Structure of weighted Hardy spaces on finitely connected domains

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Abstract

We give a complete characterization of a certain class of Hardy type spaces on finitely connected planar domains. In particular, we provide a decomposition result and give a description of such functions through their boundary values. As an application, we describe an isomorphism from the weighted Hardy space onto the classical Hardy-Smirnov space. This allows us to identify the multiplier space of the mentioned Hardy type spaces as the space of bounded holomorphic functions on the domain.

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1. Introduction

Recently, E. A. Poletsky and M. Stessin in [9] introduced a scale of Hardy and Bergman type spaces which consists of holomorphic functions on the underlying hyperconvex domain in the n-dimensional complex Euclidean space \( \mathbb{C}^n \), where \( n \geq 1 \) is an integer. The class of hyperconvex domains contain a wide range of classical domains, in particular the \( n \)-dimensional unit ball and the polydisk in \( \mathbb{C}^n \) are hyperconvex. This way, the theory of Hardy and Bergman spaces in the classical settings have been unified and generalized in [9]. Further studies in this direction will help us to construct a more unified system of methods to attack the problems of Hardy and Bergman spaces. Before the works of Poletsky and Stessin, M. A. Alan has already defined using a similar construction Hardy type spaces in hyperconvex domains in \( \mathbb{C}^n \) in [1].

In this paper we concentrate on the \( n = 1 \) case for the Hardy space construction of Poletsky and Stessin. By definition to each subharmonic function \( u \) continuous near the boundary of \( G \) corresponds a space, which is denoted by \( H^p_u \) of holomorphic functions in \( G \). Here \( G \) is a bounded regular domain in \( \mathbb{C} \). Throughout the paper these spaces will be called Poletsky-Stessin Hardy spaces. Following the motivating work of Poletsky and Stessin, the structure and first examples on the unit disk of Poletsky-Stessin Hardy spaces were further investigated in [2], [11] and [12].

Among these recent work, the author and M. A. Alan gave a complete characterization of \( H^p_u \) spaces that live in the plane domains through the boundary values of the functions in this class or through a growth description of their harmonic majorants. As an application, a Beurling's type theorem was proved in [2]. This states roughly that if \( G \) is the unit disk
Structure of weighted Hardy spaces on finitely connected domains

\( \mathbb{D} \), and \( u \) is such a subharmonic exhaustion on \( \mathbb{D} \), there exists a bounded outer function \( \varphi \) so that the space \( H^p_u \) isometrically equals to \( M_{\varphi,p} \), \( p > 0 \). Moreover, \( \varphi \) belongs to the class \( H^p_u \). Here we denote by \( M_{\varphi,p} \) the space \( \varphi^{2/p} H^p \) together with the norm defined by

\[
\|f\|_{M_{\varphi,p}} := \|f/\varphi^{2/p}\|_p, \quad f \in M_{\varphi,p}.
\]

Using this result, we were able to construct several examples of holomorphic spaces with certain properties. Let us remark that the space \( M_{\varphi,2} \) is a useful tool in the study of sub-Hardy Hilbert spaces (see [10]).

The problems and results addressed in this paper can be summarized as follows:

When \( G \) is finitely connected:

1. We provide a decomposition result of functions which belong to \( H^p_u(G) \). (Theorem 4.1).

2. We completely describe \( H^p_u(G) \) in terms of their boundary values.

3. We describe an isomorphism from the classical Hardy space \( H^p(G) \) onto the space \( H^p_u(G) \).

The last two results extend the research in [2], [11] and [12] from the disk to finitely connected planar domains. This paper is organized as follows: Section 2 is a brief summary of the previous work which are related and will be used in the paper. In section 3 we solve the inverse balayage problem for a given weight function. The main results of the paper are stated and proved in section 4.

2. Poletsky-Stessin-Hardy spaces

In this section we will give the basic definitions to be used throughout the paper and recall some earlier results. Let \( G \subset \mathbb{C} \) be a bounded domain. We say that a function \( u \leq 0 \) defined on \( G \) is an exhaustion on \( G \) if the level set \( B_{c,u} := \{ z \in G : u(z) < c \} \) is relatively compact in \( G \) for any \( c < 0 \). Let \( u \) be an exhaustion and \( c < 0 \), we set

\[
u_c := \max\{u,c\}, \quad S_{c,u} := \{ z \in G : u(z) = c \}.
\]

Let \( u \in sh(G) \) be an exhaustion function. We assume that \( u \) is continuous taking values in \( \mathbb{R} \cup \{-\infty\} \) with the extended topology on this set. Following Demailly [4] we define

\[
\mu_{c,u} := \Delta u - \chi_{G \setminus B_{c,u}} \Delta u.
\]

Here \( \chi_\omega \) denotes the characteristic function for a given set \( \omega \subset G \). We will denote the class of negative subharmonic exhaustion functions on \( G \) by \( \mathcal{E}(G) \). Also we denote by \( \mathcal{E}_0(G) \) the class of all functions \( u \in \mathcal{E}(G) \) such that \( f \Delta u < \infty \).

Following [9] we set

\[
sh_u(G) := \operatorname{sh}_u := \left\{ v \in \operatorname{sh}(G) : v \geq 0, \sup_{c<0} \int_{S_{c,u}} v \, d\mu_{c,u} < \infty \right\},
\]

and we define

\[
H^p_u(G) := H^p_u := \{ f \in \operatorname{hol}(G) : |f|^p, f \in \operatorname{sh}_u \}
\]

for \( p > 0 \). Let us write

\[
\|v\|_u := \sup_{c<0} \int_{S_{c,u}} v \, d\mu_{c,u} = \int_G (v \Delta u - u \Delta v)
\]

if \( v \) is a nonnegative subharmonic function. We set

\[
\|f\|_{u,p} := \sup_{c<0} \left( \int_{S_{c,u}} |f|^p \, d\mu_{c,u} \right)^{1/p}
\]

if \( f \) is a holomorphic function on \( G \). We list relevant recollections of basic facts below:
There exists a measure \( f \) hold true for any \( i \). Suppose that \( V \) has a sufficiently smooth boundary, then a function \( G \) belongs to \( H^p(G) \) possesses boundary values (almost everywhere non-tangential limits), which we denote by \( f^* \). The usual normalized arclength measure on \( \partial G \) is denoted by \( \nu \). Hence, \( \nu(\partial G) = 1 \).

**Theorem 2.1** ([2, Theorem 1.10]). Suppose that \( G \) is a Jordan domain with rectifiable boundary or a bounded domain with \( C^2 \) boundary. Let \( p > 1 \), \( f \in H^p(G) \) and let \( u \in \mathcal{E}(G) \). Then the following statements are equivalent:

i. \( f \in H^p_u(G) \).

ii. \( f^* \in L^p(V_u \nu), \) where

\[
V_u(\zeta) := \int_G P_G(z, \zeta) \Delta u(z), \quad \zeta \in \partial G.
\]

(2.3)

iii. There exists a measure \( \tilde{\mu}_u \) on \( \partial G \) so that \( f^* \in L^p(\tilde{\mu}_u) \). Moreover, if \( E \) is any Borel subset of \( \partial G \) with measure \( \nu(E) = 0 \), then \( \tilde{\mu}_u(E) = 0 \) and we have the equality

\[
\int_{\partial G} \gamma d\tilde{\mu}_u = \int_G P_G(\gamma) \Delta u
\]

(2.4)

for every \( \gamma \in L^1(\nu) \).

In addition, if \( f \in H^p_u(G) \), then \( \|f\|_{u,p} = \|f^*\|_{L^p(\tilde{\mu}_u)} \) and \( d\tilde{\mu}_u = V_u d\nu \).

**Remark 2.2.**

i. If \( G \) is the unit disk or a Jordan domain with rectifiable boundary, then the statements in Theorem 2.1 hold true for any \( p > 0 \).

ii. Take a compact set \( K \subset G \) so that \( \Delta u(K) > r > 0 \). Let \( m := \min_{\zeta \in \partial G} \min_{z \in K} P_G(z, \zeta) \). Then \( V_u(\zeta) \geq rm > 0 \) for every \( \zeta \in \partial G \). Hence, replacing \( u \) by \( u/(rm) \), we may assume without loss of generality that \( V_u \geq 1 \) on \( \partial G \).

iii. The weight function \( V_u \) is lower semicontinuous. In fact, by Fatou’s lemma

\[
\liminf_j V_u(\zeta_j) = \liminf_j \int_G P_G(z, \zeta_j) \Delta u(z) \geq \int_G P_G(z, \zeta) \Delta u(z) = V_u(\zeta).
\]

iv. Suppose \( G \) is a bounded domain with \( C^2 \) boundary or a Jordan domain with rectifiable boundary. Let \( u \in \mathcal{E}_0(G) \). Then

\[
u(z) = \int_G g_G(z, w) \Delta u(w), \quad z \in G.
\]

Since

\[
\frac{\partial g_G(\zeta, w)}{\partial n} = P_G(w, \zeta)
\]

when \( \zeta \in \partial G \) and \( w \in G \), \( \frac{\partial}{\partial n}(\zeta) \) exists for every \( \zeta \in \partial G \). Here \( \frac{\partial}{\partial n} \) denotes the normal derivative in the outward direction on \( \partial G \) and

\[
\frac{\partial u(\zeta)}{\partial n} = V_u(\zeta) = \int_G P_G(w, \zeta) \Delta u(w), \quad \zeta \in \partial G.
\]
Using the property (2.4) in Theorem 2.1 we see that
\[ \int_{\partial G} V_u(\zeta) d\nu(\zeta) = \int_G \Delta u = \int_{\partial G} \frac{\partial u}{\partial n}(\zeta) d\nu(\zeta). \]

Let \( \varphi \) be a nonzero holomorphic function on the unit disk. Let \( \mathcal{M}_{\varphi,p} \) denote the space \( \varphi^{2/p}H^p \) which we endow with the norm
\[ \|f\|_{\mathcal{M}_{\varphi,p}} := \|f/\varphi^{2/p}\|_p, \quad f \in \mathcal{M}_{\varphi,p}. \]

A function \( \varphi \in H^2_u \) is called a \( u \)-inner function if \( |\varphi^*(\zeta)|^2 V_u(\zeta) = 1 \) for almost every \( \zeta \in \partial \mathbb{D} \). If, moreover, \( \varphi(z) \) is zero-free, we say that the function \( \varphi \) is singular \( u \)-inner. The next result is proved in Theorem 3.2 and Corollary 3.3 in [2].

**Theorem 2.3.** Let \( u \in \mathcal{E}(\mathbb{D}) \). There exists a bounded \( u \)-inner and outer function \( \varphi \in H^2_u \) so that \( H^2_u = \mathcal{M}_{\varphi,2} \). Moreover, these spaces are isometric.

The function \( \varphi \) is uniquely determined (up to a unit constant). We have
\[ V_u(e^{i\theta}) = \frac{1}{|\varphi(e^{i\theta})|^2} = \frac{1}{\varphi^2(e^{i\theta})} \sgn \frac{1}{\varphi^2(e^{i\theta})}. \tag{2.5} \]

Here we set \( \sgn \alpha := |\alpha|/\alpha \) for any complex number \( \alpha \neq 0 \) and \( \sgn0 := 0 \).

**Theorem 2.4.** The set \( L^p(V_u d\theta) \) coincides with \( \varphi^{2/p}L^p(d\theta) \) and the map \( f \mapsto \varphi^{-2/p}f \) is an isometric isomorphism from the space \( L^p(V_u d\theta) \) onto \( L^p(d\theta) \).

The next result reveals a complete factorization of functions belonging to the space \( H^p_u(\mathbb{D}) \).

**Theorem 2.5 ([2, Theorem 3.4]).** Let \( 0 < p < \infty, \ f \in H^p_u(\mathbb{D}), \ f \neq 0, \) and \( B \) be the Blaschke product formed with the zeros of \( f \). There are zero-free \( \varphi \in H^2_u \cap H^\infty, \ S \in H^\infty \) and \( F \in H^p \) such that \( \varphi \) is singular \( u \)-inner and outer, \( S \) is singular inner, \( F \) is outer, and
\[ f = BS\varphi^{2/p}F. \tag{2.6} \]

Moreover, \( \|f\|_{p,u} = \|F\|_p \) and \( H^p_u(\mathbb{D}) = \mathcal{M}_{\varphi,p} \).

Rephrasing the last statement above we have a concrete isomorphism.

**Corollary 2.6.** The map \( f \mapsto \varphi^{-2/p}f \) is an isometric isomorphism from the space \( H^p_u \) onto \( H^p \).

The following Lemma will be useful in the next section. Its proof is a simple calculation and we outline it here.

**Lemma 2.7.** Let \( c \) be a number with \(-1 < c < 0\). Then there exists a function \( \kappa = \kappa_c \) defined on \((-\infty, 0]\) with the following properties:

i. \( \kappa : (-\infty, 0) \to (-\infty, 0] \) is non-decreasing, convex and \( C^\infty \),

ii. \( \kappa \) is real-analytic in \((c, 0]\),

iii. \( \kappa(t) \equiv c \) when \( t \leq c \), \( \kappa(0) = 0 \), and \( \kappa'(0) = 1 \).

**Proof.** Let \( a := -\frac{\ln(-c)}{c}, \ b := \frac{-1}{\ln(-c)} \), and
\[ \kappa(t) := \begin{cases} c + e^{\frac{a}{t-c}} & , \ t > c, \\ c, & , \ t \leq c. \end{cases} \]

Then
\[ \kappa'(t) = \frac{1}{e(t-c)} e^{\frac{a}{t-c}} \]
and
\[ \kappa''(t) = \frac{1}{e(t-c)^2} (1/e - (b+1)(t-c)^{b+1} e^{\frac{a}{t-c}}) \]
for \( t > c \). For \( t \leq c \), \( \kappa'(t) = \kappa''(t) = 0 \). It can be checked that \( \kappa''(t) > 0 \) for \( c < t \leq 0 \), and \( \kappa \) satisfies all properties in i., ii. and iii.

\section{Constructing subharmonic exhaustion from a given weight function}

Let \( G \) be a Jordan domain with rectifiable boundary and \( \psi \) be a given holomorphic function in \( H^1(G) \). We consider in this section the problem of finding a subharmonic exhaustion \( u \) on \( G \) so that \( V_\kappa(\zeta) = |\psi(\zeta)| \) when \( \zeta \in \partial G \). We can always suppose that \( G = \mathbb{D} \) after a conformal map of \( G \) onto \( \mathbb{D} \). The following result was obtained for the disk in [6] (see also [8]).

\begin{theorem}
Let \( G \) be a Jordan domain with rectifiable boundary, \( \psi \) be a lower semi-continuous function on \( \partial G \) so that \( \psi \geq k \) for some constant \( k > 0 \). Then there exists a function \( u \in E(G) \) so that \( \psi = V_u \).
\end{theorem}

a. \( u \) is the decreasing limit of functions in \( E_0 \cap C^\infty(\overline{G}) \) which converge uniformly to \( u \) on \( \overline{G} \).

b. \( u \in E_0(G) \) if and only if \( \psi \in L^1(du) \).

In this section we consider finitely connected domains. If \( \Gamma \) is the rectifiable boundary of a Jordan domain, let \( G^* \) and \( G \) be the unbounded and bounded components of \( \mathbb{C} \setminus \Gamma \), respectively. By a conformal map \( \Phi \) the curve \( \Gamma \) can be mapped onto the unit circle and \( G^* \) can be mapped onto \( \mathbb{D}^* \), the compliment in \( \mathbb{C} \) of the closed unit disk. We say that a function \( u^* \) defined on \( G^* \) belongs to \( E(G^*) \) if the corresponding function \( u := u^* \circ \Phi^{-1}(1/z) \) belongs to \( E(\mathbb{D}) \). A holomorphic function \( f \) on \( G^* \) belongs to \( H^p_u(G^*) \) if the function \( f \circ \Phi^{-1}(1/z) \) belongs to \( H^p_u(\mathbb{D}) \). Clearly \( H^p_u(G^*) \subset H^p(G^*) \). We write \( a \approx b \) if there exists an absolute constant \( C > 0 \) so that \( C^{-1}b \leq a \leq Cb \).

\begin{lemma}
Let \( \Gamma \) be the \( C^{1+\varepsilon} \) boundary of a Jordan domain for some \( \varepsilon > 0 \). Let \( G^* \) be the unbounded component of \( \mathbb{C} \setminus \Gamma \) and let \( \psi \) be a lower semicontinuous function on \( \Gamma \) so that \( \psi \geq k \) for some constant \( k > 0 \). Then there exists a function \( v \in E(G^*) \) so that \( \frac{\partial \psi(\zeta)}{\partial n} \approx \psi(\zeta) \) on \( \Gamma \).
\end{lemma}

\begin{proof}
Suppose first that \( \Gamma \) is the unit circle. Let \( u \in E(\mathbb{D}) \) be the function proved for \( \psi \) in Theorem 3.1. The function \( v(z) = u(1/z) \) belongs to \( E(\mathbb{D}^*) \) and satisfies \( \frac{\partial \psi(\zeta)}{\partial n} = \psi(\zeta) \) on \( \Gamma \). If \( \Gamma \) is a Jordan curve with \( C^{1+\varepsilon} \) boundary and if \( \Phi \) is a conformal map of \( G \) onto \( \mathbb{D} \), let \( v(z) := u \circ \Phi(1/z) \), where \( u \in E(\mathbb{D}) \) is so that \( \frac{\partial \psi(\zeta)}{\partial n} = \psi(1/|\zeta|) \) for \( |\zeta| = 1 \). By a classical result of Painlevé (cf. [7, Theorem 5.2.4]) \( \Phi \) extends to a \( C^1 \) map on the closed set \( G \). Thus, the estimate holds.
\end{proof}

Let \( G \) be a bounded domain so that \( \partial G = \bigcup_{j=0}^N \Gamma_j, \) where each \( \Gamma_j \) is a Jordan curve and \( \Gamma_j \cap \Gamma_k = \emptyset \) when \( j \neq k \). Let \( G_0 \) be the bounded component of \( \mathbb{C} \setminus \Gamma_0 \). We assume that each \( \Gamma_j, j \neq 0, \) is contained in \( G_0 \). Let \( G^*_j \) be the domain with boundary \( \Gamma_j \) which contains \( G \). For simplicity, in the following theorem and in the rest of the paper we will assume that all \( \Gamma_j \) are at least \( C^{1+\varepsilon} \) for some \( \varepsilon > 0 \), for all \( j = 0, 1, \ldots, N \). The following theorem gives a full description of those weight functions produced by a subharmonic exhaustion (see Remark 2.2).

\begin{theorem}
Let \( \psi \) be lower semicontinuous on \( \partial G \) so that \( \psi \geq s \) for some \( s > 0 \). Then there exists a function \( u \in E(G) \) so that \( \psi = V_u \approx \frac{\partial u}{\partial n} \). Moreover \( u \in E_0(G) \) if and only if \( \psi \in L^1(\partial G, du) \).
\end{theorem}

\begin{proof}
For any \( j \in \{0,1,\ldots,N\} \) let \( \psi_j(\zeta) = \psi(\zeta) \) if \( \zeta \in \Gamma_j \). Let \( v_j \in E(G^*_j) \) be the function for \( \psi_j \) given in Lemma 3.2 so that \( V_{v_j} = \psi_j \) on \( \partial G_j \). We can choose \( c_j < 0 \) close to 0 so that the level set \( B_{c_j,v_j} \) contains all \( \Gamma_k \) for \( k \neq j \). Let \( c = \max\{c_0,\ldots,c_N\} \). Set \( u_j(z) := c_j(v_j(z)) \) and \( u(z) := \sum_{j=0}^N u_j(z) - cN \) for \( z \in G \). If \( \zeta \in \Gamma_j \), then \( u_j(\zeta) = 0 \) while
Let $u_k(\zeta) = c$ when $k \neq j$. Hence $u(\zeta) = 0$ and $u \in \mathcal{E}(G)$. On the other hand if $k \neq j$, then $u_j(z) \equiv c$ on an open set containing $\Gamma_k$, therefore by Lemma 3.2 and Lemma 2.7

$$\frac{\partial u(\zeta)}{\partial n} = \frac{\partial u_j(\zeta)}{\partial n} = \frac{\partial v_j(\zeta)}{\partial n} \approx \psi(\zeta)$$

when $\zeta \in \Gamma_j$. Together we have that $V_u = \psi$ on $\partial G$. The equivalence of $u \in \mathcal{E}_0$ and $\psi \in L^1$ follows directly from the identity

$$\int_{\partial G} V_u(\zeta) d\nu(\zeta) = \int_G \Delta u.$$

\[ \square \]

4. Decomposition results and the multiplier algebra of $H^p_u(G)$

From the Cauchy integral formula if $f \in H(G)$, then $f$ has a unique decomposition of the form

$$f(z) = f_0(z) + \ldots + f_N(z),$$

where $f_0 \in H(G_0)$, $f_j \in H_0(G^*_j)$ if $j \in \{1, \ldots, N\}$, and $H_0(G^*_j)$ denotes the class of holomorphic functions in $H(G^*_j)$ that vanish at infinity. This holds true for bounded holomorphic functions; from [3] we see that

$$H^\infty(G) = H^\infty(G_0) + H^\infty_0(G^*_1) + \ldots + H^\infty_0(G^*_N),$$

where $H^\infty_0(G^*_j) = H^\infty(G^*_j) \cap H_0(G^*_j)$.

The next theorem tells us that if $G$ is finitely connected, the space $H^p_u(G)$ has a similar decomposition.

**Theorem 4.1.** Let $G$ be a finitely connected domain whose boundary consists of disjoint Jordan curves $\Gamma_j$, $0 \leq j \leq N$, with rectifiable boundary so that $\Gamma_0$ surrounds $G$. Let $G_0$ be the bounded component of $\mathbb{C} \setminus \Gamma_0$ and for $j \in \{1, \ldots, N\}$, let $G^*_j$ be the domain with boundary $\Gamma$ which contains $G$. Let $u \in \mathcal{E}_0(G)$ and $p \geq 0$. Then there exists $u_0 \in \mathcal{E}_0(G_0)$, $u_j \in \mathcal{E}_0(G^*_j)$, $j = 1, \ldots, N$, so that every $f \in H^p_u(G)$ can be represented in the form

$$f(z) = f_0(z) + \ldots + f_N(z),$$

where $f_0 \in H^p_u(G_0)$, $f_j \in H^p_u(G^*_j) \cap H_0(G^*_j)$ for $j \in \{1, \ldots, N\}$.

**Proof.** Let $u \in \mathcal{E}_0(G)$ and $f \in H^p_u(G)$. Then the function $V_u$ is lower semicontinuous and $V_u \geq s$ on $\partial G$ for some $s > 0$ (see Remark 2.2). By the proof of Theorem 3.3 it is deduced that there exists $\omega \in \mathcal{E}_0(G)$ so that $V_\omega = V_u$ and $\omega$ is of the form

$$\omega = u_0 + \ldots + u_N - cN,$$

where $c < 0$, $u_j \in \mathcal{E}_0(G^*_j)$ and $V_{u_j} = V_u|_{\Gamma_j}$ for $0 \leq j \leq N$.

Since $H^p_u(G) \subset H^p(G)$ it is well-known that (see for example [5, Theorem 10.12])

$$f(z) = f_0(z) + \ldots + f_N(z),$$

where $f_j \in H^p(G^*_j)$. In fact, each $f_j$ is of the form (after analytic continuation)

$$f_j(z) = \int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z} d\nu(\zeta).$$

Hence for each $j$ the functions $f_0, f_1, \ldots, f_j, \ldots, f_N$ are bounded on $\Gamma_j$. Here $\hat{f}_j$ means that $f_j$ is not in the list. Therefore, $|f_j(\zeta)|^p \leq |f(\zeta)|^p + m$ when $\zeta \in \Gamma_j$ for some constant $m$. The norm of $f_j$ is estimated as

$$\|f_j\|_{H^p_{u_j}(G^*_j)}^p = \int_{\Gamma_j} |f_j(\zeta)|^p V_u(\zeta) d\nu \leq \int_{\Gamma_j} |f(\zeta)|^p V_u(\zeta) d\nu + m\|V_u\|_{L^1}$$

$$\leq \|f\|_{H^p_u(G)}^p + m\|V_u\|_{L^1}. $$
Since \( f \in H^p_u(G) \) and \( u \in \mathcal{E}_0(G) \), \( \|f\|_{H^p_u(G)} < \infty \) and \( \|V_u\|_{L^1} < \infty \), hence \( \|f_j\|_{H^p_u(G^*_j)} < \infty \) and \( f_j \in H^p_u(G^*_j) \).

Finally, we describe the boundary values of functions from \( H^p_u(G) \) and describe an isomorphism from \( H^p_u(G) \) onto \( H^p(G) \). We use the same assumptions and notations of Theorem 4.1 in the next statement.

**Theorem 4.2.** Let \( u \in \mathcal{E}_0(G) \) and \( p > 0 \). Then there exists \( u_0 \in \mathcal{E}_0(G_0) \), \( u_j \in \mathcal{E}_0(G^*_j) \), and zero-free functions \( \varphi_0 \in H^{\infty}(G_0) \cap H^2_u(G_0) \) and \( \varphi_j \in H^{\infty}(G_0) \cap H^2_u(G^*_j), \) \( j = 1, \ldots, N \), so that

i. \( H^p_u(G) = \varphi_0^{2/p} H^p(G_0) + \varphi_1^{2/p} H^p(G^*_1) + \cdots + \varphi_N^{2/p} H^p(G^*_N) \).

ii. If \( f \in H^p_u(G) \), then \( f^{|_{\Gamma_j}} \in L^p(V_{u_j} dv) \) for \( j = 0, 1, \ldots, N \).

iii. \( \|f\|_{H^p_u(G)} \approx \|f_0\|_{H^p_u(G_0)} + \sum_{j=1}^N \|f_j\|_{H^p_u(G^*_j)} \) for \( f \in H^p_u(G) \), where \( f(z) = f_0(z) + \cdots + f_N(z) \) denotes the unique decomposition in (4.2).

iv. The mapping \( T : H^p_u(G) \to H^p(G) \) given by

\[
T f = \sum_{j=0}^N f_j/\varphi_j^{2/p}
\]

is a linear isomorphism of \( H^p_u(G) \) onto \( H^p(G) \).

**Proof.** Statements i. and ii. are direct consequences of Theorem 2.1, Theorem 2.5 and the discussions in this section. For the third statement, let \( X \) denote the vector space \( H^p_u(G) \) endowed with the norm

\[
\|f\|_X = \|f_0\|_{H^p_u(G_0)} + \sum_{j=1}^N \|f_j\|_{H^p_u(G^*_j)}.
\]

For \( p \geq 1 \), both \( H^p_u(G) \) (with the usual norm) and \( X \) are Banach spaces. For \( 0 < p < 1 \), we don’t have normed spaces, however, these are complete metric spaces. In fact, \( H^p_u(G) \) and \( X \) are closed subspaces of \( L^p(V_{u} dv) \) and several consequences of Baire category theorem, such as closed graph theorem, carry over to both of these spaces. Notice that as sets both \( H^p_u(G) \) and \( X \) are the same. By closed graph theorem, we deduce that there exists a constant \( C > 0 \) such that

\[
1/C\|f\|_X \leq \|f\|_{H^p_u(G)} \leq C\|f\|_X
\]

for every \( f \in H^p_u(G) \). This proves iii.

By Corollary 2.6, after a conformal mapping of \( G^*_j \) onto \( \mathbb{D} \), any function \( g \in H^p(G^*_j) \) is of the form \( g = \varphi_j^{2/p} h \) for some \( h \in H^p(G^*_j) \) and \( \|g\|_{H^p(G^*_j)} \approx \|h\|_{H^p(G^*_j)} \). It follows from statements i. and iii. that \( T \) gives an isomorphism of \( H^p_u(G) \) onto \( H^p(G) \). This finisishes the proof.

Let \( X \) be a Banach space of holomorphic functions on \( G \). The multiplier algebra \( M(X) \) of \( X \) is defined to be the class of holomorphic functions \( g \) on \( G \) such that \( gf \in X \) for every \( f \in X \). By the closed graph theorem a function \( g \in H(G) \) belongs to the multiplier algebra \( M(H^p_u(G)) \) if and only if the multiplication operator \( M_g \) which assigns to a function \( f \in H(G) \) the product \( gf \) is a bounded operator on \( H^p_u(G) \).

**Corollary 4.3.** For \( p > 0 \), \( M(H^p_u(G)) = H^{\infty}(G) \).

**Proof.** It is well-known that \( M(H^p(\mathbb{D})) = H^{\infty}(\mathbb{D}) \). Hence, if \( \Gamma \) is the rectifiable boundary of a Jordan domain and if \( U \) is a connected component (bounded or unbounded) of \( \Gamma \), then \( M(H^p(U)) = H^{\infty}(U) \). This can be seen by considering a conformal mapping from
\[ H_2^p(G) = \varphi_1^2p H^p(G_1) + \varphi_2^2p H^p(G_2) + \cdots + \varphi_N^2p H^p(G_N). \]

Let \( g \in M(H_2^p(G)) \). Then from Theorem 4.2 (iii), if \( f \in H_2^p(G) \), then
\[ \|gf\|_{H_2^p(G)} \approx \|gf_0\|_{H_2^p(G_0)} + \sum_{j=1}^{N} \|gf_j\|_{H_2^p(G_j^*)} < \infty \]
where \( f_j \)'s are as in (4.2). Let \( g(z) = \sum_{j=0}^{N} g_j(z) \) be the unique decomposition of \( g \) as described in (4.1). Fix \( j \in \{0, \ldots, N\} \). If \( i \neq j \), then \( g_i \) is bounded near \( \Gamma_i \), therefore,
\[ \|g_i f_j\|_{H_2^p(G_j^*)} < \infty. \]
Combining with (4.3), this means that \( \|g_i f_j\|_{H_2^p(G_j^*)} < \infty. \) Since \( g_j \in H(G_j^*) \) and \( f_j \) can be an arbitrary element of \( H_2^p(G_j^*) \), we see that \( g_j \in M(H_2^p(G_j^*)) \), therefore it is bounded on \( G_j^* \). Hence, \( g \in H^\infty(G) \) and we have proved that \( M(H_2^p(G)) \subset H^\infty(G) \). The reverse inclusion clearly holds from Theorem 4.2(iii). The proof is finished.

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