Extraordinary Invariants are Seeds that Grow
Interacting Theories Out of Free Theories

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Abstract

Extraordinary Invariants are elements of the BRST Cohomology Space which are irrevocably dependent on Zinn sources. The existence of an Extraordinary Invariant means that the symmetry is broken in that sector, and that the field equations can almost rescue the invariance. Adding the Extraordinary Invariant to the action results in a new theory with constraints on the starting theory.

So Extraordinary Invariants are seeds from which a theory can grow. For a simple example, it is shown in this paper how Yang-Mills theory is implicitly contained in the BRST Cohomology of Free Gauge Theory. It comes from an Extraordinary Invariant which can be added to the free gauge action. The Jacobi Identities are generated by requiring that the BRST Poisson Bracket be zero.

Since the mechanism is a general one, it can be used to construct new theories. Some of these, for example in Supersymmetric theories, have not yet been noticed using other methods.

1. How can we construct interesting new Actions in Quantum Field Theory?

Quantum field theory has been very successful, notably for the Standard Model [1]. Over a long period, spontaneously broken Yang-Mills gauge theory was gradually developed, starting from quantum electrodynamics, and both of these were incorporated into the Standard Model [2]. Recently it appears that yet another confirmation of this success is arising from the discovery of the Higgs particle [3]. Supersymmetry is an important development too, but its application to the real world is still very problematic, and possibly non-existent [4]. String and Superstring theory are also very interesting, but again their applicability is far from understood in a satisfactory way [5].

Of course, nobody believes that the Standard Model is the end of the story. The problem is that most efforts to extend the Standard Model result in something that is experimentally wrong. The development of new theories, like supersymmetry and superstring theory, has taken a long time. So it is natural to wonder how one can come up with new theories to test against the experimental results. Is it conceivable that there is

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a simple way to generate new theories that have some good features? Surprisingly, the answer seems to be yes.

The present paper suggests a new, simple, easy and pedestrian approach to searching for new theories, and new variations on theories. One starts with a free action. Then there is a procedure to generate a special set of interacting actions from it. Moreover, the interacting actions will have a number of desirable features.

It is well known that our understanding of many theories has been improved by the methods of BRS and T [6,7,8], and their use by many authors. In particular the application of BRST cohomology has helped to clarify renormalization, physical states, and anomalies in many contexts, both in quantum field theory [9,10], and in string theory [11]. The new approach in this paper to finding theories comes directly from the BRST formalism. The method is to start with a free quantum theory and look for its ‘Extraordinary Invariants’. Briefly, an Extraordinary Invariant is an Invariant in the BRST cohomology which is ‘irrevocably dependent’ on a Zinn source in its expression².

The approach is easily illustrated for the well-known example of Yang-Mills theory. If one looks at the BRST cohomology of free gauge theory, one can see that the interacting Yang-Mills theory is ‘lurking there’. It takes the form of the ‘Extraordinary Invariant’ written below in equation (16). This is merely a curiosity for Yang-Mills theory, but in some other theories, notably chiral scalar supersymmetric theories in 3+1 dimensions, the Extraordinary Invariants seem to take on a more important role, which has yet to be fully worked out [12]. Also, there are many free actions for which nothing is known about the Extraordinary Invariants.

So the present paper is an introduction to the method in a simple and familiar context.

Cohomology calculations have usually been done for interacting theories rather than free theories, so the Extraordinary Invariants of free theories have been ignored up to now. Moreover, Extraordinary Invariants frequently have uncontracted Lorentz indices, so that they are not Lorentz scalars, and so they have usually been ignored for that reason also, since research has been concentrated on analyzing actions rather than finding new ones. However the present results show that the analysis of Extraordinary Invariants is actually very important.

In the case of Yang-Mills theory, we can find an Extraordinary Invariant in the free theory which is also a Lorentz scalar, so Yang-Mills theory is a particularly simple case and it constitutes a nice example of what happens. In more tricky cases, one can usually couple an Extraordinary Invariant to an appropriate field to make a Lorentz scalar. This provides a route to a new action that also involves the new field.

In a rather weak way, the method recalls the old idea of the ‘Bootstrap’ [14], because the theory ‘pulls itself by its own bootstraps’ from a seemingly simple, empty and innocuous, free theory to a significant, physically interesting and interacting theory. In the process, it uses the nice properties of the BRST formalism, which incorporates symmetry, physical states, elimination of redundant or unphysical states, the equations of motion, the possibility of anomalies, the elimination of unitarity ghosts and the elimination of gauge degrees of freedom and, possibly, other things that have not yet been understood. Of course, a ‘Bootstrap’ needs to be non-linear. This requirement is supplied by the very simple (but extremely non-trivial) quadratic non-linearity of the BRST Poisson Bracket. All of this becomes clearer when one takes a simple example, and so that is what we will do.

²This means that the δLittle variation of the pure field part of the Extraordinary Invariant is proportional to the equations of motion of the theory. An example of this is set out in equation (16) of section 2 below.
2. Extraordinary Invariants in Gauge Theory

Every free quantum field theory, with any symmetry at all, can be formulated in such a way that it yields zero for some Grassmann Odd Poisson Bracket, with a corresponding ‘square root’, which is the nilpotent BRST differential $\delta_{\text{BRST}}$. Here we shall not try to be general however. We shall just discuss pure free gauge theory in four dimensions, without any matter, except that we add an index $a$ to the gauge bosons $A^a_\mu$ so that we keep the possibility of having more than one of them.

For free gauge theory, the Poisson Bracket\(^3\) is:

$$\mathcal{P} [A] = \int d^4x \left\{ \frac{\delta A}{\delta \Sigma^{a\mu}} \frac{\delta A}{\delta A^a_\mu} + \frac{\delta A}{\delta K^a} \frac{\delta A}{\delta \omega^a} \right\}$$

(1)

and the action is

$$A_{\text{Free}} = \int d^4x \left\{ -\frac{1}{4} F^{a\mu\nu} F^{a}_{\mu\nu} + \Sigma^{a\mu} \partial_\mu \omega^a \right\}$$

(2)

where

$$F^{a}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$$

(3)

Note that the source $K^a$ does not appear in the action. However, we keep it in the BRST Poisson Bracket for completeness\(^4\). We also define the corresponding BRST operator:

$$\delta_{[A]} = \int d^4x \left\{ \frac{\delta A}{\delta \Sigma^{a\mu}} \frac{\delta}{\delta A^a_\mu} + \frac{\delta A}{\delta K^a} \frac{\delta}{\delta \omega^a} + \frac{\delta A}{\delta A^a_\mu} \frac{\delta}{\delta \Sigma^{a\mu}} + \frac{\delta A}{\delta \omega^a} \frac{\delta}{\delta K^a} \right\}$$

(4)

It is easy to see that if the BRST Poisson Bracket for a given $A$ yields zero, then the corresponding $\delta_{[A]}$ is nilpotent:

$$\mathcal{P} [A] = 0 \Rightarrow (\delta_{[A]})^2 = 0$$

(5)

It is easy to verify that both of these are true for the free action (2):

$$\mathcal{P} [A_{\text{Free}}] = 0 \Rightarrow (\delta_{[A_{\text{Free}}]})^2 = 0$$

(6)

Let us use the simpler notation

$$\delta_{[A_{\text{Free}}]} \equiv \delta_{\text{Free}}$$

(7)

Then we find that

$$\delta_{\text{Free}} = \delta_{\text{Field Equation}} + \delta_{\text{Zinn}} + \delta_{\text{Little}}$$

(8)

where

$$\delta_{\text{Field Equation}} = \int d^4x \left\{ \partial^\nu \left( \partial_\nu A^a_\mu - \partial_\mu A^a_\nu \right) \frac{\delta}{\delta \Sigma^{a\mu}} \right\}$$

(9)

and

$$\delta_{\text{Zinn}} = \int d^4x \left\{ \partial^\mu \Sigma^{a\mu} \frac{\delta}{\delta K^a} \right\}$$

(10)

and

$$\delta_{\text{Little}} = \int d^4x \left\{ \partial_\mu \omega^a \frac{\delta}{\delta A^a_\mu} \right\}$$

(11)

\(^3\)The Zinn sources $\Sigma^{a\mu}$ and $K^a$ have ghost charge -1 and -2 respectively, and the ghost $\omega^a$ has ghost charge 1. We try to use Greek letters for Grassmann odd quantities like $\Sigma^{a\mu}, \omega^a$ and $\delta_{[A]}$, and Latin letters for Grassmann even quantities like $A^a_\mu, K^a$.

\(^4\)We could add another variation, for example the exterior derivative $\xi \partial$, so that $K^a$ would appear in the action in the term $K^a \xi \partial \omega^a$. 

It is simple to calculate the BRST cohomology of such a free theory for low dimensional integrated polynomials in the fields and sources. For the simple cases we are interested in here, that can be done by writing down all the possible Local Integrated Polynomials in the sector of interest.

Invariants are ghost charge zero Local Integrated Polynomials that are in the cohomology space of the theory. There are two kinds of Invariants that one may find:

1. There may be ‘Ordinary Invariants’. These are Invariants that can be written purely in terms of the gauge fields. Here is an example of an Ordinary Invariant. This is a Lorentz tensor, so it is not Lorentz invariant. One could make it Lorentz invariant by contracting the indices with $\delta^\lambda_\rho$:

$$O^\rho_\lambda = t^{abcd} \frac{1}{m^4} \int d^4x \left\{ F_{\mu \nu}^a F^{b \mu \nu} F^{c \nu \rho} F_{\rho \lambda}^d \right\}$$

(12)

2. There may be ‘Extraordinary Invariants’. These are Invariants that depend crucially on the presence of Zinn sources in the Invariant.

(a) Here is an example of an Extraordinary Invariant. It is a Lorentz vector:

$$E^\mu = \int d^4x E^\mu = t^{ab} \int d^4x \left\{ \Sigma^b_\mu \omega^a + A^{a \nu} \left( \partial_\nu A^b_\mu - \partial^\mu A^b_\nu \right) \right\}$$

(13)

It is easy to verify that

$$\delta_{\text{Free}} E^\mu = 0$$

(14)

It is also easy to see that there is no integrated local polynomial $B^\mu$ such that

$$E^\mu = \delta_{\text{Free}} B^\mu$$

(15)

This means that $E^\mu$ is indeed in the non-zero local cohomology space of $\delta_{\text{Free}}$. Also, it is easy to show that there is no possible local $Q^\mu$ such that the expression $E^\prime_\mu = E^\mu + \delta_{\text{Free}} Q^\mu$ is free of the Zinn sources, and so it follows that $E^\mu$ is an Extraordinary Invariant.

(b) Here is an example of another Extraordinary Invariant. It is a Lorentz scalar:

$$E = \int d^4x \left\{ \Sigma^a A^{b \mu} \omega^c - \frac{1}{2} \left( \partial_\mu A^a_\nu - \partial^\nu A^a_\mu \right) A^{b \nu} A^{c \mu} - \frac{1}{2} K^a_\omega \omega^b \omega^c \right\} f^{abc}$$

(16)

Again, in this case, it is easy to verify that

$$\delta_{\text{Free}} E = 0$$

(17)

provided that $f^{abc}$ is a totally antisymmetric tensor. It is also easy to verify that there is no integrated local polynomial $B$ such that

$$E = \delta_{\text{Free}} B$$

(18)

which means that $E$ is indeed in the non-zero local cohomology space of $\delta_{\text{Free}}$. Also, it is easy to verify that there is no possible local $Q$ such that the expression $E' = E + \delta_{\text{Free}} Q$ is free of the Zinn sources, and so it follows that $E$ is an Extraordinary Invariant.

It is common practice to add Ordinary Invariants, provided that they are Lorentz scalars, to the relevant free theory to get another (usually interacting) theory. But what happens if one tries to add an Extraordinary Lorentz Invariant to the free action? We shall see that adding an Extraordinary Lorentz Invariant generates a related but different theory. In some sense, the BRST Cohomology of the free theory here is ‘aware of and prepared for’ the possibility of extending the theory to Yang-Mills theory, and that arises from (16).

\(^5\)Note that (16) is constructed from (13) by adding another vector field. However (16) also needs the extra term $\frac{1}{4} K^a_\omega \omega^b \omega^c$, which uses (10) to compensate the variation, from (11), of the term in (16) which contains $\Sigma^a_\mu$. 

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3. Construction of Yang-Mills theory using the Extraordinary Lorentz Invariant

Let us verify that $\delta_{\text{Free}}$ acting on (16) yields zero. This will also give us some insights.

Integrating by parts yields the following constraints:

$$\delta_{\text{BRST}} E = 0 \Rightarrow (19)$$

$$\frac{1}{2} \int d^4 x f^{abc} (\partial^\nu A^a_{\mu} - \partial^\mu A^a_{\nu}) \omega^b (\partial_\nu A^c_{\mu} - \partial_\mu A^c_{\nu}) = 0 \quad (20)$$

This equation is true if and only if:

$$f^{abc} = -f^{cba} \quad (21)$$

A simple way to satisfy this constraint, is to ensure that $f^{abc}$ is a totally antisymmetric tensor\(^6\). The next thing to notice is that if we add this expression $E$ to the free action, and define:

$$A_1 = A_{\text{Free}} + E \quad (22)$$

then the Poisson Bracket is no longer zero, but it does reduce to:

$$\mathcal{P}[A_1] = \mathcal{P}[E] \quad (23)$$

This is a simple consequence of

$$\mathcal{P}[A_{\text{Free}}] = 0 \quad (24)$$

combined with:

$$\delta_{\text{Free}} E = 0 \quad (25)$$

However in general

$$\mathcal{P}[E] \neq 0 \quad (26)$$

To arrive at a new theory, starting from (22), which yields zero for the BRST Poisson Bracket, we must add something more to get

$$A_{\text{Complete}} = A_{\text{Free}} + E + A_{\text{Completion}} \quad (27)$$

such that the Poisson Bracket yields zero for the completed Action $A_{\text{Complete}}$:

$$\mathcal{P}[A_{\text{Completion}}] = 0 \quad (28)$$

Given the above, this means that the new term $A_{\text{Completion}}$ must satisfy:

$$\mathcal{P}[E] + \delta_{\text{Free}} A_{\text{Completion}} = 0 \quad (29)$$

and also:

$$\mathcal{P}[A_{\text{Completion}}] = 0 \quad (30)$$

To proceed further we must evaluate the expression $\mathcal{P}[E]$ and see whether there is a solution for (29). This is easy:

$$\mathcal{P}[E] = \int d^4 x \left\{ \frac{\delta E}{\delta A^a_{\mu}} \frac{\delta E}{\delta A_{\mu}^b} + \frac{\delta E}{\delta K^a} \frac{\delta E}{\delta \omega^a} \right\}$$

$$= \int d^4 x \left\{ f^{abc} A^b_{\mu} \omega^c \left( f^{ade} \left[ -\partial_\nu (A^d_{\mu} A^{e\nu}) - \partial_\nu A^d_{\mu} A^{e\nu} + \partial_\mu A^d_{\nu} A^{e\nu} \right] + f^{dac} \Sigma^d_{\mu} \omega^e \right) \right\}$$

$$- \int d^4 x \left\{ -\frac{1}{2} f^{abc} \omega^b \omega^c f^{ade} \left( -\Sigma^d_{\mu} A^{e\mu} - K^d \omega^e \right) \right\} \quad (31)$$

There are three things that happen here:

\(^6\)There are also trivial solutions of this with extra U(1) gauge fields.
1. We need to impose additional constraints on the tensor $f^{abc}$ so that this expression in (31) is in the image of $\delta_{\text{Free}}$:

(a) It is clear that the following can never arise from $\delta_{\text{Free}}$:

$$f^{abc} \omega^b \omega^c f^{ade} K^d \omega^e$$

(b) It is also clear that the following can never arise from $\delta_{\text{Free}}$:

$$\int d^4x \left\{ f^{abc} A^{b \mu} \omega^c f^{ade} A^{d \nu} - \frac{1}{2} f^{abc} \omega^b \omega^c f^{ade} \Sigma^d \mu A^{\nu \mu} \right\}$$

(c) So to proceed we must set both of the above to zero. Fortunately, this is a logical thing to do, since (32) and (33) are both satisfied if the Jacobi Identity for the tensor $f^{abc}$ is true.

2. So these constraints mean that the term (31) reduces to the following term:

$$\int d^4x \left\{ f^{abc} A^{b \mu} \omega^c \right\} f^{ade} \left\{ -\partial_\nu (A^d_\nu A^{e \mu}) - \partial_\mu A^d_\nu A^{e \nu} + \partial_\mu A^d_\nu A^{e \nu} \right\}$$

where we use the circular symmetry from the Jacobi identity on the indices (bde) to write:

$$\int d^4x \left\{ f^{abc} f^{ade} A^{b \mu} \omega^c \partial_\nu A^d_\nu A^{e \mu} \right\}$$

Now we need to try to find a form which satisfies (29). Fortunately there is such a form:

$$A_{\text{Completion}} = -\frac{1}{4} \int d^4x \left\{ f^{abc} A^b_\mu A^c_\nu f^{ade} A^{d \mu} A^{e \nu} \right\}$$

and it is easy to verify that (29) is true.

3. Finally we need to confirm that

$$\mathcal{P} [A_{\text{Completion}}] = 0$$

This is simple because $A_{\text{Completion}}$ does not have any Zinn terms in it, and $\mathcal{P} [A]$ must be zero on any expression $A$ without Zinns.

4. Conclusion

The final result for (27) is the usual Yang Mills action with the Zinn terms needed to make it satisfy the BRST Poisson Bracket

$$A_{\text{Complete}} = A_{\text{Yang Mills}} = \int d^4x \left\{ -\frac{1}{4} G^{a \mu \nu} G^a_\mu \nu + \Sigma^a_\mu D^a_\mu \omega^b + \frac{1}{2} K^a f^{abc} \omega^b \omega^c \right\}$$

where

$$G^a_\mu \nu = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu, D^a_\mu \omega^b = \partial_\mu \omega^a + f^{abc} A^b_\mu \omega^c$$

We have ignored the gauge fixing term and the ghosts–this can be done because of the usual methods used to derive the action. The essence of the action is the above.
The result is not unique however. Any Yang-Mills theory, plus any number of U(1) gauge fields, will do as the solution.

If we had added an Ordinary Invariant here, the theory would not have changed in a fundamental way. The BRST Poisson Bracket would still yield zero if one takes an action like

$$A_1 = A_{\text{Free}} + A_{\text{Ordinary Invariant}}$$

(41)

because

$$\mathcal{P}[A_1] = \mathcal{P}[A_{\text{Ordinary Invariant}}] = 0$$

(42)

holds when $A_{\text{Ordinary Invariant}}$ is free of Zinns, and that is the assumption which defines $A_{\text{Ordinary Invariant}}$. So the steps that follow from (26) do not happen for the case of adding a $A_{\text{Ordinary Invariant}}$. Note that the theory (39) has the same Poisson Bracket as the free theory, but the two are not related by a local canonical transformation, as is particularly clear from (27). More detail about the canonical transformation, and what happens when one adds terms like $A_{\text{Ordinary Invariant}}$ can be found in [13].

So why is it useful to find yet another way to derive the Yang-Mills theory? This exercise demonstrates a number of things:

1. The Extraordinary Invariant (16) that gives rise to Yang-Mills theory exists in the free gauge theory.
2. The BRST Poisson Bracket (31) that results from the action (22) is not zero, and it is also not even a cocycle of $\delta_{\text{Free}}$. To get it to be a cocycle of $\delta_{\text{Free}}$, we need to set the expressions (32) and (33) to zero. These are both equivalent to the Jacobi identity.
3. Then we find that the BRST Poisson Bracket reduces to a boundary of $\delta_{\text{Free}}$, whose BRST Poisson Bracket is zero, so that we recover the Yang-Mills theory.
4. At any stage this could have developed a problem. But it does not because Yang-Mills theory exists. What happens when we do not know where we are going?

It would be interesting to see what happens with other theories. Here is a list of possible candidates and some present progress:

1. We will see in [12] how these ideas apply to chiral scalar rigid SUSY. Many of the above steps will occur again, but the theory that emerges is new and there are an infinite number of Extraordinary Invariants in that theory, and the constraints lead us to something rather like the Supersymmetric Standard Model. So this is an example where the above procedure may lead us to something that is both useful and currently unknown.
2. The Super Yang Mills theory in D=3+1 also has plenty of Extraordinary Invariants, and work needs to be done to analyze them.
3. It seems likely that the theory of gravity would work much like the Yang-Mills theory, except that there must be interesting differences.
4. What happens here with the various theories of supergravity? Could they possibly admit new couplings through this mechanism?
5. What happens with free theories of higher spin fields with multiple indices and more than two derivatives in their kinetic terms? These seem to be needed to go beyond the analysis in [12].
This exercise has not generated a new theory in this paper, but the method introduces a new point of view. It is a different way to arrive at the Yang-Mills theory. The Extraordinary Lorentz Invariant (16) is a ‘seed’ from which we can ‘grow’ Yang-Mills theory by adding the ‘seed’ to the action and then completing the action so that it satisfies the BRST Poisson Bracket. The fact that the procedure works so nicely to generate Yang-Mills theory seems to indicate that one should see what happens in other theories, using the same notions. There are many possible candidates for that. Any free quantum field theory with a symmetry will do, and no sophisticated methods are needed to examine the low dimensional local BRST cohomology to look for Extraordinary Invariants.

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References

[1] The experimental testing of the Standard Model has been summarized in the publications of J. Beringer et al. (Particle Data Group), Phys. Rev. D86, 010001 (2012). These are also available on the Internet under ‘Particle Data Group’.

[2] See for example Wikipedia ‘Standard Model’ for a start and other references.

[3] See for example Wikipedia ‘Higgs Boson’ for a start and other references.

[4] A recent introduction is: Howard Baer and Xerxes Tata, ‘Weak Scale Supersymmetry’ Cambridge University Press 2006. There have been many ‘SUSY’ conferences which have concentrated on the application of SUSY to the Standard Model. These are easily found on the internet under ‘SUSY 2013’ conference etc.

[5] There have been many string conferences, and schools, and books, and thousands of articles on arXiv. The Wikipedia article on Superstring theory is currently quite sparse.

[6] The idea of using the cohomology of local fields for Quantum Field Theory was initiated by C. Becchi, A. Rouet, and R. Stora, Commun. Math. Phys. 42 (1975) 127. Useful contributions to gauge theory were made in [7]. The BRST Poisson Bracket was pioneered in [8]. A useful review is [10]. A parallel route has been through the many papers discussing the Batalin and Vilkovisky (‘BV’) methods, where the BRST Poisson Bracket is called the ‘master equation’ and the Zinn sources are called ‘antifields’. There are plenty of books and articles too of course. The Wikipedia articles on BRST and BV are currently in need of work, and these subjects are not easy to summarize.

[7] I. V. Tyutin, Preprint P.N. Lebedev Physical Institute, No 39, (1975); now at arXiv:0812.0580 [hep-th].

[8] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Oxford Science Publications, Reprinted 1990.

[9] For example if one tries to introduce mass for Vector Bosons, without using Yang-Mills and spontaneous breaking, all sorts of troubles appear. This is discussed in many textbooks, including J. C. Taylor: ‘Gauge Theories of Weak Interactions’, Cambridge (1975).
A review of BRST is: Glenn Barnich, Friedemann Brandt, Marc Henneaux, Physics Reports Vol 338, No 5, pp 439-569.

The book ‘Superstring Theory’ by M. Green J. Scharz and E. Witten (Cambridge University Press, 1987) is a nice early introduction, but of course much has happened since it was written.

J.A. Dixon, ‘Does SUSY Know about the Standard Model?’ To be posted on arXiv: [hep-th], April 2013.

Ibid., Nuc Phys B99 (1975) 420

This amusing old idea for the origin of the S-matrix is also closely related to the origin of supersymmetry through strings and duality. The notion of a Bootstrap in particle physics was popularized by G. Chew and others in the 60’s and 70’s.