The Hamiltonian Mean Field Model: Effect of Network Structure on Synchronization Dynamics

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Kuramoto Model\textsuperscript{1}

- Mutual synchronization: Coupled oscillators characterized by their phases
- All-to-all case:
  \[
  \dot{\theta}_n = \omega_n + \frac{K}{N} \sum_{m=1}^{N} \sin (\theta_m - \theta_n)
  \]
- K: coupling strength
- N: number of oscillators
- \theta: phase
- \omega: randomly chosen intrinsic frequency

\textsuperscript{1} Y. Kuramoto. Intl. Symp. on Math. Probs., Lecture Notes in Phys., 39, 1975.
Hamiltonian Mean Field Model

• Version of Kuramoto model that conserves energy

\[ H = \frac{1}{2} \sum_{n} p_n^2 - \frac{K}{2N} \sum_{n,m} A_{nm} \cos (\theta_m - \theta_n) \]

\[ \dot{\theta}_n = p_n \]

\[ \dot{p}_n = \frac{K}{N} \sum_{m=1}^{N} A_{nm} \sin (\theta_m - \theta_n) \]

\(^2\) Antoni, M. and S. Ruffo. Phys. Rev. E. 52, 1995
Hamiltonian Mean Field Model (2)

• Link to video—
  – [http://bit.ly/1YEmWAj](http://bit.ly/1YEmWAj)

• This video shows the evolution of phase-space for the HMF model for a high coupling constant.
Hamiltonian Mean Field Model (3)

• Recent work has focused on partial extensions to HMF for the network case:
  – Chavanis, P., J. Vatteville and F. Bouchet. Eur. Phys. J. B., 46, 2005.
  – Ciani, A., D. Fanelli and S. Ruffo. Long-range Interactions, Stochasticity and Fractional Dynamics. Springer, 2011.
  – Nigris, S. and X. Leoncini. Phys. Rev. E, 88, 2013.

• In this work, we develop a more general framework.
  – Virkar, Y. S., J. G. Restrepo and J. D. Meiss. Hamiltonian mean field model: Effect of network structure on synchronization dynamics. Phys. Rev. E 92, 052802 (2015).
Global order parameter

Desynchronized state (small $K$)

\[ R_n e^{i\psi_n} = \frac{1}{N} \sum_{m=1}^{N} A_{nm} e^{i\theta_m} \]

Synchronized state (large $K$)

\[ R = \frac{1}{\langle d \rangle} \sum_{n=1}^{N} R_n \]
Global order parameter as a function of coupling strength
Critical coupling constant $K_c$

• For initial conditions:
  – $\theta_n$ is drawn from a uniform distribution.
  – $p_n$ is drawn from a Gaussian distribution with mean $\mu$ and standard deviation $\sigma$,

we get from theory (linear stability analysis $^3$),

$$K_c = \frac{2\sigma^2N}{\lambda}$$

• $\lambda$ is the principal eigenvalue of the adjacency matrix.

$^3$ Strogatz S. and R. Mirollo, J. Stat. Phys., 63, 1991.
Numerical example in simulated network

We construct networks with degree distribution $P(d) = C d^{-\alpha}$.
Numerical example in simulated network (2)

\[ \langle R \rangle_t \]

\[ \lambda K \]

\[ \alpha \]

\[
\begin{array}{c|c|c|c|c}
\alpha & 2.5 & 2.8 & 3.1 & 3.5 & 3.8 \\
\hline
\end{array}
\]

Scale-free

?
Synchronized solutions

- Assuming that in the synchronized state all rotors rotate with a common frequency, and letting $r_n$ denote the time average of $R_n$, we obtain the following equations which can be solved numerically,

$$r_n = \frac{1}{N} \sum_{m=1}^{N} A_{nm} \frac{I_1 \left( \frac{K r_m}{\sigma^2} \right)}{I_0 \left( \frac{K r_m}{\sigma^2} \right)},$$

$I_1$ and $I_0$ are bessel functions.

$$\sigma^2 = \sigma_0^2 + \frac{K}{N} \sum_{n=1}^{N} r_n \frac{I_1 \left( \frac{K r_n}{\sigma^2} \right)}{I_0 \left( \frac{K r_n}{\sigma^2} \right)}$$
Result: Theoretical vs Simulated (R vs K)

Erdös-Renyi random network: N nodes with q the probability of connecting any two nodes.
Result: Theoretical vs Simulated (R vs K)

Scale-free

\( \langle R \rangle_t \)

\( K \)

simulated data
theory

1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
50
100
150
200
200
150
100
50
0

Result: Maximum synchrony

\[ \langle R \rangle_t \]

\( \alpha \) increasing homogeneity
Conclusions

• Network heterogeneity impacts both the onset of synchronization ($K_c$) and the path to synchrony.

• We developed a theory for quantifying the onset of synchronization and the synchronized state for the network version of HMF.

• The maximum possible synchrony is however fairly independent of network heterogeneity.
Thanks! ☺

• References:
  – Antoni, M. and S. Ruffo. Phys. Rev. E. 52, 1995.
  – Barre, J. et. al. Physica A. 365, 2006.
  – Capma, A., A. Giansanti and G. Morelli. Phys. Rev. E. 76, 2007.
  – Dauxios, T., et. al. Dynamics and thermodynamics of Systems with Long-Range Interactions, Springer, 2002.