Some Properties of Fuzzy Anti-Inner Product Spaces

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Abstract:
In this paper, the definition of fuzzy anti-inner product in a linear space is introduced. Some results of fuzzy anti-inner product spaces are given, such as the relation between fuzzy inner product space and fuzzy anti-inner product. The notion of minimizing vector is introduced in fuzzy anti-inner product settings.

Keywords: fuzzy anti-inner product, fuzzy anti-norm, fuzzy inner product.

I. Introduction
Kohli and Kumar, in 1993 [1], introduced the definition of the fuzzy inner product space and fuzzy co-inner product space. In 1997, Alsina et al [2] introduced the ideal of probabilistic inner product space. After that, in 2010, Hasankhain et al. [3] introduced some properties of fuzzy Hilbert spaces and norm of operators. In 2013, the fuzzy real inner product space and its properties were proved by Mukherjee and Bag [4]. Finally, a note on fuzzy Hilbert spaces was introduced by Daraby et al. in 2016 [5].

II. Preliminaries
This section consists of some definitions and results that will be needed later in this paper.

Definition (2.1) [6]
Assume that \( V \) is a linear space over the field \( C \) of complex numbers. A mapping \( M^*:V^2 \times C \rightarrow I \) satisfies the following conditions for all \( x, y, z \) in \( V \) and \( t, s \) in \( C \):

(FIP1) \( M^*(x + y, z, |t| + |s|) \geq \min\{M^*(x, z, |t|), M^*(y, z, |s|)\} \)

(FIP2) \( M^*(x, y, |ts|) \geq \min\{M^*(x, x, |t|^2), M^*(y, y, |s|^2)\} \)

(FIP3) \( M^*(x, y, t) = M^*(y, x, \bar{T}) \)

(FIP4) \( M^*(\alpha x, y, t) = M^*(x, y, \frac{t}{\alpha}) \), \( 0 \neq \alpha \in C \)

(FIP5) for all \( t \in C \setminus R^+, M^*(x, x, t) = 0 \)

(FIP6) \( \forall t > 0, M^*(x, x, t) = 1 \) if and only if \( x = 0 \)

(FIP7) \( M^*(x, x, \cdot): R \rightarrow I \) is a monotonic non-decreasing function of \( R \) and \( \lim_{t \rightarrow \infty} M^*(x, x, t) = 1 \), where \( M^* \) is called a fuzzy inner product function on \( V \) and \( \langle V, M^* \rangle \) is called a fuzzy inner product space.

Definition (2.2) [7]
Let \( V \) be a linear space over a field \( F \). A fuzzy set \( \mathcal{N}:V \times R \rightarrow I \) such that the following holds for all \( u, v \) in \( V \) and \( c \) in \( F \):

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\((\mathcal{N}1)\) for all \(t \in R\) with \(t \leq 0\), \(\mathcal{N}(u, t) = 1\):
\((\mathcal{N}2)\) for all \(t \in R\) with \(t > 0\), \(\mathcal{N}(u, t) = 0\) if and only if \(u = 0\):
\((\mathcal{N}3)\) for all \(t \in R\) with \(t > 0\), \(\mathcal{N}(cu, t) = \mathcal{N}(u, \frac{t}{|c|})\) if \(0 \neq c \in F\):
\((\mathcal{N}4)\) for all \(s, t \in R\), \(\mathcal{N}(u + v, s + t) \leq \max\{\mathcal{N}(u, s), \mathcal{N}(v, t)\}\):
\((\mathcal{N}5)\) \(\mathcal{N}(u, t)\) is a decreasing function of \(t \in R\) and \(\lim_{t \to \infty} \mathcal{N}(u, t) = 0\),
where \(\mathcal{N}\) is said to be a fuzzy anti-norm on \(V\) and \((V, \mathcal{N})\) is called a fuzzy anti-normed linear space.

Later on, the following condition of fuzzy norm will be required:
\((\mathcal{N}6)\) for all \(t \in R\) with \(t > 0\), \(\mathcal{N}(u, t) < 1\) implies \(u = 0\).

**Definition (2.3)** [7]
Let \(\mathcal{N}\) be a fuzzy anti-norm on \(V\) satisfying \((\mathcal{N}6)\). Define
\[
\|u\|_\alpha = \inf\{t > 0 : \mathcal{N}(u, t) < \alpha, \alpha \in (0, 1]\}.
\]

**Theorem (2.4)** [7]
Let \((V, \mathcal{N})\) be a fuzzy anti-normed linear space. Then \(\{\|u\|_\alpha : \alpha \in (0, 1]\}\) is a decreasing family of norms on \(V\).

**III. Fuzzy anti-inner product space**
The definition of fuzzy anti-inner product space on a complex linear space is introduced and some of its results are investigated.

**Definition (3.1)** [8]
Assume that \(V\) is a linear space over the filed \(C\) of complex numbers. Define
\(M^\circ : V^2 \times C \to I\) to be a mapping such that the following holds for all \(x, y\) in \(V\) and \(t, s\) in \(C\):
\(\text{Fa-IP1})\) \(M^\circ(x + y, z, |t| + |s|) \leq \max\{M^\circ(x, z, |t|), M^\circ(y, z, |s|)\}\) (\(\text{Fa-IP2})\)
\(M^\circ(x, y, |t|) \leq \max\{M^\circ(x, x, |t|^2), M^\circ(y, y, |s|^2)\}\) (\(\text{Fa-IP3})\)
\(M^\circ(x, y, t) \leq M^\circ(y, x, t)\) (\(\text{Fa-IP4})\)
\(M^\circ(ax, y, t) \leq M^\circ\left(x, y, \frac{t}{|a|}\right), 0 \neq a \in C\) (\(\text{Fa-IP5})\)
\(M^\circ(x, x, t) = 1, \forall t \in C \setminus R^+\) (\(\text{Fa-IP6})\)
\(\forall t > 0, M^\circ(x, x, t) = 0\) if and only if \(x = 0\) (\(\text{Fa-IP7})\)
\(M^\circ(x, x, \cdot) : R \to I\) is a monotonic non-decreasing function of \(R\) and \(\lim_{t \to \infty} M^\circ(x, x, t) = 0\).

Where \(M^\circ\) is called a fuzzy anti-inner product function on \(V\) and \((V, M^\circ)\) is called a fuzzy anti-inner product space.

**Example (3.2)**
Assume that \((V, <, >)\) is an inner product space over \(C\). A function \(M^\circ : V^2 \times C \to I\) is defined by
\[
M^\circ(x, y, t) = \begin{cases} 
1 & \text{if } t \leq |<x, y>| \\
0 & \text{if } t > |<x, y>| \\
1 & \forall t \in C \setminus R
\end{cases}
\]
Then \(M^\circ\) is a fuzzy anti-inner product space on \(V\).

**Proof:**
(\(\text{Fa-IP1)}\) Consider the following cases:
Case (i) if one of \(|t| \leq |<x, z>|, |s| \leq |<y, z>|\) holds, then
\[\max\{M^\circ(x, z, |t|), M^\circ(y, z, |s|)\} = 1\] and obviously
\[M^\circ(x + y, z, |t| + |s|) = 0 \leq \max\{M^\circ(x, z, |t|), M^\circ(y, z, |s|)\}\]
Case (ii) let \(|t| \leq |<x, z>|\) and \(|s| \leq |<y, z>|\)
\[\Rightarrow |t| + |s| \leq |<x + y, z>|\]
\[\Rightarrow M^\circ(x + y, z, |t| + |s|) = 0 \leq \max\{M^\circ(x, z, |t|), M^\circ(y, z, |s|)\}\]
(\(\text{Fa-IP2})\) We observe that \(|s|^2 > |<x, x>|\) and \(|t|^2 > |<y, y>|\)
\[\Rightarrow |s|^2 \cdot |t|^2 > |<x, x>| \cdot |<y, y>| = ||x||^2 \cdot ||y||^2\]
\[\Rightarrow |s| \cdot |t| > ||x|| \cdot ||y||\]
so (\(\text{Fa-IP2}\) follows.
Next (\(\text{Fa-IP3})\) (\(\text{Fa-IP5}\)) and (\(\text{Fa-IP7}\)) hold obviously.
(\(\text{Fa-IP4}\) If \(t \in (C \setminus R^+)\), then the result is obvious.
For $t \in R^+$, $0 \neq \alpha \in C$, then the property follows from the fact that

\[ \langle \alpha x, y \rangle = |\alpha| \cdot |x, y| \]

(Fa-IP6) If $x = 0 \Rightarrow |x, x| = 0 \Rightarrow \forall t > 0, |\langle x, x \rangle| > t \Rightarrow M^\circ(x, x, t) = 0$

Conversely, if $\forall t > 0, M^\circ(x, x, t) = 0 \Rightarrow \forall t > 0, |\langle x, x \rangle| > t \Rightarrow < x, x > = 0 \Rightarrow x = 0$.

This completes the proof.

**Proposition (3.3) [8]**

Let $(V, M^\circ)$ be a fuzzy anti-inner product space. Then for $x, y, z$ in $V$ and $s, t$ in $C$

(i) $M^\circ(x, y + z, |t| + |s|) \leq M^\circ(x, y, |t|) \lor M^\circ(x, z, |s|)$

(ii) For $\alpha \in C$ and $\alpha \neq 0$, $M^\circ(\alpha x, y, t) = M^\circ(x, \alpha y, t)$

(iii) $\forall t \in R$ and $t > 0$, $M^\circ(0, 0, t) \leq M^\circ(x, y, t)$

**Note (3.4) [8]**

Assume that $M^\circ$ satisfies the condition:

(Fa-IP8) $\forall t > 0, M^\circ(x, x, t^2) < 1 \Rightarrow x = 0$

Let $(V, M^\circ)$ be a fuzzy anti-inner product space, satisfying (Fa-IP8). Then $\forall \alpha \in (0, 1)$, $\|x\|^2 = \lambda \{ t > 0 : M^\circ(x, x, t^2) \leq 1 - \alpha \}$ is a crisp norm on $V$, called the $\alpha$-anti norm on $V$ generated from $M^\circ$.

In the sequel we shall consider the following condition:

(Fa-IP9) $\forall x, y$ in $V$ and $p, q$ in $R$, $M^\circ(x + y, x + y, 2q^2) \lor M^\circ(x - y, x - y, 2p^2) \leq M^\circ(x, x, p^2) \lor M^\circ(y, y, q^2)$

**Theorem (3.5) [8]**

Let $M^\circ$ be a fuzzy anti-inner product on the $V$ defined function $\mathcal{N}$, as follows:

$\mathcal{N}(x, t) = M^\circ(x, x, t^2) \forall t \in R$ and $t > 0$

where $\mathcal{N}$ is a fuzzy anti-norm on $V$.

From now on, if (Fa-IP8) and (Fa-IP9) hold for each $\alpha \in (0, 1)$, then $\|x\|^2 = \lambda \{ t > 0 : M^\circ(x, x, t^2) \leq 1 - \alpha \}$ is an ordinary anti-norm on $V$ satisfying parallelogram law.

So, by using polarization identity, one can get an ordinary inner product, called the $\alpha$-anti-inner product, as follows:

$\langle x, y \rangle_\alpha = X^*_\alpha + l Y^*_\alpha$

where $X^*_\alpha = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$

and $Y^*_\alpha = \frac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2)$, where $\alpha \in (0, 1)$.

**Definition (3.6)**

$V$ is said to be anti-level complete (AL-complete). If $(V, M^\circ)$ is a fuzzy anti-inner product space satisfying (Fa-IP8) for any $\alpha \in (0, 1)$, then every Cauchy sequence converges in $V$ w.r.t the $\alpha$-anti-norm $\|x\|^\circ$ generated by the fuzzy anti-norm $\mathcal{N}$ which is induced by fuzzy anti-inner product $M^\circ$.

**Theorem (3.7) (Minimizing vector)**

Let $(V, M^\circ)$ be a fuzzy anti-inner product space satisfying (Fa-IP8) and (Fa-IP9), let $M (\neq \emptyset)$ be the convex subset of $V$ which is anti-level complete, and let $x \in V$. Then for each $\alpha \in (0, 1)$, $\exists y^*_\alpha \in M$ such that

$m^{(\alpha)}_{y^*_\alpha} = \inf_{y \in M} \{ m^{(\alpha)}_y \}$, where

$m^{(\alpha)}_y = \lambda \{ t \in R^+, \mathcal{N}(x - y, t) \leq 1 - \alpha \}$

$M^\circ$. $\mathcal{N}$ is the fuzzy anti-norm induced by fuzzy anti-inner

**Proof:**

We note that if $(V, M^\circ)$ is a fuzzy anti-inner product space, then for each $\alpha \in (0, 1)$, $(V, \|\cdot\|^\circ_\alpha)$ is a crisp anti-normed linear space satisfying the parallelogram law. Again

$m^{(\alpha)}_y = \|x - y\|^\circ_\alpha$.

Hence the result follows from the corresponding crisp minimization vector theorem in $(V, \|\cdot\|^\circ_\alpha)$.  

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Theorem (3.8)

$M_\circ$ is a fuzzy anti-inner product space on $V$ if and only if $1 - M_\circ$ is a fuzzy inner product space on $V$.

Proof: For $x, y, z$ in $V$ and $t, s$ in $C$

(Fa-IP1) $\max \{ M_\circ(x, y, z, t) , M_\circ(y, z, s) \}$

$= 1 - \max \{ M_\circ(x, z, t), M_\circ(y, z, s) \}$

$= \min \{ M_\circ(x, z, t), M_\circ(y, z, s) \}$

$\leq M_\circ(x + y, z, t + s)$

$\geq 1 - M_\circ(x + y, z, t + s) = M_\circ(x + y, z, t + s)$

(Fa-IP2) $\max \{ M_\circ(x, x, |t|^2), M_\circ(y, y, |t|^2) \}$

$= 1 - \max \{ M_\circ(x, x, |t|^2), M_\circ(y, y, |t|^2) \}$

$= \min \{ M_\circ(x, x, |t|^2), M_\circ(y, y, |t|^2) \}$

$\leq M_\circ(y, x, |st|)$

$\geq 1 - M_\circ(y, x, |st|) = M_\circ(y, x, |st|)$

(Fa-IP3) $M_\circ(x, y, t) = 1 - M_\circ(x, y, t)$

(Fa-IP4) $M_\circ(\alpha x, y, t) = 1 - M_\circ(\alpha x, y, t)$ for $0 \neq \alpha \in C$

$= 1 - M_\circ\left(x, y, \frac{t}{|\alpha|}\right) = M_\circ\left(x, y, \frac{t}{|\alpha|}\right)$

(Fa-IP5) $M_\circ(x, x, t) = 1 - M_\circ(x, x, t) = 1 - 0 = 1 \forall t \in C \setminus R^+$

(Fa-IP6) $M_\circ(x, x, t) = 1 - M_\circ(x, x, t) \iff x = 0$

$= 1 - 1 \iff x = 0 = 0 \iff x = 0$

(Fa-IP7) We have $M_\circ(x, x, \cdot): R \rightarrow I$ is a monotonic non-decreasing function and

$\lim_{t \rightarrow \infty} M_\circ(x, x, t) = 1$

then $M_\circ(x, x, \cdot): R \rightarrow I$ is a monotonic non-increasing function and

$\lim_{t \rightarrow \infty} M_\circ(x, x, t) = 1 - \lim_{t \rightarrow \infty} M_\circ(x, x, t) = 1 - 1 = 0$

Hence $(V, M_\circ)$ is a fuzzy anti-inner product space.

Theorem (3.9)

Let $(V, \mathcal{N})$ be a fuzzy anti-normed linear space. Suppose that for $x, y, z$ in $V$ and $t, s, r$ in $C$

$\max \{ \mathcal{N}(x, |st|), \mathcal{N}(y, |st|) \} \leq \max \{ \mathcal{N}(x, |s|^2), \mathcal{N}(y, |t|^2) \}$

Define $M_\circ: V^2 \times C \rightarrow I$ as $M_\circ(x, y, s + t) = 1$

if $x = y$ and $s + t \in C \setminus R^+$ and elsewhere as

$M_\circ(x, y, s + t) = \mathcal{N}(x, |s|) \lor \mathcal{N}(y, |t|)$

Then $M_\circ$ is a fuzzy anti-inner product on $V$.

Proof: For $x, y, z$ in $V$ and $t, s$ in $C$

(Fa-IP1) $M_\circ(x + y, z, s + t) = M_\circ(x + y, z, |s + t| + 0)$

$= \mathcal{N}(x + y, |s + t|) \lor \mathcal{N}(z, 0)$

$= \mathcal{N}(x + y, |s + t|) \lor \mathcal{N}(z, 0)$

$\leq \max \{ \mathcal{N}(x, |s|^2), \mathcal{N}(y, |t|^2) \}$

(Fa-IP2) $M_\circ(x, y, |st|) = \mathcal{N}(x, |st|) \lor \mathcal{N}(y, |st|)$

$= \max \{ \mathcal{N}(x, |st|), \mathcal{N}(y, |st|) \}$

(Fa-IP3) $M_\circ(x, y, t) = \mathcal{N}(x, |t|) = \mathcal{N}\left(\frac{x}{|t|}\right)$

$= M_\circ\left(\frac{x}{|t|}\right)$

(Fa-IP4) For $\alpha \neq 0$

$M_\circ(\alpha x, y, t) = \mathcal{N}(\alpha x, |t|) = \mathcal{N}\left(\frac{x}{|\alpha|}, \frac{|t|}{|\alpha|}\right) = M_\circ\left(x, y, \frac{|t|}{|\alpha|}\right)$
(Fa-IP5) By definition $\forall t \in C \setminus R^+$, $M^\circ(x,y,t) = 1$
(Fa-IP6) $M^\circ(x,x,t) = 0 \quad \forall t > 0$
$\Leftrightarrow N(x,|t|) = 0 \quad \forall t > 0$
$\Rightarrow x = 0$
Hence $(V,M^\circ)$ is a fuzzy anti-inner product space.

**Theorem (3.10)**

Let $(V,M^\circ)$ be a fuzzy anti-inner product space satisfying (Fa-IP8) and (Fa-IP9) and $<,>_\alpha$ be $\alpha$-anti-inner product $\forall \alpha \in (0,1)$.

Define a function

$M^\circ: V^2 \times C \to I$ as $M^\circ(x,y,s+t) = 1$

if $x = y$ and $t \in C \setminus R^+$ and elsewhere as

$M^\circ(x,y,t) = \Lambda \{ \alpha \in (0,1): |<,>_\alpha | \geq |t| \}$

Then $M^\circ$ is a fuzzy anti-inner product on $V$ if $|<,>_\alpha |$ is a decreasing function of $R$.

Proof: For $x,y,z,in V$ and $t,s$ in $C$,

(Fa-IP1) To prove that $M^\circ(x + y, z, |s| + |t|) \leq \max \{ M^\circ(x, z, |s|), M^\circ(y, z, |t|) \}$

Let $p = M^\circ(x, z, |s|)$ and $q = M^\circ(y, z, |t|)$.

Without loss of generality, assume that $p \leq q$ and let $0 < r < p \leq q$

Then $\exists 0 < \alpha < r$ such that $|< x, z |_\alpha | > |s|$ and

$\exists 0 < \beta < r$ such that $|< y, z |_\beta | > |t|$

Let $0 < \gamma = \alpha \lor \beta < r$. Thus

$|< x, z |_\gamma | > |< x, z |_\alpha | > |s|$

and

$|< y, z |_\gamma | > |< y, z |_\beta | > |t|$

[ Since $|<,>_\alpha |$ is a decreasing function ]

Now $|< x + y, z |_\gamma | = |< x, z |_\gamma | + |< y, z |_\gamma |$

$\leq |< x, z |_\gamma | + |< y, z |_\gamma |$

$> |s| + |t|$

Therefore $M^\circ(x + y, z, |s| + |t|) \leq \gamma < r$, since $r > 0$, thus

$M^\circ(x + y, z, |s| + |t|) \leq \max \{ M^\circ(x, z, |s|), M^\circ(y, z, |t|) \}$

(Fa-IP2) To prove that

$M^\circ(x, y, |st|) \leq \max \{ M^\circ(x, y, |s|^2), M^\circ(x, y, |t|^2) \}$

Let $p = M^\circ(x, y, |s|^2)$ and $q = M^\circ(x, y, |t|^2)$.

Without loss of generality, assume that $p \leq q$ and let $0 < r < p \leq q$

Then $\exists 0 < \alpha < r$ such that $|< x, y |_\alpha | > |s|^2$ and

$\exists 0 < \beta < r$ such that $|< x, y |_\beta | > |t|^2$

Let $0 < \gamma = \alpha \lor \beta < r$. Thus

$|< x, y |_\gamma | > |< x, y |_\alpha | > |s|^2$

and

$|< x, y |_\gamma | > |< x, y |_\beta | > |t|^2$

[ Since $|<,>_\alpha |$ is decreasing function ]

Therefore $|< x, y |_\gamma |^2 > |s|^2.|t|^2 \Rightarrow |< x, y |_\gamma | > |st|$

therefore $M^\circ(x, y, |st|) \leq \gamma < r$, since $r > 0$ is arbitrary, thus

$M^\circ(x, y, |st|) \leq \max \{ M^\circ(x, y, |s|^2), M^\circ(x, y, |t|^2) \}$

(Fa-IP3) For $t \in C$, $M^\circ(x,y,t) = M^\circ(x,y,\overline{t}) = 1$

if $x = y$ and $\forall t \in C \setminus R^+$

Now let $t \in C$ and $x \neq y$, then

$M^\circ(x,y,t) = \Lambda \{ \alpha \in (0,1): |< x, y |_\alpha | \geq |t| \}$

$= \Lambda \{ \alpha \in (0,1): |< x, y |_\alpha | \geq |\overline{t}| \}$

$= M^\circ(x,y,\overline{t})$

(Fa-IP4) For $c \in C$,

$M^\circ(cx,y,t) = \Lambda \{ \alpha \in (0,1): |< cx, y |_\alpha | \geq |t| \}$

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\[ = \bigwedge \{ \alpha \in (0,1): |c| |<x,y>_{\alpha} | \geq |t| \} \]
\[ = \bigwedge \{ \alpha \in (0,1): |<x,y>_{\alpha} | \geq \frac{|t|}{|c|} \} \]
\[ = M^0(x,y, \frac{t}{|c|}) \]

(Fa-IP5) By definition \[ M^0(x,x,t) = 1 \quad \forall \ t \in \mathbb{C}\setminus \mathbb{R}^+ \]

(Fa-IP6) \[ \forall \ t > 0, \ M^0(x,x,t) = 0 \]
\[ \iff \bigwedge \{ \alpha \in (0,1): |<x,x>_{\alpha} | \geq |t| \} = 0 \]
\[ \iff <x,x>_{\alpha} = 0 \]
\[ \iff x = 0 \]

(Fa-IP7) \[ \forall \ t > 0, \ M^0(x,x,t) = \bigwedge \{ \alpha \in (0,1): \|x\|_{\alpha}^2 \geq |t| \} \]
\[ = \bigwedge \{ \alpha \in (0,1): \|x\|_{\alpha} \geq \sqrt{|t|} \} \]
Now \[ t_1 < t_2 \Rightarrow \sqrt{t_1} < \sqrt{t_2} \]
\[ \Rightarrow \{ \alpha \in (0,1): \|x\|_{\alpha} \geq \sqrt{t_1} \} \subset \{ \alpha \in (0,1): \|x\|_{\alpha} \geq \sqrt{t_2} \} \]
\[ \Rightarrow \bigwedge \{ \alpha \in (0,1): \|x\|_{\alpha} \geq \sqrt{t_1} \} \leq \bigwedge \{ \alpha \in (0,1): \|x\|_{\alpha} \geq \sqrt{t_2} \} \]
\[ \Rightarrow M^0(x,x,t_1) \leq M^0(x,x,t_2) \]
Therefore \[ M^0(x,x,\cdot): \mathbb{R}^+ \rightarrow I \] is decreasing and \[ \lim_{t \to \infty} M^0(x,x,t) = 0. \]
Thus \( M^0 \) is a fuzzy anti-inner product on \( V \).

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