Neutral Fermion Phenomenology With Majorana Spinors

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Abstract

We ask the question whether neutrino physics with momentum space Majorana spinors, the eigenvectors of the particle–anti-particle conjugation operator, \( C = i\gamma_2 K \) (with \( K \) standing for complex conjugation), is different but physics with Dirac spinors. First we analyze properties of Majorana spinors in great detail. We show that four dimensional, \((4d)\), Majorana spinors are unsuited for the construction of a local quantum field because \( C \) invariance does not allow for a covariant propagation in four spinor dimensions, a conduct due to \( \gamma_2 \hat{p} = -\hat{p}^* \gamma_2 \). The way out of this dilemma is finding one more discrete symmetry that respects \( C \) invariance and gives rise to covariant propagators. We construct such a symmetry in observing that the parity operator, \( \gamma_0 \), “ladders” between \((4d)\) Majorana rest frame spinors, which takes us to eight dimensional spinor spaces. We build up two types of \((8d)\) spaces—one with a symmetric- and an other with an anti-symmetric off diagonal metric and calculate traces of single beta– and neutrinoless double beta \((0\nu\beta\beta)\) decays there.

We find physics with \((8d)\) Majorana spinors in the former space to be equivalent to physics with Dirac spinors in four dimensions. In the latter space we make the rare observation that in effect of cancellations triggered by the anti-symmetric off diagonal \((8d)\) metric, the neutrino mass drops from the single beta decay trace but reappears in \(0\nu\beta\beta\), without the neutrino being massless in its free equation—a curious and in principle experimentally testable signature for a non-trivial impact of Majorana framework.

Key words: Majorana spinors, particle–anti-particle conjugation, beta decay
1 Introduction.

Virtual exchange of fermions among matter fields is a qualitatively new concept in contemporary particle physics and appears in supersymmetric theories as a process supplementary to the exchange of ordinary bosonic gauge fields.

Virtual fermions like photino, gravitino etc transport interactions and are truly neutral [1].

Further important process of the above type is the virtual neutrino line connecting two $W^- \mu^-$ currents that provides the major contribution to the spectacular neutrinoless double beta decay [2], [3] where lepton number conservation appears violated.

Truly neutral fermions, in being their own anti-particles, are invariant under particle–anti-particle (charge) conjugation, $C$, and carry well defined $C$ parity, while charged fermions are invariant under space reflection and are endowed with spatial, $P$, parity. As long as $C$ and $P$ do not commute, charged and truly neutral fermions are essentially different species.

Genuinely neutral spin-1/2 fermions are referred to as Majorana particles, while the charged ones are the Dirac particles.

The theory of Majorana fermions is based upon quantum fields that are $C$ eigenstates. The calculus of widest use for neutral spin 1/2 fermions is based upon the so called Majorana quantum field, to be denoted by $\nu(x)$ in the following. Its construction is inspired by the Dirac field,

$$\Psi_D(x) = \int \frac{d^3 p}{(2\pi)^2 \sqrt{2p_0}} \sum_h \left[ u_h(p)a_h(p)e^{-ip\cdot x} + v_h(p)b_h^\dagger(p)e^{ip\cdot x} \right]. \quad (1)$$

In defining the transformation properties of the Dirac spinors (in the convention of Ref. [4]) and the Fock operators under particle–anti-particle conjugation as [5]

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\[ Cu_h(\mathbf{p}) = \beta v_{-h}(\mathbf{p}), \quad Ca_h(\mathbf{p})C^{-1} = \beta^{-1}b_{-h}^\dagger(\mathbf{p}), \quad \beta = \delta_{h\downarrow} - \delta_{h\uparrow}, \]

one concludes for \( C \)

\[ C = i\gamma_2 K, \]

with \( K \) standing for the operation of complex conjugation. The Majorana quantum field is now defined \([6]\) in identifying in Eq. (1) the operators of particle- and anti-particle creation according to

\[ \nu(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}\sqrt{2p_0}} \sum_h [u_h(\mathbf{p})a_h(\mathbf{p})e^{-ip\cdot x} + \lambda v_h(\mathbf{p}) \ a_h^\dagger(\mathbf{p})e^{ip\cdot x}], \]

where \( h \) stands for helicity, and \( \lambda \) is the so called creation phase factor introduced in \([7]\). The freedom of having the \( \lambda \) phase in Eq. (4) is important for obtaining a real mixing matrix under CP conservation. An other option for a neutral quantum field, termed to as \( \mu(x) \) by us, would be to use in place of massive Dirac spinors, which are parity, \( P \), eigenvectors, the massive eigenvectors of the particle–anti-particle (charge) conjugation operator, \( C \). Such spinors are known as momentum space Majorana spinors, and are denoted by us as \( \Psi_M^{h(\epsilon_j)}(\mathbf{p}) \). Here, \( \epsilon_j = \pm 1, \) or \( \pm i \) fixes \( C \) parity. A \( \mu(x) \) field built upon, say, real \( C \) parity momentum spinors is expected to take the form

\[ \mu(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}\sqrt{2p_0}} \sum_h [\Psi_M^{h(1)}(\mathbf{p})a_h(\mathbf{p})e^{-ip\cdot x} + \Psi_M^{h(-1)}(\mathbf{p}) \ a_h^\dagger(\mathbf{p})e^{ip\cdot x}]. \]

Majorana spinors can be encountered in the neutrino physics chapters of a multitude of contemporary textbooks such like \([2], [5], [8], [9], [10]\). Now the main question is whether a construct like \( \mu(x) \) is reasonable, and if so, whether it predicts phenomena beyond the range of Eq. (4).

It is the goal of the present paper to answer these questions. We aim to go to the essentials of the \( C \) invariant four-spinors and unveil predictive power of truly neutral quantum fields entirely based upon momentum space \( C \) parity spinors. The preprint is organized as follows. Section 2 reveals various peculiarities of massive Majorana spinors such as twofold helicity content (in the helicity frame), and self-orthogonality. There we show that \( \mu(x) \) is unreasonable because the \( \Psi_M^{h(\epsilon_j)}(\mathbf{p}) \)'s are non-propagating. We circumvent the problem of \( \mathbf{p} \) independence of the Majorana propagators in noticing that \( p^\mu\gamma_\mu \) ladders between certain Majorana spinors, a property that reflects a discrete symmetry of Majorana spinors beyond \( C \) but in eight spinorial dimensions, (8d). We exploit the new discrete symmetries for the construction of covariant projectors and corresponding wave equations.
Throughout Section 2 we use the textbook Majorana spinors, $\Psi_M^{h; (\pm 1)} (p)$ of real $C$ parity and build up the first complete set of eight dimensional spinor degrees of freedom. It is characterized by a metric that is real, off-diagonal, and symmetric.

In Section 3 we (i) design various $(8d)$ currents (ii) calculate the neutron beta decay trace, (iii) find it to be same as if we had worked in four dimensions with Dirac spinors.

We continue in Section 4 with Majorana spinors of pure imaginary $C$ parity, $\Psi_M^{h; (\mp i)} (p)$ in our notation. While this type of Majorana spinors has same gross peculiarities in $(1/2, 0) \oplus (0, 1/2)$ as the textbook ones, it also differs from the latter in some aspects. In particular, scalar products between such Majorana spinors change sign upon reversing order of the spinors. This peculiarity shows up in the associated eight dimensional space as a metric that is off–diagonal, purely imaginary and anti–symmetric.

In the latter space we make the rare observation that in effect of cancellations triggered by the anti-symmetric metric, the neutral particle mass can drop from the neutron beta decay trace and one finds a Dirac trace with a massless neutral particle sector, an effect that should be in principle observable in $\beta$ decay with polarized sources.

In Section 5 we elaborate the trace in the neutrinoless double beta decay, $0\nu\beta\beta$, by means of $(8d)$ Majorana spinors and show it to be unaltered with respect to the standard expression based upon Dirac spinors, irrespective of the type of the spinor, or current input.

The paper closes with a brief Summary.

2 Majorana spinors of real $C$ parity.

The textbook Majorana spinors (here in momentum space) are defined as

$$\Psi_M^{h; (\epsilon_j)} (p) = \begin{pmatrix} \epsilon_j i \sigma_2 \left(\Phi_L^h (p)\right)^* \\ \Phi_L^h (p) \end{pmatrix}, \quad h = \uparrow, \downarrow, \quad \epsilon_1 = -\epsilon_2 = 1. \quad (6)$$
Here $\Phi^h_{L}(p)$ is a left handed, $(0, 1/2)$, spinor of given helicity, $^1$ $\sigma_2$ is the standard second Pauli matrix, $h = \uparrow, \downarrow$, and $\epsilon_j$ is the real relative phase between the Weyl spinors that will be identified with their $C$ parity in the following.

As is well known [5], $\sigma_2 \sigma_2 = \sigma_2 (\sigma_2^*)$, and

$$\hat{p} \cdot \sigma \epsilon_j \sigma_2 (\Phi^h_{L}(p))^* = \epsilon_j i \sigma_2 (\Phi^h_{L}(p))^* = \epsilon_j (-h) i \sigma_2 (\Phi^h_{L}(p))^* = \epsilon_j h \alpha \Phi^{-h}_{R}(p), \quad (7)$$

meaning that $i \sigma_2 (\Phi^h_{L}(p))^*$ is (up to a sign $\alpha = \pm$), a right handed field (henceforth denoted by $\Phi^{-h}_{R}(p)$) of opposite helicity to $\Phi^h_{L}(p)$.

Therefore, contrary to Dirac spinors, Majorana spinors can not be prepared as pure helicity eigenstates. Rather, they are patched together by two Weyl spinors of opposite helicities.

2.1 Rest frame properties.

The explicit expressions for the rest frame spinors resulting from Eqs. (6) and (7) read

$$\Psi^{\uparrow,(+1)}(0) = \begin{pmatrix} \Phi^\uparrow_{R}(0) \\ \Phi^\uparrow_{L}(0) \end{pmatrix}, \quad \Psi^{\downarrow,(+1)}(0) = \begin{pmatrix} -\Phi^\uparrow_{R}(0) \\ \Phi^\uparrow_{L}(0) \end{pmatrix},$$

$$\Psi^{\uparrow,(-1)}(0) = \begin{pmatrix} -\Phi^\uparrow_{R}(0) \\ \Phi^\uparrow_{L}(0) \end{pmatrix}, \quad \Psi^{\downarrow,(-1)}(0) = \begin{pmatrix} \Phi^\uparrow_{R}(0) \\ \Phi^\uparrow_{L}(0) \end{pmatrix}. \quad (8)$$

Preparing rest frame $(1/2, 0)$, and $(0, 1/2)$ helicity spinors along the direction of the intended boost is standard (compare [5])

$$\Phi^{\uparrow}_{L/R}(0) = \sqrt{m} \begin{pmatrix} \cos(\theta/2) e^{-i \varphi/2} \\ \sin(\theta/2) e^{i \varphi/2} \end{pmatrix}, \quad \Phi^{\downarrow}_{L/R}(0) = \sqrt{m} \begin{pmatrix} \sin(\theta/2) e^{-i \varphi/2} \\ -\cos(\theta/2) e^{i \varphi/2} \end{pmatrix}. \quad (9)$$

We used the convention $i \sigma_2 \Phi^{\uparrow}_{L/R}(0) = \Phi^{\uparrow}_{R/L}(0)$, and denoted azimuthal and polar angles by $\varphi$, and $\theta$, respectively.

$^1$ We prefer to perform our calculations in the helicity frame where many a statements are more obvious. Our results are however basis independent, if not emphasized differently.
It verifies directly that $\Psi^{\uparrow (+1)}_M(0)$, and $\Psi^{\uparrow (+1)}_M(0)$, are of positive–, while $\Psi^{\downarrow (-1)}_M(0)$, and $\Psi^{\downarrow (-1)}_M(0)$ are of of negative $C$ parity, 

\[
C\Psi^{h(+1)}_M(0) = \Psi^{h(+1)}_M(0),
\]

\[
C\Psi^{h(-1)}_M(0) = -\Psi^{h(-1)}_M(0).
\]  

(10)

In other words, $\epsilon_j = \pm 1$ can be viewed as a $C$ parity label. The essential difference to Dirac spinors is that $\Phi^{h}_{LR}(p) \neq i\sigma_2 (\Phi^{h}_{LR}(p))^\dagger$. It is that very difference which makes Majorana spinors so special and gives rise to several weird peculiarities, to be explored in the following.

### 2.2 Cross-Normalized Majorana spinors.

A curiosity occurs when calculating scalar products of the above Majorana spinors. In first place, all $\Psi^{h(\epsilon_j)}_M(0)$ are self-orthogonal. Second, also spinors of equal $C$ parities happen to be orthogonal. The only non-vanishing scalar products are those between Majorana spinors of opposite $C$ parities and opposite $h$ labels,

\[
\overline{\Psi}^{\uparrow (+1)}_M(0) \Psi^{\downarrow (-1)}_M(0) = \overline{\Psi}^{\downarrow (-1)}_M(0) \Psi^{\uparrow (+1)}_M(0) = 2m,
\]

\[
\overline{\Psi}^{\downarrow (+1)}_M(0) \Psi^{\uparrow (-1)}_M(0) = \overline{\Psi}^{\uparrow (-1)}_M(0) \Psi^{\downarrow (+1)}_M(0) = -2m,
\]  

(11)

with

\[
\overline{\Psi}^{h(\epsilon_j)}_M(0) = (\Psi^{h(\epsilon_j)}_M(0))^\dagger \gamma_0, \quad \gamma_0 = \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix}.
\]  

(12)

Notice, that $h$ specifies helicity of the lower Weyl spinor, $\Phi^{h}_{LR}(p)$. Majorana spinors were shown above to be of twofold helicity content. In result, the textbook Majorana spinors build two independent spaces distinct by the sign of their cross-norms. Each subspace contains a positive– and a negative $C$ parity spinor of non-vanishing cross–projections. As long as scalar products are Lorentz invariant, cross-normalization holds true in all inertial frames.

The self-orthogonality of Majorana spinors has a devastating impact on several fundamental operators in $(1/2, 0) \oplus (0, 1/2)$ such as mass term and projectors.
2.3 Absence of a diagonal mass term.

The structure of a (generic) Majorana spinor, $\Psi_M^{h; \{\varepsilon\}}$, is more transparent within the context of the group $SL(2, C)$ (consult [11] among others), where $\Phi_L^h$ is the undotted upper-, while $i\sigma_2 (\Phi_L^h)^*$ is the dotted lower index spinor. We here focus our attention onto the diagonal (Dirac) mass term $m_M \overline{\Psi_M^{h; \{\varepsilon\}}} \Psi_M^{h; \{\varepsilon\}}$. In fact, it can not be introduced at all as self-orthogonality nullifies $\overline{\Psi_M^{h; \{\varepsilon\}}} \Psi_M^{h; \{\varepsilon\}}$.

One possible way out of vanishing Dirac mass terms for Majorana particles proposed in the literature is to restrict to two component spinors and to consider the Weyl spinor components, ($\Phi_L^h$)$_a^*$ and $[i\sigma_2 (\Phi_L^h)]_b^*$ with $a, b = 1, 2$ as anti–commuting Grassmann numbers. In so doing, one produces the following mass term [5]

$$m_M \Phi_L^h \dagger [i\sigma_2 (\Phi_L^h)^*] = m_M \left( (\Phi_L^h)^*_1, (\Phi_L^h)^*_2 \right) \left( (\Phi_L^h)^*_2, - (\Phi_L^h)^*_1 \right),$$

$$\left( (\Phi_L^h)^*_1, (\Phi_L^h)^*_2 \right) = - \left( (\Phi_L^h)^*_2, (\Phi_L^h)^*_1 \right).$$

(13)

The mass term for $C$ eigenspinors in this scenario acquires purely quantum nature [12]. It is the first goal of the present study to construct classical mass terms for $C$ parity spinors.

2.4 Boosted Majorana spinors.

In this subsection we consider the effect of the $(1/2, 0) \oplus (0, 1/2)$ boost, to be referred to as $B_{R\oplus L}(p)$, upon $\Psi_M^{h; \{\varepsilon\}}(0)$. In making once again use of Eq. (7) amounts to

$$\begin{pmatrix} \epsilon_j \alpha \Phi_R^{-h}(p) \\ \Phi_L^{-h}(p) \end{pmatrix} = \sqrt{p_0 + m \over 2m} \begin{pmatrix} 1_2 + {p_0 \over p_0 + m} \sigma \cdot \hat{p} & 0_2 \\ 0_2 & 1_2 - {p_0 \over p_0 + m} \sigma \cdot \hat{p} \end{pmatrix} \begin{pmatrix} \epsilon_j \alpha \Phi_R^{-h}(0) \\ \Phi_L^{-h}(0) \end{pmatrix},$$

(14)

(compare also Ref. [13]). Identity and null matrices of dimensionality $n \times n$ are denoted in turn by $1_n$ and $0_n$, while positive and negative signs in front of the helicity operator, $\sigma \cdot \hat{p}$, correspond to $B_R(p)$, and $B_L(p)$, respectively,
Beyond the representation of the boost matrix in Eq. (14), we shall occasionally use also the following manifestly covariant expressions for

\[
B_{R\oplus L}(\mathbf{p}) = \frac{1}{\sqrt{2m(p_0 + m)}}(\not{\mathbf{p}} + m\gamma_0)\gamma_0, \tag{15}
\]

and its inverted

\[
B_{R\oplus L}(\mathbf{p})^{-1} = \frac{1}{\sqrt{2m(p_0 + m)}}\gamma_0 (\not{\mathbf{p}} + m\gamma_0). \tag{16}
\]

2.5 Spatial parity spinors and Dirac equation.

In order to obtain \(\Psi_{\mu}^{h(\epsilon)}(\mathbf{p})\) propagation one can proceed along the line of the general construction of wave equations from the discrete \(P,\) and \(C\) properties of space time. Recall, that the group-theoretical construction of the Dirac equation starts with the rest frame projector onto parity-eigenvectors,

\[
\Pi_{R}^{\pm}(0) = \frac{1}{2} (1_4 \pm \gamma_0), \tag{17}
\]

and the requirement that the spinors carry good spatial parity according to

\[
\Pi_{R}^{\pm}(0)u_h(0) = u_h(0), \quad \Pi_{R}^{\pm}(0)v_h(0) = v_h(0), \tag{18}
\]

with \(R\) labeling space reflection. Notice that the \(\gamma_0\) transformation of the \(SL(2,C)\) spinors amounts to reflections in three space, i.e. to \(\mathbf{x} \rightarrow -\mathbf{x}, \mathbf{p} \rightarrow -\mathbf{p}\). The boosted form of Eq. (17) reads

\[
\Pi_{R}^{\pm}(\mathbf{p}) = B_{R\oplus L}(\mathbf{p})\frac{1}{2} (1_4 \pm \gamma_0) B_{R\oplus L}(\mathbf{p})^{-1} = \frac{\not{\mathbf{p}} \pm m}{2m}. \tag{19}
\]

The Dirac equations for the \(u\) and \(v\) spinors are then

\[
B_{R\oplus L}(\mathbf{p})\frac{1}{2} (1_4 + \gamma_0) B_{R\oplus L}(\mathbf{p})^{-1} u_h(\mathbf{p}) = u_h(\mathbf{p}), \tag{20.1}
\]

\[
B_{R\oplus L}(\mathbf{p})\frac{1}{2} (1_4 - \gamma_0) B_{R\oplus L}(\mathbf{p})^{-1} v_h(\mathbf{p}) = v_h(\mathbf{p}). \tag{20.2}
\]

The solutions of Eqs. (20) can be found, among others, in Ref. [4]. The eigen-spinors of the totally symmetric real matrix \(\gamma_0\) are exclusively of real parities and the parity equals the relative phase between spinor and co-spinor in
In that regard, the general question on the relationship between the relative spinor–co-spinor phase and parity may be of interest. In other words, how does an imaginary spinor–co-spinor relative phase, say,

\[ U_h^{±i}(p) = \begin{pmatrix} \pm i \Phi^h_R(p) \\ \Phi^h_L(p) \end{pmatrix} \]  

relate to parity? In order to establish such a link we observe that

\[ \gamma_0 \begin{pmatrix} \pm i \Phi^h_R(-p) \\ \Phi^h_L(-p) \end{pmatrix}^* = \mp i \begin{pmatrix} \pm i \Phi^h_R(-p) \\ \Phi^h_L(-p) \end{pmatrix} \]  

(22)

In a sense, \( \gamma_0 K R \) acts as “analytical continuation” of the \( SL(2, C) \) parity operator and its eigenstates are of imaginary parity. Leaving aside the problem of a possible group structure for which \( \gamma_0 K R \) covers space reflections, one nonetheless may try to subject the projector resulting from Eq. (22) to Lorentzian boost to obtain

\[ B_{R\otimes L}(p)^{-1}U_h^{±i}(p) = U_h^{±i}(p), \]  

(23)

which is equivalent to

\[ (\hat{p}\gamma_0 + m)(\hat{p}^* + m\gamma_0)U_h^{±i}(p)^* = \mp i2m(E + m)U_h^{±i}(p), \]  

(24)

an equation that (i) invokes imaginary masses and acausal propagation, (ii) violates Lorentz invariance. Therefore, one can not expect any relevance of such “imaginary spatial parity” spinors. This contrasts the case of the charge conjugation operator which allows for both real and imaginary parities. Moreover, for Majorana spinors there is no relationship between \( C \) parity and causality, a reason for which one needs to consider both real and imaginary \( C \) parities on equal footing.

### 2.6 Non-propagating \( C \) parity spinors.

In taking above path, we first write down the rest frame projectors onto real \( C \) parities as

\[ \mathcal{P}^\pm(0) = \frac{1}{2} (14 \pm i\gamma_2 K), \quad \mathcal{P}^\pm(0)\psi^{h;(\epsilon_j)}_M(0) = \psi^{h;(\epsilon_j)}_M(0). \]  

(25)

\[(1/2, 0) \oplus (0, 1/2) [15].\]
Boosting $\mathcal{P}^\pm(0)$ in Eq. (25) amounts to

$$\mathcal{P}^\pm(p) = B_{R\oplus L}(p) \frac{1}{2} (1 \pm i\gamma_2 K) B_{R\oplus L}(p)^{-1}.$$  \hspace{1cm} (26)

In substituting Eqs. (15), and (16) for $B_{R\oplus L}(p)$, and $B_{R\oplus L}(p)^{-1}$, respectively, amounts to

$$\mathcal{P}^\pm(p) = \frac{1}{2} \left( 1 \pm \frac{1}{2m(E + m)} (\dot{p}\gamma_0 + m)i\gamma_2(\gamma_0\dot{p}^* + m) \right) = \mathcal{P}^\pm(0), \hspace{1cm} (27)$$

where one uses $\gamma_2 \dot{p} = -\dot{p}^* \gamma_2$.

In other words, one calculates momentum independence of the Majorana projector.

The consequences would be absence of propagation and impossibility to construct a local $\mu(x)$. This serious drawback of the Majorana spinors requires special attention, a subject of subsection 2.8 below.

2.7 Static versus non-covariant propagation of (4d) Majorana spinors.

A further surprise, perhaps even a pathology, associated with Majorana spinors is that when exploiting $\Psi^{h;\epsilon_j}_M(0)$ for the construction of projectors (here denoted by $\mathcal{P}^\pm(0)$) onto $C$ parity vectors, one finds them in general to be different but the analytical projector in Eq. (25). Consider

$$P^+(0) = \frac{1}{2m} \left( \Psi^{+(+1)}_M(0)\overline{\Psi}^{(-1)}_M(0) - \Psi^{+(+1)}_M(0)\overline{\Psi}^{(-1)}_M(0) \right),$$

$$P^-(0) = -\frac{1}{2m} \left( \Psi^{(-1)}_M(0)\overline{\Psi}^{(+1)}_M(0) - \Psi^{(-1)}_M(0)\overline{\Psi}^{(+1)}_M(0) \right),$$

$$P^+(0) + P^-(0) = 1.$$  \hspace{1cm} (28)

It directly verifies that $P^+(0)$ and $P^-(0)$ in turn project onto spinors of positive and negative $C$ parities according to

$$P^+(0)\Psi^{h;\epsilon_j}_M(0) = \Psi^{h;\epsilon_j}_M(0), \hspace{1cm} P^-(0)\Psi^{h;\epsilon_j}_M(0) = \Psi^{h;\epsilon_j}_M(0).$$  \hspace{1cm} (29)

Naively, one expects $P^\pm(0)$ to coincide with the analytical rest-frame projector $\mathcal{P}^\pm = \frac{1}{2}(1 \pm i\gamma_2 K)$ in Eq. (25). This is by far not so. The reason is that at rest $K\Psi^{h;\epsilon_j}_M(0)$ effectively reduces to $(\Psi^{h;\epsilon_j}_M(0))^* = \tilde{A}\Psi^{h;\epsilon_j}_M(0)$, and
where $\bar{A}$ is particular matrix. Obviously, $\bar{A}$ depends on the particular choice for the spinors and can not be frame independent as long as the operator of complex conjugation does not allow for a universal matrix representation. In case of $\Psi_M^{\alpha(\pm 1)}(0)$, and in the Cartesian frame, $\bar{A}$ is the unit matrix. For the same reason, in general

$$
\bar{A}\Psi_M^{\alpha(\epsilon_j)}(0) = \bar{A}B_{R\oplus L}(p)^{-1}B_{R\oplus L}(p)\Psi_M^{\alpha(\epsilon_j)}(0) \\
\neq [B_{R\oplus L}(p)^{-1}]^* \left(\Psi_M^{\alpha(\epsilon_j)}(p)\right)^* ,
$$

(31)

because as a rule one observes the inequality

$$
\bar{A}B_{R\oplus L}(p)^{-1}\bar{A}^{-1} \neq [B_{R\oplus L}(p)^{-1}]^*. 
$$

(32)

Next we consider projectors, in turn denoted by $\Pi^+(0)$, and $\Pi^-(0)$, onto $\Psi_M^{\alpha(\pm 1)}(0)$ vectors of positive and negative cross-norms:

$$
\Pi^+(0) = \frac{1}{2m}\left(\Psi_M^{\alpha(+1)}(0)\bar{\Psi}_M^{\beta(-1)}(0) + \Psi_M^{\alpha(-1)}(0)\bar{\Psi}_M^{\beta(+1)}(0)\right) \\
\Pi^-(0) = -\frac{1}{2m}\left(\Psi_M^{\alpha(-1)}(0)\bar{\Psi}_M^{\beta(-1)}(0) + \Psi_M^{\alpha(+1)}(0)\bar{\Psi}_M^{\beta(+1)}(0)\right),
$$

(33)

As long as according to Eq. (11) vectors of equal cross norms are of opposite C parities, the projectors $\Pi^\pm(0)$ are in general different from $P^\pm(0)$. ² An immediate and quick test of the latter statement is performed in the Cartesian frame ($\theta = \phi = 0$ in Eq. (9) ) where one calculates $\Pi^+(0) = \frac{1}{2}(1_4 + \gamma_5\gamma_1)$, while $P^+(0) = \frac{1}{2}(1_4 + \gamma_2\bar{A})$ with $\bar{A} = 1_4$. Apparently, both $B_{R\oplus L}(p)^{\frac{1}{2}}(1_4 + \gamma_2\bar{A})B_{R\oplus L}(p)^{-1}$, and $B_{R\oplus L}(p)^{\frac{1}{2}}(1_4 + \gamma_5\gamma_1)B_{R\oplus L}(p)^{-1}$ give rise to two essentially different non-covariant equations.

Certainly, one may consider the frame dependent equations and non-local $\mu(x)$ resulting from boosting $P^\pm(0)$, and/or $\Pi^\pm(0)$, and advocate arbitrary violation of Lorentz symmetry in the Universe, a path pursued in Ref. [13], and for the spinors in Section 4 below. We here take a distinct position and aim to search for covariant equations that are consistent with the boosted projectors. We circumvent the problem of static Majorana propagators in

As we shall see below, in choosing a pure imaginary $(1/2, 0)$–$(0, 1/2)$ relative phase, one at least achieves equality of the projectors onto vectors of positive/negative cross-norms and those onto vectors of positive/negative C parities but without resolving the problem of their non-covariance.

² As we shall see below, in choosing a pure imaginary $(1/2, 0)$–$(0, 1/2)$ relative phase, one at least achieves equality of the projectors onto vectors of positive/negative cross-norms and those onto vectors of positive/negative C parities but without resolving the problem of their non-covariance.
(1/2, 0) ⊕ (0, 1/2) to the cost of introducing auxiliary extra spinor dimensions. In the latter space we shall establish consistency between the covariant equations and the projectors onto the degrees of freedom under consideration and shall construct a local Majorana quantum field.

2.8 Constructing covariantly propagating Majorana spinors.

In this subsection we develop an idea how to circumvent the absence of Majorana spinor propagation in (1/2, 0) ⊕ (0, 1/2) observed above. A simple observation sheds strong light onto the problem under investigation. In looking onto Eq. (8), it is not difficult to realize that the parity operator, γ0, "ladders" between Majorana spinors of opposite charge conjugation parities and opposite helicities of the source spinor ΦhL(0), according to

\[
\begin{align*}
\gamma_0 \Psi^\dagger_{M}^{(-1)}(0) &= \Psi^\dagger_{M}^{(+1)}(0), \\
\gamma_0 \Psi^\dagger_{M}^{(+1)}(0) &= \Psi^\dagger_{M}^{(-1)}(0), \\
\gamma_0 \Psi^\dagger_{M}^{(-1)}(0) &= -\Psi^\dagger_{M}^{(-1)}(0), \\
\gamma_0 \Psi^\dagger_{M}^{(+1)}(0) &= -\Psi^\dagger_{M}^{(+1)}(0).
\end{align*}
\]

(34)

This observation takes one directly to a new discrete symmetry in the larger space of eight spinorial dimensions. The new symmetry is associated with rest frame projectors and spinors of the type

\[
\pi^\pm(0) = \frac{1}{2} \begin{pmatrix}
\gamma_0 & 0_4 \\
0_4 & \gamma_0
\end{pmatrix}, \quad \pi^+(0) \begin{pmatrix}
\Psi^\dagger_{M}^{(-1)}(0) \\
\Psi^\dagger_{M}^{(+1)}(0)
\end{pmatrix} = \begin{pmatrix}
\Psi^\dagger_{M}^{(-1)}(0) \\
\Psi^\dagger_{M}^{(+1)}(0)
\end{pmatrix}.
\]

(35)

In now defining charge conjugation in the enlarged space as diag(−iγ2K, iγ2K), one immediately realizes that (i) the blown up spinors carry a well defined C parity, (ii) the C operator commutes with π±(0). We exploit the new discrete symmetry for the construction of covariant projectors in subjecting π±(0) to similarity transformations by the boost with the following result:

\[
\frac{1}{2m} \begin{pmatrix}
m_{14} & \not{p} \\
\not{p} & m_{14}
\end{pmatrix} \begin{pmatrix}
\Psi^\dagger_{M}^{(-1)}(p) \\
\Psi^\dagger_{M}^{(+1)}(p)
\end{pmatrix} = \begin{pmatrix}
\Psi^\dagger_{M}^{(-1)}(p) \\
\Psi^\dagger_{M}^{(+1)}(p)
\end{pmatrix}.
\]

(36)

Similarly, one finds
\[
\frac{1}{2m} \begin{pmatrix} m_{14} & -\hat{p} \\ -\hat{p} & m_{14} \end{pmatrix} \begin{pmatrix} \Psi_{M}^{+(1)}(p) \\ \Psi_{M}^{+(1)}(p) \end{pmatrix} = \begin{pmatrix} \Psi_{M}^{+(1)}(p) \\ \Psi_{M}^{+(1)}(p) \end{pmatrix}.
\] (37)

Equations (36) and (37) are equivalently rewritten to

\[
\begin{pmatrix} \hat{p} & 0_4 \\ 0_4 & \hat{p} \end{pmatrix} \begin{pmatrix} \Psi_{M}^{h_e(p)}(p) \\ \Psi_{M}^{h_e(p)}(p) \end{pmatrix} = \pm m \begin{pmatrix} 0_4 & 1_4 \\ 1_4 & 0_4 \end{pmatrix} \begin{pmatrix} \Psi_{M}^{h_e(p)}(p) \\ \Psi_{M}^{h_e(p)}(p) \end{pmatrix}.
\] (38)

The off diagonal form of the \((8d)\) mass matrix in Eq. (38) expresses cross-normalization of \(\Psi_{M}^{h_e(p)}(p)\), and its symmetric character reflects the independence of the cross-norm on the order of the spinors. Equations (36) and (37) demonstrate how Majorana spinors propagate in eight dimensions and that the propagating degrees of freedom are well represented by the following complete set of eight dimensional spinors:

\[
\begin{align*}
\Lambda_1(p) &= \begin{pmatrix} \Psi_{M}^{+(1)}(p) \\ \Psi_{M}^{-(1)}(p) \end{pmatrix}, & \Lambda_2(p) &= \begin{pmatrix} \Psi_{M}^{-(1)}(p) \\ \Psi_{M}^{+(1)}(p) \end{pmatrix}, \\
\Lambda_3(p) &= \begin{pmatrix} \Psi_{M}^{+(1)}(p) \\ -\Psi_{M}^{-(1)}(p) \end{pmatrix}, & \Lambda_4(p) &= \begin{pmatrix} \Psi_{M}^{-(1)}(p) \\ -\Psi_{M}^{+(1)}(p) \end{pmatrix}, \\
\Lambda_5(p) &= \begin{pmatrix} \Psi_{M}^{+(1)}(p) \\ \Psi_{M}^{-(1)}(p) \end{pmatrix}, & \Lambda_6(p) &= \begin{pmatrix} \Psi_{M}^{-(1)}(p) \\ \Psi_{M}^{+(1)}(p) \end{pmatrix}, \\
\Lambda_7(p) &= \begin{pmatrix} \Psi_{M}^{+(1)}(p) \\ -\Psi_{M}^{-(1)}(p) \end{pmatrix}, & \Lambda_8(p) &= \begin{pmatrix} \Psi_{M}^{-(1)}(p) \\ -\Psi_{M}^{+(1)}(p) \end{pmatrix}.
\end{align*}
\] (39)

Notice that above spinors define an orthonormal basis as

\[
\begin{align*}
\bar{\Lambda}_1(p)\Lambda_1(p) &= \bar{\Lambda}_2(p)\Lambda_2(p) = \bar{\Lambda}_7(p)\Lambda_7(p) = \bar{\Lambda}_8(p)\Lambda_8(p) = 4m, \\
\bar{\Lambda}_3(p)\Lambda_3(p) &= \bar{\Lambda}_4(p)\Lambda_4(p) = \bar{\Lambda}_5(p)\Lambda_5(p) = \bar{\Lambda}_6(p)\Lambda_6(p) = -4m,
\end{align*}
\]

\[
\bar{\Lambda}_k(p) = [\Lambda_k(p)]^\dagger \Gamma_8 \Gamma^0, \quad k = 1, \ldots, 8, \quad \Gamma_8 = \begin{pmatrix} 0_4 & 1_4 \\ 1_4 & 0_4 \end{pmatrix}.
\] (40)

Here, \(\Gamma_8\) plays the role of metric in the eight dimensional space of the \(\Lambda_k(p)\) spinors. Next we check the energy-momentum dispersion relation for the \((8d)\) Majorana spinors. For this purpose we cast, say, Eq. (36) into the form
nullify the determinant of the $8 \times 8$ matrix on the lhs, and find the time-like relation, $p^2 - m^2 = 0$. Therefore, Eq. (41) describes neutral particles of real mass in terms of spinors that are eigenvectors of the particle–anti-particle conjugation operator.

2.9 Consistency of wave equations and projectors.

At that stage it is necessary to test consistency of Eq. (41) with the projector onto the $\Lambda_k(p)$ spinors. In the following $\pi^+(p)$ and $\pi^-(p)$ in turn denote projectors onto $\Lambda_k(p)$ spinors of positive, and negative norms according to:

$$
\pi^+(p) = \frac{1}{4m} \left( \Lambda_1(p)\tilde{\Lambda}_1(p) + \Lambda_2(p)\tilde{\Lambda}_2(p) + \Lambda_7(p)\tilde{\Lambda}_7(p) + \Lambda_8(p)\tilde{\Lambda}_8(p) \right),
$$

$$
\pi^-(p) = -\frac{1}{4m} \left( \Lambda_3(p)\tilde{\Lambda}_3(p) + \Lambda_4(p)\tilde{\Lambda}_4(p) + \Lambda_5(p)\tilde{\Lambda}_5(p) + \Lambda_6(p)\tilde{\Lambda}_6(p) \right).
$$

In terms of $\pi^\pm(p)$, the wave equation for the propagating $\Lambda_k(p)$ spinors reads

$$
\pi^\pm(p)\Lambda_k(p) = \Lambda_k(p), \quad k = 1, \ldots, 8,
$$

where $\pi^+(p)$ applies to $\Lambda_1(p)$, $\Lambda_2(p)$, $\Lambda_7(p)$, $\Lambda_8(p)$, while $\pi^-(p)$ applies to the rest. A direct calculation of, say, $\pi^+(p)$ leads to

$$
\pi^+(p) = \frac{1}{4m} \begin{pmatrix} 2m(\Pi^+(0) + \Pi^-(0)) & \Sigma(p) \\ \Sigma(p) & 2m(\Pi^+(0) + \Pi^-(0)) \end{pmatrix} = \frac{1}{2m} \begin{pmatrix} m_1 & \frac{1}{2}\Sigma(p) \\ \frac{1}{2}\Sigma(p) & m_1 \end{pmatrix},
$$

where we exploited completeness of the $\Psi^{h(\pi^1)}_M(p)$ degrees of freedom, and introduced $\Sigma(p) = \sum_{h_{\epsilon_j}} \Psi^{h_{\epsilon_j}}_M(p)\tilde{\Psi}^{h_{\epsilon_j}}_M(p)$. This quantity can be reduced to a combination of Dirac $u$ and $v$ spinors upon decomposing the complete set of $\Psi^{h_{\epsilon_j}}_M(p)$ spinors into the complete set of Dirac’s $\{u_h(p), v_h(p)\}$ spinors. In so doing one encounters
The latter equation shows that a $\Psi^{h(\pm 1)}_M(p)$ spinor is a linear combination of Dirac’s $u$ and $v$ spinors of opposite parities, as it should be due to the non-commutativity of the $C$ and $P$ operators.

In making use of the decomposition in Eq. (46), one calculates $\Sigma(p)$ to be the sum of the projectors onto Dirac $u$ and $v$ spinors according to

$$\Sigma(p) = u_\uparrow(p)\bar{u}_\uparrow(p) + u_\downarrow(p)\bar{u}_\downarrow(p) + v_\uparrow(p)\bar{v}_\uparrow(p) + v_\downarrow(p)\bar{v}_\downarrow(p)$$

and

$$\pi^+(p) = \frac{1}{2m} \begin{pmatrix} m & \dot{p} \\ \dot{p} & m & 1 \end{pmatrix},$$

that establishes the consistency under discussion.

Canonical quantization à la Dirac is now straightforward in the eight dimensional spinor space, denoted by $S_8$, when introducing the local $\nu_{\{8\}}(x)$ field operator as

$$\nu_{\{8\}}(x) = \int dV \left[ \sum_{k=1,2,7,8} \Lambda_k(p) a_k(p) e^{-ip \cdot x} + \sum_{j=3,4,5,6} \Lambda_j(p) a_j^\dagger(p) e^{ip \cdot x} \right].$$

Here, $dV$ is the appropriate phase volume.
3 Neutron $\beta$ decay in $S_8$ and textbook Majorana quantum field.

In this section we exploit the local truly neutral quantum field $\nu_{(8)}(x)$ in the calculation of $\beta$ decay traces. In due course we shall motivate $\nu(x)$ in Eq. (4). Our first goal on that journey will be to take a close look on neutron $\beta$ decay in $S_8$. If one wishes to consider physical processes that involve both Dirac and Majorana fermions, one needs to worry about matching dimensions of both spinor spaces. The simplest way to harmonize dimensions is to amplify the Dirac spinors similarly to Eqs. (39). In order to respect orthogonality of $P$ eigenspinors, one has to keep helicities same at top and bottom. The Dirac eight-spinors introduced in this manner are

$$U_{(j;h')}(p) = \begin{pmatrix} u_{h'}(p) \\ \epsilon_j u_{h'}(p) \end{pmatrix}, \quad V_{(j;h')}(p) = \begin{pmatrix} v_{h'}(p) \\ \epsilon_j v_{h'}(p) \end{pmatrix}, \quad h' = \uparrow, \downarrow, \quad (50)$$

respectively. To simplify notations we will suppress from now onward the momentum, $p$, as argument of spinors and operators. In order to calculate cross sections, i.e. current-current tensors, $G_{\mu\nu}$, in $S_8$, one has next to make a choice for the eight-currents. In analogy to the Dirac vector current, we here construct

$$J^\mu_k_{(j;h')} = \bar{\Lambda}_k \Gamma^\mu U_{(j;h')}, \quad \Gamma^\mu = \gamma^\mu \otimes 1_2,$$

$$J^\mu_{8k}_{(j;h')} = \bar{\Lambda}_k \Gamma_8 \Gamma^\mu U_{(j;h')}, \quad k = 1, 2, 7, 8. \quad (51)$$

Four momentum and mass of the Dirac particle will be in turn denoted as $p_1$, and $m_1$. As long as we are not gauging the theory, but are writing down ad hoc currents, one may think of the $(8d)$ model for neutron beta decay presented here as a “toy” model. Yet, as it will be shown below, it will allow for some very instructive insights into neutrino phenomenology. Above currents are conserved in the $m \to m_1$ limit and have the property to take the $U_{(j;h')}$ spinor of positive norm to $\Lambda_k(p)$ of positive norm too. The current–current tensor for, say, $J^\mu_k_{(j;h')}$, reads

$$G_{\mu\nu} = \frac{1}{2} \sum_{k(j;h')} \frac{1}{4} \bar{\Lambda}_k \Gamma^\nu U_{(j;h')} \left( \bar{\Lambda}_k \Gamma^\mu U_{(j;h')} \right)^\dagger. \quad (52)$$

In making use of the definition of $\bar{\Lambda}_k$ given in Eq. (40), i. e.

$$\left( \bar{\Lambda}_k \right)^\dagger = \left( \Lambda_k^\dagger \Gamma_8 \Gamma_0 \right)^\dagger = \Gamma^0 \Gamma^\dagger_8 \Lambda_k,$$

and the relation $\Gamma^\nu \Gamma^0 \dagger = \Gamma^0 \Gamma^\nu$, Eq. (52) amounts to
\[ G^{\mu\nu} = \frac{1}{2} \sum_{k(j;h')} \frac{1}{4} \bar{\Lambda}_k \Gamma^\mu U_{(j;h')} \bar{U}_{(j;h')} \Gamma'^\dagger_{8} \Lambda_k. \] (54)

In the following we shall introduce the notation \( \Pi^D \) as

\[
\Pi^D = \frac{1}{2m_1} \left( U_{(1;\uparrow)} \bar{U}_{(1;\uparrow)} + U_{(2;\downarrow)} \bar{U}_{(2;\downarrow)} \right) = \frac{\not{p} + m_1}{2m_1} \left( \begin{array}{c} 1 \\ 1 \\ \end{array} \right). \] (55)

Converting Eq. (54) to trace is now standard and results in

\[
G^{\mu\nu} = \frac{1}{2} tr \left( \frac{1}{4} \Gamma'^\dagger_{8} \left( 4m \pi^+ \right) \Gamma^\mu \left( 2m_1 \Pi^D \right) \Gamma'^\nu \Gamma' \right) = \frac{1}{2} \left( \begin{array}{c} 0_4 \\ 1_4 \\ \end{array} \right) \left( \begin{array}{c} m_1 & \not{p} \\ \not{p} & m_1 \\ \end{array} \right) \left( \begin{array}{c} \gamma_\mu (\not{p} + m_1) \gamma_\nu \gamma_\mu (\not{p} + m_1) \gamma_\nu \\ \gamma_\mu (\not{p} + m_1) \gamma_\nu \gamma_\mu (\not{p} + m_1) \gamma_\nu \end{array} \right) = \frac{1}{2} tr(\not{p} + m) \gamma^\mu (\not{p} + m_1) \gamma^\nu. \] (56)

Therefore, the trace entering the single beta decay is same as if we had used \( \nu(x) \). In this way we reached a further goal of our investigation, namely, understand appearance of momentum space Dirac spinors in the Majorana quantum field.

### 4 Majorana spinors of pure imaginary C parity.

#### 4.1 Spinor construct.

In this Section we present rest-frame neutral spinors which differ from the textbook ones [5]–[10] by the relative phase between the left- and right handed Weyl components. In the textbook case, the phase was real, in the currently presented one, it will be purely imaginary. The imaginary relative phase shows up as imaginary \( C \) parity. Above difference will be of profound importance for the phenomenological consequences of the theory.\(^\text{3}\)

\(^\text{3}\) The relative phase \( \zeta_j \) between the two-dimensional \((1/2,0)\) and \((0,1/2)\) should not be confused with Kayser’s creation phase factor, \( \lambda \). While \( \zeta_j \) tells something about how to stick together \((1/2,0)\) and \((1/2)\) to a four dimensional spinor of the desired transformation properties under discrete \( C, P \) space-time symmetries, the \( \lambda \) selects a particular linear superposition between four dimensional neutral particle- and anti-particle states. In the ultra-relativistic limit, \( E/|p| \rightarrow 1 \), when the particle
McLennan [16] and Case [17], constructed Majorana spinors of negative imaginary $C$ parity, while $C$ spinors of positive imaginary parity have been introduced in Refs. [13] and shown to be necessary for securing completeness in $(1/2, 0) \oplus (0, 1/2)$. Majorana spinors of that type will be denoted in turn by $\Psi^h(\zeta_j)(p)$ with $\zeta_j$ purely imaginary. The $\Psi^h(\mp i)(p)$'s correspond to $\lambda^{S/A}_{(h, -h)}(p)$ in Refs. [13], where $S$ and $A$ stand for self–conjugate (positive imaginary $C$ parity), and anti-self–conjugate (negative imaginary $C$ parity), respectively.

The rest frame spinors are therefore chosen as

$$\Psi^h(\zeta_j)(0) = \left( \begin{array}{c} \zeta_j [i\sigma_2] K \Phi^h_L(0) \\ \Phi^h_L(0) \end{array} \right), \quad \zeta_1 = -i, \quad \zeta_2 = i. \quad (57)$$

It directly verifies that the above spinors are indeed $C$ eigenvectors, i.e.,

$$C\Psi^h(-i)(0) = \zeta_1 \Psi^h(-i)(0), \quad C\Psi^h(+i)(0) = \zeta_2 \Psi^h(+i)(0). \quad (58)$$

It has been shown in Ref. [13] that following cross–normalization relations (termed to as bi-orthogonality there) hold true

$$\overline{\Psi^h(\mp i)}(0)\Psi^h(\mp i)(0) = 0, \quad \overline{\Psi^h(\mp i)}(0)\Psi_{-h}(\mp i)(0) = \pm 2im(\delta_{hL} - \delta_{hL}). \quad (59)$$

The imaginary norms have been provoked by the imaginary relative phase $\zeta_j$. As a consequence, the cross norms change sign upon inverting order of the spinors as visible from Eq. (59). At the present stage this may look as a disadvantage but on long term it will be of favor in so far as it will allow for physics different but the one related to the Majorana spinors in Eq. (6) where the relative phase has been chosen to be real.

The completeness relation for these $C$ eigenspinors is now obtained as

$$\Pi^S(0) = -\frac{1}{2im} [\Psi^h(-i)(0)\overline{\Psi^h(-i)}(0) - \Psi^h_{-i}(p)\overline{\Psi^h_{-i}}(0)],$$

$$\Pi^A(0) = +\frac{1}{2im} [\Psi^h(+i)(0)\overline{\Psi^h(+i)}(0) - \Psi^h_{+i}(0)\overline{\Psi^h_{+i}}(0)],$$

$$\Pi^S(0) + \Pi^A(0) = 1. \quad (60)$$

where $\Pi^S(0)$ and $\Pi^A(0)$ denote in turn the rest frame projection operators and anti-particle states become predominantly left- and right-handed, respectively, the $\lambda$ phase factor can conditionally be viewed as the relative phase between $(1/2, 0)$ and $(0, 1/2)$. However, in this case, the four spinor brakes down any way and the relative phase $\zeta_j$ becomes irrelevant to leading order.
onto the self- and anti-self conjugate neutral spinors. The $\Pi^S(0)$ and $\Pi^A(0)$ are simultaneously $C$ parity projectors.

The first advantage of imaginary $C$ parities is to equalize cross-norms to $C$ parity projectors. Recall, in the case of a real $\Phi^h_R(p) - \Phi^h_L(p)$ relative phase considered in Section 2, projectors onto vectors with same cross-norm did not coincide with projectors onto $C$ eigenvectors.

However, as long as Eq. (27) was deduced without any reference to the particular form of the Majorana spinors, also for $\Psi^h_{M}(\pm i)(p)$ the problem of static projectors in $(1/2, 0) \oplus (0, 1/2)$ stays same.

In order to circumvent this shortcoming we apply once again the procedure outlined in subsection 2.8 to $\Psi^h_{M}(\pm i)(p)$ and find the following new system of coupled matrix equations

\[
\begin{pmatrix}
-m_{14} - ip^\mu \gamma_\mu \\
       ip^\mu \gamma_\mu - m_{14} \\
-m_{14} + ip^\mu \gamma_\mu \\
       -ip^\mu \gamma_\mu - m_{14}
\end{pmatrix}
\begin{pmatrix}
\Psi^\dagger_{M}(-i)(p) \\
\Psi^l_{M}(-i)(p)
\end{pmatrix}
= 0,
\]

\[
\begin{pmatrix}
-m_{14} - ip^\mu \gamma_\mu \\
       ip^\mu \gamma_\mu - m_{14} \\
-m_{14} + ip^\mu \gamma_\mu \\
       -ip^\mu \gamma_\mu - m_{14}
\end{pmatrix}
\begin{pmatrix}
\Psi^\dagger_{M}(+i)(p) \\
\Psi^l_{M}(+i)(p)
\end{pmatrix}
= 0.
\]

(61)

4.2 Covariantly propagating Majorana spinors in doubled $(1/2, 0) \oplus (0, 1/2)$.

Equations (61) suggest once again to amplify dimensionality of $C$ eigenspinors from four to eight in introducing

\[
\Lambda_{(S/A;1)}(p) = \begin{pmatrix}
\Psi^\dagger_{M}(\mp i)(p) \\
\epsilon_1 \Psi^l_{M}(\mp i)(p)
\end{pmatrix},
\quad
\Lambda_{(S/A;2)}(p) = \begin{pmatrix}
\Psi^\dagger_{M}(\mp i)(p) \\
\epsilon_1 \Psi^l_{M}(\mp i)(p)
\end{pmatrix},
\]

\[
\Lambda_{(S/A;3)}(p) = \begin{pmatrix}
\Psi^\dagger_{M}(\mp i)(p) \\
\epsilon_2 \Psi^l_{M}(\mp i)(p)
\end{pmatrix},
\quad
\Lambda_{(S/A;4)}(p) = \begin{pmatrix}
\Psi^\dagger_{M}(\mp i)(p) \\
\epsilon_2 \Psi^l_{M}(\mp i)(p)
\end{pmatrix},
\quad
\epsilon_1 = -\epsilon_2 = 1.
\]

(62)

---

4 Eq. (61) has been written down for the first time (up to notational differences) in Ref. [18] though without addressing the argument about the constancy of the projector onto four dimensional $C$ eigenspinors and without exploring anyone of its phenomenological consequences.
Here, the index $S$, or $A$ of the $\Lambda$ spinors is associated in turn with the negative, or positive sign of the imaginary $C$ parity. The above spinors define an orthogonal basis as

\[
\bar{\Lambda}_{(S;1)}(p)\Lambda_{(S;1)}(p) = \bar{\Lambda}_{(S;4)}(p)\Lambda_{(S;4)}(p) =
\]

\[
\bar{\Lambda}_{(A;2)}(p)\Lambda_{(A;2)}(p) = \bar{\Lambda}_{(A;3)}(p)\Lambda_{(A;3)}(p) = +4m,
\]

\[
\bar{\Lambda}_{(S;2)}(p)\Lambda_{(S;2)}(p) = \bar{\Lambda}_{(S;3)}(p)\Lambda_{(S;3)}(p) =
\]

\[
\bar{\Lambda}_{(A;1)}(p)\Lambda_{(A;1)}(p) = \bar{\Lambda}_{(A;4)}(p)\Lambda_{(A;4)}(p) = -4m,
\]

\[
\bar{\Lambda}_{(\tau;k)}(p) = [\Lambda_{(\tau;k)}(p)]^\dagger \bar{\Gamma}_8 \Gamma^0, \quad \bar{\Gamma}_8 = \begin{pmatrix} 0 & -i1_4 \\ i1_4 & 0 \end{pmatrix}.
\]

Here, $\tau = S, A$, the new matrix $\bar{\Gamma}_8$ plays once again the role of metric in the eight-dimensional space (to be denoted by $\bar{S}_8$) but this time in the space built on top of $\Psi^h_{M}(\pm i)M(p)$.

From Eq. (61) one directly reads off that the eight-dimensional spinors satisfy the Dirac like equation

\[
\begin{pmatrix} p^\mu \gamma_{\mu} & \pm im1_4 \\ \mp im1_4 & p^\mu \gamma_{\mu} \end{pmatrix} \begin{pmatrix} \Psi^h_{M}(\pm i)(p) \\ \epsilon_j \Psi^h_{M}(\mp i)(p) \end{pmatrix} = 0,
\]

where “$-m$” and “$+m$” in turn correspond to $\Lambda_{(S/A;k)}(p)$ of positive and negative norms. In nullifying the determinant of the matrix acting upon $\Lambda_{(S/A;k)}(p)$ in Eq. (66), one finds the standard energy-momentum dispersion relation, $p^2 - m^2 = 0$. Therefore, Eq. (66) describes massive neutral particles in terms of spinors that are eigenvectors of the particle–anti-particle conjugation operator.

Comparison between Eqs. (65) and (40) shows that the $(8d)$ metric takes different form in depending on the $C$ parity. In case the $C$ parity is real, the metric, $\Gamma_8$, is real and symmetric, while in case the above parity is pure imaginary, the metric, $\bar{\Gamma}_8$, is imaginary and anti-symmetric.

The difference between $\Gamma_8$, and $\bar{\Gamma}_8$ comes about because for $\Psi^h_{M}(\mp i)(p)$ the cross-norms depend on the order of the vectors as visible from Eq. (59), while for $\Psi^h_{M}(\pm i)(p)$ they did not, in accordance to Eq. (11).

Above difference is of pivotal importance for Eq. (56). Had we used $\Lambda_{(\tau;k)}(p)$ in place of $\Lambda_k(p)$, i.e. Eq. (61) in place of Eq. (48), and substituted into Eq. (56) $\Gamma_8$ from Eq. (65), we would have observed a cancellation of mass in the neutral fermion sector of the trace. In effect, the neutral particle sector of the (single)
beta decay trace would be massless without the neutrino being massless in reality.

A different situation is obtained in considering the current (it is conserved in the $m \rightarrow m_1$ limit)

$$J_{(\tau;k),(j;j')}^{\mu,\pm} = \bar{\Lambda}_{(\tau;k)} \frac{1}{\sqrt{2}} \left( 1_s \pm \bar{\Gamma}_8 \right) \Gamma^\mu U_{(j;j')}.$$  \hspace{1cm} (67)

Here, the interference term

$$\pm \frac{1}{2} \left( \bar{\Lambda}_{(\tau;k)} \Gamma^\mu U_{(j;j')} \left( \bar{\Lambda}_{(\tau;k)} \bar{\Gamma}_8 \Gamma^\nu U_{(j;j')} \right)^\dagger + \bar{\Lambda}_{(\tau;k)} \bar{\Gamma}_8 \Gamma^\mu U_{(j;j')} \left( \bar{\Lambda}_{(\tau;k)} \Gamma^\nu U_{(j;j')} \right)^\dagger \right),$$  \hspace{1cm} (68)

contributes $\pm m_\gamma^{\mu} (p_1 + m_1) \gamma^\nu$, to the trace in Eq. (56). This happens because

$$\left( \bar{\Lambda}_{(\tau;k)} \Gamma^\nu U \right)^\dagger = \bar{U}^\dagger \Gamma^\nu \Lambda_{(\tau;k)}$$

upon accounting for $\bar{\Gamma}_8^2 = 1_s$. Therefore, the antisymmetric off diagonal metric in $\Lambda_{(\tau,k)}(p)$ goes completely away from the matrix providing the trace, and phenomenologies with $\Psi^{k(\pm i)}(p)$ and $\Psi^{k(\pm i)}(p)$ amount be same again.

5 The neutrinoless double beta decay $0\nu\beta\beta$.

The neutrinoless double beta decay ($0\nu\beta\beta$) is a process where two neutrons in a nucleus, $A(Z,N)$, are converted into two protons by the emission of two virtual $W^-$ bosons

$$A(Z,N) \rightarrow A(Z + 2, N - 2) + W^- + W^-,$$  \hspace{1cm} (69)

in such a way that the two subsequently emerging $W^- e^-$ boson-fermion currents, appear connected by a virtual neutrino line (see Ref. [2] for details). This process is associated with a second order element of the $S$ matrix and the related amplitude, here denoted by, $T_{0\nu\beta\beta}$, is given by

$$T_{0\nu\beta\beta} = W^\mu W^\eta [\bar{u}_e \gamma_\mu (1 + \gamma_5) u_\nu_e] [\bar{u}_e \gamma_\eta (1 + \gamma_5) u_\nu_e].$$  \hspace{1cm} (70)

In order to bring in the virtual neutrino line one makes use of the following identity

$$\bar{u}_e \gamma_\eta (1 + \gamma_5) u_\nu_e = \frac{1}{收(1 + \gamma_5)} \gamma_\eta (1 + \gamma_5) (u_\nu_e)^c e$$

$$= \bar{u}_e [-\gamma_\mu (1 - \gamma_5)] \nu_e.$$  \hspace{1cm} (71)
The latter expression is obtained in making use of the relations, \(\gamma_0\gamma^*_\mu = \gamma_\mu\gamma_0\), \(\gamma_2\gamma_\mu = -\gamma^*_\mu\gamma_2\), \(\gamma^*_\mu = -\gamma_\mu\), the anticommutation relations between the Dirac matrices, and \(t\) labeling the transposed. With that Eq. (70) takes the form

\[
T_{0\nu_\beta \beta} = W^\mu W^{\eta} \frac{1}{p^2_{\nu_\beta} - m^2_{\nu_\beta}} L_{\mu \eta}, \\
L_{\mu \eta} = \bar{u}_e \gamma_\mu (1 + \gamma_5) \Pi^{\nu_\epsilon} \left[-\gamma_\mu (1 - \gamma_5)\right] v_\epsilon, \quad \Pi^{\nu_\epsilon} = \sum u_{\nu_\epsilon} \bar{u}_{\nu_\epsilon}.
\] (72)

Here we suppressed helicity labeling of the Dirac spinors in order not to overload notations but \(\sum\) in \(\Pi_{\nu_\epsilon}\) expresses summation over this degree of freedom.

Finally, \(|L_{\mu \eta}|^2\) can be converted to a trace in the standard way as

\[
|L_{\mu \eta}|^2 = \left[\bar{u}_e \gamma_\mu (1 + \gamma_5) \Pi^{\nu_\epsilon} \gamma_\eta (1 - \gamma_5) v_\epsilon\right] \left[\bar{u}_e \gamma_\lambda (1 + \gamma_5) \Pi^{\nu_\epsilon} \gamma_\delta (1 - \gamma_5) v_\epsilon\right]^\dagger \\
= \text{tr} \left[\Pi^{\nu_\epsilon} \gamma_\mu (1 + \gamma_5) \Pi^{\nu_\epsilon} \gamma_\eta (1 - \gamma_5) \Pi^{\nu_\epsilon} (1 + \gamma_5) \gamma_\delta \gamma_0 \Pi^{\nu_\epsilon} \gamma_\eta (1 - \gamma_5) \gamma_\lambda\right].
\] (73)

Now we calculate above trace within the scenario of the previous section. To do so one has to perform in Eq. (73) the replacements \(\gamma_\mu \rightarrow \Gamma_\mu\), \(u_e \rightarrow U_e\), \(v_e \rightarrow V_e\), \(u_{\nu_\epsilon} \rightarrow \Lambda_{(S/A,k)}\), and

\[
\Pi^{\nu_\epsilon} \rightarrow \frac{1}{2m} \begin{pmatrix} m_{1_4} -i\hat{p} & 0_4 -i1_{4_4} \\ i\hat{p} & m_{1_4} \end{pmatrix}.
\] (74)

In this way one creates the \(8 \times 8\) version of \(|L_{\mu \eta}|^2\), where apparently, the metric matrix \(\tilde{\Gamma}_8\) enters twice. The net effect of the \(\tilde{\Gamma}_8^2\) presence in (73) is to bring back the mass to the neutral particle sector in the \(0\nu_\beta \beta\) trace. Recall, that for the type of currents in Eq. (51) and same \(\tilde{\Gamma}_8\), the neutral particle sector in the single \(\beta\) decay trace was massless. Above considerations allow to conclude that \(0\nu_\beta \beta\) phenomenology with Majorana spinors results equivalent to phenomenology with Dirac spinors.

### 6 Summary.

We demonstrated momentum independence of the projectors onto classical \(C\) eigenspinors in \((1/2, 0) \oplus (0, 1/2)\) and concluded impossibility of constructing local quantum field theory based upon four dimensional Majorana spinors. We avoided the problem of static propagators in \((1/2, 0) \oplus (0, 1/2)\) in exploiting the fact that in auxiliary eight spinorial dimensions Majorana spinors possess one more discrete symmetry beyond charge conjugation. This directed us to
the auxiliary calculus for classical Majorana spinors of eight spinorial degrees of freedom. In reference to the new symmetry we constructed related rest-frame projectors which upon boosting gave rise to covariant propagation and allowed for a field quantization \( \text{\`a la Dirac.} \)

With the aim to figure out similarities and differences between Majorana and Dirac theories for neutral fermions, we calculated the \((8d)\) trace entering the width of neutron \(\beta\) decay within such a scenario and, up to one exception, found it to be same as if we had used in four space massive Dirac spinors.

We also calculated the trace entering the neutrinoless double \(\beta\) decay and found it to be unaltered with respect to the Dirac formalism irrespective of the type of spinors and type of currents used. In other words, we showed that phenomenology with classical Majorana spinors is possible only in eight spinorial dimensions, but is by and large equivalent to phenomenology with Dirac’s \(u\) and \(v\) in four spinorial dimensions.

If this were to be the only impact of the calculus, eight spinor dimensions could be viewed only as dummy degrees of freedom. However, there is a rare exception. For \(\Psi_M^{\pm,\pm}(p)\) and the class of currents in Eq.\((51)\) the single beta decay trace was shown to contain massless Dirac spinors in the neutral fermion sector. This cancellation of the neutral particle mass was triggered by the anti-symmetric off diagonal metric in the \((8d)\) space. The latter option opens the curious possibility to have a neutral fermion theory at hand that allows polarized tritium \(\beta\) decay to drive the neutrino mass closer and closer to zero [19] without contradicting observation of possibly larger neutrino mass in oscillation–, and in 0\(\nu\beta\beta\) phenomena, thus providing an intriguing and experimentally testable signature for a potentially viable and non-trivial impact of Majorana spinors on phenomenology.

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