Gauge transformation of scalar induced tensor perturbation during matter domination

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Abstract

We study the scalar induced tensor perturbations at second order during matter domination in seven different gauges. Considering the obtained solution from the Newtonian gauge, we use the gauge transformation law of the scalar induced tensor perturbation to derive the solution in six other gauges. After identifying and eliminating the residual gauge modes in the synchronous and comoving orthogonal gauges, we obtain the same analytical results of the kernel function $I_\chi$ for these two gauges as those obtained from the gauge transformation. For the scalar induced gravitational waves oscillating as $\sin x$ and $\cos x$, we find that $\rho_{GW} \propto a^{-4}$, and $\Omega_{GW} \propto 1/a$ in the matter dominated era, so the oscillating gravitational waves behave as radiation.

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I. INTRODUCTION

The discovery of gravitational waves (GWs) from mergers of black holes (BHs) or neutron star (NS) by the Laser Interferometer Gravitational-Wave Observatory (LIGO) Scientific Collaboration and Virgo Collaboration [1–11] has marked the beginning of the era of astronomy of GWs. There are GWs with cosmological origins, such as the primordial GWs generated during inflation, the second-order GWs induced by primordial scalar perturbation, as well as GWs generated from a cosmic phase transition [12–54]. The primordial scalar perturbations at large scales are nearly scale invariant with the amplitude $A_s \approx 10^{-9}$ at the pivot scale $k = 0.05\text{Mpc}^{-1}$ [55], while the small scale ones remain to be explored. If the primordial scalar perturbations at small scales are big enough (the amplitude of the power spectrum is roughly in the order of 0.01), then they will induce sizable amount of secondary GWs after horizon reentry during the radiation domination (RD) and the matter domination (MD) due to the mixing of tensor and scalar perturbations. The scalar induced GWs (SIGWs) at second-order contribute to the stochastic gravitational-wave background, and it is possible to extract information about small scale scalar perturbations from the detection of SIGWs. Therefore, SIGWs can be used to probe the thermal history of the universe and to understand the physics during inflation [31, 38, 56].

In contrast to tensor perturbations at the first-order, the second-order tensor perturbations are gauge dependent although gauge-invariant tensor perturbation at the second-order can be constructed in a specific gauge [27, 57–63], and SIGWs depend on gauge choice too [33, 39, 41, 47, 60, 64]. However, the production of SIGWs was usually discussed in the Newtonian gauge [16, 17, 46], so it is necessary to discuss SIGWs in other gauges. During RD, the energy density of SIGWs in the Newtonian, synchronous, and uniform curvature gauge was found to be same [33]. The calculation of SIGWs in the synchronous gauge during both RD and MD was discussed in [47], and they argued that SIGWs should be freely propagating tensor perturbations oscillating like $\sin(k\eta)$ or $\cos(k\eta)$. For a general background with constant equation of state $w$, in particular, for RD with $w = 1/3$, and MD with $w = 0$, the SIGWs were calculated in the Newtonian, comoving, and uniform curvature gauges [39]. In the previous paper [41], we derived a general formula for the calculation of SIGWs during RD in an arbitrary gauge and obtained the results for SIGWs in the uniform curvature gauge, the synchronous gauge, the comoving gauge, the comoving orthogonal gauge, the uniform
density gauge, and the uniform expansion gauge from the result in the Newtonian gauge by using the coordinate transformation. In this paper, we use the same method to discuss the transformation of SIGWs under the coordinate transformation during MD and we discuss the energy density of SIGWs with the behaviour $\sin(k\eta)$ or $\cos(k\eta)$ in seven various gauges.

The paper is organized as follows. The basic formulas used to calculate SIGWs and discuss the gauge transformation are given in Sec. II. Here, we also provide the prescription to obtain the expressions in other gauges from the result in the Newtonian gauge by using the gauge transformation of the second-order tensor perturbation in MD. In Sec. III, we derive the kernels $I(u,v,x)$ analytically in different gauges. Then we discuss the energy density of SIGWs with the behaviour $\sin(k\eta)$ or $\cos(k\eta)$. The summary of our results is presented in Sec. IV.

### II. FORMULATION OF SIGWS AND GAUGE TRANSFORMATIONS

In this section, we discuss the formalism of the second-order SIGWs. We begin with the following perturbed metric:

$$ds^2 = -a^2(1 + 2\phi)d\eta^2 + 2a^2B_i dx^i d\eta + a^2 \left[ (1 - 2\psi)\delta_{ij} + 2E_{ij} + \frac{1}{2}h_{ij}^{TT} \right] dx^i dx^j,$$

(1)

where the metric perturbations include the scalar perturbations $\phi$, $\psi$, $B$, and $E$ of are the first order, and the tensor perturbation $h_{ij}^{TT}$ is of second order, which is transverse traceless: $h_{ii}^{TT} = \partial_i h_{ij}^{TT} = 0$, while the perturbations of the first-order vector and tensor are not taken into account because we discuss SIGWs only. All calculations are carried out in a matter-dominated (MD) universe. Also, in the forthcoming derivations in the next (sub)sections, we assume that the production of induced GWs begins long before the horizon reentry.

#### A. The generation of SIGWs

The equation of motion for the transverse traceless tensor mode $h_{ij}^{TT}$ at the second-order can be derived straightforwardly from the perturbed Einstein’s equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ as

$$h_{ij}^{TT''} + 2\mathcal{H} h_{ij}^{TT'} - \nabla^2 h_{ij}^{TT} = 4T_{ij}^{lm} s_{lm},$$

(2)

where $'$ stands the derivative with respect to the conformal time $\eta$, $\mathcal{H} = a'(\eta)/a(\eta)$ is the comoving Hubble parameter, and $s_{lm}$ is the source term given below in equation (12). The
projector tensor $\mathcal{T}_{ij}^{lm}$ acting on the source term extracts its transverse and trace-free part and will be discussed below. In this paper, we consider the production of SIGWs in MD only, where $a = \eta^2$, and $\mathcal{H} = 2/\eta$.

The Fourier components of transverse and traceless tensor perturbations in terms of the polarization tensors are defined as [16, 17, 41, 62, 65]

$$h_{ij}^{TT}(y, \eta) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{iky} \left[ h_k^+(\eta)e_{ij}^+ + h_k^x(\eta)e_{ij}^x \right],$$

(3)

where $y$ are spatial coordinates. In terms of the orthonormal bases $e$ and $\bar{e}$ orthogonal to $k$, with $k \cdot e = k \cdot \bar{e} = e \cdot \bar{e} = 0$ and $|e| = |\bar{e}| = 1$, the plus and cross polarization bases are defined as

$$e_{ij}^+ = \frac{1}{\sqrt{2}} \left[ e_ie_j - \bar{e}_i\bar{e}_j \right],$$

$$e_{ij}^x = \frac{1}{\sqrt{2}} \left[ e_ie_j + \bar{e}_i\bar{e}_j \right].$$

(4)

The polarization tensors (4) are transverse and traceless as $k_i e_{ij}^+ = k_i e_{ij}^x = 0$ and $e_{ii}^+ = e_{ii}^x = 0$. In the Fourier space, the projection tensor is expressed as

$$\mathcal{T}_{ij}^{lm} = [e_{ij}^+ e^{+lm} + e_{ij}^x e^{xlm}],$$

(5)

and the solution to Eq. (2) for the either polarization $e_{ij}^t$ reads

$$h_k^t(\eta) = 4 \int_0^x \frac{d\tilde{x}}{x} \frac{a(\tilde{\eta})}{a(\eta)} \frac{1}{k} G_k(\eta, \tilde{\eta}) S_k^t,$$

(6)

where the Green’s function $G_k(\tilde{\eta}, \eta)$ to Eq. (2) in MD is

$$G_k(\tilde{\eta}, \eta) = \frac{(1 + x\tilde{x}) \sin(x - \tilde{x}) - (x - \tilde{x}) \cos(x - \tilde{x})}{kx\tilde{x}},$$

(7)

$\tilde{x} = k\tilde{\eta}$, $x = k\eta$, the source $S_k^t = e_{ij}^t s^{ij}(k, \eta)$ for either polarization $t = +$ or $\times$ is

$$S_k^t(\eta) = \int \frac{d^3p}{(2\pi)^{3/2}} \zeta(p)\zeta(k - p)e_{ij}^t p^ip^j f(u, v, x),$$

(8)

$\zeta(k)$ is the primordial curvature perturbation, $u = p/k$, $v = |k - p|/k$ and $f(u, v, x)$ will be given below in the next subsection. For convenience we introduce the integral kernel [16, 18, 45]

$$I(u, v, x) = \int_0^x \frac{d\tilde{x}}{x} \frac{a(\tilde{\eta})}{a(\eta)} kG_k(\eta, \tilde{\eta}) f(u, v, \tilde{x}).$$

(9)
to express the solution $h_k^t(\eta)$,

$$
h_k^t(\eta) = 4 \int \frac{d^3p}{(2\pi)^{3/2}} e_{ij}^i p^i \zeta(p) \zeta(k-p) \frac{1}{k^2} f(u,v,x).$$

For free propagating GWs without the source, we get the solution

$$
h_k^t(\eta) = \frac{3(\sin x - x \cos x)}{x^3},
$$

with the initial condition $h_k^t(0) = 1$.

**B. The source term for SIGWs**

The source term $s_{ij}$ for SIGWs appearing in (2) coming from the second-order in the scalar perturbation is given by

$$
s_{ij} = -\psi_i \psi_j - \phi_i \phi_j + \sigma_{ij} (\phi' + \psi' - \nabla^2 \sigma) - (\psi_i' \sigma_j + \psi_j' \sigma_i) + \sigma_{ik} \sigma_{jk}
- 2 \psi_i' \phi' + \frac{2}{H' - H^2} (\phi' + H \phi)_j (\psi_i' + H \phi)_{\dot{j}}
+ 2 \psi_i \nabla^2 E - 2 E_{ij} (\psi' + 2H \psi' - \nabla^2 \psi) + E_{ik} E_{j\dot{k}} - E_{ikl} E_{jkl}
- 2 (\psi_{\dot{j}k} E_{ik} + \psi_{i\dot{k}} E_{j\dot{k}}) + 2H (\psi_i E_{j\dot{i}}' + \psi_{\dot{j}i} E_i')
+ (\psi_{\dot{i}j} E_{\dot{j}i}' + \psi_j E_{\dot{i}j}') - 2 E_{ij} \psi' - E_{i\dot{j}k} (E'' + 2H E' - \nabla^2 E)_{\dot{k}},
$$

where the anisotropic stress tensor $\Pi_{ij}$ of the matter fluid is assumed to be zero, the terms proportional to $\delta_{ij}$ are omitted because they do not contribute to the transverse and traceless part, $\sigma = E' - B$ is the shear potential, and $\rho_0$ and $P_0$ are the background values of energy density and pressure of the matter fluid. The detailed discussion of these variables is presented in the Appendix. In gauges with $E = 0$, the above equation (12) reduces to the results given in [27, 52, 64] with vanishing anisotropic stress. In general, we need to use Eq. (12) instead. In particular, we should include all the terms involving $E$ in the synchronous gauge.

Coming back to the the function $f(u,v,x)$ in Eq. (8) which contains the source information and is dependent on the gauge. The explicit expression of the source function $f(u,v,x)$ is given by

$$
f(u,v,x) = \frac{1}{2} \times \frac{9}{25} \left( \tilde{f}(u,v,x) + \tilde{f}(v,u,x) \right),
$$

5
where
\[
\hat{f}(u, v, x) = T_\psi(ux)T_\psi(vx) - T_\phi(ux)T_\phi(vx) - \frac{u}{v}T_\sigma(ux)\left[T_\phi^*(vx) + T_\psi^*(vx) + T_\sigma(vx)\right] \\
- 2\frac{u}{v}T_\psi^*(ux)T_\sigma(vx) - \frac{1 - u^2 - v^2}{2uv}T_\sigma(ux)T_\sigma(vx) + 2T_\psi(ux)T_\phi(vx) \\
+ \frac{2}{\mathcal{H}^2 - \mathcal{H}'}\left[kuT_\psi^*(ux) + \mathcal{H}T_\phi(ux)\right]\left[kvT_\psi^*(vx) + \mathcal{H}T_\phi(vx)\right] \\
+ 2T_\psi(ux)T_E(vx) + 2\frac{u^2}{v^2}T_E(vx)\left[T_\psi^*(ux) + \frac{2\mathcal{H}}{ku}T_\psi^*(ux) + T_\psi(ux)\right] \\
- \frac{1 - u^2 - v^2}{2uv}T_\psi^*(ux)T_\psi^*(vx) - \left(\frac{1 - u^2 - v^2}{2uv}\right)^2T_\psi(ux)T_E(vx) \\
+ 4\frac{\mathcal{H}}{ku}T_\psi(ux)T_E^*(vx) + 4\frac{u^2}{v^2}T_\psi^*(ux)T_E^*(vx) + 2T_\psi(ux)T_E^{**}(vx) \\
+ 2\left(\frac{1 - u^2 - v^2}{v^2}\right)T_\psi(ux)T_E(vx) \\
- \frac{1 - u^2 - v^2}{2u^2}T_E(ux)\left[T_\psi^*(vx) + 2\frac{\mathcal{H}}{ku}T_\psi^*(vx) + T_E(vx)\right],
\]

\(T^*(x) = dT(x)/dx\), and the transfer functions \(T(x)\) which relates scalar perturbations to the primordial curvature perturbation, are defined as [47]

\[
\alpha(k, x) = \frac{3}{5}\zeta(k)\frac{1}{k}T_\alpha(x),
\]
\[
\beta(k, x) = \frac{3}{5}\zeta(k)\frac{1}{k^2}T_\beta(x),
\]
\[
\sigma(k, x) = \frac{3}{5}\zeta(k)\frac{1}{k}T_\sigma(x),
\]
\[
E(k, x) = \frac{3}{5}\zeta(k)\frac{1}{k^2}T_E(x),
\]
\[
B(k, x) = \frac{3}{5}\zeta(k)\frac{1}{k}T_B(x),
\]
\[
\psi(k, x) = \frac{3}{5}\zeta(k)T_\psi(x),
\]
\[
\phi(k, x) = \frac{3}{5}\zeta(k)T_\phi(x).
\]

From Eq. (13), it is obvious that the source function \(f(u, v, x)\) is symmetric about \(u\) and \(v\).

C. The power spectrum of SIGWs

The primordial curvature perturbation induces GWs in the MD era, and the energy density of SIGWs is

\[
\rho_{GW} = \frac{1}{32\pi G} \frac{1}{4a^2} \langle h_{ij}'h_{ij}' \rangle.
\]
By using the tensor power spectrum $\mathcal{P}_h$,

$$
\langle h_{k_1}^{t_1}(\eta)h_{k_2}^{t_2}(\eta) \rangle = \frac{2\pi^2}{k_1^3}\delta_{t_1 t_2}\delta^3(k_1 + k_2)\mathcal{P}_h(k_1, \eta), \quad t_i = +, \times, \quad (23)
$$

we get the energy density parameter $\Omega_{GW}(k, x)$ of SIGWs as

$$
\Omega_{GW} = \frac{d\rho_{GW}}{\rho_c d\ln k} = \frac{1}{24}\left(\frac{x}{2}\right)^2 \mathcal{P}_h(k, x), \quad (24)
$$

where an overbar stands oscillatory average and $\rho_c = 3H^2/8\pi G$ is the critical energy density of the universe. In deriving the second equality in Eq. (24), we use the fact that either polarization contributes equally to the energy density and GWs are null waves, so we make the replacement $|h'_k(\eta)|^2 = k^2|h_k(\eta)|$ in the sub-horizon limit with $k \gg H$ [56].

Combining Eqs. (10) and (23), we get [16, 17, 45, 48]

$$
\mathcal{P}_h(k, x) = 4\int_0^\infty du \int_{[1-u]}^{1+u} dv \left[ \frac{4u^2 - (1 + u^2 - v^2)}{4uv} \right]^2 I^2(u, v, x)\mathcal{P}_\zeta(uk)\mathcal{P}_\zeta(vk), \quad (25)
$$

where $\mathcal{P}_\zeta$ is the primordial scalar power spectrum.

### D. Newtonian gauge

To calculate the energy density, we need to choose a gauge. The SIGWs in Newtonian (Poisson) gauge was studied in [16, 18, 45], and here we give a review on the results. We introduced the transfer function $T$ to separate the time evolution defined by $\phi(k, \eta) = \phi(k, 0)T(\eta)$. In the Newtonian gauge, $B = E = 0$, and if the anisotropic stress vanishes, we have $\phi_N = \psi_N = \Phi = \Psi$, where the Bardeen’s potentials $\Phi$ and $\Psi$ defined in (A11) and (A12) on superhorizon scales are given by $\Phi = \Psi = 3\zeta/5$. Therefore, $\phi_N(k, 0) = 3\zeta(k)/5$. The subscript “N” indicates that these quantities are evaluated in the Newtonian gauge.

In the Newtonian gauge, the source function in terms of the transfer functions $T$ for the gravitational potential reads [45]

$$
f_N(v, u, x) = \frac{6(1 + 3w)(w + 1)}{(3w + 5)^2} \left( vxT_N^*(vx)T_N(ux) + uxT_N^*(ux)T_N(vx) \right) + \frac{3(1 + 3w)^2(1 + w)}{(3w + 5)^2} uvx^2T_N^*(vx)T_N^*(ux) + \frac{6(w + 1)}{3w + 5} T_N(vx)T_N(ux), \quad (26)
$$

where $T_N$ is the transfer function.


where $T_N(x)$, is the transfer function of the gravitational potential. In a matter-dominated universe where $w = 0$, it is given by

$$T_N(x) = 1,$$  \hspace{1cm} (27)

and the above equation (26) reduces to

$$f_N(u, v, x) = \frac{6}{5}.$$  \hspace{1cm} (28)

Combining Eqs. (9), (7) and (28), we get the explicit expression for the kernel $I_N(u, v, x)$

$$I_N(u, v, x) = \frac{6}{5} + \frac{18(x \cos x - \sin x)}{5x^3}.$$  \hspace{1cm} (29)

Since $I_N(u, v, x \rightarrow \infty) = 6/5$, Eq. (25) tells us that at late times $\mathcal{P}_h$ is a constant and the energy density of SIGWs is proportional to $x^2$. Thus $\Omega_{GW}(k, x \rightarrow \infty) \propto a$. However, from Eq. (10), we see that the constant $6/5$ in Eq. (29) contributes a constant to $h_k$, so the contribution to $h_k'$ and the energy density is zero. Therefore, the constant $6/5$ in Eq. (29) does not contribute to the energy density $\Omega_{GW}$. In other words, the constant in Eq. (29) does not represent a wave and GWs should oscillate as $\sin x$ and $\cos x$. After dropping the constant $6/5$, we have $I_N(x \rightarrow \infty) \propto \cos x / x^2 = \cos x / a$ leading to $\Omega_{GW} \propto a^{-1}$ and $\rho_{GW} \propto a^{-4}$, which behaves, as expected, as free GWs in MD era.

**E. The gauge transformation**

Now we discuss the gauge transformation and how the energy density in other gauges can be derived from the result in Newtonian gauge [41]. We start from the infinitesimal coordinate transformation $x^\mu \rightarrow x^\mu + \epsilon^\mu$ with $\epsilon^\mu = [\alpha, \delta^{ij} \partial_j \beta]$. For the discussion of SIGWs, we do not consider the vector degrees of freedom for the coordinate transformation, and the scalars $\alpha$ and $\beta$ are of first order. Since the gauge transformation of tensor modes does not depend on the coordinate transformation of the same order, we do not need to consider the second-order coordinate transformation. For the second-order tensor perturbation, we have

$$h_{ij}^{TT} \rightarrow h_{ij}^{TT} + \chi_{ij}^{TT},$$  \hspace{1cm} (30)

where

$$\chi_{ij}^{TT}(x, \eta) = \mathcal{T}_{ij}^{lm} \chi_{lm} = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} e^{i k \cdot x} \left[ \chi_k^+(\eta) e_{ij}^+ + \chi_k^-(\eta) e_{ij}^- \right],$$  \hspace{1cm} (31)
\[ \chi_{ij}(x, \eta) = 4 \left[ \alpha (E_{ij}^I + 2HE_{ij}) + E_{ijk}\beta_{jk} - 2\psi\beta_{ij} + E_{ijk}\beta_{jk} + E_{jik}\beta_{ik} \right] \\
+ 2(B_i\alpha_{i,j} + B_j\alpha_{i,j}) + 8\mathcal{H}\alpha\beta_{ij} - 2\alpha_i\alpha_{ij} + 4\beta_{i}\beta_{jk} + 2\alpha\beta_{ij} \\
+ 2\beta_{ijk}\beta_{k} + (\beta'_{i}\beta_{j} + \beta'_{j}\alpha_{i,j}) + 2 \left[ \left( \mathcal{H}^2 + \frac{a''}{a} \right) \alpha^2 + \mathcal{H} (\alpha\alpha' + \alpha'k) \right] \delta_{ij}, \] (32)

\[ \chi^I_k(\eta) = -\int \frac{d^3p}{(2\pi)^{3/2}} e^I_{ij}p^i p^j \left[ 4\alpha(p)\sigma(k - p) + 8\mathcal{H}\alpha(p)\left( E(k - p) + \beta(k - p) \right) \\
+ p \cdot (k - p)\beta(p) \left( 4E(k - p) + 2\beta(k - p) \right) - 8\psi(p)\beta(k - p) + 2\alpha(p)\alpha(k - p) \right] \\
= 4 \int \frac{d^3p}{(2\pi)^{3/2}} e^I_{ij}p^i p^j \zeta(p)\zeta(k - p) \frac{1}{k^2} I_{\chi}(u, v, x), \] (33)

and

\[ I_{\chi}(u, v, x) = -\frac{9}{100uv} \left[ 2T_{\alpha}(ux)T_{\sigma}(vx) + 2T_{\alpha}(vx)T_{\sigma}(ux) + 2T_{\alpha}(ux)T_{\alpha}(vx) \\
- 4 \left( \frac{u}{v} T_{\psi}(ux)T_{\beta}(vx) + \frac{v}{u} T_{\psi}(vx)T_{\beta}(ux) \right) \\
+ \frac{1 - u^2 - v^2}{uv} \left( T_{\beta}(ux)T_E(vx) + T_{\beta}(vx)T_E(ux) + T_{\beta}(ux)T_{\beta}(vx) \right) \right] \\
+ 8 \left( \frac{1}{v} T_{\alpha}(ux)T_E(vx) + \frac{1}{u} T_E(ux)T_{\alpha}(vx) \\
+ \frac{1}{v} T_{\alpha}(ux)T_{\beta}(vx) + \frac{1}{u} T_{\beta}(ux)T_{\alpha}(vx) \right). \] (34)

We have symmetrized \( I_{\chi}(u, v, x) \) under \( u \leftrightarrow v \). With the gauge transformation (30) and the result for SIGWs in the Newtonian gauge, it is straightforward to derive the semi-analytic expression for SIGWs in other gauges without performing the detailed calculation in that gauge. Combining Eqs. (3), (10), (30), (31), and (33), we get the following gauge transformation [41]

\[ h^I_k \to h^I_k + \chi^I_k = 4 \int \frac{d^3p}{(2\pi)^{3/2}} e^{ij}(k)p_ip_j\zeta(p)\zeta(k - p) \frac{1}{k^2} \left( I(u, v, x) + I_{\chi}(u, v, x) \right). \] (35)

This powerful transformation rule allows us to quickly transform the solution of second-order tensor perturbation in one gauge to other gauges, that is, we just replace \( I(u, v, x) \) by \( I(u, v, x) + I_{\chi}(u, v, x) \). By setting the initial gauge to be the Newtonian gauge, we obtain \( I_{\chi}(u, v, x) \) in other gauges:

\[ I_{\chi}(u, v, x) \to I_{\chi}(u, v, x) + I_{\chi}(u, v, x), \] (36)
where
\[ I_\chi(u, v, x) = - \frac{9}{100uv} \left[ -4 \left( \frac{u}{v} T_N(ux)T_\beta(vx) + \frac{v}{u} T_N(vx)T_\alpha(ux) \right) + 2T_\alpha(ux)T_\alpha(vx) \right. \]
\[ + \left. \frac{8}{x} \left( \frac{1}{v} T_\alpha(ux)T_\beta(vx) + \frac{1}{u} T_\beta(ux)T_\alpha(vx) \right) + \frac{1 - u^2 - v^2}{uv} T_\beta(ux)T_\beta(vx) \right], \] (37)
which can be obtained by substituting the transfer functions \( T_\sigma = T_E = 0 \) and \( T_\psi = T_N \) in the Newtonian gauge into Eq. (34). The coordinate transformations from the Newtonian gauge to the other gauges give the transfer functions \( T_\alpha \) and \( T_\beta \).

### III. THE KERNEL IN DIFFERENT GAUGES

The goal of this section is to compute the analytic expressions for the kernel \( I_\chi \) in six gauges from the Poisson gauge by using the coordinate transformation.

#### A. Synchronous gauge

First, we work in the synchronous gauge with \( \phi = B = 0 \). The equation for the transfer function \( T_E \) in synchronous gauge is given by
\[ x^3 T_E^{***} + 8x^2 T_E^{**} + 8x T_E^* - 8 T_E^* = 0. \] (38)
We derive the general solution for the transfer function \( T_E \) as
\[ T_E(x) = C_1 + \frac{x^2}{2} C_2 - \frac{C_3}{x} - \frac{C_4}{3x^3}, \] (39)
where \( C_i \) are integration constants. Note that there are two gauge modes in Eq. (39) because of the residual gauge freedom in the synchronous gauge \([66–70]\). To identify these two gauge modes, we take the residual gauge transformation \([67, 68]\)
\[ \alpha = \frac{C_5}{x^2}, \]
\[ \beta = -\frac{C_5}{x} + C_6, \] (40)
and after that we are still in synchronous gauge where \( \phi = B = 0 \). From the transformation (A6), we see that the constant \( C_6 \) term in \( \beta \) contributes to the integration constant \( C_1 \) in Eq. (39) and the \( C_5 \) term in \( \beta \) contributes to the \( 1/x \) term in Eq. (39). Therefore, \( C_1 \) and \( C_3 \) terms in Eq. (39) are just pure gauge modes and we can eliminate them by substituting
\( C_1 = C_3 = 0 \). Now we determine the remaining integration constants from the initial condition. At the initial time \( x = 0 \), assume that \( \mathcal{H}E'(0) \) is finite, and we get \( C_4 = 0 \). Finally, we use the initial condition of the gauge-invariant Bardeen potential to fix the constant \( C_2 \).

The gauge-invariant Bardeen potential in synchronous gauge is

\[ \Phi = -\mathcal{H}E' - E'' \],

so the transfer function \( T_\Phi \) is \( T_\Phi = -3C_2 \). From the initial condition \( T_\Phi(0) = T_N = 1 \), we derive \( C_2 = -1/3 \). Finally, after eliminating the gauge modes, we can write the transfer function \( T_E \) as \[47\]

\[ T_E(x) = -\frac{x^2}{6}. \] (42)

Then we find the transfer function \( T_\psi \) \[47, 70\]

\[ T_\psi(x) = -T_{E^*}(x) - \frac{4}{x} T_{E^*}(x) = \frac{5}{3}. \] (43)

Substituting Eqs. (43) and (42) into Eq. (9), we get

\[ I_{\text{syn}}(u, v, x) = \frac{x^2}{400} \left[ -88 + (-1 + u^2 + v^2)x^2 \right] + \frac{6 (x^3 - 3 \sin x + 3x \cos x)}{5x^3} \] (44)

Next, we discuss the derivation of the kernel by using the gauge transformation. The coordinate transformation from the Newtonian gauge to the synchronous gauge is

\[ \alpha(k, x) = \frac{3}{5} \zeta(k) \frac{1}{k} T_\alpha(x), \]

\[ \beta(k, x) = \frac{3}{5} \zeta(k) \frac{1}{k^2} T_\beta(x), \]

where

\[ T_\alpha(x) = -\frac{x}{3}, \]

\[ T_\beta(x) = -\frac{x^2}{6}. \] (46)

Substituting Eq. (46) into Eq. (37), we get

\[ I_\chi(u, v, x) = \frac{x^2}{400} \left[ -88 + (-1 + u^2 + v^2)x^2 \right]. \] (47)

We confirm that \( I_{\text{syn}} = I_N + I_\chi \). It is noteworthy that in the above Eq. (47), there is no oscillating term to represent free GWs. Therefore, the kernel \( I_\chi(u, v, x) \) does not contribute to the free GWs and \( \Omega_{\text{GW}} \) in both the Newtonian gauge and the synchronous gauges are the same.
B. Comoving orthogonal gauge

In the comoving gauge, $\delta V = 0$ and $B = 0$ \[59\]. This gauge also retains a residual coordinate transformation $\beta = C$ which corresponds to arbitrary choice of the origin of the spatial coordinates. For the time coordinate transformation, the variable $\alpha$ is given by

$$\alpha = \frac{\mathcal{H}\phi_N + \dot{\phi}_N}{\mathcal{H}' - \mathcal{H}^2}. \quad (48)$$

From the above expression, we can find the transfer function $T_\alpha(x)$ as

$$T_\alpha(x) = -\frac{x}{3}. \quad (49)$$

The general solution of the transfer function $T_\beta$ is

$$T_\beta(x) = -\frac{x^2}{6} + C. \quad (50)$$

The last constant $C$ term is a pure gauge mode, we can choose $C = 0$ so that $T_\beta(x = 0) = 0$. Combining Eqs. (37), (49) and (50), we get

$$I_\chi(u, v, x) = -\frac{9x^2}{200}. \quad (51)$$

It is interesting to note that the kernel in the comoving orthogonal gauge is identical with that in synchronous gauge and there is no oscillation. For the free GWs, we obtain the same $\Omega_{GW}$ in the comoving orthogonal gauge as that in the Newtonian gauge.

C. Uniform curvature gauge

The uniform curvature gauge demands $\psi = E = 0$. The coordinate transformation from the Newtonian gauge to the uniform curvature gauge is

$$\alpha = \frac{\phi_N}{\mathcal{H}} = \frac{3}{5} \zeta(k) \frac{1}{k} T_\alpha(x), \quad (52)$$

$$\beta = 0, \quad (53)$$

where the transfer function $T_\alpha$ is

$$T_\alpha = x T_N/2. \quad (54)$$

Substituting these results of the transfer functions into Eq. (37), we get

$$I_\chi(u, v, x) = -\frac{9x^2}{200}. \quad (55)$$

This term is a growing mode and there is no oscillation. For the free GWs, we obtain the same $\Omega_{GW}$ in the uniform curvature gauge as that in the Newtonian gauge.
D. Comoving gauge (total matter gauge)

The comoving gauge (also referred as the total matter gauge [59]) is defined by the condition, $\delta V = E = 0$. The transfer functions for the coordinate transformation from the Newtonian gauge to the comoving gauge are

$$T_\alpha(x) = -\frac{x}{3},$$  \hspace{1cm} (56)

and $T_\beta(x) = 0$. Substituting Eq. (56) into Eq. (37), we get

$$I_\chi(u, v, x) = -\frac{x^2}{50}.$$ \hspace{1cm} (57)

This term is a growing mode and there is no oscillation. For the free GWs, we obtain the same $\Omega_{\text{GW}}$ in the uniform curvature gauge as that in the Newtonian gauge.

E. Uniform density gauge

The uniform density gauge is defined by the condition, $\delta \rho = E = 0$. The transfer functions for the coordinate transformation from the Newtonian gauge to this gauge are

$$T_\alpha(x) = -\frac{12x + x^3}{36},$$ \hspace{1cm} (58)

and $T_\beta(x) = 0$. Substituting Eq. (58) into Eq. (37), we get

$$I_\chi(u, v, x) = -\frac{x^2(12 + u^2x^2)(12 + v^2x^2)}{7200}.$$ \hspace{1cm} (59)

This term is a growing mode and there is no oscillation. For the free GWs, we obtain the same $\Omega_{\text{GW}}$ in the uniform curvature gauge as in the Newtonian gauge.

F. Uniform expansion gauge

At last, we consider the uniform expansion gauge, $3(\mathcal{H}\phi + \psi') + k^2 \sigma = 0$ and $E = 0$ [64]. The transfer functions for the coordinate transformation from the Newtonian gauge to this gauge are

$$T_\alpha(x) = -\frac{6x}{18 + x^2},$$ \hspace{1cm} (60)
and $T_\beta(x) = 0$. Substituting Eq. (60) into Eq. (37), we get

$$I_\chi(u, v, x) = -\frac{162x^2}{25(u^2x^2 + 18)(v^2x^2 + 18)}.$$  \hspace{1cm} (61)

At late times, this kernel is a decaying mode and there is no oscillation. For the free GWs, we obtain the same $\Omega_{GW}$ in the uniform curvature gauge as in the Newtonian gauge.

G. Free SIGWs

Because of the non-oscillating terms like $x^n$ in $I_\chi$, we obtain different SIGWs in different gauges. However, if we consider the free SIGWs with oscillating forms $\sin x$ and $\cos x$, the pure time increasing or decaying terms with $x^n$ in $I_\chi$ do not contribute to $\Omega_{GW}$, and we obtain $\Omega_{GW} \propto 1/a$ in MD in all considered gauges.

For the monochromatic power spectrum, $\mathcal{P}_\zeta = A_\zeta \delta (\ln k - \ln k_*)$, the energy density for free SIGWs in MD is

$$\Omega_{GW}(k, x) = \frac{1}{48x^2} A_\zeta^2 \left(1 - \left(\frac{k}{2k_*}\right)^2\right)^2 \Theta(2k_* - k),$$  \hspace{1cm} (62)

where $\Theta(x)$ is the Heaviside theta function, and $k_*$ is the wave-number of the peak in the power spectrum. Since the energy density of matter decays as $a^{-3}$ and the energy density of SIGWs decays as $a^{-4}$, so $\Omega_{GW}$ decays $1/a$ in MD.

IV. CONCLUSION

In this paper, we study the solutions for scalar induced tensor perturbations in various gauges in MD. Since the time dependence of SIGWs $h_\ell^i(\eta)$ lies in the integral kernel $I(u, v, x)$, we explicitly calculate the analytical expressions for the kernels $I_N$ in the Newtonian and $I_{syn}$ in the synchronous gauges. We also derive the relation between the kernels in different gauges under the coordinate transformation, then we obtain the analytical expressions for the kernel in six different gauges, namely the synchronous gauge, the comoving orthogonal gauge, the uniform curvature gauge, the total matter gauge, the uniform density gauge, and the uniform expansion gauge from $I_N$ in the Newtonian gauge by using the gauge transformation. The direct calculation of $I_{syn}$ in the synchronous gauge confirms that it is the same as that obtained from $I_N$ using the gauge transformation. There are two residual gauge modes in the
synchronous gauge and one residual gauge mode in the comoving orthogonal gauge. After identifying and eliminating the gauge modes, we find that the kernels in the synchronous gauge and the comoving orthogonal gauge are the same.

Although the derived kernels are different in different gauges, but the differences are either growing or decaying modes with the behaviour $x^n$ or $a^{n/2}$. For free SIGWs oscillating as $\sin x$ and $\cos x$, we find that at late times $\rho_{GW} \propto a^{-4}$, and $\Omega_{GW} \propto a^{-1}$ in all seven gauges, i.e., free SIGWs behave the same as radiation in MD. With the analytical expression for the kernel, we give the analytical result of the energy density $\Omega_{GW}$ for the monochromatic power spectrum. Our results are helpful for the probe of the thermal history of the universe by using SIGWs.

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**Appendix A: Gauge transformation**

In this appendix, we present gauge transformations of perturbations.

1. **The energy-momentum tensor**

A perfect fluid has the stress energy-momentum tensor of the form

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P g_{\mu\nu} + \Pi_{\mu\nu},$$  \hspace{1cm} (A1)

where the anisotropic stress $\Pi_{\mu\nu}$ is assumed to be zero. The first-order perturbations of the velocity $U_\mu$, the energy density, the pressure, and the anisotropic stress are $\delta U_\mu$, $\delta \rho$, $\delta P$, and $\delta \Pi_{ij}$, respectively. The first-order velocity perturbation $\delta U_\mu$ is decomposed via $\delta U_\mu = a[\delta V_0, \delta V_i + \delta V_i]$ with $\delta V_{i,i} = 0$. Notice $\delta V$ we defined here relates to the perturbations $v_M$ and $B_M$ in Ref. [59] by $\delta V = v_M + B_M$. 

2. Gauge transformations

One may perform a gauge transformation of the form $x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu(x)$ under the general infinitesimal coordinate transformation, with $\epsilon^\mu = [\alpha, \delta^{ij} \partial_j \beta]$. The first-order gauge transformations are written as

\begin{align*}
\tilde{\phi} &= \phi + \mathcal{H} \alpha + \alpha', \\
\tilde{\psi} &= \psi - \mathcal{H} \alpha, \\
\tilde{\sigma} &= \sigma + \alpha, \\
\tilde{B} &= B - \alpha + \beta', \\
\tilde{E} &= E + \beta, \\
\delta \tilde{\rho} &= \delta \rho + \rho'_0 \alpha, \\
\delta \tilde{P} &= \delta P + P'_0 \alpha, \\
\delta \tilde{V} &= \delta V - \alpha, \\
\delta \tilde{\Pi} &= \delta \Pi,
\end{align*}

where $\Pi$ is the scalar part of the (trace-free) anisotropic stress. One can obtain two gauge-invariant Bardeen potentials by using the above gauge transformation:[71]

\begin{align*}
\Phi &= \phi - \mathcal{H} \sigma - \sigma', \\
\Psi &= \psi + \mathcal{H} \sigma.
\end{align*}

The gauge transformation for the SIGWs under the infinitesimal coordinate transformation is $h_{ij}^{TT} \to h_{ij}^{TT} + \chi_{ij}^{TT}$, where

\begin{align*}
\chi_{ij} &= 2 \left( \mathcal{H}^2 + \frac{a''}{a} \right) \alpha^2 + \mathcal{H} \left( \alpha \alpha' + \alpha \epsilon^k \right) \delta_{ij} \\
&\quad + 4 \left[ \alpha \left( C'_{ij} + 2 \mathcal{H} C_{ij} \right) + C_{ij,k} \epsilon^k + C_{ik} \epsilon^j + C_{jk} \epsilon^i \right] \\
&\quad + 2 \left( B_i \alpha_j + B_j \alpha_i \right) + 4 \mathcal{H} \alpha \left( \epsilon_{ij} + \epsilon_{ji} \right) - 2 \alpha \epsilon_{ij} \alpha_j + 2 \epsilon_{k,i} \epsilon^k_j + \alpha \left( \epsilon'_{ij} + \epsilon'_{ji} \right) \\
&\quad + \epsilon_{i,jk} + \epsilon_{j,ik} \epsilon^k_j + \epsilon_{i,k} \epsilon^j_k + \epsilon_{j,k} \epsilon^i_k + \epsilon_{i,j} \alpha_j + \epsilon_{j,i} \alpha_i,
\end{align*}

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and \( C_{ij} = -\psi \delta_{ij} + E_{ij} \).

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