IS THE SYMMETRIC GROUP SPERNER?

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Abstract. An antichain $A$ in a poset $\mathcal{P}$ is a subset of $\mathcal{P}$ in which no two elements are comparable. Sperner showed that the maximal antichain in the Boolean lattice, $\mathcal{B}_n = \{0 < 1\}^n$, is the largest rank (of size $\binom{n}{\lfloor n/2 \rfloor}$). This type of problem has been since generalized, and a graded poset $\mathcal{P}$ is said to be Sperner if the largest rank of $\mathcal{P}$ is its maximal antichain. In this paper, we will show that the symmetric group $\mathcal{S}_n$, partially ordered by refinement (or by absolute order equivalently), is Sperner.

1. Introduction

A partial order, $\leq$, on $S$, is a reflexive, antisymmetric, and transitive binary relation, and a poset, $\mathcal{P} = (P, \leq)$, consists of a set, $P$, and a partial order $\leq$ on $P$. A chain is a poset in which every pair of elements is comparable.

The height of $\mathcal{P}$, $h(\mathcal{P})$, is the maximum height of a chain in $\mathcal{P}$. The Jordan-Dedekind chain condition for $\mathcal{P}$ is that all maximal chains in an interval $[x, y] = \{z \in \mathcal{P} : x \leq z \leq y\}$ have the same height. If $\mathcal{P}$ is connected and satisfies this condition, we can define a rank function: select any $x_0 \in \mathcal{P}$ and define $r(x_0) = 0$. For any $x \neq x_0$, $r(x)$ is uniquely determined by $x \leq y \Rightarrow r(y) = r(x) + 1$. A graded poset is a poset equipped with a rank function. We can define the levels $N_i = \{x \in \mathcal{P} | r(x) = i\}$.

An antichain, $A$, in $\mathcal{P}$ is a subset in which no two elements lie on a chain. Given a weighted poset, $\mathcal{P} = (P, \leq, \omega)$, the width of $\mathcal{P}$, $w$, is the maximum weight of an antichain in $\mathcal{P}$. If $\mathcal{P}$ is not explicitly weighted, the weight is implicitly the counting measure.

Given $\mathcal{P}$, Sperner’s problem is to find the width of $\mathcal{P}$. In [12], Sperner shows that the width of the (unweighted) Boolean lattice, $\mathcal{B}_n = \{0 < 1\}^n$, is $\binom{n}{\lfloor n/2 \rfloor}$, the largest binomial coefficient. For $0 \leq k \leq h(\mathcal{P})$, we can also define a $k$-antichain, $A_k$, in $\mathcal{P}$ to be a subset in which no $k+1$ elements lie on a chain. In [6], Erdős extended Sperner’s problem to finding the $k$-width, $w_k(\mathcal{P}) = \max \{w(A_k)\}$ and showed that

$$w_k(\mathcal{B}_n) = \sum_{j=1}^{k} \left(\binom{n}{(n + j - 1)/2}\right),$$

the sum of the $k$ largest binomial coefficients. In [13], Stanley used techniques from algebraic geometry to show that Weyl groups, under Bruhat order, are Sperner. Engel wrote a book [5] which presents Sperner theory from a unified point of view, bringing combinatorial techniques together with methods from programming, linear algebra, probability theory, and enumerative combinatorics.

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In \cite{11}, Rota conjectured that $\Pi_n$, the poset of partitions of $\{1, 2, \ldots, n\}$, ordered by refinement, is Sperner. The conjecture was disproved by Canfield in \cite{2} by using Graham-Harper reduction (\cite{8}) and probability theory. Canfield and Harper, in \cite{4}, went further, showing that the ratio of the size of the largest antichain to the size of the largest rank goes to infinity. Canfield, in \cite{3}, completed the resolution of Rota’s question, showing the ratio of the size of the largest antichain in $\Pi_n$ and the largest Stirling number of the second kind (the rank sizes in $\Pi_n$) is $\Omega \left(n^{\alpha} (\ln n)^{1-n^{-\frac{1}{2}}}\right)$, where $\alpha = \frac{2-\ln 2}{4} \approx \frac{1}{3.5}$. So, the ratio does go to infinity, but very slowly. In 1999, this result was designated one of ten outstanding results in order theory by the editor-in-chief of the journal Order.

One of the natural questions that arises from Rota’s conjecture is: what happens if we look at $S_n$, ordered by refinement? Given $\pi \in S_n$, we say that $\sigma$ is a refinement of $\pi$ if we can take one of the cycles of $\pi$ and slice it into two. More formally, if $\pi = \pi_1 \pi_2 \cdots \pi_k$ is the cycle decomposition of $\pi$, for any two elements $i$ and $j$ on a cycle $\pi_m$, $\pi \cdot (i j)$ is a refinement of $\pi$. In this paper, we will take a category theoretical approach to show that $S_n$, ordered by refinement, is Sperner.

It is also worth mentioning another partial order, called absolute order, on $S_n$. The absolute length of $\pi \in S_n$ is defined by

$$l_T(\pi) = n - \text{the number of cycles in } \pi.$$ 

Then, the absolute order on $S_n$ is defined by

$$\pi \leq_T \sigma \iff l_T(\sigma) = l_T(\pi) + l_T(\pi^{-1}\sigma).$$

Armstrong, in \cite{1}, showed that the absolute order is the reverse of refinement, and so, the main result of this paper implies that $S_n$, ordered by absolute order, is Sperner.

### 2. Flow morphisms

In this section, we establish the groundwork to introduce the category $FLOW$. The objects of $FLOW$ are networks in the sense of Ford-Fulkerson \cite{7}, and its morphisms preserve the Ford-Fulkerson flows (both underflows and overflows) on those networks.

A network $N$ consists of an acyclic directed graph $G = (V, E)$ and a capacity function $\nu : V \to \mathbb{R}^+$. For an edge $e \in E$, let $\partial_-(e)$ and $\partial_+(e)$ denote the head and tail of $e$, respectively. $V$ is partitioned into three sets, $R$, $S$, and $T$:

\[
S = \{ s \in V : \exists e \in E, \partial_-(e) = s \}, \text{ called sources,}
\]

\[
T = \{ t \in V : \exists e \in E, \partial_+(e) = t \}, \text{ called sinks, and}
\]

\[
R = V - S - T, \text{ called intermediate vertices.}
\]

An underflow on $N$ is a function $f : E \to \mathbb{R}^+$ such that

- for all $s \in S$, $\sum_{\partial_-(e)=s} f(e) \leq \nu(s)$,
- for all $t \in T$, $\sum_{\partial_+(e)=t} f(e) \leq \nu(t)$, and
- for all $r \in R$, $\sum_{\partial_-(e)=r} f(e) = \sum_{\partial_+(e)=r} f(e) \leq \nu(r)$.

An overflow on $N$ is defined in the same way except that the inequalities are reversed. The quantity $\text{net}(f) = \sum_{s \in S} \sum_{\partial_-(e)=s} f(e)$ is the net $S$-$T$ flow of $f$, and the MaxFlow of $N$ is defined as $\text{MaxFlow}(N) = \max_f \text{net}(f)$ over all underflows, $f$, on $N$. Similarly, MinFlow of $N$ is defined as $\text{MinFlow}(N) = \min_f \text{net}(f)$ over


all overflows, \( f \), on \( N \). By Ford-Fulkerson theory \([7]\), \( \text{MaxFlow}(N) = \text{MinCut}(N) \), where a \textit{cut} is a set of vertices intersecting any path from a source to a sink. Also, \( \text{MinFlow}(N) = \text{MaxAntichain}(N) \).

A \textit{bipartite network} is \( V = S \cup T \) with all edges \( e \in E \) directed from \( S \) to \( T \). A \textit{flow} \( f \) on a bipartite network \( V = S \cup T \) is said to be a \textit{normalized flow} if

\[
\sum_{xy \in E} f(xy) = \frac{\omega(x)}{\omega(S)} \quad \text{for all } x \in S, \text{ and}
\]

\[
\sum_{xy \in E} f(xy) = \frac{\omega(y)}{\omega(T)} \quad \text{for all } y \in T.
\]

If \( N \) is the Hasse diagram of a weighted and graded poset and every pair of consecutive ranks, \([N_k, N_{k+1}]\), accepts a normalized flow, then \( N \) is said to have the \textit{normalized flow property (NFP)}.

For \( G \) a bipartite graph with vertex sets \( A \) and \( B \), \( G \) is said to satisfy Hall’s matching condition if for all \( X \subseteq A \),

\[
|X| \leq |D(X)|
\]

holds, where \( D(X) \) is the set of vertices in \( B \) connected to vertices in \( X \). Sperner showed in his original problem that he only had to consider consecutive ranks at a time and if they satisfy Hall’s condition, then the poset under consideration is Sperner.

When trying to prove Rota’s conjecture, Graham and Harper came up with a strengthening of Hall’s matching condition. A bipartite graph \( G \) is said to satisfy \textit{normalized matching condition (NMC)} if for all \( X \subseteq A \),

\[
\frac{|X|}{|A|} \leq \frac{|D(X)|}{|B|}.
\]

The normalized matching condition is dual of the normalized flow property \([7]\). Harper has done extensive work in studying posets with NFP, and in \([9]\), he describes maps between these structures, called \textit{flow morphisms}. Let \( M \) and \( N \) be networks. Then, \( \varphi : M \to N \) is a \textit{flow morphism} if

1. \( \varphi : G_M \to G_N \) is a graph epimorphism,
2. \( \varphi^{-1}(S_N) = S_M \) and \( \varphi^{-1}(T_N) = T_M \),
3. \( \varphi \) is capacity preserving, i.e. for all \( v \in N \), \( \omega_M(\varphi^{-1}(v)) = \omega_N(v) \), and
4. the preimage of every edge \( e \in N \) has a normalized flow.

This leads us to the category \( \text{FLOW} \), whose objects are acyclic vertex-weighted networks and morphisms are precisely these flow morphisms. An important property of flow morphisms is that they preserve net \( S-T \) flow, and so, \( \text{MaxFlow} \) and \( \text{MinFlow} \) problems on \( M \) and \( N \) are equivalent. In other words, if \( M \) and \( N \) are both in \( \text{FLOW} \) and a flow morphism \( \varphi \) exists between them, then the preimage of a maximum weight antichain of \( N \) under \( \varphi \) is a maximum weight antichain of \( M \) (see \([4]\) for a fuller discussion).

3. \( S_n \) is indeed Sperner

In this section, we will prove that \( S_n \) has normalized flow property which implies that \( S_n \) is indeed Sperner.

**Theorem 3.1.** \( S_n \) \textit{has normalized flow property}.
Proof. We proceed by induction on $n$. The base case is trivial. As for the inductive step, let us assume that $S_n$ has normalized flow property.

The rank-weights of $S_n$, $|S_{n,k}| = s_{n,k}$, the Stirling numbers of the first kind, satisfy the recurrence relation

$$s_{n+1,k} = ns_{n,k} + s_{n,k-1}.$$ 

Before continuing with the proof, we give an example of using this recurrence relation to view $S_4$ as four copies of $S_3$:

![Figure 1. Viewing $S_4$ as four copies of $S_3$](image)

The copies of $S_3$ are arranged in a way that the first (blue) copy of $S_3$ has the six permutations $\pi$ with $\pi(1) = 4$, the second copy has $\pi$ with $\pi(2) = 4$, the third copy has $\pi$ with $\pi(3) = 4$, and the fourth raised copy has $\pi$ with $\pi(4) = 4$. The red edges connect permutations from the raised copy to permutations of other copies, and the gray, dashed edges connect permutations from the lower copies to other lower copies.

A direct combinatorial proof of the recurrence follows from the observation that for $\pi \in S_{n+1,k}$, there are two possibilities:

1. In the case that $\pi(n+1) = n+1$, we can remove $n+1$ from $\pi$ and have $\pi' \in S_{n,k-1}$. Conversely, adding a 1-cycle with $n+1$ to $\pi' \in S_{n,k-1}$ will give $\pi \in S_{n+1,k}$.
2. In the case that $\pi(n+1) = i$, where $1 \leq i \leq n$, we can remove $n+1$ from the cycle containing $\pi(n+1) = i$ and define $\pi'(\pi^{-1}(n+1)) = i$, which will
give \( n \) copies, \( S^{(i)}_{n,k} \) for \( 1 \leq i \leq n \). Conversely, the operation of defining \( \pi \in S_{n+1,k} \) from \( \pi' \in S_{n,k} \) can be done similarly.

There is exactly one map \( \pi \mapsto \pi' \) between the copy labeled \( i \) and \( n+1 \), by construction, and the figure below is provided to help the reader visualize.

![Figure 2](image-url)

**Figure 2.** Viewing \( S_{n+1} \) in light of the recurrence relation

By the inductive hypothesis and the regularity between the \( n \) blue copies of \( S_n \), we can collapse the \( n \) copies as in the figure below, where the collapsed copy is in bold.

![Figure 3](image-url)

**Figure 3.** “Collapsing” the \( n \) copies of \( S_n \)

We claim that this new network satisfies the normalized matching condition. To show this, we consider the two consecutive ranks \( k \) and \( k+1 \), which are shown with the corresponding vertex-weights:
The only non-trivial equivalence class to show the normalized matching condition for is the class with the lower, right vertex. In other words, we need to show that

\[
\frac{s_{n,k}}{s_{n-1,k} + ns_{n,k}} \leq \frac{s_{n,k}}{s_{n,k} + ns_{n,k+1}}.
\]

This is equivalent to \(s_{n,k} - s_{n,k+1} \leq s_{2,n,k}^2\), which is true due to the 2-positivity of \(s_{n,k}\)'s, which was proved in [10]. Hence, NMC is satisfied, which in turn implies that \(S_n\) satisfies NMC, and so, has normalized flow property. \(\square\)

**Remark.** The lattice in Figure 3 is

\[S_n \times \mathbb{1}\]

which has NFP by the Product theorem [9]. Our proof actually shows that

\[
\mathbb{1} \times \mathbb{2} \times \mathbb{3} \times \cdots \times \mathbb{1} \mathbb{2} \cdots \mathbb{n-1} \subseteq S_n.
\]

Since the former has NFP by the Product theorem, the latter has NFP also.

Now that we have shown that \(S_n\) has normalized flow property, we want to find a network we can map \(S_n\) to, via a flow morphism, which is Sperner. In fact, we can collapse the network in Figure 3 further, just by keeping the same rank:

![Diagram of collapsing the network](image)

**Figure 5.** “Collapsing” the \(n\) copies of \(S_n\)
Since the resulting network is a totally ordered set, the largest antichain is going to be the rank/vertex, say $v$, with the largest vertex weight. The composition of the collapsings is a flow morphism, and so, the preimage of $v$ in $S_n$ will be the largest antichain. By construction, the preimage of each vertex in the totally ordered network is a rank in $S_n$, and so, the largest antichain in $S_n$ is the largest rank. Thus, $S_n$ is indeed Sperner.

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