On the metric dimension of amalgamation of sunflower and lollipop graph and caveman graph

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Abstract. Let $G$ be a connected nontrivial graph with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices $u$ and $v$ in $G$ is the shortest path length between $u$ and $v$ denoted $d(u,v)$. Let $W = \{w_1, w_2, \ldots, w_k\}$ be a subset of $V(G)$. The representation of a vertex $u$ with respect to $W$ is a sequential pair of distances between $u$ and all vertices in $W$, where $u$ is a vertex in $G$. The set of $W$ is called the resolving set if the representation of each vertex is different to $W$. Resolving set with a minimum cardinality called the metric basis and the number of element from some basis is called the metric dimension, denoted by $\text{dim}(G)$. In this paper, we determine the metric dimension of amalgamation of sunflower and lollipop graph $\left(\text{SF}_{n}, v_i\right) * (L_{m,p}, u_p)$ and caveman graph $C(n,m)$. The results show that the metric dimension of amalgamation of sunflower and lollipop graph is $\text{dim}\left(\left(\text{SF}_{n}, v_i\right) * (L_{m,p}, u_p)\right) = m + 1$ for $n = 3, 4, \ldots, 7$; $\text{dim}\left(\left(\text{SF}_{n}, v_i\right) * (L_{m,p}, u_p)\right) = \left\lceil \frac{n}{2} \right\rceil + m - 2$ for $n \geq 8$, and the metric dimension of caveman graph is $\text{dim}(C(n,m)) = n$ for $m = 3, 4$; $\text{dim}(C(n,m)) = (m - 4)n$ for $m \geq 5$.

1. Introduction

One of concept in graph theory is the metric dimension. The metric dimension was first introduced by Slater in 1975, which was then continued by Harary and Melter (Caceres et al. [4]). Khuller et al. [6] has applied the concept of metric dimensions in real problems. Until now the concept of the metric dimension is still being studied and developed. Let $G$ be a connected nontrivial graph with vertex set $V(G)$ and edge set $E(G)$. Let $W = \{w_1, w_2, \ldots, w_k\}$ be a subset of the vertex set in a graph $G$. Representation of a vertex $u$ with respect to $W$ is a sequential pair of distances between $u$ and all vertices in $W$, where $u$ is a vertex of $G$ and the distance from two vertices is defined as the shortest path from one vertex to other vertices. Slater [7] introduces the concept of $W$ as locating set, while Harary and Melter [5] introduce the concept of $W$ as resolving set. The set of $W$ is called the resolving set if the representation of each vertex is different from $W$. The resolving set with minimum number of members is called the metric basis and the number of members on the basis is called the metric dimension.

According to Chartrand et al. [3] the metric dimension of $G$, denoted by $\text{dim}(G)$, is defined as the number of elements of a resolving set with the smallest cardinality at $G$. Chartrand et al. [3] and Caceres et al. [4] applied the concept of the metric dimension to a particular class of graphs. In the 2000 years, Chartrand et al. [3] discovered the metric dimensions of several graph classes including $\text{dim}(C_n) = 2$ and $\text{dim}(P_p) = 1$, where $n \geq 3$ and $p \geq 2$. In 2005, Caceres et al. [4] found the metric dimension of the fan graph $F_n$. Then in 2010, Yero et al. [10] located the metric dimension of corona graph $G \odot^k H$, $\text{dim}(G \odot^k H) \geq n_1(n_2 + 1)^{k-1}\text{dim}(H)$ for $n_1, n_2 \geq 2$. 

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and $k \geq 2$.

From existing studies the authors are motivated to investigate the metric dimensions of the amalgamation of sunflower and lollipop graph $(SF_n, v_i) \ast (L_{m,p}, u_p)$ and caveman graph $C(n, k)$ which have not been studied by the previous researchers.

2. Main Results
2.1. Metric Dimension
Following are definitions used to determine the metric dimension taken from Chartrand et al. [3].

If $r(v|W)$ for each vertex $v \in V(G)$ is different, then $W$ is called the resolving set of $V(G)$. The resolving set with minimum cardinality is called the minimum or basis, and the cardinality of the basis is called the metric dimension of $G$ denoted by $dim(G)$.

2.2. The Metric Dimension of Amalgamation of Sunflower and Lollipop Graph
Chartrand and Lesniak [1] defined that the wheel graph denoted by $W_n$, is a join of a complete graph $K_1$ and a cycle graph $C_n$ so that $W_n = K_1 + C_n$ for $n \geq 3$, vertex $c \in V(K_1)$ as the central vertex of the wheel graph.

Chartrand and Oellerman [2] defined the sunflower graph $SF_n$ as a graph obtained from wheel $W_n$ with center vertex $c$ and cycle order $c; w_1; w_2; w_3; \ldots; w_n$ and $n$ vertex additional $v_1; v_2; v_3; \ldots; v_n$ where $v_i$ is associated with edge to $w_i, w_{i+1}$ for $i = 1, 2, \ldots, n$ with $i + 1$ is modulo $n$.

Weisstein [9] defined the lollipop graph $L_{m,p}$ as a complete graph $K_m$ and the path graph $P_p$ connected to a bridge.

Chartrand et al. [3] defined that $(SF_n, v_i) \ast (L_{m,p}, u_p)$ graph is the result of amalgamation operations from vertex $u_p$ of lollipop graph $L_{m,p}$ with a vertex $v_i$ on the sunflower graph $SF_n$.

![Figure 1. $(SF_n, v_i) \ast (L_{m,p}, u_p)$ graph](image)

**Theorem 2.1** For any positive integer $n, p$ and $m \geq 3$ hold

$$dim((SF_n, v_i) \ast (L_{m,p}, u_p)) = \begin{cases} m + 1, & n = 3, 4, \ldots, 7 \\ \lceil \frac{n}{2} \rceil + m - 2, & \text{else} \end{cases}$$

Proof. Let $(SF_n, v_i) \ast (L_{m,p}, u_p)$ be a graph. We consider two cases according to the values of $n$.

- Case 1, for $n = 3, 4, \ldots, 7$.
  For $n = 3, 4, \ldots, 7$, if the vertex $v_2$ is amalgamated with vertex $u_p$ and choose $W = \{v_1, v_2, v_3, k_2, k_3, \ldots, k_{m-1}\} \subset (SF_n, v_2) \ast (L_{m,p}, u_p)$. We obtained the representation of each vertex at $(SF_n, v_2) \ast (L_{m,p}, u_p)$ with $n = 3$ with respect to $W$ is,
is,

\[ r(w_1|W) = (1, 1, 2, p + 3, p + 3, \ldots, p + 3, p + 3), \]
\[ r(w_2|W) = (2, 1, 1, p + 2, p + 2, \ldots, p + 2, p + 2), \]
\[ r(w_3|W) = (1, 2, 1, p + 2, p + 2, \ldots, p + 2, p + 2), \]
\[ r(v_1|W) = (0, 2, p + 3, p + 3, \ldots, p + 3, p + 3), \]
\[ r(v_2|W) = (2, 2, p + 1, p + 1, \ldots, p + 1, p + 1), \]
\[ r(v_3|W) = (2, 2, 0, p + 3, p + 3, \ldots, p + 3, p + 3), \]
\[ r(u_1|W) = (p + 1, p - 1, p + 1, 2, 2, \ldots, 2, 2), \]
\[ r(u_2|W) = (p, p - 2, p, 3, \ldots, 3, 3), \]
\[ r(u_3|W) = (p - 1, p - 3, p - 1, 4, 4, \ldots, 4, 4), \]

\[ \vdots \]
\[ r(u_{p-1}|W) = (3, 1, 3, p, \ldots, p, p), \]
\[ r(k_1|W) = (p + 2, p, p + 2, 1, 1, \ldots, 1, 1), \]
\[ r(k_2|W) = (p + 3, p + 1, p + 3, 0, 1, \ldots, 1, 1), \]

\[ \vdots \]
\[ r(k_{m-1}|W) = (p + 3, p + 1, p + 3, 1, 1, \ldots, 1, 0), \]
\[ r(k_m|W) = (p + 3, p + 1, p + 3, 1, 1, \ldots, 1, 1). \]

Representation of each vertex at \((SF_n, v_2) * (L_{m,p}, u_p)\) with \(n = 4, 5, 6, 7\) with respect to \(W\) is,

\[ r(w_1|W) = (1, 2, n - 2, p + 3, p + 3, \ldots, p + 3, p + 3), \]
\[ r(w_2|W) = (1, 1, 2, p + 3, p + 3, \ldots, p + 3, p + 3), \]
\[ r(w_3|W) = (2, 1, 1, p + 2, p + 2, \ldots, p + 2, p + 2), \]

\[ \vdots \]
\[ r(w_{n-1}|W) = (n - 2, n - 3, 1, p + (n - 2), p + (n - 2), \ldots, p + (n - 2), p + (n - 2)), \]
\[ r(w_n|W) = (2, n - 2, n - 3, p + (n - 1), p + (n - 1), \ldots, p + (n - 1), p + (n - 1)), \]
\[ r(v_1|W) = (0, 2, 3, p + 3, p + 3, \ldots, p + 3, p + 3), \]
\[ r(v_2|W) = (2, 0, 2, p + 1, p + 1, \ldots, p + 1, p + 1), \]
\[ r(v_3|W) = (2, 2, 0, p + 3, p + 3, \ldots, p + 3, p + 3), \]

\[ \vdots \]
\[ r(v_{n-1}|W) = (3, n - 2, n - 3, p + (n - 1), p + (n - 1), \ldots, p + (n - 1), p + (n - 1)), \]
\[ r(v_n|W) = (2, 3, n - 2, n - 3, p + (n - 1), p + (n - 1), \ldots, p + (n - 1), p + (n - 1)), \]
\[ r(c|W) = (2, 2, 2, p + 3, p + 3, \ldots, p + 3, p + 3), \]
\[ r(u_1|W) = (p + 1, p - 1, p + 1, 2, 2, \ldots, 2, 2), \]
\[ r(u_2|W) = (p, p - 2, p, 3, \ldots, 3, 3), \]
\[ r(u_3|W) = (p - 1, p - 3, p - 1, 4, 4, \ldots, 4, 4), \]

\[ \vdots \]
\[ r(u_{p-1}|W) = (3, 1, 3, p, \ldots, p, p), \]
\[ r(k_1|W) = (p + 2, p, p + 2, 1, 1, \ldots, 1, 1), \]
\[ r(k_2|W) = (p + 3, p + 1, p + 3, 0, 1, \ldots, 1, 1), \]

\[ \vdots \]
\[ r(k_{m-1}|W) = (p + 3, p + 1, p + 3, 1, 1, \ldots, 1, 0), \]
\[ r(k_m|W) = (p + 3, p + 1, p + 3, 1, 1, \ldots, 1, 1). \]

In the case of the \((SF_n, v_i) * (L_{m,p}, u_p)\) graph with \(n = 3, 4, 5, 6, 7\), we obtained the resolving set having \(m + 1\) element, so that \(\dim((SF_n, v_i) * (L_{m,p}, u_p)) = m + 1\).

- Case 2, for \(n \geq 8\).

We will show that \(\dim((SF_n, v_i) * (L_{m,p}, u_p)) = \left\lceil \frac{n}{2} \right\rceil + m - 2\). By choosing \(W = \)
\{v_1, v_3, v_7, k_2, k_3, \ldots, k_{m-1}\}. We obtained the representation for each vertex at \((SF_8, v_2) * (L_{m,p}, u_p)\) with respect to \(W\) is,

\[
\begin{align*}
    r(w_1[W]) &= (1, 3, 3, p + 3, p + 3, \ldots, p + 3, p + 3), \\
    r(w_2[W]) &= (1, 2, 3, p + 3, p + 3, \ldots, p + 3, p + 3), \\
    r(w_3[W]) &= (2, 1, 3, p + 3, p + 3, \ldots, p + 3, p + 3), \\
    r(w_4[W]) &= (3, 1, 3, p + 3, p + 3, \ldots, p + 3, p + 3), \\
    r(w_5[W]) &= (3, 2, 3, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(w_6[W]) &= (3, 3, 2, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(w_7[W]) &= (3, 3, 3, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(w_8[W]) &= (2, 3, 3, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(v_1[W]) &= (0, 3, 4, p + 3, p + 3, \ldots, p + 3, p + 3), \\
    r(v_2[W]) &= (2, 2, 4, p + 3, p + 3, \ldots, p + 3, p + 3), \\
    r(v_3[W]) &= (3, 0, 4, p + 3, p + 3, \ldots, p + 3, p + 3), \\
    r(v_4[W]) &= (4, 2, 4, p + 3, p + 3, \ldots, p + 3, p + 3), \\
    r(v_5[W]) &= (4, 3, 3, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(v_6[W]) &= (4, 4, 2, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(v_7[W]) &= (4, 4, 3, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(v_8[W]) &= (4, 4, 4, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(c[W]) &= (2, 2, 2, p + 3, p + 3, \ldots, p + 3, p + 3), \\
    r(u_1[W]) &= (p + 1, p + 1, p + 1, p + 3, 2, 2, \ldots, 2, 2), \\
    r(u_2[W]) &= (p, p, p + 2, p + 2, 3, 3, \ldots, 3, 3), \\
    \vdots \\
    r(u_{p-1}[W]) &= (3, 3, 5, 5, p, \ldots, p, p), \\
    r(k_1[W]) &= (p + 2, p + 2, p + 4, p + 4, 1, 1, \ldots, 1, 1), \\
    r(k_2[W]) &= (p + 3, p + 3, p + 5, p + 5, 0, 1, \ldots, 1, 1), \\
    r(k_3[W]) &= (p + 3, p + 3, p + 5, p + 5, 1, 0, \ldots, 1, 1), \\
    \vdots \\
    r(k_{m-1}[W]) &= (p + 3, p + 3, p + 5, p + 5, 1, 1, \ldots, 1, 1), \\
    r(k_m[W]) &= (p + 3, p + 3, p + 5, p + 5, 1, 1, \ldots, 1, 1). \\
\end{align*}
\]

For \(n = 8\), \(W\) is resolving set with \(3 + (m - 2)\) elements. For \(n = 9\), \(W\) is resolving set with \(3 + (m - 2)\) element. For \(n = 10\), \(W\) is resolving set with \(4 + (m - 2)\) element. For \(n = 11\), \(W\) is resolving set with \(4 + (m - 2)\) element. We obtained the representation for each vertex at \((SF_{12}, v_2) * (L_{m,p}, u_p)\) with respect to \(W\) is,

\[
\begin{align*}
    r(w_1[W]) &= (1, 3, 3, p + 3, p + 3, \ldots, p + 3, p + 3), \\
    r(w_2[W]) &= (1, 2, 3, p + 3, p + 3, \ldots, p + 3, p + 3), \\
    r(w_3[W]) &= (2, 1, 3, p + 3, p + 3, \ldots, p + 3, p + 3), \\
    r(w_4[W]) &= (3, 1, 3, p + 3, p + 3, \ldots, p + 3, p + 3), \\
    r(w_5[W]) &= (3, 2, 3, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(w_6[W]) &= (3, 3, 2, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(w_7[W]) &= (3, 3, 3, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(w_8[W]) &= (3, 3, 3, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(w_9[W]) &= (3, 3, 2, 1, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(w_{10}[W]) &= (3, 3, 3, 1, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(w_{11}[W]) &= (3, 3, 3, 2, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(w_{12}[W]) &= (2, 3, 3, 3, p + 4, p + 4, \ldots, p + 4, p + 4), \\
    r(v_1[W]) &= (0, 3, 4, 4, p + 3, p + 3, \ldots, p + 3, p + 3), \\
\end{align*}
\]
\begin{align*}
r(v_1^2 | W) & = (2, 2, 4, 4, p + 1, p + 1, \ldots, p + 1, p + 1), \\
r(v_3 | W) & = (3, 0, 4, 4, p + 3, p + 3, \ldots, p + 3, p + 3), \\
r(v_4 | W) & = (4, 2, 4, 4, p + 4, p + 4, \ldots, p + 4, p + 4), \\
r(v_5 | W) & = (4, 3, 3, 4, p + 5, p + 5, \ldots, p + 5, p + 5), \\
r(v_6 | W) & = (4, 4, 2, 4, p + 5, p + 5, \ldots, p + 5, p + 5), \\
r(v_7 | W) & = (4, 4, 0, 3, p + 5, p + 5, \ldots, p + 5, p + 5), \\
r(v_8 | W) & = (4, 4, 2, 2, p + 5, p + 5, \ldots, p + 5, p + 5), \\
r(v_9 | W) & = (4, 4, 3, 0, p + 5, p + 5, \ldots, p + 5, p + 5), \\
r(v_{10} | W) & = (4, 4, 4, 2, p + 5, p + 5, \ldots, p + 5, p + 5), \\
r(v_{11} | W) & = (3, 4, 4, 3, p + 5, p + 5, \ldots, p + 5, p + 5), \\
r(v_{12} | W) & = (2, 4, 4, 4, p + 4, p + 4, \ldots, p + 4, p + 4), \\
r(c | W) & = (2, 2, 2, 2, p + 3, p + 3, \ldots, p + 3, p + 3), \\
r(u_1 | W) & = (p + 1, p + 1, p + 3, p + 3, 2, 2, \ldots, 2, 2), \\
r(u_2 | W) & = (p, p, p + 2, p + 2, 3, 3, \ldots, 3, 3), \ldots \\
r(u_{p - 1} | W) & = (3, 3, 5, 5, p, p, \ldots, p, p), \\
r(k_1 | W) & = (p + 2, p + 2, p + 4, p + 4, 1, 1, \ldots, 1, 1), \\
r(k_2 | W) & = (p + 3, p + 3, p + 5, p + 5, 0, 1, \ldots, 1, 1), \\
r(k_3 | W) & = (p + 3, p + 3, p + 5, p + 5, 1, 0, \ldots, 1, 1), \ldots \\
r(k_{m - 1} | W) & = (p + 3, p + 3, p + 5, p + 5, 1, 1, \ldots, 1, 0), \\
r(k_m | W) & = (p + 3, p + 3, p + 5, p + 5, 1, 1, \ldots, 1, 1).
\end{align*}

For \( n = 12 \), \( W \) is resolving set with \( 4 + (m - 2) \) element. By choosing \( W = \{ v_1, v_3, v_7, v_9, \ldots, v_{n-8}, v_{n-6}, k_2, k_3, \ldots, k_{m-1} \} \) for \( n \geq 13 \). We obtained the representation for each vertex at \( (SF_n, v_2, (L_{m,p}, u_p) \) with respect to \( W \) is,

\begin{align*}
r(u_1 | W) & = (1, 3, 3, 3, \ldots, 3, 3, p + 3, p + 3, \ldots, p + 3, p + 3), \\
r(u_2 | W) & = (1, 2, 3, 3, \ldots, 3, 3, p + 2, p + 2, \ldots, p + 2, p + 2), \\
r(u_3 | W) & = (2, 1, 3, 3, \ldots, 3, 3, p + 2, p + 2, \ldots, p + 2, p + 2), \\
r(u_4 | W) & = (3, 1, 3, 3, \ldots, 3, 3, p + 3, p + 3, \ldots, p + 3, p + 3), \\
r(u_5 | W) & = (3, 2, 2, 3, \ldots, 3, 3, p + 4, p + 4, \ldots, p + 4, p + 4), \ldots \\
r(u_{n-1} | W) & = (3, 3, 3, 3, \ldots, 3, 3, p + 4, p + 4, \ldots, p + 4, p + 4), \\
r(u_n | W) & = (2, 3, 3, 3, \ldots, 3, 3, p + 4, p + 4, \ldots, p + 4, p + 4), \\
r(v_1 | W) & = (0, 3, 4, 4, \ldots, 4, 4, p + 3, p + 3, \ldots, p + 3, p + 3), \\
r(v_2 | W) & = (2, 2, 4, 4, \ldots, 4, 4, p + 1, p + 1, \ldots, p + 1, p + 1), \\
r(v_3 | W) & = (3, 0, 4, 4, \ldots, 4, 4, p + 3, p + 3, \ldots, p + 3, p + 3), \\
r(v_4 | W) & = (4, 2, 4, 4, \ldots, 4, 4, p + 4, p + 4, \ldots, p + 4, p + 4), \ldots \\
r(v_{n-1} | W) & = (3, 4, 4, 4, \ldots, 4, 4, p + 5, p + 5, \ldots, p + 5, p + 5), \\
r(v_n | W) & = (2, 4, 4, 4, \ldots, 4, 4, p + 4, p + 4, \ldots, p + 4, p + 4), \\
r(c | W) & = (2, 2, 2, 2, \ldots, 2, 2, p + 3, p + 3, \ldots, p + 3, p + 3), \\
r(u_1 | W) & = (p + 1, p + 1, p + 3, p + 3, 2, 2, \ldots, 2, 2), \\
r(u_2 | W) & = (p, p, p + 2, p + 2, 3, 3, \ldots, 3, 3), \\
r(u_3 | W) & = (p - 1, p - 1, p + 1, p + 1, 4, 4, \ldots, 4, 4), \ldots \\
r(u_{p-1} | W) & = (3, 3, 5, 5, p, p, \ldots, p, p), \\
r(k_1 | W) & = (p + 2, p + 2, p + 4, p + 4, 1, 1, \ldots, 1, 1), \ldots
\end{align*}
Theorem 2.2 Let $C(n, m)$ be a caveman graph. For any integer $n, m \geq 3$ hold
\[
\dim(C(n, m)) = \begin{cases} 
  n, & m = 3, 4; \\
  (m - 4)n, & m \geq 5.
\end{cases}
\]

Proof. Let $C(n, m)$ be a caveman graph. We consider two cases according to the values of $m$.

- Case 1, for $m = 3, 4$.
  In the case of $m = 3$, if $W = \{v_1^3, v_2^3, \ldots, v_n^3\} \subset C(n, 3)$, we obtained the representation of each vertex of $C(n, 3)$ with respect to $W$ as follows,
  \[
  \begin{align*}
  r(v_1^3 | W) &= (1, m, m + 2, m + 4, \ldots, m + 4, m + 2, m), \\
  r(v_2^3 | W) &= (0, m - 1, m + 1, m + 3, \ldots, m + 3, m + 1, m - 1), \\
  r(v_3^3 | W) &= (1, m - 2, m + 2, m + 4, \ldots, m + 4, m + 2, m), \\
  r(v_4^3 | W) &= (m, 1, m + 2, m + 4, \ldots, m + 4, m + 2), \\
  r(v_5^3 | W) &= (m - 1, 0, m - 1, m + 1, m + 3, \ldots, m + 3, m + 1), \\
  r(v_6^3 | W) &= (m, 1, m - 2, m + 2, m + 4, \ldots, m + 4, m + 2), \\
  &\vdots
  \end{align*}
\]

Figure 2. $C(n, m)$ graph
\[ r(v_i^1 | W) = (m, m + 2, m + 4, \ldots, m + 4, m + 2, m, 1), \]
\[ r(v_i^2 | W) = (m - 1, m + 1, m + 3, \ldots, m + 3, m + 1, m - 1, 0), \]
\[ r(v_i^3 | W) = (m - 2, m, m + 2, m + 4, \ldots, m + 4, m + 2, m, 1). \]

In the case of \( m = 4 \), if \( W = \{v_1^2, v_2^3, \ldots, v_n^3\} \subset C(n, 4) \), we obtained the representation of each vertex of \( C(n, 4) \) with respect to \( W \) as follows,

\[ r(v_i^1 | W) = (1, m + 1, m + 2, \ldots, m + 2, m + 1, m - 1), \]
\[ r(v_i^2 | W) = (0, m - 1, m + 1, \ldots, m + 2, m + 1, m - 1), \]
\[ r(v_i^3 | W) = (1, m - 1, m + 1, \ldots, m, m - 2), \]
\[ r(v_i^4 | W) = (1, m - 1, m + 1, \ldots, m + 1, m - 1), \]
\[ r(v_i^5 | W) = (m - 1, 0, m - 1, m + 1, \ldots, m + 2, m + 1), \]
\[ r(v_i^6 | W) = (m - 2, 1, m - 1, m + 1, \ldots, m), \]
\[ r(v_i^7 | W) = (m - 1, 1, m - 1, m + 1, \ldots, m + 1), \]
\[ \vdots \]
\[ r(v_i^i | W) = (m + 1, m + 2, \ldots, m + 2, m + 1, m - 1, 1), \]
\[ r(v_i^i+1 | W) = (m - 1, m + 1, \ldots, m + 2, m + 1, 1, 0), \]
\[ r(v_i^i+2 | W) = (m - 1, m + 1, \ldots, m, 1), \]
\[ r(v_i^i+3 | W) = (m - 1, m + 1, \ldots, m + 1, 1). \]

Representation for each vertex on caveman graph \( C(n, m) \) with \( m = 3, 4 \) with respect to \( W \) is different. As a result, \( W \) is the resolving set where \( |W| = n \), so that \( \dim(C(n, m)) = n \).

- Case 2, for \( m \geq 5 \).

In the case of \( m \geq 5 \), we choose \( W = \{v_1^1, v_2^2, v_3^3, \ldots, v_n^3\} \subset C(n, m) \). Suppose \( r(v_j^i | W) = (a_{1}^{i}, a_{2}^{i}, \ldots, a_{j-1}^{i}, a_{j}^{i}, a_{j+1}^{i}, \ldots, a_{m}^{i}) \) is the representation of vertex \( v_j^i \) with respect to \( W \) with \( j = 1, 2, \ldots, k \) and \( a_{1}^{i} = 1, a_{2}^{i} = 0, a_{3}^{i}, \ldots, a_{m-1}^{i} = 1, b_{1}^{i} = 3, b_{m-1}^{i} = 2 \), and \( c_{j+1}^{i} = b_{j}^{i} \) with \( i = 1, 2, \ldots, k \) and \( a_{j}^{i} = c_{j}^{i} + 2 \) then we get the representation of each vertex of \( C(n, m) \) with respect to \( W \) as follows,

\[ r(v_1^1 | W) = (1, 4, d_{1}^{1}, f_{1}^{1}, \ldots, e_{1}^{1}, 3, 1, 4, d_{1}^{1}, f_{1}^{1}, \ldots, e_{1}^{1}, 3, \ldots), \]
\[ r(v_2^1 | W) = (0, 3, e_{2}^{1}, f_{2}^{1}, \ldots, d_{3}^{1}, 3, 1, 3, e_{2}^{1}, f_{2}^{1}, \ldots, d_{3}^{1}, 3, \ldots), \]
\[ r(v_3^1 | W) = (1, 3, e_{3}^{1}, f_{3}^{1}, \ldots, d_{3}^{1}, 3, 3, 3, e_{3}^{1}, f_{3}^{1}, \ldots, d_{3}^{1}, 3, \ldots), \]
\[ \vdots \]
\[ r(v_{m-1}^1 | W) = (1, 3, e_{m-1}^{1}, f_{m-1}^{1}, \ldots, d_{m-1}^{1}, 2, 1, 3, e_{m-1}^{1}, f_{m-1}^{1}, \ldots, d_{m-1}^{1}, \ldots), \]
\[ r(v_1^2 | W) = (1, 2, e_{m}^{1}, f_{m}^{1}, \ldots, d_{m}^{1}, 3, 1, 2, e_{m}^{1}, f_{m}^{1}, \ldots, d_{m}^{1}, 3, \ldots), \]
\[ r(v_2^2 | W) = (3, 1, 4, d_{1}^{2}, f_{1}^{2}, \ldots, e_{1}^{2}, 3, 1, 4, d_{1}^{2}, f_{1}^{2}, \ldots, e_{1}^{2}, \ldots), \]
\[ r(v_3^2 | W) = (3, 0, 3, e_{2}^{2}, f_{2}^{2}, \ldots, d_{3}^{2}, 3, 1, 3, e_{2}^{2}, f_{2}^{2}, \ldots, d_{3}^{2}, 3, \ldots), \]
\[ r(v_2^3 | W) = (3, 1, 3, e_{3}^{2}, f_{3}^{2}, \ldots, d_{3}^{2}, 3, 3, 3, e_{3}^{2}, f_{3}^{2}, \ldots, d_{3}^{2}, 3, \ldots), \]
\[ \vdots \]
\[ r(v_{m-1}^2 | W) = (2, 1, 3, e_{m-1}^{1}, f_{m-1}^{1}, \ldots, d_{m-1}^{1}, 2, 1, 3, e_{m-1}^{1}, f_{m-1}^{1}, \ldots, d_{m-1}^{1}, \ldots), \]
\[ r(v_1^3 | W) = (3, 1, 2, e_{m}^{2}, f_{m}^{2}, \ldots, d_{m}^{2}, 3, 1, 2, e_{m}^{2}, f_{m}^{2}, \ldots, d_{m}^{2}, \ldots), \]
\[ r(v_2^3 | W) = (3, 0, 3, e_{2}^{3}, f_{2}^{3}, \ldots, d_{3}^{3}, 3, 0, 3, e_{2}^{3}, f_{2}^{3}, \ldots, d_{3}^{3}, 3, \ldots), \]
\[ r(v_3^3 | W) = (3, e_{3}^{3}, f_{3}^{3}, \ldots, d_{3}^{3}, 3, 1, 3, e_{3}^{3}, f_{3}^{3}, \ldots, d_{3}^{3}, 3, 0, \ldots), \]
\[ \vdots \]
\[
\begin{align*}
\forall v_{m-1} \in W, \quad & r(v_{m-1}) = (3, e_{m-1}^1, f_{m-1}^1, 1, \ldots), \\
\forall v_{m} \in W, \quad & r(v_{m}) = (2, e_{m}^1, f_{m}^1, \ldots, d_{m}^1, 3, 1, \ldots).
\end{align*}
\]

Representation for each vertex on a caveman graph \( C(n, m) \) with \( m \geq 5 \) with respect to \( W \) is different. As a result, \( W \) is the resolving set where \( |W| = (m - 4)n \). So \( \dim(C(n, m)) = (m - 4)n \).

3. Conclusion

Based on the description in the explanation above, it can be concluded that the metric dimensions of amalgamation of sunflower and lollipop graph as in the Theorem 2.1 and caveman graph \( C(n, m) \) as in Theorem 2.2.

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