Charging Symmetries and Linearizing Potentials for Einstein-Maxwell Dilaton-Axion Theory

Alfredo Herrera-Aguilar
Joint Institute for Nuclear Research,
Dubna, Moscow Region 141980, RUSSIA.
e-mail: alfa@cv.jinr.dubna.su

and

Oleg Kechkin
Institute of Nuclear Physics,
Moscow State University,
Moscow 119899, RUSSIA,
e-mail: kechkin@monet.npi.msu.su

Abstract
We derive a set of complex potentials which linearize the action of charging symmetries of the stationary Einstein-Maxwell dilaton-axion theory.
1 Introduction

Superstring theory provides a correct quantum description of gravity coupled to matter fields. In the low energy limit superstring theory leads to some modifications of General Relativity. These string gravity models preserve long-distance behaviour of the mysterious quantum gravity and in special (BPS-saturated) cases exactly reproduce it [1].

Einstein-Maxwell theory with dilaton and axion fields (EMDA) is one of the simplest string gravity models. It is described by the action

\[ S = \int d^4x \sqrt{|g|} \left\{ -R + 2(\partial \phi)^2 + \frac{1}{2} e^{2\phi} (\partial \kappa)^2 - e^{-2\phi} F^2 - \kappa F \tilde{F} \right\}, \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the Maxwell strength and \( \tilde{F}^{\mu\nu} = \frac{1}{2} F^{\mu\nu\lambda\sigma} F_{\lambda\sigma} \). Formally EMDA can be considered as an extension of the Einstein-Maxwell (EM) theory to the case of non-trivial scalar dilaton field \( \phi \) and pseudoscalar axion field \( \kappa \).

EMDA arises in the framework of the non-critical heterotic string theory (D=4, one vector field). It can be also obtained as the corresponding truncation of the critical heterotic string theory (D=10, 16 vector fields) reduced to four dimensions. The EMDA solution spectrum and symmetry structure were under extensive investigation during last several years (see [2] for a review).

In the stationary case the symmetry group of this theory consists of gauge and non-gauge parts. The first part is trivial, whereas the second one acts in the charge space of asymptotically flat field configurations. We name non-gauge transformations as charging symmetries because they generate charged solutions from neutral ones.

In this letter we derive a representation of the stationary EMDA using complex potentials which transform linearly under the action of the charging symmetries. The found potentials provide an adequate description of the EMDA field configurations possessing the flatness property at spatial infinity.

2 Charging symmetries

In this section we review the matrix Ernst potential formulation [3] and list charging symmetries of the stationary EMDA [4]. At the end of the section the main problem of the letter is formulated.

Our notations are the following. For the D=4 line element we use the decomposition

\[ ds^2 = f(dt - \omega_i dx^i)^2 - f^{-1} h_{ij} dx^i dx^j; \]
thus $f, \omega_i$ and $h_{ij}$ become scalar, vector and symmetric tensor 3-fields in the stationary case. Next, we introduce the magnetic potential $u$ on shell

$$\nabla u = \sqrt{2} \left\{ fe^{-2\phi} \left( \nabla \times \vec{A} + \nabla A_0 \times \vec{\omega} \right) + \kappa \nabla A_0 \right\}. $$

The electric potential $v$ is $\sqrt{2} A_0$. Finally, the rotational potential $\chi$ can be defined exactly as in the EM theory [5]:

$$\nabla \chi = u \nabla v - v \nabla u - f^2 \nabla \times \vec{\omega}. $$

A Kähler formulation of the theory is based on the use of three complex functions

$$Z = \kappa + ie^{-2\phi},$$
$$F = u - Zv,$$
$$E = if - \chi + vF,$$

which generalize Ernst potentials of the EM theory [5]. They can be combined into the matrix

$$E = \begin{pmatrix} E & F \\ F & -Z \end{pmatrix},$$

which we call matrix Ernst potential (MEP) [5] in view of the close analogy between EMDA in terms of MEP and Einstein gravity using the Ernst potential formulation [3]. Actually, the effective EMDA action in the stationary case reads:

$$S = \int d^3x h^{\frac{1}{2}} \left\{ -3R + 2\text{Tr} \left( JEJ^E \right) \right\}. $$

where $JE = \nabla E(E - \vec{E})^{-1}$; it concides with Einstein’s one if matrix $E$ is replaced by a function.

Using the MEP representation the EMDA symmetries adopt the “matrix valued SL(2,R) form”:

$$E \rightarrow S^T(E^{-1} + L)^{-1}S + R,$$

where $\det S \neq 0$ and matrices $L$ and $R$ are symmetric [3]. Eq. (1) describes the full symmetry group, the so-called U-duality. In this work we shall consider the charging subgroup.

Matrices $S$, $L$ and $R$ of this subgroup satisfy the relation

$$\sigma_3 + i R = S^T (\sigma_3 + i L)^{-1} S \quad (2)$$

3
It provides the conservation of the vacuum solution $E_{\text{vac}} = i\sigma_3$ (all the 4D matter fields are trivial and the metric corresponds to the Minkowski space-time).

In [4] it was shown how to solve Eq. (2). The solution consists of two commuting parts. The first part is

$$S_{U(1)} = (\cos \lambda^0)^{-1}\sigma_0, \quad R_{U(1)} = -L_{U(1)} = \sigma_3 \tan \lambda^0; \quad (3)$$

The second one is

$$S_{SU(1,1)} = \Delta^{-1}[f_1\sigma_0 + \lambda^1 f_2\sigma_1],$$
$$R_{SU(1,1)} = f_2\Delta^{-1}[\lambda^1 f_1 + \lambda^2 f_2\sigma_0 + (\lambda^2 f_1 + \lambda^1 \lambda^3 f_2)\sigma_1],$$
$$L_{SU(1,1)} = f_2\Delta^{-1}[(-\lambda^3 f_1 + \lambda^2 \lambda^1 f_2)\sigma_0 + (\lambda^2 f_1 - \lambda^1 \lambda^3 f_2)\sigma_1], \quad (4)$$

where $\Delta = f_1^2 - (\lambda^1)^2 f_2^2$, $\lambda^\mu$ are group parameters and

$$2f_1 = (1 + \sigma) \cosh \lambda + (1 - \sigma) \cos \lambda, \quad 2\lambda f_2 = (1 + \sigma) \sinh \lambda + (1 - \sigma) \sin \lambda. \quad (5)$$

Here we have introduced the parameter $\lambda = \sqrt{[(\lambda^1)^2 + (\lambda^2)^2 - (\lambda^3)^2]}$ and put $\sigma = \text{sign}[(\lambda^1)^2 + (\lambda^2)^2 - (\lambda^3)^2]$.

Let us consider field configurations near the vacuum point. Then $E_{as} = i(\sigma_3 - 2\hat{M}/r)$, where $1/r$ is the infinitesimal parameter (in what follows $r$ tends to spatial infinity), and

$$\hat{M} = \begin{pmatrix} M & Q \\ Q & -D \end{pmatrix}$$

is a constant matrix. Its components are $M = M + iN$ ($M$ is the Arnowitt-Deser-Misner mass, $N$ denotes the Newmann-Unti-Tamburino parameter), $D = D + iA$ (a combination of dilaton and axion charges) and the electromagnetic charge $Q = Q_e + iQ_m$. A remarkable fact is that $\hat{M}$ components transform linearly under the action of the charging symmetries. Actually, in [4] it was shown that

$$\hat{M} \rightarrow T^T \hat{M}T \quad (6)$$

where $T = (1 + iL\sigma_3)^{-1}S$. The operator $T$ is the commuting product of the operators

$$T_{U(1)} = e^{i\lambda^0}\sigma_0 \quad \text{and} \quad T_{SU(1,1)} = f_1\sigma_0 + f_2(\lambda^1 \sigma_1 - i\lambda^2 \sigma_2 + i\lambda^3 \sigma_3), \quad (7)$$
i.e.

\[ T = T_{U(1)} T_{SU(1,1)} = T_{U(1)} T_{SU(1,1)}; \]  

it satisfies the relation

\[ T^+ \sigma_3 T = \sigma_3. \]  

Now we are ready to formulate the problem of this work. One can see that both sets of variables \((E, Z, F)\) and \((M, D, Q)\) realize some representations of the charging subgroup. Variables of the first set transform in a highly complicated way, whereas the second set transforms linearly. It is natural to suppose that other potentials which transform like charges exist. The derivation of these potentials is presented in the next section.

3 Linearizing potentials

Let us denote the complex potentials which linearize the action of charging symmetries as \(w_1, w_2\) and \(w_3\) and combine them into the symmetric matrix

\[ \hat{W} = \begin{pmatrix} w_1 & w_3 \\ w_3 & w_2 \end{pmatrix}, \]

This matrix transforms as

\[ \hat{W} \rightarrow T^T \hat{W} T \]

under the charging subgroup.

Our plan is the following: to establish the relation between EMDA potentials and the unknown variables

\[ w_i = w_i(E, Z, F), \]  

we calculate the infinitesimal generators of the charging symmetries in both representations and identify these generators using their commutation relations. This gives a set of differential equations which defines the relations (11).

A great simplification of the solution process can be achieved if one works with appropriate variables. After some algebraic manipulations we lead to the sets \((E_+, E_-, F)\), where

\[ E_\pm = \frac{1}{2}(E \pm Z). \]
and \((r, \theta, \varphi)\), where

\[
\begin{align*}
\frac{1}{2}(w_1 + w_2) &= r \sinh \theta \cos \varphi, \\
\frac{i}{2}(w_1 - w_2) &= r \sinh \theta \sin \varphi, \\
w_3 &= r \cosh \theta.
\end{align*}
\]

(13)

The infinitesimal generators corresponding to Eq. (1) in view of Eqs. (3)-(5) read:

\[
\begin{align*}
K_1 &= \mathcal{F} \partial_- + \mathcal{E}_- \partial_{\mathcal{F}}, \\
K_2 &= -\mathcal{F}(\mathcal{E}_+ \partial_+ + \mathcal{E}_- \partial_-) + \frac{1}{2}(1 + \mathcal{E}_+^2 - \mathcal{E}_-^2 - \mathcal{F}^2)\partial_{\mathcal{F}}, \\
K_3 &= \mathcal{E}_-(\mathcal{E}_+ \partial_+ + \mathcal{F} \partial_{\mathcal{F}}) + \frac{1}{2}(1 + \mathcal{E}_+^2 + \mathcal{E}_-^2 + \mathcal{F}^2)\partial_- , \\
K_0 &= \mathcal{E}_+(\mathcal{F} \partial_{\mathcal{F}} + \mathcal{E}_- \partial_-) + \frac{1}{2}(1 + \mathcal{E}_+^2 + \mathcal{E}_-^2 - \mathcal{F}^2)\partial_+ .
\end{align*}
\]

Here generator \(K_1\) corresponds to the parameter \(\frac{\lambda_1}{2}\), etc.; \(\partial_\pm = \frac{\partial}{\partial \xi_\pm}\) and we write down only the holomorphic parts of the generators. The calculation of commutation relations gives:

\[
\begin{align*}
[K_1, K_2] &= -K_3, & [K_2, K_3] &= K_1, & [K_3, K_1] &= K_2,
\end{align*}
\]

(14)

and \(K_0\) commutes with \(K_1, K_2\) and \(K_3\).

Next, from Eqs. (7), (10) and (13) it follows that

\[
\begin{align*}
\hat{K}_1 &= \cos \varphi \partial_\theta - \coth \theta \sin \varphi \partial_\varphi, \\
\hat{K}_2 &= \sin \varphi \partial_\theta + \coth \theta \cos \varphi \partial_\varphi, \\
\hat{K}_3 &= -\partial_\varphi, \\
\hat{K}_0 &= r \partial_r.
\end{align*}
\]

One can see that the group structure becomes evident in terms of the variables \(r, \theta\) and \(\varphi\). The generators \(\hat{K}_1, \hat{K}_2\) and \(\hat{K}_3\) depend only on the complex “angles” \(\theta\) and \(\varphi\); they define “tangential” transformations. Operator \(\hat{K}_0\) generates a “transversal” movement along the “radial coordinate” \(r\). From this geometric picture the commutation of two subgroups immediately follows.

One can prove that the commutators of generators with hats are the same as the commutators of generators without hats. Using this fact we identify them correspondingly, i.e.
we put $\hat{K}_1 = K_1$, etc. We obtain an explicit form of the relations (11) in the following way. First, we identify $\hat{K}_3$ with $K_3$ (from the commutation relations we see that the third vector is special). Second, we compare $\hat{K}_1$ and $K_1$ (then the equality $\hat{K}_2 = K_2$ becomes an identity accordingly to Eqs. (14)). After the first two steps we will know the dependence of $E_\pm$ and $F$ on $\theta$ and $\varphi$. Finally, the comparison of the zero vectors gives the remaining $r$-dependence.

Now let us briefly discuss details of the solution procedure. The equation $\hat{K}_3 = K_3$ is equal to the system

$$
\begin{align*}
E_+ \varphi &= -E_- \varphi, \\
F \varphi &= -E_- F, \\
-2E_- \varphi &= 1 + E_2 + E_-^2 + F^2.
\end{align*}
$$

To solve this system it is useful to introduce the function $\Psi = \sqrt{E_2 + F^2}$. Then from Eqs. (15) one obtains simple equations for $E_-$ and $\Psi$. Solving them, it is easy to calculate $E_+$ and $F$:

$$
\begin{align*}
E_- &= -\frac{\sin(\varphi + \alpha)}{\cos \beta + \cos(\varphi + \alpha)}, \\
E_+ &= \frac{\sin \beta \cos \gamma}{\cos \beta + \cos(\varphi + \alpha)}, \\
F &= \frac{\sin \beta \sin \gamma}{\cos \beta + \cos(\varphi + \alpha)},
\end{align*}
$$

where $\alpha$, $\beta$ and $\gamma$ are functions of $\theta$ and $r$. Next, the relation $\hat{K}_1 = K_1$ leads to the following extraction of the $\theta$-dependence:

$$
\begin{align*}
\alpha &= \frac{\pi}{2}, \\
\cos \beta &= \frac{i \cos \Lambda}{\sinh \theta}, \\
\sin \beta \cos \gamma &= \frac{i \sin \Lambda}{\sinh \theta}, \\
\sin \beta \cos \gamma &= \coth \theta,
\end{align*}
$$

where $\Lambda$ is an arbitrary function of $r$.

The last step consists of a comparison of the zero vectors. However, zero vectors commute with the other ones, and commutation relations only suggest that

$$
\hat{K}_0 = C K_0, \tag{18}
$$
where $C$ is a constant. A straightforward calculation shows that the identification of the zero vectors leads to the relation $\Lambda = C \ln r + D$. Now we choose the values of $C$ and $D$ to provide the simplest (rational) form of the functions (11) and to make the trivial point $w_1 = w_2 = w_3 = 0$ correspond to the vacuum configuration. The result is:

$$C = -i, \quad D = 0. \quad (19)$$

Finally, from Eqs. (12), (13) and (16)-(19) one obtains the explicit form of the relations between MEP components and linearizing potentials:

$$\mathcal{E} = i \frac{(1 - w_1)(1 - w_2) - w_3^2}{(1 + w_1)(1 - w_2) + w_3^2},$$

$$\mathcal{Z} = i \frac{(1 + w_1)(1 + w_2) - w_3^2}{(1 + w_1)(1 - w_2) + w_3^2},$$

$$\mathcal{F} = -\frac{2i w_3}{(1 + w_1)(1 - w_2) + w_3^2}.$$

The inverse relations have the form:

$$w_1 = \frac{1 + \mathcal{E} \mathcal{Z} + \mathcal{F}^2 + i(\mathcal{E} - \mathcal{Z})}{1 - \mathcal{E} \mathcal{Z} - \mathcal{F}^2 - i(\mathcal{E} + \mathcal{Z})},$$

$$w_2 = \frac{1 + \mathcal{E} \mathcal{Z} + \mathcal{F}^2 - i(\mathcal{E} - \mathcal{Z})}{1 - \mathcal{E} \mathcal{Z} - \mathcal{F}^2 - i(\mathcal{E} + \mathcal{Z})},$$

$$w_3 = \frac{2i \mathcal{F}}{1 - \mathcal{E} \mathcal{Z} - \mathcal{F}^2 - i(\mathcal{E} + \mathcal{Z})}. \quad (20)$$

The new potentials $w_1, w_2, w_3$ seem to be the most suitable for the study of charged stationary EMDA configurations possessing asymptotic flatness property. Actually, they undergo simple transformation rules and also

$$\hat{\mathcal{W}}_{as} = \frac{\hat{\mathcal{M}}}{r},$$

i.e. these potentials are free of any non-vanishing asymptotics (from this relation we see how the formulae (6) and (10) can have identical form).

4 Concluding remarks

Thus, the stationary EMDA allows a representation which linearizes the action of its charging symmetry subgroup. The potentials that realize this representation do not form the analogy
of Kinnersley’s linearizing potentials \cite{8}. Actually, to linearize the complete symmetry group of the stationary EM theory Kinnersley introduced a new potential in addition to the present pair of potentials. In our case, the result is different: we linearize only the charging symmetry subgroup without extension of the potential space and use only the appropriate change of variables.

The close relation between linearizing potentials and charges allow us to establish one general invariant of the charging symmetry subgroup. Actually, the charge function \( I(\mathcal{M}, D, Q) = |\mathcal{M}|^2 + |D|^2 - 2|Q|^2 \equiv \text{Tr} \left( \tilde{\mathcal{M}} \sigma_3 \tilde{\mathcal{M}} \sigma_3 \right) = I(\tilde{\mathcal{M}}) \) is invariant under the action of charging symmetries in view of Eqs. (6) and (9) (this invariant vanishes for BPS-saturated configurations). Identical transformation properties of \( \tilde{\mathcal{M}} \) and \( \tilde{\mathcal{W}} \) allow us to introduce the corresponding invariant function of the linearizing potentials \( I(\tilde{W}) = \text{Tr} \left( \tilde{W} \sigma_3 \tilde{W} \sigma_3 \right) = |w_1|^2 + |w_2|^2 - 2|w_3|^2 = I(w_1, w_2, w_3) \). From Eq. (20) it follows that in terms of the EMDA Ernst potentials the charging symmetry invariant takes the form

\[
\frac{1}{2} I(\mathcal{E}, \mathcal{Z}, \mathcal{F}) = \frac{|1 + \mathcal{E} \mathcal{Z} + \mathcal{F}^2|^2 + |\mathcal{E} - \mathcal{Z}|^2 - 4|\mathcal{F}|^2}{|1 - \mathcal{E} \mathcal{Z} - \mathcal{F}^2 - i(\mathcal{E} + \mathcal{Z})|^2}.
\]

One can see that the first two terms of the \( 1/r \) expansion of this invariant vanish for the BPS-saturated fields.

**Acknowledgments**

We would like to thank our colleagues for encouraging us in our work. One of the authors (A.H.) was partially supported by CONACYT and SEP.

**References**

[1] E. Kiritsis, “Introduction to superstring theory”, CERN-TH/97-218, hep-th/9709062.

[2] D. Youm, “Black holes and solitons in string theory”, IASSNS-HEP-97/100, hep-th/9710046.

[3] D.V. Gal’tsov and O.V. Kechkin, Phys. Lett. B361 (1995) 52.

[4] O. Kechkin and M. Yurova, Gen. Rel. Grav. 29, 10, (1997) 1283.
[5] W. Israel and G.A. Wilson, J. Math. Phys. 13 (1972) 865.

[6] F. J. Ernst, Phys. Rev. 168 5 (1968) 1415.

[7] F. J. Ernst, Phys. Rev. 167 5 (1968) 1175.

[8] W. Kinnersley, J. Math. Phys. 14 (1973) 651.