Abstract

We find optimal (up to constant) bounds for the following measures for the regularity of the Schramm-Loewner evolution (SLE): variation regularity, modulus of continuity, and law of the iterated logarithm. For the latter two we consider the SLE with its natural parametrization. More precisely, denoting by $d \in (0, 2]$ the dimension of the curve, we show the following.

1. The optimal $\psi$-variation is $\psi(x) = x^d (\log \log x)^{-d-1}$ in the sense that $\eta$ is of finite $\psi$-variation for this $\psi$ and not for any function decaying more slowly as $x \downarrow 0$.

2. The optimal modulus of continuity is $\omega(s) = c s^{1/d} (\log s)^{-1-1/d}$, i.e. $|\eta(t) - \eta(s)| \leq \omega(t-s)$ for this $\omega$, and not for any function decaying faster as $s \downarrow 0$.

3. $\limsup_{t \downarrow 0} |\eta(t)| (t^{1/d} (\log \log t)^{-1-1/d})^{-1}$ is a deterministic constant in $(0, \infty)$.

1 Introduction

The Schramm-Loewner evolution (SLE) was first introduced by Schramm [Sch00] in 1999 as a candidate for the scaling limit of curves in statistical physics models at criticality. Soon afterwards it was proven that the SLE indeed describes the limiting behavior of a range of statistical physics models, including the uniform spanning tree, the loop-erased random walk [LSW04], percolation [Smi01], the Ising model [Smi10, CS12], and the discrete Gaussian free field [SS09]. Schramm argued in his work that SLE is the unique one-parameter family of processes satisfying two natural properties called conformal invariance and the domain Markov property, and he denoted the parameter by $\kappa > 0$.

In this paper we will study the regularity of SLE. We first present some measures of regularity for general fractal curves in Section 1.1 along with previous results for SLE. Then we state our main results of Section 1.2, and finally we give an outline of the paper in Section 1.3.

1.1 How to quantify the regularity of fractal curves, and previous SLE results

Let $I \subset \mathbb{R}$ be an interval and let $\eta: I \to \mathbb{C}$ be a fractal curve in the plane. A natural question is how one can quantify the regularity or fractality of $\eta$.

One approach is to view the curve as a subset of $\mathbb{C}$ by considering the set $\eta(I)$. One can study the dimension of this set, e.g. the Hausdorff or Minkowski dimension. It follows from [Bef08, RS05] that in the case of SLE, the two latter dimensions agree a.s. and are given by $(1 + \kappa / 8) \land 2$. One can also ask about the exact gauge function that gives a non-trivial Hausdorff measure or Minkowski content for SLE; this is known to be an exact power-law for the Minkowski content [LR15] while it is unknown for the Hausdorff measure [Rez18].
(1 + κ/8) ∧ 2-dimensional Minkowski content of SLE has been proven to define a parametrization of the curve known as the natural parametrization [LR15, LV21, LS11].

For conformally invariant curves in \( \mathbb{C} \) like SLE it is also natural to study the regularity of a uniformizing conformal map from the complement of the curve to some reference domain. See e.g. [BS09, GMS18, JVL12, ABV16, Sch20, GHM20b, KMS21] for results on this for SLE.

One can also quantify the regularity of \( \eta \) by viewing it as a parametrized curve rather than a subset of \( \mathbb{C} \). The modulus of continuity is natural in this regard. We say that \( \eta: I \to \mathbb{C} \) admits \( \omega: [0, \infty) \to [0, \infty] \) as a modulus of continuity if \( |\eta(t) - \eta(s)| \leq \omega(|t - s|) \) for any \( s, t \in I \). If this holds for \( \omega(t) = ct^\alpha \) for some \( c > 0 \) and \( \alpha > 0 \) then we say that the curve is \( \alpha \)-Hölder continuous.

Note that the modulus of continuity of a curve depends strongly on the parametrization of the curve. For SLE there are two commonly considered parametrizations: the capacity parametrization (see e.g. [Law05]) and the natural parametrization. The optimal Hölder exponent of SLE was computed by Lawler and Viklund [JVL11] for the capacity parametrization, and logarithmic refinements were studied in [Yua21, KMS21]. For SLE with its natural parametrization the optimal Hölder exponent is equal to the dimension of the curve (see (1) below). This was proven by Zhan for \( \kappa \leq 4 \) [Zha19] and by Gwynne, Holden, and Miller for space-filling SLE with \( \kappa > 4 \) [GHM20a].

The regularity of \( \eta \) at a fixed time \( t_0 \in I \) can be quantified via a law of the iterated logarithm which describes the magnitude of the fluctuations of \( \eta \) as it approaches \( t_0 \). One can also consider the set of exceptional times where the fluctuations are different from a typical point, e.g. so-called fast or slow points.

Finally, another important notion of regularity for a curve is the variation. For an increasing homeomorphism \( \psi: (0, \infty) \to (0, \infty) \) we say that \( \eta: I \to \mathbb{C} \) has finite \( \psi \)-variation if

\[
\sup_{t_0 < \cdots < t_r} \sum_i \psi(|\eta(t_{i+1}) - \eta(t_i)|) < \infty,
\]

where we take the supremum over all finite sequences \( t_0 < t_1 < \cdots < t_r, t_i \in I \) for \( i = 1, \ldots, r \). Equivalently, a continuous curve \( \eta \) has finite \( \psi \)-variation if and only if there exists a reparametrization \( \tilde{\eta} \) of \( \eta \) that admits modulus of continuity \( \psi^{-1} \), i.e.,

\[
|\tilde{\eta}(t) - \tilde{\eta}(s)| \leq \psi^{-1}(|t - s|).
\]

This measure of regularity is invariant under reparametrizations of the curve, and is therefore particularly natural in the setting of SLE where people use multiple different parametrizations or simply consider the curve to be defined only modulo reparametrization of time. The case \( \psi(x) = x^p \) (usually called \( p \)-variation) plays an important role in the theories of Young integration and rough paths. A curve is of finite \( p \)-variation if and only if it admits a \( 1/p \)-Hölder continuous reparametrisation. When \( \eta \) has finite \( \psi \)-variation, the following quantity is finite and, for convex \( \psi \), can be proven to define a semi-norm

\[
[\eta]_{\psi\text{-var}, I} = \inf \left\{ M > 0 \left| \sup_{t_0 < \cdots < t_r} \sum_i \psi \left( \frac{|\eta(t_{i+1}) - \eta(t_i)|}{M} \right) \leq 1 \right. \right\}.
\]

The optimal \( p \)-variation exponent of SLE was computed to be equal to its dimension \((1 + \kappa/8) \wedge 2 \) in [Bef08, FT17, Wer12], and a non-optimal logarithmic refinement of the upper bound was established by the second author [Yua21]. (Recall the general result that the \( p \)-variation exponent cannot be smaller than the Hausdorff dimension of the curve.)

\[ \textbf{1.2 Main results} \]

We assume throughout the section that \( \eta \) is either
(i) a two-sided whole-plane SLE, $\kappa \leq 8$, from $\infty$ to $\infty$ passing through 0 with its natural parametrization, or

(ii) a whole-plane space-filling SLE, $\kappa > 4$, from $\infty$ to $\infty$ with its natural parametrization (i.e., parametrized by Lebesgue area measure).

Furthermore, we let $d$ denote the dimension of the curve, namely

$$d = 1 + \frac{\kappa}{8} \quad \text{in case (i)} \quad \text{and} \quad d = 2 \quad \text{in case (ii)}.$$  \hfill (1)

The SLE variants considered in (i) and (ii) are particularly natural since it can be argued that they describe the local limit in law of an arbitrary variant of SLE with its natural parametrization zoomed in at a typical time. The reason we consider these two SLE variants is that they are self-similar processes of index $1/d$ with stationary increments, in the sense that for every $t_0 \in \mathbb{R}$ and $\lambda > 0$, the process $t \mapsto \eta(t_0 + \lambda t) - \eta(t_0)$ has the same law as $t \mapsto \lambda^{1/d} \eta(t)$ (see [Zha21, Corollary 4.7 and Remark 4.9] for case (i), and [HS18, Lemma 2.3] for case (ii)). We give a more thorough introduction to these curves in Section 2.

Throughout the paper, we write $\log^*(x) = \log(x) \vee 1$.

**Theorem 1.1** (variation regularity). Let $\psi(x) = x^d(\log^* \log^* \frac{1}{x})^{-(d-1)}$. There exists a deterministic constant $c_0 \in (0, \infty)$ such that almost surely

$$\lim_{\delta \downarrow 0} \sup_{|t_{i+1} - t_i| < \delta} \sum_i \psi(|\eta(t_{i+1}) - \eta(t_i)|) = c_0 |I|$$

for any bounded interval $I \subseteq \mathbb{R}$, where the supremum is taken over finite sequences $t_0 < \ldots < t_r$ with $t_i \in I$ and $|t_{i+1} - t_i| < \delta$.

Moreover, for any bounded interval $I \subseteq \mathbb{R}$, there exists $c > 0$ depending on the length of $I$ such that

$$\mathbb{P}((|\eta|_{\psi-\text{var},I} > u) \leq c^{-1} \exp(-cu^{d/(d-1)}).$$

Recall that the previous works [Bef08, FT17] have identified $d$ as the optimal $p$-variation exponent. Our result gives the optimal function $\psi$ up to a non-explicit deterministic factor. In other words, the best modulus of continuity among all parametrisations of $\eta$ is $\omega(t) = ct^{1/d}(\log^* \log^* t^{-1})^{1-1/d}$, and in particular, there is no reparametrisation that is $1/d$-Hölder continuous.

**Theorem 1.2** (modulus of continuity). There exists a deterministic constant $c_0 \in (0, \infty)$ such that almost surely

$$\lim_{\delta \downarrow 0} \sup_{s,t \in I, |t-s| < \delta} \frac{|\eta(t) - \eta(s)|}{|t-s|^{1/d}(\log^* |t-s|)^{1-1/d}} = c_0$$

for any non-trivial bounded interval $I \subseteq \mathbb{R}$.

Moreover, for any bounded interval $I \subseteq \mathbb{R}$ there exists $c > 0$ depending on the length of $I$ such that

$$\mathbb{P} \left( \sup_{s,t \in I} \frac{|\eta(t) - \eta(s)|}{|t-s|^{1/d}(\log^* |t-s|)^{1-1/d}} > u \right) \leq c^{-1} \exp(-cu^{d/(d-1)}).$$

The optimal Hölder exponent $1/d$ has been identified previously in [Zha19, GHM20a] (except in case (i) for $\kappa \in (4,8)$, where only the upper bound was established). We prove that $\omega(t) = t^{1/d}(\log t^{-1})^{1-1/d}$ is the optimal (up to constant) modulus of continuity.

**Theorem 1.3** (law of the iterated logarithm and maximal growth rate). There exist deterministic constants $c_0, c_1 \in (0, \infty)$ such that for any $t_0 \in \mathbb{R}$, almost surely

$$\limsup_{t \uparrow 0} \frac{|\eta(t_0 + t) - \eta(t_0)|}{t^{1/d}(\log \log t^{-1})^{1-1/d}} = c_0,$$

$$\limsup_{t \to \infty} \frac{|\eta(t)|}{t^{1/d}(\log \log t)^{1-1/d}} = c_1.$$
Remark 1.4. We expect that with some extra care one can show that the moment bounds in Theorems 1.1 and 1.2 are uniform in $\kappa$ as long as we stay away from the degenerate values, i.e., away from 4 in case (ii) and possibly away from 0 in case (i). A uniformity statement of this type is proven in [AM22] and used to construct SLE$_8$ as a continuous curve.

Remark 1.5. The results transfer to other SLE variants by conformal invariance and absolute continuity as long as we stay away from the boundary and force points, i.e., the statements of Theorems 1.1–1.3 hold true on curve segments that do not touch force points or domain boundaries. We expect that the same results also hold for entire SLE curves (including boundary intersecting segments) in bounded domains whose boundaries are not too fractal.

Remark 1.6. In view of Theorems 1.2 and 1.3, it is natural to study exceptional times where the law of the iterated logarithm fails. For $a > 0$ and a time $t_0$ call the time $a$-fast if

$$\limsup_{t \downarrow t_0} \frac{\eta(t) - \eta(t_0)}{t^{1/d}(\log t)^{1-1/d}} \geq a.$$ 

With the methods of this paper, one can show that the Hausdorff dimension of the set of $a$-fast times for SLE is bounded between $1/d - c_1a^{d/(d-1)}$ and $1 - c_2a^{d/(d-1)}$ with (non-explicit) deterministic constants $c_1, c_2 > 0$. The reason we get $1/d$ instead of 1 in our lower bound is that in our argument we only consider the radial direction of the curve instead of $d$ dimensions. We conjecture that there is a deterministic constant $c_0 > 0$ such that the dimension is exactly $1 - c_0a^{d/(d-1)}$. For comparison, the Hausdorff dimension of the set of $a$-fast times for Brownian motion is proven in [OT74] to be $1 - a^2/2$.

In the case of space-filling SLE, we show a stronger formulation of the upper bound in Theorem 1.2.

Theorem 1.7. Consider space-filling SLE$_\kappa$ as in case (ii). There exist $\delta > 0$ and $u_0 > 0$ such that the following is true. For $r, u > 0$ and $I$ a bounded interval let $E_{r,u,I}$ denote the event that for any $s, t \in I$ with $|\eta(s) - \eta(t)| \leq ur$, the set $\eta(s, t)$ contains $\delta u^2 \log(u|\eta(s) - \eta(t)|^{-1})$ disjoint balls of radius $u^{-2}|\eta(s) - \eta(t)|/\log(u|\eta(s) - \eta(t)|^{-1})$. Then for any bounded interval $I \subseteq \mathbb{R}$ there exist $c_1, c_2 > 0$ such that

$$\mathbb{P}(E_{r,u,I}^c) \leq c_1 r^{c_2u^2}$$

for any $u \geq u_0$ and $r \in (0, 1)$.

A key input to the proofs is the following precise estimate for the lower tail of the Minkowski content of SLE segments:

$$\mathbb{P}(\text{Cont}(\eta[0, \tau_r]) < t) \approx \exp\left(-c_3r^{d/(d-1)}t^{-1/(d-1)}\right), \quad (2)$$

where Cont denotes Minkowski content of dimension $d$ (with $d$ as (1)), $\tau_r = \inf\{t \geq 0 : |\eta(t)| = r\}$ denotes the hitting time of radius $r$, we write $\eta[0, \tau_r]$ instead of $\eta([0, \tau_r])$ to simplify notation, and we use $\approx$ to indicate that the left side of (2) is bounded above and below by the right side of (2) for different choices of $c$. Furthermore, building on the domain Markov property of SLE we prove “conditional” variants of (2) where we condition on part of the past curve. The conditional variant of the upper bound holds for all possible realizations of the past curve segment while the conditional variant of the lower bound requires that the tip of the past curve is sufficiently nice. See Propositions 3.10, 5.2 and 5.16 for precise statements, and see (11) and Propositions 5.15 and 5.17 for conditional variants. Note that since we parametrise $\eta$ by its Minkowski content, (2) can be equivalently formulated as

$$\mathbb{P}(\text{diam}(\eta[0, t]) > r) \approx \exp\left(-c_3r^{d/(d-1)}t^{-1/(d-1)}\right).$$
We establish the upper bounds in Theorems 1.1–1.3 for general stochastic processes whose increments satisfy a suitable moment condition (7). Several related results are available in the existing literature, see e.g. [FV10, Appendix A] and [Bed07]. We review and generalise them in Section 3.1. We then prove that the SLE variants (i) and (ii) do satisfy the required condition. In the latter step we use only the self-similarity of the SLE along with a Markov-type property satisfied by the increments, namely a conditional variant of the upper bound in (2).

To prove the lower bounds, we need to argue that the increments of the process in disjoint time intervals are sufficiently decorrelated. Given sufficient decorrelation, our proof is relatively simple to implement; see Section 4 where we have spelled out the proof for Markov processes that have uniform bounds on the transition probabilities. For SLE, we rely on the conditional variant of the lower bound in (2), which is based on the domain Markov property. Here extra care is needed due to the fact that this estimate only holds when the past curve is nice.

1.3 Outline
We give in Section 2 some basic definitions and results on conformal maps and SLE, including the precise definition of the SLE variants that we work with. In Section 3 we prove our main theorems except for the lower bounds which will be proved in Section 5. To illustrate the basic idea of the proof, we show in Section 4 the analogous results for Markov processes.

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2 Preliminaries

2.1 Conformal maps
We will always denote by $\mathbb{H}$ the upper complex half-plane $\{z \in \mathbb{C} \mid \text{Im} z > 0\}$, and by $\mathbb{D}$ the unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$. For $z_0 \in \mathbb{C}$ and $r > 0$, we denote by $B(z_0, r)$ the open ball of radius $r$ about $z$, i.e. $\{z \in \mathbb{C} \mid |z - z_0| < r\}$.

For a bounded, relatively closed set $A \subseteq \mathbb{H}$, we define its half-plane capacity to be $\lim_{y \to \infty} y \mathbb{E}[\text{Im} B_{\tau_{A \cup \mathbb{R}}}^y]$ where $B^y$ denotes Brownian motion started at $iy$ and $\tau_{A \cup \mathbb{R}}$ the hitting time of $A \cup \mathbb{R}$.

For a simply connected domain $D \subseteq \mathbb{C}$ and a prime end $x \in \partial D$, fix a conformal map $f : D \to \mathbb{H}$ with $f(x) = \infty$. For a relatively closed set $A \subseteq D$ with positive distance to $x$, we define the capacity of $A$ in $D$ relative to $x$ to be the half-plane capacity of $f(A)$.

Standard results for conformal maps include Koebe’s distortion and 1/4-theorem. See e.g. [Con95, Theorems 14.7.8 and 14.7.9] for proofs.

**Lemma 2.1** (Koebe’s distortion theorem). Let $R > 0$ and $f : B(z, R) \to \mathbb{C}$ be a univalent function. Then

$$\frac{1 - r}{(1 + r)^3} \leq \frac{|f'(w)|}{|f'(z)|} \leq \frac{1 + r}{(1 - r)^3}$$

for all $w \in B(z, R)$ where $r = \frac{|w - z|}{R} < 1$.  

\footnote{In case $\partial D$ has an analytic segment in the neighbourhood of $x$, there is a more intrinsic definition given in [DV14]. Their definition differs to ours by a fixed factor depending on the normalisation of $f$. In particular, we can pick $f$ such that both definitions agree. For our purposes, the choice of normalisation will not matter.}
The most common application in our work is that for any \( w \in B(z, r) \), \( r < R \), we have the bounds
\[
c|f'(z)| \leq |f'(w)| \leq c^{-1}|f'(z)|
\]
with \( c > 0 \) depending on \( r \).

**Lemma 2.2** (Koebe’s 1/4 theorem). Let \( R > 0 \) and \( f : B(z, R) \to D \subseteq \mathbb{C} \) be a conformal map. Then
\[
\text{dist}(f(z), \partial D) \geq \frac{R}{4}|f'(z)|.
\]

For a simply connected domain \( D \subseteq \mathbb{C} \) and \( z \in D \), the conformal radius of \( z \) in \( D \) is defined as \( |f'(0)| \) where \( f : \mathbb{D} \to D \) is a (unique up to rotations of \( \mathbb{D} \)) conformal map with \( f(0) = z \). We have the standard estimates
\[
\text{dist}(z, \partial D) \leq \text{crad}(z, D) \leq 4 \text{dist}(z, \partial D)
\]
which follow from the Schwarz lemma and Koebe’s 1/4 theorem.

Throughout the paper, we will often consider domains of the following type. Let \( (D, a) \) be such that
\[
D \subset \hat{\mathbb{C}} \text{ is a simply connected domain with } \infty \in D, \ 0 \notin D, \quad \text{and} \quad a \in \partial D \text{ with } |a| = \sup\{|z| \mid z \in \hat{\mathbb{C}} \setminus D\} > 0.
\]

Typical examples of such domains are \( D = \hat{\mathbb{C}} \setminus \text{fill}(\eta[0, \tau_r]) \), \( a = \eta(\tau_r) \) where \( \eta \) is some continuous path starting from the origin, \( \tau_r = \inf\{t \geq 0 : |\eta(t)| = r\} \) denotes the hitting time of \( \partial B(0, r) \), and \( \text{fill}(\eta[0, \tau_r]) \) denotes the union of \( \eta[0, \tau_r] \) and the set of points disconnected from \( \infty \) by this set.

To \( (D, a) \) as in (3), we associate a conformal map \( f : D \to \mathbb{D} \) as follows. Let \( z_D \in \partial B(0, |a|+2) \) be the point closest to \( a \), and \( f : D \to \mathbb{D} \) the conformal map with \( f(z_D) = 0 \) and \( f(a) = 1 \). The following property is shown within the proof of [GHM20a, Lemma 3.1].

**Lemma 2.3.** There exists \( \varepsilon_0 > 0 \) such that the following is true. Let \( (D, a) \) be as in (3) with \( |a| \geq 1 \), and \( f : D \to \mathbb{D} \) the associated conformal map. Let \( V \) be the union of \( B(z_D, 3) \cap D \) with all points that it separates from \( \infty \) in \( D \). There exists a path \( \alpha \) from 0 to 1 in \( \mathbb{D} \) whose \( \varepsilon_0 \)-neighbourhood is contained in \( f(V) \). Moreover, \( \alpha \) can be picked as a simple nearest-neighbour path in \( \varepsilon_0 \mathbb{Z}^2 \).

### 2.2 (Ordinary) SLE

In this and the next section we discuss the SLE variants that we use in the paper. All SLE variants are probability measures on curves (modulo reparametrisation) either in a simply connected domain \( D \subseteq \mathbb{C} \) or in the full plane \( \hat{\mathbb{C}} \).

Fix \( \kappa > 0 \). Let \( D \subseteq \hat{\mathbb{C}} \) be a simply connected domain, and \( a, b \in \partial D \) two distinct prime ends. Moreover, we may possibly have additional force points \( u^1, ..., u^n \in \mathbb{T} \) with weights \( \rho_1, ..., \rho_n \in \mathbb{R} \).

The chordal SLEs \( \kappa \) \( \kappa \)-\SLE(\( \rho_1, ..., \rho_n \)) in \( D \) from \( a \) to \( b \) with these force points is a probability measure on curves in \( \mathbb{T} \) ending at \( a \) with the following domain Markov property: For any stopping time \( \tau \), conditionally on \( \eta[0, \tau] \), the law of \( \eta[\tau, \infty] \) in an \( \kappa \)-\SLE(\( \rho_1, ..., \rho_n \)) in the connected component of \( D \setminus \eta[0, \tau] \) containing \( b \) from \( \eta(\tau) \) to \( b \) with the same force points. (There is a subtlety when \( \eta \) has swallowed some force points, but in this paper we will not encounter that scenario.)

Moreover, the SLE measures are conformally invariant in the sense that if \( f : D \to f(D) \) is a conformal map, then the push-forward of \( \kappa \)-\SLE(\( \rho_1, ..., \rho_n \)) in \( D \) is \( \kappa \)-\SLE(\( \rho_1, ..., \rho_n \)) in \( f(D) \) from \( f(a) \) to \( f(b) \) with force points \( f(u^1), ..., f(u^n) \).

Similarly, radial SLE is characterised by the same properties except that \( b \in D \) instead of \( \partial D \). For both chordal and radial SLE, it is sometimes convenient to consider \( b \) as an additional
force point with weight $\rho_{n+1} = \kappa - 6 - \rho_1 - \ldots - \rho_n$. By doing so, we have the following simple transformation rule (see [SW05, Theorem 3]): For any conformal map $f : D \to f(D)$, the pushforward of SLE$_{\kappa}(\rho_1, \ldots, \rho_{n+1})$ in $D$ starting from $a$ (stopped before swallowing any force point) is SLE$_{\kappa}(\rho_1, \ldots, \rho_{n+1})$ in $f(D)$ starting from $f(a)$. (Note that in this rule, the target point $b$ is not necessarily $u^{n+1}$ but can be any $u^j$. The law does not depend on the choice of a target point.)

In case $D = \mathbb{H}$ or $D = \mathbb{D}$, the laws of SLE can be spelled out explicitly (cf. [SW05]). Namely, we can describe the law of the conformal maps $g_t : \mathbb{H} \setminus \text{fill}(\eta[0, t]) \to \mathbb{H}$ with $g_t(z) = z + \theta(1/z)$, $z \to \infty$. If $\eta$ is parametrised by half-plane capacity, i.e. $\text{hcap}(\text{fill}(\eta[0, t])) = 2t$, then the families $(g_t(z))_{t \geq 0}$ satisfy

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$

with processes $(W_t, U^1_t, \ldots, U^n_t)$ satisfying the following system of SDEs

$$dW_t = \sqrt{\kappa} dB_t + \sum_j \text{Re}\frac{\rho_j}{W_t - U^j_t} dt, \quad W_0 = a,$$

$$dU^j_t = \frac{2}{U^j_t - W_t} dt, \quad U^0_0 = u^j.$$

By Girsanov’s theorem, for fixed $\kappa \geq 0$, all such SLE$_{\kappa}$ variants (with different values of $\rho_j$) are absolutely continuous with respect to each other (before any force point is swallowed). The Radon-Nikodym derivatives can be spelled out explicitly, see [SW05, Theorem 6]. One particular consequence is the following.

**Lemma 2.4.** Let $D \subseteq \mathbb{C}$ be a simply connected domain, $U \subseteq D$ a bounded subdomain, and $a \in \partial D \cap \partial U$. Let $\varepsilon > 0$ and $\rho_1, \ldots, \rho_n \in \mathbb{R}$. Then there exists $c > 0$ such that the following is true:

Let $u^1, \ldots, u^k \in U$ and let $u^{k+1}, \ldots, u^n$, and $b$ be outside the $\varepsilon$-neighbourhood of $U$. Consider the law $\nu$ of SLE$_{\kappa}$ in $D$ from $a$ to $b$ with force points $(u^1, \ldots, u^k)$ and weights $(\rho_1, \ldots, \rho_k)$, and the law $\tilde{\nu}$ of SLE$_{\kappa}$ in $D$ from $a$ to $b$ with force points $(u^1, \ldots, u^n)$ and weights $(\rho_1, \ldots, \rho_n)$. Then, as laws on curves stopped upon exiting $U$, these two SLE$_{\kappa}$ measures are absolutely continuous, and their Radon-Nikodym derivative is bounded within $[c, c^{-1}]$.

Whole-plane SLE$_{\kappa}$ and whole-plane SLE$_{\kappa}(\rho)$ are probability measures on curves in $\hat{\mathbb{C}}$ running from $a$ to $b$ where $a, b$ are two distinct points on $\hat{\mathbb{C}}$. They are characterised by an analogous domain Markov property. For any non-trivial stopping time $\tau$, conditionally on $\eta|_{(-\infty, \tau]}$, the law of $\eta|_{[\tau, \infty)}$ is a radial SLE$_{\kappa}$ (resp. SLE$_{\kappa}(\rho)$) in the connected component of $\hat{\mathbb{C}} \setminus \eta((-\infty, \tau])$ containing $b$ from $\eta(\tau)$ to $b$ (with a force point at $a$).

Two-sided whole-plane SLE$_{\kappa}$, $\kappa \leq 8$, is a probability measure on closed curves in $\hat{\mathbb{C}}$ from $a$ to $a$ passing through some $b \in \hat{\mathbb{C}}$. It is defined as follows:

- The segment $\eta|_{(\infty, 0]}$ is a whole-plane SLE$_{\kappa}(2)$ from $a$ to $b$ with force point at $a$.
- Conditionally on $\eta|_{(\infty, 0]}$, the segment $\eta|_{[0, \infty)}$ is a chordal SLE$_{\kappa}$ in $\hat{\mathbb{C}} \setminus \eta(-\infty, 0]$ from $b$ to $a$.

We will use the following facts about two-sided whole-plane SLE$_{\kappa}$ from $\infty$ to $\infty$ passing through $0$ (cf. [Zha21]). Both whole-plane SLE$_{\kappa}(2)$ and two-sided whole-plane SLE$_{\kappa}$ are reversible. In particular, the restriction $\eta|_{[0, \infty)}$ is a whole-plane SLE$_{\kappa}(2)$ from $0$ to $\infty$ with force point at $0$. Moreover, if $\eta$ is parametrised by Minkowski content, then $\eta$ is a self-similar process of index $1/d$ with stationary increments, in the sense that for every $t_0 \in \mathbb{R}$ and $\lambda > 0$, the process $t \mapsto \eta(t_0 + \lambda t) - \eta(t_0)$ has the same law as $t \mapsto \lambda^{1/d} \eta(t)$. 

7
In the remainder of this paper, we denote by $\nu_{D,a\rightarrow b}$ (chordal or radial) SLE$_\kappa$ in $D$ from $a$ to $b$, and by $\nu_{D,a\rightarrow b|a}$ SLE$_\kappa(2)$ with a force point at $u$. We denote by $\nu_{\mathcal{C};a\rightarrow b|a}$ whole-plane SLE$_\kappa(2)$.

The Minkowski content measures the size of fractal sets. It has been shown in [LR15] that SLE$_\kappa$ curves\footnote{Strictly speaking, their result applies to SLE$_\kappa$ in $\mathbb{H}$ without force points, but it transfers to other SLE$_\kappa$ variants by conformal invariance and absolute continuity, at least on segments away from force points and fractal boundaries. The result is also true for two-sided whole-plane SLE$_\kappa$, see [Zha21, Lemma 2.12].} possess non-trivial $d$-dimensional Minkowski content where $d = (1 + \frac{\kappa}{2}) \wedge 2$. Moreover, the Minkowski content of $\eta[0, t]$ is continuous and strictly increasing in $t$, hence $\eta$ can be parametrised by its Minkowski content. This is called the natural parametrisation of the curve. Finally, it is shown there that the Minkowski content is additive over SLE curve segments, i.e. $\text{Cont}(\eta[s, u]) = \text{Cont}(\eta[s, t]) + \text{Cont}(\eta[t, u])$ for $s \leq t \leq u$.

By [Zha21, Lemma 2.6], the Minkowski content satisfies the following transformation rule. If $f : U \rightarrow f(U) \subseteq \mathbb{C}$ is a conformal map and $\mu$ is the Minkowski content measure of some $S \subseteq U$, i.e. $\mu(K) = \text{Cont}(K \cap S)$ for every compact $K \subseteq U$, then

$$\text{Cont}(f(K \cap S)) = \int_K |f'(z)|^d \mu(dz) \quad (4)$$

for every compact $K \subseteq U$.

### 2.3 Space-filling SLE

In this section we introduce the whole-plane space-filling SLE$_\kappa$ which is defined via the theory of imaginary geometry. Whole-plane space-filling SLE$_\kappa$ from $\infty$ to $\infty$ for $\kappa > 4$ was originally defined in [DMS21, Section 1.4.1], building on the chordal definition in [MS17, Sections 1.2.3 and 4.3]. See also [GHS19, Section 3.6.3] for a survey. For $\kappa \geq 8$, whole-plane space-filling SLE$_\kappa$ from $\infty$ to $\infty$ is a curve from $\infty$ to $\infty$ obtained via the local limit of a regular chordal SLE$_\kappa$ at a typical point. For $\kappa \in (4, 8)$, space-filling SLE$_\kappa$ from $\infty$ to $\infty$ traces points in the same order as a curve that locally looks like (non-space-filling) SLE$_\kappa$, but fills in the “bubbles” that it disconnects from its target point with a continuous space-filling loop.

To construct whole-plane space-filling SLE$_\kappa$, from $\infty$ to $\infty$, fix a deterministic countable dense subset $\mathcal{C} \subseteq \mathbb{C}$. Let $h$ be a whole-plane GFF, viewed modulo a global additive multiple of $2\pi \chi$ where $\chi := \frac{\pi}{\sqrt{2}} - \sqrt{\frac{\pi}{2}}$; see [MS17, Section 2.2] for the definition of this variant of the GFF. It is shown in [MS17] that for each $z \in \mathcal{C}$, one can make sense of the flow lines $\eta^L_z$ and $\eta^R_z$ of $e^{ih/\chi}$ of angles $\pi/2$ and $-\pi/2$, respectively, started from $z$. These flow lines are SLE$_{16/\kappa}(2 - 16/\kappa)$ curves [MS17, Theorem 1.1]. The flow lines $\eta^L_z$ and $\eta^R_w$ (resp. $\eta^R_z$ and $\eta^R_w$) started at distinct $z, w \in \mathcal{C}$ eventually merge, such that the collection of flow lines $\eta^L_z$ (resp. $\eta^R_z$) for $z \in \mathcal{C}$ form the branches of a tree rooted at $\infty$.

We define a total ordering on $\mathcal{C}$ by saying that $w \in \mathcal{C}$ comes after $z \in \mathcal{C}$ if and only if $\eta^L_w$ merges into $\eta^L_z$ on its right side (equivalently, $\eta^R_w$ merges into $\eta^R_z$ on its left side). It can be argued that there is a unique space-filling curve $\eta : \mathbb{R} \rightarrow \mathcal{C}$ that visits the points in $\mathcal{C}$ in order, is continuous when parameterized by Lebesgue measure (i.e. $\eta[0, t]$ and $\eta[-t, 0]$ both have Lebesgue measure $t$ for any $t > 0$), satisfies $\eta(0) = 0$, and is such that $(\eta)^{-1}(\mathcal{C})$ is a dense set of times (see [MS17, Section 4.3] and [DMS21]). The curve $\eta$ does not depend on the choice of $\mathcal{C}$ and is defined to be whole-plane space-filling SLE$_\kappa$ from $\infty$ to $\infty$. For each fixed $z \in \mathcal{C}$, it is a.s. the case that the left and right outer boundaries of $\eta$ stopped at the first time it hits $z$ are given by the flow lines $\eta^L_z$ and $\eta^R_z$. Since the flow lines have zero Lebesgue measure and $(\eta)^{-1}(\mathcal{C})$ is dense, it follows that almost surely for all $s < t$ the Lebesgue measure of $\eta[s, t]$ is exactly $|t - s|$.

We remark that for $\kappa = 8$, the whole-plane space-filling SLE$_8$ as defined here is equal in law to the two-sided whole-plane SLE$_8$ defined in the previous subsection.
In our proofs in the next subsection we will consider \( \eta|_{[0,\infty)} \) and we will now describe this curve slightly more explicitly. The two flow lines \( \eta^L_0 \) and \( \eta^R_0 \) divide \( \mathbb{C} \) into two (for \( \kappa \geq 8 \)) or countably infinite (for \( \kappa \in (4,8) \)) connected components, such that the boundary of each connected component can be written as the union of a segment of \( \eta^L_0 \) and a segment of \( \eta^R_0 \). The curve \( \eta|_{[0,\infty)} \) will visit precisely the connected components that lie to the right of \( \eta^L_0 \) (i.e. \( \eta^L_0 \) traces its boundary in clockwise direction), and \( \eta|_{[0,\infty)} \) restricted to each such component has the law of a chordal space-filling SLE\(_{\kappa} \) connecting the two points of intersection of \( \eta^L_0 \) and \( \eta^R_0 \) on its boundary. For \( \kappa \geq 8 \) the chordal space-filling SLE\(_{\kappa} \) is just a regular chordal SLE\(_{\kappa} \), while for \( \kappa \in (4,8) \) the curve can be constructed by starting with a regular chordal (non-space-filling) SLE\(_{\kappa} \) and filling in the components disconnected from the target point by a space-filling SLE\(_{\kappa} \)-type loop. The SLE\(_{\kappa} \)-type loop can be obtained via an iterative construction where one first samples a regular chordal (non-space-filling) SLE\(_{\kappa} \) (\( \kappa \sim 6 \)) and then samples curves with this law iteratively in each complementary component of the curve.

The boundary data of \( h \) along an angle \( \theta \in \{ -\pi/2, \pi/2 \} \) flow line is given by \( \chi \) times the winding of the curve plus \( -\pi/\sqrt{\kappa} - \kappa \theta \) (resp. \( \pi/\sqrt{\kappa} - \kappa \theta \)) on the left (resp. right) side, where the winding is relative to a segment of the curve going straight upwards. We refer to [MS17, Section 1] for the precise description of this conditional law and in particular to [MS17, Figures 1.9 and 1.10] for the boundary data and the concept of winding.

For \( t \geq 0 \), let \( \text{fill}(\eta[0,t]) \) be the hull generated by \( \eta[0,t] \), i.e. the union of \( \eta[0,t] \) and the set of points which it disconnects from \( \infty \). For \( \kappa \geq 8 \) we have \( \text{fill}(\eta[0,t]) = \eta[0,t] \) while for \( \kappa \in (4,8) \) we have that \( \eta[0,t] \) is strictly contained in \( \text{fill}(\eta[0,t]) \). However, in the latter case it still holds a.s. that \( \eta(\tau_r) \) lies on the boundary of \( \text{fill}(\eta[0,\tau_r]) \) for \( \tau_r = \inf\{ t \geq 0 : |\eta(t)| \geq r \} \) and fixed \( r > 0 \), and that \( \eta|_{[\tau_r,\infty)} \) stays in \( \mathbb{C} \setminus \text{fill}(\eta[0,\tau_r]) \). See Figure 1.

Fix \( r > 0 \). The set \( \partial \text{fill}(\eta[0,\tau_r]) \) can be divided into four distinguished arcs, which we denote as follows.

- \( A^L \) (resp. \( A^R \)) is the arc of \( \partial \text{fill}(\eta[0,\tau_r]) \) traced by \( \eta^L_0 \) (resp. \( \eta^R_0 \)).
- \( A^L \) (resp. \( A^R \)) is the arc of \( \partial \text{fill}(\eta[0,\tau_r]) \) not traced by \( \eta^L_0 \) or \( \eta^R_0 \) which is adjacent to \( A^L \) (resp. \( A^R \)).

Define the \( \sigma \)-algebra \( \mathcal{F}_r \) by \( \mathcal{F}_r := \sigma \left( \eta|_{[0,\tau_r]}, h|_{\eta[0,\tau_r]} \right) \). The following is [GHM20a, Lemma 3.2].

**Lemma 2.5.** The set \( \eta[0,\tau_r] \) is a local set for \( h \) in the sense of [SS13, Lemma 3.9]. In particular, the boundary data for the conditional law of \( h|_{\mathbb{C} \setminus \text{fill}(\eta[0,\tau_r])} \) given \( \mathcal{F}_r \) on each of the arcs \( A^L, A^R \).
Let \((D,a)\) be as in (3), and let \(a = (u_1, u_2, u_3)\) be distinct points on \(\partial D \setminus \{a\}\) such that \(a, u_1, u_2, u_3\) are ordered counterclockwise. Let \(h\) be a GFF in \(D\) with the law in Lemma 2.5 if \(D = \mathbb{C} \setminus \text{fill}(\eta[0,\tau_r])\) and we let \(a, u_1, u_2, u_3\) describe the points of intersection of the boundary arcs \(\mathcal{A}^L, \mathcal{A}^R, \mathcal{A}^L, \text{ and } \mathcal{A}^R\). Then since the conditional law of \(\eta\) given \(\mathcal{F}_r\) depends only on \((D,a,u)\), we can define a measure \(\tilde{\nu}_{D,a \to \infty;u}\) on curves from \(a\) to \(\infty\) in \(D\) that describes this conditional law. Consider the pair \((\mathbb{C} \setminus \text{fill}(\eta[0,\tau_r]), \eta(\tau_r))\) and \(f : \mathbb{C} \setminus \text{fill}(\eta[0,\tau_r]) \to \mathbb{D}\) be the conformal map described right below (3), which in particular satisfies \(f(\eta(\tau_r)) = 1\). Define \(\tilde{h}\) by
\[
\tilde{h} := h \circ f^{-1} - \chi \arg(f^{-1})',
\]
Then \(\tilde{h}\) is a GFF on \(\mathbb{D}\) with Dirichlet boundary data determined by \(f(u_1), f(u_2), f(u_3)\) plus an arg singularity at \(f(\infty)\), where we pick an arbitrary choice of branch cut for \(\arg f\); picking a different branch cut has the effect of adding a multiple of \(4\pi \chi\) in the region between the two branch cuts. In particular, the law of \(\tilde{h}\) (modulo \(2\pi \chi\)) depends only on the location of the points \(f(u_1), f(u_2), f(u_3), f(\infty)\). On each of the four arcs \(f(\mathcal{A}^L), f(\mathcal{A}^R), f(\mathcal{A}^L), f(\mathcal{A}^R)\) on \(\partial \mathbb{D}\) the boundary data will be given by a constant (depending on which arc we consider) plus \(\chi\) times the winding of \(\partial \mathbb{D}\) viewed as a curve. The image of \(\eta_{\mathbb{R} \setminus [0,\tau_r]}\) under \(f\) can be constructed via the flow lines of \(\tilde{h}\) exactly as above. The boundary data of \(\tilde{h}\) along its flow line is given by \(\pm \chi\) plus \(\chi\) times the winding of the flow line, except that there is a jump of \(4\pi \chi\) when crossing the branch cut in clockwise direction (i.e. \(-2\chi\)-flow line boundary conditions in the terminology of \([MS17]\)).

Similarly as in the paragraph right after Lemma 2.5 we can define a measure \(\tilde{\nu}_{D,1 \to z_\infty; \tilde{u}}\) on curves in \(\mathbb{D}\) from 1 to \(z_\infty \in \mathbb{D}\) that describes the conditional law of \(f \circ \eta_{[\tau_r,\infty)}\) given \(\mathcal{F}_r\), where \(z_\infty = f(\infty)\) and \(\tilde{u} = (f(u_1), f(u_2), f(u_3))\). This conditional law can be explicitly defined in terms of the flow lines of \(\tilde{h}\). The flow lines started from \(f(u_1)\) and \(f(u_3)\) of angle \(\pi/2\) and \(-\pi/2\), respectively, will end at \(f(\infty)\), and these two flow lines divide \(\mathbb{D}\) into two (for \(\kappa \geq 8\)) connected components. The curve \(f \circ \eta_{[\tau_r,\infty)}\) visits precisely the complementary components that lie to the right of the flow line started from \(f(u_1)\) and has the law of a chordal space-filling \(\text{SLE}_{\kappa}\) in each such component.

### 3 Upper bounds

In this section we prove the upper bounds in our main results (Theorems 1.1–1.3). The upper bounds hold for general stochastic processes whose increments satisfy a suitable moment condition, and we state these general results in Section 3.1. In Sections 3.2 and 3.3 we will prove that \(\text{SLE}\) satisfies these conditions, which in particular implies the upper bound in (2). We phrase part of the argument in Section 3.2 in a more general setting for processes with a suitable scaling property and Markov-type increment condition. In Section 3.4 we prove zero-one laws for some quantities related to \(\text{SLE}\) which will imply that the constants \(c_0, c_1\) in Theorems 1.1–1.3 are deterministic. Using the earlier results in the section we get that these constants are finite (but we do not know at this point whether they are positive; this will be proved in Section 5). In Section 3.5, we prove Theorem 1.7.
3.1 Regularity upper bounds under increment moment conditions

In this section, we let \( \Phi, \varphi : [0, \infty) \rightarrow [0, \infty) \) be convex self-homeomorphisms with \( \Phi(1) = \varphi(1) = 1 \). Suppose that there exist \( R > 1 \) and \( n_0 \geq 1, n_0 \in \mathbb{N} \) such that

\[
\Phi(x)\Phi(y) \leq \Phi(R^2 xy) \quad \text{for } x, y \geq 1, \quad (6)
\]

and

\[
\sum_{k=0}^{\infty} \frac{\varphi(R^k)}{\Phi(R^{k+n_0})} < \infty.
\]

We give examples of appropriate functions \( \Phi, \varphi \) in Examples 3.5 and 3.6 below.

Let \( \alpha \in (0, 1] \), and let \( (X_t)_{t \in [0,1]} \) be a separable process with values in a separable Banach space\(^3\) that satisfies

\[
E \Phi \left( \frac{|X_s - X_t|}{|s - t|^\alpha} \right) \leq 1 \quad \text{for all } s, t \in [0,1]. \quad (7)
\]

We will give general results on the modulus of continuity, law of the iterated logarithm, and variation regularity for such processes.

3.1.1 Modulus of continuity

We review the following general result which is a special case of [Bed07, Corollary 1]. (Recall that a separable process that is uniformly continuous on a suitable countable dense subset is necessarily continuous.)

**Theorem 3.1** ([Bed07, Corollary 1]). There exists a finite constant \( K \) (depending on \( \Phi, \varphi, \alpha \)) such that every separable process as in (7) satisfies

\[
E \sup_{s, t \in [0,1]} \Phi \left( \frac{|X_s - X_t|}{2K\tau(\frac{|t - s|}{|s - t|})} \right) \leq 1
\]

where

\[
\tau(t) = \int_0^t \varphi^{-1} \left( \frac{1}{2^{u^{1/\alpha}}} \right) \, du.
\]

The above theorem in particular says that \( X \) a.s. admits a modulus of continuity given by a (possibly random) constant multiple of \( \tau \) (as long as \( \tau \) is finite).

The general result in [Bed07] allows for stochastic processes indexed by a compact metric space \( T \), and they give a modulus of continuity by a function \( \tau(s, t) \) that may depend on both variables \( s, t \) (also called “minorising metric” in the literature).

Note that for any \( t_0 > s_0 > 0 \), an application of Theorem 3.1 to the process \( t \mapsto (t_0 - s_0)^{-\alpha} X_{t_0 + (t_0 - s_0)t} \) yields

\[
E \sup_{s, t \in [s_0, t_0]} \Phi \left( \frac{|X_s - X_t|}{2K\tau(1)|s_0 - t_0|^\alpha} \right) \leq E \sup_{s, t \in [s_0, t_0]} \Phi \left( \frac{|X_s - X_t|}{2K|s_0 - t_0|^{-\frac{1}{\alpha}}|t - s||s_0 - t_0|^\alpha} \right) \leq 1. \quad (8)
\]

3.1.2 Law of the iterated logarithm

We get the following theorem via Theorem 3.1 and a union bound. In fact, we only use (8) and not the full statement of Theorem 3.1.

\( ^3 \)The result in [Bed07] is stated for real-valued processes, but it is not required in their proof.
Theorem 3.2. For every non-decreasing function \( h : (0, \infty) \rightarrow (0, \infty) \) with \( \int_1^\infty \frac{du}{h(u)} < \infty \), every separable process as in (7) satisfies
\[
\limsup_{t \downarrow 0} \frac{|X_t - X_0|}{t^{\alpha} \Phi^{-1}(h(\log \frac{1}{t}))} \leq 2 K \tau(1)
\]
with the same \( K \) and \( \tau \) as in Theorem 3.1.

Proof. Assume without loss of generality that \( X_0 = 0 \). Fix \( q < 1 \) and consider the events
\[
A_n = \{ \|X\|_{\infty; [0, q^n]} \leq 2 K \tau(1) q^n \alpha \Phi^{-1}(h(n)) \}.
\]
By (8), we have \( \mathbb{P}(A_n^c) \leq \frac{1}{n^{\alpha}} \), and therefore \( \sum_n \mathbb{P}(A_n^c) < \infty \). By the Borel-Cantelli lemma, with probability 1 all but finitely many \( A_n \) occur.

If \( A_n \) occur, then for every \( t \in [q^{n+1}, q^n] \) we have
\[
|X(t)| \leq \|X\|_{\infty; [0, q^n]} \leq 2 K \tau(1) q^{n\alpha} \Phi^{-1}(h(n)) \leq 2 K \tau(1) q^{-\alpha} t^{\alpha} \Phi^{-1}(h(\log \frac{1}{t})).
\]
The result follows by observing that for every \( C > 0 \) and \( h \) with \( \int \frac{1}{h} < \infty \) we can find \( \tilde{h} \) with \( \int \frac{1}{\tilde{h}} < \infty \) such that \( \limsup_{u \rightarrow \infty} \frac{h(Cu)}{h(u)} < 1 \), applying the estimate above with \( C = \frac{1}{\log \frac{1}{q}} \) and \( \tilde{h} \), and then sending \( q \uparrow 1 \). \( \square \)

3.1.3 Variation

We now establish variation regularity of processes satisfying (8). Such results can be found e.g. in [Tay72] and [FV10, Appendix A.4] for some exponential \( \Phi \) as described in Example 3.5. We generalise their result to the setup of Section 3.1. Moreover, we find our proof conceptually clearer since we construct a natural way of parametrising \( X \) to obtain optimal modulus of continuity \( \sigma \). (Note that this is the best modulus that we can expect, cf. Proposition 5.1.)

Theorem 3.3. Suppose that \( \Phi \) also satisfies \( \int_2^\infty \frac{\log y}{\Phi(y)} \, dy < \infty \). Let \( h : (0, \infty) \rightarrow (0, \infty) \) be a non-decreasing continuous function with \( \int_1^\infty \frac{du}{h(u)} < \infty \), and such that
\[
\sigma(t) = t^{\alpha} \Phi^{-1}(h(\log \frac{1}{t}))
\]
is increasing with \( \sigma(0^+) = 0 \). Then there exist \( c > 0 \) (depending on \( \Phi, \alpha \)) and \( C > 0 \) (depending on \( \Phi, \alpha, h \)) such that every separable process as in (7) satisfies
\[
\mathbb{P}(|X|_{\psi-\text{var}} > M) \leq \frac{C}{\Phi(cM)}
\]
with \( \psi = \sigma^{-1} \).

To prove this, we construct a parametrisation of \( X \) that has modulus \( \sigma \). Fix some \( M > 0 \). Consider the intervals \( I_{k,j} = [j2^{-k}, (j+1)2^{-k}] \), and let
\[
s_{k,j} = \sup\{s \leq 1 \mid \sup_{u,v \in I_{k,j}} |X_u - X_v| \leq M \sigma(2^{-k}/s)\},
\]
\[
s(t) = \inf\{s_{k,j} \mid t \in I_{k,j}\}.
\]
The idea is to slow down the path \( X \) at time \( t \) to speed \( s(t) \). More precisely, define
\[
T(t) = \int_0^t \frac{dt}{s(t)}.
\]
Intuitively, this describes the elapsed time after reparametrisation.
In other words, we now estimate Proof of Theorem 3.3.

Indeed, for any partition $s(t)$, we have

$$|X_{t_1} - X_{t_2}| \leq M\sigma(2^{-k}/s_{k,j})$$

and that term we get the same estimate without the factor $2^\alpha$.

We claim that

$$|X|_{\psi\text{-var}} \leq M(2^\alpha + 1)T(1)^\alpha.$$  \hfill (9)

Proof of Theorem 3.3. By (9), we have

$$\mathbb{P}([X]_{\psi\text{-var}} > 6M) \leq \mathbb{P}(T(1)^\alpha > 2) \leq \mathbb{P} \left( \int_0^1 \frac{1}{s(t)} 1_{s(t)<1} dt > 1 \right).$$

We now estimate

$$\mathbb{E} \int_0^1 \frac{1}{s(t)} 1_{s(t)<1} dt = \int_0^1 \mathbb{E} \frac{1}{s(t)} 1_{s(t)<1} dt$$

and then

$$\mathbb{E} \left[ \frac{1}{s(t)} 1_{s(t)<1} \right] = \int_0^\infty \mathbb{P} \left( \frac{1}{s(t)} 1_{s(t)<1} > y \right) dy$$

For $y \geq 1$, we get from (8)

$$\mathbb{P}(s_{k,j} < 1/y) = \mathbb{P}(\sup_{u,v \in I_{k,j}} |X_u - X_v| > M\sigma(2^{-k}y))$$

$$\leq \mathbb{P} \left( \sup_{u,v \in I_{k,j}} \frac{|X_u - X_v|}{2^{-k\alpha}} > M y^\alpha \Phi^{-1}(h(\log^*(2^k/y))) \right)$$

$$\leq \frac{1}{\Phi(2K(2^k/y)^\alpha)} \Phi(y^\alpha) \int \frac{1}{2K^2} \Phi(2K(2^k/y)^\alpha)$$
where we have applied (6) in the last step. Hence,

\[
\mathbb{P}(s(t) < 1/y) \leq \sum_{k \in \mathbb{N}} \mathbb{P}(s_{k,j} < 1/y) \leq \frac{1}{\Phi(M 2K \tau(1) R^2)} \phi(y) \sum_{k \in \mathbb{N}} h(\log^2(2^k/y)) \\
\lesssim \frac{1}{\Phi(M 2K \tau(1) R^2)} \phi(y) \left( \log y + \sum_{2^k > y} h(k \log 2 - \log y) \right) \\
\leq \frac{1}{\Phi(M 2K \tau(1) R^2)} \phi(y) (\log y + C),
\]

and finally

\[
\int_0^\infty \mathbb{P} \left( s(t) < \frac{1}{y} \wedge 1 \right) dy \leq \frac{1}{\Phi(M 2K \tau(1) R^2)} \left( C + \int_2^\infty \frac{\log y + C}{\phi(y)} dy \right) \lesssim \frac{1}{\Phi(M 2K \tau(1) R^2)}
\]

by the extra assumption on \( \Phi \). Thus we have shown

\[
\mathbb{P}(|X|_{\psi \text{-var}} > 6M) \leq \mathbb{E} \int_0^1 \frac{1}{s(t)} 1_{s(t) < 1} dt \lesssim \frac{1}{\Phi(M 2K \tau(1) R^2)}.
\]

\[\Box\]

### 3.1.4 Examples

**Example 3.5.** If the following holds for \( \beta > 0 \)

\[
\Phi(x) \asymp \exp(cx^\beta),
\]

we can pick \( \varphi = \Phi \). Hence, the modulus of continuity is given by

\[
\tau(t) \asymp t^\alpha (\log \log 1/t)^{1/\beta},
\]

the law of the iterated logarithm reads

\[
\sigma(t) \asymp t^\alpha (\log \log 1/t)^{1/\beta},
\]

and the variation regularity is

\[
\psi(x) \asymp x^{1/\alpha} (\log \log x)^{1/(\alpha \beta)}. \]

Hence, we recover the results of [FV10, Section A.4] which consider the case \( \beta = 2 \), and generalize these results to arbitrary \( \beta > 0 \).

Note that Brownian motion satisfies this with \( \alpha = 1/2, \beta = 2 \). For SLE, we will use this result with \( \alpha = 1/d, \beta = d/(d-1) \), where \( d \) is the dimension of the process as in (1).

**Example 3.6.** Suppose

\[
\Phi(x) = x^p, \quad p > \frac{1}{\alpha}.
\]

Define \( h \) by e.g. \( h(u) = u^{1+\varepsilon} \) or \( h(u) = u(\log^* u)(\log^* \log^* u) \cdots (\log^* \cdots \log^* u)^{1+\varepsilon} \), and set \( \varphi(x) = \frac{x^p}{h(\log x)} \). Then the modulus of continuity is given by

\[
\tau(t) \asymp t^{\alpha-1/p} h(\log 1/t)^{1/p},
\]
the law of the iterated logarithm reads

\[ \sigma(t) = t^\alpha h(\log \frac{1}{t})^{1/p}, \]

and the variation regularity is

\[ \psi(x) \asymp x^{1/\alpha} h(\log x)^{-1/(p\alpha)}. \]

This sharpens the results obtained from the Besov-Hölder embedding (also known as Kolmogorov’s continuity theorem) in [FV10, Corollary A.2] and the Besov-p-variation embedding in [FV10, Corollary A.3], which provide corresponding statements with Hölder and variation exponents arbitrarily close to \( \alpha - 1/p \) and \( 1/\alpha \), respectively.

### 3.2 Diameter upper bound given scale invariance and Markov-type increments

In this and the next subsection we show that SLE satisfies the conditions of Example 3.5. The argument in this subsection concerns general self-similar processes whose increments satisfy a Markov-type property. (We remark here that the argument does not apply to stable processes since they violate (10) due to large jumps.)

For what comes now, \( \eta \) can be any self-similar process of index \( 1/d < 1 \), in the sense that \( t \mapsto \eta(\lambda t) \) has the same law as \( t \mapsto \lambda^{1/d} \eta(t) \) for any \( \lambda > 0 \). Denote by \( (\mathcal{F}_t)_{t \geq 0} \) the filtration generated by \( \eta \), and \( \tau_r = \inf\{ t \geq 0 \mid |\eta(t)| \geq r \} \). Suppose there exists \( p < 1 \) and \( l > 0 \) such that

\[ \mathbb{P}(\tau_{r+6} \leq \tau_r + l \mid \mathcal{F}_{\tau_r}) < p \] (10)

for any \( r > 0 \). For such processes, we show the following statement.

**Proposition 3.7.** There exist \( l > 0 \) and \( c_1, c_2 > 0 \) such that

\[ \mathbb{P}(\tau_{r+r'} \leq \tau_r + l \mid \mathcal{F}_{\tau_r}) < c_1 \exp(-c_2(r')^{d/(d-1)}) \]

for all \( r, r' > 0 \).

The number \( 6 \) in \( \tau_{r+6} \) in (10) is not significant, and can be replaced by any \( s > 0 \). Of course, in all the statements, the constants may depend on \( s \).

**Lemma 3.8.** For any \( \varepsilon > 0 \) there exists \( c > 0 \) such that

\[ \mathbb{P}(\tau_{r+c} \leq \tau_r + l \mid \mathcal{F}_{\tau_r}) < \varepsilon \]

for any \( r > 0 \).

**Proof.** This follows by applying (10) iteratively. \( \square \)

**Lemma 3.9.** There exist \( l > 0 \) and \( c_1, c_2 > 0 \) such that

\[ \mathbb{P}(\tau_{r+r'} \leq \tau_r + lr' \mid \mathcal{F}_{\tau_r}) < c_1 \exp(-c_2 r') \]

for all \( r, r' > 0 \).

**Proof.** Pick \( \varepsilon < 1/4 \) and let \( c \) be the constant from Lemma 3.8. It suffices to show

\[ \mathbb{P}(\tau_{r+r'c} \leq \tau_r + \frac{r'}{2} \mid \mathcal{F}_{\tau_r}) < c_1 \exp(-c_2 r') \]

for \( r' \in 2\mathbb{N} \).
On the event that $\tau_{r+\epsilon'} \leq \tau_r + \frac{1}{2} r'$ there must exist integers $k_1, \ldots, k_{r'/2}$ such that $\tau_{r+k_i c} \leq \tau_{r+(k_i-1)c} + l$ for $i = 1, \ldots, r'/2$. For each such choice of $k_1, \ldots, k_{r'/2}$, we can apply Lemma 3.8 iteratively (each time conditionally on $\mathcal{F}_{\tau_{r+(k_i-1)c}}$), so that

$$\mathbb{P}(\tau_{r+k_ic} \leq \tau_{r+(k_i-1)c} + l \text{ for all } i \mid \mathcal{F}_{\tau_r}) \leq \epsilon'r'/2.$$ 

Since the number of such choices of $k_1, \ldots, k_{r'/2}$ is $(r'/2) \leq 2r'$, we get

$$\mathbb{P}(\tau_{r+\epsilon'} \leq \tau_r + \frac{l}{2} r' \mid \mathcal{F}_{\tau_r}) \leq 2r' \epsilon'r'/2.$$ 

The claim follows since we picked $\epsilon < 1/4$. 

Note that we do not need the scaling property in the proof of Lemma 3.9, so this lemma holds under only the assumption (10). Combining the lemma with the scaling property of the process we obtain Proposition 3.7.

Proof of Proposition 3.7. Let $\lambda = (r')^{1/(d-1)}$. By the self-similarity of $\eta$, we have that $t \mapsto \lambda \eta(\lambda^{-dt})$ has the same law as $\eta$. Hence the desired probability is equal to

$$\mathbb{P}(\tau_{\lambda r' + \lambda r''} \leq \tau_{\lambda r} + \lambda d l \mid \mathcal{F}_{\lambda r}).$$

We have chosen $\lambda$ such that $\lambda r' = \lambda d = (r')^{d/(d-1)}$. Therefore, by Lemma 3.9, the probability is bounded by

$$c_1 \exp(-c_2(r')^{d/(d-1)}).$$

3.3 Markov-type increment bound for SLE

In this subsection we verify (10) for SLE. Once we have done this, Proposition 3.7 and the self-similarity of $\eta$ immediately imply the following result. The proposition is a strengthening of [Zha19, Lemma 1.7] which proved that $\mathbb{E}[\text{diam}(\eta[0,1])] < \infty$ for any $a > 0$.

Proposition 3.10. Let $\eta$ be a two-sided whole-plane SLE$_\kappa$ or a whole-plane space-filling SLE$_\kappa$ as specified in Section 1.2. There exists $c > 0$ such that

$$\mathbb{E} \exp \left( c \left( \frac{\text{diam}(\eta[s,t])}{|s-t|^{1/d}} \right)^{d/(d-1)} \right) < \infty$$

for all $s < t$.

In particular, the conditions from Section 3.1 and Example 3.5 are satisfied with $\alpha = 1/d$, $\beta = \frac{d}{d-1}$. This proves the upper bounds in Theorems 1.1–1.3.

In fact, we will prove a stronger result than Proposition 3.10 below, namely that there exist finite constants $c_1, c_2 > 0$ such that for any $(D,a)$ as in (3) and $u \in \partial D \setminus \{a\}$ we have

$$v_{D,a \to \infty;u}(\text{Cont}(\eta[0,\tau_{a|u+}]]) < l) \leq c_1 \exp \left( -c_2 t^{-1/(d-1)} r^{d/(d-1)} \right)$$

(11)

for all $r > 0$. The same is true for $v_{D,a \to \infty;u}$ defined in Section 2.3.

Since we parametrise $\eta$ by its Minkowski content, we can phrase the condition (10) as follows. (Recall that the Minkowski content is additive over SLE curve segments.) As before, we write $\tau_r = \inf\{t \geq 0 \mid |\eta(t)| = r\}$.

Lemma 3.11. There exists $l > 0$ and $p < 1$ such that for any $r > 0$ we have

$$\mathbb{P} \left( \text{Cont}(\eta[\tau_r, \tau_{r+6}]) < l \ \big| \ \eta[0,\tau] \right) < p.$$
Proof of Lemma 3.11 in case of space-filling SLE$_\kappa$. In case of $r > 1$, the statement (even with $\tau_{r+1}$ instead of $\tau_{r+6}$) is precisely [GHM20a, Lemma 3.1]. In case $r \leq 1$, conditioning on $\eta|[0,\tau_2]$ we get

$$\mathbb{P}\left(\text{Cont}(\eta[\tau_2,\tau_3]) < l \mid \eta|[0,\tau_2]\right) < p.$$ 

This implies Lemma 3.11 for arbitrary $r > 0$. 

Remark 3.12. In the case of space-filling SLE$_\kappa$, the proof shows that there exist $l_0, \delta > 0$ such that

$$\mathcal{D}_{D,a}\rightarrow\infty;u \left(\eta[0,\tau_{|a|}+r]\right) \text{ contains } \delta r \text{ disjoint balls of radius } l_0 > 1 - c_1 \exp(-c_2r).$$

By scaling, we get

$$\mathcal{D}_{D,a}\rightarrow\infty;u \left(\eta[0,\tau_{|a|}+r]\right) \text{ contains } \delta l^{-1}r^2 \text{ disjoint balls of radius } l_0/l^{-1} > 1 - c_1 \exp(-c_2l^{-1}r^2).$$

Notice that on this event we have

$$\text{Cont}(\eta[0,\tau_{|a|}+r]) \gtrsim l,$$

so (12) is a stronger version of Proposition 3.10 and (11).

In the remainder of the section, we prove Lemma 3.11 for two-sided whole-plane SLE$_\kappa$. Recall from Section 2.2 that the restriction $\eta|_{[0,\infty)}$ is a whole-plane SLE$_\kappa(2)$ from 0 to $\infty$ with force point at 0. Therefore the statement is equivalent when we consider whole-plane SLE$_\kappa(2)$.

Lemma 3.13. There exists $l > 0$ and $p < 1$ such that for any $(D,a)$ as in (3) and $u \in \partial D \setminus \{a\}$ we have

$$\nu_{D,a}\rightarrow\infty;u(\text{Cont}(\eta[0,\tau_{|a|}+6])) < l < p.$$ 

Proof. We show the statement in case of $|a| \geq 1$ and with $\tau_{|a|+5}$ instead of $\tau_{|a|+6}$. In case $|a| < 1$, considering $\eta|[\tau_1,\tau_6]$ conditionally on $\eta[0,\tau_1]$ gives us

$$\nu_{D,a}\rightarrow\infty;u(\text{Cont}(\eta[\tau_1,\tau_6])) < l < p.$$ 

Let $f: D \to \mathbb{D}$ be the conformal map described in the paragraph below (3), and $\varepsilon_0 > 0$ and $\alpha$ as in Lemma 2.3. Denote by $B(\alpha,\varepsilon_0)$ the $\varepsilon_0$-neighbourhood of $\alpha$. By the definitions above, $f \circ \eta$ leaves $B(\alpha,\varepsilon_0)$ before $\eta$ hits radius $|a| + 5$.

Although $\alpha$ depends on $D$, it can be picked among a finite number of nearest-neighbour paths. Therefore it suffices to show that for given $\varepsilon_0$ and $\alpha$, there exists $l > 0$ and $q > 0$ such that the following holds. Let $w \in \mathbb{D} \setminus B(\alpha,\varepsilon_0)$ and $\nu_{D,1}\rightarrow w;u$ a radial SLE$_\kappa(2)$ with a force point $u \in \partial \mathbb{D}$. Then

$$\nu_{D,1}\rightarrow w;u(\text{Cont}(\eta[0,\sigma_{\alpha,\varepsilon_0}] \cap B(0,\varepsilon_0/2)) > l) > q$$

where $\sigma_{\alpha,\varepsilon_0}$ is the exit time of $B(\alpha,\varepsilon_0)$.

Indeed, since $|f^{-1}| \approx 1$ in a neighbourhood of 0 (due to Koebe’s distortion theorem), by the transformation rule for Minkowski content (4) this will imply $f^{-1}(\eta[0,\sigma_{\alpha,\varepsilon_0}] \cap B(0,\varepsilon_0/2))$ has Minkowski content at least a constant times $l$.

To show the claim, we need to find $l$ and $q$ such that the bound holds uniformly over all target points and force points. For concreteness, let us map $(\mathbb{D},1,0)$ to $(\mathbb{H},0,i)$. Then the image is an SLE$_\kappa$ in $\mathbb{H}$ with force points $(w,2)$, $u \in \mathbb{R}$, and $(w,\kappa - 8)$, $w \in \mathbb{H} \setminus B(\alpha,\varepsilon_0)$ (cf. [SW05, Theorem 3]). Since the force point $w$ lies outside $B(\alpha,\varepsilon_0)$, we can disregard it until the exit time of $B(\alpha,\varepsilon_0)/2$ since the density between the corresponding SLE measures is uniformly bounded, regardless of the locations of $u$ and $w$ (cf. Lemma 2.4). Hence we are reduced to proving the following statement.

\[ \square \]
Lemma 3.14. Let $\alpha$ be a simple path in $\mathbb{H}$ from 0 to $i$, and $\varepsilon_0 > 0$. There exists $l > 0$ and $q > 0$ such that the following holds. Let $\nu_{1,0 \to \infty;u}$ denote chordal $\text{SLE}_\alpha(2)$ with a force point $u \in \partial \mathbb{H}$. Then

$$\nu_{1,0 \to \infty;u}(\text{Cont}(\eta[0,\sigma_{\alpha,\varepsilon_0}] \cap B(i,\varepsilon_0)) > l) > q$$

where $\sigma_{\alpha,\varepsilon_0}$ is the exit time of $B(\alpha, \varepsilon_0)$.

Proof. Case 1: Suppose that $u \notin B(0, \varepsilon_0)$. Then the density between the laws of $\text{SLE}_\alpha(2)$ and $\text{SLE}_\alpha(0)$ is uniformly bounded until the exit of $B(\alpha, \varepsilon_0/2)$ (cf. Lemma 2.4). Therefore it suffices to consider $\text{SLE}_\alpha(0)$. There is a positive probability that $\eta$ follows $\alpha$ within $\varepsilon_0/2$ distance. Moreover, since the Minkowski content on each sub-interval is almost surely positive, there is a positive probability that also $\text{Cont}(\eta[0,\sigma_{\alpha,\varepsilon_0}] \cap B(i,\varepsilon_0)) > l$ with sufficiently small $l > 0$.

Case 2: Suppose $u$ is arbitrary. Let $t > 0$ be a small time. Denote by $f_t$ the conformal map from $\mathbb{H} \setminus \text{fill}(\eta[0,t])$ to $\mathbb{H}$ with $f_t(\eta(t)) = 0$ and $f_t(z) = z + O(1)$. Then there exist $c_1 > 0$ and $q > 0$ (independent of $u$) such that with probability at least $q$ the following occur:

1. $\eta[0,t] \subseteq B(0,\varepsilon_0/4)$,
2. $|f_t(u)| \geq c_1$.

Indeed, $u_t = f_t(u)$ is a Bessel process of positive index started at $u$ (this follows directly from the definition of $\text{SLE}_\alpha(2)$). By the monotonicity of Bessel processes in the starting point, it suffices to consider $u = 0$. The claim follows since the Bessel process stopped at a deterministic time is almost surely positive.

It follows that if $t$ is chosen small enough, we have $|f_t(z) - z| < \varepsilon_0/2$ for every $z \in \partial B(\alpha, \varepsilon_0) \subseteq \mathbb{H}$, and therefore $B(\alpha, \varepsilon_0/2) \subseteq f_t(B(\alpha, \varepsilon_0))$. Then, applying Case 1 to $f_t \circ \eta|_{[t,\infty)}$ with $\varepsilon_0$ replaced by $\varepsilon_0/2 \wedge c_1$ implies the claim.

\section{Zero-one laws and upper bounds on the regularity of SLE}

In this subsection, we show the upper bounds in our main results (Theorems 3.1–3.3), which is stated as Proposition 3.19 below. To show that the constants $c_0, c_1$ are deterministic, we prove that they satisfy zero-one laws.

We begin by proving analogues of Blumenthal’s and Kolmogorov’s zero-one laws for SLE. Define $\mathcal{F}_t = \sigma(\eta(s), s \in [0,t])$, and $\mathcal{F}_+ = \bigcap_{t \geq 0} \mathcal{F}_t$. Moreover, denote by $\mathcal{G}$ the shift-invariant $\sigma$-algebra, i.e. the sub-$\sigma$-algebra of $\bigcap_{t \geq 0} \sigma(\eta(s), s > t)$ consisting of events $A$ such that $(\eta(t))_{t \geq 0} \in A$ if and only if $(\eta(t_0 + t))_{t \geq 0} \in A$ for any $t_0 \geq 0$. Note that we are considering paths restricted to $t \in \mathbb{R}^+$. For two-sided whole-plane $\text{SLE}_\kappa$ and whole-plane space-filling $\text{SLE}_\kappa$ as in items (i) and (ii), the $\sigma$-algebras $\mathcal{F}_{0+}$ and $\mathcal{G}$ are trivial (in the sense that they contain only events of probability 0 and 1).

Proposition 3.15. For two-sided whole-plane $\text{SLE}_\kappa$ and whole-plane space-filling $\text{SLE}_\kappa$ as in items (i) and (ii), the $\sigma$-algebras $\mathcal{F}_{0+}$ and $\mathcal{G}$ are trivial (in the sense that they contain only events of probability 0 and 1).

Proof. Denote by $s(t)$ the logarithmic capacity of $\eta[0,t]$, and write $\tilde{\mathcal{F}}_u := \mathcal{F}_{s^{-1}(u)}$. Recall that $e^{s(t)}$ is comparable to $\text{diam}(\eta[0,t])$. Note also that $s^{-1}(u) = \text{Cont}(\hat{\eta}[0,u])$ where $\hat{\eta}$ is $\eta$ parametrised by capacity.

We claim that $\tilde{\mathcal{F}}_{-\infty+} = \mathcal{F}_{0+}$. Let $A \in \tilde{\mathcal{F}}_{-\infty+}$. We want to show that $A \in \mathcal{F}_t$ for any $t > 0$. Since $s(t) \downarrow -\infty$ as $t \downarrow 0$, we can write $A = \bigcup_{u \in \mathbb{Z}} A \cap \{s^{-1}(u) \leq t\}$. By definition, since $A \in \tilde{\mathcal{F}}_u$ for any $u$, we have $A \cap \{s^{-1}(u) \leq t\} \in \mathcal{F}_t$, and hence $A \in \mathcal{F}_t$. The other inclusion is analogous.

For whole-plane space-filling $\text{SLE}_\kappa$, it is shown in [HS18, Lemma 2.2] that for a whole-plane GFF modulo $2\pi \chi$, the $\sigma$-algebra $\bigcap_{t>0} \sigma(h|_{B(0,r)})$ is trivial. Since Lemma 2.5 implies $\tilde{\mathcal{F}}_{-\infty+} \subseteq \bigcap_{t>0} \sigma(h|_{B(0,r)})$, the former is also trivial.

We now prove the proposition for two-sided whole-plane $\text{SLE}_\kappa$, or rather whole-plane $\text{SLE}_\kappa(2)$ since we are restricting to $t \geq 0$. Let $\hat{\eta}$ denote whole-plane $\text{SLE}_\kappa(2)$ parametrised by capacity. Since initial segments of $\hat{\eta}$ and $\eta$ determine each other, we have the identity
\( \hat{F}_{-\infty} = \bigcap_{u \in \mathbb{R}} \sigma(\hat{\eta}(u'), u' \leq u) \). We show that \( \hat{F}_{-\infty} \) is independent of \( \sigma(\hat{\eta}(u), u \in \mathbb{R}) \) which in particular implies that \( \hat{F}_{-\infty} \) is independent of itself, and therefore trivial.

Fix arbitrary \( u_1 < ... < u_r \), and let \( g \) be some bounded continuous function. From the Markov property of the driving process and [Law05, Lemma 4.20], it follows that

\[
\mathbb{E}[g(\hat{\eta}(u_1), ..., \hat{\eta}(u_r)) \mid \hat{F}_u] \rightarrow \mathbb{E}[g(\hat{\eta}(u_1), ..., \hat{\eta}(u_r))] \quad \text{as } u \downarrow -\infty.
\]

On the other hand, by backward martingale convergence, we also have

\[
\mathbb{E}[g(\hat{\eta}(u_1), ..., \hat{\eta}(u_r)) \mid \hat{F}_u] \rightarrow \mathbb{E}[g(\hat{\eta}(u_1), ..., \hat{\eta}(u_r)) \mid \hat{F}_{-\infty}] \quad \text{as } u \downarrow -\infty
\]

which implies that \( \sigma(\hat{\eta}(u_1), ..., \hat{\eta}(u_r)) \) is independent of \( \hat{F}_{-\infty} \). Since this is true for any choice of \( u_1 < ... < u_r \), we must have that \( \sigma(\hat{\eta}(u), u \in \mathbb{R}) \) is independent of \( \hat{F}_{-\infty} \).

For the triviality of \( \mathcal{G} \) we show the following statement: Denote by \( \tilde{\eta} : (-\infty, \infty) \rightarrow \mathbb{C} \) the time-reversal of \( \eta \), parametrised by log conformal radius of its complement relative to the origin, and by \( \hat{F}_{-\infty} \) the infinitesimal \( \sigma \)-algebra of \( \tilde{\eta} \). Then we claim

\[
\mathcal{G} = \hat{F}_{-\infty} \quad \text{(modulo null sets)}.
\]

This will imply the triviality of \( \mathcal{G} \) since for whole-plane \( \text{SLE}_6(2) \), the reversibility and the previous step imply \( \hat{F}_{-\infty} \) is trivial, and for space-filling \( \text{SLE}_\kappa \), we have triviality of \( \bigcap_{R>0} \sigma(h|_{\mathbb{C} \setminus B(0,R)}) \) by [HS18, Lemma 2.2].

Now we show (13). The inclusion \( \hat{F}_{-\infty} \subseteq \mathcal{G} \) is easy to see. Indeed, for any fixed \( t_0 \) the time reversals (parametrized by log conformal radius as before) of \( \eta \) and \( \eta(t_0 + \cdot) \) agree until hitting some circle \( \partial B_R(0) \) (with \( r \) random and depending on \( t_0 \)). We are left to show \( \mathcal{G} \subseteq \hat{F}_{-\infty} \). For any \( A \in \mathcal{G} \) and \( R > 0 \), we need to show \( A \in \hat{F}_R \), where \( \hat{F}_R \) is the \( \sigma \)-algebra generated by \( \tilde{\eta} \) up to the first hitting of circle \( \partial B_R(0) \) (recall that conformal radius and radius are comparable up to a factor). Note that for any two curves \( \tilde{\eta}_1, \tilde{\eta}_2 \) starting at \( \infty \) that agree until hitting circle \( \partial B_R(0) \), their time-reversals agree after their last exit of \( B_R(0) \). Consequently, when we parametrise their time-reversals by Minkowski content (denoted by \( \eta_1, \eta_2 \)), they will agree up to a time-shift after their last exit of \( B_R(0) \). In particular, \( 1_A(\eta_1) = 1_A(\eta_2) \). This implies \( \mathbb{P}(A \mid \hat{F}_R) \) takes only the values 0, 1, or equivalently \( A \in \hat{F}_R \) (modulo null sets).

**Remark 3.16.** We believe that also the tail \( \sigma \)-algebra \( \bigcap_{t>0} \sigma(\eta(s), s > t) \) is trivial. Proving this requires extra work since the cumulative Minkowski content and hence the parametrisation of \( \eta \) at large times does depend on its initial part. An interesting consequence of the tail triviality would be that the measure-preserving maps \( T_{t_0} : \eta \mapsto \eta(t_0 + \cdot) - \eta(t_0) \) (now seen as paths on \( t \in \mathbb{R} \)) are ergodic, i.e. any event \( A \in \sigma(\eta) \) that is invariant under \( T_{t_0} \) has probability 0 or 1.

The above proposition implies that the limits \( c_0, c_1 \) in Theorem 1.3 are deterministic. We now show that the limits \( c_0 \) in Theorems 1.1 and 1.2 are also deterministic.

**Proposition 3.17.** There exist deterministic constants \( c_0, c_1 \) (possibly 0) such that almost surely the following identities hold for any non-trivial bounded interval \( I \subseteq \mathbb{R} \).

\[
(i) \lim_{\delta \downarrow 0} \sup_{s,t \in I, |t-s|<\delta} \frac{|\eta(t) - \eta(s)|}{|t-s|^{1/d}(\log |t-s|^{-1})^{-1-1/d}} = c_0.
\]

\[
(ii) \lim_{\delta \downarrow 0} \sup_{t_i \subseteq I, \delta < |t_i+1-t_i| < \delta} \sum_i \psi(|\eta(t_{i+1}) - \eta(t_i)|) = c_1|I| \quad \text{where } \psi(x) = x^d(\log \log \frac{1}{x})^{-(d-1)}.
\]

**Proof.** (i) For an interval \( I \) define

\[
\delta_I := \lim_{\delta \downarrow 0} \sup_{t,s \in I, |t-s|<\delta} \frac{|\eta(t) - \eta(s)|}{|t-s|^{1/d}(\log |t-s|^{-1})^{-1-1/d}}.
\]
The law of $S_I$ is independent of the choice of $I$ by scale-invariance in law of $\eta$ and since for any fixed $a > 0$ it holds that $\log((a \Delta t)^{-1}) = (1 + o(1)) \log((\Delta t)^{-1})$ as $\Delta t \to 0$.

We claim that $S_I = S_{I'}$ almost surely for any two intervals $I, I'$. Indeed, in case $I \subseteq I'$, we have $S_I \leq S_{I'}$ by the definition of $S_I$. But then the two random variables can only have the same law if they are already surely equal. For general $I, I'$ apply the argument iteratively.

It follows that $S_{[0,t]} = S_{[0,1]}$ almost surely for every $t$. Letting $t \downarrow 0$, we see that $S_{[0,1]}$ is (up to null sets) measurable with respect to $\mathcal{F}_{0+}$, and therefore deterministic by Proposition 3.15.

(ii) Let

$$V_I := \lim_{\delta \downarrow 0} \sup_{(t_i) \subseteq I, |t_{i+1} - t_i| < \delta} \sum_{i} \psi(|\eta(t_{i+1}) - \eta(t_i)|).$$

We claim that $V_I = |I|V_{[0,1]}$ almost surely. This will imply $V_{[0,t]} = tV_{[0,1]}$ for all rational $t$, and by continuity for all $t$. As before, we conclude that $V_{[0,1]}$ is measurable with respect to $\mathcal{F}_{0+}$, and therefore deterministic.

Note that $V$ is additive, i.e.

$$V_{[t_1,t_2]} + V_{[t_2,t_3]} = V_{[t_1,t_3]} \quad \text{for } t_1 < t_2 < t_3.$$  

Moreover, by scaling and translation-invariance of $\eta$, the random variable $V_I$ has the same law as $|I|V_{[0,1]}$. Therefore the claim follows from the lemma below. (Note that we have shown in the previous subsections that $V$ has exponential moments.)

**Lemma 3.18.** Let $X, Y, Z$ be random variables with the same law and finite second moments. If $\lambda X + (1 - \lambda)Y = Z$ for some $\lambda \neq 0, 1$, then $X = Y = Z$ a.s.

**Proof.** We have

$$\mathbb{E}[Z^2] = \lambda^2 \mathbb{E}[X^2] + (1 - \lambda)^2 \mathbb{E}[Y^2] + 2\lambda (1 - \lambda) \mathbb{E}[XY]$$

and hence (using $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = \mathbb{E}[Z^2] < \infty$)

$$\mathbb{E}[Z^2] = \mathbb{E}[XY] \leq (\mathbb{E}[X^2] \mathbb{E}[Y^2])^{1/2} = \mathbb{E}[Z^2],$$

i.e. the Cauchy-Schwarz inequality holds with equality. This means $X, Y$ are linearly dependent. 

The claim follows.

**Proposition 3.19.** The assertions of Theorems 1.1–1.3 hold, except that the constants $c_0, c_1$ may take their value in $[0, \infty)$ (instead of $(0, \infty)$).

**Proof.** By the results of this subsection there exist deterministic constants $c_0, c_1 \in [0, \infty]$ as in the theorem statements. Proposition 3.10 implies that (7) is satisfied with $\Phi$ as in Example 3.5. Theorems 3.1–3.3 imply that $c_0, c_1 < \infty$, along with the last indented equations of Theorems 1.1 and 1.2.

**3.5 Proof of Theorem 1.7**

In essence, we follow the proof of Theorem 3.1 using the stronger input given in Remark 3.12. By stationarity, it suffices to prove the result on the interval $I = [0, 1]$.

Denote by $A_{s,r}^{(k)}$ the event that $\eta[s, \tau_{s,r}]$ contains $k$ disjoint balls of radius $\ell$ where $\tau_{s,r} = \inf\{t > s \mid |\eta(t) - \eta(s)| \geq r\}$. For $n \in \mathbb{N}$, by Remark 3.12,

$$\mathbb{P}\left(\left(\sum_{s=0}^{n-1/2} A_{s,u,n/2}^{(-1, -1/2, 2 \Delta u^2 n)}\right)^c\right) \leq \exp(-c_2 u^2 n).$$

Summing over $s = j \pi \delta 2^{-n}$, $j = 0, \ldots, (\pi \delta)^{-1} 2^n - 1$, yields

$$\mathbb{P}\left(\bigcup_{s=0}^{n-1/2} A_{s,u,n/2}^{(-1, -1/2, 2 \Delta u^2 n)}\right)^c \leq 2^n \exp(-c_2 u^2 n) \leq \exp(-c_2 u^2 n)$$

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for sufficiently large \( u \). Summing over \( n \geq n_0 \) yields

\[
\mathbb{P} \left( \bigcup_{n \geq n_0} \bigcup_{s = jn\delta^2 - n} \left( A_{s, u^{2-n/2}n^{1/2}}^{u^{-1/2-n/2}n^{-1/2}, \delta u^2 n^2} \right) \right) \lesssim \exp(-\tilde{c}_2 u^2 n^2)
\]

for sufficiently large \( u \).

Let \( r \in (0, 1) \), and pick \( n_0 \in \mathbb{N} \) such that \( r \approx 2^{-n_0/2}n_0^{1/2} \). Then the estimate above reads

\[
\mathbb{P}(\ldots) \lesssim r^{\tilde{c}_2 u^2}.
\]

We claim that

\[
\mathbb{P}(\bigcup_{n \geq n_0} \bigcap_{s = jn\delta^2 - n} A_{s, u^{2-n/2}n^{1/2}}^{u^{-1/2-n/2}n^{-1/2}, \delta u^2 n^2} \subseteq E_{r,u,\{0,1\}}).
\]

Suppose \( |\eta(t) - \eta(s)| \leq ur \). Find \( n \geq n_0 \) such that \( \frac{|\eta(t) - \eta(s)|}{u^{2-n/2}n^{1/2}} \in [4, 8] \). Note that on the event \( A_{s, u^{2-n/2}n^{1/2}}^{u^{-1/2-n/2}n^{-1/2}, \delta u^2 n^2} \), we have \( \tau_{s,u^{2-n/2}n^{1/2}} > \pi \delta^2 - n^2 \) and hence \( \text{diam}(\eta[s, s + \pi \delta^2 - n^2]) \leq 2u^{2-n/2}n^{1/2} \) since \( \eta \) is parametrised by area. Therefore, by our choice of \( n \) we must have \( s \leq \pi \delta^2 - n < (j + 1) \pi \delta^2 - n \leq t \) for some \( j \). In particular, \( \eta[s, t] \supseteq \eta[j \pi \delta^2 - n, (j + 1) \pi \delta^2 - n] \) contains \( \delta u^2 n \approx \delta u^2 \log(u|\eta(t) - \eta(s)|^{-1}) \) disjoint balls of radius \( u^{-1/2-n/2}n^{-1/2} \approx u^{-2}|\eta(t) - \eta(s)|^{-1} \).

### 4 Lower bounds for Markov processes

We prove lower bounds on the regularity (corresponding to the lower bounds in Theorems 1.1–1.3) for Markov processes satisfying a uniform ellipticity condition. The arguments are elementary but they illustrate well the general idea on how to obtain lower bounds, and we have not seen them written out in earlier literature, except that (even functional versions of) laws of the iterated logarithms and rates of escape of Markov processes have been proved in [BK00]. Our arguments on SLE follow the same idea, but will be more technical since SLE is not exactly a Markov process, and we need to work with its domain Markov property.

In the following, let \( X = (X_t)_{t \geq 0} \) be a Markov process on a metric space \( (E, d) \), and let \( \mathbb{P}^x \) denote the law of the Markov process started at \( X_0 = x \). In particular, we assume the Markov property \( \mathbb{P}^x(X_{t+s} \in A \mid \mathcal{F}_s) = \mathbb{P}^{X_t}(X_s \in A) \) for every \( x \). We suppose the following uniform bounds on the transition probabilities: There exist constants \( d_w > 1, T > 0, r_0 > 0 \) such that

\[
\mathbb{P}^x(d(X_t, x) > r) \leq c_1 \exp \left( -c_2 \left( \frac{r}{T^{1/d_w}} \right)^{d_w/(d_w-1)} \right) \quad (14)
\]

for all \( r > 0, 0 < t \leq T \), and

\[
\mathbb{P}^x(d(X_t, x) > r) \geq c_3 \exp \left( -c_4 \left( \frac{r}{T^{1/d_w}} \right)^{d_w/(d_w-1)} \right) \quad (15)
\]

for \( r \leq r_0, 0 < t < r^{d_w} \). The exponent \( d_w \) is usually called the walk dimension. For instance, this is satisfied for diffusions on \( \mathbb{R}^d \) with uniformly elliptic generator (for which \( d_w = 2 \)). Other typical examples are Brownian motions on fractals (cf. [BK00] and references therein) or Liouville Brownian motion (cf. [AKM]).

For these Markov processes, the analogues of Theorems 1.1–1.3 hold with \( d = d_w \), only that we do not prove 0-1 laws for the limits but only deterministic upper and lower bounds for them (but see e.g. [BK00] for a type of 0-1 law). We only need to prove the lower bounds since the matching upper bounds follow already from the results in Section 3.1. The upper bounds hold for general stochastic processes and the Markov property is not needed.
Proposition 4.1. Under assumption (15) there exist positive deterministic constants \(a_1, a_2, a_3 > 0\) such that the following is true.

(i) Variation: For any bounded interval \(I \subseteq \mathbb{R}^+\), almost surely
\[
\inf_{\delta > 0} \sup_{|t_{i+1} - t_i| < \delta} \sum_i \psi(d(X_{t_{i+1}}, X_{t_i})) \geq a_3|I|
\]
with \(\psi(x) = x^{d_w} (\log^* \log^* x)^{-(d_w - 1)}\), and the supremum is taken over finite sequences \(t_0 < ... < t_r\) with \(t_i \in I\) and \(|t_{i+1} - t_i| < \delta\).

(ii) Modulus of continuity: For any non-trivial interval \(I \subseteq \mathbb{R}^+\), almost surely
\[
\inf_{\delta > 0} \sup_{s, t \in I, |t - s| < \delta} \frac{d(X_t, X_s)}{|t - s|^{1/d_w} (\log |t - s|)^{1 - 1/d_w}} \geq a_2.
\]

(iii) Law of the iterated logarithm: For any \(t \geq 0\), almost surely
\[
\limsup_{t \uparrow 0} \frac{d(X_{t_0 + t}, X_{t_0})}{t^{1/d_w} (\log \log t^{-1})^{1 - 1/d_w}} \geq a_1.
\]

Proof. (ii): By the Markov property, there is no loss of generality assuming \(I = [0, 1]\). For \(\varepsilon > 0\) and \(k = 1, ..., \lfloor \varepsilon^{-1} \rfloor\), we define the event
\[A_{\varepsilon, k} = \{d(X_{\varepsilon k}, X_{\varepsilon(k-1)}) \geq a_0 e^{1/d_w} (\log \varepsilon^{-1})^{1 - 1/d_w}\} \in \mathcal{F}_{\varepsilon k}\]
with \(a_0 > 0\) a constant whose value will be decided upon later. By (15) and the Markov property, we have
\[\mathbb{P}(A_{\varepsilon, k} | \mathcal{F}_{\varepsilon(k-1)}) \geq c_3 \exp \left(-c_4 a_0^{d_w/(d_w - 1)} (\log \varepsilon^{-1})\right) = c_3 \varepsilon^{1/2}\]
for a suitable choice of \(a_0\). Applying this estimate iteratively yields
\[\mathbb{P}(A_{\varepsilon, 1} \cap ... \cap A_{\varepsilon, \lfloor \varepsilon^{-1} \rfloor}) \leq \left(1 - c_3 \varepsilon^{1/2}\right)^{\lfloor \varepsilon^{-1} \rfloor} \leq \exp \left(-c_3 \varepsilon^{-1/2}\right) \to 0 \quad \text{as} \ \varepsilon \downarrow 0.
\]
This shows that for any fixed \(\delta > 0\), the event
\[\sup_{s, t \in I, |t - s| < \delta} \frac{d(X_t, X_s)}{|t - s|^{1/d_w} (\log |t - s|)^{1 - 1/d_w}} \geq a_0\]
must occur with probability 1. The claim follows.

(iii): By the Markov property, there is no loss of generality assuming \(t_0 = 0\). Define a sequence of events
\[A_k = \{d(X_{e^{-k}}, X_{e^{-(k+1)}}) \geq a_0 e^{-k/d_w} (\log k)^{1 - 1/d_w}\} \in \mathcal{F}_{e^{-k}}\]
with a constant \(a_0 > 0\) whose value will be decided upon later. We show that almost surely \(A_k\) occur infinitely often. This implies the claim since on the event \(A_k\) we have
\[d(X_{e^{-k}}, X_0) + d(X_{e^{-(k+1)}}, X_0) \geq a_0 e^{-k/d_w} (\log k)^{1 - 1/d_w}\]
and hence for either \(t = e^{-k}\) or \(t = e^{-(k+1)}\) we have
\[d(X_t, X_0) \geq a_0 t^{1/d_w} (\log \log t^{-1})^{1 - 1/d_w}.
\]
By (15) and the Markov property, we have
\[\mathbb{P}(A_k | \mathcal{F}_{e^{-(k+1)}}) \geq c_3 \exp \left(-c_4 a_0^{d_w/(d_w - 1)}(1 - e^{-1})^{1/(d_w - 1)}(\log k)\right) = c_3 k^{-1}\]
for a suitable choice of $a_0$. Applying this estimate iteratively yields

$$\mathbb{P}(A_k^c \cap \ldots \cap A_{k'}^c) \leq (1 - c_3 k^{-1}) \ldots (1 - c_3 (k')^{-1}) \leq \exp(-c_3 (k^{-1} + \ldots + (k')^{-1})) \to 0 \text{ as } k' \to \infty$$

and hence

$$\mathbb{P}\left( \bigcup_{k' > k} A_{k'} \right) = 1.$$  

Since this holds for any $k$, the claim follows.

(i): This follows from (iii) by a general result which we will state as Proposition 5.1 in the next section.

5 Lower bounds for SLE

In this section we conclude the lower bounds in our main results (Theorems 1.1–1.3). We begin in Section 5.1 by reviewing a general argument saying that the lower bound for $\psi$-variation follows from the lower bound in the law of the iterated logarithm. In Sections 5.2 and 5.3, which constitute the main part of this section, we prove the lower bound in (2) along with some conditional variants of this estimate. Finally, we use these in Section 5.4 to conclude the lower bounds for the modulus of continuity and the law of the iterated logarithm.

5.1 Law of the iterated logarithm implies variation lower bound

We review an argument for general processes that says that a "lower" law of the iterated logarithm implies a lower bound on the variation regularity. We follow [FV10, Section 13.9] where the argument is spelled out for Brownian motion (implying Taylor’s variation [Tay72]).

**Proposition 5.1.** Let $(X_t)_{t \in [0,1]}$ be a separable process such that for every fixed $t \in (0,1)$ we almost surely have

$$\lim_{s \to 0} \sup_{t} \frac{|X_{t+s} - X_t|}{\sigma(|s|)} > 1$$

where $\sigma$ is a (deterministic) self-homeomorphism of $[0, \infty)$. Then, almost surely, for any $\varepsilon > 0$ there exist disjoint intervals $[t_i, u_i]$ of length at most $\varepsilon$ such that

$$\sum_i \sigma^{-1}(|X_{t_i} - X_{u_i}|) > 1 - \varepsilon.$$

The proposition proves in particular that $X$ cannot have better $\psi$-variation regularity than $\psi = \sigma^{-1}$, i.e., $X$ has infinite $\psi$-variation if $\psi(x)/\psi(x) \to \infty$ as $x \downarrow 0$.

**Proof.** Let $E$ be the set of $t \in [0,1]$ where (16) holds. By Fubini’s theorem, we almost surely have $|E| = 1$. By definition, for each $t \in E$, there exist arbitrarily small $s$ such that $|X_{t+s} - X_t| > \sigma(|s|)$. The collection of all such intervals $[t,t+s]$ form a Vitali cover of $E$ in the sense of [FV10, Lemma 13.68]. In particular, there exist disjoint intervals $[t_i, u_i]$ with $|E \setminus \bigcup_i [t_i, u_i]| < \varepsilon$ and $|X_{t_i} - X_{u_i}| > \sigma(|t_i - u_i|)$, so

$$\sum_i \sigma^{-1}(|X_{t_i} - X_{u_i}|) > \sum_i |t_i - u_i| > 1 - \varepsilon.$$  

By picking in the Vitali cover only intervals of length at most $\varepsilon$, we get intervals $[t_i, u_i]$ of length at most $\varepsilon$. 

\[\square\]
5.2 Diameter lower bound for non-space-filling SLE

In this section we prove the matching lower bound to the result in Proposition 3.10. Our main result is the following proposition, together with a “conditional” variant of it, see Proposition 5.15 below. As before we let \( \tau_r = \inf\{ t \geq 0 : |\eta(t)| = r \} \) denote the hitting time of radius \( r \).

**Proposition 5.2.** Let \( \eta \) be a whole-plane \( \text{SLE}_\kappa(2) \) from 0 to \( \infty \), \( \kappa \leq 8 \). For some \( \tilde{c}_2 > 0 \) we have

\[
\mathbb{P}(\text{Cont}(\eta[0, \tau_r]) < \ell) \geq \exp(-\tilde{c}_2 \ell^{-1/(d-1)} r^{d/(d-1)})
\]

for any \( r, \ell > 0 \) with \( \ell \leq r^d \).

This is the matching lower bound to the upper bound in Proposition 3.10. Note that our upper and lower bounds match except for a different constant \( \tilde{c}_2 \).

We remark that the proposition can be equivalently stated as a tail lower bound on the increment of \( \eta \) when parametrised by Minkowski content. Namely, we have

\[
\mathbb{P}(\text{diam}(\eta[0, 1]) > r) \geq \exp(-c r^{d/(d-1)})
\]

for \( r \geq 1 \).

Similarly as in Section 3.3, we can use the scaling property to reduce the statement of Proposition 5.2 to the following special case.

**Lemma 5.3.** Let \( \eta \) be a whole-plane \( \text{SLE}_\kappa(2) \) from 0 to \( \infty \). Then for some \( c_1, c_2 > 0 \) we have

\[
\mathbb{P}(\text{Cont}(\eta[0, \tau_r]) < c_1 r) \geq \exp(-c_2 r)
\]

for any \( r \geq 1 \).

**Proof of Proposition 5.2 given Lemma 5.3.** Let \( \lambda = c_1^{1/(d-1)} \ell^{-1/(d-1)} r_1^{d/(d-1)} \). By the scaling property (cf. Section 2.2), \( \tilde{\eta} = \lambda \eta \) is also a whole-plane \( \text{SLE}_\kappa(2) \), and \( \text{Cont}(\tilde{\eta}[0, \tau_{\lambda r}]) = \lambda^d \text{Cont}(\eta[0, \tau_r]) \). Hence the desired probability is equal to

\[
\mathbb{P}(\text{Cont}(\tilde{\eta}[0, \tau_{\lambda r}]) < \lambda^d \ell).
\]

We have chosen \( \lambda \) such that \( \tilde{r} := \lambda r = c_1^{1/(d-1)} \ell^{-1/(d-1)} r^{d/(d-1)} \) and \( \lambda^d \ell = c_1 \tilde{r} \). Therefore, by Lemma 5.3, the probability is at least

\[
\exp(-c_2 \tilde{r}) = \exp(-\tilde{c}_2 \ell^{-1/(d-1)} r^{d/(d-1)}).
\]

The heuristic idea of Lemma 5.3 is straightforward, but requires some care to implement. We would like to show that when \( \eta \) crosses from radius \( r \) to \( r + 1 \), it has some chance \( p > 0 \) to do so while creating at most \( c_1 \) Minkowski content. If \( p > 0 \) is uniform conditionally on \( \eta[0, \tau_r] \), the lemma would follow. However, it is difficult to control the Minkowski content near the tip of \( \eta[0, \tau_r] \) due to its fractal nature, therefore we want to keep only those curves that have a sufficiently “nice” tip when hitting radius \( r \). Below we will implement this idea.

Let \( (D, a) \) be as in (3), and \( u \in \partial D \setminus \{a\} \). The point \( u \) will play the role of a force point with weight 2. (We can allow more general force points but to keep notation as simple as possible, we restrict to this case.) For \( p_N, r_N, c_N > 0 \) we say that \( (D, a, u) \) is \( (p_N, r_N, c_N) \)-nice if

\[
|f(u) - 1| \geq 2r_N
\]

and

\[
\nu_{\Delta, 1 \rightarrow 0}(\text{Cont}(f^{-1}(\eta[0, \sigma_{r_N}])) > c_N) < p_N
\]

where \( \nu_{\Delta, 1 \rightarrow 0} \) denotes radial \( \text{SLE}_\kappa \) (without force point) and \( \sigma_{r_N} \) the exit time of \( B(1, r_N) \cap \overline{D} \).
Proposition 5.4. There exist finite and positive constants $p_N, r_N, c_N, p, c$ such that the following is true. Let $(D, a, u)$ with $|a| \geq 1$ be $(p_N, r_N, c_N)$-nice. Then

$$
\nu_{D, a \to \infty; u} \left( \text{Cont}(\eta[0, \tau_{|a|+1}]) < c \quad \text{and} \quad D \setminus \eta[0, \tau_{|a|+1}] \text{ is } (p_N, r_N, c_N)\text{-nice} \right) \geq p.
$$

Lemma 5.5. There exist finite positive constants $p_N, r_N, c_N, p, c_1$ with $\varepsilon < r_N/2$ such that the following is true. Let $(D, a, u)$ with $|a| \geq 1$ be $(p_N, r_N, c_N)$-nice, and let $f: D \to \mathbb{D}$ the corresponding conformal map. Let $A = f(\overline{\mathbb{C}} \setminus B(0, 1 + 1))$, and $\tau_A$ the hitting time, and $\eta$ a radial $\text{SLE}_c(2)$ in $\mathbb{D}$ from 1 to $f(\infty)$ with force point $f(u)$. Then the following event has probability at least $p$.

(i) $\|\gamma - \eta\|_{\infty;[0, \tau_A]} < \varepsilon$ where $\gamma$ denotes the straight line from 1 to 1/64, and $\gamma$ and $\eta$ are parametrised by capacity relative to $-1$ (see Section 2 for the definition of relative capacity).

(ii) $\text{Cont}(\eta[0, \tau_A]) \leq c_1$.

(iii) $\text{Cont}(f^{-1}(\eta[0, \sigma_N])) \leq c_N$.

(iv) $D \setminus f^{-1}(\eta[0, \tau_A])$ is $(p_N, r_N, c_N)$-nice.

This lemma will be proved in several steps where we successively pick the constants that we look for. First, we recall that for any $\varepsilon > 0$, the probability of (i) can be bounded from below by some $p_\varepsilon > 0$ (cf. Lemma 5.12 below). The lemma then follows if we can guarantee that the probability of any of the other conditions failing is at most $p_\varepsilon/2$. We will pick all the required constants in a way such that this is true.

We begin by making a few general comments about the conformal maps $f: D \to \mathbb{D}$ corresponding to domains as in (3). Consider $D^* = \{1/z \mid z \in D\} \subseteq \mathbb{C}$ and $g(z) = f(1/z)$. This is the conformal map from $D^*$ to $\mathbb{D}$ with $g(1/z_D) = 0$ and $g(1/a) = 1$. Note that $\text{dist}(1/z_D, \partial D^*) = |1/z_D - 1/a| = \frac{1}{|a|} - \frac{1}{|a|+2}$. Hence, the conformal radius of $1/z_D$ in $D^*$ is between $\frac{2}{|a|(|a|+2)}$ and $\frac{8}{|a|(|a|+2)}$ (cf. Section 2.1). It follows that $|f'(z_D)| \in [\frac{|a|}{8(|a|+2)}, \frac{|a|}{2(|a|+2)}].$

Lemma 5.6. There exists $r_0 > 0$ such that for any $(D, a)$ as in (3) we have $f^{-1}(B(1, r_0) \cap \mathbb{D}) \subseteq B(0, |a| + 1)$.

Proof. When $|a|$ is not too large, we can use Koebe’s distortion theorem to argue that even $f^{-1}(\mathbb{D} \setminus B(0, 1 - r_0)) \subseteq B(0, |a| + 1)$. Indeed, considering $D^* = \{1/z \mid z \in D\} \subseteq \mathbb{C}$ and the conformal map $z \mapsto f(1/z)$ from $D^*$ to $\mathbb{D}$, we see that its derivative is comparable everywhere on $B(0, \frac{1}{|a|+1})$. In particular, it cannot map any of these points anywhere close to $\partial \mathbb{D}$.

When $|a|$ is large, let $\tilde{z} \in \partial B(0, |a| + 1/3)$ be the closest point to $a$. The argument above shows that $f(\tilde{z})$ cannot be too close to $\partial \mathbb{D}$. Let $V$ be the union of $B(\tilde{z}, 2/3) \cap D$ with all points that it separates from $\infty$ in $D$. By Lemma 2.3 there exists a universal constant $r_0$ (independent of $D$) and a path from $f(\tilde{z})$ to 1 (dependent of $D$) whose $r_0$-neighbourhood is contained in $f(V)$. Since $V \subseteq B(0, |a| + 1)$, the claim follows. \hfill $\square$

Lemma 5.7. Given $r_0 > 0$, there exists $R > 0$ such that for any $(D, a)$ as in (3) with $|a| > R$ we have $f(\infty) \notin B(0, 1 - r_0/2)$.

Proof. Consider the conformal map $z \mapsto 1/f^{-1}(z)$ from $\mathbb{D}$ to $D^* = \{1/z \mid z \in D\} \subseteq \mathbb{C}$. We saw right above the statement of Lemma 5.6 that the conformal radius of $1/z_D$ in $D^*$ is comparable to $|a|^{-2}$. Koebe’s distortion theorem implies that the derivative of $z \mapsto 1/f^{-1}(z)$ is comparable to $|a|^{-2}$ on $B(0, 1 - r_0/2)$ (up to some factor depending on $r_0$). But since the distance from $1/z_D = 1/f^{-1}(0)$ to 0 = $1/\infty$ is comparable to $|a|^{-1} \gg |a|^{-2}$, we cannot have $\infty \in f^{-1}(B(0, 1 - r_0/2))$. \hfill $\square$
Lemma 5.8. For any $R > 0$ there exists $\delta > 0$ such that the following is true. Let $(D, a)$ be as in (3) with $|a| \in [1, R]$, and $A = f(\hat{C} \setminus B(0, |a| + 1))$. Then $\text{dist}(f(\infty), \partial A) \geq \delta$.

Proof. Consider the conformal map $z \mapsto f(1/z)$ from $D^* = \{1/z \mid z \in D\}$ to $\mathbb{D}$. By the discussion right above the statement of Lemma 5.6, its derivative at $1/z_D$ is comparable to $|a|(|a| + 2) \geq 1$. Since $|1/z_D| \sim \text{dist}(1/z_D, \partial D^*) \sim 1$ (where the implicit constants depend on $R$), Koebe’s distortion theorem gives that also the derivative at 0 is bounded from below by a constant depending on $R$. Since $\partial A = \{f(1/z) \mid z \in \partial B(0, 1/(|a| + 1))\}$, the claim follows from Koebe’s 1/4-theorem.

Corollary 5.9. There exist finite constants $c' > 0$ and $\varepsilon > 0$ such that the following is true. Let $(D, a)$ be as in (3) with $|a| \geq 1$, and $A = f(\hat{C} \setminus B(0, |a| + 1))$. Let $\gamma$ be the straight line from 1 to 0. Consider either the two SLE$_c$ measures $\nu_{D,1\rightarrow u}$ and $\nu_{D,1\rightarrow f(\infty);u}$, or the two SLE$_c(2)$ measures $\nu_{D,1\rightarrow 0;u}$ and $\nu_{D,1\rightarrow f(\infty);u}$ with the same force point $u \in \partial \mathbb{D}$. Then, on the event $\{||\gamma - \eta||_{[0,\tau_A]} < \varepsilon\}$, the law of $\eta|_{[0,\tau_A]}$ under the two measures are absolutely continuous with density bounded away from $[1/c', c']$.

Proof. By Lemmas 5.7 and 5.8, if $\varepsilon$ is sufficiently small, the point $f(\infty)$ has distance at least $2\varepsilon$ from $\gamma[0, \tau_A]$. Moreover, we have $B(0, 1/64) \subseteq A$ due to Koebe’s 1/4-theorem, so 0 is also bounded away from $\gamma[0, \tau_A]$. Therefore, the claim follows from Lemma 2.4.

Lemma 5.10. Given $r_0 > 0$, there exists $r_1 > 0$ such that the following is true. Let $\alpha: [0, 1] \rightarrow \mathbb{D}$ be a curve with $\alpha(0) = 1$ and $\alpha \subseteq \{|z| \geq 1/64, \text{Re}(z) > 0\}$. Suppose also that $\alpha$ does not leave $B(0, 1 - r_0/4)$ after entering $B(0, 1 - r_0/2)$, and that $\alpha(1)$ is connected to 0 in $B(0, 1 - r_0) \setminus \alpha$. Let $D_\alpha$ denote the connected component of $\mathbb{D} \setminus \alpha$ containing 0 (so in particular $D_\alpha = \mathbb{D} \setminus \alpha$ if $\mathbb{D} \setminus \alpha$ is connected), and let $f_\alpha: D_\alpha \rightarrow \mathbb{D}$ denote the conformal map with $f_\alpha(0) = 0$ and $f_\alpha(\alpha(1)) = 1$. Then $f_\alpha^{-1}(B(1, r_1) \cap \mathbb{D}) \subseteq B(0, 1 - r_0/4)$.

Figure 2: The setup and proof of Lemma 5.10.

Proof. Notice that $\partial B(0, 1 - r_0) \setminus \alpha$ may have several connected components, each of which is an arc of $\partial B(0, 1 - r_0)$. Let $C_{r_0}$ denote the longest arc of $\partial B(0, 1 - r_0) \setminus \alpha$ (i.e. the one to the left in the figure, or, equivalently, the unique arc crossing the negative real axis). By our assumption, it does not separate $\alpha(1)$ from 0 in $D_\alpha$. Therefore $f_\alpha(C_{r_0})$ does not separate 1 from 0.

Next, let $C_{r_0/2}$ denote the longest arc of $\partial B(0, 1 - r_0/2) \setminus \alpha$. Its image $f_\alpha(C_{r_0/2})$ lies in the component of $\mathbb{D} \setminus f_\alpha(C_{r_0})$ that does not contain 0. Let $J_+, J_-$ denote the two arcs of $\partial \mathbb{D}$ that lie between $f_\alpha(C_{r_0})$ and $f_\alpha(C_{r_0/2})$. By considering a Brownian motion in the domain $D_\alpha$ starting from 0, we see that the harmonic measures of both $J_\pm$ seen from 0, and therefore their lengths,
are at least some constant depending on \( r_0 \). (More precisely, these harmonic measures are at least the probability of Brownian motion staying inside \( B(0, 1/64) \cup \{ \text{Re}(z) < 0 \} \) before entering the annulus \( \{ 1 - r_0 < |z| < 1 - r_0/2 \} \) and then making a clockwise (resp., counterclockwise) turn inside the annulus.)

We claim that \( f^{−1}_α(J_+) \subseteq B(0, 1 - r_0/4) \). The result will then follow from the fact that \( f^{−1}_α(z) \) is obtained from integrating \( f^{−1}_α \) on \( \partial \mathbb{D} \) against the harmonic measure seen from \( z \).

By symmetry, it suffices to show the claim for \( J_+ \). Let \( z_1, z_2 \) denote the endpoints of \( f^{−1}_α(J_+) \). By our assumption, any sub-curve of \( α \) from \( z_1 \) to \( z_2 \) stays inside \( B(0, 1 - r_0/4) \). Suppose \( f^{−1}_α(J_+) \) contains some point \( z' \notin B(0, 1 - r_0/4) \). Pick a simple path \( P' \) in \( D_{α} \) that begins with a straight line from \( −1 \) to \( (1 - 2/3 r_0) \), then stays between \( C_{r_0} \) and \( C_{r_0/2} \), and ends at \( z' \). As a consequence of the Jordan curve theorem, \( P' \) separates \( B(0, 1 - r_0/4) \) into at least two components, and the two halves of \( C_{r_0/2} \) lie in different components. Moreover, \( C_{r_0} \) is in the same component as the lower half of \( C_{r_0/2} \). Therefore \( z_1 \) and \( z_2 \) (which are the upper endpoints of \( C_{r_0} \) and \( C_{r_0/2} \) respectively) lie in different components of \( B(0, 1 - r_0/4) \setminus P' \). But since \( P' \subseteq D_{α} \), any sub-curve of \( α \) from \( z_1 \) to \( z_2 \) must avoid \( P' \) which is impossible while staying inside \( B(0, 1 - r_0/4) \).

\[ \square \]

**Lemma 5.11.** Given \( r_0, r_1 > 0 \) there exist \( r_N > 0 \) and \( \tilde{c} > 0 \) such that the following is true.

Let \( (D, a, u) \) be as in (3) with \( |a| \geq 1 \), and \( f: D \to \mathbb{D} \) the corresponding conformal map. Let \( α: [0,1] \to \overline{\mathbb{D}} \) be a curve with \( α(0) = 1 \), \( α(1) \in B(0, 1 - r_0) \), and \( |f^{−1}(α(t))| = |a| + 1 > |f^{−1}(α(t))| \) for \( t < 1 \). Let \( D_α \) be the connected component of \( \mathbb{D} \setminus α \) containing \( 0 \) and suppose \( α(1) \in \partial D_α \). Let \( f_α: D_α \to D \) denote the conformal map with \( f_α(0) = 0 \) and \( f_α(α(1)) = 1 \). Suppose additionally that \( f^{−1}_α(B(1, r_1) \cap \mathbb{D}) \subseteq B(0, 1 - r_0/4) \). If for some \( p_N, c_N > 0 \)

\[ r_{D;1-0}(\text{Cont}(f^{−1}_α(η[0, σ_{r_1}])) > \tilde{c} c_N) < \tilde{c} p_N, \]

then \( (D \setminus f^{−1}(α), f^{−1}(α(1)), u) \) is \((p_N, r_N, c_N)\)-nice.

**Proof.** Write \( \tilde{D} = D \setminus f^{−1}(α) \) and denote by \( \tilde{f}: \tilde{D} \to \mathbb{D} \) the conformal map corresponding to \( \tilde{D} \). Let \( z_\tilde{D} \) and \( z_\tilde{D} \) denote the points on \( \partial B(0, |a| + 2) \) and \( \partial B(0, |a| + 3) \) closest to \( a \) and \( f^{−1}(α(1)) \), respectively.

Observe that \( \tilde{f} \) and \( f \) are related via \( \tilde{f} = h^{−1} \circ f_α \circ f \) where \( h: \mathbb{D} \to \mathbb{D} \) is the conformal map with \( h(\tilde{f}(z_\tilde{D})) = 0 \) and \( h(1) = 1 \). We claim that the distance \( |z_\tilde{D} - z_\tilde{D}| \) is bounded from above by a constant depending on \( r_0 \). Indeed, if we let \( R \) be as in Lemma 5.7, then if \( |a| \geq R \), the distance \( |z_\tilde{D} - f^{−1}(α(1))| \) is bounded via Koebe’s distortion theorem, and hence also \( |z_\tilde{D} - z_\tilde{D}| \).

If \( |a| < R \), then the distance is trivially bounded by \( 2R + 5 \). We conclude that \( |z_\tilde{D} - z_\tilde{D}| \) is bounded.

**Figure 3:** The setup and proof of Lemma 5.11.
Since $|z_D - z_D'|$ is bounded from above (by a constant depending on $r_0$), we get that $\tilde{f}(z_D)$ is bounded away from $\partial \mathbb{D}$, and $|h'| > 1$ on $\mathbb{D}$, with both bounds depending only on $r_0$. Consequently there exists $r_N > 0$ such that $h(B(1, 2r_N) \cap \mathbb{D}) \subseteq B(1, r_1) \cap \mathbb{D}$.

To show that $\tilde{D}$ is nice, we need to consider an SLE$_\kappa$ in $\mathbb{D}$ from 1 to 0, stopped at hitting $\partial B(1, r_N)$. However since $\tilde{f}(z_D)$ is bounded away from $\partial \mathbb{D}$, this law is absolutely continuous with respect to SLE$_\kappa$ from 1 to $\tilde{f}(z_D)$, with Radon-Nikodym derivative bounded in some interval $[\tilde{c}, \tilde{c}^{-1}]$ with $\tilde{c} > 0$ depending on $r_0$ and $r_N$ (after possibly further decreasing $r_N$; cf. Lemma 2.4).

The first condition for niceness, i.e. $|f(u)| \geq 2r_N$ is guaranteed by $f^{-1}_\alpha(h(B(1, 2r_N) \cap \mathbb{D})) \subseteq f^{-1}_\alpha(B(1, r_1) \cap \mathbb{D}) \subseteq B(0, 1 - r_0/4)$, which is away from the boundary. The second condition for niceness is satisfied if

$$\nu_{\mathbb{D}:1 \rightarrow \tilde{f}(z_D)}(\text{Cont}(\tilde{f}^{-1}(\tilde{\eta}[0, \sigma_{r_N}])) > c_N) < \tilde{c} p_N.$$  

Suppose now that $\tilde{\eta}$ is an SLE$_\kappa$ from 1 to $\tilde{f}(z_D)$, so that $h \circ \tilde{\eta}$ has the law of an SLE$_\kappa$ $\eta$ from 1 to 0. By our assumption, with probability at least $1 - \tilde{c} p_N$ we have

$$\text{Cont}(f^{-1}_\alpha(\eta[0, \sigma_{r_1}])) \leq \tilde{c} c_N.$$  

Applying Koebe’s distortion theorem to the map $z \mapsto 1/f^{-1}(z)$ we see that on the set $B(0, 1 - r_0/4) \cap f(D \cap B(0, |a| + 2))$ the derivative $|f^{-1}'|$ is bounded from above by a constant depending on $r_0$. Therefore, using the assumption $f^{-1}_\alpha(\eta[0, \sigma_{r_1}]) \subseteq f^{-1}_\alpha(B(1, r_1) \cap \mathbb{D}) \subseteq B(0, 1 - r_0/4)$ and the transformation rule for Minkowski content (4), such $\eta$ satisfies

$$\text{Cont}(\tilde{f}^{-1}(\tilde{\eta}[0, \sigma_{r_N}])) \leq \text{Cont}(f^{-1} \circ f^{-1}_\alpha(\eta[0, \sigma_{r_1}])) \lesssim \tilde{c} c_N$$  

with a factor depending only on $r_0$. The claim follows if $\tilde{c}$ has been picked small enough. \hfill $\Box$

**Lemma 5.12.** Let $\gamma$ be a simple curve in $\mathbb{D} \setminus \{0\}$ with $h(0) = 1$, and $T \geq 0$ the capacity of $\gamma$ relative to $-1$ (i.e. the half-plane capacity of the curve after mapping to $\mathbb{H}$ as described in Section 2). For any $\varepsilon > 0$ there exists $p_\varepsilon > 0$ such that

$$\nu_{\mathbb{D}:1 \rightarrow 0}(\| \gamma - \eta \|_{\infty:[0,T]} < \varepsilon) \geq p_\varepsilon$$  

where $\gamma$ and $\eta$ are parametrised by capacity.

**Proof.** For chordal SLE from 1 to $-1$ in $\mathbb{D}$, this is [TY20, Proposition 1.4]. The result transfers to radial SLE by absolute continuity (cf. [SW03]). \hfill $\Box$

**Lemma 5.13.** Let $\gamma$ be the straight line from 1 to 0. Let $r_0, r_1, r_N > 0$ be as in Lemmas 5.6, 5.10 and 5.11. For any $\varepsilon \in (0, r_0/4]$ and $p_N > 0$ there exists $c_N > 0$ such that the following is true.

Let $(D, a, u)$ be as in (3) with $|a| \geq 1$, and $A = f(\mathbb{C} \setminus B(0, |a| + 1))$. Then

$$\nu_{\mathbb{D}:1 \rightarrow 0} \left( \| \gamma - \eta \|_{\infty:[0,\tau_A]} < \varepsilon \quad \text{but the domain} \quad D \setminus f^{-1}(\eta[0, \tau_A]) \quad \text{is not} \quad (p_N, r_N, c_N)\text{-nice} \right) < p_N$$  

where $\tau_A$ denotes the hitting time of $A$.

**Proof.** We have $B(0, 1/64) \subseteq A$ and $A \cap B(1, r_0) = \emptyset$ due to Koebe’s 1/4-theorem and Lemma 5.6. Since the Minkowski content of radial SLE$_\kappa$ (stopped before entering $B(0, 1/128)$) is almost surely finite, we can find for any $p_2 > 0$ some $c_2 > 0$ such that

$$\nu_{\mathbb{D}:1 \rightarrow 0}(\text{Cont}(\eta[0, \tau_{\partial B(0,1/128)}]) > c_2) < p_2.$$  

This gives

$$\nu_{\mathbb{D}:1 \rightarrow 0} \left( \nu_{\mathbb{D}:1 \rightarrow 0}(\text{Cont}(\eta[0, \tau_{\partial B(0,1/128)}]) > c_2 \mid \eta[0, \tau_A]) > p_2^{1/2} \right) < p_2^{1/2}$$  

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by Markov’s inequality since
\[ \nu_{D;1 \to 0} \left( \nu_{D;1 \to 0}(\text{Cont}(\eta[0, \tau_B(0.1/128)]) > c_2 \mid \eta[0, \tau_A]) \right) \]
\[ = \nu_{D;1 \to 0}(\text{Cont}(\eta[0, \tau_B(0.1/128)]) > c_2) < p_2. \]

Now, suppose \( \| \gamma - \eta \|_{\infty;[0,\tau_A]} < \varepsilon \). Apply the above with \( p_2 = \tilde{c}^2 p_N^2 \), and let \( c_N = \tilde{c}^{-1} c_2 \). We claim that if 
\[ \nu_{D;1 \to 0}(\text{Cont}(\eta[\tau_A, \tau_B(0.1/128)]) > c_2 \mid \eta[0, \tau_A]) < p_2^{1/2}, \]
then \( D \setminus f^{-1}(\eta[0, \tau_A]) \) is \((p_N, r_N, c_N)\)-nice with our choices of \( p_N, c_N \).

Indeed, conditionally on \( \eta[0, \tau_A] \), the curve \( \tilde{\eta} = f_{\tau_A} \circ \eta|_{[\tau_A, \infty)} \) is an independent \( \text{SLE}_\kappa \) in \( \mathbb{D} \). Since the capacity of \( \eta[0, \tau_B(0.1/128)] \) is much larger than of \( \eta[0, \tau_A] \), we must have \( f_{\tau_A}^{-1}(\tilde{\eta}[0, \sigma_{\tau_A}]) \subseteq \eta[\tau_A, \tau_B(0.1/128)] \) (there is no loss of generality assuming \( r_1 \) is small). Moreover, by Lemma 5.10 (the conditions are satisfied due to \( \| \gamma - \eta \|_{\infty;[0,\tau_A]} < \varepsilon < r_0/4 \)), we have \( f_{\tau_A}^{-1}(B(1, r_1) \cap \mathbb{D}) \subseteq B(0, 1 - r_0/4) \). Hence, by Lemma 5.11, \( \tilde{D} \) is \((p_N, r_N, c_N)\)-nice. □

**Lemma 5.14.** Let \( r_N > 0 \) be as in Lemma 5.13. For any \( p_N > 0 \) there exist \( c_N > 0 \) and \( p_1 > 0 \) such that the following is true. Let \((D, a, u)\) be as in (3) with \(|a| \geq 1\). Then

\[ \nu_{D,a \to \infty;u} \left( \text{the domain } \hat{\mathbb{C}} \setminus \eta[0, \tau_{|a|+1}] \text{ is } (p_N, r_N, c_N)\text{-nice} \right) \geq p_1. \]

In particular,

\[ \nu_{\hat{\mathbb{C}};0 \to \infty;0} \left( \text{the domain } \hat{\mathbb{C}} \setminus \eta[0, \tau_R] \text{ is } (p_N, r_N, c_N)\text{-nice} \right) \geq p_1 \]

for \( R \geq 2 \).

**Proof.** The second assertion follows from the first assertion by considering \( D = \hat{\mathbb{C}} \setminus \eta[0, \tau_{R-1}] \), so in the remainder of the proof we will focus only on proving the first assertion.

Let us suppose first that \( |f(u) - 1| \geq \delta \) for some \( \delta > 0 \). In that case, the claim follows from Lemmas 5.12 and 5.13 and the absolute continuity between SLE variants, cf. Lemma 2.4 and Corollary 5.9 (note that \( \tilde{\eta} = f \circ \eta \) is an \( \text{SLE}_\kappa(2) \) from 1 to \( f(\infty) \) with force point at \( f(u) \)).

In case \( f(u) \) is close to 1, we do not have uniform control over the density between the SLE variants. But picking a small \( t > 0 \) there exists some \( \tilde{p}_1 > 0 \) and \( \tilde{\delta} > 0 \) such that \( |f_t(f(u)) - 1| \geq \tilde{\delta} \) with probability at least \( \tilde{p}_1 \) (independent of \( f(u) \)). This is because \( t \mapsto \text{arg}(f_t(f(u))) \) is a radial Bessel process of positive index started at \( \text{arg}(f(u)) \) (cf. [Zha19, Section 2.1]), and by the monotonicity of Bessel processes in the starting point it suffices to compare to the case when it starts at 0. But a Bessel process stopped at a deterministic time is almost surely positive.

This allows us to consider \( \tilde{\eta}^{(t)} = f_t \circ \tilde{\eta} \), stopped at hitting \( A^{(t)} = f_t(A) \). On the event \( \| \gamma - \tilde{\eta}^{(t)} \|_{\infty;[0,\tau_{A(t)}]} < \tilde{\delta}/2 \), we now do have bounded density between the SLE variants (with a bound depending on \( \tilde{\delta} \)). In order to show \( \hat{\mathbb{C}} \setminus \eta[0, \tau_{|a|+1}] = D \setminus f^{-1}(\tilde{\eta}[0, \tau_A]) = D \setminus f^{-1}(f_t^{-1}(\tilde{\eta}^{(t)}[0, \tau_{A(t)}])) \) is nice with sufficiently positive probability, we can follow the proof of Lemma 5.13 with minor modifications. Instead of \( \tilde{\eta} \) and \( A \), we consider \( \tilde{\eta}^{(t)} \) and \( A^{(t)} \). We write \( f_t^{(s)} = f_{t+s} \circ f_t^{-1} \) for the mapping-out function of \( \tilde{\eta}^{(t)} \).

Following the proof of Lemma 5.13, we get that conditionally on \( \tilde{\eta}[0, t] \) and \( |f_t(f(u)) - 1| \geq \tilde{\delta} \), with probability at least \( p_{\varepsilon, \tilde{\delta}} > 0 \) we have

\[ \| \gamma - \tilde{\eta}^{(t)} \|_{\infty;[0,\tau_{A(t)}]} < \tilde{\delta}/2 \wedge \varepsilon \]

and

\[ \nu_{D;1 \to 0}(\text{Cont}(\tilde{\eta}^{(t)}[\tau_{A(t)}, \tau_B(0.1/128)]) > c_N \mid \tilde{\eta}^{(t)}[0, \tau_{A(t)}]) < \tilde{c} p_N. \]
We claim that on this event, \( D \setminus f^{-1}(\tilde{\eta}[0,\tau_A]) \) is \((p_N, r_N, 2c_N)\)-nice.

Conditionally on \( \tilde{\eta}^{(l)}[0,\tau_A(0)] \), consider an independent \( \text{SLE}_\kappa \) \( \tilde{\eta} \) from 1 to 0, stopped at \( \sigma_{\tau_1} \). As in the proof of Lemma 5.13, \( (f_A^{(l)})^{-1} \circ \tilde{\eta} \) has the same law as a subsegment of \( \tilde{\eta}^{(l)}[\tau_A(0), \tau_{\partial B(0,1/128)}] \), and therefore

\[
\nu_{D:1\to 0}(\text{Cont}((f_A^{(l)})^{-1}(\tilde{\eta}[0,\sigma_{\tau_1}])) > c_N \nu \ln(1/r_N).
\]

We need to map \( \tilde{\eta} \) back by \( f_A^{-1} = f_t^{-1} \circ (f_A^{(l)})^{-1} \). Since \( (f_A^{(l)})^{-1}(\tilde{\eta}[0,\sigma_{\tau_1}]) \subseteq B(0,1 - r_0/4) \) by Lemma 5.10 and \( t \) is small, the derivative \( |(f_t^{-1})'| \) is bounded on \( (f_A^{(l)})^{-1}(\tilde{\eta}[0,\sigma_{\tau_1}]) \), and therefore the transformation rule for Minkowski content, this implies

\[
\nu_{D:1\to 0}(\text{Cont}(f_A^{-1}(\tilde{\eta}[0,\sigma_{\tau_1}])) > 2c_N \nu \ln(1/r_N),
\]

and Lemma 5.11 implies the claim.

**Proof of Lemma 5.5.** We proceed as outlined below the statement of the lemma. First observe that it suffices to show the statement for radial SLE\(_\kappa\) from 1 to 0. Indeed, on the event \( \{\|\gamma - \eta\|_{\infty[0,\tau_A]} < \varepsilon\} \), the laws of the corresponding SLEs are absolutely continuous with density bounded by a constant depending on \( \varepsilon \). This is because \( f(\infty) \) and \( f(u) \) are both at distance at least \( 2\varepsilon \) from \( \gamma[0,\tau_A] \), due to Lemmas 5.7 and 5.8 and the definition of niceness of \( D \), and we may apply Lemma 2.4 to get the desired absolute continuity.

Pick \( r_0, r_1, r_N > 0 \) as in Lemmas 5.6, 5.10 and 5.11. Then pick \( \varepsilon < r_N/2 \), and \( T \) the capacity of the straight line from 1 to 1/64. For this choice of \( \varepsilon \), \( p_{\varepsilon} \) as in Lemma 5.12. Then item (i) occurs with probability at least \( p_{\varepsilon} \).

Since the Minkowski content of radial SLE\(_\kappa\) (stopped before entering \( B(0,1/64) \)) is almost surely finite, we can find \( c_1 > 0 \) such that (ii) fails with probability at most \( p_{\varepsilon}/4 \).

Next, pick \( p_N < p_{\varepsilon}/4 \). Supposing \( D \) is \((p_N, r_N, c_N)\)-nice, the probability of (iii) failing is at most \( p_N < p_{\varepsilon}/4 \).

Finally, given \( r_N, \varepsilon, \) and \( p_N \), we pick \( c_N \) according to Lemma 5.13. This will imply (iv) fails with probability at most \( p_N < p_{\varepsilon}/4 \).

**Proof of Proposition 5.4.** Pick the constants as in Lemma 5.5. As already observed in the proof of Lemma 5.13, we have \( B(0,1/64) \subseteq A \) and \( A \cap B(1, r_0) = \varnothing \). Note that \( f \circ \eta \) is an SLE\(_\kappa\) from 1 to \( f(\infty) \) with force point \( f(u) \), with probability at least \( p \) all the items (i)–(iv) occur for \( f \circ \eta \). In particular, (iv) means \( D \setminus \eta[0,\tau_{\sigma_{+1}}] \) is \((p_N, r_N, c_N)\)-nice.

To bound \( \text{Cont}(\eta[0,\tau_{\sigma_{+1}}]) \), note that due to (i) and \( B(0,1/64) \subseteq A \), \( f \circ \eta \) hits \( A \) before reaching the capacity of \( \gamma \). Moreover, also due to (i), it does not leave \( B(0,1 - r_N/4) \) after \( \tau_{\sigma_{+1}} \). Recalling from the proof of Lemma 5.5 that \( f(\infty) \) is at distance at least \( 2\varepsilon \) from \( \gamma[0,\tau_A] \), we see that \( |(f^{-1})'| \) is bounded on the points of \( (f \circ \eta)[\sigma_{\tau_A}, \tau_A] \) by a constant depending on \( \varepsilon \) and \( \sigma_{\tau_A} \) (by applying Koebe’s distortion theorem on a subdomain avoiding \( f(\infty) \)). Therefore, combining items (ii) and (iii) and the transformation rule for Minkowski content (4), \( \text{Cont}(\eta[0,\tau_{\sigma_{+1}}]) \) is bounded.

**Proof of Lemma 5.3.** It suffices to show this for large integers \( r \). By Lemma 5.14, \( \hat{\mathbb{E}} \setminus \eta[0, \tau_2] \) is \((p_N, r_N, c_N)\)-nice with positive probability. Then, by Proposition 5.4, with probability at least \( p^r \) we have

\[
\text{Cont}(\eta[\tau_2, \tau_{2+r}]) < rc.
\]

Since the Minkowski content of two-sided whole-plane SLE\(_\kappa\) and therefore also of whole-plane SLE\(_\kappa\)(2) is almost surely finite, \( \text{Cont}(\eta[0, \tau_2]) \) is also bounded above by some large constant with probability close to 1. This finishes the proof.

The proof shows also the following statement.

\[
\text{Cont}(\eta[\tau_2, \tau_{2+r}]) < rc.
\]
Proposition 5.15. There exist finite and positive constants \( p_N, r_N, c_N, c, \tilde{c}_2 \) such that the following is true. Let \((D, a, u)\) be as in (3), and \( r, \ell > 0 \) such that \( \ell \leq r^d \). Let \( \lambda = \ell^{-(d-1)} r^{1/(d-1)} \). If \( \lambda D \) is \((p_N, r_N, c_N)\)-nice and \( |a| \geq 1 \), then
\[
\nu_{\lambda D, a \to \infty, u} (\text{Cont}(\eta[0, \tau_{|a|+r}]) < cf) \geq \exp(-\tilde{c}_2 \ell^{-(d-1)} r^{d/(d-1)}).
\]

Proof. By iterating Proposition 5.4 as in the proof of Lemma 5.3, if \( D \subset \mathbb{C} \) is \((p_N, r_N, c_N)\)-nice and \( |a| \geq 1 \), then
\[
\nu_{\lambda D, a \to \infty, u} (\text{Cont}(\eta[0, \tau_{|a|+r}]) < rc) \geq p^r
\]
for any \( r \geq 1 \). For the general statement, we use the same scaling argument as in the proof of Proposition 5.2.

\section{5.3 Diameter lower bound for space-filling SLE}

In this section we will prove the counterparts of Propositions 5.2 and 5.15 in the setting of space-filling SLE, which are stated as Propositions 5.16 and 5.17 below. The proofs follow the exact same structure as in the non-space-filling case and we will therefore be brief and only highlight the differences.

The following is the counterpart of Proposition 5.2. Note in particular that the estimate takes exactly the same form as in the non-space-filling case with \( d = 2 \).

Proposition 5.16. Let \( \eta \) be a whole-plane space-filling \( \text{SLE}_\kappa \) from \(-\infty\) to \( \infty \), \( \kappa > 4 \), satisfying \( \eta(0) = 0 \). For some \( \tilde{c}_2 > 0 \) we have
\[
\mathbb{P} (\text{Cont}(\eta[0, \tau_r]) < \ell) \geq \exp(-\tilde{c}_2 \ell^{-1} r^2)
\]
for any \( r, \ell > 0 \) with \( \ell \leq r^2 \).

One difference between the space-filling and non-space-filling settings is that in the space-filling case, \( \eta|_{[0, \infty)} \) (viewed as a curve in \( \mathbb{C} \)) is not an \( \text{SLE}_\kappa(\rho) \) for any vector \( \rho \), and therefore the curve does not satisfy the domain Markov property, which plays a crucial role in the argument in the previous section. However, via the theory of imaginary geometry the curve does satisfy a counterpart of the domain Markov property if we also condition on a particular triple of marked points on the boundary of the trace, see Section 2.3. While most of the argument in the non-space-filling case carries through using this observation, we need to modify some parts of the argument, e.g. the various absolute continuity arguments comparing different variants of SLE and the proof of Lemma 5.14. There are also some parts of the proof that simplify since \( \text{Cont}(\cdot) \) is equal to Lebesgue area measure, so the natural measure of the curve while staying in a domain is deterministically bounded by the Lebesgue area measure of the domain.

Following the proof in Section 5.2, we start by giving the definition of nice for space-filling SLE. This property is now defined for tuples \((D, a, u)\), \( u = (u_1, u_2, u_3) \in (\partial D)^3 \), satisfying
\[
(D, a) \text{ is as in (3); } a, u_1, u_2, u_3 \text{ are distinct and ordered counterclockwise.}
\]
(17)

For \( r_N, c_N > 0 \) we say that \((D, a, u)\) is \((r_N, c_N)\)-nice if
\[
|f(u_1) - 1| \land |f(u_3) - 1| \geq 2r_N,
\]
and
\[
\text{Cont}(f^{-1}(B(1, r_N) \cap \overline{D})) \leq c_N.
\]
(18)
The second condition is simplified as compared to the non-space-filling case since \( \text{Cont}(\cdot) \) is equal to Lebesgue measure.

The space-filling counterpart of Proposition 5.15 is the following.
Proposition 5.17. There exist finite and positive constants $r_N, c_N, c, \tilde{c}_2$ such that the following is true. Let $(D, a, \mathbf{u})$ be as in (17), and $r, \ell > 0$ such that $\ell \leq r^2$. Let $\lambda = \ell^{-1}r$. If $\lambda D$ is $(r_N, c_N)$-nice and $\lambda |a| \geq 1$, then

$$\tilde{\nu}_{D,a \to \infty;\mathbf{u}}(\text{Cont}(\eta[0, \tau_{|a|+\ell}]) < c\ell) \geq \exp(-\tilde{c}_2\ell^{-1}r^2).$$

We now go through Section 5.2 in chronological order and point out what changes we need to make for the case of space-filling SLE. The counterparts of Lemma 5.3, Proposition 5.4, and Lemma 5.5 in the space-filling case are identical as before, except that we consider whole-plane space-filling SLE$_{\kappa}$ (restricted to the time interval $[0, \infty)$) and $(D, a, \mathbf{u})$, respectively, instead of SLE$_{\kappa}(2)$ and $(D, a, u)$. Proposition 5.2 is deduced from Lemma 5.3 as before via scaling. Lemmas 5.6, 5.7, 5.8, and 5.10 are used in precisely the same form in the space-filling case as in the non-space-filling case; note that these results only concern conformal maps and not SLE. For the counterpart of Corollary 5.9, on the other hand, we modify the statement and the proof as follows. In particular, we do not prove a uniform bound on the Radon-Nikodym derivative in this case.

Lemma 5.18. There exist finite constants $c', \varepsilon > 0$ such that the following is true. Let $(D, a, \mathbf{u})$ be as in (17) with $|a| \geq 1$ and $|f(u_1) - 1| \wedge |f(u_3) - 1| > 2\varepsilon$, and set $u_0 := (-i, -1, i)$. Let $A = f(\overline{C} \setminus B(0, |a| + 1))$ and let $\gamma$ be the straight line from 1 to 0. Then, for any event $E \subset \{||\gamma - \eta||_{[0, \tau_{|a|}]} < \varepsilon\}$ measurable with respect to the curve until time $\tau_A$ and for $\nu_1, \nu_2 \in \{\tilde{\nu}_{D,1-\tau_0;u_0}, \tilde{\nu}_{D,1-\tau_0;u_0}, \tilde{\nu}_{D,1-\tau_0;u_0}, \tilde{\nu}_{D,1-\tau_0;u_0}\}$ it holds that $\nu_1(E) \leq c'\nu_2(E)^{1/2}.$

Proof. Let $h_1, h_2$ be the variants of the GFF associated with the measures $\nu_1, \nu_2$. Then $h_1, h_2$ can be coupled together such that $h_1 = h_2 + g$ for $g$ a harmonic function that is zero along the $2\varepsilon$-neighbourhood of 1 on $\partial D$, and bounded by a constant depending only on $\varepsilon$ on $\{z \notin A : \text{dist}(z, \gamma) < 1.5\varepsilon\}$. The lemma is now immediate by the argument in [HS18, Lemma 2.1].

The space-filling counterpart of Lemma 5.11 is the following, with the same proof as before.

Lemma 5.19. In the setup of Lemma 5.11, if $\text{Cont}(f^{-1}(B(1, \tau_1) \cap \overline{D})) \leq \tilde{c}c_N$, then $(D \setminus f^{-1}(\alpha), f^{-1}(\alpha(1)), \mathbf{u})$ is $(r_N, c_N)$-nice.

Lemma 5.12 still holds in the space-filling setting, with the only modification being that we replace $\nu_{D,1-\tau_0;u}$ by $\tilde{\nu}_{D,1-\tau_0;u}$ for fixed $\mathbf{u} = (u_1, u_2, u_3)$ such that 1, $u_1, u_2, u_3$ are ordered counterclockwise. The proof of the lemma in the space-filling setting will follow by iterative applications of Lemma 5.20 right below. Notice that in this lemma we do not rule out scenarios where $\eta$ oscillates back and forth while staying close to $\gamma$, while do not want such behavior in Lemma 5.12 as we consider the $L^\infty$ distance.

Lemma 5.20. Let $\mathbf{u} = (u_1, u_2, u_3)$ be distinct points of $(\partial D) \setminus \{1\}$ such that 1, $u_1, u_2, u_3$ are ordered clockwise and let $\gamma : [0, T] \to \overline{D} \setminus \{0\}$ be a simple curve with $\gamma(0) = 1$ for some $T > 0$. For $\delta > 0$ let $A(\delta)$ denote the $\delta$-neighborhood of $\gamma([0, T])$. Define stopping times

$$\sigma_1 = \text{inf}\{t \geq 0 : |\eta(t) - \gamma(T)| < \delta\}, \quad \sigma_2 = \text{inf}\{t \geq 0 : \eta(t) \notin A(\delta)\}.$$

Then for any $\delta > 0$ we have $\tilde{\nu}_{D,1-\tau_0;u}[\sigma_1 < \sigma_2] > 0$.

Proof. The proof is identical to the proof of [MW17, Lemma 2.3]. The only difference is that the lemma treats chordal curves and we consider a radial curve, but the argument is the same. More precisely, we need the argument in the lemma for $\kappa > 4$, corresponding to the case where the curve in question is a counterflow line. It is enough to prove the claim for counterflow lines since by [MS17, Theorem 1.13], for every rational $z$, the set $\eta[0, \tau_z]$ (with $\tau_z$ the first hitting time of $z$) is the union of branches of a branching counterflow line. (Strictly speaking, [MS17, Theorem 1.13] is stated for GFF with fully branchable boundary condition, but we can perform an absolutely continuous change of measure such that the law of $h|_{A(\delta)}$ becomes the law of the restriction of a GFF with fully branchable boundary condition.)
Proof of Lemma 5.12 with \( \tilde{\nu}_{D,1 \to 0;u} \) instead of \( \nu_{D,1 \to 0} \). Note that Lemma 5.20 can also be applied to simply connected domains not equal to \( \mathbb{D} \) by applying an appropriate conformal change of coordinates to the GFF and the curves \( \eta \) and \( \gamma \). Set \( j_0 = 10/\varepsilon + 1 \), assuming without loss of generality that \( j_0 \) is an integer. We apply Lemma 5.20 iteratively for \( j = 1, 2, \ldots, j_0 \) with domain \( D = \mathbb{D} \setminus \eta_{0, \tau_{j-1}} \), \( \gamma \) the straight line from \( \eta(\tau_{j-1}) \) to \( 1 - j\varepsilon/10 \), \( \delta \ll \varepsilon \), and \( \tau_j \) defined as the stopping time \( \sigma_{\eta} \) in the lemma (with \( \tau_0 = 0 \)).

The statement and proof of Lemma 5.13 simplify as follows, since for suitable \( c_N \) the condition of Lemma 5.19 is trivially satisfied as \( f_{\alpha}^{-1} \) maps into \( \mathbb{D} \).

**Lemma 5.21.** In the setup of Lemma 5.13, if \( \| \gamma - \eta \|_{\infty;[0, \tau_A]} < \varepsilon \), then we have that \( (D \setminus f^{-1}(\eta[0, \tau_A]), f^{-1}(\eta(\tau_A)), u) \) is \( (r_N, c_N) \)-nice.

The statement and proof of Lemma 5.14 are identical to the non-space-filling case, except that we consider \( \tilde{\nu}_{D,a \to \infty;u} \) instead of \( \nu_{D,a \to \infty;u} \) and that the proof relies on the following lemma (proved at the very end of the proof of [GHM20a, Lemma 3.1]) to treat the case where both \( |f(u_1) - 1| \) and \( |f(u_3) - 1| \) are small.\(^4\) A similar lemma is used when only one of \( |f(u_1) - 1| \) and \( |f(u_3) - 1| \) is small, with the only difference being that we only grow a flow line from the point in \( \{f(u_1), f(u_3)\} \) that is close to 1.

**Lemma 5.22.** There exist positive constants \( \delta, q > 0 \) such that the following is true. Let \( u = (u_1, u_2, u_3) \) be such that 1, \( u_1, u_2, u_3 \in \partial \mathbb{D} \) are distinct and ordered clockwise. Let \( \tilde{\eta} \) denote the variant of the GFF in \( \mathbb{D} \) that is used when defining \( \tilde{\nu}_{D,1 \to 0;u} \) at the very end of Section 2.3. Suppose that \( |u_1 - 1| \lor |u_3 - 1| \leq \delta \). Let \( \eta_{u_1}^L \) (resp. \( \eta_{u_1}^R \)) denote the flow line of \( \tilde{\eta} \) started from \( u_1 \) (resp. \( u_3 \)) with angle \( \pi/2 \) (resp. \( -\pi/2 \)) targeted at 0, and let \( S_1 \) (resp. \( S_3 \)) be its exit time from \( B_{2\delta}(1) \). Let \( U \) be the connected component containing 0 of \( \mathbb{D} \setminus (\eta_{u_1}^L([0, S_1]) \cup \eta_{u_1}^R([0, S_3])) \). Let \( E \) be the event that the harmonic measure from 0 in \( U \) of each of the right side of \( \eta_{u_1}^R([0, S_1]) \) and the left side of \( \eta_{u_1}^L([0, S_3]) \) is at least \( q \). Then \( \mathbb{P}(E) \geq q \).

Finally, the proofs of the space-filling counterparts of Lemma 5.3, Proposition 5.4, Lemma 5.5, and Proposition 5.15 go through precisely as before. The only minor change is when justifying the space-filling counterpart of Lemma 5.5(iii), where we now use

\[
\text{Cont}(f^{-1}(\eta[0, \sigma_{r_N}])) \leq \text{Cont}(f^{-1}(B(1, r_N) \cap \overline{\mathbb{D})))} \leq c_N.
\]

### 5.4 Lower bounds on the regularity of SLE

In this section we conclude the proofs of Theorems 1.1–1.3. Given Proposition 3.19, it remains to prove that the constants \( c_0, c_1 \) in the theorems are positive.

A natural approach for proving lower bounds is to find disjoint intervals \([s_k, t_k]\) on which the increment of \( \eta \) is exceptionally big with a certain (small) probability. If the sum of these probabilities is infinite and there is sufficient decorrelation of these events, then a variant of the second Borel-Cantelli lemma will imply that infinitely many of these events will occur. In our case, however, the correlations are not easy to control. The probabilities of having exceptionally large increments in the non-space-filling case are given in Proposition 5.2 or its “conditional” version Proposition 5.15. But when conditioning on \( \eta[0, s_k] \), Proposition 5.15 does not apply to every realisation of \( \eta[0, s_k] \) but only the ones that are nice (as defined in Section 5.2). Another attempt would be to use the corresponding upper bound of the probability given by Proposition 3.10. But the upper and lower bounds differ not just by a factor but by a power, which is too weak to guarantee sufficient decorrelation. The exact same issues arise in the case of space-filling SLE.

\(^4\)Note that there is a typo in the published version of [GHM20a] which interchanges the role of “left” and “right” in the second-to-last sentence of the below lemma.
Our idea is to introduce another sequence of events $B_k$ on which the conditional lower bound on the interval $[s_k, t_k]$ is valid again (an example is the event that $\eta[0, s_k]$ is nice). We formulate the argument as an abstract lemma.

**Lemma 5.23.** Let $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ be events. Suppose that $P(B_j) \geq p$ for some $p > 0$ and all $j$. Let

$$p_j = P(A_j | B_j \cap (A_1 \cap B_1)^c \cap \ldots \cap (A_{j-1} \cap B_{j-1})^c).$$

Then, if $q \in [0, 1]$ satisfies

$$\exp \left( -(1 - \frac{1-p}{q})(p_1 + \ldots + p_k) \right) < q,$$

we have

$$P((A_1 \cap B_1) \cup \ldots \cup (A_k \cap B_k)) > 1 - q.$$  

**Proof.** Suppose the contrary, i.e.

$$P((A_1 \cap B_1)^c \cap \ldots \cap (A_k \cap B_k)^c) \geq q.$$  

Observe this in that case

$$P(B_j^c | (A_1 \cap B_1)^c \cap \ldots \cap (A_{j-1} \cap B_{j-1})^c) \leq \frac{P(B_j^c)}{P((A_1 \cap B_1)^c \cap \ldots \cap (A_{j-1} \cap B_{j-1})^c)} \leq \frac{1-p}{q}$$

by our assumptions. It follows that

$$P(A_j \cap B_j | (A_1 \cap B_1)^c \cap \ldots \cap (A_{j-1} \cap B_{j-1})^c) \geq (1 - \frac{1-p}{q})p_j$$

and therefore inductively

$$P((A_1 \cap B_1)^c \cap \ldots \cap (A_k \cap B_k)^c) \leq \prod_{j \leq k} \left( 1 - (1 - \frac{1-p}{q})p_j \right) \leq \exp \left( -(1 - \frac{1-p}{q}) \sum_{j \leq k} p_j \right)$$

which by (19) contradicts (20).

We now prove the lower bound in Theorem 1.2, namely that the constant $c_0$ in the theorem statement must be positive; see Proposition 3.19 for the other assertions of the theorem. We use Lemma 5.23 to show that the lower bound is satisfied with positive probability. This will imply the claim.

As in the previous sections, we define $\tau_r = \inf \{ t \geq 0 : |\eta(t)| = r \}$ to be the hitting time of radius $r$.

**Lemma 5.24.** There exist $p_0 > 0$ and $b_0 > 0$ such that the following is true. For any $(D, a)$ as in (3) with $|a| > 0$ and $u \in \partial D \setminus \{a\}$, we have

$$\nu_{D,a \to \infty; u} \left( \sup_{s, t \in [0, \tau_r|a| + r]} \frac{|\eta(t) - \eta(s)|}{|t - s|^{1/d} (\log |t - s|^{-1})^{1-1/d}} \geq b_0 \right) \geq p_0$$

for any $r > 0$.

The same holds for space-filling SLE$_\kappa$, except that we consider $(D, a, u)$ as in (17) and $\nu_{D,a \to \infty; u}$ instead of $(D, a, u)$ and $\nu_{D,a \to \infty; u}$. 

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Proof of Theorem 1.2 given Lemma 5.24. By Proposition 3.19 it remains to show $c_0 > 0$ in Proposition 3.17. Applying Lemma 5.24 with arbitrarily small $r > 0$, we see that with probability at least $p_0$ we have

$$\lim_{t \downarrow 0} \sup_{s,t \in [0,\tau_{\eta \mid \tau_{\eta} + \nu}]} \frac{|\eta(t) - \eta(s)|}{|t - s|^{1/d}(\log|t - s|^{-1})^{1-1/d}} \geq b_0.$$ 

This implies $S_I \geq b_0$ for some random interval $I$ (with $S_I$ defined in the proof of Proposition 3.17). But since $S_I$ is a deterministic constant independent of $I$, the result follows.

Proof of Lemma 5.24. We will do the proof for SLE$_\kappa(2)$ and explain at the end what modifications are necessary for the case of space-filling SLE$_\kappa$.

For $\varepsilon > 0$ and $k = 1, \ldots, \lfloor \varepsilon^{-1}r \rfloor$ we define the event

$$A_{\varepsilon,k} = \{ \text{Cont}(\eta[\tau_{|\tau_{| + (k-1)\varepsilon}}, \tau_{|\tau_{| + k\varepsilon}}]) \leq a_0 \varepsilon^{d}(\log \varepsilon^{-1})^{-(d-1)} \}$$

where $a_0 > 0$ is a constant whose value will be decided upon later. Note that on the event $A_{\varepsilon,k}$ we have

$$|\eta(\tau_{|\tau_{| + k\varepsilon}) - \eta(\tau_{|\tau_{| + (k-1)\varepsilon})| \geq \varepsilon \cdot \frac{|\tau_{|\tau_{| + k\varepsilon} - \tau_{|\tau_{| + (k-1)\varepsilon}|^{1/d}(\log|\tau_{|\tau_{| + k\varepsilon} - \tau_{|\tau_{| + (k-1)\varepsilon}|^{-1})^{(d-1)/d}}}

which is the lower bound we want to prove.

Let $p_N, r_N, c_N, \varepsilon_2$ be as in Proposition 5.15. Let $B_{\varepsilon,k}$ denote the event that $(D_k, a_k, u_k)$ is $(p_N, \tau_N, c_N)$-nice where

$$(D_k, a_k, u_k) := \left( \varepsilon^{-1}(\log \varepsilon^{-1})(D \setminus \eta[0, \tau_{|\tau_{| + (k-1)\varepsilon}]), \varepsilon^{-1}(\log \varepsilon^{-1}) \eta(\tau_{|\tau_{| + (k-1)\varepsilon}), \varepsilon^{-1}(\log \varepsilon^{-1})u \right).$$

On the event $B_{\varepsilon,k}$ we have by Proposition 5.15

$$\nu_{D,a \rightarrow \infty; \nu}(A_{\varepsilon,k} \mid \eta[0, \tau_{|\tau_{| + (k-1)\varepsilon}]) \geq \exp \left( -\tilde{c}_2 a_0^{-1/2}(\log \varepsilon^{-1}) \right) = \tilde{c}_2 a_0^{-1/2} = \varepsilon^{1/2}$$

for a suitable choice of $a_0$.

Note that $\varepsilon^{-1}(\log \varepsilon^{-1}) \eta[0, \tau_{|\tau_{| + (k-1)\varepsilon})$ is a radial SLE$_\kappa(2)$ in $\varepsilon^{-1}(\log \varepsilon^{-1})D$ stopped at hitting radius $\varepsilon^{-1}(\log \varepsilon^{-1})|\tau_{|\tau_{| + (k-1)\varepsilon}$. By Lemma 5.14, we have $\nu_{D,a \rightarrow \infty; \nu}(B_{\varepsilon,k}) \geq p_1$ where $p_1 > 0$ does not depend on $D, \varepsilon, k$.

Applying Lemma 5.23 with any choice of $q \in (1 - p_1, 1)$ and noting that

$$\exp \left( -\left(1 - \frac{1}{q} \right) r \varepsilon^{-1/2} \right) < q$$

for small $\varepsilon > 0$, this implies the result with $p_0 = 1 - q$.

In the case of space-filling SLE$_\kappa$, we instead apply Proposition 5.17 and the space-filling analogue of Lemma 5.14 as explained at the end of Section 5.3.

We finally show the lower bounds in Theorem 1.3. By Proposition 5.1, this implies also the lower bound in Theorem 1.1. Given Proposition 3.19, it remains to prove that the constants $c_0, c_1$ are positive. By the stationarity of $\eta$, it suffices to show the claim for $t_0 = 0$.

Proof of Theorem 1.3. The proof is almost identical for whole-plane SLE$_\kappa(2)$ ($\kappa \in (0, 8]$) and for whole-plane space-filling SLE$_\kappa$ ($\kappa > 4$). We begin with the former.

We first prove the $t \downarrow 0$ statement. Define the sequences

$$r_k = \exp(-k^2), \quad m_k = a_0 r_k^d(\log r_k^{-1})^{-(d-1)}$$

where the exact value of $a_0 > 0$ will be decided upon later. Note that if

$$\text{Cont}(\eta[0, \tau_{r_k}]) < m_k,$$
then for some $t < m_k$ we have

$$|\eta(t)| = r_k \times m_k^{1/d} (\log \log m_k)^{(d-1)/d}$$

which is exactly the lower bound in the law of the iterated logarithm.

By Proposition 5.2 we have

$$\mathbb{P}(\text{Cont}(\eta[0, \tau_{r_k}]) < m_k) \geq \exp(-c_2 m_k^{-1/(d-1)} r_k^{d/(d-1)}) \asymp k^{-\alpha_0/(d-1)},$$

so $a_0$ can be chosen such that the sum of the probabilities diverges. We would like to argue by Lemma 5.23 that with positive probability this happens for infinitely many $k$. Then Proposition 3.15 implies that the probability must actually be 1.

To show this, we introduce another sequence $r_k'$ with $r_{k+1} < r_k' < r_k$. Let

$$r_{k+1} = 2r_k (\log k)^{-1}$$

and define the events

$$A_k = \{\text{Cont}(\eta[\tau_{r_{k+1}}, \tau_{r_k}]) \leq m_k\}, \quad A_k' = \{\text{Cont}(\eta[0, \tau_{r_{k+1}}]) \leq m_k\}, \quad \overline{A}_k = A_k \cap A_k'.$$

As noted above, on the event $\limsup_k \overline{A}_k$ we have

$$\limsup_{t \to 0} \frac{|\eta(t)|}{t^{1/d} (\log \log t)^{(1-1)/d}} \geq b_0$$

where $b_0$ depends on the choice of $a_0$.

We are left to show $\mathbb{P}(\limsup_k A_k) > 0$. Let $B_k'$ denote the event that $\hat{\Omega} \setminus r_k^{-1}(\log k) \eta[0, \tau_{r_{k+1}}]$ is $(p_N, r_N, c_N)$-nice. On the event $B_k'$ we have by Proposition 5.15

$$\mathbb{P}(\text{Cont}(\eta[\tau_{r_{k+1}}, \tau_{r_k}]) < a_1 m_k | \eta[0, \tau_{r_{k+1}}]) \geq \exp(-c_2 m_k^{-1/(d-1)} r_k^{d/(d-1)}) \asymp k^{-\alpha_0/(d-1)} = k^{-1}$$

for suitable choice of $a_0, a_1$.

Since $r_k^{-1}(\log k) \eta[0, \tau_{r_{k+1}}]$ is again a whole-plane SLE$_\kappa(2)$ stopped at hitting radius 2, we have $\mathbb{P}(B_k') > p_1$ by Lemma 5.14 where $p_1 > 0$ does not depend on $k$.

The content $\text{Cont}(\eta[0, \tau_{r_{k+1}}])$ can be controlled by the negative moments in [Zha19, Lemma 1.7], implying

$$\mathbb{P}(\text{Cont}(\eta[0, \tau_{r_{k+1}}]) > m_k) \leq \mathbb{P}(|\eta(m_k)| < r_{k+1}) \leq (r_{k+1})^{d(1-\epsilon)} \mathbb{E}(|\eta(m_k)|)^{-d(1-\epsilon)} \lesssim (r_{k+1})^{d(1-\epsilon)} m_k^{-(1-\epsilon)} \asymp (\log k)^{-1+\epsilon}. \tag{21}$$

It follows that $\mathbb{P}(A_k' \cap B_k') > p_1/2$.

Write $B_k = A_k' \cap B_k'$. We apply Lemma 5.23 with the events $A_k', ..., A_k$ and $B_k', ..., B_k$. Pick any $q \in (1 - p_1/2, 1)$ and note that for any $k \in \mathbb{N}$ we can find $k' > k$ such that

$$\exp \left( -\left(1 - \frac{1 - p_1/2}{q}\right) ((k')^{-1} + \ldots + k^{-1}) \right) < q.$$ 

This implies $\mathbb{P}\left( \bigcup_{k' \geq k} A_{k'} \right) \geq \mathbb{P}\left( \bigcup_{k' \geq k} (A_{k'} \cap B_{k'}) \right) > 1 - q$ for all $k$, and therefore $\mathbb{P}(\limsup_k A_k) \geq 1 - q > 0$.

The proof is identical for space-filling SLE$_\kappa$, except that the proof of (21) is even easier since $\text{Cont}(\eta[0, \tau_{r_{k+1}}]) \leq \pi (r_{k+1})^2$ deterministically.

The proof of the $t \to \infty$ statement is identical when we set

$$r_k = \exp(k^2), \quad r_{k-1} = 2r_k (\log k)^{-1}, \quad m_k = a_0 r_k^d (\log \log r_k)^{-1}(d-1).$$
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