Progress Towards the Conjecture on APN Functions and Absolutely Irreducible Polynomials

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Abstract

Almost Perfect Nonlinear (APN) functions are very useful in cryptography, when they are used as S-Boxes, because of their good resistance to differential cryptanalysis. An APN function \( f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_{2^n} \) is called exceptional APN if it is APN on infinitely many extensions of \( \mathbb{F}_{2^n} \). Aubry, McGuire and Rodier conjectured that the only exceptional APN functions are the Gold and the Kasami-Welch monomial functions. They established that a polynomial function of odd degree is not exceptional APN provided the degree is not a Gold number \( (2^k + 1) \) or a Kasami-Welch number \( (2^{2k} - 2^k + 1) \). When the degree of the polynomial function is a Gold number, several partial results have been obtained [1, 7, 8, 10, 17]. One of the results in this article is a proof of the relatively primeness of the multivariate APN polynomial conjecture, in the Gold degree case. This helps us extend substantially previous results. We prove that Gold degree polynomials of the form \( x^{2^k+1} + h(x) \), where \( \deg(h) \) is any odd integer (with the natural exceptions), can not be exceptional APN.

We also show absolute irreducibility of several classes of multivariate polynomials over finite fields and discuss their applications.

Keywords: APN functions, exceptional APN functions, Janwa-McGuire-Wilson conjecture, absolutely irreducible polynomials, S-Boxes, Differential Cryptanalysis.

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1 INTRODUCTION

Block ciphers are symmetric key algorithms for performing encryption or decryption. Block ciphers map a block of bits to another block of bits in such a way that it is difficult to guess the mapping. For block ciphers, the effectiveness of the main cryptanalysis techniques can be measured by some quantities related to the round of encryption, usually named substitution box (S-box).

For differential attacks, the attacker is able to select inputs and examine outputs in an attempt to derive the secret key, more exactly, the attacker will select pairs of inputs $x, y$ satisfying a particular $a = x - y$, knowing that for that $a$ value, a particular $b = f(x) - f(y)$ value occurs with high probability. Then, one of the desired properties for an S-box to have high resistance against differential attacks is that, given any plaintext difference $x - y = a$, it provides a ciphertext difference $f(x) - f(y) = b$ with small probability.

Definition 1. Let $L = \mathbb{F}_q$, with $q = 2^n$ for some positive integer $n$. A function $f : L \to L$ is said to be almost perfect nonlinear (APN) on $L$ if for all $a, b \in L$, $a \neq 0$, the equation

$$f(x + a) - f(x) = b$$

have at most 2 solutions.

Equivalently, $f$ is APN if the set $\{f(x + a) - f(x) : x \in L\}$ has size at least $2^{n-1}$ for each $a \in L^*$. Moreover, since $L$ has characteristic 2, if $x$ is a solution of the equation (1), $x + a$ is also a solution. Then the number of solutions of (1) must be an even number.

The best known examples of APN functions are the Gold functions $f(x) = x^{2^k+1}$ and the Kasami-Welch functions $f(x) = x^{4^k-2^k+1}$, whose names are due to its exponents, the Gold and Kasami-Welch numbers respectively. These functions are APN on any field $\mathbb{F}_{2^n}$, where $k, n$ are relatively prime integers. The function $f(x) = x^{2^r+3}$ (Welch function) is also APN on $\mathbb{F}_{2^n}$, where $n = 2r + 1$.

The APN property is invariant under some transformations of functions. A function $f : L \to L$ is linear if

$$\sum_{i=0}^{n-1} c_i x^{2^i}, \quad c_i \in L.$$
The sum of a linear function and a constant is called an affine function.

Two functions \( f \) and \( g \) are called extended affine equivalent (EA-equivalent), if \( f = A_1 \circ g \circ A_2 + A \), where \( A_1 \) and \( A_2 \) are linear maps and \( A \) is a constant function. They are called CCZ-equivalence, if the graph of \( f \) can be obtained from the graph of \( g \) by an affine permutation. EA equivalence is a particular case of CCZ equivalence and two CCZ equivalent functions preserves the APN property (for more details see [6]). In general, proving CCZ equivalence is very difficult.

APN functions and their applications have become very important for the mathematicians in the last years. APN functions defined over \( \mathbb{F}_{2^n} \) are related to other mathematical objects, for example they are equivalent to binary error correcting codes \([2^n, 2^n - 2n - 1, 6]\), they are also equivalent to a certain class of dual hyperovals in the projective geometry.

Until 2006, the list of known affine inequivalent APN functions over \( K = \mathbb{F}_{2^n} \) was the families of monomial functions \( f(x) = x^d \), where the exponent \( d \) is as in the following table:

| \( x^d \)  | Exponent \( d \)                  | Constraints                 |
|-----------|----------------------------------|-----------------------------|
| Gold      | \( 2^r + 1 \)                    | \( (r, n) = 1 \)            |
| Kasami-Welch | \( 2^{2r} - 2^r + 1 \)            | \( (r, n) = 1, n \) odd    |
| Welch     | \( 2^r + 3 \)                    | \( n = 2r + 1 \)            |
| Niho      | \( 2^r + 2^{r/2} - 1 \)          | \( n = 2r + 1, r \) even   |
|           | \( 2^r + 2^{(3r+1)/2} - 1 \)     | \( n = 2r + 1, r \) odd    |
| Inverse   | \( 2^{2r} - 1 \)                 | \( n = 2r + 1 \)            |
| Dobbertin | \( 2^{4r} + 2^{3r} + 2^{2r} + 2^r - 1 \) | \( n = 5r \)               |

Mathematicians conjectured that this list was complete, up to equivalence. Motivated by this conjecture, several authors worked to find new APN functions not equivalent to the known ones. In February 2006, Y.Edel, G.Kyureghyan and A.Pott [11] published a paper with the first example of an APN function, that is a binomial of degree 36, which is not equivalent to any of the functions appeared in the above list. The function

\[
x^3 + u x^{36} \in GF(2^{10})[x]
\]

where \( u \in w GF(2^5)^* \cup w^2 GF(2^5)^* \) and \( w \) has order 3 in \( GF(2^{10}) \), is APN on \( GF(2^{10}) \).
From the emergence of this first example, there are now several more families of APN functions inequivalent to monomial functions. As some examples, Budaghyan, Carlet and Lender [4] found the following family of quadratic APN functions:

$$f(x) = x^{2^k+1} + wx^{2^{ik}+2^{mk}s}$$

where $n = 3k$, $(k, 3) = (s, 3k) = 1$, $k \geq 4$, $i = sk(mod3)$, $m = 3 - i$ and $w$ has the order $2^{2k} + 2^k + 1$.

Notice that, as shown in the above examples, the APN property depends on the extension degree of $F_2$. For any $t = 2^r + 1$ or $t = 2^{2r} - 2^r + 1$ there exist infinitely many values $m$ such that $(r, m) = 1$. That is, any fixed Gold or Kasami-Welch function which is APN on $L$ is also APN on infinitely many extensions of $L$. Such functions are called exceptional APN functions. One way to face a classification problem of APN functions is to determine which APN functions are APN infinitely often. This problem has been studied for monomials functions by Janwa, Mc.Guire, Wilson, Jedlika, Hernando [13, 14, 15] and more recently for polynomials by Aubry, McGuire, Rodier, Caullery, Delgado and Janwa [1, 7, 8, 17].

**Definition 2.** Let $L = F_q$, $q = 2^n$ for some positive integer $n$. A function $f : L \rightarrow L$ is called exceptional APN if $f$ is APN on $L$ and also on infinitely many extensions of $L$.

Aubry, McGuire and Rodier conjectured the following in [1].

**CONJECTURE:** Up to equivalence, the Gold and Kasami-Welch functions are the only exceptional APN functions.

It has been established [1] that a polynomial function of odd degree is not exceptional APN when the function is not a Gold function or a Kasami-Welch function. Although there are some results for the cases of non-monomial functions which are polynomials of Gold and Kasami-Welch degree, these cases remain open. In this paper we obtain new results which prove that a big infinite family of Gold degree polynomials can not be exceptional APN.

We make substantial progress towards the resolution of this conjecture. One of our main results in this article is a proof of the relatively primeness of the multivariate APN polynomial, in the Gold degree case (see Theorem 10). This helps us extend substantially previous results. In particular, we prove that Gold degree polynomials of the form $x^{2^k+1} + h(x)$, where $\deg(h)$ is any odd integer (with the natural exceptions), can not be exceptional
APN (section 5). We also show absolute irreducibility of several classes of multivariate polynomials over finite fields, and discuss their applications (section 5). We also give a proof of the "even case" of another theorem (section 6).

2 EXCEPTIONAL APN FUNCTIONS AND THE SURFACE $\phi(x, y, z)$

Let $L = \mathbb{F}_q$, $q = 2^n$ for some positive integer $n$. Rodier characterized APN functions as follows [16].

**Proposition 1.** A function $f : L \rightarrow L$ is APN if and only if the rational points $f_q$ of the affine surface

$$f(x) + f(y) + f(z) + f(x + y + z) = 0$$

are contained in the surface $(x + y)(x + z)(y + z) = 0$.

Given a polynomial function $f \in L[x, y, z]$, $\deg(f) = d$. We define:

$$\phi(x, y, z) = \frac{f(x) + f(y) + f(z) + f(x + y + z)}{(x + y)(x + z)(y + z)} \quad (2)$$

Then $\phi$ is a polynomial over $L[x, y, z]$ of degree $d - 3$. This polynomial defines a surface $X$ in the three dimensional affine space $L^3$.

It can be shown that if $f(x) = \sum_{j=0}^{d} a_j x^j$, then:

$$\phi(x, y, z) = \sum_{j=3}^{d} a_j \phi_j(x, y, z)$$

where

$$\phi_j(x, y, z) = \frac{x^j + y^j + z^j + (x + y + z)^j}{(x + y)(x + z)(y + z)} \quad (3)$$

is homogeneous of degree $j - 3$.

From the above proposition, one can deduce the next corollary whose proof can be found in [16].
Corollary 1. If the polynomial function $f : L \to L$ (of degree $d \geq 5$) is APN and the affine surface $X$

$$\phi(x, y, z) = \frac{f(x) + f(y) + f(z) + f(x + y + z)}{(x + y)(x + z)(y + z)} = 0$$

is absolutely irreducible, then the projective closure of $X$, $\overline{X}$ admits at most $4((d - 3)q + 1)$ rational points.

Using this corollary and the bound results of Lang-Weil and Ghorpade-Lachaud, that guarantees many rational points on a surface for all $n$ sufficiently large, we have the following theorem [16].

Theorem 1. Let $f : L \to L$ a polynomial function of degree $d$. Suppose that the surface $X$ of affine equation

$$\frac{f(x) + f(y) + f(z) + f(x + y + z)}{(x + y)(x + z)(y + z)} = 0$$

is absolutely irreducible (or has an absolutely irreducible component over $L$) and $d \geq 9$, $d < 0.45q^{1/4} + 0.5$, then $f$ is not an APN function.

Using this theorem, it can be proved that, if $X$ is absolutely irreducible (or has an absolutely irreducible factor over $L$) then $f$ is not exceptional APN.

3 RECENT RESULTS

In this section we state some families of polynomial functions that aren’t exceptional APN from [1, 7, 8, 17]

Theorem 2. (Aubry, McGuire, Rodier [1]) If the degree of the polynomial function $f$ is odd and not a Gold or a Kasami-Welch number then $f$ is not APN over $L = \mathbb{F}_q^n$ for all $n$ sufficiently large.

For the even degree case, they proved the following:

Theorem 3. If the degree of the polynomial function $f$ is $2e$ with $e$ odd, and if $f$ contains a term of odd degree, then $f$ is not APN over $L = \mathbb{F}_q^n$ for all $n$ sufficiently large.
Theorem 4. (Rodier [17]) If the degree of the polynomial function $f$ is even such that $\deg(f) = 4e$ with $e \equiv 3 \pmod{4}$ and if the polynomials of the form $(x + y)(y + z)(z + x) + P$ with
\[
P(x, y, z) = c_1(x^2 + y^2 + z^2) + c_4(xy + xz + yz) + b_1(x + y + z) + d
\]
for $c_1, c_4, b_1, d \in \mathbb{F}_{q^4}$, do not divide $\phi$ then $f$ is not APN over $\mathbb{F}_{q^n}$ for $n$ large.

Rodier proved a more precise result for polynomials of degree 12. If the degree of the polynomial defined over $\mathbb{F}_{q^n}$ is 12, then either $f$ is not APN over $\mathbb{F}_{q^n}$ for large $n$ or $f$ is CCZ equivalent to the Gold function $f(x) = x^3$.

Recently, Florian Caullery [7] obtained an analogous result for polynomials of degree 20. They are not exceptional APN or are CCZ equivalent to $f(x) = x^5$.

Aubry, McGuire and Rodier [1] also found results for Gold degree polynomials.

Theorem 5. Suppose $f(x) = x^{2k+1} + g(x) \in L[x]$ where $\deg(g) \leq 2^{k-1} + 1$. Let $g(x) = \sum_{j=0}^{2^{k-1}+1} a_j x^j$. Suppose that there exists a nonzero coefficient $a_j$ of $g$ such that $\phi_j(x, y, z)$ is absolutely irreducible. Then $\phi(x, y, z)$ is absolutely irreducible and so $f$ is not exceptional APN.

Some functions covered by this theorem are: $f(x) = x^{17} + h(x)$, where $\deg(h) \leq 9$; or $f(x) = x^{33} + h(x)$, where $\deg(h) \leq 17$.
Additionally, they also found that the bound for $g$ is best possible in the sense that if $f(x) = x^{2k+1} + g(x)$ with $\deg(g) = 2^{k-1} + 2$, then $\phi_j(x, y, z)$ is not absolutely irreducible. For being more specific they proved:

Theorem 6. Suppose $f(x) = x^{2k+1} + g(x) \in L[x]$ and $\deg(g) = 2^{k-1} + 2$. Let $k$ be odd and relatively prime to $n$. If $g(x)$ does not have the form $ax^{2^{k-1}+2} + a^2 x^3$ then $\phi$ is absolutely irreducible, while if $g(x)$ does have this form, then either $\phi$ is absolutely irreducible or $\phi$ splits into two absolutely irreducible factors that are both defined over $L$.

In [8, 10], we extended these results for the Gold degree case, we found new families of polynomials which are not exceptional APN.

Theorem 7. For $k \geq 2$, let $f(x) = x^{2k+1} + h(x) \in L[x]$, where $\deg(h) < 2^k + 1$, and $\deg(h) \equiv 3 \pmod{4}$. Then, $\phi(x, y, z)$ is absolutely irreducible.

For the case $1 \pmod{4}$, in [8, 10] we also proved:
Theorem 8. For \( k \geq 2 \), let \( f(x) = x^{2k+1} + h(x) \in L[x] \) where \( d = \deg(h) \equiv 1 \) (mod 4) and \( d < 2^k + 1 \). If \( \phi_{2^k+1}, \phi_d \) are relatively prime, then \( \phi(x, y, z) \) is absolutely irreducible.

In Theorem 10 (section 4), one of our main results is that we prove the relative primeness of the conjecture of APN polynomials \( \phi_n(x, y, z) \) and \( \phi_m(x, y, z) \) when one of them is of Gold degree. Thus proving Theorems 5 to 8 and some other results in the Gold degree polynomials \( \phi(x, y, z) \) unconditionally (see sections 5 and 6). The case when \( d = \deg(h) \equiv 5 \) (mod 8) is much simpler and has appeared in [10].

The case Kasami-Welch degree polynomials seems to be the hardest one. Rodier proved the following theorem [17].

Theorem 9. Suppose that \( f(x) = x^{2^{2k-1}+2^k+1} + g(x) \in L[x] \) where \( \deg(g) \leq 2^{2k-1} - 2^{k-1} + 1 \). Let \( g(x) = \sum_{j=0}^{2^{2k-1} - 2^{k-1} + 1} a_j x^j \). Suppose moreover that there exist a nonzero coefficient \( a_j \) of \( g \) such that \( \phi_j(x, y, z) \) is absolutely irreducible. Then \( \phi(x, y, z) \) is absolutely irreducible.

Rodier also studied the case when \( \deg(g) = 2^{2k-1} - 2^{k-1} + 2 \).

We also discuss the relatively prime case of the Kasami-Welch APN polynomials with other APN polynomials in [10], and in [9].

4 MAIN RESULTS.

From now on, let \( L = \mathbb{F}_{2^n}, \phi(x, y, z), \phi_j(x, y, z) \) as in (2) and (3).

4.1 A Proof of Relatively Prime APN Polynomial Conjecture

We first give a proof of the relatively primeness of the multivariate APN polynomial conjecture, in the Gold degree case, as stated in [10] and presented in [9]. For its statement, see Theorem 10 below.

We begin with the following fact, due to Janwa and Wilson [14], about the Gold functions.

For a Gold number \( j = 2^k + 1 \):

\[
\phi_j(x, y, z) = \prod_{\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2} (x + \alpha y + (\alpha + 1)z)
\]
Let us use the affine transformation $x \leftarrow x+1, y \leftarrow y+1$ on (5). Let’s denote $\tilde{\phi}_j(x, y) = \phi_j(x+1, y+1, 1)$. Then we have

$$\tilde{\phi}_j(x, y) = \prod_{\alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2} (x + \alpha y) \quad (6)$$

**Theorem 10.** If $d$ is an odd integer, then $\phi_{2^k+1}$ and $\phi_d$ are relatively prime for all $k \geq 1$ except when $d = 2^l + 1$ and $(l, k) > 1$.

**Proof.** Since $(\phi_n, \phi_m) = 1 \iff (\tilde{\phi}_n, \tilde{\phi}_m) = 1$, we will work with the functions $\tilde{\phi}$.

Let $n = 2^k + 1$, $m = 2^l + 1$, where $l > 1$ is an odd integer. By (6), we will prove the theorem by showing that no term $(x + ay)$ divides $\tilde{\phi}_m$, for all $a \in \mathbb{F}_{2^k}$, $a \neq 0$, $a \neq 1$. Let us suppose, by the way of contradiction, that this happen for some $a$, $a \neq 0$, $a \neq 1$. Then $(x + ay)$ divides $f(x, y) = \tilde{\phi}_m(x, y)(x+y)$ and $f(ay, y) = 0$. Writing $f(x, y)$ as a sum of homogeneous terms:

$$f(x, y) = F_{2^l+1}(x, y) + \ldots + F_{m-1}(x, y) + F_m(x, y) \quad (7)$$

Then $(x + ay) \mid f(x, y)$ if and only if $(x + ay)$ divides each homogeneous term $F_r$ in (7), implying $F_r(ay, y) = 0$.

From the expansion of $f$, we have:

$$F_{m-1}(x, y) = x^{m-1} + y^{m-1} + (x+y)^{m-1}$$

$$F_m(x, y) = x^m + y^m + (x+y)^m$$

Then

$$(ay)^{2^l} + y^{2^l} + (ay + y)^{2^l} = 0 \quad (8)$$

$$(ay)^{2^l+1} + y^{2^l+1} + (ay + y)^{2^l+1} = 0 \quad (9)$$

which respectively implies that

$$(a+1)^l + a^l + 1 = 0 \quad (10)$$

$$(a+1)^{l+1} + a^{l+1} + 1 = 0 \quad (11)$$

Substituting (10) in (11)

$$(a^l + 1)(a + 1) = a^{l+1} + 1$$
\[a^{l+1} + a^l + a + 1 = a^{l+1} + 1\]

\[a^{l-1} = 1\]  

(12)

ie, \(a\) is a \((l-1)\)-th root of unity. Furthermore, using this in (10)

\[(a + 1)^{l-1} = 1\]  

(13)

ie, \(a + 1\) is also a \((l-1)\)-th root of unity.

Now, let us consider the term \(F_{m-(2^i+1)}\) in (7) and prove that \(x + ay\) does not divide it.

\[F_{m-(2^i+1)}(x, y) = \binom{m}{2^i+1}(x^{m-(2^i+1)} + y^{m-(2^i+1)} + (x + y)^{m-(2^i+1)})\]

The classical theorem of Lucas states:

For non-negative integers \(a\), \(b\) and a prime \(p\), the following relation holds:

\[\binom{a}{b} \equiv \prod_{i=0}^{r} \binom{a_i}{b_i} \mod p,\]

where

\[a = a_r p^r + a_{r-1} p^{r-1} + \ldots + a_1 p + a_0,\]

\[b = b_r p^r + b_{r-1} p^{r-1} + \ldots + b_1 p + b_0,\]

are the base \(p\) expansions of \(a\) and \(b\) respectively. (Where by convention \(\binom{a}{b} = 0\) if \(a < b\) and \(\binom{0}{0} = 1\).)

Let \(l = a_r 2^r + a_{r-1} 2^{r-1} + \ldots + a_1 2 + 1\) be the base 2 expansion of \(l\). Then, the expansion of \(m\) is \(m = 2^i l + 1 = a_r 2^{i+r} + a_{r-1} 2^{i+r-1} + \ldots + a_1 2^{i+1} + 2^i + 1\). Using the theorem of Lucas, we have that \(\binom{m}{2^i+1} = 1\).

For \(x + ay\) to divide \(F_{m-(2^i+1)}\), it should happen that \(F_{m-(2^i+1)}(ay, y) = 0\), however:

\[F_{m-(2^i+1)}(ay, y) = (ay)^{m-(2^i+1)} + y^{m-(2^i+1)} + (ay + y)^{m-(2^i+1)}\]

\[= ((a + 1)^{m-(2^i+1)} + a^{m-(2^i+1)} + 1)y^{m-(2^i+1)}\]

\[= ((a + 1)^{2^i(l-1)} + a^{2^i(l-1)} + 1)y^{m-(2^i+1)}\]

using that \(a, a + 1\) are \((l-1)\)-th roots of unity, we get that the last equality is equal to \(y^{m-(2^i+1)}\). A contradiction.

\[\square\]

5 Application to Absolutely Irreducible Polynomials and Exceptional APN Functions

Using this result we can generalize theorems 7 and 8 in the following theorem.
Theorem 11. For \( k \geq 2 \), let \( f(x) = x^{2^k+1} + h(x) \in L[x] \) where \( \deg h < 2^k+1 \), and \( \deg(h) \) is an odd integer (not a Gold number \( 2^l+1 \) with \( (l, k) > 1 \)). Then \( \phi(x, y, z) \) is absolutely irreducible, and \( f(x) \) can not be exceptional APN.

5.1 Some Pending Cases.

From theorem 11, the missing cases are where Gold degree polynomials of the form \( f(x) = x^{2^k+1} + h(x) \), with \( \deg(h) \) any gold number. However, for polynomials of the form \( f(x) = x^{2^k+1} + h(x) \), \( \deg(h) = 2k' + 1 \), \( (k, k') = 1 \); \( \phi_{2k+1}, \phi_{2k'+1} \) are relatively primes. Then \( \phi(x, y, z) \) is absolutely irreducible by theorem 8.

For non relatively prime numbers \( k, k' \), as in the proof of the first case of theorem 8 (see [8]), we have that: \( Q_{t-1} = 0, Q_{t-2} = 0, ..., Q_1 = 0, Q_0 = 0 \) (Observe in the proof that \( t < e \), where \( e = 2^k + 1 - d \)).

Then, the hypersurface \( \phi(x, y, z) \) related to \( f \) satisfies:

\[
\sum_{j=3}^{2^k+1} a_j \phi_j(x, y, z) = (P_s + P_{s-1} + ... + P_0)(Q_t)
\]

Therefore, \( \phi(x, y, z) \) would be absolutely irreducible if \( h \) contains any term of degree \( m \) such that \( \phi_{2^k+1}, \phi_m \) are relatively primes. This condition is best possible in the sense that if \( h \) does not have such a term, then \( \phi(x, y, z) \) is not more absolutely irreducible. Theorem 10 of the previous section and the comments at the beginning of this subsection provides many examples for this condition to happen, almost for any odd number.

Until now, almost all the found families of Gold degree polynomials that fail to be exceptional APN are of the form \( x^{2^k+1} + h(x) \) for an odd degree of \( h \). Theorem 5 of section 3 justifies this fact. In the next subsection we will discuss the case when \( \deg(h) \) is an even number.

6 On the boundary of theorem 5.

Next, we prove the version of theorem 6 for the even case.

Theorem 12. For \( k \geq 2 \), let \( f(x) = x^{2^{2k}+1} + h(x) \in L[x] \) where \( \deg(h) = 2^{2k-1}+2 \). Let \( h(x) = \sum_{j=0}^{2^{2k-1}+2} a_j x^j \). If there is a nonzero coefficient \( a_j \) such that \( (\phi_{2^{2k}+1}, \phi_j) = 1 \). Then \( \phi \) is absolutely irreducible.
Proof. Suppose by contradiction that \( \phi \) is not absolutely irreducible. then \( \phi(x, y, z) = P(x, y, z)Q(x, y, z) \), where \( P, Q \) are non-constants polynomials defined on some extension of \( L \). Writing \( P, Q \) as a sum of homogeneous terms:

\[
\sum_{j=0}^{2^{2k}+1} a_j \phi_j(x, y, z) = (P_s + P_{s-1} + \ldots + P_0)(Q_t + Q_{t-1} + \ldots + Q_0) \tag{14}
\]

where \( P_i, Q_i \) are zero or homogeneous of degree \( i \), \( s + t = 2^{2k} - 2 \). Assuming without loss of generality that \( s \geq t \), then \( 2^{2k} - 2 > s > \frac{2^{2k} - 2}{2} \geq t > 0 \).

From the equation (14) we have that:

\[
P_sQ_t = \phi_{2^{2k+1}}. \tag{15}
\]

Since by (5), \( \phi_{2^{2k+1}} \) is equal to the product of different linear factors, then \( P_s \) and \( Q_t \) are relatively primes. Since \( h(x) \) is assumed to have degree \( 2^{2k-1} + 2 \), the homogeneous terms of degree \( r \), for \( 2^{2k-1} - 1 < r < 2^{2k} - 2 \), are equal to zero. Then equating the terms of degree \( s + t - 1 \) gives \( P_sQ_{t-1} + P_{s-1}Q_t = 0 \).

Hence we have that \( P_s \) divides \( P_{s-1}Q_t \) and this implies that \( P_s \) divides \( P_{s-1} \), since \( P_s \) and \( Q_t \) are relatively primes. We conclude that \( P_{s-1} = 0 \) since the degree of \( P_{s-1} \) is less than the degree of \( P_s \). Then we also have that \( Q_{t-1} = 0 \) as \( P_s \neq 0 \).

Similarly, equating the terms of degree \( s + t - 2, s + t - 3, \ldots, s + 1 \) we get:

\[
P_{s-2} = Q_{t-2} = 0, P_{s-3} = Q_{t-3} = 0, \ldots, P_{s-(t-2)} = Q_1 = 0
\]

The (simplified) equation of degree \( s \) is:

\[
P_sQ_0 + P_{s-t}Q_t = a_{s+3} \phi_{s+3} \tag{16}
\]

Lets consider two cases to prove the absolute irreducibility of \( \phi(x, y, z) \).

**First case**: \( s > t \).

Then \( s > 2^{2k-1} - 1 \) and \( \phi_{s+3} = 0 \). Then the equation (16) becomes:

\[
P_sQ_0 + P_{s-t}Q_t = 0.
\]

Then, using the previous argument, \( P_{s-t} = Q_0 = 0 \). It means that \( Q = Q_t \) is homogeneous. By the equations (17), (15) we have that for all \( j \), \( \phi_j(x, y, z) \) is divisible by \( x + \alpha y + (\alpha + 1)z \) for some \( \alpha \in \mathbb{F}_{2^k} - \mathbb{F}_2 \), which is a contradiction.
by the hypothesis of the theorem.

**Second case**: \( s = t = 2^{2k-1} - 1 \)

For this case the equation (16) becomes:

\[
P_s Q_0 + P_0 Q_t = a_{s+3} \phi_{2^{2k-1}+2}.
\]  

(17)

If \( P_0 = 0 \) or \( Q_0 = 0 \), then we have that \( Q = Q_t \) or \( P = P_s \). Then by similar arguments of the first case we have a contradiction. If both \( P_0, Q_0 \) are different from zero, let us consider the intersection of \( \phi \) with the line \( z = 0, y = 1 \). Then the equation (15) and (17) become:

\[
P_s Q_0 + P_0 Q_t = a_{s+3} (x + 1)(x) \prod_{\alpha \in F_{2^{2k-2}} - F_2} (x + \alpha)^2
\]  

(19)

This last equation comes from the fact that

\[
\phi_{2^{2k-1}+2} = (x + y)(x + z)(y + z) \phi_{2^{2k-2}+1}^2
\]

It is easy to show that \( F_{2^{2k}} \cap F_{2^{2k-2}} = F_2 \). Let \( x = \alpha_0 \in F_4, \alpha_0 \neq 0, 1 \). Then from (18) we have that \( P_s(\alpha_0) = 0 \) or \( Q_t(\alpha_0) = 0 \).

If \( P_s(\alpha_0) = 0 \), then \( Q_t(\alpha_0) \neq 0 \) (since \( P_s Q_t \) is a product of different linear factors) and from equation (19) we have \( P_0 Q_t(\alpha_0) = 0 \) that is a contradiction since both \( P_0, Q_t(\alpha_0) \) are different from zero. The case \( Q_t(\alpha_0) = 0 \) is analogous. Therefore \( \phi(x, y, z) \) is absolutely irreducible.

One of the families covered by this theorem is:

\( f(x) = x^{17} + h(x) \) where \( \deg(h) = 10 \), except the case \( f(x) = x^{17} + a_{10} x^{10} + a_5 x^5, a_{10} \neq 0, a_5 \neq 0 \).

**COMMENT:** Theorem 10 can be applied to Theorem 12 to prove absolute irreducibility of many families of this kind. We also discuss the relatively prime case the Kasami-Welch with other APN polynomials in [10], and in [9].

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