STABILITY OF THE LINE SOLITON OF THE KP-II EQUATION UNDER PERIODIC TRANSVERSE PERTURBATIONS

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Abstract. We prove the nonlinear stability of the KdV solitary waves considered as solutions of the KP-II equation, with respect to periodic transverse perturbations.

1. Introduction

Our goal here is to prove the nonlinear stability of the KdV solitary waves considered as solutions of the Kadomtsev-Petviashvili-II (KP-II) equation

\[ \partial_x (\partial_t u + \partial_x^3 u + 3 \partial_x (u^2)) + 3 \partial_y^2 u = 0 \]

with respect to periodic transverse perturbations. In this paper, we consider (1.1) for \((x, y) \in \mathbb{R} \times \mathbb{T} \), where \( \mathbb{T} = \mathbb{R} / (2\pi \mathbb{Z}) \) denotes the one dimensional torus.

The well-posedness of (1.1) is studied in [22], where it is shown that (1.1) is globally well-posed for initial data in \( H^s(\mathbb{R} \times \mathbb{T}) \), \( s \geq 0 \). Roughly speaking, it is shown that for every \( u_0 \in H^s(\mathbb{R} \times \mathbb{T}) \) with \( s \geq 0 \), there is a unique solution of (1.1) which belongs to \( C(\mathbb{R}; H^s(\mathbb{R} \times \mathbb{T})) \). Moreover the flow map is continuous (and even uniformly continuous on bounded sets) in the phase space \( H^s(\mathbb{R} \times \mathbb{T}) \). The proof is based on the ideas introduced in the purely periodic case in the work of Bourgain [5]. For other contributions on the Cauchy problem of the KP-II equation with different spatial domains, we refer to [8, 9, 10, 12, 25, 26, 28].

Let us now turn to the stability questions. Let

\[ \varphi_c(x) \equiv c \cosh^{-2} \left( \sqrt{\frac{c}{2}} x \right), \quad c > 0. \]

Then \( \varphi_c(x - 2ct) \) is a solitary wave solution of the KdV equation and also a solution of (1.1). It is well-known that \( \varphi_c(x - 2ct) \) is orbitally stable as a solution of the KdV equation (see [11, 13]). Our goal here is to show that \( \varphi_c(x - 2ct) \) remains stable as a solution of the KP-II equation subject to perturbations which are periodic in the transversal direction. Now let us introduce our result.

Theorem 1.1. For every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that if the initial data of (1.1) satisfies \( \| u_0 - \varphi_c \|_{L^2(\mathbb{R}_x \times \mathbb{T})} < \delta \), the corresponding solution of (1.1) satisfies

\[ \inf_{\gamma \in \mathbb{R}} \| u(t, x, y) - \varphi_c(x + \gamma) \|_{L^2(\mathbb{R}_x \times \mathbb{T})} < \varepsilon, \quad \forall t \in \mathbb{R}. \]
Moreover, there exists a constant \( \tilde{c} \) satisfying \( \tilde{c} - c = O(\delta) \) and a modulation parameter \( x(t) \) satisfying \( \limsup_{t \to \infty} |x(t)/t - 2c| = O(\delta) \) and such that

\[
(1.2) \quad \lim_{t \to \infty} \|u(t, x, y) - \varphi_c(x - x(t))\|_{L^2(x \geq ct \times T_y)} = 0.
\]

This result confirms the heuristic analysis of the Kadomtsev-Petviashvili seminal paper [14].

In the recent work [29], a stability result for the KP-II line soliton is studied by the inverse scattering method. There are several differences between [29] and Theorem 1.1. For instance in [29], localized perturbations belonging to weighted spaces are considered and thus one does not need to involve the modulation parameter \( x(t) \) in the stability statement. On the other hand, for the periodic perturbations considered in this paper, the modulation by translation is unavoidable (as it is for the KdV equation). In addition, our approach is apparently less dependent on the integrability features of the KP-II equation. We refer to [6] for a recent study on transverse stability for Hamiltonian PDE’s.

Let us now explain the main ideas and difficulties behind the proof of Theorem 1.1. The problem (1.1) has a Lax pair structure (see [32]) and thus it has, at least formally, an infinite sequence of conservation laws. Unfortunately, these conservation laws do not seem easy to use for dynamical issues. The only conservation law for the KP-II equation used in this paper is the \( L^2 \) conservation law, i.e. the quantity

\[
N(u(t, \cdot)) = \int_{\mathbb{R}_x \times T_y} u^2(t, x, y) dx dy
\]

is conserved (independent of \( t \)) by the flow of (1.1) established by the well-posedness result in [22]. Indeed, the first two terms of the Hamiltonian

\[
E(u(t, \cdot)) = \int_{\mathbb{R}_x \times T_y} \left( u_x^2(t, x, y) - 3(\partial_x^{-1}\partial_y u(t, x, y))^2 - 2u^3(t, x, y) \right) dx dy,
\]

which is one of the conserved quantities of (1.1), have the opposite sign. The infinite-dimensional indefiniteness is a serious obstruction to use the Hamiltonian in controlling the long time behavior of the KP-II equation. In particular, we cannot use the standard approach to prove stability based on the fact that the line soliton \( \varphi_c(x - 2ct) \) is a minimizer of the functional \( E(u) \) on the manifold \( \{ u \in H^1 \mid N(u) = N(\varphi_c) \} \).

In this paper, we aim to prove modulational stability in the \( L^2 \)-framework. For that purpose, we follow the idea of the work by Merle and Vega [16] which prove orbital/asymptotic stability of KdV 1-soliton in \( L^2 \). The idea of Merle and Vega [16] is to lift up a solution around a 1-soliton of KdV to a solution around a kink solution of the modified KdV equation by using the Miura transform. Since a kink is not in the energy class, the Miura transform eliminates the scaling freedom which generates the only direction we must be afraid of, to argue stability by using the \( L^2 \)-conservation law. The other merit of using the Miura transform is that it gains 1 more derivative and makes it possible to argue stability of kinks by a standard energy method. The Miura transform is one of the Bäcklund transformations that enable us to
observe behavior of solutions in a “simpler coordinate”. For example, it enables us to prove linear stability of solitons in a simple way (see [18]).

The Miura transform associated to the KP-II equation (1.1) is a heat operator (see [30, 13]). The main point is that it has a similar structure near the solitary wave with the Miura transform of the KdV equation which makes Merle and Vega’s approach applicable. Here we need to replace the ODE argument of [16] by a suitable Lyapunov functional argument. Once the crucial analysis of the Miura transform near a solitary wave is performed, we can argue stability of line solitons of (1.1) through stability of kink solutions of the mKP-II equation (the equation obtained from (1.1) after applying the Miura transform). This essentially explains our approach.

To obtain the asymptotic stability result, we use monotonicity coming from a Kato type smoothing effect (see [4, 13]). Our proof is simpler than a paper by Martel and Merle [15] because thanks to the Miura transform, we only need monotonicity property of small solutions to the KP-II equation. In fact, we do not need the modulation equation for the amplitude of the main line soliton because it is à priori determined through the Miura transform.

It would be interesting to extend our results to fully localized perturbations (belonging to $H^s(\mathbb{R}^2)$) of the KdV soliton under the KP-II flow. In this case the study of the linearization of the Miura transform $M_c^+$ in a neighborhood of $Q_c$ is much more delicate since the ODE analysis degenerate when the transverse frequencies tends to zero. On the other hand if one succeeds to resolve this issue then it would become possible to use the critical space analysis of [10] to get a linear behavior of the solution at the left of the solitary wave and thus give a more precise asymptotic stability statement. We plan to study this phenomenon elsewhere.

The situation changes radically if we replace (1.1) by the KP-I equation.

\begin{equation}
\partial_x(\partial_t u + \partial_x^3 u + 3\partial_x(u^2)) - 3\partial_y^2 u = 0.
\end{equation}

Indeed, it is known since the work of Zakharov [31] that $\varphi_c(x - 2ct)$ is unstable as a solution of (1.3). The proof of Zakharov is based on the integrability features of (1.3). We refer to the recent works [23, 24] for proofs of the instability of $\varphi_c(x - 2ct)$ as a solution of (1.3) independent of the integrability. These proofs have the advantage to apply to more involved Hamiltonian models. Let us also refer to [11] and the references therein for the quite intricate issues around the well-posedness in Sobolev spaces of (1.3).

Let us complete this introduction by a remark concerning possible extensions. For some bidirectional model equations such as the FPU lattice equation, the Hamiltonian is the only useful conservation law as $L^2$-norm is for the KP-II equation. Friesecke and Pego [7] and Mizumachi [17] prove stability of solitary waves using strong linear stability of solitary waves in a weighted space. However, for PDEs such as the water wave models, their approach could require smallness of higher order Sobolev norms that does not follow from conservation laws. The method of Merle and Vega we use in this paper would suggest that a lifting
Thanks to Lemma 1 in [13], one may define $\partial_x^{-1}$ via the Fourier transform for functions $u \in L^2(\mathbb{R} \times \mathbb{T})$ such that $\xi^{-1}\hat{u}(\xi, n) \in L^2(\mathbb{R} \times 
\mathbb{Z})$ (where $\hat{u}$ denotes the Fourier transform of $u$). Namely $\partial_x^{-1}u = \mathcal{F}_{\xi,n}^{-1}((i\xi)^{-1}\hat{u}(\xi, n))$, where $\mathcal{F}_{\xi,n}^{-1}$ is the inverse Fourier transform.

Let us remark that one may also consider the “integrated” form of (1.1), namely

$$(2.1) \quad \partial_t u + \partial_x^2 u + 3\partial_x^{-1}\partial_y^2 u + 3\partial_x(u^2) = 0, \quad u(0, x, y) = u_0(x, y).$$

The equation (2.1) is of the first order in $t$ but one needs to define $\partial_x^{-1}\partial_y^2 u$. Since the nonlinearity is differentiated with respect to $x$, this problem only concerns the free evolution. Thanks to Lemma 1 in [13], one may define $\partial_x^{-1}\partial_y$ of the free evolution as an $L^1_{\text{loc}}(\mathbb{R}^3)$ function even for data which does not satisfy a constraint $\int u = 0$, for example only in $L^2$ (we refer to [21] for further results in this direction).

We next introduce the space $\mathcal{E}(\mathbb{R}_x \times \mathbb{T}_y)$ which will play an important role in the analysis. Let

$$\mathcal{E}(\mathbb{R}_x \times \mathbb{T}_y) = \{ u \in L^2(\mathbb{R} \times \mathbb{T}) : \| u \|_{\mathcal{E}(\mathbb{R}_x \times \mathbb{T}_y)} < \infty \},$$

where

$$\| u \|_{\mathcal{E}(\mathbb{R}_x \times \mathbb{T}_y)}^2 = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} (1 + \xi^2 + \xi^{-2}n^2)|\hat{u}(\xi, n)|^2 d\xi$$

$$= \| u \|_{L^2}^2 + \| \partial_x u \|_{L^2}^2 + \| \partial_x^{-1}\partial_y u \|_{L^2}^2.$$

For each $n \neq 0$, we see that $u_n(x) \equiv \mathcal{F}_{\xi,n}^{-1}\hat{u}(\xi, n)$ admits an anti-derivative if $u \in \mathcal{E}$ and that $(\partial_x^{-1}\partial_y u, v) = (u, \partial_x^{-1}\partial_y v)$ if $u, v \in \mathcal{E}$. Here $(\cdot, \cdot)$ denotes the scalar product of $L^2(\mathbb{R}_x \times \mathbb{T}_y)$.

We have the following non-isotropic Sobolev inequality for functions in $\mathcal{E}(\mathbb{R}_x \times \mathbb{T}_y)$.

**Lemma 2.1.** There exists $C > 0$ such that for every $u \in \mathcal{E}(\mathbb{R}_x \times \mathbb{T}_y)$ and $p \in [2, 6],$

$$\| u \|_{L^p} \leq C\| u \|_{L^2}^{\frac{6-p}{2p}}\| \partial_x u \|_{L^2}^{\frac{p-2}{2p}}\| \partial_x^{-1}\partial_y u \|_{L^2}^{\frac{p-2}{2p}}.$$

For a proof of this lemma, we refer to [2, 27] or [20] (Lemma 2, page 783). The proof on [20] is performed for functions on $\mathbb{R}^2$ but the proof works equally well in the $\mathbb{R}_x \times \mathbb{T}_y$ setting.

In order to motivate the mKP-II equation, we now introduce the Miura transforms that we use in this paper. For $c > 0$ and $v \in \mathcal{E}$, we set

$$M^c_\pm(v) = \pm \partial_x v + \partial_x^{-1}\partial_y v - v^2 + \frac{c}{2}.$$

Observe that $M^c_\pm$ is invariant by translation, namely

$$M^c_\pm(v(x + \alpha)) = (M^c_\pm(v))(x + \alpha), \quad \forall \alpha \in \mathbb{R}.$$
Using Lemma 2.1 we obtain that if a sequence \( \{u_n\} \) converges to a limit \( u \) in \( \mathcal{E}(\mathbb{R}_x \times \mathbb{T}_y) \) the sequence \( \{M^c_\pm(u_n) - M^c_\pm(u)\} \) converges to 0 in \( L^2(\mathbb{R}_x \times \mathbb{T}_y) \).

The transformations \( M^c_\pm \) relate the KP-II equation to the mKP-II equation (mKP-II) which reads
\[
\partial_t v + \partial_x^3 v + 3\partial_x^{-1}\partial_y^2 v - 6v^2\partial_x v + 6\partial_x v\partial_x^{-1}\partial_y v = 0.
\]
At least formally, if \( v(t, x, y) \) is a solution of the mKP-II equation (2.2), then for \( c > 0 \), \( u_\pm \) defined by
\[
u_\pm(t, x, y) \equiv M^c_\pm(v)(t, x - 3ct, y)
\]
are solutions of the KP-II equation (1.1). The last statement can be directly verified (see e.g. [13], Appendix A) for sufficiently smooth solutions in \( \mathcal{E} \) and we will only use it in such a situation in this paper.

The line soliton of the KP-II equation is related to the kink \( Q_c \) defined by
\[
Q_c(x) = \sqrt{2} \tanh \left( \sqrt{\frac{c}{2}} x \right)
\]
We see that \( Q_c(x + ct) \) is a solution of (2.2) and moreover
\[
M^c_+(Q_c) = \varphi_c, \quad M^c_-(Q_c) = 0.
\]
Let \( Z = \{ u \in H^8(\mathbb{R}_x \times \mathbb{T}_y) : \partial_x^{-1}\partial_y u, \partial_x u \in H^8(\mathbb{R}_x \times \mathbb{T}_y) \} \). In this section, we will prove a global well-posedness result for (2.2) with data
\[
v(0, x, y) = Q_c(x) + w_0(x, y), \quad w_0 \in Z.
\]
It turns out that one can apply arguments similar to the work by Kenig and Martel [13] to have the following result.

**Proposition 2.2.** For every \( w_0 \in Z \), there exists a unique global in time solution of (2.2) with data (2.4) such that
\[
v(t, x, y) = Q_c(x + ct) + w(t, x, y), \quad w \in C(\mathbb{R}; \mathbb{Z}).
\]
Moreover \( v \) satisfies the conservation laws
\[
\|M^c_\pm(v)(t, x, y)\|_{L^2(\mathbb{R}_x \times \mathbb{T}_y)} = \|M^c_\pm(v)(0, x, y)\|_{L^2(\mathbb{R}_x \times \mathbb{T}_y)} < \infty.
\]

**Proof.** We need to solve the equation
\[
\partial_t w + \partial_x^3 w + 3\partial_x^{-1}\partial_y^2 w - 2\partial_x((w + \tilde{Q}_c)^3 - \tilde{Q}_c^3) + 6\partial_x w\partial_x^{-1}\partial_y w + 6\tilde{Q}_c'\partial_x^{-1}\partial_y w = 0
\]
with data
\[
w(0, x, y) = w_0(x, y), \quad w_0 \in Z,
\]
where \( \tilde{Q}_c \equiv Q_c(x + ct) \). The construction of local solutions for a regularized version of (2.6)
\[
\partial_t w + \varepsilon\partial_x^4 w + \varepsilon^5\partial_y^4 w + \partial_x^3 w + 3\partial_x^{-1}\partial_y^2 w
\]
\[
- 2\partial_x((w + \tilde{Q}_c)^3 - \tilde{Q}_c^3) + 6\partial_x w\partial_x^{-1}\partial_y w + 6\tilde{Q}_c'\partial_x^{-1}\partial_y w = 0
\]

\[
\|
\]

\[
\|M^c_\pm(v)(t, x, y)\|_{L^2(\mathbb{R}_x \times \mathbb{T}_y)} = \|M^c_\pm(v)(0, x, y)\|_{L^2(\mathbb{R}_x \times \mathbb{T}_y)} < \infty.
\]

**Proof.** We need to solve the equation
\[
\partial_t w + \partial_x^3 w + 3\partial_x^{-1}\partial_y^2 w - 2\partial_x((w + \tilde{Q}_c)^3 - \tilde{Q}_c^3) + 6\partial_x w\partial_x^{-1}\partial_y w + 6\tilde{Q}_c'\partial_x^{-1}\partial_y w = 0
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with data
\[
w(0, x, y) = w_0(x, y), \quad w_0 \in Z,
\]
where \( \tilde{Q}_c \equiv Q_c(x + ct) \). The construction of local solutions for a regularized version of (2.6)
\[
\partial_t w + \varepsilon\partial_x^4 w + \varepsilon^5\partial_y^4 w + \partial_x^3 w + 3\partial_x^{-1}\partial_y^2 w
\]
\[
- 2\partial_x((w + \tilde{Q}_c)^3 - \tilde{Q}_c^3) + 6\partial_x w\partial_x^{-1}\partial_y w + 6\tilde{Q}_c'\partial_x^{-1}\partial_y w = 0
\]
can be done as in [13], where \( v \in \{ u \in H^8(\mathbb{R}^2) \, : \, \partial_x u, \partial_x^{-1} \partial_y u \in H^8(\mathbb{R}^2) \} \) and \( \hat{Q}_c \) is replaced by 0. The main point is a variant of the Kato smoothing effect (Lemma 1 in [13]) which works equally well in the case \( \mathbb{R}_x \times \mathbb{T}_y \). Indeed the crucial effect of the change of variables used in the proof of Lemma 1 in [13] is transported directly to the \( \mathbb{R}_x \times \mathbb{T}_y \) framework. All other arguments in the analysis of \((2.8)\) are independent of the geometry of the spatial domain. Finally, the additional terms coming from the presence of \( \hat{Q}_c \) can be handled similarly to [13]. Indeed \( \hat{Q}_c' \in Z \) which makes that the term \( 6 \hat{Q}_c' \partial_x^{-1} \partial_y w \) can be treated exactly as \( 6 \partial_x w \partial_x^{-1} \partial_y w \). The new terms coming from the contributions of \((w + \hat{Q}_c)^3 - \hat{Q}_c^3 \) are \( 3 \hat{Q}_c^2 w \) and \( 3 \hat{Q}_c w^2 \) and one may readily check that they do not affect the analysis of [13] pp. 2462-2463 (they are even slightly easier to handle than \( w^3 \)).

Further one gets local solutions of \((2.8)\) on a time interval independent of the regularization parameter \( \varepsilon \) by a classical compactness argument and a reasoning similar to the proof of Lemma 2.3 below, the argument being even simpler since in Lemma 11 in [13], one needs to reason in the spirit of Lemma 2.3 below transform the energy estimates for \((2.9)\) to energy estimates for \((2.8)\). The additional term in \((2.9)\) with respect to [13] is \(-2 \varepsilon (\partial_x^{4} \hat{Q}_c) v + \varepsilon \partial_x^{5} \hat{Q}_c \) which is of lower order compared to the other terms in the right hand-side of \((2.9)\) and thus the analysis performed in [13] is not affected. The a priori uniform in \( \varepsilon \) estimates for \((2.8)\) imply the local well-posedness for \((2.6)\) by a classical compactness argument.

Once local solutions of \((2.6)\) are obtained as in [13], one has a global solution of \((2.6) - (2.7)\) thanks to the global well-posedness of KP-II posed on \( \mathbb{R}_x \times \mathbb{T}_y \) proved in [22] and the following lemma.
Lemma 2.3. Suppose that \( w \in Z \) is a solution of (2.6)-(2.7) on a time interval \([0,T]\). Let \( u = M_c^\infty(\tilde{Q}_c + w) \) and suppose that
\[
\sup_{t \in [0,T]} \| u(t, \cdot) \|_{H^s(\mathbb{R}_x \times \mathbb{T}_y)} < \infty.
\]
Then
\[
\sup_{t \in [0,T]} \| w(t, \cdot) \|_Z < \infty.
\]

Proof of Lemma 2.3. Here we will use the method of Lemma 9 in [13] by incorporating a small modification coming from the presence of \( c \) in the Miura transform \( M_c^\infty \). Since \( M_c^\infty(Q_c) = \varphi_c \), we have
\[
M_c^\infty(\tilde{Q}_c + w) = \varphi_c(x + ct) + \partial_x w + \partial^{-1}_x \partial_y w - w^2 - 2\tilde{Q}_c w.
\]
Thus
\[
(2.10) \quad \sup_{t \in [0,T]} \| \partial_x w + \partial^{-1}_x \partial_y w - w^2 - 2\tilde{Q}_c w \|_{L^2} \leq C.
\]
Combining the fact that \( (w_x, \partial^{-1}_x \partial_y w) = 0, \quad (w_x, w^2) = 0 \) and
\[
-2(w_x, \tilde{Q}_c w) = \int_{\mathbb{R}_x \times \mathbb{T}_y} Q'_c(x + ct)w^2(x, y)dxdy > 0,
\]
with (2.10), we have
\[
(2.11) \quad \sup_{t \in [0,T]} \| w_x \|_{L^2} + \sup_{t \in [0,T]} \| \partial^{-1}_x \partial_y w - w^2 - 2\tilde{Q}_c w \|_{L^2} \leq C.
\]
Using Lemma 2.1 and the bound for \( \| w_x \|_{L^2} \) we have just obtained, we have for \( t \in [0,T] \),
\[
\| \partial^{-1}_x \partial_y w \|_{L^2} \leq C(\| w \|_{L^4}^2 + \| w \|_{L^2}) \leq C(\| w \|_{L^2}^{\frac{3}{2}} \| \partial^{-1}_x \partial_y w \|_{L^2}^{\frac{1}{2}} + \| w \|_{L^2})
\]
which in turn implies that for \( t \in [0,T] \),
\[
(2.12) \quad \| \partial^{-1}_x \partial_y w \|_{L^2} \leq C \| w \|_{L^2}.
\]
We now obtain estimates for \( \| w \|_{L^2} \). We multiply (2.6) by \( w \) and integrate over \( \mathbb{R}_x \times \mathbb{T}_y \) to have after some integrations by parts
\[
\frac{1}{2} \frac{d}{dt} \| w \|_{L^2}^2 = -6 \int_{\mathbb{R}_x \times \mathbb{T}_y} \tilde{Q}_c' w \partial^{-1}_x \partial_y w + 6 \int_{\mathbb{R}_x \times \mathbb{T}_y} \tilde{Q}_c \tilde{Q}_c' w^2 + 2 \int_{\mathbb{R}_x \times \mathbb{T}_y} \tilde{Q}_c' w^3.
\]
Using Lemma 2.1, (2.11) and (2.12), we have
\[
\left| \int_{\mathbb{R}_x \times \mathbb{T}_y} \tilde{Q}_c' w^3 \right| \leq \| \tilde{Q}_c' \|_{L^2(\mathbb{R}_x \times \mathbb{T}_y)} \| w \|_{L^6}^3 \leq C \| \partial^{-1}_x \partial_y w \|_{L^2} \leq C \| w \|_{L^2}.
\]
Using the last estimate and (2.12), we have
\[
(2.13) \quad \frac{d}{dt} \| w \|_{L^2}^2 \leq C(\| w \|_{L^2} + 1)^2.
\]
Therefore thanks to Gronwall’s lemma, we have
\[
\sup_{t \in [0,T]} \| w \|_{L^2} \leq C.
\]
Therefore, we have obtained the needed bounds for \( \| w \|_{L^2} \), \( \| \partial_x w \|_{L^2} \) and \( \| \partial_x^{-1} \partial_y w \|_{L^2} \), i.e. 
\( E_0(w) \leq C \), where
\[
E_0(w) = \| w \|_{L^2} + \| \partial_x w \|_{L^2} + \| \partial_x^{-1} \partial_y w \|_{L^2}.
\]
We next estimate higher derivatives. Write
\[
\begin{align*}
\partial_x u &= \partial_x \varphi_c(x + ct) + \partial_x^2 w + \partial_y w - 2w \partial_x w - 2\partial_x (\tilde{Q}_c w), \\
\partial_y u &= \partial_x \partial_y w + \partial_x^{-1} \partial_y^2 w - 2w \partial_y w - 2\tilde{Q}_c \partial_y w.
\end{align*}
\]
Set
\[
E_1(w) = \| w \|_{L^2} + \| \partial_x^2 w \|_{L^2} + \| \partial_x \partial_y w \|_{L^2} + \| \partial_x^{-1} \partial_y^2 w \|_{L^2}.
\]
Using the orthogonality between \( \partial_x^2 w \) and \( \partial_y w \) and also between \( \partial_x \partial_y w \) and \( \partial_x^{-1} \partial_y^2 w \), we obtain that
\[
E_1(w) \leq C (1 + \| w \partial_x w \|_{L^2} + \| w \partial_y w \|_{L^2} + \| \partial_x w \|_{L^2} + \| \partial_y w \|_{L^2}).
\]
Next, thanks to an elementary interpolation inequality, we get
\[
\| \partial_x w \|_{L^2} + \| \partial_y w \|_{L^2} \leq C (E_0(w))^{\frac{1}{2}} (E_1(w))^{\frac{1}{2}} \leq C (E_1(w))^{\frac{1}{2}}.
\]
Next, we write by invoking Lemma 2.1
\[
\begin{align*}
\| w \partial_x w \|_{L^2} &\leq \| w \|_{L^5} \| \partial_x w \|_{L^6} \leq C E_0(w) \| \partial_x w \|_{L^2} \| \partial_x^2 w \|_{L^2} \| \partial_y w \|_{L^2} \\
&\leq C (E_0(w))^{\frac{5}{4}} (E_1(w))^{\frac{3}{4}} \leq C (E_1(w))^{\frac{3}{4}}.
\end{align*}
\]
Similarly
\[
\begin{align*}
\| w \partial_y w \|_{L^2} &\leq \| w \|_{L^5} \| \partial_y w \|_{L^6} \leq C E_0(w) \| \partial_y w \|_{L^2} \| \partial_x \partial_y w \|_{L^2} \| \partial_x^{-1} \partial_y^2 w \|_{L^2} \\
&\leq C (E_0(w))^{\frac{5}{4}} (E_1(w))^{\frac{3}{4}} \leq C (E_1(w))^{\frac{3}{4}}.
\end{align*}
\]
In summary, we get \( E_1(w) \leq C (1 + (E_1(w))^{\frac{3}{4}}) \) which gives \( E_1(w) \leq C \). Observe that \( \| w \|_{L^\infty} \leq CE_1(w) \), i.e. we already have a control on the \( L^\infty \) norm.
Next, we write
\[
\begin{align*}
\partial_x^8 u &= \partial_x^8 \varphi_c(x + ct) + \partial_x^9 w + \partial_x^7 \partial_y w - \partial_x^8 (w^2) - 2\partial_x^8 (\tilde{Q}_c w), \\
\partial_y^8 u &= \partial_x \partial_y w + \partial_x^{-1} \partial_y^9 w - \partial_y^8 (w^2) - 2\tilde{Q}_c \partial_y w.
\end{align*}
\]
Set
\[
E_8(w) = \| w \|_{L^2} + \| \partial_x^{-1} \partial_y w \|_{L^2} + \| \partial_x^8 w \|_{L^2} + \| \partial_x^7 \partial_y w \|_{L^2} + \| \partial_x \partial_y w \|_{L^2} + \| \partial_x^{-1} \partial_y^9 w \|_{L^2}.
\]
By invoking an orthogonality argument and the bounds we have already obtained, we can write
\[
E_8(w) \leq C (1 + \| \partial_x^8 (w^2) \|_{L^2} + \| \partial_y^8 (w^2) \|_{L^2} + \| \partial_x^8 w \|_{L^2} + \| \partial_y^8 w \|_{L^2})
\]
By invoking the elementary inequality
\begin{equation}
(2.14) \quad a^\theta b^{1-\theta} \leq a + b, \quad \forall a \geq 0, \ \forall b \geq 0, \ \forall \theta \in (0,1)
\end{equation}
in conjugation with the Fourier transform and the controls we have already obtained, we obtain that there exists \( \theta \in (0,1) \) such that
\[
\| \partial_x^8 w \|_{L^2} + \| \partial_y^8 w \|_{L^2} \leq C( E_S(w))^\theta.
\]
Next, we use a classical multiplicative inequality to get there exists \( \theta \in (0,1) \) such that
\[
\| \partial_x^8 (w^2) \|_{L^2} + \| \partial_y^8 (w^2) \|_{L^2} \leq C \| w^2 \|_{H^8} \leq C \| w \|_{L^\infty} \| w \|_{H^8}
\leq C(1 + \| \partial_x^8 w \|_{L^2} + \| \partial_y^8 w \|_{L^2}) \leq C(1 + (E_S(w))^\theta).
\]
Therefore, we obtain that \( E_S(w) \leq C(1 + (E_S(w))^\theta) \) for some \( \theta \in (0,1) \). This in turns implies that \( E_S(w) \leq C \), by a suitable use of \( (2.14) \). We finally observe that \( \| w \|_Z \leq CE_S(w) \). This completes the proof of Lemma 2.3. \( \square \)

Observe that using \( (2.3) \), we infer that if \( w \in Z \), then
\[
M^\pm_c (Q_c + w) \in H^8 (\mathbb{R}_x \times \mathbb{T}_y).
\]

Once global solutions are established, the conservation laws are obtained due to the following lemma.

**Lemma 2.4.** Let \( u \in C(\mathbb{R}; H^8(\mathbb{R}_x \times \mathbb{T}_y)) \) be a solution of the KP-II equation \( (1.1) \). Then
\[
\| u(t, \cdot) \|_{L^2(\mathbb{R}_x \times \mathbb{T}_y)} = \| u(0, \cdot) \|_{L^2(\mathbb{R}_x \times \mathbb{T}_y)}, \quad \forall t \in \mathbb{R}.
\]

The proof Lemma 2.4 can be obtained by an argument due to Molinet (see [19]). A similar argument may also be found in [20] p. 785]. This completes the proof of Proposition 2.2. \( \square \)

3. The Miura transform \( M^c_\pm \) in a neighborhood of \( Q_c \)

It turns out that the Miura transform \( M^c_\pm \) defines a bijection between a neighborhood of \((c, Q_c)\) and a neighborhood of \( \varphi_c \).

**Proposition 3.1.** For every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that if \( \| u \|_{L^2} < \delta \), there exists a unique \((k, v) \in \mathbb{R} \times \mathcal{E}(\mathbb{R}_x \times \mathbb{T}_y) \) satisfying
\begin{equation}
(3.1) \quad |k - c| < \varepsilon, \quad \| v \|_{\mathcal{E}(\mathbb{R}_x \times \mathbb{T}_y)} < \varepsilon, \quad M^c_k(Q_k + v) = \varphi_c + u.
\end{equation}
Moreover, the map \( L^2(\mathbb{R}_x \times \mathbb{T}_y) \ni u \mapsto (k, v) \in \mathbb{R} \times \mathcal{E}(\mathbb{R}_x \times \mathbb{T}_y) \) is of class \( C^1 \).

To prove Proposition 3.1, we need to investigate a linearized operator of the Miura transform \( M^c_\pm \) around \( Q_c \). Let \( (\varphi_c)^\perp \) be a subspace of \( L^2(\mathbb{R}_x \times \mathbb{T}_y) \) defined by
\[
(\varphi_c)^\perp \equiv \{ u \in L^2(\mathbb{R}_x \times \mathbb{T}_y) : (u, \varphi_c)_{L^2(\mathbb{R}_x \times \mathbb{T}_y)} = 0 \}.
\]
We will show that the Fréchet derivative \( \nabla_u M^c_+ (Q_c) : \mathcal{E} \to (\varphi_c)^\perp \) is bicontinuous and bounded from below.
Lemma 3.2. Let $c > 0$ and consider $L_c \equiv \partial_x + \partial_x^{-1} \partial_y - 2Q_c(x)$ and its formal adjoint $L_c^\ast \equiv -\partial_x + \partial_x^{-1} \partial_y - 2Q_c(x)$ as bounded operators from $\mathcal{E}(\mathbb{R}_x \times \mathbb{T}_y)$ to $L^2(\mathbb{R}_x \times \mathbb{T}_y)$. Then $\ker(L_c) = \{0\}$ and $\ker(L_c^\ast) = \text{span}\{\varphi_c\}$. Moreover $L_c$ is a Fredholm operator and $\text{Range}(L_c) = (\varphi_c)^\perp$. In addition,

$$\|w\|_{\mathcal{E}(\mathbb{R}_x \times \mathbb{T}_y)} \leq C\|L_c w\|_{L^2(\mathbb{R}_x \times \mathbb{T}_y)},$$

where $C$ is a positive constant that does not depend on $w$.

Proof of Lemma 3.2. First, we remark that if $u \in \ker(L_c)$ or $u \in \ker(L_c^\ast)$, then $\partial_x \partial_y^i u \in L^2(\mathbb{R}_x \times \mathbb{T}_y)$ for every $i \geq 0$ and $j \geq 0$ thanks to an elliptic regularity argument. We will give the proof only for $c = 2$, the case of a general $c$ being the same modulo some direct modifications. Let $Q \equiv Q_2$, $\varphi \equiv \varphi_2$, $L \equiv L_2$ and $\mathcal{L} = \mathcal{L}_2$.

Suppose $u \in \ker(L)$. Then $u$ satisfies

$$u_y + u_{xx} = 2(Qu)_x. \tag{3.3}$$

Using the last equation, we obtain that $u$ satisfies

$$\frac{1}{2} \frac{d}{dy} \int_{\mathbb{R}} u^2(x,y)dx = \int_{\mathbb{R}} (u_x^2(x,y) + Q(x)u^2(x,y))dx. \tag{3.4}$$

We next integrate the above identity over $\mathbb{T}$ to have

$$0 = \int_{\mathbb{T}} \int_{\mathbb{R}} (u_x^2(x,y) + Q(x)u^2(x,y))dxdy. \tag{3.5}$$

Combining the above with the fact that $Q'(x) > 0$ for every $x \in \mathbb{R}$, we have $u = 0$. Thus we obtain that $\ker(L) = \{0\}$.

The study of $\ker(\mathcal{L})$ is more intricate. Suppose $u \in \ker(\mathcal{L})$. Then $u$ is a solution to a heat equation

$$u_y = (u_x + 2Qu)_x, \tag{3.6}$$

and $2\pi$-periodic in $y$. A direct computation shows that (3.6) has $y$-independent solutions $\{\alpha Q'(x) | \alpha \in \mathbb{R}\}$. We will show that (3.6) has no other solution which is periodic in the $y$-variable. Let

$$V(y) = \int_{\mathbb{R}} \left( \frac{1}{2} u_x^2(x,y) - (Q'(x) - 2Q^2(x))u^2(x,y) \right)dx. \tag{3.7}$$

If $u \in L^2(\mathbb{R} \times \mathbb{T})$ is a smooth (in the Sobolev scale) solution of (3.6),

$$V'(y) = -\int_{\mathbb{R}} u_y(x,y)(u_{xx}(x,y) + 2(Q'(x) - 2Q^2(x))u(x,y))dx$$

$$= -\int_{\mathbb{R}} u_y(x,y)(u_y(x,y) - 2Q(x)(u_x(x,y) + 2Q(x)u(x,y)))dx$$

$$= -\int_{\mathbb{R}} (u_y^2(x,y) + Q'(x)(u_x(x,y) + 2Q(x)u(x,y))^2)dx \leq 0. \tag{3.8}$$

Integrating the last identity over $\mathbb{T}$, we have $u_y = u_x + 2Qu = 0$. Thus $u$ is independent of $y$ and by solving the ODE $u_x + 2Qu = 0$, we obtain that $\ker(\mathcal{L}) = \text{span}\{Q'\}$. 

Finally, we will show $R(L) = L^2(\mathbb{R} \times \mathbb{T}_y) \cap (Q')^\perp$ to prove that $L$ is Fredholm. Since $L$ is formally an adjoint operator of $\mathcal{L}$ and $\text{ker}(\mathcal{L}) = \text{span}\{Q\}$, we have $R(L) \subset L^2(\mathbb{R} \times \mathbb{T}_y) \cap (Q')^\perp$. Thus it suffices to show that $Lu = f$ has a solution $u \in \mathcal{E}(\mathbb{R} \times \mathbb{T}_y)$ for any $f \in L^2(\mathbb{R} \times \mathbb{T}) \cap (Q')^\perp$.

Suppose that $u$ satisfies $Lu = f$ with $f \in L^2(\mathbb{R} \times \mathbb{T}_y) \cap (Q')^\perp$. Let us expand $f$ and $u$ into Fourier series in $y$:

$$f(x, y) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} f_n(x)e^{iny}, \quad u(x, y) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} u_n(x)e^{iny}.$$  

Then we find that $u_n$ and $f_n$ satisfy the equations

$$u_0' = 2Qu_0 = f_0, \quad \int_{-\infty}^{\infty} f_0(x)Q'(x)dx = 0.$$  

$$u_n'' + iu_n - 2(Qu_n)' = f_n' \quad \text{if } n \neq 0,$$

and $f_n \in L^2(\mathbb{R})$ for every $n \in \mathbb{Z}$. The first equation in (3.5) can be solved explicitly as

$$u_0(x) = -\int_x^\infty \frac{\varphi(t)f_0(t)}{\varphi(x)}dt = \int_{-\infty}^x \frac{\varphi(t)f_0(t)}{\varphi(x)}dt.$$  

Note that $\varphi = 2Q'$. We have the following bound on $u_0$.

**Claim 3.1.** There exists $C$ such that for every $f_0 \in L^2(\mathbb{R})$,

$$\|u_0\|_{H^1(\mathbb{R})} \leq C\|f_0\|_{L^2}.$$  

**Proof.** Since $\varphi(t)/\varphi(x) \leq 4e^{-2|x-t|}$ for $|t| \geq |x|$, it follows that

$$|u_0(x)| \leq 4 \int_\mathbb{R} e^{-2|x-t|}\|f_0(t)\|dt.$$  

Applying Young’s inequality to the above, we obtain $\|u_0\|_{L^2} \leq C\|f_0\|_{L^2}$. The estimate for $\|u_0\|_{L^2}$ follows from the equation satisfied by $u_0$ and the previous estimate. This completes the proof of Claim 3.1.  

Next, we will show solvability of the second equation in (3.5). The key point is to show that the homogeneous problem

$$u'' + iu - 2(Qu)' = 0$$  

has no nontrivial spatially localized solution. Indeed, we have the following statement.

**Claim 3.2.** Let $n \in \mathbb{Z}$. Then (3.7) has no nontrivial solution $u$ which belongs to $L^2(\mathbb{R})$.

**Proof.** Let $u \in L^2(\mathbb{R})$ be a solution of (3.7). Then by an elliptic regularity argument, we have $u \in H^2(\mathbb{R})$. Moreover, we see that $e^{iny}u(x) \in \mathcal{E}(\mathbb{R} \times \mathbb{T}_y)$ and $L_c(e^{iny}u(x)) = 0$. Since $\text{ker}(L_c) = \{0\}$, it follows that $u = 0$. This completes the proof of Claim 3.2.  

□
By standard ODE arguments, since \( \lim_{x \to \pm \infty} Q'(x) = 0 \) and \( \lim_{x \to \pm \infty} Q(x) = \pm 1 \), the equation (3.7) has fundamental systems \( \{w^+_n(x), v^+_n(x)\} \) and \( \{w^-_n(x), v^-_n(x)\} \) such that
\[
\begin{align*}
  w^+_n(x) &\sim e^{\mu^+_n x}, \quad v^+_n(x) \sim e^{\lambda^+_n x} \quad \text{as } x \to \infty, \\
  w^-_n(x) &\sim e^{\mu^-_n x}, \quad v^-_n(x) \sim e^{\lambda^-_n x} \quad \text{as } x \to -\infty,
\end{align*}
\]
where \( \mu^+_n = 1 + \sqrt{1 - in} \) and \( \lambda^+_n = 1 - \sqrt{1 - in} \). Here and in the sequel the square roots are taken so that the real part of the result is non negative. Since \( \Re(\mu^+_n) > 0 > \Re(\lambda^+_n) \) for every \( n \neq 0 \), we see that \( v^+_n(x) \) decays exponentially as \( x \to \infty \) and \( w^-_n(x) \) decays exponentially as \( x \to -\infty \), whereas that \( w^+_n(x) \) grows exponentially as \( x \to \infty \) and \( v^-_n(x) \) grows exponentially as \( x \to -\infty \). Using Claim 3.2 we see that \( v^+_n(x) \) and \( w^-_n(x) \) are linearly independent for \( n \neq 0 \). In other words, \( v^+_n(x) = O(v^-_n(x)) \) as \( x \to -\infty \) and \( w^-_n(x) = O(w^+_n(x)) \) as \( x \to \infty \) if \( n \neq 0 \). The Green kernel is given by
\[
(3.8) \quad G_n(x, t) = \begin{cases} 
  -\frac{v^+_n(x)w^-_n(t)}{W(v^+_n, w^-_n)(t)} & \text{for } x > t, \\
  -\frac{w^-_n(x)v^+_n(t)}{W(v^+_n, w^-_n)(t)} & \text{for } x < t,
\end{cases}
\]
where
\[
W(v^+_n, w^-_n)(x) = \begin{vmatrix} v^+_n(x) & w^-_n(x) \\ \partial_x v^+_n(x) & \partial_x w^-_n(x) \end{vmatrix} = \cosh^2(x)W(v^+_n, w^-_n)(0).
\]
Observe that thanks to the above properties of the Green kernel, the kernel \( G_n \) enjoys a pointwise bound
\[
|G_n(x, t)| \lesssim \exp(-(\Re(\sqrt{1 - in} - 1)|x - t|)).
\]
Thus for every \( n \neq 0 \), the second equation of (3.5) has a solution given by
\[
u_n(x) = \int_{\mathbb{R}} G_n(x, t)f^+_n(t)dt.
\]
We now observe that
\[
\partial_t G_n(x, t) = \begin{cases} 
  2Q(t)G_n(x, t) - \frac{v^+_n(x)\partial_t w^-_n(t)}{W(v^+_n, w^-_n)(t)} & \text{for } x > t, \\
  2Q(t)G_n(x, t) - \frac{w^-_n(x)\partial_t v^+_n(t)}{W(v^+_n, w^-_n)(t)} & \text{for } x < t,
\end{cases}
\]
and
\[
|\partial_t G_n(x, t)| \lesssim \exp(-(\Re(\sqrt{1 - in} - 1)|x - t|)) .
\]
Therefore using integration by parts (and an approximation argument for \( f_n \) by \( C_0^\infty \) functions), we get
\[
u_n(x) = -\int_{\mathbb{R}} \partial_t G_n(x, t)f_n(t)dt.
\]
Differentiating the above equation, we have
\[
\partial_x \nu_n(x) = f_n(x) + \int_{\mathbb{R}} G^1_n(x, t)f_n(t)dt,
\]
where
\[
G_n^1(x, t) = \begin{cases} 
2Q(t)\partial_x G_n(x, t) - \frac{\partial_x v_n^+(x)\partial_t w_n^-(t)}{W(v_n^+, w_n^-)(t)} & \text{for } x > t, \\
2Q(t)\partial_x G_n(x, t) - \frac{\partial_x w_n^-(x)\partial_t v_n^+(t)}{W(v_n^+, w_n^-)(t)} & \text{for } x < t.
\end{cases}
\]

Obviously,
\[
|G_n^1(x, t)| \lesssim \exp(-\sqrt{1-in-1}|x-t|).
\]

Thus we obtain that \(u_n \in H^1\) and
\[
(3.9) \quad \|u_n\|_{H^1} + |n|\|\partial_x^{-1}u_n\|_{L^2} \leq C(n)\|f_n\|_{L^2}
\]
follows from the Young’s inequality and the relation \(in\partial_x^{-1}u_n = 2Q u_n - \partial_x u_n\).

Now we will show that \(C(n)\) can be chosen uniformly in \(n\). Let \(T_n(u) \equiv 2\partial_x (in + \partial_x^2)^{-1}(Qu)\)
and \(g_n \equiv \partial_x (in + \partial_x^2)^{-1}f_n\). Then the second equation of (3.5) can be rewritten as
\[
u_n = T_n(u_n) + g_n.
\]

Since \(\|T_n\|_{B(H^1)} = O(1/\sqrt{|n|})\) by the Plancherel theorem, there exists an \(n_0\) such that
\(\|T_n\|_{B(H^1)} \leq 1/2\) for every \(|n| \geq n_0\). Hence there exists a positive number \(C\) such that
for every \(|n| \geq n_0\)
\[
\|u_n\|_{H^1} \leq C\|g_n\|_{H^1} \leq C\|f_n\|_{L^2}.
\]

Furthermore, there exists a \(C' > 0\) such that for every \(|n| \geq n_0\),
\[
\|\partial_x^{-1}(T_n u_n + g_n)\|_{L^2} \leq 2\|\partial_x^2 + in\|^{-1}(Qu)\|_{L^2} + \|\partial_x^2 + in\|^{-1}f_n\|_{L^2}
\leq \frac{1}{|n|}(2\|u_n\|_{L^2} + \|f_n\|_{L^2}) \leq \frac{C'\|f_n\|_{L^2}}{|n|},
\]
whence \(\|\partial_x^{-1}u_n\| \leq \frac{C'\|f_n\|_{L^2}}{|n|}\) for \(|n| \geq n_0\). Therefore the constant \(C(n)\) involved in (3.9)
is uniform in \(n\). Combining (3.9) and Claim 3.1, we have
\[
\|u\|_{\hat{S}(\mathbb{R} \times T_y)}^2 = \sum_{n \in \mathbb{Z}}(\|u_n\|_{H^1(\mathbb{R})}^2 + n^2\|\partial_x^{-1}u_n\|_{L^2(\mathbb{R})}^2)
\leq C \sum_{n \in \mathbb{Z}}\|f_n\|_{L^2(\mathbb{R})}^2 = C\|f\|_{L^2(\mathbb{R} \times T_y)}^2.
\]

Thus we have proved that \(u \in \mathcal{E}(\mathbb{R} \times T_y)\) and \(L : \mathcal{E}(\mathbb{R} \times T_y) \rightarrow (\varphi)^\perp\) is surjective. Finally,
the estimate (3.2) follows readily from the open mapping theorem. This completes the proof
of Lemma 3.2. \(\square\)

Now we are in position to prove Proposition 3.1.

Proof of Proposition 3.1. Since \(M^c_\pm(Q_c) = \varphi_c\), (3.1) can be rewritten as
\[
(3.10) \quad \varphi_c - \varphi_k = L_c v + 2v(Q_c - Q_k) - v_c^2 - u.
\]
Let $P$ be a projection from $L^2(\mathbb{R}_x \times T_y)$ to its orthogonal subspace $(\varphi)^\perp$ defined by

$$Pf \equiv f - \frac{(f, Q_c')_{L^2}}{\|Q_c'\|_{L^2}^2}Q_c'.$$

Next, we define $F_1$ and $F_2$ by

$$F_1(u,v,k) = (\varphi_k - \varphi_c + 2v(Q_c - Q_k) - v^2 - u, \varphi_c)_{L^2},$$
$$F_2(u,v,k) = Lcv + P(\varphi_k - \varphi_c + 2v(Q_c - Q_k) - v^2 - u).$$

Since $\mathcal{L}_c \varphi_c = 0$, (3.10) holds if and only if $F_1(u,v,k) = 0$ and $F_2(u,v,k) = 0$. We consider $(F_1(u,v,k), F_2(u,v,k))$ as a $C^1$ map from $L^2(\mathbb{R}_x \times T_y) \times \mathcal{E}(\mathbb{R}_x \times T_y) \times \mathbb{R}_+$ to $\mathbb{R} \times (\varphi_c)^\perp$. Observe that $F_1(0,0,c) = 0$, $F_2(0,0,c) = 0$. Next we compute

$$\mathcal{A} \equiv \begin{pmatrix} \partial_u F_1 & \partial_v F_1 \\ \partial_u F_2 & \partial_v F_2 \end{pmatrix} \bigg|_{(u,v,k) = (0,0,c)} = \begin{pmatrix} ((\partial_k \varphi_k)|_{k = c}, \varphi_c) & 0 \\ P((\partial_k \varphi_k)|_{k = c}) & L_c \end{pmatrix}.$$

Since

$$\left((\partial_k \varphi_k)|_{k = c}, \varphi_c\right) = \frac{3}{4c}\|\varphi_c\|_{L^2}^2 \neq 0,$$

we obtain that $\mathcal{A}$ is a bicontinuous bijection from $\mathbb{R} \times \mathcal{E}(\mathbb{R}_x \times T_y)$ to $\mathbb{R} \times (\varphi_c)^\perp$ by using Lemma 3.2. Therefore the assertion of Proposition 3.1 follows from the implicit function theorem.

Next we will further investigate the linearized operator of the Miura transform $M_c$.

**Lemma 3.3.** Let $c > 0$ and $\mathcal{L}_c \equiv -\partial_x + \partial_x^{-1} \partial_y - 2Q_c(x)$ be considered as a bounded operator from $\mathcal{E}(\mathbb{R}_x \times T_y)$ to $L^2(\mathbb{R}_x \times T_y)$. Then $\mathcal{L}_c$ is Fredholm. More precisely, $\ker(\mathcal{L}_c) = \text{span}\{\varphi_c\}$, $\text{Range}(\mathcal{L}_c) = L^2$ and $\mathcal{L}_c : \mathcal{E} \cap (Q_c)^\perp \to L^2$ has a bounded inverse.

To prove Lemma 3.3, we need the following.

**Claim 3.3.** Let $I(f)(x) = \varphi(x) \int_0^x f(t)\varphi(t)^{-1}dt$. Then there exists a positive constant $C$ such that

$$\|I(f)\|_{H^1(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}.$$ 

**Proof.** Making use of

$$\varphi(x)/\varphi(t) \leq 4e^{-2|x-t|}, \quad |\varphi'(x)/\varphi(t)| \leq 8e^{-2|x-t|} \quad \text{for } t \in [-|x|, |x|],$$

we have $\|I(f)\|_{H^1} \leq C\|f\|_{L^2}$ in exactly the same way as Claim 3.1. □

**Proof of Lemma 3.3.** We will give the proof only for $c = 2$ for the sake of simplicity. Let $\mathcal{L} = \mathcal{L}_2$. We have $\ker(\mathcal{L}) = \text{span}\{Q_c\}$ from Lemma 3.2.

Next, we will show that $\mathcal{L}u = f$ has a solution $u \in \mathcal{E}$ for any $f \in L^2$. Let us expand $u$ and $f$ into Fourier series as

$$u(x,y) = \sum_{n \in \mathbb{Z}} u_n(x)e^{iny}, \quad f(x,y) = \sum_{n \in \mathbb{Z}} f_n(x)e^{iny}.$$
Then \( f_n \in L^2(\mathbb{R}) \) for every \( n \in \mathbb{Z} \) and

\[
\begin{align*}
- u_n' - 2Qu_0 &= f_0, \\
- u_n'' + inu_n - 2(Qu_n)' &= f_n' \quad \text{if } n \neq 0.
\end{align*}
\]

If \( u_0 \) is a solution of (3.12), then \( u_0(x) = \alpha \varphi(x) + I(f_0)(x) \) for an \( \alpha \in \mathbb{R} \). We remark that \( \alpha \) is uniquely determined and \( L^2(\mathbb{R}) \ni f_0 \mapsto \alpha \in \mathbb{R} \) is continuous if we impose the orthogonality condition \( \int u_0 Q' \, dx dy = 0 \).

Let us now observe that for \( n \neq 0 \), the equation (3.13) has no nontrivial solution \( u \) which belongs to \( L^2(\mathbb{R}) \). Indeed, let \( u \in L^2(\mathbb{R}) \) be a solution of (3.13). Then by elliptic regularity, we have \( u \in H^2(\mathbb{R}) \). Moreover, we see that \( e^{inx} u(x) \in \mathcal{E}(\mathbb{R}_x \times T_y) \) and \( \mathcal{L}(e^{inx} u(x)) = 0 \). Since \( \ker(\mathcal{L}) = \text{span}\{Q'\} \) it follows that \( u = 0 \) unless \( n = 0 \). Next, we will solve (3.13). By standard ODE arguments, since \( \lim_{x \to \pm \infty} Q'(x) = 0 \) and \( \lim_{x \to \pm \infty} Q(x) = \pm 1 \), the equation

\[
- u'' + inu - 2(Qu)' = 0
\]

has fundamental systems \( \{\tilde{w}_n^+(x), \tilde{v}_n^+(x)\} \) and \( \{\tilde{w}_n^-(x), \tilde{v}_n^-(x)\} \) such that

\[
\begin{align*}
\tilde{w}_n^+(x) &= e^{\tilde{\mu}_n^+ x}, & \tilde{v}_n^+(x) &= e^{\tilde{\lambda}_n^+ x} \quad \text{as } x \to \infty, \\
\tilde{w}_n^-(x) &= e^{\tilde{\mu}_n^- x}, & \tilde{v}_n^-(x) &= e^{\tilde{\lambda}_n^- x} \quad \text{as } x \to -\infty,
\end{align*}
\]

where \( \tilde{\mu}_n^\pm = +1 + \sqrt{1 + in} \) and \( \tilde{\lambda}_n^\pm = +1 - \sqrt{1 + in} \). Since \( \text{Re}(\tilde{\mu}_n^\pm) > 0 > \text{Re}(\tilde{\lambda}_n^\pm) \) for every \( n \neq 0 \),

\[
\begin{align*}
\lim_{x \to \infty} \tilde{w}_n^+(x) &= \infty, & \lim_{x \to \infty} \tilde{v}_n^+(x) &= 0, \\
\lim_{x \to -\infty} \tilde{w}_n^-(x) &= 0, & \lim_{x \to -\infty} \tilde{v}_n^-(x) &= \infty.
\end{align*}
\]

As in the proof of Claim 3.2 we see that \( \tilde{v}_n^+(x) \) and \( \tilde{w}_n^-(x) \) are linearly independent for every \( n \neq 0 \).

Thus for \( n \neq 0 \), the Green kernel of \( -u'' + inu - 2(Qu)' \) is given by

\[
\tilde{G}_n(x, t) = \begin{cases}
- \frac{\tilde{v}_n^+(x)\tilde{w}_n^-(t)}{W(\tilde{v}_n^+, \tilde{w}_n^-)(t)} & \text{for } x > t, \\
- \frac{\tilde{w}_n^-(x)\tilde{v}_n^+(t)}{W(\tilde{v}_n^-, \tilde{w}_n^+)(t)} & \text{for } x < t,
\end{cases}
\]

where

\[
W(\tilde{v}_n^+, \tilde{w}_n^-)(x) = \left| \begin{array}{ccc}
\tilde{v}_n^+(x) & \tilde{w}_n^-(x) \\
\partial_x \tilde{v}_n^+(x) & \partial_x \tilde{w}_n^-(x)
\end{array} \right| = \text{sech}^2(x)W(\tilde{v}_n^+, \tilde{w}_n^-)(0).
\]

For every \( n \neq 0 \),

\[
|\tilde{G}_n(x, t)| + |\partial_t \tilde{G}_n(x, t)| \lesssim \exp(-\text{Re}(\sqrt{1 + in} - 1)|x - t|).
\]

Repeating the arguments of the proof of Lemma 3.2, we obtain

\[
\|u_n\|_{H^1} + |n|\|\partial_x^{-1} u_n\|_{L^2} \leq C(n)\|f_n\|_{L^2}
\]
and that the constant $C(n)$ involved in (3.17) can be chosen uniformly in $n$. Therefore
\[
\|u\|_{L^2(\mathbb{R}^2)}^2 = \sum_{n \in \mathbb{Z}} (\|u_n\|^2_{H^1(\mathbb{R})} + n^2 \|\partial_x^{-1} u_n\|^2_{L^2(\mathbb{R})}) \leq C \|f\|_{L^2(\mathbb{R}^2)}^2,
\]
where $C$ is a constant independent of $f \in L^2(\mathbb{R} \times \mathbb{T}_y)$. Thus we have proved that $\mathcal{L} : \mathcal{E}(\mathbb{R} \times \mathbb{T}_y) \to L^2(\mathbb{R} \times \mathbb{T}_y)$ is surjective.

Since $\ker(\mathcal{L}) = \text{span}\{Q'\}$ and $\text{Range}(\mathcal{L}) = L^2$, it follows from the open mapping theorem that $\mathcal{L} : \mathcal{E} \cap (Q')^\perp \to L^2$ has a bounded inverse. $\square$

Finally, we will investigate a property of $\mathcal{L}_c$ in a weighted space. Let
\[
\chi_\varepsilon(x) = \frac{1 + \tanh(\varepsilon x)}{2}
\]
and let $L^2_{\varepsilon,x_0}$ and $\mathcal{E}_{\varepsilon,x_0}$ be Banach spaces equipped with norms
\[
\|u\|_{L^2_{\varepsilon,x_0}} = \left( \int_{\mathbb{R} \times \mathbb{T}_y} \chi_\varepsilon(x + x_0)|u(x,y)|^2 \, dx \, dy \right)^{\frac{1}{2}},
\]
\[
\|u\|_{\mathcal{E}_{\varepsilon,x_0}} = \|u\|_{L^2_{\varepsilon,x_0}} + \|\partial_x u\|_{L^2_{\varepsilon,x_0}} + \|\partial_{x}^{-1}\partial_y u\|_{L^2_{\varepsilon,x_0}},
\]
respectively. Roughly speaking, we will show that $\mathcal{L}_c : \mathcal{E}_{\varepsilon,x_0} \cap (Q')^\perp \to L^2_{\varepsilon,x_0}$ is bounded from below for small $\varepsilon > 0$.

**Lemma 3.4.** Let $c > 0$. Then there exist positive constants $\varepsilon_0$ and $C$ such that for every $\varepsilon \in (0, \varepsilon_0)$, $x_0 \in \mathbb{R}$ and $w \in \mathcal{E}(\mathbb{R} \times \mathbb{T}_y) \cap (Q')^\perp$,
\[
(3.18) \quad \|\mathcal{L}_c w\|_{L^2_{\varepsilon,x_0}(\mathbb{R} \times \mathbb{T}_y)} \geq C \|w\|_{\mathcal{E}_{\varepsilon,x_0}}.
\]

**Proof.** Let $c = 2$ for the sake of simplicity. Suppose $f(x,y) = \sum_{n \in \mathbb{Z}} f_n(x) e^{iny} \in L^2(\mathbb{R} \times \mathbb{T}_y)$ and that $u(x,y) = \sum_{n \in \mathbb{Z}} u_n(x) e^{iny} \in \mathcal{E}(\mathbb{R} \times \mathbb{T}_y)$ is a solution of $\mathcal{L}_c u = f$ satisfying $\int u Q' \, dx \, dy = 0$. Then
\[
u_0(x) = \alpha \varphi(x) + I(f_0)(x),
\]
\[
u_n(x) = \int_{\mathbb{R}} \tilde{G}_n(x,t) f_n'(t) \, dt \quad \text{for } n \neq 0,
\]
where $\alpha$ is a constant satisfying $\alpha = -(I(f_0), \varphi)_{L^2}/\|\varphi\|_{L^2}^2$.

By the definition of $\chi_\varepsilon$,
\[
(3.19) \quad \frac{\chi_\varepsilon(x + x_0)}{\chi_\varepsilon(t + x_0)} \leq 1 + e^{2\varepsilon|t-x|} \leq 2e^{2\varepsilon|t-x|}.
\]
Combining (3.11) and (3.19), we have for $\varepsilon \in (0, 1)$,
\[
\|\chi_\varepsilon(x + x_0) f_0\|_{L^2} + \|\chi_\varepsilon(x + x_0) \partial_x I(f_0)\|_{L^2} \leq C \|\chi_\varepsilon(x + x_0) f_0\|_{L^2},
\]
where $C$ is a positive constant depending only on $\varepsilon$. Hence there exists a $C > 0$ such that for every $f \in L^2(\mathbb{R})$ and $x_0 \in \mathbb{R}$,
\[
\|\chi_\varepsilon(x + x_0) u_0\|_{L^2(\mathbb{R})} + \|\chi_\varepsilon(x + x_0) \partial_x u_0\|_{L^2(\mathbb{R})} \leq C \|\chi_\varepsilon(x + x_0) f_0\|_{L^2(\mathbb{R})}.
\]
We shall estimate

\[ |\chi_x(x + x_0)\partial^k_t \tilde{G}_n(x, t)\chi_x(t + x_0)^{-1}| \lesssim \exp((-\Re(\sqrt{1 + \im \theta} - 1) + 2\varepsilon)|x - t|). \]

for \( k = 0, 1 \). As in the proof of Lemma 3.2 after an analysis of \( \partial_x(\chi_x(x + x_0)\partial_t \tilde{G}_n(x, t)\chi_x(t + x_0)^{-1}) \), we have that for \( 0 < 2\varepsilon < \Re(\sqrt{1 + \im \theta} - 1) \), there exist positive constants \( C(n, \varepsilon) \) such that if \( n \neq 0 \),

\[ \|\chi_x(x + x_0)u_n\|_{L^2} + \|\chi_x(x + x_0)\partial_x u_n\|_{L^2} \leq C(n, \varepsilon)\|\chi_x(x + x_0)f_n\|_{L^2}. \]

Combining the above with (3.13), we have

\[ \|\chi_x(x + x_0)\partial_x^{-1} u_n\|_{L^2} \leq \frac{2C(n, \varepsilon) + 1}{n}\|\chi_x(x + x_0)f_n\|_{L^2} \quad \text{for } n \neq 0. \]

To prove (3.18), it suffices to show that \( \sup_{n \neq 0} C(n, \varepsilon) < \infty \). Let \( \tilde{u}_n(x) = \chi_x(x + x_0)u_n(x) \), \( \tilde{g}_n(x) = \chi_x(x + x_0)(in - \partial_x^2)^{-1}f_n'(x) \) and

\[ \tilde{T}_n(u)(x) = \chi_x(x + x_0) \int_{\mathbb{R}} \frac{e^{-\sqrt{\im \theta}|x-t|}Q(t)u(t)}{\chi_x(t + x_0)}\text{sgn}(t - x)dt, \]

where \( \sqrt{\im \theta} \) is chosen with positive real part. The definition of \( \tilde{T}_n \) is obtained thanks to the explicit formula of the kernel \((in - \partial_x^2)^{-1} = \frac{1}{2\sqrt{\im \theta}}e^{-\sqrt{\im \theta}|x|}\). Then (3.13) can be rewritten as

\[ \tilde{u}_n = \tilde{T}_n \tilde{u}_n + \tilde{g}_n. \]

By (3.19),

\[ \frac{\left| \chi_x(x + x_0) e^{-\sqrt{\im \theta}|x-t|} \right|}{\chi_x(t + x_0)} \leq 2e^{-(\sqrt{|\im \theta|/2 - 2\varepsilon})|x-t|}, \]

and it follows from a convolution estimate that there exists an \( n_0 \in \mathbb{N} \) such that \( \|\tilde{T}_n\|_{B(L^2(\mathbb{R}))} \leq \frac{1}{2} \) and \( \|\tilde{g}_n\|_{L^2} \leq \frac{1}{2}\|\chi_x(x + x_0)f_n\|_{L^2} \) for \( |n| \geq n_0 \). Thus we have

\[ \|\chi_x(x + x_0)u_n\|_{L^2(\mathbb{R})} \leq C\|\chi_x(x + x_0)f_n\|_{L^2(\mathbb{R})} \quad \text{for } |n| \geq n_0. \]

We shall estimate \( \chi_x(x + x_0)\partial_x u_n(x) \) in \( L^2 \) by a similar argument. We can write

\[ \partial_x u_n(x) = \int_{\mathbb{R}} e^{-\sqrt{\im \theta}|x-t|}(Q(t)u_n(t))'\text{sgn}(t - x)dt + (\partial_x^2 + in)^{-1}f_n''(x). \]

Using a convolution estimate, as above, we obtain that there exist \( C > 0 \) and \( n_0 \in \mathbb{N} \) such that for \( |n| \geq n_0 \),

\[ \left\| \chi_x(x + x_0) \int_{\mathbb{R}} e^{-\sqrt{\im \theta}|x-t|}(Q(t)u_n(t))'\text{sgn}(t - x)dt \right\|_{L^2} \leq C|n|^{-\frac{1}{2}}\left( \|\chi_x(x + x_0)u_n\|_{L^2} + \|\chi_x(x + x_0)\partial_x u_n\|_{L^2} \right). \]

Similarly, we obtain that there exists a constant \( C \), independent of \( n \) such that for \( |n| \geq n_0 \),

\[ \|\chi_x(x + x_0)(-\partial_x^2 + in)^{-1}f_n''(x)\|_{L^2} \leq C\|\chi_x(x + x_0)f_n\|_{L^2}. \]
Therefore, we obtain that there exists a positive constant $C$ and $n_0 \in \mathbb{N}$ such that for $|n| \geq n_0$, $$\| \chi_\varepsilon(x + x_0) \partial_x u_n \|_{L^2(\mathbb{R})} \leq C \| \chi_\varepsilon(x + x_0) f_n \|_{L^2(\mathbb{R})}.$$ Thus we have $\sup_{n \neq 0} C(n, \varepsilon) < \infty$. This completes the proof of Lemma 3.4.

4. Stability of kink solutions of mKP-II

In this section, we prove a stability property of the kink solutions of the mKP-II equation which will be of crucial importance in the proof of stability of the line soliton of the KP-II equation. Recall that $Z = \{ u \in H^8 | \partial_x^{-1} \partial_y u, \partial_y u \in H^8 \}$. We will prove that kinks are stable if the perturbation belongs to $Z$ and if it is small in $\mathcal{E}$.

**Proposition 4.1.** For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if the initial data (2.4) of (2.2) satisfies $\| w_0 \|_{\mathcal{E}(\mathbb{R}^2 \times T_y)} < \delta$ and $w_0 \in Z$, then there exists a continuous function $\gamma(t)$ such that for every $t \in \mathbb{R}$, the corresponding solution of (2.2) satisfies $$\| v(t, x) - Q_c(x + \gamma(t)) \|_{\mathcal{E}(\mathbb{R} \times T_y)} < \varepsilon.$$ 

**Proof.** As long as $v(t, x, y)$ stays in a small neighborhood of $\{ Q_c(\cdot + x_0) | x_0 \in \mathbb{R} \}$, we can decompose $v(t, x, y)$ as

$$v(t, x, y) = Q_c(x + \gamma(t)) + w(t, x, y),$$

so that $w(t, x, y)$ satisfies the orthogonality condition

$$w(t, x, y), Q_c'(x + \gamma(t)) = \frac{1}{2} \langle w(t, x, y), \varphi_c(x + \gamma(t)) \rangle = 0,$$

where $(\cdot, \cdot)$ denotes the $L^2(\mathbb{R}^2 \times T_y)$ scalar product. Note that $\gamma$ depends continuously on $t$.

Let

$$\mathcal{L}_{c, \gamma(t)} \equiv -\partial_x + \partial_x^{-1} \partial_y - 2Q_c(x + \gamma(t)).$$

Let us expand the square of the $L^2$ norm of $M_c^-(v)$. Recalling that $M_c^-(Q_c) = 0$, we have

$$\| M_c^-(Q_c(x + \gamma(t)) + w(t, x, y)) \|_{L^2(\mathbb{R} \times T_y)}^2 = \int_{\mathbb{R} \times T_y} (\mathcal{L}_{c, \gamma(t)} w - w^2)^2 dx dy$$

$$= \int_{\mathbb{R} \times T_y} (\mathcal{L}_{c, \gamma(t)} w)^2 dx dy + \int_{\mathbb{R} \times T_y} (w^4 - 2w^2 \mathcal{L}_{c, \gamma(t)} w) dx dy.$$

Thanks to (4.2), we see that $w(t, x - \gamma(t), y)$ is orthogonal in $L^2(\mathbb{R} \times T_y)$ to $\varphi_c(x)$. Therefore, using Lemma 3.2, we obtain that there exists a positive constant $\nu$, independent of $t$ and $w$ such that

$$\int_{\mathbb{R} \times T_y} (\mathcal{L}_{c, \gamma(t)} w)^2 dx dy = \int_{\mathbb{R} \times T_y} (\mathcal{L}_c(w(t, x - \gamma(t), y))^2 dx dy \geq \nu \| w \|_{\mathcal{E}(\mathbb{R} \times T_y)}^2.$$

Next we invoke the anisotropic Sobolev embedding of Lemma 2.1 to have

$$\| M_c^-(v)(t, x, y) \|_{L^2(\mathbb{R} \times T_y)} \geq \frac{\nu}{2} \| w(t, \cdot) \|_{\mathcal{E}}^2 - C \| w(t, \cdot) \|_{\mathcal{E}}^3.$$
Thanks to the conservation law of Proposition 2.2, we have
\[(4.4) \quad \| M^c (v)(t, x, y) \|_{L^2(\mathbb{R} \times \mathbb{T})}^2 = \| M^c (Q_c (x) + w_0 (x, y)) \|_{L^2(\mathbb{R} \times \mathbb{T})}^2. \]

Now expanding the square of the \( L^2 \) norm of \( M^c (Q_c (x) + w_0 (x, y)) \) and using Lemma 2.1 we have
\[(4.5) \quad \| M^c (Q_c (x) + w_0 (x, y)) \|_{L^2(\mathbb{R} \times \mathbb{T})}^2 \leq C(\| w_0 \|_{L^2}^2 + \| w_0 \|_{L^4}^2). \]

Combining (4.3), (4.4) and (4.5), we get \( \| w(t) \|_{L^2} \leq C \| w_0 \|_{L^2} \) provided \( \delta \ll 1 \). This completes the proof of Proposition 4.1. \( \square \)

5. Asymptotic Stability of Kink Solutions of MKP-II

In this section, we will prove asymptotic stability of kink solutions.

**Proposition 5.1.** There exists a \( \delta > 0 \) such that if the initial data (2.4) of (2.2) satisfies
\[ \| w_0 \|_{L^2_x(\mathbb{T})} < \delta \] and \( w_0 \in Z \), then the corresponding solution of (2.2) satisfies
\[ \lim_{t \to \infty} \| (v(t, x, y) - Q_c (x + \gamma(t))) \|_{L^2(\mathbb{T})} = 0, \]
where \( \gamma(t) \) is a \( C^1 \)-function satisfying \( \gamma(t) = c + O(\| w_0 \|_{L^2}) \).

To prove Proposition 5.1 we will use the Miura transform \( u(t, x, y) = M^c (v)(t, x - 3ct, y) \) and the monotonicity property of the KP-II equation that follows from the Kato type estimate for the KP-II equation. Let us recall the Kato type identity for the KP-II equation.

**Lemma 5.2.** [3, Lemma 1] Let \( u(t) \in C(\mathbb{R}; H^8(\mathbb{R} \times \mathbb{T})) \) be a solution of (1.1) and \( \phi(x) \) be of the class \( C^3 \). Then
\[
\frac{d}{dt} \int_{\mathbb{R} \times \mathbb{T}} u^2 (t, x, y) \phi(x) dx dy = \int_{\mathbb{R} \times \mathbb{T}} (-3(\partial_x u)^2 - 3(\partial_x^{-1} \partial_y u)^2 - 4u^3) \phi' dx dy + \int_{\mathbb{R} \times \mathbb{T}} u^2 \phi'' dx dy.
\]

Next, we will show that small solutions of the KP-II equation locally tends to 0 as \( t \to \infty \) by using Lemma 5.2.

**Lemma 5.3.** Let again \( \chi_\varepsilon(x) = (1 + \tanh(\varepsilon x))/2 \) and let \( u(t) \in C(\mathbb{R}; H^8(\mathbb{R} \times \mathbb{T})) \) be a solution of the KP-II equation (1.1). For every \( c_1 > 0 \), there exist positive numbers \( \delta \) and \( \varepsilon_0 \) such that if \( \varepsilon \in (0, \varepsilon_0) \) and \( \| u(0, x, y) \|_{L^2(\mathbb{R} \times \mathbb{T})} \leq \delta \), then for every \( x_0 \in \mathbb{R} \),
\[
\frac{d}{dt} \int_{\mathbb{R} \times \mathbb{T}} u^2 (t, x, y) \chi_\varepsilon (x + x_0 - c_1 t) dx dy \leq 0,
\]
\[
\lim_{t \to \infty} \int_{\mathbb{R} \times \mathbb{T}} u^2 (t, x, y) \chi_\varepsilon (x - c_1 t) dx dy = 0.
\]

To show Lemma 5.3 we use the following.
Claim 5.1. Let \( \varepsilon > 0 \). There exists a positive constant \( C \) such that for every \( w \in \mathcal{E}(\mathbb{R}_x \times \mathbb{T}_y) \) and \( x_0 \in \mathbb{R} \),

\[
(5.2) \quad \left( \int_{\mathbb{R}_x \times \mathbb{T}_y} (\chi'_\varepsilon(x+x_0))^2 w^4 \, dx \, dy \right) \leq C \int_{\mathbb{R}_x \times \mathbb{T}_y} \left( (\partial_x w)^2 + (\partial_x^{-1} \partial_y w)^2 + w^2 \right) \chi'_\varepsilon(x+x_0) \, dx \, dy.
\]

Proof. This claim can be proved in a similar way as [20, Lemma 2]. By a density argument, we can suppose that \( w \in \mathcal{Z} \) and in particular \( w \) vanishes as \( x \to \pm \infty \).

Suppose that \( w \) vanishes at a point of \( \mathbb{T}_y \), say \( y = 0 \). Using Fubini’s theorem and integration by parts, we have

\[
\int_{\mathbb{R}} \chi'_\varepsilon(x+x_0) w^2(x,y) \, dx = 2 \int_{\mathbb{R}} \chi'_\varepsilon(x+x_0) \left( \int_{0}^{y} w(x,t) w'(x,t) \, dt \right) \, dx
\]

\[
= -2 \int_{0}^{y} \left( \int_{\mathbb{R}} \partial_x (\chi'_\varepsilon(x+x_0) w(x,t)) \partial_x^{-1} \partial_y w(x,t) \, dx \right) \, dt
\]

\[
\leq C \int_{\mathbb{R}_x \times \mathbb{T}_y} \left( (\partial_x w)^2 + (\partial_x^{-1} \partial_y w)^2 + w^2 \right) \chi'_\varepsilon(x+x_0) \, dx \, dy,
\]

where we used an inequality \( |\chi''_\varepsilon| \leq 4\varepsilon \chi'_\varepsilon \) and the inequality \( 2ab \leq a^2 + b^2 \). If \( w(x,y) \) does not vanish on \( \mathbb{T}_y \), we apply a partition of unity argument to \( w \). More precisely, let \( \psi_1(y) \) and \( \psi_2(y) \) be nonnegative smooth functions such that \( \psi_1 + \psi_2 \equiv 1 \) and \( \psi_1(y) = 0 \) for \( |y| \leq \pi/5 \) and \( \psi_2(y) = 0 \) for \( y \notin (-\pi/4, \pi/4) \). Then

\[
\int_{\mathbb{R}} \chi'_\varepsilon(x+x_0) \psi_1(y) w^2(x,y) \, dx
\]

\[
= -2 \int_{0}^{y} \psi_1(t) \left( \int_{\mathbb{R}} \partial_x (\chi'_\varepsilon(x+x_0) w(x,t)) \partial_x^{-1} \partial_y w(x,t) \, dx \right) \, dt
\]

\[
+ \int_{0}^{y} \psi_1(t) \left( \int_{\mathbb{R}} \chi'_\varepsilon(x+x_0) w^2(x,t) \, dx \right) \, dt
\]

\[
\leq C \int_{\mathbb{R}_x \times \mathbb{T}_y} \left( (\partial_x w)^2 + (\partial_x^{-1} \partial_y w)^2 + w^2 \right) \chi'_\varepsilon(x+x_0) \, dx \, dy.
\]

We can estimate \( \int_{\mathbb{R}} \chi'_\varepsilon(x+x_0) \psi_2(y) w^2(x,y) \, dx \) in the same way, by replacing 0 with another point on \( \mathbb{T}_y \), say \( \pi/3 \). Thus we have

\[
(5.4) \quad \sup_{y} \int_{\mathbb{R}} \chi'_\varepsilon(x+x_0) w^2(x,y) \, dx \leq C \int_{\mathbb{R}_x \times \mathbb{T}_y} \left( (\partial_x w)^2 + (\partial_x^{-1} \partial_y w)^2 + w^2 \right) \chi'_\varepsilon(x+x_0) \, dx \, dy.
\]

Write

\[
\chi'_\varepsilon(x+x_0) w^2(x,y) = \int_{-\infty}^{x} \left( \chi''_\varepsilon(z+x_0) w^2(z,y) + 2 \chi'_\varepsilon(z+x_0) \partial_x w(z,y) w(z,y) \right) \, dz.
\]

This yields

\[
(5.5) \quad \sup_{x} \left[ \chi'_\varepsilon(x+x_0) w^2(x,y) \right] \leq C \int_{-\infty}^{\infty} \left( w^2(z,y) + (\partial_x w)^2(z,y) \right) \chi'_\varepsilon(z+x_0) \, dz.
\]
Applying (5.3) to the first factor of \((\chi'_{\varepsilon}(x + x_0))^2w^4(x, y) = (\chi'_{\varepsilon}(x + x_0)w^2) \times (\chi'_{\varepsilon}(x + x_0)w^2)\) and integrating the resulting inequality over \(\mathbb{R}_x \times \mathbb{T}_y\), we obtain

\[
\begin{align*}
(5.6) \quad & \int_{\mathbb{R}_x \times \mathbb{T}_y} (\chi'_{\varepsilon}(x + x_0))^2w^4(x, y)dxdy \\
& \leq C \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} w^2(x, y)\chi'_{\varepsilon}(x + x_0)dx \right) \left( \int_{-\infty}^{\infty} (w^2(z, y) + (\partial_z w)^2(z, y))\chi'_{\varepsilon}(z + x_0)dz \right)dy.
\end{align*}
\]

Substituting (5.4) into (5.6), we obtain (5.2). This completes the proof of Claim 5.1.

\[\square\]

**Proof of Lemma 5.3** By taking into account the time translation and using Lemma 5.2, we can write

\[
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}_x \times \mathbb{T}_y} u^2(t, x, y)\chi_{\varepsilon}(x + x_0 - c_1 t)dxdy \\
= \int_{\mathbb{R}_x \times \mathbb{T}_y} (\chi_{\varepsilon}' x + x_0 - c_1 t)dxdy \\
+ \int_{\mathbb{R}_x \times \mathbb{T}_y} u^2(t, x, y)(\chi_{\varepsilon}''(x + x_0 - c_1 t) - c_1 \chi_{\varepsilon}'(x + x_0 - c_1 t))dxdy.
\end{align*}
\]

Using Claim 5.1, we have

\[
\begin{align*}
(5.7) \quad & \left| \int_{\mathbb{R}_x \times \mathbb{T}_y} u^2(t, x, y)\chi_{\varepsilon}'(x + x_0 - c_1 t)dxdy \right| \\
& \leq \|u(t, \cdot)\|_{L^2} \left( \int_{\mathbb{R}_x \times \mathbb{T}_y} (\chi_{\varepsilon}'(x + x_0 - c_1 t))^2u^4dxdy \right)^{\frac{1}{2}} \\
& \leq C\|u(0, \cdot)\|_{L^2} \int_{\mathbb{R}_x \times \mathbb{T}_y} \left( (\partial_x u)^2 + (\partial^{-1}_x \partial_y u)^2 + u^2 \right)(t, x, y)\chi_{\varepsilon}'(x + x_0 - c_1 t)dxdy.
\end{align*}
\]

It follows from (5.7), (5.8) and the fact that \(|\chi''_{\varepsilon}| \leq 8\varepsilon^2 \chi'_{\varepsilon}\) that

\[
\begin{align*}
& \frac{d}{dt} \int_{\mathbb{R}_x \times \mathbb{T}_y} u^2(t, x, y)\chi_{\varepsilon}(x - c_1 t)dxdy \\
& \leq -C \int_{\mathbb{R}_x \times \mathbb{T}_y} \left( (\partial_x u)^2 + (\partial^{-1}_x \partial_y u)^2 + u^2 \right)(t, x, y)\chi_{\varepsilon}'(x - c_1 t)dxdy
\end{align*}
\]

provided \(\varepsilon\) and \(\delta\) are sufficiently small. Thus we have the former part of Lemma 5.3.

Let

\[
I_{x_0}(t) = \int_{\mathbb{R}_x \times \mathbb{T}_y} u^2(t, x, y)\chi_{\varepsilon}(x - \frac{1}{2} c_1 t - x_0)dxdy.
\]

Taking \(\delta > 0\) smaller if necessary, we have \(I_{x_0}(t) \leq I_{x_0}(0)\) for every \(t \geq 0\) and \(x_0 \geq 0\). Since \(\lim_{x_0 \to \infty} I_{x_0}(0) = 0\), we have

\[
\lim_{t \to \infty} \int_{\mathbb{R}_x \times \mathbb{T}_y} u^2(t, x, y)\chi_{\varepsilon}(x - c_1 t) = \lim_{t \to \infty} I_{c_1t/2}(t) \leq \lim_{t \to \infty} \inf I_{c_1t/2}(0) = 0.
\]

Thus we complete the proof of Lemma 5.3. \(\square\)

Next, we will derive a modulation equation on the parameter \(\gamma(t)\) which describes the phase shift of the line soliton.
Similarly, we have

Combining (5.10) and (5.11) with sup

there exists a constant

Proof. By Proposition 4.1, a solution \( v(t) \) remains in a tubular neighborhood of the kink solutions for every \( t \in \mathbb{R} \) if \( \delta \) is sufficiently small. Applying the implicit function theorem and using a continuation argument, we find a \( C^1 \)-function \( \gamma(t) \) that satisfies \( |\gamma(0)| = O(\|w_0\|\varepsilon) \) and (4.1) and (4.2) for every \( t \in \mathbb{R} \).

Differentiating (4.2) with respect to \( t \) and substituting (2.2) into the resulting equation, we have

Thus we complete the proof of Lemma 5.4.

We will make use of the following statement (which will be used in the particular case \( p = 4 \)).

Claim 5.2. Let \( 2 \leq p \leq 6 \) and \( \varepsilon > 0 \). There exists a positive constant \( C \) such that for every \( w \in \mathcal{E}(\mathbb{R}_x \times T_y) \) and \( x_0 \in \mathbb{R} \),

\[
\int_{\mathbb{R}_x \times T_y} \chi_\varepsilon(x + x_0)|w(x,y)|^p \, dx \, dy \leq C\|w\|_{\mathcal{E}(\mathbb{R}_x \times T_y)}^{p-2}\|w\|_{\mathcal{E}_{\varepsilon,x_0}}^2.
\]

Proof. The proof follows the line of the proof of Claim 5.1. We only need to consider the endpoint statements, then the other cases follow by the Hölder inequality with respect to the measure \( \chi_\varepsilon(x + x_0)dx \). The case \( p = 2 \) is obvious. Let us consider the case \( p = 6 \). First, by invoking the inequality \(|\chi_\varepsilon'| \leq 2\varepsilon\chi_\varepsilon \) in place of \(|\chi_\varepsilon''| \leq 4\varepsilon^3\chi_\varepsilon^\prime \) in the proof of (5.4), we get that there exists a constant \( C \) such that for every \( x_0 \in \mathbb{R} \), every \( y \in T \) and every \( w \in \mathcal{E} \),

\[
\sup_y \int_{-\infty}^{\infty} \chi_\varepsilon(x + x_0)|w(x,y)|^2 \, dx \leq C\|w\|_{\mathcal{E}_{\varepsilon,x_0}}^2.
\]

Similarly, we have

\[
\sup_y \int_{-\infty}^{\infty} |w(x,y)|^2 \, dx \leq C\|w\|_{\mathcal{E}}^2.
\]

Combining (5.10) and (5.11) with \( \sup_x |w(x,y)|^4 \leq 2(\int_{\mathbb{R}^2} w(z,y)^2 \, dz)(\int_{\mathbb{R}} w_x(z,y)^2 \, dz) \), we obtain

\[
\int_{\mathbb{R}_x \times T_y} \chi_\varepsilon(x + x_0)|w(x,y)|^6 \, dx \, dy
\leq 2 \int_{-\infty}^{\infty} \left\{ \left( \int_{\mathbb{R}} |w(z,y)|^2 \, dz \right) \left( \int_{\mathbb{R}} |w_x(z,y)|^2 \, dz \right) \int_{\mathbb{R}} \chi_\varepsilon(x + x_0)|w(x,y)|^2 \, dx \right\} \, dy
\leq C\|w\|_{\mathcal{E}(\mathbb{R}_x \times T_y)}^4 \|w\|_{\mathcal{E}_{\varepsilon,x_0}}^2.
\]
Thus we complete the proof of Claim 5.2. □

Now we are in position to prove Proposition 5.1.

Proof of Proposition 5.1 Let \( u_-(t, x, y) = M_c^-(v)(t, x - 3ct, y) \). Then \( u_-(t, x, y) \in C(\mathbb{R}; H^8(\mathbb{R}_x \times T_y)) \) and \( u_-(t, x, y) \) is a solution of (1.1). Lemma 5.3 implies that there exist positive numbers \( \varepsilon \) and \( \delta \) such that if \( \|w_0\|_{L^2} < \delta \),

\[
\lim_{t \to \infty} \int \chi_{\varepsilon}(x + 2ct) |M_c^-(v)(t, x, y)|^2 dxdy = 0.
\]

Let us decompose \( v(t, x, y) \) as (4.1) and (4.2). Thanks to the imposed orthogonality conditions, Lemma 3.4 implies

\[
\|\sqrt{\chi_{\varepsilon}}(x + 2ct)(\mathcal{L}_{c,\gamma}(t) w)(t, x, y)\|_{L^2} \geq \nu \|w(t, x, y)\|_{\mathcal{E}_{\varepsilon,2ct}},
\]

where \( \nu \) is a positive constant depending only on \( c \). Using (5.13), the Cauchy-Schwarz inequality and Claim 5.2 with \( p = 4 \), we have

\[
\|\sqrt{\chi_{\varepsilon}}(x + 2ct)M_c^-(v)(t, x, y)\|_{L^2} \geq \frac{\nu}{2} \|w(t, x, y)\|_{\mathcal{E}_{\varepsilon,2ct}}
\]

\[
- C \left( \int_{\mathbb{R}_x \times T_y} w^4(t, x, y) \chi_{\varepsilon}(x + 2ct) \right)^{\frac{1}{2}} \geq \frac{\nu}{2} \|w(t, x, y)\|_{\mathcal{E}_{\varepsilon,2ct}} - C \|w\|_{L^2} \|w\|_{\mathcal{E}_{\varepsilon,2ct}}.
\]

Since \( \sup_{t \in \mathbb{R}} \|w(t, \cdot)\|_{L^2} = O(\delta) \) by Proposition 4.1, coming back to (5.12), we get

\[
\lim_{t \to \infty} \|w(t, \cdot)\|_{\mathcal{E}_{\varepsilon,2ct}} = 0.
\]

This completes the proof of Proposition 5.1. □

6. Proof of Theorem 1.1

Proof of Theorem 1.1 Let \( u_0(x, y) = \varphi_c(x) + \tilde{u}_0(x, y) \) and let \( u(t, x, y) \) be the solution of (1.1) with data \( u_0 \). Fix \( \varepsilon > 0 \). By Proposition 3.1, for every \( \varepsilon_1 > 0 \) there exists \( \delta_1 > 0 \) such that if \( \|\tilde{u}_0\|_{L^2} < \delta_1 \), there exists a unique couple \( (k, w_0) \in \mathbb{R} \times \mathcal{E} \) such that

\[
|k - c| < \varepsilon_1, \quad \|w_0\|_{\mathcal{E}} < \varepsilon_1, \quad M^k_+(Q_k + w_0) = \varphi_c + \tilde{u}_0.
\]

Let \( w_{0,n} \in Y \) (\( n \in \mathbb{N} \)) be such that

\[
\lim_{n \to \infty} \|w_{0,n} - w_0\|_{\mathcal{E}} = 0
\]

and let \( v_n(t) \) be a solution of (2.2) with initial data \( v_n(0) = Q_k + w_{0,n} \). Proposition 4.1, (6.1) and (6.2) imply that for any \( \varepsilon_2 > 0 \), there exist \( \delta_2 > 0 \) and \( n_0 \in \mathbb{N} \) such that if \( \|\tilde{u}_0\|_{L^2} < \delta_2 \) and \( n \geq n_0 \), then there exist continuous functions \( \tilde{\gamma}_n(t) \) such that

\[
\sup_{t \in \mathbb{R}} \sup_{n \geq n_0} \|v_n(t, x, y) - Q_k(x + \tilde{\gamma}_n(t))\|_{\mathcal{E}_{\mathbb{R}_x \times T_y}} < \varepsilon_2.
\]
Let \( u_n(t, x, y) \equiv M_k^+(v_n)(t, x - 3kt, y) \). Then \( u_n(t, x, y) \) is a solution of the KP-II equation \((1.1)\). Thanks to \((6.2)\), the sequence \( u_n(0) = M_k^+(Q_k + w_{0,n}) \) converges, as \( n \to \infty \), in \( L^2(\mathbb{R}_x \times T_y) \) to \( M_k^+(Q_k + w_0) = \varphi_c + \tilde{u}_0 \). Therefore

\[
\lim_{n \to \infty} \| u_n(0, x, y) - \varphi_c(x) - \tilde{u}_0(x, y) \|_{L^2(\mathbb{R}_x \times T_y)} = 0.
\]

(recall that \( u(0, x, y) = \varphi_c(x) + \tilde{u}_0(x, y) \)). Next, using Lemma \(2.1\) we have

\[
\begin{align*}
\| u_n(t, x, y) - \varphi_k(x + \gamma_n(t)) \|_{L^2(\mathbb{R}_x \times T_y)} &= \| M_k^+(v_n) - M_k^+(Q_k(x + \tilde{\gamma}_n(t))) \|_{L^2(\mathbb{R}_x \times T_y)} \\
&\leq C \| v_n(t, x, y) - Q_k(x + \tilde{\gamma}_n(t)) \|_{\mathcal{E}(\mathbb{R}_x \times T_y)} \\
&\quad \times (1 + \| v_n(t, x, y) \|_{\mathcal{E}(\mathbb{R}_x \times T_y)} + \| Q_k \|_{L^\infty(\mathbb{R}_x \times T_y)}),
\end{align*}
\]

where \( \gamma_n(t) = \tilde{\gamma}_n(t) - 3kt \). Coming back to \((6.3)\), we obtain that for any \( \varepsilon_3 > 0 \), there exist \( \delta_3 > 0 \) and \( n_0 \in \mathbb{N} \) such that if \( \| \tilde{u}_0 \|_{L^2} < \delta_3 \),

\[
\| u_n(t, x, y) - \varphi_k(x + \gamma_n(t)) \|_{L^2(\mathbb{R}_x \times T_y)} < \varepsilon_3, \quad \forall \ t \in \mathbb{R}, \quad \forall \ n \geq n_0.
\]

Using the triangle inequality, we obtain

\[
\begin{align*}
\inf_{\gamma \in \mathbb{R}} \| u(t, x, y) - \varphi_c(x + \gamma) \|_{L^2(\mathbb{R}_x \times T_y)} &\leq \| u(t, x, y) - \varphi_c(x + \gamma_n(t)) \|_{L^2(\mathbb{R}_x \times T_y)} \\
&\leq \| u(t, x, y) - u_n(t, \cdot) \|_{L^2(\mathbb{R}_x \times T_y)} + \| u_n(t, x, y) - \varphi_k(x + \gamma_n(t)) \|_{L^2(\mathbb{R}_x \times T_y)} + \| \varphi_k - \varphi_c \|_{L^2(\mathbb{R}_x \times T_y)}.\end{align*}
\]

For any \( t \in \mathbb{R} \), using \((6.4)\) and the \( L^2 \) well-posedness result of \([22]\), we see that there exists \( n_1 \) (depending on \( t \) and \( \varepsilon \)) such that for \( n \geq n_1 \),

\[
\| u(t, \cdot) - u_n(t, \cdot) \|_{L^2(\mathbb{R}_x \times T_y)} < \frac{\varepsilon}{3}.
\]

By \((6.5)\), there exists \( n_2 \) such that for \( \| \tilde{u}_0 \|_{L^2} \ll 1 \) and \( n \geq n_2 \),

\[
\| u_n(t, x, y) - \varphi_k(x + \gamma_n(t)) \|_{L^2(\mathbb{R}_x \times T_y)} < \frac{\varepsilon}{3}.
\]

Finally, using \((6.1)\), we obtain that for \( \| \tilde{u}_0 \|_{L^2} \ll 1 \),

\[
\| \varphi_k - \varphi_c \|_{L^2(\mathbb{R}_x \times T_y)} < \frac{\varepsilon}{3}.
\]

Summarizing, we obtain that for \( \| \tilde{u}_0 \|_{L^2} \ll 1 \),

\[
\inf_{\gamma \in \mathbb{R}} \| u(t, x, y) - \varphi_c(x + \gamma) \|_{L^2(\mathbb{R}_x \times T_y)} < \varepsilon, \quad \forall \ t \in \mathbb{R}.
\]

This completes the proof of the orbital stability.
Next, we will prove asymptotic stability of line solitons. Let \( a \) and \( \delta_4 \) be small positive numbers and \( n_0 \) be a large integer. Then if \( \|w_0\|_\varepsilon \leq \delta_4 \) and \( n \geq n_0 \),

\[
\|u_n(t, x, y) - \varphi_k(x + \gamma_n(t))\|_{L^2((x > ct) \times T_y)} = \|M^k_+(v_n)(t, x, y) - M^k_+(Q_k(x + \tilde{\gamma}_n(t)))\|_{L^2((x > (c-3k)t) \times T_y)} \\
\leq C_1 \|v_n(t, x, y) - Q_k(x + \tilde{\gamma}_n(t))\|_{\varepsilon_{a, (3k-c)t}},
\]

and we obtain

\[
\|v_n(t, x, y) - Q_k(x + \tilde{\gamma}_n(t))\|_{\varepsilon_{a, (3k-c)t}} \leq C_2 \|M^k_+(v_n)(t, x, y)\|_{L^2_{a, (3k-c)t}},
\]

in the same way as \((5.14)\). We remark that \( C \) and \( C_2 \) do not depend on \( n \). On the other hand, we can write

\[
\|M^k_+(v_n)(t, x, y)\|_{L^2_{a, (3k-c)t}}^2 = \int_{\mathbb{R}_x \times T_y} \chi_\varepsilon(x - \frac{ct}{2}) (M^k_+(v_n)(t, x, y))^2 \, dx \, dy
\]

and using Lemma \(5.3\) with \( c_1 = c/2 \) and \( x_0 = -ct/2 \), we obtain that

\[
\|M^k_+(v_n)(t, x, y)\|_{L^2_{a, (3k-c)t}}^2 \\
\leq \int_{\mathbb{R}_x \times T_y} \chi_\varepsilon(x - \frac{ct}{2}) (M^k_+(Q_k(x) + w_{0,n}(x, y)))^2 \, dx \, dy \\
\leq C \int_{\mathbb{R}_x \times T_y} \chi_\varepsilon(x - \frac{ct}{2}) \left\{ (L_k u_0)^2(x, y) + w_{0,n}^4(x, y) \right\} \, dx \, dy + C \|w_0 - w_{0,n}\|_\varepsilon^2,
\]

where \( C \) is a constant that does not depend on \( n \). Therefore using the dominated convergence, we obtain that there exist sequences \( \{\varepsilon_j\}_{j \geq 1} \) and \( \{n_0(j)\}_{j \geq 1} \) such that \( \lim_{j \to \infty} \varepsilon_j = 0 \) and

\[
\sup_{n \geq n_0(j)} \sup_{t \geq j} \|u_n(t, x, y) - \varphi_k(x + \gamma_n(t))\|_{L^2((x > ct) \times T_y)} < \varepsilon_j.
\]

On the other hand, there exists \( n_1(j) \geq n_0(j) \) such that

\[
\sup_{n \geq n_1(j)} \sup_{t \in [j, j+1]} \|u(t) - u_n(t)\|_{L^2(\mathbb{R}_x \times T_y)} < \varepsilon_j
\]

thanks to the well-posedness of \((1.1)\) for data in \( L^2(\mathbb{R}_x \times T_y) \) (see \(22\)). Letting \( \tilde{c} = k \) and \( x(t) = -\gamma_{n_1(j)}(t) \) for \( t \in [j, j+1] \), we have \((1.2)\). In view of Lemma \(5.4\) we have

\[
|\dot{\gamma}_n(t) + 2\tilde{c}| + |\gamma_n(0)| \leq C \|w_{0,n}\|_\varepsilon,
\]

where \( C \) is a constant that does not depend on \( n \) and \( t \). This completes the proof of Theorem \(11\). \( \square \)

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