REGULARITY OF THE GLOBAL ATTRACTOR FOR A NONLINEAR SCHRÖDINGER EQUATION WITH A POINT DEFECT

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Abstract. We consider a nonlinear Schrödinger equation with a delta-function impurity at the origin of the space domain. We study the asymptotic behavior of the solutions with the theory of infinite dynamical system. We first prove the existence of a global attractor in $H^1_0(-1, 1)$. We also establish that this global attractor is a compact subset of $H^{2-\epsilon}(-1, 1)$.

1. Introduction. In this article we are interested with the long time behavior of solutions to a nonlinear Schrödinger equation with a spatially localized impurity at the origin

$$i u_t + \frac{1}{2} u_{xx} - \frac{1}{2} u + u \delta_0 + |u|^2 u + i\gamma u = f,$$

supplemented with initial data

$$u_0 \in H^1_0(-1, 1).$$

This equation models the interaction of a nonlinear Schrödinger wave with a spatially localized (point) defect at the origin through which it travels. Here the unknown $u = u(t, x)$ is a complex function defined on $\mathbb{R} \times ]-1, 1[$, $\delta_0$ is the Dirac measure at the origin.

We are given $\gamma > 0$ a damping parameter and an external force $f$ which does not depend on $t$ and which belongs to $L^2(-1, 1)$.

We plan to study the dynamical system provided by this equation into the infinite dimensional dynamical system framework. We begin with recalling some classical results. Consider first the case while the equation does not feature any impurity. In this case the equation reads

$$i u_t + \frac{1}{2} u_{xx} - \frac{1}{2} u + |u|^2 u + i\gamma u = f.$$  

This equation is a weakly damped forced version of the conservative nonlinear Schrödinger equation

$$i u_t + \frac{1}{2} u_{xx} - \frac{1}{2} u + |u|^2 u = 0,$$

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that is a focusing Schrödinger equation which has been extensively studied as a fundamental equation in modern mathematical physics [6].

When dissipation and external forcing terms are considered, numerous results about the equation (3) appear in the infinite dimension dynamical systems literature. Let us provide an overview of this literature. For the case where the space variable \( x \) belongs to a finite interval, with Dirichlet homogeneous boundary conditions for instance, the first result is due to J.-M. Ghidaglia [3]. This author has proved the existence of a global attractor for the weak topology of the energy space under consideration. It turns out that this weak global attractor is actually a global attractor in the usual sense, i.e for the strong topology, due to the famous J. Ball’s argument [7]. This result was improved in the case of periodic boundary condition by O. Goubet in [4] where it was proved that this global attractor is more regular than the energy space \( H^1_{\text{per}}(0,1) \). In fact, it is a compact subset of \( H^2_{\text{per}}(0,1) \). The idea of O. Goubet in [4] is to write the trajectories in the attractor in Fourier series and to decompose it into two parts, low frequency part denoted by \( y \) and high frequency part denoted by \( z \). The author proved that \( y \) is regular and the regularity of the solution depends essentially on the regularity of the high frequency part \( z \), then to decompose it as a sum of a function which goes to 0 and an other one which is in \( H^1_{\text{per}}(0,1) \).

Our aim in this article is to construct a global attractor for the solution to the equation (1) and to prove a regularity result of such attractor.

Our first result states as follow

**Theorem 1.1.** The dynamical system provided by (1) possesses a global attractor \( A \) in \( H^1_0(-1,1) \).

Because of the existence of the delta-function impurity in equation (1) we can not expect a regularity more than \( H^{3/2-\epsilon}(-1,1) \). Actually we shall prove in the sequel

**Theorem 1.2.** The global attractor is a compact subset of \( H^{3/2-\epsilon}(-1,1) \).

To prove this result of regularity of the global attractor we essentially follow the route of O. Goubet in [4], using the splitting of the solution in the global attractor into a low-frequency part and a high frequency part. In this article, since we are dealing with Dirichlet homogeneous conditions the use of the Fourier series, used by O. Goubet in [4] to expand the solution, is no longer adequate. So, we plan first to construct a basic of \( L^2(-1,1) \) of eigenfunctions of an unbounded operator \( A \) that we define in the sequel then we follow the original proof of O. Goubet. In fact, we expand the solution in the global attractor with respect to this basis and decompose it into a low frequency part that we prove a \( H^{3/2-\epsilon}(-1,1) \) regularity and a high frequency part that we write as a sum of a function going to 0 and an other one which is in the domain \( D(A) \) of the unbounded operator \( A \).

Our article is organized as follows. In Section 2 we give more preliminary results concerning the initial value problem and dissipativity. In Section 3, we prove Theorem 1.1. In the last section we prove Theorem 1.2.

In the sequel \( C \) denotes a numerical constant, that may vary from one line to one another. We also denote by \( K \) a constant that depends on the data of the equation like \( \gamma \), \( f \) for instance. We allow \( K \) to vary from one line to one another in the computations.

2. **Initial value problem and dissipativity.** In this section, we focus on the initial value problem related to the equation (1) and the existence of absorbing
sets. In a first subsection we describe some functional spaces to be used in the sequel.

2.1. **Mathematical framework.** The $L^2(-1,1)$ scalar product reads

$$
(u,v)_{L^2(-1,1)} = \Re \int_{-1}^{1} u(x)\overline{v}(x) \, dx.
$$

The Sobolev space $H^1_0(-1,1)$ is defined by

$$
H^1_0(-1,1) = \left\{ u \in L^2(-1,1); \, u_x \in L^2(-1,1), \, u(-1) = u(1) = 0 \right\},
$$
and it is an Hilbert space for the scalar product

$$
((u,v))_{H^1_0(-1,1)} = (u_x,v_x)_{L^2(-1,1)}.
$$

The corresponding duality brackets between $H^{-1}(-1,1)$ and $H^1_0(-1,1)$ is

$$
\langle \cdot, \cdot \rangle_{H^{-1}(-1,1),H^1_0(-1,1)}.
$$

We define $H^2\mathbb{R}(-1,1) \cap H^1_0(-1,1)$ by a simple interpolation between $H^1_0(-1,1)$ and $H^1(-1,1) \cap H^3(-1,1)$.

We now recall that $\delta_0$, the Dirac mass at the origin is defined by

$$
\text{For } u \in H^1_0(-1,1), \quad \langle u\delta_0, v \rangle = \Re \int_{-1}^{1} u(x)\overline{v}(x) \, dx = \Re \left( u(0)\overline{\pi}(0) \right).
$$

We now state some standard inequalities that will be useful in the sequel.

**Lemma 2.1.** For $u \in H^1_0(-1,1)$ we have

$$
|u(0)|^2 \leq \| u \|_{L^2(-1,1)} \| u_x \|_{L^2(-1,1)}.
$$

This is a consequence of the Agmon inequality, with the optimal constant, that reads

**Lemma 2.2 (Agmon Inequality).** For $u \in H^1_0(-1,1)$ we have

$$
\| u \|_{L^\infty(-1,1)} \leq \| u \|_{L^2(-1,1)} \| u_x \|_{L^2(-1,1)}.
$$

This can be used to prove the Gagliardo-Nirenberg inequality that is

**Lemma 2.3 (Gagliardo-Nirenberg Inequality).** For $u \in H^1_0(-1,1)$ we have

$$
\| u \|_{L^4(-1,1)} \leq \| u \|_{L^2(-1,1)} \| u_x \|_{L^2(-1,1)}.
$$

**Lemma 2.4.** $\forall \epsilon > 0$ we have $\delta_0 \in H^{-\frac{1}{2}-\epsilon}(-1,1)$. Moreover, for $u \in H^1_0(-1,1)$

$$
\| u\delta_0 \|_{H^{-\frac{1}{2}}(-1,1)} \leq \| \delta_0 \|_{H^{-\frac{3}{2}}(-1,1)} \| u \|_{H^2(-1,1)}.
$$

We now define the unbounded operator $A$ on $L^2(-1,1)$ as follows

$$
Au = -\Delta u + u,
$$

with domain

$$
D(A) = \left\{ u \in H^1_0(-1,1) \cap H^2((-1,1) - \{0\}); \, u_x(0^-) - u_x(0^+) = 2u(0) \right\}.
$$

**Remark 1.** We note that

$$
D(A^{\frac{1}{2}}) = H^1_0(-1,1).
$$

We define a norm on $D(A^{\frac{1}{2}})$ by

$$
\| u \|_{D(A^{\frac{1}{2}})}^2 = \| u \|_{H^1_0(-1,1)}^2 - 2|u(0)|^2.
$$
Lemma 2.5. It is known that \( D(A^2) = H_0^1(-1,1) \) is an algebra.

We have the following embedding result

Lemma 2.6. We have the embedding
\[
D(A) \subset H^\frac{1}{2} - \epsilon(-1,1), \quad \forall \epsilon > 0.
\]  

Proof. Let \( u \in D(A) \). Let \( v = u - u(0)\sigma \) where \( \sigma \) is the function defined on \((-1,1)\) by
\[
\sigma(x) = \max(0, 1 - |x|).
\]

On one hand, \( \sigma \in H^1(-1,1) \) since \( v \) is continuous in 0 and
\[
v'(0^+) = u'(0^+) + u(0) = v'(0^-) = u'(0^-) - u(0).
\]

On the other hand, \( \sigma \in H^\frac{1}{2} - \epsilon \) since the derivative \( \sigma' = -2\delta_0 \in H^{-\frac{1}{2} - \epsilon}(-1,1) \) (here the derivative is in \( D'(-1,1) \)).

The operator \( A \) defined by (12)-(13) is a self-adjoint and positive unbounded operator with domain \( D(A) \).

Consider \( r_0 > 0 \). We can easily prove that the operator \( (A + r_0 I) \) is a self-adjoint operator and has a compact inverse. So we can deduce that \( L^2(-1,1) \) has an Hilbert basis \( (e_k)_{k \geq 0} \) made of eigenfunctions of the unbounded operator \( A \).

\[
Ae_k = \lambda_k e_k, \quad k \geq 0,
\]  

such that
\[
\lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \lambda_{k+1} \leq \ldots
\]

We note that \( \lambda_k = 1 + \alpha_k \), where \((\alpha_k)_{k \geq 0}\) is the series of eigenvalues of the operator \(-\Delta\) with domain \( D(-\Delta) \) i.e.
\[
-\Delta e_k = \alpha_k e_k,
\]  

with
\[
D(-\Delta) = D(A) = \left\{ u \in H^1_0(-1,1) \cap H^2(-1,1) - \{0\}; \ u_x(0^-) - u_x(0^+) = 2u(0) \right\}.
\]

Proposition 1. The eigenfunctions \( (e_k) \) of the unbounded operator \( A \) are either even and read
\[
e_0(x) = 1 - |x|, \quad \lambda_0 = 1, \quad (20)
\]
\[
e_k(x) = \sin(\sqrt{(\lambda_k - 1)(1 - |x|)}), \quad \tan(\sqrt{\lambda_k - 1}) = \sqrt{\lambda_k - 1}, \quad \forall k \geq 1. \quad (21)
\]

or odd and defined by
\[
e_k(x) = \sqrt{2} \sin(k \pi x), \quad k \geq 1,
\]  

with the corresponding eigenvalues \( \lambda_k = 1 + k^2 \pi^2 \).

Proof of Proposition 1. We first prove that the eigenfunctions are either even or odd.

Let \( \lambda \) be an eigenvalue of the operator \( A = -\Delta - I \) and let \( u \) be an associated eigenfunction. It is clear that \( u \) satisfies
\[
\begin{cases}
-\Delta u + u = \lambda u, \quad x \in (-1,1), \\
u(-1) = u(1) = 0, \ u_x(0^-) - u_x(0^+) = 2u(0).
\end{cases}
\]

We have,
\[
\forall x \in (-1,1) \quad -\Delta u(x) + u(x) = \lambda u(x), \quad \text{and} \quad -\Delta u(-x) + u(-x) = \lambda u(-x). \quad (24)
\]

We decompose \( u \) as follows
\[
u = u_1 + u_2,
\]
where \( u_1 \) is even and \( u_2 \) is odd. Then, we derive from (24) that
\[
\begin{align*}
-\Delta u_1(x) + \Delta u_2(x) + u_1(x) - u_2(x) &= \lambda u_1(x) - \lambda u_2(x), \\
-\Delta u_1(x) - \Delta u_2(x) + u_1(x) + u_2(x) &= \lambda u_1(x) + \lambda u_2(x).
\end{align*}
\]
(25)

So, \( u_1 \) and \( u_2 \) are two eigenfunctions associated to the same eigenvalue \( \lambda \). Hence,
\[
\begin{align*}
u_1 &= \alpha u, \\
u_2 &= \beta u.
\end{align*}
\]
Then,
\[
u = (\alpha + \beta)u_1, u_1 = \alpha u.
\]
So, \( \alpha + \beta = 1 \) and \( u_1 = \alpha u \).
\[
\begin{align*}
\text{On the one hand if } \alpha = 0 \text{ then } \beta = 1 \text{ and then the eigenfunction } u \text{ is even. On} \\
\text{the other hand, if } \alpha \neq 0 \text{ then we deduce from (26) that } \alpha = 1 \text{ and } \beta = 0. \text{ In this} \\
\text{case, the function } u \text{ is odd.}
\end{align*}
\]

Now, we give the expression of the eigenfunctions of the operator \( A \). Let \( \lambda \geq 0 \) an eigenvalue of the operator \( A \) with an associated eigenfunction \( e \).

In the case when \( e \) is even we have
\[
\begin{align*}
\begin{cases}
-\Delta e = (\lambda - 1)e, & \text{on } [0, 1], \\
e_x(0^+) + e(0) = 0, & e(1) = 0.
\end{cases}
\end{align*}
\]
(27)

We easily infer from (27) that for \( \lambda = 1 \) we have \( e(x) = 1 - |x| \) and if \( \lambda \geq 1 \) the even eigenfunctions \( e_k \) are given by
\[
\begin{align*}
\forall k \geq 1, \quad e_k(x) = \frac{-C}{\cos(\sqrt{\lambda_k - 1})} \sin(\sqrt{\lambda_k - 1} (1 - |x|)),
\end{align*}
\]
with the corresponding eigenvalues \( \lambda_k \) satifing
\[
\tan(\sqrt{\lambda_k - 1}) = (\sqrt{\lambda_k - 1}).
\]
(29)

When \( e \) is an odd eigenfunction of the unbounded operator \( A \) then \( e \) is a solution of the following problem
\[
\begin{align*}
\begin{cases}
-\Delta e = (\lambda - 1)e, & x \in [0, 1], \\
e(0) = 0, & u_x(0^-) = u_x(0^+), & u(1) = 0.
\end{cases}
\end{align*}
\]
(30)

It is clear that the solutions of the (30) are given by
\[
\tilde{e}_k = \sin(k\pi x), \quad k \geq 1
\]
and then the corresponding eigenvalues are \( \tilde{\lambda}_k = 1 + k^2 \pi^2 \). \( \square \)

**Remark 2.** From the expressions of the eigenfunctions \( (e_k)_k \) of the operator \( A \) given in Proposition 1 we infer the following embedding
\[
(e_k)_{k \geq 1} \subset D(A).
\]
(31)

Moreover, the corresponding eigenvalues \( (\lambda_k)_k \) satisfy
\[
\lambda_{2k} = 1 + k^2 \pi^2,
\]
(32)

\[
1 + \left(\frac{(2k + 1)^2 \pi^2}{16}\right) \leq \lambda_{2k+1} \leq 1 + \left(\frac{(2k + 1)^2 \pi^2}{4}\right).
\]
(33)

Then,
\[
\forall k \geq 1, \quad \lambda_k \geq k^2.
\]
(34)
2.2. Solving the initial value problem. We state

**Proposition 2.** Consider an initial data \( u_0 \in H^1_0(-1,1) \). There exists a unique solution for \((1)\)

\[
u(t) \in C_b([0,\infty), H^1_0(-1,1)).
\]

Moreover, the mapping \( u_0 \rightarrow \nu(t) \) is continuous for the \( H^1_0(-1,1) \) strong topology.

Sketch of the proof

For the existence we apply the Faedo-Galerkin method. We use \((e_n)_{n \geq 0}\) the Hilbert basis of \( L^2(-1,1) \) made of eigenfunctions of the operator \( A \) with domain \( D(A) \) and we define the finite dimensional space \( V_m \) spanned by \( e_1, \ldots, e_m \).

Then we construct a sequence of approximation \( \nu_m \in V_m \) solving the approximate equation that reads

\[
\int_{-1}^{1} \left( i \nu_{tm} + \frac{1}{2} u_{tx} - \frac{1}{2} u' + u'' \delta_0 + |u'^2 + i \gamma u'' - f \right) \nu_m \, dx = 0, \tag{35}
\]

for any \( v_m \in V_m \).

Then we prove some a priori estimates and we pass to the limit using essentially that the embedding \( H^1_0(-1,1) \subset L^2(-1,1) \) is compact. This is standard and we skip the details for the sake of conciseness.

We now seek, for later use, absorbing sets. Let us state

**Proposition 3.** There exists \( M_1 \) which depends on the data \( f, \gamma \) such that for any \( R > 0 \) and any \( u_0 \in H^1_0(-1,1) \) with \( \| u_0 \|_{H^1_0(-1,1)} \leq R \), then there exists \( T = T(R) \) such that the solution \( \nu(t) \) of \((1)\) issued from \( u_0 \) satisfied

\[
\| \nu(t) \|^2_{H^1_0(-1,1)} \leq M_1^2, \quad \forall t \geq T(R). \tag{36}
\]

**Proof of Proposition 3.** We first obtain an absorbing ball in \( L^2(-1,1) \), then we find an absorbing ball in \( H^1_0(-1,1) \).

We multiply \((1)\) by \( \overline{\nu} \) and we integrate on \((-1,1)\) the imaginary part of the resulting equation to obtain

\[
\frac{1}{2} \frac{d}{dt} \| \nu(t) \|_{L^2(-1,1)}^2 + \gamma \| \nu(t) \|_{L^2(-1,1)}^2 = \text{Im} \int_{-1}^{1} f \overline{\nu} \, dx \tag{37}
\]

\[
\| \nu(t) \|_{L^2(-1,1)}^2 \leq \| u_0 \|_{L^2(-1,1)}^2 e^{-\gamma t} + \frac{\| f \|_{L^2(-1,1)}^2}{\gamma^2} (1 - e^{-\gamma t}).
\]

So that, for \( t \geq T_0(u_0) = \frac{2}{\gamma} \ln \frac{\gamma \| u_0 \|_{L^2(-1,1)}^2}{\| f \|_{L^2(-1,1)}^2}, \)

\[
\| \nu(t) \|_{L^2(-1,1)}^2 \leq 2 \frac{\| f \|_{L^2(-1,1)}^2}{\gamma^2}. \tag{38}
\]

Without loss of generality (up to a time translation) we may assume in the sequel that \((38)\) is valid for any \( t \geq 0 \).
Next we multiply (1) by $\overline{u}_t + \gamma u$ and we integrate the real part of the resulting identity on $(-1,1)$ to obtain
\[
\frac{1}{2} \frac{d}{dt} \left[ \|u_x(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 - 2|u(t,0)|^2 - \|u(t)\|_{L^4}^4 + 4\Re \int_{-1}^1 f\overline{u} \, dx \right] + \gamma \left[ \|u_x(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 - 2|u(t,0)|^2 - \|u(t)\|_{L^4}^4 + 4\Re \int_{-1}^1 f\overline{u} \, dx \right]
= \gamma \|u(t)\|_{L^4}^4 + 2\gamma \Re \int_{-1}^1 f\overline{u} \, dx,
\]
that we write as
\[
\frac{1}{2} \frac{d}{dt} \Phi(u(t)) + \gamma \Phi(u(t)) = \gamma \|u(t)\|_{L^4}^4 + 2\gamma \Re \int_{-1}^1 f\overline{u} \, dx, \tag{40}
\]
where we have set
\[
\Phi(u(t)) = \|u_x(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 - 2|u(t,0)|^2 - \|u(t)\|_{L^4}^4 + 4\Re \int_{-1}^1 f\overline{u} \, dx. \tag{41}
\]

To begin with, using the inequality (8) and Gagliardo-Nirenberg inequality (10) we get
\[
\Phi(u) \geq \frac{1}{2} \|u_x(t)\|_{L^2}^2 - K. \tag{42}
\]

Now we will provide an upper bound for the right hand side of (41). Since $u(t); t \geq 0$ remains in the $L^2(-1,1)$ absorbing ball, we obtain using (38)
\[
\forall \ t \geq 0, \quad 2\gamma \Re \int_{-1}^1 f\overline{u} \, dx \leq 3\|f\|_{L^2}^2.
\]

We also have using Gagliardo-Nirenberg inequality (10)
\[
\gamma \|u(t)\|_{L^4(-1,1)}^4 \leq \frac{\gamma}{4} \|u_x(t)\|_{L^2}^2 + K.
\]

Therefore
\[
\frac{1}{2} \frac{d}{dt} \Phi(u) + \gamma \Phi(u) \leq \frac{\gamma}{4} \|u_x(t)\|_{L^2}^2 + K \leq \frac{\gamma}{2} \Phi(u) + K. \tag{43}
\]

So that we have a uniform bound for $u$ in $H^1_0(-1,1)$ by the Gronwall lemma. \hfill \Box

**Remark 3.** We infer from the equation (1) and the inequality (36) in Proposition 3 that for $t \geq T(R)$ we have
\[
\|u_\tau\|_{H^{-1}} \leq M'. \tag{44}
\]

3. **Existence of the global attractor.** In this section, we prove Theorem 1.1. The proof is very similar to the original proof of J. M.-Ghidaglia [3] for the NLS equation without impurity.

We first prove the existence of a weak global attractor for the dynamical system in $H^1_0(-1,1)$ restricted to the absorbing set. Then we prove that this weak global attractor is in fact a global attractor for the strong topology in $H^1_0(-1,1)$ using the famous J. Ball argument [1]. This material is very standard and then can be omitted.
4. **Regularity of the global attractor.** In this section we prove a regularity result that compares with the analog result in [4] for the NLS cubic equation without impurity. We try to mimic the proof in [4]. So that we begin with introducing the splitting using the orthonormal basis \((e_k)_k\) in Proposition 1, the analog of the Fourier basis used in [4], that we will use to expand the solution \(u(t)\) in the attractor.

4.1. **Introducing the splitting.** Let \(u(t)\) be the solution of (1)-(2). We expand \(u(t)\) with respect to \((e_k)_k\), the orthonormal basis of eigenfunctions of the unbounded operator \(A\), that reads
\[
u(t) = \sum_{k=0}^{\infty} u_k e_k(x).
\]

On the one hand, for a given level \(N\), the low frequency part of \(u\) reads
\[
y(t) = \sum_{k=0}^{N} u_k(t)e_k(x) = P_N u,
\]
is a function with regularity \(D(A)\), with respect to \(x\). Then to deduce the regularity of \(u\) with respect to \(x\) we should study the regularity of its high frequency part that is
\[
z(t) = \sum_{k>N} u_k(t)e_k(x) = Q_N u,
\]
where \(Q_N\) denotes the orthogonal projector onto
\[
Q_N D(A^{\frac{1}{2}}) = \left\{ z = \sum_{k>N} u_k(t)e_k; \ z \in D(A^{\frac{1}{2}}) \right\}.
\]

Now, we state a result that will be so useful in the splitting

**Lemma 4.1.** The unbounded operator \(A\) commutes with the projection operators \(P_N\) and \(Q_N\).

Due to Lemma 4.1 we notice that the high frequency part \(z\) is the solution of the nonautonomous partial differential equation
\[
iz_t - \frac{1}{2}Az + i\gamma z + Q_N \left( |y + z|^2(y + z) \right) = Q_N f,
\]
with the initial condition
\[
z(0) = Q_N u_0 = z_0.
\]

Since we are interested in the long time behavior of \(z(t)\), we may focus on \(z(t)\) for \(t \geq T(R)\), \(T(R)\) being as in Proposition 3.

Hence, \(z\) is solution (for \(t \geq T(R)\)) of equation (49) and of the “initial” condition
\[
z(T(R)) = Q_N u(T(R)).
\]

We now introduce \(Z: [T(R),+\infty[ \rightarrow Q_N D(A^{\frac{1}{2}})\) that is solution of
\[
\begin{cases}
iz_t - \frac{1}{2}AZ + i\gamma Z + Q_N \left( |y + Z|^2(y + Z) \right) = Q_N f, \\
Z(T(R)) = 0,
\end{cases}
\]
here \(y = P_N u = (Id - Q_N) u\) as above.

The existence result for \(Z\) will be proven later in the sequel.

Actually we shall prove for a given \(N\) large enough, depending on the data of the equation, that \(Z(t)\) exists for all \(t \geq T(R)\) and takes values in \(Q_N D(A^{\alpha})\) for \(\alpha = \frac{1}{2}\) and \(\alpha = 1\).
Then $z$ splits into

$$z = Z + (z - Z).$$

(52)

### 4.2. Existence result for $Z$.

**Remark 4.** For the sake of convenience, we assume in this section that $T(R) = 0$ in Proposition 3. Hence, $u(t)$ remains in the absorbing ball in $H^1_0(-1, 1)$ whose radius is $M_1$.

Let $N$ be fixed large enough, which will be specified later. Let $y(t) = P_N u(t)$ as above. Let $m > N$. Let $Z^m(t)$ be the solution in

$$P_m Q_N D(A^{\frac{1}{2}}) = \{ z = \sum_{k=0}^{m} u_k(t) e_k; Z \in D(A^{\frac{1}{2}}) \},$$

(53)

for the nonautonomous O. D. E

$$\left\{ \begin{array}{l}
\frac{dZ^m}{dt} - \frac{1}{2} AZ^m + P_m Q_N \left( |y + Z^m|^2(y + Z^m) \right) + i \gamma Z^m = P_m Q_N f, \\
Z^m(0) = 0.
\end{array} \right.$$

(54)

Using the Cauchy-Lipshitz Theorem, $Z^m$ exists as an application from $[0, T_m]$ into $P_m Q_N D(A^{\frac{1}{2}})$. We shall prove below a priori estimates that show that in fact $T_m = +\infty$ and that allow us to let $m$ go to infinity.

**Proposition 4.** There exists $N_0 = N_0(\gamma, f)$, such that for a given $N \geq N_0$, the solution $Z^m$ of (51) satisfies

$$\sup_{t \geq 0} \| Z^m(t) \|_{D(A^{\frac{1}{2}})} \leq K_1,$$

(55)

where $K_1$ is a constant that depends on $\gamma$ and $f$.

**Proof of Proposition 4.**

**Remark 5.** For the sake of convenience, we drop in the Proof the subscript $m$ to write $Z^m = Z$, $v = v^m = y + Z^m$. This will not introduce any confusion.

Multiplying the equation (51) by $-(\bar{Z} t + \gamma Z)$, taking the real part and integrating on $(-1, 1)$, we obtain

$$\frac{1}{2} \frac{d}{dt} J(Z) + \gamma J(Z) = K(Z),$$

(56)

where

$$J(Z) = \| Z \|_{D(A^{\frac{1}{2}})}^2 - 2 \int_{-1}^{1} |y|^2 |Z|^2 \ dx - 4 \Re \int_{-1}^{1} |y|^2 y \bar{Z} \ dx + 4 \Re \int_{-1}^{1} f \bar{Z} \ dx$$

$$- 4 \Re \int_{-1}^{1} |Z|^2 \bar{Z} \ dx - \int_{-1}^{1} |Z|^4 \ dx - 4 \int_{-1}^{1} \left( \Re (y \bar{Z}) \right)^2 \ dx,$$

(57)

and

$$K(Z) = - 2 \gamma \Re \int_{-1}^{1} \bar{Z} \ dx + \| Z \|_{L^4}^4 - 2 \gamma \Re \int_{-1}^{1} |y|^2 y \bar{Z} \ dx + 2 \gamma \Re \int_{-1}^{1} |Z|^2 \bar{Z} \ dx$$

$$- 2 \Re \int_{-1}^{1} \left( |y|^2 \right) \bar{Z} \ dx - 2 \int_{-1}^{1} \Re (y \bar{Z}) |Z|^2 \ dx - 2 \Re \int_{-1}^{1} |Z|^2 \bar{Z} \ dx$$

$$- 4 \int_{-1}^{1} \Re (\bar{Z} t \bar{Z}) \Re (\bar{Z} \bar{Z}) \ dx.$$  

(58)

We derive now from (57) a lower bound for the functional $J$.
Lemma 4.2. There exists $K = K(\gamma, f, M_1)$ such that for any $t \geq 0$
\[ J(Z) \geq \|Z\|_{D(A^{1/2})}^2 - 2\|Z\|_{L^4}^2 - K. \] (59)

Proof: On the one hand, using Hölder and Young inequalities we obtain
\[ J(Z) \geq \|Z\|_{D(A^{1/2})}^2 - 6\|y\|_{L^4}^2\|Z\|_{L^4}^2 - 4\|y\|_{L^4}^3\|Z\|_{L^4}^3 - \frac{4}{9}\|f\|_{L^4}^4\|Z\|_{L^4}^4 - \|Z\|_{L^4}^4 \] (60)

where $K$ is a constant that depends on the data of the equation, as $M_1$ for instance.

Now, an upper bound for $K(Z)$ is requested. For that purpose we establish an improved version of Poincaré inequality that reads

Lemma 4.3. For all $Z \in Q_N D(A^{1/2})$
\[ \|Z\|_{L^2} \leq \frac{1}{N} \|Z\|_{D(A^{1/2})}. \] (62)

Proof: Let $Z \in Q_N D(A^{1/2})$.
\[ \|A^{1/2}Z\|_{L^2} = \| \sum_{k>N} u_k A^{1/2} e_k(x)\|_{L^2} = \| \sum_{k>N} u_k \lambda_k^{1/2} e_k(x)\|_{L^2}, \] (63)

where $(e_k)$ is the Hilbert basis of the eigenfunctions of the operator $A$ defined in Proposition 1.

We recall that the corresponding eigenvalues $\lambda_k$ are supposed to be non decreasing then
\[ \|A^{1/2}Z\|_{L^2} \geq \lambda_N \| \sum_{k>N} u_k e_k(x)\|_{L^2} \geq \lambda_N \|Z\|_{L^2}. \] (64)

Now using the inequality (34) we get
\[ \|A^{1/2}Z\|_{L^2} \geq N\|Z\|_{L^2} \]

We also state another non-standard inequality

Lemma 4.4. For all $Z \in Q_N D(A^{1/2})$
\[ \|Z\|_{L^2} \leq C(1 + \frac{1}{N^2}) \|Z\|_{D(A^{1/2})}^2. \] (65)

Proof. Let $Z \in Q_N D(A^{1/2})$.
Using the definition of the norme in $D(A^{1/2})$ and the Agmon inequality (9), we have
\[ \|Z\|_{H^1}^2 \leq \|Z\|_{D(A^{1/2})}^2 + 2\|Z\|_{L^2} \|Z\|_{L^2} \]
\[ \leq \|Z\|_{D(A^{1/2})}^2 + 2\|Z\|_{L^2}^2 + \frac{1}{2} \|Z\|_{L^2}^2. \] (66)

Then the inequality (65) holds immediately using Lemma 4.3. □
We now establish an upper bound for the functional $K$ that reads

**Lemma 4.5.** For $Z \in Q_N D(A^{\frac{1}{2}})$

$$K(Z) \leq \frac{K}{N^2} \|Z\|_{D(A^{\frac{1}{2}})}^4 + K,$$

(67)

where $K$ is a constant that depends on the data of the equation.

**Proof:** We give an upper bound for the right hand side of (56) as

$$K(Z) \leq 2\gamma \|f\|_{L^\infty} \|Z\|_{L^4} + \gamma \|Z\|_{L^4}^2 + 2\gamma \|y\|_{L^6} \|Z\|_{L^4} + 2\gamma \|y\|_{L^4} \|Z\|_{L^4}^3$$

$$+ 2\|y_t\|_{D(A^{-\frac{1}{2}})} \|y\|_{D(A^{\frac{1}{2}})}^2 + 4\|y_t\|_{D(A^{-\frac{1}{2}})} \|\text{Re}(\bar{y} Z)\|_{D(A^{\frac{1}{2}})}$$

$$+ 2\|y_t\|_{D(A^{-\frac{1}{2}})} \|\bar{y} Z\|_{D(A^{\frac{1}{2}})}^2 + 2\|y_t\|_{D(A^{-\frac{1}{2}})} \|Z\|_{D(A^{\frac{1}{2}})}^2$$

$$+ 4\|y_t\|_{D(A^{-\frac{1}{2}})} \|Z\|_{D(A^{\frac{1}{2}})} \|\text{Re}(\bar{y} Z)\|_{D(A^{\frac{1}{2}})}.$$  

(68)

Let us majorize the third term in the right hand side of (68). On the one hand, due to (44)

$$\|y_t\|_{D(A^{-\frac{1}{2}})} \leq \|u_t\|_{D(A^{-\frac{1}{2}})} \leq M'.$$

(69)

On the other hand,

$$\|y_t\|_{D(A^{-\frac{1}{2}})} \leq C \|y\|_{L^\infty} \|Z\|_{D(A^{\frac{1}{2}})} + C \|y\|_{L^\infty} \|Z\|_{L^\infty} \|Z\|_{D(A^{\frac{1}{2}})}.$$  

(70)

Using then the embedding $D(A^{\frac{1}{2}}) \subset L^\infty$ and (69)-(70), we have

$$\|y_t\|_{D(A^{-\frac{1}{2}})} \|y\|_{D(A^{\frac{1}{2}})} \|Z\|_{D(A^{\frac{1}{2}})} \leq C \|y_t\|_{D(A^{-\frac{1}{2}})} \|y\|_{D(A^{\frac{1}{2}})} \|Z\|_{D(A^{\frac{1}{2}})} \leq K \|Z\|_{D(A^{\frac{1}{2}})}$$

$$\leq \frac{\gamma}{4} \|Z\|_{D(A^{\frac{1}{2}})}^2 + K.$$  

(71)

Similarly, we handle the eighth term of the right hand side of (68) as follows

$$\|y_t\|_{D(A^{-\frac{1}{2}})} \|Z\|_{D(A^{\frac{1}{2}})} \leq C \|y_t\|_{D(A^{-\frac{1}{2}})} \|Z\|_{L^\infty} \|Z\|_{D(A^{\frac{1}{2}})} \leq \frac{K}{N} \|Z\|_{D(A^{\frac{1}{2}})}^3,$$

(72)

using the inequalities (62)-(65) that hold on $Q_N D(A^{\frac{1}{2}})$.

$$\|Z\|_{L^\infty} \leq \|Z\|_{L^2} \|Z_x\|_{L^2} \leq \frac{C}{\sqrt{N}} \|Z\|_{D(A^{\frac{1}{2}})}.$$  

(73)

The last term in the right hand side of (68) can be majorized also in the same way:

$$4\|y_t\|_{D(A^{-\frac{1}{2}})} \|Z\|_{D(A^{\frac{1}{2}})} \|\text{Re}(\bar{y} Z)\|_{D(A^{\frac{1}{2}})} \leq C \|y_t\|_{D(A^{-\frac{1}{2}})} \|y\|_{L^\infty} \|Z\|_{L^\infty} \|Z\|_{D(A^{\frac{1}{2}})}$$

$$\leq \frac{K}{\sqrt{N}} \|Z\|_{D(A^{\frac{1}{2}})}^2.$$  

(74)

From all those computations, we infer

$$K(Z) \leq \frac{\gamma}{4} \|Z\|_{D(A^{\frac{1}{2}})}^2 + \frac{K}{\sqrt{N}} \|Z\|_{D(A^{\frac{1}{2}})}^2 + 2\gamma \|Z\|_{L^4}^2 + \frac{K'}{N} \|Z\|_{D(A^{\frac{1}{2}})}^3 + K''',$$

(75)

where $K, K', K'''$ are as in the Introduction.

We derive from Gagliardo-Nirenberg inequality and from the inequalities (62)-(65) that hold on $Q_N D(A^{\frac{1}{2}})$

$$\|Z\|_{L^4}^4 \leq C \|Z\|_{L^2}^2 \|Z_x\|_{L^2} \leq \frac{C}{N^3} \|Z\|_{D(A^{\frac{1}{2}})}^4,$$  

(76)
and
\[
\frac{K'}{N} \|Z\|_{D(A^{\frac{1}{2}})}^3 \leq \frac{\gamma}{8} \|Z\|_{D(A^{\frac{1}{2}})}^2 + \frac{K}{N^2} \|Z\|_{D(A^{\frac{1}{2}})}^4.
\]  
(77)

So, we deduce from (77) and (75) that
\[
K(Z) \leq \left( \frac{3\gamma}{8} + \frac{K}{N} \right) \|Z\|_{D(A^{\frac{1}{2}})}^2 + \frac{K'}{N^2} \|Z\|_{D(A^{\frac{1}{2}})}^4 + K''.
\]  
(78)

At this stage appears the first condition on $N_0$ (and on $N$). Let us assume in the sequel that
\[
\frac{K}{\sqrt{N}} \leq \frac{K}{\sqrt{N_0}} \leq \frac{\gamma}{8},
\]  
(79)

where the constant $K$ is as in (78). Then
\[
K(Z) \leq \frac{\gamma}{2} \|Z\|_{D(A^{\frac{1}{2}})}^2 + \frac{K}{N^2} \|Z\|_{D(A^{\frac{1}{2}})}^4 + K'.
\]  
(80)

We finally infer from (68), (56), (59), (76) and from (80) that
\[
\frac{1}{2} \frac{d}{dt} J(Z) + \gamma J(Z) \leq \frac{\gamma}{2} J(Z) + \frac{K}{N^2} \|Z\|_{D(A^{\frac{1}{2}})}^4 + K' \leq \frac{\gamma}{2} J(Z) + \gamma \|Z\|_{L^4}^4 + \frac{K}{N^2} \|Z\|_{D(A^{\frac{1}{2}})}^4 + K'.
\]  
(81)

Due to (76) we get
\[
\frac{d}{dt} J(Z) + \gamma J(Z) \leq \frac{K}{N^2} \|Z\|_{D(A^{\frac{1}{2}})}^4 + K',
\]  
(82)

that we integrate for $t$ between 0 and $T$ (observing that $J(Z(0)) = J(0) = 0$) to obtain
\[
J(Z)e^{\gamma t} \leq \frac{K}{N^2} \int_0^t \|Z(s)\|_{D(A^{\frac{1}{2}})}^4 e^{\gamma s} ds + \frac{K'}{\gamma}.
\]  
(83)

We then easily deduce from (56), (83) and (76) that
\[
\|Z\|_{D(A^{\frac{1}{2}})}^2 \leq \frac{K}{N^2} \|Z\|_{D(A^{\frac{1}{2}})}^4 + \frac{K}{N^2} \int_0^t \|Z\|_{D(A^{\frac{1}{2}})}^4 e^{\gamma (t-s)} ds + K'.
\]  
(84)

Let us introduce
\[
\xi(t) = \sup_{[0,1]} \|Z(s)\|_{D(A^{\frac{1}{2}})}^2.
\]  
(85)

The function $\xi$ is a continuous function that satisfies
\[
\xi(0) = 0.
\]  
(86)

We infer from (84) that $\xi(t)$ satisfies
\[
\xi(t) \leq \frac{K}{N^2} \xi^2(t) + K',
\]  
(87)

where $K$ and $K'$ are as in the Introduction. Let us also set
\[
\phi(\xi) = \xi - \frac{K}{N^2} \xi^2 - K'.
\]  
(88)

Let us assume that $1 - \frac{4KK'N^2}{N^2} > 0$, i.e.
\[
\frac{4KK'}{N^2} \leq \frac{4KK'}{N_0^2} < 1.
\]  
(89)

Here it is the second assumption on $N_0$. 

Therefore, since \( \phi(\xi(0)) = \phi(0) < 0 \) and \( t \mapsto \xi(t) \) is a continuous nonnegative function of \( t \) then we have
\[
\xi(t) = \sup_{[0,t]} \|Z(t)\|_{D(A^{\frac{1}{2}})}^2 \leq 2K',
\]
where \( K' \) is as above. Therefore the proof of Proposition 4 is completed. \( \Box \)

**Corollary 1.** Let \( N_0 \) be as in Proposition 4. Let \( N \geq N_0 \) be fixed. Then there exists \( Z(t) \in C_b\left(\mathbb{R}_+; Q_N D(A^{\frac{1}{2}})\right) \) that solves
\[
\begin{cases}
iZ_t + AZ + Q_N \left(|y + Z|^2(y + Z)\right) + i\gamma Z = Q_N f, \\
Z(0) = 0.
\end{cases}
\]

**Remark 6.** Actually \( Z \) satisfies (90)

**Proof:** The proof is classical and left as an exercise to the reader. We just have to pass to the limit \( m \mapsto +\infty \) in (54) and to use Proposition 4. Moreover, since the space variable is in bounded domain, strong convergence results may be used.

4.3. **A priori estimates in** \( D(A) \). We shall prove the following

**Proposition 5.** Let \( N_0 \) be as in Proposition 4. Let \( N \geq N_0 \) be fixed. There exists \( K_2 \) that depends on \( N \) such that the solution \( Z \) of the problem (91) satisfies
\[
\sup_{t \geq 0} \|Z(t)\|_{D(A)} \leq K_2.
\]

**Proof of Proposition 5.** We shall first state and prove the following

**Lemma 4.6.** Let \( y_0 \in P_N D(A^{\frac{1}{2}}) \) such that \( \|y_0\|_{D(A^{\frac{1}{2}})} \leq M_1 \). There exists a unique solution \( Z_0 \) in \( Q_N D(A^{\frac{1}{2}}) \) such that
\[
\|Z_0\|_{D(A^{\frac{1}{2}})} \leq M_1,
\]
and
\[
- \frac{1}{2} AZ_0 + i\gamma Z_0 + Q_N \left(|y_0 + Z_0|^2(y_0 + Z_0)\right) = Q_N(f).
\]

Moreover,
\[
\|Z_0\|_{D(A)} \leq 3\|f\|_{L^2(-1,1)}.
\]

**Proof of Lemma 4.6:** We shall use the fixed point theorem. Let us define a function \( Z \) in \( Q_N D(A^{\frac{1}{2}}) \) as follows

For all \( Z \in Q_N D(A^{\frac{1}{2}}), \quad Z(Z) = A^{-1} \left(2Q_N(|y_0 + Z|^2(y_0 + Z)) + 2i\gamma Z - 2Q_N(f)\right).\)

Let \( Z \in D(A^{\frac{1}{2}}) \) such that \( \|Z\|_{D(A^{\frac{1}{2}})} \leq M_1 \). We have
\[
\|Z\|_{D(A^{\frac{1}{2}})} \leq 2\|A^{-1}Q_N\|_{D(A^{\frac{1}{2}})} \|y_0 + Z\|^3_{D(A^{\frac{1}{2}})} + 2\gamma\|A^{-1}Z\|_{D(A^{\frac{1}{2}})} + 2\|A^{-1}Q_N\|\|f\|_{L^2}
\]
\[
\leq \frac{2}{\lambda_N} M_1^3 + \frac{2\gamma M_1}{\lambda_N} + \frac{2}{\sqrt{\lambda_N}} \|f\|_{L^2}.
\]

Since \( (\lambda_n)_n \) consists of nonnegative real numbers that go to infinity when \( n \) goes to infinity we observe that the right hand side of (97) goes to 0 at infinity. Then,
\[
\text{for } N \text{ large enough } \quad \|Z\|_{D(A^{\frac{1}{2}})} \leq M_1.
\]
Similarly, we prove also that $Z$ is strictly contracting. Using Banach fixed point theorem we deduce the existence of a unique point $Z_0 \in B_{Q_N D(A^{\frac{1}{2}})}(0, M_1)$ such that $Z(Z_0) = Z_0$.

On the other hand
\begin{equation}
\|Z(Z)\|_{D(A)} \leq 2\|A^{-1}Q_N\|\|y_0 + Z\|_{D(A^{\frac{1}{2}})}^3 + 2\gamma\|A^{-1}Q_N\|\|Z\|_{D(A^{\frac{1}{2}})}^3 + 2\|f\|_{L^2(-1,1)}.
\end{equation}
(99)

Then, we obtain
\begin{equation}
\|Z\|_{D(A)} \leq 3\|f\|_{L^2(-1,1)}.
\end{equation}
(100)

Remark 7. Let $Z_0$ be as in Lemma 4.6.

The problem
\begin{equation}
\begin{aligned}
&iZ_t - \frac{1}{2}AZ + i\gamma Z + Q\left(|y + Z|^2(y + Z)\right) = Qf, \\
&Z(0) = 0,
\end{aligned}
\end{equation}
(101)
has a unique solution which is bounded in $D(A^{\frac{1}{2}})$. In fact, the proof is very similar to the proof of the existence result to the problem (91) in Corollary 1.

We observe that $Z_t(0) = 0$.

Let us differentiate (101) with respect to $t$. Let us set
\begin{equation}
F(v) = f - |v|^2v.
\end{equation}
(102)

We obtain that $w = Z_t$ satisfies
\begin{equation}
iw_t - \frac{1}{2}Aw + i\gamma w - Q_N F'(v)w = Q_N F'(v)y_t,
\end{equation}
(103)
where,
\begin{align*}
F'(v)w &= -|v|^2w - 2\Re(\bar{v}w)v, \\
F'(v)y_t &= -|v|^2y_t - 2\Re(\bar{v}y_t)v.
\end{align*}

We now consider the following problem
\begin{equation}
\begin{aligned}
iw_t - \frac{1}{2}Q_N Aw &= i\gamma w - Q_N F'(v)w = G, \\
w(0) &= 0,
\end{aligned}
\end{equation}
(104)
where we have set
\begin{equation}
F'(v)w = -|v|^2w - 2\Re(\bar{v}w)v
\end{equation}
(105)
and
\begin{equation}
G = Q_N \left(-|v|^2y_t - 2\Re(\bar{v}y_t)v\right).
\end{equation}
(106)

We now state a result that will be used in the sequel

Proposition 6. Let $v$ be in $C_b\left(\mathbb{R}^+_+; D(A^{\frac{1}{2}})\right) \cap C^1_b\left(\mathbb{R}^+_+; D(A^{-\frac{1}{2}})\right)$ such that
\begin{equation}
\sup_{t \geq 0} \left(\|v\|_{D(A^{-\frac{1}{2}})} + \|v_t\|_{D(A^{-\frac{1}{2}})}\right) \leq K_0.
\end{equation}
(107)
Let $G$ be in $C_b\left(\mathbb{R}^+; D(A^{-\frac{1}{2}})\right)$. Then, there exists $N_0$ that depends on $K$ (but that is independent of $G$) such that for each $N \geq N_0$ fixed, the solution $w$ of (104) belongs to $C_b\left(\mathbb{R}^+; QD(A^{\frac{1}{2}})\right)$ and satisfies
\[
\sup_{t \geq 0} \|w(t)\|_{D(A^{\frac{1}{2}})} \leq C \sup_{t \geq 0} \left(\|G(t)\|_{D(A^{-\frac{1}{2}})} + \frac{1}{\gamma} \|G(t)\|_{D(A^{-\frac{1}{2}})}\right). \tag{108}
\]

**Proof of Proposition 6.** We first state a lemma that asserts the coercivity of the linear operator $Q_N(A + 2F'(v))$ sur $Q_N D(A^{\frac{1}{2}})$.

**Lemma 4.7.** There exists $N_0$ that depends of $K_0$ such that for $N \geq N_0$ fixed we have
\[
\left( A + 2F'(v)Z, Z \right)_{L^2(-1,1)} \geq \frac{3}{4} \|Z\|_{D(A^{\frac{1}{2}})}. \tag{109}
\]

**Proof of Lemma 4.7:** Let us first write
\[
\left( (A + 2F'(v))Z, Z \right)_{L^2(-1,1)} = \|Z\|_{D(A^{\frac{1}{2}})}^2 - 2 \int_{-1}^{1} |v|^2 |w|^2 \, dx - 4 \int_{-1}^{1} \left( \Re(\bar{v}w) \right)^2 \, dx. \tag{110}
\]
Now we observe that using (73) and the hypothesis (107)
\[
2 \int_{-1}^{1} |v|^2 |w|^2 \, dx + 4 \int_{-1}^{1} \left( \Re(\bar{v}w) \right)^2 \, dx \leq 6 \|v\|_{L^\infty(-1,1)}^2 \|Z\|_{L^2(-1,1)}^2 \leq \frac{CK_0}{N^2} \|Z\|_{D(A^{\frac{1}{2}})}^2. \tag{111}
\]
Hence (109) holds when $\frac{K_0}{N^2} \leq \frac{1}{4}$. Let us multiply (104) by $-(\bar{w}_t + \gamma \bar{w})$, take the real part and then integrate on $(-1, 1)$. This leads to
\[
\frac{1}{2} \frac{d}{dt} J(w) + \gamma J(w) = 2\gamma \Re \int_{-1}^{1} G \bar{w} \, dx - 2 \int_{-1}^{1} \Re(\bar{v}w_t) |w|^2 \, dx \tag{112}
\]
\[
-4 \int_{-1}^{1} \Re(\bar{v}w) \Re(\bar{v}_t w) \, dx + 2 \Re \int_{-1}^{1} G_t \bar{w} \, dx,
\]
where
\[
J(w) = \|w(t)\|_{D(A^{\frac{1}{2}})}^2 - 2 \int_{-1}^{1} |w|^2 |w|^2 \, dx - 4 \int_{-1}^{1} \left( \Re(\bar{v}w) \right)^2 \, dx + 4 \Re \int_{-1}^{1} G \bar{w} \, dx. \tag{113}
\]
On the one hand, due to Lemma 4.7
\[
J(w) \geq \frac{3}{4} \|w\|_{D(A^{\frac{1}{2}})}^2 - 4 \|G\|_{D(A^{-\frac{1}{2}})}^2 \|w(t)\|_{D(A^{\frac{1}{2}})}^2 \geq \frac{1}{2} \|w\|_{D(A^{\frac{1}{2}})}^2 - C \|G\|_{D(A^{-\frac{1}{2}})}^2, \tag{114}
\]
**Remark 8.** Here we assume that $N_0$ is large enough, i.e. that $N_0$ is as in Lemma 4.7.
On the other hand, the right hand side of (112) can be handled exactly as in the Proof of Proposition 4 (for global existence result). This reads

\[ \text{r.h.s of (112)} \leq \frac{C}{\sqrt{N}} \|v\|_{D(A^{\frac{1}{2}})} \|w\|_{D(A^{\frac{1}{2}})}^2 + \left( \|G\|_{D(A^{-\frac{1}{2}})} + \gamma \|G\|_{D(A^{-\frac{1}{2}})} \right) \]

\[ \leq \left( \frac{C K_0^2}{N} + \frac{\gamma}{8} \right) \|w\|_{D(A^{\frac{1}{2}})}^2 + 4 \left( \|G\|_{D(A^{-\frac{1}{2}})}^2 + \gamma \|G\|_{D(A^{-\frac{1}{2}})}^2 \right) \]. \hspace{1cm} (115)

We assume that \( N_0 \) is large enough such that

\[ \frac{\gamma}{8} + \frac{C K_0^2}{N_0} \leq \frac{\gamma}{4}. \] \hspace{1cm} (116)

We then obtain that

\[ \frac{d}{dt} J(w) + \gamma J(w) \leq C \left( \|G\|_{D(A^{-\frac{1}{2}})} + \gamma \|G\|_{D(A^{-\frac{1}{2}})} \right). \] \hspace{1cm} (117)

To obtain (107) we have just to apply the usual Gronwall lemma (observing that \( J\left(w(0)\right) = J(0) = 0 \)). \( \square \)

We now plan to apply Proposition 6. For that purpose, we begin by proving that \( v = y + Z = y + Z_m \) satisfies (107).

Using (44), (55) in Proposition 4 we observe that \( z \) is bounded respectively in \( D(A^{\frac{1}{2}}) \) and \( D(A^{-\frac{1}{2}}) \). Moreover, we have established that \( Z \) is bounded in \( \mathcal{C}(\mathbb{R}_+; D(A^{\frac{1}{2}})) \) which leads to a bound of \( Z_t \) in \( \mathcal{C}_b(\mathbb{R}_+; D(A^{-\frac{1}{2}})) \).

On the other hand, the right hand side of (112) can be handled exactly as in the

Finally, the other terms of \( G_t \) can be handled as follows

\[ \|G_t\|_{D(A^{-\frac{1}{2}})} \leq K \lambda N. \] \hspace{1cm} (122)

Hence, we have

\[ \|G_t\|_{D(A^{-\frac{1}{2}})} \leq K \lambda N. \] \hspace{1cm} (121)

\[ \|y_{tt}\|_{D(A^{-\frac{1}{2}})} \leq \lambda N \|y_{tt}\|_{D(A^{-\frac{1}{2}})} \leq \lambda N \left( C \|y_t\| + C \|u\|_{D(A^{\frac{1}{2}})} \right) \]

\[ \leq K \lambda N. \] \hspace{1cm} (120)

\[ \|y_{tt}\|_{D(A^{-\frac{1}{2}})} \]

where \( y_{tt} \) satisfies the equation

\[ y_{tt} = -\frac{1}{2} A y_t - \gamma y_t + P_N \left( 2 \Re(\bar{v} u_t + |u|^2 u_t) \right). \] \hspace{1cm} (119)

We write

\[ \left\| Q_N(-|v|^2 y_t - 2 \Re(\bar{v} y_t) v), \psi \right\| \leq C \|y_t\|_{D(A^{-\frac{1}{2}})} \|v\|_{D(A^{\frac{1}{2}})} \|\psi\|_{D(A^{\frac{1}{2}})} \] \hspace{1cm} (118)

Then we obtain an upper bound for \( G_t \).

We now consider the time derivative

\[ G_t = Q_N \left( -2 \Re(\bar{v} u_t y_t - |v|^2 y_t - 2 \Re(\bar{v}_t y_t) v_t - 2 \Re(\bar{v}_t y_t) v - 2 \Re(\bar{v} y_t) v) \right). \]

We write

\[ \|Q_N(-|v|^2 y_t) - 2 \Re(\bar{v} y_t v)\| \leq C \|y_t\|_{D(A^{-\frac{1}{2}})} \|v\|_{D(A^{\frac{1}{2}})} \leq K \|y_t\|_{D(A^{-\frac{1}{2}})}, \]

where \( y_{tt} \) satisfies the equation

\[ y_{tt} = -\frac{1}{2} A y - \gamma y + P_N \left( 2 \Re(\bar{v} u_t + |u|^2 u_t) \right). \] \hspace{1cm} (120)

We write

\[ \|y_{tt}\|_{D(A^{-\frac{1}{2}})} \leq \lambda N \|y_{tt}\|_{D(A^{-\frac{1}{2}})} \leq \lambda N \left( C \|y_t\| + C \|u\|_{D(A^{\frac{1}{2}})} \right) \]

\[ \leq K \lambda N. \] \hspace{1cm} (121)

Finally, the other terms of \( G_t \) can be handled as follows

\[ \|G_t\|_{D(A^{-\frac{1}{2}})} \leq K \lambda N. \] \hspace{1cm} (122)
4.4. Comparison of $z$ and $Z$ for large times.

Remark 9. For the sake of convenience, we again assume in this section that $T(R) = 0$ in Proposition 3 and that $u(t)$ remains bounded in the absorbing ball in $H^1_0(-1,1)$ whose radius is $M_1$.

Let $z = Q_N u$ that solves and let $Z$ that solves (51). We then state

**Proposition 7.** Let $N_0$ be as in Proposition 4. Let $N \geq N_0$ be fixed. We then have

$$\|Z(t) - z(t)\|_{D(A^2)} \leq C_0 e^{-\gamma t}, \quad (123)$$

where $C_0$ depends on $\|u_0\|_{H^1_0(-1,1)}$.

**Proof of Proposition 7.** Let $v = y + Z$ and let $\xi = Z - z = v - u$. Then $\xi$ is a solution of

$$i\xi_t - \frac{1}{2} \xi + i\gamma \xi + Q_N ((|u|^2 + |v|^2)\xi + iuv\xi) = 0. \quad (124)$$

We multiply (124) by $-\langle \xi, \xi \rangle$, we take the real part and we integrate on $(-1,1)$ to obtain

$$\frac{1}{4} \frac{d}{dt} \|A^{\frac{1}{2}} \xi(t)\|_{L^2}^2 - 2 \int_{-1}^1 (|u|^2 + |v|^2)|\xi|^2 \, dx - 2 \Re \int_{-1}^1 (uv)(\xi)^2 \, dx$$

$$+ \frac{2}{2} \|A^{\frac{1}{2}} \xi(t)\|_{L^2}^2 - 2 \int_{-1}^1 (|u|^2 + |v|^2)|\xi|^2 \, dx - 2 \Re \int_{-1}^1 (uv)(\xi)^2 \, dx \quad (125)$$

$$= -\frac{1}{2} \int_{-1}^1 (|u|^2 + |v|^2)|\xi|^2 \, dx - \frac{1}{2} \Re \int_{-1}^1 (uv)(\xi)^2 \, dx,$$

that we write as

$$\frac{1}{2} \frac{d}{dt} J(\xi(t)) + \gamma J(\xi(t)) = -\int_{-1}^1 (|u|^2 + |v|^2)|\xi|^2 \, dx - \Re \int_{-1}^1 (uv)(\xi)^2 \, dx \quad (126)$$

where

$$J(\xi(t)) = \|A^{\frac{1}{2}} \xi(t)\|_{L^2}^2 - 2 \int_{-1}^1 (|u|^2 + |v|^2)|\xi|^2 \, dx - 2 \Re \int_{-1}^1 (uv)(\xi)^2 \, dx. \quad (127)$$

We have

$$J(\xi(t)) \geq \|A^{\frac{1}{2}} \xi(t)\|_{L^2(-1,1)}^2 - 4 \left(\|u\|_{L^\infty(-1,1)}^2 + \|v\|_{L^\infty(-1,1)}^2\right) \|\xi\|_{L^2(-1,1)}^2. \quad (128)$$

On the one hand, due to (36) and (55) we have

$$J(\xi(t)) \geq \|A^{\frac{1}{2}} \xi(t)\|_{L^2(-1,1)}^2 \left(1 - C\frac{M^2 + K^2}{N^2}\right). \quad (129)$$

Hence if $N_0$ is large enough with respect to the data of the equation (Proposition 4). We obtain the coerciveness of $J$ on $Q_N D(A^{\frac{1}{2}})$ that is

$$J(\xi(t)) \geq \frac{1}{2} \|A^{\frac{1}{2}} \xi(t)\|_{L^2(-1,1)}^2. \quad (130)$$

On the other hand, we can derive an upper bound for the r. h. s of (126) as we did in the proof of Proposition 4. This reads (due to (73))

$$\text{r. h. s. of (126)} \leq C \left(\|u\|_{D(A^{\frac{1}{2}})} + \|v\|_{D(A^{\frac{1}{2}})}\right) \left(\|u\|_{D(A^{\frac{1}{2}})} + \|v\|_{D(A^{\frac{1}{2}})}\right) \times \|\xi(t)\|_{D(A^{\frac{1}{2}})} \|\xi(t)\|_{L^\infty(-1,1)}$$

$$\leq \frac{K}{\sqrt{N}} \|\xi(t)\|_{D(A^{\frac{1}{2}})}^2. \quad (131)$$
We then deduce that
\[ \frac{1}{2} \frac{d}{dt} J(\xi(t)) + \gamma J(\xi(t)) \leq \frac{K}{\sqrt{N}} \|\xi(t)\|_{D(\mathcal{A}^\frac{1}{2})} \leq \frac{2K}{\sqrt{N}} J(\xi(t)). \] (132)

Hence, for \( \frac{2K}{\sqrt{N}} \leq \frac{\gamma}{2} \) due to the Gronwall lemma we get
\[ J(\xi(t)) \leq J(\xi(0)) e^{-\gamma t}. \] (133)

Since \( T(R) \) is supposed to be 0 then
\[ \xi(0) = -z_0 = -Q_N(u_0). \] (134)

Moreover
\[ J(z_0) = \left( \| A^\frac{1}{2} \xi(t) \|_{L^2}^2 - 2 \int_{-1}^{1} (|u|^2 + |v|^2)|z_0|^2 \, dx - 2Re \int_{-1}^{1} (uv)(z_0)^2 \, dx \right) \leq C_0. \]

Finally, using (134) and the coercivity of \( J \) on \( QD(\mathcal{A}^\frac{1}{2}) \) given by (130) we easily complete the proof of estimate (123).

Now, we will conclude that the global attractor \( \mathcal{A} \) is included in \( H^\frac{1}{2} \times (-1, 1) \). We shall in fact show that it is included in \( D(\mathcal{A}) \).

Let \( u_0 \in \mathcal{A} \). On the one hand,
\[ u_0 \in D(\mathcal{A}) \iff z_0 = Q_Nu_0 \in D(\mathcal{A}). \] (135)

On the other hand due to the property of invariance of the attractor, we deduce the existence of a complete orbit \( u(t) \in C_b([m, +\infty[: QD(\mathcal{A}^\frac{1}{2}) \) such that \( u(0) = u_0 \).

Let \( u_0 \in \mathcal{A} \). Let \( T(R) = -m \) and \( Z(t) = Z^m(t) \) the solution in \( C([-m, +\infty[: Q_N D(\mathcal{A}^\frac{1}{2}) \) of the equation
\[ iZ^m_t - \frac{1}{2} A Z^m + Q_N \left( |y + Z^m|^2 (y + Z^m) \right) + i\gamma Z^m = Q_N f, \] (136)
supplemented with the data
\[ Z^m(-m) = 0 \] (137)

We notice that the problem (136) and (137) is globally well posed on \([-m, +\infty[.\)

Moreover, there exists a constant \( C_0 = C_0 \left( f, \gamma, \| u_0 \|_{H^1} \right) \) such that
\[ \sup_{t \geq -m} \| Z^m(t) \|^2_{D(\mathcal{A})} \leq C_0 N^2, \] (138)
and
\[ \| Z^m(t) - z(t) \|^2_{D(\mathcal{A}^\frac{1}{2})} \leq C_0 e^{-\gamma(t+m)}. \] (139)

Then, taking \( t = 0 \) in (139) we get
\[ \| Z_m - z_0 \|^2_{D(\mathcal{A}^\frac{1}{2})} \leq C_0 e^{-\gamma m}, \] (140)
and the regularity \( D(\mathcal{A}) \) of the attractor is proved, letting \( m \to +\infty \).

Proposition 7 implies that the attractor \( \mathcal{A} \) is a bounded subset of \( D(\mathcal{A}) \). To complete the proof of the Theorem 1.2, it remains to establish the compactness of \( \mathcal{A} \) in \( D(\mathcal{A}) \). For this purpose, we use the classical argument due to J. Ball (see [7], [5] and the references therein). We begin with proving an energy equality for a trajectory \( u(t) \) included in \( \mathcal{A} \) (and therefore in \( D(\mathcal{A}) \)) as follows:

multiply the equation
\[ iu_t - \frac{1}{2} Au + |u|^2 u + i\gamma u = f, \] (141)
by $-Au_t - \gamma \bar{u}_t$, and integrate the real part of the resulting equation to obtain
\[
\frac{1}{2} \frac{d}{dt} J(u) + \gamma J(u) = 2\gamma \Re \int_{-1}^{1} |u|^2 u A\bar{u} \, dx + 2\gamma \Re \int_{-1}^{1} f A\bar{u} \, dx
\]
\[
-2\Re \int_{-1}^{1} |u|^2 u A \, dx - \Re \int_{-1}^{1} u^2 \bar{u}_t A \, dx,
\]
where
\[
J(u) = \|Au\|_{L^2}^2 - 4\Re \int_{-1}^{1} |u|^2 \bar{u} A u \, dx + 4\Re \int_{-1}^{1} f A\bar{u} \, dx
\]
Setting $G(u)$ the right hand side of (142) and integrating (142) over $[0, T]$ we obtain
\[
J\left(S(t)u_0\right) = J(u_0) e^{-2\gamma t} + \int_{-1}^{1} e^{-2\gamma (t-s)} G\left(S(s)u_0\right) \, ds.
\]
We now consider a sequence $(x_n)_{n \in \mathbb{N}}$ which takes values in $\mathcal{A}$, and we may assume that, up to a subsequence,
\[
x_n \rightharpoonup x \quad \text{weakly in } D(A),
\]
and
\[
x_n \longrightarrow x \quad \text{strongly in } D\left(A^{1/2}\right) = H_0^1(-1, 1),
\]
since $\mathcal{A}$ is compact in $H_0^1(-1, 1)$.

We plan to prove the convergence of $x_n$ is strong in $D(A)$; this will give the compactness of $\mathcal{A}$ in $D(A)$.

For a given $T > 0$ and up to a subsequence extraction, we may assume that a.e, in $t$ over $[0, T]$
\[
S(t-T)x_n \rightharpoonup S(t-T)x \quad \text{weakly in } D(A),
\]
and
\[
S(t-T)x_n \longrightarrow S(t-T)x \quad \text{strongly in } D\left(A^{1/2}\right).
\]

On the other hand, since the mapping
\[
u \mapsto -2|u|^2 Au - u^2 Au
\]
is continuous from $D(A)$ to $L^2(-1, 1)$ (both endowed with the strong topology) and since, due to (147), (148) and to the equation
\[
iu_t - \frac{1}{2} Au = f - |u|^2 u - i\gamma u,
\]
then the Lebesgue dominated convergence theorem implies
\[
\int_0^t e^{-2\gamma (t-s)} G\left(S(t-T)x_n\right) \, ds \longrightarrow \int_{-1}^{1} e^{-2\gamma (t-s)} G\left(S(t-T)x\right) \, ds.
\]
We now infer from (144) that
\[
\limsup_{n \to \infty} J\left(S(t-T)x_n\right) \leq \limsup_{n \to \infty} J\left(S(-T)x_n\right) e^{-2\gamma t}
\]
\[
+ \int_0^t e^{-2\gamma (t-s)} G\left(S(s-t)x\right) \, ds.
\]
Choosing $t = T$ in (153) and using (144) for $u_0 = S(-T)x$, we obtain
\[
\limsup_{n \to \infty} J(x_n) \leq \limsup_{n \to \infty} J\left(S(-T)x\right) e^{-2\gamma T} + J(x) + \left|J\left(S(-T)x\right)\right|.
\]
We now let $T$ go to infinity in (154) to obtain (since $J$ is bounded on the trajectories belonging to $A$)
\[
\limsup_{n \to \infty} J(x_n) \leq J(x). \tag{155}
\]
We easily infer from (155) and from the fact that
\[
u \mapsto -\langle J(u) - \|Au\|_{L^2}^2 = -4\text{Re} \int_{-1}^{1}|u|^2 \bar{u} A u \, dx + 4\text{Re} \int_{-1}^{1} f A \bar{u} \, dx \tag{156}
\]
is weakly continuous in $D(A)$ that
\[
\limsup_{n \to \infty} \|Ax_n\|_{L^2(-1,1)}^2 \leq \limsup_{n \to \infty} J(x_n) + 4\text{Re} \int_{-1}^{1}|x|^2 \bar{x} Ax - 4\text{Re} \int_{-1}^{1} f A \bar{x}
\leq J(x) + 4\text{Re} \int_{-1}^{1}|x|^2 \bar{x} Ax - 4\text{Re} \int_{-1}^{1} f A \bar{x}
= \|Ax\|_{L^2(-1,1)}^2. \tag{157}
\]
Then, combining
\[
\limsup_{n \to \infty} \|Ax_n\|_{L^2(-1,1)}^2 \leq \limsup_{n \to \infty} J(x_n) + 4\text{Re} \int_{-1}^{1}|x|^2 \bar{x} Ax - 4\text{Re} \int_{-1}^{1} f A \bar{x}
\leq J(x) + 4\text{Re} \int_{-1}^{1}|x|^2 \bar{x} Ax - 4\text{Re} \int_{-1}^{1} f A \bar{x}
= \|Ax\|_{L^2(-1,1)}^2 \tag{158}
\]
and the weak convergence in $D(A)$ in (145) we obtain the strong convergence of $x_n$ towards $x$ in $D(A)$ which completes the proof of the theorem.

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