THE SEPARATION PROBLEM FOR REGULAR LANGUAGES
BY PIECEWISE TESTABLE LANGUAGES

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Abstract. Separation is a classical problem in mathematics and computer science. It asks whether, given two sets belonging to some class, it is possible to separate them by another set of a smaller class. We present and discuss the separation problem for regular languages. We then give a direct polynomial time algorithm to check whether two given regular languages are separable by a piecewise testable language, that is, whether a $\mathcal{B}\Sigma_1(\prec)$ sentence can witness that the languages are indeed disjoint. The proof is a reformulation and a refinement of an algebraic argument already given by Almeida and the second author.

1. Introduction

The separation problem. Separation is a classical question in mathematics and computer science. In general, one says that two sets $X, Y$ are separable by a set $U$ if $X \subseteq U$ and $Y \cap U = \varnothing$. In this case, $U$ is called a separator.

The separation problem is the following. Consider a class $\mathcal{C}$ of sets or structures, and a subclass $\mathcal{C}_0$ of $\mathcal{C}$. The problem asks whether two elements $X, Y$ of $\mathcal{C}$ can always be separated by an element of the subclass $\mathcal{C}_0$. A classical example of such a separation problem, with a positive answer, is the Hahn-Banach separation theorem. Another example that appeared recently in computer science is the proof of Leroux [14] of the decidability of the reachability problem for vector addition systems (or Petri Nets), which greatly simplifies the original proof by Mayr [15], and that of Kosaraju [12]. Namely, Leroux has shown that non-reachability can be witnessed by a class of recursively enumerable separators: from a configuration $c_1$ of such a system, one cannot reach a configuration $c_2$ if and only if the sets $\{c_1\}$ and $\{c_2\}$ can be separated by a Presburger definable set, which in addition is invariant under actions of the vector addition system. Since such sets form a recursively enumerable class, this yields a semi-algorithm for checking non-reachability.

In the case where elements of $\mathcal{C}$ cannot always be separated by an element of $\mathcal{C}_0$, several natural questions arise:

1. given elements $X, Y$ in $\mathcal{C}$, can we decide whether a separator exists in $\mathcal{C}_0$?
2. if so, what is the complexity of this decision problem?
3. can we in addition compute a separator, and what is the complexity?

In this context, it is known for example that separation of two context-free languages by a regular one is undecidable [11].

2000 Mathematics Subject Classification. Primary 68Q45,68Q70; Secondary 20M35.
In this paper, we look at the separation problem for the class $\mathcal{C}$ of regular languages, and we are looking for separators in smaller classes, such as prefix- or suffix-testable languages, locally trivial languages, and piecewise testable languages (we will define these classes below).

**The profinite approach.** Several results from the literature can be combined into an algorithm answering question (1), for all classes we are interested in. Several partial complexity results can also be derived from this approach, which we briefly explain now. This approach relies on a generic connection found by Almeida [2] between profinite semigroup theory and the separation problem, when the separators are required to belong to a given variety of regular languages.

A variety $\mathcal{V}$ of regular languages associates to each finite alphabet $A$ a class of languages $A^*\mathcal{V}$, with some closure properties (namely closure under Boolean operations, left and right residuals $L \mapsto a^{-1}L$ and $L \mapsto La^{-1}$, and inverse morphisms between free monoids). All classes of separators in this paper belong to a variety of regular languages.

Almeida [2] has shown that two regular languages over $A$ are separable by a language of $A^*\mathcal{V}$ if and only if the topological closures of these two languages inside a profinite semigroup, depending only on $\mathcal{V}$, intersect. To turn this property into an algorithm, we have therefore to be able:

− to compute representations of these topological closures, and
− to test for emptiness of intersections of such closures.

So far, these problems have no generic answer. They have been studied for a small number of specific varieties, in an algebraic context. Deciding whether the closures of two regular languages intersect is equivalent to computing the so-called 2-pointlike sets of a finite semigroup wrt. the variety we are interested in, see [2]. This question has been answered positively, in particular for the following varieties:

i) languages recognized by a finite group [5, 17, 6],

ii) star-free (that is, FO-definable) languages [10, 9],

iii) piecewise testable (that is, $\mathcal{B}\Sigma_1(<)$-definable) languages [4, 3],

iv) languages whose syntactic semigroups are $R$-trivial, that is, languages whose minimal automaton is very weak (the only cycles allowed in the graph of the automaton are self-loops) [3],

v) languages for which membership can be tested by inspecting prefixes and suffixes up to some length (folklore, see [1, Sec. 3.7]),

vi) locally testable languages, that is, languages for which membership can be tested by inspecting prefix, suffix and factors up to some length [20, 16].

For all these classes, proofs use algebraic or topological arguments. In this paper, we obtain direct polynomial time algorithms for Cases iii) and v). Our intuition is strongly lead by the proof techniques from profinite semigroup theory.

A general issue is that the topological closures cannot be described with a finite device. However, for piecewise testable languages, the approach of [4] consists in computing an automaton over an extended alphabet, which recognizes the closure of the original language. This can be performed in polynomial time wrt. the size...
of the original automaton. Since these automata admit the usual construction for intersection, and can be checked for emptiness in NLOGSPACE, we get a polynomial time algorithm wrt. the size of the original automata. The construction was presented for deterministic automata but also works for nondeterministic ones. One should mention that the extended alphabet is $2^A$ (where $A$ is the original alphabet). Therefore, these results give an algorithm which, from two NFAs, decides separability by piecewise testable languages in time polynomial in the number of states of the NFAs and exponential in the size of the original alphabet.

The improvement of the separation result for piecewise testable languages as presented in this paper is twofold: on the one hand, the algorithm presented provides better complexity as it runs in polynomial time in both the size of the automata, and in the size of the alphabet. On the other hand, our results do not make use of the theory of profinite semigroups, that is, we work only with elementary concepts. The proof follows however basically the same pattern as the original one.

The key argument is to show that non-separability is witnessed by both automata admitting a path of the same shape. In our proof, we manually extract from two non-separable automata some paths with this property, using Simon’s factorization forest Theorem [19]. Whereas in the profinite world, these witnesses are immediately obtained by a standard compactness argument.

**Organization of the paper.** After having recalled the background in Section 2, we present in Section 3 a simple toy example, to highlight the main definitions and techniques: the case of separation by prefix-testable languages. Section 4 is devoted to the question of separation by piecewise testable languages. The main algorithm and proofs are given in this section. For the interested reader, we provide some elements of profinite semigroup theory in appendix.

2. Preliminaries

Given a finite alphabet $A$, we denote by $A^*$ (resp. by $A^+$) the free monoid (resp. the free semigroup) over $A$. For a word $u \in A^*$, the smallest $B \subseteq A$ such that $u \in B^*$ is called the alphabet of $u$ and is denoted by $\text{alph}(u)$. A *nondeterministic finite automaton* (NFA) over $A$ is denoted by a tuple $A = (Q, A, I, F, \delta)$, where $Q$ is the set of states, $I \subseteq Q$ the set of initial states, $F \subseteq Q$ the set of final states and $\delta \subseteq Q \times Q$ the transition relation. If $\delta$ is a function, then $A$ is a deterministic automaton (DFA). We denote by $L(A)$ the language of words accepted by $A$. Given a word $u \in A^*$, a subset $B$ of $A$ and two states $p, q$ of $A$, we denote

- by $p \xrightarrow{u} q$ a path from state $p$ to state $q$ labeled $u$.
- by $p \xrightarrow{\subseteq B} q$ a path from $p$ to $q$ of which all transitions are labeled by letters of $B$.
- by $p \xrightarrow{= B} q$ a path from $p$ to $q$ of which all transitions are labeled by letters of $B$, with the additional demand that every letter of $B$ occurs at least once along this path.
Given a state $p$, we denote by $\text{sc}(p, A)$ the strongly connected component of $p$ in $A$ (that is, the set of states reachable from $p$), and by $\text{alph}\_\text{sc}(p, A)$ the set of labels of all transitions occurring in this strongly connected component. Finally, we define the restriction of $A$ to a subalphabet $B \subseteq A$ by $A \mid_B \overset{\text{def}}{=} (Q, A, I, F, \delta \cap(Q \times B \times Q))$.

3. A toy example: separation by prefix-testable languages

A regular language $L$ is a prefix-testable language if membership of $L$ can be tested by inspecting prefixes up to some length, that is, if $L$ is a finite Boolean combination of languages of the form $uA^*$, for a finite word $u$. Prefix-testable languages form a variety of regular languages. Therefore, as recalled in the introduction, it follows by [2] that testing whether two given languages can be separated by a prefix-testable language amounts to checking that their topological closures in some profinite semigroup have a nonempty intersection.

It turns out that for prefix-testable languages, this profinite semigroup is easy to describe (see [1, Sec. 3.7]): it is $A^+ \cup A^\infty$, where $A^\infty$ denotes the set of right infinite words over $A$. Multiplication in this semigroup is defined as follows: infinite words are left zeros ($vw = v$ if $v \in A^\infty$), and multiplication on the left by a finite word is the usual multiplication: $(a_1 \cdots a_n)(b_1 \cdots) = a_1 \cdots a_n b_1 \cdots$. Finally, the topology is the product topology: a sequence converges

- to a finite word $u$ if it is ultimately equal to $u$,
- to an infinite word $v$ if for every finite prefix $x$ of $v$, the sequence ultimately belongs to $x(A^+ \cup A^\infty)$.

Therefore, from a given NFA $A$, one can compute a Büchi automaton recognizing the language of infinite words that belong to the closure of $L(A)$, as follows:

1. Trim $A$, by removing all states from which one cannot reach a final state. This can be performed in linear time wrt. the size of $A$, and does not change the language recognized by $A$.
2. Build the Büchi automaton obtained from the resulting trim automaton by declaring all states accepting.

This yields a straightforward PTIME (actually NLOGSPACE) algorithm to decide separability by a prefix-testable language: first check that $L(A_1) \cap L(A_2) = \emptyset$. If so, compute the intersection of the languages of infinite words belonging to the closures of $L(A_1)$ and $L(A_2)$ by the usual product construction, and check that this Büchi automaton accepts at least one word.

**Proposition 1.** One can decide in PTIME whether two languages can be separated by a prefix-testable language. \(\square\)

4. A simple PTIME algorithm for separation by a piecewise testable language

**Piecewise testable languages.** Let $\prec$ be the scattered subword ordering defined on $A^*$ as follows: for $u, v \in A^*$, we have $u \prec v$ if $u = a_1 \cdots a_n$ and
\( v = v_0 a_1 v_1 \cdots v_{n-1} a_n v_n, \) with \( a_i \in A \) and \( v_i \in A^*. \) We let

\[
\text{Sub}_n(u) = \{ w \in A^* : |w| \leq n, w \prec u \}.
\]

When two words have the same scattered subwords up to length \( n, \) we say that they are \( \sim_n \)-equivalent:

\[
\text{Sub}_n(u) = \text{Sub}_n(v) \iff u \sim_n v.
\]

A regular language over an alphabet \( A \) is piecewise testable (PT) \([18]\) if it is a finite Boolean combination of languages of the form \( A^* a_1 A^* a_2 \cdots A^* a_n A^*, \) where every \( a_i \in A. \) Whether a given word belongs to a PT-language is thus determined by the set of its scattered subwords up to a certain length. In other words, a regular language \( L \) is piecewise testable if and only if there exists an \( n \in \mathbb{N} \) such that \( L \) is a union of \( \sim_n \)-classes.

The class of piecewise testable languages has been extensively studied during the last decades. It corresponds to languages that can be defined in the fragment \( \mathcal{B} \Sigma_1(\prec) \) of first-order logic on finite words. Simon has shown that piecewise testable languages are exactly those languages whose syntactic monoid is \( J \)-trivial \([18]\), and this property yields a decision procedure to check whether a language is piecewise testable. Stern has refined this procedure into a polynomial time algorithm \([21]\), whose complexity has been improved by Trahtman \([22]\).

**Separation by a piecewise testable language.** We say that two regular languages \( L_1, L_2 \) are PT-separable if there exists a piecewise testable language \( L \) that separates them, i.e.,

\[
L_1 \subseteq L \text{ and } L_2 \cap L = \emptyset.
\]

In other words, \( L_1 \) and \( L_2 \) are PT-separable if there exists a \( \mathcal{B} \Sigma_1(\prec) \) formula which is satisfied by all words of \( L_1, \) and not satisfied by any word of \( L_2. \)

Our main contribution is a simple proof of the following result, which states that one can decide in polynomial time whether two languages are PT-separable.

**Theorem 1.** Given two NFAs, one can determine in polynomial time, with respect to the number of states and the size of the alphabet, whether the languages recognized by these NFAs are PT-separable.

Note that a language is PT-separable from its complement if and only if it is piecewise testable itself. Therefore, applying Theorem 1 to a language and to its complement if they are both given by NFAs yields a polynomial time algorithm to check if a language is piecewise testable. We recover in particular the following result, proved by Stern \([21]\) using the characterization for minimal automata recognizing PT-languages as given by Simon in \([18]\) (this result has later been improved by Trahtman \([22]\)).

**Corollary 1.** One can decide in polynomial time whether a given DFA recognizes a piecewise testable language.

The rest of this section is devoted to the proof of Theorem 1. We fix a DFA \( A \) over \( A. \) For \( u_0, \ldots, u_p \in A^* \) and nonempty subalphabets \( B_1, \ldots, B_p \subseteq A, \) let
\[ \vec{u} = (u_0, \ldots, u_p) \text{ and } \vec{B} = (B_1, \ldots, B_p). \]

We call such a pair \((\vec{u}, \vec{B})\) a factorization pattern. A \((\vec{u}, \vec{B})\)-path in \(A\) is a successful path (leading from the initial state to a final state of \(A\)), of the form

\[
\begin{array}{c}
\overline{u_0} \subseteq B_1 \subseteq B_1 \overline{u_1} \ldots \overline{u_{p-1}} \subseteq B_p \subseteq B_p \overline{u_p} \\
\end{array}
\]

**Figure 1.** A \((\vec{u}, \vec{B})\)-path

Recall that edges denote sequences of transitions: an edge labeled \(\subseteq B\) denotes a path of which all transitions are labeled by letters of \(B\). An edge labeled \(= B\) denotes a path of which all transitions are labeled by letters of \(B\), with the additional demand that every letter of \(B\) occurs at least once.

**Remark 1.** The automaton \(A\) admits a \((\vec{u}, \vec{B})\)-path if and only if \(L(A)\) contains a language of the form

\[ u_0(x_1y_1^*z_1)u_1 \cdots u_{p-1}(x_py_p^*z_p)u_p, \]

where \(\text{alph}(x_i) \cup \text{alph}(z_i) \subseteq \text{alph}(y_i) = B_i.\)

Theorem 1 directly follows from the next two statements.

**Proposition 2.** Let \(A_1\) and \(A_2\) be two NFAs. Then, \(L(A_1)\) and \(L(A_2)\) are not PT-separable if and only if there exist \(\vec{u} = (u_0, \ldots, u_p)\) and \(\vec{B} = (B_1, \ldots, B_p)\) such that both \(A_1\) and \(A_2\) both have a \((\vec{u}, \vec{B})\)-path.

**Proposition 3.** Given two NFAs, one can determine in polynomial time, with respect to the number of states and the size of the alphabet, whether there exist \(\vec{u} = (u_0, \ldots, u_n)\) and \(\vec{B} = (B_1, \ldots, B_n)\) such that both NFAs admit a \((\vec{u}, \vec{B})\)-path.

As observed above, the characterization of PT-separable languages given in Proposition 2 can be applied to the minimal automata of a regular language and of its complement, to obtain a characterization for minimal automata recognizing PT-languages. It turns out that with this approach, we retrieve exactly the same characterization as given by Simon in [18].

Let us first prove Proposition 3.

**Proof of Prop. 3.** We will first show that the following problem is in PTIME: given states \(p_1, q_1, r_1\) in automaton \(A_1\) and \(p_2, q_2, r_2\) in automaton \(A_2\), determine whether there exists a nonempty alphabet \(B \subseteq A\) such that there is an \((= B)\)-loop around both \(q_1\) and \(q_2\), and \((\subseteq B)\)-paths from \(p_1\) to \(r_1\) via \(q_1\) in \(A_1\), and from \(p_2\) to \(r_2\) via \(q_2\) in \(A_2\), as pictured in Figure 2.
To do so, we compute a decreasing sequence \((C_i)\) of alphabets over-approximating the maximal alphabet \(B\) labeling the loops. Note that if there exists such an alphabet \(B\), it should be contained in

\[
C_1 \triangleq \text{alph}_\text{scc}(q_1, A_1) \cap \text{alph}_\text{scc}(q_2, A_2).
\]

Using Tarjan’s algorithm to compute strongly connected components in linear time [8], one can compute \(C_1\) in linear time as well. Then, we restrict the automata to alphabet \(C_1\), and we repeat the process to obtain the sequence \((C_i)\):

\[
C_{i+1} \triangleq \text{alph}_\text{scc}(q_1, A_1 | C_i) \cap \text{alph}_\text{scc}(q_2, A_2 | C_i).
\]

After a finite number \(n\) of iterations, we obtain \(C_n = C_{n+1}\). Note that \(n \leq |\text{alph}(A_1) \cap \text{alph}(A_2)| \leq |A|\). If \(C_n = \emptyset\), then there exists no nonempty \(B\) for which there is an \((=B)\)-loop around both \(p\) and \(q\). If \(C_n \neq \emptyset\), then it is the maximal nonempty alphabet \(B\) such that there are \((=B)\)-loops around \(q_1\) in \(A_1\) and \(q_2\) in \(A_2\). It then remains to determine whether there exist paths \(p_1 \xrightarrow{C_B} q_1 \xrightarrow{C_B} r_1\) and \(p_2 \xrightarrow{C_B} q_2 \xrightarrow{C_B} r_2\), which can be performed in linear time.

To sum up, since the number \(n\) of iterations to compute \(C_n = C_{n+1}\) is bounded by \(|A|\), and since each computation is linear wrt. the size of \(A_1\) and \(A_2\), deciding whether there is a pattern as in Figure 2 in both \(A_1\) and \(A_2\) can be done in polynomial time wrt. to both \(|A|\) and the size of the NFAs.

Now we build from \(A_1\) and \(A_2\) two new automata \(\tilde{A}_1\) and \(\tilde{A}_2\) as follows. The procedure first initializes \(\tilde{A}_i\) as a copy of \(A_i\). Denote by \(Q_i\) the state set of \(A_i\). For each 4-uple \(\tau = (p_1, q_1, r_1, p_2) \in Q_1^2 \times Q_2^2\) such that there exist an alphabet \(B\), two states \(q_1 \in Q_1, q_2 \in Q_2\) and paths \(p_1 \xrightarrow{a_r} q_i \xrightarrow{a_r} q_i \xrightarrow{a_r} r_1\) both for \(i = 1\) and \(i = 2\), we add in both \(\tilde{A}_1\) and \(\tilde{A}_2\) a new letter \(a_r\) to the alphabet, and transitions \(p_1 \xrightarrow{a_r} r_1\) and \(p_2 \xrightarrow{a_r} r_2\). Since there is a polynomial number of tuples \((p_1, q_1, q_1, r_1, p_2, q_2, q_2, r_2)\), the above shows that computing these new transitions can be performed in polynomial time. Therefore, computing \(\tilde{A}_1\) and \(\tilde{A}_2\) can be done in PTIME.

Now by construction, there exists some factorization pattern \((\overline{u}, \overline{B})\) such that \(A_1\) and \(A_2\) both have a \((\overline{u}, \overline{B})\)-path if and only if \(L(\tilde{A}_1) \cap L(\tilde{A}_1) \neq \emptyset\). Since both \(\tilde{A}_1\) and \(\tilde{A}_1\) have been built in PTIME, this can be decided in polynomial time. \(\square\)
As a side remark, let us mention that it is crucial that the \((= B)\)-paths, which are required to use exactly the same alphabets, are actually loops (occurring in Figure 2 around states \(q_1\) and \(q_2\)). The next statement shows that even for DFAs, the problem is NP-hard if we are looking for paths labeled by a common alphabet, without requesting these paths to be loops. The proof is deferred to the Appendix.

**Lemma 1.** The following problem is NP-complete:

*Input:* An alphabet \(A = \{a_1, a_2, \ldots, a_n\}\) and two DFA’s \(A_1, A_2\) over \(A\).

*Question:* Do there exist \(u \in L(A_1)\) and \(v \in L(A_2)\) such that \(\text{alph}(u) = \text{alph}(v)\)?

Let us now prove Proposition 2. Let us first prove the “if” direction. The “only if” direction is proved in Lemma 6.

**Lemma 2.** If two NFAs \(A_1\) and \(A_2\) share a common \((\tilde{u}, \tilde{B})\) path, then the languages \(L(A_1)\) and \(L(A_2)\) are not PT-separable.

*Proof.* Let \(L\) be a piecewise testable language such that \(L(A_1) \subseteq L\). Using the hypothesis and Remark 1, this implies that \(L\) contains a language

\[
u_0(x_1y_1^1z_1)u_1 \cdots u_{p-1}(x_py_p^\ast z_p)u_p,
\]

where \(\text{alph}(x_i) \cup \text{alph}(z_i) \subseteq \text{alph}(y_i) = B_i\). Similarly, \(L(A_2)\) contains a language

\[
u_0(x_1'y_1'^1z_1')u_1 \cdots u_{p-1}(x_py_p'z_p')u_p,
\]

where \(\text{alph}(x_i') \cup \text{alph}(z_i') \subseteq \text{alph}(y_i') = B_i\). We will show that for every \(n\), there is an element in this language which is \(\sim_n\)-equivalent to an element of \(u_0(x_1y_1^1z_1)u_1 \cdots u_{p-1}(x_py_p^\ast z_p)u_p\), using the following claim.

**Claim 1.** Given \(x, x', y, y', z, z' \in A^*\) that satisfy

\[
\text{alph}(x) \cup \text{alph}(z) \subseteq \text{alph}(y),
\]

\[
\text{alph}(x') \cup \text{alph}(z') \subseteq \text{alph}(y') = \text{alph}(y),
\]

then for every \(n \in \mathbb{N}\),

\[
xy^nz \sim_n x'y^mz'.
\]

Indeed, from the inclusions

\[
\text{alph}(y)^\leq_n = \text{Sub}_n(y^n) \subseteq \text{Sub}_n(xy^nz) \subseteq \text{alph}(y)^\leq_n,
\]

it follows that \(\text{Sub}_n(xy^nz) = \text{alph}(y)^\leq_n\). In the same way, \(\text{Sub}_n(x'y^mz') = \text{alph}(y')^\leq_n\), which is equal to \(\text{alph}(y)^\leq_n\). Thus \(xy^nz \sim_n x'y^mz'\). This establishes the claim.

Applying this, we obtain that \(x_iy_i^1z_i \sim_n x_i'y_i'^1z_i'\) for every \(i\). Since \(\sim_n\) is a congruence, we obtain for all \(n \in \mathbb{N}\):

\[
u_0(x_1y_1^1z_1)u_1 \cdots u_{p-1}(x_py_p^\ast z_p)u_p \sim_n u_0(x_1'y_1'^1z_1')u_1 \cdots u_{p-1}(x_py_p'^1z_p')u_p.
\]

Since \(L\) is piecewise testable, it is a union of \(\sim_n\)-equivalence classes for some \(n\), thus it cannot be disjoint from \(L(A_2)\).

To prove the other direction of Proposition 2, we introduce some notation. For \(B \subseteq A\), let us denote by \(B^\circ\) the set of words with alphabet exactly \(B\):

\[
B^\circ = \{w \in B^* \mid \text{alph}(w) = B\}.
\]
Given a factorization pattern \((\vec{u}, \vec{B})\), with \(\vec{u} = (u_0, \ldots, u_p)\) and \(\vec{B} = (B_1, \ldots, B_p)\), we let
\[
\mathcal{L}(\vec{u}, \vec{B}, n) = u_0(B_1^\oplus)^n u_1 \cdots u_{p-1}(B_p^\oplus)^n u_p.
\]

We say that a sequence \((w_n)_n\) is \((\vec{u}, \vec{B})\)-adequate if
\[
\forall n \geq 0, \ w_n \in \mathcal{L}(\vec{u}, \vec{B}, n).
\]

A sequence is called adequate if it is \((\vec{u}, \vec{B})\)-adequate for some factorization pattern \((\vec{u}, \vec{B})\).

**Lemma 3.** Every sequence \((w_n)_n\) of words admits an adequate subsequence.

**Proof.** We use Simon’s Factorization Forest Theorem, which we recall. See [19, 13, 7] for proofs and extensions of this theorem. A factorization tree of a nonempty word \(x\) is a finite ordered unranked tree \(T(x)\) whose nodes are labeled by nonempty words, and such that:
- all leaves of \(T(x)\) are labeled by letters,
- all internal nodes of \(T(x)\) have at least 2 children,
- if a node labeled \(y\) has \(k\) children labeled \(y_1, \ldots, y_k\) from left to right, then \(y = y_1 \cdots y_k\).

Given a semigroup morphism \(\varphi : A^+ \to S\) into a finite semigroup \(S\), such a factorization tree is \(\varphi\)-Ramseyan if every internal node has either 2 children, or \(k\) children labeled \(y_1, \ldots, y_k\), in which case \(\varphi\) maps all words \(y_1, \ldots, y_k\) to the same idempotent of \(S\). Simon’s Factorization Forest Theorem states that every word has a \(\varphi\)-Ramseyan factorization tree of height at most \(3|S|\).

Let now \((w_n)_n\) be a sequence of words. We use Simon’s Factorization Forest Theorem with the morphism \(\text{alph} : A^+ \to 2^A\).

Consider a sequence \((T(w_n))_n\), where \(T(w_n)\) is an \(\text{alph}\)-Ramseyan tree given by the Factorization Forest Theorem. In particular, \(T(w_n)\) has depth at most \(3 \cdot 2^{|A|}\). Therefore, extracting a subsequence if necessary, one may assume that the sequence of depths of the trees \(T(w_n)\) is a constant \(H\). We argue by induction on \(H\). If \(H = 0\), then every \(w_n\) is a letter. Hence, one may extract from \((w_n)_n\) a constant subsequence, which is therefore adequate.

We denote the arity of the root of \(T(w_n)\) by \(\text{arity}(w_n)\), and we call it the arity of \(w_n\). If \(H > 0\), two cases may arise:

1. One can extract from \((w_n)_n\) a subsequence of bounded arity. Therefore, one may extract from \(w_n\) a subsequence of constant arity, say \(K\). This implies that each \(w_n\) has a factorization in \(K\) factors
\[
w_n = w_{n,1} \cdots w_{n,K},
\]
where \(w_{n,i}\) is the label of the \(i\)-th child of the root in \(T(w_n)\). Therefore, the \(\text{alph}\)-Ramseyan subtree of each \(w_{n,i}\) is of height at most \(H - 1\). By induction, one can extract from \((w_{n,i})_n\) an adequate subsequence. Proceeding iteratively for \(i = 1, 2, \ldots, K\), one extracts from \((w_n)_n\) a subsequence \((w_{\sigma(n)})_n\) such that every \((w_{\sigma(n),i})_n\) is adequate. But a finite product of adequate sequences is obviously adequate. Therefore, the subsequence \((w_{\sigma(n)})_n\) of \((w_n)_n\) is also adequate.
The arity of $w_n$ grows to infinity. Therefore, extracting if necessary, one can assume for every $n$, \text{arity}(w_n) \geq \max(n, 3)$. Since all arities of words in the sequence are at least 3, all children of the root map to the same idempotent in $2^3$. But this says that each word from the subsequence is of the form

$$w_{\sigma(n)} = w_{n,1} \cdots w_{n,K_n},$$

with $K_n \geq n$, and where the alphabet of $w_{n,i}$ is the same for all $i$, say $B$. Therefore, $w_{\sigma(n)} \in (B^\otimes)^{K_n} \subseteq (B^\otimes)^n$. Therefore, $(w_{\sigma(n)})_n$ is adequate. \hfill \Box

We now say that a factorization pattern $(\vec{u}, \vec{B})$ is proper if

1. for all $i$, last$(u_i) \notin B_i$ and first$(u_i) \notin B_{i-1}$,
2. for all $i$, $u_i = \varepsilon \Rightarrow (B_{i-1} \notin B_i$ and $B_i \notin B_{i-1}$).

Note that if a sequence $(w_n)_n$ is adequate, then there exists a proper factorization pattern $(\vec{u}, \vec{B})$ such that $(w_n)_n$ is $(\vec{u}, \vec{B})$-adequate. This is easily seen from the following observations and their symmetric counterparts:

$$u = u_1 \cdots u_k \text{ and } u_k \in B \implies u_1 \cdots u_k B^n \subseteq u_1 \cdots u_{k-1} B^n,$$

$$B_{i-1} \subseteq B_i \implies B_{i-1} B^i \subseteq B_i B^i.$$

The following lemma gives a condition under which two sequences share a factorization pattern. This lemma is very similar to [1, Theorem 8.2.6].

**Lemma 4.** Let $(\vec{u}, \vec{B})$ and $(\vec{t}, \vec{C})$ be proper factorization patterns. Let $(v_n)_n$ and $(w_n)_n$ be two sequences of words such that

- $(v_n)_n$ is $(\vec{u}, \vec{B})$-adequate
- $(w_n)_n$ is $(\vec{t}, \vec{C})$-adequate
- $v_n \sim_n w_n$ for every $n \geq 0$

Then, $\vec{u} = \vec{t}$ and $\vec{B} = \vec{C}$.

**Proof.** For a factorization pattern $(\vec{u}, \vec{B})$, we define

$$\|(\vec{u}, \vec{B})\| := (\sum_{i=0}^{p} |u_i|) + p,$$

where $p$ is the length of the vector $\vec{u}$. Let

$$k := \max(\|(\vec{u}, \vec{B})\|, \|(\vec{t}, \vec{C})\|) + 1.$$

Consider the first word of the sequence $(v_n)_n$, i.e., $v_0 = u_0 b_1 u_1 \cdots b_p u_p$, where $\text{alph}(b_i) = B_i$. Define

$$v_0^{(k)} := u_0 b_1^k u_1 \cdots b_p^k u_p.$$

Recall that $(v_n)_n$ being a $(\vec{u}, \vec{B})$-adequate sequence means that

$$v_n \in u_0 (B_{1}^\otimes)^n u_1 \cdots u_{p-1} (B_{p}^\otimes)^n u_p$$

for every $n$. Thus, we have for every $\ell \geq k \cdot \max(|b_1|, \ldots, |b_n|)$ that $v_0^{(k)} \prec v_\ell$. Note that whenever $\ell' \geq \max(\ell, |v_0^{(k)}|)$, we have that $v_0^{(k)} \in \text{Sub}_{\ell'}(v_{\ell'})$. And, using the assumption that $v_n \sim_n w_n$ for all $n$, this gives that $v_0^{(k)} \prec w_{\ell'}$. In the same way, for
$w_0 = t_0c_1t_1 \cdots cQt_q$, we obtain an index $m$ such that for every $m' \geq \max(m, |w_0^{(k)}|)$, both $w_0^{(k)} \prec w_{m'}$ and $u_0^{(k)} \prec v_{m'}$ hold.

Let $M := \max(\ell', m')$. Then $v_0^{(k)} \prec v_M, w_M$ and $w_0^{(k)} \prec v_M, w_M$.

Now fix a factor $b_i^k$ of $v_0^{(k)}$. In particular, $b_i^k \prec w_M$. Since $k > \| (\vec{t}, \vec{C}) \|$ and $|b_i| > 0$, the pigeonhole principle gives that there is some $C_j$ with $\text{alph}(b_i) \subseteq C_j$.

Exploiting this, we want to define a bijection between the set of indexed alphabets in $\vec{B}$ and the set of those in $\vec{C}$ that will help us to show that $(\vec{u}, \vec{B}) = (\vec{t}, \vec{C})$.

Let $\mathbf{B} := \{(B_1, 1), \ldots, (B_p, p)\}$ and $\mathbf{C} := \{(C_1, 1), \ldots, (C_q, q)\}$. We define a function $f : \mathbf{B} \rightarrow \mathbf{C}$, by sending $(B_i, i)$ to that $(C_j, j)$ for which $c_j \in (C_j^{\text{alph}})_M$ is the first factor of $w_M$ used to fully read $b_i$, while reading $v_0^{(k)}$ as a scattered subword of $w_M$.

The function $g : \mathbf{C} \rightarrow \mathbf{B}$ is defined analogously. The functions $f$ and $g$ preserve the order of the indices and pointwise preserve the alphabet. If we show that $f$ and $g$ define a bijective correspondence between $\mathbf{B}$ and $\mathbf{C}$, then $p = q$. The fact that $f$ and $g$ pointwise preserve the alphabet would then imply that $B_i = C_i$, for every $i$.

To establish that $f$ and $g$ are each others inverses, we apply Lemma 8.2.5 from [1], which we shall first repeat:

**Lemma 5** ([1, Lemma 8.2.5]). Let $X$ and $Y$ be finite sets and let $P$ be a partially ordered set. Let $f : X \rightarrow Y, g : Y \rightarrow X, p : X \rightarrow P$ and $q : Y \rightarrow P$ be functions such that

1. for any $x \in X$, $p(x) \leq q(f(x))$,
2. for any $y \in Y$, $q(y) \leq p(g(y))$,
3. if $x_1, x_2 \in X, f(x_1) = f(x_2)$ and $p(x_1) = q(f(x_1))$, then $x_1 = x_2$,
4. if $y_1, y_2 \in Y, g(y_1) = g(y_2)$ and $q(y_1) = p(g(y_1))$, then $y_1 = y_2$.

Then $f$ and $g$ are mutually inverse functions and $p = q \circ f$ and $q = p \circ g$.

The functions $f$ and $g$ fulfill the conditions of this lemma, if we let $X = \mathbf{B}, Y = \mathbf{C}$, let $P$ be the set of alphabets, partially ordered by inclusion, and let $p$ and $q$ be the projections onto the first coordinate:

(1) and (2) hold since $f$ and $g$ pointwise preserve the alphabet. Suppose that $f(B_{i_1}) = f(B_{i_2})$ and that $B_{i_1} = f(B_{i_1})$. This means that a factor $b_{i_1}$ and a factor $b_{i_2}$ of $v_0^{(k)}$ are read inside the same factor $c_j'$ of $w_M$. Thus $\text{alph}(b_{i_1}, u_{i_1}, \ldots, b_{i_2}) \subseteq \text{alph}(c_j') = f(B_{i_1}) = B_{i_1} = \text{alph}(b_{i_1})$. But we assumed that $(\vec{u}, \vec{B})$ is a proper factorization pattern, so $i_1$ must be equal to $i_2$. This shows that (3) holds, and (4) is proved similarly.

It follows that indeed $f$ and $g$ define a bijective correspondence between $\mathbf{B}$ and $\mathbf{C}$, thus $p = q$ and $B_i = C_i$, for every $i$. Since we are dealing with proper factorization patterns, $v_0^{(k)} \prec w_M$ now implies that $t_i \prec u_i$, for every $i$. On the other hand, $w_0^{(k)} \prec v_M$ now implies that $u_i \prec t_i$, for every $i$. Thus, for every $i$, $u_i = t_i$. $\square$

Now we are equipped to prove the “only if” direction of Proposition 2.
Lemma 6. If the languages recognized by two DFAs $A_1$ and $A_2$ are not PT-separable, then $A_1$ and $A_2$ share a common $(\vec{u}, \vec{B})$-path.

Proof. By hypothesis, for every $n \in \mathbb{N}$, there exist $v_n \in L(A_1)$ and $w_n \in L(A_2)$ such that
\[ v_n \sim_n w_n. \]
This defines an infinite sequence of pairs $(v_n, w_n)_n$, from which we will iteratively extract infinite subsequences to obtain additional properties, while keeping (1).

By Lemma 3, one can extract from $(v_n, w_n)_n$ a subsequence whose first component forms an adequate sequence. From this subsequence of pairs, using Lemma 3 again, we extract a subsequence whose second component is also adequate (note that the first component remains adequate). Therefore, one can assume that both $(v_n)_n$ and $(w_n)_n$ are themselves adequate.

Lemma 4 shows that one can choose the same factorization pattern $(\vec{u}, \vec{B})$ such that both $(v_n)_n$ and $(w_n)_n$ are $(\vec{u}, \vec{B})$-adequate. Finally, by the following claim, we then obtain that both $A_1$ and $A_2$ admit a $(\vec{u}, \vec{B})$-path.

Claim 2. If $L(A)$ contains a $(\vec{u}, \vec{B})$-adequate sequence, then $A$ admits a $(\vec{u}, \vec{B})$-path.

Indeed, $L(A)$ contains a $(\vec{u}, \vec{B})$-adequate sequence $(v_n)_n$, i.e.
\[ \forall n \geq 0, \ v_n \in u_0(B_1^\infty)^n u_1 \cdots u_{p-1}(B_p^\infty)^n u_p \cap L(A). \]
Let $v_n$ be a sufficiently large term in this sequence, e.g. with $n > |Q(A)|$. Now the path used to read $v_n$ in $A$ must traverse loops labeled by each of the $B_i$’s and clearly, by the shape of $v_n$, this is a $(\vec{u}, \vec{B})$-path.

\begin{thebibliography}{9}

[1] J. Almeida. Finite semigroups and universal algebra, volume 3 of Series in Algebra. World Scientific Publishing Co. Inc., River Edge, NJ, 1994. Translated from the 1992 Portuguese original and revised by the author.

[2] J. Almeida. Some algorithmic problems for pseudovarieties. Publ. Math. Debrecen, 54(suppl.):531–552, 1999. Automata and formal languages, VIII (Salgótarján, 1996).

[3] J. Almeida, J. C. Costa, and M. Zeitoun. Pointlike sets with respect to $R$ and $J$. J. Pure Appl. Algebra, 212(3):486–499, 2008.

[4] J. Almeida and M. Zeitoun. The pseudovariety $J$ is hyperdecidable. RAIRO Inform. Théor. Appl., 31(5):457–482, 1997.

[5] C. J. Ash. Inevitable graphs: a proof of the type II conjecture and some related decision procedures. Internat. J. Algebra Comput., 1:127–146, 1991.

[6] K. Auinger and B. Steinberg. A constructive version of the Ribes-Zalesskiı̆ product theorem. Mathematische Zeitschrift, 250(2):287–297, 2005.

[7] T. Colcombet. Factorization forests for infinite words and applications to countable scattered linear orderings. Theor. Comput. Sci., 411(4-5):751–764, 2010.

[8] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. Introduction to Algorithms. The MIT Press, 2 edition, 2001.

[9] K. Henckell. Pointlike sets: the finest aperiodic cover of a finite semigroup. J. Pure Appl. Algebra, 55(1-2):85–126, 1988.
\end{thebibliography}
[10] K. Henckell, J. Rhodes, and B. Steinberg. Aperiodic pointlikes and beyond. *IJAC*, 20(2):287–305, 2010.
[11] H. B. Hunt, III. On the decidability of grammar problems. *J. ACM*, 29(2):429–447, Apr. 1982.
[12] S. R. Kosaraju. Decidability of reachability in vector addition systems (preliminary version). In *Proceedings of the fourteenth annual ACM symposium on Theory of computing*, STOC ’82, pages 267–281, New York, NY, USA, 1982. ACM.
[13] M. Kufleitner. The height of factorization forests. In E. Ochmanski and J. Tyszkiewicz, editors, *MFCS*, volume 5162 of *Lecture Notes in Computer Science*, pages 443–454. Springer, 2008.
[14] J. Leroux. Vector addition systems reachability problem (a simpler solution). In A. Voronkov, editor, *The Alan Turing Centenary Conference, Turing-100, Manchester UK June 22-25, 2012, Proceedings*, volume 10 of *EPiC Series*, pages 214–228. EasyChair, 2012.
[15] E. W. Mayr. An algorithm for the general petri net reachability problem. *SIAM J. Comput.*, 13(3):441–460, 1984.
[16] C. V. Nogueira. Propriedades algorítmicas envolvendo a pseudovariade LSl. PhD thesis, available at http://hdl.handle.net/1822/12277, 2010.
[17] L. Ribes and P. A. Zalesskii. On the profinite topology on a free group. *Bull. London Math. Soc.*, 25:37–43, 1993.
[18] I. Simon. Piecewise testable events. In *Proceedings of the 2nd GI Conference on Automata Theory and Formal Languages*, pages 214–222, London, UK, UK, 1975. Springer-Verlag.
[19] I. Simon. Factorization forests of finite height. *Theoretical Computer Science*, 72(1):65 – 94, 1990.
[20] B. Steinberg. A delay theorem for pointlikes. *Semigroup Forum*, 63(3):281–304, 2001.
[21] J. Stern. Complexity of some problems from the theory of automata. *Information and Control*, 66(3):163–176, 1985.
[22] A. N. Trahtman. Piecewise and local threshold testability of DFA. In *Proceedings of the 13th International Symposium on Fundamentals of Computation Theory*, FCT ’01, pages 347–358, London, UK, UK, 2001. Springer-Verlag.
Appendix A. Connection with profinite semigroup theory: overview

We show that separability of two languages by a $V$-recognizable language is equivalent to the nonemptiness of the intersection of their closures in the free profinite $V$ semigroup. This was already shown in [2]. The material of Section A.1 can be found in [1].

A.1. Background. Fix a finite alphabet $A$ and a pseudovariety $V$. A semigroup $T$ separates $u, v \in A^+$ if there exists a morphism $\varphi : A^+ \to T$ such that $\varphi(u) \neq \varphi(v)$.

Given $u, v \in A^+$, let $r_V(u, v) = \min\{|T| : T \in V$ and $T$ separates $u$ and $v\} \in \mathbb{N} \cup \{\infty\}$. Assume for simplicity that two distinct words can be separated by some semigroup of $V$. Then $d_V(u, v) = 2^{-r_V(u, v)}$, with $2^{-\infty} = 0$, defines a metric on $A^+$.

A sequence $(u_n)_n$ is Cauchy for this metric if for every morphism $\varphi : A^+ \to T$, $(\varphi(u_n))_n$ is eventually constant. Let $(\overline{\Omega}_A V, d_V)$ be the completion of the metric space $(A^+, d_V)$. By construction, $A^+$ is dense in $\overline{\Omega}_A V$. Pointwise multiplication of classes of Cauchy sequences transfers the semigroup structure of $A^+$ to $\overline{\Omega}_A V$, on which the multiplication is continuous.

Proposition 4. $(\overline{\Omega}_A V, d_V)$ is compact.

Proof. One checks that every sequence $(u_n)_n$ of elements of $\overline{\Omega}_A V$ has a converging subsequence, that is, since $\overline{\Omega}_A V$ is complete, a Cauchy subsequence. Since $A^+$ is dense in $\overline{\Omega}_A V$, one can find a word $v_n$ such that $\lim_n d_V(u_n, v_n) = 0$. This reduces the statement to the case where $u_n$ is a word. Now, since there is a finite number of morphisms from $A^+$ into a semigroup of size at most $k$, one can extract by diagonalization a subsequence $(u'_n)_n$ of $(u_n)_n$ such that for any morphism $\varphi : A^+ \to T$ with $|T| \leq k$, $(\varphi(u'_n))_{n \geq k}$ is constant. \hfill \Box

Endow $T \in V$ with the discrete topology. The definition of $d_V$ makes every morphism $\varphi : A^+ \to T \in V$ uniformly continuous. Since $A^+$ is dense in $\overline{\Omega}_A V$ compact, $\varphi$ has a unique continuous extension $\hat{\varphi} : \overline{\Omega}_A V \to T$ (which by continuity of the multiplication is also a morphism). For $L \subseteq \overline{\Omega}_A V$, denote by $\text{cl}(L)$ its topological closure in $\overline{\Omega}_A V$.

Lemma 7. Let $\varphi : A^+ \to T \in V$ and $K = \varphi^{-1}(P)$ for $P \subseteq T$. Then $\text{cl}(K) = \hat{\varphi}^{-1}(P)$.

Proof. Unions commute with inverse images and closures, so it suffices to treat the case $P = \{p\}$. Since $\hat{\varphi}$ is continuous, $\hat{\varphi}^{-1}(p)$ is clopen, and it contains $K$, so $\text{cl}(K) \subseteq \hat{\varphi}^{-1}(p)$. Conversely, for $u \in \hat{\varphi}^{-1}(p)$, pick a word $u_n$ such that $d_V(u, u_n) < 2^{-n}$ (which exists since $A^+$ is dense in $\overline{\Omega}_A V$). Then $\varphi(u_n) = p$ for $n > |T|$, hence $u_n \in K$, so $u \in \text{cl}(K)$. \hfill \Box

For $K \subseteq A^+$, we let $K^c = A^+ \setminus K$ and $\text{cl}(K)^c = \overline{\Omega}_A V \setminus \text{cl}(K)$.

Corollary 2. (1) If $K$ is $V$-recognizable, then $\text{cl}(K^c) = \text{cl}(K)^c$.

(2) If $K$ is $V$-recognizable and $L \subseteq A^+$ is such that $\text{cl}(L) \subseteq \text{cl}(K)$, then $L \subseteq K$. 

Proof. (1) Let \( \varphi : A^+ \rightarrow T \in \mathcal{V} \), with \( K = \varphi^{-1}(P) \). By Lemma 7, \( \text{cl}(K^c) = \hat{\varphi}^{-1}(T \setminus P) = \overline{\Omega}_A \mathcal{V} \setminus \hat{\varphi}^{-1}(P) = \text{cl}(K)^c \). For (2), just write \( L \cap K^c \subseteq \text{cl}(K) \cap \text{cl}(K^c) = \emptyset \) by (1).

\[ \square \]

**Proposition 5** (follows from \([1\text{, Thm. 3.6.1}]\)). **Closures of \( \mathcal{V} \)-recognizable languages form a basis of the topology of \( \overline{\Omega}_A \mathcal{V} \).**

**Proof.** By Lemma 7, the closure of a \( \mathcal{V} \)-recognizable language is of the form \( \hat{\varphi}^{-1}(P) \) for some continuous morphism \( \varphi : \overline{\Omega}_A \mathcal{V} \rightarrow T \in \mathcal{V} \), hence it is open. Conversely, for \( u \in \overline{\Omega}_A \mathcal{V} \), let \( O_u = \hat{\alpha}^{-1}(\hat{\alpha}(u)) \), where \( \alpha \) is the product of all morphisms \( \varphi : A^+ \rightarrow T \in \mathcal{V} \) for \( |T| \leq n \). By Lemma 7, \( O_u \) is the closure of the \( \mathcal{V} \)-recognizable language \( \alpha^{-1}(\hat{\alpha}(u)) \). By construction, \( O_u \) is an open containing \( u \), contained in the ball of radius \( 2^{-n} \) centered at \( u \). \[ \square \]

**A.2. Separability of languages by a \( \mathcal{V} \)-recognizable language.** Two languages \( L_1, L_2 \subseteq A^+ \) are \( \mathcal{V} \)-separable if there exists a \( \mathcal{V} \)-recognizable language \( K \) such that \( L_1 \subseteq K \) and \( K \cap L_2 = \emptyset \). Such a language \( K \) is a witness, in the given variety of languages, that \( L_1 \cap L_2 = \emptyset \), and we say that it separates \( L_1 \) and \( L_2 \).

**Proposition 6.** Two languages of \( A^+ \) are separated by a \( \mathcal{V} \)-recognizable language if and only if the intersection of their topological closures in \( \overline{\Omega}_A \mathcal{V} \) is empty.

**Proof.** Let \( L_1, L_2 \subseteq A^+ \), and let \( K \) be \( \mathcal{V} \)-recognizable such that \( L_1 \subseteq K \) and \( K \cap L_2 = \emptyset \). Then \( \text{cl}(L_1) \cap \text{cl}(L_2) \subseteq \text{cl}(K) \cap \text{cl}(K^c) = \emptyset \) by Corollary 2.

Conversely, if \( \text{cl}(L_1) \cap \text{cl}(L_2) = \emptyset \), then any \( u \in \text{cl}(L_1) \) belongs to the open set \( \text{cl}(L_2)^c \), so by Proposition 5, there exists some \( \mathcal{V} \)-recognizable language \( K_u \) whose closure \( O_u \) contains \( u \), and is such that \( O_u \cap \text{cl}(L_2) = \emptyset \). Therefore \( \text{cl}(L_1) \subseteq \bigcup_{u \in \text{cl}(L_1)} O_u \). Since \( \text{cl}(L_1) \) is a closed set in the compact space \( \overline{\Omega}_A \mathcal{V} \) (Prop. 4), it is itself compact and has a finite cover \( O_{u_1} \cup \cdots \cup O_{u_n} \). Then \( K = K_{u_1} \cup \cdots \cup K_{u_n} \) is \( \mathcal{V} \)-recognizable. We have \( \text{cl}(L_1) \subseteq \text{cl}(K) \), so by Corollary 2, \( L_1 \subseteq K \). Also, \( K \subseteq O_{u_1} \cup \cdots \cup O_{u_n} \subseteq \text{cl}(L_2)^c \). \[ \square \]

**Appendix B. Proof of Lemma 1**

**Lemma 1.** The following problem is NP-complete.

**Input:** An alphabet \( A = \{a_1, a_2, \ldots, a_n\} \) and two DFA’s \( A_1, A_2 \) over \( A \).

**Question:** Do there exist \( u \in L(A_1) \) and \( v \in L(A_2) \) such that \( \text{alph}(u) = \text{alph}(v) \)?

**Proof.** We will give a reduction from 3-SAT to this problem.

Let \( \varphi \) be a 3-SAT formula over the variables \( \{x_1, \ldots, x_n\} \). Define \( A := \{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\} \). Let \( A_1 \) be

\[ x_1 \quad \xrightarrow{-x_1} \quad x_2 \quad \xrightarrow{-x_2} \quad \ldots \quad \xrightarrow{-x_n} \quad x_n \]

and let \( A_2 \) be the serial automaton in which for every disjunct \( d \) in the \( i \)-th clause of \( \varphi \), there is an arrow from state \( i \) to \( i + 1 \) labeled \( d \), concatenated with
a copy of $A_1$. For example, if $\varphi = (x_1 \lor x_3 \lor \neg x_4) \land \ldots \land (x_4 \lor \neg x_5 \lor x_2)$, the automaton $A_2$ is

$$
\begin{array}{c}
x_1 \quad \cdots \quad x_3 \quad \neg x_4 \\
\downarrow \quad \quad \quad \quad \quad \downarrow
\end{array}
\begin{array}{c}
x_4 \\
\downarrow
\end{array}
\begin{array}{c}
x_1 \quad \cdots \quad x_2 \quad \neg x_1 \\
\downarrow \quad \quad \quad \quad \quad \downarrow
\end{array}
\begin{array}{c}
x_5 \\
\downarrow
\end{array}
\begin{array}{c}
x_2 \quad \cdots \quad x_n \quad \neg x_2 \\
\downarrow \quad \quad \quad \quad \quad \downarrow
\end{array}
\begin{array}{c}
x_3 \\
\downarrow
\end{array}
\begin{array}{c}
x_1 \\
\downarrow
\end{array}
\begin{array}{c}
x_4 \\
\downarrow
\end{array}
\begin{array}{c}
x_2 \quad \cdots \quad x_n \quad \neg x_2 \\
\downarrow \quad \quad \quad \quad \quad \downarrow
\end{array}
\begin{array}{c}
x_n \\
\downarrow
\end{array}
$$

We will show that $\varphi$ is satisfiable if and only if the question mentioned above is answered positively for these $A_1$ and $A_2$.

Suppose $\varphi$ is satisfiable. Then there is a valuation $v : \{x_1, \ldots, x_n\} \rightarrow \{0, 1\}$ such that $\overline{v}(\varphi) = 1$. Define $u := y_1 \cdots y_n$, with $y_i = x_i$ if $v(x_i) = 1$ and $y_i = \neg x_i$ if $v(x_i) = 0$. In each of the $k$ clauses of $\varphi$, there is at least one disjunct $d$ for which $v(d) = 1$. Define $v := w_1 \cdots w_k u$, where $w_i$ is any one of the disjuncts in the $i$-th clause that is evaluated to 1. Now, $u \in L(A_1)$, $v \in L(A_2)$, and by soundness of the valuation function, $\text{alph}(u) = \text{alph}(v)$.

On the other hand, suppose that for these $A_1$ and $A_2$, there are $u \in L(A_1)$, $v \in L(A_2)$ with $\text{alph}(u) = \text{alph}(v)$. By construction of $A_1$, for every $i$, $\text{alph}(u)$ contains either $x_i$ or $\neg x_i$. By construction of $A_2$ and since $\text{alph}(u) = \text{alph}(v)$, we have that $v = wu$ and that $\text{alph}(w) \subseteq \text{alph}(u)$. Define the valuation

$$
v : \{x_1, \ldots, x_n\} \rightarrow \{0, 1\}
\begin{array}{c}
x_i
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
\text{if } x_i \in \text{alph}(u)
\end{array}
\begin{array}{c}
x_i
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
0
\end{array}
\begin{array}{c}
\text{else}
\end{array}
$$

Now $v$ sends all variables occurring in $w$ to 1, which gives $\overline{v}(\varphi) = 1$. 

\[
\square
\]

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