3rd post-Newtonian higher order ADM Hamilton dynamics for two-body point-mass systems

Piotr Jaranowski

Institute of Physics, Bialystok University
Lipowa 41, 15-424 Bialystok, Poland

Gerhard Schäfer

Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität
Max-Wien-Platz 1, 07743 Jena, Germany

Abstract

The paper presents the conservative dynamics of two-body point-mass systems up to the third post-Newtonian order \(1/c^6\). The two-body dynamics is given in terms of a higher order ADM Hamilton function which results from a third post-Newtonian Routh functional for the total field-plus-matter system. The applied regularization procedures, together with making use of distributional differentiation of homogeneous functions, give unique results for the terms in the Hamilton function apart from the coefficient of the term \((\nu p_0 \partial_i)^2 r^{-3}\). The result suggests an invalidation of the binary point-mass model at the third post-Newtonian order.

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1 Introduction and summary

The calculation of general relativistic equations of motion for compact binary systems, in the recent past, was exclusively devoted to the obtention of higher order post-Newtonian gravitational radiation reaction contributions. With the works by Iyer and Will [1], Blanchet [2] and the authors [3], the radiation reaction levels have been completed up to 3.5 post-Newtonian (3.5PN) order, i.e. to the order \((1/c^2)^7/2\) beyond the Newtonian dynamics (for the 2.5PN order see, e.g., the review [4]). Quite recently, Gopakumar et al. [5] succeeded in giving first results for the gravitational radiation reaction in compact binary systems at 4.5PN order, applying balance equations between far-zone fluxes and near-zone losses of energy and angular momentum. The conservative 3PN and 4PN orders of approximation, however, have not been tackled so far (up to the 2PN level of approximation see, e.g., the review [6] and the paper [7]).

It is the aim of the present paper to develop the two-body point-mass dynamics up to the 3PN order of approximation using the canonical formalism of Arnowitt-Deser-Misner (ADM) [8]. There are several aspects which make such a calculation rather interesting: (i) the well-known problem of applicability of Dirac delta distributions for the source of the general relativistic gravitational field; (ii) the need of the 3PN dynamics for a better understanding of the innermost stable circular orbit for binary systems [8, 10, 11]; (iii) the importance of the 3PN dynamics

*E-mail: pio@alpha.fuwb.edu.pl
†E-mail: gos@tpi.uni-jena.de
to control the viability of the 2PN filters for gravitational wave measurements from inspiralling compact binaries \[12\].

Within our canonical approach, together with applying Hadamard’s “partie finie” technic, some specific analytic regularization formula and a generalization of it achieved by us — the non-generalized formula has been successfully used by Damour \[13\], Damour and Schäfer \[14\], and Kopeikin \[15\] at the 2PN level and by us \[3\] at the 3.5PN level —, and the distributional differentiation of homogeneous functions, we were able to explicitly calculate and uniquely regularize all terms which occur at the 3PN order of approximation, but one (notice: this is the only ambiguous term up to the 3.5PN order of approximation, inclusively). The coefficient of the following term turns out to be finite but ambiguous: \(-(\nu p_i \partial_i)^2 r^{-1}\), or, in non-reduced-variable form (apart from a factor 4):

\[
\frac{m_1 v_i^1 v_j^1}{2c^2} r_{s1} \partial_{i1} \partial_{j1} \left( \frac{G m_2}{r_{12}} \right) + \frac{m_2 v_i^2 v_j^2}{2c^2} r_{s2} \partial_{i2} \partial_{j2} \left( \frac{G m_1}{r_{12}} \right),
\]

where \(m_1, v_i^1, r_{s1}\) and \(m_2, v_i^2, r_{s2}\) denote the masses, velocities, and Schwarzschild radii \(r_{sa} = Gm_a/2c^2\), in isotropic coordinates) of the bodies 1 and 2, respectively, and \(r_{12}\) their relative distance. This term describes the quadrupole-quadrupole interaction of the kinetic energy tensor of each body with the gravitational tidal field of the other body scaled to the Schwarzschild radius of the former one. The bodies, in this interaction term, are obviously not tidally deformed (this is expected from Ref. \[16\] where it is shown that the tidal deformation comes in at 5PN only) but they seem to have been attributed extensions of Schwarzschild radius size. Those extensions are beyond the applicability of the 3PN approximation so the obtained result may suggest that a fully consistent 3PN compact binary model needs extended bodies as source for the gravitational field. Only within the complete theory the bodies’ Schwarzschild radii are expected to enter the equations for the binary system in a consistent and unique manner.

The paper is organized as follows. In Section 2 we introduce the point-mass model and the post-Newtonian approximation scheme and we develop the constraint equations to the 3PN order of approximation. To obtain an autonomous Hamilton function for the bodies at 3PN order, dropping the dissipative 2.5PN level, we introduce in Section 3 the Routh functional for the total field-plus-matter system and eliminate the field degrees of freedom for the bodies’ variables. The autonomous Hamilton function comes out of higher order. The Section 4 is devoted to the explicit calculation of the higher order autonomous Hamilton function applying the regularization procedures of the Appendices B.1 and B.2. In the Section 5 a thorough investigation of the obtained ambiguity is undertaken. In the Section 6 we compare our results with limiting expressions known from the literature. The Appendix A presents several explicit metric coefficients and some useful formulae for inverse Laplacians. The Appendix B.1 is devoted to the Hadamard’s “partie finie” regularization. The Appendix B.2 presents a powerful analytic regularization procedure based on an analytic formula derived by Riesz and generalized by us in this paper. In the Appendix B.3 the analytic regularization procedure devised by Riesz is given. The Appendix B.4 shows the distributional differentiation of homogeneous functions.

We use units in which \(16\pi G = c = 1\), where \(G\) is the Newtonian gravitational constant and \(c\) the velocity of light. We employ the following notation: \(x = (x^i)\) \((i = 1, 2, 3)\) denotes a point in the 3-dimensional Euclidean space \(\mathbb{R}^3\) endowed with a standard Euclidean metric and a scalar product (denoted by a dot). Letters \(a\) and \(b\) are body labels, so \(x_a \in \mathbb{R}^3\) denotes the position of the \(a\)th point mass. We also define \(r_a := x - x_a\), \(r_a := |r_a|\), \(n_a := r_a/r_a\); and for \(a \neq b\), \(r_{ab} := x_a - x_b\), \(r_{ab} := |r_{ab}|\), \(n_{ab} := r_{ab}/r_{ab}\); \(|\cdot|\) stands here for the length of a vector. The linear momentum vector of the \(a\)th body is denoted by \(p_a = (p_{ai})\), and \(m_a\) denotes its mass parameter. Indices with round brackets, like in \(\phi_{(2)}\), give the order of the object in inverse powers of the
velocity of light, in this case, $1/c^2$. We abbreviate $\delta (x - x_a)$ by $\delta_a$. An overdot, like in $\dot{x}_a$, means the total time derivative. The partial differentiation with respect to $x^i$ is denoted by $\partial_i$ or by a comma, i.e., $\partial_i \phi \equiv \phi_{,i}$; the partial differentiation with respect to $x^i_a$ we denote by $\delta_{i a}$.

Throughout this paper we extensively used the computer algebra system Mathematica [16].

2 The constraint equations up to 3PN order

We consider a many-body point-mass system which interacts with the gravitational field according to the theory of general relativity. For such a system the constraint equations in the canonical formalism of ADM read (see, e.g., Eqs. (2.8) in [17])

$$
\frac{g^{-1/2}}{2} \left[ gR + \frac{1}{2} \left( g_{ij} \pi^{ij} \right)^2 - \pi_{ij} \pi^{ij} \right] = \sum_a \left( g^{ij} p_{ai} p_{aj} + m_a^2 \right)^{1/2} \delta_a ,
$$

(1)

$$
-2 \pi_{ij}^{\dot{}} = \sum_a g^{ij} p_{aj} \delta_a .
$$

(2)

Here $g_{ij} = 4 g_{ij}$ are the field variables (the prefix $^\text{4d}$ denotes a four-dimensional quantity, all unmarked quantities are understood as three-dimensional), $g^{ij}$ is the inverse of $g_{ij}$ ($g^{ij} g_{jk} = \delta^i_j$), $g := \det \{g_{ij}\}$, ”|” indicates the covariant derivative with respect to $g_{ij}$, $R$ is the curvature scalar formed from the metric $g_{ij}$; $\pi^{ij}$ is the canonical conjugate to the field $g_{ij}$. Spatial indices are raised and lowered using $g^{ij}$ and $g_{ij}$, respectively.

We use the following coordinate conditions (see, e.g., Eqs. (2.9) in [17])

$$
g_{ij}(x) = \left( 1 + \frac{1}{8} \phi \right)^4 \delta_{ij} + h_{ij}^{\text{TT}},
$$

(3)

$$
\pi^{ii} = 0,
$$

(4)

where $h_{ij}^{\text{TT}}$ is the transverse traceless part of $g_{ij} - \delta_{ij}$. The trace-free field momentum $\pi^{ij}$ can be split into two parts, a longitudinal $\pi^{ij}$ and a transversal $\pi^{ij\text{TT}}$ one:

$$
\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij\text{TT}},
$$

(5)

where $\pi^{ij}$ can be expressed in terms of a single vector $\pi^i$ as follows:

$$
\pi^{ij} = \pi_{,j}^i + \pi_{,i}^j - \delta_{ij} \pi_k^k + \Delta^{-1} \pi_{,ijk}^k.
$$

(6)

If both the constraint equations (1)–(2) and the coordinate conditions (3)–(4) are satisfied, the Hamiltonian of the theory can be put into its reduced form, which can be written as

$$
H \left[ x_a, p_a, h_{ij}^{\text{TT}}, \pi^{ij\text{TT}} \right] = -\int d^3 x \Delta \phi \left[ x_a, p_a, h_{ij}^{\text{TT}}, \pi^{ij\text{TT}} \right].
$$

(7)

The reduced Hamiltonian contains the full information for the dynamical evolution of the canonical field and matter variables $[8, 18, 19]$.

We expand the constraint equations (1) and (2) in powers of $1/c$ where we take into account that (see, e.g., Ref. [20])

$$
m_a \sim \mathcal{O} \left( \frac{1}{c^2} \right), \quad \phi \sim \mathcal{O} \left( \frac{1}{c^2} \right), \quad p_a \sim \mathcal{O} \left( \frac{1}{c^2} \right), \quad \pi^{ij} \sim \mathcal{O} \left( \frac{1}{c^2} \right),
$$

$$
h_{ij}^{\text{TT}} \sim \mathcal{O} \left( \frac{1}{c^4} \right), \quad \pi^{ij\text{TT}} \sim \mathcal{O} \left( \frac{1}{c^4} \right).
$$

(8)
To calculate the reduced Hamiltonian \( \tilde{\mathcal{H}} \) up to 3PN order we have to expand the Hamiltonian constraint equation (1) up to \( 1/c^{10} \). Making use of Eqs. (3), (4), and (8), after long calculations, we obtain

\[
- \Delta \phi = \sum_a \left[ 1 - \frac{1}{8} \phi + \frac{1}{64} \phi^2 - \frac{1}{512} \phi^3 + \frac{1}{4096} \phi^4 + \left( \frac{1}{2} - \frac{5}{16} \phi + \frac{15}{128} \phi^2 - \frac{35}{1024} \phi^3 \right) \frac{p_a^2}{m_a^2} \right] + \left( -\frac{1}{8} + \frac{9}{64} \phi - \frac{45}{512} \phi^2 \right) \frac{p_a^2}{m_a^2} + \left( \frac{1}{16} - \frac{13}{128} \phi \right) \frac{p_a^2}{m_a^2} - \frac{5}{128} \frac{\left( p_a^2 \right)^4}{m_a^8} + \left( 1 + \frac{1}{8} \phi \right) \left( \tilde{\pi}^{ij} \right)^2 + \left( \frac{1}{2} - \frac{1}{4} \phi \right) \tilde{\pi}^{ij} \tilde{\pi}^{ijTT} + \left( \pi^{ijTT} \right)^2 \\
+ \left[ \left( \frac{1}{2} - \frac{1}{4} \phi - \frac{5}{64} \phi^2 \right) \phi,ij + \left( \frac{3}{16} - \frac{15}{128} \phi \right) \phi,ij \phi,ij + 2 \pi^{ik \pi^{jk}} \right] h^{TT}_{ij} \\
+ \left( \frac{1}{4} - \frac{7}{32} \phi \right) \left( h^{TT}_{ij,k} \right)^2 + \left( \frac{1}{2} + \frac{1}{4} \phi \right) h^{TT}_{ij,k}h^{TT}_{ij,k} + \Delta \left[ \left( \frac{1}{2} + \frac{1}{16} \phi \right) \left( h^{TT}_{ij} \right)^2 \right] \\
- \left[ \frac{1}{2} \phi h^{TT}_{ij}h^{TT}_{ik,j} + \frac{1}{4} \phi,ik \left( h^{TT}_{ij} \right)^2 \right]_{ik} + O \left( \frac{1}{c^{12}} \right). \tag{9} \]

We also need to expand the momentum constraint equations (2) up to \( 1/c^7 \). Using Eqs. (3), (4), and (8), we get

\[
\tilde{\pi}^{ij}_{\xi} = \left( \frac{-1}{2} + \frac{1}{4} \phi - \frac{5}{64} \phi^2 \right) \sum a p_{ai} \delta_a + \left( \frac{-1}{2} + \frac{1}{16} \phi \right) \phi,ij \tilde{\pi}^{ij} - \frac{1}{2} \phi,ij \tilde{\pi}^{TT}_{ijk}h^{TT}_{ij,k} + \tilde{\pi}^{jk} \left( \frac{1}{2} h^{TT}_{ij,k} - h^{TT}_{ij,k} \right) + O \left( \frac{1}{c^8} \right). \tag{10} \]

All functions entering the right-hand sides of Eqs. (8) and (10) can be written as sums of terms of different orders in \( 1/c \). To the orders needed in our calculations, they read

\[
\phi = \phi_{(2)} + \phi_{(4)} + \phi_{(6)} + \phi_{(8)} + O \left( \frac{1}{c^8} \right), \tag{11} \]

\[
\tilde{\pi}^{ij} = \tilde{\pi}^{ij}_{(3)} + \tilde{\pi}^{ij}_{(5)} + \tilde{\pi}^{ij}_{(7)} + O \left( \frac{1}{c^8} \right), \tag{12} \]

\[
h^{TT}_{ij} = h^{TT}_{(4)ij} + h^{TT}_{(5)ij} + h^{TT}_{(6)ij} + O \left( \frac{1}{c^8} \right), \tag{13} \]

\[
\pi^{ijTT} = \pi^{ijTT}_{(5)} + O \left( \frac{1}{c^8} \right). \tag{14} \]

Using Eqs. (11) – (14) we extract from Eq. (8) the Hamiltonian constraint equations valid at individual even orders in \( 1/c \), up to the order \( 1/c^8 \). They are

\[
- \Delta \phi_{(2)} = \sum_a m_a \delta_a, \tag{15} \]
\[
- \Delta \phi(4) = \sum_a \left( -\frac{1}{8} \phi(2) + \frac{1}{2} \frac{p_a^2}{m_a^2} \right) m_a \delta_a,
\]
\[
- \Delta \phi(6) = \sum_a \left( \frac{1}{64} \phi(2)^2 - \frac{1}{8} \phi(4) - \frac{5}{16} \phi(2) \frac{p_a^2}{m_a^2} - \frac{1}{8} \left( \frac{p_a^2}{m_a^2} \right)^2 \right) m_a \delta_a
+ \left( \pi_{ij}^{(3)} \right)^2 - \frac{1}{2} \phi(2) \pi^{TT}_{(4)ij},
\]
\[
- \Delta \phi(8) = \sum_a \left[ -\frac{1}{512} \phi(2)^3 + \frac{1}{32} \phi(2) \phi(4) - \frac{1}{8} \phi(6) + \left( \frac{15}{128} \phi(2)^2 - \frac{5}{16} \phi(4) \right) \frac{p_a^2}{m_a^2}
+ \frac{9}{64} \phi(2) \left( \frac{p_a^2}{m_a^2} \right)^2 + \frac{1}{16} \left( \frac{p_a^2}{m_a^2} \right)^3 - \frac{1}{2} \phi(4) \pi_{ij}^{TT} \right] m_a \delta_a
+ \frac{1}{8} \phi(2) \left( \pi_{ij}^{(3)} \right)^2 + 2 \pi_{ij}^{(3) TT} \pi_{ij}^{(5)} + 2 \pi_{ij}^{(3) TT} \pi_{ij}^{(5)}
+ \left( \frac{3}{16} \phi(2), \phi(2), j - \frac{1}{4} \phi(2) \phi(2), ij - \frac{1}{2} \phi(4), ij \right) \pi^{TT}_{(4)ij}
- \frac{1}{2} \Delta \left( \pi^{TT}_{(4)ij} \right)^2
+ \frac{1}{4} \left( \pi^{TT}_{(4)ij,k} \right)^2
+ \frac{1}{2} \pi^{TT}_{(4)ij,k} \pi^{TT}_{(4)ik,j} - \frac{1}{2} \phi(2), ij \pi^{TT}_{(4)ij}. \tag{17}
\]

Explicit solutions of the equations \((15), (16), \text{ and } (17)\) for the functions \(\phi(2), \phi(4), \text{ and } \phi(6), \) respectively, are shown in Appendix A. The full solution of the equation \((18)\) for the function \(\phi(8)\) is not known. We split \(\phi(8)\) into two parts
\[
\phi(8) = \phi(8)_1 + \phi(8)_2. \tag{19}
\]

The function \(\phi(8)_1\) is explicitly calculable and can be found in Appendix A. The unknown part of \(\phi(8)\) we call \(\phi(8)_2\). It is given by
\[
\phi(8)_2 = \Delta^{-1} S(8)_2, \tag{20}
\]

where
\[
S(8)_2 := -\frac{1}{8} \phi(2) \left( \pi_{ij}^{(3)} \right)^2 - 2 \pi_{ij}^{(3)} \pi_{ij}^{(5)} - 2 \pi_{ij}^{(3) TT} \pi_{ij}^{(5)}
+ \left( -\frac{3}{16} \phi(2), \phi(2), j - \frac{1}{4} \phi(2) \phi(2), ij \right) \pi^{TT}_{(4)ij}
- \frac{1}{4} \left( \pi^{TT}_{(4)ij,k} \right)^2
- \frac{1}{2} \pi^{TT}_{(4)ij,k} \pi^{TT}_{(4)ik,j} + \frac{1}{2} \phi(2), ij \pi^{TT}_{(4)ij}. \tag{21}
\]

Using the Eqs. \((11)\)–\((14)\), we extract from Eq. \((10)\) the momentum constraint equations valid up to the order \(1/c^4\):
\[
\pi_{ij}^{(3), j} = \frac{1}{2} \sum_a p_{ai} \delta_a, \tag{22}
\]
\[
\pi_{ij}^{(5), j} = -\frac{1}{2} \left( \phi(2) \pi_{ij}^{(3)} \right), \tag{23}
\]
\[
\pi_{ij}^{(7), j} = \Gamma_{ij}^{(7)}, \tag{24}
\]
where

\[ I^{ij}_{(7)} := -\frac{1}{2} \left[ \phi(2) \pi^{ij}_{(5)} + \left( \frac{3}{16} \phi(2) + \phi(4) \right) \tilde{\pi}^{ij}_{(3)} + \phi(2) \pi^{ij\text{TT}}_{(5)} + 2 \pi^{jk}_{(3)} h^{\text{TT}}_{(4)jk} \right. \]

\[ \left. - \left( 2 \pi^{k}_{(3)} + \Delta^{-1} \pi^{l}_{(3),kl} \right) h^{\text{TT}}_{(4)jk,l} \right] . \tag{25} \]

Explicit solutions of the equations (22) and (23) for the functions \( \tilde{\pi}^{ij}_{(3)} \) and \( \tilde{\pi}^{ij}_{(5)} \), respectively, are given in Appendix A, where one can also find the formula for the function \( \pi^{\ell}_{(3)} \) connected with \( \tilde{\pi}^{ij}_{(3)} \) by means of Eq. (6).

The TT-part of the metric in the leading order fulfills the equation (see, e.g., Eq. (14) of [3])

\[ \Delta h^{\text{TT}}_{(4)ij} = \delta_{ij}^{\text{TT}kl} A^{(4)kl}, \tag{26} \]

where

\[ A^{(4)ij} := -\sum_{a} \frac{p_{ai} p_{aj}}{m_{a}} - \frac{1}{4} \phi(2), i, j, \tag{27} \]

and where \( \delta_{ij}^{\text{TT}kl} \) is the TT-projection operator defined by (see, e.g., Eq. (4) of [3])

\[ \delta_{ij}^{\text{TT}kl} := \frac{1}{2} \left[ (\delta_{il} - \Delta^{-1} \partial_{i} \partial_{l})(\delta_{jk} - \Delta^{-1} \partial_{j} \partial_{k}) + (\delta_{ik} - \Delta^{-1} \partial_{i} \partial_{k})(\delta_{jl} - \Delta^{-1} \partial_{j} \partial_{l}) + (\delta_{kl} - \Delta^{-1} \partial_{k} \partial_{l})(\delta_{ij} - \Delta^{-1} \partial_{i} \partial_{j}) \right] . \tag{28} \]

The explicit formula for the function \( h^{\text{TT}}_{(4)ij} \) can be found in Appendix A.

In our calculations we also need the explicit formula for that part of the metric function \( h^{\text{TT}}_{(6)ij} \) which diverges linearly at infinity. So we write

\[ h^{\text{TT}}_{(6)ij} = h^{\text{TT}div}_{(6)ij} + h^{\text{TT}conv}_{(6)ij}, \tag{29} \]

where \( h^{\text{TT}div}_{(6)ij} \sim r \) and \( h^{\text{TT}conv}_{(6)ij} \sim 1/r \) as \( r \to \infty \). The function \( h^{\text{TT}div}_{(6)ij} \) is given by the integral (see, e.g., Eq. (12) in [3])

\[ h^{\text{TT}div}_{(6)ij}(x, t) = -\frac{1}{8\pi} \delta_{ij}^{\text{TT}kl} \int \delta^3 x' \frac{\partial^2 A^{(4)kl}}{\partial t^2} (x', t) \delta \left| x - x' \right| = \delta_{ij}^{\text{TT}kl} \frac{\partial^2}{\partial t^2} \left( \Delta^{-2} A^{(4)kl} \right)(x, t), \tag{30} \]

where the function \( A^{(4)ij} \) is defined in Eq. (27). We have calculated the integral from the right-hand side of Eq. (30). The result is given in Appendix A.

3 The conservative Hamilton function up to 3PN order

In the following we are interested in a conservative 3PN Hamiltonian which depends on body variables only. To achieve this goal we firstly transform our field-plus-matter Hamilton functional (7) into a Routh functional which is a Hamilton function for the bodies but a Lagrange functional for the field (notice for the following the crucial difference between a Hamiltonian and a Lagrangian: the functional derivative of the latter is zero, the one of the former not). The Routh functional up to 3PN order turns out to be

\[ R_{\leq 3PN} \left[ x_{a}, p_{a}, h^{\text{TT}}_{(4)ij}, h^{\text{TT}}_{(4)ij} \right] = H_{\leq 3PN} - \int d^3 x \pi^{ij\text{TT}}_{(5)} h^{\text{TT}}_{(4)ij}, \tag{31} \]
where \( H_{\leq 3\text{PN}} \) is the Hamiltonian up to 3PN order. The equations of motion of the point masses determined by the Routh functional \( (31) \) read

\[
\ddot{p}_a = -\frac{\partial R_{\leq 3\text{PN}}}{\partial x_a}, \quad \dot{x}_a = \frac{\partial R_{\leq 3\text{PN}}}{\partial p_a}.
\]  

(32)

We eliminate now in the Routh functional \( (31) \) \( h_{(4)ij}^\text{TT} \) and \( \dot{h}_{(4)ij}^\text{TT} \) by \( x_a, \dot{x}_a, p_a, \) and \( \dot{p}_a \) through solving the field equations which result from \( H_{\leq 3\text{PN}} \) (see Eq. (26)). After that the Routh functional \( (31) \) becomes a higher order matter Hamilton function (denoted by a tilde) of the variables \( x_a, \dot{x}_a, p_a, \) and \( \dot{p}_a \):

\[
\tilde{H}_{\leq 3\text{PN}}(x_a, p_a, \dot{x}_a, \dot{p}_a) = R_{\leq 3\text{PN}} \left[ x_a, p_a, h_{(4)ij}^\text{TT}(x_a, p_a), \dot{h}_{(4)ij}^\text{TT}(x_a, p_a, \dot{x}_a, \dot{p}_a) \right].
\]  

(33)

The equations of motion of the bodies determined by the Hamilton function \( (33) \) read

\[
\ddot{p}_a = -\frac{\partial \tilde{H}_{\leq 3\text{PN}}}{\partial x_a} + \frac{d}{dt} \left( \frac{\partial \tilde{H}_{\leq 3\text{PN}}}{\partial \dot{x}_a} \right), \quad \dot{x}_a = \frac{\partial \tilde{H}_{\leq 3\text{PN}}}{\partial p_a} - \frac{d}{dt} \left( \frac{\partial \tilde{H}_{\leq 3\text{PN}}}{\partial \dot{p}_a} \right).
\]  

(34)

In these equations the higher time derivatives may be eliminated by applying lower order equations of motion. The elimination of the higher derivatives in the Hamilton function \( (33) \) would result in a redefinition of the body variables (see, e.g., Ref. [21]).

The reduced Hamiltonian \( H_{\leq 3\text{PN}} \) of Eq. \( (34) \) can be obtained from Eq. \( (1) \). After dropping full divergences (including the term \( \tilde{x}_{ij}^\text{TT} \), which can be written as \( [(2\pi^j + \Delta^{-1}\pi^k_{ij})\pi^j_{ij}^{\text{TT}}]_i, \) cf. Eq. \( (4) \)), it reads

\[
H_{\leq 3\text{PN}} = \int d^3x \left\{ \sum_a \left[ 1 - \frac{1}{8} \phi + \frac{1}{64} \phi^2 - \frac{1}{512} \phi^3 + \frac{1}{4096} \phi^4 \right. \right.
\]

\[
+ \left( \frac{1}{2} - \frac{5}{16} \phi + \frac{15}{128} \phi^2 - \frac{35}{1024} \phi^3 \right) \frac{p_a^2}{m_a^2}
\]

\[
+ \left( -\frac{1}{8} \phi + \frac{9}{64} - \frac{45}{128} \phi^2 \right) \frac{p_a^2}{m_a^2}\left. \right] p_a m_a \left. \right] + \left( 1 - \phi \frac{3}{16} \phi \right) \frac{p_a^2}{m_a^2} \left. \right] m_a \delta_a
\]

\[
+ \left( 1 + \frac{1}{8} \phi \right) \left( \tilde{x}_{ij}^\text{TT} \right)^2 + \left( \phi \tilde{x}_{ij}^\text{TT} \right)^2 + \left( \phi \tilde{x}_{ij}^\text{TT} \right)^2
\]

\[
+ \left[ \left( \frac{1}{4} \phi - \frac{5}{64} \phi^2 \right) \phi_{ij} + \left( \frac{1}{4} \phi - \frac{15}{16} \phi \right) \phi_{ij} \phi_{ij} + 2 \pi^{ik} \tilde{x}_{ij}^\text{TT} \right]
\]

\[
\left. \right] h_{ij}^\text{TT}
\]

\[
+ \left( \frac{1}{4} - \frac{7}{32} \phi \right) \left( h_{ij,k}^\text{TT} \right)^2 + \left( \frac{1}{16} \phi \right) h_{ij,k}^\text{TT} h_{ij,k}^\text{TT} \right. \}
\]

(35)

Le us note that the integrals of the full divergences we have dropped in Eq. \( (35) \) do not contribute to the Hamiltonian because they fall off at infinity at least as \( 1/r^4 \). This can be inferred from the following asymptotic behaviour of the functions entering on the right-hand side of Eq. \( (1) \) (cf. Ref. [19]):

\[
\phi \sim \frac{1}{r}, \quad h_{ij}^\text{TT} \sim \frac{1}{r}, \quad \tilde{x}_{ij}^\text{TT} \sim \frac{1}{r^2}, \quad \pi_{ij}^\text{TT} \sim \frac{1}{r^2}, \quad \text{for} \quad r \to \infty;
\]  

(36)
see also our discussion below Eq. (39).

The higher order Hamiltonian $H_{3PN}$ of Eq. (33) can be split as follows

$$
\tilde{H}_{3PN} (x_a, p_a, \dot{x}_a, \dot{p}_a) = \tilde{H}_0 + \tilde{H}_N (x_a, p_a) + \tilde{H}_{1PN} (x_a, p_a) + \tilde{H}_{2PN} (x_a, p_a) + \tilde{H}_{3PN} (x_a, p_a, \dot{x}_a, \dot{p}_a),
$$

where the Hamiltonians $\tilde{H}_0$ through $\tilde{H}_{3PN}$ can be extracted from Eqs. (31) and (33) by means of the expansions (13)–(14). The Hamiltonians $\tilde{H}_0$ through $\tilde{H}_{2PN}$ are known. We re-calculate them below for completeness. Our aim, however, is to calculate the 3PN Hamilton function $\tilde{H}_{3PN}$.

The calculation of the Hamiltonians $\tilde{H}_0$ through $\tilde{H}_{2PN}$ can be performed directly as we explicitly know all functions needed to perform the integrations. The Hamiltonians can be written in the form

$$
\tilde{H}_0 = \int d^3x \sum_a m_a \dot{x}_a,
$$

$$
\tilde{H}_N = \int d^3x \sum_a \left( -\frac{1}{8} \phi_{(2)} + \frac{1}{2} \frac{p_a^2}{m_a} \right) m_a \dot{x}_a,
$$

$$
\tilde{H}_{1PN} = \tilde{H}_{11} + \tilde{H}_{12},
$$

$$
\tilde{H}_{11} = \int d^3x \sum_a \left( \frac{1}{64} \phi_{(2)}^2 - \frac{1}{8} \phi_{(4)} - \frac{5}{16} \phi_{(2)} \frac{p_a^2}{m_a^2} - \frac{1}{8} \frac{(p_a^2)^2}{m_a^3} \right) m_a \dot{x}_a,
$$

$$
\tilde{H}_{12} = \int d^3x \left\{ \left( \pi_{ij}^{(3)} \right)^2 \right\},
$$

$$
\tilde{H}_{2PN} = \tilde{H}_{21} + \tilde{H}_{22} + \tilde{H}_{23},
$$

$$
\tilde{H}_{21} = \int d^3x \sum a \left[ -\frac{1}{512} \phi_{(2)}^3 + \frac{1}{32} \phi_{(2)} \phi_{(4)} - \frac{1}{8} \phi_{(6)} + \frac{15}{128} \phi_{(2)}^2 - \frac{5}{16} \phi_{(4)} \right] \frac{p_a^2}{m_a^2}
$$

$$
+ \frac{9}{64} \phi_{(2)} \left( \frac{p_a^2}{m_a} \right)^2 + \frac{1}{16} \frac{(p_a^2)^3}{m_a^3} - \frac{1}{2} \frac{\pi_{ij} \pi_{ij}}{m_a} h_{TT}^{(4)ij} \right] m_a \dot{x}_a,
$$

$$
\tilde{H}_{22} = \int d^3x \left\{ \frac{1}{8} \phi_{(2)} \left( \pi_{ij}^{(3)} \right)^2 + 2 \pi_{ij}^{(3)} \pi_{ij}^{(3)} - \frac{1}{16} \phi_{(2)} \phi_{(2)} \phi_{(4)} + \frac{1}{4} \left( h_{TT}^{(4)ij} \right)^2 \right\},
$$

$$
\tilde{H}_{23} = \int d^3x \left\{ \frac{1}{4} \phi_{(2)} \phi_{(2)} \phi_{(4)} + \frac{1}{4} h_{TT}^{(4)ij} \right\}.
$$

The Hamiltonian $\tilde{H}_{3PN}$, as extracted from Eqs. (31) and (33), reads

$$
\tilde{H}_{3PN} = \int d^3x \sum a \left[ \frac{1}{4096} \phi_{(2)}^4 - \frac{3}{512} \phi_{(2)}^2 \phi_{(4)} + \frac{1}{64} \phi_{(4)}^2 + \frac{1}{32} \phi_{(2)} \phi_{(6)} - \frac{1}{8} \phi_{(8)}
$$

$$
+ \left( -\frac{35}{1024} \phi_{(2)}^3 + \frac{15}{64} \phi_{(2)} \phi_{(4)} - \frac{5}{16} \phi_{(6)} \right) \frac{p_a^2}{m_a^2}
$$

$$
+ \frac{1}{64} \frac{(p_a^2)^2}{m_a} - \frac{1}{16} \frac{(p_a^2)^3}{m_a^3} h_{TT}^{(4)ij} \right] m_a \dot{x}_a.
$$
by means of the Eqs. (24) and (25), and by the aid of the Eq. (6):

\[ \tilde{\phi} \text{[cf. Eq. (19)]}, \text{the function right-hand side of Eq. (47). There are three such functions: the part} \]

\[ \text{To eliminate the unknown function h,} \]

\[ \text{All the terms which depend on the unknown function} \]

\[ \text{we eliminate by means of the identity} \]

\[ \left( \Delta \phi(2) \right) \phi(8)_2 = \phi(2) S(8)_2 + \left( \phi(2)_i \phi(8)_2 - \phi(2) \phi(8)_2 \right)_i. \]  

For the calculation of the Hamiltonian \( \tilde{H}_{3PN} \) we perform some manipulations which allow us to do the integrations in Eq. (47) without explicit knowledge of all functions entering on the right-hand side of Eq. (47). There are three such functions: the part \( \phi(8)_2 \) of the function \( \phi(8) \) [cf. Eq. (19)], the function \( \tilde{n}^{ij} \pi^{ij}_7 \), and the part \( \tilde{h}^{TT\text{conv}} \) of the function \( \tilde{h}^{TT} \).

The function \( \phi(8)_2 \) we eliminate by means of the identity

\[ \left( \Delta \phi(2) \right) \phi(8)_2 = \phi(2) S(8)_2 + \left( \phi(2)_i \phi(8)_2 - \phi(2) \phi(8)_2 \right)_i. \]  

To eliminate the unknown function \( \tilde{n}^{ij}_7 \pi^{ij}_7 \) we use the following relation, which can be proved by means of the Eqs. (24) and (25), and by the aid of the Eq. (8):

\[ \tilde{n}^{ij}_7 \pi^{ij}_7 = \left[ \left( 2 \pi^{ij}_7 + \Delta^{-1} \pi^{ik}_7 \right) \left( \pi^{ij}_7 - \pi^{ij}_7 \right) \right]_j - \frac{3}{32} \phi(2) \left( \pi^{ij}_7 \right)^2 - \frac{1}{2} \phi(4) \left( \pi^{ij}_7 \right)^2 \\ - \frac{1}{2} \phi(2) \pi^{ij}_7 \pi^{ij}_7 - \frac{1}{2} \phi(2) \pi^{ij}_7 \pi^{ik}_7 - \left( 2 \pi^{ij}_7 + \Delta^{-1} \pi^{ik}_7 \right)_j \pi^{jk}_7 \pi^{ii}_7 \pi^{ij}_7 \pi^{ij}_7 \\ + \frac{1}{2} \left( 2 \pi^{ij}_7 + \Delta^{-1} \pi^{ij}_7 \right)_j \left( 2 \pi^{ik}_7 + \Delta^{-1} \pi^{ik}_7 \right)_j \pi^{jk}_7 \pi^{ij}_7 \pi^{ij}_7. \]  

All the terms which depend on the unknown function \( \tilde{h}^{TT} \) we were be able to write as a full divergence [see Eq. (47) below]. To do this we have used the following relation, which can be derived using the explicit formula (28) for the \( \delta^{TTkl} \) operator and the traceless property of the function \( \tilde{h}^{TT} \):

\[ \left( \delta^{TTkl} A_{4ijkl} - A_{4ijkl} \right) \tilde{h}^{TT} \]

\[ = \left\{ \left[ \frac{1}{2} \left( \Delta^{-1} A_{4kk,j} \right) - 2 \left( \Delta^{-1} A_{4jk,k} \right) + \frac{1}{2} \left( \Delta^{-2} A_{4kl,jkl} \right) \right] \tilde{h}^{TT} \right\}_j \right. \]
In the last stage we eliminate the field momentum \( \pi^{ij(5)} \) by \( x_a, \dot{x}_a, p_a, \) and \( \dot{p}_a. \) We use the field equation for the field momentum in the leading order. It reads (see, e.g., Eq. (13) of [3]):

\[
\pi^{ij(5)} = \frac{1}{2} \dot{h}^{TT(i)j} + \frac{1}{2} \left( \phi(2) \tilde{\pi}^{ij(3)} \right)^{TT}.
\]  

(51)

One can check that the following relation holds

\[
\left( \phi(2) \tilde{\pi}^{ij(3)} \right)^{TT} = \phi(2) \pi^{ij(3)} + 2\pi^{ij(5)}.
\]  

(52)

Substituting Eq. (52) into Eq. (51) we obtain

\[
\pi^{ij(5)} = \frac{1}{2} \dot{h}^{TT(i)j} + \frac{1}{2} \phi(2) \pi^{ij(3)} + \tilde{\pi}^{ij(5)}.
\]  

(53)

Using Eqs. (13), (21), (35), (36), (37), and (53) we rewrite the Hamilton function \( \tilde{H}_{3PN} \) given by Eq. (47) as a sum of terms

\[
\tilde{H}_{3PN} = \sum_{i=1}^{6} \tilde{H}_i,
\]  

(54)

where the \( \tilde{H}_i \) are defined as follows

\[
\tilde{H}_31 := \int d^3x \sum_a \left\{ \frac{1}{4096} \phi^4(2) - \frac{3}{512} \phi^2(2) \phi^2(4) + \frac{1}{64} \phi^2(4) + \frac{1}{32} \phi(2) \phi(6) - \frac{1}{8} \phi(8) \right\}
\]

\[
+ \left( \frac{35}{1024} \phi^3(2) + \frac{15}{64} \phi(2) \phi(4) - \frac{5}{16} \phi(6) \right) \frac{p^2_a}{m^2_a} + \left( \frac{45}{512} \phi^2(2) + \frac{9}{64} \phi(4) \right) \frac{(p^2_a)^2}{m^2_a}
\]

\[
- \frac{13}{128} \phi(2) \left( \frac{p^2_a}{m^2_a} \right)^3 - \frac{5}{128} \left( \frac{p^2_a}{m^2_a} \right)^4 + \left( \frac{9}{16} \phi(2) + \frac{1}{4} \frac{p^2_a}{m^2_a} \right) \frac{p_a p_a}{m^2_a} h^{TT(i)j} h^{TT(j)i} = \frac{1}{16} \left( h^{TT(i)j} \right)^2 m_a \delta_a
\]  

(55)

\[
\tilde{H}_32 := \int d^3x \left\{ \frac{7}{8} \phi(4) \left( \tilde{\pi}^{ij(3)} \right)^2 - \frac{1}{8} \phi(2) \phi(4) \right\}
\]

(56)

\[
\tilde{H}_33 := \int d^3x \left\{ \frac{1}{4} \left( h^{TT(i)j} \right)^2 - \frac{1}{2} \phi(2) \phi(4) \left( \tilde{\pi}^{ij(3)} \right)^{TT} \right\},
\]  

(57)

\[
\tilde{H}_34 := \int d^3x \left\{ \frac{35}{64} \phi^2(2) \left( \tilde{\pi}^{ij(3)} \right)^2 + 2 \pi^k(3) \pi^k(3) h^{TT(i)j} - 2 \left( 2 \pi^i(3) + \Delta^{-1} \pi^l(3) \right)_{,j} \pi^j(3) h^{TT(i)j}
\]

\[
+ \left( 2 \pi^i(3) + \Delta^{-1} \pi^l(3) \right) \Delta^{-1} \pi^m(3)_{,km} \right) \pi^j(3) h^{TT(i)j}
\]

\[
+ \left( 2 \pi^i(3) + \Delta^{-1} \pi^l(3) \right) \left( 2 \pi^k(3) + \Delta^{-1} \pi^m(3)_{,km} \right) \pi^j(3) h^{TT(i)j}
\]

\[
+ \frac{5}{64} \phi(2) \phi(2) h^{TT(i)j} - \frac{1}{4} \phi(2) \left( h^{TT(i)j} \right)^2 \right\},
\]  

(58)

\[
\tilde{H}_35 := \int d^3x \left\{ \frac{7}{64} \phi^2(2) \phi(2, j) h^{TT(i)j} + \frac{1}{4} \left( \phi(2) \phi(4) \right)_{,j} h^{TT(i)j}
\]

\[
+ \left( 2 \pi^i(5) + \Delta^{-1} \pi^j(5) \right) \left( h^{TT(i)j} + 3 \phi(2) \tilde{\pi}^{ij(3)} \right)^{TT} \right\}
\]
We also use, at the 3PN level, the Newtonian relations between the coordinate velocities ˙\(\pi^{ij}_{(7)}\) will indicate division by the reduced mass  \(\mu\) of the bodies and their relative velocity  \(v\), so we can use the relations

\[
\tilde{H}_{36} := \int d^3 x \left\{ \frac{5}{16} \phi_{(2)}^2 \phi_{(2),j}^2 h_{(6)ij}^{TT} + \frac{1}{2} h_{(4)j,k}^{TT} h_{(6)jk}^{TT} \right. \\
\left. - \frac{1}{2} \left( \frac{1}{2} \left( \Delta^{-1} A_{(4)kk,j} \right) - 2 \left( \Delta^{-1} A_{(4)jk,k} \right) + \frac{1}{2} \left( \Delta^{-2} A_{(4)kl,jkl} \right) \right) h_{(6)ij}^{TT} \right\},
\]

(60)

4 Results of regularization procedures

To diminish the number of terms we perform calculations in the center-of-mass reference frame, so we can use the relations

\[
\mathbf{p}_1 + \mathbf{p}_2 = 0, \quad \dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = 0.
\]

(61)

We also use, at the 3PN level, the Newtonian relations between the coordinate velocities  \(\dot{x}_1, \dot{x}_2\) of the bodies and their relative velocity  \(v := \dot{x}_1 - \dot{x}_2\):

\[
\dot{x}_1 = \frac{m_2}{M} v, \quad \dot{x}_2 = -\frac{m_1}{M} v,
\]

(62)

where  \(M := m_1 + m_2\) is the total mass of the system.

To shorten the formulae we introduce the following reduced variables

\[
r := \frac{16\pi}{M} (x_1 - x_2), \quad r := |r|, \quad \mathbf{n} := \frac{r}{r}, \quad \mathbf{p} := \frac{\mathbf{p}_1}{\mu} = -\frac{\mathbf{p}_2}{\mu}, \quad \mathbf{q} := \frac{\dot{\mathbf{p}}_1}{16\pi \nu} = -\frac{\dot{\mathbf{p}}_2}{16\pi \nu},
\]

(63)

where

\[
\mu := \frac{m_1 m_2}{M}, \quad \nu := \frac{\mu}{M}.
\]

(64)

We also introduce the reduced Hamilton function

\[
\hat{H} := \frac{\tilde{H}}{\mu}.
\]

(65)

\(\hat{H}\) depends on masses of the binary system only through the parameter  \(\nu\). From now on the hat will indicate division by the reduced mass  \(\mu\).

To calculate the Hamiltonians  \(\tilde{H}_0\) through  \(\tilde{H}_{2PN}\) we proceed as follows. The terms  \(\tilde{H}_0, \tilde{H}_N, \tilde{H}_{11},\) and  \(\tilde{H}_{21}\) [given by Eqs. (38), (39), (11), and (14), respectively] we regularize by means of the Hadamard’s procedure described in Appendix B.1. The terms  \(\tilde{H}_{12}\) and  \(\tilde{H}_{22}\) [given by Eqs. (42) and (43), respectively] we regularize using the procedure from Appendix B.2. The term  \(\tilde{H}_{23}\) [from Eq. (16)] is a full divergence. The integrand in  \(\tilde{H}_{23}\) falls off at infinity as  \(1/r^4\) so  \(\tilde{H}_{23}\) does not contribute to the Hamiltonian. We have checked that the regularization procedure of Appendix B.2 applied to  \(\tilde{H}_{23}\) gives zero. The final result for the Hamiltonian  \(\tilde{H}_0 + \tilde{H}_N + \tilde{H}_{1PN} + \tilde{H}_{2PN}\) coincides with that known in the literature (see, e.g., Eq. (3.1) in [8]).

The calculation of the Hamiltonian  \(\tilde{H}_{3PN}\) is much more complicated. The term  \(\tilde{H}_{31}\) of Eq. (55) we regularize using the Hadamard’s procedure described in Appendix B.1. After long calculations we obtain

\[
\hat{H}_{31} = \frac{1}{128} \left( -5 + 35\nu - 70\nu^2 + 35\nu^3 \right) (p^2)^4 \\
+ \frac{1}{16} \left\{ \left( -7 + 42\nu - 53\nu^2 - 6\nu^3 \right) (p^2)^3 + (1 - 2\nu)\nu^2 \left[ 2(n \cdot p)^2 (p^2)^2 + 3(n \cdot p)^4 p^2 \right] \right\} \frac{1}{r},
\]

(59)
\[
\begin{align*}
&+ \frac{1}{64} \left\{ (-108 + 515 \nu + 84 \nu^2) (p^2)^2 + (46 + 35 \nu) \nu (n \cdot p)^2 p^2 - (45 - 255 \nu) \nu (n \cdot p)^4 \right\} \frac{1}{r^2} \\
&+ \frac{1}{160} \left\{ (-500 + 1760 \nu - 2317 \nu^2) p^2 - 3(480 - 1897 \nu) \nu (n \cdot p)^2 \right\} \frac{1}{r^3} + \frac{1}{16} (2 - 11 \nu) \frac{1}{r^4}. \quad (66)
\end{align*}
\]

The calculation of \( \tilde{H}_{32}, \tilde{H}_{33}, \) and \( \tilde{H}_{34} \) [defined by Eqs. (54)–(58)] we do by means of the regularization procedure described in Appendix B.2. The performance of the procedure differs considerably in the case of \( \tilde{H}_{32} \) and \( \tilde{H}_{33} \) compared to \( \tilde{H}_{34} \). In \( \tilde{H}_{32} \) and \( \tilde{H}_{33} \) each term can be regularized separately. This is not the case for \( \tilde{H}_{34} \). For each term of \( \tilde{H}_{34} \) taken separately the limit (130) of Appendix B.2 does not exist. By series expansion with respect to \( \varepsilon \) one can see that for each term entering \( \tilde{H}_{34} \) the sum from Eq. (130) contains a part proportional to \( 1/\varepsilon \) (similar divergences arise in the dimensional regularization procedure in quantum field theory, see, e.g., [22]). After collecting all \( 1/\varepsilon \) terms (they are of the type (72) only) the parts cancel each other. Furthermore, after regularization, there are terms of the type (72) proportional to \( \ln r_{12} \). After collecting all such terms the logarithms cancel out, too. Unfortunately, the result of application of the regularization procedure to \( \tilde{H}_{34} \) is not unique. The ambiguity which arises can be expressed in terms of exactly one unknown number and will be discussed in detail in the next section.

The result of applying the regularization procedure of Appendix B.2 to Eqs. (56)–(58), after tedious calculations, is

\[
\tilde{H}_{32} = \frac{3}{64} \nu \left\{ (-4 + 113 \nu)(p^2)^2 + (12 - 31 \nu)(n \cdot p)^2 p^2 \right\} \frac{1}{r^2} \\
+ \frac{1}{512} \nu \left\{ [603 \pi^2 - 9632 - 4(1004 + 15 \pi^2) \nu] p^2 \\
+ 3 \left[ 4640 - 603 \pi^2 + 12 (5 \pi^2 - 44 \nu) \nu (n \cdot p)^2 \right] \frac{1}{r^3} + \frac{1}{4} \nu \frac{1}{r^4} \right\}.
\]

\[
\tilde{H}_{33} = \frac{1}{12} \nu^2 \left\{ 4(n \cdot p)(n \cdot q)(p \cdot q) - 5(n \cdot q)^2 p^2 \\
- 5(n \cdot p)^2 q^2 - (n \cdot p)^2 (n \cdot q)^2 + 13 p^2 q^2 + 2(p \cdot q)^2 \right\} r \\
+ \frac{1}{8} \nu^2 \left\{ (n \cdot v)(p \cdot q) \left[ 5p^2 + (n \cdot p)^2 \right] - (n \cdot p) \left[ p^2 (q \cdot v) + 2(p \cdot q)(p \cdot v) \right] \\
- \frac{1}{3} (n \cdot p)^3 (q \cdot v) + (n \cdot q) \left[ p^2 + (n \cdot p)^2 \right] \left[ (n \cdot p)(n \cdot v) - (p \cdot v) \right] \right\} \\
+ \frac{1}{48} \nu \left\{ 4(17 - 10 \nu)(n \cdot p)(n \cdot q)(n \cdot v) - 2(15 + 22 \nu)(n \cdot p)^2 (n \cdot q) \\
- 2(51 - 8 \nu)(n \cdot q)^2 p^2 - 4(6 - 5 \nu)(n \cdot p)(p \cdot q) - 4(1 - 2 \nu)(n \cdot v)(p \cdot q) \\
- 4(7 - 2 \nu) \left[ (n \cdot p)(q \cdot v) + (n \cdot q)(p \cdot v) \right] + 3 \nu^2 \left[ 8(n \cdot p)(n \cdot v)p^2(p \cdot v) \\
+ 8(n \cdot p)^3(n \cdot v)(p \cdot v) + 2(n \cdot p)^2 p^2 v^2 + 5(p^2)^2 v^2 + (n \cdot p)^4 v^2 - 5(n \cdot v)^2(p^2)^2 \\
- 6(n \cdot p)^2 (n \cdot v)^2 p^2 - 5(n \cdot p)^4 (n \cdot v)^2 - 4(n \cdot p)^2 (q \cdot v)^2 - 4p^2(q \cdot v)^2 \right] \frac{1}{r^2} \\
+ \frac{1}{48} \nu \left\{ 5(5 - 7 \nu)(n \cdot p)^3(n \cdot v) + 10(3 - 5 \nu)(n \cdot p)^2(n \cdot v)^2 \\
+ 3(17 - 35 \nu)(n \cdot p)(n \cdot v)p^2 - 28(3 - 8 \nu)(n \cdot p)(n \cdot v)(p \cdot v) \right\}. \right\}
\]
The number $\omega$ in Eq. (69) is unknown due to the non-uniqueness of the results of the regularization procedure, as discussed in the next section.

The integrals given by $\tilde{H}_{35}$ and $\tilde{H}_{36}$ (defined in Eqs. (59) and (60)) are integrals of full divergences. To discuss them let us first observe that the 3PN Hamiltonian given by Eqs. (54)–(60) not only describes the system of two point masses but can be applied also to the system of two dusty bodies (provided the following substitutions are made: $a_0 \delta_a \rightarrow p_a$, $p_a \delta_a \rightarrow P_a$, $(p^2_a/m_a) \delta_a \rightarrow P^2_a/p_a$, etc., where $\rho_a$ and $P_a$ are the mass and the linear momentum density of the $a$th body, respectively). It is obvious that for dusty bodies the integrands in $\tilde{H}_{35}$ and $\tilde{H}_{36}$ are locally integrable, so these terms can contribute to the Hamiltonian only if the integrands fall off at infinity slower than $1/r^4$.

Now we proceed as follows. We check whether the integrands in $\tilde{H}_{35}$ and $\tilde{H}_{36}$ fall off at infinity at least as $1/r^4$. If yes, the terms do not contribute to the Hamiltonian, if no, we must study them in more detail. Because the asymptotic behaviour of all functions entering $\tilde{H}_{35}$ and $\tilde{H}_{36}$ is the same for both point masses and dusty bodies, we use the above prescription also for the system of point masses, treating them as a limiting case of extended dusty bodies.

One can check that the whole integrand in (59) falls off at infinity as $1/r^4$, so formally $\tilde{H}_{35}$ does not contribute to the Hamiltonian. For checking the consistency of our regularization procedures we have integrated out all terms in $\tilde{H}_{35}$ except the last two ones (which contain the unknown functions $\tilde{\pi}^{ij}_{(7)}$ and $\phi^{(8)2}$). To do this we have used the regularization of Appendix B.2. We have obtained zeros for all terms but the first one. This nonzero result is connected with the ambiguity of the term $\tilde{H}_{34}$, as explained in the next section.

The integrand in $\tilde{H}_{36}$ does not fall off at infinity fast enough not to possibly contribute to the Hamiltonian. The reason is that the function $h^{TT}_{(6)ij}$ is a sum $h^{TT \text{div}}_{(6)ij} + h^{TT \text{conv}}_{(6)ij}$, where $h^{TT \text{div}}_{(6)ij} \sim r$ and $h^{TT \text{conv}}_{(6)ij} \sim 1/r$ as $r \rightarrow \infty$ (cf. Eqs. (28) and (30)). The integrand in $\tilde{H}_{36}$ for $h^{TT}_{(6)ij} = h^{TT \text{conv}}_{(6)ij}$ falls off at infinity as $1/r^4$ but it falls off only as $1/r^2$ for $h^{TT}_{(6)ij} = h^{TT \text{div}}_{(6)ij}$. Therefore we have calculated $\tilde{H}_{36}$ for $h^{TT}_{(6)ij} = h^{TT \text{div}}_{(6)ij}$. We have used the regularization procedures from Appendixes B.1 and B.2 (the need to use the Hadamard’s procedure of Appendix B.1 arises because $\Delta h^{TT}_{(4)ij}$ contains some terms with Dirac deltas). To calculate $\Delta h^{TT}_{(4)ij}$ properly we also had to employ the rule of differentiation of homogeneous functions from the Appendix B.4. The result is zero.
On the basis of the above discussion we put the integrals given by $\tilde{H}_{35}$ and $\tilde{H}_{36}$ equal to zero,
\begin{equation}
\tilde{H}_{35} = \tilde{H}_{36} = 0, \tag{70}
\end{equation}
and adjust the regularized expression different from zero to the Hamiltonian $\tilde{H}_{34}$.

Collecting the Eqs. (66)–(70) we finally obtain the autonomous higher order 3PN Hamilton function $\tilde{H}_{3PN}$. It reads
\begin{equation}
\tilde{H}_{3PN}(r, p, v, q) = \frac{1}{128} \left( -5 + 35\nu - 70\nu^2 + 35\nu^3 \right) (p^2)^4 + \frac{1}{16} \left\{ -7 + 42\nu - 53\nu^2 - 6\nu^3 \right\} (p^2)^3 + (1 - 2\nu)\nu^2 \left[ 2(n \cdot p)^2 p^2 + 3(n \cdot p)^4 p^2 \right]\frac{1}{r} + \frac{1}{48} \left\{ -27 + 140\nu + 96\nu^2 \right\} (p^2)^2 + 6(8 + 25\nu)\nu(n \cdot p)^2 p^2 - (35 - 267\nu)\nu(n \cdot p)^4 \frac{1}{r^2} + \frac{1}{1536} \left\{ -4800 - 3(8944 - 315\pi^2)\nu - 7136\nu^2 \right\} p^2 + 9 \left( 2672 - 315\pi^2 + 224\nu \right)\nu(n \cdot p)^2 \frac{1}{r^3} + \frac{1}{96} \left\{ 12 + (884 - 63\pi^2)\nu \right\} \frac{1}{r^4} + \tilde{H}_{33}(r, p, v, q) + \omega \left[ p^2 - 3(n \cdot p)^2 \right] \frac{\nu^2}{r^3}, \tag{71}
\end{equation}
where the number $\omega$ is unknown and the function $\tilde{H}_{33}(r, p, v, q)$ is given by the right-hand side of Eq. (73).

5 Ambiguity

The source of ambiguity given by the regularization procedure described in Appendix B.2 can be explained as follows. Via integration by parts, some terms in $\tilde{H}_{34}$ can be represented in different ways. The regularization method applied to both representations give different results.

As an example let us consider the integral of $\phi(2)\phi(j_1 \phi(j_2) h_{(4)ij}^{TT}$ which, by means of integration by parts, can be written as integral of $-\frac{1}{2} \phi(2)\phi(j_1 \phi(j_2) h_{(4)ij}^{TT}$ in the Appendix B.4 another such term is treated). Application of the regularization procedure to the difference between these integrals (the integrand is a full divergence) gives (in the reduced variables)
\begin{equation}
\frac{1}{\mu} \int d^3x \left( \frac{1}{2} \phi(2)\phi(j_1 \phi(j_2) h_{(4)ij}^{TT} \right)_{,i} = \frac{32}{5} \left[ p^2 - 3(n \cdot p)^2 \right] \frac{\nu^2}{r^3}. \tag{72}
\end{equation}
Only if the result (72) would be zero, application of the regularization procedure to the integrals of $\phi(2)\phi(j_1 \phi(j_2) h_{(4)ij}^{TT}$ and $-\frac{1}{2} \phi(2)\phi(j_1 \phi(j_2) h_{(4)ij}^{TT}$ would give the same results.

We can also obtain the result (72) in a different way. Let us denote the integrand from the left-hand side of (72) by $F_{i,i}$ and let us consider the volume integral
\begin{equation}
\int d^3x F_{i,i}, \tag{73}
\end{equation}
where $B(x_a, \varepsilon_a)$ ($a = 1, 2$) is a ball of radius $\varepsilon_a$ around the position $x_a$ of the $a$th body and $B(0, R)$ is a ball of radius $R$ centered at the origin of the coordinate system. We apply Gauss’s theorem to the integral (73) and then we calculate the limit
\begin{equation}
\lim_{\varepsilon_1 \to 0} \oint_{\partial B(x_1, \varepsilon_1)} d\sigma_i F_i + \lim_{\varepsilon_2 \to 0} \oint_{\partial B(x_2, \varepsilon_2)} d\sigma_i F_i + \lim_{R \to \infty} \oint_{\partial B(0, R)} d\sigma_i F_i \tag{74}
\end{equation}
with the normal vectors pointing inwards the spheres $\partial B(x_a, \varepsilon_a)$ and outwards the sphere $\partial B(0, R)$. It is easy to check that the integral over the sphere $\partial B(0, R)$ vanishes in the limit $R \to \infty$ whereas the integrals over the spheres $\partial B(x_a, \varepsilon_a)$ diverge as $\varepsilon_a \to 0$, so to calculate them we use the Hadamard’s procedure from Appendix B.1. The result coincides with that given on the right-hand side of Eq. (72).

We have checked that for all these terms entering $\tilde{H}_{34}$ for which one can use different, via integration by parts, representations the ambiguity is always a multiple of the quantity

$$\left[ p^2 - 3(n \cdot p)^2 \right] \frac{\nu^2}{r^3} = -(\nu p_i \partial_i)^2 \frac{1}{\nu},$$

or, written in non-reduced form and with the momenta substituted through the velocities (apart from a factor 4),

$$\frac{m_1 v_1^i v_1^j}{2c^2} r_s^1 \partial_1^i \partial_1^j \left( -\frac{G m_2}{r_{12}} \right) + \frac{m_2 v_2^i v_2^j}{2c^2} r_s^2 \partial_2^i \partial_2^j \left( -\frac{G m_1}{r_{12}} \right),$$

where $r_s^1$ and $r_s^2$ denote the Schwarschild radii ($r_{sa} = Gm_a/2c^2$, in isotropic coordinates) of the bodies 1 and 2, respectively. This expression indicates that the radius of the bodies might come into play already at 3PN as indicated e.g. in Ref. [20]. The interaction described by the terms (76) is the interaction of the (Newtonian) kinetic energy tensor of each body with the (Newtonian) tidal potential of the other body scaled to the respective Schwarzschild radius.

We have tried to resolve the ambiguity (75) in several ways. Firstly we have extracted the local non-integrabilities in $\tilde{H}_{34}$. Let us denote by $F$ the total integrand of the $\tilde{H}_{34}$ and by $F_{sa}$ ($a = 1, 2$) the non-integrable part (i.e., of the order $1/r^3$ or higher) of the Laurent expansion of the $F$ around the position $x_a$ of the $a$th point mass. Then we replace $F$ by the locally integrable expression $F - (F_{s1} + F_{s2})$. We have checked that the results of applying the regularization procedure of Appendix B.2 are the same for $F$ and $F - (F_{s1} + F_{s2})$ (so the regularized value of the singular part $F_{s1} + F_{s2}$ of the integrand is zero). This is so because by subtracting the local non-integrabilities we have transferred the integrand, initially locally non-integrable but integrable at infinity, to an integrand locally integrable but non-integrable at infinity.

We have studied two further possibilities to overcome the ambiguity. The first one, based on the Riesz’s kernel representation of the Dirac delta distribution, is described in detail in Appendix B.3. The Riesz’s kernel regularization of the divergence (72) gives zero, i.e. it takes into account also the contributions coming from the points $x_a$. We were yet not able to compute all terms in $\tilde{H}_{34}$ using the Riesz’s kernel regularization because of serious calculational problems, as described in Appendix B.3 (only the first and the momenta dependent part of the fifth term out of six terms in $\tilde{H}_{34}$ we could calculate, but with ill-defined (1/ε and ln r_{12}) multiplication factors for the terms of type (75)). In the second case we have tried to employ the rule of differentiation of homogeneous functions coming from the distribution theory. This is described in Appendix B.4. Here we also haven’t fully succeeded in removing the ambiguity.

We conjecture that the ambiguity has its origin in the zero extension of the bodies. We started with point-like bodies but the formalism reacted in such a manner that the Schwarzschild radii of the bodies got introduced. They, however, are far beyond the applicability of the post-Newtonian approximation scheme and thus, the result turned out to be ambiguous. In the future we shall investigate our conjecture in more detail.

We close this section with another observation. According to the applied canonical formalism one is allowed to put $h_{ij}^{TT}$ and $\pi^{ijTT}$ equal to zero at any freely chosen initial time. The energy of the initial state is given by $H_{\leq 3\text{PN}}(x_a, p_a, h_{ij}^{TT} = 0, \pi^{ijTT} = 0)$. It exactly contains the first
term of $\tilde{H}_{34}$. But this term is ill-defined even with respect to the Riesz’s kernel regularization procedure (Appendix B.3). This is a further indication that the binary point-mass model is ill-defined at the 3PN level.

6 Comparison with the known results

The 3PN Hamiltonian derived in Section 4 we now compare with the results known in the literature. We have found only two such results: the test body limit and the static part of the full Hamiltonian. Let us stress that these comparisons do not fix the ambiguity present in our paper.

The Hamiltonian (expressed in the reduced variables defined in the beginning of Section 4) describing a test body orbiting around a Schwarzschild black hole reads [10] (we restored the explicit dependence on the speed of light $c$)

$$\tilde{H}_{test} = c^2 \left[ \left( 1 - \frac{1}{2c^2r} \right) \left( 1 + \frac{1}{2c^2r} \right)^{-1} \sqrt{1 + \left( 1 + \frac{1}{2c^2r} \right)^{-4} \frac{p^2}{c^2} - 1} \right].$$

(77)

Expanding (77) with respect to $1/c$ and taking the $1/c^6$ contribution we obtain the 3PN Hamiltonian for a test body:

$$\tilde{H}_{3PN}^{test} = -\frac{5}{128} \left( \frac{p^2}{2} \right)^4 - \frac{7}{16} \left( \frac{p^2}{2} \right)^3 - \frac{27}{16} \left( \frac{p^2}{2} \right)^2 - \frac{25}{8} \frac{p^2}{2} + \frac{1}{8} \frac{1}{r^4}. $$

(78)

To obtain the test-body limit of the 3PN Hamiltonian of our two-body point-mass system we have to substitute $\nu = 0$ into the right-hand side of Eq. (71) (note that the ambiguous term and the term $\tilde{H}_{33}$ vanish for $\nu = 0$). The result coincides with (78).

The static part of the Hamiltonian is defined by means of the conditions

$$p_a = 0, \quad \pi^{ij\text{TT}} = 0, \quad h^{\text{TT}ij} = 0. $$

(79)

In the paper [23] one can find the following expression for the static part of the many-body point-mass 3PN Hamiltonian:

$$H_{3PN}^{\text{static}} = \frac{1}{8} \left( \frac{1}{16\pi} \right)^2 \sum_{a} \sum_{b} \sum_{c} \sum_{d} \sum_{e} \sum_{m} \sum_{n} m_a m_b m_c m_d m_e \int \ldots \int d^3x_1 \ldots d^3x_5 \delta (x_1 - x_a) \delta (x_2 - x_b) \delta (x_3 - x_c) \delta (x_4 - x_d) \delta (x_5 - x_e)$$

$$\times \left( \frac{2}{r_{12} r_{23} r_{34} r_{45}} + \frac{4}{r_{12} r_{23} r_{34} r_{35}} + \frac{1}{r_{12} r_{23} r_{24} r_{25}} \right),$$

(80)

where $r_{mn} := |x_m - x_n|$. We have calculated $H_{3PN}^{\text{static}}$ from (81) for a two-body system. For this we have used the Hadamard’s regularization procedure of Appendix B.1. The result is (in reduced variables)

$$\tilde{H}_{3PN}^{\text{static}} = \left( \frac{1}{8} + \frac{\nu}{2} \right) \frac{1}{r^4}. $$

(81)

To calculate the static part of the 3PN Hamiltonian in our approach, we have to implement the conditions (79). After that the quantities $\tilde{H}_{32} - \tilde{H}_{36}$ from Eqs. (56)–(60) vanish. Only some part
of the expression $\bar{H}_{31}$ of Eq. (55) survives. Applying for this part the regularization procedure from Appendix B.1 one obtains the formula (81).

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A Some explicit solutions

In this appendix we cite the known approximate solutions of the constraint equations (1) and (2). The functions $\phi(2)$, $\phi(4)$ (implicitly), $\bar{\pi}^{ij}_{(3)}$, and $h_{TT}^{(4)ij}$ can be found e.g. in Refs. [17] and [24]. The functions $\phi(81)$, $\bar{\pi}^{ij}_{(5)}$, and $h_{TT}^{(6)ij}_\text{div}$ have been calculated by us.

In most of the Poisson equations given below there are source terms of the form $\sum_a f(x)\delta_a$, where the function $f$ is singular at $x = x_a$. The Poisson integrals for these terms we have calculated as follows

$$\Delta^{-1} \left\{ \sum_a f(x)\delta_a \right\} = \Delta^{-1} \left\{ \sum_a f_{\text{reg}}(x_a)\delta_a \right\} = \sum_a f_{\text{reg}}(x_a) \Delta^{-1}\delta_a = -\frac{1}{4\pi} \sum_a f_{\text{reg}}(x_a) \frac{1}{r_a}, \quad (82)$$

where the regularized value $f_{\text{reg}}$ of the function $f$ is defined in Eq. (111) of Appendix B.1. The Poisson integrals which do not contain Dirac deltas we have computed using the explicit formulae (103)–(108) for inverse Laplacians listed at the end of this appendix.

The Hamiltonian constraint equation (1) is explicitly solved up to the $1/c^6$ order. The solutions of Eqs. (15) and (16) for the functions $\phi(2)$ and $\phi(4)$ are known for $n$-body point-mass system. They read

$$\phi(2) = \frac{1}{4\pi} \sum_a \frac{m_a}{r_a}, \quad (83)$$
$$\phi(4) = -\frac{2}{(16\pi)^2} \sum_a \sum_{b \neq a} \frac{m_am_b}{r_ar_b} + \frac{1}{8\pi} \sum_a \frac{p_a^2}{m_ar_a}. \quad (84)$$

The solution of Eq. (17) for the function $\phi(6)$ is fully known only for two-body point-mass systems. It can be written as

$$\phi(6) = \phi(6)_1 + \phi(6)_2 + \phi(6)_3, \quad (85)$$

where

$$\phi(6)_1 := \Delta^{-1} \left\{ \sum_a \left( -\frac{1}{64}\phi(2)_a + \frac{1}{8}\phi(4)_a + \frac{5}{16}\phi(2)_a \frac{p_a^2}{m_a^2} + \frac{1}{8}(p_a^2)^2 \right) m_a\delta_a \right\}, \quad (86)$$
$$\phi(6)_2 := -\Delta^{-1} \left\{ \left( \bar{\pi}^{ij}_{(3)} \right)^2 \right\}, \quad (87)$$
$$\phi(6)_3 := \frac{1}{2}\Delta^{-1} \left\{ \phi(2,ij)h_{TT}^{(4)ij} \right\} = \frac{1}{8\pi} \sum_a m_a \Delta^{-1} \left\{ \frac{1}{r_a} \right\}_{,ij} h_{TT}^{(4)ij}. \quad (88)$$
Computation of the Poisson integrals from Eqs. (86) and (87) gives the results

$$\phi_{(6)1} = -\frac{1}{32\pi} \sum_a \left( \frac{p_a^2}{m_a^3 r_a} \right)^2 - \frac{1}{(16\pi)^2} \sum_a \sum_{b \neq a} \frac{m_b p_b^2}{m_a r_{ab}} \left( \frac{5}{r_a} + \frac{1}{r_b} \right)$$

$$+ \frac{1}{(16\pi)^2} \sum_a \sum_{b \neq a} \frac{m_b^2}{r_{ab}^2} \left( \frac{1}{r_a} + \frac{1}{r_b} \right), \quad (89)$$

$$\phi_{(6)2} = -\frac{9}{8} \frac{1}{(16\pi)^2} \sum_a \left( 3 p_a^2 \rho_a^2 + p_{ai} p_{aj} (\ln r_{ai})_{ij} \right)$$

$$+ \frac{1}{(16\pi)^2} \sum_a \sum_{b \neq a} \left\{ -\frac{1}{4} (p_a \cdot \nabla_a) (p_b \cdot \nabla_b) (\nabla_a \cdot \nabla_b)^2 \left( \Delta^{-1} r_a r_b \right) \right.$$ 

$$\left. + [-8 (p_a \cdot p_b) (\nabla_a \cdot \nabla_b) + 3 (p_a \cdot \nabla_a) (p_b \cdot \nabla_b) - 8 (p_a \cdot \nabla_b) (p_b \cdot \nabla_a)] \left( \Delta^{-1} \frac{1}{r_a r_b} \right) \right\}. \quad (90)$$

The Poisson integral needed to calculate $\phi_{(6)3}$ for two-body point-mass systems reads (here $b \neq a$)

$$\Delta^{-1} \left\{ \left( \frac{1}{r_a} \right)_{ij} h_{(4)ij}^{TT} \right\} = \frac{1}{32 (16\pi)^2} \frac{m_a m_b}{r_{ab}^3} \left( 3 r_a^4 - 12 r_a^3 r_{ab} + 18 r_a^2 r_{ab}^2 - 12 r_a r_{ab}^3 + 3 r_{ab}^4 \right)$$

$$+ 28 r_a^3 r_{ab} - 12 r_a^2 r_{ab}^2 - 12 r_a r_{ab}^3 r_{ab} + 28 r_{ab}^3 r_{ab} - 30 r_{ab}^2 r_{ab}^2 + 60 r_a r_{ab} r_{ab}^2$$

$$- 30 r_{ab}^2 r_{ab}^3 - 36 r_{ab} r_{ab}^3 + 35 r_{ab}^4 \right)$$

$$+ \frac{3}{64\pi} \frac{1}{m_a r_a^2} \left( p_a^2 - 3 (n_a \cdot p_a)^2 \right) + \frac{1}{32\pi} \frac{1}{m_b} \left( \frac{2}{r_a} (p_b \cdot \nabla_b) \frac{P_a}{r_a} - 4 \frac{P_b}{r_{ab} r_{ab}} \right)$$

$$+ \left[ 2p_b^2 (\nabla \cdot \nabla_b)^2 - 4 (p_b \cdot \nabla_b) (p_b \cdot \nabla_b) (\nabla \cdot \nabla_b) \right.$$ 

$$\left. - 4 (p_b \cdot \nabla_b) (p_b \cdot \nabla_b) (\nabla \cdot \nabla_b) \right] \left( \Delta^{-1} \frac{1}{r_b} \right) + 8 (p_b \cdot \nabla_b)^2 \left( \Delta^{-1} \frac{1}{r_a r_b} \right)$$

$$+ \frac{1}{6} (p_b \cdot \nabla_b)^2 (\nabla \cdot \nabla_b)^2 \left( \Delta^{-1} \frac{r_b^4}{r_a} \right). \quad (91)$$

Only part of the solution of Eq. (88) for the function $\phi_{(8)}$ is known, namely

$$\phi_{(8)1} = \Delta^{-1} S_{(8)1} + \frac{1}{2} \Delta^{-1} \left\{ \phi_{(4)}_{ij} h_{(4)ij}^{TT} \right\} + \frac{1}{2} \left( h_{(4)ij}^{TT} \right)^2, \quad (92)$$

where

$$S_{(8)1} := \sum_a \left\{ \frac{1}{512} \phi_{(2)}^3 - \frac{1}{32} \phi_{(2)} \phi_{(4)} + \frac{1}{8} \phi_{(6)} + \left( \frac{5}{16} \phi_{(4)} - \frac{15}{128} \phi_{(2)} \right) \frac{P_a}{m_a^2} \right\}$$
The function \( \pi \) We have computed the Poisson integrals from Eq. (92). They are given by

\[
- \frac{9}{64} \phi^{(2)} \left( \frac{p_a^2}{m_a^2} \right)^2 - \frac{1}{16} \left( \frac{p_a^2}{m_a^2} \right)^3 + \frac{1}{2} \frac{m_a p_{aj} h_{TT}}{m_a^2} \left( \frac{1}{r_a} + \frac{9}{r_b} \right) \}
\]

We have computed the Poisson integrals from Eq. (92). They are given by

\[
\Delta^{-1} S_{(8)1} = \frac{1}{64 \pi} \sum_a \frac{p_a^2}{m_a} + \frac{1}{4 (16 \pi)^2} \sum_a \sum_{b \neq a} \left\{ \frac{m_a (p_a^2)^2}{m_b r_a^2} \left( \frac{1}{r_a} + \frac{9}{r_b} \right) \right\}
\]

\[
- \frac{1}{2 (16 \pi)^2} \sum_a \sum_{b \neq a} \frac{8 (p_a^2)^2 + 2 (n_{ab} \cdot p_a)^2 p_a^2 + 3 (n_{ab} \cdot p_a)^4}{m_a m_b r_a^2 r_a^2}
\]

\[
+ \frac{1}{8 (16 \pi)^3} \sum_a \sum_{b \neq a} \frac{m_a p_a}{r_a^2} \left\{ p_a^2 \left( \frac{47}{r_a} - \frac{6}{r_b} \right) + 15 (n_{ab} \cdot p_a)^2 \left( \frac{1}{r_a} + \frac{6}{r_b} \right) \right\}
\]

\[
+ \frac{5}{2 (16 \pi)^3} \sum_a \sum_{b \neq a} \frac{m_a^2 p_a^2}{m_b r_a^2} \left( \frac{1}{r_a} + \frac{3}{r_b} \right)
\]

\[
- \frac{1}{2 (16 \pi)^4} \sum_a \sum_{b \neq a} \frac{5 m_a^2 m_b^2}{r_a r_b} + \frac{m_a^3 m_b}{r_a^3} \left( \frac{1}{r_a} + \frac{1}{r_b} \right)
\},
\]

\[
\frac{1}{2} \Delta^{-1} \left\{ \phi_{(4),ij} h_{TT,ij}^{(4)} \right\} = \frac{1}{16 \pi} \sum_a \frac{p_a^2}{m_a} \Delta^{-1} \left\{ \left( \frac{1}{r_a} \right)_{ij} h_{TT,ij}^{(4)} \right\}
\]

\[
- \frac{1}{(16 \pi)^2} \sum_a \sum_{b \neq a} \frac{m_a m_b}{r_a} \Delta^{-1} \left\{ \left( \frac{1}{r_a} \right)_{ij} h_{TT,ij}^{(4)} \right\}.
\]

The momentum constraint equations \((22)-(24)\) can be written in the form

\[
\pi^{ij} = S^i.
\]

Making use of the decomposition \( \Theta \) it is not difficult to write the solution of Eq. (95) in the form

\[
\pi^{ij} = -\frac{1}{2} \Theta^{ij}_k S^k,
\]

where the operator \( \Theta^{ij}_k \) is defined as follows

\[
\Theta^{ij}_k := \left( -\frac{1}{2} \delta_{ij} \partial_k + \delta_{ik} \partial_j + \delta_{jk} \partial_i - \frac{1}{2} \partial_i \partial_j \Delta^{-1} \right) \Delta^{-1}.
\]

Using Eq. (94) we have obtained the solutions of Eqs. (22) and (23) valid for \( n \)-body point-mass systems. The solution of Eq. (22) reads

\[
\pi^{ij}_{(3)} = \frac{1}{16 \pi} \sum_a p_{ak} \left\{ -\delta_{ij} \left( \frac{1}{r_a},k \right) + 2 \left[ \delta_{ik} \left( \frac{1}{r_a},j \right) + \delta_{jk} \left( \frac{1}{r_a},i \right) \right] - \frac{1}{2} r_{a,ijkl} \right\}.
\]

The function \( \pi^{ij}_{(3)} \), connected with the function \( \pi^{ij}_{(3)} \) by means of Eq. (94), is equal to

\[
\pi^i_{(3)} = \frac{1}{8 \pi} \sum_a p_{ai} \frac{1}{r_a} - \frac{1}{32 \pi} \sum_a p_{aj} r_{a,ij}.
\]
The explicit formula for the function $\tilde{\pi}^{ij}_{(5)}$ reads

$$\tilde{\pi}^{ij}_{(5)} = \frac{3}{(16\pi)^2} \sum_a \frac{m_a}{r_a^2} \left[ (p_{ai}n_{aj}^i + p_{aj}n_{ai}^i) - (n_a \cdot p_a) \delta_{ij} + (n_a \cdot p_a) n_{ai}^i n_{aj}^j \right]$$

$$+ \frac{1}{(16\pi)^2} \sum_a \sum_{b \neq a} m_b \left\{ -\delta_{ij} p_{ab} \left( \frac{1}{r_a} \right)_k \frac{1}{r_b} + \left[ -4\delta_{ij} (p_a \cdot \nabla)(\nabla \cdot n_a) - 3\partial_i \partial_j (p_a \cdot \nabla n_a) \right] \right\}$$

$$+ 4 (p_a \cdot \nabla) (\partial_i \partial_{aj} + \partial_j \partial_{ai}) + 4 (p_{aj} \partial_i + p_{ai} \partial_j) (\nabla \cdot n_a) \right\} \left( \Delta^{-1} \frac{1}{r_a} \right)$$

$$+ \left[ \frac{1}{2} \delta_{ij} (p_a \cdot \nabla_n) (\nabla \cdot n_a)^2 - (\partial_i \partial_{aj} + \partial_j \partial_{ai}) (p_a \cdot \nabla n_a) (\nabla \cdot n_a) \right] \right\} \left( \Delta^{-1} \frac{1}{r_a} \right)$$

$$- 4\partial_i \partial_j (p_a \cdot \nabla) (\nabla \cdot n_a) \left( \Delta^{-2} \frac{1}{r_a} \right) + \frac{1}{2} \partial_i \partial_j (p_a \cdot \nabla n_a) (\nabla \cdot n_a)^2 \left( \Delta^{-2} \frac{1}{r_a} \right) \right\}.$$  (100)

The TT-part of the metric in the leading order ($1/c^4$) is the solution of Eq. (26). It is given by (here $s_{ab} := r_a + r_b + r_{ab}$; be aware that in Eq. (20) in [4] there is a misprint)

$$h^{TT}_{(4)ij} = \frac{1}{4} \frac{1}{16\pi} \sum_a \frac{1}{m_a r_a} \left\{ \left[ p_a^2 - 5(n_a \cdot p_a)^2 \right] \delta_{ij} + 2p_{ai}p_{aj} \right\}$$

$$+ \frac{1}{8} \left( \frac{r_a + r_b}{s_{ab}} + \frac{12}{s_{ab}} \right) n_{ai}^i n_{aj}^j + 6 (n_a \cdot p_a)(n_{ai}^i p_{aj} + n_{aj}^j p_{ai}) \right\}$$

$$+ 8 \left( \frac{1}{s_{ab}} - \frac{1}{s_{ab}^2} \right) \left( n_{ai}^i n_{aj}^j + n_{aj}^j n_{ai}^i \right)$$

$$+ \frac{5}{r_{ab} r_a} - \frac{1}{r_{ab}^3} \left( \frac{r_b^2}{r_a} + 3r_a \right) - \frac{8}{s_{ab}} \left( \frac{1}{r_a} + \frac{1}{s_{ab}} \right) n_{ai}^i n_{aj}^j$$

$$+ \frac{5}{r_{ab}^4} \left( \frac{r_a}{r_{ab}} - 1 \right) + \frac{17}{r_a r_b} + \frac{4}{s_{ab}} \left( \frac{1}{r_a} + \frac{4}{r_{ab}} \right) \delta_{ij} \right\}.$$  (101)

The part of the TT-metric at $1/c^6$ which diverges linearly at infinity is given by Eq. (30). After computing the double Poisson integral in (30) one finds

$$h^{TT}_{(6)ij} = \frac{1}{24} \frac{1}{16\pi} \nabla^2 \frac{r_a}{m_a} \left\{ \left[ 7(n_a \cdot p_a)^2 - 11p_a^2 \right] \delta_{ij} + 26p_{ai}p_{aj} \right\}$$

$$+ \left[ 7p_a^2 - (n_a \cdot p_a)^2 \right] n_{ai}^i n_{aj}^j - 10(n_a \cdot p_a)(n_{ai}^i p_{aj} + n_{aj}^j p_{ai}) \right\}$$

$$- \frac{1}{(16\pi)^2} \nabla^2 \frac{m_a m_b}{r_{ab}^3} \left\{ \frac{1}{8} \left[ \frac{1}{r_{ab}^3} \left( \frac{r_a^2}{r_{ab}} - \frac{r_b^2}{r_{ab}} \right) + \frac{5r_a}{r_{ab}} \right] \delta_{ij} \right\}.$$
Below we list explicit formulae for the inverse Laplacians which have been used throughout this appendix:

\[
\Delta^{-1} \frac{1}{r_a r_b} = \ln s_{ab}, \quad (103)
\]

\[
\Delta^{-2} \frac{1}{r_a r_b} = \frac{1}{36} \left( -r_a^2 + 3r_a r_b + r_b^2 - 3r_a r_b + 3r_{ab} r_a - r_b^2 \right) + \frac{1}{12} \left( r_a^2 - r_{ab}^2 + r_b^2 \right) \ln s_{ab}, \quad (104)
\]

\[
\Delta^{-1} \frac{r_a}{r_b} = \frac{1}{18} \left( -r_a^2 - 3r_a r_b - r_b^2 + 3r_a r_b + 3r_{ab} r_a + r_b^2 \right) + \frac{1}{6} \left( r_a^2 + r_{ab}^2 - r_b^2 \right) \ln s_{ab}, \quad (105)
\]

\[
\Delta^{-2} \frac{r_a}{r_b} = \frac{1}{7200} \left( -r_a^4 - 30r_a r_{ab} - 62r_a^2 r_b^2 + 90r_a^2 r_{ab} + 63r_{ab}^2 - 30r_a^2 r_b - 30r_a r_{ab} r_b \right.
\]

\[
-90r_a^2 r_{ab} r_b + 90r_a^2 r_{ab} r_b - 62r_a^2 r_b^2 - 90r_a r_{ab} r_b^2 - 126r_a^2 r_b^2 + 90r_a r_b^3
\]

\[
+90r_a^2 r_{ab}^2 + 63r_b^4 \right) + \frac{1}{240} \left( r_a^4 + 2r_a^2 r_{ab} - 3r_a^4 + 2r_a^2 r_{ab}^2 + 6r_{ab}^2 r_b^2 - 3r_{ab}^4 \right) \ln s_{ab}, \quad (106)
\]

\[
\Delta^{-1} \frac{r_a}{r_b} = \frac{1}{3600} \left( 63r_a^4 + 90r_a^2 r_{ab} - 62r_a^2 r_b^2 - 30r_a^2 r_{ab} - r_a^4 + 90r_a^2 r_{ab} - 90r_a^2 r_{ab} r_b + 30r_a^2 r_{ab} r_b \right.
\]

\[
-30r_a^2 r_{ab}^2 - 126r_a^2 r_{ab} r_b^2 - 62r_a^2 r_b^2 + 90r_a r_{ab}^2 + 90r_{ab}^2 r_b^2 + 63r_b^4 \right)
\]

\[
+ \frac{1}{120} \left( -3r_a^4 + 2r_a^2 r_{ab}^2 + r_{ab}^4 + 6r_a^2 r_{ab}^2 + 2r_a^2 r_b^2 - 3r_{ab}^4 \right) \ln s_{ab}, \quad (107)
\]

\[
\Delta^{-1} \frac{r_a^3}{r_b} = \frac{1}{1200} \left( -63r_a^4 - 90r_a^2 r_{ab} - 2r_a^2 r_{ab} r_b - 90r_a^2 r_{ab}^2 - 63r_{ab}^4 + 150r_a r_{ab}^2 \right.
\]

\[
+90r_a^2 r_{ab} r_b + 90r_a r_{ab} r_b^2 + 126r_a^2 r_{ab} r_b^2 + 90r_a r_{ab} r_b^2 \right)
\]

\[
+126r_a^2 r_{ab} r_b^2 - 90r_a^2 r_{ab} r_b^2 - 90r_a r_{ab} r_b^2 - 63r_b^4 \right)
\]

\[
+ \frac{1}{40} \left( 3r_a^4 + 2r_a^2 r_{ab}^2 + 3r_{ab}^4 - 6r_a^2 r_{ab}^2 - 6r_a^2 r_{ab} r_b^2 + 3r_{ab}^4 \right) \ln s_{ab}. \quad (108)
\]

**B Regularization procedures**

In this appendix we describe techniques which can be used to regularize integrals which appear in our paper. More details are given in Ref. [25].
B.1 Hadamard’s “partie finie” regularization

Let \( f \) be a real valued function defined in a neighbourhood of a point \( \mathbf{x}_o \in \mathbb{R}^3 \), excluding this point. At \( \mathbf{x}_o \) the function \( f \) is assumed to be singular. We define the family of auxiliary complex valued functions \( f_n \) (labelled by unit vectors \( n \)) in the following way:

\[
f_n : \mathbb{C} \ni \varepsilon \mapsto f_n(\varepsilon) := f((\mathbf{x}_o + \varepsilon \mathbf{n})) \in \mathbb{C}.
\]

We expand \( f_n \) into a Laurent series around \( \varepsilon = 0 \):

\[
f_n(\varepsilon) = \sum_{m=-N}^{\infty} a_m(n) \varepsilon^m.
\]

The coefficients \( a_m \) in this expansion depend on the unit vector \( n \). We define the regularized value of the function \( f \) at \( \mathbf{x}_o \) as the coefficient of \( \varepsilon^0 \) in the expansion (110) averaged over all directions:

\[
f_{\text{reg}}(\mathbf{x}_o) := \frac{1}{4\pi} \oint d\Omega a_0(n).
\]

We use formula (111) to compute all integrals which contain Dirac delta distribution. It means that we define

\[
\int d^3x \ f(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}_a) := f_{\text{reg}}(\mathbf{x}_a).
\]

The procedure described here was used by one of the authors (cf. Appendix B in [17]) in the calculation of the 2PN and 2.5PN ADM Hamiltonians for many-body point-mass systems. More details on the applications of the Hadamard’s regularization can be found in [25]. Also the relation of the Hadamard’s procedure to the regularization described in Appendix B.2 and the rule of differentiation of homogeneous functions of Appendix B.4 is discussed in [25].

B.2 Riesz’s formula based regularization

The following formula, firstly derived by Riesz (see Eqs. (7) and (10) in Chapter 2 of [26]), can serve as a tool to regularize a class of divergent integrals of \( r_1^\alpha r_2^\beta \) using the analytic continuation arguments:

\[
\left[ \int d^3x \ r_1^\alpha r_2^\beta \right]_{\text{reg}} := \pi^{3/2} \frac{\Gamma \left( \frac{\alpha+3}{2} \right) \Gamma \left( \frac{\beta+3}{2} \right) \Gamma \left( -\frac{\alpha+\beta+3}{2} \right)}{\Gamma \left( -\frac{3}{2} \right) \Gamma \left( -\frac{\beta}{2} \right) \Gamma \left( \frac{\alpha+\beta+6}{2} \right)} r_1^{\alpha+\beta+3}.
\]

In deriving the 3PN Hamiltonian for a two-body point-mass system more general integrals than those which can be regularized by means of the formula (113) appear. We succeeded in deriving the generalization of the formula (113) for those integrals. The generalized formula reads

\[
\left[ \int d^3x \ r_1^\alpha r_2^\beta (r_1 + r_2 + r_{12})^\gamma \right]_{\text{reg}} := R(\alpha, \beta, \gamma) r_1^{\alpha+\beta+\gamma+3},
\]

where

\[
R(\alpha, \beta, \gamma) := 2\pi \frac{\Gamma (\alpha + 2) \Gamma (\beta + 2) \Gamma (-\alpha - \beta - \gamma - 4)}{\Gamma (-\gamma)} \times \left[ I_{1/2} (\alpha + 2, -\alpha - \gamma - 2) + I_{1/2} (\beta + 2, -\beta - \gamma - 2) - I_{1/2} (\alpha + \beta + 4, -\alpha - \beta - \gamma - 4) - 1 \right].
\]
The function \( I_{1/2} \) in Eq. (115) is defined as follows:

\[
I_{1/2}(x, y) := \frac{B_{1/2}(x, y)}{B(x, y)},
\]

where \( B \) stands for the beta function (Euler’s integral of the first kind) and \( B_{1/2} \) is the incomplete beta function; it can be expressed in terms of the Gauss hypergeometric function \( _2F_1 \):

\[
B_{1/2}(x, y) = \frac{1}{2^x} \, _2F_1\left(1 - y, x; x + 1; \frac{1}{2}\right).
\]

The regularization procedure based on the formulae (114) and (115) consists of several steps. We enumerate them now.

1. By means of Eqs. (61) and (62) we replace \( p_2 \) by \(-p_1\), \( \dot{p}_2 \) by \(-\dot{p}_1\), and \( \dot{x}_1 \) and \( \dot{x}_2 \) by \( v \).

2. We eliminate the unit vector \( n_2 \) by the relation

\[
n_2 = \frac{r_1}{r_2} n_1 + \frac{r_{12}}{r_2} n_{12}.
\]

Then we expand the scalar product \( n_1 \cdot n_{12} \):

\[
n_1 \cdot n_{12} = \frac{r_2^2 - r_1^2 - r_{12}^2}{2r_1 r_{12}}.
\]

3. All integrands we reduce to integrands which depend on \( r_1 \) and \( r_2 \) only. To do this we use a set of substitutions. For the integrands of the type \( (n_1 \cdot p_1)^k f(r_1, r_2), k = 1, \ldots , 6 \), we use the rules (obtained by considering the integration in prolate spheroidal coordinates)

\[
(n_1 \cdot p_1)^k f(r_1, r_2) \rightarrow a f(r_1, r_2),
\]

\[
(n_1 \cdot p_1)^2 f(r_1, r_2) \rightarrow \left(a^2 + \frac{1}{2} b^2\right) f(r_1, r_2),
\]

\[
(n_1 \cdot p_1)^3 f(r_1, r_2) \rightarrow \left(a^3 + \frac{3}{2} ab^2\right) f(r_1, r_2),
\]

\[
(n_1 \cdot p_1)^4 f(r_1, r_2) \rightarrow \left(a^4 + 3a^2 b^2 + \frac{3}{8} b^4\right) f(r_1, r_2),
\]

\[
(n_1 \cdot p_1)^5 f(r_1, r_2) \rightarrow \left(a^5 + 5a^3 b^2 + \frac{15}{8} ab^4\right) f(r_1, r_2),
\]

\[
(n_1 \cdot p_1)^6 f(r_1, r_2) \rightarrow \left(a^6 + \frac{15}{2} a^4 b^2 + \frac{45}{8} a^2 b^4 + \frac{5}{16} b^6\right) f(r_1, r_2),
\]

where

\[
a := (n_{12} \cdot p_1) \frac{r_2^2 - r_1^2 - r_{12}^2}{2r_1 r_{12}},
\]

\[
b := \sqrt{p_1^2 - (n_{12} \cdot p_1)^2} \left[ (r_1 + r_2)^2 - r_{12}^2 \right] \frac{r_{12}^2 - (r_1 - r_2)^2}{2r_1 r_{12}}.
\]
After this step the integrand becomes the linear combination of the type
\[ \sum_{I=1}^{N} c_I r_1^{\alpha_I} r_2^{\beta_I} (r_1 + r_2 + r_{12})^{\gamma_I}, \tag{128} \]
where \( \alpha_I, \beta_I, \) and \( \gamma_I \) are integers; the constant coefficients \( c_I \) may depend on \( r_{12}, p_1^2, \dot{p}_1^2, v^2, (n_{12} \cdot p_1), (n_{12} \cdot \dot{p}_1), (v \cdot p_1), (v \cdot \dot{p}_1) \), and \( (p_1 \cdot \dot{p}_1) \).

4. We define the auxiliary function
\[ J^{\mu,\nu}_{\varepsilon}(\alpha, \beta, \gamma) := \int R(\alpha + \mu \varepsilon, \beta + \nu \varepsilon, \gamma) r_1^{\alpha+\beta+\gamma+(\mu+\nu)\varepsilon+3} \tag{129} \]
and for the combination (128) we calculate the limit
\[ \lim_{\varepsilon \to 0} \sum_{I=1}^{N} c_I J^{\mu,\nu}_{\varepsilon}(\alpha_I, \beta_I, \gamma_I). \tag{130} \]
It turns out that for all terms considered in the paper this limit is of the form
\[ A + B \frac{\mu}{\nu} + C \frac{\nu}{\mu}, \tag{131} \]
where \( A, B, C \) do not depend on \( \mu \) and \( \nu \).

5. As the regularized value of the integral of the sum (128) we take the number \( A \), i.e.
\[ \left[ \int d^3x \sum_{I=1}^{N} c_I r_1^{\alpha_I} r_2^{\beta_I} (r_1 + r_2 + r_{12})^{\gamma_I} \right]_{\text{reg}} := A. \tag{132} \]

B.3 Riesz’s kernel based regularization

Three-dimensional Dirac delta distribution can be represented as the limit
\[ \delta(x - x_a) = \lim_{\alpha \to 0} I^{\alpha}(x, x_a), \tag{133} \]
where \( I^{\alpha}(x, x') \) is the Riesz’s kernel defined as follows:
\[ I^{\alpha}(x, x') := \frac{\Gamma \left( \frac{3-\alpha}{2} \right)}{\pi^{3/2} 2^\alpha \Gamma \left( \frac{\alpha}{2} \right)} |x - x'|^{\alpha-3}. \tag{134} \]
The regularization of a class of integrals based on the kernel (134) and its relation to Hadamard’s procedure of Appendix B.1 is discussed by Riesz [26]. For us, the Riesz’s kernel regularization consists in using the kernel (134) instead of the Dirac delta in the source terms of the constraint equations, solving these equations, and performing the regularization of integrals needed to obtain the 3PN dynamics in the way described below.

For a point-mass two-body system we need two families of Riesz’s kernels, each associated with the different mass. Let the first family be labelled by the parameter \( \alpha \), and the second one by \( \beta \). After replacing the Dirac deltas by Riesz’s kernels the right-hand sides of the constraint equations (1)–(2) depend on \( \alpha \) and \( \beta \). The first step of the procedure is to solve constraint equations iteratively keeping \( \alpha \) and \( \beta \) as unspecified complex parameters.
This is a very difficult task. Out of all equations for the functions \(\phi(n)\) we have fully solved only the simplest one for the function \(\phi(2)\). Also only the leading order momentum constraint equation for \(\tilde{\varpi}^{ij}(3)\) we have solved. The TT-part of the metric is not known even in the lowest order — only the part depending quadratically on the momenta has been calculated.

The solutions obtained by us are enough to calculate that part of the full divergence (72) (discussed in Section 5) which depends quadratically on the momenta. The regularization of the full divergence by means of the procedure described in Appendix B.2 gives a non-zero result. If one uses the Riesz’s kernel generalization of the function \(\phi(2)\) and of the momenta dependent part of the function \(h^{TT}_{(4)ij}\) the result of regularization of the full divergence using the procedure described below is zero.

The procedure is similar to that described in Appendix B.2. The first three steps of the regularization procedure from Appendix B.2 apply here. After that all integrands to be regularized are functions of \(r_1, r_2\) and the parameters \(\alpha\) and \(\beta\). So any integrand can be written as

\[ f(r_1, r_2, \alpha, \beta). \quad (135) \]

We need now to use formulae analogous to the generalized Riesz’s formula (113) from Appendix B.2 to perform integration of (135) for any \(\alpha\) and \(\beta\). Let’s denote the result of integration by \(R(\alpha, \beta)\). Then the following limit is calculated:

\[ \lim_{\varepsilon \to 0} R(\alpha + \mu \varepsilon, \beta + \nu \varepsilon). \quad (136) \]

For the divergence (72), to perform the integration it is enough to use the formula (113) from Appendix B.2 for \(\gamma = 0\). The limit (136) comes out to be zero.

For all terms for which we were able to apply the above procedure we obtained the limits (136) of the form given in Eq. (131). However, we were not able to check the cancellation of all divergent terms in \(\tilde{H}_{34}\) after regularization because we were not able to calculate all terms using the Riesz’s kernel procedure. But again, the ill-defined terms we have obtained were of the type (75).

### B.4 The rule to differentiate homogeneous functions.

In distribution theory there exists a rule to differentiate some homogeneous and locally non-integrable functions under the integral sign [27]. We have studied how useful this rule is in our calculations.

Let \(f\) be a real valued function defined in a neighbourhood of the origin in \(\mathbb{R}^3\). \(f\) is said to be a positively homogeneous function of degree \(\lambda\), if for any number \(a > 0\)

\[ f(a x) = a^\lambda f(x). \quad (137) \]

Let \(k := -\lambda - 2\). If \(\lambda\) is an integer and if \(\lambda \leq -2\) (i.e. \(k\) is a non-negative integer), then the partial derivative of \(f\) with respect to the coordinate \(x^i\) has to be calculated by means of the formula (cf. Eq. (5.15) in [15])

\[ \partial_i f = (\partial_i f)_{\text{ord}} + (-1)^k \frac{\partial^k \delta(x)}{k!} \int_{\Sigma} d\sigma_i f x^{i_1} \ldots x^{i_k}, \quad (138) \]

where \((\partial_i f)_{\text{ord}}\) means the derivative computed using the standard rules of differentiations, \(\Sigma\) is any smooth close surface surrounding the origin and \(d\sigma_i\) is the surface element on \(\Sigma\).
As an example of applying the rule (138) let us consider the full divergence (connected with
the fourth term in the $\tilde{H}_{34}$ part of the 3PN Hamiltonian, see Eq. (58))

$$
\int d^3x \left\{ \left( 2\pi^i + \Delta^{-1} \pi^i_{(3),}\mu \right) \left( 2\pi^k + \Delta^{-1} \pi^m_{(3),\nu} \right) h^{TT}_{(4)ij} \right\}. 
$$

(139)

Applying the regularization procedure from Appendix B.2 to the divergence (139) gives a result
much more complicated than that given by Eq. (75). After performing in (139) differentiation
with respect to $x^k$ one finds that only in one term the rule (138) can be applied. The term reads

$$
3 \left( 2\pi^i_{(3)} + \Delta^{-1} \pi^i_{(3),\mu} \right) h^{TT}_{(4)ij} \pi^k_{(3),\nu} \pi^k_{(3),\nu}. 
$$

(140)

The rule (138) applied to $\pi^k_{(3),\nu}$ yields

$$
\int d^3x \left\{ -\frac{1}{4} \left( 2\pi^i_{(3)} + \Delta^{-1} \pi^i_{(3),\mu} \right) h^{TT}_{(4)ij} \right\} \sum_a p_{a\mu} \delta_a. 
$$

(141)

The integral (141) is calculated by means of Hadamard’s procedure from Appendix B.1. After
adding the result of regularization of the integral (141) to the result of regularization of the
integral (139) we obtain a multiple of the quantity (75).

Using similar considerations we were always able to restrict the ambiguity to a multiple of
the quantity (75). Let us also stress that the $\tilde{H}_{34}$ part of the 3PN Hamiltonian is written in
such a form that there is no need to use the rule (138) for individual derivatives appearing in
$\tilde{H}_{34}$.

A method of applying the rule (141) in regularizations of integrals can be found in Section
6 of [25]. We have yet found that this method (without modifications) is not able to give zero
for all full divergences one meets in our calculations.

References

[1] B. R. Iyer and C. M. Will, Phys. Rev. D 52, 6882 (1995).
[2] L. Blanchet, Phys. Rev. D 55, 714 (1997).
[3] P. Jaranowski and G. Schäfer, Phys. Rev. D 55, 4712 (1997).
[4] T. Damour, in Gravitation in Astrophysics, edited by B. Carter and J. B. Hartle (Plenum
Press, New York, 1987), p. 3.
[5] A. Gopakumar, B. R. Iyer, and S. Iyer, preprint (1997), gr-qc/9703075.
[6] T. Damour, in Three Hundred Years of Gravitation, edited by S. Hawking and W. Israel
(Cambridge University Press, Cambridge, 1987), p. 128.
[7] T. Damour and G. Schäfer, Nuovo Cimento B 101, 127 (1988).
[8] R. Arnowitt, S. Deser, and C. W. Misner, in Gravitation: An Introduction to Current
Research, edited by L. Witten (John Wiley, New York, 1962), p. 227.
[9] L. E. Kidder, C. M. Will, and A. G. Wiseman, Class. Quantum Grav. 9, L125 (1992); Phys.
Rev. D 47, 3281 (1993).
[10] N. Wex and G. Schäfer, Class. Quantum Grav. 10, 2729 (1993).

[11] G. Schäfer and N. Wex, in Perspectives in Neutrinos, Atomic Physics, and Gravitation, edited by J. Trân Thanh Vân, T. Damour, E. Hinds, and J. Wilkerson (Editions Frontières, 1993), p. 513.

[12] T. Damour, B. R. Iyer, and B. S. Sathyaprakash, preprint (1997), gr-qc/9705034.

[13] T. Damour, in Gravitational Radiation, edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983), p. 59.

[14] T. Damour and G. Schäfer, Gen. Rel. Grav. 17, 879 (1985).

[15] S. M. Kopeikin, Sov. Astron. 29, 516 (1985).

[16] S. Wolfram, Mathematica. A System for Doing Mathematics by Computer, 2nd ed. (Addison-Wesley, Redwood City, 1991).

[17] G. Schäfer, Annals of Physics (NY) 161, 81 (1985).

[18] T. Kimura, Prog. Theor. Phys. 26, 157 (1961).

[19] T. Regge and C. Teitelboim, Annals of Physics (NY) 88, 286 (1974).

[20] G. Schäfer, in Symposia Gaussiana, Proceedings of the 2nd Gauss Symposium, Conference A: Mathematical and Theoretical Physics, Munich, edited by M. Behara, R. Fritsch, and R. G. Lintz (Walter de Gruyter, Berlin, 1995), p. 667.

[21] T. Damour and G. Schäfer, J. Math. Phys. 32, 127 (1991).

[22] C. Itzykson and J. B. Zuber, Quantum Field Theory (McGraw-Hill, New York, 1980), p. 416.

[23] T. Kimura and T. Toiya, Prog. Theor. Phys. 48, 316 (1972).

[24] T. Ohta, H. Okamura, T. Kimura, and K. Hiida, Prog. Theor. Phys. 51, 1220 (1974); Nuovo Cimento B 27, 103 (1975).

[25] P. Jaranowski, in Mathematics of Gravitation. Part II. Gravitational Wave Detection, Banach Center Publications, Vol. 41, Part II (Institute of Mathematics, Polish Academy of Sciences, Warszawa, 1997), p. 55.

[26] M. Riesz, Acta Math. 81, 1 (1949).

[27] I. M. Gel’fand and G. E. Shilov, Generalized Functions, Vol. 1 (Academic Press, New York, 1964).