ON COHEN-MACAULAY AUSLANDER ALGEBRAS

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Dedicated to Professor Hideto Asashiba

Abstract. Cohen-Macaulay Auslander algebras are the endomorphism algebras of representation generators of the subcategory of Gorenstein projective modules over CM-finite algebras. In this paper, we study Cohen-Macaulay Auslander algebras over 1-Gorenstein algebras and \( \Omega_G \)-algebras. 1-Gorenstein algebras are those of algebras with global Gorenstein projective dimension at most one and \( \Omega_G \)-algebras are a class of algebras introduced in this paper, including some important class of algebras for example Gentle algebras and more generally quadratic monomial algebras. It will be shown how the results for Gorenstein projective representations of a quiver over an Artin algebra, including the submodule category introduced in [RS], or more generally, the (separated) monomorphism category defined in [LZh2] and [XZZ], can be applied to study the Cohen-Macaulay Auslander algebras.

1. Introduction

Gorenstein Projective modules were first introduced by Auslander and Bridger [AB] over commutative Noetherian rings (in which case they are called G-dimension zero modules) as a generalization of finitely generated projective modules; the aim was to study the better homological properties of such rings. Since then, this class of modules has found many applications in commutative algebra and algebraic geometry. It turns out that the properties of modules of Gorenstein dimension zero are very closely related to the structure of the singularities of a Gorenstein ring. The generalization of this notion to any (not necessarily finitely generated) module over any (not necessarily commutative Noetherian) ring by Enochs and Jenda [EJ1] led to the definition of Gorenstein projective modules, see 2.2 for the definition of such modules.

Nowadays, the study of Gorenstein projective modules in the representation theory of Artin algebras has been developed in various different directions. Let us explain some of them might be interesting. (1) Ringel and Zhang [RZ] showed that for a finite quiver \( Q \) there is a triangle equivalence between the stable category of Gorenstein projective modules of the path algebra of \( Q \) over the algebra of dual numbers and the orbit category of the bounded derived category of modules of the path algebra of \( Q \) over a ground field \( k \) modulo the shift functor. They also showed that the induced homology functor yields a bijection between non-projective Gorenstein projective indecomposable modules of the path algebra of \( Q \) over the algebra of dual numbers and those in mod-\( kQ \). This result explains how one can relate the concept of Gorenstein projective modules to study all the modules of a finite dimensional algebra. (2) It has been recently shown in [JKS] that the category of Gorenstein projective modules provides an additive categorification of the cluster algebra structure on the homogeneous coordinate ring of the Grassmannian of \( k \)-planes in \( n \)-space.

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For any Artin algebra $\Lambda$, denote by $\text{Gprj-}\Lambda$ the subcategory of Gorenstein projective modules of $\text{mod-}\Lambda$. If $\text{Gprj-}\Lambda$ has only finitely many isomorphism classes of indecomposable modules, then $\Lambda$ is called of finite Cohen-Macaulay type, or more simply CM-finite. It may be more reasonable to call such algebras of finite Gorenstein projective type, or GP-finite, but ‘‘of finite CM-type’’, or ‘‘CM-finite’’ is more known terminology between the experts in this area, perhaps one reason to choose this name is the equality of the subcategory of Gorenstein projective modules and the Cohen-Macaulay modules in some cases, as discussed in 2.2. Inspired by definition of Auslander algebras, the Cohen-Macaulay Auslander algebra is defined to be $\text{End}_\Lambda(G)$, where $\Lambda$ is CM-finite and $G$ is a representation generator of $\text{Gprj-}\Lambda$, i.e. $\text{add-}G = \text{Gprj-}\Lambda$. That $\text{add-}G$ consists of all summands of finite copies of $G$. The importance of Gorenstein projective module over Artin algebras motivates us to studying $\text{mod-}(\text{Gprj-}\Lambda)$, the category of finitely presented functors over $\text{Gprj-}\Lambda$. Note that in the case of $\Lambda$ being CM-finite algebra, then $\text{mod-}(\text{Gprj-}\Lambda)$ is equivalent to $\text{mod-}\Gamma$, where $\Gamma$ is the Cohen-Macaulay Auslander algebra of $\Lambda$. The studying $\text{mod-}(\text{Gprj-}\Lambda)$ goes back to the functorial approach introduced by Maurice Auslander, in where he learned us one way of studying $\text{mod-}\Lambda$ is investigation of $\text{mod-}(\text{mod-}\Lambda)$. Hence we hope studying $\text{mod-}(\text{Gprj-}\Lambda)$, or $\text{mod-}\Gamma$ when $\Lambda$ to be CM-finite, may be helpful to reflect some information for the Gorenstein projective modules, or even more for the entire of module category. Another reason of studying Cohen-Macaulay Auslander algebras in this paper can be to introduce some new classes of algebras which might be interesting. For instance, inspired by [H] we introduce $\Omega_G$-algebras which of those algebras with special property being the relative Auslander-Reiten translation in the subcategory of Gorenstein projective modules is the first syzygy functor, see Proposition 4.5 for more properties of $\Omega_G$-algebras. The section 4 is devoted to give some basic properties of such algebras. Recently, the systematic work of C.M. Ringel and M. Schmidmeier [RS] on the submodule category $\mathcal{S}_2(\Lambda)$ the category of all embedding $(A \subseteq B)$ where $B$ is a finitely generated $\Lambda$-module and $A$ is a submodule of $B$ receives more attention. D. Kussin, H. Lenzing and H. Meltzer [KLM] establish a surprising link between the stable submodule category with the singularity theory via weighted projective lines of type $(2, 3, p)$. The Ringel and Schmidmeier’s work on $\mathcal{S}_2(\Lambda)$ was generalized in [XZZ] to $\mathcal{S}_n(\Lambda)$, where we are dealing with chains of submodules with length $n - 1$. In the section 3 and 5, our results show how the results for the monomorphism categories can be related to study of (stable) Cohen-Macaulay Auslander algebras. For instance, in Corollary 3.10 we show over a self-injective $\Lambda$ of finite representation type: CM-finiteness of $T_3(\Lambda)$ ($3 \times 3$ lower triangular matrices over $\Lambda$) and representation-finiteness of $T_2(\Lambda)$ ($2 \times 2$ lower triangular matrices over $\Lambda$) are equivalent, and as an application in Corollary 3.12 will be shown the class of representation-finite self-injective algebras such that their associatedAuslander algebras are again representation-finite remains closed under derived equivalences. In addition, in Corollary 5.5 it is proved that how the computation of the Auslander-Reiten translation in $\mathcal{S}_3(\Lambda)$ can be applied to the Auslander-Reiten translation of the Auslander algebra of a self-injective Nakayama algebra $\Lambda$. The similar result for the stable case is given in Corollary 5.6.

2. Preliminaries

Throughout this paper $\Lambda$ is an Artin algebra over commutative artinian ring $R$. We denote by $D$ the $R$-dual, i.e., $D(-) = \text{Hom}_R(-, E(R/J(R)))$.

We denote by $\text{mod-}\Lambda$ the category of finitely generated (right) modules, and by $\text{prj-}\Lambda$ the category of finitely generated projective $\Lambda$-modules.
Let $\mathcal{A}$ be an abelian category, $\mathcal{X}$ a full additive subcategory of $\mathcal{A}$. Let $M \in \mathcal{A}$ be an object. A right $\mathcal{X}$-approximation of $M$ is a morphism $f : X \to M$ such that $X \in \mathcal{X}$ and any morphism $X' \to M$ from an object $X' \in \mathcal{X}$ factors through $f$. Dually one has the notion of left $\mathcal{X}$-approximation. The subcategory $\mathcal{X} \subseteq \mathcal{A}$ is said to be contravariantly finite (resp. covariantly finite) provided that each object in $\mathcal{A}$ has a right (resp. left) $\mathcal{X}$-approximation. The subcategory $\mathcal{X} \subseteq \mathcal{A}$ is said to be functorially finite provided it is both contravariantly finite and covariantly finite.

2.1. Functors category. Let $\mathcal{X}$ be a full additive subcategory of an abelian category $\mathcal{A}$. An additive contravariant functor $F : \mathcal{X} \to \text{Ab}$, where $\text{Ab}$ denotes the category of abelian groups, is called a (right) $\mathcal{X}$-module. An $\mathcal{X}$-module $F$ is called finitely presented if there exists an exact sequence

$$\text{Hom}_\mathcal{X}(-, X) \to \text{Hom}_\mathcal{X}(-, X') \to F \to 0,$$

with $X$ and $X'$ in $\mathcal{X}$. All finitely presented $\mathcal{X}$-modules and natural transformations between them form a category that will be denoted by $\text{mod-}\mathcal{X}$. It is known that if $\mathcal{X}$ is a contravariantly finite subcategory of $\text{mod-}\mathcal{A}$ then $\text{mod-}\mathcal{X}$ is an abelian category, see [AHK2, §2]. Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{X}$ consists of all projective objects of $\mathcal{A}$. We consider a category associated with $\mathcal{X}$, the stable category of $\mathcal{X}$, denoted by $\mathcal{X}^\ast$. The object of $\mathcal{X}^\ast$ are the same as the objects of $\mathcal{X}$, denoted often by $X$, when we want to consider an object $X$ in $\mathcal{X}$ as an object in $\mathcal{X}^\ast$. And the morphism are given by $\text{Hom}_\mathcal{X}(X, Y) = \text{Hom}_\mathcal{X}(X, Y)/\text{P}(X, Y)$, where $\text{P}(X, Y)$ is the subgroup of $\text{Hom}_\mathcal{X}(X, Y)$ consisting of those morphisms from $X$ to $Y$ which factor through a projective object in $\mathcal{A}$. We also denote by $f$ the residue class of $f : X \to Y$ in $\text{Hom}_\mathcal{X}(X, Y)$. Moreover, in case that the subcategory $\mathcal{X}$ is contravariantly finite in $\mathcal{A}$, then the category of finitely presented $\mathcal{X}$-modules, $\text{mod-}\mathcal{X}$, by the equivalence proved in [AHK2, Proposition 4.1], can be identified with those functors in $\text{mod-}\mathcal{X}$ such that vanish on all projective objects in $\mathcal{A}$. We use this identification completely free for some certain subcategories which we will be dealing with throughout the paper later.

In the case that $\mathcal{A} = \text{mod-}\Lambda$ for an Artin algebra $\Lambda$ and $\mathcal{X}$ a contravariantly finite subcategory of $\text{mod-}\Lambda$ containing prj-$\Lambda$, then we are be involved in three abelian categories $\text{mod-}\Lambda$, $\text{mod-}\mathcal{X}$ and $\text{mod-}\mathcal{X}^\ast$. These abelian categories can be connected via a recollement as the following

$$\text{mod-}\mathcal{X} \xrightarrow{i} \text{mod-}\mathcal{X}^\ast \xleftarrow{\sigma} \text{mod-}\Lambda,$$

in particular, the functor $\sigma$ is defined by sending $F$ in $\text{mod-}\mathcal{X}^\ast$ to the $\text{Coker}(G_1 \xrightarrow{f} G_0)$ in $\text{mod-}\Lambda$, where $(-, G_1) \xrightarrow{(-, f)} (-, G_0) \to F$ is a projective presentation of $F$, and also $\sigma \rho$ given by sending $M \in \text{mod-}\Lambda$ to the restriction of the functor $\text{Hom}_\Lambda(-, M)$ over $\mathcal{X}$, for more details we refer to [AHK2, Theorem 3.7]. To simplify for $X \in \mathcal{X}$ we show the representable functor $\text{Hom}_\Lambda(-, X)$, resp. $\text{Hom}_\Lambda(-, X)$, in $\text{mod-}\mathcal{X}$, resp. $\text{mod-}\mathcal{X}^\ast$ by $(-, X)$, resp. $(-, X)$. Let $\mathcal{X}$ be a full subcategory of $\text{mod-}\Lambda$ closed under isomorphisms and direct summands. The set of iso-classes of indecomposable modules of $\mathcal{X}$ will be denoted by $\text{Ind-}\mathcal{X}$. $\mathcal{X}$ is called of finite type if $\text{Ind-}\mathcal{X}$ is a finite set. $\Lambda$ is called of finite representation type, or simply representation-finite, if $\text{mod-}\Lambda$ is of finite type. If $\mathcal{X}$ is of finite type then it admits a representation generator, i.e. there exists $X \in \mathcal{X}$ such that $\mathcal{X} = \text{add-}\Lambda = \text{add-}X$. It is known that $\text{add-}\mathcal{X}$ is a functorially finite subcategory of $\text{mod-}\mathcal{A}$. Set $\text{Aus}(\mathcal{X}) = \text{End}_{\Lambda}(X)$. Clearly $\text{Aus}(\mathcal{X})$ is an Artin algebra. It is
known that the evaluation functor $\zeta_X : \text{mod-}\mathcal{X} \rightarrow \text{mod-Aus(}\mathcal{X}\text{)}$ defined by $\zeta_X(F) = F(X)$, for $F \in \text{mod-}\mathcal{X}$, is an equivalence of categories. It also induces an equivalence of categories $\text{mod-}\mathcal{X}$ and $\text{mod-Aus(}\mathcal{X}\text{)}$. Recall that $\text{Aus(}\mathcal{X}\text{)} = \text{End}_\Lambda(X)/P$, where $P$ is the ideal of $\text{Aus(}\mathcal{X}\text{)}$ including endomorphisms factoring through projective modules.

The Artin algebra $\text{Aus(}\mathcal{X}\text{)}$, resp. $\text{Aus(}\mathcal{X}\text{)}$, is called relative, resp. stable, Auslander algebra of $\Lambda$ with respect to the subcategory $\mathcal{X}$, which is uniquely determined by $\mathcal{X}$ up to Morita equivalence.

2.2. Gorenstein projective modules. Let $\Lambda$ be an Artin algebra. A complex

$$P^\bullet : \cdots \rightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \rightarrow \cdots$$

of finitely generated projective $\Lambda$-modules is said to be totally acyclic provided it is acyclic and the Hom complex $\text{Hom}_\Lambda(P^\bullet, \Lambda)$ is also acyclic. An $\Lambda$-module $M$ is said to be (finitely generated) Gorenstein projective provided that there is a totally acyclic complex $P^\bullet$ of finitely generated projective $\Lambda$-modules such that $M \cong \text{Ker}(d^0)$ \[EJ1\]. We denote by $\text{Gprj-}\Lambda$ the full subcategory of $\text{mod-}\Lambda$ consisting of all Gorenstein projective modules.

Definition 2.1. An Artin algebra $\Lambda$ is said to be $n$-Gorenstein if $\text{proj.dim } D(\Lambda \Lambda) = \text{inj.dim } \Lambda \Lambda = n < \infty$.

Given a $\Lambda$-module $M$, denote the kernel of the projective cover $P \rightarrow M$ by $\Omega(M)$. $\Omega(M)$ is called the first syzygy of $M$. We let $\Omega^0(M) = M$ and then inductively for each $i \geq 1$ set $\Omega^i(M) = \Omega(\Omega^{i-1}(M))$. The projective dimension of $M$, $\text{proj.dim } M$, is $m$ when $\Omega^{m+1}(M) = 0$, and $M$ has infinite projective dimension if $\Omega^i(M) \neq 0$ for every $i \geq 0$. Moreover, the injective dimension, $\text{inj.dim } M$ can be defined dually via notion of cosyzygy, or equivalently by the projective dimension of $D(M)$ as a module in $\text{mod-}\Lambda^{op}$.

For $d \geq 0$ let $\Omega^d(\text{mod-}\Lambda)$ denote the subcategory of $\text{mod-}\Lambda$ consisting of those module $M$ such that $M \cong Q \oplus N$, where $Q \in \text{prj-}\Lambda$ and $N = \Omega^d(X)$ for some $X$ in $\text{mod-}\Lambda$.

Theorem 2.2 \[EJ2\]. Let $\Lambda$ be an Artin algebra and let $d \geq 0$. Then the following statements are equivalent:

(a) the algebra $\Lambda$ is $d$-Gorenstein;

(b) $\text{Gprj-}\Lambda = \Omega^d(\text{mod-}\Lambda)$.

Definition 2.3. Let $\Lambda$ be a Gorenstein algebra. A finitely generated module $M$ is called (maximal) Cohen-Macaulay if $\text{Ext}^i(\Lambda,M) = 0$ for $i \neq 0$.

From Theorem 2.2, it is easy to see that for a Gorenstein algebra, the concept of Gorenstein projective modules coincides with the notion of Cohen-Macaulay modules.

An algebra is of finite Cohen-Macaulay type, or simply, CM-finite, if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projective modules. Clearly, $\Lambda$ is a CM-finite algebra if and only if there is a finitely generated module $E$ such that $\text{Gprj-}\Lambda = \text{add-E}$. In this way, $E$ is called to be a Gorenstein projective representation generator of $\text{Gprj-}\Lambda$, dually one can define Gorenstein injective generator for $\text{Ginj-}\Lambda$. If $\text{gldim } \Lambda < \infty$, then $\text{Gprj-}\Lambda = \text{prj-}\Lambda$, so $\Lambda$ is CM-finite. If $\Lambda$ is self-injective, then $\text{Gprj-}\Lambda = \text{mod-}\Lambda$, so $\Lambda$ is CM-finite if and only if $\Lambda$ is representation-finite.

If $E$ is a Gorenstein projective representation generator of $\text{Gprj-}\Lambda$, then the relative Auslander algebra $\text{Aus}(\text{Gprj-}\Lambda) := \text{End}(E)$ is called the Cohen-Macaulay Auslander algebra of $\Lambda$. 
2.3. Almost split sequences. Let us begin this section with some basic definitions for almost split sequences from [AS] or [ARS]. Let $C$ be a subcategory of mod-$\Lambda$ closed under direct summands and extensions. A morphism $f: A \to B$ in $C$ is a minimal left almost split morphism in $C$ if it is not a split monomorphism and every morphism $j: A \to X$ in $C$ that is not a split monomorphism factors through $f$, in addition, for all $h : B \to B$ such that $h \circ f = f$ then $h$ is an isomorphism. A minimal right almost split morphism in $C$ is defined by duality. An exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $C$ is said to be an almost split sequence in $C$ if $f$ is a minimal left almost split morphism in $C$ and $g$ is a minimal right almost split morphism in $C$. The indecomposable module $A$ is uniquely determined by $C$ and denoted by $\tau_C(C)$, and called the relative Auslander-Reiten translation of $C$ in $C$. In the case that $C$ is functorially finite, then $\tau_C(C)$ exists for non-Ext-projective modules $C$ in $C$. We say a module $C \in C$ is Ext-projective in $C$, if $\text{Ext}^1_C(C, X) = 0$ for any $X$ in $C$. Note that for $C = \text{Gprj}-\Lambda$, Ext-projective modules are exactly projective modules in mod-$\Lambda$. Further, For when $C = \text{mod}-\Lambda$, we use $\tau_\Lambda$, or more simply $\tau$, instead of $\tau_{\text{mod}-\Lambda}$.

Following [Ha], we have a triangulated version of the concept of almost split sequence called Auslander-Reiten triangle in the literature. On the other hand, from [RV] we have the notion of Serre functor for a $R$-linear triangulated category $\mathcal{T}$, that is, an auto-equivalence $S : \mathcal{T} \to \mathcal{T}$ together with an isomorphism $D\text{Hom}_\mathcal{T}(X, -) \simeq \text{Hom}_\mathcal{T}(-, \text{S}(X))$ for each $X \in \mathcal{T}$, and $D$ the usual duality. These two concepts are related via [RV, Proposition I.2.3.] which says if for an indecomposable object $C$ in $\mathcal{T}$ there exists an Auslander-Reiten triangle $A \xrightarrow{\delta} B \xrightarrow{\gamma} C \xrightarrow{\alpha} A[1]$, then $A \simeq S \circ [-1](C)$. We recall from the section 5 of [H], if Gprj-$\Lambda$ is contravariantly finite subcategory in mod-$\Lambda$, then Gprj-$\Lambda$ has Serre functor, and denoted by $S_{\text{Gprj}(\Lambda)}$, or simply $S_\Omega$ when there is no danger of confusion. Set $\tau_{\text{Gprj}(\Lambda)} = S_{\text{Gprj}(\Lambda)} \circ \Omega$, or more simply $\tau_\Omega$, where $\Omega$ is the auto-equivalence on Gprj-$\Lambda$ induced by the syzygy functor. Since an almost split sequence in Gprj-$\Lambda$ induces an Auslander-Reiten triangle in $\text{Gprj}-\Lambda$, we have $\tau_{\text{Gprj}-\Lambda}(G) \simeq \tau_\Omega(G)$ in $\text{Gprj}-\Lambda$ for a non-projective Gorenstein projective indecomposable module $G$.

2.4. Representations of a quiver over an algebra. Let $\mathcal{Q}$ be a finite quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, s, t)$, where $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are the set of arrows and vertices of $\mathcal{Q}$, respectively, and $s$ and $t$ are the starting and ending maps from $\mathcal{Q}_1$ to $\mathcal{Q}_0$, respectively. Assume that an Artin algebra $\Lambda$ is given, a representation $X$ of $\mathcal{Q}$ over $\Lambda$ is obtained by associating to any vertex $v$ a module $X_v$ in mod-$\Lambda$ and to any arrow $a : v \to w$ a morphism $X_a : X_v \to X_w$ in mod-$\Lambda$. If $\mathcal{X}$ and $\mathcal{Y}$ are two representations of $\mathcal{Q}$, then a morphism $f : X \to Y$ is determined by a family $\{f_v\}_{v \in \mathcal{Q}_0}$, so that for any arrow $a : v \to w$, the commutativity condition $Y_a \circ f_v = f_w \circ X_a$ holds. So for a given finite quiver $\mathcal{Q}$ and an Artin algebra $\Lambda$, the representations of $\mathcal{Q}$ over mod-$\Lambda$ and the morphisms between them form a category which is denoted by $\text{rep}(\mathcal{Q}, \Lambda)$. We can also for an acyclic finite quiver $\mathcal{Q}$ and an Artin algebra $\Lambda$ define the Artin algebra $\Lambda\mathcal{Q}$, which is called the path algebra of $\mathcal{Q}$ by $\Lambda$. More precisely, let $\rho$ be the set of all path in the given quiver $\mathcal{Q}$ together with the trivial paths associated to the vertices. We write the conjunction of paths from left to right. Now let $\Lambda\mathcal{Q}$ be the free $\Lambda$-module with bases $\rho$. An element of $\Lambda\mathcal{Q}$ is written as a finite sum $\sum_{\rho \in \rho} a_{\rho} \rho$, where $a_{\rho} \in \Lambda$ and $a_{\rho} = 0$ for all but finitely many $\rho$. Then $\Lambda\mathcal{Q}$ is an $R$-algebra in which multiplication is given by concatenation of paths. It is well-known for when $\Lambda$ to be a field $k$, then the category mod-$k\mathcal{Q}$ of (right) finite-dimensional $k\mathcal{Q}$-modules is equivalent to the category $\text{rep}(\mathcal{Q}, k)$ of finite-dimensional representations of $\mathcal{Q}$ over $k$. In the similar case for $\Lambda = k$, we can also prove with a simple modification that mod-$\Lambda\mathcal{Q} \simeq \text{rep}(\mathcal{Q}, \Lambda)$.
for every Artin algebra $\Lambda$. Due to this equivalence throughout this paper we will identify $\Lambda\mathcal{Q}$-modules with representations of $\mathcal{Q}$ over $\Lambda$. Here we can have for representations any notion or notation which have been defined for modules. For instance, we have the concept of Gorenstein projective representations which come from the concept of Gorenstein projective modules have been already considered. We use $\mathcal{GP}(\mathcal{Q}, \Lambda)$ to show the subcategory of Gorenstein projective representations in $\text{rep}(\mathcal{Q}, \Lambda)$, or sometimes $\text{Gprj}(\Lambda)$ because of our identification.

In the following a local characterization of Gorenstein projective representations of an acyclic finite quiver is given.

**Theorem 2.4.** ([EHS, Theorem 3.5.1] or [LZh1, Theorem 5.1]) Let $\Lambda$ be an Artin algebra and $\mathcal{Q}$ an acyclic finite quiver. Take a representation $X$ in $\text{rep}(\mathcal{Q}, \Lambda)$. Then $X$ is in $\mathcal{GP}(\mathcal{Q}, \Lambda)$ if and only if

1. For each vertex $v$, $X_v$ is a Gorenstein projective $\Lambda$-module;
2. For each vertex $v$, the $\Lambda$-morphism $g_v : \oplus_{t(a)=v}X_{s(v)} \to X_v$ is a monomorphism whose cokernel is Gorenstein projective.

In the present paper we mostly be involved in the linear quivers. Sometimes we have to put a relation on the quiver that will be explained in the place.

3. CM-finiteness versus representation-finiteness and vice versa

Throughout this section, let $(A_3, J)$ denote the quiver $A_3 : v_1 \xrightarrow{a} v_2 \xrightarrow{b} v_3$ with relation $J$ generated by $ab$. By definition, a representation $X$ of $(A_3, J)$ over $\Lambda$ is a datum

$$X = (X_i, i = 1, 2, 3, f_j^X : X_j \to X_{j+1}, j = 1, 2 \text{ such that } f_2^X \circ f_1^X = 0),$$

where $X_i$ and $f_j^X$ are modules and morphisms in $\text{mod-}\Lambda$, respectively. We will write $f_j$ instead of $f_j^X$ when no confusion can arise. A morphism from a representation $X$ to another representation $Y$ is a triplet $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_i : X_i \to Y_i$ are $\Lambda$-homomorphisms making the diagram

$$\begin{array}{ccc}
X_i & \xrightarrow{f_i^X} & X_{i+1} \\
\downarrow{\sigma_i} & & \downarrow{\sigma_{i+1}} \\
Y_i & \xrightarrow{f_i^Y} & Y_{i+1}
\end{array}$$

commute. Denote by $\text{rep}(A_3, J, \Lambda)$ the category of representations of $(A_3, J)$ over $\Lambda$. Set $A_3(\Lambda) := \Lambda A_3/J$, where $\Lambda A_3$ is the path algebra of $A_3$ over $\Lambda$ and $J$ denotes the ideal generated by the path $ab$. By an obvious modification of Theorem 1.5 of [ABS], page 57, it can be seen that $\text{mod-}A_3(\Lambda)$ is equivalent to $\text{rep}(A_3, J, \Lambda)$. This equivalence permits us to introduce the subcategory of Gorenstein projective representations in $\text{rep}(A_3, J, \Lambda)$, denoted by $\mathcal{GP}(A_3, J, \Lambda)$.

Indeed, the Gorenstein projective representations are the image of Gorenstein projective modules in $\text{mod-}A_3(\Lambda)$ subject to the equivalence. Due to [LZh2, Theorem 4.1], a Gorenstein projective representation can be explicitly described as follows.

**Lemma 3.1.** Let $X = (X_i, f_j, i = 1, 2, 3, j = 1, 2)$ be a representation in $\text{rep}(A_3, J, \Lambda)$. Then $X \in \mathcal{GP}(A_3, J, \Lambda)$ if and only if $X$ satisfies the following conditions

1. For $i = 1, 2, 3$, $X_i \in \text{Gprj}(\Lambda)$, and also $X_2/\text{Im}(f_1)$ and $X_3/\text{Im}(f_2)$ belong to $\text{Gprj}(\Lambda)$.
2. $f_1$ is a monomorphism and $\text{Ker}(f_2) = \text{Im}(f_1)$. 


In particular, when \( \Lambda \) is a 1-Gorenstein algebra, the condition of being Gorenstein projective of \( X_2/\text{Im}(f_1) \) is redundant. Furthermore, for \( \Lambda \) to be self-injective, this characterization of Gorenstein projective representations can be stated more simpler, only the condition (ii) is needed. In this case, left exact sequences in \( \text{mod-}\Lambda \) are in bijection with the Gorenstein projective representations.

Denote by \( LGP(A_3, I, \Lambda) \), (resp. \( SGP(A_3, I, \Lambda) \)) the subcategory of \( \text{rep}(A_3, J, \Lambda) \) consisting of those representations \( G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_2 \) such that \( 0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_2 \), (resp. \( 0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_2 \to 0 \)) are left exact sequence (resp. short exact sequence), with all terms Gorenstein projective modules in \( \text{mod-}\Lambda \).

**Construction 3.2.** Take a representation \( X \) in \( LGP(A_3, J, \Lambda) \), by applying the Yoneda functor over \( X \), here we consider \( X \) as a left exact sequence in \( \text{mod-}\Lambda \), then we get the following exact sequence

\[
(*) \quad 0 \to (-, X_1) \xrightarrow{(-, f_3^X)} (-, X_2) \xrightarrow{(-, f_3^X)} (-, X_3) \to F \to 0
\]

in \( \text{mod-Gprj-}\Lambda \). In fact, (*) gives us a projective resolution of \( F \) in \( \text{mod-Gprj-}\Lambda \). We now define \( \Psi : LGP(A_3, J, \Lambda) \to \text{mod-}(\text{Gprj-}\Lambda) \) by \( \Psi(X) := F \) and for morphism \( \sigma = (\sigma_1, \sigma_2, \sigma_3) : X \to Y \) in \( LGP(A_3, J, \Lambda) \), \( \Psi(\sigma) : \Psi(X) \to \Psi(Y) \) to be the unique morphism \( (-, \sigma_3) \) which makes the following diagram, obtained by applying the Yoneda functor, commute

\[
\begin{array}{ccc}
0 & \to & (-, X_1) \xrightarrow{(-, f_3^X)} (-, X_2) \xrightarrow{(-, f_3^X)} (-, X_3) \xrightarrow{(-, \sigma_3)} \Psi(X) \xrightarrow{0} \\
\downarrow{(-, \sigma_1)} & & \downarrow{(-, \sigma_2)} & & \downarrow{(-, \sigma_3)} & & \downarrow{(-, \sigma_3)} \\
0 & \to & (-, Y_1) \xrightarrow{(-, f_3^Y)} (-, Y_2) \xrightarrow{(-, f_3^Y)} (-, Y_3) \xrightarrow{\Psi(Y)} 0.
\end{array}
\]

in \( \text{mod-Gprj-}\Lambda \).

Let \( \vartheta^{-1}(\text{Gprj-}\Lambda) \) show the inverse image of \( \text{Gprj-}\Lambda \) under the functor \( \vartheta : \text{mod-}(\text{Gprj-}\Lambda) \to \text{mod-}\Lambda \), stated in 2.1.

In the following a homological characterization for this subcategory is given whenever \( \Lambda \) is a 1-Gorenstein algebra. First let us give the following Lemma.

**Lemma 3.3.** Let \( \Lambda \) be a 1-Gorenstein algebra and let \( F \) be in \( \text{mod-}(\text{Gprj-}\Lambda) \). Then for each \( P \in \text{prj-}\Lambda \), \( \text{Ext}^2_{\text{Gprj-}\Lambda}(F, (-, P)) = 0 \). Here \( \text{Ext}^i_{\text{Gprj-}\Lambda} \) denotes the \( i \)-th Ext-group in \( \text{mod-}(\text{Gprj-}\Lambda) \).

**Proof.** Since \( \Lambda \) is a 1-Gorenstein, then we have a projective resolution \( 0 \to (-, G_2) \to (-, G_1) \to (-, G_0) \to F \to 0 \). For the proof, it is enough to show that the induced map

\[
((-, G_1), (-, P)) \to ((-, G_2), (-, P))
\]

is an epimorphism, or by the Yoneda lemma, the induced map

\[
(G_1, P) \to (G_2, P)
\]

is an epimorphism. This follows by using this fact that the cokernel of monomorphism \( G_2 \to G_1 \) is a Gorenstein projective module. Indeed, the cokernel is a submodule of \( G_0 \), but Gorenstein projective modules are closed under submodules over 1-Gorenstein algebras. \( \Box \)

**Proposition 3.4.** Let \( \Lambda \) be a 1-Gorenstein algebra and let \( F \) be in \( \text{mod-}(\text{Gprj-}\Lambda) \). Then the following conditions are equivalent:

(i) \( F \) lies in \( \vartheta^{-1}(\text{Gprj-}\Lambda) \);
(ii) For each $P \in \text{prj-} \Lambda$, $\text{Ext}^{i}_{\text{prj-} \Lambda}(F, (-, P)) = 0$, $i = 1, 2$;
(iii) For each $P \in \text{prj-} \Lambda$, $\text{Ext}^{1}_{\text{prj-} \Lambda}(F, (-, P)) = 0$.

Proof. By Lemma 3.3, (ii) and (iii) are clearly equivalent, so we only prove the equivalence (i) and (iii). It follows rather similar to the proof of Lemma 3.3 and using this fact that $F$ lies in $\vartheta^{-1}(\text{Gprj-} \Lambda)$ if and only if for a projective presentation $(-, G_1) \xrightarrow{(-, f)} (-, G_0) \to F \to 0$ of $F$ then $\text{Coker}(f)$ belongs to $\text{Gprj-} \Lambda$.

Let $X$ and $Y$ be two representations in $L \mathcal{GP}(A_3, J, \Lambda)$. We define $\mathcal{R}(X, Y)$ consisting of those morphisms $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ with this property that there is a morphisms $h_3$ as the following

\[
\begin{array}{cccc}
X_1 & \xrightarrow{f_1^X} & X_2 & \xrightarrow{f_2^X} & X_3 \\
\sigma_1 & & \sigma_2 & \xrightarrow{h_3} & \\
Y_1 & \xrightarrow{f_1^Y} & Y_2 & \xrightarrow{f_2^Y} & Y_3
\end{array}
\]

such that $\sigma_3 = f_2^Y \circ h_3$. Then $\mathcal{R}$ gives a relation on $L \mathcal{GP}(A_3, J, \Lambda)$. $\Psi$ is dense and full but not in general faithful. Specially, $\Psi(\sigma) = 0$ if and only if $\sigma \in \mathcal{R}(X, Y)$. Then we get the commutative diagram in the following proposition with equivalences in the rows. This argument is essentially similar to Proposition 1.2 of p. 102 in [ARS].

**Proposition 3.5.** Let $\Lambda$ be a 1-Gorenstein algebra. Then the functor $\Psi$, defined in Construction 3.2, induces the equivalences of categories which make the following diagram

\[
\begin{array}{cccc}
L \mathcal{GP}(A_3, J, \Lambda)/\mathcal{R} & \xrightarrow{\sim} & \text{mod-}(\text{Gprj-} \Lambda) \\
\uparrow & & \uparrow \\
\mathcal{GP}(A_3, J, \Lambda)/\mathcal{R} & \xrightarrow{\sim} & \vartheta^{-1}(\text{Gprj-} \Lambda) \\
\uparrow & & \uparrow \\
S \mathcal{GP}(A_3, J, \Lambda)/\mathcal{R} & \xrightarrow{\sim} & \text{mod-}(\text{Gprj-} \Lambda)
\end{array}
\]

commute.

Based on p. 216 in [ARS] we conclude some facts which will be used later as follows. Let CM-finite algebra $\Lambda$ be given. The functor $\Psi$, defined in Construction 3.2, take all representations in $L \mathcal{GP}(A_3, J, \Lambda)$ in the form of $(G \xrightarrow{\text{id}_G} G \to 0)$ and $(0 \to G \xrightarrow{\text{id}_G} G)$ to the zero object. Denote by $\mathcal{C}$ the full additive subcategory of $L \mathcal{GP}(A_3, J, \Lambda)$ consisting of all indecomposable objects in $L \mathcal{GP}(A_3, J, \Lambda)$ not isomorphic to an object of the form $(G \xrightarrow{\text{id}_G} G \to 0)$ or $(0 \to G \xrightarrow{\text{id}_G} G)$, where $G$ is an indecomposable object in $\text{Gprj-} \Lambda$. By the definition of objects in $\mathcal{C}$ one can see for each $X$ in $\mathcal{C}$ we have $\mathcal{R}(X) \subseteq \text{rad}(\text{End}_{\mathcal{C}}(X))$, so that $\Psi(X)$ is an indecomposable module in $\text{mod-}(\text{Gprj-} \Lambda)$. Hence there is induced a one to one correspondence between the indecomposable objects in $\mathcal{C}$ and the indecomposable modules in $\text{mod-}(\text{Gprj-} \Lambda)$. Since $\Lambda$ is CM-finite, then there are only finitely many indecomposable objects in $L \mathcal{GP}(A_3, J, \Lambda)$ in the form $(G \xrightarrow{\text{id}_G} G \to 0)$ or $(0 \to G \xrightarrow{\text{id}_G} G)$, up to isomorphism. Hence $L \mathcal{GP}(A_3, J, \Lambda)$ is of finite type if and only if $\text{Aus}(\text{Gprj-} \Lambda)$ is representation-finite. Similarly, we can deduce that the subcategories $\mathcal{GP}(A_3, J, \Lambda)$ and $S \mathcal{GP}(A_3, J, \Lambda)$ of $\text{rep}(A_3, J, \Lambda)$ are of finite type if and only if the subcategories $\vartheta^{-1}(\text{Gprj-} \Lambda)$ and $\text{mod-}(\text{Gprj-} \Lambda)$ of $\text{mod-}(\text{Gprj-} \Lambda)$ are of finite type, respectively.
Let $R'$ be the image of the relation $R$ of $\text{rep}(A_3, J, \Lambda)$ under the equivalence $\text{mod-}A_3(\Lambda) \simeq \text{rep}(A_3, J, \Lambda)$ in $\text{mod-}A_3(\Lambda)$.

**Corollary 3.6.** Keeping the above notations, then we have the following.

(i) Assume that $\Lambda$ is a basic 1-Gorenstein CM-finite algebra with Gorenstein projective generator $E$. Denote by $\Gamma := \text{Aus}(\text{Gprj-}\Lambda)$ the Cohen-Macaulay Auslander algebra of $\Lambda$. Then there exists an equivalence of categories

$$\text{Gprj-}A_3(\Lambda)/R' \simeq \{X \in \text{mod-}\Gamma|\text{Ext}^1(X, e\Gamma) = 0\},$$

where $e \in \Gamma$ is the idempotent given by the projection on the summand $\Lambda$ of $\Gamma$.

(ii) Assume that $\Lambda$ is a basic representation-finite self-injective algebra. Then there exists an equivalence of categories

$$\text{Gprj-}A_3(\Lambda)/R' \simeq \text{mod-}\Gamma',$$

where $\Gamma' := \text{Aus}(\text{mod-}\Lambda)$ is the Auslander algebra of $\Lambda$.

Consider the quiver $S_3 : w_1 \xrightarrow{c} w_2 \xrightarrow{d} w_3$, without any relation, and let $\text{rep}(S_3, \Lambda)$ be the category of representations over $S_3$ by modules and morphisms in $\text{mod-}\Lambda$. By 2.4 we know that $\text{rep}(S_3, \Lambda)$ is equivalent to $\text{mod-}\Lambda S_3$, where $\Lambda S_3$ is the path algebra of $S_3$ over $\Lambda$. We can see that

$$\Lambda S_3 \simeq S_3(\Lambda) = \begin{pmatrix} \Lambda & 0 & 0 \\ \Lambda & \Lambda & 0 \\ \Lambda & 0 & \Lambda \end{pmatrix}$$

here we consider $S_3(\Lambda)$ as a subalgebra of $M_3(\Lambda)$, the algebra of $3 \times 3$ matrices over $\Lambda$.

**Lemma 3.7.** Let $\Lambda$ be an Artin algebra. Then $A_3(\Lambda)$ and $S_3(\Lambda)$ are derived equivalent.

**Proof.** Consider the following representations in $\text{rep}(S_3, \Lambda)$

$$P_1 = (\Lambda \rightarrow 0 \leftarrow 0), \quad I_2 = (0 \rightarrow 0 \leftarrow \Lambda), \quad \text{and} \quad I_3 = (0 \rightarrow \Lambda \xrightarrow{\text{Id}_{\Lambda}} \Lambda).$$

If we view them in the category of complexes over $\text{rep}(S_3, \Lambda)$ as stalk complexes concentrated at degree zero, then set $T := P_1 \oplus I_3[1] \oplus I_2[1]$. Let $\mathbb{D}^b(\text{mod-}S_3(\Lambda))$, or to simplify shown by $\mathbb{D}^b(\text{mod-}A_3(\Lambda))$, denote the derived category of the category of bounded complexes of finitely generated modules over $\text{mod-}A_3(\Lambda)$. In the reset, we will show that $T$ is a tilting complex in $\mathbb{D}^b(\text{mod-}S_3(\Lambda))$. It can be directly seen that $I_3$ is a projective object in $\text{rep}(S_3, \Lambda)$, or using the local characterization of the projective representations given in [EE]. We have the following projective resolutions for $P_1$ and $I_2$

$$0 \rightarrow (0 \rightarrow \Lambda \leftarrow 0) \rightarrow (\Lambda \xrightarrow{\text{Id}_{\Lambda}} \Lambda \leftarrow 0) \rightarrow P_1 = (\Lambda \rightarrow 0 \leftarrow 0) \rightarrow 0 \quad \text{and}$$

$$0 \rightarrow (0 \rightarrow \Lambda \leftarrow 0) \rightarrow (0 \rightarrow \Lambda \xrightarrow{\text{Id}_{\Lambda}} \Lambda) \rightarrow I_2 = (0 \rightarrow 0 \leftarrow \Lambda) \rightarrow 0.$$ 

Now, we can use the above facts to compute group homomorphisms in the homotopy category instead of the derived category, that is much easier to work with, then we reach the following isomorphisms

$$\text{Hom}_{\mathbb{D}^b(S_3(\Lambda))}(P_1, P_1) \simeq \Lambda, \quad \text{Hom}_{\mathbb{D}^b(S_3(\Lambda))}(P_1, I_2[1]) = 0, \quad \text{Hom}_{\mathbb{D}^b(S_3(\Lambda))}(P_1, I_3[1]) = \Lambda,$$

$$\text{Hom}_{\mathbb{D}^b(S_3(\Lambda))}(I_2[1], I_2[1]) \simeq \Lambda, \quad \text{Hom}_{\mathbb{D}^b(S_3(\Lambda))}(I_2[1], I_3[1]) = 0, \quad \text{Hom}_{\mathbb{D}^b(S_3(\Lambda))}(I_2[1], P_1) = 0,$$

$$\text{Hom}_{\mathbb{D}^b(S_3(\Lambda))}(I_3[1], P_1) \simeq 0, \quad \text{Hom}_{\mathbb{D}^b(S_3(\Lambda))}(I_3[1], I_2[1]) \simeq \Lambda, \quad \text{Hom}_{\mathbb{D}^b(S_3(\Lambda))}(I_3[1], I_3[1]) \simeq \Lambda.$$
and also \( \text{Hom}_{\mathcal{D}^b(\text{Gprj}-\Lambda)}(T, T[i]) = 0 \) for all \( i \neq 0 \). In addition, by using again the projective resolutions and noting that \( I_3[1] \) is a summand of \( T \), one can see in a standard argument, only applying appropriate mapping cones and shiftings, the thick subcategory \( < T > \) generated by \( T \) contains the projective representations \( (0 \to \Lambda \leftarrow 0), (\Lambda \xrightarrow{1_{\Lambda}} \Lambda \leftarrow 0) \) and \( (0 \to \Lambda \leftarrow \Lambda) \). Since the additive closure of these there projective representations is all projective representations in \( \text{rep}(S_3, \Lambda) \), then \( \mathcal{K}^b(\text{prj}-S_3(\Lambda)) = < T > \). Hence \( T \) is a tilting complex and Rickard’s theorem [R] says us \( S_3(\Lambda) \) and \( \text{End}_{\mathcal{D}^b(\text{Gprj}-\Lambda)}(T) \) are derived equivalent. Now by using the above computations of \( \text{Hom} \) we can see that

\[
\text{End}_{\mathcal{D}^b(\text{Gprj}-\Lambda)}(T) = \begin{pmatrix}
\text{Hom}_{\mathcal{D}^b(\text{Gprj}-\Lambda)}(P_1, P_1) & \text{Hom}_{\mathcal{D}^b(\text{Gprj}-\Lambda)}(I_3[1], P_1) & \text{Hom}_{\mathcal{D}^b(\text{Gprj}-\Lambda)}(I_2[1], P_1) \\
\text{Hom}_{\mathcal{D}^b(\text{Gprj}-\Lambda)}(P_1, I_3[1]) & \text{Hom}_{\mathcal{D}^b(\text{Gprj}-\Lambda)}(I_3[1], I_3[1]) & \text{Hom}_{\mathcal{D}^b(\text{Gprj}-\Lambda)}(I_2[1], I_3[1]) \\
\text{Hom}_{\mathcal{D}^b(\text{Gprj}-\Lambda)}(P_1, I_2[1]) & \text{Hom}_{\mathcal{D}^b(\text{Gprj}-\Lambda)}(I_2[1], I_2[1]) & \text{Hom}_{\mathcal{D}^b(\text{Gprj}-\Lambda)}(I_2[1], I_2[1])
\end{pmatrix}
\]

is isomorphic to the algebra

\[
\Gamma = \left\{ \begin{pmatrix} a & 0 & 0 \\
 b & c & 0 \\
 0 & e & f \end{pmatrix} \mid \text{the entries belong to } \Lambda \right\}
\]

the entries of multiplication of two elements in \( \Gamma \) is same as the entries obtained by the multiplication of two matrices except the \( (3, 1) \)-entry that is always zero. But there is trivially an isomorphism of algebra between \( \Gamma \) and \( A_3(\Lambda) \). This completes the proof. □

Recently the classical reflection functors, defined by Bernstein, Gel’fand, and Ponomarev for quiver representations over a filed, was generalized in [L] to quiver representations over arbitrary ground rings. For the case that the ground ring is Noetherian of finite global dimension, the same generalization was proved in [AHV1] as well. This implies that if \( Q \) is an oriented tree and if \( Q' \) is obtained from \( Q \) by an arbitrary orientation, then the corresponding path algebras over an given arbitrary Artin algebra are derived equivalent. In particular, by this result we can deduce that \( S_3(\Lambda) \) and \( T_3(\Lambda) \) are derived equivalent. Here \( T_3(\Lambda) \) is the algebra of \( 3 \times 3 \) lower triangular matrices with entries in \( \Lambda \), which is isomorphic to the path algebra of the quiver

\[
A_3 : v_1 \to v_2 \to v_3
\]

over algebra \( \Lambda \). Hence in view of the above lemma we have the following.

**Lemma 3.8.** Let \( \Lambda \) be an Artin algebra. Then \( T_3(\Lambda) \) and \( A_3(\Lambda) \) are derived equivalent.

If \( \Lambda \) is CM-finite, then we know that \( \text{mod-}(\text{Gprj}-\Lambda) \simeq \text{mod-Aus}(\text{Gprj}-\Lambda) \). So to state our result easier, we identify the corresponding notions and notations have been introduced before for functors and modules.

**Theorem 3.9.** Let \( \Lambda \) be a 1-Gorenstein CM-finite algebra. Then the subcategory \( \vartheta^{-1}(\text{Gprj}-\Lambda) \) of \( \text{mod-Aus}(\text{Gprj}-\Lambda) \) is of finite type if and only if \( T_3(\Lambda) \) is CM-finite.

**Proof.** In view of the discussion after Proposition 3.5, we can see that \( \mathcal{GP}(A_3, J, \Lambda) \) is of finite type if and only if the subcategory \( \vartheta^{-1}(\text{Gprj}-\Lambda) \) so is. In other words, \( A_3(\Lambda) \) is CM-finite if and only if \( \vartheta^{-1}(\text{Gprj}-\Lambda) \) is of finite type. Lemma 3.8 implies that \( T_3(\Lambda) \) and \( A_3(\Lambda) \) are derived equivalent. Now due to [AHV2, Theorem 4.1.2] we obtain the equivalence \( \text{Gprj}-A_3(\Lambda) \simeq \text{Gprj}-T_3(\Lambda) \). Note that by [AHK1, Corollary 4.3] both algebras \( T_3(\Lambda) \) and \( S_3(\Lambda) \) are Gorenstein, so also virtually Gorenstein algebras. This equivalence implies that \( A_3(\Lambda) \) is CM-finite if and only if \( T_3(\Lambda) \) is CM-finite. This finishes the proof of the theorem. □

Note that based on the proof of the above theorem we can also observe that \( \vartheta^{-1}(\text{Gprj}-\Lambda) \) is of finite type if and only if each of which algebras \( T_3(\Lambda), A_3(\Lambda) \) and \( S_3(\Lambda) \) are CM-finite.

In the sequel, we list some consequences of Theorem 3.9.
Corollary 3.10. Let $\Lambda$ be a self-injective algebra of finite representation type. Then the following conditions are equivalent.

(i) The Auslander algebra of $\Lambda$, $\text{Aus}(\text{mod-}\Lambda)$, is representation-finite;

(ii) $T_3(\Lambda)$ is representation-finite;

(iii) $T_3(\Lambda)$ is CM-finite.

Proof. By [AR2, Theorem 1.1] we can conclude that $(i) \iff (ii)$. Since $\Lambda$ is self-injective then $\text{Gprj-}\Lambda = \text{mod-}\Lambda$. By the characterization given in Lemma 3.1 for the Gorenstein projective representations over $(A_3, J)$, we conclude that $L\text{GP}(A_3, J, \Lambda) = \text{GP}(A_3, J, \Lambda)$. This equality implies another equality, that is, $\vartheta^{-1}(\text{mod-}\Lambda) = \text{mod-}\text{Aus}(\text{mod-}\Lambda)$. Now Theorem 3.9 gives the equivalence of $(i)$ and $(iii)$. The proof is completed. 

Example 3.11. For $t \geq 2$, $m \geq 1$ let $A(m, t) := k\hat{A}_m/J(k\hat{A}_m)$ be the associated self-injective Nakayama algebra, where $\hat{A}_m$ is the cyclic quiver of $n$ vertices, $k$ a field and $J(k\hat{A}_m)$ denotes the ideal generated by the arrows. In [Lu2] and [XZZ] can be seen that $T_3(A(1, 3))$ and $T_3(A(2, 2))$ are CM-finite, respectively. So by Corollary 3.10 their corresponding Auslander algebras are representation-finite. Conversely, we can decide CM-finiteness via being representation-finite. In [IPTZ] a necessary and sufficient condition was given for representation-finite algebra $\Lambda$ such that whose Auslander algebra of $\Lambda$ is of finite representation type assuming that $\Lambda$ is standard. Furthermore, Gordana Todorov and Dan Zacharia following the classification results in [IPTZ] considered some qualitative descriptions of the Auslander algebras of finite representation type. For instance, they proved that these algebras are zero-relation with Loewy length at most 4. Therefore, if we want to check that for a given self-injective of finite type $\Lambda$ when $T_3(\Lambda)$ is CM-finite, then $\Lambda$ necessarily must be zero-relation with $\text{LL}(\Lambda) \leq 4$.

As another application of Theorem 3.9, in the following we prove that the property discussed in the Example 3.11, to be representation-finite of the Auslander algebras, among representation-finite self-injective algebras is closed under derived equivalences.

Corollary 3.12. Let $\Lambda$ and $\Lambda'$ be finite dimensional algebras over a field. Assume one of algebras $\Lambda$ and $\Lambda'$ is self-injective of finite type. If $\Lambda$ and $\Lambda'$ are derived equivalent, then the Auslander algebra of $\Lambda$ is representation-finite if and only if $\Lambda'$ is as well.

Proof. Note that the properties of being self-injective and representation-finiteness over self-injective algebras are closed under derived equivalence. For the first one we refer to [AIR], and for the latter use this fact that $\text{mod-}A \simeq \text{mod-}B$ if self-injective algebras $A$ and $B$ are derived equivalent. So if we assume that one of algebras $\Lambda$ and $\Lambda'$ is self-injective and representation-finite, then both of them are self-injective and representation-finite. We know by [A, Theorem 8.5] that derived equivalence between $\Lambda$ and $\Lambda'$ implies that $T_3(\Lambda)$ and $T_3(\Lambda')$ are also derived equivalent. As we did in the proof of Theorem 3.9, from [AHV1, Theorem 4.1.2] we can deduce that $\text{Gprj-}T_3(\Lambda) \simeq \text{Gprj-}T_3(\Lambda')$. This implies that $T_3(\Lambda)$ is CM-finite if and only if $T_3(\Lambda')$ so is. Now Theorem 3.9 conclude the proof of our result. 

Another ready consequence of Theorem 3.9 is:

Corollary 3.13. Let $\Lambda$ be a 1-Gorenstein CM-finite algebra. If the Cohen-Macaulay Auslander algebra $\text{Aus}(\text{Gprj-}\Lambda)$ is representation-finite, then $T_3(\Lambda)$ is CM-finite.

The above corollary is not true in general for the algebra $T_n(\Lambda)$ of $n \times n$ lower triangular matrices, when $n > 3$. Let us give a counterexample. As mentioned in Example 3.11, $T_3(A(1, 3)) = T_3(k[x]/(x^3))$ is CM-finite then by Corollary 3.10, $\text{Aus}(\text{mod-k}[x]/(x^3))$ is representation-finite.
We know from [Lu2, Theorem 4.9], $T_n(k[x]/(x^3))$ is CM-finite if and only if $n = 1, 2, 3$ or $4$. Thus, for $n > 4$, $T_n(k[x]/(x^3))$ is not CM-finite.

4. $\Omega G$-algebras

Throughout this section we assume that $Gprj-\Lambda$ is contravariantly finite in mod-$\Lambda$. Recall that a subcategory $\mathcal{X}$ of mod-$\Lambda$ is resolving if it contains all projectives, and closed under extensions and the kernels of epimorphisms. On the other hand, since $Gprj-\Lambda$ is resolving then it also becomes a functorially finite subcategory of mod-$\Lambda$.

By definition of Gorenstein projective modules, we can see for each $G \in Gprj-\Lambda$, $G^* = \text{Hom}_\Lambda(G, \Lambda)$ is a Gorenstein projective module in mod-$\Lambda^{\text{op}}$. Then the duality $(\cdot)^* : \text{prj}-\Lambda \rightarrow \text{prj}-\Lambda^{\text{op}}$ can be naturally generalized to the duality $(\cdot)^* : Gprj-\Lambda \rightarrow Gprj-\Lambda^{\text{op}}$.

Motivated by the section 5 of [H] we have the following definition.

Definition 4.1. Let $\Lambda$ be an Artin algebra. Then $\Lambda$ is called $\Omega G$-algebra if mod-(Gprj-$\Lambda$) is a semisimple abelian category, i.e. all whose objects are projective.

Let us give some basic properties of $\Omega G$-algebras in the sequel. We start with following lemma.

Lemma 4.2. Let $\Lambda$ be an $\Omega G$-algebra. Then any short exact sequence in Gprj-$\Lambda$ is a direct sum of the short exact sequences in the form $0 \rightarrow (-, G_2) \rightarrow (-, G_1) \rightarrow (-, G_0) \rightarrow F \rightarrow 0$, $0 \rightarrow C \rightarrow 0 \rightarrow 0$ for some $A, B$ and $C$ in Gprj-$\Lambda$.

Proof. Take an exact sequence $0 \rightarrow G_2 \rightarrow G_1 \rightarrow G_0 \rightarrow 0$ in Gprj-$\Lambda$. This short exact sequence induces the sequence

\[ (1) \quad 0 \rightarrow (-, G_2) \rightarrow (-, G_1) \rightarrow (-, G_0) \rightarrow F \rightarrow 0, \]

in mod-Gprj-$\Lambda$. Since mod-(Gprj-$\Lambda$) is a semisimple abelian category then $F \simeq (-, G)$ for some $G$ in Gprj-$\Lambda$. On the other hand, we know a minimal projective resolution of $(\cdot, G)$ is as the following

\[ (2) \quad 0 \rightarrow (-, \Omega(G)) \rightarrow (-, P) \rightarrow (-, G) \rightarrow (-, G) \rightarrow 0, \]

where $P \rightarrow G$ is a projective cover of $G$. By comparing (1) and (2) as two projective resolutions of $F$ in mod-(Gprj-$\Lambda$), and using this fact that the latter is minimal then we get our result. Indeed, the fact we used here is known in the homology algebra, that is, any projective resolution of a module over an Artin algebra is a direct sum of minimal projective resolution of the module and possibly some split exact complexes. □

Definition 4.3. Let $\mathcal{X}$ be a resolving functorially finite subcategory of mod-$\Lambda$. Let $X$ and $Y$ be modules in $\mathcal{X}$. We call a morphism $f : X \rightarrow Y$ irreducible, if $f$ is neither a split monomorphism nor a split epimorphism, and whenever we have a $Z$ in $\mathcal{X}$ such that there are morphisms $h : X \rightarrow Z$ and $g : Z \rightarrow Y$ that satisfy $f = gh$, then either $h$ is a split monomorphism or $g$ split epimorphism.

We refer to [Kr] for the facts used in the proof of our next results about irreducible morphisms for subcategories.

Proposition 4.4. Let $\Lambda$ be an $\Omega G$-algebra. Then $\Lambda$ is CM-finite.

Proof. Since we assume that Gprj-$\Lambda$ is functorially finite then by [AS], the subcategory Gprj-$\Lambda$ has almost split exact sequences. Let $G$ be a non-projective Gorenstein projective indecomposable module. In view of lemma 4.2 there exists an almost split sequence $0 \rightarrow \Omega(G) \rightarrow P \xrightarrow{f} G \rightarrow 0$
in \text{Gprj}-\Lambda obtained by getting projective cover. Note that an almost split sequence must be non-split. Let \( Q \) be an indecomposable summand of \( P \). The natural injection \( f_Q : Q \to G \) is irreducible by using general facts in the theory of almost split sequences for subcategories, e.g. [Kr, Theorem 2.2.2]. Then there exits a minimal left almost split morphism \( g : Q \to Y \) in \text{Gprj}-\Lambda such that \( G \) is a summand of \( Y \). We know that \text{mod-}\Lambda contains only finitely many indecomposable projective modules up to isomorphism. Hence because of the uniqueness of minimal left almost split morphisms, we have only finitely many non-projective Gorenstein projective indecomposable modules up to isomorphism. So we are done.

We recall from [C, Lemma 3.4] that for a semisimple abelian category \( A \) and an auto-equivalence \( \Sigma \) on \( A \), there is an unique triangulated structure on \( A \) with \( \Sigma \) the translation functor. Indeed, all the triangles are split. We denote the resulting triangulated category by \((A, \Sigma)\). We call a triangulated category semisimple provided that it is triangle equivalent to \((A, \Sigma)\) for some semisimple abelian category.

We keep in the following result all notations of 2.3.

**Proposition 4.5.** Let \( \Lambda \) be an \( \Omega_G \)-algebra. Then

(i) \( \text{Gprj}-\Lambda \) is a semisimple triangulated category;

(ii) For any non-projective Gorenstein projective indecomposable \( G \), the relative Auslander-Reiten translation in \text{Gprj}-\Lambda is the first syzygy \( \Omega(G) \);

(iii) For each \( G \) in \text{Gprj}-\Lambda, there is isomorphism \( \tau_G(G) \simeq \Omega(G) \) in \text{Gprj}-\Lambda;

(iv) Serre functor over \text{Gprj}-\Lambda acts on objects by the identity, i.e. \( S_G(G) \simeq G \) for each \( G \) in \text{Gprj}-\Lambda;

Furthermore, If \( \Lambda \) is a finite dimensional algebra over algebraic closed field \( k \), then

(v) \( \tau_G \) and \( \Omega \), viewing as auto-equivalences over \text{Gprj}-\Lambda, are naturally isomorphic;

(vi) Serre functor over \text{Gprj}-\Lambda is isomorphic to the identity functor over \text{Gprj}-\Lambda.

**Proof.** Assume that \( \Lambda \) is an \( \Omega_G \)-algebra then by Lemma 4.2 we have: any short exact sequence with all terms in \text{Gprj}-\Lambda can be written as a direct sum of the short exact sequences in the form of \( 0 \to \Omega(A) \to P \to A \to 0, 0 \to 0 \to B \xrightarrow{1} B \to 0 \) and \( 0 \to C \xrightarrow{1} C \to 0 \to 0 \) for some \( A, B \) and \( C \) in \text{Gprj}-\Lambda. Hence by the structure of triangles in \text{Gprj}-\Lambda, all possible triangles in \text{Gprj}-\Lambda are a direct sums of the following trivial triangles \( K \to 0 \to K[1] \xrightarrow{1} K[1], K \xrightarrow{1} K \to 0 \to K[1] \) and \( 0 \to K \xrightarrow{1} K \to 0 \). By the axioms of triangulated categories, every morphism \( f : X \to Y \) in \text{Gprj}-\Lambda can be completed to a triangle. Now by the structure of triangles in \text{Gprj}-\Lambda, induced by the short exact sequences in \text{Gprj}-\Lambda, and in view of the shape of the short exact sequences in \text{Gprj}-\Lambda, whenever \( \Lambda \) is an \( \Omega \)-Algebra, we can write \( f = (f_1, 0) : X_1 \oplus Y_1 \to X_2 \oplus Y_2 \), where \( f_1 : X_1 \to X_2 \) is an isomorphism in \text{Gprj}-\Lambda. Define the natural injection \( i : Y_1 \to X_1 \oplus Y_1 \) and the natural projection \( X_2 \oplus Y_2 \to Y_2 \) as kernel and cokernel of \( f \), respectively. Then this gives a semisimple abelian structure over \text{Gprj}-\Lambda. Now if we consider the equivalence \( \Omega^1 : \text{Gprj}-\Lambda \to \text{Gprj}-\Lambda \), then the resulting triangulated category \((\text{Gprj}-\Lambda, \Omega)\) is triangle equivalent to \text{Gprj}-\Lambda, so we get (i). (ii) follows from Lemma 4.2 since any almost split sequence is non-split. As in 2.3, the relative Auslander-Reiten translation for a non-projective Gorenstein projective indecomposable module \( G \) is isomorphic to \( \tau_G(G) \). Hence for this case \( \tau_G(G) \simeq \Omega(G) \). However, since \( \tau_G \) is an additive functor which must preserve finite direct sums, and on the other hand, we can decompose any non-projective Gorenstein projective module to a finite direct sum of non projective Gorenstein projective indecomposable modules, so we get (iii). we can deduce
(iv) only by using this point that \( \tau_G = S_G \circ \Omega \). Therefore, the proof of the first part is given completely.

For the second part, (v) and (vi) are equivalent by the equality \( \tau_G = S_G \circ \Omega \), so to complete our proof we show that the identity functor acts as Serre functor on \( \text{Gprj}-\Lambda \). First note that by the argument given in the first part of the proof, we can deduce any morphism between two objects in \( \text{Gprj}-\Lambda \) are either split monomorphism or split epimorphism. Hence for every indecomposable \( \Lambda \) has the Krull-Schmidt property, then we only define \( \Lambda \in \text{Gprj}(\Lambda) = 0 \) if \( \Lambda \) and \( \Lambda \rightarrow \) is an indecomposable module, \( \Lambda \) in \( \text{Gprj}(\Lambda) \), Hom \( \Lambda \rightarrow \Lambda \rightarrow \) for some \( \Lambda \rightarrow \) and then extending obviously to all objects. Similarly, for \( \Lambda \rightarrow \) and check the commutativity of the above diagram only for any non-zero endomorphism \( f : \Lambda \rightarrow \) satisfies the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_\Lambda (\Lambda, -) & \xrightarrow{\eta_G} & \text{DHom}_\Lambda (-, \Lambda) \\
\downarrow \text{Hom}_\Lambda (f, -) & & \downarrow \text{DHom}_\Lambda (-, f) \\
\text{Hom}_\Lambda (\Lambda', -) & \xrightarrow{\eta_{\Lambda'}} & \text{DHom}_\Lambda (-, \Lambda')
\end{array}
\]

Since \( \text{Gprj}-\Lambda \) has the Krull-Schmidt property, then we only define \( \eta_G \) for indecomposable objects \( \Lambda \) and then extending obviously to all objects. Similarly, for \( \Lambda \rightarrow \) being an indecomposable module, to define \( \eta_G (\Lambda') : \text{Hom}_\Lambda (\Lambda, \Lambda') \rightarrow \text{DHom}_\Lambda (-, \Lambda') \) is enough to be given only for indecomposable objects \( \Lambda' \). Even more, because \( \text{Hom}_\Lambda (\Lambda, \Lambda') = 0 \), whenever \( \Lambda \not\cong \Lambda' \), we can assume that \( \Lambda = \Lambda' \) and check the commutativity of the above diagram only for any non-zero endomorphism \( f : \Lambda \rightarrow \) which is necessarily an automorphism. By the assumption over \( \Lambda \), \( \text{End}_\Lambda (\Lambda) \) can be considered naturally as a \( k \)-vector space. But \( \text{End}_\Lambda (\Lambda) \) is a division algebra containing \( k \), as \( k \) is algebraic closed, this implies that \( \text{End}_\Lambda (\Lambda) \cong k \) as \( k \)-vector space and then we obtain any endomorphism in \( \text{End}_\Lambda (\Lambda) \) is in the form of \( r1_{\Lambda} \) for some \( r \in k \). Define \( \eta_G (\Lambda) : \text{End}_\Lambda (\Lambda) \rightarrow \text{DEnd}_\Lambda (\Lambda) \), given by \( r1_{\Lambda} \rightarrow \delta_r \), where \( \delta_r : \text{End}_\Lambda (\Lambda) \rightarrow k \) given by sending \( r1_{\Lambda} \) to \( rr' \in k \). Now it is not difficult to see that by this definition of \( \eta \), the desired commutativity holds in the above. So we are done.

The above proposition explains for a justification of the terminology of \( \Omega_{G\text{-algebras}} \).

**Proposition 4.6.** Let \( \Lambda \) be an \( \Omega_G \)-algebra. Then \( \Lambda^{\text{op}} \) is also an \( \Omega_G \)-algebra.

**Proof.** It can be easily proved by using the duality \( (-)^\ast : \text{Gprj}-\Lambda \rightarrow \text{Gprj}-\Lambda^{\text{op}} \). \( \square \)

Let \( \Lambda \) be an \( \Omega_G \)-algebra. As there are only a finite number of non-isomorphic pairwise Gorenstein projective indecomposable modules and the syzygy functor preserves the indecomposable objects, then non-projective Gorenstein projective modules are \( \Omega \)-periodic modules, meaning that there exists \( n > 0 \), depending on a given module \( G \) in \( \text{Gprj}-\Lambda \), such that \( \Omega^n(G) \cong G \). On the set of non-projective Gorenstein projective indecomposable modules we define an equivalence relation, that is, \( G \sim G' \) if and only if there is some \( n \in \mathbb{Z} \) such that \( G \cong \Omega^n(G') \). Note that here \( \Omega^n(G) \) for \( n < 0 \) are defined by using right (minimal) projective resolution of \( G \). In fact, since \( \text{Gprj}-\Lambda \) is a Frobenius category with \( \text{prj}-\Lambda \) projective-injective objects, then we can construct a right projective resolution by \( \text{prj}-\Lambda \). Let \( C(\Lambda) \) denote the set of equivalence classes of \( \text{Ind}-\text{(Gprj}-\Lambda \setminus \text{prj}-\Lambda) \) respect to \( \sim \). Moreover, for a non-projective Gorenstein projective indecomposable module \( G \) we write \( l(G) \) for the size of the equivalence class \([G]\), trivially, it is the minimum number \( n > 0 \) such that \( \Omega^n(G) \cong G \).
Let $\mathcal{T}$ be an additive category and $F : \mathcal{T} \to \mathcal{T}$ an automorphism. Following [K], the orbit category $\mathcal{T}/F$ has the same objects as $\mathcal{T}$ and its morphisms from $X$ to $Y$ are $\oplus_{n \in \mathbb{Z}} \text{Hom}_\mathcal{T}(X, F^n(Y))$.

**Proposition 4.7.** Let $\Lambda$ be an $\Omega$-algebra over an algebraic closed field $k$.

If $\Lambda$ is a Gorenstein algebra, then there is an equivalence of triangulated categories

$$D_{sg}(\Lambda) \simeq \prod_{[G] \in \mathcal{C}(\Lambda)} \mathbb{D}^{b}(\text{mod}-k)_{l(G)}$$

where $\mathbb{D}^{b}(\text{mod}-k)_{l(G)}$ denotes the triangulated orbit category of $\mathbb{D}^{b}(\text{mod}-k)$ respect to the functor $F = [l(G)]$, the $l(G)$-the power of the shift functor.

**Proof.** By Buchweitz equivalence over Gorenstein algebras, $D_{sg}(\Lambda) \simeq \text{Gprj-}\Lambda$, so it is enough to describe $\text{Gprj-}\Lambda$. For a $[G] \in \mathcal{C}(\Lambda)$, we have seen that $\sum^{l(G)}(G') \simeq G'$ in $\text{Gprj-}\Lambda$ for any $G' \in [G]$. Recall that the suspension functor $\sum$ on $\text{Gprj-}\Lambda$ is a quasi-inverse of the syzygy. On the other hand, by the proof of Proposition 4.5, we have for every indecomposable modules $X$ and $Y$ in $\text{Gprj-}\Lambda$, $\text{Hom}_{\Lambda}(X, Y) = 0$ if $X \not\simeq Y$, and $\text{Hom}_{\Lambda}(X, Y) \simeq k$ as $k$-vector spaces. From our calculation on Hom-spaces, we deduce the desired equivalence. \qed

The above result is inspired by the similar equivalence in [Ka] for Gentle algebras and for quadratic monomial algebras in [CSZ].

Trivially, algebras of finite global dimension, since $\text{Gprj-}\Lambda = \text{proj-}\Lambda$, then are examples of $\Omega$-algebras. For some non-trivial examples of $\Omega$-algebras, we refer to [H, Example 5.10], in there was shown that quadratic monomial algebras, in particular Gentle algebras, are $\Omega$-algebras. Moreover, as it was mentioned in [H, Example 5.10], $\Omega$-algebras are closed under derived equivalences. There are many examples of CM-finite non-$\Omega$-algebras. Let us first give a remark. It was shown in [AR1, Theorem 10.7], for an arbitrary Artin algebra $\Lambda$, mod-(mod-$\Lambda$) is semisimple if and only if $\Lambda$ is Nakayama and with loewy length at most 2. By using this fact, we see that the self-injective Nakayama algebras $k[x]/(x^n)$, $n > 2$, are examples of CM-finite non-$\Omega$-algebras.

We will prove in the following, the property of being CM-finite of $\Omega$-algebras, shown in Proposition 4.4, can be preserved by getting path algebra over them.

**Proposition 4.8.** Let $\Lambda$ be an $\Omega$-algebra. Let $Q$ be the following linear quiver with $n \geq 1$ vertices

$$v_1 \to \cdots \to v_n.$$

Then the path algebra $\Lambda Q$ is CM-finite.

**Proof.** Since the case $n = 1$ is clear then we assume $n \geq 2$. By the local characterization given in Theorem 2.4 for the Gorenstein projective representations in $\text{rep}(Q, \Lambda)$, we can say for a given representation $X = (X_1, f_1, \cdots, X_{n-1}, f_{n-1} ; X_n)$ in $\text{rep}(Q, \Lambda)$: $X$ is a Gorenstein projective representation if and only if it satisfies the following conditions

1. For $1 \leq i \leq n$, $X_i$ are Gorenstein projective modules.
2. For $1 \leq i \leq n - 1$, $\text{Coker}(f_i)$ are Gorenstein projective modules and $f_i$ are monomorphisms.

We know by Lemma 4.2 the short exact sequences over $\Omega$-algebras with all terms in $\text{Gprj-}\Lambda$ are a direct sum of the short exact sequence in the form: $0 \to G \xrightarrow{1_{G}} G \to G \to 0$, $0 \to 0 \to G \xrightarrow{1_{G}} G \to 0$ and $0 \to \Omega(G) \to P \to G \to 0$, where $G$ is a Gorenstein projective module. By having the
characterization of the Gorenstein projective representations, given in the above, and the short exact sequences in Gprj-Λ can be checked that if $X$ is an indecomposable in rep($\mathcal{Q}, \Lambda$) then it is in the following form: Let $G$ and $G'$ be Gorenstein projective indecomposable modules and $1 \leq i \leq j \leq n$,

$$Y_{[i,j]} = (0 \rightarrow \cdots \rightarrow G \xrightarrow{\text{Id}_G} G \cdots \rightarrow G \xrightarrow{\text{Id}_G} G \xrightarrow{f} P \xrightarrow{\text{Id}_P} \cdots \xrightarrow{\text{Id}_P} P)$$

where the first $G$ is settled in the $i$-th vertex and the last in the $j$-th vertex, the map $l$ attached to the arrow coming out vertex $j$ is a monomorphism such that $\text{coker}(l) = G'$ and $P \rightarrow \text{coker}(l)$ is a projective cover.

More explanation, let $X = (X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n)$ be an indecomposable representation. Assume $i$ is the minimum natural number that $X_i \neq 0$. If $i = n$, then the case is clear. So assume $i < n$. By the characterization given in the above, we have the short exact sequence $\epsilon : 0 \rightarrow X_i \xrightarrow{f_i} X_{i+1} \rightarrow \text{coker}(f_i) \rightarrow 0$. Due to the shape of short exact sequences in Gprj-Λ over $\Omega_\mathcal{Q}$-algebra $\Lambda$, and $X$ being indecomposable, $\epsilon$ must be either $(a) : 0 \rightarrow G \xrightarrow{\text{Id}_G} G \rightarrow 0 \rightarrow 0$ or $(b) : 0 \rightarrow \Omega(G') \rightarrow P \rightarrow G' \rightarrow 0$, where $G$ and $G'$ are indecomposable Gorenstein projective modules. If the first case happens, then $f_i$ is isomorphic to $G \xrightarrow{\text{Id}_G} G$, for some Gorenstein projective indecomposable $G$. If $i+1 = n$, then we stop, but in otherwise, $i+1 < n$, in a similar argument $X_{i+1} \xrightarrow{f_{i+1}} X_{i+2}$ is in the form of the monomorphisms appeared in either $(a)$ or $(b)$. If the form $(a)$ happens, then $f_{i+1}$ is isomorphic to $G_1 \xrightarrow{\text{Id}_{G_1}} G_1$, for some $G_1 \in$ Gprj-Λ, but we may assume $G_1 = G$ as $G \simeq X_{i+1} \simeq G_1$. We continue this argument and let $j$ be the first position such that the second type $(b)$ occurs. Hence we can assume $X_j \xrightarrow{f_j} X_{j+1}$ is isomorphic to $G_2 \xrightarrow{f_j} P$, as in $(\dagger)$, but since $X_j \simeq G$ then we can assume $G_2 = G$. If $j+1 = n$, then the case is clear. So assume that $j+1 < n$. The short exact sequence $0 \rightarrow X_{j+1} \xrightarrow{f_{j+1}} X_{j+2} \rightarrow \text{coker}(f_{j+1}) \rightarrow 0$ as $X_{j+1}$ is projective then the case $(b)$ is impossible, and so it is isomorphic to $0 \rightarrow G_3 \xrightarrow{\text{Id}_{G_3}} G_3 \rightarrow 0 \rightarrow 0$, for some $G_3 \in$ Gprj-Λ, but since $X_{j+1} \simeq P$ we can assume $G_3 = P$. For the rest possible arrows, we can continue similarly and obtain all of the attached morphisms to the remaining arrows in the representation are isomorphic to $P \xrightarrow{\text{Id}_P} P$. Finally by gluing this isomorphisms together in an obvious way, we observe that $X$ is isomorphic to $Y_{[i,j]}$. Since by Proposition 4.4, $\Lambda$ is CM-finite then we have only finitely many choices for $G$ and $G'$ in $(\dagger)$. Therefore, there are finitely many isomorphisms classes of Gorenstein projective indecomposable representations in the form of $(\dagger)$. So we are done. 

\textbf{Remark 4.9.} We can produce more algebras of being CM-finite arising from $\Omega_\mathcal{Q}$-algebras. For example, by Proposition 4.8 and help of Lemmas 3.8 and 3.9 we can deduce $A_3(\Lambda)$ is also CM-finite when $\Lambda$ to be an $\Omega_\mathcal{Q}$-algebra. In general, we believe directly by using local characterization given in Theorem 2.4 for a finite acyclic quiver and even more by the similar one given in [LZh2] for acyclic quivers with monomial relations, one can find some more CM-finite algebras in this way.

\textbf{Lemma 4.10.} Let $\Lambda$ be a 1-Gorenstein algebra. If $f : G \rightarrow G'$ is an irreducible morphism in Gprj-Λ, then $f$ is either a monomorphism or an epimorphism.

\textbf{Proof.} The proof is the same as the counterpart of this statement (e.g. Lemma 5.1 p. 166 of [ARS]) for an irreducible in mod-Λ. We only need to use here that over 1-Gorenstein algebras the subcategory Gprj-Λ is closed under submodules.

\textbf{Lemma 4.11.} Let $\Lambda$ be an Artin algebra. Let $P$ be an indecomposable projective in mod-Λ.
(i) If \( G \xrightarrow{f} \text{rad}(P) \) is a minimal right \( \text{Gprj}-\Lambda \)-approximation. Then \( i \circ f \) is a minimal right almost split morphism in \( \text{Gprj}-\Lambda \). Here, \( i : \text{rad}(P) \twoheadrightarrow P \) denotes the inclusion.

(ii) If \( G' \xrightarrow{g} \text{rad}(P^*) \) is a minimal right \( \text{Gprj}-\text{Gprj}^\text{op}-\Lambda \)-approximation. Then \( (j \circ g)^* : P \rightarrow (G')^* \) is a minimal left almost split morphism in \( \text{Gprj}-\Lambda \). Here, \( j : \text{rad}(P^*) \twoheadrightarrow P^* \) denotes the inclusion.

\[ \text{Proof.} \]

(i) We only need to prove that for any non-split epimorphism \( h : G_0 \rightarrow P \), there exists morphism \( l : G_0 \rightarrow G \) so that \( i \circ f \circ l = h \). Since \( h \) is a non-split epimorphism, then \( \text{Im}(h) \) is a proper submodule in \( P \), i.e. \( \text{Im}(h) \subseteq \text{rad}(P) \). So we can write \( h = i \circ s \) for some \( s : G_0 \rightarrow \text{rad}(P) \). Now since \( f \) is a right \( \text{Gprj}-\Lambda \)-approximation, then there is \( l : G_0 \rightarrow G \) such that \( f \circ l = s \) and clearly it works for what we need.

(ii) follows from (i) and the duality \((-)^* : \text{Gprj}-\Lambda \rightarrow \text{Gprj}-\text{Gprj}^\text{op} \), introduced in the first of this section. \( \square \)

In the following we shall give a characterization of 1-Gorenstein \( \Omega_\Delta \)-algebras in terms of radical of projective modules.

**Proposition 4.12.** The following conditions are equivalent

(i) \( \Lambda \) is a 1-Gorenstein \( \Omega_\Delta \)-algebra;

(ii) \( J(\Lambda) \oplus \Lambda \) is a Gorenstein projective representation generator of \( \text{Gprj}-\Lambda \).

\[ \text{Proof.} \ (i) \Rightarrow (ii) \] 1-Gorensteiness of \( \Lambda \) implies \( \Omega^1(\text{mod}-\Lambda) = \text{Gprj}-\Lambda \), see Theorem 2.2, in particular, \( J(\Lambda) \) is a Gorenstein projective module. We observe by lemma 4.11 that for each indecomposable projective \( Q \), \( \text{rad}(Q) \hookrightarrow Q \) is a minimal right almost split morphism in \( \text{Gprj}-\Lambda \). Let \( G \) be a non-projective Gorenstein projective indecomposable. There exists a non-projective Gorenstein projective indecomposable \( G' \) such that \( \Omega(G') = G \). Then we have the almost split sequence \( 0 \rightarrow G \xrightarrow{f} P \rightarrow G' \rightarrow 0 \) in \( \text{Gprj}-\Lambda \), by using the assumption of being \( \Omega_\Delta \)-algebra. In analog with proof of Lemma 4.4, let \( Q \) be a direct summand of \( P \) then \( G \) is a summand of \( \text{rad}(Q) \), and so a summand of \( J(\Lambda) \), as required. Now we give proof \((ii) \Rightarrow (i) \). Since \( J(\Lambda \Lambda) \) is a Gorenstein projective module, we can reach that simple modules have Gorenstein projective dimension at most one. Then by induction on the length of modules, we can prove the global Gorenstein projective dimension of \( \text{mod}-\Lambda \) is at most one, equivalently, \( \Lambda \) is 1-Gorenstein. To prove \( \Lambda \) to be an \( \Omega_\Delta \)-algebra, we first claim that there is no an irreducible morphism between two non-projective Gorenstein projective indecomposable modules. Suppose, contrary to our claim, that we would have an irreducible morphism \( f : G \rightarrow G' \) with \( G \) and \( G' \) being non-projective Gorenstein projective indecomposable modules. By Lemma 4.10, \( f \) would be either a monomorphism or an epimorphism. We first assume that \( f \) is an epimorphism. Clearly, \( \tau_\Omega(G') \) is a non-projective Gorenstein projective indecomposable, hence by our assumption, there is a projective indecomposable \( P \) such that \( \tau_\Omega(G') \) is isomorphic to a summand of \( \text{rad}(P) \). Since the inclusion \( \text{rad}(P) \hookrightarrow P \) is a minimal right split sequence, then it implies that there exists the irreducible morphism \( \tau_\Omega(G') \rightarrow P \) in \( \text{Gprj}-\Lambda \). This fact follows that \( P \) must be appeared in the middle term of almost split sequence in \( \text{Gprj}-\Lambda \) ending by \( G' \). Thus we have the irreducible morphism \( g : P \rightarrow G' \), and hence there is \( h : P \rightarrow G \) such that \( f \circ h = g \). As \( g \) is an irreducible morphism, then either \( h \) is a split monomorphism or \( f \) a split epimorphism, but this is impossible, so we get a contradiction. In other case, if \( f \) is a monomorphism. By our assumption let \( Q \) be a projective indecomposable module such that \( G \) is a summand of \( \text{rad}(Q) \). Hence there is an irreducible morphism \( G \xrightarrow{f} Q \). By applying the duality \((-)^* : \text{Gprj}-\Lambda \rightarrow \text{Gprj}-\text{Gprj}^\text{op} \), we find the
following diagram

\[
\begin{array}{ccc}
Q^* & \downarrow & (G')^* \\
\downarrow & f^* & \downarrow \\
G^* & & G^*
\end{array}
\]

in \( \text{Gprj-}\Lambda^{\text{op}} \). Since \( f^* \) is an epimorphism then then there exits \( s : Q^* \to (G')^* \) such that \( f^* \circ s = l^* \). Since a duality preserves the irreducible morphisms, so either \( f^* \) is a split epimorphism or \( s \) a split monomorphism. But both cases are not impossible to happen, so we reach a contraction. Now the proof of the claim is completed.

The claim gives us that for any non-projective Gorenstein projective indecomposable \( X \), if \( g : Y \to X \) is a minimal right almost split morphism in \( \text{Gprj-}\Lambda \), then \( Y \) must be a projective module. Note that by (ii) the subcategory \( \text{Gprj-}\Lambda \) is of finite type, and so it has almost split sequences. Therefore, in view of these facts, the short exact sequence \( \eta : 0 \to \Omega(X) \to P \to X \to 0 \) acts as the almost split sequence in \( \text{Gprj-}\Lambda \) ending at \( X \). It is known that the functors induced by the almost split sequences are simple functors, see e.g. [A, Chapetr 2]. For the case of almost split sequence \( \eta \), the induced functor is \( (\cdot, X) \) in \( \text{mod-}\text{Gprj-}\Lambda \), which we can consider it as a projective indecomposable object in \( \text{mod-}(\text{Gprj-}\Lambda) \). Conversely, any projective indecomposable in \( \text{mod-}(\text{Gprj-}\Lambda) \) can be obtained in this way. So any projective indecomposable in \( \text{mod-}(\text{Gprj-}\Lambda) \) is simple. Consequently, \( \text{mod-}(\text{Gprj-}\Lambda) \) is a semisimple abelian category, and so \( \Lambda \) is also an \( \Omega_\mathcal{G} \)-algebra. Hence we are done.

An application of the above result is that a Gorenstein projective indecomposable module over a 1-Gorenstein \( \Omega_\mathcal{G} \)-algebra is isomorphic to a summand of \( J(\Lambda) \oplus \Lambda \). In the following some examples of 1-Gorenstein \( \Omega_\mathcal{G} \)-algebras are given.

**Example 4.13.**  
(i) Clearly, hereditary algebras are 1-Gorenstein \( \Omega_\mathcal{G} \)-algebras. So 1-Gorenstein \( \Omega_\mathcal{G} \)-algebras can be considered as a generalization of hereditary algebras. It might be interesting to give a complete classification of 1-Gorenstein \( \Omega_\mathcal{G} \)-algebras and to studying how the representation theory aspects of hereditary algebras can be transferred to 1-Gorenstein \( \Omega_\mathcal{G} \)-algebras.

(ii) Recently due to Ming Lu and Bin Zhu in [LZ] a criteria was given for which monomial algebras are 1-Gorenstein algebras. Hence by having in hand such criteria we can search among quadratic monomial algebras to find 1-Gorenstein \( \Omega_\mathcal{G} \)-algebras. In particular, we can specialize on Gentle algebras to find which of them are 1-Gorenstein, as done in [CL].

(iii) The cluster-tilted algebras defined in [BMR1] and [BMR2] are an important class of 1-Gorenstein algebras. So among cluster-tilted algebras we can find some examples of 1-Gorenstein \( \Omega_\mathcal{G} \)-algebras. For example the cluster-tilted algebras of type \( A \) since are Gentle algebra, so in this case we are dealing with 1-Gorenstein \( \Omega_\mathcal{G} \)-algebras. For other types \( D \) and \( E \), in [CGL] the singularity categories of cluster-tilted algebras are described by the stable categories of some self-injective algebras. In particular, those cluster-tilted algebras of type \( D \) and \( E \) which are singularity equivalent to the self-injective Nakayama algebra \( \Lambda(3, 2) \), the Nakayama algebra with cycle quiver with 3 vertices modulo the ideal generated by the paths of length 2, are other examples of \( \Omega_\mathcal{G} \)-algebras. The following
cluster-tilted algebra (taken from [ABS]) given by the quiver

\[
\begin{array}{c}
\text{2} \\
\text{1} \quad \lambda \quad \mu \\
\text{3} \\
\text{4} \\
\end{array}
\]

\[
\begin{array}{c}
\beta \swarrow \searrow \alpha \\
\delta \swarrow \searrow \gamma
\end{array}
\]

bound by the quadratic monomial relations \( \alpha \beta = 0, \gamma \delta = 0, \delta \lambda = 0, \lambda \gamma = 0, \beta \mu = 0, \) and \( \mu \alpha = 0 \) is also a 1-Gorenstein \( \Omega_G \)-algebra.

**Proposition 4.14.** Let \( \Lambda \) be a CM-finite algebra. If the Cohen-Macaulay Auslander algebra \( \text{Aus}(\text{Gprj}-\Lambda) \) is representation-finite, then \( \Lambda \) so is.

**Proof.** By 2.1 we have a fully faithful functor \( \vartheta : \text{mod-}\Lambda \rightarrow \text{mod-}(\text{Gprj-}\Lambda) \), given by \( \vartheta(M) := \text{Hom}_\Lambda(-, M)|_{\text{Gprj-}\Lambda} \). On the other hand, we know \( \text{mod-}(\text{Gprj-}\Lambda) \cong \text{mod-}\text{Aus}(\text{Gprj-}\Lambda) \), so we get an embedding functor from \( \text{mod-}\Lambda \) to \( \text{mod-}\text{Aus}(\text{Gprj-}\Lambda) \) which preserves indecomposable modules. This ends the proof. \( \square \)

There is a similar result of the following in [Lu1, Theorem 4.4].

**Theorem 4.15.** Assume that \( \Lambda \) is a 1-Gorenstein \( \Omega_G \)-algebra. Then \( \Lambda \) is representation-finite if and only if the Cohen-Macaulay Auslander algebra \( \text{Aus}(\text{Gprj-}\Lambda) \) so is.

**Proof.** The “if” part follows from Proposition 4.14.

For “only if” part assume that \( \Lambda \) is representation-finite. Based on the discussion after Proposition 3.5, we observe that the subcategory \( \mathcal{LP}(A_3, J, \Lambda) \) of \( \text{rep}(A_3, J, \Lambda) \) is of finite type if and only if \( \text{mod-}(\text{Gprj-}\Lambda) \) is as well. Thus we show \( \mathcal{LP}(A_3, J, \Lambda) \) is a subcategory of finite type of \( \text{rep}(A_3, J, \Lambda) \). To prove our result, we divide the proof in three cases as follows. Denote by \( \mathcal{C} \) the full additive subcategory of \( \mathcal{LP}(A_3, J, \Lambda) \) consisting of all indecomposable objects in \( \mathcal{LP}(A_3, J, \Lambda) \) isomorphic to an object of the form either \( (G \xrightarrow{\text{Id}} G \rightarrow 0) \) or \( (0 \rightarrow G \xrightarrow{\text{Id}} G) \) with indecomposable object \( G \) in \( \text{Gprj-}\Lambda \). Since \( \mathcal{C} \) has only finitely many indecomposable representations, up to isomorphism, hence we only consider the indecomposable representations in \( \mathcal{LP}(A_3, J, \Lambda) \) out of \( \mathcal{C} \) throughout of the proof.

**Case 1:** We will show in this step that there is only a finite number of indecomposable representations, up to isomorphism, of \( \mathcal{LP}(A_3, J, \Lambda) \) in the form \( (0 \rightarrow G \xrightarrow{f} P) \) with \( P \) projective module and \( f \) an injection. Let \( \mathcal{C}_1 \) denote the subcategory of \( \mathcal{LP}(A_3, J, \Lambda) \) which consists of all representations with the indecomposable summands in the mentioned form. We can define a functor \( \Phi : \mathcal{C}_1 \rightarrow \text{mod-}\Lambda \) by sending \( (0 \rightarrow G \xrightarrow{f} P) \) to the Coker\((f)\), which is a full and dense functor. Note that since \( \Lambda \) is 1-Gorenstein then for each module \( M \) in \( \text{mod-}\Lambda \) we have a short exact sequence \( 0 \rightarrow G \rightarrow P \rightarrow M \rightarrow 0 \) with \( P \) projective and \( G \) Gorenstein projective. Note that also only the representations \( (0 \rightarrow P \xrightarrow{1_P} P) \), for some projective module \( P \), in \( \mathcal{C}_1 \) are mapped into the zero module. In a similar way of the proof of Proposition 3.5, it can be seen that \( \mathcal{C}_1/R_1 \cong \text{mod-}\Lambda \), where \( R_1 \) is a relation on \( \mathcal{C}_1 \). Also by a same discussion after the proposition we can get that \( \mathcal{C}_1 \) is of finite type if and only if \( \Lambda \) is representation-finite. So we get the desired result by using our assumption.

**Case 2:** In this step we will prove that there is a finite number of indecomposable representations, up to isomorphism, of \( \mathcal{LP}(A_3, J, \Lambda) \) in the form \( (0 \rightarrow G \xrightarrow{f} G') \) with only \( f \) to be an
injection and no recertification on \( G' \). Take an indecomposable object \((0 \to G \overset{f}{\to} G')\) in such a form. Since \( G' \) is a Gorenstein projective module then it can be embedded in a projective module, say, \( i : G' \to P \). Now we have the representation \((0 \to G \overset{i}{\to} f P)\) which lies in \( C_1 \). Considering \((0 \to G \overset{i}{\to} f P)\) as an object in the Krull-Schmidt category \( \text{rep}(A_3, J, \Lambda) \), then we can decompose it as

\[
(\dagger) \quad (0 \to G \overset{i}{\to} f P) = \oplus (0 \to G_i \overset{i}{\to} P_i),
\]

where the \( P_i \) must be projective modules. Since \( C_1 \) is idempotent-complete, then the \((0 \to G_i \overset{i}{\to} P_i)\) are indecomposable objects in \( C_1 \). By decomposition \((\dagger)\), we also obtain the following decomposition

\[
(0 \to G \overset{f}{\to} G') = \oplus (0 \to G_i \overset{i}{\to} \text{Im}(f_i)).
\]

Clearly the \( \text{Im}(f_i) \) are in \( \text{Gprj-}\Lambda \) as \( \Lambda \) is a 1-Gorenstein. On the other hand, since \((0 \to G \overset{i}{\to} G')\) is indecomposable then there is some \( j \) such that \((0 \to G \overset{j}{\to} G') = (0 \to G_j \overset{j}{\to} \text{Im}(f_j)) \). As we have seen any indecomposable representation in this case can be uniquely obtained by the indecomposable objects in \( C_1 \). But by the Case 1, \( C_1 \) is of finite type, so this completes the proof of this step.

**Case 3:** Let \( X = (G_1 \overset{f}{\to} G_2 \overset{g}{\to} G_3) \) be an indecomposable representation in \( \text{LGp}(A_3, J, \Lambda) \). Let \( C_2 \) denote the subcategory of \( \text{LGp}(A_3, J, \Lambda) \) containing of those representations appeared in the Case 2. If we consider the representation \( X \) as a left exact sequence in \( \text{mod-}\Lambda \), then we have the following diagram

\[
\begin{tikzcd}
0 & G_1 & f \rightarrow G_2 & g \rightarrow G_3 & G \\
& & m \rightarrow & & \downarrow m
\end{tikzcd}
\]

in which an epi-mono factorization for morphism \( g \) is given. Since \( \Lambda \) is 1-Gorenstein then \( G \) belongs to \( \text{Gprj-}\Lambda \). In view of Lemma 4.2, concerning the shape of the short exact sequences in \( \text{Gprj-}\Lambda \) over an \( \Omega G \)-algebra, and being \( X \) indecomposable imply the short exact sequence \((0 \to G \overset{f}{\to} G_2 \overset{g}{\to} G \to 0)\) in the above diagram must be in the form \( 0 \to \Omega(G') \overset{f'}{\to} P \overset{g'}{\to} G' \to 0 \) for some Gorenstein projective indecomposable \( G' \), where \( g' \) is a projective cover. We remind that as said in the first of the proof we threw out the indecomposable representations in the form either \((M \overset{\text{Id}}{\to} M \to 0)\) or \((0 \to M \overset{\text{Id}}{\to} M)\). Since \( m \) is a monomorphism then we can identify it as an object in \( C_2 \) by representation \((0 \to G \overset{m}{\to} G_3) \). We decompose the representation \( m \) to the indecomposable representations as the following

\[
(0 \to G \overset{m}{\to} G_3) = \oplus (0 \to H_i \overset{m_i}{\to} M_i).
\]

Let \( p_i : P_i \to H_i \) be a projective cover of \( H_i \) for each \( i \). Because \( g' : P \to G' \) is a projective cover of \( G' \cong G \oplus H_i \), so \( P \cong \oplus P_i \) and also, due to uniqueness of a projective cover, we can identify \( g' \) with a morphism such that whose presentation as a matrix respect to the the decompositions \( P \cong \oplus P_i \) and \( H \cong H_i \) is a diagonal matrix with each \( p_i \) in the \((i, i)\)-th entry. By putting these facts together we have that the following decomposition

\[
(G_1 \overset{f}{\to} G_2 \overset{g}{\to} G_3) = \oplus (\Omega(H_i) \overset{h_i}{\to} P_i \overset{m_i \circ p_i}{\to} M_i),
\]
where for each $i$, $\Omega(H_i) \xrightarrow{h_i} P_i$ is the kernel of $p_i$. As $X$ is an indecomposable representation then it is a summand of some $(\Omega(H_j) \xrightarrow{h_j} P_j \xrightarrow{m_i g_F} M_j)$ in the above decomposition. As we have observed the representations $(\Omega(H_i) \rightarrow P_i \rightarrow M_i)$ are constructed uniquely, up to isomorphism, by the indecomposable representations in $C_2$, but there are only finitely many isomorphism classes of indecomposable objects in $C_2$ by Case 2. The proof is completed. \hfill $\Box$

As an immediate consequence of this theorem and Proposition 4.12 we have the following result.

**Corollary 4.16.** Let $\Lambda$ be a 1-Gorenstein $\Omega_2$-algebra. If $\Lambda$ is representation-finite, then $\text{End}(\Lambda \oplus J(\Lambda))$ so is.

It is interesting to see that whether Theorem 4.15 holds for a 1-Gorenstein CM-finite algebra.

5. Almost split sequences

Our purpose in this section is to make a relationship between the almost split sequences in the subcategory $\vartheta^{-1}(\text{Gprj-}\Lambda)$ of mod-$\text{Gprj-}\Lambda$, and those in the subcategory of Gorenstein projective modules over $A_3(\Lambda)$, where $\Lambda$ is 1-Gorenstein CM-finite algebra. Note that by Proposition 3.4 one can see that $\vartheta^{-1}(\text{Gprj-}\Lambda)$ is closed under extensions and direct summands. In particular, when $\Lambda$ is self-injective of finite type, then this connection is more interesting, see Corollary 5.5.

From now on, we assume that $\Lambda$ is a 1-Gorenstein CM-finite algebra throughout of this section, unless stated otherwise. By our assumption each object in mod-$\text{Gprj-}\Lambda$ admits a minimal projective resolution. So we can fix a minimal projective resolution for each $F \in \text{mod-}(\text{Gprj-}\Lambda)$ as the following

$$0 \rightarrow (-, C_F) \xrightarrow{(-, f_F)} (-, B_F) \xrightarrow{(-, g_F)} (-, A_F) \xrightarrow{\varphi_F} F \rightarrow 0.$$  

Let us show by $X_F = (C_F \xrightarrow{f_F} B_F \xrightarrow{g_F} A_F)$ the representation induced by the minimal projective resolution of $F$ in the above.

**Lemma 5.1.** Let $F$ be in mod-$\text{Gprj-}\Lambda$. Then $F$ is an indecomposable object in mod-$\text{Gprj-}\Lambda$ if and only if the induced representation $X_F$ is indecomposable in $\text{rep}(A_3, J, \Lambda)$.

**Proof.** We use this fact, that is, a module over an Artin algebra $A$ is indecomposable if and only if $\text{End}_A(X)$ is a local algebra. We know that algebra $A$ is local if and only if the non-units in $A$ are nilpotent elements. Assume that $X_F$ is an indecomposable representation and let $\gamma : F \rightarrow F$ be a non-unit element in mod-$\text{Gprj-}\Lambda$. Consider the minimal projective resolution induced by the representation $X_F$, then $\gamma$ can be lifted as the following

$$0 \rightarrow (-, C_F) \xrightarrow{(-, f_F)} (-, B_F) \xrightarrow{(-, g_F)} (-, A_F) \xrightarrow{\varphi_F} F \rightarrow 0$$

$$0 \rightarrow (-, C_F) \xrightarrow{(-, f_F)} (-, B_F) \xrightarrow{(-, g_F)} (-, A_F) \xrightarrow{\gamma} F \rightarrow 0.$$  

So by the above commutative diagram and applying the Yoneda lemma we have the endomorphism $\sigma = (\sigma_1, \sigma_2, \sigma_3) : X_F \rightarrow X_F$, in $\text{rep}(A_3, J, \Lambda)$. The endomorphism $\sigma$ can not be an automorphism, since otherwise by using again the above diagram $\gamma$ would be an isomorphism, which is a contraction. As $X_F$ is indecomposable then $\sigma$ must be nilpotent. Now by pasting the above diagram several time as needed, we obtain a commutative diagram that implies $\gamma$ is nilpotent, and consequently $F$ is indecomposable. The converse implication can be proved similarly. So we are done. \hfill $\Box$
In the following construction we illustrate how one can compute the almost split exact sequences in the subcategory $\vartheta^{-1}(\text{Gprj}-\Lambda)$ of mod-(Gprj-Λ) by using computation of the almost split sequences in the subcategory $\mathcal{GP}(A_3, J, \Lambda)$ of rep(A_3, J, Λ).

**Construction 5.2.** Let $G$ be a non-projective indecomposable object in $\vartheta^{-1}(\text{Gprj}-\Lambda)$. Then by Lemma 5.1, $X_G$ is an indecomposable representation in $\mathcal{GP}(A_3, J, \Lambda)$ and not necessarily projective. Since $\Lambda$ is 1-Gorenstein then the subcategory $\mathcal{GP}(A_3, J, \Lambda)$ is functorially finite in rep(A_3, J, Λ). Hence there exists an almost split exact sequence

\[(\dagger) \quad 0 \to Z \xrightarrow{f} Y \xrightarrow{s} X_G \to 0\]

in $\mathcal{GP}(A_3, J, \Lambda)$. Indeed, by our notation $Z = \tau_{\text{Gprj}-A_3(\Lambda)}(X_G)$. Let us again emphasis here that we identify mod-$A_3(\Lambda)$ and rep(A_3, J, Λ). The short exact sequence $(\dagger)$ induces the following commutative diagram

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & \downarrow & \downarrow \\
Z_1 & Y_1 & C_G \\
\downarrow & \downarrow & \downarrow \\
Z_2 & Y_2 & B_G \\
\downarrow & \downarrow & \downarrow \\
Z_3 & Y_3 & A_G \\
\end{array}
\]

in mod-$\Lambda$. Note that the rows in the above diagram are split short exact sequences. To see this, it is enough to consider the maps $(0, 0, 1_{A_G}) : (0 \to 0 \to A_G) \to X_G$, $(0, 1_{B_G}, g_G) : (0 \to B_G \xrightarrow{1_{A_G}} B_G) \to X_G$ and $(1_{C_G}, f_G, 0) : (C_G \xrightarrow{1_{A_G}} C_G \to 0) \to X_G$ in $\mathcal{GP}(A_3, J, \Lambda)$. Since no of these maps are split epimorphisms then by the property of almost split sequences factor through $s$, and then we get the results. By applying the Yoneda lemma on the above commutative diagram and using the snake lemma we obtain the following diagram $(\dagger\dagger)$

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & \downarrow & \downarrow \\
(-, Z_1) & (-, Y_1) & (-, C_G) \\
\downarrow & \downarrow & \downarrow \\
(-, Z_2) & (-, Y_2) & (-, B_G) \\
\downarrow & \downarrow & \downarrow \\
(-, Z_3) & (-, Y_3) & (-, A_G) \\
\downarrow & \downarrow & \downarrow \\
F & H & G \\
\end{array}
\]
but this turn in mod-(Gprj-$\Lambda$). Here two cases might be happen: either the functor $F = 0$ or $F \neq 0$. But in the following explain the case $F = 0$ never happen. Otherwise, if $F = 0$ would happen, then $G \simeq H$ and $Z$ must be either $(G' \xrightarrow{\text{id}_{G'}} G' \to 0)$ or $(0 \to G' \xrightarrow{\text{id}_{G'}} G')$, for some Gorenstein projective indecomposable $G'$. By this, in fact, we have two projective resolutions of $G \simeq H$ in the diagram (††), one in the middle column and the other in the third column. Assume $Z = (0 \to G' \xrightarrow{\text{id}_{G'}} G')$. Since the third one is minimal then we can deduce the complex

$$
Y : \cdots 0 \to (-, Y_1) \to (-, Y_2) \to (-, Y_3) \to 0 \to \cdots
$$

is isomorphic to $G \oplus X \oplus Y$ (as objects in the category of complexes), where

$$
G : \cdots 0 \to (-, C_G) \to (-, B_G) \to (-, A_G) \to 0 \to \cdots
$$

$$
X : \cdots 0 \to 0 \to (-, G_1) \xrightarrow{-, \text{Id}_{G_1}} (-, G_1) \to 0 \to \cdots
$$

$$
Z : \cdots 0 \to (-, G_2) \xrightarrow{-, \text{Id}_{G_2}} (-, G_2) \to 0 \to 0 \to \cdots
$$

for some $G_1, G_2 \in \text{Gprj-$\Lambda$},$ and $A_G$ in $G$ and the rightmost $G_1$ in $X$ are at degree 0 and the leftmost $G_2$ in $Z$ is at degree $-2$.

Since $Z_1 = 0$, in this case, then $G_2$ must be zero, and consequently $Z = 0$. On the other hand, since the rows in (††) are split, for example by the third row we have $Y_3 \simeq G' \oplus A_G$, and also by the direct sum $Y_3 \simeq G_1 \oplus A_G$. Comparing these two decomposition of $Y_3$ together, we get $G_1 \simeq G'$. Now by applying the Yoneda lemma on $Y \simeq G \oplus X$, we can deduce $Y \simeq Z \oplus X_G$ in $\mathcal{GP}(A_3, \mathcal{J}, \Lambda)$. This implies the almost split sequence (†) is isomorphic to a short exact sequence in such form $0 \to Z \to Z \oplus X_G \to X_G \to 0$. But this means that (†) is split and so we have a contradiction. Note that here for getting the contradiction, we used this fact that split exact sequences are rigid, see Proposition 2.3 of [ARS]. The other case $Z = (G' \xrightarrow{\text{id}_{G'}} G' \to 0)$ similarly is not true to be hold. Therefore, $F$ is always non-zero. In addition, since $Z$ is indecomposable, one can easily see the first column in (††) is a minimal projective resolution of $F$. Indeed, if $0 \to (-, Z_1) \to (-, Z_2) \to (-, Z_3) \to F \to 0$ would not be a minimal projective resolution. Hence, as we did before, the complex $(\cdots \to 0 \to (-, Z_1) \to (-, Z_2) \to (-, Z_3) \to 0 \to \cdots)$, where $Z_1$ is located at degree $-2$, have a summand in the form

$$
\begin{array}{ccc}
\text{deg} - 2 & \text{deg} - 1 & \text{deg} 0 \\
\cdots & 0 & (-, G_1) \xrightarrow{-, \text{Id}_{G_1}} (-, G_1) & 0 & 0 & \cdots,
\end{array}
$$

or

$$
\begin{array}{ccc}
\text{deg} - 2 & \text{deg} - 1 & \text{deg} 0 \\
\cdots & 0 & (-, G_2) \xrightarrow{-, \text{Id}_{G_2}} (-, G_2) & 0 & 0 & \cdots.
\end{array}
$$

such that $G_1 \neq 0$ and $G_2 \neq 0$. Returning to $\mathcal{GP}(A_3, \mathcal{J}, \Lambda)$ by use of the Yoneda lemma, we observe that $Z$ is decomposable, that is a contradiction.

Summing up, We have associated the short exact sequence $0 \to F \xrightarrow{\alpha} H \xrightarrow{\beta} G \to 0$ via our construction to any non-projective indecomposable object $G$ in $\vartheta^{-1}(\text{Gprj-$\Lambda$})$. To state our result later, let us denote the associated short exact sequence by $\eta_G$.

By keeping all notations in the above construction we continue.

**Lemma 5.3.** Let $G$ be a non-projective indecomposable functor in $\vartheta^{-1}(\text{Gprj-$\Lambda$})$. Then the associated short exact sequence $\eta_G$ is not split.
**Proof.** Suppose to the contrary \( \eta_G \) is split. Then \( H \cong F \oplus G \). This implies that the minimal projective resolution \( H \) is a direct sum of
\[
0 \to (-, Z_1) \to (-, Z_2) \to (-, Z_3) \to F \to 0
\]
and
\[
0 \to (-, C_G) \to (-, B_G) \to (-, A_G) \to G \to 0
\]
which both of the above sequences act as minimal projective resolutions of \( F \) and \( G \), respectively. Hence the projective resolution
\[
0 \to (-, Y_1) \to (-, Y_2) \to (-, Y_3) \to H \to 0,
\]
in the middle column of diagram (††) of Construction 5.2, contains the direct sum of minimal projective resolutions of \( F \) and \( G \). By applying the Yoneda lemma and returning to \( G \mathcal{P}(A_3, J, \Lambda) \), we have the middle term \( Y \) in the short exact sequence (†) has \( Z \oplus X_G \) as a summand. Now by comparing the length of representations appearing in the short exact sequence (†), we can deduce that (†) can be written as \( 0 \to Z \to Z \oplus X_G \to X_G \to 0 \). But this means that (†) is split and so we have a contradiction. \( \square \)

**Proposition 5.4.** Let \( G \) be a non-projective indecomposable object in \( \vartheta^{-1}(Gprj-\Lambda) \). Then the associated short exact sequence \( \eta_G \) is an almost split sequence in \( \vartheta^{-1}(Gprj-\Lambda) \).

**Proof.** First Lemma 5.3 implies that \( \eta_F \) is not split. We claim that \( \beta \) is right almost split. Let \( \delta : L \to G \) be a morphism in \( \vartheta^{-1}(Gprj-\Lambda) \), which is not a split epimorphism. Without loss of generality, we can assume that \( L \) is indecomposable, so that \( \delta \) must be a non-isomorphism. The morphism \( \delta \) can be lifted to a chain map between the corresponding minimal projective resolutions of \( L \) and \( G \) in \( \text{mod-}Gprj-\Lambda \), then, by the Yoneda lemma, this gives us a morphism \( \delta' = (\delta_1, \delta_2, \delta_3) : X_L \to X_G \) in \( G \mathcal{P}(A_3, J, \Lambda) \). But \( \delta' \) is not split epimorphism, or equivalently non-isomorphism as by Lemma 5.1 we know that \( X_L \) is an indecomposable representation. In fact, if \( \delta' \) would be an isomorphism, then by use of the following commutative diagram in \( \text{mod-}Gprj-\Lambda \), obtained by applying the Yoneda functor on \( \delta' \),

\[
\begin{array}{cccccc}
0 & \rightarrow & (-, C_L) & \rightarrow & (-, B_L) & \rightarrow & (-, A_L) & \rightarrow & L & \rightarrow & 0 \\
& & \downarrow{(-, \delta_1)} & & \downarrow{(-, \delta_2)} & & \downarrow{(-, \delta_3)} & & \downarrow{\delta} & & \\
0 & \rightarrow & (, C_G) & \rightarrow & (-, B_G) & \rightarrow & (-, A_G) & \rightarrow & G & \rightarrow & 0.
\end{array}
\]

in conjunction with 5-lemma would follow that \( \delta \) must be an isomorphism, which is a contraction. Now since \( \eta \) is an almost split sequence in \( G \mathcal{P}(A_3, J, \Lambda) \), note that since \( L \in \vartheta^{-1}(Gprj-\Lambda) \) then \( X_L \) belongs to \( G \mathcal{P}(A_3, J, \Lambda) \), then \( \delta' \) factors through \( \varsigma \), say, via \( \gamma : X_L \rightarrow Y \). Then the induced morphism \( \overline{\gamma} : L \rightarrow H \) by \( \gamma \) factors \( \delta \) through \( \beta \). The claim is proved. Also, one can prove similarly that \( \alpha \) is left almost split. Since both \( G \) and \( F \) are indecomposable, by Lemma 5.1, then it implies \( \alpha \) and \( \beta \) both are also left and right minimal morphism, respectively. Hence we get our desired result. \( \square \)

**Corollary 5.5.** Let \( A(m, t) \) be the self-injective Nakayama algebra associated to \( m \geq 1, \ t \geq 2 \), see Example 3.11. Let \( \Gamma \) denote the Auslander algebra of \( \Lambda \), i.e. \( \Gamma := \text{End}(M) \), where \( M \) is a representation generator of \( \text{mod-}A(m, t) \). Then \( \tau_{\Gamma}^{\text{fin}}(N) \simeq N \) for each non-projective indecomposable \( N \) in \( \text{mod-}\Gamma \).
Proof. First, let us give some facts which will be useful to prove the statement. As before we used from [AHV2, Theorem 4.1.2], a derived equivalence between two algebras $A$ and $A'$ induces a triangle equivalence $\text{Gprj}
olimits A \simeq \text{Gprj}
olimits A'$. On the other hand, if $0 \to \tau_{G(A)}(G) \to G' \to G \to 0$ is an almost split in $\text{Gprj}
olimits A$, then it is straightforward by the definition to check that it induces the Auslander-Reiten triangle $\tau_{G(A)}(G) \to G' \to G \to \Omega(\tau_{G(A)}(G))$ in the triangulated category $\text{Gprj}
olimits A$. Suppose $\varrho : \text{Gprj}
olimits A \to \text{Gprj}
olimits A'$ is a triangle equivalence. By using the definition of Auslander-Reiten triangle one can see the image of an Auslander-Reiten triangle in $\text{Gprj}
olimits A$ under the equivalence $\varrho$ is again an Auslander-Reiten triangle in $\text{Gprj}
olimits A'$. Therefore because of the uniqueness of Auslander-Reiten triangles we get $\varrho(\tau_{G(A)}(G)) \simeq \tau_{G(A')}(\varrho(G))$. Now we use these facts to prove. From Lemma 3.8 and in view of the facts discussed in the above we have $\text{Gprj}
olimits A_3(A(m, t)) \simeq \text{Gprj}
olimits T_3(A(m, t))$. From [XZZ, Corollary 3.6], for a non-projective indecomposable $X$ in $S_3(A(m, t))$, $\tau_3^m(X) \simeq X$, where $\tau_S$ denotes the Auslander-Reiten translation in $S_3(A(m, t))$. But thanks to the local characterization of Gorenstein projective representations given in Theorem 2.4, two subcategories $S_3(A(m, t))$ and $\mathcal{GP}(A_3, A(m, t))$ coincide. Then Proposition 5.4 finishes the proof.

The proof of Corollary 5.5 suggests us that one way of getting information for the Auslander-Reiten translation in $\text{mod}
olimits-\text{Gprj}
olimits A$ is using the computation of the relative Auslander-Reiten translation in $\mathcal{GP}(A_3, J, \Lambda)$. Let us explain this idea more. For instance, from [XZZ, Theorem 3.5] for when $\Lambda$ is self-injective of finite type can be seen that

$$\tau_{\text{Gprj}
olimits T_3(A)}^s(X) \cong \text{Mimo} \tau_\Lambda^{4s} \Omega^{-2s}(X), \quad s \geq 1$$

for each indecomposable object $X$ in $\mathcal{GP}(A_3, \Lambda)$. We refer to [XZZ] for some possible unknown notations in the above equation. Let $\varrho : \text{Gprj}
olimits A_3(\Lambda) \to \text{Gprj}
olimits T_3(\Lambda)$ be the triangle equivalence, as previously discussed in the proof of the above corollary, and let $F$ be a non-projective indecomposable in $\text{mod}
olimits-\text{mod}
olimits \Lambda$. Then the $4s$-th power of the Auslander-Reiten translation of $F$ in $\text{mod}
olimits-\text{mod}
olimits \Lambda$ is nothing but $\Psi(\tau_\Lambda^{-4s} \Omega^{-2s}(\tau_\Lambda^{-1}(F)))$, see Construction 3.2 for the definition of $\Psi$.

To end this section, we would like to give a similar method as given in this section to compute the Auslander-Reiten translations over relative stable Auslander algebras, including $\text{mod}
olimits-\text{Gprj}
olimits A$. To state this method, we skip the proofs since in general the proofs are the same as the ones previously given.

Some facts from [H] are needed as follows.

Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$ such that contains all projective objects in $\mathcal{A}$, contravariantly finite and closed under kernels of epimorphisms. Let $\mathbf{H}(\mathcal{A})$ be the morphism category over $\mathcal{A}$. That is the same as the category $\text{rep}(\Lambda_2, \Lambda)$ of representations over the quiver $\Lambda_2 : v_1 \to v_2$ by $\Lambda$-modules and $\Lambda$-homomorphisms. The subcategory $S_\mathcal{X}(\mathcal{A})$ of $\mathbf{H}(\mathcal{A})$ is defined as follows: A morphism $A \xrightarrow{f} B$ is in $S_\mathcal{X}(\mathcal{A})$ if and only if

(i) $f$ is a monomorphism;

(ii) $A$, $B$ and $\text{Coker}(f)$ belong to $\mathcal{X}$.

For example, if set $\mathcal{X} = \text{Gprj}
olimits \Lambda$ and $\mathcal{A} = \text{mod}
olimits \Lambda$, then $S_{\text{mod}
olimits \Lambda}(\text{Gprj}
olimits \Lambda)$ coincides with $\mathcal{GP}(A_2, \Lambda)$, the subcategory of Gorenstein projective representations in $\text{rep}(\Lambda_2, \Lambda)$. In [H, Construction 3.2], a functor $S_\mathcal{X}(\mathcal{A})$ to $\text{mod}
olimits \mathcal{A}$ is given by sending monomorphism $A \xrightarrow{f} B$ to the
functor $F$ appearing in the following exact sequence

$$0 \to (-, A) \xrightarrow{(-, f)} (-, B) \to (-, \text{Coker}(f)) \to F \to 0$$

in $\text{mod-}\mathcal{X}$. In [H, Theorem 3.3] was shown this functor induces an equivalence of categories $S_X(\mathcal{A})/\mathcal{V} \simeq \text{mod-}\mathcal{X}$. From now on, assume that $\mathcal{A} = \text{mod-}\Lambda$, an arbitrary Artin algebra $\Lambda$, $\text{prj-}\Lambda \subseteq \mathcal{X} \subseteq \text{mod-}\Lambda$ is of finite type, closed under kernels of epimorphisms and in addition $S_X(\text{mod-}\Lambda)$ is functionally finite subcategory in $\text{H}(\text{mod-}\Lambda)$. Take a non-projective indecomposable functor $G$ in $\text{mod-}\mathcal{X}$ and consider whose minimal projective resolution

$$0 \to (-, A_G) \xrightarrow{(-, f_G)} (-, B_G) \to (-, \text{Coker}(f_G)) \to G \to 0$$

in $\text{mod-}\mathcal{X}$. Let $Y_G$ show the monomorphism $A_G \xrightarrow{f_G} B_G$ in $S_X(\text{mod-}\Lambda)$. Similar to Lemma 5.1 we obtain that $G$ is indecomposable in $\text{mod-}\mathcal{X}$ if and only if $Y_G$ so is in $S_X(\text{mod-}\Lambda)$. Since $S_X(\text{mod-}\Lambda)$ is functorially finite in $\text{H}(\text{mod-}\Lambda)$ then there exist the almost split sequence $0 \to V \to W \to Y_G \to 0$ in the subcategory of $S_X(\text{mod-}\Lambda)$. Note that $Y_G$ can not be projective in $S_X(\text{mod-}\Lambda)$ since $G \neq 0$. Similar to the Construction 5.2, we can make the following exact sequence

$$\epsilon_G : 0 \to M \to N \to G \to 0$$

in $\text{mod-}\mathcal{X}$ for the given non-projective indecomposable $G$. Now similar to Proposition 5.4 one can prove that $\epsilon_G$ is an almost split sequence in $\text{mod-}\mathcal{X}$. In this way, we make a relationship between the results of Auslander-Reiten translation of objects of two different categories. Recently, Claus Michael Ringel and Markus Schmidmeier in [RS] provided many results for the Auslander-Reiten translation of submodule categories $S_{\text{mod-}\Lambda}(\text{mod-}\Lambda)$, or $S(\Lambda)$ on their convention. As an application of our observation in conjunction with [RS, Corollary 6.5] we have:

**Corollary 5.6.** Let $\Lambda$ be a commutative Nakayama algebra. Let $\Gamma$ denote the stable Auslander algebra of $\Lambda$, i.e. $\Gamma := \text{End}(M)$, where $M$ is a representation generator of $\text{mod-}\Lambda$. Then $\tau^R_{\mathcal{X}}(N) \simeq N$ for each non-projective indecomposable $N$ in $\text{mod-}\Gamma$.

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