Abstract—Bilevel programs are optimization problems where some variables are solutions to optimization problems themselves, and they arise in a variety of control applications, including: control of vehicle traffic networks, inverse reinforcement learning and inverse optimization, and robust control for human-automation systems. This paper develops a duality-based approach to solving bilevel programs where the lower level problem is convex. Our approach is to use partial dualization to construct a new dual function that is differentiable, unlike the Lagrangian dual that is only directionally differentiable. We use our dual to define a duality-based reformulation of bilevel programs, prove equivalence of our reformulation with the original bilevel program, and then introduce regularization to ensure constraint qualification holds. These technical results about our new dual and regularized duality-based reformulation are used to provide theoretical justification for an algorithm we construct for solving bilevel programs with a convex lower level, and we conclude by demonstrating the efficacy of our algorithm by solving two practical instances of bilevel programs.

I. INTRODUCTION

Bilevel programs are optimization problems in which some variables are solutions to optimization problems themselves. Let \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \) be vectors, and consider the following (optimistic) bilevel programming problem:

\[
\begin{align*}
\min_{x,y} & \quad F(x,y) \\
\text{s.t.} & \quad G(x) \leq 0 \\
\alpha & \quad y \in \arg\min \{ f(x,y) \mid g(x,y) \leq 0 \}
\end{align*}
\]

BLP

where \( F, f \) are scalar-valued and \( G, g \) are vector-valued functions. (Equality constraints \( G(x) = 0 \) or \( g(x,y) = 0 \) are included by replacement with \( G(x) \leq 0 \), \( G(x) \geq 0 \) or \( g(x,y) \leq 0 \), \( g(x,y) \geq 0 \).) If we call \( x \) the upper-level decision variables and \( y \) the lower-level decision variables, then \( \min_{y} \{ f(x,y) \mid g(x,y) \leq 0 \} \) is the lower level problem.

Optimization problems with the generic form given in \( \text{BLP} \) are found in a variety of control applications, including control of vehicle traffic networks [1]–[5], inverse reinforcement learning and inverse optimization [6]–[9], and robust control for human-automation systems [10]–[12]. A solution approach for \( \text{BLP} \) is to replace the lower level problem by some optimality conditions and then solve the reformulated problem. But existing algorithms [13]–[17] suffer from numerical issues [18]–[21], and so the development of new algorithms to solve \( \text{BLP} \) is an important area for research.

This work was supported in part by NSF Award CMMI-1450963 and the Philippine-California Advanced Research Institutes (PCARI).

Aurélien Ouattara and Anil Aswani are with the Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA 94720, USA, aouattara@berkeley.edu, aaswani@berkeley.edu

A. Existing Solution Approaches

One method [13]–[15] for solving \( \text{BLP} \) replaces the lower level problem with its KKT conditions, giving a mathematical program with equilibrium constraints (MPEC). The advantage of this approach is the reformulated problem can be solved using standard nonlinear optimization software. However, it uses complementarity constraints, which implies a combinatorial nature to the reformulated optimization problem and leads to numerical difficulties [18]–[21].

Another method [16], [17] for solving \( \text{BLP} \) replaces the lower level problem by \( f(x,y) \leq \varphi(x) \) and \( g(x,y) \leq 0 \), where \( \varphi(x) = \min_{y} \{ f(x,y) \mid g(x,y) \leq 0 \} \) is the value function. This introduces a non-differentiable constraint \( f(x,y) - \varphi(x) \leq 0 \) (since the value function is not differentiable), and so numerical solution needs specialized algorithms that implicitly smooth the value function [22]. This precludes use of standard nonlinear optimization software.

B. Duality-Based Solution Approach

This paper develops a duality approach to solving bilevel programs with a convex lower level. The idea is to replace the lower level problem with \( f(x,y) \leq h(\lambda, x) \), \( \lambda \geq 0 \), and \( g(x,y) \leq 0 \), where \( h(\lambda, x) \) is a dual function. Under conditions with zero duality gap, these constraints force \( y \) to be a minimizer of the lower level problem. We proposed a duality approach in a paper on inverse optimization with noisy data [6], though the prior formulation is not differentiable because of the use of Lagrangian duals. This paper constructs an alternative dual that is differentiable. We also study constraint qualification, which was not previously considered in [6].

Our reformulation of \( \text{BLP} \) is such that each term is differentiable, constraint qualification holds after regularization, and the regularization is consistent in the sense as the amount of regularization is decreased than the solution of the regularized problem approaches the solution of \( \text{BLP} \). These features allow numerical solution of our reformulation (and \( \text{BLP} \)) using standard nonlinear optimization software. Most of this paper focuses on technical properties of the new dual function and of the reformulation of \( \text{BLP} \) using this dual, and these results are used to provide theoretical justification for the algorithm that we propose for solving \( \text{BLP} \).

C. Outline

Section [II] provides preliminaries, including notation and our technical assumptions about \( \text{BLP} \), Section [III] defines a new dual function whose maximizers are equivalent to those of the Lagrangian dual function. Our dual is differentiable, unlike the Lagrangian dual (which is only directionally differentiable). We use our dual to define a duality-based
reformulation (DBP) of BLP in Section VII and the equivalence of DBP and BLP is proved. Next, we consider constraint qualification and consistency of approximation of regularized versions of DBP. In Section VIII we propose an algorithm for solving BLP and demonstrate its effectiveness by solving two instances of practical bilevel programs.

II. Preliminaries

We define some notation and concepts from variational analysis [23], and then we state our assumptions about BLP.

A. Notation

Let \( \| \cdot \| \) be the \( \ell_2 \) norm. We use \( \subseteq, \supseteq \) for subsets and supersets, respectively. All functions are extended real-valued, and the set \( \mathbb{C}^2 \) contains all twice continuously differentiable functions. Let \( C \) be a set. Then int(\( C \)) is the interior of \( C \), and the indicator function \( \delta_C(x) \) is: \( \delta_C(x) = 0 \) if \( x \in C \), and \( \delta_C(x) = +\infty \) if \( x \notin C \). \( \overline{N}_C(x) \) is the regular normal cone of \( C \) at \( x \), and recall \( v \in \overline{N}_C(x) \) if \( \langle v, x' - x \rangle \leq o(\|x' - x\|) \) for all \( x' \in C \). The normal cone of \( C \) at \( x \) is \( N_C(x) \), and note \( v \in N_C(x) \) if there are sequences \( x^n \to x \) with \( x^n \in C \), and \( v^n \to v \) with \( v^n \in \overline{N}_C(x^n) \). The non-negative orthant \( \Lambda = \{ \lambda : \lambda \geq 0 \} \) is a closed, convex set; and its (regular) normal cone is \( N_\Lambda(\lambda) = \{ x \leq 0 : \lambda x = 0 \} \).

Now let \( f \) be a function. A vector \( v \) is a regular subgradient at \( \overline{\pi} \), written \( v \in \partial f(\overline{\pi}) \), if \( f(x) \geq f(\overline{\pi}) + \langle v, x - \overline{\pi} \rangle + o(\|x - \overline{\pi}\|) \). A vector \( v \) is a subgradient at \( \overline{\pi} \), written \( v \in \partial f(\overline{\pi}) \), if there are sequences \( x^n \to \overline{\pi} \) and \( v^n \to v \) with \( v^n \in \partial f(x^n) \). These are used to define constraint qualification [24]. For a constraint set \( C = \{ g(x) \leq 0 \} \), let \( \overline{\pi} \in C \) and let \( I = \{ i : g_i(\overline{\pi}) = 0 \} \) be the indices of active constraints. This \( C \) satisfies linear independence constraint qualification (LICQ) at \( \overline{\pi} \) if all choices of \( v_i \in \partial g_i(\overline{\pi}) \), for all \( i \in I \), are linearly independent. This \( C \) satisfies Mangasarian-Fromovitz constraint qualification (MFCQ) at \( \overline{\pi} \) if there is a \( d \) such that for all choices of \( v_i \in \partial g_i(\overline{\pi}) \), for all \( i \in I \), we have \( v_i^T d < 0 \).

B. Technical Results

Our first result generalizes the boundeness theorem to set-valued mappings. Because of the technical peculiarities of continuity for set-valued mappings, we require additional assumptions beyond continuity.

Lemma 1: Let \( X \) be a compact set, and consider a set-valued mapping \( S(x) \) that is convex-valued, continuous, and bounded for each \( x \in X \). Then \( S(X) \) is bounded.

Proof: Suppose \( S(X) \) is not bounded. Then there exist sequences \( x^n \in X \) and \( s^n \in S(x^n) \) such that \( \| s^n \| \to \infty \). Since \( X \) is compact, there is some convergent subsequence by the Bolzano-Weierstrass theorem; and so by extracting this subsequence we can assume \( x^n \to \overline{x} \) for some \( \overline{x} \in X \). Now consider the sequence \( s^n / \| s^n \| \); note the norm of each term is 1. Hence there is some convergent subsequence, and so by extracting this subsequence we can assume \( s^n / \| s^n \| \to w \) for some \( w \neq 0 \). Next choose any \( t \in S(\overline{x}) \), and note that by continuity of \( S \) there exists \( t^n \in S(x^n) \) such that \( t^n \to t \). For any \( \tau \geq 0 \), there is a \( \nu \) large enough such that \( \tau / \| s^n \| < 1 \). But \( S \) is convex-valued, meaning \( (1 - \tau / \| s^n \|) t^n + \tau / \| s^n \| : s^n \in S(x^n) \) for \( \nu \) large enough. Taking the limit, we have \( t + \tau w \in S(\overline{x}) \). This is a contradiction since: \( w \neq 0 \), \( \tau \geq 0 \) is arbitrary, and \( S \) is bounded at \( \overline{x} \). Thus, we have shown by contradiction that \( S(X) \) is bounded.

C. Assumptions

For the lower level problem of BLP, we define its value function \( \phi(x) = \min_y \{ f(x,y) \mid g(x,y) \leq 0 \} \), solution set \( s(x) = \arg \min_y \{ f(x,y) \mid g(x,y) \leq 0 \} \), and feasible set \( \{ y : g(x,y) \leq 0 \} \). The Lagrangian dual function (LDF) is \( \psi(\lambda, x) = \inf_y f(x,y) + \lambda^T g(x,y) \).

We also make some assumptions about BLP. Not all assumptions are used in every result, but we list all of them here for conciseness. Let \( X = \{ x : G(x) \leq 0 \} \). Our first set of assumptions relate to the lower level problem of BLP.

A1. The functions \( f(x,y), g(x,y) \) are convex in \( y \) (for fixed \( x \)) and satisfy \( f, g \in \mathbb{C}^2 \).

A2. There exists a compact, convex set \( Y \) such that \( \{ y : \exists x \in X \text{ s.t. } g(x,y) \leq 0 \} \subseteq \text{int}(Y) \).

R1. For each \( x \in X \), there exists \( y \) such that \( g(x,y) < 0 \).

The above ensure the lower level problem and its Lagrange dual problem are solvable, meaning the minimum (maximum, respectively) is attained and the set of optimal solutions is nonempty and compact. The pointwise R1 ensures BLP has a solution under the additional assumptions below.

Our next assumptions concern BLP, and they ensure smoothness in the objective function of BLP and regularity in the constraints \( G(x) \leq 0 \). These conditions, when combined with the previous conditions, ensure BLP has a solution.

A3. The functions \( F(x,y), G(x) \) are twice continuously differentiable; or equivalently that \( F, G \in \mathbb{C}^2 \).

R2. The set \( X \) is compact and nonempty, and \( G(x) \) satisfies MFCQ for each \( x \in X \).

III. Constrained Lagrangian Dual Function

The numerical issue with the Lagrangian dual function (LDF) is that it is generally nondifferentiable in \( \lambda \).

Example 1: The example of linear programming is classical: Let \( A \in \mathbb{R}^{p \times m}, b \in \mathbb{R}^p, c \in \mathbb{R}^m \), and define \( f(x,y) = c^T y \) and \( g(x,y) = Ax - b \). Then, the LDF is

\[
\psi(\lambda, x) = \begin{cases} -b^T \lambda, & \text{if } A^T \lambda = -c \text{ and } \lambda \geq 0 \\ -\infty, & \text{otherwise} \end{cases}
\] (1)

For any \( \lambda_0 \) such that \( A^T \lambda_0 = -c \) and \( \lambda_0 \geq 0 \), this LDF is directionally differentiable in directions \( d \) such that \( A^T d = 0 \) and \( \lambda_0 + td \geq 0 \) for \( t > 0 \) small enough. However, this LDF is not differentiable because it is discontinuous in directions \( d \) such that \( A^T d \neq 0 \) or \( \lambda_0 + td \not\geq 0 \) for any \( t > 0 \).

The nondifferentiability of the LDF limits its utility in reformulating bilevel programs because in general closed-form expressions for the domain of the LDF are not available. In this section, we construct an alternative dual function that is designed to be differentiable while retaining the saddle point and strong duality properties of the LDF.
A. Definition and Solution Properties

Our approach is to perform a partial dualization. Define the Constrained Lagrangian Dual Function (CDF) to be

\[ h(\lambda, x) = \min_y \{ f(x, y) + \lambda^T g(x, y) \mid y \in Y \}. \]  (2)

The difference as compared to the (classical) LDF is the domain of minimization of the Lagrangian \( L(x, y, \lambda) = f(x, y) + \lambda^T g(x, y) \). The LDF is the infimum of the Lagrangian over \( \mathbb{R}^m \), while the CDF is the minimum of the Lagrangian over a compact, convex set \( Y \) that contains \( \{ y : \exists x \in X \text{ s.t. } g(x, y) \leq 0 \} \) strictly within its interior.

An important feature of the CDF is it maintains the strong duality of the LDF, and its solutions are a saddle point to the Lagrangian \( L(x, y, \lambda) \). Our first result establishes an equivalence between solutions of the CDF and LDF.

**Theorem 1:** Suppose A1, A2 and R1 hold. Then arg \( \max_x \{ \psi(x, \lambda) \mid \lambda \geq 0 \} \) is non-empty and compact, \( \max_x \{ \psi(x, \lambda) \mid \lambda \geq 0 \} = \max_x \{ \psi(x, \lambda) \mid \lambda \geq 0 \} \), and arg \( \max_x \{ h(x, \lambda) \mid \lambda \geq 0 \} = \arg \max_x \{ \psi(x, \lambda) \mid \lambda \geq 0 \} \).

**Proof:** We associate (see Example 11.46 of \[23\]) the generalized Lagrangian \( l(x, y, \lambda) = f(x, y) + \lambda^T g(x, y) - \delta_x(\lambda) \) for min \( \{ f(x, y) \mid g(x, y) \leq 0 \} \). But \( \psi(x, \lambda) = \inf_y \{ f(x, y) \mid g(x, y) \leq 0 \} \), and so \( \partial_y \psi(x, y, \lambda) = \nabla_y g(x, y) - \lambda \) and \( \partial_y \psi(x, y, \lambda) = -g(x, y) + N_x(\lambda) \). Since \( s(x) \) is compact and nonempty by Example 11.11 of [23], let \( y^* \in s(x) \). Theorem 11.50 and Corollary 11.51 of \[23\] give: \( \lambda^* \in \arg \max_y \{ \psi(x, \lambda) \mid \lambda \geq 0 \} \). Define \( \psi(x, \lambda) = f(x, y^*), \) and \( \psi(x, \lambda) = f(x, \lambda) \) s.t. \( \psi(x, \lambda) \in \arg \max_{\lambda, x} \{ \psi(x, \lambda) \mid \lambda \geq 0 \} \).

Next associate a generalized Lagrangian to the optimization problem min \( \{ f(x, y) \mid g(x, y) \leq 0 \} \). From Example 11.46 of [23], its generalized Lagrangian is \( \ell(x, y, \lambda) = \delta_y(\lambda) + f(x, y) + \lambda^T g(x, y) - \delta_x(\lambda) \). Note \( h(x, \lambda) = \min_y \{ \ell(x, y, \lambda) \mid y \in Y \} \) for \( \lambda \geq 0 \), and \( \partial_y \ell(x, y, \lambda) = N_y(y) + \nabla_y f(x, y) + \lambda \nabla_y g(x, y) \), and \( \partial_y \ell(x, y, \lambda) = -g(x, y) + N_x(\lambda) \). Since \( 0 \in N_y(y) \) and \( 0 \in \partial_y g(x, y, \lambda^*) \), we have \( 0 \in \partial_y \ell(x, y, \lambda) \). Similarly, \( 0 \in \partial_y \ell(x, y, \lambda^*) \) yields \( 0 \in \partial_y \ell(x, y, \lambda^*) \). Thus, we can apply Theorem 11.50 and Corollary 11.51 of [23], which gives: \( \lambda^* \in \arg \max_{\lambda, x} \{ h(x, \lambda) \mid \lambda \geq 0 \} \), and \( \lambda^*, x = f(x, y^*) \). So \( h(\lambda^*, x) = \psi(\lambda^*, x) \), proving the first part of the result.

But recall that \( \lambda^* \in \arg \max_{\lambda, x} \{ h(x, \lambda) \mid \lambda \geq 0 \} \). This means \( \arg \max_{\lambda, x} \{ h(x, \lambda) \mid \lambda \geq 0 \} \equiv \arg \max_{\lambda, x} \{ \psi(x, \lambda) \mid \lambda \geq 0 \} \). Theorem 11.50 and Corollary 11.51 of [23] give: \( \mu^* \in \arg \max_{\lambda, x} \{ h(x, \lambda) \mid \lambda \geq 0 \} \). Note \( h(x, \lambda) = \min_y \{ \ell(x, y, \lambda) \mid y \in Y \} \) for \( \lambda \geq 0 \), and \( \partial_y \ell(x, y, \lambda) = N_y(y) + \nabla_y f(x, y) + \lambda \nabla_y g(x, y) \), and \( \partial_y \ell(x, y, \lambda) = -g(x, y) + N_x(\lambda) \). Since \( 0 \in N_y(y) \) and \( 0 \in \partial_y g(x, y, \lambda^*) \), we have \( 0 \in \partial_y \ell(x, y, \lambda) \). Similarly, \( 0 \in \partial_y \ell(x, y, \lambda^*) \) yields \( 0 \in \partial_y \ell(x, y, \lambda^*) \). Thus, we can apply Theorem 11.50 and Corollary 11.51 of [23], which gives: \( \lambda^* \in \arg \max_{\lambda, x} \{ h(x, \lambda) \mid \lambda \geq 0 \} \), and \( \lambda^*, x = f(x, y^*) \). So \( h(\lambda^*, x) = \psi(\lambda^*, x) \), proving the first part of the result.

B. Differentiability

The distinguishing property of the CDF is that it is differentiable, while the LDF is only directionally differentiable.
The differentiability occurs because the CDF is defined as a minimization over a compact set that is independent of $\lambda, x$. In particular, if we define $\sigma(\lambda, x) = \arg \min_y \{ f(x, y) + \lambda^T g(x, y) \mid y \in Y \}$, then we can state the differentiability of the CDF.

Theorem 2: Suppose A1, A2 and R1 hold. If $(\lambda, x)$ is such that $\sigma(\lambda, x)$ is singleton; then the CDF is differentiable at $(\lambda, x)$, and its gradient is given by

$$
\nabla_x h(\lambda, x) = \nabla_x f(x, y) + \lambda^T \nabla_x g(x, y) \\
\nabla_\lambda h(\lambda, x) = g(x, y)
$$

(4)

where we have that $\{y\} = \sigma(\lambda, x)$.

Proof: This follows from Theorem 4.13 and Remark 4.14 of [25].

Though determining if $\sigma(\lambda, x)$ is singleton can be difficult, a simple-to-check condition ensures this is always the case:

Corollary 2: Suppose A1, A2 and R1 hold. If $\lambda \geq 0$ and $f(x, y)$ is strictly convex in $y$ for every $x \in X$; then the CDF is differentiable at $(\lambda, x)$, and its gradient is given in (4), where we have that $\{y\} = \sigma(\lambda, x)$.

Proof: Since $\lambda \geq 0$, $f(x, y) + \lambda^T g(x, y)$ is strictly convex in $y$ for every $x \in X$ (see for instance Exercise 2.18 in [23]). Example 1.11 and Theorem 2.6 of [23] imply $\sigma(\lambda, x)$ is singleton. We can then apply Theorem 2.

For the case where $f(x, y)$ is not strictly convex, we can define a regularized CDF that is guaranteed to be differentiable. In particular, we define the regularized constrained Lagrangian dual function (RDF) to be

$$
h_\mu(\lambda, x) = \min_y \{ \mu \|y\|^2 + f(x, y) + \lambda^T g(x, y) \mid y \in Y \},
$$

(5)

where $\mu \geq 0$. We can interpret this as the CDF for an optimization problem where the objective has been changed to $\mu \|y\|^2 + f(x, y)$. The benefit of adding the $\mu \|y\|^2$ term is that it makes the objective of the optimization problem defining $h_\mu(\lambda, x)$ strictly convex, and therefore ensures the RDF is differentiable as long as $\mu > 0$. More formally, if $\sigma_\mu(\lambda, x) = \arg \min_y \{ \mu \|y\|^2 + f(x, y) + \lambda^T g(x, y) \mid y \in Y \}$, then:

Corollary 3: Suppose A1, A2 and R1 hold. If $\lambda \geq 0$ and $\mu > 0$; then the RDF is differentiable at $(\lambda, x)$, and its gradient is given by

$$
\nabla_x h_\mu(\lambda, x) = \nabla_x f(x, y) + \lambda^T \nabla_x g(x, y) \\
\nabla_\lambda h_\mu(\lambda, x) = g(x, y)
$$

(6)

where we have that $\{y\} = \sigma_\mu(\lambda, x)$.

Proof: Since $\|y\|^2$ is strictly convex and $f(x, y)$ is convex, $\mu \|y\|^2 + f(x, y)$ is strictly convex in $y$ for every $x \in X$ (Exercise 2.18 in [23]). So Corollary 2 applies.

More generally, both the CDF and RDF have a strong type of regularity because of their construction. This regularity will be useful for proving subsequent results.

Proposition 1: Suppose A1, A2 and R1 hold. Then for $\mu \geq 0$, we have $[-h_\mu(\lambda, x)$ is locally Lipschitz continuous; and its subgradient is nonempty, compact, and given by

$$
\partial_x [-h_\mu(\lambda, x) = -\co \{ \nabla_x f(x, \bar{y}) + \\
\lambda^T \nabla_x g(x, \bar{y}) \mid \bar{y} \in \sigma_\mu(\lambda, x) \}
$$

(7)

$$
\partial_\lambda [-h_\mu(\lambda, x) = -\co \{ g(x, \bar{y}) \mid \bar{y} \in \sigma_\mu(\lambda, x) \}
$$

where we have that $\sigma_\mu(\lambda, x) = \arg \min_y \{ \mu \|y\|^2 + f(x, y) + \lambda^T g(x, y) \mid y \in Y \}$.

Proof: Note $[-h_\mu(\lambda, x) = \max_y \{ -\mu \|y\|^2 - f(x, y) - \lambda^T g(x, y) \mid y \in Y \}$, by rewriting the definition of $h_\mu(\lambda, x)$. So $[-h_\mu$ is lower-C$^2$ by definition (see [23], [26]). This implies local Lipschitz continuity [23], [26]. Theorem 9.13 of [23] gives nonemptiness and compactness of the subgradient, and the formula (7) is due to Theorem 2.1 of [27].

C. Convergence Properties

An important aspect of the RDF is it epi-converges to the CDF as $\mu \to 0$. Note this convergence does not require $\sigma(\lambda, x)$ to be singleton, and hence applies even when $f(x, y)$ is not strictly convex in $y$ for every $x \in X$. Also, note the epi-convergence result applies to $-h(\lambda, \mu)$ and $-h_\mu(\lambda, \mu)$ since we are typically concerned with maximizing the dual.

Proposition 2: Suppose A1, A2 and R1 hold. Then the function $[-h_\mu(\lambda, x)$ is pointwise decreasing in $\mu$, and we have that $\lim_{\mu \to 0} [-h_\mu(\lambda, x) = [-h(\lambda, x)$.

Proof: The Berge maximum theorem [28] implies $h(\lambda, x)$ and $h_\mu(\lambda, x)$ are continuous (for each fixed $\mu > 0$). Second, note Proposition 7.4.c of [23] gives that for fixed $\lambda, x$ we have $\lim_{\mu \to 0} \mu \|y\|^2 + f(x, y) + \lambda^T g(x, y) = f(x, y) + \lambda^T g(x, y)$. And so by Theorem 7.33 of [23], we have for fixed $\lambda, x$ that $\lim_{\mu \to 0} h_\mu(\lambda, x) = h(\lambda, x)$.

Finally, using Proposition 7.4.d of [23] gives the desired result: $\lim_{\mu \to 0} [-h_\mu(\lambda, x) = [-h(\lambda, x)$.

IV. DUALITY-BASED REFORMULATION

It will be more convenient to work with the approximate bilevel programming problem, which is defined as

$$
\min_{x,y} F(x, y) \\
\text{s.t.} \ G(x) \leq 0
$$

(8)

where $y \in -\arg \min_y \{ f(x, y) \mid g(x, y) \leq 0 \}$ means $f(x, y) \leq \min_y \{ f(x, y) \mid g(x, y) \leq 0 \}$ and $g(x, y) \leq \epsilon$. (Equivalently, we have that $y$ is an $\epsilon$-solution in the sense of [29], [30].) This problem is equivalent to BLP when $\epsilon = 0$.

We first define our duality-based reformulation of BLP($\epsilon$), and then show its equivalence to the approximate bilevel program. Next we study constraint qualification of our reformation and provide conditions that ensure MFCQ holds. Since the duality-based reformulation has regularization, we conclude by providing sufficient conditions that ensure convergence of solutions to the regularized duality-based reformulation to solutions of the limiting problem.
A. Definition

Our duality-based reformulation of BLP($\epsilon$) using RDF is

$$\min_{x,y,\lambda} F(x, y)$$

$$\text{s.t. } (x, y, \lambda) \in C(\epsilon, \mu)$$

where the feasible set of DBP($\epsilon, \mu$) is given by

$$C(\epsilon, \mu) = \left\{ (x, y, \lambda) : G(x) \leq 0, g(x, y) \leq \epsilon, \lambda \geq 0 \right\}$$

(8)

One useful property of the reformulation DBP($\epsilon, \mu$) is that it is convex when $x$ is fixed, and a proof of a less general version of this result is found in Proposition 6 of [6].

The next result shows that upper-bounding the objective by the RDF, which is done in the feasible set of DBP($\epsilon, \mu$), is an optimality condition for the lower level program.

Proposition 3: Suppose A1, A2 and R1 hold. Then a point $y$ is an $\epsilon$-solution to the lower level problem if and only if there exists $\lambda$ such that $(x, y, \lambda) \in C(\epsilon, \mu)$. If $\mu \geq 0$ and a point $y$ is an $\epsilon$-solution to the lower level problem, then there exists $\lambda$ such that $(x, y, \lambda) \in C(\epsilon, \mu)$.

Proof: By Proposition 5 of [6] a point $y$ is an $\epsilon$-solution to the lower level problem if and only if there exists $\lambda$ such that the following inequalities are satisfied: $f(x, y) - \psi(\lambda, x) \leq \epsilon$, $g(x, y) \leq \epsilon, \lambda \geq 0$. The result holds if we can show there exists $\lambda' \geq 0$ such that $f(x, y) - \psi(\lambda', x) \leq \epsilon$ if and only if there exists $\lambda'' \geq 0$ such that $f(x, y) - h_0(\lambda'', x) \leq \epsilon$. Let $\lambda'' \in \arg \max_x \{h_0(\lambda, x) \mid \lambda \geq 0\}$, and note $f(x, y) - h_0(\lambda'', x) \leq f(x, y) - \psi(\lambda', x)$ by Theorem 1. Similarly, let $\lambda' \in \arg \max_x \{\psi(\lambda, x) \mid \lambda \geq 0\}$, and note $f(x, y) - \psi(\lambda', x) \leq f(x, y) - h_0(\lambda'', x)$ by Theorem 1. Next recall there exists $\lambda$ such that $(x, y, \lambda) \in C(0, 0)$, and so $f(x, y) - h_0(\lambda', x) \leq f(x, y) - h_0(\lambda', x)$ since Proposition 2 shows $[-h_0(\lambda', x)]$ is decreasing.

Out next result is on the equivalence of solutions to BLP($\epsilon$) and DBP($\epsilon$). A similar result was shown in [31] for the KKT reformulation, but we cannot apply their results to our setting because feasible $\lambda$ for DBP($\epsilon$) are not necessarily Lagrange multipliers when $\epsilon > 0$.

Proposition 4: Suppose A1, A2 and R1 hold. A point $(\pi, y)$ is a minimizer of BLP($\epsilon$) if and only if for some feasible $\lambda \geq 0$ the point $(\pi, y, \lambda)$ is a minimizer of DBP($\epsilon, 0$).

Proof: We prove this by showing $(x', y')$ is not a global minimum of BLP($\epsilon$) if and only if $(x', y', \lambda')$ is not a global minimum of DBP($\epsilon, 0$) for some feasible $\lambda' \geq 0$. Suppose $(x', y')$ is not a global minimum of BLP($\epsilon$). Then there exists $(x, y)$ feasible for BLP($\epsilon$), and with $F(x, y) < F(x', y')$. By Proposition 4 there exists $\lambda \geq 0$ such that $(x, y, \lambda)$ is feasible for DBP($\epsilon, 0$), which implies $(x', y', \lambda')$ is not a global minimum of DBP($\epsilon, 0$). Similarly, suppose $(x', y', \lambda')$ is not a global minimum of DBP($\epsilon, 0$). Then there exists $(x, y, \lambda)$ feasible for DBP($\epsilon, 0$), and such that $F(x, y) < F(x', y')$. However, this $(x, y)$ is feasible for BLP($\epsilon$) by Proposition 3. Thus $(x', y')$ is not a global minimum of BLP($\epsilon$).

The issue of equivalence between local minimizers of BLP($\epsilon$) and DBP($\epsilon, 0$) is more complex. The KKT reformulation generally lacks such an equivalence [31], and [31] argues that assuming LICQ for the lower level problem provides equivalence of local minimizers since this ensures uniqueness (and hence continuity) of the Lagrange multipliers [32]. However, results for the KKT reformulation [31] cannot be applied to our setting because feasible $\lambda$ for DBP($\epsilon, 0$) are not necessarily Lagrange multipliers.

Proposition 5: Suppose A1, A2 and R1 hold. If $\epsilon > 0$, or $g(x, y)$ satisfies LICQ for each $x \in X$; then a point $(\pi, y)$ is a local minimum of BLP($\epsilon$) if and only if for some feasible $\lambda \geq 0$ the point $(\pi, y, \lambda)$ is a local minimum of DBP($\epsilon, 0$).

Proof: We show $(x', y')$ is not a local minimum of BLP($\epsilon$) if and only if $(x', y', \lambda')$ is not a local minimum of DBP($\epsilon, 0$). First suppose $(x', y', \lambda')$ is not a local minimum of DBP($\epsilon, 0$). Then there exists a feasible sequence $(x''', y'', \lambda'') \to (x', y', \lambda')$ with $F(x'', y'') < F(x', y')$, where $(x'', y'')$ is feasible for BLP($\epsilon$) by Proposition 5. This shows $(x', y')$ is not a local minimum of BLP($\epsilon$). To prove the other direction, suppose $(x', y')$ is not a local minimum of BLP($\epsilon$). Then there exists a sequence of feasible $(x', y''') \to (x, y)$ with $F(x', y''') < F(x', y')$.

We must now consider two cases. The first case is when $\epsilon > 0$. Define $\Phi_{\epsilon, \mu}(x) = \{(y, \lambda) : (x, y, \lambda) \in C(\epsilon, \mu)\}$. For each $x \in X$, choosing $\Phi$ to be a solution to the lower level problem (which exists by Example 1.11 of [23] and Theorem 2.165 of [25]) gives a corresponding $\lambda$ (by Proposition 3) that satisfies $f(x, \pi) - h_0(x, \lambda) \leq \epsilon$. But $h_0(x, \lambda)$ is decreasing in $\pi$ (Proposition 3), and so we have $f(x, \pi) - h_0(x, \lambda) \to \epsilon$. Hence there exists a sequence $\lambda'' \to \lambda'$ such that $f(x, \pi) - h_0(x, \lambda'') \to \epsilon$. Combining this with A1, Proposition 1 shows the convexity of DBP($\epsilon, \mu$) when $\epsilon$ is fixed, and Example 5.10 of [23] shows $\Phi_{\epsilon, \mu}$ is continuous when $\epsilon > 0$. Hence there exists a sequence $\lambda'' \to \lambda'$ with $(x'', y'', \lambda'')$ feasible for DBP($\epsilon, 0$). This implies that the point $(x', y', \lambda')$ is not a local minimum of DBP($\epsilon, 0$).

The second case is when $\epsilon = 0$ and LICQ holds. Theorem 4 and Corollary 1 imply $\Phi_{0, 0}(x)$ consists of saddle points to the Lagrangian $\mathcal{L}$, and hence satisfy the KKT conditions (see Corollary 11.51 of [23]) because of the constraint qualification in R1. So there is a unique $\lambda'(x)$ that makes $(x', y', \lambda'(x))$ feasible for DBP($\epsilon, 0$) [32]. By Corollary 1 we have $\lambda'(x) = \arg \max_x h_0(\lambda, x)$, and so $\lambda'(x)$ is a continuous function since it is single-valued [32] and osc by the Berge maximum theorem [28]. Hence there exists $\lambda'' \to \lambda'(x)$ with $(x'', y'', \lambda'')$ feasible for DBP($\epsilon, 0$). This implies $(x', y', \lambda')$ is not a local minimum of DBP($\epsilon, 0$).

B. Constraint Qualification

One difficulty with solving bilevel programs is reformulations do not satisfy constraint qualification [17], [20], [33]. The issue is not that the feasible region of a bilevel program usually has no interior, but rather that an inequality representing optimality must fundamentally violate constraint qualification since we can interpret constraint qualification as stating the constraints have no local optima [34]. However, one benefit of our regularization is it leads to constraint qualification of the regularized problem DBP($\epsilon, \mu$).
Theorem 3: Suppose A1–A3 and R1, R2 hold. If $\epsilon > 0$, then MFCQ holds for DBP($\epsilon, \mu$).

Proof: Consider any $(x, y, \lambda)$ feasible for DBP($\epsilon, \mu$).
Note some subset of the constraints $g(x, y) \leq \epsilon, G(x) \leq 0$, and $\lambda \geq 0$ may be active, and label the indices of the active constraints by $I, J, K$. Slater’s condition holds for $g(x, y) \leq \epsilon$ by R1, MFCQ holds for $G(x) \leq 0$ by R2, and Slater’s condition holds for $-\lambda \leq 0$ since it clearly has an interior. Since Slater’s condition is equivalent to MFCQ for convex sets [23], there exist $d_x, d_y, d_\lambda$ such that $\nabla_x G_i(x)^T d_x < 0$, $\nabla_y g_j(x, y)^T d_y < 0$, and $\nabla_\lambda [-\lambda]^T d_\lambda < 0$ for active constraints $i \in I, j \in J, k \in K$.

Next, we consider two sub-cases. The first sub-case has $f(x, y) - h_\mu(x, \lambda) \leq \epsilon$, which means this constraint cannot be active. Note we can choose $\gamma > 0$ small enough to ensure $\gamma \nabla_x G_i(x)^T d_x < 0$ and $\gamma \nabla_y g_j(x, y)^T d_y < 0$ for $i \in I$ and $j \in J$, since by the Cauchy-Schwarz inequality we have $\gamma \nabla_x G_i(x)^T d_x \leq \gamma \| \nabla_x G_i(x) \| \| d_x \|$. Thus, MFCQ holds in this sub-case. In the second sub-case, $f(x, y) - h_\mu(x, \lambda) \geq \epsilon$. Let $y^* \in \text{arg min} \{f(x, y) \mid g(x, y) \leq 0\}$ and $\lambda^* \in \text{arg max} \{h(h(x, \lambda) \mid \lambda \geq 0\}$, and note Corollary [1] gives $f(x, y^*) - h_\mu(\lambda^*, x) \leq 0$. This implies $f(x, y^*) - h_\mu(\lambda^*, x) \leq \epsilon$. But recall Proposition 2 gives that $-h_\mu$ is decreasing in $\mu$, and so we have $f(x, y^*) - h_\mu(\lambda^*, x) \leq f(x, y^*) - h_\mu(\lambda, x) < \epsilon = f(x, y) - h_\mu(x, \lambda)$. Consider any $v_2 \in \partial_x [-h_\mu(x, \lambda)]$ and $v_1 \in \partial_\lambda [-h_\mu(x, \lambda)]$, where existence and boundedness of the subgradient comes from Proposition 1. Observe that $f(x, y) - h_\mu(x, \lambda) \in y$ is convex in $y, \lambda$, and by its convexity we have $\nabla_y f(x, y)^T (y^* - y) + v_1^T (\lambda^* - \lambda) \leq f(x, y^*) - h_\mu(\lambda^*, x) - f(x, y) + h_\mu(x, \lambda) < 0$. Since the subgradient of $-h_\mu$ is bounded by the Cauchy-Schwarz inequality we can choose $\gamma > 0$ small enough to ensure $\gamma \nabla_x G_i(x)^T d_x < 0$, $\gamma \nabla_y g_j(x, y)^T d_y < 0$, and $\gamma \nabla_\lambda [-\lambda]^T d_\lambda < 0$ for $i \in I$ and $j \in J$. We next compute $\nabla_\lambda [-\lambda]^T (\lambda^* - \lambda) \leq 0$ for $k \in K$. If $-\lambda < 0$, we have $\lambda = 0$ and $\lambda^* - \lambda_k \geq 0$ because $\lambda_k^* \geq 0$. Thus, $\nabla_\lambda [-\lambda]^T (\lambda^* - \lambda) \leq 0$ for $k \in K$. Next, we can choose $\gamma^*$ such that $\nabla_\lambda [-\lambda]^T (\lambda^* - \lambda) \leq \gamma \nabla_\lambda [-\lambda]^T d_\lambda < 0$ for $k \in K$ and $\gamma \nabla_\lambda [-\lambda]^T (\lambda^* - \lambda) \leq \gamma \nabla_\lambda [-\lambda]^T d_\lambda < 0$. So MFCQ holds in this sub-case.

C. Consistency of Approximation

We show the regularized problems DBP($\epsilon, \mu$) are consistent approximations [34], [35] of the limiting problem DBP($\tau, 0$) under appropriate conditions. Our first result concerns convergence of the constraint sets $C(\epsilon, \mu)$, which leads as a corollary to convergence of optimizers of the regularized problems to optimizers of the limiting problem.

Proposition 6: Suppose A1–A3 and R1 hold. Then for any $\tau > 0$ we have that $\lim_{\epsilon \downarrow 0, \mu \uparrow 0} C(\epsilon, \mu) = C(\tau, 0)$, and $C(\epsilon_1, \mu_1) \supseteq C(\epsilon_2, \mu_2)$ whenever $\epsilon_1 \geq \epsilon_2$ and $\mu_1 \geq \mu_2$. 

Proof: For any $(x, y, \lambda) \in C(\epsilon, \mu)$, we have: $G(x) \leq 0, f(x, y) - h_\mu(x, \lambda) \leq \epsilon_2, g(x, y) \leq \epsilon_2$, and $\lambda \geq 0$. Proposition 2 shows $-h_\mu(\lambda)$ is strictly decreasing in $\mu$, and so $f(x, y) - h_\mu(x, \lambda) \leq f(x, y) - h_\mu(x, \lambda) \leq \epsilon_2$. Similarly, $g(x, y) \leq \epsilon_1 \leq \epsilon_2$. This shows $(x, y, \lambda) \in C(\epsilon_1, \mu_1)$, which proves $C(\epsilon_1, \mu_1) \supseteq C(\epsilon_2, \mu_2)$ whenever $\epsilon_1 \geq \epsilon_2$ and $\mu_1 \geq \mu_2$. But $C(\epsilon, \mu)$ is closed since $f, g, G$ are differentiable by A1,A3; and $h_\mu$ is continuous by Proposition 1. So the result follows by Exercise 4.3.b of [23].

Corollary 4: Suppose A1–A3 and R1 hold. If we have that $\epsilon \downarrow \tau, \mu \downarrow 0, \tau \downarrow 0$, then

$$\lim_{\epsilon \downarrow \tau, \mu \downarrow 0, \tau \downarrow 0} \arg \text{min} \{ EZ \arg \text{min} \{ DBP(\epsilon, \mu) \} \} \subseteq \text{arg } DBP(\tau, 0),$$

and $z \cdot \text{min } DBP(\epsilon, \mu) \rightarrow \text{min } DBP(\tau, 0)$.

Proof: Note DBP($\epsilon, \mu$) is $f'_{\epsilon, \mu}(x, y, \lambda) = f(x, y) + \delta C(\epsilon, \mu)(x, y, \lambda)$. Recall $C(\epsilon, \mu)$ is closed since $f, g, G$ are differentiable by A1,A3; and $h_\mu$ is continuous by Proposition 1. By Proposition 6 we have $C(\epsilon_1, \mu_1) \supseteq C(\epsilon_2, \mu_2)$ when $\epsilon_1 \geq \epsilon_2$ and $\mu_1 \geq \mu_2$, and so $f'_{\epsilon_1, \mu_1} \leq f'_{\epsilon_2, \mu_2}$ for $\epsilon_1 \geq \epsilon_2$ and $\mu_1 \geq \mu_2$. This means by Proposition 7.4.d of [23] that DBP($\epsilon, \mu$) epiconverges to DBP($\tau, 0$) as $\epsilon \downarrow \tau, \mu \downarrow 0$. Moreover, $C(\epsilon, \tau)$ is nonempty by R1 and Proposition 3 which implies $C(\epsilon, \mu)$ is nonempty. So $f'_{\epsilon, \mu}(x, y, \lambda)$ is lower semicontinuous and feasible. But $f'_{\epsilon, \mu}(x, y, \lambda) \leq f'_{\epsilon+\delta, \mu}(x, y, \lambda)$ for $\epsilon \leq \tau \downarrow 0$ and $\mu \downarrow 1$ by Proposition 6. Moreover, $f'_{\epsilon, \mu}(x, y, \lambda) = F(x, y) + \delta_1(x, y, \lambda) \in C(\epsilon, \mu)$ is bounded by R2, and $\delta_1(x, y, \lambda) = \{ y : g(x, y) \leq \epsilon \}$ is continuous by A1,R1 and Example 5.10 of [23]; bounded for each $x \in X$ by A1,A2,R1 and Corollary 8.7.1 of [36]; and convex for each $x \in X$ by A1. Hence applying Lemma 7 implies $\{ x : (x, y, \lambda) \in C(\epsilon, \mu) \}$ is bounded. So $\{ (x, y) : (x, y, \lambda) \in C(\epsilon, \mu) \}$ is bounded, which by Example 1.11 of [23] implies $f'_{\epsilon, \mu}(x, y, \lambda)$ is level bounded (see Definition 1.8 in [23]) for all $\epsilon \leq \tau + 1$ and $\mu \leq 1$. The result follows by Theorem 7.33 of [23].

V. NUMERICAL ALGORITHM AND EXAMPLES

Previous sections provide theoretical justification for our Algorithm 1 which uses DBP to solve BLP. We conclude with two examples that demonstrate its effectiveness in solving practical problems. The SNOPT solver [37] was used for numerical optimization. The first is a problem of inverse optimization with noisy data [6]–[8], and the second involves computing a Stackelberg strategy for routing games [1]–[5].
A. Inverse Optimization with Noisy Data

Suppose an agent decides \( y_i \) in response to a signal \( u_i \) by maximizing a utility function \( U(y, u, \theta_0) \), where \( \theta_0 \) is a vector of parameters. Statistically consistent estimation of \( \theta_0 \) given \((u_i, z_i)\) for \( i = 1, \ldots, n \) data points, where \( z_i \) are noisy measurements of \( y_i \), requires solving BLP [6]. Heuristics using convex optimization (like [7], [8]) are inconsistent [6].

If \( U(y, u, x) = -(x + u)y \) with \( x, y, u \in \mathbb{R} \), then the bilevel program for statistical estimation is

\[
\begin{align*}
\min_{x, y_i} & \frac{1}{n} \sum_{i=1}^{n} \| z_i - y_i \|^2 \\
\text{s.t.} & x \in [-1, 1] \\
& y_i \in \arg \min_{y \in [-1, 1]} \{ (x + u_i)y \} \quad \forall i = 1, \ldots, n
\end{align*}
\]

(10)

The reformulation DBP(\( \epsilon, \mu \)) for this instance is given by

\[
\begin{align*}
\min_{x, y_i} & \frac{1}{n} \sum_{i=1}^{n} \| z_i - y_i \|^2 \\
\text{s.t.} & x \in [-1, 1] \\
& (x + u_i)y_i - h_\mu(\lambda_i, x) - \epsilon \leq 0 \quad \forall i = 1, \ldots, n \\
& y_i \in [-1 - \epsilon, 1 + \epsilon] \quad \forall i = 1, \ldots, n
\end{align*}
\]

(11)

where \( \lambda_i \in \mathbb{R}^2 \), and the RDF is \( h_\mu(\lambda, x) = \min_y \{ \mu \cdot y^2 + (x + u_i) \cdot y + \lambda_{i1} \cdot (-y - 1) + \lambda_{i2} \cdot (y - 1) \mid y \in [-2, 2] \} \).

Two hundred instances of (10) with \( n = 100 \) were solved, where (a) \( u_i \) and \( \theta_0 \) were drawn from a uniform distribution over \([-1, 1]\), and (b) \( z_i = \xi_i + w_i \) with \( \xi_i \in \arg \min_y \{ (\theta_0 + u_i)y \} \mid y \in [-1, 1] \} \) and \( w_i \) drawn from a standard normal. Each instance was solved by Algorithm 1, where: \( \epsilon_0 = 1, \gamma_0 = 10^{-4}, \gamma = 0.1, \zeta = 1, \mu = 10^{-4}, \zeta = 0.1, \zeta = 1 \), and \( K = 3 \). We use \( \theta \) to refer to the value returned by Algorithm 1 to emphasize that the returned value is an estimate of \( \theta_0 \). Fig. 1 has scatter plots of the 200 solved instances; it shows (left) the initial (randomly chosen) \( x_0 \) are uncorrelated to the true \( \theta_0 \), and (right) the estimates \( \hat{\theta} \) computed using our algorithm are close to the true \( \theta_0 \).

B. Stackelberg Routing Games

A common class of routing games consists of a directed graph with multiple edges between vertices, convex delay functions for each edge, and a listing of inflows and outflows of traffic [1]–[5]. The Stackelberg strategy is a situation where a leader controls an \( \alpha \) fraction of the flow, the remaining flow is routed according to a Nash equilibrium given the flow of the leader, and the leader routes their flow to minimize the average delay in the network. This problem is a bilevel program with a convex lower level.

An example of a two edge network in this Stackelberg setting is shown below:

\[
\begin{align*}
\phi & x_1 + y_1 = 1 \\
\phi & d_2(x_2 + y_2) = (1 - \phi)/(1 - x_2 - y_2)
\end{align*}
\]

The Stackelberg strategy for this two edge network is the solution to

\[
\begin{align*}
\min_{x, y} & x_1 + y_1 + (1 - \phi) \cdot (x_2 + y_2)/(1 - x_2 - y_2) \\
\text{s.t.} & x_1 + x_2 = \alpha \cdot \phi, \quad x \geq 0 \\
& y \in \arg \min_{y} \{ x_1 + y_1 - (1 - \phi) \cdot \log(1 - x_2 - y_2) \} \quad \forall i = 1, \ldots, n
\end{align*}
\]

(12)

where (a) \( x_1, x_2 \) is the leader’s flow on the top/bottom edge, (b) \( y_1, y_2 \) is the follower’s flow on the top/bottom edge, (c) \( \phi < 1 \) is the amount of flow entering the network, and (d) \( \alpha \) is the fraction of the flow controlled by the leader. The duality-based reformulation DBP(\( \epsilon, \mu \)) for this instance is

\[
\begin{align*}
\min_{x, y} & x_1 + y_1 + (1 - \phi) \cdot (x_2 + y_2)/(1 - x_2 - y_2) \\
\text{s.t.} & x_1 + x_2 = \alpha \cdot \phi, \quad x, y, \lambda \geq 0 \\
& y_1 + y_2 + \log(1 - x_2 - y_2) + h_\mu(\lambda, \nu, x) - \epsilon \leq 0
\end{align*}
\]

(13)

where \( \lambda \in \mathbb{R}^2, \nu \in \mathbb{R}, \) and the RDF is \( h_\mu(\lambda, \nu, x) = \min_y \{ \mu \cdot ||y||^2 + x_1 + y_1 - (1 - \phi) \cdot \log(1 - x_2 - y_2) - \lambda_1 \cdot y_1 - \lambda_2 \cdot y_2 + \nu \cdot (y_1 + y_2 - (1 - \alpha) \cdot \phi) \mid y \in [-1, 2] \} \). Different instances (corresponding to different values of \( \alpha, \phi \)) were solved by Algorithm 1 where: \( \epsilon_0 = 1, \mu_0 = 10^{-4}, \gamma = 0.1, \zeta = 1, K = 3 \). The initial point provided to the algorithm was the SCALE strategy [1], [2], [5], which corresponds to computing \( x' \in \arg \min_{x \in [-1, 2]} \{ x_1 + (1 - \phi) \cdot (x_2/(1 - x_2)) \} \) and then choosing \( \alpha x' \) as the initial point.

Solution quality is evaluated by the price of anarchy (PoA) [38], which is the average delay of a solution divided by the average delay when \( \alpha = 1 \). The objective in (12) gives the average delay. A PoA close to 1 is ideal because it implies the delay of the strategy is close to the delay when the leader controls the entire flow, while a large PoA means the average delay of the strategy is much higher than when the leader controls the entire flow. The results in Fig. 2 show that our duality-based approach (initialized with SCALE) significantly improves the quality of the Stackelberg strategy.
VI. CONCLUSION

We used a new (differentiable) dual function to construct a duality-based reformulation of bilevel programs with a convex lower level, and this reformulation uses regularization to ensure constraint qualification and differentiability. We proved results about the properties of this reformulation as justification for a new algorithm to solve bilevel programs, and then we displayed the effectiveness of our algorithm by solving two practical instances of bilevel programming.

REFERENCES

[1] A. Aswani and C. Tomlin, “Game-theoretic routing of GPS-assisted vehicles for energy efficiency,” in ACC, 2011, pp. 3375–3380.
[2] V. Bonifaci, T. Harks, and G. Schäfer, “Stackelberg routing in arbitrary networks,” Math. Oper. Res., vol. 35, no. 2, pp. 330–346, 2010.
[3] W. Krichene, J. D. Reilly, S. Amin, and A. M. Bayen, “Stackelberg routing on parallel networks with horizontal queues,” IEEE Trans. Automat. Cont., vol. 59, no. 3, pp. 714–727, 2014.
[4] Y. Sharma and D. P. Williamson, “Stackelberg thresholds in network routing games or the value of altruism,” in ACM conference on Electronic commerce, 2007, pp. 93–102.
[5] C. Swamy, “The effectiveness of Stackelberg strategies and tolls for network congestion games,” ACM TALG, vol. 8, no. 4, p. 36, 2012.
[6] A. Aswani, Z.-J. M. Shen, and A. Siddiq, “Inverse optimization with noisy data,” arXiv:1507.03266, 2015.
[7] D. Bertsimas, V. Gupta, and I. C. Paschalidis, “Data-driven estimation in equilibrium using inverse optimization,” Math Prog, 2014.
[8] A. Keshavarz, Y. Wang, and S. Boyd, “Imputing a convex objective function,” in IEEE ISIC, 2011, pp. 613–619.
[9] A. Ng and S. Russell, “Algorithms for inverse reinforcement learning,” in ICML, 2000, pp. 663–670.
[10] R. Vasudevan, V. Shia, Y. Gao, R. Cervera-Navarro, R. Bajcsy, and F. Borrelli, “Safe semi-autonomous control with enhanced driver modeling,” in ACC, 2012, pp. 2896–2903.
[11] D. Sadigh, S. Sastry, S. Sheshia, and A. Dragan, “Planning for autonomous cars that leverage effects on human actions,” in RSS, 2016.
[12] Y. Mintz, A. Aswani, P. Kaminsky, E. Flowers, and Y. Fukuoka, “Behavioral analytics for myopic agents,” arXiv preprint arXiv:1702.05496, 2017.
[13] M. Antitsch, “On using the elastic mode in nonlinear programming approaches to mathematical programs with complementarity constraints,” SIAM J Optim., vol. 15, no. 4, pp. 1203–1236, 2005.
[14] M. Fukushima and J.-S. Pang, “Convergence of a smoothing continuation method for mathematical programs with complementarity constraints,” in Ill-posed Variational Problems and Regularization Techniques. Springer, 1999, pp. 99–110.
[15] A. V. d. Miguel, M. P. Friedlander, F. J. Nogales Martin, and S. Scholtes, “An interior-point method for MPECs based on strictly feasible relaxations,” Department of Decision Sciences, London Business School, Tech. Rep., 2004.
[16] J. V. Outrata, “On the numerical solution of a class of Stackelberg problems,” Zeitschrift für Operations Research, vol. 34, no. 4, pp. 255–277, 1990.
[17] J. Ye and D. Zhu, “Optimality conditions for bilevel programming problems,” Optimization, vol. 33, no. 1, pp. 9–27, 1995.
[18] C. Kanzow and A. Schwartz, “The price of inexactness: convergence properties of relaxation methods for mathematical programs with complementarity constraints revisited,” Mathematics of Operations Research, vol. 40, no. 2, pp. 253–275, 2014.
[19] G.-H. Lin and M. Fukushima, “A modified relaxation scheme for mathematical programs with complementarity constraints,” Annals of Operations Research, vol. 133, no. 1-4, pp. 63–84, 2005.
[20] S. Scholtes, “Convergence properties of a regularization scheme for mathematical programs with complementarity constraints,” SIAM Journal on Optimization, vol. 11, no. 4, pp. 918–936, 2001.
[21] S. Steffensen and M. Ulbrich, “A new relaxation scheme for mathematical programs with equilibrium constraints,” SIAM Journal on Optimization, vol. 20, no. 5, pp. 2504–2539, 2010.
[22] G.-H. Lin, M. Xu, and J. Ye, “On solving simple bilevel programs with a nonconvex lower level program,” Mathematical Programming, vol. 144, no. 1-2, pp. 277–305, 2014.
[23] R. T. Rockafellar and R. J.-B. Wets, Variational analysis, 3rd ed. Springer, 2009.
[24] J.-B. Hiriart-Urruty, “Refinements of necessary optimality conditions in nondifferentiable programming I,” Applied mathematics and optimization, vol. 5, no. 1, pp. 63–82, 1979.
[25] J. Bonnans and A. Shapiro, Perturbation Analysis of Optimization Problems. Springer, 2000.
[26] R. Rockafellar, “Favorable classes of Lipschitz continuous functions in subgradient optimization,” in Progress in Nondifferentiable Optimization. IIAAS, 1982, pp. 125–143.
[27] F. H. Clarke, “Generalized gradients and applications,” Transactions of the American Mathematical Society, vol. 205, pp. 247–262, 1975.
[28] C. Berge, Topological Spaces: including a treatment of multi-valued functions, vector spaces, and convexity. Dover, 1963.
[29] A. Nemirovski, “Interior point polynomial time methods in convex programming,” Georgia Institute of Technology, Tech. Rep., 2004.
[30] Y. Nesterov and A. Nemirovski, Interior-Point Polynomial Algorithms in Convex Programming. SIAM, 1994.
[31] S. Dempe and J. Dutta, “Is bilevel programming a special case of a mathematical program with complementarity constraints?” Mathematical programming, vol. 131, no. 1-2, pp. 37–48, 2012.
[32] G. Wachsmuth, “On LICQ and the uniqueness of Lagrange multipliers,” Operations Research Letters, vol. 41, no. 1, pp. 78–80, 2013.
[33] H. Scheel and S. Scholtes, “Mathematical programs with complementarity constraints: Stationarity, optimality, and sensitivity,” Math. Oper. Res., vol. 25, no. 1, pp. 1–22, 2000.
[34] E. Polak, Optimization: algorithms and consistent approximations. Springer Science & Business Media, 1997, vol. 124.
[35] J. O. Royset and R. J. Wets, “Optimality functions and lopsided constraints,” in Mathematical programs with complementarity constraints revisited., Mathematics of Operations Research, vol. 33, no. 1-4, pp. 6–27, 2008.
[36] R. T. Rockafellar, Convex Analysis. Princeton University Press, 1970.
[37] P. E. Gill, W. Murray, and M. A. Saunders, “SNOPT,” SIAM review, vol. 47, no. 1, pp. 99–131, 2005.
[38] T. Roughgarden, “The price of anarchy is independent of the network topology.” J. Comput. Syst. Sci., vol. 67, no. 2, pp. 341–364, 2003.