PINCHED HYPERSURFACES SHRINK TO ROUND POINTS

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Abstract. We investigate the evolution of closed strictly convex hypersurfaces in \( \mathbb{R}^{n+1} \), \( n = 3 \), for contracting normal velocities, including powers of the mean curvature, \( H \), of the norm of the second fundamental form, \( |A| \), and of the Gauss curvature, \( K \). We prove convergence to a round point for 2-pinched initial hypersurfaces. In \( \mathbb{R}^{n+1}, n = 2 \), natural quantities exist for proving convergence to a round point for many normal velocities. Here we present their counterparts for arbitrary dimensions \( n \in \mathbb{N} \).

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1. Overview

We consider the geometric flow equation

\[
\begin{cases}
\frac{d}{dt} X = -F \nu, \\
X(\cdot, 0) = M_0
\end{cases}
\]

and ask whether closed strictly convex hypersurfaces \( M_{0 \leq t < T} \) in \( \mathbb{R}^{n+1}, n = 3 \), shrink to round points.
For the cubed mean curvature, \( F = H^3 \), the answer is affirmative if the initial hypersurface \( M_0 \) is 2-pinched, i.e. the principal curvatures \((\lambda_i)_{1 \leq i \leq 3}\) fulfill

\[
\frac{\lambda_i}{\lambda_j} \leq 2
\]
everywhere on \( M_0 \) for all \( 1 \leq i, j \leq 3 \). This is our main Theorem 6.5.

Furthermore, we sketch the proof of similar results for the square of the norm of the second fundamental form, \( F = |A|^2 \), and the Gauss curvature, \( F = K \).

So far, strong pinching assumptions were needed to show convergence to a round point [3, 4, 10].

The paper is structured as follows:

- **Notation:** We give a quick introduction to differential geometric quantities used in this paper, e.g. the induced metric, the second fundamental form, and the principal curvatures.

- **Linear operator \( L \):** We calculate the linear operator \( Lw := \frac{4}{n}w - F_{ij}w_{ij} \) for a function \( w \) of the principal curvatures \( \lambda_i, i = 1, 2, 3 \), at a critical point of \( w \). To improve readability, we also choose normal coordinates at that critical point, i.e. \( g_{ij} = \delta_{ij} \) and \( (h_{ij}) = \text{diag} (\lambda_1, \lambda_2, \lambda_3) \). This lays the groundwork for subsequent calculations.

- **Vanishing functions:** In \( \mathbb{R}^{n+1}, n = 2 \), for many normal velocities \( F \) the quantity
  
  \[
  \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 \lambda_2)^2} F^2
  \]
  seems to be the natural choice when showing convergence to a round point. As in [5], we call this quantity a *vanishing function* for a normal velocity \( F \). It is used by B. Andrews for the Gauss curvature flow [1], by F. Schulze and O. Schnürrer for the \( H^\sigma \)-flow [10], by B. Andrews and X. Chen for the \( |A|^\sigma \) and the tr \( A^\sigma \)-flow [2].

  The quantity
  
  \[
  \sum_{i<j} \frac{(\lambda_i - \lambda_j)^2}{(\lambda_i \lambda_j)^2} F^2
  \]
  is the counterpart of a vanishing function for arbitrary dimensions \( n \in \mathbb{N} \). In particular, we work with this quantity in \( \mathbb{R}^{n+1}, n = 3 \).

- **\( H^3 \)-flow:** The proof of our main Theorem 6.5 is based on investigating the quantities
  
  \[
  \varphi_{H^3} = \frac{(a - b)^2 + (a - c)^2 + (b - c)^2}{(a + b + c)^2},
  \]
  and \( \psi_{H^3} = \left( \frac{(a - b)^2}{(a b)^2} + \frac{(a - c)^2}{(a c)^2} + \frac{(b - c)^2}{(b c)^2} \right) \left( H^3 \right)^2 \),
  
  which are homogeneous functions of the principal curvatures \( a \equiv \lambda_1, b \equiv \lambda_2, \) and \( c \equiv \lambda_3 \). First we show that the estimate \( \varphi_{H^3} \leq h := 1/8 \) is preserved during the \( H^3 \)-flow if the initial hypersurface \( M_0 \) is 2-pinched. Next we prove that \( \psi_{H^3} \) is bounded in time on the set where \( \varphi_{H^3} \leq h \). This involves the maximum-principle, the linear operator \( L \) and our computer program [CP]. Finally, we show convergence to a round point combining the boundedness of \( \psi_{H^3} \) and the proof of [10] Theorem A.1. by F. Schulze and O. Schnürrer.
• $|A|^2$-flow and Gauss curvature flow: We sketch the proof of results similar to our main Theorem 6.5 for the $|A|^2$-flow, and for the Gauss curvature flow.

• Appendix: Some of the Lemmas leading up to the proof our main Theorem 6.5 rely on the computer program [CP], where we use a Monte-Carlo method. For the convenience of the reader, we include the source code of [CP] in three different programming languages, namely the computer algebra systems Mathematica, Sage, and Maple.

2. Acknowledgments

We would like to thank O. Schnüer for suggesting the use of two monotone quantities instead of one. In particular, we thank O. Schnüer for proposing the quantity $\psi_{|A|^2}$, and M. Makowski for proposing the quantity $\varphi_{|A|^2}$. We are also indebted to M. Westerholt-Raum and F. Kuhl for their help in translating the computer program [CP] from Mathematica to Sage and to Maple.

3. Notation

For a quick introduction of the standard notation we adopt the corresponding chapter from [9].

We use $X = X(x, t)$ to denote the embedding vector of an $n$-manifold $M_t$ into $\mathbb{R}^{n+1}$ and $\frac{d}{dt} X = \dot{X}$ for its total time derivative. It is convenient to identify $M_t$ and its embedding in $\mathbb{R}^{n+1}$. The normal velocity $F$ is a homogeneous symmetric function of the principal curvatures. We choose $\nu$ to be the outer unit normal vector to $M_t$. The embedding induces a metric $g_{ij} := \langle X_i, X_j \rangle$ and the second fundamental form $h_{ij} := -\langle X_i, \nu \rangle$ for all $i, j = 1, \ldots, n$. We write indices preceded by commas to indicate differentiation with respect to space components, e. g. $X_{ik} = \frac{\partial X}{\partial x_k}$ for all $k = 1, \ldots, n$.

We use the Einstein summation notation. When an index variable appears twice in a single term it implies summation of that term over all the values of the index.

Indices are raised and lowered with respect to the metric or its inverse $(g^{ij})$, e. g. $h_{ij}h^{ij} = h_{ij}g^{ik}h_{kl}g^{lj} = h^2_{ij}h^j_k$.

The principal curvatures $\lambda_i$, $i = 1, \ldots, n$, are the eigenvalues of the second fundamental form $(h_{ij})$ with respect to the induced metric $(g_{ij})$. For $n = 3$, we name the principle curvatures also $a \equiv \lambda_1$, $b \equiv \lambda_2$, and $c \equiv \lambda_3$. A surface is called strictly convex if all principal curvatures are strictly positive. We will assume this throughout the paper. Therefore, we may define the inverse of the second fundamental form denoted by $(h^{ij})$.

Symmetric functions of the principal curvatures are well-defined, we will use the Gauss curvature $K = \frac{\det h_{ij}}{\det g_{ij}} = \prod_{i=1}^n \lambda_i$, the mean curvature $H = \sum_{i=1}^n \lambda_i$, the square of the norm of the second fundamental form $|A|^2 = h^{ij}h_{ij} = \sum_{i=1}^n \lambda_i^2$, and the trace of powers of the second fundamental form $\text{tr} A^n = \text{tr} (h^{ij})^n = \sum_{i=1}^n \lambda_i^n$. We write indices preceded by semi-colons to indicate covariant differentiation with respect to the induced metric, e. g. $h_{ij; k} = h_{ij,k} - \Gamma_{ik}^l h_{lj} - \Gamma_{jk}^l h_{li}$, where $\Gamma_{ij}^k = \frac{1}{2} g^{kl} (g_{il,j} + g_{jl,i} - g_{lj,i})$. It is often convenient to choose normal co-ordinates, i. e. coordinate systems such that at a point the metric tensor equals the Kronecker delta, $g_{ij} = \delta_{ij}$, in which $(h_{ij})$ is diagonal, $(h_{ij}) = \text{diag}(\lambda_1, \ldots, \lambda_n)$.
Whenever we use this notation, we will also assume that we have fixed such a coordinate system. We will only use a Euclidean metric for $\mathbb{R}^{n+1}$ so that the indices of $h_{ij,k}$ commute according to the Codazzi-Mainardi equations.

A normal velocity $F$ can be considered as a function of principal curvatures $\lambda_i$, $i = 1, \ldots, n$, or $(h_{ij}, g_{ij})$. We set $F^{ij} = \frac{\partial F}{\partial h_{ij}}$, $F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}}$. Note that in coordinate systems with diagonal $h_{ij}$ and $g_{ij} = \delta_{ij}$ as mentioned above, $F^{ij}$ is diagonal.

4. LINEAR OPERATOR $L$

We begin this chapter with Definition 4.1 of the linear operator $Lw$ for a function $w$ of the principal curvatures $\lambda_i$, $i = 1, \ldots, n$. Then we calculate the linear operator $Lw$ at a critical point of $w$ in $\mathbb{R}^{n+1}$, $n = 3$. To improve readability, we also choose normal coordinates at that critical point, i.e. $g_{ij} = \delta_{ij}$, and $(h_{ij}) = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \equiv \text{diag}(a, b, c)$. This is Lemma 4.3. In Corollary 4.4 we will see that the linear operator $Lw$ has the form

$$Lw = C_w + E_w x_0^s + x_1^T M^{Rw} x_1 + x_2^T M^{Sw} x_2 + x_3^T M^{Tw} x_3,$$

where $C_w(a, b, c)$, $E_w(a, b, c)$ are functions in $\mathbb{R}$, and $M^{Rw}(a, b, c)$, $M^{Sw}(a, b, c)$, $M^{Tw}(a, b, c)$ are functions in $\mathbb{R}^{2 \times 2}$ with

$$x_0 = h_{12;3}, \quad x_1 = \begin{pmatrix} h_{22;1} \\ h_{33;1} \end{pmatrix}, \quad x_2 = \begin{pmatrix} h_{11;2} \\ h_{33;2} \end{pmatrix}, \quad x_3 = \begin{pmatrix} h_{11;3} \\ h_{22;3} \end{pmatrix}.$$

In subsequent calculations we need the linear operator $Lw$ to be non-positive for some set $\mathcal{S} \subset \mathbb{R}^3$. We achieve this by checking the non-positivity of each of the functions $C_w$, $E_w$, $M^{Rw}$, $M^{Sw}$, and $M^{Tw}$ on $\mathcal{S} \subset \mathbb{R}_+^3$. In Remark 4.5 we state the criterion we use in our computer program [cp] to determine the negative semi-definiteness of $M^{Rw}$, $M^{Sw}$, and $M^{Tw}$.

**Definition 4.1** (Linear operator). Let $w$ be a function of the principal curvatures. Then we define the linear operator $L$ by

$$Lw = \frac{d}{dt} w - F^{ij} w_{;ij},$$

which is corresponding to the geometric flow equation (1.1).

**Lemma 4.2** (Linear operator). Let $w = w(h^i_j)$ be a function of the principal curvatures. Let $L$ be defined as in (4.1). Then we have

$$Lw = w^{ij} \left( h_{ij} F^{kl} h_{kl}^m h_{lm} + h_{ij}^m h_{jm} (F - F^{kl} h_{kl}) \right) + (w^{ij} F^{kl,rs} - F^{ij} w^{kl,rs}) h_{kl;i} h_{rs;j}.$$

**Proof.** We refer to [5] Lemma 4.5. □

**Lemma 4.3** (Second derivatives). Let $f$ be a normal velocity $F$ or a function $w$ of the principal curvatures. Then we have

$$f^{ij,kl} \eta_{ij} \eta_{kl} = \sum_{i,j} \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \eta_{ij}^2 + \sum_{i \neq j} \frac{\partial f}{\partial \lambda_i} \eta_{ij} \eta_{ij} \left( \frac{\partial \lambda_i}{\lambda_i - \lambda_j} - \frac{\partial \lambda_j}{\lambda_i - \lambda_j} \right),$$

for any symmetric matrix $(\eta_{ij})$ and $\lambda_i \neq \lambda_j$, or $\lambda_i = \lambda_j$ and the last term is interpreted as a limit.
Proof. We refer to C. Gerhardt \[6, Lemma 2.1.14\].

\[\]

**Lemma 4.4** (Linear operator at a critical point). Let \( w = w(h^i_j) \) be a symmetric function of the principal curvatures \( a, b, \) and \( c \). At a critical point of \( w \), i.e. \( w_{;i} = 0 \) for all \( i = 1, 2, 3 \), we choose normal coordinates, i.e. \( g_{ij} = \delta_{ij} \) and \( (h_{ij}) = \text{diag}(a, b, c) \). Then we have

\[
Lw = C_w(a, b, c) + E_w(a, b, c) h^2_{12,3} + R_w(a, b, c, h_{11;1}, h_{22;1}, h_{33;1}) + S_w(a, b, c, h_{11,2}, h_{22,2}, h_{33,2}) + T_w(a, b, c, h_{11;3}, h_{22;3}, h_{33;3})
\]

(4.4)

The constant terms \( C_w \) are

\[
C_w(a, b, c) = a w_a (a^2 F_a + b^2 F_b + c^2 F_c + a (F - a F_a - b F_b - c F_c)) + b w_a (a^2 F_a + b^2 F_b + c^2 F_c + b (F - a F_a - b F_b - c F_c)) + c w_c (a^2 F_a + b^2 F_b + c^2 F_c + c (F - a F_a - b F_b - c F_c)).
\]

The gradient terms \( E_w \) are

\[
E_w(a, b, c)/2 = \left( (w_c (F_a - F_b) - F_c (w_a - w_b)) / (a - b) + (w_b (F_a - F_c) - F_b (w_a - w_c)) / (a - c) + (w_a (F_b - F_c) - F_a (w_b - w_c)) / (b - c) \right).
\]

The gradient terms \( R_w \) are

\[
R_w(a, b, c, h_{11;1}, h_{22;1}, h_{33;1}) = w_a \left( (F_{aa} h_{11;1}^2 + F_{bb} h_{22;1}^2 + F_{cc} h_{33;1}^2) + 2 (F_{ab} h_{11;1} h_{22;1} + F_{ac} h_{11;1} h_{33;1} + F_{bc} h_{22;1} h_{33;1}) \right)
\]

\[
+ w_b \left( 2 \frac{F_a - F_b}{a - b} h_{22;1} \right)
\]

\[
+ w_c \left( 2 \frac{F_a - F_c}{a - c} h_{33;1} \right)
\]

\[
- F_a \left( w_{aa} h_{11;1}^2 + w_{bb} h_{22;1}^2 + w_{cc} h_{33;1}^2 + 2 (w_{ab} h_{11;1} h_{22;1} + w_{ac} h_{11;1} h_{33;1} + w_{bc} h_{22;1} h_{33;1}) \right)
\]

\[
- F_b \left( 2 \frac{w_a - w_b}{a - b} h_{22;1} \right)
\]

\[
- F_c \left( 2 \frac{w_a - w_c}{a - c} h_{33;1} \right).
\]
The gradient terms $S_w$ are

\[
S_w(a, b, c, h_{11;2}, h_{22;2}, h_{33;2}) = w_a \left( 2 \frac{F_a - F_b}{a - b} h_{11;2}^2 \right) + w_b \left( F_{aa} h_{11;2}^2 + F_{bb} h_{22;2}^2 + F_{cc} h_{33;2}^2 + 2 (F_{ab} h_{11;2} h_{22;2} + F_{ac} h_{11;2} h_{33;2} + F_{bc} h_{22;2} h_{33;2})) \right) + w_c \left( 2 \frac{b - c}{a - c} h_{11;2}^2 \right) - F_a \left( 2 \frac{w_a - w_b}{a - b} h_{11;2}^2 \right) - F_b \left( 2 \frac{b - c}{a - c} h_{11;2}^2 \right) - F_c \left( 2 \frac{w_b - w_c}{b - c} h_{11;2}^2 \right).
\]

The gradient terms $T_w$ are

\[
T_w(a, b, c, h_{11;3}, h_{22;3}, h_{33;3}) = w_a \left( 2 \frac{F_a - F_c}{a - c} h_{11;3}^2 \right) + w_b \left( 2 \frac{F_b - F_c}{b - c} h_{22;3}^2 \right) + w_c \left( F_{aa} h_{11;3}^2 + F_{bb} h_{22;3}^2 + F_{cc} h_{33;3}^2 + 2 (F_{ab} h_{11;3} h_{22;3} + F_{ac} h_{11;3} h_{33;3} + F_{bc} h_{22;3} h_{33;3})) \right) - F_a \left( 2 \frac{w_a - w_c}{a - c} h_{11;3}^2 \right) - F_b \left( 2 \frac{w_b - w_c}{b - c} h_{22;3}^2 \right) - F_c \left( 2 \frac{w_b - w_c}{b - c} h_{22;3}^2 \right).
\]

Furthermore, we have at a critical point of $w$

\[
h_{11;1} = - \frac{1}{w_a} (w_b h_{22;1} + w_c h_{33;1}),
\]

\[(4.5)\]

\[
h_{22;2} = - \frac{1}{w_b} (w_a h_{11;2} + w_c h_{33;2}),
\]

\[
h_{33;3} = - \frac{1}{w_c} (w_a h_{11;3} + w_b h_{22;3}).
\]

Proof. We use Lemma 4.2 and Lemma 4.3 at a point, where we choose normal coordinates. This way we obtain the constant terms $C_w$, and the four gradient terms $E_w$, $R_w$, $S_w$, and $T_w$. 
At a critical point of $w$, we have $w_i(a, b, c) = 0$ for $i = 1, 2, 3$. This implies

$$w_a h_{1i;1} g^{i1} + w_b h_{2i;2} g^{i2} + w_c h_{3i;3} g^{i3} = 0.$$ 

Using normal coordinates we obtain

$$w_a h_{11;1} + w_b h_{22;2} + w_c h_{33;3} = 0.$$ 

Now we obtain for $i = 1, 2, 3$ the identities

$$h_{11;1} = -\frac{1}{w_a} (w_b h_{22;1} + w_c h_{33;1}),$$

$$h_{22;2} = -\frac{1}{w_b} (w_a h_{11;2} + w_c h_{33;2}),$$

$$h_{33;3} = -\frac{1}{w_c} (w_a h_{11;3} + w_b h_{22;3}).$$

This concludes the proof. □

**Corollary 4.5** (Linear operator at a critical point). Let the gradient terms $R_w$, $S_w$, and $T_w$ be defined as in Lemma 4.4. Then we have

$$R_w(a, b, c, h_{22;1}, h_{33;1}) = \left(\begin{array}{c} h_{22;1} \\ h_{33;1} \end{array}\right)^\top M^{R_w}(a, b, c) \left(\begin{array}{c} h_{22;1} \\ h_{33;1} \end{array}\right),$$

$$S_w(a, b, c, h_{11;2}, h_{33;2}) = \left(\begin{array}{c} h_{11;2} \\ h_{33;2} \end{array}\right)^\top M^{S_w}(a, b, c) \left(\begin{array}{c} h_{11;2} \\ h_{33;2} \end{array}\right),$$

$$T_w(a, b, c, h_{11;3}, h_{22;3}) = \left(\begin{array}{c} h_{11;3} \\ h_{22;3} \end{array}\right)^\top M^{T_w}(a, b, c) \left(\begin{array}{c} h_{11;3} \\ h_{22;3} \end{array}\right).$$

The elements of the matrix $M^{R_w}(a, b, c)$ are

$$m_{11}^{R_w}(a, b, c) = 2 \frac{F_a w_b - F_b w_a}{a - b}$$

$$+ F_{aa} \frac{w_b^2}{w_a} - 2 F_{ab} \frac{w_b}{w_a} + F_{bb} w_a$$

$$- F_a \left( w_{aa} \frac{w_b^2}{w_a} - 2 w_{ab} \frac{w_b}{w_a} + w_{bb} \right),$$

$$m_{12}^{R_w}(a, b, c) = F_{aa} \frac{w_b w_c}{w_a} - F_{ab} w_c - F_{ac} w_b + F_{bc} w_a$$

$$- F_a \left( w_{aa} \frac{w_b w_c}{w_a^2} - w_{ab} \frac{w_c}{w_a^2} - w_{ac} \frac{w_b}{w_a^2} + w_{bc} \right),$$

$$m_{22}^{R_w}(a, b, c) = 2 \frac{F_a w_c - F_c w_a}{a - c}$$

$$+ F_{aa} \frac{w_c^2}{w_a} - 2 F_{ac} w_c + F_{cc} w_a$$

$$- F_a \left( w_{aa} \frac{w_c^2}{w_a^2} - 2 w_{ac} \frac{w_c}{w_a^2} + w_{cc} \right).$$
The elements of the matrix $M^{S_w}(a, b, c)$ are

$$m_{11}^{S_w} (a, b, c) = 2 \frac{F_a w_b - F_b w_a}{a - b}$$

$$+ F_{aa} w_b - 2 F_{ab} w_a + F_{bb} \frac{w_a^2}{w_b}$$

$$- F_b \left( w_{aa} - 2 w_{ab} \frac{w_a}{w_b} + w_{bb} \frac{w_a^2}{w_b^2} \right),$$

$$m_{12}^{S_w} (a, b, c) = - F_{ab} w_c + F_{ac} w_b + F_{bc} \frac{w_a w_c}{w_b} - F_{ac} w_a$$

$$- F_b \left( -w_{ac} + w_{bb} \frac{w_a}{w_b} - w_{bc} \frac{w_a}{w_b} \right),$$

$$m_{22}^{S_w} (a, b, c) = \frac{2 F_b w_c - F_c w_b}{b - c}$$

$$+ F_{bb} \frac{w_a^2}{w_b} - 2 F_{bc} w_c + F_{cc} w_b$$

$$- F_b \left( w_{bb} \frac{w_a^2}{w_b} - 2 w_{bc} w_c + w_{cc} \right).$$

The elements of the matrix $M^{T_w}(a, b, c)$ are

$$m_{11}^{T_w} (a, b, c) = 2 \frac{F_a w_c - F_c w_a}{a - c}$$

$$+ F_{aa} w_c - 2 F_{ac} w_a + F_{cc} \frac{w_a^2}{w_c}$$

$$- F_c \left( w_{aa} - 2 w_{ac} \frac{w_a}{w_c} + w_{cc} \frac{w_a^2}{w_c^2} \right),$$

$$m_{12}^{T_w} (a, b, c) = F_{ab} w_c - F_{ac} w_b - F_{bc} w_a + F_{cc} \frac{w_a w_b}{w_c}$$

$$- F_c \left( w_{ab} - w_{ac} \frac{w_b}{w_c} - w_{bc} \frac{w_a}{w_c} + w_{cc} \frac{w_a w_b}{w_c^2} \right),$$

$$m_{22}^{T_w} (a, b, c) = \frac{2 F_b w_c - F_c w_b}{b - c}$$

$$+ F_{bb} w_c - 2 F_{bc} w_b + F_{cc} \frac{w_a^2}{w_c}$$

$$- F_c \left( w_{bb} - 2 w_{bc} \frac{w_b}{w_c} + w_{cc} \frac{w_a^2}{w_c^2} \right).$$

Proof. We use identities (4.5) to replace $h_{111}, h_{2222}$, and $h_{3333}$ in $R_w, S_w$, and $T_w$ from Lemma 4.4 respectively. Now we rewrite the quadratic forms $R_w, S_w$, and $T_w$ as $x^{\top} M x$. This concludes the proof. \hfill \Box

Remark 4.6 (Sufficient conditions for the non-positivity of the linear operator). Under the assumptions of Lemma 4.4, the linear operator $L w$ is non-positive at some critical point of $w$, if $C_w, E_w$ are non-positive, and $M^{R_w}, M^{S_w}$, and $M^{T_w}$ are negative semi-definite there. This is a direct consequence of Lemma 4.4 and Corollary 4.5.
Now let $M \in \mathbb{R}^{2 \times 2}$ be a symmetric matrix. Then we have the equivalent conditions

1. $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}$ is negative semi-definite,
2. $\text{tr } M = m_{11} + m_{22} \leq 0$, and $- \det M = m_{12}^2 - m_{11} m_{22} \leq 0$.

In [CP], we check the non-positivity of the linear operator $Lw$ by checking the non-positivity of $C_w$, $E_w$, and by checking condition (2) for the matrices $M^Rw$, $M^Sw$, and $M^T_w$.

5. Vanishing functions

In $\mathbb{R}^{n+1}$, $n = 2$, for many normal velocities $F$ the quantity

$$\frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 \lambda_2)^2} F^2$$

seems to be the natural choice when showing convergence to a round point. As in [5], we call this quantity a vanishing function for a normal velocity $F$. It is used by B. Andrews for the Gauss curvature flow [1], by F. Schulze and O. Schnürer for the $H^{\sigma}$-flow [10], by B. Andrews and X. Chen for the $|A|^{\sigma}$ and the $\text{tr } A^{\sigma}$-flow [2].

First we give the Definition 5.1 of a vanishing function in $\mathbb{R}^{n+1}$, $n = 3$. In Remark 5.3 we then introduce

$$\sum_{i<j} \frac{(\lambda_i - \lambda_j)^2}{(\lambda_i \lambda_j)^2} F^2$$

as the counterpart of a vanishing function for arbitrary dimensions $n \in \mathbb{N}$. In this paper we work with the quantity in particular in $\mathbb{R}^{n+1}$, $n = 3$.

In Lemma 5.4 we deduce a simple but interesting estimate for vanishing functions for arbitrary dimensions $n \in \mathbb{N}$. We employ this Lemma 5.4 in the proof of our main Theorem 6.5.

**Definition 5.1 (Vanishing function).** Let $v(a, b, c) \in C^2(\mathbb{R}_+^3)$ with $v \not\equiv 0$. Let $C_w(a, b, c)$ be defined as in Lemma 4.4. We call $v$ a vanishing function for a normal velocity $F$ if $C_v(a, b, c) = 0$ for all $0 < a, b, c$.

**Example 5.2 (Vanishing function).** We have the following example of a vanishing function for a normal velocity $F$:

$$\left(\frac{(a-b)^2}{(a b)^2} + \frac{(a-c)^2}{(a c)^2} + \frac{(b-c)^2}{(b c)^2}\right) F^2.$$  

**Remark 5.3 (Vanishing function).** We can define a vanishing function for arbitrary dimensions $n \in \mathbb{N}$. Let $\lambda_i$, $i = 1, \ldots, n$, denote the principal curvatures of a hypersurface in $\mathbb{R}^{n+1}$. Using Lemma 4.4 we can define constant terms $C_w(\lambda_1, \ldots, \lambda_n)$ as in Lemma 4.4 for an arbitrary $n \in \mathbb{N}$. We have the following example of a vanishing function:

$$(5.1) \sum_{i<j} \frac{(\lambda_i - \lambda_j)^2}{(\lambda_i \lambda_j)^2} F^2.$$
Interestingly, we still obtain a vanishing function if we omit up to \(n - 1\) terms of the form
\[
\frac{(\lambda_i - \lambda_j)^2}{\lambda_i \lambda_j^2} F^2.
\]
This reminds us of [8, Theorem 1.5] by G. Huisken and C. Sinestrari.

**Lemma 5.4** (Vanishing function). Let \(v\) be a vanishing function as defined in Remark 5.3. Let \(v \leq C^2\) on some set \(S \subset \mathbb{R}^n_+\), and for some constant \(C > 0\).

Then we have
\[
1 \leq \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \leq 1 + C \frac{\lambda_{\text{max}}}{F} \quad \text{on } S.
\]

**Proof.** We assume \(\lambda_{\text{min}} \equiv \lambda_1 \leq \ldots \leq \lambda_n \equiv \lambda_{\text{max}}\) and obtain
\[
C^2 \geq \sum_{i<j} (\lambda_i - \lambda_j)^2 \frac{F^2}{\lambda_i \lambda_j^2} \geq \frac{(\lambda_n - \lambda_1)^2}{\lambda_1 \lambda_n^2} F^2,
\]
which implies
\[
C \frac{\lambda_n}{F} \geq \frac{\lambda_n (\lambda_n - \lambda_1)}{\lambda_1 \lambda_n} = \frac{\lambda_n}{\lambda_1} - 1 \geq 0.
\]
This concludes the proof. \(\square\)

6. \(H^3\)-flow

The proof of our main Theorem 6.5 is based on investigating
\[
\varphi_{H^3} = \frac{(a - b)^2 + (a - c)^2 + (b - c)^2}{(a + b + c)^2},
\]
and
\[
\psi_{H^3} = \left( \frac{(a - b)^2}{(a b)^2} + \frac{(a - c)^2}{(a c)^2} + \frac{(b - c)^2}{(b c)^2} \right) (H^3)^2.
\]

The quantity \(\varphi_{H^3}\) is inspired by the quantity used in [7] by G. Huisken. The other quantity \(\psi_{H^3}\) is a vanishing function. First we show that the estimate \(\varphi_{H^3} \leq h := 1/8\) is preserved during the \(H^3\)-flow if the initial hypersurface \(M_0\) is 2-pinched. This is Lemma 6.1 and Corollary 6.2. Next we prove that \(\psi_{H^3}\) is bounded in time on the set where \(\varphi_{H^3} \leq h\). This is Lemma 6.3 and Corollary 6.4.

The proofs of these Lemmas and Corollaries involve the maximum-principle, the linear operator \(L\) and our computer program [CP]. In [CP] we deal with computations of two kinds. One kind is the purely algebraic manipulation of terms, and could still be performed by pen and paper. The other kind of computations includes random numbers for a Monte-Carlo method, which appears to be very tedious to carry out with pen and paper.

Finally, we show convergence to a round point combining the boundedness of \(\psi_{H^3}\) and the proof of [10] Theorem A.1.] by F. Schulze and O. Schnürer. This is our main Theorem 6.5.

**Lemma 6.1** (Monotone quantity \(\varphi\)). Let \((M_t)_{0 \leq t < T}\) be a maximal solution of the \(H^3\)-flow, where \(M_0\) is 2-pinched. Then we have
\[
L \varphi \leq 0
\]
at a critical point of \(\varphi\), where \(0 < \varphi \leq 1/8 =: h\).
Proof. Let $\mathcal{S}_{C_2}$ be the 2-pinched cone in the positive orthant. In [CP], we compute

$$\mathcal{S}_{C_2} := \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+ : \lambda_i / \lambda_j \leq 2 \text{ for all } 1 \leq i, j \leq 3\}$$

$$\mathcal{S}_h := \{(a, b, c) \in \mathbb{R}^3_+ : 0 < \varphi \leq 1/8 = h\},$$

$$\mathcal{S}_{L\varphi} := \{(a, b, c) \in \mathbb{R}^3_+ : L\varphi \leq 0\}$$

and show that

$$\mathcal{S}_{C_2} \subset \mathcal{S}_h \subset \mathcal{S}_{L\varphi}$$

using a Monte-Carlo method. Here, we check the non-positivity of $L\varphi$ as described in Remark 4.6.

Since the functions $\lambda_i / \lambda_j$, $\varphi$, and $L\varphi$ are homogeneous in the principal curvatures, it suffices to compute the sets $\mathcal{S}_{C_2}$, $\mathcal{S}_h$, and $\mathcal{S}_{L\varphi}$ in [CP] for the radial projection

$$\pi : \mathbb{R}^3_+ \to \{a + b + c = 1\}, \quad (a, b, c) \mapsto (a, b, c)/(a + b + c).$$

This concludes the computer-based proof. □

Corollary 6.2 (Monotone quantity $\varphi$). Let $\{M_t\}_{0 \leq t < T}$ be a maximal solution of the $H^3$-flow, where $M_0$ is 2-pinched. Then we have that

$$\varphi \leq h := 1/8$$

during the $H^3$-flow.

Proof. This follows directly from Lemma 6.1 using the maximum-principle. □

Lemma 6.3 (Monotone quantity $\psi$). Let $\{M_t\}_{0 \leq t < T}$ be a maximal solution of the $H^3$-flow, where $M_0$ is 2-pinched. Then we have

$$L\psi \leq 0$$

at a critical point of $\psi$, where $\psi > 0$.

Proof. By Corollary 6.2 we have

$$\varphi \leq h$$

during the $H^3$-flow. Let $\mathcal{S}_{C_2}$, $\mathcal{S}_h$ be defined as in Lemma 6.1 in [CP], we also compute

$$\mathcal{S}_{L\psi} := \{(a, b, c) \in \mathbb{R}^3_+ : L\psi \leq 0\}$$

and show in particular the second inclusion of

$$\mathcal{S}_{C_2} \subset \mathcal{S}_h \subset \mathcal{S}_{L\psi}$$

using a Monte-Carlo method. By Lemma 6.1, we have the first conclusion.

This concludes the computer-based proof. □

Corollary 6.4 (Monotone quantity $\psi$). Let $\{M_t\}_{0 \leq t < T}$ be a maximal solution of the $H^3$-flow, where $M_0$ is 2-pinched. Then we have that

$$\max_{M_t} \psi$$

is non-increasing during the $H^3$-flow.

Proof. This follows directly from Lemma 6.3 using the maximum-principle. □

Theorem 6.5 ($H^3$-flow). Let $\{M_t\}_{0 \leq t < T}$ be a maximal solution of the $H^3$-flow, where $M_0$ is 2-pinched. Then $\{M_t\}_{0 \leq t < T}$ converges to a round point.
Proof. We closely follow proof of the corresponding [10, Theorem A.1.] by F. Schulze and O. Schnürer.

By [10, Theorem 1.1] the surfaces \( M_t \) become immediately strictly convex for \( t > 0 \). Now choose a sufficiently small \( 0 < \varepsilon < T \) such that the \( H^3 \)-flow is smooth and strictly convex on the interval \( (\varepsilon, T) \). Thus the quantity \( \psi_{H^3} \) is well-defined on this interval, and bounded from above by Corollary 6.4. By Lemma 5.4 this implies

\[
1 \leq \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \leq 1 + \frac{C}{H^2}
\]
on \( (M_t)_{\varepsilon<t<T} \).

Now the proof follows analogously to the proof of [10, Theorem 1.2]. □

7. \(|A|^2\)-flow

A result similar to our main Theorem 6.5 holds for the normal velocity \( F = |A|^2 \) and 3-pinched hypersurfaces. For a proof consider

\[
\varphi_{|A|^2} = \frac{(a^2 + b^2 + c^2)(ab + ac + bc)^2}{abc} \\
\text{and} \quad \psi_{|A|^2} = \frac{(a + b + c)^2 ((a - b)^2 + (a - c)^2 + (b - c)^2)}{abc},
\]

and O. Schnürer [9]. As in chapter on \( H^3 \)-flow using [cp] we obtain

**Lemma 7.1 (\(|A|^2\)-flow).** Let \((M_t)_{0\leq t<T}\) be a maximal solution of the \(|A|^2\)-flow, where \(M_0\) is 3-pinched. Then we have that

\[
\max_{M_t} \psi_{|A|^2}
\]
is non-increasing in time.

8. Gauss curvature flow

A result similar to our main Theorem 6.5 holds for the normal velocity \( F = K \) and 2-pinched hypersurfaces. For a proof consider

\[
\varphi_K = \frac{(a - b)^2 + (a - c)^2 + (b - c)^2}{a^2 + b^2 + c^2} \\
\text{and} \quad \psi_K = \left( \frac{(a - b)^2}{(ab)^2} + \frac{(a - c)^2}{(ac)^2} + \frac{(b - c)^2}{(bc)^2} \right) (K)^2,
\]

and B. Chow [4]. As in chapter on \( H^3 \)-flow using [cp] we obtain

**Lemma 8.1 (Gauss curvature flow).** Let \((M_t)_{0\leq t<T}\) be a maximal solution of the Gauss curvature flow, where \(M_0\) is 2-pinched. Then we have that

\[
\max_{M_t} \psi_K
\]
is non-increasing in time.
9. Outlook

Our aim is to show convergence to a round point without pinching requirements using vanishing functions in arbitrary dimensions. Instead of splitting the linear operator $L$ into constant terms and gradient terms we intend to work with integral estimates similar to G. Huisken [7]. This way we seek to prove convergence to a round point for contracting normal velocities, including powers of the Gauss curvature, $K$, of the mean curvature, $H$, and of the norm of the second fundamental form, $|A|$.

10. Appendix

Some of the Lemmas leading up to the proof our main Theorem 6.5 rely on the computer program $[\text{CP}]$. First we compute the linear operator $L$ for the corresponding quantities $\varphi$ and $\psi$. Next we use a Monte-Carlo method to compute the sets $S_{C^2}, S_h, S_{L\varphi}$, and $S_{L\psi}$. Finally, we compute the two inclusions

$$S_{C^2} \subset S_h \subset S_{L\varphi},$$

$$S_{C^2} \subset S_h \subset S_{L\psi}.\quad (10.1)$$

For the convenience of the reader, we include the source code of $[\text{CP}]$ in three different programming languages, namely for the computer algebra systems Mathematica, Sage, and Maple. The first part of the appendix is the Mathematica program, the second part is the Sage program, and the third part is the Maple program.

In the first part we also visualize the two inclusions $(10.1)$.

At www.arxiv.org we can only submit this article without the computer program $[\text{CP}]$. To download this article with the computer program $[\text{CP}]$ please go to www.martinfranzen.de

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