Congruences of multiple sums involving invariant sequences under binomial transform

ROBERTO TAURASO

Dipartimento di Matematica

Università di Roma “Tor Vergata”, Italy

tauraso@mat.uniroma2.it

http://www.mat.uniroma2.it/~tauraso

Abstract

We will prove several congruences modulo a power of a prime such as

\[ \sum_{0<k_1<\cdots<k_n<p} \left( \frac{p-k_n}{3} \right) (-1)^{k_n} k_1 \cdots k_n \equiv \begin{cases} -\frac{2^{n+1}+2}{6^{n+1}} p B_{p-n-1} \left( \frac{1}{n+1} \right) \pmod{p^2} & \text{if } n \text{ is odd} \\ -\frac{2^{n+1}+4}{n6^{n}} B_{p-n} \left( \frac{1}{3} \right) \pmod{p} & \text{if } n \text{ is even} \end{cases} \]

where \( n \) is a positive integer and \( p \) is prime such that \( p > \max(n+1,3) \).

1 Introduction

The classical binomial inversion formula states that the linear transformation of sequences

\[ T(\{a_n\}) = \left\{ \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k \right\} \]

is an involution, that is \( T \circ T \) is the identity map. Thus, \( T \) has only two eigenvalues: 1 and \(-1\). We denote by \( S_+ \) and \( S_- \) the eigenspaces corresponding respectively to the eigenvalue 1 and to the eigenvalue \(-1\). These eigenspaces contain many well known sequences:

\[ \{2^{-n}\}, \{L_n\}, \{(−1)^n B_n\}, \{(n + 1)C_n 4^{-n}\} \in S_+, \]

\[ \{0, 1, 1, \cdots\}, \{F_n\}, \{−(1)^n \left( \frac{n}{3} \right)\}, \{2^n − (−1)^n\} \in S_- \]

where \( \{F_n\}, \{L_n\}, \{B_n\}, \{C_n\} \) denote respectively the Fibonacci, Lucas, Bernoulli and Catalan numbers. For a more detailed analysis of the properties of \( S_+ \) and \( S_- \) the reader is referred to [4] and [5].

In this note we would like to present several congruences of multiple sums which involves these invariant sequences. Our main result is the following.

Theorem 1.1. Let \( \{a_n\} \in S_- \). Let \( n \) be a positive odd integer and let \( p \) be a prime such that \( p > n + 1 \) then

\[ \sum_{0<k_1<\cdots<k_n<p} \frac{a_{p-k_n}}{k_1 \cdots k_n} = \frac{p(n+1)}{2} \sum_{0<k_1<\cdots<k_{n+1}<p} \frac{a_{p-k_{n+1}}}{k_1 \cdots k_{n+1}} \pmod{p^3}. \]

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Note that since
\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k a_{k-1} = -n \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k a_k
\]
then \( \{a_n\} \in S_+ \) if and only if \( \{na_{n-1}\} \in S_+ \). Hence by the previous theorem we easily find that

**Corollary 1.2.** Let \( n \) be a positive odd integer and let \( p \) be a prime such that \( p > n + 1 \).

i) If \( \{a_n\} \in S_- \),
\[
\sum_{0 < k_1 < \cdots < k_n < p} \frac{a_{p-k_n}}{k_1 \cdots k_n} \equiv \sum_{0 < k_1 < \cdots < k_n < p} \frac{a_{k_1}}{k_1 \cdots k_n} \equiv 0 \pmod{p}.
\]

ii) If \( \{a_n\} \in S_+ \),
\[
\sum_{0 < k_1 < \cdots < k_n < p} \frac{a_{p-k_n-1}}{k_1 \cdots k_n-1} \equiv \sum_{0 < k_1 < \cdots < k_n < p} \frac{a_{k_1-1}}{k_2 \cdots k_n} \equiv 0 \pmod{p}.
\]

In the final section we consider the special class of sequences in \( S_- \) which are second-order linear recurrences and we will prove the congruence mentioned in the abstract which generalizes a result established in [6].

### 2 The main result

The starting point of our work is the following identity.

**Lemma 2.1.** Let \( \{a_n\} \in S_- \) then for \( m, n \geq 0 \)
\[
\sum_{k=0}^{n} \left[ \binom{(m-1)n+k-1}{k} + \binom{-1}{k} \right] a_{n-k} = 0.
\]

**Proof.** This identity can be obtained from
\[
\sum_{k=0}^{n} \binom{n}{k} \left[ f_k + (-1)^{n-k} \sum_{i=0}^{k} \binom{k}{i} f_i \right] a_{n-k} = 0
\]
which appears in [4] (see also [5]), by taking \( f(k) = \binom{(m-1)n+k-1}{k} / \binom{n}{k} \).

However, for completeness sake, we give here a direct proof. Let \( A(z) \) be the generating function of \( \{a_n\} \), then
\[
A(z) = -\frac{1}{1-z} A \left( \frac{z}{z-1} \right).
\]
because \( \{a_n\} \in S_- \). Since
\[
\binom{(m-1)n+k-1}{k} = \left[ z^k \right] \frac{1}{(1-z)^{(m-1)n}} = \left[ z^{-1} \right] \frac{1}{z^{n+1}(1-z)^{(m-1)n}} \cdot z^{n-k}
\]
and
\[
(-1)^{n-k} \binom{mn}{k} = \left[ z^k \right] \frac{(-1)^{n-k}}{(1-z)^{mn-k+1}} = \left[ z^{-1} \right] \frac{1}{z^{n+1}(1-z)^{(m-1)n+1}} \cdot \left( \frac{z}{z-1} \right)^{n-k},
\]

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then the right-hand side of the identity becomes
\[
[z^{-1}] \frac{1}{z^{n+1}(1-z)^{(m-1)n}} \left[ A(z) + \frac{1}{1-z} A \left( \frac{z}{z-1} \right) \right] = 0.
\]

Before proving our main result we define the *multiple harmonic sum* of order \( n > 0 \) as
\[
H^{(n)}_r = \sum_{1 \leq k_1 < k_2 < \ldots < k_n \leq r} \frac{1}{k_1 k_2 \ldots k_n} \quad \text{for } r \geq 1.
\]

**Proof of Theorem 1.1.** Since \( a_0 = 0 \) and \( p \) is odd, by the previous lemma we have that
\[
\sum_{k=1}^{p-1} \left[ \binom{(m-1)p + k - 1}{k} - (-1)^k \binom{mp}{k} \right] a_{p-k} = 0.
\]

By expanding the binomial coefficients for \( 1 < k < p \) we get
\[
\binom{(m-1)p + k - 1}{k} = \binom{mp}{k} (-1)^{k-1} \prod_{j=1}^{k-1} \left( 1 - \frac{mp}{j} \right) = \frac{(-1)^k}{k} \sum_{j \geq 1} (-mp)^j H^{(j-1)}_{k-1}.
\]

Hence,
\[
\sum_{j \geq 1} ((m-1)^j - (-m)^j) p^j S_j = 0
\]

where \( S_j = \sum_{k=1}^{p-1} H^{(j-1)}_{k-1} \frac{a_{p-k}}{k} \). The infinite matrix of the coefficients of \( \{p^j S_j\} \) is
\[
\begin{bmatrix}
1 & -1 & 1 & -1 & 1 & -1 & \cdots \\
3 & -3 & 9 & -15 & 33 & -63 & 129 & -255 & \cdots \\
5 & -5 & 35 & -65 & 275 & -665 & 2315 & -6305 & \cdots \\
7 & -7 & 91 & -175 & 1267 & -3367 & 18571 & -58975 & \cdots \\
9 & -9 & 189 & -369 & 4149 & -11529 & 94509 & -325089 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

and by performing the Gaussian elimination we obtain
\[
\begin{bmatrix}
1 & -1 & 1 & -1 & 1 & -1 & \cdots \\
0 & 0 & 1 & -2 & 5 & -10 & 21 & -42 & \cdots \\
0 & 0 & 0 & 1 & -3 & 14 & -42 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & -4 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

In general let
\[
t(i, j) = \frac{2}{(2i)!} \sum_{m=1}^{i} (-1)^{i-m} \binom{2i}{i+m} m^j
\]

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then

\[
0 = \frac{1}{(2i)!} \sum_{m=1}^{i} (-1)^{i-m}(i+m) \binom{2i}{i+m} \sum_{j \geq 1} ((m-1)^j - (-m)^j) p^j S_j
\]

\[
= \sum_{j \geq 1} \frac{p^j S_j}{(2i)!} \left[ - \sum_{m=1}^{i} (-1)^{i-(m-1)}(i-(m-1)) \binom{2i}{i+m-1} (m-1)^j \right.
\]

\[
-(-1)^j \sum_{m=1}^{i} (-1)^{i-m}(i+m) \binom{2i}{i+m} m^j \right]
\]

\[
= \sum_{j \geq 1} \frac{p^j S_j}{2} \left[ -it(i,j) + t(i,j+1) - i(-1)^j t(i,j) - (-1)^j t(i,j+1) \right]
\]

\[
= \sum_{j \geq 1} \frac{p^j S_j}{2} \left[ \frac{1+(-1)^j}{2} it(i,j) + \frac{1-(-1)^j}{2} t(i,j+1) \right]
\]

\[
= \sum_{j \geq 1} p^{2j-1} t(i,2j) (S_{2j-1} - ipS_{2j}).
\]

Note that \( t(i,2j) \) are the triangle central factorial numbers (see \[2\] p. 217) therefore \( t(i,2j) = 0 \) if \( i < 2j \) and \( t(j,2j) = 1 \). Thus

\[
S_{2i-1} \equiv ipS_{2j} + p^2 t(i,i+1) S_{2i+1} \quad (\text{mod } p^3).
\]

It follows that \( S_n \equiv 0 \) (mod \( p \)) when \( n \) is odd and finally we get

\[
S_{2i-1} \equiv ip S_{2i} \quad (\text{mod } p^3).
\]

\[\Box\]

3 A special class of invariant sequences

Let us prove first a preliminary lemma.

**Lemma 3.1.** Let \( n \) be a positive integer and let \( p \) be any prime such that \( p > n + 1 \). Then the two polynomials of \( \mathbb{Z}_p[x] \)

\[
G_n(x) = \sum_{k=1}^{p-1} H_{k-1}^{(n-1)} \frac{x^k}{k}, \quad \text{and} \quad g_n(x) = \sum_{k=1}^{p-1} \frac{x^k}{k^n},
\]

satisfy the congruence

\[
G_n(x) \equiv (-1)^{n-1} g_n(1-x) \quad (\text{mod } p).
\]

**Proof.** We prove the congruence by induction on \( n \).

For \( n = 1 \), since \( \binom{p}{k} = (-1)^{k-1} \frac{k}{k} \) (mod \( p^2 \)) for \( 0 < k < p \) then

\[
G_1(x) \equiv \frac{1}{p} \sum_{k=1}^{p-1} (-1)^{k-1} \frac{p}{k} x^k = -\frac{1}{p} \sum_{k=1}^{p-1} \binom{p}{k}(-x)^k = \frac{(1-x)^p - 1 - x^p}{p} \quad (\text{mod } p).
\]

Hence \( G_1(x) \equiv G_1(1-x) = g_1(1-x) \) (mod \( p \).
Assume that $n > 1$. The formal derivative yields
\[
\frac{d}{dx} G_n(x) = \sum_{k=1}^{p-1} H_{k-1}^{\text{(n-1)}} x^{k-1} = \sum_{k=1}^{p-1} \sum_{j=1}^{k-1} H_{j-1}^{\text{(n-2)}} x^{k-1-j} \]
\[
= \sum_{j=1}^{p-1} H_{j-1}^{\text{(n-2)}} \frac{1}{j} \sum_{k=j+1}^{p-1} x^{k-1} = \sum_{j=1}^{p-1} H_{j-1}^{\text{(n-2)}} \frac{1}{j} \left( x^{p-1} - x^j \right) \]
\[
= \frac{x^{p-1}}{x-1} H_{p-1}^{\text{(n-1)}} - \frac{G_{n-1}(x)}{x-1} \equiv G_{n-1}(x) \pmod{p} \]
where in the last step we used the fact that $H_{p-1}^{\text{(n-1)}} \equiv 0 \pmod{p}$ (see for example [7]).
Moreover
\[
\frac{d}{dx} g_n(1-x) = - \sum_{k=1}^{p-1} \frac{(1-x)^{k-1}}{k^{n-1}} = - \frac{g_{n-1}(1-x)}{1-x} \]
Hence, by the induction hypothesis
\[
(1-x) \frac{d}{dx} (G_n(x) + (-1)^n g_n(1-x)) \equiv G_{n-1}(x) + (-1)^n g_{n-1}(1-x) \equiv 0 \pmod{p} \]
Thus $G_n(x) + (-1)^n g_n(1-x) \equiv c_1 \pmod{p}$ for some constant $c_1$ since this polynomial has degree $< p$. By letting $x = 1$ we find that
\[
G_n(x) + (-1)^n g_n(1-x) \equiv c_1 \equiv G_n(1) + (-1)^n g_n(0) = H_{p-1}^{\text{(n)}} \equiv 0 \pmod{p}.
\]
\[
\square
\]
Let us consider a sequence $\{a_n\} \in S_-$ which is a second-order recurrence. Since its generating function should satisfy the identity
\[
A(z) = - \frac{1}{1-z} A\left( \frac{z}{z-1} \right)
\]
it is easy to verify that
\[
A(z) = \frac{a_1 z}{1 - z - c z^2}
\]
for some real number $c$. The following result holds for any invariant sequence of this kind.

**Theorem 3.2.** Let $n$ be a positive integer and let $p$ be a prime such that $p > n + 1$. If \[ \sum_{k \geq 0} a_k z^k = \frac{a_1 z}{1 - z - c z^2} \] for some integer $c$ then
\[
\sum_{0 < k_1 < \ldots < k_n < p} \frac{a_{p-k_n}}{k_1 \cdots k_n} \equiv \begin{cases} 
-p(n+1) \sum_{k=1}^{(p-1)/2} \frac{c^k a_{p-2k}}{k^{n+1}} \pmod{p^2} & \text{if } n \text{ is odd} \\
-2 \sum_{k=1}^{(p-1)/2} \frac{c^k a_{p-2k}}{k^n} \pmod{p} & \text{if } n \text{ is even}
\end{cases}
\]
Proof. By linearity we can assume that \( a_1 = 1 \) (the case \( a_1 = 0 \) is trivial). Since \( \{a_k\} \in S_- \) then, by Theorem 1.1, if \( n \) is odd we have that

\[
\sum_{0<k_1<\ldots<k_n<p} \frac{a_{p-k_n}}{k_1 \cdots k_n} = \frac{p(n+1)}{2} \sum_{0<k_1<\ldots<k_{n+1}<p} \frac{a_{p-k_{n+1}}}{k_1 \cdots k_{n+1}} \pmod{p^3},
\]

thus it suffices to consider the case when \( n \) is even.

Let \( \Delta = 1 + 4c \) and consider the ring \( \mathbb{Z}_p[\sqrt{\Delta}] \). Then for \( k \geq 0 \)

\[
a_k = \frac{w_k^+ + w_k^-}{\sqrt{\Delta}} \quad \text{with} \quad w_\pm = \frac{1 \pm \sqrt{\Delta}}{2}.
\]

This formula allows to extend the sequence \( \{a_k\} \) also for negative indices: \( a_{-k} = -a_k(\mp c)^{-k} \)

for \( k > 0 \) because \( w_+ + w_- = 1 \) and \( w_+ w_- = -c \). Therefore, by the previous lemma, we have that

\[
\sum_{0<k_1<\ldots<k_n<p} \frac{a_{p-k_n}}{k_1 \cdots k_n} = \sum_{k=1}^{p-1} H_{k-1}^{(n-1)} \frac{a_{p-k}}{k} \quad = \quad \frac{w_p^+}{\sqrt{\Delta}} \sum_{k=1}^{p-1} H_{k-1}^{(n-1)} \frac{w_k^-}{k} - \frac{w_p^-}{\sqrt{\Delta}} \sum_{k=1}^{p-1} H_{k-1}^{(n-1)} \frac{w_k^+}{k}
\]

\[
= -\frac{w_p^+}{\sqrt{\Delta}} \sum_{k=1}^{p-1} \frac{(1-w_k^-)^k}{k^n} + \frac{w_p^-}{\sqrt{\Delta}} \sum_{k=1}^{p-1} \frac{(1-w_k^+)^k}{k^n} \pmod{p}.
\]

Since \( 1-w_{\pm}^{-1} = cw_{\pm}^{-2} \), it follows that

\[
\sum_{0<k_1<\ldots<k_n<p} \frac{a_{p-k_n}}{k_1 \cdots k_n} = -\frac{w_p^+}{\sqrt{\Delta}} \sum_{k=1}^{p-1} \frac{c^k w_{-2k}}{k^n} + \frac{w_p^-}{\sqrt{\Delta}} \sum_{k=1}^{p-1} \frac{c^k w_{-2k}}{k^n}
\]

\[
= -\sum_{k=1}^{p-1} \frac{c^k a_{p-2k}}{k^n}
\]

\[
= -\frac{(p-1)/2}{k^n} \sum_{k=1}^{p-1} c^{p-k} a_{-(p-2k)} - \frac{(p-1)/2}{k^n} \sum_{k=1}^{p-1} c^{p-k} a_{-(p-2k)}
\]

\[
= -2 \sum_{k=1}^{p-1} \frac{c^k a_{p-2k}}{k^n} \pmod{p}.
\]

Finally, we prove are ready to prove the congruence mentioned in the abstract.

**Theorem 3.3.** Let \( n \) be a positive integer and let \( p \) be a prime such that \( p > \max(n+1, 3) \).

Then

\[
\sum_{0<k_1<\ldots<k_n<p} \frac{(p-k_n)^n}{3} \frac{(-1)^k}{k_1 \cdots k_n} \equiv \begin{cases} 
\frac{-2^{n+1}+4}{n+1} B_{p-n-1} \left( \frac{1}{3} \right) & \text{if } n \text{ is odd} \\
\frac{-2^{n+1}+4}{n+1} B_{p-n} \left( \frac{4}{3} \right) & \text{if } n \text{ is even}.
\end{cases} \pmod{p^2}.
\]
Proof. Let \( a_k = (-1)^{p-k} \left( \frac{k}{p} \right) \) then its generating function is

\[
A(z) = \frac{z}{1 - z + z^2}
\]

and by the previous theorem, for \( n \) even the right-hand side yields

\[
\sum_{0 < k_1 < \cdots < k_n < p} \frac{(p - k_n)}{3} (-1)^{k_n} k_1 \cdots k_n \equiv -2 \sum_{k=1}^{p-1} \frac{(-1)^k}{k^n} \left( \frac{p - 2k}{3} \right) \equiv \sum_{k=1}^{p-1} (-1)^{k-1} \left( \frac{p + k}{3} \right) \pmod{p}.
\]

By [3] we have that

\[
\frac{1}{k^n} \equiv \frac{1}{n6^n} \left( B_{p-n} \left( \left\{ \frac{r}{6} \right\} \right) - B_{p-n} \left( \left\{ \frac{r - p}{6} \right\} \right) \right) \pmod{p}
\]

for \( p > \max(n+1, 3) \) and for \( r = 0, 1, 2, 3, 4, 5 \). Moreover, the Bernoulli polynomials satisfy the reflection property and the multiplication formula (see for example [1] p.248):

\[
B_m(1 - x) = (-1)^m B_m(x) \text{ and } B_m(ax) = a^{m-1} \sum_{k=0}^{a-1} B_m \left( x + \frac{k}{a} \right) \text{ for } m, a > 0.
\]

Hence, since \( p - n \) is odd,

\[
B_{p-n} \left( \frac{1}{2} \right) = B_{p-n} \left( \frac{2}{3} \right) = -B_{p-n} \left( \frac{1}{3} \right)
\]

\[
B_{p-n} \left( \frac{1}{6} \right) = -B_{p-n} \left( \frac{5}{6} \right) = \left( 1 + 2^{n-(p-1)} \right) B_{p-n} \left( \frac{1}{3} \right).
\]

By these preliminary remarks it is easy to verify that

\[
\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k^n} \left( \frac{p + k}{3} \right) \equiv -\frac{2^{n+1} + 4}{n6^n} B_{p-n} \left( \frac{1}{3} \right) \pmod{p}.
\]

On the other hand, if \( n \) is odd, by Theorem 1.1, it follows from the above congruence that

\[
\sum_{0 < k_1 < \cdots < k_n < p} \frac{(p - k_n)}{3} (-1)^{k_n} k_1 \cdots k_n \equiv \frac{p(n+1)}{2} \sum_{0 < k_1 < \cdots < k_{n+1} < p} \frac{(p - k_{n+1})}{3} \left( \frac{(-1)^{k_n}}{k_1 \cdots k_{n+1}} \right) \pmod{p^3}
\]

\[
\equiv -\frac{2^{n+1} + 2}{6^{n+1}} pB_{p-n-1} \left( \frac{1}{3} \right) \pmod{p^3}.
\]

\[\square\]

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