Learning Curves for Decision Making in Supervised Machine Learning — A Survey

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Abstract—Learning curves are a concept from social sciences that has been adopted in the context of machine learning to assess the performance of a learning algorithm with respect to a certain resource, e.g. the number of training examples or the number of training iterations. Learning curves have important applications in several contexts of machine learning, most importantly for the context of data acquisition, early stopping of model training and model selection. For example, by modelling the learning curves, one can assess at an early stage whether the algorithm and hyperparameter configuration have the potential to be a suitable choice, often speeding up the algorithm selection process. A variety of approaches has been proposed to use learning curves for decision making. Some models answer the binary decision question of whether a certain algorithm at a certain budget will outperform a certain reference performance, whereas more complex models predict the entire learning curve of an algorithm. We contribute a framework that categorizes learning curve approaches using three criteria: the decision situation that they address, the intrinsic learning curve question that they answer and the type of resources that they use. We survey papers from literature and classify them into this framework.

Index Terms—learning curves, supervised machine learning

1 INTRODUCTION

Learning curves describe a system’s performance on a task as a function over some resource to solve that task. There can be a certain budget of that resource, limiting the amount of resources that can be spent. In the simplest case, the budget can be quantified by the number of examples the learner has observed before performing the task or the number of iterations or time the learner spends in an environment. The performance is a measure that captures how well a task is solved (e.g. error rate or area under the ROC curve). Learning curves are an important source of information for making decisions on the following matters in machine learning:

• Data Acquisition. The acquisition of how many more labels is (maybe economically) reasonable? [34, 29]
• Early Stopping (of training a model). If we are committed to some specific learner (a learning algorithm and its hyperparameters), we might want to minimize the training time [28, 61] or avoid overfitting [6, 23].
• Early Discarding (in model selection). If we want to select from various models, we want to stop the evaluation of a candidate when we are sure that it is not competitive to the best-known solution [16, 38, 69].

Many techniques have been proposed to address either of these problems. This is typically done by modelling (or extrapolating) the learning curves and acting upon predictions from that model. These learning curve models vary in the type of queries they are able to answer. For example, some are trained to determine the binary question whether the model produced by a learner has already attained its best value [28], whereas other, more complex, learning curve models are trained to capture the behaviour of various learners at any possible budget [16, 18]. To build such models, different data resources can be used. Besides the observations made so far for a learner on the dataset of interest, these resources can include learning curves or features of other datasets or learners.

Our contribution is a unified framework of the usage of learning curves for decision making in machine learning and an extensive review of the literature of approaches that fall within this framework. This framework contains three criteria, i.e. (i) the type of decision situation, (ii) the type of question that is being answered and (iii) the data resources that are used to model the learning curve. We focus specifically on supervised machine learning, in which learning curves describe the predictive performance of a model produced by a learning algorithm either as a function of the number of training instances or of the time or iterations spent for learning on a given dataset. We explicitly exclude learning curves that describe the performance of a learning agent in an environment over time, i.e. the learning curve of an agent in a reinforcement learning setup [76]. Similarly, this survey is not exhaustive in the sense of covering all types of performance curves. We briefly contrast learning curves to other performance curves like active learning curves, feature curves, and capacity curves, and explain why we consider them to be out of scope for the literature review. This being said, we aim to be extensive in surveying the type of approaches that use learning curves in supervised learning.

This paper is structured into three main parts. Sec. 2 presents relevant background knowledge on learning curves including formal definitions and the terminology relevant for the remainder. Sec. 3 introduces our taxonomy...
of the usage of learning curves for decision making in supervised learning, within which each approach we are aware of can be classified properly. Following this taxonomy, Sec. 2 exhaustively reviews approaches that explicitly or implicitly answer questions related to learning curves to make or recommend decisions in the context of supervised machine learning. Sec. 3 concludes our findings.

2 Background on Learning Curves

This section gives a conceptual background on learning curves. It first provides an idealized formal definition in Sec. 2.1 followed by a definition of empirical learning curves in Sec. 2.2 that can be computed in practice. The concept of utility curves is introduced in Sec. 2.3. Sec. 2.4 introduces important terminology related to the shape of learning curves such as the limit performance and the saturation point, after which Sec. 2.5 discusses how learning curves are typically modelled. Finally, Sec. 2.6 contrasts learning curves covered in this survey with other types of performance curves used in machine learning.

2.1 Observation and Iteration Learning Curves

We consider learning curves in the context of supervised machine learning. Formally, in the supervised learning context, we assume some instance space $\mathcal{X}$ and a label space $\mathcal{Y}$. A dataset $d \subseteq \{(x,y) \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$ is a finite relation between the instance space and the label space. We denote as $D$ the set of all possible datasets. A learning algorithm is a function $a : D \times \Omega \rightarrow H$, where $H = \{h \mid h : \mathcal{X} \rightarrow \mathcal{Y}\}$ is the space of hypotheses and $\Omega$ is a source of randomness.

Note that learning curves can also be considered in other machine learning setups. In fact, learning curves appeared first in reinforcement learning [15] and have also been used for unsupervised learning [46]. However, to give this survey focus, we consider learning curves for supervised learning.

The performance of a hypothesis is typically expressed as risk, which is also often called out-of-sample error:

$$R_{\text{out}}(h) = \int_{\mathcal{X} \times \mathcal{Y}} \text{loss}(y, h(x)) \, dp_{\mathcal{X} \times \mathcal{Y}}. \quad (1)$$

Here, $\text{loss}(y, h(x)) \in \mathbb{R}$ is the penalty for predicting $h(x)$ for instance $x \in \mathcal{X}$ when the true label is $y \in \mathcal{Y}$, and $p_{\mathcal{X} \times \mathcal{Y}}$ is a joint probability measure on $\mathcal{X} \times \mathcal{Y}$ from which the available dataset $d$ has been generated. As such, the out-of-sample error represents the weighted “summed” error that hypothesis $h$ makes on all possible observations, weighted by their probabilities.

The performance of a learning algorithm is simply the performance of the hypothesis it produces. In contrast to the performance of a hypothesis, the performance of a learner depends on its input, i.e. on the data provided for learning. The average performance of learner $a$ for a number of $s$ training examples can then be expressed as

$$C(a,s) = \int_{\omega \in \Omega, d_0 \in D, |d_0| = s} R_{\text{out}}(a(d_{\text{tr}}, \omega)) \, dp_{\mathcal{X} \times \mathcal{Y}} \, dp_{\Omega}, \quad (2)$$

where $d_{\text{tr}} \in D$ is the dataset of size $s$ used to induce a model using learner $a$. It is generally assumed that $d$ is a collection of i.i.d. samples from $p_{\mathcal{X} \times \mathcal{Y}}$. Since the randomness of $a$ is averaged over in the random seed $\omega$, the risk is deterministic.

When we consider Eq. $(2)$ as a function of the number of observations for a fixed learner $a$, we obtain the observation learning curve of learner $a$. That is, the observation learning curve is simply the function $C(a, \cdot) : \mathbb{N} \rightarrow \mathbb{R}$; so it is a sequence of performances, one for each training size.

Alternatively, many learning algorithms implement an iterative internal optimization process, which allows describing the learning progress over time or a number of iterations. In the formal framework, a learner can be seen more generally as a function $a : D \times \Omega \rightarrow H^t$ that maps a dataset to a sequence of hypotheses, one for each of its iterations. The above error function for learners can then be written as

$$C(a,s,t) = \int_{\omega \in \Omega, d_0 \in D, |d_0| = s} R_{\text{out}}(a(d_{\text{tr}}, \omega)_t) \, dp_{\mathcal{X} \times \mathcal{Y}} \, dp_{\Omega}, \quad (3)$$

Here, $t$ expresses some budget, for example, time or a number of iterations over the dataset, often expressed in epochs.

Based on this notion, the iteration learning curve of a learner $a$ is defined for a fixed dataset size $s$ (often around 90% of the available data) and is then the function $C(a, s, \cdot) : \mathbb{N} \rightarrow \mathbb{R}$. Examples of such learning curves occur above all in the analysis of deep learning models [16][23].

The two types of learning curves seem to be related and indeed look similar when being visualized, but they have different semantics. The crucial difference is that in the second case there is a finite number of observations, no matter how large $t$ becomes. In fact, since such iterative learners typically automatically stop the learning process as soon as no progress is observed, it typically holds that $C(a, s, t) = \lim_{s \to \infty} C(a, s, t)$.

So as $t$ grows, each observation is considered an unlimited number of times, and hence iteration learning curves show how much the learner can make out of a constant number of observations. Instead, observation learning curves show the performance of the learner as the number of examples grows. The latter typically means, for an infinite input space, that the information basis available to the learner is growing strictly bigger.

2.2 Empirical Learning Curves

The above definitions of learning curves are purely theoretical. This is because we cannot evaluate equations $(1)$ in practice. First, the out-of-sample error $R_{\text{out}}$, i.e. Eq. $(1)$ cannot be computed in practice since the measure $p_{\mathcal{X} \times \mathcal{Y}}$ is unknown. Relying on this error, the learning curve values cannot be computed either. The necessity to average over the oftentimes uncountable set of all possible train sets can add additional problems.

To compute learning curves in practice, we rely on empirical estimates of the above quantities. To this end, we estimate the out-of-sample error through the internal error:

$$R_{\text{in}}(h) = \frac{1}{|d|} \sum_{(x,y) \in d} \text{loss}(y, h(x)), \quad (4)$$

where $d$ is the dataset used for assessing the performance of hypothesis $h$. The dataset $d$ may or may not contain
instances used to create the hypothesis \( h \). If it contains exactly those instances, we refer to it as a \textit{train set} \( d_e \). If \( h \) has been built without instances of \( d \), we call it a \textit{test set} \( d_t \). The dataset \( d \) can be also a combination of both.

We consider an \textit{empirical learning curve} any set of estimates of a true learning curve for different sizes or iterations. We can use various estimation procedures to estimate the performance on a given training size, such as using a regular holdout, or cross-validation. The latter leads to various estimates, and averaging over these estimates yields an estimate of the observation learning curve in Eq. (2) at size \( s \). To obtain an empirical estimate of the iterative learning curve in Eq. (3), we do the same except that we stop the learning algorithm after \( t \) iterations.

Since empirical learning curves are the only way to gain insights about true learning curves, quite some studies have been published with the sole goal to share empirical learning curves with the community and hence to improve the understanding of how they behave. Perlich et al. \cite{59} contrast the learning curves on decision trees and logistic regression on different datasets. Notably, the authors also compare learning curves, e.g. they report whether one curve is below the other (dominates it) on all considered training set sizes or whether the two learning curves cross. Several other studies report learning curves for specific learners. Ng and Jordan \cite{55} compare logistic regression and naive Bayes. March et al. \cite{50} conduct a study similar to the one by Perlich et al. \cite{59} to compare linear vs. non-linear classifiers on a smaller scale. Recently, Bornschein et al. \cite{3} report the learning curves of different architectures on typical image classification datasets.

Empirical studies of this type do not answer generalizing questions about learning curves but rather report experimental results. This is different from contributions in which certain model assumptions are made and data is compared to those models, e.g. with the aim to compute the goodness of fit of that model. In Sec. 2.5 we briefly discuss some of such models.

### 2.3 Utility Curves

The concept of learning curves can be further generalized to a \textit{utility curve} \cite{33, 34, 79}. The utility is usually some trade-off between the performance and the computational cost of training a model. The specific details can be different per task. The utility is connected to the learning curve in so far as the utility is also a function of the dataset size and is directly influenced by the predictive performance. So one could argue that the utility curve \( U \) is obtained by passing the learning curve to the utility function alongside other parameters that influence the utility, most notably the cost of acquiring new instances and the cost to train a model on the respective dataset size.

### 2.4 Shapes and Terminology of Learning Curves

While this survey is not primarily about shapes of learning curves, the shapes of learning curves play an important role when using them to make decisions. Hence, we consider it necessary to convey some of the most important insights about basics on the shapes of learning curves. However, we refer to a recent exhaustive survey on the shapes of learning curves \cite{74} for details on this topic. Fig. 1 visualises some important concepts.

#### 2.4.1 Anchor Points

In this survey, we adopt the term \textit{anchor} to refer to a point for which the empirical learning curve carries an estimate. Throughout this paper, we refer to this as \( s \). There is no established name for such points in literature. They are called \textit{sample sizes} in \cite{28, 38, 51}, and those authors refer to a \textit{collection} of such samples sizes as a \textit{schedule} \cite{18, 28, 46, 61}. However, for iteration learning curves, the term “sample” is misleading, because the curve plots performance against “visits” of instances (out of a fixed set), which is related but not the same as sample size. Besides, the term sample size is quite overloaded in the context of machine learning, because this field deals with various types of samples in different contexts, e.g. train and validation samples, etc. Another terminology observed sometimes is the one of \textit{sample landmarks} \cite{22, 57}. However, this term is also slightly confusing, since landmarks are generally understood as the performances of cheap-to-evaluate learners, which was in fact also the motivation for this terminology in \cite{22}. A less used terminology is the term \textit{anchor} \cite{51, 38, 49}, which is not ambiguous in the machine learning context and captures the idea that analysis is based on some selected points. It serves well to immediately create an association with a particular size of a sampled training data set (observations) or a number of visited instances (iterations) that is used in the context of building an empirical learning curve.

#### 2.4.2 Limit Performance, Saturation Point, Pre-Exponential Point, Utility-Based Stopping Point, and Plateau

It is generally assumed that learning curves converge to some value. In the case of iterative learning curves, there are sometimes oscillations in the curve, but even then the curve eventually converges to some (in those cases typically a bad) value. We are not aware of a particular term that is used to describe the score to which a learner converges. In \cite{14}, this is called the \textit{limiting performance} or the \textit{asymptotic performance} of the data; i.e. it is not the property of a particular learner but the best achievable performance among “all” learners (even though only tested with two model types in the paper). In this paper, we adopt the term \textit{limit performance} of the learner (on a fixed number of training instances in the case of iteration learning curves).

Intuitively, the \textit{saturation point} is the point after which the performance convergences. That is, the point after which all values are in a distance of less than some pre-defined and typically very small \( \varepsilon \). In early works, this point has been called \textit{stopping point} \cite{18, 35, 36, 46, 61}. Provost et al. \cite{61} characterize this point as follow: “Models built with smaller training sets [than \( s_{min} \)] have lower accuracy than models built with from training sets of size \( s_{min} \), and models built with larger training sets have no higher accuracy.” In the context of those works, namely progressive sampling, the term “stopping point” makes sense because they progressively sample until they reach the convergence region and hence stop the sampling procedure. In absence of such a mechanism, the term appears a bit odd. The notion of saturation in the context of learning curves was proposed in \cite{70} and seems appropriate since the limit performance...
A related concept is the pre-exponential point and, correspondingly, the pre-exponential performance. The saturation point may be reached “late” in the sense that a lot of training data is necessary to obtain the saturation performance. We call the smallest anchor point for which an increase by a factor of $b$ leads to a performance improvement of less than some $\delta$ i.e. $C(a, s) - C(a, b \cdot s) < \delta$. Reasonable candidates for $b$ can be 2 or 10, while candidates for $\delta$ can be 0.01 or 0.001 if the metric is the error rate. Its semantic is that from the pre-exponential point on, one needs more than $b^k$ (i.e. an exponentially increasing number of) observations to improve by a low margin of $k\delta$. Correspondingly, the pre-exponential performance is the performance that can be obtained by a comparably “efficient” anchor point.

We adopt the following notation throughout the paper. We will denote the saturation point itself as $s_{sat}$, which represents a specific anchor point. The saturation performance is simply the value of the learning curve at that point. Both are defined in the context of a fixed learner $a$. Hence, we can write the saturation performance as $p_{sat} := C(a, s_{sat})$.

The utility-based stopping point is the point at which the acquisition of further datapoints has a negative impact on the utility of the data analyzing entity. As such, this concept is associated with utility curves. The utility-based stopping point is not related to the saturation point but, if at all, rather to the pre-exponential point; we denote it as $s_{sat}$.

The right-sided open interval bounded by the saturation point from the left is consistently called the plateau of the curve. However, it is noteworthy that a curve can also have intermediate plateaus, i.e. intervals of (almost) constant performance without being the final plateau.

### 2.4.3 Well Behaved Learning Curves

The notion of a well behaved learning curve is, to our knowledge first used by Provost et al. [61]. A learning curve is said to be well behaved if its slope is monotonically non-decreasing (for error-based learning curves). An even stricter criterion demanding convexity of the curve, which implies monotonicity, has been introduced recently in [48, 49]. The property of being well behaved is one of the true learning curve. The (linear interpolation of an) empirical learning curve can often violate this condition, specifically when the number of validations conducted a the anchors is small or when the learning curve has reached a plateau.

While it is known that not all learning curves are well behaved [43, 44], empirical studies suggest that such curves are rather an exception and that most learning curves are well behaved. Learning curves that are not well behaved are observed above all in the context of deep learning, where this double descent or peaking phenomenon can be observed (at times) for both observation and iteration learning curves [48]; the effect was also observed decades ago for other learners [21]. However, extensive empirical studies have shown that, most observation learning curves are even convex [48] and hence well behaved. Some recent works suggest that potential ill behaviour can be mitigated by appropriate configuration or wrapping of learners [47, 51, 75].

### 2.5 Modeling a Learning Curve

A learning curve model is a characterization of the true learning curve derived from an empirical learning curve. The empirical learning curve is the result of sampling from a stochastic process that underlies noise stemming from randomness in data splits and the learning algorithm itself. It is typically assumed [16, 18, 29, 30, 48, 69] that this stochastic process follows the distribution

$$\mu_s + N(0, \sigma_s^2),$$

where $\mu_s$ is either $C(a, s)$ as per Eq. (2) if modeling an observation learning curve or $C(a, n, s)$ as per Eq. (3) when modeling an iteration learning curve, and we assume a noise that follows a Gaussian distribution with zero mean and dispersion $\sigma_s^2$ that may vary over different anchor sizes. In this section, we write $\mu_s$ to cover both learning curve types in a single notation.

When creating a model for the learning curve, it is sensible to distinguish two types of uncertainty. First, the aleatoric uncertainty is $\sigma_s^2$, which is intrinsic to and averaged out in the true learning curve. Again, this is the uncertainty arising from random splits and randomness in the learner when computing the empirical learning curve. Second, the epistemic uncertainty is the one the model itself has about the estimate of the mean value $\mu_s$. The epistemic uncertainty can be directly or indirectly related to the variance of the random variable describing the sample and hence, by the law of large numbers, tends to 0 as the number of samples at an anchor goes to infinity.

When talking about uncertainty, it is important to understand what uncertainty is exactly expressed of. We can be uncertain about (at least) three things:

1) the gap between the predicted performance $\hat{f}(s)$ and the true value $\mu_s$ at some anchor $s$,
2) whether the current estimates of $\theta$ are the best we can get within our fixed model class, and
3) uncertainty about whether or not the model class itself is appropriate.

In the following, we will discuss different forms of addressing these types of uncertainties.
2.5.1 Point Estimates of the Learning Curve

The simplest type of learning curve model just estimates the mean curve \( \mu_s \) and ignores uncertainty aspects. Given an empirical learning curve in form of some finite samples from this process at different anchors \( S = \{s_1, ..., s_s\} \), a regression model \( f(\cdot; \theta) : \mathbb{N} \rightarrow \mathbb{R} \) is trained with respect to some model class with parameters \( \theta \). A considerable number of different parametric models has been proposed over time for this task. To our knowledge, the first proposal of such classes was made, apparently independently, by Cortes et al. [13] and John and Langley [28] with the three-parametric inverse power law (IPL)

\[
\mu_s = \alpha + \beta s^{-\gamma},
\]

where the parameters \( \alpha, \beta, \gamma > 0 \) need to be optimized to fit the learning curve. Frey and Fisher [21] took a simplified variant of that model \( (\alpha s^{-\beta}) \) and compared it to a logarithmic \( (\alpha \log s + \beta) \), and an exponential model \( (\alpha \cdot 10^{-\beta}) \). While it has been argued, at least for the power-law family, that there is a theoretical foundation for it [69], the considered model classes are typically not theoretically motivated but rather pop up in an ad-hoc manner. For example, Gu et al. [24] extended the above three classes, without a specific motivation, by a vapor pressure model, the Morgan-Mercer-Flodin (MMF) model, and a Weibull model.

Our work does not seek to give a broad overview of different model classes but rather about the usage of such models. We expose the inverse power law model because it is, by a large margin, the most prominent model class and has been advocated by many authors being a good fit for nearest neighbours, SVMs, decision trees, and neural networks [21, 24, 25, 62]. However, other models have been proposed, e.g. based on differential equations [7] or other physical laws [24], and authors have argued that other models, such as logarithmic shape can be a better fit [24, 67]. For an updated and exhaustive overview of used model classes, we refer to [74].

When talking about the “best suited” model class, it is crucial to distinguish between best-fitting and best-predictive models. As was pointed out by Gu et al. [24], the model class that can best accommodate a given set of anchor points is not always the one that will make the best predictions on a high anchor when having been fit only on some initial anchors. To understand the learning curve of a learner on some dataset, one is interested in a best-fitting model class. For extrapolation, one is interested in a best-predictive one.

2.5.2 Range Estimates of the Learning Curves

It has been recognized early in the learning curve literature that incorporating some notion of uncertainty into the model itself is important [51]. While some approaches quantify epistemic uncertainty and others aleatoric uncertainty, we are not aware that approaches make the explicit distinction between these two uncertainty types.

The uncertainty can be expressed in different forms, but all of them lead to augmented estimates that do not only estimate the mean performance \( \mu_s \) at anchor \( s \) but also some range estimate. Here, the range does not refer to the difference between maximum and minimum but simply a range of values that are believed to include the true value (epistemic uncertainty) or values that are expected to be observed with a certain probability if sampling at a specific anchor (aleatoric uncertainty). This range of possible values can be expressed through an estimate of the interquartile range [51], a confidence interval [18], the variance of the estimate [50], and possibly other criteria. The interquartile range is an estimate of the true distribution and does not converge to 0, no matter how many observations we make; it quantifies aleatoric uncertainty. Confidence intervals (both frequentist and Bayesian) estimate variances and capture the epistemic uncertainty. The typical approach for predicting ranges is to fit two models, one from the lower and upper range values at the observed anchors respectively; the range at a new point is the one spanned between the two predictions of these models.

As such, uncertainty is a complicated topic, with many aspects. Range predictions help address the first of the above types of uncertainty if the true learning curve is contained in the model class. To see the importance of the latter criterion, consider a case in which the noise \( \sigma_s^2 \) is 0. Then all of the above range measures will have size 0 as well, and also the predicted ranges will have size 0 at any point, indicating absolute certainty. But the models fitted from these data can even yield wrong predictions on the already known anchors if the model class does not allow to perfectly accommodate them; e.g. an inverse power law model fitted from four points that cannot be captured by any inverse power law function. In other words, we can have 100% certainty in the above sense but still make wrong predictions.

2.5.3 Distribution Estimates of Learning Curves

In a more ambitious case, we can try to learn a full belief model of the learning curve \( C \). In this case, we hold a distribution of the space of learning curves. Such a belief model is, for example, well defined in a Bayesian framework that defined the posterior distribution of models given the observed data and assuming a certain model class. This posterior distribution cannot be efficiently computed exactly, but one can sample from it and thereby try to get a grasp on it [16, 50].

Having an estimate of the distribution of the parameters that describe the model, we can address the second type of uncertainty discussed above. The higher the concentration of that distribution around the mode, the more certain we can be that the parameter estimates and hence the model are stable. Again, we can have very high confidence with respect to this question but still not capture the true learning curve if that curve is not covered by the model class.

Note that again we can trigger here the discussion of what the “best” parameter in the second type of uncertainty means. Implicitly, what is meant here is that we can best capture the behaviour in the observed part. However, this is not necessarily the best model when making predictions on the performance of higher anchors not observed so far.

Finally, addressing the third type of uncertainty is generally very difficult. We are not aware of any approach that does this kind of meta-reflection explicitly. The only possible way to rule out a model class is to reduce the epistemic uncertainty so much, that we can consider the observed performances as de-facto true values and then verify that there is no model in the class that can satisfy all observations. On
an implicit level, the problem can be solved by assuming a large model class that is known to contain the curve by construction. For example, Klein et al. [30] adopt a neural network based on a parametric assumption, which, at least of containing sufficient neurons, is theoretically able to be able to approximate any possible learning curve that can be modelled by a parametric function.

2.6 Relation to Other Types of Performance Curves
We briefly discuss the relation of learning curves to other types of curves and problem settings. These curves are fundamentally different from learning curves, and therefore a more detailed coverage is beyond our scope. Fig. 2 shows various types of learning curves.

2.6.1 Learning Curves in Active Learning
Active learning is a setting where the data scientist can acquire the label of arbitrary instances [65] and hence can actively increase the training set. On a concrete data source with a concrete initial dataset, a specific active learning strategy creates a deterministically extended dataset for any arbitrary anchor. While this allows drawing a curve that plots performance against the number of training instances, this curve is not the one described in Eq. (2), because the datasets are not sampled i.i.d. from \( P_{X \times Y} \) but dictated by the active learning strategy; they are sample-optimized. We do not consider this type of learning curve in this survey.

2.6.2 Learning Curves under Optimal Class Distribution
Similarly, we obtain such a biased learning curve if we do not preserve the class distribution. Weiss and Provost [77] have shown that it can be advantageous to over-sample instances of a minority class if they occur substantially more seldom than instances of a majority class. One can then indeed ask for the best class distribution for a certain anchor. Similar to the active learning case, this creates a new distribution of datasets that does not coincide with \( P_{X \times Y} \) anymore. If we optimize over the class distribution at each anchor, we obtain a curve with the same axis labels as an observation curve but a different semantics (and most likely different values). For the case of two classes, this type of learning curve is obtained by taking the budget-wise maximum of a performance surface as proposed by Forman and Cohen [20].

2.6.3 Learning Curves on Data Streams
Another type of learning curve that violates the implicit assumptions made in Eq. (2) is obtained when learning from data streams, a scenario in which observations are coming in sequentially and need to be processed under strict time and memory constraints [5]. Incremental learners like Hoeffding trees [17] and models induced by stochastic gradient descent are natural solutions to this problem domain. When applied to a data stream, these algorithms explicitly forget an instance once it has been processed. This is not only free memory but also to address the problem of concept drift, i.e. the fact that \( P_{X \times Y} \) changes over time. In such a case, we do not have that \( C(a, s) = \lim_{t \to \infty} C(a, s, t) \) but rather that \( C(a, s) = C(a, s, s) \), and each of the \( s \) instances was considered exactly once. While this produces a kind of observation learning curve, the result is clearly different from the learning curve received in the normal batch setting.

Even though syntactically equivalent to an observation learning curve, data stream learning curves are substantially different and need different treatment. This is due to the (potential) concept drift over the time dimension, the i.i.d. assumption is not guaranteed [15]. From a theoretical viewpoint, extrapolating the learning curve over time to meaningfully predict future behaviour becomes impossible without the i.i.d. assumption. Due to its special nature, the data stream setting is beyond our scope.

2.6.4 Feature Curves
Learning curves always consider a fixed number of features. Instead, one can fix the number of training instances and consider the performance as a function of the number of features. This yields so-called feature curves [25, 24].

Defining meaningful feature curves is conceptually more difficult than learning curves, because of the importance single features can play. For simplicity, consider only observation learning curves for this comparison. In such learning curves, every point \( s \) is associated with the expected performance when using \( s \) training instances. These \( s \) training instances are assumed to be drawn independently and identically distributed from the underlying distribution. So there is, at least in the context of the definition of a learning curve, barely anything like a “more informative instance” (even though this notion clearly exists in the field of active learning). In particular, the order in which instances are drawn is irrelevant. However, in the context of features, often some features are more informative than the others (and again other features that not have been measured may be even more informative). So feature curves already require some kind of averaging over all possible sets of features of size \( s \) that can be formed from a base set of (available) features. This is what is done in [26]. The fact that some features might be more important than other features makes
it hard to model and extrapolate feature curves, as there is no reasonable set of assumptions to build these models on.

Note that feature curves and learning curves can be integrated. For example, Strang et al. [33] look at the combination of the number of instances and the number of features. Since the effect of the number of features and the number of instances on the overall performance is clearly not independent, considering both together is sensible. At the same time, due to the ambiguous semantics of feature curves already discussed above, using such combined curves is not necessarily straightforward for decision making, and we are not aware that the combined curve has been used for decision making so far.

2.6.5 Capacity Curves

Cortes et al. [14] introduce a curve that plots the performance of a configurable learner as a function of the complexity of its instantiation. For example, the learner could be a neural network and the complexity would then be the number of hidden layers, perhaps, for a fixed number of units per layer. In doing this, a fixed dataset size is assumed. In that paper, this type of curve has no specific name, but we dub it the capacity curve because they plot the performance as a function of capacity.

Capacity curves are interesting from a theoretical viewpoint as they allow us to analyze the intrinsic noise level of the given data. More precisely, one can ask for the performance of a learner of some complexity level on, perhaps, an infinite number of data points. If this value can be computed for every complexity level, then we obtain a performance curve over the model complexity. If the number of data points is "large enough", this curve can be assumed to be monotonically decreasing. If we have a maximally flexible learner (such as a neural network) that can, in principle, assimilate any function, then the curve will converge towards the intrinsic noise of the data. That is, no learner can improve over that performance.

3 LEARNING CURVES FOR DECISION MAKING

In this section, we present a framework for categorizing approaches for decision making that use learning curves. We identify three orthogonal criteria along which those approaches can be categorized. The first is concerning the decision situation in which learning curves are used. We discuss these situations exhaustively in Sec. 3.1. The second dimension covers the technical question that is answered about a learning curve in order to support the decision. For example, are we interested in the saturation point or a complete model? These technical questions are sketched in Sec. 3.2 and serve as the basis to structure our literature review in Sec. 4. Finally, different data resources can be used to conduct an analysis with learning curves, e.g. other learning curves or features describing the datasets or the learning algorithms. These resources are covered in Sec. 3.3.

3.1 Types of Decision Situations

Learning curves are an important resource in at least three types of decision-making situations:

1) Quantitative Data Acquisition

The acquisition of how many more labels is economically reasonable? This is a strong link to the field of active learning, which addresses the question of which instances should be labelled next (qualitative acquisition). Another question is whether we should acquire other features instead.

2) Early Stopping (of training).

If we are committed to some specific learner (a learning algorithm and its hyperparameters), we might want to minimize the training time. Specifically, if large data is available and training is costly, we maybe do not want to consider the full data but only train until the saturation point is reached.

3) Early Discarding (in model selection).

Similarly, if we want to select from various models, we want to stop the evaluation of a candidate when we are sure that it is not competitive to the current best solution. We compare the learner performance to that of another learner instead of its own performance on more training investment.

There is a strong methodological overlap in creating a decision basis among all these decision situations. For example, the question of whether more data points would be helpful to improve performance is related to the question of the training size that should be chosen to minimize training effort. Both questions, at their core, ask for the saturation point of the learning curve.

3.1.1 Quantitative Data Acquisition

Quantitative data acquisition focuses on the question of how many training examples should be considered given that they are all sampled i.i.d. from the same source. Quantitative data acquisition does not consider or put attention to the possibility to acquire specific instances, which would be considered in qualitative data acquisition. Qualitative data acquisition is mostly studied in the field of active learning and does not ask whether or how many instances should be acquired but for which instances a label should be acquired. Since active learning undermines the i.i.d. assumption, it generates a different type of learning curve and is not covered in this survey.

The relevance of learning curves for quantitative data acquisition rises from their ability to give insights into intrinsic properties of the data source as well as into the relationship between the number of training examples and the utility of having that number of samples. On the one hand, intrinsic properties refer to the intrinsic noise of the data [14], which tells us about the best possible performance of any learner no matter how much data would be available from the source. If we know that, with the given data, we already achieve a performance close to the intrinsic noise, then data acquisition should focus on the acquisition of new features instead of new instances. On the other hand, the utility is mainly determined by the cost of acquiring (additional) examples, the performance obtained with a certain number of samples, and the cost to train a model with a number of instances [33, 34, 29]. In this economic context, there are mainly five questions that can be considered:

1) Possibility. Can the classification performance be improved by more data?
2) **Potential.** What is the best possible predictive performance given an unlimited number of training observations?

3) **Maximization Principle.** By how much can the predictive performance be improved if there is a budget for a fixed number of additional data points?

4) **Minimization Principle.** How many instances are necessary to obtain a certain degree of predictive performance?

5) **Utility maximization.** Which sample size maximizes a given utility function?

In the context of data acquisition, we typically deal with observation learning curves. Furthermore, one is often not committed to a particular learner, so the performance measure in the above questions is implicitly the best one of a portfolio of learners. That is, one assumes a set of learners that is considered admissible for the prediction task due to external restrictions. In general, due to the ability to parametrize learners, this set is usually infinite. When referring to the performance of the portfolio at a certain anchor point, we are interested in the best performing algorithm from the portfolio at that specific point at the learning curve [48]. This leads to a kind of new type of curve, which can be considered a possibility curve. Of course, if one is for some reason committed to one particular learner, then the situation just simplifies to a portfolio of size 1.

Since data acquisition is not for free, it is sensible to relate potential improvements on predictive performance with the costs to collect the additional labels. So instead of looking only at predictive performance, one now looks at utility of an anchor point. While performance typically only improves with an increased number of observations, the utility also considers acquisition costs, which negatively affect the utility. The goal is hence to decide how many instances should be labelled to maximize utility, i.e. how many additional instances are useful before the additional costs are not justified by the added value [53 79].

### 3.1.2 Early Stopping

Early stopping means interrupting the training process of a learner if the learning curve has converged. The term “early” refers to the fact that the learning process would normally be continued, e.g. because more data is available or the other stopping criteria are not yet satisfied. That is, one uses the learning curve to judge that, despite more available resources or other criteria that would encourage further training, investing more time will not improve the performance of the considered model class any further. The training process can then be stopped “early”, i.e. earlier than if that criterion would not be used. Early stopping has been applied on both observation [28 61] and iteration learning curves [6 23]; in the latter case, it is not only used to save resources but also to prevent over-fitting. In practice, we typically work on a given dataset of fixed size, which encourages the usage of iteration learning curves and therefore one typically resorts to observation learning curves only for non-incremental learners.

For non-incremental learners, early stopping can make sense if we do not already know that the saturation point is bigger than the available dataset size. If we already know that the saturation point is bigger than the dataset size, we would simply immediately train on the full data. Otherwise, we can try to analyze the observation learning curve of the learner for increasing training sizes and stop as soon as we find that performances do not change significantly between two anchors [28 61].

While this type of analysis would, in principle, also be possible for incremental learners, it would imply unnecessary re-starting of the learning process and cause waste of time. Instead, one would consider the iteration learning curve online. In this case, it is analyzed after each training iteration (or a collection of such) whether the learning curve has flattened out (or even degrades as a signal of over-fitting). The only reason that might speak against this is that, in order to build the empirical curve, we of course need to use validation data that cannot be used for training anymore. However, if we have so few instances that this could be an issue, probably no early stopping is required anyway. Hence, the motivation for early stopping mechanisms on iteration learning curves is clear and in fact, part of several standard implementations, e.g. in scikit-learn [58]. But even if an implementation does not come with such a feature, the progress can be monitored from the outside and the training mechanism interrupted if the early stopping criterion is observed.

Generally, this problem can be addressed retrospectively (i.e. stopping after observing the saturation point) and prospectively (i.e. stopping based on a model that predicts the saturation point).

#### 3.1.3 Early Discarding

In many setups, the learner itself is a matter of optimization. That is, we consider a (maybe infinite) set of learners, e.g. a finite set of algorithms each of which can be instantiated with a possibly infinite number of parametrizations. The question is which learner is the “best” for the given data. The underlying task is commonly known as model selection.

However, the question of which learner is “best” is in fact ill-posed. What is colloquially meant by this formulation is that the performance is assessed through a cross-validation technique with a high portion of training data, typically between 70% and 90%. This is typically the case in hyper-parameter optimization or Automated Machine Learning (AutoML). The question effectively answered there is which of the learners is best for the respective anchor point used for training in the cross-validation. That this concept is somewhat odd becomes apparent already from the fact that the learner that is best for a train size of 70% of the dataset could be another than the one that is best for 90%. So even when only looking at the given dataset, what is the “best” model for that dataset depends on the validation mechanism. Even more, the chosen model is eventually trained on the full dataset, and it could be the case that the model with the best out-of-sample error when using 100% of the data for training is even another one than the one that is best according to the cross-validation technique. This is an additional motivation for analyzing observation learning curves to select a model, instead of simply picking the best performing model on the validation data.

The core task of early discarding is to abort the evaluation of a candidate as soon as it becomes apparent that
it cannot be relevant in the model selection process. In the simplest case, we define some percentage of the data, e.g. 90%, and want to early drop a candidate if, during its evaluation, it turns out that it will not be better than the current best model on that training portion. Early discarding has been applied to both observation \[16, 69\] and iteration \[16, 69\] learning curves. In contrast to the early stopping situation, for early discarding, there is no clear preference whether to use observation or iteration learning curves for non-incremental learners, because using iteration learning curves might cause non-comparability with other models that are evaluated based on observation learning curves.

Early discarding is more aggressive than early stopping in that it does not even wait until the learning curve has converged. An important question is then whether the estimate of the performance at the target size, e.g. 90% of the training data, is sufficiently accurate so that the truly best model is not discarded.

Situations in which early discarding plays a role can be further classified into horizontal and vertical scenarios (and a mixture of the two):

1) **Horizontal Model Selection.** Horizontal model selection implies that a finite set of learning algorithms among one is to be selected has been fixed, and now empirical learning curves are grown iteratively for the whole set or shrinking subsets of it. A line of research that looks at horizontal decision making is that around successive halving \[27, 72\]. However, these approaches don’t look at learning curves but only at the last anchor point under consideration.

2) **Vertical Model Selection.** Vertical means that the set of learners is generally not limited to a finite set and the set of evaluated learner candidates evolves over time. Learners are evaluated one after another. Each learner is evaluated in an iterative fashion to grow a learning curve to allow for early discarding. Examples are the early discarding routine for deep networks in \[16\] or, more generally, for learning-curve based cross-validation \[48, 49\].

3) **Diagonal Model Selection.** This case is similar to the vertical decision-making situation with the difference that one does allow to continue the evaluation of a candidate at a later point of time. Hence, candidates are not evaluated one after another, but the evaluation of different candidates can be interleaved. Examples are Bayesian optimization-based approaches to pause and continue evaluations of (not necessarily iterative) learners \[29, 69\]. If learners are not iterative, then they are trained from scratch again up to an increased budget.

Another approach that addresses this kind of decision-making situation is Hyperband \[41, 51\] and Bayesian optimization based on progressive sampling \[81\]. However, both of these approaches do not consider learning curves even though they implicitly construct them. Decisions are taken just based on the observations of the largest anchor point considered so far.

### 3.2 Technical Questions Asked About Learning Curves

A plethora of questions can be asked about learning curves. Fig. 3 gives an overview of a selected set of basic such questions. The figure is organized in three layers (depth dimension) corresponding to the three types of estimates discussed in Sec. 2.5.

Each layer consists of a set of questions that can be posed about learning curves. From top to bottom, the questions are ordered by complexity, and an arrow from A to B indicates that B is a more general question and answering it implies an answer to A. In the simplest case, we can answer a binary question. There are four relevant questions, i.e. (i) whether some specific anchor point, e.g. the dataset size, is beyond the saturation point \(s_{\text{sat}} \leq s_{\text{ref}}\), (ii) whether the performance of a learner at the saturation point is better than some baseline \(p_{\text{sat}} < \tau\), (iii) whether the performance \(p_{\text{ref}} := C(a, s_{\text{ref}})\) at some reference point \(s_{\text{ref}}\) is better than some threshold \(\tau\), or (iv) whether a specific anchor point is beyond the utility-based stopping point \(s_{\text{sat}}^{\text{stop}} < s_{\text{ref}}\). The binary questions do not ask for concrete performances, which gives rise to a second level of complexity, where the concrete values of \(s_{\text{sat}}\) and \(C\) are being modelled. Here, the reference performance \(p_{\text{ref}}\) is the performance at a fixed reference point, often the number of training samples available for cross-validation. At a third level, we could ask for a model of the whole learning curve of a learner \(a\), i.e. \(C(a, \cdot)\) for observation learning curves \[18, 21, 24, 28, 51\] or \(C(a, s, \cdot)\) for iteration learning curves given a fixed (sample) anchor \(s\). We can ask a similar question for the utility curve. While the question is on the same level, it is more general, as it is based on the learning curve itself \[33, 34, 28\] and combines it with other information such as acquisition costs. Finally, at the fourth and most general level, we could ask for a model of the whole performance function \(C\), i.e. the model of the learning curves across all learners \[29, 30, 69\]. Analogously, this question could be asked for the whole utility function \(U\), which is arguably the most complex and general question that can be asked, although we are not aware of any works that have done so.

Since all of the above questions are answered based on observational statistics, we can also merge noise and uncertainty aspects into the questions. In fact, all but the binary questions aim at estimating one or more quantities of the learning curve model. In the simplest case (blue layer, cf. Sec. 2.5.1), we only get point estimates, i.e. the estimate of \(C\) at one or a set of points. Since these estimates are always afflicted with uncertainty, it is reasonable to ask for quantifications of this uncertainty. One form is to express probabilistic bounds like confidence intervals around \(C\) \[18, 32, 48, 49\] (cf. Sec. 2.5.2). In the most general form, we can ask for a full belief model of the learning curves, specifying a probability distribution over the values of \(C\) at one point or a set thereof \[16, 29, 30, 69\] (cf. Sec. 2.5.3).

It is common practice to solve rather simple questions by implicitly answering more complex ones. For example, a typical question in the context of model selection is whether the saturation performance of a candidate learner will beat a known baseline \[16, 37, 72\]. This is in fact the binary question of \(p_{\text{sat}} \leq p^{\star}\), where \(p^{\star}\) is the best-known
Fig. 3. Technical questions that can be asked on learning curves.

performance. Often this question answered by estimating $p_{sat}$ explicitly and then comparing it to $p^* [37, 73]$, which is in fact an answer to a slightly more complicated question. In [16, 69], for estimating $p_{sat}$, even a whole explicit curve model is built to estimate $p_{sat}$. The approach answers this binary question by building an entire learning curve model and then derives the binary answer from it. The rationale behind this is the idea that one often needs rather complex models to find a high-quality answer to a simple question, and it is just a side effect that one can then even answer other questions with those models.

It is likewise noteworthy that a couple of approaches implicitly create empirical learning curves without using them to answer any question. More precisely, there is a series of approaches for model selection that iteratively increase evaluation budgets on a set of competitive candidates. Examples are wrapped progressive sampling [72], successive halving [27], and Hyperband [41]. These procedures implicitly create empirical learning curves but do not analyze them; they only consider the performance of the highest budget evaluated so far. It seems sensible to extend approaches in this way to save computational resources on candidates that have converged.

3.3 Used Data Resources for Inference

Above we have discussed a series of questions that can be asked about learning curve properties, which in turn are important for decision making in a specific context. Of course, answering these questions requires certain informative resources. In a concrete decision-making situation that might involve learning curves, we are typically confronted with a dataset and a learner or a portfolio thereof. We call this the target dataset and the active learner. That is, we want to say something about the learning curve of the active learner in a domain of which we have a finite (the target) dataset $d$ available.

Fig. 4 shows the types of data resources that can be used to answer questions about learning curves. We can utilize empirical learning curves that were gathered on the target dataset, empirical learning curves that were gathered on other datasets, dataset meta-features and features describing the learners. Learning curves gathered on the target dataset come at a certain computational cost, as they have to be generated during the process. As such, typically when modelling the active learner on the target dataset, a partial empirical learning curve is constructed, which can then be further extended [16, 35, 61]. When using learning curves of other datasets, those are usually available up to the full number of available data (with a specific schedule of anchors), because these curves could be prepared long before the target dataset became available [35, 40, 73]. Both learning curves of the active learner and other learners can be utilized for this. In the context of model selection, typically various learners are being evaluated, so we can acquire various learning curves of other learners on the target dataset [3, 11, 29, 30, 69].

Another type of data resource that can be used are meta-features on the datasets and learner features. These are measurable qualities of the dataset and learner respectively, and these can indicate how similar certain datasets (or learners) are. This gives the decision-making algorithm a sense of which learning curves are more informative for the active learner and target dataset. To the best of our knowledge, the only line of research utilizing meta-features for learning curve modelling is the work of Leite and Brazdil [39, 40]. The description of learners through features for the sake of model prediction is specifically prevalent in the analysis of iteration learning curves [3, 30, 69]. The development and analysis of meta-features is a research field in its own right; for more information, we refer the reader to [9].

4 Literature Review

We organize approaches by the key problem they resolve on a rather abstract level and independent of the purpose or type of decision situation in which they were presented. As such, we organize it along the taxonomy presented in Fig. 3 i.e. the technical question asked about a learning curve. The motivation to use this axis instead of, for example, the type of decision situations, is that the same approach can be
Estimating the limit or saturation performance

4.1 Identification of Saturation Performance

alongside the approaches for question 6 in Sec. 4.6. The high-level questions by which the approaches are sorted are (the questions can be asked with respect to a single learner or a whole set of learners, the latter referring typically to the best learner in the set):

1) What is the saturation performance \( p_{sat} \)?
2) What is the saturation point \( s_{sat} \)?
3) What is the utility-based stopping point \( s_{sat}^u \)?
4) What is a lower/upper bound of the learning curve at some fixed point? (bound on \( C(a, s) \))
5) What is the value of the learning curve at a specific (fixed and known) point? \( C(a, s) \) or \( C(a, s, t) \)
6) What is the value of the learning curve at any (queryable) point? \( C(a, \cdot) \) or \( C(a, s, \cdot) \)
7) What is the utility at an arbitrary anchor point?
8) What is the value of the learning curve at any queryable point of any queryable learner? \( C(\cdot, \cdot) \)

For each approach, we always consider the most general problem that is solved by it, independently of how this approach is then used in the context of a paper. For example, Domhan et al. \[16\] “only” decide whether the saturation performance of a learner beats some threshold but develop a whole learning curve model and are hence discussed alongside the approaches for question 6 in Sec. 4.6.

4.2 Identification of Saturation Point

Identifying the saturation point \( s_{sat} \) is useful for observation learning curves in the following use cases:

1) Early-Stopping with observation learning curves: Which portion of the available data is necessary to obtain saturation performance? This applies if \( |d| > s_{sat} \) or if the relationship between \( |d| \) and \( s_{sat} \) is unknown and training on full \( d \) is potentially undesirable.
2) Early-Stopping with iteration learning curves: How many iterations are necessary until performance converges? This applies for example to neural networks and other incremental learners.
3) Data acquisition: What is the number of additional training points necessary to obtain (near-optimal) performance? This applies if \( |d| < s_{sat} \). The question can be posed for a specific learner or a portfolio.

4.2.1 Retrospective Approaches

The simplest way of determining the saturation point \( s_{sat} \) is to incrementally build a learning curve and stop as soon as it is believed that the saturation point has been exceeded. If we do this, we can estimate that the saturation point lies between the last two anchors. For iteration learning curves, determining \( s_{sat} \) comes for free as a side-product of the training procedure and is commonly done when training neural networks \[6, 23\], while it requires restarting and is potentially costly for observation learning curves.

John and Langley \[28\] define a dynamic sampling approach to determine the \( s_{sat} \) for observation learning curves. One simple approach mentioned in that paper is to observe whether the performance has become worse on the last sampled anchor. If so, one might consider to observe empirical evidence that the saturation point has been exceeded, e.g. it should be somewhere between the last two anchors. However, the authors already argue that preliminary results indicate this approach often stops too early. This can happen due to random effects in the evaluation process of the anchors, which implies that empirical learning curves are unstable. On the other hand, one could argue that this approach stops far too late because it can require quite some iterations until, by chance, the observed performance is worse than the one of the last iteration. Therefore, John and Langley \[28\] propose also a model-based approach to...
avoid this problem, which we however discuss in Sec. 4.6 because it involves a model for the complete learning curve. Of course, given such a model, we can then simply query the expected performance and compare it to the performance at the last anchor.

Provost et al. [61] address the stability issues and also (some of) the efficiency issues of the above trivial approach in a scheme they call progressive sampling. Just like dynamic sampling [28], progressive sampling induces models for each anchor in the schedule until the convergence of the learning curve is detected. There are two main differences between the two approaches. First, the authors propose to use geometrical instead of arithmetic schedules, i.e. a schedule of the form \( b^k \) instead of \( bk \), where \( b \) is a constant and \( k \) is the position of an anchor in the schedule. They prove that every geometric schedule is asymptotically optimal in terms of runtime; that is, every such schedule has the same asymptotic runtime as the schedule that knows only evaluates on \( s_{sat} \). This optimality only holds in the asymptotic calculus, and, in practice, there are better and less good geometric schedules. In fact, Provost et al. [61] propose a dynamic programming approach (called DP), which efficiently computes the cost-optimal schedule based on a prior distribution on \( s_{sat} \) and a given training runtime model. The second difference is that Provost et al. [61] check whether the saturation point has been reached using a method called linear regression with local sampling (LRLS), which samples not only at but also closely around anchors to estimate the slope at an anchor and stop if the slope is close to zero.

The practical benefit of the LRLS scheme is not entirely clear for observation-based learning curves. First, since the slope is also based on empirical values, it is not clear from a theoretical viewpoint that the criterion is necessarily better than the naive approach suggested by John and Langley [28]. Second, the approach is more expensive than the naive one, because more observations need to be sampled; this can be a substantial factor especially for large anchors. In fact, the utility of LRLS is unclear in general since it could be easily more expensive than a full evaluation. This can be seen with a simple calculation, in which we assume roughly linear training time complexity: Suppose a costly anchor at 40% of the overall data size. If we draw only 2 additional samples around this anchor, then the runtime is around 120% of the runtime we would have had if we would train on the full data once. While Provost et al. [61] argue that LRLS only adds a constant factor to the runtime, Sarkar et al. [64] reasonably argue that this factor is often prohibitive in practice.

The approaches in this section explicitly assume that more data than \( s_{sat} \) is available. This implies that they can indeed only be used for early stopping but not to answer data acquisition questions. Still, if a learner does not attain the saturation performance on the given data, the approaches can detect this at the cost of the additional evaluations at the non-final anchors.

Concerning the stability of estimates, Beleites et al. [4] point out the necessity to have an estimate for the confidence interval not only for the anchors in the training schedule but also on the test data. They argue that the confidence intervals are of particular necessity when deciding whether or not generalization statements can be made in a reasonable way for a classifier. This changes the notion of the stopping point to, perhaps, a confident stopping point. The true stopping point may be reached early, but since there is noise in the observations, one can maybe not be sure that this point has been reached. Typically, the confidence intervals are large on small anchors and then contract for increasing anchor sizes. Based on credible intervals, the authors hence propose to choose the anchor point that achieves a sufficiently narrow interval on the test data.

Ng and Dash [56] address the impact of class imbalance on the performance of a learner. That approach hypothesises that, without further knowledge, the presumably best class distribution is one in which all classes have the same distribution. They modify the aforementioned progressive sampling scheme by creating train sets at each anchor in the schedule in which all classes have the same distribution. For anchors of sizes that would require more instances of a class than are available in the existing data, random instances of that class are replicated until the class balance is established again. Note that while the work is based on the findings by Weiss and Provost [77] (cf. Sec. 2.6.2), they apply a different strategy. Instead of optimizing over the class distribution for a given anchor size, they try to find the stopping point under the premise that the training set will always be balanced.

Regarding the stopping point of iteration learning curves, a common technique is to separate some data that is not used for training but to compute an iteration learning curve online in order to detect convergence [6].

4.2.2 Projective Approaches
An different idea to obtain the saturation point \( s_{sat} \) was proposed by Leite and Brazdil [35][36] through the notion of meta-learning [9]. Similar to Provost et al. [61], a geometric schedule is used. The assumption is that we already know the performances of the active learner at all anchors in the schedule on other datasets. The idea is now to compute, on the target dataset, the performances only for the very first anchors and then to predict the saturation point by aggregating the (known) saturation point on the \( k \) most similar learning curves of the other datasets. The distance measure here is the sum of differences between the curves at the initial anchors; the concrete anchors used in their paper are \((91, 128, 181, 256, \ldots)\), corresponding to the powers of \( \sqrt{2} \). The authors consider different aggregation measures such as mean and minimum [35] and the median [36].

One issue of this approach discussed also by the authors themselves is that, among the \( k \) nearest neighbour curves, some or even all of the curves can be substantially different from the partial learning curve on the target dataset. This jeopardizes the approach of using the \( k \) nearest learning curves to predict the stopping point as such. One remedy to this problem is proposed in a follow-up work [35] discussed in Sec. 4.5.1 in which the curves are not taken directly but are adjusted via a concept called curve adaptation.

4.3 Finding the Utility-Based Stopping Point
The problem of identifying the utility-based stopping point was, to our knowledge, first addressed by Meek et al. [46] and was also independently investigated by Weiss and Tian
In these papers, a retrospective is applied. In fact, the idea is similar to the aforementioned concept of progressive sampling [61], except that the curve under examination is the utility curve and not the learning curve. In contrast to the learning curve, the utility curve does not plateau but starts to deteriorate after its peak, which is the dataset size of optimal utility for observation learning curves and the optimal number of iterations for iteration learning curves; the latter is however not investigated in the above papers.

The main difficulty with the concept of utility in the learning curve context is to find a unifying scale for the costs of data acquisition and training time on the one side and the model performance on the other side. Meek et al. [46] avoid this problem by adopting the notion of implicit utility through the comparison with a baseline. They stop the algorithm as soon as the ratio between the benefit improvement and the augmented runtime drops below a pre-defined threshold.

In contrast, Weiss and Tian [78] compute an explicit utility, and the algorithm stops as soon as the observed utility decreases for the first time, which is taken as the indicator that the utility-based stopping point has been passed. The authors adopt the concept of the net utility of a potential classifier, which is the difference (in utility) between predictive performance (under a hypothetical number of training instances) and the cost to acquire the (additional) instances. Notably, the assumption is that the user has no grasp on the instances for which labels will be acquired next, which contrasts the approach from active learning. To merge different types of inconveniences, e.g. acquisition costs and prediction errors, into a single utility measure, the user has to define costs per unit, e.g. costs for acquiring a single usable training instance and costs for making a wrong prediction. Furthermore, the authors consider the problem of deciding online whether or not to acquire more data and, in the affirmative case, how many instances should be considered in the acquisition batch before re-deciding. The latter effectively corresponds to deciding upon a progressive sampling scheme [61]. However, the paper does not analyze the effects of fixed costs per batch, which effectively implies that one could set batch sizes to 1 without any consequence.

A consecutive version of that paper adds to the costs of a point on the learning curve also the CPU cost for model induction [29]. The original paper only considered acquisition costs and the prediction error. This model seems to be directly suitable also for iteration learning curves, in which the data acquisition costs would be 0.

In all of the above approaches, the usage of the empirically gathered learning curves for decision making is minimal. In fact, the approaches ignore all except the last two points on the learning curve. In this sense, and in terms of the stopping point approaches, the above works are of a retrospective nature. No model of the learning curve is built, and no projections of errors on bigger training sizes are made, which, for example, could make sense to predict that this utility peak event will occur in the future or even only in the next iteration. Such an approach was proposed by Last [63], which we discuss in Sec. 4.7.

### 4.4 Performance Bounding at Fixed Point(s)

Performance bounding tries to give a lower or upper bound on the performance value at some specific anchor. Answering this question is important to make high confidence decisions on early discarding as discussed in Sec. 3.1.3. Formally, we denote it as $C(a, s)$, typically with $s = \lfloor \frac{d_{tr}}{t_{tr}} \rfloor$ being the size of the dataset intended for training.

Performance bounding is intuitively a simpler problem than performance prediction (Sec. 4.5), but there are also, maybe, higher expectations with respect to the accuracy of the assertion. When a performance bound is expressed, one would expect that the true value is certainly (or at least with high probability) at least or at most as high as specified. In performance prediction, there is not this kind of expectation of a guarantee; if at all, we get a value of uncertainty or variance to interpret the prediction. To this end, performance bounds are arguably better suited for early discarding with confidence than performance predictions.

Data Allocation using Upper Bounds (DAUB) addresses this problem [63]. Given a set of several configurations, it first runs all configurations on two anchors of the dataset, effectively building the initial segment of the learning curve. Based on this initial segment per configuration, it determines for each learning curve an optimistic upper bound. This upper bound is determined by calculating the linear regression slope of the last two segments and extrapolates this to the full size of the dataset. As such, there is an optimistic upper bound on the performance that a configuration can obtain. After that, it goes into the following loop: It runs the most promising configuration on a larger sample size and updates the performance bound. It reevaluates which configuration has the most potential at that moment, and continues with assigning more budget to the most promising configuration until one configuration has been run on the entire dataset. As such, this is an example of horizontal model selection.

Alternatively, learning-curve based cross-validation uses a similar approach, but address this in a, more flexible, vertical setting [48, 49]. The method explicitly assumes that observation-based learning curves have a convex behaviour. The convexity of the curve allows deriving a best-case extrapolation from a partial empirical learning curve. The convexity assumption is used to extrapolate the empirical learning curve in a linear fashion and prune learners when they can no longer improve on the best-known solution.

It is noteworthy that some approaches in the area of hyperparameter optimization adopt a kind of implicit performance bounding. This applies to so-called greedy racing algorithms. To our knowledge, the first algorithm with this purpose was introduced in the Wrapped Progressive Sampling procedure (WPS) by van den Bosch [72]. Those algorithms pretend that the performance of an algorithm at the target size can be bound by the performance of some other algorithm in a given portfolio at that size. More precisely, one takes a specific percentile $p$ at the highest evaluated anchor size $s$, often the median. It is then assumed that all algorithms with a performance at $s$ corresponding to a percentile worse than $p$ can be bound by the performance of the best candidate in the portfolio and hence be discarded.

The performance of that other algorithm at the target size is
unknown, so no explicit bound is computed. Van den Bosch even computes the percentile $p$ dynamically based on the distribution of the observed scores. Very similar algorithms have been introduced more recently with Successive Halving [22] and Hyperband [41]. While these are general hyperparameter optimization mechanisms, a similar approach specifically tailored for machine learning pipeline optimization was proposed by Zeng and Luo [51].

4.5 Performance Prediction at Fixed Point

The problem of predicting the learning curve value is at its core a regression problem. For observation learning curves it is about predicting $C(a, s)$, and for iteration learning curves it is predicting $C(a, s, t)$ for a fixed learner $a$ at a fixed anchor $s$ or $t$. The attributes are simply the performance values at different (cheap) anchors and, maybe, additional contextualizing attributes. That is, one explicitly or implicitly generalizes over datasets or over learning algorithms (or both). In the case of generalizing over datasets, one trains a model for each considered learner, and the utilized database should only contain learning curves of that learner (on other datasets). In the case of generalizing over learners, we only use a single model for the target dataset, trained from learning curves on the same dataset belonging to other learning algorithms.

We organize this section by how explicit the generalization is made over one of the two concepts. If no additional attributes are available, one implicitly assumes “suitability” of the entries in the database to predict values in a new situation. In this case, there is no explicit context, and we are generalizing implicitly either over datasets or learners. Since no explicit contextualization exists, those approaches can be used for both purposes, no matter for what they were used originally. In contrast, additional contextualizing attributes can describe the dataset, i.e. we have meta-features of the datasets available, or they can describe the learning algorithms to which the learning curve values belong. In principle, one could have both types of additional attributes, but we are not aware of any approaches that adopt both of them at a time.

4.5.1 Generalization Without Explicit Context

Generalization from learning curves without explicit context means to predict the performance of some learner on some dataset based on previously acquired learning curves that are not equipped with additional information, i.e. features describing the dataset or the algorithm used to produce them. To justify the prediction model, the existing empirical learning curves either stem from the same algorithm on other datasets, or from other algorithms on the same dataset, which implicitly qualifies them to be relevant for the new task. In other words, one simply uses a set of “unannotated” existing learning curves as aids to predict the behaviour of a new, only partially known, learning curve.

The first approach we are aware of to predict the performance of a learner at a fixed size was presented by Leite and Brazdil [37]. This approach is called meta-learning on data samples (MDS). MDS is meant to decide which of two algorithms is the better choice on a given dataset and could hence be used for early discarding in a racing-like approach. However, the fact that two algorithms are compared does not really become relevant in the approach and it can be fairly considered as an approach to predict the performance of an algorithm on a pre-defined number of data points.

The formal basis of the work is the same as the one introduced in their previous work [35] [50]. Again, they assume that empirical learning curves (with standardised anchor sizes) for the learner under examination are already known for other datasets, and also they preserve the idea of identifying the distance between the target and the other datasets based on the sum of squared distances over the already known performances at anchors of the target dataset. Once the most similar $k$ learning curves have been identified, the authors propose to predict the performance of the learner on the target dataset as the mean over the performances of the $k$ nearest neighbours (on the anchor size of interest).

Leite and Brazdil [37] acknowledge that this method can result in poor predictions because even the closest learning curve on other datasets can still be substantially different and propose learning curve adaption as a remedy. Instead of forming the mean directly over the target anchors of the nearest neighbour empirical learning curves, the authors first scale those curves to make them more similar to the shape already observed on the target dataset. To this end, they compute a scaling constant under which the overall anchor-wise distance is minimized and then multiply all the scores with this constant. This version of the MDS algorithm is called AMDS (probably for Adaptive MDS).

Interestingly, one can argue that this adaption technique could be applied both before or after determining the $k$ nearest neighbours. Doing it before would potentially lead to other (and maybe better) nearest neighbours because then the neighbours are determined more with respect to the shape of the learning curve, and the offset plays much less of a role. However, in the above paper, the adaption is done after retrieving the neighbours.

The authors extended the approach further by creating an online sampling scheme with the SetGen algorithm [38]. SetGen is an online adaption of AMDS in that, after each acquired anchor, it is decided whether and which anchor should be evaluated with each algorithm; in this, they take into account both the (believed) accuracy and the runtime. This can be seen as a way of racing between the algorithms. The potential of each “horizontal extension” is judged based on the meta-learning database.

An implicit assumption of all approaches in this line of research is that the datasets in the database from which performances are extracted need to be at least as big as the target dataset. This issue was first explicitly treated in the pairwise curve comparison approach (PCC) [73]. This algorithm builds upon the works of Leite and Brazdil [40] and implements a voting scheme to identify the best learning algorithm of a portfolio; votes are distributed based on wins, which in turn are determined based on the predicted performance at the full dataset $d$ using the aforementioned approaches. van Rijn et al. [23] explicitly discuss the issue if the sizes of datasets in the database and the target dataset are not identical. In general, the point for which predictions must be made is typically not one of the anchors, e.g. is typically not a power of 2 but simply 90% of the given
dataset size; maybe to estimate the performance of the learner in 10-fold cross-validation. A remedy is to resort to the "closest" available anchor in the schedule. However, if the highest anchor available for another dataset is much smaller than the required training size of the target dataset, then it is unclear how that curve should be used.

Chandrashekaran and Lane [11] more recently proposed similar work. While the goal here is to generalize over algorithm configurations instead of datasets, the approach essentially ignores the configurations and generalizes from learning curves without explicit context. The approach is indeed very similar to the ones presented a decade before by Leite and Brazdil [37] since it computes the most similar other learning curves in the portfolio and obtains a prediction based on the average over those curves. The three differences are that Chandrashekaran and Lane [11] (i) also consider uncertainty in the prediction based on the variance in the neighbourhood, (ii) that they adopt an affine instead of a linear transformation of the existing learning curves and apply these before selecting the most similar ones, and (iii) that they don’t use a fixed schedule but, due to the focus on iteration learning curves, simply a continuous schedule that is stopped as soon as there is enough evidence that the target performance will not be better than a current threshold. The distance between two curves here is mainly the euclidean norm between the vectors describing the performances at the anchors.

An approach that predicts values at all anchors was recently presented by Cardona-Escobar et al. [10]. The authors adopt a series of support vector regression models, one for each anchor not evaluated so far. It is not entirely clear with which data the models are trained, but we presume that it follows the same logic as Chandrashekaran and Lane [11] and use the fully known learning curves of previously evaluated neural network configurations to do so. The interesting aspect is that similar to chaining in classification, they use as inputs for the j-th future anchor not only the actually known partial learning curve values but also use the predictions for the anchor points predicted before j.

4.5.2 Generalization With an Explicit Dataset Context

The only works we are aware of that realizes an explicit generalization across datasets was proposed by Leite and Brazdil [39] [40]. Similar as before, they pick the k nearest datasets to measure the relevance of known complete learning curves of other datasets for the performance prediction on the target dataset. The main difference is that, in addition to the contribution of the partial learning curve itself to the distance, they also use the distance between the datasets in terms of their meta-features. More precisely, they compute the Manhattan distance between seven range-normalized dataset meta-features, e.g. dataset size, number of symbolic and numerical attributes, etc. The overall distance between the datasets is then the sum of the distance between the partial learning curves and the distance in terms of meta-features. This work was marginally refined in the Selection of Algorithms using Metalearning approach (SAM), which applies the same logic but assigns a weight to each of the two distance sources [40]; the weight is however implicitly assumed to be set to 0.5.

4.5.3 Generalization With an Explicit Algorithm Context

Generalization across algorithms only considers the target dataset and assumes that a number of (full) empirical learning curves on that dataset is already available for different algorithms. The explicit generalization requires that the previous learning curves are explicitly associated with features describing the algorithm to which they belong.

The first explicit generalization approach we are aware of was proposed by Baker et al. [3]. In that paper, the authors use features describing both the learning curve (including up to second-order-differences) and the algorithm. The approach is specialized in predicting the performance of neural networks, so that the features describe the architecture (number of layers and weights) as well as the hyperparameters of the learning algorithm (such as learning rate, learning rate decay, etc.). They adopt linear and kernel-based support vector regression machines, random forests, and simple linear regression based on ordinary least squares. Even though the authors advocate the use of kernel-based support vector regression machines models, they find that simple linear regression also often compares highly competitive for this prediction task. In all cases, scores of roughly 90 or more can be achieved for the $R^2$ statistic.

Following this idea, Long et al. [42] study the predictability of learning curves of networks using also textual descriptions of the architecture. Indeed, the architecture description by Baker et al. [3] is rather simplistic and only immediately well suited if all layers are of the same type, e.g. dense layers, and have the same number of neurons. Long et al. [42] report substantial prediction improvements for convolutional network architectures compare to the approach taken by Baker et al. [3].

4.6 Performance Prediction at Any Point

We now discuss approaches that build a full learning curve model for a specific learner. That is, they model the function $C(a, \cdot)$ or $C(\cdot, s, \cdot)$ for a fixed learner $a$. We divide the approaches along the layers shown in Fig. 3. In Sec. 4.6.1 we discuss approaches that simply provide point estimates of the curve. Next, Sec. 4.6.2 presents approaches that explicitly treat those estimates with uncertainty and introduce a notion of bounds on them. Finally, Sec. 4.6.3 covers the case of a belief model in the space of learning curves.

4.6.1 Point Estimates

To our knowledge, the first time that a learning curve model was fit from observed data was presented by Cortes et al. [13]. In that paper, the three-parametric inverse power law shown in Eq. (6) was used to build an iteration learning curve. The usage of the power-law is justified with the findings in the statistical mechanic framework [66] and used to predict the predictive performance at the full data set size (for two neural network architectures on the NIST dataset). The authors find that the predictive performance on 60k instances can be almost perfectly predicted using the inverse power law. Unfortunately, it does not get entirely clear from the description on which anchors the estimates are built upon. Notably, this is the only paper we are aware of that fits both a train- and test-error curve. The paper reports some
information about noise through boxplots on the curve but does not explicitly incorporate them into the model.

A similar approach was taken by John and Langley [28]. Just as in the case of Cortes et al. [13], the model is used to estimate the performance of a learner on the full data with the goal to early stop the training procedure. One difference is that the model is adopted for an observation learning curve instead of an iteration learning curve. With respect to convergence detection, John and Langley [28] employ the “Probably Close Enough criterion”, which detects convergence if the probability that the accuracy of a model trained on the full data will be at most some \( \varepsilon \) worse than the current model’s performance is less than some \( \delta \). However, the paper itself then does not adopt a notion of probabilities but simply stops if the performance on the full data predicted by an inverse power law model is not by at least \( \varepsilon \) better than the currently observed performance; they call the approach “Extrapolation of Learning Curves” (ELC). In other words, the uncertainty is not quantified.

The inverse power law has been used in a lot of applications later on and also rather recently. For example, several approaches have used the inverse power law to model the performance of neural networks in different domains [11, 12]. It is noteworthy that recent works jeopardize the usage of any type of commonly used models for neural networks, at least in the initial parts of the curve, due to the (sample-wise) double descent [53]. At least for certain combinations of architectures, datasets, and training procedures, there is empirical evidence that the learning curve exhibits non-monotonic behaviour, which contradicts all existing learning curve models like the inverse power law model.

4.6.2 Range Estimates

The above approaches basically treat the predictions \( \hat{f}(s) \) as if they were the true values \( \mu_s \) and without explicitly questioning their quality. In contrast, we now discuss approaches that do cater for some kind of uncertainty of the prediction quality. Typically, the uncertainty is either expressed through ranges, e.g. inter-quartile ranges, or confidence intervals.

Mukherjee et al. [51] build inverse power law models in the domain of DNA data. The main contribution of that paper is to analyze the appropriateness of the inverse power model on eight medical datasets. Instead of using a metric like \( R^2 \), they construct “envelopes” around the mean learning curve consisting of the q25 and q75 curves fitted from those statistics respectively. This way, a learning curve model including information about dispersion is obtained. Experiments are conducted for a support vector machine on 8 medical datasets in which the leave-one-out validation result (estimate of the learning curve on \(|d| - 1\) data points for training) is compared to the boundaries suggested by the model. Having an explicit model for the q25 and q75 curves, one can obtain for an arbitrary anchor \( s \) not only a point estimate of \( \mu_s \) but an estimate of the inter-quartile range of \( f(s) \) itself, which is much more informative.

Going into a similar direction, Figueroa et al. [18] modify the aforementioned approach in two ways. First, different anchors are associated with different weights, usually to assign higher weights to bigger anchors since they are more informative. Second, implicitly assuming a standard Gaussian distribution of the observations as in Eq. (5), they compute a 95% confidence interval to describe the uncertainty rather than the interquartile range. Based on this information, for a query point \( s \), they predict a confidence interval instead of a point estimate. Figueroa et al. [18] also applied this approach to medical data, just as Mao et al. [45] for EEG data. More recently, it was also successfully used for sensor communication [57]. The latter however seek to predict a reasonable sample size, which is perhaps more reasonably addressed by the utility-based approaches discussed in Sec. 4.3.

Recently, Koshute et al. [32] have used the inverse power law to predict the minimum anchor point on which a learner must be trained to reach near-\( p_{sat} \) performance with a given desired confidence. This can be seen as a kind of combination between the above two approaches. Similar to Figueroa et al. [18], they compute the confidence interval at all anchors. However, instead of using these to estimate confidence intervals at arbitrary points, they fit a single curve on the lower bounds of the confidence intervals at the known anchors. The resulting model is not used to make predictions on arbitrary anchors but to compute the cheapest anchor that will obtain with a pre-defined probability (size of confidence-interval) a performance that is \( \varepsilon \)-close to \( p_{sat} \), where \( \varepsilon \) is a hyperparameter controlled by the user.

The idea of computing confidence intervals is also adopted by our learning-curve based cross-validation technique (LCCV) [48, 49]. In LCCV, a three-variable inverse power law model is created to decide on whether or not to skip intermediate anchors and evaluate on the full dataset size. The usage of confidence intervals is however different than in the above cases and not used for the inverse power law model itself. In contrast, the confidence bounds are used to compute the range of possible slopes of the learning curve between two anchors.

4.6.3 Estimate Distributions

While the above approaches somehow take into account uncertainty when making predictions at new anchor, we now discuss approaches that take into account uncertainty about the model itself.

One of the first approaches in this direction was presented by Domhan et al. [16]. The approach assumes learning curves to be instances of a parametric model that is a linear combination of known model classes, such as the inverse power law, and others [24]. The main difference to the above approaches is that, instead of estimating the parameters through a maximum likelihood approach, they estimate, for each parameter, the whole posterior distribution adopting Monte-Carlo Markov Chains. The approach is successfully used to early discard neural network architectures by predicting the iteration learning curve saturation performance \( p_{sat} \) and discarding as soon as the probability that the configuration is competitive drops below a pre-defined threshold. The Bayesian model proposed by Domhan et al. [16] also explicitly estimates the noise \( \sigma^2 \), which is assumed to be homoscedastic, i.e. invariant among the anchors. In this sense, the approach quantifies both epistemic and aleatoric uncertainty (cf. Sec. 2.5).
Utility prediction combines learning curve models as discussed in Sec. 4.6 with utility models as discussed in Sec. 4.3. The first approach in this direction was presented by Last [33]. This work is very similar to the works of Weiss and Tian [78], but now works on utility forecasts rather than looking back. In the above applied terminology, it is projective instead of retrospective. The error rate that now enters the model is not anymore the empirical one of past observations but a power-law model estimated from those observations. In this framework, one can compute the optimal dataset size analytically. The main advantage of the approach over the one of Weiss and Tian [78] is that one does not need to go through several acquisition iterations, which is important if those are associated with fixed costs. Notably, the learning curve has now become a resource of decision making. While this work leaves open how the empirical learning curve has been acquired and assumes it to just be available, Last [34] embeds his idea into an algorithm that follows a progressive sampling scheme in a follow-up work.

One use case in which the above techniques have been adopted has been reported in the context of automated software configuration [64]. The context of that paper is that every instance is a parametrization of a software library, and obtaining its label requires the costly execution of a benchmark on such a configuration. The goal is to understand how many observations need to be acquired to be able to learn a reliable prediction model. To this end, the authors adopt the projective sampling approach of Last [33, 34].

The latter work raises an important issue in stating that it is (often) not enough to only know the saturation point. Instead, we often also need to know the performance at the saturation point. What Sarkar et al. [64] argue is that if we do not tell the user about the expected performance at that point, a lot of resources might be invested to obtain the observations to reach the saturation point. But if the actual performance at that point is poor, anticipating this disappointment would have stopped the user from making the resource investment. To this end, they also incorporate the utility model proposed by Weiss and Tian [79].

### 4.8 Performance at Any Point for Any Learner

The approaches discussed in this section are the most general ones developed to date as far as learning curves are concerned. They allow to create a model for the full function $C$, i.e. they allow to generalize over learners. Such a model is of course so powerful that it can be used in all types of decision situations, e.g. data acquisition, early stopping, and early discarding.

In their Freeze-Thaw-(Bayesian) Optimization (FTBO) approach, Swersky et al. [69] model the behaviour of learning curves through Gaussian processes. An important contribution of that work is a non-stationary kernel for Gaussian processes that supports exponentially decaying learning curve models; it can easily be checked that standard kernels like a linear or Gaussian kernel do not lead to meaningful learning curve models. Assuming that the kernel reflects the model class appropriately, one side effect of the usage of a Gaussian process is that one automatically obtains estimates for the noise $\sigma^2$ at an arbitrary anchor $s$. Using their kernel and the current set of observations, Swersky et al. [69] estimate the asymptotic mean performance $p_{sat}$. Since the learning curves are combined with Bayesian optimization, the uncertainty for a specific future anchor is an ingredient for the computation of their acquisition function. In a rather thin evaluation, the approach was successfully applied on Online Latent Dirichlet Allocation, Logistic Regression, and Probabilistic Matrix Factorization, considering one dataset per learner. Using a freeze-thaw logic, they used iteration learning curves, but the modelling technique can be used for observation learning curves as well.

A similar approach was presented by Klein et al. [29] in FABOLAS. Similar to the FTBO approach by Swersky et al. [69], a Gaussian process is used to model the learning curve of the learners across different hyperparameter configurations. There are three main differences between the two approaches as far as learning curves are concerned. First, FABOLAS considers observation learning curves, while FTBO considers iteration learning curves. Second and related to this point, FABOLAS uses a different kernel than FTBO to model the behaviour of the learning curve in the Gaussian process, which is defined on the relative dataset size in $[0, 1]$ instead of absolute sizes. This makes sense since the full dataset size is already known, and there is no such thing as “asymptotic behaviour”. Third, FABOLAS tries to explicitly learn the full learning curve while FTBO focuses only on the saturation performance. Similar to Swersky et al. [69], the uncertainty about the performance estimates of the learning curve is not explicitly used. Still, the mere fact that a Gaussian process is fitted from the data allows to make assertions about the certainty of the learning curve value at any point and can hence be seen as uncertainty about the model parameter itself.

Parallel to their work on FABOLAS, Klein et al. [30] proposed an approach to estimate both the mean and the noise of a learning curve through the notion of Bayesian neural networks. The neural network is used to predict the parameter of a set of basis functions. These basis functions allow to incorporate prior knowledge into the network, which is necessary to extrapolate away from the data. The main difference to the learning curve models in FABOLAS and FTBO is that this approach models the behaviour of the learning curve not through a Gaussian Process but through a neural network. This network has $d + 1$ input units ($d$ for the algorithm description and one for the anchor) and one output unit for the estimated performance and, optionally, one output unit for the noise. To our knowledge, this approach and the learning curve extrapolation proposed by Domhan et al. [16] are the only approaches in this area that explicitly model the noise of the learning curve. An important difference between the two is that Klein et al. [30] assume heteroscedastic noise, i.e. noise that changes with both different hyperparameters and anchors, while Domhan et al. [16] assume homoscedastic noise across anchor sizes (not across configurations, because the model does not generalize over different configurations). Note that, while the approach presented in the paper does not explicitly consider the uncertainty about the parameter estimates, the parameters are essentially sampled from a posterior distribution, so the uncertainty is at least implicitly available. However, it
should be noted that the number of parameters describing the model here, namely the network weights, is potentially much larger than in the approach taken by Domhan et al. [16].

Wistuba and Pedapati [80] propose to use biased matrix factorization to obtain a model for $C(\cdot,\cdot)$. The approach is settled in the context of neural architecture search. Knowledge from previous datasets and different architectures is used to estimate the performance of new architectures on the target dataset, and this estimate is used to drive a Bayesian optimization approach.

5 Outlook

Learning curves have been an ingredient to decision making in machine learning for roughly 30 years now, and they have gained significant momentum over the last decade. Learning curves have proven a suitable solution for different types of decision situations, i.e. data acquisition, early stopping and early discarding for model selection. There are various ways how these problems can be addressed, ranging in complexity from a binary decision situation to a complete model over how various learners behave when presented with more data. Some works have also extended the question towards utility curves, data distribution and probability distributions over curves.

We have provided an overview of the type of decision situations and questions that can be addressed with learning curves, and the data resources that can be used to model the learning curves. Table 1 shows an overview of the methods we have discussed in this survey, contextualizing them according to these criteria. This table immediately reveals some interesting patterns. First, many works have tried to model learning curves without using any data resource beyond the anchors of the actual learner. Indeed, several works have demonstrated that parametric models such as the inverse power law only built from the empirical learning curve can already be adequately used to model the learning curve. However, a reasonable assumption is that the methods that utilize more data resources would be more accurate. An extensive benchmark including the various learning curve questions and resource types would give the community a better understanding of which combinations work well.

Another question that arises from our survey is whether simple questions can be treated more simplistically. Fig. 3 shows four binary questions, and we are not aware of any approach that tries to answer these questions without implicitly answering a more difficult question. Some of these questions are really at the core of the discussed approaches. For example, will the learning curve intersect with the learning curve of the currently best model? Most approaches create a learning curve model for this and hence implicitly solve a much more difficult problem. While those models, if appropriate, have the potential to give additional interesting insights, the question arises whether simpler approaches could solve those problems reliably while needing much less online data.

Next, little light has been shed on more sophisticated models of uncertainty. In particular, we are not aware of any work that is able to detect that the learning curve cannot be modeled with the adopted class of functions.

Finally, none of the approaches has so far utilized all forms of meta-data in the modelling of the learning curves. Indeed, combining anchors across learners and datasets might be a complex task, but when this is done successfully, it will enormously increase our understanding of learning curves.

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