On asymptotic expansions of the density of states for Poisson distributed random Schrödinger operators

David Hasler\textsuperscript{1*} Jannis Koberstein\textsuperscript{1†}
1. Department of Mathematics, Friedrich Schiller University Jena
Jena, Germany

Abstract

We study expectation values of matrix elements for boundary values of the resolvent as well as the density of states for a random Schrödinger operator with potential distributed according to a Poisson process. Asymptotic expansions for these quantities in the limit of small disorder are derived. Explicit estimates for the expansion coefficients are given and we show that their infinite volume limits are in fact finite as the spectral parameter approaches the spectrum of the free Laplacian.

1 Introduction

Solids occur in nature in various forms and are almost always subject to disorder of some form. Sometimes they are (almost) totally ordered, sometimes they are (more or less) completely disordered. A random Schrödinger operator is a mathematical caricature to describe quantum mechanical aspects of such random systems in an approximation where many-body interactions are neglected. Anderson introduced a random Schrödinger operator where the potential was given by a sum of potentials arranged in a lattice with random strength at each lattice site [PW58]. In this paper, we study a random Schrödinger operator where the randomness is given by a Poisson distributed random potential. These random potentials are used to model amorphous materials like glass or rubber.

The physical quantities we investigate are expectations of matrix elements of the resolvent of the random Schrödinger operator and sums of such expressions, which can be used to determine the so-called density of states. The density of states determines the expected number of eigenvalues per unit volume in an energy interval. It is of fundamental importance in condensed matter physics and can be used to study physical (in particular thermodynamical) properties of disordered systems. For example vanishing density at the so called Fermi-energy explain isolating properties of solids. Furthermore, upper

\textsuperscript{*}E-mail: david.hasler@uni-jena.de
\textsuperscript{†}E-mail: jannis.koberstein@eah-jena.de
bounds for the density of states, more precisely the so called Wegner estimate, constitutes an essential input in the so called multiscale analysis proof for localization of random Schrödinger operators.

First, we consider expectations of matrix elements of the resolvent of a random Schrödinger operator with a random potential given by a Poisson measure. In particular, we investigate this quantity for small disorder and its behaviour as the disorder strength tends to zero. It is expected that in this limit one recovers the corresponding quantity for the free Schrödinger operator. We show that on a certain energy scale this is indeed the case. More explicitly, we show that the expectations have an asymptotic expansion as the disorder strength tends to zero in a certain scaling limit. Furthermore, we show that the infinite volume limit of the expansion coefficients are finite as the spectral parameter approaches the real axis, inside the spectrum of the free Laplacian. We note that a related expansion of the resolvent has been studied in [MPR97; MPR98; Poi99]. In these works the asymptotics for small disorder for a random Schrödinger operator in two dimensions with a random potential given by a Gaussian random field was studied. Some of the methods we use are inspired by the expansion techniques used in [ESY07a; ESY07b; ESY08]. The estimates of the infinite volume limit of the expansion coefficients using the method of analytic dilation, cf. [CT73; RS78] and references therein.

In the second part of the paper we extend the first result to sums of matrix elements which can be used to study the density of states, cf. [KM07] and references therein. We show that the density of states admits an asymptotic expansion for small disorder on a certain energy scale. This result is complementary to a optimal volume Wegner-type estimate [Weg81], which estimates the density of states in terms of the inverse of the disorder strength and are used to obtain estimate for the strong disorder regime. We note that for the Anderson model optimal volume Wegner estimates have been established in [CHK03; CHK07] giving a bound, which is inverse in the disorder strength. There are Wegner estimates for Poisson distributed random potentials [CH94], cf. references in [KM07], which are not optimal in the volume. However, we are not aware of an optimal volume Wegner-type estimate for a Poisson distributed random potential. In the small disorder regime much less is known. For results in this regime we want to recall the expansion results [MPR97; MPR98; Poi99]. Furthermore, for small coupling, expectations of functions of the resolvent were studied for the Anderson model on the Bethe lattice [AK92; Kle98; FHS06; ASW06; FHS07; AW13], for the discrete lattice in the special case of a Cauchy distribution [P69; KK20], and recently for quantum trees [Ana+21]. Moreover, we want to mention deterministic perturbation results about distributions of eigenvalues for periodic Schrödinger operators [FKT90; FKT91].

Our estimates are uniform in the volume, which in principle allows the investigation of the the infinite volume limit. We note that the existence of the infinite volume density has been shown for the Anderson model, for an early review see [Pas73] and references therein. For a random Schrödinger operator with Poisson distributed positive random potential this has been shown in [KM82].

Let us give a brief outline of this paper. In Section 2 we introduce the model and state the main results about expectations of matrix elements of resolvents. In Section 3
we study expectations of products of the random potential, which will be needed for the proofs. In Section 4 we prove results about the expansion coefficients for expectations of matrix elements of the resolvent and show that for their infinite volume limit the spectral parameter can be extended continuously to the real axis. In Section 5 we then show that these expansion coefficients indeed yield an asymptotic expansions of expectations of matrix values of the resolvent. In Section 6 we generalize the asymptotic expansion to the trace, i.e., we state the result about the asymptotic expansion of the density of states and give its proof.

2 Model and Expectations of Resolvents

In this section we introduce the model and state the main results about the expectation of the resolvent. We note that the definition of the model follows the one given in [ESY08] closely. We consider a finite box of size $L$, with $L > 0$, and define the box $\Lambda_L := [-\frac{1}{2}L, \frac{1}{2}L]^d \subset \mathbb{R}^d$. Let $\langle \cdot, \cdot \rangle_{L^2(\Lambda_L)}$ and $\| \cdot \|_{L^2(\Lambda_L)}$ denote the canonical scalar product and the norm of the Hilbert space $L^2(\Lambda_L)$. If it is clear from the context what the inner product or the norm is, we shall occasionally drop the subscript $L^2(\Lambda_L)$. The kinetic energy will be given by the periodic Laplacian. To introduce this operator it is convenient to work in terms of Fourier series. Let $\Lambda_L^* = (\frac{1}{L}\mathbb{Z})^d = \{(k_1, \ldots, k_d) : \forall j = 1, \ldots, d, \exists m_j \in \mathbb{Z}, : k_j = \frac{m_j}{L}\}$ denote the so-called dual lattice. We introduce the notation

$$\int_{\Lambda_L^*} f(p) dp := \frac{1}{|\Lambda_L|} \sum_{p \in \Lambda_L^*} f(p).$$ (2.1)

The sum $\int_{\Lambda_L^*} f(p) dp$ can be interpreted as a Riemann-sum, which converges to the integral $\int f(p) dp$ as $L \to \infty$, provided $f$ has sufficient decay at infinity and sufficient regularity. For any $f \in L^1(\Lambda_L)$ we define the Fourier series

$$\hat{f}(p) = \int_{\Lambda_L} e^{-2\pi ip \cdot x} f(x) dx \text{ for } p \in \Lambda_L^*.$$ 

For $g \in \ell^1(\Lambda_L^*)$ we define

$$\tilde{g}(x) = \int_{\Lambda_L^*} e^{2\pi ip \cdot x} g(p) dp \frac{1}{|\Lambda_L|} \sum_{p \in \Lambda_L^*} e^{2\pi ip \cdot x} g(p) \text{ for } x \in \Lambda_L.$$ 

Note that $\tilde{\cdot}$ extends uniquely to a continuous linear map on $\ell^2(\Lambda_L^*)$. We shall denote this extension again by the same symbol. This extension maps $\ell^2(\Lambda_L^*)$ unitarily to $L^2(\Lambda_L)$ and is the inverse of $\hat{\cdot}|_{L^2(\Lambda_L)}$, see for example [Fol13, Theorem 8.20]. From a physics background one might be familiar with the notation where $L^2(\Lambda_L)$ represents the position space, while $\ell^2(\Lambda_L^*)$ would be called momentum space, or Fourier space. For $p \in \Lambda_L^*$ and $x \in \Lambda_L$ we define

$$\varphi_p(x) = |\Lambda_L|^{-1/2} e^{i2\pi p \cdot x}.$$ (2.2)
Observe that \( \{ \varphi_p : p \in \Lambda^*_L \} \) is an orthonormal Basis (ONB) of \( L^2(\Lambda_L) \) \cite{Fol13}, Theorem 8.20] and that \( \hat{f}(p) = \| \Lambda_L \|^{1/2} \langle \varphi_p, f \rangle_{L^2(\Lambda_L)} \).

When defining the Laplacian restricted to a finite box, we have to choose boundary condition to describe how our operator behaves at the border of the box. We are presented with the choice between Dirichlet, Neumann and periodic boundary conditions. Note that in the sense of forms, periodic boundary conditions lie between Dirichlet and Neumann boundary conditions, where the latter two are better suited, when working in position space. While we will be starting in position space, the majority of this work will be performed in momentum space., where using periodic boundary conditions is technically convenient. Therefore, we are going introduce the Laplacian with periodic boundary conditions on \( L^2(\Lambda_L) \) by means of Fourier series. In this paper we shall adapt the physics convention that for a vector \( a = (a_1, ..., a_d) \) we use the notation \( a^2 := |a|^2 \) where \(|a| := \sqrt{\sum_{j=1}^d |a_j|^2} \). Using this notation, we define the energy function

\[
\nu : \mathbb{R}^d \to \mathbb{R}^+, \quad p \mapsto \nu(p) := \frac{1}{2} p^2 , \tag{2.3}
\]

which we can now use to define \( -\Delta_L \) as the linear operator with domain

\[
D(-\Delta_L) = \{ f \in L^2(\Lambda_L) : \nu \hat{f} \in \ell^2(\Lambda^*_L) \} ,
\]

and the mapping rule

\[
-\Delta_L : D(-\Delta_L) \to L^2(\Lambda_L) , \quad f \mapsto -\Delta_L f = (2\pi)^2 (\nu \hat{f})^\vee . \tag{2.4}
\]

Observe that \( -\Delta_L \) is selfadjoint, since it is unitary equivalent to a multiplication operator by a real valued function, and that we have the identity

\[
\left[ -\frac{1}{2(2\pi)^2} \Delta_L \right]^\wedge (p) = \nu(p) \hat{f}(p) . \tag{2.5}
\]

By

\[
H_L := H_{\lambda,L} := -\frac{\hbar^2}{2m} \Delta_L + \lambda V_L \tag{2.6}
\]

we denote a random Schrödinger operator acting in \( L^2(\Lambda_L) \) with a random potential \( V_L = V_{L,\omega}(x) \), defined below, and a coupling constant \( \lambda \geq 0 \). As in \cite{ESY08} we choose units for the mass \( m \) and Planck’s constant so that \( \frac{\hbar^2}{2m} = [2(2\pi)^2]^{-1} \). For a function \( h : \mathbb{R}^d \to \mathbb{C} \) we denote by \( h_\# \) the \( L \)-periodic extension of \( h|_{\Lambda_L} \) to \( \mathbb{R}^d \). The potential is given by

\[
V_{L,\omega}(x) := \int_{\Lambda_L} B_\#(x-y) d\mu_{L,\omega}(y) , \tag{2.7}
\]

where \( B \), having the physical interpretation as a single site potential profile, is assumed to be a real valued Schwartz function on \( \mathbb{R}^d \). Moreover, we assume that either \( B \) has
compact support or that $B$ is symmetric with respect to the reflections of the coordinate axis, i.e. for $j = 1, \ldots, d$

$$\mathcal{G}_j : (x_1, \ldots, x_j, \ldots, x_d) \mapsto (x_1, \ldots, -x_j, \ldots, x_d)$$

(2.8)

$$B \circ \mathcal{G}_j = B.$$ 

**Remark 2.1.** Each of the two conditions is mathematically convenient in the sense that they ensure sufficiently fast decay of the Fourier transform. While the reflection symmetry condition is satisfied by rationally invariant potentials, which occur naturally.

Furthermore, $\mu_{L,\omega}$ is a Poisson point measure on $\Lambda_L$ with homogeneous unit density and with independent identically distributed (i.i.d.) random masses. More precisely, for almost all events $\omega$, it consists of $M$ points $\{y_{L,\gamma}(\omega) \in \Lambda_L : \gamma = 1, 2, \ldots, M\}$, where $M = M(\omega)$ is a Poisson variable with expectation $|\Lambda_L|$, and $\{y_{L,\gamma}(\omega)\}$ are i.i.d. random variables uniformly distributed on $\Lambda_L$. Both are independent of the random i.i.d. weights $\{v_\gamma : \gamma = 1, 2, \ldots\}$ and the random measure is given by

$$\mu_{L,\omega} = \sum_{\gamma=1}^{M(\omega)} v_\gamma(\omega) \delta_{y_{L,\gamma}(\omega)},$$

where $\delta_y$ denotes the Dirac mass at the point $y$. Note that the case where $M(\omega) = 0$ corresponds to a vanishing potential. Furthermore we denote the the common distribution weights $\{v_\gamma\}$ by $P_v$ and assume that the moments

$$m_k := \mathbf{E} v_\gamma^k$$

(2.9)

satisfy

$$m_k < \infty, \quad \text{for all } k \in \mathbb{N}.$$ 

(2.10)

For some of the results we will use that the first moment $m_1 = \mathbf{E} v_\gamma$ vanishes. Note that for convenience the notation will not reflect the dependence on the specific choice of the random distribution $P_v$. Observe that we can write (2.7) as

$$V_{L,\omega}(x) = \sum_{\gamma=1}^{M} V_{L,\gamma}(x) \quad \text{with} \quad V_{L,\gamma}(x) := v_\gamma B_\#(x - y_{L,\gamma}).$$

(2.11)

The expectation with respect to the joint measure of $\{M, y_{L,\gamma}, v_\gamma\}$ is denoted by $\mathbf{E}_L$. Sometimes we will use the notation

$$\mathbf{E}_L = \mathbf{E}_M \mathbf{E}_{y_L}^{\otimes M} \mathbf{E}_v^{\otimes M}$$

(2.12)

referring to the expectation of $M$, $\{y_{L,\gamma}\}$ and $\{v_\gamma\}$ separately. In particular, $\mathbf{E}_y^{\otimes M}$ stands for the normalized integral

$$\frac{1}{|\Lambda_L|^M} \int_{(\Lambda_L)^M} dy_1 \cdots dy_M.$$ 

(2.13)
Since the potential is almost surely bounded it follows by standard perturbation theorems, e.g. the Kato-Rellich theorem [RS75], that the operator (2.6) is almost surely self-adjoint for all $\lambda \geq 0$.

The first object of our interest is the expectation of matrix values of the resolvent

$$E_L \langle \psi_1, (H_L - z)^{-1} \psi_2 \rangle \quad \text{for } \psi_1, \psi_2 \in L^2(\Lambda_L)$$

as the spectral parameter approaches the real axis. Note that (2.14) is the expectation of the interacting resolvent, which is difficult to study. To analyze (2.14) we will use a Neumann-Type expansion to express the interacting resolvent as the sum of products of powers of the potential and the resolvent of the periodic Laplacian, $\Delta_L$. This is the content of Lemma 2.15 below. For notational compactness we shall denote the resolvent of $-\Delta_L$ by

$$R_L(z) := \left(-\frac{\hbar^2}{2m} \Delta_L - z\right)^{-1},$$

where $z \in \mathbb{C}\setminus[0, \infty)$. Note that we use a notation for the resolvent of the free Laplacian, which one might expect for the interacting one. Iterating the second resolvent identity A.1 yields the following lemma.

**Lemma 2.2.** For $z \in \mathbb{C}\setminus[0, \infty)$, $L > 0$, and $n \in \mathbb{N}$ we have

$$(H_L - z)^{-1} = \sum_{j=0}^{n} R_L(z)[\lambda V_L R_L(z)]^j + [R_L(z)\lambda V_L]^{n+1}(H_L - z)^{-1}. \quad (2.15)$$

**Proof.** This follows directly from Lemma A.3 with $A = -\frac{\hbar^2}{2m} \Delta_L - z$ and $B = \lambda V_L$, since $\sigma(H_L) \subset [0, \infty)$ see for example [AW15].

With (2.15) on our hands, we can use the linearity of both the scalar product in each entry as well as the expectation value. With this we can now study the following expressions. For $z \in \mathbb{C}\setminus[0, \infty)$ and $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$ we define

$$T_{n,L}[z; \psi_1, \psi_2] := E_L \langle \psi_1, \psi_2 \# R_L(z)[V_L R_L(z)]^n \psi_2, \psi_2 \# \rangle. \quad (2.16)$$

By definition the potential is almost surely bounded. Thus (2.16) is well defined almost surely. We are now going to work towards Theorem 2.3 which will express (2.16) in momentum space. For this, we introduce the discrete delta function in momentum space for $u \in \Lambda_L^*$

$$\delta_s(u) = 1_{\{0\}}(u) \quad (2.17)$$

and a normalized discrete delta function

$$\delta_{s,L}(u) = |\Lambda_L|\delta_s(u), \quad (2.18)$$

where we used the notation that for a set $A$ we write $1_A(x) = 1$, if $x \in A$, and $1_A(x) = 0$, if $x \notin A$. 

6
We now need to consider the expectation of the product of the random variable \( V_L \) i.e. \( E_L \left[ \prod_{j=1}^{n} v_{\gamma_j} \right] \), for they will appear from (2.13). This topic will be discussed in more detail in Section 3. The short idea is that we can use independence of the variables as long as they are different, while for the ones that are the same, we count how often each of the variables occurs: Multiple appearances of the same random variable will result in a higher moment of the respective variable after evaluating the expected value.

To express the result of this procedure we introduce the following partition function. It (in a sense) filters all possible partition and leave only one remaining, which represents the configuration of the momenta. Meaning that every set \( a \) in the partition \( A \) has to correspond to exactly one site in the sense that for all \( a \in A \) there is a unique \( y_a \in \Lambda_L \) such that \( y_\gamma = y_a \) for all \( \gamma \in a \). To do so let \( A_n \), denote the set of partitions of the set \( \{1,...,n\} \). For \( A \in A_n \) and \( k = (k_1, ..., k_{n+1}) \) we define the partition function

\[
P_{A,L} : (\Lambda^*_L)^{n+1} \to \mathbb{R}, \quad k \mapsto P_{A,L}(k) = \prod_{a \in A} \left\{ m_{|a|} \delta_{s,L} \left( \sum_{l \in a} (k_l - k_{l+1}) \right) \prod_{l \in a} \hat{B}_\#(k_l - k_{l+1}) \right\}.
\]

(2.19)

Recall that \( m_{|a|} \) is the \(|a|\)-th moment defined in (2.9). To state the results we will define the infinite volume Fourier transform and inverse Fourier transform for \( f \in L^1(\mathbb{R}^d) \)

\[
\hat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{-2\pi ik \cdot x} dx, \quad k \in \mathbb{R}^d,
\]

and

\[
\check{f}(x) = \int_{\mathbb{R}^d} f(k) e^{2\pi ik \cdot x} dk, \quad x \in \mathbb{R}^d,
\]

respectively.

For the next theorem we transition over to the momentum space, where we give an explicit expression for the expansion coefficients. Remember that we write \( k = (k_1, ..., k_{n+1}) \in (\Lambda^*_L)^{n+1} \) and that the real energy function \( \nu \) defined in (2.3) is non-negative.

**Theorem 2.3.** For all \( z \in \mathbb{C} \setminus [0, \infty) \) and \( \psi_1, \psi_2 \in L^2(\mathbb{R}^d) \) we have

\[
T_{n,L}[z; \psi_1, \psi_2] = \int_{(\Lambda^*_L)^{n+1}} \sum_{A \in A_n} P_{A,L}((k_1, ..., k_{n+1})) \hat{\psi}_1,\#(k_1) \hat{\psi}_2,\#(k_{n+1})
\]

\[
\times \prod_{j=1}^{n+1} (\nu(k_j) - z)^{-1} d(k_1, ..., k_{n+1}).
\]

The proof of Theorem 2.3 will be given in Section 4. The next theorem shows that for each expansion coefficient the infinite volume limit exists and it gives for each expansion coefficient an upper bound dependent on the spectral parameter.
Theorem 2.4. Let $n \in \mathbb{N}$. For all $z \in \mathbb{C} \setminus [0, \infty)$, and $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$ the limit \[ \lim_{L \to \infty} T_{n,L}[z; \psi_1, \psi_2] \] exists in $\mathbb{C}$. There exists a constant $K_n$ such that the inequality \[
 T_{n,L}[z; \psi_1, \psi_2] \leq \frac{K_n \| \psi_1 \| \| \psi_2 \|}{\text{dist}(z, [0, \infty))^{n+1}} \] (2.20) holds for all $L \geq 1$, $z \in \mathbb{C} \setminus [0, \infty)$, and $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$.

Due to Theorem 2.4, which will be proven in Section 4 as well, we can define the infinite volume limit of the expansion coefficients \[ T_{n,\infty}[z; \psi_1, \psi_2] := \lim_{L \to \infty} T_{n,L}[z; \psi_1, \psi_2], \] (2.21) for any $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$ and $z \in \mathbb{C} \setminus [0, \infty)$. To state the main result about the expansion coefficients of the resolvent (2.21), we need some regularity assumptions for the profile function and the wave functions with respect to which we calculate the matrix element of the resolvent. These regularity assumptions are formulated in terms of so called analytic dilations, cf. [CT73; RS78]. Let us first introduce the group of dilations.

Definition 2.5. The group of unitary operators $u(\theta)$ on $L^2(\mathbb{R}^d)$ given by \[
 (u(\theta)\psi)(x) = e^{\frac{d \theta}{2}} \psi(e^{\theta} x) \quad \text{for } \theta \in \mathbb{R}, \] (2.22) is called the group of dilation operators on $\mathbb{R}^d$.

It is straightforward to verify that $\mathbb{R} \mapsto B(L^2(\mathbb{R}^d))$, $\theta \mapsto u(\theta)$ is indeed a strongly continuous one parameter group of unitary operators, cf. [RS72, Page 265]. Observe that from the definition of the Fourier transform and the dilation operator it is easy to see that for any $\psi \in L^2(\mathbb{R}^d)$ and all $\theta \in \mathbb{R}$ \[
 (u(\theta)\psi)^\wedge = u(-\theta)\hat{\psi}. \] (2.23)

Let us first state the hypothesis about the profile function needed for Theorem 2.6 below.

Hypothesis A. There exists $\vartheta_B > 0$ such that the Fourier transform, $\hat{B}$, of the Schwartz function $B$ satisfies that for $p = 1$ and $p = \infty$ the function $\mathbb{R} \to L^p(\mathbb{R}^d), \theta \mapsto \hat{B}_\theta := (u(\theta)B)^\wedge$ has an extension to an analytic function $D_{\vartheta_B} := \{ z \in \mathbb{C} : |z| < \vartheta_B \} \to L^p(\mathbb{R}^d)$.

Let us now state the hypothesis about the wave function with respect to which we shall calculate matrix elements of the resolvent in Theorem 2.6 below.

Hypothesis B. For the vector $\psi \in L^2(\mathbb{R}^d)$ and there exists $\vartheta_\psi > 0$ such that the transform $\hat{\psi}$ satisfies that the function $\mathbb{R} \to L^p(\mathbb{R}^d), \theta \mapsto \hat{\psi}_\theta := \hat{\psi}(e^{\theta \cdot})$ has an extension to an analytic function $D_{\vartheta_\psi} := \{ z \in \mathbb{C} : |z| < \vartheta_\psi \} \to L^2(\mathbb{R}^d)$. 

8
In Appendix B Lemma B.1 we gives a class of functions satisfying these hypotheses. For example if \( B \) or \( \psi \), respectively, is a product of a polynomial, a Gaussian, and a free wave function with any wave vector then it satisfies Hypotheses A or B, respectively. We can now state the first main result, which establishes the existence of the boundary value of the expansion coefficients.

**Theorem 2.6.** Suppose Hypothesis A holds and suppose \( \psi_1, \psi_2 \in L^2(\mathbb{R}^d) \) satisfy Hypothesis B. Then for any \( E > 0 \) the following limit exists as a finite complex number

\[
T_{n,\infty}[E \pm i\eta; \psi_1, \psi_2] := \lim_{\eta \downarrow 0} T_{n,\infty}[E \pm i\eta; \psi_1, \psi_2].
\]

Moreover, \( z \mapsto T_{n,\infty}[z; \psi_1, \psi_2] \) as a function on \( \mathbb{C}_+ = \{ z \in \mathbb{C} : \Im(z) \geq 0 \} \) has a continuous extension to a function on \( \{ z \in \mathbb{C} : \Re(z) > 0 \} \cup \{ z \in \mathbb{C} : \Re(z) = 0 \} \).

The proof of Theorem 2.6 will be given in Section 4. The next theorem is the second main result about expectations of matrix elements of the resolvent. It shows that the expectations of matrix elements of the resolvent have an asymptotic expansion with expansion coefficients given by (2.16). The estimate is uniform in the size \( L \geq 1 \) of the box and yields an asymptotic expansion as the spectral parameter approaches the positive real axis in a the weak coupling limit where \( \eta = \lambda^{2-\varepsilon} \).

**Theorem 2.7.** Assume \( m_1 = E v_\gamma = 0 \). Then there exists an \( L_0 \geq 1 \) with the following property. If \( \psi_1, \psi_2 \in L^2(\mathbb{R}^d) \) and \( E > 0 \), then the following holds.

(a) For each \( n \in \mathbb{N} \) there exists a constant \( K_{n,d,E,B} \) such that for \( \eta > 0 \) and \( L \geq L_0 \) we have

\[
\left| \mathbb{E}\langle \psi_1,\#,(H_{\lambda,L} - E \mp i\eta)^{-1}\psi_2,\#\rangle_{L^2(\Lambda_L)} - \sum_{j=0}^{n} T_{j,L}[E \pm i\eta; \psi_1, \psi_2] \right| 
\leq K_{n,d,E,B}\left( \frac{\lambda^2}{\eta} \right)^{n/2} \left( 1 + \ln(\eta^{-1} + 1) \right)^n \eta^{-3/2}\|\psi_1\|\|\psi_2\|.
\]

(b) For any \( \varepsilon \in (0, 2) \) and \( N > 0 \) we have

\[
\mathbb{E}\langle \psi_1,\#,(H_{\lambda,L} - E \mp i\lambda^{2-\varepsilon})^{-1}\psi_2,\#\rangle_{L^2(\Lambda_L)} = \sum_{n=0}^{[4(N+3)/\varepsilon]} T_{n,L}[E \pm \lambda^{2-\varepsilon}; \psi_1, \psi_2] \lambda^n + O(\lambda^N)
\]

for \( \lambda \downarrow 0 \) uniformly in \( L \geq L_0 \).

The proof of Theorem 2.7 will be given in Section 5.

**Remark 2.8.** The energy \( E > 0 \) lies inside the spectrum of the free Laplacian which presents the challenge we want to address in this paper. Note that for negative energies \( E < 0 \) outside of the free Laplacian, the corresponding estimates and limits of Theorems 2.6 and 2.7 are in fact straight forward to establish by means of the bound (2.20).
Remark 2.9. Regarding the choice of $L_0$ in Theorem 2.7 we note that in the case where
the profile function $B$ satisfies the symmetry condition (2.8) Theorem 2.7 holds for $L_0 = 1$.
In case $B$ has compact support, one chooses $L_0 \geq 1$ such that supp $B \subset (-L_0/2, L_0/2)^d$.

Remark 2.10. An estimate similar to (2.24) was obtained in [MPR98] for a related
model. In that work a Laplacian with a Gaussian random potential in two dimensions
was studied and it was shown that the difference between the expectation of the resolvent
and a renormalized free resolvent is operator norm bounded by
$$(\lambda^2 + \delta \eta)^{\frac{3}{2}} \eta$$
for some $\delta > 0$, provided $0 \leq \eta \leq \lambda^2$. Now the bound (2.24) in Theorem 2.7 can be made much smaller
than $(\lambda^2 + \delta \eta)^{\frac{3}{2}} \eta$ in the regime where $\eta = \lambda^{2-\varepsilon}$ by choosing $n$ sufficiently large.

Remark 2.11. It would be interesting to investigate to what extent the expansion (2.24)
in Theorem 2.7 could be improved using so called tadpole renormalization and an analysis
of crossing graphs, see for example [ESY08]. In that paper this was successfully used
to establish quantum diffusion in a diffusive scaling limit for similar model to the one
introduced in the present paper. We did not perform such an analysis in this work, since
the focus of it is on the expansion coefficients to any order as well as on tracial properties,
which will be discussed in Section 6.

3 Pairing Potential labels

In this section we derive formulas about expectations of products of the random potential,
which will be used in the following sections. The main formulas have been shown in
[ESY08]. We are going to work with partitions of the set $\{1, 2, ..., n\}$ of natural numbers.
We want to first give a short introduction to explain how these occur, as well as how they
are technically handled. In the definition of $T_{n,L}$ given in (2.16) we are faced with the
expectation value of powers of the random potentials, i.e., terms of the form
$$E_{v}^{\otimes M} E_{y_L}^{\otimes M} \sum_{\gamma_1, ..., \gamma_n = 1}^{M} \prod_{j=1}^{n} v_{\gamma_j}. \quad (3.1)$$

We are therefore going to consider products of the random potentials $v_{\gamma}$ at the sites
$y_L, \gamma \in \Lambda_L$. Note, that for a given configuration $\{\gamma_1, ..., \gamma_n\}$, different indices $j_1, \ldots, j_l$ may
label the same sites, i.e. $\gamma_{j_1} = \ldots = \gamma_{j_l}$. This case would produce a factor
$$\prod_{i=1}^{l} v_{\gamma_{j_i}} = v_{\gamma_{j_1}}^l,$$
since all $v_{\gamma_{j_i}}$ are the same. Thus the expectation value of this factor yields an $l$-th moment.

To control this technically, we introduce the following notation. Let $A_n$, be the set of
partitions of $\{1, 2, ..., n\}$. For a given configuration $\{\gamma_1, ..., \gamma_n\}$ we are seeking to filter for
the unique partition $A$ that sorts the indices $1, 2, ..., n$ in such a way, that the following
two conditions are satisfied.
(a) All indices in each $a \in A$ correspond to the same site.

(b) All sets $a$ correspond to different sites.

To express this, we define for $A \in \mathcal{A}_n$ the two functions $\chi_A$ and $\tilde{\chi}_A$ on $\mathbb{N}^n$ as follows. First, we set

$$\chi_A(z_1, \ldots, z_n) := \prod_{a \in A} \left[ \prod_{(j,l) \in a \times a} 1_{\{z_j = z_l\}} \right] ,$$

which ensures condition (a). Second, we define

$$\tilde{\chi}_A(z_1, \ldots, z_n) := \chi_A(z_1, \ldots, z_n) \prod_{\substack{(a,b) \in A \times A: \ a \neq b}} \left[ \prod_{(j,l) \in a \times b} 1_{\{z_j \neq z_l\}} \right] ,$$

which ensures that condition (b) is additionally met. Observe that for any $(z_1, \ldots, z_n) \in \mathbb{N}^n$ we have

$$\sum_{A \in \mathcal{A}_n} \tilde{\chi}(z_1, \ldots, z_n) = 1,$$

since there is exactly one unique partition that satisfies both conditions (i) and (ii). With this small preamble, we can now express the expected value in term (3.1) and show the following lemma.

**Lemma 3.1.** For any fixed $L > 0$, integers $n, M \in \mathbb{N}$, and any fixed momenta $q_j \in \Lambda^*_L$, $j = 1, \ldots, n$, the identity

$$E_0^\otimes M E_y^\otimes M \sum_{\gamma_1, \ldots, \gamma_n=1}^{M} \prod_{j=1}^{n} \nu_{\gamma_j} \exp(2\pi i q_j \cdot y L \gamma_j) = \sum_{A \in \mathcal{A}_n} 1_{\{|A| \leq M\}} \frac{M!}{(M - |A|)!} \prod_{a \in A} \left[ m_{|a|} \delta_{a} \left( \sum_{l \in a} q_l \right) \right]$$

holds.

**Proof.** Let $A \in \mathcal{A}_n$ and let $\chi_A$ and $\tilde{\chi}_A$ be defined as in (3.2) and (3.3) above. Inserting
the identity (3.4) into the left hand side we find
\[
E_y \otimes M \sum_{\gamma_1, \ldots, \gamma_n=1}^{M} \prod_{j=1}^{n} v_{\gamma_j} \exp(2\pi i q_j \cdot y_{L, \gamma_j}) = E_y \otimes M \sum_{\gamma_1, \ldots, \gamma_n=1}^{M} \prod_{j=1}^{n} v_{\gamma_j} \exp(2\pi i q_j \cdot y_{L, \gamma_j})
\]
\[
= \sum_{A \in \mathcal{A}_n} E_y \otimes M \sum_{\gamma_1, \ldots, \gamma_n=1}^{M} \tilde{\chi}_A(\gamma_1, \ldots, \gamma_n) \prod_{a \in A} \left[ E_y \otimes M \prod_{j \in a} v_{\gamma_j} \right] \left[ E_y \otimes M \exp \left( 2\pi i \sum_{j \in a} q_j \cdot y_{L, \gamma_j} \right) \right]
\]
\[
\overset{(*)}{=} \sum_{A \in \mathcal{A}_n} \sum_{\gamma_1, \ldots, \gamma_n=1}^{M} \tilde{\chi}_A(\gamma_1, \ldots, \gamma_n) \prod_{a \in A} m_{|a|} \delta_\star \left( \sum_{j \in a} q_j \right).
\]

Note that (*) follows from the discussion about the design of \( \tilde{\chi}_A(\gamma_1, \ldots, \gamma_n) \) prior to the lemma, as well as the fact that \( A \) is a partition. The details are as follows. Evaluating the \( \tilde{\chi}_A(\gamma_1, \ldots, \gamma_n) \) ensures by (3.2) that for a fixed \( a \in A \) all indices \( \gamma_j \) with \( j \in a \) are the same index, hence all \( y_{L, \gamma_j} \) with \( j \in a \) are the same site. Therefore the \( v_{\gamma_j} \) are the same random variables and we find
\[
E_y \otimes M \prod_{j \in a} v_{\gamma_j} = m_{|a|}.
\]

The definition of \( E_y \otimes M \) in (2.13) together with
\[
\frac{1}{|\Lambda_L|} \int_{\Lambda_L} \exp \left( 2\pi i \sum_{j \in a} q_j \cdot y \right) dy = \delta_\star \left( \sum_{j \in a} q_j \right)
\]
finally yield (*). Now the claim follows from the combinatorical identity
\[
\sum_{\gamma_1, \ldots, \gamma_n=1}^{M} \tilde{\chi}_A(\gamma_1, \ldots, \gamma_n) = 1_{|A| \leq M} \frac{M!}{(M - |A|)!},
\]
which can be seen as follows. The LHS sums over all configurations of the \( n \) indices that stand for \( M \) possible points, while evaluating the \( \tilde{\chi}_A \)-function for a given \( A \). In fact, since all \( a \in A \) - which are \( |A| \) many sets - correspond to different ones of the \( M \) possible points each. So for the first set \( a_1 \) there are \( M \) choices for the second one \( (M - 1) \) choices and so on and for the last set \( a_{|A|} \) there are \( (M - |A| + 1) \) choices left. This is exactly expressed through the RHS which coincides with a \( |A| \)-permutation of \( M \). That is, the number of ways of choosing \( |A| \) out of \( M \) elements without repetition where the order is taken in to account.
Lemma 3.2. For a Poisson distributed random variable $N$ with mean $\lambda$ we have for any $k \in \mathbb{N}$

$$E\left[\prod_{j=0}^{k-1}(N - j)\right] = \lambda^k. \quad (3.5)$$

Proof. First observe that for a Poisson distributed random variable with mean $\lambda$ we have

$$P(N = n) = e^{-\lambda} \frac{\lambda^n}{n!}. \quad (3.5)$$

We denote by $E$ the expectation. Using

$$E f(N) = \sum_{n=0}^{\infty} f(n) P(N = n)$$

we find for any $k \in \mathbb{N}$

$$E \left[\prod_{j=0}^{k-1}(N - j)\right] = e^{-\lambda} \sum_{n=k}^{\infty} \frac{\lambda^n}{n!} \prod_{j=0}^{k-1} (n - j)$$

$$= e^{-\lambda} \sum_{n=k}^{\infty} \frac{\lambda^n}{n!} \prod_{j=0}^{k-1} (n - j)$$

$$= \lambda^k e^{-\lambda} \sum_{n=k}^{\infty} \frac{\lambda^{n-k}}{(n-k)!}$$

$$= \lambda^k e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}$$

$$= \lambda^k e^{-\lambda} e^\lambda$$

$$= \lambda^k. \quad \square$$

We note that Lemma 3.2 can alternatively be proven using the probability-generating function. The following lemma is a continuation of Lemma 3.1 in the sense that we evaluate the $E_M$ expectation as well.

Lemma 3.3. Let $L > 0$, $n \in \mathbb{N}$, and $q_j \in \Lambda^*_L$, $j = 1, \ldots, n$. Then

$$E_M E_{v^M}^{\otimes M} E_{y_L^M}^{\otimes M} \sum_{\gamma_1, \ldots, \gamma_n = 1}^{M} \prod_{j=1}^{n} v_{\gamma_j} \exp(2\pi i q_j \cdot y_{L, \gamma_j}) = \sum_{A \in A_n} \prod_{a \in A} \left\{ m_{|a|} \delta_{\ast L} \left( \sum_{l \in a} q_l \right) \right\}$$

holds.

Proof. The statement follows from Lemma 3.1 and the identity

$$E_M 1_{|A| \leq M} \frac{M!}{(M - |A|)!} = |\Lambda_L|^{|A|},$$

which follows from Equation (3.5). \quad \square
To transform into momentum space we shall expand with respect to the ONB \( \varphi_p \) defined in (2.2). That is we shall insert the following identity, which hold in strong operator topology

\[
\psi = \sum_{p \in \Lambda^*_L} \langle \varphi_p, \psi \rangle \varphi_p = \int_{\Lambda^*_L} \langle e^{2\pi i(p, \cdot)}, \psi(\cdot) \rangle e^{2\pi i(p, \cdot)} dp,
\]

for all \( \psi \in L^2(\Lambda_L) \), which provides the representation via the ONB \( \{ \varphi_p : p \in \Lambda^*_L \} \).

Moreover, for \( \psi \in L^2(\Lambda_L) \) and \( W \in L^\infty(\Lambda_L) \) (since \( \Lambda_L \) is compact they are both also in \( L^1(\Lambda_L) \)) we will use the identities

\[
\langle e^{2\pi i(p, \cdot)}, \psi \rangle = \int_{\Lambda_L} e^{-2\pi ip \cdot x} \psi(x) dx = \hat{\psi}(p) \quad (3.7)
\]

\[
\langle e^{2\pi i(p, \cdot)}, W(\cdot)e^{2\pi i(q, \cdot)} \rangle = \int_{\Lambda_L} W(x)e^{-i2\pi(p-q) \cdot x} dx = \hat{W}(p-q) \quad (3.8)
\]

First we use the probabilistic structure of the potential (2.11) and calculate the Fourier transform

\[
\hat{V}_{L,\omega}(p) = \int_{\Lambda_L} \sum_{\gamma=1}^M v_\gamma B_\#(x - y_{L,\gamma}) e^{-2\pi ip \cdot x} dx
\]

\[
= \sum_{\gamma=1}^M v_\gamma \int_{\Lambda_L} B_\#(z)e^{-2\pi ip \cdot (z+y_{L,\gamma})} dz
\]

\[
= \sum_{\gamma=1}^M v_\gamma \hat{B}_\#(p)e^{-2\pi ip \cdot y_{L,\gamma}}. \quad (3.9)
\]

In the following sections we will make use of the following lemma.

**Lemma 3.4.** For the model introduced in Section 2 we have

\[
E_L \left( \prod_{j=1}^n \hat{V}_{L,\omega}(p_j - p_{j+1}) \right) = \sum_{A \in A_n} \prod_{a \in A} \left\{ m_{|a|} \delta_{s,L} \left( \sum_{l \in a} (p_l - p_{l+1}) \right) \prod_{l \in a} \hat{B}_\#(p_l - p_{l+1}) \right\}. \quad (3.10)
\]

**Proof.** Inserting (3.9) into the left hand side of (3.10) in the first, step multiplying out
the product over the sum in the second step, using Lemma 3.3 in the final step we find
\[
E_L \left( \prod_{j=1}^{n} \hat{V}_{L,\omega}(p_j - p_{j+1}) \right)
\]
\[
= E_L \left( \prod_{j=1}^{n} \sum_{\gamma=1}^{M} v_{\gamma} \hat{B}_\#(p_j - p_{j+1}) e^{-2\pi i (p_j - p_{j+1}) \cdot y_{L,\gamma}} \right)
\]
\[
= E_L \sum_{\gamma_1, \ldots, \gamma_n=1}^{M} \prod_{j=1}^{n} \left\{ v_{\gamma_j} \hat{B}_\#(p_j - p_{j+1}) e^{-2\pi i (p_j - p_{j+1}) \cdot y_{L,\gamma_j}} \right\}
\]
\[
= \sum_{A \in \mathcal{A}_n} \prod_{a \in A} \left\{ m_{\vert a \vert} \delta_{\ast,L} \left( \sum_{l \in a} (p_l - p_{l+1}) \right) \prod_{l \in a} \hat{B}_\#(p_l - p_{l+1}) \right\}.
\]

4 Expectation of the Expansion Coefficients

In this section, we prove Theorems 2.3, 2.4, and 2.6.

We introduce the notation for the canonical norm in the \( \ell^q \)-spaces. For \( f \in \ell^q(\Lambda_L^*) \) with \( q \in [1, \infty] \), we write
\[
\| f \|_{\ast, q} = \left( \int_{\Lambda_L^*} |f(k)|^q dk \right)^{1/q} \quad (1 \leq q < \infty), \quad \| f \|_{\ast, \infty} = \sup_{k \in \Lambda_L^*} |f(k)|.
\]

**Proof of Theorem 2.3** We use the definition given in (2.16) and transform into momentum space by means of the identities in Eqns. (3.6)–(3.8) as well as (2.5). This gives for \( z \in \mathbb{C} \setminus [0, \infty) \), where for reasons of lack of space we write \( dp \ := d(p_1, \ldots, p_{n+1}) \)
\[
T_{n,L}(z; \psi_1, \psi_2)
\]
\[
= E_L \langle \psi_{1,\#}, R_L(z) [V_L R_L(z)]^n \psi_{2,\#} \rangle_L
\]
\[
= E_M E_{\psi_L}^{\otimes M} E_{\hat{V}_{L,\omega}}^{\otimes M} \int_{(\Lambda_L^*)^{n+1}} \prod_{j=1}^{n} \hat{V}_{L,\omega}(p_j - p_{j+1}) \hat{\psi}_{1,\#}(p_1) \hat{\psi}_{2,\#}(p_{n+1}) \left( \prod_{j=1}^{n+1} \frac{1}{\nu(p_j) - z} \right) dp
\]
\[
= E_M \int_{(\Lambda_L^*)^{n+1}} E_{\psi_L}^{\otimes M} E_{\hat{V}_{L,\omega}}^{\otimes M} \prod_{j=1}^{n} \hat{V}_{L,\omega}(p_j - p_{j+1}) \hat{\psi}_{1,\#}(p_1) \hat{\psi}_{2,\#}(p_{n+1}) \left( \prod_{j=1}^{n+1} \frac{1}{\nu(p_j) - z} \right) dp
\]
where in the last identity we used Fubinis theorem, which we justify in the following paragraph.
For fixed $M$ we find with a canonical change of summation variables
\[
\int_{(\Lambda_v^*)} \left| \prod_{j=1}^{n+1} \hat{V}_{L,\omega} (p_j - p_{j+1}) \psi_{1,\#} (p_1) \psi_{2,\#} (p_{n+1}) \left( \prod_{j=1}^{n+1} \frac{1}{\nu(p_j) - z} \right) \right| dp 
\leq \| \hat{V}_{L,\omega} \|_{*,1} \| \hat{\psi}_{1,\#} \|_{*,2} \| \hat{\psi}_{2,\#} \|_{*,2} \frac{\text{dist}(z, [0, \infty))^{n+1}}{n+1}. \tag{4.2}
\]

Now using (3.9) we find
\[
\| \hat{V}_{L,\omega} \|_{*,L,1} \leq \sum_{j=1}^{M} |v_j| \| \hat{B}_\# \|_{*,1}. \tag{4.3}
\]

By the symmetry assumptions on $B$ we will show in Lemma 4.1 below, that $\| \hat{B}_\# \|_{*,1}$ is finite. Any power of the right hand side of (4.3) is thus integrable with respect to $E \otimes M v E \otimes M y L$ by means of the moment assumption (2.10). This justifies the use of Fubini's theorem and therefore in the last identity of (4.1).

Now using Lemma 3.1 and Lemma 3.2 as well as Fubini's theorem we can interchange $E M$ and the sum over the momentum variables and find
\[
T_{n,L}[z; \psi_1, \psi_2] = \int_{(\Lambda_v^*)} E_M E_v \otimes M E_y \otimes M \prod_{j=1}^{n} \hat{V}_{L,\omega} (p_j - p_{j+1}) \psi_{1,\#} (p_1) \psi_{2,\#} (p_{n+1}) \times \left( \prod_{j=1}^{n+1} \frac{1}{\nu(p_j) - z} \right) d(p_1, \ldots, p_{n+1})
\]
\[
= \int_{(\Lambda_v^*)} \sum_{A \in A_n} P_{A,L}(k) \psi_{1,\#} (k_1) \psi_{2,\#} (k_{n+1}) \prod_{j=1}^{n+1} (\nu(k_j) - z)^{-1} d(k_1, \ldots, k_{n+1}),
\]
where in the last identity we used Lemma 3.4 to insert (3.10) and used the definition given in (2.19).

Before we can show the other two theorems we will need some preparatory work. For this we introduce the notation
\[
\langle x \rangle = (1 + x^2)^{1/2}, \tag{4.4}
\]
which we shall use throughout the paper. The following lemma will be applied in the proof of Proposition 4.5. In it's proof, we will be making use of Lemma C.1.
Lemma 4.1. There exists a constant $C_d$ (explicitly given in the proof) and a constant $\tilde{C}_d$ such that for all $L \geq 1$ and $f \in S(\mathbb{R}^d)$ which are symmetric with respect to the reflections $\rho_j : (x_1, \ldots, x_j, \ldots, x_d) \mapsto (x_1, \ldots, -x_j, \ldots, x_d)$ for all $j = 1, \ldots, d$ (i.e. $f \circ \rho_j = f$ for all $j = 1, \ldots, d$), we have

$$\langle k_1 \rangle^2 \cdots \langle k_d \rangle^2 |\hat{f}_\#(k)| \leq C_d \sum_{\alpha \in \mathbb{N}_0^d, x \in \mathbb{R}^d, \alpha_j \leq 2} \sup_{x \in \mathbb{R}^d} |\langle x \rangle^{2d} \partial^\alpha f(x)|$$

(4.5)

for all $k = (k_1, \ldots, k_d) \in \mathbb{R}^d$. Furthermore $\hat{f}_\# \in \ell_1^1(\Lambda_L^*)$, and

$$\|\hat{f}_\#\|_{1,1} \leq \tilde{C}_d \sum_{\alpha \in \mathbb{N}_0^d, x \in \mathbb{R}^d, \alpha_j \leq 2} \sup_{x \in \mathbb{R}^d} |\langle x \rangle^{2d} \partial^\alpha f(x)|.$$ (4.6)

Proof. We will show (4.5) by induction over $d$. The integrability and the $\ell^1$-bound (4.6) will then follow by dividing by $\langle k_1 \rangle^2 \cdots \langle k_d \rangle^2$ and summing over each of the components of $k$ separately.

First consider $d = 1$. Using Lemma C.1 clearly, for $k \in \mathbb{Z}/L$ with $|k| \leq 1$, we find

$$|\hat{f}_\#(k)| = \left| \int_{-L/2}^{L/2} e^{-i2\pi kx} f(x) \, dx \right| \leq \int_{-L/2}^{L/2} |f(x)| \, dx \leq \pi \sup_{x \in \mathbb{R}} |\\langle x \rangle^{2} f(x)|.$$ (4.7)

On the other hand for $k \in \mathbb{Z}/L \setminus \{0\}$ using integration by parts and the reflection symmetry (2.8) we find

$$k \hat{f}_\#(k) = k \int_{-L/2}^{L/2} e^{-i2\pi kx} f(x) \, dx = \int_{-L/2}^{L/2} ke^{-i2\pi kx} f(x) \, dx = \frac{1}{-i2\pi} \left( \int_{-L/2}^{L/2} \frac{d}{dx} e^{-i2\pi kx} \right) f(x) \, dx = \frac{1}{-i2\pi} \left( \int_{-L/2}^{L/2} e^{-i2\pi kx} f(x) \, dx \right) = \frac{1}{2\pi i} \int_{-L/2}^{L/2} e^{-i2\pi kx} f'(x) \, dx.$$
Multiplying this by \( k \), an analogous calculation using integration by parts yields
\[
k^2 \hat{f}_\#(k) = \frac{1}{2\pi i} \int_{-L/2}^{L/2} ke^{-i2\pi kx} f'(x)dx
\]
\[
= \frac{1}{(2\pi)^2} \int_{-L/2}^{L/2} \left( \frac{d}{dx} e^{-i2\pi kx} \right) f'(x)dx
\]
\[
= \frac{1}{(2\pi)^2} \left( e^{-i2\pi kx} f'(x) \right)_{L/2}^{L/2} \left( e^{-i2\pi kx} f''(x)dx \right). \tag{4.8}
\]

Using first the triangle inequality in (4.8) and then Lemma C.1 we find
\[
k^2 |\hat{f}_\#(k)| \leq \frac{1}{(2\pi)^2} \left( 2 \sup_{x \in \mathbb{R}} |\partial f(x)| + \int_{-L/2}^{L/2} |\partial^2 f(x)|dx \right)
\]
\[
\leq c_1 \sum_{\alpha \in \mathbb{N}_0, \alpha \leq 2} \sup_{x \in \mathbb{R}} |\langle x \rangle^2 \partial^\alpha f(x)|, \tag{4.9}
\]
where \( c_1 := \frac{1}{4\pi^2} + \frac{1}{2\pi} \).

Using \( \langle k \rangle^2 \leq 2(1|k|\geq 1k^2 + 1|k|\leq 1) \) and then (4.9) for the first summand and (4.7) for the second summand shows
\[
\langle k \rangle^2 |\hat{f}_\#(k)| \leq 2(1|k|\geq 1k^2 |\hat{f}_\#(k)| + 1|k|\leq 1|\hat{f}_\#(k)|)
\]
\[
\leq \left( \frac{1}{\pi^2} + \frac{1}{2\pi} + 2\pi \right) \sum_{\alpha \in \mathbb{N}_0, x \in \mathbb{R}} \sup_{\alpha \leq 2} |\langle x \rangle^2 \partial^\alpha f(x)|, \tag{4.10}
\]
which is exactly (4.5) for \( d = 1 \) with \( C_1 := \frac{1}{\pi^2} + \frac{1}{2\pi} + 2\pi \).

Now let us assume (4.5) holds for \( d - 1 \) with some constant \( C_{d-1} \) independent of \( L \). We want to show that it also holds for \( d \). At this point, we define the Fourier transform in the last variable by
\[
\hat{f}_\#(x_1, \ldots, x_{d-1}; k_d) := \int_{-L/2}^{L/2} e^{-2\pi ik_d x_d} f(x_1, \ldots, x_d)dx_d. \tag{4.11}
\]

Let \( k_d \in \mathbb{Z}/L \). First observe that the right hand side of (4.11) is again a Schwartz function on \( \mathbb{R}^{d-1} \) which is symmetric with respect to \( \rho_j, j = 1, \ldots, d - 1 \). Using Fubini’s theorem, we find
\[
\hat{f}_\#(k', k_d) = \int_{[-L/2,L/2]^{d-1}} \hat{f}_\#(x'; k_d) e^{-2\pi ik' \cdot x'} dx'. \tag{4.12}
\]
So for \( |k_d| \leq 1 \) we find from first the induction hypothesis, then (4.12), triangle inequality
and finally Lemma C.1 applied to the function \( x_d \mapsto \partial'^\alpha f(x', x_d) \) that

\[
\langle k_1 \rangle^2 \cdots \langle k_d \rangle^2 |\tilde{f}_\#(k_1, \ldots, k_{d-1}, k_d)| \\
\leq C_{d-1} \sum_{\alpha' \in \mathbb{N}_0^{d-1} : \alpha'_j \leq 2} \sup_{x' \in \mathbb{R}^{d-1}} |\langle x' \rangle^{2(d-1)} \partial'^\alpha \tilde{f}_\#(x'; k_d) |
\]

\[
= C_{d-1} \sum_{\alpha' \in \mathbb{N}_0^{d-1} : \alpha'_j \leq 2} \sup_{x' \in \mathbb{R}^{d-1}} |\langle x' \rangle^{2(d-1)} \partial'^\alpha \int_{-L/2}^{L/2} e^{-2\pi ik_d x_d} f(x', x_d) dx_d |
\]

\[
\leq C_{d-1} \sum_{\alpha' \in \mathbb{N}_0^{d-1} : \alpha'_j \leq 2} \sup_{x' \in \mathbb{R}^{d-1}} |\langle x' \rangle^{2(d-1)} \int_{-L/2}^{L/2} e^{-2\pi ik_d x_d} \partial'^\alpha f(x', x_d) dx_d |
\]

\[
\leq C_{d-1} \sum_{\alpha' \in \mathbb{N}_0^{d-1} : \alpha'_j \leq 2} \sup_{x' \in \mathbb{R}^{d-1}} |\langle x' \rangle^{2(d-1)} \left( \pi \sup_{x_d \in \mathbb{R}} |\langle x_d \rangle^{2} \partial'^\alpha f(x', x_d) | \right) |
\]

\[
\leq C_{d-1} \pi \sum_{\alpha \in \mathbb{N}^d : \alpha \leq 2} \sup_{x \in \mathbb{R}^d} |\langle x \rangle^{2d} \partial'^\alpha f(x) | . \quad (4.13)
\]

On the other hand lets consider the estimate for any \( k_d \in \mathbb{Z}/L \) with \( k_d \neq 0 \). Then an analogous calculation as for \( d = 1 \) using integration by parts shows

\[
k_d \tilde{f}_\#(x_1, \ldots, x_{d-1} ; k_d) = k_d \int_{-L/2}^{L/2} e^{-i2\pi k_d x_d} f(x) dx_d
\]

\[
= \int_{-L/2}^{L/2} k_d e^{-i2\pi k_d x_d} f(x) dx_d
\]

\[
= \frac{1}{-i2\pi} \int_{-L/2}^{L/2} \partial_{x_d} e^{-i2\pi k_d x_d} f(x) dx_d
\]

\[
= \frac{1}{-i2\pi} \left( e^{-i2\pi k_d x_d} f(x) |_{x_d = L/2}^{x_d = -L/2} - \int_{-L/2}^{L/2} e^{-i2\pi k_d x_d} \partial_{x_d} f(x) dx_d \right)
\]

\[
= \frac{1}{2\pi i} \int_{-L/2}^{L/2} e^{-i2\pi k_d x_d} \partial_{x_d} f(x) dx_d.
\]

Multiplying this by \( k_d \) an analogous calculation using integration by parts gives

\[
k_d^2 \tilde{f}_\#(x_1, \ldots, x_{d-1} ; k_d) = \frac{1}{2\pi i} \int_{-L/2}^{L/2} k_d e^{-i2\pi k_d x_d} \partial_{x_d} f(x) dx_d
\]

\[
= \frac{1}{(2\pi)^2} \int_{-L/2}^{L/2} (\partial_{x_d} e^{-i2\pi k_d x_d}) (\partial_{x_d} f(x)) dx_d
\]

\[
= G_L(x_1, \ldots, x_{d-1} ; k_d), \quad (4.14)
\]
where we defined
\[ G_L(x_1, \ldots, x_{d-1}; k_d) := \frac{1}{(2\pi)^d} e^{-i2\pi k_d x_d} \partial_{x_d} f(x)_{x_d = L/2} - \frac{1}{(2\pi)^2} \int_{-L/2}^{L/2} e^{-i2\pi k_d x_d} \partial_{x_d}^2 f(x) dx_d. \]

Now observe that \( G_L(\cdot; k_d) \) is a Schwartz function on \( \mathbb{R}^{d-1} \) with the property that it is symmetric with respect to the reflections \( \rho_j \) with \( j = 1, \ldots, d - 1 \). Now from (4.12) and the induction hypothesis we find using (4.14) as in (4.9), while Lemma C.1 applied to \( x_d \mapsto \partial_{x_d}^2 \partial^{\alpha'} f(x', x_d) \) that
\[
\langle k_1 \rangle^2 \cdots \langle k_{d-1} \rangle^2 \langle k_d \rangle^2 |\widehat{f}_{\#}(k)| \leq C_{d-1} \sum_{\alpha' \in \mathbb{N}_0^{d-1}, \alpha' \leq 2} \sup_{x' \in \mathbb{R}^{d-1}} \langle x' \rangle^{2(d-1)} k_d^2 \partial^{\alpha'} \widehat{f}_{\#}(x', k_d) \leq C_{d-1} \sum_{\alpha' \in \mathbb{N}_0^{d-1}, \alpha' \leq 2} \sup_{x' \in \mathbb{R}^{d-1}} \langle x' \rangle^{2(d-1)} 
\times \frac{1}{(2\pi)^2} \left( 2 \sup_{x' \in \mathbb{R}} |\partial_{x_d} \partial^{\alpha'} f(x', x_d)| + \int_{-L/2}^{L/2} |\partial_{x_d}^2 \partial^{\alpha'} f(x', x_d)| dx_d \right) \leq C_{d-1} \sum_{\alpha' \in \mathbb{N}_0^{d-1}, \alpha' \leq 2} \sup_{x' \in \mathbb{R}^{d-1}} \langle x' \rangle^{2(d-1)} c_{1,1} \sum_{\alpha \in \mathbb{N}_0, \alpha \leq 2} \sup_{x \in \mathbb{R}} |\langle x \rangle^{2d} \partial^{\alpha} \partial^{\alpha'} f(x', x_d)| \leq C_{d-1} c_{1,1} \sum_{\alpha \in \mathbb{N}_0, \alpha \leq 2} \sup_{x \in \mathbb{R}} |\langle x \rangle^{2d} \partial^{\alpha} f(x)|. \tag{4.15}
\]

Finally (4.15) and (4.13) show, as we have seen in (4.10) before, that (4.5) holds for \( d \) with \( C_d = C_{d-1} C_1 \) (note \( C_d = C_d^d \)). Finally observe that (4.3) now implies (4.6). \( \square \)

**Remark 4.2.** We note that if the Schwartz function \( B \) has compact support or if it is symmetric with respect to reflections as in Lemma 4.1, then there exists an \( L_0 \geq 1 \) such that for all \( L \geq L_0 \) we have
\[
\langle k_1 \rangle^2 \cdots \langle k_d \rangle^2 |\widehat{B}_{\#}(k)| \leq C_d \sum_{\alpha \in \mathbb{N}_0, \alpha \leq 2} \sup_{x \in \mathbb{R}^d} |\langle x \rangle^{2d} \partial^{\alpha} B(x)|. \tag{4.16}
\]

In particular for \( L \geq L_0 \) we have \( \widehat{B}_{\#} \in \ell_1^d(\Lambda^*_L) \), and
\[
\|\widehat{B}_{\#}\|_{\ast, 1} \leq \widehat{C}_d \sum_{\alpha \in \mathbb{N}_0, \alpha \leq 2} \sup_{x \in \mathbb{R}^d} |\langle x \rangle^{2d} \partial^{\alpha} B(x)|. \tag{4.17}
\]

In the compact case (4.16) and (4.17) follow from choosing \( L_0 \geq 1 \) sufficiently large such that \( \text{supp} B \subset (-L_0/2, L_0/2)^d \) holds while using well known properties of the Fourier transform of Schwartz functions. In the symmetric case (4.16) and (4.17) follow from Lemma 4.1.
The remaining part of this section is devoted to the proofs of Theorems 2.4 and 2.6, which state that the infinite volume limit for each expansion coefficient exists and that for this infinite volume limit we can take the limit as the spectral parameter approaches the positive real axis from the complex upper or lower half plane. Therefore, we can write

\[ T_{n,L}[^{\cdot}^{\cdot},^{\cdot}] = \sum_{A \in \mathcal{A}_n} C_{n,A,L}[^{\cdot}^{\cdot},^{\cdot}], \tag{4.18} \]

where we defined for \( A \in \mathcal{A}_n \) and \( z \in \mathbb{C} \setminus [0, \infty) \)

\[ C_{n,A,L}[z; \psi_1, \psi_2] := \int_{(\Lambda_{L}^*)^{n+1}} \mathcal{P}_{A,L}(k) \prod_{s \in A} \delta_{\ast,L} \left( \sum_{s \in a} u_s \right) \prod_{s \in a} \delta^{\#}(-u_s) \prod_{j=1}^{n+1} (\nu(k_j) - z)^{-1} d(k_1, \ldots, k_{n+1}). \tag{4.19} \]

We will now consider the limit \( L \to \infty \). In the following we shall assume that

\[ z \in \mathbb{C} \setminus [0, \infty). \]

Technically, it is more convenient to first sum over the discrete delta functions and then take the limit \( L \to \infty \), rather than taking the limit first and then integrating out the delta distributions cf. Remark 4.6.

We introduce a change of variables given by

\[ u_0 = k_1, \quad u_s = k_{s+1} - k_s, \quad s = 1, \ldots, n \tag{4.20} \]

to be able resolve the discrete delta functions one at a time. Expressing the variables \( k \) in terms of the variables \( u = (u_0, \ldots, u_n) \in (\mathbb{R}^d)^{n+1} \) we find

\[ k_j = \sum_{l=0}^{j-1} u_l. \]

Applying this change of variables produces

\[ C_{n,A,L}[z; \psi_1, \psi_2] = \int_{(\Lambda_{L}^*)^{n+1}} \prod_{a \in A} \left\{ \delta_{a,L} \left( \sum_{s \in a} u_s \right) \prod_{s \in a} \delta^{\#}(-u_s) \right\} \prod_{j=1}^{n+1} (\nu(k_j) - z)^{-1} d(u_0, \ldots, u_n). \tag{4.21} \]

For a given partition \( A \) (hence given \( a \in A \)) we are now going to introduce a function \( M_A \) to express the dependence of indicies belonging to the same \( a \in A \) which is caused by the term

\[ \delta_{a,L} \left( \sum_{s \in a} u_s \right). \]

This term vanishes unless $\sum_{s \in a} u_s = 0$, which means that for every $a \in A$ we can express one of the variables by the negative sum of the others, or in the case, where $a$ consists of only one element, that element has to be equal to 0. We choose to express the highest number of each set $a \in A$ by the negative sum of all the other numbers, which gives us

$$u_{\text{max} a} = - \sum_{j \in a \setminus \{\text{max} a\}} u_j.$$  

In the case where $a$ consists of only one element, $a \setminus \{\text{max} a\} = \emptyset$ which produces an empty sum, which is 0 by convention. To make the notation more usable we define the set of all indices, which are the maximum of a set $a \in A$ by

$$J_A := \{\text{max} a : a \in A\} \quad (4.22)$$

as well as its complement

$$I_A := \{1, \ldots, n\} \setminus J_A. \quad (4.23)$$

Since $A$ is a partition, for any $j \in \{1, \ldots, n\}$ there is a unique set $a \in A$ such that $j \in a$, we denote this set by $a(j)$. We define the map $M_A : (\mathbb{R}^d)^{|J_A|} \to (\mathbb{R}^d)^n$ as

$$[M_A(v)]_j := \begin{cases} v_j & : j \in I_A \\ - \sum_{l \in a(j) \setminus \{j\}} v_l & : j \in J_A, \end{cases} \quad (4.24)$$

where $v = (v_1, \ldots, v_n)$ with $v_j \in \mathbb{R}^d$ for all $j \in \{1, \ldots, n\}$. Note that (4.24) contains the case in which $j$ is the only element of $a(j)$ and $[M_A(v)]_j = 0$ holds. This definition implies the following lemma.

**Lemma 4.3.** Let $A$ be a partition of $\{1, \ldots, n\}$ and $M_A$ be defined as in (4.24). Then we have

$$\sum_{j=1}^n [M_A(v)]_j = 0 \quad (4.25)$$

for all $v \in (\mathbb{R}^d)^{|J_A|}$.

**Proof.** This can easily be shown by inserting the definition

$$\sum_{j=1}^n [M_A(v)]_j = \sum_{j \in I_A} [M_A(v)]_j + \sum_{j \in J_A} [M_A(v)]_j$$

$$= \sum_{j \in I_A} v_j + \sum_{j \in J_A} \left( - \sum_{l \in a(j) \setminus \{j\}} v_l \right)$$

$$= \sum_{j \in I_A} v_j - \sum_{j \in J_A} \sum_{l \in a(j) \setminus \{j\}} v_l$$

$$\overset{(\ast)}{=} \sum_{j \in I_A} v_j - \sum_{l \in I_A} v_l$$

$$= 0.$$
It remains to show (*). First observe that by definition it is straightforward to see that the sets \( a(j) \setminus \{j\} \) for \( j \in J_A \) are disjoint. Therefore it is sufficient to show the equality of the sets to obtain the equality of the sums. Second, since by definition every \( a \in A \) contains exactly one element of \( J_A \), while \( A \) is a partition we now have

\[
\bigcup_{j \in J_A} a(j) = \bigcup_{a \in A} a = \{1, \ldots, n\}.
\]

Since for every \( j \in a(j) \) it now follows that

\[
\bigcup_{j \in J_A} (a(j) \setminus \{j\}) = \{1, \ldots, n\} \setminus J_A = I_A.
\]

This shows (*) and ends the proof. \( \square \)

With the notation introduced (4.24) summing over the variables in (4.21) and evaluating the delta function yields,

\[
C_{n,A,L}[z; \psi_1, \psi_2] = \int_{\Lambda_L} du_0 \prod_{l \in I_A} \left( \int_{\Lambda_L} dv_l \right) \prod_{j=1}^n \tilde{B}_\#(-[M_A(v)]_j) \hat{\psi}_1,\#(u_0) \hat{\psi}_4,\#(u_0) \prod_{j=1}^{n+1} \nu \left( u_0 + \sum_{l=1}^{j-1} [M_A v)_l] - z \right)^{-1} \tag{4.26}
\]

Here and henceforth we adopt a notation where we write the integration variables right after the integral sign for notational compactness. In Proposition 4.5, we will show that the limit \( L \to \infty \) of (4.26) exists. To prove this, we will use results from the following lemma. For any \( f \in L^1(\mathbb{R}^d) \) we define the Fourier series

\[
\hat{f}(p) = \int_{\mathbb{R}^d} e^{-2\pi ip \cdot x} f(x) dx \text{ for } p \in \mathbb{R}^d.
\]

**Lemma 4.4.** The following holds.

(a) Let \( f, g \in L^2(\mathbb{R}^d) \), then

\[
\int_{\Lambda_L} \hat{f}(k) \hat{g}(k) dk = \int_{\Lambda_L} \overline{f(x)} g(x) dx
\]

(b) If \( f \in L^1(\mathbb{R}^d) \), then

\[
\|\hat{f}\|_{*,\infty} \leq \int_{\Lambda_L} |f(x)| dx \leq \|f\|_1.
\]

(c) If \( f \in L^1(\mathbb{R}^d) \), then

\[
\|\hat{f} - \hat{f}_\#\|_{*,\infty} \to 0 \quad (L \to \infty).
\]
Proof. (a) is simply the Parseval theorem for Fourier series. For details we refer the reader to [Fol13, Proposition 5.30 and Theorem 8.20].

(b) follows from the triangle inequality for integrals

\[
|\hat{f}_#(k)| = \left| \int_{\Lambda_L} e^{-i2\pi k \cdot x} f(x) dx \right| \leq \int_{\Lambda_L} |f(x)| dx \leq \|f\|_1.
\]

To prove (c), we use

\[
|\hat{f}_#(k) - \hat{f}(k)| = \left| \int_{\mathbb{R}^d} (1_{\Lambda_L}(x) - 1) e^{-i2\pi k \cdot x} f(x) dx \right| \leq \int_{\mathbb{R}^d} |(1 - 1_{\Lambda_L}(x))f(x)| dx.
\]

Now the right hand side tends to zero by the dominated convergence theorem.

In view of (4.18), Theorem 2.4 will follow from the next proposition, which shows the existence of the pointwise limit

\[
\lim_{L \to \infty} T_{n,L}[z; \cdot, \cdot] = \sum_{A \in \mathcal{A}_n} \lim_{L \to \infty} C_{n,A,L}[z; \cdot, \cdot]
\]

for \( z \in \mathbb{C} \setminus [0, \infty) \).

**Proposition 4.5.** For \( z \in \mathbb{C} \setminus [0, \infty) \) and \( \psi_1, \psi_2 \in L^2(\mathbb{R}^d) \) we have, using the integral-notation introduced in (4.26)

\[
\lim_{L \to \infty} C_{n,A,L}[z; \psi_1, \psi_2] = \int_{\mathbb{R}^d} du_0 \prod_{l \in I_A} \left( \int_{\mathbb{R}^d} dv_l \right) \prod_{j=1}^n \tilde{B}([M_A(v)]_j) \psi_1(u_0) \psi_2(u_0) \prod_{j=1}^{n+1} \left( \nu \left( u_0 + \sum_{l=1}^{j-1} [M_A(v)]_l - z \right) \right)^{-1},
\]

where \( M_A \) is defined in (4.24). For every \( n \in \mathbb{N} \), there exists a constant \( K_n \) such that, for all \( L \geq 1, z \in \mathbb{C} \setminus [0, \infty) \), and \( \psi_1, \psi_2 \in L^2(\mathbb{R}^d) \), we have

\[
|C_{n,A,L}[z; \psi_1, \psi_2]| \leq \frac{K_n \|\psi_1\| \|\psi_2\|}{\text{dist}(z, [0, \infty))^{n+1}}.
\]

Proof. Step 1: \( C_{n,A,L}[z; \cdot, \cdot] \) is a sesquilinear form, which satisfies the bound (4.29).

Estimating (4.26), using (4.24), Lemma 4.1, and the elementary bound (C.1), we find
that

\[ |C_{n,A,L}[z; \psi_1, \psi_2]| \leq \int_{\Lambda_L} du_0 \prod_{l \in I_A} \left( \int_{\Lambda_L} dv_l \right) \prod_{j \in I_A} \left| \hat{B}_\#(-v_j) \right| \| \hat{B}_\# \|_{\infty}^{-|I_A|} |\psi_{1,\#}(u_0)| \times |\hat{\psi}_{2,\#}(u_0)| \text{dist}(z, [0, \infty))^{-n-1} \]

\[ \leq \| \hat{B}_\# \|_{\infty}^{-|I_A|} \| \hat{B}_\# \|_{s,1} \| \psi_{1,\#} \|_{s,2} \| \psi_{2,\#} \|_{s,2} \text{dist}(z, [0, \infty))^{-n-1} \]

\[ \leq \| B \|_{1-n|I_A|} C_B \| \psi_1 \|_2 \| \psi_2 \|_2 \text{dist}(z, [0, \infty))^{-n-1}, \]

where in the last inequality we used Lemma 4.1 and that \( \| \hat{B}_\# \|_{s,1} \) can be bounded by a constant \( C_B \), which is independent of \( L \geq 1 \) by Lemma 4.1.

Step 2: (4.28) holds for \( \psi_1, \psi_2 \in C_c^\infty(\mathbb{R}^d) \).

This will be shown using the dominated convergence theorem writing the sum as an integral over simple functions. First choose \( L \) sufficiently large such that \( \text{supp} \psi_j \subset \Lambda_L \). Then \( \hat{\psi}_{j,\#} = \hat{\psi}_j |_{\Lambda_L} \). Since the Fourier transform maps \( C_c^\infty(\mathbb{R}^d) \) to Schwartz functions \( \hat{\psi}_{j,\#} \) is the restriction of a Schwartz function. For a function \( f : (\Lambda_L^*)^m \to \mathbb{C} \), \( m \in \mathbb{N} \) we define the function \( E[f] : (\mathbb{R}^d)^m \to \mathbb{C} \) by \( E[f](x) = f(k) \) if \( x_j = k_j + \xi_j \) for \( k_j \in \Lambda_L^* \) and \( \xi_j \in [-L, L/2]^d \), \( j = 1, \ldots, m \). With this definition we can express (4.26) as an integral over simple functions. That is with \( m = 1 + |I_A| \) and \( v = (v_1, \ldots, v_{|I_A|}) \) we find that

\[ C_{n,A,L}[z; \psi_1, \psi_2] = \int_{\mathbb{R}^d} du_0 \prod_{l \in I_A} \left( \int_{\mathbb{R}^d} dv_l \right) \prod_{j=1}^n E[\hat{B}_\#(-[M_A(\cdot)]_j)](v)E[\hat{\psi}_1](u_0) \]

\[ \times E[\hat{\psi}_2](u_0)E[Q](u_0, v), \]

with

\[ Q(u_0, v) := \prod_{j=1}^{n+1} \left( \nu \left( u + \sum_{l=1}^{j-1} [M_Av]_l \right) - z \right)^{-1}. \]

First, we show that as \( L \to \infty \) the integrand of (4.30) converges to the integrand of (4.28) pointwise. To see this, observe that \( Q \) is continuous as well as \( \hat{\psi}_1 \) and \( \hat{\psi}_2 \) are continuous. Moreover, from Lemma 4.1 (c) and the continuity of \( [M_A(\cdot)]_j \) we see that \( E[\hat{B}_#(-[M_A(\cdot)]_j)] \) converges to \( \hat{B}_#(-[M_A(\cdot)]_j) \) pointwise. Finally, observe that the absolute value of the integrand of (4.30) is bounded by a constant \( C_{\psi_1,\psi_2,B,d,n} \) uniformly for
$L \geq 1$ by
\[
\prod_{j=1}^{n} E[\tilde{B}_{\#}(-[M_A(\cdot)]_j)](v)E[\psi_1](u_0)E[\tilde{\psi}_2](u_0)E[Q](u_0, v)
\leq \left\| B \right\|_1^{n-|I_A|} \prod_{j \in I_A} \left\{ c_B E[(v_j)^{-2d}] \right\} \left\| \psi_1, \psi_2 \right\|_E \left\langle (u_0)^{-d-1} \right\rangle \text{dist}(z, [0, \infty))^{-n-1}
\leq C_{\psi_1, \psi_2, B, d, n} \left\| B \right\|_1^{n-|I_A|} \prod_{j \in I_A} \left\{ (v_j)^{-2d} \right\} \left\langle (u_0)^{-d-1} \right\rangle \text{dist}(z, [0, \infty))^{-n-1},
\] (4.31)
where in the second inequality we used Lemma 4.3 (b) to bound the terms involving $B$ for $j \in J_A$, we used Lemma 4.1 to bound the terms involving $B$ for $j \in I_A$ with some constant $c_B$, we bounded the terms involving the wavefunctions $\psi_j$, $j = 1, 2$, by a constant $C_{\psi_1, \psi_2}$ (which can be done since $\hat{\psi}_1$ and $\hat{\psi}_2$ are Schwartz functions), and for the last factor we used (C.1). Note the last inequality in (4.31) follows from a simple squaring inequality.

Now (4.31) is integrable with respect to the measure $du_0 \prod_{j \in I_A} dv_j$. Now applying the dominated convergence theorem completes Step 2.

**Step 3:** (4.28) holds for $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$.

First, we observe that the right hand side of (4.28) is a bounded sesquilinear form, which we call $h$. To see this, observe that on $C_c^\infty(\mathbb{R}^d)$ it is bounded as a limit (by Step 2) of uniformly bounded forms (Step 1). Thus, for $\psi_1, \psi_2 \in C_c^\infty(\mathbb{R}^d)$ from the triangle inequality we find
\[
\left| C_{n, A, L}[z; \psi_1, \psi_2] - h(\psi_1, \psi_2) \right|
\leq \left| C_{n, A, L}[z; \psi_1, \psi_2'] - C_{n, A, L}[z; \psi_1, \psi_2'] \right|
+ \left| C_{n, A, L}[z; \psi_1', \psi_2'] - h(\psi_1', \psi_2') \right|
+ \left| h(\psi_1', \psi_2') - h(\psi_1, \psi_2) \right|
\] (4.32)

Now the right hand side of (4.32) can be made arbitrarily small by means of Step 1 and Step 2 by choosing first $\psi_j'$ sufficiently close to $\psi_j$ (by density) and then $L$ sufficiently large, respectively.

In view of the above Proposition 4.5, we shall henceforth write for $z \in \mathbb{C} \setminus [0, \infty)$ and $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$
\[
C_{n, A, \infty}[z; \psi_1, \psi_2] := \lim_{L \to \infty} C_{n, A, L}[z; \psi_1, \psi_2].
\] (4.33)

As an immediate consequence of Proposition 4.5 and the coordinate transformation (4.20) back to the original variables, we obtain the result of the following corollary, which is interesting of its own but will not be needed in the sequel.

**Corollary 4.6.** For $z \in \mathbb{C} \setminus [0, \infty)$ and $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$, we have
\[
C_{n, A, \infty}[z; \psi_1, \psi_2]
= \int_{(\mathbb{R}^d)^{n+1}} P_{A, \infty}(k) \tilde{\psi}_1(k_1) \tilde{\psi}_2(k_{n+1}) \prod_{j=1}^{n+1} \left( \nu(k_j) - z \right)^{-1} d(k_1, \ldots, k_{n+1}),
\] (4.34)
where
\[
P_{A,\infty}(k) := \prod_{a \in A} \left\{ m_{|a|} \delta \left( \sum_{l \in a} (k_l - k_{l+1}) \right) \prod_{l \in a} \hat{B}(k_l - k_{l+1}) \right\}. \tag{4.35}
\]

Here \(\delta\) denotes the usual Dirac measure at the origin in \(\mathbb{R}^d\).

In the remaining part of this section we prove Theorem 2.6. Henceforth, we assume that the spectral parameter is of the form \(z = E + i\eta\) with
\[
E > 0 \text{ and } \eta > 0,
\]
the case \(z = E - i\eta\) will follow by taking complex conjugates. Moreover we assume that Hypothesis \(A\) holds and that \(\psi_1, \psi_2 \in L^2(\mathbb{R}^d)\) satisfy Hypothesis \(B\).

The main idea of the proof is to use a so called analytic dilation (see for example [CT73; RS78]) to control the limit as \(\eta \downarrow 0\). For (4.33) using the right hand side of (4.28) and the change of variables \(u_0 \mapsto e^{\theta}u_0\) and \(v_j \mapsto e^{\theta}v_j\) for \(\theta \in \mathbb{R}\), we find that
\[
C_{n,A,\infty}[E + i\eta; \psi_1, \psi_2] \tag{4.36}
\]
\[
= e^{-d(|A|+1)\theta} \int_{(\mathbb{R}^d)^{|A|+1}} du_0 \prod_{l \in A} dv_l \prod_{j=1}^n \hat{B}(-e^{-\theta}[M_A(v)]_j) 
\times \bar{\psi}_1(e^{-\theta}u_0)\hat{\psi}_2(e^{-\theta}u_0) \prod_{j=1}^{n+1} \left( e^{-2\theta \nu} \left( u_0 + \sum_{l=1}^{j-1} [M_A v_l]_l \right) - E - i\eta \right)^{-1}.
\]

Note that the left hand side of (4.36) is independent of \(\theta \in \mathbb{R}\). By Hypothesis \(A\) the assumption that \(\psi_1\) and \(\psi_2\) satisfy Hypothesis \(B\) and the argument presented below, we will be able to analytically extend the right hand side for \(\theta\) into the open set
\[
S := \left\{ \theta \in \mathbb{C} : -\frac{1}{2} \arctan(\eta/E) < \text{Im}\theta < 3\pi/4 \right\} \cap \{ \theta \in \mathbb{C} : |\theta| < \min(\vartheta_B, \vartheta_\psi) \},
\]
which is a neighborhood of zero. As displayed in the Figure below, for \(E, \eta > 0\), the point \(E + i\eta\) is located in the first quadrant. At the same time \(\nu \left( u_0 + \sum_{l=1}^{j-1} [M_A(v)]_l \right)\) is non-negative real-valued (for any \(j = 1, \ldots, n+1\)), hence located on the non-negative real axis and gets turned by the factor \(e^{-2\theta} \nu\) in negative direction by the angle \(2\text{Im}\theta\). Since \(\arctan(\eta/E)\) represents the angle of the polar form of \(E + i\eta\) and a rotation by \(-\frac{3\pi}{2}\) marks the transition between second and first quadrant, the first set in the intersection in the definition of \(S\) ensures the invertibility of
\[
\left( e^{-2\theta \nu} \left( u_0 + \sum_{l=1}^{j-1} [M_A v_l]_l \right) - E - i\eta \right) \tag{4.37}
\]
for any \(j = 1, \ldots, n+1\).
Figure 1: The set \( \{e^{2\theta}r : r \geq 0 \} \) for \( \vartheta = \text{Im}\theta \) is drawn in red.

To estimate the resolvents we will use the following lemma.

**Lemma 4.7.** Let \( \alpha, E, \eta, \lambda \in \mathbb{R} \). Then the following holds.

(a) We have
\[
|e^{-i\alpha}\lambda - E - i\eta| = |\lambda - e^{i\alpha}(E + i\eta)| \geq |\sin(\alpha)E + \cos(\alpha)\eta|.
\]
(4.38)

(b) If \( \alpha \in [0, \pi/2] \) and \( \eta, E \in [0, \infty) \) then
\[
|e^{-i\alpha}\lambda - E - i\eta| \geq |\sin(\alpha)E|.
\]
(4.39)

**Proof.** (a) follows from
\[
|e^{-i\alpha}\lambda - E - i\eta| = |\lambda - e^{i\alpha}(E + i\eta)|
\]
\[
= \left( (\lambda - \cos(\alpha)E + \sin(\alpha)\eta)^2 + (\sin(\alpha)E + \cos(\alpha)\eta)^2 \right)^{1/2}
\]
(4.40)

and the monotonicity of the square root. Now (b) follows directly from (a).

We are now ready to prove Theorem 2.6

**Proof of Theorem 2.6.** Let
\[
\vartheta \in (0, \min(\vartheta_B, \vartheta_\psi, 3\pi/4)).
\]
(4.41)
The existence of the limit $\eta \in S$. Let $m = |I_A|$ and let a single $\int$ be a short hand notation for the collection of all the respective integrals. From (4.36) we find

$$C_{n,\Lambda}(E + i\eta; \psi_1, \psi_2)$$

$$(4.42)$$

$$= e^{-i \theta (m+1) \theta} \int du_0 \prod_{s \in I_A} \prod_{j=1}^n \hat{B}(-e^{-i \theta \cdot M_A(v)_j}) \psi_1(e^{i \theta u_0}) \psi_2(e^{-i \theta u_0})$$

$$\prod_{j=1}^{n+1} \left( e^{-2i \theta \nu} \left( u_0 + \sum_{l=1}^{j-1} [M_A(v)]_l \right) - E - i \eta \right)^{-1}.$$

Factoring out $e^{-2i \theta}$ in the denominator, we find

$$C_{n,\Lambda}(E + i\eta; \psi_1, \psi_2)$$

$$(4.43)$$

$$= e^{2(n+1)i \theta} e^{-i \theta (m+1) \theta} \int du_0 \prod_{s \in I_A} \prod_{j=1}^n \hat{B}(-e^{-i \theta \cdot M_A(v)_j}) \psi_1(e^{i \theta u_0}) \psi_2(e^{-i \theta u_0})$$

$$\prod_{j=1}^{n+1} \left( \nu \left( u_0 + \sum_{l=1}^{j-1} [M_A(v)]_l \right) - e^{2i \theta} (E + i \eta) \right)^{-1}.$$

Using assumption (4.41), we can bound the integrand in (4.43) by means of (4.38)

$$\prod_{j=1}^n \hat{B}(-e^{-i \theta \cdot M_A(v)_j}) \psi_1(e^{i \theta u_0}) \psi_2(e^{-i \theta u_0}) \prod_{j=1}^{n+1} \left( \nu \left( u_0 + \sum_{l=1}^{j-1} [M_A(v)]_l \right) - e^{2i \theta} (E + i \eta) \right)^{-1}$$

$$\leq \prod_{s \in I_A} \hat{B}(-e^{-i \theta v_s}) ||\hat{B}(e^{-i \theta \cdot \cdot \cdot})||_{A} \frac{\left| \psi_1(e^{i \theta u_0}) \psi_2(e^{-i \theta u_0}) \right|}{\sin (2 \theta) E + \cos (2 \theta) \eta}^{n+1}$$

$$\prod_{s \in I_A} \left| \hat{B}(-e^{-i \theta v_s}) ||\hat{B}(e^{-i \theta \cdot \cdot \cdot})||_{A} \right| \frac{\left| \psi_1(e^{i \theta u_0}) \psi_2(e^{-i \theta u_0}) \right|}{\sin (2 \theta) E}^{n+1}.$$  (4.44)

where in the last inequality we additionally assumed that $\theta \in (0, \pi/4)$ and applied (4.39). As a consequence of Hypothesis B and the fact that $\psi_1$ and $\psi_2$ satisfy Hypothesis B the function on the right hand side of (4.44) is integrable with respect to the measure $du_0 \prod_{s \in I_A} dv_s$ and the resulting integral is bounded by

$$\frac{||\hat{B}(-e^{-i \theta \cdot \cdot \cdot})||_{A} ||\hat{B}(e^{-i \theta \cdot \cdot \cdot})||_{A}}{\sin (2 \theta) E}^{n+1} \left[ \int dv_s \psi_1(e^{i \theta u_0}) \right]^{2} \left[ \int dv_s \psi_2(e^{-i \theta u_0}) \right]^{2}.$$  (4.45)

The existence of the limit $\eta \downarrow 0$ of (4.43) thus follows by first fixing a $\theta \in (0, \min(\theta_B, \theta_\psi, \pi/4))$ and then using the dominated convergence theorem. Note that dominated convergence is justified by the integrable majorant (4.45) and the continuity of the integrand of (4.43) in $\eta \geq 0$ for fixed $E > 0$. Analogously, it follows that $T_{n,\Lambda}(z; \psi_1, \psi_2)$ has a continuous extension to $\{w \in \mathbb{C} : \text{Im} w \geq 0, \text{Re} w > 0\}$ for $z \in \{w \in \mathbb{C} : \text{Im} w > 0, \text{Re} w > 0\}$, by choosing the same majorant and using the continuity of the integrand in $z = E + i \eta$. This completes the proof Theorem 2.6.\qed
5 Asymptotic error estimate

The goal of this section is to prove Theorem 2.7 which will take some steps as preparatory work. Let \( z \in \mathbb{C} \setminus [0, \infty) \). For \( \lambda \geq 0 \), we start of with recalling the expansion \( (2.15) \)

\[
(H_L - z)^{-1} = \sum_{j=0}^{n-1} R_L(z)[\lambda V_L R_L(z)]^j + [R_L(z)\lambda V_L]^{n}(H_L - z)^{-1}.
\]

Recalling \( T_{j,L} \) defined in \( (2.16) \), we can write the expectation of matrix elements as

\[
E_L\langle \psi_{1,\#}, (H_L - z)^{-1}\psi_{2,\#}\rangle_{L^2(\Lambda_L)} = \sum_{j=0}^{n-1} \lambda^j T_{j,L}[z; \psi_1, \psi_2] + \lambda^n U_{n,L}[z; \psi_1, \psi_2; \lambda], \tag{5.1}
\]

where we defined

\[
U_{n,L}[z; \psi_1, \psi_2; \lambda] := E_L\langle \psi_{1,\#}, [R_L(z)V_L]^{n}(H_L - z)^{-1}\psi_{2,\#}\rangle_{L^2(\Lambda_L)}. \tag{5.2}
\]

In this section we control the error terms \( U_{n,L} \). For the error estimate we use the Cauchy-Schwarz inequality twice and find

\[
|U_{n,L}[z; \psi_1, \psi_2; \lambda]| = E_L\langle [(R_L(z)V_L)^{n}]^* \psi_{1,\#}, (H_L - z)^{-1}\psi_{2,\#}\rangle_{L^2(\Lambda_L)} \tag{5.3}
\]

\[
\leq E_L\|[R_L(z)V_L]^{n}\psi_{1,\#}\|_{L^2(\Lambda_L)}\| (H_L - z)^{-1}\psi_{2,\#}\|_{L^2(\Lambda_L)} \tag{5.4}
\]

where we defined

\[
E_{n,L}[z; \psi_1] := E_L\|[R_L(z)V_L]^{n}\psi_{1,\#}\|_{L^2(\Lambda_L)} \tag{5.5}
\]

\[
= E_L\langle [(R_L(z)V_L)^{n}]^* \psi_{1,\#}\rangle_{L^2(\Lambda_L)}
\]

and used the basic inequality \( (C.1) \) together with the spectral theorem.

To estimate \( (5.3) \) transition to the Fourier space using identities \( (3.7) \) and \( (3.8) \), and note that between the middle \( V_L \)'s we have instead of a resolvent an identity, which we write for bookkeeping in terms of a delta function as in \( (2.18) \) (note that Fubinis theorem is applicable since \( z \in \mathbb{C} \setminus [0, \infty) \)) - hence the resolvents are well defined - and the assumptions on \( \hat{\psi} \) and \( \hat{V}_L \) and \( \hat{B} \) respectively by \( (4.17) \)

\[
E_{n,L}[z; \psi_1] = \int_{(\Lambda_L^*)^{n+1}} \hat{\psi}_{1,\#}(p_1) \hat{\psi}_{1,\#}(p_{n+1}) E_L \prod_{j=1}^{n} \hat{V}_L(p_j - p_{j+1}) \delta_z(p_{n+1} - \tilde{p}_1) \prod_{j=1}^{n} \hat{V}_L(\tilde{p}_j - \tilde{p}_{j+1})
\]

\[
\times \left( \prod_{j=1}^{n} \frac{1}{\nu(p_j) - z} \right) \left( \prod_{j=2}^{n+1} \frac{1}{\nu(\tilde{p}_j) - \bar{z}} \right) d(p_1, \ldots, p_{n+1}) d(\tilde{p}_1, \ldots, \tilde{p}_{n+1})
\]

\[
= \int_{(\Lambda_L^*)^{2n+1}} \hat{\psi}_{1,\#}(q_1) \hat{\psi}_{2,\#}(q_{2n+1}) E_L \prod_{j=1}^{2n} \hat{V}_L(q_j - q_{j+1})
\]

\[
\times \left( \prod_{j=1}^{n} \frac{1}{\nu(q_j) - z} \right) \left( \prod_{j=n+2}^{2n+1} \frac{1}{\nu(q_j) - \bar{z}} \right) d(q_1, \ldots, q_{2n+1}). \tag{5.6}
\]
In the second equality we integrated out \( p_{n+1} \) using the delta function and introduced the following simple relabeling of the integration variables

\[
q = ((q_1, \ldots q_n), (q_{n+1}, \ldots, q_{2n+1})) = ((p_1, \ldots, p_n), (\tilde{p}_1, \ldots, \tilde{p}_{n+1})) = (p, \tilde{p}).
\]

Now let us calculate the expectation in (5.6). Using (3.10) in (5.6), we find

\[
E_{n,L}[z; \psi_1] = \int_{(\Lambda^*_L)^{2n+1}} \sum_{A \in A_{2n}} \mathcal{P}_{A,L}(q_1, \ldots, q_{2n+1}) \overline{\psi}_1(\overline{\psi}_1(q_1)\overline{\psi}_1(q_{2n+1}))
\]

\[
\times \prod_{j=1}^{n} (\nu(q_j) - z)^{-1} \prod_{j=n+2}^{2n+1} (\nu(q_j) - \overline{z})^{-1}d(q_1, \ldots, q_{2n+1}),
\]

where we recall the definition given in (2.19)

\[
\mathcal{P}_{A,L}(q) = \prod_{a \in A} \left[ m_{|a|} \delta_{\ast,L} \left( \sum_{l \in a} (q_l - q_{l+1}) \right) \prod_{l \in a} \hat{B}_\#(q_l - q_{l+1}) \right].
\]

For \( A \in A_{2n} \) we define

\[
D_{n,A,L}[z; \psi_1] = \int_{(\Lambda^*_L)^{2n+1}} \mathcal{P}_{A,L}(q_1, \ldots, q_{2n+1}) \overline{\psi}_1(\overline{\psi}_1(q_1)\overline{\psi}_1(q_{2n+1}))
\]

\[
\times \prod_{j=1}^{n} (\nu(q_j) - z)^{-1} \prod_{j=n+2}^{2n+1} (\nu(q_j) - \overline{z})^{-1}d(q_1, \ldots, q_{2n+1}).
\]

With this definition we can write (5.7) as

\[
E_{n,L}[z; \psi_1] = \sum_{A \in A_{2n}} D_{n,A,L}[z; \psi_1].
\]

To estimate (5.10) we introduce the following change of variables similar to (4.20)

\[
u_0 = q_1 \text{ and } u_l = q_{l+1} - q_l \text{ for } l = 1, \ldots, n
\]

and \( u = (u_0, \ldots, u_{2n}) \). With this we can express the \( q_j \)'s as

\[
q_j = \sum_{l=0}^{j-1} u_l.
\]

Using the change of variables (5.12) in (5.10) we find with (5.9) expressed in the new variables that for \( A \in A_{2n} \)

\[
D_{n,A,L}[z; \psi_1] = \int_{(\Lambda^*_L)^{2n+1}} \prod_{a \in A} \left[ m_{|a|} \delta_{\ast,L} \left( \sum_{l \in a} u_l \right) \prod_{l \in a} \hat{B}_\#(-u_l) \right] \overline{\psi}_1(\overline{\psi}_1(u_0)\overline{\psi}_1(\sum_{l=0}^{2n} u_l))
\]

\[
\times \prod_{j=1}^{n} (\nu \left( \sum_{l=0}^{j-1} u_l \right) - z)^{-1} \prod_{j=n+2}^{2n+1} (\nu \left( \sum_{l=0}^{j-1} u_l \right) - \overline{z})^{-1}d(u_0, \ldots, u_{2n}).
\]

(5.13)
We are now going to resolve the $\delta_{a,L}(\sum_{l \in a} u_l)$ factors similarly to the procedure presented between (4.21) and Lemma 4.25. We are going to use the notation as introduced right before (4.24), where for $A \in A_{2n}$ we define

$$J_A := \{\max a : a \in A\} \subset \{1, \ldots, 2n\},$$

(5.14)
as the set of all the largest elements of the respective sets $a \in A$ of the partition, as well as its complement

$$I_A := \{1, \ldots, 2n\} \setminus J_A.$$

(5.15)

For $l \in \{1, \ldots, 2n\}$ we define $a(l) \in A$ as the set with $l \in a(l)$. Note that this choice is both, always possible as well as unique, since $A$ is a partition of $\{1, \ldots, 2n\}$. With this we define the map

$$M_A : (\mathbb{R}^d)^{|I_A|} \to (\mathbb{R}^d)^{2n} \quad \text{via} \quad [M_A(u)]_j = \begin{cases} u_j : j \in I_A, \\ -\sum_{l \in a(j) \setminus \{j\}} u_l : j \in J_A. \end{cases}$$

(5.16)

In the following we will be working with terms of the form $S_j = u_0 + \sum_{l=1}^{j-1} [M_A(u)]_l$.

To get some intuition on the defined objects let us discuss the following example and illustrate it with a picture: We consider the set $\{1, \ldots, 10\}$ with $n = 5$ together with the partition $A = \{\{1, 6\}, \{2, 5\}, \{3, 7, 9, 10\}, \{4, 8\}\}$. Therefore $J_A = \{5, 6, 8, 10\}$ and $I_A = \{1, 2, 3, 4, 7, 9\}$. The picture displays $S_1$ to $S_{11}$. This is illustrated in the Figure 2.

![Figure 2: The blue lines connect the points which belong to the same set $a$ of the partition $A$. The elements of the set $J_A$ are coloured with red and the elements of the set $I_A$ are black. The $S_j$ such that $j - 1 \in J_A$ are coloured with magenta.](image)
Using this notation to express (5.13) we find for $A \in \mathcal{A}_{2n}$

$$D_{n,A,L}[z; \psi_{1}] = \int_{\Lambda_{L}}^{\Lambda_{L}} du_{0} \prod_{t \in I_{A}} \int_{\Lambda_{L}}^{\Lambda_{L}} \left[ m_{|a|} \prod_{t \in a} \hat{B}_{\#}(-[M_{A}(u)]_{t}) \right] \tilde{\psi}_{1,\#}(u_{0}) \hat{\psi}_{1,\#}(u_{0})$$

$$\times \prod_{j=1}^{n} \left[ \nu \left( u_{0} + \sum_{l=1}^{j-1} [M_{A}(u)]_{l} \right) - z \right]^{-1} \prod_{j=n+2}^{2n+1} \left[ \nu \left( u_{0} + \sum_{l=1}^{j-1} [M_{A}(u)]_{l} \right) - \bar{z} \right]^{-1},$$

where we used $\sum_{j=1}^{2n} [M_{A}(v)]_{j} = 0$ by Lemma 4.3 in the argument of the second $\hat{\psi}_{1,\#}$.

We are now in a situation in which resolving the $\delta$-functions eliminated the sums over the $u_{j}$ for $j - 1 \in J_{A}$, while the sums for $j - 1 \in I_{A} \cup \{0\}$ remain. The idea is now that each “integral” $\nu_{j}$ can control with the respective $\hat{B}_{\#}(-u_{j})$ one resolvent, which we will show in Lemma 5.2. We will first estimate the other resolvents as well as the $\hat{B}_{\#}(-u_{j})$ that belong to $j \in J_{A}$.

We write $\eta = \text{Im}(z)$. We estimate all resolvents for which $j - 1 \in J_{A}$ by a factor of $\eta^{-1}$ using (C.1), assuming $0 < \eta \leq 1$. Note that this may include the case where $j = n + 1$ and $j - 1 \in J_{A}$, that is where we have an identity instead of a resolvent, in which case we estimate trivially by $\eta^{-1}$. Since $J_{A}$ contains one element of each partition set in $A$, we have $|J_{A}| = |A|$ and therefore pay a factor of $\eta^{-|A|}$. In addition, to obtain an $L^{2}$-bound for the wave function $\psi_{1}$, for convenience, we estimate the first resolvent, i.e., $j = 1$, with $\eta^{-1}$, and therefore pay a total factor of $\eta^{-|A|-1}$, since $1 \notin J_{A}$ by the assumption $m_{1} = 0$. Furthermore, for $j \in J_{A}$ we estimate $|\hat{B}_{\#}(-[M_{A}(u)]_{j})| \leq \|\hat{B}_{\#}\|_{*,\infty}$ trivially, which gives a collective upper bound for these terms by $\|\hat{B}_{\#}\|_{*,\infty}^{A}$, again since $|J_{A}| = |A|$. After these steps we arrive at

$$|D_{n,A,L}[z; \psi_{1}]| \leq \eta^{-|A|-1}\|\hat{B}_{\#}\|_{*,\infty}^{A} \prod_{a \in A} \int_{\Lambda_{L}}^{\Lambda_{L}} du_{0} \prod_{t \in I_{A}} \int_{\Lambda_{L}}^{\Lambda_{L}} du_{l} \left[ \tilde{\psi}_{1,\#}(u_{0}) \right]^{2} \prod_{l=1, l \in I_{A}}^{2n} \hat{B}_{\#}(-u_{l})$$

$$\times \prod_{j=2}^{n} \left( \nu \left( u_{0} + \sum_{l=1}^{j-1} [M_{A}(u)]_{l} \right) - z \right) \prod_{j=n+2}^{2n+1} \left( \nu \left( u_{0} + \sum_{l=1}^{j-1} [M_{A}(u)]_{l} \right) - \bar{z} \right)^{-1}.$$

To estimate the remaining resolvents, we shall use the Lemma 5.1 below. We view $u_{j}$ as the function

$$u_{j} : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}, \quad (t_{0}, \ldots, t_{2n}) \mapsto t_{j} \text{ for } j = 0, \ldots, 2n.$$

In the following lemma, we will show that the functions $\{u_{0} + \sum_{l=1}^{j-1} [M_{A}(u)]_{l} : j - 1 \in I_{A}\}$ are linearly independent. Doing so we will calculate the sums explicitly and show that for all $j = 1, \ldots, 2n + 1$ with $j - 1 \in I_{A}$ we have a relation of the form

$$u_{0} + \sum_{l=1}^{j-1} [M_{A}(u)]_{l} = u_{j-1} + u_{0} + \text{a linear combination of } \{u_{l} : l \leq j - 2, l \in I_{A}\}.$$
The identity above will later be used in Lemma 5.2 to integrate over the $du_l$ backwards (beginning with the one with the largest index) one after another to eliminate the respective $u_l$.

**Lemma 5.1.** Let $A$ be a partition of $\{1, \ldots, 2n\}$. Let $J_A, I_A$ and $M_A$ be defined as in (5.14), (5.15) and (5.16). Let $u_j$ for $j \in \{0, \ldots, 2n\}$ be the function in (5.19). Let for $A$ and $j = 1, \ldots, 2n + 1$ the function $\sigma_{A,j}$ be defined as

$$
\sigma_{A,j} : \{1, \ldots, 2n\} \to \{0, 1\} \quad \text{via} \quad \sigma_{A,j}(l) = \begin{cases} 
1 & : \max a(l) > j, \\
0 & : \text{otherwise}.
\end{cases}
$$

Then for all $j = 2, \ldots, 2n + 1$ with $j - 1 \in I_A$ we have

$$
u_0 + \sum_{l=1}^{j-1} [M_A(u)_l] = u_{j-1} + u_0 + \sum_{l \in \{1, \ldots, j-2\} \cap I_A} \sigma_{A,j-2}(l)u_l. \quad (5.20)
$$

**Proof.** To shorten the notation let $N_k = \{1, \ldots, k\}$ for $k \in \mathbb{N}$. Plugging in the definition of $M_A$ for its respective cases we get for $j - 1 \in I_A$

$$
u_0 + \sum_{l=1}^{j-1} [M_A(u)_l] = u_0 + \sum_{l \in N_{j-1} \cap I_A} [M_A(u)]_l + \sum_{l \in N_{j-1} \cap I_A} [M_A(u)]_l
$$

$$
= u_0 + \sum_{l \in N_{j-1} \cap I_A} u_l + \sum_{l \in N_{j-1} \cap I_A} \sum_{s \in a(l) \setminus \{l\}} (-u_s)
$$

$$
j-1 \notin J_A \quad u_0 + u_{j-1} + \sum_{l \in N_{j-2} \cap I_A} u_l + \sum_{l \in N_{j-2} \cap I_A} \sum_{s \in a(l) \setminus \{l\}} (-u_s)
$$

$$
\stackrel{(*)}{=} u_0 + u_{j-1} + \sum_{l \in N_{j-2} \cap I_A} u_l + \sum_{l \in N_{j-2} \cap I_A} \sum_{a(l) \leq j-2} (-u_l)
$$

$$
= u_{j-1} + u_0 + \sum_{l \in N_{j-2} \cap I_A} u_l + \sum_{l \in N_{j-2} \cap I_A} \sum_{a(l) \leq j-2} \sum_{a(l) \leq j-2} (-u_l)
$$

$$
= u_{j-1} + u_0 + \sum_{l \in N_{j-2} \cap I_A} u_l + \sum_{l \in N_{j-2} \cap I_A} (u_l - u_l)
$$

$$
= u_{j-1} + u_0 + \sum_{l \in N_{j-2} \cap I_A} \sigma_{A,j-2}(l)u_l.
$$

Now to $(*)$: We have to show

$$
\sum_{l \in N_{j-2} \cap I_A} \sum_{s \in a(l) \setminus \{l\}} (-u_s) = \sum_{l \in N_{j-2} \cap I_A} \sum_{a(l) \leq j-2} (-u_l). \quad (5.21)
$$
From the proof of Lemma 4.3, we have seen
\[ I_A = \bigcup_{l \in J_A} a(l) \setminus \{l\}. \tag{5.22} \]

Using this we will now show that
\[ I_A \cap \{l : \max a(l) \leq j - 2\} = \bigcup_{l \in N_{j-2} \cap J_A} a(l) \setminus \{l\} \tag{5.23} \]

First \(\subset\): Let \(s\) be an element of the LHS of (5.23). Since \(s \in I_A\), we have \(s \in a(l) \setminus \{l\}\) for some \(l \in J_A\) by (5.22). Now since \(s\) in \(\{t : \max a(t) \leq j - 2\}\) we find \(j - 2 \geq \max a(s) = \max a(l) = l\).

Second \(\supset\): Let \(L\) denote the set on the RHS of (5.23). Then \(L \subseteq I_A\) (by (5.22)). If \(k \in L\), then \(k \in a(l) \setminus \{l\}\) for some \(l \in N_{j-2} \cap J_A\) and so \(\max a(k) = \max a(l) = l \leq j - 2\).

Moreover, we have
\[ I_A \cap \{l : \max a(l) \leq j - 2\} \subseteq N_{j-3} \tag{5.24} \]

since for \(s \in I_A\) with \(\max a(s) \leq j - 2\), we find \(s + 1 \leq \max a(s) \leq j - 2\).

Now (5.21) follows from (5.23), (5.24), and the fact that the elements \(a \in A\) are mutually disjoint.

\[ \square \]

As a final step before we can prove Theorem 2.7, we will use the expression shown in Lemma 5.1 performing an induction to integrate over the \(u_l\) variables one after another.

**Lemma 5.2.** Let \(n \in \mathbb{N}\) and \(A\) be a partition of the set \(\{1, \ldots, 2n\}\). Let \(J_A, I_A\) and \(M_A\) be defined as in (5.14), (5.15) and (5.16). Then for \(z \in \{w \in \mathbb{C} : \operatorname{Im} w > 0\}\) we have
\[
\int_{A_L} du_0 \prod_{l \in I_A} \left( \int_{A_L} du_l \right) \left| \bar{\psi}_{1,\#}(u_0) \right|^2 \left| \prod_{l=1,l \in I_A}^{2n} \hat{B}_{\#}(-u_l) \right| \tag{5.25}
\]
\[
\times \prod_{j=2}^{n} \left| \nu \left( u_{j-1} + u_0 + \sum_{l \in \{1, \ldots, j-2\} \cap I_A} \sigma_{A,j-2}(l) u_l \right) - z \right|^{-1}
\]
\[
\times \prod_{j=n+2}^{2n+1} \left| \nu \left( u_{j-1} + u_0 + \sum_{l \in \{1, \ldots, j-2\} \cap I_A} \sigma_{A,j-2}(l) u_l \right) - z \right|^{-1}
\]
\[
\leq \| \psi_1 \|^2 [C(E, d, L, \eta, \hat{B}_{\#})]^{2n - |A|},
\]

where \(\sigma_{A,j}\) is defined in Lemma 5.1 and \(C(E, d, L, \eta, \hat{B}_{\#})\) is the constant (C.19) from Proposition C.3.
Proof. Let \( z = E + i\eta \) with \( E > 0 \) and \( \eta > 0 \) (the case \( E - i\eta \) then follows by taking complex conjugates). Moreover, since \( |a + ib| = (a^2 + b^2)^{1/2} \) for \( a, b \in \mathbb{R} \) and hence does not depend on the sign of \( b \), we can replace \( \tilde{z} \) with \( z \) in the above formula without change. To shorten the writing expenses we introduce some notation: Let \( P := P(E, d, L, \eta, \hat{B}_{\#}, \hat{\psi}) \) denote the LHS of (5.25), i.e.,

\[
P = \int_{\Lambda_{\#}} du_0 \prod_{l \in I_A} \left( \int_{\Lambda_{\#}} du_l \right) |\tilde{\psi}_{1,\#}(u_0)|^2 \left| \prod_{l=1,l \in I_A}^{2n} \hat{B}_{\#}(-u_l) \right| \times \prod_{\substack{j=2\atop j \notin n+1}}^{2n+1} \left| \nu \left( u_{j-1} + u_0 + \sum_{l \in \{1, \ldots, j-2\} \cap I_A} \sigma_{A,j-1}(l) u_l \right) - E - i\eta \right|^{-1}.
\]

Since we now want to integrate out the variables one after another, starting with the one with the largest index, we consider the set \( I_A \), but excluding it’s \( m \) largest elements, which we call \( K_m \). So, in that sense let \( K_0 = I_A \) and \( K_1 = I_A \setminus \{ \max I_A \} \) and so on, explicitly \( K_m = \{ j \in I_A : |\{ s \in I_A : s > j \}| \geq m \} \).

We are now going to prove the following statement in \( m \in \{0, \ldots, 2n+1\} \) with an induction over \( m \) [Note that we shift the index \( j \) by 1, i.e., we write \( j \) instead of \( j - 1 \)]

\[
P \leq [C(E, d, L, \eta, \hat{B}_{\#})]^{m} \int_{\Lambda_{\#}} du_0 \prod_{l \in K_m} \left( \int_{\Lambda_{\#}} du_l \right) |\tilde{\psi}_{1,\#}(u_0)|^2 \left| \prod_{l \in K_m} \hat{B}_{\#}(-u_l) \right| \times \prod_{\substack{j \in K_m\atop j \neq n}} \left| \nu \left( u_j + u_0 + \sum_{l \in \{1, \ldots, j-1\} \cap I_A} \sigma_{A,j}(l) u_l \right) - E - i\eta \right|^{-1}.
\]

Since we can not take more elements out of the set \( I_A \) other than those that are in \( I_A \), the variable \( m \) can not exceed the value \( |I_A| = 2n - |A| \). Therefore for \( 0 \leq m \leq 2n - |A| \) this statement makes sense.

First consider \( m = 0 \). In this case \([C(E, d, L, \eta, \hat{B}_{\#})]^{m} = 1 \) and \( K_m = K_0 = I_A \) hence the RHS is equal to the definition of \( P \). Now let the statement hold true for an \( m \) with \( 0 \leq m \leq 2n - |A| - 1 \). Let \( s = \max K_m \). Our plan is to use Proposition [C.4] to eliminate the “integral” over \( u_s \). Now only \( \hat{B}_{\#}(-u_s) \) and, by Lemma [5.20] the resolvent with \( j = s \) depend on \( u_s \).

To create the proper form for Proposition [C.4] to be applicable, we perform a change of variables given by

\[
v = u_s + u_0 + \sum_{l \in \{1, \ldots, s-1\} \cap I_A} \sigma_{A,s}(l) u_l
\]

and therefore

\[
-u_s = -v + u_0 + \sum_{l \in \{1, \ldots, s-1\} \cap I_A} \sigma_{A,s}(l) u_l.
\]
We also introduce the notation
\[ \hat{B}(v) := \hat{B}_\# \left( -v + u_0 + \sum_{l \in \{1, \ldots, s-1\} \cap I_A} \sigma_{A,s}(l) u_l \right) . \]

We are now able to write
\[
\int_{\Lambda_L^*} |\hat{B}_\#(-u_s)| \nu \left( u_s + u_0 + \sum_{l \in \{1, \ldots, s-1\} \cap I_A} \sigma_{A,s}(l) u_l \right) - E - i\eta \ 
= \int_{\Lambda_L^*} |\hat{B}(v)| \nu (v) - E - i\eta |^{-1} \ dv 
\leq C(E, d, L, \eta, \hat{B}) = C(E, d, L, \eta, \hat{B}_\#),
\]

where we applied Proposition C.5 in the inequality. Note that \(C(\cdots)\) being the constant
given in Proposition C.3 where the definition (C.19) of \(C\) gives the final equality. Plugging
this into the statement for \(m\), recall \(s = \max K_m\), gives us

\[
P \leq [C(E, d, L, \eta, \hat{B}_\#)]^m \int_{\Lambda_L^*} du_0 \prod_{l \in K_m} \left( \int_{\Lambda_L^*} du_l \right) |\overline{\psi}_{1,\#}(u_0)|^2 \prod_{l \in K_m} |\hat{B}_\#(-u_l)| 
\times \prod_{j \in K_m \atop j \neq n} \left| \nu \left( u_j + u_0 + \sum_{l \in \{1, \ldots, j-1\} \cap I_A} \sigma_{A,j}(l) u_l \right) - E - i\eta \right|^{-1} 
\times \int_{\Lambda_L^*} |\hat{B}_\#(-u_s)| \nu \left( u_s + u_0 + \sum_{l \in \{1, \ldots, s-1\} \cap I_A} \sigma_{A,s}(l) u_l \right) - E - i\eta \ 
\leq [C(E, d, L, \eta, \hat{B}_\#)]^{m+1} \int_{\Lambda_L^*} du_0 \prod_{l \in K_{m+1}} \left( \int_{\Lambda_L^*} du_l \right) |\overline{\psi}_{1,\#}(u_0)|^2 \prod_{l \in K_{m+1}} |\hat{B}_\#(-u_l)| 
\times \prod_{j \in K_{m+1} \atop j \neq n} \left| \nu \left( u_j + u_0 + \sum_{l \in \{1, \ldots, j-1\} \cap I_A} \sigma_{A,j}(l) u_l \right) - E - i\eta \right|^{-1} .
\]

This completes the induction over \(m\).
Finally, let \( m = 2n - |A| = |I_A| \). Using \( K_{|I_A|} = 0 \), the convention \( \prod_{j \in \emptyset} a_j = 1 \), as well as \( \int_{\Lambda L} |\psi_{1,\#}(u_0)|^2 du_0 = \|\psi_1\|^2_{L^2(\Lambda L)} \leq \|\psi_1\|^2 \), by Lemma 4.3 (a), we find

\[
P \leq [C(E, d, L, \eta, \hat{B}_\#)]^{2n - |A|} \int_{\Lambda L} du_0 \prod_{l \in \emptyset} \left( \int_{\Lambda L} du_l \right) |\psi_{1,\#}(u_0)|^2 \left| \prod_{l \in \emptyset} \hat{B}_#(-u_l) \right|
\]

\[
\times \prod_{j \in \emptyset} \nu \left( u_j + u_0 + \sum_{l \in \{1, \ldots, j-1\} \cap I_A} \sigma_{A,j}(l)u_l \right) - E - i\xi_n(j)\eta
\]

\[
\leq \|\psi_1\|^2 [C(E, d, L, \eta, \hat{B}_#)]^{2n - |A|},
\]

which completes the proof. \( \Box \)

We are now able to prove Theorem 2.7 as outlined below.

**Proof of Theorem 2.7.** We assume that \( z = E + i\eta \) (the case \( z = E - i\eta \) is obtained by taking complex conjugates). We start from (5.18). Inserting (5.20) of Lemma 5.1 into (5.18) and then using Lemma 5.2 we find

\[
|D_{n,A,L}[E + i\eta; \psi_1]| \leq \eta^{-|A|-1} \|\psi_1\|^2 \|\hat{B}_#\|_{s,\infty}^{|A|} C(E, d, L, \eta, \hat{B}_#)^{2n - |A|} \prod_{A \in \hat{A}} a_{|A|}, \quad (5.26)
\]

where \( C(E, d, L, \eta, \hat{B}_#) \) is by Lemma 5.2 the constant (C.19) given in Proposition C.5

\[
C(E, d, L, \eta, \hat{B}_#)
\]

\[
= 2\|\hat{B}_#\|_{s,\infty} \left[ C_1(2E, d) \ln \eta^{-1} + 1 \right] + 2d\sqrt{2}(4E + 1)^{d/2} \right] + 2\|\hat{B}_#\|_{s,1}.
\]

To estimate the \( \|\hat{B}_#\|_{s,\infty} \) term appearing in (5.27) we use that by Lemma 4.4 (b) we have

\[
\|\hat{B}_#\|_{s,\infty} \leq \|B\|_1 < \infty.
\]

Finally let us define \( c_B \) as the RHS of (4.17), which is an \( L \)-independent upper bound on the \( \|\hat{B}_#\|_{s,1} \) term appearing in (5.27). Now by (C.19) this shows that (5.27) is uniformly bounded in \( L \). Therefore we have

\[
C(E, d, L, \eta, \hat{B}_#)
\]

\[
\leq 2\|B\|_1 C_1(2E, d) \ln \eta^{-1} + 1 \right] + \left[ 2\|B\|_1 2d\sqrt{2}(4E + 1)^{d/2} + 2c_B \right]
\]

\[
\leq \tilde{C}(d, E, B) \left( \ln \eta^{-1} + 1 \right) + 1, \quad (5.28)
\]

38
for \( \tilde{C}(d, E, B) := \max\{2\|B\|_1C_1(2E, d), 2\|B\|_12d\sqrt{2}(4E + 1)^{d/2} + 2c_B\} \). Observe that \(|A| \leq n\) since the first moment vanishes, i.e., \(m_1 = 0\). Now inserting (5.28) in (5.20) yields

\[
\left( \sum_{A \in A_{2n}} D_{n, A, L}[E + i\eta; \psi_1] \right)^{1/2} \\
\leq \left( \sum_{A \in A_{2n}} \eta^{-|A|-1}\|\psi_1\|^2 \|\hat{B}_{\#}\|_{s, \infty} |C(E, d, \eta, B_{\#})|^{2n-|A|} \prod_{\alpha \in A} m_{|\alpha|} \right)^{1/2} \\
\leq \|\psi_1\| \left( \sum_{A \in A_{2n}} \eta^{-|A|-1}\|B\|_1^{A} \prod_{\alpha \in A} m_{|\alpha|} \left[ \tilde{C}(d, E, B) \left(1 + \ln(\eta^{-1} + 1)\right) \right]^{2n-|A|} \right)^{1/2} \\
\leq \|\psi_1\| \eta^{-(n+1)/2} \left(1 + \ln(\eta^{-1} + 1)\right)^n \left( \sum_{A \in A_{2n}} \|B\|_1^{A} \prod_{\alpha \in A} m_{|\alpha|} \tilde{C}(d, E, B)^{2n-|A|} \right)^{1/2}.
\]

We define

\[
K_{n, d, E, B} := \left( \sum_{A \in A_{2n}} \|B\|_1^{A} \prod_{\alpha \in A} m_{|\alpha|} \tilde{C}(d, E, B)^{2n-|A|} \right)^{1/2},
\]

where the notation does not reflect the dependence on the distribution \(P_{\nu}\) and the momenta \(m_{|\alpha|}\). Thus using (5.21) – (5.3), as well as (5.29) we find for the LHS of (2.21)

\[
\left| E \langle \psi_{1, \#}, (\mathcal{H}_{\lambda, L} - E \mp i\eta)^{-1}\psi_{2, \#} \rangle_{L^2(\Lambda_L)} - \sum_{j=0}^n T_{j, L} [E \pm i\eta; \psi_1, \psi_2] \right| \\
\leq \lambda^n |U_{n, L}[z; \psi_1, \psi_2; \lambda]| \\
\leq \lambda^n E_{n, L}[z; \psi_1]^{1/2} \eta^{-1} \|\psi_{2, \#}\|_{L^2(\Lambda_L)}^{1/2} \\
\leq \lambda^n \left( \sum_{A \in A_{2n}} D_{n, A, L}[z; \psi_1] \right)^{1/2} \eta^{-1} \|\psi_{2, \#}\|_{L^2(\Lambda_L)} \\
\leq \lambda^n \eta^{-n/2} K_{n, d, E, B} \left(1 + \ln(\eta^{-1} + 1)\right)^n |\eta|^{-3/2} \|\psi_1\| \|\psi_2\|,
\]

where we used \(\|\psi_{2, \#}\|_{L^2(\Lambda_L)} \leq \|\psi_2\|\), see Lemma 4.4 (a). This shows (a).

Part (b) follows for \(\varepsilon \in (0, 2)\) by inserting \(\eta = \lambda^{2-\varepsilon}\) into (a). Since then we see that the RHS of (2.21) tends to zero as \(\lambda \to 0\) for \(n\) sufficiently large. Specifically, since logarithms are bounded by any positive power, we see that there exists a constant \(C\) such that for \(\eta = \lambda^{2-\varepsilon}\) we have for small \(\lambda\)

\[
\left( \frac{\lambda^2}{\eta} \right)^{n/2} \left(1 + \ln(\eta^{-1} + 1)\right)^n \eta^{-3/2} \leq C \lambda^{n-(2-\varepsilon)\frac{4}{3} - n\frac{4}{3}-(2-\varepsilon)\frac{2}{3}} = C \lambda^{\varepsilon(\frac{4}{3}+\frac{2}{3})-3}.
\]

(5.30)
Now \( \lambda^{(n^2 + \frac{3}{2})^{-3}} \leq \lambda^N \) provided \( \varepsilon(n^2 + \frac{3}{2}) - 3 \geq N \), i.e., \( n \geq 4 \frac{N+3}{\varepsilon} - 6 \).

\[\Box\]

## 6 Expansion of the finite Volume Density of States

In this section we extend the proof of Theorem 2.7 to obtain an asymptotic expansion for the density of states for dimensions \( d \leq 3 \). We define the integrated density of states by

\[ \rho_{\lambda,L}(f) = \frac{1}{|\Lambda_L|} \mathbb{E}_L \text{tr}(f(H_{\lambda,L})), \quad (6.1) \]

where \( f \) is a continuous function decaying at infinity sufficiently fast, i.e., \( (6.2) \). For the definition of the trace we refer the reader to [RS72]. We do not discuss the limit as \( L \to \infty \) in this paper, but the estimates given in the main result, Theorem 6.4, will be uniform in \( L \geq 1 \). For details regarding the definition of the density of states we refer the reader to [KM07]. To ensure that \( (6.1) \) is meaningful, we have to show first that the trace in \( (6.1) \) is well defined. This will be the content of the next two Lemmas.

**Lemma 6.1.** Let \( f \) be a measurable function on \( \mathbb{R} \) such that for some constant \( C_f \) we have

\[ |f| \leq C_f \langle \cdot \rangle^{-2}. \quad (6.2) \]

Let \( V \) be the multiplication operator by a bounded real valued and measurable function on \( \Lambda_L \), and let \( d \leq 3 \). Then \( f(\frac{-h^2}{2m} \Delta_L + V) \) is trace class and we have the bound

\[ \text{tr} \left| f \left( \frac{-h^2}{2m} \Delta_L + V \right) \right| \leq C_f \left( 2\|V\|_\infty^2 + 2 \right) \text{tr} \left( \left( \frac{-h^2}{2m} \Delta_L \right)^2 + 1 \right)^{-1} \quad (6.3) \]

**Proof.** For simplicity we write \( \Delta \) for \( \Delta_L \). We find in the sense of quadratic forms

\[ \left( \frac{-h^2}{2m} \Delta \right)^2 = \left( \frac{-h^2}{2m} \Delta + V - V \right)^2 \]

\[ = \left( \frac{-h^2}{2m} \Delta + V \right)^2 - V \left( \frac{-h^2}{2m} \Delta + V \right) - \left( \frac{-h^2}{2m} \Delta + V \right) V + V^2 \]

\[ \leq 2 \left( \frac{-h^2}{2m} \Delta + V \right)^2 + 2V^2. \]

Thus, \( 1 \leq (-\frac{h^2}{2m} \Delta)^2 + 1 \leq (2\|V\|_\infty^2 + 2)((-\frac{h^2}{2m} \Delta + V)^2 + 1) \). By the spectral theorem, see e.g. [RS72], and \( (6.2) \) as well as operator monotonicity we obtain that

\[ 0 \leq |f| \left( \frac{-h^2}{2m} \Delta + V \right) \leq \frac{C_f}{(-\frac{h^2}{2m} \Delta + V)^2 + 1} \leq \frac{C_f(2\|V\|_\infty^2 + 2)}{(-\frac{h^2}{2m} \Delta)^2 + 1}. \]
Thus (6.3) follows. The trace class property now follows from (6.3), the identity $|f(\frac{-h^2}{2m}\Delta + V)|$, and the following estimate (using the ONB introduced in (2.2))

$$
\text{tr} \left( 1 + \left( \frac{-h^2}{2m}\Delta \right)^2 \right)^{-1} = \sum_{p \in \Lambda_L} \left( 1 + \left( \frac{1}{2D^2} \right)^2 \right)^{-1} < \infty,
$$

(6.4)

where we used (2.4) and observed that the last inequality follows since $d \leq 3$. \hfill \square

**Remark 6.2.** Note that it is inequality (6.4), where the assumption $d \leq 3$ is needed. Observe that for $d > 3$ the sum in (6.4) is not finite, which can be shown using that in this case $\int_{\mathbb{R}^d} \langle p \rangle^{-4} dp$ diverges. This restriction is caused by the choice of (6.2) on the decay of $f$. This choice is convenient when working with the real part of the resolvent, c.f. (6.8). We believe that dimensions $d > 3$ could also be treated if one considers faster decaying $f$ and using for example the almost analytic functional calculus to relate it to the resolvent \cite[HS89; HS87]{HS89, HS87}.

**Lemma 6.3.** Let $f$ be a measurable function on $\mathbb{R}$ such that for some constant $C_f$ we have (6.2). Then $\text{tr}(f(H_{\lambda,L}))$ is integrable with respect to the probability measure (2.12) and

$$
\mathbf{E}_L |\text{tr}(f(H_{\lambda,L}))| \leq \mathbf{E}_L \text{tr}|f(H_{\lambda,L})| < \infty.
$$

**Proof.** First observe that

$$
|\text{tr}(f(H_{\lambda,L}))| \leq \text{tr}|f(H_{\lambda,L})| \tag{6.5}
$$

is an elementary property of the trace, e.g. \cite{RS72}. Now from (6.3) we find

$$
\mathbf{E}_L \left| f \left( \frac{-h^2}{2m}\Delta_L + V \right) \right| \leq \mathbf{E}_L C_f \left( 2\|V\|_\infty^2 + 2 \right) \text{tr} \left( \left( \frac{-h^2}{2m}\Delta_L \right)^2 + 1 \right)^{-1}. \tag{6.6}
$$

Now the claim follows from (2.11), (2.12) and the fact that $M$ is a Poisson distributed random variable with mean $|\Lambda_L|$

$$
\mathbf{E}_L \|V\|_\infty^2 \leq \mathbf{E}_L \left( \sum_{\gamma=1}^M |v_\gamma|\|B_\#\|_\infty \right)^2 = \mathbf{E}_L \left( \sum_{l=0}^{\infty} \frac{|\Lambda_L|^l}{l!} \mathbf{E}_v \left( \sum_{\gamma=1}^l |v_\gamma|\|B_\#\|_\infty \right)^2 \right) = e^{-|\Lambda_L|} \sum_{l=0}^{\infty} \frac{|\Lambda_L|^l}{l!} |B_\#|^2 \left( \sum_{\gamma=1}^l |v_\gamma|\|B_\#\|_\infty \right)^2 \leq e^{-|\Lambda_L|} \sum_{l=0}^{\infty} \frac{|\Lambda_L|^l}{l!} l^2 \|B_\#\|_\infty^2 < \infty, \tag{6.7}
$$

where we used that $\mathbf{E}_v |v_\gamma v_\delta| \leq \frac{1}{2} \mathbf{E}_v (|v_\gamma|^2 + |v_\delta|^2) = m_2$ for any $\gamma, \delta \in \{1, \ldots, l\}$ and that trivially $\|B_\#\|_\infty \leq \|B\|_\infty$. Combining (6.5)–(6.7) yields the claim of the lemma. \hfill \square
To calculate the trace we shall use the ONB \((2.2)\). For any bounded trace class operator \(A\) in \(L^2(\Lambda_L)\) it follows from the definition of the trace that
\[
\frac{1}{|\Lambda_L|} \text{tr}(A) = \frac{1}{|\Lambda_L|} \sum_{p \in (\mathbb{Z}/L)^d} \langle \varphi_p, A \varphi_p \rangle = \int_{\Lambda_L^* L} \langle \varphi_p, A \varphi_p \rangle dp.
\]

For \(\chi\) a continuous function in \(L^1(\mathbb{R})\) and \(\gamma_\varepsilon(x) = \frac{\varepsilon}{\pi x^2 \varepsilon^2}\), with \(\varepsilon > 0\), we find
\[
\langle \varphi, \chi \ast \gamma_\varepsilon(H_{\lambda,L}) \varphi \rangle_{L^2(\Lambda_L)} = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi(x) \frac{1}{\pi (s-x)^2 + \varepsilon^2} \frac{1}{s-x-i\varepsilon} - \frac{1}{s-x+i\varepsilon} dx \, d\xi_\varphi(s) = (2\pi i)^{-1} \int_{\mathbb{R}} \chi(x) \left\langle \varphi, (H_{\lambda,L} - x - i\varepsilon)^{-1} \varphi \right\rangle - \left\langle \varphi, (H_{\lambda,L} - x + i\varepsilon)^{-1} \varphi \right\rangle_{L^2(\Lambda_L)} dx,
\]
where in the first equality we used the spectral theorem \([RS72]\), with \(\xi_\varphi\) denoting the spectral measure of \(\varphi\) with respect to the self-adjoint operator \(H_{\lambda,L}\), and in the third equality we used Fubini’s theorem.

Using \((6.8)\) we shall relate the density of states to expectation values of resolvents. Thus, results about resolvents can be used to obtain results about the density of states, which is in fact done in the proof of the following theorem. For this recall the definition of the functions \(T_{n,L}\) in \((2.16)\).

**Theorem 6.4.** Let \(d \leq 3\). Then the following holds.

(a) The map \(q \mapsto \frac{1}{\pi} \{T_{n,L}[E + i\eta; \varphi_p, \varphi_p] - T_{n,L}[E - i\eta; \varphi_p, \varphi_p]\}\) is in \(\ell^1(\Lambda_L^*)\) for \(\eta \neq 0\) and thus the sum
\[
D_{n,L}[E, \eta] := \int_{\Lambda_L^* L} \frac{1}{2\pi} \{T_{n,L}[E + i\eta; \varphi_p, \varphi_p] - T_{n,L}[E - i\eta; \varphi_p, \varphi_p]\} dp.
\]
exists and is uniformly bounded in \(L \geq 1\).

(b) Suppose the first moment vanishes, i.e., \(m_1 := \mathbb{E} v_\gamma = 0\). Then there exists an \(L_0 \geq 1\) such that the following holds. Let \(\chi \in C(\mathbb{R})\) with \(\text{supp} \chi \subset [0, \infty)\). Then for any \(\varepsilon \in (0, 2)\) and \(N > 0\) we have uniformly in \(L \geq L_0\)
\[
\rho_{\lambda,L}(\chi \ast \gamma_\eta) = \sum_{n=0}^{\lceil (24+4N)/\varepsilon \rceil} \lambda^n \frac{1}{\pi} \int \chi(E) D_{n,L}[E, \eta] dE + O(\lambda^N) \quad (\lambda \downarrow 0)
\]
with \(\eta = \lambda^{2-\varepsilon}\).
Remark 6.5. Regarding the choice of $L_0$ in Theorem 6.4 (b) we are in the same situation as in Theorem 2.7. In case the profile function $B$ satisfies the symmetry condition, (b) holds for any $L_0 \geq 1$ such that Lemma C.6 is valid. In case $B$ has compact support, one chooses $L_0 \geq 1$ such that $\text{supp } B \subset (-L_0/2, L_0/2)^d$ and Lemma C.6 is valid.

Remark 6.6. We note that the convolution with $\gamma_\eta$ in the density of states can be viewed as an averaging the density of states over an interval of size $\eta$. Thus, a Wegner type estimate gives usually a bound on the integrated density of states in the form $1/\lambda$. An average does not give an obvious improvement. Thus, in the small coupling regime the estimate (6.10) does not only give a precise description of the density of states in terms of an expansion it also gives a better bound of the density of states than a typical Wegner estimate does.

Remark 6.7. We note that the estimates in Theorem 6.4 are uniform in $L \geq 1$ and thus it can be used to obtain properties about the infinite volume limit of the density of states. We plan to address the technical details of this limit in a forthcoming paper. For the model which we consider the infinite volume Hamiltonian will become unbounded from below, and some estimates will be necessary to establish the limit.

Remark 6.8. As for Theorem 2.7 it would be interesting to investigate to what extent the expansion (6.10) in Theorem could be improved. That is whether one can prove an asymptotic expansion for $\eta = \lambda^\alpha$ for $\alpha$ smaller than 2, using so called tadpole renormalization and an analysis of crossing graphs, cf. [ESY08].

Proof of Theorem 6.4. (a) Let us first consider $n = 0$. This term consists essentially of the matrix element of the free resolvent. From the definition given in (2.16) we find

$$T_{0,L}[E + i\eta; \varphi_p, \varphi_p] = \frac{1}{\nu(p) - E - i\eta}. \quad (6.11)$$

Thus taking the imaginary part of (6.11) and using Lemma C.6 we find

$$\int_{\Lambda^L} \frac{1}{2i} \{T_{0,L}[E + i\eta; \varphi_p, \varphi_p] - T_{0,L}[E - i\eta; \varphi_p, \varphi_p]\} dp = \int_{\Lambda^L} \frac{\eta}{(\nu(p) - E)^2 + \eta^2} dp \leq \eta C_I(1 + \eta^{-2}) \quad (6.12)$$

for some constant $C_I$ uniformly in $L \geq 1$. Now consider $n \geq 1$. We treat each summand $C_{n,A,L}[\cdots]$ of $T_{n,L}[\cdots]$ in (4.18) individually. We use identity in (4.26) to express $C_{n,A,L}[\cdots]$ and insert $\hat{\varphi}_q(u) = |\Lambda_L|^{1/2}1_{(q)}(u)$. Then, we find for $z \in \mathbb{C} \setminus [0, \infty)$ and $A \in \mathcal{A}_n$, using the trivial bound (C.1) for all resolvents except the first two, $\sum_{i=1}^{1}[M_A v_i]_t = v_1$ (since 1 can never be a maximum of a subset of $\{1, ..., n\}$ containing at least two elements), and
summing over all \( v_i \) variables except \( v_1 \) that

\[
\int_{\Lambda_L} |C_{n,L}^A[z; \varphi_q, \varphi_q]| \, dq
\]

\[
= \int_{\Lambda_L} dq \left| \prod_{l \in I_A} \left( \int_{\Lambda_L} dv_l \right) \prod_{j=1}^{n} \left( \frac{\nu \left( q + \sum_{l=1}^{j-1} [M_A v_l] \right) - z}{\nu(q) - z} \right)^{-1} \right|
\]

\[
\leq |\text{Im}z|^{-n+1} \| \hat{B}_\# \|_{*, \infty}^{n-[I_A]} \| \hat{B}_\# \|_{*, 1}^{[I_A] - 1} \int_{\Lambda_L} \nu(q) - z \mid_{v_1 = \frac{1}{2}} \left( |\nu(q) - z|^{-2} + |\nu(q + v_1) - z|^{-2} \right) \, dv_1 \, dq
\]

\[
= |\text{Im}z|^{-n+1} \| \hat{B}_\# \|_{*, \infty}^{n-[I_A]} \| \hat{B}_\# \|_{*, 1}^{[I_A]} \int_{\Lambda_L} |\nu(q) - z|^{-2} \, dq
\]

\[
\leq |\text{Im}z|^{-n} \| \hat{B}_\# \|_{*, \infty}^{n-[I_A]} \| \hat{B}_\# \|_{*, 1}^{[I_A]} C_I (1 + (\text{Im}z)^{-2}),
\]

where in the fifth line we used the squaring inequality, in the second to last line a change of summation variables, and in the last line Lemma [C.6](with \( \tau = 0 \)). Note that \( \| \hat{B}_\# \|_{*, 1} \) as well as \( \| \hat{B}_\# \|_{*, \infty} \) are both dependent on \( L \) indicated by the indexed \( \# \). Now to find \( L \)-independent bounds, we use Lemma [A.1](which gives us an \( L \)-independent bound on \( \| \hat{B}_\# \|_{*, 1} \). By Lemma [A.4](b) we have \( \| \hat{B}_\# \|_{*, \infty} \leq \| B \|_1 \), which is bounded \( L \)-independently as well. Therefore, the expression in (6.13) is bounded uniformly in \( L \geq 1 \), which shows (a).

(b) For notational simplicity we shall suppress the subscript \( L^2(\Lambda_L) \). First we observe that there exists by the compactness of the support of \( \chi \) a constant \( C \) such that \( |\chi \ast \gamma_n| \leq C \gamma_n^{-2} \). To see this in detail, note that for \( K > 0 \) and \( \delta > 0 \), it follows that for all \( y \in [-K, K] \) and \( x \in \mathbb{R} \)

\[
(x-y)^2 + \delta = x^2 - 2xy + y^2 + \delta \geq (1-\kappa)x^2 + (1-\kappa^{-1})y^2 + \delta \geq (1-\kappa)x^2 + \frac{\delta}{2} = \frac{\delta}{2K^2 + \delta} x^2 + \frac{\delta}{2},
\]

where we used the squaring inequality and inserted \( \kappa := \frac{K^2}{K^2 + \delta} \in (0, 1) \). Thus \( \rho_{\lambda,L}(\chi \ast \gamma_n) \) is well defined by Lemma [6.3](Using Fubini, which is justified again by Lemma [6.3] and using (6.8) we find
where in the last line we used again Fubini’s theorem, which is justified from the elementary estimate on the resolvent by \( \eta^{-1} \) and the fact that \( \chi \) is a continuous function with compact support. Now we use (5.1), i.e.,

\[
\rho_{\lambda,L}(\chi \ast \gamma_{\eta}) (6.16)
\]

\[
= \lambda^{n} \int_{\Lambda^{*}_{L}} (2\pi i)^{-1} \int_{\mathbb{R}} \chi(x) \left( (\lambda^{n} U_{n,L}[x + i\eta; \phi_{q}, \phi_{q}] - \lambda^{n} U_{n,L}[x - i\eta; \phi_{q}, \phi_{q}]) \right) dx dq.
\]

It remains to estimate the term in last line of (6.16). For this we proceed as follows. To control the trace we will use the following bound, where \( w(p) \) is a positive weight function.
on $\mathbb{R}^d$ to be determined later
\[
|U_{n,L}[z; \varphi_p, \varphi_p; \lambda]| = \mathbb{E}_L\langle ([R_L(z)V_L])^n \varphi_p, (H_L - z)^{-1} \varphi_p \rangle \\
\leq \mathbb{E}_L\|[([R_L(z)V_L])^n \varphi_p, \#\|(H_L - z)^{-1} \varphi_p, \#\| \\
\leq \frac{1}{2} \mathbb{E}_L (w(p)\|[([R_L(z)V_L])^n \varphi_p, \#\|^2 + w(p)^{-1}\|(H_L - z)^{-1} \varphi_p, \#\|^2) \\
\leq \frac{1}{2} E_{n,L}[z; \varphi_p]w(p) + \frac{1}{2}|\text{Im}z|^{-2}w(p)^{-1}\|\varphi_p, \#\|^2, \tag{6.17}
\]
where we used the squaring inequality and the basic inequality (C.1). Here, we recall the definition given in (5.5) which reads
\[
E_{n,L}[z; \psi_1] = \mathbb{E}_L\|[([R_L(z)V_L])^n \psi_1, \#\| \\
= \mathbb{E}_L\langle \psi_1, \#, ([R_L(z)V_L])^n [V_L R_L(\overline{z})]n \psi_1, \#\rangle. \tag{6.18}
\]
For the second term in (6.17) to be summable with respect to $\int_{\Lambda^*_L}(\cdots)dp$ we will choose
\[
w(p) = |\lambda|^{n(1-\alpha)c_w} \prod_{j=1}^d \langle p_j \rangle^{1+\delta} \tag{6.19}
\]
for some $c_w > 0$, $\alpha > 0$, and $\delta > 0$ to be determined later. Recalling the definitions and relations given in Eqs. (5.10) and (5.11) we obtain the estimate
\[
\int_{\Lambda^*_L} E_{n,L}[x \pm i\eta; \varphi_q]w(p)dp \leq \sum_{A \in A_{2n}} \int_{\Lambda^*_L} |D_{n,A,L}[x \pm i\eta; \varphi_q]|w(p)dp. \tag{6.20}
\]
In particular, it suffices to investigate the following expressions. Let $A \in A_{2n}$. Using (5.17) we find
\[
D_{n,A,L}[E \pm i\eta; \varphi_q] \tag{6.21}
\]
\[
= \int_{\Lambda^*_L} du_0 \prod_{l \in I_A} \left( \int_{\Lambda^*_L} du_l \prod_{a \in A} m_{|a|} \prod_{l \in a} \hat{B}_\#([M_A(u)]_l) \right) \bar{\varphi}_q(u_0)\varphi_q(u_0) \\
\times \prod_{j=1}^n \left( \nu \left( u_0 + \sum_{l=1}^{j-1} [M_A(u)]_l \right) - E \mp i\eta \right)^{-1} \prod_{j=n+2}^{2n+1} \left( \nu \left( u_0 + \sum_{l=1}^{j-1} [M_A(u)]_l \right) - E \mp i\eta \right)^{-1}.
\]
Inserting $\varphi_q(u_0) = |\Lambda_L|^{1/2}1_q(u_0)$, therefore summing over $u_0$ has the effect of replacing the $u_0$ in the resolvents with $q$ and yields
\[
D_{n,A,L}[E \pm i\eta; \varphi_q] \tag{6.22}
\]
\[
= \prod_{l \in I_A} \left( \int_{\Lambda^*_L} du_l \prod_{a \in A} m_{|a|} \prod_{l \in a} \hat{B}_\#([M_A(u)]_l) \right) \\
\times \prod_{j=1}^n \left( \nu \left( q + \sum_{l=1}^{j-1} [M_A(u)]_l \right) - E \mp i\eta \right)^{-1} \prod_{j=n+2}^{2n+1} \left( \nu \left( q + \sum_{l=1}^{j-1} [M_A(u)]_l \right) - E \mp i\eta \right)^{-1}.
\]
To estimate this expression we proceed similarly as in Section 5 devoted to the proof of Theorem 7.1 except that we keep the first three resolvents as well as the last resolvent to obtain the necessary decay in \(q\) to show the tracial property. In view of Lemma 5.1 we know that the expression \(u_0 + \sum_{j=1}^{n-1} [M_A(u)]_t\) in the \(j\)-th resolvents for \(j = 1, \ldots, 2n + 1\) with \(j - 1 \in I_A\) are as functions of the coordinates linearly independent. The remaining resolvents except the first and last one, i.e., \(j = 2, \ldots, 2n\) for \(j - 1 \in J_A\) we estimate by \(\eta^{-1}\) using (C.1), assuming \(0 < \eta \leq 1\). Recall that this may include the case where \(j = n + 1\) and \(j - 1 \in J_A\), that is where we have an identity instead of a resolvent, we estimate trivially by \(\eta^{-1}\). Furthermore, for \(l \in J_A\) we estimate \(|\hat{B}_#(-[M_A(u)]_t)| \leq \|\hat{B}_#\|_{*\infty}\) for \(u \in \Lambda^*_L\). We arrive at

\[
|D_{n,A,L}[E \pm i\eta; \varphi_q]| \leq \eta^{-|A|+1} \|\hat{B}_#\|_{*\infty} \prod_{a \in A} \prod_{l \in I_A} \left( \int_{\Lambda^*_L} du_l \right) \prod_{l=1}^{2n} \hat{B}_#(-u_l) \times \prod_{j=1}^{2n+1} \prod_{l=1}^{j-1} \left| \nu \left( q + \sum_{l=1}^{j-1} [M_A(u)]_t - E \mp i\eta \right) \right|^{-1} \left| \nu \left( q + \sum_{l=1}^{j-1} [M_A(u)]_t - E \mp i\eta \right) \right|^{-1},
\]

where we note that compared to (5.18) the exponent of \(\eta^{-1}\) is smaller by two, since we keep the first and last resolvent. Similarly to the proof of Lemma 5.2 in (6.23) we sum over \(u_j, j \in I_A\) in decreasing order by means of Lemma 5.1 and use Proposition C.5 to estimate the product of resolvent and \(\hat{B}_#(u_j)\). We proceed until there are four resolvents left. This can always be achieved if we choose \(n\) sufficiently large, as we now argue. By assumption \(m_1 = 0\) and so each element of \(A\) has at least two elements. Thus, \(|J_A| \leq n\) and so \(|I_A| \geq n\). For \(n \geq 2\), let \(I_A^{(2)}\) denote the set of the first two elements in \(I_A\). In fact, since each element of \(A\) has at least two elements, it is straight forward to check from the definition of \(I_A\) that for \(n \geq 4\) we have \(I_A^{(2)} \subset \{1, \ldots, n\}\). Compared to the procedure in the proof of Lemma 5.2 we keep two variables, hence we obtain in the exponent of \(C(E, d, L, \eta, \hat{B}_#)\) a number which is by two smaller, i.e., we get \(C(E, d, L, \eta, \hat{B}_#)^{2n - |A| - 2}\). Using Lemma C.3 the procedure outlined above yields the following inequality for some \(\sigma \in \{0, 1\}\) (depending on \(M_A\) and a possible relabeling of the coordinates). Note that the product of \((\nu(q) - E + i\eta)^{-1}\) with \((\nu(q) - E - i\eta)^{-1}\) (the first and the last resolvent) produces the factor \((\nu(q) - E)^2 + \eta^2)^{-1}\). Moreover, we use \(C(\cdots)\) given by (C.19) and the constant \(c_B\) (given by the right hand side of (4.16)). Furthermore, we use (D.1) for
the constant $K_{d,\delta,E}$ of Lemma [D.1.1] for some $\delta \in (0, 1]$. 

$$|D_{n,A,L}[E \pm i\eta; \varphi_q]| \leq \eta^{-|A|+1}\|\hat{B}_\#\|_{*,\infty}^{|A|} \prod_{a \in A} m_{[a]} \int_{(\Lambda^*_n)^2} |\hat{B}_\#(-u_1)||\hat{B}_\#(-u_2)|((\nu(q) - E)^2 + \eta^2)^{-1}$$

$$\times |\nu(q + u_1) - E \pm i\eta|^{-1} |\nu(q + \sigma u_1 + u_2) - E \pm i\eta|^{-1} d(u_1, u_2)C(E, d, L, \eta, \hat{B}_\#)^{2n-|A|-2}$$

$$\leq \eta^{-|A|+1}\|\hat{B}_\#\|_{*,\infty}^{|A|} \prod_{a \in A} m_{[a]} \int_{(\Lambda^*_n)^2} \prod_{j=1}^d \langle u_{1,j} \rangle^{-\delta} c_B \prod_{j=1}^d \langle u_{2,j} \rangle^{-\delta} c_B ((\nu(q) - E)^2 + \eta^2)^{-1}$$

$$\times |\nu(q + u_1) - E \pm i\eta|^{-1} |\nu(q + \sigma u_1 + u_2) - E \pm i\eta|^{-1} d(u_1, u_2)C(E, d, L, \eta, \hat{B}_\#)^{2n-|A|-2}$$

$$\leq \eta^{-|A|+1}\|\hat{B}_\#\|_{*,\infty}^{|A|} \prod_{a \in A} m_{[a]} c_B^2$$

$$\times K_{d,\delta,E}(1 + \eta^{-2}) \prod_{j=1}^d \langle q_j \rangle^{-1+\delta}((\nu(q) - E)^2 + \eta^2)^{-1} C(E, d, L, \eta, \hat{B}_\#)^{2n-|A|-2}.$$ 

Now we will use the summability with respect to $q$ provided $\delta > 0$ is sufficiently small. That is, we find from (6.24) with (6.19) that

$$\int_{\Lambda^*_n} |D_{n,A,L}[E \pm i\eta; e_q]| w(q) dq \leq \eta^{-|A|+1}\|\hat{B}_\#\|_{*,\infty}^{|A|} \prod_{a \in A} m_{[a]} c_B^2 K_{d,\delta,E}(1 + \eta^{-2}) C(E, d, L, \eta, \hat{B}_\#)^{2n-|A|-2}$$

$$\times \int_{\Lambda^*_n} \prod_{j=1}^d \langle q_j \rangle^{-1+\delta}((\nu(q) - E)^2 + \eta^2)^{-1} w(q) dq$$

Now we see that using Lemma [C.6] that for $0 < \delta < 1$ there exists a constant $C_{E,\delta}$ such
Thus, (6.16) gives, inserting (6.17), (6.20), (6.27), and (6.19)

\[ \int_{\Lambda_L^*}^d \prod_{j=1}^d (q_j)^{-1+\delta}((\nu(q) - E)^2 + \eta^2)^{-1} w(q) dq \]

\[ = |\lambda|^n(1-\alpha) \int_{\Lambda_L^*} c_w \prod_{j=1}^d (q_j)^{2\delta}((\nu(q) - E)^2 + \eta^2)^{-1} dq \]

\[ \leq c_w C_{E,\delta}(1 + \eta^{-2})|\lambda|^{n-\alpha}. \quad (6.26) \]

Inserting (6.26) into (6.25) yields

\[ \int_{\Lambda_L^*} |D_{n,A,L}[E \pm i\eta; e_q]| w(p) dq \]

\[ \leq |\lambda|^{n-\alpha} \eta^{-|A|+1} \|\hat{B}_\#\|_{*\infty} \prod_{\alpha \in A} m|\alpha| \left| c_w c_B^2 K_{d,\delta,E} C_{E,\delta,\hat{B}_\#} (1 + \eta^{-2})^2 C(E, d, L, \eta, \hat{B}_\#)^{2n-|A|-2} \right| \]

Thus, (6.16) gives, inserting (6.17), (6.20), (6.27), and (6.19)

\[ \frac{\rho_{\lambda,L}(\chi \ast \chi_\eta) - \sum_{j=0}^{n-1} \lambda^j \pi^{-1} \int_{\mathbb{R}} \chi(x) D_{j,L}[x,\eta] dx}{|\lambda|^n} \]

\[ \leq |\lambda|^n \int_{\Lambda_L^*}^d (2\pi)^{-1} \int_{\mathbb{R}} \chi(x) \sum_{\sigma \in \{-1,1\}} |U_{n,L}[x + \sigma i\eta; \varphi_q, \varphi_q; \lambda]| dx dq \]

\[ \leq |\lambda|^n \int_{\Lambda_L^*}^d (2\pi)^{-1} \int_{\mathbb{R}} \chi(x) \sum_{\sigma \in \{-1,1\}} \left( \frac{1}{2} E_{n,L}[x + \sigma i\eta; \varphi_q] w(q) + \frac{1}{2} \eta^{-2} w(q)^{-1} \right) dx dq \]

\[ \leq |\lambda|^n \int_{\Lambda_L^*}^d (2\pi)^{-1} \int_{\mathbb{R}} \chi(x) \sum_{\sigma \in \{-1,1\}} \left( \frac{1}{2} \sum_{A \in A_2n} |D_{n,A,L}[x + i\sigma \eta; \varphi_q]| w(q) + \frac{1}{2} \eta^{-2} w(q)^{-1} \right) dx dq \]

\[ \leq \|\chi\| (2\pi)^{-1} |\lambda|^{2n-\alpha} \sum_{A \in A_2n} \eta^{-|A|+1} (1 + \eta^{-2})^2 \|\hat{B}_\#\|_{*\infty} \prod_{\alpha \in A} m|\alpha| \]

\[ \times c_w c_B^2 \sup_{E \in \text{supp}\chi} K_{d,\delta,E} C_{E,\delta,\hat{B}_\#}(1 + \eta^{-2})^2 C(E, d, L, \eta, \hat{B}_\#)^{2n-|A|-2} \]

\[ + \|\chi\| (2\pi)^{-1} \eta^{-2} c_w^{-1} |\lambda|^{n\alpha} \int_{\Lambda_L^*}^d \prod_{j=1}^d (q_j)^{-1-\delta} dq. \quad (6.29) \]

We recall that \( C(E, d, L, \eta, \hat{B}_\#) \) is the constant (C.19) given in Proposition C.5. To estimate the \( \|\hat{B}_\#\|_{*\infty} \) term appearing in \( C(\cdots) \), we use that by Lemma 4.4 (b) we have

\[ \|\hat{B}_\#\|_{*\infty} \leq \|B\|_1 < \infty. \]
To show that the $\|\hat{B}_#\|_{*,1}$ term appearing in $C(\cdots)$ is uniformly bounded for large $L$ we use (4.17). Since (6.28) holds for $\eta \neq 0$ and $\alpha > 0$ choosing $\eta = \lambda^{2-\varepsilon}$ as well as $\alpha = \varepsilon/2$ gives us

\[
\rho_{\lambda,L}(\chi \ast \gamma_\eta) - \sum_{j=0}^{n-1} \lambda^j \pi^{-1} \int_\mathbb{R} \chi(x) D_{j,L} [x, \eta] dx \leq \|\chi\|_1 (2\pi)^{-1} \lambda^{2n-n-3} \sum_{A \in A_{2n}} \lambda^{(-|A|+1)(2-\varepsilon)} (1 + \lambda^{-4+2\varepsilon})^2 \|\hat{B}_#\|_{*,\infty} \prod_{a \in A} m_{\{a\}} \times c_w c_B^2 \sup_{E \in \text{supp } \chi} K_{d,\delta,E} C(E, d, L, \eta, \hat{B}_#)^{2n-|A|-2} 
\]

\[
+ \|\chi\|_1 (2\pi)^{-1} \lambda^{2n-4+2\varepsilon} c_w^{-1} \int_{\Lambda_{L}} \prod_{j=1}^{d} \langle q_j \rangle^{-1-\delta} dq 
\]

\[
= \|\chi\|_1 (2\pi)^{-1} \sum_{A \in A_{2n}} \lambda^{(2-\varepsilon)(n-|A|)+n-4+2\varepsilon} (1 + 2\lambda^{-4+2\varepsilon} + \lambda^{4-2\varepsilon}) \|\hat{B}_#\|_{*,\infty} \prod_{a \in A} m_{\{a\}} \times c_w c_B^2 \sup_{E \in \text{supp } \chi} K_{d,\delta,E} C(E, d, L, \eta, \hat{B}_#)^{2n-|A|-2} 
\]

\[
+ \|\chi\|_1 (2\pi)^{-1} \lambda^{2n-4+2\varepsilon} c_w^{-1} \int_{\Lambda_{L}} \prod_{j=1}^{d} \langle q_j \rangle^{-1-\delta} dq. \quad (6.30)
\]

Finally, observe that $|A| \leq n$ since the first moment vanishes, i.e., $m_1 = 0$. Moreover, $\varepsilon \in (0, 2)$. Thus, the estimate in (b) now follows since the right hand side of (6.30) tends to zero as $\lambda \to 0$ for $n$ sufficiently large.

Specifically, since logarithms are bounded by any positive power, we see that there exits a constant $C$ such that we have for small $\lambda$ (recall $\eta = \lambda^{2-\varepsilon}$)

\[
(6.30) \leq C \left( \lambda^{n\varepsilon^{-2}+2-\varepsilon-8+4\varepsilon-n\varepsilon} + \lambda^{n-4+2\varepsilon} \right) \leq 2C\lambda^N, \quad (6.31)
\]

where the last inequality holds provided

\[
n \frac{\varepsilon}{2} + 2 - \varepsilon - 8 + 4\varepsilon - n \frac{\varepsilon}{4} \geq N,
\]

i.e., $n \geq 4\frac{N+6}{\varepsilon} - 12$.

\[\square\]

7 Acknowledgements

D.H. wants to thank Laszlo Erdős for valuable discussions about Feynman graphs. We thank Robert Hesse, Benjamin Hinrichs and Oliver Siebert for helpful comments.
A Resolvent expansion

In this chapter we want to display a very basic concept - the resolvent expansion - which we apply in Section 2. We express the perturbed resolvent as a sum of products of unperturbed resolvents and potential factors, which is the content of Lemma A.3. To proof this expansion we begin with the following underlying resolvent identity.

Lemma A.1 (Second resolvent identity). Let \( T, S \) be linear operators on the normed space \( E \) with \( D(S) = D(T) \). Let \( z \in \rho(T) \cap \rho(S) \). Then we have

\[
\frac{1}{T - z} - \frac{1}{S - z} = \frac{1}{T - z} (S - T) \frac{1}{S - z}.
\]

The following lemma is an immediate consequence.

Lemma A.2. Let \( A \) and \( B \) be linear operators on a normed space \( E \) with \( D(A) = D(A + B) \) and \( 0 \in \rho(A) \cap \rho(A + B) \). Then one has

\[
\frac{1}{A + B} = \frac{1}{A} + \frac{1}{A + B} (-B) \frac{1}{A}.
\]

Proof. By the second resolvent identity used for \( T = A + B, S = A \) and \( z = 0 \), we have

\[
\frac{1}{A + B} - \frac{1}{A} = \frac{1}{A + B} (A - (A + B)) \frac{1}{A} = \frac{1}{A + B} (-B) \frac{1}{A}.
\]

Now Lemma A.1 yields the statement.

We can now perform the resolvent expansion using an iteration of the formula of the previous lemma.

Lemma A.3. Let \( A \) and \( B \) be linear operators on a normed space \( E \) with \( D(A) = D(A + B) \) and \( 0 \in \rho(A) \cap \rho(A + B) \). Then for \( m \in \mathbb{N} \) we have

\[
\frac{1}{A + B} = \sum_{n=0}^{m-1} \frac{1}{A} \left( -\frac{B}{A} \right)^n + \frac{1}{A + B} \left( -\frac{B}{A} \right)^m.
\]

Proof. We will prove the statement by induction. For \( m = 0 \) the sum is 0 and only

\[
\left[ \frac{1}{A} (-B) \right]^0 \frac{1}{A + B} = \frac{1}{A + B}
\]

remains.

Now let

\[
\frac{1}{A + B} = \sum_{n=0}^{m-1} \frac{1}{A} \left( -\frac{B}{A} \right)^n + \frac{1}{A + B} \left( -\frac{B}{A} \right)^m
\]
hold for an $m \in \mathbb{N}$. Using Lemma A.2 we have
\begin{align*}
\frac{1}{A+B} &= \sum_{n=0}^{m-1} \frac{1}{A} \left[ (-B) \frac{1}{A} \right]^n + \left( \frac{1}{A} + \frac{1}{A+B} (-B) \frac{1}{A} \right) \left[ (-B) \frac{1}{A} \right]^m.
= \sum_{n=0}^{m-1} \frac{1}{A} \left[ (-B) \frac{1}{A} \right]^n + \frac{1}{A+B} \left[ (-B) \frac{1}{A} \right]^m + \frac{1}{A+B} \left[ (-B) \frac{1}{A} \right]^{m+1},
= \sum_{n=0}^{m} \frac{1}{A} \left[ (-B) \frac{1}{A} \right]^n + \frac{1}{A+B} \left[ (-B) \frac{1}{A} \right]^{m+1}.
\end{align*}

\[ \square \]

B \quad \text{Examples of dilation analytic functions}

In this appendix we proof Lemma B.1 to give a class of functions which satisfy Hypotheses $A$ and $B$ respectively. For this we recall the notation $\langle x \rangle = (1 + x^2)^{1/2}$.

**Lemma B.1.** Let $x_0, a \in \mathbb{R}^d$, $\sigma > 0$, $P$ be a polynomial on $\mathbb{R}^d$, and
\[ f(x) = P(x) \exp(-\pi \sigma |x - x_0|^2 + 2\pi i x \cdot a) \]
for $x \in \mathbb{R}^d$. Then $\theta \mapsto \hat{f}_\theta := (u(\theta)f)^\wedge$ - where $u(\theta)$ was defined in (2.22) - has an analytic extension from $\mathbb{R}$ to the strip $\{ z \in \mathbb{C} : |\text{Im} z| < \pi/4 \}$ as an $L^p(\mathbb{R}^d)$-valued function for all $p \in [1, \infty]$.

**Proof.** First observe that for the Gaussian on $\mathbb{R}^d$ given by $g_\sigma(x) = \exp(-\pi \sigma |x|^2)$ where $x \in \mathbb{R}^d$, we have for $k \in \mathbb{R}^d$ that $\hat{g}_\sigma(k) = \sigma^{-d/2} \exp(-\pi |k|^2/\sigma)$ (see for example [LL01, Theorem 5.2]). Thus, using that the Fourier transform is bijective, that it turns multiplication operators into differential operators, and translations into multiplication by a free wave function and vice versa [LL01, Fol13], we find $\hat{f}(k) = Q_\sigma(k - a) e^{-\frac{\pi}{4} |k-a|^2} e^{-2\pi i (k-a) \cdot x_0}$ for some polynomial $Q_\sigma$. Hence multiplying out the expressions involving $k - a$ we find that there exists a polynomial $\tilde{Q}_{\sigma, a}$ and a complex vector $u \in \mathbb{C}^d$ depending on $a$ and $x_0$ such that
\[ \hat{f}(k) = \tilde{Q}_{\sigma, a}(k)e^{-\frac{\pi}{4}|k|^2} e^{-k \cdot u}. \]
From (2.23) and the definition we see that
\[ \hat{f}_\theta(k) = e^{-d\theta/2} \hat{f}(e^{-\theta} k) = e^{-d\theta/2} \tilde{Q}_{\sigma, a}(e^{-\theta} k) e^{-\frac{\pi}{4} |k|^2} e^{-e^{-\theta} k \cdot u}. \]  

(\text{B.1})

We have to show that $\theta \mapsto \hat{f}(e^{\theta} \cdot)$ has an analytic $L^p(\mathbb{R}^d)$-extension for $p \in [1, \infty]$.

So, let $p \in [1, \infty]$. Observe that (B.1) is for each fixed $k \in \mathbb{R}^d$ analytic in $\theta$ and the pointwise analytic extension satisfies the following bound. For $r > 0$ and $w \in \mathbb{C}$ let $D_r(w) = \{ z \in \mathbb{C} : |z - w| < r \}$. Fix $\theta_0 \in (0, \pi/4)$ and $w \in \mathbb{R}$ and let $D = D_{\theta_0}(w)$. Since $\theta_0 < \pi/4$ it follows that
\[ \delta := \inf_{\theta \in D} \text{Re}(e^{-2\theta}) > 0 \]

(\text{B.2)}
holds. We write \((B.1)\) as a product

\[
\hat{f}_\theta(k) = e^{-d\theta/2} \left( \tilde{Q}_{\sigma,a}(e^{-\theta}k) e^{-\frac{\pi}{2} \frac{\delta}{k^2}} \right) \left( e^{-\frac{\pi}{2} e^{-2\theta} |k|^2} e^{\frac{\pi}{2} \frac{\delta}{|k|^2}} \right) \left( e^{-e^{-\theta}k \cdot u} e^{-\frac{\pi}{2} \frac{\delta}{|k|^2}} \right). \tag{B.3}
\]

To estimate the middle factor in brackets we observe that by the choice of \(\delta\)
\[
-\text{Re } e^{-2\theta} |k|^2 + \frac{\delta}{2} |k|^2 \leq -\frac{\delta}{2} |k|^2 \quad \text{for all } k \in \mathbb{R}^d, \theta \in D. \quad \text{(B.4)}
\]
Using \((B.3)\), we see from \((B.4)\) and the Gaussians in the first and third factor that there exists a constant \(C\) such that for all \(\theta \in D, n = 0, 1\) and \(k \in \mathbb{R}^d\) we have
\[
|\partial^n_{\theta} \hat{f}_\theta(k)| \leq C \exp \left( -\frac{\pi}{\sigma} \frac{\delta}{2} |k|^2 \right). \tag{B.5}
\]

Now let \(q \in [1, \infty)\) with \(\frac{1}{p} + \frac{1}{q} = 1\). From \((B.5)\) and dominated convergence we see that for any \(h \in L^q(\mathbb{R}^d)\) the function \(\theta \mapsto \int h(k) \hat{f}_\theta(k) dk\) is analytic (explicitly use for example [Fol13, Theorem 2.27] and the Cauchy-Riemann equations). For \(p \in [1, \infty)\) we have the duality relation \((L^p(\mathbb{R}^d))^* \cong L^q(\mathbb{R}^d)\), cf. [AF03, Theorem 2.45]. Thus for \(p \in [1, \infty)\), it follows that \(\theta \mapsto \hat{f}_\theta\) is weakly analytic in \(L^p(\mathbb{R}^d)\), and therefore, it is in fact strongly analytic in \(L^p(\mathbb{R}^d)\) by the well known fact that weak analyticity in Banach spaces implies strong analyticity, cf. [RS72, Theorem VI.4]. Hence we have shown the claim for \(p \in [1, \infty)\).

To prove analyticity in \(L^\infty(\mathbb{R}^d)\), we need a different argument, since the dual of \(L^\infty(\mathbb{R}^n)\) is larger than \(L^1(\mathbb{R}^d)\). First we show that \(\hat{f}_\theta\) is a continuous \(L^\infty(\mathbb{R}^n)\)-valued function of \(\theta \in D\) by showing this for each factor in \((B.3)\). For \(\theta \in D\) and \(w = e^{-\theta}\) we see from the mean value theorem that there exists a constant \(C\) and an \(m \in \mathbb{N}_0\) such that for all \(k \in \mathbb{R}^d\) and \(h \in \mathbb{C}\)
\[
\left| \tilde{Q}_{\sigma,a}(w+h)k) - \tilde{Q}_{\sigma,a}(wk) \right| e^{-\frac{\pi}{2} \frac{\delta}{|k|^2}} \leq |h| |C(k)|^m e^{-\frac{\pi}{2} \frac{\delta}{|k|^2}}, \tag{B.6}
\]
\[
\left| e^{-(w+h)k \cdot u} - e^{-wk \cdot u} \right| e^{-\frac{\pi}{2} \frac{\delta}{|k|^2}} \leq |h| |C(k)| \sup_{|z| \leq |h|} |e^{-(w+z)k \cdot u}| e^{-\frac{\pi}{2} \frac{\delta}{|k|^2}}, \tag{B.7}
\]
\[
\left| e^{-\frac{2}{\pi}(w+h)^2|k|^2} - e^{-\frac{2}{\pi}w^2|k|^2} \right| \left| e^{\frac{\pi}{2} \frac{\delta}{|k|^2}} \right| \leq |h| |C(k)|^2 \sup_{|z| \leq |h|} \left| (w + z)e^{-\frac{2}{\pi}(w+z)^2|k|^2} \right| e^{-\frac{\pi}{2} \frac{\delta}{|k|^2}}, \tag{B.8}
\]
where the last inequality in \((B.8)\) holds for some constant \(\tilde{C}\) provided \(|h|\) is sufficiently small, since by \((B.2)\) and continuity there exists an \(r > 0\) such that for all \(\theta \in D\) and \(z \in \mathbb{C}\) with \(|z| < r\) we have \(\text{Re}(e^\theta + z)^2 \geq \frac{2}{\pi} \delta\). Thus \(L^\infty\)-continuity of \(\theta \mapsto \hat{f}_\theta\) now follows from \((B.3)\) and \((B.6)-(B.8)\) and the fact that the exponential map \(\theta \mapsto e^{-\theta} =: w\) is a continuous function on \(\mathbb{C}\). Now let \(\gamma\) be a closed piecewise \(C^1\)-curve in \(D\). Then we
define \( \int_{\gamma} \hat{f}_\theta d\theta \) as a Riemann integral in \( L^\infty(\mathbb{R}^d) \), which exists by continuity of the integrand as a function of \( \theta \). We want to show that this integral vanishes. For this, we introduce the bilinear form \( \langle \cdot, \cdot \rangle : L^1(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \to \mathbb{C} \), \( (f, g) \mapsto \int_{\mathbb{R}^d} f(k)g(k)dk \). This bilinear form is continuous, since an elementary estimate shows that \( |\langle f, g \rangle| \leq \|f\|_1 \|g\|_\infty \). Let \( h \in L^1(\mathbb{R}^d) \) be arbitrary. By what we have shown above we know that \( \theta \mapsto \langle h, \hat{f}_\theta \rangle = \int_{\mathbb{R}^d} h(k)f(\theta(k))dk \) is analytic. Thus it follows from continuity of \( \langle \cdot, \cdot \rangle \) that

\[
\langle h, \int_{\gamma} \hat{f}_\theta d\theta \rangle = \int_{\gamma} \langle h, \hat{f}_\theta \rangle d\theta = 0.
\]  

(B.9)

Using \( (L^1)^* \cong L^\infty \), we see from (B.9) since \( h \in L^1(\mathbb{R}^d) \) was arbitrary that \( \int_{\gamma} \hat{f}_\theta d\theta = 0 \). Since this holds for any closed piecewise \( C^1 \)-curve \( \gamma \) in \( D \), strong analyticity of \( \theta \mapsto \hat{f}_\theta \) in \( L^\infty(\mathbb{R}^d) \) now follows from a straightforward generalization of Morera’s theorem to Banach space valued functions, see for example [AA02, 1.5, Exercise 2].

\[ \square \]

C  Estimates on Integrals

In this Appendix, we present a collection of estimates on integrals as well as on discrete integrals, i.e., infinite sums.

Let us first mention a basic inequality. For any \( z \in \mathbb{C} \) we have the elementary bound

\[
|z^{-1}| \leq |\text{Im}z|^{-1}.
\]  

(C.1)

The following lemma is a simple estimate, which is applied in the proof of Lemma 4.1.

Lemma C.1. For \( a, b \in \mathbb{R} \) with \( a \leq b \) and \( \langle \cdot \rangle^2 f \in L^\infty \) with \( \langle \cdot \rangle \) defined in (4.4). Then \( f \in L^1(\mathbb{R}) \) and

\[
\int_a^b |f(x)| dx \leq \pi \sup_{x \in \mathbb{R}} |\langle x \rangle^2 f(x)|.
\]

Proof. Using an estimate on the arctan and a straightforward calculation shows

\[
\int_a^b |f(x)| dx \leq \int_a^b \langle x \rangle^{-2} \langle x \rangle^2 |f(x)| dx
\]

\[
\leq \sup_{x \in \mathbb{R}} |\langle x \rangle^2 f(x)| \int_a^b \frac{1}{1 + x^2} dx
\]

\[
= \sup_{x \in \mathbb{R}} |\langle x \rangle^2 f(x)| \underbrace{[\arctan(x)]_a^b}_{\leq \pi}
\]

\[
\leq \pi \sup_{x \in \mathbb{R}} |\langle x \rangle^2 f(x)| < \infty.
\]

\[ \square \]
Moreover we shall need to estimate tracial expressions of resolvents. Let us now estimate a finite volume expression. For this we shall make use of the following elementary lemma about Riemann integrals.

**Lemma C.2.** Let $I = [-c,c]^d$ and $I_j$, $j = 1, ..., N$, be a partition (up to boundaries) of $I$ into translates of a square $Q$ which is centered at the origin. Let $\xi_j \in I_j$ and $\Delta x_j = |I_j|$. Suppose $g \geq 0$ is a Riemann integrable function on $I$ and we have

$$\sup_{\xi \in I_j} f(\xi) \leq \inf_{\xi \in I_j} g(\xi).$$

Then

$$\left| \sum_{i=1}^{N} f(\xi_j) \Delta x_j \right| \leq \int_I g(x) dx.$$

**Proof.** The statement follows directly from the theory of Riemann integration.

We now want to show Proposition [C.19] which is used in Chapter 5 to estimate integrals over resolvent-type-functions. To do so, we are first discussing the relation between the discrete sum and the Riemann integral in Lemma [C.3], to then give a bound on the Riemann integral in Lemma [C.4], with which we can then show Proposition [C.19].

**Lemma C.3.** Let $f : \Lambda_L^* \to \mathbb{C}$ and $E \geq 0$. Then

$$\int_{\Lambda_L^*} \left| \frac{f(q)}{q^2 - E - i\eta} \right| dq \leq C_0(E, d, L, \eta, f)$$

where

$$C_0(E, d, L, \eta, f) := \|f\|_* \int_{\|q\| \leq \sqrt{2E+1}} |q^2 - E - i2^{-1/2}\eta|^{-1} dq + (E + 1)^{-1}\|f\|_*.$$

where we introduced the notation $\|q\| = \max_{j=1}^{d} |q_j|$.

**Proof.** To prove the Lemma we want to use Lemma [C.2]. For this we need to verify (C.2). Let $q, \xi, \xi_1, \xi_2 \in \mathbb{R}^d$, $s \geq 1$, and $|q|^2 \leq sE$. First observe that

$$|(q + \xi)^2 - E - i\eta|^2 = |q^2 - E + \xi^2 + \xi q + q\xi - i\eta|^2$$

$$= (q^2 - E)^2 + 2(q^2 - E)(\xi q + q\xi + \xi^2) + (\xi q + q\xi + \xi^2)^2 + \eta^2.$$

From this we find for $d_j = \xi_j q + q\xi_j + \xi_j^2$ that

$$|(q + \xi_1)^2 - E - i\eta|^2 - |(q + \xi_2)^2 - E - i\eta|^2$$

$$= 2(q^2 - E)(d_1 - d_2) + d_1^2 - d_2^2 + \eta_1^2 - \eta_2^2$$

$$\geq \frac{1}{4} \eta^2 \geq 0,$$

$$\left| \sum_{i=1}^{N} f(\xi_j) \Delta x_j \right| \leq \int_I g(x) dx.$$
provided $\sqrt{2}\eta_2 = \eta_1 = \eta$ and

$$\|d_1\|,|d_2| \leq \min \left\{ \frac{1}{4} \eta, \frac{1}{2} \frac{1}{2sE} \right\} =: C(s, \eta). \quad (C.5)$$

Thus if

$$|\xi_j| \leq \max \left\{ 1, \frac{C(s, \eta)}{2\sqrt{sE} + 1} \right\} =: D(s, \eta),$$

then (C.5) holds, and hence for all $q$ with $|q| \leq \sqrt{sE}$ we find from (C.4)

$$\frac{1}{|(q + \xi_1)^2 - E - i\eta|} = \frac{1}{|(q + \xi_2)^2 - E - i2^{-1/2}\eta| + |(q + \xi_1)^2 - E - i\eta| - |(q + \xi_2)^2 - E - i2^{-1/2}\eta|} \leq \frac{1}{|(q + \xi_2)^2 - E - i2^{-1/2}\eta|}.$$  

This implies

$$\sup_{\xi:|\xi| \leq D(s, \eta)} |(q + \xi)^2 - E - i\eta|^{-1} \leq \inf_{\xi:|\xi| \leq D(s, \eta)} |(q + \xi)^2 - E - i2^{-1/2}\eta|^{-1}. $$

Hence (C.2) is satisfied and we find for $L > \frac{\sqrt{2d}}{D(s, \eta)}$ that

$$\left| \int_{\Lambda_L} (q^2 - E - i\eta)^{-1} 1_{|q| \leq \sqrt{sE/d}} dq \right| \leq \int |q^2 - E - i2^{-1/2}\eta|^{-1} 1_{|q| \leq \sqrt{sE/d}} dq.$$  

Now for $s = d(2 + E^{-1})$ we have

$$\int_{\Lambda_L} \left| f(q) (q^2 - E - i\eta)^{-1} \right| dq \leq \int_{\Lambda_L} \left| f(q) (q^2 - E - i\eta)^{-1} \right| 1_{|q| \leq \sqrt{sE/d}} dq + \int_{\Lambda_L} \left| f(q) (q^2 - E - i\eta)^{-1} \right| 1_{|q| > \sqrt{sE/d}} dq \leq \|f\|_{*,\infty} \int_{\{q \in \mathbb{R}^d : |q| \leq \sqrt{2E+1}\}} |q^2 - E - i2^{-1/2}\eta|^{-1} dq + (E + 1)^{-1} \int_{\Lambda_L} |f(q)| dq,$$

which completes the proof. □

**Lemma C.4.** For $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $E > 0$, $\eta > 0$ and $d \in \mathbb{N}$ we have

$$\left| \int_{\mathbb{R}^d} \frac{f(q)}{q^2 - E \pm i\eta} dq \right| \leq C_1(E, d) \|f\|_\infty \ln \left( \frac{1}{\eta} + 1 \right) + \sqrt{2} \|f\|_1,$$

where

$$C_1(E, d) := \sqrt{2} \left( E^{-1/2}(E + 1) \frac{d-1}{2} + E^{\frac{d-2}{2}} \right) |S_{d-1}|, \quad (C.6)$$

with $|S_{d-1}|$ being the volume of the $d - 1$-dimensional sphere with radius 1.
Proof. Let $\delta > 0$ be arbitrary. Using $\sqrt{2(|x|^2 + |y|^2) \geq (|x| + |y|)}$ we find
\begin{equation}
\left| \int \frac{f(q)}{q^2 - E + i\eta} dq \right| \leq \int \frac{\sqrt{2}|f(q)|}{|q^2 - E| + \eta} dq 
\end{equation}
\begin{equation}
= \int \frac{\sqrt{2}|f(q)|}{|q^2 - E| + \eta} dq + \int \frac{\sqrt{2}|f(q)|}{|q^2 - E| + \eta} dq. \tag{C.7}
\end{equation}

We will now consider both summands separately. For the first one we estimate
\begin{equation}
\int \frac{\sqrt{2}|f(q)|}{|q^2 - E| + \eta} dq \leq \frac{\sqrt{2}\|f\|_1}{\delta}. \tag{C.8}
\end{equation}

Independent of the dimension we can perform the following estimate. We are going over
to $d$-dimensional polar coordinates and resolve the absolute value in the condition of the
indicator function. Let $a_+ := \max\{a, 0\}$. We write
\begin{align}
&\int_{\mathbb{R}^d} 1_{|q^2 - E| \leq \delta} \frac{\sqrt{2}|f(q)|}{|q^2 - E| + \eta} dq \\
&= \int_0^{\infty} \int_{S_{d-1}} 1_{|r^2 - E| \leq \delta} \frac{\sqrt{2}|f(r\omega)|}{|r^2 - E| + \eta} d\omega r^{d-1} dr \\
&= \int_0^{\infty} \int_{S_{d-1}} 1_{0 \leq r^2 - E \leq \delta} \frac{\sqrt{2}|f(r\omega)|}{|r^2 - E| + \eta} d\omega r^{d-1} dr + \int_0^{\infty} \int_{S_{d-1}} 1_{0 \leq r^2 - E \leq \delta} \frac{\sqrt{2}|f(r\omega)|}{|r^2 - E| + \eta} d\omega r^{d-1} dr \\
&= \left( \int_{S_{d-1}} \sqrt{E} \frac{\sqrt{2}|f(r\omega)|}{|r^2 - E| + \eta} d\omega r^{d-1} dr \right)_{=: I_1} + \left( \int_{S_{d-1}} \sqrt{E} \frac{\sqrt{2}|f(r\omega)|}{|r^2 - E| + \eta} d\omega r^{d-1} dr \right)_{=: I_2}. \tag{C.9}
\end{align}

For these two summands, we can use $r^2 - E = (r - \sqrt{E})(r + \sqrt{E})$ in the denominator to get
\begin{align}
I_1 &\leq \|f\|_\infty |S_{d-1}| \int_{\sqrt{E} \delta}^{\sqrt{E} + \delta} \frac{\sqrt{2}}{(r - \sqrt{E})(r + \sqrt{E}) + \eta} r^{d-1} dr \tag{C.10} \\
as well as
I_2 &\leq \|f\|_\infty |S_{d-1}| \int_{\sqrt{(E - \delta)_+}}^{\sqrt{E}} \frac{\sqrt{2}}{(\sqrt{E} - r)(\sqrt{E} + r) + \eta} r^{d-1} dr. \tag{C.11}
\end{align}

Now using the trivial estimate $\frac{1}{r + \sqrt{E}} \leq \frac{1}{\sqrt{E}}$ lets us perform the following estimates for the
integral in (C.10)
\[
\int_{\sqrt{E}}^{\sqrt{E+\delta}} \frac{\sqrt{2}}{(r - \sqrt{E})(r + \sqrt{E}) + \eta} r^{d-1}dr \leq \int_{\sqrt{E}}^{\sqrt{E+\delta}} \frac{\sqrt{2}}{(r - \sqrt{E})\sqrt{E + \eta}} r^{d-1}dr
\]
\[
\leq \frac{1}{E^{1/2}} \int_{\sqrt{E}}^{\sqrt{E+\delta}} \frac{\sqrt{2}}{r - \sqrt{E} + \eta E^{-1/2}} (E + \delta) \frac{dr}{E^{d-1}}
\]
\[
= \frac{\sqrt{2}}{E^{1/2}} \left( \frac{E + \delta}{\eta E^{-1/2}} \right)^{d-1} \left[ \ln \left( r - \sqrt{E} + \eta E^{-1/2} \right) \right]^{\sqrt{E+\delta}}_{\sqrt{E}},
\]
(C.12)
as well as for the one in (C.11)
\[
\int_{\sqrt{(E-\delta)^+}}^{\sqrt{E}} \frac{\sqrt{2}}{(\sqrt{E} - r)(\sqrt{E} + r) + \eta} r^{d-1}dr \leq \int_{\sqrt{(E-\delta)^+}}^{\sqrt{E}} \frac{\sqrt{2}}{\sqrt{(E-\delta)^+}(\sqrt{E} - r)\sqrt{E + \eta}} r^{d-1}dr
\]
\[
\leq \frac{1}{E^{1/2}} \int_{\sqrt{(E-\delta)^+}}^{\sqrt{E}} \frac{\sqrt{2}}{\sqrt{E} - r + \eta E^{-1/2}} E^{d-1} \frac{dr}{E^{d-1}}
\]
\[
= E^{d-2} \left[ \ln \left( \sqrt{E} - r + \eta E^{-1/2} \right) \right]^{\sqrt{E}}_{\sqrt{(E-\delta)^+}},
\]
(C.13)
We can finally evaluate the logarithm-terms in the RHS of (C.12) via
\[
\left[ \ln \left( r - \sqrt{E} + \eta E^{-1/2} \right) \right]^{\sqrt{E+\delta}}_{\sqrt{E}} = \ln \left( \frac{\sqrt{E + \delta} - \sqrt{E} + \eta E^{-\frac{1}{2}}}{\eta E^{-\frac{1}{2}}} \right)
\]
\[
= \ln \left( \frac{\sqrt{E + \delta} - \sqrt{E}}{\eta E^{-1/2}} + 1 \right)
\]
\[
= \ln \left( \frac{(E + \delta) - E}{\eta(\sqrt{1 + E^{-1}d} + 1)} + 1 \right)
\]
\[
\leq \ln \left( \frac{\delta}{\eta} + 1 \right),
\]
(C.14)
as well as the one in (C.13)

\[
\left[ \ln \left( \sqrt{E} - r + \eta E^{-1/2} \right) \right]^{\sqrt{E}} = \ln \left( \frac{\sqrt{E} - \sqrt{(E - \delta)_+} + \eta E^{-\frac{1}{2}}}{\eta E^{-\frac{1}{2}}} \right) \\
= \ln \left( \frac{\sqrt{E} - \sqrt{(E - \delta)_+}}{\eta E^{-1/2}} + 1 \right) \\
= \ln \left( \frac{E - (E - \delta)_+}{\eta(1 + \sqrt{(1 - E^{-1}\delta)_+})} + 1 \right) \\
\leq \ln \left( \frac{\min\{\delta, E\}}{\eta} + 1 \right). \quad (C.15)
\]

Putting together (C.10), (C.12) and (C.14) gives us

\[
I_1 \leq \|f\|_{\infty} |S_{d-1}| \sqrt{2} \left( E + \frac{d+1}{2} \right) \ln \left( \frac{\delta}{\eta} + 1 \right), \quad (C.16)
\]

while (C.11), (C.13) and (C.15) gives

\[
I_2 \leq \|f\|_{\infty} |S_{d-1}| \sqrt{2} E^{d-2} \ln \left( \frac{\min\{\delta, E\}}{\eta} + 1 \right). \quad (C.17)
\]

Adding (C.16) and (C.17), while using \(\min\{\delta, E\} \leq \delta\) we find from (C.9) that

\[
\int_{\mathbb{R}^d} 1_{|q^2 - E| \leq \delta} \frac{\sqrt{2} |f(q)|}{|q^2 - E| + \eta} dq = I_1 + I_2 \leq \left( \frac{(E + \delta)^{d+1}}{2 E^{\frac{d}{2}}} + E^{d-2} \right) \|f\|_{\infty} |S_{d-1}| \sqrt{2} \ln \left( \frac{\delta}{\eta} + 1 \right). \quad (C.18)
\]

Now choosing \(\delta = 1\) we find with (C.7), (C.8), and (C.18) that

\[
\left| \int_{\mathbb{R}^d} f(q) \left( q^2 - E \pm i\eta \right)^{-1} dq \right| \\
\leq \int_{\mathbb{R}^d} 1_{|q^2 - E| \leq \delta} \frac{\sqrt{2} |f(q)|}{|q^2 - E| + \eta} dq + \int_{\mathbb{R}^d} 1_{|q^2 - E| > \delta} \frac{\sqrt{2} |f(q)|}{|q^2 - E| + \eta} dq \\\n\leq C_1(E, d) \|f\|_{\infty} \ln \left( \frac{1}{\eta} + 1 \right) + \sqrt{2} \|f\|_1
\]

\]

\[\square\]

The following proposition will be needed in Section 5.

**Proposition C.5.** Let \(E > 0\) and \(\eta > 0\). Then for all \(L \geq 1\) and \(f : \Lambda^*_L \to \mathbb{C}\) we have

\[
\left| \int_{\Lambda^*_L} f(q) \left( \frac{1}{2} q^2 - E \pm i\eta \right)^{-1} dq \right| \leq C(E, d, L, \eta, f)
\]

59
where
\[ C(E, d, L, \eta, f) := 2\|f\|_{*, \infty} \left[ C_1(2E, d) \ln(\eta^{-1} + 1) + 2^d \sqrt{2(4E + 1)^{d/2}} \right] + 2\|f\|_{*, 1} \]  
(C.19)

with \(C_1(E, d)\) defined in (C.6).

**Proof.** This follows by combining Lemmas C.3 and C.4. In more detail, it follows from Lemma C.3 that

\[
\left| \int_{\mathbb{R}^d} \frac{f(q)dq}{q^2 - E \pm i\eta} \right| \leq 2\|f\|_{*, \infty} \int_{\{q \in \mathbb{R}^d : \|q\|_{\infty} \leq \sqrt{4E + 1}\}} \frac{dq}{|q^2 - 2E - i2^{1/2}\eta|} + 2\|f\|_{*, 1}, \tag{C.20}
\]

recalling \(\|q\|_{\infty} = \max_{j=1}^d |q_j|\). Now using Lemma C.4 to estimate the integral on the right hand side of (C.20), we find

\[
\int_{\{q \in \mathbb{R}^d : \|q\|_{\infty} \leq \sqrt{4E + 1}\}} \frac{dq}{|q^2 - 2E - i2^{1/2}\eta|} \leq C_1(2E, d) \ln \left(2^{-1/2}\eta^{-1} + 1\right) + 2^d \sqrt{2(4E + 1)^{d/2}}. \tag{C.21}
\]

Thus inserting (C.21) into (C.20) the inequality of the lemma follows using that by monotonicity of \(\ln\) we have \(\ln \left(2^{-1/2}\eta^{-1} + 1\right) \leq \ln(\eta^{-1} + 1)\). $$\square$$

The following Lemma will be used in Section 6.

**Lemma C.6.** Let \(d \leq 3\). For all \(E > 0\) and \(\tau \in [0, 4 - d)\), there exists a constant \(C_{E, \tau}\), uniformly bounded for \(E\) in compact subsets of \((0, \infty)\), such that for all \(L \geq 1\)

\[
\int_{\Lambda_L} \frac{\langle a \rangle^{\tau}}{|a^2 - E - i\eta|^2} \, da \leq C_{E, \tau}(1 + \eta^{-2}). \tag{C.22}
\]

**Proof.** We estimate

\[
\int_{\Lambda_L} \frac{\langle a \rangle^{\tau}}{|a^2 - E - i\eta|^2} \, da = \int_{\Lambda_L} \frac{\langle a \rangle^{\tau}}{(a^2 - E)^2 + \eta^2} \, da = \int_{\Lambda_L} \frac{\langle a \rangle^{\tau} 1_{a^2 \leq 4E + 4}}{(a^2 - E)^2 + \eta^2} \, da + \int_{\Lambda_L} \frac{\langle a \rangle^{\tau} 1_{a^2 > 4E + 4}}{(a^2 - E)^2 + \eta^2} \, da \tag{C.23}
\]

For the first sum on the right hand side of (C.23) we observe that for \(L \geq 1\) we have with \(\kappa_{\tau, E} := \langle (4 + 4E)^{1/2} \rangle^{\tau}\) that

\[
\int_{\Lambda_L} \frac{\langle a \rangle^{\tau} 1_{a^2 \leq 4E + 4}}{(a^2 - E)^2 + \eta^2} \, da \leq \kappa_{\tau, E} \eta^{-2} \int_{\Lambda_L} 1_{a^2 \leq 4E + 4} \, da = \kappa_{\tau, E} \eta^{-2} \int_{\Lambda_L} 1_{|a| \leq \sqrt{4E + 4}} \, da \\
\leq \kappa_{\tau, E} \eta^{-2} \int_{\Lambda_L} 1_{|a| \leq \sqrt{4E + 4}} \, da \leq \kappa_{\tau, E} \eta^{-2} (2\sqrt{4 + 4E} + 1)^d. \tag{C.24}
\]
For the second sum on the right hand side of \((C.23)\) we will use Lemma \(C.2\). First observe for \(a^2 > 4E + 4\) that
\[
a^2 - E = \frac{1}{2}a^2 + \frac{1}{2}a^2 - E \geq \frac{1}{2}a^2 + 1. \tag{C.25}
\]

To apply Lemma \(C.2\) we need the following estimates. Let \(\xi \in \left[-\frac{1}{2}, \frac{1}{2}\right] =: Q\). Then \(|\xi| \leq \sqrt{d}/2\).

For \(0 < \delta < 1\) and \(L \geq 1\) the following inequalities hold, where we use the square inequality in the second one, while choosing \(\delta = d/(2 + d)\) in the last line
\[
\left( a + \frac{1}{L}\xi \right)^2 + 1 \geq a^2 - 2|a||\xi|L^{-1} + |\xi|^2L^{-2} + 1
\]
\[
\geq (1 - \delta)a^2 + \left( \frac{1 - \delta^{-1}}{1} \right) |\xi|^2 L^{-2} + 1
\]
\[
\geq (1 - \delta)a^2 + (1 - \delta^{-1})\frac{d}{4} + 1
\]
\[
= \frac{2}{2 + d}a^2 + \frac{1}{2} \geq \min\{1/2, 2/(d + 2)\}(a^2 + 1).
\]

On the other hand we find
\[
\left( a + \frac{1}{L}\xi \right)^2 + 1 \leq a^2 + 2|a||\xi|L^{-1} + |\xi|^2L^{-2} + 1 \leq 2a^2 + 2|\xi|^2L^{-2} + 1
\]
\[
\leq \max\{2, 1 + d/2\}(a^2 + 1).
\]

Thus by monotonicity and continuity we find for any \(\alpha \geq 0\) with \(c_d := \min\{1/2, 2/(d + 2)\}/\max\{2, 1 + d/2\}\) that
\[
\sup_{\xi \in Q} \left( a + \frac{\xi}{L} \right)^{-\alpha} = \left( \inf_{\xi \in Q} \left( a + \frac{\xi}{L} \right) \right)^{-\alpha} \leq \left( c_d \sup_{\xi \in Q} \left( a + \frac{\xi}{L} \right) \right)^{-\alpha} = c_d^{-\alpha} \inf_{\xi \in Q} \left( a + \frac{\xi}{L} \right)^{-\alpha}. \tag{C.26}
\]

Now using \((C.25)\), then Lemma \(C.2\) together with \((C.26)\) implies that the following sums and integrals are finite
\[
\int_{\Lambda_L} \frac{\langle a \rangle^\tau 1_{a^2 > 4E + 4}}{(a^2 - E)^2 + \eta^2} da \leq \int_{\Lambda_L} \frac{\langle a \rangle^\tau}{(\frac{1}{2}a^2 + 1)^2} da \tag{C.27}
\]
\[
\leq 4 \int_{\Lambda_L} \langle a \rangle^{-4 + \tau} da \tag{C.28}
\]
\[
\leq 4c_d^{-4 + \tau} \int_{\mathbb{R}^d} \langle a \rangle^{-4 + \tau} da =: C(d, \tau) < \infty.
\]

Thus inserting \((C.28)\) and \((C.24)\) into \((C.28)\) yields \((C.22)\).
D A Supremum Estimate

In this appendix we prove Lemma D.1 which will be used in the proof of Theorem 6.4 in Section 6.

Lemma D.1. For $d \leq 3$, $E > 0$, and $\varepsilon \in (0, 1]$ there exists a constant $K_{d, \varepsilon, E}$, such that the function $(0, \infty) \to (0, \infty)$, $E \mapsto K_{d, \varepsilon, E}$ is uniformly bounded on compact subsets, and for $\sigma \in \{0, 1\}$, $\eta > 0$, and $q \in \mathbb{R}^d$ we have

$$
\sup_{v_1, v_2 \in \mathbb{R}^d} \prod_{j=1}^d \{ (v_{1,j})^{-1+\varepsilon} (v_{2,j})^{-1+\varepsilon} \} \| \nu(q + v_1) - E \pm i\eta \|^{-1} \| \nu(q + \sigma v_1 + v_2) - E \pm i\eta \|^{-1} \leq K_{d, \varepsilon, E} (1 + \eta^{-2}) \prod_{j=1}^d (q_j)^{-1+\varepsilon},
$$

(D.1)

where $v$ is the function defined in (2.3).

Proof. First observe that since $(v_{1,j})^{-1+\varepsilon} \leq 1$, by its definition (4.4), and

$$
| \nu(q + v_1) - E \pm i\eta \|^{-1} \leq \eta^{-1},
$$

(D.2)

we find

$$
\text{LHS of (D.1)} \leq \eta^{-2}.
$$

(D.3)

We only show the case $d = 3$. The cases $d = 1, 2$ follow from $d = 3$ by the obvious restriction to subspaces.

To show (D.1) we proceed as follows. First we observe that by scaling it suffices to consider a fixed positive $E$. To see this, assume (D.1) holds for a fixed $E_0 > 0$. Using the inequalities $(a b)^{-1} \leq (a^{-1})(b)^{-1}$, for any $a > 0$ and $b \geq 0$, (which in turn follows using $\langle x y \rangle \leq \langle x \rangle \langle y \rangle$) we find for any $\gamma > 0$

$$
\sup_{v_1, v_2 \in \mathbb{R}^d} \prod_{j=1}^d \{ (v_{1,j})^{-1+\varepsilon} (v_{2,j})^{-1+\varepsilon} \} \| \nu(q + v_1) - \gamma^2 E_0 \pm i\eta \|^{-1} \| \nu(q + \sigma v_1 + v_2) - \gamma^2 E_0 \pm i\eta \|^{-1} = \gamma^{-4} \sup_{v_1, v_2 \in \mathbb{R}^d} \prod_{j=1}^d \{ (\gamma v_{1,j})^{-1+\varepsilon} (\gamma v_{2,j})^{-1+\varepsilon} \} \| \nu(q + v_1) - E_0 \pm i\eta \gamma^{-2} \|^{-1} \| \nu(q + \sigma v_1 + v_2) - E_0 \pm i\eta \gamma^{-2} \|^{-1} \leq \gamma^{-4} (\gamma^{-1})^{d(2-2\varepsilon)} K_{d, \varepsilon, E_0} (1 + (\gamma^{-2}\eta)^{-2}) \prod_{j=1}^d (\gamma q_j)^{-1+\varepsilon} \leq \gamma^{-4} (\gamma^{-1})^{d(2-2\varepsilon)} (\gamma)^{d(2-2\varepsilon)} K_{d, \varepsilon, E_0} (\gamma)^4 (1 + \eta^{-2}) \prod_{j=1}^d (q_j)^{-1+\varepsilon}.
$$

(D.4)

Thus we can assume without loss of generality that

$$
E = 1/(32d).
$$

(D.5)
Let \( c = \frac{1}{2\sqrt{d}} \) and \( C = \frac{3}{4\sqrt{d}} \). We assume

\[ |q| \geq 32d(c^{-1} + 1), \]

otherwise the inequality is trivial in view of (D.3). We will consider the three cases (I) \( |q + v_1| \leq C \), (III) \( |q + v_1| \geq c|q| \), and (II) the region in between.

(I) Suppose \( |q + v_1| \leq C \). For every \( j \) for which \( |q_j| \geq 2C \) holds, it follows \( |v_{1,j}| \geq |q_j| - |v_{1,j} + q_j| \geq (\frac{1}{2}|q_j| + C) - C = \frac{1}{2}|q_j| \geq \frac{1}{8}(|q_j| + 1) \geq \frac{1}{8}q_j \), where we used \( |q_j| \geq 2\frac{3}{4\sqrt{d}} \geq 1/2 \). If \( |q_j| < 2C \) we use the trivial bound \( \langle q_j \rangle \leq \langle C \rangle \langle v_{1,j} \rangle \). Thus we find in case (I) for some constant \( c_{(I),d} \)

\[ \prod_{j=1}^{d} (v_{1,j})^{-1+\epsilon} |\nu(q + v_1) - E \pm in|^{-1} \leq \eta^{-1}c_{(I),d} \prod_{j=1}^{d} \langle q_j \rangle^{-1+\epsilon}. \]  

(II) Suppose \( |q_j| < 2C \) we use the trivial bound \( \langle q_j \rangle \leq \langle C \rangle \langle v_{1,j} \rangle \).

(II) Suppose there exists \( \alpha \in [0, 1] \) such that \( \frac{3}{4\sqrt{d}}|q|^{\alpha} \leq |q + v_1| \leq \frac{3}{4\sqrt{d}}|q|^{\alpha} \) (if \( \alpha = 0 \), this case is already considered in (I), and if \( \alpha = 1 \), this case will be considered in (III)). We want to show that in case (II) there exists a constant \( c_{(II),d} \) such that

\[ \prod_{j=1}^{d} (v_{1,j})^{-1+\epsilon} |\nu(q + v_1) - E \pm in|^{-1} \leq c_{(II),d}(1 + \eta^{-1}) \prod_{j=1}^{d} \langle q_j \rangle^{-1+\epsilon}. \]

By rotational invariance of \( \nu \) we can assume by a simple relabeling of the coordinates that \( |q_1| \geq |q_2| \geq |q_3| \). Now by \( |q| \leq \sqrt{d} \max_j |q_j| \) we have \( |q_1| \geq \frac{1}{\sqrt{d}}|q| \). It follows from the assumption \( |q + v_1| \leq \frac{3}{4\sqrt{d}}|q|^{\alpha} \), that \( |v_{1,1}| \geq |q_1| - |v_{1,1} + q_1| \geq \frac{3}{4\sqrt{d}}|q| - \frac{3}{4\sqrt{d}}|q|^{\alpha} \geq \frac{1}{4\sqrt{d}}|q| \), where we used that \( |q| \geq 1 \), by (D.6). We now divide case (II) into subcases.

(II.a) If \( |q_2| \leq \frac{1}{\sqrt{d}}|q|^{\alpha} \) we find using (D.5) and \( |q| \geq 1 \), by (D.6), that \( \frac{1}{d_2}|q + v_1|^{2 \alpha} - E \geq \frac{1}{16d_2}(|q|^{2\alpha} + (2d_2)|q_2| + \frac{1}{4|q|^{\alpha}} \geq \frac{1}{64d_2}(|q_2| + 1)^{2} \geq \frac{1}{64d_2}(|q_2| + 1)^{1-\epsilon}(|q_3| + 1) \). We conclude that in this case (D.8) holds for some constant \( c_{(II),d} \).

(II, b) If \( |q_2| \geq \frac{1}{\sqrt{d}}|q|^{\alpha} \). Then we find similarly as for \( q_1 \) the estimate

\[ |v_{1,2}| \geq |q_2| - |v_{1,2} + q_2| \geq \frac{1}{4}|q_2| + \frac{3}{4}|q_2| - |v_{1,2} + q_2| \]

\[ \geq \frac{3}{4}|q_2| + \frac{3}{4\sqrt{d}}|q|^{\alpha} - \frac{3}{4\sqrt{d}}|q|^{\alpha} = \frac{3}{4}|q_2| \].

(II, b, a) if \( |q_3| \leq \frac{1}{\sqrt{d}}|q|^{\alpha} \) then as in (II, a) we find that \( \frac{1}{2}|q + v|^{2} - E \geq \cdots \geq \frac{1}{16d_2}(|q_2|^{2\alpha} + (2d_2)|q_2| + \frac{1}{4|q_2|^{\alpha}} \geq \frac{1}{64d_2}(|q_2| + 1)^{2} \geq \frac{1}{64d_2}(|q_3| + 1). \) We conclude that in this case (D.8) holds for some constant \( c_{(II),d} \).
Now let’s assume $|q_3| \geq \frac{1}{\sqrt{d}}|q|^\alpha$. Then we find similarly as for $q_2$ the estimate $|v_{1,3}| \geq |q_3| - |v_{1,3} + q_3| \geq \ldots \geq \frac{1}{4}|q_3|$. We conclude that in this case (D.8) holds for some constant $c_{(II),d}$.

Now taking the maximum for the constant $c_{(II),d}$ obtained in the cases (II,a) (II,b,a) (II,b,b), we see that (D.8) holds.

Now note that in cases (I) and (II) we obtain the desired estimate (D.1) irrespective of the value $v_2$ since the second factor can be estimated in terms of $\eta^{-1}$.

Finally let us look at the last case.

(III) Suppose $|q + v_1| \geq c|q|$. Then $\frac{1}{2}|q + v_1|^2 - E \geq \frac{c^2}{2}|q|^2 - E \geq \frac{c^2}{4}|q|^2$ (by (D.5) and (D.6)). This yields that for some constant $c_{(III),d}$ we have

$$
\prod_{j=1}^{d} (v_{1,j})^{-1+\varepsilon} |\nu(q + v_1) - E \pm i\eta|^{-1} \leq c_{(III),d}(q)^{-2}. \quad (D.9)
$$

This is not yet sufficient, and we need to make use of the factor $|\nu(q + \sigma v_1 + v_2) - E \pm i\eta|^{-1}$ in (D.1). If $\sigma = 0$, then we are in the same situation as before but now for $v_2$. Now cases (I) and (II) for $v_2$ already yield the desired bound. Now case (III) for $v_2$ yields again (D.9) but now for $v_2$. Thus case (III) for $v_1$ and case (III) for $v_2$ give a decay of the form $\langle q \rangle^{-4}$ which shows (D.1) in that case. Now suppose $\sigma = 1$. If $|q + v_1 + v_2| \leq \frac{c}{2}|q|$, then $|v_2| \geq |q + v_1| - |q + v_1 + v_2| \geq \frac{c}{2}|q|$, so in view of (D.9) we can estimate the left hand side (D.1) by a term proportional to $\eta^{-1}\langle q \rangle^{-3+\varepsilon}$, which shows (D.1) in that case. If on the other hand $|q + v_1 + v_2| \geq \frac{c}{2}|q|$, we find that $\frac{1}{2}|q + v_1 + v_2|^2 - E = \frac{c^2}{2}|q|^2 - E \geq \frac{c^2}{16}|q|^2$ (since (D.5) and (D.6)) and so, again in view of (D.9), we can estimate the left hand side (D.1) by a term proportional to $\langle q \rangle^{-4}$, which shows (D.1) in that case.

We conclude that in any case we have shown (D.1) in case of the fixed choice for $E$ as in (D.5). Finally, observe that the uniform bound on $K_{d,e,E}$ as a function of $E$ on compact subsets of $(0, \infty)$ can be seen from the inequality (D.4). \hfill \Box

References

[AA02] Y. A. Abramovich and C. D. Aliprantis. An invitation to operator theory. Vol. 50. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2002, pp. xiv+530.

[AF03] R. A. Adams and J. J. F. Fournier. Sobolev spaces. Second. Vol. 140. Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, 2003, pp. xiv+305.
[AK92] V. Acosta and A. Klein. “Analyticity of the density of states in the Anderson model on the Bethe lattice”. In: *J. Statist. Phys.* 69.1-2 (1992), pp. 277–305.

[Ana+21] N. Anantharaman, M. Ingremeau, M. Sabri, and B. Winn. “Absolutely continuous spectrum for quantum trees”. In: *Comm. Math. Phys.* 383.1 (2021), pp. 537–594.

[ASW06] M. Aizenman, R. Sims, and S. Warzel. “Absolutely continuous spectra of quantum tree graphs with weak disorder”. In: *Comm. Math. Phys.* 264.2 (2006), pp. 371–389.

[AW13] M. Aizenman and S. Warzel. “Resonant delocalization for random Schrödinger operators on tree graphs”. In: *J. Eur. Math. Soc. (JEMS)* 15.4 (2013), pp. 1167–1222.

[AW15] M. Aizenman and S. Warzel. *Random Operators*. Graduate Studies in Mathematics. American Mathematical Society, 2015.

[CH94] J.-M. Combes and P. D. Hislop. “Localization for some continuous, random Hamiltonians in d-dimensions”. In: *J. Funct. Anal.* 124.1 (1994), pp. 149–180.

[CHK03] J.-M. Combes, P. D. Hislop, and F. Klopp. “Hölder continuity of the integrated density of states for some random operators at all energies”. In: *Int. Math. Res. Not.* 4 (2003), pp. 179–209.

[CHK07] J.-M. Combes, P. D. Hislop, and F. Klopp. “An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators”. In: *Duke Math. J.* 140.3 (2007), pp. 469–498.

[CT73] J. M. Combes and L. Thomas. “Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators”. In: *Comm. Math. Phys.* 34 (1973), pp. 251–270.

[ESY07a] L. Erdős, M. Salmhofer, and H.-T. Yau. “Quantum diffusion for the Anderson model in the scaling limit”. In: *Ann. Henri Poincaré* 8.4 (2007), pp. 621–685.

[ESY07b] L. Erdős, M. Salmhofer, and H.-T. Yau. “Quantum diffusion of the random Schrödinger evolution in the scaling limit. II. The recollision diagrams”. In: *Comm. Math. Phys.* 271.1 (2007), pp. 1–53.

[ESY08] L. Erdős, M. Salmhofer, and H.-T. Yau. “Quantum diffusion of the random Schrödinger evolution in the scaling limit”. In: *Acta Math.* 200.2 (2008), pp. 211–277.

[FHS06] R. Froese, D. Hasler, and W. Spitzer. “Transfer matrices, hyperbolic geometry and absolutely continuous spectrum for some discrete Schrödinger operators on graphs”. In: *J. Funct. Anal.* 230.1 (2006), pp. 184–221.

[FHS07] R. Froese, D. Hasler, and W. Spitzer. “Absolutely continuous spectrum for the Anderson model on a tree: a geometric proof of Klein’s theorem”. In: *Comm. Math. Phys.* 269.1 (2007), pp. 239–257.
[FKT90] J. Feldman, H. Knörrer, and E. Trubowitz. “The perturbatively stable spectrum of a periodic Schrödinger operator”. In: *Invent. Math.* 100.2 (1990), pp. 259–300.

[FKT91] J. Feldman, H. Knörrer, and E. Trubowitz. “Perturbatively unstable eigenvalues of a periodic Schrödinger operator”. In: *Comment. Math. Helv.* 66.4 (1991), pp. 557–579.

[Fol13] G. Folland. *Real Analysis: Modern Techniques and Their Applications.* Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2013.

[HS87] B. Helffer and J. Sjöstrand. “Équation de Schrödinger avec champ magnétique et équation de Harper”. In: *Journées “Équations aux derivées partielles” (Saint Jean de Monts, 1987).* École Polytech., Palaiseau, 1987, Exp. No. VI, 9.

[HS89] B. Helffer and J. Sjöstrand. “Équation de Schrödinger avec champ magnétique et équation de Harper”. In: *Schrödinger operators (Sønderborg, 1988)*. Vol. 345. Lecture Notes in Phys. Springer, Berlin, 1989, pp. 118–197.

[KK20] W. Kirsch and M. Krishna. “Analyticity of density of states for the Cauchy distribution”. In: https://doi.org/10.48550/arXiv.2006.15840 (2020).

[Kle98] A. Klein. “Extended states in the Anderson model on the Bethe lattice”. In: *Adv. Math.* 133.1 (1998), pp. 163–184.

[KM07] W. Kirsch and B. Metzger. “The integrated density of states for random Schrödinger operators”. In: *Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon’s 60th birthday.* Vol. 76. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2007, pp. 649–696.

[KM82] W. Kirsch and F. Martinelli. “On the density of states of Schrödinger operators with a random potential”. In: *J. Phys. A* 15.7 (1982), pp. 2139–2156.

[LL01] E. H. Lieb and M. Loss. *Analysis.* Second. Vol. 14. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001, pp. xxii+346.

[MPR97] J. Magnen, G. Poirot, and V. Rivasseau. “The Anderson model as a matrix model”. In: vol. 58. Advanced quantum field theory (La Londe les Maures, 1996). 1997, pp. 149–162.

[MPR98] J. Magnen, G. Poirot, and V. Rivasseau. “Ward-type identities for the two-dimensional Anderson model at weak disorder”. In: *J. Statist. Phys.* 93.1-2 (1998), pp. 331–358.

[P69] L. P. “Exactly solvable model of electronic states in a three dimensional disordered hamiltonian”. In: *J. Physics (C)* 2 (1969), pp. 1717–1725.

[Pas73] L. A. Pastur. “Spectra of random selfadjoint operators”. In: *Russian Math. Surv.* 28 (1973), pp. 1–67.
[Poi99] G. Poirot. “Mean Green’s function of the Anderson model at weak disorder with an infra-red cut-off”. In: Ann. Inst. H. Poincaré Phys. Théor. 70.1 (1999), pp. 101–146.

[PW58] A. P.W. “Absence of diffusion in certain random lattices”. In: Phys. Rev. 109 (1958), pp. 1492–1505.

[RS72] M. Reed and B. Simon. Methods of modern mathematical physics. I. Functional analysis. Academic Press, New York-London, 1972, pp. xvii+325.

[RS75] M. Reed and B. Simon. Methods of modern mathematical physics. II. Fourier analysis, self-adjointness. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975, pp. xv+361.

[RS78] M. Reed and B. Simon. Methods of modern mathematical physics. IV. Analysis of operators. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978, pp. xv+396.

[Weg81] F. Wegner. “Bounds on the density of states in disordered systems”. In: Z. Phys. B44 (1981), pp. 9–15.