ON THE TRACE REGULARITY RESULTS OF MUSIELAK-ORLICZ-SOBOLEV SPACES IN A BOUNDED DOMAIN

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Abstract. Under some reasonable conditions, some trace embedding properties of Musielak-Sobolev spaces in a bounded domain are given, including the trace on the inner lower dimensional hyperplane and the trace on the boundary. Furthermore, a compact trace embedding on the boundary is given.

1. Introduction. In the study of nonlinear differential equations, it is well known that more general functional space can handle differential equations with more complex nonlinearities. If we want to study general forms of differential equations, it is very important to find a proper functional space in which the solutions may exist. Musielak-Orlicz-Sobolev space, or for short, Musielak-Sobolev space is such a general form of Sobolev space that the classical Sobolev spaces, variable exponent Sobolev spaces and Orlicz-Sobolev spaces can be interpreted as its special cases.

The properties and applications of Orlicz-Sobolev spaces and variable exponent Sobolev spaces have been studied extensively in recent years, see for example [3, 4, 10, 11]. In [1, 5], analogies for the Sobolev spaces, the Orlicz-Sobolev spaces are studied, including analogies for Sobolev embedding theorems and the trace properties on the lower dimensional hyperplane. In [6] Fan considers the trace embedding of the variable exponent Sobolev space. In a recent paper [3], the authors consider the $W^{1,p(\cdot)}$-regularity for elliptic equations with measurable coefficients in non-smooth domains. To our best knowledge, however, the properties of Musielak-Sobolev space have been studied little. In the research of the paper [2], Benkirane and Sidi prove an existence result for some class of variational boundary value problems for quasilinear elliptic equations in the Musielak-Orlicz space. And in that paper, an embedding theorem has also been provided without assuming the $\Delta_2$ condition. In two recent papers [8, 9], Fan gives some properties about this kind of functional space, including an embedding theorem and a compact embedding theorem in a bounded domain. And in [7] Fan and Guan study the uniform convexity of the Musielak-Sobolev spaces and present some applications. As an
application of the embedding results in [9], the authors in [12] give the existence of solutions to a kind of quasilinear elliptic equation. Our aim in this paper is to study the trace regularity of the Musielak-Sobolev space in a bounded domain. It is a very important part of the Musielak-Sobolev space theory for the study of the Neumann boundary problems in differential equations.

The paper is organized as follows. In Section 2, for the convenience of the readers we recall some definitions and properties of the Musielak-Sobolev space. In Section 3, based on some estimates for the $N(\Omega)$ function (Lemma 3.5 - Lemma 3.8) and the classical trace theory on the hyperplane (Lemma 3.4), we develop the embedding theory for the trace on the inner lower dimensional hyperplane (Theorem 3.3). In Section 4, the boundary trace embedding theorem is given, including a compact embedding. To prove the trace estimate on the boundary, we will not use the method in the classical theories. In the classical theory for the trace on the boundary, one should firstly extend functions defined on the bounded domain $\Omega$ to a much larger domain $\Omega^\epsilon := \{x \in \mathbb{R}^n : \text{dist}(x, \Omega) < \epsilon\}$. And $\partial\Omega$ is considered as the inner hyperplane of $\Omega^\epsilon$. Then the trace estimate on $\partial\Omega$ can be obtained by the classical inner hyperplane trace theory. But our boundary trace estimate for the Musielak-Sobolev space is much more complex now brought by the much more nonlinearity of the $N(\Omega)$-function. By some basic computations, one can figure out that the nonlinearity of the $N(\Omega)$-function does not allow us to develop such an extension theory in the Musielak-Sobolev space as similar as in the classical Sobolev space. Then the theories developed in Section 3 are not valid for the boundary trace theory in Section 4. To our best knowledge, the boundary trace embedding theory even in Orlicz-Sobolev spaces (a special case for Musielak-Sobolev spaces) is not found in the existing mathematical literature. In Section 5, two examples satisfying the conditions in our theories are provided. All of our theories are considered in a bounded domain.

2. The Musielak-Orlicz-Sobolev spaces. In this section we recall some basic definitions and properties about Musielak-Orlicz-Sobolev spaces for the readers to refer. The proofs of these properties can be found in [7, 8, 9] and the references therein. We emphasis that some of these properties and theorems are a little different in the form from the references therein in order to make an application for our case. For example, assumption $(P_3)$, Theorem 2.4 and so on.

Firstly, we give the definition of “$N$-function” and “generalized $N$-function” as follows.

**Definition 2.1.** An extended real-valued function $A : [0, +\infty) \to [0, +\infty]$ is called a $\Phi$-function, denoted by $A \in \Phi$, if $A$ is a lower semi-continuous convex function vanishing at zero but not identically 0 or $+\infty$ on $(0, +\infty)$. Let $\Omega \subset \mathbb{R}^n$ be an open set. $A : \Omega \times [0, +\infty) \to [0, +\infty]$ is called a $\Phi(\Omega)$-function, denoted by $A \in \Phi(\Omega)$, if for a.e. $x \in \Omega$, $A(x, \cdot) \in \Phi$ and for each $t \in [0, +\infty)$, $A(\cdot, t) : \Omega \to [0, +\infty]$ is measurable.

A function $A : [0, +\infty) \to [0, +\infty)$ is called an $N$-function, denoted by $A \in N$, if $A$ is convex, $A(0) = 0$, $0 < A(t) \in C^0$ for $t \neq 0$, and the following conditions hold

$$\lim_{t \to 0^+} \frac{A(t)}{t} = 0 \text{ and } \lim_{t \to +\infty} \frac{A(t)}{t} = +\infty.$$
A function \( A : \Omega \times [0, +\infty) \rightarrow [0, +\infty) \) is called a generalized \( N \)-function, denoted by \( A \in \mathcal{N}(\Omega) \), if for each \( t \in [0, +\infty) \), the function \( A(\cdot, t) \) is measurable, and for a.e. \( x \in \Omega \), we have \( A(x, \cdot) \in \mathcal{N} \).

It is clear that if \( A \in \mathcal{N}(\Omega) \) then \( A \in \Phi(\Omega) \).

Let \( A \in \mathcal{N}(\Omega) \), the Musielak-Orlicz space \( L^A(\Omega) \) is defined by

\[
L^A(\Omega) := \left\{ u : u \text{ is a measurable real function, and } \exists \lambda > 0 \text{ such that } \int_\Omega A(x, |u(x)|) \frac{dx}{\lambda} < +\infty \right\}
\]

with the (Luxemburg) norm

\[
\|u\|_{L^A(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega A(x, |u(x)|) \frac{dx}{\lambda} \leq 1 \right\}.
\]

The Musielak-Sobolev space \( W^{1,A}(\Omega) \) can be defined by

\[
W^{1,A}(\Omega) := \{ u \in L^A(\Omega) : |\nabla u| \in L^A(\Omega) \}
\]

with the norm

\[
\|u\|_{W^{1,A}(\Omega)} = \|u\|_{1,A} := \|u\|_A + \|\nabla u\|_A,
\]

where \( \|\nabla u\|_A := \|\nabla u\|_A \).

\( A \) is called locally integrable if \( A(\cdot, t_0) \in L^1_{loc}(\Omega) \) for every \( t_0 > 0 \).

For \( x \in \Omega \) and \( t \geq 0 \), we denote by \( A(x, \cdot) \) the right-hand derivative of \( A(x, \cdot) \) at \( t \). Then \( A(x, t) = \int_0^t a(x, s) \, ds \) for \( x \in \Omega \) and \( t \in [0, +\infty) \).

Define \( \tilde{A} : \Omega \times [0, +\infty) \rightarrow [0, +\infty) \) by

\[
\tilde{A}(x, s) = \sup_{t \in [0, +\infty)} (st - A(x, t)) \text{ for } x \in \Omega \text{ and } s \in [0, +\infty).
\]

\( \tilde{A} \) is called the complementary function to \( A \) in the sense of Young. It is well known that \( \tilde{A} \in \mathcal{N}(\Omega) \) and \( A \) is also the complementary function to \( \tilde{A} \).

For \( x \in \Omega \) and \( s \geq 0 \), we denote by \( a^+_1(x, s) \) the right-hand derivative of \( \tilde{A}(x, \cdot) \) at \( s \). Then for \( x \in \Omega \) and \( s \geq 0 \), we have

\[
a^+_1(x, s) = \sup\{ t \geq 0 : a(x, t) \leq s \} = \inf\{ t > 0 : a(x, t) > s \}.
\]

**Proposition 1** (See [8, 13]). Let \( A \in \mathcal{N}(\Omega) \). Then the following assertions hold.

1. \( A(x, t) \leq a(x, t) t \leq A(x, 2t) \) for \( x \in \Omega \) and \( t \in [0, +\infty) \);

2. \( A \) and \( \tilde{A} \) satisfy the Young inequality

\[
st \leq A(x, t) + \tilde{A}(x, s) \text{ for } x \in \Omega \text{ and } s, t \in [0, +\infty)
\]

and the equality holds if \( s = a(x, t) \) or \( t = a^+_1(x, s) \).

Let \( A, B \in \mathcal{N}(\Omega) \). We say that \( A \) is weaker than \( B \), denoted by \( A \preccurlyeq B \), if there exist positive constants \( K_1, K_2 \) and a nonnegative function \( h \in L^1(\Omega) \) such that

\[
A(x, t) \leq K_1 B(x, K_2 t) + h(x) \text{ for } x \in \Omega \text{ and } t \in [0, +\infty).
\]

(1)

We say that \( A \) and \( B \) are equivalent near infinity, if there exist positive constants \( t_0, k_1, k_2 \) and nonnegative functions \( h_1, h_2 \in L^1(\Omega) \) such that

\[
B(x, k_1 t) - h_1(x) \leq A(x, t) \leq B(x, k_2 t) + h_2(x) \text{ for any } x \in \Omega, t \geq t_0.
\]

**Proposition 2** (See [8, 13]). Let \( A, B \in \mathcal{N}(\Omega) \) and \( A \preccurlyeq B \). Then \( \tilde{B} \preccurlyeq \tilde{A}, L^B(\Omega) \hookrightarrow L^A(\Omega) \) and \( L^\tilde{A}(\Omega) \hookrightarrow L^\tilde{B}(\Omega) \).
Proposition 3 (See [13]). Let $A \in N(\Omega)$ be locally integrable. Then $(L^A(\Omega), \| \cdot \|)$ is a separable Banach space.

In the study of the uniform convexity of the space $L^A(\Omega)$, the following definition is useful.

Definition 2.2. A function $A \in N(\Omega)$ is called uniformly convex, if there exists a function $\sigma$ mapping the interval $(0, 1)$ into itself such that for every $s > 0$, $0 < \alpha < 1$ and $0 \leq \beta \leq \alpha$, there holds the inequality

$$A\left(x, \frac{s + \beta s}{2}\right) \leq (1 - \sigma(\alpha)) \frac{A(x, s) + A(x, \beta s)}{2} \text{ for a.e. } x \in \Omega.$$ 

We customarily call this uniformly convex condition the (UC)$_1$ condition. If an $N(\Omega)$ function satisfies (UC)$_1$ condition, then this function satisfies the classical (UC) condition, see [7].

Remark 1. One can verify that $A(x, t) = |t|^p$ with a constant $p > 1$ is uniformly convex, with

$$\sigma(\alpha) = 1 - 2^{-p+1} \frac{(1 + \alpha)^p}{1 + \alpha^p},$$

see Remark 11.4 in [13].

We extend the well-known $\Delta_2$ condition defined in [1, 5] here.

Definition 2.3. We say that a function $A : [0, +\infty) \to [0, +\infty)$ satisfies the $\Delta_2(\Omega)$ condition, denoted by $A \in \Delta_2(\Omega)$, if there exist a positive constant $K > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$A(x, 2t) \leq KA(x, t) + h(x) \text{ for } x \in \Omega \text{ and } t \in (0, +\infty).$$

If $A(x, t) = A(t)$ is an $N$-function and $h(x) \equiv 0$ in $\Omega$ in Definition 2.3, then $A \in \Delta_2(\Omega)$ if and only if $A$ satisfies the well-known $\Delta_2$ condition defined in [1, 5].

Proposition 4 (See [7]). Let $A \in N(\Omega) \cap \Delta_2(\Omega)$ be uniformly convex. Then the space $L^A(\Omega)$ is uniformly convex and the space $L^A(\Omega)$ is of the Radon-Riesz property in the modular sense, i.e., whenever $\{v_n\} \subset X$, $v_n \rightharpoonup v_0$ in $L^A(\Omega)$ and $\int_\Omega A(x, |v_n|) \, dx \to \int_\Omega A(x, |v_0|) \, dx$, it follows that $v_n \to v_0$ in $L^A(\Omega)$.

Let $A \in N(\Omega)$ be locally integrable. We will denote

$$W_0^{1,A}(\Omega) := C^\infty_0(\Omega) \| \cdot \|_{W^{1,A}(\Omega)},$$

$$D_0^{1,A}(\Omega) := C^\infty_0(\Omega) \| \nabla \cdot \|_{L^A(\Omega)}.$$ 

In the case $\| \nabla u \|_A$ is an equivalent norm in $W^{1,A}_0(\Omega)$, $W^{1,A}_0(\Omega) = D_0^{1,A}(\Omega)$.

The following assumptions will be used.

(P$_1$) $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded domain with the cone property, and $A \in N(\Omega)$;

(P$_2$) $A : \overline{\Omega} \times [0, +\infty) \to [0, +\infty)$ is continuous and $A(x, t) \in (0, +\infty)$ for $x \in \Omega$ and $t \in (0, +\infty)$.

Let $A$ satisfy (P$_1$) and (P$_2$). Denote by $A^{-1}(x, \cdot)$ the inverse function of $A(x, \cdot)$. We always suppose that the following condition holds.

(P$_3$) $A \in N(\Omega)$ satisfies

$$\int_0^1 \frac{A^{-1}(x, t)}{t^{\frac{n+1}{2}}} \, dt < +\infty, \forall x \in \overline{\Omega}, \tag{2}$$
replacing, if necessary, $A$ by another $N(\Omega)$-function equivalent to $A$ near infinity.

Similarly as mentioned in [1], since $\Omega$ is bounded, (2) places no restrictions on $A$ from the point of view of embedding theory since $N(\Omega)$-functions which are equivalent near infinity determine identical Musielak-Orlicz spaces in that case.

Under assumptions $(P_1)$, $(P_2)$ and $(P_3)$, for each $x \in \overline{\Omega}$, the function $A(x, \cdot) : [0, +\infty) \to [0, +\infty)$ is a strictly increasing homeomorphism. Define a function

$$A^{-1}_x : \overline{\Omega} \times [0, +\infty) \to [0, +\infty)$$

by

$$A^{-1}_x(x, s) = \int_0^s \frac{A^{-1}(x, \tau)}{\tau} \, d\tau \text{ for } x \in \overline{\Omega} \text{ and } s \in [0, +\infty).$$

Then under the assumption $(P_3)$, $A^{-1}_x$ is well defined, and for each $x \in \overline{\Omega}$, $A^{-1}_x(x, \cdot)$ is strictly increasing, $A^{-1}_x(x, \cdot) \in C^1([0, +\infty))$ and the function $A^{-1}_x(x, \cdot)$ is concave.

Set

$$T(x) = \lim_{s \to +\infty} A^{-1}_x(x, s), \forall x \in \overline{\Omega}. \tag{3}$$

Then $0 < T(x) \leq +\infty$. Define a function $A_* : \overline{\Omega} \times [0, +\infty) \to [0, +\infty)$ by

$$A_*(x, t) = \begin{cases} s, & \text{for } x \in \overline{\Omega}, t \in [0, T(x)) \text{ and } A^{-1}_x(x, s) = t, \\ +\infty, & \text{for } x \in \overline{\Omega} \text{ and } t \geq T(x). \end{cases}$$

Then $A_* \in \Phi(\Omega)$, and for each $x \in \overline{\Omega}$, $A_*(x, \cdot) \in C^1(0, T(x))$. In particular, if $A \in N(\Omega)$ and $T(x) = +\infty$ for any $x \in \overline{\Omega}$, it is well known that $A_* \in N(\Omega)$ (see [1]). $A_*$ is called the Sobolev conjugate function of $A$ (see [1] for the case of Orlicz functions).

Let $X$ be a metric space and $f : X \to (-\infty, +\infty]$ be an extended real-valued function. For $x \in X$ with $f(x) \in \mathbb{R}$, the continuity of $f$ at $x$ is well defined. For $x \in X$ with $f(x) = +\infty$, we say that $f$ is continuous at $x$ if given any $M > 0$, there exists a neighborhood $U$ of $x$ such that $f(y) > M$ for all $y \in U$. We say that $f : X \to (-\infty, +\infty]$ is continuous on $X$ if $f$ is continuous at every $x \in X$.

Define $\text{Dom}(f) = \{ x \in X : f(x) \in \mathbb{R} \}$ and denote by $C^{1,0}(X)$ the set of all locally Lipschitz continuous real-valued functions defined on $X$.

The following assumptions will also be used.

$(P_4)$ $T : \overline{\Omega} \to [0, +\infty)$ is continuous on $\overline{\Omega}$ and $T \in C^{1,0}(\text{Dom}(T))$;

$(P_5)$ $A_* \in C^{1,0}(\text{Dom}(A_*))$ and there exist positive constants $\delta_0 < \frac{1}{\pi}$, $C_0$ and $t_0 < \min_{x \in \overline{\Omega}} T(x)$ such that

$$|\nabla_x A_*(x, t)| \leq C_0 (A_*(x, t))^{1+\delta_1},$$

for $x \in \Omega$ and $t \in [t_0, T(x))$ provided $\nabla_x A_*(x, t)$ exists.

Let $A, B \in \Phi(\Omega)$. We say that $A \ll B$ if, for any $k > 0$,

$$\lim_{t \to +\infty} \frac{A(x, kt)}{B(x, t)} = 0 \text{ uniformly for } x \in \Omega.$$

We give an embedding theorem for Musielak-Sobolev spaces developed by Fan in [9], see also in [12] and remarks therein.

**Theorem 2.4** (See [9, 12]). Let $(P_1) - (P_5)$ hold. Then

(i) There is a continuous embedding $W^{1,A}(\Omega) \hookrightarrow L^{A_*}(\Omega)$;

(ii) Suppose that $B \in \Phi(\Omega)$, $B : \overline{\Omega} \times [0, +\infty) \to [0, +\infty)$ is continuous, and $B(x, t) \in (0, +\infty)$ for $x \in \Omega$ and $t \in (0, +\infty)$. If $B \ll A_*$, then there is a compact embedding $W^{1,A}(\Omega) \hookrightarrow L^B(\Omega)$. 

3. Trace on the inner lower dimensional hyperplane. In this section we study the trace regularity on the inner lower dimensional hyperplane in a bounded domain. We always assume the following condition holds.

\( (P^n_\partial) \) A is an \( N(\Omega) \)-function such that \( T(x) \) defined in equation (3) satisfies

\[
T(x) = +\infty, \forall x \in \overline{\Omega}.
\]

And there exists a constant \( p \) (1 ≤ \( p < n \)) such that the function \( B(x,t) = A(x,t) \) is an \( N(\Omega) \)-function.

We study the trace regularity on the inner lower dimensional hyperplane in a bounded domain. We always assume the following condition holds.

\( (\partial \Omega) \) A trace on the inner lower dimensional hyperplane.

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1. Suppose that \( \Omega \) is an open bounded subset with Lipschitz boundary \( \partial \Omega \). Then there exists a bounded operator

\[
T : W^{1,1}(\Omega) \rightarrow L^1(\Omega_k) \ (or \ L^1(\partial \Omega))
\]

such that

(i) \( Tu = u|_{\Omega_k} \) (or \( Tu = u|_{\partial \Omega} \)) if \( u \in W^{1,1}(\Omega) \cap C(\overline{\Omega}) \), and

(ii) \( \|Tu\|_{L^1(\Omega_k)} \leq C\|u\|_{W^{1,1}(\Omega)} \) (or \( \|Tu\|_{L^1(\partial \Omega)} \leq C\|u\|_{W^{1,1}(\Omega)} \)) for each \( u \in W^{1,1}(\Omega) \), with the constant \( C \) depending only on \( \Omega \).

Definition 3.2. We call \( Tu \) the trace of \( u \) on \( \Omega_k \) (or \( \partial \Omega \)) denoted by \( u|_{\Omega_k} \) (or \( u|_{\partial \Omega} \)) for any \( u \in W^{1,1}(\Omega) \).

Since by \( (P_2) \), \( \inf_{x \in \Omega} A(x,1) = S_* > 0 \). Thus, for any \( x \in \overline{\Omega} \) and \( t > 1 \), one has that \( A(x,t) \geq tA(x,1) \geq S_* t \), which implies that \( L^A(\Omega) \hookrightarrow L^1(\Omega) \) and \( W^{1,A}(\Omega) \hookrightarrow W^{1,1}(\Omega) \). Based on the above fact, for any \( u \in W^{1,A}(\Omega) \), we can define its trace in the space \( W^{1,1}(\Omega) \) in the sense of Definition 3.2.

The main theorem of this section is the following.

Theorem 3.3. Suppose that \( A \) satisfies \( (P_1) - (P_5) \) and \( (P^n_\partial) \). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) satisfying the cone condition, and let \( \Omega_k \) denote the intersection of \( \Omega \) with a \( k \)-dimensional hyperplane in \( \mathbb{R}^n \).

1. If either \( n - p < k \leq n \) or \( p = 1 \) and \( n - 1 \leq k \leq n \), then

\[
W^{1,A}(\Omega) \hookrightarrow L^A(\Omega_k),
\]

where \( A^\frac{\theta}{\tau}(x,t) = [A_\tau(x,t)]^{\frac{\theta}{\tau}} \).

2. If \( p > 1 \) and \( \theta(x,t) \) is an \( N(\Omega) \)-function such that \( \theta \ll A^\frac{\theta}{\tau} \), then the embedding

\[
W^{1,A}(\Omega) \hookrightarrow L^\theta(\Omega_k)
\]

is compact.

Remark 2. In our theorem, \( A \) satisfies \( (P^n_\partial) \) and \( (P_1) - (P_5) \). We claim that, in fact, under the assumption \( (P^n_\partial), (P_4) \) is automatically satisfied.

In the following of this paper we will use the following notations. Denote that

\[
f(y,t) = [A_\tau(y,t)]^{1/q},
\]

where \( q = np/(n-p) \) and \( (y,t) \in \Omega \times [0, +\infty) \), and denote the derivative of the function in a weak sense

\[
f_{y_j}(y,t) = \frac{\partial f}{\partial y_j}(y,t) \quad \text{and} \quad f_t(y,t) = \frac{\partial f}{\partial t}(y,t).
\]
Define
\[ \Omega_0 = \{ x \in \Omega : u(x) = 0 \} \text{ and } \Omega_+ = \{ x \in \Omega : |u(x)| > 0 \}. \]  

We give some lemmas under the assumptions of Theorem 3.3 to prove the main theorem.

Lemma 3.4 (See Theorem 4.12 in [1]). Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and, for \( 1 \leq k \leq n \), let \( \Omega_k \) be the intersection of \( \Omega \) with a hyperplane of dimension \( k \) in \( \mathbb{R}^n \). (If \( k = n \), then \( \Omega_k = \Omega \).) Suppose that \( \Omega \) satisfies the cone condition, \( p < n \) and either \( n - p < k \leq n \) or \( p = 1 \) and \( n - 1 \leq k \leq n \). Then
\[ W^{1,p}(\Omega) \hookrightarrow L^q(\Omega_k) \text{ for } p \leq q \leq p^* = \frac{kp}{n-p}. \]

Lemma 3.5 (See [9]). 1. There exists a positive constant \( C \) such that
\[ \left| \frac{\partial A_*(y,t)}{\partial y_j} \right| \leq C(A_*(y,t) + (A_*(y,t))^{1+\delta_0}), \quad j = 1, 2, \ldots, n \]
for all \( y \in \Omega \) and \( t \in [0,T(x)] \) provided \( \frac{\partial A_*(y,t)}{\partial y_j}, j = 1, 2, \ldots, n, \) exists, where \( \delta_0 < \frac{1}{n} \) is the constant appeared in assumption \((P_0)\).
2. Let \( v \in W^{1,1}(\Omega) \setminus \{0\} \) with \( v(x) \in [0,T(x)] \) for a.e. \( x \in \Omega \). Put \( g(x) = A_*(x,v(x)) \). Then for every \( j = 1, 2, \ldots, n \), the weak derivative \( g'_{x_j} \) of \( g \) exists and
\[ g'_{x_j}(x) = \frac{\partial A_*(x,v(x))}{\partial x_j} + \frac{\partial A_*(x,v(x))}{\partial t} v'_{x_j}(x) \text{ for a.e. } x \in \Omega. \]
3. Let \( A,B \in \Phi(\Omega) \). Suppose that \( B : \overline{\Omega} \times [0,\infty) \to [0,\infty) \) is continuous, \( B(x,t) \in (0,\infty) \) for \( x \in \Omega \) and \( t \in (0,\infty) \), and \( B \ll A \). If a sequence \( \{v_m\} \) is bounded in \( L^A(\Omega) \) and convergent in measure on \( \Omega \), then it is convergent in norm in \( L^B(\Omega) \).

Lemma 3.6. For any given constant \( \epsilon > 0 \), there exists a constant \( C(\epsilon) > 0 \), such that for any \( v \in W^{1,A}(\Omega) \)
\[ \|f'_{y_j}(x,|v(x)|)\|_{p,\Omega}^p \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \int_\Omega A_*(x,|v(x)|) \, dx + C(\epsilon)\|v\|^p_{p,A}. \]

Proof. By a direct calculus we have
\[ (f'_{y_j}(y,t))^p = \frac{1}{q^p}(A_*(x,t))^\frac{p(1-q)}{q}((A_*)'_{y_j}(y,t))^p. \]
Combining with assumption \((P_0^p)\)-\( (4) \), the above equation implies, by Lemma 3.5, that
\[ \left| \frac{\partial A_*(y,t)}{\partial y_j} \right| \leq C(A_*(y,t) + (A_*(y,t))^{1+\delta_0}), \forall y \in \Omega, \forall t \in \mathbb{R}^+. \]
we conclude that for any $v \in W^{1,A}(\Omega)$,
\[
\|f'_{y_j} (x, |v(x)|)\|_{p, \Omega}^p = \int_{\Omega} \left| f'_{y_j} (x, |v(x)|) \right|^p \, dx \\
= \frac{1}{q^p} \int_{\Omega} \left( A_s (x, |v(x)|) \right)^{\frac{n(1-s)}{q}} \left( (A_s)'_{y_j} (x, |v(x)|) \right)^p \, dx \\
\leq \frac{C}{q^p} \int_{\Omega} \left( \left( A_s (x, |v(x)|) \right)^{\frac{1}{q}} + \left( A_s (x, |v(x)|) \right)^{\frac{1}{q} + \delta_0} \right)^p \, dx \\
\leq \frac{C}{q^p} \int_{\Omega} \left( \left( A_s (x, |v(x)|) \right)^{\frac{1}{q}} + \left( A_s (x, |v(x)|) \right)^{p(\frac{1}{q} + \delta_0)} \right) \, dx \\
:= \frac{C}{q^p} \int_{\Omega} I(x) \, dx.
\]
Take $t_1 = t_1(\epsilon) > 0$ small enough such that $I \leq \frac{ep^p}{2C |\Omega|}$ for a.e. $x \in \Omega_1 := \{ x \in \Omega : |v(x)| \leq t_1 \}$. Then
\[
\int_{\Omega_1} I(x) \, dx \leq \frac{ep^p}{2C} \tag{8}
\]
Since $\delta_0 < \frac{1}{n}$ in assumption $\{P_b\}$, $p(\frac{1}{q} + \delta_0) < 1$. Combining with $\frac{1}{q} < 1$, we can take $M = M(\epsilon) > 0$ big enough such that $I \leq \frac{ep^p}{2C} A_s (x, |v(x)|)$ for a.e. $x \in \Omega_2 := \{ x \in \Omega : A_s (x, |v(x)|) \geq M \}$. Then we have
\[
\int_{\Omega_2} I(x) \, dx \leq \frac{ep^p}{2C} \int_{\Omega} A_s (x, |v(x)|) \, dx. \tag{9}
\]
Moreover, we have that $I(x) \leq C |v(x)|^p$ for $x \in \Omega \setminus (\Omega_1 \cup \Omega_2)$, where $C = C(\epsilon)$ is a constant. Then by Young inequality
\[
\int_{\Omega \setminus (\Omega_1 \cup \Omega_2)} I(x) \, dx \leq \frac{C}{q^p} \int_{\Omega} |v(x)|^p \, dx \leq 2C \| |v| \|_{B, \Omega} |v|_{B, \Omega} \leq C(\epsilon) \|v\|_{A, \Omega}^p. \tag{10}
\]
By (8), (9) and (10) we can see that the conclusion holds. \qed

**Lemma 3.7.** For any $t > 0$, the following estimate holds
\[
\tilde{B}(x, (q f'_t (x, t))^p) \leq A_s (x, t). \tag{11}
\]

**Proof.** By the definition of $B$, $\tilde{B}$ and $A_s$, one can verify the following equations respectively
\[
B^{-1} (x, t) = (A^{-1} (x, t))^p, \forall x \in \Omega, t \geq 0; \tag{12}
B^{-1} (x, t)(\tilde{B})^{-1} (x, t) \geq t, \forall x \in \Omega, t \geq 0; \tag{13}
A^{-1} (x, A_s (x, t))(A_s)' (x, t) = (A_s (x, t))^\frac{n+1}{p}, \forall x \in \Omega, t \geq 0. \tag{14}
\]
Hence $\forall x \in \Omega$ and $t > 0$ we have
\[
\left( f'_t (x, t) \right)^p = \frac{1}{q^p} (A_s (x, t))^p(\frac{1}{q} - 1)((A_s)' (x, t))^p \\
= \frac{1}{q^p} \frac{A_s (x, t)}{B^{-1} (x, A_s (x, t))} \leq \frac{1}{q^p} (\tilde{B})^{-1} (x, A_s (x, t)),
\]
which implies that
\[
\tilde{B}(x, (q f'_t (x, t))^p) \leq A_s (x, t). \tag{15}
\]
\qed
Lemma 3.8. For any $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that

$$
(f(x,t))^p \leq \epsilon A_*(x,t) + C(\epsilon)\,\|u\|_{1,\Omega}, \forall \, x \in \Omega.
$$

(15)

Proof. Set $g(x,t) = \frac{A_*(x,t)}{t^p}$ and $h(x,t) = \left(\frac{f(x,t)}{t}\right)^p$. Then

$$
\lim_{t \to +\infty} \frac{g(x,t)}{h(x,t)} = \lim_{t \to +\infty} \frac{A_*(x,t)}{(A_*(x,t))^p} = \lim_{t \to +\infty} (A_*(x,t))^\frac{p}{p} = +\infty, \forall \, x \in \Omega.
$$

(16)

On the other hand by the definition of $A_*$, we have

$$(h(x,t))^\frac{1}{p} = \frac{(A_*(x,t))^\frac{1}{p}}{t} = \frac{(A_*(x,t))^\frac{1}{p}}{t} = \frac{\int_0^{A_*(x,t)} \frac{A^{-1}(x,\tau)}{\tau} \, d\tau}{\int_0^{A_*(x,t)}} = \frac{\frac{1}{p}}{p}, \forall \, x \in \Omega.
$$

(17)

By assumption $(P^p_0)$ on the function $B(x,t)$ we can see that $\lim_{\tau \to 0^+} \frac{B^{-1}(x,\tau)}{\tau} = +\infty$. Then for small $t > 0$, we have

$$(h(x,t))^\frac{1}{p} \leq \frac{(A_*(x,t))^\frac{1}{p}}{t} \leq \frac{1}{p}, \forall \, x \in \Omega.
$$

(18)

By (16) and (17), for any $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that

$$
(f(x,t))^p \leq \epsilon A_*(x,t) + C(\epsilon)\,\|u\|_{1,\Omega}, \forall \, x \in \Omega,
$$

which completes the proof of the lemma.

Based on the above lemmas now we can give the proof of Theorem 3.3.

Proof. Step 1. Suppose that $u \in W^{1,A}(\Omega)$ is a bounded function. From the assumption $(A^p_0)$, it is easy to check that $A_*^\frac{1}{p}$ is an $N(\Omega)$-function. And by the definition of the norm in $L^{A_*^\frac{1}{p}}(\Omega)$, we have

$$
\int_{\Omega} A_*^\frac{1}{p} \left( x, \frac{|u(x)|}{K} \right) \, dx = 1,
$$

where $K = \|u\|_{A_*^\frac{1}{p},\Omega}$ since $u$ is bounded. We will show that

$$
K \leq C\|u\|_{1,\Omega}.
$$

(19)

with a constant $C > 0$ independent of $u$. By Theorem 2.4, there exists a constant $C > 0$ such that $\|u\|_{A_*,\Omega} \leq C\|u\|_{1,\Omega}$. Then if $\|u\|_{A_*,\Omega} \geq K$, the proof is completed. In the rest of the proof we suppose that

$$
\|u\|_{A_*,\Omega} \leq K,
$$

and without loss of generality, we assume that $u(x) \geq 0$.

It is obvious that $\frac{\partial f(x,\frac{u(x)}{K})}{\partial x_j} = 0$ for a.e. $x \in \Omega_0$. By Lemma 3.5, we can see for a.e. $x \in \Omega_+$,

$$
\frac{\partial f(x,\frac{u(x)}{K})}{\partial x_j} = f_{y_j}(x, \frac{u}{K}) + \frac{1}{K^p} f'(x, \frac{u}{K})u'_j(x).
$$

(20)
Then by Lemma 3.4, there exists a constant \( C_* > 0 \) independent of \( u \) such that
\[
1 = \left[ \int_{\Omega_k} A^k_\eta(x, \frac{|u(x)|}{K}) \, dx \right]^{\frac{n}{n-p}}
= \left\| f(x, \frac{u}{K}) \right\|^{p}_{\frac{n}{n-p}, \Omega_k}
\leq C_* \left[ \sum_{j=1}^{n} \left\| \frac{\partial f(x, \frac{u(x)}{K})}{\partial x_j} \right\|_{p, \Omega} \right]^p
+ \left\| f(x, \frac{u}{K}) \right\|_{p, \Omega}^p
= C_* \left[ \sum_{j=1}^{n} \left\| \frac{\partial f(x, \frac{u(x)}{K})}{\partial x_j} \right\|_{p, \Omega^+} \right]^p
+ \left\| f(x, \frac{u}{K}) \right\|_{p, \Omega^+}^p
\leq C_* C_1 \sum_{j=1}^{n} \left\| f'(x, \frac{u(x)}{K}) \right\|_{B, \Omega^+} \left( u'(x) \right)^p \left\| u'(x) \right\|_{B, \Omega^+} + C_* \left\| f(x, \frac{u}{K}) \right\|_{p, \Omega}^p
\leq C_* C_1 \sum_{j=1}^{n} \left\| f'(x, \frac{u(x)}{K}) \right\|_{p, \Omega} \left( f'(x, \frac{u(x)}{K}) \right)^p \left\| u'(x) \right\|_{B, \Omega^+} + C_* \left\| f(x, \frac{u}{K}) \right\|_{p, \Omega}^p
\leq C_* C_1 \sum_{j=1}^{n} \left\| f'(x, \frac{u(x)}{K}) \right\|_{p, \Omega} \left( f'(x, \frac{u(x)}{K}) \right)^p \left\| u \right\|_{1, A, \Omega}^p
+ C_* \left\| f(x, \frac{u}{K}) \right\|_{p, \Omega}^p
\leq C_* C_1 I_1 + \frac{2nC_* C_1}{K^p} I_2 + C_* I_3.
\]
By Lemma 3.6 and (19) there exists a constant \( C > 0 \), such that
\[
I_1 \leq \frac{ne}{2} + \frac{ne}{2} \int_{\Omega} A_\epsilon(x, \frac{|u(x)|}{K}) \, dx + nC(\epsilon) \frac{\| u \|_{A, \Omega}^p}{K} \leq ne + nC(\epsilon) \frac{\| u \|_{A, \Omega}^p}{K}.
\]
Taking \( \epsilon = \frac{1}{4nC_* C_1} \), we have
\[
C_* C_1 I_1 \leq \frac{1}{4} + CC_* \frac{\| u \|_{A, \Omega}^p}{K^p}.
\] (21)
By Lemma 3.7 and (19), noticing that Lemma 3.7 holds for \( t > 0(\neq 0) \), we can see that
\[
\int_{\Omega^+} B(x, \frac{f'(x, \frac{u(x)}{K})}{1/t}) \, dx \leq \int_{\Omega^+} A_\epsilon(x, \frac{u(x)}{K}) \, dx \leq \int_{\Omega^+} A_\epsilon(x, \frac{u(x)}{K}) \, dx \leq 1.
\]
So by the definition of the norm for $L^\tilde{B}_p(\Omega)$ (since $\tilde{B} \in N(\Omega)$), we conclude

$$\left\| \left(f \left( x, \frac{u(x)}{K} \right) \right)^p \right\|_{\tilde{B}, \Omega}^p \leq \frac{1}{q^p}.$$  

Then there exists a constant $C > 0$ such that

$$2nC_1C^*_1\frac{I_2}{K^p} \leq \frac{CC^*_1}{K^p} \|u\|_{1,A,\Omega}^p.$$  

(22)

Take $\epsilon = \frac{1}{4C^*_1}$ in Lemma 3.8. There exists a constant $C > 0$ such that

$$\left(f(x,t)\right)^p \leq \frac{1}{2}A_*(x,t) + C t^p, \forall x \in \Omega.$$  

Then by (19) there exists a constant $C > 0$ such that

$$C^*_1I_3 = C^*_1\int_\Omega \left(f(x, \frac{u}{K})\right)^p \, dx$$

$$\leq \frac{1}{4} \int_\Omega A_*(x, \frac{u}{K}) \, dx + \frac{CC^*_1}{K^p} \int_\Omega |u(x)|^p \, dx$$

$$\leq \frac{1}{4} + \frac{CC^*_1}{K^p} \|u\|_{1,A,\Omega}^p.$$  

(23)

Combining (21), (22) and (23) we conclude

$$1 \leq \frac{1}{2} + \frac{CC^*_1}{K^p} \|u\|_{1,A,\Omega}^p + \frac{CC^*_1}{K^p} \|u\|_{A,\Omega}^p,$$

which implies (18).

Step 2. For a general $u \in W^{1,A}(\Omega)$, define

$$u_m(x) = \begin{cases} u(x), & \text{if } |u(x)| \leq m, \\ m, & \text{if } u(x) > m, \\ -m, & \text{if } u(x) < -m. \end{cases}$$

Clearly $u_m \in W^{1,A}(\Omega)$ is bounded. By Step 1, we can see that there exists a constant $C > 0$ independent of $u_m$, such that

$$\|u_m\|^p_{A^{\frac{1}{K}}_1,\Omega_k} \leq C\|u\|_{1,A,\Omega} \leq C\|u\|_{1,A,\Omega},$$

which implies that $\|u_m\|^p_{A^{\frac{1}{K}}_1,\Omega_k}$ increases with $m$ but is bounded by $C\|u\|_{1,A,\Omega}$. Then

$$\lim_{m \to +\infty} \|u_m\|^p_{A^{\frac{1}{K}}_1,\Omega_k} := K \text{ exists and}$$

$$K \leq C\|u\|_{1,A,\Omega}.$$  

(24)

From Fatou’s lemma

$$\int_{\Omega_k} A^{\frac{1}{K}}_1(x, \frac{|u(x)|}{K}) \, dx \leq \lim_{m \to +\infty} \int_{\Omega_k} A^{\frac{1}{K}}_1(x, \frac{|u_m(x)|}{K}) \, dx \leq 1,$$

which implies that

$$\|u\|^p_{A^{\frac{1}{K}}_1,\Omega_k} \leq K.$$  

(25)

Combining (24) and (25) we can see that (18) holds for any $u_m \in W^{1,A}(\Omega)$. Part (1) of the theorem is proved.
Step 3. Compactness of the embedding. Let \( \{u_m\} \) be a bounded sequence in \( W^{1,A}(\Omega) \). From the first part of the theorem, \( \{u_m\} \) is bounded in \( L^{A^\frac{1}{n}}(\Omega) \). Since \( B(x,t) = A(x,t^\frac{1}{n}) \) is an \( N(\Omega) \)-function and \( \Omega \) is bounded, we have

\[
W^{1,A}(\Omega) \hookrightarrow W^{1,p}(\Omega) \hookrightarrow L^1(\Omega),
\]

in which the last embedding is compact for \( p > 1 \) by the Rellich-Kondrachov Theorem (Theorem 6.3 in [1]). Then there exists a subsequence \( \{u_{m_k}\} \) of \( \{u_m\} \) such that \( u_{m_k} \) converges in measure on \( \Omega \). By Lemma 3.5 (3), \( u_{m_k} \) converges in \( L^q(\Omega) \), which completes the proof of part (2) of the theorem.

\[ \square \]

4. Trace on the boundary. In this section we discuss the trace embedding on the boundary in a bounded domain. The following lemma is well known.

**Lemma 4.1** (See [1]). Let \( \Omega \subset \mathbb{R}^n \) be an open bounded domain with Lipschitz boundary. Let \( p \in [1,n) \) be a constant. Then there is a continuous boundary trace embedding \( W^{1,p}(\Omega) \hookrightarrow L^{\frac{np}{n-p}}(\partial \Omega) \). Moreover, for every \( q \in [1,\frac{(n-1)p}{n-p}) \), the trace embedding \( W^{1,p}(\Omega) \hookrightarrow L^{\frac{(n-1)p}{n-p}}(\partial \Omega) \) is compact.

In this section the following condition will be used.

\( (P_3) \quad A \) is an \( N(\Omega) \)-function such that \( T(x) \) defined in equation (3) satisfies

\[
T(x) = +\infty, \forall x \in \overline{\Omega}.
\]

(26)

A main theorem of this section is the following.

**Theorem 4.2.** Let \( \Omega \subset \mathbb{R}^n \) be an open bounded domain with Lipschitz boundary. Suppose that \( A \in N(\Omega) \) satisfies \( (P_3) \) and \( (P_3') \). Then there is a continuous boundary trace embedding \( W^{1,A}(\Omega) \hookrightarrow L^{A^\frac{n-1}{n}}(\partial \Omega) \), in which \( A^\frac{n-1}{n}(x,t) = [A_*(x,t)]^\frac{n-1}{n}, \forall x \in \Omega, t \in \mathbb{R}^+ \).

We give a lemma before proving the theorem.

**Lemma 4.3.** Let \( \Omega \subset \mathbb{R}^n \) be an open bounded domain with Lipschitz boundary. Suppose that \( A \in N(\Omega) \) satisfies \( (P_3) \) and \( (P_3') \). For every \( u \in W^{1,A}(\Omega) \), denote

\[
\varphi(x) = A^\frac{n-1}{n}(x,u(x)), \forall x \in \Omega.
\]

Then \( \varphi(x) \in W^{1,1}(\Omega) \).

**Proof.** Step 1. We show that \( \varphi(x) \in L^1(\Omega) \). By Theorem 2.4, \( u \in L^{A_*}(\Omega) \), or \( A_*(x,|u(x)|) \in L^1(\Omega) \). By the definition of \( A_* \), one can readily check \( A^\frac{1}{n} \in N(\Omega) \). Since \( A^\frac{1}{n} \ll A_* \), by Proposition 2, it is readily checked \( \varphi(x) \in L^1(\Omega) \).

Step 2. We show that \( |\nabla \varphi(x)| \in L^1(\Omega) \). Define \( \Omega_0 \) and \( \Omega_+ \) as in (6). It is readily checked that

\[
\frac{\partial \varphi(x)}{\partial x_j} = 0, \text{ for a.e. } x \in \Omega_0.
\]

(27)

By Lemma 3.5, we can see that

\[
\frac{\partial \varphi(x)}{\partial x_j} = (f_1)'(x,|u|)u_j'(x) + (f_2)'(x,|u|), \text{ for a.e. } x \in \Omega_+.
\]

(28)
where \( f_1(y,t) := [A_*(y,t)]^{\frac{n-1}{2}} \), i.e., \( f_1 \) is the special case of \( f \) in \((5)\) for \( p = 1 \). By Young inequality and \((11)\) we have
\[
|\langle f_1 \rangle_\varepsilon(x,|u|)u_{x_j}(x)| = \frac{n-1}{n} \left| \frac{n}{n-1} \langle f_1 \rangle_\varepsilon(x,|u|) \right| u_{x_j}(x) \\
\leq \frac{n-1}{n} \left( \tilde{A}(x) \left| \frac{n}{n-1} \langle f_1 \rangle_\varepsilon(x,|u|) \right| + A(x,|u_{x_j}(x)|) \right) \\
\leq \frac{n-1}{n} \left( A_*(x,|u|) + A(x,|u_{x_j}(x)|) \right).
\]

At the same time, by Lemma 3.6, we conclude for some \( \varepsilon > 0 \), there exists a \( C(\varepsilon) > 0 \) such that
\[
\int_{\Omega_\varepsilon} \left| \langle f_1 \rangle_\varepsilon(x,|u(x)|) \right| \, dx \leq \int_{\Omega} \left| \langle f_1 \rangle_\varepsilon(x,|u(x)|) \right| \, dx \\
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \int_{\Omega} A_*(x,|u(x)|) \, dx + C(\varepsilon)\|u\|_{A,\Omega}.
\]
From \((27)-(30)\) we can see
\[
\|\nabla \varphi\|_{1,\Omega} \leq C\|u\|_{A,\Omega} + C \int_{\Omega} A_*(x,|u(x)|) \, dx + C \int_{\Omega} A(x,|\nabla u|) \, dx + C,
\]
which implies that \( |\nabla \varphi(x)| \in L^1(\Omega) \). \(\square\)

Now we can give the proof of Theorem 4.2.

Proof. Let \( u \in W^{1,A}(\Omega) \) and \( \varphi \) be defined as in Lemma 4.3. By Lemma 4.3, \( \varphi(x) \in W^{1,1}(\Omega) \). By Lemma 4.1, \( W^{1,1}(\Omega) \hookrightarrow L^1(\partial \Omega) \). Then we conclude \( \varphi|_{\partial \Omega} \in L^1(\partial \Omega) \), or equivalently,
\[
u_{\partial \Omega} \in L^{A^{\frac{n-1}{n-2}}}(\partial \Omega).
\]
Define a linear operator \( T : W^{1,A}(\Omega) \rightarrow L^{A^{\frac{n-1}{n-2}}}(\partial \Omega) \) by \( T(u) = u|_{\partial \Omega} \). It is readily checked that the graph of \( T \) is closed in \( W^{1,A}(\Omega) \times L^{A^{\frac{n-1}{n-2}}}(\partial \Omega) \). Then the linear operator \( T \) is continuous by the closed graph theorem. This completes the proof. \(\square\)

Now we consider the compactness of the boundary embedding in a bounded domain.

Denote \( \Omega_d := \{ x \in \Omega : \text{dist}(x, \partial \Omega) < d \} \) for some given \( d > 0 \).

The following compact embedding theorem is one of our main results.

**Theorem 4.4.** Let \( \Omega \subset \mathbb{R}^n \) be an open bounded domain with Lipschitz boundary. Suppose that \( A \in N(\Omega) \) satisfies \((P_1)-(P_6)\) and \((P_{6}')\). There exist two constants \( \varepsilon > 0 \) and \( \delta > 0 \) such that \( A_*(x,|t|^{\frac{n-1}{2}}) \in \Delta_2(\overline{\Omega_\delta}) \) and \( A^{\frac{n-1}{n-2}} \in N(\overline{\Omega_\delta}) \). Then for any \( \theta \in N(\partial \Omega) \) satisfying
\[
\theta(x,t) \leq A^{\frac{n-1}{n-2}}(x,t), \text{ for any } t > 1 \text{ and } x \in \partial \Omega,
\]
the boundary trace embedding \( W^{1,A}(\Omega) \hookrightarrow L^\theta(\partial \Omega) \) is compact.

To give the proof of Theorem 4.4 we present several lemmas.

**Lemma 4.5.** If \( D \in N(\Omega) \), then for any positive constant \( \eta > 0 \), \( D^{1+\eta} \in N(\Omega) \) is uniformly convex in the sense of Definition 2.2.
Proof. It is easy to verify that \( D^{1+n} \in N(\Omega) \). We prove \( D^{1+n} \) is uniformly convex. In fact, by Remark 1, fixing any \( p > 1 \) for every \( s > 0, \ 0 < \alpha < 1 \) and \( 0 \leq \beta \leq \alpha \), there holds the inequality
\[
\left( \frac{s + \beta s}{2} \right)^{1+n} \leq \frac{1}{2} (1 - \sigma(\alpha)) s^{1+n} + \frac{(\beta s)^{1+n}}{2},
\]
(32)
in which, by Remark 1, \( \sigma(\alpha) = 1 - 2^{-\frac{(1+\alpha)}{1+\alpha+n}} \). Since \( D \in N(\Omega) \) is convex and \( D(x,0) = 0 \) for a.e. \( x \in \Omega \), we have
\[
D^{1+n} \left( x, \frac{s + \beta s}{2} \right) \leq \left( \frac{1}{2} D(x,s) + \frac{1}{2} D(x,\beta s) \right)^{1+n} 
\leq \left( \frac{D(x,s) + \beta D(x,s)}{2} \right)^{1+n} 
\]
(by (32)) \leq \frac{(1 - \sigma(\alpha)) D^{1+n}(x,s) + (\beta D(x,s))^{1+n}}{2} 
\leq \frac{(1 - \sigma(\alpha)) D^{1+n}(x,s) + \beta D^{1+n}(x,s)}{2}
\]
for a.e. \( x \in \Omega \), which implies that \( D^{1+n} \in N(\Omega) \) is uniformly convex in the sense of Definition 2.2. \( \square \)

Now we can give the proof of Theorem 4.4.

Proof. By the assumptions of the theorem we can take a number smaller than \( \delta \), but still denoted by \( \delta \), such that \( A^{\frac{n-1}{n}}_{\sigma^{-1}} \in N(\Omega_\delta) \). And we may assume that \( \Omega_\delta \) has a Lipschitz boundary, otherwise it is sufficient to replace \( \Omega_\delta \) by an open subset \( \Omega'_\delta \) of \( \Omega_\delta \) such that \( \Omega'_\delta \) has a Lipschitz boundary and \( \partial \Omega \subset \partial \Omega'_\delta \). By Theorem 4.2 it is easy to see that there is a continuous embedding
\[
W^{1,A}(\Omega) \hookrightarrow W^{1,A}(\Omega_\delta) \hookrightarrow L^{A^{\frac{n-1}{n}}}(\partial \Omega_\delta) \hookrightarrow L^{A^{\frac{n-1}{n}}}(\partial \Omega) \hookrightarrow L^0(\partial \Omega)
\]
(33)
since the embedding operators
\[
j : W^{1,A}(\Omega) \rightarrow W^{1,A}(\Omega_\delta)
\]
\[
u \mapsto u|_{\partial \Omega_\delta}
\]
and
\[
j_\partial : L^{A^{\frac{n-1}{n}}}(\partial \Omega_\delta) \rightarrow L^{A^{\frac{n-1}{n}}}(\partial \Omega)
\]
\[v \mapsto v|_{\partial \Omega}
\]
are continuous.

Since \( A^{\frac{n-1}{n}}_{\sigma^{-1}} \in N(\Omega_\delta) \), we can see that \( A^{\frac{n-1}{n}}_{\sigma^{-1}} \in N(\Omega_\delta) \) satisfies
\[
A^{\frac{n-1}{n}}_{\sigma^{-1}} (x,t) \geq A^{\frac{n-1}{n}}_{\sigma^{-1}} (x,t),
\]
(34)
and by Lemma 4.5, \( A^{\frac{n-1}{n}}_{\sigma^{-1}} \) is uniformly convex. Note that (34) and assumption (31) imply
\[
A^{\frac{n-1}{n}}_{\sigma^{-1}} (x,t) \geq \theta(x,t), \ \forall x \in \partial \Omega.
\]
(35)
Set \( Q(x,t) := A^{\frac{n-1}{n}}_{\sigma^{-1}} (x,t) \) for \( x \in \partial \Omega_\delta \). Then \( Q \in N(\Omega_\delta) \) and \( \theta(x,t) \leq Q(x,t) \) for \( x \in \partial \Omega_\delta \). By (35) we have
\[
L^Q(\partial \Omega) \hookrightarrow L^0(\partial \Omega).
\]
(36)
By (33) and (36), to prove the conclusion of Theorem 4.4, it is sufficient to prove that
\[ W^{1,A}(\Omega_\delta) \hookrightarrow L^q(\partial \Omega) \] (37)
is compact.

To see (37) is correct, it is readily checked out that there exists a \( \tau \in (0,1) \) small enough such that
\[ (1 + \tau) \left( \frac{n-1}{n} - \frac{\epsilon}{2} \right) < 1. \] (38)

And for any \( u \in W^{1,A}(\Omega_\delta) \) we set
\[ v(x) = Q(x, u(x)), \forall x \in \Omega_\delta. \]

By the definition of \( Q \), the equation (38) and the similar argument in the proof of Lemma 4.3, we can see that for \( x \in \Omega_\delta \),
\[ \int_{\Omega_\delta} |v(x)|^{1+\tau} \, dx \leq C \left( \int_{\Omega_\delta} A_s(x, |u|) \, dx + 1 \right) \] (39)
and
\[ \int_{\Omega_\delta} |\nabla v(x)|^{1+\tau} \, dx \leq C \left( \|u\|_{A, \Omega_\delta} + \int_{\Omega_\delta} A_s(x, |u(x)|) \, dx + \int_{\Omega_\delta} A(x, |\nabla u|) \, dx + 1 \right), \] (40)
which imply that \( v \in W^{1,1+\tau}(\Omega_\delta) \). Define a mapping \( \mathcal{S} : W^{1,A}(\Omega_\delta) \to W^{1,1+\tau}(\Omega_\delta) \) by
\[ \mathcal{S}(u) = v = Q(x, |u|) = A_s^{\frac{n-1}{n}} \zeta(x, |u|). \]

Then from (39) and (40), \( \mathcal{S} \) is continuous and bounded.

Suppose that the sequence \( \{u_n\} \subset W^{1,A}(\Omega_\delta) \) and \( u_0 \in W^{1,A}(\Omega_\delta) \) satisfy
\[ u_n \to u_0 \text{ in } W^{1,A}(\Omega_\delta). \] (41)

Since the embedding \( W^{1,A}(\Omega_\delta) \hookrightarrow L^1(\Omega_\delta) \) is compact, we can assume
\[ u_n(x) \to u_0(x) \text{, for a.e. } x \in \Omega_\delta. \] (42)

Let \( v_n := \mathcal{S}(u_n) \) and \( v_0 := \mathcal{S}(v_0) \). Then \( \{v_n\} \) is bounded in \( W^{1,1+\tau}(\Omega_\delta) \). By the reflexive property of the space \( W^{1,1+\tau}(\Omega_\delta) \), we may assume, taking a subsequence if necessary, that \( v_n \rightharpoonup w \) in \( W^{1,1+\tau}(\Omega_\delta) \) and \( v_n(x) \to w(x) \) for a.e. \( x \in \Omega_\delta \). By assumption \((P_2)\) and \((42)\),
\[ v_n(x) = A_s^{\frac{n-1}{n}} \zeta(x, |u_n(x)|) \to A_s^{\frac{n-1}{n}} \zeta(x, |u_0(x)|) = v_0(x), \text{ for a.e. } x \in \Omega_\delta, \]
which means \( w = v_0 \), or equivalently,
\[ v_n \rightharpoonup v_0 \text{ in } W^{1,1+\tau}(\Omega_\delta), \]
or once more equivalently,
\[ \mathcal{S}(u_n) \rightharpoonup \mathcal{S}(u_0) \text{ in } W^{1,1+\tau}(\Omega_\delta). \] (43)

By Lemma 4.1, the boundary trace embedding \( W^{1,1+\tau}(\Omega_\delta) \hookrightarrow L^1(\partial \Omega) \) is compact, which implies that \( v_n|_{\partial \Omega} \rightharpoonup v_0|_{\partial \Omega} \) in \( L^1(\partial \Omega) \). Then
\[ v_n(x) \to v_0(x) \text{ for a.e. } x \in \partial \Omega \] (44)
and
\[ \int_{\partial \Omega} Q(x, |u_n|) \, d\sigma \to \int_{\partial \Omega} Q(x, |u_0|) \, d\sigma \text{ as } n \to +\infty. \] (45)
From (44) it is easy to see that
\[ u_n(x) \to u_0(x) \text{ for a.e. } x \in \partial \Omega. \] (46)
By assumptions of the theorem, \( A_*^{\frac{n-1}{n}} \in \Delta_2(\partial \Omega) \) implies that \( Q = A_*^{\frac{n-1}{n}} \in \Delta_2(\partial \Omega) \). Then from Proposition 4, (46) and (45) imply
\[ u_n|_{\partial \Omega} \to u_0|_{\partial \Omega} \text{ in } L^Q(\partial \Omega), \] (47)
which implies that the embedding in (37) is compact. The proof is complete. \( \square \)

**Remark 3.** 1. From the proof of the Theorem 4.4, we can see that the property of the boundary trace embedding depends only on the property of \( A \) or \( A_* \) in any small neighborhood of \( \partial \Omega \).
2. In applications some much more stronger assumptions (e.g. global) on \( A_* \) can imply the assumptions on \( A \) in Theorem 4.4. For example: \( A_*^{\frac{n-1}{n}} \in \Delta_2(\Omega) \Rightarrow A_*^{\frac{n-1}{n}} \in \Delta_2(\Omega_s) \Rightarrow A_*^{\frac{n-1}{n}} \in \Delta_2(\Omega_s) \).

5. **Two examples.** We give two examples of the function \( A \) satisfying conditions in our theorems.

**Example 1.** Let \( p \in C^{1-0}(\overline{\Omega}) \) and \( 1 < q \leq p(x) \leq p_* := \sup_{x \in \overline{\Omega}} \frac{p(x)}{n} < n \) \((q \in \mathbb{R})\) for \( x \in \Omega \). Define \( A: \overline{\Omega} \times [0, +\infty) \to [0, +\infty) \) by
\[ A(x,t) = t^{p(x)}. \]
It is readily checked that \( A \) satisfies \((P_1),(P_2)\) and \((P_3)\). It is easy to see that \( p \in C^{1-0}(\overline{\Omega}) \) implies \( A \in C^{1-0}(\overline{\Omega}) \) and
\[ A_*^{-1}(x,s) = \frac{np(x)}{n-p(x)^q} \frac{n-p(x)}{np(x)}. \] (48)
Then \( T(x) = +\infty \), which implies that \((P_7^1)\) is satisfied \((\Rightarrow (P_4) \text{ is satisfied})\).
In addition, for \( x \in \Omega \),
\[ \nabla_x A(x,t) = t^{p(x)} \ln t \nabla p(x). \]
Since for any \( \epsilon > 0, \frac{\ln t}{t} \to 0 \) as \( t \to +\infty \), we conclude that there exist constants \( \delta_1 < \frac{1}{n}, c_1 \) and \( t_1 \) such that
\[ \left| \frac{\partial A(x,t)}{\partial x_j} \right| \leq c_1 A^{1+\delta_1}(x,t), \]
for all \( x \in \Omega \) and \( t \geq t_1 \). Combining \( A \in \Delta_2(\Omega) \), from Proposition 3.1 in [9], it is easy to see that condition \((P_3)\) is satisfied. By now conditions in Theorem 3.3 are verified.
By (48), we can see that
\[ A_*^{\frac{n-1}{n}} = \left( \frac{n-p(x)}{np(x)} \right)^{\frac{n-p(x)}{np(x)}}. \]
Then \( A_*^{\frac{n-1}{n}} \in \Delta_2(\Omega) \) and since \( p(x) < p_* < n \) we can see there exists a \( \epsilon > 0 \) such that \( A_*^{\frac{n-1}{n}} - \epsilon \in N(\overline{\Omega}_\delta) \). Then all conditions in Theorem 4.4 can be satisfied.
This example contains the conclusion of Theorem 2.2 in [6] and some of its corollaries.
Example 2. Let $p \in C^{1-0}(\mathbb{O})$ satisfy $1 < p^- \leq p(x) \leq p_+ := \sup_{x \in \Omega} p(x) < n - 1$. Define $A : \Omega \times [0, +\infty) \to [0, +\infty)$ by

$$A(x,t) = t^{p(x)} \log(1 + t), \text{ for } x \in \Omega \text{ and } t > 0.$$ 

It is obvious that $A$ satisfies $(P_1)$, $(P_2)$ and $(P_3)$. Pick $\epsilon > 0$ small enough such that $p_+ + \epsilon < n$. Then for $t > 0$ big enough, $A(x,t) \leq c t^{p_+ + \epsilon}$, which implies that $T(x) = +\infty$ for all $x \in \Omega$. Thus $(P_{0}^-)$ is satisfied ($\Rightarrow (P_4)$ is satisfied). Since $p \in C^{1-0}(\Omega)$ and $A \in C^{1-0}(\Omega \times [0, +\infty))$, by Proposition 3.1 in [9], $A_* \in C^{1-0}(\Omega \times [0, +\infty))$. Combining $A \in \Delta_2(\Omega)$, it is easy to see that condition $(P_5)$ is satisfied. Then conditions in Theorem 3.3 are verified.

Let $\epsilon > 0$ small enough such that $p(x) + \epsilon < n$ for any $x \in \Omega$. Since for $t > 0$ big enough $t^{p(x)} \leq A(x,t) \leq t^{p(x)+\epsilon}$, we can get the estimate for $A_*(x,t)$ as follows

$$\left(\frac{n - p(x)}{np(x)} t\right)^{\frac{np(x)}{n-p(x)}} \leq A_*(x,t) \leq \left(\frac{n - (p(x) + \epsilon)}{n(p(x) + \epsilon)} t\right)^{\frac{n(p(x)+\epsilon)}{n-p(x)+\epsilon}},$$

for any $x \in \Omega$. Then $A_*(x,t) \in \Delta_2(\Omega)$. The conditions in Theorem 4.4 are verified.

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