FINITE COMPACTIFICATIONS OF $\omega^* \setminus \{x\}$

MAX F. PITZ AND ROLF SUABEDISSEN

Abstract. We prove that under [CH], finite compactifications of $\omega^* \setminus \{x\}$ are homeomorphic to $\omega^*$. Moreover, in each case, the remainder consists almost exclusively of P-points, apart from possibly one point.

Similar results are obtained for other, related classes of spaces, amongst them $S_\kappa$, the $\kappa$-Parovićenko space of weight $\kappa$. Also, some parallels are drawn to the Cantor set and the Double Arrow space.

1. Introduction

Compactifications of $\omega^* \setminus \{x\}$ have received considerable attention in the literature. Fine and Gillman [3] showed that under [CH], $\omega^*$ is not the Stone-Čech compactification of any of its subspaces $\omega^* \setminus \{x\}$, whereas van Douwen, Kunen and van Mill showed later that it is consistent with the usual axioms of set theory ZFC that this situation always occurs [2].

In [9] the present authors obtained further structural results about the Stone-Čech compactification of $\omega^* \setminus \{x\}$ under [CH]. Under the assumption $\kappa = \kappa^{<\kappa}$ we also extended our results to $S_\kappa$, the Stone space of the unique $\kappa$-saturated Boolean algebra of size $\kappa$. These spaces can be seen as the natural generalisation of $\omega^*$ under [CH] to infinite cardinals $\kappa$ with the property $\kappa = \kappa^{<\kappa}$.

Yet, despite these far-reaching results regarding Stone-Čech compactifications of $\omega^* \setminus \{x\}$, to our knowledge nothing has appeared in print about finite compactifications of these spaces. In this paper we prove the somewhat surprising result that under [CH] every finite compactification of $\omega^* \setminus \{x\}$ is again homeomorphic to $\omega^*$. Moreover, each remainder consists almost exclusively of P-points, apart from possibly one point. This is accompanied by the result that there are arbitrarily large finite compactifications of $\omega^* \setminus \{x\}$.

Corresponding results also hold for spaces $S_\kappa \setminus \{x\}$. A strong emphasis is laid on Lemma 4.1, which can serve as a roadmap of how to generalise statements from $\omega^*$ to $S_\kappa$ that rely on the fact that $\omega^*$ does not contain converging sequences.

We start our investigation in Section 2 with a general framework for proving that all finite compactifications of a given space look alike, illustrated by means of the Cantor set, the Double Arrow space and other related spaces. We then proceed in Sections 3 and 4 to $\omega^*$ (under [CH]) and to $S_\kappa$ (for cardinals $\kappa$ such that $\kappa = \kappa^{<\kappa}$), and prove, after a proper introduction of these spaces, the announced results.

The case $S_\kappa$ draws together the previous cases as $S_\omega$ is homeomorphic to the Cantor set $C$, and $S_{\omega_1}$ is homeomorphic to $\omega^*$ under [CH].
2. Theory and first examples

We begin with a sufficient condition for zero-dimensional locally compact Hausdorff spaces to have only one homeomorphism type amongst their finite compactifications. Recall that a space is zero-dimensional if it has a basis of clopen (closed-and-open) sets.

Lemma 2.1. Let $X$ be a zero-dimensional compact Hausdorff space such that $X \oplus X$ is homeomorphic to $X$ and

\[(\star)\] for every point $x$ of $X$, every clopen non-compact subset $A$ of $X \setminus \{x\}$ is homeomorphic to $X \setminus \{x_A\}$ for some $x_A \in X$.

Then, for all $x$, all finite compactifications of $X \setminus \{x\}$ are homeomorphic to $X$.

Proof. Let $Z$ be a finite compactification of $X \setminus \{x\}$ with remainder consisting of points $\infty_1, \ldots, \infty_n$. By [11, 2.3], every finite compactification of a locally compact zero-dimensional space is zero-dimensional. Hence, there is a partition of $Z$ into $n$ disjoint clopen sets $A_i$ such that $\infty_i \in A_i$.

By property $(\star)$, $A_i \setminus \{\infty_i\}$ is homeomorphic to $X \setminus \{x_{A_i}\}$. Uniqueness of the one-point compactification gives $A_i \cong X$ and hence $Z$ is, after applying $X \oplus X \cong X$ iteratively, homeomorphic to $X$. □

This lemma lies at the heart of our main result regarding finite compactifications of $\omega^* \setminus \{x\}$. Surprisingly, despite its strong assumptions, it also applies to a variety of other interesting spaces.

Spaces which only have $\lambda$ different homeomorphism types amongst their open subspaces (for some cardinal $\lambda$) are said to be of diversity $\lambda$ [8]. One checks that Lemma 2.1 applies to all compact Hausdorff spaces of diversity two, which are known to be zero-dimensional [7]. In particular, it applies to the Cantor space $C$, which can be characterised as the unique compact metrizable space of diversity two [10], and to the Alexandroff Double Arrow space $D$ and also to their product $D \times C$.

In a compact Hausdorff space $X$ of diversity two, any subspace $X \setminus \{x\}$ is homeomorphic to $X \setminus \{x_1, \ldots, x_n\}$ and therefore has arbitrarily large finite compactifications. These are the cases where Lemma 2.1 is most valuable. The next lemma shows that not much is needed for this scenario to occur. The proof is a simple induction.

Lemma 2.2. Let $X$ be a topological space such that for all $x$, all finite compactifications of $X \setminus \{x\}$ have two-point compactifications, they have arbitrarily large finite compactifications. □

The following example of the Cantor cube $2^\kappa$ for uncountable $\kappa$ shows that the assumptions in Lemma 2.2 cannot be considerably weakened. Since $\beta(2^\kappa \setminus \{x\}) = 2^\kappa$ [5, Thm. 2], these spaces have a unique compactification.

The cube $2^\kappa$ is a zero-dimensional compact Hausdorff space with $2^\kappa \cong 2^\kappa \oplus 2^\kappa$. For property $(\star)$, let $A \subset 2^\kappa \setminus \{x\}$ be a clopen non-compact subset. Since $2^\kappa \setminus \{x\}$ does not have a 2-point compactification, $A \cup \{x\}$ must be clopen in $2^\kappa$. But every clopen set of $2^\kappa$ can be written as a disjoint union of finitely many product-basic open sets, which are homeomorphic to $2^\kappa$. Hence $A \cup \{x\} \cong 2^\kappa$.

We conclude that Lemma 2.1 applies, but restricts to the obvious assertion that the one-point compactification of $2^\kappa \setminus \{x\}$ is homeomorphic to $2^\kappa$. 
3. The space $\omega^*$

This section contains the proof that under [CH] all finite compactifications of $\omega^* \setminus \{x\}$ are homeomorphic to $\omega^*$. The plan for attack is clear: we want to apply Lemma 2.1 to $\omega^*$. That we may do so will be justified by Lemma 3.1. Before that, we recall some characteristics of the space $\omega^*$.

Parovićenko’s theorem says that under [CH], the Stone-Čech remainder $\omega^*$ of the countable discrete space is topologically characterised as the unique compact zero-dimensional $\text{F-space}$ of weight $\aleph$ without isolated points in which each non-empty $G_\delta$-set has non-empty interior [6, 1.2.4].

Recall that an $\text{F-space}$ is a space where all cozero sets are $C^*$-embedded. In normal spaces, the $\text{F-space}$ property is equivalent to pairs of disjoint cozero sets having disjoint closure, and is therefore closed hereditary [6, 1.2.2]. Also, it is known that infinite closed subsets of compact $\text{F-space}$ contain a copy of $\beta \omega$, and therefore have large cardinality. In particular, $\omega^*$ does not contain converging sequences.

A $P$-point is a point $p$ such that every countable intersection of neighbourhoods of $p$ contains again an open neighbourhood of $p$. In a zero-dimensional space, a point $x$ is a not a $P$-point if and only if there exists an open $F_\sigma$-set containing $x$ in its boundary.

**Lemma 3.1.** [CH]. The space $\omega^*$ has property $(*)$, i.e. the one-point compactification of a clopen non-compact subset of $\omega^* \setminus \{x\}$ is homeomorphic to $\omega^*$.

**Proof.** Let $A$ be a clopen non-compact subset of $\omega^* \setminus \{x\}$. Taking $A \cup \{x\}$, a closed subset of $\omega^*$, as representative of its one-point compactification, we see that it is a zero-dimensional compact $\text{F-space}$ of weight $\aleph$ without isolated points.

Suppose that $U \subset A \cup \{x\}$ is a non-empty $G_\delta$-set. If $U$ has empty intersection with $A$, then the singleton $U = \{x\}$ is a $G_\delta$-set, and hence has countable character in the compact Hausdorff space $A \cup \{x\}$. It follows that there is a non-trivial sequence in $\omega^*$ converging to $x$, a contradiction. Thus, $U$ intersects the open set $A$ and their intersection is a non-empty $G_\delta$-set of $\omega^*$ with non-empty interior.

An application of Parovićenko’s theorem completes the proof. $\square$

**Theorem 3.2.** [CH]. Let $x$ be a point in $\omega^*$. Every finite compactification of $\omega^* \setminus \{x\}$ is homeomorphic to $\omega^*$. Moreover, at most one point in the remainder of a finite compactification is not a $P$-point.

**Proof.** The first assertion follows immediately from Lemmas 3.1 and 2.1.

For the second assertion, suppose there some remainder of $\omega^* \setminus \{x\}$ contains two non-$P$-points $\infty_1$ and $\infty_2$. Let $A_1$ and $A_2$ be disjoint clopen neighbourhoods of $\infty_1$ and $\infty_2$ as in Theorem 2.1, i.e. such that $A'_i = A_i \setminus \{\infty_i\}$ are clopen subsets of $\omega^* \setminus \{x\}$. As the points at infinity are non-$P$-points, there are open $F_\sigma$-sets $F_1$ and $F_2$ with $F_i \subset A_i'$ such that $\infty_i \in \overline{F_i} \setminus F_i$. However, since $A'_i \cup \{x\} \cong A'_i \cup \{\infty_i\}$, we see that in $\omega^*$ the disjoint open $F_\sigma$-sets $F_1$ and $F_2$ both limit onto $x$, contradicting the $\text{F-space}$ property. $\square$

By a well-known result of Fine & Gillman using [CH], every space $\omega^* \setminus \{x\}$ splits into complementary clopen non-compact sets for all points $x$ of $\omega^*$ [4]. This gives rise to a two-point compactification of $\omega^* \setminus \{x\}$. Lemma 2.2, with Theorem 3.2, now shows that $\omega^* \setminus \{x\}$ has arbitrarily large finite compactifications. This also implies the well-known result that under [CH] the space $\omega^*$ contains $P$-points.
One may ask what of this remains true in absence of [CH]. In the above proof, [CH] is needed only in applying Parovičenko’s theorem. In ZFC, therefore, any finite compactification of \(\omega^* \setminus \{x\}\) is a Parovičenko space of weight \(\epsilon\) such that at most one point in the remainder is not a \(P\)-point.

Is it consistent with ZFC that no Parovičenko space of weight \(\epsilon\) contains \(P\)-points?

4. The space \(S_\kappa\)

The spaces \(S_\kappa\) are the natural generalisation of the Parovičenko space \(\omega^*\) under [CH] to larger cardinals \(\kappa\) with the property \(\kappa = \kappa^{<\kappa}\).

In a zero-dimensional space \(X\), the type of an open subset \(U\) of \(X\) is the least cardinal number \(\tau\) such that \(U\) can be written as a union of \(\tau\)-many clopen subsets of \(X\). A zero-dimensional space where open subsets of type less than \(\kappa\) are \(C^*\)-embedded is called \(F_\kappa\)-space [1, Ch. 14]. Note that in a zero-dimensional compact space the notions of \(F^*\) and \(F_{\omega^1}\)-space coincide. It is well-known that being an \(F_\kappa\)-space implies that disjoint open sets of type less than \(\kappa\) have disjoint closure and that under normality, the implication reverses [1, 6.5].

A \(\kappa\)-Parovičenko space is a zero-dimensional compact \(F_\kappa\)-space of weight \(\kappa^{<\kappa}\) without isolated points such that every non-empty intersection of less that \(\kappa\)-many open sets has non-empty interior. Under \(\kappa = \kappa^{<\kappa}\) there is a up to homeomorphism unique \(\kappa\)-Parovičenko space of weight \(\kappa\), denoted by \(S_\kappa\) [1, Ch. 6]. Assuming [CH], we have \(S_{\omega_1} \equiv \omega^*\).

A \(P_\kappa\)-point is a point \(p\) such that the intersection of less than \(\kappa\)-many neighbourhoods of \(p\) contains again an open neighbourhood of \(p\). By zero-dimensionality, a point \(x \in S_\kappa\) is a not a \(P_\kappa\)-point if and only if there exists an open set of type less than \(\kappa\) containing \(x\) in its boundary. Again, a \(P_{\omega_1}\)-point is simply a \(P\)-point.

In this section we generalise results from \(S_{\omega_1} = \omega^*\) to general \(S_\kappa\), assuming \(\kappa = \kappa^{<\kappa}\) throughout. The challenge lies in the fact that Lemma 3.1 does not carry through without extra work. Indeed, Lemma 3.1 rested on two corner stones: that in normal spaces, the \(F\)-space property is closed-hereditary and that every infinite closed subset of \(S_{\omega_1}\) has the same cardinality as \(S_{\omega_1}\). Both assertions do not carry over to \(S_\kappa\), as it contains a closed copy of \(\beta\omega\).

The following shows how to circumvent these obstacles.

**Lemma 4.1.** Let \(x\) be a point in \(S_\kappa\). If \(A\) is a clopen, non-compact subset of \(S_\kappa \setminus \{x\}\) then its type in \(S_\kappa\) equals \(\kappa\).

**Proof.** Suppose for a contradiction that there exists a clopen, non-compact subset \(A\) of \(S_\kappa \setminus \{x\}\) of \(S_\kappa\)-type \(\tau < \kappa\). It suffices to consider uncountable \(\kappa\). Find a representation

\[
A = \bigcup_{\alpha < \tau} A_\alpha
\]

where all \(A_\alpha\) are clopen subsets of \(S_\kappa\). We claim that there is a collection \(\{V_\alpha\}_{\alpha < \tau}\) of pairwise disjoint clopen sets of \(S_\kappa\) such that \(V_\alpha \subset A \setminus \bigcup_{\beta < \alpha} A_\beta\) for all \(\alpha < \tau\).

We proceed by transfinite induction. Choose a clopen subset \(V_0\) in the non-empty open set \(A \setminus A_0\). Now consider \(\alpha < \tau\) and suppose that \(V_\beta\) have been defined for all \(\beta < \alpha\). By [1, 14.5], the set \(U_\alpha = \bigcup_{\beta < \alpha} A_\beta \cup V_\beta\) cannot be dense in \(A\), and we may find a clopen set \(V_\alpha\) in the interior of \(A \setminus U_\alpha\). This completes the inductive construction.
Finally, let \( f \) and \( g \) be disjoint cofinal subsets of \( \tau \). We define disjoint sets

\[ V_f = \bigcup_{\alpha \in f} V_\alpha \quad \text{and} \quad V_g = \bigcup_{\alpha \in g} V_\alpha \]

and claim that both sets limit onto \( x \), contradicting the \( F_\kappa \)-space property of \( S_\kappa \).

Suppose the claim was false. Then \( \overline{V_f} \) is a subset of \( \bigcup_{\alpha < \tau} A_\alpha \). By compactness, there is a finite set \( F \subset \tau \) such that \( \overline{V_f} \subset \bigcup_{\beta \in F} A_\beta \). But there are sets \( V_\alpha \) with arbitrarily large index constituting to \( V_f \), a contradiction. \( \square \)

An interesting corollary of this is that for uncountable \( \kappa \), the boundary of every open set in \( S_\kappa \) of type less than \( \kappa \) is infinite, and hence, as a closed subset, of cardinality at least \( 2^\kappa \).

**Lemma 4.2.** Let \( x \) be a point in \( S_\kappa \). If \( A \) is a clopen, non-compact subset of \( S_\kappa \setminus \{x\} \) then its one-point compactifications is homeomorphic to \( S_\kappa \).

**Proof.** Let \( X \) be the closure of \( A \) in \( S_\kappa \), i.e. \( X = A \cup \{x\} \subset S_\kappa \). Then \( X \) is a compact zero-dimensional space of weight \( \kappa \). We check for the remaining \( \kappa \)-Parovičenko properties.

To show that \( X \) has the \( F_\kappa \)-space property, let \( U \) and \( V \) be disjoint open sets of \( X \) of type less than \( \kappa \). By normality, it suffices to show that \( U \) and \( V \) have disjoint closure in \( X \). Suppose that \( x \) belongs to \( U \cup V \). Assume \( x \in U \), so that \( x \) does not belong to the closure of \( V \). The sets \( U \cap A \) and \( V \cap A \) are of \( A \)-type less than \( \kappa \). And since \( A \) is an \( F_\kappa \)-space by [1, 14.1], they have disjoint closure in \( A \), and therefore in \( X \). Next, suppose that \( x \) does not belong to \( U \cup V \). Then \( U \) and \( V \) are subsets of \( A \), and consequently of \( S_\kappa \)-type less than \( \kappa \). Thus, \( U \) and \( V \) have disjoint closures in \( S_\kappa \), and hence in \( X \). This establishes that \( X \) is an \( F_\kappa \)-space.

To show that \( X \) has the property that every non-empty intersection of less than \( \kappa \)-many clopen sets has non-empty interior, suppose that \( U = \bigcap_{\alpha < \beta} U_\alpha \) is a non-empty set, \( \beta < \kappa \) and all \( U_\alpha \) are clopen subsets of \( X \) = \( A \cup \{x\} \). If \( U \) has empty intersection with \( A \), then all \( X \setminus U_\alpha \) are clopen subsets of \( S_\kappa \). It follows that \( A = \bigcup_{\alpha < \beta} X \setminus U_\alpha \) is a clopen non-compact subspace \( S_\kappa \setminus \{x\} \) of type less than \( \kappa \), contradicting Lemma 4.1. Thus, \( U \) intersects \( A \), and their intersection has non-empty interior in \( S_\kappa \). \( \square \)

**Theorem 4.3.** Let \( x \) be a point in \( S_\kappa \). Every finite compactification of \( S_\kappa \setminus \{x\} \) is homeomorphic to \( S_\kappa \). Moreover, at most one point in the remainder of a finite compactification is not a \( P_\kappa \)-point.

**Proof.** As in Theorem 3.2. \( \square \)

As in the case of \( \omega^* \), the spaces \( S_\kappa \setminus \{x\} \) split into complementary clopen non-compact sets [1, 14.2] and therefore have arbitrarily large finite compactifications by Lemma 2.2. Again, we obtain as a corollary that \( S_\kappa \) contains \( P_\kappa \)-points.

**References**

[1] W.W. Comfort, S. Negrepontis, The Theory of Ultrafilters, Springer-Verlag, Berlin, 1974.

[2] E.K. van Douwen, K. Kunen, J. van Mill, There can be \( C^\ast \)-embedded dense proper subspaces in \( \beta\omega - \omega \), Proc. Amer. Math. Soc. 105 (2) (1989) 462-470.

[3] N.J. Fine, L. Gillman, Extension of continuous functions in \( \beta\mathbb{N} \), Bull. Amer. Math. Soc. 66 (1960) 376-381.
[4] L. Gillman, The space $\beta\mathbb{N}$ and the continuum hypothesis, in: General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the second Prague topological symposium, Praha, 1967, pp. 144-146.

[5] I. Glicksberg, Stone-Čech Compactifications of Products, Trans. Amer. Math. Soc. 90 (3) (1959) 369-382.

[6] J. van Mill, An Introduction to $\beta\omega$, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Science, 1984, pp. 503-567.

[7] J. Mioduszewski, Compact Hausdorff spaces with two open sets, Colloq. Math. 39 (1978) 35-40.

[8] J. Norden, S. Purisch, M. Rajagopalan, Compact spaces of diversity two, Top. Appl. 70 (1996) 1-24.

[9] M.F. Pitz, R. Suabedissen, The Stone-Čech compactifications of $\omega^* \setminus \{x\}$ and $S_\kappa \setminus \{x\}$, arXiv:1310.0678.

[10] A.H. Schoenfeld, G. Gruenhage, An alternate characterization of the Cantor set, Proc. Amer. Math. Soc. 53 (1) (1975) 235-236.

[11] R.G. Woods, Zero-dimensional compactifications of locally compact spaces, Can. J. Math. 26 (4) (1974) 920-930.

Mathematical Institute, University of Oxford, Oxford OX2 6GG, United Kingdom
E-mail address, Corresponding author: pitz@maths.ox.ac.uk

Mathematical Institute, University of Oxford, Oxford OX2 6GG, United Kingdom
E-mail address: suabedis@maths.ox.ac.uk