GRAPH SPARSIFICATION
BY UNIVERSAL GREEDY ALGORITHMS

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Abstract. Graph sparsification is to approximate an arbitrary graph by a sparse graph and is useful in many applications, such as simplification of social networks, least squares problems, numerical solution of symmetric positive definite linear systems and etc. In this paper, inspired by the well-known sparse signal recovery algorithm called orthogonal matching pursuit (OMP), we introduce a deterministic, greedy edge selection algorithm called universal greedy algorithm (UGA) for graph sparsification. The UGA algorithm can output a \((1 + \epsilon)^2(1 - \epsilon)^2\)-spectral sparsifier with \(\lceil \frac{n}{2} \epsilon^2 \rceil\) edges in \(O(m + n^2/\epsilon^2)\) time with \(m\) edges and \(n\) vertices for a general random graph satisfying a mild sufficient condition. This is a linear time algorithm in terms of the number of edges that the community of graph sparsification is looking for. The best result in the literature to the knowledge of the authors is the existence of a deterministic algorithm which is almost linear, i.e. \(O(m^{1+o(1)})\) for some \(o(1) = O(\frac{\log \log(m)}{\log^2\log(m)})\). We shall point out that several random graphs satisfy the sufficient condition and hence, can be sparsified in linear time. For a general spectral sparsification problem, e.g., positive subset selection problem, a nonnegative UGA algorithm is proposed which needs \(O(mn^2 + n^3/\epsilon^2)\) time and the convergence is established.

1. Introduction

1.1. Spectral graph sparsification. Graph sparsification aims to find a sparse subgraph from a dense graph \(G\) with \(n\) vertices and \(m\) edges (typically \(m \gg n\)) so that the sparsified subgraph can serve as a proxy for \(G\) in numerical computations for graph-based applications. An influential and beautiful paper by Batson, Spielman and Srivastava [6] showed that for any undirected graph \(G\) one can find a sparse graph (sparsifier) whose graph Laplacian matrix can well preserve the spectrum of the original graph Laplacian matrix. Such a spectral graph sparsification plays increasingly important roles in many applications areas in mathematics and computer science [29, 34, 41]. A related research, known as Laplacian Paradigm, is illustrated as an emerging paradigm for the design of scalable algorithms in recent years. We refer the reader to [38, 42, 45] for excellent surveys on its background and applications.
Mathematically, we can state the graph sparsification problem as follows. Consider an undirected and weighted graph \( G = (V, E, w) \), where \( V \) is a set of vertices, \( E \) is a set of edges, and \( w \) is a weight function that assigns a positive weight to each edge. The Laplacian matrix of graph \( G \) is defined by
\[
L_G = \sum_{(u,v) \in E} w_{(u,v)} (e_u - e_v)(e_u - e_v)^\top,
\]
where \( w_{(u,v)} \geq 0 \) is the weight of edge \((u,v)\) and \( e_u \in \mathbb{R}^{|V|} \) is the characteristic vector of vertex \( u \) (with a 1 on coordinate \( u \) and zeros elsewhere). In other words, for any \( x \in \mathbb{R}^n \),
\[
x^\top L_G x = \sum_{(u,v) \in E} w_{(u,v)} (x(u) - x(v))^2 \geq 0.
\]
That is, \( L_G \) is positive semidefinite. 

**Spectral graph sparsification** is the process of approximating the graph \( G \) by a sparse (linear-sized) graph \( H = (V, \tilde{E}, \tilde{w}) \) such that
\[
ax^\top L_G x \leq x^\top L_H x \leq bx^\top L_G x
\]
for all \( x \in \mathbb{R}^{|V|} \), where \( b \geq a > 0 \). Setting \( \kappa := b/a \), \( H \) is called a \( \kappa \)-approximation of \( G \) or a \( \kappa \)-sparsifier of \( G \). Actually, if we restrict the inequality in (1) only for all \( x \in \{0, 1\}^{|V|} \), one can obtain the cut sparsification [7]. Batson, Spielman and Srivastava [6, Theorem 1.1] proved that for every weighted graph \( G \) and every \( \epsilon \in (0, 1) \) there exists a weighted graph \( H \) with at most \( \lceil (n - 1)/\epsilon^2 \rceil \) edges which is an \( (1+\epsilon)^2/(1-\epsilon)^2 \)-approximation of \( L_G \). More generally, let \( V = \{v_1, \ldots, v_m\} \subset \mathbb{R}^n \) be a collection of vectors with \( m \gg n \). We replace \( L_G \) by
\[
B = \sum_{i=1}^m v_i v_i^\top,
\]
which is clearly positive semidefinite. In [6, Theorem 1.2], the authors proved that for any \( \epsilon \in (0, 1) \), there exists a sparsifier \( s = (s_1, \ldots, s_m) \in \mathbb{R}^m_+ \) with \( \|s\|_0 \leq \lceil \text{rank}(B)/\epsilon^2 \rceil \) such that
\[
(1 - \epsilon)^2 B \preceq \sum_{i=1}^m s_i v_i v_i^\top \preceq (1 + \epsilon)^2 B,
\]
where \( \mathbb{R}^m_+ \) denotes the nonnegative orthant in \( \mathbb{R}^m \) and \( \|s\|_0 \) stands for the number of nonzero entries of vector \( s \). For convenience, we shall only discuss the isotropic case, i.e., when \( B = I_n \) is the \( n \times n \) identity matrix. Such decomposition appear in many areas of mathematics and are also called isotropic sets, tight frames, John’s decompositions (when their mean is zero), and etc..
1.2. Related work. In recent years, spectral sparsification is a widely studied topic and has applications to many areas in mathematics and theoretical computer science. Recent work includes [3–6, 9, 14, 21, 22, 26–30, 32, 33, 37, 39–41, 49].

In the seminal paper [40], Spielman and Teng first introduced the notion of spectral sparsification and showed that, for any undirected graph $G$ of $n$ vertices and $m$ edges has a spectral sparsifier with $O(n \log^c n)$ edges that can be computed in $O(m \log^c(m))$ time for some positive constant $c$. In terms of the number $n$ of vertices with assumption of the number $m$ of edges being $O(n^2)$, the Spielman and Teng’s algorithm will need $O(n^2 \log^2(n))$ time. Some progresses had been made by the work of Spielman and Srivastava [39] and Lee and Sun [33], who showed how to find the spectral sparsifiers with $O(n \log n)$ and $O(n)$ edges, respectively, in $O(m \log^c(m))$ time. We remark that all of this type of algorithms requires random sampling or random projection.

A celebrated result of Batson, Spielman and Srivastava [6] states that for any undirected graph $G$ of $n$ vertices and $m$ edges and a given parameter $\epsilon \in (0, 1)$, there exists a spectral sparsifier with $O(n/\epsilon^2)$ edges. They further provided a polynomial time, deterministic algorithm that in each iteration one edge is chosen deterministically to optimize the change of some ‘barrier’ potential functions, however such algorithm and subsequent algorithms by [50] require $O(mn^3/\epsilon^2)$ and $O(mn^2/\epsilon^2 + n^4/\epsilon^2)$ time respectively. When a graph has $m = O(n^2)$ edges, the computational time is $O(n^5)$ or $O(n^4)$. Most recently, the work [14] solves a big unsettled problem by showing that there exist a deterministic algorithm to find the spectral sparsification in almost-linear time, i.e. $O(m^{1+o(1)})$ with $o(1) = O((\log \log(m))^{2/3}/\log^{4/3}(m))$. Again, in terms of the number of vertices, the time is $O(n^{2+o(1)})$. Table 1 is a summary of various algorithms for graph sparsification.

1.3. Our contribution. In this paper, we consider the graph $G$ having $n$ vertices and $m$ edges with $m = O(n^2)$. We propose a simple and efficient algorithm called universal greedy algorithm to solve the graph sparsification problem to have at most $n/\epsilon^2$ edges for any given $\epsilon \in (0, 1)$. Our motivation can be explained as follows. Let us consider (3) as an example. Denote by $\phi_i$ the vectorization of matrix $v_i v_i^\top$ for $i = 1, \ldots, m$, and let $\Phi = [\phi_1, \ldots, \phi_m]$ be the sensing matrix and denote by $b$ the vetorization of $B$. Letting $e \in \mathbb{R}^m$ be the vector with 1 for all entries, (2) can be rewritten as the following underdetermined linear system

$$\Phi e = b.$$
Table 1. Summary of various algorithms for graph sparsification with \( n \) and \( m \) being the number of nodes and edges of the input graph, respectively. D/R stands for deterministic or randomized algorithms.

| Algorithms | Sparsifier Size | Approximation | Flops Count | D/R |
|------------|----------------|---------------|-------------|-----|
| Theorem 6.1 in [40] [28] | \( O(n \log^c n) \) | \( 1+\epsilon \) -approx. | \( O(m \log^c (m)) \) | Random. |
| Theorem 1.1 in [6] | \( O(n) \) | \( (1+\epsilon)^2 \) -approx. | \( O(mn^2 + n^4) \) | Determ. |
| Corollary 7.3 in [14] | \( O(n \log^c n) \) | \( n^{O(1)} \) -approx. | \( O(m^{1+\epsilon(1)}) \) | Determ. |
| Theorem 1 in [33] | \( O(n) \) | \( 1+\epsilon \) -approx. | \( O(m \log^c (m)) \) | Random. |
| Theorem 1 in [39] | \( O(n \log n) \) | \( 1+\epsilon \) -approx. | \( O(m \log^c (m)) \) | Random. |
| Algorithm 1 | \( O(n) \) | \( (1+\epsilon)^2 \) -approx. | \( O(m + n^2) \) | Determ. |
| Algorithm 2 | \( O(n) \) | \( (1+\epsilon)^2 \) -approx. | \( O(mn^2) \) | Determ. |

We can formulate the problem (3) as a compressive sensing [17, 25] problem: Find a sparse solution \( s \geq 0 \) with \( \|s\|_0 \ll m \) such that

\[
\min \|s\|_0 \quad \text{s.t.} \quad \|\Phi s - b\|_2 \leq \epsilon, s \geq 0.
\]

Thus, the sparsification problem in (3) is a constrained compressive sensing problem.

In the standard compressive sensing approach, i.e. the problem in (4) without \( s \geq 0 \) or the following version (5), one efficient way to solve it is to use a greedy algorithm called orthogonal matching pursuit (OMP) [36].

\[
\min \|\Phi s - b\|_2^2 \quad \text{s.t.} \quad \|s\|_0 \leq s,
\]

It is a very popular algorithm and has been studied by many researchers. See, e.g., [15, 16, 36, 43, 44, 51] and many variations of the OMP. We refer to [18] and [23] for an approach for (4). Besides, the OMP procedure is also very efficient in completing a low rank matrix with given partial known entries (see [46, 47]). To deal with the graph sparsification problem (1), we have to improve the efficiency of the OMP algorithm. To deal with positive subset selection problem (3), we have to enforce the nonnegativity in the OMP algorithm. In this paper, we mainly extend the ideas in the orthogonal rank 1 matrix pursuit (OR1MP) algorithm in [46, 47], especially, the economical OR1MP to the settings of the computation for the graph sparsification and positive subset selection. Our new algorithms will be called universal greedy algorithms (UGA) in this paper. That is, we shall use an UGA algorithm to compute graph sparsifiers and a nonnegative UGA algorithm to find positive subsets. A study of the convergence of the both UGA algorithms to compute approximations of (1) and (3) will be presented and the convergence for these algorithms will be established.
under some sufficient conditions. One major effort is to establish the desired properties (1) of the iterative solutions from the UGA algorithm. Another major effort is to establish the convergence of the nonnegativity UGA under the constraint of nonnegativity in (3).

In addition, we shall explain the computational complexity of our algorithms for graph sparsifiers and isotropic subsets. One of the major contributions in this paper is to speed up the computation of finding effective sparsifiers. Indeed, our computational cost will be $O(mn^2/\epsilon^2)$ which is linear in terms of the number $m$ of edges with the standard assumption $m = O(n^2)$. This is more efficient than the algorithms in [6] which require $O(mn^2) = O(n^4)$ or $O(m^2)$. That is, our concrete deterministic algorithm given in this paper reduces the computational time from $O(m^2)$ to $O(m)$ or from $O(n^4)$ to $O(n^2)$. To the best of our knowledge, a recent study in [14] shows that there is a deterministic algorithm for finding graph sparsification in a near-linear time $O(m^{1+o(1)})$ with $o(1) = O((\log \log(m))^{2/3}/\log^{2/3}(m))$. However, the iterative matrix from our algorithm will be a graph sparsifier if the graph Laplacian satisfies a sufficient condition. We shall point out several random graph models which will satisfy the sufficient condition. For those graphs, they can be sparsified in linear time.

1.4. Preliminary on random graphs. In order to establish the convergence of our UGA algorithm for graph sparsification, we need some knowledge on random graphs. For convenience, let us give a short preliminary explanation of random graphs considered in this paper. Four kinds of random graph models will be considered in this paper. The most simple random graph model is the classical Erdős-Rényi model which has only one simple assumption: a pair of vertices in the vertex set $V$ of graph $G = G(V,E)$ is chosen to be an edge in the edge set $E$ independently with probability $p \in (0,1)$ for some fixed $p$. In particular, a complete graph is a graph from the Erdős-Rényi model with $p = 1$. For convenience, we denote the Erdős-Rényi model by $G_{n,p} := G_{n,p}(V,E)$ with $n$ vertices in $V$ and $p$ probability for edges in $E$. As $p$ changes from 0 to 1, the evolution of $G_{n,p}$ passes through six clearly distinguishable ranges as described in [11,19,20]. When $p = c/n$ with $c = o(1), c \in (0,1)$ and $c = 1$, all connected components of $G_{n,p}$ are either trees or unicyclic components. So the graph Laplacian matrix $L_G$ is already sparsified. When $p = c/n$ with $c > 1$, except for one “giant” component, all the other components are relatively small, and most of them are trees. When $p = \omega(n) \log(n)/n$ with $\omega(n) \geq 1$, the graph $G_{n,p}(V,E)$ is almost surely connected and when $\omega(n) \to \infty$, $G_{n,p}(V,E)$ is not only almost surely connected, but the
degrees of almost all vertices are asymptotically equal. For graph Laplacian $L_G$ associated with such Erdős-Rényi model with $p = \omega(n) \log(n)/n$ with $\omega(n) \gg 1$, we will show that our UGA algorithm can find graph sparsifiers for $G_{n,p}$.

Next a general random graph $G$ can also be defined based on given expected degrees $d = (d_1, \ldots, d_n)$ among its vertices $v_1, \ldots, v_n$ [10, 12]. Indeed, $G$ has a potential edge between $v_i$ and $v_j$ subject to probability $p_{ij} = d_i d_j \rho < 1$ with $\rho = \frac{1}{\sum_{\ell=1}^{n} d_\ell}$ independent of other edges for all $i, j = 1, \ldots, n$ and $i \neq j$. It may be called Chung-Lu model as it was first introduced in [10] which is another one of the random graph models studied in the paper. Note that when all $d_i$'s are the same, $p = d_i d_j \rho = d_i/n < 1$ which is reduced to Erdős-Rényi model $G_{n,p}$ [19]. The expected degree of each vertex of the graph of interest is $pn = \omega(n) \log(n)$ with $\omega(n) \rightarrow \infty$. By Theorem 3.4 (ii) in [24], the minimal degree $d_{\text{min}}(G) = (1 - o(1))np = \omega(n) \log(n)$ and the maximal degree $d_{\text{max}}(G) = (1 + o(1))w(n) \log(n)$ almost surely and then the ratio $d_{\text{max}}/d_{\text{min}}$ is bounded from the above. Also, by Theorem 4 in [13], the minimal nonzero eigenvalue, $\lambda_2$ of the Laplacian of $G$ is close to 1. That is, $|\lambda_2 - 1| = O(\sqrt{1/w(n)}) = o(1)$. This fact will be used later in our study.

In addition, a random graph may contain a few giant components (called clusters) and may not be a clean graph which means that it has some noise (few edges between clusters). Such a graph can be defined by using a pre-planted clusters model, e.g., stochastic block model. That is, all the vertices of a graph $G$ are divided into a few clusters, say $k$ clusters, $C_1, \ldots, C_k$ and there is a probability matrix $P = [p_{ij}]_{1 \leq i, j \leq k}$ associated with $G$ with $p_{ij} \in [0, 1]$ such that possible edges among vertices in $C_i$ are subject to $p_{ii}, i = 1, \ldots, k$ and possible edge between a vertex in $C_i$ and a vertex in $C_j$ is subject to the probability $p_{ij}$. Since there are more edges within a cluster than among clusters, we have to have with $p_{ij} < \min\{p_{ii}, p_{jj}\}$ for all $j \neq i$. For example, each cluster $C_i$ is subject to Erdős-Rényi model $G_{n_i, p_i}$ for $i = 1, \ldots, k$, where $n_i$ is the size of the cluster $C_i$. Such graphs are belong to the so-called stochastic block model (SBM). See, e.g. [1] for more properties of the random graphs from SBM. For example, when $p = p_{ii}, i = 1, \ldots, k$ and $q = p_{i,j}$ for $i \neq j$ with $p = a \log(n)/n$ and $q = b \log(n)/n$, we know that the exactly recovery of the clusters of $G$ is solvable if and only if

$$\frac{1}{k} (\sqrt{a} - \sqrt{b})^2 > 1.$$
In this setting, the graph Laplacian matrix can be sparsified by using a blockly diagonal matrix (after permutation). We refer to [2] for a proof and algorithms how to recovery of clusters. More recently, a linear time algorithm (in terms of number of edges) is developed in [31] under the assumption that \( a = \omega(n) \to \infty \), e.g., \( \omega(n) = \ln(\ln(n)) \). Since each cluster is from Erdős-Rényi model, under the assumption that the number \( k \) of clusters is fixed independent of \( n \to \infty \), we have the minimal degree \( d_{\min}(G) = \omega(n) \log(n) \gg \log(n) \).

Hence, we can use Theorem 4 in [13] again to see that the minimal nonzero eigenvalue of the graph Laplacian is close to 1 almost sure. Also, we know the ratio \( d_{\max}/d_{\min} \) is bounded from the above.

Finally, it is useful to write \( G \) as a union of two edge-disjoint subgraphs \( G = G^{in} \cup G^{out} \), where \( G^{in} = (V, E^{in}) \) consists of only intra-connection edges, and \( G^{out} = (V, E^{out}) \) consists of only inter-connection edges. We will use \( d^{in}_i \) to denote the degree of vertex \( i \) in the subgraph \( G^{in} \), and \( d^{out}_i \) the degree of vertex \( i \) in the subgraph \( G^{out} \). We will also use \( A^{in} \) and \( L^{in} \) to denote the adjacency matrix and graph Laplacian of \( G^{in} \), and use \( A^{out} \) and \( L^{out} \) the adjacency matrix and graph Laplacian of \( G^{out} \). Another random graph model we consider in this paper is defined based on the four assumption given in [31].

(A1) The number \( k \) of clusters of \( G \) is \( O(1) \) as \( n \to \infty \);
(A2) For all \( i = 1, \ldots, k \), the eigenvalues \( \lambda_2(L_{C_i}) \geq 1 - \epsilon_1 \) and \( \lambda_{n_i}(L_{C_i}) \leq 1 + \epsilon_1 \) almost surely;
(A3) letting \( r_i := d^{out}_i/d^{in}_i \), \( r_i \leq \epsilon_2 \) for all \( i \in [n] \) almost surely;
(A4) Letting \( d^{in}_{av} := \mathbb{E}[d^{in}_i] \) be the expected in-degree, \( d^{in}_{\max} \leq (1 + \epsilon_3)d^{in}_{av} \) and \( d^{in}_{\min} \geq (1 - \epsilon_3)d^{in}_{av} \) almost surely,

for \( \epsilon_1, \epsilon_2, \epsilon_3 \in (0, 1) \), where \( n_i \) is the size of the cluster \( C_i \) for \( i = 1, \ldots, k \). See [31] for more detail. Besides, we refer to [11] and [24] for more graph theory.

1.5. Notation and organization. We begin with some notation and definitions. Let \( A = (a_1, \ldots, a_n) \in \mathbb{R}^{n \times n} \) be an \( n \times n \) real matrix, we use \( A(i,j) \) to denote the \( (i,j) \)-th component of \( A \). The operator norm and the Frobenius norm of \( A \) is defined as \( \|A\|_2 \) and \( \|A\|_F := \sqrt{\sum_{i,j} A(i,j)^2} \), respectively. For any \( x \in \mathbb{R}^n \), we use \( \|x\|_2 \) to denote the \( \ell_2 \)-norm of \( x \). For any subset \( \Lambda \), we use \( A_\Lambda \) to denote the sub-matrix of \( A \) obtained by extracting the columns of \( A \) indexed by \( \Lambda \). Let \( vec(A) := (a_1^\top, \ldots, a_n^\top)^\top \) denote a vector reshaped
from matrix $A$ by concatenating all its column vectors. The inner product of two matrices $A$ and $B$ is defined as $\langle A, B \rangle := \langle \text{vec}(A), \text{vec}(B) \rangle$. We call a matrix $A$ positive semidefinite if $x^\top A x \geq 0$ holds for any $x \in \mathbb{R}^n$, and a matrix $A$ positive definite if $x^\top A x > 0$ holds for any nonzero $x \in \mathbb{R}^n$. For any two matrices $A$ and $B$, we write $A \preceq B$ to represent $B - A$ is positive semidefinite, and $A \prec B$ to represent $B - A$ is positive definite. Finally, for any $b_1, b_2 \in \mathbb{R}$, we often use $b_1 = O(b_2)$ if $|b_1/b_2|$ is bounded from the above and $b_1 = o(b_1)$ if $|b_1/b_2| \ll 1$. In particular, $b_n = o(1/n)$ if $nb_n \to 0$ as $n \to +\infty$.

The paper is organized as follows. We shall first discuss the computation of (1) to show an universal greedy algorithm will find a graph sparsification under the assumption that the nonzero eigenvalue of graph Laplace $L_G$ is greater than $1/(2 - \delta)$ for $\delta \in (0, 1)$. As explained above, a random graph considered in this paper will have this property with high probability when the size of graph $L_G$ is large. Next we shall extend our study to deal with problem (3) to produce a subset $S \subset \{1, \ldots, m\}$ with nonnegative coefficients $s_i > 0, i \in S$ and establish the convergence of the nonnegative UGA algorithm under a sufficient condition on the size $K = \sum_{i \in S} s_i$ and the minimal size of the given vectors $v \in V$. The computational complexity of the nonnegative UGA is also given.

2. The UGA algorithm for Sparsifiers

In this section, we propose our algorithm to compute a graph sparsifier. The strategy follows from the ideas of the economical OR1MP in [47] which will significantly speed up the computation as stated in Theorem 2.1 below. As this technique can be used to speed up all OMP like algorithms, we will call it the universal greedy algorithm (UGA) instead. We shall first present our computational algorithm in this section. Then we explain the computational complexity for the algorithm. Next we present the convergence analysis of the algorithm and show that it has a linear convergence. Finally, we shall show that the subgraph obtained from the OMP algorithm is indeed a graph sparsifier.

2.1. An UGA for Spectral Sparsification. Let $G = (V, E, w)$ be a weighted undirected graph. In what follows, we define the weighted adjacency matrix of $G$ by

$$A = \begin{cases} w_{(u,v)}, & \text{if } (u, v) \in E, \\ 0, & \text{otherwise,} \end{cases}$$
and define the weighted degree of a vertex \( u \) by

\[
D = \sum_{u \in V} w_{(u,v)}.
\]

Then the Laplacian matrix of graph \( G \) is \( L_G = D - A \). To state conveniently, we define

\[
\phi_{(u,v)} := (e_u - e_v)(e_u - e_v)^T,
\]

where \( e_u \in \mathbb{R}^n \) is a standard basis vector which is zero everywhere except for the \( u \)-th component which is 1. Then the Laplacian \( L_G \) can be simple described as

\[
L_G = \sum_{(u,v) \in E} w_{(u,v)} \phi_{(u,v)}.
\]

Note that if \( w_{(u,v)} = 1 \), then \( L_G = D - A \) is the standard graph Laplacian for an undirected graph \( G = (V,E) \), and if \( w_{(u,v)} > 0 \), \( L_G \) is a weighted graph Laplacian.

The UGA for spectral sparsification is given in Algorithm 1.

**Algorithm 1** Universal greedy algorithm (UGA) for spectral sparsification

**Input:** Laplacian matrix \( L_G \in \mathbb{R}^{n \times n} \), \( \epsilon \in (0, 1) \).

1. \( R_0 := L_G \), \( L_{H_0} := 0 \) and \( i := 1 \).
2. Find an edge \((u_i, v_i)\) such that \( \left| \langle \phi_{(u_i,v_i)}, R_{i-1} \rangle \right| = \max_{(u,v) \in E} \left| \langle \phi_{(u,v)}, R_{i-1} \rangle \right| \).
3. Compute the optimal weights

\[
(\alpha_1^i, \alpha_2^i) = \arg \min_{(\alpha_1, \alpha_2) \in \mathbb{R}^2} \| L_G - \alpha_1 L_{H_{i-1}} - \alpha_2 \phi_{(u_i,v_i)} \|_F^2.
\]

4. Update

\[
L_{H_i} = \alpha_1^i L_{H_{i-1}} + \alpha_2^i \phi_{(u_i,v_i)}
\]

and

\[
R_i = L_G - L_{H_i}.
\]

5. If \( i > \left\lceil \frac{n}{\epsilon^2} \right\rceil \), stop and go to output. Otherwise, set \( i = i + 1 \) and return to Step 2.

**Output:** The sparsifier \( L_H \).

2.2. **Computational Complexity.** The running time of Algorithm 1 is dominated by Steps 2 and 3, whose total cost is \( O(m + n^2/\epsilon^2) \). In terms of the number \( m \) of edges, the total cost is \( O(m) \) if \( m = O(n^2) \).
Theorem 2.1. The computational complexity of Algorithm 1 for graph $G$ with $n$ vertices and $m$ edges is $O(m + n^2/\varepsilon^2)$.

Proof. We first show that the total cost of Step 2 is $O(m + n^2)$. By Step 2, we have

$$\langle \phi(u,v), R_i \rangle = \langle \phi(u,v), L_G - L_{H_i} \rangle = \langle \phi(u,v), L_G \rangle - \langle \phi(u,v), L_{H_i} \rangle$$

since each $\phi(u,v)$ has only four non-zero entries. So in the first iteration, we need $O(m)$ flops to compute all $\langle \phi(u,v), L_G \rangle$ with $(u,v) \in E$. Since

$$\langle \phi(u,v), L_{H_i} \rangle = \alpha_1^i \langle \phi(u,v), L_{H_{i-1}} \rangle + \alpha_2^i \langle \phi(u,v), \phi(u_i,v_i) \rangle,$$

hence using the incremental method, one only need to compute the last term $\langle \phi(u,v), \phi(u_i,v_i) \rangle$. Noting that

$$\langle \phi(u,v), \phi(u_i,v_i) \rangle = \begin{cases} 
0, & \text{if } u \neq u_i \text{ and } v \neq v_i, \\
4, & \text{if } u = u_i \text{ and } v = v_i, \\
1, & \text{if } u = u_i \text{ and } v \neq v_i, \\
1, & \text{if } u \neq u_i \text{ and } v = v_i,
\end{cases}$$

hence the $i$-th subsequent iteration need $O(n)$ flops to calculate all $\langle \phi(u,v), \phi(u_i,v_i) \rangle$. Thus the total computational cost of Step 2 is $O(m + n^2/\varepsilon^2)$ as the total number of iterations is $\lceil \frac{m}{\varepsilon} \rceil$.

Now let us discuss the computational cost of Step 3. Set

$$B_i = \begin{pmatrix} 
\langle L_{H_{i-1}}, L_{H_{i-1}} \rangle & \langle L_{H_{i-1}}, \phi(u_i,v_i) \rangle \\
\langle L_{H_{i-1}}, \phi(u_i,v_i) \rangle & \langle \phi(u_i,v_i), \phi(u_i,v_i) \rangle
\end{pmatrix}$$

and $b_i = (\langle L_G, L_{H_{i-1}} \rangle, \langle L_G, \phi(u_i,v_i) \rangle)^\top$. Then in Step 3, $\alpha_i$ can be compute by solving the following $2 \times 2$ system of linear equations

$$B_i \alpha_i = b_i,$$

which can be efficiently solved with $B_i$ and $b_i$ being given. Recall that every $\phi(u,v)$ only has 4-components nonzero, thus the computation of $\langle L_{H_{i-1}}, \phi(u_i,v_i) \rangle$, $\langle \phi(u_i,v_i), \phi(u_i,v_i) \rangle$ and $\langle L_G, \phi(u_i,v_i) \rangle$ shall be $O(n)$. Let us explain that $\langle L_{H_{i-1}}, L_{H_{i-1}} \rangle$ needs $O(n)$ flops. Indeed, according to Step 4 in Algorithm 1, we have

$$\langle L_{H_i}, L_{H_i} \rangle = (\alpha_1^i L_{H_{i-1}} + \alpha_2^i \phi(u_i,v_i), \alpha_1^i L_{H_{i-1}} + \alpha_2^i \phi(u_i,v_i)).$$
As we have already computed $\gamma := \langle L_{H_{i-1}}, L_{H_{i-1}} \rangle$ in the $(i-1)$th step, we can use $O(n)$ time to compute

$$\langle L_{H_{i}}, L_{H_{i}} \rangle = (\alpha_i^1)^2 \gamma + (\alpha_i^2)^2 \langle \phi(u_i,v_i), \phi(u_i,v_i) \rangle + 2\alpha_i^1 \alpha_i^2 \langle L_{H_{i-1}}, \phi(u_i,v_i) \rangle.$$ 

After running $\lceil \frac{n}{\epsilon^2} \rceil$ steps, the time complexity for Step 3 is $O(n^2/\epsilon^2)$.

### 2.3. Convergence analysis.

In this subsection, we show that Algorithm 1 is convergent and shares the linear convergence rate.

**Theorem 2.2.** Let $R_i$ be the residual matrix defined in (7) of Algorithm 1 for spectral sparsification. Then $R_i$ satisfies

$$\|R_i\|_F \leq \left(1 - \frac{1}{2|E|}\right)^i \|L_G\|_F,$$

for all $i \geq 1$.

Firstly, we state several useful properties of Algorithm 1.

**Lemma 2.3.** $\langle R_i, \phi(u_i,v_i) \rangle = 0$ and $\langle R_i, L_{H_{i-1}} \rangle = 0$.

**Proof.** Recall that $\alpha_i^j$ is the optimal solution of problem (6). By the first-order optimality condition according to $\phi(u_i,v_i)$ and $L_{H_{i-1}}$, we have

$$\langle L_G - \alpha_i^1 L_{H_{i-1}} - \alpha_i^2 \phi(u_i,v_i), L_{H_{i-1}} \rangle = 0$$

and

$$\langle L_G - \alpha_i^1 L_{H_{i-1}} - \alpha_i^2 \phi(u_i,v_i), \phi(u_i,v_i) \rangle = 0$$

which together with $R_i = L_G - \alpha_i^1 L_{H_{i-1}} - \alpha_i^2 \phi(u_i,v_i)$ implies that $\langle R_i, \phi(u_i,v_i) \rangle = 0$ and $\langle R_i, L_{H_{i-1}} \rangle = 0$. □

**Lemma 2.4.** $\|R_i\|^2_F = \|L_G\|^2_F - \|L_{H_i}\|^2_F$ for all $i \geq 0$.

**Proof.** For all $i \geq 0$,

$$\|L_G\|^2_F = \|R_i + L_{H_i}\|^2_F = \|R_i\|^2_F + \|L_{H_i}\|^2_F + 2\langle R_i, L_{H_i} \rangle = \|R_i\|^2_F + \|L_{H_i}\|^2_F,$$

the last equality follows from Lemma 2.3 that $\langle R_i, L_{H_i} \rangle = \alpha_i^1 \langle R_i, L_{H_{i-1}} \rangle + \alpha_i^2 \langle R_i, \phi(u_i,v_i) \rangle = 0$, this completes the proof of this lemma. □

**Lemma 2.5.** If $L_{H_{i-1}} = \beta \phi(u_i,v_i)$ with nonzero $\beta$, then $\|R_i\|_F = \|R_{i-1}\|_F$. 

Proof. If \( L_{H_{i-1}} = \beta \phi(u_i,v_i) \) for some \( \beta \neq 0 \), we get
\[
\| R_i \|_F^2 = \min_{\alpha \in \mathbb{R}^2} \| L_G - \alpha_1 L_{H_{i-1}} - \alpha_2 \phi(u_i,v_i) \|_F^2
\]
\[
= \min_{\alpha \in \mathbb{R}^2} \| L_G - (\alpha_1 + \alpha_2/\beta) L_{H_{i-1}} \|_F^2
\]
\[
= \min_{\gamma \in \mathbb{R}} \| L_G - \gamma L_{H_{i-1}} \|_F^2
\]
\[
= \min_{\gamma \in \mathbb{R}} \| L_G - \gamma \alpha_1^{i-1} L_{H_{i-2}} - \gamma \alpha_2^{i-1} \phi(u_{i-1},v_{i-1}) \|_F^2
\]
\[
\leq \min_{(\gamma,\gamma_2) \in \mathbb{R}^2} \| L_G - \gamma L_{H_{i-2}} - \gamma_2 \phi(u_{i-1},v_{i-1}) \|_F^2
\]
\[
= \| L_G - L_{H_{i-1}} \|_F^2 = \| R_i \|_F^2.
\]

and hence the conclusion \( \| R_i \|_F \leq \| R_{i-1} \|_F \) holds in this case. In general,
\[
\| R_{i-1} \|_F^2 = \| L_G - \alpha_1^{i-1} L_{H_{i-1}} - \alpha_2^{i-1} \phi(u_{i-1},v_{i-1}) \|_F^2
\]
\[
\geq \min_{\alpha \in \mathbb{R}^2} \| L_G - \alpha (\alpha_1^{i-1} L_{H_{i-1}} + \alpha_2^{i-1} \phi(u_{i-1},v_{i-1})) - \alpha_2 \phi(u_i,v_i) \|_F^2
\]
\[
= \min_{\alpha \in \mathbb{R}^2} \| L_G - \alpha_1 L_{H_{i-1}} - \alpha_2 \phi(u_i,v_i) \|_F^2 = \| R_i \|_F^2
\]

This completes the proof. \( \square \)

Lemma 2.6. Suppose that \( R_{i-1} \neq 0 \) for some \( i \geq 1 \). Then, \( L_{H_{i-1}} \neq \beta \phi(u_i,v_i) \) for all \( \beta \neq 0 \).

Proof. If \( L_{H_{i-1}} = \beta \phi(u_i,v_i) \) with \( \beta \neq 0 \), similar to (8) we have
\[
\| R_i \|_F^2 = \min_{\gamma \in \mathbb{R}} \| L_G - \gamma L_{H_{i-1}} \|_F^2 = \| L_G - \gamma \alpha_1^{i-1} L_{H_{i-2}} - \gamma \alpha_2^{i-1} \phi(u_{i-1},v_{i-1}) \|_F^2
\]
\[
= \| L_G - \gamma \alpha_1^{i-1} L_{H_{i-1}} - \gamma \alpha_2 \phi(u_{i-1},v_{i-1}) \|_F^2 = \| R_{i-1} \|_F^2 = \| L_G - L_{H_{i-1}} \|_F^2,
\]
where \( \gamma_{i-1}^* \) denotes the optimal solution of the minimization in terms of \( \gamma \) and the third equality follows from Lemma 2.5. As \( R_{i-1} \neq 0 \), we have \( L_{H_{i-1}} \neq 0 \). Then from the above equality, we conclude that \( \gamma_{i-1}^* = 1 \) is the unique optimal solution. While by its first-order optimality condition, we have
\[
\langle L_{H_{i-1}} - L_G, L_{H_{i-1}} \rangle = 0, \text{ i.e. } \langle R_{i-1}, L_{H_{i-1}} \rangle = 0.
\]

However, this contradicts
\[
|\langle R_{i-1}, L_{H_{i-1}} \rangle| = |\beta \langle R_{i-1}, \phi(u_i,v_i) \rangle| = |\beta| \max_{(u,v) \in E} |\langle \phi(u,v), R_{i-1} \rangle| \neq 0.
\]

This completes the proof. \( \square \)

We can build the following relationship for the residuals \( \| R_i \|_F \) and \( \| R_{i-1} \|_F \).

Lemma 2.7. \( \| R_i \|_F^2 \leq \| R_{i-1} \|_F^2 - \frac{(R_{i-1}, \phi(u_i,v_i))^2}{4} \) for all \( i \geq 1 \).
Proof. One can observe that
\[ \|R_i\|^2_F = \min_{\alpha \in \mathbb{R}^2} \|L_G - \alpha_1 L_{H_{i-1}} - \alpha_2 \phi(u_i, v_i)\|^2_F \]
\[ \leq \min_{\alpha_2 \in \mathbb{R}} \|L_G - L_{H_{i-1}} - \alpha_2 \phi(u_i, v_i)\|^2_F \]
\[ = \min_{\alpha_2 \in \mathbb{R}} \|R_i - 1 - \alpha_2 \phi(u_i, v_i)\|^2_F, \]
which has a closed form solution as \( \alpha^*_2 = \frac{\langle R_i - 1, \phi(u_i, v_i) \rangle}{\langle \phi(u_i, v_i), \phi(u_i, v_i) \rangle} \). Plugging \( \alpha^*_2 \) back into the formulation and note that \( \langle \phi(u_i, v_i), \phi(u_i, v_i) \rangle = 4 \), we get
\[ \|R_i\|^2_F \leq \|R_i - 1 - \frac{\langle R_i - 1, \phi(u_i, v_i) \rangle}{\langle \phi(u_i, v_i), \phi(u_i, v_i) \rangle} \phi(u_i, v_i)\|^2_F = \|R_i - 1\|^2_F - \frac{\langle R_i - 1, \phi(u_i, v_i) \rangle^2}{4}. \]
This completes the proof. \( \square \)

To prove Theorem 2.2, we need a technique lemma. To state conveniently, let
\[ \Phi_E = \{ \phi(u, v) : (u, v) \in E \} \]
be the collection of all matrices, we write
\[ \langle \Phi_E, \Phi_E \rangle = [\langle \phi(u, v), \phi(\hat{u}, \hat{v}) \rangle]_{(u, v), (\hat{u}, \hat{v}) \in E} \]
to be the Grammian matrix of \( \Phi_E \), where the inner product of two matrices is the standard trace of the product of two matrices. It is easy to see that the collection of matrices \( \phi(u, v), (u, v) \in E \) are linearly independent and hence, the Grammian matrix is of full rank and so, the smallest eigenvalue \( \lambda_{\min}(\langle \Phi_E, \Phi_E \rangle) > 0 \). The following lemma shows that \( \lambda_{\min}(\langle \Phi_E, \Phi_E \rangle) \geq 2 \) which is useful in our argument, and we believe that it is of independent interest.

Lemma 2.8. Let \( \langle \Phi_E, \Phi_E \rangle \) be the Grammian matrix of \( \Phi_E \). Then the smallest eigenvalue of \( \langle \Phi_E, \Phi_E \rangle \) is at least 2.

Proof. We reshape the matrices \( \phi(u, v) \) in to vectors \( m(u, v) \), let
\[ M = [\cdots, m(u, v), \cdots]_{(u, v) \in E} \]
be an \( |V|^2 \times |E| \) matrix formed by all reshaped basis vectors. For convenience, we enumerate vertices by indices \( u = 1, \ldots, n \) and \( (u, v) \in E \) is the same as the \( u^{th} \) vertex and \( v^{th} \) vertex has an edge in \( E \). By the definition of \( \langle \Phi_E, \Phi_E \rangle \), we know that
\[ \langle \Phi_E, \Phi_E \rangle = M^T M. \]
Let \(M(i, j)\) denote the \((i, j)\)-th component of \(M\), then

\[
M(i, j) = \begin{cases} 
-1, & \text{if } i = (u - 1)n + v \text{ and } j = \sum_{i=1}^{v-1} (n - i) + u - v, \\
-1, & \text{if } i = (v - 1)n + u \text{ and } j = \sum_{i=1}^{u-1} (n - i) + u - v, \\
1, & \text{if } i = (u - 1)n + u \text{ and } j = \sum_{i=1}^{v-1} (n - i) + u - v, \\
1, & \text{if } i = (v - 1)n + v \text{ and } j = \sum_{i=1}^{u-1} (n - i) + u - v, \\
0, & \text{otherwise}.
\end{cases}
\]

So for any \(x = \{x(u, v), (u, v) \in E\}\), we have

\[
(Mx)_i = \begin{cases} 
-x(u, v), & \text{if } i = (u - 1)n + v \text{ or } i = (v - 1)n + u, \\
\sum_{u, v}(x(u, j) + x(j, v)), & \text{if } i = (j - 1)n + j,
\end{cases}
\]

where \((Mx)_i\) denotes the \(i\)-th component of \(Mx\). Therefore,

\[
x^\top(\Phi_E, \Phi_E)x = \|Mx\|_2^2 = 2 \sum_{(u, v) \in E} x^2_{(u, v)} + \sum_{j=1}^n \left( \sum_{u, v} (x(u, j) + x(j, v)) \right)^2 \geq 2\|x\|_2^2.
\]

Hence, the smallest eigenvalue \(\lambda_{\min}(\langle \Phi_E, \Phi_E \rangle) \geq 2\). \(\square\)

We are now ready to prove Theorem 2.2.

**Proof of Theorem 2.2.** Writing \(\alpha(u, v) = w(u, v) - C_{i-1}(u, v)\) for \((u, v) \in E_i\) and \(\alpha(u, v) = w(u, v)\) of \((u, v) \in E \setminus E_{i-1}\), we have \(R_{i-1} = \Phi_E\alpha\) and

\[
\|R_{i-1}\|_F^2 = \sum_{(u, v) \in E} \alpha(u, v)\langle R_{i-1}, \phi_{(u, v)} \rangle \leq \|\alpha\|_1 \cdot |\langle R_{i-1}, \phi_{(u, v)} \rangle| \\
\leq \sqrt{|E|}\|\alpha\|_2 \cdot |\langle R_{i-1}, \phi_{(u, v)} \rangle|,
\]

where the first inequality follows from the Cauchy-Schwarz inequality \(|\langle a, b \rangle| \leq \|a\|_1 \cdot \|b\|_\infty\) and Step 2 in Algorithm 1, and the second inequality follows from \(\|\alpha\|_1 \leq \sqrt{|E|}\|\alpha\|_2\) as \(\alpha \in \mathbb{R}^{|E|}\). On the other hand,

\[
\|R_{i-1}\|_F^2 = \langle L_G - L_{H_{i-1}}, L_G - L_{H_{i-1}} \rangle = \alpha^\top(\Phi_E, \Phi_E)\alpha \geq 2\|\alpha\|_2^2,
\]

where the last inequality follows from Lemma 2.8. Combining (9) and (10) and using Lemma 2.7, we obtain that

\[
\|R_i\|_F^2 \leq \|R_{i-1}\|_F^2 - \frac{2}{4|E|}\|R_{i-1}\|_F^2 = (1 - \frac{1}{2|E|})\|R_{i-1}\|_F^2.
\]
In view of this relation and the fact that $R_0 = L_G$, we conclude that
\[
\|R_i\|_F \leq \left( \sqrt{1 - \frac{1}{2|E|}} \right)^i \|L_G\|_F.
\]
This completes the proof. □

2.4. Sparsification for Random Graphs. Let us first recall that the random graphs which were introduced in Section 1.4 satisfy the condition that the first nonzero eigenvalue of $L_G$ is at least $\frac{1}{(2-\eta)}$. There are at least four random graph models which enable $L_{H_i}$ from Algorithm 1 to enjoy positive semidefinite and the desired property (1): the Erdős and Rényi model with $p = \omega(n)\log(n)/n$ and $\omega(n) \gg 1$, the Chung-Lu model [10] with the minimal degree $d_{\min} = \omega(n)\log(n)$ and $\omega(n) \gg 1$, the stochastic block model with $p_{ii} = \omega(n)\log(n)/n$, $\omega(n) \gg 1$ and $p_{ij} = \beta \log(n)/n$, and the Lai-Mckenzie model in [31].

We shall use Theorem 4 in [13], i.e., Theorem 2.9 below to have the nonzero eigenvalues of $L_G$ are close to 1 as $n \to \infty$ for a random graph $G$, which satisfy the condition mentioned above.

**Theorem 2.9** ([13], Theorem 4). For the random graph $G(d)$ with a fixed expected degree vector $d$, if the minimum expected degree $d_{\min}$ satisfies $d_{\min} \gg \log(n)$, then with probability at least $1 - 1/n = 1 - o(1)$, we have that for all eigenvalues $\lambda_k(L_G(d)) > \lambda_{\min}(L_G(d)) = 0$ of the Laplacian of $G(d)$,
\[
|\lambda_k(L_G(d)) - 1| = 2\sqrt{\frac{6\ln(2n)}{d_{\min}}} = o(1).
\]

We state our main result in this subsection as follows.

**Theorem 2.10.** Let $G$ be a random graph whose first nonzero eigenvalue of $L_G$ is bigger than $\frac{1}{(2-\eta)}$ for an $\eta \in (0, 1)$. Suppose that the ratio of the maximal degree $d_{\max}$ and the minimal degree $d_{\min}$ is bounded. Let $L_{H_{\left\lceil \frac{n}{\epsilon^2} \right\rceil}}$ be the sparsifier which is obtained by Algorithm 1 after $\left\lceil \frac{n}{\epsilon^2} \right\rceil$ steps. If $\omega(n) \gg 1$ and
\[
\epsilon \leq \min \left\{ \frac{1}{2\omega(n)\log(n)}, \eta \right\},
\]
then
\[
(1 - \epsilon)^2 L_G \preceq L_{H_{\left\lceil \frac{n}{\epsilon^2} \right\rceil}} \preceq (1 + \epsilon)^2 L_G.
\]
To prove Theorem 2.10, we need the two preparatory results. In what follows, for any matrix $Q$, we will use $N(Q) := \{x : Qx = 0\}$ to denote its null space.

**Theorem 2.11.** Let $B$ be an $n \times n$ positive semidefinite matrix and $A \in \mathbb{R}^{n \times n}$. Suppose that $\|A - B\|_2 \leq \epsilon$ for some $\epsilon \in (0, 1)$ and the nonzero eigenvalues of $B$ are greater than $\frac{1}{2 - \epsilon}$. Then $A$ is positive semidefinite and satisfies

$$
(12) \quad (1 - \epsilon)^2 B \preceq A \preceq (1 + \epsilon)^2 B
$$

provided that $\mathcal{N}(B) \subseteq \mathcal{N}(A)$.

**Proof.** The assumption on the nonzero eigenvalues of $B$ implies

$$
(13) \quad (2 - \epsilon)x^\top Bx \geq \|x\|_2^2, \quad \forall \ x \notin \mathcal{N}(B),
$$

i.e.,

$$
(2\epsilon - \epsilon^2)x^\top Bx - \epsilon x^\top x \geq 0, \quad \forall \ x \notin \mathcal{N}(B).
$$

Since $x^\top (A - B)x \geq -\epsilon x^\top x$ because $\|A - B\|_2 \leq \epsilon$, we have

$$
x^\top (A - B + (2\epsilon - \epsilon^2)B)x \geq 0, \quad \forall \ x \notin \mathcal{N}(B).
$$

In other words,

$$
(14) \quad x^\top Ax \geq (1 - \epsilon)^2 x^\top Bx, \quad \forall \ x \notin \mathcal{N}(B).
$$

Clearly, the equation (14) still holds for any $x \in \mathcal{N}(B)$ since $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. We have

$$
x^\top Ax \geq (1 - \epsilon)^2 x^\top Bx \geq 0, \quad \forall \ x \in \mathbb{R}^p.
$$

Hence, $A$ is positive semidefinite and the first $\preceq$ in (12) follows.

Next, let us prove the second $\preceq$ in (12). For any $x \notin \mathcal{N}(B)$, we have

$$
x^\top Ax = x^\top (A - B)x + x^\top Bx \overset{(a)}{\leq} \epsilon x^\top x + x^\top Bx \leq \epsilon(2 - \epsilon)x^\top Bx + x^\top Bx = (1 + \epsilon)^2 x^\top Bx,
$$

where (a) follows from $x^\top (A - B)x \leq \|A - B\|_2 \|x\|_2^2 \leq \epsilon x^\top x$ and (b) follows from (13). For any $x \in \mathcal{N}(B)$, we also have the above inequality since $\mathcal{N}(B) \subseteq \mathcal{N}(A)$. That is, we have (12). \qed

**Theorem 2.12.** Assume that $H$ is a subgraph of $G$. Then the null space of $L_H$ contains the null space of $L_G$. 


Proof. It is enough to show that \( L_H x = 0 \) if \( L_G x = 0 \). Suppose that \( L_G x = 0 \) for nonzero vector \( x \). Then \( x^\top L_G x = 0 \) which implies that
\[
x^\top (e_u - e_v)(e_u - e_v)^\top x = 0
\]
for all \((u, v) \in E\). It follows that \( (e_u - e_v)^\top x = 0 \). Hence, for \( E' \subset E \), we have
\[
L_H x = \sum_{(u,v) \in E'} c_{u,v} (e_u - e_v)(e_u - e_v)^\top x = 0.
\]
This completes the proof. \( \square \)

Finally, we are ready to prove Theorem 2.10. Note that Theorem 2.12 indicates that \( N(L_H(\frac{n}{2})) \subseteq N(L_G) \), and by Theorem 2.9, for those four models of random graphs discussed in Section 1.4, we know that all the nonzero eigenvalues of the Laplacian are close to 1. Thus combining Theorem 2.11, it suffices for us to prove that \( \|R_i\|_F \leq \|L_H(\frac{n}{2}) - L_G\|_2 \leq \epsilon \).

Proof of Theorem 2.10. Recall the adjacency matrix \( A \) of \( G = (V, E, w) \) is
\[
A(u,v) := \begin{cases} w(u,v), & \text{if } (u,v) \in E, \\ 0, & \text{otherwise}, \end{cases}
\]
and the weighted degree of a vertex \( u \) is \( d(u) := \sum_{v \in V} w(u,v) \). Let \( D \) be the diagonal matrix whose diagonal contains \( d(u) \). Then the Laplacian matrix of graph \( G \) is \( L_G = D - A \). The symmetric normalized Laplacian of graph \( G = (V, E, w) \) is defined as
\[
\mathcal{L}_G := D^{-1/2}L_G D^{-1/2} = I - D^{-1/2}AD^{-1/2}.
\]
Note that the eigenvalues of \( \mathcal{L}_G \) are between 0 and 2 by the Gershgoring circle theorem, hence
\[
\|\mathcal{L}_G\|_F^2 \leq 4n. \tag{15}
\]
Also, let us normalize the residual \( R_i \) in Algorithm 1 as
\[
\mathcal{R}_i := D^{-1/2}R_i D^{-1/2} = \mathcal{L}_G - \mathcal{L}_H(\frac{n}{2}).
\]
Then be Theorem 2.2, we have
\[
d_{\min}^2 \|\mathcal{R}_i\|_F^2 \leq \|D^{1/2}\mathcal{R}_i D^{1/2}\|_F^2 \leq \left(1 - \frac{1}{2|E|}\right)^i \|D^{1/2}\mathcal{L}_G D^{1/2}\|_F^2 \leq \left(1 - \frac{1}{2|E|}\right)^i d_{\max}^2 \|\mathcal{L}_G\|_F^2.
\]
Let $w_{\min} = \min_{(u,v) \in E} \{w(u,v)\}$, then we have $|E| \leq \sum_{u \in V} \frac{d(u)}{w_{\min}} \leq \frac{nd_{\max}}{w_{\min}}$. By the assumption on the degrees of $G$, we know that there exists a constant $K$ which independent of the size of vertices of $G$ such that $\frac{d_{\max}}{d_{\min}} \leq K$. Therefore,

$$\|R_i\|_F^2 \leq K \left(1 - \frac{w_{\min}}{2nd_{\max}}\right)^i \|L_G\|_F^2 \leq 4nK \left(1 - \frac{w_{\min}}{2nd_{\max}}\right)^i,$$

where the last inequality follows from (15). Choosing $i = \lceil \frac{n}{\epsilon^2} \rceil$ with $\epsilon \leq 1/(2\omega(n) \log(n))$, we have

$$\|R_{\lceil \frac{n}{\epsilon^2} \rceil}\|_F^2 \leq 4nK \left(1 - \frac{w_{\min}}{2nd_{\max}}\right)^{2nd_{\max} - \frac{1}{2d_{\max}\epsilon^2}} \leq 4K \exp\left(-\frac{w_{\min}}{2d_{\max}\epsilon^2}\right) \exp(\log n).$$

It follows that

$$\|R_{\lceil \frac{n}{\epsilon^2} \rceil}\|_F^2 \leq 4K \exp\left(\log(n) - \frac{w_{\min}}{2d_{\max}\epsilon^2}\right),$$

Since $d_{\max} = O(\omega(n) \log(n))$ and $\epsilon \leq 1/(2\omega(n) \log(n))$, then there exist a constant $c$ such that

$$\|R_{\lceil \frac{n}{\epsilon^2} \rceil}\|_F^2 \leq 4K \exp\left(\log(n) - c\omega(n) \log(n)\right) = \frac{4K}{n^{c\omega(1)-1}} \leq \epsilon^2,$$

where the last inequality follows from $\omega(n) \gg 1$. That is,

$$\|R_{\lceil \frac{n}{\epsilon^2} \rceil}\|_2 \leq \|R_{\lceil \frac{n}{\epsilon^2} \rceil}\|_F \leq \epsilon.$$

Then we use Theorems 2.11 and 2.12 by letting $B = L_G$ and $A = L_{H_{\lceil \frac{n}{\epsilon^2} \rceil}}$ to have

$$\langle 1 - \epsilon \rangle^2 L_G \preceq L_{H_{\lceil \frac{n}{\epsilon^2} \rceil}} \preceq \langle 1 + \epsilon \rangle^2 L_G.$$  

Multiplying $D^{1/2}$ from the left and right-hand sides of the inequalities (16), we have $L_{H_{\lceil \frac{n}{\epsilon^2} \rceil}} = D^{1/2} L_{H_{\lceil \frac{n}{\epsilon^2} \rceil}} D^{1/2}$ to conclude (11).  

3. The Nonnegative UGA Algorithm for Isotropic Subsets

In this section, we shall extend the universal greedy algorithm (UGA) to find a positive subset from given set $V = \{v_1, \cdots, v_m\}$ satisfying $\sum_{i=1}^m v_i v_i^\top = I_n$. In order to select nonnegative coefficients, we have to modify the UGA accordingly. The nonnegative UGA for isotropic subsets is showed in Algorithm 2.
Algorithm 2 The Nonnegative UGA for Subset Selection

**Input:** \( V = \{v_1, v_2, \ldots , v_m\} \subset \mathbb{R}^n \) with \( \sum_{i=1}^{m} v_i v_i^\top = I_n \), and \( \epsilon \in (0, 1) \).

1: \( R_0 := I_n, L_0 := 0 \) and \( i := 1 \).
2: Find the index \( j_i \) such that.

\[
(17) \quad v_{j_i}^\top R_{i-1} v_{j_i} = \max_{v \in V} v^\top R_{i-1} v.
\]

3: Compute the optimal weights.

\[
(\alpha_1^i, \alpha_2^i) = \arg \min_{(\alpha_1, \alpha_2) \in \mathbb{R}^2} \|I_n - \alpha_1 L_{i-1} - \alpha_2 v_{j_i} v_{j_i}^\top\|_F^2.
\]

4: Update.

\[
L_i = \alpha_1^i L_{i-1} + \alpha_2^i v_{j_i} v_{j_i}^\top
\]

and

\[
R_i = I_n - L_i.
\]

5: If \( i > \lceil \frac{n}{\epsilon^2} \rceil \), stop and go to output. Otherwise, set \( i = i + 1 \) and return to Step 2.

**Output:** The sparse approximation \( L_{\lceil \frac{n}{\epsilon^2} \rceil} \).

Mainly, we update Step 2 by using (17) instead of

\[
|v_{j_i}^\top R_{i-1} v_{j_i}| = \max_{v \in V} |v^\top R_{i-1} v|.
\]

This will ensure the non-negativity of the coefficients of subsets in all iterations. See Section 3.2 after the discussion of the computational complexity. Also, this algorithm can be easily modified to deal with linear sketching problem \([8, 35, 48]\) for least square regression. For given

\[
B = AA^\top = \sum_{i=1}^{m} a_i a_i^\top,
\]

where \( A = [a_1, \ldots , a_m] \) and for any \( \epsilon > 0 \), find a sparse solution \( d_i \geq 0, i = 1, \ldots , s \) with \( s = O(n/\epsilon^2) \) such that

\[
(1 - \epsilon)^2 B \leq \sum_{i=1}^{s} d_i a_{j_k} a_{j_k}^\top \leq (1 + \epsilon)^2 B.
\]

### 3.1. Computational Complexity

We shall first establish the following result for the running time of Algorithm 2.

**Theorem 3.1.** The computational complexity of Algorithm 2 is \( O(mn^2/\epsilon^2 + n^3/\epsilon^2) \).

**Proof.** The running time of Algorithm 2 is dominated by Steps 2 and 3. We first show that the total cost of Step 2 is \( O(mn^2/\epsilon^2) \). In order to find the index \( j_{i+1} \), by Step 2 for all
Lemma 3.4. Suppose that $\theta$ is a coefficient.

Proof. Let $L_i$ be the linear combination of $v$ such that

$$\langle L_i, L_i \rangle = (\alpha_1)^2 \langle L_{i-1}, L_{i-1} \rangle + \alpha_1 \alpha_2 \langle L_{i-1}, v_{j_i}^\top v_{j_i} \rangle + (\alpha_2^2) \|v_{j_i}\|_2^4$$

which needs $O(n^2)$ flops if we have $\langle L_{i-1}, L_{i-1} \rangle$. So the cost in Step 3 is $O(n^3/\epsilon^2)$ for total iteration number $[m/\epsilon]$. $\Box$

Remark 3.2. The computational time for Algorithm 2 is $O(mn^2/\epsilon^2) + n^3/\epsilon^2$) which is a good improvement to the twice Ramanujan sparifiers in [6] based on “barrier” potential function to guide the choice of indices as its computational time is $O(mn^3)$. We remark that the computational time can be reduced if massive parallel processors are used. For example, if a GPU with $m$ processes is used, the computational time will be $O(n^2)$. In addition, if $v_i$ are sparse vectors, then the computational time can also be reduced. For example, if $\|v_i\|_0 \leq O(\log n)$, the computational time will be $O(mn \log n)$.

3.2. Nonnegativity of $L_i$. We next explain that $L_i$ obtained at each step in Algorithm 2 has nonnegative coefficients. Let us begin with the following lemma.

Lemma 3.3. $\langle R_i, v_{j_i} \rangle = 0$ and $\langle R_i, L_{i-1} \rangle = 0$. Hence, $\langle R_i, L_i \rangle = 0$.

Proof. The first two equations are simply the properties of the minimizer $L_i$. Since $L_i$ is a linear combination of $v_{j_i}$ and $L_{i-1}$, thus we have the last equation. $\Box$

Lemma 3.4. Suppose that $R_i \neq 0$ for some $i \geq 1$. Then $v_{j_{i+1}} v_{j_{i+1}}^\top$ is linearly independent of $L_i$.

Proof. Suppose that $v_{j_{i+1}} v_{j_{i+1}}^\top$ is linearly dependent of $L_i$. Then there exists a nonzero coefficient $\theta$, such that $v_{j_{i+1}} v_{j_{i+1}}^\top = \beta L_i$ which implies

$$v_{j_{i+1}}^\top R_i v_{j_{i+1}} = \langle R_i, v_{j_{i+1}} v_{j_{i+1}}^\top \rangle = \theta \langle R_i, L_i \rangle = 0,$$
where the last equality follows from Lemma 3.3. We claim that this indicates that \( R_i = 0 \). Indeed, by (17), we have \( v_{j_{i+1}}^\top R_i v_{j_{i+1}} = 0 \) which implies \( v^\top R_i v = 0 \) for all \( v \in \{v_1, \ldots, v_n\} \) and hence, \( \langle R_i, I_n \rangle = 0 \) since \( \sum_{i \leq m} v_i v_i^\top = I \). Together with \( \langle R_i, L_i \rangle = 0 \), we have \( \langle R_i, I_n - L_i \rangle = \langle R_i, R_i \rangle = 0 \), hence \( R_i = 0 \). However, this contradicts the assumption that \( R_i \neq 0 \). This completes the proof. \( \square \)

Now we will establish the nonnegativity of the coefficients of \( L_i \). Indeed, it suffices for us to show that \( \alpha_k^1, \alpha_k^2 \geq 0 \) for any \( k = 1, \ldots, i \). Our discussion is based on induction. For \( i = 1 \), we have \( L_1 = \alpha_1^1 v_{j_1} v_{j_1}^\top \) and \( \alpha_1^2 = 0 \). To see \( \alpha_2^1 \geq 0 \), we expand the minimization in Step 3 of Algorithm 2 to have

\[
\| I_n - \alpha v_{j_1} v_{j_1}^\top \|_F^2 = n - 2 \alpha \langle v_{j_1} v_{j_1}^\top, I_n \rangle + \alpha^2 \langle v_{j_1} v_{j_1}^\top, v_{j_1} v_{j_1}^\top \rangle = n - 2 \alpha \| v_{j_1} \|_2^2 + \alpha^2 \| v_{j_1} \|_2^4.
\]

Hence \( \alpha_2^1 \) satisfies \( -2 \| v_{j_1} \|_2^2 + 2 \alpha \| v_{j_1} \|_2^4 = 0 \) or \( \alpha_2^1 = 1/\| v_{j_1} \|_2^2 > 0 \).

We now assume that \( L_{i-1} = \sum_{k=1}^{i-1} c_{i-1}(k) v_{j_k} v_{j_k}^\top \) with nonnegative coefficients \( c_{i-1}(k) \) for \( k = 1, \ldots, i-1 \). Now let us take a look at the coefficients of \( L_i \) from Step 4 of Algorithm 2. The coefficients \( \alpha_1^1, \alpha_1^2 \) satisfy the following system of linear equations:

\[
\alpha_1 \| L_{i-1} \|_F^2 + \alpha_2 \langle L_{i-1}, v_{j_i} v_{j_i}^\top \rangle = \sum_{k=1}^{i-1} c_{i-1}(k) \| v_{j_k} \|_2^2 \\
\alpha_1 \langle L_{i-1}, v_{j_i} v_{j_i}^\top \rangle + \alpha_2 \| v_{j_i} \|_2^4 = \| v_{j_i} \|_2^2.
\]

(18)

If \( v_{j_i} \) is linearly dependent of \( L_{i-1} \), by Lemma 3.4, we know \( R_i = 0 \) which means that we have already found the desired subset. Otherwise, it is clear that the coefficient matrix is nonsingular since the determinant is

\[
D = \| L_{i-1} \|_F^2 \| v_{j_i} \|_2^4 - \left( \langle L_{i-1}, v_{j_i} v_{j_i}^\top \rangle \right)^2 > 0,
\]

where we have used Cauchy-Schwarz inequality as \( v_{j_i} v_{j_i}^\top \) is linearly independent of \( L_{i-1} \) by Lemma 3.4. Using the Cramer’s rule, we see that

\[
\alpha_1^i = D^{-1} \left( \| v_{j_i} \|_2^2 \sum_{k=1}^{i-1} c_{i-1}(k) \| v_{j_k} \|_2^2 - \langle L_{i-1}, v_{j_i} v_{j_i}^\top \rangle \| v_{j_i} \|_2^2 \right) \\
\alpha_2^i = D^{-1} \left( -\langle L_{i-1}, v_{j_i} v_{j_i}^\top \rangle \sum_{k=1}^{i-1} c_{i-1}(k) \| v_{j_k} \|_2^2 + \| L_{i-1} \|_F^2 \| v_{j_i} \|_2^2 \right).
\]

(19)
It is easy to see that
\[
\alpha_1^i = D^{-1}\|v_j\|^2_2 \sum_{k=1}^{i-1} c_{i-1}(k)(\|v_j\|^2_2 - \langle v_{jk}, v_{ji}v_{ji}^\top \rangle) \geq 0.
\]

It remains to show that \(\alpha_2^i\) is nonnegative. Let us take a close look at the right-hand side of \(\alpha_2^i\) in (19). By Lemma 3.3, we always have
\[
\langle I_n - L_{i-1}, L_{i-1} \rangle = 0 \text{ or } \|L_{i-1}\|^2_F = \langle L_{i-1}, I_n \rangle.
\]
Thus, we have
\[
D\alpha_2^i = -\langle L_{i-1}, v_{j_i}v_{j_i}^\top \rangle \sum_{k=1}^{i-1} c_{i-1}(k)\|v_{jk}\|^2_2 + \|L_{i-1}\|^2_F\|v_{ji}\|^2_2
\]
\[
= -\langle L_{i-1}, v_{j_i}v_{j_i}^\top \rangle\|L_{i-1}\|^2_F + \|L_{i-1}\|^2_F\langle I_n, v_{j_i}v_{j_i}^\top \rangle = v_{j_i}^\top R_{i-1}v_{ji}\|L_{i-1}\|^2_F,
\]
where we have used the fact (20). So \(\alpha_2^i\) will be nonnegative if \(v_{j_i}^\top R_{i-1}v_{ji} \geq 0\).

We claim that this is true for all \(i \geq 1\). For \(i = 1\), we have \(x^\top R_0x \geq 0\) since \(R_0 = I_n\). For \(i = 2\), it is easy to see that \(c_1(1) = 1/\|v_{j_1}\|^2_2\). We have
\[
v_{j_2}^\top R_1v_{j_2} = v_{j_2}^\top v_{j_2} - \frac{(v_{j_2}^\top v_{j_2})^2}{\|v_{j_1}\|^2_2} \geq 0.
\]
We show that \(v_{j_{i+1}}^\top R_i v_{j_{i+1}} \geq 0\) for all \(i \geq 2\). In fact, it is easy to see that
\[
v_{j_i}^\top R_i v_{j_i} = \|v_{j_i}\|^2_2 - \alpha_1^i v_{j_i}^\top L_{i-1}v_{j_i} - \alpha_2^i \|v_{ji}\|^2_2 = 0
\]
by using (18). By the definition of \(v_{j_{i+1}}\), we have
\[
v_{j_{i+1}}^\top R_i v_{j_{i+1}} = \max_{v \in V} v^\top R_i v \geq v_{j_i}^\top R_i v_{j_i} = 0.
\]
By (20), we have \(\alpha_2^i \geq 0\). We have therefore concluded the following theorem.

**Theorem 3.5.** Let \(L_i\) be the symmetric matrix obtained from Algorithm 2. Then
\[
L_i = \sum_{\ell=1}^{i} c_\ell(\ell)v_{j_\ell}v_{j_\ell}^\top
\]
with \(c_\ell(\ell) \geq 0\) for any \(i \geq 1\).

In the next subsection, we will show that \(L_i\) is the desired approximation of the identity matrix \(I_n\) satisfying (3) for \(i \gg 1\). Particularly, similar to Section 2.4, by Theorem 2.11 we will show that the residual matrix \(R_i = I_n - L_i\) in Algorithm 2 satisfying \(\|R_i\|_2 \leq \epsilon\) for \(i \gg 1\).
3.3. Convergence of Algorithm 2. To establish the convergence of Algorithm 2, let us first introduce some notations and concepts. Define
\[
    \mathcal{F} = \left\{ S : S \subset \{1, \cdots, m\}, \sum_{i \in S} s_i \mathbf{v}_i \mathbf{v}_i^\top = \mathbf{I}_n, s_i > 0, i \in S \right\}
\]
to be the feasible set of all possible isotropic subsets in \( V \). It is clear that \( \mathcal{F} \) is not empty since \( S = \{1, \cdots, m\} \in \mathcal{F} \). We will use \( K_S = \sum_{i \in S} s_i \) to denote the size of subset \( S \in \mathcal{F} \). We are now ready to state the convergence of Algorithm 2.

**Theorem 3.6.** Suppose that there exists a positive subset \( \{s_i > 0, i \in S\} \) with \( S \subset \{1, \cdots, m\} \) such that \( \mathbf{I}_n = \sum_{i \in S} s_i \mathbf{v}_i \mathbf{v}_i^\top \) and let \( K_S = \sum_{i \in S} s_i \). Then each step in Algorithm 2 satisfies
\[
    \|R_{i+1}\|_F^2 \leq \frac{K_S^2}{K_S^2/n + \sum_{k=1}^2 1/\|\mathbf{v}_{j_k}\|_2^4}, \quad \forall i \geq 1.
\]

Note that our main goal is to show that the residual \( \|R_i\|_2 \) will be less than or equal to \( \epsilon \) when \( i \geq \lceil \frac{n}{\epsilon^2} \rceil \). In fact by Theorem 3.6, we are able to show that \( \|R_{\lceil \frac{n}{\epsilon^2} \rceil}\|_F \leq \epsilon \) under some sufficient conditions.

**Theorem 3.7.** Let \( T = |S| \) be the cardinality of isotropic subset \( S \). Under the assumption of Theorem 3.6, if we further have
\[
    K_S^2 = \left( \sum_{i \in S} s_i \right)^2 \leq \min_{\mathbf{v}_{j_1}, \cdots, \mathbf{v}_{j_T} \in \mathcal{V}} \sum_{k=1}^T \frac{1}{\|\mathbf{v}_{j_k}\|_2^2}
\]
and \( T \ll \lceil \frac{n}{\epsilon^2} \rceil \). Then \( \|R_{\lceil \frac{n}{\epsilon^2} \rceil}\|_F \leq \epsilon \).

**Proof.** From the right hand of (21) and the assumption in Theorem 3.7, we have
\[
    \|R_{\lceil \frac{n}{\epsilon^2} \rceil}\|_F^2 \leq \frac{K_S^2}{K_S^2/n + \sum_{k=1}^{\lceil \frac{n}{\epsilon^2} \rceil-1} 1/\|\mathbf{v}_{j_k}\|_2^4} \leq \frac{K_S^2}{\sum_{k=1}^{\lceil \frac{n}{\epsilon^2} \rceil-1} 1/\|\mathbf{v}_{j_k}\|_2^4} \leq \frac{\sum_{k=1}^T 1/\|\mathbf{v}_{j_k}\|_2^4}{\sum_{k=1}^{\lceil \frac{n}{\epsilon^2} \rceil-1} 1/\|\mathbf{v}_{j_k}\|_2^4}.
\]

Note that \( \|\mathbf{v}_{j_k}\|_2^4 \leq 1 \). Indeed, since \( \mathbf{I}_n = \sum_{\mathbf{v} \in \mathcal{V}} \mathbf{v} \mathbf{v}^\top \), we have
\[
    \|\mathbf{v}_{j_k}\|_2^2 = \sum_{\mathbf{v} \in \mathcal{V}} (\mathbf{v}_{j_k}^\top \mathbf{v})^2 \geq \|\mathbf{v}_{j_k}\|_2^4.
\]
That is, $\|v_j\|_2^2 \leq 1$ and hence, $\|v_j\|_4^4 \leq 1$. Hence

(22) $\|R_{i+\frac{1}{2e^2}}\|_F^2 \leq \frac{\delta_0^2T}{n/e^2} - 1 \leq \epsilon^2$

provided that $T \ll \left\lceil \frac{n}{\epsilon^2} \right\rceil$. Here $\delta_0 = \min_{v \in V} \|v\|_2^2$. Thus we have $\|R_{i+\frac{1}{2e^2}}\|_F \leq \epsilon$. □

The remainder of this subsection is to prove Theorem 3.6. We begin with the following lemma.

**Lemma 3.8.** $\|R_i\|_F^2 \leq \|R_{i-1}\|_F^2 - \frac{(v_j^TR_{i-1}v_j)^2}{\|v_j\|_2^2}$ for all $i \geq 1$.

**Proof.** For all $i \geq 1$, we have

$$\|R_i\|_F^2 = \min_{\alpha_1, \alpha_2} \|R_{i-1} - (\alpha_1 - 1)L_{i-1} - \alpha_2 v_j v_j^T\|_F^2$$

$$\leq \min_{\alpha_2} \|R_{i-1} - \alpha_2 v_j v_j^T\|_F^2$$

$$= \|R_{i-1}\|_F^2 - \frac{(v_j^TR_{i-1}v_j)^2}{\|v_j\|_2^2},$$

which completes the proof. □

**Remark 3.9.** It follows from Lemma 3.8 that $\|R_i\|_F$ is monotonically decreasing and

$$\|R_i\|_F^2 + \sum_{k=1}^{i-1} \frac{(v_j^TR_{k-1}v_j)^2}{\|v_j\|_2^4} \leq \|R_0\|_F^2 = n.$$

This can be used as a stopping criterion. Indeed, we can check if the second term on the left is close to $n$ or not.

We also need the following elementary lemma.

**Lemma 3.10 (DeVore and Temlyakov, 1996 [16]).** Suppose we have two sequences of non-negative numbers $\{a_k, k \geq 1\}$ and $\{\beta_k, k \geq 1\}$ satisfying $\beta_k > 0$ and $a_1 = 1$ and

(23) $a_{k+1} \leq a_k(1 - a_k \beta_k), \quad \forall k \geq 1.$

Then

$$a_{i+1} \leq \frac{1}{1 + \sum_{k=1}^{i} \beta_k}, \quad \forall i \geq 1.$$

Now we are ready to prove Theorem 3.6.

**Proof of Theorem 3.6.** For convenience, we write $K = K_S$. By Lemma 3.3, we have

$$\|R_i\|_F^2 = \langle R_i, R_i \rangle = \langle R_i, I_n \rangle = \sum_{S_k \in S} s_k v_k^T R_i v_k.$$
As \( s_k > 0 \), by Step 2 of Algorithm 2, we have
\[
\|R_i\|^2_F \leq \sum_{s_k \in S} s_k v_{j_{i+1}}^T R_i v_{j_{i+1}} = K v_{j_{i+1}}^T R_i v_{j_{i+1}}.
\]
It follows from Lemma 3.8 that
\[
\|R_i\|^2_F \leq \|R_{i-1}\|^2_F - \frac{\|R_{i-1}\|^4}{\|v_{j_{i-1}}\|^2 K^2} = \|R_{i-1}\|^2_F \left(1 - \frac{1}{\|v_{j_{i-1}}\|^2 K^2}\right).\]
By using Lemma 3.10 and choosing \( a_k = \|R_k\|^2_F/n \) and \( \beta_k = n/(\|v_{j_k}\|^2 K^2) \), we have
\[
\|R_{i+1}\|^2_F \leq \frac{n}{1 + \sum_{k=1}^i n/(\|v_{j_k}\|^2 K^2)} = \frac{K^2}{K^2/n + \sum_{k=1}^i 1/\|v_{j_k}\|^2}
\]
for all \( i \geq 1 \). \( \square \)

### 3.4. A Strengthened Convergence Result.

Form (22) in the proof of Theorem 3.7, one can easily observe that \( \|R_i\|^2_2 = O(1/i) \) provided that \( K_S \) is bounded. This subsection aims to strengthen the convergence result. To do that, we have to improve the estimate in Lemma 3.10. This indeed can be done when \( \beta_k, k \geq 1 \) are a fixed constant, say \( \beta_k = \beta_0 > 0 \) for all \( k \geq 1 \).

**Lemma 3.11.** Suppose we have two sequences of nonnegative numbers \( \{a_k, k \geq 1\} \) and \( \{\beta_k, k \geq 1\} \) satisfying \( \beta_k \geq \beta_0 > 0 \) for all \( k \geq 1 \) and (23). Then
\[
a_{i+1}^2 = o\left(\frac{1}{i + 1}\right), \quad \forall i \geq 1.
\]

**Proof.** We rewrite the given inequality (23) as
\[
\beta_0 a_k^2 + a_{k+1} \leq a_k,
\]
for all \( i \) and hence \( \sum_{k=1}^i \beta_0 a_k^2 \leq a_1 \)
for all \( i \) and hence \( \sum_{k=1}^i a_k^2 < \infty \). Since \( a_{k+1} \leq a_k \) for all \( k \), we have
\[
\ell a_{2\ell+1}^2 \leq \ell a_{2\ell}^2 \leq \sum_{k=\ell}^{2\ell} a_k^2 \to 0
\]
and hence, \( a_{2\ell}^2 = o(1/\ell) = o(1/(2\ell)) \) and \( a_{2\ell+1}^2 = o(1/\ell) = o(1/(2\ell + 1)) \). \( \square \)

We are finally ready to establish the main result in this section.
Theorem 3.12. Suppose that the given vector set $V$ has a positive isotropic subset $S$. Then each step in Algorithm 2 satisfies

$$\|R_{i+1}\|_F^4 = o\left(\frac{1}{i+1}\right).$$

Hence, $\|R_{i+1}\|_F \leq \epsilon$ when $i \geq \lceil \frac{n}{\epsilon^2} \rceil$ with $\epsilon \geq 1/\sqrt{n}$.

Proof. By a slightly different approach from the proof of Theorem 3.6, we choose $a_k = \|R_k\|_F^2$ and $\beta_k = 1/(\|v_{jk}\|^4 K_S^2) \geq 1/(\delta^4 K_S^2)$. We use Lemma 3.11 to have

$$a_i^2 = o\left(\frac{1}{i+1}\right) \text{ or } \|R_{i+1}\|_F^4 = o\left(\frac{1}{i+1}\right).$$

So the desired result $\|R_{i+1}\|_F \leq \epsilon$ follows. \(\square\)

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