Non-commutative Cartier operator and 
Hodge-to-de Rham degeneration

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Introduction

When one thinks about differential forms and the de Rham complex in the context of homological algebra and algebraic geometry, one usually considers de Rham cohomology and treats it as one more cohomology theory for algebraic varieties, similar to Betti cohomology, étale cohomology, crystalline cohomology in positive characteristic, and so on. However, there is an alternative point of view which has been developing slowly over the years and recently became quite prominent.

Since the pioneering paper [HKR], it has been known that a differential form on the spectrum of a smooth commutative algebra can be treated as its Hochschild homology class. In 1983, it was discovered, simultaneously and independently in [C], [FT1], [LQ], that the de Rham differential also has such an algebraic interpretation. Moreover, and most surprisingly, the whole formalism makes sense for an associative, but not necessarily commutative algebra $A$. The de Rham cohomology appears in this setting as additive $K$-theory ([FT2]) or cyclic homology ([C]). Hochschild and cyclic homology are related by a spectral sequence generalizing the commutative Hodge to de Rham spectral sequence. For an introduction to the subject, we refer the reader to [L] (and also to [FT2], which contains much material not covered in [L]).

A natural next step in this direction would be to study the Hodge theory – since the standard spectral sequence relating Hodge cohomology and de Rham cohomology makes sense for a non-commutative algebra $A$, under what assumptions does this spectral sequence degenerate? Recently there has been much interest in this topic – most notably, in the work of M. Kontsevich, who has stated and popularized a conjectural non-commutative analog on the standard degeneration theorem for smooth projective algebraic varieties ([K1], [K2]). He also gave some very beautiful applications, and formulated a set of very precise finiteness assumptions which are necessary for the degeneration and play the role of both properness and smoothness in the non-commutative setting.

To the best of our knowledge, so far there has been very little progress in proving Kontsevich’s conjecture. This is perhaps not very surprising, since in the non-commutative setting, the usual analytic approach to Hodge theory makes no sense. Fortunately, there is an alternative algebraic approach. It was discovered by P. Deligne and L. Illusie [DI] back in 1987, a few years
after the cyclic homology was discovered. The argument uses reduction to positive characteristic and the Cartier isomorphism between de Rham cohomology spaces and spaces of forms for smooth varieties over a field of positive characteristic.

The goal of the present paper is to apply the Deligne-Illusie method in the non-commutative setting and to prove that the non-commutative Hodge to de Rham spectral sequence also degenerates, provided some natural finiteness conditions are met. We do this by giving a version of the Cartier isomorphism valid without any assumptions of commutativity. The final result is a direct generalization of [DI]. In particular, it reduces to [DI] for smooth proper commutative algebraic varieties (except that our general construction of the Cartier isomorphism is somewhat more canonical and does not use any explicit formulas at all).

Our approach generally follows that of Kontsevich, but differs from his in one important point. Kontsevich gave his conjecture in characteristic 0, and he used the language of $A_\infty$-algebras. Our method is reduction to positive characteristic, and the notion of $A_\infty$-algebra is not very convenient there (because it uses the “naive”, not simplicial tensor structure on the category of complexes of vector spaces). The same goes for differential-graded algebras. Instead, we give and prove the statement for sheaves of usual associative algebras over an arbitrary site. This includes both the case of usual algebraic varieties, possibly considered with some non-commutative enhancement of the structure sheaf, and the case of simplicial or cosimplicial algebras. Most likely, the proper generality for the theory is in any case that of a triangulated category equipped with some enriched structure; however, this should be the topic of further research. In one respect at least, $A_\infty$ approach is definitely better: when working with sheaves, in order to be able to reduce the problem in characteristic 0 to a problem in positive characteristic, we have to impose additional assumptions – for instance, to assume that the algebra sheaf in question is Noetherian. This is probably too strong, and indeed, the finiteness assumptions needed anyway for degeneration should also be sufficient for the reduction problem. In a subsequent paper, we will investigate the relation between our approach and the $A_\infty$ statement, and we hope to be able to prove Kontsevich’s conjecture in full generality.

We note that even in the commutative case, one might get a statement by our method for at least some proper algebraic varieties that are not smooth (the Hochschild homology in this case should be understood as the Hochschild homology of the category of perfect complexes of coherent sheaves). We do not know whether this has any geometric significance.
Also, it is known that the Deligne-Illusie method allows one to prove several strong vanishing theorems of the Kodaira type, see [EV]; we do not know whether our non-commutative version can be used in a similar way.

Another intriguing observation concerns the non-commutative Cartier isomorphism. In a sense, and this may be made quite precise, the Cartier isomorphism is only the visible part of an iceberg, which is the action of the Frobenius map on crystalline cohomology. In the non-commutative setting, there is certainly no Frobenius map. However, our results suggest that the Frobenius action on cohomology still exists. Among the many and varied applications of this Frobenius action, some might also make sense in the non-commutative world.

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1 A general overview.

1.1 Recollection on the commutative case. We start by briefly recalling the setup and the method of [DI]. Let $X$ be a smooth algebraic variety over a field $k$. Then the cotangent sheaf $\Omega(X\!/k)$ is flat, so that we have the sheaves $\Omega^i$ of higher-degree differential forms, and the de Rham differential $d : \Omega^i(X) \to \Omega^{i+1}(X)$. One considers the de Rham complex $\langle \Omega^i, d \rangle$, and one equips it with the so-called stupid filtration:

$$F^i \Omega^j(X) = \begin{cases} \Omega^j(X), & j \geq i, \\ 0, & \text{otherwise}. \end{cases}$$

It is known that if $X$ is projective over $k$, and $\text{char } k = 0$, then the spectral sequence associated to this stupid filtration degenerates, so that the de Rham cohomology groups $H^i_{dR}(X)$ of the variety $X$ are isomorphic to the Hodge cohomology groups $H^p(X)$, $H^p(X) = (X, \Omega^p(X))$.

The condition $\text{char } k = 0$ is essential: a long time ago D. Mumford constructed a counterexample, already in dim 2, which shows that the Hodge-
to-de Rham spectral sequence does not always degenerate in positive characteristic.

Nevertheless, Deligne and Illusie give a proof of the Hodge-to-de Rham degeneration by using reduction to positive characteristic; at the same time, they gain an understanding of the situation in \text{char} \, p and show that the degeneration does hold when one additional natural condition is imposed.

Namely, assume that \( k \) is a perfect field, \( \text{char} \, k = p \) is a positive prime, and assume that \( \text{dim} \, X < p \). Then the de Rham complex \( \Omega^* (X) \) is perfectly well-defined, but it is not exact even analytically, in a formal neighborhood of a point: it has been shown by P. Cartier that there exists a canonical isomorphism

\[
C : \mathcal{H}^*_{\text{DR}}(X) \cong \Omega^* (X^{(1)})
\]

between the de Rham cohomology sheaves \( \mathcal{H}^*_{\text{DR}}(X) \) of the variety \( X \) and the sheaves \( \Omega^* (X^{(1)}) \) of forms on the Frobenius twist \( X^{(1)} \) (under our assumptions, \( X^{(1)} \) is isomorphic to \( X \) equipped with the structure sheaf \( \mathcal{O}_X^p \), the subsheaf of \( p \)-th powers in \( \mathcal{O}_X \)). In particular, the kernel of the de Rham differential \( d : \mathcal{O}_X \rightarrow \Omega^1(X) \) is precisely the subsheaf \( \mathcal{O}_X^p \), so that the de Rham differential is \( \mathcal{O}_X^p \)-linear (and Zariski topology is fine enough for all computations with the de Rham complex). The Cartier isomorphism \( \mathcal{H}^0_{\text{DR}}(X) \cong \Omega_X^{(1)} \) is inverse to the Frobenius map.

The crucial part of [DI] is concerned with the following question: when does the Cartier isomorphism \( C \) or rather, the inverse map \( C^{-1} \) extend to a map of complexes? This can be rephrased as follows. Consider the canonical filtration \( \tau \) on the de Rham complex \( \Omega^*(X) \) – namely, let

\[
\tau_{\leq i} \Omega^j(X) = \begin{cases} 
\Omega^j(X), & j < i, \\
\text{Ker} \, d, & j = i, \\
0, & \text{otherwise}.
\end{cases}
\]

Then the associated graded quotient with respect to this canonical filtration is naturally quasiisomorphic to the sum \( \bigoplus_i \mathcal{H}^i_{\text{DR}}(X)[-i] \cong \bigoplus_i \Omega^i (X^{(1)})[-i] \). We want to know when the de Rham complex is quasiisomorphic to its associated graded quotient – in other words, when the canonical filtration splits.

The main result of [DI] claims that – and this is a purely local statement – this happens if and only if the variety \( X \) admits a lifting to a variety smooth over the second Witt vectors ring \( W_2(k) \) (and moreover, these splittings satisfying some natural conditions are in one-to-one correspondence with liftings of \( X \) to \( W_2(k) \)).
The rest of the proof is a surprisingly easy corollary of this basic fact. The canonical filtration induces a filtration on de Rham cohomology groups $H'_{DR}(X)$ known as conjugate filtration, and a spectral sequence called conjugate spectral sequence – its first term consists of the same Hodge cohomology groups $H^{p,q}(X) \cong H^{p,q}(X^{(1)})$ as for the Hodge spectral sequence, but the conjugate and the Hodge filtrations go in the opposite direction. By the cited main result of [DI], if $X$ can be lifted to $W_2(k)$, then the conjugate spectral sequence degenerates. But if $X$ is projective – or in fact just proper – over $k$, all these spectral sequences consist of finite-dimensional $k$-vector spaces. Thus we have two spectral sequences of finite-dimensional vector spaces with the same term $E^1$ and the same term $E^\infty$: for dimension reasons, if one degenerates, the other must degenerate, too.

1.2 Cyclic approach. Assume now that $X = \text{Spec} A$ is affine. The Hochschild homology $HH_*(A)$ of the $k$-algebra $A$ is by definition the Tor-groups of the diagonal $A$-bimodule with itself: we have

$$HH_*(A) = \text{Tor}_{A^{opp} \otimes A}(A, A).$$

It has been established in [HKR] that under the assumptions above – $X$ smooth and char $k = 0$ – we have $HH_l(A) \cong \Omega^l(A)$ for any $l \geq 0$. In positive characteristic, this is also true provided $l < \text{char} k$ – or always, if one requires dim $X < \text{char} k$.

The Hochschild homology of any algebra can be computed by the bar resolution – this results in the well-known standard complex consisting of tensor powers $A^\otimes l$, with a certain explicit differential $b : A^\otimes l+1 \to A^\otimes l$. The cyclic homology $HC_*(A)$ has several equivalent definitions; by one of them, $HC_*(A)$ is the total homology of a periodic bicomplex

$$A \xrightarrow{B} A^\otimes 2 \xrightarrow{B} A^\otimes 3 \xrightarrow{B} A^\otimes 4 \xrightarrow{B} A^\otimes 5 \xrightarrow{B} A^\otimes 6 \xrightarrow{B} A^\otimes 7 \xrightarrow{B} \cdots$$

Here $b$ is the Hochschild differential, and $B$ is the new differential introduced by A. Connes. In our situation, if one takes the vertical cohomology, one
obtains the complex

\[
\begin{array}{cccc}
A & & & A \\
\uparrow & & & \uparrow 0 \\
A & \rightarrow^{d} & \Omega^1(A) & \\
\uparrow & & \uparrow 0 & \\
A & \rightarrow^{d} & \Omega^1(A) & \rightarrow^{d} \Omega^2(A) & \\
\uparrow 0 & & \uparrow 0 & & \uparrow 0 & \\
A & \rightarrow^{d} & \Omega^1(A) & \rightarrow^{d} \Omega^2(A) & \rightarrow^{d} \Omega^3(A) & \\
\end{array}
\] (1.2)

The Connes differential $B$ becomes the usual de Rham differential, and the rows of this bicomplex are all truncation of the de Rham complex.

Both $HH_q(A)$ and $HC_q(A)$ are defined for an arbitrary associative algebra $A$, and so is the bicomplex (1.1) – but this is as far as one gets in the general situation: to split (1.2) nicely into separate rows and consider them separately, one needs to know that the algebra $A$ is commutative. Nevertheless, for some applications this is not needed. In particular, the stupid filtration on the de Rham complex can be understood as the stupid filtration on the bicomplex (1.1) taken “in horizontal direction”, and this makes perfect sense in full generality (to avoid confusion, we note that the modules of differential forms $\Omega^i(A)$ appear in the Hochschild homology theory with “wrong” degrees – $\Omega^i(A)$ appears in degree $-i$, not $i$ – and as the result of this, stupid a.k.a. Hodge filtration becomes increasing, not decreasing). Thus one gets an analog of the Hodge-to-de Rham spectral sequence and can pose the question of its degeneration (we note that this “Hochschild-to-cyclic” spectral sequence has been very prominent in cyclic homology studies from the very beginning).

Moreover, in characteristic $\text{char } k = p > 0$, one can define the conjugate filtration on (1.2) as the canonical filtration “in horizontal direction”, and repeat the argument of [DI] (with some minor modifications). The end result is the same – one shows that the Hodge-to-de Rham spectral sequence degenerates, once some natural conditions are met.

1.3 Overview of the present paper. In a nutshell, what we do in the present paper is this: we note that the conjugate filtration on (1.2), known in the commutative case, can be defined in an invariant way which makes sense for a arbitrary associative algebra or sheaf of algebras over the
base. Moreover, we identify the associated graded quotient of this filtration with the Hochschild homology – thus obtaining a version of the Cartier isomorphism valid in the non-commutative setting (or rather, as in [DI], we in fact construct the inverse map $C^{-1}$).

As an example, one can consider what happens in degree 0. Here we have $\text{HH}_0(A) = HC_0(A) = A/[A, A]$, the quotient of $A$ by the subspace spanned by commutator expressions $[a, b] = ab - ba, a, b \in A$. It is a pleasant exercise to check that, even for a non-commutative $A$, the “Frobenius” map $x \mapsto x^p$ descends to a well-defined and additive map $C^{-1} : A/[A, A] \to A/[A, A]$.

It turns out that to generalize it to all degrees, it is not convenient to use the complex (1.1), and it is better to use the second standard complex for $HC_q(A)$, which is the periodic complex:

\[
\begin{array}{cccccc}
\rightarrow & A & \rightarrow & A & \rightarrow & A \\
\uparrow b' & \uparrow b & \uparrow b' & \uparrow b \\
A^{\otimes 2} & \rightarrow & A^{\otimes 2} & \rightarrow & A^{\otimes 2} & \rightarrow & A^{\otimes 2} \\
\uparrow b' & \uparrow b & \uparrow b' & \uparrow b \\
A^{\otimes 3} & \rightarrow & A^{\otimes 3} & \rightarrow & A^{\otimes 3} & \rightarrow & A^{\otimes 3} \\
\uparrow b' & \uparrow b & \uparrow b' & \uparrow b \\
A^{\otimes 4} & \rightarrow & A^{\otimes 4} & \rightarrow & A^{\otimes 4} & \rightarrow & A^{\otimes 4} \\
\end{array}
\]

Here $b$ is the Hochschild differential, $b'$ is the contractible differential in the bar resolution, and the horizontal differential in $l$-th row is the same as in the standard periodic complex which computes the homology $H_*(\mathbb{Z}/l\mathbb{Z}, A^{\otimes l})$ of the cyclic group $\mathbb{Z}/l\mathbb{Z}$ with coefficients in $A^{\otimes l}$ equipped with the natural action. Moreover, we note that the standard bar complex which computes Hochschild homology can be modified in such a way that instead of tensor powers $A^{\otimes l}$, it only contains tensor powers $A^{\otimes pl}$ for some fixed integer $p$. This is easy to see using the Tor-interpretation of Hochschild homology: since $A = A \otimes_{A^{\text{opp}}} A$, we have

\[HH_*(A) = \text{Tor}^*_{A^{\text{opp}}} (A \otimes_{A^{\text{opp}}} A \otimes_{A^{\text{opp}}} A \cdots \otimes_{A^{\text{opp}}} A, A),\]

with an arbitrary fixed number of multiples on the left-hand side. If one uses the bar resolution and multiplies it with itself $p$ times in the correct way (see Lemma 6.3), the result is a complex computing $HH_*(A)$ and consisting of $A^{\otimes pl}$, $l \geq 0$. 

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Once one develops a version of (1.3) for this modified Hochschild complex, the Cartier isomorphism in degree $\geq 1$ boils down essentially to the following linear-algebraic fact (this is Lemma 5.2).

- For any vector space $V$ over a field $k$ of characteristic $p > 0$, the homology group $H_l(\mathbb{Z}/p\mathbb{Z}, V^\otimes p)$ is canonically isomorphic to $V$ for every $l \geq 1$.

Unfortunately, we were not able to find an explicit complex to play the role of (1.3); to overcome this difficulty, we use the more invariant technique of homology of small categories. The relevant categories are the Connes cyclic category $\Lambda$ and its $p$-fold cover $\Lambda_p$, slightly less-known but also very standard (see e.g. [BG]). The invariant definition of the conjugate filtration is obtained through considering the natural functor $\Lambda_p \to \Lambda$ – this functor is a fibration, whose fiber is the groupoid with one object and automorphism group $\mathbb{Z}/p\mathbb{Z}$, and the definition of the conjugate filtration that we use is obtained by considering the homology of this group (it turns out that the right thing to do is to consider the even terms of the canonical filtration on the standard periodic complex).

We then study the question of the splitting of the conjugate filtration. The end result, somewhat surprisingly, is again just the same as in [DI] – the filtration splits if and only if $A$ can be lifted to a flat algebra over the second Witt vector ring $W_2(k)$. Since there is no natural notion of dimension for general associative algebras, one might expect that the condition $\dim X < \text{char } k$ would disappear from the picture completely. This is not what happens, though: our method only gives splitting only “in degrees between 0 and $2p$”, and there reasons to believe that this is not a merely technical limitation (see, in particular, Remark 5.13). Nevertheless, since to deduce Hodge-to-de Rham degeneration we are allowed to consider only generic primes, this does not interfere with the proof (just as in [DI]).

Let us now describe briefly the organization of the paper. A rather long Section 2 contains all the preliminary facts that we need on homology of small categories in general and cyclic categories $\Lambda_p$ in particular; nothing here is new, and everything is contained in some form in [L]. However, we do need to recall these things in some detail to set up the notation etc. As the result, the present paper is self-contained to a large degree. The last part of Section 2, Subsection 2.4, is taken with Proposition 2.7 which essentially explains how to compute the cyclic homology of an algebra $A$ by means of a complex only containing tensor powers $A^\otimes pl$ (although it is
formulated in the invariant language of small categories and cyclic objects, and the statement sounds quite differently).

In Section 3 we introduce the Hodge and the conjugate filtration on cyclic homology of the arbitrary associative algebra $A$, and we identify the associated graded quotients of the conjugate filtration with the Hochschild homology. Then in Section 4 we study the extension data between the associated graded pieces of the conjugate filtration. It turns out that this is quite complicated technically, and moreover, that we need to exploit some additional symmetry that we call polycyclic. Partially, this technical complexity reflects the fact that the resulting description only holds in degrees between 0 and $2p$ (while in higher degrees one encounters Steenrod cohomological operations, and understanding the picture is probably far outside of what our approach allows).

We note that, for better or for worse, we have tried to separate consistently “linear-algebraic” facts, which make sense for any cyclic object, and things that involve tensor products. Therefore in Sections 3 and 4 we work in more generality than strictly needed for the computation of $HC_*(A)$ for an associative algebra $A$. Associative algebras per se and their homology only appear in Section 5 with all the work done in Section 3 and Section 4 in Subsection 5.1 it remains just to define a cyclic object $A_#$ associated to an algebra $A$ and to check that it satisfies the assumptions of previous Sections (this is very easy). Then in Subsection 5.3 we study the splitting of the conjugate filtration for an algebra $A$. With all the technical unpleasantness already done in Section 4 this boils down to a nice and compact criterion Lemma 5.8 in Lemma 5.11 we show by a reasonably simple and conceptual argument that the condition of Lemma 5.8 is satisfied if $A$ admits a lifting to $W_2(k)$.

Finally, in Section 6 we collect all of the above to prove Hodge-to-de Rham degeneration. The statement in positive characteristic, Theorem 6.1 is a precise analog of the corresponding commutative statement in [DI]. Unfortunately, the statement in char 0, while also exactly the same as in the commutative case, is not satisfactory: we need to require our algebra to be Noetherian, which is too strong a condition in the non-commutative case. It should be possible to drop this requirement by using $A_\infty$ methods; we plan to return to this in the future.
2 Cyclic objects.

In this section, we recall standard basic facts about cyclic objects and cyclic categories needed in the rest of the paper; we mostly follow [FT2], see also [L, Chapter 6].

2.0 Notation. Throughout the paper, we will need to work with filtered complexes in many places. Our conventions will be like this: if $F^*$ is a filtered complex equipped with an increasing filtration $W^*$, then $W_{[n,m]}F^*$ is the cone of the natural map $W_{n-1}F^* \to W_nF^*$, and $\text{gr}^W_{n}F^*$ is $W_{[n,n]}F^*$. For decreasing filtrations, we use dual conventions. The canonical filtration on a complex in a derived category is denoted by $\tau_{\leq n}$; $\tau_{[n,m]}$ is the cone of the map $\tau_{\leq (n-1)} \to \tau_{m}$. Sometimes we use homological degrees $F_*$ instead of cohomological degrees $F^*$ — we treat them as interchangeable notation for the same thing, related by $F_l = F_{-l}$, $l \in \mathbb{Z}$.

2.1 Homological preliminaries. We begin with homological generalities. Let $\mathcal{C}$ be a Grothendieck abelian category — or, in Grothendieck’s language [G], an abelian category which additionally satisfies $AB3$, $AB4$, $AB3'$, $AB4'$ (in applications, $\mathcal{C}$ will be the category of $B$-modules for a commutative ring $B$, or, more generally, the category of sheaves of $B$-modules for a sheaf of commutative rings $B$ on some site). For any small category $F$, denote by $\text{Fun}(F, \mathcal{C})$ the abelian category of covariant functors from $F$ to $\mathcal{C}$, and denote by $D^-(F, \mathcal{C})$ the derived category of complexes in $\text{Fun}(F, \mathcal{C})$ bounded from above. The category $\text{Fun}(F, \mathcal{C})$ is also a Grothendieck abelian category.

For any two small categories $F$, $G$ and a functor $\sigma : F \to G$, denote by $\sigma^* : \text{Fun}(G, \mathcal{C}) \to \text{Fun}(F, \mathcal{C})$ the natural restriction functor given by

$$\sigma^*(E) = E \circ \sigma \in \text{Fun}(F, \mathcal{C}), \quad E \in \text{Fun}(G, \mathcal{C}).$$

Since $\mathcal{C}$ is a Grothendieck category, $\sigma^*$ has right and left adjoint functors called Kahn extensions. Denote the left and right Kahn extensions by $\sigma_l : \text{Fun}(F, \mathcal{C}) \to \text{Fun}(G, \mathcal{C})$ and $\sigma_r : \text{Fun}(F, \mathcal{C}) \to \text{Fun}(G, \mathcal{C})$, and denote their derived functors by $L^s \sigma_l : D^-(F, \mathcal{C}) \to D^-(G, \mathcal{C})$ and $R^s \sigma_l : D^+(F, \mathcal{C}) \to D^+(G, \mathcal{C})$.

In the particular case when $F = \text{pt}$ is the category with one object and one morphism, we have $\text{Fun}(F, \mathcal{C}) \cong \mathcal{C}$, and specifying a functor $\sigma : F \to G$ is
the same as specifying an object \([f] \in G\). For any object \(A \in C\), we denote
\[
A_{/[f]} = \sigma_A,
\]
\[
A^*_{/[f]} = \sigma_* A.
\]
If \(C\) is the category of abelian groups, and \(A = \mathbb{Z}\) is the group \(\mathbb{Z}\) for any \(g \in G\) the group \(\mathbb{Z}_{/[f]}([g])\) is naturally identified with the free abelian group spanned by the set \(\text{Hom}_C([f],[g])\), and the group \(\mathbb{Z}^*_{/[f]}([g])\) is naturally identified with the abelian group of \(\mathbb{Z}\)-valued functions on the set \(\text{Hom}_C([g],[f])\).

In the particular case when \(G = \text{pt}\), there exists only one tautological functor \(\sigma : F \to G\). For any \(A \in C \cong \text{Fun}(G,C)\), we denote
\[
A_{F} = \sigma^* A,
\]
or simply \(A \in \text{Fun}(F,C)\) when there is no danger of confusion. For any \(E \in \text{Fun}(F,C)\) we denote \(L^\bullet \sigma_!(E)\) by \(H_*(F,E)\) and we call it the homology of the category \(F\) with coefficients in \(E \in \text{Fun}(F,C)\). Analogously, we denote \(R^\bullet \sigma_* E = H^*(F,E)\), and we call the cohomology of the category \(F\) with coefficients in \(E\). We note that for any \(\rho : F \to G\), and any \(E \in \text{Fun}(G,C)\), we have by adjunction a canonical map
\[
\sigma_! : H_*(F,\sigma^*C) \to H_*(G,C).
\]
If \(C = B\text{-mod}\) is the category of modules over a commutative ring \(B\), we have by adjunction
\[
H^*(F,B_F) = \text{Ext}^*(B_F,B_F),
\]
so that the cohomology \(H^*(F,B_F) = H^*(F,B)\) with coefficients in the constant functor \(B \in \text{Fun}(F,B\text{-mod})\) carries a natural algebraic structure. For any \(\mathbb{B}\)-linear Grothendieck category \(C\) and any object \(E \in \text{Fun}(F,C)\), the homology \(H_*(F,E)\) is naturally a module over the algebra \(H^*(F,B_F)\). For any \(F, G\) and a functor \(\rho : F \to G\), the restriction functor \(\rho^*\) induces an algebra map \(\rho^* : H^*(G,B) \to H^*(F,B)\). The maps \(\rho^*\) and \(\rho_!\) are, as usual, related by the projection formula: we have
\[
\sigma_!(\sigma^*(\alpha) \cdot e) = \alpha \cdot \sigma_!(e)
\]
for any \(\alpha \in H^*(G,B)\), \(e \in H_*(F,\rho^*E)\).

We will also need to use the notion of a fibered category ([SGA]); let us briefly recall it. For any functor \(\sigma : F \to G\) between small categories \(F\) and \(G\) and any object \([a] \in G\), denote by \(F_{/[a]}\) the subcategory in \(F\) of all objects \([a'] \in F\) such that \(\sigma([a']) = [a]\) and all morphisms \(f\) such that
\( \sigma(f) = \text{id}_{[a]} \). This is called the fiber of the functor \( \sigma : F \to G \) over \( [a] \in G \).

For any \( [a], [b] \in G \), any \( f : [a] \to [b] \) and any \( [a'] \in F_{[a]}, [b'] \in F_{[b]} \), denote by \( F_f([a'], [b']) \) the set of all maps \( f' : [a'] \to [b'] \) such that \( \sigma(f') = f \).

**Definition 2.1.** The functor \( \sigma : F \to G \) is called a fibration, and \( F \) is called a fibered category over \( G \), if for any \( [a], [b] \in G \), \( f : [a] \to [b] \) and \( [b'] \in F_{[b]} \) the functor

\[
[a'] \mapsto F_f([a'] \to [b'])
\]

from \( F_{[a]} \) to the category of sets is representable by an object \( f^*([b]) \in F_{[a]} \).

Passing to the opposite categories gives the dual notion of a cofibration and a cofibered category. A functor which is both a fibration and a cofibration is called a bifibration.

An example of a cofibered category is a so-called discrete cofibration: for any functor \( F \) from \( G \) to sets, pairs \( \langle [a] \in G, a \in F([a]) \rangle \) form a small category \( F \) naturally fibered over \( G \) (the fiber \( F_{[a]} \) is the set \( F([a]) \) considered as a category with no non-identical morphisms). The category \( F \) is called the total space of the functor \( F \) and denoted \( \text{Tot}(G, F) \). It is easy to check that every cofibration whose fibers have no non-identical morphisms is obtained in this way. An opposite case is a cofibration \( F \to G \) whose fibers are categories with one object. For example, every functor \( F \) from \( G \) to the category of groups defines a category \( F \) cofibered over \( G \) whose fiber \( F_{[a]} \) is a groupoid with one object with automorphism group \( F([a]) \). By abuse of notation, in this case we will also call \( F \) the total space of the functor \( F \) and denote it by \( \text{Tot}(G, F) \). More generally, a cofibered category over \( G \) whose fibers are groupoids is usually called a gerbe over \( G \). Gerbes of the form \( \text{Tot}(G, F) \) are said to be split; there are gerbes which are not of this type (we will see one example below in Remark 3.4).

We will need the following standard properties of cofibrations (and the dual properties of fibrations, which are analogous and left to the reader).

**Proposition 2.2.** Let \( \sigma : F \to G \) be a cofibration.

(i) For any small category \( G' \) and a functor \( \rho : G' \to G \), the pullback \( \sigma' : G' \times_G F \to G' \) is also a cofibration, and we have a natural base change isomorphism

\[
L \cdot \sigma_! \circ \rho^* \cong \rho^* \circ L \cdot \sigma_!.
\]

(ii) For any Grothendieck category \( \mathcal{C} \) and any object \( E \in \text{Fun}(G, \mathcal{C}) \), we have a natural isomorphism

\[
L \cdot \sigma_! \sigma^* E \cong E \otimes L \cdot \sigma_! \sigma^* \mathbb{Z}_G,
\]
Sketch of a proof. To check that $\sigma' : G' \times_G F \to G'$ is indeed a cofibration is an elementary exercise left to the reader. Moreover, for any object $[a] \in G'$ we have natural isomorphism $\pi : \sigma'_{[a']} \cong \sigma_{\pi([a'])}$. Now, by the definition of a cofibration, for any objects $M \in C$, $[a] \in G$, we have

$$\sigma^* M_{[a]} \cong \lim_{\leftarrow} M_{[b]} \cong \iota_* (M_{\sigma([a])}),$$

where the limit is taken over $[b] \in \sigma([a]) \subset F$, and $\iota : \sigma([a]) \to F$ is the embedding of the fiber $\sigma([a])$. Therefore for any $E \in \text{Fun}(F, C)$ and any $[a] \in G$ we have

$$L^\sigma \pi_!(E)([a]) \cong H_\pi(\sigma([a]), E),$$

and this is functorial in $E$ and in $[a]$. This immediately gives both the base change isomorphism (2.1) and the projection formula (2.2). □

Finally, in Section 3 we will need to use a bar resolution several times; we recall the setup. Let $F : C \to C$ be any exact functor from an abelian category $C$ to itself. Assume that $F$ is equipped with a surjective augmentation map $f : F \to \text{Id}$, and denote by $P : C \to C$ the kernel of this map ($P$ is obviously also an exact functor from $C$ to itself). The bar-resolution, or the Godement resolution associated to the pair $\langle F, f \rangle$ is constructed as follows. For any $l \geq 1$ we set

$$(2.3) \quad B_l(F, f) = F \circ P^{l-1} : C \to C,$$

and we define a map $d : B_{l+1}(F, f) \to B_l(F, f)$ as the composition of the projection $f : F \circ P^l \to \text{Id} \circ P^l = P^l$ and the embedding $P^l = P \circ P^{l-1} \to F \circ P^{l-1}$. It will be convenient for us to extend this by setting $B_0 = \text{Id}$, $d = f : B_1 = F \to \text{Id}$, and we will omit $f$ from notation when it is clear from the context what it is. Then $d^2 = 0$, and $(B_l(F), d)$ is an complex of exact functors from $C$ to itself. For any object $E \in C$, $B_\ast(F)(E)$ is a complex in $C$ with $B_0(F)(E) = E$. Moreover, it is easy to check by induction that the complex $B_\ast(F)(E)$ is actually acyclic, so that $B_{\geq 1}(F)(E)$ is a resolution of the object $E$ – indeed, by construction we have a natural embedding of complexes

$$(2.4) \quad P(B_\ast(F)(E))[1] \to B_\ast(F)(E),$$

and the cokernel of this embedding is the acyclic complex $E \to E$.

The bar-construction works without any changes in a more general situation when $F$ is functor from the category of complexes in $C$ to itself. In this
setting, for any $E \in \mathcal{C}$ the bar complex $B_*(F)(E)$ is naturally a bicomplex; by abuse of notation, we will denote by $B_*(F)(E)$ its total complex.

We note that formally, in order to define the complex $B_*(F,f)$ we do not need $P$ to be the kernel of the map $f : F \to \text{Id}$ – all we need is a sequence
\begin{equation}
0 \longrightarrow P \longrightarrow F \longrightarrow \text{Id} \longrightarrow 0
\end{equation}
of exact functors which is exact on the left and on the right. In this yet more general situation, we denote the bar complex by $B_*(F,P)$. This construction is obviously functorial with respect to sequences of the form (2.5).

If (2.5) is exact in the middle term, the complex $B_*(F,P) = B_*(F)$ is acyclic, if not, then not. However, if $F$ and $P$ in (2.5) are allowed to be exact functors from the category of complexes in $\mathcal{C}$ to itself, then the condition is weaker: the bar complex $B_*(F,P)(E)$ is acyclic for any $E \in \mathcal{C}$ if the middle homology of the sequence (2.5) is acyclic – in other words, if (2.5) gives an exact triangle after passing to the derived category.

### 2.2 Cyclic categories – definition and combinatorics.

The main small categories that we will need in the paper are the cyclic category $\Lambda$ introduced by A. Connes, and its generalizations, the so-called paracyclic categories $\Lambda_p$ introduced in [BG]. Let us recall the definitions. Consider the category $\text{Cycl}$ of linearly ordered sets $M$ equipped with an order-preserving automorphism $\sigma : M \to M$. Maps in $\text{Cycl}$ are order-preserving maps which commute with $\sigma$. Consider the set $\mathbb{Z}$ of all integers as a linearly ordered set with the natural order, and for any positive $m \in \mathbb{Z}$, denote by $[m]$ the set $\mathbb{Z}$ equipped with an automorphism $\sigma : \mathbb{Z} \to \mathbb{Z}$ given by $a \mapsto a + m$. Let $\Lambda_\infty \subset \text{Cycl}$ be the full subcategory spanned by $[m] \in \text{Cycl}$ for all positive $m \in \mathbb{Z}$. We will treat $\Lambda_\infty$ is a small category whose set of objects is the set of positive integers, and for any two positive integers $n,m \in \mathbb{Z}$, the set

\[ \Lambda_\infty([m],[n]) \]

of maps from $[m]$ to $[n]$ is the set of all order-preserving maps $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(a+m) = f(a) + n$. In particular, $\sigma$ itself gives an automorphism $\sigma \in \Lambda_\infty([m],[m])$ for any object $[m]$, and for any $f \in \Lambda_\infty([m],[n])$, we have $\sigma \circ f = f \circ \sigma$. Therefore setting $f \mapsto f \circ \sigma$ gives an action of the map $\sigma$ on $\Lambda_\infty([m],[n])$ which is compatible with compositions, and we can define a category $\Lambda$ with the same objects as $\Lambda_\infty$, and with morphism sets given by

\[ \Lambda([m],[n]) = \Lambda_\infty([m],[n]) / \sigma. \]
This is the Connes cyclic category. Moreover, for any positive \( p \in \mathbb{Z} \) we can set

\[
\Lambda_p([m], [n]) = \Lambda_\infty([m], [n]) / \sigma^p
\]

and obtain the paracyclic category \( \Lambda_p \). For any \( p \geq 1 \), we obviously have a natural embedding \( i_p : \Lambda_p \to \Lambda, [m] \mapsto [mp] \), and a natural projection \( \pi_p : \Lambda_p \to \Lambda, [m] \mapsto [m] \). The projection \( \pi_p : \Lambda_p \to \Lambda \) is a bifibration over \( \Lambda \), with all fibers isomorphic to \( \mathbb{Z}/p\mathbb{Z} \). The projection \( \rho_p : \Lambda_\infty \to \Lambda_p \) is also a bifibration, with all fibers isomorphic to \( \mathbb{Z}/p\mathbb{Z} \).

The category \( \Lambda_\infty \) is self-dual – the duality functor \( D : \Lambda_\infty \to \Lambda_\infty^o \) is identical on objects, and for any \( f \in \Lambda_\infty([m], [n]) \) we set

\[
D(f)(a) = \max\{b \in \mathbb{Z} | f(b) \leq a\}, \quad a \in \mathbb{Z}.
\]

This duality descends to the cyclic category \( \Lambda \) and to all the paracylic categories \( \Lambda_p \).

Let \( \Delta \) be the simplicial category, that is, the category of finite linearly ordered sets. Then the opposite category \( \Delta^o \) is naturally embedded into \( \Lambda_\infty \): for any \([m], [n] \), the set \( \Delta([m], [n]) \) is naturally identified with the set of those \( f \in \Lambda_\infty([m], [n]) \) that preserve \( 0 \in \mathbb{Z}, f(0) = 0 \). Denote this embedding by \( j_\infty : \Delta^o \to \Lambda_\infty \). For every \( p \geq 1 \), the embedding \( j_\infty \) induces an embedding from \( \Delta^o \) to \( \Lambda_p \), which we denote by \( j_p : \Delta^o \to \Lambda_p \) (simply \( j \) if \( p = 1 \)). By duality, we obtain the embeddings \( j_p^o : \Delta \to \Lambda_p \). The bifibration \( \pi_p : \Lambda_p \to \Lambda \) splits over \( \Delta^o \subset \Lambda \) by means of the embedding \( j_p^o \) – in other words, the embedding \( j_p^o \) extends to an embedding \( \tilde{j}_p : \Delta^o \times \text{pt}_p \to \Lambda_p \). The composition \( i_p \circ \tilde{j}_p : \Delta^o \times \text{pt}_p \to \Lambda_p \) factors through an embedding from \( \Delta^o \times \text{pt}_p \) to \( \Delta^o \) which we also denote by \( i_p \), by abuse of notation. We can collect all these data into a commutative diagram

\[
\begin{array}{ccc}
\Delta^o & \xrightarrow{j} & \Lambda \\
\downarrow{i_p} & & \downarrow{i_p} \\
\Delta^o \times \text{pt}_p & \xrightarrow{\tilde{j}_p} & \Lambda_p \\
\downarrow & & \downarrow{\pi_p} \\
\Delta^o & \xrightarrow{j} & \Lambda
\end{array}
\]

of small categories and functors, with Cartesian squares. Explicitly, the embedding \( i_p : \Delta^o \times \text{pt}_p \to \Delta^o \) sends \([m] \in \Delta \) to \([p] \times [m] \) with lexicographical
ordering; the automorphism group \( \mathbb{Z}/p\mathbb{Z} \) acts by changing the ordering on the set \([p]\).

As far as the functor categories are concerned, one can also treat the embedding \( j_p : \Delta^o \to \Lambda_p \) as a discrete cofibration, both for \( p \in \mathbb{Z} \) and \( p = \infty \). Indeed, let \( \Delta^o \) be the total space of the functor \( \Lambda_p \to \text{Sets} \) given by \([n] \mapsto \Lambda_p([1],[n])\). Explicitly, \( \Delta^o \) is the category of pairs \(([n], a)\) of the linearly ordered set \( \mathbb{Z} \) with the map \( \sigma : \mathbb{Z} \to \mathbb{Z} \) and a fixed element \( a \in \mathbb{Z} \) defined modulo \( \sigma^p \); maps in \( \Delta^o \) are maps in \( \Lambda_p \) which preserve the fixed element. Then the embedding \( j_p : \Delta^o \to \Lambda_p \) naturally factors through an embedding \( \Delta^o \to \tilde{\Delta}^o \) which sends \([n]\) to \(([n],0)\). It is easy to check that this embedding \( \Delta^o \to \tilde{\Delta}^o \) is an equivalence of categories, and in particular, it induces an equivalence \( \text{Fun}(\Delta^o, \mathcal{C}) \cong \text{Fun}(\tilde{\Delta}^o, \mathcal{C}) \).

2.3 Periodicity. Fix a Grothendieck abelian category \( \mathcal{C} \) and an integer \( p \geq 1 \), and consider the embedding \( j_p : \Delta^o \to \Lambda_p \).

Lemma 2.3. For any \( E \in \text{Fun}(\Lambda_p, \mathcal{C}) \), there exists an isomorphism

\[
j_p^* j_p^* E \cong E \otimes j_p^* j_p^* Z,
\]

this isomorphism is functorial in \( E \), and \( j_p^* j_p^* Z \) is isomorphic to \( Z_{[1]} \in \text{Fun}(\Lambda_p, \mathbb{Z}\text{-mod}) \).

Proof. Replace \( j_p : \Delta^o \to \Lambda_p \) with the discrete fibration \( \xi : \tilde{\Delta}^o \to \Lambda_p \); since \( \Delta^o \) is equivalent to \( \Delta^o \), we have \( j_p^* j_p^* \cong \xi^* \xi^* \), and both claims follow from Lemma 2.2. \( \square \)

By duality, we also obtain a functorial isomorphism \( j_p^* j_p^* E \cong E \otimes j_p^* j_p^* Z \cong E \otimes Z_{[1]} \) and since both \( Z_{[1]} \) and \( Z_{[1]}^o \) are functors into flat abelian groups, we conclude that both \( j_p^* j_p^* \) and \( j_p^* j_p^* \) are exact functors from \( \text{Fun}(\Lambda_p, \mathcal{C}) \) to itself. We tautologically have \( H_*(\Lambda_p, j_p^* j_p^* E) \cong H_* (\Delta^o, j_p^* E) \).

Lemma 2.4. For any \( E \in \text{Fun}(\Lambda_p, \mathcal{C}) \), we have

\[
H_*(\Lambda_p, j_p^* j_p^* E) = H_*(\Lambda_p, E \otimes Z_{[1]}^o) = 0.
\]

Proof. Since \( E \to E \otimes Z_{[1]}^o \) is an exact functor in \( E \), the homology that we have to study is the derived functor of the right-exact functor

\[
H_0(\Lambda_p, - \otimes Z_{[1]}^o) : \text{Fun}(\Lambda_p, \mathcal{C}) \to \mathcal{C}.
\]
Therefore it suffices to prove that this right-exact functor vanishes. Indeed, the constant functor \( Z \in \text{Fun}(\Lambda_p, \mathbb{Z}\text{-mod}) \) embeds into \( Z_{[1]}^o \) by adjunction, and the cokernel of this embedding embeds into \( Z_{[2]}^o \); by adjunction, we have a functorial exact sequence

\[
E([2]) \oplus 2p \xrightarrow{\rho} E([1]) \oplus p \rightarrow H_0(\Lambda_p, E \otimes Z_{[1]}^o) \rightarrow 0,
\]

and it remains to compute the map \( \rho \) and to notice that it is tautologically surjective (in fact, split). We leave it to the reader. \( \square \)

We can now deduce the main homological result on the embedding \( j_p : \Delta^o \to \Lambda_p \).

**Lemma 2.5.** There exists a map \( B_p : Z_{[1]}^o \to Z_{[1]} \) such that the sequence

\[
(2.6) \quad 0 \longrightarrow Z \longrightarrow Z_{[1]}^o \xrightarrow{B_p} Z_{[1]} \longrightarrow Z \longrightarrow 0
\]

is an exact sequence in \( \text{Fun}(\Lambda_p, \mathbb{Z}\text{-mod}) \). The cohomology algebra \( H^*(\Lambda_p, \mathbb{Z}) \) is the free algebra \( \mathbb{Z}[u] \) in one generator \( u = u(p) \), and this generator is represented by Yoneda by the exact sequence (2.6). The embedding \( i_p : \Lambda_p \to \Lambda \) sends the generator \( u(1) \) to \( u(p) \).

**Proof.** The construction of the map \( B_p \) for \( p = 1 \) is standard, see e.g. [FT2]. In the case \( p > 1 \), we notice that \( i^*_p(Z_{[1]}^o) \) is canonically isomorphic to \( Z_{[1]} \), and \( i^*_p(Z_{[1]}^o) \) is canonically isomorphic to \( Z_{[1]}^o \). This gives the map \( B_p \) and, by Yoneda, the universal generator \( u = u(p) \in H^2(\Lambda_p, \mathbb{Z}) \). Finally, to show that \( H^*(\Lambda_p, \mathbb{Z}) \) is freely generated by \( u \), it suffices to extend (2.6) to a resolution of the constant object \( Z \in \text{Fun}(\Lambda_p, \mathbb{Z}\text{-mod}) \) and to notice that by adjunction,

\[
H^*(\Lambda_p, Z_{[1]}^o) = H^*(\Lambda_p, \mathbb{Z}) = \mathbb{Z},
\]

while \( H^*(\Lambda_p, Z_{[1]}^o) = 0 \) by the statement dual to Lemma 2.4. \( \square \)

**Corollary 2.6.** For any object \( E \in \text{Fun}(\Lambda_p, C) \), we have a functorial exact sequence

\[
(2.7) \quad 0 \longrightarrow E \longrightarrow j_p^* j_p^* E \longrightarrow j_p^* j_p^* E \longrightarrow E \longrightarrow 0
\]

and a functorial exact triangle

\[
(2.8) \quad H_*(\Delta^o, j_p^* E) \longrightarrow H_*(\Lambda_p, E) \xrightarrow{u} H_*(\Lambda_p, E)[2] \longrightarrow ,
\]

where \( u : H_*(\Lambda_p, E) \to H_*(\Lambda_p, E)[2] \) is given by the action of the universal generator \( u \in H^2(\Lambda_p, \mathbb{Z}) \). \( \square \)
It will be convenient for us to rewrite (2.7) as a functorial short exact sequence

\[(2.9) \quad 0 \longrightarrow E[1] \longrightarrow j_p^! E \longrightarrow E \longrightarrow 0\]

in the category of complexes of objects in Fun(\(\Lambda_p, C\)), where \(E[1]\) is, as usually, the complex consisting of \(E\) placed in degree \(-1\), and \(j_p^! E\) is the complex \(j_p^! \circ j_p^\ast E \to j_p^! \circ j_p^\ast E\) placed in degrees 0 and \(-1\). Then (2.8) amounts to a canonical isomorphism \(H_q(\Delta^\circ, j_p^\ast E) \cong H_q(\Lambda_p, j_p^! E)\) and the long exact sequence associated to (2.9). We note that \(j_p^!\) is an exact functor from Fun(\(\Lambda_p, C\)) to the category of complexes in Fun(\(\Lambda_p, C\)); in particular, it trivially extends to a functor \(j_p^!: \mathcal{D}^-(\Lambda_p, C) \to \mathcal{D}^-(\Lambda_p, C)\) on the derived categories. If \(p = 1\), one usually denotes

\[(2.10) \quad H_q(\Delta^\circ, j^* E) = \text{HH}_q(E), \quad H_q(\Lambda, E) = \text{HC}_q(E),\]

and calls them the Hochschild and cyclic homology of the cyclic object \(E\). The map \(u\) is the Connes' periodicity map, and the exact triangle (2.8) is known as the Connes' exact sequence.

### 2.4 Compatibility for paracyclic embeddings.

We finish this section with a somewhat surprising result which shows that for any cyclic object \(E\) and any integer \(p\), one can find a \(p\)-cyclic object with the same homology (this is the key point in our construction of the non-commutative Cartier operator).

**Proposition 2.7.** For any Grothendieck abelian category \(C\) and any object \(E \in \text{Fun}(\Lambda, C)\), the natural map

\[i_p!: H_q(\Lambda_p, i_p^! E) \to H_q(\Lambda, E)\]

is a quasiisomorphism.

**Proof.** By Lemma 2.5, the map \(i_p!\) is compatible with the periodicity map \(u\); therefore by (2.8) it suffices to prove that the natural map

\[i_p!: H_q(\Delta^\circ, i_p^* E') \to H_q(\Delta^\circ, E')\]

is a quasiisomorphism, where \(E' = j^* E\), and \(i_p\) is now treated as the embedding from \(\Delta^\circ\) to itself. We will in fact prove this claim for any
$E' \in \text{Fun}(\Delta^o, C)$. To do this, it suffices to consider the case when $E'$ goes over a family of generators of the category $\text{Fun}(\Delta^o, C)$; to construct such a family, it suffices to take a generator $S$ of the category $\text{Fun}(\Delta^o, \mathbb{Z}\text{-mod})$ and tensor it with all objects in $\mathcal{C}$ in turn. Thus we are reduced to the case $\mathcal{C} = \mathbb{Z}\text{-mod}$, $E' = S$ is a generator of $\text{Fun}(\Delta^o, \mathbb{Z}\text{-mod})$.

Consider the category of bimodules over the polynomial algebra $B = \mathbb{Z}[t]$ in one variable $t$ over $\mathbb{Z}$, and let $S$ be the standard bar-resolution of the diagonal bimodule $B$ (we have $S([n]) = B^{\otimes(n+1)}$). This is a simplicial abelian group which in addition inherits the standard grading from the polynomial algebra $B$. It is easy to check that the degree-$l$ component $S^l$ of the simplicial $\mathbb{Z}$-module $S$ is isomorphic to the functor $\mathbb{Z}^l$ (the standard $l$-simplex); therefore $S = \bigoplus S^l$ is indeed a generator of the category $\text{Fun}(\Delta^o, \mathbb{Z}\text{-mod})$. On the other hand, it is well-known that the homology of $\Delta^o$ with coefficients in a simplicial abelian group can be computed by the standard simplicial complex, and since $S$ is a resolution, we have

$$H_0(\Delta^o, S) = B$$

and $H_i(\Delta^o, S) = 0$ for $i \geq 1$. It remains to notice that $S$ is a simplicial flat $B$-bimodule, and

$$i_p^* S \cong S \otimes_B S \otimes_B \cdots \otimes_B S,$$

where the product contains $p$ terms. Since $B \otimes_B B \cong B$, we conclude that $i_p^* S$ is another resolution for the same bimodule $B$, and the natural map $H_i(\Delta^o, i_p^* S) \to H_i(\Delta^o, S)$ is indeed a quasiisomorphism, as required. \qed

3 Filtrations on cyclic homology.

3.1 The Hodge filtration. Fix a Grothendieck abelian category $\mathcal{C}$, and consider an arbitrary object $E \in \text{Fun}(\Lambda_p, \mathcal{C})$. Iterating (2.7), one obtains a canonical resolution $E_*$ of the object $E$; using this resolution, one can refine (2.9) in the following way. Consider the stupid filtration on the complex $E_*$, and let $F^l E_*, \ l \geq 0$, denote the $2l$-th term of this stupid filtration. Then $\langle E, F^* \rangle$ is a filtered complex, and it defines an object in the filtered derived category $\mathcal{DF}^-(\Lambda_p, \mathcal{C})$. The following is a reformulation of Corollary 2.6.

**Lemma 3.1.** The correspondence $E \mapsto \langle E_*, F^* \rangle$ extends to a functor

$$\mathcal{D}^- (\Lambda_p, \mathcal{C}) \to \mathcal{DF}^-(\Lambda_p, \mathcal{C}),$$

20
and we have a functorial exact triangle

\[ j^! \mathcal{E} \longrightarrow \mathcal{E} \xrightarrow{u} \mathcal{E}[2] \longrightarrow , \]

where the filtration in the right-hand side is shifted by 1, and \( u \) is the canonical periodicity map.

**Definition 3.2.** The filtration \( F^* \) on the object \( E \in \mathcal{D}^-(\Lambda_p, \mathcal{C}) \) is called the Hodge filtration.

By the standard formalism of filtered complexes, the Hodge filtration induces a spectral sequence in homology which starts with \( HH^q(E)[v] \), the space of polynomials in one variable \( v = u^{-1} \) of degree \(-2\), and converges to \( HC^q(E) \). We call it the Hodge-to-de Rham, or Hochschild-to-cyclic spectral sequence. It is this spectral sequence whose degeneration we are going to study in the rest of the paper.

Assume now that the integer \( p \geq 1 \) is actually an odd prime, and assume that the category \( \mathcal{C} \) is \( k \)-linear over a field \( k \) of characteristic \( p \). In this case, there is another way of looking at the Hodge filtration. Namely, consider the projection \( \pi = \pi_p : \Lambda_p \to \Lambda \) (from now on, we will fix \( p \) and drop it from the notation whenever there is no danger of confusion). By definition, for any \( E \in \text{Fun}(\Lambda_p, \mathcal{C}) \) we have

\[ H_*(\Lambda_p, E) \cong H_*(\Lambda, L^* \pi_!(E)), \]

where \( L^* \pi_! : \mathcal{D}^-(\Lambda_p, \mathcal{C}) \to \mathcal{D}^-(\Lambda, \mathcal{C}) \) is the derived functor of the direct image functor \( \pi_! \). We also have \( H^*(\Lambda, R^* \pi_* k) \cong H^*(\Lambda_p, k) \); consider the associated spectral sequence (the Hochschild-Serre spectral sequence for the fibration \( \pi : \Lambda_p \to \Lambda \)). Recall that the fibers of the fibration \( \pi : \Lambda_p \to \Lambda \) are all isomorphic to the groupoid \( \mathbf{pt}_p \), so that taking \( L^* \pi_! \) amounts to taking homology of the finite group \( \mathbb{Z}/p\mathbb{Z} \). Recall also that the cohomology algebra \( H^*(\mathbb{Z}/p\mathbb{Z}, k) \) of the cyclic group \( \mathbb{Z}/p\mathbb{Z} \) with coefficients in the trivial module \( k \) is the free graded-commutative algebra generated by a generator \( \varepsilon \) of degree 1 and a generator \( u \) of degree 2.

**Lemma 3.3.** The graded cyclic \( k \)-vector space \( R^* \pi_* k \) is a free module over the cohomology algebra \( H^*(\mathbb{Z}/p\mathbb{Z}, k) = k[u, \varepsilon] \); in particular, for any \( l \geq 0 \) we have \( R^l \pi_* k \cong k \in \text{Fun}(\Lambda, k \text{-mod}) \). The Hochschild-Serre spectral sequence for the fibration \( \pi : \Lambda_p \to \Lambda \) degenerates at \( E_3 \), we have

\[ E_2^{r,s} = H^s(\Lambda, R^r k) = \begin{cases} k, & \text{s is even,} \\ 0, & \text{s is odd,} \end{cases} \]
and the differential \( d_2 : E^{r,s}_2 \to E^{-1,s+2}_2 \) vanishes for even \( r \) and induces an isomorphism for odd \( r \).

**Proof.** To prove the first claim, it suffices to evaluate \( \mathbb{R}^r \pi_* k([n]) \) for all \([n] \in \Lambda\) and to notice that by base change, we have

\[
\mathbb{R}^r \pi_* k([n]) = H^r(\mathbb{Z}/p\mathbb{Z}, k([n])) = H^r(\mathbb{Z}/p\mathbb{Z}, k).
\]

To compute \( E^{r,s}_2 \), combine this with Lemma 2.5. Again by Lemma 2.5, the class \( \varepsilon \in E_1^{1,0} \) cannot survive in the \( E_\infty \)-term of the spectral sequence; therefore \( d_2(\varepsilon) \) generates \( E_0^{2,2} \). Once again by Lemma 2.5, the class \( u \in E_2^{2,0} \) must survive in \( E_\infty \), so that in particular \( d_2(u) = 0 \). The claim about \( d_2 \) follows by multiplicativity. This implies that 

\[
E^{r,s}_3 = 0 \quad \text{unless} \quad s = 0, 1 \quad \text{and} \quad r = 2l \quad \text{is even, and the spectral sequence degenerates at} \quad E_3 \quad \text{by dimension reasons.}
\]

**Remark 3.4.** This Lemma shows, in particular, that \( \Lambda_p \) considered as a gerbe over \( \Lambda \) is not split – conversely, its class in \( H^2(\Lambda, \mathbb{Z}/p\mathbb{Z}) \) is the generator class \( u \).

**Lemma 3.5.** For any \( E \in \text{Fun}(\Lambda_p, C) \), the complex \( j^!_p(E) \in \text{Fun}(\Lambda_p, C) \) consists of objects acyclic for the functors \( \pi_! \), \( \pi_* : \text{Fun}(\Lambda_p, C) \to \text{Fun}(\Lambda, C) \).

**Proof.** By the projection formula, it suffices to prove that \( j^!_p(k), j^*_p(k) \in \text{Fun}(\Lambda_p, k\text{-mod}) \) are acyclic for \( \pi_! \) and \( \pi_* \). By Proposition 2.2, this has to be checked for every fiber \( \pi_{[n]} \) of the bifibration \( \pi : \Lambda_p \to \Lambda \). This fiber is equivalent to \( \text{pt}_p \), so that we have to check that \( j^!_p(k)([n]) \) and \( j^*_p(k)([n]) \) are regular representations of the group \( \mathbb{Z}/p\mathbb{Z} \). This is immediate.

By virtue of Lemma 3.5, one can compute \( L^r \pi_! E \) by using the standard periodic resolution \( E_* \) assembled out of the complexes \( j^!_p(E) \). This defines a filtration \( F^* \) on \( L^r \pi_! E \) and equips it with the structure of a filtered complex in \( \text{Fun}(\Lambda, C) \); by abuse of terminology, we will also call the filtration \( F^* L^r \pi_! E \) the Hodge filtration.

**3.2 The conjugate filtration.** It turns out, however, that the complex \( L^r \pi_! E \) carries another canonical filtration, which goes in the opposite direction; we call it the conjugate filtration. We start with the following.

**Lemma 3.6.** An object \( F \in \text{Fun}(\Lambda_p, k\text{-mod}) \) is acyclic for the functor \( \pi_! \) if and only if it is acyclic for the functor \( \pi_* \); moreover, there exists a functorial
map \( T : \pi_! F \to \pi_* F \) which is an isomorphism for acyclic \( F \). For any \( k \)-linear Grothendieck category \( \mathcal{C} \), any \( E \in \text{Fun}(\Lambda_p, \mathcal{C}) \) and any acyclic \( F \in \text{Fun}(\Lambda_p, k\text{-mod}) \), \( F \otimes E \in \text{Fun}(\Lambda_p, \mathcal{C}) \) is acyclic for both \( \pi_! \) and \( \pi_* \), and 
\[ T : \pi_!(E \otimes F) \to \pi_*(E \otimes F) \] is an isomorphism.

Proof. As in the proof of Lemma 3.5, it suffices by Proposition 2.2 to check the statement for every fiber \( \pi_![n] \) of the bifibration \( \pi : \Lambda_p \to \Lambda \); since \( \pi_![n] \cong \text{pt} \), this reduces to standard facts about the homology of finite groups – in this case, the group is \( \mathbb{Z}/p\mathbb{Z} \). The map \( T \) is the Tate trace map:
\[ T([n]) = 1 + \sigma + \cdots + \sigma^{p-1} \]
is the averaging over \( \mathbb{Z}/p\mathbb{Z} \), where \( \sigma \in \mathbb{Z}/p\mathbb{Z} \) is the generator. \( \square \)

Therefore we can take a left acyclic resolution \( I_\ast \) and a right acyclic resolution \( I^\ast \) of the constant functor \( k \in \text{Fun}(\Lambda_p, k\text{-mod}) \) and combine them into an unbouded complex \( I_\# \) by taking the cone of the natural map \( I_\ast \to kI^\ast \) (we normalize the grading so that \( I_\# \) is an extension of \( I_\ast[1] \) by \( I^\ast \)).

Definition 3.7. For \( k \)-linear Grothendieck category \( \mathcal{C} \) and for any \( E \in \text{Fun}(\Lambda_p, \mathcal{C}) \), the unbounded complex
\[ \pi_!(E) = \pi_*(E \otimes I_\#) \in D(\Lambda, \mathcal{C}) \]
is called the relative Tate homology complex of \( E \) with respect to \( \pi : \Lambda_p \to \Lambda \).

The Tate homology complex \( \pi_!(E) \in D(\Lambda, \mathcal{C}) \) is obviously independent of the choice of resolutions \( I_\ast, I^\ast \) and functorial in \( E \). We have a functorial exact triangle
\[ L^\ast \pi_!(E)[1] \to \pi_!(E) \to R^* \pi_*(E) \to \]
in \( D(\Lambda, k\text{-mod}) \) and a periodicity map \( u : \pi_!(E) \to \pi_!(E)[2] \) which is compatible with the periodicity maps on \( L^\ast \pi_!(E) \) and \( R^* \pi_*(E) \).

Definition 3.8. The conjugate filtration \( W \) on the complex \( \pi_!(E) \) is defined as its canonical truncation
\[ W_l \pi_!(E) = \tau_{-2l-1} \pi_!(E). \]

This is also functorial and independent of choices. Moreover, by construction we see that \( W_l \) for \( l \geq 1 \) is actually a filtration on \( L^\ast \pi_!(E)[1] \cong \pi_!(E \otimes I_\ast)[1] \). We will adopt this point of view to avoid dealing with unbounded complexes. However, for various reasons it will be more convenient
to consider also the canonical truncation $\tau_{\leq 0} \pi_\#(E)$ equipped with the filtration induced by $W_\ast$, and we will denote this filtered complex by

$$\pi_\ast(E) = \tau_{\leq 0} \pi_\#(E).$$

The complex $\pi_\ast(E)$ is an extension of $L^\ast \pi_\!(E)[1]$ by $\pi_\ast(E)$.

We note that if we compute the complex $\pi_\#(E)$ by using the standard periodic resolution $E_\#$, then the periodicity map $u$ is induced by an invertible map of complexes $u : E_\# \to E_\#[2]$. We denote the inverse map by

$$v = u^{-1} : E_\#[2] \to E_\#.$$

This map shifts the conjugate filtration by 1 – we actually have a series of maps

$$v : W_l E_\#[2] \to W_{l+1} E_\#.$$ 

### 3.3 Periodicity in the bar resolutions.

To analyse the conjugate filtration, we will need to use several acyclic resolution in Lemma 3.6. We use the machinery of bar resolutions described in Subsection 2.1. Consider the category $\text{Fun}(\Lambda_p, C)$ of $p$-cyclic objects in a Grothendieck abelian category $C$.

There are two natural exact functors from $\text{Fun}(\Lambda_p, C)$ to itself that can play the role of $F$ in (2.3): firstly, we can take the functor $j_p j_p^!$, secondly, we can take the functor $j_p^b$ (the second one takes values in complexes in $\text{Fun}(\Lambda_p, C)$). Moreover, in the second case we can take $P = \text{Id}[1]$, with the natural embedding $\text{Id}[1] \to j_p j_p^! [1] \to j_p^b$. This gives three possible functorial resolution related by the following natural maps

$$B_\ast(j_p j_p^!)(E) \longrightarrow B_\ast(j_p^b)(E) \longleftarrow B_\ast(j_p^b, \text{Id}[1])(E).$$

All these maps are quasiisomorphisms, and all the complexes are acyclic. To simplify notation, denote $B_\ast(j_p j_p^!)(E) = B_1^\ast, B_\ast(j_p^b)(E) = B_1^b$ and $B_\ast(j_p^b, \text{Id}[1])(E) = \overline{B}_1^b$. By Lemma 3.5 for any $l \geq 1$, all three objects $B_1^\ast(E), B_1^b(E)$ and $\overline{B}_1^b(E)$ are acyclic for the functor $\pi_\ast$, so that all three resolutions can be used in Lemma 3.6 to compute the relative Tate homology complex $\pi_\ast(E)$. For any $E \in \text{Fun}(\Lambda_p, C),$

$$\pi_\ast(B_1^\ast(E)) \cong \pi_\ast(B_1^b(E)) \cong \pi_\ast(\overline{B}_1^b(E)) \cong \pi_\ast(E).$$

The complex $\overline{B}_1^b(E)$ is nothing but the standard periodic resolution of $E$ composed of the complexes $j_p^b(E)[2l], l \geq 0$, and the filtration on $\overline{B}_1^b(E)$ is
the conjugate filtration. The canonical map (2.4) for this complex is a map

\[ v : B^1(E)[2] \to B^1(E), \]

which coincides with the periodicity map (3.1). For the complex \( B^1(E) \), the map (2.4) is the map

\[ j^\dagger_p(B^1(E))[1] \to B^1(E), \]

where we denote by \( j^\dagger_p \) the kernel of the natural map \( j^\dagger_p \to \text{Id} \). Since the natural map \( \text{Id}[1] \to j^\dagger_p \) is a quasiisomorphism, the natural map \( B^1(E) \to B^1(E) \) is a quasiisomorphism for every \( E \in \text{Fun}(\Lambda_p, C) \). To see the periodicity map \( v \), we have to compose the map (3.5) with the natural quasiisomorphism \( B^1(E) \to j^\dagger_p \).

In Section 4, we will need to use the complex \( B^1_p \) and in particular, to interpret the periodicity in terms of this complex. To do this, we note that \( \pi_* \circ j_p = j_* \), and by Lemma 3.5 and Lemma 3.6 we have \( \pi_* \circ j^p \cong \pi_* \circ j^p \cong j_* \). Therefore \( \pi_*(j^p) \cong j^p \), and by adjunction, we have a map

\[ \varphi : \pi^*(j^p) \to j^p. \]

Both sides are complexes of length 2 with homology in degree 0 and 1 equal to the constant functor \( k \); the right-hand side is a Yoneda representation of the periodicity class \( u(p) \in H^2(\Lambda_p, k) \), and the left-hand side represents \( \pi^*(u) \in H^2(\Lambda_p, k) \), which is equal to 0 by Lemma 3.5. The map \( \varphi \) is an isomorphism on homology in degree 1, and since \( \pi^*(u) = 0 \), it is trivial on homology in degree 0. Thus the map \( \varphi \) actually maps \( \pi^*(j^p) \) into \( j^p \), \( j^p \subset j^p \), and thus extends the canonical embedding \( k[1] \to j^p \). Composing \( \varphi \) with the canonical map (3.5), we extend the periodicity map (3.4) to a map

\[ v : \pi^*(j^p(k)) \otimes B^1(E)[1] \to B^1(E). \]

We will now denote by \( k(1) \subset j^1 k \in \text{Fun}(\Lambda_p, k\text{-mod}) \) the kernel of the canonical map \( j^1 k \to k \), and for every \( E \in \text{Fun}(\Lambda, C) \), or \( E \in \text{Fun}(\Lambda_p, C) \), we will denote \( E(1) = E \otimes k(1) \), resp. \( E(1) = E \otimes \pi^*k(1) \). The map \( \varphi \) in (3.6) induces in particular a map \( \varphi : E(1) \to j^p \pi^*E \), which actually goes into the
kernel of the canonical map $j_p j^*_p E \to E$. Composing this map $\varphi$ with the canonical map $\text{2.4}$ for the complex $B^i_\cdot(E)$, we obtain a functorial map

$\varpi : B^i_\cdot(E)(1)[1] \to B^i_\cdot(E)$.

(3.8)

Now, the object $k(1) \in \text{Fun}(\Lambda, k\text{-mod})$ is also the cokernel of the canonical map $k \to j_\ast k$. Therefore by Lemma $\text{2.4}$ we have

$H_\ast(\Lambda, E(1)) \cong H_\ast(\Lambda, E)[1]$

for any $E \in \text{Fun}(\Lambda, C)$. More generally, by the projection formula, we have

(3.9) $H_\ast(\Lambda, \pi_\ast (E(1))) \cong H_\ast(\Lambda, \pi_\ast (E)(1)) \cong H_\ast(\Lambda, \pi_\ast E)[1]$

for any $E \in \text{Fun}(\Lambda_p, C)$;

**Lemma 3.9.** For any $E \in \text{Fun}(\Lambda_p, C)$, the map

$H_\ast(\Lambda, \pi_\ast B^i_\cdot(E)(1)[1]) \to H_\ast(\Lambda, \pi_\ast B^i_\cdot(E))$

induced by the map $\varpi$ from (3.8) and the map

$H_\ast(\Lambda, \pi_\ast \overline{B}^i_\cdot(E)[2]) \to H_\ast(\Lambda, \pi_\ast \overline{B}^i_\cdot(E))$

induced by the periodicity map $v$ from (3.4) become equal under the identification (3.9).

**Proof.** By construction, the diagram (3.2) extends to a diagram

$B^i_\cdot(E)[1](1) \longrightarrow B^i_\cdot[1](E) \otimes \pi^* \overline{j}^! k \longleftarrow \overline{B}^i_\cdot(E)[2]$

(3.10)

$\varpi \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow v$

$B^i_\cdot(E) \longrightarrow B^i_\cdot(E) \longleftarrow \overline{B}^i_\cdot(E)$,

and by the projection formula, it suffices to prove that for any complex $E'$ in $\text{Fun}(\Lambda, C)$ – in particular, for $E' = \pi_\ast B^i_\cdot(E)$ and such – the natural maps

$H_\ast(\Lambda, E'(1)) \longrightarrow H_\ast(\Lambda, E' \otimes \overline{j}^!(k)) \longleftarrow H_\ast(\Lambda, E'[1])$

are quasiisomorphisms. Indeed, the map on the right-side is a quasiisomorphism already in $\text{Fun}(\Lambda, C)$, while the cone of the map on the left-hand side is $H_\ast(\Lambda, j_\ast j^* E')$, which is trivial by Lemma $\text{2.4}$.

□

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As an application, Lemma 3.9 allows to see the conjugate filtration in the bar resolutions $B^i_q(E)$ and $B^j_q(E)$. Namely, by definition $B^j_q(E)$ carries the conjugate filtration $W^j_q$, and in particular, we have $W^j_1B^j_q(E) = \tau^{\leq -1}B^j_q(E)$. Set $W^i_1B^i_q(E) = \tau^{\leq -1}B^i_q(E)$, and note that all the vertical maps in (3.10) are injective. Therefore we can define inductively

$$W^lB^i_q(E) = v(W^{l-1}B^i_q(E)[1] \pi^* k),$$

$$W^lB^j_q(E) = \pi^* j^* k).$$

This turns $B^i_q(E)$ and $B^j_q(E)$ into filtered complexes, and all the maps in (3.11) are filtered maps.

**Corollary 3.10.** The natural map $\pi_*B^i_q(E) \to \pi_*B^i_q(E)$ is a filtered quasiisomorphism, while the natural map $\pi_*B^i_q(E) \to \pi_*B^i_q(E)$ becomes a filtered quasiisomorphism after applying the cyclic homology functor $HC_*$. 

**Proof.** By Lemma 3.9 it suffices to notice that the map $k[1] \to j^* \pi$ is a quasiisomorphism in Fun($\Lambda, k$-mod), while the map $k[1] \to j^* k$ becomes a filtered quasiisomorphism after applying $HC_*$.

3.4 Tight $\mathbb{Z}/p\mathbb{Z}$-modules. We will now use the bar resolutions to compute, under some assumptions on the $p$-cyclic object $E \in \text{Fun}(\Lambda_p, C)$, the associated graded quotients of the conjugate filtration on $\pi_5(E)$.

Let $V$ be a vector space over the field $k$ – or, more generally, an object in a $k$-linear Grothendieck category $C$ – equipped with an action of the group $\mathbb{Z}/p\mathbb{Z}$. Denote by $K_*\langle V \rangle$ the complex

$$V_{\mathbb{Z}/p\mathbb{Z}} \xrightarrow{T} V_{\mathbb{Z}/p\mathbb{Z}}$$

placed in degrees 0 and $-1$ (here $T$ is the Tate trace map from coinvariants to invariants with respect to the group $\mathbb{Z}/p\mathbb{Z}$). The natural map $V_{\mathbb{Z}/p\mathbb{Z}} \to V \to V_{\mathbb{Z}/p\mathbb{Z}}$ factors through a map

$$\varphi : H_0(K_*\langle V \rangle) \to H_1(K_*\langle V \rangle).$$

The complex $K_*\langle V \rangle$ is actually a piece of the standard periodic complex which computes the homology $H_*(\mathbb{Z}/p\mathbb{Z}, V)$, so that we have

$$H_0(K_*\langle V \rangle) \cong H_{\text{odd}}(\mathbb{Z}/p\mathbb{Z}, V),$$

$$H_1(K_*\langle V \rangle) \cong H_{\text{even}}(\mathbb{Z}/p\mathbb{Z}, V),$$

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and the map $\varphi$ is given by the action of the standard generator $\varepsilon$ of the first cohomology group $H^1(\mathbb{Z}/p\mathbb{Z}, k)$.

**Definition 3.11.** The representation $V$ of the group $\mathbb{Z}/p\mathbb{Z}$ is called *tight* if the map $\varphi$ is an isomorphism.

For example, a trivial representation $k$ is tight, and so is the regular representation $k[\mathbb{Z}/p\mathbb{Z}]$. In other words, a representation is tight if $\varepsilon : H_{\text{odd}}(\mathbb{Z}/p\mathbb{Z}, V) \to H_{\text{even}}(\mathbb{Z}/p\mathbb{Z}, V)$ is an isomorphism. We note that since $\varepsilon^2 = 0$, this automatically implies that $\varepsilon : H_{\text{even}}(\mathbb{Z}/p\mathbb{Z}, V) \to H_{\text{odd}}(\mathbb{Z}/p\mathbb{Z}, V)$ is a trivial map. For any tight $\mathbb{Z}/p\mathbb{Z}$-module $V$ in a category $C$, we will denote by $\pi_0(V)$ the object $H_0(K, \langle V \rangle) \cong H_1(K, \langle V \rangle) \in C$.

Assume given an object $E \in \text{Fun}(\Lambda, C)$; then $E([m])$ is a $\mathbb{Z}/p\mathbb{Z}$-module in $C$ for every $[m] \in \Lambda$, and all the complexes $K_0(K, \langle E([m]) \rangle)$ fit together into a single complex 

$$\pi_1E \to \pi_0E,$$

which we denote by $K_0(E) \in \text{Fun}(\Lambda, C)$. The maps $\varphi$ for various $E([m])$ fit into a single map

$$(3.12) \quad \varphi = \varphi_E : H_0(K_0(E)) \to H_1(K_0(E)).$$

**Definition 3.12.** An object $E \in \text{Fun}(\Lambda, C)$ is called *tight* if the map $\varphi_E$ is an isomorphism – or, equivalently, if for any $[m] \in \Lambda$ the object $E([m])$ is tight with respect to $pt_p \times [m] = \pi_0[E] \subset \Lambda$. For a tight object $E \in \text{Fun}(\Lambda, C)$, we denote by $l(E) \in \text{Fun}(\Lambda, C)$ the object $H_0(K_0(E)) \cong H_1(K_0(E))$.

**Lemma 3.13.** For any tight $E \in \text{Fun}(\Lambda, C)$, the periodicity map $v$ induces a filtered exact triangle

$$(3.13) \quad \pi_0(E)[1] \xrightarrow{v} \pi_0(E) \longrightarrow K_0(E) \longrightarrow$$

in $\mathcal{DF}^-(\Lambda, C)$, where the conjugate filtration on the left is shifted by 1, and the complex $K_0(E)$ is equipped with the canonical filtration. Moreover, for any $l \geq 1$ the map $v$ induces a filtered exact triangle

$$(3.14) \quad W_{l+1} \pi_0(E) \longrightarrow W_l \pi_0(E) \longrightarrow j^l l(E)[2l] \longrightarrow ,$$

where again the conjugate filtration on the left is shifted by 1, and the complex $j^l l(E)$ is equipped with a trivial one-step filtration of degree $l$. 

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Proof. Recall that we have filtered quasiisomorphisms
\[ \pi_0(E) \cong \pi_*(\overline{B}^1_0(E)) \cong \pi_*(B^1_0(E)). \]
By definition, \( \pi_*(\overline{B}^1_0(E)) \cong \pi_*(E) \), and \( H_1(\pi_*(\overline{B}^1_{\geq 1}(E)) \cong \pi_1(E) \), so that we
have a natural projection \( \pi_*(\overline{B}^1_{\geq 1}(E)) \to K_\langle E \rangle \), and it obviously fits into a
sequence
\[ 0 \longrightarrow \pi_*(\overline{B}^1_{\geq 1}(E))[2] \stackrel{v}{\longrightarrow} \pi_*(\overline{B}^1(E)) \longrightarrow K_\langle E \rangle \longrightarrow 0 \]
of filtered complexes. This sequence is not exact in the middle term, but its
homology is an acyclic complex with trivial one-step filtration, so that after
passing to the filtered derived category we obtain an exact triangle. This is
the triangle (3.13). To prove (3.14), we may assume by induction that \( l = 1 \).
The periodicity map \( v \) is filtered by construction and identical in degrees
\( \neq 1 \); we have to prove that \( \text{gr}^W_0 \pi_*(E) \cong j^1 I_\langle E \rangle \).
Indeed, by the projection formula, the canonical filtered map (3.7) induces a map
\[ j^1 k \otimes \pi_*(\overline{B}^1(E)) \cong j^1 \pi_*(\overline{B}^1(E)) \to \pi_*(\overline{B}^1(E)), \]
which in particular gives a map
\[ \varphi : j^1 (\text{gr}^W_0 \pi_*(\overline{B}^1(E))) \to \text{gr}^W_1 \pi_*(\overline{B}^1(E)). \]
Since \( \text{gr}^W_0 \pi_*(\overline{B}^1(E)) \cong H_0(K_\langle E \rangle) \) by (3.13), it suffices to prove that this
map \( \varphi \) is a quasiisomorphism. Both sides are complexes with non-trivial
homology in degrees 0 and 1 only, and, again by (3.13), the homology in
degree 1 on both sides is equal to \( H_0(K_\langle E \rangle) \) and identified by \( \varphi \). In degree
0, the homology on the right-hand side is \( H_1(K_\langle E \rangle) \), and the homology on
the left-hand side is again \( H_0(K_\langle E \rangle) \). It is elementary to check that the
induced map \( \varphi : H_0(K_\langle E \rangle) \to H_1(K_\langle E \rangle) \) is the same map as in (3.12);
since \( E \) is assumed tight, \( \varphi \) is an isomorphism. \( \square \)

As a corollary, we see that for any tight \( E \in \text{Fun}({\Lambda}_p,C) \), the associated
graded quotients \( \text{gr}^W_1 HC_\langle E \rangle \) are isomorphic to the cyclic homol-
ogy \( HC_\langle (j^1 I_\langle E \rangle) \rangle \), which is by definition equal to the Hochschild homol-
ogy \( HH_\langle (1_\langle E \rangle) \rangle \). This is the linear-algebraic origin of our non-commutative
Cartier isomorphism.

4 Conjugate filtration in detail.

We will now investigate the conjugate filtration on cyclic homology in some
detail. Unfortunately, it seems that apart from Lemma 6.13 nothing can
be said for general $p$-cyclic objects, even for those which are tight. Thus in general, we can identify the associated graded quotients of the conjugate filtration but cannot study the extension data between these quotients. In order to have any control over the extension data, we need to impose an additional symmetry on our objects, which we will call polycyclic.

4.1 Polycyclic groups. For any finite set $S$, denote by $G_S$ the abelian group $\mathbb{Z}/p\mathbb{Z}[S]$ – the free $\mathbb{Z}/p\mathbb{Z}$-module generated by $S$. Fix a finite set $S$ and assume given a $G_S$-module in a $k$-linear abelian category $C$ over a field $k$ of characteristic $p$. For every subset $I \subset S$, denote

\[ V_I = H_0(G_I, H^0(G_{\bar{T}}, V)), \]

where $\bar{T} = S \setminus I$ is the complement to $I \subset S$. For any two subsets $I \subset I' \subset S$,

we have natural transition maps

\[ \tau_{I, I'} : V_{I'} \to V_I \quad \sigma_{I, I'} : V_I \to V_{I'} \]

induced by the natural embedding $H^0(G_{\bar{T}}, V) \hookrightarrow H^0(G_{\bar{T}'}, V)$ and the trace map $H^0(G_{\bar{T}'}, V) \to H^0(G_{\bar{T}}, V)$. Fix a linear order on $S$, so that $S = \{i, 1 \leq i \leq n\}$, where $n = |S|$ is the number of elements in $S$. For any $I \subset S$, $i \in \bar{T}$, let $l(i, I)$ be the number of elements in $I$ which are less than $i$. Then we define a map

\[ d_I = \sum_{i \in \bar{T}} (-1)^{l(i, I)} \tau_{I, I \cup \{i\}} : V_I \to \bigoplus_{i \in \bar{T}} V_{I \cup \{i\}}, \]

and if we set

\[ K_I(V) = \bigoplus_{|I|=l} V_I, \quad d_I = \bigoplus_{|I|=l} d_I, \]

then $d_{l+1} \circ d_l = 0$, so that $\langle K_I(V), d_I \rangle$ becomes a complex of length $|S| + 1$. For any $i \in S$, define a map $v_i : K_i(V) \to K_{i+1}(V)$ by

\[ \varphi_i = \sum_{i \in \bar{T}} (-1)^{l(i, \bar{T})} \sigma_{I\setminus\{i\}, I}. \]

Then $\varphi_i, i \in S$ anticommute with each other and with the differential $d$, so that the homology $H_*(K_i(V))$ becomes a module over the exterior algebra $\Lambda^* \langle \varphi_1, \ldots, \varphi_n \rangle$ generated by $\varphi_1, \ldots, \varphi_n$.

**Definition 4.1.** A $G_S$-module $V$ is called tight if the homology $H_*(K_i(V))$ is the free $\Lambda^* \langle \varphi_1, \ldots, \varphi_n \rangle$-module generated by $H_0(K_i(V))$. 

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It is easy to see that when \( n = |S| = 1 \), this reduces to Definition 3.11 so that the terminology and notation is consistent with Subsection 3.4.

We will now give a different construction of the complex \( K_\ast(V) \) which is more canonical (and in particular, manifestly independent of the choice of an order on \( S \)). Namely, let \( \mathcal{C}' \) be the category of \( G_S \)-modules in \( \mathcal{C} \). For any \( i \in S \), let \( R_i = k[\mathbb{Z}/p\mathbb{Z}] \) be the regular representation of \( \mathbb{Z}/p\mathbb{Z} \) over the field \( k \) considered as a \( G_S \)-module by means of the natural projection \( G_S \to G_{\{i\}} = \mathbb{Z}/p\mathbb{Z} \). Define a functor \( F : \mathcal{C}' \to \mathcal{C}' \) by

\[
F(V) = \bigoplus_{i \in S} V \otimes R_i.
\]

This is obviously an exact functor; the trace projections \( R_i \to k \) induce a map \( F \to \text{Id} \). Therefore we can set up a bar resolution as described in Subsection 2.1 and obtain a functorial resolution \( B_\ast(F)(V) \). We form a canonical complex \( C_\ast(V) \) by setting

\[
(4.2) \quad C_\ast(V) = H^0(G_S, B_\ast(F)(V)).
\]

Explicitly, we have

\[
B_l(F)(V) = V \otimes \left( \bigoplus_{i} R_i \right)^{\otimes l} = V \otimes \bigoplus_{i_1, \ldots, i_l \in S} R_{i_1} \otimes \cdots \otimes R_{i_l},
\]

and this space is naturally graded by the number of different multiples on the right-hand side: we set

\[
(4.3) \quad B_{l-m,m}(V) = V \otimes \bigoplus_{|\{i_1, \ldots, i_l\}|=m} R_{i_1} \otimes \cdots \otimes R_{i_l}.
\]

It is easy to see that this is a bicomplex – the differential \( d \) splits into a sum of a component \( d^{1,0} \) of bidegree \((1,0)\) and a component \( d^{0,1} \) of bidegree \((0,1)\). This grading and the bicomplex structure descend to the complex \( C_\ast(V) \). The grading can be further refined by specifying precisely the subset \( \{i_1, \ldots, i_l\} \subset S \) – for any \( I \subset S \), we set

\[
B_{l-|I|,|I|}(V) = V \otimes \bigoplus_{\{i_1, \ldots, i_l\}=I} R_{i_1} \otimes \cdots \otimes R_{i_l}.
\]

**Lemma 4.2.** For every \( I \subset S \), the complex \( B_{\ast,I}(V) \) is a free resolution of \( V \) considered as a \( G_I \)-module. Moreover, we have

\[
H_0(C_\ast(V), d^{0,1}) \cong K_\ast(V)
\]

– in other words, \( K_\ast(V) \) is the 0-th homology complex of the bicomplex \( C_\ast(V) \) with respect to the differential \( d^{0,1} \).
Proof. Every product

\[ R = R_{i_1} \otimes \cdots \otimes R_{i_l} \]

is obviously a free \( G_I \)-module, where \( I = \{ i_1, \ldots, i_l \} \), while \( G_T \) acts trivially on \( R \). Therefore the complex \( B_{*,I}(V) \) indeed consists of free \( G_I \)-modules, and moreover,

\[ H^0(G_S, B_{*,I}(V)) \cong H^0(G_I, B_{*,I}(H^0(G_T, V))) \cong H_0(G_I, B_{*,I}(H^0(G_T, V))). \]

Thus the second claim follows from the first, and to prove the first, it suffices to prove that the complex \( B_{*,I}(V) \) is indeed a resolution of \( V \). It obviously suffices to consider \( C = k\text{-mod}, V = k \) with the trivial \( G_S \)-action. By induction on \( n = |S| \), it suffices to consider \( I = S \), and we may assume the claim proved for all proper subsets \( I \subset S \). Then instead of proving that \( B_{*,n} \to k \) is a quasiisomorphism, we may prove that \( B_{*,} \to k \), is a quasiisomorphism, where \( k_{*} \) is the complex given by

\[ k_m = \bigoplus_{|I|=m} k, \quad d = \sum_{i \in I} (-1)^{|i|} l(i,I) \text{id}. \]

This latter complex is easily seen to be acyclic, while the bar-complex \( B_{*} \) is acyclic by construction. This finishes the proof.

We see that the order on \( S \) that we used to construct the complex \( K_{*,\langle V \rangle} \) is an artefact of the particular explicit construction: the bar resolution does not depend on the order and gives the same thing. To see the maps \( \varphi_i \) in this language, we note that the subgroup of \( G_S \)-invariants in \( R_i \) is the trivial \( G_S \)-module \( \mathbb{Z} \); this gives a canonical embedding \( k \to R_i \) and a functorial map \( \varphi_i : \text{Id} \to F \). As in (2.4), this gives rise to functorial maps

\[ \varphi_i : B_{*,}(V)[1] \to B_{*}, \quad \varphi_i : C_{*,\langle V \rangle} \to C_{*,\langle V \rangle} \]

and on the level of homology \( H_0(C_{*,\langle V \rangle}, d^{0,1}) \), this is the same map \( \varphi_i \) as in (4.4). We can also collect all the maps \( \varphi_i \) together into a single map

\[ \varphi : k[S] \otimes B_{*,}(V)[1] \to B_{*}(V). \]

One final result we will need is the following. Consider the multiplicative group \((\mathbb{Z}/p\mathbb{Z})^*\), and let it act on \( G_S = \mathbb{Z}/p\mathbb{Z}[S] \) by dilations. Assume that a \( G_S \)-module \( V \) is tight, and that the \( G_S \)-action on \( V \) is extended to an action of the semidirect product \( G_S \rtimes (\mathbb{Z}/p\mathbb{Z})^* \). Moreover, assume that the induced \((\mathbb{Z}/p\mathbb{Z})^*\)-action on the homology \( H_0(K_{*,\langle V \rangle}) \) is trivial.
Lemma 4.3. In the assumptions above, the canonical map
\[ \tau_{[0,l]} C_q \langle V \rangle^{(\mathbb{Z}/p\mathbb{Z})^*} \to \tau_{[0,l]} K_q \langle V \rangle \]
is a quasiisomorphism whenever \( l \leq 2(p - 2) \).

Proof. The canonical maps \( \varphi_i : H_q(K_q \langle V \rangle) \to H_{q+1}(K_q \langle V \rangle) \) are obviously \((\mathbb{Z}/p\mathbb{Z})^*\)-equivariant; therefore in the assumptions above, \((\mathbb{Z}/p\mathbb{Z})^*\) acts trivially on the homology \( H_l(K_q \langle V \rangle) \) for any \( l \). By Lemma 4.2, it suffices to prove that for every \( I \subset S \), the natural map
\[ H_q(G_I, H^0(G_T, V))^{(\mathbb{Z}/p\mathbb{Z})^*} \to V_I \]
is a quasiisomorphism in degrees \( \leq 2(p - 2) \) – in other words, we have to check that \( H_l(G_I, H^0(G_T, V)) \) has no \((\mathbb{Z}/p\mathbb{Z})^*\)-invariant elements when \( 0 < l \leq 2(p - 2) \). But indeed, by the same assumption of tightness, we have
\[ H_l(G_I, H^0(G_T, V)) \cong V_I \otimes H_l(G_I, k), \]
and \((\mathbb{Z}/p\mathbb{Z})^*\) acts through the second factor on the right-hand side, so that it suffices to check that \( H^0((\mathbb{Z}/p\mathbb{Z})^*, H_l(G_I, k)) = 0 \) for \( 0 < l \leq 2(p - 1) \). This is obvious: the homology space \( H_*(G_I, k) \) is the free module of rank 1 over the free graded-commutative algebra – or rather, coalgebra – \( k[v_1, \ldots, v_m, \varphi_1, \ldots, \varphi_m] \) generated by \( m = |I| \) elements \( v_1, \ldots, v_m \) of degree \(-2\) and \( m \) elements \( \varphi_1, \ldots, \varphi_m \) of degree \(-1\), and all the generators are \((\mathbb{Z}/p\mathbb{Z})^*\)-eigenvectors with weight \(1\). □

Remark 4.4. It is rather unfortunate that we have to define the complex \( C_q \langle V \rangle \) as an explicit complex, by fixing an explicit resolution \( B_q(V) \). It would be much nicer to be able to define it as a derived functor of some type. Then \( |S| = 1 \), it is possible, and we indeed do this in Lemma 3.6: the complex \( C_q \langle V \rangle \) can be equivalently defined as the negative part of the canonical filtration in the Tate homology complex \( H_\#(G_S, V) \); if \( V \) is tight, the homology of the complex \( C_q \langle V \rangle \) is isomorphic to
\[ H_\#(G_S, V)/H^*(G_S, V) = H^*(G_S, V) \otimes_{k[[u]]} k[u^{-1}], \]
where \( u \in H^2(G_S, \mathbb{Z}) \) is the periodicity generator, and none of this depends on any choices of a free resolution. When \( n = |S| > 1 \), we still have
\[ H_*(C_q \langle V \rangle) \cong H^*(G_S, V) \otimes_{k[[u_1, \ldots, u_n]]} k[u_1^{-1}, \ldots, u_n^{-1}], \]
but this is not a good description from the general homological point of view (in particular, it makes no sense in the derived category and it is not sufficiently functorial in $S$). A derived category interpretation is unknown, and none of the obvious candidates give the correct answer. A similar situation occurs, for instance, in the representation theory of affine Kac-Moody algebras (see e.g. [FF]): one can construct a certain type of homology by an explicit resolution, and while it is intuitively clear that the choice of a resolution should be irrelevant, there is no general categorical framework which makes it precise.

4.2 Polycyclic categories. For any set $X$, define a natural contravariant functor $X^\# : \Lambda^{opp} \to \text{Sets}$ by

$$(4.5) \quad X^\#([m]) = \text{Maps}(\Lambda([1],[m]), X) = X^m.$$ 

If $X = G$ is a group, then $G^\#$ can be treated as a functor from $\Lambda^{opp}$ to the category of groups.

**Definition 4.5.** The wreath product $G \wr \Lambda$ is the fibered category over $\Lambda$ which is the total space of the functor $G^\#$. The polycyclic categories $B_p$, $p \geq 1$, and $B_\infty$ are the wreath products $B_p = (\mathbb{Z}/p\mathbb{Z}) \wr \Lambda$, $B_\infty = \mathbb{Z} \wr \Lambda$.

Explicitly, objects of $G \wr \Lambda$ are $[n]$, $n \geq 1$, and for any $[n],[m]$ the set of morphisms from $[m]$ to $[n]$ is given by

$$(G \wr \Lambda)([m],[n]) = G^m \times \Lambda([m],[n])$$

$$= \{ (f', f) \mid f' \in \text{Hom}_\sigma([m],G), f \in \text{Hom}_{\text{Cycl}}([m],[n]) / \sigma \} ,$$

with composition defined by $(g', g) \circ (f', f) = (g' \circ f')f' \circ f$. Here we recall that Cycl is the category of linearly ordered sets equipped with an order-preserving automorphism $\sigma$, for any $l \geq 1$, $[l] \in \text{Cycl}$ is the linearly ordered set $\mathbb{Z}$ with $\sigma : x \mapsto x + l$, and we interpret $G^m$ as the set of $\sigma$-invariant maps from $[m] = \mathbb{Z}$ to $G$.

In the particular case $G = \mathbb{Z}$, $G \wr \Lambda = B_\infty$, we define for any $[m],[n]$ a subset $\overline{B}_\infty([m],[n]) \subset B_\infty([m],[n])$ by

- $\langle f', f \rangle \in \overline{B}_\infty([m],[n])$ if and only if for any $l$, $0 \leq l < m$, we have

$$(4.6) \quad 0 \leq f(l) + nf'(l) < n.$$ 

It is easy to see that $(4.6)$ is preserved by the composition law, so that it defines a subcategory $\overline{B}_\infty \subset B_\infty$. Moreover, for any $f \in \text{Hom}_{\text{Cycl}}([m],[n])$
there is exactly one $f'$ such that $\langle f', f \rangle$ satisfy (4.6). We conclude that $\mathcal{B}_\infty \cong \Lambda_\infty$, so that we have a canonical embedding $\lambda_\infty : \Lambda_\infty \to \mathcal{B}_\infty$. Reducing this modulo $p$, we obtain a canonical embedding $\lambda_p : \Lambda_p \to \mathcal{B}_p$.

For any small category $\Sigma$ equipped with a functor $\Sigma \to \Lambda$, we denote $G \int \Sigma = \Sigma \times_\Lambda G \int \Lambda$. In particular, we have the simplicial category $\Delta^o$ equipped with the functor $j : \Delta^o \to \Lambda$, and we can form the wreath product $G \int \Delta^o$. Denote $\Delta^o_p = (\mathbb{Z}/p\mathbb{Z}) \int \Delta^o$. We have a natural embedding $\Delta^o_p \to \mathcal{B}_p$, and a Cartesian diagram

$$
\begin{array}{ccc}
\Delta^o \times \text{pt}_p & \xrightarrow{j_p} & \Lambda_p \\
\downarrow & & \downarrow \lambda_p \\
\Delta^o_p & \longrightarrow & \mathcal{B}_p.
\end{array}
$$

(4.7)

But by definition, for any $[m] \in \Delta^o$, the set $\Lambda([1], j([m]))$ has a distinguished element; therefore for any set $X$ the pullback $j^* X^# \to \Delta^o \to \Lambda$ admits a canonical projection $X^# \to X\Delta^o$ onto the constant functor $X\Delta^o : \Delta^o \to \text{Sets}$. If $X = \mathbb{Z}/p\mathbb{Z}$, then this is compatible with the group structure and induces a projection $\Delta^o_p \to \Delta^o \times \text{pt}_p$. Denote

$$
\overline{\Delta}^o_p = \Delta^o_p \times_\Delta^o \times \text{pt}_p \Delta^o.
$$

Then the Cartesian diagram (4.7) extends to a diagram

$$
\begin{array}{ccc}
\Delta^o & \longrightarrow & \Delta^o \times \text{pt}_p \\
\downarrow & & \downarrow \lambda_p \\
\overline{\Delta}^o_p & \longrightarrow & \Delta^o_p \\
\longrightarrow & & \longrightarrow \mathcal{B}_p
\end{array}
$$

(4.8)

with Cartesian squares. By abuse of notation, we will denote the composition $\overline{\Delta}^o_p \to \Delta^o_p \to \mathcal{B}_p$ of the embeddings in the bottom row by the same letter $j_p$.

Finally, note that the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$ acts on $\mathbb{Z}/p\mathbb{Z}$ and consequently on $\mathcal{B}_p$, so that we can form the semidirect product $\mathcal{B}_p = \mathcal{B}_p \rtimes (\mathbb{Z}/p\mathbb{Z})^*$: it has the same objects $[n]$, $n \geq 1$, we set

$$
\mathcal{B}_p([m], [n]) = \{ \langle f, f' \rangle | f \in \mathcal{B}_p([m], [n]), f' \in (\mathbb{Z}/p\mathbb{Z})^* \},
$$

and the composition law is $\langle g, g' \rangle \circ \langle f, f' \rangle = \langle f'(g)f, g'f' \rangle$. We have a natural embedding $\mathcal{B}_p \subset \mathcal{B}_p$, and the embedding $\lambda_p : \Lambda_p \to \mathcal{B}_p$ extends to an embedding $\overline{\lambda}_p : \Lambda_p \to \mathcal{B}_p$. We will call Fun($\mathcal{B}_p, \mathcal{C}$) extended polycyclic category, and we will call its objects extended polycyclic.
4.3 Tight polycyclic objects. For any $E \in \text{Fun}(B_p, C)$, we can form the bar resolution $B^i_p(\Lambda^*_p E) = B_*(j_pq_p^*)(\Lambda^*_p E) \in \text{Fun}(\Lambda^*_p, C)$ as in Subsection 3.3. Applying the base change to the diagram (4.8), we see that the functor $j_pq_p^*$ extends to a functor $j_pq_\ast : \text{Fun}(B_p, C) \to \text{Fun}(B_p, C)$, so that the resolution $B^i_p(\Lambda^*_p E)$ comes from a polycyclic resolution $B^i_*(E) \in \text{Fun}(B_p, C)$. For any $[n] \in B_p$, the complex $B^i_*(E)([n])$ is the functorial bar resolution $B_*(E)(E([n]))$ of the polycyclic module $E([n])$ that we have considered in Subsection 4.1. If we apply the direct image functor $\chi_* : \text{Fun}(B_p, C) \to \text{Fun}(\Lambda, C)$, then we have

$$\chi_* (B^i_*(E))([n]) \cong C_\ast (E([n])),$$

where $C_\ast (E([n]))$ is as in (4.2). The additional grading (4.3) on $B^i_*(E)$ is not compatible with maps $[n] \to [n']$ in the category $\Lambda$. However, one checks easily that the associated filtration is preserved by the maps: if we set

$$F_q B^i_*(E)([n]) = \bigoplus_{m \geq q} B_{m,l-m}(E([n]))$$

for every $[n]$, then these fit together into a subobject $F_q B^i_*(E) \subset B^i_*(E)$. Denote

$$K_\ast (E) = \chi_*(B^i_*(E))/(\chi_*(F_1 B^i_*(E)) + d\chi_*(F_1 B^i_*(E))),$$

where $d$ is the differential in the complex $B^i_*(E)$. Then $K_\ast (E)$ is a well-defined complex in $\text{Fun}(\Lambda, C)$, functorial in $E$, and Lemma 4.2 immediately shows that we have

$$K_\ast (E)([n]) = K_\ast (E([n]))$$

for any $[n] \in \Lambda$.

Recall that we have defined in Subsection 3.3 a canonical map $\pi^*(jik) \to j_pq_k$ of $p$-cyclic objects; the reader will check easily that this map is compatible with the polycyclic stucture on $j_pq_k$ and gives a map $\varphi : \chi^*(jik) \to j_pq_k$ of objects in $\text{Fun}(B_p, k\text{-mod})$. Composing this with (4.1), we obtain a functorial map

$$\varphi : \chi^* jik \otimes B^i_*(E)[1] \to B^i_*(E)$$

for any $E \in \text{Fun}(B_p, C)$, and as in Subsection 3.3, this allows to introduce the conjugate filtration $W_*$ on $B^i_*(E)$ and on $\chi_* (B^i_*(E))$. We note that if we equip $K_\ast (E)$ with the canonical filtration, then the natural map $\chi_*(B^i_*(E)) \to K_\ast (E)$ is a filtered map. By the projection formula, the map $\varphi$ induces a map

$$\varphi : jik \otimes \chi_* (B^i_*(E))[1] \to \chi_* (B^i_*(E)).$$
We note that for any \([n] \in \Lambda\), we have \(j_i k([n]) = k[S]\), where the finite set \(S\) is the set of maps \(\text{Maps}_\Lambda([1],[n])\), and \(\varphi\) evaluated at \([n]\) is just the canonical map \((4.4)\). In particular, \(\varphi\) induces a map

\[
\varphi : j_i k \otimes K_\bullet \langle E \rangle [1] \to K_\bullet \langle E \rangle.
\]

Restricting this to \(k(1) \subset j_i k\), we obtain a map

\[
v : K_\bullet \langle E \rangle (1)[1] \to K_\bullet \langle E \rangle.
\]

It turns out that a statement completely analogous to Lemma 3.13 is true for the complex \(K_q \langle E \rangle\). Namely, for any \(E \in \text{Fun}(B_p,C)\) and any \([n] \in B_p\), the object \(E([n]) \in C\) by definition carries a representation of the polycyclic group \((\mathbb{Z}/p\mathbb{Z})^n\) or, more precisely, of the group \(\mathbb{Z}/p\mathbb{Z}[S]\), where \(S = \text{Maps}([1],[n])\).

**Definition 4.6.** The object \(E \in \text{Fun}(B_p,C)\) is called tight if

(i) for any object \([n] \in B_p\), the \((\mathbb{Z}/p\mathbb{Z})^n\)-module \(E([n])\) is tight in the sense of Definition 4.1, and,

(ii) the object \(\lambda_p^* E \in \text{Fun}(\Lambda_p,C)\) is tight in the sense of Definition 3.12, and the natural map

\[
\lambda_p^* : l(E) \to l(\lambda_p^* E)
\]

is an isomorphism (where we denote \(l(E) = H_0(K_\bullet \langle E \rangle)\)).

**Remark 4.7.** The condition (ii) of Definition 4.6 is probably redundant, but we were not able to prove it, and in practical applications, it is very easy to check the condition by hand.

**Lemma 4.8.** Let \(E \in \text{Fun}(B_p,C)\) be tight in the sense of Definition 4.6. Then there exists a triangle

\[
K_\bullet \langle E \rangle (1)[1] \overset{v}{\longrightarrow} K_\bullet \langle E \rangle \longrightarrow K_\bullet (\lambda_p^* E) \longrightarrow
\]

which, if we equip all the terms with the canonical filtration, becomes a filtered exact triangle after applying the cyclic homology functor \(HC_\bullet\).

**Proof.** By definition, we have to prove that for any \(l \geq 0\), the cyclic homology functor \(HC_\bullet\) turns the sequence

\[
(4.9) \quad 0 \longrightarrow K_{l-1} \langle E \rangle (1) \overset{v}{\longrightarrow} K_l \langle E \rangle \longrightarrow K_l (\lambda_p^* E) \longrightarrow 0
\]
into an exact triangle. By Corollary 2.6 we may equally well use the Hochschild homology functor $HH$. For $l = 0$, the sequence is exact already in $\text{Fun}(\Lambda, C)$ by Definition 4.6 (ii). By Definition 4.1 we have

$$K_l(E) \cong l(E) \otimes \Lambda^l j_1 k,$$

and the map $v$ is induced by the exterior multiplication map $m : j k \otimes \Lambda^{l-1} j_1 k \to \Lambda^l j_1 k$ restricted to $k(1) \subset j_1 k$ (here, sadly, $\Lambda$ with an upper index has to mean exterior power in the sense of the pointwise tensor product in $\text{Fun}(\Lambda, k\text{-mod})$, and we hope that this does not cause any confusion). By Definition 3.11, $K_l(\lambda^* E)$ is isomorphic to $l(E)$ for $l = 1$ and trivial for $l \geq 2$.

Consider the multiplication map

$$(4.10) \quad m : \Lambda^{l-1} j_1 k \otimes k(1) \to \Lambda^l j_1 k.$$ 

If $l = 1$, then this is just the embedding map $k(1) \to j_1 k$; it is injective, its cokernel is by definition the constant functor $k$, and we conclude that $(4.10)$ is exact for $l = 1$, also already in $\text{Fun}(\Lambda, C)$. For $l \geq 2$, we restrict terms to $\text{Fun}(\Delta^o, C)$-objects and use the Dold-Thom equivalence $DT$ between simplicial objects in an abelian category and non-positively graded complexes in the same category. Under this equivalence, $DT(j^* k(1)) \cong k[1]$, $DT(j_1 k) \cong k[1] \oplus k$, and moreover, $DT(\Lambda^l j_1 k) \cong k[l] \oplus k[l-1]$. Therefore $DT$ would be an isomorphism, were the functor $DT$ compatible with the tensor product. Since it is only compatible with the tensor product up to a quasiisomorphism, we conclude that $DT(m)$ is a quasiisomorphism. Therefore for any $E' \in \text{Fun}(\Delta^o, C)$, in particular for $E' = j^* l(E)$, the map

$$DT(m) : DT(E' \otimes \Lambda^{l-1} j_1 k \otimes k(1)) \to DT(E' \otimes \Lambda^l j_1 k)$$

is a quasiisomorphism. Since for any $E' \in \text{Fun}(\Delta^o, C)$, $H_*(\Delta^o, E')$ is the homology of the complex $DT(E')$, we conclude that $v$ in $(4.10)$ becomes a quasiisomorphism after applying $H_*(\Delta^o, -) = HH_*(-)$, so that $(4.9)$ indeed becomes exact.

### 4.4 Comparison maps

Below in Section 5 in our applications to cyclic homology for associative algebras, the computation at some point passes through a tight polycyclic object $E$, and it turns out that the associated complex $K_*(E)$ is rather easy to control. However, it is the complex $\pi_\gamma \lambda_p^* E$ that is related to the cyclic homology. Both $HC_*(K_*(E))$ and $HC_*(\pi_\gamma \lambda_p^*)$ have very similar structure — in particular, both are equipped with filtrations whose associated graded quotients are the same (and isomorphic to
$HH_\ast(\langle E \rangle)[v]$]. It would be very nice to know that the two complexes themselves are quasiisomorphic, possibly under additional natural assumptions on $E$. Unfortunately, we were not able to prove it, and there are reasons to believe that it is not true: the extension data between associated graded pieces of the corresponding filtrations are different. In particular, there is no natural map from one complex to the other. We had to settle for a weaker comparison result: there is a third complex which maps both into $K_q\langle E \rangle$ and into $\pi_\ast \lambda_p^\ast E$, and after we take cyclic homology, both maps become quasiisomorphisms in degrees $\geq -2(p - 2)$. Consequently, $K_q\langle E \rangle$ and $\pi_\ast \lambda_p^\ast E$ do have equal cyclic homology in low degrees, which turns out to be enough in application to the Hodge-to-de Rham degeneration. An instance of this phenomenon occurs already in the commutative situation considered in [DI] (where the characteristic of the base field had to be greater than the dimension of the algebraic variety in question).

The comparison result that we prove is as follows. Assume given a polycyclic object $E \in \text{Fun}(B_p, C)$ which is tight in the sense of Definition 4.6. Moreover, assume that $E$ comes from an extended polycyclic object $\tilde{E} \in \text{Fun}(\tilde{B}_p, C)$. It is easy to check that the bar resolution $B^1(E)$ together with its conjugate filtration $W_\ast$ is compatible with the extended polycyclic structure, so that we have a natural comparison map

$$\tilde{\chi}_\ast(B^1(E)) \to \chi_\ast(B^1(E)) \to K_q\langle E \rangle,$$

which becomes a map of filtered complexes in $\text{Fun}(\Lambda, C)$ if we equip $K_q\langle E \rangle$ with the canonical filtration. On the other hand, we have a natural filtered map

$$\tilde{\chi}_\ast(B^1(E)) \to \pi_\ast(B^1(\lambda_p^\ast E)).$$

Let us say that a map $f : E_\ast \to E'_\ast$ of (filtered) complexes in $\text{Fun}(\Lambda, C)$ is a (filtered) quasiisomorphism up to homology in degrees $\geq m$ if the corresponding map $f : HC_\ast(E_\ast) \to HC_\ast(E'_\ast)$ is a (filtered) quasiisomorphism in degrees $\geq m$.

**Proposition 4.9.** In the assumptions above, the comparison maps (4.11) and (4.12) are filtered quasiisomorphisms up to homology in degrees $\geq -2(p - 2)$.

**Proof.** The map (4.11) is a quasiisomorphism in degrees $\geq -2(p - 2)$ by Lemma 4.3 and since it is compatible with the periodicity map, it is a filtered quasiisomorphism up to homology in the said degrees by Lemma 4.8. The
map \( (4.12) \) is also compatible with the periodicity map, and by the definition of the conjugate filtration, it suffices to prove that the induced map

\[
\gr^W_1 \tilde{\chi}^*(B^i_1(E)) \to \gr^W_1 \pi^*_s(B^i_1(\lambda^*_pE))
\]

is a quasiisomorphism up to homology in degrees \( \geq -2(p - 2) \). By definition, both sides are complexes concentrated in degrees \( \leq -1 \). In degree \(-1\), the homology of the left-hand side isomorphic to \( j! j^*_i I_{\langle E \rangle} \), while the homology of the right-hand side is \( j! j^*_i I_{\langle \lambda^*_pE \rangle} \), and the map is an isomorphism by Definition \( 4.6 \) (ii); thus it suffices to prove that \( \tau \leq -2 \) \( \gr^W_1 \tilde{\chi}^*(B^i_1(E)) \) and \( \tau \leq -2 \) \( \gr^W_1 \pi^*_s(B^i_1(\lambda^*_pE)) \) are trivial up to homology in degrees \( \geq -2(p - 2) \). For the first complex, we can apply the map \( (4.11) \) and deduce the statement. For the second complex, we note that for any tight \( p \)-cyclic \( E' \in \Fun(\Lambda_p, \mathcal{C}) \), by Lemma \( 3.9 \) and Corollary \( 3.10 \) \( \tau \leq -2 \) \( \gr^W_1 \pi^*_s(B^i_1(E')) \) is trivial up to homology in all degrees. \( \square \)

5 **Associative algebras.**

We now fix a perfect field \( k \) of odd positive characteristic \( p \), and we assume given a \( k \)-linear Grothendieck abelian category \( \mathcal{C} \). Moreover, we assume that \( \mathcal{C} \) is equipped with a symmetric tensor product with unit object \( k \in \mathcal{C} \).

5.1 **Definitions.** Let \( A \) be an associative unital algebra object in the tensor category \( \mathcal{C} \). We define a canonical cyclic object \( A_\# \in \Fun(\Lambda, \mathcal{C}) \) in the following way. For any \( m \geq 1 \), we set

\[
A_\#([m]) = A^\otimes m = \bigotimes_{i \in [m]} A,
\]

where we number the factors in the tensor product by elements of the set \( [m] = \Lambda([1], [m]) \). For any map \( f \in \Lambda([m], [n]) \), we set

\[
f_\# = \bigotimes_{i \in [n]} f_i,
\]

where

\[
f_i : \bigotimes_{j \in [\overline{[i]}]^{-1}} A \cong A^\otimes [\overline{[i]}] \to A
\]

is the multiplication map given by the algebra structure on \( A \), and \( \overline{f} : [m] \to [m] \) is the natural map induced by \( f \in \Lambda([m], [n]) \) (if \( \overline{[i]}^{-1} \) is empty, we...
let $A^\otimes 0 = k$ be the unit object in $\mathcal{C}$, and we take $f^i : k \to A$ to be the unit embedding.

This is a well-defined cyclic object in $\mathcal{C}$, so that we have the Hochschild homology complex $HH_*(A_\#) \in D^-(\mathcal{C})$ and the cyclic homology complex $HC_*(A_\#) \in D^-(\mathcal{C})$.

To proceed further, we specialize to the following situation. Assume fixed a small category $Z$ equipped with a Grothendieck topology $J$. From now on, and until Section 6, let $\mathcal{C}$ be the category of sheaves of $k$-vector spaces on $\langle Z, J \rangle$. Moreover, assume given and fixed a cohomological functor $H^*(-)$ from $\mathcal{C}$ to the category of $k$-vector spaces (for example, this may be the cohomology of the site $\langle Z, J \rangle$, or the cohomology with some fixed supports).

**Definition 5.1.** The Hochschild and cyclic homology of the algebra $A \in \mathcal{C}$ is given by

$$HH_*(A) = H^*(\langle Z, J \rangle, HH_*(A_\#)) \quad HC_*(A) = H^*(\langle Z, J \rangle, HC_*(A_\#)),$$

where $HH_*(A_\#)$ and $HC_*(A_\#)$ are as in (2.10).

The Hodge and the conjugate filtrations on $HC_*(A_\#)$ induce filtrations on $HC_*(A)$. We will be interested in the conjugate filtration $W_*$. By abuse of notation, for any $l \geq 0$ we will denote

$$W_lHC_*(A) = H^*(\langle Z, J \rangle, W_l \pi_!\pi^* A_\#), \quad \text{gr}_l^W HC_*(A) = H^*(\langle Z, J \rangle, \text{gr}_l^W \pi_!\pi^* A_\#),$$

although the map $W_lHC_*(A) \to HC_*(A)$ does not have to be injective, so that $W_*HC_*(A)$ is only a filtration in the generalized sense.

For any $V \in \mathcal{C}$, denote by $V^{(1)} = Fr^*(V)$ the sheaf $V$ with $k$-vector space structure twisted by the Frobenius map $Fr : k \to k$.

**Lemma 5.2.** For any $V \in \mathcal{C}$, the tensor power $V^\otimes p$ equipped with the natural action of $\mathbb{Z}/p\mathbb{Z}$ by transpositions is tight in the sense of Definition 3.11 and there is a canonical isomorphism $I(V^\otimes p) \cong V^{(1)}$.

**Proof.** Since taking the associated sheaf is an exact functor, it suffices to prove the statement for the trivial topology $J$ on $Z$, so that $\mathcal{C}$ is the category of presheaves of $k$-vector spaces on $Z$. This in turn reduces to proving the statement for the vector spaces $V(X)$ for all objects $X \in Z$. Thus we may assume that $Z$ is trivial and $V = k[S]$ is just a $k$-vector space with some
basis $S$. Then $V^\otimes p = k[S^p] = k[S] \oplus V'$, where $S \subset S^p$ is the diagonal, and $V' = k[S^p \setminus S]$ is the natural complement to $V = k[S] \subset V^\otimes p$. The group $\mathbb{Z}/p\mathbb{Z}$ acts freely on $S^p \setminus S$; therefore $V'$ is a free $\mathbb{Z}/p\mathbb{Z}$-$k$-module, it is trivially tight, and we have $I(V') = 0$. The action on $V \subset V^\otimes p$ is trivial, so that $V$ also tight, and $I(V) \cong V$. Therefore $V^\otimes p = V \oplus V'$ is tight, and $I(V^\otimes p) \cong V$.

Finally, to construct a functorial isomorphism $V^{(1)} \cong I(V^\otimes p)$, we note that the embedding $V \to V^\otimes p$ defined by the basis $S$ sends $V$ into the subspace of $\mathbb{Z}/p\mathbb{Z}$-invariant vectors, and moreover, coincides modulo $T(V^\otimes p)$ with $v \mapsto v \otimes \cdots \otimes v$ mod $T(V^\otimes p) \in V^\otimes p/T(V^\otimes p)$.

This is a well-defined map, it is explicitly independent of any bases, it is a posteriori additive, and the $k$-vector space structure on both sides obviously differs by the Frobenius map $Fr : k \to k$, $\lambda \mapsto \lambda^p$. □

**Corollary 5.3.** For any associative unital algebra $A$ in the tensor category $C$, the object $i^*_p A# \in \text{Fun}(\Lambda_p, C)$ is tight, and we have $I(i^*_p A#) \cong A^{(1)}_#$, so that

$$ \text{gr}_W^l \pi^* i^*_p A# \cong j^1 A^{(1)}_# [2l] $$

for any $l \geq 1$, and $\text{gr}_l^W HC_*(A) \cong HH_*(A^{(1)})[2l]$.

**Proof.** By definition, for any $m \geq 1$ we have

$$ i^*_p A#([m]) \cong A#([pm]) \cong (A#([m]))^\otimes p , $$

this is tight by Lemma 5.2 and $I(i^*_p A#([m])) \cong A#([m])^{(1)}$. We leave it to the reader to check that the isomorphism is compatible with the action of maps $f \in \Lambda([m], [n])$, $n, m \geq 1$. The last claim is (3.14) of Lemma 3.13. □

**Remark 5.4.** The isomorphism 5.1 and the induced isomorphism

$$ \text{gr}_l^W HC_*(A) \cong HH_*(A^{(1)}) $$

is probably the closest analog of the usual Cartier isomorphism in our non-commutative theory.
5.2 Polycyclic structure. We now note that if the algebra $A$ is equipped with an action of a group $G$, then the object $A_{\#} \in \text{Fun}(\Lambda, C)$ has a natural structure of a functor from $G \int \Lambda$ to $C$. In particular, if $G = \mathbb{Z}/p\mathbb{Z}$, we have a polycyclic object $A_{\#} \in \text{Fun}(B_p, C)$. On the other hand, the length-2 complex $K_{\#}(A)$ has a natural structure of a DG algebra in $C$.

**Lemma 5.5.** In the assumptions above, we have

$$K_{\#}(A_{\#}) \cong K_{\#}(A).$$

**Proof.** Evaluating on $[n] \in B_p$, we have $A_{\#}([n]) = A^{\otimes n}$, and the action of the group $(\mathbb{Z}/p\mathbb{Z})^n$ on this object is induced by the $(\mathbb{Z}/p\mathbb{Z})$-action on each factor in $A^{\otimes n} = A \otimes \cdots \otimes A$. Therefore for any $I \subset S$, $S = \Lambda([1], [n])$, we have

$$A_{\#}([n])_I \cong \left( \bigotimes_{i \in I} H_0(\mathbb{Z}/p\mathbb{Z}, A) \right) \otimes \left( \bigotimes_{i \notin I} H^0(\mathbb{Z}/p\mathbb{Z}, A) \right).$$

This identification is by definition compatible with the differentials and gives an isomorphism $K_{\#}(A_{\#}([n])) \cong K_{\#}(A)^{\otimes n}$. It remains to check that these isomorphisms are compatible with the maps $[m] \to [n]$; this is easy and left to the reader. \hfill \square

In particular, for any associative unital algebra $A$ in $C$, the $p$-th power $A^{\otimes p}$ has a natural algebra structure, and this algebra $A^{\otimes p}$ is acted upon by the cyclic group $\mathbb{Z}/p\mathbb{Z}$ — and in fact, by the whole symmetric group $S_p$ on $p$ letters. Therefore we can form a polycyclic object $A^{\otimes p}_{\#} \in \text{Fun}(B_p, C)$, and this object is in fact extended polycyclic.

**Lemma 5.6.** The polycyclic object $A^{\otimes p}_{\#}$ is tight in the sense of Definition 4.6; we have $\lambda^*_{\#}A^{\otimes p}_{\#} \cong i^*_pA_{\#}$ and $l(\langle A^{\otimes p}_{\#} \rangle) \cong l(\langle i^*_pA_{\#} \rangle) \cong A^{(1)}_{\#}$.

**Proof.** The isomorphism $\lambda^*_{\#}A^{\otimes p}_{\#} \cong i^*_pA_{\#}$ immediately follows from the definitions, and the condition (i) of Definition 4.6 is an immediate corollary of Lemma 5.5. Moreover, by the same lemma we have $l(\langle A^{\otimes p}_{\#} \rangle) \cong l(\langle A^{\otimes p} \rangle) \cong l(\langle i^*_pA_{\#} \rangle) \cong A^{(1)}_{\#}$; together with Corollary 5.3 this gives an isomorphism $l(\langle A^{\otimes p}_{\#} \rangle) \cong l(\langle i^*_pA_{\#} \rangle) \cong A^{(1)}_{\#}$, which yields Definition 4.6 (ii). \hfill \square

**Definition 5.7.** For any $V \in C$, denote $V^\dagger = K_{\#}(V^{\otimes p})$. 43
Lemma 5.8. Assume that for an associative algebra $A$ in $\mathcal{C}$, the DG algebra $A^\dagger$ is formal – in other words, $A^\dagger$ is quasiisomorphic to its homology, the trivial square-zero extension of $A^{(1)}$ by $A^{(1)}[1]$. Then the spectral sequence for the conjugate filtration computing $W_1 HC_\ast(A)$ degenerates up to the term $E^{p-2}$.

Proof. Since the conjugate filtration is periodic, it suffices to analyse the top terms of the spectral sequence – that is, we have to show that

$$W_{[1,(p-2)]}HC_\ast(A) \cong \bigoplus_{1 \leq l \leq p-2} \text{gr}_l^W HC_\ast(A).$$

In other words, we have to analyse the cyclic homology of $W_{[1,(p-2)]}\pi_{\ast p}A^\#$. By Proposition 4.9, we may replace it with $\tau_{[-1, -(p-2)]}K(\langle A^\# \rangle)$, and by Lemma 5.6 this is isomorphic to $\tau_{[-1, -(p-2)]}A^\#$. In other words, we have to show that the canonical filtration on the complex $A^\#$ splits in a certain range of degrees. But by assumption, we have $A^\dagger \cong A^{(1)} \oplus A^{(1)}[1] –$ which means that the complex $A^\dagger$ is quasiisomorphic to the sum of its homology in all degrees. □

5.3 Splittings. We will now study the formality of the DG algebra $A^\dagger$.

To do this, we need to refine (and explain) Lemma 5.2.

Let $V$ be a $k$-vector space. Denote by

$$(5.2) \quad \rho^0 : V \to H^0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$$

the map which sends $v$ to $v \otimes \cdots \otimes v$. This map, although not additive, is functorial with respect to $V$. Therefore for any small category $Z$ and a presheaf $V \in \text{Fun}(Z^\circ, k\text{-mod})$ of $k$-vector spaces on $Z$, we have a natural map $\rho : V \to H^0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$. In particular, we may take $Z = \Delta$, the category of linearly ordered finite sets. By the Dold-Thom Theorem [DT], the category $\text{Fun}(\Delta^0, k\text{-mod})$ is equivalent to the category of non-positively graded complexes of $k$-vector spaces. Take a $k$-vector space $V$, consider the complex $V[1]$, denote the associated simplicial vector space by $V(1)$, and consider the map

$$\rho^1 : V(1) \to H^0(\mathbb{Z}/p\mathbb{Z}, V(1)^{\otimes p}).$$

This map extends to an additive map

$$(5.3) \quad \rho^1 : k[V(1)] \to H^0(\mathbb{Z}/p\mathbb{Z}, V(1)^{\otimes p}).$$
Apply now the Dold-Thom equivalence $\text{DT}$. The left-hand side becomes the standard bar complex $C_\ast(V,k)$ which computes the homology of the vector space $V$ – considered as an abelian group – with coefficients in the field $k$. On the other hand, one checks easily that we have an isomorphism of $(\mathbb{Z}/p\mathbb{Z})$-modules

$$\text{DT}_\ast(V(1)^{\otimes p}) \cong V^{\otimes p} \otimes \text{DT}_\ast(k(1)^{\otimes p}),$$

and the complex $\text{DT}(k(1)^{\otimes p})$ is quasiisomorphic to $k[p]$ and concentrated in degrees $\leq -1$, $\text{DT}_1(k(1)^{\otimes p})$ is the trivial $(\mathbb{Z}/p\mathbb{Z})$-module $k$, and all the $\text{DT}_l(k(1)^{\otimes p})$, $l \geq 2$ are regular representations of $(\mathbb{Z}/p\mathbb{Z})$. Therefore we can take the standard periodic resolution $I_\ast(k)$ of the trivial $(\mathbb{Z}/p\mathbb{Z})$-module $k$, set $I_0(k) = \tau_{\leq 1}I_\ast(k)$ as in Subsection 3.2 and choose a $(\mathbb{Z}/p\mathbb{Z})$-module map $\psi : \text{DT}(k(1)^{\otimes p}) \to I_0(k)[2]$ which is a quasiisomorphism in degrees $\geq -p$. Composing this with $\text{DT}(\rho)$, we obtain a canonical map

$$C_\ast(V,k) \to H^0(\mathbb{Z}/p\mathbb{Z}, \text{DT}_\ast(V(1)^{\otimes p})) \to H^0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p} \otimes I_0(k))[2] \cong H^0(\mathbb{Z}/p\mathbb{Z}, I_0(V))[2].$$

Note that that the only choice of the whole construction was the choice of the map $\psi$, which does not depend on $V$ at all. Therefore (5.4) is completely functorial in $V$. In particular, we may just as well let $V$ be a presheaf on some category $Z$, and moreover, by applying the associated sheaf functor we may even consider sheaves with respect to some topology $J$.

Truncating the right-hand side of (5.4), we obtain a canonical map

$$(5.5) \quad C_\ast(V,k) \to V^\dagger[1]$$

which is again completely functorial in $V$. In degree 1, this is the canonical map $V \cong H_1(V,k) \to H_0(V^\dagger)$ constructed in Lemma 5.2. In degree 2, we obtain a canonical cocycle $\rho_V \in C^2(V, H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})).$

**Definition 5.9.** Denote by $\tilde{V}$ the group obtained as an extension of $V$ by $H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$ given by the cocycle $\rho$.

One checks easily that the 2-cocycle $\rho_V$ is in fact symmetric, so that the group $\tilde{V}$ is commutative. The group $\tilde{V}$ is not a $k$-vector space and not a group of characteristic $p$: multiplication by $p$ is given by the map

$$\tilde{V} \to V \xrightarrow{\varphi} H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \to \tilde{V}.$$ 

Assume from now on that the field $k$ is perfect; then one checks easily that if we twist the $k$-module structure on $V$ by the Frobenius map – in other
words, replace $V$ with $V^{(1)}$ – then the cocycle $\rho_V$ is compatible with multiplication by constants, and $\tilde{V}$ is in fact a module over the second Witt vectors ring $W_2(k)$. Since the cocycle $\rho_V$ comes from a map of complexes \(\ref{5.5}\), it reduction modulo $V \subset H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$ is the cocycle for the extension

$$H^0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \to V^{(1)}$$

split as a map of sets by \(\ref{5.2}\), so that we have $\tilde{V}/p \cong H^0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$.

To sum up: we have a natural three-step filtration

$$V^{(1)} \subset H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \subset \tilde{V}$$
on the $W_2(k)$-module $\tilde{V}$; the quotient $\tilde{V}/V^{(1)}$ is naturally identified with $H^0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$, and the quotient $\tilde{V}/H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$ is naturally identified with $V^{(1)}$. Multiplication by $p \in W_2(k)$ isomorphically sends the top quotient $V^{(1)}$ of this filtration into the bottom subobject $V^{(1)} \subset \tilde{V}$.

The construction of the cocycle $\varphi_V$ is also compatible with tensor products in the following sense.

**Lemma 5.10.** Assume given two $k$-vector spaces $V, W$, and let $m$ be the natural map

$$H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \otimes H^0(\mathbb{Z}/p\mathbb{Z}, W^{\otimes p}) \to H_0(\mathbb{Z}/p\mathbb{Z}, (V \otimes W)^{\otimes p}).$$

Then $m(\rho_V \otimes \rho^0_W) = \rho_{V \otimes W}$, where $\rho^0_W$ is the map $\rho^0$ in \(\ref{5.2}\) for the vector space $W$. Consequently, there exists a functorial map

$$\tilde{V} \otimes_{W_2(k)} \tilde{W} \to \tilde{V \otimes W}.$$

**Proof.** To obtain $\rho_V \otimes \rho^0_W$, one uses the same procedure as for the cocycle $\rho_V$, but for the simplicial abelian group $V(1) \otimes W$ instead of $V(1)$. The map $m$ is induced by the natural map $V(1) \otimes W \to (V \otimes W)(1)$. Since \(\ref{5.2}\) is functorial with respect to any maps of simplicial groups, we get the desired compatibility. \(\square\)

As before, all of the above obviously works not only for $k$-vector spaces, but also for presheaves and sheaves of $k$-vector spaces on a site $\langle Z, J \rangle$. Take a $k$-vector space or a sheaf $V$. Denote by $\tilde{V}^\dagger$ the complex

$$H_0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \xrightarrow{T} \tilde{V}$$

placed in degrees 0 and $-1$. This is a complex quasiisomorphic to $V^{(1)}$, and we have a natural map $\tilde{V}^\dagger \to V^\dagger$ which induces an isomorphism on $H_0$. If
V = A is an associative unital algebra, then by Lemma 5.10 \( \hat{A} \) is a DG algebra, and the canonical map \( \hat{A} \to A \) is a DG algebra map. Adding the tautological embedding \( A^{(1)}[1] \cong H_1(A^{\dagger})[1] \to A^{\dagger} \), we obtain a DG algebra quasiisomorphism

\[
\hat{A} \oplus A^{(1)}[1] \to A^{\dagger}.
\]

Therefore the DG algebra \( A^{\dagger} \) is always quasiisomorphic to the sum of its homology, but in the wrong category: \( A^{\dagger} \) is a DG algebra over \( W_2(k) \), not over \( k \).

**Lemma 5.11.** Assume that there exists an associative flat \( W_2(k) \)-algebra \( \hat{A} \) in \( \text{Shv}(Z, J) \) such that \( \hat{A}/p \cong A^{(1)} \). Then the DG \( k \)-algebra \( A^{\dagger} \) is formal.

**Proof.** We have two extensions of the algebra \( A^{(1)} \): \( \hat{A} \) and \( \hat{A}^{\dagger} \). Let \( \overline{A^{\dagger}} \) be their Baer difference – that is, the middle cohomology of the complex

\[
A^{(1)} \longrightarrow \hat{A} \oplus \hat{A}^{\dagger} \longrightarrow A^{(1)},
\]

where the left-hand side map is the sum, and the right-hand side map is the difference of the natural maps. Then \( \overline{A^{\dagger}} \) is a DG algebra over \( W_2(k) \), and moreover, \( p \) acts trivially on it, so that \( \overline{A^{\dagger}} \) is in fact a DG algebra over \( k \). On the other hand, \( \overline{A^{\dagger}} \) is quasiisomorphic to \( A^{(1)} \), and we have a natural map \( \overline{A^{\dagger}} \to A^{\dagger} \). Adding the embedding \( A^{(1)}[1] \to A^{\dagger} \), we obtain the desired quasiisomorphism \( A^{(1)} \oplus A^{(1)}[1] \cong \overline{A^{\dagger}} \oplus A^{(1)}[1] \to A^{\dagger} \). \( \square \)

**Remark 5.12.** One can show without much difficulty that the converse to this statement is also true: liftings of \( V \) to a flat \( W_2(k) \)-module are in functorial one-to-one correspondence with splittings of the complex \( V^{\dagger} \) (understood in appropriate way), and this is compatible with algebra structures, if they are present. We do not go into this to save space.

**Remark 5.13.** It is very interesting to repeat our construction of the map \( \rho^l \) in (5.3) for different shifts – one takes \( V[l] \), \( l \geq 0 \), instead of \( V[1] \), and obtains a map \( \rho^l \). If one goes to the limit \( l \to \infty \), then the left-hand side of (5.3) becomes the stable homology of the group \( V \) – this is a complex which depends functorially on \( V \), and whose homology is isomorphic to \( \hat{V} \) tensored with the dual to the Steenrod algebra. The right-hand side stabilizes in a straightforward manner and becomes quasiisomorphic to the truncated Tate homology \( H_{> -1}(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \). The limit map \( \rho^{\infty} \) is essentially just the Steenrod \( p \)-th power map. The cocycle \( \rho^V \) survives in the stable situation.
and becomes the Bokstein homomorphism (this is the topological underly-
ing reason for Lemma 5.11 and Remark 5.12). We note that the structure of
Steenrod algebra is well-known; in particular, after the Bokstein homomor-
phism class in degree 1, there is a non-trivial class in degree $p$ (and in other
higher degrees). We suspect that $\rho^\infty$ sends these classes to some non-trivial
classes in $H^1(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p})$, so that even if $V$ is lifted to a $W_2(k)$-module, this
Tate homology complex does not split as a whole in a functorial way - it only
splits in degrees $\geq -(p - 1)$. Thus the restriction on degree in Lemma 5.8
and Proposition 4.9 is unavoidable: this is how things really are. We note
that our trick of using the additional $(\mathbb{Z}/p\mathbb{Z})^*$-symmetry is lifted out the
standard computation of Steenrod powers found in any topology textbook.

6 Hodge to de Rham degeneration.

We are now ready to study the Hodge-to-de Rham spectral sequence and its
degeneration. Assume given a field $k$ and a small category $Z$ equipped with
a Grothendieck topology $J$. Denote by $C$ the category of sheaves of $k$-vector
spaces on $(Z,J)$. Assume also given a cohomological functor $H^\ast(-)$ from $C$
to the category of $k$-vector spaces.

We start with the positive characteristic case.

**Theorem 6.1.** Assume given an associative unital algebra $A \in C$ over a
field $k$ in the category $C$ of sheaves of $k$-vector spaces on a site $(Z,J)$. As-
sume that $k$ is a perfect field of positive odd characteristic $\text{char } k = p > 2$.
Moreover, assume that $A$

\begin{enumerate}
\item the diagonal $A$-bimodule $A$ admits a finite flat resolution of length $n$, and
\item for some integer $m > 0$ and every sheaf $E \in C$, the cohomology groups
$H^l((Z,J), E)$ are trivial whenever $l > m$, and
\item we have $p > n + m$, and
\item for any integer $l$, the Hochschild homology group $HH_l(A)$ is a finite-
dimensional vector space over $k$.
\end{enumerate}

Finally, assume that there exists a flat algebra $\tilde{A} \in \text{Shv}(Z,J)$ over the ring
$W_2(k)$ of second Witt vectors of the field $k$ such that $\tilde{A}/p \cong A$.

Then the Hochschild-to-cyclic spectral sequence $HH_\ast(A)[v] \Rightarrow HC_\ast(A)$
degenerates.

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Lemma 6.2. In the assumptions of Theorem 6.1, the spectral sequence associated to the conjugate filtration on $W_1HC_*(A)$ degenerates, so that we have an isomorphism $W_1HC_*(A) \cong HH_0(A^{(1)})[v]$.

Proof. By Lemma 5.8, the conjugate spectral sequence for $W_1HC_*(A)$ degenerates up to term $E_p^{p-2}$. By the assumption (iii) of Theorem 6.1, it degenerates in all the following terms for dimension reasons. □

We now recall that by definition, we have an exact triangle

$$W_1HC_*(A) \rightarrow HC_*(A) \rightarrow H^*(\langle Z, J \rangle, HC_*(\overline{A}_#)) \rightarrow$$

of complexes of $k$-vector spaces, where the cyclic object $\overline{A}_# \in \text{Fun}(\Lambda, \mathcal{C})$ is the quotient

$$\overline{A}_# = \pi i_p^* A#/A^{(1)}_#.$$

Lemma 6.3. In the assumptions of Theorem 6.1, the Hochschild homology sheaf $HH_l(\overline{A}_#) \in \mathcal{C}$ is trivial whenever $l > mp$.

Proof. As in the proof of Lemma 5.2, we may assume that $\langle Z, J \rangle$ is trivial, so that $\mathcal{C}$ is the category of $k$-vector spaces. Moreover, the homology of $A^{(1)}_#$ is bounded by Theorem 6.1 (i), so that we might just as well prove our statement for $\pi i_p^* A#$ instead of its quotient $\overline{A}_#$.

For any $A$-bimodule $M$, denote by $M_\bullet$ the cokernel of the commutator map $A \otimes M \rightarrow M$ – equivalently, we have $M_\bullet = M \otimes_A A_{opp} A$. Moreover, denote

$$M_p = (M \otimes_A M \otimes_A \cdots \otimes_A M)^{\otimes_p},$$

where we have $p$ multiples on the right-hand side. The reader will check easily that this construction is actually cyclically symmetric in all $p$ multiples (but since paper is two-dimensional, we cannot represent this symmetry in convenient notation). Moreover, $M \mapsto M_p$ is obviously functorial in $M$.

Recall that by the Dold-Thom Theorem [DT], for any abelian category $\mathcal{C}$ the category $\text{Fun}(\Delta^o, \mathcal{C})$ of simplicial objects in $\mathcal{C}$ is equivalent to the category of complexes in $\mathcal{C}$ concentrated in non-positive degree. For any $M \in \text{Fun}(\Delta^o, \mathcal{C})$, the corresponding complex represents the homology $H_*(\Delta^o, M)$. If the category $\mathcal{C}$ is equipped with a symmetric tensor product, then the category $\text{Fun}(\Delta^o, \mathcal{C})$ also has a natural symmetric tensor structure, and this structure is homotopy-invariant in the following strong sense. For any group $G$, denote by $\mathcal{C}[G]$ the category of objects in $\mathcal{C}$ equipped with an action of $G$. For any $M \in \text{Fun}(\Delta^o, \mathcal{C})$, the $p$-fold tensor product $M^{\otimes p}$ equipped with the transposition action of the symmetric group $S_n$ is naturally an object in
Fun(Δ₀, C[S_n]) – or, by Dold-Thom equivalence, a complex in C[S_n]. Then if M₀, M₁ ∈ Fun(Δ₀, C) are homotopy-equivalent – that is, the corresponding complexes are equivalent up to chain homotopy of complexes – then the complexes M₀ ⊗ p M₀, M₁ ⊗ p M₁ are homotopy-equivalent in C[S_n].

If the category C is tensor but not symmetric, there is no natural group action on tensor powers M⊗p. However, if C = A-bimod is the category of A-bimodules, then for any simplicial A-bimodule M ∈ Fun(Δ₀, C) we can form M⊗p ∈ Fun(Δ₀, C[Z/pZ]), and this is also homotopy-invariant in the above sense.

Take now a flat resolution P₁ of the diagonal A-bimodule A of length m whose existence is assumed by Theorem 6.1 (i), and treat it as a simplicial A-bimodule P₁ ∈ Fun(Δ₀, A-bimod). Moreover, let P₂ ∈ Fun(Δ₀, A-bimod) be the standard simplicial bar-resolution of A. Then by definition of the cyclic object A# ∈ Fun(Λ, k-mod), we have

\[ j^*i^*pA# \cong (P₂) \in Fun(Δ₀, k[Z/pZ]-mod) = Fun(Δ₀ × pt, k-mod), \]

and since P₁ must be homotopy-equivalent to P₂, this is homotopy-equivalent to (P₁) as a complex of k[Z/pZ]-modules. We conclude that

\[ j^*πi^*pA# \cong H₀ \left( Z/pZ, (P₂) \right) \]

is homotopy equivalent to H₀(Z/pZ, (P₂)) ∈ Fun(Δ₀, k-mod). Applying the Dold-Thom equivalence, we see that the latter complex is manifestly trivial in degrees > mp.

□

Proof of Theorem 6.1. Consider the exact triangle (6.1). We have

HC₁(A) ≅ H⁺((Z, J), H⁺(Λ, Lpπi^*pA#)) ≅ H⁺((Z, J), H⁺(Λ, Lpπi^*pA#)),

and by Lemma 3.3, the generator u ∈ H²(Λ, k) of the cohomology algebra H⁺(Λ, k) acts trivially on the right-hand side. Therefore the cokernel of the natural map W₁HC₁(A) → HC₁(A) lies inside the subspace in H⁺((Z, J), HC₁(A)) annihilated by u. This subspace is a quotient of H⁺((Z, J), HH₁(A#)), and by Lemma 6.3 the latter space is trivial in degrees ≪ 0. We conclude that the map W₁HC₁(A) → HC₁(A) is trivial in degrees ≪ 0. By Lemma 6.2 this implies that

\[ \dim HC₁(A) \leq \bigoplus_m \dim HH_{l+2m}(A^{(1)}) = \bigoplus_m \dim HH_{l+2m}(A) \]

for l ≪ 0 (the sum on the right-hand side is over all integers m, and it is bounded and finite by our assumptions). By the standard criterion [D],...
this implies that the Hochschild-to-cyclic spectral sequence $HH_q(A)[v] \Rightarrow HC_q(A)$ degenerates in degrees $\ll 0$. Since the sequence is periodic, it degenerates everywhere.

This finishes the positive characteristic case. The characteristic 0 statement would follow from this by a standard and well-known procedure, but there is a problem: interesting non-commutative algebras are often not Noetherian. Therefore it is unclear how to “spread out” an algebra given over a field of characteristic 0 so that it acquires fibers over fields of positive characteristic. In this paper, we have decided to settle for the following compromise: we give a statement “in mixed characteristic” – that is, we assume that the algebra is already defined and flat over a $\mathbb{Z}$-algebra of finite type – and we give a second statement which shows that such a such a $\mathbb{Z}$-model exists in the Noetherian situation. This is enough for some applications but certainly not for all of them. This is a technical limitation, since the finiteness assumptions we impose in order to have the degeneration should also be sufficient to construct a $\mathbb{Z}$-model. In a separate paper, we will handle this problem by using the $A_\infty$-methods.

**Theorem 6.4.** Assume given an associative unital algebra $A \in \text{Shv}(\mathbb{Z}, J)$ flat over an integral domain $O$ of finite type over $\mathbb{Z}$. Moreover, assume that

(i) the diagonal $A$-bimodule $A$ admits a finite flat resolution,

(ii) for some integer $m > 0$ and every sheaf $E \in \text{Shv}(\mathbb{Z}, J)$ of $K$-vector spaces on $(\mathbb{Z}, J)$, the cohomology groups $H^l((\mathbb{Z}, J), E)$ are trivial whenever $l > m$, and

(iii) for any integer $l$, the Hochschild homology group $HH_l(A)$ is a finitely generated $O$-module.

Then the Hochschild-to-cyclic spectral sequence $HH_q(A)[v] \Rightarrow HC_q(A)$ degenerates.

**Proof.** Denote by $n$ the length of the resolution in (i). It suffices to show that the differentials in the spectral sequence vanish after reduction modulo all maximal ideals $m \subset O$, and moreover, it suffices to consider all the maximal ideals outside of a closed subset in $\text{Spec } O$. Therefore we may assume that $\text{char } O/m > n + m$. Since $O$ is of finite type over $\mathbb{Z}$, the field $O/m$ is perfect, and we are done by Theorem 6.1.
**Theorem 6.5.** Assume given a field $K$ of characteristic 0, and assume given an associative unital Noetherian $K$-algebra $A \in \text{Shv}(Z,J)$ which is generated by a finite number of local sections. Moreover, assume that

(i) the diagonal $A$-bimodule $A$ admits a finite flat resolution,

(ii) for some integer $m > 0$ and every sheaf $E \in \text{Shv}(Z,J)$ of $K$-vector spaces on $\langle Z,J \rangle$, the cohomology groups $H^l(\langle Z,J \rangle, E)$ are trivial whenever $l > m$, and

(iii) for any integer $l$, the Hochschild homology group $HH_l(A)$ is a finite-dimensional $K$-vector space.

Then the Hochschild-to-cyclic spectral sequence $HH_*(A)[v] \Rightarrow HC_*(A)$ degenerates.

**Proof.** Since $A$ is generated by a finite number of local sections, there exists a subalgebra $O \subset K$ of finite type over $\mathbb{Z}$ such that there exists an $O$-algebra $A_O \in \text{Shv}(Z,J)$ with $A \cong A_O \otimes_K K$. Moreover, we may assume that the finite flat resolution of the diagonal bimodule $A$ is defined over $O$ (as a complex, possibly not acyclic). Since $A$ is Noetherian, the homology of this complex consists of finitely generated torsion $O$-modules; localizing $O$, we may assume that the complex is also a resolution over $O$. Finally, again since $O$ is Noetherian, we may assume after localizing $O$ that the resolution is flat, and that $A_O$ itself is flat over $O$. Now $A_O$ satisfies all the assumptions of Theorem 6.4. □

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