ON MOTIVIC ZETA FUNCTIONS AND THE MOTIVIC
NEARBY FIBER

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Abstract. We collect some properties of the motivic zeta functions and the
motivic nearby fiber defined by Denef and Loeser. In particular, we calculate
the relative dual of the motivic nearby fiber. We give a candidate for a nearby
cycle morphism on the level of Grothendieck groups of varieties using the
motivic nearby fiber.

1. Introduction

Let \( k \) be an algebraically closed field of characteristic zero. One parameter Taylor
series of length \( n \) in a smooth variety \( X \) over \( k \) are called arcs of order \( n \) on \( X \).
The set of all these arcs are the \( k \)-valued points of a \( k \)-variety \( \mathcal{L}_n(X) \).

Suppose that we are given a function \( f : X \to \mathbb{A}^1 \) on a smooth connected variety
\( X \) of dimension \( d \). Denef and Loeser have associated to this data the so-called
motivic zeta function \( S(f)(T) \), which is a formal power series with coefficients in a
localized equivariant Grothendieck group of varieties over the zero locus of \( f \). The
\( n \)-th coefficient of this series (for \( n \geq 1 \)) is given as \( L^{-nd} \) times the class of the variety
of arcs \( \gamma(t) \) of order \( n \) on \( X \) satisfying \( f \circ \gamma(t) = t^n \), which carries a \( \mu_n \)-action induced
by \( t \mapsto \zeta t \). Here \( L \) denotes the class of the affine line. Using the transformation rule
for motivic integrals, they have given a formula for \( S(f)(T) \) in terms of an embedded
resolution of the zero locus, which shows that it is in fact a rational function which
is regular at infinity. Let us denote \( -S(f)(\infty) \), the motivic nearby fiber, by \( \psi_f \),
as it is supposed to be a virtual motivic incarnation of the nearby cycle sheaf.
We establish some identities for the motivic nearby fiber which are analogues of
identities known for the nearby cycle sheaf. In particular, we calculate the relative
dual over the zero locus of \( f \). It turns out to be \( L^{1-d}\psi_f \), which means that \( \psi_f \)
behaves as the class of a smooth variety, proper over the zero locus. Furthermore
we give a functional equation for the motivic zeta function. Unfortunately, to be
able to define e.g. a relative dual, we have to work in Grothendieck groups which
are coarser than the ones considered by Denef and Loeser.

For the purpose of investigating the zeta function and the nearby fiber, we first
have to generalize slightly the presentations from [1] in the equivariant setting.
Essentially, we also allow free actions on the base varieties. Then we calculate the
relative dual of an affine toric variety given by a simplicial cone which is proper
over the base variety.

Using the motivic nearby fiber, we end with defining a nearby cycle morphism
on the level of Grothendieck groups of varieties and listing some properties of this
morphism.

1991 Mathematics Subject Classification. 14F42, 32S30.
Key words and phrases. Motivic zeta function, motivic nearby fiber.
I am indebted to Eduard Looijenga, my thesis advisor. I thank Jan Denef and Wim Veys for their comments and questions.

Conventions: In the sequel $k$ denotes an algebraically closed field of characteristic zero. By a variety over $k$ we mean a reduced scheme of finite type over $k$.

2. Motivic zeta functions and the motivic nearby fiber

The aim of this section is to recall the definition and properties of motivic zeta functions and the motivic nearby fiber as they can be found e.g. in [3] and in [7]. We also use this opportunity to fix some notations.

2.1. Arc spaces. We denote Spec$(k[t]/t^{n+1})$ by $\mathbb{D}_n$. An order $n$ arc in a $k$-variety $X$ is a morphism $\gamma : \mathbb{D}_n \to X$. There is a $k$-scheme $\mathcal{L}_n(X)$, the space of arcs of order $n$ in $X$, whose $k$-valued points are the arcs of order $n$ on $X$. Actually, it is the scheme representing the functor from $k$-schemes to sets which sends $Z$ to $X(Z \times \mathbb{D}_n)$. There are natural projections $\mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ which are $\mathbb{A}^{\dim X}$-bundles in case $X$ is smooth and equidimensional.

2.2. Grothendieck groups of varieties. We denote by $K_0(\text{Var}_k)$ the free abelian group on isomorphism classes of $k$-varieties modulo the relations $[X] = [X-Y] + [Y]$, where $Y \subset X$ is a closed subvariety. It is also called the naive Grothendieck group of varieties over $k$ and carries a ring structure induced by the product of varieties. More generally, for a variety $S$ over $k$, we denote by $K_0(\text{Var}_S)$ the free abelian group on isomorphism classes $[X]_S$ of varieties over $S$ modulo the relations $[X]_S = [X-Y]_S + [Y]_S$, where $Y \subset X$ is a closed subvariety. It is naturally a $K_0(\text{Var}_k)$-module. We denote by $\mathbb{L}$ the class of the affine line $[\mathbb{A}^1] \in K_0(\text{Var}_k)$. Let $\mathcal{M}_S$ be the localization $K_0(\text{Var}_S)[\mathbb{L}^{-1}]$ and $\mathcal{M}_k$ the ring $K_0(\text{Var}_k)[\mathbb{L}^{-1}]$. A morphism $f : S \to S'$ naturally induces $f_! : \mathcal{M}_S \to \mathcal{M}_{S'}$ by composition and $f^* : \mathcal{M}_{S'} \to \mathcal{M}_S$ by pulling back. The presentation for $K_0(\text{Var}_S)$ given in [1] allows to define a relative dualization endomorphism $D_S$ on $\mathcal{M}_S$ characterized by the property $D_S[X]_S = \mathbb{L}^{-\dim X}[X]_S$ for a smooth equidimensional variety $X$ which is proper over $S$.

Let $S$ be a variety over $k$ with a good action of a finite group $G$. Here by a good action we mean an action such that every orbit is contained in an affine open subvariety. By $K_0^G(\text{Var}_S)$ we denote the free group on isomorphism classes $[X]_S$ of varieties $X \to S$ with a good $G$-action over the action on $S$ modulo relations for closed $G$-subvarieties. It has the structure of a $K_0^G(\text{Var}_k)$-module provided by the product of varieties with the diagonal action.

We define $K_0^G(\text{Var}_S)$ to be $K_0^G(\text{Var}_S)$ modulo the submodule generated as a group by expressions of the form $[G \circ \mathbb{P}(V)]_S - [\mathbb{P}^n \times (G \circ X)]_S$, where $V \to X$ is a vector bundle of rank $n+1$ with a $G$-action which is linear over the action on $X$. Here $G \circ \mathbb{P}(V)$ denotes the projectivization of this action, whereas $\mathbb{P}^n \times (G \circ X)$ denotes the action of $G$ on the right factor only. For $S = \text{Spec } k$ this is an ideal, and $K_0^G(\text{Var}_S)$ is a $K_0^G(\text{Var}_k)$-module. For an affine $G$-bundle $F \to X$ with an affine $G$-action over the action on the base we have $[G \circ F]_S = [\mathbb{A}^{\dim X} \times (G \circ X)]_S$ in $K_0^G(\text{Var}_S)$. We obtain restriction and induction morphisms from group homomorphisms. In particular, $K_0^G(\text{Var}_S)$ is a $K_0(\text{Var}_k)$-module. Denote by $\mathcal{M}_S^G$ the localization $K_0^G(\text{Var}_S)[\mathbb{L}^{-1}]$. An equivariant morphism $f : S \to S'$ induces $f_! : \mathcal{M}_S^G \to \mathcal{M}_{S'}^G$ and $f^* : \mathcal{M}_{S'}^G \to \mathcal{M}_S^G$. In case of a trivial action of $G$ on the
base variety, the presentation in \cite{1} allows to define a relative dualization endomorphism $D_S$ on $M^S_k$ characterized by the property $D_S[X]_S = L^{-\dim X}X[X]_S$ for a smooth equidimensional variety $X$ which is proper over $S$.

Now let $\hat{\mu} = \lim_{\mu_m}$ (where in the projective limit we take the natural surjections $\mu_{dm} \rightarrow \mu_m$, $\zeta \mapsto \zeta^d$). Following \cite{3}, we call an action of $\hat{\mu}$ which comes from a good action of a finite quotient $\mu_n$ of $\hat{\mu}$ a good $\hat{\mu}$-action.

As in the case of a finite group for a base variety $S$ (with a trivial $\hat{\mu}$-action) we can then define the equivariant group $K^\mu_R(Var_S)$. Actually, it is just the direct limit $\lim_{\mu_m} K^{\mu_m}_R(Var_S)$, where $K^{\mu_m}_R(Var_S) \rightarrow K^{\mu_{dm}}_R(Var_S)$ is restriction along the $d$-th power map $\mu_{dm} \rightarrow \mu_m$. We denote $K^\mu_R(Var_S)[L^{-1}]$ by $M^\mu_S$.

Remark 2.1. The equivariant groups which Denef and Loeser consider are finer. The groups we use allow us to define dualization, induction morphisms, quotient morphisms by groups acting freely on the base variety and zeta-functions in an equivariant setting.

2.3. Motivic zeta functions. Let $X$ be a smooth connected variety over $k$ of dimension $d$, let $f : X \rightarrow \mathbb{A}^1$ be a function. For a natural number $n \geq 0$ we define

$$S_n(f) := L^{-nd}[\{\gamma \in L_n(X) \mid \ord_t(f\gamma) = n\}]_X \in M_X.$$ 

Remark 2.2. Sometimes, for example in \cite{2}, there is an extra factor $L^{-d}$.

Following Denef and Loeser, we define the naive motivic zeta function of $f$ as

$$S(f)(T) := \sum_{n \geq 0} S_n(f)T^n \in M_X[[T]].$$

Remark 2.3. Sometimes (for example in \cite{3}) the constant term $S_0(f)$ is omitted. We keep it, like in \cite{2} and \cite{4}.

Denef and Loeser established a formula for $S(f)(T)$ in terms of an embedded resolution $H : Y \rightarrow X$ of $f^{-1}(0)$. Let us assume that $f$ is non-zero. Let $E = (fH)^{-1}(0)$ be a simple normal crossings divisor with irreducible components $E_i$ (where $i \in \text{irr}(E)$). The zero divisor of $fH$ can then be written as $\sum m_iE_i$, and the Jacobian ideal $J_H$ of $H$ (which is the principal ideal of $O_Y$ characterized by $H^*\Omega^1_Y = J_H\Omega^1_Y$) can be expressed as $\sum(n_i - 1)E_i$, where $n_i > 0$. For $I \subset \text{irr}(E)$ we define $E_I$ as the intersection $\bigcap_{i \in I} E_i$ and $E_I^n$ as $E_I - \bigcup_{j \notin I} E_j$ (for $I = \emptyset$ we get $E_\emptyset = Y$ and $E_\emptyset^n = Y - E$).

Proposition 2.4. We have the identity

$$S(f)(T) = \sum_{I \subseteq \text{irr}(E)} [E_I]_X \prod_{i \in I} \frac{L - 1}{T^{m_i}L^{n_i} - 1}.$$ in $M_X[[T]]$.

In particular, $S(f)$ lies in $M_X[(T^{-N}L^n - 1)^{-1} \mid n, N \in \mathbb{N}_{>0}]$.

The motivic zeta function also carries information about the monodromy.

Again, let $X$ be a smooth connected variety of dimension $d$ over $k$, and let $f : X \rightarrow \mathbb{A}^1$ be a function on $X$. We will always denote the zero set of $f$ by $X_0$. For a natural number $n \geq 1$ we define

$$S_n(f) := L^{-nd}[\{\gamma \in L_n(X) \mid f\gamma(t) = t^n\}]_X \in M^\mu_{X_0}. $$
Here the $\mu$-action is induced by the $\mu_n$-action on $\mathbb{D}_n$ given by $t \mapsto \zeta t$. Following Denef and Loeser, we define the \textit{motivic zeta function} of $f$ as $$S(f)(T) := \sum_{n \geq 1} S_n(f)T^n \in \mathcal{M}_{\mathcal{X}_0}[\langle T \rangle].$$

There is also a formula for $S(f)(T)$ in terms of an embedded resolution of $X_0$ (for non-zero $f$): Let $Y$, $n_i$, and $m_i$ be like above. For $I \subseteq \text{irr}(E)$ we denote the greatest common divisor $(m_i)_{i \in I}$ by $m_I$. For a point in $E_I^0$ in a neighborhood $U$ the function $f \in H$ can be written as $u \prod_{i \in I} x_i^{m_i}$, where $u$ is a unit and $x_i$ is a local (analytic) coordinate defining $E_i$. We define $\tilde{E}_I^0$ as the $\mu_{m_I}$-covering of $E_I^0$ given over $U \cap E_I^0$ by $\{(z, p) \in \mathbb{A}^1 \times U \cap E_I^0 \mid z^{m_i} = u(p)^{-1}\}$. These patch to a $\mu_{m_I}$-covering of $E_I^0$: If $y_i = \eta_i x_i$ are other local (analytic) coordinates with $\eta_i$ units, and $f$ is written as $v \prod_{i \in I} y_i^{m_i}$, then $u = v \prod_{i \in I} \eta_i^{a_i}$ and 

$$\{(z, p) \in \mathbb{A}^1 \times U \cap E_I^0 \mid z^{m_i} = u(p)^{-1}\} \cong \{(z, p) \in \mathbb{A}^1 \times U \cap E_I^0 \mid z^{m_i} = v(p)^{-1}\}$$

via $(z, p) \mapsto (\prod_{i \in I} \eta_i^{a_i} z, p)$, where $a_i := \frac{m_i}{m_I}$.

The $\mu_{m_I}$-operation on $\tilde{E}_I^0$ induces a good $\bar{\mu}$-action over $E_I^0$.

There is also an intrinsic description of these coverings, compare also [7], Lemma 5.3 and [15], Lemma 2.5:

Denote by $\nu_{E_i}$ the normal bundle of $E_i$ in $Y$, denote the complement of the zero section by $U_{E_i}$, and the fiber product of the restrictions of the $(U_{E_i})_{i \in I}$ over $E_I^0$ by $U_I$. As $fH$ is a section of $\mathcal{O}_Y(\sum_{i \in I} -m_iE_i)$, it induces a morphism

$$\bigotimes_{i \in I} \nu_{E_i} \otimes^{m_i} |E_I^0| \to \mathbb{A}^1.$$ 

The composition of this map with the morphism $\prod_{i \in I} \nu_{E_i} |E_I^0| \to \bigotimes_{i \in I} \nu_{E_i} |E_I^0|$ which sends $(v_i)$ to $\bigotimes_i v_{E_i}^{m_i}$, restricted to $U_I$ induces $U_I \to \mathbb{G}_m$.

Let us spell this out in the analytic coordinates used above: An element of the fiber of $U_I$ over $p$ given by $(\lambda_i, \frac{\eta_i y_i}{x_i})_{i \in I}$ is mapped to $u(p) \prod \lambda_i^{a_i}$ (where $\lambda_i \in \mathbb{G}_m$).

Define $U_I(1)$ as the preimage of 1 under the morphism $U_I \to \mathbb{G}_m$. Denote $\sum_{i \in I} m_i$ by $c_I$. Then $\bar{\mu}$ acts on $U_I(1)$ via the scalar action of $\mu_{c_I}$.

We get a $\mu$-equivariant mapping $U_I(1) \to \tilde{E}_I^0$, where in the local coordinates the $z$-coordinate is given as

$$\prod_{i \in I} \lambda_i^{a_i}$$

(note that this is well defined, as for different coordinates $y_i = \eta_i x_i$ with $\eta_i$ a unit we have $\frac{\partial}{\partial x_i} = \eta_i \frac{\partial}{\partial y_i}$ on $E_i$).

As the $\alpha_i$ are relatively prime, the vector $(\alpha_i)$ can be completed by $|I| - 1$ more vectors to a basis of $\mathbb{Z}^I$, so we conclude that $U_I(1) \to \tilde{E}_I^0$ is a torus bundle with fiber $\mathbb{G}_m^{|I| - 1}$, which in turn establishes $\tilde{E}_I^0 \to E_I^0$ as the covering obtained from $U_I(1) \to \tilde{E}_I^0$ by passing (fiber-wise) to connected components.

\textit{Remark 2.5.} Note that for the zero function $f = 0$ we get $S(0)(T) = 0 \in \mathcal{M}_X[[T]]$

If on the other hand $f$ is a unit, $X_0 = \emptyset$ and consequently $\mathcal{M}_{X_0} = 0$.

Denef and Loeser prove the
Theorem 2.6. We have the identity
\[ S(f)(T) = \sum_{\emptyset \neq I \subseteq \text{irr}(E)} (L - 1)^{|I|-1} [E_I]_{X_0} \prod_{i \in I} \frac{1}{T - m_i, n_i - 1} \]
in \( \mathcal{M}^g_{X_0}[[T]] \).

In particular, \( S(f) \) lies in \( \mathcal{M}^g_{X_0}[[T^{-N}L^n - 1]^{-1} | n, N \in \mathbb{N}_{>0}] \) and can hence be evaluated at \( T = \infty \). Following Denef and Loeser, we call \( \psi_f := -S(f)(\infty) \in \mathcal{M}^g_{X_0} \) the motivic nearby fiber of \( f \). In terms of an embedded resolution of \( X_0 \) it is given by
\[ (2.2) \quad \psi_f = \sum_{\emptyset \neq I \subseteq \text{irr}(E)} (1 - L)^{|I|-1} [E_I]_{X_0}. \]

Remark 2.7. Note that if \( \pi : Y \to X \) is an isomorphism outside \( X_0 \), we have \( \psi_f = (\pi_0)_* \psi_{f|X} \), where \( \pi_0 \) denotes the morphism \( Y_0 \to X_0 \) between the zero loci of \( f\pi \) and \( f \).

3. Relative equivariant Grothendieck groups of varieties

In this section we collect some properties of relative equivariant Grothendieck groups of varieties.

Remark 3.1. Suppose that \( V \to S \) is a vector bundle of rank \( n + 1 \) which carries a linear \( G \)-action over the action on \( S \). Denote \( \mathbb{P}(V) \to S \) by \( \nu \). Then the endomorphism \( \nu \nu^* \) of \( K^G_0(\text{Var}_S) \) is multiplication with \( [\mathbb{P}^n] \).

The presentation for \( K^G_0(\text{Var}_S) \) given in \( \mathbb{P} \) in case of a trivial \( G \)-action on the base variety \( S \) can be generalized slightly.

Lemma 3.2. Suppose we are given a good \( G \times H \)-action on a variety \( S \), such that \( G \) acts trivially and \( H \) acts freely. Then the morphism
\[ K^G_0(\text{Var}_S) \to K^G_0(\text{Var}_{H \setminus S}) \]
which maps \( [X]_S \) to \( [H \setminus X]_{H \setminus S} \) is a \( K^G_0(\text{Var}_k) \)-linear isomorphism. Furthermore it induces a \( K^G_0(\text{Var}_k) \)-linear isomorphism
\[ K^G_0(\text{Var}_S) \to K^G_0(\text{Var}_{H \setminus S}). \]

Proof. The \( K^G_0(\text{Var}_k) \)-linearity is quite clear. Suppose we are given a variety \( X \to S \) with a good \( G \times H \)-action over the action on \( S \). Then the \( G \times H \)-equivariant mapping \( X \to H \setminus X \times H \setminus S \) is an isomorphism (as both are étale of the same degree over \( H \setminus X \)). Hence pulling back along \( S \to H \setminus S \) is an inverse for \( K^G_0(\text{Var}_S) \to K^G_0(\text{Var}_{H \setminus S}) \).

A vector bundle \( V \to X \) with a linear \( G \times H \)-action over a good action on \( X \) descends to the vector bundle \( H \setminus V \) over \( H \setminus X \). And as \( \mathbb{P}(H \setminus V) \cong H \setminus \mathbb{P}(V) \) in this case, the above morphism induces an isomorphism \( K^G_0(\text{Var}_S) \to K^G_0(\text{Var}_{H \setminus S}) \) which is obviously \( K^G_0(\text{Var}_k) \)-linear. \( \square \)

Remark 3.3. Applied to the case that \( G \) is the trivial group this yields \( K^H_0(\text{Var}_S) \cong K^H_0(\text{Var}_S) \) for a free good \( H \)-action on \( S \).
Remark 3.4. The morphism \( K_0^{G \times H}(\text{Var}_S) \to K_0^G(\text{Var}_{H \setminus S}) \) induces an \( \mathcal{M}^G \)-linear morphism

\[
\mathcal{M}^{G \times H}_S \to \mathcal{M}^G_{H \setminus S}
\]

which we will sometimes denote by \( A \to \overline{A} \).

For convenience we spell out the presentations for \( K_0^{G \times H}(\text{Var}_S) \) and \( K_0^{G \times H}(\text{Var}_S) \) we obtain from the above lemma.

Corollary 3.5. Suppose that we are given a good \( G \times H \)-action on \( S \) such that \( G \) acts trivially and \( H \) acts freely on \( S \). The group \( K_0^{G \times H}(\text{Var}_S) \) has a presentation as the abelian group generated by the isomorphism classes of \( S \)-varieties with good \( G \times H \)-action over \( S \) which are smooth over \( k \) and proper over \( S \) subject to the relations \( [\emptyset]_S = 0 \) and \([\text{Bl}_Y X]_S - [E]_S = [X]_S - [Y]_S \), where \( X \) is smooth over \( k \) and proper over \( S \) and carries a good \( G \times H \)-action over \( S \), \( Y \subset X \) is a closed smooth \( G \times H \)-invariant subvariety, \( \text{Bl}_Y X \) is the blow-up of \( X \) along \( Y \) and \( E \) is the exceptional divisor of this blow-up. Moreover, we get the same group if we restrict to varieties which are projective over \( S \). We can also restrict to varieties such that \( G \times H \) acts transitively on the connected components.

Corollary 3.6. The group \( K_0^{G \times H}(\text{Var}_S) \) is the free abelian group on smooth varieties, projective (respectively, proper) over \( S \) with good \( G \times H \)-action over \( S \) (transitive on the connected components), modulo blow-up relations and the subgroup generated by expressions of the form \([G \times H \cap \mathbb{P}(V)]_S - [\mathbb{P}^n \times (G \times H \cap X)]_S\), where \( X \) is a smooth variety, projective over \( S \), with a \( G \times H \)-action transitive on the connected components, and \( V \to X \) is a vector bundle of rank \( n + 1 \) over \( X \) with a linear action over the action on \( X \).

Corollary 3.7. We can define the duality endomorphism \( \mathcal{D}^{G \times H}_S \) on \( \mathcal{M}^{G \times H}_S \) as in [11] and get the same formulae as developed there. Dualizing commutes with restriction and induction and with the morphism induced by dividing out the free \( H \)-action.

Remark 3.8. We have not used that \( k \) is algebraically closed here.

Instead of finite groups, we will also consider \( \hat{\mu} \) and finite products of \( \hat{\mu} \) with itself and with finite groups:

For a natural number \( l \), we define a good \( \hat{\mu}^l \)-action to be an action coming from a good \((\mu_n)^l\)-action. For a finite group \( G \), we can also consider \( G \times \hat{\mu}^l \)-actions, which we call good if they come from a good \( G \times (\mu_n)^l \)-action. We define \( K_0^{G \times \hat{\mu}^l}(\text{Var}_S) \), \( K_0^{G \times \hat{\mu}^l}(\text{Var}_S) \) and \( K_0^{G \times \hat{\mu}^l}(\text{Var}_S) \) as above. As they are direct limits of groups of the kind considered above, we get analogous presentations. We only spell out one.

Corollary 3.9. Suppose \( G \) and \( H \) are finite groups, and \( l \in \mathbb{N} \). Suppose that \( S \) carries a good \( G \times H \times \hat{\mu}^l \)-action, such that \( H \) acts freely and \( G \) and \( \hat{\mu}^l \) acts trivially. Then the group \( K_0^{G \times H \times \hat{\mu}^l}(\text{Var}_S) \) is the free abelian group on smooth varieties, projective (respectively, proper) over \( S \) with (good) \( G \times H \times \hat{\mu}^l \)-action over \( S \) (transitive on the connected components), modulo blow-up relations and the subgroup generated by expressions of the form \([G \times H \times \hat{\mu}^l \cap \mathbb{P}(V)]_S - [\mathbb{P}^n \times (G \times H \times \hat{\mu}^l \cap X)]_S\), where \( X \) is a smooth variety, projective over \( S \), with a \( G \times H \times \hat{\mu}^l \)-action transitive on the connected components, and \( V \to X \) is a vector bundle of rank \( n + 1 \) over \( X \) with a linear action over the action on \( X \).
Remark 3.10. As above, dividing out by $H$ induces an $\mathcal{M}_k^G \times \hat{\mu}_l$-linear morphism
\[ \mathcal{M}_S^{G \times H \times \hat{\mu}_l} \rightarrow \mathcal{M}_{H \setminus S}^{G \times \hat{\mu}_l} \]
which we will also denote by $A \mapsto \overline{A}$.

Furthermore we also get a duality endomorphism $\mathcal{D}_S^{G \times H \times \hat{\mu}_l}$ of $\mathcal{M}_S^{G \times H \times \hat{\mu}_l}$ which satisfies the same relations as developed above.

4. The relative dual of an affine simplicial toric variety

The aim of this section is the following

Lemma 4.1. Let $X$ be an affine toric variety associated to a simplicial cone, let $X \rightarrow S$ be proper, let $G$ be a finite group acting on $X$ over $S$ via the torus, where $S$ carries the trivial $G$-action. Then $\mathcal{D}_S[X]_S = \mathbb{L}^{-\dim X}[X]_S \in \mathcal{M}_S^G$.

Corollary 4.2. If $X$ is a toric variety associated to a simplicial fan, we have $\mathcal{D}_X[X]_X = \mathbb{L}^{-\dim X}[X]_X \in \mathcal{M}_X$.

Proof. As for an open cover $\{U_i\}$ of $X$ the map $\mathcal{M}_X \rightarrow \prod_i \mathcal{M}_{U_i}$ is injective and commutes with dualizing, we may assume that $X$ is defined by a simplicial cone. □

Remark 4.3. Note that if $X$ is complete, in particular $\mathcal{D}_k[X] = \mathbb{L}^{-\dim X}[X] \in \mathcal{M}_k^G$.

We need Lemma 4.7 on triangulations of simplices. It is probably well known and perhaps should be proven by means of toric geometry as it is derived from the Dehn-Sommerville equations which are the combinatorial counterpart of Poincaré duality for toric varieties defined by complete simplicial fans. First an auxiliary

Lemma 4.4. \[ \sum_{l=0}^{n} \binom{n+1}{n-l}(t-1)^l = t^n + \cdots + 1. \]

Proof. This follows from
\[ \sum_{l=0}^{n} \binom{n+1}{l+1}(t-1)^{l+1} = ((t-1) + 1)^{n+1} - 1 = t^{n+1} - 1. \]

□

For convenience we recall the Dehn-Sommerville equations — see for example [5], page 126.

Theorem 4.5 (Dehn-Sommerville equations). Suppose we are given a triangulation of an $(m-1)$-sphere with $f_i$ faces of dimension $i$. Let $f_{-1} := -1$. For $0 \leq p \leq m$ we set
\[ h_p = \sum_{i=p}^{m} (-1)^{i-p} \binom{i}{p} f_{m-1-i}. \]

Then
\[ h_p = h_{m-p} \text{ for } 0 \leq p \leq m. \]

Suppose we are given a linear triangulation $S$ of an $n$-simplex $\Delta$ which refines the standard triangulation $T$. For $\sigma \in S$ we define $\sigma_\Delta$ to be the smallest simplex in $T$ which contains $\sigma$. We denote the dimension of a simplex $\sigma$ by $|\sigma|$ and define the star of $\tau$ in $S$ by $\text{Star}^S(\tau) = \{ \sigma \in S \mid \tau \subseteq \sigma \}$. 
Lemma 4.6. For a fixed $\tau \in S$ consider the polynomial

$$g^S_\tau(t) = \sum_{\sigma \in \text{Star}^S(\tau)} (-1)^{|\sigma\Delta|} (t - 1)^{|\sigma\Delta| - |\sigma|}.$$  

Then $g^S_\tau(t^{-1}) = t^{|\tau| - n} g^S_\tau(t)$, in other words, $g^S_\tau$ is a polynomial of degree $\leq n - |\tau|$ with symmetric coefficients.

Proof. We proceed by induction on $n - |\tau\Delta|$.

Suppose $|\tau\Delta| = n$. Then also $|\sigma\Delta| = n$ for all $\sigma \in \text{Star}^S(\tau)$, hence

$$(-1)^n g^S_\tau(t) = \sum_{\sigma \in \text{Star}^S(\tau)} (t - 1)^{n - |\sigma|} = \sum_{k = |\sigma|}^n c_k (t - 1)^{n - k} = \sum_{i = 0}^{n - |\tau|} c_{n-i} (t - 1)^i,$$

where $c_k$ denotes the number of $k$-simplices in $\text{Star}^S(\tau)$. Without loss of generality we may assume that that $\Delta \subset \mathbb{R}^n$ and that $0 \in \tau$. Denote the subspace generated by $\tau$ by $W_\tau$. The projection of $\text{Star}^S(\tau)$ to $\mathbb{R}^n/W_\tau$ generates a complete fan whose intersection with a sphere around the origin gives a triangulation with $f_i = c_{i+|\tau|+1}$ simplices of dimension $i$. If we set $f_{-1} = 1$ we have $f_i = c_{i+|\tau|+1}$ for all $-1 \leq i \leq n - |\tau| - 1$. Hence

$$(-1)^n g^S_\tau(t) = \sum_{i = 0}^{n - |\tau|} f_{n - |\tau| - 1 - i} (t - 1)^i = \sum_{i = 0}^{n - |\tau|} \sum_{p = 0}^{i} \binom{i}{p} t^p (-1)^{i-p} f_{n - |\tau| - 1 - i}.$$

The Dehn-Sommerville equations then imply that $g^S_\tau(t^{-1}) = t^{|\tau| - n} g^S_\tau(t)$.

Now suppose $k := |\tau\Delta| < n$. Then $\tau$ is contained in a $k$-dimensional face of $\Delta$. Without loss of generality we can assume that $\Delta \subset \mathbb{R}^n$ is the simplex spanned by 0 and the $n$ standard basis vectors $e_1, \ldots, e_n$, and that $\tau$ is contained in the facet spanned by 0 and $e_1, \ldots, e_k$. Denote by $H$ the hyperplane spanned by $e_1, \ldots, e_n$. Denote the reflection at $H$ by $\rho$. We get a new simplex $\Delta' = \Delta \cup \rho(\Delta)$ (spanned by $e_1, \ldots, e_{n-1}, e_n, -e_n$) with the linear triangulation $S' = S \cup \rho(S)$ refining the standard triangulation of $\Delta'$ such that $|\tau\Delta'| = |\tau\Delta| + 1$.

The star of $\tau$ in $S'$ decomposes as

$$\text{Star}^S(\tau) = \{ \sigma \in \text{Star}^S(\tau) \mid \sigma \subseteq H \}$$

$$\cup \{ \sigma \in \text{Star}^S(\tau) \mid \sigma \not\subseteq H \} \cup \rho\{ \sigma \in \text{Star}^S(\tau) \mid \sigma \not\subseteq H \}.$$

For $\sigma \in \text{Star}^S(\tau)$ such that $\sigma \subseteq H$ we get $|\sigma\Delta| = |\sigma\Delta| + 1$ and $|\sigma \Delta \cap \Delta'| = |\sigma\Delta|$, while for $\sigma \in \text{Star}^S(\tau)$ such that $\sigma \not\subseteq H$ we have $|\sigma\Delta'| = |\sigma\Delta|$. Furthermore
By the induction hypothesis we have $g^\sigma (t) = \sum_{\sigma \in \text{Star}^\tau (\tau)} (-1)^{|\sigma \Delta|} (t - 1)^{|\sigma \Delta| - |\sigma| + 1}$,

\[
g^\sigma (t) = \sum_{\sigma \in \text{Star}^\tau (\tau) \subseteq H} (-1)^{|\sigma \Delta|} (t - 1)^{|\sigma \Delta| - |\sigma|} + 2 \sum_{\sigma \in \text{Star}^\tau (\tau) \ni H} (-1)^{|\sigma \Delta|} (t - 1)^{|\sigma \Delta| - |\sigma|}
\]

\[
= 2g^\sigma (t) - 2g^\Delta H \cap S (t) + (1 - t)g^\Delta H \cap S (t)
\]

By the induction hypothesis we have $g^\sigma (t - 1) = t^{\tau} \cdot g^\sigma (t)$ and $g^\Delta H \cap S (t - 1) = t^{\tau} \cdot n g^\Delta H \cap S (t)$. Using the above equation we conclude $g^\sigma (t - 1) = t^{\tau} \cdot n g^\sigma (t)$. \qed

**Lemma 4.7.** Consider the polynomial

\[
h^\sigma (t) := \sum_{\sigma \in S} (-1)^{|\sigma \Delta|} (t - 1)^{|\sigma \Delta| - |\sigma|} - 1.
\]

Then $h^\sigma (t - 1) = t^{-(n + 1)} h^\sigma (t)$.

**Proof.** The cone $R$ on $S$ is a linear triangulation of an $(n + 1)$-simplex. If $\tau$ is the top, we have $g^\tau (t) = -h^\tau (t)$. \qed

**Proof of Proposition 4.4.** Denote the fun defining $X$ by $T$. We proceed by induction on $\dim T$.

For $\dim T = 0$ the claim holds, because in this case $X$ is smooth.

Now suppose $\dim T \geq 1$. Note that all toric constructions will be compatible with the action of $G$. In the sequel we will denote the dimension of a cone $\tau$ by $|\tau|$. We have the orbit stratification $X = \bigsqcup_{\tau \in T} O_{\tau}$, where $O_{\tau}$ is $(k - |\tau|)$-dimensional.

Furthermore, if $V_{\tau}$ denotes the closure of $O_{\tau}$, we have the equation $[O_{\tau}]_S = \sum_{\tau \geq \tau, \tau \geq \tau} (-1)^{|\tau| - |\tau|} [V_{\tau}]_S$ in $K^G_0 (\text{Var}_S)$.

We choose a toric resolution of singularities $Y \to X$. It is given by a certain simplicial refinement $\Sigma$ of $T$ and carries the orbit stratification $Y = \bigsqcup_{\sigma \in \Sigma} O_\sigma$.

For $\sigma \in \Sigma$ we denote by $\varphi (\sigma) \in T$ the smallest facet of $T$ which contains $\sigma$.

Then $O_{\sigma} \to O_{\varphi (\sigma)}$ is a (trivial) $G_{\text{tor}}^{[\varphi (\sigma) - |\sigma|]}$-bundle. As $G$ acts via the torus, $[O_\sigma]_S = (\mathbb{L} - 1)^{|\varphi (\sigma) - |\sigma|} [O_{\varphi (\sigma)}]_S$ in $K^G_0 (\text{Var}_S)$. Thus

\[
[Y]_S = \sum_{\tau \in T} \sum_{\varphi (\sigma) = \tau} (\mathbb{L} - 1)^{|\tau| - |\sigma|} [O_{\tau}]_S
\]

\[
= \sum_{\tau \in T} \left( \sum_{\varphi (\sigma) = \tau} (\mathbb{L} - 1)^{|\tau| - |\sigma|} \right) (-1)^{|\tau|} [V_{\tau}]_S
\]

\[
= \sum_{\tau \in T} \left( \sum_{\varphi (\sigma) = \tau} (-1)^{|\tau|} (\mathbb{L} - 1)^{|\tau| - |\sigma|} \right) (-1)^{|\tau|} [V_{\tau}]_S + [X]_S
\]
For \( \tau' \in T \) such that \(|\tau'| \geq 1 \) consider
\[
p^{\tau'}(t) = \sum_{\tau \subseteq \tau', \varphi(\sigma) = \tau} (-1)^{|\tau|}(t-1)^{|\tau|-|\sigma|}
\]
\[
= \sum_{\tau \subseteq \tau', \varphi(\sigma) = \tau, |\tau| \geq 1} (-1)^{|\tau|}(t-1)^{|\tau|-|\sigma|} + 1
\]

Now from Lemma 4.7 (applied to the intersection of \( \tau' \) with a transversal hyperplane and the triangulations induced by \( \Sigma \) and \( T \)) we know that \( p^{\tau'}(t^{-1}) = t^{-|\tau'|}p^{\tau'}(t) \). Furthermore \( V_{\tau'} \) is an affine toric variety defined by a simplicial cone of dimension \( \dim T - |\tau'| \), hence for \(|\tau'| \geq 1 \) we deduce from the induction hypothesis that \( D_{S}([V_{\tau'}]_S) = L^{-\dim X - |\tau'|}([V_{\tau'}]_S) \). Thus
\[
D_{S}([Y]_S - [X]_S) = L^{-\dim X}([Y]_S - [X]_S),
\]
which completes the induction step.

\[\square\]

5. A Completion

Let \( f : X \rightarrow \mathbb{A}^1 \) be a non-zero function on a smooth connected variety \( X \), let \( Y \) be an embedded resolution of \( X_0 = f^{-1}(0) \) and let \( E_I \) and \( E_I^\circ \) be as in Section 2.

**Definition 5.1.** Let \( \emptyset \neq I \subset \text{irr}(E) \). Define \( \widetilde{E}_I \) as the normalization of \( E_I \) in \( E_I^\circ \).

**Lemma 5.2.** We have \( \widetilde{E}_I|_{E_J} \cong \widetilde{E}_J \) for \( I \subseteq J \) and \( D_{E_I}(\widetilde{E}_I)E_J = L^{\dim X}([\widetilde{E}_I]_{E_J}) \) in \( M^{\hat{\mu}}_{E_I} \) (and in particular \( D_{X_0}(\widetilde{E}_I)|_{X_0} = L^{\dim X}([\widetilde{E}_I]_{X_0}) \) in \( M^{\hat{\mu}}_{X_0} \)).

**Proof.** Both statements are local on the embedded resolution of \( X_0 \), hence we can assume that \( f = ux^{{m_1}}_1 \cdots x^{{m_k}}_k \), with \( k \geq 1 \) and \( m_i > 0 \), where \( x_1, \ldots, x_k, \ldots, x_n \) are local analytic coordinates, and \( I = \{1, \ldots, l\} \subseteq J = \{1, \ldots, l'\} \), where \( 1 \leq l \leq l' \leq k \).

Adjoining an \( m_1 \)-th root of \( u \) we again get a \( \mu_m \)-cover
\[
\pi : Y := \{(p,t) \in X \times \mathbb{A}^1 \mid t^{m_1} = u(p)\} \rightarrow X
\]
with analytic coordinates \( y_1 = tx_1, y_2 = x_2, \ldots, y_n = x_n \) around \( \{x_1 = 0\} \) such that \( f\pi = y_1^{m_1} \cdots y_k^{m_k} \). Here \( \xi \in \mu_m \) acts on \( y_1 \) by multiplication with \( \xi \) and trivially on \( y_j \) for \( j \geq 2 \). Shrinking \( X \) (outside \( \{x_1 = 0\} \)), we may assume that \( \nu = (y_1, \ldots, y_n) : Y \rightarrow \mathbb{A}^n \) is étale. Denote \( \{y_1 = \cdots = y_l = 0\} \) by \( F_I \) and \( F_{l-I} = \bigcup_{j=l+1}^k \{y_j = 0\} \) by \( F_I^\circ \) (these are the pullbacks of \( E_I \) and \( E_I^\circ \) to \( Y \)). Let
\[
\widetilde{F}_I^\circ := \{(s,q) \in \mathbb{A}^1 \times F_I^\circ \mid s^{m_1} = \prod_{j=l+1}^k y_j(q)^{-m_j}\}
\]
. Recall that
\[
\widetilde{E}_I^\circ = \{(z,p) \in \mathbb{A}^1 \times E_I^\circ \mid z^{m_1} = u(p) \prod_{j=l+1}^k x_j(p)^{-m_j}\}
\]
and note that the pullback of \( \widetilde{E}_I^\circ \) to \( \widetilde{F}_I^\circ \) is isomorphic to \( \widetilde{F}_I^\circ \) via \( (z,p,t) \rightarrow (t^{m_1}z, (p,t)) \) (and that the same holds for \( \widetilde{F}_I^\circ \)). Under this isomorphism the \( \hat{\mu} \)-action comes from
a $\mu_{m_1}$-action, where $\zeta \in \mu_{m_1}$ acts on $s$ by multiplication with $\zeta$, while $\xi \in \mu_{m_1}$ acts on $s$ by multiplication with $\xi$. Denote the normalization of $F_I$ in $\widetilde{F}_I$ by $\widetilde{F}_I$. Consider the following cartesian diagram:

\[
\begin{array}{ccc}
\pi^*\widetilde{E}_I & \longrightarrow & \widetilde{E}_I \\
\downarrow & & \downarrow \\
F_I & \longrightarrow & E_I \\
\end{array}
\]

As $\pi$ is smooth and $\widetilde{E}_I$ is normal, $\pi^*\widetilde{E}_I$ is normal. Furthermore, it is finite and surjective over $F_I$ and hence isomorphic to $\widetilde{F}_I$. Thus it suffices that $\widetilde{E}_I \cong \mu_{m_1}\widetilde{F}_I$ and similarly for $E_I$. Hence it suffices that $\widetilde{F}_I|_{F_J} \cong \widetilde{F}_I$ for $I \subseteq J$ and that $\mathcal{D}_{F_I}[\widetilde{F}_I|_{F_J}] = \mathbb{L}^{J|\dim \chi}[\widetilde{F}_I|_{F_I}] \in \mathcal{M}^{\mu_{m_1}\times\mu}_{F_I}$. As the projection $p : \mathbb{A}^n \rightarrow \mathbb{A}^k$ is smooth and $\mathcal{D}p_* = \mathbb{L}^{k-n}p^*\mathcal{D}$, we may assume without loss of generality that $k = n$. In this case the claim follows from Lemma 5.3 and Lemma 5.4.

**Lemma 5.3.** The restriction of the normalization $\widetilde{S}$ of $S := \{s^d = x_1^{p_1} \cdots x_k^{p_k}\}$ in $\mathbb{A}^1 \times \mathbb{A}^k$ to $\{x_1 = 0\} \subset \mathbb{A}^k$ is isomorphic to the normalization $\widetilde{S'}$ of $S' := \{s'^{d'} = x_2^{p_2} \cdots x_k^{p_k}\}$ in $\mathbb{A}^1 \times \mathbb{A}^{k-1}$, where $d' = (d, p_1)$. The $\mu_{d}$-action on $\widetilde{S}$ which is given by $\zeta(s, x_1, \ldots, x_k) = (\zeta^{-1}s, x_1, \ldots, x_k)$ restricts to the action induced by $\zeta(s', x_2, \ldots, x_k) = (\zeta^{-1}s', x_2, \ldots, x_k)$. In other words, it is given via the canonical surjection $\mu_d \twoheadrightarrow \mu_{d'}$.

**Proof.** Let us first assume that the greatest common divisor $(p_1, \ldots, p_k, d)$ equals 1 or equivalently that $S$ is irreducible.

Let $M$ be the lattice in $\mathbb{R}^k$ spanned by $\mathbb{Z}^k$ and by $v := (\frac{p_1}{d}, \ldots, \frac{p_k}{d})$. Then $\widetilde{S} \cong \text{Spec } k[M^+]$, where $M^+ = M \cap (\mathbb{R}_{\geq 0})^k$. The restriction to $\{x_1 = 0\}$ is then given as $\text{Spec } k[M_1^+]$, where $M_1 := \{\alpha \in M|x_1 = 0\}$ and $M_1^+ = M_1 \cap (\mathbb{R}_{\geq 0})^k$ (compare [3], page 16).

But $M'$ is generated by $e_2, \ldots, e_k$ and $v' := (0, \frac{p_2}{d}, \ldots, \frac{p_k}{d})$: First note that $v' = \frac{d}{d'}v - \frac{d}{d'}e_1 \in M'$.

On the other hand, if $(\lambda_1, \ldots, \lambda_k) + \mu v \in M'$, where $\lambda_i, \mu \in \mathbb{Z}$, then $\lambda_1 + \mu \frac{p_1}{d} = 0$. 


Write \( p_1 = fd' \) and \( d = cd' \) such that \((c, f) = 1\). As \( \mu \frac{L}{d} \in \mathbb{Z} \) also \( \mu' = \frac{L}{d'} \in \mathbb{Z} \) and hence \((\lambda_1, \ldots, \lambda_k) + \mu v = (0, \lambda_2, \ldots, \lambda_k) + \mu' v' \) lies in the \( \mathbb{Z} \)-module generated by \( c_2, \ldots, c_k \) and \( v' \).

Hence indeed \( \widetilde{S} |_{\{x_1 = 0\}} \cong \widetilde{S}' \). Note that \( s' = s \frac{d'}{d} x_1 \frac{d}{d} \), hence the \( \mu_d \)-action on \( \widetilde{S}' \) comes from the \( \mu_d \)-action induced by \( s' \mapsto \zeta^{-1} s' \) via the homomorphism \( \mu_d \rightarrow \mu_d' \) which maps \( \zeta \) to \( \zeta \frac{d}{d} \), and the \( \mu_p \)-action comes from the \( \mu_d \)-action via \( \xi \mapsto \xi \frac{d}{d} \).

Now let us drop the assumption that \( c = (p_1, \ldots, p_k, d) \) equals 1. Note that \( c = (d', p_2, \ldots, p_k) \). Let \( e = \frac{d}{c}, \, q_i = \frac{d}{c} e, \, e' = (e, q_1) \). Note that \( e' = \frac{d}{c} \). Let \( \widetilde{T} \) be the normalization of \( T = \{ t e = x_1^q \cdots x_k^q \} \) and \( \widetilde{T}' \) the normalization of \( T' = \{ t e' = x_2^{q_2} \cdots x_k^{q_k} \} \). They both carry a \( \mu_c \)-action as above. The mapping

\[
\mu_d \times \mu_c \rightarrow S, \quad (\eta, t, x) \mapsto (\eta^{-1} t, x)
\]

induces an isomorphism

\[
\mu_d \times \mu_c \rightarrow \widetilde{T} \cong \widetilde{T}'
\]

over \( \mathbb{A}^k \). The \( \mu_d \)-action on \( \widetilde{S} \) corresponds to the action on \( \mu_d \times \mu_c \rightarrow \widetilde{T} \) given by (left) multiplication on \( \mu_d \).

Furthermore the mapping

\[
\mu_d \times \mu_c \rightarrow \widetilde{T}' \cong \widetilde{S}', \quad (\eta, t', x) \mapsto (\eta^{-1} t', x)
\]

induces an isomorphism

\[
\mu_d \times \mu_c \rightarrow \widetilde{T}' \cong \widetilde{S}'.
\]

As

\[
(\mu_d \times \mu_c \rightarrow \widetilde{T}) |_{\{x_1 = 0\}} = \mu_d \times \mu_c \rightarrow \widetilde{T} |_{\{x_1 = 0\}} = \mu_d \times \mu_c \rightarrow \widetilde{T}'
\]

this establishes \( \widetilde{S} |_{\{x_1 = 0\}} \cong \widetilde{S}' \).

Now let us calculate the action of \( \mu_d \) on \( \mu_d \times \mu_c \rightarrow \widetilde{T}' \) obtained from the action on \( \mu_d \times \mu_c \rightarrow \widetilde{T} \) and the corresponding action on \( \widetilde{S}' \).

The action of \( \mu_d \) is given by (left) multiplication on the first factor. It corresponds to the \( \mu_d \)-action on \( \widetilde{S}' \) induced by \( \zeta(s', x_2, \ldots, x_k) = (\zeta^{-1} s', x_2, \ldots, x_k) \).

\[\square\]

**Lemma 5.4.** Let \( p_1, \ldots, p_k, d \) be natural numbers, not necessarily relatively prime. Again denote by \( \widetilde{S} \) the normalization of \( S := \{ t e = x_1^{p_1} \cdots x_k^{p_k} \} \subset \mathbb{A}^1 \times \mathbb{A}^k \). Then

\[
D_{\mathbb{A}^k}(S |_{\mathbb{A}^k}) = L^{-k} \mathbb{S}_\mathbb{A}^k \text{ in } M_{\mathbb{A}^k}^{\mu_d \times \mu_p}. 
\]

Here the \( \mu_d \)-action is induced by \( \zeta(t, x_1, \ldots, x_k) = (\zeta^{-1} t, x_1, \ldots, x_k) \), and the \( \mu_p \)-action is given by the \( \mu_d \)-action on \( \widetilde{S} \) via \( \mu_d \rightarrow \mu_d, \xi \mapsto \xi \frac{d}{d} \).

**Proof.** As dualization commutes with restriction, it is enough to prove the claim in \( M_{\mathbb{A}^k}^{\mu_d} \).

If the greatest common divisor \((p_1, \ldots, p_k, d)\) equals 1, the claim follows directly from Lemma 5.3.

The general case follows from the fact that dualizing commutes with induction as in the proof of Lemma 5.3. \[\square\]
6. The dual of the motivic nearby fiber and two functional equations

Let $X$ be a smooth connected variety of dimension $d$, let $f : X \to \mathbb{A}^1$ be a (non-zero) function.

**Theorem 6.1.** We have $D_{X_0} \psi_f = \mathbb{L}^{1-d} \psi_f$ in $\mathcal{M}_{X_0}^\theta$.

So $\psi_f$ behaves like a smooth $(d-1)$-dimensional variety, proper over $X_0$.

**Remark 6.2.** Let $J$ be a finite nonempty set. Then

$$\sum_{\emptyset \neq I \subseteq J} (\mathbb{L} - 1)^{|I|-1} = [\mathbb{P}^{|J|-1}].$$

**Proof of Theorem 6.1.** According to Formula (2.2) we have

$$\psi_f = \sum_{\emptyset \neq J \subseteq \text{irr}(E)} (1 - \mathbb{L})^{|J|-1} [\widetilde{E}_J]_{X_0}.$$

Due to Lemma 5.2 $[\widetilde{E}_J]_{X_0} = \sum_{J \supseteq I} (-1)^{|J|-|I|} [\widetilde{E}_J]_{X_0}$, and hence

$$\psi_f = \sum_{\emptyset \neq J \subseteq \text{irr}(E)} (1 - \mathbb{L})^{|J|-1} \sum_{J \supseteq I} (-1)^{|J|-|I|} [\widetilde{E}_J]_{X_0} = \sum_{\emptyset \neq J \subseteq \text{irr}(E)} (-1)^{|J|} [\mathbb{P}^{|J|-1}] [\widetilde{E}_J]_{X_0}$$

due to Remark 6.2.

Again using Lemma 5.2 we conclude that

$$D_{X_0} (\psi_f) = \sum_{\emptyset \neq J \subseteq \text{irr}(E)} (-1)^{|J|} \mathbb{L}^{-|J|+1} [\mathbb{P}^{|J|-1}] \mathbb{L}^{-d+|J|} [\widetilde{E}_J]_{X_0} = \mathbb{L}^{-d+1} \psi_f.$$

□

Recall that Proposition 2.4 shows that $S(f)$ lies in $\mathcal{M}_X \{(T^{-m} \mathbb{L}^n - 1)^{-1} \mid n, m \in \mathbb{N}_{>0}\}$.

Consider the ring $\mathcal{P}_k := \mathcal{M}_k[T, T^{-1}, (T^{-m} \mathbb{L}^n - 1)^{-1} \mid n, m \in \mathbb{N}_{>0}]$. The duality involution on $\mathcal{M}_k$ can be extended to a ring involution $D_k^P$ of $\mathcal{P}_k$ by setting $D_k^P(T) = T^{-1}$.

We set $\mathcal{P}_X := \mathcal{M}_X \otimes_{\mathcal{M}_k} \mathcal{P}_k = \mathcal{M}_X[T, T^{-1}, (T^{-m} \mathbb{L}^n - 1)^{-1} \mid n, m \in \mathbb{N}_{>0}]$. The duality involution $D_X$ induces a $D_k^P$-linear involution $D_X^P$.

**Claim 6.3.** The functional equation $D_X^P S(f) = \mathbb{L}^{-d} S(f)$ holds in $\mathcal{P}_X$.

**Proof.** This is much easier to see than the functional equation in [2]. The calculation given here can already be found there.

Note that $[E_J]_{X} = \sum_{J \supseteq I} (-1)^{|J|-|I|} [E_J]_{X}$.

Furthermore for a finite collection of elements $(a_l)_{l \in L}$ in a commutative ring the identity $\prod_{l \in L} (a_l - 1) = \sum_{K \subseteq L} (-1)^{|K|} \prod_{k \in K} a_k$ holds.

This together with Proposition 2.4 yields

$$S(f) = \sum_{J \subseteq \text{irr}(E)} [E_J]_X \prod_{j \in J} B_j.$$
where \( B_j = \frac{1}{T^n - \nu_j - 1} = \frac{1}{T^{m_j} - \nu_j} - 1 \). Now note that \( D_k(B_j) = L^{-1}B_j \) and that \( D_X(E_j|_X) = \mathbb{L}^{|J|-d|E_j|X} \).

Theorem 2.4.6 and Lemma 5.2 yield
\[
S'(f)(T) := (L - 1)S(f)(T) + \sum_{\emptyset \neq J \subseteq \text{irr}(E)} (-1)^{|J|}[\tilde{E}_J|_{X_0}] = \sum_{\emptyset \neq J \subseteq \text{irr}(E)} [\tilde{E}_J|_{X_0}] \sum_{I \subseteq J} (-1)^{|J|-|I|} \prod_{i \in I} \mathbb{L} - 1^{T-m_i \mathbb{L}^{n_i} - 1},
\]
and as in the proof of Claim 5.3, we conclude that
\[
D_{X_0}^\mu S'(f) = \mathbb{L}^{-d}S'(f)
\]
in \( \mathcal{M}_{X_0}^\mu[T, T^{-1}, (T^{-m_i \mathbb{L}^{n_i} - 1})^{-1}] | n, m \in \mathbb{N}_{>0} \), where \( D_{X_0}^\mu(T) = T^{-1} \).

7. Some properties of motivic zeta functions and the motivic nearby fiber

Let again \( X \) be smooth connected variety over \( k \) and \( f : X \to \mathbb{A}^1 \) a function on \( X \). We denote the zero locus of \( f \) by \( X_0 \).

**Definition 7.1.** For a morphism \( \pi : X' \to X \) of varieties we will denote the zero locus of \( f \pi \) by \( X'_0 \) and the induced morphism \( X'_0 \to X_0 \) by \( \pi_0 \).

We collect some properties of the motivic zeta function and the motivic nearby fiber. For this purpose we first introduce a notation.

**Definition 7.2.** Let \( m \geq 1 \) be a natural number. Let \( X \) be a variety with good \( \hat{\mu} \)-action. Then we denote by \( \text{Ind}^{(m)}_{\mu_n^m} : \mathcal{M}^\mu_X \to \mathcal{M}^\mu_{X_0} \) the morphism induced by \( \text{Ind}^{(m)}_{\mu_n^m} \), where \( \mu_n \to \mu_n^m \) is the inclusion \( \zeta \to \zeta^m \).

**List of properties 7.3.** Suppose \( u : X \to \mathbb{G}_m \) is a morphism. Then after a finite étale base change \( \pi : \tilde{X} \to X \) (taking a sufficiently high root of \( u \)) we get \( S((f \circ \pi)(T)) = S((u(f) \circ \pi)(T)) \) and in particular
\[
\psi_{f \circ \pi} = \psi_{u(f) \circ \pi} \cdot \psi_f.
\]

Suppose \( \pi : X' \to X \) is a smooth morphism of smooth connected varieties. Then \( S(f \pi)(T) = \pi_0^* S(f)(T) \) and in particular
\[
\psi_{f \pi} = \pi_0^* \psi_f.
\]

For a natural number \( m \geq 1 \) we have \( S((f^m)(T)) = \text{Ind}^{(m)} S(f)(T^m) \) and in particular
\[
\psi_{f^m} = \text{Ind}^{(m)} \psi_f.
\]

**Proof.** The first equality follows directly from Theorem 2.4.6.

For the second identity, note that \( \mathcal{L}_n(X') \to \pi^* \mathcal{L}_n(X) \) is a locally trivial fibration with fiber \( \mathbb{A}^{nd} \), where \( d = \dim X' - \dim X \). Therefore \( S_n(f \pi) = \pi_0^* S_n(f) \).

Finally,
\[
S_{nm}(f^m) = \mathbb{L}^{-nm \dim X} \gamma | \gamma \in \mathcal{L}_{nm}(X) | f^m \gamma(t) = t^{nm} | X_0 \]
\[
= \mathbb{L}^{-nm \dim X} \mathbb{L}^{n(m-1) \dim X} \gamma | \gamma \in \mathcal{L}_n(X) | f^m \gamma(t) = t^n \zeta^n \text{ where } \zeta \in \mu_m | X_0 \]
\[
= \text{Ind}^{(m)} S_n(f).
\]
Remark 7.4. Equality (7.1) does not always hold before taking a base change. For example, let \( k = \mathbb{C} \). Consider the open subvariety \( U \subset \mathbb{A}^1 \) which is the complement of the zero locus of \( g(x) = x^3 + ax + b \), where we assume that \( g \) has no multiple roots. On \( U \times \mathbb{A}^1 \) consider the functions \( f(x, y) = y^2 \) and \( u(x, y) = g(x) \). Then \( \psi_f = \mu_2 \times U \in \mathcal{M}_U^\hat{\mu} \) and \( \psi_u = \{ t^2 = u(x) \} \in \mathcal{M}_U^\hat{t} \) (set \( t = z^{-1} \) in Formula 2.2). Their images in \( \mathcal{M}_C \) are not equal: They are distinguished by the Hodge character.

We will need the following lemma later on to define a nearby cycle morphism.

Lemma 7.5. Let \( X \) be a smooth connected variety and \( Y \subset X \) a smooth closed subvariety, let \( f : X \to \mathbb{A}^1 \) be a function on \( X \). Let \( \pi : \text{Bl}_Y X \to X \) be the blow-up of \( X \) along \( Y \), let \( E \) be the exceptional divisor of the blow-up. Denote the inclusion \( Y \subset X \) by \( \iota \) and the inclusion \( E \subset \text{Bl}_Y X \) by \( \iota' \). Let \( g := f \iota \), \( f' = f \pi \) and \( g' = f' \iota' \). Then

\[
\psi_f - \iota_0 \psi_g = \pi_0 \psi_{f'} - \pi_0 \iota'_0 \psi_{g'}.
\]

Proof. Note that if \( Y \subset f^{-1}(0) \) the claim follows from Remarks 2.5 and 2.7. Hence we may assume that \( Y \) is not contained in the zero locus of \( f \).

If \( f^{-1}(0) \) is a simple normal crossings divisor which has normal crossings with \( Y \), the same holds for \( f'^{-1}(0) \) and \( E \), and Formula 2.2 yields

\[
\psi_g = \iota_0 \psi_f \quad \text{and} \quad \psi_{g'} = \iota'_0 \psi_{f'}.
\]

Hence \( \psi_f - \iota_0 \psi_g = j_0 j_0^* \psi_f \), where \( j \) denotes the inclusion \( X - Y \subset X \). Now \( j_0^* \psi_f = \psi_{fj} \), hence \( \psi_f - \iota_0 \psi_g = j_0 \psi_{fj} \) and similarly for \( \psi_{f'} - \iota'_0 \psi_{g'} \), thus the claim follows from the fact that \( \pi \) is an isomorphism outside \( Y \).

In the general case we first choose an embedded resolution \( X^\natural \) of the zero locus of \( f \). Denote the closure of the inverse image of \( Y - f^{-1}(0) \) by \( Y^\natural \), denote the function \( X^\natural \to X \to \mathbb{A}^1 \) by \( f^\natural \). Now we choose an embedded resolution \( \hat{Y} \subset \hat{X} \) of \( Y^\natural \) which is compatible with the zero divisor of \( f^\natural \).

The situation is as follows:

\[
\begin{array}{ccc}
\hat{Y} & \xrightarrow{\hat{g}} & \hat{X} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & X
\end{array}
\]

\[
\begin{array}{ccc}
\hat{Y} & \xrightarrow{\hat{r}} & \hat{X} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g} & X
\end{array}
\]

Here \( \hat{f}^{-1}(0) \) is a simple normal crossings divisor and has normal crossings with \( \hat{Y} \). Note that \( \hat{r} \) induces an isomorphism outside the zero locus of \( f \), hence in the diagram

\[
\begin{array}{ccc}
\text{Bl}_{\hat{Y}} \hat{X} & \xrightarrow{\hat{s}} & \text{Bl}_{Y} X \\
\downarrow & & \downarrow \\
\hat{X} & \xrightarrow{s} & X
\end{array}
\]
the proper birational map $\varphi$ induces an isomorphism outside the zero locus of $f'$. Thus we can find $\tilde{X}$ smooth making

\[
\begin{array}{c}
\tilde{X} \\
\downarrow p \\
\Bl_Y \tilde{X} \\
\downarrow q \\
\Bl_Y X
\end{array}
\]

commutative such that the closure $\tilde{E}$ of the inverse image of $E - (f')^{-1}(0)$ is smooth and such that $p$ and $q$ are proper and isomorphisms outside the zero loci of $f'$ and $\tilde{f'} = \tilde{f}\pi$ (we can take a resolution of singularities of the graph of $\varphi$ and then an embedded resolution of the closure of the inverse image of $E - (f')^{-1}(0)$). We denote the inclusion $\tilde{E} \hookrightarrow \tilde{X}$ by $\tilde{\iota}$, $f'q$ by $\tilde{f}$ and $\tilde{f}\pi$ by $\tilde{g}$. Furthermore we define $\hat{\mu}'$ as the inclusion $\hat{E} \hookrightarrow \hat{X}$ and $\hat{g}' := \hat{f}'\hat{\iota}$. From the commutative diagram

\[
\begin{array}{c}
\hat{X} \\
\downarrow \hat{\pi} \\
\hat{E} \\
\downarrow \hat{f}' \\
\hat{X} \\
\downarrow \hat{f} \\
\hat{X} \\
\downarrow \hat{\pi} \\
X \\
\downarrow f \\
X \\
\downarrow f' \\
A^1
\end{array}
\]

we conclude:

\[
\begin{align*}
\psi_f - \iota_0 \psi_{\hat{g}} &= r_0 \psi_{\hat{f}} - r_0 \iota_0 \psi_{\hat{g}} \\
&= r_0 \hat{\pi} \psi_{\hat{f}} - r_0 \iota_0 \hat{\pi} \psi_{\hat{g}} & \text{due to Remark 2.7} \\
&= r_0 \hat{\pi} \psi_{\hat{f}} - r_0 \iota_0 \hat{\pi} \psi_{\hat{g}} & \text{due to the above discussion} \\
&= r_0 \hat{\pi} \psi_{\hat{f}} - r_0 \iota_0 \hat{\pi} \psi_{\hat{g}} & \text{due to Remark 2.7} \\
&= \pi_0 \psi_f - \pi_0 \iota' \psi_{\hat{g}} & \text{due to Remark 2.7}
\end{align*}
\]

Now suppose there is a good action of a finite group $G$ on $X$ which is transitive on the connected components of $X$ and leaves $f$ invariant.

Hence we can regard $S(f)(T)$ as an element of $\mathcal{M}_X^G[[T]]$. The transformation formula also holds in the equivariant setting (as an element of $g \in G$ induces an affine action over the base in the fibrations of the Key lemma 9.2 and on the $A^{\dim X}$-bundle $\mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$). Hence we get a formula analogous to Proposition 2.4 in the equivariant context if we choose an $G$-equivariant embedded resolution of the zero locus (where the summation runs over the orbits of finite subsets of $\text{irr} E$).

The $\tilde{\mu}$-action on $\mathcal{L}_n(X)$ induced by $\mu_n$ commutes with the action of $G$ and hence $S(f)(T)$ can be regarded as an element of $\mathcal{M}_{X_0}^{G \times \tilde{\mu}}[[T]]$ and $\psi_f$ as an element of $\mathcal{M}_{X_0}^{G \times \tilde{\mu}}$. 

\[\square\]
Also Theorem \ref{thm:extension} and Formula \ref{formula:extension} have analogues in this context, if we choose an equivariant embedded resolution of the zero locus (from the intrinsic description of $E_\delta^\circ$ it is also clear that $\bigcup_{g \in G/\text{Stab}_G(I)} E_{\delta g}^\circ$ carries a $G$-action, where $\text{Stab}_G(I)$ denotes the stabilizer of $I$ in $G$).

Lemma \ref{lemma:extension} and properties \ref{property:extension} also hold in the equivariant setting.

We can replace $G$ by $G \times \mu^t$ and get the same identities as before.

**Remark 7.6.** Suppose we are given a good free action of a finite group $H$ which is transitive on the connected components on a smooth variety $X$. Suppose we have a $H$-invariant function $f$ on $X$. It then induces a function $\overline{f}$ on $\overline{X} = H \backslash X$. Furthermore $\mathcal{L}_n(X) = H \backslash \mathcal{L}_n(X)$ and $S_n(f) = H \backslash S_n(f)$, hence $S(\overline{f})(T) = \overline{S(f)(T)}$ (we extend the quotient morphism by $T \mapsto T$) and in particular

$$\psi_\overline{f} = \overline{\psi_f}.$$  

8. The nearby cycle morphism

Let $X$ be a (not necessarily smooth) variety over $k$, let $f : X \to \mathbb{A}^1$ be a function. Let $X_0 := f^{-1}(0)$. We want to define a nearby cycle morphism

$$\Psi_f : M_X \to M_{X_0}$$

such that for a proper morphism $\pi : X' \to X$ we get $\pi_0 \Psi_{f,\pi} = \Psi_{f,\pi}$ and furthermore in the case of a smooth connected variety $X$ the image of $1_X$ is $\psi_f$. For this purpose we define $\Psi_f$ on $K_0(\text{Var}_k)$ first.

**Definition 8.1.** Let $p : Y \to X$ be a proper morphism, where $Y$ is a smooth connected $k$-variety. Then we set $\Psi_f([Y]_X) := p_0(\psi_f)$.

**Claim 8.2.** The morphism $\Psi_f$ is compatible with the blow-up relations (and hence well defined).

**Proof.** Let $p : Y \to X$ as above, let $Z \subset Y$ be a smooth connected closed subvariety. Denote by $\pi : Y' \to Y$ the blow-up of $Y$ along $Z$, denote the exceptional divisor by $E$. Denote the inclusion $Z \hookrightarrow Y$ by $\iota$ and the inclusion $E \hookrightarrow Y'$ by $\iota'$. Then we have

$$\Psi_f([Y]_X) - \Psi_f([Z]_X) = p_0 \psi_f - (p_0 \pi_0) \psi_f, \quad (p_0) \psi_f = (p_0 \pi_0) \psi_f, \quad (p_0 \pi'_0) \psi_f, \quad \text{due to Lemma \ref{lemma:blow-up}.}$$

$$\Psi_f([Y]_X) - \Psi_f([Z]_X) = \Psi_f([Y']_X) - \Psi_f([E]_X).$$

□

**Claim 8.3.** The morphism $\Psi_f$ is $K_0(\text{Var}_k)$-linear.

**Proof.** Let $W$ be a smooth complete variety over $k$, let $p : Y \to X$ be as above. Denote the projection $W \times Y \to Y$ by $\pi$. Then

$$\Psi_f([W \times Y]_X) = (p_0 \pi_0) \psi_f = (p_0 \pi_0 \pi'_0) \psi_f \quad \text{due to Formula \ref{formula:blow-up}.}$$

$$\Psi_f([W \times Y]_X) = p_0([W]_X) \psi_f = [W] \Psi_f([Y]_X).$$

□

Hence $\Psi_f$ can be extended to an $M_k$-linear morphism

$$\Psi_f : M_X \to M_{X_0}.$$
List of properties 8.4. For \( \pi : X' \to X \) proper we get \( \pi_0! \Psi_f \pi = \Psi_f \pi_1! \).

If \( \pi : X' \to X \) is a smooth morphism, then \( \Psi_f \pi^* = \pi_0^* \Psi_f \).

If \( \iota : X_0 \hookrightarrow X \) denotes the inclusion, \( \Psi_f \iota_! = 0 \).

For a natural number \( m \geq 1 \) we get \( \Psi_f \pi^m = \text{Ind}^{(m)}(\Psi_f) \).

Dualizing and the nearby cycle morphism commute up to a factor: \( D_{X_0} \Psi_f = L \Psi_f D_X \).

Proof. The first identity holds by construction of \( \Psi_f \).

For the second formula, suppose that \( \pi : X' \to X \) is a smooth morphism. Let \( p : Y \to X \) be a proper morphism, where \( Y \) is a smooth variety over \( k \). Let

\[
\begin{array}{ccc}
Y' & \xrightarrow{p'} & X' \\
\downarrow \pi' & & \downarrow \pi \\
Y & \xrightarrow{p} & X
\end{array}
\]

be cartesian. Then \( \pi' \) is smooth, and \( p' \) is proper, furthermore

\[
\begin{array}{ccc}
Y_0' & \xrightarrow{p_0'} & X_0' \\
\downarrow \pi_0' & & \downarrow \pi_0 \\
Y_0 & \xrightarrow{p_0} & X_0
\end{array}
\]
is cartesian, too. We therefore get

\[
\Psi_f (\pi^*([Y]_X)) = p_{01}! \psi_{fp\pi'} = p_{01}! \pi_0'^* \psi_{fp} \quad \text{due to Formula (7.2)}
\]

\[
= \pi_0^* \psi_{fp} = \pi_0^* \Psi_f ([Y]_X).
\]

The third identity follows from the fact that \( S(0)(T) = 0 \) (see Remark 2.5).

The last two identities follow from Formula (7.3) and Theorem 6.1 respectively.

We can also define \( \Psi_f \) in the equivariant setting. Let \( G \) be a group of the form \( H \times H' \times \hat{\mu}^l \), where we assume that \( H \) acts freely and \( H' \times \hat{\mu}^l \) acts trivially on the base variety \( X \). We assume that we have a \( G \)-invariant function \( f : X \to \mathbb{A}^1 \). We first get a \( K_0^G(\text{Var}_k) \)-linear morphism

\[
\Psi_f : K_0^G(\text{Var}_X) \to \mathcal{M}_{X_0}^{G \times \hat{\mu}}
\]
as before.

Claim 8.5. Suppose \( Y \) is a smooth variety, projective over \( X \) and that \( V \to Y \) is a vector bundle of rank \( n + 1 \) with a linear \( G \)-action over the action on \( X \). Then

\[
\Psi_f ([G \circ \mathbb{P}(V)]_X) = \Psi_f ([\mathbb{P}^n \times (G \circ Y)]_X).
\]

Proof. Let \( \nu : \mathbb{P}(V) \to Y \) be induced by the structure map of \( V \). Denote the morphism \( Y \to X \) by \( p \). We have

\[
\Psi_f ([\mathbb{P}(V)]_X) = (p_0 \nu_0)! \psi_{fp
u} = (p_0 \nu_0)! (\nu_0)^* \psi_{fp} \quad \text{due to Formula (7.2)}
\]

\[
= p_0! (\mathbb{P}^n) \psi_{fp} \quad \text{due to Remark 8.2}
\]

\[
= \mathbb{P}^n \Psi_f ([Y]_X) = \Psi_f ([\mathbb{P}^n \times Y]_X).
\]

\[ \square \]
Hence $\Psi_f$ induces a $K^G_0(\text{Var}_k)$-linear morphism
\[
\Psi_f : K^G_0(\text{Var}_X) \rightarrow \mathcal{M}^{G \times \hat{\mu}}_{X_0}
\]
and an $\mathcal{M}^G_k$-linear morphism
\[
\Psi_f : \mathcal{M}^G_k(X) \rightarrow \mathcal{M}^{G \times \hat{\mu}}_{X_0}.
\]

The first four identities of the List 8.4 also hold in the equivariant setting. We do not know the relationship between dualizing and the nearby cycle morphism in this context.

Furthermore, from Remark 8.6 we conclude

**Proposition 8.6.** Suppose we have a good $G \times H$-action (transitive on the connected components) on the smooth variety $X$ and a $G \times H$-invariant function $f : X \rightarrow \mathbb{A}^1$. Suppose that $H$ is a finite group and acts freely. Denote the function induced on $\overline{X} = H \setminus X$ by $\overline{f}$. Then for $A \in \mathcal{M}^{G \times H}_X$ we have
\[
\overline{\Psi_f(A)} = \overline{\Psi_f(A)} \in \mathcal{M}^{G \times \hat{\mu}}_{X_0}.
\]

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