Static and spherically symmetric general relativity solutions in Minimal Theory of Bigravity

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We investigate static and spherically symmetric solutions in the Minimal Theory of Bigravity (MTBG). First, we show that a pair of Schwarzschild-de Sitter spacetimes with different cosmological constants and black hole masses written in the spatially-flat Gullstrand-Painlevé (GP) coordinates is a solution in the self-accelerating branch of MTBG, while it cannot be a solution in the normal branch. We then illustrate how Schwarzschild-de Sitter solutions can become compatible with the normal branch when using different coordinates. We also confirm that the self-accelerating branch of MTBG admits static and spherically symmetric general relativity solutions with matter written in the spatially-flat coordinates, including neutron stars with arbitrary matter equations of state. Finally, we show that in the self-accelerating branch nontrivial solutions are given by the Schwarzschild-de Sitter metrics written in nonstandard coordinates.

I. INTRODUCTION

While general relativity (GR) has passed all the experimental tests in the weak-gravity field regime [1], with the dawn of gravitational-wave astronomy and other experiments associated with black holes (BHs) and compact objects a new frontier for testing GR has opened [2–5]. A part of scalar-tensor theories which have passed Solar System tests on GR could still have large deviations from GR in the vicinity of BHs and compact stars. Such a mechanism is known as spontaneous scalarization [6–19], which is triggered by a tachyonic instability on a BH or compact star background in GR. A key assumption for successful spontaneous scalarization is the existence of GR solutions, namely, the solutions that share the same metric and matter profiles with those in GR. While certain non-GR metric gravitational theories could allow GR solutions [20, 21], perturbations on top of them can behave very differently from those in GR [22].

There were several attempts to extend the idea of spontaneous scalarization to other field species [23–30], for instance a vector field. However, it has been argued that spontaneous vectorization [23–27] can be realized from selected initial conditions, rather than from the tachyonic instabilities of GR solutions. Since the branch of vectorized solutions is disconnected from the GR branch, spontaneous vectorization would not proceed as a continuous evolution from a GR solution. Indeed, in GR solutions the scalar mode suffers from ghost instabilities, rather than tachyonic ones [28, 30, 31]. A similar problem has also been pointed out for the models of spontaneous spinorization [31]. One important lesson from such models is that for a successful higher-spin extension of spontaneous scalarization the gravitational theory should, on top of having GR solutions, not contain any extra scalar modes.

We expect that spontaneous tensorization which is analogous to spontaneous scalarization in the spin-2 sector could occur in the fiducial sector of a bigravity theory, if the theory does not contain any extra scalar and vector degrees of freedom (DOFs) besides the two metric sectors. The de Rham-Gabadadze-Tolley (dRGT) massive gravity [32] which was the first nonlinear massive gravity theory free from the Boulware–Deser (BD) ghost [33] has been extended to bigravity by Hassan and Rosen (HR) [34] by promoting the metric of the fiducial sector to a dynamical field. In HR bigravity, however, the BD ghost can be generically revived when the matter is coupled to both the physical and fiducial metrics [35, 36]. The Minimal Theory of Massive Gravity (MTMG) [37, 38] is an extension of dRGT massive gravity with only two tensorial DOFs as in GR, instead of 5 DOFs in dRGT. In the original formulation of MTMG, the four-dimensional diffeomorphism invariance is completely broken as the unitary gauge for both time and space directions is chosen. Cosmology in MTMG has been studied in Refs. [38–41], where cosmological solutions in MTMG have been classified into the normal and self-accelerating branches. BH and stellar solutions in MTMG have been investigated in Ref. [42], where it has been shown that any GR solution which can be written in terms of the spatially-flat Gullstrand–Painlevé (GP) coordinates can be a solution in the self-accelerating branch of MTMG. Up to now, still spherically symmetric solutions of MTMG in the normal branch are not known, as in general the normal
branch is harder to study than the self-accelerating branch.\textsuperscript{1}

The ideas behind MTMG have been applied to the case of bigravity, which led to the Minimal Theory of Bigravity (MTBG) \cite{43}, where this time the joint four-dimensional diffeomorphism invariance is broken down to the three-dimensional one. While MTBG shares, by construction, the same background cosmological dynamics with HR bigravity, the number of propagating DOFs are down to four, namely, two tensorial DOFs in the physical metric and the other two in the fiducial metric, at least in the absence of matter. The absence of the extra scalar and vector DOFs in MTBG means the absence of ghost/gradient instabilities associated with them, which is one of the desired features to successfully realize spontaneous tensorization, and motivates us to study BHs and stars in MTBG. In order to see whether spontaneous tensorization can be successfully realized in MTBG, in this work we are going to clarify the existence of static and spherically symmetric GR solutions without and with matter in MTBG. Of course, even if spontaneous tensorization cannot be realized, these solutions would also remain interesting per se, as one can hope to test MTBG in the strong gravity regime by investigating perturbations around them.

Therefore, we will study static and spherically symmetric solutions in MTBG with and without matter. We will clarify the conditions under which static and spherically symmetric GR solutions written in the spatially-flat coordinates are also solutions in MTBG, and explicitly show that Schwarzschild-de Sitter solutions written in the spatially-flat GP coordinates can also be solutions in the self-accelerating branch of MTBG. We will also show that in the self-accelerating branch of MTBG with two matter sectors coupled to the two metrics separately the static and spherically symmetric physical and fiducial metrics written in the spatially-flat coordinates satisfy the Einstein equations with matter in GR, namely, GR solutions with two individual matter sectors can also be solutions in MTBG. Finally we will show how Schwarzschild-de Sitter solutions, when written in other non-spatially-flat coordinates, may become solutions in the normal branch as well, provided that the two metrics are parallel to each other. This last condition, which does not need to hold in the self-accelerating branch, shows indeed that far fewer solutions are known for MTBG in the normal branch.

On the other hand, in order to investigate whether MTBG admits nontrivial static and spherically symmetric solutions besides GR solutions, we will construct static and spherically symmetric vacuum solutions perturbatively in the small mass limit of MTBG. We will regard the graviton mass squared as the expansion parameter, and expand the metric variables and the Lagrange multipliers. We will show that in the self-accelerating branch the Schwarzschild-de Sitter metrics written in nonstandard coordinates can be obtained perturbatively, while in the normal branch these metrics are not compatible with the massless limit of MTBG. We will also show that the results in the perturbative analysis can be naturally extended to the fully nonperturbative analysis, and the Schwarzschild-de Sitter solutions written in the nonstandard coordinates satisfying certain conditions can be solutions in the self-accelerating branch of MTBG.

The structure of this paper is as follows: In Sec. II, we review the formulation of MTBG theory. In Sec. III, we derive the conditions under which static and spherically symmetric vacuum solutions, i.e., the Schwarzschild-de Sitter solutions written in the spatially-flat GP coordinates are also solutions in MTBG. In Sec. IV, we consider the case where two individual matter sectors are coupled to the two metrics separately, and show that the static and spherically symmetric physical and fiducial metrics written in the spatially-flat coordinates satisfy the Einstein equations with matter in GR, and hence GR solutions with matter coupled to each sector separately can also be solutions in MTBG. In Sec. V, we investigate static and spherically symmetric vacuum solutions with the perturbative and nonperturbative approaches, and show the existence of the Schwarzschild-de Sitter solutions written in nonstandard coordinates. Finally, Sec. VII is devoted to a brief summary and conclusion.

\section{II. Minimal Theory of Bigravity}

We consider bigravity composed of the physical and fiducial metrics, $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. We introduce the Arnowitt-Deser-Misner (ADM) decomposition of $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively,

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt) (dx^j + N^j dt),$$

$$f_{\mu\nu}dx^\mu dx^\nu = -M^2 dt^2 + \phi_{ij} (dx^i + M^i dt) (dx^j + M^j dt),$$

where $t$ and $x^i$ ($i = 1, 2, 3$) are the temporal and spatial coordinates, $(N, N^i, \gamma_{ij})$ and $(M, M^i, \phi_{ij})$ are the sets of the lapse function, shift vector, and spatial metric on the constant $t$ hypersurfaces for the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively.

\textsuperscript{1}The property that normal branch solutions are more difficult to be found, as we will see, is also shared by the theory we will discuss here.
The extrinsic curvature tensors on constant \( t \) hypersurfaces are given by
\[
K_{ij} := \frac{1}{2N} \left( \partial_t \gamma_{ij} - D_i N_j - D_j N_i \right), \quad \Phi_{ij} := \frac{1}{2M} \left( \partial_t \phi_{ij} - D_i M_j - D_j M_i \right),
\]
where \( D_i \) and \( \tilde{D}_i \) are covariant derivatives for \( \gamma_{ij} \) and \( \phi_{ij} \), respectively.

The action of MTBG is then given by
\[
S = \frac{1}{2\kappa^2} \int d^4x \mathcal{L},
\]
\[
\mathcal{L} := \mathcal{L}_g \left[ N, N^i, \gamma_{ij}; M, M^i, \phi_{ij}; \lambda, \bar{\lambda}, \lambda' \right] + \mathcal{L}_m \left[ N, N^i, \gamma_{ij}; M, M^i, \phi_{ij}; \Psi \right],
\]
where \( \mathcal{L}_g \) and \( \mathcal{L}_m \) represent the gravitational and matter parts of the Lagrangian, respectively. \( \lambda, \bar{\lambda}, \lambda' \) are the Lagrange multipliers, and \( \Psi \) represents matter. The gravitational Lagrangian \( \mathcal{L}_g \) of MTBG in the unitary gauge is further decomposed into the *precursor* and *constraint* parts as
\[
\mathcal{L}_g := \mathcal{L}_{\text{pre}} \left[ N, N^i, \gamma_{ij}; M, M^i, \phi_{ij} \right] + \mathcal{L}_{\text{con}} \left[ N, N^i, \gamma_{ij}; M, M^i, \phi_{ij}; \lambda, \bar{\lambda}, \lambda' \right],
\]
with
\[
\mathcal{L}_{\text{pre}} := \sqrt{-g} R[g] + \tilde{a}^2 \sqrt{-f} R[f] - m^2 \left( N \sqrt{\gamma} \mathcal{H}_0 + M \sqrt{\phi} \tilde{\mathcal{H}}_0 \right),
\]
\[
\mathcal{L}_{\text{con}} := \sqrt{-\gamma} \alpha_{1\gamma} \left( \lambda + \Delta \lambda \right) + \sqrt{-\phi} \alpha_{1\phi} \left( \lambda - \Delta \phi \right) + \sqrt{-\gamma} \alpha_{2\gamma} \left( \lambda + \Delta \lambda \right)^2 + \sqrt{-\phi} \alpha_{2\phi} \left( \lambda - \Delta \phi \right)^2
\]
\[ - m^2 \left[ \sqrt{-\gamma} U^i S D_i \lambda^k - \beta \sqrt{-\phi} \tilde{U}^i S \tilde{D}_i \lambda^k \right],
\]
and
\[
\alpha_{1\gamma} := -m^2 U^p q K^q p, \quad \alpha_{1\phi} := m^2 U^p q \Phi^q p,
\]
\[
\alpha_{2\gamma} := -m^4 4N \left( U^p q - \frac{1}{2} U^p q \delta^p q \right) U^q p, \quad \alpha_{2\phi} := -m^4 4M \tilde{\alpha}^2 \left( \tilde{U}^p q - \frac{1}{2} \tilde{U}^p q \delta^p q \right) \tilde{U}^q p,
\]
where the constant \( \tilde{a} \) is the ratio of the two gravitational constants, \( m \) is a parameter with dimensions of mass which can be related with the graviton mass, \( \beta \) is a constant, \( \gamma := \det(\gamma_{ij}) \) and \( \phi := \det(\phi_{ij}) \) are the determinants of the three-dimensional spatial metrics \( \gamma_{ij} \) and \( \phi_{ij} \) respectively. Furthermore, \( \mathcal{H}_0 \) and \( \tilde{\mathcal{H}}_0 \) are defined by \( \mathcal{H}_0 := \sum_{n=0}^{3} c_{4-n} e_n(\mathcal{K}) \) and \( \tilde{\mathcal{H}}_0 := \sum_{n=0}^{3} \bar{c}_{4-n} e_n(\bar{\mathcal{K}}) \) with
\[
e_0(\mathcal{K}) = 1, \quad e_1(\mathcal{K}) = [\mathcal{K}], \quad e_2(\mathcal{K}) = \frac{1}{2} \left( [\mathcal{K}]^2 - [\mathcal{K}^2] \right), \quad e_3(\mathcal{K}) = \det(\mathcal{K}),
\]
and similarly for \( e_n(\bar{\mathcal{K}}) \) with \( \mathcal{K}_k^i \) and \( \bar{\mathcal{K}}_k^i \) characterized by
\[
\mathcal{K}_k^i \mathcal{K}^k_j = \gamma^{ik} \phi_{kj}, \quad \bar{\mathcal{K}}_k^i \bar{\mathcal{K}}^k_i = \gamma_{jk} \phi^{kj},
\]
\( \Delta \gamma := \gamma^{ij} D_i D_j \) and \( \Delta \phi := \phi^{ij} \tilde{D}_i \tilde{D}_j \) are the Laplacian operators, the spatial tensors \( U^i j \) and \( \tilde{U}^i j \) are defined by
\[
U^i j := \frac{1}{2} \sum_{n=1}^{3} c_{4-n} \left( U(n)^i j + \gamma^{ik} \gamma_{jk} U(n)^k \right),
\]
\[
\tilde{U}^i j := \frac{1}{2} \sum_{n=1}^{3} \bar{c}_n \left( \tilde{U}(n)^i j + \phi^{ik} \phi_{jk} \tilde{U}(n)^k \right),
\]
with \( U(n)^i k := \frac{\partial \gamma(\mathcal{K})}{\partial \gamma_k^n} \) and \( \tilde{U}(n)^i k := \frac{\partial \phi(\bar{\mathcal{K}})}{\partial \phi_k^n} \), and \( c_j \) (\( j = 0, 1, 2, 3, 4 \)) being constants.

Variation of the action (3) with respect to \( N, N^i, \gamma_{ij}, M, M^i, \) and \( \phi_{ij} \) provides the gravitational equations of motion, and variation with respect to \( \Psi \) provides the matter equation of motion, which we do not show explicitly. Finally, variation with respect to the Lagrange multipliers \( \lambda, \bar{\lambda}, \lambda' \) gives the constraint equations
\[
\sqrt{-\gamma} \alpha_{1\gamma} + \sqrt{-\phi} \alpha_{1\phi} + 2 \sqrt{-\gamma} \alpha_{2\gamma} \left( \lambda + \Delta \lambda \right) + 2 \sqrt{-\phi} \alpha_{2\phi} \left( \lambda - \Delta \phi \right) = 0,
\]
\[
\sqrt{-\gamma} \Delta \gamma \alpha_{1\gamma} - \sqrt{-\phi} \Delta \phi \alpha_{1\phi} + 2 \sqrt{-\gamma} \Delta \gamma \left( \alpha_{2\gamma} \left( \lambda + \Delta \lambda \right) \right) - 2 \sqrt{-\phi} \Delta \phi \left( \alpha_{2\phi} \left( \lambda - \Delta \phi \right) \right) = 0,
\]
\[
\sqrt{-\gamma} \mathcal{D}_p U^p_q - \beta \sqrt{-\phi} \tilde{\mathcal{D}}_p \tilde{U}^p_q = 0.
\]
It should be noted that adding constraints to a theory in general shrinks the space for the allowed solutions. As for MTBG, these same constraints are, however, necessary in order to remove the unwanted unstable degrees of freedom, while keeping the tensor modes for both metrics propagating on any background solutions of the theory. On a homogeneous and isotropic background, it was shown that in general nontrivial solutions do exist. However, on a different background, things could go differently, and we would like to investigate in the following the presence of solutions in spherically symmetric configurations.

III. SCHWARZSCHILD-DE SITTER SOLUTIONS IN MTBG

First, we consider the vacuum case by setting the matter action to be zero $\mathcal{L}_m = 0$ in the action (3). We derive the conditions under which static and spherically symmetric solutions written in the spatially-flat coordinates in vacuum GR, namely, the Schwarzschild-de Sitter solutions in the GP coordinates, are also the solutions in MTBG. We consider the static and spherically symmetric physical and fiducial metrics written in the following coordinates

$$g_{\mu\nu}dx^\mu dx^\nu = -A_0(r)dt^2 + A_1(r)(dr + N'(r)dt)^2 + A_2(r)r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

$$f_{\mu\nu}dx^\mu dx^\nu = C_0^2\left[-A_0f(r)dt^2 + A_1f(r)(dr + N'(r)dt)^2 + A_2f(r)r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \quad (13)$$

where $C_0 > 0$ is constant, $r$ and $(\theta, \varphi)$ represent the radial and angular coordinates, and $A_0(r)$, $A_1(r)$, $A_2(r)$, $A_0f(r)$, $A_1f(r)$, and $A_2f(r)$ are functions of $r$.

A. Self-accelerating branch in GP coordinates

The Schwarzschild-de Sitter metric in the spatially-flat GP coordinates is expressed as

$$A_0(r) = A_1(r) = A_2(r) = 1, \quad N'(r) = \pm \sqrt{\frac{2M}{r} + \frac{\Lambda}{3}r^2},$$

$$A_0f(r) = A_1f(r) = A_2f(r) = 1, \quad N'_f(r) = \pm \sqrt{\frac{2M_f}{r} + \frac{C_0^2\Lambda_f}{3}r^2}, \quad (14)$$

where $M$ and $M_f$ represent the mass parameters, and $\Lambda$ and $\Lambda_f$ are effective cosmological constants in the physical and fiducial sectors, respectively. In general, the masses and effective cosmological constants in the two sectors may be different, $M \neq M_f$ and $\Lambda \neq \Lambda_f$.

First, focusing on the precursor part of the vacuum MTBG $\mathcal{L}_{\text{pre}}$ in Eq. (5), the conditions under which the Schwarzschild-de Sitter solutions exist are given by

$$2\Lambda = (C_0^3c_1 + 3C_0^2c_2 + 3C_0c_3 + c_4)m^2, \quad 2\bar{\Lambda}^2\Lambda_f = (c_0 + 3C_0^{-1}c_1 + 3C_0^{-2}c_2 + C_0^{-3}c_3)m^2. \quad (15)$$

We then consider the full vacuum MTBG $\mathcal{L}_g$ including the constraint part of the Lagrangian $\mathcal{L}_{\text{con}}$ given by Eq. (5). Because of the symmetries in the physical and fiducial sectors, from the beginning we may choose

$$\lambda^\theta(r) = \lambda^\varphi(r) = 0, \quad (16)$$

while still assume that $\lambda'(r)$ is a priori a nontrivial function of $r$. For the Schwarzschild-de Sitter solutions written in the spatially-flat GP coordinates (14), we find that the constraint equation (12) is trivially satisfied. The coefficients in the remaining constraint equations (10) and (11) are given by

$$\alpha_1 = \frac{(C_0^2c_1 + 2C_0c_2 + c_3)m^2(3M + r^3\Lambda)}{r^2\sqrt{2M + \frac{\Lambda r^2}{3}}}, \quad \alpha_1 = \frac{(C_0^2c_1 + 2C_0c_2 + c_3)m^2(3M_f + C_0^2r^3\Lambda_f)}{r^2\sqrt{2M_f + \frac{\Lambda_f r^2}{3}}}, \quad (17)$$

and then Eqs. (10) and (11) have the structure

$$\left(C_0^2c_1 + 2C_0c_2 + c_3\right)\mathcal{F}\left(\lambda(r), \tilde{\lambda}'(r), \tilde{\lambda}''(r)\right) = 0,$$

$$\left(C_0^2c_1 + 2C_0c_2 + c_3\right)\mathcal{G}\left(\lambda(r), \lambda'(r), \lambda''(r), \lambda'''(r), \lambda^{(3)}(r), \lambda^{(4)}(r)\right) = 0, \quad (18)$$
where $\mathcal{F}$ and $\mathcal{G}$ are linear combinations of the variables in the argument. From Eq. (18), we see that there are two possible branches, i.e., the self-accelerating and normal branches.

The self-accelerating branch is given by

$$C_0^2 c_1 + 2C_0 c_2 + c_3 = 0. \quad (19)$$

Substituting the Schwarzschild-de Sitter metrics written in the spatially-flat coordinates (14) with the conditions (15) and (19) into the gravitational equations of motion of MTBG, we find

$$\lambda(r) = 0, \quad \bar{\lambda}'(r) + \frac{2}{r} \bar{\lambda}'(r) = 0, \quad \lambda' r = 0, \quad (20)$$

which allows us to choose

$$\lambda(r) = \lambda'(r) = 0, \quad \bar{\lambda}'(r) = d_0 + \frac{d_1}{r}, \quad (21)$$

with $d_0$ and $d_1$ being constants, as a solution. Thus, Eqs. (15) and (19) provide the condition for the Schwarzschild-de Sitter metrics written in the spatially-flat coordinates to be solutions in MTBG. We note that the choice (19) coincides with the self-accelerating branch in cosmology [43].

### B. Normal branch in GP coordinates

If a consistent solution of

$$\mathcal{F}(\lambda(r), \bar{\lambda}'(r), \lambda''(r)) = 0, \quad \mathcal{G}(\lambda(r), \lambda'(r), \lambda''(r), \bar{\lambda}'(r), \lambda^{(3)}(r), \bar{\lambda}^{(4)}(r)) = 0, \quad (22)$$

exists for $\lambda(r)$ and $\bar{\lambda}(r)$, this should correspond to the normal branch. Within the Schwarzschild-de Sitter ansatz (13) and (14), for simplicity we focus on the asymptotically flat Schwarzschild spacetimes with $\Lambda = \Lambda_f = 0$. The condition $\mathcal{F} = 0$ in Eq. (22) then reduces to

$$\lambda(r) = \frac{1}{C_0^2 r^2 (1 + C_0^2 \alpha^2)} \left[ \frac{2 \sqrt{2} C_0^2 \alpha^2}{m^2 (C_0^2 c_1 + 2C_0 c_2 + c_3)} \left( \sqrt{M_f r} - \sqrt{M r} \right) - r \left(1 + C_0^4 \alpha^2 \right) (2 \bar{\lambda}'(r) + r \lambda''(r)) \right]. \quad (23)$$

Then, imposing $\mathcal{G} = 0$ in (22) yields

$$\bar{\lambda}(r) = -\frac{8C_0^2 \sqrt{2r} \left( \sqrt{M + C_0^2 \alpha^2 \sqrt{M_f}} \right)}{3(1 + C_0^2)(C_0^2 c_1 + 2C_0 c_2 + c_3) m^2} + \frac{q_1}{r} + q_2 + q_3 r + q_4 r^2, \quad (24)$$

where $q_1$, $q_2$, $q_3$, and $q_4$ are integration constants. We note that the integration constants $q_1$ and $q_2$ are associated with the solutions in the Laplace equation in the three-dimensional flat space and do not physically contribute to the solution of $\lambda(r)$, Eq. (23). On the other hand, the terms of $q_3$ and $q_4$ physically contribute to Eq. (23). However, the Euler-Lagrange equations for $N^r(r)$ and $N_f^r(r)$ provide, respectively,

$$-3 \sqrt{\frac{2M}{r}} + \frac{2(1 + C_0^2)(c_3 + 2C_0 c_2 + c_1C_0^2) m^2 q_3}{C_0^2 (1 + \alpha^2 C_0^2)} = 0,$$

$$-3 \sqrt{\frac{2M_f}{r}} + \frac{2(1 + C_0^2)(c_3 + 2C_0 c_2 + c_1C_0^2) m^2 q_3}{C_0^2 (1 + \alpha^2 C_0^2)} = 0, \quad (25)$$

whose combination provides $M = M_f$. Assuming that $c_3 + 2C_0 c_2 + c_1C_0^2 \neq 0$ for the non-self-accelerating branch, they cannot be satisfied unless $q_3 = 0$ and $M = M_f = 0$. Adding nonzero $\Lambda$ and $\Lambda_f$ does not change the results. This means that in the normal branch, Schwarzschild BHs written in the spatially-flat GP coordinates cannot be embedded into the vacuum spacetimes.

The same conclusion could be derived for a regular matter distribution. Thus, we conclude that the normal branch of MTBG cannot accommodate static and spherically symmetric GR solutions written in the spatially-flat GP coordinates, irrespective of the presence of the matter sector (see Appendix A for a similar result in the massless limit of the normal branch of MTBG).
C. Normal branch in slicing with $D_iD^iK = 0$

The fact that the GP choice for the slicing of the metric does not satisfy the equations of motion does not necessarily mean that there are no solutions in the normal branch. In fact, even if the constraints imposed on MTBG do not allow the GP slicing in the normal branch, in principle there could be other slicings which could instead lead to some nontrivial solutions for the equations of motion. Indeed, in MTBG, since the four-dimensional diffeomorphism invariance is broken, different time-slicings in general lead to different physical configurations. Then, at least in principle, one is supposed to investigate the most general ansatz for the differential equations, i.e. to consider all the different time-slicings which are compatible with a given background. And this attempt should be done, in principle, also for general nonzero values of $\lambda$, $\bar{\lambda}$ and $\lambda'$.

Although this kind of general configurations would manifestly show the whole background possibilities for the theory \textsuperscript{2}, in practice to do so turns out to be an analytically formidable problem. So, let us try to find instead a class of time-slicings in the normal branch which can shed some light on the space of the solutions. In the remaining part of this section, we will also try to be as independent as possible from a given choice of the background, but only assume that the chosen slicing is compatible with the ADM splitting. Although at the beginning we will let the slicing to be general, still we aim to find a particular class of solutions (how particular will be discussed later on).

Hence, we focus our attention to a vacuum configuration solution which satisfies the following ansatz

$$
\phi_{ij} = C_0^2 \gamma_{ij}, \quad M = C_0 N, \quad M^i = N^i, \\
\lambda = 0, \quad \bar{\lambda} = \text{constant}, \quad \lambda^i = 0,
$$

(26)

where $C_0$ is again a numerical constant. Then this ansatz leads to the following relations on a general background

$$
\bar{D}_j A^i = D_j A^i, \quad \bar{D}_j B_i = D_j B_i, \quad \bar{D}_k \gamma_{ij} = 0, \quad M_i = C_0^2 N_i,
$$

(27)

$$
\bar{R}^i_{ij} = (3) R^i_{ij}, \quad \bar{R} = \frac{1}{C_0^2} R, \quad \phi^{ij} = \frac{1}{C_0^2} \gamma^{ij}, \quad \phi = \frac{C_0^6}{\lambda} \gamma,
$$

(28)

where $A^i$ ($B_i$) is a general three dimensional vector (covector), and evidently $(3) R^i_{ij}$ and $(3) R$ represent the three dimensional Ricci tensor and scalar for the metric $\gamma_{ij}$ (leaving a clear interpretation for $(3) \bar{R}^i_{ij}$ and $(3) \bar{R}$). On using the properties of the chosen ansatz, we find

$$
\bar{K}^i_k \bar{K}^k_j = \gamma^i_k \phi_{kj} = C_0^2 \delta^i_j,
$$

(29)

so that

$$
\bar{K}^i_j = C_0 \delta^i_j, \quad \bar{K}^i_j = \frac{1}{C_0} \delta^i_j,
$$

(30)

since $\bar{K}^i_j$ must be the inverse of $\bar{K}^i_j$. These expressions lead to the following relation for the extrinsic curvature for the two three dimensional metrics

$$
\Phi_{ij} = C_0 K_{ij}.
$$

(31)

We note that, so far, the value of $C_0$ is still a free numerical parameter.

In order to simplify and to solve all the constraints of MTBG in the normal branch, we impose the following condition on top of the ansatz (26),

$$
D_i D^i K = 0.
$$

(32)

At this point, one is left to solve the equations of motion for the metric. Hence, on considering the equation of motion for the lapse $N$, in MTBG, we find that

$$
K_{ij} K^{ij} - K^2 - R + 2 \Lambda_{\text{eff}} = 0,
$$

(33)

where we identify

$$
\Lambda_{\text{eff}} = \frac{1}{2} m^2 \left( c_1 C_0^3 + 3 c_2 C_0^2 + 3 c_3 C_0 + c_4 \right).
$$

(34)

\textsuperscript{2} For MTBG, and for other theories which break four dimensional diffeomorphism, a Birkhoff theorem does not hold in general, but one can still try to find out general solutions to the equations of motion.
Here we remind the reader that the term $c_4$ is a pure cosmological constant in the physical sector, whereas $c_0$ is a pure cosmological constant in the fiducial sector. As for the equation of motion for the lapse function in the fiducial sector $M$, we have that, on combining it with the previous equation of motion for the lapse function in the physical sector $N$, the following condition must hold, namely

$$c_0 C_0^3 + c_1 C_0^2 (3 - \alpha^2 C_0^2) + 3 c_2 C_0 (1 - \alpha^2 C_0^2) + c_3 (1 - 3 \alpha^2 C_0^2) - \alpha^2 c_4 C_0 = 0. \tag{35}$$

This constraint is not to be understood as a fine-tuning condition for the fiducial cosmological constant $c_0$ (or vice versa for the physical cosmological constant $c_4$), rather it is the equation which determines $C_0$ for the solution. So at this level, whatever the cosmological constant are in the two metric frames, a solution will in general exist, provided a real $C_0$ satisfies Eq. (35).

Now, let us consider the remaining nontrivial Hamilton equations of motion for the two metrics, namely

$$\dot{\pi}^{ij} = \{\pi^{ij}, H\}, \tag{36}$$
$$\dot{\sigma}^{ij} = \{\sigma^{ij}, H\}, \tag{37}$$

where $\pi^{ij}$ and $\sigma^{ij}$ are the canonical momenta of the metric $\gamma^{ij}$ and $\phi^{ij}$ respectively, whereas $H$ is the total Hamiltonian of MTBG (given in Eq. (11) of [43]). Then, for the considered ansatz, we can show that

$$\sigma^{ij} = \alpha^2 \pi^{ij}, \tag{38}$$

together with

$$\{\sigma^{ij}, H\} - \alpha^2 \{\pi^{ij}, H\} = [\ldots] \gamma^{ij} = 0, \tag{39}$$

because the right hand side of this equation is proportional to Eq. (35). Finally, the Hamilton equations of motion for the physical momenta $\pi^{ij}$ coincide with those of GR with cosmological constant.

In summary, what we find here is that any solutions of GR with cosmological constant in a slicing satisfying the condition (32), such as a constant mean curvature slicing, are also solutions of the normal branch of MTBG (with untuned cosmological constants), provided that the two metric are parallel to each other (and $\lambda$, $\partial_{t}\lambda$, and $\lambda'$ all vanish) in the sense of (26).

As a consequence, if we consider a static and spherically symmetric background, the solutions which satisfy the imposed ansatz for MTBG, having $K = -3b_0 = \text{constant}$, coincide with the general solutions given in [44] for a theory of minimally modified gravity, namely VCDM and VCCDM$^{3}$ [45–48], which can now be written as

$$g_{\mu\nu} \, dx^\mu \, dx^\nu = -\frac{N_0^2}{F(r)^2} \, dt^2 + \left[ F(r) \, dr + \left( b_0 r - \frac{\kappa_0}{r^2} \right) N_0 \, dt \right]^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \tag{40}$$
$$f_{\mu\nu} \, dx^\mu \, dx^\nu = C_0^2 \left\{ -\frac{N_0^2}{F(r)^2} \, dt^2 + \left[ F(r) \, dr + \left( b_0 r - \frac{\kappa_0}{r^2} \right) N_0 \, dt \right]^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right\}, \tag{41}$$

where $N_0$, $b_0$ and $\kappa_0$ are free constants$^{4}$, whereas

$$\frac{1}{F(r)^2} = 1 - \frac{2\mu_0}{r} - \frac{1}{3} \Lambda_0 \, r^2 + \frac{\kappa_0^2}{r^4}, \quad \text{with} \quad \Lambda_0 = \Lambda_{\text{eff}} - 3 \kappa_0^2. \tag{42}$$

Several properties of the solutions were already discussed in [44], e.g. they represent the Schwarzschild-de Sitter solution written in a constant-$K$ (in space and time) slicing, so that we here would like to discuss instead this solution in the context of MTBG. We note that the case of $b_0 = \kappa_0 = 0$ corresponds to a pair of the Schwarzschild-de Sitter spacetimes sharing the same ADM mass and effective cosmological constant written in the Schwarzschild coordinates.

Then, although this class of solutions do exist in the normal branch and they can successfully provide a description for a BH, still we have to elaborate on the choice, in MTBG, for the chosen ansatz. Indeed, both metrics $f_{\mu\nu}$ and $g_{\mu\nu}$ are parallel to each other. This choice is legitimate, but we should ask ourselves whether it is physically motivated or not, more explicitly, if this configuration is fine-tuned or not. For instance, we know that this strong condition for the two metrics was not necessary in the self-acceleration branch. Furthermore, it is not clear whether a generic collapse in the normal branch would end up in this parallel configuration or not.

---

$^3$ Although there is no link between these theories (e.g. the number of the degrees of freedom are different), still it is interesting that they share same solutions on this background. In the following we also keep the same notation of [44] for an immediate comparison.

$^4$ The constant parameter $N_0$, because of time-reparametrization invariance, can be safely set to unity.
We note that the constraint (26), in particular a proportionality between the fiducial and the physical metrics on the background, namely \( f_{\mu \nu} = C_0^2 g_{\mu \nu} \), provided that a slicing satisfying \( D_i D^i K = 0 \) exists, and the condition determining \( C_0 \) written in Eq. (35) holds true. In fact, this same procedure can be successfully extended to other vacuum backgrounds provided a constant-\( K \) slicing exists, such as the Kerr-de Sitter solution written in Boyer–Lindquist coordinates.

Once more, although an existence proof of non-perturbative solutions is now given in the normal branch of MTBG — a result which is not trivial given the existence of constraints — how to make sense of this parallel configuration for the two metrics remains. In this case as well, it would be interesting to either understand the collapse dynamics, or at least extend the solution by e.g. perturbatively detuning the condition (32). Whether more generic solutions exist in the normal branch (in particular for the case \( \lambda \neq 0 \)), or the constraint equations are too restrictive and render the normal branch unviable for realistic situations remains to be studied.

### IV. Static and spherically symmetric GR solutions with two distinct matter sectors

We then introduce the matter sector \( \Psi \), which is divided into the two distinct sectors \( \Psi_{m,g} \) and \( \Psi_{m,f} \) minimally coupled to the physical and fiducial metrics \( g_{\mu \nu} \) and \( f_{\mu \nu} \) separately,

\[
\mathcal{L}_m [N, N^i, \gamma_{ij}; M, M^i, \phi_{ij}; \Psi] = \mathcal{L}_{m,g} [N, N^i, \gamma_{ij}; \Psi_{m,g}] + \mathcal{L}_{m,f} [M, M^i, \phi_{ij}; \Psi_{m,f}].
\]  

#### A. Static and spherically symmetric GR solutions with matter

Before studying MTBG, we first consider GR with the gravitational constant \( \kappa^2 \) and the cosmological constant \( \Lambda \) and assume that the metric \( g_{\mu \nu} \) represents a static and spherically symmetric solution in GR, which is written in terms of the spatially-flat GP coordinates as

\[
g_{\mu \nu} dx^\mu dx^\nu = -A_0(r) dt^2 + (dr + N^r(r) dt)^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).
\]  

We also assume that \( g_{\mu \nu} \) satisfies the Einstein equations in GR with the matter energy-momentum tensor

\[
T_{(m,g)\mu \nu} := -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{m,g}}{\delta g^{\mu \nu}} = (\rho + p) u_\mu u_\nu + pg_{\mu \nu},
\]  

where \( u^\mu = \left( \frac{1}{\sqrt{A_0(r)-(N^r(r))^2}}, 0, 0, 0 \right) \) represents the four velocity, and \( \rho \) and \( p \) are the energy density and pressure of the perfect fluid, respectively. The Einstein equations are explicitly given by

\[
\begin{align*}
&-N^r(r)^2 \frac{A'_0(r)}{2r A_0(r)} + \frac{N^r(r)^2 A'_0(r) A_0(r)}{2r A_0(r)} - \frac{2N^r(r)N'^r(r)}{r A_0(r)} = -\kappa^2 \rho(r) - \Lambda, \\
&-N^r(r)^2 \frac{A'_0(r)}{2r A_0(r)} + \frac{A'_0(r)}{r A_0(r)} - \frac{2N^r(r)N'^r(r)}{r A_0(r)} = \kappa^2 p(r) - \Lambda, \\
&\frac{A'_0(r)}{2r A_0(r)} + \frac{N^r(r)^2 A'_0(r)}{2r A_0(r)^2} - \frac{A'_0(r)}{4A_0(r)^2} - \frac{2N^r(r)N'^r(r)}{r A_0(r)} + \frac{N^r(r)A'_0(r)N'^r(r)}{2A_0(r)^2} \frac{A_0(r)}{2} = 0. \\
\end{align*}
\]  

#### B. Static and spherically symmetric GR solutions in MTBG

In MTBG, we assume that the physical and fiducial metrics are given by Eq. (13) with the spatially-flat form

\[
A_1(r) = 1, \quad A_2(r) = 1, \quad A_{1f}(r) = 1, \quad A_{2f}(r) = 1.
\]  

We note that the constraint (12) is trivially satisfied for the spatially-flat coordinates (47). We also assume that the matter in the physical and fiducial sectors (43) is given by the perfect fluids with the energy-momentum tensors, Eq. (45) and

\[
T_{\mu \nu}^{(m,f)} := -\frac{2}{\sqrt{-f}} \frac{\delta \mathcal{L}_{m,f}}{\delta f^{\mu \nu}} = (\rho_f + p_f) u_{(f)\mu} u_{(f)\nu} + pg_{\mu \nu},
\]  


with the matter four velocity \( u_{(f)\mu} = \left( \frac{1}{C_0 \sqrt{A_{0f}(r)-(N_f^r(r))}}, 0, 0, 0 \right) \), \( \rho_f \), and \( p_f \) being the four velocity, energy density, and pressure of the perfect fluid, respectively, in the fiducial sector. The coefficients in the constraint equations (10) and (11) are given by

\[
\begin{align*}
\alpha_{1\gamma} &= \frac{(C_0^2 c_1 + 2C_0 c_2 + c_3) m^2}{r \sqrt{A_0(r)}} [r N^\gamma(r) + 2 N^\gamma] , \\
\alpha_{1\phi} &= -\frac{(C_0^2 c_1 + 2C_0 c_2 + c_3) m^2}{C_0 r \sqrt{A_{0f}(r)}} [r N^\gamma(r) + 2 N^\gamma] , \\
\alpha_{2\gamma} &= 3 \left( \frac{C_0^2 c_1 + 2C_0 c_2 + c_3}{8 \sqrt{A_0(r)}} \right) m^4 , \\
\alpha_{2\phi} &= 3 \left( \frac{C_0^2 c_1 + 2C_0 c_2 + c_3}{8 \sqrt{A_{0f}(r)}} \right) m^4 .
\end{align*}
\]

We choose the self-accelerating branch \((19)\)** and impose the conditions \((15)\). We also assume that the physical and fiducial metrics satisfy the Einstein equations in GR coupled to the matter energy-momentum tensors \((45)\) and \((48)\) respectively, which are explicitly given by Eq. \((46)\) in the physical frame and

\[
\begin{align*}
-\frac{N_f^r(r)^2}{r^2 A_{0f}(r)} + \frac{N_f^r(r) A'_{0f}(r)}{r A_{0f}(r)^2} - \frac{2 N_f^r(r) N_f^r(r)}{r A_{0f}(r)} &= C_0^2 \left( \frac{-\kappa^2}{\alpha^2} p_f(r) - \Lambda_f \right) , \\
-\frac{N_f^r(r)^2}{r^2 A_{0f}(r)} + \frac{A'_{0f}(r)}{r A_{0f}(r)} - \frac{2 N_f^r(r) N_f^r(r)}{r A_{0f}(r)} &= C_0^2 \left( \frac{-\kappa^2}{\alpha^2} p_f(r) - \Lambda_f \right) , \\
\frac{A''_{0f}(r)}{2 r A_{0f}(r)} + \frac{N_f^r(r) N_f''(r)}{A_{0f}(r)} &= C_0^2 \left( \frac{-\kappa^2}{\alpha^2} p_f(r) - \Lambda_f \right) ,
\end{align*}
\]

in the fiducial frame. Substituting all the metric conditions into the metric and constraint equations in MTBG, we find the solution \((21)\). Thus, as well as the vacuum case, the self-accelerating branch in MTBG with Eqs. \((15)\) and \((19)\) allows the static and spherically symmetric GR solutions with matter fields.

C. Regularity at the center of the star

We check the regularity at the center of the star. Since the four-dimensional diffeomorphism invariance of the spacetime is explicitly broken in MTBG in the unitary gauge, even if two metrics in a coordinate system describe a solution in MTBG, these expressed in another coordinate system may not be a solution. Thus, if a coordinate singularity exists in one coordinate system where MTBG allows a solution, it may be a physical singularity in contrast to the case of GR where a coordinate singularity is removed via a coordinate transformation.

A regular static and spherically symmetric metric written in the Schwarzschild coordinates, \( g_{\mu\nu} dx^\mu dx^\nu = -f(T,r) dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^2 \), can be brought to the spatially-flat form \([42]\]

\[
g_{\mu\nu} dx^\mu dx^\nu = -\frac{f}{1-\frac{2m}{r}} dt^2 + \left( dr - \frac{\sqrt{2m f}}{\sqrt{r-2m}} dt \right)^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) ,
\]

by the coordinate transformation \(dT = dt + \frac{\sqrt{2m f}}{\sqrt{r-2m}} dr\). For the regularity in the spatially-flat coordinates, we require the regularity and isotropy at the center of the star \( r = 0 \) in both sectors. Computing the traceless part of the extrinsic curvature tensors \((2)\), we obtain

\[
K^\gamma_{,\gamma} - \frac{1}{3} K = \frac{2}{3N} \left( N^r_{,r} - \frac{N^r}{r} \right) , \quad N^r = -\frac{\sqrt{2m f}}{\sqrt{r-2m}}
\]

Assuming a stellar solution in each sector whose metric and matter field are regular at the center of the star \( r = 0 \) in the Schwarzschild coordinates, the leading behavior around \( r = 0 \) are given by \( f(r) = f_0 + f_2 r^2 + O(r^4) \), and \( m(r) = m_3 r^3 + O(r^5) \) where \( f_0, f_2, \) and \( m_3 \) are constants and hence \( K^\gamma_{,\gamma} - \frac{1}{3} K \sim r^2 \) in the spatially-flat coordinates. Thus, the extrinsic curvature is regular and isotropic in the physical frame. Since the two sectors are decoupled, the same conclusion also holds in the fiducial sector.

**5** We choose the self-accelerating branch since, as in the vacuum case discussed in Sec. III, there is no consistent solution for static and spherically symmetric stars compatible with the constraint conditions satisfying the equivalent of Eq. \((22)\).
V. SOLUTIONS IN THE SMALL $m^2$ EXPANSION

In this section, we construct static and spherically symmetric vacuum solutions perturbatively in the small mass limit of MTBG, $m \to 0$. We regard the graviton mass squared $m^2$ as an expansion parameter, and expand the metric variables and the Lagrange multipliers. We expect that at the leading order $\mathcal{O}(m^0)$ the Schwarzschild-de Sitter metric written in the spatially-flat GP coordinates with the cosmological constants $V$ and $V_f$ and the masses $M$ and $M_f$ in the physical and fiducial frames, respectively, are realized, and at the next-to-leading order $\mathcal{O}(m^2)$ the nontrivial corrections to the GR solutions may be obtained. We investigate whether and how the finite nonzero values $m^2$ can provide the deviation from the Schwarzschild-de Sitter metrics in GR in each sector.

We modify the precursor action (5), so that the GR part of the action also contains the cosmological constants $V$ and $V_f$ in the physical and fiducial sectors, respectively, as

$$\mathcal{L}_{\text{pre}} := \sqrt{-g} (R[g] - 2V) + \bar{\alpha}^2 \sqrt{-f} (R[f] - 2V_f) - m^2 \left( N\sqrt{gH_0} + M\sqrt{\phi H_0} \right),$$

where $V$ and $V_f$ are assumed to be positive constants. We extend the metric ansatz (13) to the case of the spherically symmetric but time-dependent spacetimes, where all functions $A_0$, $A_0f$, $N^r$, $N_f^r$, $A_1$, $A_1f$, $A_2$, and $A_2f$ in Eq. (13) are promoted to the functions of $t$ as well as $r$, such as $A_0(r) \to A_0(t, r)$. Substituting the time-dependent and spherically symmetric metrics of the physical and fiducial sectors into the action (3) and varying it with respect to $A_0$, $A_0f$, $N^r$, $N_f^r$, $A_1$, $A_1f$, $A_2$, and $A_2f$, respectively, we obtain the Euler-Lagrange equations for these metric components. We also take the constraint equations (10), (11), and (12) into consideration. Because of the spherical symmetry of the background spacetimes, we may set $\lambda^0 = \lambda^r = 0$ as Eq. (16). We also promote the nontrivial Lagrange multipliers $\lambda$, $\bar{\lambda}$, and $\lambda'$ to functions of $(t, r)$.

We then expand the metric variables with respect to the Schwarzschild-de Sitter metrics written in the spatially-flat coordinates in terms of $m^2$ as

$$A_0(t, r) = 1 + m^2a_0(t, r) + \mathcal{O}(m^4), \quad N^r(t, r) = \pm \sqrt{\frac{2M}{r} + V + m^2n^r(t, r)} + \mathcal{O}(m^4),$$

$$A_1(t, r) = 1 + m^2a_1(t, r) + \mathcal{O}(m^4), \quad A_2(t, r) = 1 + m^2a_2(t, r) + \mathcal{O}(m^4),$$

$$A_{0f}(t, r) = 1 + m^2a_{0f}(t, r) + \mathcal{O}(m^4), \quad N^r_f(t, r) = \pm \sqrt{\frac{2M_f}{r} + \frac{C_0^2V_f}{3} + m^2n^r_f(t, r)} + \mathcal{O}(m^4),$$

$$A_{1f}(t, r) = 1 + m^2a_{1f}(t, r) + \mathcal{O}(m^4), \quad A_{2f}(t, r) = 1 + m^2a_{2f}(t, r) + \mathcal{O}(m^4),$$

where $a_0(t, r)$, $n^r(t, r)$, $a_1(t, r)$, $a_2(t, r)$, $a_{0f}(t, r)$, $n^r_f(t, r)$, $a_{1f}(t, r)$, and $a_{2f}(t, r)$ are functions of $t$ and $r$. We also expand the Lagrange multipliers $\lambda(t, r)$, $\bar{\lambda}(t, r)$, and $\lambda'(t, r)$ in terms of $m^2$ as

$$\lambda(t, r) = \frac{\Lambda_{-2}(t, r)}{m^2} + \Lambda_0(t, r) + \mathcal{O}(m^2),$$

$$\bar{\lambda}(t, r) = \frac{\bar{\Lambda}_{-2}(t, r)}{m^2} + \Lambda_0(t, r) + \mathcal{O}(m^2),$$

$$\lambda'(t, r) = \frac{\Lambda'_{-2}(t, r)}{m^2} + \Lambda_0(t, r) + \mathcal{O}(m^2),$$

where the leading dependence on $m$ is chosen as $m^{-2}$, since the Lagrange multipliers always appear with $m^2$ in the Lagrangian (5) and may in principle contribute nontrivially at $\mathcal{O}(m^0)$. We consider the radial component of the constraint (12). At $\mathcal{O}(m^2)$, the radial component of Eq. (12) is satisfied for

$$((c_2 + C_0c_1) + \beta(c_3 + c_2C_0))[a_1(t, r) - a_2(t, r) - a_{1f}(t, r) + a_{2f}(t, r) - ra_{2f,r}(t, r) + ra_{2f,r}(t, r)] = 0,$$

which provides the following three cases

1. $a_{1f}(t, r) = a_1(t, r) - a_2(t, r) + a_{2f}(t, r) - ra_{2f,r}(t, r) + ra_{2f,r}(t, r)$,
2. $(c_2 + C_0c_1) + \beta(c_3 + c_2C_0) = 0$ with $\beta C_0 \neq 1$,
3. $(c_2 + C_0c_1) + \beta(c_3 + c_2C_0) = 0$ with $\beta C_0 = 1$, yielding the degenerate condition (19).

In order to satisfy the constraint equations (10) and (11), we consider the self-accelerating branch (See Appendix A for the corresponding analysis of the normal branch). The self-accelerating branch is given by Eq. (19) which automatically satisfies the constraint equations (10) and (11).
In Case 1, at $O(m^2)$, the Euler-Lagrange equations provide the solution for the metrics

$$
\begin{align*}
    a_0(t,r) &= q_{n_0}(t) - b_1(r), \\
    a_1(t,r) &= b_1(r) + a_2(t,r) + ra_{2,r}(t,r), \\
    a_0(t,r) &= q_{n_0}(t) - b_1(r), \\
    a_1(t,r) &= b_1(r) + a_2(t,r) + ra_{2,r}(t,r), \\
    n^r(t,r) &= \sqrt{\frac{3}{12\sqrt{r}(6M + r^3V)}} \\
    &\times r^3(-2C_0^3c_1 - 3C_0^2c_2 + c_4) - 2(6M + 3r + r^3V)b_1(t,r) - 18Ma_2(t,r) - 2r(6M + r^3V)a_{2,r}(t,r) \\
    &+ \frac{\sqrt{3}Mq_{n_r}}{r(6M + r^3V)} + \frac{q_{n_0}(r)}{2} \left[ \frac{2M}{r} + \frac{V}{3r^2} \right] + \frac{r}{2}a_{2,t}(t,r), \\
    n^r_f(t,r) &= \sqrt{\frac{3}{12\alpha^2 r(6M_f + r^3C_0^2V_f)}} \\
    &\times r(3C_0^2c_0 + 2C_0c_1 + c_2) - 2\alpha^2(6M_f + 3r + r^3V_f)b_1(t,r) - 18\alpha^2Ma_2(t,r) - 2\alpha^2r(6M_f + C_0^2r^3V_f)a_{2,r}(t,r) \\
    &+ \frac{\sqrt{3}Mf_{q_{n_r}}}{r(6M_f + r^3C_0^2V_f)} + \frac{q_{n_0}(r)}{2} \left[ \frac{2M_f}{r} + \frac{C_0^2V_f}{3r^2} \right] + \frac{r}{2}a_{2,t}(t,r), \\
    (57)
\end{align*}
$$

and that for the Lagrange multipliers

$$
\begin{align*}
    \Lambda_{-2}(t,r) &= 0, \\
    \bar{\Lambda}_{-2}(t,r) &= d_0(t), \\
    \Lambda_{-2}(t,r) &= 0, \\
    \bar{\Lambda}_0(t,r) &= e_0(t) + \frac{e_1(t)}{r}, \\
    \Lambda_0(t,r) &= 0, \\
    \bar{\Lambda}_0(t,r) &= e_0(t) + \frac{e_1(t)}{r}, \\
    \Lambda_5(t,r) &= 0, \\
    \bar{\Lambda}_5(t,r) &= e_0(t) + \frac{e_1(t)}{r}, \\
    (58)
\end{align*}
$$

where $(q_{n_0}(t), q_{n_0}(t), d_0(t), e_0(t), e_1(t))$ are pure functions of $t$, $b_1(r)$ is a pure function of $r$, and $(q_{n_r}, q_{n_r})$ are integration constants, respectively. We note that the coefficients $e_0(t)$ and $e_1(t)$ are those for the solutions of the Laplace equation in the Euclid space and do not play a physical role. Up to $O(m^2)$, $a_2(t,r)$ and $a_{2,r}(t,r)$ are not determined. The solution describes the Schwarzschild-de Sitter metric with the effective cosmological constants and masses

$$
\begin{align*}
    V_{\text{eff}} &:= V + \frac{m^2}{2} (-2C_0^3c_1 - 3C_0^2c_2 + c_4), \\
    V_{f,\text{eff}} &:= V_f + \frac{m^2}{2\alpha^2} (c_0 + 2c_1C_0^{-1} + c_2C_0^{-2}), \\
    M_{\text{eff}} &:= M (1 + m^2q_{n_r}), \\
    M_{f,\text{eff}} &:= M_f \left(1 + m^2q_{n_r} \right), \\
    (59)
\end{align*}
$$

respectively, written in the non-standard coordinates. $q_{n_r}$ and $q_{n_r}$ correspond to the constant shifts of masses, which can be absorbed into a redefinition of $M$ and $M_f$. The functions $q_{n_0}(t)$ and $q_{n_0}(t)$ correspond to the reparametrization of time in the physical and fiducial sectors, $dt \to \left(1 + \frac{m^2}{2}q_{n_0}(t)\right) dt$ and $dt \to \left(1 + \frac{m^2}{2}q_{n_0}(t)\right) dt$, respectively, and only one of them can be set to vanish. Similarly, up to $O(m^2)$, the functions $a_2(t,r)$ and $a_{2,r}(t,r)$ correspond to the reparametrization of the radial coordinates $r \to r \left(1 + \frac{m^2}{2}a_2(t,r)\right)$ and $r \to r \left(1 + \frac{m^2}{2}a_{2,r}(t,r)\right)$.

In Case 2 with $\beta C_0 \neq 1$, the compatibility with the self-accelerating condition (19) yields

$$
\begin{align*}
    c_2 &= -C_0c_1, \\
    c_3 &= C_0^2c_1, \\
    (60)
\end{align*}
$$
At $O(m^2)$, the Euler-Lagrange equations provide the solution for the metrics

\[
a_0(t, r) = q_{n_0}(t) - b_1(r), \\
a_1(t, r) = b_1(r) + a_2(t, r) + r a_{2,r}(t, r), \\
a_0f(t, r) = q_{n_0f}(t) - b_1f(r), \\
a_1f(t, r) = b_1f(r) + a_{2f}(t, r) + r a_{2f,r}(t, r),
\]

\[
n^r(t, r) = \pm \left[ \frac{\sqrt{3}}{12 \sqrt{r} (6M + r^3 V)} \right] \\
\times \left( r^3 \left( C_0^3 c_1 + c_4 \right) - 2 \left( 6M + 3r + r^3 V \right) b_1(r) - 18Ma_2(t, r) - 2r \left( 6M + r^3 V \right) a_{2,r}(t, r) \right) \\
+ \frac{\sqrt{3} M q_{nr}}{\sqrt{r} (6M + r^3 V)} + \frac{q_{n_0f}(t)}{2} \left[ \frac{2M}{r} + \frac{V}{3r^2} \right] + \frac{r}{2} a_{2,f}(t, r),
\]

\[
n^r_f(t, r) = \pm \left[ \frac{\sqrt{3}}{12 \sqrt{r} (6M_f + r^3 C_0^3 V_f)} \right] \\
\times \left( r^3 \left( C_0^3 c_0 + C_0 c_1 \right) - 2\sqrt{3} (6M_f + 3r + C_0^3 r^3 V_f) b_1f(r) - 18\sqrt{3} M a_{2f}(t, r) - 2\sqrt{3} r \left( 6M_f + C_0^3 r^3 V_f \right) a_{2f,r}(t, r) \right) \\
+ \frac{\sqrt{3} M q_{nrf}}{\sqrt{r} (6M_f + r^3 C_0^3 V_f)} + \frac{q_{n_0f}(t)}{2} \left[ \frac{2M}{r} + \frac{V}{3r^2} \right] + \frac{r}{2} a_{2f,f}(t, r),
\]

and that for the Lagrange multipliers

\[
\Lambda_{-2}(t, r) = 0, \quad \bar{\Lambda}_{-2}(t, r) = c_0(t) + \frac{e_1(t)}{r}, \quad \Lambda_{-2}(t, r) = 0,
\]

(62)

where $(q_{n_0}(t), q_{n_0f}(t), d_0(t), c_0(t), e_1(t))$ are pure functions of $t$, $(b_1(r), b_{1f}(r))$ are pure functions of $r$, and $(q_{nr}, q_{nrf})$ are integration constants, respectively. We note that the coefficients $c_0(t)$ and $e_1(t)$ are those for the solutions of the Laplace equation in the Euclidean space and do not play a physical role. The functions $a_2(t, r), a_{2f}(t, r), \Lambda_0(t, r), \bar{\Lambda}_0(t, r), \Lambda_0(t, r)$, and $\bar{\Lambda}_0(t, r)$ are not determined by the Euler-Lagrange equations up to $O(m^2)$. At the next order of $O(m^6)$, the constraint (12) yields

\[
b_1(r) = b_{1f}(r), \quad \text{or} \quad a_2(t, r) = a_{2f}(t, r).
\]

(63)

The solution describes the Schwarzschild-de Sitter metrics with the effective cosmological constants and masses

\[
V_{\text{eff}} := V + \frac{m^2}{2} \left( C_0^3 c_1 + c_4 \right), \quad V_{f,\text{eff}} := V_f + \frac{m^2}{2\alpha^2} \left( C_0^2 c_0 + C_0 c_1 \right),
\]

\[
M_{\text{eff}} := M \left( 1 + m^2 q_{nr} \right), \quad M_{f,\text{eff}} := M_f \left( 1 + m^2 q_{nrf} \right),
\]

(64)

respectively, written in the non-standard coordinates. On the other hand, the functions $b_1(r)$ and $b_{1f}(r)$ correspond to the freedom to choose radial coordinates for $g_{\mu \nu}$ and $f_{\mu \nu}$, respectively.

In Case 3, all the constraints (10), (11), and (12) are satisfied under the same condition as Eq. (19). At $O(m^2)$,
the Euler-Lagrange equations provide the solution for the metrics
\[ a_0(t, r) = q_{a_0}(t) - b_1(r), \]
\[ a_1(t, r) = b_1(r) + a_2(t, r) + ra_2, r(t, r), \]
\[ a_{0f}(t, r) = q_{a_{0f}}(t) - b_{1f}(r), \]
\[ a_{1f}(t, r) = b_{1f}(r) + a_{2f}(t, r) + ra_{2f, r}(t, r), \]
\[ n^r(t, r) = \pm \left[ \frac{3}{12 \sqrt{r (6M + r^3 V)}} \times \left( r^3 (-2C^2_0C_1 - 3C^2_0c_2 + c_4) - 2 (6M + 3r + r^3 V) b_1(r) - 18M a_2(t, r) - 2r (6M + r^3 V) a_{2, r}(t, r) \right) \right. \]
\[ + \frac{\sqrt{3M q_{av}}}{\sqrt{r (6M + r^3 V)}} + \frac{q_{nu}(t)}{2} \left[ \sqrt{\frac{2M}{r} + \frac{V}{3r^2}} \right] + \frac{r}{2} a_{2, l}(t, r), \]
\[ n^\gamma(t, r) = \pm \left[ \frac{\sqrt{3M q_{av}}}{\sqrt{r (6M_f + r^3 C^2_0V_f)}} \times \left( r^3 (C^2_0c_0 + 2C_0C_1 + c_2) - 2\bar{\alpha}^2 (6M_f + 3r + r^3 V_f) b_1(f(r) - 18\bar{\alpha}^2 M_f a_2(t, r) - 2\bar{\alpha}^2 (6M_f + C^2_0r^3 V_f) a_{2, r}(t, r) \right) \right. \]
\[ + \frac{\sqrt{3M f_{q_{av}}}}{\sqrt{r (6M_f + r^3 C^2_0V_f)}} + \frac{q_{nu}(t)}{2} \left[ \sqrt{\frac{2M_f}{r} + \frac{C^2_0V_f}{3r^2}} \right] + \frac{r}{2} a_{2, l}(t, r), \]
\[ (65) \]
and that for the Lagrange multipliers
\[ \Lambda_\pm(t, r) = 0, \quad \bar{\Lambda}_\pm(t, r) = d_0(t), \quad \Lambda_{\pm}^r(t, r) = 0, \]
\[ \Lambda_0(t, r) = 0, \quad \bar{\Lambda}_0(t, r) = e_0(t) + \frac{e_1(t)}{r}, \]
\[ (66) \]
respectively, where \((q_{a_0}(t), q_{a_{0f}}(t), d_0(t), e_0(t), c_1(t))\) are pure functions of \(t\), \((b_1(r), b_{1f}(r))\) are pure functions of \(r\), and \((q_{av}, q_{av})\) are integration constants, respectively. We note that the coefficients \(e_0(t)\) and \(e_1(t)\) are those for the solutions of the Laplace equation in the Euclid space and do not play a physical role. We also note that \(a_2(t, r), a_{2f}(t, r), \Lambda_{\pm}^r(t, r), \) and \(\Lambda_{\pm}^0(t, r)\) are not determined by the Euler-Lagrange equations up to \(O(m^2)\). The solution describes the Schwarzschild-de Sitter metrics with the effective cosmological constants and masses
\[ V_{\text{eff}} := V + \frac{m^2}{2} (-2C^2_0c_1 - 3C^2_0c_2 + c_4) , \quad V_{f, \text{eff}} := V_f + \frac{m^2}{2\bar{\alpha}^2} (c_0 + 2c_1 C_0^{-1} + c_2 C_0^{-2}) , \]
\[ M_{\text{eff}} := M \left( 1 + m^2 q_{av} \right) , \quad M_{f, \text{eff}} := M_f \left( 1 + m^2 q_{av} \right) , \]
\[ (67) \]
respectively, written in the non-standard coordinates. On the other hand, the functions \(b_1(r)\) and \(b_{1f}(r)\) correspond to the freedom to choose radial coordinates for \(g_{\mu\nu}\) and \(f_{\mu\nu}\), respectively.

VI. NONPERTURBATIVE STATIC AND SPHERICALLY SYMMETRIC VACUUM SOLUTIONS

Having constructed the vacuum static and spherically symmetric solutions perturbatively in the small \(m^2\) limit in the self-accelerating branch \((19) \)\(^{6}\), the perturbative metrics agreed with the Schwarzschild-de Sitter metric written in nonstandard coordinates. This makes us expect that beyond the perturbative construction the Schwarzschild-de Sitter metric written in the nonstandard coordinates should be a solution in the self-accelerating branch of MTBG for a generic value of \(m^2\). Here, we confirm this.

We start with the Schwarzschild-de-Sitter metrics written in the Schwarzschild coordinates,
\[ g_{\mu\nu}dx^\mu dx^\nu = - \left( 1 - \frac{2M}{r} + \frac{V_{\text{eff}}}{3} r^2 \right) dt(t, r)^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{V_{\text{eff}}}{3} r^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) , \]
\[ f_{\mu\nu}dx^\mu dx^\nu = C^2_0 \left[ - \left( 1 - \frac{2M_f}{r} + \frac{C^2_0V_{f, \text{eff}}}{3} r^2 \right) dt_f(t, r)^2 + \frac{dr^2}{1 - \frac{2M_f}{r} + \frac{C^2_0V_{f, \text{eff}}}{3} r^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] , \]
\[ (68) \]
\(^6\)In the normal branch the small \(m^2\) expansion does not allow for the spatially-flat Schwarzschild-de Sitter metrics as the zero-th order solutions. See Appendix A for details.
where the Schwarzschild times $T(t,r)$ and $T_f(t,r)$ in the physical and fiducial sectors, respectively, are related to the MTBG coordinates $(t,r)$ by
\[
T(t,r) := \int h(t) dt + \int g(r) dr, \quad T_f(t,r) := \int h_f(t) dt + \int g_f(r) dr.
\]
(69)

Here, $g(r)$ and $g_f(r)$ are pure functions of $r$, and $h(t)$ and $h_f(t)$ are pure functions of $t$. The Schwarzschild-de Sitter metrics (68) are then written in the form of Eq. (13) by
\[
A_0(t,r) = \frac{3r(3r - 6M - r^3V_{\text{eff}})}{9r^2 - (3r - 6M - r^3V_{\text{eff}})^2} h(t)^2, \quad N'(t,r) = -\frac{(3r - 6M - r^3V_{\text{eff}})^2 g(r) h(t)}{9r^2 - (3r - 6M - r^3V_{\text{eff}})^2 g(r)^2},
\]
\[
A_1(t,r) = \frac{h(t)^2}{A_0(t,r)}, \quad A_2(t,r) = 1,
\]
\[
A_{0f}(t,r) = \frac{3r(3r - 6M_f - r^3C_0^2V_{\text{eff}})}{9r^2 - (3r - 6M_f - r^3C_0^2V_{\text{eff}})^2} g_f(r)^2, \quad N'_f(t,r) = -\frac{(3r - 6M_f - r^3C_0^2V_{\text{eff}})^2 g_f(r) h_f(t)}{9r^2 - (3r - 6M_f - r^3C_0^2V_{\text{eff}})^2 g_f(r)^2},
\]
\[
A_{1f}(t,r) = \frac{h_f(t)^2}{A_{0f}(t,r)}, \quad A_{2f}(t,r) = 1,
\]
(70)

which describe the Schwarzschild-de Sitter solutions written in the nonstandard coordinates. Choosing the self-accelerating branch (19), for $(c_2 + C_0c_1) + \beta(c_3 + C_0c_2) \neq 0$ the radial component of the constraint equation (12) yields
\[
g_f(r) = \pm \sqrt{\frac{9r^2(-6M + 6M_f + r^3(C_0^2V_{\text{eff}} - V_{\text{eff}})) + (3r - 6M_f - r^3C_0^2V_{\text{eff}})(3r - 6M - r^3V_{\text{eff}})^2 g(r)^2}{(3r - 6M_f - C_0^2r^3V_{\text{eff}})\sqrt{3r - 6M - r^3V_{\text{eff}}}}},
\]
(71)

which relates $g_f(r)$ and $g(r)$. We note that $h(t)$ and $h_f(t)$ are not constrained at all by the constraint and metric equations. All the nontrivial components of the metric equations and the rest of the constraint equations are consistently satisfied for
\[
V_{\text{eff}} := V + \frac{m^2}{2C_0^3 c_1 - 3C_0^2 c_2 + c_4}, \quad V_f, V_{\text{eff}} := V_f + \frac{m^2}{2C_0^3 c_1 - 3C_0^2 c_2 + c_4},
\]
(72)

together with the condition (19), which agree with Eq. (59) as well as
\[
\Lambda = 0, \quad \bar{\Lambda} = d_0(t), \quad \Lambda' = 0,
\]
(73)

where $d_0(t)$ is a pure function of $t$.

We now show that the perturbative solution can be recovered in the limit of small $m^2$. We assume the perturbative expansion of the free function $g(r)$ in the $m \to 0$ limit
\[
g(r) = g_0(r) + g_2(r)m^2 + \mathcal{O}(m^4),
\]
\[
h(t) = 1 + \frac{m^2}{2} q_0(t) + \mathcal{O}(m^4), \quad h_f(t) = 1 + \frac{m^2}{2} q_{0f}(t) + \mathcal{O}(m^4),
\]
(74)

where $g_0(r)$ and $g_2(r)$ are pure functions of $r$, and $q_0(t)$ and $q_{0f}(t)$ are functions of $t$, respectively. Requiring that the $\mathcal{O}(m^0)$ part of the metrics can be expressed by the Schwarzschild solution in the spatially-flat coordinates, we find
\[
g_0(r) = \frac{1}{3} \sqrt{\frac{3(6Mr + r^3V_f)}{3r - 6M - r^3V_f}},
\]
\[
g_2(r) = \pm \frac{\sqrt{3r}}{4\sqrt{6Mr + r^3V_f}} \left[ \frac{(2C_0^3 c_1 + 3C_0^2 c_2 - c_4)(6M + 3r + r^3V_f)r^3}{(3r - 6M - r^3V_f)^2} + 2b_1(r) \right],
\]
(75)

where $b_1(r)$ is a pure function of $r$, and the nontrivial components of the Schwarzschild-de Sitter metrics written in
the nonstandard coordinates can be expanded in terms of \( m^2 \) as

\[
A_0(r) = A_{0f}(r) = 1 - b_1(r)m^2 + \mathcal{O}(m^4),
\]

\[
A_1(r) = A_{1f}(r) = 1 + b_1(r)m^2 + \mathcal{O}(m^4),
\]

\[
N^r(r) = \pm \sqrt{\frac{2M}{r} + \frac{r^2V}{3}} + m^2 \left[ \frac{\sqrt{3}}{12} \left( \frac{c_1}{r^3} \right) - 3C_0^2c_1 - 3C_0^2c_2 + c_4 \right] - 2 \left( 6M + 3r + r^3V \right) b_1(r) + \frac{g_{\alpha\beta}(t)}{2} \sqrt{\frac{2M}{r} + \frac{V}{3} + \mathcal{O}(m^4)},
\]

\[
N^f_j(r) = \pm \sqrt{\frac{2M_f}{r} + \frac{r^2C_0^2V_f}{3}} + m^2 \left[ \frac{\sqrt{3}}{12} \left( \frac{c_1}{r^3} \right) - 3C_0^2c_1 + c_2 \right] - 2 \left( 6M_f + 3r + r^3V_f \right) b_{1f}(r) + \frac{g_{\alpha\beta}(t)}{2} \sqrt{\frac{2M_f}{r} + \frac{C_0^2V_f}{3} + \mathcal{O}(m^4)},
\]

which agrees with Eq. (57) with \( q_{\alpha\beta} = q_{\alpha\beta} = 0 \) and \( a_2(t, f) = a_{2f}(t, r) = 0 \). Thus, the solution (70) is the nonperturbative extension of the perturbative solution (57) discussed in the previous section.

As an extension of our analysis in this section, we could start from even more general description of the Schwarzschild-de Sitter metrics in the physical and fiducial sectors

\[
g_{\mu\nu} dx^\mu dx^\nu = - \left( 1 - \frac{2M}{R(t,r)} + \frac{V_{\text{eff}}}{3} R(t,r) \right) dt(t,r)^2 + \frac{dR(t,r)^2}{1 - \frac{2M}{R(t,r)} + \frac{V_{\text{eff}}}{3} R(t,r)^2} + R(t,r)^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),
\]

\[
f_{\mu\nu} dx^\mu dx^\nu = C_0^2 \left[ - \left( 1 - \frac{2M_f}{R_f(t,r)} + \frac{C_0^2V_{\text{eff}}}{3} R_f(t,r) \right) dt_f(t,r)^2 + \frac{dR_f(t,r)^2}{1 - \frac{2M_f}{R_f(t,r)} + \frac{C_0^2V_{\text{eff}}}{3} R_f(t,r)^2} + R_f(t,r)^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],
\]

where \( R(t,r) \) and \( R_f(t,r) \) are the functions of \((t, r)\). We expect that in the self-accelerating branch (19), the transformed metrics written in terms of \((t, r)\) satisfy all the metric and constraint equations with the conditions (71), (72), and (73). Then, \( R(t,r) \) and \( R_f(t,r) \) would correspond to \( a_2(t, r) \) and \( a_{2f}(t, r) \) at \( \mathcal{O}(m^2) \) in the perturbative approach, respectively.

**VII. CONCLUSIONS**

We have investigated static and spherically symmetric solutions in the Minimal Theory of Bigravity (MTBG). Our main focus was whether static and spherically symmetric solutions in general relativity (GR) written in the standard or nonstandard coordinates, e.g., the spatially-flat Gullstrand-Painlevé (GP) coordinates, can also be solutions in MTBG. We have considered both the vacuum solutions and the solutions with matter.

First, we have investigated the existence of vacuum Schwarzschild-de Sitter solutions written in the spatially-flat GP coordinates in MTBG. In the general static and spherically symmetric backgrounds written in the spatially-flat coordinates (47), we have found that all the components of the constraint equation (12) were trivially satisfied. In static and spherically symmetric backgrounds, there could be two branches, namely, the self-accelerating and normal branches. We have shown that in the self-accelerating branch the substitution of the algebraic conditions (15) and (19) and the Schwarzschild-de Sitter metrics written in the spatially-flat coordinates into the gravitational equations of motion in MTBG results in a consistent solution, provided Eq. (21). These results could be extended to the presence of matter in both sectors, thus allowing for static and spherically-symmetric stellar solutions in the self-accelerating branch of MTBG. On the other hand, in the normal branch the Lagrange multiplier \( \lambda \) could not be zero when we chose \( \lambda = 0 \), and hence the Schwarzschild solutions written in the GP coordinates could not satisfy the Euler-Lagrange equations in MTBG, unless the masses of black holes (BHs) vanish.

Since MTBG in the unitary gauge does not enjoy two copies of the four-dimensional diffeomorphism invariance, the absence of certain GR solutions in a particular coordinate system does not necessarily mean their absence in the other coordinates. For instance, when written in the Schwarzschild coordinates, the Schwarzschild-de Sitter solutions with equal ADM masses do satisfy the Euler-Lagrange equations in vacuum MTBG under the condition (35), even in the normal branch. A second example, this one regular at the horizon, is the Schwarzschild-de Sitter solution
written in the slicing (32) as in [44], which is also a solution in the normal branch under the parallel metrics ansatz, i.e. \( f_{\mu\nu} \propto g_{\mu\nu} \) (together with vanishing \( \lambda, \partial_t \lambda \) and \( \lambda^i \)). This normal-branch ansatz provides a solution not only for spherically symmetric configurations, but coincide with vacuum-GR solutions provided a slicing satisfying (32) such as a constant mean curvature slicing is adopted. It remains to be investigated whether these solutions can be obtained via a standard matter collapse, and in such a sense are connected to more generic solutions. Aside from studying matter collapse directly, an avenue of research could be a perturbative detuning of the conditions (32). More in general, although we have provided a large class of physically interesting solutions (especially for the self-accelerating branch), still the most general solutions of MTBG in a spherically symmetric configuration are not known. For instance, non-perturbative solutions with \( \lambda \neq 0 \) (and/or \( \partial_t \lambda \neq 0 \), and/or \( \lambda^i \neq 0 \)) are not known. Although a mathematical general solution seems hard to be found, what we can guess is that these solutions, if existing, would probably differ from vacuum GR solutions.

Finally, we have investigated whether MTBG admits nontrivial static and spherically symmetric solutions besides the GR solutions. We have constructed static and spherically symmetric vacuum solutions perturbatively in the small mass limit of MTBG. We have regarded the graviton mass squared as the expansion parameter, and expanded the metric variables and the Lagrange multipliers. We have shown that in the self-accelerating branch the nontrivial solutions are given by the Schwarzschild-de Sitter metrics written in nonstandard coordinates, while in the normal branch the spatially-flat Schwarzschild-de Sitter solutions are not compatible with the set up of MTBG in the massless limit. We have also confirmed that in the self-accelerating branch the Schwarzschild-de Sitter solutions written in terms of the nonstandard coordinates with a single free function of \( r \) satisfy all the metric and constraint equations in MTBG and correspond to the straightforward nonperturbative extension of the perturbative Schwarzschild-de Sitter solutions in the small mass expansion.

Although we have obtained the Schwarzschild-de Sitter and static spherically symmetric GR stellar solutions in MTBG, the behavior of the linear perturbations against these solutions could be different from that in GR. A linear instability of a GR solution may signal the realization of spontaneous tensorization, which could provide a new way to probe the existence of the graviton mass in the strong gravity regime. It would also be interesting to investigate the existence of GR solutions beyond the cosmological and the spherically symmetric configurations in MTBG, for instance, stationary and axisymmetric solutions. We hope that we will come back to these issues in our future work.

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Appendix A: Non-existence of the spatially-flat Schwarzschild (-de Sitter) solutions in the massless limit of the normal branch of MTBG

In this Appendix, we employ the small \( \lambda^2 \) expansion in the normal branch of MTBG with \( C_2^0 c_1 + 2C_0 c_2 + c_3 \neq 0 \) and show the nonexistence of Schwarzschild BH solutions written in the spatially-flat coordinates in the \( \lambda^2 \to 0 \) limit. While we mainly focus on the case where the background spacetimes at \( O(m^0) \) are given by the Schwarzschild metrics with \( V = V_f = 0 \) in Eq. (54), the extension to the Schwarzschild-de Sitter metrics with \( V \neq 0 \) and \( V_f \neq 0 \) is straightforward.

At \( O(\lambda^2) \), the combinations of the Euler-Lagrange equations for \( N^r \) and \( N^f \) provide the general solution

\[
\Lambda_{-2}(t, r) = \lambda_0(t), \quad \bar{A}_{-2}(t, r) = d_1(t) + \frac{d_2(t)}{r} + d_3(t) r^2,
\]

where \( \lambda_0(t), d_1(t), d_2(t), d_3(t) \) are pure functions of the time \( t \). At \( O(m^0) \), the Euler-Lagrange equations for \( \lambda_0 \) and \( A_{0f} \) uniquely fix

\[
\lambda_0(t) = 0, \quad d_3(t) = 0.
\]
At $\mathcal{O}(m^0)$, a combination of the Euler-Lagrange equations for $A_1$ and $A_{1f}$ reduces to
\begin{equation}
(C_0^2 c_1 + 2C_0 c_2 + c_3) \left(\sqrt{M} + C_0 \sqrt{M_f}\right) d_2(t) = 0,
\end{equation}
which with $C_0^2 c_1 + 2C_0 c_2 + c_3 \neq 0$ imposes
\begin{equation}
d_2(t) = 0.
\end{equation}
Then, at $\mathcal{O}(m^0)$, the Euler-Lagrange equations for $A_1$ and $A_{1f}$ reduce to
\begin{equation}
(c_2 + c_3 + C_0 (c_1 + c_2 \beta)) \Lambda_{-2}(t, r) = 0.
\end{equation}
In Case 1, because of $c_2 + c_3 \beta + C_0 (c_1 + c_2 \beta) \neq 0$, we have to impose
\begin{equation}
\Lambda_{-2}(t, r) = 0.
\end{equation}
At $\mathcal{O}(m^0)$, the above equations (A2), (A4), and (A6) also satisfy the Euler-Lagrange equations for $A_2$ and $A_{2f}$. Thus, at $\mathcal{O}(m^0)$, all the metric Euler-Lagrange equations are satisfied. Finally, at $\mathcal{O}(m^2)$ the constraint equations (10) and (11) provide
\begin{equation}
m^2 (C_0^2 c_1 + 2C_0 c_2 + c_3) \left(\sqrt{M} - \sqrt{M_f}\right) = 0,
\end{equation}
\begin{equation}
m^2 (C_0^2 c_1 + 2C_0 c_2 + c_3) \left(C_0^2 \sqrt{M} + \sqrt{M_f}\right) = 0,
\end{equation}
which results in the no-go result for the existence of the Schwarzschild metrics written in the GP coordinates in the $m^2 \to 0$ limit,
\begin{equation}
M = M_f = 0.
\end{equation}
We note that adding the nonzero cosmological constants $V \neq 0$ and $V_f \neq 0$ also provides the no-go result in the $m^2 \to 0$ limit
\begin{equation}
M = M_f = 0, \quad \sqrt{V} = C_0 \sqrt{V_f}.
\end{equation}
In Case 2, Eq. (A5) is automatically satisfied. Then, at $\mathcal{O}(m^2)$, the constraint equations (10) and (11) provide
\begin{equation}
m^2 \left(\sqrt{M_f} - \sqrt{M}\right) (C_0 \beta - 1) = 0,
\end{equation}
\begin{equation}
m^2 \left(\sqrt{M_f} + C_0^2 \sqrt{M}\right) (C_0 \beta - 1) = 0,
\end{equation}
which for $C_0 \beta \neq 1$ again results in the no-go result for the Schwarzschild solution (A8) written in the spatially flat coordinates. Again, adding the nonzero cosmological constants $V \neq 0$ and $V_f \neq 0$ also provides the no-go result (A9) in the $m^2 \to 0$ limit.

On the other hand, Case 3 with $C_0 \beta = 1$, for which all the constraints (10), (11) and (12) are automatically satisfied, corresponds to the special case of the self-accelerating branch and needs not to be considered here separately. Thus, in the normal branch, both Case 1 and Case 2 result in the no-go result that does not allow the existence of Schwarzschild and Schwarzschild-de Sitter metrics written in the spatially-flat coordinates in the massless limit of MTBG.

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[1] C. M. Will, The Confrontation between General Relativity and Experiment, Living Rev. Rel. **17**, 4 (2014), arXiv:1403.7377 [gr-qc].
[2] E. Berti et al., Testing General Relativity with Present and Future Astrophysical Observations, Class. Quant. Grav. **32**, 243001 (2015), arXiv:1501.07274 [gr-qc].
[3] E. Berti, K. Yagi, H. Yang, and N. Yunes, Extreme Gravity Tests with Gravitational Waves from Compact Binary Coalescences: (II) Ringdown, Gen. Rel. Grav. **50**, 49 (2018), arXiv:1801.03587 [gr-qc].
[4] E. Berti, K. Yagi, and N. Yunes, Extreme Gravity Tests with Gravitational Waves from Compact Binary Coalescences: (I) Inspiral-Merger, Gen. Rel. Grav. **50**, 46 (2018), arXiv:1801.03208 [gr-qc].
[5] R. Abbott et al. (LIGO Scientific, VIRGO, KAGRA), Tests of General Relativity with GWTC-3, (2021), arXiv:2112.06861 [gr-qc].
[6] T. Damour and G. Esposito-Farèse, Nonperturbative strong field effects in tensor - scalar theories of gravitation, Phys.Rev.Lett. **70**, 2220 (1993).
[7] T. Damour and G. Esposito-Farèse, Tensor-scalar gravity and binary pulsar experiments, Phys.Rev. **D54**, 1474 (1996), arXiv:gr-qc/9602056 [gr-qc].
[8] T. Harada, Stability analysis of spherically symmetric star in scalar-tensor theories of gravity, Prog.Theor.Phys. **98**, 359 (1997), arXiv:gr-qc/9706014 [gr-qc].
[9] T. Harada, Neutron stars in scalar tensor theories of gravity and catastrophe theory, Phys.Rev. D57, 4802 (1998), arXiv:gr-qc/9801049 [gr-qc].
[10] J. Novak, Neutron star transition to strong scalar field state in tensor scalar gravity, Phys.Rev. D58, 064019 (1998), arXiv:gr-qc/9806022 [gr-qc].
[11] C. Palenzuela, E. Barausse, M. Ponce, and L. Lehner, Dynamical scalarization of neutron stars in scalar-tensor gravity theories, Phys.Rev. D89, 044024 (2014), arXiv:1310.4481 [gr-qc].
[12] L. Sampson, N. Yunes, N. Cornish, M. Ponce, E. Barausse, A. Klein, C. Palenzuela, and L. Lehner, Projected Constraints on Scalarization with Gravitational Waves from Neutron Star Binaries, Phys. Rev. D 90, 124091 (2014), arXiv:1407.7038 [gr-qc].
[13] P. Pani and E. Berti, Slowly rotating neutron stars in scalar-tensor theories, Phys.Rev. D90, 024025 (2014), arXiv:1405.4547 [gr-qc].
[14] H. O. Silva, C. F. B. Macedo, E. Berti, and L. C. B. Crispino, Slowly rotating anisotropic neutron stars in general relativity and scalar–tensor theory, Class. Quant. Grav. 32, 145008 (2015), arXiv:1411.6286 [gr-qc].
[15] G. Antoniou, A. Bakopoulos, and P. Kanti, Evasion of No-Hair Theorems and Novel Black-Hole Solutions in Gauss-Bonnet Theories, Phys. Rev. Lett. 120, 131102 (2018), arXiv:1711.03390 [hep-th].
[16] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou, and E. Berti, Spontaneous scalarization of black holes and compact stars from a Gauss-Bonnet coupling, Phys. Rev. Lett. 120, 131104 (2018), arXiv:1711.02080 [gr-qc].
[17] D. D. Doneva and S. S. Yazadjiev, New Gauss-Bonnet Black Holes with Curvature-Induced Scalarization in Extended Tensor-Scalar Theories, Phys. Rev. Lett. 120, 131103 (2018), arXiv:1711.01187 [gr-qc].
[18] C. A. R. Herdeiro, E. Radu, N. Sanchís-Gual, and J. A. Font, Spontaneous Scalarization of Charged Black Holes, Phys. Rev. Lett. 121, 101102 (2018), arXiv:1806.05190 [gr-qc].
[19] P. V. P. Cunha, C. A. R. Herdeiro, and E. Radu, Spontaneously Scalarized Kerr Black Holes in Extended Scalar–Gauss–Bonnet Gravity, Phys. Rev. Lett. 123, 011101 (2019), arXiv:1904.09997 [gr-qc].
[20] D. Psaltis, D. Perrodin, K. R. Dienes, and I. Mocioiu, Kerr Black Holes are Not Unique to General Relativity, Phys.Rev.Lett. 100, 091101 (2008), arXiv:0710.4564 [astro-ph].
[21] H. Motolashvili and M. Minamitsuji, General Relativity solutions in modified gravity, Phys. Lett. B781, 728 (2018), arXiv:1804.01731 [gr-qc].
[22] E. Barausse and T. P. Sotiriou, Perturbed Kerr Black Holes can probe deviations from General Relativity, Phys.Rev.Lett. 101, 099001 (2008), arXiv:0803.3433 [gr-qc].
[23] F. M. Ramazanoğlu, Spontaneous growth of vector fields in gravity, Phys. Rev. D96, 064009 (2017), arXiv:1706.01056 [gr-qc].
[24] F. M. Ramazanoğlu, Spontaneous tensorization from curvature coupling and beyond, Phys. Rev. D99, 084015 (2019), arXiv:1901.10009 [gr-qc].
[25] L. Annulli, V. Cardoso, and L. Gualtieri, Electromagnetism and hidden vector fields in modified gravity theories: spontaneous and induced vectorization, Phys. Rev. D100, 084026 (2019), arXiv:1910.02801 [gr-qc].
[26] R. Kase, M. Minamitsuji, and S. Tsujikawa, Neutron stars with a generalized Proca field and spontaneous vectorization, Phys.Rev. D102, 024067 (2020), arXiv:2001.10701 [gr-qc].
[27] M. Minamitsuji, Spontaneous vectorization in the presence of vector field coupling to matter, Phys. Rev. D 101, 104044 (2020), arXiv:2003.11885 [gr-qc].
[28] M. Minamitsuji, Stealth spontaneous spinorization of relativistic stars, Phys. Rev. D 102, 044048 (2020), arXiv:2008.12758 [gr-qc].
[29] E. S. Demirboğa, A. Coates, and F. M. Ramazanoğlu, Instability of vectorized stars, Phys. Rev. D 105, 024057 (2022), arXiv:2112.04269 [gr-qc].
[30] H. O. Silva, A. Coates, F. M. Ramazanoğlu, and T. P. Sotiriou, Ghost of vector fields in compact stars, Phys. Rev. D 105, 024046 (2022), arXiv:2111.13212 [hep-th].
[31] C. de Rham, G. Gabadadze, and A. J. Tolley, Resummation of Massive Gravity, Phys.Rev.Lett. 106, 231101 (2011), arXiv:1011.1232 [hep-th].
[32] D. G. Boulware and S. Deser, Can gravitation have a finite range?, Phys. Rev. D 6, 3368 (1972).
[33] S. Hassan and R. Rosen, Bimetric gravity from ghost-free massive gravity, J. High Energy Phys. 2012 (02), 126, arXiv:1109.3515 [hep-th].
[34] Y. Yamashita, A. De Felice, and T. Tanaka, Appearance of Boulware–Deser ghost in bigravity with doubly coupled matter, Int. J. Mod. Phys. D 23, 1443003 (2014), arXiv:1408.0487 [hep-th].
[35] C. de Rham, L. Heisenberg, and R. H. Ribeiro, Ghosts and matter couplings in massive gravity, bigravity and multigravity, Phys. Rev. D 90, 124042 (2014), arXiv:1409.3834 [hep-th].
[36] A. De Felice and S. Mukohyama, Minimal theory of massive gravity, Phys. Lett. B 752, 302 (2016), arXiv:1506.01594 [hep-th].
[37] A. De Felice and S. Mukohyama, Phenomenology in minimal theory of massive gravity, JCAP 4, 028, arXiv:1512.04008 [hep-th].
[38] A. De Felice and S. Mukohyama, Graviton mass might reduce tension between early and late time cosmological data, Phys. Rev. Lett. 118, 091104 (2017), arXiv:1607.03368 [astro-ph.CO].
[39] R. Hagala, A. D. Felice, D. F. Mota, and S. Mukohyama, Non-linear dynamics of the minimal theory of massive gravity, Astron. Astrophys. 653, A148 (2021), arXiv:2011.14697 [astro-ph.CO].
[41] A. De Felice, S. Mukohyama, and M. C. Pookkillath, Minimal theory of massive gravity and constraints on the graviton mass, JCAP 12 (12), 011, arXiv:2110.01237 [astro-ph.CO].
[42] A. De Felice, F. Larrouturou, S. Mukohyama, and M. Oliosi, Black holes and stars in the minimal theory of massive gravity, Phys. Rev. D 98, 104031 (2018), arXiv:1808.01403 [gr-qc].
[43] A. De Felice, F. Larrouturou, S. Mukohyama, and M. Oliosi, Minimal Theory of Bigravity: construction and cosmology, JCAP 04, 015, arXiv:2012.01073 [gr-qc].
[44] A. De Felice, A. Doll, F. Larrouturou, and S. Mukohyama, Black holes in a type-II minimally modified gravity, JCAP 03, 004, arXiv:2010.13067 [gr-qc].
[45] A. De Felice, S. Mukohyama, and M. C. Pookkillath, Addressing $H_0$ tension by means of VCDM, Phys. Lett. B 816, 136201 (2021), arXiv:2009.08718 [astro-ph.CO].
[46] A. De Felice and S. Mukohyama, Weakening gravity for dark matter in a type-II minimally modified gravity, JCAP 04, 018, arXiv:2011.04188 [astro-ph.CO].
[47] A. De Felice, S. Mukohyama, and M. C. Pookkillath, Static, spherically symmetric objects in Type-II minimally modified gravity, (2021), arXiv:2110.14496 [gr-qc].
[48] A. De Felice, K.-i. Maeda, S. Mukohyama, and M. C. Pookkillath, to appear, .