Topological $\sigma$-Models and Large-$N$ Matrix Integral

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ABSTRACT

In this paper we describe in some detail the representation of the topological $CP^1$ model in terms of a matrix integral which we have introduced in a previous article. We first discuss the integrable structure of the $CP^1$ model and show that it is governed by an extension of the 1-dimensional Toda hierarchy. We then introduce a matrix model which reproduces the sum over holomorphic maps from arbitrary Riemann surfaces onto $CP^1$. We compute intersection numbers on the moduli space of curves using geometrical method and show that the results agree with those predicted by the matrix model. We also develop a Landau-Ginzburg (LG) description of the $CP^1$ model using a superpotential $e^X + e^{t_0 q} e^{-X}$ given by the Lax operator of the Toda hierarchy ($X$ is the LG field and $t_{0,Q}$ is the coupling constant of the Kähler class). The form of the superpotential indicates the close connection between $CP^1$ and $N = 2$ supersymmetric sine-Gordon theory which was noted some time ago by several authors. We also discuss possible generalizations of our construction to other manifolds and present a LG formulation of the topological $CP^2$ model.
1. Introduction

In our attempts at understanding the geometrical principles behind the string theory, the approach of topological field theories [1] seems to offer an important clue. String theory exposes its geometrical structures in a most transparent manner in its topological formulation. Thus we may gain geometrical insights from the study of the topological version of string theories. When a string is compactified on a Kähler manifold $M$, the theory is described by an $N = 2$ supersymmetric non-linear $\sigma$-model. After suitable twistings an $N = 2$ $\sigma$-model yields a pair of topological field theories, topological A- and B-models. B-models are essentially classical and easy to solve. A-model, on the other hand, is given by a sum over the holomorphic maps (instantons) from the Riemann surface to the target manifold and in general difficult to evaluate.

When the target space $M$ is a Calabi-Yau manifold, a special situation arises; A-model may be replaced by the B-model associated with the mirror manifold of $M$ and the genus=0 A-model correlation functions are obtained from those of the B-model using the mirror map [2,3,4]. Recently the A-model partition functions are determined further at higher genera $g = 1, 2$ using the method of holomorphic anomaly [5].

When we come to target manifolds $M$ with a positive 1st Chern class $c_1(M) > 0$ (Fano varieties), the situation becomes quite different from that of Calabi-Yau manifolds which are characterized by the condition $c_1(M) = 0$ and scale invariance. It is well-known that the one-loop $\beta$-function of a supersymmetric non-linear $\sigma$-model is proportional to the 1st Chern class of $M$. When $c_1(M) > 0$, the theory is asymptotically free and develops a mass gap due to the dimensional transmutation. Examples of manifolds with $c_1(M) > 0$ are given by complex projective spaces $CP^n$ and Grassmannians.

Recently an algorithm based on the associativity of the operator-product-expansion has been used to recursively determine the instanton sum in the genus $g = 0$ free energy of the A-model with the target manifolds $CP^2, CP^3, \cdots$ etc.
This method is efficient and general, however, is limited to the genus $g = 0$ case at the moment.

In a previous paper [11] we have considered $CP^1$, the simplest manifold with $c_1(M) > 0$, and constructed a matrix model which reproduces holomorphic maps from Riemann surfaces of arbitrary genera onto $CP^1$. A characteristic feature of the $CP^1$ model is the presence of the logarithmic terms in the action which implies the scaling violation in the system. In this paper we would like to describe in some details the properties of the model and its possible extensions. In section 2 we first briefly recall the basic facts about the topological $CP^1$ model and steps in deriving the matrix action. In section 3 we obtain a set of Ward identities of the model which form a Virasoro algebra. We check these identities against the geometrical data of the moduli space of holomorphic maps in the case of $g = 0, 1$. Intersection numbers on the moduli space are calculated using the method of algebraic geometry and we find a complete agreement with the predictions of the $CP^1$ model. In section 4 we discuss a Landau-Ginzburg (LG) description of the $CP^1$ model. It turns out that all the $g = 0$ correlation functions are reproduced by residue integrals if we use a superpotential of the form $\cos X$ where $X$ is the LG field. Thus the topological $CP^1$ model may be identified with the (topological version of the) $N = 2$ supersymmetric sine-Gordon theory. The close relation between the $CP^1$ and $N = 2$ sine-Gordon models has been noted sometime ago based on the comparison of the particle spectra and $S$-matrices of these theories [12,13]. In Section 5 we discuss a possible generalization of our construction to other manifolds and formulate a LG description of the topological $CP^2$ model. We also present comments and discussions.
2. The $CP^1$ model

Physical observables in the topological A-model arise from the de Rham classes of the target manifold. In the case of $CP^1$, there are just two classes, 1 (identity) and $\omega$ (Kähler class), and the corresponding physical observables are denoted as $P$ and $Q$, respectively. The integrable structure of the system is described using 2-point functions

$$\langle PP \rangle = u, \quad \langle PQ \rangle = v.$$  \hspace{1cm} (2.1)

The genus zero free energy in the small phase space is given by

$$F_0 = \frac{1}{2} t_{0,P}^2 t_{0,Q} + e^{t_{0,Q}},$$  \hspace{1cm} (2.2)

where the parameters $t_{0,P}, t_{0,Q}$ are coupled to $P, Q$. The second term in the RHS of (2.2) comes from the contribution of the degree-1 instanton. Combining (2.1) and (2.2) we note that

$$\langle QQ \rangle = e^{t_{0,Q}} = e^{\langle PP \rangle}.$$  \hspace{1cm} (2.3)

holds in the small phase space. Using the topological recursion relation it is possible to show that (2.3) in fact holds in a large phase space where couplings \{${t_{n,P}, t_{n,Q}, n = 1, 2, \cdots}$\} to the gravitational descendants of $P, Q$ do not vanish [14]. It is then easy to see that the integrable structure of the $CP^1$ model is described by the 1-dimensional Toda hierarchy

$$\frac{\partial^2 u}{\partial t_{0,Q}^2} = \frac{\partial}{\partial t_{0,Q}} \frac{\partial}{\partial t_{0,P}} \langle PQ \rangle = \frac{\partial}{\partial t_{0,P}} \langle PQQ \rangle = \frac{\partial^2}{\partial t_{0,P}^2} e^u.$$  \hspace{1cm} (2.4)

(2.4) in fact is the Toda-lattice equation when we identify $t_{0,P}$ as the continuum version of the index $n$ of a field $u_n = u(t_{0,P})$.

The Lax formalism of the Toda-lattice hierarchy is well-known. We introduce a Lax operator (at genus $g = 0$ or dispersionless limit, it becomes a number rather
than an operator)

\[ L = p + v + e^u p^{-1} \tag{2.5} \]

and construct Hamiltonians

\[ H^Q_n = [L^n]_+ , \quad n = 1, 2, \ldots \tag{2.6} \]

(+ means to take terms with non-negative powers of \( p \)). (2.6) generate flows in the Toda times \( t_n \) \((n = 1, 2, \ldots)\)

\[ \frac{\partial L}{\partial t_n} = \{ H^Q_n, L \}, \quad n = 1, 2, \ldots \tag{2.7} \]

where \( \{ A, B \} \) denotes the Poisson bracket of the Toda theory defined by

\[ \{ A, B \} = p \left( \frac{\partial A}{\partial p} \frac{\partial B}{\partial t_{0,P}} - \frac{\partial B}{\partial p} \frac{\partial A}{\partial t_{0,P}} \right). \tag{2.8} \]

It is easy to identify the Toda times \( t_n \) as the descendant times \( t_{n-1,Q} \) \( n = 1, 2, \ldots \) of the operator \( Q \). For instance, putting \( n = 1, \phi_\alpha = Q, \) and \( X = Y = P \) or \( X = P, Y = Q \) in Witten’s topological recursion relation [15]

\[ \langle \sigma_n(\phi_\alpha)XY \rangle = n \langle \sigma_{n-1}(\phi_\alpha)\phi_\beta \rangle \langle \phi_\beta XY \rangle \tag{2.9} \]

we obtain flow equations

\[ \frac{\partial u}{\partial t_{1,Q}} = \langle \sigma_1(Q)PP \rangle = \left( \frac{1}{2} v^2 + e^u \right)' , \]
\[ \frac{\partial v}{\partial t_{1,Q}} = \langle \sigma_1(Q)PQ \rangle = (ve^u)' . \tag{2.10} \]

Comparing (2.10) with

\[ \frac{\partial L}{\partial t_2} = 2(ve^u)' + (v^2 + 2u)'e^u p^{-1} , \tag{2.11} \]

we find \( t_2 = t_{1,Q}/2 \). In general a relation

\[ t_n = \frac{1}{n} t_{n-1,Q}, \quad n = 1, 2, \ldots \tag{2.12} \]

holds.
Analysis of the flows in the parameters \( \{ t_n, p, n = 1, 2, \cdots \} \) is more involved. Flow equations can again be written down using the recursion relation (2.9). We have, for instance,

\[
\begin{align*}
\frac{\partial u}{\partial t_1, p} &= \langle \sigma_1(P) PP \rangle = (uv)', \\
\frac{\partial v}{\partial t_1, p} &= \langle \sigma_1(P) PQ \rangle = \left( \frac{1}{2} v^2 + (u-1)e^u \right)'.
\end{align*}
\]  

(2.13)

It is somewhat non-trivial to find Hamiltonians for these flows. It turns out [11] that Hamiltonians involving logarithms of the Lax operator

\[
H^P_n = 2[L^n(\log L - c_n)]_+, \quad n = 0, 1, \cdots,
\]

(2.14)

c_n = \sum_{j=1}^{n} 1/j, \quad c_0 = 0

generate the flows in the \( t_n, p \) variables. In (2.14) the logarithm of \( L \) is defined by taking the average

\[
\log L = \log(p + v + e^u p^{-1}) = \frac{1}{2} \log p(1 + vp^{-1} + e^u p^{-2}) + \frac{1}{2} \log e^u p^{-1}(1 + ve^{-u} p + e^{-u} p^2)
\]

(2.15)

\[
= \frac{u}{2} + \frac{1}{2} \log(1 + vp^{-1} + e^u p^{-2}) + \frac{1}{2} \log(1 + ve^{-u} p + e^{-u} p^2).
\]

We note in particular an important relation

\[
\frac{\partial L}{\partial t_{0, p}} = \{H^P_0, L\} = 2\{[\log L]_+, L\}.
\]  

(2.16)

Note that \( \{H^P_n\} \) mutually commute with each other and also with \( \{H^Q_n\} \).

So far we have considered the \( CP^1 \) model at genus \( g = 0 \). Higher genus structure of the theory can be described by a matrix model where \( L \) becomes a matrix acting on the space of orthogonal polynomials and Poisson brackets are replaced
by matrix commutators. Let us recall the system of orthogonal polynomials of a matrix model with an action $S$,

$$
\int d\lambda \varphi_n(\lambda) \varphi_m(\lambda) e^{NS(\lambda)} = \delta_{nm} h_n, \quad n, m = 0, 1, \cdots, N - 1.
$$

(2.17)

Here $\varphi_n(\lambda) = \lambda^n + (\text{lower order terms})$ are degree-$n$ polynomials which are orthogonal to each other with respect to the weight $\exp NS(\lambda)$ ($N$ is the genus expansion parameter). Then the multiplication by $\lambda$ is represented by a matrix $Q$

$$
\lambda \varphi_n(\lambda) = \sum_{m=n-1}^{n+1} Q_{nm} \varphi_m(\lambda)
$$

(2.18)

which has non-vanishing elements along 3 diagonal lines $m = n, n \pm 1$. We parametrize the matrix $Q$ as

$$
Q_{nm} = \delta_{n+1,m} + v_n \delta_{n,m} + e^{N(\phi_n - \phi_{n-1})} \delta_{n-1,m}, \quad h_n = e^{N \phi_n}.
$$

(2.19)

In the continuum limit, $n/N$ is replaced by $t_{0,p}$ and (2.19) becomes the dispersionless Lax operator $L$ (2.5). Matrix commutators are replaced by Poisson brackets (2.8).

Now the partition function of the $CP^1$ model is given by an matrix integral [11]

$$
Z = \int dMe^{N\text{Tr}S(M)},
$$

(2.20)

where $M$ is an $(N \times t_{0,p})^2$ hermitian matrix and the action $S(M)$ is defined by

$$
S(M) = -2M(\log M - 1) + \sum_{n=1} t_{n-1,Q} M^n + \sum_{n=1} 2t_{n,p} M^n (\log M - c_n).
$$

(2.21)

Note that the 1st term in the action has the same form as the piece proportional to $t_{1,p}$ and hence we may consider $t_{1,p}$ having a “background value” $-1$. The structure of $S(M)$ may be intuitively inferred from the forms of the Hamiltonians $H_n^Q$ (2.6) and $H_n^P$ (2.14).
The characteristic feature of (2.21) is the appearance of logarithms which generate additive terms under the scale transformation of the matrix $M$. As we see in the next section, this is the mechanism by means of which we reproduce the sum over instantons.

We now recall the ghost number conservation laws. The (virtual) dimension of the moduli space of maps of degree $d$ from the genus $g$ Riemann surface with $s$ punctures onto $CP^1$ is given by

$$\dim \mathcal{M}_{g,s}(CP^1; d) = 2d + 2(g - 1) + s. \quad (2.22)$$

The factor 2 in front of $d$ stands for $c_1(CP^1) = 2$. When all couplings vanish, genus $g$ correlation functions

$$\langle \prod_{i=1}^{s} \sigma_{n_i}(\phi_{\alpha_i}) \rangle_g \quad (2.23)$$

receive contributions from instantons which satisfy the conservation law

$$\sum_{i=1}^{s} (n_i + q_{\alpha_i}) = 2d + 2(g - 1) + s, \quad (2.24)$$

where $q_{\alpha}$ is the $U(1)$ charge of the field $\phi_{\alpha}$. Thus at each genus $g$ contributions come from instantons of a definite degree unlike the Calabi-Yau manifold case where the dimension of the moduli space is independent of the degree of instantons and instantons of all possible degrees contribute to a correlation function.
3. Intersection numbers

Let us first derive Ward identities of our model (2.20) by varying the matrix eigenvalues as $\lambda_i \rightarrow \lambda_i + \varepsilon \lambda_i^{m+1}$. We find

$$L_{-1}Z = \left( - \frac{\partial}{\partial t_{0,P}} + \sum_{n=1}^{N_t} nt_{n,\alpha} \frac{\partial}{\partial t_{n-1,\alpha}} + N^2 t_{0,P} t_{0,Q} \right) Z = 0, \quad (3.1)$$

$$L_0Z = \left( \sum_{n=1}^{N_t} nt_{n-1,Q} \frac{\partial}{\partial t_{n-1,Q}} - \frac{\partial}{\partial t_{1,P}} + \sum_{n=1}^{N_t} nt_{n,P} \frac{\partial}{\partial t_{n,P}} \right. \\
\left. - 2 \frac{\partial}{\partial t_{0,Q}} + 2 \sum_{n=1}^{N_t} nt_{n,P} \frac{\partial}{\partial t_{n-1,Q}} + N^2 t_{0,P} \right) Z = 0, \quad (3.2)$$

$$L_mZ = \left( - \frac{\partial}{\partial t_{m+1,P}} + \sum_{n=1}^{N_t} nt_{n,P} \frac{\partial}{\partial t_{n+m,P}} + \sum_{n=1}^{N_t} (n+m)t_{n-1,Q} \frac{\partial}{\partial t_{n+m-1,Q}} \right. \\
\left. - 2(m+1)c_{m+1} \frac{\partial}{\partial t_{m,Q}} + 2 \sum_{n=0}^{N_t} (n+m)(1+n(c_{n+m} - c_n)) t_{n,P} \frac{\partial}{\partial t_{n+m-1,Q}} \right. \\
\left. + \frac{1}{N^2} \sum_{r=1}^{m-1} r(m-r) \frac{\partial^2}{\partial t_{m-r-1,Q} \partial t_{r-1,Q}} \right) Z = 0, \quad m = 1, 2, \ldots \quad (3.3)$$

The operators $\{L_m\}$ form a Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n}, \quad m, n \geq -1. \quad (3.4)$$

The $L_0$ equation for the $CP^1$ model has also been derived by Hori using the method of intersection theory without referring to the matrix model [16]. The special feature of the above operators is the presence of the mixing terms of the form $t_{n,P} \partial / \partial t_{n+m-1,Q}$ which arise due to the lack of scale invariance of the action (2.21). The factor 2 in front of these terms is identified as the 1st Chern class $c_1(CP^1) = 2$. 

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If one switches off the coupling constants except \( t_{0,P} \), \( t_{0,Q} \) and \( t_{1,P} \), one finds the genus \( g = 0 \) free energy
\[
F_0 = \frac{1}{2} \frac{t_{0,P}^2 t_{0,Q}}{1 - t_{1,P}} + (1 - t_{1,P})^2 e^{i t_{0,Q} / t_{1,P}}
\] (3.5)
by solving \( L_{-1}, L_0 \) equations and recovers (2.2) in the small phase space.

Let us next consider other Virasoro operators and check them in a simple case where we put all couplings to zero except \( t_{0,P} \). We find, for instance,
\[
L_1 Z = 0 \implies -\langle \sigma_2(P) \rangle_0 - 6\langle \sigma_1(Q) \rangle_0 + 2t_{0,P} \langle Q \rangle_0 = 0,
\] (3.6)
\[
L_2 Z = 0 \implies -\langle \sigma_3(P) \rangle_0 - 11\langle \sigma_2(Q) \rangle_0 + 4t_{0,P} \langle \sigma_1(Q) \rangle_0 + \langle Q \rangle_0 \langle Q \rangle_0 = 0,
\] (3.7)
\[
L_3 Z = 0 \implies -\langle \sigma_4(P) \rangle_0 - \frac{50}{3}\langle \sigma_3(Q) \rangle_0 + 6t_{0,P} \langle \sigma_2(Q) \rangle_0 + 4\langle \sigma_1(Q) \rangle_0 \langle Q \rangle_0 = 0,
\] (3.8)
where \( \langle \cdots \rangle_0 \) denotes the \( g = 0 \) expectation value. We may simply set \( t_{0,P} = 0 \) or take a suitable number of derivatives in \( t_{0,P} \) and set \( t_{0,P} = 0 \) in the above equations. \( \frac{\partial}{\partial t_{0,P}} \big|_{t_{0,P}=0} \) of (3.6) gives, for instance,
\[
-\langle \sigma_2(P) \rangle_0 - 6\langle \sigma_1(Q) \rangle_0 + 2\langle Q \rangle_0 = -2\langle \sigma_1(P) \rangle_0 - 6\langle Q \rangle_0 + 2\langle Q \rangle_0
\]
\[
= -2 \cdot (-2) - 4 = 0,
\] (3.9)
where we have used \( \langle Q^n \rangle_0 = 1 \) for \( n = 0, 1, \cdots \) and the puncture equation
\[
\langle P \prod_{i=1}^s \sigma_{n_i}(\phi_{\alpha_i}) \rangle_0 = \sum_{i} n_i \langle \sigma_{n_1}(\phi_{\alpha_1}) \cdots \sigma_{n_{i-1}}(\phi_{\alpha_{i-1}}) \sigma_{n_i}(\phi_{\alpha_i}) \cdots \sigma_{n_s}(\phi_{\alpha_s}) \rangle_0.
\] (3.10)
The dilaton equation
\[
\langle \sigma_1(P) \prod_{i=1}^s \sigma_{n_i}(\phi_{\alpha_i}) \rangle_g = (2g - 2 + s)\langle \prod_{i=1}^s \sigma_{n_i}(\phi_{\alpha_i}) \rangle_g
\] (3.11)
has also been used (at $g = 0$). Similarly $\frac{\partial^2}{\partial t_0^2}\bigg|_{t_0, P = 0}$ of (3.7) gives

$$-\langle \sigma_3(P)PP\rangle_0 - 11\langle \sigma_2(Q)PP\rangle_0 + 8\langle \sigma_1(Q)P\rangle_0 + 2\langle QPP\rangle_0(Q)_0 = -3 \cdot 2 \cdot (-2) - 11 \cdot 2 + 8 + 2 = 0.$$  

(3.12)

We may also check

$$\frac{\partial^3}{\partial t^3_{0, P}}\bigg|_{t_0, P = 0} \text{ of (3.8)}$$

$$= - \langle \sigma_4(P)PPP\rangle_0 - \frac{50}{3}\langle \sigma_3(Q)PPP\rangle_0 + 6 \cdot 3 \langle \sigma_2(Q)PP\rangle_0$$

$$+ 3 \cdot 4 \langle \sigma_1(Q)P\rangle_0\langle PPP\rangle_0 + 4\langle \sigma_1(Q)PPP\rangle_0(Q)_0$$

$$= - 4 \cdot 3 \cdot 2 \cdot (-2) - \frac{50}{3} \cdot 3 \cdot 2 \cdot 1 + 6 \cdot 3 \cdot 2 \cdot 1 + 12 \cdot 1 + 4 \cdot 1$$

$$= 0.$$  

(3.13)

If one wants to verify (3.6)-(3.8) directly without taking derivatives in $t_{0, P}$, one has to prepare more geometrical data on correlation functions. At $g = 0$ intersection numbers are calculated by means of a set of recursion relations which are derived by combining topological recursion relation (2.9) and Hori’s equation for the intersection of the Kähler class. Hori’s relation [16] reads as

$$\langle Q \prod_{i=1}^s \sigma_{n_i}(\phi_{\alpha_i})\rangle_{0, d} = d\langle \prod_{i=1}^s \sigma_{n_i}(\phi_{\alpha_i})\rangle_{0, d} + \sum_{i=1}^s n_i \langle \sigma_{n_1}(\phi_{\alpha_1}) \cdots \sigma_{n_{i-1}}(\phi_{\alpha_{i-1}}) \cdots \sigma_{n_s}(\phi_{\alpha_s})\rangle_{0, d},$$  

(3.14)

where $\langle \cdots \rangle_{0, d}$ denotes the contribution of the degree-$d$ instanton to genus zero correlation functions. By integrating (3.14) one finds

$$\langle Q \rangle_{0, d} = d\langle 1 \rangle_{0, d} + \sum_m m t_{m, \alpha} \frac{\partial}{\partial t_{m-1, \alpha+1}}\langle 1 \rangle_{0, d},$$  

(3.15)

or

$$\sum_m m t_{m, \alpha} \frac{\partial}{\partial t_{m-1, \alpha+1}}\langle 1 \rangle_{0, d} = -d\langle 1 \rangle_{0, d},$$  

(3.16)
where we have defined $\tilde{t}_{m,\alpha} = t_{m,\alpha} - \delta_{m,1}\delta_{\alpha,P}$. Then we have

$$
\left( \sum_{m} m \tilde{t}_{m,\alpha} \frac{\partial}{\partial t_{m-1,\alpha+1}} \right)^2 \langle 1 \rangle_{0,d} = d^2 \langle 1 \rangle_{0,d}
$$

$$
= \sum m \tilde{t}_{m,1,\gamma} m_2 \tilde{t}_{m,2,\delta} \langle \sigma_{m-1}(\phi_{\gamma+1}) \sigma_{m-1}(\phi_{\delta+1}) \rangle_{0,d}
+ \sum m(m+1) t_{m,\gamma} \langle \sigma_{m-2}(\phi_{\gamma+2}) \rangle_{0,d}. \tag{3.17}
$$

The last term in the above equation can be dropped in the $CP^1$ case (only two primaries exist). By taking the derivative in $t_{n,\alpha}$ of (3.17)

$$
d^2 \langle \sigma_n(\phi_{\alpha}) \rangle_{0,d}
= \sum m_1 \tilde{t}_{m_1,\gamma} m_2 \tilde{t}_{m_2,\delta} \langle \sigma_{n-1}(\phi_{\alpha}) \sigma_{m-1}(\phi_{\gamma+1}) \sigma_{m-1}(\phi_{\delta+1}) \rangle_{0,d}
+ 2n \sum m \tilde{t}_{m,\gamma} \langle \sigma_{n-1}(\phi_{\alpha+1}) \sigma_{m-1}(\phi_{\gamma+1}) \rangle_{0,d}
= \sum \sum_{d_1+d_2=d} m_1 \tilde{t}_{m_1,\gamma} m_2 \tilde{t}_{m_2,\delta} \langle \sigma_{n-1}(\phi_{\alpha}) \phi_{\beta} \rangle_{0,d_1} \langle \phi_{\beta} \sigma_{m-1}(\phi_{\gamma+1}) \sigma_{m-1}(\phi_{\delta+1}) \rangle_{0,d_2}
+ 2n \sum m \tilde{t}_{m,\gamma} \langle \sigma_{n-1}(\phi_{\alpha+1}) \sigma_{m-1}(\phi_{\gamma+1}) \rangle_{0,d}
= \sum (d_2)^2 n \langle \sigma_{n-1}(\phi_{\alpha}) \phi_{\beta} \rangle_{0,d_1} \langle \phi_{\beta} \rangle_{0,d_2} - 2nd \langle \sigma_{n-1}(\phi_{\alpha+1}) \rangle_{0,d}. \tag{3.18}
$$

Going from the 1st to 2nd line we have used the topological recursion relation and used (3.16) in going from the 2nd to 3rd line. Thus we have obtained a basic recursion relation

$$
d^2 \langle \sigma_n(\phi_{\alpha}) \rangle_{0,d} = -2nd \langle \sigma_{n-1}(\phi_{\alpha+1}) \rangle_{0,d} + \sum_{d_1+d_2=d} n(d_2)^2 \langle \sigma_{n-1}(\phi_{\alpha}) \phi_{\beta} \rangle_{0,d_1} \langle \phi_{\beta} \rangle_{0,d_2}. \tag{3.19}
$$

valid in the large phase space. (3.19) may be derived in a more intrinsic manner using the method of algebraic geometry. Derivation is given in Appendix B.

If one puts all couplings to zero and recalls $\langle P \rangle_{0,d} = \langle Q \rangle_{0,d} = 0$ except $\langle Q \rangle_{0,1} =$
1, one finds

$$(m + 1)^2 \langle \sigma_{2m}(Q) \rangle_{0,m+1} = 2m \langle \sigma_{2m-1}(Q)P \rangle_{0,m} = 2m(2m - 1) \langle \sigma_{2m-2}(Q) \rangle_{0,m},$$

$$(m + 1)^2 \langle \sigma_{2m+1}(P) \rangle_{0,m+1} = -2(2m + 1)(m + 1) \langle \sigma_{2m}(Q) \rangle_{0,m+1} + (2m + 1)2m \langle \sigma_{2m-1}(P) \rangle_{0,m}.$$  \tag{3.20}$$

(Note that due to the ghost number conservation $\langle \sigma_{2m}(Q) \rangle_{0,d}$ and $\langle \sigma_{2m+1}(P) \rangle_{0,d}$ are non-vanishing only for $d = m + 1$). These equations are easily solved and yield the data

$$\langle \sigma_{2m}(Q) \rangle_{0,m+1} = \frac{(2m)!}{(m + 1)!(m + 1)!},$$

$$\langle \sigma_{2m+1}(P) \rangle_{0,m+1} = -2c_{m+1} \frac{(2m + 1)!}{(m + 1)!(m + 1)!}. \tag{3.21}$$

Now one may check (3.7) at $t_0,P = 0$,

$$-\left( -\frac{9}{2} \right) - 11 \cdot \frac{1}{2} + 1 \cdot 1 = 0. \tag{3.22}$$

One may also verify $\partial / \partial t_{0,P}$ of (3.8) at $t_0,P = 0$,

$$-4 \cdot \left( -\frac{9}{2} \right) - \frac{50}{3} \cdot 3 \cdot \frac{1}{2} + 6 \cdot \frac{1}{2} + 4 \cdot 1 \cdot 1 = 0. \tag{3.23}$$

We can also discuss our model at genus $g = 1$ and compare its predictions with geometrical data. If we put all couplings to zero except $t_{0,P}$, we find

$$L_{-1}Z = 0 \implies \langle P \rangle_1 = 0, \tag{3.24}$$

$$L_0Z = 0 \implies \langle \sigma_1(P) \rangle_1 + 2\langle Q \rangle_1 = 0. \tag{3.25}$$

Here $\langle \cdots \rangle_1$ denotes the genus $g = 1$ expectation value. (3.25) is consistent with
the geometrical data [16]

\[
\langle \sigma_1(P) \rangle_1 = \frac{1}{24} \chi(CP^1) = \frac{1}{12},
\]

\[
\langle Q \rangle_1 = -\frac{1}{24} \int_{C P^1} \omega = -\frac{1}{24}.
\]

(3.26)

Other Virasoro operators predict

\[
L_1 Z = 0 \implies -\langle \sigma_2(P) \rangle_1 - 6 \langle \sigma_1(Q) \rangle_1 + 2 t_{0,P} \langle Q \rangle_1 = 0,
\]

(3.27)

\[
L_2 Z = 0 \implies -\langle \sigma_3(P) \rangle_1 - 11 \langle \sigma_2(Q) \rangle_1 + \langle Q Q \rangle_0 + 2 \langle Q \rangle_1 \langle Q \rangle_0 + 4 t_{0,P} \langle \sigma_1(Q) \rangle_1 = 0,
\]

(3.28)

\[
L_3 Z = 0 \implies -\langle \sigma_4(P) \rangle_1 - \frac{50}{3} \langle \sigma_3(Q) \rangle_1 + 4 \langle \sigma_1(Q) \rangle Q_0
\]

\[
+ 4 \langle Q \rangle_0 \langle \sigma_1(Q) \rangle_1 + 4 \langle Q \rangle_1 \langle \sigma_1(Q) \rangle_0 + 6 t_{0,P} \langle \sigma_2(Q) \rangle_1,
\]

(3.29)

etc.. It is easy to see that these relations are all satisfied as in the \( g = 0 \) case.

There is in fact a subtlety associated with the \( g = 1 \) free energy. From the definition of the partition function (2.20) and the fact that the size of the matrix \( M \) being equal to \( (N \times t_{0,P})^2 \), it is possible to derive a relation

\[
\left( D - N \frac{\partial}{\partial N} \right) F = 0,
\]

(3.30)

where \( F = \log Z \) and \( D = \sum_{n=0} \hat{t}_{n,\alpha} \frac{\partial}{\partial \hat{m}_{\alpha}} \). If the free energy has an expansion without a term proportional to \( \log N \), (3.30) would imply \( \langle \sigma(P) \rangle_1 = 0 \) at zero couplings in contradiction to the data (3.26). Thus \( F \) must contain a \( \log N \) term as

\[
F = \sum_{g=0} N^{2-2g} F_g - \frac{1}{12} \log N
\]

(3.31)

in the \( 1/N \) expansion. (3.31) reproduces (3.26).
The presence of the log term is related to the issue of whether the genus-1 free energy has an expression suggestive of a string field theory

\[ F_1 = \frac{1}{24} \log \det u_{\alpha\beta}, \quad (3.32) \]

where \( u_{\alpha\beta} = \partial^3 F_0 / \partial t_{0,P} \partial t_{0,\alpha} \partial t_{0,\beta} \). Unlike the case of minimal models where (3.32) holds [14,17,18], it may not be valid in the case of \( \sigma \)-models [19]. In the \( CP^1 \) model we have instead a relation

\[ F_1 = \frac{1}{24} \log \det u_{\alpha\beta} - \frac{1}{24} u, \quad (\alpha, \beta = 1, 2) \quad (3.33) \]

where the extra term reproduces \( \langle Q \rangle_1 = -\frac{1}{24} \). In our previous article [11] we wrote down flow equations which did not contain the 2nd term in the RHS of (3.33). We present corrected flow equations in the Appendix A.

4. Landau-Ginzburg formulation

In the case of topological minimal models coupled to topological gravity at genus \( g = 0 \) Landau-Ginzburg (LG) description of the system has been developed where the scalar Lax operator of the dispersionless KP hierarchy plays the role of the superpotential [20]. In the following we shall show that it is also possible to develop a LG formulation in the case of the topological \( CP^1 \) model using the Lax operator of the Toda hierarchy as the superpotential. Throughout this section we restrict ourselves to the small phase space, i.e. \( u = t_{0,Q} \) and \( v = t_{0,P} \). The Lax operator is then given by

\[ L = p + t_{0,P} + e^{t_{0,Q}} p^{-1}. \quad (4.1) \]

If one regards \( L \) as the superpotential, its extrema give the vacuum states

\[ \partial_p L = 0 \implies p^2 = e^{t_{0,Q}}. \quad (4.2) \]

If the Kähler class \( Q \) is described by \( p \), this gives the relation \( Q \cdot Q = e^{t_{0,Q}} \) of the
quantum cohomology ring of $CP^1$. We thus make an identification

$$P = 1, \quad Q = p.$$  \hfill (4.3)

Note that the variable $p$ itself is not the Landau-Ginzburg field since the Poisson bracket of the Toda theory (2.8) has an unusual form with an extra factor of $p$ in the RHS. We may instead regard $X = \log p$ as the LG variable. The superpotential (4.1) is then rewritten as

$$L = \exp X + t_{0,P} + e^{t_{0,Q}} \exp(-X).$$ \hfill (4.4)

It is of the form of the sine-Gordon potential. Because of the change of variable $p \to X$ there appear Jacobian factors which one must take into account. The residue formula is now given by

$$\langle ABC \rangle = \oint dp \frac{A(p)B(p)C(p)}{p^2 \partial_p L},$$ \hfill (4.5)

where the integration contour is taken around $p = \infty$. Note that the denominator equals

$$p^2 \partial_p L = p^2 - e^{t_{0,Q}}$$ \hfill (4.6)

which looks as if we were dealing with the case of the $A_1$ minimal model. It is easy to check the flatness of the metric

$$\langle P_\alpha \phi_\beta \rangle = \eta_{\alpha\beta},$$ \hfill (4.7)

where $\eta_{PQ} = \eta_{QP} = 1, \; \eta_{PP} = \eta_{QQ} = 0$.

Our next step is to describe gravitational descendants $\sigma_\alpha(\phi_\alpha), \; \phi_\alpha = P, Q$. In the case of the topological minimal models coupled to topological gravity it is possible to adopt the matter picture and express gravitational descendants using
only the matter fields \([21,22]\). It turns out that also in the case of \(CP^1\) model it is possible to describe gravitational descendants using only the LG field. We postulate

\[
\sigma_n(P) = 2n[L^{n-1}(\log L - c_{n-1})p^0L]_+ ,
\]

\[
\sigma_n(Q) = [L^np^0L]_+ ,
\]

with \(n = 0, 1, 2, \cdots\). Note that \(\sigma_n(Q)\) is a polynomial of order \(n + 1\) in \(p\), while \(\sigma_n(P)\) is an infinite series in the variable \(p\). We remark that \(\sigma_0(Q) = p\) from \((4.8)\) in agreement with \((4.3)\). On the other hand, \(\sigma_0(P)\) differs from 1, however, the difference is a BRST exact term.

In order to check our LG formulation we look at the flow equations of the \(CP^1\) model at genus zero. Let us recall the flow equations of Section 2,

\[
\frac{\partial L}{\partial t_{n,P}} = 2\{[L^n(\log L - c_n)]_+, L\},
\]

\[
\frac{\partial L}{\partial t_{n,Q}} = \frac{1}{n+1}\{[L^{n+1}]_+, L\}.
\]

Comparing the coefficients of \(p^{-1}\) on both sides we find

\[
\frac{\partial u}{\partial t_{n,\alpha}} = \frac{\partial}{\partial t_{0,P}} \langle \sigma_n(\phi_\alpha)P \rangle ,
\]

where

\[
\langle \sigma_n(P)P \rangle = 2\text{res}(L^n(\log L - c_n)/p),
\]

\[
\langle \sigma_n(Q)P \rangle = \frac{1}{n+1}\text{res}(L^{n+1}/p).
\]

These are the analogues of the Gelfand-Dikii potentials of the KP hierarchy. Here “res” means to take the coefficient of \(p^{-1}\). We denote \(\langle \sigma_n(\phi_\alpha)P \rangle = R_{n,\alpha}\) henceforth. Integrating \((4.11)\) over \(t_{0,P}\) we find

\[
\langle \sigma_n(P) \rangle = \frac{1}{n+1}R_{n+1,P} = \frac{2}{n+1}\text{res}(L^{n+1}(\log L - c_{n+1})/p),
\]

\[
\langle \sigma_n(Q) \rangle = \frac{1}{n+1}R_{n+1,Q} = \frac{1}{(n+2)(n+1)}\text{res}(L^{n+2}/p).
\]

3-point functions \(\langle \sigma_n(\phi_\alpha)P\phi_\beta \rangle\) are calculated either by taking derivatives of \((4.11)\)
or by using the residue formula (4.5). We find the same results.

We now discuss how various types of recursion relations can be recovered in the LG approach. First of all, notice that

\[
\frac{\partial}{\partial t_0, P} \sigma_n(\phi_\alpha) = n \sigma_{n-1}(\phi_\alpha), \quad n \geq 1. \tag{4.13}
\]

This relation leads to the puncture equation

\[
\langle P \sigma_{n_1}(\phi_\alpha) \sigma_{n_2}(\phi_\beta) \sigma_{n_3}(\phi_\gamma) \rangle = \frac{\partial}{\partial t_0, P} \langle \sigma_{n_1}(\phi_\alpha) \sigma_{n_2}(\phi_\beta) \sigma_{n_3}(\phi_\gamma) \rangle
\]

\[
= \frac{\partial}{\partial t_0, P} \int dp \frac{\sigma_{n_1}(\phi_\alpha(p)) \sigma_{n_2}(\phi_\beta(p)) \sigma_{n_3}(\phi_\gamma(p))}{p^2 \partial_p L} \tag{4.14}
\]

\[
= n_1 \langle \sigma_{n_1-1}(\phi_\alpha) \sigma_{n_2}(\phi_\beta) \sigma_{n_3}(\phi_\gamma) \rangle + n_2 \langle \sigma_{n_1}(\phi_\alpha) \sigma_{n_2-1}(\phi_\beta) \sigma_{n_3}(\phi_\gamma) \rangle
\]

\[
+ n_3 \langle \sigma_{n_1}(\phi_\alpha) \sigma_{n_2}(\phi_\beta) \sigma_{n_3-1}(\phi_\gamma) \rangle.
\]

In order to verify the topological recursion relation we first make a decomposition of gravitational descendants into primary components and BRST exact pieces

\[
\sigma_n(P) = 2n p^2 \partial_p L [L^{n-1}(\log L - c_{n-1})/p]_+ + \sum_{\alpha, \beta} \eta^{\alpha \beta} \frac{\partial R_{n,P}}{\partial t_0, \alpha} \phi_\beta, \tag{4.15}
\]

\[
\sigma_n(Q) = p^2 \partial_p L [L^n/p]_+ + \sum_{\alpha, \beta} \eta^{\alpha \beta} \frac{\partial R_{n,Q}}{\partial t_0, \alpha} \phi_\beta.
\]

Derivation of (4.15) is in parallel with that explained in [21]. Thus

\[
\langle \sigma_n(\phi_\alpha) XY \rangle = \int dp \frac{\sigma_n(\phi_\alpha) XY}{p^2 \partial_p L} = \sum_{\beta, \gamma} \frac{\partial}{\partial t_0, \beta} R_{n, \alpha} \eta^{\beta \gamma} \langle \phi_\gamma XY \rangle \tag{4.16}
\]

\[
= \sum_{\beta, \gamma} n \frac{\partial}{\partial t_0, \beta} \langle \sigma_{n-1}(\phi_\alpha) \rangle \eta^{\beta \gamma} \langle \phi_\gamma XY \rangle = \sum_{\beta, \gamma} n \langle \sigma_{n-1}(\phi_\alpha) \phi_\beta \rangle \eta^{\beta \gamma} \langle \phi_\gamma XY \rangle,
\]

where we have used (4.12). This is the desired relation.
Now, turn off $t_{0,P}$ and make a change of variable $p = e^{t_{0,Q}/\bar{p}}$, then

$$\sigma_n(\phi_\alpha) = e^{(n+q_\alpha)t_{0,Q}/2} \bar{\sigma}_n(\phi_\alpha),$$  \hspace{1cm} (4.17)

where the $U(1)$ charges are $q_P = 0, q_Q = 1$ and

$$\bar{\sigma}_n(P) = 2n[\bar{L}^{(n-1)}(\log L - c_{n-1})\bar{p}\partial_{\bar{p}}\bar{L}]_+, \hspace{1cm} (4.18)$$

$$\bar{\sigma}_n(Q) = [\bar{L}^n\bar{p}\partial_{\bar{p}}\bar{L}]_+ ,$$

with

$$L = \bar{p} + \bar{p}^{-1},$$

$$\log L = \frac{1}{2}t_{0,Q} + \frac{1}{2}\log(1 + \bar{p}^2) + \frac{1}{2}\log(1 + \bar{p}^{-2}).$$  \hspace{1cm} (4.19)

Notice that

$$\frac{\partial}{\partial t_{0,Q}} \bar{\sigma}_n(P) = n\bar{\sigma}_{n-1}(Q), \hspace{1cm} \frac{\partial}{\partial t_{0,Q}} \bar{\sigma}_n(Q) = 0. \hspace{1cm} (4.20)$$

The degree-$d$ instanton contribution to the 3-point function $\langle \sigma_{n_1}(\phi_{\alpha_1})\sigma_{n_2}(\phi_{\alpha_2})\sigma_{n_3}(\phi_{\alpha_3}) \rangle_d$ is given by

$$\langle \sigma_{n_1}(\phi_{\alpha_1})\sigma_{n_2}(\phi_{\alpha_2})\sigma_{n_3}(\phi_{\alpha_3}) \rangle_d = e^{d_{t_{0,Q}}} \int d\bar{p} \frac{\bar{\sigma}_{n_1}(\phi_{\alpha_1})\bar{\sigma}_{n_2}(\phi_{\alpha_2})\bar{\sigma}_{n_3}(\phi_{\alpha_3})}{\bar{p}^2\partial_{\bar{p}}\bar{L}},$$  \hspace{1cm} (4.21)

where we have used the ghost number conservation $\sum_{i=1}^{3}(n_i + q_{\alpha_i} - 1) = 2d - 2$. Combining (4.21) and (4.20) we find

$$\langle Q\sigma_{n_1}(\phi_{\alpha_1})\sigma_{n_2}(\phi_{\alpha_2})\sigma_{n_3}(\phi_{\alpha_3}) \rangle_d = \frac{d}{d_{t_{0,Q}}} \langle \sigma_{n_1}(\phi_{\alpha_1})\sigma_{n_2}(\phi_{\alpha_2})\sigma_{n_3}(\phi_{\alpha_3}) \rangle_d$$

$$= de^{d_{t_{0,Q}}} \int \frac{d\bar{p}}{\bar{p}^2\partial_{\bar{p}}\bar{L}} \prod_{i=1}^{3}\bar{\sigma}_{n_i}(\phi_{\alpha_i}) + e^{d_{t_{0,Q}}} \int \frac{d\bar{p}}{\bar{p}^2\partial_{\bar{p}}\bar{L}} \frac{\partial}{\partial t_{0,Q}} \prod_{i=1}^{3}\bar{\sigma}_{n_i}(\phi_{\alpha_i})$$

$$= d \langle \sigma_{n_1}(\phi_{\alpha_1})\sigma_{n_2}(\phi_{\alpha_2})\sigma_{n_3}(\phi_{\alpha_3}) \rangle_d + n_1 \langle \sigma_{n_1-1}(N)\sigma_{n_2}(\phi_{\alpha_2})\sigma_{n_3}(\phi_{\alpha_3}) \rangle_d \delta_{\alpha_1} P$$

$$+ n_2 \langle \sigma_{n_1}(\phi_{\alpha_1})\sigma_{n_2-1}(Q)\sigma_{n_3}(\phi_{\alpha_3}) \rangle_d \delta_{\alpha_2} P + n_3 \langle \sigma_{n_1}(\phi_{\alpha_1})\sigma_{n_2}(\phi_{\alpha_2})\sigma_{n_3-1}(Q) \rangle_d \delta_{\alpha_3} P.$$  \hspace{1cm} (4.22)

This is the relation (3.14) for the insertion of the Kähler class.
Finally we present some sample calculations. First we compute \( \langle \sigma_3(P)\sigma_1(P)\sigma_1(P) \rangle \) at \( t_{0,P} = 0, \, t_{0,Q} \neq 0 \). We write

\[
\sigma_3(P) = A(p) + B(p), \quad \sigma_1(P)\sigma_1(P) = C(p) + D(p),
\]

where

\[
A(p) = \frac{1}{2}(e^{-t_{0,Q}p^2} + t_{0,Q} - c_2)(p^3 + e^{t_{0,Q}}p), \quad B(p) = \sum_{\ell=1}^{\infty} B_{\ell} p^\ell;
\]

\[
C(p) = t_{0,Q}^2 p^2, \quad D(p) = \sum_{\ell=1}^{\infty} D_{\ell} p^\ell
\]

and the coefficients \( B_{\ell}, D_{\ell} \) are easily read off from (4.8). Expanding \( 1/p^2 \partial_p L \) around \( p = \infty \) we have

\[
\langle \sigma_3(P)\sigma_1(P)\sigma_1(P) \rangle = \oint dp \sum_{k=0}^{\infty} \frac{e^{kt_{0,Q}}}{p^{2k+2}}[A(p)C(p) + A(p)D(p) + B(p)C(p) + B(p)D(p)].
\]

Explicit calculations show that the contributions from the last three terms cancel out among themselves. The first term then yields

\[
\langle \sigma_3(P)\sigma_1(P)\sigma_1(P) \rangle = 6t_{0,Q}^2(t_{0,Q} - 2)e^{2t_{0,Q}}.
\]

This result can be reproduced by a computation using (4.9) and (4.16).

Setting \( t_{0,P} = t_{0,Q} = 0 \) we next examine 1-point functions. Using the binomial expansion of powers of \( L = p + p^{-1} \) in (4.12) we immediately recover the previous result (3.21)

\[
\langle \sigma_{2m}(Q) \rangle = \frac{(2m)!}{(m+1)!(m+1)!}.
\]
Similarly $\langle \sigma_{2m+1}(P) \rangle$ can be evaluated as

$$
\langle \sigma_{2m+1}(P) \rangle = \frac{1}{m+1} \int \frac{dp}{p} L^{2m+2} (\log L - c_{2m+2})
$$

$$
= \frac{1}{m+1} \int \frac{dp}{p} L^{2m+2} (\log(1 + p^{-2}) - c_{2m+2})
$$

$$
= \frac{1}{m+1} \int \frac{dx}{(x-1)^{2m+2}} (\log x - c_{2m+2})
$$

$$
= \frac{2(2m+1)!}{(m+1)!(m+1)!} \int dx \frac{x^{m+1}}{x-1} (\log x - c_{m+1}),
$$

(4.28)

where we have made a change of variable $1 + p^{-2} = x$ and repeated partial integrations. Then by picking up a pole at $x = 1$ we reproduce (3.21)

$$
\langle \sigma_{2m+1}(P) \rangle = -2c_{m+1} \frac{(2m+1)!}{(m+1)!(m+1)!}.
$$

(4.29)

As we have noted earlier, the superpotential of the $CP^1$ model has the form of the sine-Gordon theory when expressed in terms of the LG field $X$. This seems to indicate the equivalence of the $CP^1$ and the $N = 2$ supersymmetric sine-Gordon theories at least in their topological versions. The correspondence between these theories have been known for some time based on the agreement of the particle spectra and scattering matrices [12,13]. Our results seem to provide further evidence for their equivalence. It will be interesting to see if we can make the correspondence more precise and find out, in particular, how a super-renormalizable sine-Gordon theory may be turned into an asymptotically free theory possibly at some special value of the coupling constant.

The relation between our coupling constant $t_{0,Q}$ and those of the physical theory $g^2/4\pi$, $\theta$ is given by

$$
t_{0,Q} = -\left(\frac{4\pi}{g^2} + i\theta\right).
$$

(4.30)

In asymptotically free theories the coupling constant is replaced by the $\Lambda$-parameter via the dimensional transmutation $e^{-4\pi/g^2} = \Lambda^b$ with $b$ being the one-loop $\beta$-function. For the $N = 2 CP^n \sigma$-model $b$ is equal to the 1st Chern class, $b = n + 1$
[23]. Thus we have (putting \( \theta = 0 \))

\[
e^{t_0, Q} = \Lambda^2
\]

(4.31)

and the superpotential reads after shifting \( X \rightarrow X + t_0 Q / 2 \)

\[
L = \Lambda(e^X + e^{-X}).
\]

(4.32)

Clearly \( \Lambda \) corresponds to the sine-Gordon mass parameter.

5. Generalizations and Discussions

In this paper we have discussed the topological \( CP^1 \) model in some details. We have described its integrable structure, matrix model realization and Landau-Ginzburg formulation. Predictions of the matrix model on the intersection numbers (Gromov-Witten invariants) on the moduli space \( \mathcal{M}_{g,s}(CP^1; d) \) agree with those obtained by using the method of algebraic geometry and also by residue integrals of the LG superpotential at \( g = 0, 1 \). Thus the model in fact seems consistent and reproduces the sum over instantons from the Riemann surfaces to \( CP^1 \). It will be important to verify the predictions of the model at higher genera.

It is somewhat unexpected that the Virasoro type constraints also hold in the \( CP^1 \) theory although the Virasoro operators have a peculiar form reflecting the lack of scale invariance in the system. Unlike the case of minimal models, however, we can not construct negative-index operators \( \{L_{-n}, n = 2, 3, \ldots \} \) to supplement the original ones \( \{L_m, m \geq -1 \} \) into a full-fledged Virasoro algebra. In the \( CP^1 \) case only “half” of the algebra exists. A related issue is a possible free boson/fermion description of the \( CP^1 \) system. While the form of the Virasoro operators suggests free bosons associated with each set of variables \( \{t_{n,P}\}, \{t_{n,Q}\} \), it differs from that of the canonical stress tensor and does not seem to have an obvious interpretation. It will be an interesting problem to provide a space-time description of the \( CP^1 \) model.
Another important problem is to extend our construction to $CP^n$ or other Fano varieties. It turns out that in the case of $CP^2$ and $CP^3$ it is also possible to develop a LG formulation; in the $CP^2$ case, for instance, we assume that the primary fields $P, Q, R$ are represented as

$$P = 1, \quad Q = p + a, \quad R = p^2 + bp + c$$

and satisfy the algebra

$$\phi_i \phi_j = \langle \phi_i \phi_j \phi^k \rangle \phi_k \mod W'.$$  \hfill (5.2)

It is also assumed that the potential $W$ has a form

$$W = p^3 + up^2 + vp + w.$$ \hfill (5.3)

It is then possible to find a solution to the above set of equations (by shifting $p$ we may generate other solutions)

$$a = \langle QQQ \rangle, \quad b = \langle QQQ \rangle, \quad c = -\langle QQR \rangle,$$

$$u = 2\langle QQQ \rangle, \quad v = \langle QQQ \rangle^2 - 2\langle QQR \rangle, \quad w = -2\langle QQQ \rangle \langle QQR \rangle - \langle QRR \rangle$$  \hfill (5.4)

and we further predict

$$\langle RRR \rangle = -\langle QQQ \rangle \langle QRR \rangle + \langle QQR \rangle^2.$$ \hfill (5.5)

(5.5) is the equation which comes from the associativity of the operator-product-expansion [6]. Note that when we switch off $t_P, t_R, \langle QQQ \rangle, \langle QQR \rangle, \langle RRR \rangle$ vanishes and $\langle QRR \rangle = e^{t_Q}$ and the above construction reduces to the familiar relations in quantum cohomology [24,25],

$$P = 1, \quad Q = p, \quad R = p^2$$

$$W = p^3 - e^{t_Q}.$$ \hfill (5.6)

A similar construction works also for the $CP^3$ model and we deform quantum cohomology relations. However, we do not yet know the integrability structure.
behind these constructions. (Recently Kanno-Ohta [26] have introduced an interesting model whose superpotential has a form $ap^2 + \cdots + bp^{-1}$. The model, however, does not describe $CP^2$ but a modified $CP^1$ theory coupled with some minimal matter).

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APPENDIX A

Flow equations at higher genera of \([11]\) do not reproduce the \(g = 1\) free energy (3.33) (the relation between the 2-point function \(u = \langle PP\rangle\) and the free energy was not chosen properly at higher genera). We present in the following the revised version of the flow equations.

\[
\frac{\partial u}{\partial t_{0,Q}} = \langle PQ\rangle' = [v]'
\]

\[
\frac{\partial v}{\partial t_{0,Q}} = \langle QQ\rangle' = \left[ e^u + \frac{1}{\mathcal{N}^2} \frac{1}{12} u'' e^u + \frac{1}{\mathcal{N}^4} \left( \frac{1}{360} u^{(4)} + \frac{1}{288} u^{r2} \right) e^u + \cdots \right]' \tag{A.2}
\]

\[
\frac{\partial u}{\partial t_{1,P}} = \langle \sigma_1 (P) P \rangle' = \left[ uv + \frac{1}{\mathcal{N}^2} \left( \frac{1}{6} v'' - \frac{1}{12} v' u' \right) + \frac{1}{\mathcal{N}^4} \left( - \frac{1}{360} v^{(4)} + \frac{1}{720} v'' u' + \frac{1}{180} v'' u'' + \frac{1}{720} v' u''' \right) + \cdots \right]' \tag{A.3}
\]

\[
\frac{\partial v}{\partial t_{1,P}} = \langle \sigma_1 (P) Q \rangle' = \left[ \frac{1}{2} v^2 + (u - 1) e^u + \frac{1}{\mathcal{N}^2} \left( \frac{1}{12} uu'' + \frac{1}{6} u'' + \frac{1}{12} u^{r2} \right) e^u - \frac{1}{24} v^{r2} \right] + \frac{1}{\mathcal{N}^4} \left( \frac{1}{360} uu^{(4)} + \frac{1}{90} u^{(4)} + \frac{1}{120} u' u''' - \frac{1}{720} u^{r2} u'' \right. \\
\left. - \frac{1}{720} u^4 + \frac{1}{288} u u^{r2} + \frac{19}{1440} u'' u' \right) e^u + \frac{1}{720} v'' v' + \frac{1}{360} v^{r2} + \cdots \right]' \tag{A.4}
\]

\[
\frac{\partial u}{\partial t_{1,Q}} = \langle \sigma_1 (Q) P \rangle' = \left[ \frac{1}{2} v^2 + e^u + \frac{1}{\mathcal{N}^2} \left( \frac{1}{12} u^2 + \frac{1}{6} u'' \right) e^u - \frac{1}{24} v^{r2} \right] + \frac{1}{\mathcal{N}^4} \left( \frac{1}{120} u^{(4)} + \frac{1}{120} u' u''' - \frac{1}{720} u^{r2} u'' - \frac{1}{720} u^4 + \frac{1}{160} u'' u'' \right) e^u \\
+ \frac{1}{720} v' v''' + \frac{1}{360} v'' v' + \cdots \right]' \tag{A.5}
\]
\[
\frac{\partial v}{\partial t_1 Q} = \langle \sigma_1(Q)Q \rangle'
\]
\[
= \left[ ve^u + \frac{1}{N^2} \left( \frac{1}{12} vu'' + \frac{1}{6} v'' \right) e^u 
+ \frac{1}{N^4} \left( \frac{1}{360} vu^{(4)} + \frac{1}{288} vu''^2 + \frac{1}{72} v'' u'' + \frac{1}{120} v^{(4)} \right) e^u + \cdots \right]'
\]

APPENDIX B

Any holomorphic map \( f : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1 \) of degree \( d \) is expressed as
\[
f(z) = a \frac{(z - b_1) \cdots (z - b_d)}{(z - c_1) \cdots (z - c_d)},
\]  
where \( b_i \neq c_j \) for every \( i, j \), or is obtained as the limit of such an expression as \( b_i \rightarrow \infty, a \rightarrow 0 \) (or \( c_j \rightarrow \infty, a \rightarrow \infty \)). Two maps \( f \) and \( f' \) are equivalent when there is a fractional linear transformation \( g \in PSL(2, \mathbb{C}) \)
\[
g(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \quad \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in SL(2, \mathbb{C}),
\]  
such that \( f = f' \circ g \). This induces the action of \( PSL(2, \mathbb{C}) \) on the parameters \( b_i, c_j \):

\[
b_i \mapsto g(b_i), \quad c_j \mapsto g(c_j)
\]  
while \( a \) is transformed in a complicated way. A marked point \( x \) is also transformed as
\[
x \mapsto g(x).
\]

Let \( \mathcal{M}_{0,1}(\mathbb{C}P^1, d) \) be the moduli space of degree \( d \) holomorphic maps \( \mathbb{C}P^1 \rightarrow \mathbb{C}P^1 \) with one marked point on the world sheet. The expression
\[
s = \bigotimes_{1 \leq i, j \leq d} \left( \frac{dx}{x - b_i} - \frac{dx}{x - c_j} \right)
\]
is \( PSL(2, \mathbb{C}) \)-invariant and is symmetric under permutations \( b_i \leftrightarrow b_i', c_j \leftrightarrow c_j' \). Therefore, \( s \) is a meromorphic section of the line bundle \( L^d \) over \( \mathcal{M}_{0,1}(\mathbb{C}P^1, d) \). 

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where \( \mathcal{L} \) is the bundle of cotangent space at the marked point. It extends over the compactified moduli space \( \overline{\mathcal{M}}_{0,1}(\mathbb{CP}^1, d) \) introduced in [6,27]. One can figure out the first Chern class \( c_1(\mathcal{L} \otimes d^2) = d^2 c_1(\mathcal{L}) \) by looking at the locus of zeroes and poles of \( s \).

### Poles

The section \( s \) has poles of order \( d \) at the divisors \( D_b \) and \( D_c \) where \( D_b \) (resp. \( D_c \)) is the locus of \( x = b_i \) (resp. \( x = c_i \)) for some \( i \). In other words, \( D_b \) (resp. \( D_c \)) is the locus of configurations such that \( x \) is mapped by \( f \) to \( 0 \in \mathbb{CP}^1 \) (resp. \( \infty \in \mathbb{CP}^1 \)). The Poincaré dual is thus \([D_b] = [D_c] = \phi^* \omega\) where \( \phi \) is the evaluation map and \( \omega \) is the Kähler form of volume 1.

### Zeroes

The section \( s \) goes to zero as \( b_i \) approaches to \( c_j \). Let us see what happens to the configuration when the \( 2k \) points \( b_1, \ldots, b_k, c_1, \ldots, c_k \) converge into one point \( x_* \):

\[
\begin{align*}
    b_i &= x_* + \beta_i \epsilon \\
    c_i &= x_* + \gamma_i \epsilon \\
\end{align*}
\]

\( i = 1, \ldots, k \) \hspace{1cm} (B.6)

If we use \( z \) as the coordinate on the world sheet, \( f \) converges for \( z \neq x_* \) as

\[
f(z) \xrightarrow{\epsilon \rightarrow 0} a \frac{(z - b_{k+1}) \cdots (z - b_d)}{(z - c_{k+1}) \cdots (z - c_d)} =: f_0(z). \tag{B.7}
\]

\( f_0 \) is generically a map of degree \( d - k \). If we use instead the coordinate \( \zeta \) defined by

\[
z - x_* = \epsilon \zeta, \tag{B.8}
\]

\( f(z) = \tilde{f}(\zeta) \) converges for \( \zeta \neq \infty \) as

\[
\tilde{f}(\zeta) \xrightarrow{\epsilon \rightarrow 0} f_0(x_*) \frac{(\zeta - \beta_1) \cdots (\zeta - \beta_k)}{(\zeta - \gamma_1) \cdots (\zeta - \gamma_k)} =: \tilde{f}_0(\zeta). \tag{B.9}
\]

\( \tilde{f}_0 \) is generically a map of degree \( k \). The limit can be identified as the configuration of genus zero Riemann surface \( \Sigma_0 \cup \tilde{\Sigma}_0 \) with a double point \( \Sigma_0 \cap \tilde{\Sigma}_0 \) where \( \Sigma_0 \) is
mapped to $\mathbb{CP}^1$ by a map of degree $d-k$ and $\tilde{\Sigma}_0$ is mapped to $\mathbb{CP}^1$ by a map of degree $k$. (Generically, $\Sigma_0$ is $\mathbb{CP}^1$ with coordinate $z$ that is mapped to $\mathbb{CP}^1$ by $f_0$ and $\tilde{\Sigma}_0$ is $\mathbb{CP}^1$ with coordinate $\zeta$ that is mapped by $\tilde{f}_0$. The double point $\Sigma_0 \cap \tilde{\Sigma}_0$ ($z = x_*$ or $\zeta = \infty$) is mapped to $f_0(x_*)$.) We denote by $D_{d-k,k}$ the moduli space of such configurations with a marked point in $\Sigma_0$ (the branch of degree $d-k$). The moduli space $\overline{M}_{0,1}(\mathbb{CP}^1, d)$ includes $D_{d-k,k}$ as a compactification divisor to which the coordinate $\epsilon$ is transversal. Since $s$ is proportional to $\epsilon^{k^2}$ as $\epsilon \to 0$, it has zero of order $k^2$ at $D_{d-k,k} \subset \overline{M}_{0,1}(\mathbb{CP}^1, d)$.

Thus, we see that

$$d^2 c_1(\mathcal{L}) = -2d \phi^* \omega + \sum_{k=1}^d k^2 [D_{d-k,k}].$$  \hspace{1cm} (B.10)

The same argument shows the similar relation in the case with extra marked points. Let $S$ be the set of marks for them. Denoting by $\sum_{X \cup Y = S}$ the sum over disjoint unions, we have

$$d^2 c_1(\mathcal{L}) = -2d \phi^* \omega + \sum_{X \cup Y = S} \sum_{k=1}^d k^2 [D^{X,Y}_{d-k,k}].$$  \hspace{1cm} (B.11)

$D^{X,Y}_{d-k,k}$ is the locus of configurations such that the point $x$ and the points marked by $X$ belong to the branch of degree $d-k$, while the points marked by $Y$ belong to the branch of degree $k$. This yields the recursion relation of intersection numbers

$$d^2 \langle \sigma_n(\phi_\alpha) \prod_{i \in S} \sigma_n(\phi_{\alpha_i}) \rangle_{0,d} = -2dn \langle \sigma_{n-1}(\phi_{\alpha+1}) \prod_{i \in S} \sigma_n(\phi_{\alpha_i}) \rangle_{0,d}$$

$$+ \sum_{X \cup Y = S} \sum_{k=1}^d k^2 n \langle \sigma_{n-1}(\phi_\alpha) \prod_{i \in X} \sigma_n(\phi_{\alpha_i}) \phi_\beta \prod_{j \in Y} \sigma_{n_j}(\phi_{\alpha_j}) \rangle_{0,k}$$

\hspace{1cm} (B.12)

which leads to (3.19).
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