integral, and they can be used to obtain various identities. However, it appears to be difficult to derive (4.8) directly from them, even if we restrict ourselves to the infinitesimal change $\xi \sim 0$, because the present formulation does not use the action $S_0$ transparently.

**Effects on $\hat{A}$**

In a similar manner we can estimate in a similar manner the effect of the change of variables on the normal ordered module algebra $\hat{A}$. However, the situation is somewhat different from that of the twisted module algebra. This is because the former is highly background-dependent.

The action of an element $u = e^\xi$ of the classical Hopf algebra $H$ on the normal ordering element is given by $N'_0 = u \triangleright N_0 = uN_0u^{-1}$. Then, a single local vertex operator $F[X] := N_0 \triangleright F[X] \in \hat{A}$ is transformed under the classical diffeomorphism in the same manner as (4.19),

$$N'_0 \triangleright F[X'] = uN_0 \triangleright F[X] = u \triangleright F[X] : ,$$

which is simply a classical action of $u$ when $F[X] :$ is considered as a functional of $X$ after the normal ordering is performed. Note that this is not the $\hat{H}$-action $\hat{u} \triangleright :F[X]: =: u \triangleright F[X] :$. Correspondingly, a transformed functional is not well-defined in $\hat{A}$, that is, it is divergent in terms of $\hat{A}$. It should be a well-defined element of the new normal ordered module algebra $\hat{A}'$, which is isomorphic to the twisted module
algebra $A_{\mathcal{F}_0}$ with the product $\ast_{\mathcal{F}_0}$. The relation between $A_{\mathcal{F}_0}$ and $\hat{A}'$ is the same as that discussed in §3. For example, the analogue of (3-12), $\mathcal{F}_0' = (\mathcal{N}_0' \otimes \mathcal{N}_0') \Delta (\mathcal{N}_0')$, holds. We denote this new normal ordering with respect to $\mathcal{N}_0'$ as $\hat{\circ} F \hat{\circ}$. Therefore, the VEV of the product of the vertex operators should also be defined through that of $A_{\mathcal{F}_0}$. To see this, by rewriting the product (4-19), we obtain the relation between $A_{\mathcal{F}_0}$ and $\hat{A}'$ given by (3-21) in §3

$$F[X'] \ast_{\mathcal{F}_0} G[X'] = \mathcal{N}_0'^{-1} \triangleright m \circ (\mathcal{N}_0' \otimes \mathcal{N}_0')(u \otimes u) \triangleright (F \otimes G)$$

$$= \mathcal{N}_0'^{-1} \triangleright (\hat{\circ} F[X'] \hat{\circ} G[X']) . \tag{4-24}$$

This and (4-23) indicate that the VEV on $\hat{A}'$ should be the map $\langle \cdots \rangle_0' = \tau' \mathcal{N}_0'^{-1} \triangleright \hat{A}' \rightarrow \mathbb{C}$. For example, we have an $\mathcal{F}_0'$-version of (3-13) as

$$\langle \hat{\circ} F[X'] \hat{\circ} G[X'] \rangle_0' = \tau'(F[X'] \ast_{\mathcal{F}_0'} G[X']) . \tag{4-25}$$

Of course, it also coincides with the VEV without the prime through (4-22). Therefore, the transformation of the normal ordered module algebra under the change of variables requires the change of the normal ordered module algebra itself to be consistent with that of the twisted module algebras.

There is also a direct correspondence between $\hat{A}'$ and $\hat{A}$. For the product, we obtain from (4-23)

$$\hat{\circ} F[X'] \hat{\circ} G[X'] \hat{\circ} = (N_0' \triangleright F[X']) (N_0' \triangleright G[X'])$$

$$= m \circ (u \otimes u)(N_0 \otimes N_0) \triangleright (F \otimes G)$$

$$= u \triangleright (\hat{\circ} F[X] : : G[X] : ) , \tag{4-26}$$

which is also derived from (4-24). The corresponding VEVs on $\hat{A}'$ and $\hat{A}$ coincide

$$\langle \hat{\circ} F[X'] \hat{\circ} G[X'] \hat{\circ} \rangle_0 = \langle \hat{\circ} F[X] : : G[X] : \rangle_0 , \tag{4-27}$$

which follows from each definition of the VEV and (4-26).

The whole structure together with the isomorphism of twisted module algebras is as follows. We define a map $\hat{\rho} : \hat{A} \rightarrow \hat{A}'$ as $\hat{\rho}(\hat{\circ} F[X] :) = u \triangleright : F[X] :$. Then (4-23) is written as $\mathcal{N}_0' \triangleright \rho(F) = \hat{\rho}(\mathcal{N}_0 \triangleright F)$ and (4-26) is written as $\mathcal{N}_0' \triangleright \rho(F \ast_{\mathcal{F}_0} G) = \hat{\rho}(\mathcal{N}_0 \triangleright (F \ast_{\mathcal{F}_0} G))$. This means that $\hat{\rho}$ is a (formal) algebra isomorphism, and the following diagram commutes,

$$A_{\mathcal{F}_0} \xrightarrow{\rho} A_{\mathcal{F}_0}'$$

$$\mathcal{N}_0' \triangleright \downarrow \quad \downarrow \mathcal{N}_0' \triangleright \cdot$$

$$\hat{A} \xrightarrow{\hat{\rho}} \hat{A}' \tag{4-28}$$

This shows the consistency of the isomorphism $\hat{\rho}$ between two normal ordered module algebras, but it is formal, that is, the relation is between two different divergent series. Contrary to the case of twisted module algebras, changing the background requires the change of the normal ordering, which corresponds to the new mode expansion in the operator formulation.
Note that in (4.26) \( u \) acts as a group like element. This agrees with the transformation law inside the path integral, where the variation is taken as if classical functionals are being considered, but each insertion is understood as being normal ordered.

Of course, if \( \xi \) is in a twist-invariant Hopf subalgebra, such as \( U(\mathcal{P}) \) in our example, the change of variables does not change the twisted \( \mathcal{A}_{\mathcal{F}_0} = \mathcal{A}_{\mathcal{F}_0} \) or the normal ordered \( \hat{\mathcal{A}}' = \hat{\mathcal{A}} \) module algebras. In this case, the transformation is closed within \( \hat{\mathcal{A}} \) and has a well-defined meaning even in the operator formulation.

In the low-energy effective theory derived from the worldsheet theory, fields in space-time are associated with normal ordered vertex operators rather than with twisted module algebras. If we consider the diffeomorphism beyond the Poincaré invariant theory, we should be careful when considering the above change of the normal ordering. Even in such an application, the twist is the only simple way to treat these changes systematically. In any case, the space-time symmetry is governed by a single twist defining a background and its infinitesimal change under the diffeomorphism.

4.3. Ward-like identities

The difference between the path integral identities (4.8) and the Hopf algebra counterpart (4.22) or (4.27) is the explicit appearance or absence of the action \( S_0 \). The VEV in the Hopf algebra (3.17) is based on the twist element, while the action \( S_0 \) is not needed. In this sense in the Hopf algebra approach, the twist element has a more fundamental role than the action. However, it is also useful if we can compare it with the path integral expression, and we also obtain the relation to the action from the Hopf algebra viewpoint.

Actually, in the problem considered here, using the fact that we are dealing with a free theory, we can directly derive identities concerning \( S_0 \) in terms of the Hopf algebra action. Using them, we attempt to derive the same type of identity as (4.8).

To this end, we start with the standard relation derived from (2.1),

\[
\frac{\delta S_0}{\delta X^\mu(z)} = -\frac{1}{\pi \alpha'} \bar{\partial} \partial X^\mu(z) .
\]  

(4.29)

Then we find an example of an identity: The action \( S_0 \) itself satisfies

\[
S_0 = \frac{1}{2} \int d^2z d^2w \left( \frac{1}{\pi \alpha'} \partial \bar{\partial} X^\mu(z) \right) G_0^{\mu\nu}(z, w) \left( \frac{1}{\pi \alpha'} \bar{\partial} \partial X^\nu(w) \right),
\]

\[
= \frac{1}{2} m \left( \int d^2z d^2w \ G_0^{\mu\nu}(z, w) \frac{\delta}{\delta X^\mu(z)} \otimes \frac{\delta}{\delta X^\nu(w)} \right) \triangleright \left( S_0 \otimes S_0 \right),
\]

\[
= -\frac{1}{2} m \circ F_0 \triangleright \left( S_0 \otimes S_0 \right),
\]  

(4.30)

where in the first line \( \partial_z \bar{\partial}_z G_0^{\mu\nu}(z, w) = -\pi \alpha' \eta^{\mu\nu} \delta^{(2)}(z - w) \) and integration by parts is used, in the second line (4.29) is inserted, and in the last line \( F_0 \) is defined through \( \mathcal{F}_0 = e^{F_0} \).

In a similar manner, the functional derivative is also rewritten using (4.29) and
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By a direct calculation, we find

\[
\begin{align*}
\frac{\delta}{\delta X^\mu(z)} \triangleright I[X] &= -m \circ F_0 \triangleright \left( \frac{\delta S_0}{\delta X^\mu(z)} \otimes I[X] \right) \\
&= -m \circ \left( \frac{\delta}{\delta X^\mu(z)} \otimes 1 \right) F_0 \triangleright \left( S_0 \otimes I[X] \right).
\end{align*}
\]

(4.31)

Here in the second line we used the commutativity between the functional derivative and \( F_0 \). This is proved by noting that

\[
\begin{align*}
F_0 \triangleright \left( S_0 \otimes I[X] \right) &= -\frac{1}{\pi \alpha'} \int d^2z \int d^2w G_0^{\mu\nu}(z, w) \left( \partial \bar{\partial} X_\mu(z) \otimes \frac{\delta I[X]}{\delta X^\nu(w)} \right) \\
&= \frac{1}{\pi \alpha'} \int d^2z \int d^2w \partial \bar{\partial} G_0^{\mu\nu}(z, w) X_\mu(z) \otimes \frac{\delta I[X]}{\delta X^\nu(w)} \\
&= \int d^2z X_\mu(z) \otimes \frac{\delta I[X]}{\delta X^\mu(z)},
\end{align*}
\]

(4.32)

where integration by parts and the defining relation of the Green function are again used. It is straightforward to extend to the action of a vector field \( \xi \in \mathfrak{X} \) as

\[
\xi \triangleright I[X] = -m \circ (\xi \otimes 1) F_0 \triangleright \left( S_0 \otimes I[X] \right).
\]

(4.33)

Therefore, the action of any vector field on a functional can be rewritten using the action \( S_0 \). From the Hopf algebra viewpoint, we do not start with the action \( S_0 \) but with the twist element \( F_0 \). In this context, the identities (4.33) and (4.30) are regarded as defining the action \( S_0 \) of the theory from the Hopf algebra action. Note that they simply reflect the fact that the Green function is the inverse of the second-order differential operator defining the equation of motion (4.29). Therefore, these relations and the following argument can be generalized to any theory on the worldsheet with a quadratic action, and are not limited to (2.1). This corresponds to any twist element of the type (3.9) with an appropriate propagator. Although we do not derive the boundary contribution explicitly, (4.33) also holds for the worldsheet with boundaries.

One can also obtain a similar identity to (4.33) with \( F_0^{-1} \) instead of \( F_0 \). Because \( S_0 \) is quadratic in \( X \), we can explicitly calculate it as

\[
m \circ (\xi \otimes 1) F_0^{-1} \triangleright \left( S_0 \otimes I[X] \right) = m \circ (\xi \otimes 1) \left\{ 1 - F_0 + \frac{1}{2} F_0^2 \right\} \triangleright \left( S_0 \otimes I[X] \right),
\]

(4.34)

where in the second line (4.33) is used for the \( F_0 \) term. The \( F_0^2 \) term is given by

\[
F_0^2 \triangleright \left( S_0 \otimes I[X] \right) = \int d^2z \int d^2w G_0^{\mu\nu}(z, w) \left( 1 \otimes \frac{\delta^2 I[X]}{\delta X^\mu(z) \delta X^\nu(w)} \right) = -2 (1 \otimes N_0 \triangleright I[X]).
\]

(4.35)

but it has no contribution since it vanishes when acting \( \xi \otimes 1 \) further on r.h.s., because \( \xi \triangleright 1 = 0 \). Note that (4.35) itself is divergent owing to the coincident point. Applying
the map \(\tau\) to both sides of (4.34), the first term in the r.h.s. of (4.34) vanishes, and we have an identity for the VEV:

\[
\tau \circ m \circ (\xi \otimes 1) F_0^{-1} \triangleright (S_0 \otimes I [X]) = \tau (\xi \triangleright I [X]). \tag{4.36}
\]

We can consider that the identity (4.36) corresponds to (4.8) in the path integral. To see this more explicitly, let us rewrite the l.h.s. of (4.34) as

\[
m \circ F_0^{-1} (\xi \triangleright S_0 \otimes I [X]) + m \circ [\xi \otimes 1, F_0^{-1}] \triangleright (S_0 \otimes I [X]). \tag{4.37}
\]

Note that each term in (4.37) is potentially divergent by the same reasoning as above, thus care must be taken. Using the relations obtained in §3,

\[
\tau (\xi \triangleright I [X]) = \langle :\xi \triangleright I [X] : \rangle_0 = \langle \check{\xi} \triangleright :I [X] : \rangle_0,
\]

\[
\tau \circ m \circ F_0^{-1} (\xi \triangleright S_0 \otimes I [X]) = \tau ((\xi \triangleright S_0) \ast \mathcal{F}_0 I [X]) = \langle :\xi \triangleright S_0 : :I [X] : \rangle_0. \tag{4.38}
\]

the identity (4.36) reduces to

\[
0 = \langle \check{\xi} \triangleright :I [X] : \rangle_0 - \langle :\xi \triangleright S_0 : :I [X] : \rangle_0 - \tau (m \circ [\xi \otimes 1, F_0^{-1}] \triangleright (S_0 \otimes I [X])). \tag{4.39}
\]

The identity (4.39) is obtained by a simple rewriting of the Hopf algebra action, but remarkably, it appears to be similar to (4.8). This also suggests that the last term can be identified with the variation of the measure \(\mathcal{D}X\), but we cannot conclude it at this stage. Note also that \(\check{A}\) is a suitable description of the change of variable as argued in §4.2, while here \(\check{A}\) is used.\(^*)

Furthermore, (4.39) contains the same type of information about identities (4.8) derived in the path integral formalism as follows:

(i) If \(\xi \in \mathcal{C}\), then the last term vanishes since \([\xi \otimes 1, F_0^{-1}] = 0\) and we obtain a Schwinger-Dyson type equation similar to (4.9).

\[
\tau (\xi \triangleright I [X]) = \tau ((\xi \triangleright S_0) \ast \mathcal{F}_0 I [X]) \tag{4.40}
\]

or equivalently

\[
0 = \langle \check{\xi} \triangleright :I [X] : \rangle_0 - \langle :\xi \triangleright S_0 : :I [X] : \rangle_0. \tag{4.41}
\]

In this case, \(\xi\) is not affected by the twist since \(\Delta_{\mathcal{F}_0} (\xi) = \Delta (\xi)\), or equivalently, \(\check{\xi} = \xi\). Therefore, the action of \(\xi\) here satisfies the Leibniz rule (we do not need to consider the difference between \(\check{A}\) and \(\check{A}'\)).

(ii) If \(\xi\) is a classical symmetry of the theory, we have \(\xi \triangleright S_0 = 0\). Moreover, if the last term vanishes, then we obtain the Ward identity

\[
0 = \tau (\xi \triangleright I [X]) = \langle \check{\xi} \triangleright :I [X] : \rangle_0. \tag{4.42}
\]

With the same reasoning as above, \(\xi\) is still primitive under the twist so that the action of \(\xi\) splits into the sum of the transformations for each local functional contained in the functional \(I [X]\).

\(^*)\) These two descriptions are related by the divergent series, which is related to the potential divergence noted above.
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(iii) As in the path integral case, considering the variation $\rho \xi$ instead of $\xi$ in the above derivation, we obtain the relation including the Noether current.

(iv) For general $\xi$, there are contributions from the variation of the action $S_0$ as well as the last term, and the variation of the insertion $I[X]$ is not split into individual variations. Nevertheless, it is compactly written as the twisted Hopf algebra action $\xi \triangleright I[X]$.

We do not derive the path integral identities (4.8) using the change of variables argument in §4.2. However, as we have seen above, the symmetry is characterized again as the twist-invariant Hopf subalgebra of $\mathcal{H}$ which keeps $S_0$ invariant. In this case, the transformation law of the twisted Hopf subalgebra is given in the same form as the classical transformation. In our model, this subalgebra is the universal enveloping algebra of the Poincaré-Lie algebra $U(\mathcal{P})$, as already remarked. The action of $u \in U(\mathcal{P})$ keeps $S_0$ invariant, and thus the quantized transformation is the same as the classical transformation. It also satisfies the Leibniz rule and leads to the ordinary Ward identity of the form (2.9).

§5. Conclusion and discussion

We have investigated the Hopf algebra structure in the quantization of the string worldsheet theory in the target space $\mathbb{R}^d$. It gives a unified description of both the quantization and the space-time symmetry simply as a twist of the Hopf algebra.

In the functional description of the string, we found that the module algebra $\mathcal{A}$ of classical functionals as well as the Hopf algebra $\mathcal{H}$ of functional derivatives correspond to space-time diffeomorphisms and worldsheet variations. They are background-metric-independent in nature, but the choice of a twist element $\mathcal{F}_0$ fixes the background. Twisting them by $\mathcal{F}_0$ gives the covariant quantization on this background. We have seen that the twist is formally trivial and that it also characterizes the normal ordering. Therefore, the twist is equivalent to the description in the path integral as well as the operator formulations. On the other hand, the twist also characterizes the broken and unbroken space-time symmetry. In our fixed Minkowski background, the symmetry of the Poincaré transformations remains unbroken as a twist-invariant Hopf subalgebra $U(\mathcal{P})$. The remaining transformations, which are broken in the Minkowski background, are still retained as a twisted Hopf algebra. We have explicitly seen that the classical diffeomorphism in $\mathbb{R}^d$ is realized as a twisted diffeomorphism in such a way that the background $\eta_{\mu\nu}$ is fixed. Therefore, it is a good starting point for discussing the background independence in full generality.

We give an outlook regarding this work. Our consideration is limited to the worldsheet theory of strings, but it is also important to relate it to the low-energy effective theory including gravity, where there is the classical general covariance. For that purpose, we have to further investigate particular correlation functions and the S-matrix. Note that this is merely the on-shell equivalence and there is always a difficulty of field redefinition ambiguities.

Another issue that we did not treat in this paper is the local symmetries on the worldsheet, in particular the conformal symmetry. It restricts the possible back-
grounds and it is also necessary to obtain the spectrum of the theory. At the level of our treatment in this paper, the conformal symmetry should be additionally imposed. However, it would be possible to incorporate it by enlargement of the Hopf algebra, which probably touches upon the Hopf algebraic structure in conformal field theory.\(^{20}\)

Our scheme of quantization with normal ordering works at least for any twist \(F_0 \in U(\mathfrak{c}) \otimes U(\mathfrak{c})\) given by a Green function, corresponding to free theories. In this context, a background with a non-zero \(B\)-field can be considered in the same manner. This will be discussed in Ref.13). On the other hand, a twist element can be any element in \(H \otimes H\) satisfying the cocycle condition and counital condition. Thus, the twist element is not necessarily given by a Green function. Such a nonabelian twist would correspond to the interacting theory on the worldsheets. We would also like to consider target spaces other than \(\mathbb{R}^d\). In that case, the general strategy proposed in this paper, the unified treatment of the worldsheets and target space variations as a Hopf algebra, is also expected to work. This would shed new light on the understanding of the quantization of strings in more general backgrounds and also on the geometric structure of strings as quantum gravity.

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Appendix A

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Hopf Algebra

---

Here we introduce some definitions and our conventions regarding the Hopf algebra and its action.

A Hopf algebra \((H;\mu,\iota,\Delta,\epsilon,S)\) over a field \(k\) is a \(k\)-vector space \(H\) equipped with the following linear maps

\[
\begin{align*}
\mu &: H \otimes H \to H \quad \text{(multiplication)}, \\
\iota &: k \to H \quad \text{(unit)}, \\
\Delta &: H \to H \otimes H \quad \text{(coproduct)}, \\
\epsilon &: H \to k \quad \text{(counit)}, \\
S &: H \to H \quad \text{(antipode)},
\end{align*}
\]

(We also denote \(\mu(h \otimes g) = hg\) and \(\iota(k) = k1_H\)) and satisfying the following relations:

\[
\begin{align*}
(fg)h &= f(gh), & h1_H &= h = 1_Hh, \\
(\Delta \otimes \text{id}) \circ \Delta(h) &= (\text{id} \otimes \Delta) \circ \Delta(h), & (\epsilon \otimes \text{id}) \circ \Delta(h) &= h = (\text{id} \otimes \epsilon) \circ \Delta(h) \\
\mu \circ (S \otimes \text{id}) \circ \Delta(h) &= \epsilon(h) = \mu \circ (\text{id} \otimes S) \circ \Delta(h) \\
\Delta(gh) &= \Delta(g)\Delta(h), & \epsilon(gh) &= \epsilon(g)\epsilon(h), & S(gh) &= S(h)S(g)
\end{align*}
\]
for \( f, g, h \in H \). An universal enveloping algebra \( U(g) \) of a Lie algebra \( g \) is a Hopf algebra: it is a tensor algebra generated by the elements of \( g \) modulo the Lie algebra relation. The remaining maps are defined for \( g \in U(g) \) by

\[
\Delta(g) = g \otimes 1 + 1 \otimes g, \quad \epsilon(g) = 0, \quad S(g) = -g,
\]

and are extended for arbitrary elements \( u \in U(g) \) by the (anti)homomorphism property of \( \Delta \) and \( \epsilon \) (S).

A (left) \( H \)-module \( A \) of a Hopf algebra \( H \) is a module of \( H \) that is an algebra, i.e., a \( k \)-vector space equipped with a map \( \alpha : H \otimes A \to A \) called an \( H \)-action (we denote it as \( \alpha(h \otimes a) = h \triangleright a \)) such that

\[
(gh) \triangleright a = g \triangleright (h \triangleright a), \quad 1_H \triangleright a = a.
\]

If in addition \( A \) is a unital algebra with a multiplication \( m : A \otimes A \to A \) such that

\[
h \triangleright m(a \otimes b) = m \circ \Delta(h) \triangleright (a \otimes b), \quad h \triangleright 1 = \epsilon(h)1,
\]

then \( A = (A;m) \) is called an \( H \)-module algebra.

The Drinfeld twist of a Hopf algebra \( H \) is given by an invertible element \( F \in H \otimes H \) such that

\[
\Delta_F(h) = F \Delta(h) F^{-1}, \quad S_F(h) = U S(h) U^{-1}, \quad \text{where} \quad U := \mu \circ (\text{id} \otimes S)(F)
\]

for \( \forall h \in H \), satisfies all the axioms. We denote this as \( H_F \).

For an \( H \)-module algebra \( A = (A;m) \), there is an associated \( H_F \)-module algebra \( A_F = (A;m_F) \) with a twisted multiplication \( m_F : A_F \otimes A_F \to A_F \), which is also denoted as \( *_F \). For \( a, b \in A \) it is given by

\[
a *_F b = m_F(a \otimes b) := m \circ F^{-1} \triangleright (a \otimes b)
\]

and is associative owing to the cocycle condition. Condition (A.7) is proved as

\[
h \triangleright m_F(a \otimes b) = m \circ \Delta(h) F^{-1} \triangleright (a \otimes b) = m_F \circ \Delta_F(h) \triangleright (a \otimes b).
\]

#### Appendix B

---

**Proof of the Cocycle Condition**

Here, we show that an element in \( U(\mathcal{C}) \otimes U(\mathcal{C}) \subset \mathcal{H} \otimes \mathcal{H} \) of the form

\[
\mathcal{F} = \exp \left( -\int d^2z \int d^2w G^\mu\nu(z,w) \delta \frac{\delta}{\delta X^\mu(z)} \otimes \delta \frac{\delta}{\delta X^\nu(w)} \right), \quad (B.1)
\]
is a twist element and can be used to obtain the twist Hopf algebra $\mathcal{H}_F$. It is clearly invertible and counital, $(\text{id} \otimes e)F = 1$. The 2-cocycle condition, $\mathcal{F}_{12}(\Delta \otimes \text{id})F = \mathcal{F}_{23}(\text{id} \otimes \Delta)F$, is satisfied because the two sides can be written as

$$\begin{cases}
\mathcal{F}_{12}(\Delta \otimes \text{id})F &= e^{-\int d^2z d^2w G^{\mu \nu}(z,w) \left\{ 1 \otimes \frac{\delta}{\delta X^\mu}(z) + \frac{\delta}{\delta X^\mu}(w) \right\} \otimes \frac{\delta}{\delta X^\nu}(w)} \\
&= e^{-\int d^2z d^2w G^{\mu \nu}(z,w) \left\{ \frac{\delta}{\delta X^\mu}(z) \otimes \frac{\delta}{\delta X^\nu}(w) + \frac{\delta}{\delta X^\mu}(z) \otimes \frac{\delta}{\delta X^\nu}(w) + \frac{\delta}{\delta X^\mu}(z) \otimes \frac{\delta}{\delta X^\nu}(w) \right\}} \\
\mathcal{F}_{23}(\text{id} \otimes \Delta)F &= e^{-\int d^2z d^2w G^{\mu \nu}(z,w) \left\{ 1 \otimes \frac{\delta}{\delta X^\mu}(z) \otimes \frac{\delta}{\delta X^\nu}(w) + \frac{\delta}{\delta X^\mu}(z) \otimes \frac{\delta}{\delta X^\nu}(w) \right\}}
\end{cases}$$

where we used $\Delta(\frac{\delta}{\delta X}) = 1 \otimes \frac{\delta}{\delta X} + \frac{\delta}{\delta X} \otimes 1$ and the fact that $\Delta$ is an algebra homomorphism.

Next let us assume that the “propagator” in the exponent is symmetric, $G^{\mu \nu}(z,w) = G^{\nu \mu}(w,z)$. Then, from the same argument as that in §3, the twist element $F$ is a coboundary, $\mathcal{F} = (N^{-1} \otimes N^{-1})\Delta(N)$. On the other hand, the antipode should be twisted, $S \neq S_F = USU^{-1}$, in general. In fact, $U$ is not 1 and is given explicitly by

$$U = \mu \circ (\text{id} \otimes S)F = e^{\int d^2z d^2w G^{\mu \nu}(z,w) \frac{\delta}{\delta X^\mu}(z) \frac{\delta}{\delta X^\nu}(w) \Delta^{-1}} = N^{-2},$$

where we used $S(\frac{\delta}{\delta X}) = -\frac{\delta}{\delta X}$ and the fact that $S$ is an algebra antihomomorphism.

**Appendix C**

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**Trivial Twists**

---

In this appendix, we consider the case when a Hopf algebra twist is trivial in the cohomologous sense as discussed in Ref.2). Let $H$ be a Hopf algebra. For any invertible element $\gamma \in H$ s.t. $e\gamma = 1$, the corresponding element $\partial \gamma \in H \otimes H$

$$\partial \gamma = (1 \otimes \gamma)(\gamma \otimes 1)\Delta \gamma^{-1} = (\gamma \otimes \gamma)\partial \gamma \gamma^{-1} \quad (C.1)$$

is a counital 2-cocycle, where $\partial$ is defined in Ref.2). However, since $\partial \partial \gamma = 0$, it is a trivial 2-cocycle (called a coboundary). More generally, two 2-cocycles $\psi, \chi$ are said to be cohomologous if they are related by a coboundary $\gamma$ as

$$\psi = (\gamma \otimes \gamma)\chi \gamma \gamma^{-1}. \quad (C.2)$$

Then, it is shown that two Hopf algebras $H_\chi$ and $H_\psi$ twisted by $\chi, \psi$ are isomorphic as Hopf algebras under an inner automorphism. This isomorphism is given by $\pi : H_\psi \rightarrow H_\chi : h \mapsto \pi(h) = \gamma^{-1} h \gamma$. Here, the coproduct in $H_\psi$ can be written for $\gamma h \in H_\psi$ as

$$\Delta_\psi(h) = \psi(\Delta h)\psi^{-1} = (\gamma \otimes \gamma)\chi((\Delta \gamma)(\Delta h)(\Delta \gamma^{-1})(\gamma^{-1} \otimes \gamma^{-1})) = (\gamma \otimes \gamma)((\Delta \chi(\gamma^{-1} h \gamma)))(\gamma^{-1} \otimes \gamma^{-1}).$$

Because $(\pi \otimes \pi)(h_1 \otimes h_2) \mapsto (\gamma^{-1} \otimes \gamma^{-1})(h_1 \otimes h_2)(\gamma \otimes \gamma)$, it implies the coalgebra isomorphism $(\pi \otimes \pi)(\Delta_\psi(h)) = \Delta_\chi(\pi(h))$. The other structures are also easily verified to be isomorphic.
In the same way, if an $H_\chi$-module algebra $A_\chi$ and an $H_\psi$-module algebra $A_\psi$ are obtained by twisting the same $H$-module algebra $A$, then they are isomorphic as module algebras. Let us define $\tilde{\pi}$ also relates the twisted products $\psi$ (3.9). Since are obtained by twisting the same $H_{\chi 30}$, we have used this fact and $\pi$ (3.9). For this, it is sufficient to show that none of the coproducts of the generators $\pi$ of the Poincare-Lie algebra $P$ are modified. Recalling that the coproduct of the generators $\pi$ (3.9). For the Lorentz generators, by using the fact that the propagator is of the form $\int d^2z d^2w G_0(z, w) (\epsilon_{\nu} + \epsilon^\nu_\lambda) \delta_{\lambda}(z) \delta_{\lambda}(w) = 0$ (D.1)

where we have used this fact and $G_0(z, w) = G_0(w, z)$ in the last step.

Appendix D

Poincare Symmetry and $F_0$

Here we prove that $U(P)$ is the invariant Hopf subalgebra under the twist $F_0$ (3.9). For this, it is sufficient to show that none of the coproducts of the generators of the Poincare-Lie algebra $P$ are modified. Recalling that $F_0 = e^{F_0}$, we must show that the coproduct of the generators $P_\mu$ and $L_{\mu\nu}$ in (2.11) commutes with $F_0$ in (3.9). Since $P_\mu \in C$, it is apparent that $\Delta(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu$ commutes with $F_0$. For the Lorentz generators, by using the fact that the propagator is of the form $G_0(z, w) = \gamma^{\mu\nu} G_0(z, w)$, we can verify this as $[F_0, \Delta(\epsilon_{\mu\nu} L^{\mu\nu})] = [F_0, (\epsilon_{\mu\nu} L^{\mu\nu} \otimes 1 + 1 \otimes \epsilon_{\mu\nu} L^{\mu\nu})]$

$= - \int d^2z d^2w G_0(z, w) \left( \epsilon_{\mu\nu} \frac{\delta}{\delta X_{\nu}(z)} \right) \left( \epsilon_{\nu\lambda} \frac{\delta}{\delta X_{\lambda}(w)} \right)$

$= - \int d^2z d^2w G_0(z, w) (\epsilon_{\nu} + \epsilon^\nu_\lambda) \frac{\delta}{\delta X_{\nu}(z)} \frac{\delta}{\delta X_{\lambda}(w)} = 0$ (D.1)

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