Recent progress in applying lattice QCD to Kaon physics

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Our visible universe
How much we know about the nucleon?

What we have known

- Nucleon (proton and neutron) has internal structure ⇒ quarks and gluons
- Strong interaction between quarks and gluons is described by fundamental theory ⇐ Quantum Chromodynamics (QCD)

We have difficulty to study nucleon’s internal structure

- **asymptotic freedom**
  ⇒ non-perturbative at low energy scale
- **color confinement**
  ⇒ quarks and gluons cannot be isolated singularly

\[
\alpha_s(Q) = \text{QCD} \quad \alpha_s(M_Z) = 0.1185 ± 0.0006
\]

![Graph showing QCD coupling constant \(\alpha_s\) vs. energy scale Q.](image)
QCD: perturbative and non-perturbative aspects

Experimental measurements ⇒ mixture of SD and LD physics

**QCD factorization**

- separate the physics into short-distance, high-energy part and long-distance, low-energy part
- the former can be calculated perturbatively
- the latter can be determined with global fit to experiments

**Lattice QCD**

- high-performance computing, simulate QCD using Monte Carlo methods
- non-perturbatively study long-distance, low-energy QCD effects
- first-principle calculation, the input parameters, $\alpha_s$ and quark masses are fixed by physical mass of pion and other hadrons
- once SM parameters set, no requirement of other inputs from experiments

Lattice QCD – a virtual laboratory on super computer
Lattice QCD: a virtual laboratory on super computer

credit by Meifeng Lin
Introduction to lattice QCD
Inventied by Kenneth G. Wilson in 1973

1st numerical implementation by M. Creutz in 1979

QCD computers 1983 – 2011 [credit by N. Christ]

QCD computers start to enter in the Eflops generation, $10^{18}$ floating point operation per second
QCD on the lattice

Lattice discretization

- quark fields live on the lattice sites, $\psi(x), x_\mu = n_\mu a$
- gluons represented as links between lattice sites, $U_\mu(x) = e^{iagA_\mu(x)}$

With finite $a$ and $L$, quarks and gluons can be simulated on supercomputer.

Euclidean path integral:

- Minkowski time replaced by $x_0 \to -it \quad \Rightarrow \quad e^{-iHx_0} \to e^{-Ht} = e^{-S[\psi,\bar{\psi},A]}$
- Same Hamiltonian $H$ for Minkowski space and Euclidean space

$$\langle O \rangle \sim \int [d\psi][d\bar{\psi}][dA]O \, e^{-S[\psi,\bar{\psi},A]}$$
Integrate out the quark fields using Grassmann Algebra

\[ \langle O \rangle \sim \int [dU] O[U] \det(\Phi + m) e^{-S_g[U]} \]

**Importance sampling:** generate gauge configurations with probability distribution

\[ p[U] \propto \det(\Phi + m) e^{-S_g[U]} \]

this can be achieved by **hybrid Monte Carlo simulation:** Monte Carlo + Molecular Dynamics

**Integration is approximated** by average over gauge configurations

\[ \int [dU] \det(\Phi + m) e^{-S_g[U]} \rightarrow \frac{1}{N} \sum_{\{U\}} \]

statistical error is reduced by \(1/\sqrt{N}\)
Experiment vs Lattice QCD

**HEP Experiment**

- BEPC collider (Energy, Luminosity)
- Collision, Events
- BES III Detector, measurement

**LQCD simulation**

- Super Computer (Performance, Memory)
- Simulation, QCD vacuum
- Lattice QCD calculation
Systematic effects

- **Lattice spacing** $a$ brings in UV cutoff $\Lambda_{lat} \sim \frac{1}{a}$
  - in 4-flavor theory, we include dynamical charm quark
  - then we want $\frac{1}{a} \gg m_c$, otherwise a large lattice artifact from $am_c$

- **Lattice size** $L$ brings in IR cutoff
  - $L$ should be larger than the Compton wave length of pion, $L \gg \frac{1}{m_\pi}$
  - otherwise shape of pion distorted by $L \to$ large finite volume effect

- **Pion mass** $m_\pi$ used in lattice QCD are heavier than 140 MeV
  - cheaper for simulation but unphysical effects

Various $a$, $L$, $m_\pi$ to control the systematic effects

Can we make use of the systematic effects?

- $m_\pi$ dependence, ChPT, LECs
Milestone: mass spectrum

Hadron spectrum from lattice QCD

- input: $\alpha_s$, quark masses; set by $\pi$, $K$, … (empty symbols in the plot)
- output: hadron spectrum vs experiment

![Graph showing hadron spectrum vs experiment](image-url)
Applying lattice QCD to Kaon physics
Lattice Kaon physics

- How lattice QCD works for Kaon physics ⇒ powerful for standard hadronic matrix elements
  \[ \langle h_2(p_2)|O(0)|h_1(p_1) \rangle \text{ or } \langle 0|O(0)|h(p) \rangle \]

- single local operator insertion: \( O(0) \)

- only single stable hadron \( |h\rangle \) or vacuum \( |0\rangle \) in the initial/final state

- spatial momenta \( p_1, p_2 \) need to be small compared to \( 1/a \)
  (not a problem for Kaon physics, but essential for \( B \) decays)
"standard" quantities in Kaon physics: $f_K^\pm/f_{\pi^\pm}$ and $f_+(0)$

**Flavor Lattice Averaging Group (FLAG): arXiv:1607.00299**

$$f_K^\pm/f_{\pi^\pm} = 1.1933(29) \quad \Rightarrow \quad 0.25\% \text{ error}$$

$$f_+(0) = 0.9704(33) \quad \Rightarrow \quad 0.34\% \text{ error}$$

**Experimental information: arXiv:1411.5252, 1509.02220**

$$K_{\ell 3} \quad \Rightarrow \quad |V_{us}| f_+(0) = 0.2165(4) \quad \Rightarrow \quad |V_{us}| = 0.2231(9)$$

$$K_{\mu 2}/\pi_{\mu 2} \quad \Rightarrow \quad \left| \frac{V_{us}}{V_{ud}} \right| f_K^\pm = 0.2760(4) \quad \Rightarrow \quad \left| \frac{V_{us}}{V_{ud}} \right| = 0.2313(7)$$
Test the CKM unitarity

\[ |V_u|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.980(9) \Rightarrow 2\sigma \text{ deviation from 1} \]

Experiment + CKM unitarity + \( f_K^\pm / f_{\pi^\pm} \) or \( f_+(0) \) \( \Rightarrow \) \( V_{us} \) and \( V_{ud} \)

### Diagram

**FLAG2016**

- \( |V_{us}| \)
- \( |V_{ud}| \)

**Legend**

- FLAG average for \( N_f = 2 + 1 \)
- ETM 15C
- ETM 14E
- FNAL/MILC 14A
- FNAL/MILC 13E
- FNAL/MILC 13C
- ETM 13F
- HPQCD 13A
- MILC 13A
- MILC 11 (stat. err. only)
- ETM 10E (stat. err. only)

- FLAG average for \( N_f = 2 + 1 \)
- RBC/UKQCD 15A
- RBC/UKQCD 14B
- RBC/UKQCD 13
- RBC/UKQCD 12
- FNAL/MILC 12I
- LHRD 12
- LEGC 11
- MILC 10
- MILC 9A
- MILC 9B
- JQCD 08
- JQCD 07
- MILC 04

- FLAG average for \( N_f = 2 \)
- ETM 14D (stat. err. only)
- ALPHA 13A
- SGU 13
- ETM 10B (stat. err. only)
- ETM 10D (stat. err. only)
- ETM 09A
- ETM 09B
- QCD02/UKQCD 07
- QCD02 09 (stat. err. only)
- RBC 06
- LHRD 05

- HFAG 14 r decay
- Marfat 09 r decay and \( \sigma^* \sigma^* \)
- Gamiz 08 r decay

- Hardy 15 nuclear \( \beta \) decay
**$V_{us}$: comparison with other determinations**

- $K_{l3}$ decay + FLAG average for $f_+(0)$
- $K_{\mu2}$ decay + FLAG average for $f_K/f_\pi +$ CKM unitarity
- $|V_{ud}|$ from nuclear $\beta$ decays + CKM unitarity, Hardy 15
- $\tau$ decay + flavor breaking sum rules, HFAG 14
- $\tau$ decay + flavor breaking sum rules, HLMZ 17

- $>3\sigma$ deviation from semi-inclusive $\tau$ decay (HFAG 14) seems resolved by a new implementation of sum rules (HLMZ 17, arXiv:1702.01767)
- $<0.5\%$ uncertainty requires the inclusion of $O(\alpha_e)$ EM corrections
"standard" quantities in Kaon physics: $B_K$

Short distance dominance $\Rightarrow$ OPE $\Rightarrow$ Wilson coeff. $C(\mu) \times$ operator $Q^{\Delta S=2}(\mu)$

![Diagram](image)

$\mathcal{H}_{\text{eff}}^{\Delta S=2} = \frac{G_F^2 M_W^2}{16\pi^2} C(\mu) Q^{\Delta S=2}(\mu)$

- Serve as a dominant contribution to the indirect CP violation $\epsilon_K$

\[
\epsilon_K = \exp(i\phi_\epsilon) \sin(\phi_\epsilon) \left[ \frac{\text{Im}[\langle K^0|\mathcal{H}_{\text{eff}}^{\Delta S=2}|K^0\rangle]}{\Delta M_K} + \frac{\text{Im}[M_{00}^{LD}]}{\Delta M_K} + \frac{\text{Im}[A_0]}{\text{Re}[A_0]} \right]
\]

- Within Standard Model, only single operator with $V-A$ structure

$Q^{\Delta S=2} = [\bar{s}_a \gamma_\mu (1-\gamma_5)d_a][\bar{s}_b \gamma_\mu (1-\gamma_5)d_b]$

- Beyond SM, 4 other operator possible

$Q_2^{\Delta S=2} = [\bar{s}_a (1-\gamma_5)d_a][\bar{s}_b (1-\gamma_5)d_b]$

$Q_3^{\Delta S=2} = [\bar{s}_a (1-\gamma_5)d_b][\bar{s}_b (1-\gamma_5)d_a]$

$Q_4^{\Delta S=2} = [\bar{s}_a (1-\gamma_5)d_a][\bar{s}_b (1+\gamma_5)d_b]$

$Q_5^{\Delta S=2} = [\bar{s}_a (1-\gamma_5)d_b][\bar{s}_b (1+\gamma_5)d_a]$
**FLAG average for Standard Model $B_K$**

- $B_K$ in NDR-$\overline{\text{MS}}$ scheme: $B_K(\mu) = \frac{\langle K^0|Q^{\Delta S=2}(\mu)|K^0\rangle}{\frac{8}{3} f_K^2 m_K^2}$

- Renormalization group independent $B$ parameter $\hat{B}_K$:
  
  $\hat{B}_K = \left(\frac{\bar{g}(\mu)^2}{4\pi}\right)^{-\gamma_0/(2\beta_0)} \exp \left\{ \int_0^{\bar{g}(\mu)} dg \left( \frac{\gamma(g)}{\beta(g)} + \frac{\gamma_0}{\beta_0 g} \right) \right\} B_K(\mu)$

---

**Diagram**

- $N_f = 2+1+1$: $\hat{B}_K = 0.717(24)$
- $N_f = 2+1$: $\hat{B}_K = 0.763(10)$
- $N_f = 2$: $\hat{B}_K = 0.727(25)$
Status for BSM $B_i$

$$B_i(\mu) = \frac{\langle K^0 | Q_i(\mu) | K^0 \rangle}{N_i \langle K^0 | \bar{s}\gamma_5 d | 0 \rangle \langle 0 | \bar{s}\gamma_5 d | K^0 \rangle}, \quad \{N_2, \ldots, N_5\} = \{-5/3, 1/3, 2, 2/3\}$$

$B_i(\mu)$ at $\mu_{\text{MS}} = 3$ GeV

| $B_2$ | $B_3$ | $B_4$ | $B_5$ |
|-------|-------|-------|-------|
| $N_f=2+1+1$ |       |       | ETM 15 |
|       |       |       | SWME 15A |
|       |       |       | SWME 14C |
|       |       |       | RBC/UKQCD 12E |
| $N_f=2+1$ |       |       |       |
|       |       |       |       |
| $N_f=2$ |       |       |       |

Uncontrolled systematic effects may cause the observed deviations:

- NPR renormalization
- Matching procedure
- Continuum limit

No FLAG average yet
Go beyond “standard” quantities in lattice Kaon physics

- $K \to \pi\pi$ decays and direct CP violation

Final state involve $\pi\pi$ (multi-hadron system)

- Long-distance contributions to flavor changing processes
  - $\Delta M_K$ and $\epsilon_K$

- Rare kaon decays: $K \to \pi\nu\bar{\nu}$ and $K \to \pi\ell^+\ell^-$

Hadronic matrix element for bilocal operators

$$\int d^4x \langle f| T[Q_1(x)Q_2(0)]|i \rangle$$
$K \to \pi\pi$ decays and direct CP violation
Parity violation (\(\tau-\theta\) puzzle) \(\Rightarrow\) Lee and Yang’s Nobel prize (1957)

- \(\tau \to \pi^+ \pi^0\) (\(P = +\))
- \(\theta \to \pi^+ \pi^+ \pi^-\) (\(P = -\))
- Both are \(K^+\)

**Charge-Parity** violation in neutral Kaon decays

- CP eigenstates
  - Under CP transform: \(CP|K^0\rangle = -|\bar{K}^0\rangle\)
  - Define CP eigenstates: \(K^0_{\pm} = (K^0 \mp \bar{K}^0)/\sqrt{2}\)
- Weak eigenstates
  - \(K_S \to 2\pi\) (\(CP = +\))
  - \(K_L \to 3\pi\) (\(CP = -\))
- Neglecting CP violation, we have \(K_S = K^0_+\) and \(K_L = K^0_-\)

1964, BNL discovered \(K_L \to 2\pi\) \(\Rightarrow\) CP violation \(\Rightarrow\) Nobel prize (1980)
Direct and indirect CP violation

- $K_{L/S}$ are not CP eigenstates

$$|K_{L/S}\rangle = \frac{1}{\sqrt{1 + \epsilon}} \left(|K^0_+\rangle + \bar{\epsilon}|K^0_-\rangle\right)$$

- $K_L \rightarrow 2\pi$ (CP = +)
  - $K^0_+ \rightarrow 2\pi$ (indirect CP violation, $\epsilon$ or $\epsilon_K$)
  - $K^0_- \rightarrow 2\pi$ (direct CP violation, $\epsilon'$)

- Experimental measurement

$$\epsilon + \epsilon' = \frac{A(K_L \rightarrow \pi^+\pi^-)}{A(K_S \rightarrow \pi^+\pi^-)}$$

$$\epsilon - 2\epsilon' = \frac{A(K_L \rightarrow \pi^0\pi^0)}{A(K_S \rightarrow \pi^0\pi^0)}$$

- $\epsilon'$ is 1000 times smaller than the indirect CP violation $\epsilon$

  PDG: $\ |\epsilon'| = 3.70(53) \times 10^{-6}, \ |\epsilon| = 2.228(11) \times 10^{-3}$

Thus direct CP violation $\epsilon'$ is very sensitive to New Physics
Theoretically, Kaon decays into the isospin $I = 2$ and $0$ $\pi\pi$ states

\[
\Delta I = 3/2 \text{ transition: } \langle \pi\pi(I = 2) | H_W | K^0 \rangle = A_2 e^{i\delta_2}
\]
\[
\Delta I = 1/2 \text{ transition: } \langle \pi\pi(I = 0) | H_W | K^0 \rangle = A_0 e^{i\delta_0}
\]

If CP symmetry were protected $\Rightarrow$ $A_2$ and $A_0$ are real amplitudes

$\epsilon$ and $\epsilon'$ can be given by $K \to \pi\pi(I)$ amplitude $A_I$

\[
\epsilon = \bar{\epsilon} + i \left( \frac{\text{Im}[A_0]}{\text{Re}[A_0]} \right)
\]
\[
\epsilon' = \frac{ie^{i(\delta_2 - \delta_0)}}{\sqrt{2}} \frac{\text{Re}[A_2]}{\text{Re}[A_0]} \left( \frac{\text{Im}[A_2]}{\text{Re}[A_2]} - \frac{\text{Im}[A_0]}{\text{Re}[A_0]} \right)
\]

The target for lattice QCD is to calculate the amplitude $A_2$ and $A_0$
Recent results for $K \rightarrow \pi\pi (I = 2)$

**Results for $A_2$ [RBC-UKQCD, PRD91 (2015) 074502]**

- Use two ensembles (both at $m_\pi = 135$ MeV) for continuum extrapolation
  
  $48^3 \times 96, \quad a = 0.11$ fm, \quad $L = 5.4$ fm
  
  $64^3 \times 128, \quad a = 0.084$ fm, \quad $L = 5.4$ fm

- After continuum extrapolation:
  
  \[
  \text{Re}[A_2] = 1.50(4)_{\text{stat}}(14)_{\text{syst}} \times 10^{-8} \text{ GeV}
  
  \text{Im}[A_2] = -6.99(20)_{\text{stat}}(84)_{\text{syst}} \times 10^{-13} \text{ GeV}
  \]

- Experimental measurement:
  
  \[
  \text{Re}[A_2] = 1.479(3) \times 10^{-8} \text{ GeV}
  
  \text{Im}[A_2] \text{ is unknown}
  \]

- Scattering phase at $E_{\pi\pi} = M_K$
  
  $\delta_2 = -11.6(2.5)(1.2)^\circ$

  consistent with phenomenological analysis [Schenk, '91]
Resolve the puzzle of $\Delta I = 1/2$ rule

$\Delta I = 1/2$ rule: $A_0 = 22.5 \times A_2 \Rightarrow$ a 50 year puzzle

- Wilson coefficient only contributes a factor of 2
- $\text{Re}[A_2]$ is dominated by diagrams $C_1$ (left) and $C_2$ (right)

Opposite sign in $C_1$ and $C_2$ leads to large cancellation in $\text{Re}[A_2] \propto C_1 + C_2$

- Such cancellation is first observed in an earlier calculation
  [RBC-UKQCD, PRL110 (2013) 152001]
- It is further confirmed in the latest calculation of $A_2$
  [RBC-UKQCD, PRD91 (2015) 074502]

Puzzle of $\Delta I = 1/2$ rule is resolved from first principles
Recent results for $K \to \pi\pi (I = 0)$

Results for $A_0$ [RBC-UKQCD, PRL115 (2015) 212001]

- Use a $32^3 \times 64$ ensemble, $a = 0.14$ fm, $L = 4.53$ fm

  
  \[ M_\pi = 143.1(2.0) \text{ MeV}, \quad M_K = 490(2.2) \text{ MeV}, \quad E_{\pi\pi} = 498(11) \text{ MeV} \]

- G-boundary condition is used: non-trivial to tune the volume $\Rightarrow M_K = E_{\pi\pi}$

- The largest contributions to $\text{Re}[A_0]$ and $\text{Im}[A_0]$ come from $Q_2$ (current-current) and $Q_6$ (QCD penguin) operator
Results for $\text{Re}[A_0]$, $\text{Im}[A_0]$ and $\text{Re}[\epsilon'/\epsilon]$

- Determine the $K \rightarrow \pi\pi (I=0)$ amplitude $A_0$
  - Lattice results
    \[
    \text{Re}[A_0] = 4.66(1.00)_{\text{stat}}(1.26)_{\text{syst}} \times 10^{-7} \text{ GeV} \\
    \text{Im}[A_0] = -1.90(1.23)_{\text{stat}}(1.08)_{\text{syst}} \times 10^{-11} \text{ GeV}
    \]
  - Experimental measurement
    \[
    \text{Re}[A_0] = 3.3201(18) \times 10^{-7} \text{ GeV} \\
    \text{Im}[A_0] \text{ is unknown}
    \]

- Determine the direct CP violation $\text{Re}[\epsilon'/\epsilon]$
  \[
  \text{Re}[\epsilon'/\epsilon] = 0.14(52)_{\text{stat}}(46)_{\text{syst}} \times 10^{-3} \quad \text{Lattice} \\
  \text{Re}[\epsilon'/\epsilon] = 1.66(23) \times 10^{-3} \quad \text{Experiment}
  \]

2.1 $\sigma$ deviation $\Rightarrow$ require more accurate lattice results
Long-distance contributions to flavor changing processes
$\Delta M_K$ and $\epsilon_K$
$K^0 - \bar{K}^0$ mixing

$K^0 - \bar{K}^0$ mixing: time evolution

$$i \frac{d}{dt} \left( \frac{K^0}{\bar{K}^0} \right) = \left[ \begin{pmatrix} M_{00} & M_{0\bar{0}} \\ M_{0\bar{0}}^* & M_{\bar{0}\bar{0}} \end{pmatrix} - i \frac{1}{2} \begin{pmatrix} \Gamma_{00} & \Gamma_{0\bar{0}} \\ \Gamma_{\bar{0}0}^* & \Gamma_{\bar{0}\bar{0}} \end{pmatrix} \right] \left( \frac{K^0}{\bar{K}^0} \right)$$

- To 2nd-order in $H_W$

$$M_{ij} = M_K \delta_{ij} + \langle i| H_W |j \rangle + \mathcal{P} \int_\alpha \frac{\langle i| H_W |\alpha \rangle \langle \alpha| H_W |j \rangle}{M_K - E_\alpha}$$

$$\Gamma_{ij} = 2\pi \int_\alpha \langle i| H_W |\alpha \rangle \langle \alpha| H_W |j \rangle \delta(E_\alpha - M_K)$$

- $\Delta M_K$ and $\epsilon_K$ is related to $\text{Re}[M_{0\bar{0}}]$ and $\text{Im}[M_{0\bar{0}}]$, respectively

$$\Delta M_K = M_{K_S} - M_{K_L} = 2\text{Re}[M_{0\bar{0}}]$$

$$\epsilon_K = e^{i\phi_\epsilon} \sin(\phi_\epsilon) \left[ \frac{\text{Im}[M_{0\bar{0}}]}{\Delta M_K} + \frac{\text{Im}[A_0]}{\text{Re}[A_0]} \right], \quad \phi_\epsilon = \arctan \frac{\Delta M_K}{\Delta \Gamma_K / 2} \approx 45^\circ$$
Long-distance contribution: $\Delta M_K$ and $\epsilon_K$

- $\Delta M_K \Rightarrow \text{Re}[M_{0\bar{0}}] \Rightarrow \text{CP conserving part of } K^0-\bar{K}^0 \text{ mixing}$

- Dominant contribution from charm-charm loop: $\lambda_c^2 \frac{m_c^2}{M_W^2} > \lambda_t^2 \frac{m_t^2}{M_W^2}$
  $\Rightarrow$ historically led to the predication of the mass scale of charm quark

- $\epsilon_K \Rightarrow \text{Im}[M_{0\bar{0}}] \Rightarrow \text{CP violating part of } K^0-\bar{K}^0 \text{ mixing}$

Top-top, top-charm and charm-charm loops compete in size
$\Rightarrow$ important top-top loop, thus $\epsilon_K$ is sensitive to the CKM input $V_{cb}$
Results for $\Delta M_K$

- Use $32^3 \times 64$ ensemble: $1/a = 1.37$ GeV, $m_\pi = 170$ MeV and $m_c = 750$ MeV
  [Preliminary results, from Z. Bai, N. Christ]

| Types        | $\Delta M_K \times 10^{12}$ MeV |
|--------------|---------------------------------|
| Types 1-4    | 3.26(63)                        |
| Types 1-2    | 4.19(15)                        |
| $\eta$       | 0                               |
| $\pi$        | 0.27(14)                        |
| $\pi\pi$, $I=0$ | -0.097(49)                   |
| $\pi\pi$, $I=2$ | -6.56(6) $\times 10^{-4}$     |
| $D_{FV}$     | 0.029(19)                       |
| Expt.        | 3.483(6)                        |

- New project: $64^3 \times 128$, $1/a = 2.38$ GeV, $m_c = 1.2$ GeV, $m_\pi = 140$ MeV
  - Based on 60 configurations: $\Delta M_K = 4.0(2.4) \times 10^{-12}$ MeV

- Double GIM cancellation $\Rightarrow$ No SD divergence
$\lambda_t \lambda_u$ contribution to $\epsilon_K$ [calculated by Z. Bai, RBC-UKQCD]

- Without top quark in the lattice QCD calculation, logarithmic divergence

\[ -X_{ij}(\mu) \times \quad \]

- Subtract $X_{ij}(\mu) \left[ (\bar{d}s)_{V-A} (\bar{s}d)_{V-A} \right]$ to remove the lattice cutoff effects
- We thus define the bilocal operator in the RI/SMOM scheme

Preliminary results at $m_\pi = 340$ MeV and $m_c = 970$ MeV

| $\mu_{RI}$ | $\text{Im} M_{00}^{ut,RI}$ | $\text{Im} M_{00}^{ut,RI \rightarrow \overline{MS}}$ | $\text{Im} M_{00}^{ut,ld \ corr}$ | contribution to $\epsilon_K$ |
|----------------|----------------|----------------|----------------|----------------|
| 1.54 | -1.30(69) | 0.352 | -0.95(69) | 0.186(135) $\times 10^{-3}$ |
| 1.92 | -1.49(69) | 0.476 | -1.01(69) | 0.199(135) $\times 10^{-3}$ |
| 2.11 | -1.58(69) | 0.537 | -1.04(69) | 0.205(135) $\times 10^{-3}$ |
| 2.31 | -1.65(69) | 0.599 | -1.05(69) | 0.206(135) $\times 10^{-3}$ |
| 2.56 | -1.73(69) | 0.674 | -1.06(69) | 0.207(135) $\times 10^{-3}$ |

Experimental value for $|\epsilon_K| = 2.228 \times 10^{-3}$
Rare Kaon decays
$K^+ \to \pi^+ \nu \bar{\nu}$: Experiment vs Standard model

$K^+ \to \pi^+ \nu \bar{\nu}$: largest top quark contribution, thus theoretically clean

$$H_{\text{eff}} \sim \frac{G_F}{\sqrt{2}} \cdot \frac{\alpha_{\text{EM}}}{2\pi \sin^2 \theta_W} \cdot \frac{\lambda_t X_t(x_t) \cdot (\bar{s}d)_{V-A} (\bar{\nu} \nu)_{V-A}}{\mathcal{N} \sim 2 \times 10^{-5}}$$

Probe the new physics at scales of $\mathcal{N}^{-\frac{1}{2}} M_W = O(10 \text{ TeV})$

**Past experimental measurement** is 2 times larger than SM prediction

$$\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})_{\text{exp}} = 1.73^{+1.15}_{-1.05} \times 10^{-10} \quad [\text{BNL E949, '08}]$$

$$\text{Br}(K^+ \to \pi^+ \nu \bar{\nu})_{\text{SM}} = 9.11 \pm 0.72 \times 10^{-11} \quad [\text{Buras et. al., '15}]$$

but still consistent with $> 60\%$ exp. error
New experiments

**New generation of experiment:** NA62 at CERN

- aims at observation of $O(100)$ events in 2-3 years
- 10%-precision measurement of $\text{Br}(K^+ \rightarrow \pi^+ \nu \bar{\nu})$

$K_L \rightarrow \pi^0 \nu \bar{\nu}$

- even more challenging since all the particles involved are neutral
- only upper bound was set by KEK E391a in 2010
- new **KOTO** experiment at J-PARC designed to observe $K_L$ decays
OPE: integrate out the heavy fields, $Z$, $W$, $t$, ...

\[ \langle \pi^+ \nu \bar{\nu} | Q_A(x) Q_B(0) | K^+ \rangle: \text{need lattice QCD} \]
Bilocal $C^\text{MS}_A(\mu)C^\text{MS}_B(\mu)r^\text{MS}_{AB}(\mu)$ vs Local $C^\text{MS}_0(\mu)$

[Buras, Gorbahn, Haisch, Nierste, ‘06]

At $\mu = 2.5$ GeV, 50% charm quark contribution from bilocal term
Lattice results

First results at $m_\pi = 420$ MeV, $m_c = 860$ MeV [RBC-UKQCD, arXiv:1701.02858]

W-W

Z-exchange

W-W + Z-exchange

P_{cu} = \text{Lattice} - X + \text{physical SD part} (Y)

Lattice - unphysical SD part (X)

P_{cu} \text{ (PT)}

Unrenormalized
RI-renormalized
\Delta P_{c,u}
\sum P_{c,u}

$\mu_{DT} = \mu_{MS}$ [GeV]
Results for charm quark contribution

Charm quark contribution $P_c$

$$P_c = P^{SD}_c + \delta P_{c,u}$$

NNLO QCD [Buras, Gorbahn, Haisch, Nierste, ‘06]:

$$P^{SD}_c = 0.365(12)$$

Phenomenological ansatz [Isidori, Mescia, Smith, ‘05]

$$\delta P_{c,u} = 0.040(20)$$

Lattice results

$$\Delta P_{c,u} = 0.0040(\pm13)_{\text{stat}}(\pm32)_{\text{syst}}(-45)_{\text{FV}}$$

- Cancellation in $W$-$W$ and $Z$-exchange diagrams leads to small $\Delta P_{c,u}$
- Important to check whether such large cancellation also occurs for physical quark masses
CP conserving decays $K^+ \to \pi^+ \ell^+ \ell^-$ and $K_S \to \pi^0 \ell^+ \ell^-$ are dominated by long-distance contribution induced by photon exchange

$$T_{+}^{\mu} = \int d^4 x e^{iqx} (\pi(p)) | T \{ J_{\text{em}}(x) H^{\Delta S=1}(0) \} | K_+, S(k) \rangle$$

$$= \frac{G_F M_K^2}{(4\pi)^2} V_+ , S(z) \left[ z(k+p)^{\mu} - (1 - r_{\pi}^2) q^{\mu} \right]$$

with $q = k - p$, $z = q^2 / M_K^2$, $r_{\pi} = M_{\pi} / M_K$

Calculation of $K^+ \to \pi^+ \ell^+ \ell^-$ can be compared with Exp. + ChPT analysis

$V_+(z)$ useful for test of lepton flavor universality violation in rare K decays

Results for $K_S \to \pi^0 \ell^+ \ell^-$ can be used for the evaluation of the significant interference between direct and indirect CP violation in $K_L \to \pi^+ \ell^+ \ell^-$ decay

- Even the sign of $a_S = V_S(0)$ is useful
Analysis of experimental measurement of $K^+ \rightarrow \pi^+ \ell^+ \ell^-$

summarized by Jorge Portoles at Kaon 2016

$V_j(z) = a_j + b_j z + \frac{\alpha_j r^2 + \beta_j (z-z_0)}{G_F M_K^2 r^4} \left[ 1 + \frac{z}{r^2_V} \right] \left[ \Phi \left( \frac{z}{r^2_V} \right) + \frac{1}{6} \right]$

| Process          | $\text{Br} \times 10^8$ | $a$             | $b$             | $b/a$ |
|------------------|--------------------------|-----------------|-----------------|-------|
| $K^+ \rightarrow \pi^+ e^+ e^-$ | $31.4 \pm 1.0$ | $-0.578 \pm 0.016$ | $-0.779 \pm 0.066$ | $\sim 1.35$ |
| $K^+ \rightarrow \pi^+ \mu^+ \mu^-$ | $9.62 \pm 0.25$ | $-0.575 \pm 0.039$ | $-0.813 \pm 0.145$ | $\sim 1.41$ |
Lattice results for $K^+ \rightarrow \pi^+ \ell^+ \ell^-$

Use $24^3 \times 64$ ensemble [RBC-UKQCD, PRD94 (2016) 114516]

$1/a = 1.78$ GeV, $m_\pi = 430$ MeV, $m_K = 625$ MeV, $m_c = 530$ MeV

Momentum dependence of $V_+(z)$

$$V_+(z) = a_+ + b_+ z, \quad \Rightarrow \quad a_+ = 1.6(7), \ b_+ = 0.7(8)$$

Compare with experimental data + phenomenological analysis

| Process     | $\text{Br} \times 10^8$ | $a$            | $b$            | $b/a$  |
|-------------|-----------------|----------------|----------------|--------|
| $K^+ \rightarrow \pi^+ e^+ e^-$ | $31.4 \pm 1.0$ | $-0.578 \pm 0.016$ | $-0.779 \pm 0.066$ | $\sim 1.35$ |
| $K^+ \rightarrow \pi^+ \mu^+ \mu^-$ | $9.62 \pm 0.25$ | $-0.575 \pm 0.039$ | $-0.813 \pm 0.145$ | $\sim 1.41$ |
Conclusion

- For “standard” quantities such as $f_K/f_\pi$, $f_+(0) B_K$, lattice calculation reach the precision of $O(1\%)$ or better.

- It’s time to go beyond “standard”
  - $K \rightarrow \pi\pi$ and $\epsilon'$
  - $\Delta M_K$ and $\epsilon_K$
  - rare kaon decays: $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K^+ \rightarrow \pi^+ \ell^+ \ell^-$

- Lattice QCD is now capable of first-principles calculation of the above “beyond-standard” quantities.

- Realistic calculation await for the next generation of super-computers.