**Trans-Planckian Censorship and \( k \)-inflation**

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We propose a more general version of the Trans-Planckian Censorship Conjecture (TCC) which can apply to models of inflation with varying speed of sound. We find that inflation models with \( c_s < 1 \) are in general more strongly constrained by censorship of trans-Planckian modes than canonical inflation models, with the upper bound on the tensor/scalar ratio reduced by as much as three orders of magnitude for sound speeds consistent with bounds from data. As a concrete example, we apply the constraint to Dirac-Born-Infeld inflation models motivated by string theory.

**I. INTRODUCTION**

Over the past several years, a series of so-called “swampland” conjectures have been proposed concerning the consistency of effective scalar field models of inflation with a UV completion in a theory of quantum gravity, including the de Sitter Swampland Conjecture and modified versions \([1,3]\). These conjectures have been widely studied, and place strong constraints on models of cosmological inflation \([3,10]\), in particular placing canonical single-field inflation models in tension with current observational bounds \([11,12]\).

More recently proposed is the Trans-Planckian Censorship Conjecture (TCC), which postulates that any consistent theory of quantum gravity must forbid quantum fluctuations with wavelengths shorter than the Planck length from being redshifted by cosmological expansion to wavelengths where they become classical perturbations \([13]\). It is straightforward to show that if inflation continues beyond a minimal number of e-folds, modes with wavelengths on observable astrophysical scales today must have been shorter than the Planck length during inflation. This “trans-Planckian problem” has been well studied in the literature \([14,21]\), with the generic expectation being that such modes are adiabatically rotated into a Bunch-Davies state, and have little if any effect on cosmological observables unless the scale of inflation is very high, and the scale of quantum gravity is very low \([22]\). The TCC by contrast, postulates that such modes are forbidden in any self-consistent UV-complete theory. In this paper, we investigate the consequences of this conjecture in a general class of string-inspired inflation models.

In canonical single-field inflation theories, quantum-to-classical “freezeout” of perturbations happens at approximately the Hubble length. For modes freezing out at the Hubble length, the TCC can then be formulated as

\[
a_f / a_i = e^N \leq H_f^{-1} / P_c = M_P / H_f, \tag{1}
\]

where the subscript \( i \) represents the onset of inflation, and the subscript \( f \) represents the moment when inflation ended in a period of reheating, initiating hot radiation-dominated expansion. The implication of the TCC on canonical-single field inflation was first studied in \([23]\), in which a stringent bound on the potential energy was found

\[
V^{1/4} < 6 \times 10^8 GeV \sim 3 \times 10^{-10} M_P. \tag{2}
\]

This low energy scale leads to upper bounds on the first slow roll parameter, \( \epsilon < 10^{-31} \), and the associated tensor/scalar ratio, \( r = 16 \epsilon < 10^{-30} \). Under the assumption that the TCC holds, these bounds strongly constrain the inflationary model space \([4]\).

**II. GENERALIZED TRANS-PLANCKIAN CENSORSHIP CONJECTURE**

While the bounds \([1,2]\) apply for the case of inflation generated by a canonical scalar field, in the case of a more general scalar Lagrangian, in particular in theories with a time-varying equation of state or speed of sound, the Hubble length is not the length at which quantum fluctuations freeze out and become classical \([29,30]\). In this case, a general quadratic action for the curvature perturbation \( \zeta \) can be written in terms of the dynamical variable \( dy \equiv c_s d \tau \) as

\[
S_2 = {M}_P^2 \int dx^3 dy q^2 \left[ \left( \frac{d \zeta}{dy} \right)^2 - (\nabla \zeta)^2 \right], \tag{3}
\]

where \( \tau \) is the conformal time, \( ds^2 = a^2(\tau) [d\tau^2 - dx^2] \), and \( q \) is a function of the speed of sound \( c_s \) and the slow roll parameter \( \epsilon \equiv -H / H^2 \),

\[
q \equiv \frac{a \sqrt{2 \epsilon}}{\sqrt{c_s}}. \tag{4}
\]

\(^1\) Warm Inflation \([24,26]\) and modified initial states for perturbations \([27,28]\) have been proposed as means to avoid these constraints.
Defining a canonically normalized scalar mode function \( v \equiv M_p q \zeta \), the associated mode equation becomes

\[
v'' + \left( k^2 - \frac{q''}{q} \right) v_k = 0, \tag{5}\]

where a prime denotes differentiation with respect to \( y \). The freezeout radius \( R_\zeta \) for which the quantum modes cease oscillation and become effectively classical is then

\[
R_\zeta^{-2} = \frac{q''}{q}. \tag{6}\]

Approximate scale invariance is obtained for \( R_\zeta \simeq -y/\sqrt{2} \), which is in general dynamically completely independent of the Hubble length, and may be larger or smaller depending on the details of the dynamics. The usual limit of canonical single-field inflation can be obtained by taking \( c_S = 1 \) and \( \epsilon \simeq \text{const.} \ll 1 \), so that

\[
\frac{q''}{q} \simeq \frac{a''}{a} \simeq 2a^2 H^2, \tag{7}\]

so that the freezeout horizon corresponds to the comoving Hubble length \( R_\zeta \simeq (aH)^{-1} \). Mode freezing occurs as long as the freezeout horizon shrinks in comoving coordinates. (This can happen even in the case of non-inflationary expansion, for example in models with \( c_S > 1 \).)

In this paper, we consider the generalization of the TCC to the case of non-canonical inflation models, with time-varying speed of sound \( c_S \), also known generically as \( k \)-inflation \([34]\), for which the freezeout length is given not by the Hubble length, but by the acoustic length,

\[
R_\zeta \simeq \frac{c_S}{aH}. \tag{8}\]

For \( c_S < 1 \) the acoustic horizon can be much smaller than the Hubble length, and sub-Planckian fluctuations can become classical without violating the Hubble TCC condition \([1]\). A schematic diagram of this possible scenario in \( k \)-inflation is shown in Fig. 1.

In order to incorporate models with a non-canonical kinetic term, here we propose the Generalized Trans-Planckian Censorship Conjecture (GTCC) as

\[
N_{\text{tot}} < \ln \frac{c_S(a_e) M_P}{H_e}, \tag{9}\]

where \( N_{\text{tot}} \) is the number of e-folds during inflation, while \( c_S(a_e) \) and \( H_e \) are the values of the speed of sound and Hubble parameter at the end of inflation respectively. To simplify the discussion, we assume that the Hubble parameter is a constant during inflation, \( H \simeq H_{\text{inf}} \), and instantaneous reheating to a radiation-dominated phase, similar to the analysis of Ref. \([35]\). (We consider the more complicated case of power-law \( k \)-inflation in Sec. III.) To derive a modified upper bound on the \( H_{\text{inf}} \) for models with decreasing speed of sound, we consider the cases having \( c_S(a_e) = 1 \), see Fig 2. First, we saturate the condition of solving the horizon problem to get the relation

\[
\ln \frac{H_{\text{inf}}^{-1}}{H_0^{-1}} = 2N_{\text{tot}}, \tag{10}\]

which can be rewritten as

\[
\ln \frac{M_P}{H_0} + \ln \frac{H_{\text{inf}}^{-1}}{M_P} = 2N_{\text{tot}}. \tag{11}\]

Saturating the GTCC \([9]\) together with the condition \( H_e \sim H_{\text{inf}} \) gives us

\[
N_{\text{tot}} = \ln \frac{c_S(a_e) M_P}{H_{\text{inf}}}. \tag{12}\]

Substituting Eq. (12) into Eq. (11), we have

\[
\frac{1}{3} \ln \frac{M_P}{H_0} - \frac{2}{3} \ln c_S(a_e) = \ln \frac{M_P}{H_{\text{inf}}}. \tag{13}\]
Using $H_0 = h \times 2.13 \times 10^{-42} \text{GeV}$ with $h = 0.7$ and $M_P = 2.435 \times 10^{18} \text{GeV}$, we obtain the upper bound of $H_{\text{inf}}$ scale as

$$\frac{H_{\text{inf}}}{M_P} \sim 10^{-20} (c_S(a_e))^2/3,$$

which shows that the bound on the energy scale of inflaton energy scale can be more severe for k-inflation models with small speed of sound in the end of inflation.

Under the same simplified setting, we next consider k-inflation models with increasing speed of sound and $c_S(a_e) = 1$. Although the condition $c_S(a_e) = 1$ makes Eq. (9) reduce back to the original TCC, the maximal value of $H_{\text{inf}}$ is still different from that of canonical inflaton fields due to the fact that this type of model requires more e-folds to solve the horizon problem (Fig. 3). The condition of solving the horizon problem for k-inflation models with increasing speed of sound becomes

$$\ln \frac{H_0^{-1}}{H_{\text{inf}}} - 2 \ln c_S(a_i) < 2N_{\text{tot}}.$$  

(15)

Saturating Eq. (9) and Eq. (15) with $c_S(a_e) = 1$, we have

$$\frac{1}{3} \ln \frac{M_P}{H_0} - \frac{2}{3} \ln c_S(a_i) = \ln \frac{M_P}{H_{\text{inf}}},$$

(16)

which has the same form of Eq. (13) with $c_S(a_e)$ replaced by $c_S(a_i)$. Then, the upper bound of $H_{\text{inf}}$ also scales as

$$\frac{H_{\text{inf}}}{M_P} \sim 10^{-20} (c_S(a_i))^2/3.$$  

(17)

Then, we can consider the general cases that both $c_S(a_e) \neq 1$ and $c_S(a_i) \neq 1$ in k-inflation with $H = H_{\text{inf}}$

by using Eq. (9) and Eq. (15), and we obtain the general form

$$\frac{1}{3} \ln \frac{M_P}{H_0} - \frac{2}{3} \ln [c_S(a_e)c_S(a_i)] = \ln \frac{M_P}{H_{\text{inf}}},$$

(18)

which leads to

$$\frac{H_{\text{inf}}}{M_P} \sim 10^{-20} (c_S(a_e)c_S(a_i))^2/3.$$  

(19)

This is the first main result of this paper.

Next, we use this result, Eq. (19), to derive the upper bound of the first slow roll parameter $\epsilon$ and the tensor/scalar ratio $r$ for k-inflation with $H \sim H_{\text{inf}}$. The power spectrum of curvature perturbations for k-inflation is given by

$$\mathcal{P}_\zeta(k) = \frac{1}{8\pi M_P^2 c_S\epsilon} \frac{H^2}{k_{\text{TCC}}},$$

(20)

and the power spectrum of tensor modes is

$$\mathcal{P}_t(k) = \frac{2}{\pi M_P^2} \frac{H^2}{k_{\text{TCC}}^2}.$$  

(21)

For k-inflation, the tensor/scalar ratio is given by

$$r = \frac{\mathcal{P}_t(k_*)}{\mathcal{P}_\zeta(k_*)} \sim 16c_S^{(1+\epsilon)/(1-\epsilon)} \approx 16c_S\epsilon,$$

(22)

with $c_S$ and $\epsilon$ evaluated at the time when the $k_*$ mode exits the sound horizon, and approximating the Hubble parameter as constant between $k_* = H$ and $k_*c_S = H$. 

FIG. 2. A $k$-inflation model with decreasing speed of sound saturates both bounds from trans-Planckian Censorship Conjecture and horizon problem.

FIG. 3. A schematic diagram of a $k$-inflation model with increasing speed of sound and $c_S(a_e) = 1$. Comparing to the canonical inflation model beginning at $a_i$, the $k$-inflation model needs to start form an earlier time $a'_i$ to solve the horizon problem.
Substituting the upper bound Eq. (19) into Eq. (20) with $P_\zeta(k_*) \sim 10^{-9}$, we have the upper bound for the first Hubble slow roll parameter as

$$\epsilon < 10^{-32} \frac{(c_S(a_e) c_S(a_i))^{4/3}}{c_S(a_*)},$$  

(23)

where $a_*$ is the time when the pivot $k_*$ mode exits the sound horizon during inflation. Also notice that $c_S(a_e)$ or $c_S(a_i)$ might be significantly different from $c_S(a_*)$, depending on the particular model. From this upper bound on $\epsilon$, the upper bound for the tensor/scalar ratio is given as

$$r < 10^{-31} (c_S(a_e) c_S(a_i))^{4/3},$$  

(24)

which is the second main result of this paper. We can give a rough estimate by using the current bound on the speed of sound from CMB non-Gaussianity, $c_S(a_*) > 0.079$ (95% from temperature data) [29], and assuming that both $c_S(a_e), c_S(a_i) \sim 0.08$, which lowers the upper bound on $r$ by a factor of $10^{-3}$. Thus, the bound on the tensor/scalar ratio can be tighter by several orders of magnitude when we consider a general $k$-inflation model with $c_S < 1$.

III. POWER-LAW MODELS

Next, we relax the approximation that $H \approx \text{const.}$ during inflation by considering a special class of models of $k$-inflation with a power-law background expansion. In power-law models, the Hubble parameter during inflation is given by

$$H(N) = H_e e^{\epsilon N},$$  

(25)

where $\epsilon = \text{const.}$ is the first Hubble slow roll parameter. Incorporating the change of Hubble parameter, $H_i = H_e e^{\epsilon N_{tot}}$, the condition for sufficient inflation becomes

$$\ln \frac{H_0}{H_e} - 2 \ln c_S(a_i) < 2(1 - \epsilon) N_{tot}.$$  

(26)

Saturating Eqs. (9) and (26), we obtain

$$(3 - 2\epsilon) \ln \frac{M_P}{H_e} = \ln \frac{M_P}{H_0} - 2(1 - \epsilon) \ln[c_S(a_e)] - 2 \ln[c_S(a_i)],$$  

(27)

which can be rewritten as

$$\frac{H_e}{M_P} = \left(\frac{H_0}{M_P}\right)^{\frac{1 - 2\epsilon}{2 - 2\epsilon}} [c_S(a_e)]^{\frac{1}{2 - 2\epsilon}} [c_S(a_i)]^{\frac{1}{2 - 2\epsilon}}.$$  

(28)

Next, we consider the simplest case of non-trivial speed of sound in the flow formalism, in which the speed of sound is given by the relation

$$c_S = c_S(a_e) e^{-s N},$$  

(29)

where $s = \text{const.}$ is the first flow parameter of speed of sound [40, 41]. Such evolution occurs, for example, in the Dirac-Born-Infeld (DBI) scenario for inflation in string theory [42]. Using Eq. (29), we can write $c_S(a_e)$ as

$$c_S(a_e) = c_S(a_i) e^{s N_{tot}}.$$  

(30)

From Eqs. (9), (26) and (30), we can derive the relation

$$\frac{3 - 2\epsilon - s}{1 - s} \ln \frac{M_P}{H_e} = \ln \frac{M_P}{H_0} - 4 - 2\epsilon - 2s \ln[c_S(a_i)],$$  

(31)

which can be rewritten as

$$\frac{H_e}{M_P} = \left(\frac{H_0}{M_P}\right)^{\frac{1 - 2\epsilon}{2 - 2\epsilon}} [c_S(a_i)]^{\frac{1}{2 - 2\epsilon}} [c_S(e)]^{\frac{1}{2 - 2\epsilon}}.$$  

(32)

To apply the upper bound on $H_e$, Eq. (32), to the DBI-Power-Law inflationary models, we can expand the exponent factors to the first order of $\epsilon, s$, since the bound from the tilt of scalar power spectrum already limits the size of them to be small [13]. The first order approximation of Eq. (32) is given by

$$\frac{H_e}{M_P} = 10^{-20(1 + \frac{1}{2}(e - s))} (0.079)^{\frac{1}{2} + \frac{1}{2}(e - s)},$$  

(33)

Notice that since we have saturated the lower bound of $N_{tot}$ from solving the horizon problem and both $\epsilon, s < 1$, we can use $N_{tot} - N_s > 2.7$ and $N_{tot} > 46.2$, which are calculated from the $H = H_{inf}$ canonical case with the pivot scale $k_* = 0.05 Mpc^{-1}$. Therefore, we have $c_S(a_i) \sim c_S(a_i) e^{-2.7s} \sim c_S(a_*) > 0.079$. Substituting $\frac{H_0}{M_P} \sim 10^{-60}$ and $c_S(a_i) \sim 0.079$, we obtain

$$\frac{H_e}{M_P} = 10^{-20(1 + \frac{1}{2}(e - s))} (0.079)^{\frac{1}{2} + \frac{1}{2}(e - s)},$$  

(34)

which qualitatively shows that especially for the decreasing speed of sound case in the DBI-Power-Law models, i.e. $s < 0$, the constraint on $H_e$ is more severe.

To evaluate the bound on $\epsilon$ and $r$, we first substitute eqs. (29) into Eq. (20) to obtain

$$P_\zeta(k) = \frac{1}{8 \pi M_P^2 c_S(e^2 N)} \bigg|_{k = 3 H},$$  

(35)

and then use the approximation mentioned above: $c_S(a_i) \sim c_S(a_*) > 0.079$ and $N_s \sim N_{tot} - 2.7 \sim 45$ to have

$$\epsilon < 10^{-32}(0.079)^{\frac{3}{2} + \frac{3}{2}(e + s)} \sim 10^{-32}(0.079)^{\frac{3}{2} 10^{0.046}},$$  

(36)

In [43], the value of $2\epsilon + s$ is bound by the measurement of the tilt of scale power spectrum in this special model $0.045 > 2\epsilon + s > 0.0263$, which incorporated with Eq. (30) should become $0.045 > s > 0.0263$. This constraint on the parameter $s$ rules out all the UV models in
the DBI-Power-Law inflation. Substituting the maximal value of $s$ into Eq. (36), the bound on $\epsilon$ is

$$\epsilon < 10^{-32}(0.079)^{7/2}10^{1.2} \sim 10^{-33},$$

(37)

which gives a bound on the tensor/scalar ratio of $r < 10^{-33}$. By substituting $s = 0.045$ into Eq. (34), we can also estimate the corresponding energy

$$\frac{H_e}{M_P} \sim 10^{-21},$$

(38)

which is a free parameter in the original phenomenological model. This result shows that although the DBI-Power-Law models still has region in parameter space satisfying GTCC, the corresponding energy scale at the end of inflation is strongly tightened by Eq. (38); meanwhile, the decreasing speed of sound scenario in this particular model is completely ruled out if we assume the GTCC.

IV. CONCLUSIONS

In this paper we propose a generalization of the Trans-Planckian Censorship Conjecture to include models with variable speed of sound, since in such models mode freezing and the quantum-to-classical transition happens at the acoustic horizon rather than the Hubble length,

$$R_\zeta \simeq \frac{c_S}{aH}.$$  

(39)

The condition that trans-Planckian quantum modes never be redshifted into classical states is then

$$N_{tot} < \ln \frac{c_S(a_0)M_P}{H_e},$$

(40)

which reduces to the usual Censorship Conjecture in the limit that $c_S = 1$. For $c_S < 1$, inflationary evolution is more tightly constrained than for the canonical limit, since the sound horizon can be several orders of magnitude smaller than the Hubble length. The upper bound on the Hubble parameter in the general case is

$$\frac{H_e}{M_P} \sim 10^{-20}[c_S(a_e)c_S(a_i)]^{2/3},$$

(41)

and the corresponding upper bound on the tensor/scalar ratio is

$$r < 10^{-31}[c_S(a_e)c_S(a_i)]^{4/3},$$

(42)

We apply the generalized bound to power-law DBI inflation models, taking into account the time-evolution of the Hubble parameter, and derive an upper bound on $H$ at the end of inflation,

$$\frac{H_e}{M_P} \sim \left( \frac{H_0}{M_P} \right)^{\frac{1-\epsilon}{2-\epsilon}} [c_S(a_i)]^{\frac{4-2\epsilon}{2-\epsilon}}.$$  

(43)

where $\epsilon$ is the first slow roll parameter, and $c_S \propto e^{-sN}$. ACKNOWLEDGMENTS

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