SQCD, Superconducting Gaps and Cyclic RG Flows

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Abstract

We consider the relation between the Ω deformed $N = 2$ SQCD with the single deformation parameter $\epsilon$ and integrable models of the BCS-like superconductivity. It is argued that the vortex string worldsheet theory is related to the Russian Doll(RD) model of the truncated BCS superconductivity. We argue that the Ω deformed gauge theory manifests the interesting cyclic RG behavior with the period of the RG cycle proportional to $\epsilon^{-1}$. The deformed gauge theory can develop several non-perturbative scales. We conjecture on the monopole bound state interpretation of the Efimov tower.
1 Introduction

A confinement of electric degrees of freedom implies a condensation of some magnetic degrees of freedom. They certainly emerge from D-branes however the precise pattern of the D-brane condensation responsible for confinement in the asymptotically free gauge theories is still unclear. One could imagine the condensation of the monopoles, strings, domain walls or complicated webs designed from these ingredients. Therefore it is useful to look at the situations where the gap generation could be understood from the first principles. The simplified models of superconductivity provide the proper background to analyze the different aspects of these issues.

The well-known Seiberg-Witten solution \cite{1} yields the pattern for the monopole condensation in the perturbed $N = 2$ SUSY YM theories with the different matter content. It was argued that the low-energy effective action and the spectrum of the stable BPS states are governed by the classical finite-dimensional integrable system \cite{2}. In particular the theory which we shall mainly focus on, namely $N_f = 2N_c$ SQCD, is governed by the twisted XXX spin chain \cite{3}. The vacua in the deformed theory correspond to the classical equilibrium states in the corresponding integrable system. The relation between the finite dimensional integrable systems and supersymmetric YM theories has been generalized to the quantum case \cite{4}. The parameter of the $\Omega$ deformation plays the role of the Planck constant and the Bethe anzatz equation in the finite dimensional integrable system corresponds to the extremization of the superpotential in the deformed gauge theory.

On the other hand the finite-dimensional integrable systems in some cases provide the exact solution to the truncated BCS - like models of superconductivity. The generic arguments explaining the relation between the integrable systems and the superconducting models go as follows. Consider the fermionic system with a kind of attractive four-fermion interaction. It can be derived upon the integration over the ”phonon”-like degrees of freedom. The system develops the fermionic BCS condensate of the Cooper pairs which is generically inhomogeneous. It is assumed that the single fermion or Cooper pair propagating on the top of the condensate should not ruin it. This is a kind of consistency condition for the combined system ”inhomogeneous condensate + excitation”. It is this consistency condition that provides some integrable system behind the scene. The space-time dependence of the condensate is governed by the integrable equation itself while the equation for the one-particle fermion wave function in the condensate background is interpreted as the Lax equation for the corresponding dynamical system. The condensate can be represented via the continuous gap function or some lattice of Abrikosov strings for the type II superconductivity. In the first case one has the continuous integrable system of KdV
type while in the second case the discrete systems of Toda type are involved.

The analogy with the Peierls superconductivity has been developed in [5] for the pure perturbed $N = 2$ SYM theory whose low-energy sector is described by the periodic Toda chain [2]. The Peierls model involves the fluctuating one-dimensional crystal and fermions propagating in such background [6]. The Toda system governs the interaction between phonons induced by fermions while the Lax equation describes the propagation of the fermion in the background of the emerging condensate. The Seiberg-Witten curve is identified with the dispersion law for the fermions and the nonperturbative $\Lambda_{QCD}$ scale in SYM corresponds to the superconducting gap in the Peierls model.

The Toda system emerges as the particular limit of the XXX spin chain when the Casimirs of the representations at each site of the chain tend to infinity. Hence we could expect some superconducting model behind the SQCD as well generalizing the arguments from [5]. It turns out that such model exists indeed and we shall describe it in the paper. The Richardson model [7] and the RD model [8] are proper generalizations for pure SQCD and $\epsilon$-deformed SQCD correspondingly. The Richardson model can be also considered as the XY model in the inhomogeneous external field.

On the other hand the Richardson model links up with the generalized Gaudin system [9] and the RD model with the twisted inhomogeneous XXX spin chain correspondingly [10]. It turns out that the relation between the quantum integrable systems and the vacua of the supersymmetric gauge theories [4] is quite convenient to clarify the underlying physical degrees of freedom. The Bethe anzatz equation describing the spectrum of the RD model coincides with the equation describing the ground state in the worldsheet theory of several nonabelian strings. These semilocal nonabelian strings with rich worldsheet theory [11, 12] are the classical solution to the equations of motion in the SQCD.

To some extend the integrability plays the role of consistency condition in more general setting. Recall that the most familiar example of such phenomenon is provided by the strings propagating in the external metric or the gauge field. It is well-known that the condition of the vanishing of Fradkin-Tseytlin beta-function of the string worldsheet theory corresponds to the classical equations of motion for background fields. Speaking differently this is the consistency condition for the propagation of the string in its own condensate.

In the problem under consideration the situation is more subtle since the worldsheet theory of the nonabelian string is not conformal. Hence the consistency condition is more involved and the quantum properties of the bulk fields have to be matched in this case. The interplay between the $D = 4$ bulk theory and the $D = 2$ worldsheet theory on the nonabelian string has been examined in [13, 14] where it
was argued that the $\beta$-functions and the spectrum of the BPS states match. In this paper we shall argue that the additional ingredient can be added to the generic matching condition that is the system of superconducting fermions. The corresponding Richardson of RD models are conjectured to describe the magnetic degrees of freedom and the consistency condition implies that their dispersion law fits with the spectral curve of the proper integrable system. These integrable spin chain models correspond to the dynamics of the nonabelian strings per se.

Due to the mapping of $\epsilon$-deformed SQCD into the RD model it is possible to gain interesting observations from the known results concerning the superconducting model. The most unusual feature of RD model is the cyclic RG behavior yielding the multiple condensates instead of the single one [8]. The period of the RG cycle is determined by the RG invariant and the spectrum of the model is reshuffled upon each cycle. There are several systems enjoying cyclic RG flows both in the quantum mechanics [15] and in the field theory with the several couplings [16]. In particular such behavior has been found in the sin-Gordon model analytically continued in the coupling constant [16]. In that example the cyclic RG behavior amounts to the set of the unusual resonances with the Regge-type stringy spectrum. In general the origin of the cyclic RG behavior is some resonance-like behavior in two-body system which amounts into the hierarchy of the Efimov-like states. The most surprising aspect of the cyclic RG is its sensitivity to the UV scale of the theory under consideration. Note that it was shown recently that the cyclic RG flow is consistent with the c- and a-theorems [17].

The RG step in RD model corresponds to the change of the XXX spin chain length. Being translated to the conformal gauge theory side it corresponds to the decoupling of two flavors sending the corresponding mass to infinity with the simultaneous change $N \rightarrow (N - 1)$. The theory remains conformal with the different rank of the gauge group. However in the deformed theory we have additional dimensionless parameter which is the ratio $\frac{m}{\epsilon}$ hence decoupling of the heavy flavor is described via two-coupling RG flow. Using the known results concerning RD model we shall argue that decoupling of a heavy flavor occurs in a cyclic manner and the period of a cycle is proportional to the inverse of the deformation parameter therefore in the undeformed case the period is infinite.

The remarkable AGT relation [19] makes manifest the connection between the SU(2) Nekrasov partition function of the deformed SYM theory and the conformal blocks in the Liouville theory [19]. The interplay between the four-dimensional and two-dimensional theories reflects the complimentary viewpoints of observers at the M5 brane worldvolume where the SYM lives on. For the higher rank group the Liouville theory gets substituted by the 2d Toda theory [20]. The nonabelian strings
with large tension in the gauge theory are identified as the surface operators which on the other hand are represented as the proper vertex operators in the Liouville-Toda theory [21]. In particular the D2 branes representing the surface operators in Liouville-Toda theory yield the degrees of freedom for the Toda-Calogero type integrable models.

Hence we could look at the possible interpretation of the cyclic RG flows for the conformal blocks in the Liouville/Toda theory and the possible interpretation of the superconducting models on the gravity side of the AGT correspondence. The key point is the interpretation of the fermions and their Cooper pairs from the first principles. We shall make some conjectures however the complete analysis is still to be done.

The paper is organized as follows. In Section 2 we briefly review the Peierls model while in Section 3 the Richardson model and the RD model are considered. In Section 4 we comment on the generic properties of the cyclic RG flows. In Section 5 the mapping to the vortex nonabelian states in the SQCD is considered. In Section 6 we conjecture on the interpretation of the fermions in the superconducting model as the monopoles in the Higgs branch. The main observations of the paper and the list of the open questions can be found in the last Section.

2 The Peierls model

In this section we briefly review the main facts concerning the Peierls model relevant for the further consideration. Originally it was formulated to describe the selfconsistent behavior of 1d fermions interacting with the fluctuating lattice and has been applied to the description of the 1d superconductivity [6]. The Coulomb interaction between the fermions is neglected, the fermions are assumed to be in the external field determined by the lattice degrees of freedom while the lattice dynamics itself gets modified by the fermions. In what follows we will mention the integrable structure both in the continuum [22] and discrete Peierls models [6]. Hamiltonian density for the simplest continuum model looks as follows

$$H_{con} = \Psi^+ \sigma_3 \partial_x \Psi + \Psi^+ (\sigma_- \Delta(x)^\ast - \sigma_+ \Delta(x)) \Psi + \Delta(x)^2$$

where $\Delta(x)$ represents the lattice potential. The saddle point solution for $\Delta(x)$ provides the four-fermion interaction Hamiltonian induced by the phonon exchange. To some extend the lattice potential $\Delta(x)$ can be interpreted as the inhomogeneous fermionic condensate.
For the discrete version one has

\begin{equation}
H_{\text{dis}} = \sum_n \left( \Psi_n^+ v_n \Psi_n + \Psi_n^+ c_n \Psi_{n+1} + \Psi_n^+ c_{n-1} \Psi_{n-1} \right) + \sum_i \kappa_i I_i \tag{2.2}
\end{equation}

\begin{equation}
c_n = \exp(x_{n+1} - x_n), \quad I_0 N = \sum_n \ln c_n, \quad I_2 N = \sum_n (c_n^2 + v_n^2) \tag{2.3}
\end{equation}

where \( x_n, v_n \) are the lattice coordinate and momentum while \( I_n \) are identified as the Toda chain Hamiltonians.

To identify the ground state of the model we minimize the Hamiltonian with respect to the fermionic and lattice variables. The variation over the fermionic variables yields the Lax equation for the Toda chain

\begin{equation}
c_n \Psi_{n+i} + c_{n-1} \Psi_{n-1} + v_n \Psi_n = E \Psi_n \tag{2.4}
\end{equation}

Variation over the lattice degrees of freedom gives rise to the system of the finite number of algebraic equations describing the Riemann surface. In the Peierls model we assume the periodicity of the fermionic wave function on the lattice

\begin{equation}
\Psi_{n+N}(E) = e^{iNp(E)} \Psi_n(E). \tag{2.5}
\end{equation}

The solution is described in terms of the hyperelliptic Riemann surface

\begin{equation}
y + y^{-1} = P_N(E) \tag{2.6}
\end{equation}

where \( P_N(E) \) is the polynomial. This Riemann surface plays the double role. First, its Jacobian is identified as the complex Liouville torus for the periodic Toda chain describing the dynamics of phonons. On the other hand it is nothing but the Fermi surface for the fermions. The moduli of this surface are fixed by the minimization equations in terms of the physical parameters like the fermionic density \( \rho = N/e \).

The key feature of the solution is the appearance of the fermionic mass gap which is a \( \Lambda_{QCD} \). Fermionic wave function is uniquely defined on this surface and the number of its zeros coincides with the genus of the curve. It was proved that only one gap vacuum configuration is stable, therefore the generic polynomial degenerates. The very similar one gap vacuum configuration emerges in the Gross-Neveu model [18].

The spectrum of the excitations above the ground state involves fermionic and bosonic quasiparticles. The phonon and the charge density wave represent the bosonic excitations [22, 23]. The fermionic excitations strongly depend on the
fermionic density \( \rho \). At large \( \rho \) the polyaron type state is localized at the lattice configuration

\[
\Delta(x) = \text{const} - \frac{2\chi}{\text{ch}^2(\sqrt{\chi}x)}
\] (2.7)

In the opposite limit at small density one has the delocalized fermionic state and the gap potential

\[
\Delta(x) = \text{const} + \chi \cos(2\sqrt{\chi}x + \phi)
\] (2.8)

where \( \chi \) is some constant. Exact solution provides the temperature dependence of the mass gap. It was shown [22] that the fermion mass gap gets renormalized and disappears at some critical value \( T_c \). Being translated into the form of the dispersion law it tells us that the Riemann surface degenerates to a sphere above the phase transition point.

Let us compare the data governing the vacuum structure of the pure \( N = 2 \) SUSY YM theory and the one from the Peierls model. Low energy effective action in SYM theory is fixed by the Riemann surface and holomorphic differential defined on it [1]. Prepotential \( \mathcal{F} \) can be derived from the relations

\[
a_{D_i} = \frac{d\mathcal{F}}{d\alpha_i} \quad a_i = \int_{A_i} \lambda \quad a_{D_i} = \int_{B_i} \lambda \quad \lambda = dS = Edp
\] (2.9)

where \( A_i \) and \( B_i \) are the cycles on the Seiberg-Witten spectral curve. The number of cites in the Peierls model corresponds to the rank of the gauge group while the total length can be identified with the coupling constant \( \tau_0 = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \) taken at the UV scale. The density of the fermions turns out to be related to the renormalized coupling constant in the gauge theory [5]. The equations selecting the ground state of the Peierls model correspond to the selection of the point at the Coulomb branch of the moduli space.

It what follows we shall generalize the relation between the low-energy sector of SYM theory and models of superconductivity for the theories with matter in the fundamental representation.

### 3 Truncated models of BCS superconductivity

#### 3.1 Richardson model versus Gaudin with irregular singularities

Let us recall the truncated BCS-like Richardson model of superconductivity [7] with some number of doubly degenerated fermionic levels with the energies \( \epsilon_{j\sigma} \ldots j = \)
1...N. It describes the system of a fixed number of the Cooper pairs. It is assumed that several energy levels are populated by Cooper pairs while levels with the single fermions are blocked. The Hamiltonian reads as

\[ H_{BCS} = \sum_{j,\sigma=\pm} \epsilon_{j\sigma} c_{j\sigma}^{+} c_{j\sigma} - G \sum_{jk} c_{j+}^{+} c_{k-} c_{k} - c_{jk+} \]  

(3.10)

where \( c_{j\sigma}^{+} \) are the fermion operators and \( G \) is the coupling constant providing the attraction leading to the formation of the Cooper pairs. It terms of the hard-core boson operators it reads as

\[ H_{BCS} = \sum_{j} \epsilon_{j} b_{j}^{+} b_{j} - G \sum_{jk} b_{j}^{+} b_{k} \]  

(3.11)

where

\[ [b_{j}^{+} b_{k}] = \delta_{jk}(2N_{j} - 1), \quad b_{j} = c_{j-} c_{j+}, \quad N_{j} = b_{j}^{+} b_{j} \]  

(3.12)

The eigenfunctions of the Hamiltonian can be written as

\[ |M> = \prod_{i} B_{i}(E_{i})|vac>, \quad B_{i} = \sum_{j} \frac{1}{\epsilon_{j} - E_{i}} b_{j}^{+} \]  

(3.13)

provided the Bethe anzatz equations are fulfilled

\[ G^{-1} = - \sum_{j} \frac{2}{\epsilon_{j} - E_{i}} + \sum_{j} \frac{1}{E_{j} - E_{i}} \]  

(3.14) \{BA\}

The energy of the corresponding states reads as

\[ E(M) = \sum_{i} E_{i} \]  

(3.15)

It was shown in [9] that the Richardson model is exactly solvable and closely related to the particular generalization of the Gaudin model known as the model with irregular singularities [26]. To describe this relation it is convenient to introduce the so-called pseudospin \( sl(2) \) algebra in terms of the creation-annihilation operators for the Cooper pairs

\[ t^{-} = b \quad t^{+} = b^{+} \quad t^{0} = N - 1/2 \]  

(3.16)
The Richardson Hamiltonian commutes with the set of operators \( R_j \)

\[
R_i = -t_i^0 - 2G \sum_{i \neq j}^{N} \frac{t_i t_j}{\epsilon_i - \epsilon_j}
\]  

(3.17)

which are identified as the Gaudin Hamiltonians.

\[
[H_{BCS}, R_j] = [R_i, R_j] = 0
\]

(3.18)

Moreover the Richardson Hamiltonian itself can be expressed in terms of the operators \( R_i \) as

\[
H_{BCS} = \sum_j \epsilon_i R_i + G(\sum_i R_i)^2 + \text{const}
\]

(3.19)

The number of the fermionic levels \( N \) coincides with the number of sites in the Gaudin model and the coupling constant in the Richardson Hamiltonian corresponds to the ”twisted boundary condition” in the Gaudin model. The Bethe anzatz equations for the Richardson model (3.14) exactly coincides with the ones for the generalized Gaudin model. Note that the Gaudin model is the Hitchin system on the marked sphere which plays the role of the Fermi surface for the fermions very much in the same manner as we have seen in the Peierls model. It was argued in [8] that the Bethe roots corresponds to the excited Cooper pairs that is natural to think about the solution to the Baxter equation as the wave function of the condensate. In terms of the conformal field theory Cooper pairs correspond to the screening operators [25].

For the nontrivial degeneracies of the energy levels \( d_j \) the BA equations read as

\[
G^{-1} = -\sum_j^{N} \frac{d_j}{\epsilon_j - E_i} + \sum_j^{M} \frac{2}{E_j - E_i}
\]

(3.20)

### 3.2 Russian Doll model of superconductivity and twisted XXX spin chains

The important generalization of the Richardson model describing the reduced BCS superconductivity as well is the so-called RD model [8]. It involves the additional dimensionless parameter \( \alpha \) and the RD Hamiltonian reads as

\[
H_{RD} = 2 \sum_i^N (\epsilon_i - G)N_i - G \sum_{j < k} (e^{-i\alpha} b_k^+ b_j + e^{-i\alpha} b_j^+ b_k)
\]

(3.21)
with two dimensionful parameters $G, \eta$ and $\bar{G} = \sqrt{G^2 + \eta^2}$. In terms of these variables the dimensionless parameter $\alpha$ has the following form

$$\alpha = \arctanh\left(\frac{\eta}{G}\right)$$

(3.22)

It is also useful to consider two dimensionless parameters $g, \theta$ defined as $G = gd$ and $\eta = \theta d$ where $d$ is the level spacing. The RD model reduces to the Richardson model in the limit $\eta \to 0$.

The RD model turns out to be integrable as well. Now instead of the Gaudin model the proper counterpart is the generic quantum twisted XXX spin chain [10]. The transfer matrix of such spin chain model $t(u)$ commutes with the $H_{RD}$ which itself can be expressed in terms of the spin chain Hamiltonians.

The equation defining the spectrum of the RD model reads as

$$\exp(-2i\alpha) \prod_j^N \frac{E_j - \epsilon_k - i\eta/2}{E_j - \epsilon_k + i\eta/2} = \prod_j^M \frac{\epsilon_j - \epsilon_k - i\eta}{\epsilon_j - \epsilon_k + i\eta}$$

(3.23)

and coincides with the BA equations for the spin chain. It reduces to the BA equation of the Richardson model (3.14) in the limit $\eta \to 0$.

The key feature of the RD model is the multiple solutions to the gap equation. The gaps are parameterized as follows

$$\Delta_n = \frac{\omega}{\sinh t_n}, \quad t_n = t_0 + \frac{\pi n}{\theta}, \quad n = 0, 1, \ldots$$

(3.24)

where $t_0$ is solution to the following equation

$$\tan(\theta t_0) = \frac{\theta}{g}, \quad 0 < t_0 < \frac{\pi}{\theta}$$

(3.25)

and $\omega = dN$ for equal level spacing. This behavior can be derived via the mean field approximation [24]. Different solutions to the gap equation yield the different superconducting states. In the limit $\theta \to 0$ the gaps $\Delta_{n>0} \to 0$ and

$$t_0 = \frac{1}{g}, \quad \Delta_0 = 2\omega e^{\frac{1}{g}}$$

(3.26)

therefore the standard BCS expression for the gap is recovered. At the weak coupling limit the gaps behave as

$$\Delta_n \propto \Delta_0 e^{-\frac{\pi n}{\sigma}}$$

(3.27)

In terms of the solutions to the BA equations the multiple gaps correspond to the choices of the different branches of the logarithms.
If the degeneracy of the levels is $d_n$ then the RD model gets modified a little bit and is related to the higher spin XXX spin chain. The local spins $s_i$ are determined by the corresponding higher pair degeneracy $d_i$ of the i-th level

$$s_i = d_i/2$$

and the corresponding BA equations read as

$$\exp(-2i\alpha) \prod_{j=1}^{N} \frac{E_j - \epsilon_k - i\eta/2 + i\eta s_i}{E_j - \epsilon_k + i\eta/2 - i\eta s_i} = \prod_{j=1}^{M} \frac{\epsilon_j - \epsilon_k - i\eta}{\epsilon_j - \epsilon_k + i\eta}$$

When the local spin tends to infinity the twisted XXX chain degenerates into the periodic Toda chain hence the large degeneracy of the levels is necessary at the superconducting side. It is this limit which corresponds to the Peierls model discussed earlier.

## 4 Cyclic RG flows and RD model

Let us explain the key points concerning the cyclic RG phenomena. It was anticipated for a while however the first clear-cut example has been elaborated only in [15] for the finite-dimensional system. The method developed in [15] formulates the step in the RG flow in the finite-dimensional system as removing the single highest energy level with simultaneous renormalization of the couplings. This approach has a lot in common with the renormalization procedure in the matrix models considered in [28]. The different approach to RG in the finite-dimensional system concerns the introducing the UV cutoff at the small scales and looking at the cutoff dependence. This approach has been successfully applied to the rational Calogero model [27] and it was argued that conformal quantum mechanics manifests the cyclic RG which reflects a kind of ”anomalous” violation of the conformal group down to the discrete one.

In fact the important phenomena which has the cyclic RG origin has been discovered long time ago by Efimov in the three-body system. His key observation concerns the emergence of the specific bound states in the three-body system involving two very different scales. It was argued that in such situation when there are the two-body bound state near threshold there is the tower of the bound states with the RD scaling behavior of energies

$$E_{n+1} = e^\delta E_n$$
The presence of so-called Efimov states with the RD scaling is the common feature of the models with cyclic RG phenomena. The spectrum is reorganized in the universal way during the single cycle and the total number of Efimov states is of order $\log(\frac{r_{UV}}{r_{IR}})$. The review on the cyclic RG in the finite-dimensional systems can be found in [29].

There are examples of the cyclic RG flows in the 2D field theories with the several couplings where the RG is formulated in the standard manner as the dependence on the log of the renormalization scale [16]. It was argued that generically there should be at least two couplings and usually one of them does not run at all. In the field theory one also has the RD scaling for the Efimov-like resonances which in some examples manifest the Regge like structure. Moreover it was argued that the S-matrix behaves universally under the cyclic RG flows. The total number of Efimov states scales in the same manner as in the quantum mechanical case.

The RD model of truncated superconductivity enjoys the cyclic RG behavior [8]. The RG flows can be treated as the integrating out the highest fermionic level with appropriate scaling of the parameters using the procedure developed in [30,15]. The RG equations read as

$$g_{N-1} = g_N + \frac{1}{N}(g_N^2 + \theta^2), \quad \theta_{N-1} = \theta_N$$

(4.31)

At large N limit the natural RG variable is identified with $\log N$ and the solution to the RG equation is

$$g(s) = \theta\tan(\theta s + \tan^{-1}(\frac{g_0}{\theta}))$$

(4.32)

Hence the running coupling is cyclic

$$g(s + \lambda) = g(s), \quad g(e^{-\lambda}N) = g(N)$$

(4.33)

with the RG period

$$\lambda = \frac{\pi}{\theta}$$

(4.34)

and the total number of the independent gaps in the model is

$$N_{\text{cond}} \propto \frac{\theta}{\pi} \log N$$

(4.35)

The multiple gaps are the manifestations of the Efimov-like states. The sizes of the Cooper pairs in the n-th condensates also have the RD scaling. The cyclic RG can be derived even for the single Cooper pair.

What is going on with the spectrum of the model during the period? It was shown in [24] that it gets reorganized. The RG flows possesses the discontinuities
from \( g = +\infty \) to \( g = -\infty \) when a new cycle gets started. At each jump the lowest condensate disappears from the spectrum

\[
\Delta_{n+1}(g = +\infty) = \Delta_n(g = -\infty)
\]  

indicating that \((N+1)\)-th state wave function plays the role of \(N\)-th state wave function at the next cycle.

The same behavior can be derived from the BA equation \[24\]. To identify the multiple gaps it is necessary to remind that the solutions to the BA equations are classified by the integers \( m_i, i = 1, \ldots M \) parameterizing the branches of the logarithms. If one assumes that \( m_i = m \) for all Bethe roots then this quantum number gets shifted by one at each RG cycle and was identified with the integer parameterizing the solution to the gap equations. At the large \( N \) limit the BA equations of the RD model reduce to the BA equation of the Richardson-Gaudin model with the rescaled coupling

\[
G_m^{-1} = \eta^{-1}(\alpha + \pi m)
\]  

which can be treated as the shifted boundary condition in the generalized Gaudin model parameterized by the integer. Let us emphasize that the unusual cyclic RG behavior in due to the presence of two couplings in the RD model.

5 Nonabelian string versus BA equation

5.1 BA equations at the string worldvolume

Let us consider the \( \Omega \) deformed \( N = 2 \) SQCD with \( SU(L) \) gauge group, \( L \) fundamental hypermultiplets with masses \( m_{f_i} \) and \( L \) antifundamental hypermultiplets with masses \( m_{a_{f_i}} \). In the NS limit with the single deformation parameter \( \epsilon \) the Coulomb branch of the vacuum manifold parameterized by the vev of the adjoint scalar \( a_i \) gets reduced to the set of points since the theory develops the twisted superpotential \[4\] determined by Nekrasov partition function. The isolated vacua are determined by the equation

\[
\frac{\partial W(a_i)}{\partial a_i} = n_i
\]  

which yields

\[
\vec{a} = \vec{m}_f - \vec{N}\epsilon
\]  

where \( \vec{n} = (n_1 \ldots n_L) \). The integers can be attributed to the selection of the branch of logarithm in the twisted superpotential.
The bulk $D=4$ theory develops the Higgs branch where the scalar components of the fundamental are condensed. The Higgs branch admits the stable nonabelian string solution with the rich worldsheet theory. The physics of the non-abelian strings in the nondeformed case is described in the reviews [11].

In the bulk theory in the NS limit the worldsheet theory on the nonabelian string involves the $L$ fundamental chiral multiplets with twisted masses $M_{Fi}$ and $L$ antifundamental multiplets with twisted masses $M_{AFi}$. The additional chiral multiplet in the adjoint representation gets non-vanishing mass $\epsilon$ due to the the background graviphoton field. It can be integrated out amounting to the twisted superpotential for the scalars in the vector multiplet. The minimization of the worldsheet theory at $N$ nonabelian strings yields the BA equations where Bethe roots $\lambda_i$ correspond to the values of the scalars in the vector multiplet [4, 32, 31]. The solutions to the BA equations are parameterized by the set of integers $\tilde{n}_i$ obeying the condition $N = \sum_{i=1}^{L} n_i$.

It turns out that on-shell values of the twisted superpotentials in the bulk theory and the worldvolume theory of $N$ nonabelian strings coincide

$$W^{4D}(a_l = m_l - n_l \epsilon) - W^{4D}(a_l = m_l - \epsilon) = W^{2D}(\tilde{n}_L)$$

upon the following identifications of parameters [32]

$$\tilde{M}_f = \tilde{m}_f - 3/2 \epsilon, \quad \tilde{M}_{af} = \tilde{m}_{af} - 1/2 \epsilon$$

The rank of the worldvolume theory $N$ which corresponds to the number of non-abelian strings is defined via the relation

$$N + L = \sum_i n_i, \quad \tilde{n}_l = n_l - 1$$

The modular parameter in the worldsheet theory

$$\tau_{2D} = i r + \frac{\theta_{YM}}{2 \pi}$$

where $r$ is the FI parameter related to the $D = 4$ coupling constant [13]. The $\theta$ term penetrates the worldsheet theory from the bulk one [43]. The modular parameters of the theories are related as

$$\tau_{2D} = \tau_{4D} + \frac{1}{2}(N + 1)$$

In the $D = 4$ bulk theory the BA equations emerge in the saddle point calculation of the instanton partition function [31]. The number of the Bethe roots coincides
with the number of the nonabelian strings which is in the perfect agreement with the interpretation of the nonabelian strings as the excitations above the root of the Higgs branch in the bulk theory.

The BA equations for the deformed SQCD exactly coincide with the BA equations involved into the solution to the RD model. The asymmetry parameter of the RD model $\eta$ is identified with the deformation parameter of the $\Omega$ background

$$\eta = \epsilon$$

(5.45)

The fermionic energies in the RD model are identified with the masses of fundamen-
tals in $D = 4$ bulk theory or the twisted masses in the worldsheet theory

$$E_i = M_i$$

(5.46)

The modular parameter in the worldsheet theory provides the twisted boundary conditions in the BA equations for the inhomogeneous XXX spin chain.

Geometrically the Bethe root or vev of the scalars corresponds to the position of
the D2 brane representing nonabelian string in the transverse coordinates. Note that from
the spin chain viewpoint the Bethe roots are zeros of the polynomial solution

$$Q(\lambda) = \prod(\lambda - \lambda_i)$$

(5.47)

to the Baxter equation for the twisted spin chain.

5.2 Cyclic RG in the deformed gauge theory

Let us turn to the important observation. Since the BA equation for the deformed
SQCD coincides with the one for the twisted XXX spin chain we can use the results
concerning its cyclic RG behavior. First remind once again that there are two di-

mensionless couplings in the $\Omega$ deformed SQCD required for cyclicity. One of them is
the conventional complexified coupling in the bulk gauge theory which plays the role
of the twist in the spin chain. The second dimensionless parameter in the simplest
case can be identified with the ratio

$$\theta = \frac{\epsilon}{\delta m}$$

(5.48)

where $\delta m$ is the difference between the masses of the fundaments. We assume that
the masses are almost equidistant. It is useful following [24] to introduce the second
dimensionful parameter $G$ similar to the $\Lambda_{QCD}$ scale

$$\tan \tau = \frac{G}{\epsilon}$$

(5.49)
This scale is evidently nonperturbative with respect to the gauge coupling.

To describe the cyclic RG behavior in SQCD let us identify the step of the RG flow in terms of the finite-dimensional system. As we have mentioned the step in the RD model corresponds to the decoupling of the highest energy level. Since the energy level in RD model corresponds to the mass of fundamental the RG step corresponds to the decoupling of the heavy flavor $N_F \rightarrow N_F - 2$. Simultaneously the rank of the group is changed $N_c \rightarrow N_c - 1$ and the form of the BA equation holds the same.

In the non-deformed case this procedure yields the RG flow without any cycles. On the other hand if we just decouple the heavy flavor without changing the rank of the group the nonperturbative scale emerges in the asymptotically free theory.

$$\Lambda = M_{reg} \exp\left(\frac{2\pi}{\alpha(M_{reg})\beta_0}\right)$$

(5.50)

where $M_{reg}$ is the UV scale and $\beta_0$ is the coefficient of the $\beta$-function. In the deformed case the situation is more involved since the decoupling of the heavy flavor is described by two-coupling RG equations. As we have seen before the $\epsilon$ parameter is not deformed however the modular parameter enjoys the cyclic RG solutions. Indeed the complexified gauge coupling is expressed as a function of the scales via (??) hence the RG flow of $G$ yields the RG flow of $\tau$. Let us emphasize that we consider the conformal theories and the cyclicity involves the rank of the gauge group. The cyclicity of the RG flow breaks the conformal group down to the discrete subgroup selected by the deformation parameter.

As we have described above the key feature is the emergence of the multiple Efimov-like scales in the problem. In the RD model these Efimov-like scales correspond to the multiple gaps with the Efimov scaling. In the deformed SQCD this scales emerge as the multiple nonperturbative $\Lambda$-like scales whose number is defined by the ratio

$$N_{cond} \propto \frac{\epsilon}{\delta m}$$

(5.51)

At the weak coupling the scales behave as

$$\Delta_n \propto \Delta_0 e^{-\frac{n \Delta m}{\epsilon}}$$

(5.52)

which are certainly nonperturbative in the parameter of the $\Omega$ background.

Our observations provide the qualitative picture behind the RG flows in the deformed QCD and more detailed analysis is required. In particular we have considered the situation with the single massive parameter only. In the generic situation we have the set of the dimensionless parameters

$$\theta_i = \frac{m_i}{\epsilon}$$

(5.53)
hence the RG flow involves the multiple couplings.

6 On the Cooper pair interpretation

So far we have not discussed the interpretation of the fermions developing the superconductivity. In this Section we make some conjecture concerning their identification and present the several evidences supporting it. Namely we shall conjecture that the relevant degrees of freedom which form the Cooper pairs are the monopoles in the Higgs branch localized at the nonabelian string.

6.1 On the monopole interpretation from the knot homologies

First let us make a few comments concerning the another appearance of the BA equations for the generalized Gaudin model. Attempting to get the field theory realization of the knot homologies the counting of the solutions to the BPS equations in the proper gauge theory was considered in \[33\]. The problem of counting of the BPS equations was reformulated in terms of the counting of the solutions to the BA equations in the particular integrable system. The mapping of the gauge system to the BA equations goes as follows. The inhomogeneities in the BA equation correspond to the t’Hooft lines or the singular abelian monopoles. On the other hand the Bethe roots correspond to the positions of the BPS monopoles at the particular plane. The BA equations reflect the interaction between the t’Hooft lines and the BPS monopoles which is S-dual to the process of the W-boson exchange between two Wilson lines.

The second ingredient of the generalized Gaudin concerns the nontrivial symmetry breaking in the gauge theory which yields the constant term in the BA equation. It is this term that gets renormalized when the \(\Omega\) deformation is added. We expect that upon the deformation the theory enjoys the cyclic RG behavior. In the brane terms the relevant five-dimensional theory involves the set of M2 branes whose coordinates are partially fixed by minimizing the superpotential which is generated by the string M2 instanton. The positions of the BPS monopoles are fixed by the BA equations.

The relationship between the knot homologies and the cyclic RG seems to be not accidental. The point is that cyclic RG can be discovered in the quantum mechanics of Calogero-like potential \[27\] when the Calogero coupling constant is subject of renormalization. On the other hand the Calogero model at the rational coupling
constant is tied intrinsically to the toric knots.

6.2 Cyclic RG in Liouville-Toda

Since there is the AGT relation between the Nekrasov partition function of conformal SQCD and the 2d conformal block one could ask about the interpretation of the Cooper pairs at the Liouville/Toda side. The relation between the wave functions in the Richardson model and the conformal blocks in the perturbed WZW model defined on the spectral plane of the fermionic model has been analyzed in [25].

Let us briefly comment on the dictionary between the BSC-like model and the perturbed conformal theory found in [25]. The Richardson wave function is defined by the number of the Bethe roots and can be treated as the conformal block in the $\beta - \gamma$ system in the perturbed conformal theory. The coupling constant is introduced into the conformal model via the operator

$$V_g = \exp\left(-\frac{i\alpha_0}{g} \oint_C z \partial\phi(z)\right)$$

(6.54)

where $\phi(z)$ is the conventional scalar boson. The Richardson wave function enters the integral representation of the perturbed conformal block in the same manner as the Gaudin wave function enters the integral representation of the Liouville conformal block (see, for example [39]). The BA equation for the Richardson wave function corresponds to the saddle point equation in the integral representation for the conformal block.

The most important observation from the dictionary obtained in [25] is that the Cooper pair operator is attributed to the screening operator and the SL(2) algebra in the Gaudin model was identified with the algebra of screenings. Semiclassically these screening operators are attached to the surface operators hence we have no contradiction with the conjectured monopole interpretation. The generalization of the dictionary above to the RD model should be similar since the Gaudin model is just substituted by the twisted inhomogeneous XXX spin chain. We believe that the interpretation of the Cooper pair as screenings works in this case as well.

Since the Nekrasov partition function for superconformal QCD is related to the conformal blocks in Liouville and Toda theories is it natural to ask what is the possible interpretation of the cyclic RG flows in the Liouville/Toda side. We restrict ourself by the simple remarks postponing the analysis for the separate study. The cyclicity could be looked at in the conformal blocks of the Toda theory corresponding to the conformal SQCD. It would correspond to the decoupling of the single flavor with the largest mass which enters the conformal dimensions of the vertex operators.
Simultaneously the gauge coupling constant which corresponds to the position of the particular vertex operator should be renormalized. One could expect that the Efimov states in the Liouville/Toda theory could manifest themselves as the particular resonant states similar to the example elaborated in [16].

6.3 Bion condensates

If the fermions are identified as the monopoles in the Higgs phase one could concern on the physical mechanism providing the formation of the bound states with monopole charge \( Q_M = 2 \). The possible mechanism providing the bound states of monopoles has been suggested in [40, 41]. It is based on the consideration of the SYM theory at \( R^3 \times S^1 \) geometry where the interesting solution with the several quantum numbers exists. The most relevant solutions involve both fractional topological and magnetic charges. The compact dimension provides the breaking of the gauge symmetry and the finite number of the vacuum states.

In this geometry there are also so-called KK monopoles which provide the closeness of the monopole array. The instanton itself gets interpreted as the bound state of the array of monopoles and KK monopole with vanishing magnetic and unit topological charge. The bound state found in [40] involves the monopole with the fractional topological charge and the KK-antimonopole with the opposite topological charge. Hence the state does not have topological charge at all.

The bion condensation has been considered in 3+1 dimensions however the bulk monopole has the kink counterpart at the string worldsheet. Hence we could conjecture that there are the bound states with the kink charge two similar to the bion condensate in the bulk. The naive analysis of the zero modes responsible for the attraction supports this possibility however the detailed analysis is required.

The RG analysis of the model involving the gas of bions and electrically charged W-bosons has been considered in [41] where the RG flows involves the fugacities for electric and magnetic components and the coupling constant. The coupled set of the RG equations has been solved explicitly in the self-dual case and the solution to the RG equations for the fugacities obtained in [41] is identical to the solution for the coupling in the RD model upon the analytic continuation. The period of the RG in the solution above is fixed by the RG invariant which has been identified with the product of the UV values of the electric and magnetic fugacities \( y_e \times y_m \). The similarity between the RG behavior is not accidental since the mapping of the gauge theory and the perturbed XY model has been found in [41].

In the previous sections we have argued that in the deformed SQCD the period of the cycle is fixed by graviphoton field hence one could wonder if the product of fugac-
itiest has any relation with the graviphoton background. The possible answer could be as follows. The presence of the magnetic and electric components in the plasma simultaneously implies the possible angular momentum of the field. On the other hand it is possible to identify the analogue of the gravimagnetization of the gauge theory in the RD model. In the gauge theory it is related to the projection of the angular momentum in the Euclidean 4d space-time \[45\]. The similar differentiation of the RD Hamiltonian with respect to \(\epsilon\) at \(\epsilon = 0\) yields

\[
\frac{dH_{RD}}{d\epsilon} = \sum_{i<j}(b_ib_j^+ - b_j^+b_i)
\]  \(\text{(6.55)}\)

which is related to the projection of angular momentum as well upon use of the commutation relation in \(SL(2, R)\).

Such interpretation suggests the possible place of the full Efimov tower. As we have mentioned the new state which appear(disappear) during the RG cycle corresponds to the shift in the branch of the logarithm. This is usually attributed to the additional flux of the global symmetry charge. Hence the natural conjecture is that the higher Efimov states are the bound states of the dyons instead of the monopoles. Such states exist only in some region of the moduli space decaying at the corresponding curves of the marginal stability. Note that the relation of the bion ensembles and the perturbed XY model and interesting interplay of different scales has been recently discussed in \([42]\). We plan to discuss the relation between perturbed XY models in \([42]\) and in our paper elsewhere.

7 Discussion

In this paper we have focused on the two aspects of the relation between the models of truncated BCS-like superconductivity and supersymmetric gauge theories in the graviphoton background. It was demonstrated that the vacuum structure of the \(D = 4\) bulk theories and \(D = 2\) worldsheet theory at the nonabelian string is inherited also by the third dynamical system - some version of the superconducting system. The Bethe equations defining the vacuum structure of the gauge theory yield the spectrum of the excited states of the Cooper pairs in the superconducting RD model. This can be considered as an complicated condition of the self-consistency between the several components of the whole system. The consistency between the worldvolume theories at D4 and D2 branes with monopole excitation generalizes the 4d/2d duality for the nonabelian strings.

The second surprising ingredient of the correspondence concerns the RG behavior. It was known for a while that RG behavior in the bulk and worldsheet theories are
consistent. Here we have the third finite-dimensional RD model subsystem which enjoys the peculiar cyclic RG behavior. This cyclic behavior can be recognized in the $D = 2$ worldsheet theory and $D = 4$ bulk theory. In worldsheet theory the cycle corresponds to the shift of the branch of the solution. This corresponds to the additional flux along the nonabelian string. In the $D = 4$ theory it can be attributed to the change of the branch of the superpotential as well. In the bulk theory the change of the branch corresponds to the additional flux string in the bulk theory which fits with the transformation of the magnetic string into dyonic one. Let us emphasize that the cyclic RG implies the possibility of the multiple nonperturbative $\Lambda_i$ scales like the multiple gaps in the superconducting model. There is some similarity with the scenario with the several nonperturbative scales discussed in [3].

It is natural to ask if there are other possible manifestations of the cyclic RG behavior in the SUSY gauge theories. Let us mention two possible candidates. First note that since we have observed the cyclicity in the number of flavors the Seiberg duality can be looked at from the new viewpoint. Indeed generically the Seiberg dual system has the additional mesonic degrees of freedom and the spectrum is reordered. One could assume that the "RG time" in this case is

$$t_{RG} = \log \left( \frac{N_F}{N_C} \right)$$

(7.56)

and the transition from the initial to the Seiberg dual system could be treated as the period of the cyclic RG flow. Naively one needs the second dimensionless parameter to get the two-coupling flow. In this case the possible candidate is the ratio of adjoint and fundamental masses. Recently the bulk Seiberg duality has been recognized in the worldsheet theory [44] where it corresponds to peculiar interchange of the scale and orientational moduli of the semilocal nonabelian string. The spectrum in the dual worldsheet theory has the additional degrees of freedom as required.

Secondly it was found recently that the spectrum of the stable BPS particles in the worldsheet theory manifests the interesting periodicity [37]. Namely the curves of the marginal stability are organized as concentric curves and the transition from one curve to the next one corresponds to the shift of the branch of the superpotential. The curve of the marginal stability is the very promising place to look at for the whole Efimov tower. Indeed as we have discussed above the Efimov states appear when the third degree of freedom is added to the two-body system near threshold. It is this situation which is realized near the CMS where the bound state of two BPS constituents disappears. It seems that phenomena observed in [37] could be the manifestation of the RG cycles and we plan to discuss this issue elsewhere.
The are a few evident directions of the generalization. First, it would be interesting to generalize the analysis above to the asymmetric XXZ and XYZ twisted spin chains corresponding to the 5d and 6d gauge theories [38]. One could expect the corresponding superconducting systems for each case where the anisotropy parameters should be the particular coupling constants. It would be also interesting to discover the superconducting model involving two independent parameters of the Ω deformation.

It is evident that the investigation of the cyclic RG flow in the field theories is at the very beginning and much more work is required. Among the most immediate questions a few could be mentioned.

- The anomalies can be equally considered as the IR and UV phenomena since they have interpretation as the spectrum flow at any scale. This means that anomalies are a kind of invariants of the RG cycle. How this could be formulated in an invariant manner?

- What is the holographic image of the cyclic RG flow and the cycle step?

- Is it possible to get the bound state of regulator(UV) and physical (IR) degrees of freedom similar to the higher states in the Efimov tower?

- Is it possible to get the refined partition function involving the Efimov states or resonances and is there any relation between such partition functions and the knot invariants? The natural candidate to look at is the monodromy of the spectrum under the RG cycle.

We hope to investigate these questions elsewhere.

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