1. Introduction

Geodesic acoustic modes (GAMs) are basically electrostatic potential fluctuations with finite radial wavenumber and exist naturally in tokamak plasmas [1, 2]. They play an important role in moderating plasma turbulence and turbulent transport by affecting the radial electric field \( E_r \) as well as the flow \( \propto E_B \). There have been plentiful experimental observations (see, for example, [6–8]), theoretical analysis [9, 10], and numerical simulation [11–14] on GAMs. As the high-frequency branch of the zonal flows, GAMs have the toroidally and poloidally symmetrical structures \((m = n = 0)\) and a typical frequency \( \omega^2 = \nu_{ti}^2 \Gamma (1 + \tau) (1 + 1/(2q^2))/R^2 \) derived from the ideal magnetohydrodynamic (MHD) model, where \( q \) is the safety factor, \( R \) is the major radius of the tokamak, and \( \nu_{ti} = (2T_i/m_i)^{1/2} \) is the ion thermal speed, \( \Gamma \) is the adiabatic index, and \( \tau = T_e/T_i \) is the temperature ratio. Meanwhile, the kinetic theory [15, 16] and two fluid model [17–19] predict \( \omega^2 = (7/4 + \tau)\nu_{ti}^2/R^2 \).

GAMs can be significantly affected by the toroidal rotation (TR) flow [20] which generally exists in a tokamak plasma, especially in the case with strong tangential injection of neutral beams [21]. Owing to the significant applications to the shear flow control of anomalous transport and turbulence, the magnitude, radial profile, and evolution of the TR are important issues in ITER [22]. Similarly, energetic particles (EPs) are becoming a key issue in tokamak plasma physics as well [23]. EPs are shown to drive a new sort of GAM known as the energetic particle driven geodesic acoustic mode (EGAM), which was originally theoretically predicted by Fu in 2008.
[24], and then experimentally observed [25], and reported by a recent flux-driven 5D gyrokinetic simulation providing direct evidence of the impact on turbulent transport of EGAMs [26]. Because radial inhomogeneity can lead to a continuous spectrum of GAMs [27, 28], the nonlocal theory of EGAMs has also been developed taking into account the nonuniformity of EPs radial density profile in [28, 29].

During the intense neutral beam injection (NBI) heating process, not only is the plasma TR driven, but EPs are also generated. Sometimes, the bulk plasma anisotropy is also excited and should be taken into account, especially for GAMs. We know that the TR and bulk plasma anisotropy play an important role in the oscillation frequency and collisionless damping rate of GAMs [30]. It is then, quite natural to ask how they affect the EGAMs—does the bulk ions’ anisotropy interact with the EPs’ anisotropy? In this paper, we try to answer these questions and find out the dispersion relation of EGAMs in a toroidally rotating anisotropic plasma. The TR flow speed is allowed to be of the order of the ion thermal speed.

The remaining content of the present paper is organized as follows. The rigid equilibrium condition and self-consistent distributions of EPs and plasma are discussed first in section 2. In section 3, we use a hybrid fluid-kinetic model to investigate the EGAMs in a toroidally rotating anisotropic tokamak plasma. The EPs are described by gyrokinetic equations and the bulk plasma by double adiabatic equations. Next, we review this issue in the full kinetic framework in section 4. It is found that keeping only leading-order terms and ignoring the two equations yields the hybrid one. Section 5 is devoted to the detailed discussion about the dispersion relation. The importance of ion Landau damping is demonstrated. Finally, a brief summary is presented in section 6.

2. Equilibrium analysis

The force balance of plasmas in the presence of EPs can be described by the following hybrid kinetic-fluid momentum equation as [23],

\[
\rho \frac{d \vec{v}}{dt} = -\nabla \cdot \vec{p} - \nabla \cdot \vec{p}_h + (\nabla \times \vec{B}) \times \vec{B},
\]

in which \(\vec{p} (\vec{p}_h)\) is the pressure tensor of the bulk ions (EPs) with a Chew–Goldberger–Low (CGL) form [31], \(\vec{p} = p_h \hat{I} + (p_h - p_i) \hat{b} \hat{b}\). Other symbols have their usual meanings. We consider a large-aspect-ratio tokamak plasma with a toroidal symmetric magnetic field \(\vec{B} = I(\psi, \theta) \nabla \zeta_T + \nabla \zeta_T \times \nabla \psi\), and work in the \((r, \theta, \zeta_T)\) coordinate system, where \(\psi\) is the magnetic flux, \(\zeta_T\) and \(\theta\) are the toroidal and poloidal angles respectively. \(\tilde{f}_0\) denotes the equilibrium profile and \(\tilde{f}\) the perturbed one but the subscript is omitted and the equilibrium magnetic field is referred to by \(\vec{B}\) directly. The equilibrium condition should be discussed first carefully since the poloidal dependence of the equilibrium profiles is of great importance to the GAM as well as the EGAM. From the equation above, we find

\[
\nabla \cdot \left( p_{\parallel 0} + p_{\perp 0} \right) - (\Delta_b + \Delta) B \nabla \cdot \vec{B} - \rho_0 \omega_T^2 R \nabla \cdot \vec{R} = 0,
\]

\[
\nabla \cdot \left( p_{\parallel 0} + p_{\perp 10} \right) + (\Delta_b + \Delta) B \nabla \cdot \vec{B} - \rho_0 \omega_T^2 R \nabla \cdot \vec{R} + \frac{B_t^2}{\tau} \left( 1 - (\Delta_b + \Delta) h \right) \nabla \cdot \vec{B} = 0.
\]

Here, the toroidal rotation velocity \(\vec{u}_0 = \omega_T R^2 \nabla \zeta_T\) is assumed, with \(\omega_T = \omega_T (\psi)\) being the toroidally rotational frequency, and \(\Delta_b\) is short for \((p_{\parallel 0} - p_{\perp 0}) B^2\). Combining the two expressions yields \(I(1 - (\Delta_b + \Delta)) = f(\psi)\), which is similar to the result in a non-rotating anisotropic tokamak without EPs [32]. Since we are restricted to the case of \(\Delta_b \sim \beta_p, \Delta \sim \beta, \beta_0 \sim \epsilon^2\), the corrections to the equilibrium magnetic field are very small and invariably ignored in the low-beta limit. So the magnetic field can be considered as unaffected by the anisotropy of both bulk ions and EPs. As a result, we have the following self-consistent solution

\[
\nabla \cdot \vec{p}_{\parallel 0} = \Delta b \nabla \cdot \vec{B} + \rho_0 \omega_T^2 R \nabla \cdot \vec{R},
\]

\[
\nabla \cdot \vec{p}_{\perp 0} = \Delta_b B \cdot \nabla \vec{B}.
\]

This means the pressures of bulk ions and of EPs do not affect each other and the TR plays no effect on the EPs.

According to the previous analysis [32], we know that the bulk ions have the following equilibrium distribution function

\[
F_0 = n_i(\psi, \theta) \frac{(2\pi/m_i)^{3/2}}{T_{i,0}^{1/2} T_{L,0}} e^{-m_i \left( \psi \vec{v}_i \hat{b} \hat{b} - \hat{\psi} \vec{v}_i \hat{b} \hat{b} \right)^2 / 2 T_{i,0}},
\]

with unperturbed number density \(n_i = N(\psi) T_i^{3/2} e^{M^2 / 2} / \hbar^2\), where the Mach number \(M \) is defined as \(\omega_T R / \gamma_i T_i = T_i(\psi) / (T_i M^2 (1 + \tau))\).

In the presence of TR and temperature anisotropy, the unperturbed electrostatic potential in the lab reference frame is \(\Phi = \Phi_{-1} + \Phi_0\). Correspondingly, the electric field can be expressed as \(\vec{E} = \vec{E}_{-1} - \vec{E}_0\), where \(\vec{E}_{-1} = -\vec{u}_0 \times \vec{B}\) leads to \(\Phi_{-1}(\psi) = -\int \omega_T d\psi\) and \(\vec{E}_0\) is related to \(\Phi_{-1}\) which reads [30]

\[
e^{\Phi_0} = \frac{T_i}{1 + \tau} \ln \left( \frac{T_i}{T_{i,0}} \right) + \frac{T_i}{1 + \tau} M^2.
\]

Here, we write \(T_i = \langle T_{i,0} \rangle + \vec{T}_{i,0} \rangle\), where \(\langle T_{i,0} \rangle\) is the magnetic surface average of \(T_{i,0}\) and \(\vec{T}_{i,0}\) is the part depending on the poloidal angle. Here and below, we simplify the discussion by assuming \(\tau \ll 1\). Then the plasma pressures in equations (2) and (3) mainly come from the ions. That is, \(p_{\parallel 0} = n_i T_{i,0}\)

\(p_{\perp 0} = n_i T_{i,0}\), and \(n_i = N(\psi) \sigma e M^2\). Here, \(\sigma \equiv T_{i,0} / T_{i,0}\) is the anisotropy parameter. Keep in mind that \(\sigma \) and \(M\) both depend on the poloidal angle.

As for the EPs, we adopt the slowing-down beam distribution by ignoring the pitch-angle scattering,

\[
F_{\perp 0} = c_0(\psi) b(\lambda - \lambda_0) H(E_0 - E) \frac{1}{E^{3/2} + (E_0)^{3/2}},
\]

\[
\frac{1}{E^{3/2} + (E_0)^{3/2}}.
\]
Here, \( c_0(\psi) = \frac{m_e^2 l - \lambda B m_e v_B^2}{2 e^2 \alpha B n E_0} \) is a flux function with EPs number density \( n_h = n_0(\psi, B) \), \( E \) is the kinetic energy of EPs, \( E_0 \) is the inertial energy, \( E_r \) is the critical energy of the slowing beam ions, \( \lambda = \mu \alpha B \) is the pitch angle, and \( \mu = m_e v_B^2 / 2 \) is the magnetic moment. Using \( \int d^4v = \sum_i \sqrt{2 \pi} h m_i^{-3/2} v_i E_i^{1/2} (1 - \lambda B)^{-1/2} \lambda dE \) with \( \nabla \equiv \text{Sign}(v_i) \), we find \( p_{h,0} = 2(1 - \lambda B) n_0 E_0 / \ln(E_0/E_r) \) and \( p_{h,0} = \lambda_0 B n_0 E_0 / \ln(E_0/E_r) \). It is easy to verify equation (5).

### 3. Hybrid description

Equilibrium analysis is based on the fluid momentum equation. It is naturally for us to study the EGAM in the framework of fluid model. Since electrons temperature has been assumed to be much less than the ions, we will use double adiabatic CGL mode to derive the dispersion relation. The basic equations of CGL model are

\[
\frac{d\rho}{dt} + \rho \nabla \cdot \vec{u} = 0, \tag{9}
\]

\[
\partial_t \vec{B} = \nabla \times (\vec{u} \times \vec{B}), \tag{10}
\]

\[
\frac{d}{dt} \left( \frac{B^2}{\rho^3} \right) = 0, \tag{11}
\]

\[
\frac{d}{dt} \left( \frac{p_{\parallel}}{p_B} \right) = 0, \tag{12}
\]

as well as equation (1). Here and below, we restricted ourselves to the case of high safety factor, i.e. \( q \gg 1 \).

With the aid of \( \nabla \rho_{\parallel,0} = \rho_{\parallel,0} \delta(1 - \sigma - 2M^2) \nabla \ln B, \nabla \rho_{\parallel,0} = 2 \nabla B \delta(1 - \sigma - 2M^2) \nabla \ln B \), we arrive at the following linearized equations:

\[
\delta p_{\parallel} = -3 \rho_{\parallel,0} \phi - \xi_0 K_0 g_1^4 \rho_{\parallel,0} (3 - \sigma - 2M^2), \tag{13}
\]

\[
\delta p_\parallel = -p_{\perp,0} \sigma \phi + \xi_0 K_0 g_1^4 \rho_{\perp,0} (2 \sigma - 1 + 2M^2), \tag{14}
\]

\[
-\rho_0 \omega_0^2 \vec{\xi}_0 + 2 \rho_0 \omega_1 \omega_0 R K_0 g_1^4 \vec{\xi}_0 + \nabla_\parallel (\delta p_{\parallel} + \delta p_{\parallel,0}) = 0, \tag{15}
\]

\[
-\rho_0 \omega_0^2 \vec{\xi}_0 + 2 \rho_0 \omega_1 \omega_0 R K_0 g_1^4 \vec{\xi}_0 + \nabla_\parallel (\delta p_{\parallel} + \delta p_{\perp,0}) = 0, \tag{16}
\]

\[
-\rho_0 \omega_0^2 \vec{\xi}_0 + 2 \rho_0 \omega_1 \omega_0 R K_0 g_1^4 \vec{\xi}_0 + \nabla_\parallel (\delta p_{\parallel} + \delta p_{\perp,0}) = 0, \tag{17}
\]

Here, \( \vec{\xi} \) is the displacement defined as \( \delta \vec{u} (\sim \omega) \), not the Lagrangian displacement. Since strictly speaking, the Lagrangian perturbation \( \vec{\xi}_0 \) determines the perturbed velocity as \( \delta \vec{u} = \partial_t \vec{\xi}_0 + \vec{u} \cdot \nabla \vec{\xi}_0 - \vec{\xi}_0 \cdot \nabla \vec{u} \). \( \phi \) stands for \( \nabla \cdot \vec{\xi}_0 \), \( \xi_0 \) is the poloidal projection of \( \vec{\xi}_0 \), \( K_0 = \vec{B} \times \nabla r \cdot (\vec{B} \cdot \nabla) b (g_1^4 \) is the geodesic component of magnetic curvature, and \( g_1^1 = \nabla r \cdot \nabla r \sim 1 \) in the circular cross-section limit. Under the low-beta condition, the set of equations above eventually leads to

\[
\Omega^2 = \Omega_G^2 + \frac{R}{2 \rho_\parallel 0} (\sin(\theta \delta p_{\parallel} + \delta p_{\parallel,0})), \tag{18}
\]

in which \( \Omega \) is the frequency normalized by \( R/v_{\gamma B} \) where \( v_{\gamma B} = \sqrt{2 T_B \mu_i} \), and

\[
\omega_0^2 = \frac{3}{4} + \frac{\sigma}{2} (1 + \sigma) + \left[ (2 + 3 \sigma) M^2 + \left( 3 \sigma - 1 \right) M^4 \right], \tag{19}
\]

is the normalized frequency of GAM to the leading order in the absence of EPs [30].

The perturbed pressures of EPs are given by the perturbed distribution function, which is yielded from the gyro-kinetic equations [33–35]:

\[
\delta F_h = \frac{\partial F_{h,0}}{\partial U} e\Phi + \frac{1 - J_0(k_i \rho_h)}{B} \frac{\partial F_{h,0}}{\partial \mu} e\Phi + \frac{J_0 \delta H_h}{(v_\parallel - \omega \delta \phi_h) / \omega}, \tag{20}
\]

where \( U = E + e\Phi \) is the particle total energy, \( e \) is the charge, \( Q \) is short for \( \omega \partial F_{h,0} / \partial U + \hat{k} \times \vec{B} \cdot \nabla F_{h,0}(\vec{B}) \), \( \rho_h \) is the particle Larmor radius, \( k_i \) is the radial wave number, \( v_{\parallel,0} = v_{\parallel,0} + \gamma \delta \phi_h / \omega \), and \( v_{\parallel,0} \) is the zeroth-order drift velocity defined as \( v_{\parallel,0} = v_{\parallel,0} + \Omega_{\parallel,0} \vec{B} \times (\mu_0 \nabla B_0 + \nu_{\parallel,0} \vec{K}) \) with \( \Omega_{\parallel,0} = e B_0 / m_0 c \). Recall that in the lab frame, the unperturbed electrostatic potential is \( \Phi = \Phi_{h,1} + \Phi_{h,0} \), the 0th-order \( \vec{E} \times \vec{B} \) drift is therefore \( v_{\parallel,0} = v_{\parallel,0} + \gamma c / B + \nu_{\parallel,0} / B \). By using equation (7) and recalling \( F_{h,0} = F_{h,0}(\psi, E, \mu) \), the second gyro-kinetic (GK) equation goes to

\[
\partial_\psi \delta H_h - \frac{\omega_i}{\omega} \delta H_h - ik \frac{v_{\parallel,0}}{\omega_i} \sin \theta \delta H_h = \frac{i \omega}{\omega_i} \frac{\partial F_{h,0}}{\partial U} e\Phi, \tag{21}
\]

with \( \omega_i = (\nu - \omega \gamma R) / (\nu R) \) and \( \nu_i = \frac{1}{\Omega_{\parallel,0}} (v_{\parallel,0}^2 + v_{\parallel,0}^2 / 2) \). Apparently, the TR effect on the EPs can be disregarded considering that \( \Omega_{\parallel,0} \ll 1 \). Hence, for EPs, the particle transit frequency can be considered as \( \omega_i / (\nu R) \), the radial drift velocity is reduced to the classical one \( v_{\parallel,0} = \frac{1}{\Omega_{\parallel,0}} (v_{\parallel,0}^2 + v_{\parallel,0}^2 / 2) \), and the total energy \( U \sim E \). The perturbed distribution function is solved as

\[
\delta F_h = [1 - J_0(k_i \rho_h)] \left( \frac{\partial F_{h,0}}{\partial E} + \frac{1}{B} \frac{\partial F_{h,0}}{\partial \mu} \right) e\Phi + \frac{eJ_0 \partial F_{h,0}}{\partial E} \times \sum_{l,k} \frac{j^{l+k} \partial F_{h,0}}{(k_l + \nu \omega_i) (l_k - \nu \omega_i)} \tag{22}
\]

Here, we have assumed \( \delta \Phi = \sum \delta \Phi e^{i\theta} \). Since \( \tau \ll 1 \) is used in our model, it is reasonable that \( \partial \Phi = \delta \Phi_0 \) is a flux function.

The finite-orbit-width (FOW) of EPs is taken into account by supposing \( k_i \rho_h \ll 1 \), which means that we can write \( \delta F_h = \delta F_{h,0} e^{i\theta} + \delta F_{h,0} e^{-i\theta} \). As a result,
\begin{equation}
\langle \sin(\theta_{p||} + \delta_{p\perp}) \rangle = \frac{1}{2} \int d^3 \nu E(2 - \lambda B) (\delta_{F_{\perp,1} - \delta_{F_{\perp,-1}}}).
\end{equation}

According to Ohm’s law, one has \( \xi_0 = -k_i/(\omega B) \delta \Phi_0 \). In view of equation (22), the dispersion relation equation (18) becomes

\begin{equation}
\Omega^2 = \Omega_G^2 + \frac{R \omega^2 e B}{4 p_{\perp,0}} \int d^3 \nu E(2 - \lambda B) \frac{\partial F_{\|0}}{\partial E} \omega_t \frac{\nu_t}{\omega_t^2 - \omega^2}.
\end{equation}

4. Full kinetic description

In this section, we use full kinetic equations to derive the dispersion relation of EGAMs. Different from the EPs, the effect of TR on ions perturbed distribution function is remarkable. Let us focus on the local reference frame moving with \( \vec{u}_0 \) relative to the lab frame by defining the local particle velocity \( \vec{w} = \vec{v} - \vec{u}_0 \). In the local frame, particles feel only the potential \( \Phi_0 \) but not \( \Phi_1 \), so we can define \( E = m w^2/2 - m \vec{u}_0 \vec{v} + e \Phi_0 \). The leading order drift velocity is

\begin{equation}
\tilde{\nu}_D = \left[ (w_{\parallel}^2 + w_{\perp}^2)/\Omega_{ci} \right] \hat{b} \times \nabla \ln B + \frac{\hat{b}}{\Omega_{ci}} \nabla \Phi_0 + \vec{u}_0 \cdot \nabla \vec{w} + \omega_t (\hat{b} \cdot \nabla \hat{b} + \vec{u}_0 \cdot \nabla \vec{b}).
\end{equation}

According to [30], the ion perturbed distribution has the same expression as \( \delta F_{\|} \) in equation (22) (for more details of derivation, please see [30]), in which the transit frequency is defined as \( \omega_t = w_{\parallel}/(qR) \) and \( \nu_t \) is modified to

\begin{equation}
\nu_t = \frac{1}{R \Omega_{ci}} \left[ w_{\parallel}^2 + \frac{1}{2} w_{\perp}^2 + \frac{M^2 \nu_t^2}{1 + \tau} + 2 w_{\parallel} \nu_t \eta \right] + \frac{\tau \nu_t^2}{2(1 + \tau)} (1 - \sigma).
\end{equation}

Now using charge quasi-neutrality condition \( \int d^3 \nu \delta F_{\|0} + \int d^3 \nu \delta F_{\perp,0} = 0 \), we obtain

\begin{equation}
\Omega^2 (1 + q^2 S) = \frac{eB^2 R^2 + \omega_t^2}{2 p_{\perp,0} k_f^2} \int d^3 \nu \left( \frac{\partial F_{\|0}}{\partial E} + \frac{1}{B} \frac{\partial F_{\|0}}{\partial \mu} \right) (1 - J_0) + \frac{eBR \omega_t^2}{4 p_{\perp,0}} \int d^3 \nu \frac{\partial F_{\|0}}{\partial E} (2 - \lambda B) \frac{\nu_t}{\omega_t^2 - \omega^2}.
\end{equation}

with

\[ S = \zeta^2 + \frac{1}{2} + \sigma + 6 M^2 + \frac{Z(\zeta)}{\zeta} \times \left[ \zeta^4 + \zeta^2 (\sigma + 6 M^2) + \frac{1}{2} \sigma^2 + M^2 (\sigma + M^2) \right], \]

where \( \nu_t = \frac{E_0 - \nu_t}{qB} \) is used for EPs, \( \zeta \) is short for \( q \Omega \), and \( Z \) is the plasma dispersion function. Ordering analysis shows the first term on the right-hand side of equation (27) is about \( 1/q^2 \) of the second term when \( \omega/\omega_t \sim O(1) \) is adopted for EPs and thus can be neglected in the large \( q \) limit. If we ignore the ion Landau damping effect by expanding \( S \) to the leading order, we have \( S \approx -\frac{eB}{q^2 \mu_0} \). Substituting it into the formula above, we find that the kinetic dispersion relation above is identical with the hybrid one of equation (24).

5. Discussion

For simplicity of notation, let us define the last term on the right-hand side of equation (27) as \( H_E \). With the help of equation (8) and in view of \( E_c \ll E_0 \), one has

\begin{equation}
\frac{\partial F_{\|0}}{\partial E} = -c_0 E \left[ \frac{1}{2} \left( \delta(E_0 - E) \delta(\lambda - \lambda_0) \right.ight.
\end{equation}

Integrating over the velocity space, we obtain

\[ H_E = \frac{\omega^2}{\omega_{\|0}} \frac{\nu_t}{\omega_{\|0}} \int \left[ 1 - \frac{\omega^2}{\omega_{\|0}^2} \right] - \frac{\omega^2}{\omega_{\|0}^2} \frac{\nu_t}{\omega_{\|0}^2} \int \left[ 1 - \frac{\omega^2}{\omega_{\|0}^2} \right] \]

where \( D_0 = \frac{5 \lambda B - 2}{3(1 - \lambda B)} \), \( C_0 = \frac{2 - \lambda B}{1 - \lambda B} \), \( \beta_{\|} \) stands for \( (p_{\|,0} + p_{\perp,0})/(4 p_{\perp,0}) \), and \( \omega_{\|0} \) is short for \( \sqrt{2 E_0 (1 - \lambda_0 B)m_b/(qR)} \). The previous result reported in [29] is recovered when zeroing the boxed term. As discussed in [29], the critical condition responsible for the local instability is \( D_0 > 0 \). The boxed term does not change the condition essentially. Defining a new normalized frequency \( X = \omega/\omega_{\|0} \) and \( \varpi = \nu_t/(R^2 \omega_{\|0}^2) \), the kinetic dispersion relation can be rearranged as

\begin{equation}
X^2 \left( 1 + q^2 S \right) = \varpi \left[ X^2 D_0 \int \left( 1 - \frac{1}{X^2} \right) + \frac{C_0 X^2}{X^2 - 1} \right] \]

5.1. Hybrid results

Let us skip the ions Landau damping term and keep only the leading order of \( S \). The kinetic dispersion relation (30) is then simplified to the hybrid one as

\begin{equation}
X^2 = \varpi \left[ X^2 D_0 \int \left( 1 - \frac{1}{X^2} \right) + \frac{C_0 X^2}{X^2 - 1} \right] \]

This hybrid dispersion relation is similar to the previous one [29]. There are three branches of solutions of the equation above. One is the high-frequency branch (HFB) with a frequency larger than \( \omega_{\|0} \), and the other two are low-frequency branches (LFBs) with a frequency smaller than \( \omega_{\|0} \). We see
from the equation above that for $X^2 > 1$, $\ln(1 - 1/X^2)$ is real and negative and the last term on the right-hand side is positive and real. Thus there exists a real solution of equation (31). That is to say, the HFB is always stable without growth rate or damping rate. For the LFBs with $X < 1$, $\ln(1 - 1/X^2)$ is a complex number. Two solutions give two complex conjugate modes, one of which is unstable and the other is damped. So in this section we just focus on the HFB and the unstable LFB (uLFB) by disregarding the damped LFB. Let us assume $X = \Omega + i \gamma$ with $\gamma < \Omega$, and insert it into equation (31). We find $\Omega_\gamma$ is determined by $\Omega_\gamma^2 = \varpi_\gamma^2 + \varpi_\gamma^2 / \theta^2 = \left[ \Omega_\theta^2 D_0 \ln(\Omega_{\gamma}^2 - 1) + C_0 \right] / (1 - \Omega_{\gamma}^2)$ and the imaginary part reads

$$\gamma \simeq \frac{1}{2} \varpi_\gamma D_0 r \pi \Omega_\gamma \Theta^{-1}, \quad (32)$$

in which

$$\Theta = 1 - \varpi_\gamma D_0 \ln(\Omega_{\gamma}^2 - 1) - \frac{D_0}{1 - \Omega_{\gamma}^2} - \frac{C_0}{(1 - \Omega_{\gamma}^2)^2} \quad (33)$$

and $\eta = \text{Sign}(\gamma)$. Recall that $\varpi_\gamma D_0 \sim \frac{n_e}{n_i}$, in which $n_e/n_i \ll 1$ should always hold. For $\varpi_\gamma D_0 \ll 1$, it naturally has $\gamma < \Omega$. For $\varpi_\gamma D_0 \sim O(1)$, it is fortunately the case that $\Theta^{-1}$ is much less than unit. So $\gamma/\Omega < 1$ still comes into being. As shown below, a typical $\Omega_\gamma$ is about 0.6 and a typical growth rate is about 0.02. Such a growth rate is coincident with the previous one found in [29]. Equation (32) indicates clearly that the LFB is unstable when $D_0 > 0$ as reported in [29]. The anisotropy and TR of bulk ions do not play a role in the instability criterion of EGAM. When $D_0$ is negative, there is no self-consistent complex solution. Although we obtain the instability criterion, the growth rate cannot be analytically evaluated directly, since it is difficult to get a transparent expression for $\Omega_\gamma$. In the special case of $\varpi \ll 1$, one has $\Omega_\gamma \simeq \varpi^{1/2} \Omega_C$. Generally, $\varpi \ll 1$ does not hold, noting that $\varpi = \frac{n_e}{n_i} q^2 \beta \frac{\Omega}{D \theta}$. We suppose the atomic masses of the plasma ions and EPs are the same. For $q \simeq 4$ and $\lambda B \simeq 0.5$, one has $\varpi = 0.8$ for $T_i / E_0 = 1/40$ and $\varpi = 0.32$ for $T_i / E_0 = 1/100$. Thereby, numerical evaluation is needed.

Figure 1 illustrates the dependence of $\Omega_\gamma$ on the Mach number $M$. It is shown that the Mach number enlarges both frequencies of uLFB and HFB. In the case of $\varpi = 0.32$, the Mach number will decrease the growth rate. While for $\varpi = 0.1$, the growth rate increases with $M$ first and then decreases with $M$ when $M$ is sufficiently large, as demonstrated in figure 2. Actually, the analytical expression for the growth rate (see equation (32)) implies that

$$\frac{\partial \gamma}{\partial M} = \gamma^2 \frac{\partial \Omega_\gamma^2}{\partial M} \frac{G_1}{\Omega_\gamma^2 \Theta} \quad (34)$$

where

$$G_1 = \frac{1}{2} \left[ \varpi_\gamma D_0 \Theta^{-1} \right] [(2C_0 - D_0) \Omega_\gamma^2 + D_0]. \quad (35)$$

Hence there exists a critical $\Omega_{\text{cri}}$. Below this critical value, the growth rate increases with $M$, which enlarges $\Omega_\gamma$. When $\Omega_\gamma$ is greater than $\Omega_{\text{cri}}$ as the Mach number increases, the growth rate decreases with $M$. $\Omega_{\text{cri}}$ is determined by the real solution of $G_1 = 0$. Taking $\varpi = 0.1$ and $\beta h = 0.5$ for example, one finds $\Omega_{\text{cri}}^2 = 0.652$. Meanwhile, as shown in figures 1 and 2, the critical point appears at $M = 0.87$. The corresponding real frequency of uLFB is $\Omega_\gamma = 0.653$. In the case of $\varpi = 0.32$ and $\beta h = 0.5$, one has $\Omega_{\text{cri}}^2 = 0.549$. According to figure 1 we know the real frequency in this case is always larger than this critical value. As a result, the growth rate decreases with $M$.

According to equation (31), the dependence of $X$ on $\sigma$ is only through $\Omega_G$, just like the dependence of $X$ on $M$. Equation (19) indicates that $\partial \Omega_G / \partial \sigma$ is always as positive as $\partial \Omega_G / \partial M$. It is easy to obtain $\partial \gamma / \partial \sigma = \gamma \varpi G_1 \Omega_{\text{cri}}^{-2} \Theta^{-1} \partial \Omega_{\text{cri}} / \partial \sigma$. So it is predicted that the $\sigma$ scans of $\Omega_\gamma$ and $\gamma$ are similar to figures 1 and 2 respectively. The critical point $\Omega_{\text{cri}}^2$ is still determined by $G_1 = 0$, which does not depend on $\sigma$ and $M$. Figures 3 and 4 confirm that. The turning point of the growth rate curve for $\varpi = 0.1$ in figure 4 appears at $\sigma \simeq 2.11$, corresponding to $\Omega_\gamma = 0.654$, as illustrated in figure 3.
Figures 5 and 6 illustrate the dependence of frequencies on the pressure ratio \( \beta_h \). When \( \varpi \Omega > 1 \), the \( \beta_h \) scan of HFB stays smooth and continuous, which is actually the classical GAM modified by the presence of EPs. Meanwhile, the LFB excited by the EPs remains unstable when \( \lambda_b \beta > 2/5 \). Without EPs, the HFB reduces to the classical GAM. On the contrary, when \( \varpi \Omega < 1 \), the classical GAM turns into the unstable EGAM and the HFB excited purely by the EPs keeps stable. From the two figures we also know that when \( \beta_h \) exactly equals zero, only classical GAM mode with \( \varpi \epsilon \leq \Omega - \Omega G \) exists with \( \gamma = 0 \). Once \( \beta_h \) does not equal zero, even just a little more than zero, an unstable mode comes into being. In the case of \( \varpi \Omega < 1 \), the original mode is driven to be unstable by EPs and, meanwhile, a HF stable mode is resonantly excited by the EPs, while in the case of \( \varpi \Omega > 1 \) the original mode keeps stable and a new unstable LF mode is excited by the EPs. In both cases, the real frequency of HFB increases with \( \beta_h \) whereas the real frequency of LFB decreases with \( \beta_h \). Also, their growth rates both increase as \( \beta_h \) increases.

5.2. Kinetic results

To compare the kinetic result with the hybrid one, let us keep both the leading term and damping term at first. The kinetic dispersion relation (30) is then simplified to

\[
\varpi \varpi = \varpi \Omega G^2 + \varpi H_G = \varpi e^{-qX^{2\gamma - 1}} \frac{1}{\pi q^2 X^5 \varpi} \Omega G^{x \cdot \frac{3}{2}}. \tag{36}
\]

The ion damping term \( \gamma_i = \frac{1}{\varpi} e^{-qX^{2\gamma - 1}} \sqrt{\varpi} \varpi q^2 \Omega G^{x \cdot \frac{3}{2}} \) is on the order of \( 10^{-28} \) at \( \varpi = 0.32 \) and \( q = 4 \) for the HFB with \( \Omega \approx 1.2 \). That means for the HFB, the ion damping effect is totally ignorable. According to the expression of \( \gamma_i \), it is expected that the ion Landau damping plays a role only for small \( X \). That is to say, the ion collisionless damping may be of great importance to the LFBs. As stated in the section above, there are two complex conjugate modes of LFBs. In the presence of ion Landau damping, the conjugate modes are not conjugate anymore. Let us define the two low-frequency solutions of the kinetic dispersion relation (30), \( X = \Omega_{r_{\pm}} + i \gamma_{l_{\pm}} \), as kinetic LFB\(_+\) (kLFB\(_+\)) and kinetic
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The low-frequency solutions of the simplified dispersion relation equation (36) are defined as $LFB_+$ and $LFB_-$ respectively. Correspondingly, the low-frequency solutions of the simplified dispersion relation equation (36) are defined as $LFB_+$ and $LFB_-$ respectively. Figure 7 shows the real frequencies, the growth rate of $u_{LFB}$ (the unstable solution of equation (31)) and $k_{LFB}^+$ and $k_{LFB}^-$, and the damping rate of $k_{LFB}^-$ and $LFB_-$ respectively, on the Mach number. From this figure we arrive at the following items:

1. Due to the presence of ion Landau damping, the two LFBes are not conjugate anymore. The discrepancy between $\Omega_+ - \Omega_-^r$ is tiny and can almost be neglected. However, the discrepancy between $\gamma_+ - \gamma_-^r$ is significant. For the unstable branch, its growth rate is decreased by the ion Landau damping. As for the damped one, its damping rate is enlarged by the ion Landau damping.

2. The real frequency $\Omega_+$ increases with $M$, shown in the upper half of figure 7, similarly as displayed in figure 1. Since $\gamma_+$ is dramatically decreased as $\Omega_+$ increases, the discrepancy between $\gamma_+$ and $\gamma_-^r$ is diminished as $M$ increases. The two modes become almost conjugate in the region of $M \gtrsim 0.7$ for $\beta_n = 1$ or in the region of $M \gtrsim 0.5$ for $\beta_n = 0.5$, respectively. The LFB$_+$ is, not surprisingly, reduced to $u_{LFB}$.

3. Both the growth rate of kLFB$_+$ and damping rate of kLFB$_-$ are smaller than the growth rate of $u_{LFB}$. Correspondingly, the real frequency of kLFB$_+$ is larger than that of $u_{LFB}$. Such a discrepancy between kLFB$_+$ and $u_{LFB}$ at large $M$ is introduced by asymptotic expansion of $S$ and is not related to the EPs. In the simplified dispersion relation (36), we keep the term of $\Omega^2 S \sim -\frac{\beta_n}{2}$. Actually, $q^2 S \approx -\frac{\beta_n}{2} - G_2(q^2 \Omega^4) + ...$, where $G_2 = \frac{15}{8} + \frac{3}{2} (9 + \sigma) + \frac{\sigma^2 + 3\sigma}{4} + \frac{6\sigma}{7}$. When $\Omega$ is larger than unit, it is reasonable to disregard the terms on the order of $O(q^2 \Omega^4)$ and other higher-order terms in the large safety factor limit. However, for the LFBes with a typical $\Omega = 0.7$ at $\omega = 0.8, M = 0.5, \sigma = 1$ and $q = 4$, noting that $\Omega^2 \approx \frac{\beta_n}{2}$, we find that the $q^2 \Omega^4$-dependent term is about 0.69, which definitely cannot be dropped directly without consideration. That means although we can take advantage of large safety factor approximation to neglect the first EP-related term on the right-hand side of equation (27), we should keep more higher-order terms when expanding $S$. Similarly, one should keep terms proportional to $1/q$ and $1/q^2$ when deriving equation (18). The $q$ scan of low-frequency solutions of equations (30)
and (31), plotted in figure 8, shows that only for $q \gtrsim 7$, the hybrid dispersion relation obtained in the large safety factor limit is of sufficient accuracy.

The growth rate of LFB$_+$ increases with $M$ at first and then decreases with $M$, similar to uLFB. We also note that the damping rate of LFB$_-$ keeps decreasing with $M$. There is no turning point on the curve of LFB$_-$ in figure 7. The explanation is that the ion Landau damping term is remarkable for small $\Omega_r$, i.e. small $M$ when other variables are fixed. The damping rate is lifted up at first and then keeps going down as $M$ increases. After some similar calculation, we obtain the analog to equation (32) on the basis of equation (36) as

$$
\frac{\partial \gamma}{\partial M} = \frac{\omega}{\Theta} \left[ \frac{1}{2} \frac{\partial \beta_B \partial \beta_B}{\partial \omega} \frac{G_1}{\Omega_B} + \frac{\gamma_d (q^2 / \omega - \frac{2}{\Omega_r^2})}{G_3} \right],
$$

The critical value $\Omega^\text{cri}_r$ is now determined by $G_1 = 0$. For the unstable branch with $\eta = 1$ and $\beta_B = 1$, we find $\Omega^\text{cri}_r = 0.646$. According to figure 7, the turning point of LFB$_-$ appears at $M = 0.47$, leading to $\Omega_r = 0.646$. For the damped branch with $\eta = -1$, we find $\Omega^\text{cri}_r = 0.318$. Figure 7 shows $\Omega_r$ of the damped branches is always larger than this critical value, so the damping rate keeps decreasing with $M$. Such a critical point also exists for kLFB$_+$ (see the lower left figure in figure 7), but it is hard to obtain a concise equation like $G_1$ or $G_2$. Besides, the effects of ion Landau damping on $\Omega_r$ is ignorable, leading to $\Omega_r^\text{cri} \simeq \Omega_r$. Hence it is easy to find $\gamma_+ < -\gamma_-$. It should be specifically pointed out that equation (36) has a root with $\Omega_r > 1$ and $\gamma > 1$. But one should be careful about this root, which actually is an extraneous root and is not a solution of equation (30). This root is also introduced by expanding the plasma dispersion function $Z(\zeta)$ into $i \sqrt{\pi} e^{-\zeta^2} - 1/[1 + 1/(2\zeta^2) + 3/(4\zeta^4) + ...]$. This expansion requires the real part of $\zeta$ should be much greater than the imaginary part. As a result, the unstable EGAM branch has a typical growth rate $\gamma/\Omega_k \sim O(10^{-2})$, and there is only one stable HFB. These are different from [24] but are coincident with [29]. From figures 5 and 6 we can see the real frequency and growth rate of uLFB at large $\beta_B$ for different $\varpi$ do not significantly differ from each other. The real frequency curves become flat at large $\beta_B$. For $\beta_B \simeq 1$, the linear growth rate $\gamma/\Omega_k$ is up to 7.5%, which is consistent with the simulation on DIII-D (see figure 3(b) in [25]).

6. Summary

Due to TR and temperature anisotropy, the pressure of tokamak bulk ions is shifted out toward the magnetic surface and then depends on the poloidal angle via the magnetic field strength $B$, but is independent of the anisotropy of EP distribution generated by auxiliary heating, such as NBI heating. Rigid equilibrium analysis is presented and shows that the TR affects only the bulk ion equilibrium rather than that of EPs, eventually leading to the self-consistent equilibrium distributions for plasma ions and EPs.

Based on the self-consistent distributions, we use a hybrid kinetic-fluid model and full gyrokinetic equations to investigate the energetic particle driven geodesic acoustic mode (EGAM) in a toroidally rotating anisotropic tokamak plasma. In the large safety factor limit $q \gg 1$, the hybrid dispersion relation is derived (see equation (24)) and the kinetic one is presented in equation (27). By restricting terms to the leading order and ignoring the ion Landau damping effect, the kinetic relation can be reduced to the hybrid one.

Slowing down beam distribution is adopted for EPs with initial energy $E_0$, which is much greater than the ion parallel temperature $T_k$. A normalized dispersion relation of EGAMs is presented in equation (30). According to this dispersion relation, it is found that for $\lambda_B > 2.5$, there are three branches of EGAMs co-existing. One is the high-frequency branch (HFB), which is shown to be always stable. The other two are low-frequency branches (LFBs). In the hybrid approximation, the two LFBs are complex conjugate. The unstable one is abbreviated as uLFB for simplicity of notation. Their growth/damping rate is given in equation (32). As shown in figures 2 and 4, there is a critical $\Omega^\text{cri}_r$ determined by $G_1 = 0$. The growth rate of uLFB increases with $M$ when $\Omega < \Omega^\text{cri}_r$ and decreases with $M$ when $\Omega > \Omega^\text{cri}_r$. After taking into account ion Landau damping in the simplified dispersion relation (36), the two LFBs (now defined as LFB$_+$ and LFB$_-$) are not conjugate any more. Similar to the uLFB, there is a critical value $\Omega^\text{cri}_r$, which is now determined by $G_3 = 0$. For the damped branch LFB$_-$, the damping rate always decreases with $M$, since its real frequency is larger than $\Omega^\text{cri}_r$. The anisotropy of bulk ions plays a similar role in the dispersion relation to TR (see figures 1 and 3, or figures 2 and 4).

The ion Landau damping term becomes ignorable for large $M$, as shown in figure 7. Then the simplified dispersion relation equation (36) shows no difference from the hybrid one.
However, the discrepancy between the kinetic result and hybrid one still exists. Figure 8 implies that for $q \gtrsim 7$ with other variables fixed, the hybrid dispersion relation is of sufficient accuracy. This reminds us that the case of $q \sim O(1)$ is of possible importance to the EGAM in a small $q$ tokamak, which will be the focus of our future work.

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