On the independence of Heegner points associated to distinct quadratic imaginary fields

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Abstract

Let $E/\mathbb{Q}$ be an elliptic curve with a fixed modular parametrization $\Phi_E : X_0(N) \to E$ and let $P_1, \ldots, P_r \in E(\overline{\mathbb{Q}})$ be Heegner points attached to the rings of integers of distinct quadratic imaginary field $k_1, \ldots, k_r$. We prove that if the odd parts of the class numbers of $k_1, \ldots, k_r$ are larger than a constant $C = C(E, \Phi_E)$ depending only on $E$ and $\Phi_E$, then the points $P_1, \ldots, P_r$ are independent in $E(\overline{\mathbb{Q}})/E_{\text{tors}}$. We also discuss a possible application to the elliptic curve discrete logarithm problem.

Introduction

The theory of Heegner points provides a fundamental method for creating algebraic points on modular curves and on the elliptic curves that they parametrize. The work of Wiles et.al. [BCDT01, CDT99, TW95, Wil95] says that every elliptic curve $E/\mathbb{Q}$ of conductor $N$ admits a modular parametrization $\Phi_E : X_0(N) \to E$, so in particular there is a theory of Heegner points on elliptic curves defined over $\mathbb{Q}$. Heegner points appear prominently in the work of Gross, Kohnen, and Zagier [GKZ87, GZ86] on the Birch-Swinnerton-Dyer conjecture and in the work of Kolyvagin [Kol88a, Kol88b] on Mordell-Weil ranks and Shafarevich-Tate finiteness. (A nice survey of this material may be found in [Dar04].)

Of particular importance in the work of Kolyvagin and others is the construction of Euler systems of Heegner points. (See [Kol90] for a general formulation.) These are collections of Heegner points $(P_n)$ defined over a tower of ring class fields lying over a single quadratic imaginary field and satisfying various trace and Galois compatibility conditions. In this paper we consider the orthogonal problem of collections of Heegner points $(P_n)$ defined over class fields of different quadratic imaginary fields.

Our main theorem says that under a fairly mild class number condition, such a set of Heegner points on $E$ corresponding to distinct quadratic imaginary fields has maximal rank in $E(\overline{\mathbb{Q}})/E_{\text{tors}}$. We briefly state the result here and refer the reader to Sections 1 and 2 for definitions and to Section 8 for a more precise statement.

Theorem 1. Let $E/\mathbb{Q}$ be an elliptic curve (with a given modular parametrization). There is a constant $C = C(E)$ so that the following is true.

Let $k_1, \ldots, k_r$ be distinct quadratic imaginary fields whose class numbers satisfy

$$h(k_i)^{\text{odd}} \geq C$$

for all $1 \leq i \leq r$,

where $h^{\text{odd}}$ denotes the odd part of the integer $h$. Let $P_1, \ldots, P_r \in E(\overline{\mathbb{Q}})$ be Heegner points associated

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to (the ring of integers of) $k_1, \ldots, k_r$, respectively. Then

$$P_1, \ldots, P_r \text{ are independent in } E(\bar{Q})/E_{\text{tors}}.$$  

Remark 2. Theorem 1 is possibly not surprising, and indeed the statement may be true with no (or a much weaker) class number hypothesis. On the other hand, since the compositum of the quadratic imaginary fields $k_1, \ldots, k_r$ may have degree as small as $r$, as opposed to the maximal value of $2^r$, there seems no obvious reason why the associated Heegner points must be completely independent. Thus elementary considerations might lead to an estimate of the form

$$2^{\text{rank}(\mathbb{Z}P_1 + \cdots + \mathbb{Z}P_r)} \geq [k_1 \cdots k_r : Q],$$

but the proof of the stronger statement given in Theorem 1 requires a blend of class field theory, Galois theory, linear algebra (over $\mathbb{Z}/n\mathbb{Z}$), and Serre’s theorem on the image of Galois in $\text{Aut}(E_{\text{tors}})$.

Remark 3. Other authors have considered the behavior of Heegner points associated to different quadratic imaginary fields. In particular, we mention the fundamental work of Gross, Kohnen and Zagier [GKZ87]. In the notation of Theorem 1, the Heegner point $P_i \in E(\bar{Q})$ is defined over the Hilbert class field $K_i$ of $k_i$. We can obtain points defined over $\mathbb{Q}$ by taking the trace,

$$Q_i = \text{Trace}_{K_i/\mathbb{Q}}(P_i) \in E(\mathbb{Q}).$$

Gross, Kohnen, and Zagier [op.cit.] compute the canonical height pairing $\langle Q_i, Q_j \rangle$ of these points and prove that $Q_1, \ldots, Q_r$ generate a subgroup of $E(\mathbb{Q})$ of rank at most 1. More precisely, they show that

$$\text{rank}(\mathbb{Z}Q_1 + \cdots + \mathbb{Z}Q_r) = \begin{cases} 1 & \text{if } \text{ord}_{s=1} L(E/\mathbb{Q}, s) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This is in accordance with the predictions of the Birch-Swinnerton-Dyer conjecture.

Aside from its intrinsic interest, the independence result of Theorem 1 has a (negative) application to the elliptic curve discrete logarithm problem (ECDLP). If the theorem were false and Heegner points had a tendency to be dependent, then potentially there would be an algorithm to solve the ECDLP on elliptic curves with small coefficients by using Deuring lifts and Heegner points. We briefly sketch the idea in Section 8 and refer the reader to [RS] for a more detailed description.

Finally, in Section 10 we make some brief remarks and raise a question concerning the distribution of quadratic imaginary fields whose class numbers have bounded odd parts.

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1. Heegner points on $X_0(N)$

In this section we briefly review the theory of Heegner points on the modular curve $X_0(N)$ and in the next section we discuss Heegner points on elliptic curves. We refer the reader to [Dar04 §§3.1,3.3] and [Gro84] for further details.

Recall that the noncuspidal points of the modular curve $X_0(N)$ classify isomorphism classes of triples $(A, A', \phi)$ consisting of two elliptic curves $A$ and $A'$ and an isogeny $\phi : A \to A'$ whose kernel is cyclic of order $N$. Heegner points are associated to orders in quadratic imaginary fields, so we set

$$k/\mathbb{Q} \quad \text{a quadratic imaginary field},$$
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$\mathcal{O}_k$  the ring of integers of $k$,
$\mathcal{O}$  an order in $\mathcal{O}_k$.

Every order has the form $\mathcal{O} = \mathbb{Z} + c\mathcal{O}_k$ for a unique integer $c \geq 1$ called the conductor of $\mathcal{O}$. The discriminant of $\mathcal{O}$ is given by

$$\text{Disc}(\mathcal{O}) = c^2 \text{Disc}(\mathcal{O}_k), \quad \text{where } c^2 = (\mathcal{O}_k : \mathcal{O}).$$

In order to describe the Heegner points on $X_0$, we follow that notation used in [Dar04] and define:

- Pic($\mathcal{O}$) the Picard group (or class group) of $\mathcal{O}$, defined to be the group of isomorphism classes of rank 1 projective $\mathcal{O}$-modules. If $\mathcal{O} = \mathcal{O}_k$, then Pic($\mathcal{O}$) is the usual ideal class group of $\mathcal{O}_k$.

- Ell$^{(N)}$(O) the set of isomorphism classes of triples $(A, A', \phi)$ such that $A$ and $A'$ are elliptic curves satisfying $\text{End}(A) \cong \text{End}(A') \cong \mathcal{O}$ and $\phi : A \to A'$ is an isogeny with $\text{ker}(\phi) \cong \mathbb{Z}/N\mathbb{Z}$.

- CM$^{(N)}$(O) the set of points in the noncuspidal part of $X_0(N)$ corresponding to the triples $(A, A', \phi)$ in Ell$^{(N)}$(O).

**Definition.** We will generally identify without further comment the two sets

$$\text{Ell}^{(N)}(\mathcal{O}) \leftrightarrow \text{CM}^{(N)}(\mathcal{O}).$$

The points in either set are called Heegner points of $X_0(N)$.

It is clearly important to determine conditions on $\mathcal{O}$ that ensure that there exist nontrivial Heegner points.

**Proposition 4.** Assume that the discriminant of $\mathcal{O}$ is prime to $N$. Then the set Ell$^{(N)}$(O) is nonempty if and only if every prime dividing $N$ is split in $\mathcal{O}$.

**Proof.** See [Dar04, Proposition 3.8], or see [Gro84, §3] for a stronger statement in which it is only required that $N$ be relatively prime to the conductor of $\mathcal{O}$.. \( \square \)

**Definition.** We say that $\mathcal{O}$ satisfies the Heegner condition for $N$ if the following two conditions are satisfied:

1. \( \gcd(\text{Disc}(\mathcal{O}), N) = 1. \)
2. Every prime dividing $N$ is split in $\mathcal{O}$.

We say that $k$ satisfies the Heegner condition for $N$ if its ring of integers satisfies the condition.

Let $\mathcal{O}$ be an order in $k$. Class field theory associates to $\mathcal{O}$ a finite abelian extension $K_{\mathcal{O}}/k$, called the ring class field of $k$ attached to $\mathcal{O}$. The extension $K_{\mathcal{O}}/k$ is unramified outside the primes dividing the conductor of $\mathcal{O}$, and the Artin reciprocity map gives an isomorphism

$$\left( \cdot , K_{\mathcal{O}}/k \right) : \text{Pic}(\mathcal{O}) \xrightarrow{\sim} \text{Gal}(K_{\mathcal{O}}/k).$$

In particular, if $\mathfrak{p} \in \text{Pic}(\mathcal{O})$ corresponds to a prime ideal of $k$ that does not divide $\text{Disc}(\mathcal{O})$, then $(\mathfrak{p}, K_{\mathcal{O}}/k)$ is the inverse of the Frobenius element at $\mathfrak{p}$.

**Theorem 5.** Let $\mathcal{O}$ be an order in $k$ that satisfies the Heegner condition for $N$.

(a) The points in CM$^{(N)}$(O) are defined over $K_{\mathcal{O}}$, i.e.,

$$\text{CM}^{(N)}(\mathcal{O}) \subset X_0(N)(K_{\mathcal{O}}).$$
(b) The points of $\text{Ell}^{(N)}(O)$ are in one-to-one correspondence with the set of pairs

$$\{(n, \bar{a}) : \bar{a} \in \text{Pic}(O), \ n \text{ is a proper } O\text{-ideal, } O/n \cong \mathbb{Z}/N\mathbb{Z}\}.$$  

The correspondence is given explicitly by associating to a pair $(n, \bar{a})$ the cyclic $N$-isogeny

$$C/\bar{a} \rightarrow C/an^{-1}.$$  

(c) There is a natural action of $\text{Pic}(O)$ on $\text{Ell}^{(N)}(O)$ (and thus also on $\text{CM}^{(N)}(O)$) which we denote by $\star$. In terms of pairs $(n, \bar{a})$, it is given by the formula

$$\bar{b} \star (n, \bar{a}) = (n, \bar{a} \bar{b}).$$  

(d) The $\star$-action is compatible with the action of Galois via the reciprocity map in the sense that

$$y^{(bK_O/k)} = \bar{b}^{-1} \star y \quad \text{for all } y \in \text{CM}^{(N)}(O).$$  

Proof. See [Dar04, Chapter 3] or [Gro84].

For our purposes, the importance of Theorem 5 is that it allows us to conclude that every point in $\text{CM}^{(N)}(O)$ generates a large extension of $k$, as in the following result.

**Corollary 6.** Let $O$ be an order in $k$ that satisfies the Heegner condition for $N$ and let $y \in \text{CM}^{(N)}(O)$. Then

$$k(y) = K_O.$$

Proof. We know from Theorem 5(a) that $y$ is defined over $K_O$. Further, if we identify $y$ with a pair $(n, \bar{a})$ as in Theorem 5(b), then (c) and (d) tell us that the full set of Galois conjugates of $y$ is given by

$$\{y^\sigma : \sigma \in \text{Gal}(K_O/k)\} = \{\bar{b} \star y : \bar{b} \in \text{Pic}(O)\} = \{(n, \bar{a} \bar{b}^{-1}) : \bar{b} \in \text{Pic}(O)\} = \{(n, \bar{b}) : \bar{b} \in \text{Pic}(O)\}.$$  

The points $(n, \bar{b})$ are distinct for distinct $\bar{b} \in \text{Pic}(O)$, so we see that

$$[K_O : k] \geq [k(y) : k] = \#\{y^\sigma : \sigma \in \text{Gal}(K_O/k)\} \geq \#\text{Pic}(O).$$  

Class field theory [11] tells us that $\#\text{Pic}(O) = [K_O : k]$. Hence all of the inequalities in [2] are equalities, which proves that $k(y) = K_O$.  

2. Heegner points on elliptic curves

Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N$. The theorem of Wiles et.al. [BCDT01, CDT99, TW95, Wil95] says that there exists a modular parametrization

$$\Phi_E : X_0(N) \rightarrow E.$$  

The map $\Phi_E : X_0(N) \rightarrow E$ is a finite covering defined over $\mathbb{Q}$.

**Definition.** Let $k$ be a quadratic imaginary field and let $O$ be an order in $k$ that satisfies the Heegner condition for $N$. The set of Heegner points of $E$ (associated to $O$) is the set

$$\{\Phi_E(y) : y \in \text{CM}^{(N)}(O)\}.$$  

The action of $\text{Pic}(O)$ and $\text{Gal}(K_O/K)$ on $\text{CM}^{(N)}(O)$ as described in Theorem 5 translates directly into analogous actions on Heegner points on $E$, see [Dar04, Theorems 3.6, 3.7]. All that we will require is the following elementary consequence.
Proposition 7. Let $\mathcal{O}$ be an order in $k$ that satisfies the Heegner condition for $N$, let $y \in \text{CM}^{(N)}(\mathcal{O})$, and let $P_y = \Phi_E(y)$ be the associated Heegner point. Then

$$[k(P_y) : k] \geq \frac{[K_\mathcal{O} : k]}{\deg \Phi_E}.$$ 

Proof. To ease notation, let $d = \deg \Phi_E$, $n = [K_\mathcal{O} : k]$, $m = [k(P_y) : k]$.

From Corollary 6 we know that $k(y) = [K_\mathcal{O} : k]$, so $y$ has exactly $n$ Galois conjugates, say $y_1, \ldots, y_n$. Further, Galois acts transitively on the collection of points $y_1, \ldots, y_n$, so it acts transitively on their images $\Phi_E(y_1), \Phi_E(y_2), \ldots, \Phi_E(y_n)$. Since $\Phi_E$ is at most $d$-to-1, it follows that $\Phi(y)$ has at least $n/d$ distinct conjugates, and hence that $m \geq n/d$.

The ring class field $K_\mathcal{O}$ is an abelian extension of $k$. It is not abelian extension of $\mathbb{Q}$, but it is a Galois extension, and there is an exact sequence

$$1 \longrightarrow \text{Gal}(K_\mathcal{O}/k) \longrightarrow \text{Gal}(K_\mathcal{O}/\mathbb{Q}) \longrightarrow \text{Gal}(k/\mathbb{Q}) \longrightarrow 1.$$ 

The elements in $\text{Gal}(K_\mathcal{O}/\mathbb{Q})$ with nontrivial image in $\text{Gal}(k/\mathbb{Q})$ are called reflections. The structure of $\text{Gal}(K_\mathcal{O}/\mathbb{Q})$ and its action on Heegner points is described in the following proposition.

Proposition 8. Let $\mathcal{O}$ be an order in the quadratic imaginary field $k$, let $K_\mathcal{O}$ be the associated ring class field, and let $\rho \in \text{Gal}(K_\mathcal{O}/\mathbb{Q})$ be a reflection.

(a) The extension $K_\mathcal{O}/\mathbb{Q}$ is Galois with group equal to a generalized dihedral group. More precisely,

$$\rho \sigma = \sigma^{-1} \rho \quad \text{for all } \sigma \in \text{Gal}(K_\mathcal{O}/k).$$

(b) Let $w(E/\mathbb{Q})$ be the sign of the functional equation of $E/\mathbb{Q}$, let $y \in \text{CM}^{(N)}(\mathcal{O})$, and let $P_\tau = \Phi_E(\tau)$ be the associated Heegner point on $E$. Then there exists a $\sigma \in \text{Gal}(K_\mathcal{O}/k)$, depending on $P_y$ and $\rho$, so that

$$P_\rho y = -w(E/\mathbb{Q})P_\sigma y \quad \text{mod } E(K_\mathcal{O})_{\text{tors}}.$$ 

Proof. See [Dar04, Proposition 3.11] or [Gro84].

3. A linear algebra estimate

The intuition behind the following proposition is that a large subgroup of $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$ cannot act in an abelian fashion on a large subgroup of $(\mathbb{Z}/n\mathbb{Z})^2$. The key for our application is to quantify this statement in such a way that it is uniform with respect to $n$.

Proposition 9. Suppose that the following quantities are given:

- $n$ a positive integer.
- $V$ a free $\mathbb{Z}/n\mathbb{Z}$ module of rank 2.
- $\Gamma$ a subgroup of $\text{Aut}(V) \cong \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$.
- $W$ a $\Gamma$-invariant $\mathbb{Z}/n\mathbb{Z}$-submodule of $V$, i.e., $\Gamma W = W$.

Let

$$I(\Gamma) = ([\text{Aut}(V) : \Gamma]) = \text{the index of } \Gamma \text{ in } \text{Aut}(V).$$

Suppose that the action of $\Gamma$ on $W$ is abelian in the sense that $\Gamma|_W$ is an abelian subgroup of $\text{Aut}(W)$. Then

$$|W| \leq I(\Gamma)^3. \quad (3)$$
Remark 10. For our purposes it suffices to know that $|W|$ is bounded in terms of $I(\Gamma)$, independent of $n$, but it is an interesting question to ask to what extent the inequality (3) might be improved. A more elaborate argument (which we omit) gives exponent 2, and one might hope for an estimate of the form $|W| \ll I(\Gamma)^{1+\epsilon}$. However, the following example shows that an exponent of at least $\frac{4}{3}$ is necessary.

Example 11. Let $\ell$ be a prime, let $n = \ell^{2k}$, and let $W = V$. Then one easily checks that the action of

$$\Gamma = \{A \in \text{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) : A \text{ mod } \ell^k \text{ is diagonal}\}$$

on $(\mathbb{Z}/\ell^2\mathbb{Z})^2$ is abelian. Then

$$|\Gamma| = |(\mathbb{Z}/\ell^k\mathbb{Z})^*| \cdot |M_2(\mathbb{Z}/\ell^k\mathbb{Z})| = \ell^{5k}(1 - \ell^{-1}),$$

$$|\text{GL}(\mathbb{Z}/\ell^2\mathbb{Z})| = \ell^{8k}(1 - \ell^{-1})(1 - \ell^{-2}),$$

$$I(\Gamma) = \ell^{3k}(1 - \ell^{-2}).$$

Since $|W| = \ell^{4k}$, this yields

$$\frac{\log |W|}{\log I(\Gamma)} \xrightarrow{k\to\infty} \frac{4}{3}.$$

Proof of Proposition 11. By the standard structure theorem on modules over PIDs [Lan02, Theorem VI.2.7] we can find a basis for $V$ so that

$$V \cong \frac{\mathbb{Z}}{n\mathbb{Z}} \times \frac{\mathbb{Z}}{n\mathbb{Z}}$$

and

$$W \cong \frac{m_1\mathbb{Z}}{n\mathbb{Z}} \times \frac{m_2\mathbb{Z}}{n\mathbb{Z}}$$

with $m_1|m_2|n$.

We begin by doing the case that $n = \ell^e$ is a power of a prime, so

$$V \cong \frac{\mathbb{Z}}{\ell^e\mathbb{Z}} \times \frac{\mathbb{Z}}{\ell^e\mathbb{Z}}$$

and

$$W \cong \frac{\ell^i\mathbb{Z}}{\ell^e\mathbb{Z}} \times \frac{\ell^j\mathbb{Z}}{\ell^e\mathbb{Z}}$$

with $i,j \geq 0$ and $i + j \leq e$.

This allows us to identify $\Gamma$ with a subgroup of $\text{GL}_2(\mathbb{Z}/\ell^e\mathbb{Z})$. Further, the condition that $\Gamma W = W$ implies that every matrix $(a, b \begin{smallmatrix} c & d \end{smallmatrix}) \in \Gamma$ satisfies $c \equiv 0 \pmod{\ell^j}$. In terms of the classical modular groups, the condition $\Gamma W = W$ is equivalent to the requirement

$$\Gamma \subset \text{Image}(\Gamma_0(\ell^j) \twoheadrightarrow \text{GL}_2(\mathbb{Z}/\ell^e\mathbb{Z})).$$

(4)

It may happen that (4) is true for a larger value of $j$, so we define

$$J = \min\{e, \max\{\text{ord}_\ell(c) : \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \in \Gamma\}\}$$

In other words, $J$ is the largest integer less than or equal to $e$ such that every matrix in $\Gamma$ is congruent to $(a, b \begin{smallmatrix} c & d \end{smallmatrix})$ modulo $\ell^J$.

In particular, we can find a matrix

$$A = \begin{pmatrix} a & b \\ p^jc & d \end{pmatrix} \in \Gamma$$

with $c \neq 0 \pmod{\ell}$.

(Notice that if $J = e$, we can simply take $c = 1$.) We fix the matrix $A$ and consider another matrix $B \in \Gamma$. The definition of $J$ tells us that $B$ has the form

$$B = \begin{pmatrix} \alpha & \beta \\ p^j\gamma & \delta \end{pmatrix}.$$ 

If we were working over a field, we might hope that the condition $AB|_W = BA|_W$ implies that $B$ has the form $xI + yA$ for some scalars $x$ and $y$. This is not quite true in our case, but we will now prove the validity of a similar statement up to a carefully chosen power of $\ell$. 


We consider first the case that
\[ e > i + J \]
and apply the assumption that \( A \) and \( B \) commute in their action on \( W \). Note that this is not the same as saying that \( AB = BA \) in \( \text{GL}_2(\mathbb{Z}/\ell^e\mathbb{Z}) \). We obtain the correct statement by requiring that \( BA - AB \) kills a basis of \( W \). Thus
\[
(BA - AB) \begin{pmatrix} \ell^i & 0 \\ 0 & \ell^{i+j} \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \pmod{\ell^e}.
\]
Multiplying this out and doing some algebra yields
\[
\begin{pmatrix} \ell^{i+J}(c\beta - b\gamma) & \ell^{i+j}(b(\alpha - \delta) - (a - d)\beta) \\ \ell^{i+j}((a - d)\gamma - c(\alpha - \delta)) & -\ell^{i+j}(c\beta - b\gamma) \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \pmod{\ell^e}. \tag{5}
\]
We use the congruences in the matrix equation (5) to compute
\[
\ell^{i+J}((c\alpha - a\gamma)I + \gamma A) = \ell^{i+J} \begin{pmatrix} c\alpha & b\gamma \\ \ell^J c\gamma & c\alpha - \gamma(a - d) \end{pmatrix} \equiv \ell^{i+J} \begin{pmatrix} c\alpha & c\beta \\ \ell^J c\gamma & c\delta \end{pmatrix} \pmod{\ell^e} = \ell^{i+J}cB.
\]
Using the fact that \( \ell \nmid c \), we have shown that every \( B \in \Gamma \) satisfies
\[ B \equiv xI + yA \pmod{\ell^{e-i-J}} \quad \text{for some } x, y \in \mathbb{Z}/\ell^{e-i-J}\mathbb{Z}, \]
i.e., every \( B \in \Gamma \) has the form
\[ B = xI + yA + \ell^{e-i-J}Z \quad \text{with} \quad \begin{cases} x, y \in \mathbb{Z}/\ell^{e-i-J}\mathbb{Z} \quad \text{and} \\ Z \in M_2(\mathbb{Z}/\ell^{i+J}\mathbb{Z}). \end{cases} \]
Of course, the group \( \Gamma \) will not contain all of these matrices (e.g., we cannot have \( \ell | x \) and \( \ell | y \)), but in any case we obtain an upper bound
\[ |\Gamma| \leq (\# \text{ of } (x, y)) \cdot (\# \text{ of } Z) = \ell^{2(e-i-J)} \cdot \ell^{4(i+J)} = \ell^{2e+2i+2J}. \]
The order of \( \text{GL}_2(\mathbb{Z}/\ell^e\mathbb{Z}) \) is well known, so we obtain a lower bound for the index
\[
I(\Gamma) = \frac{|\text{GL}_2(\mathbb{Z}/\ell^e\mathbb{Z})|}{|\Gamma|} \geq \frac{\ell^{4e}(1 - \ell^{-1})(1 - \ell^{-2})}{\ell^{2e+2i+2J}} = \ell^{2e-2i-2J}(1 - \ell^{-1})^2(1 + \ell^{-1}). \tag{6}
\]
This estimate is helpful provided that \( J \) is not too large. However, in the case that \( J \) is large, we can instead use the fact that \( \Gamma \) is contained in the image of \( \Gamma_0(\ell^J) \) to estimate
\[ |\Gamma| \leq \left| \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/\ell^e\mathbb{Z}) : \ell^J | c \right\} \right| = \ell^{4e-J}(1 - \ell^{-1})^2,
\]
so
\[
I(\Gamma) \geq \frac{\ell^{4e}(1 - \ell^{-1})(1 - \ell^{-2})}{\ell^{4e-J}(1 - \ell^{-1})^2} = \ell^{J} \frac{1 + \ell^{-1}}{1 - \ell^{-1}}. \tag{7}
\]
Multiplying (6) by the square of (7) yields
\[
I(\Gamma)^3 \geq \ell^{2e-2i}(1 + \ell^{-1})^3,
\]
which proves that \( I(\Gamma)^3 \) is larger than

\[
|W| = \ell^{2e-2i-j}.
\]

Next we consider the case that

\[ i + J \geq e. \quad (8) \]

The commutativity relation (5) then gives little information, but the fact that \( \Gamma \) is contained in the image of \( \Gamma_0(\ell^J) \) still gives the lower bound (7), which we use in the weaker form \( I(\Gamma) \geq \ell^J \). Combining this with the assumption (8) yields

\[
I(\Gamma)^2 \geq \ell^{2J} \geq \ell^{2e-2i} \geq \ell^{2e-2i-j} = |W|,
\]

which is stronger than the desired result. This completes the proof of the proposition in the case that \( n \) is a power of a prime.

Finally, suppose that \( n \) is arbitrary. Let

\[
V_\ell = V \otimes \mathbb{Z}_\ell = \ell\text{-primary part of } V,
\]

and similarly let \( W_\ell = W \otimes \mathbb{Z}_\ell \). Then

\[
V = \bigoplus_{\ell \mid n} V_\ell \quad \text{and} \quad W = \bigoplus_{\ell \mid n} W_\ell
\]

by the Chinese remainder theorem. Further, we have

\[
\text{Aut}(V) = \bigoplus_{\ell \mid n} \text{Aut}(V_\ell) \quad \text{and} \quad \Gamma = \bigoplus_{\ell \mid n} \Gamma_\ell \quad \text{with } \Gamma_\ell = \text{Image} \left( \Gamma \to \text{Aut}(V_\ell) \right).
\]

Applying the \( \ell \)-primary case to this direct sum decomposition yields

\[
|W| = \prod_{\ell \mid n} |W_\ell| \leq \prod_{\ell \mid n} I(\Gamma_\ell)^3 = I(\Gamma)^3,
\]

which completes the proof of Proposition 9.

\[ \square \]

4. Multiples of points and abelian extensions

Our eventual goal is to prove the independence of points that are defined over fields that are “sufficiently large, sufficiently disjoint, and sufficiently abelian.” The content of the next theorem is to quantify the meaning of the word “sufficiently” in this statement. Its proof combines the elementary linear algebra result from Section 3 with Serre’s deep theorem on the image of Galois.

**Theorem 12.** Let \( F/\mathbb{Q} \) be a number field, let \( E/F \) be an elliptic curve that does not have complex multiplication, and let \( d \geq 1 \). There is an integer \( M = M(E/F,d) \) so that for any field \( k/F \) and any \( P \in E(\bar{F}) \) satisfying

\[
[k : F] \leq d \quad \text{and} \quad k(P)/k \text{ is abelian},
\]

the following estimate is true:

\[
[k(P) : k] \text{ divides } M[k(nP) : k] \quad \text{for all } n \geq 1.
\]

**Proof.** Fix an integer \( n \geq 1 \). Writing

\[
[k(P) : k] = [k(P) : k(nP)] [k(nP) : k],
\]

it suffices to find a bound of the form

\[
[k(P) : k(nP)] \leq C = C(E/F,d),
\]
since then we can take $M$ to equal the least common multiple of the integers less than $C$.

Consider the following set of points in $E$,

$$S(P, n) = \{P^\sigma - P^\tau : \sigma \in \text{Gal}(k(P)/k(nP)), \tau \in \text{Gal}(k(P)/k)\}. \tag{9}$$

Clearly $S(P, n) \subset E(k(P))$, since $k(P)/k$ is Galois.

**Claim 1.** $S(P, n) \subset E[n] \cap E(k(P))$.

We have $nP \in E(k(nP))$, and also $k(nP)/k$ is Galois since $k(P)/k$ is abelian, from which it follows that every $\text{Gal}(k(nP)/k)$-conjugate of $nP$ is defined over $k(nP)$. Hence for all $\tau \in \text{Gal}(k(P)/k)$ and all $\sigma \in \text{Gal}(k(P)/k(nP))$,

$$nP^\sigma - P^\tau = (nP^\tau)^\sigma - nP^\tau = 0,$$

which shows that $S(P, n)$ is contained in $E[n]$. The inclusion of $S(P, n)$ in $E(k(P))$ follows from the assumption that $k(P)/k$ is Galois, so every $\bar{F}/k$ conjugate of $P$ is in $E(k(P))$.

**Claim 2.** $S(P, n)$ is $\text{Gal}(k(P)/k)$-invariant.

Let $\tau, \lambda \in \text{Gal}(k(P)/k)$ and $\sigma \in \text{Gal}(k(P)/k(nP))$. Then $\tau \lambda \in \text{Gal}(k(P)/k)$, and also $\lambda^{-1}\sigma \lambda \in \text{Gal}(k(P)/k(nP))$, since $k(nP)/k$ is Galois (in fact, abelian). Hence

$$(P^\tau - P^\gamma)^\lambda = P^{\tau \lambda} - P^\gamma = P^{\tau \lambda (\lambda^{-1}\sigma \lambda)} - P^\tau \lambda \in S(P, n).$$

so $\lambda$ maps $S(P, n)$ to itself.

**Claim 3.** $|S(P, n)| \geq [k(P) : k(nP)]$.

The set $S(P, n)$ contains in particular all points of the form $P^\sigma - P$ with $\sigma \in \text{Gal}(k(P)/k(nP))$ (i.e., take $\tau = 1$), and these points are distinct. Hence

$$|S(P, n)| \geq |\text{Gal}(k(P)/k(nP))| = [k(P) : k(nP)].$$

We set the following notation, where note that Claim 2 tells us that $W$ is contained in $V$:

- $V = E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$,
- $\text{Aut}(V) = \text{Aut}(E[n]) \cong \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$,
- $W = (\mathbb{Z}\text{-span of } S(P, n)) \subset V$,
- $\Gamma(k) = \text{Image}(\text{Gal}(\bar{F}/k) \to \text{Aut}(V))$.

Then we are in the following situation:

- $V$ is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank 2.
- $\Gamma(k)$ is a subgroup of $\text{Aut}(V)$.
- $W$ is a $\Gamma(k)$-invariant submodule of $V$ (from Claim 2).
- The action of $\Gamma(k)$ on $W$ is abelian (since $W \subset E(k(P))$ from Claim 1 and $k(P)/k$ is abelian).

These four conditions are exactly the assumptions needed to apply Proposition \ref{prop:main} which yields the estimate

$$|W| \leq \text{Index}(\Gamma(k))^3. \tag{10}$$

The group $\Gamma(k)$ is the image of $\text{Gal}(\bar{F}/k)$ in $\text{Aut}(V)$, but we would like to replace it by the possibly larger group

$$\Gamma(F) = \text{Image}(\text{Gal}(\bar{F}/F) \to \text{Aut}(V)).$$
Clearly $\Gamma(k) \subset \Gamma(F)$, so
\[
\text{Index}(\Gamma(k)) = (\text{Aut}(V) : \Gamma(k)) \\
= (\text{Aut}(V) : \Gamma(F)) \cdot (\Gamma(F) : \Gamma(k)) \\
\leq (\text{Aut}(V) : \Gamma(F)) \cdot |\text{Gal}(k/F)| \\
\leq (\text{Aut}(V) : \Gamma(F)) \cdot d \quad \text{since } [k : F] \leq d, \\
= d \cdot \text{Index}(\Gamma(F)).
\] (11)

Combining (10) and (11) yields
\[
|W| \leq d^3 \cdot \text{Index}(\Gamma(F))^3.
\] (12)

We now apply Serre’s deep and fundamental theorem on the image of Galois.

**Theorem 13.** (Serre [Ser72, Ser98]) Let $E/F$ be an elliptic curve defined over a number field. For any prime $\ell$, let
\[
\rho_\ell : \text{Gal}(\overline{F}/F) \longrightarrow \text{Aut}(T_\ell(E))
\]
be the $\ell$-adic representation attached to $E/F$. Assume that $E$ does not have complex multiplication.

(a) The image of $\rho_\ell$ is of finite index in $\text{Aut}(T_\ell(E))$ for all $\ell$.

(b) The image of $\rho_\ell$ equals $\text{Aut}(T_\ell(E))$ for all but finitely many $\ell$.

Serre’s theorem is easily seen to be equivalent to the statement that there exists a constant $C_1(E/F) > 0$ such that
\[
\frac{|\text{Aut}(E[N])|}{|\text{Image}(\text{Gal}(\overline{F}/F) \to \text{Aut}(E[N]))|} \leq C_1(E/F) \quad \text{for all } N \geq 1.
\]

The crucial point here is that the constant $C_1(E/F)$ is independent of $N$, so the index is bounded by a constant depending only on $E/F$. (Note that this is where we are using the assumption that $E$ does not have complex multiplication.) Applying Serre’s estimate with $N = n$ yields
\[
\text{Index}(\Gamma(F)) \leq C_1(E/F). \quad (13)
\]

We combine the inequalities (12) and (13) with Claim 3 to obtain the estimate
\[
[k(P) : k(nP)] \leq |S(P,n)| \leq |W| \leq d^3 \cdot \text{Index}(\Gamma(F))^3 \leq d^3 \cdot C_1(E/F)^3.
\]
This this completes the proof of Theorem 12. 

\section{5. A direct sum decomposition via an idempotent relation}

Various versions of the results in this section are well-known, see for example [Cor91] or [Wa79, Theorem 6.3]. For the convenience of the reader we include proofs of the specific statements that we require.

Let $G$ be a finite group. For each subgroup $H \subset G$, the associated idempotent $\epsilon_H$ in the group ring of $G$ is the element
\[
\epsilon_H = \frac{1}{|H|} \sum_{\sigma \in H} \sigma \in \mathbb{Q}[G].
\]
One easily verifies that $\epsilon_H^2 = \epsilon_H$.

**Lemma 14.** Let $p$ be a prime, let $G = \mathbb{F}_p^r$, let $N = (p^{r-1} - 1)/(p - 1)$, and let $G_1, G_2, \ldots, G_N$ be the subgroups of $G$ of index $p$. 

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(a) \[ \sum_{i=1}^{N} \epsilon_{G_i} = \frac{p^r - p}{p - 1} \epsilon_G + e, \quad \text{where } e \in G \text{ is the identity element.} \]

(b) \[ \epsilon_{G_i} \cdot \epsilon_{G_j} = \epsilon_G \quad \text{for all } i \neq j. \]

Proof. Let \( \hat{G} = \text{Hom}(G, \mathbb{F}_p) \) be the dual group to \( G \) and let \( \hat{G}^* \) denote the nonzero elements of \( G \). Then the kernel of each \( \chi \in \hat{G}^* \) is an index \( p \) subgroup of \( G \), and \( \chi \) and \( \chi' \) give the same subgroup if and only if \( \chi' = c\chi \) for some \( c \in \mathbb{F}_p^* \). In other words, the index \( p \) subgroups of \( G \) are in one-to-one correspondence with the points of \( \hat{G}^*/\mathbb{F}_p^* = \mathbb{F}^{p-1}(\mathbb{F}_p) \). We also let \( G^* \) denote the nonzero elements of \( G \), i.e., \( G^* = \{ \sigma \in G : \sigma \neq e \} \), and compute

\[ p^{r-1} \sum_{i=1}^{N} \epsilon_{G_i} = p^{r-1} \sum_{\chi \in \hat{G}^*/\mathbb{F}_p^*} \epsilon_{\ker(\chi)} \]

\[ = \frac{1}{p-1} \sum_{\chi \in \hat{G}^*} \sum_{\sigma \in G} \sigma \]

\[ = \frac{1}{p-1} \sum_{\sigma \in G} \left| \{ \chi \in \hat{G}^* : \chi(\sigma) = 0 \} \right| \sigma \]

\[ = \frac{1}{p-1} \left( |\hat{G}^*| e + \sum_{\sigma \in G^*} \left| \{ \chi \in \hat{G}^* : \chi(\sigma) = 0 \} \right| \sigma \right) \]

\[ = \frac{1}{p-1} \left( (p^r - 1)e + (p^{r-1} - 1) \sum_{\sigma \in G^*} \sigma \right) \]

\[ = \frac{1}{p-1} \left( (p^r - p^{r-1})e + (p^{r-1} - 1) \sum_{\sigma \in G} \sigma \right) \]

\[ = p^{r-1}e + \frac{p^{r-1} - 1}{p-1} p^r \epsilon_G. \]

This proves (a).

Next let \( i \neq j \) be distinct indices. Let \( \chi \) and \( \lambda \) be elements of \( \hat{G}^* \) corresponding to \( G_i \) and \( G_j \), respectively. Then

\[ p^{2r-2} \epsilon_{G_i} \cdot \epsilon_{G_j} = \sum_{\sigma \in G_i} \sum_{\tau \in G_j} \sigma \tau = \sum_{\sigma \in G} \sum_{\tau \in G} \sigma \tau = \sum_{\rho \in G} \sum_{\rho \in G} \rho = \sum_{\rho \in G} \sum_{\rho \in G} 1. \]

The fact that \( \chi \) and \( \lambda \) are distinct nonzero homomorphisms, i.e., distinct nonzero linear maps, from \( \mathbb{F}_p^r \) to \( \mathbb{F}_p \) means that the pair of equations

\[ \chi(\sigma) = c_1 \quad \text{and} \quad \lambda(\sigma) = c_2 \]

has exactly \( p^{r-2} \) solutions for any given \( c_1, c_2 \in \mathbb{F}_p \). (This is simply the number of points on the intersection of two transversal hyperplanes in \( \mathbb{A}^r(\mathbb{F}_p) \).) Hence

\[ p^{2r-2} \epsilon_{G_i} \cdot \epsilon_{G_j} = p^{r-2} \sum_{\rho \in G} \rho = p^{2r-2} \epsilon_G. \]

Dividing by \( p^{2r-2} \) gives (b). \[ \square \]

**Lemma 15.** Let \( p, G, N, \) and \( G_1, \ldots, G_N \) be as in the statement of Lemma 13. Let \( M \) be a finite \( G \)-module whose order is prime to \( p \), so in particular the group ring \( \mathbb{Z}[p^{-1}][G] \) acts on \( M \).
(a) \( \epsilon_H M = M^H \), i.e., \( \epsilon_H M \) is the subgroup of \( M \) fixed by \( H \).

(b) The “norm map”

\[
\frac{M}{M^G} \xrightarrow{\text{norm}} \bigoplus_{i=1}^N \frac{\epsilon_G M}{\epsilon_G^i M} \cong \bigoplus_{i=1}^N \frac{M^{G_i}}{M^G_b}
\]

is an isomorphism. Its inverse is the summation map

\[
(m_1, \ldots, m_N) \mapsto m_1 + \cdots + m_N.
\]

Proof. First let \( \epsilon_H m \in \epsilon_H M \) and let \( \tau \in H \). Then

\[
\tau \epsilon_H m = \frac{1}{|H|} \sum_{\sigma \in H} \tau \sigma m = \frac{1}{|H|} \sum_{\sigma \in H} \sigma m = \epsilon_H m,
\]

so \( \epsilon_H m \in M^H \). Conversely, if \( m \in M^H \), then

\[
\epsilon_H m = \frac{1}{|H|} \sum_{\sigma \in H} \sigma m = \frac{1}{|H|} \sum_{\sigma \in H} m = m,
\]

so \( m = \epsilon_H m \in \epsilon_H M \). This proves that \( \epsilon_H M = M^H \), which completes the proof of (a).

Let \( \Phi \) be the map

\[
\Phi : M \xrightarrow{\text{norm}} \bigoplus_{i=1}^N \frac{\epsilon_G M}{\epsilon_G^i M}, \quad m \mapsto (\epsilon_G m_1, \ldots, \epsilon_G m_N).
\]

It is clear that \( \epsilon_G M \) is contained in the kernel of \( \Phi \). Conversely, let \( m \in \ker(\Phi) \). This means that there are \( m_i \in M \) satisfying \( \epsilon_G m = \epsilon_G m_i \). Summing over \( i \) and using Lemma 14(a) yields

\[
\epsilon_G \sum_{i=1}^N m_i = \sum_{i=1}^N \epsilon_G m_i = \left( \frac{p^r - 1}{p - 1} \epsilon_G + 1 \right) m.
\]

Thus

\[
m = \epsilon_G \left( \frac{p^r - 1}{p - 1} m + \sum_{i=1}^r m_i \right) \in \epsilon_G M,
\]

which gives the other inclusion. Hence \( \ker(\Phi) = \epsilon_G M \).

Next we show that \( \Phi \) is surjective. Let \( m_1, \ldots, m_N \in M \). We need to prove that the point

\[
(\epsilon_G m_1, \ldots, \epsilon_G m_N)
\]

is in the image of \( \Phi \). Let \( m = \sum \epsilon_G m_i \). Then for each \( j \) we use Lemma 14(b) to compute

\[
\epsilon_G m = \sum_{i=1}^N \epsilon_G \epsilon_G m_i = \epsilon_G m_j + \sum_{i \neq j} \epsilon_G m_i \equiv \epsilon_G m_j \pmod{\epsilon_G M}.
\]

This proves that

\[
\Phi (\epsilon_G m_1 + \cdots + \epsilon_G m_N) = (\epsilon_G m_1, \ldots, \epsilon_G m_N) \pmod{\epsilon_G M},
\]

completes the proof that \( \Phi \) induces an isomorphism \((14)\) and that the inverse isomorphism is the summation map.

6. An elementary Galois theory estimate

In this section we prove a basic estimate on the degree of successive composita of Galois extensions, cf. [Lan02, Corollary VI.1.15].
Proposition 16. Let $K_1, K_2, \ldots, K_r$ be Galois extensions of a field $k$, and for each $2 \leq i \leq r$, let

$$K'_i = K_i \cap (K_1 \cdots K_{i-1}).$$

Then

$$\prod_{i=1}^{r} [K_i : k] = [K_1 \cdots K_r : k] \prod_{i=2}^{r} [K'_i : k]. \quad (15)$$

Proof. Let $E_1/k$ and $E_2/k$ be any two Galois extensions of $k$. We claim that

$$[E_1 E_2 : k] \cdot [E_1 \cap E_2 : k] = [E_1 : k] \cdot [E_2 : k]. \quad (16)$$

If $E_1 \cap E_2 = k$, then [Lan02, Theorem VI.1.14] implies that (16) is true. The general case of (16) can be reduced to this case as follows:

$$[E_1 E_2 : k] = [E_1 : k] \cdot [E_2 : k] \cdot [E_1 \cap E_2 : k].$$

We prove (15) by induction on $r$. If $r = 1$, then both sides are equal to $[K_1 : k]$. Assume now that (15) is true for $r$. We use (16) with $E_1 = K_{r+1}$ and $E_2 = K_1 \cdots K_r$ and note that $E_1 \cap E_2 = K'_{r+1}$ to obtain

$$[K_{r+1} : k] = \frac{[K_1 \cdots K_{r+1} : k]}{[K_1 \cdots K_r : k]} \cdot [K'_{r+1} : k].$$

Multiplying (15) by this quantity gives (15) with $r+1$ in place of $r$, which completes the induction. \qed

7. Ideal class groups in multiquadratic fields

For this section we fix the following notation.

- $k/\mathbb{Q}$ a Galois extension with group $\text{Gal}(k/\mathbb{Q}) = (\mathbb{Z}/2\mathbb{Z})^r$.
- $K$ the Hilbert class field of $k$.
- $N = 2^r - 2^r - 1$.
- $k_1, \ldots, k_N$ the distinct quadratic subfields of $k$.
- $K_1, \ldots, K_N$ the Hilbert class fields of $k_1, \ldots, k_N$, respectively.

Further, for any finite abelian group $G$, we let $G^{\text{odd}}$ denote the largest subgroup of $G$ of odd order, and similarly $h^{\text{odd}}$ denotes the odd part of the integer $h$.

Proposition 17. With notation as set above, the natural restriction map

$$\text{Gal}(K/k)^{\text{odd}} \longrightarrow \prod_{i=1}^{N} \text{Gal}(K_i/k_i)^{\text{odd}}$$

is an isomorphism.

Proof. Let $H_k$ be the ideal class group of $k$, and similarly for each $i$, let $H_{k_i}$ denote the ideal class group of $k_i$. Standard properties of the Artin map [Lan94, X §1, A2 & A4] give a commutative
diagram

\[
\begin{array}{ccc}
H_k & \xrightarrow{\text{Artin}} & H_k \\
\downarrow & & \downarrow \\
\text{Gal}(K/k) & \rightarrow & \text{Gal}(kK_i/k) \\
\end{array}
\]

Combining these maps for \(i = 1, 2, \ldots, r\) and taking the odd parts of each of the groups gives us a commutative diagram

\[
\begin{array}{ccc}
H_{k}^{\text{odd}} & \xrightarrow{\text{Norm}} & \prod_{i=1}^{N} H_{k_i}^{\text{odd}} \\
\downarrow & & \downarrow \\
\text{Gal}(K/k)^{\text{odd}} & \rightarrow & \prod_{i=1}^{N} \text{Gal}(K_i/k_i)^{\text{odd}}
\end{array}
\]

(17)

The Galois group \(\text{Gal}(k/Q) = (\mathbb{Z}/2\mathbb{Z})^r\) acts on \(H_{k}^{\text{odd}}\), the odd part of ideal class group of \(k\). Also note that the subgroups of \(\text{Gal}(k/Q)\) of index 2 are exactly the groups \(\text{Gal}(k/k_i)\) for \(1 \leq i \leq N\). This is exactly the situation needed to apply Lemma 15(b), which tells us that there is an isomorphism

\[
\begin{array}{ccc}
H_{k}^{\text{odd}} & \xrightarrow{\text{Norm}} & \prod_{i=1}^{N} (H_{k_i}^{\text{odd}})^{\text{Gal}(k/k_i)} \\
\downarrow & & \downarrow \\
(\text{H}^{\text{odd}}_{k})^{\text{Gal}(k/Q)} & \rightarrow & \bigoplus_{i=1}^{N} (H_{k_i}^{\text{odd}})^{\text{Gal}(k/k_i)}
\end{array}
\]

(18)

(The norm maps \(N_{k/k_i} : H_k \rightarrow H_{k_i}^{\text{Gal}(k/k_i)}\) appearing in (18) are actually the \(2^{N-1}\)-power of the “norm maps” in Lemma 15(b). However, raising to the \(2^{N-1}\) power is an automorphism of \(H_{k}^{\text{odd}}\), so it is still valid to conclude that (18) is an isomorphism.)

In order to complete the proof of the proposition, we use the following elementary lemma.

**Lemma 18.** Set the following quantities.

- \(k/F\) a Galois extension of number fields of degree \(n\).
- \(H'_k = H_k \otimes \mathbb{Z}[1/n]\), the prime-to-\(n\) part of the class group of \(k\).
- \(H'_F = H_F \otimes \mathbb{Z}[1/n]\), the prime-to-\(n\) part of the class group of \(F\).

Then the natural map \(H_F \rightarrow H_k\) induces an isomorphism

\(H'_F \rightarrow (H'_k)^{\text{Gal}(k/F)}\).

**Proof.** Consider the compositions

\[
H'_F \rightarrow (H'_k)^{\text{Gal}(k/F)} \xrightarrow{N_{k/F}} H'_F
\]

and

\[
(H'_k)^{\text{Gal}(k/F)} \xrightarrow{N_{k/F}} H'_F \rightarrow (H'_k)^{\text{Gal}(k/F)}.
\]

Both compositions have the effect of raising to the \(n\)th power. Hence they are both isomorphisms, since \(H'_F\) and \(H'_k\) have order prime to \(n\). Therefore each of the individual arrows is also an isomorphism.

We now resume the proof of Proposition 17. We apply Lemma 18 with \(F = k_i\) and \(F = Q\) to obtain

\[
(\text{H}^{\text{odd}}_{k})^{\text{Gal}(k/k_i)} \cong H_{k_i}^{\text{odd}} \quad \text{and} \quad (\text{H}^{\text{odd}}_{k})^{\text{Gal}(k/Q)} \cong H_{Q}^{\text{odd}} = 1.
\]
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Figure 1: A tower of fields used in the proof of Proposition 19. Various extensions of 2-power degree are indicated. The proposition proves that the extension marked with a ? is also a 2-power extension.

Substituting these values into the isomorphism (18) yields

\[ H_k^{\text{odd}} \xrightarrow{\text{Norm}} \bigoplus_{i=1}^{N} H_{k_i}^{\text{odd}}. \]

Thus the top horizontal arrow in the commutative diagram (17) is an isomorphism, and the vertical arrows are Artin isomorphisms, hence the bottom arrow is also an isomorphism. This completes the proof of Proposition 17.

We now show that up to extensions of 2-power degree, the Hilbert class fields of distinct quadratic fields are maximally disjoint.

**Proposition 19.** With the notation set at the beginning of this section, for every \( 2 \leq i \leq r \) the degree

\[ [K_i \cap k_{i} \prod_{j<i} K_j : k_{i}] \]

is a power of 2.

**Proof.** The diagram given in Figure 1 should assist in keeping track of the various fields under consideration.
We look first at the maps

$$\text{Gal}(K/k) \twoheadrightarrow \text{Gal}(K_1 \cdots K_N/k) \hookrightarrow \prod_{i=1}^{N} \text{Gal}(K_i/k_i).$$ (19)

Proposition 17 tells us that if we restrict to the odd parts of these groups, then the composition (19) is an isomorphism. Since the first map is surjective and the second map is injective, we conclude that

$$[K:k]_{\text{odd}} = [K_1 \cdots K_N : k]_{\text{odd}} = \prod_{i=1}^{N} [K_i : k_i]_{\text{odd}}.$$ (20)

Next we apply Proposition 16 to the base field $k$, the Galois extensions $kK_1, \ldots, kK_N$ of $k$, and their compositum $L = K_1 \cdots K_N$. (Note that $k$ is already contained in $K_1 \cdots K_N$. For each $2 \leq i \leq N$ we obtain the formula

$$\prod_{i=1}^{N} [kK_i : k_i] = [K_1 \cdots K_N : k] \prod_{i=2}^{N} [kK_i \cap (kK_1 \cdots K_{i-1}) : k]$$ (21)

Every extension $k/k_i$ has degree a power of 2, so taking the odd part of (21) and replacing $k$ by $k_i$ as appropriate yields

$$\prod_{i=1}^{N} [K_i : k_i]_{\text{odd}} = [K_1 \cdots K_N : k]_{\text{odd}} \prod_{i=2}^{N} [K_i \cap (k_i K_1 \cdots K_{i-1}) : k_i]_{\text{odd}}$$ (22)

Comparing (20) and (22) gives the formula

$$\prod_{i=2}^{N} [K_i \cap (k_i K_1 \cdots K_{i-1}) : k_i]_{\text{odd}} = 1,$$

and hence every factor in the product must equal 1. □

8. The independence of Heegner points

In this section we combine all of the previous material in order to prove our principal result (Theorem 1), which we restate here in a more precise form.

**Theorem 20.** Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N$ and let

$$\Phi_E : X_0(N) \rightarrow E$$

be a modular parametrization of $E$. There is a constant $C = C(E, \Phi_E)$ so that the following is true.

Suppose that the following are given:

- $k_1, \ldots, k_r$ distinct quadratic imaginary fields satisfying the Heegner condition for $N$.
- $h_1, \ldots, h_r$ the class numbers of $k_1, \ldots, k_r$.
- $y_1, \ldots, y_r$ points $y_i \in \text{CM}^{(N)}(O_{k_i})$.
- $P_1, \ldots, P_r$ the associated Heegner points, $P_i = \Phi_E(y_i)$.

Assume further that the class numbers satisfy

$$h_i^{\text{odd}} \geq C \quad \text{for all } 1 \leq i \leq r.$$ (23)

Then

$P_1, \ldots, P_r$ are independent in $E(\overline{\mathbb{Q}})/E_{\text{tors}}$. 16
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Proof. We will assume that $P_1, \ldots, P_r$ are dependent and deduce an upper bound for $\min\{h_i^{\text{odd}}\}$ that depends only on $E$ and $\Phi_E$. Thus we assume that there is a relation

$$n_1 P_1 + \cdots + n_r P_r = 0 \quad \text{with } n_i \in \mathbb{Z} \text{ not all } 0. \quad (24)$$

Relabeling the points if necessary, we may assume that $n_r \neq 0$. In order to complete the proof, it then suffices to find a bound for $h_r^{\text{odd}}$

We first apply Theorem 12 (with $d = 2$) to deduce that

$$[k_r(P_r) : k_r] \text{ divides } M[k_r(n_r P) : k_r], \quad (25)$$

where the constant $M = M(E/F, 2)$ is independent of $P_r$ and $n_r$.

Next we let $k = k_1 \cdots k_r$. Then $\text{Gal}(k/\mathbb{Q}) = (\mathbb{Z}/2\mathbb{Z})^t$ for some $t \leq r$. We set $N = 2^t - 1$ and extend the list of $k_i$ to be the complete list $k_1, \ldots, k_N$ of quadratic fields contained in $k$. Continuing with the notation from Section 7, we set

$$K = \text{Hilbert class field of } k,$$

$$K_i = \text{Hilbert class field of } k_i \text{ for } 1 \leq i \leq N.$$

Each point $P_i \in E(K_i)$, so the assumed linear dependence (24) tells us that

$$n_r P_r = - \sum_{i < r} n_i P_i \in E\left(\prod_{i < r} K_i\right).$$

Thus

$$k_r(n_r P_r) \subset K_r \cap k_r \prod_{i < r} K_i,$$

from which we conclude that $[k_r(n_r P) : k_r]$ divides

$$\left[K_r \cap k_r \prod_{i < r} K_i : k_r\right] = \left[K_r \cap k_r \prod_{i < r} K_i : k\right] \cdot [k : k_r]. \quad (26)$$

Proposition 19 tells us that the first factor on the right-hand side of (26) is a power of 2, and the second factor is clearly a factor of 2. This proves that

$$[k_r(n_r P) : k_r] \text{ is a power of 2.}$$

Then (25) tells us that $[k_r(P_r) : k_r]^{\text{odd}}$ divides $M^{\text{odd}}$, so it is bounded by a constant depending only on $E$.

Finally, we use Proposition 7 and the fact that $P_r$ is a Heegner point to deduce that

$$[K_r : k_r]^{\text{odd}} \leq (\deg \Phi_E)[k_r(P_r) : k_r]^{\text{odd}} \leq (\deg \Phi_E)M^{\text{odd}}.$$

The quantity $[K_r : k_r]^{\text{odd}}$ is the odd part of the class number of $k_r$, so this contradicts (23). Hence $P_1, \ldots, P_r$ are independent.

9. Deuring lifts, Heegner points, and the ECDLP

In this section we briefly sketch an initially plausible approach to solving the elliptic curve discrete logarithm problem using Heegner points and explain why Theorem 20 makes it unlikely that this approach will yield anything better than an algorithm with $O(\sqrt{p})$ running time. We refer the reader to [RS] for further details.

Definition. An Elliptic Curve Discrete Logarithm Problem (ECDLP) over $\mathbb{F}_p$ starts with a known elliptic curve $E/\mathbb{F}_p$ and two points $\bar{S}, \bar{T} \in E(\mathbb{F}_p)$ and asks for the (smallest positive) integer $m$ such
that
\[ \bar{S} = n \bar{T}. \]
(We use a bar to denote quantities defined over \( \mathbb{F}_p \) or to denote the reduction of a quantity modulo \( p \).) It may be assumed that one knows
\[ n_E = \# \bar{E}(\mathbb{F}_p). \]

Suppose that it is possible to lift \( \bar{E} \) to an elliptic curve \( E/\mathbb{Q} \) of small conductor \( N \) and to find, in some reasonably explicit fashion, a modular parametrization \( \Phi_E : X_0(N) \to E \). (If \( \bar{E} \) is a “random” elliptic curve over \( \mathbb{F}_p \), it is unlikely that this is possible; but for reasons of efficiency in cryptographic applications, it is not uncommon to take \( \bar{E} \) to have very small coefficients.)

A standard approach to solving the ECDLP is to choose many random pairs of number \((a_i,b_i)\) modulo \( n_E \) and try to find a nontrivial relation among the points \( \bar{P}_i = a_i \bar{S} - b_i \bar{T} \). If \( \sum c_i \bar{P}_i = 0 \) is such a relation, then there is a good chance that the resulting relation
\[ \left( \sum_i c_i a_i \right) \bar{S} = \left( \sum_i c_i b_i \right) \bar{T} \]
between \( \bar{S} \) and \( \bar{T} \) can be inverted to express \( \bar{S} \) as a multiple of \( \bar{T} \). We thus look for a way of generating relations among a given list \( \bar{P}_1, \bar{P}_2, \ldots \) of points in \( E(\mathbb{F}_p) \).

The map \( \Phi_E : X_0(N) \to E \) has small degree, so there is a reasonable chance that a randomly chosen point in \( E(\mathbb{F}_p) \) will lift to an \( \mathbb{F}_p \)-rational point on \( X_0(N) \). (The exact probability, which we do not need, can be computed using the function field version of the Tchebotarev density theorem.) Hence taking a subsequence, we may assume that every point \( \bar{P}_i \in E(\mathbb{F}_p) \) lifts via \( \Phi_E \) to a point \( \bar{y}_i \in X_0(N)(\mathbb{F}_p) \).

In general, a point \( \bar{y} \in X_0(N)(\mathbb{F}_p) \) corresponds to a triple \((\bar{A}, \bar{A}', \bar{\phi})\) consisting of a pair of elliptic curves \( \bar{A}/\mathbb{F}_p \) and \( \bar{A}'/\mathbb{F}_p \) and an isogeny
\[ \bar{\phi} : \bar{A} \to \bar{A}' \]
defined over \( \mathbb{F}_p \), with kernel \( \ker(\bar{\phi}) \cong \mathbb{Z}/N\mathbb{Z} \). Since \( \bar{A} \) and \( \bar{A}' \) are \( \mathbb{F}_p \)-isogenous, they have the same number of points, so we set the notation
\[ n_{\bar{y}} = \# \bar{A}(\mathbb{F}_p) = \# \bar{A}'(\mathbb{F}_p) \quad \text{and} \quad a_{\bar{y}} = p + 1 - n_{\bar{y}}. \]
(Note that these quantities can be computed in polynomial time by the SEA variant of Schoof’s algorithm [Sch85, Sch95].) Then the endomorphism rings of \( \bar{A} \) and \( \bar{A}' \) have discriminant
\[ \Delta(\bar{y}) = a_{\bar{y}}^2 - 4p. \]
Note that \( \Delta(\bar{y}) < 0 \) by Hasse’s theorem [Sil86, Theorem V.1.1].

We perform this computation for each of the points \( \bar{y}_1, \bar{y}_2, \ldots \). Taking a subsequence, we may assume that every integer \( \Delta(\bar{y}_i) \) is a fundamental discriminant. (A negative integer \( D \) is a fundamental discriminant if either it is odd, squarefree, and \( D \equiv 1 \) (mod 4) or if it is divisible by 4 with \( D/4 \) squarefree and \( D/4 \equiv 2 \) or 3 (mod 4).) This ensures that
\[ \text{End}(\bar{A}_i) = \text{End}(\bar{A}'_i) = \mathcal{O}_{k_i} \]
is the full ring of integers
in the quadratic imaginary field \( k_i = \mathbb{Q}(\sqrt{\Delta(\bar{y}_i)}) \).
In other words, both \( \bar{A}_i \) and \( \bar{A}'_i \) have CM by the full ring of integers of \( k_i \). (We also discard \( \bar{y}_i \) in the unlikely event that \( \bar{A}_i \) is supersingular, i.e., if \( a_{\bar{y}_i} = 0 \).)

Remark 21. In practice, we would like the fields \( k_1, k_2, \ldots \) to be “not too independent,” in the sense that we would like the compositum \( k_1 k_2 \cdots \) to stop growing as we take more and more points. This
can be accomplished by keeping only those $\bar{y}_i$ whose associated discriminant is $B$-smooth for an appropriately chosen value of $B$. For example, taking $B = O(e^{\log p \log \log p})$ as usual, suppose that the discriminants are reasonably randomly distributed (a theorem of Birch [Bir68] says that they follow a Sato-Tate distribution). Then we can collect $O(MB)$ points $y_i$ and fields $k_i$ in time $O(MB)$, while the compositum $k_1 k_2 k_3 \cdots$ is always contained in the field $\mathbb{Q}(\sqrt{\ell} : \ell \leq O(B))$, independent of $M$.

We next use a variant of Deuring’s lifting theorem [Deu41] to lift $\bar{A}$ and $\bar{A}'$ to CM elliptic curves defined over the Hilbert class field $K_i$ of $k_i$ with the property that

$$\text{End}(\bar{A}_i) = \text{End}(A_i) \quad \text{and} \quad \text{End}(\bar{A}'_i) = \text{End}(A_i).$$

This can be done so that the cyclic $N$-isogeny $\tilde{\phi} : \bar{A} \to \bar{A}'$ lifts to a cyclic $N$-isogeny $\phi : A \to A'$. (For modern expositions of Deuring’s theorem, see [Lan87], 13 §5, Theorem 14 or [Oor73]). The triple $(A_i, A'_i, \phi)$ then corresponds to a point $y_i \in X_0(N)(K_i)$ that lifts $\bar{y}_i$.

Pushing these points forward, we obtain Heegner points

$$P_i = \Phi_E(y_i) \in E(K_i) \quad \text{satisfying} \quad P_i \mod p = \bar{P}_i.$$

(More precisely, there is a degree 1 prime ideal $p_i$ in $K_i$ so that $P_i \mod p_i$ equals $\bar{P}_i$.)

Our goal is to generate a list $P_1, \ldots, P_r$ of points that are dependent and to find an equation of dependency. There are at least two plausible methods to check whether $P_1, \ldots, P_r$ are dependent, despite the fact that we generally cannot write down explicitly any reasonable representation of their fields of definition $K_1, \ldots, K_r$. (Note that $[K_i : k_i] = h_{k_i} \approx \sqrt{|\text{Disc} k_i|}$.) First, we can try to use the modular interpretation of the points $y_i$ to compute the canonical height pairing $\langle y_i, y_j \rangle$ in terms of other, more easily computable, quantities. The algebraic and analytic formulas developed by Gross, Kohnen and Zagier [GKZ87] might be useful for this approach. Second, we can use the standard proof of the weak Mordell-Weil theorem to map a putative linear relation from $E(K_1 \cdots K_r)$ into the Selmer group and attempt to derive information about the coefficients of the relation. Note that there is never any trouble checking if a potential relation is correct, since it is easy to verify if $\sum c_i \bar{P}_i$ is zero, and if it is, then we do not actually care if the relation lifts.

Thus if the Deuring-Heegner lifts of the points $\bar{P}_1, \bar{P}_2, \ldots \in \bar{E}(\mathbb{F}_p)$ had a reasonable probability of being dependent, then the algorithm outlined above might yield a theoretical (and conceivably even a practical) algorithm to solve the ECDLP on curves with small coefficients. However, Theorem 20 provides fairly convincing evidence that this approach will not work, and indeed our initial motivation in attempting to prove a theorem such as Theorem 20 was our desire to assess the effectiveness of Deuring-Heegner lifts for solving the ECDLP.

10. Remarks on the 2-part of the ideal class group

Let

$$D(X) = \{\text{fundamental discriminants } -D \text{ with } 1 \leq D \leq X\},$$

and for each $-D \in D(X)$, let $h_D$ denote the class number of the quadratic imaginary field $\mathbb{Q}(\sqrt{-D})$. In view of Theorem 20 it would be of considerable interest to have some knowledge of the growth rate of the counting function

$$N(C; X) = \#\{-D \in D(X) : h_D^{\text{odd}} \leq C\}.$$

Genus theory says that $h_D$ is divisible by $2^{\nu(D)}$, where $\nu(D)$ is the number of odd primes dividing $D$. More precisely, genus theory tells us that the 2-rank of the class group is (essentially)
Table 1: Counting quadratic imaginary fields whose class number has small odd part and whose discriminant \(-D\) satisfies \(500,000 \leq D \leq 1,000,000\) (computations performed with PARI-GP).

| \# of D | \(A^{\text{odd}} \leq 1\) | \(A^{\text{odd}} \leq 3\) | \(A^{\text{odd}} \leq 5\) | \(A^{\text{odd}} \leq 7\) | \(A^{\text{odd}} \leq 9\) |
|--------|----------------|----------------|----------------|----------------|----------------|
| 5655   | 184           | 312           | 499           | 614           | 783           |
| 11110  | 264           | 589           | 901           | 1183          | 1486          |
| 16665  | 407           | 899           | 1381          | 1793          | 2273          |
| 22220  | 525           | 1202          | 1825          | 2375          | 3012          |
| 27775  | 667           | 1511          | 2281          | 3012          | 3730          |
| 33330  | 775           | 1789          | 2699          | 3499          | 4427          |
| 38885  | 893           | 2062          | 3114          | 4052          | 5154          |
| 44440  | 1037          | 2358          | 3544          | 4599          | 5847          |
| 55550  | 1163          | 2670          | 3999          | 5140          | 6532          |
| 61105  | 1278          | 2999          | 5073          | 7225          | 8610          |
| 66660  | 1395          | 3329          | 5556          | 8348          | 10061         |
| 72215  | 1515          | 3665          | 6039          | 9465          | 11939         |
| 77770  | 1635          | 3995          | 6541          | 10184         | 13813         |
| 83325  | 1755          | 4329          | 7012          | 10635         | 15697         |
| 88880  | 1875          | 4665          | 7435          | 11188         | 17529         |
| 94440  | 1995          | 5009          | 7869          | 11741         | 20161         |
| 99990  | 2115          | 5359          | 8289          | 12294         | 22783         |
| 105545 | 2235          | 5719          | 8719          | 12848         | 25405         |
| 111100 | 2355          | 6089          | 9149          | 13401         | 28027         |
| 116650 | 2475          | 6459          | 9579          | 13954         | 30649         |
| 122210 | 2595          | 6839          | 9989          | 14504         | 33271         |
| 127765 | 2715          | 7219          | 10419         | 15064         | 35913         |
| 133320 | 2835          | 7599          | 10849         | 15624         | 38535         |
| 138875 | 2955          | 7979          | 11279         | 16184         | 41157         |
| 144430 | 3075          | 8359          | 11709         | 16745         | 43779         |
| 149985 | 3195          | 8739          | 12139         | 17305         | 46391         |
| 155540 | 3315          | 9119          | 12569         | 17866         | 48913         |
| 161105 | 3435          | 9499          | 13009         | 18426         | 51515         |
| 166660 | 3555          | 9879          | 13439         | 18986         | 54117         |
| 172215 | 3675          | 10259         | 13869         | 19546         | 56719         |
| 177770 | 3795          | 10639         | 14299         | 20106         | 59321         |
| 183325 | 3915          | 11019         | 14739         | 20666         | 61923         |
| 188880 | 4035          | 11399         | 15179         | 21226         | 64525         |
| 194430 | 4155          | 11779         | 15619         | 21786         | 67127         |
| 200000 | 4275          | 12159         | 16059         | 22346         | 69729         |

Linear correlation 99.926% 99.932% 99.943% 99.949% 99.955%

equal to \(\nu(D)\). Since the \(n\)th prime is \(O(n \log n)\), this immediately implies that

\[
(2\text{-rank of } H_D) \ll \frac{\log D}{\log \log D},
\]

so the 2-part of \(h_D\) coming from the rank is negligible compared to \(D\). However, little seems to be known about the expected growth of the 2-exponent of \(H_D\), see for example [BK72, Ear89, Wal79].

If we look at all integers, then it is easy to see that

\[
\# \{n \leq N : n^{\text{odd}} \leq C\} = \left\lfloor \frac{C + 1}{2} \right\rfloor \cdot \log_2(N) + O(1) \quad \text{as } N \to \infty.
\]

The Brauer-Siegel theorem [Lan94, Chapter XVI, Theorem 4] says that the magnitude of \(h_D\) is approximately \(\sqrt{D}\), so \(\{h_D : D \in \mathcal{D}(X)\}\) contains \(O(X)\) integers roughly less than \(\sqrt{X}\). This leads to the following question, which we do not honor with the name of “conjecture” because there seems to be little evidence either for or against it.

**Question 22.** Fix a constant \(C\). Is it true that

\[
\# \{-D \in \mathcal{D}(X) : h^{\text{odd}}_D \leq C\} \gg \ll \sqrt{X} \log(X) \quad \text{as } X \to \infty,
\]

where the implied constants depend on \(C\)?

The numerical evidence is far from compelling. Indeed, the data in Table 1 seems to suggest that \(N(C; X) \sim \kappa_C X\) for some \(\kappa_C > 0\), but it seems unlikely that this is true, and indeed the value of \(\kappa_C\) shows a slow, but steady, decrease as we eliminate the smaller data from the top of the table. However, the data also does not suggest that \(N(C; X) \ll X^{1-\delta}\) for any particular \(\delta > 0\).

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