TIME-INCONSISTENT RECURSIVE ZERO-SUM STOCHASTIC
DIFFERENTIAL GAMES

QINGMENG WEI
School of Mathematics and Statistics, Northeast Normal University
Changchun 130024, China

ZHIYONG YU*
School of Mathematics, Shandong University
Jinan 250100, China

Dedicated to Professor Jiongmin Yong’s 60 Birthday

Abstract. In this paper, a kind of time-inconsistent recursive zero-sum stochastic differential game problems are studied by a hierarchical backward sequence of time-consistent subgames. The notion of feedback control-strategy law is adopted to constitute a closed-loop formulation. Instead of the time-inconsistent saddle points, a new concept named equilibrium saddle points is introduced and investigated, which is time-consistent and can be regarded as a local approximate saddle point in a proper sense. Moreover, a couple of equilibrium Hamilton-Jacobi-Bellman-Isaacs equations are obtained to characterize the equilibrium values and construct the equilibrium saddle points.

1. Introduction. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a $d$-dimensional standard Brownian motion $W(\cdot)$ is defined, and $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is its natural filtration (augmented by all the $\mathbb{P}$-null sets). Let $T > 0$ be a given terminal time. For any $t \in [0, T]$ regraded as an initial time, we denote the set of all possible initial states by

$$L^{2}_{\mathcal{F}_t}(\Omega; \mathbb{R}^n) = \left\{ \xi : \Omega \to \mathbb{R}^n \mid \xi \text{ is } \mathcal{F}_t\text{-measurable}, \ E|\xi|^2 < \infty \right\}.$$ 

Moreover, we denote

$$\mathcal{D} = \left\{ (t, \xi) \mid t \in [0, T], \ \xi \in L^{2}_{\mathcal{F}_t}(\Omega; \mathbb{R}^n) \right\},$$

and each element $(t, \xi) \in \mathcal{D}$ is called an initial pair. In our zero-sum game problems, there are two players (named Player $i$, $i = 1, 2$). We denote the set of all admissible

2010 Mathematics Subject Classification. Primary: 49N70; Secondary: 60H10.
Key words and phrases. Time-inconsistency, stochastic differential game, equilibrium saddle point, equilibrium Hamilton-Jacobi-Bellman-Isaacs equation, recursive criterion functional.

This work is supported in part by the National Natural Science Foundation of China (11471192, 11401091, 11571203), the Nature Science Foundation of Shandong Province (JQ201401), the Fundamental Research Funds of Shandong University (2017JC016), and the Fundamental Research Funds for the Central Universities (2412017FZ008).

* Corresponding author: Zhiyong Yu.
control processes for Player \( i \) (\( i = 1, 2 \)) on \([t, T]\) by
\[
\mathcal{U}_i[t, T] = \left\{ u_i : [t, T] \times \Omega \to U_i \mid u_i(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable,} \quad \mathbb{E} \int_t^T |u_i(r)|^2 dr < \infty \right\}
\]
with \( U_i \subseteq \mathbb{R}^{m_i} \) being a nonempty set which could be bounded or unbounded.

For any initial pair \((t, \xi) \in \mathcal{D}\), and any admissible control processes \( u_i(\cdot) \in \mathcal{U}_i[t, T] \) \( (i = 1, 2) \), we consider the following controlled stochastic differential equation (SDE, for short):
\[
\begin{cases}
    dX(r) = b(r, X(r), u_1(r), u_2(r))dr + \sigma(r, X(r), u_1(r), u_2(r))dW(r), & r \in [t, T], \\
    X(t) = \xi,
\end{cases}
\]
(1.1)
where \( b : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}^n \) and \( \sigma : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \to \mathbb{R}^{n \times d} \) are suitable deterministic mappings. Under some mild conditions, (1.1) admits a unique strong solution \( X(\cdot) = X(\cdot; t, \xi, u_1(\cdot), u_2(\cdot)) \), which is called the state process. Additionally, we also introduce a backward stochastic differential equation (BSDE, for short) as follows:
\[
\begin{cases}
    dY^0(r) = g^0(r, X(r), u_1(r), u_2(r), Y^0(r), Z^0(r))dr + Z^0(r)dW(r), & r \in [t, T], \\
    Y^0(T) = h^0(X(T))
\end{cases}
\]
(1.2)
with deterministic mappings \( g^0 : [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \times \mathbb{R} \times \mathbb{R}^{1 \times d} \to \mathbb{R} \) and \( h^0 : \mathbb{R}^n \to \mathbb{R} \). From the theory of BSDEs, (1.2) admits a unique strong solution \( (Y^0(\cdot), Z^0(\cdot)) \equiv (Y^0(\cdot; t, \xi, u_1(\cdot), u_2(\cdot)), Z^0(\cdot; t, \xi, u_1(\cdot), u_2(\cdot))) \) under some mild conditions. With the help of BSDE (1.2), a criterion functional is introduced:
\[
J^0(t, \xi; u_1(\cdot), u_2(\cdot)) = Y^0(t) \equiv Y^0(t; t, \xi, u_1(\cdot), u_2(\cdot)).
\]
(1.3)
In the game, Player 1 wants to maximize the functional \( J^0 \) and Player 2 aims to minimize it. Therefore, the criterion functional \( J^0 \) can be regarded as a payoff for Player 1 and a cost for Player 2.

In a special case where the mapping \( g^0 \) is independent of \((y, z)\), i.e.
\[
g^0(r, x, u_1, u_2, y, z) = g^0(r, x, u_1, u_2),
\]
the BSDE (1.2) is reduced to be a trivial one and the criterion functional reads
\[
J^0(t, \xi; u_1(\cdot), u_2(\cdot)) = \mathbb{E}_t \left[ h^0(X(T)) + \int_t^T g^0(r, X(r), u_1(r), u_2(r)) dr \right],
\]
where \( \mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t] \) for simplicity. This is the classical Bolza type criterion functional. While, in the general case, \( J^0 \) defined through the BSDE (1.3) is called a recursive criterion functional. Using BSDEs to describe recursive functionals was originated from the financial background. For more details about recursive criterion functionals, please refer to Duffie-Epstein [4], Wei-Yong-Yu [15], and the references therein.

In the above, the admissible controls are defined in the open-loop form. For the requirement of numerous practical problems, some kinds of closed forms of admissible controls and strategies are desired. Now we give some definitions.

- Firstly, as the same as the one given in the optimal control theory (see Yong-Zhou [20]), for any \((t, \xi) \in \mathcal{D}\), an admissible feedback control law for Player 1 is a
measurable mapping $u_1 : [t,T] \times \mathbb{R}^n \to U_1$ such that for each $u_2(\cdot) \in \mathcal{U}_2[t,T]$, there exists a unique solution to the following SDE:

$$
\begin{cases}
    d\bar{X}(r) = b(r, \bar{X}(r), u_1(r, \bar{X}(r)), u_2(r))dr \\
    + \sigma(r, \bar{X}(r), u_1(r, \bar{X}(r)), u_2(r))dW(r), \quad r \in [t,T],
\end{cases}
$$

and $u_1(\cdot) = u_1(\cdot, \bar{X}(\cdot)) \in \mathcal{U}_1[t,T]$. In this case, although there is a bit of ambiguity, for simplicity of notations, the corresponding recursive criterion functional is denoted by $J^0(t, \xi; u_1, u_2(\cdot)) \equiv J^0(t, \xi; u_1(\cdot, \bar{X}(\cdot)), u_2(\cdot)) = \bar{Y}^0(t)$, where $(\bar{Y}^0(\cdot), \bar{Z}^0(\cdot))$ is the unique solution to the following BSDE:

$$
\begin{cases}
    d\bar{Y}^0(r) = -g^0(r, \bar{X}(r), u_1(r, \bar{X}(r)), u_2(r), \bar{Y}^0(r), \bar{Z}^0(r))dr \\
    + \bar{Z}^0(r)dW(r), \quad r \in [t,T],
\end{cases}
$$

Similarly, we can define an admissible feedback control law $u_2$ for Player 2.

- Secondly, as the same as Yu [21], for any $(t, \xi) \in \mathcal{D}$, an admissible feedback strategy law for Player 2 is a measurable mapping $\varphi_2 : [t,T] \times \mathbb{R}^n \to U_1 \to U_2$ such that, for each $u_1(\cdot) \in \mathcal{U}_1[t,T]$, there exists a unique solution to

$$
\begin{cases}
    d\bar{X}(r) = b(r, \bar{X}(r), u_1(r), \varphi_2(r, \bar{X}(r), u_1(r)))dr \\
    + \sigma(r, \bar{X}(r), u_1(r), \varphi_2(r, \bar{X}(r), u_1(r)))dW(r), \quad r \in [t,T],
\end{cases}
$$

and $u_2(\cdot) = \varphi_2(\cdot, \bar{X}(\cdot), u_1(\cdot)) \in \mathcal{U}_2[t,T]$. Similarly, in this case, we can define $(\bar{Y}^0(\cdot), \bar{Z}^0(\cdot))$ as the solution to a BSDE (we omit its detailed expression), and denote $J^0(t, \xi; u_1, u_2(\cdot)) \equiv J^0(t, \xi; u_1(\cdot, \bar{X}(\cdot)), u_2(\cdot)) = \bar{Y}^0(t)$.

An admissible feedback strategy law $\varphi_1$ for Player 1 can be analogously defined.

- Thirdly, for any $(t, \xi) \in \mathcal{D}$, $(u_1, \varphi_2)$ is called an admissible feedback control-strategy law if $u_1$ is an admissible feedback control law, $\varphi_2$ is an admissible feedback strategy law, and the following SDE

$$
\begin{cases}
    d\bar{X}(r) = b(r, \bar{X}(r), u_1(r, \bar{X}(r)), \varphi_2(r, \bar{X}(r), u_1(r, \bar{X}(r))))dr \\
    + \sigma(r, \bar{X}(r), u_1(r, \bar{X}(r)), \varphi_2(r, \bar{X}(r), u_1(r, \bar{X}(r))))dW(r), \quad r \in [t,T],
\end{cases}
$$

admits a unique solution, and $u_1(\cdot, \bar{X}(\cdot)) \in \mathcal{U}_1[t,T]$, $\varphi_2(\cdot, \bar{X}(\cdot), u_1(\cdot, \bar{X}(\cdot))) \in \mathcal{U}_2[t,T]$. Once again, we can introduce a corresponding BSDE with its unique solution $(\bar{Y}^0(\cdot), \bar{Z}^0(\cdot))$ and use the following simplified notation:

$J^0(t, \xi; u_1, u_2(\cdot)) \equiv J^0(t, \xi; u_1(\cdot, \bar{X}(\cdot)), u_2(\cdot, \bar{X}(\cdot), u_1(\cdot, \bar{X}(\cdot))) = \bar{Y}^0(t)$.

An admissible feedback control-strategy law $(u_1, u_2)$ can be defined in a similar way.

The notion of strategy is widely used in the game theory to characterize the changing of control of a player when the control of his/her opponent changes. A more general Elliott-Kalton type non-anticipative strategies ([7]) and some related literatures are recalled in the next section. In this paper, we combine the “strategy against control” setting and “feedback” mechanism to constitute a kind of closed-loop form.
Now let us state a family of two-person zero-sum stochastic differential game problems which are parameterized by the initial pairs $(t, \xi) \in \mathcal{D}$, precisely.

**Problem (C-SDG).** (i). For any $(t, \xi) \in \mathcal{D}$, find an admissible feedback control-strategy law $(u_1, \varphi_2)$ such that

$$J^0(t, \xi; u_1, \varphi_2) = \text{essinf}_{u_2(\cdot) \in \mathcal{U}[t, T]} J^0(t, \xi; u_1(\cdot), u_2(\cdot)) = \text{esssup}_{u_1(\cdot) \in \mathcal{U}[t, T]} J^0(t, \xi; u_1(\cdot), \varphi_2).$$

(1.4)

(ii). For any $(t, \xi) \in \mathcal{D}$, find an admissible feedback control-strategy law $(u_1, \varphi_2)$ such that

$$J^0(t, \xi; \varphi_1, u_2) = \text{esssup}_{u_1(\cdot) \in \mathcal{U}[t, T]} J^0(t, \xi; u_1(\cdot), u_2) = \text{essinf}_{u_2(\cdot) \in \mathcal{U}[t, T]} J^0(t, \xi; \varphi_1, u_2(\cdot)).$$

(1.5)

The control-strategy law $(u_1, \varphi_2)$ (resp. $(\varphi_1, u_2)$) satisfying (1.4) (resp. (1.5)) is called a saddle point with form (I) (resp. form (II)).

In the next section, we shall derive that Problem (C-SDG) has an important property: there exists a saddle point $(u_1, \varphi_2)$ with form (I) (resp. $(\varphi_1, u_2)$ with form (II)) on a given time interval $[t, T]$ which is still a saddle point on the small time interval $[s, T]$ for any $s \in (t, T)$. Such a property is called the time-consistency of Problem (C-SDG). For a precise statement and a proof, please see Remark 1 and Theorem 2.4.

Although time-consistency is very good for mathematical treatments, but it is too ideal to be satisfied by most practical problems. Among various reasons leading to the time-inconsistency, a typical one is people’s subjective time preference. As a matter of fact, people usually discount more on the utility for the outcome of immediate future events. Mathematically, such a situation can be described by the so-called non-exponential discounting. As suggested in [15, 16, 17, 18, 19], to incorporate the non-exponential discounting into the game problem, one may consider the following recursive criterion functional:

$$J(t, \xi; u_1(\cdot), u_2(\cdot)) = Y(t) \equiv Y(t; t, \xi, u_1(\cdot), u_2(\cdot)),$$

(1.6)

where $(Y(\cdot), Z(\cdot))$ is the unique solution to the following BSDE:

$$\begin{cases}
dY(r) = -g(t, r, X(r), u_1(r), u_2(r), Y(r), Z(r))dr + Z(r)dW(r), \quad r \in [t, T], \\
Y(T) = h(t, X(T)).
\end{cases}$$

(1.7)

Here, we introduce a notation

$$D[0, T] = \{(t, r) \in [0, T]^2 \mid 0 \leq t \leq r \leq T\},$$

(1.8)

and the deterministic mappings $g$ and $h$ are defined on $D[0, T] \times \mathbb{R}^n \times U_1 \times U_2 \times \mathbb{R} \times \mathbb{R}^{1 \times d}$ and $[0, T] \times \mathbb{R}^n$, respectively. We notice that, comparing with (1.2), the initial time $t$ is introduced into the new BSDE (1.7) as a parameter in order to characterize the time preference in the criterion functional. A typical example of functional $J$ defined by (1.6) is a kind of recursive utility/disutility involving the so-called hyperbolic discounting, in which

$$g(t, r, u_1, u_2, y, z) = \frac{1}{1 + \lambda(r - t)} \left[\tilde{g}(r, u_1, u_2) - \beta y - \gamma z\right],$$

$$h(t, x) = \frac{1}{1 + \lambda(T - t)} \tilde{h}(x),$$
for some mappings \( \tilde{g}(\cdot, \cdot, \cdot) \) and \( \tilde{h}(\cdot) \) and some positive constants \( \lambda, \beta \) and \( \gamma \).

Subsequently, Problem (C-SDG) is replaced by the following

**Problem (InC-SDG).** (i). For any \( (t, \xi) \in \mathcal{D} \), find an admissible feedback control-strategy law \((u_1, \omega_2)\) such that

\[
J(t, \xi; u_1, \omega_2) = \min_{u_2(\cdot) \in \mathcal{W}_2[t,T]} \max_{\omega_1(\cdot) \in \mathcal{W}_1[t,T]} J(t, \xi; u_1(\cdot), \omega_2). \tag{1.9}
\]

(ii). For any \( (t, \xi) \in \mathcal{D} \), find an admissible feedback control-strategy law \((\omega_1, u_2)\) such that

\[
J(t, \xi; \omega_1, u_2) = \max_{u_1(\cdot) \in \mathcal{W}_1[t,T]} \min_{\omega_2(\cdot) \in \mathcal{W}_2[t,T]} J(t, \xi; u_1(\cdot), \omega_2). \tag{1.10}
\]

Similarly, the control-strategy law \((u_1, \omega_2)\) (resp. \((\omega_1, u_2)\)) satisfying \((1.9)\) (resp. \((1.10)\)) is called a saddle point with form (I) (resp. form(II)).

Let us do a simple analysis. If there exists a saddle point satisfying the time-consistent property, then, when we fixed the feedback control laws or the feedback strategy laws, Problem (InC-SDG) is reduced to be a family of recursive stochastic optimal control problems which was studied in Wei-Yong-Yu [15] (see also Yong [17] for a special case), and the time-consistent saddle point is reduced to be a time-consistent optimal control. However, the results in [17, 15] point out the family of optimal control problems are time-inconsistent in general. This contradiction shows that Problem (InC-SDG) is also time-inconsistent.

By now, the time-inconsistent optimal control problems have attracted many researches. Instead of the time-inconsistent optimal controls, the time-consistent equilibrium controls are introduced to deal with time-inconsistent optimal control problems. One major method to investigate such problems is the Stackelberg type multi-person differential games approach, which can be traced back to Pollak [14] in 1968. Later, this approach was further developed by Ekeland-Lazrak [5], Yong [16, 17, 18, 19], Hu-Jin-Zhou [10], Björk-Murgoci [1], Björk-Murgoci-Zhou [2], and so on for various kinds of time-inconsistent optimal control problems.

In this paper, we aim to study time-inconsistent differential game problems (Problem (InC-SDG)). Since the saddle point in the classical sense is no longer time-consistent, inspired by the concept of equilibrium control proposed for the time-inconsistent optimal control problems, we shall suggest a new concept named equilibrium saddle point which is time-consistent and has some properties of local saddle point to characterize Problem (InC-SDG). Meanwhile, we shall develop the multi-person differential games approach (which is for time-inconsistent optimal control problems) to a new one called backward sequence of time-consistent subgames to investigate Problem (InC-SDG).

We explain the new method briefly. Firstly, we divide the whole time interval \([t, T]\) into \(N\) subintervals: \([t_0, t_1]\), \([t_1, t_2]\), \ldots, \([t_{N-1}, t_N]\) with \(t_0 = t\) and \(t_N = T\). For each \(k = 1, 2, \ldots, N\), there is a pair of players (the \(k\)-th pair of players, which can be regarded as the future selves of Player 1 and Player 2) who control the system and play a subgame on \([t_{k-1}, t_k]\). In the subgames sequence, the \(k\)-th pair of players take over the system at time \(t_{k-1}\) from the \((k-1)\)-th pair of players, and hand it over to the \((k+1)\)-th pair of players at \(t_k\). Although the \(k\)-th pair of players will not control the system on \([t_k, T]\), they will still “discount” the future payoffs/costs in their own way, which can be interpreted as the time-preference feature of the problem ([17, 18, 19]). Therefore, the criterion functional for the
The $k$-th pair of players is “sophisticated” and recursive. In detail, the sophisticated recursive criterion functional for the $k$-th pair of players is defined by a BSDE on $[t_{k-1}, t_k]$, whose coefficient/generator depends on its initial pair $(t_{k-1}, X_{k-1}(t_{k-1}))$ with $X_k(t_{k-1})$ equaling to $X_{k-1}(t_{k-1})$ (the terminal state of the $(k - 1)$-th pair of players) and whose terminal value at $t_k$ equals to $\Theta_k(t_k, X_k(t_k))$ with $X_k(t_k)$ being the terminal state of the $k$-th pair of players. The function $\Theta_k(\cdot, \cdot)$ is constructed based on the assumption that later players will play at the saddle point with respect to their sophisticated recursive criterion functionals. There will be a standard time-consistent recursive stochastic differential game (SDG, for short) problem for each pair of players on each subinterval. The verification theorem for time-consistent recursive SDG problems allows us to find a saddle point on each subinterval. These saddle points on all subintervals constitute a partition-dependent equilibrium saddle point for the whole sequence of subgames. At the same time, we also obtain the partition-dependent equilibrium lower and upper value functions for the sequence of subgames.

Next, by letting the mesh size of the partition tend to zero, the partition-dependent equilibrium saddle point will bring us a time-consistent equilibrium saddle point of Problem (InC-SDG), and the partition-dependent equilibrium lower (resp. upper) value function will bring us an equilibrium lower (resp. upper) value function of Problem (InC-SDG) (The definitions of equilibrium lower and upper value functions will be given in Section 3). A couple of so-called equilibrium (lower and upper) Hamilton-Jacobi-Bellman-Isaacs equations (HJBI equations, for short) are obtained to characterize the couple of value functions. We also establish the well-posedness of the equilibrium HJBI equations when the control processes $u_1(\cdot)$ and $u_2(\cdot)$ do not enter the diffusion term $\sigma$ of the state equation (1.1). The general case that $\sigma$ depends on $u_1(\cdot)$ and $u_2(\cdot)$ is still under our investigation.

We summarize the main innovations and difficulties that have been overcome in this paper. (1) As far as we know, it is the first time to study the time-inconsistent SDG problems. (2) As the basis of studying time-inconsistent problems, the theory of the corresponding time-consistent problems plays an important role. The main difficulties in the paper arise when we consider the time-consistent zero-sum SDG problems. To overcome them, we employ a control-strategy law framework and focus on the existence of saddle points instead of values of the games. With the help of some delicate analysis techniques, we establish successfully a verification theorem (see Theorem 2.4) for the time-consistent zero-sum SDG involving recursive/differential utility. (3) A backward sequence of time-consistent subgames approach is introduced to deal with Problem (InC-SDG). We believe this method can also be used to solve some other time-inconsistent SDG problems.

The rest of this paper is organized as follows. We make some preliminaries in Section 2. We first recall the zero-sum SDGs involving Elliott-Kalton type admissible strategies. Then a stochastic verification theorem is established for Problem (C-SDG). In Section 3, for Problem (InC-SDG), a new concept named equilibrium saddle point is proposed. In Section 4, we introduce a backward sequence of time-consistent subgames, and get the local saddle point for each subgame problem. In Section 5, by letting the mesh size of the partition go to zero, we obtain the time-consistent equilibrium saddle points and the equilibrium HJBI equations characterizing the equilibrium value functions. Sections 3, 4 and 5 mainly focus on the equilibrium lower saddle points and the equilibrium lower HJBI equation. A similar analysis could lead to the corresponding results on the equilibrium upper...
saddle points and the equilibrium upper HJBI equation, which are presented in Section 6.

2. Preliminaries. Firstly, we present some notations and assumptions which will be frequently used in the rest of this paper. For any Euclidean space $\mathbb{M}$, we introduce a couple of spaces:

\[
L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{M})) = \left\{ f : \Omega \times [t,T] \to \mathbb{M} \mid f(\cdot) \text{ is an } \mathbb{F}\text{-progressively measurable process with continuous paths and satisfies } \mathbb{E}\left[ \sup_{r \in [t,T]} |f(r)|^2 \right] < \infty \right\},
\]

\[
L^2_{\mathbb{F}}(t,T;\mathbb{M}) = \left\{ f : \Omega \times [t,T] \to \mathbb{M} \mid f(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable and satisfies } \mathbb{E}\int_t^T |f(r)|^2 dr < \infty \right\}.
\]

For the mappings $b$, $\sigma$, $g$, and $h$ appearing in (1.1) and (1.7), we introduce the following assumptions.

\textbf{(A1).} For any $(\tau, r, u_1, u_2) \in D[0,T] \times U_1 \times U_2$, the mappings $b$, $\sigma$, $h$ are Lipschitz continuous with respect to $x$, and $g$ is Lipschitz continuous with respect to $(x, y, z)$. Moreover,

\[
|b(r,0,u_1,u_2)| + |\sigma(r,0,u_1,u_2)| + |g(\tau, r, 0, u_1, u_2, 0, 0)| \leq L(1 + |u_1| + |u_2|),
\]

for any $(\tau, r, u_1, u_2) \in D[0,T] \times U_1 \times U_2,$

where $L > 0$ is a constant.

Let $S^n \subseteq \mathbb{R}^{n \times n}$ be the collection of all $(n \times n)$ symmetric matrices. Let

\[
\begin{align*}
H(t, r, \theta, p, P) &= \text{tr} \left[ a(r, x, u_1, u_2) P + b(r, x, u_1, u_2) \right] + g(\tau, r, 0, u_1, u_2, 0, 0)
\end{align*}
\]

where the superscript $\top$ is the transpose of matrices. Similar to [17, 15], we give the following assumptions on the function $H$.

\textbf{(A2).} (i) There exist a pair of mappings $\psi_2 : D[0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times S^n \to U_2$ and $\varphi_1 : D[0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times S^n \to U_1$ with needed regularity such that

\[
\psi_2(\tau, r, x, u_1, \theta, p, P) = \arg\min_{u_2 \in U_2} H(\tau, r, x, u_1, u_2, \theta, p, P),
\]

\[
\varphi_1(\tau, r, x, \theta, p, P) = \arg\max_{u_1 \in U_1} H(\tau, r, x, u_1, \theta, p, P)
\]
\[ \begin{align*}
&= \max_{u_1 \in U_1} \mathbb{H}(\tau, r, x, u_1, \psi_2(\tau, r, x, u_1, \theta, p, P), \theta, p, P) \\
&= \max_{u_1 \in U_1} \left( \min_{u_2 \in U_2} \mathbb{H}(\tau, r, x, u_1, \theta, p, P) \right). \end{align*} \]

(ii) There exist a pair of mappings \( \psi_1 : D[0, T] \times \mathbb{R}^n \times U_2 \to \mathbb{R} \) and \( \psi_2 : D[0, T] \times \mathbb{R}^n \times U_2 \to U_1 \) with needed regularity such that

\[ \begin{align*}
\psi_1(\tau, r, x, u_2, \theta, p, P) &= \arg\max_{\tau, r, x, \psi_1(\tau, r, x, \cdot, \psi_2, \theta, p, P), \cdot, \theta, p, P} \mathbb{H}(\tau, r, x, u_2, \theta, p, P), \\
\psi_2(\tau, r, x, \theta, p, P) &= \arg\max_{\tau, r, x, \psi_1(\tau, r, x, \cdot, \theta, p, P), \cdot, \theta, p, P} \mathbb{H}(\tau, r, x, \psi_1, \theta, p, P),
\end{align*} \]

and

\[ \psi_1(\tau, r, x, u_2, \theta, p, P), \psi_2(\tau, r, x, \theta, p, P) \in \arg\max \mathbb{H}(\tau, r, x, u_2, \theta, p, P), \psi_1(\tau, r, x, \cdot, \theta, p, P) \]

Now we give some preliminary results on Problem (C-SDG) as a basis of studying Problem (InC-SDG). Since Problem (C-SDG) is time-consistent, then the functions \( g, h, \) and \( \mathbb{H} \) are independent of the first time variable \( \tau \) in the rest of this section.

For convenience, we omit the superscript 0 in (1.2), (1.3), (1.4), and (1.5) to keep the same forms with our main Problem (InC-SDG). According to the classical theory of SDEs and BSDEs, under Assumption (A1), for any \((t, \xi) \in \mathcal{D}\) and any \((u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1[t, T] \times \mathcal{U}_2[t, T]\), SDE (1.1) admits a unique solution \( X(\cdot) \equiv X(\cdot; t, \xi, u_1(\cdot), u_2(\cdot)) \in L^2(\Omega; C([t, T]; \mathbb{R}^n)) \) and BSDE (1.2) admits a unique solution \( (Y(\cdot), Z(\cdot)) \equiv (Y(\cdot; t, \xi, u_1(\cdot), u_2(\cdot)), Z(\cdot; t, \xi, u_1(\cdot), u_2(\cdot))) \in L^2(\Omega; C([t, T]; \mathbb{R})) \times L^2(t, T; \mathbb{R}^{1 \times d}) \), respectively. We notice that Assumption (A1) can be substantially relaxed such that the existence and uniqueness of solutions to SDE (1.1) and BSDE (1.2) still hold.

Besides the saddle points defined in the statement of Problem (C-SDG), we also recall some other definitions in game theory including admissible strategies, values, and value functions.

**Definition 2.1.** An admissible strategy (also called an Elliott-Kalton type non-anticipative strategy [7]) for Player 1 is a mapping \( \alpha_1 : \mathcal{U}_2[t, T] \to \mathcal{U}_1[t, T] \) such that for any \( \mathcal{F}\)-stopping time \( \rho : \Omega \to [t, T] \) and any \( u_2(\cdot), u_2'(\cdot) \in \mathcal{U}_2[t, T] \) with \( u_2(\cdot) \equiv u_2'(\cdot) \) on \([t, \rho]\), it holds that \( \alpha_1[u_2(\cdot)] \equiv \alpha_1[u_2'(\cdot)] \) on \([t, \rho]\). An admissible strategy \( \alpha_2 : \mathcal{U}_1[t, T] \to \mathcal{U}_2[t, T] \) for Player 2 is defined in the same way. The set of all admissible strategies for Player \( i \) is denoted by \( \mathcal{A}_i[t, T] \) \((i = 1, 2)\).

In the Elliott-Kalton “strategy against control” setting, when the maximizing Player 1 chooses an admissible control \( u_1(\cdot) \), the minimizing Player 2 will choose an admissible strategy \( \alpha_2(\cdot) \). Asymmetrically, when the minimizing Player 2 chooses an admissible control \( u_2(\cdot) \), the maximizing Player 1 will choose an admissible strategy \( \alpha_1(\cdot) \). Additionally, if \( \alpha_i \) is selected to be an admissible feedback strategy law for Player \( j \), it is easy to check that \( u_i(\cdot) \to \alpha_j(\cdot, X(\cdot), u_i(\cdot)) \) is an admissible strategy for Player \( j \) \((j = 1, 2)\).

**Definition 2.2.** For any given \((t, \xi) \in \mathcal{D}\), if the \( \mathcal{F}_t \)-measurable random variable

\[ \inf_{\alpha_2(\cdot) \in \mathcal{A}_2[t, T]} \sup_{u_2(\cdot) \in \mathcal{U}_2[t, T]} J(t, \xi; u_1(\cdot), \alpha_2[u_1(\cdot)]) \]

exists, then it is called the lower value of Problem (C-SDG) with the initial pair \((t, \xi)\). If the \( \mathcal{F}_t \)-measurable random variable

\[ \sup_{\alpha_1(\cdot) \in \mathcal{A}_1[t, T]} \inf_{u_1(\cdot) \in \mathcal{U}_1[t, T]} J(t, \xi; \alpha_1[u_1(\cdot)], u_2(\cdot)) \]

exists, then it is called the upper value of Problem (C-SDG) with the initial pair \((t, \xi)\).
exists, then it is called the upper value of Problem (C-SDG) with \((t, \xi)\). If both the lower value and the upper value exist and equal, we call this common value the value of Problem (C-SDG) with \((t, \xi)\). Moreover, if there exists a function \(V^\pm : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}\) such that

\[
V^-(t, \xi) = \text{essinf}_{\alpha_1 \in \mathcal{U}_1[t, T]} \text{esssup}_{u_1(\cdot) \in \mathcal{G}_1[t, T]} J(t, \xi; u_1(\cdot), \alpha_2[1(\cdot)])/ \mathbb{P}-\text{a.s., } \forall (t, \xi) \in \mathcal{D} \tag{2.2}
\]

(resp. \(V^+(t, \xi) = \text{esssup}_{\alpha_1 \in \mathcal{U}_1[t, T]} \text{essinf}_{u_2(\cdot) \in \mathcal{G}_2[t, T]} J(t, \xi; \alpha_1[1(\cdot)], u_2(\cdot))/ \mathbb{P}-\text{a.s., } \forall (t, \xi) \in \mathcal{D} \).

Then \(V^-\) (resp. \(V^+\)) is called the lower (resp. upper) value function of Problem (C-SDG).

**Proposition 2.3.** (i). For any given \((t, \xi) \in \mathcal{D}\), if there exists a saddle point with form (I), i.e. there exists an admissible feedback control-strategy law \((u_1, \omega_2)\) satisfying (1.4), then the corresponding lower value exists. Moreover,

\[
\text{essinf}_{\alpha_2[1] \in \mathcal{U}_2[t, T]} \text{esssup}_{u_1(\cdot) \in \mathcal{G}_1[t, T]} J(t, \xi; u_1(\cdot), \alpha_2[1(\cdot)]) = J(t, \xi; u_1, \omega_2). \tag{2.4}
\]

Due to this, a saddle point with form (I) is also called a lower saddle point.

(ii). For any given \((t, \xi) \in \mathcal{D}\), if there exists a saddle point with form (II), i.e. there exists an admissible feedback control-strategy law \((\alpha_1, u_2)\) satisfying (1.5), then the corresponding upper value exists. Moreover,

\[
\text{esssup}_{\alpha_1[1] \in \mathcal{U}_1[t, T]} \text{essinf}_{u_2(\cdot) \in \mathcal{G}_2[t, T]} J(t, \xi; \alpha_1[1(\cdot)], u_2(\cdot)) = J(t, \xi; \alpha_1, u_2). \tag{2.5}
\]

Similarly, a saddle point with form (II) is also called an upper saddle point.

**Proof.** We only prove the conclusion (i), and the proof of (ii) is the same.

On the one hand, by the first equation in (1.4),

\[
J(t, \xi; u_1, \omega_2) \leq J(t, \xi; u_1, u_2(\cdot)) \leq \sup_{u_2(\cdot) \in \mathcal{G}_2[t, T]} J(t, \xi; u_1(\cdot), u_2(\cdot)), \forall u_2(\cdot) \in \mathcal{G}_2[t, T].
\]

Therefore,

\[
J(t, \xi; u_1, \omega_2) \leq \sup_{u_2(\cdot) \in \mathcal{G}_2[t, T]} \inf_{u_1(\cdot) \in \mathcal{G}_1[t, T]} J(t, \xi; u_1(\cdot), \alpha_2[1(\cdot)], \forall \alpha_2[1] \in \mathcal{U}_2[t, T].
\]

Then,

\[
J(t, \xi; u_1, \omega_2) \leq \inf_{\alpha_2[1] \in \mathcal{U}_2[t, T]} \sup_{u_1(\cdot) \in \mathcal{G}_1[t, T]} J(t, \xi; u_1(\cdot), \alpha_2[1(\cdot)]). \tag{2.6}
\]

On the other hand, by the second equation in (1.4), and noticing that \(u_1(\cdot) \mapsto \omega_2(\cdot, \bar{X}(\cdot), \alpha_2[1(\cdot)])\) is a special admissible strategy for Player 2, we have

\[
J(t, \xi; u_1, \omega_2) = \sup_{u_1(\cdot) \in \mathcal{G}_1[t, T]} J(t, \xi; u_1(\cdot), \omega_2) \geq \inf_{\alpha_2[1] \in \mathcal{U}_2[t, T]} \sup_{u_1(\cdot) \in \mathcal{G}_1[t, T]} J(t, \xi; u_1(\cdot), \alpha_2[1(\cdot)]). \tag{2.7}
\]

We finish the proof. \(\square\)

The time-consistent differential games have been extensively researched. Among the rich literatures, we would like to mention the following results which are more related to our present work. In 1989, Fleming-Souganidis [9] adopted the Elliott-Kalton type “strategy against control” setting to study zero-sum SDGs for the first time. They proved the celebrated dynamic programming principle holds true and the lower value and upper value functions are the unique viscosity solutions to the
associated HJBI equations. Their work generalized that of Evans-Souganidis [8] from the deterministic framework to the stochastic one. Later, Buckdahn-Li [3] further generalized the work in [9] to the zero-sum SDGs with recursive criterion functionals. The researches in the above mentioned works focused on the existence of lower and upper values. Clearly, in the view of Proposition 2.3, when a lower (resp. upper) saddle point exists, the lower (resp. upper) value must exist. On the other hand, in general, one should not expect the existence of the lower (resp. upper) value implies the existence of a lower (resp. upper) saddle point. Recently, for the time-consistent zero-sum SDGs in the linear-quadratic case, Yu [21] constructed explicitly a lower (resp. upper) saddle point in the control-strategy law form by virtue of an associated Riccati equation. In the rest of this section, we will focus on the existence and presentation of saddle points of Problem (C-SDG) which is much more general comparing with the model studied in [21].

Now, we introduce a couple of partial differential equations (PDEs, for short) named HJBI equations as follows:

\[
\begin{align*}
V^{-}_r(r,x) + \sup_{u_1 \in U_1} \inf_{u_2 \in U_2} & \mathbb{H}(r,x,u_1,u_2,V^{-}(r,x),V^{-}_x(r,x),V^{-}_{xx}(r,x)) = 0, \\
V^{-}(T,x) = h(x), & \quad x \in \mathbb{R}^n, \\
\end{align*}
\]

and

\[
\begin{align*}
V^{+}_r(r,x) + \inf_{u_2 \in U_2} \sup_{u_1 \in U_1} & \mathbb{H}(r,x,u_1,u_2,V^{+}(r,x),V^{+}_x(r,x),V^{+}_{xx}(r,x)) = 0, \\
V^{+}(T,x) = h(x), & \quad x \in \mathbb{R}^n.
\end{align*}
\]

Let

\[
C^{1,2}([0,T] \times \mathbb{R}^n) = \left\{ v : [0,T] \times \mathbb{R}^n \to \mathbb{R} \mid v(\cdot, \cdot) \text{ is continuous}, \right. \]

\[
\left. v_r(\cdot, \cdot), v_x(\cdot, \cdot), v_{xx}(\cdot, \cdot) \text{ exist and are also continuous} \right\}.
\]

The following theorem establishes a relationship between Problem (C-SDG) and the above HJBI equations.

**Theorem 2.4 (Verification Theorem).** (i) Let Assumptions (A1) and (A2)-(i) hold true. Suppose HJBI equation (2.8) admits a classical solution \( V^{-}(\cdot, \cdot) \in C^{1,2}([0,T] \times \mathbb{R}^n) \). For any \((t, \xi) \in \mathcal{D}\), let \( u_1 : [t,T] \times \mathbb{R}^n \to U_1 \) and \( u_2 : [t,T] \times \mathbb{R}^n \times U_1 \to U_2 \) be given by

\[
\begin{align*}
\{ u_1(r,x) = \varphi_1(r,x,V^{-}(r,x),V^{-}_x(r,x),V^{-}_{xx}(r,x)) & , \quad (r,x) \in [t,T] \times \mathbb{R}^n, \\
\{ u_2(r,x,u_1) = \varphi_2(r,x,u_1,V^{-}(r,x),V^{-}_x(r,x),V^{-}_{xx}(r,x)) & , \quad (r,x,u_1) \in [t,T] \times \mathbb{R}^n \times U_1.
\end{align*}
\]

Then, \((u_1, u_2)\) is a lower saddle point to Problem (C-SDG) with the initial pair \((t, \xi)\). Moreover,

\[
V^{-}(t, \xi) = J(t, \xi; u_1, u_2), \quad \mathbb{P}\text{-a.s.}, \tag{2.11}
\]

i.e. \( V^{-}(\cdot, \cdot) \) is the lower value function of Problem (C-SDG).

(ii) Let Assumptions (A1) and (A2)-(ii) hold true. Suppose HJBI equation (2.9) admits a classical solution \( V^{+}(\cdot, \cdot) \in C^{1,2}([0,T] \times \mathbb{R}^n) \). For any \((t, \xi) \in \mathcal{D}\), let
\( u_2 : [t, T] \times \mathbb{R}^n \rightarrow U_2 \) and \( \alpha_1 : [t, T] \times \mathbb{R}^n \times U_2 \rightarrow U_1 \) be given by

\[
\begin{cases}
  u_2(r, x) = \varphi_2(r, x, V^+(r, x), V^+_x(r, x), V^+_{xx}(r, x)), \quad (r, x) \in [t, T] \times \mathbb{R}^n, \\
  \alpha_1(r, x, u_2) = \psi_1(r, x, u_2, V^+(r, x), V^+_x(r, x), V^+_{xx}(r, x)), \quad (r, x, u_2) \in [t, T] \times \mathbb{R}^n \times U_2.
\end{cases}
\]  

(2.12)

Then, \((\alpha_1, u_2)\) is an upper saddle point to Problem \((C-SDG)\) with the initial pair \((t, \xi)\). Moreover,

\[ V^+(t, \xi) = J(t, \xi; \alpha_1, u_2), \quad \mathbb{P}\text{-a.s.,} \]  

(2.13)
i.e. \( V^+(\cdot, \cdot) \) is the upper value function of Problem \((C-SDG)\).

**Proof.** Since the proofs of (i) and (ii) are similar, then we only prove (i).

**Step 1.** For any \((t, \xi) \in \mathcal{D}\), fix the feedback strategy law \(\alpha_2\) defined by (2.10). For any \(u_1(\cdot) \in \mathcal{U}[t, T]\), let \((X(\cdot), Y(\cdot), Z(\cdot))\) be the solution to the following decoupled forward-backward stochastic differential equation (FBSDE, for short):

\[
\begin{align*}
  d\hat{X}(r) &= b(r, \hat{X}(r), u_1(r), \omega_2(r, \hat{X}(r), u_1(r)))dr \\
  &\quad + \sigma(r, \hat{X}(r), u_1(r), \omega_2(r, \hat{X}(r), u_1(r)))dW(r), \quad r \in [t, T], \\
  d\hat{Y}(r) &= -g(r, \hat{X}(r), u_1(r), \omega_2(r, \hat{X}(r), u_1(r)), \hat{Y}(r), \hat{Z}(r))dr \\
  &\quad + \hat{Z}(r)dW(r), \quad r \in [t, T],
  \\
  \hat{X}(t) &= \xi, \quad \hat{Y}(T) = h(\hat{X}(T)).
\end{align*}
\]  

(2.14)

Here the forward equation (with the initial condition) is just the game system (1.1) under the control \(u_1(\cdot)\) and the feedback strategy law \(\omega_2\), and the backward equation (with the terminal condition) is just the corresponding BSDE (1.2) which is used to define the recursive criterion functional. In this paper, many times, we would like to write them in a compact form like (2.14). By applying Itô’s formula to \(V^{-}(\cdot, \hat{X}(\cdot))\), we have

\[
V^{-}(s, \hat{X}(s)) = h(\hat{X}(T))
\]

\[
- \int_s^T \left\{ V^-_x(r, \hat{X}(r)) + \text{tr} \left[ a(r, \hat{X}(r), u_1(r), \omega_2(r, \hat{X}(r), u_1(r)))V^-_{xx}(r, \hat{X}(r)) \right] \right. \\
\left. + \left\langle b(r, \hat{X}(r), u_1(r), \omega_2(r, \hat{X}(r), u_1(r))), V^-_x(r, \hat{X}(r)) \right\rangle \right\} dr
\]

\[
- \int_s^T V^-_x(r, \hat{X}(r)) \sigma(r, \hat{X}(r), u_1(r), \omega_2(r, \hat{X}(r), u_1(r)))dW(r), \quad s \in [t, T].
\]

By the definition of function \(H\) in (2.1), the above equation can be rewritten as

\[
V^{-}(s, \hat{X}(s)) = h(\hat{X}(T))
\]

\[
+ \int_s^T \left\{ g(r, \hat{X}(r), u_1(r), \omega_2(r, \hat{X}(r), u_1(r)), V^{-}(r, \hat{X}(r)), \\
V^-_x(r, \hat{X}(r)) \sigma(r, \hat{X}(r), u_1(r), \omega_2(r, \hat{X}(r), u_1(r))) \right\} dr
\]

\[
- V^-_x(r, \hat{X}(r)) - H\left(r, \hat{X}(r), u_1(r), \omega_2(r, \hat{X}(r), u_1(r))\right), \quad s \in [t, T].
\]  

(2.15)
\[ V^-(r, \bar{X}(r)), V_x^-(r, \bar{X}(r)), V_{xx}^-(r, \bar{X}(r)) \} \, dr \]

\[ - \int_s^T V_x^-(r, \bar{X}(r))^\top \sigma(r, \bar{X}(r), u_1(r), \omega_2(r, \bar{X}(r), u_1(r))) \, dW(r), \quad s \in [t, T]. \]

From the definition of \( \omega_2 \) in (2.10) and the equation (2.8), we have, for any \( r \in [t, T] \),

\[ - V_r^-(r, \bar{X}(r)) - \mathbb{H}(r, \bar{X}(r), u_1(r), \omega_2(r, \bar{X}(r), u_1(r)), V^-_r(r, \bar{X}(r)), V^-_x(r, \bar{X}(r)), V^-_{xx}(r, \bar{X}(r)) \]

\[ = - V_r^-(r, \bar{X}(r)) \]

\[ - \inf_{u_2 \in U_2} \mathbb{H}(r, \bar{X}(r), u_1(r), u_2, V^-_r(r, \bar{X}(r)), V^-_x(r, \bar{X}(r)), V^-_{xx}(r, \bar{X}(r)) \]

\[ \geq - V_r^-(r, \bar{X}(r)) \]

\[ - \sup_{u_1 \in U_1} \inf_{u_2 \in U_2} \mathbb{H}(r, \bar{X}(r), u_1, u_2, V^-_r(r, \bar{X}(r)), V^-_x(r, \bar{X}(r)), V^-_{xx}(r, \bar{X}(r)) \]

\[ = 0. \]  

(2.16)

From the comparison theorem for BSDEs (see El Karoui-Peng-Quenez [6]), we have

\[ V^-(s, \bar{X}(s)) \geq \bar{Y}(s), \quad s \in [t, T]. \]

Especially, taking \( s = t \),

\[ V^-(t, \xi) \geq \bar{Y}(t) = J(t, \xi; u_1(\cdot), \omega_2), \quad \text{for any } u_1(\cdot) \in \mathcal{U}_1[t, T]. \]  

(2.17)

Next, we further fix \( u_1(\cdot) \) by (2.10), and let \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))\) be the solution to the following decoupled FBSDE:

\[
\begin{aligned}
 d\bar{X}(r) &= b(r, \bar{X}(r), u_1(r, \bar{X}(r)), \omega_2(r, \bar{X}(r), u_1(r, \bar{X}(r)))) \, dr \\
 &\quad + \sigma(r, \bar{X}(r), u_1(r, \bar{X}(r)), \omega_2(r, \bar{X}(r), u_1(r, \bar{X}(r)))) \, dW(r), \quad r \in [t, T], \\
 d\bar{Y}(r) &= -g(r, \bar{X}(r), u_1(r, \bar{X}(r)), \omega_2(r, \bar{X}(r), u_1(r, \bar{X}(r)))) \, dr \\
 &\quad + \bar{Z}(r) \, dW(r), \quad r \in [t, T], \\
 \bar{X}(t) &= \xi, \quad \bar{Y}(T) = h(\bar{X}(T)).
\end{aligned}
\]

(2.18)

Then the “\( \geq \)” in (2.16) becomes “\( = \)”, and the uniqueness of the solution to BSDE leads to

\[ V^-(t, \xi) = \bar{Y}(t) = J(t, \xi; u_1, \omega_2). \]  

(2.19)

Combining (2.17) and (2.19), we obtain

\[ V^-(t, \xi) = J(t, \xi; u_1, \omega_2) = \text{esssup}_{u_1(\cdot) \in \mathcal{U}_1 [t, T]} J(t, \xi; u_1(\cdot), \omega_2). \]

(2.20)

**Step 2.** For any \((t, \xi) \in \mathcal{D}\), fix the feedback control law \( u_1 \) defined by (2.10). For any \( u_2(\cdot) \in \mathcal{U}_2[t, T] \), let us denote by \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))\) the solution to the following
FBSDE:

\[
\begin{cases}
    d\bar{X}(r) = b(r, \bar{X}(r), u_1(r, \bar{X}(r)), u_2(r))dr \\
    \quad + \sigma(r, \bar{X}(r), u_1(r, \bar{X}(r)), u_2(r))dW(r), \quad r \in [t, T], \\
    d\bar{Y}(r) = -g(r, \bar{X}(r), u_1(r, \bar{X}(r)), u_2(r), \bar{Y}(r), \bar{Z}(r))dr \\
    \quad + \bar{Z}(r)dW(r), \quad r \in [t, T], \\
    \bar{X}(t) = \xi, \quad \bar{Y}(T) = h(\bar{X}(T)).
\end{cases}
\] (2.21)

Similar with Step 1, applying Itô’s formula to \( V^-(\cdot, \bar{X}(\cdot)) \) leads to

\[
V^-(s, \bar{X}(s)) = h(\bar{X}(T)) \\
+ \int_s^T \left\{ g(r, \bar{X}(r), u_1(r, \bar{X}(r)), u_2(r), V^-(r, \bar{X}(r)),
\begin{align*}
    &V_x^-(r, \bar{X}(r))^\top \sigma(r, \bar{X}(r), u_1(r, \bar{X}(r)), u_2(r)) \\
    &- V_r^-(r, \bar{X}(r)) - \mathbb{H}(r, \bar{X}(r), u_1(r, \bar{X}(r)), u_2(r))
\end{align*}
\right\} dr
\]

\[
- \int_s^T V_x^-(r, \bar{X}(r))^\top \sigma(r, \bar{X}(r), u_1(r, \bar{X}(r)), u_2(r))dW(r), \quad s \in [t, T].
\] (2.22)

By the definition of \( u_1 \) in (2.10) and the HJBI equation (2.8), for any \( r \in [t, T] \), we have

\[
- V^-(r, \bar{X}(r)) - \mathbb{H}(r, \bar{X}(r), u_1(r, \bar{X}(r)), u_2(r)) \\
\leq -V^-(r, \bar{X}(r)) \\
- \inf_{u_2 \in U_2} \mathbb{H}(r, \bar{X}(r), u_1(r, \bar{X}(r)), u_2, V^-(r, \bar{X}(r)), V_x^-(r, \bar{X}(r)), V_{xx}^-(r, \bar{X}(r)))
\]

\[
= -V^-(r, \bar{X}(r)) \\
- \sup_{u_1 \in U_1, u_2 \in U_2} \mathbb{H}(r, \bar{X}(r), u_1, u_2, V^-(r, \bar{X}(r)), V_x^-(r, \bar{X}(r)), V_{xx}^-(r, \bar{X}(r)))
\]

\[
= 0. \tag{2.23}
\]

The comparison theorem for BSDEs works again to derive

\[
V^-(t, \xi) \leq \bar{Y}(t) = J(t, \xi; u_1, u_2(\cdot)), \quad \text{for any } u_2(\cdot) \in \mathcal{U}_2[t, T]. \tag{2.24}
\]

If we select \( u_2(\cdot) = u_2(\cdot, \bar{X}(\cdot), u_1(\cdot, \bar{X}(\cdot))) \), then FBSDE (2.21) coincides with F-Bsde (2.18). Furthermore, with the relationship (2.19) obtained in Step 1, we get

\[
V^-(t, \xi) = J(t, \xi; u_1, u_2) = \text{essinf}_{u_2(\cdot) \in \mathcal{U}_2[t, T]} J(t, \xi; u_1, u_2(\cdot)). \tag{2.25}
\]

Then, we obtain all the conclusions, and complete the proof. \( \Box \)

**Remark 1.** For a given initial pair \((t, \xi) \in \mathcal{D}\), let \((u_1, u_2)\) be defined by (2.10), and let \( \bar{X}(\cdot) \) be the corresponding state process with the initial pair \((t, \xi)\) under the
feedback control-strategy law \((u_1, \varnothing_2)\). For any \(s \in [t, T]\), we denote
\[
\begin{cases}
u_1[s, T](r, x) = u_1(r, x), & (r, x) \in [s,T] \times \mathbb{R}^n, \\
\varnothing_2[s, T](r, x, u_1) = \varnothing_2(r, x, u_1), & (r, x, u_1) \in [s, T] \times \mathbb{R}^n \times U_1.
\end{cases}
\]
It follows from the above verification theorem that, for any \(s \in (t, T)\),
\[
J(s, \tilde{X}(s); u_1|_{[s,T]}, \varnothing_2|_{[s,T]}) = V^{-}(s, \tilde{X}(s)) = \underset{u_2(\cdot) \in \varnothing_2|_{[s,T]}}{\text{essinf}} J(s, \tilde{X}(s); u_1[\cdot], u_2(\cdot))
\leq \underset{u_1(\cdot) \in \varnothing_1|_{[s,T]}}{\text{esssup}} J(s, \tilde{X}(s); u_1(\cdot), \varnothing_2|_{[s,T]}).
\]
This means that the restriction \((u_1|_{[s,T]}, \varnothing_2|_{[s,T]})\) of the lower saddle point \((u_1, \varnothing_2)\) with the initial pair \((t, \xi)\) on any later time interval \([s, T]\) is still a lower saddle point with the corresponding initial pair \((s, \tilde{X}(s))\). Such a property is called the time-consistency of the lower saddle point \((u_1, \varnothing_2)\).

Similarly, we can define and obtain the time-consistency of the upper saddle point \((\varnothing_1, u_2)\) (defined by (2.12)). Naturally, a game problem is called time-consistent if both time-consistent lower saddle points and time-consistent upper saddle points exist. In particular, the verification theorem implies Problem (C-SDG) is time-consistent.

In the special case where \(U_1\) and \(U_2\) are the sets containing only one element, the verification theorem is reduced to be the following result, named a nonlinear Feynman-Kac formula, which was introduced and well studied by Peng [13], Pardoux-Peng [12], Ma-Protter-Yong [11], and so on.

**Corollary 1 (Nonlinear Feynman-Kac Formula).** Let Assumption (A1) hold true. Suppose \(\Theta(\cdot, \cdot) \in C^{1,2}([0, T] \times \mathbb{R}^n)\) is a classical solution to the following PDE:
\[
\begin{cases}
\Theta_r(r, x) + \mathbb{H}(r, x, \Theta(r, x), \Theta_x(r, x), \Theta_{xx}(r, x)) = 0, & r \in [0, T] \times \mathbb{R}^n, \\
\Theta(T, x) = h(x), & x \in \mathbb{R}^n,
\end{cases}
\]
where, for simplicity, we use the notation \(\mathbb{H}\) in (2.1) omitting \(\tau, u_1\) and \(u_2\). We also introduce a family of FBSDEs parameterized by \((t, \xi) \in \mathcal{D}\):
\[
\begin{cases}
dX(r) = b(r, X(r))dr + \sigma(r, X(r))dW(r), & r \in [t, T], \\
dY(r) = -g(r, X(r), Y(r), Z(r))dr + Z(r)dW(r), & r \in [t, T],
\end{cases}
\]
Then,
\[
\Theta(t, \xi) = Y(t, t, \xi), \quad \text{a.s.}, \quad \forall (t, \xi) \in \mathcal{D}.
\]

3. **Time-inconsistent zero-sum stochastic differential game.** This section focuses on Problem (InC-SDG). We recall from Section 1 that, the difference between Problem (C-SDG) and Problem (InC-SDG) is the appearance of two time variables in the functions \(g\) and \(h\), which is the resource of time-inconsistent. In Problem (InC-SDG), the classical saddle point does not keep the time-consistency as time goes by. Therefore, instead of the classical saddle point, we shall propose a new notion called the equilibrium saddle point (which will be proved to be time-consistent) to fit Problem (InC-SDG). The new notion can be regarded as the counterpart of the equilibrium control in the study of time-inconsistent optimal control problem (see [17, 15] for example).
Problem (InC-SDG)-(i) and Problem (InC-SDG)-(ii) are symmetric, then the ways to solve them have no essential differences. Due to this, we shall focus on the details of Problem (InC-SDG)-(i) below.

Now we extend the notation \( D[0, T] \) (see (1.8)). For any \( t \in [0, T] \), denote
\[
D[t, T] = \left\{ (s, r) \in [0, T]^2 \mid 0 \leq s \leq r \leq T \right\}.
\]

Let \( \Pi = \{ t_k \mid 0 \leq k \leq N \} \) be a partition of time interval \([t, T]\) with \( t_0 < t_1 < \cdots < t_{N-1} < t_N = T \), and let \( \mathcal{P}[t, T] \) denote the collection of all partitions of \([t, T]\). We denote \( \|\Pi\| = \max_{1 \leq k \leq N} (t_k - t_{k-1}) \) as the mesh size of partition \( \Pi \).

**Definition 3.1.** An equilibrium lower saddle point to Problem (InC-SDG)-(i) with a given initial pair \((t, \xi) \in \mathcal{D}\) is an admissible feedback control-strategy law \((u_1, \omega_2)\) such that, there exist a family of partitions \( \mathcal{P}_0[t, T] \subseteq \mathcal{P}[t, T] \) with \( \inf_{\Pi \in \mathcal{P}_0[t, T]} \|\Pi\| = 0 \) and a family of admissible feedback control-strategy laws \((u_1^\Pi, \omega_2^\Pi)\) parameterized by the partition \( \Pi \in \mathcal{P}_0[t, T] \) satisfying:

1. \( \lim_{\|\Pi\| \to 0} \sup_{(t, x) \in \mathcal{K}} |u_1^\Pi(t, x) - u_1(t, x)| = 0, \quad \forall \text{ compact set } \mathcal{K} \subseteq [t, T] \times \mathbb{R}^n, \)
2. \( \lim_{\|\Pi\| \to 0} \sup_{(t, x, u_1) \in \mathcal{K}'} |\omega_2^\Pi(t, x, u_1) - \omega_2(t, x, u_1)| = 0, \quad \forall \text{ compact set } \mathcal{K}' \subseteq [t, T] \times \mathbb{R}^n \times U_1. \)

Let \((\bar{X}(\cdot), \bar{Y}(\cdot, \cdot), \bar{Z}(\cdot, \cdot))\) denote the solution to the following closed-loop system:
\[
\begin{aligned}
d\bar{X}(r) &= b(r, \bar{X}(r), u_1(r, \bar{X}(r)), \omega_2(r, \bar{X}(r), u_1(r, \bar{X}(r))))dr \\
&\quad + \sigma(r, \bar{X}(r), u_1(r, \bar{X}(r)), \omega_2(r, \bar{X}(r), u_1(r, \bar{X}(r))))dW(r), \quad r \in [t, T], \\
d\bar{Y}(s, r) &= -g(s, r, \bar{X}(r), u_1(r, \bar{X}(r)), \omega_2(r, \bar{X}(r), u_1(r, \bar{X}(r))))dW(r), \quad (s, r) \in D[t, T], \\
\bar{X}(t) &= \xi, \quad \bar{Y}(s, T) = h(s, \bar{X}(T)), \quad s \in [t, T].
\end{aligned}
\] (3.1)

For any \( \Pi \in \mathcal{P}_0[t, T] \), let \((X^\Pi(\cdot), Y^\Pi(\cdot, \cdot), Z^\Pi(\cdot, \cdot))\) denote the solution to the following system:
\[
\begin{aligned}
dX^\Pi(r) &= b(r, X^\Pi(r), u_1^\Pi(r, X^\Pi(r)), \omega_2^\Pi(r, X^\Pi(r), u_1^\Pi(r, X^\Pi(r))))dr \\
&\quad + \sigma(r, X^\Pi(r), u_1^\Pi(r, X^\Pi(r)), \omega_2^\Pi(r, X^\Pi(r), u_1^\Pi(r, X^\Pi(r))))dW(r), \quad r \in [t, T], \\
dY^\Pi(t_k, r) &= -g(t_k, r, X^\Pi(r), u_1^\Pi(r, X^\Pi(r))), Y^\Pi(t_k, r), Z^\Pi(t_k, r) \right)dr \\
&\quad + Z^\Pi(t_k, r)dW(r), \quad r \in [t_k, T], \quad 0 \leq k \leq N - 1, \\
X^\Pi(t) &= \xi, \quad Y^\Pi(t_k, T) = h(t_k, X^\Pi(T)), \quad 0 \leq k \leq N - 1.
\end{aligned}
\] (3.2)

Then, for any \( s \in [t, T] \),
\[
\lim_{\|\Pi\| \to 0} \left( |X^\Pi(s) - \bar{X}(s)| + |Y^\Pi(l^\Pi(s), l^\Pi(s)) - \bar{Y}(s, s)| \right) = 0, \quad a.s., \quad (3.3)
\]
where
\[ t^\Pi(s) = \sum_{k=1}^{N-1} t_{k-1} \mathbb{1}_{[t_{k-1}, t_k)}(s) + t_{N-1} \mathbb{1}_{[t_{N-1}, t_N]}, \quad s \in [t, T]. \]  \hfill (3.4)

(iii). For any \( \Pi \in \mathcal{P}_0[t, T], \) \((u_1^\Pi, \omega_2^\Pi)\) is a local lower saddle point in the following sense: for each \( k = 1, 2, \ldots, N, \)
\[
J(t_{k-1}, X^\Pi(t_{k-1}); u_1^\Pi|_{[t_{k-1}, t_k]}, \omega_2^\Pi|_{[t_{k-1}, t_k]})
= \text{essinf}_{u_2^k(\cdot) \in \mathcal{U}_2[t_{k-1}, t_k]} J(t_{k-1}, X^\Pi(t_{k-1}); u_1^\Pi|_{[t_{k-1}, t_k]}, u_2^k(\cdot) + \omega_2^\Pi|_{[t_k, T]})
= \text{esssup}_{u_1^k(\cdot) \in \mathcal{U}_1[t_{k-1}, t_k]} J(t_{k-1}, X^\Pi(t_{k-1}); u_1^k(\cdot) + u_1^\Pi|_{[t_k, T]}, \omega_2|_{[t_{k-1}, t_k]}),
\]
where \( u_2^k(\cdot) + \omega_2^\Pi|_{[t_k, T]} \) means that Player 2 takes the admissible control \( u_2^k(\cdot) \) on \([t_{k-1}, t_k]\) and obeys the strategy law \( \omega_2^\Pi|_{[t_k, T]} \) on \([t_k, T]\), and similarly, \( u_1^k(\cdot) + u_1^\Pi|_{[t_k, T]} \) means Player 1 takes the admissible control \( u_1^k(\cdot) \) on \([t_{k-1}, t_k]\) and obeys the control law \( u_1^\Pi|_{[t_k, T]} \) on \([t_k, T]\).

In the above, \( \bar{X}(\cdot) \) is called the corresponding equilibrium state process of Problem (InC-SDG) with the initial pair \((t, \xi)\). \((u_1^\Pi, \omega_2^\Pi)\) is called an approximate equilibrium lower saddle point of Problem (InC-SDG) with the initial pair \((t, \xi)\) and the partition \( \Pi \).

Furthermore, in the case that, for any \((t, \xi) \in \mathcal{D}, \) an equilibrium lower saddle point \((u_1, \omega_2)\) exists, if a function \( V^- : [0, T] \times \mathbb{R}^n \to \mathbb{R} \) satisfies
\[
V^-(t, \xi) = J(t, \xi; u_1, \omega_2), \quad a.s., \quad \forall (t, \xi) \in \mathcal{D}, \hfill (3.6)
\]
then it is called an equilibrium lower value function of Problem (InC-SDG).

**Remark 2.** (i). It is easy to see that (3.3) implies
\[
\lim_{\|\Pi\| \to 0} J\left( t^\Pi(s), X^\Pi(t^\Pi(s)); u_1^\Pi|_{[l^\Pi(s), T]}, \omega_2^\Pi|_{[l^\Pi(s), T]} \right) = J\left( \bar{X}(s); u_1|_{[s, T]}, \omega_2|_{[s, T]} \right),
\quad a.s., \quad s \in [t, T].
\]

(ii). We have the following facts: the first one is the time-consistent lower saddle point does not exist; the second one is that, in the definition of equilibrium lower saddle point, the approximate \((u_1^\Pi, \omega_2^\Pi)\) is a local lower saddle point in a proper sense (see (3.5)). Later, we shall prove the third one: there exists indeed a time-consistent equilibrium lower saddle point (see Remark 3). Due to these three points, for the time-inconsistent Problem (InC-SDG), instead of the time-inconsistent lower saddle point, to seek a time-consistent equilibrium lower saddle point might be a better choice.

**4. A backward sequence of time-consistent subgames.** In this section, we shall carry out the idea proposed in Section 1. Let \( \Pi : t = t_0 < t_1 < \cdots < t_{N-1} < t_N = T \) be a partition of \([t, T]\), and denote the forthcoming associated backward sequence of time-consistent subgames by Problem \((G^\Pi)\). There are \( N \) pairs of players performing in the \( N \) stochastic differential game problems on \([t_{k-1}, t_k]\), \( 1 \leq k \leq N, \) in total. We will denote by Player \( k_1 \) and Player \( k_2 \) the first and second one in the \( k \)-th pair of players \((1 \leq k \leq N), \) respectively.

Let us begin with the last game problem on the last time interval \([t_{N-1}, t_N].\)
4.1. The \( N \)-th pair of players — a classical zero-sum differential game.

On the last interval \([t_{N-1}, t_N]\), the controlled system for the \( N \)-th pair of players is described by

\[
\begin{align*}
    dX^N(r) &= b(r, X^N(r), u_1^N(r), u_2^N(r))dr \\
    &\quad + \sigma(r, X^N(r), u_1^N(r), u_2^N(r))dW(r), \quad r \in [t_{N-1}, t_N], \\
    dY^N(r) &= -g(t_{N-1}, r, X^N(r), u_1^N(r), u_2^N(r), Y^N(r), Z^N(r))dr \\
    &\quad + Z^N(r)dW(r), \quad r \in [t_{N-1}, t_N], \\
    X^N(t_{N-1}) &= \xi_{N-1} \in L^2_{\mathcal{F}_{t_{N-1}}} (\Omega; \mathbb{R}^n), \quad Y^N(T) = h(t_{N-1}, X^N(t_N)),
\end{align*}
\]

with \( u_i^N(\cdot) \in \mathcal{U}_i[t_{N-1}, t_N] \) controlled by Player \( N_i, i = 1, 2 \). The \( N \)-th pair of players take over the system from the previous \((N-1)\)-th pair of players. Therefore, the initial state \( \xi_{N-1} \) of the \( N \)-th pair of players is just the terminal state of the \((N-1)\)-th pair of players. The recursive criterion functional is given by

\[
J(t_{N-1}, \xi_{N-1}; u_1^N(\cdot), u_2^N(\cdot)) = Y^N(t_{N-1}; t_{N-1}, \xi_{N-1}, u_1^{N-1}(\cdot), u_2^{N-1}(\cdot))
\]

\[
= \mathbb{E}_{t_{N-1}} \left[ \int_{t_{N-1}}^T g(t_{N-1}, r, X^N(r), u_1^N(r), u_2^N(r), Y^N(r), Z^N(r))dr \\
+ h(t_{N-1}, X^N(t_N)) \right].
\]

The differential game problem for the \( N \)-th pair of players is the following

**Problem (C\( N \)).** For any \( \xi_{N-1} \in L^2_{\mathcal{F}_{t_{N-1}}} (\Omega; \mathbb{R}^n) \), find a lower saddle point, i.e. an admissible feedback control-strategy law \((u_1^N, u_2^N)\) such that

\[
J(t_{N-1}, \xi_{N-1}; u_1^N, u_2^N) = \text{essinf}_{u_1^N(\cdot) \in \mathcal{U}_1[t_{N-1}, t_N]} \text{esssup}_{u_2^N(\cdot) \in \mathcal{U}_2[t_{N-1}, t_N]} J(t_{N-1}, \xi_{N-1}; u_1^N, u_2^N(\cdot)) = \text{esssup}_{u_1^N(\cdot) \in \mathcal{U}_1[t_{N-1}, t_N]} \text{essinf}_{u_2^N(\cdot) \in \mathcal{U}_2[t_{N-1}, t_N]} J(t_{N-1}, \xi_{N-1}; u_1^N(\cdot), u_2^N). \]

Here, \( t_{N-1} \) appearing in the functions \( g \) and \( h \) is a time parameter, which does not lead to any essential difference from the classical Problem (C-SDG). Then we can apply the verification theorem (see Theorem 2.4) developed in Section 2 to solve Problem (C\( N \)). Under some mild conditions, the following HJBI equation

\[
\begin{align*}
    V_{\mathcal{U}}^N^- (r, x) &= \sup_{u_1 \in U_1, u_2 \in U_2} \inf_{H(t_{N-1}, r, x, u_1, u_2, V_{\mathcal{U}}^N^-(t, x), V_{\mathcal{U}}^N^-(t, x))} = 0, \quad (r, x) \in [t_{N-1}, t_N] \times \mathbb{R}^n, \\
    V_{\mathcal{U}}^N^- (t_{N-1}, x) &= h(t_{N-1}, x), \quad x \in \mathbb{R}^n
\end{align*}
\]

admits a unique smooth solution \( V_{\mathcal{U}}^N^- (\cdot, \cdot) \in C^{1,2}([t_{N-1}, t_N] \times \mathbb{R}^n) \). Let the functions \( \varphi_1 \) and \( \psi_2 \) be given in Assumption (A2)-(i), and define

\[
\begin{align*}
    u_1^N(r, x) &= \varphi_1(t_{N-1}, r, x, V_{\mathcal{U}}^N^-(r, x), V_{\mathcal{U}}^N^-(r, x), V_{\mathcal{U}}^N_\varepsilon^-(r, x)), \quad (r, x) \in [t_{N-1}, t_N] \times \mathbb{R}^n, \\
    u_2^N(r, x, u_1) &= \psi_2(t_{N-1}, r, x, u_1, V_{\mathcal{U}}^N^-(r, x), V_{\mathcal{U}}^N^-(r, x), V_{\mathcal{U}}^N_\varepsilon^-(r, x)), \quad (r, x, u_1) \in [t_{N-1}, t_N] \times \mathbb{R}^n \times U_1.
\end{align*}
\]

By Theorem 2.4, \((u_1^N, u_2^N)\) is a lower saddle point to Problem (C\( N \)) with the initial pair \((t_{N-1}, \xi_{N-1})\). More detailed, let \((X^N(\cdot), Y^N(\cdot), Z^N(\cdot))\) be the solution to the
following FBSDE:

\[
\begin{align*}
\frac{dX^N(r)}{dr} &= b(r, X^N(r), u_1^I, \alpha_2^I)dr \\
&\quad + \sigma(r, X^N(r), u_1^I, \alpha_2^I)dW(r), \quad r \in [t_{N-1}, t_N], \\
\frac{dY^N(r)}{dr} &= -g(t_{N-1}, r, X^N(r), u_1^I, \alpha_2^I)dr \\
&\quad + \bar{Z}^N(r)dW(r), \quad r \in [t_{N-1}, t_N], \\
\bar{X}^N(t_{N-1}) &= \xi_{N-1}, \quad \bar{Y}^N(T) = h(t_{N-1}, \bar{X}^N(t_N)),
\end{align*}
\]

(4.5)

where, for simplicity of notations, we denote by

\[
b(r, X^N(r), u_1^I, \alpha_2^I) \equiv b\left(r, X^N(r), u_1^I(r, X^N(r)), \alpha_2^I(r, X^N(r))\right),
\]

and so on, then \(\bar{X}(\cdot)\) is the corresponding state under \((u_1^I, \alpha_2^I)\), and the corresponding lower value is given by

\[
J(t_{N-1}, \xi_{N-1}; u_1^I, \alpha_2^I) = \bar{Y}^N(t_{N-1}).
\]

4.2. The \((N-1)\)-th pair of players — a sophisticated differential game.

The \((N-1)\)-th pair of players only control the system on \([t_{N-2}, t_{N-1}]\), and they will hand the system over to the \(N\)-th pair of players at time \(t_{N-1}\). Moreover, although it is known by the \((N-1)\)-th pair of players that the \(N\)-th pair of players will act by carrying out the lower saddle point \((u_1^I, \alpha_2)\) in (4.4), due to the subjective time-preference, the \((N-1)\)-th pair of players still “discount” the future payoffs (or costs) in their own way.

Therefore, based on the above viewpoint, the controlled system of the \((N-1)\)-th pair of players is

\[
\begin{align*}
\frac{dX^{N-1}(r)}{dr} &= b(r, X^{N-1}(r), u_1^{N-1}(r), u_2^{N-2}(r))dr \\
&\quad + \sigma(r, X^{N-1}(r), u_1^{N-1}(r), u_2^{N-2}(r))dW(r), \quad r \in [t_{N-2}, t_{N-1}], \\
\frac{dX^{N-1}(r)}{dr} &= b(r, X^{N-1}(r), u_1^{N-1}(r), \alpha_2^I)dr \\
&\quad + \sigma(r, X^{N-1}(r), u_1^{N-1}(r), \alpha_2^I)dW(r), \quad r \in [t_{N-1}, t_N], \\
\frac{dY^{N-1}(r)}{dr} &= -g(t_{N-2}, r, X^{N-1}(r), u_1^{N-1}(r), u_2^{N-2}(r), Y^{N-1}(r), Z^{N-1}(r))dr \\
&\quad + Z^{N-1}(r)dW(r), \quad r \in [t_{N-2}, t_{N-1}], \\
\frac{dY^{N-1}(r)}{dr} &= -g(t_{N-2}, r, X^{N-1}(r), u_1^{N-1}(r), \alpha_2^I, Y^{N-1}(r), Z^{N-1}(r))dr \\
&\quad + Z^{N-1}(r)dW(r), \quad r \in [t_{N-1}, t_N], \\
X^{N-1}(t_{N-2}) &= \xi_{N-2} \in L^2_{\mathcal{F}_{t_{N-2}}} (\Omega; \mathbb{R}^n), \quad Y^{N-1}(t_N) = h(t_{N-2}, X^{N-1}(t_{N-1})),
\end{align*}
\]

(4.6)

where \(u_1^{N-1}(\cdot) \in \mathscr{U}[t_{N-2}, t_{N-1}]\) is the control process of Player \((N-1)_i, i = 1, 2\), respectively. Different from Problem \((C_N)\), the criterion functional for the \((N-1)\)-th pair of players is a bit complex. Precisely, we define the sophisticated recursive criterion functional for the \((N-1)\)-th pair of players as follows:

\[
\tilde{J}(t_{N-2}, \xi_{N-2}; u_1^{N-1}(\cdot), u_2^{N-1}(\cdot)) = J(t_{N-2}, \xi_{N-2}; u_1^{N-1}(\cdot) \oplus u_1^I, u_2^{N-1}(\cdot) \oplus \alpha_2)
\]

(4.7)

\[
= Y^{N-1}(t_{N-2}; t_{N-2}, \xi_{N-2}, u_1^{N-1}(\cdot) \oplus u_1^I, u_2^{N-1}(\cdot) \oplus \alpha_2).
\]

The differential game problem for the \((N-1)\)-th pair of players is posed as
Problem (C\(_{N-1}\)). For any \(\xi_{N-2} \in L^2_{\mathcal{F}_{t_{N-2}}} (\Omega; \mathbb{R}^n)\), find a lower saddle point, i.e., an admissible feedback control-strategy law \((u^\Pi_1, \phi^\Pi_2)\) such that

\[
\tilde{J}(t_{N-2}, \xi_{N-2}; u^\Pi_1, \phi^\Pi_2) = \inf_{u^\Pi_2 \in \mathcal{U}} \tilde{J}(t_{N-2}, \xi_{N-2}; u^\Pi_1, u^\Pi_2(N^{-1}(\cdot)))
\]

Since on the time interval \([t_{N-1}, t_N]\), the controls are fixed to obey the law \((u^\Pi_1, \phi^\Pi_2)\), then we would like to use the nonlinear Feynman-Kac formula to simplify Problem (C\(_{N-1}\)). For this aim, we introduce the following PDE:

\[
\begin{aligned}
\Theta_r^{-1}(r, x) + \text{H}(t_{N-2}, r, x, u^\Pi_1(r, x), \omega_2(r, x, u^\Pi_1(r, x)), \\
\Theta^{-1}(r, x), \Theta^{-1}(r, x), \Theta^{-1}(r, x))) = 0,
\end{aligned}
\]

If the above PDE admits a classical solution \(\Theta^{-1}(\cdot, \cdot) \in C^{1,2}([t_{N-1}, t_N] \times \mathbb{R}^n)\), then \(\Theta^{-1}(t_{N-1})\) admits the following representation:

\[
\Theta^{-1}(t_{N-1}) = \Theta^{-1}(t_{N-1}, X^{-1}(t_{N-1})).
\]

Consequently, the controlled system can be restricted on \([t_{N-2}, t_{N-1}]\) as follows:

\[
\begin{aligned}
dX^{-1}(r) = & b(r, X^{-1}(r), u^\Pi_1(r), u^\Pi_2(r))dr \\
& + \sigma(r, X^{-1}(r), u^\Pi_1(r), u^\Pi_2(r))dW(r), \quad r \in [t_{N-2}, t_{N-1}], \\
dY^{-1}(r) = & -g(t_{N-2}, r, X^{-1}(r), u^\Pi_1(r), u^\Pi_2(r), Y^{-1}(r), Z^{-1}(r))dr \\
& + Z^{-1}(r)dW(r), \quad r \in [t_{N-2}, t_{N-1}],
\end{aligned}
\]

\[
\begin{aligned}
X^{-1}(t_{N-2}) = & \xi_{N-2} \in L^2_{\mathcal{F}_{t_{N-2}}} (\Omega; \mathbb{R}^n), \\
Y^{-1}(t_{N-1}) = & \Theta^{-1}(t_{N-1}, X^{-1}(t_{N-1})).
\end{aligned}
\]

and Problem (C\(_{N-1}\)) with the initial state \(\xi_{N-2}\) turns out to be a standard recursive zero-sum stochastic differential game problem on \([t_{N-2}, t_{N-1}]\).

We suppose that the following HJBI equation:

\[
\begin{aligned}
V^\Pi_1(r, x) + \sup_{u_1 \in U_1} \inf_{u_2 \in U_2} \text{H}(t_{N-2}, r, x, u_1, u_2, V^\Pi_1(r, x), V^\Pi_2(r, x)) = 0, \\
V^\Pi_1(t_{N-1}, x) = \Theta^{-1}(t_{N-1}, x), \quad x \in \mathbb{R}^n
\end{aligned}
\]

admits a classical solution \(V^\Pi_1(\cdot, \cdot) \in C^{1,2}([t_{N-2}, t_{N-1}] \times \mathbb{R}^n)\). Similar to (4.4), let

\[
\begin{aligned}
u^\Pi_1(r, x, u_1) = & \varphi_1(t_{N-2}, r, x, V^\Pi_1(r, x), V^\Pi_2(r, x)), \\
\phi^\Pi_2(r, x, u_1) = & \psi_2(t_{N-2}, r, x, u_1, V^\Pi_1(r, x), V^\Pi_2(r, x)),
\end{aligned}
\]

According to the verification theorem (Theorem 2.4), \((u^\Pi_1, \phi^\Pi_2)\) defined by (4.11) is a lower saddle point for the \((N-1)\)-th pair of players on \([t_{N-2}, t_{N-1}]\). Moreover,
let \((\bar{X}^{N-1}(-), \bar{Y}^{N-1}(-), \bar{Z}^{N-1}(-))\) be the solution to the following FBSDE:

\[
\begin{aligned}
&d\bar{X}^{N-1}(r) = b(r, \bar{X}^{N-1}(r), \bar{u}^{N}_{1}(r), \bar{u}^{N}_{2}(r))dr \\
&\quad + \sigma(r, \bar{X}^{N-1}(r), \bar{u}^{N}_{1}(r), \bar{u}^{N}_{2}(r))dW(r), \quad r \in [t_{N-2}, t_{N-1}),
\end{aligned}
\]

\[
\begin{aligned}
&d\bar{Y}^{N-1}(r) = -g(t_{N-2}, r, \bar{X}^{N-1}(r), \bar{u}^{N}_{1}(r), \bar{u}^{N}_{2}(r), \bar{Y}^{N-1}(r), \bar{Z}^{N-1}(r))dr \\
&\quad + \bar{Z}^{N-1}(r)dW(r), \quad r \in [t_{N-2}, t_{N-1}),
\end{aligned}
\]

\[
\begin{aligned}
&\bar{X}^{N-1}(t_{N-2}) = \xi_{N-2}, \quad \bar{Y}^{N-1}(t_{N-1}) = \Theta^{N-1}(t_{N-1}, \bar{X}^{N-1}(t_{N-1})),
\end{aligned}
\]

then \(\bar{X}^{N-1}(\cdot)\) is the corresponding state under \((\bar{u}^{N}_{1}, \bar{u}^{N}_{2})\), and the corresponding lower value is given by

\[
\bar{J}(t_{N-2}, \xi_{N-2}; \bar{u}^{N}_{1}, \bar{u}^{N}_{2}) = \bar{Y}^{N-1}(t_{N-2}).
\]

Now we collect the problems for both the \(N\)-th pair and the \((N-1)\)-th pair of players together. Noticing in (4.4) and (4.11), \((\bar{u}^{N}_{1}, \bar{u}^{N}_{2})\) is defined on \([t_{N-1}, t_{N}]\) and \([t_{N-2}, t_{N-1})\), respectively. Now, we would like to rewrite them in a unified form:

\[
\begin{aligned}
&\bar{u}^{N}_{1}(r, x) = \varphi_{1}(\bar{t}^{N}(r), r, x, V^{N-1}(r, x), V_{x}^{N-1}(r, x), V_{xx}^{N-1}(r, x)), \\
&\bar{u}^{N}_{2}(r, x, u_{1}) = \psi_{2}(\bar{t}^{N}(r), r, x, u_{1}, V^{N-1}(r, x), V_{x}^{N-1}(r, x), V_{xx}^{N-1}(r, x)),
\end{aligned}
\]

where \(\bar{t}^{N}(\cdot)\) is defined by (3.4). Obviously, the feedback control-strategy law \((\bar{u}^{N}_{1}, \bar{u}^{N}_{2})\) is a lower saddle point for the \(N\)-th pair and the \((N-1)\)-th pair of players when it is restricted on \([t_{N-1}, t_{N}]\) and \([t_{N-2}, t_{N-1})\), respectively. However, it is not a lower saddle point on the whole interval \([t_{N-2}, t_{N}]\) in general. We call \((\bar{u}^{N}_{1}, \bar{u}^{N}_{2})\) an equilibrium lower saddle point of Problem (G\(^{N}\)) on \([t_{N-2}, t_{N}]\).

4.3. The \(k\)-th pair of players and equilibrium saddle point of Problem (G\(^{N}\)). Generally, the \(k\)-th pair of players, who take over the system from the \((k-1)\)-th pair of players, control the system on \([t_{k-1}, t_{k}]\) and then hand it over to the \((k+1)\)-th pair of players at time \(t_{k}\).

Suppose that the equilibrium lower saddle point \((u_{1}^{k}(\cdot), u_{2}^{k}(\cdot))\) of Problem (G\(^{N}\)) on \([t_{k}, t_{N}]\) has been constructed. Although the \(k\)-th pair of players know that the future players will control the system obeying the law \((u_{1}^{k}, u_{2}^{k})\) on \([t_{k}, t_{N}]\), but they “discount” in their own way. According to this viewpoint, for any pair of admissible controls \((u_{1}^{k}(\cdot), u_{2}^{k}(\cdot))\) \(\in \mathcal{U}_{1}[t_{k-1}, t_{k}] \times \mathcal{U}_{2}[t_{k-1}, t_{k}]\), we have the following FBSDE:

\[
\begin{aligned}
&dX^{k}(r) = b(r, X^{k}(r), u_{1}^{k}(r), u_{2}^{k}(r))dr \\
&\quad + \sigma(r, X^{k}(r), u_{1}^{k}(r), u_{2}^{k}(r))dW(r), \quad r \in [t_{k-1}, t_{k}),
\end{aligned}
\]

\[
\begin{aligned}
&dY^{k}(r) = b(r, X^{k}(r), u_{1}^{k}(r), u_{2}^{k}(r))dr + \sigma(r, X^{k}(r), u_{1}^{k}(r), u_{2}^{k}(r))dW(r), \quad r \in [t_{k}, t_{N}],
\end{aligned}
\]

\[
\begin{aligned}
&dY^{k}(r) = -g(t_{k-1}, r, X^{k}(r), u_{1}^{k}(r), u_{2}^{k}(r), Y^{k}(r), Z^{k}(r))dr \\
&\quad + Z^{k}(r)dW(r), \quad r \in [t_{k-1}, t_{k}),
\end{aligned}
\]

\[
\begin{aligned}
&dY^{k}(r) = -g(t_{k-1}, r, X^{k}(r), u_{1}^{k}(r), u_{2}^{k}(r), Y^{k}(r), Z^{k}(r))dr \\
&\quad + Z^{k}(r)dW(r), \quad r \in [t_{k}, t_{N}],
\end{aligned}
\]

\[
\begin{aligned}
X^{k}(t_{k-1}) = \xi_{k-1} \in L_{F}^{2}[t_{k-1}] (\Omega; \mathbb{R}^{n}), \quad Y^{k}(t_{N}) = h(t_{k-1}, X^{k}(t_{N})).
\end{aligned}
\]
The associated sophisticated criterion functional of the $k$-th pair of players is given by

$$\tilde{J}(t_{k-1}, \xi_{k-1}; u_1^k(), u_2^k()) = J(t_{k-1}, \xi_{k-1}; u_1^k() + u_1^\Pi, u_2^k() + u_2^\Pi) = Y^k(t_{k-1}; t_{k-1}, \xi_{k-1}, u_1^k() + u_1^\Pi, u_2^k() + u_2^\Pi).$$  \hfill (4.15)

And then the zero-sum game problem for the $k$-th pair of players is formulated as

**Problem (C$_k$).** For any $\xi_{k-1} \in L^2_{X_{t_{k-1}}} (\Omega; \mathbb{R}^n)$, find a lower saddle point, i.e. an admissible feedback control-strategy law $(u_1^\Pi, u_2^\Pi)$ such that

$$\tilde{J}(t_{k-1}, \xi_{k-1}; u_1^\Pi, u_2^\Pi) = \text{essinf}_{u_2^\Pi(\cdot) \in \mathcal{U}_2[t_{k-1}, t_k]} \tilde{J}(t_{k-1}, \xi_{k-1}; u_1^\Pi, u_2^\Pi(\cdot))$$

$$= \text{esssup}_{u_1^\Pi(\cdot) \in \mathcal{U}_1[t_{k-1}, t_k]} \tilde{J}(t_{k-1}, \xi_{k-1}; u_1^\Pi(\cdot), u_2^\Pi).$$  \hfill (4.16)

Now, by the same method in Subsection 4.2, we give the solution to Problem (C$_k$). Let $\Theta^k(\cdot, \cdot) \in C^{1,2}([t_k, t_N] \times \mathbb{R}^n)$ be a classical solution to the following PDE:

$$\begin{cases}
\Theta^k(r, x) + \mathbb{H}(t_{k-1}, r, x, u_1^u(r, x), u_2^\Pi(r, x, u_1^u(r, x)), \\
\quad \Theta^k(r, x), \Theta_x^k(r, x), \Theta_{xx}^k(r, x)) = 0,
(r, x) \in [t_k, t_N] \times \mathbb{R}^n,
\end{cases}$$

$$\Theta^k(t_N, x) = h(t_{k-1}, x), \quad x \in \mathbb{R}^n.$$  \hfill (4.17)

Let $V^{-k}(\cdot, \cdot) \in C^{1,2}([t_k, t_k] \times \mathbb{R}^n)$ be a classical solution to the following HJBI equation:

$$\begin{cases}
V^{-k}_{r}(r, x) + \sup_{u_1 \in U_1} \inf_{u_2 \in U_2} \mathbb{H}(t_{k-1}, r, x, u_1, u_2, V^{\Pi^{-k}}(r, x), V^{\Pi^{-k}}(r, x)) = 0,
(r, x) \in [t_k, t_k] \times \mathbb{R}^n,
V^{-k}(t_k, x) = \Theta^k(t_{k-1}, x), \quad x \in \mathbb{R}^n.
\end{cases}$$

Define

$$\begin{cases}
u_1^\Pi(r, x) = \varphi_1(t_{k-1}, r, x, V^{\Pi^{-k}(r, x)}), V^{\Pi}(r, x), V^{\Pi^{-k}}(r, x)),
(r, x) \in [t_k, t_k] \times \mathbb{R}^n,
\varphi_2^\Pi(r, x, u_1) = \psi_2(t_{k-1}, r, x, u_1, V^{\Pi^{-k}(r, x)}, V^{\Pi^{-k}}(r, x), V^{\Pi^{-k}}(r, x)),
(r, x, u_1) \in [t_k-1, t_k] \times \mathbb{R}^n \times U_1.
\end{cases}$$

The nonlinear Feynman-Kac formula (Corollary 1) and the verification theorem (Theorem 2.4) imply that $(u_1^\Pi, u_2^\Pi)$ is a lower saddle point for the $k$-th pair of players on $[t_{k-1}, t_k]$. Let $(\hat{X}^k(\cdot), \hat{Y}^k(\cdot), \hat{Z}^k(\cdot))$ be the solution to the following FBSDE:

$$\begin{cases}
d\hat{X}^k(r) = b(r, \hat{X}^k(r), u_1^\Pi, u_2^\Pi) dr + \sigma(r, \hat{X}^k(r), u_1^\Pi, u_2^\Pi) dW(r), \quad r \in [t_{k-1}, t_k],
\end{cases}$$

$$\begin{cases}
d\hat{Y}^k(r) = -g(t_{k-1}, r, \hat{X}^k(r), u_1^\Pi, u_2^\Pi, \hat{Y}^k(r), \hat{Z}^k(r)) dr + \hat{Z}^k(r) dW(r), \quad r \in [t_{k-1}, t_k],
\end{cases}$$

$$\hat{X}^k(t_{k-1}) = \xi_{k-1}, \quad \hat{Y}^k(t_k) = \Theta^k(t_k, \hat{X}^k(t_k)).$$
then $\bar{X}^k(\cdot)$ is the corresponding state under $(u^\Pi_1, \alpha^\Pi_2)$ on $[t_{k-1}, t_k]$, and the corresponding lower value is

$$\bar{J}(t_{k-1}, \xi_{k-1}; u^\Pi_1, \alpha^\Pi_2) = \bar{V}^k(t_{k-1}).$$

Additionally, the equilibrium lower saddle point of Problem (G$^\Pi$) is extended to the time interval $[t_{k-1}, t_N]$:

$$
\begin{align*}
&\begin{cases}
  u^\Pi_1(r, x) = \varphi_1(t^\Pi(r), r, x, V^{\Pi-}(r, x), V_x^{\Pi-}(r, x), V_{xx}^{\Pi-}(r, x)), \\
  \alpha^\Pi_2(r, x, u_1) = \psi_2(t^\Pi(r), r, x, u_1, V^{\Pi-}(r, x), V_x^{\Pi-}(r, x), V_{xx}^{\Pi-}(r, x)),
\end{cases} \\
&\quad (r, x) \in [t_{k-1}, t_N] \times \mathbb{R}^n, \\
&\Theta^k(r, x, u_1) = \psi_2(t^\Pi(r), r, x, u_1, V^{\Pi-}(r, x), V_x^{\Pi-}(r, x), V_{xx}^{\Pi-}(r, x)), \\
&\quad (r, x, u_1) \in [t_{k-1}, t_N] \times \mathbb{R}^n \times U_1.
\end{align*}
$$

By induction, the equilibrium lower saddle point $(u^\Pi_1, \alpha^\Pi_2)$ and the lower value function $V^{\Pi-}(\cdot, \cdot)$ are well defined on the whole time interval $[t, T]$. In the rest of this subsection, let us summarize the results for Problem (G$^\Pi$) on $[t, T]$ with partition $\Pi$.

In the above, functions $V^{\Pi-}(\cdot, \cdot)$ and $\Theta^k(\cdot, \cdot)$ ($1 \leq k \leq N - 1$) are constructed recursively. Through a careful observation on (4.17), (4.18), and (4.19), we find $V^{\Pi-}(\cdot, \cdot)$ and $\Theta^k(\cdot, \cdot)$ can be represented in a unified form. In detail, firstly, for any $1 \leq k \leq N$, we extend $\Theta^k(\cdot, \cdot)$ from $[t_k, T]$ to $[t_{k-1}, T]$ by setting

$$\Theta^k(r, x) = V^{\Pi-}(r, x), \quad (r, x) \in [t_{k-1}, t_k] \times \mathbb{R}^n,$n

then the extended function $\Theta^k: [t_{k-1}, t_N] \times \mathbb{R}^n$ satisfies the following PDE:

$$
\begin{align*}
&\begin{cases}
  \Theta^k(r, x) + \mathbb{H}(t_{k-1}, r, x, u^\Pi_1(r, x), \alpha^\Pi_2(r, x, u^\Pi_1(r, x))), \\
  \Theta^k(r, x), \Theta^k_x(r, x), \Theta^k_{xx}(r, x) = 0,
\end{cases} \\
&\quad (r, x) \in [t_{k-1}, t_N] \times \mathbb{R}^n,
\end{align*}
$$

Secondly, we sum up all $\Theta^k(\cdot, \cdot)$ from $k = 1$ to $N$ by introducing a new time variable $\tau$ in the following way:

$$\Theta^\Pi(\tau, r, x) = \sum_{k=1}^N \Theta^k(r, x)1_{[t_{k-1}, t_k)}(\tau), \quad (\tau, r, x) \in D[t, T] \times \mathbb{R}^n. \tag{4.24}$$

With the help of the following notations

$$
\begin{align*}
&h^\Pi(\tau, x) = \sum_{k=1}^N h(t_{k-1}, x)1_{[t_{k-1}, t_k)}(\tau), \quad (\tau, x) \in [t, T] \times \mathbb{R}^n, \\
g^\Pi(\tau, r, x, y, z) = \sum_{k=1}^N g(t_{k-1}, r, x, y, z)1_{[t_{k-1}, t_k)}(\tau), \\
&\quad (\tau, r, x, y, z) \in D[t, T] \times \mathbb{R}^n \times U_1 \times U_2 \times \mathbb{R} \times \mathbb{R}^d, \\
&\mathbb{H}^\Pi(\tau, r, x, u_1, u_2, u, \theta, p, P) = \operatorname{tr} [a(r, x, u_1, u_2)P] + \langle b(r, x, u_1, u_2), p \rangle \\
&\quad + g(\tau, r, x, u_1, u_2, \theta, p^\top \sigma(r, x, u_1, u_2)), \\
&\quad (\tau, r, x, u_1, u_2, u, \theta, p, P) \in D[t, T] \times \mathbb{R}^n \times U_1 \times U_2 \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n,
\end{align*}
$$

$$
\begin{align*}
&\Theta^\Pi(\tau, r, x) = \sum_{k=1}^N \Theta^k(r, x)1_{[t_{k-1}, t_k)}(\tau), \quad (\tau, r, x) \in D[t, T] \times \mathbb{R}^n.
\end{align*}
$$
Using the above new notations, the control law \(X\) corresponding equilibrium state process follows BSDE:
\[
\begin{aligned}
\Theta^\Pi(t,r,x) \Theta^\Pi_x(t,r,x) & = 0, \\
\Theta^\Pi_x(t,r,x) & \in D[t,T] \times \mathbb{R}^n, \\
\Theta^\Pi(t,T,x) & = \bar{h}(r,x), \quad (r,x) \in [t,T] \times \mathbb{R}^n.
\end{aligned}
\tag{4.26}
\]

Putting (4.29) and (4.30) together, we obtain the decoupled FBSDE system (3.2) actually.

\[
\begin{align}
\varphi_1 & = \sum_{k=1}^N \varphi_1(t_{k-1}, r, x, \Theta^k(r,x), \Theta^k_x(r,x), \Theta^k_x(r,x)) \mathbb{I}_{[t_{k-1},t_k)}(r), \\
& = \varphi_1 \left( \sum_{k=1}^N t_{k-1} \mathbb{I}_{[t_{k-1},t_k)}(r), r,x, \sum_{k=1}^N \Theta^k(r,x) \mathbb{I}_{[t_{k-1},t_k)}(r), \\
& \sum_{k=1}^N \Theta^k_x(r,x) \mathbb{I}_{[t_{k-1},t_k)}(r), \sum_{k=1}^N \Theta^k_x(r,x) \mathbb{I}_{[t_{k-1},t_k)}(r), \mathbb{I}_{[t_{k-1},t_k)}(r) \right), \\
& = \varphi_1 \left( \mathbb{I}^\Pi(r), r,x, \Theta^\Pi(r,x), \Theta^\Pi_x(r,x), \Theta^\Pi_x(r,x), \Theta^\Pi_x(r,x) \right), \\
& (r,x) \in [t,T] \times \mathbb{R}^n.
\end{align}
\tag{4.27}
\]

Similarly, the strategy law \(u^\Pi\) in the lower saddle point can be rewritten as
\[
\begin{align}
\psi_2 & = \psi_2 \left( \mathbb{I}^\Pi(r), r,x, u_1, \Theta^\Pi(r,x), \Theta^\Pi_x(r,x), \Theta^\Pi_x(r,x) \right), \\
& (r,x,u_1) \in [t,T] \times \mathbb{R}^n \times U_1.
\end{align}
\tag{4.28}
\]

Obeying the control-strategy law \((u^\Pi, \alpha^\Pi)\) given by (4.27) and (4.28), the corresponding equilibrium state process \(X^\Pi(\cdot)\) satisfies the following SDE:
\[
\begin{aligned}
dX^\Pi(r) & = \bar{b}(r, X^\Pi(r), u^\Pi, \alpha^\Pi)dr + \bar{\sigma}(r, X^\Pi(r), u^\Pi, \alpha^\Pi)dW(r), \quad r \in [t,T], \\
X^\Pi(t) & = \bar{\zeta}.
\end{aligned}
\tag{4.29}
\]

We notice that, for any \(1 \leq k \leq N\),
\[
X^\Pi(r) = \bar{X}^k(r; t_{k-1}, X^\Pi(t_{k-1}), u^\Pi, \alpha^\Pi), \quad r \in [t_{k-1}, t_k).
\]

Moreover, for any \(t_k \in \Pi \setminus \{t_N\}\), let \((Y^\Pi(t_k, \cdot), Z^\Pi(t_k, \cdot))\) be the solution to the following BSDE:
\[
\begin{aligned}
dY^\Pi(t_k, r) & = -g(t_k, r, X^\Pi(r), u^\Pi, \alpha^\Pi, Y^\Pi(t_k, r), Z^\Pi(t_k, r))dr \\
& \quad + Z^\Pi(t_k, r)dW(r), \quad r \in [t_k,T], \\
Y^\Pi(t_k, T) & = \bar{h}(t_k, X^\Pi(T)),
\end{aligned}
\tag{4.30}
\]

and then the corresponding lower value is
\[
J(t_k, X^\Pi(t_k); u^\Pi, \alpha^\Pi) = Y^\Pi(t_k, t_k).
\tag{4.31}
\]

Similarly, the following relationship holds:
\[
\begin{align}
Y^\Pi(t_k, r) & = \bar{Y}^{k+1}(r; t_k, X^\Pi(t_k), u^\Pi, \alpha^\Pi), \\
Z^\Pi(t_k, r) & = \bar{Z}^{k+1}(r; t_k, X^\Pi(t_k), u^\Pi, \alpha^\Pi), \quad r \in [t_k,T].
\end{align}
\tag{4.32}
\]

Putting (4.29) and (4.30) together, we obtain the decoupled FBSDE system (3.2) actually.
5. Equilibrium saddle points and equilibrium HJBI equations. In Section 4, for any initial pair \((t, \xi) \in \mathcal{D}\), any partition \(\Pi\) of the time interval \([t, T]\), a backward sequence of time-consistent subgames (Problem \((G^\Pi)\)) associated with Problem \((\text{InC-SDG})\) was studied. In this section, we focus on the limit behaviors of Problem \((G^\Pi)\) when the mesh size of partition \(\Pi\) tends to zero, which will provide a solution to Problem \((\text{InC-SDG})\).

5.1. The formal limits. In this subsection, we study the limit behaviors formally to get the limit equations. In Subsection 5.2, we will show the formal limits can be made rigorously under some conditions. Temporarily, we assume the following assumption.

\((\text{TA})\). There exists some \(\Theta(\cdot, \cdot, \cdot) \in C^{0,0,2}(D[0, T] \times \mathbb{R}^n)\) such that

\[
\lim_{\|\Pi\| \to 0} \left( |\Theta^\Pi(\tau, r, x) - \Theta(\tau, r, x)| + |\Theta^\Pi_x(\tau, r, x) - \Theta_x(\tau, r, x)| + |\Theta^\Pi_{xx}(\tau, r, x) - \Theta_{xx}(\tau, r, x)| \right) = 0
\]

uniformly for \((\tau, r, x)\) in any compact set of \(D[t, T] \times \mathbb{R}^n\) with any \(t \in [0, T]\).

With the help of Assumption (TA) and some standard estimates for SDEs and BSDEs, the following convergences are easily to be obtained.

**Lemma 5.1.** Let Assumptions \((A1), (A2), \) and \((\text{TA})\) hold true.

(i). We have

\[
\lim_{\|\Pi\| \to 0} u^\Pi_1(r, x) = u_1(r, x), \quad (r, x) \in [t, T] \times \mathbb{R}^n,
\]

\[
\lim_{\|\Pi\| \to 0} \alpha^\Pi_2(r, x, u_1) = \alpha_2(r, x, u_1), \quad (r, x, u_1) \in [t, T] \times \mathbb{R}^n \times U_1,
\]

uniformly for \((r, x, u_1)\) in any compact set, where

\[
\begin{aligned}
&u_1(r, x) = \varphi_1(r, x, \Theta(r, r, x), \Theta_x(r, r, x), \Theta_{xx}(r, r, x)), \quad (r, x) \in [t, T] \times \mathbb{R}^n, \\
&\alpha_2(r, x, u_1) = \psi_2(r, x, u_1, \Theta(r, r, x), \Theta_x(r, r, x), \Theta_{xx}(r, r, x)), \\
&(r, x, u_1) \in [t, T] \times \mathbb{R}^n \times U_1.
\end{aligned}
\]

(ii). Moreover,

\[
\lim_{\|\Pi\| \to 0} \|X^\Pi(\cdot) - \bar{X}(\cdot)\|_{L^2(\Omega; C([t, T]; \mathbb{R}^n))} = 0,
\]

\[
\lim_{\|\Pi\| \to 0} \left( \|Y^\Pi(t^\Pi(s), \cdot) - \bar{Y}(s, \cdot)\|_{L^2(\Omega; C([t, T]; \mathbb{R}^n))} + \|Z^\Pi(t^\Pi(s), \cdot) - \bar{Z}(s, \cdot)\|_{L^2(t, T; \mathbb{R}^n \times \mathbb{R}^n)} \right) = 0, \quad \forall \ s \in [t, T],
\]

where \((\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))\) is the solution to FBSDE system \((3.1)\).

Lemma 5.1 and \((4.16)\) imply that, the limit function \((u_1, \alpha_2)\) defined by \((5.1)\) is an equilibrium lower saddle point to Problem \((\text{InC-SDG})\) with the initial pair \((t, \xi) \in \mathcal{D}\), and \((u^\Pi_1, \alpha^\Pi_2)\) is an approximate equilibrium lower saddle point.

Furthermore, by the verification theorem (applying to Problem \((C_1)\)), \((4.31), (4.22), \) and \((4.24)\), we have

\[
Y^\Pi(t, t) = J(t, \xi; u^\Pi_1, \alpha^\Pi_2) = V^\Pi(t, \xi) = \Theta^1(t, \xi) = \Theta^\Pi(t, t, \xi).
\]
Taking limits on the both sides, by Lemma 5.1 and Assumption (TA),
\[ \hat{Y}(t, t) = J(t, \xi; u_1, \alpha_2) = \Theta(t, t, \xi). \]
By the arbitrariness of \((t, \xi)\), from Definition 3.1,
\[ V^-(t, x) \equiv \Theta(t, t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^n \]
is an equilibrium lower value function to Problem (InC-SDG).

Next, we want to derive the limit equation of (4.26) to characterize the equilibrium lower value function. For this aim, the following assumption is needed:

\textbf{(A3).} There exists a constant \(K > 0\) such that
\[ |h_\tau(\tau, x)| + |g_\tau(\tau, r, x, u_1, r, y, z)| \leq K, \quad (\tau, r, x, u_1, r, y, z) \in D[0, T] \times \mathbb{R}^n \times U_1 \times \mathbb{R}^1 \times \mathbb{R}^d. \]

Under Assumption (A3), for any \(t \in [0, T]\), we have
\[
\begin{align*}
& \lim_{||\Pi|| \to 0} \left[ \Theta^H(\tau, r, x, u_1, \alpha_2), \Theta^H_x(\tau, r, x, \alpha_2), \Theta^H_{xx}(\tau, r, x) \right] \\
& = g(\tau, r, x, u_1, \alpha_2), \quad \Theta(\tau, r, x) \in D[0, T] \times \mathbb{R}^n.
\end{align*}
\]

Then, by letting \(||\Pi|| \to 0\) in (4.26), we find function \(\Theta(\cdot, \cdot, \cdot)\) satisfies the following equation:
\[
\begin{align*}
\Theta(\tau, r, x) + H(\tau, r, x, u_1, \alpha_2, \Theta(\tau, r, x)) \\
& = 0, \quad (\tau, r, x) \in D[0, T] \times \mathbb{R}^n, \quad (5.3)
\end{align*}
\]
with \((u_1, \alpha_2)\) being in (5.1) with \(t = 0\). Similar with the time-inconsistent optimal control theory, (5.3) is called the \textit{equilibrium lower Hamilton-Jacobi-Isaacs equation} (equilibrium lower HJBI equation, for short). To sum up, we give the following result.

\textbf{Theorem 5.2.} Let Assumptions (A1)-(A3) hold. Suppose that the equilibrium lower HJBI equation (5.3) admits a solution denoted by \(\Theta(\cdot, \cdot, \cdot)\) satisfying Assumption (TA). Then, for any initial pair \((t, \xi) \in \mathcal{D}\), \((u_1, \alpha_2)\) defined by (5.1) is an equilibrium lower saddle point for Problem (InC-SDG) with \((t, \xi)\). Moreover, \(V^-(\cdot, \cdot)\) defined by (5.2) is an equilibrium lower value function to Problem (InC-SDG).

\textbf{Remark 3.} Similar to the analysis in Remark 1, since the equilibrium lower saddle point \((u_1, \alpha_2)\) (see (5.1)) is defined through the equilibrium lower HJBI equation, then it is time-consistent!
5.2. Well-posedness of the equilibrium HJBI equation. Let us make some observations on (5.3). By substituting (5.1) into (5.3), we get an equivalent form as follows:

\[
\begin{align*}
\Theta_r(\tau, r, x) &+ \text{tr} \left( \tilde{a}(r, x, \Theta(\tau, r, x), \Theta_x(r, r, x), \Theta_{xx}(r, r, x)) \Theta_{xx}(\tau, r, x) \right) \\
+ \tilde{b}(r, x, \Theta(\tau, r, x), \Theta_x(r, r, x), \Theta_{xx}(r, r, x)) &+ \tilde{g}(\tau, r, x, \Theta(\tau, r, x), \Theta_x(r, r, x), \Theta_{xx}(r, r, x)) = 0, \\
(\tau, r, x) &\in D[0, T] \times \mathbb{R}^n,
\end{align*}
\]

where

\[
\begin{align*}
\tilde{b}(r, x, \bar{\theta}, \bar{p}, \bar{P}) &= b \left( r, x, \varphi_1(r, r, x, \bar{\theta}, \bar{p}, \bar{P}), \psi_2(r, r, x, \varphi_1(r, r, x, \bar{\theta}, \bar{p}, \bar{P}), \bar{\theta}, \bar{p}, \bar{P}) \right), \\
\tilde{\sigma}(r, x, \bar{\theta}, \bar{p}, \bar{P}) &= \sigma \left( r, x, \varphi_1(r, r, x, \bar{\theta}, \bar{p}, \bar{P}), \psi_2(r, r, x, \varphi_1(r, r, x, \bar{\theta}, \bar{p}, \bar{P}), \bar{\theta}, \bar{p}, \bar{P}) \right), \\
\tilde{a}(r, x, \bar{\theta}, \bar{p}, \bar{P}) &= \frac{1}{2} \sigma \left( r, x, \bar{\theta}, \bar{p}, \bar{P} \right) \sigma \left( r, x, \bar{\theta}, \bar{p}, \bar{P} \right)^\top, \\
\tilde{g}(\tau, r, x, \bar{\theta}, \bar{p}, \bar{P}) &= g \left( \tau, r, x, \varphi_1(r, r, x, \bar{\theta}, \bar{p}, \bar{P}), \psi_2(r, r, x, \varphi_1(r, r, x, \bar{\theta}, \bar{p}, \bar{P}), \bar{\theta}, \bar{p}, \bar{P}) \right), \\
&\quad \left( \Theta, \bar{\theta}, \bar{p}, \bar{P} \right) \in D[0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n \times \mathbb{R}^n \times \mathbb{R}^n.
\end{align*}
\]

It is clear that system (5.4) is no longer a classical PDE, which turns out to not only be fully nonlinear, but also includes both \( \Theta(\tau, r, x) \) and \( \Theta(\tau, r, x) \) at the same time. Note that \( \Theta(\tau, r, x) \) is the restriction of \( \Theta(\tau, r, x) \) on \( \tau = r \). Therefore, we can not apply the classical results of PDEs directly to get the well-posedness.

In fact, the appearance of \( \Theta(\tau, r, x) \), \( \Theta_x(r, r, x) \) and \( \Theta_{xx}(r, r, x) \) in (5.4) will bring us some difficulties in the studying of the regularity properties which are necessary for the well-posedness of the equilibrium HJBI equation. At the moment, we are not able to overcome the difficulty from \( \Theta_{xx}(r, r, x) \). To avoid this difficulty, we need the following Assumption

(A4). \( \sigma(r, x, u_1, u_2) = \sigma(r, x), \quad (r, x, u_1, u_2) \in [0, T] \times \mathbb{R}^n \times U_1 \times U_2 \).

Noticing the definition of functions \( \varphi_1 \) and \( \psi_2 \), Assumption (A4) implies \( \varphi_1 \) and \( \psi_2 \) are independent of \( \bar{P} \in \mathbb{S}^n \). In other words, in this case, \( \tilde{a}(r, x, \bar{\theta}, \bar{p}, \bar{P}) = a(r, x), \quad (r, x) \in [0, T] \times \mathbb{R}^n \), and the functions \( \tilde{b} \) and \( \tilde{g} \) are independent of \( \bar{P} \in \mathbb{S}^n \). Therefore (5.4) is reduced to

\[
\begin{align*}
\Theta_r(\tau, r, x) &+ \text{tr} \left[ a(r, x) \Theta_{xx}(r, r, x) \right] + \tilde{b}(r, x, \Theta(\tau, r, x), \Theta_x(r, r, x)) \Theta_x(\tau, r, x) \\
+ \tilde{g}(\tau, r, x, \Theta(\tau, r, x), \Theta_x(r, r, x), \Theta_{xx}(r, r, x)) &= 0, \\
(\tau, r, x) &\in D[0, T] \times \mathbb{R}^n,
\end{align*}
\]

Comparing with (5.4), \( \Theta_{xx}(r, r, x) \) does not appear in the equation (5.6), which reduces significantly the difficulty in the issue of the solvability of equations. Moreover, the well-posedness of (5.6) was established already in [15], and the result will
be recalled below. For the proof and more details, the interested readers please refer to [15].

The following assumption is also introduced for obtaining some estimates in the proof of the unique solvability for (5.6).

\((A5)\). The functions \(a, \tilde{b}, \tilde{g}\) and \(h\) are continuous and bounded. Moreover, there exists a constant \(L > 0\) such that

\[
|a_x(r, x)| + |\tilde{b}_x(r, x, \tilde{\theta}, \tilde{p})| + |\tilde{g}_x(\tau, r, x, \tilde{\theta}, p)| + |\tilde{\tilde{g}}_x(t, x, \tilde{\theta}, \tilde{p})| + |\tilde{\tilde{g}}_\tilde{p}(\tau, r, x, \tilde{\theta}, \tilde{p}, \tilde{\theta}_\tilde{p})|
\]

Furthermore, \(a(r, x)^{-1}\) exists for all \((r, x) \in [0, T] \times \mathbb{R}^n\), and there exist constants \(\lambda_0, \lambda_1 > 0\) such that

\[
\lambda_0 I \leq a(r, x)^{-1} \leq \lambda_1 I, \quad (r, x) \in [0, T] \times \mathbb{R}^n.
\]

**Theorem 5.3** (Wei-Yong-Yu [15]). Under Assumption \((A5)\), (5.6) admits a unique solution \(\Theta(\cdot, \cdot, \cdot) \in C^{0,1,2}(D[0, T] \times \mathbb{R}^n)\).

In the previous subsection, we introduce Assumption (TA) temporarily to ensure some associated convergences. Now it is the time to get rid of it. Let us introduce the following

\((TA')\). There exists some \(\Theta(\cdot, \cdot, \cdot) \in C^{0,0,2}(D[0, T] \times \mathbb{R}^n)\) such that

\[
\lim_{||\Pi|| \to 0} \left( |\Theta^\Pi(\tau, r, x) - \Theta(\tau, r, x)| + |\Theta^\Pi_x(\tau, r, x) - \Theta_x(\tau, r, x)| \right) = 0
\]

uniformly for \((\tau, r, x)\) in any compact set of \(D[t, T] \times \mathbb{R}^n\) with any \(t \in [0, T]\).

Due to Assumption \((A4)\), the functions \(\varphi_1\) and \(\psi_2\) are independent of the variable \(P \in \mathbb{S}^n\). Therefore, if we replace Assumption (TA) by the new Assumption (TA’), all the convergences and results in Subsection 5.1 still hold. Furthermore, a similar analysis with [15, Theorem 6.2] leads to the following result. Here, we omit the proof.

**Lemma 5.4.** Under Assumptions \((A1)-(A5)\), the unique solution \(\Theta(\cdot, \cdot, \cdot)\) to the equilibrium lower HJBI equation (5.3) satisfies Assumption \((TA')\).

Combining Theorem 5.2, Theorem 5.3, and Lemma 5.4, we give the following verification theorem for Problem (InC-SDG).

**Theorem 5.5.** Let Assumptions \((A1)-(A5)\) hold. Then the equilibrium lower HJBI equation (5.3) admits a unique solution \(\Theta(\cdot, \cdot, \cdot)\). Moreover, for any initial pair \((t, \xi) \in \mathcal{D}, (\tau_1, \tau_2)\) defined by (5.1) is an equilibrium lower saddle point for Problem (InC-SDG) with \((t, \xi)\). Furthermore, \(V^-(\cdot, \cdot)\) defined by (5.2) is the corresponding equilibrium lower value function to Problem (InC-SDG).

6. **Conclusions on equilibrium upper saddle points.** By now, Problem (InC-SDG)-(i) has been solved completely. The same approach can be applied to solve Problem (InC-SDG)-(ii). In this section, we state the corresponding conclusions on Problem (InC-SDG)-(ii).
We introduce the following PDE called the \textit{equilibrium upper HJBI equation}:\[
\begin{align*}
\Lambda(\tau, r, x) + \mathbb{H}(\tau, r, x, u_1(\tau, r, x), u_2(\tau, r, x)), u_1(\tau, r, x), u_2(\tau, r, x),
\Lambda(\tau, r, x), \Lambda_x(\tau, r, x), \Lambda_{xx}(\tau, r, x)) = 0, \\
(\tau, r, x) \in D[0, T] \times \mathbb{R}^n,
\Lambda(\tau, T, x) = h(\tau, x), \quad (\tau, x) \in [0, T] \times \mathbb{R}^n,
\end{align*}
\]
where
\[
\begin{align*}
u_1(\tau, r, u_2) = \psi_1(r, r, x, u_1, \Lambda(r, r, x), \Lambda_x(r, x), \Lambda_{xx}(r, x)), \\
u_2(\tau, r, x) = \varphi_2(r, r, x, \Lambda(r, r, x), \Lambda_x(r, r, x), \Lambda_{xx}(r, r, x)), \quad (r, x) \in [0, T] \times \mathbb{R}^n.
\end{align*}
\]

For the corresponding assumptions, firstly, we notice that Assumption (A2)-(i) has been applied to solve Problem (InC-SDG)-(i). Now for Problem (InC-SDG)-(ii), we need introduce a couple of new functions \(\tilde{b}\) and \(\tilde{y}\) depending on \(\psi_1\) and \(\varphi_2\), and then propose a new Assumption (A5'). However, in order to avoid repeat, we would like to omit the detailed statement of Assumption (A5').

With the above preparation, now we give the corresponding verification theorem for the equilibrium upper saddle points.

\textbf{Theorem 6.1.} Let Assumptions (A1)-(A4) and (A5') hold. Then the equilibrium upper HJBI equation (6.1) admits a unique solution \(\Lambda(\cdot, \cdot, \cdot)\). Moreover, for any initial pair \((t, \xi) \in \mathcal{D}\), \((u_1(t, \xi), u_2(t, \xi))\) (where \((u_1, u_2)\) defined by (6.2)) is an equilibrium upper saddle point for Problem (InC-SDG) with \((t, \xi)\). Furthermore,
\[
V^+(t, x) = \Lambda(t, t, x), \quad (t, x) \in [0, T] \times \mathbb{R}
\]
is the corresponding equilibrium upper value function to Problem (InC-SDG).

\textbf{Remark 4.} Similar with the classical time-consistent zero-sum differential game theory, we can introduce similarly the following Isaacs condition:
\[
\inf_{u_2 \in U_2} \sup_{u_1 \in U_1} \mathbb{H}(\tau, r, x, u_1, u_2, \theta, p, P) = \sup_{u_2 \in U_2} \inf_{u_1 \in U_1} \mathbb{H}(\tau, r, x, u_1, u_2, \theta, p, P),
(\tau, r, x, u_1, u_2, \theta, p, P) \in D[0, T] \times \mathbb{R}^n \times U_1 \times U_2 \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n.
\]
Under the above Isaacs condition, it turns out that (5.3) and (6.1) are the same equation. Therefore, \(\Theta(\cdot, \cdot, \cdot) \equiv \Lambda(\cdot, \cdot, \cdot)\). Consequently, \(V^{-}(\cdot, \cdot) \equiv V^{+}(\cdot, \cdot)\). Then, for any \((t, \xi) \in \mathcal{D}\), the equilibrium lower value and the equilibrium upper value are equal. We say that Problem (InC-SDG) with \((t, \xi)\) has an equilibrium value
\[
V(t, \xi) = V^{-}(t, \xi) = \Theta(t, t, \xi) = V^{+}(t, \xi) = \Lambda(t, t, \xi).
\]
Moreover, \(V(\cdot, \cdot)\) is called an equilibrium value function of Problem (InC-SDG).

\textbf{REFERENCES}

[1] T. Björk and A. Murgoci, A theory of Markovian time-inconsistent stochastic control in discrete time, \textit{Finance Stoch.}, \textbf{18} (2014), 545–592.
[2] T. Björk, A. Murgoci and X. Zhou, Mean-variance portfolio optimization with state-dependent risk aversion, \textit{Math. Finance}, \textbf{24} (2014), 1–24.
[3] R. Buckdahn and J. Li, Stochastic differential games and viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs equation, \textit{SIAM J. Control Optim.}, \textbf{47} (2008), 444–475.
[4] D. Duffie and L. G. Epstein, Stochastic differential utility, Econometrica, 60 (1992), 353–394.
[5] I. Ekeland and A. Lazrak, The golden rule when preferences are time inconsistent, Math. Financ. Econ., 4 (2010), 29–55.
[6] N. El Karoui, S. Peng and M. C. Quenez, Backward stochastic differential equations in finance, Math. Finance, 7 (1997), 1–71.
[7] R. Elliott and N. J. Kalton, Values in differential games, Bulletin of the American Mathematical Society, 78 (1972), 427–431.
[8] L. C. Evans and P. E. Souganidis, Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations, Indiana Univ. Math. J., 33 (1984), 773–797.
[9] W. H. Fleming and P. E. Souganidis, On the existence of value functions of two-player, zero-sum stochastic differential games, Indiana Univ. Math. J., 38 (1989), 293–314.
[10] Y. Hu, H. Jin and X. Zhou, Time-inconsistent stochastic linear-quadratic control, SIAM J. Control Optim., 50 (2012), 1548–1572.
[11] J. Ma, P. Protter and J. Yong, Solving forward-backward stochastic differential equations explicitly - a four step scheme, Probab. Theory Related Fields, 98 (1994), 339–359.
[12] E. Pardoux and S. Peng, Backward stochastic differential equations and quasi-linear parabolic partial differential equations, in: Stochastic Partial Differential Equations and their Applications. Lect. Notes in Control & Info. Sci. (eds. B.L. Rozovskii and R.S. Sowers), 176, Springer, Berlin, Heidelberg, 1992, 200–217.
[13] S. Peng, Probabilistic interpretation for systems of quasilinear parabolic partial differential equations, Stochastics Stochastics Rep., 37 (1991), 61–74.
[14] R. A. Pollak, Consistent planning, Review of Economic Studies, 35 (1968), 201–208.
[15] Q. Wei, J. Yong and Z. Yu, Time-inconsistent recursive stochastic optimal control problems, SIAM J. Control Optim., 55 (2017), 4156–4201.
[16] J. Yong, A deterministic linear quadratic time-inconsistent optimal control problem, Math. Control Relat. Fields, 1 (2011), 83–118.
[17] J. Yong, Time-inconsistent optimal control problems and the equilibrium HJB equation, Math. Control Relat. Fields, 2 (2012), 271–329.
[18] J. Yong, Time-inconsistent optimal control problems, Proceedings of 2014 ICM, Section 16. Control Theory and Optimization, 4 (2014), 947–969.
[19] J. Yong, Linear-quadratic optimal control problems for mean-field stochastic differential equations — time-consistent solutions, Trans. Amer. Math. Soc., 369 (2017), 5467–5523.
[20] J. Yong and X. Zhou, Stochastic Controls. Hamiltonian Systems and HJB Equations, Applications of Mathematics (New York), 43, Springer-Verlag, New York, 1999.
[21] Z. Yu, An optimal feedback control-strategy pair for zero-sum linear-quadratic stochastic differential game: The Riccati equation approach, SIAM J. Control Optim., 53 (2015), 2141–2167.

Received August 2017; revised March 2018.

E-mail address: weiqm100@nenu.edu.cn
E-mail address: yuzhiyong@sdu.edu.cn