Parameterized Algorithms for Diverse Multistage Problems

Leon Kellerhals
Algorithmics and Computational Complexity, Faculty IV, Technische Universität Berlin, Germany

Malte Renken
Algorithmics and Computational Complexity, Faculty IV, Technische Universität Berlin, Germany

Philipp Zschoche
Algorithmics and Computational Complexity, Faculty IV, Technische Universität Berlin, Germany

Abstract

The world is rarely static — many problems need not only be solved once but repeatedly, under changing conditions. This setting is addressed by the “multistage” view on computational problems. We study the “diverse multistage” variant, where consecutive solutions of large variety are preferable to similar ones, e.g. for reasons of fairness or wear minimization. While some aspects of this model have been tackled before, we introduce a framework allowing us to prove that a number of diverse multistage problems are fixed-parameter tractable by diversity, namely Perfect Matching, s-t Path, Matroid Independent Set, and Plurality Voting. This is achieved by first solving special, colored variants of these problems, which might also be of independent interest.

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1 Introduction

In the multistage setting, given a sequence of instances of some problem, one asks whether there is a corresponding sequence of solutions such that consecutive solutions relate in some way to each other. Often the aim is to find consecutive solutions that are very similar [5, 6, 19–21, 26]. This is reasonable when changing between distinct solutions incurs some form of cost. In other settings, the opposite goal is more reasonable, that is, consecutive solutions should be very different. This is a natural goal when wear minimization, load distribution, or resilience against failures or attacks are of interest. This “diverse multistage” setting is what we want to focus on in this paper. Here, given a sequence of instances of some decision problem, the task is to find a sequence of solutions such that the diversity, i.e., the size of the symmetric difference of any two consecutive solutions is at least \( \ell \).

This problem has already received some attention in the literature: Fluschnik et al. [22] studied the problem of finding diverse s-t paths and Bredereck et al. [11] considered series of committee elections. In a similar setting, but aiming for large symmetric difference between every two (i.e., not just consecutive) solutions, Baste et al. [7] provide a framework for parameterization by treewidth, while Fomin et al. [24, 25] focus on the case that all problems are defined on the same graph and study matching, independent set, and matroids.
We briefly give a formal definition. Assume $\Pi$ to be some decision problem which asks whether the family of solutions $R(I) \subseteq 2^B(I)$ of an instance $I$ of $\Pi$ is non-empty, where $B(I)$ is some base set encompassing all possible solutions. For example, for an instance $I$ of Vertex Cover, the set $B(I)$ is the set of all vertices and $R(I)$ is the set of all vertex covers within the size bound. The problem Diverse Multistage $\Pi$ is now the following.

**Diverse Multistage $\Pi$**

**Input:** A sequence $\{I_i\}_{i=1}^r$ of instances of $\Pi$ and an integer $\ell \in \mathbb{N}_0$.

**Question:** Is there a sequence $\{S_i\}_{i=1}^r$ of solutions $S_i \in R(I_i)$ such that $|S_i \Delta S_{i+1}| \geq \ell$ for all $i \in [r - 1]$?

**Our contributions.** We present a general framework which allows us to prove fixed-parameter tractability of Diverse Multistage $\Pi$ parameterized by the diversity $\ell$ for several problems $\Pi$. This includes finding diverse matchings, but also diverse committees (answering an open question by Bredereck et al. [11]), diverse $s$-$t$ paths, and diverse independent sets in matroids such as spanning forests. Finally, we show that similar results cannot be expected for finding diverse vertex covers.

Generally, our framework can be applied to Diverse Multistage $\Pi$ whenever one can solve a 4-colored variant of $\Pi$ efficiently. Formally, this variant is defined as follows.

**4-Colored Exact $\Pi$**

**Input:** An instance $I$ of $\Pi$, a coloring $c: B(I) \rightarrow [4]$, and $n_i \in \mathbb{N}_0$, $i \in [4]$.

**Output:** A solution $S \in R(I)$ such that $|\{x \in S \mid c(x) = i\}| = n_i$ for all $i \in [4]$ or “no” if no solution exists.

Our main result reads as follows.

**Theorem 1.** If an instance $I$ of 4-Colored Exact $\Pi$ can be solved in $f(r) \cdot |J|^{O(1)}$ time, then an instance $J$ of Diverse Multistage $\Pi$ of size $n$ can be solved in $2^{|O(\ell)} \cdot f(r_{\text{max}}) \cdot |J|^{O(1)}$ time, where $r_{\text{max}}$ is the maximum of parameter $r$ over all instances of $\Pi$ in $J$.

We prove Theorem 1 in Section 3 in a more general form which also allows solving 4-Colored Exact $\Pi$ by a Monte Carlo algorithm. We then apply our framework to the following problems:

- **Committee Election (Section 4).** In Diverse Multistage Plurality Voting, we are given a set $A$ of agents, a set $C$ of candidates, and $\tau$ many voting profiles $u_i: A \rightarrow C$. The goal is to find a sequence $(C_i)_{i=1}^\tau$ of committees $C_i \subseteq C$ such that each committee $C_i$ is of size at most $k$ and gets at least $x$ votes in the voting profile $u_i$ (i.e., $|u_i^{-1}(C_i)| \geq x$), and $|C_i \Delta C_{i+1}| \geq \ell$ for all $i \in [\tau - 1]$. We show that there is a $2^{|O(\ell)} \cdot |J|^{O(1)}$-time algorithm to solve a Diverse Multistage Plurality Voting instance $J$. This answers an open question of Bredereck et al. [11]. Later, in Section 7 we generalize the algorithm used to solve 4-Colored Exact Plurality Voting to matroids.

- **Perfect Matching (Section 5).** In the multistage setting, Perfect Matching is among the problems most intensively studied [3, 4, 13, 26, 39]. Given a sequence of graphs $(G_i)_{i=1}^\tau$ and an integer $\ell$, Diverse Multistage Perfect Matching asks whether there is a sequence $(M_i)_{i=1}^\tau$ such that each $M_i$ is a perfect matching in $G_i$, and $|M_i \Delta M_{i+1}| \geq \ell$ for all $i \in [\tau - 1]$. We show that there is a randomized $2^{|O(\ell)} \cdot |J|^{O(1)}$-time algorithm to solve a Diverse Multistage Perfect Matching instance $J$ with constant error probability.

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1 For example, if the input is a sequence of graphs and $r$ is the treewidth, then $r_{\text{max}}$ is the maximum treewidth over all graphs in the input.
This stands in remarkable contrast to the \( \text{W}[1] \)-hardness of the (non-diverse) \textsc{Multistage Perfect Matching}, when parameterized by \( \ell + \tau \). To apply our framework, we establish an algebraic algorithm using the Pfaffian of a specific variant of the Tutte matrix to solve \( s \)-\textsc{Colored Exact Perfect Matching} on an \( n \)-vertex graph in \( n^{O(s)} \) time with low error probability.

\subsection*{3-t Path (Section 4)}

 Studying \( s \)-\textsc{t Path} in the multistage setting was already suggested in the seminal work of Gupta et al. [21]. In \textsc{Diverse Multistage s-t Path} one is given a sequence of graphs \( \{G_i\}_{i=1}^t \), two distinct vertices \( s \) and \( t \), and an integer \( \ell \), and asks whether there is a sequence \( \{P_i\}_{i=1}^t \) such that each \( P_i \) is an \( s \)-\textsc{t Path} in \( G_i \), and \( |V(P_i)\Delta V(P_{i+1})| \geq \ell \) for all \( i \in [\tau-1] \). Fluschnik et al. [22] provided a comprehensive study of finding \( s \)-\textsc{t Path}s of bounded length in the multistage setting from the viewpoint of parameterized complexity. Among other results, they showed that \textsc{Diverse Multistage s-t Path} is \textsc{NP}-hard but fixed-parameter tractable when parameterized by the maximum length of an \( s \)-\textsc{t Path} in the solution. We show that \textsc{Diverse Multistage s-t Path} parameterized by \( \ell \) is fixed-parameter tractable. At first glance, using our framework seems unpromising since \textsc{4-Colored Exact s-t Path} can presumably not be solved in polynomial time (it is \textsc{NP}-hard by a straight-forward reduction from \textsc{Hamiltonian Path}). However, we develop a win/win strategy around a generalization of the Erdős-Pósa theorem for long cycles due to Mousset et al. [32] so that we have to solve \textsc{4-Colored Exact s-t Path} only on graphs on which the treewidth is upper-bounded in the parameter \( \ell \).

In Section 5 we complement our fixed-parameter tractability results with a \( \text{W}[1] \)-hardness for \textsc{Diverse Multistage Vertex Cover} when parameterized by \( \ell \).

\section{Preliminaries}

We denote by \( \mathbb{N} \) and \( \mathbb{N}_0 \) the natural numbers excluding and including zero, respectively. For \( n \in \mathbb{N} \), let \( [n] := \{1, 2, \ldots, n\} \). For two sets \( A \) and \( B \), we denote by \( A \Delta B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B) \) the symmetric difference of \( A \) and \( B \), and by \( A \uplus B \) the disjoint union of \( A \) and \( B \). For a function \( c : A \to B \), let \( c(A') := \bigcup_{a \in A'} c(a) \) and \( c^{-1}(b) := \{ a \in A | c(a) = b \} \), where \( A' \subseteq A \). We also use the notations \( b \) and \( b' \) as shorthands for \( c^{-1}(b) \) and \( c^{-1}(b) \cup c^{-1}(b') \), respectively.

A Monte Carlo algorithm, or an algorithm with error probability \( p \), is a randomized algorithm that returns a correct answer with probability \( 1 - p \).

Let \( \Sigma \) be a finite alphabet. A parameterized problem \( L \) is a subset \( L \subseteq \{(x, k) \in \Sigma^* \times \mathbb{N}_0\} \). An instance \( (x, k) \in \Sigma^* \times \mathbb{N}_0 \) is a \textit{yes-instance} of \( L \) if and only if \( (x, k) \in L \) (otherwise, it is a \textit{no-instance}). A parameterized problem \( L \) is fixed-parameter tractable (in \textsc{FPT}) if for every input \( (x, k) \) one can decide in \( f(k) \cdot |x|^{O(1)} \) time whether \( (x, k) \in L \), where \( f \) is some computable function only depending on \( k \). A \textsc{W}[1]-hard parameterized problem is not fixed-parameter tractable unless \textsc{FPT} = \textsc{W}[1]. We refer to Downey and Fellows [18] and Cygan et al. [15] for more material on parameterized complexity. We use standard notation from graph theory [17]. Throughout this paper, we assume graphs to be simple and undirected.

\section{The General Framework}

In this section, we introduce a general framework to show (for some decision problem II) fixed-parameter tractability of \textsc{Diverse Multistage II} parameterized by \( \ell \). Recall that, for every instance \( I \) of decision problem II, we denote the family of solutions by \( \mathcal{R}(I) \subseteq 2^{B(I)} \) and the input size \( |I| \) of \( I \) is at least \( |B(I)| \). For the reminder of this section we assume
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that \(|B(I)| \geq 2\) for all instances \(I\) of \(\Pi\). The framework is applicable to Diverse Multistage \(\Pi\) if there is an efficient algorithm for 4-Colored Exact \(\Pi\). Formally, we use the following prerequisite, which is slightly more general than in Theorem 1.

**Assumption 2.** There are computable functions \(f, g\) such that for every \(0 \leq p \leq 1\) for which \(g(\tau)\) is defined, there is a Monte-Carlo algorithm \(A\) with error probability \(p\) and running time \(f(\tau) \cdot |I|^{O(1)} \cdot g(\tau)\), that solves an instance \(I\) of 4-Colored Exact \(\Pi\), where \(\tau \in \mathbb{N}_0\) is some parameter of \(I\) and \(g\) is monotone non-increasing.

We allow an error probability in Assumption 2 because for one of our applications (in Section 5), no other polynomial-time algorithm is known. The goal is to prove the following.

**Theorem 3.** Let Assumption 2 be true. Then any size-\(n\) instance \(I\) of Diverse Multistage \(\Pi\) can be solved in \(2^{O(n)} \cdot f(\tau_{\text{max}}) \cdot n^{O(1)} \cdot g(\tau_{\text{max}})\) time by a Monte-Carlo algorithm with error probability \(\tau\), where \(\tau_{\text{max}}\) is the maximum of parameter \(\tau\) over all instances of \(\Pi\) in \(I\), and \(0 \leq \tau \leq 1\) is an arbitrary probability for which the above expression is defined.

The proof of Theorem 3 is deferred to the end of this section. Note that, if we have a non-randomized algorithm in Assumption 2 (that is, \(g(0)\) is defined and \(g\) maps always to one), then Theorem 1 follows directly from Theorem 3.

The underlying strategy of the algorithm for a Diverse Multistage \(\Pi\)-instance \(J\) behind Theorem 3 is to compute for each instance \(I\) of \(\Pi\) in \(J\) a solution family such that the Cartesian product of these families contains a solution for \(J\) if and only if \(J\) is a yes-instance. Once these families are obtained, we can check whether \(J\) is a yes-instance by dynamic programming. To this end, we compute a small subset of \(R(I)\) satisfying the following definition.

**Definition 4.** Let \(F\) be a set family. A subfamily of \(\hat{F} \subseteq F\) is called an \(\ell\)-diverse representative of \(F\) if, for any \(S \in \hat{F}\) and sets \(A, B\) with \(\min\{|AΔS|, |BΔS|\} \geq \ell\), there is an \(\hat{S} \in \hat{F}\) such that \(\min\{|AΔ\hat{S}|, |BΔ\hat{S}|\} \geq \ell\).

First of all, we note that \(\ell\)-diverse representatives can be rather small.

**Lemma 5.** Let \(F\) be a set family and \(S_1, S_2, S_3 \in F\). If \(|S_iΔS_j| \geq 2\ell\) for all distinct \(i, j \in [3]\), then \(\{S_1, S_2, S_3\}\) is an \(\ell\)-diverse representative of \(F\).

**Proof.** Assume for contradiction that there exist sets \(A\) and \(B\) with \(\min\{|AΔS_i|, |BΔS_i|\} < \ell\) for all \(i\). Without loss of generality, assume that \(|AΔS_1| < \ell\). Then for \(j \in \{2, 3\}\) we have \(|AΔS_j| \geq |S_1ΔS_j| - |S_1ΔA| > 2\ell - \ell = \ell\) by the triangle inequality. Therefore, \(|BΔS_2| < \ell\). Again, by the triangle inequality \(|BΔS_3| \geq |S_2ΔS_3| - |S_2ΔB| > 2\ell - \ell = \ell\), i.e., \(\min\{|AΔS_3|, |BΔS_3|\} \geq \ell\) — a contradiction.

In the following, we measure the distance of two solutions by the size of the symmetric difference. In a nutshell, we compute an \(\ell\)-diverse representative of the family of solutions by first trying to compute three solutions which are far apart from each other (that is, size of symmetric difference at least \(2\ell\)). If this succeeds, then by Lemma 5 we are done. Otherwise, we distinguish between three cases.

**No solution.** If there is no solution at all, then trivially \(\emptyset\) is an \(\ell\)-diverse representative of the family of solutions.

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2 For example, if we only have an algorithm with non-zero error probability, then \(p = 0\) is excluded.
One solution. If we only find one solution $S_1$ to the instance of $\Pi$, then each other solution is close to $S_1$. Hence, for any two sets $A, B$, if one of them is far away from $S_1$, then by the triangle inequality it is also far away from every other solution and can be safely ignored. For those sets which are close to $S_1$, we can exploit the upper bound on the symmetric difference by using color-coding \[2\] and then applying Assumption \[2\] to compute an $\ell$-diverse representative of the family of solutions. This case is handled in Lemma \[6\].

Two solutions. If we find two diverse solutions $S_1$ and $S_2$ such that no other solution is far away from both, then $S_1$ and $S_2$ partition the solution space into two parts: the solutions close to $S_1$ and those close to $S_2$. Again, given two sets $A, B$, if either of them is far away from $S_1$ and $S_2$, then we may ignore it. By including $S_1$ and $S_2$ in our family, we may further assume that $A$ is similar to $S_1$ and $B$ is similar to $S_2$. We distinguish two subcases. If the distance between $S_1$ and $S_2$ is very large, then $A$ is far away from all solutions in the second part and $B$ is far away from all solutions in the first part. We can thus ignore one of them (say $B$) and exploit the fact that $A, S_1$, and all solutions of interest are close to each other to use color-coding and then apply Assumption \[2\]. In the other subcase where the distance between $S_1$ and $S_2$ is bounded, we can utilize that fact similarly. This case is handled in Lemma \[10\].

Hereafter, the details. Before we dive into the case distinction outlined above, we need to prove two technical lemmata, telling us how to build a diverse representative set that works for all sets obeying some given coloring of the elements of $B(I)$. These will later work as building blocks in the construction of proper diverse representatives. In the first lemma, only two colors are used, and we are only concerned with one arbitrary set $A$ instead of two.

\[ \text{Lemma 6.} \] Let Assumption \[2\] be true. Given an instance $I$ of $\Pi$ of size $n$, a coloring $c: B(I) \to [2]$, and a solution $M \in R(I)$, one can compute in $O(n^{O(1)}g(pn^{-4})$ time and with error probability at most $p$ a family $F \subseteq R(I)$ of size at most $n^4$ such that for any $S \in R(I)$ and any $A \subseteq B(I)$ with $S \setminus A \subseteq c_1$ and $A \setminus S \subseteq c_2$, there is $\hat{S} \in F$ with $|A \Delta \hat{S}| \geq |A \Delta S|$ and $|M \Delta \hat{S}| = |M \Delta S|$.

Proof. Let $F'_1 := c_1 \cap M$, $F'_2 := c_2 \cap M$, $F'_3 := c_1 \setminus M$, and $F'_4 := c_2 \setminus M$.

Start with $F = \emptyset$. Then, for each $m \leq n$ and each partition $\sum_{i=1}^4 m_i = m$, use algorithm $A$ to search in $O(n^{O(1)}g(pn^{-4})$ time and with error probability at most $pn^{-4}$ for a set $N \in R(I)$ such that $|N \cap F'_i| = m_i$ for all $i \in [4]$. If this succeeds, then we add $N$ to $F$.

Since there are $\binom{n^4}{\frac{1}{4}} \leq n^4$ possibilities for $m_1, \ldots, m_4$, the probability of an error occurring is upper-bounded by $p$. Moreover, the size of $F$ is upper-bounded by $n^4$ and hence the time required is bounded by $O(n^{O(1)}g(pn^{-4})$.

It remains to be proven that $F$ has the desired properties. Let $S \in R(I)$ be arbitrary and set $m_i := |S \cap F'_i|$ for all $i \in [4]$. By construction, $F$ contains a set $\hat{S} \in R(I)$ such that $|\hat{S} \cap F'_i| = m_i$. We then have $|\hat{S} \Delta M| = m_3 + m_4 + |M| - m_1 - m_2 = |S \Delta M|$.

Let $A \subseteq B(I)$ be a set with $(S \setminus A) \subseteq c_1$ and $(A \setminus S) \subseteq c_2$. Since $A \setminus S \subseteq c_2$ we have

\[ |A \cap S \cap c_1| = |A \cap c_1| \geq |A \cap \hat{S} \cap c_1| \tag{1} \]

and since $S \setminus A \subseteq c_1$, we have that

\[ |A \cap S \cap c_1| = |S \cap c_1| = m_2 + m_4 = |\hat{S} \cap c_1| \geq |A \cap \hat{S} \cap c_1|. \tag{2} \]

By adding (1) and (2) we obtain $|A \cap S| \geq |A \cap \hat{S}|$ which in turn implies $|A \Delta S| \leq |A \Delta \hat{S}|$ since $|S| = |\hat{S}|$. \[ \square \]
The next lemma extends the approach of Lemma 6 to the case where we have four colors and two arbitrary sets \(A, B\).

**Lemma 7.** Let Assumption 2 be true. Given an instance \(I\) of \(\Pi\) of size \(n\), a coloring \(c: B(I) \to \{1, 2, 3, 4\}\), one can compute in \(f(r)n^{O(1)}g(pm^{-4})\) time and with error probability at most \(p\) a family \(\mathcal{F} \subseteq \mathcal{R}(I)\) of size at most \(n^4\) such that for any \(S \in \mathcal{R}(I)\) and all sets \(A, B \subseteq B(I)\) with \(A \setminus (B \cup S) \subseteq c_i\) for all \(i \in [4]\), \(S \subseteq (A \cap B) \subseteq c_i\), and \(S \setminus (A \cap B) \subseteq c_i\), there is \(\tilde{S} \in \mathcal{F}\) with \(|C \Delta \tilde{S}| \geq |C \Delta S|\) for all \(C \in \{A, B\}\).

**Proof.** Begin with \(\mathcal{F} = \emptyset\). Then, for each \(m \leq n\) and each partition \(\sum_{i=1}^4 m_i = m\), use algorithm \(\mathcal{A}\) to search in \(f(r)n^{O(1)}g(pm^{-4})\) time and with error probability at most \(p\) for an \(M \in \mathcal{R}(I)\) such that \(|M \cap c_i| = m_i\) for all \(i \in [4]\). If this succeeds, then add \(M\) to \(\mathcal{F}\). Since there are \(\binom{n+4}{4} \leq n^4\) possibilities for \(m_1, \ldots, m_4\), the probability of an error occurring is upper-bounded by \(p\). Moreover, the size of \(\mathcal{F}\) is at most \(n^4\) and thus the overall running time is \(f(r)n^{O(1)}g(pm^{-4})\).

Now let \(S \in \mathcal{R}(I)\) be arbitrary. Set \(m_i := |S \cap c_i|\) for all \(i \in [4]\). By construction there is \(\tilde{S} \in \mathcal{F}\) such that \(|\tilde{S} \cap c_i| = m_i\) for all \(i\). It remains to be proven that \(\tilde{S}\) has the desired properties. To this end, let \(A, B \subseteq B(I)\) be two sets as stated in the lemma. By symmetry, it suffices to show that \(|A \Delta \tilde{S}| \geq |A \Delta S|\).

Since \(S \setminus A \subseteq c_3\) we have

\[
|S \cap A \cap c_3| = |S \cap c_3| = m_1 + m_3 = |\tilde{S} \cap c_3| \geq |\tilde{S} \cap A \cap c_3|
\]

and since \(S \setminus A \subseteq c_3\), we have

\[
|S \cap A \cap c_3| = |A \cap c_3| \geq |\tilde{S} \cap A \cap c_3|.
\]

By adding \(3\) and \(4\), we obtain \(|S \cap A| \geq |\tilde{S} \cap A|\) and thus \(|S \Delta A| \leq |\tilde{S} \Delta A|\) since \(|S| = |\tilde{S}|\).

We now describe how we generate the colorings required for using Lemmata 6 and 7. Color-coding \([2]\) is well-established in the toolbox of parameterized algorithms. While color-coding was initially described as a randomized technique, we use universal sets \(\{35\}\) to derandomize this technique as shown in the next lemma. Interestingly, without this derandomization the error probability of the color-coding step would later propagate through the dynamic program and consequently also depend on the number of instances of \(\Pi\) in the input instance of Diverse Multistage \(\Pi\). The derandomization works as follows.

**Lemma 8.** For any set \(A\) of size \(n\) and any \(b \leq n\) one can compute in \(2^{b + o(b)} \cdot \log n \cdot n\) time a family of functions \(c_j: A \to [4] | j \in [2^{b + o(b)} \cdot \log n]\) such that for any \(\bigcup_{i=1}^4 B_i \subseteq A\) with \(\|\bigcup_{i=1}^4 B_i\| \leq b\) there is a \(j\) such that \(c_j(B_i) = \{i\}\), for all \(i \in [4]\).

**Proof.** Let \(A \equiv \{a_1, \ldots, a_n\}\). By a result of Naor et al. \((35)\), one can compute in \(2^{b\cdot \log(b)} \cdot \log n \cdot n \leq 2^{b + o(b)} \cdot \log n \cdot n\) time a so-called \((2n, 2b)\)-universal set which is a family \(\mathcal{U} \subseteq \binom{[2n]}{2b}\) such that for every \(B^* \subseteq A\) with \(|B^*| = 2b\) the family \(\{B^* \cap U | U \in \mathcal{U}\}\) contains all \(2^{2b}\) subsets of \(B^*\). Let \(\mathcal{U} \equiv \{U_i\}_{i=1}^{2^{b + o(b)} \cdot \log n}\). We then define \(c_j, j \in [2^{b + o(b)} \cdot \log n]\), by

\[
c_j(a_i) := 
\begin{cases} 
1, & \text{if } i, i + n \in U_j, \\
2, & \text{if } i \in U_j \text{ and } i + n \not\in U_j, \\
3, & \text{if } i \not\in U_j \text{ and } i + n \in U_j, \text{ and} \\
4, & \text{if } i, i + n \not\in U_j.
\end{cases}
\]
Now let $B_1 \cup B_2 \cup B_3 \cup B_4 \subseteq A$ be an arbitrary 4-partition of a subset of $A$ of size at most $b$. Consider $B' := \{i \mid a_i \in \bigcup_{j=1}^{4} B_j\}$. We assume that $B'$ is of size $b$, otherwise we add arbitrary elements from $[2n]$. Since $B'' := \{i \mid a_i \in B_1\} \cup \{i \mid a_i \in B_2\} \cup \{i + n \mid a_i \in B_3\} \subseteq B'$ there is an $U_j \in \mathcal{U}$ such that $B' \cap U_j = B''$. Hence, $c_j(B_i) = \{i\}$, for all $i \in [4]$. ▶

We now show how to generate an $\ell$-diverse representative of the family of solutions if there is one solution $M^*$ from which no other solution differs by more than $2\ell$.

**Lemma 9.** Let Assumption 3 be true. Given an instance $I$ of $\Pi$ of size $n$, and a solution $M^* \in \mathcal{R}(I)$ such that each $M \in \mathcal{R}(I)$ satisfies $|M \Delta M^*| \leq 2\ell$, one can compute in $2^{16\ell+o(\ell)} \log n \cdot f(r) \cdot n^{O(1)} \cdot g(n/2^{16\ell+o(\ell)} \log n \cdot n^3)$ time and with error probability $p$ an $\ell$-diverse representative of $\mathcal{R}(I)$ of size at most $2^{16\ell+o(\ell)} \log n \cdot n^3$.

**Proof.** For simplicity, let $J := [2^{10\ell+o(\ell)} \log n]$. Apply Lemma 8 with $b = 8\ell$ to compute in $2^{16\ell+o(\ell)} \log n \cdot n^3$ time a family of colorings $\{c_j : B(I) \rightarrow [4] \mid j \in J\}$. By Lemma 8 this family has size $|J|$. For each $j \in J$, apply Lemma 7 to $I$ and $c_j$ to compute a family $\mathcal{F}_j \subseteq \mathcal{R}(I)$ with error probability $p \cdot |J|^{-1}$. Observe that the probability of an error occurring at any of the $|J|$ steps is bounded by $p$. Choose $F := \{M^*\} \cup \bigcup_{j \in J} \mathcal{F}_j$. According to Lemma 7, the size of $F$ is upper-bounded by $|J| \cdot n^3$ and the time required is bounded by $|J| \cdot f(r) \cdot n^{O(1)} \cdot g(\log n \cdot n^3)$.

We now show that $F$ is an $\ell$-diverse representative of $\mathcal{R}(I)$. To this end, let $S \in \mathcal{R}(I)$ and let $A,B$ be two arbitrary sets such that $|A \Delta S| \geq \ell$ and $|B \Delta S| \geq \ell$. Since $M^* \in F$, we may assume by symmetry that, say, $|M^* \Delta A| < \ell$, otherwise we are done. Note that $|M^* \Delta S| \leq 2\ell$ and that $|A \Delta S| \leq |A \Delta M^*| + |M^* \Delta S| < 3\ell$. We say that some coloring $c$ is good for $A,B,S$ if the conditions of Lemma 7 are satisfied, i.e. if

$A \setminus (B \cup S) \subseteq c^1, \ B \setminus (A \cup S) \subseteq c^2, \ (A \cap B) \setminus S \subseteq c^3, \ and \ S \setminus (A \cap B) \subseteq c^4.$

We distinguish between two cases.

**Case 1:** $|M^* \Delta B| < 3\ell$. Then $|B \Delta S| \leq |B \Delta M^*| + |M^* \Delta S| \leq 5\ell$. According to Lemma 8 there is an $i \in J$ such that coloring $c_i$ is good for $A,B,S$, since $|B \Delta S| + |A \Delta S| < 8\ell$. By Lemma 7 and the construction of $\mathcal{F}_i$, there is an $\tilde{S} \in \mathcal{F}_i \subseteq \mathcal{F}$ such that $|\tilde{S} \Delta A| \geq |S \Delta A| \geq \ell$ and $|\tilde{S} \Delta B| \geq |S \Delta B| \geq \ell$.

**Case 2:** $|M^* \Delta B| \geq 3\ell$. Set $B' := A$. According to Lemma 8 there is an $i \in J$ such that coloring $c_i$ is good for $A,B',S$, since $|B' \Delta S| + |A \Delta S| < 6\ell$. Thus, by Lemma 7 and the construction of $\mathcal{F}_i$ there is an $\tilde{S} \in \mathcal{F}_i \subseteq \mathcal{F}$ such that $|\tilde{S} \Delta A| \geq |S \Delta A| \geq \ell$. Finally, we observe that $|\tilde{S} \Delta B| \geq |M^* \Delta B| - |M^* \Delta \tilde{S}| \geq 2\ell - 2\ell \geq \ell$ by the triangle inequality.

This completes the proof. ▶

Next, we show how to generate an $\ell$-diverse representative of the family of solutions if there are two solutions such that no other solution differs from both by more than $2\ell$.

**Lemma 10.** Let Assumption 3 be true. Let $I$ be an $\Pi$-instance of size $n$, and $M_1, M_2 \in \mathcal{R}(I)$ such that $|M_1 \Delta M_2| \geq 2\ell$ and each $M \in \mathcal{R}(I)$ has min$\{|M \Delta M_1|, |M \Delta M_2| \leq 2\ell$. Then one can compute, in $2^{20\ell+o(\ell)} \log n \cdot f(r) n^{O(1)} \cdot g(n/2^{20\ell+o(\ell)} \log n)$ time and with error probability $p$, an $\ell$-diverse representative of $\mathcal{R}(I)$ of size $2^{20\ell+o(\ell)} \log n \cdot n^4$.

**Proof.** For simplicity, let $J := [2^{20\ell+o(\ell)} \log n]$. Apply Lemma 8 with $b = 10\ell$ to compute in $2^{20\ell+o(\ell)} \log n \cdot n^3$ time a family of colorings $\{c_j : B(I) \rightarrow [4] \mid j \in J\}$. By Lemma 8 this family has size $|J|$. For each $j \in J$, apply Lemma 7 to $I$ and $c_j$ to compute a family $\mathcal{F}_j \subseteq \mathcal{R}(I)$ of size at most $n^4$ with error probability $p/3 \cdot n^{-4}|J|^{-1}$. Observe that the probability of an error
occurring at any of the $n^4|J|$ steps is upper-bounded by $p/3$ and the computation of all $F_j$ takes $|J| |f(r)n^{O(1)}g(\eta/n^4, |I|)|$ time.

Next, define another family of colorings $\{c_j': B(I) \rightarrow [2] \mid j \in J\}$ by setting $c_j'(x) := \lfloor c_j(x)/2 \rfloor$. Then, for each $j \in J$, apply Lemma 6 to $I$, $c_j'$ and $M_1$ to compute a family $F_j' \subseteq R(I)$, with the same error probability and time bound as before. Repeat with $M_2$ instead of $M_1$ to obtain $F_j''$.

Set $F := \{M_1, M_2\} \cup \bigcup_{j \in J} (F_j \cup F_j' \cup F_j'')$. Then $F$ has size at most $3|J||n^4 + 2 \leq 2^{20|\ell + o(|\ell)|} \log n \cdot n^4$. Computing $F$ takes $2^{20|\ell + o(|\ell)|} \log n \cdot f(r)n^{O(1)}g(\eta/n^4, |I|)$ time. The probability of an error occurring at any step while computing $F$ is upper-bounded by $p$.

We now show that $F$ is an $\ell$-diverse representative of $R(I)$. To this end, let $S \in R(I)$ and $A, B$ be two arbitrary sets such that $|A\Delta S| \geq \ell$ and $|B\Delta S| \geq \ell$. We may assume for each $i \in [2]$ that $|M_i\Delta A| < \ell$ or $|M_i\Delta B| < \ell$, otherwise we are done. By symmetry, we may assume $|M_1\Delta A| < \ell$. Then $|M_2\Delta A| \geq |M_2\Delta M_1| - |M_1\Delta A| \geq \ell$ by the triangle inequality and thus we must have $|M_2\Delta B| < \ell$. By assumption, $\min(|S\Delta M_1|, |S\Delta M_2|) \leq 2\ell$, so let without loss of generality $|S\Delta M_1| \leq 2\ell$. Note that $|A\Delta S| \leq |A\Delta M_1| + |M_1\Delta S| < 3\ell$. We distinguish the following two cases.

**Case 1:** $|M_2\Delta M_2| \leq 4\ell$. Then, $|B\Delta S| \leq |B\Delta M_2| + |M_2\Delta M_1| + |M_1\Delta S| < \ell$. We say that some coloring $c$ is good for $A, B, S$ if the conditions of Lemma 7 are satisfied, i.e. if $A \setminus (B \cup S) \subseteq c_1$, $B \setminus (A \cup S) \subseteq c_2$, $(A \cap B) \setminus S \subseteq c_3$, and $S \setminus (A \cap B) \subseteq c_4$. According to Lemma 8 there is an $i \in J$ such that coloring $c_i$ is good for $A, B, S$, since $|B\Delta S| + |A\Delta S| \leq 10\ell$. By Lemma 9, there is $\tilde{S} \in F_i \subseteq F$ such that $|\tilde{S}\Delta A| \geq |S\Delta A| \geq \ell$ and $|\tilde{S}\Delta B| \geq |S\Delta B| \geq \ell$.

**Case 2:** $|M_2\Delta M_2| > 4\ell$. Since $|S\Delta A| \leq 3\ell \leq 10\ell$, there is $j \in J$ such that $S \setminus A \subseteq c_j^1$ and $A \setminus S \subseteq c_j^2$. By Lemma 6 there is $\hat{S} \in F_j$ such that $|\hat{S}\Delta M_1| = |S\Delta M_1| \leq 2\ell$ and $|\hat{S}\Delta A| \geq |S\Delta A| \geq \ell$. Finally, observe that by the triangle inequality $|\hat{S}\Delta B| \geq |M_1\Delta M_2| - |M_1\Delta \hat{S}| - |B\Delta M_2| > \ell$.

This completes the proof.

With Lemmata 5, 6, and 9 at hand we can formalize the case distinction outlined in the beginning of the section. This gives us a way to efficiently compute an $\ell$-diverse representative in general.

**Lemma 11.** Let Assumption 3 be true. Let $I$ be an instance of $\Pi$ of size $n$. One can compute an $\ell$-diverse representative of $R(I)$ of size $2^{20|\ell + o(|\ell)|} \log n \cdot n^4$ in $2^{20|\ell + o(|\ell)|} \log n \cdot f(r)n^{O(1)}g(\eta/n^4, 2^{20|\ell + o(|\ell)|} \log n)$ time with error probability at most $p$.

**Proof.** Our procedure to compute an $\ell$-diverse representative of $R(I)$ works in four steps.

**Step 1.** We use $A$ with a monochromatic coloring and error probability $p/4n$ to search for some $M_1 \in R(I)$ in $f(r)n^{O(1)}g(\eta/n)$ by guessing the size of $|M_1| \leq n$. Observe that the probability of an error occurring in any of the searches is upper-bounded by $p/4$ if we do not succeed, then output the empty set and we are done. Otherwise, we proceed with the next step.

**Step 2.** For each pair $m_1, m_2$ with $m_1 + m_2 \leq n$ and $m_2 + |M_1| - m_1 > 2\ell$, try to compute $M_2 \in R(I)$ with $|M_2 \cap M_1| = m_1$ and $|M_2 \cap (B(I) \setminus M_1)| = m_2$ in $f(r)n^{O(1)}g(\eta/n^4)$ time and with error probability $p/4n^2$ using $A$ with a 2-coloring where elements in $M_1$ are assigned one color and elements in $B(I) \setminus M_1$ are assigned the second color. If no such $M_2$ is found for any pair $m_1, m_2$, then for every $M \in R(I)$ the symmetric difference $|M \Delta M_1| \leq 2\ell$. In that case we may apply Lemma 9 with error probability $p/2$ and are...
Step 3. We have $M_1, M_2 \in \mathcal{R}(I)$ with $|M_1 \Delta M_2| \geq 2\ell$. Define the coloring $c: \mathcal{B}(I) \rightarrow [4]$ by

$$c(v) := \begin{cases} 
  i & \text{if } v \in M_1 \setminus M_2 \text{ for } \{i, j\} = \{1, 2\}, \\
  3 & \text{if } v \in M_1 \cap M_2, \text{ and} \\
  4 & \text{otherwise.}
\end{cases}$$

For all $m'_1, m'_2, m'_3, m'_4$ with $m'_1 + m'_2 + m'_3 + m'_4 \leq n$ and $m'_1 + m'_2 + |M_1| - m'_3 - m'_4 \geq 2\ell$ and $m'_1 + m'_2 - m'_3 \geq 2\ell$, search for a solution $M_3 \in \mathcal{R}(I)$ with $|M_3 \cap c^i| = m'_3$, for all $i \in [4]$, using $A$ with $c$ and error probability $p/n^4$. For all these combined, we thus have error probability $p/4$ and need $f(r)n^{O(1)}g(p/n^4)$ time. If no such $M_3$ is found for any choice of $m'_1, m'_2, m'_3, m'_4$, then any $M \in \mathcal{R}(I)$ must have $\min \{|M \Delta M_1|, |M \Delta M_2|\} < 2\ell$. In that case we may apply Lemma 10 with error probability $p/4$ and are done. Observe that the probability of an error occurring at any step until here is upper-bounded by $p$ and the overall running time is $2^{20\ell+o(\ell)}\log n \cdot f(r)n^{O(1)}g(p/n^4)2^{20\ell+o(\ell)}\log n)$. In case that we found such an $M_3$, we proceed with the next step.

Step 4. We have $M_1, M_2, M_3 \in \mathcal{R}(I)$ such that $|M_i \Delta M_j| \geq 2\ell$ for all distinct $i, j \in [3]$. Hence, by Lemma 5 we can output $\{M_1, M_2, M_3\}$. This completes the proof.

Finally, Lemma 11 allows us to formulate a dynamic program for Diverse Multistage II and prove Theorem 3.

Proof of Theorem 3. Let $J := ((I_i)_{i=1}^\ell, \ell)$ be an instance of Diverse Multistage II, where $n := \max_{i \in [\tau]} |I_i|$. For each $i \in [\tau]$ we apply Lemma 11 to obtain an $\ell$-diverse representative $\mathcal{F}_I$ of $\mathcal{R}(I_i)$ that has size at most $2^{20\ell+o(\ell)}\log n \cdot n^4$ in $2^{20\ell+o(\ell)}\log n \cdot f(r)n^{O(1)}g(p/n^4)2^{20\ell+o(\ell)}\log n)$ time with error probability $p/\tau$. Observe that the probability of an error occurring at any step is upper-bounded by $p$. Now we use the following dynamic program to check whether $J$ is a yes-instance.

$$\forall i \in \{2, 3, \ldots, \tau\}, S \in \mathcal{F}_I: D[i, S] := \begin{cases} 
  \top & \text{if } \exists \hat{S} \in \mathcal{F}_{i-1}: D[i-1, \hat{S}] = \top \text{ and } |S \Delta \hat{S}| \geq \ell, \\
  \bot & \text{otherwise,}
\end{cases}$$

where $D[1, \hat{S}] = \top$ if and only if $\hat{S} \in \mathcal{F}_1$. We report that $J$ is a yes-instance if and only there is an $S \in \mathcal{F}_I$ such that $D[\tau, S] = \top$. Note that this takes $2^{20\ell+o(\ell)}\log n \cdot n^4(1)^\tau = 2^{20\ell+o(\ell)}\log n^{O(1)}\tau$ time. Hence our overall running time is $2^{20\ell}f(r_{\text{max}})n^{O(1)}\tau \cdot g(p/n^4)2^{20\ell+o(\ell)}\log n)$, where $r_{\text{max}}$ is the maximum of parameter $r$ over all instances of $\Pi$ in $J$.

($\Leftarrow$): We show by induction over $i \in [\tau]$ that if $D[i, S] = \top$, then there is a sequence $(S_j)_{j \in [i]}$ such that $S_i = S$, $S_j \in \mathcal{R}(I_j)$ for all $j \in [i]$ and $|S_{j-1} \Delta S_j| \geq \ell$ for all $j \in \{2, 3, \ldots, i\}$.

By definition of $D$ this is clearly the case for $i = 1$. Now let $1 \leq i \leq \tau$ and $D[i, S] = \top$. Since $D[i, S] = \top$, $S \in \mathcal{F}_I$, and thus $S \in \mathcal{R}(I_i)$. By definition of $D$ there is an $\hat{S} \in \mathcal{F}_{i-1}$ with $D[i-1, \hat{S}] = \top$ and $|S \Delta \hat{S}| \geq \ell$. By induction hypothesis, there is a sequence $(S_j)_{j \in [i-1]}$ such that $S_{i-1} = \hat{S}$, $S_j \in \mathcal{R}(I_j)$ for all $j \in [i-1]$ and $|S_{j-1} \Delta S_j| \geq \ell$ for all $j \in \{2, 3, \ldots, i-1\}$. Hence, the sequence $(S_1, \ldots, S_{i-1} = \hat{S}, S_i)$ completes the induction. Thus, if we report that $J$ is a yes-instance, then this is true.

($\Rightarrow$): Now let $(S_j)_{j \in [\tau]}$ be a solution for $J$. To simplify the proof let $S_{i+1}$ be a set of $\ell$ elements that are disjoint from $S_r$. We show by induction that for all $i \in [\tau]$ there is a $Z \in \mathcal{F}_I$ such that $D[i, Z] = \top$ and $|Z \Delta S_{i+1}| \geq \ell$. 
Let \( i = 1 \). Then there is a \( Z \in \mathcal{F}_1 \) such that \( |S_2 \Delta Z| \geq \ell \) since \( \mathcal{F}_1 \) is an \( \ell \)-diverse representative of \( \mathcal{R}(I_1) \). Hence, \( D[1, Z] = \top \).

Now let \( 1 < i \leq \tau \). By induction hypothesis, there is a \( Z_{i-1} \in \mathcal{F}_{i-1} \) such that \( D[i - 1, Z_{i-1}] = \top \) and \( |S_i \Delta Z_{i-1}| \geq \ell \). Since \( S_i \in \mathcal{R}(I_i) \) and we have \( |S_i \Delta Z_{i-1}|, |S_i \Delta S_{i+1}| \geq \ell \) and \( \mathcal{F}_i \) is an \( \ell \)-diverse representative of \( \mathcal{R}(I_i) \), there is a \( Z \in \mathcal{F}_i \) such that \( |Z \Delta Z_{i-1}|, |Z \Delta S_{i+1}| \geq \ell \). By definition of \( D \), we also have \( D[i, Z] = \top \). This completes the induction step. Thus, there is a \( Z \in \mathcal{F}_\tau \) such that \( D[\tau, Z] = \top \) and if \( J \) is a yes-instance, then we report that.

\[ \blacktriangleleft \]

4 Application: Committee Election

Bredereck et al. [11] studied the following problem under the name Revolutionary Multistage Plurality Voting.

**Diverse Multistage Plurality Voting**

**Input:** A set \( A \) of agents, a set \( C \) of candidates, a sequence \( (u_i)_{i=1}^\tau \) of voting profiles \( u_i: A \to C \cup \{\emptyset\} \), and integers \( k, x, \ell \in \mathbb{N} \).

**Question:** Is there a sequence \( (C_1, C_2, \ldots, C_\tau) \) such that for all \( i \in [\tau] \) it holds that \( |C_i| \leq k \) and \( |u^{-1}(C_i)| \geq x \), and for all \( i \in [\tau - 1] \) it holds true that \( |C_i \Delta C_{i+1}| \geq \ell \)?

In this section, we affirmatively answer the question of Bredereck et al. [11] whether Diverse Multistage Plurality Voting parameterized by \( \ell \) or \( k \) is in \( \text{FPT} \).

**Theorem 12.** An instance \( J \) of Diverse Multistage Plurality Voting can be solved in \( 2^{O(\ell)} \cdot |J|^{O(1)} \) time.

To prove Theorem 12 we use Theorem 1. In the notation of our framework, we deal with the following problem \( \Pi \): given an instance \( I = (A, C, u, k, x) \) consisting of a set \( A \) of agents, a set \( C = B(I) \) of candidates, a voting profile \( u: A \to C \), and two integers \( k, x \), decide whether \( \mathcal{R}(I) := \{S \subseteq C \mid k \geq |S| \text{ and } |u^{-1}(S)| \geq x\} \) is non-empty. Hence, to apply Theorem 1 we consider the following problem.

**4-Colored Exact Plurality Voting**

**Input:** A set \( A \) of agents, a set \( C \) of candidates, a voting profile \( u: A \to C \cup \{\emptyset\} \), a coloring \( c: C \to [4] \), and integers \( n_i, x, k \in \mathbb{N}, i \in [4] \).

**Output:** A set \( C' \subseteq C \) of at most \( k \) candidates so that \( |u^{-1}(C')| \geq x \) and \( |c^{-1}(i) \cap C'| = n_i \) for all \( i \in [4] \) or "no" if no such set exists.

This problem is polynomial-time solvable and hence the following observation together with Theorem 1 proves Theorem 12. In Section 7 we will generalize this application to independent sets in matroids.

**Observation 13.** 4-Colored Exact Plurality Voting is polynomial-time solvable.

**Proof.** Given an instance \( I = (A, C, u, c, n_1, n_2, n_3, n_4, x, k) \) of 4-Colored Exact Plurality Voting. We may assume that \( \sum_{i=1}^4 n_i \leq k \), otherwise we can terminate without an output. For each candidate \( v \in C \), we compute its score \( s(v) := |u^{-1}(v)| \). Let \( C_i := c^{-1}(i) \), for all \( i \in [4] \). For each \( i \in [4] \), sort the candidates in \( C_i \) by their scores. Compute the set \( C'_i \subseteq C \) containing the \( n_i \) candidates in \( C_i \) with the highest score. If \( \sum_{i=1}^4 \sum_{v \in C'_i} s(v) \geq x \), then output \( \bigcup_{i=1}^4 C_i \). Otherwise, we terminate without an output. It is easy to verify that this procedure is correct.

\[ \blacktriangleleft \]

\(^3\) Note that \( \ell \leq 2k \) for all non-trivial instances, so it suffices to prove this for \( \ell \).
5 Application: Perfect Matching

In this section, we apply our framework from Section 3 to find a sequence of diverse perfect matchings.

**Diverse Multistage Perfect Matching**

**Input:** A sequence \((G_\tau)_{\tau=1}^\ell\) of graphs and an integer \(\ell \in \mathbb{N}_0\).

**Question:** Is there a sequence \((M_\tau)_{\tau=1}^\ell\) of perfect matchings \(M_\tau \subseteq E(G_\tau)\) such that \(|M_\tau \Delta M_{\tau+1}| \geq \ell\) for all \(\tau \in [\tau - 1]?)

There are two closely related variants of this problem which were studied extensively. The first variant is the non-diverse variant, where one seeks to bound the symmetric differences (in some way) from above \[3, 4, 13, 26, 39\]. Steinhau \[39\] proved that if the size of the symmetric difference of two consecutive perfect matchings shall be at most \(\ell\), then this problem variant is NP-hard even if \(\ell\) is constant, and \(W[1]\)-hard when parameterized by \(\ell + \tau\).

The second variant is the non-multistage variant, where one is given a single graph and is asked to compute a set of pairwise diverse perfect matchings \[24, 25\]. Fomin et al. \[24\] proved that this variant is NP-hard even if one asks only for two diverse matchings. This directly implies NP-hardness for **Diverse Multistage Perfect Matching** even when \(\tau = 2\).

Our goal is to show fixed-parameter tractability of **Diverse Multistage Perfect Matching** when parameterized by \(\ell\). This stands in contrast to the NP-hardness for the non-diverse problem variant with constant \(\ell\).

▶ **Theorem 14.** An instance \(J\) of **Diverse Multistage Perfect Matching** can be solved in \(2^{O(\ell)} \cdot |J|^{O(1)}\) time with a constant error probability.

We will prove Theorem 14 by means of Theorem 3 at the end of this section. To this end we need to consider the following problem.

**s-Colored Exact Perfect Matching**

**Input:** A graph \(G = (V,E)\), a coloring \(c: E \rightarrow [s]\), and \(k_i \in \mathbb{N}, i \in [s]\).

**Output:** (if exists) A perfect matching \(M\) in \(G\) such that \(|c_i \cap M| = k_i\), for all \(i \in [s]\)?

For \(s = 2\), this problem is known as **Exact Matching**, and Mulmuley et al. \[34\] showed that this special case is solvable by a randomized polynomial-time algorithm. We generalize this result by showing that **s-Colored Exact Perfect Matching** can be solved in polynomial time for any constant \(s\) by a randomized algorithm with constant error probability. While we only need this for \(s = 4\) in order to prove Theorem 14, we believe that the general case may be of independent interest. We remark that it is open whether **Exact Matching** can be solved in (deterministic) polynomial time.

▶ **Lemma 15.** For every \(0 < p < 1\) there is an \(n^{O(s)} \cdot \log 1/p\)-time algorithm which, given an instance of **s-Colored Exact Perfect Matching**, finds a solution with probability at least \(1 - p\) if one exists, and concludes that there is no solution otherwise.

The proof of Lemma 15 is deferred for a moment. To determine whether a given **s-Colored Exact Perfect Matching** has a solution we use the following algorithm.

▶ **Algorithm 16.** Let \(0 < p < 1\) and let \(I = (G,c,k_1,\ldots,k_s)\) be an instance of **s-Colored Exact Perfect Matching** where \(G = (V,E)\) has \(n\) vertices.

**Step 1.** Set \(\gamma := \lceil n/(2p) \rceil\) and draw \(w_{ij} \in [\gamma]\) for all \(\{i,j\} \in E\) uniformly at random.
Step 2. Construct an $n \times n$ matrix $A'$ with entries $a_{ij} \in \mathbb{Z}[y_1, \ldots, y_s]$, $1 \leq i \leq j \leq n$, where

$$a_{ij} := \begin{cases} 0 & \text{if } \{i, j\} \notin E, \\ w_{ij} y_q & \text{if } \{i, j\} \in c^q \cap E, q \in [s]. \end{cases}$$

Afterwards we compute the skew-symmetric matrix $A := A' - (A')^T$.

Step 3. Compute the polynomial $P := \sqrt{\det(A)} \in \mathbb{Z}[y_1, \ldots, z_s]$.

Step 4. If $P$ contains a monomial $b^* y_1^{k_1} y_2^{k_2} \cdots y_s^{k_s}$ such that $b^* \neq 0$ then, output yes. Otherwise, output no.

Before studying the running time of Algorithm 16 we first focus on its correctness.

Lemma 17. Let $I$ and $p$ be the input of Algorithm 16. If Algorithm 16 returns yes, then there is a solution for $I$. Conversely, if $I$ is a yes-instance, then Algorithm 16 returns yes with probability at least $1 - p$.

To show Lemma 17 we need the following well-known lemma.

Lemma 18 (DeMillo and Lipton [16], Schwartz [38], Zippel [40]). Let $P \in \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial of total degree $d \geq 0$ over a field $\mathbb{F}$. Let $S$ be a finite subset of $\mathbb{F}$ and let $r_1, \ldots, r_n$ be selected uniformly and independently at random from $S$. Then the probability that $P(r_1, \ldots, r_n) = 0$ is at most $d/|S|$.

Proof of Lemma 17. Let $P$ be the set of all partitions of $V$ into unordered pairs. For $\sigma \in P$ with $\sigma = \{(i_1, j_1), (i_2, j_2), \ldots, (i_{n/2}, j_{n/2})\}$ with $i_k < j_k$ for $k \in [n/2]$ and $i_1 < i_2 < \cdots < i_{n/2}$, let

$$\pi_\sigma := \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n-1 & n \\ i_1 & j_1 & i_2 & j_2 & \cdots & i_{n/2} & j_{n/2} \end{bmatrix}$$

be the corresponding permutation. Let $\text{val}(\sigma) := \text{sgn}(\pi_\sigma) \prod_{i,j} a_{ij}$, where $\text{sgn}(\pi_\sigma) \in \{+1, -1\}$ is the signum of $\pi_\sigma$. The Pfaffian of $A$ (computed by Algorithm 16) is defined as $p(A) := \sum_{\sigma \in \mathcal{P}} \text{val}(\sigma)$ [29]. Note that $A$ is skew-symmetric, hence, $p(A) = \sqrt{\det(A)} = P$ [29, 33]. As $\text{val}(\sigma) = 0$ whenever $\sigma$ contains a non-edge, we have $P = \sum_{M \in \mathcal{P} \mathcal{M}} \text{val}(M)$, where $\mathcal{P} \mathcal{M}$ is the set of perfect matchings in $G$. Let $M$ be a perfect matching and let $z_q = |c^q \cap M|$, $q \in [s]$. Then $\text{val}(M) = \text{sgn}(\pi_M) \prod_{q \in [s]} \prod_{(i,j) \in c^q} w_{ij} y_q = b^* y_1^{k_1} y_2^{k_2} \cdots y_s^{k_s}$, where $b \in \mathbb{Z}$. Let $\mathcal{P} \mathcal{M} \subseteq \mathcal{P} \mathcal{M}$ be the family of perfect matchings $M^*$ which have exactly $k_i$ edges of color $i$, for all $i \in [s]$. Then the coefficient $b^*$ of the monomial $b^* y_1^{k_1} y_2^{k_2} \cdots y_s^{k_s}$ of $P$ is $b^* = \sum_{M^* \in \mathcal{P} \mathcal{M}} \text{sgn}(\pi_{M^*}) \prod_{(i,j) \in M^*} w_{ij}$. Hence, if Algorithm 16 returns yes (i.e., $b^* \neq 0$), then $\mathcal{P} \mathcal{M}^* \neq \emptyset$.

Now conversely assume $I$ to be a yes-instance, i.e., $\mathcal{P} \mathcal{M}^* \neq \emptyset$. We analyze the probability of the event $b^* = 0$ occurring. Note that $b^*$ can be seen as a polynomial of degree at most $n/2$ over the indeterminates $\{w_{ij} \mid \{i, j\} \in E\}$. As we have drawn the $w_{ij}$ independently and uniformly at random from $[\gamma]$ with $\gamma \geq n/(2p)$, by the DeMillo-Lipton-Schwartz-Zippel lemma (Lemma 18) the probability that $b^* = 0$ is at most $n/(2\gamma) \leq p$. \hfill \qed

Now we show that Algorithm 16 can be executed efficiently.

Lemma 19. Algorithm 16 runs in $O^{(*)}(\log(1/p))$ time.

As for the running time of Algorithm 16 note that computing the determinant as well as its square root are the most expensive operations. For completeness, we first show that we can compute the square root of a polynomial efficiently.
Lemma 20. Let $P \in \mathbb{Z}[x_1, \ldots, x_s]$ be a polynomial of degree $2n > s$ such that there exists a polynomial $Q \in \mathbb{Z}[x_1, \ldots, x_s]$ with $P(x) = (Q(x))^2$. Computing $Q$ from $P$ takes $n^{O(s)}$ algebraic operations.

Proof. For $\alpha \in \mathbb{N}^s$, we write $x^\alpha$ to denote $x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_s^{\alpha_s}$. Let $P(x) = \sum_{\alpha \in \mathbb{N}^s} c_\alpha x^\alpha$ and $Q(x) = \sum_{\alpha \in \mathbb{N}^s} d_\alpha x^\alpha$ with coefficients $c_\alpha, d_\alpha \in \mathbb{Z}$. Then

$$P(x) = \sum_{\alpha, \beta \in \mathbb{N}^s} d_\alpha d_\beta x^{\alpha+\beta}$$

and thus for all $\kappa \in \mathbb{N}^s$

$$c_\kappa = \sum_{\alpha, \beta \in \mathbb{N}^s \atop \alpha + \beta = \kappa} d_\alpha d_\beta.$$

We will compute the $d_\alpha$ by induction on $\|\kappa\|_1$. Clearly, $d_{(0,0,\ldots,0)} = \sqrt{c_{(0,0,\ldots,0)}}$, this is our base case. Now let $\kappa \in \mathbb{N}^s$ and suppose we have computed all $d_{\alpha}$ with $\|\alpha\|_1 < \|\kappa\|_1$. We have

$$c_\kappa = \sum_{\alpha + \beta = \kappa \atop \|\alpha\|_1 \neq 0} d_\alpha d_\beta = 2 \cdot d_{(0,0,\ldots,0)} \cdot d_\kappa.$$

This is equivalent to

$$d_\kappa = \frac{1}{2 \cdot d_{(0,0,\ldots,0)}} \cdot \left( c_\kappa - \sum_{\alpha + \beta = \kappa \atop \|\alpha\|_1 \neq 0} d_\alpha d_\beta \right).$$

(5)

We already computed $d_{(0,0,\ldots,0)}$ as well as all $d_\alpha$ and $d_\beta$ occuring in (5). Therefore we can use (5) to compute $d_\kappa$ and thus $Q(x)$.

Note that the sum in (5) contains one summand for each $d_\alpha$ with $0 < \|\alpha\|_1 < \|\kappa\|_1$. Since $1/(2d_{(0,0,\ldots,0)})$ only needs to be computed once, computing $d_\kappa$ requires

$$1 + \sum_{j=1}^{\|\kappa\|_1-1} \binom{s+j}{j} = \sum_{j=0}^{\|\kappa\|_1-1} \binom{s+j}{j} = \binom{s + \|\kappa\|_1}{s + 1}$$

algebraic operations. As we need to do this for every $\kappa \in \mathbb{N}^s$ with $\|\kappa\|_1 \leq n$, we require

$$O(1) + \sum_{i=1}^{n} \binom{s + i}{s} \cdot \binom{s + i}{s + 1} \leq O(n \cdot (s + n)^s \cdot (s + n)^{s+1}) \leq O(n^{2s+2}) \leq n^{O(s)}$$

algebraic operations overall.

Proof of Lemma 20. Note that, without loss of generality, $s \leq n/2$. Moreover, we need at most $O(\log \gamma)$ cells to store $w_{ij}$ for all $\{i, j\} \in \mathcal{E}$. Hence, computing $A$ takes $O(n^2 \log \gamma)$ time. As $\det(A)$ is a polynomial of degree at most $n$ with $s$ variables, it consists of at most $n^{s+1}$ terms $O((2n)^s)$ coefficients. Hence, computing $\det(A)$ takes at most $n^{O(s)}$ algebraic operations, e.g., using Gaussian elimination. As we need at most $O(\log \gamma)$ cells for the initial values $w_{ij}$ for all $\{i, j\} \in \mathcal{E}$, and we need only $n^{O(s)}$ algebraic operations to compute $\sqrt{\det(A)}$ (see Lemma 20), we have an overall running time of $n^{O(s)} \log(\gamma) = n^{O(s)} \log(1/p)$.

---

4 That is, additions, subtractions, multiplications, divisions.
We are now ready to put all parts together and prove Lemma 15. In a nutshell, we use Algorithm 16 to check whether there is a solution. If this is the case, then we try to delete as many edges as possible from the instance until the whole edge set is a solution.

Proof of Lemma 15 Let \( I = (G = (V, E), c, k_1, \ldots, k_s) \) be an instance of s-COLORED EXACT PERFECT MATCHING. Let \( m = |E| \) and \( n = |V| \). We check whether \( I \) has a solution by applying Algorithm 16 with error probability \( p/(m+1) \). If the answer is no, then we output that there is no solution. Otherwise we initialize \( M := \emptyset \).

We iterate over all edges \( e \in E \) and apply Algorithm 16 with error probability \( p/(m+1) \) to the instance \( ((V, M \cup (E \setminus \{e\})), c, k_1, \ldots, k_s) \). Afterwards, we delete \( e \) from \( E \). If the result is no, then we add \( e \) to \( M \). In any case we proceed with the next edge. If we reached \( M = E \), then we output the solution \( M \).

By Lemma 17, the probability that at some step an error occurs is at most \( p/(m+1) \). Since we execute Algorithm 16 at most \( m+1 \) times, the overall error probability is \( p \).

Overall, we execute Algorithm 16 at most \( m+1 \) times, each of which can be computed in \( n^{O(s)} \log(1/p) \) time due to Lemma 19. Hence, the overall running time can be bounded by \( n^{O(s)} \log(1/p) \).

Putting Lemma 15 and Theorem 3 together, we can prove the main theorem of this section.

Proof of Theorem 14 Lemma 15 with \( s = 4 \) fulfills Assumption 2 wherein \( g(p) = \mathcal{O}(\log 1/p) \) and \( f(r) = 1 \). We aim for a constant error probability, say \( p = 1/4 \). Hence, by Theorem 3 and Lemma 15 we have an algorithm with error probability \( 1/4 \) for an instance \( J \) of DIVERSE MULTISTAGE PERFECT MATCHING with running time \( 2^{O(1)} \cdot |J|^{O(1)} \).

6 Application: s-t Path

In this section, we apply our framework to the task of finding a sequence of diverse s-t paths. This has obvious applications e.g. in convoy routing [22].

Diverse Multistage s-t Path

Input: A sequence of graphs \( (G_i)_{i=1}^{\ell} \), two distinct vertices \( s, t \in \bigcap_{i=1}^{\ell} V(G_i) \), and \( \ell \in \mathbb{N}_0 \).

Question: Is there a sequence \( (P_1, P_2, \ldots, P_{\ell}) \) such that \( P_i \) is an s-t path in \( G_i \) for all \( i \in [\ell] \), and \( |S_i \Delta S_{i+1}| \geq \ell \) for all \( i \in [\ell-1] \)?

Our goal is to show that DIVERSE MULTISTAGE s-t Path parameterized by \( \ell \) is in FPT.

Theorem 21. DIVERSE MULTISTAGE s-t Path parameterized by \( \ell \) is in FPT.

We will prove Theorem 21 by means of Theorem 1 at the end of this section. To this end, we need to consider the following problem.

4-Colored Exact s-t Path

Input: A graph \( G \), distinct vertices \( s, t \in V(G) \), coloring \( c: V(G) \to [4] \), and \( n_i \in \mathbb{N}_0, i \in [4] \).

Output: (if exists) An s-t path \( P \) such that \( |c^{-1}(i) \cap V(P)| = n_i \) for all \( i \in [4] \).

Unfortunately, 4-Colored Exact s-t Path is unlikely to be polynomial-time solvable, as it is NP-hard even if only a single color is used, by a trivial reduction from HAMILTONIAN PATH. However, as we will see in the proof of Theorem 21, by a result of Mousset et al. [32] we can actually reduce 4-Colored Exact s-t Path to the case that all graphs have small treewidth. In this setting, we then employ dynamic programming.
Lemma 22. 4-COLORED EXACT s-t Path is solvable $k^{O(k)} \cdot |I|^{O(1)}$ time, where $k$ is the treewidth of the input graph $G$.

While some techniques [8, 14, 23] seem applicable to improve the running time of Lemma 22 slightly, for our needs a straight-forward dynamic program on a nice tree decomposition suffices. We introduce nice tree decompositions before we prove Lemma 22.

Definition 23. A tree decomposition of a graph $G$ is a pair $\mathcal{T} = (T, \{X_v\}_{v \in V(T)})$, where $T$ is a tree whose every node $v$ is assigned a bag $X_v \subseteq V(G)$ such that
1. $\bigcup_{v \in V(T)} X_v = V(G)$,
2. $\forall e \in E(G), \exists v \in V(T): e \subseteq X_v$,
3. $\forall u \in V(G): T[\{v \in V(T) \mid u \in X_v\}]$ is a tree.

The width of $\mathcal{T}$ is $\max_{v \in V(T)} |X_v| - 1$. The treewidth of a graph $G$ is the minimum width of a tree decomposition of $G$. A tree decomposition $\mathcal{T}$ is called nice if $T$ is rooted at a vertex $r$ such that
- $X_r = \emptyset = X_v$ for all leaves $v \in V(T)$, and
- every non-leaf node $v \in V(T)$ of $T$ is of one of the following three types:
  - **Introduce node:** $v$ has exactly one child $w$ in $T$ and $X_v = X_w \cup \{u\}$, for some vertex $u \notin X_w$.
  - **Forget node:** $v$ has exactly one child $w$ in $T$ and $X_v = X_w \setminus \{u\}$, for some vertex $u \in X_w$.
  - **Join node:** $v$ has exactly two children $u, w$ in $T$ and $X_v = X_w = X_u$.

Lemma 24 ([9] and [15, Lemma 7.4]). Given graph $G$ of treewidth $k$, one can compute in $2^{O(k)} \cdot n^{O(1)}$ time a nice tree decomposition $\mathcal{T} = (T, \{X_v\}_{v \in V(T)})$ for $G$ of width $O(k)$ such that $|V(T)| \in O(|V(G)|)$.

Proof of Lemma 22: Let $I = (G, s, t, c, n_1, n_2, n_3, n_4)$ be an instance of 4-COLORED EXACT s-t Path. Let $n := |V(G)|$ and $k$ be the treewidth of $G$. By Lemma 24, we compute a nice tree decomposition $\mathcal{T} = (T, \{X_v\}_{v \in V(T)})$ for $G$ of width $O(k)$ such that $|V(T)| \in O(n)$. As a first step we add $\{s, t\}$ to every bag of $\mathcal{T}$. Henceforth, we say a node $v \in V(T)$ is an introduce/forget/join node if it was such a node before we added $\{s, t\}$ to every bag. Let $T_v$ be the subtree of $T$ rooted at $v$ and $X(T_v) := \bigcup_{u \in V(T_v)} X_u$. We are going to compute a dynamic program $D$ such that the following assumption is true for all $v \in V(G)$.

Assumption 25. Let $v \in V(T)$ be fixed. For every set of vertex pairs $\Lambda_1, \Lambda_2, \ldots, \Lambda_p \subseteq X_v$ and every vector $\gamma \in [n]^4$, assume that $D_v[\{\Lambda_i\}_{i=1}^p, \gamma] = \top$ if and only if there are internally vertex-disjoint paths $P_1, \ldots, P_q$ in $G[X(T_v)]$ with $X_v \cap V(P_i) = \Lambda_i$ and $\bigcup_{i=1}^q V(P_i) \cap c^p = \gamma_p$ for all $p \in [4]$.

Hence, we would like to know whether $D_v[\{\{s, t\}\}, (n_1, n_2, n_3, n_4)] = \top$. Note that we could store instead of $\top$ a set of vertex disjoint paths $P_1, \ldots, P_q$ as specified in Assumption 25 to compute an actual solution, but we omit this for the sake of simplicity. Thus, it is only left to show that we can compute $D$ in $k^{O(k)}n^{O(1)}$ time such that Assumption 25 holds for all $v \in V(T)$. We do this by induction over the structure of the tree $T$. In the following, $\widetilde{c}_i$ denotes the $i$-th canonical unit vector.

Leaf node. Let $v$ be a leaf in $T$, hence $X_v = \{s, t\}$. We set $D_v[\{\{s, t\}\}, \widetilde{c}_{\gamma(s)} + \widetilde{c}_{\gamma(t)}] = \top$ if and only if $\{s, t\} \in E(G)$. Then it is easy to verify that Assumption 25 holds for $v$ and all values of $D_v$ can be computed in $n^{O(1)}$ time.
We are now ready to prove Theorem 21.

We set \( D_w[\{\Lambda_i\}_{i=1}^q, \gamma] = \top \) if and only if any of the following two conditions holds.

(i) \( D_w[\{\Lambda_i\}_{i=1}^q, \gamma] = \top \).

(ii) There is an index \( j \in [q] \) such that \( D_w[\Psi_j(\{\Lambda_i\}_{i=1}^q), \gamma] = \top \) where \( \Psi_j \) is the operation of replacing \( \Lambda_j := \{s_j, t_j\} \) by the two pairs \( \{s_j, u\}, \{u, t_j\} \).

Note that \( X(T_v) = X(T_w) \) and \( X_v = X_w \setminus \{u\} \). Assume that paths \( P_1, \ldots, P_q \) as required by Assumption 25 exist for \( v \). Either none of them contains \( u \), in which case \( \Psi \) must be true, or one of them (say \( P_j \)) does, in which case \( \Psi \) holds. Conversely, if one of these conditions holds, then such paths do exist. Clearly, all values of \( D_v \) can be computed in \( k^{O(k)}n^{O(1)} \) time.

Join node. Let \( v \) be a join node in \( T \) and \( w \) the children of \( v \). Hence \( X_v = X_u = X_w \). Assume that Assumption 25 holds for \( w \) and \( u \). Let \( \Lambda_1, \ldots, \Lambda_q \subseteq X_v \) be vertex pairs and \( \gamma \in [n]^4 \). We set \( D_v[\{\Lambda_i\}_{i=1}^q, \gamma] = \top \) if and only if \( \{\Lambda_i\}_{i=1}^q \) can be partitioned into two sets \( \{\Lambda_{j,v}\}_{i=1}^{q_v}, \{\Lambda_{j,w}\}_{i=1}^{q_w} \) and \( \gamma \) can be written as a sum \( \gamma = \gamma_v + \gamma_w \) such that

\[
D_w[\{\Lambda_{j,v}\}_{i=1}^{q_v}, \gamma_v] = \top \quad \text{and} \quad D_w[\{\Lambda_{j,w}\}_{i=1}^{q_w}, \gamma_w] = \top
\]

Since \( X(T_v) = X(T_u) \cup X(T_w) \) and no two vertices of \( (X(T_u) \cup X(T_w)) \setminus X_v \) are connected by an edge, any path in \( G[X(T_v)] \) without internal vertices from \( X_v \) is either a path in \( G[X(T_u)] \) or in \( G[X(T_w)] \). This proves the correctness of this step. Again, it is easy to see that all values of \( D_v \) can be computed in \( k^{O(k)}n^{O(1)} \) time.

Introduce node. Let \( v \) be an introduce node in \( T \) and \( w \) the child of \( v \). Hence, \( X_v = X_w \cup \{u\} \) for some \( u \not\in X_w \). Assume that Assumption 25 holds for \( w \). Let \( \Lambda_1, \ldots, \Lambda_q \subseteq X_v \) be vertex pairs and \( \gamma \in [n]^4 \). Let \( J := \{j \in [q] : u \in \Lambda_j\} \) and \( \gamma' := \gamma - \sum_{j \in J} \sum_{x \in \Lambda_j} e_{c(x)} \).

We set \( D_v[\{\Lambda_i\}_{i=1}^q, \gamma] = \top \) if and only if \( D_w[\{\Lambda_i\}_{i=1}^q \setminus \gamma', \gamma''] = \top \) and each pair \( \Lambda_j \) with \( j \in J \) forms an edge of \( G \). Since \( X(T_v) = X(T_u) \cup \{u\} \) and \( u \) does not have an edge to any vertex of \( X(T_u) \setminus X_v \), any pair \( \Lambda_j \) containing \( u \) must be connected by a direct edge, while all other pairs must be connected by paths in \( G[X(T_u)] \). From this, the correctness of the above follows. All values of \( D_v \) can clearly be computed in \( k^{O(k)}n^{O(1)} \) time.

Having now proven the inductive step for all types of nodes, we conclude that Assumption 25 is true for the root node \( r \). Since \( V(T) \in \mathcal{O}(n) \), we have an overall running time of \( k^{O(k)}n^{O(1)} \). This completes the proof of Lemma 22. \( \blacksquare \)

We are now ready to prove Theorem 21.

Proof of Theorem 21. Let the instance \( J \) of DIVERSE MULTISTAGE \( s \)-\( t \) Path be given in the form of graphs \( G_1, \ldots, G_r \), two vertices \( s, t \in \bigcap_{i=1}^r G_i \) and \( \ell \in \mathbb{N} \). We may assume that every vertex \( v \) of every graph \( G_i \) is contained in at least one \( s \)-\( t \) path in \( G_i \), since otherwise we may delete \( v \). This is equivalent to the assumption that the graph \( G_i \) obtained from adding the edge \( \{s, t\} \) to \( G_i \) is biconnected.

By a result of Mousset et al. 32, there is a universal constant \( \gamma > 0 \), such that each \( G_i \) with treewidth \( tw(G_i) \geq \gamma \ell \) contains two vertex-disjoint cycles of size at least \( 4\ell \).

If two such cycles \( C, C' \) exist in \( G_i \), then let \( P_1 \) be an \( s \)-\( t \) path containing at least one edge of \( C \). To see that such a path exists, construct a biconnected graph by simply attaching
a new degree-two vertex \( s' \) to both \( s \) and \( t \), create another new vertex \( t' \) by subdividing some edge of \( C \), and take two disjoint paths between \( s' \) and \( t' \).

Without loss of generality, \( P_1 \) enters \( C \) and \( C' \) at most once each. Construct another \( s-t \) path \( P_2 \) from \( P_1 \) by setting \( E(P_2) := E(P_1) + E(C) \). If \( P_1 \) contains any edge of \( C' \), then define \( P_3 \) by \( E(P_3) := E(P_1) + E(C') \). Otherwise, let \( P_3 \) be any \( s-t \) path containing at least half of the edges of \( C' \) (this can be achieved analogously to the construction of \( P_1 \) resp. \( P_2 \)). Note that \( P_1, P_2, P_3 \) have pairwise symmetric differences at least \( 2\ell \). Thus, \( \{P_1, P_2, P_3\} \) is an \( \ell \)-diverse representative of all \( s-t \) paths in \( G \) by Lemma 5.

We can then solve the subinstances given by \((G_j)_{j<i} \) and \((G_j)_{j>i} \), separately and pick a suitable path from \( \{P_1, P_2, P_3\} \) afterwards.

All subinstances in which every graph \( G_i \) has \( tw(G_i) < \gamma \ell \) can be solved by Theorem 1 in combination with Lemma 22 in \( 2^{O(\ell)} f(\gamma \ell)|J|^{O(1)} \) time, where \( f \) is given by Lemma 22. □

7 Application: Spanning Forests and Other Matroids

In this section, we apply our framework in the context of matroid theory which abstracts the notion of linear independence in vector spaces and finds applications in geometry, topology, combinatorial optimization, network theory, and coding theory [28, 36]. In the classical non-diverse multistage setting, matroids have already been studied [26]. We first introduce some standard notations and then define the problem we apply our framework to.

A pair \((U, \mathcal{I})\), where \( U \) is the ground set and \( \mathcal{I} \subseteq 2^U \) is a family of independent sets\(^5\) is a matroid if the following holds:

- \( \emptyset \in \mathcal{I} \).
- If \( A' \subseteq A \) and \( A \in \mathcal{I} \), then \( A' \in \mathcal{I} \).
- If \( A, B \in \mathcal{I} \) and \( |A| < |B| \), then there is an \( x \in B \setminus A \) such that \( A \cup \{x\} \in \mathcal{I} \).

Throughout this section, we assume to have access to an oracle which tells us in polynomial time whether a given set \( A \subseteq U \) is an element of \( \mathcal{I} \). An inclusion-wise maximal independent set \( A \in \mathcal{I} \) of a matroid \( M = (U, \mathcal{I}) \) is a basis. The cardinality of the bases of \( M \) is called the rank of \( M \). A cycle matroid of an undirected graph \( G \) is a matroid \((E(G), \mathcal{I})\), where an \( A \subseteq E(G) \) is in \( \mathcal{I} \) if and only if \((V, A) \) is a forest. A partition matroid is a matroid \((U, \mathcal{I})\) such that \( \mathcal{I} := \{S \subseteq U \mid |U_i \cap S| \leq r_i, \ i \in [m]\} \), where \( \bigcup_{i=1}^m U_i \) is a partition of \( U \) and \( r_i \in \mathbb{N}_0, \ i \in [m], m \in \mathbb{N} \). A partition matroid is called uniform matroid if \( m = 1 \). Later we will use partition matroids to encode constraints of type “at most \( r_i \) elements of color \( i \)”.

We now can formulate the central problem of this section.

Diverse Multistage Matroid Independent Set

**Input:** A sequence of matroids \( (M_i, (U_i, \mathcal{I}_i))_{i=1}^\tau \) with \( \tau \) many weight functions \( \omega_i : U_i \to \mathbb{N}_0 \), and integers \( x_i, \ell \in \mathbb{N}_0 \), for all \( i \in [\tau] \).

**Question:** Is there a sequence \((S_1, S_2, \ldots, S_\tau)\) such that for all \( i \in [\tau] \) the set \( S_i \) is in \( \mathcal{I}_i \) and is of weight \( \omega_i(S_i) \geq x_i \), and for all \( i \in [\tau - 1] \) it holds true that \( |S_i \Delta S_{i+1}| \geq \ell \)?

Note that Diverse Multistage Plurality Voting is a special case of Diverse Multistage Matroid Independent Set, where matroid \( M_i \) is a uniform matroid of rank 1 with the set of candidates as ground set and the weight functions map to the number of agents approving the candidate. We show fixed-parameter tractability Diverse Multistage Matroid Independent Set parameterized by \( \ell \).

---

\(^5\) Note that this is not a set of pairwise non-adjacent vertices in a graph.
Theorem 26. **Diverse Multistage Matroid Independent Set** parameterized by \( \ell \) can be solved in \( 2^{O(\ell)} \cdot n^{O(1)} \) time, where \( n \) is the input size.

In the context of our framework, II here is the following: given an instance \( I = (M, w, x) \) consisting of a matroid \( M = (U, \mathcal{I}) \), a weight function \( \omega: U \to \mathbb{N}_0 \), and an integer \( x \), is \( R(I) = \{ S \in \mathcal{I} \mid \omega(S) \geq x \} \) non-empty, where \( B(I) = U \)? Hence, to apply Theorem 3, we study the following problem.

**s-Colored Exact Matroid Independent Set**

**Input:** A matroid \( M = (U, \mathcal{I}) \), a weight function voting profile \( \omega: U \to \mathbb{N} \), a coloring \( c: U \to [s] \), and integers \( n_i, x, \in \mathbb{N}, i \in [s] \).

**Output:** (if exists) An independent set \( S \in \mathcal{I} \) such that \( \omega(S) \geq x \) and \( |c^{-1}(i) \cap S| = n_i, i \in [s] \).

We show that **s-Colored Exact Matroid Independent Set** can be solved efficiently.

**Lemma 27.** **s-Colored Exact Matroid Independent Set** is polynomial time solvable.

**Proof.** Given an instance \( I = (M = (U, \mathcal{I}), \omega, c, n_1, n_2, \ldots, n_s, x, k) \) of **s-Colored Exact Matroid Independent Set**. Partition the ground set \( U := \bigcup_{i=1}^s U_i \), where \( U_i := \{ v \in U \mid c(v) = i \} \), for all \( i \in [s] \). Construct the partition matroid \( M' := (U, \mathcal{I}' := \{ S \subseteq U \mid |S \cap U_i| \leq n_i \}) \). Now compute in polynomial time a set \( S \in \mathcal{I} \cap \mathcal{I}' \) of size \( \sum_{i=1}^s n_i \) maximum-weight with respect to \( \omega \) [37, Section 4.1 and 41.3a]. Thus, we clearly have \( S \in \mathcal{I} \). Observe that \( S \in \mathcal{I}' \) implies \( |c^{-1}(i) \cap S| \leq n_i, i \in [4] \). Since the set \( S \) is of size \( \sum_{i=1}^s n_i \), we have \( |c^{-1}(i) \cap S| = n_i, i \in [s] \). Hence, if \( w(S) \geq x \), then we correctly output \( S \). Otherwise, we terminate without an output. Observe that this is correct since we have for any set \( S \in \mathcal{I} \) with \( |c^{-1}(i) \cap S| = n_i, i \in [s] \) that \( S \in \mathcal{I} \cap \mathcal{I}' \).

Now Theorem 26 follows from Theorem 1 and Lemma 27. Among others, Theorem 26 implies that the following problem is fixed-parameter tractable when parameterized by \( \ell \).

**Diverse Multistage Spanning Forests**

**Input:** A sequence of graphs \( (G_i)_{i=1}^\tau \) and \( \ell \in \mathbb{N}_0 \).

**Question:** Is there a sequence \( (S_1, S_2, \ldots, S_\tau) \) such that for all \( i \in [\tau] \) the graph \( (V(G_i), S_i) \) is a spanning forest of \( G_i \), and for all \( i \in [\tau-1] \) it holds true that \( |S_i \Delta S_{i+1}| \geq \ell \)?

Spanning forests have been studied by Gupta et al. [26] in the non-diverse multistage setting.

## 8 Hardness of Vertex Cover

We finally present a problem where our framework from Section 3 is not applicable, unless FPT = W[1]. The non-diverse variant of the following problem was studied by Fluschnik et al. [21]. Among others, they showed W[1]-hardness when parametrized by the vertex cover size \( k \) or by the maximum number of edges over all instances in the input.

**Diverse Multistage Vertex Cover**

**Input:** A sequence of graphs \( (G_i)_{i=1}^\tau \) and \( k, \ell \in \mathbb{N} \).

**Question:** Is there a sequence \( (S_1, S_2, \ldots, S_\tau) \) such that for all \( i \in [\tau] \) the set \( S_i \subseteq V(G_i) \) is a vertex cover of size at most \( k \) in \( G_i \) and \( |S_i \Delta S_{i+1}| \geq \ell \) for all \( i \in [\tau-1] \)?

The framework from Section 3 is presumably not applicable to **Diverse Multistage Vertex Cover** because of the following result.

**Theorem 28.** **Diverse Multistage Vertex Cover** parameterized by \( \ell \) is W[1]-hard, even if \( \tau = 2 \).
We introduced a versatile framework to show fixed-parameter tractability for a variety of
problems. Thus, improving the time or space constraints could be a fruitful research direction.

In particular, a broad systematic study of the multistage setting in elections was proposed
independent interest from a technical and motivational point of view, see Sections 5 and 6.

We believe that our framework can be applied to a broad spectrum of multistage problems.
In particular, a broad systematic study of the multistage setting in elections was proposed
by Boehmer and Niedermeier [10]. Herein, diversity is a natural goal. From a motivational
point of view, an interesting direction for future research is to combine the diverse multistage
setting with time windows, known from other temporal domains [12, 30, 31, 41]. Here, a
solution to the $i$-th instance should be sufficiently different from the $\delta$ previous solutions
in the sequence; our work covers the case $\delta = 1$. In some multistage scenarios a “global
view” [27] on the symmetric differences is desired. In context of this paper this means that
two consecutive solutions can have a small symmetric difference as long as the sum of all
consecutive symmetric differences is at least $\ell$. We believe that our framework (Section 3)
can be extended to this setting. To see this, we have to realize that for an $\ell$-diverse representative
$F$ of a family of solutions the following holds: For all sets $A$ and $B$ and integers $\ell_a, \ell_b \leq \ell$, if
there is an $S \in F$ such that $|A \Delta S| \geq \ell_a$ and $|B \Delta S| \geq \ell_b$, then there is an $\hat{S} \in F$ such that
$|A \Delta \hat{S}| \geq \ell_a$ and $|B \Delta \hat{S}| \geq \ell_b$. We leave the details for further research. Finally, the presented
time and space constraints to compute $\ell$-diverse representatives seem to be suboptimal.
Hence, improving the time or space constraints could be a fruitful research direction.

**Proof.** We reduce from **Independent Set:** Given an graph $G = (V,E)$, and $k \in \mathbb{N}$, is
there a vertex set $S \subseteq V$, $|S| \geq k$, such that the vertices in $S$ are pairwise nonadjacent?
**Independent Set** is $\mathsf{W[1]}$-hard with respect to $k$ [18].

Let $I := (G = (V,E), k)$ be an instance of **Independent Set** and let $|V| = n$. Without
loss of generality, we assume that $k > 1$. We construct an instance $J := ((G_1, G_2), k', \ell')$ of
**Diverse Multistage Vertex Cover** as follows. The first graph $G_1$ is a complete graph
on the vertex set $V \cup \{v\}$. The second graph $G_2$ consists of the vertex set $V \cup \{v\}$ and
the edge set $E \cup \{(u, v) \mid u \in V\}$, that is, $G_2$ is a copy of $G$ to which we add a vertex $v$ which
is adjacent to every other vertex. Lastly, we set $k' := n$ and $\ell' := k + 1$. Clearly, $J$ can be
constructed in polynomial time. We now show that $J$ is a yes-instance if and only if $I$ is a
yes-instance.

$(\Rightarrow)$: Let $S$ be an independent set of size at least $k$ in $G$. Let $S_1 := V$ and $S_2 := \{v\} \cup V \setminus S$.
Note that $|S_1 \Delta S_2| \geq k + 1$ and $|S_2| \leq |S_1| = n$, and $S_i$ is a vertex cover in $G_i$ for $i \in [2]$.
Thus $(S_1, S_2)$ is a valid solution for our instance of **Diverse Multistage Vertex Cover**.

$(\Leftarrow)$: Let $(S_1, S_2)$ be a solution for our instance of **Diverse Multistage Vertex
Cover**. As $G_1$ is a complete graph, we have $|S_1| \geq n$. Without loss of generality, we assume
that $S_1 = V$. Then $S_1 \Delta S_2 = \{v\} \cup V \setminus S_2$. Note that $v \in S_2$, otherwise $S_2$ must be equal
to $V$ in order to be a vertex cover, and $|S_1 \Delta V| < \ell$. As $S_2$ is a vertex cover of $G_2$, the
set $S := (V \cup \{v\}) \setminus S_2$ is an independent set of $G_2$. Note that $v \notin S$, hence $S$ is also an
independent set of $G$. Finally, as $S = (S_1 \Delta S_2) \setminus \{v\}$, we have $|S| \geq \ell - 1 = k$, and we are
done. \qed

## Conclusion

We introduced a versatile framework to show fixed-parameter tractability for a variety of
diverse multistage problems when parameterized by the diversity $\ell$. The only requirement
for applying our framework is that a four-colored variant of the base problem can be solved
efficiently. We presented four applications of our framework, one of which resolving an open
question by Bredereck et al. [11]. Two other applications revealed problems which may be of
independent interest from a technical and motivational point of view, see Sections 5 and 6.

We believe that our framework can be applied to a broad spectrum of multistage problems.

L. Kellerhals, M. Renken, and P. Zschoche
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