DECOUPLING AND SCHröDINGER MAXIMAL ESTIMATES FOR FINITE TYPE PHASES IN HIGHER DIMENSIONS

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ABSTRACT. In this article, we establish an \( \ell^2 \) decoupling inequality for the hypersurface

\[
\left\{ (\xi_1, \ldots, \xi_{n-1}, \xi_m^n + \ldots + \xi_{n-1}^m) : (\xi_1, \ldots, \xi_{n-1}) \in [0,1]^{n-1} \right\},
\]

where \( m \geq 4 \) is an even number, associated with the decomposition adapted to finite type hypersurfaces from our previous work [18]. The key ingredients of the proof include an \( \ell^2 \) decoupling inequality for the hypersurfaces

\[
\left\{ (\xi_1, \ldots, \xi_{n-1}, \phi_1(\xi_1) + \ldots + \phi_s(\xi_s) + \xi_m^s + \ldots + \xi_{n-1}^m) : (\xi_1, \ldots, \xi_{n-1}) \in [0,1]^{n-1} \right\},
\]

where \( 0 \leq s \leq n-1 \), with \( \phi_1, \ldots, \phi_s \) being non-degenerate. As an application, we generalize the Schrödinger maximal estimates in [19] to higher dimensions.

Key Words: decoupling inequality; Schrödinger maximal estimate; finite type; induction.

AMS Classification: 42B10

1. INTRODUCTION AND MAIN RESULT

Decoupling inequalities was introduced by Wolff [21] in connection with the local smoothing estimates for the solution to the wave equation. In 2015, Bourgain and Demeter [5] proved the sharp \( \ell^2 \) decoupling inequality for compact hypersurfaces with positive definite second fundamental form, which has a wide range of important consequences [5, 7, 14]. The general decoupling theory for hypersurfaces with vanishing Gaussian curvature is still under-developed. For partial results in this direction, we refer to the work in [3, 22, 15, 17, 20].

In this paper, we will study the \( \ell^2 \) decoupling problem for certain hypersurfaces of finite type in \( \mathbb{R}^n \). More precisely, we consider the hypersurface in \( \mathbb{R}^n \) given by

\[
F_{2,m}^{n-1}(0, n-1) := \left\{ (\xi_1, \ldots, \xi_{n-1}, \xi_m^n + \ldots + \xi_{n-1}^m) : (\xi_1, \ldots, \xi_{n-1}) \in [0,1]^{n-1} \right\},
\]

where \( m \geq 2 \).

For any function \( g \in L^1([0,1]^{n-1}) \) and each subset \( Q \subset [0,1]^{n-1} \), we denote the corresponding Fourier extension operator by

\[
\mathcal{E}_Q g(x) := \int_Q g(\xi_1, \ldots, \xi_{n-1}) e(x_1 \xi_1 + \ldots + x_n \xi_{n-1} + \xi_m^n + \ldots + \xi_{n-1}^m) \, d\xi_1 \ldots d\xi_{n-1},
\]

where \( e(t) = e^{2\pi i t} \) for \( t \in \mathbb{R} \), and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \).

For \( m = 2 \), the hypersurface \( F_{2,2}^{n-1}(0, n-1) \) is exactly a paraboloid over the region \([0,1]^{n-1}\). For \( n = 3 \), \( m = 4 \), the hypersurface \( F_{2,4}^{3}(0,2) \) is exactly the surface studied in Theorem 1.2 from [20].

By making use of Bourgain-Guth methods in [8], parabolic rescaling and induction on scales, Bourgain and Demeter [5] have established the sharp \( \ell^2 \) decoupling inequality for the paraboloid in \( \mathbb{R}^n \) as follows.
Theorem 1.1. For $2 \leq p \leq \frac{2(n + 1)}{n - 1}$ and all $\varepsilon > 0$, there exists a constant $C(\varepsilon, p)$ such that
\[
\|E_{[0,1]}^{\text{Par}}g\|_{L^p(B^n_R)} \leq C(\varepsilon, p)R^{\varepsilon}\left(\sum_{\theta; R^{-1/2}-\text{cubes in } [0,1]^{n-1}} \|E_{\theta}^{\text{Par}}g\|_{L^p(w_{B^n_R})}^2\right)^{1/2},
\]
(1.1)
where $E_{[0,1]}^{\text{Par}}$ denotes the Fourier extension operator associated with the paraboloid $P^{n-1}$ and $w_{B^n_R}(x) = \left(1 + \frac{|x - x_0|^2}{R^2}\right)^{-100n}$ denotes the standard weight function adapted to the ball $B^n_R$ in $\mathbb{R}^n$ centered at $x_0$ with radius $R$.

We will use the following equivalent version of (1.1)
\[
\|F\|_{L^p(\mathbb{R}^n)} \leq C(\varepsilon, p)R^{\varepsilon}\left(\sum_{\theta} \|F_{\theta}\|_{L^p(\mathbb{R}^n)}^2\right)^{1/2}, \quad 2 \leq p \leq \frac{2(n + 1)}{n - 1},
\]
(1.2)
where supp $\hat{F} \subset \mathcal{N}_{\frac{n}{m}}(P^{n-1})$, the $\theta$’s are finitely overlapping $R^{-1/2}$-slabs in $\mathbb{R}^n$ which cover $\mathcal{N}_{\frac{n}{m}}(P^{n-1})$ (the $\frac{n}{m}$-neighborhood of $P^{n-1}$) and $F_{\theta} := \mathcal{F}^{-1}(\hat{F}\chi_{\theta})$. For the equivalence of (1.1) and (1.2), we refer to [6].

For $m > 2$, the Gaussian curvature of the surface $F_{x, m}(0, n - 1)$ vanishes if there is at least one $j$ such that $\xi_j = 0$. From now on, we focus on the case that $m \geq 4$ is an even number. We adopt the notations as in the previous work [18]. At first, given $R \gg 1$, we divide $[0, 1]$ into
\[
[0, 1] = \bigcup_k I_k,
\]
where $I_0 = [0, R^{-\frac{1}{2}}]$ and
\[
I_k = [2^{k-1}R^{-\frac{1}{2}}, 2^kR^{-\frac{1}{2}}], \quad \text{for } 1 \leq k \leq \left\lfloor \frac{1}{m} \log_2 R \right\rfloor.
\]
For each $k \geq 1$, we divide $I_k$ further into
\[
I_k = \bigcup_{\mu = 1}^{2^{2(k-1)}} I_{k, \mu},
\]
with
\[
I_{k, \mu} = [2^{k-1}R^{-\frac{1}{2}} + (\mu - 1)2^{-(k-1)}R^{-\frac{1}{2}}, 2^{k-1}R^{-\frac{1}{2}} + \mu2^{-(k-1)}R^{-\frac{1}{2}}].
\]
Thus, we have the following decomposition
\[
[0, 1]^2 = \bigcup_{\theta \in \mathcal{F}_n(R; 2, m; 0, n - 1)} \theta,
\]
(1.3)
where
\[
\mathcal{F}_n(R; 2, m; 0, n - 1) := \{I_{k_1, \mu_1} \times I_{k_1, \mu_2} \times \cdots \times I_{k_{n-1}, \mu_{n-1}}, I_0 \times I_{k_2, \mu_2} \times \cdots \times I_{k_{n-1}, \mu_{n-1}}, \ldots; I_{k_1, \mu_2} \times I_{k_2, \mu_2} \times \cdots \times I_0, I_0 \times I_0 \times \cdots \times I_0 : 1 \leq k_j \leq \left\lfloor \frac{1}{m} \log_2 R \right\rfloor, 1 \leq \mu_j \leq 2^{2(k-1)}, j = 1, 2, \ldots, n - 1\}.
\]
We refer to [18] for the detailed decomposition (1.3). Buschenhenke [9] utilized the analogous decomposition to study the restriction estimates for certain conic surfaces of finite type.
Our main result is the following decoupling inequality associated with the surface $F_{2,m}^{n-1}(0, n-1)$ based on the decomposition (1.3).

**Theorem 1.2.** For $2 \leq p \leq \frac{2(n+1)}{n-1}$ and any $\varepsilon > 0$, there exists a constant $C(\varepsilon, p)$ such that

$$\|E_{[0,1]^{n-1}}g\|_{L^p(B_R)} \leq C(\varepsilon, p)R^\varepsilon \left( \sum_{\theta \in \mathcal{F}_n(\mathbb{R}^2, m; 0, n-1)} \|E_{\theta}g\|_{L^p(w_{B_R})}^2 \right)^{1/2}, \quad (1.4)$$

with $w_{B_R}(x)$ being the standard weight function as in Theorem 1.1.

We will prove Theorem 1.2 via an induction on dimension argument. Our main inductive proposition is formulated in Section 2.

**Notations:** For nonnegative quantities $X$ and $Y$, we will write $X \lesssim Y$ to denote the estimate $X \leq CY$ for some $C > 0$. If $X \lesssim Y \lesssim X$, we simply write $X \sim Y$. Dependence of implicit constants on the power $p$ or the dimension will be suppressed; dependence on additional parameters will be indicated by subscripts. For example, $X \lesssim_u Y$ indicates $X \leq CY$ for some $C = C(u)$. We denote $e(t) = e^{2\pi i t}$. We denote $|x|$ to be the greatest integer not larger than $x$. We use $B(x_0, r)$ to denote an arbitrary ball centred at $x_0$ with radius $r$ in $\mathbb{R}^3$ and abbreviate it by $B_r$ in the context. For any region $\Omega \subset \mathbb{R}^n$, we denote the characteristic function on $\Omega$ by $\chi_\Omega$. In $\mathbb{R}^n$, we denote $R^{-1/2} \times \cdots \times R^{-1/2} \times R^{-1}$-rectangle to be an $R^{-1/2}$-slab in $\mathbb{R}^n$. Define the Fourier transform on $\mathbb{R}^n$ by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) \, dx,$$

and the inverse Fourier transform by

$$\mathcal{F}^{-1} f(x) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} f(\xi) \, d\xi.$$

**2. Formulation of the inductive proposition**

We say that a smooth function $\phi(t)$, $t \in [0, 1]$ is non-degenerate if

$$\phi'' \sim 1, \quad 0 \leq \phi^{(\ell)} \lesssim 1 \quad \text{for} \quad 3 \leq \ell \leq m, \quad \phi^{(k)} \equiv 0 \quad \text{for} \quad k \geq m + 1 \quad \text{on} \quad [0, 1].$$

To prove Theorem 1.2, we need to consider a family of hypersurfaces in $\mathbb{R}^n$ as follows

$$\mathcal{F}^{n-1} := \left\{ F_{2,m}^{n-1}(s, n - 1 - s) : \ 0 \leq s \leq n - 1 \right\},$$

where $F_{2,m}^{n-1}(s, n - 1 - s) := \left\{ (\xi_1, \ldots, \xi_{n-1}, \phi_1(\xi_1) + \cdots + \phi_s(\xi_s) + \xi_{s+1}^m + \cdots + \xi_{n-1}^m) : \ (\xi_1, \ldots, \xi_{n-1}) \in [0, 1]^{n-1} \right\}, \ 0 \leq s \leq n - 1$

with each function $\phi_j$, $0 \leq j \leq s$ being non-degenerate.

For each hypersurface $F_{2,m}^{n-1}(s, n - 1 - s)$ in the family $\mathcal{F}^{n-1}$, one can write down the corresponding decomposition $\mathcal{F}_n(\mathbb{R}^2, m; s, n - 1 - s)$ as similar as the decomposition $\mathcal{F}_n(\mathbb{R}^2, m; 0, n - 1)$ adapted to the hypersurface $F_{2,m}^{n-1}(R; 0, n - 1)$ in the introduction. We shall establish decoupling inequalities for all the hypersurfaces in the family $\mathcal{F}^{n-1}$ associated with the corresponding decompositions. Our main inductive proposition is the following:
Proposition 2.1. Suppose that $2 \leq p \leq \frac{2(n+1)}{n-1}$. For any $\epsilon > 0$, there exists a constant $C_\epsilon$ (uniform in $F^{n-1}$) such that

$$\|E_{[0,1]}^{n-1}f\|_{L^p(B_R)} \leq C_\epsilon R^\epsilon \left( \sum_{\theta \in F_n(R;2,m;0,n-1)} \|\hat{E}_\theta f\|_{L^p(w_{BR})}^2 \right)^{1/2},$$

(2.1)

where $E_{[0,1]}^{n-1}$ denotes the Fourier extension operator associated with the hypersurface $F^{n-1}_{2,m}(s,n-1-s)$.

We shall prove Proposition 2.1 by induction on the dimension, the radius and the number $n-1-s$. The base case is $d = 3$, which is proved in [20], $1 \leq r \leq 10$ and $n-1-s = 0$. Our inductive hypothesis is that Proposition 2.1 holds for dimension $3 \leq d \leq n-1$ or the radius $1 \leq r \leq \frac{n}{2}$ or $d = n$, $r = R$, $0 \leq n - 1 - s \leq n - 2$. To prove Proposition 2.1 it suffices to consider the case $s = 0$.

3. PROOF OF PROPOSITION 2.1

In this section, we shall prove Proposition 2.1.

Lemma 3.1. For $2 \leq p \leq \frac{2(n+1)}{n-1}$ and any $\epsilon > 0$, there exists a constant $C_\epsilon$ such that

$$\|E_{[0,1]}^{n-1}f\|_{L^p(B_R)} \leq C_\epsilon R^\epsilon \left( \sum_{\theta \in F_n(R;2,m;0,n-1)} \|\hat{E}_\theta f\|_{L^p(w_{BR})}^2 \right)^{1/2},$$

(3.1)

where $E_{[0,1]}^{n-1}$ denotes the Fourier extension operator associated with the hypersurface $F^{n-1}_{2,m}(0,n-1)$.

To prove Lemma 3.1 we divide $[0,1]^{n-1}$ into $[0,1]^{n-1} = \bigcup_b \Omega_b$, where

$$b := (b_1, \ldots, b_{n-1}), \quad b_j \in \{0, 1\} \text{ for } 1 \leq j \leq n - 1,$$

and

$$\Omega_b := \prod_{j=1}^{n-1} J_{b_j},$$

and

$$J_0 := [0, K^{-1/m}], \quad J_1 := [K^{-1/m}, 1].$$

For technical reasons, $K^{-1/m}, R^{-1/m}$ and $(\frac{R}{K})^{-1/m}$ should be dyadic numbers satisfying $1 \ll K \ll R^\epsilon$ for any fixed $\epsilon > 0$. Therefore, we choose $K = 2^n s$ and $R = 2^m l$ ($s, l \in \mathbb{N}$) to be large numbers satisfying $K \approx \log R$.

We observe that the hypersurface $F^{n-1}_{2,m}(0,n-1)$ has positive second fundamental form if $(\xi_1, \ldots, \xi_{n-1}) \in \Omega_{(1,\ldots,1)}$. In this region, one can adopt Bourgain-Demeter’s decoupling for the perturbed paraboloid. For the region $\Omega_b$ ($b_1 = b_2 = \ldots = b_{n-1} = 0$), we will use the rescaling technique directly. While for the regions $\Omega_b$ ($b_1 = \ldots = b_s = 1, b_{s+1} = \ldots = b_{n-1} = 0$), $1 \leq s \leq n-2$, we reduce them to lower dimensional decoupling problems. By the Minkowski inequality and Cauchy-Schwartz inequality, we have

$$\|E_{[0,1]}^{n-1}g\|_{L^p(B_R)} \leq 2^{\frac{2\epsilon}{p-1}} \left( \sum_b \|E_{\Omega_b} g\|_{L^p(B_R)}^2 \right)^{1/2}. \quad (3.2)$$

Let $D_p(R)$ denote the least number such that

$$\|E_{[0,1]}^{n-1}g\|_{L^p(B_R)} \leq D_p(R) \left( \sum_{\theta \in F_n(R;2,m;0,n-1)} \|\hat{E}_\theta g\|_{L^p(w_{BR})}^2 \right)^{1/2}. \quad (3.3)$$
We are going to treat different regions with different approaches. Firstly, we estimate the contribution from \( \Omega_b \)-part, where \( b_1 = b_2 = \ldots = b_{n-1} = 1 \).

3.1. Decoupling for \( \Omega_{(1,\ldots,1)} \). In this subsection, we will establish the decoupling inequality for \( \Omega_{(1,\ldots,1)} \)-part. Recall

\[
\begin{align*}
\Omega_{(1,\ldots,1)} &= \bigcup_{\lambda_1,\ldots,\lambda_{n-1}} \Omega_{\lambda_1,\ldots,\lambda_{n-1}}, \\
\Omega_{\lambda_1,\ldots,\lambda_{n-1}} &= \bigcup_{\lambda_1,\ldots,\lambda_{n-1}} r_{\lambda_1,\ldots,\lambda_{n-1}},
\end{align*}
\]

where

\[
\Omega_{\lambda_1,\ldots,\lambda_{n-1}} := \prod_{j=1}^{n-1} [\lambda_j, 2\lambda_j]
\]

and

\[
[\lambda_j, 2\lambda_j] = \bigcup_{\lambda_j + (t_j - 1)\frac{m-2}{m} K^{-1/2}, \lambda_j + t_j\lambda_j^{\frac{m-2}{m}} K^{-1/2}]
\]

for \( 1 \leq t_j \leq \lambda_j^{1+\frac{m-2}{m}} K^{1/2} \) and \( \lambda_j \in [K^{-1/m}, \frac{1}{2}] \) is a dyadic number. Combining this decomposition with the Minkowski inequality and Cauchy-Schwartz inequality, we get a trivial decoupling at scale \( K \) for \( 2 \leq p \leq \frac{2(n+1)}{n-1} \).

\[
\| \mathcal{E}_{\Omega_{(1,\ldots,1)}} g \|_{L^p(B_R)} \lesssim K^{\frac{n-1}{2}} \left( \sum_{\tau \subseteq \Omega_{(1,\ldots,1)}} \| \mathcal{E}_\tau g \|_{L^p(B_R)}^2 \right)^{1/2}. \tag{3.4}
\]

Summing over all the balls \( B_K \subset B_R \), we obtain

\[
\| \mathcal{E}_{\Omega_{(1,\ldots,1)}} g \|_{L^p(B_R)} \lesssim K^{\frac{n-1}{2}} \left( \sum_{\tau \subseteq \Omega_{(1,\ldots,1)}} \| \mathcal{E}_\tau g \|_{L^p(B_R)}^2 \right)^{1/2}. \tag{3.5}
\]

For any given \( \tau \subseteq \Omega_{(1,\ldots,1)} \) of size \( \lambda_1^{\frac{m-2}{m}} K^{1/2} \times \ldots \times \lambda_{n-1}^{\frac{m-2}{m}} K^{1/2} \), we have

**Lemma 3.2.** For \( 2 \leq p \leq \frac{2(n+1)}{n-1} \) and each \( \varepsilon > 0 \), there exists a positive constant \( C_\varepsilon \) such that

\[
\| \mathcal{E}_\tau g \|_{L^p(B_R)} \leq C_\varepsilon |R|^\varepsilon \left( \sum_{\theta \subseteq \tau} \| \mathcal{E}_\theta g \|_{L^p(wB_R)}^2 \right)^{1/2}, \quad \tag{3.6}
\]

where \( \theta \in \mathcal{F}_n(R; 2, m; 0, n-1) \).

With Lemma 3.2 in hand, plugging (3.6) into (3.5), we get the decoupling inequality for \( \Omega_{(1,\ldots,1)} \)

\[
\| \mathcal{E}_{\Omega_{(1,\ldots,1)}} g \|_{L^p(B_R)} \leq C_\varepsilon K^{O(1)} |R|^\varepsilon \left( \sum_{\theta \subseteq \tau} \| \mathcal{E}_\theta g \|_{L^p(wB_R)}^2 \right)^{1/2}. \tag{3.7}
\]

**Proof of Lemma 3.2.** Without loss of generality, we may assume that \( B_R \) is centered at the origin and

\[
\tau = \bigcup_{j=1}^{n-1} [\lambda_j, \lambda_j + \lambda_j^{\frac{m-2}{m}} K^{-1/2}].
\]

By a change of variables, we see that

\[
\| \mathcal{E}_\tau g \|_{L^p(B_R)}^p = \lambda_1^\frac{m-2}{m} \cdots \lambda_{n-1}^\frac{m-2}{m} K^\frac{m-2}{m} \| \mathcal{E}^{parp}_{[0,1]_{n-1}} g \|_{L^p(L_0(B_R))}^p.
\]
where $\mathcal{L}_0$ denotes the map
\[(x_1, \ldots, x_{n-1}, x_n) \mapsto (\lambda_1^{-\frac{m-2}{2}} K^{-\frac{1}{2} x_1} + m \lambda_1^{\frac{m-2}{2}} K^{-\frac{1}{2} x_n}, \ldots, \lambda_{n-1}^{-\frac{m-2}{2}} K^{-\frac{1}{2} x_{n-1}} + m \lambda_1^{\frac{m-2}{2}} K^{-\frac{1}{2} x_n}, K x_n)\]
and $\mathcal{L}_0(R)$ denotes the image of $B_R$ with size roughly as follows
\[\lambda_1^{-\frac{m-2}{2}} K^{-1/2} R \times \ldots \times \lambda_{n-1}^{-\frac{m-2}{2}} K^{-1/2} R \times K^{-1} R.\]

We divide it into a finitely overlapping union of balls as follows
\[\mathcal{L}_0(B_R) = \bigcup B_{R/K}.
\]
For a given $\theta \subset \mathbb{T}$ such as
\[\theta = [\lambda_1, \lambda_1 + \lambda_1^{\frac{m-2}{2}} R^{-1/2}] \times \ldots \times [\lambda_{n-1}, \lambda_{n-1} + \lambda_{n-1}^{\frac{m-2}{2}} R^{-1/2}],\]
and under a change of variables, we deduce that the image of $\theta$ is
\[\hat{\theta} = [0, K^{1/2} R^{-1/2}]^{n-1}.
\]
We use Bourgain-Demeter’s decoupling inequality \((1.1)\) on each $B_{R/K}$
\[\|E_{[0,1]}^{\text{Parp}} \hat{g}\|_{L^p(B_{R/K})} \leq C_{\varepsilon} R^\varepsilon \left( \sum_{\hat{\theta} : K^{1/2} R^{-1/2} \text{-cube}} \|E_{[0,1]}^{\text{Parp}} \hat{g}\|_{L^p(w_{B_{R/K}})}^2 \right)^{1/2}. \tag{3.8}\]
This can be done by the argument in Section 7 of \([5]\). In fact, we denote $K_p(R)$ to be the least number such that
\[\|F\|_{L^p(\mathbb{R}^n)} \leq K_p(R) \left( \sum_{\hat{\theta} : R^{-1/2} \text{-slab}} \|F_{\hat{\theta}}\|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2}. \tag{3.9}\]
holds for each $F$ with Fourier support in $\mathcal{N}_{R^{-1}}(S)$, where
\[S := \{(\eta_1, \ldots, \eta_{n-1}, \phi_1(\eta_1) + \ldots + \phi_{n-1}(\eta_{n-1})) : (\eta_1, \ldots, \eta_{n-1}) \in [0,1]^{n-1}\}\]
with $\phi_j$ being non-degenerate for $1 \leq j \leq n-1$. It follows from \((3.9)\) that
\[\|F\|_{L^p(\mathbb{R}^n)} \leq K_p(R^{2/3}) \left( \sum_{\hat{\alpha} : R^{-1/3} \text{-slab}} \|F_{\hat{\alpha}}\|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2}. \tag{3.10}\]
Furthermore, from Taylor’s formula, we know that on each $\hat{\alpha}$, $S$ is contained in $\mathcal{N}_{R^{-1}}(P^{n-1})$. By invoking \((1.2)\) for this paraboloid and parabolic rescaling, we get
\[\|F_{\hat{\alpha}}\|_{L^p(\mathbb{R}^n)} \leq C_{\varepsilon} R^{\varepsilon} \left( \sum_{\hat{\theta}, \hat{\theta} \subset \hat{\alpha}} \|F_{\hat{\theta}}\|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2}. \tag{3.11}\]
Hence, we conclude
\[K_p(R) \leq C_{\varepsilon} R^{\varepsilon} K_p(R^{2/3}),\]
which immediately leads to $K_p(R) \lesssim \varepsilon R^{\varepsilon}$ by iteration. This verifies the inequality \((3.8)\).

Summing over all the balls $B_{R/K} \subset \mathcal{L}_0(B_R)$ on both sides of \((3.8)\) and using Minkowski’s inequality, we have
\[\|E_{[0,1]}^{\text{Parp}} \hat{g}\|_{L^p(\mathcal{L}_0(B_R))} \leq C_{\varepsilon} R^{\varepsilon} \left( \sum_{\hat{\theta} : K^{1/2} R^{-1/2} \text{-cube}} \|E_{[0,1]}^{\text{Parp}} \hat{g}\|_{L^p(\mathcal{L}_0(B_R))}^2 \right)^{1/2}. \]
Taking the inverse change of variables, it follows that
\[
\|E_{\tau}g\|_{L^p(B_R)} \leq C_\varepsilon R^\varepsilon \left( \sum_{\theta \subset \tau} \|E_{\theta}g\|^2_{L^p(w_{B_R})} \right)^{1/2}.
\]

Therefore, we complete the proof of Lemma 3.2. □

Next, we turn to discuss the contribution from \(\Omega_{(0,\ldots,0)}\)-part.

### 3.2. Decoupling for \(\Omega_{(0,\ldots,0)}\)

By rescaling and induction on scales, we can show

**Lemma 3.3.** For \(2 \leq p \leq \frac{2(n+1)}{n-1}\), there holds
\[
\|E_{\Omega_{(0,\ldots,0)}}g\|_{L^p(B_R)} \leq \mathcal{D}_p \left( \frac{R}{K} \right) \left( \sum_{\theta \subset \Omega_{(0,\ldots,0)}} \|E_{\theta}g\|^2_{L^p(w_{B_R})} \right)^{1/2},
\] (3.12)

where \(\theta \in \mathcal{F}_n(R;2,m;0,n-1)\).

**Proof.** Taking the change of variables
\[
\eta_j = K^{1/m} \xi_j \quad (j = 1, \ldots, n-1),
\]
it follows
\[
|E_{\Omega_{(0,\ldots,0)}}g(x)| = |E_{[0,1]}^n \tilde{g}(\tilde{x})|,
\]
where
\[
\tilde{x} := (K^{-1/m} x_1, \ldots, K^{-1/m} x_{n-1}, K^{-1} x_n),
\]
and
\[
\tilde{g}(\eta) := K^{-\frac{1}{2m}} g(K^{-1/m} \eta_1, K^{-1/m} \eta_2).
\]
The \(\tilde{x}\)-variables belong to a rectangular box \(\square\) with size
\[
\frac{R}{K^{1/m}} \times \ldots \times \frac{R}{K^{1/m}} \times \frac{R}{K},
\]
which can be written into a finitely overlapping union of balls
\[
\square = \bigcup B_{R/K}.
\]

By the definition of \(\mathcal{D}_p(\cdot)\), we obtain
\[
\|E_{[0,1]}^n \tilde{g}\|_{L^p(B_{R/K})} \leq \mathcal{D}_p \left( \frac{R}{K} \right) \left( \sum_{\theta \in \mathcal{F}_n(\frac{R}{K};2,m;0,n-1)} \|E_{\theta}\tilde{g}\|^2_{L^p(w_{B_{R/K}})} \right)^{1/2}.
\]

Summing over all the cubes \(B_{R/K} \subset \square\) and using Minkowski’s inequality, we have
\[
\|E_{[0,1]}^n \tilde{g}\|_{L^p(\square)} \leq \mathcal{D}_p \left( \frac{R}{K} \right) \left( \sum_{\theta \in \mathcal{F}_n(\frac{R}{K};2,m;0,n-1)} \|E_{\theta}\tilde{g}\|^2_{L^p(\square)} \right)^{1/2}.
\]

Taking the inverse change of variables, one has
\[
\|E_{\Omega_{(0,\ldots,0)}}g\|_{L^p(B_R)} \leq \mathcal{D}_p \left( \frac{R}{K} \right) \left( \sum_{\theta \subset \Omega_{(0,\ldots,0)}} \|E_{\theta}g\|^2_{L^p(w_{B_R})} \right)^{1/2}.
\]

Thus, we complete the proof of Lemma 3.3. □

Finally, we deal with the contribution from \(\Omega_\theta\)-part for \(b_1 = \ldots = b_s = 1, \ b_{s+1} = \ldots = b_{n-1} = 0, \) where \(1 \leq s \leq n-2.\)
3.3. Decoupling for $\Omega_b$ $(b_1 = \ldots = b_s = 1, b_{s+1} = \ldots = b_{n-1} = 0, 1 \leq s \leq n-2)$. First, by reduction of dimension arguments, we are able to prove the following result.

**Lemma 3.4.** For $2 \leq p \leq \frac{2n}{n-2}$ and any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ such that
\[ \| \mathcal{E}_{\Omega_b} g \|_{L^p(B_R)} \leq C_\varepsilon K^\varepsilon \left( \sum_{\tau \subset \Omega_b} \| \mathcal{E}_\tau g \|_{L^p(w_{B_R})}^2 \right)^{1/2}. \tag{3.13} \]

To prove Lemma 3.4, we employ the induction hypothesis on dimensions from Section 2.

**Lemma 3.5.** For $2 \leq p \leq \frac{2n}{n-2}$ any any $\varepsilon > 0$, there holds
\[ \| G \|_{L^p(\mathbb{R}^{n-1})} \leq C_\varepsilon K^\varepsilon \left( \sum_{\tau \in \mathcal{F}_{n-1}(K;2,m,0,n-2)} \| G_\tau \|_{L^p(\mathbb{R}^2)}^2 \right)^{1/2}, \]
where $\text{supp} \hat{G} \subset \mathcal{N}_{K^{-1}}(F_{2,m}^{n-2}(0,n-2))$.

With Lemma 3.5 in hand, we prove Lemma 3.4 by freezing the $x_{n-1}$ variable as follows. Fix a bump function $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\text{supp} \varphi \subset B(0,1)$ and $|\varphi(x)| \geq 1$ for all $x \in B(0,1)$. Define $F := \varphi_{K^{-1}} \mathcal{E}_{\Omega_b} g$, where $\varphi_{K^{-1}}(\xi) := K^n \varphi(K\xi)$, $\xi \in \mathbb{R}^n$. We denote $F(\cdot, x_{n-1}, \cdot)$ by $G$. From Guth [13], it is easy to see that $\text{supp} \hat{G}$ is contained in the projection of $\text{supp} \hat{F}$ on the plane $\xi_{n-1} = 0$, that is, in the $K^{-1}$-neighborhood of $F_{2,m}^{n-2}(s, n-2 - s)$. We employ Lemma 3.5 to get
\[ \| G \|_{L^p(\mathbb{R}^{n-1})} \leq C_\varepsilon K^\varepsilon \left( \sum_{\tau \in \mathcal{F}_{n-1}(K;2,m,0,n-2)} \| G_\tau \|_{L^p(\mathbb{R}^2)}^2 \right)^{1/2}, \]
i.e.,
\[ \| F(\cdot, x_{n-1}, \cdot) \|_{L^p(\mathbb{R}^{n-1})} \leq C_\varepsilon K^\varepsilon \left( \sum_{\tau \in \mathcal{F}_{n-1}(K;2,m,0,n-2)} \| F_\tau(\cdot, x_{2}, \cdot) \|_{L^p(\mathbb{R}^{n-1})}^2 \right)^{1/2}, \]
where $F_\tau(x) := \varphi_{K^{-1}} \mathcal{E}_{\tau} g$. Integrating on both sides of the above inequality with respect to $x_{n-1}$-variable from $-\infty$ to $\infty$, we derive
\[ \| F \|_{L^p(\mathbb{R}^n)} \leq C_\varepsilon K^\varepsilon \left( \sum_{\tau \subset \Omega_b} \| F_\tau \|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2}. \]

Thus, we have
\[ \| \mathcal{E}_{\Omega_b} g \|_{L^p(B_K)} \lesssim \| F \|_{L^p(\mathbb{R}^n)} \leq C_\varepsilon K^\varepsilon \left( \sum_{\tau \subset \Omega_b} \| F_\tau \|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2} \leq C_\varepsilon K^\varepsilon \left( \sum_{\tau \subset \Omega_b} \| \mathcal{E}_\tau g \|_{L^p(w_{B_K})}^2 \right)^{1/2}. \]

It follows that
\[ \| \mathcal{E}_{\Omega_b} g \|_{L^p(B_K)} \leq C_\varepsilon K^\varepsilon \left( \sum_{\tau \subset \Omega_b} \| \mathcal{E}_\tau g \|_{L^p(\mathbb{R}^n)}^2 \right)^{1/2} \]

Summing over all the balls $B_K \subset B_R$, we get the inequality (3.13), as required.
To do this, we need verify that for \( \theta \) out loss of generality, we may assume that

\[
\text{where } \tilde{\theta} \in \mathcal{F}_n(R; 2, m; s, n - 1 - s) \text{ for } 1 \leq s \leq n - 2.
\]

**Lemma 3.6.** Let \( 1 \leq s \leq n - 2 \). For \( 2 \leq p \leq \frac{2(n+1)}{n-1} \) and any \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon \) such that

\[
\| \hat{\mathcal{E}}_{[0,1]^{n-1}} f \|_{L^p(B_R)} \leq C_\varepsilon R^\varepsilon \left( \sum_{\theta \in \mathcal{F}_n(R; 2, m; s, n - 1 - s)} \| \hat{\mathcal{E}}_\theta f \|_{L^p(w_{BR})}^2 \right)^{1/2},
\]

where \( \hat{\mathcal{E}}_{[0,1]^{n-1}} \) denotes the Fourier extension operator associated with the hypersurface

\[
\{(\xi_1, \cdots, \xi_{n-1}, \phi_1(\xi_1) + \cdots + \phi_s(\xi_s) + \xi_{n+1}^m + \cdots + \xi_{n-1}^m) : (\xi_1, \cdots, \xi_{n-1}) \in [0,1]^{n-1}\}
\]

with \( \phi_j \) being non-degenerate for \( 1 \leq j \leq s \).

Note that \( n - 1 - s \leq n - 2 \) when \( 1 \leq s \leq n - 2 \). Lemma 3.6 holds because of our induction hypothesis on \( n - 1 - s \).

For each \( \tau \in \Omega_b \), we claim that

\[
\| \mathcal{E}_\tau g \|_{L^p(B_R)} \leq C_\varepsilon \left( \sum_{\theta \in \tau} \| \mathcal{E}_\theta g \|_{L^p(w_{BR})}^2 \right)^{1/2}, \quad 2 \leq p \leq \frac{2(n+1)}{n-1},
\]

where \( \theta \in \mathcal{F}_n(R; 2, m; 0, n - 1) \). We take the change of variables associated with \( \tau \) and denote the image of \( B_R \) by \( \mathcal{L}_\lambda(B_R) \). We rewrite \( \mathcal{L}_\lambda(B_R) \) into a finitely overlapping union of balls as follows

\[
\mathcal{L}_\lambda(B_R) = \bigcup B_{R/K}.
\]

Now, we employ Lemma 3.6 on each \( B_{R/K} \) to estimate the

\[
\| \hat{\mathcal{E}}_{[0,1]^{n-1}} \tilde{g} \|_{L^p(B_{R/K})}.
\]

To do this, we need verify that for \( \theta \in \tau \), its image \( \tilde{\theta} \) belongs to the decomposition \( \mathcal{F}_n(\tilde{K}; 2, m; s, n - 1 - s) \) under the change of variables associated with \( \tau \). Without out loss of generality, we may assume that

\[
\theta = \prod_{j=1}^{s} [\lambda_j, \lambda_j + \lambda_j^m R^{-1/2}] \times \prod_{k=s+1}^{s+t} [\sigma_k, \sigma_k + \sigma_k^m R^{-1/2}] \times [0, R^{-1/m}]^{n-1-s-t}.
\]

Under the change of variables associated with \( \tau \), we see that

\[
\tilde{\theta} = [0, \tilde{K}; 0, R^{-1/2}] \times \prod_{k=s+1}^{s+t} [\tilde{\sigma}_k, \tilde{\sigma}_k + \tilde{\sigma}_k^m (\tilde{K}/R)^{-1/2}] \times [0, \tilde{K}^{-1/m}]^{n-1-s-t}
\]

where \( \tilde{\sigma}_k := K^{1/m} \sigma_k \) is also a dyadic number.

By Lemma 3.6, we obtain

\[
\| \hat{\mathcal{E}}_{[0,1]^{n-1}} \tilde{g} \|_{L^p(B_{R/K})} \leq C_\varepsilon \left( \sum_{\tilde{\theta} \in \mathcal{F}_n(\tilde{K}; 2, m; s, n - 1 - s)} \| \hat{\mathcal{E}}_{\tilde{\theta}} \tilde{g} \|_{L^p(w_{BR/K})}^2 \right)^{1/2}.
\]
Taking the inverse change of variables, we deduce that
\[ R \text{ curvature in } \ell_0 \text{ decoupling inequalities for general smooth hypersurfaces with vanishing Gaussian } \]
This concludes the proof of Proposition 2.1.

We shall prove an \( L^p \) maximal estimates in \( \ell_n \) to higher dimensions. We shall prove an\( L^p \) maximal estimates in \( \ell_n \) to higher dimensions. We shall prove an

Iterating the above inequality
\[ \sum \text{over all the balls } B_{R/K} \subset \ell_n \text{ and using Minkowski's inequality, we have } \]
\[ \| \tilde{\mathcal{E}}' \|_{ \ell_0 (\ell_n)} \leq C_\varepsilon (\frac{R}{K})^\varepsilon \left( \sum_{\theta \in \mathcal{F}_n (\frac{R}{K}, \theta)} \| \tilde{\mathcal{E}}' \|_{ \ell_0 (\ell_n)}^2 \right)^{1/2}. \]

Taking the inverse change of variables, we deduce that
\[ \| \mathcal{E}_\tau g \|_{ \ell_0 (\ell_n)} \leq C_\varepsilon (\frac{R}{K})^\varepsilon \left( \sum_{\theta \subset \tau} \| \tilde{\mathcal{E}}' \|_{ \ell_0 (\ell_n)}^2 \right)^{1/2}. \]

Thus, we prove the claim (3.16).

Plugging (3.16) into (3.13), one has
\[ \| \tilde{\mathcal{E}}_{\Omega_\theta} g \|_{ \ell_0 (\ell_n)} \leq C_\varepsilon (\frac{R}{K})^\varepsilon \left( \sum_{\tau \subset \Omega_\theta} \| \tilde{\mathcal{E}}' \|_{ \ell_0 (\ell_n)}^2 \right)^{1/2} \]
\[ = C_\varepsilon R^\varepsilon \left( \sum_{\theta \subset \Omega_\theta} \| \tilde{\mathcal{E}}' \|_{ \ell_0 (\ell_n)}^2 \right)^{1/2}, \]

where \( \theta \in \mathcal{F}_n (R; 2; m; 0, n - 1) \).

Thus, we obtain the decoupling inequality for \( \Omega_\theta \)-part. Therefore, we have
\[ \| \tilde{\mathcal{E}}_{[0,1]^{n-1}} f \|_{ \ell_0 (\ell_n)} \leq \left( C(\varepsilon) K^{O(1)} R^\varepsilon + D_p (\frac{R}{K}) + \sum_{b \neq (1, \ldots, 1)} C_\varepsilon R^\varepsilon \right) \left( \sum_{\theta \subset \mathcal{F}_n (R; 2; m; 0, n - 1)} \| \tilde{\mathcal{E}}' f \|_{ \ell_0 (\ell_n)}^2 \right)^{1/2}. \]

This inequality together with the definition of \( D_p (R) \) yields
\[ D_p (R) \leq C(\varepsilon) K^{O(1)} R^\varepsilon + D_p (\frac{R}{K}) + 2\varepsilon C_\varepsilon R^\varepsilon. \]

Iterating the above inequality \( m = \lfloor \log_K R \rfloor \) times, we derive that \( D_p (R) \lesssim \varepsilon R^\varepsilon \).

This concludes the proof of Proposition 2.1

**Remark 3.7.** We don’t know how to apply the arguments in this paper to the more general phase functions studied in [16, 15, 17]. It is still open to prove sharp decoupling inequalities for general smooth hypersurfaces with vanishing Gaussian curvature in \( \mathbb{R}^n \) for \( n \geq 4 \).

### 4. Applications on Schrödinger maximal estimates

In this section, we will apply Proposition 2.1 to generalized the Schrödinger maximal estimates in [19] to higher dimensions. We shall prove an \( L^2 \) fractal restriction estimate associated with the surfaces
\[ F_m (0, n) := \{ (\xi_1, \ldots, \xi_n, \phi(\xi_1, \ldots, \xi_n)) : (\xi_1, \ldots, \xi_n) \in [0, 1]^n \}. \]

Let \( m \geq 4 \) be an even number. The solution to the following generalized Schrödinger equation
\[
\begin{cases}
  i \partial_t u - (\partial_1^m u + \ldots + \partial_n^m u) = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\
  u(x, 0) = f(x)
\end{cases}
\]
is given by
\[ e^{it\phi(D)} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi_1 x_1 + \ldots + x_n \xi_n + t\phi(\xi)} d\xi, \]
where \( \phi(\xi_1, \ldots, \xi_n) := \xi_1^m + \ldots + \xi_n^m \).
For the study of necessary conditions, we refer to [11], Z. Li, J. Zhao and T. Zhao [19] give a sufficient condition for the almost everywhere convergence problem associated with this degenerate phase in \( \mathbb{R}^2 \) via the methods in [11]. We will establish the following result.

**Theorem 4.1.** For every \( f \in H^s(\mathbb{R}^n) \) with \( s > s(n, m) := \frac{2n^2+2n+2}{2(n+1)(n+2)} - \frac{n^2-n+2}{m(n+1)(n+2)} \),
\[
\lim_{\varepsilon \to 0} e^{it\phi(D)} f(x) = f(x) \text{ almost everywhere.}
\]

We use \( B_d(x, r) \) to denote a ball centered at \( x \) with radius \( r \) in \( \mathbb{R}^d \). By a standard approximation argument, Theorem 4.1 is a consequence of the following Schrödinger maximal estimate associated with \( \phi \):

**Theorem 4.2.** For any \( s > s(n, m) \), there holds
\[
\| \sup_{0 < t \leq 1} |e^{it\phi(D)} f| \|_{L^2(B^n(0,1))} \leq C_s \| f \|_{H^s(\mathbb{R}^n)}
\]
for any function \( f \in H^s(\mathbb{R}^n) \).

By Lemma 2.1 from [10], Littlewood-Paley decomposition and rescaling, Theorem 4.2 is reduced to the following result.

**Theorem 4.3.** For any \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon \) such that
\[
\| \sup_{0 < t \leq R} |e^{it\phi(D)} f| \|_{L^2(B^2(0,R))} \leq C_\varepsilon R^{\varepsilon(n,m)+\varepsilon} \| f \|_2
\]
holds for all \( R \geq 1 \) and all \( f \) with \( \text{supp} \hat{f} \subset A(1) := [0,1]^n - [0,\frac{1}{2}]^n \).

Cubes of the form \( m + [0, M]^{n+1} \) with \( m \in (M\mathbb{Z})^{n+1} \) are called lattice \( M \)-cubes. We will prove the following \( L^2 \) restriction estimate as the main result, from which Theorem 4.3 follows.

**Theorem 4.4.** Suppose that \( X = \bigcup_k B_k \) is a union of lattice unit cubes in \( Q_R := [0,R]^{n+1} \) and each \( R^s \)-cube intersecting \( X \) contains \( \sim \nu \) many unit cubes in \( X \). Let \( \lambda \) be given by
\[
\lambda := \max_{B^{n+1}(x', r) \subset Q_R, x' \in \mathbb{R}^{n+1}, r \geq 1} \frac{\# \{ B_k : B_k \subset B^{n+1}(x', r) \}}{r^n}, \tag{4.1}
\]
For any \( \varepsilon > 0 \), there exists a positive constant \( C_\varepsilon \) such that the following holds for all \( R \geq 1 \)
\[
\| e^{it\phi(D)} f \|_{L^2(X)} \leq C_\varepsilon \lambda^{\frac{\varepsilon}{\nu+1}} R^{\frac{n^2+2n+2}{n(n+1)} + \frac{n^2-n+2}{m(n+1)(n+2)} + \varepsilon} \| f \|_2 \tag{4.2}
\]
for all \( f \) with \( \text{supp} \hat{f} \subset A(1) \).

Clearly, one has \( \nu \leq \lambda \lambda^{n/2} \) in Theorem 4.4. As a direct result of Theorem 4.4, there holds a weaker \( L^2 \) restriction estimate:

**Corollary 4.5.** Suppose that \( X = \bigcup_k B_k \) is a union of lattice unit cubes in \( Q_R \). Let \( \lambda \) be given by
\[
\lambda := \max_{B^{n+1}(x', r) \subset Q_R, x' \in \mathbb{R}^3, r \geq 1} \frac{\# \{ B_k : B_k \subset B^{n+1}(x', r) \}}{r^n}, \tag{4.3}
\]
For any \( \varepsilon > 0 \), there exists a positive constant \( C_\varepsilon \) such that the following hold for all \( R \geq 1 \)
\[
\| e^{it\phi(D)} f \|_{L^2(X)} \leq C_\varepsilon \lambda^{\frac{\varepsilon}{\nu+1}} R^{\frac{n^2+2n+2}{n(n+1)} + \frac{n^2-n+2}{m(n+1)(n+2)} + \varepsilon} \| f \|_2
\]
for all \( f \) with \( \text{supp} \hat{f} \subset A(1) \).

Our main inductive proposition is the following:

**Proposition 4.6.** Let \( p = \frac{2(n+1)}{n-1} \). Suppose that \( \phi \) is the corresponding phase function to the hypersurface \( F_{n,m}^{-1}(s,n-1-s), 0 \leq s \leq n-1 \), \( Y = \bigcup_{k=1}^{N} B_k \) is a union of lattice \( K \)-cubes in \( Q_R \) and each \( R^k \)-cube intersecting \( Y \) contains \( \sim \nu \) many \( K \)-cubes in \( Y \), where \( K = R^k \). Suppose that \( \| e^{it\phi(D)} f \|_{\mathcal{L}^p(B_k)} \) is essentially constant in \( k = 1, 2, ..., N \). Given \( \lambda \) by

\[
\lambda := \max_{B^{n+1}(x',r) \subset Q_n, x' \in \mathbb{R}^{n+1}, r \geq K} \mathcal{Z}\{B_k : B_k \subset B^{n+1}(x',r)\}. \tag{4.4}
\]

For any \( \varepsilon > 0 \), there exists a positive constant \( C_\varepsilon \) and \( \delta = \varepsilon^{100} \) such that the following hold for all \( R \geq 1 \)

\[
\| e^{it\phi(D)} f \|_{\mathcal{L}^p(Y)} \leq C_\varepsilon N^{-\frac{1}{2}} \left( \frac{2^{n+1}}{n+1} \right)^{\frac{2}{n} \lambda} \cdot \delta^{\frac{n^2 + 2}{2(n+1)(n+2)}} R^k(n,m,\alpha) + \varepsilon \| f \|_2
\]

for all \( f \) with \( \text{supp} \hat{f} \subset A(1) \), where

\[
h(n,m,\alpha) := \frac{n^2 + n + 2}{2(n+1)(n+2)} - \frac{n^2 + n + 2 - 2\alpha}{m(n+1)(n+2)}
\]

and the above estimate is uniform for the hypersurfaces in the family \( \mathcal{F}^{n-1} \).

Here a collection of quantities are said to be essentially constant provided that all the quantities lie in the same interval of the form \([r^k, 2^{k+1}]\), where \( n \in \mathbb{Z} \).

We are going to prove Proposition 4.6 by induction on the dimension, the radius and the number \( n-t \), where \( t \) denotes the number of non-degenerate terms in the phase functions as in the previous sections.

Recall that \([0,1]^n\) is divided into \( \bigcup_b \Omega_b \). For technical reasons, \( K^{-\frac{1}{m}}, R^{-\frac{1}{m}} \) and \( (\frac{2}{3})^{-1/m} \) should be dyadic numbers. Therefore, we choose \( K = 2^mt \) and \( R = 2^n \) for \( l,t \in \mathbb{N} \) to be large numbers satisfying \( K \approx R^k \) with \( \delta \) as in Proposition 4.6. By Cauchy-Schwartz inequality, we have

\[
\| e^{it\phi(D)} f \|_{\mathcal{L}^p(Y)} \leq \left( \sum_b \| e^{it\phi(D)} f_{\Omega_b} \|_{\mathcal{L}^p(Y)}^2 \right)^{1/2}.
\]

We will estimate \( \| e^{it\phi(D)} f \|_{\mathcal{L}^p(Y)} \) in several cases. Roughly speaking, if the contribution from the region \( \Omega_b \) dominates, we call it \( \Omega_b \) case. More precisely, given a \( K \)-cube \( B \), we write \( \| e^{it\phi(D)} f \|_{\mathcal{L}^p(B)} \approx c_b \| e^{it\phi(D)} f_{\Omega_b} \|_{\mathcal{L}^p(B)} \) \( \approx c_b \). By the triangle inequality, we have \( c_b \geq 1 \) for at least one of the \( b \)'s.

We sort the \( K \)-cubes in \( Y \) as follows: Denote \( \{B \subset Y : c_b \geq \frac{1}{4} \} \) \( Y^b \). Clearly, one has \( Y = \bigcup_b Y^b \). If \( \mathcal{Z}\{B : B \subset Y^b\} \geq \frac{N}{2} \), we call it \( \Omega_b \)-case. When \( b = (1, ..., 1) \), we abbreviate \( \Omega_{(1, ..., 1)} \) by \( \Omega_0 \).

For the \( \Omega_0 \)-case, we sort the \( K \)-cubes in \( Y^0 \) as follows: (1) For any dyadic number \( A^{(0)} \), let \( Y^{(0)}_{A^{(0)}} \) be the union of the \( K \)-cubes in \( Y^{(0)} \) satisfying

\[
\| e^{it\phi(D)} f_{\Omega_{(0)}} \|_{\mathcal{L}^p(B)} \approx A^{(0)}.
\]

(2) Fix \( A^{(0)} \), for any dyadic number \( \nu^{(0)} \), let \( Y^{(0)}_{A^{(0)},\nu^{(0)}} \) be the union of the \( K \)-cubes in \( Y^{(0)}_{A^{(0)}} \) such that for each \( B \subset Y^{(0)}_{A^{(0)},\nu^{(0)}} \), the \( R^{1/2} \)-cube intersecting \( B \) contains \( \approx \nu^{(0)} \) cubes from \( Y^{(0)}_{A^{(0)}} \).
Without loss of generality, we may assume $\|f\|_2 = 1$. Then the dyadic numbers $A^{(0)}$, $\nu^{(0)}$ making significant contributions can be assumed to be between $R^{-C}$ and $R^C$. Therefore, there exist some dyadic numbers $A^{(0)}$, $\nu^{(0)}$ such that $\sharp\{B : B \subset Y^{(0)}_{A^{(0)},\nu^{(0)}}\} =: N^{(0)} > \frac{\lambda}{(\log R)^2}$. Fix a choice of $A^{(0)}$, $\nu^{(0)}$ and denote $Y^0_{A^{(0)},\nu^{(0)}}$ by $Y^0$ for convenience. Then, in the $\Omega_0$-case, we have
\[ \|e^{it\phi(D)}f\|_{L^p(Y)} \lesssim_{\epsilon} R^\epsilon \|e^{it\phi(D)}f\|_{L^p(Y^0)}, \]
and
\[ \frac{N}{(\log R)^2} \lesssim N^{(0)} \leq N, \nu^{(0)} \leq \nu, \lambda^{(0)} \leq \lambda, \]
where
\[ \lambda^{(0)} := \max_{B^{n+1}(x',r) \subset Q_R, x' \in \mathbb{R}^{n+1}, r \geq 1} \sharp\{B_k \subset Y^0 : B_k \subset B^{n+1}(x',r)\}. \]

Using the arguments from [11] as well as the rescaling techniques from [18, 20], we have
\[ \|e^{it\phi(D)}f\|_{L^p(Y^0)} \leq C_\epsilon K^{O(1)} \left( \frac{\lambda^{(0)}}{\nu^{(0)}} \right)^{\frac{2}{n+2}} \left( \lambda^{(0)} \right)^{\frac{m}{n+2}} \left( \nu^{(0)} \right)^{\frac{n}{n+2}} R^{h(n,2,\alpha)+\epsilon} \|f\|_2. \] (4.5)

Note that $h(n,2,\alpha) \leq h(n,m,\alpha)$ for all $(0,n+1)$ and $\forall m \geq 2$. The factor $K^{O(1)}$ appears on the right-hand side of the last inequality because the principle curvatures of the hypersurface
\[ \Sigma_0 := \{(\xi_1, \xi_2, \phi(\xi_1, \xi_2)) : (\xi_1, \xi_2) \in \Omega_0\} \]
have lower bounds $K^{-C}$.

When $b \neq (1, \ldots, 1)$ and $b \neq (0, \ldots, 0)$, we abbreviate $\Omega_0$ by $\Omega_1$. For the $\Omega_1$-case, we will employ the following lemma. Let
\[ \psi(\xi_1, \ldots, \xi_n) := \phi_1(\xi_1) + \ldots + \phi_t(\xi_t) + \xi_{t+1}^m + \ldots + \xi_n^m, \quad 1 \leq t \leq n - 1 \]
with $\phi_k$, $1 \leq k \leq t$ being non-degenerate smooth functions as defined in Section 2.

**Lemma 4.7.** For any $0 < \epsilon < \frac{1}{100}$, there exist constants $C_\epsilon$ and $\delta = \epsilon^{100}$ such that the following holds for all $R \geq 1$ and all $f$ with $\text{supp}\hat{f} \subset [0,1]^n$. Suppose that $Y = \bigcup_{k=1}^N B_k$ is a union of lattice $K$-cubes in $Q_R$ and each $R^\delta$-cube intersecting $Y$ contains $\sim \nu$ many $K$-cubes in $Y$, where $K = R^\delta$. Suppose that $\|e^{it\hat{\psi}(D)}f\|_{L^p(B_k)}$ is essentially constant in $k = 1, 2, \ldots, N$. Given $\lambda$ by
\[ \lambda := \max_{B^{n+1}(x',r) \subset Q_R, x' \in \mathbb{R}^{n+1}, r \geq K} \sharp\{B_k : B_k \subset B^{n+1}(x',r)\}. \] (4.6)

Then
\[ \|e^{it\hat{\psi}(D)}f\|_{L^p(Y)} \leq C_\epsilon N^{-\frac{1}{\lambda^{1/2}}} \lambda^{\frac{2}{n+2}} \nu^{\frac{2}{n+2}} R^{h(n,m,\alpha)+\epsilon} \|f\|_2. \]

Note that $n - t < n$ when $t \geq 1$. Lemma 4.7 holds because of our hypothesis on the number $n - t$ we estimate $\|e^{it\phi(D)}f\|_{L^p(Y^1)}$ as follows.

Firstly, we divide the region $\Omega_1$ into a family of subregions
\[ \Omega_1 = \bigcup_{\sigma_2, \ldots, \sigma_t} \Omega_{\sigma_2, \ldots, \sigma_t}, \]
where
\[ \Omega_{\sigma_2,\ldots,\sigma_t} := \left[ \frac{1}{2}, 1 \right] \times \prod_{k=2}^{t} I_k \times \left[ 0, K^{-\frac{1}{m}} \right]^{n-t} \]
with \( I_k = [\sigma_k, 2\sigma_k] \). Here \( \sigma_k \) is a dyadic number in \( [K^{-\frac{1}{m}}, \frac{1}{2}] \). We divide each \( I_k \) further into
\[ I_k = \bigcup_{j=1}^{\sigma_k^{m\frac{m-2}{m}}} K^{1/2} I_{k,j}. \]
Each \( I_{k,j} \) has length of \( \sigma_k^{m-2} K^{-1/2} \). We abbreviate \( \Omega_{\sigma_2,\ldots,\sigma_t} \) by \( \Omega_{\sigma} \), and write
\[ \Omega_{\sigma} = \bigcup \tau, \]
where each \( \tau \) in the above union has the form
\[ [a, a + K^{-1/2}] \times \prod_{k=2}^{t} I_{k,k_j} \times \left[ 0, K^{-\frac{1}{m}} \right]^{n-t}. \]

Therefore, we have
\[ e^{it\phi(D)} f_{\Omega_1} = \sum_{\sigma} e^{it\phi(D)} f_{\Omega_{\sigma}} \]
and
\[ e^{it\phi(D)} f_{\Omega_{\sigma}} = \sum_{\tau \subset \Omega_{\sigma}} e^{it\phi(D)} f_{\tau}. \]

We will estimate \( \|e^{it\phi(D)}f\|_{L^p(Y_1)} \) in several cases. Roughly speaking, if the contribution from the region \( \Omega_{\sigma} \) dominates, we call it \( \Omega_{\sigma} \) case. More precisely, given a \( K \)-cube \( B \), we write \( \|e^{it\phi(D)}f_{\Omega_{\sigma}}\|_{L^p(B)} \approx c_{\sigma}, \|e^{it\phi(D)}f_{\Omega_{\sigma}}\|_{L^p(B)} \approx c_{\sigma}. \) By the triangle inequality, we have \( c_{\sigma} \geq \frac{c_{\lambda}}{\log K} \) for at least one of the \( \sigma \)'s.

We sort the \( K \)-cubes in \( Y^1 \) as follows: Denote \( \{B \subset Y^1 : c_{\sigma} \geq \frac{c_{\lambda}}{\log K}\} \) by \( Y^\sigma \). Clearly, one has \( Y = \bigcup_{\sigma} Y^\sigma \). If \( \sharp\{B : B \subset Y^\sigma\} \geq \frac{N}{\log K (\log R) \eta} \), we call it \( \Omega_{\sigma} \)-case.

For the \( \Omega_{\sigma} \)-case, we sort the \( K \)-cubes in \( Y^\sigma \) as follows:

(1) For any dyadic number \( A^{(\sigma)} \), let \( Y^\sigma_{A^{(\sigma)}} \) be the union of the \( K \)-cubes in \( Y^\sigma \) satisfying
\[ \|e^{it\phi(D)}f_{\Omega_{\sigma}}\|_{L^p(B)} \approx A^{(\sigma)}. \]

(2) Fix \( A^{(\sigma)} \), for any dyadic numbers \( \nu^{(\sigma)} \), let \( Y^\sigma_{A^{(\sigma)}, \nu^{(\sigma)}} \) be the union of the \( K \)-cubes in \( Y^\sigma_{A^{(\sigma)}} \) such that for each \( B \subset Y^\sigma_{A^{(\sigma)}, \nu^{(\sigma)}} \), the \( R^{1/2} \)-cube intersecting \( B \) contains \( \approx \nu^{(\sigma)} \) cubes from \( Y^\sigma_{A^{(\sigma)}} \).

The dyadic numbers \( A^{(\sigma)}, \nu^{(\sigma)} \) making significant contributions can be assumed to be between \( R^{-C} \) and \( R^C \). Therefore, there exist some dyadic numbers \( A^{(\sigma)}, \nu^{(\sigma)} \) such that \( \sharp\{B : B \subset Y^\sigma_{A^{(\sigma)}, \nu^{(\sigma)}}\} = N^{(\sigma)} \geq \frac{N}{\log K (\log R) \eta} \), many cubes \( B \). Fix a choice of \( A^{(\sigma)}, \nu^{(\sigma)} \) and denote \( Y^\sigma_{A^{(\sigma)}, \nu^{(\sigma)}} \) by \( Y^\sigma \) for convenience. Then, in the \( \Omega_{\sigma} \)-case we have
\[ \|e^{it\phi(D)}f\|_{L^p(Y)} \lesssim_{\epsilon} R^C \|e^{it\phi(D)}f_{\Omega_{\sigma}}\|_{L^p(Y^\sigma)}, \]
and
\[ \frac{N}{\log K (\log R) \eta} \lesssim N^{(\sigma)} \leq N, \nu^{(\sigma)} \leq \nu, \lambda^{(\sigma)} \leq \lambda, \]
where
\[
\lambda(\sigma) := \max_{B^{n+1}(x',r) \subset Q_{\rho}, x' \in \mathbb{R}^{n+1}, r \geq K} \frac{\sharp\{B_k \subset Y^{\sigma} : B_k \subset B^{n+1}(x',r)\}}{r^{\omega}}.
\]

We are going to estimate \(\|e^{it\phi(D)}f_{\Omega_{\sigma}}\|_{L^p(Y^{\sigma})}\). Recall that we have partitioned \(\Omega_{\sigma}\) into disjoint rectangles \(\tau\) of dimensions \(K^{-1/2} \times \sigma_k^{-m/2} K^{-1/2} \times K^{-1/2}\) and write \(f_{\Omega_{\sigma}} = \sum_{\tau} f_{\tau}\).

For each \(K\)-cube \(B\), we employ the following decoupling inequality, which has root in the original paper of Bourgain and Demeter [5].

**Proposition 4.8.** Suppose that \(B\) is a \(K\)-cube in \(\mathbb{R}^{n+1}\). Then for any \(\varepsilon > 0\), there exists a constant \(C_{\varepsilon}\) such that
\[
\|e^{it\phi(D)}f_{\Omega_{\sigma}}\|_{L^p(B)} \leq C_{\varepsilon} K^n \left( \sum_{\tau} \|e^{it\phi(D)}f_{\tau}\|^2_{L^p(\omega_B)} \right)^{1/2},
\]
where \(\omega_B\) is a weight function essentially supported on \(B\).

For the proof of Proposition 4.8 we refer to [22].

For each \(\tau\), we will handle \(e^{it\phi(D)}f_{\tau}\) by rescaling and induction on scales. To do it, we further decompose \(f_{\Omega_{\sigma}}\) in physical space and perform dyadic pigeonholing several times.

Firstly, we divide the physical square \([0, R]^n\) into \(\frac{R}{K^{1/2}} \times \frac{R}{\sigma_k^{-m/2} K^{1/2}} \times \frac{R}{K^{1-1/2}}\) rectangles \(D\). For each pair \((\tau, D)\), let \(f_{\square, \tau, D}\) be the function formed by cutting off \(f\) on the rectangle \(D\) (with a Schwartz tail) in physical space and the rectangle \(\tau\) in Fourier space. Note that \(e^{it\phi(D)}f_{\square, \tau, D}\) is essentially supported on an \(\frac{R}{K^{1/2}} \times \frac{R}{\sigma_k^{-m/2} K^{1/2}} \times \frac{R}{K^{1-1/2}}\) box, which is denoted by \(\square, \tau, D\). The box \(\square, \tau, D\) is parallel to the normal direction of the surface \(\Sigma_{\tau}\) at the left bottom corner of \(\tau\). For a fixed \(\tau\), the different boxes \(\square, \tau, D\) tile \(Q_{\rho}\). In particular, for each \(\tau\), a given \(K\)-cube \(B\) lies in exactly one box \(\square, \tau, D\). Hence, we have
\[
f_{\Omega_{\sigma}} = \sum_{\tau} \sum_{D} f_{\square, \tau, D},
\]
and write \(f_{\Omega_{\sigma}} = \sum_{\square} f_{\square}\) for abbreviation.

By Proposition 4.8, for each \(K\)-cube \(B\), there holds
\[
\|e^{it\phi(D)}f_{\Omega_{\sigma}}\|_{L^p(B)} \leq C_{\varepsilon} K^n \left( \sum_{\square} \|e^{it\phi(D)}f_{\square}\|^2_{L^p(\omega_B)} \right)^{1/2}. \tag{4.8}
\]

We denote by
\[
R_1 := \frac{R}{K} = R^{1-\delta}, K_1 = R_1^{\delta}, B_1 = R_1^{\delta - 2}.
\]
Tile \(\square\) by \(\rho^{-m/2} K^{1/2} K_1 \times K^{1/2} K_1 \times K_1\)-tube \(S\), and also tile \(\square\) by \(\frac{R^{1/2}}{\rho^{1/2}} \times \frac{R^{1/2}}{K^{1/2}} \times K^{1/2} R_1^{1/2}\)-tubes \(S'\) (all running parallel to long axis of \(\square\)). After rescaling the \(\square\) becomes an \(R_1\)-cube, the tubes \(S'\) and \(S\) become lattice \(R_1^{1/2}\)-cubes and \(K_1\)-cubes, respectively.

We regroup the tubes \(S\) and \(S'\) inside each \(\square\) as follows:
1) Sort those tubes \(S\) which intersect \(Y^{\sigma}\) according to the value \(\|e^{it\phi(D)}f_{\square}\|_{L^p(S)}\) and the number of \(K\)-cubes contained in it. For dyadic numbers \(\eta, \beta_1\), we use
S[□,η,β] to stand for the collection of tubes \( S \subset □ \) each of which containing \( \sim \eta \) K-cubes and \( ∥e^{i\theta(D)}f□∥_{L^p(S)} \sim β_1 \).

2) For fixed \( η, β_1 \), we sort the tubes \( S' \subset □ \) according to the number of the tubes \( S \in S[□,η,β_1] \) contained in it. For dyadic number \( ν_\sigma \), let \( S[□,η,β_1,ν_σ] \) be the sub-collection of \( S[□,η,β_1] \) such that for each \( S \in S[□,η,β_1,ν_σ] \), the tube \( S' \) containing \( \sim \nu_\rho \) tubes from \( S[□,η,β_1] \).

3) For fixed \( η, β_1, ν_σ \), we sort the boxes \( □ \) according to the value \( □f□_2 \), the number \( □S[□,η,β_1,ν_σ] \) and the value \( λ_σ \), defined below. For dyadic numbers \( β_2, N_σ, λ_σ \), let \( B[η,β_1,ν_σ,β_2,N_σ,λ_σ] \) denote the collection of boxes \( □ \) each of which satisfying that

\[
\|f□\|_2 \sim β_2, \|S[□,η,β_1,ν_σ]\|_N_σ \sim N_σ
\]

and

\[
\max_{T_r \subset □, r ≥ K_f} \|\{S \in S[□,η,β_1,ν_σ]: S \subset T_r\}\|_r^α \sim λ_σ,
\]

where \( T_r \) are \( K^{1/2}r \times K^{1/2}2^{−m_2}r \times K^{1/2}r \times K\)-tubes in \( □ \) running parallel to the long axis of \( □ \).

Let \( Y[□,η,β_1,ν_σ] \) denote \( \{S: S \subset S[□,η,β_1,ν_σ]\} \) and \( χ_Y[□,η,β_1,ν_σ] \) be the corresponding characteristic function. Thus, there are only \( O(\log R) \) significant choices for each dyadic number. By pigeonholing, we can choose \( η, β_1, ν_σ, β_2, N_σ, λ_σ \) so that

\[
\|e^{i\theta(D)}fΩ_σ\|_{L^p(B)} \lesssim (\log R)^6 K^{α/2} \left( \sum_{□ \in B[η,β_1,ν_σ,β_2,N_σ,λ_σ], B \subset Y[□,η,β_1,ν_σ]} \|e^{i\theta(D)}f□\|_{L^{p}(ω_B)}^2 \right)^{1/2}
\]

(4.10)

holds for a fraction \( \gtrsim (\log R)^{-6} \) of all \( K \)-cubes \( B \).

For brevity, we denote by

\[
Y[□] := Y[□,η,β_1,ν_σ], B := B[η,β_1,ν_σ,β_2,N_σ,λ_σ].
\]

Finally, we sort the \( K \)-cubes \( B \) satisfying (4.10) by \( □ \{□ \in B : B \subset Y[□]\} \). Let \( Y' \subset Y[□] \) be a union of \( K \)-cubes \( B \) each of which obeying

\[
\|e^{i\theta(D)}fΩ_σ\|_{L^p(B)} \lesssim (\log R)^6 K^{α/2} \left( \sum_{□ \in B, B \subset Y'[□]} \|e^{i\theta(D)}f□\|_{L^{p}(ω_B)}^2 \right)^{1/2}
\]

(4.11)

and

\[
\{□ \in B : B \subset Y'[□]\} \sim μ
\]

(4.12)

for some dyadic number \( 1 ≤ μ ≤ R^C \). Moreover, the number of \( K \)-cubes \( B \) in \( Y' \) is \( \gtrsim (\log R)^{-7}N(σ) \).

By our assumption that \( ∥e^{i\theta(D)}fΩ_σ∥_{L^p(B)} \) is essentially constant in \( k = 1, 2, ..., N(σ) \), we have

\[
\|e^{i\theta(D)}fΩ_σ\|^p_{L^p(Y')} \lesssim (\log R)^C \left( \sum_{B \subset Y'} \|e^{i\theta(D)}fΩ_σ\|^p_{L^p(B)} \right).
\]

(4.13)

For each \( B \subset Y' \), it follows from (4.11), (4.12) and Hölder’s inequality that

\[
\|e^{i\theta(D)}fΩ_σ\|^p_{L^p(B)} \lesssim (\log R)^C K^{pα/2} μ^{2/p} \left( \sum_{□ \in B, B \subset Y'[□]} \|e^{i\theta(D)}f□\|^p_{L^{p}(ω_B)} \right)^{1/2}.
\]

(4.14)

Putting (4.13) and (4.14) together, one has

\[
\|e^{i\theta(D)}fΩ_σ\|_{L^p(Y')} \lesssim (\log R)^C K^{α/2} μ^{1−p/p} \left( \sum_{□ \in B} \|e^{i\theta(D)}f□\|^p_{L^p(Y'[□])} \right)^{1/p}
\]

(4.15)
Next, we apply rescaling to each \(\|e^{it\phi(D)}f\|_{L^p(Y)}\) and run induction on scales. For each \(K^{-1/2} \times \sigma_k^{-m/2} K^{-1/2} \times K^{-1/m}\)-rectangle \(\tau = \tau_k\) in \(\Omega_\sigma\), we write

\[
\begin{align*}
\xi_1 &= a + K^{-\frac{1}{2}}\eta_1, \\
\xi_k &= \sigma_k + \sigma_k^{-\frac{m}{2}} K^{-\frac{1}{2}}\eta_k, \quad 2 \leq j \leq t, \\
\xi_j &= K^{-\frac{1}{m}}\eta_j, \quad t + 1 \leq j \leq n
\end{align*}
\]

for some \(\frac{1}{t} \leq a \leq 1\). For brevity, we denote the corresponding change of variables in the \((x,t)\)-space by \((\tilde{x}, \tilde{t}) = \mathcal{L}(x,t)\). Therefore, one has

\[
\|e^{it\phi(D)}f\|_{L^p(Y)} = K^{1/p} \left( K^{t/2} K^{(n-t)/m} \right)^{\frac{1}{p} - \frac{1}{2}} \left( \prod_{k=2}^{t} \sigma_k \right)^{-\frac{m-2}{2}} \left( \prod_{j=t+1}^{n} \sigma_j \right)^{-\frac{1}{2}} \|e^{i\tilde{\phi}(D)}g(\tilde{x})\|_{L^p(\tilde{Y})},
\]

(4.16)

where \(\tilde{Y} = \mathcal{L}(Y)\). Hence, by (4.16) and applying Lemma 4.7 to the term

\[
\|e^{i\tilde{\phi}(D)}g(\tilde{x})\|_{L^p(\tilde{Y})}
\]

at scale \(R_1\), we deduce

\[
\|e^{it\phi(D)}f\|_{L^p(Y)} \lesssim K^{1/p} \left( K^{t/2} K^{(n-t)/m} \right)^{\frac{1}{p} - \frac{1}{2}} \left( \prod_{k=2}^{t} \sigma_k \right)^{-\frac{m-2}{2}} \left( \prod_{j=t+1}^{n} \sigma_j \right)^{-\frac{1}{2}} \times \lambda_\sigma^{m+1}\nu^{m+1} \nu^\sigma \|f\|_2.
\]

Consider the cardinality of the set \(\{ (\Box, B) : \Box \in \mathcal{B}, B \subseteq Y^\sigma \cap Y' \} \). By our choice of \(\mu\) as in (4.12), there is a lower bound

\[
\# \{ (\Box, B) : \Box \in \mathcal{B}, B \subseteq Y^\sigma \cap Y' \} \gtrsim (\log R)^{-7} N^\sigma \mu.
\]

On the other hand, by our choices of \(N_\sigma\) and \(\eta\), for each \(\Box \in \mathcal{B}, Y^\sigma\) contains \(\sim N_\sigma\) tubes \(S\) and each \(S\) contains \(\sim \eta\) cubes in \(Y^\sigma\), so

\[
\# \{ (\Box, B) : \Box \in \mathcal{B}, B \subseteq Y^\sigma \cap Y' \} \lesssim (\# \mathcal{B}) N_\sigma \eta.
\]

Therefore, we get

\[
\mu \lesssim \frac{(\log R)^7 N_\sigma \eta}{N^\sigma}.
\]

(4.17)

Next, by our choices of \(\lambda_\sigma\) as in (4.9) and \(\eta\),

\[
\begin{align*}
\lambda_\sigma \eta &\sim \max_{T_r \subset \Box, r \geq K_1} \frac{\# \{ S : S \subseteq Y^\sigma \cap T_r \}}{r^a} \cdot \# \{ B : B \subseteq S \cap Y^\sigma \} \\
&\lesssim \max_{T_r \subset \Box, r \geq K_1} \frac{\# \{ B : B \subseteq Y^\sigma \cap T_r \}}{r^a} \leq \left( \prod_{k=2}^{t} \sigma_k \right)^{\frac{m-2}{2}} K^{\frac{1}{2} - \frac{1}{2t}} \lambda^\sigma \left( K^{1 - \frac{1}{m}} r \right)^a.
\end{align*}
\]

Hence, we obtain

\[
\eta \lesssim \frac{\lambda^\sigma \left( \prod_{k=2}^{t} \sigma_k \right)^{\frac{m-2}{2}} K^{\frac{1}{2} - \frac{1}{2t}} \lambda^\sigma \left( K^{1 - \frac{1}{m}} r \right)^a}{\lambda_\sigma}.
\]

(4.18)

Finally, we relate \(\nu^\sigma\) and \(\nu^\sigma\) by considering the number of \(K\)-cubes in each relevant \(R^{1/2} \times \sigma_k^{-1/2} \times K^{1/2} \times K^{1/2} \)-tube \(S^\prime\). Recall that each relevant \(S^\prime\) contains \(\sim \nu^\sigma\) tubes \(S\) in \(Y\) and each such \(S\) contains \(\sim \eta\) \(K\)-cubes. On the other
hand, we can cover $S'$ by $\sim K^{1/2}$ finitely overlapping $R^\perp$-cubes and by assumption each $R^\perp$-cube contains $\ll \nu^\sigma$ many $K$-cubes in $Y^\sigma$. Thus, it follows that

$$\nu_\sigma \ll \frac{K^{\perp} \nu^\sigma}{\eta}. \quad (4.19)$$

By inserting (4.17), (4.18) and (4.19) into (4.15), we derive

$$\|e^{it\phi(D)} f_{\Omega_\sigma} \|_{L^p(Y^\sigma)} \ll K^{2^{\perp} - \varepsilon} \left( N^\sigma \right)^{-\frac{1}{\sigma+1} \left( \nu^\sigma \right) \left( \lambda^\sigma \right) \left( \frac{1}{\sigma+1} \left( \nu^\sigma \right) \left( \lambda^\sigma \right) \right)} R^{h(n,m,\alpha) + \varepsilon} \| f \|_2. \quad (4.20)$$

Since $K = R^\delta$ and $R$ can be assumed to be sufficiently large compared to any constant depending on $\varepsilon$, we have $K^{2^{\perp} - \varepsilon} \ll 1$ and the induction closes. Recall that $N \log K \left( \log R \right)^4 \ll N, \nu^\sigma \ll \nu$ and $\lambda^\sigma \ll \lambda$. This yields

$$\|e^{it\phi(D)} f_{\Omega_\sigma} \|_{L^p(Y^\sigma)} \ll N^{-\frac{1}{\sigma+1} \left( \nu^\sigma \right) \left( \lambda^\sigma \right) \left( \frac{1}{\sigma+1} \left( \nu^\sigma \right) \left( \lambda^\sigma \right) \right)} R^{h(n,m,\alpha) + \varepsilon} \| f \|_2.$$

This concludes the proof of Proposition 4.6.

Acknowledgements.

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