New Multicritical Random Matrix Ensembles

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February 13, 2022

Abstract

In this paper we construct a class of random matrix ensembles labelled by a real parameter $\alpha \in (0, 1)$, whose eigenvalue density near zero behaves like $|x|^\alpha$. The eigenvalue spacing near zero scales like $1/N^{1/(1+\alpha)}$ and thus these ensembles are representatives of a continuous series of new universality classes. We study these ensembles both in the bulk and on the scale of eigenvalue spacing. In the former case we obtain formulas for the eigenvalue density, while in the latter case we obtain approximate expressions for the scaling functions in the microscopic limit using a very simple approximate method based on the location of zeroes of orthogonal polynomials.

1 Introduction

Random matrix ensembles arise in an overwhelming number of diverse applications. Their utility stems from the fact that even though one can construct...
a multitude of different random matrix models, when one studies properties
on the scale of eigenvalue spacing the results become universal and indepen-
dent of the detailed structure of the random matrix measure. Hence one
can extract predictions even without knowing the details of the appropriate
microscopic model of the phenomenon being studied.

The type of universality class depends in general on the symmetry prop-
erties of the random matrix model and, of course, on the scaling proper-
ties of eigenvalue spacing. Thus one has different bulk and edge unive-
sality regimes, as well as a discrete series of multicritical universality regimes.
Consequently only a discrete series of (rational) scaling exponents could be
obtained.

In this paper we would like to show that there exists a real continuous
range of universality regimes with generic real scaling exponents. We will
perform an explicit construction for ensembles with eigenvalue spacing be-
having like $1/N^{1/(1+\alpha)}$ with $\alpha \in (0, 1)$.

A similar continuous series of universality classes appeared naturally in
[1], for random matrix models with noncompact eigenvalue support and
power-law tails in the eigenvalue density. There the universal behaviour ap-
ppeared for large eigenvalues. In general, however, the appropriate measures
were difficult to obtain in an explicit way. This was the main motivation for
this investigation.

In this paper we would like to perform a construction of a more conven-
tional class of models with compact support and where the universal regime
is located close to $\lambda = 0$.

We strongly suspect that a mapping of this behaviour to infinity would
correspond to the universal scaling for the Lévy ensembles of [1]. Another
possible application of similar random matrix models (in their chiral variant)
might be to model the behaviour of low lying eigenvalues of the QCD Dirac
operator at the chiral phase transition. In this context the exponent $\alpha$ in
$\rho(\lambda) \sim |\lambda|^\alpha$ is the inverse of the $\delta$ critical exponent of the chiral phase
transition [2, 3, 4].

The plan of this paper is as follows. In section 2 we briefly recall the main
features of the old and new universality regimes in random matrix theory. In
section 3 we construct the random matrix measures of the new multicritical
ensembles and compute the eigenvalue density in the bulk of the spectrum. In
section 4 we move to a discussion of the microscopic (presumably universal)
scaling behaviour and propose a simple approximate method based on the
distribution of zeroes of orthogonal polynomials. In section 5 we compare
these approximate formulas with exact expressions calculated numerically for \( \alpha = 1/3 \). We close the paper with a discussion.

## 2 Universality regimes

The universality of a correlation function in a random matrix model means that it does not change (up to a trivial rescaling) when the probability measure is modified. Typically the universal quantities involve properties of the eigenvalues and their correlations considered on the scale of eigenvalue spacing in the large \( N \) limit. For a multicritical point (see below), the modifications of the probability measure are restricted to those that do not destroy multicriticality. However, in general, this still leaves an infinite dimensional space of possible deformations.

The type of universal behaviour in random matrix models depends on the part of the spectrum that one is studying. In general the scaling of eigenvalue spacing with \( N \), in the vicinity of a fixed eigenvalue \( \lambda_0 \), can be easily obtained by looking at the local behaviour of the (bulk) eigenvalue density (normalized to \( N \)):

\[
N \rho(\lambda) \sim N(\lambda - \lambda_0)^\alpha
\]

Consequently the number of eigenvalues between \( \lambda_0 \) and \( \Lambda \) is approximately \( n \sim N(\Lambda - \lambda_0)^{\alpha+1} \). Reexpressing \( \Lambda \) in terms of \( n \) shows that the eigenvalue spacing in the vicinity of \( \lambda_0 \) scales like \( 1/N^{1/(1+\alpha)} \).

In the bulk the eigenvalue spacing is of the order \( 1/N \) — this is the classical Wigner-Dyson regime \[1\]. At the edges of the spectrum the spacing is like \( 1/N^{2/3} \) and one observes universal Airy-like oscillations \[2\] \[4\] \[6\]. When the random matrix possesses an additional chiral structure as in applications to QCD i.e.

\[
\mathcal{M} = \begin{pmatrix} 0 & \mathcal{A} \\ \mathcal{A}^\dagger & 0 \end{pmatrix}
\]

there appears a different universal regime close to \( \lambda = 0 \) (with eigenvalue spacing \( \sim 1/N \)) \[7\].

Apart from these universal regimes, when one finetunes the potential so that \( \rho(\lambda) \sim \lambda^{2m} \), a discrete series of multicritical points appear (with scaling \( 1/N^{1/(2m+1)} \)). The behaviour on the scale of eigenvalue spacing in this regime is very difficult to extract \[8\] \[10\] \[11\]. A different (but also discrete) class of
multicritical models appears when adding an appropriate fixed matrix to the random matrix. This behaviour and novel scaling has been analyzed both for ordinary [12] and chiral [4] random matrix models.

From the above discussion we see that in order to construct a class of random matrix models with arbitrary real scaling exponents we have to find a random matrix ensemble whose bulk eigenvalue spacing behaves like

\[ \rho(\lambda) \sim |\lambda|^\alpha \] (3)

for real \( \alpha \). We will perform the construction for \( \alpha \in (0, 1) \) but an extension to other \( \alpha \) should not be difficult.

3 Eigenvalue density in the bulk

For a hermitian random matrix model with measure

\[ e^{-N \operatorname{tr} V(M)} \] (4)

we consider the following class of potentials

\[ V(M) = -\frac{t}{p} |M|^p + \frac{g}{2p} |M|^{2p} \] (5)

For hermitian matrices we may define \( |M|^p \) in the following way. By a unitary transformation \( M \) can be rewritten in the form \( M = U \Lambda U^\dagger \) with \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \) a diagonal matrix. Then we may define \( |M|^p \equiv U |\Lambda|^p U^\dagger \) where \( |\Lambda|^p = \text{diag}(|\lambda_1|^p, \ldots, |\lambda_N|^p) \). In fact we only need its trace which is given by

\[ \operatorname{tr}|M|^p = \sum_{i=1}^N |\lambda_i|^p \] (6)

We note that although random matrix models with potentials of the type (5) are perfectly well defined, they seem to lack an evident diagrammatic perturbative expansion. Potentials with a single power-like term were first considered in the mathematical literature [13] and in [14]. The parameter \( p \) will be related to \( \alpha \) by

\[ p = 1 + \alpha \] (7)

In the bulk we solve for the eigenvalue density by standard saddle point method. We are looking for a solution with a single cut \((-a, a)\). Later we
will choose \( a = 1 \). The expression for the Green’s function is then

\[
G(z) = \frac{1}{2\pi} \sqrt{z^2 - a^2} \int \frac{d\lambda}{z - \lambda} \frac{V'(\lambda)}{\sqrt{a^2 - \lambda^2}}
\]  

(8)

We note that for \( \alpha > 0 \), the derivative exists. The cut endpoint is fixed through the constraint

\[
\frac{1}{\pi} \int_0^a d\lambda \frac{\lambda V''(\lambda)}{\sqrt{a^2 - \lambda^2}} = 1
\]  

(9)

The eigenvalue density can be extracted from the imaginary part through

\[
\rho(z) = -\frac{1}{2\pi^2} \sqrt{a^2 - z^2} \cdot PV \int_{-a}^a d\lambda \frac{V'(\lambda)}{z - \lambda} \frac{\sqrt{a^2 - \lambda^2}}{z^2 - \lambda^2}
\]  

(10)

Because of the nonanalytic power like structure of the potential (5), it is convenient to get rid of the principal value and rewrite the formula as an ordinary integral. First using the fact that the potential is even we may use

\[
\frac{1}{z - \lambda} - \frac{1}{z + \lambda} = \frac{2\lambda}{z^2 - \lambda^2}
\]  

(11)

to obtain

\[
\rho(z) = -\frac{1}{\pi^2} \sqrt{a^2 - z^2} \cdot PV \int_0^a d\lambda \frac{\lambda V'(\lambda)}{z^2 - \lambda^2 \sqrt{a^2 - \lambda^2}}
\]  

(12)

Now we use the fact that

\[
PV \int_0^a d\lambda \frac{1}{z^2 - \lambda^2} \frac{1}{\sqrt{a^2 - \lambda^2}} = 0
\]  

(13)

Therefore we may rewrite (12) as

\[
\rho(z) = \frac{1}{\pi^2} \sqrt{a^2 - z^2} \int_0^a \frac{z V'(z)}{z^2 - \lambda^2} \frac{1}{\sqrt{a^2 - \lambda^2}} - \frac{\lambda V'(\lambda)}{z^2 - \lambda^2} \frac{1}{\sqrt{a^2 - \lambda^2}}
\]  

(14)

where we could erase the principal value as the integrand is now nonsingular.

We will now fix the endpoint of the cut to \( a = 1 \). Substitution of (1) into (9) yields

\[
-t \frac{\Gamma \left( \frac{1+p}{2} \right)}{\sqrt{\pi p \Gamma \left( \frac{p}{2} \right)}} + g \frac{\Gamma \left( \frac{1+2p}{2} \right)}{\sqrt{\pi 2p \Gamma \left( p \right)}} = 1
\]  

(15)
Since we want to locate a multicritical point we have to require that \( \rho(0) = 0 \). This gives the second equation for \( t \) and \( g \):

\[
\int_0^1 \frac{V'(\lambda)}{\lambda \sqrt{1 - \lambda^2}} d\lambda = 0
\]

(16)

i.e.

\[-t \frac{\Gamma \left( \frac{p-1}{2} \right)}{\Gamma \left( \frac{p}{2} \right)} + g \frac{\Gamma \left( \frac{2p-1}{2} \right)}{\Gamma (p)} = 0 \]

(17)

The solution is

\[
t = \frac{2 \sqrt{\pi} p \Gamma \left( p - \frac{1}{2} \right) \Gamma \left( \frac{p}{2} \right)}{\Gamma \left( p + \frac{1}{2} \right) \Gamma \left( \frac{p-1}{2} \right) - 2 \Gamma \left( p - \frac{1}{2} \right) \Gamma \left( \frac{1+p}{2} \right)}
\]

(18)

\[
g = \frac{2 \sqrt{\pi} p \Gamma \left( \frac{p-1}{2} \right) \Gamma (p)}{\Gamma \left( p + \frac{1}{2} \right) \Gamma \left( \frac{p-1}{2} \right) - 2 \Gamma \left( p - \frac{1}{2} \right) \Gamma \left( \frac{1+p}{2} \right)}
\]

(19)

**Bulk eigenvalue density**

Once the potential is fixed let us compute explicitly the bulk eigenvalue density

\[
\rho(z) = \frac{1}{\pi^2} \sqrt{1 - z^2} \left[ -t \phi_p(z) + g \phi_{2p}(z) \right]
\]

(20)

where

\[
\phi_p(z) \equiv \int_0^1 \frac{z^p - \lambda^p}{z^2 - \lambda^2 \sqrt{1 - \lambda^2}} \frac{1}{\sqrt{1 - \lambda^2}}
\]

(21)

This integral can be explicitly expressed in terms of hypergeometric functions

\[
\phi_p(z) = \frac{\sqrt{\pi} \Gamma \left( \frac{p-1}{2} \right)}{2 \Gamma \left( \frac{p}{2} \right)} \, _2F_1 \left( 1, 1 - \frac{p}{2}, \frac{3-p}{2}; z^2 \right) + \pi \tan \frac{\pi p}{2} \, \frac{z^{p-1}}{\sqrt{1 - z^2}}
\]

(22)

We see that it is the last term which gives the multicritical behaviour that we wanted to obtain \( \rho(z) \sim z^{p-1} = z^\alpha \). Owing to equation (16) the constant terms cancel. We could have chosen of course a different power for the second term in (5), however with this choice the numerical construction of the relevant orthogonal polynomials is made easier.

Let us note what happens when we reach the point \( \alpha = 1 \). Then \( p = 2 \) and the potential obtained here is the standard multicritical quartic one.
$V(x) = -4x^2 + 4x^4$. However due to the vanishing of the coefficient of the last term in (22) the eigenvalue density behaves like $\rho(z) \sim z^2$ and not $z$. When we increase $\alpha$ the expression for the eigenvalue density ceases to be positive and thus the construction fails. Presumably a modification of the power in the second term of the potential might cure the problem but we will not consider that here. To sum up, the expression (20) is nonnegative when $\alpha \in (0, 1)$ as considered in this paper.

4 Behaviour near the origin — orthogonal polynomials

The ultimate interest in constructing the random matrix models with the new scaling properties is to extract universal properties which typically occur in the microscopic regime i.e. on the scale of eigenvalue spacing. It is well known that all the relevant properties are encoded in the orthogonal polynomials

$$\int_{-\infty}^{\infty} dx P_n(x) P_m(x) e^{-N V(x)} = \delta_{nm} \quad (23)$$

It is convenient to introduce the wavefunctions

$$\psi_n(x) = P_n(x) e^{-N V(x)/(2)} \quad (24)$$

Then the kernel which allows for the determination of all correlation functions is given by the expression

$$K(z, w) = \sqrt{R_N} \frac{\psi_N(z) \psi_{N-1}(w) - \psi_{N-1}(z) \psi_N(w)}{z - w} \quad (25)$$

where $R_N$ is the recursion coefficient entering $z \psi_n(z) = \sqrt{R_{n+1}} \psi_{n+1}(z) + \sqrt{R_n} \psi_{n-1}(z)$.

The microscopic limit is defined through a rescaling

$$x = \frac{y}{N^{1/\alpha}} \quad (26)$$

and a limit $N \to \infty$ with $y$ kept fixed. The key quantity that determines the universal properties is the rescaled wavefunction

$$\lim_{N \to \infty} C_N \psi_N \left( \frac{y}{N^{1/(1+\alpha)}} \right) = F(y) \quad (27)$$

7
(as well as a similar expression with $\psi_{N-1}$), $C_N$ is a normalization constant chosen in such a way that the limit exists. In fact the existence of such a limit is a nontrivial and nongeneric property.

In general the determination of the scaling function $F(y)$ is a very difficult problem c.f. [8, 10, 11]. One can formulate a differential equation satisfied by the wavefunction $\psi_n(x)$, but taking the scaling limit is extremely difficult and e.g. for the case of quartic multicritical ensemble involves data from an auxiliary Painlevé equation [8, 10]. Moreover the starting point of such considerations requires a polynomial potential, which is not the case for our class of models. It would be interesting in the future to explore the possibility of applying the Riemann-Hilbert methods of [11].

Here we would like to adopt a different approach and give a very simple but approximate method of constructing the scaling function $F(y)$. In section 5 we will compare the approximate solution with the numerically obtained exact result for $\alpha = 1/3$, and with the mesoscopic approximation of [8].

**Zeroes of the orthogonal polynomials**

The starting point of our construction is the elementary fact that we may reconstruct the orthogonal polynomial from the knowledge of its zeroes. Since in any case we normalize the (even) wave functions through $\psi_{2n}(0) = 1$ the appropriate formula is

$$\psi_{2n}(x) = e^{-N V(x)} \prod_{i=1}^{n} \left(1 - \frac{x^2}{\lambda_i^2}\right)$$

(28)

Interestingly enough there is a theorem due to Ismail [15] which states that for an (almost) arbitrary potential $V(x)$ (see [15]) the zeroes of the $n^{th}$ orthogonal polynomial $\lambda_i$ solve the equations

$$\sum_{1 \leq k \leq n, i \neq k} 1 \frac{1}{\lambda_i - \lambda_k} = \frac{N V'(\lambda_i)}{2} + \frac{1}{2} \left(\log \frac{A_n(\lambda_i)}{\sqrt{R_n}}\right)'$$

(29)

i.e. these are electrostatic equilibrium positions in an external field. The additional assumptions present in [15] serve only to prove the uniqueness of a solution to (29).

\footnote{We always consider even potentials.}
The correction term involves the function $A_n(x)$, given by the formula
\[
\frac{A_n(x)}{\sqrt{R_n}} = N \int_{-\infty}^{\infty} \frac{V'(x) - V'(y)}{x - y} P_n^2(y) e^{-NV(y)} dy \quad (30)
\]
For polynomial potentials of degree $m$, $A_n(x)$ is a polynomial of at most degree $m - 2$. Note that the above equations are exact and valid for any $N$. We see that the last term in (29) is suppressed w.r.t. the ordinary potential term.

Now in order to develop our approximation scheme let us neglect the correction factor $(\log A_n(x))'$ (this is an approximation, the $A_n$ term may indeed give some contributions in the scaling limit). From (29) we see that the zeroes are distributed with a continuum density identical to the eigenvalue density $\rho(\lambda)$ of the corresponding random matrix model. The basis of our approximation scheme is the assumption that locally the zeroes are distributed uniformly w.r.t. $\rho(\lambda)$ (see the examples below).

**Classical Wigner-Dyson scaling functions**

Before we consider the case of immediate interest to us, in order to illustrate the method let us first rederive the formulas for the standard Wigner-Dyson scaling functions. To this end we assume that the eigenvalue density at $\lambda = 0$ is nonvanishing and $N$ is very large. For simplicity we take the potential to be even. It is convenient to normalize the continuum density of zeroes by
\[
\int_{-\infty}^{\infty} \rho_{\text{cont}}(\lambda) d\lambda = N \quad (31)
\]
In our approximation $\rho_{\text{cont}}(\lambda)$ is simply given by $N \cdot \rho(\lambda)$ where $\rho(\lambda)$ is the bulk eigenvalue density of the random matrix model (normalized to 1). If the number of zeroes is even then they are distributed symmetrically around 0 and there is no zero at $\lambda = 0$. The assumption of ‘uniform distribution’ means that the $i-th$ zero is located at $\lambda_i$ where
\[
\int_{0}^{\lambda_i} \rho_{\text{cont}}(\lambda) d\lambda = i + \frac{1}{2} \quad (32)
\]
Since we are interested only in zeroes which lie close to $\lambda = 0$, in the above expression we may substitute $\rho_{\text{cont}}(\lambda) \rightarrow N\rho(0)$. Consequently $\lambda_i = (i + 1/2)/(N\rho(0))$. Therefore the wavefunction is
\[
\psi(x) = e^{-N\frac{V(x)}{2}} \prod_{i=0}^{\infty} \left( 1 - \left( \frac{\rho(0)xN}{i + 1/2} \right)^2 \right) \quad (33)
\]
Here we extended the upper limit of the product to infinity. After introducing the scaling variable \( y = xN \) we obtain immediately

\[
F(y) = \cos(\pi \rho(0)y)
\]  
(34)
the exact scaling function in the bulk. The odd case is similar but then there is a zero at \( \lambda = 0 \) and consequently the relevant equation is

\[
\int_0^{\lambda_i} \rho(\lambda)d\lambda = i
\]  
(35)
From the infinite product representation one obtains then the sine function. The kernel (25) in the scaling limit follows immediately

\[
K(y_1, y_2) \sim \frac{\sin(\pi \rho(0)y_1) \cos(\pi \rho(0)y_2) - \cos(\pi \rho(0)y_1) \sin(\pi \rho(0)y_2)}{y_1 - y_2} = \frac{\sin(\pi \rho(0)(y_1 - y_2))}{y_1 - y_2}
\]  
(36)
The fact that we obtained here the exact scaling functions does not mean, however, that we should expect to get exact results in general.

**Approximate scaling function for \( \alpha \in (0, 1) \)**

The analysis of the scaling function for the new multicritical ensembles is very similar. For definiteness we will just consider the even case. We start from the behaviour of the eigenvalue density near zero:

\[
\rho(\lambda) = c_\alpha |\lambda|^\alpha
\]  
(37)
where \( c_\alpha \) can be easily extracted from (20). The positions of the zeroes according to the ‘uniform distribution’ approximation are determined by

\[
\int_0^{\lambda_i} Nc_\alpha \lambda^\alpha d\lambda = \frac{1}{\alpha + 1} \frac{1}{c_\alpha} N\lambda_i^{\alpha + 1} \equiv \frac{1}{\alpha + 1} c_\alpha y_i^{\alpha + 1} = i + \frac{1}{2}
\]  
(38)
where we introduced the rescaled variable \( y = \lambda N^{1/(1+\alpha)} \). Hence

\[
y_i = \left[ \frac{\frac{1 + \alpha}{c_\alpha} \left( i + \frac{1}{2} \right)}{\frac{1}{1+\alpha}} \right]^{\frac{\alpha}{1+\alpha}}
\]  
(39)
Consequently the scaling function normalized by $F(0) = 1$ is

$$F(y) = \exp \left\{ \frac{t}{1 + \alpha} \frac{y^{1+\alpha}}{2} \right\} \cdot \prod_{i=0}^{\infty} \left( 1 - \frac{y^2}{y_i^2} \right)$$

(40)

In the next section we will compare this approximate result with the exact orthogonal polynomials for $\alpha = 1/3$. However because the convergence properties of the infinite product are not very good we will truncate it at some $n_{\text{max}}$ (for numerical computation we use $n_{\text{max}} = 200$) and approximate the rest of the terms through

$$\prod_{i=n_{\text{max}}+1}^{N} \left( 1 - \frac{y^2}{y_i^2} \right) \sim \exp \left\{ \sum \log \left( 1 - \frac{y^2}{y_i^2} \right) \right\} \sim \exp \left\{ -y^2 \int_{y_{n_{\text{max}}+1}}^{\infty} \frac{\rho(\lambda)}{\lambda^2} \right\}$$

(41)

The result is

$$\exp \left( -\frac{1}{1 - \alpha} \frac{y^2}{y_{n_{\text{max}}+1}^{1-\alpha}} \right)$$

(42)

As we see when taking the $n_{\text{max}} \to \infty$ limit this term becomes equal to unity (because $y_{n_{\text{max}}+1} \to \infty$). However we include it just because of the slow numerical convergence of the infinite product.

5 An example — $\alpha = \frac{1}{3}$

The orthogonal polynomials for any potential can be constructed using the determinant formula:

$$P_n(x) \sim \det \begin{pmatrix} \Delta_0 & \Delta_1 & \Delta_2 & \cdots & \Delta_n \\ \Delta_1 & \Delta_2 & \Delta_3 & \cdots & \Delta_{n+1} \\ \Delta_2 & \Delta_3 & \Delta_4 & \cdots & \Delta_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x^0 & x^1 & x^2 & \cdots & x^n \end{pmatrix}$$

(43)

where $\Delta_i$ are the moments

$$\Delta_i = \int_{-\infty}^{\infty} x^i e^{-NV(x)} dx$$

(44)

The advantage of the specific form of (5) is that the moments $\Delta_i$ can be expressed analytically in terms of the confluent hypergeometric functions $_1F_1$.
We constructed orthogonal polynomials for \( \alpha = 1/3 \) \((p = 4/3)\) for \( N = 40, 120 \) and \( N = 240 \). In Fig. 1 we show the wavefunctions expressed in terms of the scaling variable \( y = xN^{3/4} \)

\[
\psi_N^{sc}(y) \equiv \psi_N \left( y/N^{3/4} \right) = P_N \left( y/N^{3/4} \right) \cdot e^{-\frac{2}{N} V\left( y/N^{3/4} \right)} \quad (45)
\]

for the above values of \( N \), normalized by \( \psi_N(0) = 1 \). We see convergence towards a well defined scaling function.

Figure 1: The wavefunctions \( \psi_N^{sc}(y) \) for \( \alpha = 1/3 \) expressed as a function of the scaling variable \( y \) for \( N = 40, 120 \) and \( N = 240 \) (thick line).

In Fig. 2 we compare the exact wavefunction for \( N = 240 \) with the approximate scaling function \( F(y) \) obtained in the previous section. There is no free parameter in \( F(y) \). The agreement is indeed surprisingly good. Even the small ‘bump’ close to \( y = 0 \) is correctly reproduced. In fact it is difficult to judge from the numerical comparison whether the approximation is exact or not in this case. The small deviations might be caused either by true corrections which go beyond our approximation or by finite size effects. The reason why such a simple approximation scheme works so well certainly deserves further study.

Finally let us say a few words about the comparison with the ‘mesoscopic approximation’ of [8]. An analysis of the differential equations for \( \psi_N(x) \) for the potentials considered here is still lacking, so we will just take a suitable analytical continuation of the mesoscopic approximation from the discrete multicritical points considered in [8] to our case. The result is just

\[
F_{meso}(y) = \cos \left( \frac{\pi}{\alpha + 1} c_\alpha y^{\alpha+1} \right) \quad (46)
\]
We see that the zeroes of this function on the real axis coincide with the zeroes of our approximation, but the analytical structure in the complex plane is certainly different. There are spurious cuts and zeroes coming from \( y^{4/3} \). Nevertheless this approximation is also quite good but the substructure close to \( y = 0 \) is not captured by the mesoscopic approximation (similarly as in the quartic multicritical case considered in [3]). The comparison is presented in Fig. 3.
6 Discussion

In this paper we constructed a continuous class of random matrix ensembles which exhibit a new type of universal behaviour close to the zero eigenvalue. The eigenvalue spacing in this region behaves like $1/N^{1/(\alpha+1)}$ with a real exponent $\alpha \in (0, 1)$. We derived the bulk eigenvalue density and studied approximately the behaviour of the scaling functions in the microscopic limit. Our method for obtaining the scaling function was approximate and based just on the local behaviour of the bulk eigenvalue distribution. Nevertheless the approximation seems to be very good and captures even the fine structure of the scaling functions near the origin.

It would be very interesting to see if these new universality classes could appear in some physical systems. We conjecture that the scaling behaviour can be directly related to the large eigenvalue behaviour of Lévy random matrix models [1].

There are numerous open questions and directions for further study. Firstly it would be very interesting to try to obtain, even implicitly through e.g. a differential equation, the exact scaling function, especially as the vast majority of techniques dealing with multicritical random matrix ensembles requires from the outset a polynomial potential. Perhaps the most promising approach would be the Riemann-Hilbert method. On the technical side a more systematic analysis of the properties of the orthogonal polynomial zeroes starting from (29), in particular the nature and scaling properties of corrections would be very welcome. This is especially interesting in order to better understand the effectiveness and limitations of the approximate method which seems to work so well here. In general, however, this seems to be quite a formidable problem. Secondly other quantities of interest, such as eigenvalue spacing distributions, would be important in view of possible applications. Other possible directions are an extension of the above considerations to chiral random matrix models (in view of possible applications to QCD), and a rigorous treatment of universality properties.

Note added: An independent study of a very similar class of multicritical random matrix model with real exponents was undertaken by Gernot Akemann and Graziano Vernizzi [16].

Acknowledgments. I would like to thank Graziano Vernizzi for interesting discussion on the bulk properties, Poul Damgaard for discussion on [8] and
Maciej A. Nowak for discussions and comments. This work was supported in part by KBN grant 2P03B01917.

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