Chapter 1

On the Scalar Curvature of 4-Manifolds

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Dimension four provides a peculiarly idiosyncratic setting for the interplay between scalar curvature and differential topology. Here we will explain some of the peculiarities of the four-dimensional realm via a careful discussion of the Yamabe invariant (or “sigma constant”). In the process, we will also prove some new results, and point out open problems that continue to represent key challenges in the subject.

Much of modern Riemannian geometry concerns the relationship between curvature and differential topology. In keeping with the overall theme of this book, the present chapter will describe some of what this dialog has taught us about the scalar curvature of Riemannian manifolds. Our specific objective will be to explain how the interplay between scalar curvature and differential topology in dimension four hinges on phenomena that are quite unlike anything seen in other dimensions.

Recall that the scalar curvature $s : M \rightarrow \mathbb{R}$ of a Riemannian $n$-manifold $(M,g)$ is by definition the full trace

$$s = R^{ij} = g^{jk} R_{ij}^k$$

of the Riemann curvature tensor. This has a nice synthetic interpretation in terms of the volumes of the Riemannian distance balls $B_\varepsilon(p)$ of small radius $\varepsilon$ about a point $p \in M$, because

$$\frac{\text{vol}_g(B_\varepsilon(p))}{v_n \varepsilon^n} = 1 - \frac{s(p)}{6(n + 2)} \varepsilon^2 + O(\varepsilon^4),$$

where

$$v_n = \frac{\pi^{n/2}}{(n/2)!} := \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}$$
is the volume of the unit ball in $\mathbb{R}^n$. Thus, in a region where $s > 0$, a tiny ball in the manifold has smaller volume than it would in Euclidean space, while in a region where $s < 0$, tiny balls instead have larger volumes than they would in Euclidean space. Still, because the scalar curvature does not similarly control the volume of moderately large balls in dimension $n \geq 3$, this synthetic interpretation does not immediately allow one to prove much of anything, and the search for a deeper synthetic understanding of the scalar curvature therefore remains an active area of current research. Fortunately, however, the study of geometric partial differential equations provides an alternative source of insights into the global differential-topological meaning of the scalar curvature, and this article will largely focus on results that have come to light by dint of such means.

One of the most surprising lessons these methods have taught us is that the global behavior of the scalar curvature in dimension four is wildly different from what occurs in other dimensions. As a simple illustration, given a smooth compact manifold $M^n$, we might ask how small we can make the $L^p$ norm of the scalar curvature, among all possible metrics on $M$. For most values of $p$, this is of course a silly question, because one can usually make the $L^p$ norm of the scalar curvature tend to zero by simply choosing a suitable sequence of constant rescalings $cg$ of some fixed metric $g$; the question is therefore only interesting when $p = n/2$, as this is the unique value of $p$ for which the $L^p$-norm of the scalar curvature is actually scale-invariant. We are thus led to consider the scale-invariant, non-negative Riemannian functional

$$\mathcal{S}(g) = \int_M |s_g|^{n/2} d\mu_g = \|s_g\|_{L^{n/2}}^{n/2},$$  \hspace{1cm} (1)$$

along with the associated real-valued diffeomorphism invariant given by

$$\mathcal{I}_s(M^n) = \inf_g \mathcal{S}(g),$$  \hspace{1cm} (2)$$

where the infimum is taken over all smooth Riemannian metrics $g$ on $M$. Of course, the $n = 2$ case is highly atypical, because the classical Gauss-Bonnet theorem tells us that $\mathcal{I}_s(M^2) = 4\pi|\chi(M^2)|$, and that the infimum is achieved by any metric $g$ whose Gauss curvature $K = s/2$ doesn’t change sign. We will therefore restrict our attention henceforth to the setting of $n \geq 3$, where the problem takes on an entirely different character. In this range, the functional $\mathcal{S}(g)$ depends differentiably on $g$, and the critical

\[\text{In this article, the term manifold is construed in the restrictive sense, and so should always be understood to mean a manifold without boundary.}\]
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points of (1) are precisely the Einstein metrics, which is to say the metrics of constant Ricci curvature, and the scalar-flat metrics, meaning the metrics with $s \equiv 0$. Although this might make it seem tempting to look for Einstein metrics by trying to minimize the non-negative functional $\mathcal{G}$, such hopes are usually destined to be disappointed. Indeed, in dimensions other than four, the differential-topological invariant $I_s$ defined by (2) turns out to be trivial, at least for simply-connected manifolds:

**Theorem 1.** Let $M$ be a smooth compact manifold of dimension $n \geq 3$. If $M$ is simply-connected, and if $n \neq 4$, then $I_s(M) = 0$.

By contrast, however, the invariant $I_s$ is highly non-trivial in dimension 4:

**Theorem 2.** There are sequences $\{M_j\}$ of smooth compact simply connected 4-manifolds with $\lim_{j \to \infty} I_s(M_j) = +\infty$. Moreover, the manifolds $M_j$ may be chosen so that, for each $j$, the infimum (2) is achieved by an Einstein metric $g_j$ on $M_j$.

While this pair of results suffices to show that dimension four is sui generis when it comes to questions about the scalar curvature, we will soon see that the invariant $I_s(M)$ is merely a pale shadow of the Yamabe invariant $Y(M)$, which requires more work to define, but which often encodes significantly more geometrical information about a manifold. After defining $Y(M)$ in §1 below, and explaining how it related to $I_s(M)$, we will immediately see why Theorem 1 is a direct consequence of results due to Gromov, Lawson, Stolz, Petean, and Perelman. By contrast, as will be explained in §2, Theorem 2 is proved by means of Seiberg-Witten theory, and fits into a larger story about 4-manifolds with negative Yamabe invariant. Dimension four is also the arena for unusual results about Yamabe invariants in the zero and positive cases, and the rather different methods used to prove such results are explored and explained in §3 and §4 respectively. In the process, we will encounter some of the unsolved mysteries of the subject. One can only hope that drawing attention to these questions will stimulate new activity along this beautiful interface between Riemannian geometry and differential topology.

1. Yamabe Constants and Yamabe Invariants

In this section, we describe the Yamabe invariant $Y(M) \in \mathbb{R}$ of a smooth compact $n$-manifold $M$, $n \geq 3$, and discuss some of its basic properties.
This differential-topological invariant quantitatively refines the question of whether a given manifold admits positive-scalar-curvature metrics, because

$$\mathcal{Y}(M) > 0 \iff M \text{ admits metrics with } s > 0.$$ (3)

On the other hand, we will also see that the Yamabe invariant also refines the invariant defined by (2), because

$$I_s(M) = \begin{cases} 0 & \text{if } \mathcal{Y}(M) \geq 0, \\ \frac{|\mathcal{Y}(M)|^{n/2}}{n} & \text{if } \mathcal{Y}(M) \leq 0. \end{cases}$$ (4)

The Yamabe invariant is a natural outgrowth of the study of the total scalar curvature, by which we mean the integral of the scalar curvature with respect to the Riemannian volume measure. In dimension 2, where the scalar curvature coincides with twice the Gauss curvature, the total scalar curvature just computes the Euler characteristic, because the classical Gauss-Bonnet theorem tells us that

$$\int_{M^2} s_g d\mu_g = 4\pi\chi(M^2)$$

for any Riemannian metric $g$ on a compact surface $M^2$. By contrast, when $n \geq 3$, the total scalar curvature is ridiculously far from being a topological invariant, because it is not even scale invariant; instead, under constant rescaling $g \rightsquigarrow cg$ of the metric, the total scalar curvature rescales by a factor of $c^{(n-2)/2}$. The standard way of remedying this last pathology is to divide by an appropriate power of the volume, which then results in the so-called normalized Einstein-Hilbert action

$$\mathcal{E}(M^n, g) = \frac{\int_{M^n} s_g d\mu_g}{(\int_{M^n} d\mu_g)^{1-\frac{n}{2}}}.$$ (5)

However, when $n \geq 3$, this action $\mathcal{E}$ still depends quite sensitively on the metric. Indeed, it turns out to neither be bounded above nor below, and its critical points turn out to exactly be the Einstein metrics [16 Theorem 4.21].

Although $\mathcal{E}$ has no lower bound when $n \geq 3$, Yamabe nonetheless discovered that its restriction to any conformal class of metrics is always bounded below. To see this, we set $p = \frac{2n}{(n-2)} > 2$, which is the unique real number so that a metric $\hat{g}$ conformally related to $g$ and its corresponding metric volume measure can be simultaneously expressed as

$$\hat{g} = u^{p-2} g \quad \text{and} \quad \hat{d\mu} = u^p d\mu.$$
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With this choice, Yamabe then discovered that the scalar curvature \( \hat{s} \) of the rescaled metric \( \hat{g} \) is given by the remarkably simple equation

\[
\hat{s} u^{p-1} = \left( (p+2)\Delta + s \right) u,
\]

where \( \Delta = d^* d = -\nabla \cdot \nabla \). Given this, it then follows that

\[
\mathcal{E}(\hat{g}) = \int_m \left( (p+2)|\nabla u|^2 + su^2 \right) \, d\mu
\]

and, by focusing on functions of the form \( u = 1 + tf \), where \( t \in \mathbb{R} \) and \( \int_M f \, d\mu = 0 \), one therefore easily sees that \( g \) is a critical point of \( \mathcal{E}|_g \) iff its scalar curvature \( s \) is constant; and since we could have just as easily chosen any other metric in the conformal class as our reference metric, it now immediately follows that \( \hat{g} \in [g] := \{ u^{p-2}g \} \) is a critical point of \( \mathcal{E}|_g \) iff \( \hat{s} \) is constant. In light of this, Yamabe therefore set out to prove that each conformal class \( \gamma = [g] \) contains a metric of constant scalar curvature by showing that minimizers of \( \mathcal{E}|_\gamma \) always exist. This claim is actually correct, and we therefore now call such minimizers Yamabe metrics. Moreover, Yamabe’s strategy for proving their existence was largely on target. Namely, he proved the existence of a unit-norm minimizing functions \( u_\epsilon > 0 \) for the modifications of (6) obtained by replacing \( p \) with \( p - \epsilon \), and then asserted that these \( u_\epsilon \) converge to a smooth positive function \( u \) as \( \epsilon \downarrow 0 \). However, while Yamabe’s posthumously-published proof\(^9\) hinged on the claim that the \( u_\epsilon \) are uniformly bounded in \( C^0 \), Neil Trudinger\(^9\) observed that this fails for metrics in the conformal class of the usual sectional-curvature (+1) metric \( g_0 \) on \( S^n \). Fortunately, Trudinger went on to show that Yamabe’s outline does indeed work whenever the Yamabe constant

\[
Y(M, [g]) := \inf_{\hat{g} \in [g]} \mathcal{E}(\hat{g})
\]

is non-positive, and then laid out a general approach to the more difficult positive case, based on trying to show that the \( u_\epsilon \) instead converge in a suitable Sobolev space. By carefully analyzing the differential-geometric meaning of the best Sobolev constant for \( L^2 \hookrightarrow L^p \), Aubin\(^11\) then discovered that Trudinger’s method actually works whenever

\[
Y(M^n, [g]) < \mathcal{E}(S^n, g_0),
\]

while also observing (see Figure\(^1\) that one always at least has

\[
Y(M^n, [g]) \leq \mathcal{E}(S^n, g_0) \tag{7}
\]

for any compact Riemannian manifold \( (M^n, g) \), \( n \geq 3 \). Having thereby reduced Yamabe’s problem to that of showing the equality case in (7) only
occurs when \((M^n, [g])\) is the standard \(n\)-sphere, Aubin went on to prove that this is automatically true except perhaps when \(n \leq 5\) or \([g]\) is locally conformally flat. Schoen then completed the proof of Yamabe’s claim by using the Schoen-Yau positive mass theorem to eliminate all the remaining cases.

![Fig. 1.](image)

**Fig. 1.** Because any Riemannian metric is nearly Euclidean on a sufficiently tiny scale, any compact Riemannian manifold has conformal rescalings that resemble a large, nearly spherical balloon with a tiny, topology-laden gondola attached. In any dimension \(n \geq 3\), this means that, for any \(g\) and any \(\epsilon > 0\), one can always construct conformal rescalings \(\tilde{g} = u^{p-2}g\) with \(\mathcal{E}(M^n, \tilde{g}) < \mathcal{E}(S^n, g_0) + \epsilon\).

An oral tradition alleges that Yamabe’s ultimate goal was to construct Einstein metrics by next maximizing \(Y(M^n, \gamma)\) over the set of conformal classes \(\gamma\), as illustrated in Figure 2. While this program is now definitely known to fail on many low-dimensional manifolds, Yamabe’s dream does at least give rise to a fascinating differential-topological invariant. Indeed, the **Yamabe invariant** of any smooth compact \(n\)-manifold \(M\), \(n \geq 3\), is defined as

\[
\mathcal{Y}(M) = \sup_{\gamma} Y(M, \gamma) = \sup_{\gamma} \inf_{g \in \gamma} \mathcal{E}(M, g),
\]

where the supremum is taken over all conformal classes \(\gamma = [g]\) of smooth
Riemannian metrics on $M$, and so automatically satisfies

$$\mathcal{Y}(M^n) \leq \mathcal{E}(S^n, g_0) = \mathcal{Y}(S^n)$$

by (7). This invariant was apparently introduced independently by Schoen and Kobayashi, who respectively called $\mathcal{Y}(M)$ the *sigma constant* and the *mu invariant* of $M$. While both of these alternative terminologies continue to have proponents, I personally feel that it is usually preferable to name objects after mathematicians rather than after commonly-used Greek letters. Nonetheless, the terminology adopted here still requires a bit of disambiguation, because one must be careful not to confuse Yamabe invariants (of smooth manifolds) with Yamabe constants (of specific conformal classes).

![Fig. 2:](image)

Fig. 2.: Yamabe apparently dreamt of finding Einstein metrics on compact $n$-manifolds, $n \geq 3$, by minimizing $\mathcal{E}$ over each conformal class, and then maximizing over conformal classes. While a direct implementation of this scheme is usually destined to fail, this idea nonetheless allows us to attach an important real number, called the *Yamabe invariant*, to every smooth compact manifold.

While the normalized Einstein-Hilbert functional $\mathcal{E}$ is technically simpler than the functional $\mathcal{S}$ of (1), the two are actually closely related. Indeed, applying the Hölder inequality to (5) immediately yields

$$[\mathcal{S}(g)]^{2/n} \geq \mathcal{E}(g)$$

for any metric $g$, with equality iff $s_g = \text{const} \geq 0$. On the other hand, if $g$ and $\hat{g} = u^{n-2}g$ are any two conformally related metrics, applying the same
Hölder inequality to (6), after first suppressing the $|\nabla u|^2$ term, implies that 
\[ \mathcal{E}(\tilde{g}) \geq -|\mathcal{S}(g)|^{2/n} \] 
with equality iff $u = \text{const}$ and $s_g = \text{const} \leq 0$. Combining (10) and (11), we therefore conclude that any Yamabe metric $g$ can alternatively be characterized as a minimizer of $\mathcal{S}$ in its conformal class $\gamma = [g]$.

Notice that the equality case of (11) not only implies that any metric $g$ with $s_g = \text{const} \leq 0$ is a Yamabe metric, but moreover shows that, up to constant rescaling, such a $g$ is actually the unique Yamabe metric in its conformal class. By contrast, however, a metric $g$ with $s_g = \text{const} > 0$ may often not be a Yamabe metric, and this nuisance can in practice make it quite difficult to calculate the exact value of $\mathcal{Y}(M)$ in the positive case. One of the few tools available to help address this issue is a result of Obata\textsuperscript{73} which implies that any Einstein metric $g$ is a Yamabe metric. Indeed, Obata’s theorem says that, except in the case of $(S^n, [g_0])$, any Einstein metric $g$ is, modulo constant rescalings, the only constant-scalar-curvature metric, and hence the unique Yamabe metric, in its conformal class.

Let us now prove assertions (3) and (4). To show (3), first notice that the functional (6) is manifestly positive on the conformal class $[g] = \{u^{-2}g\}$ of any metric $g$ with $s_g > 0$. The existence of Yamabe minimizers therefore tells us, in particular, that $\mathcal{Y}(M, [g]) > 0$ iff and only if $[g]$ contains a metric $\tilde{g}$ with $s_{\tilde{g}} > 0$. It thus follows that $\mathcal{Y}(M) > 0$ iff $M$ carries a metric $g$ of positive scalar curvature, which is exactly statement (3).

To prove (4), it is helpful to first recall the standard fact that $\mathcal{E}$ is unbounded below on any compact $n$-manifold $M$, $n \geq 3$. A particularly simple proof of this fact goes as follows. Let $f : \mathbb{R}^n \to \mathbb{R}$ be any smooth function that is supported in the ball of radius $1/2$, and then observe that the metric 
\[ g_f := e^f (dx^1)^2 + e^{-f} (dx^2)^2 + (dx^3)^2 + \cdots + (dx^n)^2 \]
has the same volume form as the Euclidean metric, but satisfies 
\[ \int_{\mathbb{R}^n} s_{g_f} \, d\mu_{g_f} = -\frac{1}{2} \int_{\mathbb{R}^n} |d^2 f|^2 \, d\mu_{g_f}, \] 
(12)

\textsuperscript{8}For example, one can prove (12) by using O’Neill’s formulas to show that $s_{g_f}$ is everywhere equal to $-\frac{1}{2} |df|^2$ plus the scalar curvature of the relevant fiber of the Riemannian submersion $(x^1, x^2, x^3, \cdots, x^n) \mapsto (x^3, \cdots, x^n)$. Indeed, (16) equation (9.37) simplifies dramatically in our case, because the fibers of the submersion are minimal and orthogonal to a flat transverse foliation. Using classical Gauss-Bonnet to integrate out the scalar curvature of the fibers then proves (12).

Two other proofs involve viewing the restriction of $g_f$ to $[-\frac{1}{2}, \frac{1}{2}]^n$ as defining a smooth metric $\tilde{g}_f$ on the $n$-torus $\mathbb{R}^n/\mathbb{Z}^n$. For example, after multiplying by $\mathbb{R}/\mathbb{Z}$ if necessary, one
where \( d^4 f := \sum_{j=3}^{n} \frac{\partial f}{\partial x_j} \, dx_j \). Thus, if we start with an otherwise arbitrary metric \( h \) on \( M^n \) that contains a Euclidean unit ball, and then replace \( h \) on this ball by \( g_f \), the total volume will remain unchanged, while the total scalar curvature will be reduced by an arbitrarily large amount provided we take \( f \) to be sufficiently oscillatory. Since \( \mathcal{Y}(g) \geq Y(M, [g]) \) for any metric \( g \), this immediately implies that any given \( M^n, n \geq 3 \), admits sequences of conformal classes \([g_j]\) with \( Y(M, [g_j]) \to -\infty \).

If \( \mathcal{Y}(M) > 0 \), it therefore follows that \( M \) admits a pair of conformal classes \([h_+]\) and \([h_-]\) such that \( Y(M, [h_+]) > 0 \) and \( Y(M, [h_-]) < 0 \). On the other hand it is not difficult to show, by direct inspection of (6), that \( Y(M) \) is a continuous function of \( g \) in the \( C^2 \) topology. Since the space of metrics is connected, the intermediate value theorem therefore implies the existence of conformal classes \([h_0]\) on \( M \) with \( Y(M, [h_0]) = 0 \), and hence of Yamabe metrics \( h_0 \) on \( M \) with \( s \equiv 0 \). Hence \( \mathcal{L}_s(M) = 0 \) whenever \( \mathcal{Y}(M) > 0 \). On the other hand, (11) implies that \( \inf_{g \in \gamma} \mathcal{Y}(g) = [\mathcal{Y}(M, \gamma)]^{n/2} \) whenever \( Y(M, \gamma) \leq 0 \). Assertion (4) now follows from these two observations.

Since we have now observed that a smooth compact manifold \( M \) of dimension \( n \geq 3 \) has \( \mathcal{Y}(M) > 0 \) iff \( M \) admits some metric \( g \) of positive scalar curvature, let us next recall that not every such manifold \( M \) has this property. Indeed, the first obstruction to the existence of positive-scalar-curvature metrics on compact manifolds was discovered by Lichnerowicz, who observed that the Dirac operator \( \not\partial : \Gamma(S) \to \Gamma(S) \) on a Riemannian spin manifold satisfies the so-called Weitzenböck formula

\[
\not\partial^2 = \nabla^* \nabla + \frac{s}{4},
\]

and therefore has trivial kernel and co-kernel if the scalar curvature \( s \) is everywhere positive. However, when \( n \equiv 0 \mod 4 \), the full spinor bundle decomposes as a Whitney sum \( S = S_+ \oplus S_- \) of the so-called chiral spinor bundles, and the Dirac operator correspondingly decomposes as \( \not\partial = \not\partial_+ \oplus \not\partial_- \), where the chiral Dirac operator

\[
\not\partial : \Gamma(S_+) \to \Gamma(S_-)
\]

obtains an almost-Kähler metric on an even-dimensional torus \( \mathbb{T}^{2m} \) that is adapted to the standard symplectic form \( \omega = dx^1 \wedge dx^2 + \cdots + dx^{2m-1} \wedge dx^{2m} \), and Blair’s formula for the total Hermitian scalar curvature then yields a conceptual proof of (12). Alternatively, one may instead apply the second variation formula inductively to the minimal hypersurfaces \( T_m \subset \mathbb{T}^{m+1} \), \( m = 2, \ldots, n-1 \), for the variation \( s \equiv 1 \) with \( \mathcal{Y}' = 0 \), while integrating out the auxiliary variables at each stage of the induction.
is an elliptic operator whose index $\tilde{A}(M)$ had previously been shown by Atiyah and Singer\cite{Atiyah-Singer} to be a specific linear combination of Pontryagin numbers, and thus in particular a cobordism invariant. This allowed Lichnerowicz to prove that a smooth compact spin-manifold $M^{4k}$ cannot admit metrics of positive scalar curvature if $\tilde{A}(M) \neq 0$. In his thesis, Hitchin\cite{Hitchin} then generalized Lichnerowicz’s result by noticing that \cite{Lichnerowicz} also gives an obstruction to the existence of positive-scalar-curvature metrics on spin manifolds of dimension $n \equiv 1$ or $2 \mod 8$. Indeed, in these dimensions there is, for each spin structure on a smooth compact manifold, a $\mathbb{Z}_2$-valued invariant $\alpha$ given by $\dim \ker \partial \mod 2$ when $n \equiv 1 \mod 8$, or by $\dim \ker D \mod 2$ when $n \equiv 2 \mod 8$. Since this element of $\mathbb{Z}_2$ is independent of the choice of a Riemannian metric $g$ on $M$, Hitchin was therefore able to prove that a necessary condition for the existence of a positive-scalar-curvature metric is that $\alpha$ must vanish for every spin structure. When $M$ is simply connected, it can then have at most one spin structure, so this discussion then only involves an invariant $\alpha(M) \in \mathbb{Z}_2$ of any smooth compact simply connected manifold of dimension $n \equiv 1$ or $2 \mod 8$ with $w_2 = 0$. To keep the notation as simple as possible, one extends the definition of $\alpha(M)$ to smooth compact spin manifolds $M$ of other dimensions $n$ by setting $\alpha(M^n) := \tilde{A}(M) \in \mathbb{Z}$ if $n \equiv 0 \mod 4$ and $\alpha(M^n) := 0$ if $n \equiv 3, 5, 6$ or $7 \mod 8$. Hitchin’s generalization of Lichnerowicz’s theorem then tells us that a simply connected spin manifold $M$ cannot admit a metric of positive scalar curvature if $\alpha(M) \neq 0$. Remarkably, $\alpha(M)$ is actually invariant under spin cobordisms, and so only depends on the spin-cobordism class $[M] \in \Omega_n^{\text{Spin}}$ of the $n$-manifold $M$.

The role of spin cobordism in this story means that the obstruction $\alpha(M)$ is invariant under elementary surgeries in a suitable range of dimensions. Conversely, Gromov-Lawson\cite{Gromov-Lawson} and Schoen-Yau\cite{Schoen-Yau} independently proved that the existence of a positive-scalar-curvature metric on $M$ is invariant under elementary surgeries in codimension $\geq 3$. Using this, Gromov and Lawson then went on to show that every compact simply-connected non-spin manifold of dimension $\geq 5$ admits metrics of positive scalar curvature, just by showing that every such manifold is obtained by a sequence of such surgeries on products and disjoint unions of specific positive-scalar-curvature generators of the oriented cobordism ring $\Omega^{SO}$. For simply connected spin manifolds, they also conjectured that Hitchin’s obstruction $\alpha : \Omega_n^{\text{Spin}} \to KO^{-n} (pt)$ was the only obstruction to the existence of positive-scalar-curvature metrics, and observed that this would follow from their surgery result if one could show that $\ker \alpha \subset \Omega^{\text{Spin}}$ was generated by spin
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manifolds of positive scalar curvature. Stolz then proved this conjecture by showing that every cobordism class in $\ker \alpha$ can actually be represented by the total space of an $\mathbb{HP}_2$-bundle over spin a manifold. Consequently, every simply connected $n$-manifold $M$, $n \geq 5$, satisfies exactly one of the following:

- either $\mathcal{Y}(M) > 0$; or else
- $M$ is a spin manifold with $\alpha(M) \neq 0$.

It was Petean who first seriously considered the results of Gromov, Lawson, and Stolz from the perspective of the Yamabe invariant, and realized that more could be proved in this context. First, Petean showed that the Gromov-Lawson surgery arguments also imply that, for any $\varepsilon > 0$, the condition $\mathcal{Y}(M) > -\varepsilon$ is preserved under elementary surgeries in codimension $\geq 3$. Second, Petean discovered that adjoining a well-chosen collection of Ricci-flat manifolds of special holonomy to Stolz’s $\mathbb{HP}_2$-bundles shows that the spin-cobordism ring $\Omega^{\text{Spin}}$ is generated by manifolds with non-negative Yamabe invariant. Putting these facts together, he obtained

**Theorem 3 (Petean).** Any compact simply-connected $n$-manifold $M$, $n \geq 5$, has Yamabe invariant $\mathcal{Y}(M) \geq 0$. Moreover, such a manifold has $\mathcal{Y}(M) = 0$ iff $M$ is a spin manifold with $\alpha(M) \neq 0$.

On the other hand, since the 3-dimensional Poincaré conjecture follows from Perelman’s proof of Thurston’s geometrization conjecture, any simply-connected compact 3-manifold $M^3$ is necessarily diffeomorphic to $S^3$, and therefore has $\mathcal{Y}(M) > 0$. In conjunction with Theorem 3 and assertion (4), this then immediately implies Theorem 1.

For simply connected manifolds of dimension $n \neq 4$, Theorem 3 provides a complete understanding of the sign of the Yamabe invariant, but usually says nothing at all about its precise value. On the other hand, (9) gives us a universal upper bound, while Obata’s theorem provides a non-trivial lower bound for $\mathcal{Y}(M)$ whenever $M$ admits an Einstein metric of positive scalar curvature. In conjunction with Kobayashi’s inequality $\mathcal{Y}(M^n) \geq 0$, $\mathcal{Y}(N^n) \geq 0 \implies \mathcal{Y}(M \# N) \geq \min[\mathcal{Y}(M), \mathcal{Y}(N)]$ (14), this confines the Yamabe invariants of many manifolds to specific (and often reasonably narrow) ranges. Of course, (14) is a statement about surgery in codimension $n$, but generalizing it to surgeries in other codimension involves extra complications. In this direction, the best available analogue of the Gromov-Lawson-Petean surgery result is a theorem of Ammann,
Dahl, and Humbert which says that, for every \( n \), there is a constant \( \Lambda_n > 0 \) such that, whenever \( \varepsilon \leq \Lambda_n \), the condition \( \mathcal{Y}(M^n) > \varepsilon \) is invariant under elementary surgeries in codimension \( \geq 3 \). One consequence is the following gap theorem: for each \( n \geq 5 \), there is a \( \delta_n > 0 \) such that every compact simply-connected \( n \)-manifold \( M \) with \( \mathcal{Y}(M) > 0 \) actually satisfies \( \mathcal{Y}(M^n) > \delta_n \).

It must however be emphasized that simple-connectivity plays a key role in the proofs of Theorems 1 and 3, and that the story is known to become far more complicated when the fundamental group is non-trivial. For example, Perelman’s geometrization of 3-manifolds involves monotonicity results for the Ricci-flow that allow one to compute the Yamabe invariants of many 3-manifolds, owing to the intimate relationship between Perelman’s \( \bar{\lambda} \) invariant and the Yamabe invariant. Indeed, Anderson showed that Perelman’s results imply that the hyperbolic metric \( h_0 \) on any compact quotient \( M^3 = H^3/\Gamma \) of hyperbolic 3-space realizes the Yamabe invariant, in the sense that \( \mathcal{Y}(M) = \mathcal{Y}(M, h_0) \); thus, in particular, \( \mathcal{Y}(M) < 0 \) and \( \mathcal{Z}_0(M) > 0 \) for any hyperbolic 3-manifold \( M \). On the other hand, geometrization also tells us that that any compact 3-manifold \( M^3 \) with \( |\pi_1(M^3)| < \infty \) is a spherical space-form \( S^3/\Gamma, \Gamma \subset SO(4) \), and therefore has \( \mathcal{Y}(M) > 0 \). In this context, an argument due to Bray and Neves shows that the Yamabe invariant of \( \mathbb{RP}^3 = S^3/\mathbb{Z}_2 \) is actually realized by the constant-curvature metric, so that \( \mathcal{Y}(\mathbb{RP}^3) = \mathcal{Y}(S^3)/2^{2/3} = 6\pi^{4/3} \). While this method unfortunately does not allow one to similarly compute the Yamabe invariant for other spherical space-forms, it does at least yield a beautiful gap theorem: if \( \mathcal{Y}(M^3) < \mathcal{Y}(S^3) \), then \( \mathcal{Y}(M^3) \leq \mathcal{Y}(\mathbb{RP}^3) \).

Related issues persist in higher dimensions. While Gromov, Lawson, and Stolz showed, for \( n \geq 5 \), that the Dirac operator is the source of the only obstruction to a simply-connected \( n \)-manifold \( M \) having \( \mathcal{Y}(M^n) > 0 \), this precept no longer holds true when \( \pi_1(M) \neq 0 \). Perhaps the first indication that something else might be at play was provided by the discovery that the \( n \)-torus \( \mathbb{T}^n \) does not admit metrics of positive scalar curvature, even though its Dirac operator has index zero. Indeed, notice that, by multiplying by \( S^1 \) if necessary, it suffices to prove it when \( n = 2m \) is even. Now, while \( \alpha(\mathbb{T}^{2m}) = 0 \), there are nonetheless spin\(^\ast\) Dirac operators of non-zero index on any covering \( \mathbb{T}^{2m} \to \mathbb{T}^{2m} \), and that one can arrange for the curvature of the line bundle to which the spin\(^\ast\) Dirac operator is coupled to be the pull-back of an arbitrarily small multiple of the standard symplectic form simply by choosing a suitable covering of sufficiently high degree; thus, a variation on the Lichnerowicz argument therefore shows that \( \mathbb{T}^{2m} \)
cannot admit a metric of scalar curvature $s \geq \varepsilon$ for any $\varepsilon > 0$. However, versions of this argument works equally well if we instead couple the Dirac operator to vector bundles of higher rank, just as long as we can ensure that of the coupled operator has non-zero index, and that the bundle curvature can be chosen to converge point-wise to zero as we pass to covers of higher and higher degree. By systematizing this idea, Gromov and Lawson\cite{GromovLawson1983} were able to more generally prove the non-existence of positive-scalar-curvature metrics on enlargeable manifolds. Here an $n$-manifold $M$ is said to be enlargeable if for some Riemannian metric $g$ and every $r > 0$, there is a finite spin covering $\tilde{M} \to M$ that, when equipped with pull-back $\tilde{g}$ of $g$, admits a distance-decreasing map of positive degree to the standard $n$-sphere of radius $r$. This condition turns out to be to be metric-independent, and moreover depends only on the homotopy-type of $M$. While we will find these results on enlargeable manifolds very useful in $\S3$, an inherent limitation of the method is that it only applies to manifolds whose universal covers are spin.

However, around the same time, Schoen and Yau\cite{SchoenYau1979} discovered an entirely different method that, for example, also proves the non-existence of positive-scalar-curvature metrics on manifolds like $T^5 \# [SU(3)/SO(3)]$, where the universal cover has $W_3 \neq 0$, and so is not even spin$^c$. Since will also make essential use of this technique in $\S3$ we will now carefully review the Schoen-Yau method in the context of manifolds of dimension $n \leq 7$.

For recent progress on extending these arguments to higher dimensions, see Schoen and Yau’s recent preprint\cite{SchoenYau2021}.

The Schoen-Yau method depends in part on Jim Simons’ second-variation formula for a minimal hypersurface. Let $(\Sigma^m, h) \subset (M^{m+1}, g)$, $m \geq 2$, be a compact oriented minimal hypersurface in an oriented Riemannian manifold, and let $\Sigma \subset M$ be any smooth 1-parameter variation of $\Sigma = \Sigma_0$ with with normal variation vector vector field $v = \nabla_u n$, where $n$ is the unit normal vector of $\Sigma$. The second-variation formula\cite{Simons1968} Theorem 3.2.2] then asserts that the $m$-dimensional volume $\mathcal{A}(t)$ of $\Sigma_t$ satisfies

$$
\mathcal{A}''(0) = \int_{\Sigma} \left[ |\nabla u|^2 - r_g(n,n)u^2 - |\Pi|^2 u^2 \right] d\mu_h
$$

where $r_g$ is the Ricci tensor of the ambient metric and $\Pi$ is the second fundamental form of $\Sigma \subset M$. However, the Gauss-Codazzi equations imply that the scalar curvatures of $h$ and $g$ are related along $\Sigma$ by

$$
s_h = s_g - 2r_g(n,n) + H^2 - |\Pi|^2,
$$
where the mean curvature $H = h^{ij} \mathbb{I}_{ij}$ of $\Sigma$ vanishes in our case because $\Sigma$ is assumed to be minimal. Consequently, (15) can be rewritten as
\[
\int_{\Sigma} \left[ 2|\nabla u|^2 + s_h u^2 \right] d\mu_h = 2s''(0) + \int_{\Sigma} (s_g + |\mathbb{I}|^2) u^2 d\mu_h
\] (16)
for every 1-parameter variation of $\Sigma$.

If we now assume that $(M, g)$ has positive scalar curvature $s_g > 0$ and that $\Sigma \subset M$ is volume-minimizing in its homology class, it then follows that $\Sigma$ carries a positive-scalar-curvature metric $\hat{h}$ conformal to $h$. Indeed, since our volume-minimizing hypothesis on $\Sigma$ forces $A''(0) \geq 0$ for any 1-parameter variation, plugging the positivity of $s_g$ into (16) now forces
\[
\int_{\Sigma} \left[ 2|\nabla u|^2 + s_h u^2 \right] d\mu_h > 0
\] (17)
for any smooth function $u \not\equiv 0$. If $m \geq 3$, now let $\hat{h} = u^{p-2} h$ be a Yamabe metric, where $p = \frac{2m}{m-2} > 2$, and notice that (17) then tells us that
\[
Y(\Sigma, [h]) = \frac{\int_{\Sigma} \left( (p+2)|\nabla u|^2 + s_h u^2 \right) d\mu_h}{||u||_{L^p}^2} > 0,
\]
which shows that $(\Sigma, \hat{h})$ has positive scalar curvature. On the other hand, if $m = 2$, setting $u \equiv 1$ in (17) yields $\chi(\Sigma) > 0$ by Gauss-Bonnet, so one therefore has $(\Sigma, [h]) \cong (S^2, [g_0])$ by classical uniformization.

The Schoen-Yau strategy now proceeds by downward induction on the dimension of the manifold. Suppose a smooth compact oriented $n$-manifold, $n \leq 7$, admits a metric of positive scalar curvature, and let $a \in H^1(M, \mathbb{Z})$ be a non-trivial cohomology class. Compactness results in geometric measure theory [33, §5.1.6] guarantee that there is a mass-minimizing rectifiable current that represents the Poincaré dual homology class $a \in H_{n-1}(M, \mathbb{Z})$, and our assumption that $n \leq 7$ then guarantees, by a regularity result that Federer [33, Theorem 5.4.15] deduced from a lemma of Simons [85, Lemma 6.1.7], that this current is moreover a sum of disjoint smooth compact oriented hypersurfaces, with positive integer multiplicities. Any one of these hypersurfaces $\Sigma^{n-1} \subset M^n$ then admits a metric of positive scalar curvature by the above argument, and one may then try to repeat the same argument to further reduce the dimension.

For example, if there is a map $M^n \to \mathbb{T}^n$ of non-zero degree, pulling back the generators of $H^1(\mathbb{T}^n, \mathbb{Z})$ yields a collection of classes $a_1, \ldots, a_n \in H^1(M, \mathbb{Z})$ with $a_1 \cup \cdots \cup a_n \neq 0$. We now apply the above argument to $a = a_n$, and then choose the hypersurface $\Sigma^{n-1}$ so that $\langle a_1 \cup \cdots \cup a_{n-1}, [\Sigma^{n-1}] \rangle \neq 0$. Iterating the same argument with respect to conformal
rescalings of the induced metrics then produces a compact surface $\Sigma^2 \subset \Sigma^3 \subset \cdots \subset \Sigma^{n-1}$ with positive scalar curvature and $\langle a_1 \cup a_2, [\Sigma^2] \rangle \neq 0$, contradicting the fact that $\chi(\Sigma^2) > 0$ implies $b_1(\Sigma^2) = 0$. Thus, whenever a smooth compact oriented $n$-manifold $M$, $n \leq 7$, admits a map $M \to \mathbb{T}^n$ of non-zero degree, it necessarily satisfies $\mathcal{Y}(M) \leq 0$. As a corollary, we thus deduce that examples like the previously-mentioned connected sum $M^5 = \mathbb{T}^5 \# [\text{SU}(3)/\text{SO}(3)]$ cannot admit metrics of positive scalar curvature.

While the above discussion emphasizes the use of homologically volume-minimizing hypersurfaces, the Schoen-Yau method is also applicable in contexts where one can prove the existence of stable minimal hypersurfaces for other reasons. As we will also see in §3, the inductive step in the Schoen-Yau approach can also be applied in substantially different ways. Variations on this approach thus allow one to prove the non-positivity of the Yamabe invariant for many interesting manifolds with large fundamental group that do not neatly conform to the simple paradigm we’ve just described.

2. Dimension Four: Yamabe Negative Case

Having carefully discussed the behavior of the Yamabe invariant in other dimensions, we now shift our focus to dimension four, where we will see that the relationship between scalar curvature and differential topology is strangely different.

The peculiar character of 4-dimensional geometry is largely ascribable to a single Lie-theoretic fluke: while the rotation group $\text{SO}(n)$ is a simple Lie group for every other $n \geq 3$, this fails in dimension four. Indeed, since $\text{Spin}(4) = \text{Sp}(1) \times \text{Sp}(1) \cong \text{Spin}(3) \times \text{Spin}(3)$, the adjoint action of $\text{SO}(4)$ on $\mathfrak{so}(4)$ is consequently reducible:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3), \quad (18)$$

This has an immediate and powerful impact on the geometry of 2-forms, because, for any $n$, $\mathfrak{so}(n)$ and $\Lambda^2(\mathbb{R}^n)$ are isomorphic as $\text{SO}(n)$-modules. Thus, the decomposition $(18)$ implies that the rank-6 bundle of 2-forms on an oriented Riemannian 4-manifold $(M,g)$ invariantly decomposes as the Whitney sum of two rank-3 bundles

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^- , \quad (19)$$

a phenomenon without analogue in other dimensions. Here the summands $\Lambda^\pm$ just turn out to be the the $(\pm 1)$-eigenspaces of the Hodge star operator

$$*: \Lambda^2 \to \Lambda^2, \quad (20)$$
and, for this reason, $\Lambda^+$ is called the bundle of self-dual 2-forms, while $\Lambda^-$ is called the bundle of anti-self-dual 2-forms. Of course, the distinction between the two depends on a choice of orientation; reversing the orientation of $M$ simply interchanges $\Lambda^+$ and $\Lambda^-$.  

On any oriented Riemannian 4-manifold $(M,g)$, the bundle $\Lambda^+ \to M$ carries a natural inner product and orientation, so every fiber of its unit sphere bundle $Z = S(\Lambda^+)$ carries both a metric and orientation. This allows us to consider the so-called twistor space $Z$ as a bundle of complex projective lines $\mathbb{CP}^1$.Remarkably, this $\mathbb{CP}^1$-bundle can always be realized as the projectivization of a rank-2 complex vector bundle $V^+ \to M$. What’s more, the choice of such a $V^+$ is equivalent to choosing a spin$^c$ structure on $M$. This stems from the fact that $Z$ can naturally be expressed as $S(\Lambda^+) = F/\mathbb{U}(2)$, where $F$ is the principal $\text{SO}(4)$-bundle of oriented orthonormal frames.

Indeed, according to the usual definition, a spin$^c$ structure on $(M,g)$ is a choice of principal $\text{Spin}^c(4)$-bundle $\widehat{F} \to M$, where $\text{Spin}^c(4) := [\text{Spin}(4) \times \text{U}(1)]/\langle (-1, -1, -1) \rangle$, together with a fixed isomorphism $\widehat{F} \to F/\mathbb{U}(2)$. Up to isomorphism, such a structures is determined by the Chern class $\widehat{c} \in H^2(\widehat{F}, \mathbb{Z})$ of the circle bundle $\widehat{F} \to F$, and $\widehat{c}$ can in principle be any element of $H^2(\widehat{F}, \mathbb{Z})$ whose restriction to a fiber yields the non-trivial element of $H^2(\text{SO}(4), \mathbb{Z}) \cong \mathbb{Z}_2$. On the other hand, expressing $Z$ as $\mathbb{P}(V^+)$ gives rise to an $\mathbb{O}(1)$ line-bundle $L \to Z$, and so gives us a cohomology class $c = c_1(L) \in H^2(Z, \mathbb{Z})$ that satisfies $\langle c, [S^2] \rangle = 1$, where $[S^2] \in H_2(Z, \mathbb{Z})$ is the the fiber homology class. This allows us to associate a unique spin$^c$ structure to any choice of $V^+$ by setting $\widehat{c} = q^*c$, where $q : \widehat{F} \to F/\mathbb{U}(2)$ is the quotient map. Conversely, one can construct $V^+$ from a principal $\text{Spin}^c(4)$-bundle $\widehat{F} \to M$ by applying the associated bundle construction to the representation of $\text{Spin}^c(4)$ on $\mathbb{C}^2$ $[\text{Sp}(1) \times \text{Sp}(1) \times \text{U}(1)]/\langle (-1, -1, -1) \rangle \to [\text{Sp}(1) \times \text{U}(1)]/\langle (-1, -1) \rangle = \mathbb{U}(2)$ gotten by dropping the second $\text{Sp}(1)$ factor. Of course, dropping the first $\text{Sp}(1)$ factor instead gives us a second rank-2 complex vector bundle $V_-$ with $\mathbb{P}(V_-) = S(\Lambda^-)$. 
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The relationship between these two representations then guarantees that

$$\text{Hom}(\mathbb{V}_+, \mathbb{V}_-) = \mathbb{C} \otimes T^* M,$$  \hspace{1cm} (23)

while the Hermitian line-bundle $L$ associated to the representation

$$[\text{Sp}(1) \times \text{Sp}(1) \times \text{U}(1)]/((-1, -1, -1)) \rightarrow \text{U}(1)/(-1) = \text{U}(1)$$

automatically satisfies

$$L := \wedge^2 \mathbb{V}_+ = \wedge^2 \mathbb{V}_-.$$ \hspace{1cm} (24)

This in particular makes $\mathcal{F}$ into a double cover of the fiber-wise product $\mathcal{F} \otimes S(L)$ of the oriented Riemannian frame-bundle and the unitary frames for $L$, so any $\text{U}(1)$ connection $\theta$ on $L$ induces a uniquely-defined principal $\text{Spin}^c(4)$-connection on $\mathcal{F}$, and this in turn induces Hermitian connections $\nabla_\theta$ on the bundles $\mathbb{V}_\pm$. From these, one can of course recover the Riemannian connection on $TM$ via the isomorphism (23).

Now, using the Gysin sequence and Poincaré duality, the Euler class $e(\Lambda^+)$ $\in H^3(M, \mathbb{Z})$ can be shown to vanish for any compact oriented Riemannian 4-manifold $(M, g)$, because a twistor-lift construction allows one to show that the torsion subgroup $\mathcal{T}_2(Z) \subset H_2(Z, \mathbb{Z})$ surjects onto the torsion subgroup $\mathcal{T}_2(M) \subset H_2(M, \mathbb{Z})$ under the twistor projection $Z \rightarrow M$, and that the torsion subgroup $\mathcal{T}^3(M) \subset H^3(M, \mathbb{Z})$ therefore injects into $\mathcal{T}^3(Z) \subset H^3(Z, \mathbb{Z})$. This is of course in perfect agreement with the conventional approach\textsuperscript{49},\textsuperscript{57} to the existence of spin$^c$ structures, because $e(\Lambda^+)$ can also be shown \textit{a priori} to coincide with the third integral Stiefel-Whitney class $W_3(TM)$ $\in H^3(M, \mathbb{Z})$, and the latter can in turn be shown to vanish for any oriented 4-manifold by an argument due to Hirzebruch and Hopf\textsuperscript{48} Thinking of a spin$^c$ structure as an element $c$ $\in H^2(Z, \mathbb{Z})$ with fiber integral +1 also gives such structures a metric-independent meaning, because the 2-sphere bundles associated with any two metrics are naturally bundle-equivalent. Moreover, the Gysin sequence of $\Lambda^+ \rightarrow M$ implies that $H^2(M, \mathbb{Z})$ acts freely and transitively on the set of spin$^c$ structures by pull-back; and, in terms of the vector bundles $\mathbb{V}_\pm$ $\rightarrow M$, the effect of this is that

$$\mathbb{V}_+ \sim E \otimes \mathbb{V}_+, \quad \mathbb{V}_- \sim E \otimes \mathbb{V}_-, \quad \text{as } E \text{ ranges over all complex line-bundles } E \rightarrow M.$$ 

One consequence of (22) is that $Z$ can be canonically identified\textsuperscript{9} with the set of all metric-compatible point-wise almost-complex structures that also determine the given orientation of $M$. Concretely, this amounts to the observation that if $J$ $\in \text{End}(T_p M)$, $J^2 = -I$, is an almost-complex
structure at \( p \) that preserves \( g \) and determines the fixed orientation, and if we then associate a tensor \( \omega \) with \( J \) via the prescription
\[
\omega(\cdot, \cdot) = g(J \cdot, \cdot),
\]
then \( \omega/\sqrt{2} \) is a unit-norm self-dual 2-form, and every unit-norm self-dual form at \( p \) conversely arises this way from a unique almost-complex structure \( J \) at \( p \). This gives rise to yet another way to understand spin\(^c\) structures. Indeed, for any given spin\(^c\) structure, the vector bundle \( V_+ \) has real rank equal to the dimension of \( M \), so one can find smooth sections of \( V_+ \) that are non-zero everywhere except at a chosen base-point \( q \in M \) for our connected oriented compact 4-manifold. Since this gives us a section of \( \mathbb{P}(V_+) \) on the complement of the base-point, equations (21) and (22) together give the punctured manifold \( M - \{q\} \) an associated almost-complex structure. On the other hand, it is a standard fact\(^{57}\) that an almost-complex structure on \( M - \{q\} \) determines a spin\(^c\) structure on \( M - \{q\} \), and it is also easy to see that any such spin\(^c\) structure on \( M - \{q\} \) uniquely extends to a spin\(^c\) structure on all of \( M \). These two constructions are actually inverses of each other, so the spin\(^c\) structure determined by the constructed almost-complex structure on the punctured manifold is exactly the one we started with. Thus, a spin\(^c\) structure on \( M \) may also be thought of as an equivalence class of almost-complex structures on the punctured manifold \( M - \{q\} \), where \( q \in M \) is an arbitrary base-point. Since (23) moreover allows us to identify the restriction of \( V_- \) to \( M - \{q\} \) with the \((1,0)\)-tangent bundle of \( J \), the line-bundle \( L = \det V_- \) restricted to \( M - \{q\} \) is actually the anti-canonical line bundle of the almost-complex structure \( J \). Since \( H^2(M, \mathbb{Z}) = H^2(M - \{q\}, \mathbb{Z}) \), this in particular shows that
\[
c_1(L) \equiv w_2(M) \mod 2. \tag{25}
\]
Also notice that the action of \( H^2(M, \mathbb{Z}) \) on spin\(^c\) structures is manifested in this context by
\[
L \sim L \otimes E^2,
\]
so that (25) is actually the only constraint on \( c_1(L) = c_1(V_+) \). Indeed, if (but only if) \( H_1(M, \mathbb{Z}) \) has trivial 2-torsion, spin\(^c\) structures are then in one-to-one correspondences with integer cohomology classes \( c_1(L) \) satisfying (25).

While the above discussion certainly shows that (21) has significant geometric consequences, the intimate relationship between the bundles \( V_+ \) and \( \Lambda^+ \) has a more vivid realization that is ultimately far more consequential. Indeed, (21) can be enriched into a natural real-quadratic map
\[
\sigma : V_+ \to \Lambda^+
\]
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that plays a key role in the theory of spin$^c$ Dirac operators. Indeed, since $V_+$ and $\Lambda^+$ are, respectively, the bundles associated with the defining and adjoint representations of $U(2) = \text{Spin}^c(4)/(1 \times \text{Sp}(1) \times 1)$, the relationship between these representations gives us a (Clifford multiplication) isomorphism

$$\text{End}_0(V_+) = \mathbb{C} \otimes \Lambda^+,$$

where $\text{End}_0$ indicates trace-free endomorphisms, and where complex conjugation in $\mathbb{C} \otimes \Lambda^+$ corresponds to $A \mapsto -A^*$ in $\text{End}_0(V_+)$. Thus, $\Lambda^+$ is identified with the skew-adjoint trace-free endomorphisms of $V_+$, while $i\Lambda^+$ is identified with the self-adjoint trace-free endomorphisms. For any $\Phi \in V_+$, we may thus uniquely define $\sigma(\Phi) \in \Lambda^+$ by declaring that the corresponding trace-free skew-adjoint map $V_+ \to V_+$ is to be given by

$$\sigma(\Phi) \cdot \Psi = i \left[ \langle \Phi, \Phi \rangle \Phi - \frac{1}{2} |\Phi|^2 \Psi \right].$$

With this convention, one then has

$$|\sigma(\Phi)| = \frac{|\Phi|^2}{2 \sqrt{2}}$$

for any $\Phi \in V_+$, and

$$\langle \omega \cdot \Phi, \Phi \rangle = 4i \langle \omega, \sigma(\Phi) \rangle$$

for any $\omega \in \Lambda^+$.

Now, given a spin$^c$ structure, we have already observed that every unitary connection $\theta$ on $L$, in conjunction with the Riemannian connection on $TM$, induces a unitary connection

$$\nabla_\theta : \Gamma(V_+) \to \Gamma(\Lambda^1 \otimes V_+)$$

on $V_+$. Composing this with the (Clifford multiplication) map

$$\Lambda^1 \otimes V_+ \to V_-$$

induced by (23) then gives us a spin$^c$ analogue

$$\rho_\theta : \Gamma(V_+) \to \Gamma(V_-)$$

of the chiral Dirac operator. In this setting, the Lichnerowicz Weitzenböck formula then generalizes as

$$\rho_\theta^* \rho_\theta = \nabla_\theta^2 + \frac{s}{4} - \frac{1}{2} F_\theta^+. $$
where the self-dual part $F^+_{\theta} \in i\Lambda^+$ of the curvature of $L$ acts on $\mathbb{V}_+$ via (26). In light of (28), we therefore have

$$\langle \Phi, \partial^*_\theta \partial_\theta \Phi \rangle = \frac{1}{2} \Delta |\Phi|^2 + |\nabla_\theta \Phi|^2 + \frac{s}{4} |\Phi|^2 + 2 \langle -i F^+_{\theta}, \sigma(\Phi) \rangle$$  \hspace{1cm} (29)$$

for any $\Phi \in \Gamma(\mathbb{V}_+)$. It is this last form of the spin$^c$ Weitzenböck formula that we will generally use in what follows.

On a spin manifold, the standard Dirac operator is entirely determined by the relevant Riemannian metric, and we previously saw in §1 that (13) then implies a non-existence result for positive-scalar-curvature metrics on spin manifolds $M$ with $\alpha(M) \neq 0$. However, this approach does not obviously generalize to the spin$^c$ setting, because the curvature of the connection $\theta$ on the line bundle $L$ appears in (29), and is in principle entirely independent of the Riemannian geometry of $(M,g)$. To overcome this, is therefore necessary to impose conditions on $\theta$ that somehow tie it more closely to the Riemannian geometry of $(M,g)$. One interesting choice, which we will discuss in detail in §4 below, is to simply demand that the curvature of $\theta$ be a harmonic 2-form. However, a more subtle condition, originally introduced by Witten\textsuperscript{97} instead requires that $\Phi$ and $\theta$ together solve a coupled system of PDE.

Given a spin$^c$ structure on a compact oriented Riemannian 4-manifold $(M,g)$, the Seiberg-Witten equations thus ask for the Hermitian connection $\theta$ on $L$ and the generalized spinor $\Phi \in \Gamma(\mathbb{V}_+)$ to satisfy the coupled equations

$$\partial_\theta \Phi = 0$$  \hspace{1cm} (30)$$

$$F^+_{\theta} = i\sigma(\Phi)$$  \hspace{1cm} (31)$$

and a solution $(\Phi, \theta)$ of this system is then called irreducible if $\Phi \neq 0$. However, (27) and (29) together imply that $\Phi$ satisfies

$$0 = 2\Delta |\Phi|^2 + 4|\nabla_\theta \Phi|^2 + s|\Phi|^2 + |\Phi|^4,$$  \hspace{1cm} (32)$$

so one immediately sees that the Seiberg-Witten equations cannot admit an irreducible solution relative to a metric $g$ with $s > 0$. But why would one ever expect for such solutions to exist? The Seiberg-Witten system is non-linear, so one certainly cannot merely rely on an index calculation to predict the existence of solutions. Instead, Witten had the remarkable insight that one can instead define a new invariant of a smooth compact oriented 4-manifold with fixed spin$^c$ structure by “counting” solutions of these equations, in a manner that can then be shown to be metric-independent.
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However, whenever the solution space of the Seiberg-Witten equations is non-empty, it is automatically infinite-dimensional, because the gauge group

\[ \mathcal{G} = \{ \text{smooth maps } f : M \to S^1 \subset \mathbb{C} \} \]

of circle-valued functions acts on solutions of (30–31) by

\[ (\Phi, \theta) \mapsto (f \Phi, \theta + 2d \log f), \]

thereby carrying solutions of (30–31) into new solutions that essentially just differ by automorphisms of \( L \), and so are geometrically really the same. Given a spin\(^c\) structure \( c \) on \( M \), we are thus led to consider, for each Riemannian metric \( g \), the Seiberg-Witten moduli space

\[ M_c(g) = \{ \text{solutions of } (30–31) \} / \mathcal{G}, \]

and this moduli space can always be shown to be compact. However, \( M_c(g) \) is not necessarily a manifold, for reasons we will need to overcome in order to define Witten’s invariant.

Fortunately, however, there is a simple remedy for this difficulty whenever \( b_+(M) \neq 0 \). Here \( b_+(M) \) is the oriented homotopy invariant of \( M \) that may be calculated by diagonalizing the intersection form

\[ \bullet : H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \longrightarrow \mathbb{R} \]

\[ ( [\varphi], [\psi] ) \mapsto \int_M \varphi \wedge \psi. \]

of \( M \) with real coefficients, and then counting

\[
\begin{bmatrix}
1 \\
\vdots \\
1 \\
b_+(M) \\
b_-(M) \\
-1 \\
\vdots \\
-1
\end{bmatrix}
\]

the number of positive directions of \( \bullet \) in the second cohomology; similarly, the number of negative directions is called \( b_-(M) \), and we automatically have \( b_+(M) + b_-(M) = b_2(M) \) because Poincaré duality guarantees that the intersection pairing \( \bullet \) is non-degenerate. This may be made more concrete by identifying \( H^2(M, \mathbb{R}) \) with the space of harmonic 2-forms

\[ \mathcal{H}^2_g = \{ \varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, \ d \ast \varphi = 0 \} \]
via the Hodge Theorem; since the Hodge star operator $\star$ defines an involution of the right-hand side, decomposing $\mathcal{H}_g^2$ into the $(\pm 1)$-eigenspaces of $\star$ then yields

$$\mathcal{H}_g^2 = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-,$$

(34)

where

$$\mathcal{H}_g^\pm = \{ \varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0 \}$$

is the space of closed (and hence harmonic) self-dual (respectively, anti-self-dual) 2-forms. We then have

$$b_\pm(M) = \dim \mathcal{H}_g^\pm,$$

because $\bullet$ may concretely be diagonalized, in the above manner, by choosing an $L^2$-orthonormal basis for $\mathcal{H}_g^+$ consisting of an $L^2$-orthonormal basis for $\mathcal{H}_g^+$, followed by an $L^2$-orthonormal basis for $\mathcal{H}_g^-$. With this in mind, we now generalize the Seiberg-Witten equations by replacing (31) with the “perturbed” equation

$$iF_\theta^+ + \sigma(\Phi) = \eta$$

(35)

for some self-dual 2-form $\eta \in \Gamma(\Lambda^+)$. If the harmonic part $\eta_H$ of $\eta$ satisfies

$$\eta_H \neq 2\pi [c_1(L)]^+,\quad (36)$$

where $[c_1(L)]^+$ is the image of $c_1(L)$ under the projection $H^2(M, \mathbb{R}) \to \mathcal{H}_g^+$ defined by (34), then any solution of (30) and (35) is irreducible, in the sense that $\Phi \neq 0$. The Smale-Sard theorem then allows one to show that, for a set of $\eta$ of the second Baire category, the moduli space

$$\mathcal{M}_c(g, \eta) = \{ \text{solutions of (30) and (35)} \}/G$$

is a compact manifold of dimension

$$\dim \mathcal{M}_c(g, \eta) = \frac{c_1^2(L) - (2\chi + 3\tau)(M)}{4},$$

(37)

where $\chi(M) = (2 - 2g_1 + b_2)(M)$ and $\tau(M) = (b_+ - b_-)(M)$ respectively denote the Euler characteristic and signature of our smooth compact oriented connected 4-manifold, and where $c_1^2(L) := c_1(L) \bullet c_1(L)$.

To prove this claim, one imposes the harmless gauge-fixing condition

$$d^*(\theta - \theta_0) = 0,$$

(38)

\footnote{When the dimension predicted by (37) is negative, this statement is defined to mean that $\mathcal{M}_c(g, \eta) = \emptyset$ for generic $\eta$.}
relative to some an arbitrarily chosen reference connection $\theta_0$ on $L$, and notices that this simply cuts down the action of the infinite-dimensional gauge group $\mathcal{G}$ to that of the 1-dimensional Lie group

$$\mathcal{G}_0 = \{\text{harmonic maps } f : M \to S^1\} = S^1 \times H^1(M, \mathbb{Z}).$$

This reduces the problem to understanding the fibers of the “monopole map”

$$L^2_k(V_+^\perp) \oplus L^2_k(V_+^\perp) \to L^2_{k-1}(V_-^\perp) \oplus L^2_{k-1}(\Lambda^1_{\Lambda}^+ \oplus L^2_{k-1}/\mathbb{R})$$

for any sufficiently large $k$. By ellipticity, this is a Fredholm map, and the index theorem, applied to the linearization, tells us that the Fredholm index of (39) equals the right-hand side of (37) plus one, where the +1 arises from our having modded out by the constant functions $\mathbb{R}$ in the codomain.

The moduli space $\mathcal{M}_c(g, \eta)$ is then the fiber over $(0, \eta, 0)$, modulo the action of $\mathcal{G}_0$. Because the linearization of $\mathcal{P}_0 \oplus d^*$ at an irreducible solution always maps surjectively onto $L^2_{k-1}(V_-^\perp) \oplus L^2_{k-1}/\mathbb{R}$, the Smale-Sard theorem then tells us that $(0, \eta, 0)$ is a regular value for generic $\eta$, and the corresponding fiber is therefore a smooth manifold by the implicit function theorem. On the other hand, compactness follows from the generalization

$$0 = 2\Delta |\Phi|^2 + 4|\nabla \Phi|^2 + s|\Phi|^2 + |\Phi|^4 - 8\langle \eta, \sigma(\Phi) \rangle$$

of (32) arising from (30) and (35), because this Weitzenböck formula implies that any irreducible solution satisfies the point-wise bound

$$|\Phi|^2 \leq \max(2\sqrt{2}|\eta| - s)$$

everywhere. Since the action of $\mathcal{G}_0$ also allows us to assume that the harmonic part of $\vartheta$ is bounded, boot-strapping then shows that $(\Phi, \vartheta)$ belongs to a bounded subset of $L^2_{k+1}(V_+^\perp) \oplus L^2_{k+1}(\Lambda^1_{\Lambda})$ for any large $k$, and the Rellich theorem then says that its image in $L^2_k(V_+^\perp) \oplus L^2_k(\Lambda^1_{\Lambda})$ is necessarily compact. Since (36) again implies that every solution is irreducible, $\mathcal{G}_0$ acts freely and properly, and $\mathcal{M}_c(g, \eta)$ is therefore a smooth compact manifold, with dimension given by (37), for “most” choices of $\eta$.

To define Witten’s invariant, we now restrict ourselves to the case when the “expected dimension” (37) of the moduli space is zero. Now, remarkably enough, the right-hand side of (37) equals $c_2(V_+) = c(V_+)$ for any spin$^c$ structure, so this expected dimension vanishes if and only if the spin$^c$ structure $c$ is the one determined by some orientation-compatible almost-complex structure $J$ defined on all of $M$. If $b_+(M) \geq 2$, any two regular-value choices of $\eta$ satisfying (36) can be joined by a smooth path $\eta(t)$
satisfying (36) for all \( t \), and such a path then has an arbitrarily small deformation that is transverse to the monopole map (39). Taking the inverse image and modding out by \( \mathbb{G}_0 \) then gives a 1-dimensional cobordism between 0-dimensional moduli spaces associated with our two points. One may therefore define two invariants that are infinite-dimensional analogues of the un-oriented and oriented degrees of a proper map. The first of these is thus defined\(^{23}\) by just setting \( n_c(M) \in \mathbb{Z}_2 \) equal to \( \# \mathcal{M}_c(g, \eta) \) mod 2 when \( (0, \eta, 0) \) a regular value of (39). The more sophisticated second version\(^{24}\) depends on first defining a consistent orientation of the moduli spaces, and then defines \( \text{SW}_c(M) \in \mathbb{Z} \) to be signed count of the points of the discrete set \( \mathcal{M}_c(g, \eta) \). In either case, the resulting invariant is actually metric-independent, because one can more generally construct cobordisms of the moduli spaces by considering paths \( (g(t), \eta(t)) \) where the metric also varies. When the invariant \( n_c(M) \) or \( \text{SW}_c(M) \) is non-zero, it then follows that the Seiberg-Witten equations (30–31) must have a solution, relative to the given spin\(^c\) structure \( c \), for any metric \( g \). Indeed, if there were no solution for a metric \( g \) satisfying \( |c_1(L)|^+ \neq 0 \), then \( \eta = 0 \) would satisfy (36), and the absence of solutions would then make \( (0, 0, 0) \) a regular value of the monopole map (39); thus, the count of solutions would then say that \( n_c(M) \) and \( \text{SW}_c(M) \) both vanished, thereby contradicting our hypothesis. On the other hand, when \(|c_1(L)|^+ = 0\) with respect to a given metric \( g \), we may at least produce a reducible solution of the equations by setting \( \Phi \equiv 0 \) and then choosing \( \theta \) so as to make the 2-form \( F_\theta \) is harmonic; thus, the assertion remains true in this “bad” case, albeit for trivial reasons.

The above discussion also works reasonably well when \( b_+(M) = 1 \), but one must remember to pay careful attention to a key additional subtlety. Indeed, when \( b_+(M) = 1 \), the vector space \( \mathcal{H}_g^+ \) is 1-dimensional, so removing a point from \( \mathcal{H}_g^+ \) therefore disconnects it into two open rays. Consequently, the self-dual 2-forms \( \eta \) satisfying (36) then fall into precisely two connected components. Furthermore, for each spin\(^c\) structure \( c \), the set of pairs \( (g, \eta) \), where \( g \) is a Riemannian metric, and where \( \eta \) is a self-dual 2-form satisfying (36) with respect to \( g \), consists of exactly two connected components, \( \prec^+ \) and \( \prec^- \), called chambers. Applying the previous discussion to each chamber then produces two distinct invariants \( n_c(M, \prec^+) \in \mathbb{Z}_2 \), and two distinct invariants \( \text{SW}_c(M, \prec^+) \in \mathbb{Z} \); these are typically different, and an expression for their difference is then called a wall-crossing formula. As long as \( |c_1(L)|^+ \neq 0 \) for a given metric \( g \) and spin\(^c\) structure \( c \), the same arguments used when \( b_+(M) \geq 2 \) will then guarantee the existence of an irreducible solution of the “unperturbed” Seiberg-Witten equations (30–31) provided...
that \( n_\alpha \neq 0 \) or \( \text{SW}_\epsilon \neq 0 \) for the chamber containing \( \eta = 0 \).

![Diagram](image_url)

**Fig. 3.** When \( b_+(M) = 1 \), the Seiberg-Witten invariant depends on a choice of chamber, and whether the invariant forces the Seiberg-Witten equations \( (30-31) \) to admit an irreducible solution hinges on whether the self-dual-harmonic projection \( c_1^+ = [c_1(L)]^+ \in H^2(M, \mathbb{R}) \) of \( c_1 = c_1(L) \) is past-pointing or future-pointing. If \( c_1^2(L) < 0 \), the answer to this question genuinely depends on the metric.

To clarify this point, notice that if \( b_+(M) = 1 \) and \( b_-(M) \neq 0 \), then \( (H^2(M, \mathbb{R}), \bullet) \) is essentially a copy of \( b_2(M) \)-dimensional Minkowski space. The set of “timelike” cohomology classes \( \alpha \in H^2(M, \mathbb{R}) \) with \( \alpha^2 := \alpha \bullet \alpha > 0 \) is thus an open double cone consisting of two connected components, or nappes. The choice of a “time orientation” for \( (H^2(M, \mathbb{R}), \bullet) \) then amounts to labeling one of these nappes, henceforth denoted by \( \mathcal{C}^+ \), as the set of “future-pointing” time-like vectors, while declaring that the remaining nappe \( \mathcal{C}^- \) consists of “past-pointing” time-like vectors. The impact of the chamber-dependent invariants on the unperturbed Seiberg-Witten equations \( (30-31) \) is then entirely governed by the image \( [c_1(L)]^+ \in \mathcal{H}_g \) of the first Chern class under Minkowski-space-orthogonal projection to \( \mathcal{H}_g^+ \subset H^2(M, \mathbb{R}) \). Since \( \mathcal{H}_g^+ \cup \{0\} \subset \mathcal{C}^+ \cup \mathcal{C}^- \) for any Riemannian metric \( g \), the “no reducible solutions” condition \( [c_1(L)]^+ \neq 0 \) implies that either \( [c_1(L)]^+ \in \mathcal{C}^+ \) or \( [c_1(L)]^+ \in \mathcal{C}^- \). In the first case, the self-dual 2-form \( \eta = 0 \) belongs to the chamber \( \mathcal{C}^+ \) containing the open-ray component of \( \mathcal{H}_g^+ \cup \{0\} \) that terminates in \( \mathcal{C}^+ \), while in the second case \( \eta = 0 \) belongs to the chamber \( \mathcal{C}^- \) containing the open-ray component of \( \mathcal{H}_g^+ \cup \{0\} \) that terminates in \( \mathcal{C}^- \). Thus, in order to use the invariant \( n_\epsilon(M, \mathcal{C}^\pm) \) or \( \text{SW}_\epsilon(M, \mathcal{C}^\pm) \) to predict the the existence of solutions to \( (30-31) \), one must simply keep track of whether \( [c_1(L)]^+ \) is future-pointing or past-pointing for a given
metric $g$. If $c_2^2(L) \geq 0$ and $c_1(L)$ is not a torsion class, the answer to this question is actually independent of $g$. However, when $c_2^2(L) < 0$, the answer becomes dependent on the metric, as indicated by Figure 3. Fortunately, this technical inconvenience can often be overcome by carefully playing several spin$^c$ structures off against one another.

After outlining the definition of the Seiberg-Witten invariant in the $b_4(M) \geq 2$ case, Witten then went on to argue that the invariant is non-zero when $M$ is also the underlying smooth oriented 4-manifold of a compact-complex surface of Kähler type, equipped with the spin$^c$ structure determined by the complex structure. Recall that a compact complex surface $(M^4, J)$ admits compatible Kähler metrics iff its first Betti number $b_1(M)$ is even, and that this happens iff $b_4(M)$ is odd. To gain some insight into Witten’s claim regarding Kähler manifolds, let us first prove a general technical result that indicates the sense in which the Seiberg-Witten equations generalize key aspects of Kähler geometry to a general Riemannian setting:

**Proposition 1.** Let $(M, g)$ be a smooth compact oriented Riemannian 4-manifold, let $c$ be a spin$^c$ structure on $M$, and let $c^+_1 = [c_1(L)]^+$ denote the self-dual part of the harmonic 2-form representing the first Chern class $c_1(L)$ of $c$. If there is a solution of the Seiberg-Witten equations (30–31) on $M$ for $g$ and $c$, then the scalar curvature $s_g$ of $g$ satisfies

$$\int_M s_g^2 d\mu_g \geq 32\pi^2 [c^+_1]^2.$$

Moreover, when $[c^+_1] \neq 0$, equality can occur only if $g$ is a Kähler metric of constant negative scalar curvature that is compatible with a complex structure $J$ such that $c_1(M, J) = c_1(L)$.

**Proof.** Integrating the Weitzenböck formula (32), we have

$$0 = \int [4|\nabla \Phi|^2 + s|\Phi|^2 + |\Phi|^4] d\mu,$$

and it follows that

$$\int (-s)|\Phi|^2 d\mu \geq \int |\Phi|^4 d\mu.$$

Applying the Cauchy-Schwarz inequality to the left-hand side therefore yields

$$\left( \int s^2 d\mu \right)^{1/2} \left( \int |\Phi|^4 d\mu \right)^{1/2} \geq \int |\Phi|^4 d\mu.$$
We therefore have
\[ \int s^2 d\mu \geq \int |\Phi|^4 d\mu = 8 \int |F^+_{\theta}|^2 d\mu, \]
and the inequality is strict unless \( \nabla_\theta \Phi \equiv 0 \) and \( s \) is a non-positive constant. However, \( F^+_{\theta} - 2\pi c_1^+ \) is an exact form plus a co-exact form, and so is \( L^2 \)-orthogonal to the harmonic forms. This gives us the inequality
\[ \int |F^+_{\theta}|^2 d\mu \geq 4\pi^2 \int |c_1^+|^2 d\mu = 4\pi^2 \int c_1^+ \wedge c_1^+, \]
and the last expression may be re-interpreted as the intersection pairing \( [c_1^+]^2 \) of the de Rham class of \( c_1^+ \) with itself. This gives us the desired inequality
\[ \int s^2 d\mu \geq 32\pi^2 |c_1^+|^2, \]
and, when the right-hand side is non-zero, equality can only happen if \( \sigma(\Phi) \) is parallel and \( g \) has constant negative scalar curvature. \( \square \)

Conversely, suppose that \((M, g, J)\) is a compact Kähler surface of constant negative scalar curvature; and let \( \omega = g(J \cdot , \cdot ) \) denote the corresponding Kähler form. For the spin\( ^c \) structure \( c \) determined by \( J \), one then has
\[ \nabla_+ = \Lambda^{0,0} \oplus \Lambda^{0,2}, \quad \nabla_- = \Lambda^{0,1}, \]
and the spin\( ^c \) Dirac operator \( \partial_\theta \) determined by the Chern connection \( \theta \) on the anti-canonical line bundle \( K^{-1} = L = \Lambda^{0,2} \) is just
\[ \sqrt{2}(\bar{\partial} + \partial^* ) : \Lambda^{0,0} \oplus \Lambda^{0,2} \to \Lambda^{0,1}. \]
Moreover, the bundle of real self-dual 2-forms of \( g \) is just
\[ \Lambda^+ = \Re \omega \oplus \Re(\Lambda^{0,2}) \]
while \( \sigma : \nabla_+ \to \Lambda^+ \) is exactly given by
\[ \sigma(f, \phi) = (|f|^2 - |\phi|^2)\omega + 3m(f\phi). \]
Because the curvature of \((K^{-1}, \theta)\) is exactly \(-i\rho \), where the Ricci form \( \rho \) has self-dual part \( \rho^+ = \frac{s}{8}\omega \), it therefore follows that \( \Phi = (\sqrt{-s}, 0) \) and the Chern connection \( \theta \) together solve the Seiberg-Witten equations \( \text{[30][31]} \). Moreover, because the Ricci form \( \rho \) of any constant-scalar-curvature Kähler metric \( g \) is harmonic, the harmonic representative of \( c_1 = c_1(L) \) is \( \frac{s}{8\pi} \) in the present setting, and its self-dual piece is therefore \( c_1^+ = \frac{s}{8\pi} \omega \). It follows that any Seiberg-Witten solution exactly saturates the inequalities in the proof of Proposition \( \text{[4]} \) so that \( \Phi \) must be parallel with respect to \( \theta \). This then
implies that any solution is gauge-equivalent to the explicit one we have displayed. Moreover, direct calculation shows that the derivative of the monopole map (39) is surjective at our explicit solution. This shows that $SW_c(M) = \pm 1$ and $n_c(M) \neq 0$ if $b_+(M) > 1$, while $SW_c(M, <^+) = \pm 1$ and $n_c(M, <^+) \neq 0$ for the appropriate chamber $<^+$ if $b_+(M) = 1$.

As a consequence, we therefore obtain the following result:

**Theorem 4.** Let $(M^4, J)$ be a compact complex surface that admits a compatible Kähler-Einstein metric $g$ with scalar curvature $s < 0$. Then
\[ I_s(M) = 32\pi^2 c_1^2(M, J) > 0 \]

and
\[ \mathcal{Y}(M) = -4\pi \sqrt{2c_1^2(M, J)} < 0 \]

are both achieved by $g$. Moreover, if $g'$ is a Riemannian metric on $M$ that achieves one of these critical values for the relevant Riemannian functional, then $g'$ is also Kähler-Einstein, and is compatible with an integrable complex structure $J'$ such that $c_1(M, J') = c_1(M, J) \in H^2(M, \mathbb{Z})$.

**Proof.** Since $(M, J)$ is of Kähler type, $b_+ = 1 + h^{2,0}(M, J) \geq 1$, and the above discussion of Seiberg-Witten invariants therefore applies. On the other hand, the Kähler-Einstein metric $g$ has Ricci-form $\rho = s/4\omega$, and hence
\[ c_1^2(M, J) = \frac{1}{4\pi^2} |\rho|^2 = \frac{s^2}{64\pi^2} |\omega|^2 > 0. \]

Thus, for the spin$^c$ structure determined by $J$, the mod-2 count of solutions of (30–31) mod gauge is metric independent, even if $b_+ = 1$. However, since $g$ is Kähler, with constant negative scalar curvature, the count of solutions is 1 mod 2 for $g$ and the spin$^c$ structure determined by $J$, so there must therefore be a solution for any other metric. Proposition [1] therefore tells us that
\[ \int s^2 d\mu \geq 32\pi^2 |c_1^+|^2 \geq 32\pi^2 c_1^2(M, J) \]

for any Riemannian metric $g'$ on $M$. Thus
\[ I_s(M) = \inf_{g'} \mathcal{G}(g') = 32\pi^2 c_1^2(M, J) \]

and the infimum is moreover attained by $g$. Since this means, in particular, that $I_s(M) > 0$, we therefore have
\[ \mathcal{Y}(M) = -\sqrt{I_s(M)} = -4\pi \sqrt{2c_1^2(M, J)} < 0. \]

Moreover, if $g'$ saturates (42), then $g'$ is a Kähler metric of constant negative scalar curvature by Proposition [1] and because $|c_1^+|^2 = c_1^2(M, J)$, the Ricci form of $g'$, which is the harmonic representative of $2\pi c_1$ with respect to this metric, must also be self-dual, thus implying that $g'$ is Kähler-Einstein.
On the Scalar Curvature of 4-Manifolds

Theorem 2 is now an immediate consequence. Indeed, consider the complex surfaces $M_\ell \subset \mathbb{CP}^3 = \{[z_0 : z_1 : z_2 : z_3] \mid z_j \in \mathbb{C}\}$ defined by

$$z_0^\ell + z_1^\ell + z_2^\ell + z_3^\ell = 0.$$  

Each is the zero set of a section of a positive line bundle, namely $O(\ell)$, that is transverse to the zero section, so each is simply connected by Lefschetz’s theorem on hyperplane sections. On the other hand, the canonical line bundle of $M_\ell$ is exactly $O(\ell - 4)$ by the adjunction formula, and so is ample if $\ell > 4$. The Aubin-Yau theorem therefore implies that for each $\ell \geq 5$, the complex surface $M_\ell$ admits a negative-scalar-curvature compatible Kähler-Einstein metric, and Theorem 4 therefore tells us that $I_s(M_\ell) = \frac{32\pi^2 c_1^2(M_\ell)}{\ell(\ell - 4)^2}$, $\forall \ell \geq 5$, thus providing a sequence of 4-manifolds for which $I_s \to +\infty$, as claimed.

Moreover, for the sequence 4-manifolds we have just displayed, the infimum $I_s = \inf S$ is achieved in each instance by an Einstein metric (that, for these purposes, just happens to be Kähler).

An important consequence of this discussion is that the Yamabe invariant can often be used to distinguish between different smooth structures on a fixed topological 4-manifold. For example, consider the underlying smooth compact oriented 4-manifolds represented by the above complex surfaces $M_\ell$. These manifolds are simply connected and have

$$b_+(M_\ell) = \frac{(\ell - 1)(\ell - 2)(\ell - 3)}{3} + 1, \quad b_-(M_\ell) = \frac{(\ell + 2)(\ell - 2)}{3} + b_+(M).$$

However, a deep theorem due to Freedman implies that two compact connected oriented smooth 4-manifolds are orientedly homeomorphic if and only if they have the same invariants $b_+$ and $b_-$, and either both are spin, or both are non-spin. For $\ell = 2k + 1$ odd, with $k \geq 2$, this for instance tells us that the negative-Yamabe-invariant non-spin 4-manifold $M_{2k+1}$ is homeomorphic to the positive-Yamabe-invariant connected sum $m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$, where $m = 1 + \frac{1}{3}k(k - 1)(2k - 1)$, and $n = \frac{1}{2}k(8k^2 + 1)$. Similarly, when $\ell = 2k$ is even and $k \geq 3$, the negative-Yamabe-invariant spin manifold $M_{2k}$ is homeomorphic to the zero-Yamabe-invariant connected sum $pK3\# q(S^2 \times S^2)$, where $p = \binom{k+1}{3}$, $q = \frac{1}{12}(k - 2)(13k^2 - 22k + 3)$, and where $K3 := M_4$ is one of the essential building blocks of 4-dimensional topology.

The fact that these connected sums have $\mathcal{Y} \leq 0$ follows from the fact that they are spin and have non-zero $\tilde{A} = -r/8$. The fact that they have $\mathcal{Y} \geq 0$ follows from Petean’s surgery lemma, together with the fact that $K3$ admits a Ricci-flat (and hence scalar flat) metric by Yau’s solution of the Calabi conjecture.
While the role of the scalar curvature in this story was a stunning consequence of the advent of Seiberg-Witten theory, it is worth remembering that it was previously known that the complex surfaces $M_\ell$, $\ell \geq 4$, were certainly not diffeomorphic to the connect sums considered above; this had been proved by Donaldson\cite{Donaldson32} using new polynomial invariants that he had defined using moduli spaces of Yang-Mills instantons. It should also be noted Donaldson’s Yang-Mills-based thesis result\cite{Donaldson31} is actually needed to streamline the smooth-manifold case of Freedman’s theorem into the user-friendly statement given above. Witten’s ostensible justification for introducing the Seiberg-Witten equations in the first place was a physics argument indicating that his new theory should encode exactly the same information as the Donaldson polynomials. While partial results\cite{Witten34} and large amounts of practical experience strongly indicate that Witten’s claim is actually true, the intuition linking the two theories continues to largely elude the mathematical community. In particular, the Yang-Mills equations, unlike the Seiberg-Witten equations, are conformally invariant, and so, at least locally, are utterly insensitive to the scalar curvature!

Now, “most” compact complex surfaces of Kähler type are deformation-equivalent to surfaces which admit compatible Kähler metrics of constant negative scalar curvature\cite{LeBrun88}, so citing this fact while appealing to the converse of Proposition 1 proved in the discussion above provides a correct-but-inefficient way of showing that complex surfaces of Kähler type typically carry non-trivial Seiberg-Witten invariants. However, one can easily prove more by instead just considering well-chosen perturbations of the Seiberg-Witten equations for an arbitrary Kähler metric. Indeed, if $t$ is any positive constant, then $\Phi = (t, 0)$ and the Chern connection $\theta$ together solve (30) and (35) for the perturbation $\eta = \frac{t^2 + s}{4} \omega$, and the linearization of the monopole map is moreover surjective at this solution; careful inspection of (40) then shows that, up to gauge equivalence, this is the unique solution of (30) and (35) for this specific $\eta$. When $b_+ (M) > 1$, this shows that $\text{SW}_c (M) = \pm 1$ on any Kähler-type complex surface, and hence that $n_c (M) = 1 \pmod 2$, where $c$ is or the the spin$^c$ structure determined by the complex structure by $J$; when $b_+ (M) = 1$, one similarly concludes that $\text{SW}_c (M, \langle \omega^+ \rangle) = \pm 1$, and hence that $n_c (M, \langle \omega^+ \rangle) = 1 \pmod 2$, for the chamber $\langle \omega^+ \rangle$ containing large positive multiples of the Kähler form $\omega$. In particular, when $b_+ (M) > 1$, there are solutions of the unperturbed equations (30–31) for any metric on $M$ for this specific spin$^c$ structure; when $b_+ (M) = 1$, one instead gets solutions of the unperturbed equations whenever the self-dual projection $c_1^+ \otimes \eta_1 (M, J)$ belongs to the nappe $C^- \subset H^2 (M, \mathbb{R})$ that does
not contain the Kähler class $[\omega]$ of a reference Kähler metric on $(M, J)$.

One of the fundamental operations of complex surface theory\cite{13,41} is the blow-up operation, which replaces a point of a complex surface $N$ with a $\mathbb{CP}_1$ of normal bundle $O(-1)$; this then produces a new complex surface $M$ that is diffeomorphic to $N#\mathbb{CP}_2$, where $\mathbb{CP}_2$ is the oriented manifold obtained from $\mathbb{CP}_2$ by reversing its orientation. Conversely, any complex surface $M$ containing a $\mathbb{CP}_1$ of normal bundle $O(-1)$ can be “blown down” to produce a complex surface $N$ such that $M$ becomes its blow-up. This operation can in principle be iterated, but the process must terminate after finitely many steps, because each blow-down decreases $b_2$ by 1. When a complex surface $X$ cannot be blown-down, it is called minimal, and the upshot is that any complex surface $M$ can be obtained from a minimal complex surface $\overline{X}$ by blowing up finitely many times. In this situation, one then says that $X$ is a minimal model of $M$.

With this said, we are now ready to introduce one of the most important complex-analytic invariants of a compact complex surface, namely its Kodaira dimension\cite{13,41} This is defined in terms of positive powers of the canonical line bundle $K := \Lambda^{2,0}$, and is given by

$$\text{Kod}(M, J) = \limsup_{j \to +\infty} \frac{\log \dim H^0(M, K^\otimes j)}{\log j}.$$ 

The only possible values of this invariant are $-\infty$, 0, 1 and 2, because the Kodaira dimension is actually just the largest complex dimension\cite{41} of the image of $M \to \mathbb{P}(H^0(M, K^\otimes j))$ under all the various “pluricanonical” maps associated with the line bundles $K^\otimes j$, $j \in \mathbb{Z}^+$. Blowing up or down always leaves the Kodaira dimension unchanged, and the minimal model of a complex surface is moreover unique whenever $\text{Kod} \neq -\infty$. As an illustration, for the surfaces $M_\ell \subset \mathbb{CP}_3$ discussed above, it is not hard to show that

$$\text{Kod}(M_\ell) = \begin{cases} -\infty & \text{if } \ell \leq 3 \\ 0 & \text{if } \ell = 4 \\ 2 & \text{if } \ell \geq 5. \end{cases}$$

This sequence of examples begins to explain why complex surfaces with $\text{Kod} = 2$ are said to be of general type. Notice that, among these specific examples, the ones of general type are exactly those for which seen that the

\footnote{Unfortunately, this means that the term minimal surface is widely used by algebraic geometers to mean something that has nothing whatsoever to do with soap bubbles!}

\footnote{In this context, we conventionally set $\dim \emptyset := -\infty$.}
Yamabe invariant is also negative. This is simply the first glimmer of the following general pattern: \(^{51}\)

**Theorem 5.** Let \(M\) be the smooth 4-manifold underlying a compact complex surface \((M^4, J)\) with \(b_1(M)\) even. Then

\[
\begin{align*}
\mathcal{V}(M) > 0 & \iff \text{Kod}(M, J) = -\infty \\
\mathcal{V}(M) = 0 & \iff \text{Kod}(M, J) = 0 \text{ or } 1 \\
\mathcal{V}(M) < 0 & \iff \text{Kod}(M, J) = 2.
\end{align*}
\]

Here the assumption that \(b_1(M)\) is even is equivalent to \((M, J)\) admitting a compatible Kähler metric; thus, complex surfaces with \(b_1\) even are said to be of Kähler type. In \(\S\) below will we see that part, but only part, of this pattern still holds true for complex surfaces with \(b_1\) odd. In particular, Theorem 11 below and previously-known results together imply the following result, which does not depend on the parity of \(b_1\):

**Theorem 6.** Let \((M, J)\) be a compact complex surface with \(\text{Kod}(M, J) \geq 0\), and let \((X, \tilde{J})\) be its minimal model. Then

\[
\mathcal{V}(M) = \mathcal{V}(X).
\]

When \(\text{Kod}(M, J) = 2\), this takes the following quantitative form: \(^{51}\)

**Theorem 7.** Let \((M^4, J)\) be a compact complex surface of general type, and let \((X, \tilde{J})\) be its minimal model. Then

\[
\mathcal{I}_s(M) = \mathcal{I}_s(X) = 32\pi^2 c_1^3(X) > 0
\]

and

\[
\mathcal{V}(M) = \mathcal{V}(X) = -4\pi \sqrt{2c_1^2(X)} < 0.
\]

The proof first uses Seiberg-Witten theory to show that \(\mathcal{I}_s(M) \geq 32\pi^2 c_1^3(X)\); this is made possible by the fact that every surface of general type is Kähler (and indeed projective algebraic), for reasons related to the fact that the minimal model \(X\) satisfies \(c_1^3(X) > 0\). If \(b_+(M) = b_+(X) > 1\), the real Chern class

\[
c_1^R(M, J) \in H^2(M, \mathbb{Z})/\text{torsion} \subset H^2(M, \mathbb{R})
\]

is a so-called basic class, meaning that it is the Chern class of a spin\(^c\) structure \(c\) with non-zero Seiberg-Witten invariant. If \(M = X\) is minimal, the desired lower bound follows from Proposition \(^{[1]}\). Otherwise, \(M \approx X \# k\mathbb{C}P^2\) has \(H^2(M) = H^2(X) \oplus H^2(k\mathbb{C}P^2)\), and \(M\) has a self-diffeomorphism
Let $M^4$ be a compact symplectic 4-manifold, and let $\varphi$ be the spin$^c$ structure determined by the associated integrable almost-complex structure $J$, we have already seen that the Seiberg-Witten invariant is non-zero if $b_+ > 1$, and that the Seiberg-Witten invariant is similarly non-zero in a suitable chamber if $b_+ = 1$. However, Taubes proved a dramatic generalization of this result:

**Theorem 8 (Taubes).** Let $(M^4, \omega)$ be a compact symplectic 4-manifold, and let $\varphi$ be the spin$^c$ structure on $M$ determined by an almost-complex structure $J$ that acts by $+1$ on $H^2(X)$, but as $-1$ on $H^2(k\mathbb{CP}^2)$. Since $\varphi$ sends basic classes to basic classes, this means that $c_1(X)$ is the average of two basic classes $a_1 = c_1(M, J)$ and $a_2 = c_1(M, \varphi_* J)$, and its projection $c_1(X)_+ \in H^+_g$ is therefore the average of the orthogonal projections $a_1^+, a_2^+ \in H^+_g$ of these classes with respect to the intersection form $\bullet$. In particular, we must have

$$\max([a_1^+]^2, [a_2^+]^2) \geq [c_1(X)_+]^2 \geq c_1^2(X)$$

because $\bullet$ is positive-definite on $H^+_g$. Proposition 1 therefore implies

$$\mathcal{S}(g) = \int_M s^2 d\mu_g \geq 32\pi^2 \max([a_1^+]^2, [a_2^+]^2) \geq 32\pi^2 c_1^2(X).$$

When $b_+(M) = b_+(X) = 1$, the proof is similar, but also depends on the fact that $\max([a_1^+]^2, [a_2^+]^2)$ is achieved by a spin$^c$ structure for which $a_1^+$ is past-pointing. Either way, we now see that $\mathcal{L}_s(M) = \inf \mathcal{S} \geq 32\pi^2 c_1^2(X)$.

To finish the proof, we now just need to construct a sequence of metrics $g_j$ on $M$ for which $\mathcal{S}(g_j)$ tends to this lower bound. This is made possible by the fact that the **pluricanonical model** $\tilde{X}$ of $M$ is a complex orbifold with $c_1 < 0$, so that the proof of the Aubin-Yau theorem therefore endows it with an orbifold Kähler-Einstein metric satisfying $\int s^2 d\mu = 32\pi^2 c_1^2(\tilde{X})$. However, the natural map $X \rightarrow \tilde{X}$ relating the minimal and pluricanonical models is a so-called crepant resolution, meaning that $c_1(\tilde{X})$ is just the pull-back of $c_1(X)$. Because the orbifold singularities of $\tilde{X}$ are all modeled on $\mathbb{C}^2/\Gamma$ for finite groups $\Gamma \subset \text{SU}(2)$, each can be replaced with a gravitational instanton (asymptotically-locally-Euclidean Ricci-flat Kähler manifold) to obtain a Riemannian metric on the minimal model $X$, while increasing $\int s^2 d\mu$ by an arbitrarily small amount arising from the transition region. Similarly, to pass from the minimal model $X$ to the given complex surface $M \approx X\#k\mathbb{CP}^2$, one can use the fact that $\mathbb{CP}^2 - \{pt\}$ admits an asymptotically-flat scalar-flat metric called the **Burns metric** gluing in rescaled copies of this model then gives metrics on $M$ that with $\mathcal{S} < 32\pi^2 c_1^2(X) + \varepsilon$ for any given $\varepsilon > 0$.

If $M^4$ admits a Kähler metric, and if $\varphi$ is the spin$^c$ structure determined by the associated integrable almost-complex structure $J$, we have already seen that the Seiberg-Witten invariant is non-zero if $b_+ > 1$, and that the Seiberg-Witten invariant is similarly non-zero in a suitable chamber if $b_+ = 1$. However, Taubes proved a dramatic generalization of this result:
structure $J$ compatible with $\omega$. If $b_+(M) > 1$, then $\text{SW}_c(M) = \pm 1$. Similarly, if $b_+(M) = 1$, then $\text{SW}_c(M, \langle \cdot, \cdot \rangle) = \pm 1$, where $\langle \cdot, \cdot \rangle$ is the chamber containing large positive multiples of the symplectic form $\omega$.

The strategy of the proof is similar to what has been described in the Kähler case, but is technically far more subtle. One chooses a so-called almost-Kähler metric $g$ that is related to the symplectic form by $\omega = g(J \cdot, \cdot)$ for some almost-complex structure $J$, and then writes down an explicit solution of (30–35) for perturbations $\eta$ of the form $t^2 \omega + \eta_0$. When $t \gg 0$ is sufficiently large, one then uses the Weitzenböck formula to show that, up to gauge equivalence, the explicit solution is actually the only solution.

In a stunning pair of sequels, Taubes then extended these ideas in ways that completely revolutionized 4-dimensional symplectic geometry, and gave the trail-blazing earlier contributions of Gromov and McDuff surprising new meanings. While McDuff had previously shown that blowing up and down had natural generalizations to symplectic 4-manifolds, Taubes now showed that every symplectic 4-manifold $M$ with $b_+(M) > 1$ has a unique minimal model $X$ with $c_1^2(X) \geq 0$, and that symplectic minimality or non-minimality for such manifolds depends only on diffeomorphism type. Taubes’ also showed that the anti-canonical class $-c_1$ on any such symplectic manifold is always represented by one of Gromov’s pseudo-holomorphic curves, so that every symplectic 4-manifold with $b_+ > 1$ in particular satisfies $c_1 \cdot [\omega] < 0$. When $b_+ = 1$, versions of Taubes’ results still hold, but in this context one must carefully check whether the Seiberg-Witten invariant is non-zero in chamber needed for any given application.

Taubes’ results immediately made it easy to extend part of Theorem 7 to 4-manifolds that support symplectic structures. The key definition needed to make this possible was the following:

**Definition 1.** A symplectic 4-manifold $(M, \omega)$ is said to be of **general type** iff its minimal model $(X, \tilde{\omega})$ satisfies

- $c_1^2(X) > 0$; and
- $c_1(X) \cdot [\tilde{\omega}] < 0$.

Kodaira classification then implies that a Kähler manifold $(M^4, J, \omega)$ is of general type as a complex surface iff it is of general type as a symplectic 4-manifold, and essentially the same proof as before then yields the
following partial generalization of Theorem 7:

**Theorem 9.** Let \((M^4, \omega)\) be a symplectic manifold of general type, and let \((X, \tilde{\omega})\) be its minimal model. Then

\[
\mathcal{I}_s(M) \geq 32 \pi^2 c_1^2(X) > 0 \quad \text{and} \quad \mathcal{V}(M) \leq -4 \pi \sqrt{2c_1^2(X)} < 0. \quad (43)
\]

Together with constructions by Gompf, Fintushel-Park-Stern, and others, Theorem 9 implies that there are many symplectic 4-manifolds with negative Yamabe invariant that do not arise as complex surfaces. On the other hand, Theorem 9 merely gives an upper bound for the Yamabe invariant, rather than actually calculating its actual value, and my guess at the time was that this bound would never be sharp in the non-Kähler case. However, this turned out to be wrong. Indeed, Ioana Suvaină later succeeded in proving that certain symplectic manifolds arising from Gompf’s rational-blow-down construction have Yamabe invariants that do exactly saturate (43), even though they do not admit Kähler metrics. In her examples, the rational-blow-down operation can be carried out by starting with a Kähler-Einstein orbifold, and then gluing in finite quotients of Gibbons-Hawking gravitational instantons. Whether this phenomenon is rare or common seems to be largely open, and would seem to deserve a thorough and systematic investigation.

It is also worth pointing out that Definition 1 was eventually rediscovered by Li, who went on to provide a systematic definition of Kodaira dimension for symplectic 4-manifolds. This of course makes it tempting to ask whether, with Li’s definitions, Theorems 5 and/or 6 also hold for symplectic 4-manifolds. While this might seem unlikely, the issue is one with enough intrinsic interest to definitely make it worth settling, one way or the other.

Even though Witten’s invariant leads to so many compelling results, our applications of Seiberg-Witten theory to the Yamabe invariant have really only depended on being guaranteed a solution of the Seiberg-Witten equation for every metric on a given 4-manifold. As it turns out, this can happen even in contexts where Witten’s invariant vanishes for every \(\text{spin}^c\) structure. For example, Bauer and Furuta discovered a generalization of the \(\mathbb{Z}_2\)-valued Seiberg-Witten invariant \(n_e\) that implies such an existence statement in contexts where the expected dimension (37) of the moduli space is positive. In the simply-connected case, their invariant is the element of the equivariant stable cohomotopy groups \(\pi^b_{S^1}(\text{Ind } \mathcal{V})\) represented,
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via a finite-dimensional-approximation scheme, by the monopole map (39), thought of as a map between one-point compactifications of Hilbert spaces. When this invariant is non-zero, one then concludes that there is a solution of (30–31) for every Riemannian metric \( g \) on \( M \). Because this property is useful in and of itself, it is worth codifying, as follows:

**Definition 2.** Let \( M \) be a smooth compact oriented 4-manifold with \( b_+ \geq 2 \). An element \( a \in H^2(M, \mathbb{Z})/\text{torsion} \), \( a \neq 0 \), is called a monopole class of \( M \) iff there is some spin\(^c\) structure \( c \) on \( M \) with first Chern class

\[
c_1(L) \equiv a \mod \text{torsion}
\]

for which the Seiberg-Witten equations (30–31) have a solution for every Riemannian metric \( g \) on \( M \).

While this is a completely “soft” definition\(^8\) it nonetheless has some useful immediate consequences.\(^65\)

**Proposition 2.** If \( M \) is a smooth compact oriented manifold with \( b_+ \geq 2 \), then

\[
\mathcal{C} = \{\text{monopole classes on } M \} \subset H^2(M, \mathbb{R})
\]

is a diffeomorphism invariant of \( M \). Moreover, \( \mathcal{C} \) is a finite set, and is invariant under reflections \( a \mapsto -a \) through the origin.

As a consequence, if we define \( \text{Hull} \mathcal{C} \subset H^2(M, \mathbb{R}) \) to be the convex hull of the monopole classes, then \( \text{Hull} \mathcal{C} \) is compact, and necessarily contains 0 if \( \mathcal{C} \neq \emptyset \). This allows us to define a non-negative numerical invariant of any smooth compact oriented 4-manifold that that provides a transparent distillation of many key arguments in the subject:

**Definition 3.** Let \( M \) be a smooth compact oriented 4-manifold with \( b_+ \geq 2 \). If \( \mathcal{C} = \emptyset \), set \( \beta(M) = 0 \). Otherwise, let

\[
\beta^2(M) = \max\{ a \cdot a | a \in \text{Hull} \mathcal{C} \}
\]

and set \( \beta(M) := \sqrt{\beta^2(M)} \geq 0 \).

When \( \mathcal{C} \neq \emptyset \), the compactness of \( \text{Hull} \mathcal{C} \) ensures that the above maximum is actually achieved, while the fact that that 0 \( \in \text{Hull} \mathcal{C} \) guarantees

\(^8\)Here the hypothesis \( b_+ \geq 2 \) is primarily imposed as a matter of taste rather than of necessity, because the related notion of a retroactive class better encapsulates the essential features of the usual Seiberg-Witten invariant when \( b_+ = 1 \). For details, see.\(^65\)
that $\beta^2(M) \geq 0$. Here it is important to remember that, since the intersection form $\bullet$ is typically indefinite, the function $a \mapsto a^2 := a \bullet a$ is usually not convex, and that the maxima of $a^2$ on Hull $\mathcal{C}$ therefore can (and often do) occur away from the vertices $\mathcal{C}$ of the convex hull.

Using essentially the same Weitzenböck arguments as before, one now deduces the following general result:

Proposition 3. Let $M$ be a smooth compact connected oriented 4-manifold with $b_+ \geq 2$ and $\mathcal{C} \neq \emptyset$. Then

$$I_\alpha(M) \geq 32\pi^2 \beta^2(M) \quad \text{and} \quad \mathcal{Y}(M) \leq -4\pi\sqrt{2} \beta(M).$$

(44)

Here one might object that our soft definition of a monopole class means that we may well not be able to directly determine the full collection $\mathcal{C}$ of all monopole classes on a given 4-manifold. Nonetheless, knowing any subset of $\mathcal{C}$ gives a lower bound for $\beta^2$, while [44] gives us an upper bound for $\beta^2$ in terms of $\inf \mathcal{S}(g_j)$ for any sequence of Riemannian metrics $g_j$ on $M$. By combining these two facts, one can in practice exactly calculate $\beta$ for many interesting 4-manifolds. For example, using the results of Bauer and Furuta to prove the existence of specific monopole classes on suitable connected sums, and then constructing specific minimizing sequences $g_j$ by pasting together Kähler building blocks, one can prove the following [51, 52]:

Theorem 10. Let $X, Y,$ and $Z$ be minimal simply-connected complex surfaces with

$$b_+ \equiv 3 \text{ mod } 4.$$

Then, for any $k \geq 0$,

$$\beta^2(X \# Y \# k\mathbb{CP}_2) = c_1(X) + c_1(Y)$$

and

$$\beta^2(X \# Y \# Z \# k\mathbb{CP}_2) = c_1(X) + c_1(Y) + c_1(Z).$$

Moreover, equality holds in (44) for each of these connected sums.

For example, for any positive integer $m$, the complex surface $M_{4m} \subset \mathbb{CP}_3$ of degree $4m$ is simply connected and has $b_+ \equiv 3 \text{ mod } 4$. Choosing $X, Y,$ and $Z$ to be of this form, where $X$ is not a quartic, we thus obtain a horde of concrete simply connected 4-manifolds whose Yamabe invariants are negative and explicitly calculable. These examples are significantly different from any previously discussed here. In particular, these
smooth manifolds never admit symplectic structures, because an argument first sketched by Witten\textsuperscript{97} shows that $SW_c = 0$ for any spin$^c$ structure $c$ of almost-complex type on a connected sum of two 4-manifolds with $b_+ \neq 0$.

Seiberg-Witten theory thus reveals that an astonishing profusion of compact simply connected topological 4-manifolds admit smooth structures for which the Yamabe invariant is negative. We will now use other techniques to explore the cases where the Yamabe invariant is zero or positive.

3. Dimension Four: the Yamabe Zero Case

If $(M^4, J)$ is a compact complex surface of Kähler type, Theorem \textsuperscript{5} says that the sign of the Yamabe invariant $\mathcal{Y}(M)$ is completely determined by its Kodaira dimension $\text{Kod}(M, J)$. But does this pattern also hold true when $(M^4, J)$ is of non-Kähler type, or equivalently, when $b_1(M)$ is odd? Since every complex surface with $\text{Kod}(M) = 2$ is of Kähler type, the non-Kähler case naturally breaks up into the sub-cases of $\text{Kod}(M) = 1$, $\text{Kod}(M) = 0$, and $\text{Kod}(M) = -\infty$. However, my earlier paper\textsuperscript{64} also settled the non-Kähler case when $\text{Kod}(M) = 0$, thereby only leaving the problem open for complex surfaces $(M^4, J)$ with $b_1(M)$ odd and $\text{Kod}(M, J) = 1$ or $\text{Kod}(M, J) = -\infty$. Meanwhile, Michael Albanese\textsuperscript{5} more recently showed that the simple pattern linking Kodaira dimension and the Yamabe invariant is sometimes violated for complex surfaces with $\text{Kod} = -\infty$ and $b_1$ odd.

In this section, we will clarify his arguments by putting them in a somewhat more general setting, and then show that these same ideas nonetheless imply that the original pattern does hold true for complex surfaces with $\text{Kod} = 1$ and $b_1$ odd.

Any complex surface $(M, J)$ of Kodaira dimension 1 is property elliptic\textsuperscript{13} because holomorphic sections of some some power $K^{\otimes \ell}$ of the canonical line bundle $K = \Lambda^{2,0}$ define a holomorphic map $M \to C$ onto a smooth connected Riemann surface $C$ such that any regular fiber is an elliptic curve $E \approx T^2$. By deforming the complex structure, one can then show that such manifolds can always be obtained by starting with a 4-orbifold that fibers over a 2-orbifold with flat fibers, then gluing in gravitation instantons, and finally blowing up. By a generalization of arguments of

\textsuperscript{5}This inescapable piece of traditional terminology unfortunately clashes with most uses of the term “elliptic” in geometry and topology, where it is usually reserved for spaces that are positively curved, rather than flat. The historical origin of this peculiar usage is that elliptic curves first arose in connection with the “elliptic integral” that expresses the arclength of a generic ellipse in the plane.
On the Scalar Curvature of 4-Manifolds

Cheeger-Gromov\textsuperscript{26} this allows one\textsuperscript{26} to show that these manifolds admit sequences of Riemannian metrics with $\int s^2 d\mu \downarrow 0$. It therefore follows that every properly elliptic complex surface $(M^4, J)$ has $\mathcal{I}_s(M) = 0$, and hence $\mathcal{Y}(M) \geq 0$. To show that these manifolds have $\mathcal{Y}(M) = 0$, it therefore suffices to show that they never admit metrics of positive scalar curvature.

In the Kähler case, Seiberg-Witten theory rules out the existence of positive-scalar-curvature metrics on properly elliptic surfaces, so their Yamabe invariants must indeed vanish. In the non-Kähler case, a systematic calculation by Biquard\textsuperscript{18} also revealed that many properly elliptic surfaces with $b_1$ odd do still carry non-trivial Seiberg-Witten basic classes, thereby making it plausible that the pattern might be independent of the parity of $b_1$. Unfortunately, however, Biquard’s results do not suffice to settle the problem, because there do exist properly elliptic surfaces with $b_1$ odd that do not carry Seiberg-Witten basic classes. For example, let $C$ be a Riemann surface, let $L \to C$ be a complex line bundle of Chern class 1, let $L^\times \subset L$ be the complement of the zero section, and let $M = L^\times / \mathbb{Z}$, where the action of $\mathbb{Z}$ on $L^\times$ is generated by, say, scalar multiplication by 2. If $C$ has genus $g \geq 2$, this elliptic surface has Kod = 1 and $b_1 = 2g + 1$, but Biquard’s calculation reveals that its Seiberg-Witten invariants are all zero. Finishing the job therefore requires a different method for excluding the existence of positive-scalar-curvature metrics.

Fortunately, combining the Schoen-Yau and Gromov-Lawson methods yields the following useful generalization of a theorem of Albanese\textsuperscript{4,5}

**Proposition 4.** Let $N^3$ be a smooth compact oriented connected enlargeable 3-manifold, and let $X$ be a smooth compact oriented 4-manifold that admits a smooth submersion $\varpi : X \to S^1$ with fiber $N$. Let $P$ be any smooth compact oriented 4-manifold, and let $M = X \# P$. Then $\mathcal{Y}(M) \leq 0$. In other words, $M$ does not admit metrics of positive scalar curvature.

**Proof.** Proceeding by contradiction, let us suppose that $M$ admits a metric $g$ of positive scalar curvature. Let $\delta : X \# P \to X$ be the smooth “blowing down” map that collapses $(P - B^4) \subset M$ to a point, and let 

$$f = \varpi \circ \delta : M \to S^1 = \mathbb{U}(1)$$

be the induced projection. The pre-image $f^{-1}(z)$ of a regular value is thus a copy of $N$ in $M$ whose homology class $[N] \in H_3(M, \mathbb{Z}) \cong H^1(M, \mathbb{Z})$ is non-zero, because $M - N$ is connected. Now recall that

$$H^1(M, \mathbb{Z}) = C^\infty(M, \mathbb{U}(1))/ \exp [2\pi i C^\infty(M, \mathbb{R})],$$

(45)
because there is short exact sequence of sheaves of Abelian groups
\[ 0 \to \mathbb{Z} \to C^\infty(\_ , \mathbb{R}) \xrightarrow{\exp 2\pi i} C^\infty(\_ , \mathbb{U}(1)) \to 0, \]
where \( C^\infty(\_ , \mathbb{R}) \) is a fine sheaf. We may now represent the homology class \([N]\) by a mass-minimizing rectifiable current, and recall that, since \( M \) has dimension \( 4 < 8 \), this current is then a sum of smooth oriented connected hypersurface \( \Sigma_1, \ldots, \Sigma_k \) with positive integer multiplicities; in particular,
\[ [N] = \sum_{i=1}^k n_i [\Sigma_i], \quad n_i \in \mathbb{Z}^+. \quad (46) \]
By the Schoen-Yau argument recounted in § 1, the \( g \)-induced conformal class \([h]\) on each hypersurface \( \Sigma_i \) necessarily has positive Yamabe constant. On the other hand, each hypersurface \( \Sigma_i \) can then be written as \( f_i^{-1}(1) \) for some smooth map \( f_i : M \to \mathbb{U}(1) \), where \([f_i] \in H^1(M, \mathbb{Z})\) is the Poincaré dual of \([\Sigma_i]\) and \( 1 \in \mathbb{U}(1) \) is a regular value of \( f_i \). It then follows that
\[ \hat{f} := \prod_i f_i^{n_i} : M \to \mathbb{U}(1) \]
represents the Poincaré dual of \([\Sigma]\), where the product of course means the point-wise product of \( \mathbb{U}(1) \)-valued functions. Since \( f \) and \( \hat{f} \) therefore represent the same class in \( H^1(M, \mathbb{Z}) \), equation (45) therefore tells that
\[ f = e^{2\pi i u} \hat{f} \]
for some smooth real valued function \( u : M \to \mathbb{R} \), and we thus obtain an explicit homotopy of \( \hat{f} \) to \( f \) by simply setting \( f_t = e^{2\pi i u(t)} \hat{f} \) for \( t \in [0, 1] \). Since \( \hat{f} \) is constant on each \( \Sigma_i \), it follows that \( f|_{\Sigma_i} \) induces the zero homomorphism \( \pi_1(\Sigma_i) \to \pi_1(S^1) \), and the inclusion map \( j_i : \Sigma_i \hookrightarrow M \) therefore lifts to an embedding \( \tilde{j}_i : \Sigma_i \hookrightarrow \tilde{M} \) of this hypersurface in the covering space \( \tilde{M} \to M \) corresponding to the kernel of \( f_* : \pi_1(M) \to \pi_1(S^1) \).

However, we can identify \( X \) with the mapping torus
\[ \mathcal{O}_\varphi := (N \times \mathbb{R})/\langle (x, t) \mapsto (\varphi(x), t + 2\pi) \rangle \]
of some diffeomorphism \( \varphi : N \to N \) by simply choosing a vector field on \( X \) that projects to \( \partial/\partial \theta \) on \( S^1 \), and then following the flow. Since the diffeotype of \( \mathcal{O}_\varphi \to S^1 \) moreover only depends on the isotopy class of \( \varphi \), we may also assume that \( \varphi \) has a fixed-point \( p \in N \). The flow-line \( \{p\} \times \mathbb{R} \) then covers an embedded circle \( S^1 \hookrightarrow X \), which we may moreover take to avoid the ball where surgery is to be performed to construct \( M = X \# P \). This
circle in $X$ then also defines an embedded circle in $M$, which we will call the reference circle; and by first making a small perturbation, if necessary, we may assume that this reference circle $S^1 \hookrightarrow M$ is also transverse to the $\Sigma_i$. Since $N$ has intersection $+1$ with the reference circle, equation [15] tells us that at least one of the hypersurfaces $\Sigma_i$ has non-zero intersection with the reference circle. Setting $\Sigma = \Sigma_i$ for some such $i$, we will henceforth denote the corresponding inclusions map $j_i : \Sigma \hookrightarrow M$ and its lift $\tilde{j}_i : \Sigma \hookrightarrow \tilde{M}$. Our mapping torus model of $X$ now gives us a diffeomorphism $\tilde{M} = (N \times \mathbb{R}) \# (\bigoplus_{\ell=1}^{\infty} P)$, along with a blow-down map $\hat{b} : \tilde{M} \to N \times \mathbb{R}$ that lifts $b : M \to X$. However, $\{p\} \times \mathbb{R} \subset N \times \mathbb{R}$ now meets $\hat{b} \circ \tilde{j}(\Sigma)$ transversely in a set whose oriented count exactly computes the homological intersection number $n \neq 0$ of our reference circle $S^1$ with $\Sigma$. Thus, if $\mathcal{P} : N \times \mathbb{R} \to N$ denotes the first-factor projection, the smooth map $\mathcal{P} \circ \hat{b} \circ \tilde{j} : \Sigma \to N$ has degree $n \neq 0$. However, since $N$ is enlargeable by hypothesis, this implies that $\Sigma$ is enlargeable, too, and therefore does not admit a metric of positive scalar curvature. But we have also already noticed that $\Sigma = \Sigma_i$ must admit a metric $h$ of positive scalar curvature conformal to the restriction of the putative positive-scalar-curvature metric $g$ on $M$! This contradiction therefore shows that $M$ cannot actually admit such a metric $g$, and hence that $\mathcal{Y}(M) \leq 0$, as claimed.

Using this, one can now prove the following satisfying result, which was discovered in the course of conversations with Michael Albanese:

**Theorem 11.** Let $M$ be the underlying smooth 4-manifold of a compact complex surface $(M^4, J)$ of Kodaira dimension 0 or 1. Then $\mathcal{Y}(M) = 0$.

**Proof.** Because the other cases are covered by my earlier paper [64] we may henceforth assume that $b_1(M)$ is odd and that Kod$(M_J) = 1$. Since these previous results also showed that $\mathcal{Y}(M) \geq 0$, we also merely need to show that $M$ does not admit metrics of positive scalar curvature. If $X$ is the minimal model of $(M, J)$, then, after normalization, the pluricanonical system defines a holomorphic map $X \to C$ to a smooth connected complex curve $C$. Because $b_1$ is odd by assumption, there cannot [23] be any fibers that are just unions of rational curves. Thus, $X$ has Euler characteristic zero, and $X \to C$ has at worst multiple fibers. Now equip $C$ with an orbifold structure by giving each point a weight equal to the multiplicity of the corresponding fiber. Because Kod = 1 by assumption, we must have $\chi^{\text{orb}}(C) \leq 0$, because $(M, J)$ would otherwise [38] §2.7 be a Hopf surface, and thus have Kod = $-\infty$. In particular, $C$ must be a good orbifold in
the sense of Thurston, so that \( C = \hat{C}/\Gamma \), where \( \hat{C} \) is a smooth complex curve of positive genus, and where the finite group \( \Gamma \) acts biholomorphically on \( \hat{C} \). Pulling \( X \) back to \( \hat{C} \) then produces an (unramified) cover \( \hat{X} \rightarrow X \) with a smooth fibration \( \hat{X} \rightarrow \hat{C} \) that makes it into a bundle of elliptic curves. Since \( \text{Kod}(\hat{X}) = 1 \), the base \( \hat{C} \) must have genus \( \geq 2 \), and, at the price of perhaps replacing \( \hat{C} \) with an unbranched cover, we can then kill the rotational monodromy, and thereby arrange for \( \hat{X} \rightarrow \hat{C} \) to just be a principal \( \mathbb{C}^* \)-bundle over \( \hat{C} \). Moreover, this cover \( \hat{X} \) must still have \( b_1 \) odd, because if \( \hat{X} \) admitted a Kähler metric, the fiberwise-average of its local push-forwards would then produce a forbidden Kähler metric on \( X \). The Chern classes of the two \( \mathbb{C}^* \)-factors must therefore be linearly dependent over \( \mathbb{Q} \), and by again passing to a cover and then changing basis if necessary, one may arrange for exactly one of these Chern classes to be non-zero. This reduces the problem to the case of \( \hat{X} \approx N \times S^1 \), where \( N \rightarrow \hat{C} \) is a circle bundle of non-zero degree over a Riemann surface of genus \( \geq 2 \). However, this \( N^3 \) is enlargeable, because it contains a homologically non-trivial 2-torus, namely the preimage of any homologically non-trivial \( S^1 \subset \hat{C} \). Proposition \ref{prop:enlarge} therefore says that \( \hat{X} \# k\mathbb{CP}_2 \) cannot admit positive-scalar-curvature metrics for any \( k \). But since \( M \) has a covering space that is precisely of this form, it follows that \( M \) cannot admit positive-scalar-curvature metrics either. Since we also know that \( \mathcal{Y}(M) \geq 0 \), it therefore follows that \( \mathcal{Y}(M) = 0 \), as claimed. 

This immediately implies Theorem \ref{thm:main} and so might give one the false impression that the parity of \( b_1 \) should have little effect on results like Theorem \ref{thm:main}. However, life becomes significantly more complicated when \( \text{Kod} = -\infty \). When \( b_1 \) is even, the surfaces of Kodaira dimension \( -\infty \) are all rational or ruled, meaning that they are exactly \( \mathbb{C}P_2 \), \( \mathbb{C}P_1 \)-bundles over Riemann surfaces, and their blow-ups; and Theorem \ref{thm:main} is made possible, in part, by the fact that every manifold on this list admits Riemannian metrics of positive scalar curvature. By contrast, no complete classification is currently available for the complex surfaces with \( b_1 \) odd and Kodaira dimension \( -\infty \), which are known, for historical reasons, as surfaces of class VII. The most familiar class-VII surfaces are the (primary) Hopf surfaces \( (\mathbb{C}^2 - \{0\})/\mathbb{Z} \), which are diffeomorphic to \( S^3 \times S^1 \); and since these and their blow-ups \( \approx (S^3 \times S^1) \# k\mathbb{CP}_2 \) obviously also admit metrics of positive scalar curvature, one might hope for the same pattern to continue to hold even when \( b_1 \) is odd. However, Inoue discovered families of minimal class-VII surfaces that are topologically quite unlike Hopf surfaces, and Michael
Albanese recently showed that this topological difference has a major impact on the Yamabe invariant:

**Theorem 12 (Albanese).** Let $X$ be an Inoue surface, in the above sense, and let $M$ be obtained from $X$ by blowing up $k \geq 0$ points. Then $\mathcal{Y}(M) = 0$.

**Proof.** Inoue’s examples, which were all constructed as quotients of a half-space in $\mathbb{C}^2$, are grouped into three families, but each such complex surface $X$ is diffeomorphic to the mapping torus $\mathcal{T}_\phi$ of some self-diffeomorphism $\phi$ of a 3-manifold $N$. For one family, $N$ is the 3-torus $\mathbb{T}^3$, and $\phi \in SL(3, \mathbb{Z})$ is an affine map, while for the others, $N \to \mathbb{T}^2$ is a non-trivial circle bundle over the 2-torus, and $\phi$ is an automorphism of its nilgeometry. Since any such $N$ is enlargeable, and since any blow-up $M$ is diffeomorphic to $X \# k\mathbb{CP}^2$ for some $k \geq 0$, Proposition 4 therefore tells us that $\mathcal{Y}(M) \leq 0$.

On the other hand, each such $X$ admits an $F$-structure of positive rank, in the sense of Cheeger-Gromov, and so admits families of metrics which collapse to zero volume with bounded curvature. Modifying such metrics by inserting scaled-down Burns metrics on $\mathbb{CP}^2 - \{pt\}$, we thus obtain sequences of metrics on any blow-up $M$ with $\mathcal{S} \searrow 0$. This shows that $\mathcal{I}_s(M) = 0$, and hence $\mathcal{Y}(M) \geq 0$. Putting these two observations together, we therefore have $\mathcal{Y}(M) = 0$, as claimed. \(\square\)

Theorem 12 at least implies that all known class-VII surfaces therefore have non-negative Yamabe invariant, and the global-spherical-shell conjecture credibly claims that no other class-VII surfaces remain to be discovered. See Dloussky and Teleman for an overview of this classification problem.

Finally, while Theorem 11 focused on the notion of enlargeability because it behaves well under maps of non-zero degree, it is perhaps also worth noting that this hypothesis is in fact **optimal**. Indeed, Ricci-flow methods show that a smooth compact oriented 3-manifold $N$ admits a positive-scalar-curvature metric if and only if it is a connected sum of spherical space forms $S^3/\Gamma_j$ and/or copies of $S^2 \times S^1$. Geometrization goes on to tell us that any other 3-fold either contains an incompressible 2-torus or has an enlargeable connect-summand that is a compact $K(\pi_1)$ modeled on one of the other six Thurston geometries. Because any 3-manifold that contains an incompressible torus is also enlargeable, and because the connect sum of an enlargeable manifold with any spin manifold is also enlargeable, it therefore follows that a 3-manifold $N$ has $Y(N) \leq 0$ iff it is enlargeable. Nonetheless, when $Y(N^3) \leq 0$, the same reasoning also says that one can detect this fact using the results of Schoen-Yau, because Agol’s proof
of the virtual Haken conjecture implies that any such $N$ has a finite cover that, for any metric, must contain a stable minimal surface of genus $\geq 1$.

4. Dimension Four: the Yamabe Positive Case

If $(M^4, g, J)$ is a compact Kähler-Einstein manifold with negative scalar curvature, Theorem 4 asserts that $g$ actually realizes $\mathcal{Y}(M)$. The same conclusion also holds in the Ricci-flat case, because such an $M$ is finitely covered by either by $K3$ or $T^4$, and so cannot admit positive-scalar-curvature metrics. Unfortunately, however, life becomes significantly more complicated in the positive case. Nonetheless, the same phenomenon does still occur\cite{62} in one interesting case here:

**Theorem 13.** The Fubini-Study metric on $\mathbb{CP}^2$ realizes the Yamabe invariant $\mathcal{Y}(\mathbb{CP}^2)$. In other words,

$$\mathcal{Y}(\mathbb{CP}^2) = 4\pi\sqrt{2}c_1^2(\mathbb{CP}^2) = 12\pi\sqrt{2}.$$  

Moreover, modulo diffeomorphisms and rescalings, the Fubini-Study metric is the unique Yamabe metric that achieves this minimax.

While this was originally discovered\cite{62} using the perturbed Seiberg-Witten equations, we will instead describe the simpler proof later given in my joint paper with Gursky\cite{62}. Observe that, for the spin$^c$ structure on $\mathbb{CP}^2$ induced by the usual complex structure, the spin$^c$ Dirac operator

$$\mathcal{D}_\theta : \Gamma(V^+) \to \Gamma(V^-)$$

has index 1, because this is the Todd genus $h^{0,0} - h^{0,1} + h^{0,2}$ of $\mathbb{CP}^2$. Now, for an arbitrary metric $g$ on $\mathbb{CP}^2$, choose the connection $\theta$ so that $\frac{i}{\sqrt{2}} F_\theta$ is the unique harmonic 2-form representing $c_1(L) = c_1(\mathbb{CP}^2, J)$. Because $\mathbb{CP}^2$ has positive-definite intersection form $\cdot$, this harmonic form is therefore self-dual, and we therefore have

$$\|F_\theta\|^2_{L^2} = 4\pi^2 c_1^2(\mathbb{CP}^2) = 36\pi^2.$$  

(47)

However, since $\text{Ind} \mathcal{D}_\theta > 0$, the twisted spin bundle $V_+$ must have a smooth section $\Phi \neq 0$ with $\mathcal{D}_\theta \Phi = 0$. Applying the Weitzenböck formula \cite{20}, we therefore have

$$0 = 2\Delta |\Phi|^2 + 4|\nabla_\theta \Phi|^2 + s|\Phi|^2 + 8\langle -i F_\theta^+, \sigma(\Phi) \rangle$$

$$\geq 2\Delta |\Phi|^2 + (s - 2\sqrt{2}|F_\theta|)|\Phi|^2$$
so that

$$0 \geq \int_M (s_g - 2\sqrt{2}|F_\theta|_g)|\Phi|^2 d\mu,$$

(48)

with equality only if $\Phi$ is parallel. However, this would then force $\sigma(\Phi)$ to consequently be a non-zero parallel section of $\Lambda^+$, so equality in (48) can only happen if $g$ is Kähler.

Let’s now see what this tells us about conformal rescalings $\hat{g} = u^2 g$ of our metric $g$. One key observation is that harmonic 2-forms are conformally invariant in dimension four, so our prescription for $F_\theta$ is unchanged by a conformal change of metric. On the other hand, the point-wise norm of $F_\theta$ is of course not conformally invariant; instead, its norms with respect to $g$ and $\hat{g} = u^2 g$ are related by

$$|F_\theta|_{\hat{g}} = u^{-2} |F_\theta|_g.$$  

Thus, the Yamabe equation

$$\hat{s} = u^{-3} (6\Delta + s) u$$

implies that the “perturbed scalar curvature”

$$s := s - 2\sqrt{2}|F_\theta|$$

(49)

satisfies a perfect analog

$$\hat{s} = u^{-3} (6\Delta + s) u$$

of the Yamabe equation. (Here, of course, $s_g$ has been abbreviated as $s$, while $\hat{s}$ is used as short-hand for $s_{\hat{g}}$.) However, if we now let $\lambda_u$ denote the smallest eigenvalue of $6\Delta + s$, then the Rayleigh-quotient characterization of $\lambda_u$ implies that corresponding eigenspace is 1-dimensional, and is spanned by an everywhere-positive function [39, Theorem 8.38]. Thus, there is a $C^2$ function $u > 0$ on $M$ with

$$(6\Delta + s) u = \lambda_u u,$$

and the corresponding rescaled metric $\hat{g} = u^2 g$ therefore satisfies $\hat{s} = \lambda_u u^{-2}$ everywhere. In particular, this means that $\hat{s}$ has the same sign at every point! But [48], applied to $\hat{g}$, tells us that this rescaled metric must also satisfy

$$0 \geq \int_M \hat{s} |\hat{\Phi}|^2 d\mu_{\hat{g}}$$
for some section $\hat{\phi} \neq 0$ of $V_+$, with equality only if $\hat{g}$ is Kähler. Thus, either $\hat{s} < 0$ everywhere, or else $\hat{s} \equiv 0$ and $\hat{g}$ is Kähler. In either case, and the Cauchy-Schwarz inequality now tell us that

$$\int_M \hat{s} \hat{d} \mu_{\hat{g}} \leq 2\sqrt{2} \left( \int_M |F_{\phi}|^2 \hat{d} \mu_{\hat{g}} \right)^{1/2} \left( \int_M 1 \hat{d} \mu_{\hat{g}} \right)^{1/2}$$

so that

$$\mathcal{E}(\hat{g}) = \frac{\int_M s_2 \hat{d} \mu_{\hat{g}}}{\sqrt{\text{Vol}(M, \hat{g})}} \leq 2\sqrt{2} \|F_{\phi}\|_{L^2} = 12\pi \sqrt{2}$$

by (47). Minimizing $\mathcal{E}$ over the conformal class $[g]$, we consequently have

$$Y(\mathbb{CP}_2, [g]) \leq 12\pi \sqrt{2}.$$ (50)

Since any Einstein metric is a Yamabe metric by Obata’s theorem, and since the Fubini-Study metric is an Einstein metric with $\mathcal{E} = 12\pi \sqrt{2}$, the conformal class of the Fubini-Study metric exactly saturates the upper bound (50), and we therefore have $\mathcal{E}(\mathbb{CP}_2) = 12\pi \sqrt{2}$, as claimed.

If equality held in (50), the constructed conformal metric $\hat{g}$ would have to be both a Yamabe minimizer and a Kähler metric. In particular, $\hat{g}$ would necessarily be a Kähler metric of constant positive scalar curvature. But since any constant-scalar-curvature Kähler metric has harmonic Ricci form, and since $b_2(\mathbb{CP}_2) = 1$, this would force the Ricci form to be a constant times the Kähler form, thus making $\hat{g}$ a Kähler-Einstein metric of positive Einstein constant. In particular, Obata’s theorem would then say that $\hat{g}$ is the unique Yamabe minimizer in the given conformal class $[g]$.

Let us now normalize $\hat{g}$ by a constant rescaling, and thereby give it Ricci curvature 6 and scalar curvature 24. The assumption of equality in (50) then says that the volume of $(\mathbb{CP}_2, \hat{g})$ is $\pi^2/2$. However, if we now give the unit circle bundle $S \to \mathbb{CP}_1$ in $K^{1/3} \cong O(-1)$ the Riemannian submersion metric $\hat{h}$ for which the standard vertical $\partial/\partial \theta$ vector field is assigned length 1, this construction then yields [16, Theorem 9.76] an Einstein metric on $S \approx S^5$; indeed, this is the standard Sasaki-Einstein metric associated with the normalized Kähler-Einstein metric $\hat{g}$. However, since $(S, \hat{h})$ then has same volume $2\pi(\pi^2/2) = \pi^3$ and the same Ricci curvature 4 as the standard metric on $S^5$, it saturates the Bishop-Gromov inequality and must therefore be isometric to the standard unit 5-sphere. Since the $S^1$ orbits of the Killing field $\partial/\partial \theta$ moreover all have the same length, the group of isometries it generates must just be a diagonally embedded $S^1$ in the maximal torus $S^1 \times S^1 \times S^1$ of $SO(6)$. The Riemannian submersion $S^5 \to S^5/S^1$ is therefore standard, and the submersion metric on the base
$S^5/S^1$ must therefore be isometric to the usual Fubini-Study metric on $\mathbb{CP}^2$. This shows that equality holds in (50) if and only if $[g]$ is the conformal class of the Fubini-Study metric, albeit typically pulled back by some self-diffeomorphism of $\mathbb{CP}^2$.

It is not difficult to generalize this to other 4-manifolds with strictly positive intersection form $\cdot$. Indeed, essentially the same argument proves the following:

**Theorem 14.** Let $k \in \{1, 2, 3\}$, and let $\ell$ be any natural number. Then

$$12\pi \sqrt{2} \leq \mathcal{V}(k\mathbb{CP}^2\#\ell(S^1 \times S^3)) \leq 4\pi \sqrt{2k + 16}.$$  

In particular, these connected sums of copies of $\mathbb{CP}^2$ and $S^1 \times S^3$ all have Yamabe invariant strictly less than $\mathcal{V}(S^4) = 8\pi \sqrt{6}$.

Here the upper bound is proved by applying the previous argument to a spin$^c$ structure with $c_1(L) = (3), (3, 1), \text{ or } (3, 1, 1)$ relative to a basis for $H^2(\mathbb{Z}) \cong \mathbb{Z}^k$ in which the intersection form $\cdot$ is diagonalized. The corresponding spin$^c$ Dirac operator then again has index 1, so the proof is essentially the same as before, except that $c_2^2(L) = 8 + k$, and that, when $\ell \geq 1$ or $k \geq 2$, equality can immediately be ruled out in (48) because these Yamabe-positive 4-manifolds are never even homotopy-equivalent to rational or ruled surfaces, and so cannot admit Kähler metrics by Theorem 5. On the other hand, the lower bound for the Yamabe constant follows from the Kobayashi inequality (14), together with the fact, discovered independently by Kobayashi$^{54}$ and Schoen$^{80}$ that $\mathcal{V}(S^1 \times S^3) = \mathcal{V}(S^4)$.

One surprising consequence of Theorem 14 is the following:

**Corollary 1.** The Kod $\geq 0$ hypothesis in Theorem 6 is essential, because a primary Hopf surface $\approx S^3 \times S^3$ and its one-point blow-up $\approx [S^1 \times S^3]#\mathbb{CP}^2$ have different Yamabe invariants.

Indeed, since the Yamabe invariant is manifestly independent of orientation, Theorem 14 tells us that

$$\mathcal{V}([S^1 \times S^3]#\mathbb{CP}^2) = 12\sqrt{2}\pi < 8\sqrt{6}\pi = \mathcal{V}(S^1 \times S^3).$$

Of course, this example involves complex surfaces with $b_1$ odd, and so says nothing about the case where $b_1$ is even. One peculiarity is that, while we could have just as well used the blow-up of a Hopf surface at 2 or 3 points as the key exhibit in Corollary 1, the argument would fail if we instead blew up
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4 or more points. It is an open problem to determine whether this limitation of Theorem 14 is a mere technical artifact, or whether it genuinely reflects of some undiscovered geometric phenomenon. Settling this question either way might easily shed new light on other open problems in the subject.

For complex surfaces with $b_1$ even and $\text{Kod} = -\infty$, the behavior of the Yamabe invariant under blowing up still remains to be elucidated, but could turn out to be rather more satisfying. These complex surfaces are exactly the rational or ruled ones, and their underlying smooth compact oriented 4-manifolds are characterized by the positivity of the Yamabe invariant and the existence of an orientation-compatible symplectic structure. Among these, the non-spin, simply connected ones are just the 4-manifolds $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ arising from blow-ups of $\mathbb{CP}^2$. For these, the Kobayashi inequality immediately tells us that

$$\mathcal{Y}(\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}) \geq \mathcal{Y}(\mathbb{CP}^2) = 12\pi \sqrt{2},$$

so blowing up $\mathbb{CP}^2$ certainly never decreases the Yamabe invariant; rather, the open question is whether blowing up could increase it in this context. While the method used to prove Theorem 13 does not allow us to settle this, it nonetheless does provide some interesting partial information.

Indeed, if $(M, g)$ is a smooth compact oriented Riemannian 4-manifold with indefinite intersection form $\bullet$, we saw in (34) that

$$H^2(M, \mathbb{R}) = H_g^+ \oplus H_g^-,$$

where $H_g^\pm$ consists of cohomology classes whose harmonic representative with respect to $g$ is self-dual (respectively, anti-self-dual). In particular, for any metric $g$ and any spin$^c$ structure $c$ on $M$, we thus obtain a cohomology class $c_1^+ = [c_1(L)]^+ \in H_g^+$ as the orthogonal projection of $c_1 = c_1(L)$ with respect to $\bullet$. We can then repeat the argument previous by again taking $iF_\theta$ to be the harmonic representative of $2\pi c_1$, but now carefully noting that $iF_\theta^+$ is now the harmonic representative of $2\pi c_1^+$, and that only $F_\theta^+$ appears in (29). Assuming that the index

$$\text{Ind} \, D_\theta = \frac{c_1^2(L) - \tau(M)}{8}$$

is positive, much the same argument then shows that any conformal class $[g]$ on $M$ then satisfies

$$Y(M, [g]) \leq 4\pi \sqrt{2(c_1^+)^2}. \quad (51)$$

This should have a familiar ring to it, because we previously saw $(c_1^+)^2$ arise in a Seiberg-Witten estimate that we used to bound the Yamabe invariant from above. But in the Seiberg-Witten setting, we got $-4\pi \sqrt{2(c_1^+)^2}$
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as an upper bound for $Y(M, [g])$, whereas in (51) this expression instead occurs with the opposite sign. However, while

$$(c_1^+)^2 \geq c_1^2(L)$$

for any metric $g$, this is much less useful in the context of (51) than in the setting of (42), because $(c_1^+)^2$ will become larger and larger as $\mathcal{H}_g^+$ tilts further and further away from $c_1 = c_1(L)$, so that, when $b_- \neq 0$, the upper bound (51) will actually be weaker than the Aubin bound (7) for many conformal classes $[g]$. Of course, one can partially remedy this by considering many different spin$^c$ structures on $M$, but this does not seem to ever lead to definitive results on the Yamabe invariant.

Nonetheless, (51) does still have interesting ramifications for Yamabe invariants of specific conformal classes. For example, on a smooth compact oriented 4-manifold $M$ with $b_+ = 1$, one can fix some specific $[\omega] \in H^2(M, \mathbb{R})$ with $[\omega]^2 > 0$, and only consider those metrics $g$ for which $\mathcal{H}_g^+ = \text{span} \ [\omega]$. For any such metric and any spin$^c$ structure $c$ for which the Dirac operator has positive index, (51) then takes the more concrete form

$$Y(M, [g]) \leq 4\pi \frac{|c_1(L) \bullet [\omega]|}{\sqrt{[\omega]^2}/2},$$

with equality iff some Yamabe minimizer $g' \in [g]$ is a Kähler metric of non-negative-constant scalar curvature that is compatible with a complex structure $J$ for which $c_1(M, J) = c_1(L)$. In particular, if $g$ is a metric such that $c_1(L) \in \mathcal{H}_g^+$, then

$$Y(M, [g]) \leq 4\pi \sqrt{2c_1^2(L)},$$

with equality iff the (unique) Yamabe minimizer of $[g]$ is Kähler-Einstein. This puts Theorem 13 in a broader context, and should also be compared with Theorems 4 and 14.

Nonetheless, when the intersection form $\bullet$ is indefinite, Kähler-Einstein metrics with positive Einstein constant never achieve the Yamabe invariant; that is, they are never supreme Einstein metrics. Indeed, the non-spin 4-manifolds that carry such metrics are exactly $\mathbb{CP}_2 \# k\mathbb{CP}_2$, $k = 3, \ldots, 8$, and on any of these the Einstein-Hilbert action of a Kähler-Einstein metric has $\mathcal{E} = 4\pi \sqrt{2(9-k)} < Y(\mathbb{CP}_2)$, so the Kobayashi inequality (14) says that one can do better via a sequence that tends toward the Fubini-Study conformal class. Similarly, while $\mathbb{CP}_2 \# k\mathbb{CP}_2$ admits an Einstein metric for $k = 1, 2$ that is conformally Kähler (but not Kähler), these Einstein metrics have $\mathcal{E} < 4\pi \sqrt{2(9-k)}$, and so, again, do not achieve the Yamabe
invariant. Indeed, [31] shows that any maximizing sequence must involve metrics for which $H^1 \cong \mathbb{R}$ becomes as far away from $c_1(J)$ as the pull-back $H^2(\mathbb{C}P^2, \mathbb{R}) \hookrightarrow H^2(\mathbb{C}P^2 \# k\mathbb{C}P^2, \mathbb{R})$. Moreover, since [45, 47] any oriented non-symmetric 4-dimensional Einstein manifold $(M, g)$ must have Einstein-Hilbert action $\mathcal{E}(g) \leq 4\pi \sqrt{2(2\chi + 3\tau)}(M)$, the existence of a supreme Einstein metric on $\mathbb{C}P^2 \# k\mathbb{C}P^2$ is a priori impossible for any $k \neq 0$.

The classification of del Pezzo surfaces [29] tells us that there is only one other 4-manifold that can admit a positive-scalar-curvature Kähler-Einstein metric, namely the spin manifold $S^2 \times S^2 = \mathbb{C}P^1 \times \mathbb{C}P^1$. Here, the Kobayashi inequality (14) tells us nothing at all about the Yamabe invariant, but a closer examination of this example allows us to draw similar, albeit weaker, conclusions. Indeed, given a positive real number $t$, consider the homogeneous Kähler metric $g_t$ on $S^2 \times S^2$ given by the Riemannian product of two standard metrics on the 2-sphere $S^2 \subset \mathbb{R}^3$, one of Euclidean radius $t^{1/2}$ and the other of Euclidean radius $(t - 1)^{1/2}$. This Kähler metric is then Einstein iff $t = 1$. Our choice of radii guarantees that the volume of $(S^2 \times S^2, g_t)$ is always $16\pi^2$, but that its scalar curvature is $2(t + t^{-1}) \geq 4$. Thus, the Einstein-Hilbert action of $(S^2 \times S^2, g_t)$ is $\mathcal{E} = 8\pi(t + t^{-1})$, which has a unique minimum of $16\pi$ at $t = 1$, corresponding to the Kähler-Einstein metric. A beautiful argument of Böhm, Wang, and Ziller [21, Theorem 5.1] now shows, for $t$ in some interval $(1 - \epsilon, 1 + \epsilon)$, the metrics $g_t$ are, up to constant scale, the only constant-scalar-curvature metrics in their conformal classes. Thus, the $g_t$ must be (unique) Yamabe minimizers when $t$ is sufficiently close to 1. In particular, the Einstein metric $g_1$ is not supreme, and $\mathcal{F}(S^2 \times S^2) > 16\pi$.

This makes it irresistible to ask precisely which of the homogeneous metrics $g_t$ are actually Yamabe minimizers. However, separation of variables reveals that the first positive eigenvalue of the Laplacian $\Delta$ of $g_t$ is given by $\lambda_1 = \min(2t, 2/t)$. This now allows us to show that some $g_t$ are not Yamabe minimizers. Indeed, let $f$ be a $\lambda_1$-eigenfunction of unit $L^2$-norm, and let us now compute the Yamabe energy of $\hat{g} = u^2 g$, where $u = 1 + \epsilon f$
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for $\varepsilon$ small. We obtain

$$\mathcal{E}'((1+\varepsilon f)^2 g_t) = \frac{\int |\nabla f|^2 + s(1+\varepsilon f)^2 d\mu}{\sqrt{\int (1+\varepsilon f)^4 d\mu}} = sV^{1/2} + 6(\lambda_1 - \frac{8}{3})\varepsilon^2 V^{-1/2} + O(\varepsilon^3)$$

where $V = 16\pi^2$ is the volume of $g_t$. Thus, if $g_t$ is a Yamabe metric, we must have $\lambda_1 \geq \frac{s}{3}$, and our computations of $s$ and $\lambda_1$ therefore reveal that a necessary condition for $g_t$ to be a Yamabe metric is that $t \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$. Amazingly enough, however, we also have

$$\mathcal{E}(g_t) = 8\pi(t + t^{-1}) \leq 12\pi\sqrt{2} = \mathcal{Y}(\mathbb{CP}^2) \iff t \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right].$$

This suggests a continuity argument to show that $\mathcal{Y}(S^2 \times S^2) \geq \mathcal{Y}(\mathbb{CP}^2)$. Indeed, the set of $t \in [1, \sqrt{2}]$ for which $g_t$ is a Yamabe minimizer is a priori closed, because the Yamabe constant is a continuous function on the space of metrics. On the other hand, a straightforward generalization of the Böhm-Wang-Ziller argument shows that the set of $t \in (1, \sqrt{2})$ for which $g_t$ is the unique Yamabe minimizer is a priori open, because the inverse-function-theorem part of the argument works as long as $\lambda_1 > \frac{s}{3}$. This reduces the problem to showing that this smaller set is also closed, which sounds plausible, but could be quite difficult.

Finally, Petean and Ruiz [78] have shown that $\mathcal{Y}(S^2 \times \Sigma_g) > \frac{8}{3}\mathcal{Y}(S^4)$, where $\Sigma_g$ is a Riemann surface of any genus $g$; and the same proof moreover appears to also work for twisted products. In conjunction with [14], this implies that the Yamabe invariants of all rational or ruled surfaces actually lie in a comparatively narrow range. However, determining the precise value of the Yamabe invariants for these manifolds continues to represent the sort of daunting challenge that illustrates just how much of this territory still remains mysterious and largely unexplored.

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