TRACES ON LOCALLY COMPACT GROUPS

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Abstract. We conduct a systematic study of traces on locally compact groups, in particular traces on their universal and reduced C*-algebras. We introduce the trace kernel, and examine its relation to the von Neumann kernel and to small-invariant neighbourhood (SIN) quotients. In doing so, we introduce the class of residually-SIN groups, which contains both SIN and maximally almost periodic groups. We examine in detail the trace kernel for connected groups. We study traces on reduced C*-algebras, giving a simple proof for compactly generated groups that existence of such a trace is equivalent to having an open normal amenable subgroup, and we display non-discrete groups admitting unique trace.

We finish by examining amenable traces and the factorization property. We show for property (T) groups that amenable trace kernels coincide with von Neumann kernels. We show for totally disconnected groups that amenable trace separation implies the factorization property. We use amenable traces to give a simple proof that amenability of the group is equivalent to simultaneous nuclearity and possessing a trace of its reduced C*-algebra. As a final application of the results obtained in the paper, we address the embeddability of group C*-algebras into simple AF algebras. As a consequence, if a locally compact group is amenable and tracially separated (trace kernel is trivial), then its reduced C*-algebra is quasi-diagonal.

1. Introduction and background

Tracial states play an indispensable role when studying C*-algebras and are intimately related to many important structural properties. For example, unital quasidiagonal C*-algebras admit amenable traces and every tracial state on a C*-algebra with the weak expectation property (WEP) is amenable. As such, obtaining an understanding of the set of tracial states on a C*-algebra may allow one to deduce further
structural properties of the C*-algebra. In this paper, we initiate a systematic study of tracial states on group C*-algebras. Being one of the most prolifically studied classes of C*-algebras, it is unsurprising that this problem has been considered in these special cases before. If G is a discrete group, then it is well-known that its reduced C*-algebra $\mathcal{C}_r(G)$ admits a tracial state. Moreover, the uniqueness of this trace on $\mathcal{C}_r(G)$ is linked to the simplicity $\mathcal{C}_r(G)$.

A locally compact group is said to be C*-simple if the reduced C*-algebra $\mathcal{C}_r(G)$ is simple, or equivalently if every continuous unitary representation $\pi$ of $G$ that is weakly contained in the left regular representation $\lambda$ is weakly equivalent to $\lambda$. The literature concerning C* simplicity for locally compact groups is very rich. Since a non-trivial connected group is never C*-simple, the focus in studying C*-simple groups has been on discrete groups. Interest in such groups can be traced back to a question of Dixmier who asked in 1967 whether every simple C*-algebra is generated by its projections. Later Kadison conjectured that the reduced C*-algebra of $\mathbb{F}_2$, the free group on two generators, might provide an example of a simple C*-algebra with no non-trivial projections. The simplicity of $\mathcal{C}_r(\mathbb{F}_2)$ was shown by Powers in [51], but he was unable to determine whether or not $\mathcal{C}_r(\mathbb{F}_2)$ was without non-trivial projections. This was verified several years later by Pimsner and Voiculescu [48] completing the proof of Kadison’s Conjecture. Following the work of Powers, W. L. Paschke and N. Salinas [46] showed that the reduced C*-algebra of the free product of two groups not both of order two is simple and has a unique tracial state. P. de la Harpe, together with M. Bekka and M. Cowling gave additional example of C*-simple discrete groups with unique traces including non-solvable subgroups of $\text{PSL}(2, \mathbb{R})$ and non-almost solvable subgroups of $\text{PSL}(2, \mathbb{C})$. See [17], [3] and [4] as well as [18]. In an unpublished manuscript [52], T. Poznansky showed that a linear group $G$ is C*-simple if and only if $\mathcal{C}_r(G)$ admits a unique trace. More recently Kalantar and Kennedy [31] showed that if $\mathcal{C}_r(G)$ is simple, then the trace on $\mathcal{C}_r(G)$ is always unique. From there a complete characterization of when $\mathcal{C}_r(G)$ admits a unique tracial state for a discrete group $G$ was given by Breuillard, Kalantar, Kennedy and Ozawa [12]. In particular, they show that a discrete group has the unique trace property if and only if the amenable radical, the largest normal amenable subgroup, is trivial. They also show that a discrete group with the unique trace property that has non-trivial bounded cohomology or non-vanishing $\ell^2$-Betti numbers is C*-simple. Finally, Le Boudec [36] settled the question of the possible equivalence of the unique trace property with
C*-simplicity by exhibiting an example of discrete group $G$ with the unique trace property that is not C*-simple.

In contrast with the discrete case, for non-discrete groups $C^*_r(G)$ need not always have a tracial state, though it does when $G$ is amenable. For locally compact groups, the problem of when the reduced group $C^*$-algebra admits a tracial state has been studied by various authors. This problem was first solved in the case of separable and connected groups by Ng (see [43]) and later solved in the compactly generated case by the present authors (see [25]) where we showed that when $G$ is compactly generated, $C^*_r(G)$ admits a tracial state if and only if $G$ has an open amenable subgroup. We also speculated that the assumption of $G$ being compactly generated was not necessary. While [25] was being considered for publication, Kennedy and Raum [32], using entirely different techniques were able, to remove the assumption that $G$ was almost compactly generated to show that our speculation was correct. While the result of Kennedy and Raum is more general, our approach had the advantage in that it gave more detailed information about the structure of groups for which $C^*_r(G)$ possesses a trace. Moreover, it gave us tools to address the more general question concerning the nature of the traces on the full group $C^*$-algebra $C^*(G)$.

Our approach to systematically studying tracial states on the full group $C^*$-algebra of a locally compact group relies on the introduction of the “trace kernel”, which is the closed and normal subgroup $N_{\text{Tr}}$ of the locally compact group $G$ corresponding to group elements that are not separated from the identity by tracial states. The introduction of the trace kernel allows us to relate the set of tracial states on the full group $C^*$-algebra with the structure of the corresponding locally compact group. In general, the smaller that $N_{\text{Tr}}(G)$ is the richer the collection of traces. Moreover, we show amongst other things that $G/N_{\text{Tr}}(G)$ is almost SIN. In particular, if $G$ is connected, then $G/N_{\text{Tr}}(G) = V \times K$ where $V$ is a vector group and $K$ is compact.

The question of when a $C^*$-algebra can be embedded into an AF-algebra has a long history. It can be traced back to Elliott’s original classification of AF-algebras. Early work in this direction includes Spielberg’s proof that a separable, residually finite, type I $C^*$-algebra can be embedded into an AF algebra [58]. Later Pimsner and Voiculescu [49] showed that the irrational rotation algebra is AF-embeddable, prompting Effros [23] to ask about a possible abstract classification of such embeddable algebras.

Despite the considerable effort of many people, the question of embeddability into a generic AF-algebra is difficult with as of yet no satisfactory general theory to rely on. At this point the working conjecture
is that a C*-algebra is AF-embeddable if and only it is separable, exact, and quasidiagonal. In contrast, if we restrict our attention to embeddability into a unital, simple AF algebra, we can take advantage of recent deep work of C. Schafhauser [55]. amongst other things, Schafhauser showed that if $A$ is a separable, exact C*-algebra which satisfies the Universal Coefficient Theorem (UCT) and has a faithful, amenable trace, then $A$ admits a trace-preserving embedding into a simple, unital AF-algebra with unique trace. In particular, for any countable, discrete, amenable group $G$, the reduced group C*-algebra $C^*_r(G)$ admits a trace-preserving embedding into the universal UHF-algebra. For $G$ non-discrete, I. Beltiță and D. Beltiță [9] show that if $G$ is connected, second countable and solvable, that embeddability into an AF-algebra is possible if and only if $G$ is abelian.

**Notation and summary of results.** For a locally compact group $G$, we let

$$P(G) = \{ u : G \to \mathbb{C} \mid u \text{ is continuous and positive definite} \}$$

$$P_1(G) = \{ u \in P(G) : u(e) = 1 \}$$

$$T(G) = \{ u \in P_1(G) : u(st) = u(ts) \text{ for each } s,t \in G \}$$

denote, respectively, the positive definite functions, states, and the traces, each on $G$. By a well-known correspondence these sets may be identified with the positive functionals on the enveloping C*-algebra $C^*(G)$, respectively, the states, and the tracial states. See, for example [50, §7.1].

In Section 2 we introduce the trace kernel $N_T$. We place this in context against two other kernels, the small-invariant neighbourhood kernel, $N_{\text{SIN}}$; and the maximally almost periodic (MAP), or von Neumann, kernel, $N_{\text{MAP}}$. We introduce the new class of residually small invariant neighbourhood (SIN) groups [RSIN]. We show that this class properly contains the union of classes $[\text{MAP}] \cup [\text{SIN}]$, and that $N_T \subseteq N_{\text{SIN}} \subseteq N_{\text{MAP}}$, with $N_T = N_{\text{SIN}}$ when $G$ is compactly generated. We also show that $G/N_T$ is almost-SIN, and hence quasi-SIN.

In Section 3 we apply results from the previous section to learn about the structure of $N_T$, and of $G/N_T$, for connected groups, with a primary focus on connected Lie groups.

Sections 2 and 3 contain many examples that illustrate the complexities and limitations of the results.

In Section 4 we consider reduced traces, those on the reduced group C*-algebra. Using some observations from Section 2, we simplify the proof from our preprint [25] that for compactly generated groups, the
existence of a reduced trace is equivalent to \(G\) admitting an open normal amenable subgroup. As previously mentioned, this fact was simultaneously shown in [32] without the compact generation assumption. Our methods complement theirs and both can be used in tandem to learn about the structures of reduced traces in many examples. For example, we can produce examples of non-discrete groups admitting a unique reduced trace.

In Section 5 we consider amenable traces. We give a (mostly) self-contained function-theoretic approach to their definition. We introduce the amenable trace kernel \(N_{amTr}\). If \(G\) has property (T), then \(N_{amTr}\) coincides with the von Neumann kernel \(N_{MAP}\), hence \(G\) is amenably tracially separated exactly when it is maximally almost periodic. We then proceed to examine the relationship of amenable trace separation with the factorization property of Kirchberg [33]. In particular, we learn that for totally disconnected groups amenable trace separation implies the factorization property. Finally, we use amenable traces on the reduced \(C^*\)-algebra to gain a simple proof of the characterization, first due to Ng [43], that \(G\) is amenable if and only if \(C^*_r(G)\) is nuclear and admits a trace.

In Section 6, we turn our attention to studying structural properties of group \(C^*\)-algebras, in particular embeddability into simple, unital AF algebras. Appealing to deep results arising from classification theory of \(C^*\)-algebras, we show the reduced group \(C^*\)-algebra of a second countable locally compact group embeds inside of a simple, unital AF algebra if and only if the group is amenable and the tracial kernel \(N_{Tr}\) is trivial. As a consequence, we find the reduced \(C^*\)-algebras of tracially separated, amenable groups are quasidiagonal. We additionally demonstrate that the full \(C^*\)-algebra of a non-compact property (T) group cannot be embedded inside of a unital, simple AF algebra.

1.1. First observations. We note two obvious cases:

\[
\begin{align*}
G \text{ abelian:} & \quad T(G) = P_1(G) \cong \text{Prob}(\hat{G}) \\
G \text{ compact:} & \quad T(G) = \text{conv} \left\{ \frac{1}{d_\pi} \chi_\pi : \pi \in \hat{G} \right\}
\end{align*}
\]

where \(\hat{G}\) denotes the dual object, a locally compact group in the abelian case, and in the compact case a full set of irreducible representations, each of necessarily finite dimension \(d_\pi\) and with character \(\chi_\pi = \text{Tr} \circ \pi\).

If \(N\) is a closed normal subgroup of \(G\), we let \(q_N : G \to G/N\) denote the quotient map.
Given \( u \in P_1(G) \) we let \((\pi_u, \mathcal{H}_u, \xi)\) denote the Gelfand-Naimark-Segal (GNS) representation associated with \( u \), so \( u = \langle \pi_u(s)\xi|\xi \rangle \). We record an observation that is well-known to specialists.

**Lemma 1.1.**

(i) If \( u \in P_1(G) \) then \( N_u = u^{-1}(\{1\}) \) is a closed subgroup of \( G \), \( u \) is constant on double cosets \( N_u tN_u \), and \( \ker \pi_u = \bigcap_{t \in G} tN_u t^{-1} \).

(ii) If \( u \in T(G) \) then \( N_u = \ker \pi_u \) is normal, and \( u \in T(G/N_u) \circ q_{N_u} \).

**Proof.** (i) By uniform convexity \( N_u = \{ s \in G; \pi_u(s)\xi = \xi \} \), which is easily shown to be a subgroup. Furthermore \( \pi \) is constant on double cosets \( N_u tN_u \). If \( s \in \bigcap_{t \in G} tN_u t^{-1} \) and \( t \in G \) then \( \pi_u(s)\pi_u(t)\xi = \pi_u(t)\pi_u(t^{-1}st)\xi = \pi_u(t)\xi \), and cyclicity of \( \xi \) provides that \( \pi_u(s) = I \), so \( N_u \subseteq \ker \pi_u \). The converse inclusion is obvious.

(ii) The trace condition provides normality of \( N_u \). Hence \( N_u = \ker \pi \). Furthermore, \( \pi \) is constant on cosets of \( N_u \) and hence defines a representation \( \tilde{\pi} \) of \( G/N_u \). Then \( u = \langle \tilde{\pi}_u \circ q_{N_u}(\cdot)\xi, \xi \rangle \). \( \square \)

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**2. The tracial kernel**

Let \( G \) be a locally compact group. We define the *tracial kernel* of \( G \) by

\[
N_{Tr} = N_{Tr}(G) = \bigcap_{u \in T(G)} N_u.
\]

We say that \( G \) is *tracially separated* if \( N_{Tr} = \{ e \} \). It is evident that

\[
(2.1) \quad T(G) = T(G/N_{Tr}) \circ q_{N_{Tr}}
\]

and that \( G/N_{Tr} \) is the maximal tracially separated quotient of \( G \).

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**2.1. \( P \)-kernels.** We first wish to position the trace kernel in relation to other classes of kernels. Let \([P]\) be a class of locally compact groups. For example we let \([\text{MAP}]\) denote the class of *maximally almost periodic* groups and \([\text{SIN}]\) that of *small invariant neighbourhood* groups. We let \( N_P(G) \) denote the set of “co-P” closed normal subgroups \( N \) in \( G \), i.e. those for which \( G/N \in [P] \), and define the \( P \)-kernel by

\[
N_P = N_P(G) = \bigcap_{N \in N_P(G)} N.
\]

The most well-known example is the \textit{von Neumann kernel}, \( N_{\text{MAP}} \). Furthermore, \( G \) is called *minimally almost periodic* if \( N_{\text{MAP}} = G \) or, equivalently, if the trivial representation is the unique irreducible finite dimensional representation of \( G \).
We do not necessarily have that \( G/N \in \mathcal{P} \), but \( G/N \) is residually-P, i.e. the intersection of co-P subgroups. In fact, if \( N \) is a residually-P closed normal subgroup then
\[
N = \bigcap_{M \in \mathcal{N}(G/N)} (q_M \circ q_N)^{-1}(M) \subseteq N_{\mathcal{P}}(G)
\]
so \( N_{\mathcal{P}} \) is the smallest closed normal subgroup for which \( G/N \) is residually-P. If we let \([\mathcal{R}P]\) denote the class of residually-P groups, then \( N_{\mathcal{R}P} = N_{\mathcal{P}} \). In particular, we have an apparently novel class \([\mathcal{R}SIN]\).

**Theorem 2.1.** We have inclusion of classes \([\mathcal{M}AP] \subseteq [\mathcal{R}SIN]\).

**Proof.** Let \( G \in [\mathcal{M}AP] \). We shall show for each finite dimensional representation \( \pi : G \to U(d) \), that \( G_{\pi} = G/\ker \pi \in [\mathcal{S}IN] \). As the intersection of such kernels separates points of \( G \), this will show that \( G \in [\mathcal{R}SIN] \).

The connected component \( G_0 \in [\mathcal{M}AP] \), and the Fruedenthal-Weil Theorem (see [15, 12.4.8] or [20, 16.4.4]) shows that \( G_0 \cong V \times K \) for a vector group \( V \) and compact \( K \). Then \( K \) is characteristic in \( G_0 \), hence normal in \( G \), while \( V \) can arranged to be normal by Robertson and Wilcox [53, Theo. 2]. Furthermore, [53, Theo. 1] shows that the centralizer \( C = C_G(V) \) is of finite index, hence open, in \( G \). Notice that \( G_0 \subseteq C \). We first establish that the open image of \( C \) in \( G_{\pi} = G/\ker \pi \), which is isomorphic to \( C_{\pi} = C/\ker \pi|_C \), is in \([\mathcal{S}IN]\).

We let \( \mathcal{B} \) be a neighbourhood base at the identity for \( C/G_0 \) consisting of compact open subgroups. Each \( H_B = q_{G_0}^{-1}(B) \subseteq C \) is almost connected, hence by [27, (2.9)] of the form \( V \times K_B \), which is a direct product as \( V \) is central in \( C \). Notice that each \( K_B \) is open in \( C \). We have that \( \bigcap_{B \in \mathcal{B}} K_B = K \), and each \( \pi(K_B) \) is a compact Lie group. For \( B \) in \( \mathcal{B} \) we have that \( \pi(K_B)/\pi(K) \) is a Lie image of totally disconnected compact group, hence finite. It follows that \( \pi(K_B) = \pi(K) \) for \( B \) in \( \mathcal{B} \) sufficiently small, and we fix such a \( B \). Then \( \pi(K_B) = \pi(K) \), and hence \( q_{\pi}(K) = q_{\pi}(K_B) \) is open in \( C_{\pi} \). Now \( q_{\pi}(K) \), being homeomorphic to the normal subgroup \( \pi(K) \) in the compact group \( \pi(C) \) in \( U(d) \), admits a base \( \mathcal{C} \) of \( C_{\pi} \)-invariant neighbourhoods of the identity, while \( q_{\pi}(V) \) is central in \( C_{\pi} \). Consider the open subgroup of \( C_{\pi} \) given by
\[
q_{\pi}(G_0) = q_{\pi}(V \times K) = q_{\pi}(V)q_{\pi}(K).
\]
Any neighbourhood \( U \) of the identity in \( q_{\pi}(G_0) \) admits \( W \) in \( \mathcal{C} \) such that \( \pi(W) \subseteq U \), and \( (U \cap \pi(V)) \cap \pi(W) \) is a \( C_{\pi} \)-invariant neighbourhood of the identity. In other words, \( C_{\pi} \in [\mathcal{S}IN] \).

Now \( C_{\pi} \) is open and of finite index in \( G_{\pi} \). If \( U \) is a relatively compact conjugation-invariant neighbourhood of the identity in \( C_{\pi} \), then
\[ \bigcap_{t \in G_t/C} tUt^{-1} \] is a well-defined and conjugation-invariant neighbourhood of the identity for \( G_\pi \). It follows that \( G_\pi \in [\text{SIN}] \). \hfill \Box

The following may be known, but may easily seen from our proof.

**Corollary 2.2.** If \( G \in [\text{MAP}] \) and totally disconnected, then \( G \) is residually discrete. If, additionally, \( G \) is compactly generated then \( G \) is residually finite.

**Proof.** Let \( \pi : G \to U(d) \) be a representation. The proof above shows that \( \ker \pi \) contains some open subgroup \( B \). Hence if \( G \) is compactly generated, then \( \pi(G) \) is finitely generated, and [41, 1] shows that \( \pi(G) \) is residually finite. \hfill \Box

We let \([K]\) denote the class of compact groups and we obtain inclusions:

\[
\begin{array}{cccc}
[K] & \longrightarrow & [RK] & \longrightarrow [\text{MAP}] = [\text{RMAP}] \\
\downarrow & & \downarrow & \\
[\text{SIN}] & \longrightarrow & [\text{RSIN}].
\end{array}
\]

We get accordingly, the following containment of kernels:

\[
N_{\text{SIN}} \subseteq N_{\text{MAP}} \subseteq N_K.
\]

Proper inclusions for the top row in (2.2) are shown, respectively, by \( \mathbb{R} \), and the divisible discrete group \( \mathbb{Q} \). The following construction can be fine-tuned to create examples which show that no inclusion of classes is omitted in (2.2).

**Example 2.3.** We consider a sequence of pairs \((\Gamma_n, F_n)\) where each \( \Gamma_n \) is discrete, and each \( F_n \) is a finite group of automorphisms which contains one element \( \alpha_n \) for which the commutator \( \{s\alpha_n(s^{-1}) : s \in \Gamma_n\} \) is infinite. Hence in each semi-direct product \( \Gamma_n \rtimes F_n \), there is an element with infinite conjugacy class. Let

\[
\Gamma = \bigoplus_{n=1}^{\infty} \Gamma_n, \quad F = \prod_{n=1}^{\infty} F_n \quad \text{and} \quad G = \Gamma \rtimes F
\]

where \( \Gamma \) is restricted direct product (catagorical direct sum), and the product group \( F \) is compact and acts on \( \Gamma \) coordinate-wise. It is then easy to see that

- each neighbourhood of the identity on \( G \) contains a neighbourhood \( \{e\} \times \prod_{n=m}^{\infty} F_n \) and hence an element with infinite conjugacy class in a discrete subgroup, so \( G \notin [\text{SIN}] \); and
• each normal subgroup $N_m = (\bigoplus_{n=m}^{\infty} \Gamma_n) \rtimes (\prod_{n=m}^{\infty} F_n)$ has discrete quotient, so $G \in [RSIN]$.

(i) The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is of order 4 but squares to a central element, and is easily checked the have an infinite conjugacy class in $\text{SL}_2(\mathbb{Z})$, thus in $\text{SL}_2(\mathbb{Q})$, and hence its inner automorphism $\alpha$ has infinite commutator. We form $G$, as above, with choices $\Gamma_n = \text{SL}_2(\mathbb{Q})$ and $F_n = \langle \alpha_n \rangle$, where $\alpha_n = \alpha$. Since $G$ contains (closed) copies of $\text{SL}_2(\mathbb{Q})$, which is minimally almost periodic (as first shown by von Neumann and Wigner in [64]), $G \not\in [MAP]$.

(ii) If in contrast to above, we let each $\Gamma_n = \text{SL}_2(\mathbb{Z})$ instead of $\text{SL}_2(\mathbb{Q})$, then $\alpha$-invariant congruence subgroups of $\text{SL}_2(\mathbb{Z})$ show that each $\text{SL}_2(\mathbb{Z}) \rtimes \langle \alpha \rangle$ is residually finite. We obtain that $G \in [RK]$.

(iii) The classical example of Murakami [42] (also [45, 12.6.4]), with each $\Gamma_n = \mathbb{Z}$ and $F_n = \{1, \alpha\}$ with $\alpha(n) = -n$, is also in $[RK] \setminus [SIN]$ and is, furthermore, amenable. If here, we instead let each $\Gamma_n = \mathbb{Q}$ we get a group in $[MAP] \setminus ([RK] \cup [SIN])$.

We now position our trace kernel with respect to the kernels of (2.3), and gain information about the size of $T(G)$ in the process. We shall say that $T(G)$ is infinite dimensional, if it contains an infinite linearly independent set.

**Proposition 2.4.** (i) We have that $N_{Tr} \subseteq N_{SIN}$.

(ii) If $N_{SIN}$ is non-open, then $T(G)$ is infinite dimensional.

(ii') If $N_{MAP}$ is of infinite index in $G$, then $T(G)$ is infinite dimensional.

**Proof.** (i) Let $q : G \to H$ be a quotient homomorphism where $H \in [SIN]$. Let $\lambda : H \to \text{U}(L^2(H))$ denote the left regular representation. Given any $r \in H \setminus \ker \varphi$ there is a conjugation-invariant symmetric neighbourhood $U$ of $e$ in $H$ for which $q(r) \not\in U^2$. Then for $s$ in $H$

$$u_U(s) = \frac{1}{m(U)} \langle \lambda(s)1_U | 1_U \rangle = \frac{m(sU \cap U)}{m(U)}$$

defines a trace, since $H$ is unimodular and $t^{-1}(tsU \cap U)t = sUt \cap t^{-1}Ut = stU \cap U$ for $s, t$ in $H$. Also $u_U \circ q(r) = 0 \neq 1$. Hence $\ker q \supseteq N_{Tr}$.

(ii) First, suppose there is $N$ in $N_{SIN}$ be non-open, so $G/N$ is a non-discrete SIN-group. Then there is a sequence $U_1, U_2, \ldots$ of relatively compact symmetric open neighbourhoods of the identity in $G/N$ for which $\overline{U_{k+1}^2} \subseteq U_k$. Then the set of elements $u_{U_n} \circ q_N$, as above, is linearly independent.
If each element $N$ in $N_{\text{SIN}}$ is open, but $N_{\text{SIN}}$ is not open, then we may find a strictly decreasing sequence $N_1, N_2, \ldots$ of elements of open normal subgroups. Then $1_{N_1}, 1_{N_2}, \ldots$ is a linearly independent sequence in $T(G)$.

(ii') We have that $Q = G/N_{\text{MAP}}$ is infinite and maximally almost periodic, since it is residually maximally almost periodic. Then the almost periodic compactification $Q^{ap}$ is infinite. Let $\iota : Q \to Q^{ap}$ be the compactification map. With $U_1, U_2, \ldots$ chosen in $Q^{ap}$ as above, the set of elements $u_{U_n} \circ \iota \circ q_{N_{\text{MAP}}}$ is linearly independent. □

2.2. The role of compact generation. The next observation is of fundamental importance. The main aspect of the argument is very similar to one partially attributed by Hofmann and Mostert [30, Prop. 12.2] to Freudenthal (see [45, 12.4.16], for example), where it is shown for compactly generated $G$ that $G \in [\text{MAP}]$ implies that $G \in [\text{SIN}]$.

Proposition 2.5. Suppose that $G$ is compactly generated, and admits a continuous homomorphism $\varphi : G \to H$, where $H$ is a topological group admitting a family of conjugation invariant open sets $B$ with $\cap_{B \in B} B = \{ e \}$. Then $G \in [\text{SIN}]$.

In particular if $G$ is compactly generated we have

(i) for each $u$ in $T(G)$ that $N_u \in N_{\text{SIN}}$; and

(ii) if $G \in [\text{RSIN}]$, then $G \in [\text{SIN}]$.

Proof. The assumption of compact generation provides a compact symmetric neighbourhood $K$ of $e$ in $G$ for which $G = \bigcup_{n=1}^{\infty} K^n$. Let $W$ be an open neighbourhood of $e$ with $W \subseteq K^3$. We have $K^3 \setminus W \subseteq \bigcup_{B \in B} (G \setminus \varphi^{-1}(B))$, and hence $K^3 \setminus W \subseteq G \setminus \varphi^{-1}(B) \subseteq G \setminus \varphi^{-1}(B)$ for some $B$. Let $U = K^3 \cap \varphi^{-1}(B) \subseteq W$. If $x \in K$ we have on one hand that $xUx^{-1} \subseteq \varphi^{-1}(B)$, and on the other that $xUx^{-1} \subseteq K^3$, and hence $xUx^{-1} \subseteq U$. As $W$ is arbitrarily small, and $K$ generates $G$, this construction provides a conjugation invariant base of neighbourhoods of $e$, i.e. $G \in [\text{SIN}]$.

(i) We let $(\pi_u, H_u, \xi)$ be the GNS representation associated with $u$. Consider the group of unitaries $H = \pi_u(G)$ which is a topological group in the relativized weak operator topology (equivalently, strong operator topology). The trace condition shows that for $\varepsilon > 0$ that each $B_\varepsilon = \{ x \in H : |\langle x\xi | \xi \rangle - 1 | < \varepsilon \}$, so $B_\varepsilon \subseteq \{ x \in H : |\langle x\xi | \xi \rangle - 1 | \leq \varepsilon \}$. As in Lemma [14] we see that $\bigcap_{\varepsilon > 0} B_\varepsilon = \{ I \}$. Let $\varphi = \tilde{\pi}_u : G/N_u \to H$.

(ii) Here we let $H = \prod_{N \in N_{\text{SIN}}(G)} G/N$ be the product topological group; $B$ be the family of open neighbourhoods of $e$ which are proper conjugation-invariant sets on a finite set of coordinates, and the whole...
group on the remaining ones; and \( \varphi : G \to H \) be given by \( \varphi(s) = (q_N(s))_{N \in N_{\text{SIN}}(G)}. \)

**Remark 2.6.** Conclusion (i) can fail if \( G \) is not compactly generated. Consider \( G = \bigoplus_{n=1}^{\infty} \Gamma_n \rtimes \prod_{n=1}^{\infty} F_n \), as in Example 2.3. Each quotient \( \Gamma_n \rtimes F_n \) admits trace 1, so \( u((x_n, \sigma_n))_{n=1}^{\infty} = \sum_{n=1}^{\infty} \frac{1}{2^n} 1_{\{(e,e)\}}(x_n, \sigma_n) \) defines a trace with \( N_u = \{e\}. \)

Proposition 2.5 shows that for compactly generated groups we get containment diagram

\[
(RK) \longrightarrow [\text{MAP}] \longrightarrow [\text{SIN}] = [\text{RSIN}]
\]

If \( G \) is either connected, or totally disconnected, then \( G \in [\text{MAP}] \) implies that \( G \in [\text{RK}] \). Indeed this follows from the Freudenthal-Weil Theorem and Corollary 2.2, respectively.

**Example 2.7.** The first two inclusions of (2.4) remain proper amongst compactly generated groups.

(i) Let \( C_3 = \mathbb{Z}/3\mathbb{Z} \) act on \( \mathbb{R}^2 \), by the obvious rotations. Any normal subgroup of \( G = \mathbb{R}^2 \rtimes C_3 \) must intersect \( \mathbb{R}^2 \) as a \( C_3 \)-invariant subgroup. Any non-trivial such subgroup must be dense in \( \mathbb{R}^2 \). It follows that \( N_K = \mathbb{R}^2 \), though this group is maximally almost periodic.

(ii) Certain Baumslag-Solitar groups, such as \( G = BS(2,3) \), seem to be amongst the most famous examples of finitely generated, non-residually finite, or equivalently non-maximally almost periodic, groups (e.g. Corollary 2.2). In fact, in [2] it is indicated that for \( G \) above that the solvable group \( G/G'' \) also satisfies a condition which entails that it is not residually finite.

(iii) In [16, 2.7] de Cornulier presents a construction of a finitely generated linear group with a quotient by a certain proper subgroup of its centre that is not residually finite. This illustrates some limitations of [38], where it is shown that a MAP-group modulo its centre is in [MAP], and also of [111, 11] that finitely generated linear groups are residually finite.

**Corollary 2.8.** If \( G \) is compactly generated, then \( N_{TV} = N_{\text{SIN}}. \)

**Proof.** Proposition 2.5 (i) shows that \( N_{\text{SIN}} \subseteq N_{TV} \). Combine with Proposition 2.4 (i). \( \square \)

As of yet, we do not have an example of a tracially separated non-RSIN locally compact group \( G \), hence no example for which containment \( N_{TV} \subseteq N_{\text{SIN}} \) is proper. Example 3.8 below will show for an infinite group that we can have \( N_{TV} \) proper and of finite index, and hence \( T(G) \) is finite dimensional. It would be interesting to know if there are any
discrete $G$ for which $T(G)$ is non-trivial, but finite-dimensional. A plausible candidate is $G = \text{SL}_2(\mathbb{Q})$.

We say that $G$ is an almost-SIN group if there is a net $(U_\alpha)_{\alpha}$ of relatively compact open sets, satisfying that $U_\alpha \searrow \{e\}$ (i.e. is eventually contained in any neighbourhood of $e$), and satisfies the asymptotic invariance property

$$
\limsup_{\alpha \in K} \frac{m(tU_\alpha t^{-1} \Delta U_\alpha)}{m(U_\alpha)} = 0
$$

for each compact $K \subseteq G/N_{\text{Tr}}$, where $m$ denotes Haar measure. A mildly weaker form of this condition is noted by Stokke in [59, §3], where sets $U_\alpha$ are assumed to be of finite measure, but can be replaced by relatively compact open sets thanks to regularity; also see Thom [61, Rem. 3.1]. In both of the above references convergence is assumed to be pointwise, rather than uniform on compact sets.

**Proposition 2.9.** The group $G/N_{\text{Tr}}$ is almost-SIN.

**Proof.** Every compactly generated open subgroup $H$ of $G/N_{\text{Tr}}$ satisfies $T(H) \subseteq T(G/N_{\text{Tr}})|_H$, so $H \in [\text{SIN}]$, and hence $H$ is unimodular.

Let $W$ be a relatively compact open neighbourhood of $eN_{\text{Tr}}$, and let $K \subseteq G/N_{\text{Tr}}$ be compact. Then $H = \langle WK \rangle \in [\text{SIN}]$. Hence, there is an $H$-conjugation-invariant neighbourhood $U = U_{K,W}$ of $eN_{\text{Tr}}$ contained in $W$. It is now clear that we can build the desired net accordingly. \qed

Notice that the net $f_\alpha = \frac{1}{m(U_\alpha)}1_{U_\alpha}$ shows that $G/N_{\text{Tr}}$ enjoys the quasi-SIN property of Losert and Rindler [39]. It is noted in [59] that the almost-SIN condition implies unimodularity, and is equivalent to a strong quasi-SIN property: $L^1(G)$ admits a bounded approximate identity consisting of normalized indicator functions.

Summarizing results of this section, we get the following implications for $G$:

$$
[\text{MAP}] \Rightarrow [\text{RSIN}] \Rightarrow \text{tracial separation}
\Rightarrow \text{unimodular \& almost-SIN} \Rightarrow \text{quasi-SIN}.
$$

Non-abelian solvable connected groups, being amenable, are quasi-SIN, but frequently non-unimodular, and always lack trace separation by Remark 3.5 below. We do not know if either of the middle two implications are equivalences.

### 3. Connected groups

Now let us suppose that $G$ is connected, hence compactly generated. Then Corollary 2.8 and Freudenthal-Weil Theorem [45, 12.4.8] tell us...
that

\[ (3.1) \quad N_{TV} = N_{SIN} = N_{MAP} \text{ and } G/N_{TV} = V \times K \]

where \( V \) is a vector group and \( K \) is compact. Hence \( G \) admits a unique trace, i.e. \( T(G) = \{1\} \), if and only if it is \emph{minimally almost periodic}, i.e. \( N_{MAP} = G \). Notice that (2.1) and (1.1) tell us that

\[ (3.2) \quad T(G) \cong \text{conv} \left\{ P_1(V) \otimes \left\{ \frac{1}{d_\pi} \chi_\pi : \pi \in \hat{K} \right\} \right\}. \]

Indeed, we consider the GNS triple \((\pi_u, \xi, H_u)\) of \( u \) in \( T(V \times K) \). Factor \( \pi_u(v, k) = \pi_u(v, e) \pi_u(0, k) \), and then \( \langle \pi_u(0, \cdot) \xi | \xi \rangle \) is in the norm-closed convex hull of normalized characters on \( K \). The associated reducing projections each commute with \( \pi_u(v, e) \) for each \( v \) in \( V \).

We aim to get more precise descriptions of \( V \) and \( K \).

If \( G \) is a connected Lie group, it admits a maximal solvable normal connected subgroup \( R \), which is always closed. There is a semisimple connected subgroup \( S \), called a Levi complement, satisfies \( G = RS \) and that \( R \cap S \) is central and discrete. Though \( S \) need not be closed, it admits a Lie group structure by which the inclusion \( S \hookrightarrow G \) is continuous. Two Levi complements are conjugate by an element from the reductive radical \([R, G]\), which is connected and normal. A Levi complement \( S \) contains a maximal connected normal compact subgroup \( S_c \), which is the integral subgroup of the compact ideal \( s_c \) in its semisimple Lie algebra \( \mathfrak{s} \). Let \( S_{nc} \) denote the integral subgroup of the complementary ideal of \( s_c \) in \( \mathfrak{s} \). Then \( S_{nc} \) commutes with \( S_c \), the intersection \( S_{nc} \cap S_c \) is finite and central in \( S \), and \( S = S_c S_{nc} \). Using (3.1), and in mildly differing notation, it is proved by Shtern [56] that

\[ (3.3) \quad N_{TV} = [R, G]S_{nc}. \]

Notice that if \( G \) is solvable, then \( R = G \), \( N_{TV} \) is the closure of the derived subgroup, and \( G/N_{TV} \) is abelian.

\textbf{Proposition 3.1.} \textit{Let \( G \) be a connected Lie group. Then in (2.1) we have that \( V \) is the maximal vector quotient of \( G \), which is also a (not necessarily maximal) vector subgroup of \( R/[R, G] \), and \( K \) is the maximal compact quotient of \( G \) whose semisimple part is a quotient of \( S_c \) by a finite group.}

\textit{If, moreover, \( G \) is simply connected, then \( V \cong R/[R, G] \) and \( K \cong S_c \).}

\textbf{Proof.} For \( r, r' \) in \( R \) and \( s, s' \) in \( G \) we have

\[ rs[R, G] = sr[R, G] \quad \text{and hence} \quad rsr's' [R, G] = rr's's' [R, G]. \]
Thus the map 

\[(r[R,G], sS_{nc}) \mapsto rs[R,G]S_{nc} : (R/[R,G]) \times (S/S_{nc}) \to G/N_{Tr}\]

is a continuous homomorphism with kernel 

\[D = \{(s[R,G], s^{-1}S_{nc}) : s \in R \cap S\} .\]

We let 

\[L = S/S_{nc} = S_{c}S_{nc}/S_{nc} = S_{c}/(S_{c} \cap S_{nc}) .\] The abelian group 

\[R/[R,G]\]

admits trivial action by \(S\) and decomposes as a direct product 

\(WT\), where \(W\) is a vector group and \(T\) is compact. Thus \(G/N_{Tr}\) is the continuous isomorphic image of 

\[(W \times T \times L)/D = (V \times U \times T \times L)/D \cong V \times K\]

where \(V\) is a complementary subspace to \(U = \text{span}_{R}(W \cap S_{nc})\) in \(W\) and 

\[K = (U \times T \times L)/D .\]

If \(G\) is simply connected, then so too are \(R\) and \(S\), and the latter is a direct product \(S_{c}S_{nc}\). Furthermore, \([R,G]\) is a normal integral subgroup hence itself closed, with \(R/[R,G]\) a vector group (see [29, §11.2,§14.5]). Thus \(K = S_{c}\). □

**Remark 3.2.** In [56], a further remarkable fact is shown. If we consider the discretization \(G_{d}\) of a connected Lie group, then 

\[N_{MAP}(G_{d}) = [R,G]S_{nc}\].

Notice that Proposition 2.4 above, shows that 

\[N_{Tr}(G_{d}) = N_{SIN}(G_{d}) = \{e\} .\] We shall indicate in our examples, below, some situations in which this differs from 

\(N_{Tr}(G)\).

We now address minimal almost periodicity of \(G\).

**Corollary 3.3.** A connected Lie group \(G\) is minimally almost periodic if and only if \(S_{c} = \{e\}\) and the image of \(S\) in \(G/[R,G]\) is dense. In particular, this holds if \(S_{c} = \{e\}\) and \([R,G] = R\).

**Proof.** If \(S_{c} \neq \{e\}\), then \(G/R\) admits a non-trivial quotient of \(S_{c}\), as a quotient. The proof of Proposition 3.1 shows that 

\[Q = G/[R,G]\]

is a continuous isometric image of 

\[(R/[R,G] \times S)/D .\]

If the image of \(S\) is not dense in \(Q\), then \(Q\) has a non-trivial abelian quotient. These two situations are the only pair of obstructions to minimal almost periodicity. □

**Example 3.4.** We consider examples to illustrate the complications of situation above.
We let $\widetilde{\text{SL}}_2(\mathbb{R})$ be the simply connected covering group of $\text{SL}_2(\mathbb{R})$. Let $Z$ denote the centre of $\widetilde{\text{SL}}_2(\mathbb{R})$ which admits an isomorphism $j : Z \to Z$, and satisfies that $\widetilde{\text{SL}}_2(\mathbb{R})/Z \cong \text{SL}_2(\mathbb{R})$.

(i) Let $G = [\mathbb{R} \times \widetilde{\text{SL}}_2(\mathbb{R})]/D$ where $D = \{(n, j(n)) : n \in \mathbb{Z}\}$. Then $R = [\mathbb{R} \times \{I\}]/D \cong R$ with $[R, G] = \{e\}$ and $S = S_{nc} = \{(0) \times \widetilde{\text{SL}}_2(\mathbb{R})\}/D \cong \text{SL}_2(\mathbb{R})$. Notice that $[\mathbb{Z} \times \{I\}]/D = R \cap S$, i.e. $(0, j(n))D = (-n, I)D$. Hence, we have

$$G/N_T = RS/S = R/(R \cap S) \cong \mathbb{R}/\mathbb{Z} \cong \mathbb{T}.$$ 

This shows that $V$ can be smaller than the maximal vector subgroup of $R/[R, G]$.

(ii) If $\xi_1, \ldots, \xi_m$ are rationally independent in $\mathbb{R}$, then the map $\alpha_m : \mathbb{Z} \to \mathbb{T}^m$, $\alpha_m(n) = (e^{i\xi_k})_{k=1}^m$, has dense range. Indeed, $\mathbb{Z}^m + \mathbb{Z}(\xi_k)_{k=1}^m$ is dense in $\mathbb{R}^m$, and projects onto this range.

We consider

$$G = (\mathbb{T}^m \times \widetilde{\text{SL}}_2(\mathbb{R}))/D \text{ where } D = \{(\alpha_m(n), j(n)) : n \in \mathbb{Z}\}.$$ 

Here $R = (\mathbb{T}^m \times \{I\})/D \cong \mathbb{T}^m$ is central, so $[R, G] = \{e\}$ and $S = \{(e) \times \widetilde{\text{SL}}_2(\mathbb{R})\}/D \cong \widetilde{\text{SL}}_2(\mathbb{R})$ is dense. Hence $G$ is minimally almost periodic.

Notice that $N_{\text{MAP}}(G_d) = S$, and $G_d/N_{\text{MAP}}(G_d) \cong \mathbb{T}_d^m/\alpha_m(\mathbb{Z})$.

This is an easy modification of a well-known example, see [29, 14.5.10]. We have increased dimension of the torus to support Example 3.6 below.

(iii) We let $\xi$ be irrational and let $H = \mathbb{R}^4$ with Heisenberg-type product

$$(x, y, z, \zeta)(x', y', z', \zeta') = (x + x', y + y', z + z' + xy' - x'y, \zeta + \zeta' + \xi(xy' - x'y)).$$

We see that $s = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the connected group $\text{GL}_2(\mathbb{R})_0$ acts on $H$ by

$$s \cdot (x, y, z, \zeta) = (ax + by, cx + dy, (\det s) z, (\det s) \zeta).$$

We let $G = H \rtimes \text{GL}_2(\mathbb{R})_0$. Then $R = H \rtimes (\mathbb{R}_{>0}I)$ (recall that $R$ is connected so we use only the connected component of the centre of $\text{GL}_2(\mathbb{R})_0$). $[R, G] = (\mathbb{R}^2 \times \mathbb{R}(1, \xi)) \times \{I\}$ and $S = \{0\} \times \text{SL}_2(\mathbb{R})$. Thus $G/N_T \cong \mathbb{R} \times \mathbb{R}^2_{>0} \cong \mathbb{R}^2$. This example and its successor were inspired by Rothman [54, 2.5].

(iii') The centre of $H$, above, is $Z(H) = \{0\} \times \mathbb{R}^2$, and is acted upon trivially by $\text{SL}_2(\mathbb{R})$. We let $\overline{H} = H/(\{0\} \times \mathbb{Z}^2) \cong \mathbb{R}^2 \times \mathbb{T}^2$. This time
let \( G = \mathbb{T} \times \text{SL}_2(\mathbb{R}) \). Here we find that \([R, G] = \mathbb{T} \times \{I\}\) and we get that \( G \) is minimally almost periodic.

Notice, in this case, that \([R, G] = H \rtimes \{I\}\) and we get that \( G \) is minimally almost periodic.

Remark 3.5. Suppose \( G \) is connected, and let \( \mathcal{L} \) be a downward directed collection of compact normal subgroups \( L \) for which \( G/L \) is Lie, and \( \bigcap_{L \in \mathcal{L}} L = \{e\} \).

(i) For \( L \supseteq L' \) in \( \mathcal{L} \) we have \( G/L = (G/L')/(L/L') \). It follows that \((G/L)/\mathcal{N}_T(G/L) = V \times K_L\) for the same vector group \( V \) while \( K_L \) is a quotient of \( K_L \). Hence \( G/\mathcal{N}_T(G) = V \times K \) where \( K \) is the inverse limit of the compact Lie groups \( K_L \).

(ii) In [57] it is shown that \( \mathcal{N}_T = \mathcal{N}_{\text{MAP}} \) has form similar to (3.3), and thus is connected.

(iii) If each \( G/L \) is solvable, then each \( K_L \) is abelian, and hence so too is \( K \). Thus \( \mathcal{N}_T \) contains the derived subgroup \([G, G]\), and hence its closure. However, \( G/\mathcal{N}_T(G) = V \times K \) where \( K \) is the inverse limit of the compact Lie groups \( K_L \).

We note that this captures aspects of Beltiţă and Beltiţă [9, Theo. 2.8], which were proved there by much different means.

Example 3.6. Let \( \xi_1, \xi_2, \ldots \) be rationally independent in \( \mathbb{R} \), \( \alpha : \mathbb{Z} \to \mathbb{T}^\mathbb{N} \) be given by \( \alpha(n) = (e^{i\pi k})_{k=1}^\infty \). Any non-empty open set in \( \mathbb{T}^\mathbb{N} \) meets \( \alpha(\mathbb{Z}) \), thanks to Example 3.4 (ii), so \( \alpha(\mathbb{Z}) = \mathbb{T}^\mathbb{N} \). We then consider

\[
G = [\mathbb{T}^\mathbb{N} \times \mathbb{SL}_2(\mathbb{R})]/D \text{ where } D = \{(\alpha(n), j(n)) : n \in \mathbb{Z}\}.
\]

By the last remark, the family \( \mathcal{L} = \{\mathbb{T}^{m+1, m+2, \ldots} : m \in \mathbb{N}\} \), with each Lie quotient of the form given in Example 3.4 (ii), shows that \( G \) is minimally almost periodic.

We have only one interesting thing to say about the trace kernels of non-connected Lie groups.

Proposition 3.7. Let \( G \) be an almost connected Lie group. Then \( \mathcal{N}_T(G) = \mathcal{N}_T(G_0) \). In particular, \( \mathcal{N}_T(G) \) is connected.

Proof. We have that \( G_0 \) is open. Since \( T(G)|_{G_0} \subseteq T(G_0) \) we see that \( \mathcal{N}_T(G_0) \subseteq \mathcal{N}_T(G) \). To see the converse, suppose \( r \in G_0 \setminus \mathcal{N}_T(G_0) \). Then there is a \( u \) in \( T(G_0) \) for which \( u(r) \neq 1 \). Hence if \( v = \frac{1}{2}(u + \bar{u}) \) then \( v \in T(G_0) \) with \(-1 \leq v(r) < 1\). Define

\[
\hat{v}(s) = \begin{cases} 
\frac{1}{\|G_0\|} \sum_{tG_0 \in G_0} u(t^{-1}st) & \text{if } s \in G_0 \\
0 & \text{otherwise.}
\end{cases}
\]
Then $\dot{v}$ is well-defined, and continuous since $G_0$ is open, and thus in $T(G)$, with $\dot{v}(r) < 1$. \qed

Example 3.8. Let $\alpha$ be an inner automorphism on $\text{SL}_2(\mathbb{R})$, as given in Example 2.3 (i), above. Then the last proposition shows for $G = \text{SL}_2(\mathbb{R}) \rtimes \langle \alpha \rangle$ that $N_{\text{Tr}} = \text{SL}_2(\mathbb{R}) \rtimes \{e\}$, and hence is of finite index in $G$.

Remark 3.9. Since an almost connected compactly generated group $G$ is compactly generated, we know that $G/N_{\text{Tr}}$ is almost connected and in [SIN] and hence of the form $V \rtimes K$ where the action of $K$ on $V$ factors through a finite group. The vector group $V$, being connected, is a vector quotient of $G_0$, which is also the same vector quotient of any quotient $G_0/L$ for any compact co-Lie subgroup $L$ of $G_0$.

Hence, much like (3.2) we have that

$$T(G) \cong \mathbb{T}(V \rtimes K) = \text{conv} \left[ P^K_1(V) \otimes \left\{ \frac{1}{d_{\pi}} \chi_{\pi} : \pi \in \hat{G} \right\} \right]$$

where $P^K_1(V) = \{ u \in P_1(V) : u(k \cdot v) = u(v) \text{ for } v \in V \text{ and } k \in K \}$, which is isomorphic to the space of probabilities on the dual group, averaged over (finite) orbits of $K$, $\text{Prob}^K(\hat{V})$.

4. Reduced traces

We let $\lambda = \lambda_G : G \to U(L^2(G))$ denote the left regular representation $P_\lambda(G) = \{ \langle \lambda(\cdot)f, f \rangle : f \in L^2(G) \}$. This space is naturally isomorphic to the space of normal positive functionals on the group von Neumann algebra $\text{VN}(G) = \lambda(G)^\prime\prime$, as this algebra is in standard form. We let $P_r(G) = \overline{P_\lambda(G)}^{w^*} = \overline{P_\lambda(G)}^{uc} \subseteq P(G)$ where $w^*$ represents the weak* topology of $C^*(G)$ and $uc$ the topology of uniform convergence on compact sets; see [20] 13.5.2. Then $P_r(G)$ is naturally identified with the space of positive linear functionals on the reduced $C^*$-algebra $C^*_r(G)$ and consists of all positive definite matrix coefficients of representations which are weakly contained in $\lambda$.

We recall Hulanicki's Theorem:

$G$ is amenable $\iff 1 \in P_r(G) \iff P_r(G) = P(G) \iff C^*_r(G) \cong C^*(G)$.

Let us review two properties of these spaces, which are familiar to specialists.

Lemma 4.1.  

(i) If $H$ is a closed subgroup of $G$ then $P_r(G)|_H \subseteq P_r(H)$.
(ii) If $N$ is an amenable closed normal subgroup of $G$ then $P_r(G/N) \circ q_N \subseteq P_r(G)$.

Proof. (i) We have that $P_\lambda(G)|_H \subseteq P_\lambda(H)$. Indeed, $P_\lambda(G)$ is the closure of the compactly supported positive definite functions; see [20, 13.8.6] or [50, 7.2.5].

(ii) Amenability provides the weak containment $1_N \prec \lambda_N$. Then Fell’s continuity of induction provides

$$\lambda_{G/N} \circ q_N = \text{ind}_{N}^{G} 1_N \prec \text{ind}_{N}^{G} \lambda_N = \lambda_G.$$  

Thus $P_\lambda(G/N) \circ q_N \subseteq P_r(G)$. \hfill $\square$

The following is surely known, admitting the same proof as the fact that totally disconnected compact groups are pro-finite, but is necessary for the main result of this section.

**Lemma 4.2.** If $G$ in [SIN] is totally disconnected, then $G$ is pro-discrete, i.e. there is a base at the identity consisting of compact open normal subgroups.

Proof. Let $U$ be a conjugation-invariant neighbourhood of the identity. Then $U$ contains a compact open subgroup $K$ which, in turn, contains another conjugation invariant neighbourhood $V$, which contains a compact open subgroup $L$. Then the subgroup generated by $\bigcup_{x \in G} xLx^{-1}$ is open, normal and contained in $K$. \hfill $\square$

We let the *reduced traces* be given by

$$T_r(G) = T(G) \cap P_r(G).$$

These are the tracial states on $C^*_r(G)$. We then consider the *reduced tracial kernel*

$$N^r_r = N^r_r(G) = \bigcap_{u \in T_r(G)} N_u$$

which we deem to be all of $G$ if $T_r(G) = \emptyset$.

The following was the main result of an earlier version of our work [25].

**Theorem 4.3.** If $G$ is compactly generated then the following are equivalent:

(i) $T_r(G) \neq \emptyset$;

(ii) $N_{SIN}$ admits an amenable element;

(iii) $G$ admits an open normal amenable subgroup; and

(iv) $N^r_r$ is amenable.
Proof. (i) ⇒ (ii) If \( u \in T_r(G) \), then Lemma 4.1 (i) gives that \( 1 = u|_{N_u} \in P_\lambda(N_u) \) so \( N_u \) is amenable. Proposition 2.5 provides that \( N_u \in N_{\text{SIN}} \).

(ii) ⇒ (iii) If \( N \in N_{\text{SIN}} \) is amenable, then the Frueadenthal-Weil Theorem provides that \( (G/N)_0 \) is amenable, and \( \tilde{N} = q_N^{-1}((G/N)_0) \) is also amenable with \( G/\tilde{N} \cong (G/N)/(G/N)_0 \) a totally disconnected SIN-group. Lemma 4.2 provides a compact open normal subgroup \( M \) in \( G/\tilde{N} \), so \( q_N^{-1}(M) \) is open, normal and amenable.

(iii) ⇒ (iv) Let \( N \) be an open, normal amenable subgroup of \( G \). Then Lemma 4.1 (ii) shows that \( 1 = (\lambda_{G/N} \circ q_N(\cdot) \delta_e N, \delta_e N) \in T_r(G) \). Hence \( N \in T_r \subseteq N \) and is thus amenable.

(iv) ⇒ (i) If \( N \in T_r \) = \( G \), then \( G \) is amenable, so \( T_r(G) = T(G) \). If \( N \in T_r \neq G \), then \( T_r(G) \neq \emptyset \). □

Our proof of (ii) ⇒ (iii) ⇒ (iv) ⇒ (i), holds in absence of the assumption of compact generation. Our proof of (iii) ⇒ (i) generalizes and conceptually simplifies [46, Prop. 1.6].

The amenable radical, \( AR(G) \) of a locally compact group \( G \) is the largest amenable closed normal subgroup. For second countable \( G \), the existence is given in [66, 4.1.2]. Let us briefly outline the main steps to show its existence more generally.

If \( M, N \) are amenable normal subgroups then \( M/(M \cap N) \) embeds continuously and densely into \( MN/N \), showing that \( MN \) is amenable. Let

\[
AR(G) = \bigcup \{ N : \text{closed normal amenable subgroup} \} \subseteq G.
\]

Any cluster point of the net of \( N \)-invariant means, \( \tilde{M}_N(\psi) = M_N(\psi|_N) \) for bounded left uniformly continuous \( \psi \) on \( AR(G) \), indexed over increasing closed normal subgroups, admits an invariant mean as any cluster point. This is less delicate if some such a subgroup is open. Condition (iii) above is clearly equivalent to

(iii') \( AR(G) \) is open.

Remark 4.4. Suppose \( G \) is a connected Lie group, and recall the notation of Proposition 3.1. We have that \( AR(G) = RS_c Z(S_{nc}) \). Hence it is immediate that

\[
T_r(G) \neq \emptyset \iff AR(G) = G \iff G/R \text{ is compact}.
\]

In this case [30, 10.25, 10.28 & 10.29] and Remark 3.5 (ii), the same holds for any connected group.
We note the following beautiful result.

**Theorem 4.5.** (Kennedy and Raum [32]) We have that \( T_r(G) \neq \emptyset \) if and only if \( AR(G) \) is open in \( G \). Furthermore, in this case we have

\[
T_r(G) \cong \left\{ u \in T(AR(G)) : \frac{u(trt^{-1})}{u(r) \text{ for } r \in AR(G) \text{ and } t \in G} \right\}.
\]

Indeed, the authors in [32] show that any element of \( \lambda(C_c(G)) \) supported in \( G \setminus AR(G) \) is annihilated by each reduced trace. Hence traces are supported in \( AR(G) \). A trace on \( AR(G) \) admits a trivial extension (i.e. 0 outside of \( AR(G) \)) to a reduced trace on \( G \) exactly when it is conjugation invariant by actions from \( G \).

**Remark 4.6.** We note that our Theorem 4.3 gives, for compactly generated groups, a much simpler proof of the first statement of Theorem 4.5. Combined with Proposition 3.1 we are now in a position to have a clear understanding of the nature of traces in the case for connected groups; see Remark 4.4, above.

It is worth noting that we proved our main result in [25] before we were aware of the work of [32]. Happily, we can combine both methods to gain a deeper understanding of the structure of reduced traces.

As in (2.1), we see that if \( AR(G) \) is open, then (4.1) becomes

\[
T_r(G) \cong \left\{ u \in T(AR(G)/N_{Tr}^r) : \frac{u(trt^{-1}N_{Tr}^r)}{u(rN_{Tr}^r) \text{ for } r \in AR(G) \text{ and } t \in G} \right\}.
\]

We summarize some immediate consequences of this observation.

**Proposition 4.7.** If \( AR(G)/N_{Tr}^r \) is finite, then \( T_r(G) \) is finite dimensional. If \( AR(G) = N_{Tr}^r \) and is open, we will get unique reduced trace.

By way of contrast, with aid of Lemma 4.1 (ii) and Theorem 4.3, we may recycle the proof of Proposition 2.4 (ii) to see the following:

**Proposition 4.8.** If \( G \) admits a non-open amenable element of \( N_{\text{SIN}} \), then \( T_r(G) \) is infinite dimensional. This happens, in particular, if \( G \) is compactly generated and \( N_{Tr}^r \) is not open.

We shall use the following to help devise and inspect examples:

**Theorem 4.9.** Let \( \Gamma \) be a finitely generated group which acts irreducibly on a non-discrete abelian group \( A \), in the sense that \( A \) admits no non-trivial \( \Gamma \)-invariant closed subgroups. Then for \( G = A \rtimes \Gamma \) we have that \( N_{Tr}^r \cap (A \rtimes \{e\}) \) is either \( A \rtimes \{e\} \) or the trivial subgroup. The latter case is equivalent to the action of \( \Gamma \) factoring through a compact group of automorphisms acting continuously on \( A \).
Proof. The irreducibility condition tells us that \( N_{Tr}^r \cap A \) is either \( A \) or \( \{ e \} \). Let us suppose the latter.

Since \( A_0 \) is characteristic in \( A \), the irreducibility condition entails either that \( A \) is connected or totally disconnected. In the latter case, let \( K \) be any compact open subgroup of \( A \). Then \( \Gamma(K) = A \) so \( G = \langle (K \rtimes \{ e \}) \cup (\{ e \} \rtimes \Gamma) \rangle \) is compactly generated. Hence we see generally that \( G \) is compactly generated. Since \( \{ e \} = N_{Tr}^r \supseteq N_{Tr} \), Corollary 2.8 tells us that \( G \in [\text{SIN}] \), and hence \( A \in [\text{SIN}] \) and admits a neighbourhood base at the identity consisting of \( \Gamma \)-invariant sets.

Notice, in particular, that each \( \Gamma \)-orbit in \( A \) is relatively compact. Indeed, let \( K \) be either a compact generating set (in the case that \( A \) is connected), or an open compact subgroup (in the case that \( A \) is totally disconnected), contained in a relatively compact \( \Gamma \)-invariant neighbourhood of the identity. Then any element of \( \Gamma(K^n) \), for \( n \in \mathbb{N} \) has relatively compact \( \Gamma \)-orbit.

The conditions gathered in the last two paragraphs imply that the image of \( \Gamma \) in the automorphism group of \( A \) (with compact open topology) is relatively compact, thanks to the Ascoli Theorem of Grosse and Moskowitz, [26, Theo. 4.1].

Conversely, suppose there is a compact group \( \Sigma \) of automorphisms on \( A \) which contains the image of \( \Gamma \). For \( \chi \in \hat{A} \setminus \{ 1 \} \), we average over the normalized Haar measure to get \( u = \int_{\Sigma} \chi \circ \sigma \, d\sigma \), which is a \( \Gamma \)-invariant trace on \( A \). For this \( u \) we have \( N_u = \bigcap_{\sigma \in \Sigma} \ker(\chi \circ \sigma) \), which, thanks to the irreducibility assumption, is \( \{ e \} \). \( \square \)

A semi-direct product with a compact group is simpler.

**Proposition 4.10.** Let \( \Gamma \) be a discrete group which acts on a compact group \( K \). Then for \( G = K \rtimes \Gamma \), \( N_{Tr}^r \subseteq K \) if and only if \( X_K = \{ \chi_\pi : \pi \in \hat{K} \} \) admits a non-trivial finite orbit for the adjoint action.

Proof. If \( \sigma \) is a continuous automorphism of \( K \) and \( \pi \in \hat{K} \), then it is evident that \( \pi \circ \sigma \in \hat{K} \) with \( \chi_\pi \circ \sigma = \chi_{\pi \circ \sigma} \).

Since \( T(K) = \text{conv} X_K \), each element \( u \) in \( T(K) \) admits decomposition \( u = \sum_{\pi \in \hat{K}} \bar{u}(\pi) \chi_\pi \) where \( \bar{u} \in \text{Prob}(\hat{K}) \). Then \( u \) is \( \Gamma \)-invariant if and only if each element of \( \{ \chi_\pi \in X_K : \bar{u}(\pi) > 0 \} \) admits finite orbit. If such \( u \) has \( \bar{u}(\pi) > 0 \) for \( \pi \neq 1 \), then \( N_u = \bigcap_{\sigma \in \Gamma} \ker(\pi \circ \sigma) \subseteq K \). \( \square \)

**Remark 4.11.** Theorem 4.9 and Proposition 4.10 also hold if \( G \) is an extension: \( H \to G \to \Gamma \), where \( H = A \) or \( K \). However, these results are intended to aid in the building of examples, for which the semi-direct product formulation is adequate.
Example 4.12. Here we shall always write $G = A \rtimes \Gamma$ or $K \rtimes \Gamma$, and $N^r_{\text{Tr}} = N^r_{\text{Tr}}(G).

(i) Let $\Gamma$ be a non-commutative free group, which is dense inside $\text{SO}(d)$, and $A = \mathbb{R}^d$. Then $N^r_{\text{Tr}} = \{e\}$. We have that $AR(G) = A \rtimes \{I\}$, thanks to \[51\].

(ii) Let $M$ in $\text{GL}_2(\mathbb{R})$ have non-real eigenvalues, of modulus not 1. Let $q : \Gamma = F_2 \to \mathbb{Z}^2$ be the quotient map onto the abelianization, and let $\eta : \mathbb{Z}^2 \to \mathbb{R}$ be given by $\eta(m, n) = m + \xi n$ where $\xi$ is irrational. Let $\Gamma$ act on $A = \mathbb{R}^2$ by $\sigma \cdot x = \exp(\eta \circ q(\sigma) M)x$. This is an irreducible action with non-relatively compact non-trivial $\Gamma$-orbits. Hence $N^r_{\text{Tr}} = A \rtimes \{I\}$. Notice that $\hat{A}$ will support a $\Gamma$-invariant mean on the bounded uniformly continuous functions.

As in (i), above, $AR(G) = A \rtimes \{I\}$.

(iii) Let $\Gamma = \text{SL}_d(\mathbb{Z})$ act on $A = \mathbb{T}^d$. Then Proposition 4.10 shows that $N^r_{\text{Tr}} = A \rtimes \{I\}$.

It is shown in [3] that $\text{PSL}_d(\mathbb{Z})$ admits unique reduced trace. This group is $\Gamma$ when $d$ is odd and $\Gamma/\langle -I \rangle$ when $d$ is even. It follows that

$$AR(G) = \begin{cases} A \rtimes \{I\} & \text{if } d \text{ is odd, and} \\ A \rtimes \langle -I \rangle & \text{if } d \text{ is even.} \end{cases}$$

(iii') Let $\Gamma = \text{SL}_d(\mathbb{Z})$ act on $A = \mathbb{R}^d$. Then $A$ is not irreducible for the action of $\Gamma$, but $G$ is compactly generated. Hence, as in the proof of Theorem 4.9, if we had $N^r_{\text{Tr}} \cap (A \rtimes \{I\}) = \{(0, I)\}$, then $A$ would have a base of $\Gamma$-invariant neighbourhoods at the identity, which is clearly false. The only other closed $\Gamma$-invariant subgroups are lattices $L \rtimes \{I\}$, and $A \rtimes \{I\}$. If we had $N^r_{\text{Tr}} = L \rtimes \{I\}$, then (2.1) shows that we would violate (iii). Hence $N^r_{\text{Tr}} = A \rtimes I$.

In this example we may instead take $\Gamma = \text{SL}_d(\mathbb{Q})$. Though this group is not finitely generated, we just observed a subgroup which allows no non-trivial invariant traces on $A$. Since $\text{PSL}_d(\mathbb{Q}) = \Gamma/\langle -I \rangle$ is simple, we get descriptions of $AR(G)$, similar to those in (iii) above.

(iii'') Consider a free group $\Gamma = F_2$ embedded into $S = \text{SL}_2(\mathbb{Z})$ with finite index. Again for $A = \mathbb{R}^2$ or $A = \mathbb{T}^2$ we see that $N^r_{\text{Tr}} = A \rtimes \{e\}$. As in (i), above, $AR(G) = A \rtimes \{I\}$.

(iii'') We let $\Gamma = F_2$ as above, act on $A = H$ where $H$ is from Example 3.4 (ii). Here $A$ is nilpotent with closed derived subgroup $A/A' \cong \mathbb{R}^2$. Then it follows (3.3) that $T(A) = T(A/A') \circ q_{A'}$. Thus this reduces to (iii''), above, and we see that $N^r_{\text{Tr}} = A \rtimes \{e\}$.
(iv) Let $\Gamma$ be any infinite discrete group and let it act on a product group $K = L^G$, where $L$ is compact, by shifting index. Each non-trivial member of $\hat{K}$ is a finite Kroenecker product $\pi_1 \times \cdots \times \pi_n$ where each element $\pi_j$ acts on a distinct copy of $L$ in $K$. Then non-trivial adjoint orbits in $X_K$ are infinite, so Proposition 4.10 shows that $N^r_{\Gamma} = K \times \{e\}$.

(v) Let $K$ be a semi-simple compact Lie group, so $K = \prod_{i \in 1}^n S_i/D$ where each $S_i$ is a simple compact Lie group and $D$ is a finite central subgroup. By looking at automorphisms of the associated Lie algebra, we see that $\text{Aut}(K)$ is a subgroup of $I \ltimes P$, where $I = \prod_{i \in 1}^n (S_i/Z_i)$ is the group of inner automorphisms where each $Z_i$ is the centre of $S_i$, and $P$ is a discrete group of permutations of isomorphic constituents $S_i$. Elements of $I$ fix elements of $X_K$, so a group $\Gamma \subset \text{Aut}(K)$, acts on $X_K$ as does $\Gamma/(\Gamma \cap I) \subseteq P$, and hence has finite orbits. It follows that $N^r_{\Gamma} = \{(I, e)\}$.

(vi) We let $S = \text{SL}_2(\mathbb{Z}/p)$ act on $M_2(\mathbb{Q}_p)$ by $\sigma \cdot x = \sigma x \sigma^T$, which leaves invariant the subspace $L = \mathbb{Q}_p \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and induces an action of $\Gamma = \text{PSL}_2(\mathbb{Z}/p) \cong S/\langle -I \rangle$ on $A = M_2(\mathbb{Q}_p)/L$. Here, no non-trivial $\Gamma$-orbit in $A$ is relatively compact, so $N^r_{\Gamma} = A \times \{e\}$. Since $\Gamma$ admits unique reduced trace $\text{Tr} = (\Gamma)$, we see that $\text{AR}(G) = A \times \{e\}$, as well.

(vii) Let $\Gamma = \text{SL}_d(\mathbb{Z})$ act on $A = \mathbb{Q}_p^d$. This fails the irreducibility hypothesis of Theorem 4.9, as subgroups $p^n \mathbb{Q}_p^d$ are each $\Gamma$-invariant. Here the action factors through $\text{SL}_d(\mathbb{Q}_p)$, so conclusion (b) of the theorem still holds.

Recall that we say that $G$ if $C^*-simple$ of $C^*_r(G)$ is simple. This is well-known to imply that $\text{AR}(G) = \{e\}$. [Indeed, the method of Lemma 4.11(ii) shows that $C^*_r(G/\text{AR}(G))$ is a proper quotient of $C^*_r(G)$ if $\text{AR}(G) \neq \{e\}$.] In each example above, $C^*_r(G)$ admits as a proper quotient $C^*_r(\Gamma)$, so none of these examples give a $C^*$-simple group. Le Boudec $[30]$ produces examples of discrete groups with $\text{AR}(G) = \{e\}$, but are not $C^*$-simple.

We saw that connected groups admitting non-compact semi-simple quotients have non-open amenability radical. The same extends to certain semi-simple algebraic groups over non-discrete local fields, such as $\text{SL}_n(k)$ and $\text{Sp}_n(k)$ for general $k$, and $\text{SO}_n(k)$ when $k \neq \mathbb{R}$. Each of these is admits only the trivial group as a discrete quotient. Let us next consider further examples that admit no reduced traces, but many quotients.

Example 4.13. We recall the notation of Example 2.3. It is straightforward to see that $\text{AR}(G)$ is open if and only if $\{e\} \times F_n \subseteq \text{AR}(\Gamma_n \times F_n)$ for all $n \geq m$ for some $m$. We produce two examples where $\text{T}_r(G) = \emptyset$. 


(i) As we saw in Example 4.12 (i), $AR(\text{SL}_2(\mathbb{Z})) = \langle -I \rangle$. Let $\alpha$ and $G$ be as in Example 2.3 (ii). We have that $AR(\text{SL}_2(\mathbb{Z}) \rtimes \langle \alpha \rangle) = \langle -I \rangle \rtimes \{e\}$.

(ii) The following example is from Suzuki [60]. Each $G_n = \mathbb{Z} \ast (\mathbb{Z}/k_n \mathbb{Z})$ with $k_n \geq 2$ has $AR(G_n) = \{e\}$, by [46]. If $\Gamma_n$ is the normalizer of $\mathbb{Z}$ in $G_n$, then $G_n = \Gamma_n \rtimes F_n$ where $F_n = \mathbb{Z}/k_n \mathbb{Z}$. Notice that this group is residually finite, thanks to Gruenberg [28]. That $\text{T}_r(G) = \emptyset$, in this case, was shown by different means in [25].

The goal of this example was to have a non-discrete C*-simple group with unique reduced trace. Example 4.12, above, gives non-C*-simple groups with unique trace.

5. Amenable Traces

In the spirit of Brown [13, 3.1.6] (really Kirchberg [33, Prop. 3.2]) we shall say that a tracial state $\tau$ on $A$ is amenable (or liftable) provided that $a \otimes b \mapsto \tau(ab)$ extends to a state on the minimal tensor product $A \otimes_{\text{min}} A^\text{op}$.

To study amenable traces on groups we begin by recording the well-known facts that $C^*(G)$ and $C^r_v(G)$ are symmetric, i.e. each isomorphic to its opposite algebra. This allows us to establish concepts and notation.

Let $\pi : G \to \text{U}(\mathcal{H})$ be a unitary representation. We let its $C^*$-algebra and positive definite cone be given by

$$C^*_\pi = \pi(L^1(G)) \quad \text{and} \quad P_\pi = \text{cone}\{\langle \pi(\cdot)\xi,\xi \rangle : \xi \in \mathcal{H}\}.$$ 

We let $\bar{\pi}$ denote the contragradient representation, given by $\langle \bar{\pi}(s)\xi^*,\eta^* \rangle = \langle \eta,\pi(s)\xi \rangle$, where $\xi^*(\zeta) = \langle \zeta,\xi \rangle$ for $\zeta$ in $\mathcal{H}$. We let

$$\varpi = \bigoplus_{u \in \text{P}_1(G)} \pi^\infty_u$$

be the direct sum of GNS representations from all states, with infinite ampliation; so, for example $\varpi^\infty \cong \varpi$. Then $C^*_\varpi(G) \cong C^*_\varpi$.

**Proposition 5.1.** We have an isomorphism $(C^*_\pi)^{\text{op}} \cong C^*_\bar{\pi}$. Hence if we have unitary equivalence, $\pi \cong \bar{\pi}$, then $C^*_\bar{\pi}$ is symmetric.

In particular, each of $C^*(G)$ and $C^*_v(G)$ are symmetric.

**Proof.** Let $F_\pi = \text{span}P_\pi = \text{span}\{\langle \pi(\cdot)\xi,\eta \rangle : \xi,\eta \in \mathcal{H}\}$, where the second equality follows from polarization identity. We have $\langle \pi(\cdot)\xi,\eta^* \rangle = \langle \bar{\pi}(\cdot)\xi^*,\eta^* \rangle$. This shows that if $\pi \cong \bar{\pi}$, then $F_\pi$, equivalently $P_\pi$, is closed under conjugation.
For \( f \in L^1(G) \) define \( \tilde{f}(s) = \Delta(s^{-1})f(s^{-1}) \) for almost every \( s \) in \( G \), so \( f \mapsto \tilde{f} \) is an anti-homomorphism. Then for \( \xi, \eta \) in \( \mathcal{H} \) we have
\[
\langle \pi(\tilde{f})\xi, \eta \rangle = \int_G f(s)\langle \pi(s^{-1})\xi, \eta \rangle \, ds = \langle \pi(f)^*\xi^*, \eta^* \rangle.
\]

Hence \( \|\pi(\tilde{f})\| = \|\pi(f)\| \) for each \( f \) in \( L^1(G) \), so \( f \mapsto \tilde{f} \) induces an anti-isomorphism from \( C^*_\pi \) onto \( C^*_\bar{\pi} \). Furthermore, if \( F_\pi \) is closed under conjugation, then \( \|\pi(f)\| = \|\pi(f)\| \) for each \( f \) in \( L^1(G) \), so \( C^*_\pi \) is symmetric.

We have unitary equivalences
\[
\varpi = \bigoplus_{u \in P_1(G)} \pi_u^\infty = \bigoplus_{u \in P_1(G)} \bar{\pi}_{\bar{u}}^\infty \cong \bigoplus_{u \in P_1(G)} \bar{\pi}_{\bar{u}}^\infty = \varpi.
\]

Meanwhile, we have for \( h \) in \( L^2(G) \) that \( \langle \lambda(\cdot)h, h \rangle = \langle \lambda(\cdot)\bar{h}, \bar{h} \rangle \). These observations provide symmetry for \( C^*(G) \) and \( C^*_r(G) \).

**Remark 5.2.** The algebras \( C^*_\pi \) need not be symmetric. If \( \mathcal{A} \) is a non-symmetric unital \( C^* \)-algebra, let \( \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) be a faithful representation and \( \Gamma = U(\mathcal{A})_d \), the discretized group of unitaries. Then \( \mathcal{A} \cong \pi(\mathcal{A}) = C^*_\pi|_\Gamma \).

If \( u \in T(G) \), define \( \tilde{u} : G \times G \to \mathbb{C} \) by
\[
\tilde{u}(s,t) = u(st^{-1}).
\]
Notice that if if \( (s_1, t_1), \ldots, (s_n, t_n) \in G \times G \) then for \( i, j = 1, \ldots, n \) we have
\[
\tilde{u}(s_i^{-1}s_j, t_i^{-1}t_j) = u(s_i^{-1}s_jt_i^{-1}t_j) = u(t_is_i^{-1}(t_jt_j)^{-1})
\]
from which it follows that \( \tilde{u} \in P_1(G \times G) \).

The universal representation \( \varpi : G \to U(\mathcal{H}_\varpi) \) has Kronecker product given by \( \varpi \times \varpi : G \times G \to U(\mathcal{H}_\varpi \otimes \mathcal{H}_\varpi) \). We let
\[
P_{\text{min}}(G \times G) = \overline{P_{\varpi \times \varpi}^{u*, u}} = \overline{P_{\varpi \times \varpi}^{uc}} \subseteq P(G \times G).
\]
Notice that \( P_{\text{min}}(G \times G) \) corresponds to the positive linear functionals on \( C^*(G) \otimes_{\text{min}} C^*(G) \), and hence comprises all of the positive definite matrix coefficients of representations weakly contained in \( \varpi \times \varpi \). Furthermore
\[
P_{\text{min}}(G \times G) = \text{cone}\{u \times v : u, v \in P(G)\}^{uc}
\]
where each \( u \times v(s,t) = u(s)v(t) \).

Since Proposition 5.1 provides that \( C^*(G) \cong C^*(G)^{\text{op}} \) we can define the set of amenable traces on \( G \) by
\[
T_{\text{am}}(G) = \{u \in T(G) : \tilde{u} \in P_{\text{min}}(G \times G)\}.
\]
Accordingly, we define the \textit{amenable tracial kernel} by
\[ N_{\text{amTr}} = N_{\text{amTr}}(G) = \bigcap_{u \in T_{\text{am}}(G)} N_u \]
and say that \( G \) is \textit{amenable tracially separated} if \( N_{\text{amTr}} = \{ e \} \).

\textbf{Remark 5.3.} If \( u \in T(G) \), then \( \tilde{u} \) corresponds to a state on \( C^*_{\pi_u} \otimes_{\max} (C^*_{\pi_u})^\text{op} \). If \( C^*_{\pi_u} \) is nuclear, then we employ Proposition 5.1 along the way to see that
\[ C^*_{\pi_u} \otimes_{\max} (C^*_{\pi_u})^\text{op} \cong C^*_{\pi_u} \otimes_{\min} C^*_{\bar{\pi}_u} \cong C^*_{\pi_u \times \bar{\pi}_u} \]
so we get the following weak containments:
\[ \pi_{\bar{u}} \prec \pi_u \times \pi_{\bar{u}} \prec \varpi \times \varpi. \]

That is, \( u \in T_{\text{am}}(G) \).

(i) Characters \( \chi_\pi \) of finite dimensional representations \( \pi \) show that \( N_{\text{amTr}} \subseteq N_{\text{MAP}} \).

(ii) If \( G \) is either amenable, almost connected or type I, then \( C^*(G) \), and hence any quotient, is nuclear; see the survey \cite{47} and references therein. Hence \( T(G) = T_{\text{am}}(G) \), and \( N_{\text{Tr}} = N_{\text{amTr}} \).

It has recently been shown by Bekka and Echterhoff \cite{6} that any algebraic group over a local field, \( G(k) \), is type I.

(iii) If \( G \) is almost connected, then it is compactly generated so Proposition 2.5 and the result of Grosser and Moskowitz \cite{27}, (2.9)] show that \( G/N_{\text{amTr}} \in [\text{MAP}] \), and hence \( N_{\text{amTr}} = N_{\text{MAP}} \).

5.1. Property (T). The next result stems from Ozawa \cite{44}, Theo. 7.2], where it is shown that each discrete group with both property (T) and the factorization property (see definition before Theorem 5.7 below) is residually finite.

\textbf{Theorem 5.4.} If \( G \) has property (T), then \( N_{\text{amTr}} = N_{\text{MAP}} \).

\textbf{Proof.} That \( N_{\text{amTr}} \subseteq N_{\text{MAP}} \) is given in the last remark.

If \( u \in T_{\text{am}}(G) \), then since \( \varpi^\infty = \varpi \), we can find a net of unit vectors \( (\xi_i) \) in \( \mathcal{H}_\varpi \otimes \mathcal{H}_\varpi \) for which
\[ \tilde{u} = \text{uc-lim}_i (\varpi \otimes \varpi(\cdot) \xi_i, \xi_i). \]

Restricting to the diagonal subgroup \( G_D = \{(s, s) : s \in G\} \cong G \) we see that
\[ 1 = \tilde{u}|_{G_D} = \text{uc-lim}_i (\varpi \otimes \varpi(\cdot) \xi_i, \xi_i) \]
and hence
\[ 0 = \text{uc-lim}_i \| \varpi \otimes \varpi(\cdot) \xi_i - \xi_i \| \]
The assumption of property (T) allows us to find unit vectors \((\xi'_i)\) tending asymptotically to \((\xi_i)\) with

\[(5.2) \quad 1 = \langle \varpi \otimes \varpi(\cdot) \xi'_i, \xi'_i \rangle \text{ for each } i.\]

Let \(P\) be the projection onto the almost periodic part of \(H_\varpi\), so by Berglund and Rosenblatt [10, 1.14], \(P \otimes P\) is the projection onto the almost periodic part of \(H_\varpi \otimes H_\varpi\). Then (5.2) tells us that \(\xi'_i = P \otimes P \xi'_i\) for each \(i\). We have

\[\tilde{u} = uc-\lim_i (\varpi \times \varpi(\cdot) \xi'_i, \xi'_i) \quad \text{so } u = uc-\lim_i u_i\]

where each \(u_i = \langle \varpi \times \varpi(\cdot, e) \xi'_i, \xi'_i \rangle\) is almost periodic. Thus if \(s \in G \setminus N_u\), \(\langle \varpi \times \varpi(s, e) \xi'_i, \xi'_i \rangle \neq 1\) for some \(i\), so \(s \in G \setminus N_{MAP}\), i.e. \(N_{MAP} \subseteq N_u\). \(\Box\)

**Example 5.5.** (i) The following is motivated by [15, Prop. 2.6.5].

Let \(F\) be a finite simple group, and consider the wreath product \(G = F^{\oplus \mathbb{Z}} \rtimes \mathbb{Z}\). This is readily seen to be amenable and finitely generated. The only subgroups of \(N = F^{\oplus \mathbb{Z}} \rtimes \{0\}\) that are normal in \(G\) are \(\{e\}\) and \(N\) itself. Hence \(N\) is in the kernel of any homomorphism into a finite group. By way of Corollary 2.2 (ii), we see that \(N = N_{MAP}\).

However this is an amenable discrete group so \(N_{amTr} = \{e\}\).

(ii) Let \(k\) be a local field and \(G\) be one of \(SL_n(k)\) for \(n \geq 3\), or \(Sp_{2n}(k)\) for \(k \geq 2\). Then \(G\) has property (T); see [5, §§1.4-1.5]. The only proper normal subgroup of \(G\) is the centre and \(G\) contains a subgroup \(S\) which is isomorphic to \(SL_2(k)\). If \(k\) has characteristic 0, then the proof of [64] shows that \(S \subseteq N_{MAP}\), hence \(N_{MAP} = G(k)\). As noted in Remark 5.3 (ii), \(C^*(G)\) is nuclear. Hence \(N_{Tr} = N_{amTr} = N_{MAP} = G(k)\), and \(T(G) = \{1\}\).

**5.2. Property (F).** The following is of independent interest, and is for use in the next theorem.

**Lemma 5.6.** Let \(u \in T(G)\) and \(K\) be a compact normal subgroup of \(G\) for which \(\int_K u(k) \, dk \neq 0\). Then

\[u_K = \frac{1}{\int_K u(k) \, dk} \int_K u(k) \, dk \in T(G) \text{ with } N_{u_K} = KN_u.\]

If, further, \(u \in T_{am}(G)\), then \(u_K \in T_{am}(G)\) too.

**Proof.** We let \(m_K\) denote the Haar probability measure on \(K\), regarded as a measure on \(G\). Then \(P_K = \pi_u(m_K) = \int_K \pi(k) \, dk\) is a projection on \(H_u\) in the centre of \(\pi_u(G)^0\). Then \(u_K\) is the trace \(u\) compressed by \(P_K\) and normalized, provided the compression is non-zero. Then Lemma 4.1 shows that \(N_{u_K} = \ker \pi_{u_K}\). Here \(\pi_{u_K} = P_K \pi_u(\cdot)|_{P_K H_u}\).
so ker $\pi_{u_K} \supseteq KN_u$. Now if $s \in G \setminus KN_u$, then $\pi_u(s) \not\in \pi_u(K)$, so $\pi_{u_K}(s) \neq P_K$, whence $s \not\in \ker \pi_{u_K}$.

We then have that $\tilde{u}_K$ is the normalized compression by $P_K \otimes P_K$. Hence if we have weak containment $\pi_{\tilde{u}} \prec \varpi \otimes \varpi$, then we have weak containments

$$\pi_{\tilde{u}_K} \prec Z_K \varpi(\cdot)|_{Z_K H_w} \times Z_K \varpi(\cdot)|_{Z_K H_w} \prec \varpi \otimes \varpi$$

where $Z_K = \varpi(m_K)$. In other words, $u_K$ is amenable provided that $u$ is.

We say that $G$ has the factorization property, or property (F), if the left-right regular representation $\lambda \cdot \rho : G \times G \to \mathcal{U}(L^2(G))$, given by

$$\lambda \cdot \rho(s,t)h(x) = \lambda(s)\rho(t)h(x) = h(s^{-1}xt)\Delta(t)^{1/2}$$

satisfies that $\langle \lambda \cdot \rho(\cdot,\cdot)h| h \rangle \in P_{\min}(G \times G)$ for each $h$; i.e. we have weak containment $\lambda \cdot \rho \prec \varpi \times \varpi$ so $\lambda \cdot \rho$ induces a representation of $C^*(G) \otimes_{\min} C^*(G)$. In the case when $G$ is amenable, this is equivalent to the point mass $\delta_e$ being an amenable trace on $C^*(G)$.

**Theorem 5.7.**

(i) If $G \in [\text{SIN}]$ and has property (F), then $G$ is amenable tracially separated.

(ii) If $G$ is totally disconnected and amenable tracially separated, then $G$ has property (F).

**Proof.** (i) If $U$ is a relatively compact conjugation-invariant symmetric neighbourhood of the identity then as in the proof of Proposition 2.3 (i), we see that

$$\tilde{u}_U(s,t) = \frac{m(st^{-1}U \cap U)}{m(U)} = \frac{m(sUt^{-1} \cap U)}{m(U)} = \frac{\langle \lambda(s)\rho(t)1_U,1_U \rangle}{m(U)}.$$

Property (F) implies that $u_U \in T_{\text{am}}(G)$. Furthermore the proof of Proposition 2.4 (i) also shows that this set of such traces establishes amenable tracial separation.

(ii) We let $H$ be a compactly generated open subgroup. Being tracially separated we have $H \in [\text{SIN}]$ by Corollary 2.8.

Fix a compact open normal subgroup $K$ in $H$. If $s \in H \setminus K$, then $sK \subset H \setminus \{e\} = \bigcup_{u \in T_{\text{am}}(H)} (H \setminus N_u)$, there are $u_1, \ldots, u_n$ in $T_{\text{am}}(H)$ so $sK \subset H \setminus N_u$ where $u = \frac{1}{n}(u_1 + \cdots + u_n)$. In particular, $s \not\in KN_u$. Lemma 5.6 then shows that $s \not\in N_{u_K}$.

Let $L$ be a compact subset of $H \setminus K$. For each $s$ in $L$, find as above $u_s$ in $T_{\text{am}}(G)$ which is constant on cosets of $K$, $u_s(e) = 1$ and $u_s(s) \neq 1$. Find $s_1, \ldots, s_n$ in $L$ so $L \subseteq \bigcup_{j=1}^n s_j K$ and let $u_L = \frac{1}{n+1}(1 + u_{s_1} + \cdots + u_{s_n})$ which is in $T_{\text{am}}(H)$ with $u_L|_K = 1$ and $|u_L(s)| < 1$ for any $s$ in $L$. It follows from (5.1) that is closed under pointwise multiplication. Thus
$T_{am}(G)$ is closed under pointwise multiplication. Hence we can form
a net $(u^L_k)$ indexed over the directed product of $L$ in the increasing
set of compact subsets of $H \setminus K$ and $k$ in $\mathbb{N}$. This net converges,
uniformly on compact sets to $1_K$, showing that $1_K \in T_{am}(H)$, hence
$\tilde{1}_K \in P_{\min}(H \times H)$. Recalling that $G$ is unimodular as noted in the
proof of Proposition 2.9 as in the proof of (i), above, we see that

$$\tilde{1}_K = \frac{1}{m(K)} \langle \lambda \cdot \rho(\cdot, \cdot) 1_K, 1_K \rangle \in P_{\min}(H \times H) \subseteq P_{\min}(G \times G).$$

The latter containment holds as $T(H) \cong 1_H T(G) \subseteq T(G)$, and we use
(5.1), above.

It follows from Lemma 4.2 that translates of $1_K$, for all compact open
normal subgroups $K$ uniformly densely span $C_c(H)$, hence norm densely
$L^2(H)$. But the union of $L^2(H)$ where $H$ ranges over the compactly
generated open subgroups of $G$ is dense in $L^2(G)$. Hence for each $h$
in $L^2(G)$, $\langle \lambda \cdot \rho(\cdot, \cdot) h, h \rangle$ can be uniformly approximated on compact
sets by sums of translations of elements $\tilde{1}_K$, above. Thus $G$ admits
property (F).

\begin{proof}
To see (i) we apply Theorems 5.7 (i) and 5.4. To see (ii) we
appeal to Lemma 4.2, then Theorem 5.7 (ii).
\end{proof}

\begin{remark}
Wiersma has shown in [65, Cor. 4.4], that MAP-groups,
enjoy property (F). In light of Remark 5.3 (i), part (ii) generalizes t his
fact for totally disconnected groups, and with a simpler proof.

Part (i) of the next observation captures for discrete groups the tit-ral property of [33], long since recovered by many other means.

\begin{corollary}
(i) If $G \in [\text{SIN}]$ with both properties (T) and (F),
then $G \in [\text{MAP}]$.

(ii) If $G$ is pro-discrete, then property (F) is equivalent to being
amenably tracially separated.
\end{corollary}

\begin{proof}
To see (i) we apply Theorems 5.7 (i) and 5.4. To see (ii) we
appeal to Lemma 4.2, then Theorem 5.7 (ii).
\end{proof}

\begin{example}
(i) We display an example of a non-amenable, non-
discrete, group which is amenably tracially separated, but not maxi-
mal almost periodic.

The group $SL_2(\mathbb{Q}) \ltimes \langle \alpha \rangle$ of Example 2.3 (i) may be written $\bigcup_{n=1}^\infty S_n \ltimes \langle \alpha \rangle$ where each $S_n = SL_2(\mathbb{Z}[\frac{1}{m}])$, i.e. as an increasing union of residually
finite groups. That an increasing union of discrete property (F) groups
admits property (F) is noted in [44, Prop. 7.3]. The group $G = \Gamma \ltimes F$ of
Example 2.3 (i), admits a separating family of quotients, each discrete
with property (F). Hence Theorem 5.7 (i) and 5.11 show that $G$ is
amenably tracially separated, while Theorem 5.7 (ii) (or [65, Cor. 5.5])
shows that $G$ has property (F).

\end{example}
5.3. Amenable reduced traces. Since $L^2(G) \otimes L^2(G) \cong L^2(G \times G)$, we have that $\lambda \times \lambda = \lambda_{G \times G}$ and
\[
C^*_r(G \times G) = C^*_G \times C^*_G = C^*_G \otimes C^*_G.
\]
We consider the set of \textit{reduced amenable traces}
\[
T_{r,\text{am}}(G) = \{ u \in T_r(G) : \tilde{u} \in P_r(G \times G) \}.
\]
For discrete $G$, Lance [32] Theo. 4.2 showed that $C^*_r(G)$ is nuclear if and only if $G$ is amenable. More recently, Ng [43] proved for general locally compact $G$ that $G$ is amenable if and only if $C^*_r(G)$ is nuclear and admits a trace. The following captures the latter result, but the use of amenable traces allows a simpler proof.

**Theorem 5.11.** The following are equivalent:

(i) $G$ is amenable;
(ii) $T_{r,\text{am}}(G) \neq \emptyset$; and
(iii) $C^*_r(G)$ is nuclear and $T_r(G) \neq \emptyset$.

\textit{Proof.} Suppose (i) holds. Then $1 \in T(G) = T_r(G)$ satisfies $\bar{1} = 1 \times 1$ which gives (ii). Also $C^*_r(G) = C^*_r(G)$ is nuclear (see, for example, [47]), so (iii) holds.

Suppose (ii) holds. Let $G_D = \{(s,s) : s \in G\} \cong G$. If $u \in T_{r,\text{am}}(G)$, then Lemma 4.1 provides that $1 = \tilde{u}|_{G_D} \in P_r(G)$, so $G$ is amenable, i.e. we have (i).

If (iii) holds, then $T_{r,\text{am}}(G) = T_r(G)$, so we get (ii).

We can also use Theorem 4.5 ([32]) to prove that (iii) implies (i). Indeed $T_r(G) \neq \emptyset$ implies that $R = R(G)$ is open. If $C^*_r(G)$ is nuclear, then so to is its quotient $C^*_r(G/R)$. We then appeal to [34] Theo. 4.2 to see that $G/R$ must also be amenable, hence $G = R$. \qed

Notice that if $G$ is any non-amenable residually finite discrete group, then [43, Prop. 7.3] (see also Theorem 5.7) provides that $1\{e\} \in T_r \cap
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T_{am}(G)$, while $T_{ram}(G) = \emptyset$. Hence a reduced and amenable trace need not be reduced amenable trace.

6. Embeddability of group C*-algebras into simple AF algebras

In this section, we apply our results on traces to study structural properties of group C*-algebras. We will primarily be interested in AF embeddability of the reduced group C*-algebra. Determining whether a C*-algebra is AF embeddable can be a very difficult problem since no abstract characterization of C*-subalgebras of AF algebras is known. As such, we will satisfy ourselves by studying embeddability in unital, simple AF algebras, where a deep theorem of Schafhauser can be applied [55, Theorem A].

Since AF algebras are quasidiagonal, the problem of determining when the reduced group C*-algebra embeds into an AF algebra is deeply related to when the reduced group C*-algebra is quasidiagonal. In 2017, Tikuisis, White and Winter proved a discrete group $G$ is amenable if and only if $C^*_r(G)$ is quasidiagonal [62, Corollary C], solving the Rosenberg conjecture in the affirmative. Moreover, Tikuisis, White and Winter show that if $G$ is countable and amenable, then $C^*_r(G)$ embeds into an AF-algebra [62, Corollary 6.6]. Schafhauser further strengthened this result by showing that $C^*_r(G)$ embeds into the universal UHF algebra $\mathcal{Q} = \otimes_{n=1}^{\infty} M_n$, which is a unital, simple AF algebra, for all countable, discrete groups $G$ [55, Theorem B]. Building upon Schafhauser’s result, we characterize all second countable locally compact groups whose reduced group C*-algebra embeds into a unital, simple AF algebra.

We will require the following lemma, whose proof is similar to that of [7, Theorem 1.3].

Lemma 6.1. Let $G$ be a locally compact group. For each $u \in T(G)$, let $\pi_u$ denote the GNS representation of $u$.

(i) $S_{Tr} = \{ \pi_u : u \in T(G) \}$ weakly contains $\lambda_{G/N_{Tr}}$;
(ii) $S'_{Tr} = \{ \pi_u : u \in T_r(G) \}$ weakly contains $\lambda_{G/N_{Tr}'}$;
(iii) $S_{amTr} = \{ \pi_u : u \in T_{am}(G) \}$ weakly contains $\lambda_{G/N_{amTr}}$.

Proof. The proofs of these three claims are nearly identical. We shall only prove (i).

Let $q: G \to G/N_{Tr}$ denote the canonical quotient map. For each $u \in T(G)$, we let $u'$ be the unique element of $P(G/N_{Tr})$ such that $u = u' \circ q$ and $\pi_u'$ denote the unique representation of $G/N_{Tr}$ such that $\pi_u = \pi_u' \circ q$. Further, we set $T'(G) = \{ u' : u \in T(G) \}$. Since $T'(G)$
is closed under pointwise multiplication, Lemma 1.1] implies there exists a net \( \{v'_\alpha\} \) contained in the positive cone generated by \( T'(G), \) \( P_{S_1}(G/N_{\Theta}) \), so that \( v'_\alpha dm_{G/N_{\Theta}} \) converges to the point mass \( \delta_q(e) \) with respect to the topology \( \sigma(M(G/N_{\Theta}), C_\sigma(G/N_{\Theta})) \). Let \( f \in C_\sigma(G/N_{\Theta}) \).

If we consider the natural action of \( C^*(G/N_{\Theta}) \) on \( B(G/N_{\Theta}) \), we then have \( f \cdot v'_\alpha \cdot f^* \in P_{S_1}(G/N_{\Theta}) \) for every \( \alpha \). Since

\[
\lim_\alpha \|f \cdot v'_\alpha \cdot f^*\|_{B(G/N_{\Theta})} = \lim_\alpha f \cdot v'_\alpha \cdot f^*(q(e)) = \lim_\alpha \langle f \cdot v'_\alpha \cdot f^*, \delta_q(e) \rangle \\
= \lim_\alpha \langle v'_\alpha, f^* f \rangle = f^* f(g(e))
\]

we may assume that the net \( \{f \cdot v'_\alpha \cdot f^*\} \) is bounded in \( B(G/N_{\Theta}) \). We also calculate that for all \( g \in C_c(G/N_{\Theta}) \) that

\[
\lim_\alpha (f \cdot v'_\alpha \cdot f^*, g) = \lim_\alpha \langle v'_\alpha, f^* g f \rangle \\
= f^* g f(e) = \langle \lambda_{G/N_{\Theta}}(g)f, f \rangle.
\]

It follows that \( (f \cdot v'_\alpha \cdot f^*) \) converges to the element \( u' = \langle \lambda_{G/N_{\Theta}}(\cdot)f, f \rangle \) in the weak* topology on \( B(G/N_{\Theta}) \). Hence, \( \lambda_{G/N_{\Theta}} \) is weakly contained in \( S'_1 = \{\pi' : u \in T(G)\} \), viewed as representations of \( G/N_{\Theta} \), and \( u' = \lim_\alpha f \cdot v'_\alpha \cdot f^* \) uniformly on compact subsets of \( G/N_{\Theta} \). But then \( u = u' \circ q \) is the uniform limit on compact subsets of \( G \) of \( ((f \cdot v'_\alpha \cdot f^*) \circ q) \), and this implies (i).

The following theorem greatly generalizes Cor. 2.10], which asserts the C*-algebra of a connected, second countable, locally compact, solvable group embeds into a simple, unital AF algebra if and only if the group is abelian.

**Theorem 6.2.** Let \( G \) be a second countable locally compact group. The following are equivalent.

1. \( C^*_r(G) \) embeds into a simple, unital AF algebra.
2. \( C^*_r(G) \) admits a faithful amenable trace.
3. \( G \) is amenable and tracially separated.

**Proof.** (i) \( \Rightarrow \) (ii). A unital AF algebra is always nuclear and has a faithful trace. Hence any C*-subalgebra admits an amenable trace.

(ii) \( \Rightarrow \) (iii). Suppose \( C^*_r(G) \) admits a faithful amenable trace \( u \). It is immediate from Theorem 5.11 that \( G \) is amenable. Since \( u \) is faithful, \( N_u = \{e\} \) and \( G \) is tracially separated.

(iii) \( \Rightarrow \) (i): Suppose \( G \) is amenable and tracially separated. Since \( G \) is second countable, the group algebra \( L^1(G) \) is separable. Let \( \{f_n\}_{n=1}^\infty \) be a sequence in \( L^1(G) \) with dense range. Since \( \lambda \) is weakly contained in \( \{\pi_u : u \in T(G)\} \) by the previous lemma, for each natural number \( n \), we can find \( u_n \in T(G) \) such that \( \|\pi_{u_n}(f)\| \geq \frac{1}{2}\|\lambda(f_n)\| \). Then
\[ \{ \pi_n : n \in \mathbb{N} \} \text{ separates points of } C^*_r(G) \text{ and, so, } u := \sum_{n=1}^{\infty} \frac{1}{2^n} u_n \text{ is a faithful trace on the separable, nuclear C*-algebra } C^*_r(G). \] As the reduced group C*-algebra of an amenable, second countable locally compact group satisfies the UCT by a result of Tu [63], we conclude that \( C^*_r(G) \) embeds into a AF algebra by Schafhauser’s theorem [55, Theorem A].

If we further restrict our attention to only second countable, compactly generated groups \( G \), the results obtained in Section 2 combined with Theorem 6.2 imply \( C^*_r(G) \) embeds into a simple, unital AF algebra if and only if \( G \) is amenable and SIN.

As an immediate consequence of Theorem 6.2, one obtains that if \( G \) is an amenable, tracially separated, second countable locally compact group, then \( C^*_r(G) \) is quasidiagonal. We can remove the assumption of second countability. We begin by mildly expanding on an observation of Lau and Losert [35, Rem. 14(b)] which gives a variant of the Kakutani-Kodaira Theorem.

**Lemma 6.3.** Let \( H \) be a \( \sigma \)-compact locally compact group. Then there is decreasing net \( (K_i) \) of compact normal subgroups for which \( \bigcap_i K_i = \{e\} \) and each \( H/K_i \) is metrizable, hence second countable.

**Proof.** For any \( f \) in \( C_c(H) \), \( \sigma \)-compactness of \( H \) shows that the set of translates \( \{ f(r \cdot t) \}_{r,t \in G} \) is separable in \( C_0(H) \). For any \( s \in H \setminus \{e\} \), let \( f_s \in C_c(G) \) be so \( f_s(s) = 0 \) while \( f_s(e) = 1 \). Then for each finite subset \( F \subset H \), the smallest closed subalgebra containing all translates of \( \{ f_s \}_{s \in F} \) in \( C_0(H) \) is separable, and, by [35, Lem. 12], isomorphic to \( C_0(G/K_F) \) for a compact normal subgroup \( K_F \), so \( H/K_F \) is \( \sigma \)-compact. Bounded sets in the dual space are hence metrizable, so \( G/K_F \) is metrizable. With our choices, \( K_F \supseteq K_{F'} \) if \( F \subseteq F' \). In a metrizable space any compact set is second countable, so \( \sigma \)-compact metrizable spaces are second countable.

Comments in [37, Ex. 1.19] show that we cannot relax the assumption of \( \sigma \)-compactness, above.

**Corollary 6.4.** If \( G \) is an amenable, tracially separated locally compact group, then \( C^*_r(G) \) is quasidiagonal.

**Proof.** Let \( H \) be a compactly generated, hence \( \sigma \)-compact subgroup of \( G \). Let \( (K_i) \) be as in Lemma 6.3. Then each \( P_i = \lambda(m_{K_i}) \) is a central projection in the multiplier algebra of \( C^*_r(H) \) with \( C^*_r(H)P_i \cong C^*_r(H/K_i) \), which is quasi-diagonal as a consequence of Theorem 6.2(i). Furthermore \( \bigcup_i C^*_r(H)P_i \) is dense in \( C^*_r(H) \), showing that the latter is quasi-diagonal, thanks to, for example, [13, Prop. 7.1.9]. But then
the union $\bigcup_H C_r^*(H)$, over all compactly generated open subgroups, is
norm dense in $C_r^*(G)$, giving the desired result. \hfill \Box

Interestingly, the condition of $G$ being tracially separated in Corol-
lar 6.4 cannot be dropped. Indeed, examples of amenable Lie groups
whose reduced $C^*$-algebra is not quasidiagonal are given in [8].

As a further application of the techniques we have developed in this
paper, we show that full $C^*$-algebras of property (T) groups do not
embed inside of simple, unital AF algebras.

**Proposition 6.5.** Let $G$ be a non-compact locally compact group with
property (T). Then $C^*(G)$ does not embed inside of a simple, unital AF
algebra.

*Proof.* Suppose towards a contradiction that $C^*(G)$ embeds into a sim-
ple, unital AF algebra. Then $C^*(G)$ admits a faithful, amenable trace.
Consequently, $N_{\text{amTr}} = \{e\}$ implying that $G$ is maximally almost peri-
odic by Theorem 5.4. So $G$ has the factorization property by [65, Cor.
4.4]. Since groups with property (T) are compactly generated and $G$
tracially separated, we also have that $G$ is a SIN group. It now follows
from [65, Prop. 3.1] that $C^*(G)$ is not exact since SIN groups are inner
amenable, contradicting the fact that $C^*$-subalgebras of AF algebras
are exact. \hfill \Box

7. Questions

Proposition 2.4 shows that $N_{\text{Tr}} \subseteq N_{\text{SIN}}$, and these coincide if $G$
is compactly generated.

**Question 7.1.** Is there a locally compact group for which $N_{\text{Tr}} \subset N_{\text{SIN}}$?

Theorem 5.7 shows that an amenable tracially separated totally dis-
connected group admits property (F). Furthermore, if $G$ is (amenable)
tracially separated, then $G_0 = V \times K$ is amenable.

**Question 7.2.** If $G$ is amenable tracially separated, does it admit prop-
erity (F)?
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