GEOMETRIC INEQUALITIES AND SYMMETRY
RESULTS FOR ELLIPTIC SYSTEMS

SERENA DIPIERRO
SISSA - International School for Advanced Studies
Sector of Mathematical Analysis
Via Bonomea, 265
34136 Trieste, Italy

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Abstract. We obtain some Poincaré type formulas, that we use, together with the level set analysis, to detect the one-dimensional symmetry of monotone and stable solutions of possibly degenerate elliptic systems of the form
\[
\begin{align*}
\operatorname{div} (a(|\nabla u|) \nabla u) &= F_1(u,v), \\
\operatorname{div} (b(|\nabla v|) \nabla v) &= F_2(u,v),
\end{align*}
\]
where \( F \in C^{1,1}_{\text{loc}}(\mathbb{R}^2) \).

Our setting is very general, and it comprises, as a particular case, a conjecture of De Giorgi for phase separations in \( \mathbb{R}^2 \).}

1. Introduction. In this paper we consider a class of quasilinear (possibly degenerate) elliptic systems in \( \mathbb{R}^n \). We prove that, under suitable assumptions, the solutions have one-dimensional symmetry, showing that the results obtained in \([1, 2, 8]\) hold in a more general setting.

In \([1]\) the following problem has been studied:
\[
\begin{align*}
\Delta u &= uv^2, \\
\Delta v &= vu^2, \\
u, v &> 0.
\end{align*}
\] (1.1)

The authors proved the existence, symmetry and nondegeneracy of the solution to problem (1.1) in \( \mathbb{R} \); in particular, they showed that entire solutions are reflectionally symmetric, namely that there exists \( x_0 \) such that \( u(x - x_0) = v(x - x_0) \). Moreover, they established a result that may be considered the analogue of a famous conjecture of De Giorgi for problem (1.1) in dimension 2, that is they proved that monotone solutions of (1.1) in \( \mathbb{R}^2 \) have one-dimensional symmetry under the additional growth condition
\[
u(x) + v(x) \leq C(1 + |x|).
\] (1.2)

On the other hand, in \([9]\), it has been proved that the linear growth is the lowest possible for solutions to (1.1); in other words, if there exists \( \alpha \in (0, 1) \) such that
\[
u(x) + v(x) \leq C(1 + |x|)^\alpha,
\]

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then \( u = v \equiv 0 \).

In [2] the authors proved that the above mentioned one-dimensional symmetry still holds in \( \mathbb{R}^2 \) when the monotonicity condition is replaced by the stability of the solutions (which is a weaker assumption). Moreover, they showed that there exist solutions to (1.1) which do not satisfy the growth condition (1.2), by constructing solutions with polynomial growth.

Moreover, we mention the paper [14], where the author proved that, for any \( n \geq 2 \), a solution to (1.1) which is a local minimizer and satisfies the growth condition (1.2) has one-dimensional symmetry.

In this paper we consider a more general setting, that is we take \( F \in C^{1,1}_{loc}(\mathbb{R}^2) \), and we study the following elliptic system in \( \mathbb{R}^n \)

\[
\begin{align*}
\text{div}(a(|\nabla u|)\nabla u) &= F_1(u,v), \\
\text{div}(b(|\nabla v|)\nabla v) &= F_2(u,v),
\end{align*}
\]

(1.3)

where \( F_1 \) and \( F_2 \) denote the derivatives of \( F \) with respect to the first and the second variable respectively.

We suppose that \( a, b \in C^1((0, +\infty)) \) satisfy the following conditions:

\[
a(t) > 0, \quad b(t) > 0 \quad \text{for any } t \in (0, +\infty),
\]

(1.4)

\[
a(t) + a'(t)t > 0, \quad b(t) + b'(t)t > 0 \quad \text{for any } t \in (0, +\infty).
\]

(1.5)

We define \( A, B : \mathbb{R}^n \setminus \{0\} \rightarrow \text{Mat}(n \times n) \) by setting, for any \( 1 \leq h, k \leq n \),

\[
A_{hk}(\xi) := \frac{a'(|\xi|)}{|\xi|} \xi_h \xi_k + a(|\xi|)\delta_{hk},
\]

and

\[
B_{hk}(\xi) := \frac{b'(|\xi|)}{|\xi|} \xi_h \xi_k + b(|\xi|)\delta_{hk}.
\]

Now, for any \( t > 0 \), we introduce the following notation:

\[
\lambda_1(t) := a(t) + a'(t)t, \quad \lambda_2(t) = \ldots = \lambda_n(t) := a(t),
\]

(1.6)

\[
\gamma_1(t) := b(t) + b'(t)t, \quad \gamma_2(t) = \ldots = \gamma_n(t) := b(t),
\]

(1.7)

and we define

\[
\Lambda_i(t) := \int_0^t \lambda_i(|s|) s \, ds, \quad \Gamma_i(t) := \int_0^t \gamma_i(|s|) s \, ds
\]

for \( i = 1, 2 \) and \( t \in \mathbb{R} \).

We will require that \( a \) satisfies (A1) or (A2), where:

(A1) \( \{\nabla u = 0\} = \emptyset \) and

\[
t^2 \lambda_1(t) \in L^\infty_{loc}([0, +\infty)).
\]

(A2) We have that

\[
a \in C([0, +\infty))
\]

and

the map \( t \mapsto ta(t) \) belongs to \( C^1([0, +\infty)) \).

Moreover, we require the same properties for \( b \):

(B1) \( \{\nabla v = 0\} = \emptyset \) and

\[
t^2 \gamma_1(t) \in L^\infty_{loc}([0, +\infty)).
\]
(B2) We have that 
\[ b \in C([0, +\infty)) \]
and
\[ \text{the map } t \mapsto tb(t) \text{ belongs to } C^1([0, +\infty)). \]

In case (A2) and (B2) hold, we define \( A_{hk}(0) := a(0)\delta_{hk} \) and \( B_{hk}(0) := b(0)\delta_{hk} \).

These assumptions may look rather technical at a first glance, but they are the standard conditions that comprise as particular cases the classical elliptic degenerate and nonlinear operators, such as the p-Laplacian and the mean curvature operator.

In order to state our main result, we give the definition of monotone and stable solution.

**Definition 1.1.** We say that a solution \((u, v)\) of (1.3) satisfies a **monotonicity condition** if
\[ u_n > 0, \quad v_n < 0. \]  
(1.8)

**Definition 1.2.** When \( F \in C^2_{\text{loc}}(\mathbb{R}^2) \) we say that a solution \((u, v)\) of (1.3) is **stable** if the linearization is weakly positive definite, that is, for any \( \phi, \psi \in C^0_0(\mathbb{R}^n) \),
\[
\int_{\mathbb{R}^n} \left( A(\nabla u(x)) \nabla \phi(x) \right) \cdot \nabla \phi(x) + \left( B(\nabla v(x)) \nabla \psi(x) \right) \cdot \nabla \psi(x) + F_{11}(u, v)\phi^2(x) + F_{22}(u, v)\psi^2(x) + 2F_{12}(u, v)\phi(x)\psi(x) \, dx \geq 0. \]  
(1.9)

In our general framework, since \( F_1 \) and \( F_2 \) may not be everywhere differentiable, the integral in (1.9) may not be well defined. Therefore it is convenient to introduce the sets
\[ \mathcal{D} := \left\{ (t, s) \in \mathbb{R}^2 : F_{11}(t, s), F_{12}(t, s), F_{22}(t, s) \text{ exist} \right\}, \]
and
\[ \mathcal{N} := \mathbb{R}^2 \setminus \mathcal{D}. \]

It is known that
\[ \text{the set } \mathcal{N} \text{ is Borel and with zero Lebesgue measure} \]  
(1.10)
(see pages 81–82 in [4]). Moreover, we consider the sets
\[ \mathcal{N}_{uv} := \{ x \in \mathbb{R}^n : (u(x), v(x)) \in \mathcal{N} \}, \]
and
\[ \mathcal{D}_{uv} := \mathbb{R}^n \setminus \mathcal{N}_{uv}. \]

So we say that \((u, v)\) is a stable solution to (1.3), if for any \( \phi, \psi \in C^\infty_0(\mathbb{R}^n) \),
\[
\int_{\mathbb{R}^n} \left( A(\nabla u(x)) \nabla \phi(x) \right) \cdot \nabla \phi(x) + \left( B(\nabla v(x)) \nabla \psi(x) \right) \cdot \nabla \psi(x) \, dx + \int_{\mathcal{D}_{uv}} F_{11}(u, v)\phi^2(x) + F_{22}(u, v)\psi^2(x) + 2F_{12}(u, v)\phi(x)\psi(x) \, dx \geq 0. \]  
(1.11)

Of course, (1.11) reduces to (1.9) when \( F \) is in \( C^2_{\text{loc}}(\mathbb{R}^2) \).

Then, we state our symmetry result. For this, we denote by \( \Im(u, v) \) the image of the map \((u, v) : \mathbb{R}^n \to \mathbb{R}^2 \), i.e. \( \Im(u, v) := \{(u(x), v(x)), x \in \mathbb{R}^n\} \).

**Theorem 1.3.** Let \((u, v)\) be a solution of (1.3). Suppose that \( u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\}) \), \( v \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla v \neq 0\}) \), and \( \nabla u, \nabla v \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) \).

Suppose that either (A2) holds or that \( \{\nabla u = 0\} = \emptyset \), and that either (B2) holds or that \( \{\nabla v = 0\} = \emptyset \).

Assume that either
\[
\text{the monotonicity condition (1.8) holds, and } F_{12}(u, v) \geq 0 \text{ in } \Im(u, v), \]  
(1.12)
or
\((u,v)\) is stable, and \(F_{12}(u,v) \leq 0\) in \(\mathcal{S}(u,v)\).
\(\text{(1.13)}\)

If
\[
\liminf_{R \to +\infty} \frac{1}{\log^2 R} \int_{B_R \setminus B_{\sqrt{2}}} \frac{|A(\nabla u(x))||\nabla u(x)|^2 + |B(\nabla v(x))||\nabla v(x)|^2}{|x|^2} \, dx = 0,
\(\text{(1.14)}\)

then \((u,v)\) has one-dimensional symmetry, in the sense that there exist \(\overline{\pi}, \overline{\nu}: \mathbb{R} \to \mathbb{R}\) and \(\omega_u, \omega_v \in S^{n-1}\) in such a way that \((u(x), v(x)) = (\overline{\pi}(\omega_u \cdot x), \overline{\pi}(\omega_v \cdot x))\), for any \(x \in \mathbb{R}^n\).

Moreover, if we assume in addition that either

\(\text{the monotonicity condition (1.8) holds, and there exists a non-empty open set } \Omega' \subseteq \mathbb{R}^n \text{ such that } F_{12}(u(x), v(x)) > 0 \text{ for any } x \in \Omega', \\text{(1.15)}\)

or

\((u,v)\) is stable, and there exist two open intervals \(I_u, I_v \subseteq \mathbb{R}\)

such that \((I_u \times I_v) \cap \mathcal{S}(u,v) \neq \emptyset\) and \(F_{12}(\overline{\pi}, \overline{\nu}) > 0\) for any \((\overline{\pi}, \overline{\nu}) \in I_u \times I_v, \\text{(1.16)}\)

then \((u,v)\) has one-dimensional symmetry, and \(\omega_u = \omega_v\).

Remark 1.4. Notice that the hypothesis that \(F_{12}(u,v)\) is not identically zero cannot be removed if we want to conclude that \(u,v\) have one-dimensional symmetry with the same unit vector \(\omega\). Indeed, in \(\mathbb{R}^2\) one can consider the system in (1.3) with the Laplace operator and \(F \equiv 0\). Then, if one take the functions \(u(x_1, x_2) = x_2\) and \(v(x_1, x_2) = x_1 - x_2\), it is easy to see that \((u,v)\) is a monotone and stable solution to (1.3) and (1.12), (1.13) and (1.14) are satisfied, but \(u\) and \(v\) have one-dimensional symmetry with a different vector \(\omega\).

Notice also that one can consider a more general function \(F\) such that \(F_{12}(u,v) = 0\), that is a system with two independent equations, and there is no reason why \(u\) and \(v\) should have one-dimensional symmetry with the same vector.

We notice that, as paradigmatic examples satisfying the assumptions of Theorem 1.3, one may take the \(p\)-Laplacian, with \(p \in (1, +\infty)\) if \(\{\nabla u = 0\} = \emptyset\) and any \(p \in [2, +\infty)\) if \(\{\nabla u = 0\} \neq \emptyset\) (in this case, for instance, \(a(t) = t^{p-2}\) or the mean curvature operator (in this case, \(a(t) = (1 + t^2)^{-1/2}\)). Moreover, we observe that Theorem 1.3 holds even if \(a\) and \(b\) are two different functions satisfying the hypotheses (e.g., one can take \(a\) to be of \(p\)-Laplacian type and \(b\) of mean curvature type).

To prove Theorem 1.3 we borrow a large number of ideas from [5] and [6], and exploit some techniques of [12, 13]. In particular, we will show that a formula proved in [12, 13] and its extension obtained in [6] for elliptic equations still hold for systems (see Corollaries 3.3 and 4.4). Since this formula bounds a weighted \(L^2\)-norm of any test function by a weighted \(L^2\)-norm of its gradient, we may see it as a weighted Poincaré type inequality. Such a formula is geometric in spirit, since it bounds tangential gradients and curvatures of level sets of monotone and stable solutions in terms of suitable energy integrals.

Our result extends the one obtained in [8], where the authors studied problem (1.3) in the case \(a = b = Id\), and use this kind of geometric Poincaré inequality to show that in \(\mathbb{R}^2\) any stable solution has a one-dimensional symmetry. Of course in our setting several technical and conceptual complications arise due to the possible
of the operators considered and to the nonlinear dependence on the gradient terms.

Moreover, as a particular case, Theorem 1.3 comprises a conjecture of De Giorgi for phase separations in $\mathbb{R}^2$ (see the end of Section 7).

We refer the reader to [7] for a recent review on the conjecture of De Giorgi and related topics.

The paper is organized as follows. In Section 2 we collect some preliminary material. Sections 3 and 4 are devoted to show that some geometric Poincaré type inequalities hold for monotone and stable solutions to (1.3) respectively. In Section 5 we develop the level set analysis. In Section 6 we provide the proof of Theorem 1.3, by using the results obtained in the previous sections. Finally, in Section 7, we give an application of Theorem 1.3, namely we prove that a conjecture of De Giorgi holds in $\mathbb{R}^2$ for systems like (1.3), and in particular for phase separations.

2. Some useful results. In this section we collect some results that we will use in the sequel.

First, we have the following lemma (see Lemma 2.1 in [6] for a simple proof):

**Lemma 2.1.** For any $\xi \in \mathbb{R}^n \setminus \{0\}$, the matrices $A(\xi)$, $B(\xi)$ are symmetric and positive definite, and their eigenvalues are $\lambda_1(\xi), \ldots, \lambda_n(\xi)$ and $\gamma_1(\xi), \ldots, \gamma_n(\xi)$ respectively.

Moreover

$$A(\xi)\xi \cdot \xi = |\xi|^2 \lambda_1(\xi), \quad B(\xi)\xi \cdot \xi = |\xi|^2 \gamma_1(\xi).$$

It follows from Lemma 2.1 that, for any $t \in \mathbb{R} \setminus \{0\}$,

$$\Lambda_i(-t) = \Lambda_i(t) > 0, \quad \Gamma_i(-t) = \Gamma_i(t) > 0.$$ Moreover, for any $V, W \in \mathbb{R}^n$, and any $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$0 \leq A(\xi) (V - W) \cdot (V - W) = A(\xi) V \cdot V + A(\xi) W \cdot W - 2 A(\xi) V \cdot W, \quad (2.1)$$

$$0 \leq B(\xi) (V - W) \cdot (V - W) = B(\xi) V \cdot V + B(\xi) W \cdot W - 2 B(\xi) V \cdot W. \quad (2.2)$$

**Lemma 2.2.** Let $(u, v)$ be a weak solution of (1.3) such that $u \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla u \neq 0\})$, $v \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla v \neq 0\})$, with $\nabla u, \nabla v \in W^{1,2}_{{\text{loc}}} (\mathbb{R}^n)$. Suppose that either (A2) holds or that $\{\nabla u = 0\} = \emptyset$, and that either (B2) holds or that $\{\nabla v = 0\} = \emptyset$.

Then, for any $j = 1, \ldots, n$, $(u_j, v_j)$ is a weak solution of

$$\begin{cases}
\text{div} (A(\nabla u) \nabla u_j) = F_{11}(u, v) u_j + F_{12}(u, v) v_j, \\
\text{div} (B(\nabla v) \nabla v_j) = F_{21}(u, v) u_j + F_{22}(u, v) v_j.
\end{cases} \quad (2.3)$$

**Proof.** First of all, we observe that

the map $x \mapsto A(x) := a(|\nabla u(x)|) \nabla u(x)$ belongs to $W^{1,1}_{{\text{loc}}} (\mathbb{R}^n, \mathbb{R}^n)$, \quad (2.4)

and

the map $x \mapsto B(x) := b(|\nabla u(x)|) \nabla u(x)$ belongs to $W^{1,1}_{{\text{loc}}} (\mathbb{R}^n, \mathbb{R}^n)$. \quad (2.5)

Let us show (2.4). It is obvious if $\{\nabla u = 0\} = \emptyset$, while, if (A2) holds, we have that the map

$$\xi \in \mathbb{R}^n \mapsto A(\xi) := a(\|\xi\|) \xi$$

belongs to $W^{1,\infty}_{{\text{loc}}} (\mathbb{R}^n)$, and so (2.4) follows by writing $A(x) = A(\nabla u(x))$. In the same way one shows (2.5).
From (2.4) and (2.5), we have that, for any \( \phi, \psi \in C_0^{\infty}(\mathbb{R}^n, \mathbb{R}^n) \),
\[
- \int_{\mathbb{R}^n} \partial_j \left( a (|\nabla u|) \nabla u \right) \cdot \phi \, dx = \int_{\mathbb{R}^n} a (|\nabla u|) \nabla u \cdot \partial_j \phi \, dx,
\]
and
\[
- \int_{\mathbb{R}^n} \partial_j \left( b (|\nabla v|) \nabla v \right) \cdot \psi \, dx = \int_{\mathbb{R}^n} b (|\nabla v|) \nabla v \cdot \partial_j \psi \, dx.
\]
Moreover, by (2.4) and Stampacchia’s Theorem (see, for instance, Theorem 6.19 of [10]), we get that \( \partial_j \mathcal{A}(x) = 0 \) for almost any \( x \in \{ \mathcal{A} = 0 \} \), that is
\[
\partial_j \left( a (|\nabla u(x)|) \nabla u(x) \right) = 0
\]
for almost any \( x \in \{ \nabla u = 0 \} \).

Similarly, by using again Stampacchia’s Theorem and (A2), we conclude that
\( \nabla u_j(x) = 0 \), and then \( A (\nabla u(x)) \nabla u_j(x) = 0 \), for almost any \( x \in \{ \nabla u = 0 \} \).

A direct computation also shows that on \( \{ \nabla u \neq 0 \} \)
\[
\partial_j \left( a (|\nabla u|) \nabla u \right) = A (\nabla u) \nabla u_j.
\]
As a consequence,
\[
\partial_j \left( a (|\nabla u|) \nabla u \right) = A (\nabla u) \nabla u_j
\]
almost everywhere.

Reasoning in the same way, we conclude also that
\[
\partial_j \left( b (|\nabla v|) \nabla v \right) = B (\nabla v) \nabla v_j
\]
almost everywhere.

Let now \( \phi, \psi \in C_0^{\infty}(\mathbb{R}^n) \). We use the above observations to obtain that
\[
- \int_{\mathbb{R}^n} A (\nabla u) \nabla u_j \cdot \nabla \phi + F_{11}(u, v) u_j \phi + F_{12}(u, v) v_j \phi \, dx
= - \int_{\mathbb{R}^n} \partial_j \left( a (|\nabla u|) \nabla u \right) \cdot \nabla \phi + \partial_j (F_1(u, v)) \phi \, dx
= \int_{\mathbb{R}^n} a (|\nabla u|) \nabla u \cdot \nabla \phi_j + F_1(u, v) \phi_j \, dx,
\]
and
\[
- \int_{\mathbb{R}^n} B (\nabla v) \nabla v_j \cdot \nabla \psi + F_{21}(u, v) u_j \psi + F_{22}(u, v) v_j \psi \, dx
= - \int_{\mathbb{R}^n} \partial_j \left( b (|\nabla v|) \nabla v \right) \cdot \nabla \psi + \partial_j (F_2(u, v)) \psi \, dx
= \int_{\mathbb{R}^n} b (|\nabla v|) \nabla v \cdot \nabla \psi_j + F_2(u, v) \psi_j \, dx,
\]
which vanish, since \( (u, v) \) is a weak solution of (1.3). \( \square \)

We observe that in the proof of Lemma 2.2 it is sufficient to assume that \( \nabla u, \nabla v \in W^{1,1}_{\text{loc}}(\mathbb{R}^n) \). Since such a generality is not needed here, we assumed, for simplicity, \( \nabla u, \nabla v \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) \) in order to use the above result in the sequel.

Let us notice that (2.3) means that, for any \( \phi, \psi \in C_0^{\infty}(\mathbb{R}^n) \), and for any \( j = 1, \ldots, n \),
\[
\int_{\mathbb{R}^n} A (\nabla u) \nabla u_j \cdot \nabla \phi + F_{11}(u, v) u_j \phi + F_{12}(u, v) v_j \phi \, dx = 0,
\]
\[
\int_{\mathbb{R}^n} B (\nabla v) \nabla v_j \cdot \nabla \psi + F_{21}(u, v) u_j \psi + F_{22}(u, v) v_j \psi \, dx = 0.
\]
(2.6)
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Since the integrals in (2.6) may not be well defined, recalling the definitions of the sets $\mathcal{D}, \mathcal{N}, \mathcal{M}_w, \mathcal{P}_w$ given in the Introduction and using (1.10) we can say that $(u_j, v_j)$ satisfies

$$
\int_{\mathbb{R}^n} A(\nabla u) \nabla u_j \cdot \nabla \phi dx + \int_{\mathcal{P}_w} F_{11}(u, v) u_j \phi + F_{12}(u, v) v_j \phi dx = 0,
$$

$$
\int_{\mathbb{R}^n} B(\nabla v) \nabla v_j \cdot \nabla \psi dx + \int_{\mathcal{P}_w} F_{12}(u, v) u_j \psi + F_{22}(u, v) v_j \psi dx = 0. \tag{2.7}
$$

In the sequel we will need to use (2.7) for a less regular test functions. To do this, we prove the following:

**Lemma 2.3.** Under the assumptions of Lemma 2.2, we have that (2.7) holds for any $j = 1, \ldots, n$, any $\phi, \psi \in W^{1,2}_0(B)$ and any ball $B \subset \mathbb{R}^n$.

**Proof.** Let us prove the first equality in (2.7). Given $\phi \in W^{1,2}_0(B)$, we consider a sequence of functions $\phi_k \in C^\infty_0(B)$ which converge to $\phi$ in $W^{1,2}_0(B)$. Let $m_n$ and $M_n$ (respectively $m_v$ and $M_v$) be the minimum and the maximum of $|\nabla u|$ (respectively $|\nabla v|$) on the closure of $B$. Moreover, let

$$
K_A := \sup_{m_n \leq |\xi| \leq M_n} |A(\xi)|, \quad K_B := \sup_{m_v \leq |\xi| \leq M_v} |B(\xi)|.
$$

Notice that $K_A < +\infty$, since $0 \leq m_u \leq M_u < +\infty$; in fact, if $\{\nabla u = 0\} = \emptyset$, then $m_u > 0$, whereas, if (A2) holds, then $A \in L^\infty_{\text{loc}}(\mathbb{R}^n)$. In the same way, one has that also $K_B < +\infty$.

Now, since the assumptions of Lemma 2.2 hold, we deduce from (2.7)

$$
\int_{\mathbb{R}^n} A(|\nabla u|) \nabla u_j \cdot \nabla \phi_k dx + \int_{\mathcal{P}_w} F_{11}(u, v) u_j \phi_k + F_{12}(u, v) v_j \phi_k dx = 0. \tag{2.8}
$$

Also,

$$
\left| \int_{\mathbb{R}^n} A(|\nabla u|) \nabla u_j \cdot (\nabla \phi_k - \nabla \phi) \ dx \right|
\leq K_A \left( \int_B |\nabla u_j|^2 \ dx \right)^{1/2} \left( \int_B |\nabla (\phi_k - \phi)|^2 \ dx \right)^{1/2}
+ \left( \int_{B \cap \mathcal{P}_w} |F_{11}(u, v) u_j|^2 \ dx \right)^{1/2} \left( \int_{B \cap \mathcal{P}_w} |\nabla (\phi_k - \phi)|^2 \ dx \right)^{1/2}
+ \left( \int_{B \cap \mathcal{P}_w} |F_{12}(u, v) v_j|^2 \ dx \right)^{1/2} \left( \int_{B \cap \mathcal{P}_w} |\nabla (\phi_k - \phi)|^2 \ dx \right)^{1/2},
$$

which tends to zero as $k$ tends to infinity, because of the assumptions on $u, v$. The latter consideration and (2.8) give the first equality in (2.7). Reasoning in a similar way, we obtain also the second equality in (2.7). \qed

We will now consider the tangential gradient with respect to a regular level set. Given $w \in C^1(\mathbb{R}^n)$, we define the level set of $w$ at $x$ as

$$
L_{w,x} := \{ y \in \mathbb{R}^n \text{ s. t. } w(y) = w(x) \}. \tag{2.9}
$$

If $\nabla w(x) \neq 0$, $L_{w,x}$ is a hypersurface near $x$ and one can consider the projection of any vector onto the tangent plane: in particular, the tangential gradient, which
will be denoted as $\nabla_{L_{w,x}}$, is the projection of the gradient. This means that, given $f \in C^1 (B_r(x))$, for $r > 0$, the tangential gradient is

$$\nabla_{L_{w,x}} f(x) := \nabla f(x) - \left( \nabla f(x) \cdot \frac{\nabla w(x)}{|\nabla w(x)|} \right) \frac{\nabla w(x)}{|\nabla w(x)|}. \quad (2.10)$$

We will use the following lemma (see Lemma 2.3 in [6] for a simple proof):

**Lemma 2.4.** Let $U \subseteq \mathbb{R}^n$ be an open set, $w \in C^2 (U)$ and $x \in U$ such that $\nabla w(x) \neq 0$. Then

$$a (|\nabla w(x)|) \left[ |\nabla|\nabla w(x)|^2 - \sum_{j=1}^n |\nabla w_j(x)|^2 \right]$$

$$= (A (\nabla w(x)) (\nabla|\nabla w(x)|) \cdot (\nabla|\nabla w(x)|) - (A (\nabla w(x)) \nabla w_j(x)) \cdot \nabla w_j(x),$$

and

$$b (|\nabla w(x)|) \left[ |\nabla|\nabla w(x)|^2 - \sum_{j=1}^n |\nabla w_j(x)|^2 \right]$$

$$= (B (\nabla w(x)) (\nabla|\nabla w(x)|) \cdot (\nabla|\nabla w(x)|) - (B (\nabla w(x)) \nabla w_j(x)) \cdot \nabla w_j(x).$$

Given $y \in L_{w,x} \cap \{ \nabla w \neq 0 \}$, let $k_1, w(y), ..., k_{n-1}, w(y)$ denote the principal curvatures of $L_{w,x}$ at $y$.

By using formula (2.1) of [12], tangential gradients and curvatures may be conveniently related in the following way:

$$\sum_{j=1}^n |\nabla w_j(x)|^2 - |\nabla_{L_{w,x}} \nabla w(x)|^2 - |\nabla \nabla w(x)|^2 = |\nabla w(x)|^2 \sum_{l=1}^{n-1} k_{l,w}, \quad (2.11)$$

on $\{ \nabla w \neq 0 \}$, for any $w \in C^2 (\{ \nabla w \neq 0 \})$.

3. **Monotone solutions.** Recalling the definition of monotone solution given in (1.8), in this section we obtain some geometric inequalities.

**Proposition 3.1.** Let $\Omega \subseteq \mathbb{R}^n$ be open (not necessarily bounded). Let $(u, v)$ be a solution of (1.3), with $u, v \in C^2 (\Omega)$, and $\nabla u, \nabla v \in W^{1,2}_{\text{loc}} (\Omega)$. Suppose that the monotonicity condition (1.8) holds.

Then,

$$\int_{\Omega} (A (\nabla u(x)) \nabla \phi(x) \cdot \nabla \phi(x) \, dx$$

$$+ \int_{\Omega \cap \partial u} F_{11}(u, v) \phi^2(x) + F_{12}(u, v) \frac{\partial^2 \phi^2}{\partial u} \phi^2(x) \, dx \geq 0,$$

and

$$\int_{\Omega} (B (\nabla v(x)) \nabla \psi(x) \cdot \nabla \psi(x) \, dx$$

$$+ \int_{\Omega \cap \partial v} F_{12}(u, v) \frac{\partial \psi^2}{\partial u} + F_{22}(u, v) \psi^2(x) \, dx \geq 0,$$

for any locally Lipschitz functions $\phi, \psi : \Omega \to \mathbb{R}$ whose supports are compact and contained in $\Omega$. 
Proof. By Lemma 2.3, we have that $u_n$ satisfies (2.7). We use $\phi^2_{u_n}$ as test function in the first equality in (2.7):

$$
\int_{\Omega \cap \Omega_u} F_{11}(u, v) \phi^2 + F_{12}(u, v) \frac{u_n}{u_n} \phi^2 \, dx
$$

$$
= - \int_{\Omega} (A(\nabla u) \nabla u_n) \cdot \nabla \left( \frac{\phi^2}{u_n} \right) \, dx
$$

$$
= - \int_{\Omega} (A(\nabla u) \nabla u_n) \cdot \left( \frac{2 \phi \nabla \phi u_n - \phi^2 \nabla u_n}{u_n^2} \right) \, dx
$$

$$
+ \int_{\Omega} (A(\nabla u) \nabla \phi) \cdot \nabla \phi - (A(\nabla u) \nabla \phi) \cdot \nabla \phi \, dx
$$

$$
= \int_{\Omega} A(\nabla u) \left( \nabla u_n \frac{\phi}{u_n} - \nabla \phi \right) \cdot \left( \nabla u_n \frac{\phi}{u_n} - \nabla \phi \right) - (A(\nabla u) \nabla \phi) \cdot \nabla \phi \, dx
$$

$$
\geq - \int_{\Omega} (A(\nabla u) \nabla \phi) \cdot \nabla \phi \, dx,
$$

since (2.1) holds. This implies the first inequality in (3.1).

Using $\psi^2_{u_n}$ as test function in the second equality in (2.7), and reasoning as above, we obtain also the second inequality in (3.1). □

In the subsequent Theorem 3.2 and Corollary 3.3 we obtain some inequalities which involve the principal curvature of the level sets and the tangential gradient of the solution.

**Theorem 3.2.** Let $\Omega \subseteq \mathbb{R}^n$ be open (not necessarily bounded). Let $(u, v)$ be a weak solution of (1.3), with $u, v \in C^2(\Omega)$, and $\nabla u, \nabla v \in W^{1,2}_0(\Omega)$. Suppose that the monotonicity condition (1.8) holds.

For any $x \in \Omega$ let $L_u, x$ and $L_v, x$ denote the level set of $u$ and $v$ respectively at $x$, according to (2.9).

Let also $\lambda_1(\|\xi\|), \lambda_2(\|\xi\|), \gamma_1(\|\xi\|), \gamma_2(\|\xi\|)$ be as in (1.6) and (1.7).

Then,

$$
\int_{\Omega} \left[ \lambda_1(\|\nabla u(x)\|) |\nabla_{L_u,x}| \nabla u(x)|^2 + \lambda_2(\|\nabla u(x)\|) |\nabla u(x)|^2 \sum_{l=1}^{n-1} k^2_{l,u} \right] \phi^2(x) \, dx
$$

$$
\leq \int_{\Omega} |\nabla u(x)|^2 (A(\nabla u(x)) \nabla \phi(x)) \cdot \nabla \phi(x) \, dx
$$

$$
+ \int_{\Omega} F_{12}(u, v) \left( \frac{u_n}{u_n} |\nabla u(x)|^2 - \nabla u(x) \cdot \nabla v(x) \right) \phi^2(x) \, dx,
$$

and

$$
\int_{\Omega} \left[ \gamma_1(\|\nabla v(x)\|) |\nabla_{L_v,x}| \nabla v(x)|^2 + \gamma_2(\|\nabla v(x)\|) |\nabla v(x)|^2 \sum_{l=1}^{n-1} k^2_{l,v} \right] \psi^2(x) \, dx
$$

$$
\leq \int_{\Omega} |\nabla v(x)|^2 (B(\nabla v(x)) \nabla \phi(x)) \cdot \nabla \psi(x) \, dx
$$

$$
+ \int_{\Omega} F_{12}(u, v) \left( \frac{v_n}{v_n} |\nabla v(x)|^2 - \nabla u(x) \cdot \nabla v(x) \right) \psi^2(x) \, dx,
$$
for any locally Lipschitz functions $\phi, \psi : \Omega \to \mathbb{R}$ whose supports are compact and contained in $\Omega$.

**Proof.** We prove first (3.2). By using the first inequality in (3.1) with $|\nabla u|\phi$ as test function, we have that

$$0 \leq \int_{\Omega} (A(\nabla u(x)) \nabla (|\nabla u(x)|\phi(x))) \cdot \nabla (|\nabla u(x)|\phi(x)) \, dx$$

$$+ \int_{\Omega \cap \partial u} F_{11}(u, v)|\nabla u(x)|^2\phi^2(x) + F_{12}(u, v) \frac{v_n}{u_n} |\nabla u(x)|^2 \phi^2(x) \, dx$$

$$= \int_{\Omega} \phi^2(x) (A(\nabla u(x)) \nabla (|\nabla u(x)|)) \cdot \nabla (|\nabla u(x)|)$$

$$+ |\nabla u(x)|^2 (A(\nabla u(x)) \nabla \phi(x)) \cdot \nabla \phi(x)$$

$$+ \frac{1}{2} (A(\nabla u(x)) \nabla (\phi^2(x))) \cdot \nabla (|\nabla u(x)|^2) \, dx$$

$$+ \int_{\Omega \cap \partial u} F_{11}(u, v)|\nabla u(x)|^2\phi^2(x) + F_{12}(u, v) \frac{v_n}{u_n} |\nabla u(x)|^2 \phi^2(x) \, dx. \quad (3.4)$$

Now, since Lemma 2.3 holds, we can use $u_j \phi^2$ as test function in the first equality in (2.7):

$$\int_{\Omega \cap \partial u} F_{11}(u, v)u_j^2(x)\phi^2(x) + F_{12}(u, v)u_j(x)u_j(x)\phi^2(x) \, dx$$

$$= - \int_{\Omega} (A(\nabla u(x)) \nabla u_j(x)) \cdot \nabla (u_j(x)\phi^2(x)) \, dx$$

$$= - \int_{\Omega} \phi^2(x) (A(\nabla u(x)) \nabla u_j(x)) \cdot \nabla u_j(x)$$

$$+ \frac{1}{2} (A(\nabla u(x)) \nabla (\phi^2(x))) \cdot \nabla (u_j^2(x)) \, dx.$$

We sum over $j$ and use (3.4) to see that

$$\int_{\Omega} \phi^2(x) \sum_{j=1}^{n} (A(\nabla u(x)) \nabla u_j(x)) \cdot \nabla u_j(x)$$

$$+ \frac{1}{2} (A(\nabla u(x)) \nabla (\phi^2(x))) \cdot \nabla (|\nabla u|^2) \, dx$$

$$= - \int_{\Omega \cap \partial u} F_{11}(u, v)|\nabla u(x)|^2\phi^2(x) + F_{12}(u, v)\nabla u(x) \cdot \nabla (u_j^2(x)) \, dx$$

$$\leq \int_{\Omega} \phi^2(x) (A(\nabla u(x)) \nabla (|\nabla u(x)|)) \cdot \nabla (|\nabla u(x)|)$$

$$+ |\nabla u(x)|^2 (A(\nabla u(x)) \nabla \phi(x)) \cdot \nabla \phi(x)$$

$$+ \frac{1}{2} (A(\nabla u(x)) \nabla (\phi^2(x))) \cdot \nabla (|\nabla u(x)|^2) \, dx$$

$$+ \int_{\Omega \cap \partial u} F_{11}(u, v)|\nabla u(x)|^2\phi^2(x) + F_{12}(u, v) \frac{v_n}{u_n} |\nabla u(x)|^2 \phi^2(x) \, dx$$

$$- \int_{\Omega \cap \partial u} F_{11}(u, v)|\nabla u(x)|^2\phi^2(x) + F_{12}(u, v)\nabla u(x) \cdot \nabla (u_j^2(x)) \, dx.$$
\[
\begin{align*}
&= \int_{\Omega} \phi^2(x) \left( A (\nabla u(x)) \nabla (|\nabla u(x)|) \cdot \nabla (|\nabla u(x)|) \\
&\quad + |\nabla u(x)|^2 \left( A (\nabla u(x)) \nabla \phi(x) \right) \cdot \nabla (\phi(x)) \\
&\quad + \frac{1}{2} \left( A (\nabla u(x)) \nabla (\phi^2(x)) \right) \cdot \nabla (|\nabla u(x)|^2) \right) \, dx \\
&\quad + \int_{\Omega \cap D^v} F_{12}(u,v) \left( \frac{v_n}{u_n} |\nabla u(x)|^2 - \nabla u(x) \cdot \nabla v(x) \right) \phi^2(x) \, dx.
\end{align*}
\]

By Lemma 2.4, we have that
\[
\int_{\Omega} \phi^2(x) \, a(|\nabla u(x)|)|\nabla u(x)||\nabla_{L_{u,v}}|\nabla u(x)||^2 \phi^2(x) \, dx \\
- \phi^2(x) a(|\nabla u(x)|) \left( |\nabla \nabla u(x)|^2 - \sum_{j=1}^{n} |\nabla u_j(x)|^2 \right) \, dx \\
\leq \int_{\Omega} |\nabla u(x)|^2 \left( A (\nabla u(x)) \nabla \phi(x) \right) \cdot \nabla (\phi(x)) \, dx \\
+ \int_{\Omega \cap D^v} F_{12}(u,v) \left( \frac{v_n}{u_n} |\nabla u(x)|^2 - \nabla u(x) \cdot \nabla v(x) \right) \phi^2(x) \, dx.
\]

That is, by using (1.6),
\[
\int_{\Omega} \phi^2(x) \left( |\nabla u(x)| \right) |\nabla_{L_{u,v}}|\nabla u(x)||^2 \, dx \\
+ \phi^2(x) \lambda_2(|\nabla u(x)|) \left( \sum_{j=1}^{n} |\nabla u_j(x)|^2 - |\nabla_{L_{u,v}}|\nabla u(x)||^2 \right) \, dx \\
\leq \int_{\Omega} |\nabla u(x)|^2 \left( A (\nabla u(x)) \nabla \phi(x) \right) \cdot \nabla (\phi(x)) \, dx \\
+ \int_{\Omega \cap D^v} F_{12}(u,v) \left( \frac{v_n}{u_n} |\nabla u(x)|^2 - \nabla u(x) \cdot \nabla v(x) \right) \phi^2(x) \, dx.
\]

Notice that (1.10) and Theorem 6.19 of [10] give that
\[
\nabla u = 0 = \nabla v \text{ almost everywhere on } N_{uc},
\]
and therefore
\[
\int_{\Omega \cap D^v} F_{12}(u,v) \left( \frac{v_n}{u_n} |\nabla u(x)|^2 - \nabla u(x) \cdot \nabla v(x) \right) \phi^2(x) \, dx \\
= \int_{\Omega} F_{12}(u,v) \left( \frac{v_n}{u_n} |\nabla u(x)|^2 - \nabla u(x) \cdot \nabla v(x) \right) \phi^2(x) \, dx.
\]

This and (2.11) imply the desired result.

Arguing in a similar way we obtain also (3.3). \( \square \)

**Corollary 3.3.** Let \( \Omega \subseteq \mathbb{R}^n \) be open (not necessarily bounded). Let \((u,v)\) be a weak solution of (1.3), with \(u,v\in C^2(\Omega)\), and \(\nabla u, \nabla v \in W_{loc}^{1,2}(\Omega)\). Suppose that the monotonicity condition (1.8) holds and that \(F_{12}(u,v)\geq 0\).

For any \(x\in \Omega\) let \(L_{u,x}\) and \(L_{v,x}\) denote the level set of \(u\) and \(v\) respectively at \(x\), according to (2.9).

Let also \(\lambda_1(|\xi|), \lambda_2(|\xi|), \gamma_1(|\xi|), \gamma_2(|\xi|)\) be as in (1.6) and (1.7).
Proof. By summing up the inequalities in (3.2) and (3.3), we have that, for any \( \varphi \) as in the corollary,

\[
\int_{\Omega} \left[ \lambda_1 (|\nabla u(x)|) |\nabla_{L,x} \nabla u(x)|^2 + \lambda_2 (|\nabla u(x)|) |\nabla u(x)|^2 \sum_{l=1}^{n-1} k_{l,u}^2 \right] \varphi^2(x) \, dx
\]

\[
+ \int_{\Omega} \left[ \gamma_1 (|\nabla v(x)|) |\nabla_{L,x} \nabla v(x)|^2 + \gamma_2 (|\nabla v(x)|) |\nabla v(x)|^2 \sum_{l=1}^{n-1} k_{l,v}^2 \right] \varphi^2(x) \, dx
\]

\[
\leq \int_{\Omega} |\nabla u(x)|^2 (A (\nabla u(x)) \nabla \varphi(x)) \cdot \nabla \varphi(x) \, dx
\]

\[
+ \int_{\Omega} |\nabla v(x)|^2 (B (\nabla v(x)) \nabla \varphi(x)) \cdot \nabla \varphi(x) \, dx
\]

\[
+ \int_{\Omega} F_{12}(u,v) \left( \frac{u_n}{u} |\nabla u(x)|^2 - 2\nabla u(x) \cdot \nabla v(x) + \frac{u_n}{v_n} |\nabla v(x)|^2 \right) \varphi^2(x) \, dx
\]

\[
= \int_{\Omega} |\nabla u(x)|^2 (A (\nabla u(x)) \nabla \varphi(x)) \cdot \nabla \varphi(x) \, dx
\]

\[
+ \int_{\Omega} |\nabla v(x)|^2 (B (\nabla v(x)) \nabla \varphi(x)) \cdot \nabla \varphi(x) \, dx
\]

\[
- \int_{\Omega} F_{12}(u,v) \left( \frac{-v_n}{u_n} |\nabla u(x)|^2 + 2\nabla u(x) \cdot \nabla v(x) + \frac{u_n}{-v_n} |\nabla v(x)|^2 \right) \varphi^2(x) \, dx
\]

\[
= \int_{\Omega} |\nabla u(x)|^2 (A (\nabla u(x)) \nabla \varphi(x)) \cdot \nabla \varphi(x) \, dx
\]

\[
+ \int_{\Omega} |\nabla v(x)|^2 (B (\nabla v(x)) \nabla \varphi(x)) \cdot \nabla \varphi(x) \, dx
\]

\[
- \int_{\Omega} F_{12}(u,v) \left( \frac{-v_n}{u_n} \nabla u(x) + \frac{u_n}{-v_n} \nabla v(x) \right)^2 \varphi^2(x) \, dx,
\]

which gives the conclusion, since \( F_{12}(u,v) \geq 0 \).

4. Stable solutions. In this section we obtain some geometric inequalities for stable solutions of (1.3). Since we will use the stability condition (1.11) with a less regular test functions, we need to state the following:
Lemma 4.1. Let \((u, v)\) be a stable weak solution of (1.3) such that \(u, v \in C^1(\mathbb{R}^n)\). Suppose that either (A2) holds or that \(\{\nabla u = 0\} = \emptyset\), and that either (B2) holds or that \(\{\nabla v = 0\} = \emptyset\). Then, the stability condition (1.11) holds for any \(\phi, \psi \in W^{1,2}_0(B)\), and any ball \(B \subset \mathbb{R}^n\).

Proof. As in the proof of Lemma 2.3, we introduce \(m_u, M_u, m_v, M_v, K_A, K_B\), and notice that, under the hypotheses of Lemma 4.1, \(K_A, K_B < +\infty\). Moreover, given \(\phi, \psi \in W^{1,2}_0(B)\), we consider two sequences \(\phi_k, \psi_k \in C_0^\infty(B)\) which converge to \(\phi, \psi\) respectively in \(W^{1,2}_0(B)\).

Therefore,
\[
\left| \int_{\mathbb{R}^n} (A(\nabla u) \nabla \phi_k) \cdot \nabla \phi_k \, dx - \int_{\mathbb{R}^n} (A(\nabla u) \nabla \phi) \cdot \nabla \phi \, dx \right| \\
\leq \int_B |A(\nabla u)| |\nabla (\phi_k - \phi)||\nabla \phi_k| + |A(\nabla u)||\nabla (\phi_k - \phi)| \, dx \\
\leq K_A \left( \int_B |\nabla (\phi_k - \phi)|^2 \, dx \right)^{1/2} \left[ \left( \int_B |\nabla \phi_k|^2 \, dx \right)^{1/2} + \left( \int_B |\nabla \phi|^2 \, dx \right)^{1/2} \right],
\]
which tends to zero as \(k\) tends to infinity.

Similarly, one obtains
\[
\left| \int_{\mathbb{R}^n} (B(\nabla v) \nabla \psi_k) \cdot \nabla \psi_k \, dx - \int_{\mathbb{R}^n} (B(\nabla v) \nabla \psi) \cdot \nabla \psi \, dx \right| \\
\leq K_B \left( \int_B |\nabla (\psi_k - \psi)|^2 \, dx \right)^{1/2} \left[ \left( \int_B |\nabla \psi_k|^2 \, dx \right)^{1/2} + \left( \int_B |\nabla \psi|^2 \, dx \right)^{1/2} \right],
\]
which again tends to zero.

Moreover, one has that, as \(k\) tends to infinity,
\[
\int_{\mathcal{D}_{uv}} F_{11}(u, v) \phi_k^2 \, dx \to \int_{\mathcal{D}_{uv}} F_{11}(u, v) \phi^2 \, dx,
\]
and
\[
\int_{\mathcal{D}_{uv}} F_{22}(u, v) \psi_k^2 \, dx \to \int_{\mathcal{D}_{uv}} F_{22}(u, v) \psi^2 \, dx.
\]

Finally,
\[
\left| \int_{\mathcal{D}_{uv}} F_{12}(u, v) \phi_k \psi_k \, dx - \int_{\mathcal{D}_{uv}} F_{12}(u, v) \phi \psi \, dx \right| \\
\leq C \int_B |\phi_k \psi_k - \phi \psi| \, dx \\
\leq C \int_B |\phi_k| |\psi_k - \psi| + |\psi| |\phi_k - \phi| \, dx \\
\leq C \left( \int_B |\phi_k|^2 \, dx \right)^{1/2} \left( \int_B |\psi_k - \psi|^2 \, dx \right)^{1/2} \\
+ \left( \int_B |\psi|^2 \, dx \right)^{1/2} \left( \int_B |\phi_k - \phi|^2 \, dx \right)^{1/2},
\]
which converges to zero as \(k\) tends to infinity. This concludes the proof.

We prove next that, under suitable assumptions, a monotone solution of (1.3) is also stable.
Proposition 4.2. Let \((u,v)\) be a weak solution of (1.3), with \(u, v \in C^2(\mathbb{R}^n)\), and \(\nabla u, \nabla v \in W^{1,2}_{\text{loc}}(\mathbb{R}^n)\). Suppose that the monotonicity condition (1.8) holds, and that \(F_{12}(u,v) \geq 0\). Then \((u,v)\) is a stable solution.

Proof. By summing up the inequalities in (3.1), we have

\[
0 \leq \int_{\mathbb{R}^n} \left( A (\nabla u(x)) \nabla \phi(x) \right) \cdot \nabla \phi(x) + (B (\nabla v(x)) \nabla \psi(x)) \cdot \nabla \psi(x) \, dx
\]

\[
+ \int_{\mathcal{G}_{u,v}} F_{11}(u,v) \phi^2(x) + F_{22}(u,v) \psi^2(x)
\]

\[
+ F_{12}(u,v) \left( \frac{u_n}{v_n} \phi^2(x) + \frac{u_n}{v_n} \psi^2(x) \right) \, dx
\]

\[
= \int_{\mathbb{R}^n} \left( A (\nabla u(x)) \nabla \phi(x) \right) \cdot \nabla \phi(x) + (B (\nabla v(x)) \nabla \psi(x)) \cdot \nabla \psi(x) \, dx
\]

\[
+ \int_{\mathcal{G}_{u,v}} F_{11}(u,v) \phi^2(x) + F_{22}(u,v) \psi^2(x)
\]

\[
- F_{12}(u,v) \left( \frac{-v_n}{u_n} \phi^2(x) + \frac{u_n}{-v_n} \psi^2(x) \right) \, dx
\]

\[
\leq \int_{\mathbb{R}^n} \left( A (\nabla u(x)) \nabla \phi(x) \right) \cdot \nabla \phi(x) + (B (\nabla v(x)) \nabla \psi(x)) \cdot \nabla \psi(x) \, dx
\]

\[
+ \int_{\mathcal{G}_{u,v}} F_{11}(u,v) \phi^2(x) + F_{22}(u,v) \psi^2(x) + 2F_{12}(u,v) \phi(x) \psi(x) \, dx,
\]

where we have used the monotonicity condition, the fact that \(F_{12}(u,v) \geq 0\), together with

\[
0 \leq \left( \sqrt{-\frac{-v_n}{u_n}} \phi(x) + \sqrt{-\frac{u_n}{-v_n}} \psi(x) \right)^2 = -\frac{v_n}{u_n} \phi^2(x) + 2\phi(x)\psi(x) + \frac{u_n}{-v_n} \psi^2(x).
\]

This concludes the proof. \(\square\)

In the subsequents Theorem 4.3 and Corollary 4.4, we prove that a formula obtained in [12, 13] and its extension obtained in [6] hold also for a system of the form (1.3). These formulas relate the stability of the system with the principal curvatures of the corresponding level sets and with the tangential gradient of the solution.

Theorem 4.3. Let \(\Omega \subseteq \mathbb{R}^n\) be open (not necessarily bounded). Let \((u,v)\) be a stable weak solution of (1.3), with \(u \in C^1(\Omega) \cap C^2(\Omega \cap \{\nabla u \neq 0\})\), \(v \in C^1(\Omega) \cap C^2(\Omega \cap \{\nabla v \neq 0\})\), and \(\nabla u, \nabla v \in W^{1,2}_{\text{loc}}(\Omega)\). Suppose that either (A2) holds or that \(\{\nabla u = 0\} = \emptyset\), and that either (B2) holds or that \(\{\nabla v = 0\} = \emptyset\).

For any \(x \in \Omega \) let \(L_{u,x}\) and \(L_{v,x}\) denote the level set of \(u\) and \(v\) respectively at \(x\), according to (2.9).

Let also \(\lambda_1(\varepsilon), \lambda_2(\varepsilon), \gamma_1(\varepsilon), \gamma_2(\varepsilon)\) be as in (1.6) and (1.7).
Then,
\[
\int_{\Omega\setminus\{\nabla u \neq 0\}} \left[ \lambda_1 \left( |\nabla u(x)| \right) |\nabla_{L_u,x} \nabla u(x)|^2 \right.
+ \lambda_2 \left( |\nabla u(x)| \right) |\nabla u(x)|^2 \sum_{l=1}^{n-1} k_l^2 u_l^2 \right] \varphi^2(x) \, dx \\
+ \int_{\Omega\setminus\{\nabla v \neq 0\}} \left[ \gamma_1 \left( |\nabla v(x)| \right) |\nabla_{L_v,x} \nabla v(x)|^2 \right.
+ \gamma_2 \left( |\nabla v(x)| \right) |\nabla v(x)|^2 \sum_{l=1}^{n-1} k_l^2 v_l^2 \right] \varphi^2(x) \, dx \\
\leq \int_{\Omega} |\nabla u(x)|^2 \left( A(\nabla u(x)) \nabla \varphi(x) \right) \cdot \nabla \varphi(x) \, dx \\
+ \int_{\Omega} |\nabla v(x)|^2 \left( B(\nabla v(x)) \nabla \varphi(x) \right) \cdot \nabla \varphi(x) \, dx \\
+ 2 \int_{\Omega} F_{12}(u,v) \left( |\nabla u(x)| |\nabla v(x)| - \nabla u(x) \cdot \nabla v(x) \right) \varphi^2(x) \, dx,
\]
for any locally Lipschitz function \( \varphi : \Omega \to \mathbb{R} \) whose support is compact and contained in \( \Omega \).

Proof. Since the maps \( x \mapsto u_j(x) \) and \( x \mapsto |\nabla u(x)| \) belong to \( W^{1,2}_{loc}(\mathbb{R}^n) \), by using Stampacchia’s Theorem (see Theorem 6.19 in [10]) we have that
\[
|\nabla u| = 0 \text{ almost everywhere on } \{ |\nabla u| = 0 \}
\]
and, for any \( j = 1, \ldots, n \),
\[
\nabla u_j = 0 \text{ almost everywhere on } \{ |\nabla u| = 0 \} \subseteq \{ u_j = 0 \}.
\]

Now, we take \( \phi = u_j \varphi^2 \) in the first equality in (2.7) and we sum over \( j \) to obtain
\[
\sum_j \int_{\mathbb{R}^n} (A(\nabla u) \nabla u_j) \cdot \nabla (u_j \varphi^2) \, dx \\
+ \int_{\mathcal{S}_{uv}} F_{11}(u,v) |\nabla u|^2 \varphi^2 + F_{12}(u,v) \nabla u \cdot \nabla v \varphi^2 \, dx = 0.
\]
(4.1)

Notice that (1.10) and Theorem 6.19 of [10] give that
\[
\nabla u = 0 = \nabla v \text{ almost everywhere on } \mathcal{N}_{uv},
\]
and therefore
\[
\int_{\mathcal{S}_{uv}} F_{11}(u,v) |\nabla u|^2 \varphi^2 + F_{12}(u,v) \nabla u \cdot \nabla v \varphi^2 \, dx \\
= \int_{\mathbb{R}^n} F_{11}(u,v) |\nabla u|^2 \varphi^2 + F_{12}(u,v) \nabla u \cdot \nabla v \varphi^2 \, dx.
\]

Taking \( \psi = v_j \varphi^2 \) in the second equality in (2.7) and summing over \( j \), we obtain
\[
\sum_j \int_{\mathbb{R}^n} (B(\nabla v) \nabla v_j) \cdot \nabla (v_j \varphi^2) \, dx \\
+ \int_{\mathbb{R}^n} F_{12}(u,v) \nabla u \cdot \nabla v \varphi^2 + F_{22}(u,v) |\nabla v|^2 \varphi^2 \, dx = 0.
\]
(4.2)
Now, we exploit the stability condition (1.11) with $\phi = |\nabla u|\varphi$ and $\psi = |\nabla v|\varphi$. Note that this choice is possible, thanks to Lemma 4.1, and gives

$$0 \leq \int_{\mathbb{R}^n} (A(\nabla u)\nabla (|\nabla u|\varphi)) \cdot \nabla (|\nabla u|\varphi) + (B(\nabla v)\nabla (|\nabla v|\varphi)) \cdot \nabla (|\nabla v|\varphi)$$

$$+ F_{11}(u,v)|\nabla u|^2\varphi^2 + F_{22}(u,v)|\nabla v|^2\varphi^2 + 2F_{12}(u,v)|\nabla u||\nabla v|^2 \, dx$$

$$= \int_{\mathbb{R}^n} |\nabla u|^2 (A(\nabla u)\nabla \varphi) \cdot \nabla \varphi + |\nabla v|^2 (B(\nabla v)\nabla \varphi) \cdot \nabla \varphi$$

$$+ \varphi^2 (A(\nabla u)\nabla |\nabla u|) \cdot \nabla |\nabla u| + \varphi^2 (B(\nabla v)\nabla |\nabla v|) \cdot \nabla |\nabla v|$$

$$+ 2\varphi |\nabla u| (A(\nabla u)\nabla \varphi) \cdot \nabla |\nabla u| + 2\varphi |\nabla v| (B(\nabla v)\nabla \varphi) \cdot \nabla |\nabla v|$$

$$+ F_{11}(u,v)|\nabla u|^2\varphi^2 + F_{22}(u,v)|\nabla v|^2\varphi^2 + 2F_{12}(u,v)|\nabla u||\nabla v|^2 \, dx. \quad (4.3)$$

By using (4.1) and (4.2) in (4.3), we get

$$0 \leq \int_{\mathbb{R}^n} |\nabla u|^2 (A(\nabla u)\nabla \varphi) \cdot \nabla \varphi + |\nabla v|^2 (B(\nabla v)\nabla \varphi) \cdot \nabla \varphi \, dx$$

$$+ \int_{\{\nabla u \neq 0\}} \varphi^2 (A(\nabla u)\nabla |\nabla u|) \cdot \nabla |\nabla u| \, dx$$

$$+ \int_{\{\nabla v \neq 0\}} \varphi^2 (B(\nabla v)\nabla |\nabla v|) \cdot \nabla |\nabla v| \, dx$$

$$+ \int_{\{\nabla u \neq 0\}} 2\varphi |\nabla u| (A(\nabla u)\nabla \varphi) \cdot \nabla |\nabla u| - \sum_j (A(\nabla u)\nabla u_j) \cdot (u_j\varphi^2) \, dx$$

$$+ \int_{\{\nabla v \neq 0\}} 2\varphi |\nabla v| (B(\nabla v)\nabla \varphi) \cdot \nabla |\nabla v| - \sum_j (B(\nabla v)\nabla v_j) \cdot (v_j\varphi^2) \, dx$$

$$+ \int_{\mathbb{R}^n} 2F_{12}(u,v) (|\nabla u||\nabla v| - \nabla u \cdot \nabla v) \varphi^2 \, dx$$

$$= \int_{\mathbb{R}^n} |\nabla u|^2 (A(\nabla u)\nabla \varphi) \cdot \nabla \varphi + |\nabla v|^2 (B(\nabla v)\nabla \varphi) \cdot \nabla \varphi \, dx$$

$$+ \int_{\{\nabla u \neq 0\}} \varphi^2 \left[ (A(\nabla u)\nabla |\nabla u|) \cdot \nabla |\nabla u| - \sum_j (A(\nabla u)\nabla u_j) \cdot u_j \right] \, dx$$

$$+ \int_{\{\nabla v \neq 0\}} \varphi^2 \left[ (B(\nabla v)\nabla |\nabla v|) \cdot \nabla |\nabla v| - \sum_j (B(\nabla v)\nabla v_j) \cdot v_j \right] \, dx$$

$$+ \int_{\mathbb{R}^n} 2F_{12}(u,v) (|\nabla u||\nabla v| - \nabla u \cdot \nabla v) \varphi^2 \, dx.$$
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\[ + \int_{\{\nabla v \neq 0\}} \varphi^2 \left[ b(\left\{ |\nabla v| \right\}) \left( |\nabla v| \right)^2 - \sum_j |\nabla v_j|^2 \right) \\
- b'(\left\{ |\nabla v| \right\}) |\nabla v| |\nabla_{L,v}^e| |\nabla v|^2 \right] \, dx \\
+ \int_{\mathbb{R}^n} 2F_{12}(u, v) \left( |\nabla u| |\nabla v| - \nabla u \cdot \nabla v \right) \varphi^2 \, dx. \]

That is, using (1.6) and (1.7)

\[ \int_{\mathbb{R}^n} |\nabla u|^2 (A(\nabla u) \nabla \varphi) \cdot \nabla \varphi + |\nabla v|^2 (B(\nabla \varphi) \cdot \nabla \varphi) \, dx \]
\[ + \int_{\mathbb{R}^n} 2F_{12}(u, v) \left( |\nabla u| |\nabla v| - \nabla u \cdot \nabla v \right) \varphi^2 \, dx \]
\[ \geq \int_{\{\nabla u \neq 0\}} \varphi^2 \left[ \lambda_1 (|\nabla u|) |\nabla_{L,u,x}^e| |\nabla u|^2 \right. \\
+ \left. \lambda_2 (|\nabla u|) \left( \sum_j |\nabla v_j|^2 - |\nabla_{L,v}^e| |\nabla u|^2 - |\nabla |\nabla u| |\right) \right] \, dx \]
\[ + \int_{\{\nabla v \neq 0\}} \varphi^2 \left[ \gamma_1 (|\nabla v|) |\nabla_{L,v,x}^e| |\nabla v|^2 \right. \\
+ \left. \gamma_2 (|\nabla v|) \left( \sum_j |\nabla v_j|^2 - |\nabla_{L,v}^e| |\nabla v|^2 - |\nabla |\nabla v| |\right) \right] \, dx. \]

This and (2.11) imply the desired result. \qed

An immediate consequence of Theorem 4.3 is the following:

**Corollary 4.4.** Let \( \Omega \subseteq \mathbb{R}^n \) be open (not necessarily bounded). Let \((u, v)\) be a stable weak solution of (1.3), with \( u \in C^1(\Omega) \cap C^2(\Omega \cap \{ \nabla u \neq 0 \}) \), \( v \in C^1(\Omega) \cap C^2(\Omega \cap \{ \nabla v \neq 0 \}) \), and \( \nabla u, \nabla v \in W^{1,2}_{\text{loc}}(\Omega) \). Suppose that either (A2) holds or that \( \{ \nabla u = 0 \} = \emptyset \), and that either (B2) holds or that \( \{ \nabla v = 0 \} = \emptyset \). Moreover, assume that \( F_{12}(u, v) \leq 0 \).

For any \( x \in \Omega \) let \( L_{u,x} \) and \( L_{v,x} \) denote the level set of \( u \) and \( v \) respectively at \( x \), according to (2.9).

Let also \( \lambda_1(|\xi|), \lambda_2(|\xi|), \gamma_1(|\xi|), \gamma_2(|\xi|) \) be as in (1.6) and (1.7).

Then,

\[ \int_{\Omega \cap \{ \nabla u \neq 0 \}} \left[ \lambda_1 (|\nabla u(x)|) |\nabla_{L,u,x}^e| |\nabla u(x)| \right]^2 \]
\[ + \lambda_2 (|\nabla u(x)|) |\nabla u(x)|^2 \sum_{l=1}^{n-1} k_{l,u}^2 \varphi^2(x) \, dx \]
\[ + \int_{\Omega \cap \{ \nabla v \neq 0 \}} \left[ \gamma_1 (|\nabla v(x)|) |\nabla_{L,v,x}^e| |\nabla v(x)| \right]^2 \]
\[ + \gamma_2 (|\nabla v(x)|) |\nabla v(x)|^2 \sum_{l=1}^{n-1} k_{l,v}^2 \varphi^2(x) \, dx \]
By (5.1), for any locally Lipschitz function \( \varphi : \Omega \to \mathbb{R} \) whose support is compact and contained in \( \Omega \).

5. Level set analysis. We recall here the geometric analysis performed in Subsection 2.4 in [6]. In order to make this paper self-contained, we include the proofs in full detail.

We consider connected components of the level sets (in the inherited topology).

**Lemma 5.1.** Let \( w \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla w \neq 0\}) \). Fix \( \pi \in \mathbb{R}^n \), and suppose that for any \( x \in L_{w,\pi} \cap \{\nabla w \neq 0\} \), we have that \( \nabla_{L_{w,x}}|\nabla w(x)| = 0 \).

Then, \( |\nabla w| \) is constant on every connected component of \( L_{w,\pi} \cap \{\nabla w \neq 0\} \).

**Proof.** Since any connected components of \( L_{w,\pi} \cap \{\nabla w \neq 0\} \) is a regular hypersurface, any two points in it may be joined by a \( C^1 \) path.

We notice that, if \( t_1 > t_0 \in \mathbb{R} \) and \( \sigma \in C^1([t_0, t_1], L_{w,\pi} \cap \{\nabla w \neq 0\}) \), then

\[
\frac{d}{dt}|\nabla w(\sigma(t))| = \nabla|\nabla w(\sigma(t))| \cdot \dot{\sigma}(t) = \nabla_{L_{w,\pi}}|\nabla w(\sigma(t))| \cdot \dot{\sigma}(t),
\]

thanks to (2.10). As a consequence, if \( \sigma \in C^1([t_0, t_1], L_{w,\pi} \cap \{\nabla w \neq 0\}) \), then \( |\nabla w(\sigma(t))| \) is constant for \( t \in [t_0, t_1] \).

Now, we take \( a \) and \( b \) in \( L_{w,\pi} \cap \{\nabla w \neq 0\} \) and \( \sigma \in C^1([0, 1], L_{w,\pi}) \) such that \( \sigma(0) = a \) and \( \sigma(1) = b \). Then \( |\nabla w(a)| = |\nabla w(b)| \).

**Corollary 5.2.** Under the assumptions of Lemma 5.1, every connected component of \( L_{w,\pi} \cap \{\nabla w \neq 0\} \) is closed in \( \mathbb{R}^n \).

**Proof.** Let \( M \) be any connected component of \( L_{w,\pi} \cap \{\nabla w \neq 0\} \). With no loss of generality, we suppose that \( M \neq \emptyset \) and take \( z \in M \).

Let \( y \in \partial M \). Then there is a sequence \( x_n \in M \) approaching \( y \), thus

\[
w(y) = \lim_{n \to +\infty} w(x_n) = w(z).
\]

Then, by Lemma 5.1, we have that \( |\nabla w(x_n)| = |\nabla w(z)| \). So, since \( z \in M \),

\[
|\nabla w(y)| = \lim_{n \to +\infty} |\nabla w(x_n)| = |\nabla w(z)| \neq 0.
\]

By (5.1) and (5.2), we have that \( y \in M \).

**Corollary 5.3.** Let the assumptions of Lemma 5.1 hold. Let \( M \) be a connected component of \( L_{w,\pi} \cap \{\nabla w \neq 0\} \). Suppose that \( M \neq \emptyset \) and \( M \) is contained in a hyperplane \( \pi \). Then, \( M = \pi \).

**Proof.** We show that \( M \) is open in the topology of \( \pi \).

For this, let \( z \in M \). Then, there exists an open set \( U_1 \) of \( \mathbb{R}^n \) such that \( z \in U_1 \subseteq \{\nabla w \neq 0\} \). Also, by the Implicit Function Theorem, there exists an open set \( U_2 \) in \( \mathbb{R}^n \) for which \( z \in U_2 \) and \( L_{w,x} \cap U_2 \) is a hypersurface. Since \( M \subseteq \pi \), we have that \( L_{w,x} \cap U_2 \subseteq \pi \), hence \( L_{w,x} \cap U_2 \) is open in the topology of \( \pi \).

Then, \( z \in L_{w,x} \cap U_1 \cap U_2 \), which is an open set in \( \pi \).

This proves (5.3).
Also, $M$ is closed in $\mathbb{R}^n$ and so $M = M \cap \pi$ is closed in $\pi$.
Hence, $M$ is open and closed in $\pi$.

**Lemma 5.4.** Let $w \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla w \neq 0\})$ be such that $\nabla_{L_{w,x}}|\nabla w(x)| = 0$ for every $x \in \{\nabla w \neq 0\}$, and let $\pi \in \mathbb{R}^n$.

Suppose that a non-empty connected component $\overline{L}$ of $L_{w,\pi} \cap \{\nabla w \neq 0\}$ has zero principal curvatures at all points.
Then, $\overline{L}$ is a flat hyperplane.

**Proof.** We use a standard differential geometry argument (see, for instance, page 311 in [11]). Since the principal curvatures vanish identically, the normal of $\overline{L}$ is constant, thence $\overline{L}$ is contained in a hyperplane.
Then, the claim follows from Corollary 5.3.

**Lemma 5.5.** Let $w \in C^1(\mathbb{R}^n) \cap C^2(\{\nabla w \neq 0\})$. Suppose that

any connected component of $L_{w,x} \cap \{\nabla w \neq 0\}$
has zero principal curvatures at all points

\[\nabla_{L_{w,x}}|\nabla w(x)| = 0.\] (5.5)

Then, $w$ possesses one-dimensional symmetry, in the sense that there exists $\overline{w} : \mathbb{R} \to \mathbb{R}$ and $\omega \in S^{n-1}$ in such a way that $w(x) = \overline{w}(\omega \cdot x)$, for any $x \in \mathbb{R}^n$.

**Proof.** If $\nabla w(x) = 0$ for any $x \in \mathbb{R}^n$, the one-dimensional symmetry is trivial.
If $\nabla w(x) \neq 0$, then the connected component of $L_{w,\pi} \cap \{\nabla w \neq 0\}$ passing through $\pi$ is a hyperplane, thanks to Lemma 5.4.
We notice that all these hyperplanes are parallel, since connected components cannot intersect. Moreover, $w$ is constant on these hyperplanes, because each of them lies on a level set.
On the other hand, $w$ is also constant on any other possible hyperplane parallel to the ones of the above family, since the gradient vanishes identically there.
From this, the one-dimensional symmetry of $w$ follows by noticing that $w$ only depends on the orthogonal direction with respect to the above family of hyperplanes.
6. **Proof of Theorem 1.3.** Since either (1.12) or (1.13) holds, from Corollaries 3.3 and 4.4, we have

\[
\int_{\{\nabla u \neq 0\}} \left[ \lambda_1 (|\nabla u(x)|) |\nabla_{L_{u,x}} |\nabla u(x)|^2 
+ \lambda_2 (|\nabla u(x)|) |\nabla u(x)|^2 \sum_{l=1}^{n-1} k^2_{l,u} \right] \varphi^2 (x) \, dx 
+ \int_{\{\nabla v \neq 0\}} \left[ \gamma_1 (|\nabla v(x)|) |\nabla_{L_{v,x}} |\nabla v(x)|^2 
+ \gamma_2 (|\nabla v(x)|) |\nabla v(x)|^2 \sum_{l=1}^{n-1} k^2_{l,v} \right] \varphi^2 (x) \, dx
\]

\[
\leq \int_{\mathbb{R}^n} |\nabla u(x)|^2 (A (\nabla u(x)) \nabla \varphi(x)) \cdot \nabla \varphi(x) \, dx 
+ \int_{\mathbb{R}^n} |\nabla v(x)|^2 (B (\nabla v(x)) \nabla \varphi(x)) \cdot \nabla \varphi(x) \, dx
\]

\[
\leq \int_{\mathbb{R}^n} (|A(\nabla u(x))| |\nabla u(x)|^2 + |B(\nabla v(x))| |\nabla v(x)|^2) |\nabla \varphi(x)|^2 \, dx. \quad (6.1)
\]

Now, we chose conveniently \( \varphi \) in (6.1). For any \( R > 1 \), we define the function \( \varphi_R \) as

\[
\varphi_R(x) := \begin{cases} 
2 \frac{\log R - \log |x|}{\log R} & \text{if } x \in B_{\sqrt{\pi} R}, \\
0 & \text{if } x \in B_R \setminus B_{\sqrt{\pi} R}, \\
-1 & \text{if } x \in \mathbb{R}^n \setminus B_R. 
\end{cases} \quad (6.2)
\]

We denote by

\[
\chi_R := \chi_{B_R \setminus B_{\sqrt{\pi} R}}.
\]

Notice that

\[
|\nabla \varphi_R(x)| = \frac{\chi_R(x)}{2|x| \log R}.
\]

Therefore, by using \( \varphi_R \) in (6.1), we have

\[
\int_{B_{\sqrt{\pi} R} \setminus \{\nabla u \neq 0\}} \left[ \lambda_1 (|\nabla u(x)|) |\nabla_{L_{u,x}} |\nabla u(x)|^2 
+ \lambda_2 (|\nabla u(x)|) |\nabla u(x)|^2 \sum_{l=1}^{n-1} k^2_{l,u} \right] \varphi^2 (x) \, dx 
+ \int_{B_{\sqrt{\pi} R} \setminus \{\nabla v \neq 0\}} \left[ \gamma_1 (|\nabla v(x)|) |\nabla_{L_{v,x}} |\nabla v(x)|^2 
+ \gamma_2 (|\nabla v(x)|) |\nabla v(x)|^2 \sum_{l=1}^{n-1} k^2_{l,v} \right] \varphi^2 (x) \, dx
\]

\[
\leq \frac{C}{\log^2 R} \int_{B_R \setminus B_{\sqrt{\pi} R}} \frac{|A(\nabla u(x))| |\nabla u(x)|^2 + |B(\nabla v(x))| |\nabla v(x)|^2}{|x|^2} \, dx. \quad (6.3)
\]
Letting $R \to +\infty$ in (6.3), by the hypothesis (1.14), we obtain

\[
\int_{\{\nabla u \neq 0\}} \left[ \lambda_1(|\nabla u(x)|)|\nabla L_{u,x}|\nabla u(x)|^2 + \lambda_2(|\nabla u(x)|)|\nabla u(x)|^2 \sum_{l=1}^{n-1} k^2_{l,u} \right] dx \\
+ \int_{\{\nabla v \neq 0\}} \left[ \gamma_1 (|\nabla v(x)|) |\nabla L_{v,x}|\nabla v(x)|^2 + \gamma_2 (|\nabla v(x)|) |\nabla v(x)|^2 \sum_{l=1}^{n-1} k^2_{l,v} \right] dx = 0,
\]

which implies that, for any $x \in \{\nabla u \neq 0\}$,

\[
\lambda_1(|\nabla u(x)|)|\nabla L_{u,x}|\nabla u(x)|^2 + \lambda_2(|\nabla u(x)|)|\nabla u(x)|^2 \sum_{l=1}^{n-1} k^2_{l,u} = 0,
\]

and, for any $x \in \{\nabla v \neq 0\}$,

\[
\gamma_1 (|\nabla v(x)|) |\nabla L_{v,x}|\nabla v(x)|^2 + \gamma_2 (|\nabla v(x)|) |\nabla v(x)|^2 \sum_{l=1}^{n-1} k^2_{l,v} = 0.
\]

Recalling the definition of $\lambda_1, \lambda_2, \gamma_1, \gamma_2$ given in (1.6) and (1.7), and the assumptions (1.4) and (1.5), the last two equalities imply that

\[
\nabla L_{u,x}|\nabla u(x) = 0, \quad k_{1,u} = \ldots = k_{n-1,u} = 0,
\]

for any $x \in \{\nabla u \neq 0\}$, and that

\[
\nabla L_{v,x}|\nabla v(x) = 0, \quad k_{1,v} = \ldots = k_{n-1,v} = 0,
\]

for any $x \in \{\nabla v \neq 0\}$. This means that $u, v$ satisfy (5.4) and (5.5). Hence, by Lemma 5.5 we obtain that there exist $\pi, \varphi : \mathbb{R} \to \mathbb{R}$ and $\omega_\ell, \omega_\ell' \in S^{n-1}$ in such a way that $(u(x), v(x)) = (\pi(\omega_\ell \cdot x), \varphi(\omega_\ell' \cdot x))$, for any $x \in \mathbb{R}^n$, which proves the first part of Theorem 1.3.

Now, we assume that condition (1.15) holds. Since $(u, v)$ has a one dimensional symmetry, by summing up (3.2) and (3.3) we obtain that

\[
\int_{\mathbb{R}^n} F_{12}(u, v) \left| \sqrt{\frac{u_n}{u_n - v_n}} \nabla u(x) + \sqrt{\frac{u_n}{u_n - v_n}} \nabla v(x) \right|^2 \varphi(x) dx, \\
\leq \int_{\mathbb{R}^n} |\nabla u(x)|^2 (A (\nabla u(x)) \nabla \varphi(x)) \cdot \nabla \varphi(x) dx \\
+ \int_{\mathbb{R}^n} |\nabla v(x)|^2 (B (\nabla v(x)) \nabla \varphi(x)) \cdot \nabla \varphi(x) dx.
\]

Choosing the test function $\varphi$ as in (6.2) and reasoning as above, we obtain

\[
\int_{\mathbb{R}^n} F_{12}(u, v) \left| \sqrt{\frac{u_n}{u_n - v_n}} \nabla u(x) + \sqrt{\frac{u_n}{u_n - v_n}} \nabla v(x) \right|^2 dx = 0,
\]

which implies that

\[
F_{12}(u, v) \left| \sqrt{\frac{u_n}{u_n - v_n}} \nabla u(x) + \sqrt{\frac{u_n}{u_n - v_n}} \nabla v(x) \right|^2 = 0 \quad \text{a.e.}
\]
Since (1.15) holds, there exists $x_0 \in \Omega'$ such that $F_{12}(u(x_0), v(x_0)) > 0$. Therefore,

$$\sqrt{-v_n(x_0) \nabla u(x_0)} + \sqrt{\frac{u_n(x_0)}{-v_n(x_0)}} \nabla v(x_0) = 0,$$

which gives that $\nabla u(x_0) = h(x_0) \nabla v(x_0)$, for some function $h$. Since we know that $(u, v)$ has a one dimensional symmetry, this implies that $\omega_u = \omega_v$.

Finally, we assume that condition (1.16) holds. Arguing as in the proof of Theorem 1.8 in [3] (see the comments after formula (8.5)), one can prove that

there exists a non-empty open set $\Omega'' \subset \mathbb{R}^2$ such that

$u(x) \in I_u, v(x) \in I_v, \nabla u(x) \neq 0$ and $\nabla v(x) \neq 0$ for all $x \in \Omega''$.

(6.5)

Now, reasoning as above, from Theorem 4.3 we obtain that

$$-2 \int_{\mathbb{R}^n} F_{12}(u, v)(|\nabla u(x)| |\nabla v(x)| - \nabla u(x) \cdot \nabla v(x)) \varphi^2(x) \, dx$$

$$\leq \int_{\mathbb{R}^n} |\nabla u(x)|^2 \left( A(\nabla u(x)) \nabla \varphi(x) \right) \cdot \nabla \varphi(x) \, dx$$

$$+ \int_{\mathbb{R}^n} |\nabla v(x)|^2 \left( B(\nabla v(x)) \nabla \varphi(x) \right) \cdot \nabla \varphi(x) \, dx.$$

We choose the test function $\varphi$ as in (6.2) and we use that fact that $F_{12}(u, v) \leq 0$ to get

$$F_{12}(u, v)(|\nabla u(x)| |\nabla v(x)| - \nabla u(x) \cdot \nabla v(x)) = 0 \quad \text{a.e.}$$

By (6.5) and (1.16), there exists $x_1 \in \Omega''$ such that $F_{12}(u(x_1), v(x_1)) < 0$. Hence

$$|\nabla u(x_1)| |\nabla v(x_1)| - |\nabla u(x_1)| \cdot \nabla v(x_1)$$

$$= |\nabla u(x_1)| |\nabla v(x_1)| - |\nabla u(x_1)| \frac{\nabla u(x_1)}{|\nabla u(x_1)|} \cdot \frac{\nabla v(x_1)}{|\nabla v(x_1)|} = 0,$$

which implies that

$$\frac{\nabla u(x_1)}{|\nabla u(x_1)|} \cdot \frac{\nabla v(x_1)}{|\nabla v(x_1)|} = 1.$$

Since we know that $(u, v)$ has a one dimensional symmetry, this implies that $\omega_u = \omega_v$. This concludes the proof of Theorem 1.3.

7. An application. In this section, we use the result stated in Theorem 1.3 to obtain a proof of a conjecture of De Giorgi for the system (1.3) in $\mathbb{R}^2$.

**Theorem 7.1.** Let $n = 2$, and let $(u, v)$ be a weak solution of (1.3), with $u \in C^1(\mathbb{R}^2) \cap C^2(\{\nabla u \neq 0\})$, $v \in C^1(\mathbb{R}^2) \cap C^2(\{\nabla v \neq 0\})$, and $\nabla u, \nabla v \in L^\infty(\mathbb{R}^2) \cap W^{1,2}_{loc}(\mathbb{R}^2)$.

Suppose that either (A1) or (A2) holds, and that either (B1) or (B2) holds.

Assume that either

the monotonicity condition (1.8) holds, and $F_{12}(u, v) \geq 0$ in $\mathbb{S}(u, v)$,

or

$(u, v)$ is stable, and $F_{12}(u, v) \leq 0$ in $\mathbb{S}(u, v)$.

Then $(u, v)$ has one-dimensional symmetry, in the sense that there exist $\uppi, \upvarpi : \mathbb{R} \to \mathbb{R}$ and $\omega_u, \omega_v \in S^{n-1}$ in such a way that $(u(x), v(x)) = (\uppi(\omega_u \cdot x), \upvarpi(\omega_v \cdot x))$, for any $x \in \mathbb{R}^n$. 

Moreover, if we assume in addition that either
the monotonicity condition (1.8) holds, and there exists a non-empty open set \( \Omega' \subseteq \mathbb{R}^n \) such that \( F_{12}(u(x), v(x)) > 0 \) for any \( x \in \Omega' \),
or
\((u, v)\) is stable, and there exist two open intervals \( I_u, I_v \subseteq \mathbb{R} \) such that \((I_u \times I_v) \cap \mathbb{S}(u, v) \neq \emptyset\) and \( F_{12}(\Xi, \Xi) > 0 \) for any \((\Xi, \Xi) \in I_u \times I_v\),
then \((u, v)\) has one-dimensional symmetry, and \( \omega_u = \omega_v \).

**Proof.** By the assumptions of Theorem 7.1, \(|\nabla u|\) and \(|\nabla v|\) are taken to be bounded. Moreover, thanks to either (A1) or (A2) and either (B1) or (B2), the maps \( t \mapsto t^2 \lambda_1(t) + t^2 \lambda_2(t) \), \( t \mapsto t^2 \gamma_1(t) + t^2 \gamma_2(t) \) belong to \( L^\infty_{\mathrm{loc}}([0, +\infty)) \). Therefore, we have that
\[
|A(\nabla u(x))| |\nabla u(x)|^2 + |B(\nabla v(x))| |\nabla v(x)|^2 \leq C,
\]
for some positive constant \( C \).

Then,
\[
\frac{1}{\log^2 R} \int_{B_R \setminus B_{\sqrt{R}} \Pi} \frac{|A(\nabla u(x))| |\nabla u(x)|^2 + |B(\nabla v(x))| |\nabla v(x)|^2}{|x|^2} \, dx \leq \frac{C}{\log R},
\]
Therefore, letting \( R \to +\infty \), we have that the condition (1.14) is satisfied. Hence, by Theorem 1.3, we obtain the desired result. \( \square \)

Notice that, as a particular case of (1.3), we can consider the following system, which arises in phase separation for multiple states Bose-Einstein condensates:
\[
\begin{cases}
\Delta u = uv^2, \\
\Delta v = vu^2,
\end{cases}
\quad u, v > 0. \tag{7.1}
\]
In fact, in this case, the operators in (1.3) reduce to the standard Laplacian and \( F(u, v) = \frac{1}{2} u^2 v^2 \). Under the assumptions of Theorem 7.1 (notice that \( F_{12}(u, v) = 2uv > 0 \)), one has that the monotone solutions of (7.1) have one-dimensional symmetry. This result has been proved in [1].

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E-mail address: dipierro@sissa.it