Quintics with Finite Simple Symmetries

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Abstract
We construct all quintic invariants in five variables with simple Non-Abelian finite symmetry groups. These define Calabi-Yau three-folds which are left invariant by the action of $A_5$, $A_6$ or $PSL_2(11)$.

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1 Introduction

There has long been interest in the study of hypersurfaces defined by a quintic polynomial in five variables. Quintic polynomials in five complex variables characterize a class of Calabi-Yau three-folds over $\mathbb{CP}^4$, see Ref. [1]. A special subclass of such manifolds is that where the polynomial is invariant under discrete groups.

The identification of those hypersurfaces which are left invariant under the action of Non-Abelian finite groups proceeds in two steps. First, identify Non-Abelian finite groups with five-dimensional irreducible representations. Second, single out those which can sustain a quintic invariant.

Unlike the case of two, three and four variables [2, 3, 4, 5], there is no catalog of all finite groups with five-dimensional representations. However it is an easy matter to find the simple groups which have such representations. The ATLAS [6] lists just a handful of simple groups:

- The 60 element symmetry group of the isocahedron and dodecahedron, aka the Icosahedral group $\mathcal{I} = A_5 = P\mathcal{S}\mathcal{L}_2(5)$ with one real five-dimensional representation.

- The Alternating group of even permutations on six objects, $A_6$ with two inequivalent real five-dimensional irreps.

- The group of projective linear transformations of unit determinant over the Galois field of order 11, $P\mathcal{S}\mathcal{L}_2(11)$ with 660 elements has one complex five-dimensional irrep $\bar{5}$ and its conjugate $\bar{5}$.

- The group of $(4 \times 4)$ unitary matrices over the Galois field of order two, $U_4(2) = Sp_4(3)$ with 25920 elements also has one complex irrep $\bar{5}$ and the $\bar{5}$.

It is known that the real $A_5$ quintet has two quintic invariants, and two cubic invariants. Their explicit construction can be found in the beautiful paper of Cummins and Patera [7], which serves as a starting point for our investigations.

Our new results are simply stated:

- We identify and construct one quintic and one cubic $A_6$ invariant from its quintet representation.

- We find only one $P\mathcal{S}\mathcal{L}_2(11)$ quintic invariant on five complex variables, and give its explicit form.

- We show that the $U_4(2) = Sp_4(3)$ complex quintet does not have a quintic invariant, by constructing its Molien polynomial (see below).

We present these results in the hope that they will motivate the study of these special hypersurfaces, in particular the unique $P\mathcal{S}\mathcal{L}_2(11)$ Calabi-Yau manifold.
1.1 Finite Group Invariants

The construction of finite group invariants [8, 9] is a complicated but tractable problem. Finite groups have, up to conjugation, one irreducible representation whose Kronecker products generate all other irreducible representations. One can in this way find all invariants made out of one such irreducible representation. The procedure can be duplicated for any other irreducible representation. The same technique can be continued in principle to the building of invariants constructed out of several irreducible representations, although it becomes quite unwieldy.

Consider a discrete group \( G \) of order \( N \), with \( l \) irreducible representations \( r_a \), with \( r_1 = 1 \) the singlet. The \( k \)-fold symmetric Kronecker product

\[
(r_a \otimes r_a \otimes \cdots \otimes r_a)_k = \sum_b D^{[k]}(r_b; r_a) r_b ,
\]

is a reducible representation, the sum of the group’s irreps. The integer coefficient \( D^{[k]}(r_b; r_a) \) is the multiplicity of the \( r_b \) representation in the product. These coefficients are determined from Molien’s remarkable generating function [10]

\[
F(r_b; r_a; \lambda) = \sum_{i=1}^{l} n_i \frac{\chi_i^{[r_b]}}{\det(1 - \lambda A_i^{[r_a]})} ,
\]

where \( i \) labels the class \( C_i \) of \( G \), with \( n_i \) elements. In the \( i \)th class, \( A_i^{[r_a]} \) is any group element in the \( r_a \) representation, and \( \chi_i^{[r_b]} \) is the character of the \( r_b \) representation. Its power series

\[
F(r_b; r_a; \lambda) = \sum_k D^{[k]}(r_b; r_a) \lambda^k ,
\]

yields the desired coefficients. The Molien function is an important tool in determining all invariants constructed out of one irreducible representation

\[
F(1; r; \lambda) = \sum_{i=1}^{l} \frac{n_i \chi_i^{[r]}}{\det(1 - \lambda A_i^{[r]})} = \sum_{k=0}^{\infty} c_k \lambda^k ,
\]

where \( c_k \) denotes the number of invariants of order \( k \). The evaluation of the finite sum over the classes yields an expression which can always be written in the form

\[
F(1; r; \lambda) = \frac{(1 + \sum d_k \lambda^k)}{(1 - \lambda^{a_1})^{n_1}(1 - \lambda^{a_2})^{n_2} \cdots} ,
\]

where the numerator is a finite polynomial in \( \lambda \), and the \( d_k, a_k \) and \( n_k \) are positive integers. Expanding each factor in the denominator yields \( d_k \) invariants of order \( k \), as well as \( n_k \) invariants of order \( a_k \), etc. This infinite number of
invariants can be expressed as products of a finite set of basic invariants, some of which satisfy non-linear relations called syzygies among themselves.

In some simple cases, the invariants of order \( a_1, a_2, \) etc..., as indicated by the denominator in the Molien series do not satisfy syzygies among themselves. They are called free invariants by the authors of reference [7]. The powers in the numerator refer to constrained invariants which satisfy syzygies with the free invariants. Unfortunately, this neat distinction among invariants is of limited validity.

To see how this works, consider the Tetrahedral group \( A_4 \), which has four irreps, \( 1, 1_1, 1_\bar{1}, \) and \( 3 \). Using its character table, it is an easy matter to compute the generating function for each. For the singlet irrep, we have of course

\[
F(1; 1; \lambda) = \frac{1}{1 - \lambda},
\]
corresponding to the trivial invariant. The situation is a bit less trivial for the other one-dimensional representations,

\[
F(1; 1_1; \lambda) = F(1; 1_\bar{1}; \lambda) = \frac{1}{1 - \lambda^3},
\]
which means that there is one invariant of cubic order. If \( z \) spans the irrep, then \( z^3 \) is an invariant. For the triplet, the situation becomes more complicated. From

\[
F(1; 3; \lambda) = \frac{1 + \lambda^6}{(1 - \lambda^2)(1 - \lambda^3)(1 - \lambda^4)},
\]
we deduce that there are three free invariants, of order 2, 3 and 4, and one sixth order constrained invariant. Assume that the triplet irrep is spanned by the three real coordinates \( x_i, i = 1, 2, 3 \). The quadratic invariant is the same as for \( SO(3) \)

\[
I_3^{[2]} = (x_1^2 + x_2^2 + x_3^2).
\]
The cubic and quartic invariants are given by

\[
I_3^{[3]} = x_1 x_2 x_3, \quad I_3^{[4]} = x_1^4 + x_2^4 + x_3^4.
\]
The constrained sixth-order invariant is

\[
E_3^{[6]} = (x_1^2 - x_2^2)(x_2^2 - x_3^2)(x_3^2 - x_1^2).
\]
The syzygy, too long to write here, expresses the square of the sixth-order invariant in terms of polynomials in the free invariants.
2 Quintics from $A_5$

We begin our study with the Icosahedral group, $A_5 = PSL_2(5)$. It is generated by two elements with presentation

$$< A, B \mid A^2 = B^3 = (AB)^5 = 1 > ,$$

and character table

| $A_5$ | $C_1$ | $12C_2^{[3]}$ | $12C_3^{[3]}$ | $15C_4^{[2]}$ | $20C_5^{[3]}$ |
|-------|-------|----------------|----------------|----------------|----------------|
| $\chi^{[1]}$ | 1 | 1 | 1 | 1 | 1 |
| $\chi^{[3_1]}$ | 3 | $-b_5$ | $-\bar{b}_5$ | -1 | 0 |
| $\chi^{[3_2]}$ | 3 | $-\bar{b}_5$ | $-b_5$ | -1 | 0 |
| $\chi^{[4]}$ | 4 | -1 | -1 | 0 | 1 |
| $\chi^{[5]}$ | 5 | 0 | 0 | 1 | -1 |

where

$$b_5 = \frac{1}{2}(-1 + \sqrt{5}) , \quad \bar{b}_5 = \frac{1}{2}(-1 - \sqrt{5}) .$$

In the Cummins-Patera basis [7], the quintic representation generators are given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} ; \quad B = -\frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & -\omega^2 & -\omega \\ 0 & 1 & 1 & -1 & -1 \\ -1 & -1 & 0 & \omega & \omega^2 \\ \omega & 1 & \omega^2 & 0 & -\omega \\ \omega^2 & 1 & \omega & -\omega^2 & 0 \end{pmatrix} ,$$

where $\omega = \exp(2i\pi/3)$ is the cubic root of unity,

$$\omega^3 = 1 , \quad 1 + \omega + \omega^2 = 0 , \quad \omega = -\frac{1}{2} + \frac{i}{2}\sqrt{3} .$$

These matrices act on the column vector

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ z \end{pmatrix} , \quad z = \frac{1}{\sqrt{2}}(x_4 + ix_5) ,$$

where the $x_i$ are real. The Molien generating function for the quintet is
There are two quintic invariants. One is odd under conjugation or in terms of the vector components

\[ I_{\lambda}^{[2]} = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = x_1^2 + x_2^2 + x_3^2 + 2z \tau . \] (11)

One cubic invariant

\[ I_{\alpha}^{[3]} = \frac{1}{\sqrt{2}} \left\{ x_5^2 + 3\sqrt{3} x_4 (x_2^2 - x_3^2) + 3x_3 (2x_1^2 - x_2^2 - x_3^2 - x_4^2) \right\}, \] (12)

is pure imaginary, that is odd under complex conjugation

\[ z \rightarrow \tau, \quad \tau \rightarrow z; \quad \omega \rightarrow \omega^2, \quad \omega^2 \rightarrow \omega, \]
or in terms of the vector components

\[ z \rightarrow \tau, \quad \tau \rightarrow z; \quad x_2 \rightarrow x_3, \quad x_3 \rightarrow x_2. \]

The second cubic

\[ I_{\delta}^{[3]} = \frac{1}{\sqrt{2}} \left\{ \sqrt{3} x_5 (x_2^2 - x_3^2) + x_4 (2x_1^2 - x_2^2 - x_3^2 + x_4^2 - 3x_5^2) - 4\sqrt{2} x_1 x_2 x_3 \right\}. \] (13)

is real. The one quartic invariant is

\[ I_{\lambda}^{[4]} = \frac{1}{\sqrt{2}} \left( 2x_4^2 + 2x_2^2 - 3x_5^2 + (x_2^2 + x_3^2 + x_5^2) \sqrt{3} x_4 x_5 (x_2^2 - x_3^2) + 2x_5^2 (4x_1^2 + x_2^2 + x_3^2) + 6x_4^2 (x_2^2 + x_3^2) \right). \] (14)

There are two quintic invariants. One is odd under conjugation

\[ I_{\alpha}^{[5]} = (x_1^4 + \omega x_2^4 + \omega^2 x_3^4) \tau - (x_1^4 + \omega x_2^4 + \omega^2 x_3^4) z \]

\[ -2x_1^2 (x_2^4 + 3x_3^4 + 2x_5^4) \left[ (x_2^2 + \omega x_3^2 + \omega^2 x_1^2) \tau - (x_2^2 + \omega x_3^2 + \omega^2 x_1^2) z \right] - 2\tau (z^3 - \tau^3) , \]
or explicitly

\[ I_{a5}^{[5]} = \frac{i}{\sqrt{2}} \left\{ \sqrt{3}x_4^3 - \sqrt{3}x_1^3x_2^3 - \sqrt{3}x_1^3x_2 + \sqrt{3}x_1^3x_2^4 + \sqrt{3}(2x_1^3x_2 - 2x_2^3x_1) x_1^2 \right. \\
+ x_5 \left[ \frac{-3}{2} x_1^4 - x_2^3 x_2^2 - x_3^2 x_2 - x_4^2 x_2^2 - 4 x_1^2 x_3^3 - x_1^2 x_4^3 + (2 x_1^2 + 2 x_2^2 + 2 x_2^2) x_1^2 + 2 x_1^4 \right] \right. \\
+ \sqrt{3} x_5 \left[ x_3^2 x_4 - x_2^3 x_4 \right] - x_5 \left[ x_2^3 - 2 x_2^3 + x_4^3 + x_3^3 \right] + \frac{1}{2} x_5^2 \right\} . \tag{15}

The “real” quintic invariant is given by

\[ I_{s5}^{[5]} = (x_1 + \omega x_1 + \omega^2 x_4)\sqrt{3} + (x_1 + \omega^2 x_1 + \omega x_4)\sqrt{3} z \right. \\
- (2x_1^2 + 2x_2^2 + 2x_2^2 - 2z) \left[ (x_1 + \omega x_2 + \omega^2 z)^2 \sqrt{3} + (x_2 + \omega^2 x_2 + \omega z)^2 \right] \right. \\
+ 16 x_1 x_2 x_3 (z - z^3 + \sqrt{3}) \right),

or in terms of the five real variables

\[ I_{s5}^{[5]} = \frac{1}{2} \left\{ \frac{3}{2} x_1 x_2^3 + \sqrt{3} x_3^3 [x_2^3 - x_2^3] - x_2^3 \left[ x_3^4 x_2 - 2 x_1^2 x_4 - x_2^4 - 8 \sqrt{2} x_1 x_2 x_3 + x_2^2 x_1 \right] \right. \\
+ \sqrt{3} x_5 \left[ x_4^3 (2 x_1^2 - x_3^3 + x_3^3) + x_2^2 (x_1^2 - x_2^2 - 2 x_1^2) \right] - \frac{1}{2} x_5^2 + 8 \sqrt{2} x_1 x_2 x_3 x_4^2 \right. \\
+ x_4 x_5 [4x_3^3 + x_4^3 - 2 x_1^2 - x_3^2] + x_2^2 (x_3^3 + x_3^3 + 2 x_1^2 + x_4^2 x_4^3 - x_2^3 - 2 x_2^3) \right\} . \tag{16}

The most general quintic invariant can therefore be written as

\[ I_{a5}^{[5]}(\alpha, \beta, \gamma, \delta; \vec{x}) = \alpha I_{s5}^{[5]} + \beta I_{a5}^{[5]} + \left( \gamma I_{s5}^{[3]} + \delta I_{a5}^{[5]} \right) I_{s5}^{[2]} , \tag{17}\]

with arbitrary coefficients \( \alpha, \beta, \gamma, \delta, \) and variables

\[ \vec{x} = (x_1, x_2, x_3, x_4, x_5) . \]

The higher order invariants and their syzygies can be found in compact notation in Ref. [7].

### 3 A5 Quartic

Manifolds defined by quartic polynomials in four variables generate K3 surfaces. It might be instructive to use our group methods to catalog all such cases, and compare it with the classification [11] of K3 manifolds with Non-Abelian discrete symmetries.

The \( \mathcal{A}_5 \) generators in its quartic representation are given by

\[
\mathcal{A}_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}; \quad B_4 = \frac{1}{4} \begin{pmatrix}
-1 & -\sqrt{5} & -\sqrt{5} & -\sqrt{5} \\
\sqrt{5} & 1 & 1 & -3 \\
-\sqrt{5} & -1 & 3 & -1 \\
\sqrt{5} & -3 & 1 & 1
\end{pmatrix} . \tag{18}
\]
These matrices act on the real variables
\[
\begin{pmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4
\end{pmatrix}.
\]

From the Molien function of the four-dimensional representation \[7\]
\[
\frac{1 + \lambda^{10}}{(1 - \lambda^2)(1 - \lambda^3)(1 - \lambda^4)(1 - \lambda^5)},
\]
we see that there are four free and one constrained invariants. The quadratic invariant is
\[
I^{[2]}_4 = (x_1^2 + x_2^2 + x_3^2 + x_4^2).
\] (19)

The cubic invariant is
\[
I^{[3]}_4 = -2\sqrt{5}x_2x_3x_4 - x_1(x_2^2 + x_3^2 + x_4^2) + x_1^3.
\] (20)

The quartic invariant is
\[
I^{[4]}_4 = 5(x_2^4 + x_3^4 + x_4^4) - 12\sqrt{5}x_1x_2x_3x_4 + x_1^4 + 12x_1^2(x_2^2 + x_3^2 + x_4^2).
\] (21)

Hence the most general quartic invariant takes the form
\[
I^{[4]}_{A_6}(\kappa) = I^{[4]}_4 + \kappa \left(I^{[2]}_4\right)^2.
\] (22)

This invariant therefore defines a K3 surface which is left invariant under the action of $\mathcal{PSL}_2(5)$.

4 Quintics from $A_6$

This is the group of even permutations on six objects. It is simple, and its 360 elements are products of two generators $c$ and $d$ with the presentation
\[
< c, d \mid c^2 = d^4 = (cd)^5 = (cd^2)^5 = 1 >.
\] (23)

Its character table is
The ATLAS gives the form of the generators in the quintet representation

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} ;
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1
\end{pmatrix}
\] (24)

The Molien function for the quintet representation can be easily computed. The result is

\[
\frac{1 + \lambda^{15}}{(1 - \lambda^2)(1 - \lambda^3)(1 - \lambda^4)(1 - \lambda^5)(1 - \lambda^6)},
\]

which shows quadratic, cubic, quartic, quintic, sextic and fifteenth-order invariants. The building of these invariants is facilitated by our knowledge of the A\(_5\) invariants.

We first express the A\(_6\) generators in the Cummins-Patera basis. We identify an A\(_6\) element of order three,

\[
b' = dcd^3(cd)^2 =
\begin{pmatrix}
-1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix},
\]

and fix the similarity transformation by requiring

\[
A = U^{-1}cU , \quad B = U^{-1}b'U,
\]

where A and B are the A\(_5\) generators in the Cummins-Patera basis, see Eq. (8). This yields

The A\(_6\) gives the form of the generators in the quintet representation

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 2 & -1 & -1 & 0 & 0 \\
5 & 1 & -1 & 2 & -1 & 0 & 0 \\
8 & 0 & -1 & -1 & 0 & -b_5 & -\tilde{b}_5 \\
8 & 0 & -1 & -1 & 0 & -\tilde{b}_5 & -b_5 \\
9 & 1 & 0 & 0 & 1 & -1 & -1 \\
10 & -2 & 1 & 1 & 0 & 0 & 0
\end{pmatrix};
\]

The Molien function for the quintet representation can be easily computed. The result is

\[
\frac{1 + \lambda^{15}}{(1 - \lambda^2)(1 - \lambda^3)(1 - \lambda^4)(1 - \lambda^5)(1 - \lambda^6)},
\]

which shows quadratic, cubic, quartic, quintic, sextic and fifteenth-order invariants. The building of these invariants is facilitated by our knowledge of the A\(_5\) invariants.

We first express the A\(_6\) generators in the Cummins-Patera basis. We identify an A\(_6\) element of order three,
The $A_6$ order-four generator $d$ in the Cummins-Patera basis is given by

$$d_{CP} = \mathcal{U}^{-1} d \mathcal{U}.$$  

Explicitly

$$d_{CP} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \end{pmatrix}.$$  

Since $A_5 \subset A_6$, the most general $A_6$ quintic invariant can be expressed in terms of the $A_5$ invariants of Section 2 as

$$\alpha I_{A_5}^{[5]} + \beta I_{a_5}^{[5]} + (\gamma I_{A_5}^{[3]} + \delta I_{a_5}^{[3]} ) I_{A_5}^{[2]}.$$  

The unknown coefficients $\alpha, \beta, \gamma, \delta$, are determined by requiring that this expression be unaltered by the action of $d_{CP}$. This yields $\alpha = \gamma = 0$; the quintic $A_6$ invariant

$$I_{A_6}^{[5]}(\kappa) = I_{a_5}^{[5]} + \kappa I_{a_5}^{[3]} I_{a_5}^{[2]},$$  

depends only on one parameter. It is an invariant for both five-dimensional irreps of $A_6$.

## 5 Quintics from $\mathcal{PSL}_2(11)$

This group has 660 elements generated by two elements $C$ and $D$, with presentation

$$< C, D | C^2 = D^3 = (CD)^{11} = [C, DCDCD]^2 = 1 >.$$  

Its character table is
\[\chi^{[11]}\]  \begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]
\[\chi^{[5]}\]  \begin{array}{cccccccc}
5 & 1 & -1 & 0 & 0 & 1 & b_{11} & \overline{b}_{11}
\end{array}
\]
\[\chi^{[5]}\]  \begin{array}{cccccccc}
5 & 1 & -1 & 0 & 0 & 1 & \overline{b}_{11} & b_{11}
\end{array}
\]
\[\chi^{[10]}\]  \begin{array}{cccccccc}
10 & -2 & 1 & 0 & 0 & 1 & -1 & -1
\end{array}
\]
\[\chi^{[10]}'\]  \begin{array}{cccccccc}
10 & 2 & 1 & 0 & 0 & -1 & -1 & -1
\end{array}
\]
\[\chi^{[11]}\]  \begin{array}{cccccccc}
11 & -1 & -1 & 1 & 1 & -1 & 0 & 0
\end{array}
\]
\[\chi^{[12]}\]  \begin{array}{cccccccc}
12 & 0 & 0 & b_5 & \tilde{b}_5 & 0 & 1 & 1
\end{array}
\]
\[\chi^{[12]}'\]  \begin{array}{cccccccc}
12 & 0 & 0 & \tilde{b}_5 & b_5 & 0 & 1 & 1
\end{array}

where
\[
b_{11} = \frac{1}{2}(-1 + i\sqrt{11}) = \eta + \eta^3 + \eta^4 + \eta^5 + \eta^9,
\]
\[
\overline{b}_{11} = \frac{1}{2}(-1 - i\sqrt{11}) = \eta^{10} + \eta^8 + \eta^7 + \eta^6 + \eta^2,
\]  \hspace{1cm} (30)

\(\eta\) being the eleventh root of unity, \(\eta^{11} = 1\).

The ATLAS expresses the generators as \((5 \times 5)\) matrices

\[
C = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & -1 & b_{11} & 1 & -b_{11} \\
0 & 0 & 1 & 0 & 0
\end{pmatrix} ;
D = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 \\
-\overline{b}_{11} & 0 & 0 & -1 & 2 + b_{11} \\
1 & 0 & 0 & -1 & 1
\end{pmatrix} ,
\]  \hspace{1cm} (31)

acting on five complex numbers \((z_1, z_2, z_3, z_4, z_5)\), the components of its complex quintet irrep. In this basis the order-eleven element \(H\) is not diagonal,

\[
H = CD = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & -1 & 1 \\
1 & -b_{11} & \overline{b}_{11} & b_{11} & 2 \\
0 & -1 & -1 & 0 & 0
\end{pmatrix} .
\]  \hspace{1cm} (32)

We take \(C\) to be the order-two \(A_5\) generator. \(\mathcal{PSL}_2(11)\) has 110 order-three generators, twenty of which are \(A_5\) elements. We can choose the second \(A_5\) generator to be
\[ B' = DH^2D \], \quad \text{Tr } B' = -1, \]

since it satisfies \( A_5 \)'s presentation

\[ (B')^3 = 1, \quad (CB')^5 = 1. \]

Explicitly,

\[
B' = \begin{pmatrix}
-\frac{b_{11}}{b_{11} + 2} & -2 & \frac{2}{b_{11} - 1} & \frac{2}{b_{11} - 1} \\
\frac{1}{b_{11} - 1} & -2 & -\frac{2}{b_{11} - 1} & 1 - \frac{1}{b_{11}} \\
0 & 0 & 1 & 0 \\
1 & 2 & 1 - \frac{1}{b_{11}} & -1 + \frac{1}{b_{11}} \end{pmatrix}.
\]

The next step is to find the similarity transformation \( S \) which expresses these generators in the Cummins-Patera basis

\[ A = \tau C \tau^{-1}, \quad B = \tau B' \tau^{-1}, \quad (33) \]

where \( A \) and \( B \) were defined in Eq. (8). We find

\[
S = \begin{pmatrix}
0 & 2(\omega + 2) & -b_{11} - (\omega + 2) & -2(\omega + 2) & -\frac{b_{11}}{b_{11} + 2} \\
-2(\omega + 2) & 2(\omega + 2) & -b_{11}(\omega + 2) & 0 & b_{11}(\omega + 2) \\
0 & 0 & -\omega - 2 & 0 & b_{11}(\omega + 2) \\
-\frac{2b_{11}}{b_{11} - 1} & 6 & -3b_{11} - \omega + 1 & 2(b_{11} - 2) & 3b_{11} - \omega + 7 \\
\frac{2b_{11}}{b_{11} - 1} & -6\omega & 2\omega + 3b_{11} \omega^2 + 1 & -2(b_{11} - 2) \omega^2 & 8\omega - 3b_{11} \omega^2 + 7 \end{pmatrix}.
\]

This allows us to calculate the order-three \( \text{PSL}_2(11) \) generator in the Cummins-Patera basis:

\[ D_{CP} = \tau D \tau^{-1}, \quad (34) \]

giving

\[
D_{CP} = \frac{1}{12} \times \begin{pmatrix}
3b_{11} & 3 + 3b_{11} & 3 & -3b_{11}\omega & -3b_{11}\omega^2 \\
-3 & -6 - 3b_{11} & -3 - 3b_{11} & 3 + 3b_{11} + 3\omega & 3(b_{11} - \omega) \\
9 + 3b_{11} & -3 & -b_{11}(\omega + 1) + 3\omega & -b_{11}(1 + b_{11}\omega^2) \\
b_{11}(\omega + 2) - 2\omega - 1 & 5\omega + 4 + b_{11} - b_{11}\omega & b_{11} - 5\omega - 7 - 2b_{11}\omega & -3b_{11}(2 + \omega) & -2b_{11}(1 + 2\omega) \\
1 + 2\omega + b_{11}(1 - \omega) & b_{11}(2 + \omega) - 1 - 5\omega & b_{11}(2\omega + 1) - 2 + 5\omega & 2b_{11}(1 + 2\omega) & -3 - b_{11}(\omega - 1) \end{pmatrix}.
\]
5.1 Quintic Invariant

The Kronecker products of the eight irreducible representations of $\mathcal{P}SL_2(11)$ can be found in an unpublished note by the late B. Wybourne [12]. They can be used to show the existence of one quintic invariant built out of its five-dimensional representation. The product

$$10' \otimes 10' = (1 + 5 + 5 + 10' + 10' + 10' + 12 + 12') + (10 + 11 + 12 + 12')\,,$$

shows the existence of the invariant coupling

$$(10' \cdot 10' \cdot 5).$$

The second Kronecker product

$$5 \otimes 5 = (5 + 10') + 10,$$

allows us to decompose it in terms of quintets

$$5 \otimes 5 \otimes 5 = 10', 10' \otimes 5 \otimes 5 = 5,$$

Finally,

$$5 \otimes 5 = 1 + 12 + 12',$$

leads to the quintic invariant built out of one quintet. Examination of the other Kronecker products shows that it is unique.\(^1\) This procedure is more convenient than computing the Molien function for this representation.

Consider the most general $\mathcal{P}SL_2(5)$ quintic invariant

$$I_{\mathcal{P}SL_2(11)}^{[5]} = \alpha I_{s5}^{[5]} + \beta I_{a5}^{[5]} + \gamma (\gamma I_{s5}^{[3]} + \delta I_{a5}^{[3]}) I_{5}^{[2]},$$

written in terms of the five real variables $x_i$ which span its real $5$ irreducible representation. However, in the embedding

$$\mathcal{P}SL_2(11) \supset \mathcal{P}SL_2(5),$$

the quintets of the group and its subgroup match, that is

$$5 = 5, \quad \bar{5} = 5.$$

This allows us to replace in the invariant the real $x_i$ with the complex $z_i$ which span the complex $5$ of $\mathcal{P}SL_2(11)$.

---

\(^1\) Even though there exists a cubic invariant $I_{\mathcal{P}SL_2(11)}^{[3]} = 3 (1 + 2\omega) I_{a5}^{[3]} + (3 - 2b_{11}) I_{a5}^{[3]}$, no quintic invariant can be built from this due to the absence of a quadratic invariant.
It is now a matter of acting $D_{CP}$ on $j^{(5)}_{PSL_2(11)}$, and determine the coefficients in Eq. (36) from invariance. This is an algebraically demanding exercise, which involves a combination of numerical and analytical tricks and the use of symbolic mathematical software. However, a tedious calculation yields a simple answer

\[
\alpha = 1 + 13b_{11}, \quad \beta = 3(1 + 2\omega)(1 - 9b_{11}), \\
\gamma = -8(2 - b_{11}), \quad \delta = -8(1 + 2\omega)b_{11},
\]

which gives the $PSL_2(11)$ quintic invariant in the Cummins-Patera basis.

### 5.2 A Convenient Basis

This invariant can be considerably simplified when expressed in the basis where the order-eleven element is diagonal. If we perform the following similarity transformation on the generators

\[
\tilde{C} = T^{-1}CT, \quad \tilde{D} = T^{-1}DT,
\]

where

\[
T = \begin{pmatrix}
\eta^4 + \eta^6 & \eta + \eta^7 & \eta^2 + \eta^5 & \eta^8 + \eta^9 & \eta^3 + \eta^{10} \\
-\eta^2 & -\eta^6 & -\eta^2 & -\eta^8 & -\eta^4 \\
-\eta^2 & -\eta^6 & -\eta^2 & -\eta^8 & -\eta^4 \\
-\eta^2 & -\eta^6 & -\eta^2 & -\eta^8 & -\eta^4 \\
-\eta^2 & -\eta^6 & -\eta^2 & -\eta^8 & -\eta^4
\end{pmatrix}, \quad \det T = 11,
\]

we find that

\[
\tilde{H} = \tilde{C} \tilde{D},
\]

becomes diagonal

\[
\tilde{H} = \text{Diag}(\eta, \eta^3, \eta^4, \eta^5, \eta^9).
\]

$\tilde{H}$ generates the $Z_{11}$ subgroup. The second order-eleven class of $PSL_2(11)$ is generated by its conjugate.

The order-two generator $\tilde{C}$ is real. Applying the notation

\[
c_k = \cos \left( \frac{2\pi k}{11} \right),
\]

its explicit form reads

\[
\tilde{C} = \frac{1}{11} \times
\]

\[
\begin{pmatrix}
-4c_1 - 2c_2 + 4c_3 + 2 & 6c_2 + 2c_4 - 4c_5 - 4 & 2c_1 + 10c_3 + 8c_4 + 2 & 2c_1 + 10c_3 + 8c_4 + 2 & -10c_2 - 6c_3 - 12c_4 - 5 \\
8c_1 + 10c_2 + 2c_3 + 2 & -4c_3 + 4c_4 - 2c_5 + 2 & 8c_1 + 10c_2 + 2c_3 + 2 & -12c_1 - 6c_2 - 10c_3 - 5 & 2c_1 - 4c_4 + 6c_5 - 4 \\
-4c_2 + 6c_3 + 2c_5 - 4 & -6c_1 - 10c_3 - 12c_5 - 5 & 4c_2 - 4c_3 - 4c_4 + 2 & -10c_1 + 2c_4 + 8c_5 + 2 & 10c_1 + 2c_4 + 8c_5 + 2 \\
-10c_1 - 12c_2 - 6c_3 - 4 & 8c_2 + 10c_4 + 2c_5 + 2 & 4c_2 + 2c_4 - 4c_5 - 4 & -2c_1 + 4c_3 - 4c_5 + 2 & 8c_2 + 10c_4 + 2c_5 + 2 \\
2c_2 + 8c_3 + 10c_5 + 2 & 2c_2 + 8c_3 + 10c_5 + 2 & -12c_2 - 10c_4 - 6c_5 - 5 & -4c_1 + 2c_3 + 6c_4 - 4 & 4c_1 - 4c_2 - 6c_4 + 2
\end{pmatrix},
\]

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Since $\tilde{C}$ is an involution, the second $\mathcal{PSL}_2(11)$ generator is easily obtained from

$$\tilde{D} = \tilde{C} \tilde{H}.$$  \hspace{1cm} (42)

The subgroup $A_5$ is now generated by the order-two and order-three elements

$$\tilde{C}, \quad \tilde{B}' = \tilde{D} \tilde{H}^2 \tilde{D}.$$  \hspace{1cm}

It is known since Galois, see e.g. Ref. [13], that every element $g \in \mathcal{PSL}_2(11)$ can be written uniquely as

$$g = k \cdot h,$$

where $k \in A_5$ and $h \in Z_{11}$. In this basis, where $h$ is diagonal, it is easy to determine the eleven monomials which are invariant under $Z_{11}$. Written in terms of the components of the quintet $z_i$, $i = 1, 2, \ldots, 5$, they are

$$\prod_{i=1}^{5} z_i^{a_i},$$

with

$$a_1 + 3a_2 + 4a_3 + 5a_4 + 9a_5 = 0 \mod (11).$$

For the fifth-order monomials that is

$$z_1^3 z_2^2, \quad z_1^3 z_3^2, \quad z_1^3 z_4^2, \quad z_1^3 z_5^2, \quad z_1^2 z_2 z_3, \quad z_1^2 z_2 z_4, \quad z_1^2 z_2 z_5, \quad z_1^2 z_3 z_4, \quad z_1^2 z_3 z_5, \quad z_1^2 z_4 z_5,$$

The $A_5$ elements $\tilde{C}$ and $\tilde{B}'$ then rotate these monomials into one another so as to achieve invariance. The quintic invariant is a linear combination of these eleven monomials. A laborious calculation yields

$$I^{[5]}_{\mathcal{PSL}_2(11)} = z_1^3 z_2^2 (-5 - 2\eta + 6\eta^2 + 10\eta^3 + 12\eta^4 + 15\eta^5 + 10\eta^6 + 3\eta^7 - 6\eta^8 - 8\eta^9)$$

$$+ z_5^3 z_2^2 (-5 + 15\eta - 8\eta^2 + 12\eta^3 - 6\eta^4 + 10\eta^5 + 3\eta^6 + 2\eta^7 + 10\eta^8 + 9\eta^9 + 10\eta^{10})$$

$$+ z_3^3 z_2^2 (-5 - 10\eta + 10\eta^2 - 6\eta^3 - 2\eta^4 + 12\eta^5 + 3\eta^6 - 8\eta^7 + 6\eta^8 + 15\eta^9)$$

$$+ z_2^3 z_2^2 (-5 + 12\eta - 2\eta^2 + 15\eta^3 - 6\eta^4 + 6\eta^5 + 10\eta^6 - 8\eta^7 + 10\eta^8 + 9\eta^9 + 3\eta^{10})$$

$$+ z_2^3 z_4^2 (-5 - 6\eta + 3\eta^2 + 15\eta^3 + 10\eta^4 - 2\eta^5 - 8\eta^6 + 10\eta^8 + 9\eta^9 + 3\eta^{10})$$

$$+ z_1^3 z_2 z_4 (-1 - 3\eta + 2\eta^2 + 3\eta^3 + 3\eta^4 + 5\eta^5 + 3\eta^6 - 4\eta^7 - 4\eta^8 - 4\eta^{10})$$

$$+ z_3^3 z_4 z_1 (-1 + 5\eta - 4\eta^2 + 3\eta^3 - 4\eta^4 + 3\eta^5 + 9\eta^7 + 3\eta^9 + 3\eta^{10})$$

$$+ z_3^3 z_4 z_2 (-1 + 3\eta + 3\eta^2 - 3\eta^3 - 3\eta^4 + 3\eta^5 - 4\eta^7 + 2\eta^8 + 5\eta^9)$$

$$+ z_3^3 z_5 z_3 (-1 + 3\eta - 3\eta^2 + 3\eta^3 - 4\eta^4 + 2\eta^5 + 3\eta^7 - 4\eta^8 + 3\eta^9)$$

$$+ z_4^3 z_2 z_2 (-1 - 4\eta + 5\eta^2 + 3\eta^3 - 4\eta^4 - 4\eta^5 + 3\eta^6 + 3\eta^9 + 2\eta^{10})$$

$$- 3 z_1 z_2 z_3 z_4 z_5 (-2 + \eta + \eta^3 + \eta^4 + \eta^5 + \eta^9).$$ \hspace{1cm} (43)
In order to understand this expression a little bit better, note that a particular conjugation of the order-five generator $\tilde{C}\tilde{B}'$ takes a rather simple form:

$$\left( \tilde{B}'\tilde{H}^4 \right)^{-1} \left( \tilde{C}\tilde{B}' \right) \left( \tilde{B}'\tilde{H}^4 \right) =
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 + \eta^2 + \eta^6 + \eta^7 \\
0 & 1 + \eta^2 + \eta^6 + \eta^8 & 0 & 0 & 0 \\
0 & 0 & 1 + \eta^7 + \eta^8 + \eta^{10} & 0 & 0 \\
1 + \eta^2 + \eta^8 + \eta^{10} & 0 & 0 & 0 & 0
\end{pmatrix}.$$ 

Its action on the complex variables $z_i$ relates the first five rows of Eq. (43) among each other as well as the next five rows. For instance, replacing

$$z_1 \rightarrow z'_1 = z_5 (1 + \eta^2 + \eta^6 + \eta^7),$$
$$z_3 \rightarrow z'_3 = z_2 (1 + \eta^2 + \eta^6 + \eta^8),$$

in the first line of Eq. (43) yields the second one.

Of course, the quintic invariant can be written in many different ways. Arranging the monomials in the diagonal matrices

$$\mathcal{E} = \text{Diag} \left( z_5^3 z_2, z_4^3 z_5, z_2^3 z_1, z_1^3 z_4, z_3^3 z_4 \right),$$
$$\mathcal{F} = \text{Diag} \left( z_5^3 z_4 z_1, z_4^3 z_4 z_2, z_2^3 z_5 z_3, z_1^3 z_2 z_4, z_3^3 z_1 z_5 \right),$$

the quintic invariant $I_{PSL_2(11)}^{[5]}$ of Eq. (43) can be rewritten as

$$I_{PSL_2(11)}^{[5]} = 3 z_1 z_2 z_3 z_4 z_5 (2 - b_{11})$$
$$+ \text{Tr} \left( \mathcal{E} \left[ 15\tilde{H} - 8\tilde{H}^2 + 12\tilde{H}^3 - 6\tilde{H}^4 + 6\tilde{H}^5 + 6\tilde{H}^7 + 3\tilde{H}^8 - 2\tilde{H}^9 + 10\tilde{H}^{10} - 5 \right] \right)$$
$$+ \text{Tr} \left( \mathcal{F} \left[ 5\tilde{H} - 4\tilde{H}^2 + 3\tilde{H}^3 - 4\tilde{H}^4 + 3\tilde{H}^5 + 2\tilde{H}^7 - 3\tilde{H}^9 + 3\tilde{H}^{10} - 1 \right] \right).$$

Setting this invariant to zero determines a unique Calabi-Yau three-fold with the largest simple Non-Abelian finite symmetry group. Its group, $\mathcal{PSL}_2(11)$, encodes the geometry of the truncated icosahedron [13, 14] (football, $C_{60}$, quasicrystals), which has twelve pentagonal faces, and 20 hexagonal faces. Its detailed study should be of some interest to both physicists and mathematicians.

## 6 No Quintics from $U_4(2) = Sp_4(3)$

The simple group of $(4 \times 4)$ unitary matrices over the Galois field of order two contains 25920 elements, and has a complex five-dimensional representation as well as its conjugate. It is generated by two elements $a$ and $b$ with presentation

$$< a, b | a^2 = b^5 = (ab)^9 = [a, b]^3 = [a, bab]^2 = 1 > .$$
In the quintet, the ATLAS shows that they are given by
\[
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\omega^2 & 1 & -\omega^2 & \omega & -\omega \\
0 & -\omega^2 & 0 & 0 & -\omega
\end{pmatrix}.
\]
\hspace{1cm} (47)

This group has twenty irreducible representations. In order to find out the
degree of the invariants made out of quintets, we need to compute the Molien
function. For that we need a representative from each class.

- One class of order one.
- Two classes of order two:
  1. generated by \(a\), with trace = \(-3\);
  2. generated by \([a, bab]\), with trace = 1.
- Four classes of order three:
  1. generated by \([a, b]\), with trace = 2;
  2. generated by \(b^3ab[a, bab]\), with trace = \(-b_{27}\);
  3. generated by \(a[a, bab][a, b]b^2\), with trace = \(-b_{27}\);
  4. generated by \(b^4ab^3ab^3a\), with trace = 1.
- Two classes of order four:
  1. generated by \(ab^3ab^3ab^3\), with trace = 1;
  2. generated by \(a[a, bab][a, b]\), with trace = 2.
- One class of order five: generated by \(b\), with trace = 0.
- Six classes of order six:
  1. generated by \(ab^2ab^2\), with trace = \(\overline{\omega} - 1\);
  2. generated by \(b^2a[a, bab][a, b]a\), with trace = \(\omega - 1\);
  3. generated by \(b^2[a, bab]\), with trace = \(i\sqrt{3}\);
  4. generated by \(babab[a, b]\), with trace = \(-i\sqrt{3}\);
  5. generated by \(b^2ab^3ab^3[a, b][a, bab]\), with trace = 0;
  6. generated by \(ab^4[a, bab][a, b]\), with trace = 1.
- Two classes of order nine:
  1. generated by \(ab\), with trace = \(-\omega\);
  2. generated by \(ab^4\), with trace = \(-\overline{\omega}\).
Two classes of order twelve:

1. generated by \( b[a, b] \), with trace = \( \omega \);
2. generated by \( bab \), with trace = \( \overline{\omega} \).

Using the method outlined in Section 1.1 we compute the Molien generating function for the invariants constructed out of one quintic

\[
F(1; 5; \lambda) = \frac{1}{25920} \sum_{i=1}^{20} n_i \det(1 - \lambda A_i^{(5)}) ,
\]

where \( A_i^{(5)} \) is any element in the \( i \)th class. The result from MAPLE is

\[
\frac{(\lambda^{24} + \lambda^{21} - \lambda^{15} - \lambda^{12} - \lambda^9 + \lambda^6 + \lambda + 1)(\lambda^8 + \lambda^7 - \lambda^5 - \lambda^4 - \lambda^3 + \lambda + 1)}{\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1}(\lambda^6 + \lambda^4 + 1)(\lambda^6 + \lambda^2 + 1)(1 + \lambda^2 + \lambda^4)(\lambda^2 + \lambda + 1)(\lambda + 1)^4(\lambda^2 + 1)^4(\lambda - 1)^5.
\]

This function can be written in a suggestive form with the denominator as a product of functions of the form \( (1 - \lambda^n) \), and the numerator as a finite polynomial with positive coefficients, that is

\[
F(1; 5; \lambda) = \frac{(1 + \lambda^{10} + \lambda^{20})(1 + \lambda^{30} + \lambda^{60})}{(1 - \lambda^4)(1 - \lambda^6)(1 - \lambda^{12})(1 - \lambda^{18})(1 - \lambda^{45})} .
\]

It clearly contains no term of order \( \lambda^5 \), showing that there is no quintic invariant.

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