Analysis of Interaction Scattering Cross Sections and Their Physical Bounds for Multiple-Dipole Stimulation of a Three-Dimensional Layered Medium

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ABSTRACT  A three-dimensional layered and isotropic medium is excited by primary spherical waves due to \( N \) magnetic dipoles radiating inside or outside the medium. Interaction scattering cross sections (ISCS) are defined as the differences between the overall scattering cross section and the sum of the individual cross sections generated by all dipoles within a layer or by all \( N \) dipoles. Optical theorems and physical bounds for the ISCS are established. Extensive numerical investigations are performed for the variations of the ISCS and their physical bounds with respect to the geometrical and physical characteristics of the layered medium. Conditions for which ISCS contribute significantly in the overall cross section are analyzed. It is also demonstrated that the number of excitation layers and the total number \( N \) of dipoles can be determined by means of the individual scattering cross sections.

INDEX TERMS  Electromagnetic scattering, layered medium, dipoles, physical bounds, scattering cross sections.

I. INTRODUCTION

EXCITATION of a three-dimensional layered (piecewise homogeneous) medium by \( N \) electric or magnetic dipoles, located in different internal layers or in the medium’s exterior, constitutes a realistic model for applications spreading from low frequencies to the visible range. Representative applications include, e.g., stimulation of the brain by the neurons currents \([1],[2]\), cancer-treatment techniques like ablation and interstitial hyperthermia \([3],[4]\), radiation by multiple sensors in 5G networks \([5]\), multilayer optical diffusion \([6]\), and design of nonplanar microstrip antennas \([7]\). Moreover, in the context of inverse problems, excitation by \( N \) dipoles was employed in field-splitting techniques \([8]\), identifications of fields on spherically-symmetric conductors \([9]\), reconstruction of obstacles buried in layered media \([10]\), and dipoles localization using electromagnetic induction sensors \([11]\). Besides, scattering by layered uniaxial objects was investigated in \([12]\) by employing a methodology involving electric and magnetic infinitesimal dipoles located in different layers.

In this work, we, first, formulate the boundary-value problems for the generated electric fields corresponding to the excitation of a piecewise homogeneous medium composed of annuli-like layers by \( N \) internal and external magnetic dipoles. Then, we adopt specific fields’ decompositions and introduce interaction scattering cross sections (ISCS) as the differences between the overall scattering cross section and the sum of the individual cross sections due to all dipoles of a certain layer or due to all \( N \) dipoles exciting the scatterer. ISCS quantify the energy flux rate which is induced by the interaction of the individual fields and is not directly connected to an actual dipole involved in the boundary-value problem.

Next, we distinguish two different cases: single-layer excitation when all dipoles lie in the same layer (or in the medium’s exterior) and mixed excitation when dipoles are located in more than one layers. For each of the two cases, we derive optical theorems determining the individual and overall scattering cross sections by means of the secondary fields at the dipoles’ positions. Optical theorems relating the ISCS with the partial fields (defined as the sum of the individual fields due to all dipoles in an excitation layer except one) are also established. Moreover, we derive physical bounds for the ISCS with respect to the minimum and maximum...
The considered three-dimensional layered medium is also analyzed. Additionally, it is shown that the materials of the layers affects significantly the ISCS in reducing the expected energy flux rate. Furthermore, changing the materials of the layers significantly affects the additivity of the scattering cross sections. Mainly for mixed excitation, it is shown that the ISCS become negative in some cases, which means that fields interactions reduce the expected energy flux rate. Therefore, changing the materials of the layers affects significantly the ISCS in mixed excitation, but does not result in significant variations in single-layer excitation. The influence on the ISCS of the excitation dipoles’ distance from the medium’s boundary is also analyzed. Additionally, it is shown that the number of excitation layers and the total number N of dipoles can be determined by the derived physical bounds involving the individual scattering cross sections. It is also demonstrated that for external excitation the ISCS ratios in the low-frequency regime are very close to $1 - \frac{1}{3}$.

This paper is organized as follows. The formulation of the layered medium is included in Section II. Optical theorems and physical bounds on the ISCS together with numerical results with respect to changes in the layered medium and the excitation dipoles are presented in Sections III and IV, for the single-layer and mixed excitation cases, respectively. Selected conclusions from the numerical results are summarized in Section V.

II. MATHEMATICAL FORMULATION

The considered three-dimensional layered medium $V$ is shown in Fig. 1. It has a $C^2$ boundary $S_1$ and is divided by $P - 1$ $C^2$ surfaces $S_p$ ($p = 2, \ldots, P$) into $P$ annuli-like layers $V_p$ ($p = 1, \ldots, P$). Surface $S_p$ includes surfaces $S_q$, for all $q > p$ and $p = 1, \ldots, P$. The first $P - 1$ layers $V_p$ are homogeneous, isotropic and dielectric with real wavenumbers $k_p$, dielectric permittivities $\epsilon_p$, and magnetic permeabilities $\mu_p$. The scatterer’s core $V_0$ can be a perfect electric conductor (PEC), perfect magnetic conductor (PMC) or isotropic dielectric with wavenumber $k_p$, permittivity $\epsilon_p$, and permeability $\mu_p$. The exterior $V_0$ of $V$ has respective physical parameters $\epsilon_0$, $\epsilon_0$, and $\mu_0$.

The layered medium $V$ is excited by $N$ magnetic dipoles which are distributed arbitrarily in its interior or its exterior. These $N$ dipoles are located at $r_i$ and possess unit dipole moments $\hat{p}_i$, with $i = 1, \ldots, N$. In particular, we suppose that the dipoles are contained in $Q$ of the medium’s layers, called excitation layers and denoted by $V^x_q$, with $Q \leq P + 1$. When dipoles lie in $V_0$, then $V^x_0$ (for $q = 1$) coincides with $V_0$. Each excitation layer $V^x_q$ contains $n_q$ dipoles, of strength $A^q$, position vector $r^q$, and dipole moment $\hat{p}^q$, for $j = 1, \ldots, n_q$. Hence, it holds that $n_1 + n_2 + \cdots + n_Q = N$.

Each magnetic dipole radiates a primary electric field (under an $\exp(-i\omega t)$ time dependence with $\omega$ as the angular frequency, $t$ as time, and $i = \sqrt{-1}$)

$$E^p(\mathbf{r}; \mathbf{r}^q) = A^q \nabla \times \left( \frac{\exp(ik_q |\mathbf{r} - \mathbf{r}^q|)}{|\mathbf{r} - \mathbf{r}^q|} \hat{p}_q \right), \quad \mathbf{r} \neq \mathbf{r}^q, \quad (1)$$

where $j = 1, \ldots, n_q$ and $q = 1, \ldots, Q$. According to the scattering superposition method [13], [14], the total electric field in $V^x_q$ due to a dipole at $\mathbf{r}^q \in V^x_q$ is expressed as

$$E^t_q(\mathbf{r}; \mathbf{r}^q) = E^p(\mathbf{r}; \mathbf{r}^q) + E^s_q(\mathbf{r}; \mathbf{r}^q), \quad \mathbf{r} \in V^x_q \setminus \{\mathbf{r}^q\}, \quad (2)$$

where $E^s_q(\mathbf{r}; \mathbf{r}^q)$ is the secondary field in layer $V_q$ due to a dipole at $\mathbf{r}^q$. If $V_p$ is not an excitation layer, then

$$E^t_p(\mathbf{r}; \mathbf{r}^q) = E^p_p(\mathbf{r}; \mathbf{r}^q).$$

Fields due to a single excitation dipole will be referred to as individual fields.

Next, we define the secondary and total $q$-excitation fields as the superpositions of the corresponding individual fields due to all dipoles in $V^x_q$, i.e.,

$$E^s_q(\mathbf{r}; \mathbf{r}^q, \ldots, \mathbf{r}^{q_n}) = \sum_{j=1}^{n_q} E^s_q(\mathbf{r}; \mathbf{r}^q), \quad (3)$$

where $\ell \in \{\text{sec}, \text{t}\}$. If $V_p$ is an excitation layer $V^x_q$, then the $q$-excitation field is given by

$$E^t_q(\mathbf{r}; \mathbf{r}^q, \ldots, \mathbf{r}^{q_n}) = E^p(\mathbf{r}; \mathbf{r}^q, \ldots, \mathbf{r}^{q_n}) + E^s_q(\mathbf{r}; \mathbf{r}^q, \ldots, \mathbf{r}^{q_n}), \quad \mathbf{r} \in V^x_q \setminus \{\mathbf{r}^q, \ldots, \mathbf{r}^{q_n}\}. \quad (4)$$
where the primary q-excitation field is defined as
\[
E_q^{pr}(r; r_q^1, \ldots, r_q^N) = \sum_{j=1}^{n_q} E_q^{pr}(r; r_q^j).
\] (5)

The overall secondary \(E_q^{sec}(r; r_1^1, \ldots, r_1^N)\) and overall total field \(E_q^t(r; r_1^1, \ldots, r_1^N)\) of \(V_p\) are defined, accordingly, as the superpositions of all corresponding fields due to all \(N\) dipoles.

Individual, q-excitation, and overall fields satisfy the vector Helmholtz equations
\[
\nabla^2 E_q^t(r; :) + k^2_q E_q^t(r; :) = 0,
\] (6)
in \(V_p\), if \(V_p\) is not an excitation layer, and in \(V_q^{ex}(r_1^1, \ldots, r_1^N)\) if \(V_p\) is an excitation layer \(V_q^{ex}\). These fields also satisfy the transmission conditions
\[
\hat{n} \times E_{p-1}^t(r; :) = \hat{n} \times E_{q}^t(r; :),\]
(7)
\[
\frac{1}{\mu_{p-1}} \hat{n} \times \nabla \times E_{p-1}^t(r; :) = \frac{1}{\mu_q} \hat{n} \times \nabla \times E_{q}^t(r; :)
\] (8)
on the boundaries of each dielectric layer \(V_p (p = 1, \ldots, P)\). If the core \(V_p\) is PEC or PMC, then on its boundary the following conditions hold [15]
\[
\hat{n} \times E_{p-1}^t(r; :) = 0,
\] (9)
\[
\hat{n} \times \nabla \times E_{p-1}^t(r; :) = 0.
\] (10)

Moreover, the total individual fields in \(V_0\) satisfy the Silver-Müller radiation condition [16], and are expressed as
\[
E_0^t(r; r_q^j) = g_q^j(\hat{r})h_0(k_0r) + O(r^{-2}), \quad r = |r| \to \infty,
\] (11)
where \(g_q^j\) is the individual far-field due to a dipole at \(r_q^j \in V_q^{ex}\) and \(h_0\) is the zero-th order and first-kind spherical Hankel function. The q-excitation \(g_q\) and overall far-field \(g\) are defined as the superpositions of the individual far-fields due to all \(n_q\) dipoles in \(V_q^{ex}\) and to all \(N\) dipoles, respectively, i.e.,
\[
g_q(\hat{r}) = \sum_{j=1}^{n_q} g_q^j(\hat{r}),
\] (12)
\[
g(\hat{r}) = \sum_{q=1}^{Q} g_q(\hat{r}).
\] (13)

Next, the individual \(\sigma_q^j\), q-excitation \(\sigma_q\), and overall cross section \(\sigma\) are the scattering cross sections by a dipole at \(r_q^j \in V_q^{ex}\) all dipoles in \(V_q^{ex}\), and all \(N\) dipoles, respectively, i.e.,
\[
\sigma_q^j = \frac{1}{k_0^2} \int_{S^2} |g_q^j(\hat{r})|^2 d\sigma(\hat{r}),
\] (14)
\[
\sigma_q = \frac{1}{k_0^2} \int_{S^2} |g_q(\hat{r})|^2 d\sigma(\hat{r}),
\] (15)
\[
\sigma = \frac{1}{k_0^2} \int_{S^2} |g(\hat{r})|^2 d\sigma(\hat{r}),
\] (16)
where \(S^2\) is the unit sphere of \(\mathbb{R}^3\). The overall cross section is not—in general—equal to the sum of the individual cross sections. This was elaborated in [17] for acoustic point-source excitation of a layered medium, and in [18] for acoustic plane-wave multiple scattering. For plane-wave light scattering by a small number of particles, the additivity of the cross sections was investigated in [19] under the condition of sufficiently large distance between each particle. Non-additive properties of the cross sections in conjunction with validity conditions of the Discrete Dipole Approximation (DDA) were studied in [20].

The difference between the sum of the individual cross sections and the overall cross section expresses the rate of induced energy flux, which stems from the interactions between the individual fields, and is not connected directly to an actual exciting dipole. This induced energy flux rate is quantified by the following interaction scattering cross sections (ISCS)
\[
\tilde{\sigma}_q = \sigma_q - \sum_{j=1}^{n_q} \sigma_q^j
\] (17)
\[
\tilde{\sigma}_q = \sigma_q - \sum_{j=1}^{n_q} \sigma_q^j
\] (18)
\[
\sigma^D = \sum_{q=1}^{Q} \tilde{\sigma}_q.
\] (19)
\[
\sigma^T = \sigma - \sum_{j=1}^{Q} \tilde{\sigma}_q
\] (20)
The q-ISCS \(\tilde{\sigma}_q\) quantifies the energy flux rate due to the interaction between the dipoles in layer \(V_q^{ex}\). The direct ISCS \(\sigma^D\) is the sum of the q-ISCS for all \(Q\) excitation layers. The indirect ISCS \(\sigma^T\) accounts for the flux rate induced by the interaction between total fields generated in different excitation layers. The total ISCS \(\sigma^T\) quantifies the flux rate due to the interaction between all dipoles exciting the scatterer and, thus, measures all possible interactions between the participating fields. The ISCS are related by
\[
\sigma^T = \sigma^D + \sigma^I.
\] (21)
in determining the additivity of the cross sections, and, moreover, elaborate that the energy flux quantified by the ISCS contributes significantly to the overall flux—especially when strong near-field interaction occurs between fields generated by the N dipoles.

### III. SINGLE-LAYER EXCITATION

Single-layer excitation concerns the case of all dipoles lying in the same layer (internal excitation) or in the scatterer’s exterior (external excitation); hence, it holds that \( n_q = N \) and \( Q = 1 \). In this case, \( \sigma^\text{I} = \sigma^\text{D} = \tilde{\sigma}_q \), \( \sigma^\text{a} = 0 \), since there is no indirect interaction between the participating fields and \( \sigma_q = \sigma \).

#### A. OPTICAL THEOREMS AND PHYSICAL BOUNDS

Using [21, Th. 5.1], we arrive at the following optical theorems relating the \( q \)-excitation cross section and the \( q \)-ISCS with their corresponding secondary fields

\[
\sigma_q = 4\pi \frac{\mu_0}{k_0} \left[ \sum_{j=1}^{n_q} \text{Re} \left( i\lambda q_j \left( \nabla \times \mathbf{E}_{q}^\text{sec}(\mathbf{r}_j; r^1, \ldots, r^n_q) \right) \cdot \mathbf{p}_j \right) ight] + \frac{1}{2} \sum_{l=1}^{n_q} \sum_{v=1}^{n_q} \tilde{\sigma}_{q,l,v}^\text{pr},
\]

\[
\tilde{\sigma}_q = 4\pi \frac{\mu_0}{k_0} \left[ \sum_{j=1}^{n_q} \text{Re} \left( i\lambda q_j \left( \nabla \times \mathbf{E}_{q}^\text{sec}(\mathbf{r}_j; r^1, \ldots, r^n_q) \right) \cdot \mathbf{p}_q \right) ight] + \sum_{v=1}^{n_q-1} \sum_{j=1}^{n_q} \tilde{\sigma}_{q,l,v}^\text{pr},
\]

where \( \lambda q_j = \lambda q / \mu_0 \), while \( \mathbf{E}_{q}^\text{sec}(\mathbf{r}; \mathbf{r}_j) \) denotes the secondary partial field of \( V^\text{ex} \) with respect to a single dipole, i.e.,

\[
\mathbf{E}_{q}^\text{sec}(\mathbf{r}; \mathbf{r}_j) = \sum_{v=1}^{n_q} \mathbf{E}_{q}^\text{pr}(\mathbf{r}; \mathbf{r}_v).
\]

Quantity \( \tilde{\sigma}_{q,l,v}^\text{pr} \) denotes the average flux rate per surface unit area induced by the interaction of the primary far-fields generated by dipoles at \( \mathbf{r}_j \) and \( \mathbf{r}_v \) under the absence of the scatterer, and it is calculated as

\[
\tilde{\sigma}_{q,l,v}^\text{pr} = \frac{1}{2\pi} \int_{S^2} \mathbf{g}_{q,l,v}^\text{pr}(\mathbf{r}_j) \cdot \mathbf{g}_{q,l,v}^\text{pr}(\mathbf{r}) dS(\mathbf{r}).
\]

Employing Hölder’s inequality in the definitions of the involved scattering cross sections and ISCS, we obtain the following bounds for the \( q \)-ISCS

\[
1 - n_q \frac{\sigma_{q}^\text{max}}{\sigma_q} \leq \tilde{\sigma}_q \leq \min \left\{ 1 - n_q \frac{\sigma_{q}^\text{min}}{\sigma_q}, 1 - \frac{1}{n_q} \right\},
\]

where \( \sigma_{q}^\text{min} \) and \( \sigma_{q}^\text{max} \) are the minimum and maximum individual cross sections for the dipoles located in \( V^\text{ex} \). When

\[
n_q^2 \sigma_{q}^\text{min} \leq \sigma_q,
\]

then the minimum involved in (25) is \( 1 - \frac{1}{n_q} \). Considering that \( \sigma_q \leq n_q \sigma_{q}^\text{max} \) and combining with (25), we conclude that condition (26) holds if and only if

\[
\sqrt{\frac{\sigma_q}{\sigma_{q}^\text{max}}} \leq n_q \leq \sqrt{\frac{\sigma_q}{\sigma_{q}^\text{min}}}.
\]

A detailed proof of (25) is given in the Appendix. Optical theorems for a single dipole in the exterior of a homogeneous medium were established in [22]. Physical bounds for the differential radar cross sections with respect to the number of closely-spaced isotropic radiators excited by a plane wave were derived in [23].

#### B. PARAMETRIC ANALYSIS AND NUMERICAL RESULTS

The variations of the \( q \)-ISCS and their associated physical bounds with respect to the input parameters of the scattering problem are investigated numerically. The presented results correspond to a layered spherical scatterer \( V \) with all excitation dipoles lying either in the exterior of \( V \) (external excitation) or in a certain spherical layer (internal excitation). Precisely, a 2-layered spherical scatterer \( V \) (i.e., \( P = 2 \)) is considered with external radius \( a_1 \) and core’s radius \( a_2 \), excited by external dipoles in \( V_0 (r > a_1) \) or internal dipoles in the spherical shell \( V_1 (a_2 < r < a_1) \). The core \( V_2 \) is PEC or dielectric.

Concerning the choices of physical parameters and thicknesses of the shells, connections can be established to potential applications, among the ones identified in the Introduction above. For example, in applications involving biological tissues, the magnetic permeability is considered to be that of vacuum, and hence the relative permeability is \( \mu_r = 1 \) [9], [11]. However, the relative dielectric permittivities of biological tissues depends on many factors, including excitation frequency, e.g., the dielectric permittivity of human lung tissue is 2.2 at 1 GHz [27]. On the other hand, in hyperthermia techniques, a multi-slot coaxial antenna, with central conductor’s (core’s) radius less than 1/5 of the antenna (catheter) radius is frequently employed [3], [4]. Besides, in the spherical three-shell model of the brain, the brain’s (core’s) radius is at least half the head’s external radius [2].

The associated boundary-value problems are solved by employing the methodology developed in [24]–[26], which combines the Sommerfeld’s and T-matrix methods in conjunction with suitable eigenfunction expansions. This methodology is entirely analytical and does not require any restrictions on the problem’s parameters. Moreover, its validity was tested with respect to other solutions having appeared in the literature as well as to special cases of the considered spherical geometry and involved materials (see [24, Secs. 5 and 6] and [25, Secs. 4 and 6]).
First, we consider the external excitation of a homogeneous dielectric spherical scatterer by \( N = 4 \) external dipoles lying on the z-axis at \( r_j = (1 + 0.25) a_1 \), for \( j = 1, 2, 3, 4 \). For the scatterer’s dielectric permittivity \( \epsilon_{r_1} \) different values are examined, while its relative magnetic permeability is \( \mu_{r_1} = 1 \). Table 1 presents the values of the individual cross sections \( \sigma_j \), the total ISCS \( \sigma \), and the overall cross section \( \sigma \) for different \( \epsilon_{r_1} \) and electric radii \( k_0 a_1 \). For the computation of the individual cross sections, we used the exact solution of the direct scattering problem due to a single source developed in [24]. For the overall cross section and the total ISCS, we used the optical theorems (22) and (23). From this table, it is readily observed that \( \sigma = \sigma^T + \sum_{j=1}^{4} \sigma^j \) for all depicted cases.

Then, in Fig. 2, we depict the variations of \( \sigma^T/\sigma \) versus \( k_0 a_1 \) for a spherical scatterer with \( a_1 = 2 a_2, \epsilon_{r_1} = 2, \mu_{r_1} = 1.5 \) and a PEC core (top panel) and dielectric core with \( \epsilon_{r_2} = 3, \mu_{r_2} = 2.5 \) (bottom panel). The scatterer is excited by three sets of \( N = 4 \) external sources with distance 0.2a1 between successive sources.

In Fig. 4, we depict the ISCS ratios and the physical bounds indicated by (25) for external excitation by \( N = 4 \) dipoles at \( r_j = (1.25 + 0.25 j) a_1 \), with \( j = 0, 1, 2, 3 \). In the lower frequencies \( k_0 a_1 < 3.1 \) for the PEC and \( k_0 a_1 < 2.1 \) for the dielectric core), the upper bound of (25) is \( 1 - 1/N \), which implies that in these regions holds \( \sigma^\text{min}_q \leq \sigma/N^2 \). Furthermore, we observe that for all examined frequencies and both types of cores the differences between the upper bound and the actual ISCS ratio are less than 1%. On the other hand, the differences between the lower and the upper bounds of (25) are less than 4% for \( k_0 a_1 \geq 1 \). In the insets, we show the variations of the ISCS ratios in the low-frequency region, in particular for \( k_0 a_1 \leq 1 \). In this region, the \( q \)-interaction ISCS ratios are much closer to the upper than to the lower bounds of (25). This is due to the fact that the ratios of the minimum and maximum cross sections over the \( q \)-excitation cross section differ substantially. Precisely, for \( k_0 a_1 \leq 0.5 \), their differences can exceed 5%.
which yields at least a 20% difference between the physical bounds.

The ISCS ratios and associated physical bounds from (25) are shown in Fig. 5 for the case of $N = 4$ internal dipoles located at $r_j = (0.65 + 0.05j)a_1$, with $j = 0, 1, 2, 3$. A steeper descent of the $q$-excitation ISCS ratio is now observed compared to the external excitation case of Fig. 4. For lower frequencies ($k_0a_1 < 3.5$ for the PEC and $k_0a_1 < 2.5$ for the dielectric core), the upper bound of (25) is $1 - 1/N$. The differences between the lower and the upper bounds of (25) are larger compared to the corresponding differences for external excitation; in some cases they now reach 15%. In the low-frequency region (i.e., $k_0a_1 \leq 1$), the ISCS ratio almost coincides to its upper bound. Another difference between the behavior of the ISCS ratios for external and internal excitation is that in external excitation, for the frequencies where $\sigma^\text{min} \geq \sigma/N^2$, all quantities show a uniform behavior, while in internal excitation, the ISCS ratios seem to act as a “mirror” between the lower and upper bounds.

Figure 6 depicts the physical bounds for the number $N$ of dipoles that excite the spherical scatterer. All dipoles are external and lie at $r_1 = 2.5a_1$, $r_2 = 2.8a_1$, $r_3 = 3.1a_1$, $r_4 = 3.5a_1$. We observe the almost identical behavior of the physical bounds for both cores, especially in the higher frequencies. Some slight differences occur in the lower frequencies. In particular, for $0.2 < k_0a_1 < 1.5$ for the dielectric core and for $0.1 < k_0a_1 < 2.5$ for the PEC core, we see that the physical bounds are valid and determine accurately the number of dipoles exciting the scatterer. For $k_0a_1 < 0.2$, for the dielectric core, and for $k_0a_1 < 0.1$, for the PEC core, the physical bounds remain valid, but they cannot be used to accurately determine the number of dipoles; this is explained by the significant difference between the minimum and maximum individual cross sections in the low-frequency region. However, for $k_0a_1 > 1.5$, for the dielectric core, and for $k_0a_1 > 2.5$, for the PEC core, where the minimum and maximum individual cross sections are very close, we observe that it holds $N = [\sqrt{\sigma/\sigma^\text{min}}] + 1$, where [x] denotes the integer part of x. We note, that even sparser or denser dipole distributions have been found to exhibit similar patterns with respect to the physical bounds.

In Fig. 7, we depict the variations of $\sigma^T/\sigma$ versus the relative permittivity $\epsilon_{r1}$ of the first spherical shell, for a scatterer with PEC or dielectric core, excited by $N = 4$ external dipoles located at $r_1 = 1.3a_1$, $r_2 = 1.8a_1$, $r_3 = 2.3a_1$, $r_4 = 2.8a_1$ on the $z$-axis. For the higher frequency, $\sigma^T/\sigma$ change only slightly with $\epsilon_{r1}$, i.e., less than 2% for the PEC core and less than 3% for the dielectric core. Furthermore, for both types of cores, $\sigma^T/\sigma$ decreases in an oscillating manner with increasing $\epsilon_{r1}$. For the lower frequency, the behavior is...
KALOGEROPOULOS and TSITSAS: ANALYSIS OF ISCS AND THEIR PHYSICAL BOUNDS

FIGURE 5. As in Fig. 4, but for excitation due to $N = 4$ internal dipoles.

FIGURE 6. Physical bounds for the number $N$ of dipoles exciting a 2-layered sphere with $a_1 = 2a_2$, $\mu_{r1} = 1.5$ and $k_0a_1 = 2$ (top panel) or $k_0a_1 = 0.5$ (bottom panel) with $\epsilon_{r1} = 1.5\epsilon_{r2}$ and $\mu_{r2} = 2.5$. The scatterer is excited by $N = 4$ external dipoles.

different: $\sigma^T/\sigma$ increases with $\epsilon_{r1}$ and then stabilizes when $\epsilon_{r1} = 2$ for the PEC and $\epsilon_{r1} = 3$ for the dielectric core. The ranges of $\sigma^T/\sigma$ are less than 2% and 8% for the PEC and dielectric core, respectively.

In Fig. 8, we show the variations of $\sigma^T/\sigma$ for $N = 4$ dipoles located on the $z$-axis and in the first shell $V_1$ of a sphere $V$ with a PEC core $V_2$ of radius $a_2 = a_1/5$. The following three dipoles distributions are considered: “core side” where the dipoles are located closer to the core $V_2$, “middle side” where the dipoles are in the middle of $V_1$, and “boundary side” where the dipoles are closer to the boundary of $V$. In the top panel, the distance between successive dipoles is $0.05a_1$, while in the bottom panel the corresponding distance is $0.1a_1$. We notice that for $k_0a_1 \leq 5$ the ISCS ratios are smooth and descending for all distributions. For $k_0a_1 \geq 5$, oscillations appear for the distributions away from the core, but not for the distribution closer to the core. For the denser distribution, not so significant changes occur in the ISCS ratios with respect to the placement of the dipoles. The situation is different for the less dense distribution, where for higher frequencies rapid oscillations occur for the distribution closer to the scatterer’s boundary. Particularly, for $k_0a_1 \geq 7$, the ISCS ratios obtain also negative values, which implies the reduction of the energy flux rate. The ranges of the ISCS ratios are smaller for the dense dipoles (less than 35%) and larger for the less dense dipoles (more than 120%). A sparser dipole distribution will lead to an even less-predictable ISCS ratios’ behavior.

Figure 9 depicts the variations of the ISCS values for the same distributions of Fig. 8. We see that the values follow a somewhat different pattern than the ratios.
distributions show large ISCS ranges, but small ranges in the corresponding ratios (as we have seen above). For dipoles groups nearer the sphere’s core, the ISCS values decrease and approach zero in higher frequencies. Less dense distributions show significantly smaller ISCS values, and negative values in higher frequencies. Besides, when the dipoles are closer to the sphere’s core, the differences in the ISCS values between dense and less dense distributions are small.

In Fig. 10, we depict the overall cross section \( \sigma \) and total ISCS ratio \( \sigma_T/\sigma \) versus the radius \( k_0 a_1 \) in case of a sphere with dielectric core of different radii \( a_2 \) excited by \( N = 4 \) internal sources in shell \( V \). For all the examined core’s radii, the values of \( \sigma \) remain fairly the same and oscillate rapidly (after \( k_0 a_1 > 1.2 \)); the latter is expected from the discussions of [21], [26]. The ratios \( \sigma_T/\sigma \) descent smoothly for \( k_0 a_1 < 5 \), oscillate rapidly for \( k_0 a_1 > 5 \), and are not significantly affected by changes in the core’s radii.

Figure 11 shows the overall cross section \( \sigma \) and total ISCS \( \sigma_T \) at the fixed frequency \( k_0 a_1 = 1 \) for a spherical scatterer with a PEC core. We consider the cross sections variations as one, two, and three dipoles move from their original positions at a distance \( R \) (the moving dipoles are each time the ones being further away from the scatterer). In particular, the initial positions are \( r_j = (1.3+0.2j) \) with \( j = 0, 1, 2, 3 \), while the moving dipoles’ locations are given by \( r_j(R) = r_j R \) with \( j = 1, 2, 3 \). A notable similarity is observed in these figures since both \( \sigma \) and \( \sigma_T \) follow a similar pattern: they decrease as more dipoles move further away from their original to a sparser distribution. This fact indicates that the ratio \( \sigma_T/\sigma \) remains fairly unchanged. The ranges of \( \sigma \) and \( \sigma_T \) increase with the number of moving dipoles. The above conclusions have been found to be similar for the dielectric core (not shown here) with the only difference that the ranges of \( \sigma \) and \( \sigma_T \) are much smaller than for the PEC core.

IV. MIXED EXCITATION

Mixed excitation refers to the case of dipoles located in more than one excitation layers. Then, the indirect ISCS \( \sigma^I \) is— in general— non zero.

A. OPTICAL THEOREMS AND PHYSICAL BOUNDS

First, we provide two optical theorems relating the overall cross section and the indirect ISCS with the corresponding secondary fields

\[
\sigma = \frac{4\pi \mu_0}{k_0} \left[ \text{Re} \left( \sum_{q=1}^{Q} \sum_{j=1}^{n_q} i k_{q,j}^r \left( \nabla \times \mathbf{E}_{sec}^r \left( \mathbf{r}_q^j, \mathbf{r}_1^1, \ldots, \mathbf{r}_N \right) \right) \cdot \hat{p}_q^j \right) \right] + \frac{1}{2} \sum_{q=1}^{Q} \sum_{j=1}^{n_q} \sum_{i=1}^{n_q} \tilde{\sigma}_{q,j,i}^r \hat{p}_q^j \mu_q^r, \tag{28}
\]

FIGURE 8. ISCS ratios \( \sigma_T/\sigma \) for \( N = 4 \) dipoles lying in the first shell of a 2-layered sphere with \( \mu_1 = 1, 5, \mu_1 = 2 \) and PEC core of radius \( a_2 = a_1 / 5 \). Top panel: dense dipole distribution with \( r_1^1 - r_2^1 = 0.05a_1 \) (middle) and \( r_3^1 = 0.25a_1 \) (core). \( r_1^1 = 0.55a_1 \) (middle), \( r_2^1 = 0.8a_1 \) (boundary side). Bottom panel: a less dense dipole distribution with \( r_1^1 - r_2^1 = 0.1a_1 \) (middle) and \( r_3^1 = 0.5a_1 \) (core), \( r_1^1 = 0.6a_1 \) (boundary side).

FIGURE 9. Total ISCS \( \sigma_T \) values for the same setup of Fig. 8.
is the \( q \)-partial field with respect to excitation layer \( V_q^{\text{ex}} \), i.e., the sum of all \( q \)-excitation fields except those due to the dipoles of \( V_q^{\text{ex}} \). Combining (21), (23), and (29) an optical theorem for the total ISCS \( \sigma^T \) can be also obtained.

Radiation from the primary fields does not appear in (29), since all interactions between primary fields are included in the direct ISCS \( \sigma^D \). Indirect ISCS \( \sigma^I \) is, in fact, a measure of the energy flux rate due to interactions of secondary fields induced by dipoles within different layers.

The following physical bounds for \( \sigma^I \) and \( \sigma^T \) are derived

\[
1 - Q \frac{Q_{\sigma_{\text{ex}}}}{\sigma} \leq \frac{\sigma^I}{\sigma} \leq \min \left\{ 1 - Q \frac{Q_{\sigma_{\text{ex}}}}{\sigma}, 1 - \frac{1}{Q} \right\}, \tag{31}
\]

\[
1 - N \frac{Q_{\sigma_{\text{max}}}}{\sigma} \leq \frac{\sigma^T}{\sigma} \leq \min \left\{ 1 - N \frac{Q_{\sigma_{\text{min}}}}{\sigma}, 1 - \frac{1}{N} \right\}, \tag{32}
\]

where \( Q_{\sigma_{\text{min}}^{\text{ex}}} \) and \( Q_{\sigma_{\text{max}}^{\text{ex}}} \) are the minimum and maximum \( q \)-excitation cross sections and \( Q_{\sigma_{\text{min}}}^{\text{ex}} \) and \( Q_{\sigma_{\text{max}}}^{\text{ex}} \) are the minimum and maximum individual cross sections of all dipoles.

We note that (25) holds in the mixed excitation case as well. For the proofs of (31) and (32), we refer to the Appendix. When

\[
\frac{Q^2}{\sigma_{\text{ex}}} \leq \sigma \leq \frac{N^2}{\sigma_{\text{min}}} \tag{33}
\]

\[
\frac{N^2}{\sigma_{\text{min}}} \leq \sigma \leq \frac{Q^2}{\sigma_{\text{ex}}} \tag{34}
\]

then the minima involved in (31), (32) are \( 1 - \frac{1}{Q} \) and \( 1 - \frac{1}{N} \), respectively. Taking into account that \( \sigma \leq Q \frac{Q_{\sigma_{\text{max}}}}{\sigma} \) and \( \sigma \leq N \frac{Q_{\sigma_{\text{max}}}}{\sigma} \) and combining with (31), (32), we conclude that conditions (33), (34) hold respectively, if and only if

\[
\frac{\sigma}{\sigma_{\text{ex}}} \leq Q \leq \frac{\sigma}{\sigma_{\text{ex}}}, \tag{35}
\]

\[
\frac{\sigma}{\sigma_{\text{max}}} \leq N \leq \frac{\sigma}{\sigma_{\text{min}}}, \tag{36}
\]
Scattering relations and optical theorems for a layered magneto-dielectric medium excited by two internal or external dipoles and a layered acoustic medium excited by \( N \) internal or external point sources were derived in [21] and [17], respectively.

**B. PARAMETRIC ANALYSIS AND NUMERICAL RESULTS**

Now, we consider that the scatterer \( V \) is excited by two dipoles in the external region \( V_0 \) \( (r > a_1) \) and two dipoles in the first spherical shell \( V_1 \) \( (a_2 < r < a_1) \); hence we have \( Q = 2 \) excitation layers.

In Fig. 12, we depict the overall cross section \( \sigma \) and the sum of 0-excitation \( \sigma_0 \) and 1-excitation \( \sigma_1 \) cross sections as well as the total \( \sigma^I/\sigma \), indirect \( \sigma^T/\sigma \), and direct \( \sigma^D/\sigma \) ISCS ratios. We observe that \( \sigma \) gradually converges to the sum \( \sigma_0 + \sigma_1 \), and that \( \sigma_0 + \sigma_1 > \sigma \) for \( k_0a_1 > 8 \). This is explained from the behavior of the ISCS ratios, where, for \( k_0a_1 > 8 \), we see that \( \sigma^I/\sigma < 0 \), which in turn means that \( \sigma^I < 0 \). In particular, it holds \( \sigma = \sigma_0 + \sigma_1 + \sigma^I \); see (19).

Hence, larger ratios \( \sigma^I/\sigma \) lead to larger differences between the sum of \( q \)-excitation and the overall cross sections.

Figure 13 shows the variations of the total \( \sigma^T/\sigma \), the indirect \( \sigma^I/\sigma \), and the direct \( \sigma^D/\sigma \) ISCS ratios versus the relative dielectric permittivity \( \epsilon_{r1} \) of a 2-layered sphere with \( k_0a_1 = 2 \) and a PEC core. The ratio \( \sigma^I/\sigma \), and hence the indirect ISCS \( \sigma^I \), becomes negative for \( \epsilon_{r1} > 3.1 \). For \( \epsilon_{r1} = 4 \), ratios \( \sigma^T/\sigma \) and \( \sigma^D/\sigma \) are minimized, while the direct \( \sigma^D/\sigma \) ISCS ratio (the sum of the 0- and 1-ISCS) is maximized; see (18). Corresponding results for the variations of the ISCS with respect to the magnetic permeability \( \mu_{r1} \) have been also derived and the conclusions are the same with the ones drawn above with the only difference being that the range of ISCS variations is now smaller.

In Fig. 14, we depict the physical bounds for the number \( Q \) of excitation layers indicated by (35). The considered number \( Q = 2 \) is depicted with a straight red line. For \( k_0a_1 > 1 \), the physical bounds can be used to determine \( Q \) for a wide range of the examined frequencies. The upper physical bound for \( k_0a_1 > 1 \) remains very close to the number \( Q \) of excitation layers even when the upper bound is not valid. In fact, we see that \( Q = \lfloor \sqrt{\sigma/\sigma_{\text{max}}} \rfloor + 1 \). The insets demonstrate the variations in the low-frequency region. For the PEC core, the bounds remain valid, but for \( k_0a_1 \leq 0.5 \) cannot be safely used for the determination of \( Q \), since the minimum \( q \)-excitation cross section–\( \sigma_1 \) in this case–is significantly smaller than the overall cross section. For the dielectric core, the physical bounds remain valid for \( k_0a_1 < 0.7 \). Besides, for both types of cores a change in the minimum and maximum \( q \)-excitation cross sections occurs at \( k_0a_1 = 1 \). Precisely, for \( k_0a_1 < 1 \) it holds \( \sigma_0 < \sigma_1 \), while for \( k_0a_1 > 1 \) it holds \( \sigma_0 > \sigma_1 \).

Figure 15 depicts the variations of \( \sigma^T/\sigma \) and \( \sigma^D/\sigma \) for \( k_0a_1 = 1 \) and \( k_0a_1 = 2.5 \) versus the distance \( k_0R \) between the internal group of \( n_1 = 2 \) dipoles, initially located at \( r_j^1 = 0.8a_1 \), \( r_2^1 = 0.9a_1 \), and the external group of \( n_0 = 2 \) dipoles, initially located at \( r_j^0 = 1.2a_1 \), \( r_2^0 = 1.3a_1 \). In the top panel, the internal group moves towards the sphere’s core and the external group moves away from the scatterer’s boundary with increasing \( R \). Precisely, the moving dipoles’ locations are given by \( r_j^0(R) = r_j^0 + R \) for the external group.
and \( r_1' (R) = r_1' / R \) for the internal group. The initial setup (before moving the dipoles) corresponds to a negative \( \sigma^1 / \sigma \). We observe that for both frequencies the ISCS ratios \( \sigma^T / \sigma \) and \( \sigma^1 / \sigma \) follow a similar pattern: they first increase until a certain value of \( k_0 R \) and then decrease. For the lower frequency, the variations of the ISCS ratios are smaller. In the bottom panel, the meaning of \( R \) is slightly different: the initial locations are \( r_1 = 0.2 a_1 \), \( r_2 = 0.2475 a_1 \) for the internal group and \( r_0 = 1.2 a_1 \) for the external group. The dipoles located at \( r_1 = 0.2 a_1 \), \( r_0 = 1.2 a_1 \) remain fixed, while the one at \( r_1 = 0.2475 a_1 \) moves towards the scatterer’s boundary, and the one at \( r_0 = 1.3 a_1 \) moves away from it. The moving dipoles’ locations are given by \( r_q (R) = r_q R \) for \( q = 0, 1 \). The behavior of the ISCS is different now: for the lower frequency, we see a steeper decrease in \( \sigma^T / \sigma \) and a sharper increase in \( \sigma^1 / \sigma \). Thus, the direct ISCS will decrease more rapidly than the total ISCS. Furthermore, a crossover is observed at \( k_0 R = 1.6 \) between \( \sigma^T / \sigma \) and \( \sigma^1 / \sigma \) for the two examined frequencies. This is due to that the decrease in \( \sigma^T / \sigma \) is steeper for the higher than the lower frequency. However, for the higher frequency, \( \sigma^1 / \sigma \) is maximized at \( k_0 R = 2.2 \) and follows a descending behavior after that point.

This comes in stark contrast with the ascending behavior of \( \sigma^1 / \sigma \) for the lower frequency.

In Fig. 16, we depict \( \sigma^T / \sigma \) and \( \sigma^1 / \sigma \) versus \( k_0 a_1 \) for different radii \( a_2 \) of the PEC core. A remarkable similarity is observed in the ISCS curves for all examined radii \( a_2 \). For \( k_0 a_1 < 2 \), larger radii \( a_2 \) yield larger ISCS ratios, while, on the contrary, for \( k_0 a_1 > 4 \), larger \( a_2 \) yield smaller ISCS ratios. Larger cores have larger ISCS ranges, e.g., for \( a_2 = a_1 / 5 \) we have \( \sigma^T / \sigma \in (0.35, 0.61) \), \( \sigma^1 / \sigma \in (0.1, 0.26) \) whereas for \( a_2 = a_1 / 2 \) we have that \( \sigma^T / \sigma \in (0.25, 0.7) \), \( \sigma^1 / \sigma \in (-0.15, 0.41) \). For \( 2 \leq k_0 a_1 < 4 \), a more steady behavior is observed. The ISCS are not significantly affected by the changes in the core’s radius, except for the larger core \( a_2 = a_1 / 2 \), which yields larger variations. Both \( \sigma^T / \sigma \) and \( \sigma^1 / \sigma \) exhibit more oscillatory behaviors for larger cores.

The variations of the ISCS ratios and values as well as the overall cross section and the sum of individual cross sections (denoted by \( \hat{\sigma} \)) for different distributions of \( N = 4 \) dipoles in the high-frequency zone are depicted in Fig. 17. The dipoles’ distributions are those of the top panel of Fig. 15. A notable similarity is observed between the ISCS ratios and the values of all involved cross sections. For \( k_0 R > 2 \) (i.e., when the
distance between the external and the internal dipole groups (larger than the sphere’s diameter), the values and ratios begin to stabilize, which implies that the overall cross section \( \sigma \) develops a more stable behavior. Besides, all \( \sigma^T/\sigma \) ratios remain positive, except for \( k_0R \in (1.25, 1.35) \) for \( k_0a_1 = 10 \).

This fact is readily explained by the bottom panel, where we see that for these frequencies, the sum of individual cross sections is greater than the overall cross section.

In Fig. 18, we depict the same quantities as in Fig. 17, but in the low-frequency regime. The dipoles’ distributions are those of the top panel of Fig. 15. The ISCS ratios and values remain very close for each of the two examined frequencies. The indirect ISCS remain negative for all \( k_0R \); this fact implies that the interaction between the 0-excitation and 1-excitation fields, reduces the rate of the energy flux. Since the total ISCS remains positive, it is concluded that the sum of the 0- and 1-excitation cross sections is greater than the overall cross section but the sum of individual cross sections remains smaller than the overall cross section, as demonstrated by the bottom panel. Another interesting observation is the ascending behavior of the ISCS ratios as the distance \( k_0R \) between the dipoles’ groups increases. Indirect ISCS values exhibit an ascending behavior as well—only steeper than their corresponding ratios. This is readily explained by the descending behavior of the total ISCS values with increasing \( k_0R \). The contradiction between the ascending behavior of the indirect ISCS and the descending behavior of the total ISCS is explained from the bottom panel, where we see that the overall scattering cross section approaches zero for large distances \( R \). Additionally, we see that the sum of the individual cross sections is very close to the overall cross section with both quantities following a similar descent pattern.

To explain the reduction of the energy flux rate, we point out that the overall energy flux is generated from the individual energy fluxes, which are quantified by the sum of the individual cross sections, and the energy flux caused by the interactions between the scattered fields, which are quantified by the total ISCS. Utilizing the Poynting vectors of the overall and individual fields, the Silver-Müller radiation condition (11) and the transmission boundary conditions (7) and (8) on \( S_1 \), we find that

\[
\sigma = Z_0 \int_{S_1} \hat{n} \cdot \mathbf{S}_1 \, ds(\mathbf{r}),
\]
with $\mathbf{S}_1$ denoting the overall energy flux in $V_1$, and $Z_0 = \sqrt{\mu_0/\epsilon_0}$ the free-space impedance. From a physical standpoint, since $\sigma$ is always positive, (37) shows that the overall scattering cross section is equal to the overall energy flow over all directions through the surface of the scatterer directed towards its exterior. By employing each individual Poynting vector $\mathbf{S}_j^1$, for $j = 1, \ldots, N$, using similar techniques as in the derivation of (37), and considering the definition (20), we obtain the following expression of the total ISCS

$$\sigma^T = Z_0 \int_{S_1} \mathbf{n} \cdot S^1_t \, ds(\mathbf{r}),$$

(38)

where $S^1_t = S_1 - \sum_{j=1}^{N} S^j_1$ denotes the total interaction energy flux in $V_1$. Eq. (38) implies that the total ISCS is equal to the energy flow through the surface $S_1$ of the scatterer, caused by the interactions between the individual fields in $V_1$. When $\sigma^T > 0$, this energy flow is directed “outwards”. However, when $\sigma^T < 0$, (38) implies that the interactions between the participating fields, produce energy flow that is directed towards the interior of the scatterer (“inwards”), and, therefore, a portion of the energy flow returns back to the scatterer. This fact results in a reduction of the overall energy flux rate. We note, that $\sigma^T < 0$ does not necessarily imply that all ISCS are negative; see Table 2, below, for $k_0a_1 = 0.1$ with $a_1 = 4a_2$, $a_1 = 10a_2$, where $\sigma^T > 0$, $\sigma^I < 0$, $\sigma^D > 0$. Besides, a similar analysis to the above one is also

| $k_0a_1 = 0.1$ | $a_1 = 1.25a_2$ | $a_1 = 2a_2$ | $a_1 = 4a_2$ | $a_1 = 10a_2$ |
|----------------|----------------|-------------|-------------|-------------|
| $\sigma^T$    | 0.0320         | 0.0123      | 0.0027      | 0.0037      |
| $\sigma^I$    | 0.0103         | 0.0051      | 0.0039      | 0.0037      |
| $\sigma^D$    | 0.2776         | 0.1139      | 0.0284      | 0.0046      |
| $\sigma_0$    | 0.5031         | 0.2054      | 0.0517      | 0.0083      |
| $\sigma_1$    | 0.0786         | 0.0382      | 0.0292      | 0.0281      |
| $\sigma^T_1$  | 1.5276         | 0.6206      | 0.1568      | 0.0251      |
| $\sigma^I_1$  | 0.2979         | 0.0316      | -0.0371     | -0.0248     |
| $\sigma^D_1$  | 0.7832         | 0.3211      | 0.0902      | 0.0253      |
| $\sigma_1$    | -0.6533        | -0.2895     | -0.1273     | -0.0500     |
| $\sigma_0$    | 0.9529         | 0.3693      | 0.0587      | 0.0032      |

| $k_0a_1 = 10$ | $a_1 = 1.25a_2$ | $a_1 = 2a_2$ | $a_1 = 4a_2$ | $a_1 = 10a_2$ |
|----------------|----------------|-------------|-------------|-------------|
| $\sigma^T$    | 5.4177         | 3.6491      | 4.7598      | 4.8259      |
| $\sigma^I$    | 4.8377         | 3.5577      | 4.6977      | 4.7678      |
| $\sigma^D$    | 7.4435         | 1.7532      | 2.0484      | 0.0196      |
| $\sigma_0$    | 5.0671         | 1.3939      | 0.3361      | 0.0356      |
| $\sigma_1$    | 19.3743        | 14.2296     | 18.7399     | 19.0205     |
| $\sigma^T_1$  | 20.4900        | 4.7437      | 1.0450      | 0.1077      |
| $\sigma^I_1$  | 46.3710        | 21.0969     | 19.0745     | 19.9368     |
| $\sigma^D_1$  | 17.0983        | 9.1973      | 9.7864      | 9.4756      |
| $\sigma_1$    | 29.2727        | 11.8996     | 5.2804      | 4.6611      |
| $\sigma_0$    | 49.1370        | 30.8728     | 23.0730     | 20.5894     |
valid for the direct and indirect ISCS and their corresponding energy flows.

Table 2 presents the scattering cross sections and ISCS for the case where \( N_0 = 2 \) external dipoles, lying at \( r_{11}^\text{i} = 1.5a_1, r_{21}^\text{i} = 2a_1 \), and \( N_1 = 2 \) internal dipoles, lying at \( r_{11}^\text{i} = 1.01a_2, r_{21}^\text{i} = 1.24a_2 \), excite a 2-layered spherical scatterer with a dielectric core and \( \mu_1 = 1.5, \mu_2 = 2.5 \) and \( \epsilon_1 = 2, \epsilon_2 = 3 \). For the individual cross sections, we used the exact solution of the direct scattering problem for a single dipole [25], while for the rest of the cross sections involved, we used the optical theorems (22), (28) and (29).

From the table, we see that all relations concerning ISCS are validated. Precisely, it holds: \( \sigma = \sigma^T + \hat{\sigma} = \sigma^1 + \sigma_0 + \sigma_1 \) and \( \sigma^T = \sigma^D + \sigma^I \). In general, the findings of the figures and tables above suggest that the behavior of the total ISCS in the mixed excitation case is not easily predictable, since it is affected by various factors, like shells’ thicknesses, core sizes, material parameters as well as external and internal dipoles distributions.

V. CONCLUSION

Excitation of a layered medium by \( N \) arbitrarily distributed magnetic dipoles was investigated. Two cases were considered and analyzed: single layer excitation when all dipoles lie in the same layer (or in the scatterer’s exterior) and mixed excitation when dipoles are located in more than one layers. Interaction scattering cross sections (ISCS) were introduced which quantify the energy flux rate due to interactions between the individual and \( q \)-excitation fields by dipoles lying in the same layer or in different layers. Optical relations for the overall and \( q \)-excitation ISCS were derived. Physical bounds for the ISCS ratios, the number of excitation layers and the number of exciting dipoles were also established.

Numerical parametric analysis was performed for a layered spherical medium with a PEC or dielectric core, excited by 4 dipoles. From the presented numerical results, we concluded that the ISCS contribute significantly in the overall cross section. Therefore, when spherical waves excite a scatterer, the additivity of the cross sections must be examined with caution. Furthermore, we showed that in some cases the ISCS can also become negative; this was mainly observed for mixed excitation. In such cases, interactions of the participating fields may reduce the anticipated energy flux rate. Moderate changes in the geometrical characteristics of the medium do not result in significant changes of the ISCS ratios. Changes in the physical parameters of the medium affect significantly the ISCS ratios in the mixed excitation, but not so significantly in the single-layer excitation.

The ISCS ratios in external-excitation cases are very close to \( 1 - \frac{1}{N} \), especially in the low-frequency regime. For higher frequencies, the dipoles’ strength is crucial for the ISCS ratios. Moreover, in higher frequencies, the established physical bounds can be used to determine accurately the medium’s excitation layers and the number of exciting dipoles.

Finally, the distance of the excitation dipoles from the medium’s boundary plays a pivotal role in the behavior of ISCS. For single-layer excitation, distributions away from the boundary yield larger values and more stable variations of the ISCS ratios—in contrast with distributions at close proximity to the boundary. For mixed excitation, the behavior of the ISCS is more erratic and depends on the scatterer’s parameters and characteristics of the dipoles’ distributions.

The presented numerical results in this work correspond to layered spherical scatterers. Examining the effect of the shells’ shape on the ISCS is an interesting direction for future work. Spheroidal or ellipsoidal boundaries can be considered by developing combinations of analytical and numerical methodologies for the solution of the associated excitation problems by internal or external dipoles.

APPENDIX

Here we state relation (25) in the form of a theorem and prove it.

**Theorem 1:** The ratio \( \tilde{\sigma}_q/\sigma_q \) of the \( q \)-interaction over the \( q \)-excitation cross section satisfies

\[
1 - n_q \frac{\sigma_q^{\text{max}}}{\sigma_q} \leq \frac{\tilde{\sigma}_q}{\sigma_q} \leq \min \left\{ 1 - n_q \frac{\sigma_q^{\text{min}}}{\sigma_q}, 1 - \frac{1}{n_q} \right\}, \tag{A.1}
\]

where \( \sigma_q^{\text{max}}, \sigma_q^{\text{min}} \), respectively, denote the maximum and minimum individual cross sections of the dipoles lying in the excitation layer \( V_q \).

**Proof:** For \( \sigma_q^{\text{min}} \) and \( \sigma_q^{\text{max}} \), we have

\[
- n_q \sigma_q^{\text{max}} \leq - \sum_{j=1}^{n_q} \sigma_j \leq - n_q \sigma_q^{\text{min}}, \tag{A.2}
\]

which, combined with the definition (17), yields

\[
1 - n_q \frac{\sigma_q^{\text{max}}}{\sigma_q} \leq \frac{\tilde{\sigma}_q}{\sigma_q} \leq 1 - n_q \frac{\sigma_q^{\text{min}}}{\sigma_q}. \tag{A.3}
\]

By the definition of the \( q \)-excitation cross section (15), we find

\[
\sigma_q \leq \frac{1}{k_0^2} \left[ \sum_{j=1}^{n_q} \int_{S^2} \left| g_j(\hat{r}) \right|^2 ds(\hat{r}) \right] + \frac{2}{k_0^2} \left[ \sum_{j=1}^{n_q-1} \sum_{q=j+1}^{n_q} \left| \int_{S^2} g_j(\hat{r}) \cdot g_q(\hat{r}) ds(\hat{r}) \right| \right]. \tag{A.4}
\]

Hölder’s inequality in conjunction with (14) and (A.4) imply

\[
\sigma_q \leq \sum_{j=1}^{n_q} \left| \sigma_j \right|^2 + \sum_{j=1}^{n_q-1} \sum_{q=j+1}^{n_q} \left( \sigma_j^{1/2} \sigma_q^{1/2} \right)^2. \tag{A.5}
\]

Now, since it holds

\[
2 \left( \sigma_q^{1/2} \sigma_q^{1/2} \right)^2 \leq \sigma_q + \sigma_q',
\]

\[
\sum_{j=1}^{n_q-1} \sum_{q=j+1}^{n_q} \left( \sigma_j^{1/2} \sigma_q^{1/2} \right)^2 \leq \sigma_q + \sigma_q'.
\]

\[
\sigma_q \leq \sum_{j=1}^{n_q} \sigma_j + \sum_{j=1}^{n_q-1} \sum_{q=j+1}^{n_q} \left( \sigma_j^{1/2} \sigma_q^{1/2} \right)^2. \tag{A.6}
\]
from (A.5), we get

\[ \sigma_q \leq \sum_{j=1}^{n_q} \sigma_j^0 + \sum_{j=1}^{n_q-1} \sigma_j^0(n_q - j) + \sum_{j=2}^{n_q} \sigma_j^0(j - 1) = n_q \sum_{j=1}^{n_q} \sigma_j^0. \]  
\[ (A.6) \]

The last inequality together with (17) imply that

\[ \frac{\sigma_q}{\sigma_q^0} \leq 1 - \frac{1}{n_q}. \]  
\[ (A.7) \]

Eq. (A.1) is derived from (A.3) and (A.7). Implication (26) is obvious.

Inequality \( \sigma_q \leq n_q \sigma_q^0 \), which is used in the derivation of (27), is a direct consequence of (A.6). Finally, relations (31), (32) as well as (35), (36), are proved by similar techniques.

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