Entanglement fidelity of the standard quantum teleportation channel

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Abstract

We consider the standard quantum teleportation protocol where a general bipartite state is used as entanglement resource. We use the entanglement fidelity to describe how well the standard quantum teleportation channel transmits quantum entanglement and give a simple expression for the entanglement fidelity when it is averaged on all input states.

Keywords: Quantum teleportation, entanglement fidelity

1. Introduction

Nowadays the protocol of quantum teleportation [1] plays an important role in quantum information science [2]. Quantum teleportation can naturally be related to quantum channels since there are an input and an output state involved. Mathematically, a quantum channel is a completely positive and trace-preserving (CPTP) operator that maps an input density operator to an output density operator, and it can be represented in an operator-sum form [2, 3].

The property of a quantum teleportation channel is dependent on both the entanglement resource and the particular local operations and classical communication (LOCC) we used [4, 5, 6]. In a realistic quantum teleportation the sender and the receiver usually share a mixed entangled state, instead of a maximally entangled pure state, accounted to the decoherence. Quantum teleportation using a mixed entangled state is equivalent to a noisy quantum channel. In 2001, it was shown that the standard quantum teleportation protocol using a mixed entangled resource is the same as a generalized depolarizing channel [6].

In this paper we consider the standard quantum teleportation protocol where a general bipartite state is used as the entanglement resource. It is known that the ordinary fidelity [2, 3] between the input state and the output state is usually used to measure the quality of a quantum teleportation channel [4, 5]. However, people may be interested in how well a quantum teleportation channel preserves quantum entanglement in the case that the particle to be teleported is entangled with some other particle. To answer this question, we will consider entanglement fidelity [2, 3] instead of the ordinary fidelity. To our knowledge, entanglement fidelity has not yet been used to measure the quality of quantum teleportation channel. The main result of this paper is to give a simple expression for the entanglement fidelity of the standard quantum teleportation channels when it is averaged on all input states.

2. The standard quantum teleportation

A general quantum teleportation protocol is as follows. Suppose the sender Alice and the receiver Bob share an entangled state $\chi_{34}$, where 3 and 4 stand for the particles shared by Alice and Bob respectively, and Alice is given another particle 1 in an unknown state $\rho_1$ to be teleported to Bob. We assume each particle is associated with a $d$-dimensional Hilbert space. To start quantum teleportation, Alice first performs a measurement on particles 1 and 3, which is described by a collection of measurement operators $M_{13}$ with $\sum_i M_{13}^{\dagger}M_{13} = I_{13}$, where $i$ denotes measurement result. The state of Bob’s particle after the measurement will change to

$$\rho_{i} = \frac{1}{p_i}Tr_{13}\left[(M_{13}^{\dagger} \otimes I_4)(\rho_1 \otimes \chi_{34})(M_{13} \otimes I_4)\right]$$

(1)

if the result $i$ occurs, where

$$p_i = Tr_{13}[(M_{13}^{\dagger} \otimes I_4)(\rho_1 \otimes \chi_{34})(M_{13} \otimes I_4)]$$

(2)

is the probability of obtaining the measurement result $i$. After obtaining the measurement result $i$, Alice tells Bob the result $i$ via a classical channel. Then Bob applies a quantum operation $\epsilon^i$, a completely positive and trace-preserving (CPTP) operator that maps the input density operator $\rho_1$ to the output density operator $\gamma_4$ [10]. In an operator-sum form the quantum teleportation channel $\epsilon$ can be written as $[11, 12, 13]

$$\gamma_4 = \epsilon (\rho_4) = \sum_i A_i^\dagger \rho_4 A_i$$

(3)

with $\sum_i A_i A_i^\dagger = I_4$ and $\rho_4$ being the same state as $\rho_1$. It is obvious that the operators $A_i$ depend on the entanglement resource $\chi_{34}$, the sender’s measurement operators $M_{13}$ and the receiver’s corresponding CPTP maps $\epsilon^i$. 
In the standard quantum teleportation, the maximally entangled state
\[ |\Omega^{0,0}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle, \]
is usually assumed to be used as the entanglement resource to teleport a d-dimensional state. And the sender’s measurement is a generalized Bell measurement with measurement operators \([|\Omega^{n,m}\rangle \langle \Omega^{n,m}|]\), which are defined as
\[ |\Omega^{n,m}\rangle = (U^{n,m} \otimes I)|\Omega^{0,0}\rangle, \]
where \(n, m = 0, 1, \ldots, d-1\).

Here the unitary operator
\[ U^{n,m} = e^{2\pi in/d}|j\rangle\langle j| (j \equiv m \mod d). \]

When a measurement result denoted by index \((n, m)\) is obtained, i.e., the state of sender’s particles is mapped to state \(|\Omega^{n,m}\rangle\), the receiver’s corresponding CPTP map is defined to be the unitary operator \(U^{n,m}\). This protocol of quantum teleportation can teleport any d-dimensional state perfectly. However, it can only be viewed as a noisy quantum channel when a general entanglement resource, it has been shown that
\[
\mathcal{F}(\epsilon) = \frac{d}{d+1} f + \frac{1}{d+1},
\]
where
\[ f = p_{00} = \langle \Omega^{0,0}| \chi |\Omega^{0,0}\rangle \]
is the generalized singlet fraction. We can also make use of the entanglement fidelity to characterize the similarity between the input state and the entangled state preserved by this standard quantum teleportation channel \(\epsilon\). The entanglement fidelity \(F_\epsilon(\rho, \epsilon(\rho))\) measures how well the entangled state \(|\varphi\rangle\) is preserved. We note that any purification of the input state \(\rho\) can be used in Eq. (14) and it always gets the same result. The entanglement fidelity \(F_\epsilon(\rho, \epsilon(\rho))\) is defined as
\[ F_\epsilon(\rho, \epsilon(\rho)) = \langle \varphi | I \otimes \epsilon(|\varphi\rangle \langle \varphi|) |\varphi\rangle, \]
where \(|\varphi\rangle\) is a d \times d bipartite state and is a purification of the input state \(\rho\). The entanglement fidelity \(F_\epsilon(\rho, \epsilon(\rho))\) measures how well quantum entanglement is preserved by this standard quantum teleportation channel. The standard quantum teleportation channel \(\epsilon\) in Eq. (5) generally cannot transport quantum state perfectly and a way is needed to measure how well the output state \(\epsilon(\rho)\) is similar to the input state \(\rho\). The fidelity of the input state \(\rho\) and the output state \(\epsilon(\rho)\) can be used to do this, which is defined as
\[
F(\rho, \epsilon(\rho)) = \left(\text{Tr} \sqrt{\sqrt{\epsilon(\rho) \rho \epsilon(\rho)}}\right)^2. \]
When the input state is a pure state \(|\psi\rangle\) the fidelity will be
\[
F(|\psi\rangle, \epsilon(|\psi\rangle \langle \psi|)) = \langle \psi | \epsilon(|\psi\rangle \langle \psi|) |\psi\rangle. \]

It measures the similarity between the output state \(\epsilon(|\psi\rangle \langle \psi|)\) and the input state \(|\psi\rangle\) in the way that it will be zero when the output state \(\epsilon(|\psi\rangle \langle \psi|)\) is orthogonal to the input state \(|\psi\rangle\) and be the unit when the output state \(\epsilon(|\psi\rangle \langle \psi|)\) is the same as the input state \(|\psi\rangle\).

However, there are many possible input states and different input states can lead to different fidelities. The average of the fidelity \(F(\rho, \epsilon(\rho))\) over all input pure state \(|\psi\rangle\) is usually introduced to characterize the quality of the standard quantum teleportation channel \(\epsilon\). Precisely, the quantity
\[
\mathcal{F}_\epsilon(\epsilon) = \int d\psi \langle \psi | \epsilon(|\psi\rangle \langle \psi|) |\psi\rangle
\]
is used to measure how well the standard quantum teleportation channel \(\epsilon\) is similar to a perfect channel, where the integral is performed with respect to the uniform distribution \(d\psi\) over all input pure states. For the standard quantum teleportation channel \(\epsilon\) using a general \(d \times d\) bipartite state \(\chi\) as the entanglement resource, it has been shown that
\[
\mathcal{F}_\epsilon(\epsilon) = \frac{d}{d+1} f + \frac{1}{d+1},
\]
where
\[ f = p_{00} = \langle \Omega^{0,0}| \chi |\Omega^{0,0}\rangle \]
is the generalized singlet fraction. We can also make use of the entanglement fidelity to characterize the similarity between the input \(\rho\) and the output state \(\epsilon(\rho)\) of the standard quantum teleportation channel \(\epsilon\). The entanglement fidelity \(F_\epsilon(\rho, \epsilon(\rho))\) is defined as
\[ F_\epsilon(\rho, \epsilon(\rho)) = \langle \varphi | I \otimes \epsilon(|\varphi\rangle \langle \varphi|) |\varphi\rangle, \]
where \(|\varphi\rangle\) is a d \times d bipartite state and is a purification of the input state \(\rho\). The entanglement fidelity \(F_\epsilon(\rho, \epsilon(\rho))\) measures how well the entangled state \(|\varphi\rangle\) is preserved. We note that any purification of the input state \(\rho\) can be used in Eq. (14) and it always gets the same result. The entanglement fidelity \(F_\epsilon(\rho, \epsilon(\rho))\) is dependent on the input state \(\rho\), but we can use
\[
\mathcal{F}_\epsilon(\epsilon) = \int d\psi \langle \psi | \epsilon(|\psi\rangle \langle \psi|) |\psi\rangle
\]
to measure how well the standard quantum teleportation channel \(\epsilon\) preserve quantum entanglement, where the integral is performed with respect to the uniform distribution \(d\psi\) over all \(d \times d\) bipartite pure states, which is equal to sample mixed state \(\rho\) uniformly with respect to Hilbert-Schmidt measure. Our main result is to give an expression for \(\mathcal{F}_\epsilon(\epsilon)\), which is summarized in the following theorem:

**Theorem 1.** In the standard quantum teleportation channel \(\epsilon\) where a general \(d \times d\) bipartite state \(\chi\) is used as the entanglement resource, the average of the entanglement fidelity \(\mathcal{F}_\epsilon(\epsilon)\) defined in Eq. (15) is given by
\[
\mathcal{F}_\epsilon(\epsilon) = \frac{d^2}{d^2 + 1} f + \frac{1}{d^2 + 1},
\]
where
\[ f = \langle \Omega^{0,0}| \chi |\Omega^{0,0}\rangle \]
is the generalized singlet fraction defined in Eq. (13).
Proof. The standard quantum teleportation channel $\varepsilon$ has an operator-sum form as shown in Eq. (8), and we can submit it to $F_\varepsilon(\varepsilon)$ in Eq. (15) to get

$$F_\varepsilon(\varepsilon) = \sum_{n,m} p_{nm} \int_\varphi d\varphi \lambda_{nm}(\varphi)$$

where

$$\lambda_{nm}(\varphi) = |\langle \varphi | (I \otimes U_{n-m}) | \varphi \rangle|^2.$$  \hspace{1cm} (17)

We can also write $\lambda_{nm}(\varphi)$ as

$$\lambda_{nm}(\varphi) = ((\varphi \otimes \langle \varphi |) \mu_{nm} (\varphi \otimes |\varphi \rangle),$$

where

$$\mu_{nm} = I \otimes U^{n-m} \otimes I \otimes U^{n-m}.$$  \hspace{1cm} (20)

Our next step is to compute $\int_\varphi d\varphi \lambda_{nm}(\varphi)$ using Eq. (19). We first note that

$$\int_\varphi d\varphi \lambda_{nm}(\varphi) = \langle 00 | \int_\varphi (V^\dagger \otimes V^\dagger) \mu_{nm} (V \otimes V) d\varphi | 00 \rangle,$$

where $V$ are unitary operators defined on a Hilbert space of dimension $d^2$ and the integral is performed with respect to the uniform distribution $d\varphi$ over all unitary operators. Using Schur’s lemma [10, 17], we can find

$$\int_\varphi d\varphi \lambda_{nm}(\varphi) = \alpha_{nm} + \beta_{nm},$$  \hspace{1cm} (22)

with

$$\alpha_{nm} = \frac{Tr(\mu_{nm})}{d^4 - 1} - \frac{Tr(\mu_{nm}F)}{d^2(d^4 - 1)}$$

$$\beta_{nm} = \frac{Tr(\mu_{nm}F)}{d^2 - 1} - \frac{Tr(\mu_{nm})}{d^2(d^4 - 1)}$$

where $F$ is the exchange operator. Using the identity [3, 18]

$$Tr(U_{nm}^m) = d\delta_{n0}\delta_{m0},$$  \hspace{1cm} (25)

$$Tr((A \otimes B) F) = Tr(AB).$$  \hspace{1cm} (26)

We have the following results

$$Tr(\mu_{nm}) = d^2\delta_{n0}\delta_{m0}, Tr(\mu_{nm}F) = d^2$$

Then we have

$$\alpha_{nm} = \frac{d^2\delta_{n0}\delta_{m0}}{d^4 - 1} - \frac{1}{d^4 - 1},$$

$$\beta_{nm} = d^2 - \frac{d^2\delta_{n0}\delta_{m0}}{d^4 - 1}.$$  \hspace{1cm} (29)

Then we have

$$\bar{F}_\varepsilon(\varepsilon) = \sum_{n,m} p_{nm} (\alpha_{nm} + \beta_{nm}) = \frac{d^2}{d^2 + 1} + \frac{1}{d^2 + 1}$$

Here $p_{00} = \langle \Omega_{00} | X | \Omega_{00} \rangle$ = $f$ is the generalized singlet fraction.

We note that the expressions of the average $\bar{F}_\varepsilon(\varepsilon)$ of the entanglement fidelity and the average $\bar{F}(\varepsilon)$ of the ordinary fidelity are very similar, which is due to the fact that both quantities can be deduced via Schur’s lemma [10, 17].

4. Conclusion

We use the entanglement fidelity to measure the quality of the standard quantum teleportation channel where a general $d \times d$ bipartite state $x$ instead of the maximally entangled pure state $|\Omega_{00}\rangle$ is used as the entanglement resource. We obtain an explicit expression for the average $\bar{F}_\varepsilon(\varepsilon)$ of the entanglement fidelity, which is only dependent on the generalized singlet fraction. Our obtained $\bar{F}_\varepsilon(\varepsilon)$ quantifies how well the teleportation channel $\varepsilon$ preserves quantum entanglement.

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