Van der Waals quintessence stars

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The van der Waals quintessence equation of state is an interesting scenario for describing the late universe, and seems to provide a solution to the puzzle of dark energy, without the presence of exotic fluids or modifications of the Friedmann equations. In this work, the construction of inhomogeneous compact spheres supported by a van der Waals equation of state is explored. These relativistic stellar configurations shall be denoted as van der Waals quintessence stars. Despite of the fact that, in a cosmological context, the van der Waals fluid is considered homogeneous, inhomogeneities may arise through gravitational instabilities. Thus, these solutions may possibly originate from density fluctuations in the cosmological background. Two specific classes of solutions, namely, gravastars and traversable wormholes are analyzed. Exact solutions are found, and their respective characteristics and physical properties are further explored.

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1. INTRODUCTION

The universe is presently undergoing an accelerated phase of expansion. Several candidates have been proposed in the literature to explain this cosmic accelerated expansion, such as dark energy models, the generalized Chaplygin gas, modified gravity and scalar-tensor theories, tachyon scalar fields and specific braneworld models, such as the Dvali-Gabadadze-Porrati (DGP) model. However, it was pointed out that assuming that dark energy is governed by a perfect fluid equation of state may systematically induce wrong results and be misleading in inferring the nature of dark energy [1]. In this spirit, an alternative model was recently proposed without the presence of exotic fluids and modifications of the Friedmann equations, using a more complicated equation of state, namely, the van der Waals (VDW) equation of state [2, 3], given by

\[ p = \frac{\gamma \rho}{1 - \beta \rho} - \alpha \rho^2, \]

where \( \rho \) is the energy density and \( p \) the pressure of the VDW fluid. The accelerated and decelerated periods depend on the parameters, \( \alpha, \beta \) and \( \gamma \) of the equation of state, and in the limiting case \( \alpha, \beta \to 0 \), one recovers the dark energy equation of state, with \( \gamma = p/\rho < -1/3 \). It was also stressed that the perfect fluid equation of state \( p = \gamma \rho \) reflects an approximation of cosmic epochs describing stationary situations, where phase transitions are not considered [4]. Thus, one of the advantages of the VDW model is that it describes the transition from a matter field dominated era to a scalar field dominated epoch, without introducing scalar fields. Furthermore, it is useful to explain the universe with a minimal number of ingredients, and the VDW gas actually treats dark matter and dark energy as a single fluid. The VDW quintessence scenario has also been confronted successfully with a wide variety of observational tests, by constraining the free parameters [4].

The success of this model has stimulated several different approaches. Recently, a model with a binary mixture whose constituents are described by a VDW fluid and by a dark energy density was also proposed [5]. It is interesting to note that this model can simulate several aspects, namely, an inflationary period where the acceleration grows exponentially; and a present accelerated period where the dark energy density dominates over the energy density of the VDW fluid. The construction of a general scheme where the VDW ideal fluid has a mathematically equivalent representation as a scalar-tensor theory with a specific potential, as well as explicit examples were also explored in Ref. [6]. Another approach has been analyzed in the context of brane cosmology [7], where the cosmological fluid on the brane is modeled by the VDW equation of state. It was shown that this model reproduces several features, namely, an initial accelerated epoch where the VDW fluid behaves like a scalar field with a negative pressure, and a present accelerated phase due to a cosmological constant on the brane.

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Despite of the fact that the VDW equation of state represents a spatially homogeneous cosmic fluid and is assumed not to cluster, due to gravitational instabilities inhomogeneities may arise. Thus, the solutions outlined in this paper may possibly originate from density fluctuations in the cosmological background, resulting in the nucleation through the respective density perturbations. Therefore, the pressure in the VDW equation of state may be regarded a radial pressure, and the transverse pressure is determined through the Einstein field equations. The justification of this approach may be considered as an extension of the construction of inhomogeneous wormhole spacetimes supported by phantom energy [3]. The latter construction was motivated by the analysis carried out in Ref. [9], where time-dependent solutions were constructed with ghost scalar fields. In this model, it was shown that the radial pressure is negative everywhere and equals the transverse pressure far from the throat, showing that ghost scalar fields behave essentially as dark energy. Now, in the VDW context, one may also consider, in principle, that for large radial distances in these inhomogeneous solutions the radial pressure equals the transverse pressure, and thus behaving essentially as a VDW cosmological fluid, consequently justifying the above approach. The transverse pressure obtained from the field equations should not be considered typical of the cosmic fluid driving the accelerated expansion, as it is only a feature of these inhomogeneous VDW configurations. It is also important to emphasize that the equation of state leading to the acceleration of the Universe on large scales, is an average equation of state corresponding to a background fluid. So in the inhomogeneities giving rise to the configurations discussed in this work, it is also important to justify if one may really use an equation of state that in fact corresponds to a background fluid. For this, it is important to understand the evolution of structure formation in the VDW scheme. It was shown that for large redshifts the VDW model is similar to the ΛCDM model [4], where the VDW dark matter plays the role of a cosmological constant, showing that structure formation could evolve in a similar manner as in the ΛCDM model, although these points need to be further investigated [4]. Now, it was also shown that due to a negative adiabatic sound speed, perturbations are unstable in the VDW quintessence scenario, and inevitably the role of anisotropic stresses and entropy production has to be taken into account. Thus, as a first approximation and as a working hypothesis, one may argue that this also justifies the above approach of using an equation of state that corresponds to a background fluid, with a radial pressure. We shall denote these inhomogeneous spherically symmetric solutions as van der Waals quintessence stars. It is also interesting to note that the thermodynamic properties of a general relativistic isotropic pressure self-gravitating ball of fluid, undergoing a generic first order phase transition and governed by the VDW equation of state, was studied in Ref. [10]. In this work, we shall generalize the latter to an anisotropic pressure general relativistic model, and we shall consider two specific class of solutions of VDW quintessence stars, namely, gravastars and traversable wormholes.

The “gravastar” (gravitational vacuum star) picture developed by Mazur and Mottola [11], initially arouse from a considerable amount of scepticism related to spacetime singularities and event horizons in the Schwarzschild black hole solution (see Ref. [12] for a review on specific misconceptions and ambiguities related to these issues). In the gravastar model, the interior Schwarzschild solution is replaced with a de Sitter condensate, and thus, does away with the problem of the singularity at the origin and the event horizon. In this model, the quantum vacuum undergoes a phase transition at or near the location where the event horizon is expected to form. In this spirit, it is interesting to note that several models have been proposed to some extent in the literature [13]. Recently, motivated by dark energy quintessence, which is a possible candidate responsible for the late-time cosmic accelerated expansion, a generalization of the gravastar model has been proposed [14, 15], by considering an interior solution governed by the dark energy equation of state, $\gamma = p/\rho < -1/3$. Stable dark energy stellar models were analyzed in Ref. [16], and it was found that large stability regions exist that are sufficiently close to where the event horizon is expected to form. It was also emphasized that it would be difficult to distinguish the exterior geometry of the dark energy stars from an astrophysical black hole. In this context, it is interesting to note that stars supported by a generalized Chaplygin gas equation of state have been explored [16].

In this work, we shall also be interested in investigating the construction of traversable wormholes [17] supported by the VDW equation of state. An analogous approach was recently carried out in the presence of phantom energy [3] and the generalized Chaplygin gas [18]. We will show that traversable wormhole solutions may be constructed using the VDW equation of state, which are either asymptotically flat or possess finite dimensions, where the exotic matter is confined to the throat neighborhood. The latter solutions are constructed by matching an interior wormhole geometry to an exterior Schwarzschild vacuum. Analogously to their phantom and Chaplygin counterparts, these VDW quintessence wormholes have far-reaching astrophysical and cosmological consequences, such as the production of closed timelike curves and the consequent violation of causality.

This paper is outlined in the following manner: In Sec. 2, the equations of structure for van der Waals quintessence stellar models are presented. In Sec. 3 an exact solution of a gravastar is found, and the respective properties and characteristics are analysed. In Sec. 4 an exact solution of a traversable wormhole geometry is also found. In Sec. 5 we conclude.
2. VDW quintessence stars: equations of structure

Consider a static and spherically symmetric spacetime, given by the following metric, in curvature coordinates

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - 2m(r)/r} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2),$$

(2)

where $\Phi(r)$ and $m(r)$ are arbitrary functions of the radial coordinate, $r$. The function $m(r)$ is the quasi-local mass, and is denoted as the mass function.

The Einstein field equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ the stress-energy tensor, provides the following relationships

$$m' = 4\pi r^2 \rho,$$

(3)

$$\Phi' = \frac{m + 4\pi r^3 p_r}{r(r - 2m)},$$

(4)

$$p_r' = \frac{(\rho + p_r)(m + 4\pi r^3 p_r)}{r(r - 2m)} + \frac{2}{r}(p_t - p_r),$$

(5)

where the prime denotes a derivative with respect to the radial coordinate, $r$. $\rho(r)$ is the energy density, $p_r(r)$ is the radial pressure, and $p_t(r)$ is the lateral pressure measured in the orthogonal direction to the radial direction. Equation (5) corresponds to the anisotropic pressure Tolman-Oppenheimer-Volkoff (TOV) equation, and may be obtained using the conservation of the stress-energy tensor, $T^\mu_{\nu,\nu} = 0$.

The four-velocity of a static observer, at rest at constant $r, \theta, \phi$, is $U^\mu = dx^\mu/d\tau = (U^t, 0, 0, 0) = (e^{-\Phi(r)}, 0, 0, 0)$. The observer’s four-acceleration is $a^\mu = U^\mu_{,\nu} U^\nu$, so that taking into account metric (2), we have $a^t = 0$ and

$$a^r = \Gamma^r_{tt} \left( \frac{dt}{d\tau} \right)^2 = \Phi' (1 - b/r).$$

(6)

Note that from the geodesic equation, a radially moving test particle, which initially starts at rest, obeys the following equation of motion

$$\frac{d^2 r}{d\tau^2} = -\Gamma^r_{tt} \left( \frac{dt}{d\tau} \right)^2 = -a^r.$$

(7)

$a^r$ is the radial component of proper acceleration that an observer must maintain in order to remain at rest at constant $r, \theta, \phi$. One may consider that the geometry is attractive if $a^r > 0$, i.e., observers must maintain an outward-directed radial acceleration to keep from being pulled into the star; and repulsive if $a^r < 0$, i.e., observers must maintain an inward-directed radial acceleration to avoid being pushed away from the star. This distinction depends on the sign of $\Phi'$, as is transparent from Eq. (6). In particular, for a constant redshift function, $\Phi'(r) = 0$, static observers are also geodesic. Thus, the convention used is that $\Phi'(r)$ is positive for an inwardly gravitational attraction, and negative for an outward gravitational repulsion.

Despite of the fact that, in a cosmological context, the van der Waals fluid is considered homogeneous, inhomogeneities may arise through gravitational instabilities, resulting in a nucleation of the cosmic fluid due to the respective density perturbations. Thus, these solutions may possibly originate from density fluctuations in the cosmological background. Consider the VDW equation of state for an inhomogeneous spherically symmetric spacetime, given by

$$p_r = \frac{\gamma \rho}{1 - \beta \rho} - \alpha \rho^2.$$

(8)

In the present context of relativistic stellar models, we shall consider the pressure in Eq. (8) as the radial pressure, $p_r$, and the tangential pressure may be obtained via Eq. (5). Note that the limiting case $\alpha, \beta \to 0$, reduces to the dark energy equation of state, with $\gamma < -1/3$. Using Eqs. (2)-(4), then Eq. (5) provides the following relationship

$$r \left( 1 - \frac{2m}{r} \right) \Phi' = \frac{m}{r} + \frac{\gamma m'}{1 - \frac{3m}{4\pi r^2}} - \frac{\alpha m'^2}{4\pi r^2}.$$

(9)

Note that now we have four equations, Eqs. (3)-(5) and Eq. (8), with five unknown functions of $r$, i.e., $\Phi(r)$, $m(r)$, $\rho(r)$, $p_r(r)$ and $p_t(r)$. To solve the system, one may adopt different strategies, namely, one may model an appropriate
spacetime geometry by imposing \( m(r) \) and/or \( \Phi(r) \) by hand and consequently determine the stress-energy tensor components. In counterpart, one may consider an adequate source of the spacetime geometry by imposing the stress-energy components, and consequently determine the metric fields.

Another useful and interesting relationship may be obtained from Eqs. (13), (16) and Eq. (18), which provide

\[
\Delta = \frac{1}{8\pi r^2} \left\{ (m''r - 2m') \left[ \frac{\gamma}{(1 - \frac{\alpha m'}{4\pi r^2})^2} - \frac{\alpha m'}{2\pi r^2} \right] + m'r \left[ 1 + \frac{\gamma}{1 - \frac{\alpha m'}{4\pi r^2}} - \frac{\alpha m'}{4\pi r^2} \right] \Phi' \right\},
\]

\[\Delta = p_t - p_r \] is denoted the anisotropy factor, as it is a measure of the pressure anisotropy of the fluid comprising the VDW quintessence star. The factor \( \Delta/r \) represents a force due to the anisotropic nature of the stellar model, which is repulsive, i.e., being outward directed if \( p_t > p_r \), and attractive if \( p_t < p_r \). \( \Delta = 0 \) corresponds to the particular case of an isotropic pressure VDW quintessence star.

One may, in principle, construct asymptotically flat spacetimes, where \( \Phi(r) \to 0 \), and \( m(r)/r \to 0 \), as \( r \to \infty \). An alternative approach is to consider a cut-off of the stress-energy tensor at a junction radius \( a \). For instance, consider for simplicity that the exterior solution is the Schwarzschild spacetime, given by

\[
ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2).
\]

\( M \) may be interpreted as the VDW quintessence star’s total mass. In this case the spacetimes given by the metrics Eq. (2) and (11) are matched at \( a \), and one has a thin shell surrounding the star. Using the Darmois-Israel formalism, the surface stresses are given by

\[
\sigma = -\frac{1}{4\pi a} \left( \sqrt{1 - \frac{2M}{a} + \dot{a}^2} - \sqrt{1 - \frac{2m(a)}{a} + \dot{a}^2} \right),
\]

\[
\mathcal{P} = \frac{1}{8\pi a} \left[ \sqrt{1 - \frac{2M}{a} + \dot{a}^2} + \sqrt{1 - \frac{2m(a)}{a} + \dot{a}^2} \right] \frac{(1 + a\Phi')(1 - \frac{2m}{a} + \dot{a}^2) + a\ddot{a} - \frac{\dot{a}^2(m - m')}{(a - 2m)}}{\sqrt{1 - \frac{2m(a)}{a} + \dot{a}^2}},
\]

where the overdot denotes a derivative with respect to the proper time, \( \tau \); \( \sigma \) is the surface energy density and \( \mathcal{P} \) the surface pressure (see Refs. 20 for details). The static case is given by taking into account \( \dot{a} = \ddot{a} = 0 \). The total mass of the VDW quintessence star, for the static case, is given by

\[
M = m(a_0) + m_s(a_0) \left[ \sqrt{1 - \frac{2m(a_0)}{a_0}} - \frac{m_s(a_0)}{2a_0} \right],
\]

where \( m_s \) is the surface mass of the thin shell, and is defined as \( m_s = 4\pi a^2\sigma \).

### 3. VDW QUINTESSENCE GRAVASTARS

The Mazur-Mottola model is constituted by an onion-like structure with five layers, including two thin-shells, with surface stresses \( \sigma_{\pm} \) and \( \mathcal{P}_{\pm} \), where \( \sigma \) is the surface energy density and \( \mathcal{P} \) the surface tangential pressure. The interior of the solution is replaced by a segment of de Sitter space, which is then matched to a finite thickness shell of stiff matter with the equation of state \( p = \rho \). The latter is further matched to an external Schwarzschild vacuum with \( p = \rho = 0 \). A simplification of the Mazur-Mottola configuration, and its dynamic stability, was considered in Ref. 21 using a three-layer solution 21, i.e., a de Sitter interior solution was matched to a Schwarzschild exterior solution at a junction surface, comprising of a thin shell with surface stresses \( \sigma \) and \( \mathcal{P} \).

Consider the specific case of a constant energy density, \( \rho(r) = \rho_0 \), so that Eq. (3) provides the following mass function

\[
m(r) = Ar^3,
\]

where for simplicity, the definition \( A = 4\pi \rho_0/3 \) is used. Using Eq. (19), one finds the following expression

\[
\Phi'(r) = \frac{Ar\chi}{1 - 2Ar^2}.
\]
where, for notational convenience, the constant $\chi$ is defined by

$$\chi = \left(1 + \frac{3\gamma}{1 - \frac{3\alpha A}{4\pi}} - \frac{9\alpha A}{4\pi}\right). \tag{17}$$

Now, in order to obtain an accelerated behaviour of the Universe, the condition $\rho + 3p < 0$, should be obeyed. We shall use this condition, in the present inhomogeneous spacetime, which yields the following inequality

$$\rho \left(1 + \frac{3\gamma}{1 - 3\beta p} - 3\alpha \rho\right) < 0. \tag{18}$$

Note that in order to provide a positive energy density, one may also impose the following simultaneous conditions, in terms of the VDW equation of state parameters

$$\rho > \frac{1}{\beta}, \quad \beta + \frac{3\alpha}{\alpha \beta} > 0, \quad \frac{(\beta + 3\alpha)^2}{12\alpha \beta} > (1 - 3\gamma). \tag{19}$$

See Ref. [2] for further details. Taking into account the mass function given by Eq. (15), and using Eq. (3), we have $\rho = \rho_0 = 3A/(4\pi)$, so that inequality (18) takes the following form

$$\frac{3A\chi}{4\pi} < 0. \tag{20}$$

from which one readily verifies, using Eq. (16), that $\Phi'(r) < 0$, providing the necessary repulsive character, which is a fundamental property of gravastar models. Note that as we are considering positive energy densities, so that $\lambda > 0$, in the gravastar models, we have $\chi < 0$.

Equation (16), can be integrated to provide the following spacetime metric

$$ds^2 = -(1 - 2Ar^2)^{-\chi/2} dr^2 + \frac{dr^2}{1 - 2Ar^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{21}$$

The stress-energy scenario is given by $\rho = \rho_0 = 3A/(4\pi)$ and

$$p_r(r) = \frac{A\chi(1 + Ar^2\chi)}{4\pi}, \quad p_t(r) = \frac{1}{8\pi} \left[\frac{A\chi(1 + Ar^2\chi)}{1 - 2Ar^2} + A(\chi - 2)\right]. \tag{22}$$

The anisotropy factor is provided by the following relationship

$$\Delta = \frac{A^2r^2(2 + \chi)}{8\pi(1 - 2Ar^2)}, \tag{23}$$

which is plotted in Fig. [1]. Note that $\Delta = 0$, i.e., $p_r = p_t$, at the center $r = 0$ as was to be expected.

### 4. VDW QUINTESSENCE TRAVERSABLE WORMHOLES

The spacetime metric representing a spherically symmetric and static wormhole is given by Eq. (22), where $\Phi(r)$ and $b(r) = 2m(r)$ are arbitrary functions of the radial coordinate, $r$, denoted as the redshift function, and the form function, respectively [17]. The radial coordinate has a range that increases from a minimum value at $r_0$, corresponding to the wormhole throat, to infinity. To be a wormhole solution a flaring out condition of the throat is imposed, i.e., $(b - b')/b^2 > 0$ [17]. From this we verify that at the throat $b(r_0) = r = r_0$, the condition $b'(r_0) < 1$ is imposed to have wormhole solutions. For the wormhole to be traversable, one must demand that there are no horizons present, which are identified as the surfaces with $e^{2\Phi} \to 0$, so that $\Phi(r)$ must be finite everywhere. Note that the condition $1 - b/r > 0$ is also imposed.

Consider, for instance, the specific case of $b(r) = r_0^2/r$, which obeys the conditions of a wormhole, so that Eq. (22) provides the following solution

$$\Phi(r) = -\frac{1}{8\pi r_0^2 + \beta} \left[\frac{\alpha - \beta}{4} + \frac{3\alpha}{32\pi r_0^2} - 2\pi r_0^2(1 - \gamma)\right] \ln(r^2 - r_0^2) + \frac{\pi r_0^2 \gamma}{8\pi r_0^2 + \beta} \ln(8\pi r^4 + \beta r_0^2)$$

$$+ \left(\frac{\alpha}{16\pi r_0^2} - \frac{1}{2}\right) \ln(r) - \frac{\alpha}{32\pi r^2} - \frac{\alpha r_0^2}{64\pi r^4} - \frac{\sqrt{2\pi \beta r_0 \gamma}}{2(8\pi r_0^2 + \beta)} \arctan\left(\frac{2r^2}{r_0^2/\sqrt{3\beta}}\right) + C, \tag{25}$$
FIG. 1: Plot of the anisotropy factor for a VDW quintessence star with a constant energy density. We have defined the following dimensionless parameters: $\delta = \Delta / A$ and $\alpha = \sqrt{A r}$.

where $C$ is a constant of integration. We verify the existence of an event horizon, $r = r_0$, from the first term in the right-hand-side. Thus, fine-tuning the $\gamma$ parameter as

$$\gamma = 1 - \frac{1}{(2 \pi r_0^2)} \left( \frac{\alpha - \beta}{4} + \frac{\alpha \beta}{32 \pi r_0^4} \right),$$

Eq. (25) reduces to

$$\Phi(r) = \left( \frac{\alpha}{16 \pi r_0^2} - \frac{1}{2} \right) \ln(r) + \frac{\pi r_0^2 \gamma}{8 \pi r_0^4 + \beta} \ln(8 \pi r^4 + \beta r_0^2) - \frac{\sqrt{2 \pi \beta r_0 \gamma}}{2(8 \pi r_0^2 + \beta)} \arctan \left( \frac{2r^2}{r_0 \sqrt{2 \pi}} \right) - \frac{\alpha}{32 \pi r^2} - \frac{\alpha r_0^2}{64 \pi r^4} + C. \quad (27)$$

This now corresponds to a traversable wormhole, as one verifies the absence of an event horizon. Note that this solution is not asymptotically flat, so that one needs to match the latter to an exterior vacuum spacetime. If we consider a further imposition, namely, $\alpha = 8 \pi r_0^2$, then Eq. (26) becomes zero, $\gamma = 0$, so that the redshift function assumes a particularly simple form, given by

$$\Phi(r) = -\frac{r_0^2}{4 r^2} \left( 1 + \frac{r_0^2}{2 r^2} \right) + C. \quad (28)$$

The VDW equation of state, Eq. (9), for this particular choice of parameters is $p_r = -8 \pi r_0^2 \rho$, implying a negative radial pressure. Note that this specific choice of a wormhole is asymptotically flat. The stress-energy components take the following form

$$\rho(r) = -\frac{r_0^3}{8 \pi r^4}, \quad p_r(r) = -\frac{r_0^6}{8 \pi r^8}, \quad p_t(r) = -\frac{r_0^3 (4 r^8 + r_0^6 r^6 - 15 r_0^4 r^4 + r_0^6 r^2 + r_0^8)}{32 \pi r^{12}}. \quad (29)$$

This solution corresponds to a rather exotic form of a VDW fluid, due to the presence of a negative energy density. The null energy condition, defined as $T_{\mu\nu} k^\mu k^\nu > 0$, where $k^\mu$ any null vector, is also violated, i.e, $\rho + p_r < 0$, which is a necessary condition of wormholes.

It is also of interest to note that the VDW cosmic fluid with an isotropic pressure may be completely determined by only one parameter in the equation of state, so that one may consider $\gamma$ as the only independent parameter needed to describe the VDW fluid. Therefore, using a wide variety of values of $\gamma$ constrained by observations considered in Ref. [3], one may obtain restrictions on the parameters $\alpha$ and $\beta$, through Eq. (26), in the inhomogeneous solutions considered here. On the other hand, although the particular case of $\gamma = 0$ is not included in the best fits parameter range [3], it is not excluded from the observational constraints. This should not be considered a serious shortcoming, as the parameters $\alpha$, $\beta$ and $\gamma$ are defined using the critical values of the energy density, the isotropic pressure and
the critical volume \[3\], and in the presence of anisotropic pressures the respective definitions should be extended to include the transverse pressure. Despite of this fact, and due to the analytical complexity of Eq. \[27\], we shall, for simplicity, consider the case \(\gamma = 0\) and further analyze the physical properties and characteristics of this specific wormhole solution below, which prove to be extremely interesting.

One may also quantify the “total amount” of energy condition violating matter, which amounts to calculating the “volume integral quantifier” defined as \(\int T_{\mu\nu}k^\mu k^\nu dV\) \[22\]. The amount of violation is defined as the extent to which this integral becomes negative. Using the “volume integral quantifier”, which provides information about the “total amount” of averaged null energy condition (ANEC) violating matter in the spacetime (see Ref. \[22\] for details), given by \(I_V = \int [\rho(r) + p_r(r)] dV\), with a cut-off of the stress-energy at \(a\), we have

\[
I_V = \int_{r_0}^a (r - b) \left[ \ln \left( \frac{e^{2\Phi}}{1 - b/r} \right) \right]' dr.
\]  

(31)

Now, using the form function given by \(b(r) = r_0^2/r\), and the redshift function provided by Eq. \[28\], and evaluating the integral, one finally ends up with the following simplified expression for the “volume integral quantifier"

\[
I_V = \frac{r_0(r_0^5 + 5r_0a^4 - 6a^5)}{5a^5}.
\]  

(32)

By taking the limit \(a \to r_0\), one readily verifies that \(I_V \to 0\). It is also interesting to note that in the limit \(a \to \infty\), the volume integral tends to a finite value, i.e., \(I_V = -r_0^4/5\). This proves that as in the specific case of phantom \[8\] and Chaplygin \[13\] wormholes, one may theoretically construct a wormhole with arbitrarily small amounts of a VDW quintessence fluid. As emphasized in Ref. \[8\], this result is not unexpected, however, it is interesting to note the relative ease with which one may theoretically construct wormholes supported by infinitesimal amounts of exotic fluids used in cosmology to explain the present accelerated cosmic expansion.

It is also of a particular interest to analyze the traversability conditions. We will be interested in specific solutions for traversable wormholes and assume that a traveller of an absurdly advanced civilization, with human traits, begins the trip in a space station in the lower universe, at proper distance \(l = -l_1\), and ends up in the upper universe, at \(l = l_2\). The proper distance is given by \(dl = \pm(1 - b/r)^{-1/2} dr\). Assume that the traveller has a radial velocity \(v(r)\), as measured by a static observer positioned at \(r\). One may relate the proper distance travelled \(dl\), radius travelled \(dr\), coordinate time lapse \(dt\), and proper time lapse as measured by the observer \(d\tau\), by the following relationships

\[
v = e^{-\Phi} \frac{dl}{dt} = \mp e^{-\Phi} \left(1 - \frac{b}{r}\right)^{-1/2} \frac{dr}{dt},
\]  

(33)

\[
v\gamma = \frac{dl}{d\tau} = \mp \left(1 - \frac{b}{r}\right)^{-1/2} \frac{dr}{d\tau}.
\]  

(34)

See \[17\] for details.

For a convenient trip through the wormhole, certain conditions should also be imposed \[17\]. Firstly, the entire journey should be done in a relatively short time as measured both by the traveller, \(\Delta t_{tr}\), and by observers who remain at rest at the stations, \(\Delta t_{st}\). For simplicity, we shall take into account non-relativistic and constant traversal velocities, \(\gamma \approx 1\). Thus, \(\Delta t_{tr}\) and \(\Delta t_{st}\) are given by

\[
\Delta t_{tr} = \int_{-l_1}^{+l_2} \frac{dl}{v\gamma} = 2\int_{r_0}^a \frac{dr}{v\gamma (1 - b/r)^{1/2} v\gamma} \geq 2\int_{r_0}^a \frac{dr}{v\gamma} \approx \frac{2(a - r_0)}{v},
\]  

(35)

\[
\Delta t_{st} = \int_{-l_1}^{+l_2} \frac{dl}{ve^\Phi} = 2\int_{r_0}^a \frac{dr}{ve^\Phi (1 - b/r)^{1/2} ve^\Phi} \geq 2\int_{r_0}^a \frac{dr}{ve^\Phi} = \frac{2}{v} \int_{r_0}^a e^{-\Phi} dr,
\]  

(36)

Considering the redshift function provided by Eq. \[28\], and \(a = 2r_0\), so that \(\int_{r_0}^a e^{-\Phi} dr \approx 1.18r_0\), we verify that both \(\Delta t_{tr}\) and \(\Delta t_{st}\) are bounded from below by \(\sim 2r_0/v\).

An important traversability condition required is that the acceleration felt by the traveller should not exceed Earth’s gravity \[17\]. Thus, the traveller’s four-acceleration expressed in his proper reference frame, considering once again non-relativistic and constant traversal velocities, \(\gamma \approx 1\), yields the following restriction (see Ref. \[17\] for details)

\[
|\ddot{a}| \approx \left| \left(1 - \frac{b}{r}\right)^{1/2} \Phi \right| \leq g_E,
\]  

(37)

which is readily verified at the throat, where the conditions are most severe.
Another important condition is that an observer traversing through the wormhole should not be ripped apart by enormous tidal forces. Thus, it is required that the tidal accelerations felt by the traveller should not exceed, for instance, the Earth’s gravitational acceleration \[ g_\oplus \]. The constraint \( |\Delta u^\alpha| \leq g_\oplus \) provides the tidal acceleration restrictions as measured by a traveller moving radially through the wormhole, given by the following inequalities

\[
\begin{align*}
\left(1 - \frac{b}{r}\right) \left[ \Phi'' + \left(\Phi'\right)^2 - \frac{b'r - b}{2r(r-b)\Phi'} \right] |\eta^1| \leq g_\oplus, \\
\frac{\gamma^2}{2r^2} \left[ v^2 \left( b' - \frac{b}{r} \right) + 2(r-b)\Phi' \right] |\eta^2| \leq g_\oplus.
\end{align*}
\]

We refer the reader to Ref. [17] for details related to the deduction of these conditions. The factors \( |\eta^1| \) and \( |\eta^2| \) appearing in inequalities (38)-(39) are the spatial separations between radial and lateral parts of the traveller’s body, as measured in his proper reference frame. For computational purposes we may assume \( |\eta^1| \approx |\eta^2| \approx 2 \text{ m} \).

The radial tidal constraint, Eq. (45), constrains the redshift function, and the lateral tidal constraint, Eq. (39), constrains the velocity with which observers traverse the wormhole. These inequalities are particularly simple at the throat, \( r_0 \), and reduce to

\[
\begin{align*}
|\Phi'(r_0)| &\leq \frac{2g_\oplus r_0}{\left(1 - b'\right)|\eta^1|}, \\
\gamma^2 v^2 &\leq \frac{2g_\oplus r_0^2}{\left(1 - b'\right)|\eta^2|}.
\end{align*}
\]

Taking into account the redshift function given by Eq. (28), and the form function considered above, i.e., \( b(r) = r_0^2/r \), we verify that evaluated at the throat we have \( \Phi'(r_0) = 1/r_0 \) and \( b'(r_0) = -1 \). Thus, inequality (40) provides the restriction \( 1 \leq g_\oplus r_0^2/|\eta^1| \). Considering the minimum value, i.e., \( g_\oplus r_0^2/|\eta^1| \approx 1 \), and that \( |\eta^1| \approx |\eta^2| \approx 2 \text{ m} \), we verify that the minimum value of the throat is given by \( r_0 \approx 2 \times 10^6 \text{ m} \). Using the latter approximation, and taking into account non-relativistic and constant traversal velocities, \( \gamma \approx 1 \), then inequality (41) provides the following restriction for the velocity, \( v \leq c \). For instance, considering that the traversal velocity be of the order \( v \approx 10^{-2}c \), and assuming that the traversal times take their minimum value, we have \( \Delta t_{tr} \sim \Delta t_{st} \sim 10^2 \text{ s} \). This provides an extremely reasonable traversable wormhole, useful for travellers with human traits.

### 5. CONCLUSION

In conclusion, noting that the VDW quintessence equation of state is an interesting scenario for describing the late universe, we have explored the construction of relativistic stellar models, in particular gravastars and traversable wormholes, supported by a VDW equation of state. These solutions may possibly originate from density fluctuations in the cosmological background, resulting in the nucleation through the respective density perturbations. We have found an exact solution of a gravastar model, by considering the specific case of a constant energy density. Relatively to traversable wormhole geometries, we also found an exact solution of an asymptotically flat wormhole, as well as solutions, where the exotic matter is constrained to the throat neighborhood, by considering a matching of an interior wormhole geometry to an exterior Schwarzschild vacuum solution. It was also found that one may theoretically construct a wormhole with arbitrarily small amounts of a VDW quintessence fluid. Several characteristics and properties of these traversable wormholes using the traversability conditions were also explored.

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[1] V. F. Cardone, C. Tortora, A. Troisi and S. Capozziello, “Beyond the perfect fluid hypothesis for dark energy equation of state,” Phys. Rev. D 73 043508 (2006), [arXiv:astro-ph/0511528](http://arxiv.org/abs/astro-ph/0511528).

[2] S. Capozziello, S. De Martino and M. Falanga, “Van der Waals quintessence,” Phys. Lett. A 299, 494 (2002).
[3] S. Capozziello, S. Carloni and A. Troisi, “Quintessence without scalar fields,” arXiv:astro-ph/0303041.
[4] S. Capozziello, V. F. Cardone, S. Carloni, D. De Martino, M. Falanga, A. Troisi and M. Bruni, “Constraining van der Waals quintessence by observations,” JCAP 0504, 005 (2005). arXiv:astro-ph/0410503.
[5] G. M. Kremer, “Cosmological models described by a mixture of van der Waals fluid and dark energy,” Phys. Rev. D 68, 123507 (2003) arXiv:gr-qc/0309111.
[6] S. Capozziello, S. Nojiri and S.D. Odintsov, “Dark Energy: the equation of state description versus scalar-tensor or modified gravity,” Phys. Lett. B 634, 93 (2006). arXiv:hep-th 0512118.
[7] G. M. Kremer, “Brane cosmology with a van der Waals equation of state,” Gen. Rel. Grav. 36 1423-1432 (2004) arXiv:gr-qc/0404037.
[8] S. Sushkov, “Wormholes supported by a phantom energy,” Phys. Rev. D 71, 043520 (2005) arXiv:gr-qc/0502084; F. S. N. Lobo, “Phantom energy traversable wormholes,” Phys. Rev. D 71, 084011 (2005) arXiv:gr-qc/0502099; F. S. N. Lobo, “Stability of phantom wormholes,” Phys. Rev. D 71, 124022 (2005) arXiv:gr-qc/0506001.
[9] S. V. Sushkov and S. W. Kim, “Cosmological evolution of a ghost scalar field,” Gen. Rel. Grav. 36, 1671 (2004) arXiv:gr-qc/0404049.
[10] J. D. Polanco, P. S. Letelier and M. Ujevic, “Space-time geometry and thermodynamic properties of a self-gravitating ball in phase transition,” Phys. Rev. D 70, 064006 (2004).
[11] P. O. Mazur and E. Mottola, “Gravitational Condensate Stars: An Alternative to Black Holes,” arXiv:gr-qc/0109035; P. O. Mazur and E. Mottola, “Dark energy and condensate stars: Casimir energy in the large,” arXiv:gr-qc/0405111; R. Doran, F. S. N. Lobo and P. Crawford, “Interior of a Schwarzschild black hole revisited,” arXiv:gr-qc/0609042.
[12] I. Dymnikova, “Vacuum nonsingular black hole,” Gen. Rel. Grav. 24, 235 (1992); I. Dymnikova, “Spherically symmetric space-time with the regular de Sitter center,” Int. J. Mod. Phys. D 12, 1015-1034 (2003) arXiv:gr-qc/0304110; I. Dymnikova and E. Galaktionov, “Stability of a vacuum nonsingular black hole,” Class. Quant. Grav. 22, 2331-2358 (2005) arXiv:gr-qc/0409049.
[13] G. Chapline, “Dark energy stars,” arXiv:astro-ph/0503200.
[14] F. S. N. Lobo, “Stable dark energy stars,” Class. Quant. Grav. 23, 1525 (2006) arXiv:astro-ph/0508115.
[15] O. Bertolami and J. J. Figueroa, “The Chaplygin dark star,” Phys. Rev. D 72, 123512 (2005) arXiv:astro-ph/0509547; N. Bilic, G. B. Tupper and R. D. Viollier, “Born-Infeld Phantom Gravastars,” JCAP 0602, 013 (2006) arXiv:astro-ph/0508347.
[16] M. S. Morris and K. S. Thorne, “Wormholes in spacetime and their use for interstellar travel: A tool for teaching General Relativity,” Am. J. Phys. 56, 395 (1988).
[17] F. S. N. Lobo, “Chaplygin traversable wormholes,” Phys. Rev. D 73 064028 (2006). arXiv:gr-qc/0511003.
[18] T. A. Roman, “Inflating Lorentzian wormholes,” Phys. Rev. D 47, 1370 (1993) arXiv:gr-qc/9211012.
[19] J. P. S. Lemos and F. S. N. Lobo and S. Q. de Oliveira, “Morris-Thorne wormholes with a cosmological constant,” Phys. Rev. D 68, 064004 (2003) arXiv:gr-qc/0302049; F. S. N. Lobo, “Surface stresses on a thin shell surrounding a traversable wormhole,” Class. Quant. Grav. 21 4811 (2004) arXiv:gr-qc/0409013; F. S. N. Lobo, “Energy conditions, traversable wormholes and dust shells,” Gen. Rel. Grav. 37, 2023 (2005) arXiv:gr-qc/0410087; F. S. N. Lobo and P. Crawford, “Stability analysis of dynamic thin shells,” Class. Quant. Grav. 22, 4869 (2005), arXiv:gr-qc/0507063.
[20] M. Visser and D. L. Wiltshire, “Stable gravastars - an alternative to black holes?,” Class. Quant. Grav. 21 1135 (2004) arXiv:gr-qc/0310107.
[21] M. Visser, S. Kar and N. Dadhich, “Traversable wormholes with arbitrarily small energy condition violations,” Phys. Rev. Lett. 90, 201102 (2003) arXiv:gr-qc/0301003.