CP Violating Phase Originated from Right-handed Neutrino Mixing

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Abstract—We propose an idea that the observed CP violation in neutrino oscillation is originated from the phase in right-handed neutrino mixing by seesaw mechanism. We add small breaking terms \( M_{ij} N_R N_{R_i} \) in the model based on \( A_4 \) symmetry to generate non-diagonal right-handed neutrino mixing, which will give a small correction to TBM mixing via seesaw mechanism with nonzero reactor angle and CP violating phase. We estimate the CP violating phase by investigating the process of leptogenesis due to the decay of right-handed neutrino.

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1. INTRODUCTION

Neutrino oscillation [1–4] is one of the greatest discoveries in particle physics, which confirms neutrino mass and mixing. To explain the generation of neutrino mass, physicists consider neutrino as Majorana particle, which could generate mass term without right-handed neutrino which has not been sought up to now. Majorana mass term violates lepton number conservation which is required in the standard model (SM), thus, the discovery of neutrino oscillation is physics beyond SM.

Seesaw mechanism is the most natural and elegant theory to explain the smallness of neutrino mass, which introduces very heavy Majorana right-handed neutrino and leads to the following relation

\[
M_{\nu} \approx -\frac{m_D m_D^T}{M_R},
\]

where \( M_\nu, m_D, M_R \) are neutrino mass matrix, Dirac neutrino mass matrix, and Majorana right-handed neutrino mass matrix, respectively. From Eq. (1), it is obvious that neutrino mass matrix \( M_\nu \) will be diagonalized when \( m_D \) and \( M_R \) are both diagonalized.

Neutrino mixing is defined as

\[
\begin{pmatrix}
v_e \\
v_\mu \\
v_\tau
\end{pmatrix}
= U
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix},
\]

where \( v_i (i = e, \mu, \tau) \) is flavor state which is defined by charged weak current, \( v_i (i = 1, 2, 3) \) is neutrino mass eigenstate and \( U \) is neutrino mixing matrix which is generally standard parameterized as PMNS matrix

\[
U_{\text{PMNS}} =
\begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\
s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23}
\end{pmatrix},
\]

where \( s_{ij} = \sin \theta_{ij}, c_{ij} = \cos \theta_{ij}, \) and \( \delta = \delta_{CP} \) which is CP violating phase. In 2002, P.F. Harrison et al. proposed TBM mixing pattern [5] as follows

\[
U_{\text{TBM}} =
\begin{pmatrix}
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}
\end{pmatrix},
\]
which described neutrino oscillation very well. TBM mixing Eq. (3) indicates two large mixing angles \( \tan^2 \theta_{12} = 1/2, \tan^2 \theta_{23} = 1 \), and zero reactor angle \( \sin \theta_{13} = 0 \). In 2012, Double Chooz [3] and Daya Bay [4] experiments detected nonzero reactor angle \( \theta_{13} \), which implied TBM mixing pattern should be modified to accord with experiments, while TBM matrix is still considered as the best zeroth-order neutrino mixing. The authors exerted perturbation in various methods on TBM matrix to attempt to predict nonzero \( \theta_{13} \). For example, literature [6] generated deviation from TBM matrix with two seesaw mechanisms. Moreover, nonzero reactor angle \( \theta_{13} \) will give rise to the \( CP \) violation of neutrino oscillation. Cabibbo [7] and other physicists [8] pointed out that three-flavor mixing would generate inevitable \( CP \) violating phase.

For TBM mixing pattern with zero reactor angle, \( CP \) violating phase which accompanying \( \sin \theta_{13} \) would not act on neutrino oscillation. The detection of nonzero reactor angle urges experiments of detecting \( CP \) violating phase [9, 10] and theories [11–21] to predict it.

Generally, in the scenarios predicting \( CP \) violation of neutrino oscillation, the \( CP \) phase is introduced in neutrino mixing matrix directly. For example, \( CP \) violating phase was introduced in mixing matrix by exponential parameterization of mixing matrix [12, 15–19, 22, 23]. In this paper, we propose an idea that the observed nonzero reactor angle and \( CP \) violation is due to the entry with \( CP \) phase \( e^{i\delta} \) in right-handed neutrino mixing via seesaw mechanism. Similar ideas were discussed in literature [13, 14, 24].

In our scheme, we will generate TBM mixing matrix in model based on discrete symmetry as the zeroth-order mixing. According to Eq. (1), this TBM mixing could be considered as mixing in left-handed neutrino sector in Dirac mass term with diagonal right-handed neutrino mixing matrix, which is only dependent on the way of left-handed neutrino coupling to right-handed neutrino and independent of right-handed neutrino mixing. Then we will add softly breaking mass terms to generate non-diagonal right-handed neutrino mixing matrix, which will give a small correction to TBM matrix via seesaw mechanism. The resultant neutrino mixing will present nonzero reactor mixing angle and \( CP \) violating phase. At the end, the \( CP \) violating phase will be estimated by investigating leptogenesis due to the decay of right-handed neutrino.

2. FLAVOR SYMMETRY
 FOR TBM MIXING MATRIX

The discrete flavor symmetry can control the structure of lepton mixing matrix. TBM mixing matrix Eq. (3) is well known for its good fit for experiments and the neutrino mass matrix with TBM mixing can be decomposed into the sum of three simple matrices with integer elements which conform to certain discrete symmetry

\[
M_v = U_{\text{TBM}}^* M_{\text{TBM}} U_{\text{TBM}}^T \tag{4}
\]

where \( m_1, m_2, m_3 \) are neutrino masses. The three matrices at the r. h. s of the second equality in Eq. (4) are all \( S_3 \)-symmetric. So any group including \( S_3 \) subgroup and triplet representation could generate TBM neutrino mixing matrix, in which the minimal groups are \( S_4 \) and \( A_4 \).

2.1. \( S_4 \) Symmetry

\( S_4 \) group covers all permutations among four objects \( (x_1, x_2, x_3, x_4) \), i.e. \( (x_1, x_2, x_3, x_4) \rightarrow (x_i, x_j, x_k, x_l) \), which includes 24 elements and non-Abelian subgroups: \( S_3, A_4 \), and \( \Sigma(8) \) which is \( (Z_2 \times Z_2) \times Z_2 \). The representations of \( S_4 \) include two singlets, one doublet, and two triplets. In \( S_4 \) model, \( S_4 \) is spontaneously broken to one of subgroups, in which \( Z_2 \times Z_2 \) is arranged for neutrinos which is called Klein four group and \( Z_3 \) (included in \( S_3 \) and \( A_4 \)) for charged leptons. The generators for these two subgroups, i.e. \( Z_2 \times Z_2 \) and \( Z_3 \), could be chosen \( a_1, a_2, b_1 \) (symbols used in Appendix), which can be transformed by unitary matrix \( U_w \) as

\[
X = U_w^* d_1 U_w \tag{5}
\]

\[
= \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 \\
1 & w^* & w^{2*} \\
1 & w^{2*} & w^*
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
\times \begin{pmatrix}
1 & 1 & 1 \\
1 & w & w^2 \\
1 & w^2 & w
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]
where $w^3 = 1$ and $U_w$ is called magic matrix (refers to equation B.20 in literature [25])

$$U_w = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix}. \quad (8)$$

The above two matrices can both be diagonalized by TBM matrix, thus, TBM mixing matrix can be generated in $A_4$ model. $A_4$ group includes three singlets and a single triplet, thus, we could put three left-handed neutrinos and charged leptons into the triplet, and put three right-handed charged leptons into three singlets as in literature [26]. More details are elaborated in Appendix.

3. TBM MIXING MODIFIED BY RIGHT-HANDED NEUTRINO MIXING VIA SEESAW MECHANISM

We adopt the $A_4$ model in literature [26] to generate the zeroth-order TBM neutrino mixing. The model introduced very heavy right-handed neutrinos and the assignment of leptons was

$$\begin{pmatrix} v_i, l_i \end{pmatrix}_L, N_i_R, (3, 1) \quad l_{1R} (1, 1), l_{2R} (1', 1), l_{3R} (1'', 1) \quad \Phi_i = (\phi_i^+, \phi_i^0) (3, 0) \quad \eta = (\eta^+, \eta^0) (1, -1). \quad (9)$$

Lepton masses are generated by the following Lagrangian

$$\mathcal{L} = \frac{1}{2} M_{N_R}^2 \Phi_i + F N_{iR} (v_i l_i \eta^0 - l_i l_i \eta^+), \quad (10)$$

where right-handed neutrino mass matrix $M_N^R$ could be considered diagonal. As a real unitary matrix $U_{TB}^T = U_{TB}$, TBM matrix $U_{TB}$ could be considered as mixing in left-handed neutrino sector in Dirac mass term, which was only dependent on the way of left-handed neutrino coupling to right-handed neutrino and independent of right-handed neutrino mixing.

Then we further add very small breaking terms $M_{ij} N_{R_i} N_{R_j} (i \neq j)$ as perturbation to generate non-diagonal right-handed neutrino mixing, in which we assume $\sin \theta_{12} (N_R) \sim \sin \theta_{23} (N_R) \sim \sin \theta_{13} (N_R) = \sin \alpha \sim 0$. Now Eq. (11) becomes non-diagonal again

$$\text{Diag} M^R_v \approx -\frac{U_{TB}^T m_{D}^T m_{D}^T U_{TB}}{M_R}, \quad (11)$$

which will be diagonalized by $U_R$ as

$$\text{Diag} M_v = U^T M_v U$$
where we have used

\[ M_0 \approx -\frac{m_D^T m_D}{M_R}. \]  

Then neutrino mixing matrix will be \( U = U_{\text{TBM}} U_R^+ \), where we have used \( U_{\text{TBM}}^T = U_{\text{TBM}}^+ \).

For right-handed neutrino mixing is approximately diagonal, we adopt the matrix in literature \([23]\) Eq. (37) as approximation (more things about this matrix can refer to \([12, 15-19, 22]\))

\[ U_R \approx \begin{pmatrix} \cos \alpha & 0 & -\sin \alpha e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin \alpha e^{i\delta} & 0 & \cos \alpha \end{pmatrix}. \]  

Here angle \( \alpha \) is corresponding to \( \theta_{13}(R) \) in right-handed neutrino mixing. The nondiagonal phase in Eq. (15) cannot be absorbed in \( N_R \) due to its Majorana property, i.e. \( N_R \equiv N_R^\dagger \) (for Majorana phases do not attribute to \( CP \) violation \([27]\), we ignore them in Eq. (15)). Then neutrino mixing matrix can be written approximately as

\[
U \approx U_{\text{TBM}} U_R^+ = \begin{pmatrix}
\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
\cos \alpha & 0 & -\sin \alpha e^{-i\delta} \\
0 & 1 & 0 \\
\sin \alpha e^{i\delta} & 0 & \cos \alpha
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{2}{\sqrt{6}} \cos \alpha & \frac{1}{\sqrt{3}} \sin \alpha e^{i\delta} & -\sqrt{\frac{2}{3}} \sin \alpha e^{-i\delta} \\
-\frac{1}{\sqrt{6}} \cos \alpha + \frac{1}{\sqrt{2}} \sin \alpha e^{i\delta} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \cos \alpha \\
-\frac{1}{\sqrt{6}} \cos \alpha - \frac{1}{\sqrt{2}} \sin \alpha e^{i\delta} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \sin \alpha e^{i\delta} - \frac{1}{\sqrt{2}} \cos \alpha
\end{pmatrix}.
\]  

In Eq. (16), TBM mixing matrix is slightly modified with nonzero reactor angle and \( CP \) violating phase, from which we can quickly obtain relation about parameter \( \alpha \)

\[
\sin \theta_{13} = -\sqrt{\frac{2}{3}} \sin \alpha.
\]  

We notice that in our scheme, the \( CP \) violating phase in neutrino mixing is the same as the one in right-handed neutrino mixing.

4. \( CP \) VIOlATING PHASE ESTIMATED BY INVESTIGATING LEPTOGENESIS DUE TO THE DECAY OF RIGHT-HANDED NEUTRINO

In 1986, Fukugita et al. \([28]\) proposed that the origin of cosmological baryon number asymmetry or lepton number asymmetry was leptogenesis due to the \( CP \) violation in the decay of Majorana right-handed neutrino. By analogy with the opinion in literature \([24]\), we assume that \( CP \) violation in decay of Majorana right-handed neutrino is due to \( CP \) violating phase in right-handed neutrino mixing.
The Lagrangian of leptogenesis Eq. (18) leads to
\[ f_{ij} \tilde{\mathcal{N}}_{iR} \mathcal{L}_{jL} + H.C. \]
\[ = \left( \mathcal{N}_{eR} \mathcal{N}_{\mu R} \mathcal{N}_{\tau R} \right) F \eta^+ \left( \begin{array}{c} l_e L \\ l_\mu L \\ l_\tau L \end{array} \right), \quad (22) \]
where \( F \) is the coupling matrix as
\[ F = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}, \quad (23) \]
which indicates the couplings to \( N_{eR} \) are \( f_{11}, f_{12}, f_{13} \), and the rest can be done in a similar fashion. Comparing Eq. (22) with Eq. (21), coupling matrix \( F \) could be rewritten as coupling strength multiplied by \( U_R \)
\[ F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} U_R, \quad (24) \]
where \( f_1, f_2, f_3 \) are coupling strengths. We assume the Majorana masses of right-handed neutrinos \( M_1 < M_2 < M_3 \). The strength of a fermion coupling to certain scalar field is proportional to mass of the fermion and then we have \( f_1 < f_2 < f_3 \). According to literature [28], we choose \( f_1 \sim 10^{-5} \) and \( M_1 \sim 10^4 \) GeV, then \( 1 \geq f_2, f_3 > 10^{-5} \).

The net lepton number produced by the decay of right-handed neutrino can be calculated by
\[ \epsilon = \frac{9}{4\pi} \text{Im}(f_{ij} f_{jk} f_{kl} f_{lk}^*) I(M_{ij}^2/M_{jk}^2)/(FF^+)_{11}, \quad (25) \]
where
\[ I(x) = x^{1/2} \{ 1 + (1 + x) \ln[x/(1 + x)] \}. \quad (26) \]
Eq. (25) can lead to
\[ \epsilon = \frac{9}{4\pi} \text{Im}(f_{12} f_{23} f_{13}^*) I(M_2^2/M_3^2)/(FF^+)_{11} \]
\[ \rightarrow \epsilon = \frac{9}{4\pi} f_2^2 f_3 f_4 \text{Im}((U_R)_{12} (U_R)_{23} (U_R)_{13}^*)(U_R)_{23}^*) \]
\[ \times I(1)/(FF^+)_{11}, \quad (27) \]
where
\[ I(M_2^2/M_3^2) = I(1) = -0.3863. \quad (28) \]

Taking \( \epsilon \lesssim 10^{-6}, (FF^+)_{11} \sim 10^{-10} \) [28], Eq. (27) will be
\[ -f_2 f_3 \text{Im}((U_R)_{12} (U_R)_{23} (U_R)_{13}^*)(U_R)_{23}^*) \]
\[ \lesssim 3.6 \times 10^{-6} \rightarrow -f_2 f_3 s_{12}(N_R) s_{23}(N_R) s_{13}(N_R) \]
\[ \times c_{13}^2(N_R) c_{23}^2(N_R) \sin \delta \lesssim 3.6 \times 10^{-6}. \quad (29) \]
As mentioned above, we assume \( \sin \theta_{12}(N_R) \sim \sin \theta_{23}(N_R) \sim \sin \theta_{13}(N_R) = \sin \alpha = -\sqrt{3/2} \sin \theta_{13} \), where the last equality comes from Eq. (17) and \( \cos \alpha \sim 1 \). Then Eq. (29) will become
\[ f_2 f_3 \sin^3 \theta_{13} \sin \delta \lesssim 1.96 \times 10^{-6}. \quad (30) \]
Taking the reactor angle obtained by Daya Bay Collaboration [29], i.e., \( \sin \theta_{13} \approx 0.1479 \), Eq. (30) will become
\[ f_2 f_3 \sin \delta \lesssim 6.06 \times 10^{-4}. \quad (31) \]
Choosing \( f_2 f_3 \sim 10^{-2} \), we will obtain \( \sin \delta \lesssim 0.0606 \), i.e. \( 0 \leq \delta \lesssim 0.02 \pi \) or \( 0.98 \pi \lesssim \delta \lesssim 2 \pi \), which is in the range of the data of NOvA experiment [9], i.e. \( \delta_{CP} \in [0, 0.12 \pi] \cup [0.91 \pi, 2 \pi] \).

5. NEUTRINO MIXING MATRIX WITH CP VIOLATING PHASE

By employing seesaw mechanism and TBM matrix as well as the simple matrix proposed by literature [23], we obtain the simple neutrino mixing matrix Eq. (16). Comparing the matrix elements with the standard parameterization of PMNS matrix, one can quickly obtain simple relation about reactor mixing angle \( \sin \theta_{13} \) and the simplest form of \( CP \) violating phase. By decomposing coupling factors into product of coupling strengths and elements of right-handed neutrino mixing matrix, we analyze the Lagrangian of leptogenesis due to the decay of right-handed neutrino and obtain relation Eq. (27) for right-handed neutrino mixing matrix elements in terms of right-handed neutrino mixing angles \( \theta_{12}(R), \theta_{23}(R), \theta_{13}(R) = \alpha \) and \( CP \) violating phase \( \sin \delta_{CP} \). Taking into account experimental data \( \sin \theta_{13} \approx 0.1479 \) (published in 2018) [29], we can obtain \( \sin \alpha = -\sqrt{3/2} \sin \theta_{13} = -0.1811, \cos \alpha = 0.9834 \) (Here, we assume reactor neutrino mixing angle and solar mixing angle are both in the first quadrant which leads to \( \cos \alpha > 0 \)). Then neutrino mixing matrix Eq. (16) will be
non-diagonal right-handed neutrino mixing. With \( CP \) violation, the TBM matrix has been estimated as consistent with experimental data Eq. (35). Thus the predicting values in Eq. (34) are consistent with experimental data published in 2018 \[30\], neutrino mixing matrix with best-fit values at 3\( \sigma \) level will be

\[
U \approx \begin{pmatrix}
\frac{2}{\sqrt{6}} \cos \alpha & \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{2} \sin \alpha e^{-i\delta} \\
-\frac{1}{\sqrt{6}} \cos \alpha + \frac{1}{\sqrt{2}} \sin \alpha e^{i\delta} & -\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{2} \sin \alpha e^{i\delta} \\
-\frac{1}{\sqrt{6}} \cos \alpha - \frac{1}{\sqrt{2}} \sin \alpha e^{i\delta} & -\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{2} \sin \alpha e^{-i\delta}
\end{pmatrix} \\
= \begin{pmatrix}
0.8029 & 0.5773 & 0.1479 e^{-i\delta} \\
-0.4015 - 0.1280 e^{i\delta} & 0.5773 & -0.0739 e^{-i\delta} + 0.6954 \\
-0.4015 + 0.1280 e^{i\delta} & 0.5773 & -0.0739 e^{-i\delta} - 0.6954
\end{pmatrix}.
\]

(32)

Before counting in \( CP \) violation of neutrino oscillation, consider first the following two formulas:

\[
|a + be^{i\pm\delta}| = \sqrt{a^2 + b^2 + 2ab \cos \delta} \in [ |a - b|, |a + b| ],
\]
\[
|a - be^{i\pm\delta}| = \sqrt{a^2 + b^2 - 2ab \cos \delta} \in [ |a + b|, |a - b| ],
\]

(33)

Then neutrino mixing matrix (32) will be

\[
|U| \approx \begin{pmatrix}
0.8029 & 0.5773 & 0.1479 \\
\{0.2735, 0.5295\} & 0.5773 & \{0.7693, 0.6215\} \\
\{0.5295, 0.2735\} & 0.5773 & \{0.6215, 0.7693\}
\end{pmatrix}.
\]

(34)

Taking experimental data published in 2018 \[30\], neutrino mixing matrix with best-fit values at 3\( \sigma \) level will be

\[
|U|_{3\sigma} = \begin{pmatrix}
0.796 \leftrightarrow 0.843 & 0.518 \leftrightarrow 0.586 & 0.143 \leftrightarrow 0.156 \\
0.214 \leftrightarrow 0.533 & 0.425 \leftrightarrow 0.703 & 0.639 \leftrightarrow 0.784 \\
0.246 \leftrightarrow 0.505 & 0.451 \leftrightarrow 0.721 & 0.603 \leftrightarrow 0.755
\end{pmatrix}.
\]

(35)

Obviously, our predicting values in Eq. (34) are consistent with experimental data Eq. (35). Thus the matrix in Eq. (16) describes neutrino mixing well with the \( CP \) violating phase valued in the range \( 0 \leq \delta \leq 0.02\pi \) or \( 0.98\pi \leq \delta \leq 2\pi \).

6. CONCLUSIONS

TBM mixing matrix has been corrected by right-handed neutrino mixing with \( CP \) violating phase via seesaw mechanism. With the model proposed by literature \[26\], TBM matrix has been obtained as zero-order neutrino mixing. We have added small breaking terms \( M_{ij}N_{Ri}N_{Rj} \) in the model to generate non-diagonal right-handed neutrino mixing. With the aid of the matrix proposed by literature \[23\] as the approximation of right-handed neutrino mixing matrix with \( CP \) violating phase, corrected TBM neutrino mixing matrix with \( CP \) violating phase Eq. (16) has been presented in a simple form which retains the advantages of TBM mixing. By means of decomposing coupling factors into product of coupling strengths and elements of right-handed neutrino mixing matrix, we have analyzed the Lagrangian of leptogenesis due to the decay of right-handed neutrino. With the aid of results of leptogenesis, we have estimated the range of the \( CP \) violating phase which matches the experimental data of NOvA (2018) \[9\] very well.
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Appendix A

$S_4$ SYMMETRY

$S_4$ group covers all permutations among four objects $(x_1, x_2, x_3, x_4)$, i.e. $(x_1, x_2, x_3, x_4) \rightarrow (x_i, x_j, x_k, x_l)$ and includes 24 elements as in literature [25]

\[
\begin{align*}
    a_1 &: (x_1, x_2, x_3, x_4), \\
    a_2 &: (x_2, x_1, x_4, x_3), \\
    a_3 &: (x_3, x_4, x_1, x_2), \\
    a_4 &: (x_4, x_3, x_2, x_1), \\
    b_1 &: (x_1, x_4, x_2, x_3), \\
    b_2 &: (x_4, x_1, x_3, x_2), \\
    b_3 &: (x_2, x_3, x_1, x_4), \\
    b_4 &: (x_3, x_2, x_4, x_1), \\
    c_1 &: (x_1, x_3, x_4, x_2), \\
    c_2 &: (x_3, x_1, x_2, x_4), \\
    c_3 &: (x_4, x_2, x_3, x_1), \\
    c_4 &: (x_2, x_4, x_1, x_3), \\
    d_1 &: (x_1, x_2, x_4, x_3), \\
    d_2 &: (x_2, x_1, x_3, x_4), \\
    d_3 &: (x_3, x_4, x_1, x_2), \\
    d_4 &: (x_4, x_3, x_2, x_1), \\
    e_1 &: (x_1, x_3, x_2, x_4), \\
    e_2 &: (x_3, x_1, x_4, x_2), \\
    e_3 &: (x_2, x_4, x_1, x_3), \\
    e_4 &: (x_4, x_2, x_3, x_1), \\
    f_1 &: (x_1, x_4, x_3, x_2), \\
    f_2 &: (x_4, x_1, x_2, x_3), \\
    f_3 &: (x_3, x_2, x_1, x_4), \\
    f_4 &: (x_2, x_3, x_4, x_1),
\end{align*}
\]

which are written in cycle representation as

\[
\begin{align*}
    a_1 &: e, \\
    a_2 &: (12)(34), \\
    a_3 &: (13)(24), \\
    a_4 &: (14)(32), \\
    b_1 &: (234), \\
    b_2 &: (124), \\
    b_3 &: (132), \\
    b_4 &: (314), \\
    c_1 &: (234), \\
    c_2 &: (231), \\
    c_3 &: (134), \\
    c_4 &: (142), \\
    d_1 &: (34), \\
    d_2 &: (12), \\
    d_3 &: (1324), \\
    d_4 &: (2314), \\
    e_1 &: (23), \\
    e_2 &: (2431), \\
    e_3 &: (1342), \\
    e_4 &: (14), \\
    f_1 &: (24), \\
    f_2 &: (1234), \\
    f_3 &: (13), \\
    f_4 &: (1432).
\end{align*}
\]

$S_4$ is cube symmetry group and the elements can also be written as

\[
\begin{align*}
    a_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
    a_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
    a_3 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
    a_4 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
    b_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
    b_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
    b_3 &= \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
    b_4 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
    c_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
    c_2 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
    c_3 &= \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
    c_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
    d_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
    d_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
    d_3 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \\
    d_4 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \\
    e_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
    e_2 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
    e_3 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \\
    e_4 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \\
    f_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \\
    f_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \\
    f_3 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \\
    f_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

Generally elements are classified by symmetry operations as follows: $T$ represents the element gen-
erated by rotating around coordinate axes which includes 3 axes \( x, y, z \): rotating \( \pi \) for 2-order element and \( \pm \frac{\pi}{2} \) for 4-order element; \( S \) represents 2-order element generated by rotating around axes connecting the midpoints in two opposite edges which include 6 axes. \( R \) represents 3-order element generated by rotating around body diagonals which include 4 axes.

The elements are classified by axes as follows

\[
T \text{ axis: } \{a_2, a_3, a_4, d_3, d_4, e_2, e_3, f_2, f_4, f_3\},
\]

\[
R \text{ axis: } \{b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4\},
\]

\[
S \text{ axis: } \{d_1, d_2, e_1, e_4, f_1, f_3\}, \quad (A.4)
\]

and by the order of element as

\[
h = 1 : \{a_1\},
\]

\[
h = 2 : \{a_2, a_3, a_4, d_1, d_2, e_1, e_4, f_1, f_3\},
\]

\[
h = 3 : \{b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4\},
\]

\[
h = 4 : \{d_3, d_4, e_2, e_3, f_2, f_4\}. \quad (A.5)
\]

Elements are classified into conjugate classes as

\[
C_1(h = 1) : \{a_1\},
\]

\[
C_2(h = 2) : \{a_2, a_3, a_4\},
\]

\[
C_3(h = 2) : \{d_1, d_2, e_1, e_4, f_1, f_3\},
\]

\[
C_4(h = 3) : \{b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4\},
\]

\[
C_5(h = 4) : \{d_3, d_4, e_2, e_3, f_2, f_4\}. \quad (A.6)
\]

Permutation group or symmetric group \( S_n \) includes two generators which could be chosen one cycle of \( n \) length, e.g. \( (123\cdots n) \) and one cycle of its neighboring objects e.g. \( (12) \). It could be easily verified by proving \( (j, j + 1) = (123\cdots n)(j - 1, j)(123\cdots n)^{-1} \). For example, from Eq. (32), we could choose \( d_1 \) and corresponding \( n \)-length cycle \( f_2 \) as generators of \( S_1 \).

Appendix B

**A\(_4\) SYMMETRY**

\( A_4 \) group covers all even permutations of four objects, which is subgroup of \( S_4 \) and includes 12 elements as follows

\[
a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

\[
a_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
b_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},
\]

\[
b_3 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad b_4 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[
c_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix},
\]

\[
c_3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad c_4 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (B.1)
\]

\( A_4 \) group is the symmetry group of regular tetrahedron which generators generally are called \( S, T \). \( T \) represents 3-order element generated by rotating around axes passing through vertices which include 4 axes, which can be classified as: element by rotating \( \frac{2\pi}{3} \) clockwise and element by rotating \( \frac{2\pi}{3} \) anticlockwise. \( S \) represents 2-order element generated by rotating around axes connecting the midpoints in two opposite edges which include 3 axes.

All 12 elements can be classified by axes as follows

\[
S \text{ axis: } \{a_2, a_3, a_4\},
\]

\[
T \text{ axis: } \{b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4\}. \quad (B.2)
\]

The generators could be chosen \( S = a_2, T = b_1 \) which satisfy \( T^2 = T^3 = 1 \).

More can refer to literature [25].

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