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A robust study of a piecewise fractional order COVID-19 mathematical model

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Abstract In the current manuscript, we deal with the dynamics of a piecewise covid-19 mathematical model with quarantine class and vaccination using SEIQR epidemic model. For this, we discussed the deterministic, stochastic, and fractional forms of the proposed model for different steps. It has a great impact on the infectious disease models and especially for covid-19 because in start the deterministic model played its role but with time due to uncertainty the stochastic model takes place and with long term expansion the use of fractional derivatives are required. The stability of the model is discussed regarding the reproductive number. Using the non-standard finite difference scheme for the numerical solution of the deterministic model and illustrate the obtained results graphically. Further, environmental noises are added to the model for the description of the stochastic model. Then take out the existence and uniqueness of positive solution with extinction for infection. Finally, we utilize a new technique of piecewise differential and integral operators for approximating Caputo-Fabrizio fractional derivative operator for the purpose of constructing the fractional-order model. Then study the dynamics of the models such as positivity and boundedness of the solutions and local stability analysis. Solved numerically fractional-order model used Newton Polynomial scheme and present the results graphically.

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1. Introduction

Understanding and predicting natural phenomenon is one of the powerful tools used by human for predicting and avoiding many worst scenarios in the recent few decades. Indeed, mathematics proved its usefulness in this regards, which it can be
seen through the huge involvement in the recent scientific fields, as physics [1], ecology [2,3], and epidemiology [4–7].

Our planet witnessed the recent outbreak of the new COVID-19 disease. Indeed, COVID-19 abbreviation used for the infectious disease coronavirus, is a contagious disease which spread throughout the world since Dec 2019. This viral disease is mostly connected with the closed contact rates of publics. Although WHO and researchers have taken all the necessary precautions to curb the speed of this virus spreading see [8–13]. COVID-19 has affected more than 21 million confirmed copulative cases around the world and more deaths due to it occurred. In this scenario WHO, all other health organizations throughout the countries, all governments, all doctors and researchers are in struggle for the controlling of the rapid spread of COVID-19 using stopping migrations from one country to another country, masks, social distances, hand wash and specially vaccination.

While mathematical models have the capability to describe the spreading nature and controlling of infectious diseases. In mathematical models, compartments are always taken in different groups of the population with respect to their epidemiological conditions. These groups are generally modeled using ordinary differential equation as the papers [14,15]. Indeed, other equations are used for describing different aspect as age dependence in [16–18], fractional order systems [19–22] and other approximations [23–25].

The transfer rates among these compartments described mathematically via derivatives (integer-order derivatives and fractional-order derivatives) see [26–28,30–32]. In the real world, every thing is the subject of the random effect, where the movement of persons, animals. This effect is interpreted mathematically using stochastic models. Stochastic models have been studied [33–37] and still [8,38–41] the subject of interest of the recent research activities. In literature, till now several models have been done with respect to the nature of different diseases, but the start was a simple SIR model. The development and investigation of this type of model provide us tools for describing and characterizing its transmission, and thus, we are able to propose successful techniques to foresee, prevent, and control infections, as well as to ensure population well-being. Up to presently, numerous mathematical models have been considered and analyzed to ponder the spreading of infections including (COVID-19).

Besides, based on the paper [42] it has been mentioned that the fractional-time-derivative has a important relationship with the memory which it has been applied to various fields as physics [42], biology and ecology [44,45], economy [46], chemistry [43], which highlights the huge relevance in epidemiology and other fields. In fact, it is mentioned that the order of the time derivative highlights the memory rate, and the kernel of the fractional-order derivative is the memory function. Indeed, the fractional derivative is very useful to examine the rate of gradual change in the growth of the individual by revealing every slight change in the dynamic system. The application of the fractional derivative covers a huge part of the recent research activities as epitome the researches [9,10,12,13,47–50].

Motivated from the above literature works, this paper is done to continue this research line by introducing a COVID-19 model that takes into account the existence of vaccines in stepwise form. Our paper is organized as follows: Section 2, is related to provide some basics definitions and results that will be later needed. Section 3 is devoted to the description of the piecewise model. In Section 4, we provide the extinction scenario of the determinist model in terms of the basic reproduction number. Next, in Section 5, we will analyze the fractional order system, where we will show the existence and the uniqueness of the solution next to some results concerning the asymptotic behavior of the solution. In Section 6, the solution of the piecewise fractional-order model is presented. In Section 7, we provide the existence result and the well posedness of the stochastic model. Section 8 is the responsible for the graphical representation of different models considered in this paper, with different methods, and we highlight the difference between them. Some concluding and remarks ends the paper.

2. Preliminaries

Here we recall some definitions and results for further uses.

Definition. [51] The Caputo-Fabrizio (CF) fractional derivative of order \( \alpha \) for a function \( \theta \in H^1(c, d) \) and \( 0 < \alpha < 1 \), is given by:

\[
\text{\text{CF}}D^\alpha \theta(t) = \frac{1}{1-\alpha} \int^t_c \frac{d\theta(\lambda)}{d\lambda} \exp[-\sigma(t-\lambda)]d\lambda
\]

where \( \sigma = \frac{1}{1-\alpha} \).

The respective CF fractional integral is defined by

\[
\text{\text{CF}}I^\alpha \theta(t) = (1-\alpha)\theta(t) + \alpha \int^t_c \theta(\lambda)d\lambda.
\]

Theorem 1. [52] Let \( \mathcal{M} \) be a compact metric space and \( C(\mathcal{M}, \mathbb{R}) \) denotes the space of continuous functions when endowed with the supremum norm metric. A set \( \mathcal{E} \subset C(\mathcal{M}, \mathbb{R}) \) is compact if and only if \( \mathcal{E} \) is bounded, closed and equicontinuous.

Definition. [29] The modified Caputo fractional derivative operator, \( D^\alpha_{d} \), of order \( \alpha > 0 \) is given by:

\[
(D^\alpha_{d} \Psi)(\xi) = \frac{\sigma^{\alpha-\alpha+1}}{1-\alpha} \int^\xi_d s^{\alpha-1}(\xi-s)^{\alpha-\alpha+1} (s^{d/ds})^\alpha \Psi(s)ds, \quad \xi > d
\]

where \( \sigma > 0, d \geq 0, \sigma \alpha > 1, \sigma \alpha + 1 \leq \alpha n \).

Definition. [53] We consider Ito’s process \( X_t \), processus stochastique de la forme

\[
(X_t) = \int^t_0 \theta(s)ds + \int^t_0 \int^t_s \sigma(s) dW_r dr
\]

where \( \theta(s) \) is a deterministic function, \( \sigma(s) \) is a \( \mathcal{F}_s \)-adapted \( \mathcal{P} \)-square integrable function, \( \{W_t : t \geq 0\} \) is a \( \mathcal{F} \)-Brownian motion, \( \mathcal{F}_t \) is the filtration generated by \( \{W_t : t \geq 0\} \) and \( \mathcal{P} \) is the probability measure.
formulated otherwise, we get
\[ dX = \mu_1 \, dt + \sigma_1 \, dB_t, \]
with \( \mu_1 \) and \( \sigma_1 \) are two random processes satisfying some adaptation theoretical hypothesis of \( B_t \) (Brownien movement).

If \( f(X, t) \) is a function of class \( C^2([0, \infty) \times \mathbb{R}, \mathbb{R}) \), then Ito’s formula can be expressed as
\[ \frac{d}{dt}f(X, t) = \frac{\partial f}{\partial t}(X, t) + \sum_i \frac{\partial f}{\partial x_i}(X, t) \cdot \frac{dX^i}{dt} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(X, t) \sigma_{ij}(X, t) \, dt. \]

3. Mathematical models

Efforts have been made by several scientists in the last two years to develop vaccines that will be used to reduce the spread of covid-19. WHO has now validated some vaccines including Pfizer-BioNTech, Moderna, Johnson and Johnson Janssen, and other. With no doubt the mortality rate and infection rate have reduced significantly, although nowadays a fourth wave has emerged. While vaccine has played a very important role in reducing the mortality rate, it has been noticed that, the infection rate is high during cold seasons, thus beside the used vaccine, different governments have put measure in place to isolate infected persons, additionally, some additional measures like lockdown have been implemented [54-58]. Prediction of vaccine, different governments have put measure in place to isolate infected persons, additionally, some additional measures like lockdown have been implemented [54-58]. Prediction of future possible scenarios help law makers to take some decisions in order to avoid high spread, these predictions are achieved using mathematical models. Due to complexity of the spread, a \( SEIR \) mathematical model may not be able to replicate spread of covid-19 among humans. Therefore, in addition to the equations of \( SEIR \) model, we consider adding a quarantined class with vaccination. For this purpose, we consider that the susceptible population is represented as \( S \), second one is exposed individual \( E \), third is infectious class \( I \), fourth one is quarantined \( Q \) (in which the infectious peoples are placed for isolation), and last one is the recovered class \( R \), with temporary immunity. The flow of the population is described in the following system of differential equations as:
\[
\begin{align*}
\frac{dS}{dt} &= (1 - q)b - \beta S E - dS + \delta Q, \\
\frac{dE}{dt} &= \beta S E - (\eta_1 + \eta_2 + d) E, \\
\frac{dI}{dt} &= \eta_1 E - (\eta + \gamma + d + \sigma_1) I, \\
\frac{dQ}{dt} &= \eta_2 E + \eta I - (\rho + d + \sigma_2) Q, \\
\frac{dR}{dt} &= \gamma I + \rho Q - (d + \delta) R + qb, \\
\end{align*}
\]
where \( b \) describes the enrollment rate of the population that directly joins the susceptible class \( S \), \( \beta \) stands for the contact rate incidence rate at which susceptible class joins exposed class \( E \), \( \eta_1 \) and \( \eta_2 \) are the rates at which the exposed class population joint the infected class and quarantine class respectively. \( d \) denotes the out going rate of each class in the form of natural death or migration rate from each class. \( \gamma \) is the recovered rate of infected class to join recovered class \( R \), \( \eta \) is the rate at which the infectious class comes to quarantine class \( Q \), and \( p \) is the recovered rate of quarantine class people. Moreover, \( \sigma_1 \) and \( \sigma_2 \) are the disease related deaths rates for infected class and quarantined class, \( \delta \) shows the relapse rate at which the recovered class \( R \) moves to susceptible class and \( q \) represents the vaccine rate, that is, the proportion of the susceptible class that becomes vaccinated with \( 0 \leq q \leq 1 \). To include in the model (2) the past history or hereditary properties, here we adopt a new technique piecewise differential and integral operators developed by Atangana– Seda for the Caputo-Fabrizio fractional derivative in [26]. Motivated by the discussion done in the introduction section, we replace the classical derivative by the piecewise derivatives and follow the model as
\[
\begin{align*}
\frac{D_t^\alpha S}{S} &= (1 - q)b - \beta S E - dS + \delta Q, \\
\frac{D_t^\alpha E}{E} &= \beta S E - (\eta_1 + \eta_2 + d) E, \\
\frac{D_t^\alpha I}{I} &= \eta_1 E - (\eta + \gamma + d + \sigma_1) I, \\
\frac{D_t^\alpha Q}{Q} &= \eta_2 E + \eta I - (\rho + d + \sigma_2) Q, \\
\frac{D_t^\alpha R}{R} &= \gamma I + \rho Q - (d + \delta) R + qb, \\
\end{align*}
\]
where \( 0 < \alpha < 1 \) and \( \frac{D_t^\alpha}{D_t^\alpha} \) denotes the fractional derivative in the piecewise sense.

While by including the environmental noise to model (2) follows a stochastic model. Let \( \sigma_1, \sigma_2, \sigma_3, \sigma_4 \) and \( \sigma_5 \) are the intensities, while \( B_1(t), B_2(t), B_3(t), B_4(t) \) and \( B_5(t) \) are environmental noise functions. The stochastic model is given by:
\[
\begin{align*}
\frac{dS}{dt} &= ((1 - q)b - \beta S E - dS + \delta Q) + \sigma_1 S dB_1(t), \\
\frac{dE}{dt} &= (\beta S E - (\eta_1 + \eta_2 + d) E) + \sigma_2 E dB_2(t), \\
\frac{dI}{dt} &= (\eta_1 E - (\eta + \gamma + d + \sigma_1) I) + \sigma_3 I dB_3(t), \\
\frac{dQ}{dt} &= (\eta_2 E + \eta I - (\rho + d + \sigma_2) Q) + \sigma_4 Q dB_4(t), \\
\frac{dR}{dt} &= (\gamma I + \rho Q - (d + \delta) R + qb) + \sigma_5 R dB_5(t). \\
\end{align*}
\]

4. Extinction scenario for the model (3)

4.1. Free virus equilibrium point and reproduction number

Disease free equilibrium (DFE) point of model (2) is given by
\[
F_0 = (S_0, E_0, I_0, Q_0, R_0) = \left( \frac{b((1-q)d+\delta)}{d(\delta+d)}, 0, 0, 0, \frac{qb}{\delta+d} \right) \tag{5}
\]
For the reproductive number of model (2), suppose that \( y = (E, I, \sigma) \) and using next generation matrix approach [6], we have
\[
\frac{dy}{dt} = \mathcal{F}(y) - \mathcal{V}(y), \tag{6}
\]
where Jacobian of \( \mathcal{F} \) and \( \mathcal{V} \) are
\[
\begin{align*}
\mathcal{F}(y) &= \left( \begin{array}{c}
\frac{\partial \beta S E}{\partial \sigma} \\
(\eta_1 + \eta_2 + d) \frac{\partial \sigma}{\partial \sigma} \\
(\eta + \gamma + d + \sigma_1) \frac{\partial \sigma}{\partial \sigma} \\
\rho + d + \sigma_2 \frac{\partial \sigma}{\partial \sigma} \\
\end{array} \right), \\
\mathcal{V}(y) &= \left( \begin{array}{c}
-\eta_1 \frac{\partial \sigma}{\partial \sigma} \\
\eta_1 \frac{\partial \sigma}{\partial \sigma} + (\eta + \gamma + d + \sigma_1) \frac{\partial \sigma}{\partial \sigma} \\
(1-q) b + \beta S E - d \sigma_1 \frac{\partial \sigma}{\partial \sigma} - \delta \sigma_1 \frac{\partial \sigma}{\partial \sigma} \\
\end{array} \right). \tag{7}
\end{align*}
\]
At $F_0$, we have
\[
V(F_0) = \begin{pmatrix} \frac{b((1-q)d+\delta)}{d+\delta} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
V(F_0) = \begin{pmatrix} \eta_1 + \eta_2 + d & 0 & 0 \\ -\eta_1 & \eta + \gamma + d + \sigma_1 & 0 \\ 0 & \frac{b((1-q)d+\delta)}{d+\delta} & 0 \end{pmatrix}.
\]
Hence, reproductive number for model (2) is
\[
\psi_0 = \rho(V^{-1}) = \frac{b((1-q)d+\delta)}{d(\delta + d)(\eta_1 + \eta_2 + d)}.
\]
The result about positive endemic equilibrium point is contained in the next theorem.

**Theorem 2.** There exists a unique positive endemic equilibrium point $F^*$ for system (2) if $\psi_0 > 1$.

**Proof.** Endemic equilibrium point is obtained from system (2) which is the positive solution of the following system
\[
0 = (1-q)b - \beta \mathcal{S}, \quad \mathcal{I} = d \mathcal{S}, \quad \mathcal{R} = \delta \mathcal{S},
\]
\[
0 = \beta \mathcal{S} - (\eta_1 + \eta_2 + d) \mathcal{E},
\]
\[
0 = \eta_1 \mathcal{E} - (\eta + \gamma + d + \sigma_1) \mathcal{I},
\]
\[
0 = \eta_2 \mathcal{E} + \eta \mathcal{I} - (\rho + d + \sigma_2) \mathcal{R},
\]
\[
0 = \gamma \mathcal{I} + \rho \mathcal{R} - (d + \delta) \mathcal{S} + qb.
\]
by a straightforward calculation, we deduce that the system (9) can be expressed as
\[
\mathcal{S} = \begin{pmatrix} \frac{\eta_1 + \eta_2 + d}{\beta} \\ \eta + \gamma + d + \sigma_1 \\ \eta \mathcal{I} + \frac{\eta_1 + \eta_2 + d}{\rho + d + \sigma_2} \mathcal{R} \end{pmatrix},
\]
\[
\mathcal{I} = \begin{pmatrix} \eta_1 \eta_2 + \eta (\eta + \gamma + d + \sigma_1) \\ \eta_1 \eta_2 + \eta (\eta + \gamma + d + \sigma_1) \mathcal{R} + \eta \mathcal{I} \end{pmatrix},
\]
\[
\mathcal{R} = \begin{pmatrix} \eta_1 \eta_2 + \eta (\eta + \gamma + d + \sigma_1) \mathcal{R} + \eta \mathcal{I} \\ \eta_1 \eta_2 + \eta (\eta + \gamma + d + \sigma_1) \mathcal{R} + \eta \mathcal{I} \end{pmatrix},
\]
\[
\mathcal{E} = \mathcal{S} \mathcal{I} + \gamma + \rho \mathcal{R}, \quad \mathcal{I} = \begin{pmatrix} \eta_1 \eta_2 + \eta (\eta + \gamma + d + \sigma_1) \mathcal{R} + \eta \mathcal{I} \end{pmatrix}.
\]
where
\[
\Phi(\mathcal{S}) = \frac{b((1-q)d+\rho)}{d+\delta} + \frac{\delta (\gamma + \rho \mathcal{I})}{d+\delta} = \frac{\eta_1 (\eta_1 + \eta_2 + d)}{\eta + \gamma + d + \sigma_1}.
\]
It is obvious from the values of $\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{R}$ and $\mathcal{P}$ that there exists a unique positive endemic equilibrium point $F^*$ if $\psi_0 > 1$.

**Theorem 3.** The system (2) is locally stable at $F_0$ for $\psi_0 < 1$ and unstable for $\psi_0 > 1$.

**Proof.** The Jacobian of system (2) is
\[
J = \begin{pmatrix} -\beta \mathcal{E}_1 - d - \beta \mathcal{I}_1 - \delta & 0 & 0 & 0 \\ \beta \mathcal{E}_1 & \beta \mathcal{I}_1 - k_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]
where
\[
k_1 = \eta_1 + \eta_2 + d, \quad k_2 = \eta_1 \gamma + d + \sigma_1, \quad k_3 = \rho + d + \sigma_2, \quad k_4 = \delta + d.
\]
Along $F_0$, it implies that
\[
J(F_0) = \begin{pmatrix} -d - \beta (1-q)d+\delta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]
which follows that all the eigenvalues are negative if $\psi_0 < 1$ and eigenvalue $\lambda_2$ is positive for $\psi_0 > 1$. Hence, we conclude that the system (2) is locally stable under the condition $\psi_0 < 1$ and unstable for $\psi_0 > 1$.

**Theorem 4.** The system (2) is globally stable, if $\psi_0 > 1$ at $F_0$.

**Proof.** First, we construct the Lyapunov function $\mathcal{V}(t)$, for the system as:
\[
\mathcal{V}(t) = 1 + \mathcal{E}_1 + \ln \mathcal{E}_1.
\]
Then differentiating the Eq. (14) with respect to time, we have
\[
\dot{\mathcal{V}}(t) = \left( \begin{array}{c} 1 + \frac{\mathcal{E}_1}{\mathcal{E}_1} \\ \mathcal{I}_1 \\ \mathcal{R}_1 \\ \mathcal{P}_1 \end{array} \right) = \dot{\mathcal{V}} + \mathcal{E}_1 \mathcal{E}_1 - (\eta_1 + \eta_2 + d).
\]
By manipulating along the point $F_0$, we get
\[
\dot{\mathcal{V}}(t) = -(\beta \mathcal{I}_1 - (\eta_1 + \eta_2 + d))
\]
\[
= -(\beta (1-q)d+\delta) - (\eta_1 + \eta_2 + d)
\]
\[
= -\eta_1 \eta_2 \rho d + (\beta (1-q)d+\delta) - (\eta_1 + \eta_2 + d)
\]
\[
\leq 0 \text{ for } \psi_0 > 1.
\]
Therefore, if $\psi_0 > 1$, then $\dot{\mathcal{V}}(t) < 0$, which implies that the system (2) is globally stable for $\psi_0 > 1$ at $F_0$.

**Remark 1.** The analysis of stability of $F^*$ is an interesting mathematical problem, but in this work, we mainly focus on the case $\psi_0 < 1$ to find effective strategy to prevent the disease.

5. Analysis of the fractional order system

5.1. Existence and uniqueness and positivity of solution

Suppose that
\[
\mathbb{R}_+ = \{(\mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{R}) \mid \mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{R} \geq 0\}.
\]
From [29] and utilizing a generalized mean value theorem and a fractional comparison principle, the proof of the following theorem is achieved. We state the analysis for the Caputo-Fabrizio fractional model (3).

**Theorem 5.** [Positivity and boundedness] Let \((\mathcal{S}_0, \mathcal{I}_0, \mathcal{F}_0, 0, \mathcal{R}_0)\) be any initial data belonging to \(\mathbb{R}_+^5\) and \((\mathcal{S}_t, \mathcal{I}_t, \mathcal{F}_t, 2, \mathcal{R}_t)\) the corresponding solution of model (3) to the given initial data. The set \(\mathcal{R}_+^5\) is positively invariant. Furthermore, we have

\[
\begin{align*}
\limsup_{t \to \infty} \mathcal{S}_t & \leq \mathcal{S}_\infty := \frac{1}{\eta_1^* + \eta_2^* + d^*}, \\
\limsup_{t \to \infty} \mathcal{I}_t & \leq \mathcal{I}_\infty := \frac{1}{\eta_1^* + \eta_2^* + d^*}, \\
\limsup_{t \to \infty} \mathcal{F}_t & \leq \mathcal{F}_\infty := \frac{1}{\eta_1^* + \eta_2^* + d^*}, \\
\limsup_{t \to \infty} \mathcal{R}_t & \leq \mathcal{R}_\infty := \frac{1}{\eta_1^* + \eta_2^* + d^* + q^* b^*}.
\end{align*}
\]

**Proof.** From model (3), we have

\[
\frac{d}{dt} \mathcal{S}_t = (1 - q^*) b^* - d^* \mathcal{S}_t + \delta^* \mathcal{R}_t, \quad \mathcal{S}_0 > 0,
\]

\[
\frac{d}{dt} \mathcal{I}_t = 0,
\]

\[
\frac{d}{dt} \mathcal{F}_t = \eta_1^* \mathcal{I}_t > 0,
\]

\[
\frac{d}{dt} \mathcal{R}_t = \gamma^* \mathcal{I}_t + \rho^* \mathcal{I}_t + q^* b^* \mathcal{R}_t \geq 0.
\]

For all \(t \geq 0\), with the help of generalized mean value theorem [29] and system (16), we can conclude that \(\mathcal{S}_t, \mathcal{I}_t, \mathcal{F}_t, 2, \mathcal{R}_t \geq 0\). First equation of system (3), implies that

\[
\mathcal{S}_t < \mathcal{S}_\infty := \frac{1}{\eta_1^* + \gamma^* + d^*},
\]

From the second equation of system (3), we have

\[
\mathcal{I}_t < \mathcal{I}_\infty := \frac{1}{\eta_1^* + \gamma^* + d^*},
\]

\[
\mathcal{F}_t < \mathcal{F}_\infty := \frac{1}{\eta_1^* + \gamma^* + d^*},
\]

\[
\mathcal{R}_t < \mathcal{R}_\infty := \frac{1}{\eta_1^* + \gamma^* + d^*}. \qquad \text{(15)}
\]

Consequently, we have the second estimate of (15). From the third equation of the system (3), we get

\[
\mathcal{S}_t < \mathcal{S}_\infty := \frac{1}{\eta_1^* + \eta_2^* + \gamma^* + d^* + \sigma_1^*},
\]

for \(t \) large enough. This follows the third estimate of (15). Now with the help of fourth equation of the system (3), we have

\[
\mathcal{I}_t < \mathcal{I}_\infty := \frac{1}{\eta_1^* + \eta_2^* + \gamma^* + d^*},
\]

for \(t \) large enough. This implies the fourth estimate of (15).

Finally, from the last equation of system (3), we have

\[
\mathcal{R}_t < \mathcal{R}_\infty := \frac{1}{\eta_1^* + \eta_2^* + \gamma^* + d^* + q^* b^*},
\]

for \(t \) large enough and the fourth estimate of (15) holds. Therefore, we deduce the existence and the uniqueness of solution, and the boundedness of the solution [30]. The proof is achieved.

### 5.2. Mathematical analysis of the fractional model (3)

#### 5.2.1. Free virus equilibrium point and reproduction number

Diseases free equilibrium (DFE) point of model (3) is given by

\[
F_0 = \left( \frac{b^*}{\alpha^* (d^* + c^*)} \right).
\]

Reproductive number of model (3), is

\[
\psi_0 = \frac{\beta b^* (1 - q^*) d^* + \delta^*}{d^* (d^* + d^* + \alpha^* + \sigma_1^*)}.
\]

**Theorem 6.** The system (3) is locally stable at \(F_0\) for \(\psi_0 < 1\) and unstable for \(\psi_0 > 1\).

**Proof.** The Jacobian of system (3) is

\[
\begin{bmatrix}
-\beta^* & -\beta^* & 0 & 0 & 0 \\
\beta^* I_1 & 0 & 0 & 0 & 0 \\
\beta^* I_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
k_1^* \\
k_2^* \\
0 \\
0 \\
0
\end{bmatrix}
\]

along \(F_0\), it implies that

\[
\begin{bmatrix}
-\delta^* + \frac{\beta b^* (1 - q^*) d^* + \delta^*}{d^* (d^* + c^* + \alpha^*)} \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

which follows that all the eigenvalues are negative if \(\psi_0 < 1\) and eigenvalue \(\lambda_2\) is positive for \(\psi_0 > 1\). Hence, we conclude that the system (3) is locally stable under the condition \(\psi_0 < 1\) and unstable for \(\psi_0 > 1\).

**Theorem 7.** The system (3) is globally stable, if \(\psi_0 > 1\) at \(F_0\).

**Proof.** First, we construct the Lyapunov function \(\mathcal{L}(t)\), for the system as:

\[
\mathcal{L}(t) = 1 + \mathcal{I}_t - \ln \mathcal{I}_0.
\]

Then differentiating the Eq. (21) with respect to time, we have

\[
\frac{d}{dt} \mathcal{L}(t) = \left( 1 - \frac{1}{\mathcal{I}_0} \right) \frac{d\mathcal{I}_t}{dt} = \frac{d\mathcal{I}_t}{dt} - \beta S_t + (\eta_1^* + \eta_2^* + d^*).$

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By manipulating along the point $P_0^n$, we get
\[
\frac{d}{dt} (\mathbf{Y}(t)) = -\left( \beta^{*} \mathcal{F}_t - (\eta^n_t + \eta^*_t + d^*) \right) \\
= -\left( \beta^{*} \mathcal{F}_t - (\eta^n_t + \eta^*_t + d^*) \right) \\
= \left( \beta^{*} \mathcal{F}_t - (\eta^n_t + \eta^*_t + d^*) \right) \\
\leq 0, \quad \text{for } \psi_0^n > 1.
\]
Therefore, if $\psi_0^n > 1$, then $\frac{d}{dt} (\mathbf{Y}(t)) < 0$, which implies that the system (3) is globally stable for $\psi_0^n > 1$ at $P_0^n$.

6. Solution of the fractional model (3)

In this section we present the numerical solution of adopted fractional order model, which is inspired by [8]. By applying the piecewise integral, we have
\[
\mathcal{L}_0(t) = \left\{ \begin{array}{l}
\mathcal{L}_0(0) + \int_0^t ((1-q)b - \beta \mathcal{F}_t, \mathcal{F}_t - d \mathcal{F}_t + \delta \mathcal{R}_t) dt, \\
\mathcal{L}_0(t) + \int_0^t ((1-q)b - \beta \mathcal{F}_t, \mathcal{F}_t - d \mathcal{F}_t + \delta \mathcal{R}_t) g(t) dt,
\end{array} \right.
\]
\[
\mathcal{L}_0(t) = \left\{ \begin{array}{l}
\mathcal{L}_0(0) + \int_0^t ((1-q)b - \beta \mathcal{F}_t, \mathcal{F}_t - d \mathcal{F}_t + \delta \mathcal{R}_t) g(t) dt, \\
\mathcal{L}_0(t) + \int_0^t ((1-q)b - \beta \mathcal{F}_t, \mathcal{F}_t - d \mathcal{F}_t + \delta \mathcal{R}_t) g(t) dt,
\end{array} \right.
\]
\[
\mathcal{L}_0(t) = \left\{ \begin{array}{l}
\mathcal{L}_0(0) + \int_0^t ((1-q)b - \beta \mathcal{F}_t, \mathcal{F}_t - d \mathcal{F}_t + \delta \mathcal{R}_t) g(t) dt, \\
\mathcal{L}_0(t) + \int_0^t ((1-q)b - \beta \mathcal{F}_t, \mathcal{F}_t - d \mathcal{F}_t + \delta \mathcal{R}_t) g(t) dt,
\end{array} \right.
\]
\[
\mathcal{L}_0(t) = \left\{ \begin{array}{l}
\mathcal{L}_0(0) + \int_0^t ((1-q)b - \beta \mathcal{F}_t, \mathcal{F}_t - d \mathcal{F}_t + \delta \mathcal{R}_t) g(t) dt, \\
\mathcal{L}_0(t) + \int_0^t ((1-q)b - \beta \mathcal{F}_t, \mathcal{F}_t - d \mathcal{F}_t + \delta \mathcal{R}_t) g(t) dt,
\end{array} \right.
\]
Now time $t = t_{n+1}$
\[
\mathcal{L}_0(t_{n+1}) = \left\{ \begin{array}{l}
\mathcal{L}_0(0) + \int_0^t ((1-q)b - \beta \mathcal{F}_t, \mathcal{F}_t - d \mathcal{F}_t + \delta \mathcal{R}_t) g(t) dt, \\
\mathcal{L}_0(t) + \int_0^t ((1-q)b - \beta \mathcal{F}_t, \mathcal{F}_t - d \mathcal{F}_t + \delta \mathcal{R}_t) g(t) dt,
\end{array} \right.
\]
By applying Newton Polynomial interpolation scheme we have
\[
\mathcal{L}_0(t_{n+1}) = \left\{ \begin{array}{l}
\mathcal{L}_0(0) + \sum_{i=1}^{n+1} \left( \begin{array}{l}
\mathcal{L}_0(0) + \int_0^t ((1-q)b - \beta \mathcal{F}_t, \mathcal{F}_t - d \mathcal{F}_t + \delta \mathcal{R}_t) g(t) dt, \\
\mathcal{L}_0(t) + \int_0^t ((1-q)b - \beta \mathcal{F}_t, \mathcal{F}_t - d \mathcal{F}_t + \delta \mathcal{R}_t) g(t) dt,
\end{array} \right),
\end{array} \right.
\]
(22)

(23)
7. Analysis of the Piecewise model (4)

This section is related to the attempt for the proof of existence and uniqueness of the solution of proposed system. For this purpose we use the well-known method of Banach fixed-point theorem. For the general form of given system, let us consider \( X = (S, E, I, Q, R) \), so we can write system (4) as

\[
\begin{align*}
\frac{dX}{dt} &= g(X, t), \quad \text{for } 0 \leq t \leq T_1, \\
\mathcal{P}_0 D^0_t X &= g(X, t), \quad \text{for } T_1 \leq t \leq T_2, \\
dX &= g(X, t)dt + \sigma dB(t), \quad \text{for } T_2 \leq t \leq T,
\end{align*}
\]

where \( g = (g_1, g_2, g_3, g_4, g_5) \), \( \sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \), \( B(t) = (B_1(t), B_2(t), B_3(t), B_4(t), B_5(t)) \).

Here, we investigate the existence and uniqueness of the positive solution for the model (4) \( \forall t \geq 0 \).

**Theorem 8.** Let \( \Omega(0) = (\mathcal{F}_1(0), \mathcal{F}_2(0), \mathcal{F}_3(0), \mathcal{F}_4(0), \mathcal{F}_5(0)) \) be the initial conditions with \( \Omega(0) \in \mathbb{R}^5_+ \), therefore there exists a nonnegative solution \( \Omega(t) = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5) \) of the stochastic model (4) \( \forall t \geq 0 \), the solution will be maintain in \( \mathbb{R}^5_+ \) with unit probability.

**Proof.** Since the parameters used in the model (4) are locally continuous for a given \( \Omega(0) \in \mathbb{R}^5_+ \), there exist unites this condition a unique locally solution \( \Omega(t) \) on \( t \in [0, \lambda_e) \), where \( \lambda_e \) denotes the explosion time. Let us consider a positive real number \( k_0 \) and large enough so that all initial conditions fit in \( \left\{ \frac{1}{k_0}, k_0 \right\} \). Additionally, we defined a stopping time.

Fig. 1 Shows the numerical simulation proposed system (2). (a) Graph of susceptible population. (b) Graph of infected people. (c) Graph of quarantined people. (d) Graph of recovered people. Initial values are \( S_0 = 100, E_0 = 10, I_0 = 5, Q_0 = 0 \), and parameters are \( \delta = 0.001, \beta = 0.003, \gamma = 0.002, b = 10, \eta = 0.05 \), \( \rho = 0.003 \), \( \sigma_1 = 0.003 \), \( \sigma_2 = 0.002 \) and vaccination fraction \( q = 0 \). Further, the figures are plotted using three different codes, the first uses the Matlab command ‘ode45’, and the second is Runge–Kutta of the fourth order, and NSDF is the Non standard finite difference scheme.
\[
\lambda_k = \left\{ t \in \left[ 0, \lambda_k \right] : \frac{1}{k} \geq \min \Omega(t) \text{ or } \max \Omega(t) \geq k \right\}, \forall k > k_0.
\]

For this to show that, the actual solution \( \Omega(t) \) is global, we have to show that \( \lambda_k = \infty \). To start, we assume that \( \inf \Omega = \infty \), where \( \Omega \) is the empty set. By the definition of stopping time \( \lambda_k \) is monotonically increasing as \( k \to \infty \). We now set
\[
\lim_{k \to \infty} \lambda_k = \lambda_\infty,
\]
with \( \lambda_\infty \geq \lambda_\infty \).

Nevertheless if \( \forall t \geq 0 \), we show that \( \lambda_\infty = \infty \), then we can say \( \lambda_\infty = \infty \) and \( \Omega(t) \in \mathbb{R}^3 \). Thus we have demonstrated that that \( \lambda_k = \infty \). However, if the condition is not true, then there exists \( T > 0 \) and \( r \in (0,1) \) such that
\[
P\{ T \geq \lambda_\infty \} > r.
\]

We define a function \( F : \mathbb{R}_+^3 \to \mathbb{R}_+^4 \), \( F \in \mathcal{C}^2 \) such that
\[
F(S, E, I, R) = S + E + I + R - \left( \log S + \log E + \log I + \log R \right).
\]

It is known that \( \forall x > 0, x - 1 - \log x \geq 0 \). Hence \( L \geq 0 \). Furthermore, it is assumed \( k_0 < k \) and \( 0 < T \) by further applying Itô's formula, we obtain that
\[
\begin{align*}
\frac{dF}{dt} &= \left( 1 - \frac{1}{\lambda} \right)dS_t + \frac{1}{\lambda}S_t dS_t - 1 dB_S(t) \\
&\quad + \left( 1 - \frac{1}{\lambda_E} \right)dE_t + \frac{1}{\lambda_E}E_t dE_t - 1 dB_E(t) \\
&\quad + \left( 1 - \frac{1}{\lambda_I} \right)dI_t + \frac{1}{\lambda_I}I_t dI_t - 1 dB_I(t) \\
&\quad + \left( 1 - \frac{1}{\lambda_R} \right)dR_t + \frac{1}{\lambda_R}R_t dR_t - 1 dB_R(t) \\
&= K(S, E, I, R) dS_t + \sum_{j=1}^{5} \frac{1}{\lambda_j}(X_j - 1) dB_j(t).
\end{align*}
\]

Fig. 2 Shows the numerical simulation of proposed system (2). (a) Graph of susceptible population. (b) Graph of infected people. (c) Graph of quarantined people. (d) Graph of recovered people. Initial values are \( S_0 = 100, E_0 = 10, I_0 = 5, R_0 = 0 \), and parameters are \( d = 0.001, \beta = 0.003, \delta = 0.003, \gamma = 0.002, b = 0.10, \eta = 0.05, \rho = 0.003, \sigma_1 = 0.003, \sigma_2 = 0.002 \) and vaccination fraction \( q = 0.5 \). Further, the figures are plotted using three different codes, the first uses the Matlab command 'ode45', and the second is Runge–Kutta of the fourth order, and NSDF is the Non standard finite difference scheme.
where $X_1 = \mathcal{S}, X_2 = \mathcal{I}, X_3 = \mathcal{F}, X_4 = \mathcal{R}$, and $X_5 = \mathcal{B}$. Here

$$K(\mathcal{S}, \mathcal{I}, \mathcal{F}, \mathcal{R}, \mathcal{B}) = \left(1 - \frac{1}{\mathcal{S}}\right)((1 - q)b - \beta \mathcal{S} \mathcal{I} - d \mathcal{S} + \delta \mathcal{R}) + \sigma_1(\mathcal{S} - 1)d \mathcal{B}(t) + \left(1 - \frac{1}{\mathcal{I}}\right)(\beta \mathcal{S} \mathcal{I} - (\eta_1 + \eta_2 + d) \mathcal{I} + \sigma_2(\mathcal{I} - 1)d \mathcal{B}(t) + \left(1 - \frac{1}{\mathcal{F}}\right)(\eta \mathcal{F} - (\rho + d + \sigma_1) \mathcal{F} + \sigma_3(\mathcal{F} - 1)d \mathcal{B}(t) + \left(1 - \frac{1}{\mathcal{R}}\right)(\gamma \mathcal{R} + \rho \mathcal{R} - (d + \delta) \mathcal{R} + q b + \sigma_5(\mathcal{R} - 1)d \mathcal{B}(t)\right) + \frac{\sigma_1^2}{2},$$

$\leq (1 - q)b - \beta \mathcal{S} \mathcal{I} - d \mathcal{S} + \delta \mathcal{R} + \beta \mathcal{S} \mathcal{I} - (\eta_1 + \eta_2 + d) \mathcal{I} + \eta \mathcal{F} - (\rho + d + \sigma_1) \mathcal{F} + \sigma_3(\mathcal{F} - 1)d \mathcal{B}(t) + \gamma \mathcal{R} + \rho \mathcal{R} - (d + \delta) \mathcal{R} + q b + \sigma_5(\mathcal{R} - 1)d \mathcal{B}(t)\right) + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{\sigma_5^2}{2} = \Pi.$

**Fig. 3** Shows the numerical simulation of proposed system (2). (a) Graph of susceptible population. (b) Graph of infected people. (c) Graph of quarantined people. (d) Graph of recovered people. Initial values are $S_0 = 100$, $F_0 = 10$, $R_0 = 5$, $B_0 = 0$, and parameters are $d = 0.001$, $b = 0.003$, $\delta = 0.002$, $\gamma = 0.002$, $b = 10$, $\eta = 0.05$, $\rho = 0.003$, $\sigma_1 = 0.003$, $\sigma_2 = 0.002$ and vaccination fraction $q = 1$. Further, the figures are plotted using three different codes, the first uses the Matlab command 'ode45', and the second is Runge–Kutta of the fourth order, and NSDF is the Non standard finite difference scheme.
Clearly $\Pi$ shows that it is positive but also independent of variables as well

$$dF \leq \Pi dt + \sum_{j=1}^{s} \sigma_j(X_i - 1)dB_j(t).$$

A direct integration from 0 to $\lambda_k \wedge T$ yields

$$E(F(Y(\lambda_k \wedge T), \xi_1(\lambda_k \wedge T), \xi_2(\lambda_k \wedge T), \xi_3(\lambda_k \wedge T), \xi_4(\lambda_k \wedge T)))$$

$$\leq F(\Omega(0)) + E\left(\int_{0}^{\lambda_k \wedge T} \Pi \right)$$

$$\leq \Omega(0) + 7\Pi.$$ 

Now, let us set

$$\Gamma_k = T \geq \lambda_k,$$

for $k_1 \leq k$ and thus $P(\Gamma_k) \geq r$. However $\forall \Phi \in \Gamma_k$, there must be at least $\Omega(t, \Phi)$ which equals $\frac{r}{k}$ or $k$. Hence $\Omega(\lambda_k)$ is not less than $k - \log k - 1$ or $\log k - 1 + \frac{1}{k}$ a.s.

Thus

$$\log(k - 1 + \frac{1}{k}) \times E(k - \log k - 1) \leq F(\Omega(\lambda_k)),$$

From the above, we can concluded

$$F(\Omega(0)) + \Pi T \geq E(1_{\Gamma_k} F(\Omega(\lambda_k)))$$

$$\geq r(\log k - 1) \wedge \log k - 1 + \frac{1}{k}.$$  

Hence the $1_{\Gamma_k}$ indicates the indication function of $\Gamma_k$. By taking $k \to \infty$ yields

$$\infty > F(\Omega(0)) + 7\Pi = 0,$$

which is contradiction, therefore we can conclude that $\lambda_\infty = \infty$, which complete the proof, see [33,38].

Now for $\forall [0, T_2]$, we need to prove that

$$|g_i(x, t)|^2 \leq c_i\left(1 + |x|^2\right),$$

and

$$|g_i(x_1, t) - g_i(x_2, t)|^2 \leq c_i|x_1^2 - x_2^2|^2.$$

For all $t \in [0, T_2]$, we have

$$g_i(Y, \xi, \xi, \xi, \xi, \xi) = \frac{1 - q|h - \beta Y|dY + \delta|\xi|^2}{\|\xi\|_\infty + \sup_{\xi \in [0, \infty]}|\xi|^2 + \|\xi\|_\infty + \delta \sup_{\xi \in [0, \infty]}|\xi|^2}$$

$$\leq \frac{1}{\|\xi\|_\infty + \sup_{\xi \in [0, \infty]}|\xi|^2 + \|\xi\|_\infty + \delta \sup_{\xi \in [0, \infty]}|\xi|^2}.$$ 

(a) The Susceptible population by COVID-19

Fig. 4 Shows the numerical simulation of proposed system (4). (a) Graph of susceptible population. (b) Graph of infected people. (c) Graph of quarantined people. (d) Graph of recovered people. Initial values are $\xi_0 = 100, \xi_0 = 10, h = 5, \xi_0 = 0$, and parameters are $d = 0.001, \beta = 0.003, \delta = 0.003, \gamma = 0.002, b = 10, \eta = 0.05, \rho = 0.003, \sigma_1 = 0.003, \sigma_2 = 0.002, \tau_1 = 0.06, \tau_2 = 0.06, \tau_3 = 0.04, \tau_4 = 0.06$ and vaccination fraction $q = 0$, with the help of Euler’s method (Sauer, 2006).
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\[
g_2(\mathcal{F}, \mathcal{E}, \mathcal{I}, \mathcal{R}) = \beta \mathcal{E} - (\eta_1 + \eta_2 + d \rho) \mathcal{S}, \quad \leq 2 \left[ \beta^2 \mathcal{S}^2 \mathcal{E}^2 + (\eta_1 + \eta_2 + d \rho)^2 \mathcal{E}^2 \right]^{-1} < 1.
\]

\[
g_3(\mathcal{F}, \mathcal{E}, \mathcal{I}, \mathcal{R}) = \eta_1 \mathcal{E} - (\eta + \gamma + d + \sigma_1) \mathcal{I}, \quad \leq 2 \left[ \eta_1^2 \mathcal{S}^2 \mathcal{E}^2 + (\eta + \gamma + d + \sigma_1)^2 \mathcal{I}^2 \right]^{-1} < 1.
\]

\[
\frac{\|x(t) - x(t_0)\|_{\infty}}{\|x(t_0)\|_{\infty}} < 1.
\]

\[
g_4(\mathcal{F}, \mathcal{E}, \mathcal{I}, \mathcal{R}) = \eta_2 \mathcal{I} - (\rho + \sigma_2) \mathcal{R}, \quad \leq 3 \left( \eta_2^2 \mathcal{E}^2 + \eta_2^2 \mathcal{I}^2 + (\rho + \sigma_2)^2 \mathcal{R}^2 \right)^{-1} < 1.
\]

\[
\frac{\|x(t) - x(t_0)\|_{\infty}}{\|x(t_0)\|_{\infty}} < 1.
\]

\[
\frac{\|x(t) - x(t_0)\|_{\infty}}{\|x(t_0)\|_{\infty}} < 1.
\]

Fig. 5  Shows the numerical simulation of proposed system (4).  ((a) Graph of susceptible population.  (b) Graph of infected people.  (c) Graph of quarantined people.  (d) Graph of recovered people.  Initial values are \( \mathcal{F}_0 = 100, \mathcal{E}_0 = 10, \mathcal{I}_0 = 5, \mathcal{R}_0 = 0 \), and parameters are \( d = 0.001, \beta = 0.003, \delta = 0.002, \gamma = 0.002, h = 10, \eta = 0.05, \rho = 0.003, \sigma_1 = 0.003, \sigma_2 = 0.002, \sigma_3 = 0.06, \sigma_4 = 0.06 \), and vaccination fraction \( q = 0.5 \), with the help of Euler’s method (Sauer, 2006).
under \( \frac{(d + b)^2}{\gamma^2} \leq 1 \).

\[
|g_5(t, \xi_1) - g_5(t, \xi_2)|^2 = |(1 - q) h - p \xi_1 \xi_1 - d \xi_1 + \delta \xi - (1 - q) h + p \xi_2 \xi_2 + d \xi_2 - \delta \xi|^2,
\]
\[
\leq 2(b^2) \xi_1^2 + 2b^2 \xi_2^2
\]
\[
\leq k_1|\xi_1 - \xi_2|^2
\]
(34)

\[
|g_6(t, \xi_1) - g_6(t, \xi_2)|^2 = |1 - q) h - p \xi_1 \xi_1 - d \xi_1 + \delta \xi - (1 - q) h + p \xi_2 \xi_2 + d \xi_2 - \delta \xi|^2,
\]
\[
\leq 2(b^2) \xi_1^2 + 2b^2 \xi_2^2
\]
\[
\leq k_1|\xi_1 - \xi_2|^2
\]
(35)

\[
|g_7(t, \xi_1) - g_7(t, \xi_2)|^2 = |1 - q) h - p \xi_1 \xi_1 - d \xi_1 + \delta \xi - (1 - q) h + p \xi_2 \xi_2 + d \xi_2 - \delta \xi|^2,
\]
\[
\leq (b^2) \xi_1^2 + (1 - q) h + p \xi_2 \xi_2 - d \xi_2 - \delta \xi|^2
\]
\[
\leq k_1|\xi_1 - \xi_2|^2
\]
(36)

\[
|g_8(t, \xi_1) - g_8(t, \xi_2)|^2 = |1 - q) h - p \xi_1 \xi_1 - d \xi_1 + \delta \xi - (1 - q) h + p \xi_2 \xi_2 + d \xi_2 - \delta \xi|^2,
\]
\[
\leq (b^2) \xi_1^2 + (1 - q) h + p \xi_2 \xi_2 - d \xi_2 - \delta \xi|^2
\]
\[
\leq k_1|\xi_1 - \xi_2|^2
\]
(37)

\[
|g_9(t, \xi_1) - g_9(t, \xi_2)|^2 = |1 - q) h - p \xi_1 \xi_1 - d \xi_1 + \delta \xi - (1 - q) h + p \xi_2 \xi_2 + d \xi_2 - \delta \xi|^2,
\]
\[
\leq (b^2) \xi_1^2 + (1 - q) h + p \xi_2 \xi_2 - d \xi_2 - \delta \xi|^2
\]
\[
\leq k_1|\xi_1 - \xi_2|^2
\]
(38)

8. Numerical Simulation

Figs. 1–9 are depictions of numerical solution of the studied models. In this simulation, we did not plot them as piecewise

(a) The Susceptible population by COVID-19

Fig. 6 Shows the numerical simulation of proposed system (4). (a) Graph of susceptible population. (b) Graph of infected people. (c) Graph of quarantined people. (d) Graph of recovered people. Initial values are \( S_0 = 100, I_0 = 10, \xi_0 = 5, R_0 = 0, \) and parameters are \( d = 0.001, b = 0.003, \delta = 0.003, \beta = 0.002, \eta = 0.05, \rho = 0.003, \sigma_1 = 0.003, \sigma_2 = 0.002, \tau_1 = 0.06, \tau_2 = 0.06, \tau_3 = 0.04, \tau_4 = 0.06 \) and vaccination fraction \( q = 1, \) with the help of Euler’s method (Sauer, 2006).
rather than single pattern starting by deterministic model, fol-
lowing by stochastics and later by fractional.

9. Conclusion

A simple mathematical model with four classes was considered
here to replicate the spread of covid-19 within a given settl-
ment. The existence model was found unsuitable to accurately
replicate nonlocalities of the spread for example a passage
from deterministic to stochastic with no steady state. To solve
these limitations, the model was modified by adding compo-
nents that account for quarantine class as well as vaccinated
class. Addition to this, we have successfully applied the concept
of piecewise differentiation to capture crossover behaviors of
the spread. The modified model was subjected to several anal-
ysis, for example, existence and uniqueness of a positive system
solutions, derivation of conditions under which extinction will
occur and finally derivation of numerical. Some simulations
were presented for different values of fractional orders and
density of randomness.

Fig. 7 Shows the numerical simulation of proposed system (22). (a) Graph of susceptible population. (b) Graph of infected people. (c) Graph of quarantined people. (d) Graph of recovered people. Initial values are $S_0 = 100$, $I_0 = 10$, $Q_0 = 5$, $R_0 = 0$, and parameters are $d = 0.001$, $b = 0.003$, $\delta = 0.003$, $\gamma = 0.002$, $b = 10$, $\eta = 0.05$, $\rho = 0.003$, $\sigma_1 = 0.003$, $\sigma_2 = 0.002$ and vaccination fraction $q = 0.5$, and multi values of the fractional order derivative, which shows the impact of the memory on the spread of covid-19 disease. Further, the trapezoidal product-integration rule [?] is used for building the scheme for the graphical representations.
Fig. 8 Shows the numerical simulation of proposed system (??). (a) Graph of susceptible population. (b) Graph of infected people. (c) Graph of quarantined people. (d) Graph of recovered people. Initial values are $S_0 = 100$, $I_0 = 10$, $Q_0 = 5$, $R_0 = 0$, and parameters are $d = 0.001$, $\beta = 0.003$, $\delta = 0.003$, $\gamma = 0.002$, $b = 10$, $\eta = 0.05$, $\rho = 0.003$, $\sigma_1 = 0.003$, $\sigma_2 = 0.002$ and vaccination fraction $q = 0.5$, and multi values of the fractional order derivative, which shows the impact of the memory on the spread of covid-19 disease. Further, the trapezoidal product-integration rule [?] is used for building the scheme for the graphical representations.
Declarations

Availability of data and materials
The authors confirm that the data supporting the findings of this study are available within the article cited there in.

Authors Contribution

Authors are equally contributed in preparing this manuscript.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] H. Rudolf, Applications of fractional calculus in physics, (2000).
[2] S. Kumar, R. Kumar, R.P. Agarwal, B. Samet, A study of fractional Lotka-Volterra population model using Haar wavelet and Adams-Bashforth-Moulton methods, Math. Methods Appl. Sci. 43 (8) (2020) 5564–5578.
[3] S. Kumar, R. Kumar, C. Cattani, B. Samet, Chaotic behaviour of fractional predator-prey dynamical system, Chaos, Solitons & Fractals 135 (2020) 109811.
[4] S. Bekiros, D. Kouloumpou, SBDiEM: A new mathematical model of infectious disease dynamics, Chaos, Solitons & Fractals 136 (2020) 109828.
[5] G. Bocharov, V. Volpert, B. Ludewig, A. Meyerhans, Mathematical Immunology of Virus Infections, Springer International Publishing, 2018.
[6] F. Brauer, P. Van den Driessche, J. Wu, Mathematical Epidemiology, Springer, Berlin Heidelberg, 2008.
[7] F. Brauer, Mathematical epidemiology: Past, present, and future, Infect. Disease Modell. 2 (2017) 113–127.
A. Atangana, A. Iqret Araz, Modeling and forecasting the spread of COVID-19 with stochastic and deterministic approaches: Africa and Europe, Adv. Difference Equ. 2021 (57) (2021) 107.

M.S. Abdo, K.S. Hanan, A.W. Satish, K. Pancha, On a comprehensive model of the novel coronavirus (COVID-19) under Mittag-Leffler derivative, Chaos, Solitons & Fractals 135 (2020) 109867.

S. Cakmak, Dynamic analysis of a mathematical model with health care capacity for COVID-19 pandemic, Chaos, Solitons & Fractals 139 (2020) 110033.

W. Ming, J.V. Huang, C.J.P. Zhang, Breaking down of the healthcare system: Mathematical modelling for controlling the novel coronavirus (2019-nCoV) outbreak in Wuhan, China (2020) DOI: 10.1101/2020.01.27.922443.

A. Atangana, Modelling the spread of COVID-19 with new fractal-fractal operators: can the lockdown save mankind before vaccination?, Chaos Solitons Fractals 136 (2020) 109860.

S. Kumar, J. Cao, M. Abdel-Ary, A novel mathematical approach of COVID-19 with non-singular fractional derivative, Chaos Solitons Fractals 139 (2020) 110048.

Zhang, Zizhen, Zeb, Anwar, Alzahrani, Ebraheme, Iqbal, Sohail, Crowding effects on the dynamics of COVID-19 mathematical model, Adv. Diff. Eqs. 020(1) (2020) 1–13.

K.M. Saad, J.F. Gomez-Aguilar, A.A. Almady, A fractional numerical study on a chronic hepatitis C virus infection model with immune response, Chaos, Solitons & Fractals 139 (2020) 110062.

S. Bentout, Y. Chen, S. Djilali, Global dynamics of an SEIR model with two age structures and a nonlinear incidence, Acta Applicandae Math. 171 (1) (2021) 1–27.

S. Djilali, B. Ghanbari, Coronavirus pandemic: A predictive analysis of the peak outbreak epidemic in South Africa, Turkey, and Brazil, Chaos, Solitons & Fractals 138 (2020) 109971.

S. Bentout, A. Tridane, S. Djilali, T.M. Touaoula, Age-structured modeling of COVID-19 epidemic in the USA, UAE and Algeria, Alexandria Eng. J. 60 (1) (2021) 401–411.

H.M. Srivastava, K.M. Saad, Numerical simulation of the fractal-fractional Ebola virus, Fractal Fractional 4 (4) (2020) 49.

M.M. Khader et al, A spectral collocation method for fractional chemical clock reactions, Comput. Appl. Mathe. 39 (4) (2020) 1–12.

M.K. Saad, Ma. Alqhtani, Numerical simulation of the fractal-fractional reaction diffusion equations with general nonlinear, AIMS Mathe. 6 (4) (2021) 3788–3804.

Al.Sami et al, Numerical Solutions of Certain New Models of the Time-Fractional Gray-Scott, J. Function Spaces 2021 (2021).

A. Akgul, A novel method for a fractional derivative with non-local and non-singular kernel, Chaos, Solitons & Fractals 114 (2018) 478–482.

A. Atangana, A. Akgul, K.M. Owolabi, Analysis of fractal fractional differential equations, Alexandria Eng. J. 59 (3) (2020) 1117–1134.

K.M. Owolabi, A. Atangana, A. Akgul, Modelling and analysis of fractal-fractional partial differential equations: application to reaction-diffusion model, Alexandria Eng. J. 59 (4) (2020) 2477–2490.

A. Atangana, I.A. Sedag, New numerical method for ordinary differential equations: Newton polynomial, J. Comput. Appl. Math. 372 (2020) 112622.

I. Podlubny, Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, (1998).

A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, 204, (2006).

Z. Odibat, D. Baleanu, Numerical simulation of initial value problems with generalized Caputo-type fractional derivatives, Appl. Num. Mathe. (2020).

B. Ghanbari, S. Kumar, R. Kumar, A study of behaviour for immune and tumor cells in immunogenetic tumour model with non-singular fractional derivative, Chaos, Solitons & Fractals 133 (2020) 109619.

Sunil Kumar, A new analytical modelling for fractional telegraph equation via Laplace transform, Appl. Math. Model. 38 (13) (2014) 3154–3163.

S. Kumar, K.S. Nisar, R. Kumar, C. Cattani, B. Samet, A new Rabotnov fractional-exponential function-based fractional derivative for diffusion equation under external force, Mathe. Methods Appl. Sci. 43 (7) (2020) 4460–4471.

A. Linda, An introduction to stochastic epidemic models. Mathematical epidemiology, Springer, Berlin, Heidelberg, 2008, pp. 81–130.

S. Norman, Generalizations of some stochastic epidemic models, Math. Biosci. 4 (3-4) (1969) 395–402.

P.D. O’Neill, A tutorial introduction to Bayesian inference for stochastic epidemic models using Markov chain Monte Carlo methods, Mathe. Biosci. 180 (1-2) (2002) 103–114.

T. Britton et al, Five challenges for stochastic epidemic models involving global transmission, Epidemics 10 (2015) 54–57.

M. Holhe, E. Jorgensen, D. O’Neill, Inference in disease transmission experiments by using stochastic epidemic models, J. Roy. Stat. Soc.: Ser. C (App. Stat.) 54 (2) (2005) 349–366.

Yanli Zhou, Weiguo Zhang, Threshold of a stochastic SIR epidemic model with Lévy jumps, Phys. A: Stat. Mech. Appl. 446 (2016) 204–216.

Xinhong Zhang, Daqing Jiang, Tasawar Hayat, Bashir Ahmad, Dynamics of a stochastic SIS model with double epidemic diseases driven by Lévy jumps, Physica A 471 (2017) 767–777.

Zhang, Zizhen, Zeb, Anwar, Alzahrani, Ebraheme, Iqbal, Sohail, Crowding effects on the dynamics of COVID-19 mathematical model, Adv. Difference Eqs. 2020(1) (2020) 1–13.

Y. Zhang, Zizhen, Zeb, Anwar, Hussain, Sultan, Alzahrani, Ebraheme, Dynamics of COVID-19 mathematical model with stochastic perturbation, Adv. Diff. Eqs. 2020(1) (2020) 1–12.

Ebraheem, Iqbal, Sohail, Dynamics of COVID-19 mathematical model with stochastic perturbation, Adv. Diff. Eqs. 2020(1) (2020) 1–12.

M. Du, Z. Wang, H. Hu, Measuring memory with the order of fractional derivative, Sci. Rep. 3 (2013) 3431, https://doi.org/10.1038/srep03431.

A. Flores-Tlacuahuac, L.T. Biegler, Optimization of fractional order dynamic chemical processing systems, Ind. Eng. Chem. Res. 53 (13) (2014) 5110–5127.

M. Javadi, B. Ahmad, Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton-zooplankton system, Ecol. Modell. 318 (2015) 8–18.

J. Singh, D. Kumar, Z. Hammouch, A. Atangan, A fractional epidemiological model for computer viruses pertaining to a new fractional derivative, Appl. Math. Comput. 316 (2018) 450–515.

V.G. Tarasova, V.E. Tarasov, Concept of dynamic memory in economics, Commun. Nonlinear Sci. Numer. Simul. 55 (2018) 127–145.

M. A. Abbas-Bakar, P. Kumar, V.S. Erturk, A. Kumar, A mathematical study of a Tuberculosis model with fractional derivatives, Int. J. Modeling, Simulat. Scientific Comput. 12 (4) (2021) 2150037.

P. Kumar, N. Rangaik, H. Abboubakar, S. Kumar, A Malaria Model with Caputo-Fabrizio and Atangana-Baleanu Derivatives, Int. J. Model. Simulat. Sci. Comput. 12 (02) (2021) 2150013.

W. Gao, P. Veeresha, H.M. Baskonus, D.G. Prakasha, P. Kumar, A New Study of Unreported Cases of 2019-nCoV Epidemic Outbreaks, Chaos, Solitons & Fractals 138 (2020) 109929.

P. Kumar, V.S. Erturk, A case study of Covid-19 epidemic in India via new generalised Caputo type fractional derivatives, Mathe. Methods Appl. Sci. 1–14 (2021), https://doi.org/10.1002/mma.7284.
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[51] M. Caputo, F. Mauro, A new definition of fractional derivative without singular kernel, Progr. Fract. Differ. Appl 1 (2) (2015) 1–13.

[52] P. Verma, M. Kumar, Analysis of a novel coronavirus (2019-nCOV) system with variable Caputo-Fabrizio fractional order, Chaos, Solitons & Fractals 142 (2020) 110451.

[53] C.G. Gardiner, Handbook of Stochastic Methods, 3rd ed., Springer, (2004). Ji C, Jiang D, Shi N: Multigroup SIR epidemic model with stochastic perturbation. Physica A 2011, 390: 1747–1762. 10.1016/j.physa.2010.12.042.

[54] S Kumar, RP Chauhan, S Momani, Numerical investigations on COVID-19 model through singular and non-singular fractional operators, Num. Methods Partial Diff. Eqs. (2020). doi: 10.1002/num.22707.

[55] S. Kumar, R. Kumar, S. Momani, S. Hadid, A study on fractional COVID-19 disease model by using Hermite wavelets, Math. Meth. Appl. Sci. (2021), https://doi.org/10.1002/mma.7065.

[56] K.M. Safare, V.S. Betageri, D.G. Prakasha, et al, A mathematical analysis of ongoing outbreak COVID-19 in India through nonsingular derivative, Num. Methods Partial Diff. Eqs. 37 (2) (2021) 1282–1298.

[57] M.A. Khan, S. Ullah, S. Kumar, A robust study on 2019-nCOV outbreaks through non-singular derivative, Eur. Phys. J. Plus (2021), https://doi.org/10.1140/epjp/s13360-021-01159-8.

[58] M. Du, Z. Wang, H. Hu, Measuring memory with the order of fractional derivative, Sci. Rep. 3 (2013) 3431, https://doi.org/10.1038/srep03431.