Uniform error estimates for artificial neural network approximations for heat equations

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Abstract

Recently, artificial neural networks (ANNs) in conjunction with stochastic gradient descent optimization methods have been employed to approximately compute solutions of possibly rather high-dimensional partial differential equations (PDEs). Very recently, there have also been a number of rigorous mathematical results in the scientific literature which examine the approximation capabilities of such deep learning based approximation algorithms for PDEs. These mathematical results from the scientific literature prove in part that algorithms based on ANNs are capable of overcoming the curse of dimensionality in the numerical approximation of high-dimensional PDEs. In these mathematical results from the scientific literature usually the error between the solution of the PDE and the approximating ANN is measured in the $L^p$-sense with respect to some $p \in [1, \infty)$ and some probability measure. In many applications it is, however, also important to control the error in a uniform $L^\infty$-sense. The key contribution of the main result of this article is to develop the techniques
to obtain error estimates between solutions of PDEs and approximating ANNs in the uniform $L^\infty$-sense. In particular, we prove that the number of parameters of an ANN to uniformly approximate the classical solution of the heat equation in a region $[a, b]^d$ for a fixed time point $T \in (0, \infty)$ grows at most polynomially in the dimension $d \in \mathbb{N}$ and the reciprocal of the approximation precision $\varepsilon > 0$. This verifies that ANNs can overcome the curse of dimensionality in the numerical approximation of the heat equation when the error is measured in the uniform $L^\infty$-norm.

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1 Introduction

Artificial neural networks (ANNs) play a central role in machine learning applications such as computer vision (cf., e.g., [36, 40, 58]), speech recognition (cf., e.g., [27, 35, 61]), game intelligence (cf., e.g., [56, 57]), and finance (cf., e.g., [7, 12, 59]). Recently, ANNs in conjunction with stochastic gradient descent optimization methods have also been employed to approximately compute solutions of possibly rather high-dimensional partial differential equations (PDEs); cf., for example, [4, 5, 6, 7, 8, 10, 13, 14, 17, 18, 19, 22, 23, 26, 32, 33, 34, 37, 41, 48, 49, 50, 53, 60, 63, 64] and the references mentioned therein. The numerical simulation results in the above named references indicate that such deep learning based approximation
methods for PDEs have the fundamental power to overcome the curse of dimensionality (cf., e.g., Bellman [9]) in the sense that the precise number of the real parameters of the approximating ANN grows at most polynomially in both the dimension \( d \in \mathbb{N} = \{1, 2, 3, \ldots \} \) of the PDE under consideration and the reciprocal \( \varepsilon^{-1} \) of the prescribed approximation precision \( \varepsilon > 0 \). Very recently, there have also been a number of rigorous mathematical results examining the approximation capabilities of these deep learning based approximation algorithms for PDEs (see, e.g., [11, 20, 28, 30, 38, 43, 47, 51, 60]). These works prove in part that algorithms based on ANNs are capable of overcoming the curse of dimensionality in the numerical approximation of high-dimensional PDEs. In particular, the works [11, 20, 28, 30, 38, 47, 51] provide mathematical convergence results of such deep learning based numerical approximation methods for PDEs with dimension-independent error constants and convergence rates which depend on the dimension only polynomially.

Except of in the article Elbrächter et al. [20], in each of the approximation results in the above cited articles [11, 20, 28, 30, 38, 43, 47, 51, 60] the error between the solution of the PDE and the approximating ANN is measured in the \( L^p \)-sense with respect to some \( p \in [1, \infty) \) and some probability measure. In many applications it is, however, also important to control the error in a uniform \( L^\infty \)-sense. This is precisely the subject of this article. More specifically, it is the key contribution of Theorem 5.4 in Subsection 5.2 below, which is the main result of this article, to prove that ANNs can overcome the curse of dimensionality in the numerical approximation of the heat equation when the error is measured in the uniform \( L^\infty \)-norm. The arguments used to prove the approximation results in the above cited articles, where the error between the solution of the PDE and the approximating ANN is measured in the \( L^p \)-sense with respect to some \( p \in [1, \infty) \) and some probability measure, can not be employed for the uniform \( L^\infty \)-norm approximation and the article Elbrächter et al. [20] is concerned with a specific class of PDEs so that the PDEs can essentially be solved analytically and the error analysis in [20] strongly exploits this explicit solution representation. The key contribution of the main result of this article, Theorem 5.4 in Subsection 5.2 below, is to develop the techniques to obtain error estimates between solutions of PDEs and approximating ANNs in the uniform \( L^\infty \)-sense. To illustrate the findings of the main result of this article in more detail, we formulate in the next result a particular case of Theorem 5.4.

**Theorem 1.1.** Let \( a \in \mathbb{R}, b \in (a, \infty), c, T \in (0, \infty), a \in C^1(\mathbb{R}, \mathbb{R}), \) let \( A_d : \mathbb{R}^d \to \mathbb{R}^d, d \in \mathbb{N}, \) satisfy for all \( d \in \mathbb{N}, x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) that \( A_d(x) = (a(x_1), \ldots, a(x_d)) \), let \( N = \bigcup_{L \in \mathbb{N} \setminus \{2, \infty\}} \bigcup_{l_0, \ldots, l_L \in \mathbb{N}} (\times_{k=1}^L (\mathbb{R}^{l_k} \times \mathbb{R}^{l_k})) \), let \( \mathcal{P} : N \to \mathbb{N} \) and \( \mathcal{R} : N \to \bigcup_{m, n \in \mathbb{N}} C(\mathbb{R}^m, \mathbb{R}^n) \) satisfy for all \( L \in \mathbb{N} \cap [2, \infty), l_0, \ldots, l_L \in \mathbb{N}, \Phi = ((W_1, B_1), \ldots, (W_L, B_L)) \in \bigtimes_{k=1}^L (\mathbb{R}^{l_k} \times \mathbb{R}^{l_k})) \), \( x_0 \in \mathbb{R}^{l_0}, \ldots, x_{L-1} \in \mathbb{R}^{l_{L-1}} \) with \( \forall k \in \mathbb{N} \cap (0, L) : x_k = A_{l_k}(W_{k}x_{k-1} + B_k) \) that \( \mathcal{P}(\Phi) = \sum_{k=1}^L l_k(l_{k-1} + 1), (\mathcal{R}\Phi) \in C(\mathbb{R}^{l_0}, \mathbb{R}^{l_{L}}), \) and \( (\mathcal{R}\Phi)(x_0) = W_Lx_{L-1} + B_L, \) let \( \varphi_d \in C(\mathbb{R}^{d'}, \mathbb{R}), d \in \mathbb{N}, \) let \( \varphi_{d}(x) \in (0, 1) \times \mathbb{N} \subseteq \mathbb{N}, \) let \( |||| : \bigcup_{d \in \mathbb{N}} C(\mathbb{R}) \to [0, \infty) \) satisfy for all \( d \in \mathbb{N}, x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) that \( ||x|| = \left[ \sum_{j=1}^d |x_j|^2 \right]^{1/2}, \)
and assume for all \( \varepsilon \in (0, 1] \), \( d \in \mathbb{N} \), \( x \in \mathbb{R}^d \) that

\[
(\mathcal{R}\phi_{\varepsilon,d}) \in C(\mathbb{R}^d, \mathbb{R}), \quad \|(\mathcal{R}\phi_{\varepsilon,d})(x)\| + \|(\nabla(\mathcal{R}\phi_{\varepsilon,d}))(x)\| \leq cd^r(1 + \|x\|^c),
\]

and

\[
\mathcal{P}(\phi_{\varepsilon,d}) \leq cd^{r}\varepsilon^{-c}, \quad |\varphi_d(x) - (\mathcal{R}\phi_{\varepsilon,d})(x)| \leq \varepsilon cd^r(1 + \|x\|^c).
\]

Then

(i) there exist unique at most polynomially growing \( u_d \in C([0, T] \times \mathbb{R}^d, \mathbb{R}) \), \( d \in \mathbb{N} \), which satisfy for all \( d \in \mathbb{N} \), \( t \in (0, T] \), \( x \in \mathbb{R}^d \) that \( u_d(0, t, x) = \varphi_d(x) \), and

\[
\left(\frac{\partial}{\partial t} u_d\right)(t, x) = (\nabla_x u_d)(t, x)
\]

and

(ii) there exist \( (\psi_{\varepsilon,d})_{(\varepsilon,d) \in (0,1] \times \mathbb{N}} \subseteq \mathbb{N} \) and \( \kappa \in \mathbb{R} \) such that for all \( \varepsilon \in (0, 1] \), \( d \in \mathbb{N} \) we have that \( \mathcal{P}(\psi_{\varepsilon,d}) \leq \kappa d^c \varepsilon^{-c} \), \( (\mathcal{R}\psi_{\varepsilon,d}) \in C(\mathbb{R}^d, \mathbb{R}) \), and

\[
\sup_{x \in [a,b]^d} |u_d(T, x) - (\mathcal{R}\psi_{\varepsilon,d})(x)| \leq \varepsilon.
\]

Theorem 1.1 follows directly from Corollary 5.5 in Subsection 5.2 below. Corollary 5.5 in turn, is a consequence of Theorem 5.1 in Subsection 5.2, the main result of the article. Let us add a few comments on some of the mathematical objects appearing in Theorem 1.1 above. The real number \( T \in (0, \infty) \) denotes the time horizon on which we consider the heat equations in (3). The function \( a \in C^1(\mathbb{R}, \mathbb{R}) \) describes the activation function which we employ for the considered ANN approximations. In particular, Theorem 1.1 applies to ANNs with the standard logistic function as the activation function in which case the function \( a \in C^1(\mathbb{R}, \math{R}) \) in Theorem 1.1 satisfies that for all \( x \in \mathbb{R} \) we have that \( a(x) = (1 + e^{-x})^{-1} \). The set \( \mathbb{N} \) contains all possible ANNs, where each ANN is described abstractly in terms of the number of hidden layers, the number of nodes in each layer, and the values of the parameters (weights and biases in each layer), the function \( \mathcal{P} : \mathbb{N} \to \mathbb{N} \) maps each ANN \( \Phi \in \mathbb{N} \) to its total number of parameters \( \mathcal{P}(\Phi) \), and the function \( \mathcal{R} : \mathbb{N} \to \bigcup_{m,n \in \mathbb{N}} C(\mathbb{R}^m, \mathbb{R}^n) \) maps each ANN \( \Phi \in \mathbb{N} \) to the actual function \((\mathcal{R}\Phi)\) (its realization) associated to \( \Phi \) (cf., e.g., Grohs et al. [29] Section 2.1 and Petersen & Voigtlaender [32] Section 2). Item (ii) in Theorem 1.1 above establishes under the hypotheses of Theorem 1.1 that the solution \( u_d : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) of the heat equation can at time \( T \) be approximated by means of an ANN without the curse of dimensionality. To sum up, roughly speaking, Theorem 1.1 verifies that if the initial conditions of the heat equations can be approximated well by ANNs, then the number of parameters of an ANN to uniformly approximate the classical solution of the heat equation in a region \([a, b] \times \mathbb{R}^d \) for a fixed time point \( T \in (0, \infty) \) grows at most polynomially in the dimension \( d \in \mathbb{N} \) and the reciprocal of the approximation precision \( \varepsilon > 0 \).

In our proof of Theorem 1.1 and Theorem 5.1 respectively, we employ several probabilistic and analytic arguments. In particular, we use a Feynman-Kac
type formula for PDEs of the Kolmogorov type (cf., for example, Hairer et al. [31 Corollary 4.17]), Monte Carlo approximations (cf. Proposition 2.3), Sobolev type estimates for Monte Carlo approximations (cf. Lemma 2.16), the fact that stochastic differential equations with affine linear coefficient functions are affine linear in the initial condition (cf. Grohs et al. [28 Proposition 2.20]), as well as an existence result for realizations of random variables (cf. Grohs et al. [28 Proposition 3.3]).

The rest of this research paper is structured in the following way. In Section 2 below preliminary results on Monte Carlo approximations together with Sobolev type estimates are established. Section 3 contains preliminary results on stochastic differential equations. In Section 4 we employ the results from Sections 2–3 to obtain uniform error estimates for ANN approximations. In Section 5 these uniform error estimates are used to prove Theorem 5.4 in Subsection 5.2 below, the main result of this article.

2 Sobolev and Monte Carlo estimates

2.1 Monte Carlo estimates

In this subsection we recall in Lemma 2.2 below an estimate for the \( p \)-Kahane–Khintchine constant from the scientific literature (cf., for example, Cox et al. [16 Definition 5.4] or Grohs et al. [28 Definition 2.1]). Lemma 2.2 in particular, ensures that the \( p \)-Kahane–Khintchine constant grows at most polynomially in \( p \). Lemma 2.2 will be employed in the proof of Corollary 4.2 in Subsection 4.1 below. Our proof of Lemma 2.2 is based on an application of Hytönen et al. [40 Theorem 6.2.4] and is a slight extension of Grohs et al. [28 Lemma 2.2]. For completeness we also recall in Definition 2.1 below the notion of the Kahane–Khintchine constant (cf., e.g., Cox et al. [16 Definition 5.4]). Proposition 2.3 below is a \( L^p \)-approximation result for Monte-Carlo approximations. This \( L^p \)-approximation result for Monte-Carlo approximations is one of the main ingredients in our proof of Lemma 2.16 in Subsection 2.4 below. Proposition 2.3 is well-known in the literature and is proved, e.g., as Corollary 5.12 in Cox et al. [16].

Definition 2.1. Let \( p \in (0, \infty) \). Then we denote by \( \mathcal{R}_p \in [0, \infty] \) the quantity given by

\[
\mathcal{R}_p = \sup_{c \in [0, \infty]} \left\{ \begin{array}{l}
\exists \text{ probability space } (\Omega, \mathcal{F}, \mathbb{P}):
\exists \text{ } \mathbb{R} \text{-Banach space } (E, \| \cdot \|_E):
\exists \text{ } k \in \mathbb{N} : \exists x_1, \ldots, x_k \in E \setminus \{0\}:
\exists \text{ } \mathbb{P} \text{-Rademacher family } r_j : \Omega \to \{-1, 1\}, j \in \mathbb{N}:
\left( \mathbb{E} \left[ \left\| \sum_{j=1}^k r_j x_j \right\|_E^p \right] \right)^{1/p} = c \left( \mathbb{E} \left[ \left\| \sum_{j=1}^k r_j x_j \right\|_E^2 \right] \right)^{1/2}
\end{array} \right\}
\] (5)
and we refer to $\mathcal{R}_p$ as the $p$-Kahane–Khintchine constant.

**Lemma 2.2.** Let $p \in [1, \infty)$. Then $\mathcal{R}_p \leq \sqrt{\max\{1, p - 1\}}$ (cf. Definition 2.1).

**Proof of Lemma 2.2.** Throughout this proof let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(E, \|\cdot\|_E)$ be a $\mathbb{R}$-Banach space, let $k \in \mathbb{N}$, $x_1, \ldots, x_k \in E \setminus \{0\}$, and let $r_j : \Omega \to \{-1, 1\}$, $j \in \mathbb{N}$, be i.i.d. random variables which satisfy that $\mathbb{P}(r_1 = -1) = \mathbb{P}(r_1 = 1) = \frac{1}{2}$.

Observe that Jensen’s inequality ensures for all $q \in [1, 2]$ that

$$
\left( \mathbb{E} \left[ \left\| \sum_{j=1}^{k} r_j x_j \right\|^q_E \right] \right)^{1/q} = \left( \mathbb{E} \left[ \sum_{j=1}^{k} r_j x_j \right]^{q \cdot 2^{q/2}}_E \right)^{1/q} \leq \left( \mathbb{E} \left[ \sum_{j=1}^{k} r_j x_j \right]^{2}_E \right)^{1/2}.
$$

In addition, observe that \cite[Theorem 6.2.4]{90} (applied with $q \leftarrow q$, $p \leftarrow 2$ for $q \in (2, \infty)$ in the notation of \cite[Theorem 6.2.4]{90}) ensures for all $q \in (2, \infty)$ that

$$
\left( \mathbb{E} \left[ \left\| \sum_{j=1}^{k} r_j x_j \right\|^q_E \right] \right)^{1/q} \leq \sqrt{q - 1} \left( \mathbb{E} \left[ \left\| \sum_{j=1}^{k} r_j x_j \right\|^{2q/2}_E \right] \right)^{1/2}.
$$

Combining this with (7) demonstrates that for all $q \in [1, \infty)$ we have that

$$
\left( \mathbb{E} \left[ \left\| \sum_{j=1}^{k} r_j x_j \right\|^q_E \right] \right)^{1/q} \leq \sqrt{\max\{1, q - 1\}} \left( \mathbb{E} \left[ \left\| \sum_{j=1}^{k} r_j x_j \right\|^{2}_E \right] \right)^{1/2}.
$$

This completes the proof of Lemma 2.2. \qed

**Proposition 2.3.** Let $d, n \in \mathbb{N}$, $p \in [2, \infty)$, let $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X_i : \Omega \to \mathbb{R}^d$, $i \in \{1, \ldots, n\}$, be i.i.d. random variables with $\mathbb{E}[\|X_1\|] < \infty$. Then

$$
\left( \mathbb{E} \left[ \left\| \mathbb{E}[X_1] - \frac{1}{n} \left( \sum_{i=1}^{n} X_i \right) \right\|^{p}_E \right] \right)^{1/p} \leq \frac{2 \mathcal{R}_p (\mathbb{E}[\|X_1\|^{2p}]^{1/p})}{\sqrt{n}}.
$$

(cf. Definition 2.1).

**2.2 Volumes of the Euclidean unit balls**

In this subsection we provide in Corollary 2.8 below an elementary and well-known upper bound for the volumes of the Euclidean unit balls. Corollary 2.8 will be used in our proof of Corollary 2.13 in Subsection 2.3 below. Our proof of Corollary 2.8 employs the elementary and well-known results in Lemmas 2.4, 2.7 below. For completeness we also provide in this subsection detailed proofs for Lemmas 2.4, 2.7 and Corollary 2.8.
Lemma 2.4. Let $\Gamma: (0, \infty) \to (0, \infty)$ and $B: (0, \infty)^2 \to (0, \infty)$ satisfy for all $x, y \in (0, \infty)$ that $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt$ and $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt$ and let $x, y \in (0, \infty)$. Then

(i) we have that $\Gamma(x + 1) = x\Gamma(x)$,

(ii) we have that $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$,

(iii) we have that $\Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, and

(iv) we have that

$$\sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x \leq \Gamma(x) \leq \sqrt{\frac{2\pi}{x}} e^{-1/2x}. \quad (11)$$

Proof of Lemma 2.4. Throughout this proof let $\Phi: (0, \infty) \times (0, 1) \to (0, \infty)^2$ satisfy for all $s \in (0, \infty)$, $t \in (0, 1)$ that

$$\Phi(s, t) = (s(1-t), st) \quad (12)$$

and let $f: (0, \infty)^2 \to (0, \infty)$ satisfy for all $s, t \in (0, \infty)$ that

$$f(s, t) = s^{x-1}t^{y-1}e^{-(s+t)}. \quad (13)$$

Observe that integration by parts verifies that

$$\Gamma(x + 1) = \int_0^\infty t^x e^{-t} \, dt = \left[-t^x e^{-t}\right]_t=0^\infty + x \int_0^\infty t^{x-1}e^{-t} \, dt = x\Gamma(x). \quad (14)$$

This establishes item (i). Next note that Fubini’s theorem ensures that

$$\Gamma(x)\Gamma(y) = \int_0^\infty s^{x-1}e^{-s} \, ds \int_0^\infty t^{y-1}e^{-t} \, dt = \int_0^\infty \int_0^\infty s^{x-1}t^{y-1}e^{-(s+t)} \, ds \, dt = \int_0^\infty \int_0^\infty f(s, t) \, ds \, dt. \quad (15)$$

Moreover, note that for all $s \in (0, \infty)$, $t \in (0, 1)$ we have that

$$\det(\Phi'(s, t)) = s \in (0, \infty). \quad (16)$$

This, (15), the integral transformation theorem (cf., for example, [15, Theorem 6.1.7]), and Fubini’s theorem prove that

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^1 f(\Phi(s, t)) |\det(\Phi'(s, t))| \, dt \, ds$$

$$= \int_0^\infty \int_0^1 (s(1-t))^{x-1}(s(1-t)+st)^{y-1}e^{-(s(1-t)+st)} \, ds \, dt \quad (17)$$

$$= \int_0^\infty s^{x+y-1}e^{-s} \, ds \int_0^1 t^{y-1}(1-t)^{x-1} \, dt = \Gamma(x+y)B(y, x).$$
Next note that the integral transformation theorem with the diffeomorphism 
\( (0, 1) \ni t \mapsto (1 - t) \in (0, 1) \) ensures that
\[
B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} dt = \int_0^1 (1 - t)^{x-1} t^{y-1} dt = B(y, x).
\] (18)
Combining this with (17) establishes item (ii). Next note that
\[
\Gamma(1) = \int_0^\infty t^{-1} e^{-t} dt = \int_0^\infty e^{-t} dt = 1.
\] (19)
Item (ii) and the integral transformation theorem with the diffeomorphism 
\( (0, \pi/2) \ni t \mapsto \sin(t) \in (0, 1) \) therefore verify that
\[
\left[ \frac{\Gamma(1)}{\Gamma(1/2)} \right]^2 = B\left( \frac{1}{2}, \frac{1}{2} \right) = \int_0^1 t^{-1/2} (1 - t)^{-1/2} dt = \int_0^{\pi/2} 2 dt = \pi.
\] (20)
Combining this with (19) establishes item (ii). Next note that Artin [3, Chapter 3, (3.9)] ensures that there exists \( \mu : (0, \infty) \to \mathbb{R} \) which satisfies for all \( t \in (0, \infty) \) that \( 0 < \mu(t) < \frac{1}{2\sqrt{t}} \) and
\[
\Gamma(t) = \sqrt{2\pi t^{1/2} e^{-t/2}} e^{\mu(t)}.
\] (21)
Hence, we obtain that
\[
\sqrt{2\pi x^{1/2} e^{-x}} \leq \Gamma(x) \leq \sqrt{2\pi x^{1/2} e^{-x}} e^{1/2}.
\] (22)
This establishes item (iv). This completes the proof of Lemma 2.4.

**Lemma 2.5.** Let \( B : (0, \infty)^2 \to (0, \infty) \) satisfy for all \( x, y \in (0, \infty) \) that \( B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} dt \). Then it holds for all \( p \in [0, \infty) \) that
\[
\int_0^{\pi/2} [\sin(t)]^p dt = \frac{B\left( \frac{p+1}{2}, \frac{1}{2} \right)}{2}.
\] (23)

**Proof of Lemma 2.5.** First, note that for all \( t \in (0, 1) \) we have that
\[
\arcsin'(t) = (1 - t^2)^{-1/2}.
\] (24)
This and the integral transformation theorem with the diffeomorphism \( (0, 1) \ni t \mapsto \arcsin(t) \in (0, \pi/2) \) ensure for all \( p \in [0, \infty) \) that
\[
\int_0^{\pi/2} [\sin(t)]^p dt = \int_0^1 t^p (1 - t^2)^{-1/2} dt.
\] (25)
The integral transformation theorem with the diffeomorphism \( (0, 1) \ni t \mapsto \sqrt{t} \in (0, 1) \) hence implies for all \( p \in [0, \infty) \) that
\[
\int_0^{\pi/2} [\sin(t)]^p dt = \frac{1}{2} \int_0^1 t^{p/2-1/2} (1 - t)^{-1/2} dt
\[
= \frac{1}{2} \int_0^1 t^{p/2+1/2-1} (1 - t)^{1/2-1} dt = \frac{B\left( \frac{p+1}{2}, \frac{1}{2} \right)}{2}.
\] (26)
This completes the proof of Lemma 2.5. \( \square \)
Lemma 2.6. Let $R \in (0, \infty]$, for every $d \in \mathbb{N}$ let $\| \cdot \|_{\mathbb{R}^d} : \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, for every $d \in \{2, 3, \ldots\}$ let $B_d = \{ x \in \mathbb{R}^d : \| x \|_{\mathbb{R}^d} < R \}$ and

$$S_d = \begin{cases} (0, 2\pi) & \text{for } d = 2 \\ (0, 2\pi) \times (0, \pi)^{d-2} & \text{for } d \in \{3, 4, \ldots\} \end{cases}$$ (27)

and let $T_d : (0, R) \times S_d \to \mathbb{R}^d$, $d \in \{2, 3, \ldots\}$, satisfy for all $d \in \{2, 3, \ldots\}$, $r \in (0, R)$, $\varphi \in (0, 2\pi)$, $\vartheta_1, \ldots, \vartheta_{d-2} \in (0, \pi)$ that if $d = 2$ then $T_2(r, \varphi) = r(\cos(\varphi), \sin(\varphi))$ and if $d \geq 3$ then

$$T_d(r, \varphi, \vartheta_1, \ldots, \vartheta_{d-2}) = r \left( \cos(\varphi) \left[ \prod_{i=1}^{d-2} \sin(\vartheta_i) \right], \sin(\varphi) \left[ \prod_{i=1}^{d-2} \sin(\vartheta_i) \right] \right),$$

$$\cos(\vartheta_1) \left[ \prod_{i=2}^{d-2} \sin(\vartheta_i) \right] \ldots, \cos(\vartheta_{d-3}) \sin(\vartheta_{d-2}), \cos(\vartheta_{d-2}) \right).$$ (28)

Then

(i) it holds for all $r \in (0, R)$, $\varphi \in (0, 2\pi)$ that

$$| \det((T_2)'(r, \varphi)) | = r,$$ (29)

(ii) it holds for all $d \in \{3, 4, \ldots\}$, $r \in (0, R)$, $\varphi \in (0, 2\pi)$, $\vartheta_1, \ldots, \vartheta_{d-2} \in (0, \pi)$ that

$$| \det((T_d)'(r, \varphi, \vartheta_1, \ldots, \vartheta_{d-2})) | = r^{d-1} \left[ \prod_{i=1}^{d-2} \sin(\vartheta_i) \right],$$ (30)

and

(iii) it holds for all $d \in \{2, 3, \ldots\}$ and all $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}([0, \infty))$-measurable functions $f : \mathbb{R}^d \to [0, \infty)$ that

$$\int_{B_d} f(x) \, dx = \int_0^R \int_{S_d} f(T_d(r, \varphi)) | \det((T_d)'(r, \varphi)) | \, d\varphi \, dr.$$ (31)

Proof of Lemma 2.6. Throughout this proof for every $d \in \{2, 3, \ldots\}$ let $\lambda_{\mathbb{R}^d} : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$ be the Lebesgue–Borel measure on $\mathbb{R}^d$. Observe that for all $r \in (0, R)$, $\varphi \in (0, 2\pi)$ we have that

$$(T_2)'(r, \varphi) = \begin{pmatrix} \cos(\varphi) & -r \sin(\varphi) \\ \sin(\varphi) & r \cos(\varphi) \end{pmatrix}.$$ (32)

Hence, we obtain that for all $r \in (0, R)$, $\varphi \in (0, 2\pi)$ we have that

$$| \det((T_2)'(r, \varphi)) | = r |\cos(\varphi)|^2 + r |\sin(\varphi)|^2 = r.$$ (33)

This establishes item (i). Next observe that Amann & Escher \cite{2} Ch. X, Lemma 8.8] establishes item (ii). To establish item (iii) we distinguish between the case $d = 2$ and the case $d \in \{3, 4, \ldots\}$. First, we consider the case $d = 2$. Note that $T_2 : (0, R) \times (0, 2\pi) \to T_2((0, R) \times (0, 2\pi))$ is a bijective function. This and item
verify that \( T_2 : (0, R) \times (0, 2\pi) \to T_2((0, R) \times (0, 2\pi)) \) is a diffeomorphism. Next observe that

\[
T_2((0, R) \times (0, 2\pi)) = B_2 \setminus \{(x, 0) : x \in [0, \infty)\}. \tag{34}
\]

The fact that \( T_2 : (0, R) \times (0, 2\pi) \to T_2((0, R) \times (0, 2\pi)) \) is a diffeomorphism, the fact that \( \lambda_{\mathbb{R}^2}(\{(x, 0) : x \in [0, \infty)\}) = 0 \), and the integral transformation theorem hence yield that for all \( \mathcal{B}((0, \infty)) \)-measurable functions \( f : \mathbb{R}^2 \to [0, \infty) \) we have that

\[
\int_{B_2} f(x) \, dx = \int_0^R \int_{S_2} f(T_2(r, \phi)) \, |\det ((T_2)'(r, \phi))| \, d\phi \, dr. \tag{35}
\]

This establishes item (iii) in the case \( d = 2 \). Next we consider the case \( d \in \{3, 4, \ldots\} \). Note that Amann & Escher [2, Ch. X, Lemma 8.8] implies that \( T_d : (0, R) \times (0, 2\pi) \times (0, \pi)^{d-2} \to T_d((0, R) \times (0, 2\pi) \times (0, \pi)^{d-2}) \) is a diffeomorphism with

\[
T_d((0, R) \times (0, 2\pi) \times (0, \pi)^{d-2}) = B_d \setminus (0, \infty) \times \{0\} \times \mathbb{R}^{d-2}. \tag{36}
\]

The fact that \( \lambda_{\mathbb{R}^d}(\{0, \infty\} \times \{0\} \times \mathbb{R}^{d-2}) = 0 \) and the integral transformation theorem hence verify that for all \( \mathcal{B}(\mathbb{R}^d)/\mathcal{B}([0, \infty)) \)-measurable functions \( f : \mathbb{R}^d \to [0, \infty) \) we have that

\[
\int_{B_d} f(x) \, dx = \int_0^R \int_{S_d} f(T_d(r, \phi)) \, |\det ((T_d)'(r, \phi))| \, d\phi \, dr. \tag{37}
\]

This establishes item (iii) in the case \( d \in \{3, 4, \ldots\} \). This completes the proof of Lemma 2.6 \( \Box \)

Lemma 2.7. Let \( d \in \mathbb{N}, \) let \( \|\cdot\| : \mathbb{R}^d \to [0, \infty) \) be the standard norm on \( \mathbb{R}^d, \) let \( \lambda : \mathcal{B}(\mathbb{R}^d) \to [0, \infty) \) be the Lebesgue–Borel measure on \( \mathbb{R}^d, \) let \( \mathcal{B} \subseteq \mathbb{R}^d \) be the set given by \( \mathcal{B} = \{x \in \mathbb{R}^d : \|x\| < 1\}, \) and let \( \Gamma : (0, \infty) \to (0, \infty) \) satisfy for all \( x \in (0, \infty) \) that \( \Gamma(x) = \int_0^\infty t^{d-1} e^{-t} \, dt. \) Then

(i) for \( d \in \{2, 3, \ldots\} \) we have that

\[
\lambda(\mathcal{B}) = \frac{2\pi}{d} \left[ \prod_{i=1}^{d-2} \int_0^\pi [\sin(\vartheta_i)]^i \, d\vartheta_i \right]. \tag{38}
\]

and

(ii) we have that

\[
\lambda(\mathcal{B}) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)}. \tag{39}
\]
Proof of Lemma 2.7. To establish (38) and (39) we distinguish between the case \(d = 1\), the case \(d = 2\), and the case \(d \geq 3\). First, we consider the case \(d = 1\). Note that items (i) and (iii) in Lemma 2.4 verify that

\[
\Gamma\left(\frac{1}{2} + 1\right) = \Gamma\left(\frac{3}{2}\right) = \frac{\pi^{1/2}}{2}. \tag{40}
\]

This implies that

\[
\frac{\pi^{1/2}}{\Gamma\left(\frac{1}{2} + 1\right)} = 2. \tag{41}
\]

Combining this and the fact that \(\lambda(B) = \lambda((-1,1)) = 2\) establishes (38) in the case \(d = 1\). Next we consider the case \(d = 2\). Note that items (i) and (iii) in Lemma 2.6 and Fubini’s theorem prove that

\[
\lambda(B) = \int_B dx = \int_0^{2\pi} \int_0^1 r \, dr \, d\phi = \pi. \tag{42}
\]

Next note that items (i) and (iii) in Lemma 2.4 verify that

\[
\Gamma(2) = \Gamma(1 + 1) = \Gamma(1) = 1 \tag{43}
\]

This implies that

\[
\frac{\pi}{\Gamma(1 + 1)} = \pi. \tag{44}
\]

Combining this with (42) establishes (38) and (39) in the case \(d = 2\). Next we consider the case \(d \geq 3\). Note that items (ii)–(iii) in Lemma 2.6 and Fubini’s theorem ensure that

\[
\lambda(B) = \int_B dx = \int_0^{2\pi} \int_0^1 \int_0^{d-2} \prod_{i=1}^{d-2} \left[\sin(\theta_i)\right]^i \, dr \, d\phi \, d\theta_{d-2} \tag{45}
\]

\[
= \frac{1}{d} \int_0^{2\pi} d\phi \left[\prod_{i=1}^{d-2} \int_0^{\pi} \left[\sin(\vartheta_i)\right]^i \, d\vartheta_i \right].
\]

This establishes (38) in the case \(d \in \{3,4,\ldots\}\). Moreover, note that (45) and the fact that for all \(k \in \mathbb{N}\) we have that

\[
\int_0^\pi [\sin(t)]^k \, dt = 2 \int_0^{\frac{\pi}{2}} [\sin(t)]^k \, dt
\]

verifies that

\[
\lambda(B) = \frac{4}{d} \int_0^{\frac{\pi}{2}} d\vartheta \left[\prod_{i=1}^{d-2} \int_0^{\vartheta_i} [\sin(\vartheta_i)]^i \, d\vartheta_i \right]. \tag{46}
\]

Combining this, Lemma 2.5 (applied with \(p \leftarrow i\) for \(i \in \{0,\ldots,d - 2\}\) in the
notation of Lemma 2.5, and item (ii) in Lemma 2.4 yields that
\[
\lambda(\mathbb{B}) = \frac{2}{d} B\left(\frac{1}{2}, \frac{d}{2}\right) \left[\prod_{i=1}^{d-2} B\left(i+\frac{3}{2}, \frac{1}{2}\right)\right]
\]
\[
= \frac{2}{d} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{d}{2}\right)}{\Gamma(1)} \left[\prod_{i=1}^{d-2} \frac{\Gamma\left(i+\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}\right] = \frac{2}{d} \left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right]^d. \tag{47}
\]

Items (i) and (iii) in Lemma 2.4 hence verify that
\[
\lambda(\mathbb{B}) = \left[\frac{\Gamma\left(\frac{1}{2}\right)}{\frac{d}{2} \Gamma\left(\frac{d}{2}\right)}\right]^d = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)}. \tag{48}
\]

This establishes (39) in the case \(d \in \{3, 4, \ldots\}\). This completes the proof of Lemma 2.7. \(\square\)

**Corollary 2.8.** Let \(d \in \mathbb{N}\), let \(\|\cdot\|: \mathbb{R}^d \to [0, \infty)\) be the standard norm on \(\mathbb{R}^d\), and let \(\lambda: \mathcal{B}(\mathbb{R}^d) \to [0, \infty]\) be the Lebesgue–Borel measure on \(\mathbb{R}^d\). Then
\[
\lambda\left(\left\{x \in \mathbb{R}^d: \|x\| < 1\right\}\right) \leq \frac{1}{\sqrt{d\pi}} \left[\frac{2\pi e}{d}\right]^{d/2}. \tag{49}
\]

**Proof of Lemma 2.8.** Throughout this proof let \(\Gamma: (0, \infty) \to (0, \infty)\) satisfy for all \(x \in (0, \infty)\) that \(\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt\). Note that Lemma 2.7 verifies that
\[
\lambda\left(\left\{x \in \mathbb{R}^d: \|x\| < 1\right\}\right) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)}. \tag{50}
\]

Moreover, note that items (i) and (iv) in Lemma 2.4 imply that for all \(x \in (0, \infty)\) we have that
\[
\Gamma(x+1) = x\Gamma(x) \geq \sqrt{2\pi x} \left[\frac{x}{e}\right]^x. \tag{51}
\]
Hence, we obtain that
\[
\frac{1}{\Gamma\left(\frac{d}{2} + 1\right)} \leq \frac{1}{\sqrt{d\pi}} \left[\frac{2\pi e}{d}\right]^{d/2}. \tag{52}
\]
Combining this with (50) yields that
\[
\lambda\left(\left\{x \in \mathbb{R}^d: \|x\| < 1\right\}\right) \leq \frac{1}{\sqrt{d\pi}} \left[\frac{2\pi e}{d}\right]^{d/2}. \tag{53}
\]
This completes the proof of Lemma 2.8. \(\square\)
2.3 Sobolev type estimates for smooth functions

In this subsection we present in Corollary 2.15 below a Sobolev type estimate for smooth functions with an explicit and dimension-independent constant in the Sobolev type estimate. Corollary 2.15 will be employed in our proof of Corollary 4.2 in Subsection 4.1 below. Corollary 2.15 is a consequence of the elementary results in Lemma 2.9 and Corollary 2.14 below. Corollary 2.14, in turn, follows from Proposition 2.13 below. Proposition 2.13 is a special case of Mizuguchi et al. [51, Theorem 2.1]. For completeness we also provide in this subsection a proof for Proposition 2.13. Our proof of Proposition 2.13 employs the well-known results in Lemmas 2.10–2.12 below. Results similar to Lemmas 2.10–2.12 can, e.g., be found in Mizuguchi et al. [51, Lemma 3.1, Theorem 3.3, and Theorem 3.4] and Gilbarg & Trudinger [25, Lemma 7.16].

Lemma 2.9. Let \( \Phi \in C^1((0,1),\mathbb{R}) \) satisfy that \( \int_0^1 (|\Phi(x)| + |\Phi'(x)|) \, dx < \infty \). Then
\[
\sup_{x \in (0,1)} |\Phi(x)| \leq \int_0^1 (|\Phi(x)| + |\Phi'(x)|) \, dx. \tag{54}
\]

Proof of Lemma 2.9. First, note that the fundamental theorem of calculus ensures that for all \( x \in (0,1) \) we have that
\[
\Phi(x) = \int_0^1 \left[ \Phi(s) - \int_x^s \Phi'(t) \, dt \right] \, ds. \tag{55}
\]
The triangle inequality and the hypothesis that \( \int_0^1 (|\Phi(x)| + |\Phi'(x)|) \, dx < \infty \) hence verify that for all \( x \in (0,1) \) we have that
\[
|\Phi(x)| = \left| \int_0^1 \left[ \Phi(s) - \int_x^s \Phi'(t) \, dt \right] \, ds \right|
\leq \int_0^1 \left[ |\Phi(s)| + \int_x^s |\Phi'(t)| \, dt \right] \, ds
\leq \int_0^1 |\Phi(s)| \, ds + \int_0^1 |\Phi'(t)| \, dt < \infty.
\]
This implies (54). This completes the proof of Lemma 2.9. \( \square \)

Lemma 2.10. Let \( d \in \{2,3,\ldots\} \), \( p \in (d,\infty) \), let \( ||\cdot|| : \mathbb{R}^d \to [0,\infty) \) be the standard norm on \( \mathbb{R}^d \), let \( \lambda : \mathcal{B}(\mathbb{R}^d) \to [0,\infty] \) be the Lebesgue–Borel measure on \( \mathbb{R}^d \), and let \( W \subseteq \mathbb{R}^d \) be a non-empty, bounded, and open set. Then
\[
\int_{\bigcup_{x \in W} \{ y \in W : y-x \in W \} } \|z\|^{(1-d)p-1} \, dz
\leq \frac{d(p-1)}{p-d} \left[ \sup_{v, w \in W} \|v - w\| \right]^{\frac{p-d}{p-1}} \lambda(\{x \in \mathbb{R}^d : \|x\| < 1\}). \tag{57}
\]
Proof of Lemma 2.10. Throughout this proof let $\rho \in (0, \infty)$ satisfy that $\rho = \sup_{v, w \in W} \|v - w\|$, let $V \subseteq \mathbb{R}^d$ be the set given by $V = \cup_{x \in W} \{y - x: y \in W\}$, let $S \subseteq \mathbb{R}^{d-1}$ be the set given by

$$S = \begin{cases} (0, 2\pi) & : d = 2 \\ (0, 2\pi) \times (0, \pi)^{d-2} & : d \in \{3, 4, \ldots\} \end{cases}$$  \tag{58}$$

let $B_r \subseteq \mathbb{R}^d$, $r \in (0, \infty)$, be the sets which satisfy for all $r \in (0, \infty)$ that $B_r = \{x \in \mathbb{R}^d: \|x\| < r\}$. and let $T_R: (0, R) \times S \to \mathbb{R}^d$, $R \in (0, \infty)$, satisfy for all $R \in (0, \infty)$, $r \in (0, R)$, $\varphi \in (0, 2\pi)$, $\vartheta_1, \ldots, \vartheta_{d-2} \in (0, \pi)$ that if $d = 2$ then $T_R(r, \varphi) = r(\cos(\varphi), \sin(\varphi))$ and if $d \in \{3, 4, \ldots\}$ then

$$T_R(r, \varphi, \vartheta_1, \ldots, \vartheta_{d-2}) = r\left( \cos(\varphi) \left[ \prod_{i=1}^{d-2} \sin(\vartheta_i) \right], \sin(\varphi) \left[ \prod_{i=1}^{d-2} \sin(\vartheta_i) \right], \right.$$

$$
\cos(\vartheta_1) \left[ \prod_{i=1}^{d-2} \sin(\vartheta_i) \right], \ldots, \cos(\vartheta_{d-3}) \sin(\vartheta_{d-2}), \sin(\vartheta_{d-2}) \left. \right) \tag{59}$$

Observe that

$$\frac{1}{1 - d} \frac{p}{p - 1} + d = \frac{(1 - d)p + d(p - 1)}{p - 1} = \frac{p - d}{p - 1} \tag{60}$$

Next note that items (i)-(iii) in Lemma 2.6 and the fact that for all $r \in (0, \rho)$, $\phi \in S$ we have that $\|T_R(r, \phi)\| = r$ verify that

$$\int_{B_r} \|x\|^{(1-d)\frac{p}{p-1}} \, dx = \int_{0}^{\rho} \int_{S} r^{(1-d)\frac{p}{p-1}} |\det \left( (T_R)'(r, \phi) \right) | \, d\phi \, dr$$

$$= \int_{0}^{\rho} \int_{S} r^{(1-d)\frac{p}{p-1} - d} |\det \left( (T_\infty)'(1, \phi) \right) | \, d\phi \, dr \tag{61}$$

The fact that $V \subseteq B_\rho$, Fubini’s theorem, and (60) hence yield that

$$\int_{V} \|x\|^{(1-d)\frac{p}{p-1}} \, dx \leq \int_{B_\rho} \|x\|^{(1-d)\frac{p}{p-1}} \, dx$$

$$= \int_{S} \left( \int_{0}^{\rho} r^{(1-d)\frac{p}{p-1} - d} \, dr \right) |\det \left( (T_\infty)'(1, \phi) \right) | \, d\phi \tag{62}$$

$$= \rho^{\frac{p-d}{d}} \frac{(p-1)}{p-d} \int_{S} |\det \left( (T_\infty)'(1, \phi) \right) | \, d\phi.$$ 

This, Lemma 2.6, and Lemma 2.7 hence prove that

$$\int_{V} \|x\|^{(1-d)\frac{p}{p-1}} \, dx \leq \rho^{\frac{p-d}{d}} \frac{(p-1)}{p-d} \int_{S} |\det \left( (T_\infty)'(1, \phi) \right) | \, d\phi$$

$$\rho^{\frac{p-d}{d}} \frac{(p-1)}{p-d} \int_{0}^{\pi} \cdots \int_{0}^{\pi} \left[ \prod_{i=1}^{d-2} |\sin(\vartheta_i)| \right] \, d\vartheta_1 \cdots d\vartheta_{d-2} \, d\varphi$$

$$= \rho^{\frac{p-d}{d}} \frac{(p-1)}{p-d} \int_{0}^{\pi} \cdots \int_{0}^{\pi} \left[ \prod_{i=1}^{d-2} |\sin(\vartheta_i)| \right] \, d\vartheta_1 \cdots d\vartheta_{d-2} \, d\varphi \tag{63}$$

This completes the proof of Lemma 2.10. \qed

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Lemma 2.11. Let \( d \in \{2, 3, \ldots, p \in (d, \infty) \), let \( W \subseteq \mathbb{R}^d \) be a non-empty, open, bounded, and convex set, let \( \Phi \in C^1(W, \mathbb{R}) \), let \( \| \cdot \| : \mathbb{R}^d \rightarrow [0, \infty) \) be the standard norm on \( \mathbb{R}^d \), let \( \lambda : \mathcal{B}(\mathbb{R}^d) \rightarrow [0, \infty) \) be the Lebesgue–Borel measure on \( \mathbb{R}^d \), and assume that \( \int_W (|\Phi(x)|^p + \|\nabla \Phi(x)\|^p) \, dx < \infty \). Then it holds for all \( x \in W \) that

\[
\left| \lambda(W) \Phi(x) - \int_W \Phi(y) \, dy \right| \leq \frac{1}{d} \left[ \sup_{v, w \in W} \|v - w\| \right]^d \int_W \|\nabla \Phi(y)\| \|x - y\|^{1-d} \, dy. \tag{64}
\]

Proof of Lemma 2.11. Throughout this proof let \( x \in W \), let \( \rho = \sup_{v, w \in W} \|v - w\| \), let \( \langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) be the \( d \)-dimensional Euclidean scalar product, let \( S \subseteq \mathbb{R}^{d-1} \) be the set given by

\[
S = \begin{cases} 
(0, 2\pi) & : d = 2 \\
(0, 2\pi) \times (0, \pi)^{d-2} & : d \in \{3, 4, \ldots\},
\end{cases}
\tag{65}
\]

let \( \omega_{v, w} \in \mathbb{R}^d \) satisfy for all \( v, w \in W \) that

\[
\omega_{v, w} = \begin{cases} 
\frac{w - v}{\|w - v\|} & : v \neq w \\
0 & : v = w,
\end{cases}
\tag{66}
\]

let \( B_r \subseteq \mathbb{R}^d \), \( r \in (0, \infty) \), satisfy for all \( r \in (0, \infty) \) that \( B_r = \{ y \in \mathbb{R}^d : \|y - x\| < r \} \), let \( E : \mathbb{R}^d \rightarrow \mathbb{R}^d \) satisfy for all \( y \in \mathbb{R}^d \) that

\[
E(y) = \begin{cases} 
\nabla \Phi(y) & : y \in W \\
0 & : y \in \mathbb{R}^d \setminus W,
\end{cases}
\tag{67}
\]

and let \( T_R : (0, R) \times S \rightarrow \mathbb{R}^d \), \( R \in (0, \infty) \), satisfy for all \( R \in (0, \infty) \), \( r \in (0, R) \), \( \varphi \in (0, 2\pi) \), \( \vartheta_1, \ldots, \vartheta_{d-2} \in (0, \pi) \) that if \( d = 2 \) then \( T_R(r, \varphi) = r(\cos(\varphi), \sin(\varphi)) \) and if \( d \geq 3 \) then

\[
T_R(r, \varphi, \vartheta_1, \ldots, \vartheta_{d-2}) = r \left( \cos(\varphi) \left[ \prod_{i=1}^{d-2} \sin(\vartheta_i) \right] \right), \sin(\varphi) \left[ \prod_{i=1}^{d-2} \sin(\vartheta_i) \right], \cos(\vartheta_1) \left[ \prod_{i=2}^{d-2} \sin(\vartheta_i) \right], \ldots, \cos(\vartheta_{d-3}) \sin(\vartheta_{d-2}), \cos(\vartheta_{d-2}) \right). \tag{68}
\]

Observe that the hypothesis that \( \int_W (|\Phi(y)|^p + \|\nabla \Phi(y)\|^p) \, dy < \infty \), the hypothesis that \( W \) is bounded, and Hölder’s inequality ensure that

\[
\int_W |\Phi(y)| + \|\nabla \Phi(y)\| \, dy 
\leq |\lambda(W)|^{(p-1)/p} \left( \int_W |\Phi(y)|^p \, dy \right)^{1/p} + \left( \int_W \|\nabla \Phi(y)\|^p \, dy \right)^{1/p} < \infty. \tag{69}
\]
Next note that the assumption that \( W \) is convex and the fundamental theorem of calculus yield that for all \( y \in W \) we have that
\[
\Phi(x) - \Phi(y) = \left( \Phi(x + r\omega_{x,y}) \right)_{r=0}^{r=\|y-x\|} = - \int_0^{\|y-x\|} \frac{d}{dr} \Phi(x + r\omega_{x,y}) \, dr.
\] (70)

The fact that \( \lambda(W) < \infty \), (69), the Cauchy-Schwarz inequality, and the fact that for all \( y \in W \setminus \{x\} \) we have that \( \|\omega_{x,y}\| = 1 \) hence prove that
\[
\left| \lambda(W)\Phi(x) - \int_W \Phi(y) \, dy \right| = \left| \int_W \int_0^{\|y-x\|} \frac{d}{dr} \Phi(x + r\omega_{x,y}) \, dr \, dy \right|
\]
\[
= \left| \int_W \int_0^{\|y-x\|} \langle (\nabla \Phi)(x + r\omega_{x,y}), \omega_{x,y} \rangle \, dr \, dy \right|
\]
\[
\leq \int_W \int_0^{\|y-x\|} \| (\nabla \Phi)(x + r\omega_{x,y}) \| \| \omega_{x,y} \| \, dr \, dy
\]
\[
= \int_W \int_0^{\|y-x\|} \| (\nabla \Phi)(x + r\omega_{x,y}) \| \, dr \, dy.
\] (71)

The fact that \( W \subseteq B_\rho \) and Fubini’s theorem therefore verify that
\[
\left| \lambda(W)\Phi(x) - \int_W \Phi(y) \, dy \right| \leq \int_0^\infty \int_{B_\rho} \| E(x + r\omega_{x,y}) \| \, dy \, dr.
\] (72)

Next observe that the integral transformation theorem with the diffeomorphism \( \{ v \in \mathbb{R}^d : \| v \| < \rho \} \ni y \mapsto y + x \in B_\rho \), items (i)–(iii) in Lemma 2.6, the fact that for all \( r \in (0, \rho), \phi \in S \) we have that \( \| T_\rho(r, \phi) \| = r \), and (68) imply that for all \( r \in (0, \rho) \) we have that
\[
\int_{B_\rho} \| E(x + r\omega_{x,y}) \| \, dy = \int_{\{ v \in \mathbb{R}^d : \| v \| < \rho \}} \| E(x + r\omega_{0,y}) \| \, dy
\]
\[
= \int_0^\rho \int_S \| E(x + r \frac{T_\rho(s,\phi)}{\|T_\rho(s,\phi)\|}) \| \| \det (T_\rho)'(s, \phi) \| \, d\phi \, ds
\]
\[
= \int_0^\rho \int_S \| E(x + T_\infty(r, \phi)) \| \| \det (T_\infty)'(s, \phi) \| \, d\phi \, ds.
\] (73)
Proof of Lemma 2.12. Throughout this proof let 

\[ \rho = \sup_{v, w \in W} \|v - w\|, \]

let \( W_x \subseteq \mathbb{R}^d, x \in W \), satisfy for all \( x \in W \) that \( W_x =\)

1. 

\[ \lambda(W)\Phi(x) - \int_W \Phi(y) \, dy \]

\[ \leq \frac{\rho^d}{d} \int_0^\infty \int_S \|E(x + T_\infty(r, \phi))\| \left| \det \big((T_\infty)'(s, \phi)\big) \right| ds \, d\phi \, dr 
\]

\[ = \frac{\rho^d}{d} \int_0^\infty \int_S \|E(x + T_\infty(r, \phi))\| \left| \det \big((T_\infty)'(1, \phi)\big) \right| s^{d-1} \, ds \, d\phi \, dr 
\]

\[ = \frac{\rho^d}{d} \int_0^\infty \int_S \|E(x + T_\infty(r, \phi))\| \left| \det \big((T_\infty)'(1, \phi)\big) \right| \, d\phi \, dr . \] 

Combining this, (68), items (i)-(iii) in Lemma 2.6, Fubini’s theorem, and (72) therefore prove that

\[ \left| \lambda(W)\Phi(x) - \int_W \Phi(y) \, dy \right| \]

\[ \leq \frac{\rho^d}{d} \int_0^\infty \int_S \|E(x + T_\infty(r, \phi))\| \left| \det \big((T_\infty)'(1, \phi)\big) \right| r^{1-d} \, r^{d-1} \, d\phi \, dr 
\]

\[ = \frac{\rho^d}{d} \int_0^\infty \int_S \|E(x + T_\infty(r, \phi))\| \|T_\infty(r, \phi)\|^{1-d} \left| \det \big((T_\infty)'(r, \phi)\big) \right| \, d\phi \, dr 
\]

\[ = \frac{\rho^d}{d} \int_{\mathbb{R}^d} \|E(x + y)\| \|y\|^{1-d} \, dy 
\]

\[ = \frac{\rho^d}{d} \int_{\mathbb{R}^d} \|E(y)\| \|x - y\|^{1-d} \, dy = \frac{\rho^d}{d} \int_W \|\nabla \Phi(y)\| \|x - y\|^{1-d} \, dy . \]

This completes the proof of Lemma 2.11 \( \square \)

\textbf{Lemma 2.12.} \( d \in \{2,3,\ldots\}, \ p \in (d, \infty), \) let \( \lambda : \mathcal{B}(\mathbb{R}^d) \rightarrow [0, \infty] \) be the Lebesgue–Borel measure on \( \mathbb{R}^d \), let \( W \subseteq \mathbb{R}^d \) be an open, bounded, and convex set with \( \lambda(W) > 0 \), let \( \Phi \in C^1(W, \mathbb{R}) \), let \( \|\cdot\| : \mathbb{R}^d \rightarrow [0, \infty) \) be the standard norm on \( \mathbb{R}^d \), and assume that \( \int_W (\Phi(x))^p + \|\nabla \Phi(x)\|^p \, dx < \infty \). Then

\[ \sup_{x \in W} \left| \Phi(x) - \frac{1}{\lambda(W)} \int_W \Phi(y) \, dy \right| \]

\[ \leq \left[ \frac{\sup_{v, w \in W} \|v - w\|^d}{\lambda(W)d} \right] \left[ \int_{\cup_{x \in W} \{x - y : y \in W\}} \|z\|^{(1-d)\frac{p-d}{d}} \, dz \right]^{(p-1)/p} \]

\[ \cdot \left[ \int_W \|\nabla \Phi(y)\|^p \, dy \right]^{1/p} < \infty . \]

\textbf{Proof of Lemma 2.12} Throughout this proof let \( \rho \in [0, \infty) \) satisfy that \( \rho = \sup_{v, w \in W} \|v - w\|, \) let \( W_x \subseteq \mathbb{R}^d, x \in W \), satisfy for all \( x \in W \) that \( W_x =\)
that λ

Moreover, the assumption that x

formation theorem prove that for all

Observe that the hypothesis that \( E(x) \) and let \( x \) satisfy for all \( x \in \mathbb{R}^d \) that

Let Proposition 2.13.

\[
\begin{aligned}
\Phi \left( x - \frac{1}{\lambda(W)} \int_W \Phi(y) \, dy \right) & \leq \frac{\rho^d}{\lambda(W) d} \int_W \| \nabla \Phi(y) \| \| x - y \|_{1-d} \, dy \\
& = \frac{\rho^d}{\lambda(W) d} \int_W \| \nabla \Phi(x - y) \| \| y \|_{1-d} \, dy \\
& \leq \frac{\rho^d}{\lambda(W) d} \int_V \| E(x - y) \| \| y \|_{1-d} \, dy = \frac{\rho^d}{\lambda(W) d} \int_{\mathbb{R}^d} \| E(x - y) \| \| \psi(y) \| \, dy.
\end{aligned}
\]

Lemma 2.10 (79), and Hölder’s inequality therefore yield that for all \( x \in W \) we have that

\[
\begin{aligned}
\Phi(x) - \left[ \Phi \left( x - \frac{1}{\lambda(W)} \int_W \Phi(y) \, dy \right) \right] & \leq \frac{\rho^d}{\lambda(W) d} \left[ \int_{\mathbb{R}^d} \| E(x - y) \| \, dy \right]^{1/p} \left[ \int_{\mathbb{R}^d} \| \psi(y) \|^{p(q-1)} \, dy \right]^{(q-1)/p} \\
& = \frac{\rho^d}{\lambda(W) d} \left[ \int_W \| \nabla \Phi(y) \| \, dy \right]^{1/p} \left[ \int_V \| y \|_{1-d} \, dy \right]^{(q-1)/p} < \infty.
\end{aligned}
\]

This completes the proof of Lemma 2.12.

Proposition 2.13. Let \( d \in \{ 2, 3, \ldots \} \), \( p \in (d, \infty) \), let \( W \subseteq \mathbb{R}^d \) be an open, bounded, and convex set, let \( \Phi \in C^1(W, \mathbb{R}) \), let \( \| \cdot \| : \mathbb{R}^d \to [0, \infty) \) be the standard norm on \( \mathbb{R}^d \), let \( \lambda : \mathcal{B}(\mathbb{R}^d) \to [0, \infty) \) be the Lebesgue–Borel measure on \( \mathbb{R}^d \), let \( I \) be a finite and non-empty set, let \( W_i \subseteq \mathbb{R}^d \), \( i \in I \), be open and convex sets, assume for all \( i \in I \), \( j \in I \setminus \{ i \} \) that \( \lambda(W_i) > 0 \), \( W_i \cap W_j = \emptyset \), and \( \overline{W} = \bigcup_{i \in I} \overline{W_i} \),
assume that $\int_W (|\Phi(x)|^p + \|\nabla \Phi(x)\|^p) \, dx < \infty$, and let $(D_i)_{i \in I} \subseteq [0, \infty)$ satisfy for all $i \in I$ that
\[ D_i = \frac{\sup_{v, w \in W} \|v - w\|^d}{\lambda(W_i)^d} \left[ \int_{\bigcup_{x \in W_i} \{x \in W_i \}} \|z\|^p \frac{(1-d)\|v\|^{p-1} \, dz}{p-1} \right]^{(p-1)/p}. \] (82)

Then
\[ \sup_{x \in W} |\Phi(x)| \leq 2^{(p-1)/p} \max \left\{ \lambda(W_i)^{-1/p} \cdot \max(D_i), \max(\lambda(W_i)) \right\} \cdot \left[ \int_W (|\Phi(x)|^p + \|\nabla \Phi(x)\|^p) \, dx \right]^{1/p}. \] (83)

**Proof of Proposition 2.13** Note that the hypothesis that $W \subseteq \mathbb{R}^d$ is open and convex and \[\text{[55 Theorem 6.3]}\] imply that for all $i \in I$ we have that $W_i \subseteq W$. Next observe that the hypothesis that $\int_W (|\Phi(x)|^p + \|\nabla \Phi(x)\|^p) \, dx < \infty$, the hypothesis that for all $i \in I$ we have that $W_i$ is bounded, and Hölder’s inequality yield that for all $i \in I$ we have that
\[ \frac{1}{\lambda(W_i)} \int_{W_i} |\Phi(y)| \, dy = \int_{W_i} \frac{1}{\lambda(W_i)} |\Phi(y)| \, dy \]
\[ \leq \left[ \int_{W_i} \left[ \lambda(W_i)^{-p/(p-1)} \right]^{1/(p-1)} \right]^{1/p} \left[ \int_{W_i} |\Phi(y)|^p \, dy \right]^{1/p} \]
\[ \leq \left[ \lambda(W_i)^{-1/p} \right] \left[ \int_{W_i} |\Phi(y)|^p \, dy \right]^{1/p} < \infty. \] (84)

The hypothesis that $\overline{W} = \bigcup_{i \in I} \overline{W_i}$, the triangle inequality, and Lemma 2.12 (applied with $W \leftarrow W_i$ for $i \in I$ in the notation of Lemma 2.12) hence verify that
\[ \sup_{x \in W} |\Phi(x)| = \max_{i \in I} \left( \sup_{x \in W_i} |\Phi(x)| \right) \]
\[ \leq \max_{i \in I} \left( \sup_{x \in W_i} \left| \Phi(x) - \frac{1}{\lambda(W_i)} \int_{W_i} \Phi(y) \, dy \right| + \frac{1}{\lambda(W_i)} \int_{W_i} |\Phi(y)| \, dy \right) \]
\[ \leq \max_{i \in I} \left( D_i \left[ \int_{W_i} \|\nabla \Phi(y)\|^p \, dy \right]^{1/p} + \left[ \lambda(W_i)^{-1/p} \right] \left[ \int_{W_i} |\Phi(y)|^p \, dy \right]^{1/p} \right) \]
\[ \leq \max_{i \in I} \left( \left[ \int_{W_i} |\Phi(y)|^p \, dy \right]^{1/p} + \left[ \int_{W_i} \|\nabla \Phi(y)\|^p \, dy \right]^{1/p} \right). \] (85)

Next note that for all $(x_i)_{i \in I} \subseteq \mathbb{R}$ we have that
\[ \max_{i \in I} |x_i| \leq \left[ \sum_{i \in I} |x_i|^p \right]^{1/p}. \] (86)
Combining this, the hypothesis that for all $i \in I$, $j \in I \setminus \{i\}$ we have that $W_i \cap W_j = \emptyset$ and $W = \bigcup_{i \in I} W_i$, and the fact that for all $a, b \in [0, \infty)$ we have that $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ with (85) yields that

$$\sup_{x \in W} |\Phi(x)| \leq \max \left\{ \max_{i \in I} \left( \lambda(W_i)^{-\frac{1}{p}} \right), \max(D_i) \right\}$$

$$\cdot \left[ \sum_{i \in I} \left[ \int_{W_i} |\Phi(y)|^p \, dy \right]^{\frac{1}{p}} + \left[ \int_{W_i} \|\nabla \Phi(y)\|^p \, dy \right]^{\frac{1}{p}} \right]^{\frac{1}{2}}$$

$$\leq 2 \frac{p-1}{p} \max \left\{ \max_{i \in I} \left( \lambda(W_i)^{-\frac{1}{p}} \right), \max(D_i) \right\}$$

$$\cdot \left[ \sum_{i \in I} \int_{W_i} (|\Phi(y)|^p + \|\nabla \Phi(y)\|^p) \, dy \right]^{\frac{1}{p}}$$

$$= 2 \frac{p-1}{p} \max \left\{ \max_{i \in I} \left( \lambda(W_i)^{-\frac{1}{p}} \right), \max(D_i) \right\}$$

$$\cdot \left[ \int_{W} (|\Phi(y)|^p + \|\nabla \Phi(y)\|^p) \, dy \right]^{\frac{1}{p}} \cdot (87)$$

This completes the proof of Proposition 2.13.

**Corollary 2.14.** Let $d \in \{2, 3, \ldots\}$, $\Phi \in C^1((0, 1)^d, \mathbb{R})$, let $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, and assume that $\int_{(0,1)^d} (|\Phi(x)|^d + \|\nabla \Phi(x)\|^d) \, dx < \infty$. Then

$$\sup_{x \in (0,1)^d} |\Phi(x)| \leq 8 \sqrt{e} \left[ \int_{(0,1)^d} \left( |\Phi(x)|^d + \|\nabla \Phi(x)\|^d \right) \, dx \right]^{\frac{1}{4d}} \cdot (88)$$

**Proof of Corollary 2.14** Throughout this proof let $\lambda : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$ be the Lebesgue–Borel measure on $\mathbb{R}^d$, let $m \in \mathbb{N}$ satisfy that

$$m = \min \left( \mathbb{N} \cap \left[ \frac{d}{2 \ln(2)} (\ln(2) + \ln(\pi) + 1), \infty \right) \right),$$

(89)

let $B \subseteq \mathbb{R}^d$ be the set given by $B = \{ x \in \mathbb{R}^d : \|x\| < 1 \}$, let $I$ be the set given by $I = \{1, \ldots, 2^m\}$, let $W_i, V_i \subseteq \mathbb{R}^d$, $i \in I$, be the sets which satisfy for all $i = (i_1, \ldots, i_d) \in I$ that

$$W_i = (x_{i_1-1}^2, (i_2-1)2^{-m}, i_32^{-m}) \quad \text{and} \quad V_i = \cup_{x \in W_i} \{ x - y : y \in W_i \}, \quad (90)$$

and let $(D_i)_{i \in I}, (\rho_i)_{i \in I} \subseteq (0, \infty)$ satisfy for all $i \in I$ that

$$\rho_i = \sup_{v, w \in W_i} \|v - w\| \quad \text{and} \quad D_i = \frac{\rho_i^d}{\lambda(W_i)^d} \left[ \int_{V_i} \|x\|^d \, dx \right]^{\frac{d^2-1}{d^2}} \cdot (91)$$

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Observe that (89) ensures that

\[ 2\pi e^{\sqrt{2} - 1} = e^{\sqrt{2} \ln(2) + \ln(\pi) + 1} = 2^{\frac{\sqrt{2} \ln(2) + \ln(\pi) + 1}{2}} \leq 2^m. \]  

(92)

Hence, we obtain that

\[ [2\pi e]^{\sqrt{2} 2^{-m}} \leq 1. \]  

(93)

Next note that (90) verifies that for all \( i \in I \) we have that

\[ \lambda(W_i) = 2^{-d^m} \quad \text{and} \quad \rho_i = \sqrt{d} 2^{-m}. \]  

(94)

Moreover, note that (89) implies that

\[ \frac{d}{2 \ln(2)} (\ln(2) + \ln(\pi) + 1) \leq m \leq \frac{d}{2 \ln(2)} (\ln(2) + \ln(\pi) + 1) + 1. \]  

(95)

This, (94), and (92) hence verify that

\[ \max_{i \in I} \left( \lambda(W_i) \right)^{-1/\sqrt{2}} = 2^{m/\sqrt{2}} \leq \frac{d}{2 \ln(2)} \left( 2^{\ln(2) + \ln(\pi) + 1} \right)^{1/\sqrt{2}} = 2^{\sqrt{2} \sqrt{2} \pi e}. \]  

(96)

Next note that Lemma 2.10 (applied with \( p \leftarrow d^2, W \leftarrow W_i \) for \( i \in I \) in the notation of Lemma 2.10) assures that for all \( i \in I \) we have that

\[
\left[ \int_{V_i} \|x\|^{(1-d) \sqrt{2^d+1}} dx \right] \frac{\sqrt{2^d+1}}{\sqrt{2^d}} \leq \rho_i^{\sqrt{2^d+1}} \left( \frac{d(d^2-1)}{d^2-d} \lambda(B) \right)^{\sqrt{2^d+1}} \rho_i \frac{\sqrt{2^d+1}}{(d+1) \lambda(B)} \frac{\sqrt{2^d+1}}{\sqrt{2^d+1}}.
\]

(97)

Corollary 2.8, the fact that \( (d+1) \frac{1}{\sqrt{d \pi}} (2\pi e)^{d/2} \geq 1 \), the fact that \( (d+1) \leq 2d \), the fact that \( \frac{\sqrt{2^d+1}}{\sqrt{2^d+1}} \leq 1 \), and (94) hence prove that for all \( i \in I \) we have that

\[
\left[ \int_{V_i} \|x\|^{(1-d) \sqrt{2^d+1}} dx \right] \frac{\sqrt{2^d+1}}{\sqrt{2^d}} \leq (\sqrt{d} 2^{-m}) \frac{\sqrt{2^d+1}}{\sqrt{d \pi}} \left( d+1 \frac{1}{\sqrt{d \pi}} (2\pi e)^{d/2} \right)^{\sqrt{2^d+1}} \frac{\sqrt{2^d+1}}{\sqrt{2^d+1}} d^{-\frac{d}{2}} \frac{\sqrt{2^d+1}}{\sqrt{2^d+1}}
\]

\[ \leq (\sqrt{d} 2^{-m}) \frac{\sqrt{2^d+1}}{\sqrt{d \pi}} 2d \frac{\sqrt{2^d+1}}{\sqrt{d \pi}} (2\pi e)^{d/2} d^{-\frac{d}{2}} \frac{\sqrt{2^d+1}}{\sqrt{2^d+1}}
\]

\[ = 2^{1+\frac{d}{2} - m} d^{1+\frac{d}{2} - \frac{d}{2}} \frac{1}{\sqrt{\pi}} (2\pi e)^{d/2}
\]

\[ = 2^{1+\frac{d}{2} - m} d^{1+\frac{d}{2} - \frac{d}{2}} \frac{1}{\sqrt{\pi}} (2\pi e)^{d/2} = \frac{2^{1+\frac{d}{2} - m} d^{1+\frac{d}{2} - \frac{d}{2}} (2\pi e)^{d/2}}{\sqrt{\pi}}.
\]

(98)
This, (95), (92), and (93) therefore yield that for all $i \in I$ we have that
\[
\left[ \int_{V_i} \|x\|^{(1-d)/2} d^{x-1} dx \right]^{2^{x-1}} \leq \frac{2^{1+d/2}d^{-d/2}}{\sqrt{\pi}} 
\leq \frac{2^{1+\ln(2)+\ln(d)+1/2}d^{-d/2}}{\sqrt{\pi}} = 2^{1+1/2} \sqrt{2e} d^{-d/2}.
\] (99)

Next note that (94) ensures that for all $i \in I$ we have that
\[
\frac{d_i^d}{\lambda(W_i)d} = d^{d/2-1}2^{-d} = d^{d/2-1}.
\] (100)

Combining this with (99) yields that
\[
\max_{i \in I} D_i = \max_{i \in I} \left( \frac{\rho_i^d}{\lambda(W_i)d} \left[ \int_{V_i} \|x\|^{(1-d)/2} d^{x-1} dx \right]^{2^{x-1}} \right) \leq 2^{1+1/2} \sqrt{2e}.
\] (101)

Combining this and (100) with the hypothesis that $d \in \{2, 3, \ldots\}$ yields that
\[
\max \left\{ \max_{i \in I} \left[ \lambda(W_i) \right]^{-1/2} , \lambda(D_i) \right\} = \max \left\{ 2^{1+1/2} \sqrt{2e} , 2^{1+1/2} \sqrt{2e} \right\} = 2^{1+1/2} \sqrt{2e} \leq 2 \sqrt{2} \sqrt{2e} = 4 \sqrt{e}.
\] (102)

Next note that (94) ensures that for all $i \in I$, $j \in I \setminus \{i\}$ we have that
\[
W_i \cap W_j = \emptyset \quad \text{and} \quad [0,1]^d = \cup_{i \in I} W_i.
\] (103)

This, (102), (94), Proposition 2.13 (applied with $p \leftarrow d^2$, $I \leftarrow I$, $W \leftarrow (0,1)^d$, $W_i \leftarrow W_i$ for $i \in I$ in the notation of Proposition 2.13), and the hypothesis that $d \in \{2, 3, \ldots\}$ hence imply that
\[
\sup_{x \in (0,1)^d} |\Phi(x)| \leq 2^{1-d/2} \max \left\{ \frac{\lambda(W_i)}{2^{1+d/2}d^{-d/2}} , \lambda(D_i) \right\} \cdot \left[ \int_{(0,1)^d} \left( |\Phi(x)|^2 + \|\nabla \Phi(x)\|^2 d^2 \right) dx \right]^{1/2} 
\leq 8 \sqrt{e} \left[ \int_{(0,1)^d} \left( |\Phi(x)|^2 + \|\nabla \Phi(x)\|^2 d^2 \right) dx \right]^{1/2}.
\] (104)

This completes the proof of Corollary 2.14.

□
Corollary 2.15. Let \( d \in \mathbb{N}, \Phi \in C^1((0,1)^d, \mathbb{R}), \) let \( \|\cdot\| : \mathbb{R}^d \to [0,\infty) \) be the standard norm on \( \mathbb{R}^d \), and assume that \( \int_{(0,1)^d} (|\Phi(x)|^{\max(2,d^2)} + \|\nabla\Phi(x)\|^{\max(2,d^2)}) \, dx < \infty \). Then

\[
\sup_{x \in (0,1)^d} |\Phi(x)| \leq 8\sqrt{e} \left[ \int_{(0,1)^d} \left( |\Phi(x)|^{\max(2,d^2)} + \|\nabla\Phi(x)\|^{\max(2,d^2)} \right) \, dx \right]^{1/\max(2,d^2)}.
\] (105)

Proof of Corollary 2.15. To establish (105) we distinguish between the case \( d = 1 \) and the case \( d \in \{2,3,\ldots\} \). First, we consider the case \( d = 1 \). Note that Lemma 2.9 ensures that

\[
\sup_{x \in (0,1)} |\Phi(x)| \leq \int_0^1 \left( |\Phi(x)| + |\Phi'(x)| \right) \, dx
\leq 8\sqrt{e} \left[ \int_0^1 \left( |\Phi(x)|^2 + |\Phi'(x)|^2 \right) \, dx \right]^{1/2}.
\] (106)

This establishes (105) in the case \( d = 1 \). Next we consider the case \( d \in \{2,3,\ldots\} \). Note that Corollary 2.14 verifies that

\[
\sup_{x \in (0,1)^d} |\Phi(x)| \leq 8\sqrt{e} \left[ \int_{(0,1)^d} \left( |\Phi(x)|^{\max(2,d^2)} + \|\nabla\Phi(x)\|^{\max(2,d^2)} \right) \, dx \right]^{1/d^2}.
\] (107)

This establishes (105) in the case \( d \in \{2,3,\ldots\} \). This completes the proof of Corollary 2.15.

2.4 Sobolev type estimates for Monte Carlo approximations

In this subsection we provide in Lemma 2.16 below a Sobolev type estimate for Monte Carlo approximations. Lemma 2.10 is one of the main ingredients in our proof of Lemma 4.1 in Subsection 4.1 below.

Lemma 2.16. Let \( d,n \in \mathbb{N}, \zeta, a \in \mathbb{R}, b \in (a,\infty), p \in [1,\infty) \), let \( \|\cdot\| : \mathbb{R}^d \to [0,\infty) \) be the standard norm on \( \mathbb{R}^d \), let \( \mathcal{R} \in (0,\infty) \) be the max\{2,p\}-Kahane-Khintchine constant (cf. Definition 2.7 and Lemma 2.2), let \( (\Omega,\mathcal{F},\mathbb{P}) \) be a probability space, let \( \xi_i : \mathbb{R}^d \times \Omega \to \mathbb{R}, i \in \{1,\ldots,n\}, \) be i.i.d. random fields satisfying for all \( i \in \{1,\ldots,n\}, \omega \in \Omega \) that \( \xi_i(\cdot,\omega) \in C^1(\mathbb{R}^d,\mathbb{R}) \), let \( \xi : \mathbb{R}^d \times \Omega \to \mathbb{R} \) be the random field satisfying for all \( x \in \mathbb{R}^d, \omega \in \Omega \) that \( \xi(x,\omega) = \xi_1(x,\omega), \) assume for all \( \Phi \in C^1((0,1)^d,\mathbb{R}) \) with \( \int_{(0,1)^d} (|\Phi(x)|^{\max(2,p)} + \|\nabla\Phi(x)\|^{\max(2,p)}) \, dx < \infty \) that

\[
\sup_{x \in (0,1)^d} |\Phi(x)| \leq \zeta \left[ \int_{(0,1)^d} \left( |\Phi(x)|^{\max(2,p)} + \|\nabla\Phi(x)\|^{\max(2,p)} \right) \, dx \right]^{1/\max(2,p)},
\] (108)
and assume that for all $x \in [a, b]^d$ we have that
\[
\inf_{\delta \in (0, \infty)} \sup_{v \in [-\delta, \delta]^d} \mathbb{E} \left[ |\xi(x + v)|^{1 + \delta} + \| (\nabla \xi)(x + v) \|^{1 + \delta} \right] < \infty. \tag{109}
\]

Then

(i) we have that
\[
\sup_{x \in [a, b]^d} \left| \mathbb{E} \left[ \xi(x) \right] - \frac{1}{n} \left( \sum_{i=1}^{n} \xi_i(x) \right) \right|^p \tag{110}
\]
is a random variable and

(ii) we have that
\[
\left( \mathbb{E} \left[ \sup_{x \in [a, b]^d} \mathbb{E} \left[ |\xi(x)|^{\max\{2, p\}} \right] \right] \right)^{\frac{1}{\max\{2, p\}}} \leq \frac{4K^{\frac{1}{2}}}{\sqrt{n}} \left( \sup_{x \in [a, b]^d} \left[ \mathbb{E} \left[ \| \nabla \xi(x) \|^{\max\{2, p\}} \right] \right] \right)^{\frac{1}{\max\{2, p\}}} \tag{111}
\]

\[
+ (b - a) \left[ \mathbb{E} \left[ \| \nabla \xi(x) \|^{\max\{2, p\}} \right] \right]^{\frac{1}{\max\{2, p\}}} \right). \tag{112}
\]

**Proof of Lemma 2.16** Throughout this proof let $\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be the $d$-dimensional Euclidean scalar product, let $q \in [2, \infty)$ satisfy that $q = \max\{2, p\}$, let $\rho : \mathbb{R}^d \to \mathbb{R}^d$ satisfy for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ that
\[
\rho(x) = ((b - a)x_1 + a, (b - a)x_2 + a, \ldots, (b - a)x_d + a), \tag{113}
\]
let $e_1, \ldots, e_d \in \mathbb{R}^d$ satisfy that $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$, let $Y : [0, 1]^d \times \Omega \to \mathbb{R}$ be the random field which satisfies for all $x \in [0, 1]^d$ that
\[
Y(x) = \mathbb{E} \left[ \xi(\rho(x)) \right] - \frac{1}{n} \left( \sum_{i=1}^{n} \xi_i(\rho(x)) \right), \tag{114}
\]
let $Z : [0, 1]^d \times \Omega \to \mathbb{R}^d$ be the random field which satisfies for all $x \in [0, 1]^d$ that
\[
Z(x) = (b - a) \left[ \mathbb{E} \left[ \| \nabla \xi(\rho(x)) \|^{\max\{2, p\}} \right] \right]^{\frac{1}{\max\{2, p\}}}, \tag{115}
\]
and let $E : \Omega \to [0, \infty)$ be the random variable given by
\[
E = \sup_{x \in [a, b]^d \cap Q^d} \left| \mathbb{E} \left[ \xi(x) \right] - \frac{1}{n} \left( \sum_{i=1}^{n} \xi_i(x) \right) \right|. \tag{116}
\]
Note that (112) ensures that \( \rho([0,1]^d) = [a,b]^d \). Furthermore, note that (109) ensures that for all \( x \in [a,b]^d \) there exists \( \delta_x \in (0,\infty) \) such that

\[
\sup_{v \in [-\delta_x, \delta_x]^d} \mathbb{E} \left[ \| (\nabla \xi)(x + v) \|^{1 + \delta_x} \right] < \infty.
\] (116)

Hölder’s inequality therefore verifies that for all \( x \in [a,b]^d \) there exists \( \delta_x \in (0,\infty) \) which satisfies that

\[
\sup_{v \in [-\delta_x, \delta_x]^d} \mathbb{E} \left[ \| (\nabla \xi)(x + v) \| \right] \leq \sup_{v \in [-\delta_x, \delta_x]^d} \left( \mathbb{E} \left[ \| (\nabla \xi)(x + v) \|^2 \right] \right)^{1/(1 + \delta_x)} < \infty.
\] (117)

Next note that the collection \( S_x = \{ y \in [a,b]^d : y - x \in (-\delta_x, \delta_x)^d \} \), \( x \in [a,b]^d \), is an open cover of \( [a,b]^d \). The fact that \( [a,b]^d \) is compact hence ensures that there exists \( N \in \mathbb{N} \) and \( x_k \in [a,b]^d \), \( k \in \{1, \ldots, N\} \) which satisfies that the collection \( S_{x_k}, k \in \{1, \ldots, N\} \) is a finite open cover of \( [a,b]^d \). Combining this with (117) yields that

\[
\max_{k \in \{1, \ldots, N\}} \mathbb{E} \left[ \| (\nabla \xi)(x_k + v) \| \right] < \infty.
\] (118)

Moreover, note that the fact that for all \( \omega \in \Omega \) we have that the functions \( (a, b)^d \ni x \mapsto \xi(x, \omega) \in \mathbb{R} \) and \( (a, b)^d \ni x \mapsto (\nabla \xi)(x, \omega) \in \mathbb{R}^d \) are continuous ensures that \( [a,b]^d \times \Omega \ni (x, \omega) \mapsto \xi(x, \omega) \in \mathbb{R} \) and \( [a,b]^d \times \Omega \ni (x, \omega) \mapsto (\nabla \xi)(x, \omega) \in \mathbb{R}^d \) are Carathéodory functions. This implies that \( [a,b]^d \times \Omega \ni (x, \omega) \mapsto \xi(x, \omega) \in \mathbb{R} \) is \((B([a,b]^d) \otimes \mathcal{F})/B(\mathbb{R})\)-measurable and \( [a,b]^d \times \Omega \ni (x, \omega) \mapsto (\nabla \xi)(x, \omega) \in \mathbb{R}^d \) is \((B([a,b]^d) \otimes \mathcal{F})/B(\mathbb{R}^d)\)-measurable, see, e.g., Aliprantis and Border [11, Lemma 4.51]). Next note that the fundamental theorem of calculus ensures that for all \( x, y \in \mathbb{R}^d \) we have that

\[
\xi(x) - \xi(y) = \int_0^1 \langle (\nabla \xi)(y + t(x-y)), x - y \rangle \, dt.
\] (119)

This reveals that for all \( x, y \in [a,b]^d \) we have that

\[
|\xi(x) - \xi(y)| \leq \| x - y \| \int_0^1 \| (\nabla \xi)(y + t(x-y)) \| \, dt.
\] (120)

Combining this with Fubini’s theorem verifies that for all \( x, y \in [a,b]^d \) we have
that

\[
|E[\xi(x)] - E[\xi(y)]| \leq E[|\xi(x) - \xi(y)|] \\
\leq \|x - y\| E \left[ \int_0^1 \|\nabla \xi\| (y + t(x - y)) \, dt \right] \\
= \|x - y\| \int_0^1 E \left[ \|\nabla \xi\| (y + t(x - y)) \right] \, dt \\
\leq \|x - y\| \sup_{v \in [a, b]} E \left[ \|\nabla \xi\| \right].
\]

This and (118) prove that \([a, b]^d \ni x \mapsto E[\xi(x)] \in \mathbb{R}\) is a Lipschitz continuous function. Hence, we obtain for all \(\omega \in \Omega\) that \([0, 1]^d \ni x \mapsto Y(x, \omega) \in \mathbb{R}\) is a continuous function. Combining this with (115) implies that

\[
E = \sup_{x \in [0, 1]^d} |Y(x)|.
\]

This establishes item (i). Next note that (112) implies that for all \(j \in \{1, \ldots, d\}\), \(x \in [0, 1]^d\), \(h \in \mathbb{R}\) we have that

\[
\rho(x + he_j) - \rho(x) = (b - a) he_j.
\]

This and (119) verify that for all \(j \in \{1, \ldots, d\}\), \(x \in [0, 1]^d\), \(h \in \mathbb{R} \setminus \{0\}\) we have that

\[
\frac{\xi(\rho(x + he_j)) - \xi(\rho(x))}{h} = (b - a) \int_0^1 \langle (\nabla \xi)(\rho(x) + t(b - a) he_j), e_j \rangle \, dt.
\]

Moreover, note that (109) implies that for all \(x \in [0, 1]^d\) there exists \(\delta_x \in (0, \infty)\) such that

\[
\sup_{v \in [-\delta_x, \delta_x]^d} E \left[ \|\nabla \xi\| (\rho(x) + v) \right]^{1+\delta_x} < \infty.
\]

This, Hölder’s inequality, and Fubini’s theorem verify that for all \(j \in \{1, \ldots, d\}\), \(x \in [0, 1]^d\) there exists \(\delta_x \in (0, \infty)\) such that for all \(h \in \mathbb{R} \setminus \{0\}\) we have that

\[
E \left[ \left\| \int_0^1 (\nabla \xi)(\rho(x) + t(b - a) he_j) \, dt \right\|^{1+\delta_x} \right] \\
\leq E \left[ \int_0^1 \left\| (\nabla \xi)(\rho(x) + t(b - a) he_j) \right\|^{1+\delta_x} \, dt \right] \\
= \int_0^1 E \left[ \left\| (\nabla \xi)(\rho(x) + t(b - a) he_j) \right\|^{1+\delta_x} \right] \, dt \\
\leq \sup_{v \in [-\delta_x, \delta_x]^d} E \left[ \left\| (\nabla \xi)(\rho(x) + v) \right\|^{1+\delta_x} \right] < \infty.
\]
This and \[124\] verify that for all \(j \in \{1, \ldots, d\}, x \in [0, 1]^d\) there exists \(\delta_x \in (0, \infty)\) such that for all \(h' \in \mathbb{R} \setminus \{0\} : |(b-a)h'| < \delta_x\) we have that

\[
E \left[ \frac{\xi(\rho(x + he_j)) - \xi(\rho(x))}{h} \right]^{1+\delta_x} \leq (b-a)^{1+\delta_x} E \left[ \left\| \int_0^1 (\nabla \xi)(\rho(x) + t(b-a)h) \, dt \right\|^{1+\delta_x} \right]
\]

\[
\leq (b-a)^{1+\delta_x} \sup_{v \in [-\delta_x, \delta_x]^d} E \left[ \left\| (\nabla \xi)(\rho(x) + v) \right\|^{1+\delta_x} \right] < \infty. \tag{127}
\]

This, the theorem of de la Vallée–Poussin (see, e.g., \[45\, \text{Theorem 6.19}\]), and Vitali’s convergence theorem (see, e.g., \[45\, \text{Theorem 6.25}\]) verify that for all \(x \in [0, 1]^d, j \in \{1, \ldots, d\}\) there exists \(\delta_x \in (0, \infty)\) such that for all \((h_m)_{m \in \mathbb{N}} \subseteq \{h' \in \mathbb{R} \setminus \{0\} : |(b-a)h'| < \delta_x\}\) with \(\lim_{m \to \infty} h_m = 0\) we have that

\[
\lim_{m \to \infty} E \left[ \frac{\xi(\rho(x + h_me_j)) - \xi(\rho(x))}{h_m} \right] = E \left[ \lim_{m \to \infty} \frac{\xi(\rho(x + h_me_j)) - \xi(\rho(x))}{h_m} \right]. \tag{128}
\]

Therefore, we obtain that for all \(x \in [0, 1]^d, j \in \{1, \ldots, d\}\) there exists \(\delta_x \in (0, \infty)\) such that for all \((h_m)_{m \in \mathbb{N}} \subseteq \{h' \in \mathbb{R} \setminus \{0\} : |(b-a)h'| < \delta_x\}\) with \(\lim_{m \to \infty} h_m = 0\) we have that

\[
\lim_{m \to \infty} E \left[ \frac{\xi(\rho(x + h_me_j)) - \xi(\rho(x))}{h_m} \right] = (b-a)E \left[ \langle \nabla \xi(\rho(x)), e_j \rangle \right]. \tag{129}
\]

Furthermore, the theorem of de la Vallee–Poussin, Vitali’s convergence theorem, and \[125\] prove that for all \(x \in [0, 1]^d, j \in \{1, \ldots, d\}\) we have that

\[
\limsup_{R^d \setminus \{0\} \ni h \to 0} |E[\langle \nabla \xi(\rho(x) + h), e_j \rangle] - E[\langle \nabla \xi(\rho(x)), e_j \rangle]| = 0. \tag{130}
\]

This and \[129\] imply that for all \(\omega \in \Omega, x \in (0, 1)^d\) we have that \((0, 1)^d \ni y \mapsto Y(y, \omega) \in \mathcal{C}^1((0, 1)^d, \mathbb{R})\) and \((\nabla Y)(x, \omega) = Z(x, \omega)\). Combining this, \[108\], and \[122\] yields that

\[
E = \sup_{x \in (0, 1)^d} |Y(x)| \leq \zeta \left[ \int_{(0, 1)^d} (|Y(x)|^q + \|Z(x)\|^q) \, dx \right]^{1/q} \tag{131}
\]

Next observe that \[108\] ensures that \(\zeta \in [0, \infty)\). Hölder’s inequality, \[131\], and
Fubini’s theorem hence verify that

\[
\left( E\left[ E^p \right] \right)^{1/p} \leq \left( E\left[ E^q \right] \right)^{1/q} \leq \zeta \left( E\left[ \int_{(0,1)^d} |Y(x)|^q + \|Z(x)\|^q \, dx \right] \right)^{1/q} 
\]

\[
= \zeta \left[ \int_{(0,1)^d} E[|Y(x)|^q + \|Z(x)\|^q] \, dx \right]^{1/q} \tag{132}
\]

\[
\leq \zeta \left[ \sup_{x \in [0,1]^d} E[|Y(x)|^q + \|Z(x)\|^q] \right]^{1/q} .
\]

Next note that (109) and Proposition 2.3 prove that for all \( x \in [0,1]^d \) we have that

\[
\left( E\left[ |Y(x)|^q \right] \right)^{1/q} \leq \frac{2 \mathcal{R}}{\sqrt{n}} \left( E\left[ |\xi(\rho(x)) - E[\xi(\rho(x))]|^q \right] \right)^{1/q} \tag{133}
\]

and

\[
\left( E\left[ \|Z(x)\|^q \right] \right)^{1/q} \leq \frac{2 \mathcal{R} (b - a)}{\sqrt{n}} \left( E\left[ \left\| (\nabla \xi)(\rho(x)) - E[(\nabla \xi)(\rho(x))] \right\|^q \right] \right)^{1/q} . \tag{134}
\]

This and (132) imply that

\[
\left( E\left[ E^p \right] \right)^{1/p} \leq \frac{2 \mathcal{R} \zeta}{\sqrt{n}} \left[ \sup_{x \in [0,1]^d} \left( E\left[ |\xi(\rho(x)) - E[\xi(\rho(x))]|^q \right] \right. \right.

\[
+ (b - a)^q \left\| (\nabla \xi)(\rho(x)) - E[(\nabla \xi)(\rho(x))] \right\|^q \right) \right]^{1/q} \tag{135}
\]

Hence, we obtain that

\[
\left( E\left[ E^p \right] \right)^{1/p} \leq \frac{2 \mathcal{R} \zeta}{\sqrt{n}} \left[ \sup_{x \in [a,b]^d} \left( E\left[ |\xi(x) - E[\xi(x)]|^q \right] \right. \right.

\[
+ (b - a)^q \left\| (\nabla \xi)(x) - E[(\nabla \xi)(x)] \right\|^q \right) \right]^{1/q} . \tag{136}
\]

The fact that for all \( r, s \in [0, \infty) \) we have that \( (r + s)^{1/s} \leq r^{1/s} + s^{1/s} \) and the
triangle inequality therefore yield that
\[
(\mathbb{E}[|E|^p])^{1/p} \leq \frac{2R\zeta}{\sqrt{n}} \left( \sup_{x \in [a,b]^d} \left[ \left( \mathbb{E}[|\xi(x) - \mathbb{E}[\xi(x)]|^q]\right)^{1/q} + (b - a) (\mathbb{E}[|\nabla\xi(x) - \mathbb{E}[\nabla\xi(x)]|^q])^{1/q} \right) \right)
\]
\[
\leq \frac{4R\zeta}{\sqrt{n}} \left( \sup_{x \in [a,b]^d} \left[ (\mathbb{E}[|\xi(x)|^q])^{1/q} + (b - a) (\mathbb{E}[|\nabla\xi(x)|^q])^{1/q} \right) \right).
\] (137)

This establishes item (ii). This completes the proof of Lemma 2.16. □

3 Stochastic differential equations with affine coefficient functions

3.1 A priori estimates for Brownian motions

In this subsection we provide in Lemma 3.1 below essentially well-known a priori estimates for standard Brownian motions. Lemma 3.1 will be employed in our proof of Corollary 3.5 in Subsection 3.2 below. Our proof of Lemma 3.1 is a slight adaption of the proof of Lemma 2.5 in Hutzenthaler et al. [39].

Lemma 3.1. Let \(d,m \in \mathbb{N}, T \in [0,\infty), p \in (0,\infty), A \in \mathbb{R}^{d \times m}\), let \(\|\cdot\| : \mathbb{R}^d \to [0,\infty)\) be the standard norm on \(\mathbb{R}^d\), let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, and let \(W : [0,T] \times \Omega \to \mathbb{R}^m\) be a standard Brownian motion. Then it holds for all \(t \in [0,T]\) that
\[
(\mathbb{E}[\|AW_t\|^p])^{1/p} \leq \sqrt{\max\{1, p - 1\}} \text{Trace}(A^*A) \frac{1}{T}.
\] (138)

Proof of Lemma 3.1. Throughout this proof for every \(n \in \mathbb{N}\) let \(\|\cdot\|_{\mathbb{R}^n} : \mathbb{R}^n \to [0,\infty)\) be the standard norm on \(\mathbb{R}^n\), let \((q_r)_{r \in [0,\infty)} \subseteq \mathbb{N}_0\) satisfy for all \(r \in [0,\infty)\) that \(q_r = \max(\mathbb{N}_0 \cap [0, r/2])\), let \(f_r : \mathbb{R}^m \to \mathbb{R}, r \in [0,\infty),\) satisfy for all \(r \in [0,\infty), x \in \mathbb{R}^m\) that
\[
f_r(x) = \|Ax\|_{\mathbb{R}^d}^{q_r},
\] (139)
and let \(\beta^{(i)} : [0,T] \times \Omega \to \mathbb{R}, i \in \{1,\ldots,m\},\) be the stochastic processes which satisfy for all \(t \in [0,T]\) that
\[
W_t = (\beta_t^{(1)}, \ldots, \beta_t^{(m)}).
\] (140)

Note that for all \(r \in [2,\infty), x \in \mathbb{R}^m\) we have that
\[
(\nabla f_r)(x) = r \|Ax\|_{\mathbb{R}^d}^{r-2} A^*Ax.
\] (141)
This implies that for all \( r \in [2, \infty), x \in \mathbb{R}^m \) we have that
\[
(Hess f_r)(x) = r \| Ax \|^{(r-2)}_2 A^* A + \mathbb{1}_{\{Ax \neq 0\}} r (r-2) \| Ax \|^{(r-4)}_2 (A^* Ax) (A^* Ax)^* .
\]
(142)

The fact that for all \( B \in \mathbb{R}^{m\times d}, x \in \mathbb{R}^d \) we have that \( \| Br \|^{2}_\mathbb{R}_m \leq \text{Trace}(B^* B) \| x \|^{2}_\mathbb{R}_d \)
and \( \text{Trace}(B^* B) = \text{Trace}(BB^*) \) hence verifies that for all \( r \in [2, \infty), x \in \mathbb{R}^m \) we have that
\[
\text{Trace}(Hess f_r)(x)
= \text{Trace}
\left(
  r \| Ax \|^{(r-2)}_2 A^* A + \mathbb{1}_{\{Ax \neq 0\}} r (r-2) \| Ax \|^{(r-4)}_2 (A^* Ax) (A^* Ax)^*
\right)
= r \| Ax \|^{(r-2)}_2 \text{Trace}(A^* A) + \mathbb{1}_{\{Ax \neq 0\}} r (r-2) \| Ax \|^{(r-4)}_2 \| A^* Ax \|^{2}_\mathbb{R}_m
\leq r \| Ax \|^{(r-2)}_2 \text{Trace}(A^* A) + \mathbb{1}_{\{Ax \neq 0\}} r (r-2) \| Ax \|^{(r-4)}_2 \text{Trace}(AA^*) \| A^* A \|^{2}_\mathbb{R}_d
= r (r-1) \text{Trace}(A^* A) f_{r-2}(x) .
\]
(143)

Moreover, note that the fact that \( W: [0, T] \times \Omega \rightarrow \mathbb{R}^m \) is a stochastic process
with continuous sample paths (w.c.s.p.) ensures that \( W: [0, T] \times \Omega \rightarrow \mathbb{R}^m \) is a
\((\mathcal{B}([0, T]) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R}^m)\)-measurable function. The fact for all \( r \in [2, \infty) \) we have that \( f_r \in C^2(\mathbb{R}^m, \mathbb{R}) \) hence implies that for all \( r \in [2, \infty), i \in \{1, \ldots, m\} \) we have that
\[
[0, T] \times \Omega \ni (t, \omega) \mapsto (\frac{\partial}{\partial x_i} f_r)(W_t(\omega)) \in \mathbb{R}
\]
(144)
is a \((\mathcal{B}([0, T]) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})\)-measurable function. Combining this and (141) yields that for all \( r \in [2, \infty), i \in \{1, \ldots, m\} \) we have that
\[
\int_0^T \mathbb{E}
\left[
  \left( \frac{\partial}{\partial x_i} f_r \right)(W_t)
\right]^2 dt
\leq \int_0^T \mathbb{E}
\left[
  \| (\nabla f_r)(W_t) \|^{2}_\mathbb{R}_m
\right] dt
= \int_0^T \mathbb{E}
\left[
  r^2 \| AW_t \|^{2r-4}_L(\mathbb{R}^d, \mathbb{R}^m) \| A^* AW_t \|^{2r-2}_\mathbb{R}_m
\right] dt
\leq \int_0^T \mathbb{E}
\left[
  r^2 \| A^* A \|^{2r-2}_\mathbb{R}_d \| AW_t \|^{2r-2}_\mathbb{R}_d
\right] dt
\leq r^2 \text{Trace}(A^* A) T \left( \sup_{t \in [0, T]} \mathbb{E}
\left[
  \| AW_t \|^{2r-2}_\mathbb{R}_d
\right] \right) ,
\]
(145)

Next note that the fact that for all \( r \in [2, \infty) \) we have that \( 2r - 2 \in [2, \infty) \)
ensures that for all \( r \in [2, \infty) \) we have that
\[
\sup_{t \in [0, T]} \mathbb{E}
\left[
  \| AW_t \|^{2r-2}_\mathbb{R}_d
\right] = \left( \sup_{t \in [0, T]} t^{r-1} \right) \mathbb{E}
\left[
  \| AW_1 \|^{2r-2}_\mathbb{R}_d
\right] < \infty.
\]
(146)
Combining this with (145) demonstrates that for all \( r \in [2, \infty) \), \( i \in \{1, \ldots, m\} \) we have that
\[
\int_0^T \mathbb{E}\left[\left|\left(\frac{\partial}{\partial x^i} f_r\right)(W_t)\right|^2\right] dt < \infty. \tag{147}
\]
This proves that for all \( r \in [2, \infty) \), \( i \in \{1, \ldots, m\} \), \( t \in [0, T] \) we have that
\[
\mathbb{E}\left[\int_0^t \left(\frac{\partial}{\partial x^i} f_r\right)(W_s) d\beta_s^{(i)}\right] = 0. \tag{148}
\]
Itô’s formula, Fubini’s theorem, (139), and (143) hence verify that for all \( r \in [2, \infty) \), \( t \in [0, T] \) we have that
\[
\mathbb{E}\left[f_r(W_t)\right] = \mathbb{E}\left[\left\|AW_t\right\|_2^r\right] \leq \frac{2(2 - 1)}{2} \text{Trace}(A^*A) \mathbb{E}\left[f_0(W_0)\right] \leq \text{Trace}(A^*A) t. \tag{150}
\]
Hölder’s inequality therefore proves that for all \( r \in [0, 2) \), \( t \in [0, T] \) we have that
\[
\mathbb{E}\left[f_r(W_t)\right] = \mathbb{E}\left[\left\|AW_t\right\|_2^r\right] \leq \left(\mathbb{E}\left[\left\|AW_t\right\|_2^2\right]\right)^{r/2} \leq \left(\text{Trace}(A^*A) t\right)^{r/2}. \tag{151}
\]
This reveals that for all \( r \in (0, 2] \), \( t \in [0, T] \) we have that
\[
\left(\mathbb{E}\left[\left\|AW_t\right\|_2^r\right]\right)^{1/r} \leq \sqrt{\text{Trace}(A^*A) t}. \tag{152}
\]
Next note that (149), the fact that for all \( r \in (2, \infty) \) we have that \( r - 2q_r \in [0, 2) \),
and \((151)\) imply that for all \(r \in (2, \infty), s_0 \in [0, T]\) we have that

\[
\mathbb{E}[\|AW_{s_0}\|_{\mathbb{R}^d}] \leq \left[\prod_{i=0}^{q_r-1} (r-2i)(r-1-2i)\right] \frac{\text{Trace}(A^*A)^{q_r}}{2q_r} \cdot \int_0^{s_0} \cdots \int_0^{s_{q_r-1}} \mathbb{E}[f_{r-2q_r}(W^{s_{q_r}})] \, ds_{q_r} \cdots ds_1
\]

\[
\leq \left[\prod_{i=0}^{q_r-1} (r-2i)(r-1-2i)\right] \frac{\text{Trace}(A^*A)^{q_r + \frac{r-2q_r}{2}}}{2q_r} \cdot \int_0^{s_0} \cdots \int_0^{s_{q_r-1}} (s_{q_r})^{r-2q_r} ds_{q_r} \cdots ds_1
\]

\[
= \left[\prod_{i=0}^{q_r-1} (r-2i)(r-1-2i)\right] \frac{2q_r}{\left[\prod_{i=0}^{q_r-1} (r-2i)\right]} [\text{Trace}(A^*A)]^{r/2} s_0^{r/2}
\]

\[
= \left[\prod_{i=0}^{q_r-1} (r-1-2i)\right] [\text{Trace}(A^*A)]^{r/2} s_0^{r/2}.
\]

The fact that for all \(r \in (2, \infty)\) we have that \(q_r \leq \frac{r}{2}\) hence yields that for all \(r \in (2, \infty), t \in [0, T]\) we have that

\[
(E[\|AW_t\|_{\mathbb{R}^d}])^{1/r} \leq \left[\prod_{i=0}^{q_r-1} (r-1-2i)\right]^{1/r} \sqrt{\text{Trace}(A^*A) t}
\]

\[
\leq (r-1)^{\frac{q_r}{r}} \sqrt{\text{Trace}(A^*A) t}
\]

\[
\leq (r-1)^{\frac{r}{2r}} \sqrt{\text{Trace}(A^*A) t}
\]

\[
= \sqrt{(r-1) \text{Trace}(A^*A) t}.
\]

Combining this with \((152)\) establishes \((138)\). This completes the proof of Lemma 3.1. \(\square\)

### 3.2 A priori estimates for solutions

In this subsection we present in Lemma 3.4 and Corollary 3.5 below essentially well-known a priori estimates for solutions of stochastic differential equations with at most linearly growing drift coefficient functions and constant diffusion coefficient functions. Corollary 3.5 is one of the main ingredients in our proof of Lemma 3.4 in Subsection 4.1 below and is a straightforward consequence of Lemma 3.3 above and Lemma 3.4 below. Our proof of Lemma 3.3 is a slight adaption of the proof of Lemma 2.6 in Beck et al. [5]. In our formulation of the statements of Lemma 3.4 and Corollary 3.5 below we employ the elementary result in Lemma 3.3 below. In our proof of Lemma 3.3 we employ the elementary result in Lemma 3.2 below. Lemma 3.2 and Lemma 3.3 study measurability properties for time-integrals of suitable stochastic processes.
Lemma 3.2. Let $T \in [0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $Y : [0, T] \times \Omega \to \mathbb{R}$ be $\mathcal{B}([0, T]) \otimes \mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable, and assume that for all $\omega \in \Omega$ we have that $\int_0^T |Y_t(\omega)| \, dt < \infty$. Then the function $\Omega \ni \omega \mapsto \int_0^T Y_t(\omega) \, dt \in \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable.

Proof. Throughout this proof let $Y^+ : [0, T] \times \Omega \to \mathbb{R}$ and $Y^- : [0, T] \times \Omega \to \mathbb{R}$ satisfy for all $t \in [0, T]$, $\omega \in \Omega$ that $Y^+_t(\omega) = \max\{Y_t(\omega), 0\}$ and $Y^-_t(\omega) = -\min\{Y_t(\omega), 0\}$. Note that for all $\omega \in \Omega$ we have that

$$\int_0^T Y_t(\omega) \, dt = \int_0^T Y^+_t(\omega) \, dt - \int_0^T Y^-_t(\omega) \, dt. \quad (155)$$

Moreover, observe that Tonelli’s theorem implies that $\Omega \ni \omega \mapsto \int_0^T Y^+_t(\omega) \, dt \in \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable and $\Omega \ni \omega \mapsto \int_0^T Y^-_t(\omega) \, dt \in \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable. This and $\tag{155}$ prove that $\Omega \ni \omega \mapsto \int_0^T Y_t(\omega) \, dt \in \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable. This completes the proof of Lemma 3.2. \hfill\square

Lemma 3.3. Let $d \in \mathbb{N}$, $c, C, T \in [0, \infty)$, let $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, let $\mu : \mathbb{R}^d \to \mathbb{R}^d$ be a $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d)$-measurable function which satisfies for all $x \in \mathbb{R}^d$ that $\|\mu(x)\| \leq C + c\|x\|$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X : [0, T] \times \Omega \to \mathbb{R}^d$ be a stochastic process w.c.s.p. Then for all $t \in [0, T]$ the function $\Omega \ni \omega \mapsto \int_0^t \mu(X_s(\omega)) \, ds \in \mathbb{R}^d$ is $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable.

Proof. The fact that $X : [0, T] \times \Omega \to \mathbb{R}^d$ is a stochastic process w.c.s.p. and Aliprantis and Border \cite{AlpBan07} Lemma 4.51 ensure that for all $t \in [0, T]$ the function $[0, t] \times \Omega \ni (s, \omega) \mapsto X_s(\omega) \in \mathbb{R}^d$ is $\mathcal{B}([0, t]) \otimes \mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable. This and the hypothesis that $\mu : \mathbb{R}^d \to \mathbb{R}^d$ is $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d)$-measurable imply that for all $i \in \{1, \ldots, d\}$, $t \in [0, T]$ we have that the function $[0, t] \times \Omega \ni (s, \omega) \mapsto \mu_i(X_s(\omega)) \in \mathbb{R}$ is $(\mathcal{B}([0, t]) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R})$-measurable. Moreover, note that the hypothesis that for all $x \in \mathbb{R}^d$ we have that $\|\mu(x)\| \leq C + c\|x\|$ and the fact that for all $\omega \in \Omega$ the function $[0, T] \ni t \mapsto X_t(\omega) \in \mathbb{R}^d$ is continuous imply that for all $t \in [0, T]$, $\omega \in \Omega$, $i \in \{1, \ldots, d\}$ we have that $\int_0^t \|\mu_i(X_s(\omega))\| \, ds < \infty$. Lemma 3.2 (applied with $T \leftarrow t$, $Y \leftarrow \mu(X)$ for $t \in [0, T]$, $i \in \{1, \ldots, d\}$ in the notation of Lemma 3.2) hence proves that for all $i \in \{1, \ldots, d\}$, $t \in [0, T]$ the function $\Omega \ni \omega \mapsto \int_0^t \mu_i(X_s(\omega)) \, ds \in \mathbb{R}^d$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable. This implies that for all $t \in [0, T]$ the function $\Omega \ni \omega \mapsto \int_0^t \mu(X_s(\omega)) \, ds \in \mathbb{R}^d$ is $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable. This completes the proof of Lemma 3.3. \hfill\square

Lemma 3.4. Let $d, m \in \mathbb{N}$, $x \in \mathbb{R}^d$, $p \in [1, \infty)$, $c, C, T \in [0, \infty)$, $A \in \mathbb{R}^{d \times m}$, let $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W : [0, T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion, let $\mu : \mathbb{R}^d \to \mathbb{R}^d$ be a $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d)$-measurable function which satisfies for all $y \in \mathbb{R}^d$ that $\|\mu(y)\| \leq C + c\|y\|$, and let $X : [0, T] \times \Omega \to \mathbb{R}^d$ be a stochastic process w.c.s.p. which satisfies for all $t \in [0, T]$ that

$$\mathbb{P}\left( X_t = x + \int_0^t \mu(X_s) \, ds + AW_t \right) = 1 \quad (156)$$

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(cf. Lemma 3.3). Then
\[
\left( \mathbb{E}[\|X_T\|^p] \right)^{1/p} \leq \left( \|x\| + CT + \left( \mathbb{E}[\|AW_T\|^p] \right)^{1/p} \right) e^{CT}.
\] (157)

Proof of Lemma 3.4. Throughout this proof for every \( n \in \mathbb{N} \) let \( ||\cdot||_{\mathbb{R}^n} : \mathbb{R}^n \rightarrow [0, \infty) \) be the standard norm on \( \mathbb{R}^n \), let \( \beta^{(i)} : [0, T] \times \Omega \rightarrow \mathbb{R}, i \in \{1, \ldots, m\} \), be the stochastic processes which satisfy for all \( t \in [0, T] \) that
\[
W_t = (\beta_t^{(1)}, \ldots, \beta_t^{(m)})
\] (158)
and let \( B \subseteq \Omega \) be the set given by
\[
B = \bigcap_{t \in [0, T]} \left\{ X_t = x + \int_0^t \mu(X_s) \, ds + AW_t \right\}
\] = \( \{ \omega \in \Omega : (\forall t \in [0, T] : X_t(\omega) = x + \int_0^t \mu(X_s(\omega)) \, ds + AW_t(\omega) \} \).
\] (159)
Observe that the fact that \( X : [0, T] \times \Omega \rightarrow \mathbb{R}^d \) and \( W : [0, T] \times \Omega \rightarrow \mathbb{R}^m \) are stochastic processes w.c.s.p. yields that
\[
B = \left( \bigcap_{t \in [0, T] \cap Q} \left\{ X_t = x + \int_0^t \mu(X_s) \, ds + AW_t \right\} \right) \in \mathcal{F}.
\] (160)
Combining this and (156) proves that
\[
\mathbb{P}(B) = \mathbb{P}\left( \bigcap_{t \in [0, T] \cap Q} \left\{ X_t = x + \int_0^t \mu(X_s) \, ds + AW_t \right\} \right)
\] = 1 - \( \mathbb{P}\left( \Omega \setminus \bigcap_{t \in [0, T] \cap Q} \left\{ X_t = x + \int_0^t \mu(X_s) \, ds + AW_t \right\} \right) \)
\] = 1 - \( \mathbb{P}\left( \bigcup_{t \in [0, T] \cap Q} \left\{ X_t \neq x + \int_0^t \mu(X_s) \, ds + AW_t \right\} \right) \)
\] \geq 1 - \left[ \sum_{t \in [0, T] \cap Q} \mathbb{P}\left( X_t \neq x + \int_0^t \mu(X_s) \, ds + AW_t \right) \right] = 1.
\] (161)

Next note that the triangle inequality and the hypothesis that for all \( y \in \mathbb{R}^d \) we
have that \( \| \mu(y) \| \leq C + c \| y \| \) ensure that for all \( \omega \in B, t \in [0, T] \) we have that

\[
\| X_t(\omega) \| \leq \| x \| + \| A W_t(\omega) \| + \int_0^t \| \mu(X_s(\omega)) \| \, ds
\]

\[
\leq \| x \| + \| A W_t(\omega) \| + C t + c \int_0^t \| X_s(\omega) \| \, ds
\]

\[
\leq \| x \| + \left( \sup_{s \in [0, T]} \| A W_s(\omega) \| \right) + C T + c \int_0^t \| X_s(\omega) \| \, ds. \tag{162}
\]

Moreover, note that the assumption that \( X : [0, T] \times \Omega \to \mathbb{R}^d \) is a stochastic process w.c.s.p. assures that for all \( \omega \in \Omega \) we have that

\[
\int_0^T \| X_s(\omega) \| \, ds < \infty. \tag{163}
\]

Grohs et al. [28, Lemma 2.11] (applied with \( \alpha \leftarrow \| x \| + \sup_{t \in [0, T]} \| A W_t(\omega) \| + C T, \beta \leftarrow c, f \leftarrow (0, T] \) \( t \mapsto \| X_t(\omega) \| \in [0, \infty) \)) for \( \omega \in B \) in the notation of Grohs et al. [28, Lemma 2.11]) and (162) hence prove that for all \( \omega \in B, t \in [0, T] \) we have that

\[
\| X_t(\omega) \| \leq \left( \| x \| + \left( \sup_{s \in [0, T]} \| A W_s(\omega) \| \right) + C T \right) e^{cT}. \tag{164}
\]

Next note that

\[
\left( \mathbb{E} \left[ \sup_{s \in [0, T]} \| A W_s \|^p \right] \right)^{1/p} \leq \| A \|_{L(\mathbb{R}^m, \mathbb{R}^d)} \left( \mathbb{E} \left[ \sup_{s \in [0, T]} \| W_s \|^p \right] \right)^{1/p}. \tag{165}
\]

Moreover, if \( p \in [1, 2] \), then we have that

\[
\sup_{s \in [0, T]} \| W_s \|_{\mathbb{R}^m}^p = \sup_{s \in [0, T]} \left( \sum_{i=1}^m |\beta_s^{(i)}|^2 \right)^{p/2} \leq \sup_{s \in [0, T]} \left( 1 + \sum_{i=1}^m |\beta_s^{(i)}|^2 \right)^{p/2}
\]

\[
\leq 1 + \sum_{i=1}^m \left( \sup_{s \in [0, T]} |\beta_s^{(i)}|^2 \right). \tag{166}
\]

Hence, if \( p \in [1, 2] \), then the Burkholder–Davis–Gundy inequality (see, e.g., Karatzas and Shreve [44, Theorem 3.28]) implies that

\[
\mathbb{E} \left[ \sup_{s \in [0, T]} \| W_s \|_{\mathbb{R}^m}^p \right] \leq 1 + m \mathbb{E} \left[ \sup_{s \in [0, T]} |\beta_s^{(1)}|^2 \right] < \infty. \tag{167}
\]

Next note that if \( p \in (2, \infty) \), then Hölder’s inequality implies that

\[
\sup_{s \in [0, T]} \| W_s \|_{\mathbb{R}^m}^p = \sup_{s \in [0, T]} \left( \sum_{i=1}^m |\beta_s^{(i)}|^2 \right)^{p/2} \leq m^{p/2-1} \sup_{s \in [0, T]} \left( \sum_{i=1}^m |\beta_s^{(i)}|^p \right)^{1/p}
\]

\[
\leq m^{p/2-1} \sum_{i=1}^m \left( \sup_{s \in [0, T]} |\beta_s^{(i)}|^p \right). \tag{168}
\]
Hence, if \( p \in (2, \infty) \), then the Burkholder–Davis–Gundy inequality (see, e.g., Karatzas and Shreve [44, Theorem 3.28]) implies that
\[
\mathbb{E} \left[ \sup_{s \in [0,T]} \|W_s\|_p^p \right] \leq m^{p/2 - 1} m \mathbb{E} \left[ \sup_{s \in [0,T]} |\beta_s^{(1)}|^p \right] < \infty. \tag{169}
\]
This, (165), and (167) hence imply that
\[
\left( \mathbb{E} \left[ \sup_{s \in [0,T]} \|AW_s\|_p^p \right] \right)^{1/p} < \infty. \tag{170}
\]
Combining this, (164), and (161) yields that
\[
\int_0^T \left( \mathbb{E} \left[ \|X_t\|_p^p \right] \right)^{1/p} dt
\leq T \left[ \sup_{t \in [0,T]} \left( \mathbb{E} \left[ \|X_t\|_p^p \right] \right)^{1/p} \right]
\leq T \left[ \mathbb{E} \left[ \|x\| + \sup_{s \in [0,T]} \|AW_s\| + CT |pe^{ct}T|^p \right] \right]^{1/p}
\leq T \left[ \|x\| + \left( \mathbb{E} \left[ \sup_{s \in [0,T]} \|AW_s\| \right]^p \right)^{1/p} + CT \right] e^{cT} < \infty. \tag{171}
\]
Next observe that the hypothesis that \( X : [0,T] \times \Omega \to \mathbb{R}^d \) is a stochastic process w.c.s.p. ensures that \( X : [0,T] \times \Omega \to \mathbb{R}^d \) is a \((\mathcal{B}([0,T]) \otimes \mathcal{F})/\mathcal{B}(\mathbb{R}^d)\)-measurable function. This reveals that for all \( t \in [0,T] \) we have that
\[
\left( \mathbb{E} \left[ \left\| \int_0^t \|X_s\|_p^p \, ds \right\|^{1/p} \right] \right)^{1/p} \leq \int_0^t \left( \mathbb{E} \left[ \|X_s\|_p^p \right] \right)^{1/p} ds \tag{172}
\]
(cf., for example, Garling [24, Corollary 5.4.2] or Jentzen & Kloeden [42, Proposition 8 in Appendix A]). The triangle inequality, the fact that for all \( t \in [0,T] \) we have that \( W_t \) has the same distribution as \( \sqrt{\frac{t}{T}} W_T \), (161), and (162) therefore verify that for all \( t \in [0,T] \) we have that
\[
\left( \mathbb{E} \left[ \|X_t\|_p^p \right] \right)^{1/p}
\leq \|x\| + \left( \mathbb{E} \left[ \|AW_t\|_p^p \right] \right)^{1/p} + Ct + c \int_0^t \left( \mathbb{E} \left[ \|X_s\|_p^p \right] \right)^{1/p} ds
\leq \|x\| + \sqrt{\frac{T}{t}} \left( \mathbb{E} \left[ \|AW_T\|_p^p \right] \right)^{1/p} + Ct + c \int_0^t \left( \mathbb{E} \left[ \|X_s\|_p^p \right] \right)^{1/p} ds
\leq \|x\| + \left( \mathbb{E} \left[ \|AW_T\|_p^p \right] \right)^{1/p} + CT + C \int_0^t \left( \mathbb{E} \left[ \|X_s\|_p^p \right] \right)^{1/p} ds. \tag{173}
\]
Combining Grohs et al. [28, Lemma 2.11] (applied with $\alpha \leftarrow \|x\| + (\mathbb{E}[\|AW_T\|^p])^{1/p} + CT$, $\beta \leftarrow c$, $f \leftarrow (0, T) \ni t \mapsto (\mathbb{E}[\|X_t\|^p])^{1/p} \in [0, \infty)$) in the notation of Grohs et al. [28, Lemma 2.11]) and (171) hence establishes that for all $t \in [0, T]$ we have that

$$(\mathbb{E}[\|X_t\|^p])^{1/p} \leq \left(\|x\| + (\mathbb{E}[\|AW_T\|^p])^{1/p} + CT\right) e^{ct}. \quad (174)$$

This completes the proof of Lemma 3.4.

**Corollary 3.5.** Let $d, m \in \mathbb{N}$, $x \in \mathbb{R}^d$, $p \in [1, \infty)$, $c, C, T \in [0, \infty)$, $A \in \mathbb{R}^{d \times m}$, let $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W : [0, T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion, let $\mu : \mathbb{R}^d \to \mathbb{R}^d$ be a $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d)$-measurable function which satisfies for all $y \in \mathbb{R}^d$ that $\|\mu(y)\| \leq C + c\|y\|$, and let $X : [0, T] \times \Omega \to \mathbb{R}^d$ be a stochastic process w.c.s.p. which satisfies for all $t \in [0, T]$ that

$$\mathbb{P}\left(X_t = x + \int_0^t \mu(X_s) \, ds + AW_t\right) = 1 \quad (175)$$

(cf. Lemma 3.3). Then

$$(\mathbb{E}[\|X_T\|^p])^{1/p} \leq \left(\|x\| + CT + \sqrt{\max\{1, p-1\} \text{Trace}(A^*A)T}\right) e^{cT}. \quad (176)$$

**Proof of Corollary 3.5.** Observe that Lemma 3.1 and Lemma 3.4 establish (176). This completes the proof of Corollary 3.5.

### 3.3 A priori estimates for differences of solutions

In this subsection we provide in Lemma 3.6 below well-known a priori estimates for differences of solutions of stochastic differential equations with Lipschitz continuous drift coefficient functions and constant diffusion coefficient functions. Lemma 3.6 is one of the main ingredients in our proof of Lemma 4.1 in Subsection 4.1 below. Our proof of Lemma 3.6 is a slight adaption of the proof of Lemma 2.6 in Beck et al. [5].

**Lemma 3.6.** Let $d, m \in \mathbb{N}$, $p \in [1, \infty)$, $l \in [0, \infty)$, $T \in [0, \infty)$, $A \in \mathbb{R}^{d \times m}$, let $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W : [0, T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion, let $\mu : \mathbb{R}^d \to \mathbb{R}^d$ be a $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R}^d)$-measurable function which satisfies for all $x, y \in \mathbb{R}^d$ that $\|\mu(x) - \mu(y)\| \leq l\|x - y\|$, and let $X^x : [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be stochastic processes w.c.s.p. which satisfy for all $x \in \mathbb{R}^d$, $t \in [0, T]$ that

$$\mathbb{P}\left(X_t^x = x + \int_0^t \mu(X_s^x) \, ds + AW_t\right) = 1. \quad (177)$$

Then it holds for all $x, y \in \mathbb{R}^d$, $t \in [0, T]$ that

$$(\mathbb{E}[\|X_t^x - X_t^y\|^p])^{1/p} \leq e^{ct}\|x - y\|. \quad (178)$$
Proof of Lemma 3.6. Throughout this proof let \( B_x \subseteq \Omega, x \in \mathbb{R}^d \), be the sets which satisfy for all \( x \in \mathbb{R}^d \) that

\[
B_x = \bigcap_{t \in [0,T]} \left\{ X_t^x = x + \int_0^t \mu(X_s^x) \, ds + AW_t \right\}
\]

\[
= \left\{ \omega \in \Omega : \forall t \in [0,T] : X_t^x(\omega) = x + \int_0^t \mu(X_s^x(\omega)) \, ds + AW_t(\omega) \right\}.
\] (179)

Observe that the fact that for all \( x \in \mathbb{R}^d \) we have that \( X^x : [0,T] \times \Omega \to \mathbb{R}^d \) and \( W : [0,T] \times \Omega \to \mathbb{R}^m \) are stochastic processes w.c.s.p. yields that for all \( x \in \mathbb{R}^d \) we have that

\[
B_x = \left( \bigcap_{t \in [0,T] \cap Q} \left\{ X_t^x = x + \int_0^t \mu(X_s^x) \, ds + AW_t \right\} \right) \in \mathcal{F}.
\] (180)

Combining this and (177) proves that for all \( x \in \mathbb{R}^d \) we have that

\[
P(B_x) = P \left( \bigcap_{t \in [0,T] \cap Q} \left\{ X_t^x = x + \int_0^t \mu(X_s^x) \, ds + AW_t \right\} \right)
\]

\[
= 1 - P \left( \Omega \setminus \bigcap_{t \in [0,T] \cap Q} \left\{ X_t^x = x + \int_0^t \mu(X_s^x) \, ds + AW_t \right\} \right)
\]

\[
= 1 - P \left( \bigcup_{t \in [0,T] \cap Q} \left\{ X_t^x \neq x + \int_0^t \mu(X_s^x) \, ds + AW_t \right\} \right)
\]

\[
\geq 1 - \sum_{t \in [0,T] \cap Q} P \left( X_t^x \neq x + \int_0^t \mu(X_s^x) \, ds + AW_t \right) = 1.
\] (181)

This reveals that for all \( x, y \in \mathbb{R}^d \) we have that

\[
P(B_x \cap B_y) = 1 - P(\Omega \setminus [B_x \cap B_y])
\]

\[
= 1 - P(B_x^c \cup B_y^c) \geq 1 - [P(B_x^c) + P(B_y^c)] = 1.
\] (182)

Next note that the triangle inequality and the assumption that for all \( x, y \in \mathbb{R}^d \) we have that \( \|\mu(x) - \mu(y)\| \leq \|x - y\| \) ensure that for all \( x, y \in \mathbb{R}^d, \omega \in B_x \cap B_y, t \in [0,T] \) we have that

\[
\|X_t^x(\omega) - X_t^y(\omega)\| \leq \|x - y\| + \int_0^t \|\mu(X_s^x(\omega)) - \mu(X_s^y(\omega))\| \, ds
\]

\[
\leq \|x - y\| + t \int_0^t \|X_s^x(\omega) - X_s^y(\omega)\| \, ds.
\] (183)
Moreover, note that the assumption that for all $x \in \mathbb{R}^d$ we have that $X^x : [0, T] \times \Omega \to \mathbb{R}^d$ is a stochastic process with continuous sample paths assures that for all $x, y \in \mathbb{R}^d$, $\omega \in \Omega$ it holds that

$$\int_0^T \|X^x_t(\omega) - X^y_t(\omega)\| \, ds < \infty. \quad (184)$$

Grohs et al. [28, Lemma 2.11] (applied with $\beta \mapsto \|t \in \omega \times x, y\| \in [0, \infty]$) hence prove that for all $x, y \in \mathbb{R}^d$, $\omega \in B_x \cap B_y$ in the notation of Grohs et al. [28 Lemma 2.11] and (183) hence prove that for all $x, y \in \mathbb{R}^d$, $\omega \in B_x \cap B_y$, $t \in [0, T]$ we have that

$$\|X^x_t(\omega) - X^y_t(\omega)\| \leq \|x - y\| e^{\|x\| t}. \quad (185)$$

This and (182) prove that for all $x, y \in \mathbb{R}^d$, $t \in [0, T]$ we have that

$$(\mathbb{E}[\|X^x_t - X^y_t\|^p])^{1/p} \leq c^{1/t} \|x - y\|. \quad (186)$$

This completes the proof of Lemma 3.6.

4 Error estimates

4.1 Quantitative error estimates

In this subsection we establish in Corollary 4.3 below a quantitative approximation result for viscosity solutions (cf., for example, Hairer et al. [31]) of Kolmogorov PDEs with constant coefficient functions. Our proof of Corollary 4.3 employs the quantitative approximation results in Lemma 4.1 and Corollary 4.2 below. Corollary 4.3 is one of the key ingredients which we use in our proof of Proposition 4.4 and Proposition 4.6 below, respectively, in order to construct ANN approximations for viscosity solutions of Kolmogorov PDEs with constant coefficient functions.

Lemma 4.1. Let $d, n \in \mathbb{N}$, $\varphi \in C(\mathbb{R}^d, \mathbb{R})$, $c, l, a \in \mathbb{R}$, $b \in (a, \infty)$, $\zeta, \varepsilon, T \in (0, \infty)$, $v, w, w, z, z \in [0, \infty)$, $p \in [1, \infty)$, let $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, let $\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be the $d$-dimensional Euclidean scalar product, let $\mathfrak{R} \in (0, \infty)$ be the max{$2, p$}-Kahane–Khintchine constant (cf. Definition 2.1 and Lemma 2.2), assume for all $\Phi \in C^1((0, 1)^d, \mathbb{R})$ that

$$\sup_{x \in (0, 1)^d} |\Phi(x)| \leq \zeta \left[ \int_{(0, 1)^d} \left( |\Phi(x)|^{\max\{2, p\}} + |(\nabla \Phi)(x)|^{\max\{2, p\}} \right) \, dx \right]^{1/\max\{2, p\}}, \quad (187)$$

let $\phi \in C^1(\mathbb{R}^d, \mathbb{R})$ satisfy for all $x \in \mathbb{R}^d$ that

$$|\phi(x)| \leq cd^\varepsilon(1 + \|x\|^z), \quad |(\nabla \phi)(x)| \leq cd^\varepsilon(1 + \|x\|^w), \quad (188)$$

and

$$|\varphi(x) - \phi(x)| \leq d^\varepsilon(1 + \|x\|^z), \quad (189)$$

and
let $\mu: \mathbb{R}^d \to \mathbb{R}$ satisfy for all $x, y \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$ that $\mu(\lambda x + y) = \lambda \mu(x) + \mu(y)$ and

$$\|\mu(x) - \mu(y)\| \leq l\|x - y\|. \quad (190)$$

let $A = (A_{i,j})_{i,j} \in \{1, \ldots, d\} \in \mathbb{R}^{d \times d}$ be a symmetric and positive semi-definite matrix, let $u \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$, assume for all $x \in \mathbb{R}^d$ that $u(0, x) = \varphi(x)$, assume that $\inf_{\gamma \in (0, \infty)} \sup_{(t,x) \in [0, T] \times \mathbb{R}^d} \left( \frac{|u(t,x)|}{1 + \|x\|^2} \right) < \infty$, and assume that $u|_{(0, T) \times \mathbb{R}^d}$ is a viscosity solution of

$$\left( \frac{\partial}{\partial t} u(t, x) = (\mu(x), (\nabla_x u)(t, x)) + \frac{1}{2} \sum_{i,j=1}^d A_{i,j} \left( \frac{\partial^2}{\partial x_i \partial x_j} u \right)(t, x) \right) \quad (191)$$

for $(t, x) \in (0, T) \times \mathbb{R}^d$. Then there exist $W_1, \ldots, W_n \in \mathbb{R}^{d \times d}$, $B_1, \ldots, B_n \in \mathbb{R}^d$ such that

$$\sup_{x \in [a, b]^d} \left| u(T, x) - \left[ \frac{1}{n} \sum_{k=1}^n \phi(W_k x + B_k) \right] \right| \leq c_d \left( 1 + e^{c_T} \right) T \|\mu(0)\| + \sqrt{\max \{1, \gamma - 1\}} T \text{Trace}(A)$$

$$+ \sup_{x \in [a, b]^d} \|x\|^2 \cdot (b - a) c_d^{n+1/2} e^{c_T} \left( 1 + e^{c_T} \right) T \|\mu(0)\|$$

$$+ \sqrt{\max \{1, \gamma \max \{2, p\} + 1\}} T \text{Trace}(A) + \sup_{x \in [a, b]^d} \|x\| \right)^w \right). \quad (192)$$

Proof of Lemma 4.1. Throughout this proof let $e_1, \ldots, e_d \in \mathbb{R}$ satisfy that $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$, let $m_r: (0, \infty) \to [r, \infty)$, $r \in \mathbb{R}$, satisfy for all $r \in \mathbb{R}$, $x \in (0, \infty)$ that $m_r(x) = \max \{r, x\}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0, T]}$, let $W^i: [0, T] \times \Omega \to \mathbb{R}^d$, $i \in \mathbb{N}$, be independent standard $(\mathcal{F}_t)_{t \in [0, T]}$-Brownian motions, and let $X^i_t: [0, T] \times \Omega \to \mathbb{R}^d$, $i \in \mathbb{N}$, be $(\mathcal{F}_t)_{t \in [0, T]}$-adapted stochastic processes w.c.s.p. which satisfy that

(a) for all $i \in \mathbb{N}$, $x \in \mathbb{R}^d$, $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that

$$X^i_t = x + \int_0^t \mu(X^i_s) \, ds + \sqrt{A} W^i_t \quad (193)$$

and

(b) for all $i \in \mathbb{N}$, $x, y \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$, $t \in [0, T]$, $\omega \in \Omega$ we have that $X^i_t, \lambda x + y(\omega) + \lambda X^i_t, 0(\omega) = \lambda X^i_t, x(\omega) + X^i_t, y(\omega)$

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(cf. Grohs et al. [28, Proposition 2.22]). Note that item [13] and Grohs et al. [28, Corollary 2.8] (applied with $d \leftarrow d$, $m \leftarrow d$, $\varphi \leftarrow (\mathbb{R}^d \ni x) \mapsto X_T^{i,x}(\omega) \in \mathbb{R}^d$) for $i \in \mathbb{N}$, $\omega \in \Omega$ in the notation of Grohs et al. [28, Corollary 2.8]) ensure that for all $i \in \mathbb{N}$, $\omega \in \Omega$ there exist $\mathcal{W}_{i,\omega} \in \mathbb{R}^{d \times d}$ and $\mathcal{B}_{i,\omega} \in \mathbb{R}^d$ which satisfy that for all $x \in \mathbb{R}^d$ we have that
\[
X_T^{i,x}(\omega) = \mathcal{W}_{i,\omega} x + \mathcal{B}_{i,\omega}.
\] (194)
Combining this with the assumption that $\phi \in C^1(\mathbb{R}^d, \mathbb{R})$ proves that for all $\omega \in \Omega$, $i \in \mathbb{N}$ we have that
\[
\left(\mathbb{R}^d \ni x \mapsto \phi\left(X_T^{i,x}(\omega)\right) \in \mathbb{R}\right) \in C^1(\mathbb{R}^d, \mathbb{R}).
\] (195)
Next note that [188] and [189] ensure that $\varphi : \mathbb{R}^d \to \mathbb{R}$ is an at most polynomially growing function. This, [190], and the Feynman-Kac formula (cf., for example, Grohs et al. [28, Proposition 2.22] or Hairer et al. [31, Corollary 4.17]) imply that for all $x \in \mathbb{R}^d$ we have that
\[
u(T, x) = \mathbb{E}\left[\phi(X_T^{1,x})\right].
\] (196)
This verifies that
\[
\sup_{x \in [a, b]^d} \left|\nu(T, x) - \mathbb{E}\left[\phi(X_T^{1,x})\right]\right| = \sup_{x \in [a, b]^d} \left|\mathbb{E}\left[\phi(X_T^{1,x}) - \phi(X_T^{1,x})\right]\right|
\leq \sup_{x \in [a, b]^d} \mathbb{E}\left[\left|\phi(X_T^{1,x}) - \phi(X_T^{1,x})\right|\delta^v(1 + \|X_T^{1,x}\|)^\gamma\right]
\leq \delta^v \sup_{y \in \mathbb{R}^d} \left(\frac{|\phi(y) - \phi(y)|}{\delta^v(1 + \|y\|)^\gamma}\right) \sup_{x \in [a, b]^d} \mathbb{E}\left[1 + \|X_T^{1,x}\|\right].
\] (197)
Jensen’s inequality and [189] therefore verify that
\[
\sup_{x \in [a, b]^d} \left|\nu(T, x) - \mathbb{E}\left[\phi(X_T^{1,x})\right]\right|
\leq \delta^v \sup_{y \in \mathbb{R}^d} \left(\frac{|\phi(y) - \phi(y)|}{\delta^v(1 + \|y\|)^\gamma}\right) \left[1 + \sup_{x \in [a, b]^d} \mathbb{E}\left[\|X_T^{1,x}\|\right]\right]
\leq \varepsilon \delta^v \left[1 + \sup_{x \in [a, b]^d} \mathbb{E}\left[\|X_T^{1,x}\|^{m_1(v)}\right]\right]^{\gamma/m_1(v)}.
\] (198)
Item [71], Corollary [33] (applied with $m \leftarrow d$, $C \leftarrow \|\mu(0)\|$, $c \leftarrow l$, $W \leftarrow W^1$, $A \leftarrow \sqrt{A}$, $X \leftarrow X^{1,x}$, $p \leftarrow m_1(v)$ for $x \in [a, b]^d$ in the notation of Corollary [5.4],
and the fact that $m_1(m_1(v) - 1) = m_1(v - 1)$ hence yield that

$$
\sup_{x \in [a, b]^d} \left| u(T, x) - E[\phi(X_T^{1,x})] \right|
\leq \varepsilon d^2 \left( 1 + e^{x^T \mu(0)} T + \sqrt{m_1(v - 1) T \text{Trace}(A)} \right)
\quad \text{(199)}
$$

Next note that

$$
\sup_{x \in [a, b]^d} \left| \phi(X_T^{1,x}) \right|^{m_2(p)}
= \sup_{x \in [a, b]^d} \left[ E \left[ \left| \phi(X_T^{1,x}) \right|^{m_2(p)} \right] \right]^{1/m_2(p)}
\leq d^2 \left( \sup_{y \in \mathbb{R}^d} \frac{|\phi(y)|}{d^2(1 + \|y\|^{2})} \right) \left( \sup_{x \in [a, b]^d} \left| \left( 1 + \|X_T^{1,x}\|^{2} \right)^{m_2(p)} \right| \right)^{1/m_2(p)}
\quad \text{(200)}
$$

This, (188), the triangle inequality, and Hölder’s inequality verify that

$$
\sup_{x \in [a, b]^d} \left| E \left[ \phi(X_T^{1,x}) \right]^{m_2(p)} \right|
\leq d^2 \left[ 1 + \sup_{x \in [a, b]^d} \left| E \left[ \|X_T^{1,x}\|^{m_2(p)} \right] \right| \right]^{1/m_2(p)}
\quad \text{(201)}
$$

Item (3), Corollary 3.3 (applied with $m \leftarrow d$, $C \leftarrow \|\mu(0)\|$, $c \leftarrow l$, $A \leftarrow \sqrt{A}$, $W \leftarrow W^1$, $X \leftarrow X^{1,x}$, $p \leftarrow m_1(m_2(p))$) for $x \in [a, b]^d$ in the notation of Corollary 3.3, and the fact that $m_1(m_1(m_2(p)) - 1) = m_1(m_2(p) - 1)$ hence yield that

$$
\sup_{x \in [a, b]^d} \left| E \left[ \phi(X_T^{1,x}) \right]^{m_2(p)} \right|
\leq d^2 \left( 1 + e^{x^T \mu(0)} T + \sqrt{m_1(m_2(p) - 1) T \text{Trace}(A)} \right)
\quad \text{(202)}
$$
Next note that
\[
\sup_{x \in [a, b]^d} \left| E \left[ \limsup_{\mathbb{R}^d \setminus \{0\} \ni h \to 0} \left( \frac{\phi(X_1^T x + h) - \phi(X_1^T x)}{\|h\| m_2(p)} \right) \right] \right|^{1/m_2(p)}
\]
\[
= \sup_{x \in [a, b]^d} \left| E \left[ \left( \frac{\partial}{\partial x_1^T} \right)^* (\nabla \phi)(X_1^T x) \right] \right|^{1/m_2(p)}
\]
\[
\leq \sup_{x \in [a, b]^d} \left| E \left[ \left( \nabla \phi \right)(X_1^T x) \right] \right|^{1/m_2(p)}.
\]

Moreover, observe that for all \( x \in [a, b]^d \) we have that
\[
\left| E \left[ \left( \nabla \phi \right)(X_1^T x) \right] \right|^{1/m_2(p)}
\]
\[
= \left| E \left[ \left( \nabla \phi \right)(X_1^T x) \right] \right|^{1/m_2(p)}
\]
\[
\leq d^w \left[ \sup_{y \in \mathbb{R}^d} \frac{\| (\nabla \phi)(y) \|}{d^w (1 + \|y\|^{m_2(p)})} \right] \left| E \left[ \left( 1 + \|X_1^T x\|^{m_2(p)} \right) \|\frac{\partial}{\partial x_1^T} X_1^T x \|^{m_2(p)} \right] \right|^{1/m_2(p)}.
\]

This, (203), and (205) prove that
\[
\sup_{x \in [a, b]^d} \left| E \left[ \limsup_{\mathbb{R}^d \setminus \{0\} \ni h \to 0} \left( \frac{\phi(X_1^T x + h) - \phi(X_1^T x)}{\|h\| m_2(p)} \right) \right] \right|^{1/m_2(p)}
\]
\[
\leq cd^w \sup_{x \in [a, b]^d} \left| E \left[ \left( 1 + \|X_1^T x\|^{m_2(p)} \right) \frac{\partial}{\partial x_1^T} X_1^T x \right] \left|^{1/m_2(p)} \right.
\]

This, Hölder's inequality, and the triangle inequality verify that
\[
\sup_{x \in [a, b]^d} \left| E \left[ \limsup_{\mathbb{R}^d \setminus \{0\} \ni h \to 0} \left( \frac{\phi(X_1^T x + h) - \phi(X_1^T x)}{\|h\| m_2(p)} \right) \right] \right|^{1/m_2(p)}
\]
\[
\leq cd^w \sup_{x \in [a, b]^d} \left( E \left[ \left( \frac{\partial}{\partial x_1^T} X_1^T x \right)^{m_2(p)(m_2(p)+1)} \right] \right|^{1/(m_2(p)(m_2(p)+1))}
\]
\[
\cdot E \left[ \left( 1 + \|X_1^T x\|^{m_2(p)+1} \right) \right|^{1/(m_2(p)+1)} \right]
\]
\[
\leq cd^w \left( E \left[ \left( \frac{\partial}{\partial x_1^T} X_1^T x \right)^{m_2(p)(m_2(p)+1)} \right] \right|^{1/(m_2(p)(m_2(p)+1))}
\]
\[
\cdot \left( 1 + E \left[ \left( \|X_1^T x\|^{m_2(p)+1} \right) \right|^{1/(m_2(p)+1)} \right).
\]
Jensen’s inequality therefore yields that

\[
\sup_{x \in [a, b]^d} \left| \mathbb{E} \left[ \limsup_{\mathbb{R}^d \setminus \{0\} : h \to 0} \left( \frac{\| \phi(X_T^{1,x+h}) - \phi(X_T^{1,x}) \|_{m_2(p)}}{\|h\|^{m_2(p)}} \right) \right] \right|^{1/m_2(p)} \leq c d^\omega \sup_{x \in [a, b]^d} \left( \mathbb{E} \left[ \left\| \left( \frac{\partial}{\partial x} X_T^{1,x} \right) e_i \right\|_{m_2(p)(m_2(p)+1)}^{m_2(p)(m_2(p)+1)} \right] \right)^{1/(m_2(p)(m_2(p)+1))} \cdot \left( 1 + \mathbb{E} \left[ \left\| X_T^{1,x} \right\|_{m_1(w(m_2(p)+1))}^{w/m_1(w(m_2(p)+1))} \right] \right).
\]

(207)

Next note that (193) implies that for all \( \omega \in \Omega, x, y \in \mathbb{R}^d \) we have that

\[
\left( \frac{\partial}{\partial x} X_T^{1,x} (\omega) \right) y = X_T^{1,y} (\omega) - X_T^{1,0} (\omega).
\]

(208)

This and Hölder’s inequality verify that for all \( x \in \mathbb{R}^d \) we have that

\[
\mathbb{E} \left[ \left\| \left( \frac{\partial}{\partial x} X_T^{1,x} \right) e_i \right\|_{m_2(p)(m_2(p)+1)}^{m_2(p)(m_2(p)+1)} \right]^{1/(m_2(p)(m_2(p)+1))} \leq \mathbb{E} \left[ \left\| \left( \frac{\partial}{\partial x} X_T^{1,x} \right) e_i \right\|_{m_2(p)(m_2(p)+1)/2}^{m_2(p)(m_2(p)+1)/2} \right]^{1/(m_2(p)(m_2(p)+1))} \leq \mathbb{E} \left[ \left\| \left( \frac{\partial}{\partial x} X_T^{1,x} \right) e_i \right\|_{m_2(p)(m_2(p)+1)}^{m_2(p)(m_2(p)+1)} \right]^{1/(m_2(p)(m_2(p)+1))} = d^{m_2(p)(m_2(p)+1) - 2} \sum_{i=1}^d \mathbb{E} \left[ \left\| \left( \frac{\partial}{\partial x} X_T^{1,x} \right) e_i \right\|_{m_2(p)(m_2(p)+1)}^{m_2(p)(m_2(p)+1)} \right]^{1/(m_2(p)(m_2(p)+1))} \leq d^{1/2} \max_{i \in \{1, \ldots, d\}} \left( \mathbb{E} \left[ \left\| \left( \frac{\partial}{\partial x} X_T^{1,x} \right) e_i \right\|_{m_2(p)(m_2(p)+1)}^{m_2(p)(m_2(p)+1)} \right]^{1/(m_2(p)(m_2(p)+1))} \right) \leq d^{1/2} \max_{i \in \{1, \ldots, d\}} \left( \mathbb{E} \left[ \left\| X_T^{1,e_i} - X_T^{1,0} \right\|_{m_2(p)(m_2(p)+1)}^{m_2(p)(m_2(p)+1)} \right]^{1/(m_2(p)(m_2(p)+1))} \right).
\]

(209)

Combining this with (207) verifies that

\[
\sup_{x \in [a, b]^d} \left| \mathbb{E} \left[ \limsup_{\mathbb{R}^d \setminus \{0\} : h \to 0} \left( \frac{\| \phi(X_T^{1,x+h}) - \phi(X_T^{1,x}) \|_{m_2(p)}}{\|h\|^{m_2(p)}} \right) \right] \right|^{1/m_2(p)} \leq c d^\omega \sup_{i \in \{1, \ldots, d\}} \max_{i \in \{1, \ldots, d\}} \left( \mathbb{E} \left[ \left\| X_T^{1,e_i} - X_T^{1,0} \right\|_{m_2(p)(m_2(p)+1)}^{m_2(p)(m_2(p)+1)} \right]^{1/(m_2(p)(m_2(p)+1))} \right) \cdot \left[ 1 + \sup_{x \in [a, b]^d} \mathbb{E} \left[ \left\| X_T^{1,x} \right\|_{m_1(w(m_2(p)+1))}^{w/m_1(w(m_2(p)+1))} \right] \right].
\]

(210)
Item (iii). Corollary 3.5, (applied with \( m \leftarrow d, C \leftarrow \|\mu(0)\|, c \leftarrow l, A \leftarrow \sqrt{A}, W \leftarrow W^1, X \leftarrow X^1, p \leftarrow m_1(w(m_2(p) + 1)) \)) for \( x \in [a, b]^d \) in the notation of Corollary 3.5, Lemma 3.6 (applied with \( W \leftarrow W \)). Hence verifies that there exists \( \Omega \) which satisfies that \( \sum_{i=1}^{\infty} \sup_{\Omega} \|\phi(X^1_{\tau} - \phi(X^1_{\tau}))\|^{m_2(p)} \rightarrow 0 \) for all \( \tau \). Combining this, (202), and (195) with item (ii) in Lemma 2.16 (applied with \( W \leftarrow W \)) hence yield that

\[
\begin{align*}
&\sup_{x \in [a, b]^d} \left| \mathbb{E} \left[ \lim_{h \to 0} \sup_{x \in [a, b]^d} \left| \left( \frac{\phi(X^1_{\tau} + h) - \phi(X^1_{\tau})}{\|\mu\|/m_2(p)} \right)^{1/m_2(p)} \right| \right] \right| \\
&\leq c_d^{w+1/2} \max_{i \in \{1, \ldots, d\}} \left( e^{IT} \|e_i\| \right) \left( 1 + e^{W^T T} \|\mu(0)\| \right) \\
&\quad + \sqrt{m_1(w(m_2(p) + 1) - 1)} T \text{Trace}(A) + \sup_{x \in [a, b]^d} \|x\|^{w}
\end{align*}
\]

(211)

Combining this, (202), and (195) with item (ii) in Lemma 2.16 (applied with \( n \leftarrow n, \xi \leftarrow ((\mathbb{R}^d \times \Omega) \ni (x, \omega) \mapsto \phi(X^1_{\tau}(\omega)) \in \mathbb{R}) \)) for \( i \in \{1, \ldots, n\} \) in the notation of Lemma 2.16 implies that

\[
\begin{align*}
&\left| \mathbb{E} \left[ \sup_{x \in [a, b]^d} \left| \mathbb{E} \left[ \phi(X^1_{\tau}) - \frac{1}{n} \sum_{k=1}^{n} \phi(X^1_{\tau}) \right] \right|^{p} \right] \right|^{1/p} \\
&\leq \frac{4R \sqrt{n}}{\sqrt{n}} \left( c_d^{w+1/2} \left( 1 + e^{w^T T} \|\mu(0)\| \right) + \sqrt{m_1(zm_2(p) - 1)} T \text{Trace}(A) \right) \\
&\quad + \sup_{x \in [a, b]^d} \|x\|^{w} + (b - a) c_d^{w+1/2} e^{IT} \left( 1 + e^{W^T T} \|\mu(0)\| \right) \\
&\quad + \sqrt{m_1(zw_2(p) + w - 1) T \text{Trace}(A) + \sup_{x \in [a, b]^d} \|x\|^{w}}
\end{align*}
\]

(212)

Grohs et al. 28 Proposition 3.3] hence verifies that there exists \( \omega_n \in \Omega \) which satisfies that

\[
\begin{align*}
&\sup_{x \in [a, b]^d} \left| \mathbb{E} \left[ \phi(X^1_{\tau}) - \left( \frac{1}{n} \sum_{k=1}^{n} \phi(X^1_{\tau}(\omega_n)) \right) \right] \right| \\
&\leq \frac{4R \sqrt{n}}{\sqrt{n}} \left( c_d^{w+1/2} \left( 1 + e^{w^T T} \|\mu(0)\| \right) + \sqrt{m_1(zm_2(p) - 1)} T \text{Trace}(A) \right) \\
&\quad + \sup_{x \in [a, b]^d} \|x\|^{w} + (b - a) c_d^{w+1/2} e^{IT} \left( 1 + e^{W^T T} \|\mu(0)\| \right) \\
&\quad + \sqrt{m_1(zw_2(p) + w - 1) T \text{Trace}(A) + \sup_{x \in [a, b]^d} \|x\|^{w}}
\end{align*}
\]

(213)
Combining this, (194), (196), and (199) with the triangle inequality yields that

\[
\sup_{x \in [a,b]^d} \left| u(T, x) - \left[ \frac{1}{n} \sum_{k=1}^{n} \phi(W_{k, \omega_n} x + B_{k, \omega_n}) \right] \right| \\
= \sup_{x \in [a,b]^d} \left| E[\varphi(X_{T}^{1,x})] - \left[ \frac{1}{n} \sum_{k=1}^{n} \phi(X_{T}^{k,x}(\omega_n)) \right] \right| \\
\leq \sup_{x \in [a,b]^d} \left| E[\varphi(X_{T}^{1,x})] - E[\phi(X_{T}^{1,x})] \right| \\
+ \sup_{x \in [a,b]^d} \left| E[\phi(X_{T}^{1,x})] - \left[ \frac{1}{n} \sum_{k=1}^{n} \phi(X_{T}^{k,x}(\omega_n)) \right] \right| \\
\leq \varepsilon d^w \left( 1 + e^{wT} [T \|\mu(0)\| + \sqrt{m_1(v - 1)T \text{Trace}(A)}} \\
+ \sup_{x \in [a,b]^d} \|x\|^w \right) \\
+ \frac{4\varepsilon K}{\sqrt{n}} \left( c d^z \left( 1 + e^{zT} [T \|\mu(0)\| + \sqrt{m_1(zm_2(p) - 1)T \text{Trace}(A)}} \\
+ \sup_{x \in [a,b]^d} \|x\|^z \right) + (b-a)cd^{w+1/2}e^{lT} \left( 1 + e^{wT} \|T \|\mu(0)\|} \\
+ \sqrt{m_1(wm_2(p) + w - 1)T \text{Trace}(A)}} + \sup_{x \in [a,b]^d} \|x\|^w \right) \right).
\]

This completes the proof of Lemma 4.1. □

**Corollary 4.2.** Let \(d, n \in \mathbb{N}, \varphi \in C(\mathbb{R}^d, \mathbb{R}), \alpha, a \in \mathbb{R}, \beta \in [0, \infty), b \in (a, \infty), \varepsilon, T, c \in (0, \infty), v, w, z, v \in [0, \infty), \mu \in \mathbb{R}^d, \) let \(\|\cdot\| : \mathbb{R}^d \to [0, \infty)\) be the standard norm on \(\mathbb{R}^d, \) let \(\phi \in C^1(\mathbb{R}^d, \mathbb{R}), \) let \(A = (A_{i,j})_{(i,j) \in \{1, \ldots, d\}^2} \in \mathbb{R}^{d \times d}\) be a symmetric and positive semi-definite matrix, assume for all \(x \in \mathbb{R}^d\) that

\[
|\phi(x)| \leq cd^w (1 + \|x\|^z), \quad \|\nabla \phi(x)\| \leq cd^w (1 + \|x\|^w),
\]

(215)

\[
|\varphi(x) - \phi(x)| \leq \varepsilon d^z (1 + \|x\|^y), \quad \sqrt{\text{Trace}(A)} \leq cd^\beta,
\]

(216)

and \(\|\mu\| \leq cd^w, \) let \(u \in C([0, T] \times \mathbb{R}^d, \mathbb{R}), \) assume for all \(x \in \mathbb{R}^d\) that \(u(0, x) = \varphi(x), \) assume that \(\inf_{\gamma \in (0, \infty)} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \left( \frac{|u(t,x)|}{1+\|x\|} \right) < \infty, \) and assume that \(u|_{(0,T) \times \mathbb{R}^d}\) is a viscosity solution of

\[
\left( \frac{\partial}{\partial t} u \right)(t, x) = \sum_{i,j=1}^{d} A_{i,j} \left( \frac{\partial^2}{\partial x_i \partial x_j} u \right)(t, x) + \sum_{i=1}^{d} \mu_i \left( \frac{\partial}{\partial x_i} u \right)(t, x)
\]

(217)

for \((t, x) \in (0, T) \times \mathbb{R}^d. \) Then there exist \(W_1, \ldots, W_n \in \mathbb{R}^{d \times d}, B_1, \ldots, B_n \in \mathbb{R}^d\)
such that
\[
\sup_{x \in [a,b]^d} \left| u(T, x) - \left[ \frac{1}{n} \sum_{k=1}^{n} \phi(W_k x + B_k) \right] \right| \\
\leq \varepsilon d^{\nu + w \max\{\alpha, \beta, 1/2\}} \\
\cdot \left( 1 + \left[ \sqrt{2} \max\{1, T\} \max\{1, \sqrt{\nu}\} (2c + \max\{|a|, |b|\}) \right]^\nu \right) \\
+ d^{1 + \max\{z + z \max\{\alpha, \beta + 1, w + 1/2 + w \max\{\alpha, \beta + 1\}\} \}
\]
\cdot \frac{32\sqrt{\nu} c}{\sqrt{n}} \left[ 1 + (b - a) \right]
\cdot \left( 1 + \left[ \sqrt{\nu} \max\{1, T\} \max\{1, \sqrt{z}, \sqrt{w}\} (2c + \max\{|a|, |b|\}) \right]\max\{z, w\} \right].
\]

Proof of Corollary 4.2. Throughout this proof let \( \mathcal{K} \in (0, \infty) \) be the \( \max\{2, d^2\} \)-Kahane–Khintchine constant (cf. Definition 2.1 and Lemma 2.2). Observe that for all \( x \in [a,b]^d \) we have that
\[
\|x\| = \left[ \sum_{i=1}^{d} |x_i|^2 \right]^{1/2} \leq \left[ \sum_{i=1}^{d} \max\{|a|, |b|\} \right]^{1/2} = d^{1/2} \max\{|a|, |b|\}.
\]
This proves that
\[
\sup_{x \in [a,b]^d} \|x\| \leq d^{1/2} \max\{|a|, |b|\}.
\]
Next note that Lemma 2.2 (applied with \( p \leftarrow \max\{2, d^2\} \) in the notation of Lemma 2.2) ensures that
\[
\mathcal{K} \leq \sqrt{\max\{1, \max\{2, d^2\} - 1\}} \leq d.
\]
Combining this, (220), Corollary 2.15, (216), and the hypothesis that \( \|\mu\| \leq cd^\alpha \) with Lemma 2.1 (applied with \( l \leftarrow 0, n \leftarrow n, \zeta \leftarrow 8\sqrt{c}, p \leftarrow d^2, \mu \leftarrow (\mathbb{R}^d \ni x \mapsto \mu \in \mathbb{R}^d), A \leftarrow 2A \) in the notation of Lemma 4.1) yields that there exist \( W_1, \ldots, W_n \in \mathbb{R}^{d \times d}, B_1, \ldots, B_n \in \mathbb{R}^d \) which satisfy that
\[
\sup_{x \in [a,b]^d} \left| u(T, x) - \left[ \frac{1}{n} \sum_{k=1}^{n} \phi(W_k x + B_k) \right] \right| \\
\leq \varepsilon d^\nu \left( 1 + \left[ Tcd^\alpha + \sqrt{2} \max\{1, \nu - 1\} Tcd^\beta \right. \right.
\]
\[
+ d^{1/2} \max\{|a|, |b|\})^\nu \right) \\
+ \frac{32\sqrt{\nu} c}{\sqrt{n}} \left[ cd^\nu \left( 1 + \left[ Tcd^\alpha + \sqrt{2} \max\{1, z \max\{2, d^2\} - 1\} Tcd^\beta \right. \right.ight.
\]
\[
+ d^{1/2} \max\{|a|, |b|\})^z \right) \left. \left. + (b - a)cd^{w + 1/2} \left( 1 + \left[ Tcd^\alpha \right. \right. \right.
\]
\[
+ \sqrt{2} \max\{1, w \max\{2, d^2\} + w - 1\} Tcd^\beta + d^{1/2} \max\{|a|, |b|\})^w \right) \right).\]
Hence, we obtain that
\[
\max\{1, \max\{2, \max\{d^2, -1\}\} \leq \max\{1, \max\{2, d^2\}\} \leq \max\{1, \max\{d^2, 1\}\}
\]
\[
\leq 2 \max\{1, \max\{d\}^2\}.
\]

Moreover, note that
\[
\max\{1, \max\{2, d^2\} + w - 1\}
\]
\[
\leq \max\{1, \max\{2, d^2\} + \max\{1, w\}\} \leq \max\{1, \max\{2, d^2\}\}(d^2 + 1) + \max\{1, \max\{1, w\}\}
\]
\[
\leq 2 \max\{1, \max\{d^2, 1\}\}d^2 + \max\{1, \max\{1, w\}\} \leq 3 \max\{1, \max\{w\}\}d^2.
\]

Combining this, the fact that \(\max\{1, v - 1\} \leq \max\{1, v\}\), (222), and (223) yields that
\[
\sup_{x \in [a, b]^d} \left| u(T, x) - \frac{1}{n} \sum_{k=1}^{n} \phi(W_k x + B_k) \right|
\]
\[
\leq \varepsilon d^{\nu + \max\{\alpha, \beta, \gamma\}} \left( 1 + \left[ T c + \sqrt{2} \max\{1, v\} T c + \max\{|a|, |b|\}\right]^{\nu} \right)
\]
\[
+ d^{1 + \max\{z + \zeta \max\{\alpha, \beta, \gamma\}, v + \frac{1}{2} + w \max\{\alpha, \beta, \gamma\}\}}
\]
\[
\cdot \frac{32 \sqrt{c e}}{\sqrt{n}} \left( 1 + \left[ T c + \sqrt{4} \max\{1, z\} T c + \max\{|a|, |b|\}\right]^{z} \right)
\]
\[
+ (b - a) \left( 1 + \left[ T c + \sqrt{6} \max\{1, w\} T c + \max\{|a|, |b|\}\right]^{w} \right).
\]

Hence, we obtain that
\[
\sup_{x \in [a, b]^d} \left| u(T, x) - \frac{1}{n} \sum_{k=1}^{n} \phi(W_k x + B_k) \right|
\]
\[
\leq \varepsilon d^{\nu + \max\{\alpha, \beta, \gamma\}} \left( 1 + \left[ \sqrt{2} \max\{1, T\} \max\{1, \sqrt{v}\} \left( 2c + \max\{|a|, |b|\}\right)\right]^{\nu} \right)
\]
\[
+ d^{1 + \max\{z + \zeta \max\{\alpha, \beta, \gamma\}, v + \frac{1}{2} + w \max\{\alpha, \beta, \gamma\}\}}
\]
\[
\cdot \frac{32 \sqrt{c e}}{\sqrt{n}} \left( 1 + (b - a) \right)
\]
\[
\cdot \left( 1 + \left[ \sqrt{6} \max\{1, T\} \max\{1, \sqrt{z}, \sqrt{w}\} \left( 2c + \max\{|a|, |b|\}\right)\right]^{\max\{z, w\}} \right).
\]

This completes the proof of Corollary 4.2. \(\square\)

**Corollary 4.3.** Let \(d, n \in \mathbb{N}, \varphi \in C(\mathbb{R}^d, \mathbb{R}), \alpha, a \in \mathbb{R}, \beta \in [0, \infty), b \in (a, \infty), \varepsilon, T \in (0, \infty), c \in [\frac{1}{2}, \infty), v, v, w, w, z \in [0, \infty), \mu \in \mathbb{R}^d, \text{let } \|\cdot\| : \mathbb{R}^d \to [0, \infty)\)
be the standard norm on \(\mathbb{R}^d\), let \(A = (A_{i,j})_{i,j\in\{1,\ldots,d\}} \in \mathbb{R}^{d \times d}\) be a symmetric and positive semi-definite matrix, let \(\phi \in C^1(\mathbb{R}^d, \mathbb{R})\), assume for all \(x \in \mathbb{R}^d\) that

\[
|\phi(x)| \leq cd^x(1 + \|x\|^y), \quad \|\nabla \phi(x)\| \leq cd^w(1 + \|x\|^w),
\]

(227)

\[
|\varphi(x) - \phi(x)| \leq \varepsilon d^x(1 + \|x\|^y), \quad \sqrt{\text{Trace}(A)} \leq cd^\beta,
\]

and \(\|\mu\| \leq cd^\alpha\), let \(u \in C([0, T] \times \mathbb{R}^d, \mathbb{R})\), assume for all \(x \in \mathbb{R}^d\) that \(u(0, x) = \varphi(x)\), assume that \(\inf_{\gamma \in (0, \infty)} \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \left(\frac{\|u(t, x)\|}{1 + \|x\|^5}\right) < \infty\), and assume that \(u|_{(0, T) \times \mathbb{R}^d}\) is a viscosity solution of

\[
\left(\frac{\partial}{\partial t} u\right)(t, x) = \sum_{i,j=1}^d A_{i,j} \left(\frac{\partial^2}{\partial x_i \partial x_j}\right) u(t, x) + \sum_{i=1}^d \mu_i \left(\frac{\partial}{\partial x_i}\right) u(t, x)
\]

(229)

for \((t, x) \in (0, T) \times \mathbb{R}^d\). Then there exist \(W_1, \ldots, W_n \in \mathbb{R}^{d \times d}, B_1, \ldots, B_n \in \mathbb{R}^d\) such that

\[
\sup_{x \in [a, b]^d} \left| u(T, x) - \left(\frac{1}{n} \sum_{k=1}^n \phi(W_k x + B_k)\right) \right|
\]

\[
\leq \varepsilon d^v + \max\{\alpha, \beta, 1/2\} \left[5c + T + \sqrt{v} + |a| + |b|\right]^{4v+1}
\]

\[
+ d^l + \max\{z + \max\{\alpha, \beta + 1\}, w + 1/2 + w + \max\{\alpha, \beta + 1\}\}
\]

\[
\cdot \frac{1}{\sqrt{n}} \left[7c + T + \sqrt{z} + \sqrt{w} + |a| + |b|\right]^{5+4(z+w)}.
\]

Proof of Corollary 4.3. Observe that Corollary 4.2 ensures that there exist \(W_1, \ldots, W_n \in \mathbb{R}^{d \times d}, B_1, \ldots, B_n \in \mathbb{R}^d\) which satisfy that

\[
\sup_{x \in [a, b]^d} \left| u(T, x) - \left(\frac{1}{n} \sum_{k=1}^n \phi(W_k x + B_k)\right) \right|
\]

\[
\leq \varepsilon d^v + \max\{\alpha, \beta, 1/2\} \cdot \left[1 + \left(\sqrt{\frac{2}{\pi}} \max\{1, T\} \max\{1, \sqrt[3]{v}\} (2c + \max\{|a|, |b|\})\right)^y\right]
\]

(231)

\[
+ d^l + \max\{z + \max\{\alpha, \beta + 1\}, w + 1/2 + w + \max\{\alpha, \beta + 1\}\} \frac{32\sqrt{\varepsilon c}}{\sqrt{n}} \left[1 + (b - a)\right]
\]

\[
\cdot \left(1 + \left[\sqrt[5]{\frac{2}{\pi}} \max\{1, T\} \max\{1, \sqrt[3]{w}\} (2c + \max\{|a|, |b|\})\right]^{\max\{z, w\}}\right).
\]

Next note that the assumption that \(c \in [1/2, \infty)\) implies that

\[
1 + \left[\sqrt[5]{\frac{2}{\pi}} \max\{1, T\} \max\{1, \sqrt[3]{w}\} (2c + \max\{|a|, |b|\})\right]^{\max\{z, w\}}
\]

\[
\leq 1 + \max\{\sqrt[5]{\frac{2}{\pi}} T, \sqrt[3]{w}, 2c + \max\{|a|, |b|\}\}^{4v}
\]

\[
\leq 1 + [2\sqrt[5]{2} c + 2c + T + \sqrt{v} + |a| + |b|]^{4v}
\]

(232)

\[
\leq 2\left[5c + T + \sqrt{v} + |a| + |b|\right]^{4v}
\]

\[
\leq [5c + T + \sqrt{v} + |a| + |b|]^{4v+1}.
\]
Moreover, note that the assumption that \( c \in [1/2, \infty) \) verifies that
\[
32\sqrt{\epsilon} c \left( 1 + (b - a) \right) \\
\cdot \left( 1 + \left[ \sqrt{6} \max\{1, T\} \max\{1, \sqrt{z}, \sqrt{w}\} (2c + \max\{|a|, |b|\}) \right]^{\max\{z, w\}} \right) \\
\leq 52c (1 + |a| + |b|) \left[ 1 + (2\sqrt{6}c + 2c + T + \sqrt{z} + \sqrt{w} + |a| + |b|)^{4\max\{z, w\}} \right] \\
\leq 52c (1 + |a| + |b|) \left[ 1 + (7c + T + \sqrt{z} + \sqrt{w} + |a| + |b|)^{4\max\{z, w\}} \right].
\]
(233)

Hence, we obtain that
\[
32\sqrt{\epsilon} c \left( 1 + (b - a) \right) \\
\cdot \left( 1 + \left[ \sqrt{6} \max\{1, T\} \max\{1, \sqrt{z}, \sqrt{w}\} (2c + \max\{|a|, |b|\}) \right]^{\max\{z, w\}} \right) \\
\leq \left[ 7c + T + \sqrt{z} + \sqrt{w} + |a| + |b| \right]^{5 + 4\max\{z, w\}}.
\]
(234)

Combining this with (231) and (232) establishes (230). This completes the proof of Corollary 4.3.

4.2 Qualitative error estimates

In this subsection we provide in Proposition 4.4 below a qualitative approximation result for viscosity solutions (cf., for example, Hairer et al. [31]) of Kolmogorov PDEs with constant coefficient functions. Informally speaking, we can think of the approximations in Proposition 4.4 as linear combinations of realizations of ANNs with a suitable continuously differentiable activation function. Proposition 4.4 will be employed in our proof of Proposition 4.6 in Subsection 4.3 below.

**Proposition 4.4.** Let \( d \in \mathbb{N} \), \( \varphi \in C(\mathbb{R}^d, \mathbb{R}) \), \( \alpha, a \in \mathbb{R} \), \( \beta \in [0, \infty) \), \( b \in (a, \infty) \), \( r, T \in (0, \infty) \), \( c \in \left( \frac{1}{2}, \infty \right) \), \( v, v, w, w, z \in [0, \infty) \), \( C = \frac{1}{2}[5c + T + \sqrt{v} + |a| + |b|]^{-\frac{1}{4} - 1} \), \( C = 4[7c + T + \sqrt{z} + \sqrt{w} + |a| + |b|]^{10 + 8(z + w)} \), \( p = v + v \max\{\alpha, \beta, 1/2\} \), \( p = 2 + \max\{2z + 2w \max\{a, \beta + 1\} \} \), \( 2w + 1 + 2w \max\{a, \beta + 1\} \), \( \mu \in \mathbb{R}^d \), let \( |||: \mathbb{R}^d \to [0, \infty) \) be the standard norm on \( \mathbb{R}^d \), let \( \phi_\epsilon \in C^{1}(\mathbb{R}^d, \mathbb{R}) \), \( \epsilon \in (0, r] \), let \( A = (A_{i,j})_{i,j} \in \{1, \ldots, d\}^2 \in \mathbb{R}^{d \times d} \) be a symmetric and positive semidefinite matrix, and assume for all \( \epsilon \in (0, r] \), \( x \in \mathbb{R}^d \) that
\[
|\phi_\epsilon(x)| \leq cd^2 (1 + ||x||^2), \\
|||\nabla \phi_\epsilon(x)||| \leq cd^3 (1 + ||x||^2), \\
\varphi(x) - \phi_\epsilon(x) \leq \epsilon cd^3 (1 + ||x||^2), \\
\sqrt{\text{Trace}(A)} \leq cd^3, \quad \text{and} \quad |||\mu||| \leq cd^3.
\]
and (235)

and (236)

(i) there exists a unique \( u \in C([0, T] \times \mathbb{R}^d, \mathbb{R}) \) which satisfies for all \( x \in \mathbb{R}^d \)

and (237)

\( u(0, x) = \varphi(x) \), which satisfies that \( \inf_{\gamma \in (0, \infty)} \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \left( \frac{u(t, x)}{1 + ||x||^\gamma} \right) < \)
Corollary 4.3 (applied with \( \varepsilon \) notation of Corollary 4.3) hence yields that for all \( \varepsilon \) prove that for all \( t,x \) for (235) and (236) ensure that \( \varphi: \mathbb{R}^d \to \mathbb{R} \) is an at most polynomially growing function. Grohs et al. [28, Corollary 2.23] (see also Hairer et al. [31, Corollary 4.17]) hence implies that there exists a unique continuous function \( u: [0, T] \times \mathbb{R}^d \to \mathbb{R} \) which satisfies for all \( x \in \mathbb{R}^d \) that \( u(0, x) = \varphi(x) \), which satisfies that inf \( \gamma \in (0, \infty) \sum_{i=1}^{d} (1 + \|x\|^\gamma) < \infty \), and which satisfies that \( u|_{(0, T) \times \mathbb{R}^d} \) is a viscosity solution of

\[
\left( \frac{\partial}{\partial t} u \right)(t, x) = \sum_{i,j=1}^{d} A_{i,j} \left( \frac{\partial^2}{\partial x_i \partial x_j} u \right)(t, x) + \sum_{i=1}^{d} \mu_i \left( \frac{\partial}{\partial x_i} u \right)(t, x) \tag{237}
\]

for \( (t, x) \in (0, T) \times \mathbb{R}^d \).

This establishes item (i). Next note that the hypothesis that \( c \in [1/2, \infty) \) verifies that

\[
\frac{C}{c} = \frac{1}{2c} \frac{1}{c + T + \sqrt{\nu} |a| + |b|^2 + \nu} \leq 1. \tag{240}
\]

This reveals that for all \( \varepsilon \in (0, r] \), \( n \in \mathbb{N} \cap [\varepsilon^{-2} d^p, \infty) \) that there exist \( W_1, \ldots, W_n \in \mathbb{R}^{d \times d}, B_1, \ldots, B_n \in \mathbb{R}^d \) such that

\[
\sup_{x \in [a,b]^d} \left| u(T, x) - \frac{1}{n} \sum_{k=1}^{n} \phi_{\varepsilon^{-1} d^{-p} C}(W_k x + B_k) \right| \leq \varepsilon. \tag{242}
\]

This establishes item (ii). This completes the proof of Proposition 4.4. \( \square \)
4.3 Qualitative error estimates for artificial neural networks (ANNs)

In this subsection we prove in Corollary 4.7 below that ANNs with continuously differentiable activation functions can overcome the curse of dimensionality in the uniform approximation of viscosity solutions of Kolmogorov PDEs with constant coefficient functions. Corollary 4.7 is an immediate consequence of Proposition 4.6 in turn, follows from Proposition 4.4 above and the well-known fact that linear combinations of realizations of ANNs are again realizations of ANNs. To formulate Corollary 4.7 we introduce in Setting 4.5 below a common framework from the scientific literature (cf., e.g., Grohs et al. [29, Section 2.1] and Petersen & Voigtlaender [52, Section 2]) to mathematically describe ANNs.

**Setting 4.5.** Let \( N \) be the set given by

\[
N = \bigcup_{L \in \mathbb{N} \cap [2, \infty)} \bigcup_{(l_0, \ldots, l_L) \in \{(N_L \times \{1\}\} \times \{L\}} \Big( \times_{k=1}^L (\mathbb{R}^{l_k} \times \mathbb{R}^{l_k-1} \times \mathbb{R}^{l_k}) \Big), \tag{243}
\]

let \( a \in C^{1}([\mathbb{R}, \mathbb{R}]), \) let \( A_n : \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( n \in \mathbb{N}, \) satisfy for all \( n \in \mathbb{N}, \) \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) that \( A_n(x) = (a(x_1), \ldots, a(x_n)) \), and let \( N, L, P, \mathcal{Y} : N \rightarrow \mathbb{N} \) and \( R : \mathcal{N} \rightarrow \bigcup_{d \in [2, \infty)} C(\mathbb{R}^d, \mathbb{R}) \) satisfy for all \( L \in \mathbb{N} \cap [2, \infty), \) \( (l_0, \ldots, l_L) \in ((N_L \times \{1\}) \times \{L\}) \), \( \Phi = \{(W_1, B_1), \ldots, (W_L, B_L)\} = \{(W_L^{(i,j)}), \ldots, (W_1^{(i,j)})\} \in (\mathbb{R}^{l_k} \times \mathbb{R}^{l_k-1} \times \mathbb{R}^{l_k}) \), \( x_0 \in \mathbb{R}^{l_0} \), \( \ldots, \) \( x_{L-1} \in \mathbb{R}^{l_{L-1}} \) with \( \forall k \in \mathbb{N} \cap (0, L) : x_k = A_{l_k} (W_k x_{k-1} + B_k) \) that \( N(\Phi) = \sum_{k=0}^L l_k, \mathcal{L}(\Phi) = L+1, \) \( P(\Phi) = \sum_{k=0}^L l_k(l_k-1) + 1, \) \( R(\Phi) = C(\mathbb{R}^{l_0}, \mathbb{R}) \), \( (\mathcal{Y}(x_0)) = W_L x_{L-1} + B_L, \) and

\[
\mathcal{Y}(\Phi) = \sum_{k=1}^L \sum_{i=1}^{l_k} \left[ \mathbb{I}_{\mathbb{R}\setminus\{0\}} (B_k^i) + \sum_{j=1}^{l_{k-1}} \mathbb{I}_{\mathbb{R}\setminus\{0\}} (W_k^{(i,j)}) \right]. \tag{244}
\]

**Proposition 4.6.** Assume Setting 4.5, let \( d \in \mathbb{N}, \mu \in \mathbb{R}^d, \varphi \in C(\mathbb{R}^d, \mathbb{R}), \alpha, a \in \mathbb{R}, \beta \in [0, \infty), \) \( b \in (a, \infty), \) \( r, T \in (0, \infty), \) \( c \in \left[ \frac{1}{2}, \infty \right), \) \( v, w, z \in (0, \infty), \) let

\[
C = (4[7c + T + \sqrt{z} + \sqrt{w} + |a| + |b|]^{10} + w^{10} + 1 + r^2), \tag{245}
\]

\[
p = 2 + \max \{ 2z + 2a \max \{ \alpha, \beta + 1 \}, 2w + 1 + 2w \} \max \{ \alpha, \beta + 1 \},
\]

\[
C = \frac{1}{2}[5c + T + \sqrt{z} + |a| + |b|]^{-4v}, \tag{246}
\]

\[
p = v + w \max \{ \alpha, \beta, 1/2 \},
\]

let \( \| \cdot \| : \mathbb{R}^d \rightarrow [0, \infty) \) be the standard norm on \( \mathbb{R}^d, \) let \( (\phi_\varepsilon)_{\varepsilon \in (0, r]} \subseteq N, \) let \( A = (A_{i,j})_{i,j \in \{1, \ldots, d\}} \subseteq \mathbb{R}^{d \times d} \) be a symmetric and positive semi-definite matrix, and assume for all \( \varepsilon \in (0, r], \) \( x \in \mathbb{R}^d \) that \((\mathcal{R}_\varepsilon) \in C(\mathbb{R}^d, \mathbb{R}), \)

\[
|\mathcal{R}_\varepsilon(x)| \leq \varepsilon d^2 (1 + \|x\|^2), \quad \| \nabla (\mathcal{R}_\varepsilon(x)) \| \leq \varepsilon d^2 (1 + \|x\|^2), \tag{246}
\]

\[
|\varphi(x) - (\mathcal{R}_\varepsilon(x))| \leq \varepsilon d^2 (1 + \|x\|^2), \quad \sqrt{\text{Trace}(A)} \leq \varepsilon d^2, \tag{247}
\]

and \( \| \mu \| \leq \varepsilon d^2. \) Then
(i) there exists a unique $u \in C([0,T] \times \mathbb{R}^d, \mathbb{R})$ which satisfies for all $x \in \mathbb{R}^d$ that $u(0, x) = \varphi(x)$, which satisfies that $\inf_{y \in \mathbb{R}^d} \sup_{(t,x) \in \mathbb{R}^d} \left( \frac{1}{1+\|x\|} \right) < \infty$, and which satisfies that $u|_{(0,T)} \times \mathbb{R}^d$ is a viscosity solution of
\[
(\frac{\partial}{\partial t}) u(t, x) = \sum_{i,j=1}^{d} A_{i,j} (\frac{\partial^2}{\partial x_i \partial x_j}) u(t, x) + \sum_{i=1}^{d} \mu_i (\frac{\partial}{\partial x_i}) u(t, x) \tag{248}
\]
for $(t,x) \in (0, T) \times \mathbb{R}^d$ and

(ii) there exists $(\psi_\varepsilon)_{\varepsilon \in (0,r]} \subseteq \mathbb{N}$ such that for all $\varepsilon \in (0,r]$ we have that $N(\psi_\varepsilon) \leq C \varepsilon^{-2} N(\phi_{\varepsilon^{-1}d-p\varepsilon}), \mathcal{L}(\psi_\varepsilon) = \mathcal{L}(\phi_{\varepsilon^{-1}d-p\varepsilon}), \mathcal{P}(\psi_\varepsilon) \leq C \varepsilon^{-4} \mathcal{P}(\phi_{\varepsilon^{-1}d-p\varepsilon}), \mathcal{P}(\psi_\varepsilon) \leq C \varepsilon^{-2} \mathcal{P}(\phi_{\varepsilon^{-1}d-p\varepsilon}), (\mathcal{R}\psi_\varepsilon) \in C(\mathbb{R}^d, \mathbb{R})$, and
\[
\sup_{x \in [a,b]^d} |u(T, x) - (\mathcal{R}\psi_\varepsilon)(x)| \leq \varepsilon. \tag{249}
\]

Proof of Proposition 4.6 Throughout this proof let $\varepsilon \in (0,r]$, let
\[
C = 4[7c + T + \sqrt{x} + \sqrt{w} + |a| + |b|]^{10 + 8(z + w)}, \tag{250}
\]
let $n = \min(N \cap [4, \infty))$, let $\gamma_1, \ldots, \gamma_n \in \mathbb{R}^{d \times d}$, $\delta_1, \ldots, \delta_n \in \mathbb{R}^d$ satisfy
\[
\sup_{x \in [a,b]^d} \left| u(T, x) - \left\lfloor \frac{1}{n} \sum_{k=1}^{n} (\mathcal{R}\phi_{\varepsilon^{-1}d-p\varepsilon})(\gamma_k x + \delta_k) \right\rfloor \right| \leq \varepsilon \tag{251}
\]
(cf. item (ii) in Proposition 4.4), let $L \in \mathbb{N} \cap [2, \infty)$, $(l_0, \ldots, l_L) \in (\{d\} \times (\mathbb{N} \setminus \{1\}) \times \{1\}), ((W_1, B_1), \ldots, (W_L, B_L)) \in (\times_{k=1}^{L} (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}))$ satisfy
\[
\phi_{\varepsilon^{-1}d-p\varepsilon} = ((W_1, B_1), \ldots, (W_L, B_L)), \tag{252}
\]
and let
\[
\psi = ((W_1 W_0, W_1 B_0 + B_1), (W_2, B_2), \ldots, (W_L-1, B_{L-1}), (W_L, B_L)) \in (\mathbb{R}^{n l_0 \times l_0} \times \mathbb{R}^{n l_1}) \times (\times_{k=1}^{L-1} (\mathbb{R}^{n l_k \times n l_{k-1}} \times \mathbb{R}^{n l_k})) \times (\mathbb{R}^{l_L \times l_{L-1}} \times \mathbb{R}^{l_L}) \tag{253}
\]
satisfy for all $k \in \{1, \ldots, L-1\}$ that
\[
W_0 = \begin{pmatrix} \gamma_1 \\
\gamma_2 \\
\vdots \\
\gamma_n 
\end{pmatrix}, \quad W_k = \text{diag}(W_k, \ldots, W_k), \quad W_L = \frac{1}{n} (W_L \ldots W_L), \tag{254}
\]
\[
B_0 = \begin{pmatrix} \delta_1 \\
\vdots \\
\delta_n 
\end{pmatrix}, \quad \text{and} \quad B_k = \begin{pmatrix} B_k \\
\vdots \\
B_k 
\end{pmatrix}. \tag{255}
\]
Observe that item (ii) in Proposition 4.4 (applied with $r \leftarrow r$, $\phi_\varepsilon \leftarrow (\mathcal{R}\phi_\varepsilon)$ for $\varepsilon \in (0,r]$ in the notation of Proposition 4.4) establishes item (ii). Next note that
the fact that \( n \in [Cd^p \varepsilon^{-2}, Cd^p \varepsilon^{-2} + 1] \) and the fact that \( r^2 \varepsilon^{-2} \in [1, \infty) \) prove that
\[
\begin{align*}
n &\leq Cd^p \varepsilon^{-2} + 1 \leq (C + 1)d^p \max\{1, \varepsilon^{-2}\} \\
&\leq (C + 1) \max\{1, r^{-2}\} r^2 d^p \varepsilon^{-2} \\
&\leq Cd^p \varepsilon^{-2}.
\end{align*}
\] (256)

This and (253) verify that
\[
\begin{align*}
N(\psi) &= l_0 + \sum_{k=1}^{L-1} nl_k + l_L \leq n \sum_{k=0}^{L} l_k \\
&= nN(\phi_{\varepsilon^{-1},d-pC}) \leq Cd^p \varepsilon^{-2} N(\phi_{\varepsilon^{-1},d-pC}).
\end{align*}
\] (257)

Next note that (253) and (256) yield that
\[
\begin{align*}
P(\psi) &= nl_1 l_0 + nl_1 + \sum_{k=2}^{L-1} nl_k (nl_{k-1} + 1) + nl_LL_{L-1} + l_L \\
&\leq n^2 \left[ l_1(l_0 + 1) + \sum_{k=2}^{L-1} l_k(l_{k-1} + 1) + l_L(l_{L-1} + 1) \right] \\
&= n^2 P(\phi_{\varepsilon^{-1},d-pC}) \leq C^2 d^p \varepsilon^{-4} P(\phi_{\varepsilon^{-1},d-pC}).
\end{align*}
\] (258)

Moreover, note that (253) and (260) ensure that
\[
\begin{align*}
P(\psi) &\leq nl_1(l_0 + 1) + n \sum_{k=2}^{L} l_k(l_{k-1} + 1) \\
&= nP(\phi_{\varepsilon^{-1},d-pC}) \leq Cd^p \varepsilon^{-2} P(\phi_{\varepsilon^{-1},d-pC}).
\end{align*}
\] (259)

Furthermore, (253), (251), and (256) imply that for all \( x \in \mathbb{R}^d \) we have that
\[
(R\psi)(x) = \frac{1}{n} \sum_{k=1}^{n} (R\phi_{\varepsilon^{-1},d-pC})(\gamma_k x + \delta_k).
\] (260)

Combining this and (261) yields that
\[
\sup_{x \in [a,b]^d} |u(T,x) - (R\psi)(x)| \leq \varepsilon.
\] (261)

Next observe that (252) and (253) verify that \( L(\psi) = L + 1 = L(\phi_{\varepsilon^{-1},d-pC}) \). Combining this with (257), (258), and (261) establishes item (ii). This completes the proof of Proposition 4.6.

**Corollary 4.7.** Assume Setting 4.5, let \( \alpha, c, a \in \mathbb{R}, \beta \in [0, \infty), b \in (a, \infty), r, T \in (0, \infty), p, q, v, w, z \in [0, \infty), p = 2 + \max\{2z + 2z \max\{\alpha, \beta + 1\}, 2w + 1 + 2w \max\{\alpha, \beta + 1\}\}, q = v + v \max\{\alpha, \beta, 1/2\}, for every \( d \in \mathbb{N} \) let
Throughout this proof let \( \| \cdot \|_{d : \mathbb{R}^d} \rightarrow [0, \infty) \) be the standard norm on \( \mathbb{R}^d \), let \((\phi_{\varepsilon,d})_{(\varepsilon,d)\in (0,\gamma)\times N} \subseteq N \), let \( \mu_d \in \mathbb{R}^d \), \( d \in \mathbb{N} \), let \( A_d = (A^{(i,j)}_{d})_{(i,j)\in \{1,\ldots,d\}^2} \in \mathbb{R}^{d\times d} \), \( d \in \mathbb{N} \), be symmetric and positive semi-definite matrices, let \( \varphi_d \in C(\mathbb{R}^d, \mathbb{R}) \), \( d \in \mathbb{N} \), and assume for all \( \varepsilon \in (0, r] \), \( d \in \mathbb{N} \), \( x \in \mathbb{R}^d \) that \((R\phi_{\varepsilon,d}) \in C(\mathbb{R}^d, \mathbb{R})\),
\[
(\mathcal{R}\phi_{\varepsilon,d})(x) \leq c d^{\gamma} (1 + \|x\|_{\mathbb{R}^d}^\gamma), \quad \|\nabla (\mathcal{R}\phi_{\varepsilon,d})\|_{\mathbb{R}^d} \leq c d^{w} (1 + \|x\|_{\mathbb{R}^d}^w), \quad (262)
\[
|\varphi_d(x) - (\mathcal{R}\phi_{\varepsilon,d})(x)| \leq \varepsilon c d^{\gamma} (1 + \|x\|_{\mathbb{R}^d}^\gamma), \quad \text{Trace}(A_d) \leq c d^\gamma, \quad (263)
\[
\|\mu_d\|_{\mathbb{R}^d} \leq c d^\gamma, \quad \text{and} \quad \mathcal{P}(\phi_{\varepsilon,d}) \leq c d^{\gamma - q}. \quad (264)
\]
Then
\((i)\) there exist unique \( u_d \in C([0,T] \times \mathbb{R}^d, \mathbb{R}) \), \( d \in \mathbb{N} \), which satisfy for all \( d \in \mathbb{N} \), \( x \in \mathbb{R}^d \) that \( u_d(0,x) = \varphi_d(x) \), which satisfy for all \( d \in \mathbb{N} \) that \( \inf_{\gamma \in (0,\gamma)} \sup_{(t,x)\in [0,T] \times \mathbb{R}^d} \left( \frac{\|u_d(t,x)\|}{1 + \|x\|_{\mathbb{R}^d}} \right) < \infty \), and which satisfy that for all \( d \in \mathbb{N} \) we have that \( u_d(0,T,x) \times \mathbb{R}^d \) is a viscosity solution of
\[
\left( \frac{\partial}{\partial t} u_d \right)(t,x) = \left( \frac{\partial}{\partial x} u_d \right)(t,x) \mu_d + \sum_{i,j=1}^{d} A^{(i,j)}_{d} \left( \frac{\partial^2}{\partial x_i \partial x_j} u_d \right)(t,x) \quad (265)
\]
for \((t,x) \in (0,T) \times \mathbb{R}^d \) and \((ii)\) there exist \( (\psi_{\varepsilon,d})_{(\varepsilon,d)\in (0,\gamma)\times N} \subseteq N \), \( C \in \mathbb{R} \) such that for all \( \varepsilon \in (0, r] \), \( d \in \mathbb{N} \) we have that \( N(\psi_{\varepsilon,d}) \leq C d^{p+2} + p \varepsilon^{-(q+2)} \), \( \mathcal{P}(\psi_{\varepsilon,d}) \leq C d^{p+2} + p \varepsilon^{-(q+2)} \), \( \mathcal{R}(\psi_{\varepsilon,d}) \in C(\mathbb{R}^d, \mathbb{R}) \), and
\[
\sup_{x \in [a,b]^d} |u_d(T,x) - (\mathcal{R}\psi_{\varepsilon,d})(x)| \leq \varepsilon. \quad (266)
\]

Proof of Corollary \[ 4.3 \] Throughout this proof let \( C = \{4|\gamma| \max \{1/2, c \} + T + \sqrt{\gamma + \sqrt{\theta}} + |a| + |b| \}, C = \max \{1/2, c \}, C = \frac{1}{2} \gamma \varepsilon \}\] Note that item \( \dd{5} \) in Proposition \[ 4.4 \] (applied with \( c \leq \max \{1/2, c \} \), \( \mu \leq \mu_d \), \( \varphi \leq \varphi_d \), \( \phi_{\varepsilon} \in (0,\gamma) \times \{ \cdot \} \), \( A \leq A_d \), \( d \in \mathbb{N} \) in the notation of Proposition \[ 4.4 \] implies that for every \( d \in \mathbb{N} \) there exists a unique continuous function \( u_d : [0,T] \times \mathbb{R}^d \rightarrow \mathbb{R} \) which satisfies for all \( x \in \mathbb{R}^d \) that \( u_d(0,x) = \varphi_d(x) \), which satisfies that \( \inf_{\gamma \in (0,\gamma)} \sup_{(t,x)\in [0,T] \times \mathbb{R}^d} \left( \frac{\|u_d(t,x)\|}{1 + \|x\|_{\mathbb{R}^d}} \right) < \infty \), and which satisfies that \( u_d(0,T,x) \times \mathbb{R}^d \) is a viscosity solution of
\[
\left( \frac{\partial}{\partial t} u_d \right)(t,x) = \left( \frac{\partial}{\partial x} u_d \right)(t,x) \mu_d + \sum_{i,j=1}^{d} A^{(i,j)}_{d} \left( \frac{\partial^2}{\partial x_i \partial x_j} u_d \right)(t,x) \quad (267)
\]
for \((t,x) \in (0,T) \times \mathbb{R}^d \). This establishes item \( \dd{5} \). Next note that for all \( L \in \mathbb{N} \cap \{2, \infty \}, (l_0, \ldots, l_L) \in \mathbb{N}^L \times \{1 \}, \Phi \in (\times_{k=1}^{L} (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k})) \) we have that
\[
N(\Phi) = \sum_{k=0}^{L} l_k \leq l_1 l_0 + \sum_{k=1}^{L} l_k + \sum_{k=2}^{L} l_k l_{k-1} = \mathcal{P}(\Phi). \quad (268)
\]
Moreover, note that Proposition 4.6 (applied with $c \leftarrow \max \{1/2, c\}$, $\mu \leftarrow \mu_d$, $\varphi \leftarrow \varphi_d$, $(\phi_\varepsilon,c)_{\varepsilon \in [0,r]} \leftarrow (\phi_\varepsilon,d)_{\varepsilon \in [0,r]}$, $A \leftarrow A_d$, for $d \in \mathbb{N}$ in the notation of Proposition 4.6) assures that there exist $\psi_\varepsilon,c \in \mathbb{N}$, $d \in \mathbb{N}$, $\varepsilon \in (0,r]$, which satisfy that for every $d \in \mathbb{N}$, $\varepsilon \in (0,r]$ we have that $N(\psi_\varepsilon,c) \leq C d^p \varepsilon^{-2} N(\phi_\varepsilon^{-1} d^{-1/2} c, d)$, $P(\psi_\varepsilon,c) \leq C d^p \varepsilon^{-4} P(\phi_\varepsilon^{-1} d^{-1/2} c, d)$, $P(\psi_\varepsilon,c) \leq C d^p \varepsilon^{-2} P(\phi_\varepsilon^{-1} d^{-1/2} c, d)$, $(\mathcal{R} \psi_\varepsilon,c) \in C(\mathbb{R}^d, \mathbb{R})$, and

$$\sup_{x \in [a,b]^d} |u_d(T,x) - (\mathcal{R} \psi_\varepsilon,c)(x)| \leq \varepsilon. \quad (269)$$

Next note that $(264)$ implies that for all $\varepsilon \in (0,r], d \in \mathbb{N}$ we have that

$$P(\phi_\varepsilon^{-1} d^{-1/2} c, d) \leq c d^p (\varepsilon^{-1} d^{-p} c)^{-q} = c c^q d^p + p q c^{-q} \varepsilon^{-q}. \quad (270)$$

Combining this, $(268)$, and $(269)$ hence yields that for all $d \in \mathbb{N}$, $\varepsilon \in (0,r]$ we have that $N(\psi_\varepsilon,c) \leq C c^q d^p + p q c^{-q} \varepsilon^{-q+2}$, $P(\psi_\varepsilon,c) \leq C^2 c^q d^p + p q c^{-q} \varepsilon^{-q+4}$, and $P(\psi_\varepsilon,c) \leq C c^q d^p + p q c^{-q} \varepsilon^{-q+2}$. This and $(269)$ establish item III. This completes the proof of Corollary 4.7.

5 Artificial neural network approximations for heat equations

5.1 Viscosity solutions for heat equations

In this subsection we establish in Lemma 5.3 below a well-known connection between viscosity solutions and classical solutions of heat equations with at most polynomially growing initial conditions. Lemma 5.3 will be employed in our proof of Theorem 5.4 below, the main result of this article. Lemma 5.3 is a simple consequence of Lemma 5.2 and the Feynman-Kac formula for viscosity solutions of Kolmogorov PDEs (cf., for example, Hairer et al. [31]). Lemma 5.2 in turn, is an elementary and well-known existence result for solutions of heat equations (cf., for example, Evans [21, Theorem 1 in Subsection 2.3.1]). For completeness we also provide in this subsection a detailed proof for Lemma 5.2. Our proof of Lemma 5.3 employs the elementary and well-known result in Lemma 5.1 below.

Lemma 5.1. Let $p \in [0,\infty)$, $d \in \mathbb{N}$, and let $\| \cdot \| : \mathbb{R}^d \rightarrow [0,\infty)$ be the standard norm on $\mathbb{R}^d$. Then it holds for all $t \in (0,\infty)$, $x \in \mathbb{R}^d$ that

$$\int_{\mathbb{R}^d} \| y \|^p e^{-\frac{\|x-y\|^2}{t}} dy < \infty. \quad (271)$$

Proof of Lemma 5.1. Throughout this proof let $S$ be the set given by

$$S = \begin{cases} (-1,1) & : d = 1 \\ (0,2\pi) & : d = 2 \\ (0,2\pi) \times (0,\pi)^{d-2} & : d \in \{3,4,\ldots\} \end{cases} \quad (272)$$

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and for every \( n \in \{1,2,\ldots\} \) let \( T_n : (0, \infty) \times (0, 2\pi) \times (0, \pi)^{n-2} \to \mathbb{R} \) satisfy for all \( n \in \{1,2,\ldots\} \), \( r \in (0, \infty) \), \( \varphi \in (0, 2\pi) \), \( \theta_1, \ldots, \theta_{n-2} \in (0, \pi) \) that if \( n = 2 \) then \( T_2(r, \varphi) = r \) and if \( n \geq 3 \) then

\[
T_n(r, \varphi, \theta_1, \ldots, \theta_{n-2}) = r^{n-1} \left[ \prod_{i=1}^{n-2} [\sin(\theta_i)]^i \right].
\]  

(273)

Observe that the integral transformation theorem with the diffeomorphism \((0, \infty) \ni r \mapsto \sqrt{r} \in (0, \infty)\) implies that

\[
\int_0^\infty r^{p+d-1} e^{-r^2} \, dr = \int_0^\infty r^{(p+d-1)/2} e^{-r^2} \frac{1}{2^{p/2}} \, dr = \frac{1}{2} \int_0^\infty r^{(p+d-1)/2-1} e^{-r} \, dr.
\]  

(274)

Item \( \text{(iv)} \) in Lemma 2.4 (applied with \( x \leftarrow \frac{p+d}{d} \), in the notation of Lemma 2.4) hence verifies that

\[
\int_0^\infty r^{p+d-1} e^{-r^2} \, dr \leq \frac{1}{2} \sqrt{\frac{4\pi}{p+d}} \left[ \frac{p+d}{2e} \right]^{p+d} e^{-\pi(p+d)} < \infty.
\]  

(275)

Next note that the integral transformation theorem with the diffeomorphism \( \mathbb{R}^d \ni y \mapsto 2\sqrt{t}y \in \mathbb{R}^d \) for \( t \in (0, \infty) \), the triangle inequality, and the fact that for all \( a, b \in [0, \infty) \) we have that \((a+b)^p \leq \max\{1, 2^{p-1}\}(a^p + b^p)\) ensure that for all \( t \in (0, \infty) \), \( x \in \mathbb{R}^d \) we have that

\[
\int_{\mathbb{R}^d} \|y\|^{p} e^{-\frac{\|x-y\|^2}{4t}} \, dy = \int_{\mathbb{R}^d} \|x - y\|^{p} e^{-\frac{\|y\|^2}{4t}} \, dy
\]

\[
= \int_{\mathbb{R}^d} \|x - 2\sqrt{t}y\|^{p} e^{-\frac{\|y\|^2}{4t}} (2\sqrt{t})^d \, dy
\]

\[
\leq \max\{1, 2^{p-1}\} (2\sqrt{t})^d \|x\|^p \int_{\mathbb{R}^d} e^{-\frac{\|y\|^2}{4t}} \, dy
\]

\[
+ \max\{1, 2^{p-1}\} (2\sqrt{t})^p \int_{\mathbb{R}^d} \|y\|^p e^{-\frac{\|y\|^2}{4t}} \, dy.
\]  

(276)

To establish \( (277) \) we distinguish between the case \( d = 1 \) and the case \( d \in \mathbb{N} \cap [2, \infty) \). First, we consider the case \( d = 1 \). Note that

\[
\int_{\mathbb{R}} \|y\|^p e^{-\frac{\|y\|^2}{4t}} \, dy = 2 \int_0^\infty y^p e^{-y^2} \, dy.
\]  

(277)

Combining this with \( (276) \) and \( (277) \) establishes \( (276) \) in the case \( d = 1 \). Next we consider the case \( d \in \{2,3,\ldots\} \). Note that \( (272) \), \( (273) \), item \( \text{(iii)} \) in Lemma 2.6 and Fubini’s theorem ensure that

\[
\int_{\mathbb{R}^d} \|y\|^p e^{-\frac{\|y\|^2}{4t}} \, dy = \int_0^\infty \int_{\mathbb{S}} r^{p+d-1} e^{-r^2} T_d(1, \phi) \, d\phi \, dr
\]

\[
= \int_{\mathbb{S}} \int_0^\infty r^{p+d-1} e^{-r^2} T_d(1, \phi) \, dr \, d\phi
\]

\[
\leq 2\pi^{d-1} \int_0^\infty r^{p+d-1} e^{-r^2} \, dr.
\]  

(278)
Combining this with (275) and (276) establishes (271) in the case $d \in \{2, 3, \ldots\}$. This completes the proof of Lemma 5.1. \hfill $\square$

**Lemma 5.2.** Let $d \in \mathbb{N}$, $\varphi \in C(\mathbb{R}^d, \mathbb{R})$, let $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, assume that $\inf_{\gamma \in (0, \infty)} \sup_{x \in \mathbb{R}^d} \left(\left|\frac{\varphi(x)}{1 + \|x\|}\right|\right) < \infty$, and let $\Phi: (0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ satisfy for all $t \in (0, \infty)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ that

$$\Phi(t, x) = \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{d/2}} e^{-\frac{(x_1 - y_1)^2 + \ldots + (x_d - y_d)^2}{4t}} \varphi(y) \, dy. \tag{279}$$

Then it holds for all $t \in (0, \infty)$, $x \in \mathbb{R}^d$ that $\Phi(t, x) \in C^1(\mathbb{R}^d, \mathbb{R})$ and

$$(\frac{\partial}{\partial t}) \Phi(t, x) = (\Delta_t \Phi)(t, x). \tag{280}$$

**Proof of Lemma 5.2.** Throughout this proof let $\rho: (0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ satisfy for all $t \in (0, \infty)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ that

$$\rho(t, x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{x_1^2 + \ldots + x_d^2}{4t}}. \tag{281}$$

Observe that for all $t \in (0, \infty)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have that

$$(\frac{\partial}{\partial t}) \rho(t, x) = \left[\frac{x_1^2 + \ldots + x_d^2}{4t^2} - \frac{d}{2t}\right] \rho(t, x). \tag{282}$$

Next note that for all $i \in \{1, \ldots, d\}$, $t \in (0, \infty)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have that

$$(\frac{\partial}{\partial x_i}) \rho(t, x) = -\frac{x_i}{2t} \rho(t, x). \tag{283}$$

This implies that for all $i, j \in \{1, \ldots, d\}$, $t \in (0, \infty)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have that

$$(\frac{\partial^2}{\partial x_i \partial x_j}) \rho(t, x) = \begin{cases} \frac{x_i^2 - 1}{4t^2} \rho(t, x) & : i = j \\ \frac{x_i x_j}{4t^2} \rho(t, x) & : i \neq j. \end{cases} \tag{284}$$

This reveals that for all $t \in (0, \infty)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have that

$$(\Delta_t \rho)(t, x) = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} \rho(t, x) = \left[\frac{x_1^2 + \ldots + x_d^2}{4t^2} - \frac{d}{2t}\right] \rho(t, x). \tag{285}$$

Combining this with (282) yields that for all $t \in (0, \infty)$, $x \in \mathbb{R}^d$ we have that

$$(\frac{\partial}{\partial t}) \rho(t, x) - (\Delta_t \rho)(t, x) = 0. \tag{286}$$

Next note that the hypothesis that $\inf_{\gamma \in (0, \infty)} \sup_{x \in \mathbb{R}^d} \left(\left|\frac{\varphi(x)}{1 + \|x\|}\right|\right) < \infty$ ensures that there exist $\gamma \in (0, \infty)$, $C \in \mathbb{R}$ which satisfy that for all $x \in \mathbb{R}^d$ we have that

$$|\varphi(x)| \leq C(1 + \|x\|^\gamma). \tag{287}$$
This and Lemma 5.1 verify that for all $t \in (0, \infty)$, $x \in \mathbb{R}^d$ we have that

$$\left| \Phi(t, x) \right| \leq \int_{\mathbb{R}^d} |\rho(t, x-y)\varphi(y)| \, dy \leq C \int_{\mathbb{R}^d} \rho(t, x-y)(1 + \|y\|^\gamma) \, dy < \infty. \quad (288)$$

Next note that (282), (284), and Lemma 5.1 demonstrate that for all $t \in (0, \infty)$, $x \in \mathbb{R}^d$ we have that

$$\int_{\mathbb{R}^d} \left| \left( \frac{\partial}{\partial t} \right) \rho \right|(t, x-y)\varphi(y) \, dy \leq C \int_{\mathbb{R}^d} \left( \frac{\|x-y\|^2}{4t^2} + \frac{d}{2t} \right) \rho(t, x-y)(1 + \|y\|^\gamma) \, dy < \infty. \quad (289)$$

Combining this, (288), and (282) with Amann & Escher [2, Ch. X, Theorem 3.18] verifies that for all $t \in (0, \infty)$, $x \in \mathbb{R}^d$ we have that $\Phi \in C^{1,0}((0, \infty) \times \mathbb{R}^d, \mathbb{R})$ and

$$\left( \frac{\partial}{\partial t} \Phi \right)(t, x) = \int_{\mathbb{R}^d} \left( \frac{\partial}{\partial t} \right) \rho(t, x-y)\varphi(y) \, dy. \quad (290)$$

Next observe that (282), (284), and Lemma 5.1 ensure that for all $i \in \{1, \ldots, d\}$, $t \in (0, \infty)$, $x \in \mathbb{R}^d$ we have that

$$\int_{\mathbb{R}^d} \left| \left( \frac{\partial}{\partial x_i} \right) \rho \right|(t, x-y)\varphi(y) \, dy \leq C \int_{\mathbb{R}^d} \|x-y\|\rho(t, x-y)(1 + \|y\|^\gamma) \, dy < \infty. \quad (291)$$

Combining this, (288), and (282) with (290) and Amann & Escher [2, Ch. X, Theorem 3.18] verifies that for all $i \in \{1, \ldots, d\}$, $t \in (0, \infty)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have that $\Phi \in C^{1,1}((0, \infty) \times \mathbb{R}^d, \mathbb{R})$ and

$$\left( \frac{\partial}{\partial x_i} \Phi \right)(t, x) = \int_{\mathbb{R}^d} \left( \frac{\partial}{\partial x_i} \right) \rho(t, x-y)\varphi(y) \, dy. \quad (292)$$

Next note that (282), (284), the fact that for all $a, b \in \mathbb{R}$ we have that $ab \leq a^2 + b^2$, and Lemma 5.1 ensure that for all $i, j \in \{1, \ldots, d\}$, $t \in (0, \infty)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have that

$$\int_{\mathbb{R}^d} \left| \left( \frac{\partial^2}{\partial x_i \partial x_j} \right) \rho \right|(t, x-y)\varphi(y) \, dy \leq C \int_{\mathbb{R}^d} \left[ \frac{\|x-y\|^2}{4t^2} + \frac{1}{2t} \right] \rho(t, x-y)(1 + \|y\|^\gamma) \, dy < \infty. \quad (293)$$

Combining this, (290), and (282) with (290) and Amann & Escher [2, Ch. X, Theorem 3.18] verifies that for all $i, j \in \{1, \ldots, d\}$, $t \in (0, \infty)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have that $\Phi \in C^{1,2}((0, \infty) \times \mathbb{R}^d, \mathbb{R})$ and

$$\left( \frac{\partial^2}{\partial x_i \partial x_j} \Phi \right)(t, x) = \int_{\mathbb{R}^d} \left( \frac{\partial^2}{\partial x_i \partial x_j} \right) \rho(t, x-y)\varphi(y) \, dy. \quad (294)$$
Hence, we obtain that for all $t \in (0, \infty)$, $x \in \mathbb{R}^d$ we have that

$$\begin{align*}
(\Delta_x \Phi)(t, x) &= \frac{d}{dt}(\partial_x \Phi)(t, x) \\
&= \int_{\mathbb{R}^d} \left[ \sum_{i=1}^{d} \left( \frac{\partial^2}{\partial x_i^2} \Phi \right)(t, x - y) \right] \phi(y) \, dy \\
&= \int_{\mathbb{R}^d} \left( \Delta_x \rho \right)(t, x - y) \phi(y) \, dy.
\end{align*}$$

Combining this and (290) with (286) establishes (280). This completes the proof of Lemma 5.3.

**Lemma 5.3.** Let $d \in \mathbb{N}$, $T \in (0, \infty)$, $\varphi \in C(\mathbb{R}^d, \mathbb{R})$, $u \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$, let $\| \cdot \| : \mathbb{R}^d \to [0, \infty)$ be a norm, assume for all $x \in \mathbb{R}^d$ that $u(0, x) = \varphi(x)$, assume that $\inf_{\gamma \in (0, \infty)} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \left( \frac{\|u(t,x)\|}{1 + \|x\|^2} \right) < \infty$, and assume that $u|_{(0,T) \times \mathbb{R}^d}$ is a viscosity solution of

$$\left( \frac{\partial}{\partial t} u \right)(t, x) = (\Delta_x u)(t, x)$$

for $(t, x) \in (0, T) \times \mathbb{R}^d$. Then it holds for all $t \in (0, T]$, $x \in \mathbb{R}^d$ that $u|_{(0,T) \times \mathbb{R}^d} \in C^{1,2}((0,T] \times \mathbb{R}^d, \mathbb{R})$ and

$$\left( \frac{\partial}{\partial t} u \right)(t, x) = (\Delta_x u)(t, x).$$

**Proof of Lemma 5.3.** Throughout this proof let $\| \cdot \|_2 : \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathbb{F}_t)_{t \in [0,T]}$, and let $W : [0, T] \times \Omega \to \mathbb{R}^d$ be a standard $(\mathbb{F}_t)_{t \in [0,T]}$-Brownian motion. Observe that there exist $C \in \mathbb{R}$, $c \in (0, \infty)$ such that for all $x \in \mathbb{R}^d$ we have that

$$c\|x\|_2 \leq \|x\| \leq C\|x\|_2.$$  

This and the fact that $\inf_{\gamma \in (0, \infty)} \sup_{x \in \mathbb{R}^d} \left( \frac{\|\varphi(x)\|}{1 + \|x\|^2} \right) < \infty$ verify that

$$\inf_{\gamma \in (0, \infty)} \sup_{x \in \mathbb{R}^d} \left( \frac{\|\varphi(x)\|}{1 + \|x\|^2} \right) < \infty.$$  

Hence, we obtain that $\varphi : \mathbb{R}^d \to \mathbb{R}$ is an at most polynomially growing function. The Feynman-Kac formula (cf., for example, Grohs et al. [28 Proposition 2.22(iii)] and Hairer et al. [31 Corollary 4.17]) hence ensures that for all $t \in [0, T]$, $x \in \mathbb{R}^d$ we have that

$$u(t, x) = \mathbb{E}[\varphi(x + \sqrt{2}W_t)].$$

Next note that the fact that for all $t \in (0, T]$ we have that $W_t$ is a $\mathcal{N}_0, \mathcal{T}_{x,t}$-distributed random variable implies that for all $t \in (0, T]$, $x \in \mathbb{R}^d$ we have that

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Then let hypotheses in (303)–(304) below, all coincide. 

Lemma 5.2 hence proves that for all $t \in (0, T]$, $x \in \mathbb{R}^d$ we have that $u_{|[0,T] \times \mathbb{R}^d} \in C^{1,2}((0, T) \times \mathbb{R}^d, \mathbb{R})$ and

$$
(\frac{\partial}{\partial t} u)(t, x) = (\Delta_x u)(t, x).
$$

This completes the proof of Lemma 5.3.

5.2 Qualitative error estimates for heat equations

It is the subject of this subsection to state and prove Theorem 5.4 below, which is the main result of this work. Theorem 5.4 establishes that ANNs do not suffer from the curse of dimensionality in the uniform numerical approximation of heat equations. Corollary 5.5 below specializes Theorem 5.4 to the case in which the constants $c \in \mathbb{R}, p, q, v, v, w, w, z, z \in [0, \infty)$, which are used to formulate the hypotheses in (300)–(304) below, all coincide.

Theorem 5.4. Assume Setting 4.5, let $c, a \in \mathbb{R}$, $b \in (a, \infty)$, $r, T \in (0, \infty)$, $p, q, v, v, w, w, z, z \in [0, \infty)$, $p = 2 + \max\{2z + 3z, 2w + 3w + 1\}$, for every $d \in \mathbb{N}$ let $\|\|_{\mathbb{R}^d} : \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, let $\varphi_d \in C(\mathbb{R}^d, \mathbb{R})$, $d \in \mathbb{N}$, let $(\phi_{\varepsilon, d})_{\varepsilon \in (0, r]} \subseteq \mathbb{N}$, and assume for all $\varepsilon \in (0, r]$, $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ that $(\mathcal{R}\phi_{\varepsilon, d}) \in C(\mathbb{R}^d, \mathbb{R})$,

$$
||| \mathcal{R}\phi_{\varepsilon, d} |||_{\mathbb{R}^d} \leq C ||x||_{\mathbb{R}^d}^p, \quad ||| (\nabla \mathcal{R}\phi_{\varepsilon, d}) |||_{\mathbb{R}^d} \leq C ||x||_{\mathbb{R}^d}^q,
$$

(303)

Then

(i) there exist unique $u_d \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$, $d \in \mathbb{N}$, which satisfy for all $d \in \mathbb{N}$, $t \in (0, T]$, $x \in \mathbb{R}^d$ that $u_d |_{[0, T] \times \mathbb{R}^d} \in C^{1,2}((0, T] \times \mathbb{R}^d, \mathbb{R})$, $u_d(0, x) = \varphi_d(x)$, inf$_{t \in (0, T]}$ sup$_{(s, y) \in [0, T] \times \mathbb{R}^d}$ $|u_d(s, y)| (1 + ||y||_{\mathbb{R}^d}) < \infty$, and

$$
(\frac{\partial}{\partial t} u_d)(t, x) = (\Delta_x u_d)(t, x)
$$

(305)

and

(ii) there exist $(\psi_{\varepsilon, d})_{\varepsilon \in (0, r]} \subseteq \mathbb{N}$, $C \in \mathbb{R}$ such that for all $\varepsilon \in (0, r]$, $d \in \mathbb{N}$ we have that $(\mathcal{R}\psi_{\varepsilon, d}) \in C(\mathbb{R}^d, \mathbb{R})$, $N(\psi_{\varepsilon, d}) \leq C d^{p+q+q(1+\frac{1}{2})} \varepsilon^{-(q+2)}$, $\mathcal{P}(\psi_{\varepsilon, d}) \leq C d^{p+q+q(1+\frac{1}{2})} \varepsilon^{-(q+2)}$, $\mathcal{Q}(\psi_{\varepsilon, d}) \leq C d^{p+q+q(1+\frac{1}{2})} \varepsilon^{-(q+2)}$, and

$$
\sup_{x \in [a, b]^d} |u_d(T, x) - (\mathcal{R}\psi_{\varepsilon, d})(x)| \leq \varepsilon.
$$

(306)
Proof of Theorem 5.4. First, observe that for all \( d \in \mathbb{N} \) we have that
\[
\sqrt{\text{Trace}(I_{\mathbb{R}^d})} = \left[ \sum_{i=1}^{d} \right]^{1/2} = d^{1/2} \leq \max\{1, c\} d^{1/2}.
\]

Corollary 4.7 (applied with \( \alpha \leftarrow 0, \beta \leftarrow \frac{1}{2}, c \leftarrow \max\{1, c\}, \mu_d \leftarrow 0, A_d \leftarrow I_{\mathbb{R}^d} \) for \( d \in \mathbb{N} \) in the notation of Corollary 4.7) hence implies that there exist unique \( u_d \in C([0, T] \times \mathbb{R}^d, \mathbb{R}) \), \( d \in \mathbb{N} \), which satisfy \( u_d(0, x) = \varphi_d(x) \), which satisfy for all \( d \in \mathbb{N} \), \( x \in \mathbb{R}^d \) that
\[
\inf_{\gamma \in (0, \infty)} \sup_{(t,x) \in [0,T] \times \mathbb{R}^d}(\varphi_d(t,x)) < \infty,
\]
and which satisfy that for all \( d \in \mathbb{N} \) we have that \( u_d(0, T) \times \mathbb{R}^d \) is a viscosity solution of
\[
\left( \frac{\partial}{\partial t} u_d \right)(t, x) = (\Delta_x u_d)(t, x)
\]
for \((t, x) \in (0, T) \times \mathbb{R}^d \) and there exist \( (\psi_{x,d})_{(x,d) \in (0, r] \times \mathbb{N}} \subseteq \mathbb{N}, C \in \mathbb{R} \) such that for all \( \varepsilon \in (0, r), d \in \mathbb{N} \) we have that
\[
N(\psi_{x,d}) \leq Cd^{\rho(q+(\frac{1}{2})q)} \varepsilon^{-(q+2)},
\]
\[
\mathcal{P}(\psi_{x,d}) \leq Cd^{\rho(2p+(\frac{1}{2})q)} \varepsilon^{-(q+4)}, \quad \mathcal{Q}(\psi_{x,d}) \leq Cd^{\rho(p+(\frac{1}{2})q)} \varepsilon^{-(q+2)}, \quad (R_{\psi_{x,d}}) \in C(\mathbb{R}^d, \mathbb{R}),
\]
\[
\sup_{x \in [0, d]} |u_d(T, x) - (R_{\psi_{x,d}})(x)| \leq \varepsilon.
\]
This proves item (ii). Next note that \( (303) \) and \( (304) \) ensure that for all \( d \in \mathbb{N} \) we have that \( \varphi_d : \mathbb{R}^d \rightarrow \mathbb{R} \) is an at most polynomially growing function. This reveals that for all \( d \in \mathbb{N} \) we have that
\[
\inf_{\gamma \in (0, \infty)} \sup_{x \in \mathbb{R}^d} (\varphi_d(x)) < \infty.
\]
Lemma 5.3 (applied with \( T \leftarrow T, \varphi \leftarrow \varphi_d, u \leftarrow u_d \) for \( d \in \mathbb{N} \) in the notation of Lemma 5.3) hence shows that for all \( d \in \mathbb{N}, t \in (0, T], x \in \mathbb{R}^d \) we have that
\[
\sup_{\gamma \in (0, \infty)} \sup_{x \in \mathbb{R}^d} \left| \frac{\varphi_d(x)}{1 + \|x\|_{\mathbb{R}^d}} \right| < \infty.
\]
This, \( (305) \), and Hairer et al. 31 Remark 4.1) prove item (i). This completes the proof of Theorem 5.4. \( \square \)

Corollary 5.5. Assume Setting 5.4, let \( a \in \mathbb{R}, b \in (a, \infty), c, T \in (0, \infty), \) for every \( d \in \mathbb{N} \) let \( \| \cdot \|_{\mathbb{R}^d} : \mathbb{R}^d \rightarrow [0, \infty) \) be the standard norm on \( \mathbb{R}^d \), let \( \varphi_d \in C(\mathbb{R}^d, \mathbb{R}), d \in \mathbb{N}, (\varphi_{x,d})_{(x,d) \in (0,1] \times \mathbb{N}} \subseteq \mathbb{N}, \) and assume for all \( \varepsilon \in (0, 1], d \in \mathbb{N}, x \in \mathbb{R}^d \) that \( (R_{\varphi_{x,d}}) \in C(\mathbb{R}^d, \mathbb{R}), \)
\[
\mathcal{P}(\varphi_{x,d}) \leq cd^\varepsilon, |\varphi_d(x) - (R_{\varphi_{x,d}})(x)| \leq \varepsilon cd^\varepsilon(1 + \|x\|_{\mathbb{R}^d}),
\]
and
\[
|(R_{\varphi_{x,d}})(x)| + \|\nabla(R_{\varphi_{x,d}}))(x)\|_{\mathbb{R}^d} \leq cd^\varepsilon(1 + \|x\|_{\mathbb{R}^d}).
\]
Then
(i) there exist unique \( u_d \in C([0,T] \times \mathbb{R}^d, \mathbb{R}) \), \( d \in \mathbb{N} \), which satisfy for all \( d \in \mathbb{N} \), \( t \in (0,T] \), \( x \in \mathbb{R}^d \) that \( u_d(t,x) \in C^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R}) \), \( u_d(0,x) = \varphi_d(x) \), inf\( \gamma \in (0,\infty) \) sup\( (s,y) \in [0,T] \times \mathbb{R}^d \) \( \left| \frac{u_d(s,y)}{1 + \|y\|^{1+d}} \right| < \infty \), and

\[
\left( \frac{\partial}{\partial t} u_d \right)(t,x) = (\Delta_x u_d)(t,x)
\]  

(314)

and

(ii) there exist \((\psi_{\varepsilon,d})(\varepsilon,d) \subseteq N, \kappa \in \mathbb{R} \) such that for all \( \varepsilon \in (0,1] \), \( d \in \mathbb{N} \) we have that \( P(\psi_{\varepsilon,d}) \leq \kappa d^2 \varepsilon - \kappa \), \( (R \psi_{\varepsilon,d}) \in C(\mathbb{R}^d, \mathbb{R}) \), and

\[
\sup_{x \in [a,b]^d} \left| u_d(T,x) - (R \psi_{\varepsilon,d})(x) \right| \leq \varepsilon.
\]  

(315)

Proof of Corollary 5.5. First, observe that \((313), (314)\), and item \((312)\) in Theorem 5.3 applied with \( r \leftarrow 1, c \leftarrow c, p \leftarrow c, q \leftarrow c, v \leftarrow c, v \leftarrow c, w \leftarrow c, w \leftarrow c, z \leftarrow c, z \leftarrow c \) in the notation of Theorem 5.3, establish item \((312)\). Next note that \((312), (313)\), and item \((312)\) in Theorem 5.3 applied with \( r \leftarrow 1, c \leftarrow c, p \leftarrow c, q \leftarrow c, v \leftarrow c, v \leftarrow c, w \leftarrow c, w \leftarrow c, z \leftarrow c, z \leftarrow c \) in the notation of Theorem 5.3 ensure that there exist \((\psi_{\varepsilon,d})(\varepsilon,d) \subseteq N, C \in \mathbb{R} \) which satisfy that for all \( \varepsilon \in (0,1], d \in \mathbb{N} \) we have that \( P(\psi_{\varepsilon,d}) \leq C d^2 e^{2+11c+6} e^{-(c+4)}, (R \psi_{\varepsilon,d}) \in C(\mathbb{R}^d, \mathbb{R}) \), and

\[
\sup_{x \in [a,b]^d} \left| u_d(T,x) - (R \psi_{\varepsilon,d})(x) \right| \leq \varepsilon.
\]  

(316)

This reveals that for all \( \varepsilon \in (0,1], d \in \mathbb{N} \) we have that

\[
P(\psi_{\varepsilon,d}) \leq \max\{C, \frac{3}{2} C^2 + 11c + 6\} d^2 \max\{C, \frac{3}{2} e^{2+11c+6}\} e^{-\max\{C, \frac{3}{2} e^{2+11c+6}\}} e^{-(c+4)}.
\]  

(317)

Combining this with \((316)\) establishes item \((312)\). This completes the proof of Corollary 5.5. \( \square \)

### 5.3 ANN approximations for geometric Brownian motions

In this subsection we specialize Theorem 5.3 above in Corollary 5.6 below to an example in which the activation function is the softplus function \((\mathbb{R} \ni x \mapsto \ln(1 + e^x) \in (0,\infty))\).

**Corollary 5.6.** Let \( c, a \in \mathbb{R}, b \in (a,\infty), p \in [0,\infty), T \in (0,\infty), \) \( x \) \in \( \mathbb{R}^d \) be the standard norm on \( \mathbb{R}^d \), let \( \mathbb{N} \) be the set given by

\[
\mathbb{N} = \cup_{L \in \mathbb{N}\cap[2,\infty)} \cup_{l_0,...,l_k \in \mathbb{N}} \left( \times_{k=1}^{l_k-1} (\mathbb{R}^{l_k} \times \mathbb{R}^{l_k-1} \times \mathbb{R}^{l_k}) \right),
\]  

(318)

let \( A_d : \mathbb{R}^d \rightarrow \mathbb{R}^d \), \( d \in \mathbb{N} \), satisfy for all \( d \in \mathbb{N} \), \( x = (x_1,...,x_d) \in \mathbb{R}^d \) that \( A_d(x) = (\ln(1 + e^{x_1}),...\ln(1 + e^{x_d})) \), let \( P : \mathbb{N} \rightarrow \mathbb{N} \) and \( R : \mathbb{N} \rightarrow \cup_{m,n \in \mathbb{N}} C(\mathbb{R}^m, \mathbb{R}^n) \) satisfy for all \( L \in \mathbb{N}\cap[2,\infty) \), \( l_0,...,l_L \in \mathbb{N} \), \( \Phi = ((W_1,B_1),...,(W_L,B_L)) \in \)
Observe that (323) assures that for all $d \phi$ and let $(K_d)_{d \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $d \in \mathbb{N}$ such that $|K_d| \leq cd^p$. Then

(i) there exist unique $u_d \in C([0, T] \times \mathbb{R}^d, \mathbb{R})$, $d \in \mathbb{N}$, which satisfy for all $d \in \mathbb{N}$, $t \in (0, T]$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ that $u_d|_{[0, T] \times \mathbb{R}^d} \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$, $u_d(0, x) = \ln(1 + e^{x_1 + \ldots + x_d - K_d}) + K_d$, $\inf_{t \in (0, \infty)} \sup_{(s,y) \in [0, T] \times \mathbb{R}^d} (\frac{\partial}{\partial t} u_d(s,y)) < \infty$, and

\begin{equation}
(\frac{\partial}{\partial t} u_d)(t, x) = (\Delta_x u_d)(t, x) \tag{320}
\end{equation}

and

(ii) there exist $(\psi_{d,\epsilon})_{\epsilon \in (0,1)}$ such that for all $\epsilon \in (0, 1)$, $d \in \mathbb{N}$ we have that $P(\psi_{d,\epsilon}) \leq c d^{1+4} \sup_{(s,y) \in [0, T] \times \mathbb{R}^d} (|u_d(s,y)|) < 1$, $R \phi_d \in C(\mathbb{R}^d, \mathbb{R})$, and

\begin{equation}
\sup_{x \in [a,b]^d} |u_d(T, x) - (R \psi_{d,\epsilon})(x)| \leq \epsilon. \tag{321}
\end{equation}

Proof of Corollary 5.6. Throughout this proof let $\varphi_d: \mathbb{R}^d \rightarrow \mathbb{R}$, $d \in \mathbb{N}$, satisfy for all $d \in \mathbb{N}$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ that

\begin{equation}
\varphi_d(x) = \ln(1 + e^{x_1 + \ldots + x_d - K_d}) + K_d \tag{322}
\end{equation}

and let $(\phi_d)_{d \in \mathbb{N}} \subseteq \mathbb{N}$ satisfy for all $d \in \mathbb{N}$ that

\begin{equation}
\phi_d = (((1, \ldots, 1), -K_d), (1, K_d)) \in (\mathbb{R}^{1 \times d} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R}). \tag{323}
\end{equation}

Observe that (322) assures that for all $d \in \mathbb{N}$ we have that

\begin{equation}
P(\phi_d) = 1(d + 1) + 1 + 1 = d + 1 \leq \max\{4, c\} d. \tag{324}
\end{equation}

Next note that the fact that $(\mathbb{R} \ni x \mapsto \ln(1 + e^x)) \in C^{1}(\mathbb{R}, \mathbb{R})$, (319), and (323) imply that for all $d \in \mathbb{N}$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have that $(R \phi_d) \in C(\mathbb{R}^d, \mathbb{R})$ and

\begin{equation}
(R \phi_d)(x) = \ln(1 + e^{x_1 + \ldots + x_d - K_d}) + K_d = \varphi_d(x). \tag{325}
\end{equation}

Next note that for all $d \in \mathbb{N}$ we have that $\ln(1 + e^{-K_d}) \leq \ln 2 + |K_d|$ and for any $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ we have that $\|\nabla \varphi_d(x)\|_{\mathbb{R}^d} \leq d^{1/2}$. This and the hypothesis that for all $d \in \mathbb{N}$ we have that $|K_d| \leq cd^p$ hence yield that for all $d \in \mathbb{N}$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we have that

\begin{equation}
\|(R \phi_d)(x)\| = |\varphi_d(x)| \leq \sup_{y \in \mathbb{R}^d} \|\nabla \varphi_d(y)\|_{\mathbb{R}^d} \|x\|_{\mathbb{R}^d} + |\varphi_d(0)| \\
\leq d^{1/2} \|x\|_{\mathbb{R}^d} + \ln 2 + |K_d| \leq d^{1/2} \|x\|_{\mathbb{R}^d} + 1 + 2cd^p \tag{326}
\end{equation}

\begin{equation}
\leq \max\{1, 2c\}(1 + d^p + d^{1/2} \|x\|_{\mathbb{R}^d}) \\
\leq 2 \max\{1, 2c\} d^{\max\{p, 1/2\}}(1 + \|x\|_{\mathbb{R}^d}).
\end{equation}
Next note that for all $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ we have that

$$\left\| (\nabla (R\phi_d))(x) \right\|_{\mathbb{R}^d} = \left\| (\nabla \phi_d)(x) \right\|_{\mathbb{R}^d} \leq d^{1/2} \leq 2 \max\{1, 2c\}d^{1/2}(1 + \|x\|_0^0).$$

(327)

Combining this, (324), (325), (326), and the fact that $(\mathbb{R} \ni x \mapsto \ln(1 + e^x) \in \mathbb{R}) \in C^1(\mathbb{R}, \mathbb{R})$ with Theorem 5.4 (applied with $c \leftarrow \max\{4, 4c\}$, $r \leftarrow 1$, $p \leftarrow 1$, $q \leftarrow 0$, $v \leftarrow 0$, $v \leftarrow 1/2$, $w \leftarrow 0$, $z \leftarrow \max\{p, 1/2\}$, $w \leftarrow 1$, $a \leftarrow (\mathbb{R} \ni x \mapsto \ln(1 + e^x) \in \mathbb{R})$ in the notation of Theorem 5.4) establishes items (i)–(ii). This completes the proof of Corollary 5.6.

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