GENERICITY OF FILLING ELEMENTS

BRENT B. SOLIE

ABSTRACT. An element of a finitely generated non-Abelian free group \( F(X) \) is said to be filling if that element has positive translation length in every very small minimal isometric action of \( F(X) \) on an \( \mathbb{R} \)-tree. We give a proof that the set of filling elements of \( F(X) \) is exponentially \( F(X) \)-generic in the sense of Arzhantseva and Ol’shanski˘ı. We also provide an algebraic sufficient condition for an element to be filling and show that there exists an exponentially \( F(X) \)-generic subset of filling elements whose membership problem is solvable in linear time.

1. Introduction

Statistical group theory is a fairly new area of mathematics that has generated a great deal of active research in recent years. A typical result in this area usually concerns the probability with which a random group element satisfies a given property. Properties which are held by almost all sufficiently long elements of a given finitely generated group are especially interesting, and we informally refer to such properties as generic.

The earliest occurrence of the notion of genericity appears to be due to Guba [15]. Shortly afterwards, Gromov gave a formal definition in the context of finitely presented groups [14]. In the same paper, Gromov asserts that almost every finitely presented group is hyperbolic, a fact first proved by Ol’shanski˘ı [28] and later by Champetier [10, 11]. Subsequent results in statistical group theory include the work of Arzhantseva [1, 2], Arzhantseva and Ol’shanski˘ı [3], and Ollivier [24, 23, 25, 26]. The surveys by Ghys [13] and Ollivier [27] provide an excellent overview of genericity with a focus on random groups. More recent results in statistical group theory apply the notion of genericity to computational group theory. Some group-theoretic decision problems with high worst-case complexity have been shown to have low complexity on a generic set of inputs [20, 21]. These results have furthered the understanding of the average-case complexity of these problems [18, 19].

The focus of the present paper is to investigate the generic geometric behavior of elements of a finitely generated non-Abelian free group. Our investigation is motivated by the notion of a filling element, first introduced by Kapovich and Lustig as a free group analogue of a filling curve on a surface [17]. Filling curves have been important in the theory of surface groups and may be characterized by a number of equivalent geometric and algebraic properties.

Let \( \Sigma \) be a closed, orientable surface of genus at least two. By a surface group we mean the fundamental group of such a surface \( \Sigma \). Let \( \alpha \) and \( \beta \) be closed curves
The geometric intersection number, denoted \( i(\alpha, \beta) \), is the least number of intersections between members of the free homotopy classes of \( \alpha \) and \( \beta \). If \( \beta \) is such that \( i(\alpha, \beta) > 0 \) for every essential simple closed curve \( \alpha \), then we say that \( \beta \) is a filling curve.

Recall that the dual tree associated to an essential simple closed curve \( \alpha \) on \( \Sigma \) is a simplicial tree equipped with a small minimal isometric action by \( \pi_1(\Sigma) \). (From now on, we will assume all of our group actions on trees are minimal and isometric.) It is well-known that if \( \beta \) is a (not necessarily simple) closed curve on \( \Sigma \), then the translation length of \( \beta \) on \( T_\alpha \), denoted \( ||\beta||_{T_\alpha} \), is equal to \( i(\alpha, \beta) \). Therefore, a closed curve \( \beta \) is filling if and only if it has positive translation length on \( T_\alpha \) for every essential simple closed curve \( \alpha \).

As a consequence of Skora’s duality theorem, any simplicial tree equipped with a small action by \( \pi_1(\Sigma) \) can be collapsed down to a tree \( T_\alpha \) for some essential simple closed curve \( \alpha \). Therefore, a closed curve \( \beta \) is filling if and only if it has positive translation length in every small action of \( \pi_1(\Sigma) \) on \( \mathbb{R} \)-trees.

In [17], Kapovich and Lustig introduce the notion of a filling element as a free group analogue for a filling curve. A filling element is an element \( w \in F(X) \) that has positive translation length in every very small action of \( F(X) \) on an \( \mathbb{R} \)-tree. In the same paper, Kapovich and Lustig prove that almost every prefix of almost every freely reduced right-infinite word on the letters \( X^\pm := X \cup X^{-1} \) is a filling element of \( F(X) \) [17, Theorem 13.6]. This result echoes the work of Bonahon showing that filling is the typical behavior of closed curves on a closed orientable hyperbolic surface [5, 6, 7].

However, the proof of Kapovich and Lustig’s theorem is non-constructive; we would still like to find an algorithmically verifiable sufficient condition for a free group element to be filling. Further, we would like to find such a condition which is also generic in the following formal sense of Arzhantseva and Ol’shanski. Let \( S \) be the set of elements of \( F(X) \) satisfying a given condition. We say that \( S \) is \( F(X) \)-generic if

\[
\lim_{n \to \infty} \frac{|S \cap B_n|}{|B_n|} = 1,
\]

where \( B_n := \{ w \in F(X) : |w|_X \leq n \} \) [21]. If the limit converges to 1 exponentially fast, we say that \( S \) is exponentially \( F(X) \)-generic. We will give a slightly more general definition of genericity in Definition 2.3.

The main result of this paper is:

**Theorem 3.7.** Let \( F(X) \) be a finitely generated non-Abelian free group.

1. Let \( w \in F(X) \). If the stabilizer of \( w \) in \( \text{Aut} F(X) \) is infinite cyclic, then \( w \) is filling.
The set of filling elements of $F(X)$ is exponentially $F(X)$-generic.

There exists an exponentially $F(X)$-generic subset $S$ of $F(X)$ such that every element of $S$ is filling and the membership problem for $S$ is solvable in linear time.

The proof of Theorem 3.7 relies on the construction of an exponentially $F(X)$-generic set $T S'$ constructed to study the generic-case complexity of Whitehead’s algorithm [21]. Let $T S$ be the set of all cyclically reduced elements of $F(X)$ which are not proper powers, whose conjugacy classes are fixed by no relabeling Whitehead automorphism, and whose cyclic length is strictly increased by every non-inner, non-relabeling Whitehead automorphism. (Whitehead automorphisms will be defined formally in Definition 2.4.) The set $T S'$ is defined as the set of all elements of $F(X)$ whose cyclic reductions are elements of $T S$. The set $T S'$ satisfies a number of important properties, outlined in Proposition 3.2.

The other main ingredient in the proof of Theorem 3.7 is an analysis of the structure of vertex groups in elementary cyclic splittings of $F(X)$. The work of Guirardel [16] implies that a non-filling element must be conjugate into such a vertex group. The algebraic structure of these vertex groups, detailed in in Proposition 3.1, implies part (1) of Theorem 3.7. Parts (2) and (3) then follow from part (1) and the properties of the set $T S'$.

2. Preliminaries

By $X$ we will always denote a finite set with $\#X \geq 2$. Let $F(X)$ be the free group on the set of letters $X$.

An $\mathbb{R}$-tree is a geodesic metric space in which any two points are connected by a unique simple path. We continue to assume that every action of $F(X)$ on an $\mathbb{R}$-tree is isometric and minimal. An action of $F(X)$ on an $\mathbb{R}$-tree is very small if the stabilizer of any tripod is trivial and the stabilizer of any arc is either trivial or maximal cyclic in the stabilizers of the endpoints of the arc. For $w \in F(X)$ and $T$ an $\mathbb{R}$-tree on which $F(X)$ acts, then the translation length of $w$ is defined to be $||w||_T := \inf_{p \in T} d_T(p, w(p))$.

**Definition 2.1.** Let $w \in F(X)$. We say that $w$ is a filling element if $||w||_T > 0$ for every very small action of $F(X)$ on an $\mathbb{R}$-tree $T$. We say that $w$ is a non-filling element if $||w||_T = 0$ for some very small action of $F(X)$ on an $\mathbb{R}$-tree $T$.

The work of Guirardel allows us to approximate the very small action of $F(X)$ on a given $\mathbb{R}$-tree by very small actions on a simplicial trees [16]. In particular, if $w \in F(X)$ fixes a point in a very small action on an $\mathbb{R}$-tree, it must also fix a point in a very small action on a simplicial tree. This implies:

**Proposition 2.1.** An element $w \in F(X)$ is non-filling if and only if $||w||_T = 0$ for some very small action of $F(X)$ on a simplicial tree $T$.

Recall that a very small action of $F(X)$ on a simplicial tree gives a decomposition of $F(X)$ as the fundamental group of a graph of groups with cyclic edge groups. We briefly review some of the associated terminology.

**Definition 2.2.** A cyclic splitting of $F(X)$ is the decomposition of $F(X)$ as the fundamental group of a graph of groups with cyclic edge groups. A boundary map in a cyclic splitting is a map from an edge group to a vertex group. A cyclic splitting
of $F(X)$ is \textit{elementary} if the corresponding graph of groups is connected and has exactly one edge. An elementary cyclic splitting of $F(X)$ is a \textit{segment of groups} if it has two distinct vertices and is a \textit{loop of groups} if it has a single vertex. An elementary cyclic splitting is \textit{nontrivial} if it is either a loop of groups, or it is a segment of groups in which neither boundary map is an isomorphism. Given any splitting of $F(X)$, we say that $w \in F(X)$ is \textit{elliptic} with respect to this splitting if $w$ is conjugate to an element of a vertex group. Elements of $F(X)$ which are not elliptic in a given splitting are said to be \textit{hyperbolic}.

We will find it useful to slightly generalize the notion of $F(X)$-genericity presented in the introduction. The following notion of genericity referred to as the \textit{Arzhantseva-Ol’shanskii model of genericity}, and the specific terminology is due to Kapovich, Schupp, and Shpilrain \cite{KSS}.

\textbf{Definition 2.3.} Let $S \subseteq T \subseteq F(X)$. We say that $S$ is $T$-\textit{generic} if

$$\lim_{n \to \infty} \frac{\#(S \cap B_n)}{\#(T \cap B_n)} = 1,$$

where $B_n := \{w \in F(X) : |w|_X \leq n\}$ and $|w|_X$ denotes the word length of the element $w$ with respect to the basis $X$. If the above limit converges exponentially fast, we say that $S$ is \textit{exponentially} $T$-generic. If $T - S$ is (exponentially) $T$-generic, then we say that $S$ is \textit{(exponentially) $T$-negligible}.

Recall that for the standard basis $X$ of $F(X)$, we have a corresponding set of Whitehead automorphisms which act as a finite generating set for $\text{Aut} F(X)$. Each of these Whitehead automorphism falls into one of the two following categories.

\textbf{Definition 2.4.} An automorphism $\sigma : F(X) \to F(X)$ is called a \textit{type I Whitehead automorphism} (or a \textit{relabeling Whitehead automorphism}) if $\sigma$ is induced by a permutation on the set $X^\pm := X \cup X^{-1}$.

An automorphism $\sigma : F(X) \to F(X)$ is called a \textit{type II Whitehead automorphism} if there is an element $a \in X^\pm$ such that for all $x \in X^\pm$, we have $\sigma(x) \in \{x, xa, a^{-1}x, x^a\}$, where $x^a := z^{-1}xa$.

\section{Main Results}

\subsection{Cyclic Splittings of $F(X)$}

The structure of elementary cyclic splittings of free groups has been well-studied. A topological lemma due to Bestvina and Feighn \cite[Lemma 4.1]{BF} can be used to characterize one-relator presentations of $F(X)$, resulting in the following characterization of vertex groups of cyclic splittings. The following proposition may also be obtained from earlier results in \cite{KSS}, \cite{KSS2}, or \cite{KSS3}.

\textbf{Proposition 3.1.} If $F(X)$ has the structure of a nontrivial segment of groups with cyclic edge group, then its vertex groups must have the form $\langle A, b \rangle$ and $\langle B \rangle$, where $A \cup B$ is a basis for $F(X)$, $\#B \geq 2$, and $b \in \langle B \rangle$.

If $F(X)$ has the structure of a nontrivial loop of groups with cyclic edge group, then its vertex group must have the form $\langle U, u^v \rangle$, where $U \cup \{v\}$ is a basis for $F(X)$ and $u \in \langle U \rangle$.

\subsection{The Set $TS'$}

In \cite{KSS}, Kapovich, Schupp, and Shpilrain construct an exponentially $F(X)$-generic set with several important properties related to Whitehead’s algorithm.
**Definition 3.1.** Let \( C \subseteq F(X) \) be the set of cyclically and freely reduced elements of \( F(X) \). The set \( TS \) is the set of \( w \in C \) which are not proper powers, whose cyclic length is increased by every non-inner type II Whitehead automorphism, and whose conjugacy class is fixed by no type I Whitehead automorphism. The set \( TS' \) is the set of elements \( w \in F(X) \) whose cyclic reductions are in \( TS \).

**Proposition 3.2 (21 Theorem 8.5).** Let \( \#X \geq 2 \) and let \( TS' \subseteq F(X) \) be as above.

1. The set \( TS' \) is exponentially \( F(X) \)-generic.
2. For any nontrivial \( w \in TS' \), the stabilizer of \( w \) in \( \text{Aut} F(X) \) is the infinite cyclic group generated by right-conjugation by \( w \).
3. The membership problem for \( TS' \) is solvable in linear time.

We very briefly summarize the arguments presented in [21]. Let \( N := \#X \). Let \( w \in C \) have cyclic length \( n \). For every \( x \in X^\pm \), define \( w_x \) to be the number of occurrences of \( x \) in \( w \). Likewise, for every \( x, y \in X^\pm \) such that \( x \neq y^{-1} \), define \( w_{xy} \) to be the number of occurrences of \( xy \) as a subword of \( w \) written as a cyclic word.

Let \( \epsilon > 0 \). Let \( L(\epsilon) \) be the set of all \( w \in C \) such that for all \( x, y \in X^\pm \) where \( x \neq y^{-1} \), we have

\[
\frac{w_x}{n} \in \left( \frac{1}{2N} - \epsilon, \frac{1}{2N} + \epsilon \right) \quad \text{and} \quad \frac{w_{xy}}{n} \in \left( \frac{1}{2N(2N-1)} - \epsilon, \frac{1}{2N(2N-1)} + \epsilon \right).
\]

If \( \tau \) is a type II Whitehead automorphism, the difference in cyclic length between \( w \) and \( \tau(w) \) is easily calculated in terms of the quantities \( w_x \) and \( w_{xy} \) (see [22, 29]). Let \( 0 < \epsilon < (2N-3)/N(2N-1)(4N-3) \). If \( w \in L(\epsilon) \) for such an \( \epsilon \), the quantities \( w_x \) and \( w_{xy} \) nearly uniformly distributed. It is then fairly straightforward to see that any non-inner type II Whitehead automorphism will necessarily increase the cyclic length of \( w \in L(\epsilon) \).

Using some deep results from large deviation theory, one can further show that the set \( L(\epsilon) \) itself is \( C \)-generic. Since \( L(\epsilon) \) is \( C \)-generic, the set \( L'(\epsilon) \) consisting of elements of \( F(X) \) whose cyclic reductions are in \( L(\epsilon) \) is \( F(X) \)-generic. Although \( L'(\epsilon) \) may contain elements whose conjugacy class is fixed by a type I Whitehead automorphism, the set of such elements is \( F(X) \)-negligible and may be safely discarded to obtain a subset of \( TS' \) which is \( F(X) \)-generic. We conclude that \( TS' \) itself is also \( F(X) \)-generic.

Suppose now that \( w \in TS \). If \( \alpha \in \text{Aut} F(X) \) is such that \( \alpha(w) = w \), recall that Whitehead’s theorem states that \( \alpha \) can be written as a sequence of Whitehead automorphisms \( \alpha = \tau_m \circ \cdots \circ \tau_1 \) such that \( \tau_{i+1} \) does not increase the cyclic length of \( \tau_i \circ \cdots \tau_1(w) \). Since \( w \in TS \) and \( TS \) is closed under type I Whitehead automorphisms and cyclic permutations, each \( \tau_i \) must be either a type I Whitehead automorphism or it must cyclically permute \( w \). We may therefore write \( \alpha = \gamma \circ \sigma \), where \( \sigma \) is a type I Whitehead automorphism and \( \gamma \) is an inner automorphism.

Since the conjugacy class of \( w \) cannot be fixed by a type I Whitehead automorphism, \( \sigma \) must be trivial, and so \( \alpha \) itself must be inner. It follows directly that \( \alpha \) can only be conjugation by some power of \( w \), since \( w \) is not a proper power itself. A slight modification to this argument shows that any \( w' \) whose cyclic reduction is \( w \) also has a cyclic stabilizer in \( \text{Aut} F(X) \).

To algorithmically decide whether an element \( w' \) belongs to \( TS' \), we first pass to the cyclic reduction of \( w' \), denoted by \( w \), in time linear in the length of \( w' \). We then decide whether \( w \in TS \). It is well-known how to check whether \( w \) is a proper power
Define \( \tau \) elementary cyclic splitting if and only if \( \tau \) fixes exactly \( \langle w \rangle \), while \( \tau \) fixes \( \langle w, B \rangle \). Thus \( \tau \) must be distinct from every power of \( \tau \), so the Aut \( F(X) \) stabilizer of \( w \) cannot be cyclic.

If \( w = z^r \) where \( r > 1 \) and \( z \) is not a proper power, then \( z \) is elliptic in an elementary cyclic splitting if and only if \( w \) is elliptic in that same splitting. We may therefore pass from \( w \) to its root \( z \), which is also elliptic in the given splitting. The argument above shows that \( z \) has a non-cyclic stabilizer, and since the stabilizer of \( w \) contains that of \( z \), \( w \) must have non-cyclic stabilizer in Aut \( F(X) \) as well. \( \square \)

Since the set \( TS' \) is an exponentially \( F(X) \)-generic set whose elements all have cyclic stabilizers in Aut \( F(X) \), any set consisting of elements with non-cyclic stabilizers is exponentially \( F(X) \)-negligible.

**Corollary 3.4.** The set of elements of \( F(X) \) which lie in a proper free factor of \( F(X) \) is exponentially \( F(X) \)-negligible.

**Remark.** This is a slight generalization of results appearing in [8] and [9], which show that the set of primitive elements of \( F(X) \) is \( F(X) \)-negligible.

**Lemma 3.5.** Let \( w \in F(X) \) be elliptic in some elementary cyclic splitting of \( F(X) \). Then \( w \) has a non-cyclic stabilizer in Aut \( F(X) \).

**Proof.** Suppose that \( w \) is not a proper power.

Let \( w \in F(X) \) be elliptic in a segment of groups. Then there must exist a basis \( A \sqcup B \) of \( F(X) \) such that \( \#A \geq 1, \#B \geq 2, b \in \langle B \rangle \), and either \( w \in \langle A, b \rangle \) or
GENERICITY OF FILLING ELEMENTS

Note that if $b$ is a proper power of some $c \in F(X)$, then we would have $w \in \langle A, c \rangle$, so $w$ would remain elliptic in a splitting of the same type. Hence we may assume that $b$ is not a proper power. Define an automorphism $\sigma : F(X) \to F(X)$ by

$$\sigma(y) = \begin{cases} y, & \text{if } y \in A \\ y^b, & \text{if } y \in B. \end{cases}$$

Any power of $\sigma$ fixes the rank 2 subgroup $\langle A, b \rangle$ pointwise and so also fixes $w$, whereas right-conjugation by $w$ fixes exactly the cyclic subgroup $\langle w \rangle$. Right-conjugation by $w$ must therefore differ from every power of $\sigma$, so the stabilizer of $w$ in $\text{Aut} \ F(X)$ cannot be cyclic.

If $w \in \langle B \rangle$, since $|A| \geq 1$, $w$ lies in a proper free factor of $F(X)$. Lemma 3.3 states that such an element has a non-cyclic stabilizer in $\text{Aut} \ F(X)$.

Let $w \in F(X)$ be elliptic in a loop of groups. There then exists a basis $U \sqcup \{v\}$ of $F(X)$ such that $w \in \langle U, u^v \rangle$ for some $u \in \langle U \rangle$. We define the map $\tau : F(X) \to F(X)$ by

$$\tau(y) = y, \text{ if } y \in U$$

$$\tau(v) = uv.$$

Since $u \in \langle U \rangle$, $\tau$ is an automorphism. In particular, $\tau$ fixes the subgroup $\langle U, u^v \rangle$ pointwise, so no power of $\tau$ equals right-conjugation by $x$, which fixes only the cyclic subgroup $\langle w \rangle$. Again, the stabilizer of $w$ in $\text{Aut} \ F(X)$ therefore cannot be cyclic.

We handle the case where $w$ is a proper power in the same way it was handled in the proof of Lemma 3.3.

Lemma 3.6. If $w \in TS'$, then $w$ is filling.

Proof. We consider instead the set of non-filling elements in $F(X)$. By Proposition 2.1, a non-filling element $w$ fixes a point in some simplicial tree equipped with a very small action by $F(X)$. This in turn gives a splitting of $F(X)$ over infinite cyclic and trivial groups. Since $w$ fixes a point in the action, $w$ must be elliptic in this splitting. We may choose any edge in the splitting and collapse the components of its complement down to vertices, thereby obtaining an elementary splitting of $F(X)$ over an infinite cyclic or trivial group in which $w$ is elliptic. Since $w$ is elliptic in such a splitting, by Lemmas 3.3 and 3.5, the stabilizer of $w$ in $\text{Aut} \ F(X)$ does not contain a non-cyclic element. Hence $w$ cannot be filling.

Theorem 3.7. Let $F(X)$ be a finitely generated non-Abelian free group.

(1) Let $w \in F(X)$. If the stabilizer of $w$ in $\text{Aut} \ F(X)$ is infinite cyclic, then $w$ is filling.

(2) The set of filling elements of $F(X)$ is exponentially $F(X)$-generic.

(3) There exists an exponentially $F(X)$-generic subset $S$ of $F(X)$ such that every element of $S$ is filling and the membership problem for $S$ is solvable in linear time.

Proof. Part (1) follows from Lemmas 3.3 and 3.5. Since every element of $TS'$ has a cyclic stabilizer in $\text{Aut} \ F(X)$ (Proposition 3.2 part (1)), every element of $TS'$ must be filling. Part (2) then follows from the fact that $TS'$ is exponentially
$F(X)$-generic (Proposition 3.2, part (2)). Finally, Part (3) follows from part (3) of Proposition 3.2 taking $S$ to be exactly $TS'$.

4. Acknowledgements

The author would like to thank Ilya Kapovich for his guidance and many valuable comments and Enric Ventura for a helpful discussion via email.

References

[1] G. N. Arzhantseva. Generic properties of finitely presented groups and Howson’s theorem. Comm. Algebra, 26(11):3783–3792, 1998.
[2] G. N. Arzhantseva. A property of subgroups of infinite index in a free group. Proc. Amer. Math. Soc., 128(11):3205–3210, 2000.
[3] G. N. Arzhantseva and A. Yu. Ol’shanskii. Generality of the class of groups in which subgroups with a lesser number of generators are free. Mat. Zametki, 59(4):489–496, 638, 1996.
[4] M. Bestvina and M. Feighn. Outer limits (preprint), 1994.
[5] Francis Bonahon. Bouts des variétés hyperboliques de dimension 3. Ann. of Math. (2), 124(1):71–158, 1986.
[6] Francis Bonahon. The geometry of Teichmüller space via geodesic currents. Invent. Math., 92(1):139–162, 1988.
[7] Francis Bonahon. Geodesic currents on negatively curved groups. In Arboreal group theory (Berkeley, CA, 1988), volume 19 of Math. Sci. Res. Inst. Publ., pages 143–168. Springer, New York, 1991.
[8] Alexandre V. Borovik, Alexei G. Myasnikov, and Vladimir Shpilrain. Measuring sets in infinite groups. In Computational and statistical group theory (Las Vegas, NV/Hoboken, NJ, 2001), volume 298 of Contemp. Math., pages 21–42. Amer. Math. Soc., Providence, RI, 2002.
[9] J. Burillo and E. Ventura. Counting primitive elements in free groups. Geom. Dedicata, 93:143–162, 2002.
[10] Christophe Champetier. Petite simplification dans les groupes hyperboliques. Ann. Fac. Sci. Toulouse Math. (6), 3(2):161–221, 1994.
[11] Christophe Champetier. Propriétés statistiques des groupes de présentation finie. Adv. Math., 116(2):197–262, 1995.
[12] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. Invent. Math., 84(1):91–119, 1986.
[13] Étienne Ghys. Groupes aléatoires (d’après Misha Gromov, . . .). Astérisque, (294):viii, 173–204, 2004.
[14] M. Gromov. Hyperbolic groups. In Essays in group theory, volume 8 of Math. Sci. Res. Inst. Publ., pages 75–263. Springer, New York, 1987.
[15] V. S. Guba. Conditions under which 2-generated subgroups in small cancellation groups are free. Izv. Vyssh. Uchebn. Zaved. Mat., (7):12–19, 87, 1986.
[16] Vincent Guirardel. Approximations of stable actions on $\mathbb{R}$-trees. Comment. Math. Helv., 73(1):89–121, 1998.
[17] Ilya Kapovich and Martin Lustig. Intersection form, laminations and currents on free groups. Geom. Funct. Anal., 19(5):1426–1467, 2010.
[18] Ilya Kapovich, Alexei Miasnikov, Paul Schupp, and Vladimir Shpilrain. Generic-case complexity, decision problems in group theory, and random walks. J. Algebra, 264(2):665–694, 2003.
[19] Ilya Kapovich, Alexei Miasnikov, Paul Schupp, and Vladimir Shpilrain. Average-case complexity and decision problems in group theory. Adv. Math., 190(2):343–359, 2005.
[20] Ilya Kapovich and Paul Schupp. Genericity, the Arzhantseva-Ol’shanskii method and the isomorphism problem for one-relator groups. Math. Ann., 331(1):1–19, 2005.
[21] Ilya Kapovich, Paul Schupp, and Vladimir Shpilrain. Generic properties of Whitehead’s algorithm and isomorphism rigidity of random one-relator groups. Pacific J. Math., 223(1):113–140, 2006.
[22] Roger C. Lyndon and Paul E. Schupp. Combinatorial group theory. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.
[23] Y. Ollivier. Sharp phase transition theorems for hyperbolicity of random groups. Geom. Funct. Anal., 14(3):595–679, 2004.
[24] Yann Ollivier. Critical densities for random quotients of hyperbolic groups. C. R. Math. Acad. Sci. Paris, 336(5):391–394, 2003.
[25] Yann Ollivier. Cogrowth and spectral gap of generic groups. Ann. Inst. Fourier (Grenoble), 55(1):289–317, 2005.
[26] Yann Ollivier. Effondrement de quotients aléatoires de groupes hyperboliques avec torsion. C. R. Math. Acad. Sci. Paris, 341(3):137–140, 2005.
[27] Yann Ollivier. A January 2005 invitation to random groups, volume 10 of Ensaios Matemáticos [Mathematical Surveys]. Sociedade Brasileira de Matemática, Rio de Janeiro, 2005.
[28] A. Yu. Ol’shanskiĭ. Almost every group is hyperbolic. Internat. J. Algebra Comput., 2(1):1–17, 1992.
[29] Abdó Roig, Enric Ventura, and Pascal Weil. On the complexity of the Whitehead minimization problem. Internat. J. Algebra Comput., 17(8):1611–1634, 2007.
[30] Abe Shenitzer. Decomposition of a group with a single defining relation into a free product. Proc. Amer. Math. Soc., 6:273–279, 1955.
[31] J. R. Stallings. Free groups which are free products with cyclic amalgamations. Notices A.M.S., 1(1):49, 1980.
[32] G. A. Swarup. Decompositions of free groups. J. Pure Appl. Algebra, 40(1):99–102, 1986.

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green St., Urbana, IL 61801
E-mail address: solie@math.uiuc.edu