DEFORMATION OF MORPHISMS, VARIETIES OF LOW CODIMENSION AND ASYMPTOTIC LIMITS

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Dedicated to our dear friend Miguel González on his 60th birthday

ABSTRACT. In this article we study the deformations of a finite morphism $\varphi$ to a projective space factoring through an abelian cover of a complete intersection subvariety $Y$ and the intimately connected problem of existence or non existence of multiple structures called ropes on $Y$. We show that these ropes exist in many cases and in many cases they do not exist. From our more general main theorems we obtain the following interesting applications:

1. We construct infinitely many ropes of right multiplicity, supported on complete intersection subvarieties, embedded in the projective space. We smooth these ropes to obtain infinitely many families of smooth subvarieties with codimension in the range of Hartshorne's conjecture. Therefore these smooth subvarieties occur as general members of a one-parameter family whose central member is not a complete intersection, not even locally so. In the codimension two case the smooth subvarieties are complete intersections due to the existence of rank two vector bundles and results concerning them. We also construct infinitely many families of examples of subvarieties of dimension $m$ in $\text{P}^N$ for which the ratio $m/N$ is an arbitrary rational number between 0 and 1. In addition, we systematically construct smooth non–complete intersection subvarieties, embedded by complete linear series, outside the range of Hartshorne's conjecture, thereby showing many ropes that are not in the boundary of a component of the Hilbert scheme parameterizing smooth complete intersections subvarieties.

2. We go beyond Enriques' original question on constructing simple canonical surfaces in projective spaces, and construct simple canonical varieties in all dimensions. An infinite subset of these simple canonical varieties have finite birational canonical map which is not an embedding.

The deformations of the finite morphisms studied in this article display a wide variety of behavior, in sharp contrast to earlier results on morphisms whose images are projective spaces, projective bundles and blow ups of the projective plane (see [GGP13b], [GGP16a], [GGP16b], [GGP10]). In particular we show that there are infinitely many families of examples for which a general deformation of the associated finite morphism of degree $n$ remains of degree $n$, infinitely many families of examples for which degree $n$ becomes degree $n/2$ (when $n$ is even), and infinitely many families for which degree $n$ becomes degree 1. This has potential applications to finding components of moduli spaces of varieties of general type in higher dimensions, as the first two authors and González showed for surfaces in [GGP10].

1. INTRODUCTION

Deformation of morphisms and the related multiple structures on varieties are topics of interest. Their study is motivated by a number of interesting issues in geometry (see [Bân96], [BE95], [EG95], [Fon93], [HV85], [Man95], [Man04], [Vat09]). The first compelling example of a deformation of morphism that gives rise to interesting geometry is the deformation of canonical morphism of the hyperelliptic curve of genus greater than 2. It is well known that the general deformation of the canonical morphism of such a curve is an embedding, whereas for a curve of genus 2, any deformation of the canonical morphism again remains 2:1. The former case gives rise to the so–called rational ribbons, that is, local complete intersection double structures on the rational normal curve. This object has been studied quite a bit in recent decades. The situation in higher dimensions are much more complex and gives rise to a number
of results that have compelling applications in the following topics: interesting multiple structures on higher dimensional varieties, construction of varieties with given invariants, construction of surfaces of general type with birational canonical maps addressing a question of Enriques in 1940’s, and syzygies among others (see for example [BMR20], [GGP10], [GGP08a], [GGP08b], [GP97], [MR20], [RS19]).

In this article, we study deformations of finite morphisms \( \varphi \) to projective space that factor through an abelian cover of a complete intersection subvariety. These deformations have very diverse behavior and the degree of the morphisms so obtained varies greatly. The most interesting case is when the morphism deforms to a morphism of degree 1 and, even more, when it deforms to an embedding. In this case, we are interested in one-parameter deformations of \( \varphi \) which produce a special kind of everywhere nonreduced multiple structure called *ropes*. Thus, the smooth subvarieties we construct by this method are smoothings of these ropes. The case when \( \varphi \) has degree 2 has been studied in great detail in [BG20]; the focus of the present article is the case when the degree of the finite cover is bigger than 2. The results we obtain have several applications. We sketch some of the compelling ones.

1. **Deformations of finite morphisms.** We study finite morphisms onto complete intersections of varying degrees. We study the deformations of arbitrary abelian covers of complete intersections in Section 3, and we show that they have wide ranging behavior. We produce families of examples in Section 4, and in Section 5, where the finite morphisms associated to a Galois cover deforms to finite morphisms of admissible lower degrees and families where the degree remains the same. In fact, we show that there are many families where the finite morphisms deform to one to one map and embeddings.

2. **Construction of canonical varieties in projective space.** Enriques in 1943 raised the question of the existence of canonical surfaces in projective spaces, i.e., those surfaces for which the canonical map is birational. We constructed various new families of such in [GGP10] for the case of algebraic surfaces. Our methods also enabled us to exhibit “hyperelliptic and trigonal components” in infinitely many moduli spaces of surfaces of general type. Here we systematically construct canonical varieties of general type in all dimensions. More generally, we construct \( s \)-subcanonical varieties \( X \), of \( \dim(X) = m \), appearing as a simple cyclic \( n \)-cover \( \pi : X \to Y \), branched along a smooth divisor \( D \in |\mathcal{O}_Y(kn)| \), over a complete intersection \( i : Y \hookrightarrow \mathbb{P}^N \) of multidegree \( d \), such that there exists a flat family of deformations \( \Phi : \mathcal{X} \to \mathbb{P}^N_T \) of \( \varphi := i \circ \pi \), for which \( \Phi_t \) is *birational* onto its image. We present some of the varieties below (see §4.3): the light blue rows indicate the varieties for which the \( \Phi_t \) is birational but not an embedding; the white rows indicate the varieties for which \( \Phi_t(X_t) \) is not a complete intersection if \( \Phi_t \) is an embedding.

| \( m \) | \( n \) | \( k \) | \( N \) | \( s \) | \( d \) | \( K_y^{N-1} \) |
|---|---|---|---|---|---|---|
| 17 | 4 | 2 | 19 | -1 | (6,7) | -168 |
| 16 | 4 | 2 | 18 | 0 | (6,7) | 0 |
| 15 | 4 | 2 | 17 | 1 | (6,7) | 168 |
| 17 | 4 | 2 | 20 | -1 | (2,6,6) | -288 |
| 16 | 4 | 2 | 19 | 0 | (2,6,6) | 0 |
| 15 | 4 | 2 | 18 | 1 | (2,6,6) | 288 |

We produce similar examples by starting with \( \mathbb{Z}_n \times \mathbb{Z}_2 \) covers, see §5.3 for details. We also construct varieties for which the \( s \)-subcanonical morphisms are embeddings, and that allow us to produce varieties with small codimension in the projective spaces.

3. **Construction of varieties with low codimension and asymptotic limits.** We construct infinitely many families of examples of smooth varieties with small codimension in projective spaces, all of which are embedded by complete linear series. Our methods provide constructions of smooth varieties whose codimension are in the range of Hartshorne’s conjecture and also smooth varieties outside this range. To do so, first we exhibit families of examples of complete intersections subvarieties that support ropes, then we smooth these ropes.
Ropes of multiplicity bigger than 2 are not complete intersections, not even locally so. Therefore we exhibit non-complete intersection subschemes in the range of the Hartshorne’s conjecture. We then prove that these embedded ropes, that arise from a first order deformation of a general abelian cover, can be deformed to smooth subvarieties. It can be shown that these ropes in the range of Hartshorne’s conjecture, when obtained while deforming simple cyclic covers or, more generally, compositions of simple cyclic covers, are in an irreducible component of the Hilbert scheme that parametrizes smooth complete intersection subvarieties. In codimension 2 any one-parameter deformation of these ropes are indeed complete intersections. This follows from Serre construction and a result of [KPR03] (see Corollary 3.11). However, if codimension is bigger than 2, there are no such methods to assure that any one-parameter deformation of the ropes is a complete intersection, even in the case of a simple cyclic cover, where it is shown a general deformation is a complete intersection. Many abelian covers of complete intersections whose codimension is in the Hartshorne’s range cannot be deformed to produce smooth subvarieties. This is due to the non-existence of appropriate embedded rope structures.

In contrast, we also construct in a systematic way, infinitely many examples of non complete intersection subvarieties, embedded by complete linear series, outside the range of Hartshorne’s conjecture. These examples show ropes which do not lie in any irreducible component of the Hilbert scheme parameterizing smooth complete intersection subvarieties. Our methods to deform abelian covers to embeddings via smoothing of ropes, demonstrate a way to construct examples of smooth subvarieties in a systematic manner. We give a detailed recipe on how to do it in general and we compute it for the case of simple cyclic covers and the case of a composition of simple cyclic cover and a double cover. These results also yield families with interesting asymptotic limits. We construct families of smooth varieties $X_i$ of $\dim X_i = m_i$ embedded in $\mathbb{P}^{N_i}$ of the following types:

1) (§ 4.2.2) $\lim m_i/N_i = 1/2$, where $X_i$ embedded by complete linear series in $\mathbb{P}^{N_i}$ are not complete intersections,

2) (§ 4.1.3, 4.1.2) Infinitely many varieties with $m_i/N_i = 2/3, 3/4$, and 1, where the special member is not a local complete intersection, but whose general member, which is a smooth variety, may or may not be a complete intersection. In fact we can get any rational number less than one as limit, see Example 4.1.1 for details.

We illustrate a few of those infinite classes of examples we construct. The entire family of examples in all of the cases can be found in Section 4 and 5 in the paper. In the following tables, $X$ is a smooth projective $s$–subcanonical variety with $m = \dim(X)$, and $X$ is embedded inside $\mathbb{P}^N$ by a complete linear series $|L|$, and $X$ is obtained by deforming an $n$–simple cyclic cover of branch degree $nk$ of a complete intersection of multidegree $d$.

(a) (See § 4.1.1) We present some of the first examples of smooth varieties of low codimension. In these examples, the general deformation of the finite morphism we are considering is an embedding whose image is a complete intersection (see Proposition 3.12).

| $m$ | $n$ | $k$ | $N$ | $s$ | $d$ | $K_X^m$ |
|-----|-----|-----|-----|-----|-----|----------|
| 16  | 4   | 2   | 19  | -2  | (2, 4, 6) | 192 \cdot 2^{16} |
| 15  | 4   | 2   | 18  | -1  | (2, 4, 6) | -192 |
| 14  | 4   | 2   | 17  | 0   | (2, 4, 6) | 0 |
| 13  | 4   | 2   | 16  | 1   | (2, 4, 6) | 192 |
| 25  | 5   | 2   | 29  | -2  | (2, 4, 6, 8) | -1920 \cdot 2^{25} |
| 24  | 5   | 2   | 28  | -1  | (2, 4, 6, 8) | 1920 |
| 23  | 5   | 2   | 27  | 0   | (2, 4, 6, 8) | 0 |
| 22  | 5   | 2   | 26  | 1   | (2, 4, 6, 8) | 1920 |
(b) (See § 4.2.1) The following table gives the first few examples of smooth varieties that are not complete intersections. These varieties are indeed embedded by complete linear series, but are outside the Hartshorne range.

| $m$ | $n$ | $k$ | $N$ | $s$ | $d$ | $K_X^m$ |
|-----|-----|-----|-----|-----|-----|--------|
| 3   | 3   | 2   | 8   | 15  | (4,4,4,4) | 3072 · 15³ |
| 3   | 3   | 2   | 8   | 16  | (4,4,4,4,5) | 3840 · 16³ |
| 3   | 4   | 2   | 9   | 32  | (6,6,6,6,6,6) | 186624 · 32⁵ |
| 3   | 4   | 2   | 9   | 33  | (6,6,6,6,6,7) | 217728 · 33³ |

The construction of a family of non–complete intersections, embedded by complete linear series, with asymptotic limit 1/2 has been constructed in Theorem 4.7.

(c) (See § 4.1.3) The following table illustrates some of the first sequences where $m_i/N_i$ is equal to 2/3 and 3/4. As stated above these are coming from smoothing ropes of multiplicity bigger than 2, which are not locally a complete intersection. For the light blue rows, any smooth subvariety obtained by a one-parameter deformation of such a rope is a complete intersection (see Corollary 3.11). For the white rows, the general deformation of the finite morphism we are considering is an embedding whose image is a complete intersection (see Proposition 3.12).

| $m$ | $n$ | $k$ | $N$ | $s$ | $d$ | $K_X^m$ |
|-----|-----|-----|-----|-----|-----|--------|
| 4   | 3   | 2   | 6   | 3   | (2,4) | 24 · 3⁴ |
| 6   | 4   | 2   | 9   | 8   | (2,4,6) | 192 · 8⁶ |
| 8   | 5   | 2   | 12  | 15  | (2,4,6,8) | 1920 · 15⁸ |

| $m$ | $n$ | $k$ | $N$ | $s$ | $d$ | $K_X^m$ |
|-----|-----|-----|-----|-----|-----|--------|
| 6   | 3   | 2   | 8   | 1   | (2,4) | 24 |
| 9   | 4   | 2   | 12  | 5   | (2,4,6) | 192 · 5³ |
| 12  | 5   | 2   | 16  | 11  | (2,4,6,8) | 1920 · 11² |

We produce analogous examples by working with $\mathbb{Z}_n \times \mathbb{Z}_2$ covers in Section 5. More generally, this article shows a way to find similar examples by deforming more general abelian covers of complete intersections and smoothing special multiple structures on them. This process is producing infinitely many families of examples of smooth subvarieties of small codimension with interesting invariants. Many of them are in relatively small dimensional projective spaces, as well. These include examples of Calabi-Yau varieties, Fano varieties and varieties of general type embedded by subcanonical linear systems.

Among these ropes, the ones arising when deforming simple cyclic and iterated simple cyclic covers and satisfying the conditions of Proposition 3.12 are limits of complete intersections. However, it is not clear whether the general member of every one-parameter deformation of these ropes is a smooth, complete intersection subvariety. Moreover, it is also not clear whether all ropes of small codimension that arise when we deform an arbitrary abelian cover to an embedding belong to a component of the Hilbert scheme whose general point parametrizes complete intersection subvarieties. It can be shown that the embedded ropes appearing in § 4.2.1, § 4.2.2, and in Subsection 5.2, that deform to non complete intersections, are not arithmetically Cohen-Macaulay. However, it is not clear whether the ropes whose codimension is in the Hartshorne’s range are arithmetically Cohen-Macaulay or not. The situation is further complicated by the existence of examples—see Remark 6.4—where we show a finite morphism that is unobstructed in its deformation space, but the corresponding rope is obstructed, in some of the cases because it lies in the intersection of two components of the Hilbert scheme.

In Section 6 of this article, we show that results we proved for the smooth case in Section 3, hold for normal, Cohen Macaulay varieties with canonical singularities. This would enable one to construct more low codimensional subvarieties of projective space with canonical singularities. The tables we compile throughout are very small samples of the examples that come from the main theorems in this article.

**Acknowledgements.** We sincerely thank Madhav Nori and Mohan Kumar for some motivational discussions and for generously sharing their valuable time listening to the first author talk about our results. In particular we thank Madhav Nori for insightful discussions on deformations of iterated simple cyclic
covers from a point of view different from one taken here, and Mohan Kumar on codimension two sub-varieties. We also thank Miguel González for helpful and motivating conversations.

2. Preliminaries

We work over the field of complex numbers. Our objects of study are morphisms to projective space that factor through an abelian cover, so we start recalling the definition and reviewing the properties of abelian covers we will use:

**Definition 2.1.** Let $Y$ be a variety and let $G$ be a finite abelian group. A Galois cover of $Y$ with Galois group $G$ is a finite flat morphism $\pi : X \to Y$ together with a faithful action of $G$ on $X$ that exhibits $Y$ as a quotient of $X$ via $G$. We call a Galois cover abelian, if $G$ is abelian.

Although in Sections 3, 4 and 5 the varieties $X$ are smooth, in Section 6 we will consider varieties $X$ with canonical singularities, so in the following discussion $\pi : X \to Y$ will be an abelian covers with Galois group $G$, $Y$ will be a smooth variety and $X$ will be a normal variety. The first crucial fact we will use is that $\pi_*\mathcal{O}_X$ splits as a direct sum of line bundles on $Y$. More precisely (see e.g. [Par91], (1.1))

\[ (2.1) \quad \pi_*\mathcal{O}_X = \bigoplus_{\chi \in G^*} L_\chi^{-1}, \]

where $L_\chi$ is a line bundle on $Y$, $G$ acts on $L_\chi$ via the character $\chi$ and the invariant summand $L_1$ is isomorphic to $\mathcal{O}_Y$.

The other key fact we need to know is the structure of $\pi_*\mathcal{N}_\pi$. For this we need to introduce some notation (see [Par91] for further details). Let $D$ be the branch divisor of $\pi$. Let $\mathcal{C}$ be the set of cyclic subgroups of $G$ and for all $H \in \mathcal{C}$, denote by $S_H$ the set of generators of the group of characters $H^*$. Then, we may write

\[ D = \sum_{H \in \mathcal{C}} \sum_{\psi \in S_H} D_{H,\psi}, \]

where $D_{H,\psi}$ is the sum of all the components of $D$ that have inertia group $H$ and character $\psi$. For every $\chi \in G^*$, and $H \in \mathcal{C}$, and for every $\psi \in S_H$, one may write $\chi|_H = \psi^{i_\chi}$, $i_\chi \in \{0, \ldots, m_H - 1\}$, where $m_H$ is the order of $H$. For every character $\chi \in G^*$, let

\[ S_\chi = \{ (H, \psi) : \chi|_H \neq \psi^{m_H-1} \}. \]

For a variety $Z$, $T_Z$ denotes its tangent sheaf. Furthermore, if $Z$ is smooth and $D$ is a divisor on $Z$, $\Omega_Z^p(\log D)$ denotes the sheaf of logarithmic differential $p$-forms.

The following is a generalization of [Par91], Corollary 4.1 to the case $X$ normal.

**Proposition 2.2.** Let $\pi : X \to Y$ be an abelian cover with Galois group $G$, $X$ normal and $Y$ smooth. Assume the branch divisor $D$ of $\pi$ is normal crossing. Then,

\[ (\pi_*\mathcal{N}_\pi)^X \cong \bigoplus_{(H,\psi) \in S_\chi} \mathcal{O}_{D_{H,\psi}}(D_{H,\psi}) \star L_\chi^{-1}. \]

**Proof.** We have an exact sequence

\[ 0 \to T_X \to \pi^*T_Y \to \mathcal{N}_\pi \to 0. \]

Now since $\pi$ is finite and all sheaves in the exact sequence are quasi-coherent we have that the pushforward is an exact functor and consequently we get the following exact sequence

\[ 0 \to \pi_*T_X \to \pi_*\pi^*T_Y \to \pi_*\mathcal{N}_\pi \to 0. \]

Now observe that $(\pi_*\pi^*T_Y)^X = T_Y \star L_\pi^{-1}$. Also note that $\pi_*T_X$ is a reflexive sheaf and hence it is enough to determine $\pi_*T_X$ in an open subset of codimension $\geq 2$. Consider the set $S$ which is the image of the singular locus of $X$ under $\pi$. Since $X$ is normal we have that $S$ is of codimension at least two in $Y$. Removing $S$ and $\pi^{-1}(S)$ (which is also of codimension 2 in $X$ since $\pi$ is finite) from $X$ and $Y$ respectively,
we can assume that $X$ and $Y$ are smooth hence by [Par91], Proposition 4.1(b) we have that $(\pi_\ast T_X)^T = T_Y (-(\log D_{H,\psi}),(H,\psi) \in S_T) \otimes L^{-1}_X$. Since $Y$ is smooth and $D_{H,\psi}$ is a normal crossing divisor (since $D$ is normal crossing) we have that $T_Y (-(\log D_{H,\psi}),(H,\psi) \in S_T)$ is locally free and the conclusion follows.

One of the central tools for deforming a finite morphism to a morphism of smaller degree is to construct a suitable multiple structure, called rope, on the image of the morphism.

**Definition 2.3.** Let $Y$ be a reduced connected scheme and let $\mathcal{E}$ be a vector bundle of rank $m - 1$ on $Y$. A rope of multiplicity $m$ on $Y$ with conormal bundle $\mathcal{E}$ is a scheme $\tilde{Y}$ with $\tilde{Y}_{\text{red}} = Y$ such that

1. $\mathcal{F}_{\tilde{Y}/Y} = 0$,
2. $\mathcal{F}_{\tilde{Y}/Y} = \mathcal{E}$ as $\mathcal{O}_Y$ modules.

If $\mathcal{E}$ is a line bundle then $\tilde{Y}$ is called a ribbon on $Y$.

Recall that, for a morphism $\varphi : X \to \mathbb{P}^N$ from a smooth, projective variety which is finite onto a smooth variety $Y \to \mathbb{P}^N$, the space $H^0(\mathcal{N}_\varphi)$ parametrizes the space of infinitesimal deformations of $\varphi$. Suppose $\mathcal{E}$ is the trace zero module of the induced morphism $\pi : X \to Y$. It is shown in [Gon06], Proposition 2.1, that the space $H^0(\mathcal{N}_{Y/p}\otimes \mathcal{E})$ parametrizes the pairs $(\tilde{Y}, \tilde{i})$ where $\tilde{Y}$ is a rope on $Y$ with conormal bundle $\mathcal{E}$ and $\tilde{i} : \tilde{Y} \to \mathbb{P}^N$ is a morphism that extends $i$. The relation between these two cohomology groups is given by the following proposition.

**Proposition 2.4.** ([Gon06], Proposition 3.7) Let $X$ be a smooth variety and let $\varphi : X \to \mathbb{P}^N$ be a morphism that factors as $\varphi = i \circ \pi$, where $\pi$ is a finite cover of a smooth variety $Y$ and $i : Y \to \mathbb{P}^N$ is an embedding. Let $\mathcal{E}$ be the trace zero module of $\pi$ and let $\mathcal{F}$ be the ideal sheaf of $i(Y)$. There exists a homomorphism

$$H^0(\mathcal{N}_\varphi) \xrightarrow{\psi} \text{Hom}(\pi^*(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_X)$$

that appears when taking cohomology on the commutative diagram [Gon06] (3.3.2). Since

$$\text{Hom}(\pi^*(\mathcal{I}/\mathcal{I}^2), \mathcal{O}_X) = H^0(\mathcal{N}_{Y/p}\otimes \mathcal{E}),$$

the homomorphism $\psi$ has two components;

$$H^0(\mathcal{N}_\varphi) \xrightarrow{\psi_1} H^0(\mathcal{N}_{Y/p}\otimes \mathcal{E}), \quad \text{and} \quad H^0(\mathcal{N}_\varphi) \xrightarrow{\psi_2} H^0(\mathcal{N}_{Y/p}\otimes \mathcal{E}).$$

The following is our main tool that we will use throughout this article.

**Theorem 2.5.** ([GGP10], Theorem 1.4) Let $X$ be a smooth projective variety and let $\varphi : X \to \mathbb{P}^N$ be a morphism that factors through an embedding $Y \to \mathbb{P}^N$ with $Y$ smooth. Let $\pi : X \to Y$ be the induced morphism which we assume to be finite of degree $n \geq 2$. Let $\tilde{\varphi} : \tilde{X} \to \mathbb{P}^N (\Delta = \text{Spec} \left\langle \frac{e_1d_1}{e} \right\rangle)$ be a locally trivial first order infinitesimal deformation of $\varphi$ and let $\nu \in H^0(\mathcal{N}_{\tilde{\varphi}})$ be the class of $\tilde{\varphi}$. If

(a) the homomorphism $\psi_2(\nu)$ has rank $k > \frac{2}{3} - 1$,

(b) there exists an algebraic formally semiuniversal deformation of $\varphi$ and $\varphi$ is unobstructed,

then there exists a flat family of morphisms, $\Phi : \mathcal{X} \to \mathbb{P}^N$ over $T$, where $T$ is a smooth irreducible algebraic curve with a distinguished point $0$, such that

1. $\mathcal{X}_t$ is a smooth irreducible projective variety,
2. the restriction of $\Phi$ to the first order infinitesimal neighbourhood of $0$ is $\tilde{\varphi}$, and
3. for $t \neq 0$, $\Phi_t$ is finite one-to-one onto its image in $\mathbb{P}^N_t$.

The first and the second author, along with González, in fact gave a criterion under which a finite map deforms to an embedding.

**Theorem 2.6.** ([GGP13a], Theorem 1.5) Under the assumption of Theorem 2.5, suppose moreover $\psi_2(\nu)$ is a surjective homomorphism. Then there exists a flat family of morphisms, $\Phi : \mathcal{X} \to \mathbb{P}^N$ over $T$, where $T$ is a smooth irreducible algebraic curve with a distinguished point $0$, such that

1. $\mathcal{X}_t$ is a smooth irreducible projective variety,
2. the restriction of $\Phi$ to the first order infinitesimal neighbourhood of $0$ is $\tilde{\varphi}$, and
(3) for \( t \neq 0 \), \( \Phi_t \) is a closed immersion into \( \mathbb{P}^N_t \).

The authors of [GGP13a] in fact showed that under the assumptions above, \((\text{Im} \Phi)_0\) is an embedded rope \( \tilde{Y} \) corresponding to \( \psi_2(v) \). It is clear from the above theorems that we need to produce homomorphisms of appropriate rank between the vector bundles \( \mathcal{I}, \mathcal{I}^2 \) and \( \mathcal{E} \). In order to do that, we need a theorem of Bănică. We need the following definition in order to state the theorem.

**Definition 2.7.** Given a morphism \( v: \mathcal{E} \rightarrow \mathcal{F} \) between vector bundles of rank \( e \) and \( f \) respectively, on an irreducible complex projective variety \( X \). For any positive integer \( k \leq \min(e, f) \), we define the \( k \)-th degeneracy locus \( D_k(v) \) as the subscheme cut out by the minors of order \( k + 1 \) of the matrix locally representing \( v \).

Now we state the result of Bănică. We will use this result to deform a finite morphism to a birational morphism.

**Theorem 2.8.** ([Bănică96], §4.1) Let \( \mathcal{E} \) and \( \mathcal{F} \) be vector bundles on a projective variety \( X \) of rank \( e \) and \( f \) respectively. Assume \( \mathcal{E}^* \otimes \mathcal{F} \) is globally generated. Then, for a general morphism \( v: \mathcal{E} \rightarrow \mathcal{F} \), the subschemes \( D_k(v) \) either are empty or have pure codimension \( (e - k)(f - k) \) in \( X \) and the singular locus \( \text{Sing}(D_k(v)) = D_{k-1}(v) \).

The double covers of complete intersections are studied in [BG20] in more detail. In case of double covers, there are two possibilities, namely the general deformation is either birational, or it is generically finite of degree 2. However, for higher degree covers, the degree of general deformation might drop, but it might not be birational onto its image. We will show that this case indeed appears. In order to prove that, the following remark, and the proposition after that, are crucial.

**Remark 2.9.** Let \( \pi: X \rightarrow Y \) be an abelian cover with \( X \) and \( Y \) smooth projective varieties. Suppose \( p: \mathcal{Y} \rightarrow T \) be a deformation of \( Y \) (\( p \) is proper, flat and surjective) over a smooth curve \( T \). Then there exists a deformation \( \Pi: \mathcal{X} \rightarrow \mathcal{Y} \rightarrow T \) of \( \pi \) over \( T \).

The following result is the consequence of [Weh86], Proposition 1.10.

**Proposition 2.10.** Let \( \pi: X \rightarrow Y \) be a finite, flat morphism between smooth projective varieties with trace zero module \( \mathcal{E} \) and let \( \psi: Y \rightarrow Z \) be a non-degenerate morphism to a smooth variety \( Z \). Let \( \varphi = \psi \circ \pi \) be the composed morphism. If \( H^0(\mathcal{N}_\psi \otimes \mathcal{E}) = 0 \) then the natural map between the functors \( \text{Def}(\pi / Z) \rightarrow \text{Def}_\varphi \) is smooth.

**Proof.** We apply [Weh86], Proposition 1.10 to the following commutative diagram.

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & Y \\
\downarrow{\varphi} & \nearrow{\psi} \\
Z
\end{array}
\]

The maps \( \beta_1 \) and \( \beta_2 \) of [Weh86], Proposition 1.10 become the following:

\[
\beta_1: H^0(\mathcal{N}_\varphi) \rightarrow H^0(\pi^* \mathcal{N}_\psi), \quad \beta_2: H^1(\mathcal{N}_\varphi) \rightarrow H^1(\pi^* \mathcal{N}_\psi).
\]

The assertion follows since the map \( \beta_2 \) is always injective and \( \beta_1 \) is surjective if \( H^0(\mathcal{N}_\psi \otimes \mathcal{E}) = 0 \).

We will mainly study \( s \)-subcanonical covers, we include the definition below.

**Definition 2.11.** Let \( X \) be a smooth projective variety and let \( \varphi: X \rightarrow \mathbb{P}^N \) be a morphism. Let \( L = \varphi^* \mathcal{O}_{\mathbb{P}^N}(1) \) and \( s \in \mathbb{Z} \).

The polarized variety \( (X, L) \) is said to be \( s \)-subcanonical if \( K_X = \varphi^* \mathcal{O}_{\mathbb{P}^N}(s) \). Moreover, \( \varphi \) is called \( s \)-subcanonical if \( (X, L) \) is \( s \)-subcanonical and \( \varphi \) is induced by the complete linear series \( |L| \).

**Remark 2.12.** We note that, an \( s \)-subcanonical polarized scheme \( (X, L) \) is
(1) a Fano polarized scheme of index \(-s\), in the sense of Fujita (see [Fuj83], Definition 1.5), if \(s < 0\);
(2) a polarized scheme of Calabi–Yau, if \(s = 0\);
(3) a canonically polarized scheme of general type, if \(s = 1\);
(4) an \(s\)-subcanonical polarized scheme of general type, if \(s > 0\).

3. Deformations of abelian covers of complete intersections

In this section, we will prove the main results that we need to construct subvarieties in \(\mathbb{P}^N\). We start from a given complete intersection \(i : Y \hookrightarrow \mathbb{P}^N\), construct abelian cover \(\pi : X \to Y\) over \(Y\), and we will deform the composed morphism \(\varphi = i \circ \pi\). Here is our set-up.

**Set-up 3.1.** Throughout the article, unless otherwise stated, we will use the following set-up.

1. Let \(i : Y \hookrightarrow \mathbb{P}^N\) be a smooth complete intersection of multidegree \(d = (d_1, d_2, \ldots, d_r)\), with \(r \geq 1\) and \(2 \leq d_1 \leq d_2 \leq \cdots \leq d_r\). We will assume that \(\dim(Y) = N - r \geq 3\). Set

\[
\delta = \sum_{i=1}^r d_i \quad \text{and} \quad d = \prod_{i=1}^r d_i.
\]

2. Let \(Y\) be as in (1) and let \(\pi : X \to Y\) be a finite abelian Galois cover from a smooth variety \(X\). Let \(\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}\), where \(\mathcal{O}_Y(1) := i^* \mathcal{O}_{\mathbb{P}^N}(1)\), and

\[
\mathcal{E} = \bigoplus_{i=1}^{n-1} \mathcal{O}_Y(-k_i) \quad \text{i.e.,} \quad \pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \bigoplus_{i=1}^{n-1} \mathcal{O}_Y(-k_i)
\]

with \(1 \leq k_1 \leq k_2 \leq \cdots \leq k_{n-1}\) where \(n = \deg(\pi) \geq 2\). Set \(\varphi = i \circ \pi\).

**Remark 3.2.** We make a note of the following facts that we will use without explicitly stating.

1. Since \(Y\) is a complete intersection in \(\mathbb{P}^N\), it follows from the Grothendieck–Lefschetz theorem (recall that \(\dim(Y) \geq 3\)) that any line bundle on \(Y\) is the restriction of a line bundle on \(\mathbb{P}^N\). It follows that \(\text{Pic}(Y) = \mathbb{Z}\).

2. Since \(Y\) is a complete intersection of multidegree \(d = (d_1, \cdots, d_r)\), it follows that

\[
\mathcal{N}_{Y/\mathbb{P}^N} = \bigoplus_{i=1}^r \mathcal{O}_Y(d_i).
\]

3. For \(a \in \mathbb{Z}\), \(H^0(\mathcal{O}_Y(a)) \neq 0 \iff a \geq 0 \iff \mathcal{O}_Y(a)\) is globally generated.

4. For \(a \in \mathbb{Z}\), \(H^i(\mathcal{O}_Y(a)) = 0\), for all \(1 \leq i \leq N - r - 1\).

5. \(\varphi\) is induced by the complete linear series \(|\pi^* \mathcal{O}_Y(1)|\) if and only if \(k_1 \geq 2\).

We need the following important lemma.

**Lemma 3.3.** In the situation of Set-up 3.1, \(\varphi\) has an algebraic formally semiuniversal deformation space and it is unobstructed. Moreover, \(H^1(\mathcal{N}_\varphi) = 0\) and the map (see Proposition 2.4)

\[
\psi_2 : H^0(\mathcal{N}_\varphi) \to H^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E})
\]

is surjective, where \(\mathcal{E}\) is the trace zero module of \(\pi\).

**Proof.** Notice that \(H^2(\mathcal{O}_X) = H^2(\mathcal{O}_Y) \oplus H^2(\mathcal{E})\). Consequently, by [BGG20], Proposition 1.5, the morphism \(\varphi\) has an algebraic formally semiuniversal deformation space since by Remark 3.2, (4) \(H^2(\mathcal{O}_X) = 0\). It follows from Proposition 2.2 that \(H^1(\mathcal{N}_\varphi) = H^1(\pi_* \mathcal{N}_\varphi) = 0\). In particular, \(\psi_2\) surjects. By projection formula and Remark 3.2, (4), we get that \(H^1(\pi^* \mathcal{N}_{Y/\mathbb{P}^N}) = H^1(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E}) = 0\). Then, from the following exact sequence (see [Gon06], Lemma 3.3)

\[
0 \to \mathcal{N}_\pi \to \mathcal{N}_\varphi \to \pi^* \mathcal{N}_{Y/\mathbb{P}^N} \to 0,
\]

we get \(H^1(\mathcal{N}_\varphi) = 0\).

The following proposition gives a criterion under which the degree of a finite morphism remains unchanged for any of its deformation.
Proposition 3.4. In the situation of Set-up 3.1, if \( d_r < k_1 \) then any deformation of \( \varphi \) is finite of degree \( n \) onto its image, which is a smooth, complete intersection subvariety of \( \mathbb{P}^N \) of multidegree \( \underline{d} = (d_1, d_2, \ldots, d_r) \).

Proof. By hypothesis, and Remark 3.2, (2), we obtain \( H^0(\mathcal{N}_{Y/\mathbb{P}^N} \otimes \mathcal{E}) = 0 \). The conclusion follows from Proposition 2.10 and [Ser75].

Our methods of deformations of the finite morphism to an embedding is intimately related to the existence of some special rope structure on \( Y \). We construct ropes of suitable ranks on \( Y \) and we smooth them. This in turn deforms the finite morphism to an embedding or a birational map. One can explicitly construct smoothing family of split multiple structures, but our constructions are general.

Now we state our main theorem that gives a criterion to determine when the general deformation of the covering morphism is of degree one. We study the two cases possible, namely the cases in which the deformed morphism is birational onto its image, and the cases in which the deformed morphism is an embedding. Our result is a consequence of the existence of appropriate ropes.

Theorem 3.5. In the situation of Set-up 3.1, the following happens.

1. Let \( r > \left\lfloor \frac{n}{2} \right\rfloor - 1 \) and \( d_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \geq k_{n-1} \), then a general element of the algebraic formally semiuniversal deformation space of \( \varphi \) is finite and birational onto its image.

2. Let either of the following conditions (a) or (b) hold:
   a. \( r \geq n-1 \) and there exists \( 1 \leq l_1 < l_2 < \cdots < l_{n-1} \), such that \( d_{l_j} = k_j \) for all \( 1 \leq j \leq n-1 \);
   b. \( r > \frac{n-2}{2} \), and \( d_{2r+2-n-N} \geq k_{n-1} \).

Then there exist embedded ropes \( Y \hookrightarrow \mathbb{P}^N \) on \( Y \) with conormal bundle \( \mathcal{E} \), which are non-split if \( \varphi \) is induced by a complete linear series.

For any embedded rope \( Y \hookrightarrow \mathbb{P}^N \) on \( Y \) with conormal bundle \( \mathcal{E} \), there exists a flat family \( \Phi : \mathcal{X} \rightarrow \mathbb{P}^N_T \) over \( T \), where \( T \) is a smooth irreducible algebraic curve with a distinguished point \( 0 \), such that

1. \( \mathcal{X}_0 \) is a smooth irreducible projective variety,
2. \( \mathcal{X}_0 = X, \Phi_0 = \varphi \), and
3. for \( t \neq 0, \Phi_t \) is a closed immersion onto \( \mathbb{P}^N_T \) and \( \text{Im} \Phi_0 \).

Consequently, a general element of the algebraic formally semiuniversal deformation space of \( \varphi \) is an embedding.

Remark 3.6. Observe that \( 2r + 2 - n - N = r - (N - r) - (n - 2) < r \) as \( n \geq 2 \) and \( N - r \geq 3 \).

Proof of Theorem 3.5. First we prove (1). Let \( \mathcal{N}' = \mathcal{O}_Y(d_a) \oplus \mathcal{O}_Y(d_{a+1}) \oplus \cdots \oplus \mathcal{O}_Y(d_r) \) where \( a = r - \left\lfloor \frac{n}{2} \right\rfloor + 1 \) and let \( \mathcal{E} \) be the trace zero module of \( \pi \). Notice that \( \text{rank}(\mathcal{N}') = \left\lfloor \frac{n}{2} \right\rfloor \). The condition \( d_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \geq k_{n-1} \) guarantees that \( \mathcal{N}' \otimes \mathcal{E} \) is globally generated. It follows from Theorem 2.8 that, for a general homomorphism \( \sigma : \mathcal{N}'^+ \rightarrow \mathcal{E}, D_{\left\lfloor \frac{n}{2} \right\rfloor - 1}(\sigma) \) is a proper closed subvariety inside \( Y \). Consequently, there exists a rank \( \left\lfloor \frac{n}{2} \right\rfloor \) homomorphism in \( \text{Hom}(\mathcal{N}'^+, \mathcal{E}) \), which in turn implies the existence of a rank \( \left\lfloor \frac{n}{2} \right\rfloor \) homomorphism in \( \text{Hom}(\mathcal{N}'^*_{Y/\mathbb{P}^N}, \mathcal{E}) \). The assertion follows from Theorem 2.5 and Lemma 3.3, since birationality is an open condition.

Now we prove (2). We aim to show that if (a) or (b) holds, then there exists a surjective homomorphism in \( \text{Hom}(\mathcal{N}'^*_{Y/\mathbb{P}^N}, \mathcal{E}) \). Clearly that is the case if (a) holds. Now, assume (b) holds. As before, let \( a = 2r + 2 - n - N \) and set \( \mathcal{N}'' = \mathcal{O}_Y(d_a) \oplus \mathcal{O}_Y(d_{a+1}) \oplus \cdots \oplus \mathcal{O}_Y(d_r) \). Notice that \( \mathcal{N}'' \) is a vector bundle of rank \( (N - r) + (n - 1) \) and the condition \( d_{2r+2-n-N} \geq k_{n-1} \) guarantees that \( \mathcal{N}' \otimes \mathcal{E} \) is globally generated. It follows from Theorem 2.8 that, for a general homomorphism \( \sigma' : \mathcal{N}''^+ \rightarrow \mathcal{E}, D_{n-2}(\sigma') \) is empty. Indeed, otherwise it has expected codimension

\[
((N - r) + (n - 1) - (n - 2))((n - 1) - (n - 2)) = N - r + 1 > N - r
\]

which is impossible. Thus, \( \sigma' \) can be extended to a surjective homomorphism in \( \text{Hom}(\mathcal{N}'^*_{Y/\mathbb{P}^N}, \mathcal{E}) \).

Now, any surjective homomorphism of \( \text{Hom}(\mathcal{N}'^*_{Y/\mathbb{P}^N}, \mathcal{E}) \) corresponds to an embedded rope \( Y \) on \( Y \), with conormal bundle \( \mathcal{E} \). Assume \( \varphi \) is induced by the complete linear series \( |\pi^* \mathcal{O}_Y(1)| \) (equivalently,
$k_1 \geq 2$). Then any embedded rope $\tilde{Y} \hookrightarrow \mathbb{P}^N$ on $Y$ with conormal bundle $\mathcal{E}$ is non–split. Indeed, it follows from the long exact sequence associated to the restricted Euler sequence twisted by $\mathcal{E}$:
$$0 \to \mathcal{E} \to \mathcal{E}(1)^{\oplus N+1} \to T_{\mathbb{P}^N|Y} \otimes \mathcal{E} \to 0$$
that $H^0(T_{\mathbb{P}^N|Y} \otimes \mathcal{E}) = H^1(T_{\mathbb{P}^N|Y} \otimes \mathcal{E}) = 0$. Consequently, by the long exact sequence associated to the following exact sequence,
$$0 \to T_Y \otimes \mathcal{E} \to T_{\mathbb{P}^N|Y} \otimes \mathcal{E} \to \mathcal{N}_{\mathbb{P}^N/Y} \otimes \mathcal{E} \to 0,$$
it follows that $H^0(\mathcal{N}_{\mathbb{P}^N/Y} \otimes \mathcal{E}) \cong H^1(T_Y \otimes \mathcal{E})$. Thus, the assertion follows from the fact that the class of a surjective homomorphism in $H^0(\mathcal{N}_{\mathbb{P}^N/Y} \otimes \mathcal{E})$ is non–zero.

The rest of the assertions of the statement of the theorem are consequences of Theorem 2.6, Lemma 3.3, [GGP13a], Theorem 2.2, and the fact that being an embedding is an open condition. 

Next theorem gives a criterion under which the degree of the deformed morphism is one half of the initial degree. We will see examples of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_3 \times \mathbb{Z}_2$ covers whose general deformation is finite of degree 2 and 3 onto its image which is a smooth variety.

**Theorem 3.7.** In the situation of Set-up 3.1, assume $n$ is even, $n \geq 4$, and the following holds;

1. $\pi = p_1 \circ \pi_1$ where $\pi_1 : X \to X_1$ is a $\frac{n}{2}$ cover with trace zero module 
$$\mathcal{E}_2 = p_1^* \left( \mathcal{O}_Y(-k'_1) \oplus \cdots \oplus \mathcal{O}_Y(-k'_{\frac{r}{2} - 1}) \right)$$
and $p_1 : X_1 \to Y$ is a double cover with trace zero module $\mathcal{E}_2 = \mathcal{O}_Y(-l)$,
2. $k'_l > \max \{ 2l, d_r \}$ and $d_r \geq l$.

Then, a general element of the algebraic formally semiuniversal deformation space of $\varphi$ is generically finite of degree $\frac{n}{2}$. If, moreover, one of the following holds;

1'. $d_s = l$ for some $1 \leq s \leq r$, or
2'. $r > \frac{N}{2}$, and $d_{2r-N} \geq l$,

then, a general element of the algebraic formally semiuniversal deformation space of $\varphi$ is finite of degree $\frac{n}{2}$, onto a smooth variety.

**Proof.** We have the following short exact sequence where $\varphi_1 = i \circ p_1$ (see [Gon06], Lemma 3.3);

(3.2) $$0 \to \mathcal{N}_{\varphi_1} \to \mathcal{N}_{\varphi_1} \to p_1^* \mathcal{N}_{\mathbb{P}^N/Y} \to 0.$$ 

Note that $h^0(\mathcal{N}_{p_1} \otimes \mathcal{E}_2) = \sum_{i=1}^{\frac{n}{2}-1} h^0(\mathcal{O}_B(2l - k'_i))$, where $B \in |\mathcal{O}_Y(2l)|$ is the branch divisor of $p_1$. It is easy to see from the following exact sequence;

$$0 \to \mathcal{O}_Y(-k'_l) \to \mathcal{O}_Y(2l - k'_l) \to \mathcal{O}_B(2l - k'_l),$$

that $h^0(\mathcal{N}_{p_1} \otimes \mathcal{E}_2) = \sum_{i=1}^{\frac{n}{2}-1} h^0(\mathcal{O}_Y(2l - k'_i))$. By assumption, $k'_i > 2l$, and consequently, $k'_j > 2l$, for all $i$. Thus, $h^0(\mathcal{N}_{p_1} \otimes \mathcal{E}_2) = 0$. Also, since $k'_j > d_j$, we obtain

$$h^0(p_1^*(\mathcal{N}_{\mathbb{P}^N/Y} \otimes \mathcal{E}_2)) = \sum_{i=1}^{\frac{n}{2}-1} \sum_{j=1}^{\frac{n}{2}-1} h^0(\mathcal{O}_Y(d_i - k'_j - l)) = 0.$$ 

Consequently, by tensoring the exact sequence (3.2) by $\mathcal{E}_2$ and taking the long exact sequence of cohomology, one finds that $h^0(\mathcal{N}_{\varphi_1} \otimes \mathcal{E}_2) = 0$. It follows from Proposition 2.10 that any deformation of $\varphi$ is of degree greater than or equal to $\frac{n}{2}$, since it factors through a deformation of $\pi_1$. 

Since $\varphi$ is unobstructed (Lemma 3.3), $h^1(\mathcal{N}_{\varphi_1}) = 0$ and $h^0(\mathcal{N}_{\mathbb{P}^N/Y} \otimes \mathcal{O}_Y(-l)) \neq 0$, it follows that there exists a deformation $\Psi : \mathcal{X}_1 \to \mathbb{P}_T^N$ over a smooth pointed affine algebraic curve $(T, 0)$ for which $\Psi_0 = \varphi_1$ and $\Psi_t$ is of degree 1 for $t \neq 0$. It follows from Remark 2.9, that, possibly after shrinking $T$, there exists $\Pi : \mathcal{X} \to \mathcal{X}_1$ such that $\Pi_0 = \pi_1$. Thus, $\Phi = \Psi \circ \Pi : \mathcal{X} \to \mathbb{P}_T^N$ is a deformation of $\Phi_0 = \varphi$ such that $\Phi_t$ of
degree $\frac{\delta}{2}$ for all $t \neq 0$. Thus, a general element of the algebraic formally semiuniversal deformation space of $\varphi$ will have degree less than or equal to $\frac{\delta}{2}$. The conclusion follows since we have already showed that any deformation of $\varphi$ is of degree greater than or equal to $\frac{n}{2}$.

The last assertion is clear from Theorem 3.5, since under the assumption, $\varphi_1$ deforms to an embedding, and being an embedding is an open condition. That completes the proof. \hfill $\square$

We will systematically produce examples of covers of complete intersections that will deform to embeddings and the embedded varieties will be non–complete intersections. In order to do that, we need the following important lemma.

**Lemma 3.8.** In the situation of Set-up 3.1, let $K_X = \varphi^* \mathcal{O}_{\mathbb{P}^N}(\delta - N - 1 + a)$. Assume $\Phi : \mathcal{X} \to \mathbb{P}^N$ is a flat family of deformations of $\varphi : X \to \mathbb{P}^N$ over a smooth curve $(T, 0)$, such that $\Phi_t : \mathcal{X}_t \to \mathbb{P}^N_t$ is an embedding for $t \neq 0$, and $X_t$ is a complete intersection of multidegree $d = (d'_1, d'_2, \ldots, d'_r)$. Then,

$$\begin{align*}
(a) \sum_{i=1}^{r} d'_i &= \delta + a, \quad \text{and} \\
(b) \prod_{i=1}^{r} d'_i &= n \prod_{i=1}^{r} d_i.
\end{align*}$$

**Proof.** Notice that a general deformation of the canonical bundle of $X$ remains canonical. Since, $K'_X = \varphi_1^* \mathcal{O}_{\mathbb{P}^N_1}(\sum_{i=1}^{r} d'_i - N - 1)$, we get that $\sum_{i=1}^{r} d'_i - N - 1 = \delta - N - 1 + a$, consequently (a) follows. To see (b), notice that

$$n \prod_{i=1}^{r} d_i = \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)^{N-r} = \varphi_1^* \mathcal{O}_{\mathbb{P}^N_1}(1)^{N-r} = \prod_{i=1}^{r} d'_i.$$

The proof is now complete. \hfill $\square$

**Remark 3.9.** In the situation of Set-up 3.1, assume that $\varphi$ is induced by the complete linear series $|L| = \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)$ (this condition is equivalent to $k_i \geq 2$, for all $i$). Then, for any deformation $\Phi : \mathcal{X} \to \mathbb{P}^N$ of $\varphi$ over a smooth curve $(T, 0)$, we may assume (possibly after shrinking $T$), that there is a line bundle $\mathcal{L}$ on $\mathcal{X}$, such that $\Phi_t$ is induced by the complete linear series $|\mathcal{L}|$. This is a consequence of semicontinuity and the fact that being non–degenerate morphism (in the sense that the image is not contained in any hyperplane) is an open condition.

We now show that codimension two examples produced by Theorem 3.5 (2) (a) are always complete intersections. We illustrate this fact by means of the following lemma and its corollary. In what follows, for a subvariety $j : Z \hookrightarrow \mathbb{P}^M$, we will have $\mathcal{O}_Z(1) = j^* \mathcal{O}_{\mathbb{P}^M}(1)$, and $H^i_s(\mathcal{F}) := \bigoplus_{v \in Z} H^i(\mathcal{F}(v))$.

**Lemma 3.10.** In the situation of Set-up 3.1, assume the hypothesis of Theorem 3.5 (2) (a) or (b) is satisfied. Let $(T, 0)$ be a smooth irreducible curve satisfying the conditions (I), (II), and (III) of Theorem 3.5, (2). Then, possibly after shrinking $T$, $H^i_s(\mathcal{I}_{X_t}) = 0$ for all $2 \leq i \leq m$ and $t \neq 0$, where $\mathcal{I}_{X_t}$ is the ideal sheaf of $X_t$ inside $\mathbb{P}^N_t$.

**Proof.** We have the short exact sequence $0 \to \mathcal{I}_{X_t} \to \mathcal{O}_{\mathbb{P}^N_t} \to \mathcal{O}_{X_t} \to 0$. It is easy to see that $H^i(\varphi^* \mathcal{O}_{\mathbb{P}^N}(k)) = 0$ for all $1 \leq i \leq m - 1, k \in \mathbb{Z}$. The conclusion follows by semicontinuity and the fact that $H^i_s(\mathcal{O}_{\mathbb{P}^N}) = 0$ for all $2 \leq i \leq m$. \hfill $\square$

**Corollary 3.11.** In the situation of Set-up 3.1, assume $r = 2$. Assume the hypothesis of Theorem 3.5 (2) (a) is satisfied, and let $(T, 0)$ be a smooth irreducible curve satisfying the conditions (I), (II), and (III) of Theorem 3.5, (2). Then (shrinking $T$ if necessary), $\varphi_t : X_t \hookrightarrow \mathbb{P}^N_t$ embeds $X_t$ as a complete intersection, for $t \neq 0$.

**Proof.** Notice that codim$(X_t/\mathbb{P}^N_t) = 2$. The following exact sequence follows from [OSS80], Theorem 5.1.1; $0 \to \mathcal{O}_{\mathbb{P}^N_t} \to \mathcal{E}'_t \to \mathcal{I}_{X_t}(1) \to 0$, where $\mathcal{E}'_t$ is the rank 2 bundle associated to $X_t$, and $\text{det}(\mathcal{N}_{X_t/\mathbb{P}^N_t}) = \mathcal{O}_{\mathbb{P}^N_t(t)}|_{X_t}$ (see for example the theorem of Barth, [Laz80]). We also know from [OSS80], Lemma 5.2.1 that $\mathcal{E}'_t$ is split if and only if $X_t$ is a complete
intersection. To this end, we apply [KPR03], Theorem 1. We obtain from Lemma 3.10 that $H^i_s(\mathcal{A}_X(k)) = 0$ for $2 \leq i \leq N - 2$. Thus, $H^i_s(\mathcal{E}_t^i) = 0$ for all $2 \leq i \leq N - 2$, since $H^i_s(\mathcal{O}_{P'}(n)) = 0$ when $i$ is in the same range. □

Proposition 3.12 below, which was inspired by conversations with Nori, shows that some of the ropes that appear in Theorem 3.5 (2) (only the ones satisfying (i) and (ii) below) correspond to points that lie in an irreducible component of the Hilbert scheme whose general points correspond to smooth complete intersections. However, the general arguments below cannot determine if the general members of every one-parameter smoothing of those ropes are complete intersections (see Question 6.2).

**Proposition 3.12.** Let $X, Y, \varphi, \pi, n$ and $\mathcal{E}$ be as in Set-up 3.1. Assume that

1. $\pi = \pi_1 \circ \cdots \circ \pi_1$ is the composition of simple cyclic covers $\pi_1, \ldots, \pi_1$ such that, for each $1 \leq \mu' \leq \mu$, $\pi_{\mu'}$ is branched along the pull back by $\pi_1 \circ \cdots \circ \pi_1$ of a divisor of $|\mathcal{O}_Y(n_k\mathcal{E})|$ ($n = n_1 \cdots n_1$);

2. the unordered multidegree of $Y$ is

$$d_{\text{unord}} = (\kappa_1, \ldots, \kappa_1, \beta_1, \ldots, \beta_{T-1}).$$

Furthermore, assume the hypotheses of Theorem 3.5 (2) are satisfied. Then:

1. A general member of the algebraic formally semiuniversal deformation of $\varphi$ is an embedding whose image is a complete intersection of unordered multidegree

$$d'_{\text{unord}} = (n_1\kappa_1, \ldots, n_1\kappa_1, \beta_1, \ldots, \beta_{T-1}).$$

2. If $\tilde{Y} \hookrightarrow \mathbb{P}^N$ is a embedded rope on $Y$ with conormal bundle $\mathcal{E}$ and $\Phi$ is a flat family of morphisms satisfying (I), (II) and (III) of Theorem 3.5, then $\tilde{Y}$ and, for any $t \neq 0$, $\Phi_t(\mathcal{X}_t)$, correspond to points of an irreducible component of the Hilbert scheme whose general point corresponds to a smooth, complete intersection subvariety of unordered multidegree $d'$. 

Proof. We do in detail the proof when $\pi$ is simple cyclic, the general case follows from iterating the arguments used to prove the simple cyclic case.

Thus, let $\pi$ be a simple cyclic cover branched along a (smooth) divisor of $|\mathcal{O}_Y(n)|$, for some $k \in \mathbb{Z}$, $k > 0$. Recall that $Y$ is a complete intersection $H_1 \cap \cdots \cap H_r$ of multidegree

$$d_{\text{unord}} = (k, \beta_1, \ldots, \beta_{r-1})$$

and let $k$ be the degree of $H_1$. Let $Y' = H'_1 \cap H_2 \cap \cdots \cap H_r$ be a smooth complete intersection of unordered multidegree

$$d'_{\text{unord}} = (nk, k\beta_1, \ldots, \beta_{r-1}),$$

where $H'_1$ has degree $nk$. By letting $H'_1$ degenerate to $n$ times $H_1$, we obtain a smooth algebraic curve $S$, with a distinguished point $0 \in S$ and a flat family $\mathcal{Y}$ of subschemes of $\mathbb{P}^N$ over $S$, whose general member of $\mathcal{Y}$ is a smooth complete intersection of unordered multidegree $d'$ and whose member at $0$ is a primitive multiple structure $\tilde{Y}$ of multiplicity $n$, supported on $Y$. In fact, $H'_1$ can be chosen in such a way that, after base change, normalization and, if necessary, a linear automorphism of $Y$, we obtain a flat family $\Psi$ of morphisms to $\mathbb{P}^N$, over an algebraic curve $T'$ with a distinguished point $0 \in T'$, such that $\Psi_0 = \varphi$ and $\Psi_t$ is an embedding whose image is a smooth, complete intersection subvariety of unordered multidegree $d'$. Then there is a point in the base $Z$ of the algebraic formally semiuniversal deformation of $\varphi$ (see Lemma 3.3) which corresponds to an embedding whose image is a smooth, complete intersection variety. Since being a complete intersection is an open property (see [Ser75]), then there is a non empty open set $U$ of $Z$ consisting of points that correspond to embeddings (see Theorem 3.5 (2)), whose images are smooth, complete intersection subvarieties. This proves (1).

Recall that $Z$ is irreducible (see Lemma 3.3). Hence there is a rational map $\rho$ from $Z$ to the Hilbert scheme. Now Im($\rho$) is irreducible and is hence contained inside a unique irreducible component $H$ of the Hilbert scheme. Any rope $\tilde{Y}$ as in the statement of (2) corresponds to a point in the closure of Im($\rho$) and is hence contained in $H$. Also, for any $t \neq 0$, any $\Phi_t(X_t)$ as in the statement of (2) corresponds to a point in Im($\rho$), which is therefore a point of $H$. By part (1), Im($\rho$) contains at least one subvariety which
is a complete intersection subvariety of multidegree $d'$. Since, by [Ser75], being a complete intersection is an open property in the Hilbert scheme, a general point of $H$ corresponds to a complete intersection subvariety of multidegree $d'$. This proves (2).  

**Remark 3.13.** To prove Proposition 3.12 (1) we use, among other things, an elementary construction from which we obtain the flat family $\Psi$. This construction is a particular case of the one in [CL19] (see [CL19], Definition 2.3, Theorem 2.4 and Remark 2.5). M. Nori also showed us a similar example.

**Question 3.14.** Let $\mathcal{E}$ be as in Proposition 3.12. If $n > 2$, there is one irreducible component of the Hilbert containing points corresponding to two different kinds of multiple structures, namely, ropes embedded in $\mathbb{P}^N$, supported on $Y$ with conormal bundle $\mathcal{E}$ on the one hand, and complete intersection multiple structures obtained by intersecting multiple hypersurfaces and smooth hypersurfaces on the other hand. In the case in which $\pi$ is simple cyclic, then the latter multiple structures are primitive (and are like $\hat{Y}$). Thus, a natural question to ask is how, in the Hilbert scheme, the loci parameterizing each kind of multiple structure are related.

**Remark 3.15.** In the situation of Set-up 3.1, assume the hypothesis of Theorem 3.5 (2) (a) is satisfied and $N \geq 2r + 2$. Then $\text{Pic}(X) = \mathbb{Z}$. Indeed, let $(T, 0)$ be an affine algebraic curve and let $\Phi : \mathcal{X} \to \mathbb{P}^N_t$ be a deformation for which $\Phi_t$ is an embedding for all $t \neq 0$. Since $H^2(\Omega_X^1) = 0$, it follows that $\text{Pic}(X) \hookrightarrow \text{Pic}(\mathcal{X}_t)$. However, by the theorem of Barth (see [Laz80]), we know that $\text{Pic}(\mathcal{X}_t) = \mathbb{Z}$. The conclusion follows from the projectivity of $X$. (Compare [Laz80], Theorem 1)

4. **Deformations of simple cyclic covers**

In this section, we study the deformations of the simple cyclic covers of complete intersections. We will use the results proven in the previous section, and as a consequence, we will produce examples of varieties with small codimension in $\mathbb{P}^N$. We first describe our set-up.

**Set-up 4.1.** In the situation of Set-up 3.1, assume $\pi$ is simple cyclic. Consequently, there is $k$ in $\mathbb{Z}_{>0}$ such that $k_i = ik$ for all $i = 1, \ldots, n - 1$. Let $\text{dim}(X) = m$.

We make the following two important remarks. The first one describes the $s$–subcanonical cover. The next one shows that, if we fix the dimension and the degree, canonical morphisms (these are the cases in which $s = 1$) have bounded geometric genus.

**Remark 4.2.** Notice that $K_Y = \pi^* \Omega_Y (\delta - N - 1)$ and $K_X = \pi^* \Omega_Y (\delta + (n - 1)k - N - 1)$, by the ramification formula (see e.g. [BHPV04], Lemma I.17.1). Let $L = \pi^* \Omega_Y (1)$. It is easy to see that $(X, L)$ is $s$–subcanonical if and only if $N + 1 + s = \delta + (n - 1)k$. We also make a note of the following facts:

1. $X$ is Fano variety if and only if $\delta + (n - 1)k \leq N$ (then, $Y$ is also Fano). In this case $(X, L)$ is a Fano polarized variety of index $-s$ if and only if $N + 1 + s = \delta + (n - 1)k$.
2. $X$ is a Calabi–Yau variety if and only if $\delta + (n - 1)k = N + 1$ (in this case, $Y$ is Fano).
3. $X$ is a variety of general type if and only if $\delta + (n - 1)k \geq N + 2$. The morphism $\varphi$ (respectively $(X, L)$) is:
   a. Canonical if and only if $\delta + (n - 1)k = N + 2$ and $k \geq 2$ (resp. $\delta + (n - 1)k = N + 2$); in this case, $Y$ is Fano (resp. $Y$ is Fano, unless $n = 2$ and $k = 1$, in which case $Y$ is Calabi–Yau).
   b. Subcanonical if and only if $\delta + (n - 1)k \geq N + 3$ and $k \geq 2$ (resp. $\delta + (n - 1)k \geq N + 3$).

**Remark 4.3.** In the situation of Set-up 4.1, the following happens.

1. If $(X, L)$ is $s$–subcanonical, then $m + 1 \leq N \leq 2m + s + 1 - (n - 1)k \leq 2m + s - n + 2$.
2. If $\varphi$ is $s$–subcanonical, then $m + 1 \leq N \leq 2m + s - 2n + 3$.

We will mostly deal with $s$–subcanonical morphisms $\varphi$. We provide an example for which $(X, L)$ is $s$–subcanonical but $\varphi$ is not, and see the cases in which Theorem 3.5 applies.
Example 4.4. Set $k = 1$, and assume $N \leq \frac{nm + s}{n - 1}$. Let $Y$ be a complete intersection of a hypersurface of degree $\alpha = nm + N(1 - n) + 2 + s$ and $N - m - 1$ hypersurfaces of degree $n$. By assumption, $\alpha \geq 2$. If one wants to write the multidegree of $Y$ that is consistent with the convention of Set-up 3.1, it will be

$$d = \begin{cases} 
\frac{N - m - 1}{n} + \ldots + n, & \text{if } N \leq \frac{nm + 2 + s - n}{n - 1}, \\
\frac{n - 1}{n}, \ldots, n, & \text{if } N > \frac{nm + 2 + s - n}{n - 1}.
\end{cases}$$

Recall that $\pi : X \to Y$ is a simple cyclic cover of $Y$ branched along a smooth divisor in $|\mathcal{O}_Y(n)|$ and $\varphi$ is the corresponding morphism (which is induced by an incomplete linear series).

(a) Assume one of the following conditions hold;

$$\left\lfloor \frac{n}{2} \right\rfloor + m \leq N \leq \min\left\{ \frac{nm + 2 + s}{n - 1} - 1, 1, 2m + s + 2 - n \right\}, \text{ or,}$$

$$\left\lfloor \frac{n}{2} \right\rfloor + m + 1 \leq N \leq 2m + s + 2 - n.$$  

Then the general deformation of $\varphi$ is finite and birational onto its image. Indeed, (4.1) guarantees that $\alpha \geq n - 1$ and (4.2) guarantees that $N - m - 1 \geq \left\lfloor \frac{n}{2} \right\rfloor$ so that Theorem 3.5 (1) applies.

(b) If one of the following conditions hold;

$$2m + n - 1 \leq N \leq \min\left\{ \frac{nm + 2 + s}{n - 1} - 1, 1, 2m + s + 2 - n \right\} \text{ or}$$

$$2m + n \leq N \leq 2m + s + 2 - n,$$

then, since Theorem 3.5 (2) (b) applies, a general deformation of $\varphi$ is an embedding.

Now on, we will only look at the cases in which $\varphi$ is $s$–subcanonical, in particular, it is induced by a complete linear series. Recall that in these cases $k_1 \geq 2$ (see Set-up 3.1 for notations).

4.1. Subvarieties with small codimension and their asymptotic limits. First, we study the cases for which Theorem 3.5 (2) (a) is applicable. Recall that in these cases $\varphi$ deforms to an embedding.

Lemma 4.5. In the situation of Set-up 4.1, assume $\varphi$ is $s$–subcanonical. If the hypothesis of Theorem 3.5 (2) (a) holds, then,

$$m + n - 1 \leq N \leq 2(m + n) + s - (n - 1)(n + 2) - 1 \text{ and } s \geq (n - 1)(n + 2) - (m + n).$$

Further, if $s = (n - 1)(n + 2) - (m + n)$, then $N = m + n - 1$ and $d = (2, 4, \ldots, 2(n - 1)).$

Proof. Assume the hypothesis of Theorem 3.5 (2) (a) holds. Then, obviously $r = N - m \geq n - 1$, which gives the lower bound of $N$.

Under the assumption, the unordered multidegree of $Y$ is the following;

$$d_{\text{unord}} = (k, 2k, \ldots, (n - 1)k, \beta_1, \beta_2, \ldots, \beta_{N - m - n + 1}),$$

where each $\beta_i \geq 2$. Thus, $\delta = \frac{k(n - 1)n}{2} + 2(N - m - n + 1)$. Since $\delta + k(n - 1) = N + s + 1$, we get the following;

$$N + s + 1 \geq \frac{k(n - 1)n}{2} + 2(N - m - n + 1) + k(n - 1).$$

An elementary computation completes the proof of the first assertion since $k \geq 2$. The second assertion is an obvious consequence of the first. \qed
4.1.1. **Small codimensional subvarieties.** First we describe the small codimension varieties $X$ as in Setup 4.1 for which Theorem 3.5 (2) (a) applies and such that $s = (n-1)(n+2)-(m+n)$ (i.e., $r = n-1$ and $k = 2$). In particular, there exist deformations $\Phi: \mathcal{X} \to \mathbb{P}^N_k$ of $\varphi$, over smooth curves $(T, 0)$, such that $\Phi_t$ is an embedding for $t \neq 0$. Let $\tilde{Y}$ be the central fiber of the image of $\Phi$, which is a rope of multiplicity $n$. Notice that $\tilde{Y}$ is not a complete intersection (abbreviated as n.c.i), because, since $n \geq 3$, it is not locally a complete intersection. We recall that the embedded deformed varieties obtained this way can still be complete intersections, which happens for the rows shaded with light blue color, thanks to Corollary 3.11. For the white rows, the general deformation of $\varphi$ is an embedding whose image is a complete intersection, due to Proposition 3.12. The number of different invariants of such varieties $X$ is infinite, so we will just list below a few of them, restricting ourselves to codimension 2, 3 and 4 and $-3 \leq s \leq 2$.

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
m & n & k & N & s & d & K^m_X \\
\hline
10 & 2 & 12 & -3 & 2 & 4 & 24 \cdot 3^{10} \\
14 & 2 & 20 & -3 & 2, 4, 6 & -192 \cdot 2^{17} & 0 \\
26 & 5 & 2 & 30 & -3 & 2, 4, 6, 8 & 1920 \cdot 2^{26} & 0 \\
9 & 3 & 2 & 11 & -2 & 2, 4 & -24 \cdot 2^{9} & 0 \\
16 & 4 & 2 & 19 & -2 & 2, 4, 6 & 192 \cdot 2^{16} & 0 \\
25 & 5 & 2 & 29 & -2 & 2, 4, 6, 8 & -1920 \cdot 2^{25} & 0 \\
8 & 3 & 2 & 10 & -1 & 2, 4 & 24 & 0 \\
15 & 4 & 2 & 18 & -1 & 2, 4, 6 & -192 & 0 \\
24 & 5 & 2 & 28 & -1 & 2, 4, 6, 8 & 1920 & 0 \\
7 & 3 & 2 & 9 & 0 & 2, 4 & 0 & 1 \\
14 & 4 & 2 & 17 & 0 & 2, 4, 6 & 0 & 1 \\
23 & 5 & 2 & 27 & 0 & 2, 4, 6, 8 & 0 & 1 \\
6 & 3 & 2 & 8 & 1 & 2, 4 & 24 & 9 \\
13 & 4 & 2 & 16 & 1 & 2, 4, 6 & 192 & 17 \\
22 & 5 & 2 & 26 & 1 & 2, 4, 6, 8 & 1920 & 27 \\
5 & 3 & 2 & 7 & 2 & 2, 4 & 24 \cdot 2^{5} & 36 \\
\hline
\end{array}
$$

We construct a few more small codimensional subvarieties of the projective spaces. Let $m, n, s \in \mathbb{Z}$, with $m \geq 3$, $n \geq 2$ and

$$m + n \leq N \leq 2(m + n) + s - (n - 1)(n + 2) - 1 \text{ and } s \geq (n - 1)(n + 2) - (m + n) + 1.$$

Then there are integers $\beta_1, \beta_2, \cdots, \beta_{N-m-n+1} \geq 2$ satisfying the following equation;

$$\sum \beta_i + n(n - 1) + 2(n - 1) = N + s + 1.$$

For any such choices of $\beta_i$'s, let $Y$ be a complete intersection in $\mathbb{P}^N$ of multidegree

$$d_{\text{multord}} = (2, 4, \cdots, 2(n - 1), \beta_1, \cdots, \beta_{N-m-n+1}).$$

Let $\pi: X \to Y$ be a simple cyclic cover branched along a smooth member of $|\mathcal{O}_Y(2n)|$. Then the corresponding morphism $\varphi$ is $s$-subcanonical and satisfies the hypothesis of Theorem 3.5 (2) (a). Thus, a general deformation of $\varphi$ is an embedding.

The following table describes the varieties of codimension 3 and 4 and with $s = (n-1)(n+2)-(m+n)+1$, $-1 \leq s \leq 1$, we obtain this way (these are the ones for which the inequalities of Lemma 4.5 are strict).

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
m & n & k & N & s & d & K^m_X \\
\hline
9 & 3 & 2 & 12 & -1 & 2, 4 & -48 & 0 \\
16 & 4 & 2 & 20 & -1 & 2, 4, 6 & 384 & 0 \\
8 & 3 & 2 & 11 & 0 & 2, 4, 6 & 0 & 1 \\
\hline
15 & 4 & 2 & 19 & 0 & 2, 4, 6 & 0 & 1 \\
7 & 3 & 2 & 10 & 1 & 2, 4, 6 & -48 & 11 \\
14 & 4 & 2 & 18 & 1 & 2, 4, 6 & 384 & 19 \\
\hline
\end{array}
$$
4.1.2. A family of codimension 3 subvarieties with asymptotic limit of $m/N$ equal to 1. Choose integers $N$ and $k \geq 2$ that satisfy $2N = 9k$. Let $i : Y \hookrightarrow \mathbb{P}^N$ be a complete intersection of multidegree $(k,2k,3k)$, and let $\pi : X \to Y$ be a simple cyclic quadruple cover branched along a smooth member of $|\mathcal{O}_Y(4k)|$. Then, there exists a deformation $\Phi : \mathcal{X} \to \mathbb{P}_T^N$ of $\varphi$, over a smooth curve $(T,0)$, such that $\Phi_I$ is an embedding, and let $\tilde{Y}$ be the central fibre of the image family of $\Phi$. Thus, for arbitrarily large $N$, we get smooth subvarieties of dimension $m$, codimension 3 and subcanonicity index $s = N - 1$. Therefore we get a collection of subvarieties for which $\frac{m}{N}$ is arbitrarily close to 1 as $N$ goes to infinity. We present the first few such varieties in the following table (recall $L = \pi^*\mathcal{O}_Y(1)$).

| $m$ | $n$ | $k$ | $N$ | $s$ | $L^m$ | $Y$ |
|-----|-----|-----|-----|-----|-------|-----|
| 6   | 4   | 2   | 9   | (2,4,6) | 192   | n.c.i |
| 15  | 4   | 4   | 18  | 17     | (4,8,12)| 1536 | n.c.i |
| 24  | 4   | 6   | 27  | 26     | (6,12,18)| 5184 | n.c.i |

4.1.3. Family of varieties such that $m/N$ is any positive rational number $< 1$. Choose any positive rational number $\frac{a}{b} < 1$. Fix an integer $k \geq 2$, and let $l$ be any integer, $l \geq 2$. Let $N_l = bl$ and let $i : Y \hookrightarrow \mathbb{P}^{N_l}$ be a complete intersection of multidegree $(k,2k,\ldots, (b-a)k)$, and let $\pi : X \to Y$ be a simple cyclic $(bl-al+1)$-cover branched along a smooth member of $|\mathcal{O}_Y((bl-al+1)k)|$. Then, there exists a family $\Phi : \mathcal{X} \to \mathbb{P}_T^{N_l}$ over a smooth curve $(T,0)$, such that $\Phi_I$ is an embedding, and let $\tilde{Y}$ be the central fibre of the image of $\Phi$. Then, we get embedded varieties $\mathcal{X}_l$, of dimension $m_l = al$ in $\mathbb{P}^{N_l}$, with subcanonicity index $s_l = \frac{l(b-a)(bl-al+3)k}{2} - bl - 1$. Notice that $\frac{m_l}{N_l} = \frac{a}{b}$. Let $L = \varphi^*\mathcal{O}_Y(1)$.

(I) We take $a = 2$, $b = 3$, $k = 2$, and present the first few such varieties in the following table. Notice that the image of $\mathcal{X}_l$ is a complete intersection for the first (light blue) row, by Corollary 3.11.

| $m$ | $n$ | $k$ | $N$ | $s$ | $L^m$ | $Y$ |
|-----|-----|-----|-----|-----|-------|-----|
| 4   | 3   | 2   | 6   | 3   | (2,4) | 24   | n.c.i |
| 6   | 4   | 2   | 9   | 8   | (2,4,6) | 192  | n.c.i |
| 8   | 5   | 2   | 12  | 15  | (2,4,6,8) | 1920 | n.c.i |

(II) We take $a = 3$, $b = 4$, $k = 2$, and present the first few varieties in the following table. Notice that the image of $\mathcal{X}_l$ is a complete intersection for the first (light blue) row, by Corollary 3.11.

| $m$ | $n$ | $k$ | $N$ | $s$ | $L^m$ | $Y$ |
|-----|-----|-----|-----|-----|-------|-----|
| 6   | 3   | 2   | 8   | 1   | (2,4) | 24   | n.c.i |
| 9   | 4   | 2   | 12  | 5   | (2,4,6) | 192  | n.c.i |
| 12  | 5   | 2   | 16  | 11  | (2,4,6,8) | 1920 | n.c.i |

4.2. Construction of non–complete intersection subvarieties. In what follows, we study the cases where Theorem 3.5 (2) (b) applies, in particular, $\varphi$ deforms to an embedding. This case is interesting in the sense that now we will be able to produce varieties *embedded by complete linear series* inside projective space which will not be complete intersections any more.

**Lemma 4.6.** In the situation of Set-up 4.1, assume $\varphi$ is $s$–subcanonical. If the hypothesis of Theorem 3.5 (2) (b) holds, then,

$$2m + n - 1 \leq N \leq 2(2m + n - 1) - 2(n - 1)(m + n) + s + 1 \text{ and } s \geq 2m(n - 2) + n(2n - 3).$$

Furthermore, if $s = 2m(n - 2) + n(2n - 3)$, then $N = 2m + n - 1$, $k = 2$, and $Y$ has multidegree

$$\left(\frac{2(n - 1), \ldots, 2(n - 1)}{m + n - 1}\right).$$

**Proof.** We just prove the inequality of $N$, the other assertions will easily follow from that. Assume the hypothesis of Theorem 3.5 (2) (b) holds. Then, $2r \geq N + n - 1$ and consequently $N \geq 2m + n - 1$ since
\(r = N - m\). Now, under the assumption, \(\delta\) will be least if the unordered multidegree is the following:

\[
d_{\text{unord}} = (k(n-1), \ldots, k(n-1), 2, \ldots, 2, N-2m-n+1)
\]

consequently, \(\delta \geq k(n-1)(m+n-1) + 2(N-2m-n+1)\). Since \(\delta + k(n-1) = N + s + 1\), we get,

\[N + s + 1 \geq k(n-1)(m+n) + 2(N-2m-n+1)\]

An elementary computation completes the proof since \(k \geq 2\).

We systematically construct examples of this case, in particular, we show that the converse of the above lemma is also true. With the help of Lemma 3.8, we will find examples of embedded varieties inside projective spaces which are not complete intersections.

4.2.1. **Construction of non–complete intersections with high subcanonicity indices.** Assume \(m, n \in \mathbb{Z}_{\geq 0}\) and \(s \in \mathbb{Z}\), with \(m \geq 3, n \geq 2\) and

\[2m + n - 1 \leq N \leq 2(2m + n - 1) - 2(n-1)(m+n) + s + 1\]

and \(s \geq 2m(n-2) + n(2n-3)\). Then there are integers \(\beta_1, \beta_2, \cdots, \beta_{N-2m-n+1} \geq 2\), and \(\alpha_1, \cdots, \alpha_{m+n-1} \geq 2n - 1\) satisfying the following equation:

\[
\sum \beta_i + \sum \alpha_j + 2(n-1) = N + s + 1.
\]

For any such choices of \(\alpha_i\)'s and \(\beta_j\)'s, let \(Y\) be a complete intersection in \(\mathbb{P}^N\) of multidegree

\[
d_{\text{unord}} = (\alpha_1, \cdots, \alpha_{m+n-1}, \beta_1, \cdots, \beta_{N-2m-n+1}).
\]

Let \(\pi : X \to Y\) be a simple cyclic cover branched along a smooth member of \(|\mathcal{O}_Y(2n)|\). Then \(\pi\) is an \(s\)--subcanonical cover of degree \(n\) satisfying the hypothesis of Theorem 3.5 (2) (b). Thus, a general deformation of \(\varphi\) is an embedding.

The following table describes the first few varieties with \(m = 3, n = 3,4\) and \(k = 2\) we obtain this way. It follows from Lemma 3.8 that the deformed varieties in the following examples are not complete intersections.

| \(m\) | \(n\) | \(k\) | \(N\) | \(s\) | \(d\) | \(L^m\) |
|---|---|---|---|---|---|---|
| 3 | 3 | 2 | 8 | 15 | \(4,4,4,4\) | 3072 |
| 3 | 3 | 2 | 8 | 16 | \(4,4,4,5\) | 3840 |
| 3 | 4 | 2 | 9 | 32 | \(6,6,6,6,6\) | 186624 |
| 3 | 4 | 2 | 9 | 33 | \(6,6,6,6,7\) | 217728 |

4.2.2. **Family of non–complete intersections with \(1/2\) as the asymptotic limit of \(m/N\).** In the next theorem we construct non–complete intersections of dimension \(m\), embedded inside a projective space \(\mathbb{P}^N\) which are embedded by complete linear series, and \(\lfloor \frac{m}{N} \rfloor \to \frac{1}{2}\) from below as \(m\) goes to infinity.

**Theorem 4.7.** Fix an integer \(n \geq 3\). For all integer \(m \geq 3\), there exists varieties \(X'_m\) of dimension \(m\) and embeddings \(\alpha_m : X'_m \to \mathbb{P}^{2m+n-1}\), satisfying the following properties;

1. \(\alpha_m : X'_m \to \mathbb{P}^{2m+n-1}\) is not a complete intersection.
2. \(\alpha_m\) is induced by a complete linear series of a line bundle.

**Proof.** Fix an integer \(m \geq 3\). Let \(Y_m\) be a complete intersection of multidegree

\[
d = (2(n-1), \cdots, 2(n-1)) \overline{m+n-1}
\]

inside \(\mathbb{P}^{2m+n-1}\) and let \(\pi_m : X_m \to Y_m\) be an \(n\) cyclic double branched along a smooth member of \(|\mathcal{O}_Y(2n)|\). We know by Theorem 3.5 (2) (b) that \(i_m \circ \pi_m\) deforms to an embedding, where \(i_m : Y_m \to \mathbb{P}^{2m+n-1}\) is the
embedding. Let the embedded variety be $X'_m$. Assume $X'_m$ is a complete intersection of multidegree $(d'_1, d'_2, \cdots, d'_{m+n-1})$. Then, by Lemma 3.8, we know that

$$\sum d'_i = 2(n-1)(m+n), \text{ and } \prod d'_i = n((2(n-1))^{m+n-1}.$$

By arithmetic mean–geometric mean inequality, we know that

$$\left( \frac{\sum d'_i}{m+n-1} \right)^{m+n-1} \geq \prod d'_i$$

$$\implies \left( \frac{m+n}{m+n-1} \right)^{m+n-1} = \left( 1 + \frac{1}{m+n-1} \right)^{m+n-1} \geq n,$$

which is a contradiction since $(1 + \frac{1}{m+n-1})^{m+n-1}$ is an increasing function of $m+n$ and the limit at infinity is e, but $n \geq 3$. The claim about complete linear series follows from Remark 3.9.

4.3. Varieties with birational subcanonical morphisms. Now we study the cases for which Theorem 3.5 (1) is applicable. Recall that, in this case $\phi$ deforms to a birational morphism, which a priori, is not an embedding. We will actually produce some examples in which the deformed morphism is birational, but not an embedding.

Lemma 4.8. In the situation of Set-up 4.1, assume $\phi$ is $s$–subcanonical. If the hypothesis of Theorem 3.5 (1) holds, then,

$$m + \lfloor n/2 \rfloor \leq N \leq 2(m + \lfloor n/2 \rfloor) - 2(n-1)(\lfloor n/2 \rfloor + 1) + s + 1 \text{ and } s \geq 2(n-1)(\lfloor n/2 \rfloor + 1) - (m + \lfloor n/2 \rfloor) - 1.$$

Proof. Assume the hypothesis of Theorem 3.5 (1) holds. Then $r = N - m \geq \lfloor n/2 \rfloor$ and that gives the lower bound.

Now, under the assumption, $\delta$ will be least if the unordered multidegree is the following:

$$d_{\text{unord}} = (k(n-1), \cdots, k(n-1), \beta_1, \cdots, \beta_{N-m-\lfloor n/2 \rfloor}),$$

where $\beta_i \geq 2$ for all $i$. Since $\delta + k(n-1) = N + s + 1$, we obtain;

$$N + s + 1 \geq \lfloor n/2 \rfloor k(n-1) + 2(N - m - \lfloor n/2 \rfloor) + k(n-1).$$

An elementary computation completes the proof since $k \geq 2$.

Example 4.9. Assume $m, n \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}$, with $m \geq 3, n \geq 2$ and

$$m + \lfloor n/2 \rfloor \leq N \leq 2(m + \lfloor n/2 \rfloor) - 2(n-1)(\lfloor n/2 \rfloor + 1) + s + 1 \text{ and } s \geq 2(n-1)(\lfloor n/2 \rfloor + 1) - (m + \lfloor n/2 \rfloor) - 1.$$

Then there are integers $\beta_1, \beta_2, \cdots, \beta_{N-m-\lfloor n/2 \rfloor} \geq 2$, and $\alpha_1, \cdots, \alpha_{\lfloor n/2 \rfloor} \geq 2(n-1)$ satisfying the following equation;

$$\sum \beta_i + \sum \alpha_i + 2(n-1) = N + s + 1.$$

For any such choices of $\alpha_i$’s, and $\beta_j$’s, let $Y$ be a complete intersection in $\mathbb{P}^N$ of multidegree

$$d_{\text{unord}} = (\alpha_1, \cdots, \alpha_{\lfloor n/2 \rfloor}, \beta_1, \cdots, \beta_{N-m-\lfloor n/2 \rfloor}).$$

Let $\pi : X \rightarrow Y$ be a simple cyclic cover branched along a smooth member of $|\Omega_Y (2n)|$. Then $\phi$ is an s–subcanonical cover of degree $n$ satisfying the hypothesis of Theorem 3.5 (1). Thus, a general deformation of $\phi$ is birational onto its image.

Suppose we already know, that if $\phi$ deforms to an embedding then the embedded variety must be a complete intersection (which is the case for codimension 1, and for some codimension 2 cases thanks to Corollary 3.11). Then one may use Lemma 3.8 to check, if at all it is possible for $\phi$ to deform to an embedding, or not. The following table describes the first few varieties with $k = 2$ we obtain the way described in this example. The rows with light blue color deform to birational morphisms which are not embeddings. For the white rows, if $\phi$ deforms to an embedding, then the images are not complete intersections.
4.4. **Degree $n$ subcanonical morphisms whose deformations are of degree $n$.** Finally, we study the cases for which Proposition 3.4 is applicable, in particular, the degree of any deformation of $\varphi$ in these cases remain **unchanged**.

**Lemma 4.10.** In the situation of Set-up 4.1, assume $\varphi$ is $s$–subcanonical and the hypothesis of Proposition 3.4 is satisfied. Then,

1. $k \geq 3$ and

$$\max \left\{ \frac{s + 1 + m(k - 1) - k(n - 1)}{k - 2}, m + 1 \right\} \leq N \leq 2m + s + 1 - k(n - 1).$$

2. If $k = 3$, then $N = 2m + s + 1 - 3(n - 1)$ and $d = (2, \cdots, 2)$.

**Proof.** Since $\delta + k(n - 1) = N + s + 1$ and $\delta \geq 2(N - m)$, the upper bound follows. Since $d = (d_1, \cdots, d_r)$ and $d_i \leq k - 1$, it follows that $\delta \leq (N - m)(k - 1)$. Consequently,

$$N + s + 1 \leq (N - m)(k - 1) + k(n - 1).$$

An easy computation completes the proof of (1), and (2) is a consequence of (1). $\square$

We provide concrete examples below for which Proposition 3.4 applies. We actually prove that the converse of the above lemma is also true.

**Example 4.11.** Let $k, m, n$ be integers satisfying $k \geq 3$, $m \geq 3$ and $n \geq 2$. Let $s \in \mathbb{Z}$. Assume,

$$\max \left\{ \frac{s + 1 + m(k - 1) - k(n - 1)}{k - 2}, m + 1 \right\} \leq N \leq 2m + s + 1 - k(n - 1).$$

Then, there are integers $\beta_r \geq \cdots \geq \beta_1 \geq 2$ such that $\beta_i \leq k - 1$ and $\sum \beta_i + k(n - 1) = N + s + 1$. Let $Y$ be of multidegree $d = (\beta_1, \cdots, \beta_r)$ inside $\mathbb{P}^N$. Let $\pi : X \to Y$ be a simple cyclic $n$ cover branched along a smooth divisor in $|\mathcal{O}_Y(nk)|$. Then $\varphi$ is $s$–subcanonical and satisfies the hypothesis of Proposition 3.4, consequently any deformation of $\varphi$ has degree $n$.

The following table describes the first few varieties with $k = 3$ that we obtain in this way.

| $m$ | $n$ | $k$ | $N$ | $s$ | $d$ | $K^m_X$ | $p_g(X)$ |
|-----|-----|-----|-----|-----|-----|--------|--------|
| 7   | 3   | 3   | 8   | −1  | 2   | −6     | 0      |
| 8   | 3   | 3   | 10  | −1  | (2, 2)| 12     | 0      |
| 10  | 4   | 3   | 11  | −1  | 2   | 8      | 0      |
| 11  | 4   | 3   | 13  | −1  | (2, 2)| −16    | 0      |

| $m$ | $n$ | $k$ | $N$ | $s$ | $d$ | $K^m_X$ | $p_g(X)$ |
|-----|-----|-----|-----|-----|-----|--------|--------|
| 5   | 3   | 3   | 6   | 1   | 2   | 6      | 7      |
| 6   | 3   | 3   | 8   | 1   | (2, 2)| 12     | 9      |
| 8   | 4   | 3   | 9   | 1   | 2   | 8      | 10     |
| 9   | 4   | 3   | 11  | 1   | (2, 2)| 16     | 12     |

| $m$ | $n$ | $k$ | $N$ | $s$ | $d$ | $K^m_X$ | $p_g(X)$ |
|-----|-----|-----|-----|-----|-----|--------|--------|
| 6   | 3   | 3   | 7   | 0   | 2   | 0      | 1      |
| 7   | 3   | 3   | 9   | 0   | (2, 2)| 0      | 1      |
| 9   | 4   | 3   | 10  | 0   | 2   | 0      | 1      |
| 10  | 4   | 3   | 12  | 0   | (2, 2)| 0      | 1      |
5. DEFORMATIONS OF \( \mathbb{Z}_n \times \mathbb{Z}_2 \) COVERS

In this section, we will produce examples of \( \mathbb{Z}_n \times \mathbb{Z}_2 \) covers, and apply the results of Section 3 on them. In the previous section, we have used all the results of Section 3, except Theorem 3.7, on simple cyclic covers. We will see now that Theorem 3.7 is applicable to some of \( \mathbb{Z}_n \times \mathbb{Z}_2 \) covers. As before, we start by describing our set-up.

**Set-up 5.1.** Let \( Y \) be as in Set-up 3.1 (l). Let \( p_1 : X_1 \to Y \) be a double cover branched along a smooth divisor \( D_2 \) in \( |O_Y(l)| \) for some \( l \in \mathbb{Z}_{\geq 0} \). Let \( p_2 : X_2 \to Y \) be a simple cyclic cover of degree \( n \) branched along a smooth divisor \( D_1 \) in \( |O_Y(nk)| \) for some \( n, k \in \mathbb{Z}_{\geq 0} \), \( n \geq 2 \). Assume \( D_1 \) and \( D_2 \) intersect transversally. Set \( X := X_1 \times_Y X_2 \) and let \( \pi : X \to Y \) be the natural morphism from the fiber product to \( Y \). Note that in this section the degree of \( \pi \) is not \( n \) but \( 2n \). Let \( \dim(X) = m \).

**Remark 5.2.** In the situation of Set-up 5.1, \( X \) is smooth, \( K_X = \pi^*O_Y(\gamma - N - 1 + \delta + l + k(n - 1)) \). Consequently, \( (X, L) \) is s–subcanonical if and only if \( \delta + l + k(n - 1) = N + s + 1 \). We also make a note of the following facts:

1. \( X \) is a Fano variety if and only if \( \delta + l + (n - 1)k \leq N \) (in this case, \( Y \) is also Fano). \( (X, L) \) is a Fano polarized variety of index \( -s \) if and only if \( N + 1 + s = \delta + l + (n - 1)k \).
2. \( X \) is a Calabi–Yau variety if and only if \( N + 1 = \delta + l + (n - 1)k \) (in this case, \( Y \) is Fano).
3. \( X \) is a variety of general type if and only if \( \delta + l + (n - 1)k \geq N + 2 \). The morphism \( \varphi \) (respectively \( (X, L) \)) is:

   a. Canonical if and only if \( \delta + l + (n - 1)k = N + 2 \) and \( k, l \geq 2 \) (resp. \( \delta + l + (n - 1)k = N + 2 \)); in this case \( Y \) is Fano.
   b. Subcanonical if and only if \( \delta + l + (n - 1)k \geq N + 3 \) and \( k, l \geq 2 \) (resp. \( \delta + l + (n - 1)k \geq N + 3 \)).

5.1. **Subvarieties with small codimension.** We study the cases in which Theorem 3.5 (2) (a) applies.

**Lemma 5.3.** In the situation of Set-up 5.1, assume \( \varphi \) is s–subcanonical. If the hypothesis of Theorem 3.5 (2) (a) holds, then,

\[
m + 2n - 1 \leq N \leq 2(m + 2n - 1) - 2N + s + 1 \quad \text{and} \quad s \geq 2n^2 - m.
\]

Further, if \( s = 2n^2 - m \), then \( N = m + 2n - 1 \) and the unordered multidegree of \( Y \) is

\[
d_{\text{unord}} = (2, 4, \ldots, 2(n - 1), 2, 4, \ldots, 2n).
\]

**Proof.** Assume the hypothesis of Theorem 3.5 (2) (a) holds. Then, \( r \geq 2n - 1 \) and consequently \( N \geq m + 2n - 1 \) since \( r = N - m \).

Now, under the assumption, \( \delta \) will be least if the unordered multidegree of \( Y \) is the following:

\[
d_{\text{unord}} = (k, 2k, \ldots, (n - 1)k, k, l, k + l, \ldots, (n - 1)k + l, 2, \ldots, 2),
\]

consequently, \( \delta \geq n(n - 1)k + nl + 2(N - m - 2n + 1) \). Since \( \delta + k(n - 1) + l = N + s + 1 \) and \( k, l \geq 2 \) we get that,

\[
N + s + 1 \geq 2n(n - 1) + 2(N - m - 2n + 1) + 4n,
\]

so the first assertion follows. The second assertion is clear. \( \square \)

Now we describe the first few varieties as in Set-up 5.1, with \( \varphi \) s–subcanonical, and for which the hypothesis of Theorem 3.5 (2) (a) holds and \( s = 2n^2 - m \). Recall that in these cases, \( \varphi \) deforms to an embedding.

| \( m \) | \( n \) | \( k \) | \( l \) | \( N \) | \( s \) | \( d \) | \( K^m_X \) | \( p_{\varphi}(X) \) |
|---|---|---|---|---|---|---|---|---|
| 9 | 2 | 2 | 2 | 12 | -1 | (2, 2, 4) | -64 | 0 |
| 19 | 3 | 2 | 2 | 24 | -1 | (2, 2, 4, 4, 6) | -2304 | 0 |
| 8 | 2 | 2 | 2 | 11 | 0 | (2, 2, 4) | 0 | 1 |

| \( m \) | \( n \) | \( k \) | \( l \) | \( N \) | \( s \) | \( d \) | \( K^m_X \) | \( p_{\varphi}(X) \) |
|---|---|---|---|---|---|---|---|---|
| 18 | 3 | 2 | 2 | 23 | 0 | (2, 2, 4, 4, 6) | 0 | 1 |
| 7 | 2 | 2 | 2 | 10 | 1 | (2, 2, 4) | 64 | 11 |
| 17 | 3 | 2 | 2 | 22 | 1 | (2, 2, 4, 4, 6) | 2304 | 23 |
We construct a few more subvarieties with small codimension, for which the inequalities of Lemma 5.3 are strict. Assume \( m, n \in \mathbb{Z}_{>0} \), and \( s \in \mathbb{Z} \), with \( m \geq 3 \), \( n \geq 2 \) and
\[
m + 2n \leq N \leq 2(m + 2n - 1) - 2n(m + n + 1) + s + 1 \text{ and } s \geq 2n^2 - m + 1.
\]
Then there are integers \( \beta_1, \beta_2, \ldots, \beta_{N-m-2n+1} \geq 2 \) satisfying the following equation;
\[
\sum \beta_i + 2n(n - 1) + 2n + 2(N - m - 2n + 1) = N + s + 1.
\]
For any such choices of \( \beta_i \)'s, let \( Y \) be a complete intersection in \( \mathbb{P}^N \) of multidegree
\[
\mathbf{d}_{\text{unord}} = (2, 4, \ldots, 2(n - 1), 2, 4, \ldots, 2n, \beta_1, \ldots, \beta_{N-m-2n+1}).
\]
Let \( \varphi: X \to Y \) be the natural morphism from the fiber product \( X := X_1 \times_Y X_2 \) of a double cover \( p_1: X_1 \to Y \) branched along a smooth member \( D_2 \) of \( |\mathcal{O}_Y(4)| \), and a simple cyclic cover \( p_2: X_2 \to Y \) of degree \( n \), branched along a smooth member \( D_1 \) of \( |\mathcal{O}_Y(2n)| \) such that \( D_1 \) and \( D_2 \) intersect transversally. Then \( X \) is smooth, \( \varphi \) is an \( s \)-subcanonical cover of degree \( n \) satisfying the hypothesis of Theorem 3.5 (2) (a). Thus, a general deformation of \( \varphi \) is an embedding.

The following table describes the first few varieties with \( m = l = 3 \) we obtain this way.

| \( m \) | \( n \) | \( k \) | \( l \) | \( N \) | \( s \) | \( \mathbf{d} \) | \( K^m_X \) | \( p_g(X) \) |
|---|---|---|---|---|---|---|---|---|
| 10 | 2 | 2 | 2 | 14 | -1 | (2, 2, 4) | -128 | 0 |
| 20 | 3 | 2 | 2 | 26 | -1 | (2, 2, 2, 4, 4, 6) | -4608 | 0 |
| 9 | 2 | 2 | 2 | 13 | 0 | (2, 2, 2, 4) | 0 | 1 |
| 19 | 3 | 2 | 2 | 25 | 0 | (2, 2, 2, 4, 4, 6) | 0 | 1 |
| 8 | 2 | 2 | 2 | 12 | 1 | (2, 2, 2, 4) | 128 | 13 |
| 18 | 3 | 2 | 2 | 24 | 1 | (2, 2, 2, 4, 4, 6) | 4608 | 25 |

5.2. Construction of non-complete intersection subvarieties. We prove the following basic lemma.

**Lemma 5.4.** In the situation of Set-up 5.1, assume \( \varphi \) is \( s \)-subcanonical. If the hypothesis of Theorem 3.5 (2) (b) holds, then,
\[
2m + 2n - 1 \leq N \leq 2(2m + 2n - 1) - 2m(m + n + 1) + s + 1 \text{ and } s \geq 2m(n - 1) + n(2n - 1).
\]

**Proof.** Assume the hypothesis of Theorem 3.5 (2) (b) holds. Then, \( 2r \geq N + 2n - 1 \) and consequently \( N \geq 2m + 2n - 1 \) since \( r = N - m \).

Now, under the assumption, \( \delta \) will be least if the unordered multidegree is the following;
\[
\mathbf{d}_{\text{unord}} = (k(n - 1) + l, \ldots, k(n - 1) + l, \underbrace{2, \ldots, 2}_{m + 2n - 1}, \underbrace{2, \ldots, 2}_{N - 2m - 2n + 1}),
\]
consequently, \( \delta \geq (k(n - 1) + l)(m + 2n - 1) + 2(N - 2m - 2n + 1) \). Since \( \delta + k(n - 1) + l = N + s + 1 \), we get that,
\[
N + s + 1 \geq (m + 2n - 1)(k(n - 1) + l) + 2(N - 2m - 2n + 1) + l + k(n - 1).
\]

An elementary computation completes the proof since \( k, l \geq 2 \).

In these cases, as before, we will find examples in which \( \varphi \) deforms to an embedding, but the embedded variety is not a complete intersection anymore, thanks to Lemma 3.8. Assume \( m, n \in \mathbb{Z}_{>0} \) and \( s \in \mathbb{Z} \), with \( m \geq 3 \), \( n \geq 2 \) and
\[
2m + 2n - 1 \leq N \leq 2(2m + 2n - 1) - 2n(m + 2n) + s + 1 \text{ and } s \geq 2m(n - 1) + n(2n - 1).
\]
Then there are integers \( \beta_1, \beta_2, \ldots, \beta_{N-2m-n+1} \geq 2 \), and \( \alpha_1, \ldots, \alpha_{m+2n-1} \geq 2 \) satisfying the following equation;
\[
\sum \beta_i + \sum \alpha_j + 2n = N + s + 1.
\]
For any such choices of \( \alpha_i \)'s and \( \beta_j \)'s, let \( Y \) be a complete intersection in \( \mathbb{P}^N \) of multidegree
\[
\mathbf{d}_{\text{unord}} = (\alpha_1, \ldots, \alpha_{m+2n-1}, \beta_1, \ldots, \beta_{N-2m-2n+1}).
\]
As before, let $\varphi : X \rightarrow Y$ be the morphism from the fiber product $X := X_1 \times_Y X_2$ of a double cover $p_1 : X_1 \rightarrow Y$ branched along a smooth member $D_2$ of $|\mathcal{O}_Y(4)|$, and a simple cyclic cover $p_2 : X_2 \rightarrow Y$ of degree $n$, branched along a smooth member $D_1$ of $|\mathcal{O}_Y(2n)|$ such that $D_1$ and $D_2$ intersect transversally. Then $X$ is smooth, $\varphi$ is an $s$–subcanonical cover of degree $n$ satisfying the hypothesis of Theorem 3.5 (2) (b). Thus, a general deformation of $\varphi$ is an embedding.

The following table describes the first few varieties with $n = 3$ we obtain this way. Moreover, for the following varieties, the deformed embedded varieties are not complete intersections.

| $m$ | $n$ | $k$ | $l$ | $N$ | $s$ | $d$ | $L^m$ |
|-----|-----|-----|-----|-----|-----|-----|-------|
| 3   | 2   | 2   | 2   | 9   | 18  | (4,4,4,4,4,4) | 4'    |
| 2   | 2   | 2   | 9   | 19  | (4,4,4,4,4,5) | 5·4°  |

5.3. Varieties with birational subcanonical morphisms. We study the cases for which Theorem 3.5 (1) applies.

**Lemma 5.5.** In the situation of Set-up 5.1, assume $\varphi$ is $s$–subcanonical. If the hypothesis of Theorem 3.5 (1) holds, then,

$$m + n \leq N \leq 2m - 2n^2 + s + 1 \text{ and } s \geq 2n^2 + n - m - 1.$$  

**Proof.** Assume the hypothesis of Theorem 3.5 (1) holds. Then, $r \geq n$ and consequently $N \geq m + n$.

Now, under the assumption, $\delta$ will be least if the unordered multidegree is the following:

$$d_{\text{unord}} = (k(n-1) + l, \ldots, k(n-1) + l, 2, \ldots, 2),$$

consequently, $\delta \geq 2(N - m - n) + n(k(n-1) + l)$. Since $\delta + k(n-1) + l = N + s + 1$, we get that,

$$N + s + 1 \geq 2(N - m - n) + n(k(n-1) + l) + l + k(n-1).$$

An elementary computation completes the proof since $k, l \geq 2$.  \qed

We construct examples of this case below. In some of the cases, we will actually be able to say that $\varphi$ deforms to birational maps which are definitely not embeddings. In other cases, we can not conclude if $\varphi$ deforms to embeddings or not. However, in those cases, we will be able to say that if $\varphi$ deforms to an embedding, then the embedded variety can not be a complete intersection.

**Example 5.6.** Assume $m, n \in \mathbb{Z}_0$ and $s \in \mathbb{Z}$, with $m \geq 3$, $n \geq 2$ and

$$m + n \leq N \leq 2m - 2n^2 + s + 1 \text{ and } s \geq 2n^2 + n - m - 1.$$  

Then there are integers $\beta_1, \beta_2, \ldots, \beta_{N-m-n} \geq 2$, and $\alpha_1, \ldots, \alpha_n \geq 2n$ satisfying the following:

$$\sum \beta_i + \sum \alpha_j + 2n = N + s + 1.$$  

For any such choices of $\alpha_i$’s and $\beta_j$’s, let $Y$ be a complete intersection in $\mathbb{P}^N$ of multidegree

$$d_{\text{unord}} = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{N-m-n}).$$

Let $\varphi : X \rightarrow Y$ be the natural morphism from the fiber product $X := X_1 \times_Y X_2$ of a double cover $p_1 : X_1 \rightarrow Y$ branched along a smooth member $D_2$ of $|\mathcal{O}_Y(4)|$, and a simple cyclic cover $p_2 : X_2 \rightarrow Y$ of degree $n$, branched along a smooth member $D_1$ of $|\mathcal{O}_Y(2n)|$, such that $D_1$ and $D_2$ intersect transversally. Then $X$ is smooth, $\varphi$ is an $s$–subcanonical cover of degree $n$ satisfying the hypothesis of Theorem 3.5 (1). Thus, a general deformation of $\varphi$ is birational onto its image.

The following table describes the first few varieties with $k = l = 2$ we obtain this way. Notice that the equations described in Lemma 3.8 has no solution for any of the following varieties. Light blue rows of the tables indicate the varieties for which the morphisms $\varphi$ deform to birational morphisms which are not embeddings, thanks to Corollary 3.11. For white rows, $\varphi$ deforms to birational morphism whose image is not a complete intersection, if it is an embedding.
5.4. **Degree 2n subcanonical morphisms whose deformations are of degree 2n.** Next, we study the cases in which Proposition 3.4 is applicable.

**Lemma 5.7.** In the situation of Set-up 5.1, assume \( \varphi \) is \( s \)-subcanonical and the hypothesis of Proposition 3.4 is satisfied. Then,

1. If \( k, l \geq 3 \) and

\[
\max \left\{ \frac{m(k-1) + s + 1 - l - k(n-1)}{k-2}, \frac{m(l-1) + s + 1 - l - k(n-1)}{l-2}, m + 1 \right\} \leq N \leq 2m + s + 1 - l - k(n-1).
\]

2. If \( k = l = 3 \), then \( N = 2m + s + 1 - 3n \) and \( d = (2, \ldots, 2) \).

**Proof.** Since \( \delta + l + k(n-1) = N + s + 1 \) and \( \delta \geq 2(N - m) \), the upper bound follows.

Since \( d = (d_1, \ldots, d_l) \) and \( d_i \leq \min\{k-1, l-1\} \), it follows that

\[
\delta \leq \min\{(N-m)(k-1), (N-m)(l-1)\}.
\]

Consequently, \( N + s + 1 \leq \min\{(N-m)(k-1)+k(n-1)+l, (N-m)(l-1)+k(n-1)+l\} \). An easy computations completes the proof of (1), and (2) is a consequence of (1). \( \square \)

We will provide concrete examples for which Proposition 3.4 is applicable, in particular the following example proves the converse of the above lemma.

**Example 5.8.** Let \( k, m, n \) be integers satisfying \( k, l \geq 3 \), \( m \geq 3 \) and \( n \geq 2 \). Let \( s \in \mathbb{Z} \). Assume,

\[
\max \left\{ \frac{m(k-1) + s + 1 - l - k(n-1)}{k-2}, \frac{m(l-1) + s + 1 - l - k(n-1)}{l-2}, m + 1 \right\} \leq N \leq 2m + s + 1 - l - k(n-1).
\]

Then, there are integers \( \beta_r \geq \cdots \geq \beta_1 \geq 2 \) such that \( \beta_i \leq \min\{k-1, l-1\} \) and

\[
\sum \beta_i + l + k(n-1) = N + s + 1.
\]

Let \( Y \) be a smooth complete intersection in \( \mathbb{P}_1 \) of multidegree \( d = (\beta_1, \ldots, \beta_r) \). Let \( \varphi : X \rightarrow Y \) be the natural morphism from \( X := X_1 \times_X X_2 \) where \( p_1 : X_1 \rightarrow Y \) is a double cover, branched along a smooth member \( D_2 \) of \( |\mathcal{O}_Y(2l)| \), \( p_2 : X_2 \rightarrow Y \) is a simple cyclic cover of degree \( n \), branched along a smooth member \( D_2 \) of \( |\mathcal{O}_Y(2n)| \), \( D_1 \) and \( D_2 \) intersect transversally. Then \( X \) is smooth, \( \varphi \) is \( s \)-subcanonical, satisfies the hypothesis of Proposition 3.4, consequently any general deformation of \( \varphi \) is a \( 2n \)-cover.

The following table describes the first few varieties with \( k = l = 3 \) that we obtain in this way.
5.5. Varieties with degree $n$ subcanonical morphisms. Finally, we study the cases for which Theorem 3.7 applies. This is a new case that did not appear in the previous section. This case does not appear for double covers of complete intersections, studied in [BG20] as well.

**Lemma 5.9.** In the situation of Set-up 5.1, assume $\varphi$ is $s$–subcanonical, and the hypothesis (1) and (2) of Theorem 3.7 are satisfied. Then,

$$m + 1 \leq N \leq 2(m + 1) + s + 1 - 2l - (n - 1)(2l + 1),$$

$$2l + 1 \leq k \leq \frac{2(m + 1) + s + 1 - N}{n - 1}, \quad s \geq 2l + (n - 1)(2l + 1) - (m + 2).$$

Further, the following happens.

1. If $s = 2l + (n - 1)(2l + 1) - (m + 2)$, then $N = m + 1$, $k = 2l + 1$ and $d = l$.
2. If $N \geq m + 2$, then $s \geq 2l + (n - 1)(2l + 1) - (m + 1)$ and $N \geq \frac{md_r - l(k(n - 1) + s + 1)}{d_r - 1}$.

**Proof.** We just prove the first assertion, the remaining ones are straightforward. The lower bound of $N$ is obvious. By our assumption $k \geq 2l + 1$. Also, $\delta \geq l + 2(N - m - 1)$. Since $\delta + l + k(n - 1) = N + s + 1$, we obtain, $N + s + 1 \geq 2(N - m - 1) + 2l + (2l + 1)(n - 1)$. An easy computation gives the upper bound of $N$. The other inequalities are obvious. \hfill $\square$

**Example 5.10.** (a) First we give examples of varieties as in Set-up 5.1, with $\varphi$ $s$–subcanonical, and for which Theorem 3.7 applies, i.e., the degree of a general deformation is $n$.

| $m$ | $n$ | $k$ | $l$ | $N$ | $s$ | $d$ | $K_X^m$ | $p_g(X)$ |
|-----|-----|-----|-----|-----|-----|-----|-------|-------|
| 8   | 2   | 5   | 2   | 9   | -1  | 2   | -8    | 0     |
| 13  | 3   | 5   | 2   | 14  | -1  | 2   | -12   | 0     |
| 7   | 2   | 5   | 2   | 8   | 0   | 2   | 0     | 1     |
| 12  | 3   | 5   | 2   | 13  | 0   | 2   | 0     | 1     |
| 6   | 2   | 5   | 2   | 7   | 1   | 2   | 8     | 8     |
| 11  | 3   | 5   | 2   | 12  | 1   | 2   | 12    | 13    |

(b) Assume $m, n, N, l \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}$, with $m \geq 3$, $n, l \geq 2$ and

$$\max\left\{ m + 2, \frac{ml - l - (2l + 1)(n - 1) + s + 1}{l - 1} \right\} \leq N \leq 2(m + 1) + s + 1 - 2l - (n - 1)(2l + 1)$$

Then there are integers $l \geq \beta_1, \beta_2, \ldots, \beta_{N-m-1} \geq 2$ satisfying the following equation;

$$\sum \beta_i + l + (2l + 1)(n - 1) = N + s + 1.$$ 

For any such choices of $\beta_i$'s, let $Y$ be a complete intersection in $\mathbb{P}^N$ of multidegree

$$d_{\text{unord}} = (l, \beta_1, \ldots, \beta_{N-m-1}).$$

Let $\varphi : X \rightarrow Y$ be the morphism from the fiber product $X := X_1 \times_Y X_2$ of a double cover $p_1 : X_1 \rightarrow Y$ branched along a smooth member $D_2$ of $|\mathcal{O}_Y(2l)|$, and a simple cyclic cover $p_2 : X_2 \rightarrow Y$ of degree $n$, branched along a smooth member $D_1$ of $|\mathcal{O}_Y((2l + 1)(n - 1))|$, such that $D_1$ and $D_2$ intersect transversally. Then $X$ is smooth, $\varphi$ is an $s$–subcanonical cover of degree $n$ satisfying the hypothesis (1) and (2) of Theorem 3.7. Thus, a general deformation of $\varphi$ of degree $n$.

The following table describes the first few varieties with $l = 2$ and $k = 5$ we obtain this way.
In this section, we add some remarks about the case when $X$ is singular, and the case when $\dim(X) = \dim(Y) = 2$, i.e., when $X$ and $Y$ are surfaces. We also list the open questions that we have encountered.

### 6.1. The singular case

The results proved in Section 3 crucially depends on the structure of the pushforward of the normal bundle of $\pi$. In this sense, recall that Proposition 2.2 holds for $X$ normal.

Furthermore, the conclusion of Proposition 2.4 remains unchanged if instead of smoothness, we only assume that $X$ is a normal, Cohen–Macaulay variety (see [Gon06], Proposition 3.7). Now we remove the smoothness of $X$ in Theorem 2.5 and in Theorem 2.6; and let $X$ to be a normal, Cohen–Macaulay projective variety with at worst canonical singularities. In that case, all the conclusions remain the same, except (1) for both theorems, where $\mathcal{Z}_i$ becomes a normal, Cohen–Macaulay projective variety with at worst canonical singularities, thanks to [Kaw99]. Moreover, Remark 2.9, and Lemma 2.10 can be proven when $X$ is normal, Cohen–Macaulay projective variety; Definition 2.11 can obviously be extended for this case as well.

Now we let $X$ be a normal, Cohen–Macaulay projective variety in the hypothesis of Set-up 3.1 (2) ($\pi_*\mathcal{O}_X$ still splits, see (2.1)). One still has the exact sequence (3.1), in fact one can prove the following.

**Lemma 6.1.** Let $Z_1, Z_2, Z_3$ be normal Cohen–Macaulay varieties. Let $f : Z_1 \to Z_2$ be a non-degenerate morphism for which $f^*$ is an exact functor (this happens if $f$ is finite and flat) and let $g : Y \to Z$ be a non-degenerate morphism. Suppose $h := g \circ f$. Then there is an exact sequence:

$$0 \to \mathcal{N}_f \to \mathcal{N}_h \to f^*\mathcal{N}_g \to 0.$$

Thanks to Lemma 6.1, the proof of Lemma 3.3 goes through after slight modifications, and consequently Proposition 3.4 follows under our modified set-up. Furthermore, Theorem 3.5 also follows; with the obvious change, namely in (I), $\mathcal{Z}_i$ will be a normal, Cohen–Macaulay projective variety.

Thus, we will be able to produce many more low codimensional projective varieties with at worst canonical singularities by deforming appropriate covers, using the procedures described in Sections 4, and 5. One can construct these covers by choosing suitable branch divisors which we now allow to be singular! We will describe one such construction when $X$ and $Y$ are surfaces.

### 6.2. The case of an algebraic surface

In the situation of Set-up 3.1, we drop the assumption that $\dim(Y) \geq 3$ and replace it with $\dim(Y) = 2$. Notice that, in this case $\text{Pic}(Y) \neq \mathbb{Z}$ in general, thus the splitting described in Set-up 3.1 is indeed an assumption in this case, which was automatic before.

Furthermore, the existence and unobstructedness of the algebraic formally semuniversal deformation space of $\varphi$ (first part of Lemma 3.3) should be checked in a case by case manner, since second cohomology group is not intermediate anymore, even though the second part of Lemma 3.3 goes through. For example, if $X$ is a surface of general type, then $X$ has an algebraic formally universal deformation by [Sei95], Theorem 3.1. Then, arguing as in [BGG20], Proposition 1.5, so does $\varphi$. In this case, all of Proposition 3.5 goes through. One can handle the cases when $X$ is Del Pezzo, or when $X$ is K3 in a similar fashion, see [BG20], proof of Proposition 1.5 for details.
6.2.1. **Low codimensional surfaces of general type.** It follows from the argument we have just given that when \( X \) is of general type, and \( \pi \) is simple cyclic, one recovers Lemma 4.5. We list the first few small codimensional smooth surfaces below.

| \( m \) | \( n \) | \( k \) | \( N \) | \( s \) | \( d \) | \( K^c_\chi \) |
|-----|-----|-----|-----|-----|-----|-----|
| 2   | 4   | 2   | 4   | 13  | (2,4,6) | 192 \cdot 13^c |
| 2   | 4   | 3   | 4   | 22  | (3,6,9) | 148 \cdot 22^c |
| 2   | 5   | 2   | 4   | 23  | (2,4,6,8) | 1152 \cdot 23^c |
| 2   | 5   | 3   | 4   | 37  | (3,6,9,12) | 5832 \cdot 37^c |

6.2.2. **Non–complete intersection surfaces of general type.** We also construct non–complete intersection surfaces of general type as in §4.2.1, we list the first few of them below.

| \( m \) | \( n \) | \( k \) | \( N \) | \( s \) | \( d \) | \( K^c_\chi \) |
|-----|-----|-----|-----|-----|-----|-----|
| 2   | 3   | 2   | 6   | 13  | (4,4,4,4) | 768 \cdot 13^c |
| 2   | 3   | 2   | 6   | 14  | (4,4,4,5) | 960 \cdot 14^c |

6.2.3. **Construction of singular surfaces.** Here we construct singular \( \mathbb{Z}_4 \) covers of complete intersection surfaces following the work of the first two authors in [GP08]. Assume \( Y \) is a complete intersection surface of multidegree \( d = (a, b, a + b) \) with \( a, b \geq 2 \), and let \( \pi : X \rightarrow Y \) be a \( \mathbb{Z}_4 \) cover that is a composition of two double covers \( p_1 : X_1 \rightarrow Y \) branched along a smooth divisor \( D_2 \in |\mathcal{O}_Y(2a)| \) and \( p_2 : X \rightarrow X_1 \) branched along the ramification of \( p_1 \) and \( p_1^*D_1 \) where \( D_1 \in |\mathcal{O}_Y(b)| \) is a smooth divisor, with trace zero module \( p_1^*\mathcal{O}_Y(-b) \). Assume \( D_1 \) and \( D_2 \) intersect transversally. Then \( X \) has \( ab \) singular points of type \( A_1 \). Furthermore, the canonical bundle \( \omega_X = \pi^*\mathcal{O}_Y(3a + 3b - 6) \) is ample, consequently \( \varphi \) has smooth algebraically semiuniversal deformations space. By our discussions \( \varphi \) can be deformed to an embedding.

6.3. **Open questions and final remarks.** We end this article by asking the questions that are crucial to understanding the small codimensional subvarieties which we have constructed in this article.

**Question 6.2.** We recall that the subvarieties appearing in the white rows of the tables constructed in §4.1.1, §4.1.2, §4.1.3, Section 5.1, and §6.2.1 were constructed as one–parameter smoothing of ropes. We again emphasize that Proposition 3.12 does not imply that the general members of every one–parameter smoothing of ropes are complete intersections. Then, in which of the above cases is every one–parameter smoothing of the corresponding rope a complete intersection?

On the other hand, our results also raise the following

**Question 6.3.** Is \( \Phi_t \) an embedding for varieties appearing in the white rows of the tables that are constructed in Section 4.3, and in Section 5.3, for \( t \neq 0 \)?

**Embedded ropes that are obstructed.** Next remark shows the subtleties of the situation we handle, as it introduces an example of a unobstructed morphism \( \varphi \) to which we can associate an embedded rope which is obstructed. In some of the cases, even if the deformation space of \( \varphi \) is smooth, the rope lies in at least two components of the Hilbert scheme:

**Remark 6.4.** With the notation of Set-up 3.1, we give an example of the following situation:

1. a morphism \( \varphi : X \rightarrow \mathbb{P}^N \) with \( Y \) smooth such that there exists a smooth curve \( T \) with a distinguished point \( 0 \) and a flat family of morphisms \( \Phi : \mathcal{X} \rightarrow \mathbb{P}^N \) over \( T \) such that \( \Phi_0 = \varphi \) and \( \Phi_t \) is an embedding for all \( t \neq 0 \);
2. \( \text{Im}\Phi_0 \) is a rope \( \tilde{Y} \) in \( \mathbb{P}^N \) supported on \( Y \) with conormal bundle \( \mathcal{E} \);
3. the morphism \( \varphi \) is unobstructed and, by (1), a general element of the algebraic formally semiuniversal deformation of \( \varphi \) is an embedding;
4. However, the rope \( \tilde{Y} \) does not correspond to a smooth point of its Hilbert scheme.

For such an example, consider the rational normal scroll \( Y = F_e \hookrightarrow \mathbb{P}^N \) with \( e = 3 \) or \( e = 4 \). By [Rei76], there exists a double \( K3 \) cover \( \pi : X \rightarrow Y \), branched along a reasonably singular curve \( C \in |−2K_Y| \) (see
[Rei76], Theorem 2.2). Then $H^1(N_T) = H^1(\pi_* N_T) = H^1(-2K_Y|_C) = H^1(2K_C) = 0$ by Proposition 2.2. Since $H^1(N_{Y\otimes N }):= H^1(N_{Y\otimes N } \otimes K_Y) = 0$, we have $H^1(N_0) = 0$ and $\phi$ is unobstructed. But $\bar{Y}$ is a singular point of its Hilbert scheme by [GP97], Theorem 4.1. Moreover, if $N \geq 10$ and is congruent with 1 modulo 4, then Hilbert point of $\bar{Y}$ lies in two irreducible components of the Hilbert scheme (see [GP97], Theorem 4.3).

**References**

[Bâ96] Bănică, Constantin. *Smooth reflexive sheaves*. Proceedings of the Colloquium on Complex Analysis and the Sixth Romanian-Finnish Seminar. Rev. Roumaine Math. Pures Appl. 36 (1991), no. 9-10, 571–593.

[BGG20] Bangere, Purnaprajna; Gallego, Francisco Javier; González. *Deformations of hyperelliptic and generalized hyperelliptic polarized varieties*. Preprint. https://arxiv.org/abs/2005.00342.

[RG20] Bangere, Purnaprajna; Gallego, Francisco Javier. *Deformation of double covers of complete intersections*. Preprint.

[BM20] Bangere, Purnaprajna; Mukherjee, Jayan; Raychaudhury, Debadiya. *K3 carpets on minimal rational surfaces and their smoothings*. Preprint. https://arxiv.org/abs/2006.16448.

[BHPV04] Barth, Wolf P.; Hulek, Klaus; Peters, Chris A. M.; Van de Ven, Antonius. *Compact complex surfaces*. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 4. Springer-Verlag, Berlin, 2004.

[BE95] Bayer, Dave; Eisenbud, David. *Ribbons and their canonical embeddings*. Trans. Amer. Math. Soc. 347 (1995), no. 3, 719–756.

[CL19] Catanese, Fabrizio; Lee, Yongnam. *Deformation of a generically finite map to a hypersurface embedding*. J. Math. Pures Appl. (9) 125 (2019), 175–188.

[EG95] Eisenbud, David; Green, Mark. *Clifford indices of ribbons*. Trans. Amer. Math. Soc. 347 (1995), no. 3, 757–765.

[Fon93] Fong, Lung-Ying. *Rational ribbons and deformation of hyperelliptic curves*. Algebraic Geom. 2 (1993), no. 2, 295–307.

[Fuj83] Fujita, Takao. *On hyperelliptic polarized varieties*. Tôhoku Math. Journ. 35 (1983), 1–44.

[GGP13a] Gallego, Francisco Javier; González, Miguel; Purnaprajna, Bangere P. *An infinitesimal condition to smooth ropes*. Rev. Mat. Complut. 26 (2013), no. 1, 253–269.

[GGP08a] Gallego, Francisco Javier; González, Miguel; Purnaprajna, Bangere P. *K3 double structures on Enriques surfaces and their smoothings*. J. Pure Appl. Algebra 212 (2008), no. 5, 981–993.

[GGP08b] Gallego, Francisco Javier; González, Miguel; Purnaprajna, Bangere P. *Deformation of finite morphisms and smoothing of ropes*. Compos. Math. 144 (2008), no. 3, 673–688.

[GGP13b] Gallego, Francisco Javier; González, Miguel; Purnaprajna, Bangere P. *Canonical double covers of minimal rational surfaces and the non-existence of carpets*. J. Algebra 374 (2013), 231–244.

[GGP08] Gallego, Francisco Javier; González, Miguel; Purnaprajna, Bangere P. *Deformations of canonical double covers*. J. Algebra 463 (2016), 23–32.

[GGP16a] Gallego, Francisco Javier; González, Miguel; Purnaprajna, Bangere P. *Deformations of canonical triple covers*. J. Algebra 463 (2016), 1–9.

[GGP10] Gallego, Francisco Javier; González, Miguel; Purnaprajna, Bangere P. *Deformation of canonical morphisms and the moduli of surfaces of general type*. Invent. Math. 182 (2010), no. 1, 1–46.

[GP97] Gallego, Francisco Javier; Purnaprajna, B. P. *Degenerations of K3 surfaces in projective space*. Trans. Amer. Math. Soc. 349 (1997), no. 6, 2477–2492.

[GP08] Gallego, Francisco Javier; Purnaprajna, Bangere P. *Classification of quadruple Galois canonical covers*. J. Trans. Amer. Math. Soc. 360 (2008), no. 10, 5489–5507.

[Gon06] González, Miguel. *Smoothing of ribbons over curves*. J. Reine Angew. Math. 591 (2006), 201–235.

[Har80] Hartshorne, Robin. *Stable Reflexive Sheaves*. Math. Ann. 254 (1980), no. 2, 121–176.

[HV85] Hulek, K.; Van de Ven, A. *The Horrocks-Mumford bundle and the Ferrand construction*. Manuscripta Math. 50 (1985), 313–335.

[Kaw99] Kawamata, Yujiro. *Deformations of canonical singularities*. J. Amer. Math. Soc. 12 (1999), no. 1, 85–92.

[KPR03] Kumar, N. Mohan; Peterson, Chris; Rao, A. Prabhakar. *Monads on projective spaces*. Manuscripta Math. 112 (2003), no. 2, 183–189.

[Laz80] Lazarsfeld, Robert. *A Barth-type theorem for branched coverings of projective space*. Math. Ann. 249 (1980), no. 2, 153–162.

[Man95] Manolache, Nicolae. *Nilpotent ICI structures on global complete intersections*. Math. Z. 219 (1995), no. 3, 403–411.

[Man04] Manolache, Nicolae. *Cohen-Macaulay nilpotent schemes*. Recent advances in geometry and topology, 235–248, Cluj Univ. Press, Cluj-Napoca, 2004.

[MR20] Mukherjee, Jayan; Raychaudhury, Debadiya. *Smoothing of multiple structures on embedded Enriques manifolds*. Preprint. https://arxiv.org/abs/2002.05846.

[Rei76] Reid, M. *Hyperelliptic linear systems on a K3 surface*. J. London Math. Soc. (2) 13 (1976), 427–437. MR 55:8044
Okonek, Christian; Schneider, Michael; Spindler, Heinz. *Vector bundles on complex projective spaces*. Progress in Mathematics, 3. Birkhäuser, Boston, Mass., 1980. vii+389 pp.

Pardini, Rita. *Abelian covers of algebraic varieties*. J. Reine Angew. Math. 417 (1991), 191–213.

Raicu, Claudiu; Sam, Steven. *Bi-graded Koszul modules, K3 carpets, and Green’s conjecture*. Preprint, https://arxiv.org/abs/1909.09122.

Seiler, Wolfgang K. *Moduli of surfaces of general type with a fibration by genus two curves*. Math. Ann. 301 (1995), no. 4, 771–812.

Sernesi, Edoardo. *Deformations of Algebraic Schemes*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 334. Springer-Verlag, Berlin, 2006.

Sernesi, Edoardo. *Small deformations of global complete intersections*. Boll. Un. Mat. Ital. (4) 12 (1975), 138—146.

Vatne, Jon Eivind. *Multiple structures and Hartshorne’s conjecture*. Comm. Algebra 37 (2009), no. 11, 3861–3873.

Wehler, Joachim. *Cyclic coverings: deformation and Torelli theorem*. Math. Ann. 274 (1986), no. 3, 443–472.

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