Remarks on Some Compact Symplectic Solvmanifolds

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Abstract  We study the hard Lefschetz property on compact symplectic solvmanifolds, i.e., compact quotients $M = \Gamma \setminus G$ of a simply-connected solvable Lie group $G$ by a lattice $\Gamma$, admitting a symplectic structure.

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1 Introduction

Let $(M,\omega)$ be a compact symplectic $2n$-manifold, that is, $M$ is a $2n$-dimensional smooth manifold endowed with a closed non-degenerate 2-form $\omega$, where $\omega$ is called the symplectic structure. We say that a symplectic manifold $(M,\omega)$ satisfies the Hard Lefschetz Condition, shortly the HLC, if for any $k \in \{0, 1, \ldots, n\}$, the homomorphism

$$L^k : H^{n-k}_{dR}(M) \to H^{n+k}_{dR}(M)$$

$$[\alpha] \mapsto [\alpha \wedge \omega^k]$$

is surjective (cf. [23]). As a classical result, compact Kähler manifolds satisfy HLC; nevertheless, there are compact symplectic manifolds satisfying HLC, with no Kähler structure. In analogy with Riemannian Geometry, starting with the symplectic form $\omega$, one can define a symplectic codifferential $d^\Lambda$ operator; it turns out (see [15, 17, 23]) that a compact symplectic manifold satisfies the HLC if and only if the $dd^\Lambda$-Lemma holds, or, equivalently, any de Rham class $a$ of $M$ contains a symplectic harmonic representative $\alpha$, i.e., $a = [\alpha]$ and $\alpha \in \ker d \cap \ker d^\Lambda$.

The aim of this paper is to study the Hard Lefschetz Condition on compact symplectic solvmanifolds, i.e., compact quotients $M = \Gamma \setminus G$ of a simply-connected solvable Lie group $G$ by a lattice $\Gamma$ endowed with a symplectic structure. We will show the following

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Theorem (see Theorem 4.2) Let $G$ be a simply-connected 4-dimensional Lie group admitting a uniform lattice $\Gamma$ and let $M = \Gamma \backslash G$. Let $\omega$ be a symplectic structure on $M$. Then, if $\omega$ satisfies the HLC, any other symplectic structure on $M$ satisfies the HLC.

Furthermore, for several examples of compact 6-dimensional symplectic solvmanifolds, we show that the same conclusion of the above theorem holds (see Theorems 5.1, 6.1 and Remark 6.2).

2 Preliminaries

Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold. An almost complex structure $J$ on $M$, i.e., a smooth $(1, 1)$-tensor field on $M$ satisfying $J^2 = -\text{id}$, is said to be $\omega$-compatible, if at any given $x \in M$, for every pair of tangent vectors $u, v$ and any non-zero tangent vector $w$, the following hold

$$\omega_x(Ju, Jv) = \omega(u, v), \quad \omega_x(w, Jw) > 0.$$ 

In other words, $g_x(u, v) = \omega_x(u, Jv)$ is an almost Kähler metric on $M$ and $(M, J, g, \omega)$ is an almost Kähler manifold.

Let $(M, J)$ be a $2n$-dimensional almost complex manifold and let $\Lambda^k(M)$ be the bundle of $k$-forms on $M$; denote by $A^k(M) = \Gamma(M, \Lambda^k(M))$ the set of smooth sections of $\Lambda^k(M)$. Then $J$ acts on $A^2(M)$ as an involution, by setting $J\alpha(u, v) = \alpha(Ju, Jv)$, for every pair of vector fields $u, v$ on $M$. Denote by $A^+_J(M)$, (resp., $A^-_J(M)$) the spaces of $J$-invariant, (resp., $J$-anti-invariant) forms, i.e.,

$$A^+_J(M) = \{ \alpha \in A^2(M) \mid J\alpha = \pm \alpha \}.$$ 

and by

$$Z^+_J = \{ \alpha \in A^+_J(M) \mid d\alpha = 0 \}.$$ 

Then, following T.-J. Li and W. Zhang [14], define

$$H^+_J(M) = \{ \alpha \in H^2_{dR}(M) \mid \alpha = [\alpha] \text{ for some } \alpha \in Z^+_J \}.$$ 

Then by [14, Definition 4.12], $J$ is said to be $C^\infty$-pure-and-full if

$$H^2_{dR}(M) = H^+_J(M) \oplus H^-_J(X).$$ 

If $(M, J, g, \omega)$ is a $2n$-dimensional almost Kähler manifold, then the space $A^2_0(M)$ of primitive 2-forms is given by

$$A^2_0(M) = \{ \alpha \in A^2(M) \mid \omega^{n-1} \wedge \alpha = 0 \}$$ 

and the primitive $J$-invariant cohomology group $H^+_{J,0}(M)$ [19] by

$$H^+_{J,0}(M) = \{ \alpha \in H^2_{dR}(M) \mid \alpha = [\alpha] \text{ for some } \alpha \in Z^+_J \cap A^2_0(M) \}.$$ 

We will denote by $P_J : A^2_0(M) \to A^2_0(M)$ the generalized Lejmi differential operator (see [13] and [19]) defined on the space $A^2_0(M)$ as

$$P_J(\psi) = \Delta g \psi - \frac{1}{n} g(\Delta g \psi, \omega) \omega,$$ 

where $\Delta_g$ is the Hodge Laplacian and $g(\cdot , \cdot )$ is the metric induced by $g$ on the space of 2-forms.

In the sequel we will assume that $M$ is a compact symplectic solvmanifold, that is a compact quotient $M = \Gamma \backslash G$ of a simply-connected solvable Lie group $G$ by a lattice $\Gamma$, endowed with a
symplectic structure $\omega$. A compact solvmanifold $M = \Gamma \backslash G$ is said to be completely solvable if the adjoint representation $\text{ad}_X$ of the Lie algebra $\mathfrak{g}$ of $G$ has real eigenvalues for every $X \in \mathfrak{g}$.

3 The $\dd^A$-Lemma on Symplectic Manifolds and Symplectic Cohomologies

Let $(M, \omega)$ be a compact symplectic manifold of dimension $2n$. Then the non degenerate 2-form $\omega$ induces a $C^\infty$-bilinear form $\omega^{-1}$ on $A^k(M)$, setting pointwise on simple elements

$$\omega^{-1}(\alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k) = \det(\omega^{-1}(\alpha_i, \beta_j)),$$

where $\omega^{-1}$ is the natural bilinear form induced by $\omega$ on $T^*M$ and then extending $\omega^{-1}$ linearly on $A^k(M)$. Then the symplectic star operator $*_{s}: A^k(M) \to A^{2n-k}(M)$ is defined by the following representation formula: given any $\beta \in A^k(M)$, for every $\alpha \in A^k(M)$, set

$$\alpha \wedge *_{s} \beta = \omega^{-1}(\alpha, \beta) \frac{\omega^n}{n!}.$$

It turns out that $*_{s}^2 = \text{id}$. Denote as usual by $L$, $\Lambda$ and $H$ the three basic operators defined respectively as

$$L : A^k(M) \to A^{k+2}(M), \quad L := \omega \wedge \cdot,$$

$$\Lambda : A^k(M) \to A^{k-2}(M), \quad \Lambda := *_{s}^{-1}L *_{s},$$

$$H : A^k(M) \to A^k(M), \quad H := [L, \Lambda].$$

Then $\{L, \Lambda, H\}$ gives rise to an action of $\mathfrak{sl}_2$ on $A^\ast(M) = \bigoplus_{k \geq 0} A^k(M)$, namely an $\mathfrak{sl}_2$-triple on $A^\ast(M)$.

The symplectic codifferential $d^A$ is defined, acting on $k$-forms, by means of the following formula

$$d^A |_{A^k(M)} = (-1)^{k+1} *_{s} d *_{s}. \quad (3.1)$$

In view of the basic symplectic identity (see e.g., [5]), it turns out that the symplectic codifferential is expressed by

$$d^A = [d, \Lambda]. \quad (3.2)$$

Since $*_{s}^2 = \text{id}$, it is $(d^A)^2 = 0$; then Brylinski defined the following natural symplectic cohomology

$$H^k_{d^A}(M) := \frac{\ker d^A \cap A^k(M)}{\text{Im} d^A \cap A^k(M)},$$

showing that the symplectic star operator $*_{s}$ induces an isomorphism between $H^k_{d^A}(M)$ and $H^{2n-k}_{d^A}(M)$. Later, Tseng and Yau [21] introduced the Bott–Chern and Aeppli symplectic cohomologies respectively as

$$H^k_{d^A + d}(M) := \frac{\ker (d + d^A) \cap A^k(M)}{\text{Im} (d + d^A) \cap A^k(M)},$$

and

$$H^k_{d^A + d}(M) := \frac{\ker (dd^A) \cap A^k(M)}{(\text{Im} d + \text{Im} d^A) \cap A^k(M)}.$$ 

Then such cohomologies groups are the symplectic counterpart of the Bott–Chern and Aeppli cohomology groups respectively defined in the complex setting. Tseng and Yau developed a
Hodge theory for such cohomologies, showing that Bott–Chern and Aeppli symplectic cohomologies on a compact symplectic manifold are isomorphic to the kernel of suitable 4-order elliptic self-adjoint differential operators. Consequently, the symplectic cohomology groups are finite-dimensional vector spaces on a compact symplectic manifold.

By definition, a compact symplectic manifold \((M, \omega)\) is said to satisfy the \(dd^\Lambda\)-Lemma (see [21, Definition 3.12]) if the natural map \(H^*_d + d\Lambda(M) \to H^*_d \omega(M)\) is injective, i.e., every \(d\Lambda\)-closed, \(d\)-exact form is also \(dd^\Lambda\)-exact, that is

\[
\ker d \Lambda \cap \text{Im } d = \text{Im } dd^\Lambda.
\]  

Since

\[
* s (\ker d \Lambda \cap \text{Im } d) = \ker d \cap \text{Im } d^\Lambda,
\]

then (3.3) holds if and only if the following holds

\[
\ker d \cap \text{Im } d^\Lambda = \text{Im } dd^\Lambda.
\]  

We collect all the known results just recalling the following

**Theorem 3.1** Let \((M, \omega)\) be a compact symplectic manifold. Then the following facts are equivalent:

i) the Hard Lefschetz Condition holds on \((M, \omega)\), i.e., for every \(k \in \mathbb{Z}\), the maps

\[
L^k : H^{n-k}_d \omega(M) \to H^{n+k}_d \omega(M)
\]

are isomorphisms;

ii) any de Rham cohomology class \(a\) has a symplectic harmonic representative, i.e., \(a = [\alpha]\), where \(da = 0\) and \(d^\Lambda \alpha = 0\);

iii) the \(dd^\Lambda\)-Lemma holds;

iv) the natural maps induced by the identity \(H^*_d + d\Lambda(M) \to H^*_d \omega(M)\) are injective;

v) the natural maps induced by the identity \(H^*_d + d^\Lambda(M) \to H^*_d \omega(M)\) are isomorphisms.

For the proof see [5, Conjecture 2.2.7], [15, Corollary 2], [17, Proposition 1.4], [23, Theorem 0.1], [6, Theorem 5.4], [21, Proposition 3.13]. Therefore, according to the above theorem, starting with a compact symplectic manifold \((M, \omega)\) satisfying the Hard Lefschetz Condition, then the de Rham cohomology algebra \(H^*_d \omega(M) = \bigoplus_{k\geq 0} H^k_d \omega(M)\) carries an \(sl_2\)-action. Indeed, by ii), every de Rham cohomology class of \(M\) contains a symplectic harmonic representative. Hence, the symplectic star operator \(*_s\), and consequently, the operator \(\Lambda = *_s^{-1} L *_s\) are well defined on \(H^*_d \omega(M)\), by taking symplectic harmonic representatives.

Summing up, the de Rham cohomology of compact symplectic manifolds satisfying HLC shares with that of Kähler manifolds an action of the Lie algebra \(sl_2\). In the latter case and also in the Hyper-Kähler setting, Figueroa-O’Farrill, Köhl, and Spence [9], following an idea of Witten [22], showed that such an action and the Hodge–Lefschetz theory of compact (Hyper)-Kähler manifolds derive from the supersymmetry, more in particular, from the symmetries of certain supersymmetric sigma models (see also [24]). Finally, it has to be remarked that HLC, or equivalently, the notion of \(dd^\Lambda\)-Lemma on compact symplectic manifolds, is a special case of the notion of the \(dd^J\)-Lemma on generalized complex manifolds. For general results of such a notion and for other results in the context of supersymmetry we refer to [6] and [20] respectively.
4 Hard Lefschetz Condition on 4-dimensional Compact Symplectic Solvmanifolds

Let \((M, \omega)\) be a compact symplectic manifold. Then, it is easy to see that the set of all symplectic forms on \(M\), \(\text{SF}(M)\), is an open set in \(\mathbb{R}^{b_2(M)}\), where \(b_2(M)\) is the second Betti number of \(M\). Let

\[\text{SF}_{HLC}(M) \triangleq \{\Omega \in \text{SF}(M) : (M, \Omega) \text{ has HLC}\} .\]

Since every Kähler manifold has the Hard Lefschetz Condition (cf., [10]), then we have the following proposition:

**Proposition 4.1** Every symplectic form on torus of dimension \(2n\) is cohomologous to a Kähler form and thus has the Hard Lefschetz Condition.

**Proof** Let \(\omega\) be a symplectic form on the torus, then it cohomologous to a symplectic form \(\omega_0\) which can be expressed as a constant coefficient combination of the standard bases \(\{dx^i \wedge dx^j\}\). For such an \(\omega_0\) there exists a calibrated almost complex structure \(J_0\), which is in fact integrable. Therefore, \(\omega_0\) is Kähler, and consequently it satisfies the HLC. The same holds for \(\omega\). \(\square\)

Hasegawa has proven the following result in [11]: A compact solvmanifold admits a Kähler structure if and only if it is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus. In particular, a compact solvmanifold of completely solvable type has a Kähler structure if and only if it is a complex torus.

In the rest of this section we focus on Hard Lefschetz Condition 4-dimensional compact homogeneous manifolds \(M = \Gamma \backslash G\), where \(G\) is a simply-connected 4-dimensional Lie group and \(\Gamma\) is a uniform lattice in \(G\), endowed with a symplectic structure.

**Theorem 4.2** Let \(G\) be a simply-connected 4-dimensional Lie group admitting a uniform lattice \(\Gamma\) and let \(M = \Gamma \backslash G\). Let \(\omega\) be a symplectic structure on \(M\). Then, if \(\omega\) satisfies the HLC, any other symplectic structure on \(M\) satisfies the HLC.

**Proof** Let \(\mathfrak{g}\) be the Lie algebra of \(G\). First of all, recall that according to [7, Theorem 9], a 4-dimensional symplectic Lie algebra is solvable. Therefore, \(\mathfrak{g}\) is a unimodular and symplectic 4-dimensional Lie algebra. According to [18], we have the following list:

0) \(\mathbb{R}^4\);
1) \(\text{nil}^3 \times \mathbb{R}\);
2) \(\text{nil}^4\);
3) \(\text{sol}^3 \times \mathbb{R}\);
4) \(\tau_{3,0}' \times \mathbb{R}\).

Denoting by \(\{e^1, \ldots, e^4\}\) a basis of the dual space \(\mathfrak{g}^*\), we can present the Lie algebras above by the following Maurer-Cartan structure equations:

0) \(\mathfrak{g} = \mathbb{R}^4,\ de^i = 0,\ i = 1, \ldots, 4,\)
1) \(\mathfrak{g} = \text{nil}^3 \times \mathbb{R},\)
\[
de^1 = 0, \quad de^2 = 0, \quad de^3 = -e^{12}, \quad de^4 = 0.\]
2) \(\mathfrak{g} = \text{nil}^4,\)
\[
de^1 = e^{24}, \quad de^2 = e^{34}, \quad de^3 = 0, \quad de^4 = 0.\]
3) \(\mathfrak{g} = \text{sol}^3 \times \mathbb{R},\)
\[
de^1 = 0, \quad de^2 = e^{12}, \quad de^3 = -e^{13}, \quad de^4 = 0.\]
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4) \( g = r_{3,0}' \times \mathbb{R} \),

\[ de^1 = 0, \quad de^2 = -e^{13}, \quad de^3 = e^{12}, \quad de^4 = 0, \]

where \( e^{ij} := e^i \wedge e^j \) and so on. By assumption \( \omega \) is a symplectic structure on \( M \) satisfying the HLC. Therefore, in view of Benson and Gordon Theorem (cf., [3, 4]), every symplectic structure on any compact quotient corresponding to Cases 1) and 2) does not satisfy the HLC. The compact quotients corresponding to Cases 0) and 4) are complex solvmanifolds, namely Complex Tori and Hyperelliptic Surfaces respectively, so that any symplectic structure on such manifolds satisfies the HLC. Finally, if \( M = \Gamma \backslash G \) is a compact quotient corresponding to case 3), we easily compute

\[
\begin{align*}
H^1_{\text{dR}}(M) &= \text{Span}_\mathbb{R} \langle [e^1], [e^4] \rangle; \\
H^2_{\text{dR}}(M) &= \text{Span}_\mathbb{R} \langle [e^{14}], [e^{23}] \rangle; \\
H^3_{\text{dR}}(M) &= \text{Span}_\mathbb{R} \langle [e^{123}], [e^{234}] \rangle.
\end{align*}
\]

Let \( \omega_0 = Ae^{14} + Be^{23} \), for \( A, B \in \mathbb{R}, AB \neq 0 \). Then \( \omega_0 \) is a symplectic structure on \( M \) and it is immediate to check that it satisfies the Hard Lefschetz condition. Let now \( \omega \) be an arbitrary symplectic form on \( M \). Then

\[ \omega = \omega_0 + d\eta \]

and, consequently, \( \omega \) satisfies the HLC too.

5 Almost Kähler Structures and Hard Lefschetz Condition on Nakamura Manifolds

The construction of completely solvable Nakamura manifolds (cf., [16]) is well known. For the sake of completeness we briefly recall it. Let \( A \in \text{SL}(2,\mathbb{Z}) \) have two real positive distinct eigenvalues

\[ \mu_1 = e^\lambda, \quad \mu_2 = e^{-\lambda}. \]

Set

\[
\Lambda = \begin{pmatrix}
  e^{-\lambda} & 0 \\
  0 & e^{\lambda}
\end{pmatrix}
\]

and let \( P \in M_{2,2}(\mathbb{R}) \) be such that

\[ \Lambda = PAP^{-1} \]

Define \( \Gamma := P\mathbb{Z}^2 + iP\mathbb{Z}^2 \); then \( \Gamma \) is a uniform discrete subgroup in \( \mathbb{C}^2 \), so that

\[ T^2_{\mathbb{C}} = \mathbb{C}^2 / \Gamma \]

is a 2-dimensional complex torus and the map

\[ F : \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad F(z) = \Lambda z, \quad \text{where } z = (z^1, z^2)^t \]

induces a biholomorphism of \( T^2_{\mathbb{C}} \) by setting \( \tilde{F}([z]) = [F(z)] \). Indeed, it is immediate to check that \( \tilde{F} \) is well defined and that \( \tilde{F} \) is a biholomorphism, with \( \tilde{F}^{-1}([z]) = [F^{-1}(z)] \).

By identifying \( \mathbb{R} \times \mathbb{C}^2 \) with \( \mathbb{R}^5 \) by \( (s, z^1, z^2) \mapsto (s, x^1, x^2, x^3, x^4) \), where \( z^1 = x^1 + ix^3, z^2 = x^2 + ix^4 \), set

\[ T_1 : \mathbb{R}^5 \rightarrow \mathbb{R}^5, \]
\[ T_1(s, x^1, x^2, x^3, x^4) = (s + \lambda, e^{-\lambda}x^1, e^{\lambda}x^2, e^{-\lambda}x^3, e^{\lambda}x^4), \]

then \( T_1(s, x^1, x^2, x^3, x^4) = T_1(s, z^1, z^2) = (s + \lambda, F(z^1, z^2)) \). Therefore \( T_1 \) induces a transformations of \( \mathbb{R} \times T^2_C \), by setting

\[ T_1(s, [(z^1, z^2)]) = (s + \lambda, [F(z^1, z^2)]). \]

Define

\[ N^6 := S^1 \times \frac{\mathbb{R} \times T^2_C}{\langle T_1 \rangle}. \]

Then \( N^6 \) is a compact 6-dimensional solvmanifold of completely solvable type.

We give a numerical example. Let

\[ A = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}, \]

\( A \in \text{SL}(2, \mathbb{Z}) \). Then \( \mu_{1,2} = \frac{3 \pm \sqrt{5}}{2} \). We set

\[ \mu_1 = \frac{3 - \sqrt{5}}{2} = e^{-\lambda} \quad \text{and} \quad \mu_2 = \frac{3 + \sqrt{5}}{2} = e^{\lambda}, \]

i.e., \( \lambda = \log(\frac{3 + \sqrt{5}}{2}) \). Then

\[ P^{-1} = \begin{pmatrix} \frac{3 - \sqrt{5}}{2} & \frac{3 + \sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}, \]

and

\[ P = \begin{pmatrix} 1 & -\frac{3 + \sqrt{5}}{2} \\ -1 & \frac{3 - \sqrt{5}}{2} \end{pmatrix}, \]

and the uniform lattice \( \Gamma \) is given by

\[ \Gamma = \text{Span}_\mathbb{Z} \left\{ \begin{bmatrix} -\sqrt{5} \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 + 3\sqrt{5} \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 - 3\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} \sqrt{5} \\ 0 \\ 5 + 3\sqrt{5} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 - 3\sqrt{5} \end{bmatrix} \right\}. \]

By using previous notations, it is straightforward to check that

\[
\begin{align*}
\begin{cases}
e^1 := ds, \\
e^2 := dt, \\
e^3 := esdx^1, \\
e^4 := e^{-s}dx^2, \\
e^5 := esdx^3, \\
e^6 := e^{-s}dx^4.
\end{cases}
\end{align*}
\]
gives rise to a global coframe on $N^6$, where $dt$ is the global 1-form on $S^1$. Therefore, with respect to $\{e^i\}_{i \in \{1, \ldots, 6\}}$ the structure equations are the following:

$$
\begin{cases}
  de^1 = 0, \\
  de^2 = 0, \\
  de^3 = e^{13}, \\
  de^4 = -e^{14}, \\
  de^5 = e^{15}, \\
  de^6 = -e^{16}.
\end{cases}
$$

Then $(J, \omega, g)$ defined respectively as

$$
\begin{cases}
  J e^1 := -e^2, \\
  J e^3 := -e^4, \\
  J e^5 := -e^6,
\end{cases}
$$

$$\omega := e^{12} + e^{34} + e^{56},$$

and $g(\cdot, \cdot) = \omega(\cdot, J \cdot)$ give rise to an almost Kähler structure on $N^6$. Furthermore,

$$
\begin{cases}
  \psi^1 := e^1 + ie^2, \\
  \psi^2 := e^3 + ie^4, \\
  \psi^3 := e^5 + ie^6.
\end{cases}
$$

is a complex co-frame of $(1, 0)$-forms on $(M^6, J)$; one can compute

$$
\begin{cases}
  d\psi^1 = 0, \\
  d\psi^2 = \frac{1}{2}(\psi^{12} + \psi^{21}), \\
  d\psi^3 = \frac{1}{2}(\psi^{13} + \psi^{31}).
\end{cases}
$$

Since $b_1(N^6) = 2$, $b_2(N^6) = 5$ (see [2]), we obtain

$$
H^1_{dR}(N^6) \cong \text{Span}_\mathbb{R}(\psi^1 + \psi^1, i(\psi^1 - \psi^1)),
$$

$$
H^2_{dR}(N^6) \cong \text{Span}_\mathbb{R}(i(\psi^{11}, i\psi^{22}, i(\psi^{23} + \psi^{32})), i(\psi^{23} - \psi^{23})),
$$

and consequently,

$$
H^+_J(N^6) = \mathbb{R}\langle i(\psi^{11}, i\psi^{22}, i\psi^{33}) \rangle,
$$

$$
H^-_J(N^6) = \mathbb{R}\langle i(\psi^{23} - \psi^{32}) \rangle,
$$

and the primitive $J$-invariant cohomology

$$
H^+_J(0)(N^6) = \langle i(\psi^{11} - \psi^{33}), i(\psi^{22} - \psi^{33}), i(\psi^{23} + \psi^{32}) \rangle,
$$

so that the dimensions of such groups are

$$
h^+_J = 4, \quad h^-_J = 1, \quad h^+_{J,0} = 3.
$$

Then according to [19, Proposition 2.3]

$$
\dim_{\mathbb{R}} \ker P_J = h^+_J + h^-_J = 3 + 1 = 4 = b_2 - 1,
$$
and $J$ is $C^\infty$-pure-and-full. We can show the following

**Theorem 5.1** Let $\Omega$ be any symplectic form on a Nakamura manifold $N^6$. Then $\Omega$ satisfies the HLC. Thus, $\text{SF}_{\text{HLC}}(N^6) = \text{SF}(N^6)$.

**Proof** The first and second de Rham cohomology groups of $N^6$ can be expressed in terms of the real coframe $\{e^1, \ldots, e^6\}$ as

\[
\begin{align*}
H^1_{\text{dR}}(N^6) &\cong \text{Span}_\mathbb{R}\{e^1, e^2\} \\
H^2_{\text{dR}}(N^6) &\cong \text{Span}_\mathbb{R}\{e^{12}, e^{34}, e^{56}, e^{36}, e^{45}\}.
\end{align*}
\]

Let $\Omega$ be a symplectic form on $N^6$. Then $\Omega$ can be written as

\[
\Omega = c_1 e^{12} + c_2 e^{34} + c_3 e^{56} + c_4 e^{36} + c_5 e^{45} + d\eta,
\]

where $c_i \in \mathbb{R}$, $i = 1, \ldots, 5$ and $\eta$ is a suitable 1-form on $N^6$. We obtain that

\[
\Omega^3 = 6c_1(c_2c_3 + c_4c_5)e^{123456} + d\eta'.
\]

Since by assumption $\Omega$ is a symplectic structure on $N^6$, we get

\[
c_1(c_2c_3 + c_4c_5) \neq 0.
\]  \hspace{1cm} (5.4)

A direct computation shows that

\[
\begin{align*}
[\Omega]^2 \cup [e^1] &= [2(c_2c_3 + c_4c_5)e^{13456}], \\
[\Omega]^2 \cup [e^2] &= [2(c_2c_3 + c_4c_5)e^{23456}],
\end{align*}
\]

and

\[
\begin{align*}
[\Omega] \cup [e^{12}] &= [c_2e^{1234} + c_3e^{1256} + c_4e^{1236} + c_5e^{1245}], \\
[\Omega] \cup [e^{56}] &= [c_1e^{1256} + c_2e^{3456}], \\
[\Omega] \cup [e^{34}] &= [c_1e^{1234} + c_3e^{3456}], \\
[\Omega] \cup [e^{45}] &= [c_1e^{1245} + c_4e^{3456}], \\
[\Omega] \cup [e^{36}] &= [c_1e^{1236} + c_5e^{3456}].
\end{align*}
\]

Therefore, by the above computations it turns out that

\[
\begin{align*}
[\Omega]^2 : H^1_{\text{dR}}(N^6) &\rightarrow H^5_{\text{dR}}(N^6), \\
[\Omega] : H^2_{\text{dR}}(N^6) &\rightarrow H^4_{\text{dR}}(N^6)
\end{align*}
\]

are isomorphisms if and only if

\[
c_1(c_2c_3 + c_4c_5) \neq 0,
\]

which is exactly the condition (5.4). This ends the proof. \hfill \Box
6 A Six-dimensional Cohomologically Kähler Manifold with No Kähler Metrics

Let $G(c)$ be the connected completely solvable Lie group of dimension 5 consisting of matrices of the form

$$a = \begin{pmatrix}
e^{cz} & 0 & 0 & 0 & x_1 \\
0 & e^{cz} & 0 & 0 & y_1 \\
0 & 0 & e^{cz} & 0 & x_2 \\
0 & 0 & 0 & e^{cz} & y_2 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 
\end{pmatrix},$$

where $x_i, y_i, z \in \mathbb{R}$ ($i = 1, 2$) and $c$ is a nonzero real number. Then a global system of coordinates $x_1, y_1, x_2, y_2$ and $z$ for $G(c)$ is given by $x_i(a) = x_i, y_i(a) = y_i$ and $z(a) = z$. A standard calculation shows that a basis for the right invariant 1-forms on $G(c)$ consists of

$$\{dx_1 - cx_1dz, dy_1 - cy_1dz, dx_2 - cx_2dz, dy_2 - cy_2dz, dz\}.$$ Alternatively, the Lie group $G(c)$ may be described as a semidirect product $G(c) = \mathbb{R} \ltimes \psi \mathbb{R}^4$, where $\psi(z)$ is the linear transformation of $\mathbb{R}^4$ given by the matrix

$$\begin{pmatrix}e^{cz} & 0 & 0 & 0 \\
0 & e^{-cz} & 0 & 0 \\
0 & 0 & e^{cz} & 0 \\
0 & 0 & 0 & e^{-cz} \end{pmatrix},$$

for any $z \in \mathbb{R}$. Thus, $G(c)$ has a discrete subgroup $\Gamma(c) = \mathbb{Z} \ltimes \psi \mathbb{Z}^4$ such that the quotient space $\Gamma(c) \backslash G(c)$ is compact. Therefore, the forms

$$dx_i - cx_idz, \quad dy_i - cy_idz, \quad dz, \quad i = 1, 2$$

descend to 1-forms $\alpha_i, \beta_i$ and $\gamma, i = 1, 2$ on $\Gamma(c) \backslash G(c)$.

De Andrés, Fernández, De León, and Mencia considered the manifold $M^6(c) = G(c)/\Gamma(c) \times S^1$ which is a compact completely solvmanifold (see [1]). Moreover, Fernández, Muñoz and Santisteban have proven that $M^6(c)$ does not admit any Kähler metric (cf., [8]). Here, there are 1-forms $\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma$ and $\eta$ on $M^6(c)$ such that

$$d\alpha_i = -c\alpha_i \wedge \gamma, \quad d\beta_i = -c\beta_i \wedge \gamma, \quad d\gamma = d\eta = 0, \quad (6.1)$$

where $i = 1, 2$ and such that at each point of $M^6(c)$, $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma, \eta\}$ is a basis for the 1-forms on $M^6(c)$. Using Hattori’s theorem [12], they compute the real cohomology of $M^6(c)$:

$$H^0_{dR}(M^6(c)) \simeq \text{Span}_\mathbb{R}(1),$$

$$H^1_{dR}(M^6(c)) \simeq \text{Span}_\mathbb{R}(\gamma, [\eta]),$$

$$H^2_{dR}(M^6(c)) \simeq \text{Span}_\mathbb{R}(\{\alpha_1 \wedge \beta_1, [\alpha_1 \wedge \beta_2], [\alpha_2 \wedge \beta_1], [\alpha_2 \wedge \beta_2], [\gamma \wedge \eta]\}),$$

$$H^3_{dR}(M^6(c)) \simeq \text{Span}_\mathbb{R}(\{\alpha_1 \wedge \beta_1 \wedge \gamma, [\alpha_1 \wedge \beta_2 \wedge \gamma], [\alpha_2 \wedge \beta_1 \wedge \gamma], [\alpha_2 \wedge \beta_2 \wedge \gamma]\}).$$
Therefore, the Betti number of $M^6(c)$ are

\[
\begin{align*}
\beta^0 &= b^0 = 1, \\
\beta^1 &= b^5 = 2, \\
\beta^2 &= b^4 = 5, \\
\beta^3 &= 8.
\end{align*}
\]

We denote by $(g, J, \omega)$ be an almost Kähler structure on $M^6(c)$, where we choose

\[g = \alpha_1 \otimes \alpha_1 + \beta_1 \otimes \beta_1 + \alpha_2 \otimes \alpha_2 + \beta_2 \otimes \beta_2 + \gamma \otimes \gamma + \eta \otimes \eta\]

and

\[\omega = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 + \gamma \wedge \eta.\]

So $J$ is given by

\[J\alpha_1 = -\beta_1, \quad J\alpha_2 = -\beta_2, \quad J\gamma = -\eta.\]

It is clear that the maps

\[\omega : H_{\text{dr}}^2(M^6(c)) \rightarrow H_{\text{dr}}^4(M^6(c))\]

and

\[\omega^2 : H_{\text{dr}}^1(M^6(c)) \rightarrow H_{\text{dr}}^5(M^6(c))\]

are isomorphisms. Thus, $(M^6(c), \omega)$ satisfies the Hard Lefschetz Condition. By simple calculation, we can get

\[H_J^0 = \text{Span}_R \{[\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1]\},\]

\[H_J^1 = \text{Span}_R \{[\alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1], [\alpha_1 \wedge \beta_1], [\alpha_2 \wedge \beta_2], [\gamma \wedge \eta]\},\]

\[\ker P_J = \text{Span}_R \{\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1, \alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1, \alpha_1 \wedge \beta_1 - \gamma \wedge \eta, \alpha_2 \wedge \beta_2 - \gamma \wedge \eta\}.\]

Hence, $\dim \ker P_J = 4 = b^2 - 1$. Of course, $J$ is $C^\infty$ pure and full (cf., [19]).

In the following, we will study the other symplectic structures on $M^6(c)$. Set

\[
\xi_1 = \alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1, \quad \xi_2 = \alpha_1 \wedge \beta_1 - \gamma \wedge \eta, \quad \xi_3 = \alpha_2 \wedge \beta_2 - \gamma \wedge \eta
\]

and $\theta = \alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1$. Let $\Omega = c \omega + c_1 \xi_1 + c_2 \xi_2 + c_3 \xi_3 + a \theta$, where $c, c_i, a \in \mathbb{R}$. By direct calculation,

\[
\Omega^3 = 6(c^3 - cc_1^2 - cc_2^2 - cc_3^2 + ca^2 - cc_2c_3
\]

\[+ c_1^2c_2 + c_1^2c_3 - c_2c_3^2 - c_2a^2 - c_2c_3 - c_3a^2)\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \gamma \wedge \eta.\]
Then \( \Omega \) is a symplectic form if and only if
\[
c^3 - cc_1^2 - cc_2^2 - cc_3 + ca^2 - cc_2c_3 + c_1^2c_2 + c_2^2c_3 - c_2c_3^2 - c_2a^2 - c_2^2c_3 - c_3a^2 \neq 0. \tag{6.2}
\]
A direct computation shows that
\[
[\Omega]^2 \cup [\gamma] = 2(c^2 - c_1^2 + a^2 + cc_2 + cc_3 + c_2c_3)\alpha_1 \land \beta_1 \land \alpha_2 \land \beta_2 \land \gamma,
\]
\[
[\Omega]^2 \cup [\eta] = 2(c^2 - c_1^2 + a^2 + cc_2 + cc_3 + c_2c_3)\alpha_1 \land \beta_1 \land \alpha_2 \land \beta_2 \land \eta,
\]
and
\[
[\Omega] \cup [\alpha_1 \land \beta_1] = [(c - c_2 - c_3)\alpha_1 \land \beta_1 \land \gamma \land \eta + (c + c_3)\alpha_1 \land \beta_1 \land \alpha_2 \land \beta_2],
\]
\[
[\Omega] \cup [\alpha_1 \land \beta_2] = [(c - c_2 - c_3)\alpha_1 \land \beta_2 \land \gamma \land \eta + (a - c_1)\alpha_1 \land \beta_1 \land \alpha_2 \land \beta_2],
\]
\[
[\Omega] \cup [\alpha_2 \land \beta_1] = [(c - c_2 - c_3)\alpha_2 \land \beta_1 \land \gamma \land \eta - (a + c_1)\alpha_1 \land \beta_1 \land \alpha_2 \land \beta_2],
\]
\[
[\Omega] \cup [\alpha_2 \land \beta_2] = [(c - c_2 - c_3)\alpha_2 \land \beta_2 \land \gamma \land \eta + (c + c_2)\alpha_1 \land \beta_1 \land \alpha_2 \land \beta_2],
\]
\[
[\Omega] \cup [\gamma \land \eta] = [(c + c_2)\alpha_1 \land \beta_1 \land \gamma \land \eta + (c + c_3)\alpha_2 \land \beta_2 \land \gamma \land \eta
\]
\[
+ (c_1 + a)\alpha_1 \land \beta_2 \land \gamma \land \eta + (c_1 - a)\alpha_2 \land \beta_1 \land \gamma \land \eta].
\]
It turns out that
\[
\begin{cases}
[\Omega]^2 : H^{1}_{dR}(M^6(c)) \to H^{5}_{dR}(M^6(c)), \\
[\Omega] : H^{2}_{dR}(M^6(c)) \to H^{4}_{dR}(M^6(c))
\end{cases}
\]
are isomorphisms if and only if
\[
(c - c_2 - c_3)(c^2 - c_1^2 + a^2 + cc_2 + cc_3 + c_2c_3) \neq 0, \tag{6.3}
\]
which is exactly the condition (6.2). It means that all symplectic forms on \( M^6(c) \) satisfy the Hard Lefschetz Condition. Therefore, we proved the following

**Theorem 6.1** Let \( \Omega \) be any symplectic form on \( M^6(c) \). Then \( \Omega \) satisfies the HLC. Thus, \( SF_{HLC}(M^6(c)) = SF(M^6(c)) \).

**Remark 6.2** In addition, we still calculate two other compact completely solvmanifolds \( N^6(c) \) and \( P^6(c) \) in [8]. We have found that all symplectic forms on these two manifolds satisfy the Hard Lefschetz Condition.

At last, we want to propose the following conjecture:

**Conjecture 6.3** Suppose that \( M \) is a compact completely solvable manifold. Let \( \omega \) be a symplectic structure on \( M \). If \( \omega \) satisfies the HLC, then any other symplectic structure on \( M \) satisfies the HLC.

**Conflict of Interest**
The authors declare no conflict of interest.

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