STOCHASTIC HOMOGENISATION OF NONCONVEX FUNCTIONALS
IN THE SPACE OF $A$-WEAKLY DIFFERENTIABLE MAPS

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Abstract. We prove the $\Gamma$-convergence of sequences of differentially constrained, random integral functionals of the form

$$\int_{U} f(\omega, x/\varepsilon, Au) \, dx$$

for the class of vectorial differential operators $A$ with finite-dimensional nullspaces. This work is intended to generalise results for the full gradient and to cover the cases of symmetric gradients and the deviatoric operator. The homogenisation procedure is carried out by employing a variant of the blow-up method in the setting of $A$-weakly differentiable maps along with the Akcoglu-Krengel subadditive ergodic theorem.

Keywords: Homogenization, $\Gamma$-convergence, $C$-elliptic operators, $BV^A$ spaces, ergodic theory, blow-up method, linear and superlinear growth functionals

1. Introduction

This paper is aimed at deriving homogenisation limits of sequences of random integral functionals that act on vectorial differential operators. Studying the asymptotics of random nonlinear systems is an instrumental and highly effective apparatus for modelling macroscopic progressions of materials, particularly in the context of elasticity theory. In fact, essential information on measuring energies of states at small scales is on many occasions captured from the analysis of rapidly oscillating stochastic integral functionals. Eventual homogenised formulae give rise to energy densities that describe arrangements of statistically distributed heterogeneities in a medium. In terms of variational formulations, a prototypical construction is demonstrated by considering a bounded region $U \subset \mathbb{R}^n$ along with scale-dependent stochastic functionals evaluated on an assigned class of deformations $u : U \subset \mathbb{R}^n \to \mathbb{R}^N$ which take the form:

(1.1) $$\mathcal{F}_\varepsilon(\omega)[u; U] := \int_{U} f(\omega, x/\varepsilon, Au) \, dx.$$ 

Here $\omega$ is a random parameter in a probability space $\Omega$, the mapping $A$ symbolises a (vectorial) linear differential operator and the integral is evaluated on stochastic densities $\omega \mapsto f(\omega, \cdot, \cdot)$ subject to appropriate growth bounds. The main theme revolves around finding an explicit description of the asymptotic limit of the sequence $(\mathcal{F}_\varepsilon(\omega))$ through $\Gamma$-convergence and thereby proving almost sure convergence of minima in (1.1) as $\varepsilon \to 0$ to a minimum of a precisely framed functional. We emphasize that introducing probabilistic compositions into the variation, grants the possibility to address statistically posed hypotheses linked to concrete physical experimentations. For that matter a predominant example relates to randomised chessboard-like structures where by selecting a specific process, the homogenised densities correspond to cell formulae that encode locations of impurities in a region, see Figure 1

![Figure 1. Examples of random chessboards.](image-url)
There have appeared numerous contributions serving detailed analysis of the above problems and to indicate an almost certainly incomplete list we reference exemplarily [15, 28, 34] for deterministic motifs and [15, 1, 28, 34] for the ones with stochastic realizations. As for the former variant, a customary condition assumed on integrands is the periodicity in the space variable i.e. \( f(x + z, \xi) = f(x, \xi) \) for all \( (x, \xi) \in \text{Dom}(f) \), \( z \in \mathbb{Z}^n \). The resulting cell-formulae is then implied to satisfy homogeneity i.e. it relies only on the phase variable \( \xi \). Physically this is ought to confirm a homogenous nature of recurring structured materials in the macroscopic perspective. In the context of random media, the notion of stochastic periodicity may be conceptualised by asserting that the spatial displacement \( \xi \) is periodic in law or \( \xi \) is stationary provided:

\[
f(\omega, x + z, \xi) = f(\tau_z(\omega), x, \xi)
\]

for all \( (x, \xi) \in \text{Dom}(f(\omega, \cdot, \cdot)) \), all \( z \in \mathbb{Z}^n \) and all \( \omega \in \Omega \). We notice that it is entirely consistent with the spatial periodicity for deterministic (i.e. not dependent on \( \omega \)) integrands.

In addition, a standard assumption imposed on densities \( f(\omega, \cdot, \cdot) \) in such context is the \( p \)-growth condition, namely: for \( p \in [1, \infty) \) there exist constants \( 0 < \alpha \leq \beta < \infty \) such that

\[
\alpha |\xi|^p \leq f(\omega, x, \xi) \leq \beta (1 + |\xi|^p)
\]

for all \( (x, \xi) \in \text{Dom}(f(\omega, \cdot, \cdot)) \) and all \( \omega \in \Omega \). In the literature early occurrences of stochastic homogenisation of integral functionals in the above setting are seen for instance in the work of Dal Maso and Modica [14, 15]. There the authors treat the cases of periodicity in \( \mathbb{R} \) for the full gradient, that is, \( A = \nabla = (\partial_1, \ldots, \partial_n) \) with densities subject to coerciveness and superlinear growth bounds \( p > 1 \).

It must be noted that a wealth of instances exposing considerable complexities is manifested in the borderline cases of the density \( f(\omega, \cdot, \cdot) \) in [14, 15] satisfying the linear growth condition \( p = 1 \). Naturally the problem is posed on \( W^{1,1}(U; \mathbb{R}^m) \) and in contrast to higher integrability exponents, the nonreflexivity of \( W^{1,1} \)-spaces does not guarantee the weak-convergence of minimising sequences. Therefore the domain of any apparent \( \Gamma \)-limiting functional begs to expand the set of admissible competitors in order to attribute sufficient boost in compactness. Applying relaxation techniques in the space of maps of bounded variation \( \text{BV}(U; \mathbb{R}^m) \) yields the usual homogenised limit. The explicit formula is subsequently obtained with aid of the integral representation results in \( \text{BV} \). A detailed scheme of such regimes is conveyed as the main subject of the paper [1].

On the other hand to account a broad spectrum of possible models where energies often do not depend on full gradients, one may look at resembling problems with differential constrains being posed. This is inscribed by themes where the full gradient in [14, 15] is replaced by a potentially default linear differential operator. From the perspective of physical models examples of high interest thereof include the symmetric gradient \( e(u) := \frac{1}{2} (\nabla u + \nabla u^T) \) or the deviatoric/trace-free symmetric gradient \( e^D(u) := e(u) - \frac{1}{n} \text{div}(u) I_n \) which act on vector fields of the form \( u : U \to \mathbb{R}^n \) where \( I_n \) is the \( (n \times n) \)-identity matrix. More concretely for a pair of finite-dimensional vector spaces \( V, W \)
(V ≃ ℝ^k, W ≃ ℝ^l) we consider a linear differential operator \( A \) from V to W defined as

\[
Au := \sum_{j=1}^{d} A_j \partial_j u, \quad u : U \to V
\]

where each \( A_j : V \to W \) is a fixed linear map.

The desired goal is to establish a corresponding \( \Gamma \)-convergence of energies in (1.1) acting on operators as above up to a set of probability zero. The major difference between our formulation and the nominal BV-set up is that any conclusive information on the full gradients of sequences involved cannot be accessed from uniform bounds on \( (F_v(\omega)) \). This can be seen as a consequence of the so called Ornstein’s non-inequality [33] which essentially says that nontrivial \( L^1 \)-inequalities for generic operators \( A \) are unattainable. Therefore it is inevitable to tackle separately the coercivity issues and lacking information on the partial derivatives. For that reason by introducing more diverse differential constellations, an independent argumentation is demanded. A known instance of the borderline case is the one of BD (the space of maps of bounded deformation) covered in [26] exhibits a resembling conclusion in comparison to the full gradient. The contrasting part is in the limiting functionals that are instead governed by the distributional symmetric gradient. As the problem may not be directly transposed to the full gradient theory, all quintessential derivations are conditioned upon incorporating some key properties of BD-spaces which are strictly contained in BV.

Thereafter in view of the above noted contributions the emerging intention is to advance the existing results and investigate analogous behaviour of energies defined on a categorised family of differential operators. This objective on the one hand is principally to cover numerous relevant models occurring in the related literature and on the other to give a profound insights into the universal notions involved thereby unifying the theory.

1.1. Main Result and Strategy. In this section we shall give an outline of the central theorems to which the discourse of our paper is devoted. The aforementioned aim is to prove existence of \( \text{almost sure} \) \( \Gamma \)-limit of stochastic functionals as given in (1.1) and determine the homogenised densities. Whilst the superlinear regimes \( (p > 1) \) arguably require a fair amount of care, it will become apparent that the core strategy can be recast to the \( W^{1,p} \)-theory led by a simplified technique devised for the linear growth case. Indeed once the borderline \( p = 1 \) has been established, to obtain the precise \( \Gamma \)-limit as integral functional, for \( p > 1 \) is of routine manner. The only disparity lies in detecting key features of Sobolev maps within the proposed constructions. We dedicate Section 4 for a more rigorous elaboration on the matter. As to the main concern we shall largely focus on integrands that satisfy linear growth in the phase variable. This assumption substantially affects the methodology since the spaces of weakly \( A \)-differentiable maps \( W^{1,1}(U) \) are in general not isomorphic to \( W^{1,1}(U; V) \) as normed spaces. In particular all argument have got to be carried out in the setup of \( A \)-Sobolev spaces, see Section 2.3 whose properties heavily depend on the type of differential operators \( A \) they are coupled up with. Interestingly a number of essential features such as variants of Poincaré and Korn-type inequalities are known to be valid for operators \( A \) with finite-dimensional nullspace i.e.

\[
\dim \{ u \in \mathcal{D}'(\mathbb{R}^n; V) : Au = 0 \} < +\infty
\]
where $\mathcal{D}'(\mathbb{R}^n; V)$ denotes the class of all $V$-valued distributions on $\mathbb{R}^n$. Throughout our work all objects of consideration are going to assume this property. This confining feature is shared among many widely studied operators including the full or symmetric gradient, (see Example 2.2 for a larger list). It must be said that finite-dimensionality of nullspaces is strongly linked to the algebraic notion of $C$-ellipticity (Definition 2.1) being a determining condition in conceptualising relevant machinery to successfully proceed with homogenisation analysis. These consist of especially the mentioned Korn-Poincaré inequalities but also integrability of traces and $L^p$-approximate differentiability of $A$-Sobolev maps. Inspite of present counterparts in $\mathbb{R}^2$ such as in [22], up to author’s awareness it is not know whether there is a unifying way of verifying Poincaré inequalities for operators whose distributional kernel have possibly infinite dimension. Therefore it seems as the $C$-elliptic class is up to now the most optimal family of operators. The other confronting factor is the lack coercivity incurred by nonreflexive behaviour of $W^{k,1}$-spaces. Consequently just as in the symmetric gradient instance the homogenisation procedure must be conducted in a strictly larger space to boost compactness. For our theme the optimal choice is $BV^h(U)$, the maps of bounded $A$-variation. These spaces comprise of all $u \in L^1(U; V)$ such that $Au$ understood distributionally is a finite Radon measure. We shall discuss the right notions of relaxation and point out the associated integral representations. For a discussion on relation between functional theoretic details and the nullity of $A$ we refer to Section 2.3

Regarding the question of $\Gamma$-convergence, based on the foundational works of Dal Maso and Modica [14, 15] we will contemplate the subadditive process defined on a probability space $(\Omega, \mathcal{T}, \mathbb{P})$ that assumes the form

$$\mu^A(\omega, U) := \inf \left\{ \int_U f(\omega, x, A + Av) \, dx : v \in W_0^{h,p}(U) \right\}$$

where $U \subset \mathbb{R}^n$ is a bounded and open Lipschitz set, $A \in W$ and $W_0^{h,p}(U)$ denotes the closure of $C_c^\infty(U; V)$ under the seminorm $||A(\cdot)||_{L^p(U; W)}$. In view of the ergodic theorem of Akcoglu and Krengel [2] this process is instrumental in deriving existence of homogenised densities. In particular subject to all said conceptions the hypotheses of Theorem 3.1 and Theorem 4.1 respectively state that the energies ($\mathcal{F}_\epsilon(\omega)$) $\Gamma$-converge almost surely in the local $L^1$-topology in the following accord:

1. if $f$ is of linear growth $p = 1$ in the phase variable, the $\Gamma$-limit is given by

$$\mathcal{F}_{\text{hom}}(\omega)[u; U] := \int_U f_{\text{hom}}(\omega, \frac{dAu}{d\mathcal{L}^n}) \, dx + \int_U f_{\text{hom}}(\omega, \frac{dA^n}{d\mathcal{L}^n}) \, d|A^n| \text{ for } u \in BV^h(U)$$

where $f_{\text{hom}}(\omega, \cdot)$ is the recession function of $f_{\text{hom}}(\omega, \cdot)$ defined as

$$f_{\text{hom}}(\omega, A) := \lim_{k \to \infty} k^{-n} \inf \left\{ \int_{(0,k)^n} f(\omega, x, A + Av) \, dx : v \in W_0^{h,1}((0,k)^n) \right\}.$$ 

2. if $f$ is of $p$-growth for $p > 1$ in the phase variable, the $\Gamma$-limit is given by:

$$\mathcal{E}_{\text{hom}}(\omega)[u; U] := \int_U f_{\text{hom}}(\omega, Au) \, dx \text{ for } u \in W^{1,p}(U; V),$$

where

$$f_{\text{hom}}(\omega, A) := \lim_{k \to \infty} k^{-n} \inf \left\{ \int_{(0,k)^n} f(\omega, x, A + Av) \, dx : v \in W_0^{h,p}((0,k)^n) \right\}.$$
We observe that this result yields the homogenised limit in the $\text{BV}^A$-regimes and that reciprocates preceding situations for $\text{BV}$ and $\text{BD}$. As already detectable at the level of the symmetrised gradient, the strategy of $W^{1,1}$ has to be altered due to evident coercivity issues. Nevertheless inspired by the works of [1] we will contemplate the suitable relaxation techniques for differentially constrained functionals. The recently established theory [24, 25, 23, 10] for $\text{BV}^A$-maps goes hand in hand with the requisite apparatus to generalise the prototypical method for $\text{BV}$. In that respect we divide the proof into two main parts. Firstly in verifying the lim-inf inequality we will implement the method of blow-up [7, 20]. We will use the weak*-compactness along with the Radon-Nikodym differentiation to separately compare measure theoretic derivatives of the absolutely continuous and singular parts. In doing so the key steps rely on the use of a modified Poincaré inequality on annuli, (see Proposition 2.5) as well as an exploitation of approximate differentiability of sequence of functions in $W^{k,1}$. Secondly the lim-sup inequality is settled by applying approximations of $W^{k,1}$ by piecewise-affine maps. Regarding the degree one assumption, while we do obtain convolution-type decomposition of $W^{A,1}$-maps (cf. [25, Thm. 2.1]) with the $A$-gradient dependence, the appearance of $(k-n)$-homogeneous kernels ($k > 1$) for higher orders seems to be insufficient for the essential $L^p$-differentiability validity. To avoid having to deal such obstructions and maintain a holistic narrative we will abide by $A$ as in [13].

1.2. Structure of the paper. In Section 2 we gather some key concepts of $C$-ellipticity with examples of such operators and introduce the notion of $A$-weakly differentiable maps. Crucially there we present Poincaré inequalities, approximate differentiability and approximations results for $\text{BV}^A$ spaces. Moreover we dedicate a paragraph lay out the stochastic features and go over relevant aspects of ergodic theory being used. In Section 3 we address the $\Gamma$-convergence of $(\mathcal{F}_\varepsilon(\omega))$ for $p = 1$ and give a proof of Theorem 3.1. In the concluding Section 4 we discuss the method aligned for the superlinear growth that covers the cases of $W^{A,p}$ for $p > 1$.

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2. Preliminaries and Contextualisation

2.1. Notation and general conventions. Let us concisely mention the notation present in the paper. Throughout $n \geq 2$ is a fixed integer. We set $Q := (-\frac{1}{2}, \frac{1}{2})^n$ to be the open unit cube. The set $Q_r(x) := rQ + x$ represents the open cube centred at $x \in \mathbb{R}^n$ with side length $r > 0$. We denote by $\mathcal{L}^n$ the $n$-dimensional Lebesgue measure. The collection $\mathcal{A}(\mathbb{R}^n)$ denotes all open and bounded subsets of $\mathbb{R}^n$ with Lipschitz boundary. For a topological space $\mathcal{S}$, the symbol $\mathcal{B}(\mathcal{S})$ signifies the Borel sigma algebra of $\mathcal{S}$. For $u \in L^p_{\text{loc}}(\mathbb{R}^n; V)$ and $O \subset \mathbb{R}^n$ s.t. $\mathcal{L}^n(O) < +\infty$ we set $(u)|_O := \int_O u \, dx = \mathcal{L}^n(O)^{-1} \int_O u \, dx$. For a matrix $\xi \in \mathbb{R}^{K \times n}$, by $\ell_\xi$ we mean the affine map $x \in \mathbb{R}^n \mapsto \ell_\xi x$. We denote by $\langle \cdot, \cdot \rangle$ the euclidean inner product and by $| \cdot |$ the induced norm. The letter $c > 0$ is designated for absorbing constants where any significant dependence will be specified.
Let us fix once and for all a pair of finite-dimensional vector spaces \( V, W \) of dimension at least two (i.e. \( V \cong \mathbb{R}^k \) and \( W \cong \mathbb{R}^l \) for some integers \( k, l \geq 2 \)).

### 2.2. On differential constrains

A vectorial differential operator \( \mathcal{A} \) from \( V \) to \( W \) of order one and with constant coefficients is determined via the action

\[
\mathcal{A}u := \sum_{j=1}^{n} A_j \partial_j u, \quad u : U \subset \mathbb{R}^n \to V
\]

for \( A_j \in \mathcal{L}(V; W) \) linear maps from \( V \) to \( W \). An operator \( \mathcal{A} \) of such form can be interpreted through the linear coupling \( \mathcal{A}u = \iota_{\mathcal{A}}(\nabla u) \) where \( \iota_{\mathcal{A}} \in \mathcal{L}(V \otimes \mathbb{R}^n; W) \). Explicitly we illustrate the relation by means of a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{R}^n \times V & \mathcal{A} & V \otimes \mathbb{R}^n \\
\otimes & & \downarrow \iota_{\mathcal{A}} \\
& \mathbb{R}^n \otimes_{\mathcal{A}} V &
\end{array}
\]

Here \( \otimes \) stands for the usual euclidean tensor pairing whereas for \( \xi \in \mathbb{R}^n \) and \( \sigma \in V \) we declare \( \xi \otimes_{\mathcal{A}} \sigma := \sum_{j=1}^{n} \xi_j A_j \sigma \). The space \( \mathcal{C}(\mathcal{A}) := \{ \xi \otimes_{\mathcal{A}} \sigma : \xi \in \mathbb{R}^n, \sigma \in V \} \) is called the \( \mathcal{A} \)-rank one cone. Moreover we associate a Fourier symbol mapping to \( \mathcal{A} \) by writing the linear combination

\[
A[\xi] \sigma := \xi \otimes_{\mathcal{A}} \sigma = \sum_{j=1}^{n} \xi_j A_j \sigma, \quad \text{for } \xi \in \mathbb{R}^n \text{ and } \sigma \in V.
\]

An operator \( \mathcal{A} \) is thought of being real elliptic as long as for all \( \xi \in \mathbb{R}^n \setminus \{0\} \), the Fourier symbol \( A[\xi] : \sigma \mapsto A[\xi] \sigma \) is an injective map from \( V \) to \( W \). By the nullspace of \( \mathcal{A} \) we mean the vector subspace \( \ker(\mathcal{A}) := \{ u \in \mathcal{D}'(\mathbb{R}^n; V) : \mathcal{A}u = 0 \} \). Here \( \mathcal{D}'(\mathbb{R}^n; V) \) denotes the class of all \( V \)-valued tempered distributions on \( \mathbb{R}^n \) i.e. all bounded functionals on the space \( \mathcal{D}(\mathbb{R}^n; V) := C_c^\infty(\mathbb{R}^n; V) \). The equality in the parenthesis is regarded in the distributional sense.

A differential operator \( \mathcal{A} \) from \( W \) to some finite-dimensional vector space \( Z \) is an annihilator of \( \mathcal{A} \) if the corresponding Fourier symbol maps satisfy the equality \( \ker A[\xi] = A[\xi](V) \) for any \( \xi \in \mathbb{R}^n \setminus \{0\} \). In other words one requires exactness of the symbol complex:

\[
V \xrightarrow{A[\xi]} W \xrightarrow{A[\xi]} 0.
\]

In such an occurrence we also say that \( \mathcal{A} \) is the potential of \( \mathcal{A} \). Existence of annihilators for real elliptic operators is reassured, see e.g. [36]. However let us emphasize that annihilators of first order potentials do not necessarily have to be of the same order e.g the annihilator of the symmetrised gradient, see Example 2.2 (ii), is curl \( \circ \) curl. In terms of the tensoric formulation, by combining with the exact sequence condition it is immediate that the \( \mathcal{A} \)-rank one cone \( \mathcal{C}(\mathcal{A}) \) must coincide with the characteristic cone of \( \mathcal{A} \), that is \( \Lambda_{\mathcal{A}} := \bigcup_{\xi \neq 0} \ker A[\xi] \):

\[
\mathcal{C}(\mathcal{A}) = \Lambda_{\mathcal{A}}
\]

**Definition 2.1** \((\mathbb{C}\text{-ellipticity})\). Let \( V_\mathbb{C} := V + iV \) and \( W_\mathbb{C} := W + iW \) be the usual complexifications of the vector spaces \( V \) and \( W \) respectively. For \( \xi \in \mathbb{C}^n \) let \( A[\xi] \) be the canonical extension as a \( \mathbb{C} \)-linear map from \( V_\mathbb{C} \) to \( W_\mathbb{C} \). We say that an operator \( A \) is complex elliptic or \( \mathbb{C} \)-elliptic if for all \( \xi \in \mathbb{C}^n \setminus \{0\} \) the Fourier symbol \( A[\xi] \) is injective as a mapping between \( \mathbb{C} \)-vector spaces.
This algebraic notion of elliptic behaviour is due to Aronszajn [5]. An equivalent formulation from an analytic perspective is constituted in the finite-dimensionality of \(\ker(\mathcal{A})\) see [24, Prop. 3.1]. In fact we may distinguish the following correspondence:

\[
\dim \ker(\mathcal{A}) < +\infty \iff \{\text{complex ellipticity of } \mathcal{A}\} \Rightarrow \{\text{real ellipticity of } \mathcal{A}\}.
\]

Notice that in the latter relation one can only afford the forward implication, for there exist \(\mathbb{R}\)-elliptic operators which do not admit finite-dimensional nullspace. For instance if we identify \(\mathbb{R}^2 \cong \mathbb{C}\), the trace-free symmetric gradient \(\mathcal{E}^D\) contains all holomorphic maps in its kernel.

To become apparent shortly, the algebraic assertion of \(\mathbb{C}\)-ellipticity is quintessential in carrying out the homogenisation arguments as it permits the requisite functional relaxations and pertains to fine properties of functions of bounded \(\mathcal{A}\)-variations.

Let us fix throughout the course of our analysis the target vector space \(W\) to be \(\mathcal{R}(\mathcal{A}) := \text{span}\{\mathcal{E}(\mathcal{A})\}\), the effective range associated to the operator \(\mathcal{A}\). This represents a linear combination of all \(\mathcal{A}\)-tensoric pairings which in turn encapsulates the minimal cone containing the codomain of the Fourier symbol of \(\mathcal{A}\) as linear mapping. Let us conclude this section by enumerating a selection of differential operators that satisfy the \(\mathbb{C}\)-elliptic property.

**Example 2.2.** (i) **Full gradient:** for \(V = \mathbb{R}^n\) and \(W = \mathbb{R}^m\), the set \(\ker(\nabla)\) is made out of all real \((m \times n)\)-matrices.

(ii) **Symmetric gradient:** \(e(u) := \frac{1}{n}(\nabla u + (\nabla u)^\top)\) for \(V = \mathbb{R}^n\) and \(W = \mathbb{R}^{n \times n}\). The nullspace \(\ker(e)\) is given by the space of rigid deformations \(\mathcal{R} := \{x \mapsto Ax + b : A \in \mathbb{R}^{n \times n}_{\text{skew}}, b \in \mathbb{R}^n\}\) which is finite-dimensional.

(iii) **Trace-free symmetric gradient:** \(e^D u := e(u) - \frac{1}{n} \text{div}(u) I_n\) where \(I_n\) is the \((n \times n)\)-identity matrix. This is a degenerate in the sense that \(\mathbb{C}\)-ellipticity persists for \(n \geq 3\). The nullspace is given by the so called conformal killing vectors. However when inspecting \(n = 2\) the map \(z \mapsto (1 - z)^{-1}\) on the disc \(B_1(0)\) with \(\mathbb{R}^2 \cong \mathbb{C}\) sends all holomorphic maps to \(\ker(e^D)\).

(iv) For \(V = \mathbb{R}^2\) consider the operator \(\mathcal{B}u = (\partial_1 u_2 + \partial_2 u_1, \partial_2 u_2, \partial_1 u_1)\). Then by inspecting the nullity \(\mathbb{B}[\xi][v] = 0\) it follows that \(\mathbb{B}\) is \(\mathbb{C}\)-elliptic.

(v) **Hodge-type operator:** \(\mathcal{D} u := (du, d^*u)\) for \(u : \Lambda^2 \mathbb{R}^n \to \Lambda^3 \mathbb{R}^n \times \Lambda^{n-1} \mathbb{R}^n\) is \(\mathbb{C}\)-elliptic as the Fourier symbol in this case is given by \(\mathcal{D}[\xi][\sigma] = (\xi \wedge \sigma, \star(\xi \wedge \star\sigma))\) where \(\star\) denotes the usual Hodge-star operator.

(vi) If \(\mathbb{L}\) is any of the above operators and \((\lambda_1, \ldots, \lambda_N) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}^{N-1}\), then the operator \(\mathbb{K} := (\lambda_1 \mathbb{L}, \ldots, \lambda_N \mathbb{L})\) has a finite-dimensional nullspace.

### 2.3 Function spaces.

Let \(U \subset \mathbb{R}^n\) be open and bounded and let \(\mathcal{A}\) be a fixed \(\mathbb{C}\)-elliptic differential operator from \(V\) to \(W\). The generalised \(\mathcal{A}\)-Sobolev spaces are formulated as:

\[
W^{\mathcal{A},p}(U) := \{ u : U \to V : u \in L^p(U; V), \mathcal{A}u \in L^p(U; W) \}.
\]

Endowed with the norm \(\| \cdot \|_{W^{\mathcal{A},p}(U)} := \| \cdot \|_{L^p(U; V)} + \|\mathcal{A}(\cdot)\|_{L^p(U; W)}\), this space becomes Banach and separable i.e. we have an isometry \(W^{\mathcal{A},p}(U) \to L^p(U; V) \times L^p(U; W)\) given by \(u \mapsto (u, \mathcal{A}u)\). Let us denote by \(W^{\mathcal{A},p}_0(U)\) the closure of \(C_0^\infty(U; V)\) in the \(\|\mathcal{A}(\cdot)\|_{L^p(U; W)}\) seminorm.

We define the space of maps of bounded \(\mathcal{A}\)-variations as

\[
\text{BV}^\mathcal{A}(U) := \{ u \in L^1(U; V) : \mathcal{A}u \in \mathcal{M}(U; W) \}.
\]
where $\mathcal{M}(U; W)$ is the collection of finite $W$-valued Radon measures. The associated total variation of the measure $Au$ is understood through the duality pairing

$$|Au|(U) = \sup \left\{ \int_U \langle u, A^* \phi \rangle \, dx : \phi \in C^1_c(U, W), \|\phi\|_{L^\infty} \leq 1 \right\},$$

where $A^* = \sum_{j=1}^n A^*_j \partial_j$ is the formal adjoint of $A$. Importantly equipping $BV^\Lambda(\Omega)$ with the natural norm

$$\|u\|_{BV^\Lambda(U)} := \|u\|_{L^1(U; V)} + |Au|(U)$$

renders it a Banach space, this being an analogous feature to the full gradient case. For any $\nu \in \mathcal{M}(U; W)$ the Lebesgue-Radon-Nikodym decomposition is written as $\nu = \nu^a + \frac{d\nu}{d|\nu^s|} |\nu^s|$ for $\nu^a \ll \mathcal{L}^n$ and $\nu^s \perp \mathcal{L}^n$. Likewise for the differentially constrained measures $Au$, we may perform the decomposition with respect to the Lebesgue measure $\mathcal{L}^n$ as follows:

$$A^a u = A^{ac} \nu d\mathcal{L}^n \mathcal{L} U + \frac{dA^a u}{d\mathcal{L}^n} d\mathcal{L}^n \mathcal{L} U + \mathcal{A} u \mathcal{L} u$$

where $\mathcal{A} u = \{ x \in U : \lim_{\varrho \to 0^+} \varrho^{-n} |Au|(B_\varrho(x)) = +\infty \}$ and $B_\varrho(x)$ is a ball centred at $x$ with radius $\varrho > 0$.

Let us enumerate a couple of essential function theoretic results that are reminiscent of the BV-theory. Firstly we bring up the statement describing approximation of $W^{1,1}$-spaces by smooth maps up to the boundary, see also [24 Lem 5.5].

**Proposition 2.3** (Global smooth approximation). Let $U \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary. Then $C^\infty(\overline{U}; V)$ is a dense subset of $W^{1,1}(U)$ in the norm topology.

**Proof.** We begin by observing that $\partial U$ can be covered by a finite collection of balls $\{B_i\}_{i=1}^\ell$. We choose an associated partition of unity $\varphi_i \in C^\infty_c(B_i; [0, 1])$ and $\varphi_0 \in C^\infty_c(U; [0, 1])$ such that $\sum_{i=0}^\ell \varphi_i = 1$. Let $\delta_k > 0$ be such that $\delta_k \to 0$ as $k \to \infty$ and let $\eta_{\delta_k} : \mathbb{R}^n \to \mathbb{R}$ be a standard mollifying kernel. Setting $u^i_k := \eta_{\delta_k} * (\varphi_i u) \in C^\infty_c(\overline{U}; V)$ for $i = 1, \ldots, \ell$ it follows that $\text{supp}(u^i_k) \subset \overline{U} \cap B_i$ and $\|u^i_k - \varphi_i u\|_{W^{1,1}(U \setminus B_i)} < 2^{-k/\ell}$ for $k$ large enough. On the other hand from the local smooth approximation [10] Thm 2.8 (a) we may find a map $u^0_k \in C^\infty_c(U; V)$ such that $\|u^0_k - \varphi_i u\|_{W^{1,1}(U)} < 2^{-k/\ell}$. Altogether if we define $u_k = \sum_{i=0}^\ell u^i_k \in C^\infty_c(U; V)$, then

$$\|u - u_k\|_{W^{1,1}(U)} < \frac{1}{2^k}$$

and this finishes the proof by passing to $k \to \infty$. \hfill $\Box$

Following the theorem of Banach-Alaoglu for reflexive normed spaces there is a corresponding compactness result phrased for $A$-variations proof of which is omitted.

**Proposition 2.4.** Compactness: Let $U \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary. If $(u_k) \subset BV^\Lambda(U)$ is uniformly bounded in the $BV^\Lambda$-norm, then there exists $u \in BV^\Lambda(U)$ and a subsequence $(u_{k_j})$ such that $u_{k_j} \to u$ in $L^1(U; V)$ and $Au_{k_j} \rightharpoonup^* Au$ weakly* as Radon measures i.e.

$$\int_U \phi \, dAu_{k_j} \to \int_U \phi \, dAu$$

for any $\phi$ in the closure of $C^0_c(U; V)$ with respect to the supremum norm.
Subsequently let us present a pair of key estimates designed as specifically rectified Poincaré inequalities for \(BV^A\)-maps, a concept extensively studied for instance in [23, 24]. These shall turn out instrumental for the forthcoming construction of blow-up sequences. Thereupon we recall the algebraic interpretation in [10, Theorem 2.6] explicitly shows that \(\mathcal{C}\)-ellipticity property of the operator \(A\) results in the finite-dimensionality of its nullspace. More concretely there exists an integer \(m = m(A) \in \mathbb{N}\) such that \(\ker(A)\) is a subspace of \(\mathcal{P}_m\), the space of polynomials of order at most \(m\) valued in \(V\) for which we select a finite polynomial basis \(p_1, \ldots, p_{M_k}\). Given an open, bounded and connected set \(O \subset \mathbb{R}^n\), we define \(\pi_O : L^2(O; V) \to \ker(A)\) the \(L^2\)-projection onto \(\ker(A)\) by

\[
\pi_O u = \sum_{i=1}^{M_k} \langle u, \hat{p}_i \rangle_{L^2(O; V)} \hat{p}_i, \quad \text{for } u \in L^2(O; V)
\]

where \(\hat{p}_1, \ldots, \hat{p}_{M_k}\) are the orthonormal basis elements with respect to the \(L^2(O; V)\) inner product obtained from \(p_1, \ldots, p_{M_k}\). Using the inverse estimates [23, Eq.(2.6), (2.7)] and boundedness of the set \(O\), it is possible to extend \(\pi_O\) for it to act boundedly on \(L^1(O; V)\). Additionally such an extended projection satisfies the \(L^1\)-stability, meaning there exists a constant \(c = c(n, A) > 0\) such that

\[
\int_O |\pi_O u| \, dx \leq c \int_O |u| \, dx \quad \forall u \in L^1(O; V).
\]

Notice also that if \(U \subset O\) is an open subset, then \(\pi_O \circ \pi_U = \pi_U\).

**Proposition 2.5.** (A-Poincaré inequality for annuli) Let \(t \in (0, 1)\) be a fixed number. Let us define the \(t\)-annulus by \(Q^t := Q \setminus (1-t)Q\). Then there exist a constant \(c = c(n, A)\) such that

\[
\int_{Q^t} |u - \pi_{Q^t} u| \, dx \leq c |A \mathcal{H}^n(Q^t) / L^n(Q^t)|
\]

for all \(u \in BV^A(Q)\).

**Proof.** We divide the annulus \(Q^t\) into cubes \(\{Q_i\}_{i \in \mathcal{I}}\) of side length proportional to \(t\) in a manner that \(L^n(Q_i \cap Q_j) = 0\) for \(i \neq j\). Notice that each \(Q_i\) is star-shaped with respect to a ball of radius comparable to \(t\). Simultaneously we may choose a finite union of rectangles \(\{R_1, \ldots, R_M\}\) where \(M\) depends only on dimension \(n\) such that for each \(j\) the quantity \(\mathcal{L}^n(R_j)\) is proportional to \(\mathcal{L}^n(Q^t)\) and \(\mathcal{L}^n(Q^t \setminus \bigcup_{j=1}^M R_j) = 0\). Incorporating these decompositions one has

\[
\int_{Q^t} |u - \pi_{Q^t} u| \, dx \leq \mathcal{L}^n(Q^t)^{-1} \sum_{i \in \mathcal{I}} \left( \int_{Q_i} |u - \pi_{Q_i} u| \, dx + \int_{Q_i} |\pi_{Q^t} u - \pi_{Q_i} u| \, dx \right).
\]

In view of \(L^1\)-stability in (2.6) and the fact that \(\pi_Q \circ p = p\) for all \(p \in \ker(A)\), the hypotheses of [24, Prop. 4.2], [23, Cor. 2.2] ensure existence of a constant \(C = C(n, A) > 0\) such that

\[
\|u - \pi_{Q_i} u\|_{L^1(Q_i; V)} \leq C t |A \mathcal{H}^n(Q_i)|.
\]

With regards to the second summand, for every \(i \in \mathcal{I}\) and every \(j \in \{1, \ldots, M\}\) let \(\sigma_{ij} : Q_i \to R_j\) be a spatial transformation mapping and select \(R_k \in \{R_1, \ldots, R_M\}\) such that \(\|\tilde{u} - \pi_{Q_i} \tilde{u}\|_{L^1(R_k; V)} = \max_j \|\tilde{u} - \pi_{Q_i} \tilde{u}\|_{L^1(R_k; V)}\) where \(\tilde{u} := u \circ \sigma^{-1}_{ik}\). Since \(\pi_{Q^t} \circ \pi_{Q_i} = \pi_{Q_i}\), we therefore see that

\[
\int_{Q_i} |\pi_{Q^t} u - \pi_{Q_i} u| \, dx = \int_{Q_i} |\pi_{Q^t} (u - \pi_{Q_i} u)| \, dx \leq c_n \mathcal{L}^n(Q_i) \int_{R_k} |\pi_{Q^t} (\tilde{u} - \pi_{Q_i} \tilde{u})| \, dx
\]

\[
\leq c_n, A \mathcal{L}^n(Q_i) \mathcal{L}^n(Q^t) \int_{R_k} |\tilde{u} - \pi_{Q_i} \tilde{u}| \, dx \leq c_n, A \int_{Q_i} |u - \pi_{Q_i} u| \, dx.
\]
In the penultimate inequality we have used the $L^1$-stability property \eqref{2.6} of $\pi_{Q'}$ as well as the change of variable twice in between. Altogether we conclude our calculation with the estimate:

$$\int_{Q'} |u - \pi_{Q'} u| \, dx \leq \frac{1 + c_n \Lambda}{\mathcal{L}^n(Q')} \sum_{i \in I} \int_{Q_i} |u - \pi_{Q_i} u| \, dx \leq c\ell \sum_{i \in I} \|A_u(Q_i)\|_1 \mathcal{L}^n(Q').$$

□

**Proposition 2.6** (Poincaré for $W^{1,1}_0$). Let $U \subset \mathbb{R}^n$ be open and bounded. Then there exists a constant $C = C(n, \Lambda)$ such that

$$\|u\|_{L^1(U; V)} \leq C\text{diam}(U)\|A u\|_{L^1(U; W)}.$$  

for all $u \in W^{1,1}_0(U)$.

*Proof.* By smooth approximation it suffices to consider $u \in C_c^{\infty}(U; V)$. Applying the Fourier transform we may write $\hat{u}(\xi) = k_\Lambda(\xi) \Lambda[\xi]\hat{u}(\xi)$ where $k_\Lambda(\xi) = \Lambda[\xi]^{*} \circ (\Lambda[\xi] \circ \Lambda[\xi]^{*})^{-1}$ and thus $u = K_\Lambda \ast A u$ where $K_\Lambda$ is an $(1-n)$-homogeneous kernel induced from the Fourier inversion. Therefore using the Young convolution inequality along with $L^p$-boundedness of Riesz potentials on bounded domains we calculate the $L^1$-bounds through

$$\|u\|_{L^1(U; V)} = \|K_\Lambda \ast A u\|_{L^1(U; V)} \leq \|K_\Lambda\|_{L^1(U)}\|A u\|_{L^1(U; W)} \leq C(n, \Lambda)\|\cdot|^{1-n}\|_{L^1(U)}\|A u\|_{L^1(U; W)} \leq C(n, \Lambda)\text{diam}(U)\|A u\|_{L^1(U; W)}.$$

□

When dealing with the blow-up type techniques in the context of homogenisations of integral functionals, it is essential to address a notion of differentiability of the blow-up sequence elements. Building upon Alberti’s contributions \cite{4}, the works presented in \cite{25} accurately indicate the coherent framework to interpret Lebesgue differentiability in the setup of $BV^{A}$. For our purposes we recount \cite{25} Thm 2.3, Lem. 3.1]

**Proposition 2.7** ($L^p$-differentiability). Let $u \in BV^{A}_{loc}(\mathbb{R}^n)$ and $p \in [1, 1^*)$. Then

1. $u$ is $L^p$-differentiable for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$, that is,

$$u(y) = \nabla u(x)(y - x) + u(x) + \mathcal{R}(x, y)$$

such that $\langle |\mathcal{R}(x, \cdot)|^p \rangle_{Q_r(x)} = o(r^p)$ as $r \to 0^+$, the first order term Taylor expansion in $L^p$.

2. for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$ there holds

$$\frac{dA u}{d\mathcal{L}^n}(x) = A[\nabla]u(x).$$

The term $A[\nabla]u$ symbolises the induced linear map from the constant coefficients of $A$ acting on the approximate gradient of $u$ i.e. $A[\nabla]u = t_A(\nabla u)$.

2.4. **On the $A$-tensorial rank-one counterparts.** A foundational result of Alberti \cite{3} states that in $BV$-spaces the densities of singular parts of distributional gradients are represented pointwisely up to equivalence as the euclidean tensor pairing of two vectors in the underlying spaces. In the context of more general differential classes, De Philippis and Rindler \cite{17} recently derived other counterparts of alike phenomenon and discovered that there is a strong connection of singular densities with nullities of differential operators:
Theorem 2.8. [17 Thm 1.2] Let $U \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary and let $K, N \in \mathbb{N}$. Suppose that $A$ is a linear differential operator of order $K$ i.e. $A := \sum_{|\alpha| \leq K} A_\alpha \partial^\alpha$ for linear maps $A_\alpha : W \to \mathbb{R}^N$. If $\nu \in \mathcal{M}(U; W)$ is such that $A\nu = 0$ in the sense of distributions, then

$$
\frac{d\nu}{d|\nu|}(x_0) \in \left( \bigcup_{\xi \neq 0} \ker A[\xi] \right) \cap S^{n-1}
$$

holds for $|\nu^\gamma|\cdot\text{a.e. } x_0 \in U$.

That is to say, the singular density of $\nu$ is representable by unit vectors contained in the wave cone $\Lambda_A \cap S^{n-1}$ of $A$. Crucially transmitting this relation into the realm of the exact sequence chains of symbols $\Lambda[\xi]$ and $A[\xi]$, we obtain an algebraic characterisation of the singular part of measure $A\nu$.

More explicitly in view of (2.4) and the homological equality $\ker A[\xi] = \Lambda[\xi](V)$ we may assert the existence of measurable maps $\xi, \sigma : U \to S^{n-1}$ such that for $|\Lambda^\gamma u|\cdot\text{a.e. } x \in U$

$$
(2.7) \quad \frac{dA^\gamma u}{d|A^\gamma u|}(x) = \xi(x) \otimes_\Lambda \sigma(x),
$$

and in consequence expand on the overall pointwise description of $A$-variations.

2.5. Admissible integrands and relaxations in $\text{BV}^A$. Recall we have declared $W := \mathcal{K}(\Lambda)$ which is the smallest cone containing the image of all admissible vector fields $u : U \to V$ under $\Lambda$. The family of integrands within our interest is manifested as follows. We say that a function $g : \mathbb{R}^n \times \mathcal{K}(\Lambda) \to [0, +\infty)$ is in the class $\mathcal{F}$ if:

(A) $g$ is of linear growth: there exist constants $\alpha, \beta > 0$ such that

$$
\alpha |A| \leq g(x, A) \leq \beta (1 + |A|)
$$

for all $x \in \mathbb{R}^n$, $A \in \mathcal{K}(\Lambda)$,

(B) it satisfies modulus of continuity: there exist a constant $c_1 > 0$ as well as a continuous, non-decreasing function $\rho : [0, +\infty) \to [0, +\infty)$ such that $\rho(0) = 0$ and

$$
|g(x, A_1) - g(x, A_2)| \leq \rho(|A_1 - A_2|) (g(x, A_1) + g(x, A_2)) + c_1 |A_1 - A_2|
$$

for all $x \in \mathbb{R}^n$, $A_1, A_2 \in \mathcal{K}(\Lambda)$.

(C) the function $g^\infty(x, A) := \lim_{t \to \infty} t^{-1} g(x, tA)$ is well-defined and there exists $\gamma \in (0, 1)$ such that for all $x \in \mathbb{R}^n$, all $A \in \mathcal{K}(\Lambda)$ with $|A| = 1$ and for every $\lambda > 0$ there exists $c_2 > 0$ such that

$$
|g^\infty(x, A) - t^{-1} g(x, tA)| \leq \frac{c_2}{t^{\gamma}}
$$

for all $t \geq \lambda$.

On the account of linear growth assumption, the lack of coercivity eventually triggers any apparent $\Gamma$-limit of a sequence of linear growth functionals to attain its domain in a space strictly larger than $W^{\Lambda,1}(U)$. Suppose that $h : \mathcal{K}(\Lambda) \to \mathbb{R}$ be a representative of $\mathcal{F}$ additionally satisfying the $\Lambda$-quasiconvexity property:

$$
h(A) = \inf \left\{ \int_{(0,1)^n} h(A + \Lambda \varphi(y)) \, dy : \varphi \in W^{\Lambda,1}_0((0,1)^n) \right\}
$$

for all $A \in \mathcal{K}(\Lambda)$. In the light of exactness in (2.3) and existence of potentials the above definition is in fact equivalent to the foundational notion of $A$-quasiconvexity of Fonseca and Müller [21].
The growth bounds on $h$ view of [10, Prop. 5.1] for any $h$ given by $\text{semicontinuous envelope of } H$ implies that $\text{minimising sequences are relatively compact.}$ In precise terms we define $H^* : BV^h(U) \to [0, +\infty]$ by

$$H^*[u] = \inf \{ \liminf_{k \to \infty} H[u_k] : (u_k) \subset W^{1,1}(U), u_k \rightharpoonup u \text{ in } BV^h(U) \} \quad \forall u \in BV^h(U).$$

The growth bounds on $h$ imply that $H^*$ is in fact the same as the customarily used $L_1$-lower semicontinuous envelope of $H$. Among a plethora of contributions, in the work of Breit, Dening and Gmeineder [10], see also earlier accounts in BD by Kristensen and Rindler [30], we find an explicit integral representation of $H^*$. In short the assembling terms assimilate the Lebesgue-Radon-Nikodym decomposition of $\Lambda$-variations with augmentation by the recession function $h^\infty : \mathcal{R}(\Lambda) \to \mathbb{R}$ given by $h^\infty(A) = \lim_{t \to \infty} t^{-1} h(tA)$ to entail the behaviour of singular parts of $\Lambda$. In particular in view of [10] Prop. 5.1 for any $u \in BV^h(U)$ the following equality holds

$$H^*[u] = \int_U h(\Lambda[\nabla]u) \, dx + \int_U h^\infty(\frac{d\Lambda^s}{d|\Lambda^s|} u) \, d|\Lambda^s|.$$  

In consequence the expression on the right-hand side of the above equality is automatically sequentially weak*-lower semicontinuous on $BV^h(U)$.

In case when $h : \mathcal{R}(\Lambda) \to [0, +\infty)$ is not assumed $\Lambda$-quasiconvex, then the weak*-envelope of $H$ is instead represented by

$$H^*[u] = \int_U \mathcal{L}_h h(\Lambda[\nabla]u) \, dx + \int_U \mathcal{L}_h h^\infty(\frac{d\Lambda^s}{d|\Lambda^s|} u) \, d|\Lambda^s|,$$

where $\mathcal{L}_h h : \mathcal{R}(\Lambda) \to [0, +\infty)$ is the $\Lambda$-quasiconvex envelope of $h$ given by

$$\mathcal{L}_h h(A) := \inf \left\{ \int_{(0,1)^n} h(A + \Lambda \varphi(y)) \, dy : \varphi \in W^{1,1}_0((0,1)^n) \right\}$$

for all $A \in \mathcal{R}(\Lambda)$.

2.6. Probabilistic setting. Let $(\Omega, \mathcal{T}, \mathbb{P})$ be a probability space and let $(\tau_z)_{z \in \mathbb{Z}^n}$ be a group of $\mathbb{P}$-preserving transformations. More precisely we consider a collection of maps $\tau_z : \Omega \to \Omega$ such that for all $z \in \mathbb{Z}^n$

- $\tau_z$ is $\mathcal{T}$-measurable
- $\tau_z$ is a bijection
- $\mathbb{P}(\tau_z(A)) = \mathbb{P}(A)$ for all $A \in \mathcal{T}$
- $\tau_z \circ \tau_{z'} = \tau_{z+z'}$ and $\tau_0 = \text{id}_\Omega$.

We say that the group $(\tau_z)_{z \in \mathbb{Z}^n}$ is ergodic if every element $E \in \mathcal{T}$ such that $\tau_z(E) = E$ for all $z \in \mathbb{Z}^n$ attains probability 0 or 1. The prescribed variety of functions for homogenisation shall additionally exhibit selected stochastic properties.
Definition 2.9 (Stationary random integrands). We say that \( f : \Omega \times \mathbb{R}^n \times \mathfrak{H}(\mathbb{A}) \rightarrow [0, +\infty) \) is a random integrand if

- \( f \) is \((\mathcal{T} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathfrak{H}(\mathbb{A}))\)-measurable.
- For \( \mathbb{P}\)-a.e. \( \omega \in \Omega \) we have that \( f(\omega, \cdot, \cdot) \in \mathcal{F} \).

Moreover \( f \) is a stationary random integrand if along with the conditions above, it also satisfies stationarity:

- \( f(\omega, x+z, A) = f(\tau_z(\omega), x, A) \) for all \( x \in \mathbb{R}^n \), \( A \in \mathfrak{H}(\mathbb{A}) \), \( z \in \mathbb{Z}^n \) and \( \omega \in \Omega \).

Definition 2.10 (Subadditive process). A subadditive process with respect to \(((\tau_z)_{z \in \mathbb{Z}^n})\) is a set function \( \mu : \mathcal{A}(\mathbb{R}^n) \rightarrow L^1(\Omega, \mathcal{T}, \mathbb{P}) \) satisfying:

- Additivity: for any \( U \in \mathcal{A}(\mathbb{R}^n) \) and any finite, pairwise disjoint collection of subsets of \( U \), \( \{ U_i \}_{i=1}^M \in \mathcal{A}(\mathbb{R}^n) \) such that \( \mathcal{L}^n(U \setminus \bigcup_{i=1}^M U_i) = 0 \) there holds \( \mu(\omega, U) \leq \sum_{i=1}^M \mu(\omega, U_i) \) for all \( \omega \in \Omega \).

- Covariance: for all \( U \in \mathcal{A}(\mathbb{R}^n) \), \( \omega \in \Omega \) and \( z \in \mathbb{Z}^n \), \( \mu(\omega, U+z) = \mu(\tau_z \omega, U) \).

(c) there exists a constant \( C > 0 \) such that \( 0 \leq \mu(\omega, U) \leq C \mathcal{L}^n(U) \) for every \( U \in \mathcal{A}(\mathbb{R}^n) \) and \( \omega \in \Omega \). In case \(((\tau_z)_{z \in \mathbb{Z}^n})\) is ergodic, then we say that the process \( \mu \) is ergodic.

The following result due to \([2, 15]\) depicts the pointwise asymptotic behaviour of subadditive processes.

Theorem 2.11. \([15]\) Prop. 1\] Let \( \mu : \mathcal{A}(\mathbb{R}^n) \rightarrow L^1(\Omega, \mathcal{T}, \mathbb{P}) \) be a subadditive process with respect to \(((\tau_z)_{z \in \mathbb{Z}^n})\). Then there exists a \( \mathcal{T}\)-measurable function \( h : \Omega \rightarrow [0, +\infty) \) such that for every \( x \in \mathbb{R}^n \), \( r > 0 \) and for \( \mathbb{P}\)-a.e. \( \omega \in \Omega \) we have that

\[
h(\omega) = \lim_{\varepsilon \to 0} \frac{\mu(\omega, Q_{\varepsilon}(x))}{\mathcal{L}^n(Q_{\varepsilon}(x))}.
\]

If \( \mu \) is ergodic, then the map \( h \) is constant.

For each \( A \in \mathfrak{H}(\mathbb{A}) \) and \( U \in \mathcal{A}(\mathbb{R}^n) \) we define \( \mu^A(\cdot, U) \in L^1(\Omega, \mathcal{T}, \mathbb{P}) \) by

\[
\mu^A(\omega, U) := m_f(\omega, A, U) := \inf \left\{ \int_U f(\omega, x, A + \lambda v) \, dx : v \in W_{0,1}^A(U) \right\}
\]

Crucially for admissible integrands \( f \) the volume density process is subadditive:

Proposition 2.12. Let \( f : \Omega \times \mathbb{R}^n \times \mathfrak{H}(\mathbb{A}) \rightarrow [0, +\infty) \) be a stationary random integrand. Then for all \( A \in \mathfrak{H}(\mathbb{A}) \) the mapping \( \mu^A : \mathcal{A}(\mathbb{R}^n) \rightarrow L^1(\Omega, \mathcal{T}, \mathbb{P}) \) is a subadditive process and in addition we have the bound \( 0 \leq \mu^A(\cdot, U) \leq \beta(1 + |A|) \mathcal{L}(U) \).

Proof. Let \( A \in \mathfrak{H}(\mathbb{A}) \) and \( U \in \mathcal{A}(\mathbb{R}^n) \). Firstly observe that \( \omega \mapsto m_f(\omega, A, U) \) is measurable since the separability of \( W_{0,1}^A \)-spaces ensures existence of a set \( \mathcal{D} \subset W_{0,1}^A(U) \) such that

\[
\mu^A(\cdot, U) = \inf \left\{ \int_U f(\cdot, x, A + \lambda v) \, dx : v \in \mathcal{D} \right\}.
\]

We now proceed in verifying all conditions of Definition 2.10.

For property (a) we take a collection \( U_1, \ldots, U_M \in \mathcal{A}(\mathbb{R}^n) \) of pairwise disjoint subsets of \( U \) such that...
\( \mathcal{L}^n(U \setminus \bigcup_{i=1}^M U_i) = 0 \). Now for any \( \delta > 0 \) let \( u^\delta_i \in W^{1,1}_0(U_i) \) be such that \( \int_{U_i} f(\omega, x, Au_i + A) \, dx \leq m_f(\omega, A, U_i) + \delta/M \). Then defining \( u := \sum_{i=1}^M u_i \chi_{U_i} \), it follows that
\[
\mu^A(\omega, U) \leq \sum_{i=1}^M \int_{U_i} f(\omega, x, Au_i + A) \, dx \leq \sum_{i=1}^M \mu^A(\omega, U_i) + \delta.
\]
and the claim follows by passing to \( \delta \to 0 \). As to (b), applying the change of variable and invoking stationarity property of \( f \) we see that
\[
m_f(\omega, A, U + z) = \inf \left\{ \int_U f(\tau_z \omega, x, A + Av) \, dx : v \in W^{1,1}_0(U) \right\} = m_f(\tau_z \omega, A, U)
\]
and therefore \( \mu^A(\cdot, U) \) is covariant. Lastly the pointwise bounds are clear from the integral functional appearing in the parenthesis of the above infimum. Hence \( \mu^A(\cdot, U) \) is a subadditive process. \( \square \)

We denote \( T' \) the \( \sigma \)-subalgebra of \( T \) containing all \( \{\tau_z\} \)-invariant sets, meaning that \( \mathbb{P}(\tau_z(A)) = \mathbb{P}(A) \) \( \forall A \in T' \). As \( m_f \) satisfies the hypotheses of Theorem \( 2.11 \) a direct consequence emerges:

**Proposition 2.13.** With the assertions of Proposition 2.12 there exists a homogeneous random integrand \( f_{\text{hom}} : \Omega \times \mathcal{H}(\mathbb{A}) \to [0, +\infty) \) realised by the limit
\[
f_{\text{hom}}(\omega, A) = \lim_{\epsilon \to 0} \frac{m_f(\omega, A, \epsilon^{-1}Q_r(x))}{\mathcal{L}^n(\epsilon^{-1}Q_r(x))} = \lim_{\epsilon \to 0} \frac{m_f(\omega, A, Q_{\sqrt{\epsilon}})}{\mathcal{L}^n(Q_{\sqrt{\epsilon}})}
\]
for all \( r > 0, x \in \mathbb{R}^n \) and for \( \mathbb{P} \text{-a.e.} \ \omega \in \Omega \). Moreover the volume integrand \( f_{\text{hom}} \) exhibits the following characterisation:
\[
f_{\text{hom}}(\omega, A) = \inf_{k \in \mathbb{N}} k^{-n} \mathbb{E}[m_f(\omega, A, kQ)|T']
\]
for \( \mathbb{P} \text{-a.e.} \ \omega \in \Omega \) where the term \( \mathbb{E}[m_f(\omega, A, kQ)|T'] \) denotes the conditional expectation of the random variable \( \omega \mapsto m_f(\omega, A, kQ) \) given \( T' \).

**Proof.** Firstly let \( A \in \mathcal{H}(\mathbb{A}) \). Then as \( \mu^A(\cdot, U) \) is a subadditive process, cf. Proposition 2.12, we may apply Theorem 2.11 to find a set \( \Omega_A \in T \) such that \( \mathbb{P}(\Omega_A) = 1 \) as well as a \( T \)-measurable function \( h_A : \Omega \to [0, +\infty) \) with
\[
h_A(\omega) = \lim_{\epsilon \to 0} \frac{m_f(\omega, A, Q_{\sqrt{\epsilon}})}{\mathcal{L}^n(Q_{\sqrt{\epsilon}})} \quad \forall \omega \in \Omega_A.
\]
Consequently let us define \( f_{\text{hom}} : \Omega \times \mathcal{H}(\mathbb{A}) \to [0, +\infty) \) by
\[
f_{\text{hom}}(\omega, A) = \limsup_{\epsilon \to 0} \frac{m_f(\omega, A, Q_{\sqrt{\epsilon}})}{\mathcal{L}^n(Q_{\sqrt{\epsilon}})}.
\]
As for property (B) let \( A, B \in \mathcal{H}(\mathbb{A}) \) and \( u \in W^{1,1}_0(Q_{\sqrt{\epsilon}}) \). Then
\[
\int_{Q_{\sqrt{\epsilon}}} f(\omega, x, Au + B) \, dx \leq \int_{Q_{\sqrt{\epsilon}}} f(\omega, x, Au + A) \, dx + c_1 |A - B| \mathcal{L}^n(Q_{\sqrt{\epsilon}})
\]
\[
+ \rho(|A - B|) \left( \int_{Q_{\sqrt{\epsilon}}} f(\omega, x, Au + A) \, dx + \int_{Q_{\sqrt{\epsilon}}} f(\omega, x, Au + B) \, dx \right).
\]
In the case of $\rho(|A - B|) < 1$ rearranging the terms and taking the infimum through $W_0^{1,1}(Q_+^1)$ we obtain
\[
(1 - \rho(|A - B|))m_f(\omega, B, Q_+^1) \leq (1 + \rho(|A - B|))m_f(\omega, A, Q_+^1) + c_1\mathcal{L}^n(Q_+^1)|A - B|.
\]
or in other words
\[
m_f(\omega, B, Q_+^1) - m_f(\omega, A, Q_+^1) \leq \rho(|A - B|)(m_f(\omega, B, Q_+^1) + m_f(\omega, A, Q_+^1)) + c_1\mathcal{L}^n(Q_+^1)|A - B|.
\]
(2.15)

Since (2.15) is clear for $\rho(|A - B|) \geq 1$ we may exchange the roles of $A, B$ and divide through $\mathcal{L}^n(Q_+^1)$ to arrive at
\[
\left|\frac{m_f(\omega, A, Q_+^1)}{\mathcal{L}^n(Q_+^1)} - \frac{m_f(\omega, B, Q_+^1)}{\mathcal{L}^n(Q_+^1)}\right| \leq \rho(|A - B|)\left(\frac{m_f(\omega, A, Q_+^1)}{\mathcal{L}^n(Q_+^1)} + \frac{m_f(\omega, B, Q_+^1)}{\mathcal{L}^n(Q_+^1)}\right) + c_1|A - B|
\]

Letting $\varepsilon \to 0$ by Proposition 2.13 verifies the continuity condition (B) for $f_{\text{hom}}$.

Now since $\mathcal{R}(\mathcal{A})$ is a finite-dimensional real vector space, we may find a countable subset $W \subset \mathcal{R}(\mathcal{A})$ such that $W$ which is a bijection with $Q_{\dim \mathcal{R}(\mathcal{A})}$. Define $\tilde{\Omega} := \cap_{A \in W} \Omega$ so that (2.13) is valid for every $\omega \in \tilde{\Omega}$ and every $w \in W$. Now take an arbitrary $A \in \mathcal{R}(\mathcal{A})$ and consider a sequence $(A_k) \subset W$ such that $A_k \to A$. Then
\[
\left|f_{\text{hom}}(\omega, A) - \frac{m_f(\omega, A, Q_+^1)}{\mathcal{L}^n(Q_+^1)}\right| \leq |f_{\text{hom}}(\omega, A) - f_{\text{hom}}(\omega, A_k)|
+ \left|f_{\text{hom}}(\omega, A_k) - \frac{m_f(\omega, A_k, Q_+^1)}{\mathcal{L}^n(Q_+^1)}\right|
+ \left|m_f(\omega, A, Q_+^1)\right|\left|\frac{m_f(\omega, A_k, Q_+^1)}{\mathcal{L}^n(Q_+^1)}\right| + \left|m_f(\omega, A_k, Q_+^1)\right|\left|\frac{m_f(\omega, A, Q_+^1)}{\mathcal{L}^n(Q_+^1)}\right|
\]
\begin{align}
\leq c'\rho(|A - A_k|)\left|\frac{m_f(\omega, A, Q_+^1)}{\mathcal{L}^n(Q_+^1)}\right| + \left|\frac{m_f(\omega, A_k, Q_+^1)}{\mathcal{L}^n(Q_+^1)}\right|
\end{align}
\begin{align}
\leq c'|A - A_k| + c'|A - A_k|
\end{align}

for some constant $c' > 0$. Passing to $\varepsilon \to 0$ followed by $k \to \infty$ gives the desired identification of $f_{\text{hom}}$ for all $A \in \mathcal{R}(\mathcal{A})$ and all $\omega \in \tilde{\Omega}$.

From property (C) of $f$ it follows that for $s, t \geq \lambda$
\[
|s^{-1}f(x, sA) - t^{-1}f(x, tA)| \leq \frac{c_2}{t^\gamma} + \frac{c_2}{s^\gamma}
\]
(2.17)

Let $\delta > 0$ be arbitrary and choose $u \in W_0^{1,1}(Q_+^1)$ such that
\[
\int_{Q_+^1} f(\omega, x, tA)\,dx \leq m_f(\omega, tA, Q_+^1) + \delta\varepsilon^{-n}.
\]
With that at hand using (2.17) and dividing through by $\mathcal{L}^n(Q_1^\omega)$ we have
\[
\frac{1}{t} m_f(\omega, sA, Q_1^\omega) \leq \int_{Q_1^\omega} f(\omega, x, sA u + sA) \, dx \leq \int_{Q_1^\omega} f(\omega, x, tA u + tA) \, dx + \frac{c_2}{t^\gamma} + \frac{c_2}{s^\gamma}.
\]
Taking the limit as $\varepsilon \to 0$ we notice that
\[
s^{-1} f_{\text{hom}}(x, sA) \leq t^{-1} f_{\text{hom}}(x, tA) + \frac{c_2}{t^\gamma} + \frac{c_2}{s^\gamma} + \delta.
\]
Exchanging the roles of $s, t$ and taking the limits $s \to \infty, \delta \to 0$ ultimately yields the (C) property for $f_{\text{hom}}$.

As to the linear growth condition of $f_{\text{hom}}$ observe that since $\int_U A u \, dx = 0$ for all $u \in W^{1,1}(U)$
\[(2.18) \quad \int_U f(\omega, x, A u + A) \, dx \geq \alpha \int_U |A u + A| \, dx \geq \alpha |A|.
\]
Likewise the upper bound is a direct consequence of the second assertion in the statement of Proposition 2.12.

**3. The Main Theorem for Linear Growth**

Let $f : \Omega \times \mathbb{R}^n \times \mathcal{A}(\mathbb{A}) \to [0, +\infty)$ be an integrand as in Definition 2.9. The associated parametrised functionals in the interest of homogenisation are given by $F(z) : L^1_{\text{loc}}(\mathbb{R}^n; V) \times \mathcal{A}(\mathbb{A}) \to [0, +\infty]$ such that
\[(3.1) \quad F(z)[u; U] := \begin{cases} \int_U f \left( \omega, \frac{x}{\varepsilon}, A u \right) \, dx, & \text{for } u \in W^{1,1}(U) \\ +\infty & \text{otherwise in } L^1_{\text{loc}}(\mathbb{R}^n; V). \end{cases}
\]
for $\omega \in \Omega$. Subject to all said assumptions let us now state the main theorem in the interest of our discourse.

**Theorem 3.1** (Almost sure $\Gamma$-convergence). Let $A$ be a constant coefficient differential operator as in (2.1) with finite-dimensional nullspace. Suppose that $f$ is a stationary random integrand with conditions specified in Definition 2.9. Then the sequence $(F(z))_{\varepsilon > 0}$ defined in (3.1) $\Gamma$-converges almost surely in $\Omega$ as $\varepsilon \to 0$ in the $L^1_{\text{loc}}(\mathbb{R}^n; V)$-topology to an integral functional $F_{\text{hom}}(\omega) : L^1_{\text{loc}}(\mathbb{R}^n; V) \times \mathcal{A}(\mathbb{A}) \to [0, +\infty]$ given by
\[(3.2) \quad F_{\text{hom}}(\omega)[u; U] := \begin{cases} \int_U f_{\text{hom}} \left( \omega, \frac{dA u}{dZ^n} \right) \, dx + \int_U f_{\text{hom}} \left( \omega, \frac{dA^* u}{d|A^* u|} \right) \, d|A^* u| & \text{if } u \in \text{BV}(U) \\ +\infty & \text{otherwise in } L^1_{\text{loc}}(\mathbb{R}^n; V). \end{cases}
\]
for all $u \in \text{BV}(U)$ and $\mathbb{P}$-a.e. $\omega \in \Omega$. The homogenised density is represented as
\[(3.3) \quad f_{\text{hom}}(\omega, A) := \lim_{k \to \infty} k^{-n} \mathbb{E} [m_f(\omega, A, (0, k)^n) | T']
\]
where $A \in \mathcal{A}(\mathbb{A})$ and the recession function $f_{\text{hom}}(\omega, \cdot) := \lim_{t \to \infty} t^{-1} f_{\text{hom}}(\omega, t(\cdot))$ is well-defined. Further, if $(\tau_\omega)_{\omega \in \mathbb{Z}^n}$ is in addition ergodic, then $F_{\text{hom}}$ is deterministic (independent of the random parameter $\omega$) and
\[(3.3) \quad f_{\text{hom}}(A) := \lim_{k \to \infty} k^{-n} \int_{\Omega} m_f(\omega, A, (0, k)^n) \, d\mathbb{P}.
\]
We shall divide the proof of Theorem 3.1 into two parts.

3.1. The lim-inf inequality. Here we announce the lower bound. The line of proof follows the argument by ”blow-up” also implemented in [11, 20] the scheme of which is thoroughly explained for instance in [7, 20]. Let $(\varepsilon_k) \subset (0, 1)$ be a sequence converging to zero as $k \to \infty$.

**Theorem 3.2.** For any $u \in L^1_{\text{loc}}(\mathbb{R}^n; V)$ and any sequence $(u_k) \subset L^1_{\text{loc}}(\mathbb{R}^n; V)$ such that $u_k \overset{L^1_{\text{loc}}}{\to} u$, the lim-inf inequality

$$\mathcal{F}_{\text{hom}}(\omega)[u; U] \leq \liminf_{k \to \infty} \mathcal{F}_{\varepsilon_k}(\omega)[u_k; U]$$

holds for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $U \in \mathcal{A}(\mathbb{R}^n)$.

**Proof.** Let us assume that the right hand side of (3.3) is finite as otherwise there is nothing to prove. Up to extraction of a subsequence let us suppose that the limit $\lim_{k \to \infty} \mathcal{F}_{\varepsilon_k}(\omega)[u_k; U]$ exists so that $(u_k) \subset BV^h(U)$. We begin by contemplating the induced sequence of Radon measures $\nu_k := \mathcal{F}_{\varepsilon_k}(\omega)[u_k; :]$. Taking the asserted boundedness into account forces $(\nu_k)$ to be uniformly bounded in $\mathcal{M}(U; W)$ and whence one may select a subsequence $(\nu_{k_j})$ such that $\nu_{k_j} \rightharpoonup \nu$ in $\mathcal{M}(U; W)$. In particular the bound (3.3) may be congruently rephrased by inquiring that $\nu(\cdot) \geq \mathcal{F}_{\text{hom}}(\omega)[u; :]$ as measure theoretic functions. To this end notice that by the Lebesgue-Radon-Nikodym decomposition one has the representation $\nu = \nu^a + \nu^c$ with respect to the Lebesgue measure. In turn the claimed inequality is accessible by comparing Radon-Nikodym densities of the two terms:

$$\frac{d\nu}{d\mathcal{L}^n}(x) \geq f_{\text{hom}}(\omega, A|\nabla u(x)) \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in U$$

and

$$\lim_{r \to 0} \frac{\nu^c(Q_r(x))}{|A^c u|(Q_r(x))} \geq f_{\text{hom}}^\infty \left( \omega, \frac{dA^c u}{d|A^c u|}(x) \right) \quad \text{for } |A^c u|\text{-a.e. } x \in U.$$

**Step 1:** Firstly we tackle the absolutely continuous part in (3.3). By a variant of the Besicovitch Differentiation Theorem [19, Thm 1.30] for $\mathcal{L}^n\text{-a.e. } x \in U$

$$\frac{d\nu}{d\mathcal{L}^n}(x) = \lim_{r \to 0^+} \frac{\nu(Q_r(x))}{\mathcal{L}^n(Q_r(x))}.$$

Furthermore in view of Proposition 2.7 we regard $x$ as an differentiability point of $u$. As $\nu$ is a finite Radon measure, we have that $\nu(\partial Q_r(x)) = 0$ for a.e. $r > 0$. Thereby for such $r > 0$ the equality

$$\nu(Q_r(x)) = \lim_{k \to \infty} \nu_k(Q_r(x))$$

holds true. Let $\delta, \tau \in (0, 1)$, $m \in \mathbb{N}$ be arbitrary and label $Q_i = Q_{\tau r + \delta r (1 - \tau)}(x)$ for indices $i \in \{0, \ldots, m\}$. Fix one such $i$ and take a cut-off function $\eta_i \in C_c^\infty(Q_i; [0, 1])$ such that $\eta_i = 1$ in $Q_{i-1}$ and $|\nabla \eta_i| \leq m/(r(1 - \tau))$. Now set $u_0(y) = u(x) + \nabla u(x)(y - x)$ and define $u_{i,k} := u_0 + \eta_i(u_k - u_0)$. Then clearly $u_{i,k} \in W^{1,1}(Q_r(x))$ and crucially the $A$-gradients split as

$$A u_{i,k} = \begin{cases} A_{\varepsilon_k}, & \text{in } Q_{i-1} \\ A_{\varepsilon_k} |\nabla u(x) + \eta_i A_{\varepsilon_k}(u_k - u_0) + \nabla \eta_i \otimes A (u_k - u_0), & \text{in } Q_i \setminus Q_{i-1} \\ A_{\varepsilon_k} |\nabla u(x), & \text{in } Q_r(x) \setminus Q_i. \end{cases}$$
Thus \( u_{i,k} - u_0 \) is an admissible test map for infimisation in (2.11) and whence inserting it as a competitor we compute
\[
\inf \left\{ \int_{Q_{\varepsilon_k}(x)} f\left( \omega, y, A[\nabla u(x)] + A \nu \right) \, dy : v \in W^{1,1}_0(Q_{\varepsilon_k}(x)) \right\}
\leq \frac{1}{\mathcal{L}^n(Q_{\varepsilon_k}(x))} \int_{Q_{\varepsilon_k}(x)} f\left( \omega, \frac{y}{\varepsilon_k}, A u_{i,k} \right) \, dy
\leq \frac{1}{\mathcal{L}^n(Q_{\varepsilon_k}(x))} \int_{Q_{\varepsilon_k}(x)} f\left( \omega, \frac{y}{\varepsilon_k}, A u_{i,k} \right) \, dy
+ \frac{1}{\mathcal{L}^n(Q_{\varepsilon_k}(x))} \int_{Q_{\varepsilon_k}(x)} f\left( \omega, \frac{y}{\varepsilon_k}, A[\nabla u(x)] \right) \, dy
= I + II + III.
\]

Regarding bounds on I, II, III we argue as follows:
\[
I \leq \int_{Q_{\varepsilon_k}(x)} f\left( \omega, \frac{y}{\varepsilon_k}, A u_{i,k} \right) \, dy = \frac{\nu_k(Q_{\varepsilon_k}(x))}{\mathcal{L}^n(Q_{\varepsilon_k}(x))}
\]
\[
II \leq \beta (1 + A[\nabla u(x)](1 - \tau^n)) + \frac{\beta}{\mathcal{L}^n(Q_{\varepsilon_k}(x))} \int_{Q_{\varepsilon_k}(x)} |A(u_{i,k} - u_0)| \, dy
+ \frac{\beta m}{(1 - \tau)\mathcal{L}^n(Q_{\varepsilon_k}(x))} \int_{Q_{\varepsilon_k}(x)} \frac{|u_{i,k} - u_0|}{r} \, dy
\]
\[
III \leq \beta (1 + A[\nabla u(x)](1 - \tau^n)).
\]

Applying the change of variable on the right-hand side of II, because \( Q_{i} \setminus Q_{i-1} \) has its annular "thickness" comparable to \( m^{-1} \) the expression transforms to
\[
II \leq \beta (1 + A[\nabla u(x)](1 - \tau^n)) + \frac{\beta}{m\mathcal{L}^n(Q_{\varepsilon_k}(x))} \int_{Q_{\varepsilon_k}(x) \setminus Q_{\varepsilon_k}(x)} |A(u_{i,k} - u_0)| \, dy
+ \frac{\beta}{(1 - \tau)\mathcal{L}^n(Q_{\varepsilon_k}(x))} \int_{Q_{\varepsilon_k}(x)} \frac{|u_{i,k} - u_0|}{r} \, dy.
\]

Altogether passing to \( m \to \infty \) first, followed by \( k \to \infty, r \to 0 \) as well as \( \tau \to 1 \), in view of Proposition 2.13 the terms II and III vanish. Hence invoking Proposition 2.13 in conjunction with 3.8 it amounts
\[
f_{\text{hom}}(\omega, A[\nabla u(x)]) \leq \lim_{r \to 0} \lim_{k \to \infty} \frac{\nu_k(Q_{\varepsilon_k}(x))}{\mathcal{L}^n(Q_{\varepsilon_k}(x))} = \frac{d\nu}{d\mathcal{L}^n}(x).
\]

**Step 2:** To complete the claim let us verify the bounds on the singular part. For \( |A^*u| \)-a.e. \( x \in U \) one notices that
\[
\lim_{r \to 0} \frac{\nu^*(Q_{\varepsilon_k}(x))}{|A^*u|(Q_{\varepsilon_k}(x))} = \lim_{r \to 0} \frac{\nu(Q_{\varepsilon_k}(x))}{|A^*u|(Q_{\varepsilon_k}(x))} = \lim_{r \to 0} \frac{\nu(Q_{\varepsilon_k}(x))}{|A^*u|(Q_{\varepsilon_k}(x))}
\]
where the first equality is true since for the choice of \( x \in U \), \( r^{-n}|A^*u|(Q_{\varepsilon_k}(x)) \to \infty \) as \( r \to 0 \). In this regard keeping in mind the weak*-convergence of \( (\nu_k) \) we shall focus on showing that
\[
\lim_{r \to 0} \lim_{k \to \infty} \frac{\nu_k(Q_{\varepsilon_k}(x))}{|A^*u|(Q_{\varepsilon_k}(x))} \geq f_{\text{hom}}\left( \omega, \frac{dA^*u}{d|A^*u|}(x) \right).
\]
We define the blow-up sequences

\begin{equation}
\label{eq:blow-up-sequences}
v_r(y) := \frac{r^{n-1}}{|A u|(|Q_r(x)|)} \left( u(x + ry) - \pi_{Q_r(x)} u(x) \right)
\end{equation}

and

\[ v_{r,k}(y) := \frac{r^{n-1}}{|A u|(|Q_r(x)|)} \left( u_k(x + ry) - \pi_{Q_r(x)} u_k(x) \right) \]

for \( y \in Q \) where \( \pi_{Q_r(x)} u, \pi_{Q_r(x)} u_k \) are the elements from the hypotheses of Proposition 2.5 (variant for star-shaped domains, see [10, Thm 3.2], [23, Cor. 2.2]) for \( u \) and \( u_k \) as perceived in \( BV^A(Q_r(x)) \). With that being proposed in the context of the densities of singular parts in (2.7), the A-gradations of \( v_r \) satisfy the convergence (cf. Theorem 2.8):

\[ A v_r(Q) = \frac{A u(Q_r(x))}{|A u|(|Q_r(x)|)} \rightarrow \xi(x) \otimes_A \sigma(x) \]

as \( r \to 0 \). From the very construction we also infer that \( \|v_{r,k} - v_r\|_{L^1(U;V)} \to 0 \) as \( k \to \infty \).

Moreover from the Poincaré inequality again, \((v_r)\) are uniformly bounded in \( BV^A(Q) \). Due to weak* compactness by Proposition 2.4 there exists a subsequence (not relabelled) \( v_r \rightharpoonup v \) for some \( v \) in \( BV^A(\Omega) \) as \( r \to 0 \). At this point we examine the behaviour of sequences close to \( x \) by arguing in a similar manner as in Step 1. For an integer \( m \in \mathbb{N} \) consider the scaled cubes \( Q_i = Q_{(1-s_r) + \delta s_r, \delta} \) for indices \( i \in \{0, \ldots, m\} \). Take a cut-off function \( \eta_i \in C^\infty(Q) \) with \( \chi_{Q_{-1}} \leq \eta \leq \chi_{Q} \), such that \( \|\nabla \eta_i\|_{L^\infty} \leq c m s_r^{-1} \) where \( s_r := \|v_r - v\|_{L^1(Q_r(x);V)}^{1/2} \). Additionally let \( \psi \in C^\infty(\mathbb{R}^n) \) be such that \( \psi = 1 \) in \( Q \) and set \( \Phi_\delta(y) = \delta^{-n}(\xi(x) \otimes_A \sigma(x)) \psi(y) \). Let \( w_\delta \in \mathcal{S}(\mathbb{R}^n; V) \) (the space of Schwartz functions valued in \( V \)) be a fundamental solution to \( A w_\delta = \Phi_\delta \) in \( \mathbb{R}^n \), that is \( w_\delta = K_\delta * \Phi_\delta \), cf. Prop. 2.6 where \( \Phi_\delta \) is the Fourier inversion of \( \Phi_\delta \). Let \( \Psi_{\delta,r} \in \ker(A) \) be chosen as in the Poincaré inequality Proposition 2.5 for \( v - w_\delta \) in \( \delta Q_{s_r} := \delta(Q \setminus (1-s_r)Q) \) i.e. \( \Psi_{\delta,r} := \tau_{\delta Q_{s_r}}(v - w_\delta) \). Consequently we define \( \Theta_{\delta,r} := w_\delta + \Psi_{\delta,r} \) and construct a sequence of admissible competitors by

\[ u_{r,k}^i := (1 - \eta_i)\tau_r \Theta_{\delta,r} + \eta_i u_{r,k}, \quad u_{r,k} := \tau_r v_{r,k}, \quad \tau_r := \frac{|A u|(|Q_r(x)|)}{r^n}. \]

In terms of A-gradations this gives

\begin{equation}
\label{eq:A-gradients}
A u_{r,k}^i = \begin{cases}
\frac{A u_{r,k}}{r^n} & \text{in } Q_{i-1}, \\
\frac{\tau_r \xi(x) \otimes_A \sigma(x)}{r^n} + \eta_i A(u_{r,k} - \tau_r \Theta_{\delta,r}) + \nabla \eta_i \otimes_A (u_{r,k} - \tau_r \Theta_{\delta,r}) & \text{in } Q_{i} \setminus Q_{i-1}, \\
\frac{\tau_r \xi(x) \otimes_A \sigma(x)}{r^n} & \text{in } Q_{\delta} \setminus Q_i.
\end{cases}
\end{equation}

Observe that \( s_r \to 0 \) and \( \tau_r \to \infty \) as \( r \to 0 \) by construction of the very parameters (since \( x \in \operatorname{supp}(|A^* u|) \)). By the change of variable, with the above notation we may write

\[ \frac{1}{|A u|(|Q_{b_r}(x)|)} \int_{Q_{b_r}(x)} f(\omega, \frac{z}{\varepsilon_k}, A u_k) \, dz = \frac{1}{\tau_r} \int_{Q_{\delta}} f(\omega, \frac{x + ry}{\varepsilon_k}, A u_{r,k}) \, dy. \]
where \( Q_{i-1}(x) := Q_{i-1} + x \). Expanding out the integral on the right-hand side of the above equality applied to \( u_{r,k} \) in accord with (3.15) amounts in

\[
\frac{1}{\tau_r} \int_{\partial Q} f(\omega, x + ry, \kappa u_{r,k}) \, dy \leq \frac{1}{\tau_r} \int_{Q_{i-1}} f(\omega, \frac{x}{\varepsilon_k}, \kappa u_{r,k}) \, dy \\
+ \frac{1}{\tau_r} \int_{Q \setminus Q_{i-1}} f(\omega, \frac{x + ry}{\varepsilon_k}, \kappa u_{r,k}) \, dy \\
+ \frac{1}{\tau_r} \int_{\partial Q \setminus Q_i} f(\omega, \frac{x + ry}{\varepsilon_k}, \tau_r \xi(x) \otimes \sigma(x)) \, dy \\
=: \text{I} + \text{II} + \text{III}.
\]

(3.16)

Using the pointwise gradient bounds of the cut-off function \( \eta_i \) as well as the linear growth condition of \( f \) we derive bounds on \( \text{II} \) and \( \text{III} \) leads to the estimates:

\[
\text{III} \leq \frac{\beta \delta^n}{\tau_r} (1 + \frac{\tau_r}{\delta^n}) \delta^n (1 - (1 - s_r)^n) =: \tilde{\Pi}(r, \delta) \xrightarrow{r \to 0} 0
\]

\[
\text{II} \leq \tilde{\Pi}(r, \delta) + \frac{\beta}{\tau_r} \int_{Q \setminus Q_{i-1}} |\kappa(u_{r,k} - \tau_r \Theta_{\delta,r})| \, dy + \frac{\beta \delta}{s_r} \int_{Q \setminus Q_{i-1}} |v_{r,k} - \Theta_{\delta,r}| \, dy
\]

(3.17)

\[
\leq \tilde{\Pi}(r, \delta) + \frac{\beta}{\tau_r} \int_{Q \setminus Q_{i-1}} |\kappa(u_{r,k} - \tau_r \Theta_{\delta,r})| \, dy + \frac{\beta \delta}{s_r} \int_{Q \setminus Q_{i-1}} |v_{r,k} - v| \, dy
\]

If we incorporate the above bounds into the limit in (3.13) along with Proposition 2.13 we find that after taking the averages in \( m \) and passing to \( m \to 0 \) as in Step 1:

\[
f_{\text{hom}}(\omega, \xi(x) \otimes \kappa v(x)) \xrightarrow{\text{Prop. 2.13(C)}} \frac{\delta^n}{\tau_r} f_{\text{hom}}(\omega, \frac{\tau_r \delta^n \xi(x) \otimes \kappa v(x)}{\varepsilon_k}) + \frac{c_2}{\tau_r} \leq \frac{\delta^n}{\tau_r} \sup_{k \to \infty} \frac{f_{\text{hom}}(\omega, \frac{\tau_r \delta^n \xi(x) \otimes \kappa v(x)}{\varepsilon_k})}{\tau_r \mathcal{L}^n(Q_{\varepsilon_k}^{\delta Q})} + \frac{c_2}{\tau_r}
\]

(3.18)

\[
\leq \lim_{k \to \infty} \frac{1}{\tau_r} \int_{\partial Q} f(\omega, \frac{x + ry}{\varepsilon_k}, \kappa u_{r,k}) \, dy + \frac{c_2}{\tau_r}
\]

\[
\leq \lim_{k \to \infty} \frac{1}{|\kappa u|(Q_r(x))} \int_{Q_r(x)} f(\omega, \frac{y}{\varepsilon_k}, \kappa u_{r,k}) \, dy
\]

\[
+ 2 \tilde{\Pi}(r, \delta) + \beta s_r + \frac{\beta}{s_r} \int_{\partial Q \setminus (1 - s_r) \delta Q} |v - \Theta_{\delta,r}| \, dy + \frac{c_2}{\tau_r}
\]

\[
= \lim_{k \to \infty} \frac{\nu_k(Q_r(x))}{|\kappa u|(Q_r(x))} + 2 \tilde{\Pi}(r, \delta) + \beta s_r + \frac{\beta}{s_r} \int_{\partial Q \setminus (1 - s_r) \delta Q} |v - \Theta_{\delta,r}| \, dy + \frac{c_2}{\tau_r}
\]

In the first inequality we have used the continuity condition (B) cf. Proposition 2.13 of the recession function of \( f_{\text{hom}} \). Recall now since \( \Theta_{\delta,r} = u_{\delta} + \Psi_{\delta,r} \) we may invoke the Poincaré inequality from Proposition 2.45 to bound the integral in the final equality:

\[
\int_{\partial Q \setminus (1 - s_r) \delta Q} |v - \Theta_{\delta,r}| \, dy \leq c s_r \int_{\partial Q \setminus (1 - s_r) \delta Q} |\kappa(v - w_{\delta})| \, dy =: \tilde{\Pi}(r, \delta).
\]

(3.19)
Passage to $r \to 0$ and then $\delta \to 1$ in view of (3.19), (3.21), (3.23), (3.25) as well as the definition of recession function ultimately settle the desired bound

$$\lim_{r \to 0} \lim_{k \to \infty} \frac{\mu_k(Q_r(x))}{|A_k|(|Q_r(x)|)} \geq f^\text{hom}_1(\omega, \xi(x) \otimes \sigma(x))$$

and this concludes the proof. \hfill \Box

3.2. Proof of the upper bound. In this final section we verify the almost sure existence of recovery sequences for the functionals ($F_{\epsilon_k}$). Let us recount a result based on [18, Prop. 2.1] that illustrates approximation by piecewise-affine maps in the vectorial setting. We say that $v \in Aff^{pc}(U; V)$ if there exists a finite collection of subsets $\{U_i\}_{i \in I}$ such that $\bigcup_{i \in I} U_i = U$, $\mathcal{L}^n(U_i \cap U_j) = 0$ for $i \neq j$ such that each restriction $v|_{U_i}$ is an affine map and $v \in C^0(U; V)$.

Lemma 3.3. Suppose that $u \in C^\infty(U; V) \cap C^0(U; V)$. Then there exists $(u_k) \subset Aff^{pc}(U; V)$ such that

- $\|\nabla u_k\|_{L^\infty(U)} \leq \|\nabla u\|_{L^\infty(U)}$
- $u_k \to u$ as $k \to \infty$ in $L^\infty(U; V)$
- $\partial_i u_k \to \partial_i u$ as $k \to \infty$ for $1 \leq i \leq n$ in $L^\infty_{\text{loc}}(U; V)$.

From these conditions we infer by linearity that $\mathcal{A}u_k \to \mathcal{A}u$ holds locally in $L^\infty(U; W)$ as well. Consequently in view of Proposition 2.3 the density of piecewise-affine maps manifests itself in the norm topology of $W^{1,1}$.

Lemma 3.4. The space $Aff^{pc}(U; V)$ is dense in $W^{1,1}(U)$ with respect to the $W^{1,1}$-norm.

Proof. Let $u \in W^{1,1}(U)$. By Proposition 2.3 (b) there exists $(v_k) \subset C^\infty(U)$ such that $\|v_k - u\|_{W^{1,1}(U)} \to 0$ as $k \to \infty$. However for every $v_k$ there exists a sequence $(u^k) \subset Aff^{pc}(U; V)$ satisfying hypotheses of Lemma 3.3. From this we conclude that $\mathcal{A}u^k \to \mathcal{A}v_k$ as $m \to \infty$ locally in $L^\infty(U; V)$ and so the dominated convergence, since $W^{1,\infty}(U) \subset W^{1,1}(U)$, yields $\|v_k - u^k\|_{W^{1,1}(U)} \to 0$ as $m \to \infty$. Finally extracting a diagonal subsequence $(u^k)_{m_k}$ eventually leads to

$$\|u - u^k_{m_k}\|_{W^{1,1}(U)} \leq \|v_k - u^k_{m_k}\|_{W^{1,1}(U)} + \|v_k - u\|_{W^{1,1}(U)} \to 0$$

as $k \to \infty$. \hfill \Box

Such an approximation procedure is instrumental because in conjunction with the continuity of $F_{\text{hom}}$ in the norm topology of $W^{1,1}$ it allows one to transpose the question of the upper bound in the way that it is sufficient to find recovery sequences for maps in $Aff^{pc}(U; V)$.

Theorem 3.5. Let $U \in \mathcal{A}(\mathbb{R}^n)$ and $u \in L^1(U; V)$. There exists sequence $(u_k) \subset BV^k(U)$ such that $u_k \to u$ in $L^1(U)$ and

$$\limsup_{k \to \infty} F_{\epsilon_k}(\omega)[u_k; U] \leq F_{\text{hom}}(\omega)[u; U]$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Proof. Step 1: Assume first that $u = \ell_A$ is an affine map $x \mapsto Ax$ for $A \in V \otimes \mathbb{R}^n$ so that $\mathcal{A}u = \ell_A(A) \in \mathcal{C}(\mathcal{A})$. Let $\delta > 0$ be fixed and let $\{Q_i\}_{i \in I}$ be a subcollection of $U$ of pairwise-disjoint
cubes such that each \( Q_i \) has side length comparable to \( \delta \) and \( \mathcal{L}^n(U \setminus \cup_{i \in \mathcal{I}_\delta} Q_i) \leq \delta \). For each \( i \in \mathcal{I}_\delta \) take \( w^i_{k, \delta} \in W^{0,1}_0(Q_i) \) such that

\[
\int_{\varepsilon_k^{-1} Q_i} f(x, \ell_A(A) + A w^i_{k, \delta}; \omega) \, dx \leq \frac{m_f(\omega, \varepsilon_k^{-1} Q_i)}{\mathcal{L}^n(\varepsilon_k^{-1} Q_i)} + \frac{\delta}{|\mathcal{I}_\delta|}
\]

where \( |\mathcal{I}_\delta| \) denotes the cardinality of \( \mathcal{I}_\delta \) and define \( u^i_{k, \delta} := \ell_A + \varepsilon_k w^i_{k, \delta}(x) \). After applying the change of variable and letting \( k \to \infty \), in view of the subadditive theorem Prop. 2.13 the estimate reads

\[
(3.20) \quad \limsup_{k \to \infty} \mathcal{F}_{\varepsilon_k}(\omega)[u^i_{k, \delta}; Q_i] \leq \mathcal{F}_{\text{hom}}(\omega)[u; Q_i] + \frac{\delta}{|\mathcal{I}_\delta|}.
\]

We let \( u_{k, \delta} \in W^{0,1}(U) \) be defined by

\[
\begin{cases}
  u_{k, \delta} = u^i_{k, \delta} & \text{in } Q_i, \, i \in \mathcal{I}_\delta \\
  u_{k, \delta} = \ell_A & \text{otherwise in } U \setminus Q_i.
\end{cases}
\]

Thus using the fact that both \( f \) as well as \( f_{\text{hom}} \) are of linear growth and due to the almost sure convergence in Proposition 2.13 we deduce that

\[
\limsup_{k \to \infty} \mathcal{F}_{\varepsilon_k}(\omega)[u_{k, \delta}; U] \leq \limsup_{k \to \infty} \sum_{i \in \mathcal{I}_\delta} \mathcal{F}_{\varepsilon_k}(\omega)[u^i_{k, \delta}; Q_i] + \beta(1 + |\ell_A(A)|)\delta + \delta 
\]

\[
\leq \mathcal{F}_{\text{hom}}(\omega)[\ell_A; U] + \beta(1 + |\ell_A(A)|)\delta + \delta.
\]

Since \( \delta > 0 \) was arbitrary we may apply the diagonalisation argument with some \( \delta = \delta_k \to 0 \) as \( k \to \infty \) and verify the convergence \( u_{k, \delta_k} \to \ell_A \) in \( L^1(U; V) \). Observe that from (3.20) and the linear growth of \( f \) there holds

\[
\|A u_{k, \delta_k}\|_{L^1(Q_i; W)} = \|A w^i_{k, \delta_k} + \ell_A(A)\|_{L^1(Q_i; W)} \leq \frac{1}{\alpha} \mathcal{F}_{\varepsilon_k}(\omega)[w^i_{k, \delta_k} + \ell_A; Q_i; \omega]
\]

\[
\leq \beta \left(1 + |\ell_A(A)| \right) \mathcal{L}^n(Q_i) + \frac{\delta_k}{|\mathcal{I}_\delta|}.
\]

At the same time, since \( w^i_{k, \delta_k} \in W^{0,1}_0(Q_i) \), Proposition 2.6 tells us that \( \|w^i_{k, \delta_k}\|_{L^1(Q_i; W)} \leq c_{n, \beta} \delta_k \|A u^i_{k, \delta_k}\|_{L^1(Q_i; W)} \) and therefore

\[
\|u_{k, \delta_k} - \ell_A\|_{L^1(Q_i; V)} \leq C \delta_k \left( \mathcal{L}^n(Q_i) + \frac{\delta_k}{|\mathcal{I}_\delta|} \right).
\]

This bound ultimately implies

\[
\|u_{k, \delta_k} - \ell_A\|_{L^1(U; V)} \leq \sum_{i \in \mathcal{I}_\delta} C \delta_k \left( \mathcal{L}^n(Q_i) + \frac{\delta_k}{|\mathcal{I}_\delta|} \right)
\]

\[
\leq C \delta_k \left( \mathcal{L}^n(\Omega) + \delta_k \right)
\]

from which the claim follows if we let \( k \to \infty \).

**Step 2:** Suppose now that \( u \in \text{Aff}^{pc}(U; V) \) so that there is a finite partition \( \{U_j\}_{j \in \mathcal{J}} \) of \( U \) such that \( u = \ell_{A_j} \) in \( U_j \) for \( A_j \in V \otimes \mathbb{R}^n \). Hence by **Step 1** for each subset \( U_j \) there exist \( w^j_k \in u + W^{0,1}_0(U_j) \) such that \( w^j_k \to u|_{U_j} \) in \( L^1(U; V) \) and \( \limsup_{k \to \infty} \mathcal{F}_{\varepsilon_k}(\omega)[w^j_k; U_j] \leq \mathcal{F}_{\text{hom}}(\omega)[u; U_j] \). We define
Let \( u_k := u + \sum_{j \in J} u^j - u \) in which case \( u_k \in W^{1,1}(U) \) and \( u_k \to u \) in \( L^1(U; V) \). Because \( \mathcal{L}^n(U_i \cap U_j) = 0 \) if \( i \neq j \), it follows that

\[
\limsup_{k \to \infty} \mathcal{F}_{e_k}(\omega)[u_k; U] \leq \mathcal{F}_{\text{hom}}(\omega)[u; U].
\]

**Step 3:** Let \( u \in W^{1,1}(U) \), by Lemma 5.3 for any \( \tau > 0 \) we find \( u_\tau \in \text{Aff}^c(U; V) \) such that \( \|u - u_\tau\|_{W^{1,1}(U)} \leq \tau \) where \( \tau > 0 \). As every \( u_\tau \) is piecewise-affine there exist recovery sequences \( (u^k) \) for \( \mathcal{F}_{\text{hom}}(\omega)[u_\tau; U] \). But now from the continuity condition of \( f_{\text{hom}} \), cf. Proposition 2.13 the bounds

\[
|\mathcal{F}_{\text{hom}}(\omega)[w; U] - \mathcal{F}_{\text{hom}}(\omega)[v; U]| \leq \int_U \rho'(|\hat{\kappa} w - \hat{\kappa} v|) \left( f_{\text{hom}}(\omega, \hat{\kappa} w) + f_{\text{hom}}(\omega, \hat{\kappa} v) \right) \, dx
+ c' \mathcal{L}^n(U) \|\hat{\kappa} w - \hat{\kappa} v\|_{L^1(U; W)}
\]

hold true for all \( w, v \in W^{1,1}(U) \). With that being brought forward, extracting a diagonal sequence \( v^k_{\tau_k} = u^k_{\tau_k} + (u - u_{\tau_k}) \) that converges to \( u \) in \( L^1(U; V) \) and passing to a subsequence such that \( v^k_{\tau_k} \to u \) pointwise \( \mathcal{L}^n \)-a.e. yields

\[
\limsup_{k \to \infty} \mathcal{F}_{e_k}(\omega)[v^k_{\tau_k}; U] \leq \limsup_{k \to \infty} \left( \mathcal{F}_{e_k}(\omega)[v^k_{\tau_k}; U] + C \tau_k \right)
\leq \mathcal{F}_{\text{hom}}(\omega)[u; U].
\]

**Step 4:** Finally let \( u \in L^1_{\text{loc}}(\mathbb{R}^n; V) \). Consider

\[
\Gamma \limsup_{k \to \infty} \mathcal{F}_{e_k}(\omega)[u; U] := \inf \{ \limsup_{k \to \infty} \mathcal{F}_{e_k}(\omega)[u_k; U] : (u_k) \subset W^{1,1}(U) \, u_k \to u \text{ in } L^1(U; V) \}
\]

the upper \( \Gamma \)-envelope of \( \{ \mathcal{F}_{e_k}(\omega)[.; U] \} \). By **Step 3** it holds \( \Gamma \limsup_{k \to \infty} \mathcal{F}_{e_k}(\omega)[.; U] \leq \mathcal{F}_{\text{hom}}(\omega)[u; U] \) in \( W^{1,1}(U) \) and in consequence defining \( \mathcal{F}_{\text{hom}}(\omega) : L^1_{\text{loc}}(\mathbb{R}^n; V) \to [0, +\infty] \) by

\[
\mathcal{F}_{\text{hom}}(\omega)[u; U] := \begin{cases} \int_U f_{\text{hom}}(\omega, \hat{\kappa} u) \, dx & \text{for } u \in W^{1,1}(U) \\ +\infty & \text{otherwise in } L^1(U; V) \end{cases}
\]

we conclude from the above argumentation that

\[
\Gamma \limsup_{k \to \infty} \mathcal{F}_{e_k}(\omega) \leq \mathcal{F}_{\text{hom}}(\omega).
\]

for all \( \mathcal{P} \)-a.e. \( \omega \in \Omega \). Moreover we note that from **Step 3** it is possible to infer the \( \bar{\kappa} \)-quasiconvexity of \( f_{\text{hom}} \). This is seen as follows. Let \( A \in \mathcal{A}(\bar{\kappa}) \) and let \( v \in W^{1,1}(\mathbb{R}^n) \) be the fundamental solution cf. Prop 2.6 to \( \hat{\kappa} v = \Psi \) where \( \Psi(x) = \bar{\Psi}(x) \) and \( \bar{\Psi} \in C_c^\infty(\mathbb{R}^n) \) is such that \( \chi_{(0,1)^n} \leq \bar{\Psi} \leq \chi_{(-1,2)^n} \). Further for \( \varphi \in C_c^\infty((0,1)^n; V) \) we set \( v_k(x) := v(x) + k^{-1} \varphi(kx) \). Then \( \hat{\kappa} v_k \to \hat{\kappa} v \) as Radon measures and up to a subsequence \( v_k \to v \) \( L^1((0,1)^n; V) \). The \( L^1 \)-lower-semicontinuity of the \( \Gamma \)-limit in \( W^{1,1} \) obtained in **Step 3** ensures the bound

\[
f_{\text{hom}}(\omega, A) = \int_{(0,1)^n} f_{\text{hom}}(\omega, \hat{\kappa} v) \, dx \leq \liminf_{k \to \infty} \int_{(0,1)^n} f_{\text{hom}}(\omega, \hat{\kappa} v_k) \, dx.
\]

Let \( h \in L^1_{\text{loc}}(\mathbb{R}^n) \) be the \( (0,1)^n \)-periodic extension of \( x \mapsto f_{\text{hom}}(\omega, A + \varphi(x)) \) and define \( h_k(x) := h(kx) \). Then the Riemann-Lebesgue lemma implies the convergence \( h_k \to \int_{(0,1)^n} h \, dx \) in \( L^1((0,1)^n) \)
and hence
\[
\hat{f}_{\text{hom}}(\omega, A) \leq \liminf_{k \to \infty} \int_{(0,1)^n} f_{\text{hom}}(\omega, \frac{A_{
u_k}}{n}) \, dx = \liminf_{k \to \infty} \int_{(0,1)^n} h_k(x) \, dx
\]
\[
= \int_{(0,1)^n} \int_{(0,1)^n} h(x) \, dx = \int_{(0,1)^n} f_{\text{hom}}(\omega, A + A_{\varphi(x)}) \, dx.
\]
which ultimately concludes that \( f_{\text{hom}} \) is \( \mathbb{A} \)-quasiconvex.

Now since \( f_{\text{hom}} \) is an admissible integrand, see Section 2.5 \( \mathcal{F}_{\text{hom}}(\omega)[\cdot; U] \) assumes an integral representation as in (2.10) so that
\[
\Gamma\text{-lim sup}_{k \to \infty} \mathcal{F}_{\epsilon_k}(\omega)[u; U] \leq \mathcal{F}_{\text{hom}}(\omega)[u; U].
\]
Now since \( f_{\text{hom}} \) is an admissible integrand, see Section 2.5 \( \mathcal{F}_{\text{hom}}(\omega)[\cdot; U] \) assumes an integral representation as in (2.10) so that
\[
\Gamma\text{-lim sup}_{k \to \infty} \mathcal{F}_{\epsilon_k}(\omega)[u; U] \leq \int_U f_{\text{hom}}(\omega, \frac{A[\nabla] u}{n}) \, dx + \int_U f_{\text{hom}}^{\infty}(\omega, \frac{\partial h_k u}{\partial A^* u}) \, d|A^* u| = \mathcal{F}_{\text{hom}}(\omega)[u; U]
\]
as required. \( \square \)

4. Discussion on the case \( W^{k,p} \) for \( p > 1 \)

Let us complete our discourse by commenting on the case of superlinear growth bounds i.e. \( p > 1 \). The core argument is a direct adaptation of the blow-up technique employed in Section 2.3 for the linear growth setup. Under higher integrability assumptions \( p > 1 \), the singular part of the homogenised functional becomes obsolete thereby verifying lower/upper bound is simpler to handle in comparison. Suppose that \( f : \Omega \times \mathbb{R}^n \times \mathcal{B}(\mathbb{A}) \to [0, \infty] \) is an integrand satisfying the following:

1. \( f \) is \( (T \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{A})) \)-measurable
2. for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) we have that \( f(\omega, \cdot, \cdot) \) satisfies the \( p \)-growth condition: there exist \( 0 < c_3 \leq c_4 < \infty \) such that
\[
c_3 |A|^p \leq f(\omega, x, A) \leq c_4 (1 + |A|^p)
\]
for all \( x \in \mathbb{R}^n \) and all \( A \in \mathcal{B}(\mathbb{A}) \),
3. for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) \( f(\omega, \cdot, \cdot) \) satisfies the continuity condition (B) in Section 2.3
4. stationarity: \( f(\omega, x + z, A) = f(\tau_z(\omega), x, A) \) for \( \mathcal{L}^n \)-a.e. \( x \in \mathbb{R}^n \), all \( A \in \mathcal{B}(\mathbb{A}) \), \( z \in \mathbb{Z}^n \) and \( \mathbb{P} \)-a.e. \( \omega \in \Omega \).

Consider the parametrised integral functionals given by
\[
(4.1) \quad u \mapsto \int_U f\left(\omega, \frac{x}{\epsilon}, A u\right) \, dx \quad \text{for } u \in W^{k,p}(U).
\]
Before embarking on the core statement, we bring up a key remark on a phenomenon that distinguishes higher integrability exponents, namely the presence of \( A \)-Korn inequalities. Let \( p > 1 \) and let \( O \subset \mathbb{R}^n \) be a bounded domain, star-shaped with respect to a ball \( B \). According to [5] [24] [27] there exists a projection \( \pi_B^O : L^1(O; V) \to \ker(A) \) and a constant \( c_{n,O,A} > 0 \) such that
\[
(4.2) \quad \|
\nabla(u - \pi_B^O u)
\|_{L^p(O; V)} \leq c_{n,O,A} \|Au\|_{L^p(O; W)}
\]
\[
\|
\nabla(\pi_B^O u)
\|_{L^p(O; V)} \leq c_{n,A} \text{diam}(O)^{-1} \|\pi_B^O u\|_{L^p(O; V)}.
\]
for all $W^{A,p}(O)$. Recalling the fact that any bounded Lipschitz set can be written as a finite union of star-shaped domains see e.g. [9], in conjunction with the pair of estimates in [12], we may readily deduce the functional correspondence

$$W^{1,p}(U; V) \simeq W^{A,p}(U)$$

where the isomorphism is understood via the equivalence of $W^{1,p}$- and $W^{A,p}$-norms. In turn such a congruence of function spaces allows one to reiterate the question concerning homogenisation of (4.1) as the task of contemplating the $\Gamma$-convergence of:

$$E_{\varepsilon_k}(\omega)[u; U] = \begin{cases} \int_U f\left(\omega, \frac{x}{\varepsilon_k}, A u\right) \, dx & \text{for } u \in W^{1,p}(U; V) \\ +\infty & \text{otherwise in } L^1(U; V). \end{cases}$$

Accordingly let us define the process

$$\tilde{m}_f(\omega, A, U) = \inf \left\{ \int_U f(\omega, x, A + A v) : v \in W^{1,p}_0(U; V) \right\}$$

for all $\omega \in \Omega$, $A \in \mathcal{A}(\mathbb{R})$ and $U \in \mathcal{A}(\mathbb{R}^n)$. Following closely the lines of Proposition 2.12 Proposition 2.13 we notice that $\tilde{m}_f$ defines a subadditive process and moreover the density $\tilde{f}_{\text{hom}} : \Omega \times \mathcal{A}(\mathbb{R}) \to [0, +\infty)$ given by

$$\tilde{f}_{\text{hom}}(\omega, A) := \lim_{r \to \infty} \frac{\tilde{m}_f(\omega, A, Q_r)}{\mathcal{L}^n(Q_r)}$$

is a well-defined random variable satisfying conditions (1)-(3) above.

With the parametrised functionals (4.1) being tailored to the Sobolev space regime, the existing mechanisms grant the possibility of computing the anticipated $\Gamma$-limit thereby.

**Theorem 4.1.** The functionals $\mathcal{E}_{\varepsilon_k}(\omega) : \Omega \times \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n; V) \times \mathcal{A}(\mathbb{R}^n) \to [0, +\infty]$ $\Gamma$-converge almost surely in $\Omega$ as $k \to \infty$ to $\mathcal{E}_{\text{hom}}(\omega) : \Omega \times \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n; V) \times \mathcal{A}(\mathbb{R}^n) \to [0, +\infty]$ in the underlying topology of $\mathcal{L}^1_{\text{loc}}(\mathbb{R}^n; V)$ where

$$\mathcal{E}_{\text{hom}}(\omega)[u; U] = \begin{cases} \int_U \tilde{f}_{\text{hom}}(\omega, A u) \, dx & \text{for } u \in W^{1,p}(U; V) \\ +\infty & \text{otherwise in } \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n; V). \end{cases}$$

In case of $(\tau_z)_{z \in \mathbb{Z}^n}$ being ergodic, $\mathcal{E}_{\text{hom}}$ becomes deterministic and

$$\tilde{f}_{\text{hom}}(A) := \lim_{k \to \infty} k^{-n} \int_{\Omega} \tilde{m}_f(\omega, A, (0, k)^n) \, d\mathbb{P}.$$

**Proof.** We will give a sketch as the methodology is very much the same as for the proof contained in Section 3. Let $u \in W^{1,p}(U; V)$. Resorting to the blow-up sequence construction in Theorem 3.2 for $u_k \rightharpoonup u$ in $W^{1,p}(U; V)$ (passing to a subsequence if necessary) and $\nu_k \rightharpoonup \nu$ as in (3.3) it suffices to show

$$\frac{d\nu}{d\mathcal{L}^n}(x) \geq f(\omega, Au(x))$$
for $\mathcal{L}^n$-a.e. $x \in U$ to settle the lower bound. In doing so, mimicking the calculations in Step 1 of Theorem 3.2 cf. (3.9), (3.10), (3.11) we obtain

\begin{align*}
\tilde{m}_f(\omega, A u(x), Q_r(x)) \leq & \frac{\nu_k(Q_r(x))}{\mathcal{L}^n(Q_r(x))} + 2c_4(1 + |A u(x)|^p)(1 - r^n) \\
&+ \frac{c_2 c_{r,h}}{\mathcal{L}^n(Q_r(x))} \int_{Q_r \setminus Q_{r-1}} |\nabla(u_k - u_0)|^p \, dx + \frac{c_3}{(1 - r)} \int_{Q_r(x)} \frac{|u_k - u_0|^p}{r^p} \, dx.
\end{align*}

From the properties of Sobolev spaces e.g. [37] Thm. 3.4.2] the fourth term on the right-hand side tends to zero as $k \to \infty$ and $r \to 0$ whilst the remaining terms behave in the same way as in Theorem 3.2. Since $\tilde{m}_f$ defines a subadditive process applying Theorem 2.11 to the right hand side of the above expression gives the desired bound.

Concerning the upper bound, in view of approximation of $W^{1,p}$-spaces by piecewise affine maps, cf. Lemma 3.4 reproducing the proof of Theorem 3.5 Step 1-3 yields the existence of $(u_k) \subset L^1(U; V)$ converging to $u$ strongly in $L^1(U; V)$ such that

\[ \limsup_{k \to \infty} E_{\varepsilon_k}(\omega)[u_k; U] \leq E_{\text{hom}}(\omega)[u; U]. \]

\[ \square \]

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