HERMITE-HADAMARD’S MID-POINT TYPE INEQUALITIES FOR GENERALIZED FRACTIONAL INTEGRALS

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Abstract. Some Hermite-Hadamard’s mid-point type inequalities related to Katugampola fractional integrals are obtained where the first derivative of considered mappings is Lipschitzian or convex. Also some mid-point type inequalities are given for Lipschitzian mappings, with the aim of generalizing the results presented in previous works. Finally as an application, some generalized inequalities in connection with special means are provided.

1. Introduction

Recently U. N. Katugampola in [12], introduced an Erdélyi-Kober type fractional integral operator which now is known as Katugampola fractional integral. The Katugampola fractional integral is a generalization of Riemann-Liouville and Hadamard fractional integrals simultaneously. Let’s review these concepts.

The following definition is modified version of Definition 4.3 in [12].

Definition 1.1. Let $[a, b] \subset \mathbb{R}$ be a finite interval. The left and right side Katugampola fractional integrals of order $\alpha > 0$ are defined respectively by

\[\rho I_{a^+}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)dt,\]

and

\[\rho I_{b^-}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t)dt,\]

where $a < x < b$, $\rho > 0$, $\Gamma(\alpha)$ is Gamma function and the integrals exist.

The relation between Riemann-Liouville fractional integrals i.e.,

\[J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt\]

and

\[J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt\]

and Hadamard fractional integrals i.e.,

\[H_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \ln \frac{x}{t} \right)^{\alpha-1} f(t)dt\]

and

\[H_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left( \ln \frac{t}{b} \right)^{\alpha-1} f(t)dt,\]

has been shown in the following result:

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Theorem 1.2. Let $\alpha > 0$ and $\rho > 0$. Then for $x > a$

(a) $\lim_{\rho \to 1} \rho I_a^\alpha f(x) = J_a^\alpha f(x)$ and $\lim_{\rho \to 1} \rho I_b^\alpha f(x) = J_b^\alpha f(x)$,

(b) $\lim_{\rho \to 0^+} \rho I_a^\alpha f(x) = H_a^\alpha f(x)$ and $\lim_{\rho \to 0^+} \rho I_b^\alpha f(x) = H_b^\alpha f(x)$.

For basic and fundamental information about fractional integrals and operators we refer an interested reader to [6, 14, 16, 22].

In [2], the authors obtained two important inequalities in connection with Hermite-Hadamard inequality and Katugampola fractional integrals. The first is Hermite-Hadamard type inequality related to Katugampola fractional integrals:

Theorem 1.3. Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in X_\rho^p(a^\rho, b^\rho)$. If $f$ is also a convex function on $[a, b]$, then the following inequalities hold:

$$f\left(\frac{a^\rho + b^\rho}{2}\right) \leq \frac{\alpha \rho \Gamma(\alpha + 1)}{2(b^\rho - a^\rho)^\alpha} \left[\rho I_a^\alpha f(b^\rho) + \rho I_b^\alpha f(a^\rho)\right] \leq \frac{f(a^\rho) + f(b^\rho)}{2}. \quad (1)$$

Note that inequalities obtained in [1], generalize the Hermite-Hadamard inequality related to Riemann-Liouville fractional integrals presented by M. Z. Sarikaya et al. [23](also see [25]):

$$f\left(\frac{a + b}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_a^\alpha f(b) + J_b^\alpha f(a)] \leq \frac{f(a) + f(b)}{2}. \quad (2)$$

If in [2] we consider $\alpha = 1$, then we recapture classic Hermite-Hadamard inequality [7, 8, 17] for a convex function $f$ on $[a, b]$:

$$f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$ 

For more results about Hermite-Hadamard inequality and fractional integrals see [11, 10, 11, 18, 20, 21, 24, 25] and references therein.

The second is the following inequality in connection with [1]:

Theorem 1.4. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on $(a^\rho, b^\rho)$ with $0 \leq a < b$. If $|f'|$ is convex on $[a^\rho, b^\rho]$, then the following inequality holds:

$$\left|\frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\alpha \rho \Gamma(\alpha + 1)}{2(b^\rho - a^\rho)^\alpha} [\rho I_a^\alpha f(b^\rho) + \rho I_b^\alpha f(a^\rho)]\right| \leq \frac{b^\rho - a^\rho}{2\rho(\alpha + 1)} (1 - \frac{1}{2^\alpha}) \|f'(a^\rho)\| + \|f'(b^\rho)\|. \quad (3)$$

We call (3) as trapezoid type inequality in connection with [1], because of the geometric interpretation contained in the following interesting classic inequality obtained by S. S. Dragomir et al. in [3].

Theorem 1.5. Let $f : I^\rho \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\rho$, $a, b \in I^\rho$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:

$$\left|\frac{f(a) + f(b)}{2} - \int_a^b f(x)dx\right| \leq \frac{(b - a)^2}{8} \left(\left|f'(a)\right| + \left|f'(b)\right|\right).$$

Also U. S. Kirmaci, in [15] obtained another classic inequality related to Hermite-Hadamard inequality as the following:
Mid-Point Type Inequalities

In this section we obtain three Hermite-Hadamard’s mid-point type theorems related to Katugam- pola fractional integrals by considering the concepts of Lipschitzian and convex mappings. The following lemma is of importance to achieve our main results.

Lemma 2.1. Let $f : I \to \mathbb{R}$ be a differentiable function on $I^*$. For $0 \leq a < b$ and $\rho > 0$, suppose that $a^\rho, b^\rho \in I^*$ and $f' \in L[a^\rho, b^\rho]$. Then for $\alpha > 0$, the following identities for fractional integrals hold:

$$f\left(\frac{a^\rho + b^\rho}{2}\right) - \rho(b^\rho - a^\rho)\left\{\int_0^{t^\alpha} t^{(\alpha+1)\rho-1} f'(t^\rho a^\rho + (1-t^\rho)b^\rho) dt + \int_1^{t^\alpha} (t^\alpha - 1)t^{(\alpha-1)\rho-1} f'(t^\rho a^\rho + (1-t^\rho)b^\rho) dt\right\} =$$

$$-\rho(b^\rho - a^\rho)\left\{\int_0^{t^\alpha} t^{(\alpha+1)\rho-1} f'(t^\rho b^\rho + (1-t^\rho)a^\rho) dt + \int_1^{t^\alpha} (t^\alpha - 1)t^{(\alpha-1)\rho-1} f'(t^\rho b^\rho + (1-t^\rho)a^\rho) dt\right\} =$$

$$f\left(\frac{a^\rho + b^\rho}{2}\right) - \frac{\rho\rho^\alpha \Gamma(\alpha + 1)}{(b^\rho - a^\rho)^\alpha} \frac{\rho^\alpha I_{a^\rho}^\alpha f(a^\rho)}{2}.$$

Furthermore

$$f\left(\frac{a^\rho + b^\rho}{2}\right) - \frac{\rho\rho^\alpha \Gamma(\alpha + 1)}{(b^\rho - a^\rho)^\alpha} \frac{\rho^\alpha I_{a^\rho}^\alpha f(a^\rho)}{2}.$$
Proof. By the use of integration by parts we get
\[
\int_0^\frac{1}{\sqrt[\rho]{\alpha}} t^\alpha \rho^{\alpha-1} f'(t^\rho a^\rho + (1-t^\rho)b^\rho)dt = t^\alpha f\left(\frac{t^\rho a^\rho + (1-t^\rho)b^\rho}{\rho(a^\rho - b^\rho)}\right)\bigg|_0^1 - \frac{1}{\frac{\alpha}{\rho(b^\rho - a^\rho)}} f\left(\frac{a^\rho + b^\rho}{2}\right) + \frac{\alpha}{b^\rho - a^\rho} \int_0^\frac{1}{\sqrt[\rho]{\alpha}} t^\alpha \rho^{\alpha-1} f(t^\rho a^\rho + (1-t^\rho)b^\rho)dt.
\]
Similarly we have
\[
\int_0^1 (t^\alpha - 1) t^\alpha - 1 f'(t^\rho a^\rho + (1-t^\rho)b^\rho)dt = \left(\frac{1}{\frac{\alpha}{\rho(b^\rho - a^\rho)}} f\left(\frac{a^\rho + b^\rho}{2}\right) + \frac{\alpha}{b^\rho - a^\rho} \int_0^1 t^\alpha \rho^{\alpha-1} f(t^\rho a^\rho + (1-t^\rho)b^\rho)dt.
\]
Now merging (8) and (9) with applying the change of variable \(x^\rho = t^\rho a^\rho + (1-t^\rho)b^\rho\) imply that
\[
\int_0^\frac{1}{\sqrt[\rho]{\alpha}} t^\alpha \rho^{\alpha-1} f'(t^\rho a^\rho + (1-t^\rho)b^\rho)dt + \int_0^1 (t^\alpha - 1) t^\alpha - 1 f'(t^\rho a^\rho + (1-t^\rho)b^\rho)dt = \frac{\alpha}{(b^\rho - a^\rho)} \int_0^b \left(\frac{b^\rho - x^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} f(x^\rho) \frac{x^\rho - 1}{b^\rho - a^\rho} dx - \frac{1}{\rho(b^\rho - a^\rho)} f\left(\frac{a^\rho + b^\rho}{2}\right) = \frac{\alpha}{(b^\rho - a^\rho)^{\alpha+1}} \rho f(a^\rho) - \frac{1}{\rho(b^\rho - a^\rho)} f\left(\frac{a^\rho + b^\rho}{2}\right).
\]
Note that for identity (10), the proof is similar. To prove (11), it is enough to add identity (10) to (10).

2.1. \(f'\) and \(f\) are Lipschitzian Mappings.

Definition 2.2. \((19\)) A function \(f : [a, b] \rightarrow \mathbb{R}\) is said to satisfy a Lipschitz condition on interval \([a, b]\) (M-Lipschitzian) if there is a constant \(M\) so that, for any two points \(x, y \in [a, b]\),
\[
|f(x) - f(y)| \leq M|x - y|.
\]

By the use of Lemma 2.1 we can obtain a new mid-point type theorem in the case that first derivative of considered function is Lipschitzian.

Theorem 2.3. Let \(f : I \rightarrow \mathbb{R}\) be a differentiable function on \(I^\rho\). For \(0 \leq a < b\) and \(\rho > 0\), suppose that \(a^\rho, b^\rho \in I^\rho\) and \(f'\) satisfies a Lipschitz condition on \([a^\rho, b^\rho]\) with respect to \(M\). Then for \(\alpha > 0\), the following mid-point type inequality holds:
\[
\left| f\left(\frac{a^\rho + b^\rho}{2}\right) - \frac{\alpha f(a^\rho) + \rho f(b^\rho)}{2(b^\rho - a^\rho)\alpha} \right| \leq \frac{M(b^\rho - a^\rho)^2(\alpha^2 - \alpha + 2)}{8(\alpha + 1)(\alpha + 2)}.
\]
Proof. From identity (7) we have

\[
\frac{\rho(b^\rho - a^\rho)}{2}\left\{ \int_0^{\sqrt[\rho]{t}} t^{(\alpha+1)\rho-1} \left| f'(t^\rho b^\rho + (1-t^\rho)a^\rho) - f'(t^\rho a^\rho + (1-t^\rho)b^\rho) \right| dt + \int_0^{\sqrt[\rho]{t}} (t^\rho - 1) \left| f'(t^\rho b^\rho + (1-t^\rho)a^\rho) - f'(t^\rho a^\rho + (1-t^\rho)b^\rho) \right| dt \right\} \leq M \frac{\rho(b^\rho - a^\rho)}{2} \left\{ \int_0^{\sqrt[\rho]{t}} t^{(\alpha+1)\rho-1} \left| (2t^\rho - 1)(b^\rho - a^\rho) \right| dt + \int_0^{\sqrt[\rho]{t}} (1-t^\rho) t^{\rho-1}(2t^\rho - 1) dt \right\} = \frac{M \rho(b^\rho - a^\rho)^2}{2} \left\{ \frac{\sqrt[\rho]{t}}{(\alpha+1)\rho} - \frac{\sqrt[\rho]{t}}{(\alpha+1)\rho} + \frac{2}{(\alpha+2)\rho} + \frac{1}{(\alpha+2)\rho} + \frac{1}{4\rho} \right\} = \frac{M \rho(b^\rho - a^\rho)^2}{8(\alpha+1)(\alpha+2)}.
\]

The details are omitted in calculating of above integrals. \(\square\)

Corollary 2.4. Let \( f : I \to \mathbb{R} \) be a differentiable function on \( I^2 \) with \( a, b \in I^\circ \). If \( f' \) satisfies a Lipschitz condition on \([a, b]\) with respect to \( M \), then the following mid-point type inequality holds:

\[
\left| f\left( \frac{a+b}{2} \right) - \frac{\alpha \Gamma(\alpha+1)}{2(b-a)\alpha} \int_a^b f(x) dx \right| \leq \frac{M(b-a)^2}{8(\alpha+1)(\alpha+2)} \left( \alpha^2 - \alpha + 2 \right).
\]

(11)

Also if we consider \( \alpha = 1 \) in (11) we get

\[
\left| f\left( \frac{a+b}{2} \right) - \frac{b-a}{2} \int_a^b f(x) dx \right| \leq \frac{M(b-a)^2}{24},
\]

(12)

which is new in literature.

Furthermore if in Theorem 2.3 we consider that \( f \) is twice differentiable on \([a^\rho, b^\rho]\), \( f' \) is convex on \([a^\rho, b^\rho]\) and \( M = \sup_{t \in [a^\rho, b^\rho]} |f''(t)| < \infty \), then by using Lagrange's theorem for any \( x, y \in (a^\rho, b^\rho) \), there exists a \( t \in (x, y) \) such that

\[
|f'(x) - f'(y)| = |x - y||f''(t)| \leq M|x - y|,
\]

which shows that \( f' \) satisfy a Lipschitz condition on \([a^\rho, b^\rho]\) and so again we have (10).

Finally if \( f \) is twice differentiable on \([a^\rho, b^\rho]\), the functions \( f \) and \( f' \) are convex on \([a^\rho, b^\rho]\) and \( M = \sup_{t \in [a^\rho, b^\rho]} |f''(t)| < \infty \), then from (11) we get

\[
0 \leq \frac{\alpha \rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)\alpha} \left[ \rho \int_a^b f(b^\rho) + \rho \int_b^a f(a^\rho) \right] - \frac{f\left( \frac{a^\rho + b^\rho}{2} \right)}{2} \leq \frac{M(b^\rho - a^\rho)^2}{8(\alpha+1)(\alpha+2)} \left( \alpha^2 - \alpha + 2 \right).
\]

Example 2.5. Consider \( f(x) = \sin x \), \( x \in [a, b] \) with \( 0 \leq a < b \). From the fact that \( |\cos x - \cos y| \leq |x - y| \), we have that \( f'(x) = \cos x \) satisfies a Lipschitz condition with respect to \( M = 1 \). Then from Theorem 2.3 we have

\[
\left| \sin\left( \frac{a^\rho + b^\rho}{2} \right) - \frac{\alpha \rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)\alpha} \left[ \rho \int_a^b \sin(b^\rho) + \rho \int_b^a \sin(a^\rho) \right] \right| \leq \frac{(b^\rho - a^\rho)^2}{8(\alpha+1)(\alpha+2)} \left( \alpha^2 - \alpha + 2 \right),
\]
holds:

\[ f \text{ satisfies a Lipschitz condition on } [a, b]. \]

Now by letting \( \rho \to 1 \), we obtain that

\[ \left| \sin \left( \frac{a + b}{2} \right) - \frac{(b - a)^\alpha}{2(b - a)^\alpha} \sin a + J_a^{\alpha + 1} \cos b + \frac{(b - a)^\alpha}{\alpha} \sin b - J_b^{\alpha + 1} \cos a \right| \leq \frac{(b - a)^2}{8(\alpha + 1)(\alpha + 2)}. \]  

If in (13), we set \( \alpha = 1 \) we get

\[ \left| \sin \left( \frac{a + b}{2} \right) - \frac{\sin a + \sin b}{2} - \frac{J_a^2 \cos b + J_b^2 \cos a}{2(b - a)} \right| \leq \frac{(b - a)^2}{24}. \]  

It follows that

\[ J_a^2 \cos b = \int_a^b (b - t) \cos t \, dt = -(b - a) \sin a + \cos a - \cos b, \]

and

\[ J_b^2 \cos a = \int_a^b (t - a) \cos t \, dt = (b - a) \sin b + \cos b - \cos a, \]

which along with (14) we deduce that

\[ \left| \sin \left( \frac{a + b}{2} \right) - \frac{\cos a - \cos b}{b - a} \right| \leq \frac{(b - a)^2}{24}. \]

Remark 2.6. If in Example 2.5 we consider \( f(x) = a^n (a > 0) \), \( f(x) = x^n (n \geq 2) \), \( f(x) = \ln x \) and \( f(x) = -\frac{1}{x} \), then with the fact that \( f' \) satisfies a Lipschitz condition with respect to some \( M \) we can obtain some new mid-point estimation type inequalities for \( f \) with new bounds.

To prove the following result, we use the structure presented in [4] where the considered functions are Lipschitzian.

**Theorem 2.7.** Suppose that for \( \rho > 0 \) and \( 0 \leq a < b \), the function \( f : [a^\rho, b^\rho] \to \mathbb{R} \) satisfies a Lipschitz condition on \( [a^\rho, b^\rho] \) with respect to \( M \). Then the following mid-point type inequality holds:

\[ \left| f \left( \frac{a^\rho + b^\rho}{2} \right) - \frac{\alpha \rho M \Gamma(\alpha + 1)}{(b^\rho - a^\rho)^\alpha} \left[ \left( \frac{1}{2} \right)^\alpha I_{a^\rho}^\alpha f(b^\rho) + \left( 1 - \left( \frac{1}{2} \right)^\alpha \right)^\rho I_{b^\rho}^\alpha f(a^\rho) \right] \right| \leq M(b^\rho - a^\rho) \left( \frac{(1/2)^\alpha + \alpha - 1}{2(\alpha + 1)} \right). \]
Proof. For any \( t \in [0, 1] \), we have
\[
\left| t^{\alpha \rho} f(a^{\rho}) + (1 - t^{\alpha \rho}) f(b^{\rho}) - f\left( (1-t) a^{\rho} + (1-t) b^{\rho} \right) \right| \leq \tag{16}
\]
\[
t^{\alpha \rho} \left| f(a^{\rho}) - f\left( t^{\alpha \rho} a^{\rho} + (1-t^{\alpha \rho}) b^{\rho} \right) \right| + \left| 1 - t^{\alpha \rho} \right| \left| f\left( b^{\rho} \right) - f\left( t^{\alpha \rho} b^{\rho} + (1-t^{\alpha \rho}) b^{\rho} \right) \right| \leq
\]
\[
M t^{\alpha \rho} \left| (1-t^{\rho})(a^{\rho} - b^{\rho}) \right| + M \left| 1 - t^{\alpha \rho} \right| t^{\rho} \left| b^{\rho} - a^{\rho} \right| = M \left| b^{\rho} - a^{\rho} \right| \left[ t^{\alpha \rho} + t^{\rho} - 2t^{\alpha+1}\rho \right].
\]
Now if in (16) consider \( t = \frac{1}{2^{\rho}} \), then we deduce that
\[
\left| \frac{1}{2}^{\alpha \rho} f(a^{\rho}) + \left( 1 - \frac{1}{2}^{\alpha \rho} \right) f\left( \frac{a^{\rho} + b^{\rho}}{2} \right) \right| \leq \tag{17}
\]
\[
M \left| b^{\rho} - a^{\rho} \right| \left[ \left( \frac{1}{2} \right)^{\alpha \rho} + \frac{1}{2} - 2\left( \frac{1}{2} \right)^{\alpha+1} \right] = M \frac{1}{2} \left| b^{\rho} - a^{\rho} \right|.
\]
If in (17) we replace \( a^{\rho \cdot -1} \) with \( t^{\alpha \rho} a^{\rho} + (1 - t^{\rho}) b^{\rho} \) and replace \( b^{\rho \cdot -1} \) with \( t^{\alpha \rho} b^{\rho} + (1 - t^{\rho}) a^{\rho} \), then we obtain
\[
\left| \frac{1}{2}^{\alpha \rho} f\left( t^{\alpha \rho} a^{\rho} + (1 - t^{\rho}) b^{\rho} \right) + \left( 1 - \frac{1}{2}^{\alpha \rho} \right) f\left( t^{\alpha \rho} b^{\rho} + (1 - t^{\rho}) a^{\rho} \right) \right|
\]
\[
\leq \frac{M}{2} \left| t^{\alpha \rho} a^{\rho} + (1 - t^{\rho}) a^{\rho} - (1-t^{\rho}) b^{\rho} \right|
\]
\[
M \left| b^{\rho} - a^{\rho} \right| \left| 2t^{\rho - 1} \right|.
\]
Multiplying above inequality with \( t^{\rho} a^{\rho -1} \) and then integrating with respect to \( t \) on \([0,1]\) imply that
\[
\left| \frac{1}{2}^{\alpha \rho} \int_{0}^{1} t^{\alpha \rho -1} f\left( t^{\alpha \rho} a^{\rho} + (1 - t^{\rho}) b^{\rho} \right) dt + \left( 1 - \frac{1}{2}^{\alpha \rho} \right) \int_{0}^{1} t^{\alpha \rho -1} f\left( t^{\alpha \rho} b^{\rho} + (1 - t^{\rho}) a^{\rho} \right) dt - \right|
\]
\[
\int_{0}^{1} t^{\alpha \rho -1} f\left( \frac{a^{\rho} + b^{\rho}}{2} \right) \right| \leq \frac{M}{2} \left| b^{\rho} - a^{\rho} \right| \int_{0}^{1} t^{\alpha \rho -1} \left| 2t^{\rho - 1} \right| dt.
\]
So it follows that
\[
\left| \frac{1}{2}^{\alpha \rho} \rho^{\alpha -1} \Gamma(\alpha+1) \rho f_{a}^{\alpha} + \left( 1 - \frac{1}{2}^{\alpha \rho} \right) \rho^{\alpha -1} \Gamma(\alpha+1) \rho f_{b}^{\alpha} - \frac{1}{\alpha \rho} f\left( \frac{a^{\rho} + b^{\rho}}{2} \right) \right| \leq \tag{18}
\]
\[
\frac{M}{2} \left| b^{\rho} - a^{\rho} \right| \int_{0}^{1} t^{\alpha \rho -1} \left| 2t^{\rho - 1} \right| dt = \frac{M}{2} \left| b^{\rho} - a^{\rho} \right| \left[ \int_{0}^{\frac{1}{2}^{\rho}} t^{\alpha \rho -1} dt - 2 \int_{\frac{1}{2}^{\rho}}^{1} t^{\alpha \rho -1} dt + \right.
\]
\[
2 \int_{\frac{1}{2}^{\rho}}^{1} t^{\alpha \rho -1} dt - \int_{0}^{1} t^{\alpha \rho -1} dt \right] = \frac{M}{2} \left| b^{\rho} - a^{\rho} \right| \left[ \left( \frac{1}{2} \right)^{\alpha -1} + \alpha - 1 \right].
\]
Finally by multiplying (18) with \( \alpha \rho \) we obtain (15). This completes the proof. \( \square \)

**Corollary 2.8.** Similar to Corollary 2.4 we have that
\[
\left| f\left( \frac{a^{\rho} + b^{\rho}}{2} \right) - \frac{\alpha \Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[ \left( \frac{1}{2} \right)^{\alpha} J_{a}^{\alpha} f(b) + \left( 1 - \frac{1}{2}^{\alpha \rho} \right) J_{b}^{\alpha} f(a) \right] \right| \leq \tag{19}
\]
\[
\frac{M\left( b-a \right) \left( \frac{1}{2} \right)^{\alpha -1} + \alpha - 1}{2(\alpha + 1)}.
\]
and

\[
|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx| \leq \frac{M(b-a)}{4}.
\]  

(20)

Inequality (20) originally obtained in [4]. Also we can get (15) without using absolute value symbol if \( f \) is differentiable, convex on \([a, b]\) and \( M = \sup_{t \in [a, b]} |f'(t)| < \infty \).

**Example 2.9.** In (19), consider \( f(x) = \tan x \), \( x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \). For any \( a, b \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \), there exists \( t \in (a, b) \) such that

\[
1 + \tan^2 t = \frac{\tan b - \tan a}{b - a},
\]

showing that

\[
|\tan b - \tan a| \leq 4|b - a|.
\]

So for \( 0 \leq a < b \leq \frac{\pi}{3} \) we have

\[
|\tan \left(\frac{a+b}{2}\right) - \frac{\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} \left[\left(\frac{1}{2}\right)^\alpha J_a^{\alpha+} \tan(b) + \left(1 - \left(\frac{1}{2}\right)^\alpha \right) J_b^{\alpha-} \tan(a)\right]| \leq \quad \quad (21)
\]

\[
\frac{4(b-a)}{2(\alpha+1)} \left(\frac{1}{\alpha} - 1\right)
\]

where

\[
J_a^{\alpha+} \tan(b) = \frac{1}{\Gamma(\alpha)} \int_a^b (x-t)^{\alpha-1} \tan(t) dt,
\]

and

\[
J_b^{\alpha-} \tan(a) = \frac{1}{\Gamma(\alpha)} \int_a^b (t-x)^{\alpha-1} \tan(t) dt.
\]

Now if in (21) we set \( \alpha = 1 \), then we get

\[
|\tan \left(\frac{a+b}{2}\right) - \frac{1}{b-a} \ln \frac{\sec b}{\sec a}| \leq b - a.
\]

**Remark 2.10.** (1) For functions \( f(x) = \frac{1}{x} \), \( f(x) = e^x \) and \( f(x) = -\ln x \), we can obtain some inequalities which generalize the corresponding inequalities obtained in Corollary 2.3 in [4].

(2) Suppose that \( f' \) is a Lipschitzian mapping with respect to \( M_1 \) and \( f \) is a Lipschitzian mapping with respect to \( M_2 \). Comparing two inequalities (12) and (20) implies that in the case \( M_1 < \frac{6M_2}{b-a} \), we have better estimation for mid-point type inequalities.

(3) If \( f' \) is an \( M \)-Lipschitzian mapping, then from inequality \( ||f'(x)| - |f'(y)|| \leq |f'(x) - f'(y)| \) we have \( |f'| \) is Lipschitzian with respect to \( M \). So in this case, we can replace \( f \) in (15) with \( |f'| \).

2.2. \( |f'| \) is Convex. Now we obtain Hermite-Hadamard's mid-point type inequality related to Katugampola fractional integrals for functions whose the absolute values of first derivative are convex. The Hermite-Hadamard's trapezoid type inequality of this kind is presented in Theorem 1.4.
Theorem 2.11. Let \( f : I \to \mathbb{R} \) be a differentiable function on \( I^\circ \). For \( 0 \leq a < b \) and \( \rho > 0 \), suppose that \( |f'| \) is convex and integrable on \([a^\rho, b^\rho]\). Then in the case that \( 0 < \alpha \rho \leq 1 \), the following mid-point type inequality holds:

\[
\left| f\left( \frac{a^\rho + b^\rho}{2} \right) - \alpha \rho \frac{\Gamma(\alpha + 1)}{(b^\rho - a^\rho)\alpha} \left[ \rho I_{a^\rho}^f f(b^\rho) + \rho I_{b^\rho}^f f(a^\rho) \right] \right| \leq \frac{b^\rho - a^\rho}{2^\alpha + 1(\alpha + 1)} \left( |f'(a^\rho)| + |f'(b^\rho)| \right).
\]

Proof. From (20) and (21) we have

\[
\begin{align*}
&\left| 2f\left( \frac{a^\rho + b^\rho}{2} \right) \right| - \alpha \rho \frac{\Gamma(\alpha + 1)}{(b^\rho - a^\rho)\alpha} \left[ \rho I_{a^\rho}^f f(b^\rho) + \rho I_{b^\rho}^f f(a^\rho) \right] \right| \\
&\left| f\left( \frac{a^\rho + b^\rho}{2} \right) \right| - \alpha \rho \frac{\Gamma(\alpha + 1)}{(b^\rho - a^\rho)\alpha} \left[ \rho I_{a^\rho}^f f(b^\rho) + \rho I_{b^\rho}^f f(a^\rho) \right] \right| \\
&\rho(b^\rho - a^\rho) \left\{ \int_0^{t(a^\rho + b^\rho)/2} t^\alpha \left[ |t| f'(b^\rho) \right] \right\} dt + \\
&\int_0^{t(a^\rho)} t^\alpha \left[ |t| f'(b^\rho) \right] dt + \int_{\rho(a^\rho - t^\rho)}^{t(a^\rho)} t^\alpha \left[ |t| f'(b^\rho) \right] dt + \\
&\int_{t(a^\rho)}^{t(a^\rho + b^\rho)} t^\alpha \left[ |t| f'(b^\rho) \right] dt \\
&\int_0^{t(a^\rho)} t^\alpha \left[ |t| f'(b^\rho) \right] dt + \\
&\int_0^{t(a^\rho + b^\rho)} t^\alpha \left[ |t| f'(b^\rho) \right] dt \\
&\int_0^{t(a^\rho + b^\rho)} t^\alpha \left[ |t| f'(b^\rho) \right] dt + \\
&\int_0^{t(a^\rho + b^\rho)} t^\alpha \left[ |t| f'(b^\rho) \right] dt \\
&\rho(b^\rho - a^\rho) \left\{ \int_0^{t(a^\rho + b^\rho)/2} t^\alpha \left[ |t| f'(b^\rho) \right] \right\} dt + \\
&\frac{1}{2^\alpha + 1(\alpha + 2)\rho} |f'(b^\rho)| + \frac{\alpha + 3}{2^\alpha + 1(\alpha + 2)\rho} |f'(b^\rho)| + \\
&\frac{1}{2^\alpha + 1(\alpha + 2)\rho} |f'(b^\rho)| + \frac{\alpha + 3}{2^\alpha + 1(\alpha + 2)\rho} |f'(b^\rho)| + \\
&\frac{1}{2^\alpha + 1(\alpha + 2)\rho} |f'(b^\rho)| + \frac{\alpha + 3}{2^\alpha + 1(\alpha + 2)\rho} |f'(b^\rho)| \right\} = \frac{b^\rho - a^\rho}{2\alpha(\alpha + 1)} \left( |f'(a^\rho)| + |f'(b^\rho)| \right).
\end{align*}
\]

Note that in calculations of integrals (22) and (23) we used the fact that \(|1 - t^\alpha| \leq |1 - t|^\alpha\), where \(0 < \alpha \leq 1\). Some other details are omitted. \( \Box \)

Remark 2.12. Theorem 2.11 is a generalized form of Theorem 2 in \([9]\) (consider \( \rho = 1 \)) and so is a generalization for Theorem 1.6 (consider \( \alpha = \rho = 1 \)).
3. Special Means

In this section as an application of our results we obtain some generalized inequalities related to two well known special means:

\[ A(a, b) = \frac{a + b}{2} \quad \text{arithmetic mean,} \]

\[ L_n(a, b) = \left[ \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right]^{\frac{1}{n}} \quad \text{generalized log-mean, } n \in \mathbb{N}, a < b. \]

In fact we give some generalized estimation type results for the difference of two means. For more concepts and results about special means, see [5] and references therein. Consider \( f(t) = t^n \) for \( t \geq 0, n \in \mathbb{N} \). Now \( M = nb^{(n-1)\rho} \), and so from Theorem 2.7 we have

\[
\begin{align*}
\left| \left( \frac{a^{\rho} + b^{\rho}}{2} \right)^n - \frac{nb^{(n-1)\rho}(b^\rho - a^\rho)}{2(\alpha + 1)} \left( \frac{1}{2(n-1)\rho} \right) \right| \leq & \quad (24) \\
\frac{\rho^n \int_a^{b^{\rho}} \left( 1 - t^\rho \right)^{\alpha n^2} dt}{\rho^n \prod_{i=0}^{n-1}(\alpha + i)}.
\end{align*}
\]

By using integration by parts for \( n \) times we have

\[
\begin{align*}
\rho^{\alpha n} & \rho^{\alpha n} \frac{1}{\Gamma(\alpha)} \left[ a^n(b^\rho - a^\rho)^\alpha - \frac{na^n - \rho a^n(\rho a^\rho - a^\rho)^\alpha}{\alpha \rho} \right] + \frac{\rho^n \prod_{i=0}^{n-1}(\alpha + i)}{\alpha \rho \prod_{i=0}^{n-1}(\alpha + i)} \int_a^{b^{\rho}} \left( 1 - t^\rho \right)^{\alpha n^2} dt \leq (24).
\end{align*}
\]

Also

\[
\begin{align*}
\rho^{\alpha n} & \rho^{\alpha n} \frac{1}{\Gamma(\alpha)} \left[ b^n(b^\rho - a^\rho)^\alpha - \frac{nb^n - \rho b^n(\rho b^\rho - b^\rho)^\alpha}{\alpha \rho} \right] + \frac{\rho^n \prod_{i=0}^{n-1}(\alpha + i)}{\alpha \rho \prod_{i=0}^{n-1}(\alpha + i)} \int_a^{b^{\rho}} \left( 1 - t^\rho \right)^{\alpha n^2} dt \leq (24).
\end{align*}
\]
Now letting $\rho \to 1$ in (25) and (26), along with some calculations, implies that:

\[
\left| \left( \frac{a+b}{2} \right)^n - \frac{a^2}{(b-a)^\alpha} \left[ \frac{1}{2} \alpha \sum_{m=0}^n \frac{a^{n-m}(b-a)^{\alpha+m} P(n,m)}{\prod_{i=0}^m (\alpha+i)} \right] \right| \leq \frac{nh(b^{(n-1)}(b-a)\left(\frac{1}{2}\right)^{\alpha-1} + \alpha - 1)}{2(\alpha+1)}.
\]

(27)

where

\[
\sum_{m=0}^n \frac{a^{n-m}(b-a)^{\alpha+m} P(n,m)}{\prod_{i=0}^m (\alpha+i)} = \Gamma(\alpha) J_{a+}^\alpha (b^n) = \lim_{\rho \to 1} \rho J_{a+}^\alpha (b^n),
\]

and

\[
\sum_{m=0}^n \frac{(-1)^m b^{n-m}(b-a)^{\alpha+m} P(n,m)}{\prod_{i=0}^m (\alpha+i)} = \Gamma(\alpha) J_{b-}^\alpha (a^n) = \lim_{\rho \to 1} \rho J_{b-}^\alpha (a^n),
\]

which is the number of possible permutations of $k$ objects from a set of $n$.

In special case if we consider $\alpha = 1$, then it is not hard to see that

\[
J_{a+}^1 f(b^n) + J_{b-}^1 f(a^n) = \frac{2(b^{n+1}-a^{n+1})}{n+1}.
\]

So from inequality (27) we obtain that

\[
\left| A^n(a, b) - L^n(a, b) \right| \leq \frac{n!}{(n-m)!}.
\]

(28)

So we conclude that inequalities (24) and (27) are generalization of inequality (28), which has been obtained in [4].

Also with similar argument as above, from Theorem 2.11 we have

\[
\left| \left( \frac{a^\rho + b^\rho}{2} \right)^n - \frac{\alpha^\rho \Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha} \left[ \rho J_{a+}^\alpha (b^\rho) + \rho J_{b-}^\alpha (a^\rho) \right] \right| \leq \frac{n(b^\rho-a^\rho)}{2^{\alpha+1}(\alpha+1)} (a^{(n-1)\rho} + b^{(n-1)\rho}),
\]

(29)

and if $\rho \to 1$, then

\[
\left| \left( \frac{a+b}{2} \right)^n - \frac{\alpha^2}{2(b-a)^\alpha} \sum_{m=0}^n \frac{[a^{n-m} + (-1)^m b^{n-m}](b-a)^{\alpha+m} P(n,m)}{\prod_{i=0}^m (\alpha+i)} \right| \leq \frac{n(b-a)(a^{n-1}+b^{n-1})}{2^{\alpha+1}(\alpha+1)}.
\]

(30)

Now if in (30) we consider $\alpha = 1$, then we recapture inequality (3.1) in [15]:

\[
\left| A^n(a, b) - L^n(a, b) \right| \leq \frac{n!}{(n-m)!} A(a^{n-1}, b^{n-1}),
\]

(31)

showing that (29) and (30) generalize (31).
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