Gowdy waves as a test-bed for constraint-preserving boundary conditions

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Abstract. Gowdy waves, one of the standard ’apples with apples’ tests, is proposed as a test-bed for constraint-preserving boundary conditions in the non-linear regime. As an illustration, energy-constraint preservation is separately tested in the Z4 framework. Both algebraic conditions, derived from energy estimates, and derivative conditions, deduced from the constraint-propagation system, are considered. The numerical errors at the boundary are of the same order than those at the interior points.

Constraint-preserving boundary conditions is a very active research topic in Numerical Relativity [1, 2, 3]. During this decade, many conditions have been proposed, adapted in each case to some specific evolution formalism: Fritelli-Reula [4], Friedrich-Nagy [5], KST [6, 7], Z4 [8], Generalized-Harmonic [9, 10, 1, 2], or BSSN [3]. Cross-comparison among different evolution formalisms has been carried out (’apples with apples’ initiative [11, 12]). But only periodic boundary conditions have been considered up to now.

We endorse some recent claims (by Winicour and others) that the cross-comparison effort should be extended to the boundaries treatment. In this paper, we show that Gowdy waves [13], one of the ’apples with apples’ tests, is suitable for boundary conditions cross-comparison in the non-linear regime. As an illustration, we test separately the energy-constraint preservation in the Z4 framework. We compare algebraic conditions, derived from energy estimates, with derivative conditions, deduced from the constraint-propagation system. The resulting numerical errors at the boundary are of the same order-of-magnitude than those at interior points.

1. The Gowdy waves metric
Let us consider the Gowdy solution [13], which describes a space-time containing plane polarized gravitational waves. The line element can be written as

\[ ds^2 = \frac{1}{t^{1/2}} e^{Q/2} \left( -dt^2 + dz^2 \right) + t \left( e^{P} \, dx^2 + e^{-P} \, dy^2 \right) \] (1)

where the quantities \( Q \) and \( P \) are functions of \( t \) and \( z \) only, and periodic in \( z \). The initial slice \( t = t_0 \) is usually chosen so that the simulations can start with an homogeneous lapse.

Let us now perform the following time coordinate transformation

\[ t = t_0 \, e^{-\tau / \tau_0}, \] (2)

so that the expanding line element (1) is seen in the new time coordinate \( \tau \) as collapsing towards the \( t = 0 \) singularity, which is approached only in the limit \( \tau \to \infty \). This “singularity avoidance”
property of the $\tau$ coordinate is due to the fact that the resulting slicing by $\tau = \text{constant}$ surfaces is harmonic [14]. We will run our simulations in normal coordinates, starting with a constant lapse $\alpha_0 = 1$ at $\tau = 0$ ($t = t_0$).

Standard cross-comparison tests [11, 12] are currently done with periodic boundary conditions. But one gets basically the same results by setting up algebraic boundary conditions, which take advantage of the symmetries of the Gowdy line element. For a rectangular grid, planar symmetry allows trivial boundary conditions along the $x$ and $y$ directions. Also, allowing for the fact that the $z$ dependence in (1) is only through $\cos(2\pi z)$, one can set reflecting boundary conditions for the interval $0 \leq z \leq 1$. In this way, the Gowdy waves metric is obtained as a sort of stationary gravitational wave in a cavity with perfectly reflecting walls.

We can then set up a full set of algebraic boundary conditions, which are consistent with the Gowdy line element (1) for all times. This opens the door to a selective testing procedure, where one could for instance try some constraint-preserving condition for the longitudinal and transverse-trace modes, while keeping the exact condition for the transverse traceless ones. Or, as we will do below, testing just some energy-constraint preserving boundary conditions while dealing with all the remaining modes in an exact way.

2. Characteristic decomposition

We will consider here the the first-order version in normal coordinates, as described in refs. [15, 16]. For further convenience, we will recombine the basic first-order fields ($K_{ij}, D_{ijk}, A_i, \Theta, Z_i$) in the following way:

$$\Pi_{ij} = K_{ij} - (\text{tr}K - \Theta)\gamma_{ij}, \quad V_i = \gamma^{rs}(D_{irs} - D_{ris}) - Z_k \quad (3)$$

$$\mu_{ijk} = D_{ijk} - (\gamma^{rs}D_{irs} - V_i)\gamma_{jk}, \quad W_i = A_i - \gamma^{rs}D_{irs} + 2V_i \quad (4)$$

so that the new basis is ($\Pi_{ij}, \mu_{ijk}, W_i, \Theta, V_i$). Note that the vector $Z_i$ can be recovered easily from this new basis as

$$Z_i = -\mu_{ik}^k. \quad (5)$$

In order to compute the characteristic matrix, we will consider the standard form of (the principal part of) the evolution system as follows

$$\partial_t \mathbf{u} + \alpha \partial_n \mathbf{F}^n(\mathbf{u}) = \cdots, \quad (6)$$

where $\mathbf{u}$ stands for the array of dynamical fields and $\mathbf{F}^n$ is the array of fluxes along the direction given by the unit vector $\mathbf{n}$. With this choice of basic dynamical fields, the principal part of the evolution system gets a very simple form in the harmonic slicing case:

$$F^n(W_i) = 0 \quad F^n(\Theta) = V^n \quad F^n(V_i) = n_i \Theta \quad (7)$$

$$F^n(\Pi_{ij}) = \lambda_{nij} \quad F^n(\mu_{nij}) = n_k \Pi_{ij} \quad (8)$$

where the index $n$ means a projection along $n_i$, and we have noted for short

$$\lambda_{nij} = \mu_{nij} + n_i W_j - W_n \gamma_{nij}, \quad (9)$$

where round brackets denote index symmetrization.

We can now identify the constraint modes, by looking at the Fluxes of $\Theta$ and $Z_i$ in the array (7-8). It follows from (7) that the energy-constraint modes are given by the pair

$$E^\pm = \Theta \pm V_n \quad (10)$$
with propagation speed $\pm \alpha$. Also, allowing for (5,8), we can easily recover the flux of $Z_i$:

$$F^n(Z_i) = -\Pi^n_i$$

so that we can identify the momentum-constraint modes with the three pairs

$$M_i^\pm = \Pi_{ni} \pm \lambda_{nni},$$

with propagation speed $\pm \alpha$. Note that, allowing for (3), the longitudinal component $\Pi_{nn}$ does correspond with the transverse-trace component of the extrinsic curvature $K_{ij}$. We give now the remaining modes: the fully transverse ones, with propagation speed $\pm \alpha$,

$$T_{AB}^\pm = \Pi_{AB} \pm \lambda_{nAB},$$

(13)

(the capital indices denote a projection orthogonal to $n_i$), and the standing modes (zero propagation speed):

$$W_i, \quad V_A, \quad \mu_{Aij}. $$

Note that the standing modes (14), the energy modes (10) and the transverse momentum modes $M_A^\pm$ actually vanish for the Gowdy line element (1).

3. Energy-constraint preserving boundary conditions

In refs. [15, 16], the system above was shown to be symmetric hyperbolic, by providing a suitable energy estimate. We can rewrite it here as

$$\Pi_{ij} \Pi^{ij} + \lambda_{kij} \lambda^{kij} + \Theta^2 + V_k V^k + W^k W^k$$

This leads to the following sufficient condition for stability

$$\left(\Pi^{ij} \lambda_{nij} + \Theta V_n\right)_{|\Sigma} \geq 0$$

(16)

where $\Sigma$ stands for the boundary surface ($n$ being here the outward normal).

Let us consider for instance the boundary at $z = 1$. We can enforce there the partial set of exact (reflection) boundary conditions $\lambda_{nij} = 0$, so that the requirement (16) reduces to

$$\left(\Theta V_n\right)_{|\Sigma} \geq 0$$

(17)

which can be used for a separate test of energy-constraint preserving boundary conditions.

As an illustration, we will test two such conditions. The first one is given in the form of a logical gate:

$$\left(\Theta V_n\right)_{|\Sigma} < 0 \quad \Rightarrow \quad \Theta_{|\Sigma} = 0$$

(18)

($\Theta$-gate), so that it only acts when condition (17) is violated. The second one is an advection equation

$$\partial_t \Theta_{|\Sigma} = -\alpha \partial_n \Theta - \eta \Theta,$$

(19)

where we have included a suitable damping term. Note that the principal part of the constraint-preserving condition (19) coincides with the ‘maximal dissipation’ one, $\partial_t E^- = 0$, which is not constraint-preserving in the generic case. Condition (19) can be understood as a sort of maximal dissipation condition for the energy-constraint evolution equation

$$\partial_{tt}^\Sigma \Theta - \alpha^2 \Delta \Theta = \cdots,$$

(20)

which follows from (the time component of) the covariant divergence of the Z4 field equations [15].
Figure 1. Θ profiles along the z direction for a Gowdy waves simulation ending at τ = 100. From bottom to top, results for the pure advection condition (19), damped advection (with η = 0.2), Θ-gate (18), and exact reflection (included here for comparison).

We show in Fig. 1 our results for some numerical simulations ending at τ = 100. We have included in the plot the exact (reflection) results for comparison. It is clear that the Θ-gate condition gets very close to (actually slightly better than) the exact result in this case. The pure advection condition (19) is off by half-an-order of magnitude. However, a suitable damping term (we have taken η = 2) greatly improves this, leading in this case to even less error than the pure reflection condition. Note that we are showing here the Θ profiles, giving the cumulated energy-constraint deviation. The average energy-constraint error in these strong-field simulations is actually smaller.

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