Exact results in two dimensional chiral hydrodynamics with diffeomorphism and conformal anomalies

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Abstract

An exact formulation of two dimensional chiral hydrodynamics with diffeomorphism and conformal anomalies is provided. The constitutive relation involving the stress tensor is computed. This relation is analogous to the corresponding relation for an ideal fluid, appropriately modified to include the chirality property.

1 Introduction

A universal effective picture of any finite temperature quantum field theory on large time and length scales is provided by hydrodynamics [1]. This has led to a revival of interest in this topic. Also, the fluid-gravity map of AdS/CFT has further bolstered this activity. The fundamental equations governing the dynamics in this description are the conservation laws manifesting the global symmetries of the underlying theory. While the space-time symmetry leads to the conservation of the stress tensor, the charge symmetry leads to the conservation of charge current. Furthermore, the conservation laws are supplemented by constitutive relations expressing the stress tensor and the current in terms of the basic fluid variables which are velocity, temperature and chemical potential. These constitutive relations, naturally, have to be consistent with the conservation laws and are thus usually obtained by requiring compatibility with the second law of thermodynamics [1]. This standard picture is nontrivially modified in the presence of quantum anomalies.

In this paper we investigate the structure of constitutive relations in the presence of gravitational anomalies in two dimensions. Using techniques peculiar to two dimensions we obtain new results with novel interpretations and implications. We consider a chiral theory having both diffeomorphism and trace anomalies. The constitutive relation which is exact and not any order by order expansion, accounts for both these anomalies. Incidentally, two dimensions are very special and merit a separate study that is distinct from an arbitrary dimensional analysis. The metric in two dimensions is conformally flat which ensures that the effective action is exactly obtainable. A derivative expansion approach that is appropriate for calculating the effective action in higher dimensional fluid dynamics [2-4] now becomes redundant. In fact much of the two dimensional physics is hidden if one adopts the standard derivative expansion method or its variations [5-8].

In section 2 we provide a brief overview of chiral gravitational anomalies in two dimensions highlighting the role of the diffeomorphism anomaly vis-a-vis the conformal(trace) anomaly.
is necessary to appreciate the distinction between a chiral anomalous theory and a nonchiral anomalous theory. Section 3 contains the central result of the paper. The constitutive relation involving the stress tensor is calculated. An exact relation is obtained that is put in the form of the corresponding relation for an ideal chiral fluid. It may be observed that the constitutive relation for an ideal chiral fluid is different from a normal one. This is explained and the relevant relation is derived in section 4. The last section contains our concluding remarks.

2 General notions on gravitational anomalies

Consider a relativistic quantum field theory on a manifold with a time like killing vector. By an appropriate coordinate redefinition, the metric on such a manifold can be expressed as,

\[ ds^2 = -e^{2\sigma(\bar{x})}(dt + a_i(\bar{x}) \, dx^i)^2 + g_{ij}dx^i dx^j \]  

where the coordinates \( \bar{x} = x^i \) (\( i = 1, 2, 3, ..., p \)) specify (p) spatial degrees of freedom. For the special case of (1+1) dimensions, the Riemann manifold is conformally flat and we may write

\[ ds^2 = -e^{2\sigma(r)}dt^2 + g_{11}(r)dr^2 \]  

where the above form is taken for its resemblance with (1) and also for comparing with results given earlier. Precisely because two dimensional Riemann spaces may be put in the form (2), it is feasible to explicitly calculate the effective action. Variation of the effective action immediately yields the exact expression for the energy momentum tensor. This should be contrasted with higher dimensions where the effective action and hence, the stress tensor, is obtained approximately as a power series expansion. Nevertheless even though the stress tensor may not have a closed form, its conservation law (whether normal or anomalous) is obtained as an exact result.

This naturally leads us to the topic of anomalies. An anomaly is a breakdown of classical symmetry upon quantization. Generally this is manifested as a clash of symmetries. For instance, in QED if a gauge invariant regularisation is chosen, then axial symmetry is violated leading to non-conservation of the axial current. Likewise, it is possible to preserve conservation of axial current by using a different regularisation but then gauge invariance gets violated. The essence of the anomaly is that there is no regularisation that simultaneously preserves both the vector and axial symmetries.

A similar feature holds for the gravitational case. Here there is a clash between general coordinate invariance and conformal invariance. Preserving the first yields covariant conservation of the stress tensor whose trace, however, is non zero. A vanishing trace follows by involving a conformal invariant regularisation but this leads to a non conservation of the stress tensor. Since general coordinate invariance is regarded as more fundamental, this is usually kept intact at the expense of a trace anomaly. The above comments which are valid for a vector theory, have to be modified if the theory is chiral(i.e. chiral fermions moving in a curved background ). In this case both general coordinate invariance and conformal invariance are violated. There is a diffeomorphism anomaly (non conservation of the stress tensor) as well as a trace anomaly. This point is usually not appreciated leading to confusion. Moreover, for a chiral theory, there is a further subtlety. Chiral expressions may be regularised in various ways but there are particularly two which are useful for different reasons. Using a covariant regularisation one may obtain covariant expressions. However these do not satisfy the so called Wess-Zumino consistency condition or the integrability condition and hence cannot be directly obtained from a variation of the effective action. Expressions that
are derived in this manner are termed ‘consistent’. They satisfy the Wess-Zumino consistency condition but do not transform covariantly under a general coordinate transformation. For gravitational theories covariant expressions are preferred otherwise the basic transformation laws of tensors are violated.

Since we will be dealing with a chiral theory, let us analyse it in more details. For chiral fermions in a two dimensional gravitational background the possible structures of the covariant diffeomorphism anomaly and the trace anomaly are uniquely fixed, up to some normalisation, from general considerations as,

\[ \nabla_\mu T^\mu_\nu = N_d \epsilon_\nu R, \quad T^\mu_\mu = N_t R \]  

(3)

where \( N_d \) and \( N_t \) are normalization constants for the diffeomorphism anomaly and trace anomaly, respectively. Here \( R \) is the Ricci scalar which for the metric (2) is given by,

\[ R = \frac{1}{g^{11}} \left( g'_{11} \sigma' - 2g_{11} \sigma'' - 2g_{11} \sigma'' \right) \]  

(4)

and \( \bar{\epsilon}_{\mu\nu} = \sqrt{-g} \epsilon_{\mu\nu} \), with \( g = \text{det} g_{\mu\nu} \) and \( \epsilon_{\mu\nu} \) is the antisymmetric numerical tensor \( \epsilon_{01} = -\epsilon_{10} = -1 \). The functional form for the anomalies given in (3) is dictated by dimensionality, covariance and parity. The stress tensor \( T_{\mu\nu} \), due to its chiral nature, satisfies the property \[ J_{\mu} = -\bar{\epsilon}_{\mu\nu} J^\nu \]  

(6)

The chiral property is best manifested in the null coordinates,

\[ u = t - r^*, \quad v = t + r^*, \quad \frac{dr^*}{dr} = -\frac{e^\sigma}{\sqrt{g_{11}}} \]  

(7)

In these coordinates the metric is given by,

\[ ds^2 = -\frac{1}{2}e^{2\sigma}(duv + dvu) \]  

(8)

and the antisymmetric tensor \( \bar{\epsilon}_{\mu\nu} \) is given by

\[ \bar{\epsilon}_{uv} = \frac{e^{2\sigma}}{2} \]  

(9)

In our chosen system of coordinates it is easy to check from (5) that chirality is manifested by the vanishing of \( vv \)-component while, expectedly, the \( uv \)-component yields the trace

\[ T_{vv} = 0, \quad T_{uv} = g_{uv} T^\alpha_\alpha = -\frac{e^{2\sigma}}{4} T^\alpha_\alpha \]  

(10)

It is interesting to observe that the above results follow from general considerations. Subsequently we shall reproduce these from a direct calculation of the stress tensor obtained from the chiral effective action.

\[ ^1 \text{Covariant and consistent forms are related by local counterterms [9,10].} \]
We now reveal a connection between the normalization constants $N_d$ and $N_t$ appearing in (3).

For $\nu = v$, the left hand side of the first relation in (3) becomes,

$$\nabla_\mu T_\mu^v = \nabla_u T_u^v + \nabla_v T_v^u = \nabla_u(g^{uv}T_{uv}) + \nabla_v(g^{vu}T_{uv}) = \nabla_v[(-\frac{2}{e^{2\sigma}})(-\frac{e^{2\sigma}}{4})T_\alpha^\alpha]$$

(11)

where the results (10) have been exploited. Substituting the expression for the trace $T_\alpha^\alpha$ from (3) we find,

$$\nabla_\mu T_\mu^v = \frac{N_t}{2} \nabla_v R$$

(12)

Now the right hand side of the first relation in (3) for $\nu = v$, simplifies to,

$$\nabla_\mu T_\mu^v = N_d \bar{\epsilon} \nabla_\mu \omega_\mu = -N_d \frac{e^{2\sigma}}{2} \nabla^v R = N_d \nabla_v R$$

(13)

Equating (12) and (13) immediately yields the cherished connection,

$$N_t = 2N_d$$

(14)

It is now important to stress that chirality enforces both the trace and diffeomorphism anomalies. Note that the trivial(anomaly free) case $N_t = N_d = 0$ is ruled out since, using the unidirectional property of chirality, it is feasible to prove the existence of the diffeomorphism anomaly in 1 + 1 dimensions [12]. This implies that both $N_d$ and $N_t$ are non zero, connected by the relation (14).

For a non chiral theory, on the contrary, the relation (14) does not hold and the absence of a unidirectional property leads to a scenario where there is no diffeomorphism anomaly but a trace anomaly exists or vice-versa. This is the crucial distinction between a chiral and a non chiral theory. In any formulation of the chiral theory, therefore, it is imperative to properly account for both diffeomorphism and trace anomalies - a feature that we claim has not been accounted for or emphasized in the literature on hydrodynamics with anomalies.

The general analysis is now supplemented by explicit structures. The two dimensional chiral effective action is defined as, [13, 14]

$$Z(\omega) = -\frac{1}{12\pi} \int d^2x d^2y \epsilon_{\mu\nu}(x) \nabla^{-1}(x, y) \delta_{\mu\nu}(\epsilon^{\rho\sigma} + \sqrt{-g} g^{\rho\sigma})\omega_\sigma(y)$$

(15)

where $\nabla^{-1}$ is the inverse of the d’alembertian $\triangle = (\nabla^\mu \nabla_\mu) = \frac{1}{\sqrt{-g}} \partial_\mu[\sqrt{-g} g^{\mu\nu} \partial_\nu]$ and $\omega_\mu$ is the spin connection. This is an exact result.

The stress tensor may now be obtained by a variation of this effective action. That would yield the consistent expression. The covariant form that is relevant for our purpose is obtained by adding a local polynomial. This is allowed since energy momentum tensors and currents are only defined modulo local polynomials which manifest the regularisation freedom. We obtain,

$$\delta Z = \int d^2x \sqrt{-g} \left(\frac{1}{2} \delta g_{\mu\nu} T^{\mu\nu}\right) + l$$

(16)

where the local polynomial is given by,

$$l = \frac{1}{12\pi} \int d^2x \epsilon_{\mu\nu}[\omega_\mu \delta \omega_\nu + \frac{1}{8} Re_\mu^\alpha \delta e^\alpha_\nu]$$

(17)

where the zweibein vectors $e^\alpha_\mu$ fix the metric $g_{\mu\nu}$.
The covariant energy momentum tensor is easily read off from the above relations as,

\[ T^\mu_\nu = \frac{1}{96\pi} \left( \frac{1}{2} D^\mu G D_\nu G - D^\mu D_\nu G + \delta^\mu_\nu R \right) \]  \hspace{1cm} (18)

The chiral nature is highlighted by the presence of the chiral covariant derivative \( D_\mu \) defined in terms of the ordinary covariant derivative \( \nabla_\mu \),

\[ D_\mu = \nabla_\mu - \epsilon_{\mu\nu} \nabla^\nu = -\epsilon_{\mu\nu} D^\nu \]  \hspace{1cm} (19)

while \( R \) is defined in (4). The auxiliary field \( G \) in (18) is given by,

\[ G(x) = \int d^2y \sqrt{-g} \Delta^{-1}(x,y) R(y) \]  \hspace{1cm} (20a)

so that

\[ \Delta G(x) = R(x) \]  \hspace{1cm} (20b)

Calculating the covariant divergence and the trace of \( T^\mu_\nu \) from (18) we find,

\[ \nabla_\mu T^\mu_\nu = \frac{\epsilon_{\nu\mu}}{96\pi} \nabla^\mu R, \quad T^\mu_\mu = \frac{R}{48\pi} \]  \hspace{1cm} (21)

These are the covariant expressions for the diffeomorphism anomaly and the trace anomaly. They agree with the structures (3,14) given on general grounds. Furthermore, exploiting (19) it is simple to establish the chirality property (5) following from (18).

Solving (20b) we obtain,

\[ G = G_0(r) - 4pt + q, \quad \partial_r G_0 = \frac{\sqrt{g_{11}}}{e^\sigma} \left( \frac{2\sigma' e^\sigma}{\sqrt{g_{11}}} + z \right) \]  \hspace{1cm} (22)

where \( p,q \) and \( z \) are constants. This solution is exploited to obtain the explicit form for \( T^\mu_\nu \). In the null coordinates we have already found (see (10)) that \( T_{vv} = 0 \) and \( T_{uv} \) is proportional to the trace. Using (10) and (21) we get,

\[ T_{uv} = -\frac{e^{2\sigma}}{4} \times \frac{R}{48\pi} = -\frac{e^{2\sigma} R}{192\pi} \]  \hspace{1cm} (23)

The only other component \( T_{uu} \) is obtained from (18) and (22) after a slight calculation,

\[ T_{uu} = \frac{1}{96\pi} \frac{e^{2\sigma}}{g_{11}^2} (2g_{11}\sigma'' - \sigma' g_{11}') + C \]  \hspace{1cm} (24)

where \( C \) is a constant made out of \( p,q \) and \( z \). It may be fixed by choosing a boundary condition. A reasonable one in this context is the Boulware vacuum. We require the vanishing of the stress tensor at asymptotic infinity where the metric is assumed flat. In that case at \( r \to \infty \), \( e^{2\sigma} = g_{11} = 1 \), \( \sigma' = \sigma'' = g_{11}' = T_{uu} = 0 \) so that \( C = 0 \). However to keep the discussion general, we retain \( C \). Relations (23,24) along with \( T_{vv} = 0 \) yield the exact expressions for the covariant stress tensor in the null coordinates.
3 Anomaly induced constitutive relation

The results for the stress tensor will now be used to deduce the constitutive relation. Consider the standard ansatz for the velocity \( u^\mu \) of the time independent equilibrium fluid fields in the background metric (2),

\[
\begin{align*}
u^\mu = e^{-\sigma}(1, 0), \quad u_\mu = -e^\sigma(1, 0), \quad (\mu = t, r) 
\end{align*}
\] (25)
satisfying the criterion \( u^\mu u_\mu = -1 \). Correspondingly, in the null coordinates, we have,

\[
\begin{align*}
u_\mu = -e^{-2}(1, 1), \quad u^\mu = e^{-\sigma}(1, 1), \quad (\mu = u, v)
\end{align*}
\] (26)
which has a more symmetrical structure. Introduce the dual vectors,

\[
\begin{align*}	ilde{v}_\mu &= \bar{\epsilon}^{\mu\nu} u^\nu = e^{-2}(1, -1), \quad \tilde{u}^\mu u_\nu = e^{-\sigma}(1, -1), \quad (\mu = u, v)
\end{align*}
\] (27)
where we have used (9) to obtain the explicit values. Observe that the dual velocity fields satisfy a different normalization from the original ones, since \( \tilde{v}_\mu \tilde{v}^\mu = 1 \).

To obtain the constitutive relation let us first write certain relations that express covariant combinations of the fluid variables in terms of the metric coefficients,

\[
\begin{align*}
u^\mu \nabla_\rho \nabla^\rho v_\mu = -\frac{1}{2} R, \quad \nu^\rho \nabla_\mu \nabla_\mu v_\rho = \frac{\sigma^2}{g_{11}}
\end{align*}
\] (28)

Using the above equalities, the stress tensor (23, 24) in the null coordinates is expressed as

\[
\begin{align*}T_{uu} &= \frac{e^{2\sigma}}{48\pi}(u^\alpha \nabla_\beta u_\alpha - u^\beta \nabla_\alpha u_\beta) + C, \quad T_{uv} = \frac{e^{2\sigma}}{96\pi}(u^\alpha \nabla_\beta u_\alpha)
\end{align*}
\] (29)
These relations along with \( T_{vv} = 0 \), are expressible as follows:

\[
\begin{align*}
T_{\mu\nu} = \left\{ \frac{1}{48\pi} [(u^\alpha \nabla_\beta - u^\beta \nabla_\alpha) \nabla_\alpha u_\beta] + e^{-2\sigma} C \right\} \left( 2u_\mu u_\nu - \tilde{u}_\mu u_\nu - \tilde{u}_\mu \tilde{u}_\nu \right) \\
&\quad - \left\{ \frac{1}{48\pi} [u_\beta \nabla_\alpha \nabla_\alpha u_\beta] - e^{-2\sigma} C \right\} g_{\mu\nu}
\end{align*}
\] (30)
Introducing the temperature \( T \) by the usual ansatz \( T = T_0 e^{-\sigma} \), we obtain,

\[
\begin{align*}
T_{\mu\nu} = \left\{ \frac{1}{48\pi} [(u^\alpha \nabla_\beta - u^\beta \nabla_\alpha) \nabla_\alpha u_\beta] + \tilde{C} T^2 \right\} \left( 2u_\mu u_\nu - \tilde{u}_\mu u_\nu - \tilde{u}_\mu \tilde{u}_\nu \right) - \\
&\quad \left\{ \frac{1}{48\pi} [u_\beta \nabla_\alpha \nabla_\alpha u_\beta] - \tilde{C} T^2 \right\} g_{\mu\nu}
\end{align*}
\] (31)
where \( \tilde{C} = C T_0^{-2} \) is an arbitrary constant. This is the cherished constitutive relation in the presence of chiral(covariant) anomalies. It is an exact relation and is a new result. There are no derivative corrections. Moreover, the specific functional form of the factor involving the velocity is unique since it is the only one that ensures chirality. This will be clarified in the next section.

In the fluid variables it is straightforward to show that (30) correctly reproduces both the diffeomorphism anomaly (21) as well as the trace anomaly (21) obtained earlier from the equilibrium
partition function. For example the trace anomaly is trivially seen by contracting and using \( u^\mu u_\mu = -1 \), \( \tilde{u}^\mu u_\mu = 0 \) to yield,

\[
T_\mu^\nu = -\frac{1}{24\pi} (u^\alpha \nabla^\beta \nabla_\alpha u_\beta)
\]  

(32)

This reproduces, on using (28), the expression (21) for the trace anomaly. Likewise the diffeomorphism anomaly may also be reproduced.

4 Correspondence with ideal chiral fluid constitutive relation

We now show that the relation (32) can be put exactly in the form governing an ideal chiral fluid. Let us recall that the familiar (textbook) ideal fluid constitutive relation is given by the well known formula

\[
T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + pg_{\mu\nu}
\]  

(33)

where \( \epsilon \) and \( p \) are the energy density and pressure, respectively.

The above relation, as it stands, does not properly account for chirality and has to be modified. In two dimensions chiral vectors or tensors must satisfy properties similar to (6) or (5). Note that chirality is an algebraic property which should be manifested at the classical level itself. To incorporate these features in (33), replace the velocity \( u_\mu \) by the chiral velocity vector,

\[
u^c_\mu = u_\mu - \tilde{u}^\mu = u_\mu - \bar{\epsilon}^{\mu\nu}u_\nu
\]  

(34)

It is easy to see that (34) satisfies the chiral condition

\[ u^c_\mu u^c_\mu = 0 \]  

(38)

following from the basic definition (34) or (35), it follows from (36) that \( p \) gets identified with the trace of the stress tensor, \( p = \frac{1}{2}T^\mu_\mu \)  

(39)

Putting this back in (37) immediately yields the standard chiral property (5) for \( T_{\mu\nu} \).

Now using (34) the relation (39) is expressed as

\[
T_{\mu\nu} = (\epsilon + p)(2u_\mu u_\nu - u_\mu \tilde{u}_\nu - \tilde{u}_\mu u_\nu) + (\epsilon + 2p)g_{\mu\nu}
\]  

(40)

where, at an intermediate step, the identity,

\[ \tilde{u}_\mu \tilde{u}_\nu - u_\mu u_\nu = g_{\mu\nu} \]  

(41)
has been exploited. We regard (40) as the appropriate constitutive relation for an ideal chiral fluid in two dimensions.

As announced earlier the constitutive relation (30) in the presence of anomalies has precisely the same form (40) as the ideal case once we identify,

$$\epsilon + p = \frac{1}{48\pi} (u^\alpha \nabla_\beta - u^\beta \nabla_\alpha) \nabla_\alpha u_\beta$$

$$\epsilon + 2p = -\frac{1}{48\pi} u^\beta \nabla^\alpha \nabla_\alpha u_\beta$$

(42)

Solving (42) yields,

$$\epsilon = \frac{1}{48\pi} (2u^\alpha \nabla_\beta - u^\beta \nabla_\alpha) \nabla_\alpha u_\beta$$

$$p = -\frac{1}{48\pi} u^\alpha \nabla^\beta \nabla_\alpha u_\beta$$

(43)

Recalling from (39) that $p$ is half the trace of the stress tensor we see, in the anomalous context, that it exactly corresponds to half the trace anomaly (32). Furthermore, computing the $0-0$ component of (30) yields

$$T_{00} = \epsilon \sigma$$

(44)

where $\epsilon$ is defined in (43). This shows that $\epsilon$ is interpretable as the energy density. The same result also follows by contracting (40) with $u^\mu u^\nu$ to get,

$$\epsilon = u^\mu u^\nu T_{\mu\nu}$$

(45)

and then using the ansatz (25).

We conclude this section by comparing our results with those existing in the literature [6, 7]. Also, this will help to put the present work in a proper perspective.

Let us begin by considering [6], which is specifically devoted to two dimensions. The constitutive relation for the covariant stress tensor ($T_{\mu\nu}$) of a chiral theory found there involves the structure $u_\mu \tilde{u}_\nu + \tilde{u}_\mu u_\nu$. As discussed earlier, there is a unique structure for the velocity factor that ensures chirality (see 30). Since this is not reproduced the results of [6] are incorrect. To see this more clearly, the stress tensor has a vanishing trace since $T_{\mu\mu} \sim u^\mu \tilde{u}_\mu = 0$ implying that there is no conformal (trace) anomaly. However, as shown here in the discussion surrounding (14), chirality simultaneously enforces both diffeomorphism and trace anomalies. A constitutive relation, like the one given in [6], that cannot account for this feature is incorrect.

The analysis of [7] is more general where the results for two dimensions are considered separately. In so far as [7] uses the findings of [6] it is subjected to the same criticism. However the section on the thermodynamics in two dimensions contains results which have a parallel with those presented here. This will now be expedited.

The constitutive relation found in [7] by the gradient expansion approach is quite involved and does not reveal the chiral structure. To obtain the final form where this structure becomes manifest it is mandatory to use a couple of highly nontrivial relations that fix the values of certain constants in terms of the coefficients of the diffeomorphism and trace anomalies that coexist in the chiral theory. Incidentally, the principal objective of [7] was to study these relations.\footnote{It is actually possible to invert the argument by demanding chirality of the stress tensor in order to reproduce such relations.}
use of this additional information obscures the derivation of the constitutive relation. As shown here, no extra input is either required or used to obtain the constitutive relation which follows naturally from the effective action. The other point is that the use of extra relations to obtain the constitutive equation automatically fixes the constant $\bar{C}$ in (31) to be $\frac{\pi}{12}$. In this sense the result of [7] is restrictive and yields the constitutive relation corresponding to a particular boundary condition.

Let us wind up by stating that we have developed a novel way of analyzing chiral hydrodynamics where special properties of chirality in two dimensions were exploited. Apart from other features discussed above, this explains the functional structure of the velocity factor in (30,31). It is the unique form that ensures the chirality of $T_{\mu\nu}$. This was elaborated by explicitly working out the constitutive relation for an ideal(chiral) fluid. Given in (40) this is a new structure that is different from the familiar relation (33) for a nonchiral ideal fluid.

5 Conclusion

Gravitational anomalies in 1+1 dimensions have played a ubiquitous role in physics. Besides producing surprising connections between physics and mathematics, they have been used extensively to understand the Hawking effect in black holes [12,15,16] or the thermal Hall effect in topological insulators [18]. Very recently their influence in hydrodynamics has generated considerable interest [2–7]. The two dimensional example, in particular, is somewhat unique, because of its special features.

In this paper we have developed a new formulation for two dimensional chiral hydrodynamics with anomalies. The special characteristics of both chirality and two dimensions have been taken into account. While the specific nature of two dimensions ensures that the formulation yields exact results, chirality puts additional restrictions on the structure of the constitutive relations. These restrictions are algebraic in nature and are thus valid even for an ideal(chiral) fluid. We have explicitly given the form for the constitutive relation (see 40) for an ideal(chiral) fluid. This new structure helps to understand the eventual nature of the constitutive relation for quantum chiral hydrodynamics. At the quantum level, chirality implies the presence of both diffeomorphism and conformal(trace) anomalies. It was reassuring to note that the constitutive relation (31) for quantum chiral hydrodynamics, which was the central result of the paper, was compatible with both types of anomalies as well as the algebraic restriction imposed by chirality in two dimensions. We have also discussed the novelty of this relation, both in its content as well as in its derivation, vis-a-vis a similar relation in [7].

This analysis illuminates a similarity between hydrodynamics and quantum field theory. Two dimensional models in QFT are exactly solvable and using the technique of bosonisation, may be put in a form that resembles the free theory but with a different normalisation of the parameters [19]. Something analogous happens here also. The exact constitutive relation for two dimensional anomalous chiral hydrodynamics is put in the corresponding form of an ideal chiral fluid but with a different normalisation of the parameters.

It is possible to extend or generalise the present work. For example, one can consider the case of a charged chiral fluid so that the analysis has to be performed in the presence of gravitational as well as gauge anomalies. Another aspect would be to drop the chirality condition and treat a nonchiral fluid in the presence of anomalies. This case is fundamentally different from chiral hydrodynamics.
considered here because the lack of unidirectional property ensures that both diffeomorphism and coformal anomalies do not coexist. There would be a diffeomorphism anomaly but no trace anomaly or vice-versa. Since for gravitational theories diffeomorphism invariance is more fundamental it is retained at the expense of a conformal anomaly. Its consequences for hydrodynamics have been considered, albeit briefly, in\[7,8\] within the derivative expansion scheme. Nevertheless, here also one expects exact results in the two dimensional case. Implications of this analysis to higher dimensions is yet another possibility to be explored. Work in these directions is in progress and will soon be reported \[20\].

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