ON MAXIMUM-PRINCIPLE FUNCTIONS FOR FLOWS BY POWERS OF THE GAUSS CURVATURE

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Abstract. We consider flows with normal velocities equal to powers strictly larger than one of the Gauss curvature. Under such flows closed strictly convex surfaces converge to points. In his work on $|A|^2$, Schn"urer proposes criteria for selecting quantities that are suitable for proving convergence to a round point. Such monotone quantities exist for many normal velocities, including the Gauss curvature, some powers larger than one of the mean curvature, and some powers larger than one of the norm of the second fundamental form. In this paper, we show that no such quantity exists for any powers larger than one of the Gauss curvature.

1. Introduction

We consider closed strictly convex surfaces $M_t$ in $\mathbb{R}^3$ that contract with normal velocities equal to the positive powers of the Gauss curvature,

$$\frac{d}{dt}X = -K^\sigma \nu.$$  

(1.1)

For all $\sigma > 0$, this is a parabolic flow equation. We have a solution on a maximal time interval $[0, T)$, $0 < T < \infty$. Chow \cite{Chow} proves that the surfaces converge to a point as $t \to T$.

For $\sigma = 1$, this is the Gauss curvature flow. It was introduced by Firey as a model for the shape of wearing stones on beaches \cite{Firey}. Firey conjectured that, after appropriate rescaling, the surfaces converge to spheres. This is also referred to as convergence to a "round point". The conjecture was confirmed by Andrews in \cite{Andrews}. Andrews and Chen \cite{AndrewsChen} extended this result to all powers $\frac{1}{2} \leq \sigma \leq 1$. The crucial step in their proof, Theorem 2.2, is to show that the quantity

$$\max_{M_t} \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1^2 \lambda_2^2} \cdot K^{2\sigma}$$

(1.2)

is non-increasing in time. $\lambda_1$, $\lambda_2$ denote the principal curvatures of the surfaces $M_t$.

We give a more detailed introduction to the standard notation in Section 2.

For many other normal velocities $F$ monotone quantities $w$ like (1.2) are known, which are monotone during the corresponding flows and vanish precisely for spheres. For $F = |A|^2$ Schn"urer \cite{Schnuerer} obtains

$$w = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2} \cdot H$$
and for $F = H^\sigma$, $1 \leq \sigma \leq \sigma_*$, $\sigma_* \approx 5.17$, Schulze and Schnürer [11] use

$$w = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1^2 \lambda_2^2} \cdot H^{2\sigma}.$$ 

For $F = \text{tr} A^\sigma$, $1 \leq \sigma < \infty$, Andrews and Chen [4] get

$$w = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1^2 \lambda_2^2} \cdot (\text{tr} A^\sigma)^2.$$ 

This is the first example where flows of an arbitrarily high degree of homogeneity converge surfaces to round points.

In [8], Schnürer proposes criteria for selecting monotone quantities like (1.2). To the author’s knowledge, to date, all known quantities which fulfill these criteria can be used to prove convergence to a round point. This is why we work with these criteria as a definition in this paper. Our question is whether such monotone quantities exist for equation (1.1) if $\sigma > 1$. Their monotonicity is proven using the maximum-principle so we name these quantities maximum-principle functions.

**Definition 1.1. (Maximum-principle function)** Let $w$ be a symmetric rational function of the principal curvatures,

$$w = \frac{p(\lambda_1, \lambda_2)}{q(\lambda_1, \lambda_2)},$$

(1.3)

Here, $p \not\equiv 0$ and $q \not\equiv 0$ are homogeneous polynomials. $w$ is called a maximum-principle function for a normal velocity $F$, if

1. (a) $p(\lambda_1, \lambda_2) \geq 0$, $q(\lambda_1, \lambda_2) > 0$ for all $0 < \lambda_1, \lambda_2$,
   (b) $p(\lambda_1, \lambda_2) = 0$ for $\lambda_1 = \lambda_2 > 0$.
2. $\deg p > \deg q$.
3. $\frac{\partial w(\rho, \lambda_1)}{\partial \rho} < 0$ for all $0 < \rho < 1$ and $\frac{\partial w(\rho, \lambda_1)}{\partial \rho} > 0$ for all $\rho > 1$.
4. $L(w) := \frac{d}{dt}w - F_{ij}w_{;ij} \leq 0$ for all $0 < \lambda_1, \lambda_2$.

We achieve this by assuming

(a) terms without derivatives of $(h_{ij})$ are nonpositive, and
(b) terms involving derivatives of $(h_{ij})$ at a critical point of $w$, i.e. $w_{;i} = 0$ for all $i = 1, 2$, are nonpositive.

As in [8, 9], we motivate conditions (1) to (4). For all flow equations considered, spheres contract to round points. So we can only find monotone quantities if $\deg p \leq \deg q$ or $p(\lambda, \lambda) = 0$ for all $\lambda > 0$.

If $\deg p < \deg q$, we obtain that $w$ is non-increasing on any self-similarly contracting surface. So this does not imply convergence to a round point.

Condition (3) ensures that the quantity decreases, if the ratio of the principal curvatures $\lambda_1/\lambda_2$ approaches one.

By condition (4) we check that we can apply the maximum-principle to prove monotonicity.

The linear operator $L(w)$, which corresponds to the general flow equation

$$\frac{d}{dt}X = -F\nu,$$

fulfills an identity of the form

$$L(w) = C_w(\lambda_1, \lambda_2) + G_w(\lambda_1, \lambda_2) h_{11;1}^2 + G_w(\lambda_2, \lambda_1) h_{22;2}^2.$$
at a critical point of \( w \). This is Lemma 4.14. We name the rational function \( C_w(\lambda_1, \lambda_2) \) the constant terms and the rational function \( G_w(\lambda_1, \lambda_2) \) the gradient terms of the evolution equation. To fulfill condition (4) the constant terms \( C_w(\lambda_1, \lambda_2) \) and the gradient terms \( G_w(\lambda_1, \lambda_2) \) simultaneously have to be nonpositive for all \( 0 < \lambda_1, \lambda_2 \). Here, we obtain a contradiction for \( F = K^\sigma \) if \( \sigma > 1 \).

Our main theorem is

**Theorem 1.2.** For a family of smooth closed strictly convex surfaces \( M_t \) in \( \mathbb{R}^3 \) flowing according to
\[
\frac{d}{dt}X = -K^\sigma \nu, \quad \sigma > 1,
\]
there exist no maximum-principle functions.

Despite this fact, it remains an open question whether for any powers \( \sigma > 1 \), closed strictly convex surfaces converge to round points. Due to Andrews we already know that this does not necessarily happen for all powers \( \frac{1}{4} \leq \sigma \leq \frac{1}{2} \). For \( \sigma = \frac{1}{4} \), they converge to ellipsoids [1]. For all powers \( \frac{1}{4} < \sigma \leq \frac{1}{2} \), surfaces contract homothetically in the limit [3].

In Section 3 we explain the proof strategy. In Section 4 and 5 we outline the proof of our main Theorem 1.2.

## 2. Notation

For this paper, we adopt the chapter on standard notation from [5].

The linear operator \( L \) corresponding to the general flow equation
\[
\frac{d}{dt}X = -F \nu
\]
is defined by
\[
L(w) := \frac{d}{dt}w - F^{ij}w_{,ij}.
\]

We use \( X = X(x, t) \) to denote the embedding vector of a manifold \( M_t \) into \( \mathbb{R}^3 \) and \( \frac{\partial}{\partial t}X = \dot{X} \) for its total time derivative. It is convenient to identify \( M_t \) and its embedding in \( \mathbb{R}^3 \). The normal velocity \( F \) is a homogeneous symmetric function of the principal curvatures. We choose \( \nu \) to be the outer unit normal vector to \( M_t \). The embedding induces a metric \( g_{ij} := \langle X_i, X_j \rangle \) and the second fundamental form \( h_{ij} := -\langle X_{,ij}, \nu \rangle \) for all \( i, j = 1, 2 \). We write indices preceded by commas to indicate differentiation with respect to space components, e.g. \( X_k = \frac{\partial X}{\partial x_k} \) for all \( k = 1, 2 \).

We use the Einstein summation notation. When an index variable appears twice in a single term it implies summation of that term over all the values of the index. Indices are raised and lowered with respect to the metric or its inverse \( (g^{ij}) \), e.g. \( h_{ij}h^{ij} = h_{ij}g^{ik}h_{kj}g^{lj} = h_{ij}^k h^{lj}_k \).

The principal curvatures \( \lambda_1, \lambda_2 \) are the eigenvalues of the second fundamental form \( (h_{ij}) \) with respect to the induced metric \( (g_{ij}) \). A surface is called strictly convex, if all principal curvatures are strictly positive. We will assume this throughout the paper. Therefore, we may define the inverse of the second fundamental form denoted by \( (h^{ij}) \).
Symmetric functions of the principal curvatures are well-defined, we will use the mean curvature $H = q^{ij} h_{ij} = \lambda_1 + \lambda_2$, the square of the norm of the second fundamental form $|A|^2 = h^{ij} h_{ij} = \lambda_1^2 + \lambda_2^2$, the trace of powers of the second fundamental form $tr A^\sigma = tr (h^i_j)^\sigma = \lambda_1^\sigma + \lambda_2^\sigma$, and the Gauss curvature $K = \det h_{ij}/\det g_{ij} = \lambda_1 \lambda_2$. We write indices preceded by semi-colons to indicate covariant differentiation with respect to the induced metric, e.g., $h_{ij;k} = h_{ij,k} - \Gamma^l_{jk} h_{il} - \Gamma^l_{ik} h_{jl}$, where $\Gamma^l_{jk} = \frac{1}{2} g^{kl} (g_{il,j} + g_{jl,i} - g_{ij,l})$. It is often convenient to choose normal coordinates, i.e., coordinate systems such that at a point the metric tensor equals the Kronecker delta, $g_{ij} = \delta_{ij}$, and $(h_{ij})$ is diagonal, $(h_{ij}) = \text{diag}(\lambda_1, \lambda_2)$. Whenever we use this notation, we will also assume that we have fixed such a coordinate system. We will only use Euclidean coordinate systems for $\mathbb{R}^3$ so that the indices of $h_{ij,k}$ commute according to the Codazzi-Mainardi equations.

A normal velocity $F$ can be considered as a function of $(\lambda_1, \lambda_2)$ or $(h_{ij}, g_{ij})$. We set $F^{ij} = \frac{\partial F}{\partial h_{ij}}$, $F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}}$. Note that in coordinate systems with diagonal $h_{ij}$ and $g_{ij} = \delta_{ij}$ as mentioned above, $F^{ij}$ is diagonal. For $F = K^\sigma$, we have $F^{ij} = \sigma K^\sigma h^{ij}$.

3. Proof strategy

To prove our main Theorem 1.2, we use an elementary fact about polynomials in one variable. If a polynomial in one variable $\rho$, which is not constantly zero, is nonpositive for all $\rho > 0$, then the coefficient of its leading term has to be negative. As mentioned before, we focus on condition (4) in the Definition 1.1 of the maximum-principle functions. We use an indirect proof and assume the existence of a maximum-principle function. We calculate the constant terms $C_w (\lambda_1, \lambda_2)$ and the gradient terms $G_w (\lambda_1, \lambda_2)$ for general homogeneous symmetric polynomials $p (\lambda_1, \lambda_2)$ and $q (\lambda_1, \lambda_2)$. We state them in the algebraic basis $\{H, K\}$, where $H$ is the mean curvature and $K$ is the Gauss curvature, i.e., $p (H, K) := \sum_{i=0}^{[s/2]} c_{i+1} H^{s-2i} K^i$. Using the identity $q^2 L (p/q) = q L (p) - p L (q)$ at a critical point of $p/q$, where we also choose normal coordinates, we easily see that $C_w (\lambda_1, \lambda_2)$ and $G_w (\lambda_1, \lambda_2)$ differ from a polynomial only by a nonnegative factor. Dividing by this nonnegative factor, we transform $C_w (\lambda_1, \lambda_2)$ and $G_w (\lambda_1, \lambda_2)$ into polynomial versions of the constant terms and the gradient terms. The next step is to dehomogenize the polynomial version of $C_w (\lambda_1, \lambda_2)$ and $G_w (\lambda_1, \lambda_2)$ by setting $\lambda_1 = \rho$, $\lambda_2 = 1$ and vice versa. Since the polynomial version of the constant terms is symmetric and the polynomial version of the gradient terms is asymmetric, we obtain only three instead of four polynomials in one variable $C (\rho)$, $G_1 (\rho)$ and $G_2 (\rho)$. Due to the property of a maximum-principle function, all three polynomials in $\rho$ have to be nonpositive for all $\rho \geq 0$.

Now we calculate the leading terms of all three polynomials in one variable. Here, for technical reasons we have to distinguish nine cases. However, each case results in a contradiction. As it turns out, the coefficients of the leading terms of $C (\rho)$, $G_1 (\rho)$ and $G_2 (\rho)$ never can be simultaneously negative. This concludes the proof of our main Theorem 1.2.

4. Evolution equations, constant terms and gradient terms

4.1. Evolution equations. In the first part of Section 1.1 we do some preliminary work. We calculate the covariant derivatives of the mean curvature $H$ and the Gauss
curvature \( K \). Furthermore, we present the evolution equations, corresponding to the general flow equation (2.1) of the following geometric quantities

- induced metric \( g_{ij} \),
- inverse of the induced metric \( g^{ij} \),
- second fundamental form \( h_{ij} \),
- mean curvature \( H \),
- Gauss curvature \( K \),
- and general function \( w(H, K) \) depending on \( H \) and \( K \).

**Lemma 4.1.** The covariant derivative of the mean curvature \( H \) is given by

\[

H_{;k} = g^{ij} h_{ij; k}.

\]

**Proof.** Direct calculations yield

\[

H_{;k} = (g^{ij} h_{ij})_{;k} = (g^{ij})_{;k} h_{ij} + g^{ij} h_{ij; k} = 0 = g^{ij} h_{ij; k}.
\]

**Lemma 4.2.** The covariant derivative of the Gauss curvature \( K \) is given by

\[

K_{;k} = K \tilde{h}^{ij} h_{ij; k}.
\]

**Proof.** Direct calculations yield

\[

K_{;k} = \left( \frac{\det h_{ij}}{\det g_{ij}} \right)_{;k} = \frac{1}{(\det g_{ij})^2} \left( (\det h_{ij})_{;k} (\det g_{ij}) - (\det h_{ij}) (\det g_{ij})_{;k} \right) = \frac{1}{(\det g_{ij})^2} \left( \partial_{h_{ij}} \det h_{ij} (h_{ij; k}) (\det g_{ij}) - (\det h_{ij}) \left( \partial_{g_{ij}} \det g_{ij} \right) \left( g_{ij; k} \right) \right) = 0 = \frac{1}{\det g_{ij}} (\det h_{ij}) \tilde{h}^{ij} h_{ij; k} = K \tilde{h}^{ij} h_{ij; k}.
\]

**Lemma 4.3.** The metric \( g_{ij} \) evolves according to

\[

\frac{d}{dt} g_{ij} = -2 F h_{ij}.
\]

**Proof.** We refer to [10].

**Corollary 4.4.** The inverse metric \( g^{ij} \) evolves according to

\[

\frac{d}{dt} g^{ij} = 2 F h^{ij}.
\]

**Proof.** Direct calculations yield

\[

\frac{d}{dt} g^{ij} = - g^{ik} g^{l} \frac{d}{dt} g_{kl} = 2 F g^{ik} h_{kl} g^{lj}.
\]
Lemma 4.5. The second fundamental form $h_{ij}$ evolves according to
\begin{equation}
L(h_{ij}) = F^{kl} h^a_k h_{al} \cdot h_{ij} - F^{kl} h_{kl} \cdot h^a_{aj} - F^k h^k_{kj} + F^{kl,rs} h_{kl;i} h_{rs;j}. 
\end{equation}
\[ (4.5) \]

Proof. We refer to [10].

Lemma 4.6. The mean curvature $H$ evolves according to
\begin{equation}
L(H) = F^{kl} h^a_k h_{al} \cdot H + (F - F^{kl} h_{kl}) |A|^2 + g^{ij} F^{kl,rs} h_{kl;i} h_{rs;j}. 
\end{equation}
\[ (4.6) \]

Proof. This is a straightforward calculation.

$$
L(H) = \frac{d}{dt} H - F^{kl} H_{kl} \quad (\text{see (2.2)}) 
= \frac{d}{dt} \left( g^{ij} h_{ij} - F^{kl} \left( \frac{H_{kj}}{h_{ij}} \right) \right) 
= \left( \frac{d}{dt} g^{ij} \right) h_{ij} + g^{ij} \left( \frac{d}{dt} h_{ij} \right) - F^{kl} \left( g^{ij} h_{ij:k} + g^{ij} h_{ij:kl} \right) 
= 2F h_{ij} h_{ij} + g^{ij} \left( \frac{d}{dt} h_{ij} \right) - F^{kl} \left( g^{ij} h_{ij:k} + g^{ij} h_{ij:kl} \right) 
= 2F |A|^2 + g^{ij} L(h_{ij}) \quad (\text{see (4.5)}) 
= 2F |A|^2 + g^{ij} \left( F^{kl} h^a_k h_{al} \cdot h_{ij} - F^{kl} h_{kl} \cdot h^a_{aj} - F^k h^k_{kj} + F^{kl,rs} h_{kl;i} h_{rs;j} \right) 
= 2F |A|^2 + F^{kl} h^a_k h_{al} \cdot g^{ij} h_{ij} - F^{kl} h_{kl} \cdot h^a_{aj} g^{ij} h_{kj} - F^k h^k_{kj} g^{ij} h_{kj} + g^{ij} F^{kl,rs} h_{kl;ij} h_{rs;j} 
= 2F |A|^2 + F^{kl} h^a_k h_{al} \cdot H - F^{kl} h_{kl} \cdot |A|^2 - |A|^2 + g^{ij} F^{kl,rs} h_{kl;ij} h_{rs;j} 
= F^{kl} h^a_k h_{al} \cdot H + (F - F^{kl} h_{kl}) |A|^2 + g^{ij} F^{kl,rs} h_{kl;ij} h_{rs;j}. 
$$

Lemma 4.7. The Gauss curvature $K$ evolves according to
\begin{equation}
L(K) = K \left( F^{kl} h^a_k h_{al} \cdot h_{ij} - (F^{kl} h_{kl} + F) h^a_{aj} - 2F H + \left( \tilde{h}_{ij} F^{kl,rs} + F^{ij} \left( \tilde{h}^{kr} \tilde{h}^{ls} - \tilde{h}^{kl} \tilde{h}^{rs} \right) \right) h_{kl;ij} h_{rs;j} \right). 
\end{equation}
\[ (4.7) \]

Proof. This is a straightforward calculation.

$$
L(K) = \frac{d}{dt} K - F^{kl} K_{,kl} \quad (\text{see (2.2)}) 
$$
\[
= \frac{1}{(\det g_{ij})^2} \left( \left( \frac{d}{dt} \det h_{ij} \right) (\det g_{ij}) - (\det h_{ij}) \left( \frac{d}{dt} \det g_{ij} \right) \right)
- F^{kl} \left( K \tilde{h}^{ij} h_{ij;k} \right) ;
\]
\[
= K \left( \tilde{h}^{ij} \left( \frac{d}{dt} h_{ij} \right) - (g^{ij})(-2F h_{ij}) \right)
- F^{kl} \left( K \tilde{h}^{r;kl} h_{ij;k} + K \left( -\tilde{h}^{ij} h_{rs;kl} \right) h_{ij;k} + K \tilde{h}^{ij} h_{ij;kl} \right)
= K \left( \tilde{h}^{ij} L (h_{ij}) + 2F g^{ij} h_{ij} - F^{kl} \tilde{h}^{ij} \tilde{h}^{r;kl} h_{ij;k} h_{rs;kl} + F^{kl} \tilde{h}^{ij} \tilde{h}^{s;kl} h_{ij;k} h_{rs;kl} \right)
= K \left( \tilde{h}^{ij} L (h_{ij}) + 2FH + F^{ij} \left( \tilde{h}^{kr} \tilde{h}^{ls} - \tilde{h}^{kl} \tilde{h}^{rs} \right) h_{kl;ir;rs;kl} \right)
= K \left( \tilde{h}^{ij} \left( F^{kl} \tilde{h}^{a;kl} h_{al} \cdot h_{ij} - F^{kl} \tilde{h}^{a;kl} \tilde{h}^{a;kl} h_{aj} - F \tilde{h}^{kl} h_{ij;k} + F^{kl;rs} h_{kl;ir;rs;kl} \right)
+ 2FH + F^{ij} \left( \tilde{h}^{kr} \tilde{h}^{ls} - \tilde{h}^{kl} \tilde{h}^{rs} \right) h_{kl;ir;rs;kl} \right)
= K \left( \tilde{h}^{ij} \left( F^{kl} \tilde{h}^{a;kl} h_{al} \cdot \tilde{h}^{ij} h_{ij} - F^{kl} \tilde{h}^{a;kl} \tilde{h}^{a;kl} h_{aj} - F \tilde{h}^{ij} h_{ij;k} + \tilde{h}^{ij} F^{kl;rs} h_{kl;ir;rs;kl} \right)
+ 2FH + F^{ij} \left( \tilde{h}^{kr} \tilde{h}^{ls} - \tilde{h}^{kl} \tilde{h}^{rs} \right) h_{kl;ir;rs;kl} \right)
= K \left( \tilde{h}^{ij} \left( F^{kl} \tilde{h}^{a;kl} h_{al} \cdot \tilde{h}^{ij} h_{ij} - (F^{kl} h_{kl} + F) \tilde{h}^{ij} h_{aj} \right)
+ 2FH + \left( \tilde{h}^{ij} F^{kl;rs} + F^{ij} \left( \tilde{h}^{rs} \tilde{h}^{ls} - \tilde{h}^{kl} \tilde{h}^{rs} \right) \right) h_{kl;ir;rs;kl} \right).
\]

\textbf{Lemma 4.8.} \textit{The function } \text{w}(H, K) \text{ evolves according to}
\[
L(\text{w}(H, K)) = L(H) \, w_H + L(K) \, w_K
- F^{ij} \left( H_{ij} H_{ij} w_{HH} + (H_{ij} K_{ij} + K_{ij} H_{ij}) \, w_{HK} + K_{ij} K_{ij} \, w_{KK} \right),
\]
where \text{H} is the mean curvature and \text{K} is the Gauss curvature. \text{H} and \text{K} form an algebraic basis of the symmetric homogeneous polynomials in two variables.
Here, the $w$-terms are defined as

\[ w_H := \frac{\partial w}{\partial H}, \ w_K := \frac{\partial w}{\partial K}, \]

\[ w_{HH} := \frac{\partial^2 w}{\partial H^2}, \ w_{HK} := \frac{\partial^2 w}{\partial H \partial K}, \ w_{KK} := \frac{\partial^2 w}{\partial K^2}. \]

**Proof.** We use the chain rule.

\[ L(w(H,K)) = \frac{d}{dt} w(H,K) - F_{ij} (H_i w_H + K_i w_K), \]

\[ = \frac{d}{dt} H w_H + \frac{d}{dt} K w_K - F_{ij} (H_i w_H + K_i w_K) \]

\[ - F_{ij} (H_i H_j w_{HH} + H_i K_j w_{HK} + H_j K_i w_{HK} + K_i K_j w_{KK}) \]

\[ = L(H) w_H + L(K) w_K \]

\[ - F_{ij} (H_i H_j w_{HH} + (H_i K_j + H_j K_i) w_{HK} + K_i K_j w_{KK}). \]

4.2. **Evolution equations at a critical point, where we also choose normal coordinates.** In the second part of Section 4, we calculate the evolution equations of the following geometric quantities at a critical point of the general function $w(H,K)$, where we also choose normal coordinates,

- mean curvature $H$,
- Gauss curvature $K$,
- and general function $w(H,K)$ depending on $H$ and $K$.

**Lemma 4.9.** The covariant derivatives of the second fundamental form $h_{ij}$ fulfill these identities at a critical point of $w(H,K)$ (CP), i.e. $w(H,K)_{;i} = 0$ for $i = 1, 2$, where we also choose normal coordinates (NC), i.e. the metric tensor equals the Kronecker delta, $g_{ij} = \delta_{ij}$, and $(h_{ij})$ is diagonal, $(h_{ij}) = \text{diag}(\lambda_1, \lambda_2)$,

\[ h_{22;1} = -\frac{w_H + \lambda_2 w_K}{w_H + \lambda_1 w_K} h_{11;1} \equiv a_1 \cdot h_{11;1}, \]

\[ h_{11;2} = -\frac{w_H + \lambda_1 w_K}{w_H + \lambda_2 w_K} h_{22;2} \equiv a_2 \cdot h_{22;2}. \]

**Proof.** Let $i = 1$.

\[ w(H,K)_{;1} = H_{11} w_H + K_{11} w_K \]

see (4.1) see (4.2)

\[ = (g^{ij} h_{ij;1}) w_H + (K h_{ij;1}) w_K \]

\[ = (h_{11;1} + h_{22;1}) w_H + K \left( \frac{1}{\lambda_1} h_{11;1} + \frac{1}{\lambda_2} h_{22;1} \right) w_K \]  

(NC)

\[ = (w_H + \lambda_2 w_K) h_{11;1} + (w_H + \lambda_1 w_K) h_{22;1} \]
This implies
\[ h_{22,1} = -\frac{w_H + \lambda_2 w_K}{w_H + \lambda_1 w_K} \cdot h_{11,1} \equiv a_1 \cdot h_{11,1}. \]

Let \( i = 2 \).

\[ w(H, K)_{;2} = H_{;2} w_H + K_{;2} w_K \]

\[ = (g^{ij}h_{ij;2}) w_H + (K\bar{h}^{ij}h_{ij;2}) w_K \]

\[ = (h_{11;2} + h_{22;2}) w_H + K \left( \frac{1}{\lambda_1} h_{11;2} + \frac{1}{\lambda_2} h_{22;2} \right) w_K \quad \text{(NC)} \]

\[ = (w_H + \lambda_2 w_K) h_{11;2} + (w_H + \lambda_1 w_K) h_{22;2} \]

\[ = 0 \quad \text{(CP)}. \]

This implies
\[ h_{11;2} = -\frac{w_H + \lambda_1 w_K}{w_H + \lambda_2 w_K} \cdot h_{22;2} \equiv a_2 \cdot h_{22;2}. \]

\[ \square \]

**Lemma 4.10.** The covariant derivatives of the mean curvature \( H \) fulfill these identities at a critical point of \( w(H, K) \), where we also choose normal coordinates,

\[ H_{;1} = (1 + a_1) h_{11;1}, \]

\[ H_{;2} = (1 + a_2) h_{22;2}. \]

**Proof.**

\[ H_{;1} = g^{ij}h_{ij;1} \quad \text{(see (4.1))} \]

\[ = h_{11;1} + h_{22;1} \quad \text{(NC)} \]

\[ = (1 + a_1) h_{11;1} \quad \text{(CP)}, \]

\[ H_{;2} = g^{ij}h_{ij;2} \quad \text{(see (4.1))} \]

\[ = h_{11;2} + h_{22;2} \quad \text{(NC)} \]

\[ = (1 + a_2) h_{22;2} \quad \text{(CP)}. \]

\[ \square \]

**Lemma 4.11.** The covariant derivatives of the Gauss curvature \( K \) fulfill these identities at a critical point of \( w(H, K) \), where we also choose normal coordinates,

\[ K_{;1} = (\lambda_2 + \lambda_1 a_1) h_{11;1}, \]

\[ K_{;2} = (\lambda_1 + \lambda_2 a_2) h_{22;2}. \]

**Proof.**

\[ K_{;1} = K\bar{h}^{ij}h_{ij;1} \quad \text{(see (4.2))} \]

\[ = K \left( \frac{1}{\lambda_1} h_{11;1} + \frac{1}{\lambda_2} h_{22;1} \right) \quad \text{(NC)} \]
Lemma 4.12. The evolution equation of the mean curvature $H$ fulfills this identity at a critical point of $w(H,K)$, where we also choose normal coordinates,

\begin{equation}
L(H) = C_H(\lambda_1, \lambda_2) + G_H(\lambda_1, \lambda_2) h^2_{1;1} + G_H(\lambda_2, \lambda_1) h^2_{2;2},
\end{equation}

where

\begin{equation}
C_H(\lambda_1, \lambda_2) = F(\lambda_1^2 + \lambda_2^2) + \left( \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} \right)(\lambda_1 - \lambda_2) \lambda_1 \lambda_2,
\end{equation}

and

\begin{equation}
G_H(\lambda_1, \lambda_2) = \frac{\partial^2 F}{\partial \lambda_1^2} + 2 \frac{\partial^2 F}{\partial \lambda_1 \partial \lambda_2} a_1^2 + \frac{\partial^2 F}{\partial \lambda_2^2} a_1^2 + 2 \frac{\partial F}{\partial \lambda_1} \frac{\partial F}{\partial \lambda_2} a_1^2.
\end{equation}

Proof.

\[ L(H) = F^{kl} h^a_{k} h^{al} \cdot H + (F - F^{kl} h_{kl}) |A|^2 + g^{ij} F^{kl} h_{kl} ; h_{rs} ; i j ; (\text{see (4.10)}) \]

\[ = \left( \frac{\partial F}{\partial \lambda_1} \lambda_1^2 + \frac{\partial F}{\partial \lambda_2} \lambda_2^2 \right)(\lambda_1 + \lambda_2) \]

\[ + \left( F - \frac{\partial F}{\partial \lambda_1} \lambda_1 - \frac{\partial F}{\partial \lambda_2} \lambda_2 \right)(\lambda_1^2 + \lambda_2^2) \]

\[ + \left( \sum_{i,j=1}^{2} \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} h_{ii;1} h_{jj;1} + \sum_{i,j=1}^{2} \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j} h^2_{ij;1} \right) \]

\[ + \left( \sum_{i,j=1}^{2} \frac{\partial F}{\partial \lambda_i} h_{ii;2} h_{jj;2} + \sum_{i,j=1}^{2} \frac{\partial F}{\partial x_i} \frac{\partial F}{\partial x_j} h^2_{ij;2} \right) \quad \text{(NC)} \]

\[ = \left( \frac{\partial F}{\partial \lambda_1} \lambda_1^2 + \frac{\partial F}{\partial \lambda_2} \lambda_2^2 \right)(\lambda_1 + \lambda_2) \]

\[ + \left( F - \frac{\partial F}{\partial \lambda_1} \lambda_1 - \frac{\partial F}{\partial \lambda_2} \lambda_2 \right)(\lambda_1^2 + \lambda_2^2) \]

\[ + \left( \frac{\partial^2 F}{\partial \lambda_1^2} h^2_{1;1} + \frac{\partial^2 F}{\partial \lambda_1 \partial \lambda_2} h_{11;1} h_{22;1} + \frac{\partial^2 F}{\partial \lambda_2^2} h^2_{2;1} + 2 \frac{\partial F}{\partial \lambda_1} \frac{\partial F}{\partial \lambda_2} h^2_{1;1} \right) \]

\[ + \left( \frac{\partial^2 F}{\partial \lambda_1^2} h^2_{1;2} + \frac{\partial^2 F}{\partial \lambda_1 \partial \lambda_2} h_{11;2} h_{22;2} + \frac{\partial^2 F}{\partial \lambda_2^2} h^2_{2;2} + 2 \frac{\partial F}{\partial \lambda_1} \frac{\partial F}{\partial \lambda_2} h^2_{1;2} \right). \]
According to [7], the terms

\[ F^{ij, kl} \eta_{ij} \eta_{kl} = \sum_{i,j=1}^{2} \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} \eta_{ij} \eta_{jj} + \sum_{i,j=1}^{2} \frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j} (\eta_{ij})^2 \]

are well-defined for symmetric matrices (\( \eta_{ij} \)) and \( \lambda_1 \neq \lambda_2 \) or \( \lambda_1 = \lambda_2 \), when we interpret the last term as a limit.

We get the constant terms

\[ C_H (\lambda_1, \lambda_2) = F (\lambda_1^2 + \lambda_2^2) + \left( \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} \right) (\lambda_1 - \lambda_2) \lambda_1 \lambda_2, \]

and at a critical point of \( w (H, K) \) (CP), using identities (4.9) and (4.10), we get the gradient terms

\[ G_H (\lambda_1, \lambda_2) = \frac{\partial^2 F}{\partial \lambda_1^2} + 2 \frac{\partial^2 F}{\partial \lambda_1 \partial \lambda_2} a_1 + \frac{\partial^2 F}{\partial \lambda_2^2} a_1^2 + 2 \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} a_1^2. \]

\( \square \)

Lemma 4.13. The evolution equation of the Gauss curvature \( K \) (4.7) fulfills this identity at a critical point of \( w (H, K) \), where we also choose normal coordinates,

\[ (4.18) \quad L (K) = C_K (\lambda_1, \lambda_2) + G_K (\lambda_1, \lambda_2) h_{11;1} + G_K (\lambda_2, \lambda_1) h_{22;2}, \]

where

\[ (4.19) \quad C_K (\lambda_1, \lambda_2) = \left( F (\lambda_1 + \lambda_2) + \left( \frac{\partial F}{\partial \lambda_1} \lambda_1 - \frac{\partial F}{\partial \lambda_2} \lambda_2 \right) (\lambda_1 - \lambda_2) \right) \lambda_1 \lambda_2, \]

\[ (4.20) \quad G_K (\lambda_1, \lambda_2) = 2 \left( \frac{\partial F}{\partial \lambda_1} a_1 + \frac{\partial F}{\partial \lambda_2} a_1^2 \right) + \left( \frac{\partial^2 F}{\partial \lambda_1^2} + 2 \frac{\partial^2 F}{\partial \lambda_1 \partial \lambda_2} a_1 + \frac{\partial^2 F}{\partial \lambda_2^2} a_1^2 \right) \lambda_2 + 2 \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} a_1^2 \lambda_1. \]

Proof.

\[ L (K) = K \left( F^{kl} h^{\alpha \beta} h_{\alpha \beta} \cdot \tilde{h}^{ij} h_{ij} - (F^{kl} h_{kl} + F) \tilde{h}^{ij} h_{ij} \right) + 2FH \]

\[ + \left( \tilde{h}^{ij} F^{kl, rs} + F^{ij} \left( \tilde{h}^{kr} \tilde{h}^{ls} - \tilde{h}^{kl} \tilde{h}^{rs} \right) \right) h_{kl;i} h_{rs;j} \]

see (4.7).
\[ \begin{align*}
&= \lambda_1 \lambda_2 \left( 2 \left( \frac{\partial F}{\partial \lambda_1} \lambda_1^2 + \frac{\partial F}{\partial \lambda_2} \lambda_2^2 \right) + \left( F - \frac{\partial F}{\partial \lambda_1} \lambda_1 - \frac{\partial F}{\partial \lambda_2} \lambda_2 \right) (\lambda_1 + \lambda_2) \right) \\
&+ \frac{1}{\lambda_1} \left( \sum_{i,j=1}^{2} \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} h_{ii;1} h_{jj;1} + \sum_{i,j=1}^{2} \frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j} \lambda_i - \lambda_j \right) h_{ij;1}^2 \\
&+ \frac{1}{\lambda_2} \left( \sum_{i,j=1}^{2} \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} h_{ii;2} h_{jj;2} + \sum_{i,j=1}^{2} \frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j} \lambda_i - \lambda_j \right) h_{ij;2}^2 \\
&+ \frac{\partial F}{\partial \lambda_1} \left( \frac{1}{\lambda_1} h_{11;2}^2 \right) - \frac{\partial F}{\partial \lambda_1} \left( \frac{1}{\lambda_1} \lambda_1 h_{11;1} h_{22;1} \right) \\
&+ \frac{\partial F}{\partial \lambda_2} \left( \frac{1}{\lambda_2} h_{22;1}^2 \right) - \frac{\partial F}{\partial \lambda_2} \left( \frac{1}{\lambda_2} \lambda_2 h_{11;2} h_{22;2} \right) \quad \text{(NC)}
\end{align*} \]

According to [1], the terms

\[ F^{ij,kl} \eta_{ij} \eta_{kl} = \sum_{i,j=1}^{2} \frac{\partial^2 F}{\partial \lambda_i \partial \lambda_j} \eta_{ii} \eta_{jj} + \sum_{i,j=1}^{2} \frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j} \frac{\lambda_i - \lambda_j}{\left( \eta_{ij} \right)^2} \]

are well-defined for symmetric matrices \((\eta_{ij})\) and \(\lambda_1 \neq \lambda_2\) or \(\lambda_1 = \lambda_2\), when we interpret the last term as a limit.

We get the constant terms

\[ C_K (\lambda_1, \lambda_2) = \left( F(\lambda_1 + \lambda_2) + \left( \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} \right) (\lambda_1 - \lambda_2) \right) \lambda_1 \lambda_2, \]

and at a critical point of \(w(H, K)\) \((\text{CP})\), using identities [4.9] and [4.10], we get the gradient terms

\[ G_K (\lambda_1, \lambda_2) = 2 \left( - \frac{\partial F}{\partial \lambda_1} a_1 + \frac{\partial F}{\partial \lambda_2} u_1^2 \right) \\
+ \left( \frac{\partial^2 F}{\partial \lambda_1^2} + \frac{\partial^2 F}{\partial \lambda_1 \partial \lambda_2} a_1 + \frac{\partial^2 F}{\partial \lambda_2^2} u_1^2 \right) \lambda_2 + 2 \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} \frac{\lambda_1 - \lambda_2}{a_1^2} \lambda_1. \]

\( \square \)
Lemma 4.14. The evolution equation of the function $w(H, K)$ fulfills this identity at a critical point of $w(H, K)$, where we also choose normal coordinates,

$$L(w(H, K)) = C_w(\lambda_1, \lambda_2) + G_w(\lambda_1, \lambda_2) h_{11;1} + G_w(\lambda_1, \lambda_2) h_{22;2},$$

where

$$C_w(\lambda_1, \lambda_2) = C_H(\lambda_1, \lambda_2) w_H + C_K(\lambda_1, \lambda_2) w_K$$

$$= \left( F(\lambda_1^2 + \lambda_2^2) + \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} \right) (\lambda_1 - \lambda_2) \lambda_1 \lambda_2 \lambda_1 \lambda_2,$$

$$+ \left( F(\lambda_1 + \lambda_2) + \left( \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} \right) (\lambda_1 - \lambda_2) \right) \lambda_1 \lambda_2 w_K,$$

$$G_w(\lambda_1, \lambda_2) = G_H(\lambda_1, \lambda_2) w_H + G_K(\lambda_1, \lambda_2) w_K$$

$$= \left( F(\lambda_1^2 + \lambda_2^2) + \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} \right) \lambda_1 \lambda_2,$$

$$\left( F(\lambda_1 + \lambda_2) + \left( \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} \right) \right) \lambda_1 \lambda_2,$$

$$\frac{\partial F}{\partial \lambda_1} \left( (1 + a_1)^2 w_{HH} + 2 (1 + a_1) (\lambda_2 + \lambda_1 a_1) w_{HK} + (\lambda_2 + \lambda_1 a_1)^2 w_{KK} \right)$$

$$\left( (1 + a_1)^2 w_{HH} + 2 (1 + a_1) (\lambda_2 + \lambda_1 a_1) w_{HK} + (\lambda_2 + \lambda_1 a_1)^2 w_{KK} \right).$$

Here, the $w$-terms are defined as

$$w_H := \frac{\partial w}{\partial H}, \quad w_K := \frac{\partial w}{\partial K},$$

$$w_{HH} := \frac{\partial^2 w}{\partial H^2}, \quad w_{HK} := \frac{\partial^2 w}{\partial H \partial K}, \quad w_{KK} := \frac{\partial^2 w}{\partial K^2}.$$

Proof.

$$L(w(H, K))$$

$$= L(H) w_H + L(K) w_K$$

$$= \left( F_{ij} (H_i H_j w_{HH} + (H_i K_j + K_j H_i) w_{HK} + K_i K_j w_{KK}) \right) \text{(see 1.8)}$$

$$= \left( F_{ij} (H_i^2 w_{HH} + 2 H_i K_j w_{HK} + K_i K_j w_{KK}) \right) \quad \text{(NC)}$$
\[
(C_H(\lambda_1, \lambda_2) + G_H(\lambda_1, \lambda_2) h^2_{11;1} + G_H(\lambda_1, \lambda_2) h^2_{22;2}) w_H \\
+ (C_K(\lambda_1, \lambda_2) + G_K(\lambda_1, \lambda_2) h^2_{11;1} + G_K(\lambda_1, \lambda_2) h^2_{22;2}) w_K \\
- \frac{\partial F}{\partial \lambda_1} \left( (1 + a_1)^2 w_{HH} + 2 (1 + a_1) (\lambda_2 + \lambda_1 a_1) w_{HK} + (\lambda_2 + \lambda_1 a_1)^2 w_{KK} \right) h^2_{11;1} \\
- \frac{\partial F}{\partial \lambda_2} \left( (1 + a_2)^2 w_{HH} + 2 (1 + a_2) (\lambda_1 + \lambda_2 a_2) w_{HK} + (\lambda_1 + \lambda_2 a_2)^2 w_{KK} \right) h^2_{22;2}
\]

(see (4.11), (4.12), (4.13), (4.14), (4.15), (4.16)).

We get the constant terms
\[
C_w(\lambda_1, \lambda_2) = C_H(\lambda_1, \lambda_2) w_H + C_K(\lambda_1, \lambda_2) w_K \\
= \left( F(\lambda_1^2 + \lambda_2^2) + \left( \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} \right)(\lambda_1 - \lambda_2) \lambda_1 \lambda_2 \right) w_H \\
+ \left( F(\lambda_1 + \lambda_2) + \left( \frac{\partial F}{\partial \lambda_1} \lambda_1 - \frac{\partial F}{\partial \lambda_2} \lambda_2 \right)(\lambda_1 - \lambda_2) \right) \lambda_1 \lambda_2 w_K
\]

(see (4.10), (4.11)),

and at a critical point of \( w(H, K) \) (CP), we get the gradient terms
\[
G_w(\lambda_1, \lambda_2) = G_H(\lambda_1, \lambda_2) w_H + G_K(\lambda_1, \lambda_2) w_K \\
- \frac{\partial F}{\partial \lambda_1} \left( (1 + a_1)^2 w_{HH} + 2 (1 + a_1) (\lambda_2 + \lambda_1 a_1) w_{HK} + (\lambda_2 + \lambda_1 a_1)^2 w_{KK} \right) \\
= \left( \frac{\partial^2 F}{\partial \lambda_1^2} + 2 \frac{\partial^2 F}{\partial \lambda_1 \partial \lambda_2} a_1 + \frac{\partial^2 F}{\partial \lambda_2^2} a_1^2 \right) + 2 \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} \lambda_1 - \lambda_2 \right) \right) \lambda_1 \lambda_2 w_K
\]

\[
- \frac{\partial F}{\partial \lambda_2} \left( (1 + a_2)^2 w_{HH} + 2 (1 + a_2) (\lambda_1 + \lambda_2 a_2) w_{HK} + (\lambda_1 + \lambda_2 a_2)^2 w_{KK} \right) \\
(see (4.17), (4.20)).
\]

4.3. Constant terms and gradient terms at a critical point, where we also choose normal coordinates. The normal velocity is equal to powers of the Gauss curvature. In the third and last part of Section 4, we calculate the constant terms \( C_w(\lambda_1, \lambda_2) \) and the gradient terms \( G_w(\lambda_1, \lambda_2) \) from Definition 4.14 of the maximum-principle functions. As before, we calculate these terms at a critical point of the general function \( w(H, K) \), where we also choose normal coordinates. Furthermore, we set the normal velocity to powers of the Gauss curvature, \( F = K^\sigma \).

So far, the constant terms and the gradient terms are rational functions. Now we divide each of them by some nonnegative factor in order to turn them into polynomials in two variables. For the constant terms we get a symmetric polynomial and for the gradient terms an asymmetric polynomial. Finally, we dehomogenize both polynomials by setting \( \lambda_1 = \rho \) and \( \lambda_2 = 1 \) and vice versa. We obtain for the
Lemma 4.15. Calculating the evolution equation of the quotient of two functions, \( w = \frac{p}{q} \), we obtain the following identity
\[
q^2 L \left( \frac{p}{q} \right) = q \, L(p) - p \, L(q)
\]
at a critical point of \( w \), i.e. \( w_{,i} = 0 \) for \( i = 1, 2 \).

Proof.
\[
L \left( \frac{p}{q} \right) = \frac{d}{dt} \left( \frac{p}{q} \right) - F_{ij} \left( \frac{p}{q} \right)_{;ij}
\]
\[
= \frac{\dot{p} q - p \dot{q}}{q^2} - F_{ij} \frac{1}{q^4} \left( (p_i q - p q_i)_j \right) q^2 - \left( (p_i q - p q_i)_j \right) q^2
\]
\[
= \frac{1}{q^2} \left( \dot{p} q - p \dot{q} - F_{ij} \left( p_{;ij} q + p q_{;ij} - p q_{;ij} \right) \right)
\]
\[
= \frac{1}{q^2} \left( q \, L(p) - p \, L(q) \right).
\]

Lemma 4.16. We calculate the following constant terms at a critical point, where we also choose normal coordinates. The normal velocity is equal to powers of the Gauss curvature, \( F = K^\sigma \).

First we calculate the constant terms for the mean curvature \( H \)
\[
C_H (\lambda_1, \lambda_2) = K^\sigma \left( \lambda_1^2 + \lambda_2^2 - \sigma (\lambda_1 - \lambda_2)^2 \right),
\]
and the constant terms for the Gauss curvature \( K \)
\[
C_K (\lambda_1, \lambda_2) = K^{\sigma + 1} (\lambda_1 + \lambda_2) .
\]

Then we calculate the constant terms for a rational function \( w = \frac{p}{q} \)
\[
q^2 C_r (\lambda_1, \lambda_2) = C_H (\lambda_1, \lambda_2) r_H + C_K (\lambda_1, \lambda_2) r_K
\]
\[
= K^\sigma \left( \left( \lambda_1^2 + \lambda_2^2 \right) - \sigma (\lambda_1 - \lambda_2)^2 \right) r_H + K (\lambda_1 + \lambda_2) r_K.
\]

Now we divide the previous constant terms by a nonnegative factor and get a polynomial in two variables
\[
C_r (\lambda_1, \lambda_2) := \frac{q^2}{K^\sigma} C_r (\lambda_1, \lambda_2).
\]

We dehomogenize the previous polynomial setting \( \lambda_1 = \rho, \lambda_2 = 1 \) and get a polynomial in one variable
\[
C (\rho) := C_r (\rho, 1) = \left( (1 - \sigma) \rho^2 + 2 \sigma \rho + (1 - \sigma) \right) r_H + \rho (\rho + 1) r_K.
\]

Here, the \( r \)-terms are defined as
\[
r_H := q \frac{\partial p}{\partial H} - p \frac{\partial q}{\partial H}.
\]
$$r_K := q \frac{\partial p}{\partial K} - p \frac{\partial q}{\partial K}.$$  

Proof. We calculate the constant terms $C_H(\lambda_1, \lambda_2)$ for $F = K^\sigma$.

\[
C_H(\lambda_1, \lambda_2) = F(\lambda_1^2 + \lambda_2^2) + \left( \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} \right) (\lambda_1 - \lambda_2) \lambda_1 \lambda_2 \quad \text{(see (4.19))}
\]

\[
= K^\sigma \left( \lambda_1^2 + \lambda_2^2 + (\sigma K^\sigma - 1) \lambda_1 - \sigma K^\sigma - 1 \lambda_2 \right) (\lambda_1 - \lambda_2) \lambda_1 \lambda_2
\]

We calculate the constant terms $C_K(\lambda_1, \lambda_2)$ for $F = K^\sigma$.

\[
C_K(\lambda_1, \lambda_2) = \left( F(\lambda_1 + \lambda_2) + \left( \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} \right) (\lambda_1 - \lambda_2) \right) \lambda_1 \lambda_2 \quad \text{(see (4.19))}
\]

\[
= (K^\sigma (\lambda_1 + \lambda_2) + (\sigma K^\sigma - 1) \lambda_1 \lambda_2 - \sigma K^\sigma - 1 \lambda_1 \lambda_2) (\lambda_1 - \lambda_2) \lambda_1 \lambda_2
\]

\[
= K^\sigma + 1 (\lambda_1 + \lambda_2) \lambda_1 \lambda_2.
\]

Using the identity $q^2 L \left( \frac{\partial}{\partial q} \right) = q L(p) - p L(q)$ (4.24) we calculate the constant terms $C_{\frac{q}{q}}(\lambda_1, \lambda_2)$.

\[
q^2 C_{\frac{q}{q}}(\lambda_1, \lambda_2) = C_H(\lambda_1, \lambda_2) r_H + C_K(\lambda_1, \lambda_2) r_K \quad \text{(see (4.24), (4.22))}
\]

\[
= K^\sigma \left( \left( \lambda_1^2 + \lambda_2^2 - \sigma (\lambda_1 - \lambda_2)^2 \right) r_H + K (\lambda_1 + \lambda_2) r_K \right).
\]

Dividing by the nonnegative factor $\frac{K^\sigma}{\sigma}$ we get the polynomial in two variables $C_r(\lambda_1, \lambda_2)$.

\[
C_r(\lambda_1, \lambda_2) = \frac{q^2}{K^\sigma} C_{\frac{q}{q}}(\lambda_1, \lambda_2)
\]

\[
= \left( \lambda_1^2 + \lambda_2^2 - \sigma (\lambda_1 - \lambda_2)^2 \right) r_H + K (\lambda_1 + \lambda_2) r_K.
\]

Now we dehomogenize the previous polynomial setting $\lambda_1 = \rho$, $\lambda_2 = 1$. We get the polynomial in one variable $C(\rho)$.

\[
C(\rho) := C_r(\rho, 1)
\]

\[
= \left( \rho^2 + 1 - \sigma (\rho - 1)^2 \right) r_H + \rho (\rho + 1) r_K
\]

\[
= \left( (1 - \sigma) \rho^2 + 2 \sigma \rho + (1 - \sigma) \right) r_H + \rho (\rho + 1) r_K.
\]

\[\square\]

Lemma 4.17. We calculate the following gradient terms at a critical point, where we also choose normal coordinates. The normal velocity is equal to powers of the Gauss curvature, $F = K^\sigma$. 


First we calculate the gradient terms for the mean curvature $H$

$$G_H (\lambda_1, \lambda_2) = \frac{\sigma K^{\sigma - 2}}{(w_H + \lambda_1 w_K)^2} \left( \left( -(\lambda_1 + \lambda_2)^2 + \sigma (\lambda_1 - \lambda_2)^2 \right) w_H^2 - 2(\lambda_1 + 3\lambda_2)K w_H w_K - 2(\lambda_1 + \lambda_2)\lambda_2 K w_K^2 \right),$$

(4.30)

the gradient terms for the Gauss curvature $K$

$$G_K (\lambda_1, \lambda_2) = \frac{\sigma K^{\sigma - 2}}{(w_H + \lambda_1 w_K)^2} (\sigma - 1)(\lambda_1 - \lambda_2) w_H^2,$$

(4.31)

and the mixed terms (compare (4.23))

$$-\frac{\partial F}{\partial \lambda_1} \left( (1 + a_1)^2 w_{HH} + 2(1 + a_1)(\lambda_2 + \lambda_1 a_1) w_{HK} + (\lambda_2 + \lambda_1 a_1)^2 w_{KK} \right)$$

$$= \frac{\sigma K^{\sigma - 2}}{(w_H + \lambda_1 w_K)^2} \lambda_1 (\lambda_1 - \lambda_2)^2 \lambda_2^2 \left( -w_H^2 w_{HH} + 2 w_H w_K w_{HK} - w_K^2 w_{KK} \right).$$

(4.32)

Then we calculate the gradient terms for a rational function $w = \frac{p}{q}$

$$q^2 G_r (\lambda_1, \lambda_2) = G_H (\lambda_1, \lambda_2) r_H + G_K (\lambda_1, \lambda_2) r_K$$

$$-\frac{\partial F}{\partial \lambda_1} \left( (1 + a_1)^2 w_{HH} + 2(1 + a_1)(\lambda_2 + \lambda_1 a_1) w_{HK} + (\lambda_2 + \lambda_1 a_1)^2 w_{KK} \right)$$

$$= \frac{\sigma K^{\sigma - 2}}{(r_H + \lambda_1 r_K)^2} \left( ((\sigma - 1) \lambda_1^2 - 2(\sigma + 1)\lambda_1 \lambda_2 + (\sigma - 1) \lambda_2^2) r_H^2 \right.$$

$$\left. ((\sigma - 3) \lambda_1^2 - 2(\sigma + 2)\lambda_1 \lambda_2 + (\sigma - 1) \lambda_2^2) \lambda_2 r_H r_K^2 \right.$$ 

$$- 2\lambda_1 (\lambda_1 + \lambda_2) \lambda_2^2 r_H r_K^2$$

$$- \lambda_1 (\lambda_1 - \lambda_2)^2 \lambda_2^2 r_K^2 r_{HH}$$

$$+ 2\lambda_1 (\lambda_1 - \lambda_2)^2 \lambda_2^2 r_H r_K r_{HK}$$

$$- \lambda_1 (\lambda_1 - \lambda_2)^2 \lambda_2^2 r_H^2 r_{KK} \right).$$

(4.33)

Now we divide the previous gradient terms by a nonnegative factor and get a polynomial in two variables

$$G_r (\lambda_1, \lambda_2) := \frac{q^2 (w_H + \lambda_1 w_K)^2}{\sigma K^{\sigma - 2}} G_r (\lambda_1, \lambda_2).$$

(4.34)
We dehomogenize the previous polynomial setting \( \lambda_1 = \rho, \lambda_2 = 1 \) and \( \lambda_1 = 1, \lambda_2 = \rho \), respectively. We get two polynomials in one variable

\[
G_1 (\rho) := G_r (\rho, 1) = ((\sigma - 1) \rho^2 - 2(\sigma + 1) \rho + (\sigma - 1)) r_H^3
+ ((\sigma - 3) \rho^2 - 2(\sigma + 2) \rho + (\sigma - 1)) r_H^2 r_K
- 2\rho(\rho + 1) r_H r_K^2
- \rho(\rho - 1)^2 r_K^2 r_{HH}
+ 2\rho(\rho - 1)^2 r_H r_K r_{HK}
- \rho(\rho - 1)^2 r_H^2 r_{KK},
\]

\[ (4.35) \]

\[
G_2 (\rho) := G_r (1, \rho) = ((\sigma - 1) \rho^2 - 2(\sigma + 1) \rho + (\sigma - 1)) r_H^3
+ \rho((\sigma - 1) \rho^2 - 2(\sigma + 2) \rho + (\sigma - 3)) r_H^2 r_K
- 2\rho^2(\rho + 1) r_H r_K^2
- \rho^2(\rho - 1)^2 r_K^2 r_{HH}
+ 2\rho^2(\rho - 1)^2 r_H r_K r_{HK}
- \rho^2(\rho - 1)^2 r_H^2 r_{KK}.
\]

\[ (4.36) \]

Here, the \( r \)-terms are defined as

\[
r_H := q \frac{\partial p}{\partial H} - p \frac{\partial q}{\partial H}, r_K := q \frac{\partial p}{\partial K} - p \frac{\partial q}{\partial K},
\]

\[
r_{HH} := q \frac{\partial^2 p}{\partial H^2} - p \frac{\partial^2 q}{\partial H^2}, r_{HK} := q \frac{\partial^2 p}{\partial H \partial K} - p \frac{\partial^2 q}{\partial H \partial K}, r_{KK} := q \frac{\partial^2 p}{\partial K^2} - p \frac{\partial^2 q}{\partial K^2}.
\]

**Proof.** We calculate the gradient terms \( G_H (\lambda_1, \lambda_2) \) for \( F = K^\sigma \).

\[
G_H (\lambda_1, \lambda_2)
= \frac{\partial^2 F}{\partial \lambda_1^2} + 2 \frac{\partial^2 F}{\partial \lambda_1 \partial \lambda_2} a_1 + \frac{\partial^2 F}{\partial \lambda_2^2} a_1^2 + \frac{\partial^2 F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_1} - \frac{\partial F}{\partial \lambda_2} a_1^2
= \sigma(\sigma - 1) K^{\sigma - 2} \lambda_2^2 + 2\sigma^2 K^{\sigma - 1} a_1 + \sigma(\sigma - 1) K^{\sigma - 2} \lambda_1^2 a_1^2
+ 2 \frac{\sigma K^{\sigma - 1} \lambda_2 - \sigma K^{\sigma - 1} \lambda_1}{\lambda_1 - \lambda_2} a_1^2
= \sigma K^{\sigma - 2} ((\sigma - 1) \lambda_2^2 + 2\sigma K a_1 + (\sigma - 1) \lambda_1^2 a_1^2 - 2K a_1^2)
= \sigma K^{\sigma - 2} ((\sigma - 1) \lambda_2^2 + 2\sigma K \left( \frac{w_H + \lambda_2 w_K}{w_H + \lambda_1 w_K} \right)
+ ((\sigma - 1) \lambda_1^2 - 2K) \left( \frac{w_H + \lambda_2 w_K}{w_H + \lambda_1 w_K} \right)^2
\]

(see \[4.17\])
We calculate the gradient terms $G_K(\lambda_1, \lambda_2)$ for $F = K^\sigma$.

\[
G_K(\lambda_1, \lambda_2) = 2 \left( -\frac{\partial F}{\partial \lambda_1} a_1 + \frac{\partial F}{\partial \lambda_2} a_1^2 \right) + \left( \frac{\partial^2 F}{\partial \lambda_1^2} + \frac{\partial^2 F}{\partial \lambda_1 \partial \lambda_2} a_1 + \frac{\partial^2 F}{\partial \lambda_2^2} a_1^2 \right) \lambda_2 + 2 \frac{\partial F}{\lambda_1 - \lambda_2} \lambda_2 \lambda_1 a_1^2
\]

(see (4.20))

\[
= 2 \left( -\sigma \lambda_2^{-1} a_1 + \sigma K^{-1} \lambda_1 a_1^2 \right)
+ \left( \sigma (\sigma - 1) K^{-2} \lambda_2^2 + 2 \sigma a_1^2 + \sigma (\sigma - 1) K^{-2} \lambda_2 a_1^2 \right) \lambda_2
+ 2 \frac{\sigma K^{-1} \lambda_2 - \sigma K^{-1} \lambda_1}{\lambda_1 - \lambda_2} \lambda_2 \lambda_1 a_1^2
\]

\[
= \sigma K^{-2} \left( -2 \lambda_2^2 a_1 + 2 \lambda_1 a_1^2 + ((\sigma - 1) \lambda_2^2 + 2 \sigma K a_1 + (\sigma - 1) \lambda_2 a_1^2) \lambda_2
- 2 \lambda_1 a_1^2 \right)
\]

\[
= \sigma K^{-2} \left( (\sigma - 1) \lambda_2^2 + (-2 \lambda_2 + 2 \sigma K \lambda_2) a_1 + (\sigma - 1) \lambda_2^2 \lambda_2 a_1^2 \right)
\]

\[
= \sigma (\sigma - 1) K^{-2} \left( \lambda_2^2 + 2 K \lambda_2 a_1 + \lambda_2^2 a_1^2 \right)
= \sigma (\sigma - 1) K^{-2} \lambda_2 \left( \lambda_2^2 + 2 K a_1 + \lambda_2 a_1^2 \right)
= \sigma (\sigma - 1) K^{-2} \lambda_2 \left( \lambda_2 + \lambda_1 a_1 \right)^2
= \sigma (\sigma - 1) K^{-2} \lambda_2 \left( \frac{w_H + \lambda w_K}{w_H + \lambda w_K} \right)^2
\]

(see (4.9))
\[
\frac{\sigma K^{\alpha-2}}{(w_H + \lambda_1 w_K)^2} (\sigma - 1) \lambda_2 \left( \lambda_2 (w_H + \lambda_1 w_K) - \lambda_1 (w_H + \lambda_2 w_K) \right)^2
\]
\[
= \frac{\sigma K^{\alpha-2}}{(w_H + \lambda_1 w_K)^2} (\sigma - 1) \lambda_2 \left( \lambda_2 w_H + K w_K - \lambda_1 w_H - K w_K \right)^2
\]
\[
= \frac{\sigma K^{\alpha-2}}{(w_H + \lambda_1 w_K)^2} (\sigma - 1) (\lambda_1 - \lambda_2)^2 \lambda_2 w_H^2.
\]

We calculate the mixed terms \( \text{(compare (4.23))} \)

\[- \frac{\partial F}{\partial \lambda_1} \left( (1 + a_1)^2 w_{HH} + 2 (1 + a_1) (\lambda_2 + \lambda_1 a_1) w_{HK} + (\lambda_2 + \lambda_1 a_1)^2 w_{KK} \right) \]

for \( F = K^\alpha \).

\[- \frac{\partial F}{\partial \lambda_1} \left( (1 + a_1)^2 w_{HH} + 2 (1 + a_1) (\lambda_2 + \lambda_1 a_1) w_{HK} + (\lambda_2 + \lambda_1 a_1)^2 w_{KK} \right) \]

\[- \frac{\sigma K^{\alpha-1} \lambda_2}{(w_H + \lambda_1 w_K)^2} \left( (w_H + \lambda_1 w_K - w_H - \lambda_2 w_K)^2 w_{HH} + 2 (w_H + \lambda_1 w_K - w_H - \lambda_2 w_K) \right) \cdot \left( \lambda_2 w_H + \lambda_1 \lambda_2 w_K - \lambda_1 w_H - \lambda_1 \lambda_2 w_K \right) w_{HK} + (\lambda_2 w_H + \lambda_1 \lambda_2 w_K - \lambda_1 w_H - \lambda_1 \lambda_2 w_K)^2 w_{HK}^2 (w_{HH}) \]

\[- \frac{\sigma K^{\alpha-1} \lambda_2}{(w_H + \lambda_1 w_K)^2} \left( (\lambda_1 - \lambda_2)^2 w_{HK}^2 w_{HH} - 2 (\lambda_1 - \lambda_2)^2 w_{HK} w_{HK} \right.
\]
\[+ (\lambda_1 - \lambda_2)^2 w_{HK}^2 \]
\[= \frac{\sigma K^{\alpha-2}}{(w_H + \lambda_1 w_K)^2} \lambda_1 (\lambda_1 - \lambda_2)^2 \lambda_2 \left( - w_{HK}^2 w_{HH} + 2 w_{HK} w_{KK} - w_{HK}^2 \right). \]

Using \( q^2 L \left( \frac{\hat{E}}{\hat{F}} \right) = q \ L(p) - p \ L(q) \) \((4.24)\) we calculate the gradient terms \( G_{\hat{F}}(\lambda_1, \lambda_2). \)

\[q^2 G_{\hat{F}}(\lambda_1, \lambda_2) = G_H(\lambda_1, \lambda_2) r_H + G_K(\lambda_1, \lambda_2) r_K \]

\[- \frac{\partial F}{\partial \lambda_1} \left( (1 + a_1)^2 r_{HH} + 2 (1 + a_1) (\lambda_2 + \lambda_1 a_1) r_{HK} + (\lambda_2 + \lambda_1 a_1)^2 r_{KK} \right) \]

\[(\text{see (4.24), (4.21))} \]

\[= \frac{\sigma K^{\alpha-2}}{(w_H + \lambda_1 w_K)^2} \lambda_1 (\lambda_1 - \lambda_2)^2 \lambda_2 \left( - w_{HK}^2 w_{HH} + 2 w_{HK} w_{KK} - w_{HK}^2 \right). \]
$$
\frac{\sigma K^{\sigma-2}}{(r_H + \lambda_1 r_K)^2} \left( \left( - (\lambda_1 + \lambda_2)^2 + \sigma (\lambda_1 - \lambda_2)^2 \right) r_H^3 \right.
- 2 (\lambda_1 + 3 \lambda_2) K r_H^2 r_K - 2 \lambda_1 (\lambda_1 + \lambda_2) \lambda_2^2 r_H^2 r_K^2 \\
+ (\sigma - 1) (\lambda_1 - \lambda_2)^2 \lambda_2 r_H^2 r_K \\
- \lambda_1 (\lambda_1 - \lambda_2)^2 \lambda_2^2 r_H^2 r_K \\
+ 2 \lambda_1 (\lambda_1 - \lambda_2)^2 \lambda_2^2 r_H r_K r_K \\
- \lambda_1 (\lambda_1 - \lambda_2)^2 \lambda_2^2 r_H^2 r_K \right) \quad \text{(see (4.30), (4.31), (4.32))}
$$

Dividing by the nonnegative factor $\frac{\sigma K^{\sigma-2}}{q^2(r_H + \lambda_1 r_K)^2}$ we get the polynomial in two variables $G_r(\lambda_1, \lambda_2)$.

$$
G_r(\lambda_1, \lambda_2) := \frac{q^2 (r_H + \lambda_1 r_K)^2}{\sigma K^{\sigma-2}} G_4(\lambda_1, \lambda_2)
= \left( (\sigma - 1) \lambda_1^2 - 2 (\sigma + 1) \lambda_1 \lambda_2 + (\sigma - 1) \lambda_2^2 \right) r_H^3 \\
+ (\sigma - 3) \lambda_1^2 - 2 (\sigma + 2) \lambda_1 \lambda_2 + (\sigma - 1) \lambda_2^2 \lambda_2 r_H^2 r_K \\
- 2 \lambda_1 (\lambda_1 + \lambda_2) \lambda_2^2 r_H^2 r_K^2 \\
- \lambda_1 (\lambda_1 - \lambda_2)^2 \lambda_2^2 r_H^2 r_K \\
+ 2 \lambda_1 (\lambda_1 - \lambda_2)^2 \lambda_2^2 r_H r_K r_K \\
- \lambda_1 (\lambda_1 - \lambda_2)^2 \lambda_2^2 r_H^2 r_K \\
$$

Now we dehomogenize the previous polynomial setting $\lambda_1 = \rho$, $\lambda_2 = 1$ and $\lambda_1 = 1$, $\lambda_2 = \rho$, respectively. We get the two polynomials in one variable $G_1(\rho)$.
and \( G_2 (\rho) \).

\[
G_1 (\rho) := G_r (\rho, 1) \\
= ((\sigma - 1) \rho^2 - 2 (\sigma + 1) \rho + (\sigma - 1)) r_H^3 \\
+ (\sigma - 3) \rho^2 - 2 (\sigma + 2) \rho + (\sigma - 1)) r_H^2 r_K \\
- 2 \rho (\rho + 1) r_H r_K^2 \\
- \rho (\rho - 1)^2 r_K r_H H \\
+ 2 \rho (\rho - 1)^2 r_H r_K r_H K \\
- \rho (\rho - 1)^2 r_K^2 r_K K,
\]

\[
G_2 (\rho) := G_r (1, \rho) \\
= ((\sigma - 1) \rho^2 - 2 (\sigma + 1) \rho + (\sigma - 1)) r_H^3 \\
+ (\sigma - 3) \rho^2 - 2 (\sigma + 2) \rho + (\sigma - 3)) r_H^2 r_K \\
- 2 \rho^2 (\rho + 1) r_H r_K^2 \\
- \rho^2 (\rho - 1)^2 r_K^2 r_H H \\
+ 2 \rho^2 (\rho - 1)^2 r_H r_K r_H K \\
- \rho^2 (\rho - 1)^2 r_K^2 r_K K.
\]

5. Dehomogenized polynomials, leading terms and nine cases

5.1. Dehomogenized polynomials, leading terms. In the first part of Section 5 we define homogeneous symmetric polynomials in the algebraic basis \( \{ H, K \} \). We also state their first and second derivatives with respect to \( H \) and \( K \). Furthermore, we calculate their dehomogenized versions setting \( \lambda_1 = \rho, \lambda_2 = 1 \). So we obtain several polynomials in one variable.

Then we define an operator \( L \) that determines the leading terms of a given polynomial in one variable. Now we present the leading terms of the above polynomials in one variable. Here, we have to distinguish three distinct cases.

Due to the form of the \( r \)-terms this means that we have nine different cases to explore. In the second part of Section 5 we determine the leading terms of the \( r \)-terms. In each case we continue with the calculation of the leading terms of the polynomial constant terms \( C (\rho) \) and the calculation of the leading terms of the polynomial gradient terms \( G_1 (\rho) \) and \( G_2 (\rho) \). All nine cases result in a contradiction. This concludes the proof of our main Theorem 1.2.

Lemma 5.1. We define two homogeneous symmetric polynomials

\[
p(H, K) := \sum_{i=0}^{\lfloor g/2 \rfloor} c_{i+1} H^{g-2i} K^i,
\]

\[
q(H, K) := \sum_{j=0}^{\lfloor h/2 \rfloor} d_{j+1} H^{h-2j} K^j,
\]

□
Lemma 5.2. We calculate the dehomogenized version of the polynomial $p(H, K)$, setting $\lambda_1 = 1$, $\lambda_2 = \rho$.

We calculate the derivatives of the polynomial $p(H, K)$
\[
\frac{\partial p}{\partial H}(H, K) := \sum_{i=0}^{[g/2]} c_{i+1}(g - 2i)H^{g - 2i - 1}K^i,
\]
\[
\frac{\partial p}{\partial K}(H, K) := \sum_{i=0}^{[g/2]} c_{i+1}iH^{g - 2i - 1},
\]
\[
\frac{\partial^2 p}{\partial H^2}(H, K) := \sum_{i=0}^{[g/2]} c_{i+1}(g - 2i)(g - 2i - 1)H^{g - 2i - 2}K^i,
\]
\[
\frac{\partial^2 p}{\partial H \partial K}(H, K) := \sum_{i=0}^{[g/2]} c_{i+1}(g - 2i)iH^{g - 2i - 1}K^{i-1},
\]
\[
\frac{\partial^2 p}{\partial K^2}(H, K) := \sum_{i=0}^{[g/2]} c_{i+1}i(i - 1)(g - 2i)H^{g - 2i}K^{i-2}.
\]

Lemma 5.2. We calculate the dehomogenized version of the polynomial $p(H, K)$ and its derivatives from Lemma 5.1 setting $\lambda_1 = 1$, $\lambda_2 = \rho$.

Remark 5.3. To determine the leading term of a polynomial $p \in \mathbb{R}[\rho]$ we write
\[
L(p) = c_g \rho^g,
\]
if $p = c_g \rho^g + \sum_{i=0}^g c_i \rho^i$ for some $c_i \in \mathbb{R}$. Note that $c_g = 0$ is possible.

Furthermore, we set
\[
\mathcal{P}_\rho(g) := \{ q \text{ polynomial in } \rho : \text{degree of } q \leq g \}.
\]

Lemma 5.4. We apply the operator $L$ from Definition 5.3 to the polynomials in Lemma 5.2 in all three distinct cases.
Case A. $c_1 > 0$

\[
L(p) = c_1 \rho^g, \\
L(p_H) = c_1 \rho^{g-1}, \\
L(p_K) = c_2 \rho^{g-2}, \\
L(p_{HH}) = c_1 (g - 1) \rho^{g-2}, \\
L(p_{HK}) = c_2 (g - 2) \rho^{g-3}, \\
L(p_{KK}) = c_3 2 \rho^{g-4}.
\]

Terms with negative powers of $\rho$ do not occur for $g \geq 4$. If $g \leq 3$ the terms with negative powers of $\rho$ are 0.

Case B. $c_1 = 0, c_2 > 0$

\[
L(p) = c_2 \rho^{g-1}, \\
L(p_H) = c_2 (g - 2) \rho^{g-2}, \\
L(p_K) = c_2 \rho^{g-2}, \\
L(p_{HH}) = c_2 (g - 2) (g - 3) \rho^{g-3}, \\
L(p_{HK}) = c_2 (g - 2) \rho^{g-3}, \\
L(p_{KK}) = c_3 2 \rho^{g-4}.
\]

Terms with negative powers of $\rho$ do not occur for $g \geq 4$. If $g \leq 3$ the terms with negative powers of $\rho$ are 0.

Case C. $c_1 = 0, \ldots, c_{k-1} = 0, c_k > 0$ for all $k \geq 3$

\[
L(p) = c_k \rho^{g-(k-1)}, \\
L(p_H) = c_k (g - 2 (k - 1)) \rho^{g-k}, \\
L(p_K) = c_k (k - 1) \rho^{g-k}, \\
L(p_{HH}) = c_k (g - 2 (k - 1)) (g - 2k + 1) \rho^{g-(k+1)}, \\
L(p_{HK}) = c_k (g - 2 (k - 1)) (k - 1) \rho^{g-(k+1)}, \\
L(p_{KK}) = c_k (k - 2) (k - 1) \rho^{g-(k+1)}.
\]

Terms with negative powers of $\rho$ do not occur. Since $c_1 = 0, \ldots, c_{k-1} = 0, c_k > 0$ for all $k \geq 3$, we have $3 \leq k \leq \#\{c_i\} = \lfloor g/2 + 1 \rfloor$. Thus, $2 (k - 1) \leq g$. Therefore, we get

\[
g - (k + 1) \geq 2 (k - 1) - (k + 1) = k - 3 \geq 0.
\]

Furthermore, we have $g - k \geq 1$. We will use this implicitly in the second part of Section 3.

5.2. Nine cases.

Remark 5.5. We recall from Lemma 4.17 that the $r$-terms are defined as

\[
r_H := q \frac{\partial p}{\partial H} - p \frac{\partial q}{\partial H}, \quad r_K := q \frac{\partial p}{\partial K} - p \frac{\partial q}{\partial K},
\]

\[
r_{HH} := q \frac{\partial^2 p}{\partial H^2} - p \frac{\partial^2 q}{\partial H^2}, \quad r_{HK} := q \frac{\partial^2 p}{\partial H \partial K} - p \frac{\partial^2 q}{\partial H \partial K}, \quad r_{KK} := q \frac{\partial^2 p}{\partial K^2} - p \frac{\partial^2 q}{\partial K^2}.
\]
Therefore, we have to distinguish these nine cases in order to calculate the leading terms of the $ r $-terms:

- Case I: $ c_1 > 0, d_1 > 0 $,
- Case II: $ c_1 > 0, d_2 > 0 $,
- Case III: $ c_1 > 0, d_l > 0 $,
- Case IV: $ c_2 > 0, d_2 > 0 $,
- Case V: $ c_2 > 0, d_l > 0 $,
- Case VI: $ c_k > 0, d_l > 0 $,
- Case VII: $ c_2 > 0, d_1 > 0 $,
- Case VIII: $ c_k > 0, d_1 > 0 $,
- Case IX: $ c_k > 0, d_2 > 0 $,

for all $ k, l \geq 3 $.

**Remark 5.6.** We recall the constant terms $ C(\rho) $ from Lemma 4.16:

$$ C(\rho) = \left( (1 - \sigma) \rho^2 + 2\sigma \rho + (1 - \sigma) \right) r_H + \rho (\rho + 1) r_K, $$

the gradient terms $ G_1(\rho) $ from Lemma 4.17:

$$ G_1(\rho) = \left( (\sigma - 1) \rho^2 - 2(\sigma + 1) \rho + (\sigma - 1) \right) r_H^3 $$
$$ + (\sigma - 3) \rho^2 - 2(\sigma + 2) \rho + (\sigma - 1) \right) r_H^2 r_K $$
$$ - 2\rho (\rho + 1) r_H r_K^2 $$
$$ - \rho (\rho - 1)^2 r_K^2 r_{HH} $$
$$ + 2\rho (\rho - 1)^2 r_H r_K r_{HK} $$
$$ - \rho (\rho - 1)^2 r_K^2 r_{KK}, $$

and the gradient terms $ G_2(\rho) $ from Lemma 4.17:

$$ G_2(\rho) = \left( (\sigma - 1) \rho^2 - 2(\sigma + 1) \rho + (\sigma - 1) \right) r_H^3 $$
$$ + \rho (\sigma - 1) \rho^2 - 2(\sigma + 2) \rho + (\sigma - 3) \right) r_H^2 r_K $$
$$ - 2\rho^2 (\rho + 1) r_H r_K^2 $$
$$ - \rho^2 (\rho - 1)^2 r_K^2 r_{HH} $$
$$ + 2\rho^2 (\rho - 1)^2 r_H r_K r_{HK} $$
$$ - \rho^2 (\rho - 1)^2 r_K^2 r_{KK}. $$

5.3. **Case I.** $ c_1 > 0, d_1 > 0 $  
First we calculate the leading terms or the maximal order of the $ r $-terms using Lemma 5.4:

$$ L(r_H) = c_1 d_1 (g - h) \rho^{g+h-1}, $$
$$ L(r_K) \in P_{\rho}(g + h - 2), $$
$$ L(r_{HH}) \in P_{\rho}(g + h - 2), $$
$$ L(r_{HK}) \in P_{\rho}(g + h - 3), $$
$$ L(r_{KK}) \in P_{\rho}(g + h - 4). $$
Now we calculate the leading terms of $G_1(\rho)$ using \([5.6]\)

\[
L(G_1(\rho)) = c_1^3 d_3^3 (\sigma - 1)(g - h)^3 \rho^{3(g+h)-1}.
\]

For maximum-principle functions \([1.1]\) we have $g \geq 2, g-h > 0$ and $L(G_1(\rho)) \leq 0$ for all $\rho \geq 0$. Since $\sigma - 1 > 0$ Case I, results in a contradiction.

5.4. Case II. $c_1 > 0, d_2 > 0$
First we calculate the leading terms or the maximal order of the $r$-terms using Lemma \([5.4]\)

\[
\begin{align*}
L(r_H) &= c_1 d_2 (g - h + 2) \rho^{g+h-2}, \\
L(r_K) &= -c_1 d_2 \rho^{g+h-2}, \\
L(r_{HH}) &= c_1 d_2 (g - h + 2) (g + h - 3) \rho^{g+h-3}, \\
L(r_{HK}) &= -c_1 d_2 (h - 2) \rho^{g+h-3}, \\
L(r_{K}) &\in P_\rho(g + h - 4).
\end{align*}
\]

Now we calculate the leading terms of $G_1(\rho)$ using \([5.6]\)

\[
L(G_1(\rho)) = -c_1^3 d_3^3 (g - h + 1)(g - h + 2)((g - h)(1 - \sigma) + 1 - 2\sigma) \rho^{3(g+h)-4}.
\]

For maximum-principle functions \([1.1]\) we have $g \geq 2, g-h > 0$ and $L(G_1(\rho)) \leq 0$ for all $\rho \geq 0$. Since $\frac{2\sigma - 1}{\sigma - 2} > 2$ for all $\sigma > 1$, we get $g-h+\frac{2\sigma - 1}{\sigma - 1} > 0$ which is equivalent to $(g - h)(1 - \sigma) + 1 - 2\sigma < 0$. Therefore, Case II results in a contradiction.

5.5. Case III. $c_1 > 0, d_l > 0$ for all $l \geq 3$
First we calculate the leading terms of the $r$-terms using Lemma \([5.4]\)

\[
\begin{align*}
L(r_H) &= c_1 d_l (g - h + 2(l - 1)) \rho^{g+h-l}, \\
L(r_K) &= -c_1 d_l (l - 1) \rho^{g+h-l}, \\
L(r_{HH}) &= c_1 d_l (g - h + 2(l - 1)) (g + h - 2l + 1) \rho^{g+h-(l+1)}, \\
L(r_{HK}) &= -c_1 d_l (h - 2 (l - 1)) (l - 1) \rho^{g+h-(l+1)}, \\
L(r_{K}) &\in P_\rho(l - 2 (l - 1) \rho^{g+h-(l+1)}).
\end{align*}
\]

Now we calculate the leading terms of $G_1(\rho)$ using \([5.6]\)

\[
L(G_1(\rho)) = -c_1^3 d_3^3 (g - h + (l - 1))(g - h + 2(l - 1)) \cdot \\
\cdot ((g - h)(1 - \sigma) + (l - 1)(1 - 2\sigma)) \rho^{3(g+h-l)+2}.
\]

For maximum-principle functions \([1.1]\) we have $h_l \geq 1, g-h > 0$ and $L(G_1(\rho)) \leq 0$ for all $\rho \geq 0$. Since $(l - 1) \frac{2\sigma - 1}{\sigma - 1} > 4$ for all $l \geq 3, \sigma > 1$, we get $g - h + (l - 1) \frac{2\sigma - 1}{\sigma - 1} > 0$ which is equivalent to $(g - h)(1 - \sigma) + (l - 1)(1 - 2\sigma) < 0$. Therefore, Case III results in a contradiction.

5.6. Case IV. $c_2 > 0, d_2 > 0$
First we calculate the leading terms or the maximal order of the $r$-terms using
Lemma 5.4

\begin{align*}
L(r_H) &= c_2 d_2 (g - h) \rho^{g+h-3}, \\
L(r_K) &= c_2 d_2 (g + h - 4), \\
L(r_{HH}) &= c_2 d_2 (g + h - 4), \\
L(r_{HK}) &= c_2 d_2 (g + h - 4), \\
L(r_{KK}) &= c_2 d_2 (g + h - 5).
\end{align*}

Now we calculate the leading terms of \( G_1(\rho) \) using (5.6)

\[ L(G_1(\rho)) = c_2^3 d_2^3 (\sigma - 1) (g - h)^3 \rho^{3(g+h)-7}. \]

For maximum-principle functions we have \( g \geq 2, g-h > 0 \) and \( L(G_1(\rho)) \leq 0 \) for all \( \rho \geq 0 \). Due to \( d_1 = 0, d_2 > 0 \) we have \( h \geq 2 \). Since \( \sigma - 1 > 0 \), Case IV results in a contradiction.

5.7. Case V. \( c_2 > 0, d_l > 0 \) for all \( l \geq 3 \)

First we calculate the leading terms of the \( r \)-terms using Lemma 5.4

\begin{align*}
L(r_H) &= c_2 d_l (g - h + 2(l - 2)) \rho^{g+h-(l+1)}, \\
L(r_K) &= -c_2 d_l (l - 2) \rho^{g+h-(l+1)}, \\
L(r_{HH}) &= c_2 d_l (g - h + 2(l - 2)) (g + h - (2l + 1)) \rho^{g+h-(l+2)}, \\
L(r_{HK}) &= c_2 d_l ((g - 2) - (h - 2(l - 1)) (l - 1)) \rho^{g+h-(l+2)}, \\
L(r_{KK}) &= -c_2 d_l (l - 2) (l - 1) \rho^{g+h-(l+2)}.
\end{align*}

Now we calculate the leading terms of \( G_1(\rho) \) using (5.6)

\[ L(G_1(\rho)) = -c_2^3 d_l^3 (g - h + (l - 2)) (g - h + 2(l - 2)) \cdot ((g - h)(1 - \sigma) + (l - 2)(1 - 2\sigma)) \rho^{3(g+h-l)-1}. \]

For maximum-principle functions we have \( h-l \geq 1, g-h > 0 \) and \( L(G_1(\rho)) \leq 0 \) for all \( \rho \geq 0 \). Since \((l-2) \frac{2\sigma - 1}{\sigma - 1} > 2 \) for all \( l \geq 3, \sigma > 1 \), we get \( g-h + (l-2) \frac{2\sigma - 1}{\sigma - 1} > 0 \) which is equivalent to \((g-h)(1-\sigma)+(l-2)(1-2\sigma) < 0 \). Therefore, Case V results in a contradiction.

5.8. Case VI. \( c_k > 0, d_l > 0 \) for all \( k, l \geq 3 \)

First we calculate the leading terms of the \( r \)-terms using Lemma 5.4

\begin{align*}
L(r_H) &= c_k d_l ((g - h) - 2(k - l)) \rho^{g+h-(k+l-1)}, \\
L(r_K) &= c_k d_l (k - l) \rho^{g+h-(k+l-1)}, \\
L(r_{HH}) &= c_k d_l (g + h + 3 - 2(k + l)) (g + h + 2(k - l)) \rho^{g+h-(k+l)}, \\
L(r_{HK}) &= c_k d_l ((g - 2)(k - 1)) (k - 1) - (h + 2(l - 1)) (l - 1)) \rho^{g+h-(k+l)}, \\
L(r_{KK}) &= c_k d_l ((k - 2)(k - 1) - (l - 2)(l - 1)) \rho^{g+h-(k+l)}.\end{align*}
Now we calculate the leading terms of \( C(\rho) \), \( G_1(\rho) \), \( G_2(\rho) \) using (5.3), (5.6), (5.7):

\[
L(C(\rho)) = c_k d_1 ((g - h)(1 - \sigma) + (l - k)(1 - 2\sigma)) \rho^{\sigma+h-(k+l)+3},
\]

\[
L(G_1(\rho)) = -c_k d_1^3 (g - h + (l - k)(g - h + 2(l - k)),
\]

\[
\cdot ((g - h)(1 - \sigma) + (l - k)(1 - 2\sigma)) \rho^{\sigma+h-(k+l)+5},
\]

\[
L(G_2(\rho)) = -c_k d_1^3 (l - k)(g - h + 2(l - k)),
\]

\[
\cdot ((l - k) + (g - h + 2(l - k)) \rho^{\sigma+h-(k+l)+6}.
\]

For maximum-principle functions (1.11) we have \( g - k \geq 1 \), \( h - l \geq 1 \), \( g - h > 0 \) and \( L(C(\rho)) \leq 0, L(G_1(\rho)) \leq 0, L(G_2(\rho)) \leq 0 \) for all \( \rho \geq 0 \).

We assume \((g - h)(1 - \sigma) + (l - k)(1 - 2\sigma) = 0\) which is equivalent to the identity 
\[ g - h = (k - l) \frac{2\sigma - 1}{\sigma - 1}. \]

Since \( \frac{2\sigma - 1}{\sigma - 1} > 2 \), we get \( k - l > 0 \). Using this identity we get
\[
L(G_2(\rho)) = c_k d_1^3 (k - l)^3 \left( \frac{1}{\sigma - 1} \right)^2 \rho^{\sigma+h-(k+l)+6}
\]
which results in a contradiction. So we have \((g - h)(1 - \sigma) + (l - k)(1 - 2\sigma) < 0\) which is equivalent to \( g - h + (l - k) \frac{2\sigma - 1}{\sigma - 1} > 0 \). Furthermore, we assume \( k - l > 0 \) which implies
\[
g - h > (k - l) \frac{2\sigma - 1}{\sigma - 1} > 2(k - l) > k - l
\]
and
\[
g - h + (l - k) > g - h + 2(l - k) > 0.
\]

Thus, the condition \( L(G_1(\rho)) \leq 0 \) for all \( \rho \geq 0 \) results in a contradiction. For \( l - k \geq 0 \) the same condition also results in a contradiction. Therefore, Case VI results in a contradiction.

5.9. Case VII: \( c_2 > 0, d_1 > 0 \)

First we calculate the leading terms of the \( r \)-terms using Lemma (5.4):

\[
L(r_H) = c_2 d_1 (g - h - 2) \rho^{\sigma+h-2},
\]

\[
L(r_K) = c_2 d_1 \rho^{\sigma+h-2},
\]

\[
L(r_{H\bar{H}}) = c_2 d_1 (g - h - 2)(g + h - 3) \rho^{\sigma+h-3},
\]

\[
L(r_{H\bar{K}}) = c_2 d_1 (g - h - 2) \rho^{\sigma+h-3},
\]

\[
L(r_{K\bar{K}}) \in \mathcal{P}_\rho (g + h - 4).
\]

Now we calculate the leading terms of \( C(\rho), G_1(\rho), G_2(\rho) \) using (5.3), (5.6), (5.7):

\[
L(C(\rho)) = c_2 d_1 ((g - h)(1 - \sigma) - 1 + 2\sigma) \rho^{\sigma+h},
\]

\[
L(G_1(\rho)) = -c_2 d_1^3 (g - h - 2)(g - h - 1)((g - h)(1 - \sigma) - 1 + 2\sigma) \rho^{3(g+h)-4},
\]

\[
L(G_2(\rho)) = c_2 d_1^3 \left( - (g - h - 2) - (g - h - 2)^2 \sigma \right) \rho^{3(g+h)-3}.
\]

For maximum-principle functions (1.11) we have \( g \geq 2, g - h > 0 \) and \( L(C(\rho)) \leq 0, L(G_1(\rho)) \leq 0, L(G_2(\rho)) \leq 0 \) for all \( \rho \geq 0 \).

We assume \((g - h)(1 - \sigma) - 1 + 2\sigma = 0\) which is equivalent to the identity \( g - h = \frac{2\sigma - 1}{\sigma - 1} \). Using this identity we get
\[
L(G_2(\rho)) = c_2 d_1^3 \left( \frac{1}{\sigma - 1} \right)^2 \rho^{3(g+h)-3}.
\]
which results in a contradiction. So we have \((g - h) (1 - \sigma) - 1 + 2 \sigma < 0\) which is equivalent to \(g - h > \frac{2 \sigma - 1}{\sigma - 1}\). This implies

\[
g - h > \frac{2 \sigma - 1}{\sigma - 1} > 2 > 1 \text{ and } g - h - 1 > g - h - 2 > 0.
\]

Thus, the condition \(L(G_1 (\rho)) \leq 0\) for all \(\rho \geq 0\) results in a contradiction. Therefore, Case VII results in a contradiction.

5.10. **Case VIII.** \(c_k > 0, d_1 > 0\) for all \(k \geq 3\)

First we calculate the leading terms of the \(r\)-terms using Lemma 5.4.

\[
L(r_H) = c_k d_1 (g - h - 2 (k - 1)) \rho^{g + h - k},
\]

\[
L(r_K) = c_k d_1 (k - 1) \rho^{g + h - k},
\]

\[
L(r_{HH}) = c_k d_1 (g - h - 2 (k - 1)) (g + h - 2 k + 1) \rho^{g + h - (k+1)},
\]

\[
L(r_{HK}) = c_k d_1 (g - 2 (k - 1)) (k - 1) \rho^{g + h - (k+1)},
\]

\[
L(r_{KK}) = c_k d_1 (k - 2) (k - 1) \rho^{g + h - (k+1)}.
\]

Now we calculate the leading terms of \(C(\rho), G_1(\rho), G_2(\rho)\) using (5.5), (5.6), (5.7)

\[
L(C(\rho)) = c_k d_1 ((g - h) (1 - \sigma) + (k - 1) (-1 + 2 \sigma)) \rho^{g + h - k + 2},
\]

\[
L(G_1(\rho)) = -c_k^3 d_1^3 (g - h - 2 (k - 1)) (g - h - (k - 1)) \cdot ((g - h) (1 - \sigma) + (k - 1) (-1 + 2 \sigma)) \rho^{3(g + h - k) + 2},
\]

\[
L(G_2(\rho)) = c_k^3 d_1^3 (g - h - 2 (k - 1)) (k - 1) \cdot (- (k - 1) + (g - h - 2 (k - 1)) \sigma) \rho^{3(g + h - k) + 3}.
\]

For maximum-principle functions (114) we have \(g - k \geq 1, g - h > 0\) and \(L(C(\rho)) \leq 0, L(G_1(\rho)) \leq 0, L(G_2(\rho)) \leq 0\) for all \(\rho \geq 0\).

We assume \((g - h) (1 - \sigma) + (k - 1) (-1 + 2 \sigma) = 0\) which is equivalent to the identity \(g - h = (k - 1) \frac{2 \sigma - 1}{\sigma - 1}\). Using this identity we get

\[
L(G_2(\rho)) = c_k^3 d_1^3 (k - 1)^3 \left(\frac{1}{\sigma - 1}\right)^2 \rho^{3(g + h - k) + 3}
\]

which results in a contradiction. So we have \((g - h) (1 - \sigma) + (k - 1) (-1 + 2 \sigma) < 0\) which is equivalent to \(g - h > (k - 1) \frac{2 \sigma - 1}{\sigma - 1}\). This implies

\[
g - h > (k - 1) \frac{2 \sigma - 1}{\sigma - 1} > 2 (k - 1) > k - 1 \text{ and } g - h - (k - 1) > g - h - 2 (k - 1) > 0.
\]

Thus, the condition \(L(G_1(\rho)) \leq 0\) for all \(\rho \geq 0\) results in a contradiction. Therefore, Case VIII results in a contradiction.
5.11. **Case IX.** \( c_k > 0, \ d_2 > 0 \) for all \( k \geq 3 \)

First we calculate the leading terms of the \( r \)-terms using Lemma 5.4.

\[
\begin{align*}
L(r_H) &= c_k d_2 (g - h - 2(k - 2)) \rho^{g+h-(k+1)}, \\
L(r_K) &= c_k d_2 (k - 2) \rho^{g+h-(k+1)}, \\
L(r_{HH}) &= c_k d_2 (g - h - 2(k - 2)) (g + h - (2k + 1)) \rho^{g+h-(k+2)}, \\
L(r_{HK}) &= c_k d_2 ((g - 2(k - 1)) (k - 1) - (h - 2)) \rho^{g+h-(k+2)}, \\
L(r_{KK}) &= c_k d_2 (k - 2) (k - 1) \rho^{g+h-(k+2)},
\end{align*}
\]

Now we calculate the leading terms of \( C(\rho), \ G_1(\rho), \ G_2(\rho) \) using (5.3), (5.0), (5.7).

\[
\begin{align*}
L(C(\rho)) &= c_k d_1 ((g - h) (1 - \sigma) + (k - 2) (-1 + 2\sigma)) \rho^{g+h-(k-1)}, \\
L(G_1(\rho)) &= -c_k^2 d_2^2 (g - h - 2(k - 2)) (g - h - (k - 2)) \cdot \\
&\quad ((g - h) (1 - \sigma) + (k - 2) (-1 + 2\sigma)) \rho^{3(g+h-k)-1}, \\
L(G_2(\rho)) &= c_k^2 d_2^3 (g - h - 2(k - 1)) (k - 1) \cdot \\
&\quad (- (k - 1) + (g - h - 2(k - 1)) \rho \rho^{3(g+h-k)}.
\end{align*}
\]

For maximum-principle functions (1.1) we have \( g - k \geq 1, \ g - h > 0 \) and \( L(C(\rho)) \leq 0, \ L(G_1(\rho)) \leq 0, \ L(G_2(\rho)) \leq 0 \) for all \( \rho \geq 0 \).

We assume \((g - h) (1 - \sigma) + (k - 2) (-1 + 2\sigma) = 0 \) which is equivalent to the identity \( g - h = (k - 2) \frac{2\sigma - 1}{\sigma} \). Using this identity we get

\[
L(G_2(\rho)) = c_k^2 d_2^3 (k - 2) \left( \frac{1}{\sigma - 1} \right)^2 \rho^{3(g+h-k)}
\]

which results in a contradiction. So we have \((g - h) (1 - \sigma) + (k - 2) (-1 + 2\sigma) < 0 \) which is equivalent to \( g - h > (k - 2) \frac{2\sigma - 1}{\sigma} \). This implies

\[
\begin{align*}
g - h &> (k - 2) \frac{2\sigma - 1}{\sigma - 1} > 2(k - 2) > k - 2 \quad \text{and} \\
g - h - (k - 2) &> g - h - 2(k - 2) > 0.
\end{align*}
\]

Thus, the condition \( L(G_1(\rho)) \leq 0 \) for all \( \rho \geq 0 \) results in a contradiction. Therefore, Case IX results in a contradiction.

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