Decoupling Braided Tensor Factors *

Gaetano Fiore, ¹,² Harold Steinacker ³, Julius Wess ³,⁴

¹Dip. di Matematica e Applicazioni, Fac. di Ingegneria
Università di Napoli, V. Claudio 21, 80125 Napoli

²I.N.F.N., Sezione di Napoli,
Complesso MSA, V. Cintia, 80126 Napoli

³Sektion Physik, Ludwig-Maximilian Universität,
Theresienstraße 37, D-80333 München

⁴Max-Planck-Institut für Physik
Föhringer Ring 6, D-80805 München

Abstract

We briefly report on our result [9] that the braided tensor product algebra of two module algebras \( A_1, A_2 \) of a quasitriangular Hopf algebra \( H \) is equal to the ordinary tensor product algebra of \( A_1 \) with a subalgebra isomorphic to \( A_2 \) and commuting with \( A_1 \), provided there exists a realization of \( H \) within \( A_1 \). As applications of the theorem we consider the braided tensor product algebras of two or more quantum group covariant quantum spaces or deformed Heisenberg algebras.

*Talk given at the 23-rd International Conference on Group Theory Methods in Physics, Dubna (Russia), August 2000
1 Introduction and main theorem

As is well known, given two associative unital algebras \( A_1, A_2 \) (over the field \( \mathbb{C} \), say), there is an obvious way to build a new algebra \( \mathcal{A} \) which is as a vector space the tensor product \( \mathcal{A} = A_1 \otimes_{\mathbb{C}} A_2 \) of the two vector spaces (over the same field) and has a product law such that \( A_1 \otimes 1 \) and \( 1 \otimes A_2 \) are subalgebras isomorphic to \( A_1 \) and \( A_2 \) respectively: one just completes the product law by postulating the trivial commutation relations

\[
(1 \otimes a_2)(a_1 \otimes 1) = (a_1 \otimes 1)(1 \otimes a_2) \tag{1}
\]

for any \( a_1 \in A_1, a_2 \in A_2 \). The resulting algebra is the ordinary tensor product algebra. With a standard abuse of notation we shall denote in the sequel \( a_1 \otimes a_2 \) by \( a_1 a_2 \) for any \( a_1 \in A_1, a_2 \in A_2 \); consequently (1) becomes

\[
a_2 a_1 = a_1 a_2. \tag{2}
\]

If \( A_1, A_2 \) are module algebras of a Lie algebra \( \mathfrak{g} \), and we require \( \mathcal{A} \) to be too, then (2) has no alternative, because any \( g \in \mathfrak{g} \) acts as a derivation on the (algebra as well as tensor) product of any two elements, or, in Hopf algebra language, because the coproduct \( \Delta(g) = g^{(1)} \otimes g^{(2)} \) (at the rhs we have used Sweedler notation) of the Hopf algebra \( H \equiv U\mathfrak{g} \) is cocommutative. In this paper we shall work with right-module algebras (instead of left ones), and denote by \( \triangleleft : (a, g) \in \mathcal{A} \times H \rightarrow a \triangleleft g \in \mathcal{A} \) the right action; the reason is that they are equivalent to left comodule algebras, which are used in much of the literature. In Ref. [9] we give also the corresponding formulae for the left module algebras. We recall that a right action \( \triangleleft : (a, g) \in \mathcal{A} \times H \rightarrow a \triangleleft g \in \mathcal{A} \) by definition fulfills

\[
a \triangleleft (gg') = (a \triangleleft g) \triangleleft g', \tag{3}
\]

\[
(aa') \triangleleft g = (a \triangleleft g^{(1)}) \circ (a' \triangleleft g^{(2)}). \tag{4}
\]

If we take as Hopf algebra \( H \) a quasitriangular noncocommutative one like the quantum group \( U_q \mathfrak{g} \), as \( \mathcal{A} \), some \( H \)-module algebras, and we require \( \mathcal{A} \) to be a \( H \)-module algebra too, then (4) has to be replaced by one of the formulae

\[
a_2 a_1 = (a_1 \triangleleft \mathcal{R}^{(1)}) (a_2 \triangleleft \mathcal{R}^{(2)}), \tag{5}
\]

\[
a_2 a_1 = (a_1 \triangleleft \mathcal{R}^{-1(2)}) (a_2 \triangleleft \mathcal{R}^{-1(1)}). \tag{6}
\]

\[2\]
This yields instead of $\mathcal{A}$ two different \textit{braided} tensor product algebras \cite{[11]}, which we shall call $\mathcal{A}^+ = \mathcal{A}_1 \otimes^+ \mathcal{A}_2$ and $\mathcal{A}^- = \mathcal{A}_1 \otimes^- \mathcal{A}_2$ respectively. Here $\mathcal{R} \equiv \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in H^+ \otimes H^-$ denotes the so-called universal $R$-matrix of $H$ \cite{[3]}, $\mathcal{R}^{-1}$ its inverse, and $H^\pm$ denote the Hopf positive and negative Borel subalgebras of $H$. If in particular $H$ is triangular, then $\mathcal{R}^{-1} = \mathcal{R}_{21}$, $\mathcal{A}^+ = \mathcal{A}^-$, and one has just one braided tensor product algebra. In any case, both $\mathcal{A}^+$ and $\mathcal{A}^-$ go to the ordinary tensor product algebra $\mathcal{A}$ in the limit $q \to 1$, because in this limit $\mathcal{R} \to 1 \otimes 1$.

The braided tensor product is a particular example of a more general notion, that of a \textit{crossed (or twisted) tensor product} \cite{[1]} of two unital associative algebras.

In view of (5) or (6) studying representations of $\mathcal{A}^\pm$ is a more difficult task than just studying the representations of $\mathcal{A}_1, \mathcal{A}_2$ and taking their tensor products. The degrees of freedom of $\mathcal{A}_1, \mathcal{A}_2$ are so to say “coupled”. One might ask whether one can “decouple” them by a transformation of generators. As shown in Ref. \cite{[9]}, the answer is positive if there respectively exists an algebra homomorphism $\varphi_1^+$ or an algebra homomorphism $\varphi_1^-$ acting as the identity on $\mathcal{A}_1$, namely for any $a_1 \in \mathcal{A}_1$

$$\varphi_1^\pm(a_1) = a_1. \quad (8)$$

(Here $\mathcal{A}_1 \rtimes H^\pm$ denotes the cross product between $\mathcal{A}_1$ and $H^\pm$). In other words, this amounts to assuming that $\varphi_1^+\left(H^+\right)$ [resp. $\varphi_1^\pm\left(H^-\right)$] provides a \textit{realization} of $H^+$ (resp. $H^-$) within $\mathcal{A}_1$. In this report we summarize the main results of Ref. \cite{[9]}. The basic one is

\textbf{Theorem 1} \cite{[9]}. Let \{\(H, \mathcal{R}\)\} be a quasitriangular Hopf algebra and $H^+, H^-$ be Hopf subalgebras of $H$ such that $\mathcal{R} \in H^+ \otimes H^-$. Let $\mathcal{A}_1, \mathcal{A}_2$ be respectively a $H^+$- and a $H^-$-module algebra, so that we can define $\mathcal{A}^+$ as in (5), and $\varphi_1^+$ be a homomorphism of the type (2), (3), so that we can define the map $\chi^+: \mathcal{A}_2 \to \mathcal{A}^+$ by

$$\chi^+(a_2) := \varphi_1^+\left(\mathcal{R}(1)\right) \left(a_2 \triangleleft \mathcal{R}(2)\right). \quad (9)$$

Alternatively, let $\mathcal{A}_1, \mathcal{A}_2$ be respectively a $H^-$- and a $H^+$-module algebra, so that we can define $\mathcal{A}^-$ as in (6), and $\varphi_1^-$ be a homomorphism of the type (2), (3), so that we can define the map $\chi^-: \mathcal{A}_2 \to \mathcal{A}^-$ by

$$\chi^-(a_2) := \varphi_1^-\left(\mathcal{R}^{-1}(2)\right) \left(a_2 \triangleleft \mathcal{R}^{-1}(1)\right). \quad (10)$$

3
In either case $\chi^\pm$ are then injective algebra homomorphisms and

$$[\chi^\pm(a_2), A_1] = 0,$$

namely the subalgebras $\tilde{A}_2^\pm := \chi^\pm(A_2) \approx A_2$ commute with $A_1$. Moreover $A^\pm = A_1 \otimes \tilde{A}_2^\pm$.

The last equality means that $A^\pm$ are respectively equal to the ordinary tensor product algebra of $A_1$ with the subalgebras $\tilde{A}_2^\pm \subset A_2$ which are isomorphic to $A_2! \chi^+, \chi^-$ will be called “unbraiding” maps.

We recall the content of the hypotheses stated in the theorem. The algebra $A_1 \rtimes H^\pm$ as a vector space is the tensor product $A_1 \otimes C H^\pm$, as an algebra it has subalgebras $A_1 \otimes 1, 1 \otimes H$ and has cross commutation relations

$$a_1 g = g(1) (a_1 \triangleleft g(2)),$$

for any $a_1 \in A_1$ and $g \in H^\pm$. $\varphi_1^\pm$ being an algebra homomorphism means that for any $\xi, \xi' \in A_1 \rtimes H^\pm$ $\varphi_1^\pm(\xi \xi') = \varphi_1^\pm(\xi) \varphi_1^\pm(\xi')$. Applying $\varphi_1^\pm$ to both sides of (12) we find $a \varphi_1^\pm(g) = \varphi_1^\pm(g(1)) (a \triangleleft g(2))$.

Of course, we can use the above theorem iteratively to completely unbraid the braided tensor product algebra of an arbitrary number $M$ of copies of $A_1$. We end up with

**Corollary 1** If $A_1$ is a (right-) module algebra of the Hopf algebra $H$ and there exists an algebra homomorphism $\varphi_1^+$ of the type (7), (8), then there is an algebra isomorphism

$$\underbrace{A_1 \otimes^+ \ldots \otimes^+ A_1}_{M \text{ times}} \approx \underbrace{A_1 \otimes \ldots \otimes A_1}_{M \text{ times}}.$$  

An analogous claim holds for the second braided tensor product if there exists a map $\varphi_1^-$.

2 The unbraiding under the $*$-structures

$A^+$ (as well as $A^-$) is a $*$-algebra if $H$ is a Hopf $*$-algebra, $A_1, A_2$ are $H$-module $*$-algebras (we shall use the same symbol $*$ for the $*$-structure on all algebras $H, A_1$, etc.), and

$$R^* = R^{-1}$$

(14)
(here $R^*$ means $R^{(1)} \ast \otimes R^{(2)}$). In the quantum group case \cite{14} requires $|q| = 1$. Under the same assumptions also $A_1 \triangleright H$ is a $\ast$-algebra. If $\varphi_1^\pm$ exist setting $\varphi_1^{\prime \pm} := \ast \circ \varphi_1^{\pm} \circ \ast$ we realize that also $\varphi_1^\prime \pm$ are algebra homomorphisms of the type \cite{9}, \cite{8}. If such homomorphisms are uniquely determined, we conclude that $\varphi_1^\pm$ are $\ast$-homomorphisms. More generally, one may be able to choose $\varphi_1^\pm$ as $\ast$-homomorphisms. How do the corresponding $\chi^\pm$ behave under $\ast$?

**Proposition 1** \cite{9}. Assume that the conditions of Theorem 1 for defining $\chi^+$ (resp. $\chi^-$) are fulfilled. If $R^* = R^{-1}$ and $\varphi_1^+$ (resp. $\varphi_1^-$) is a $\ast$-homomorphism then $\chi^+$ (resp. $\chi^-$) is, too. Consequently, $A_1$, $\tilde{A}_2^\pm$ are closed under $\ast$.

### 3 Applications

In this section we illustrate the application of Theorem 1 and Corollary 1 to some algebras $H$, $A_i$ for which homomorphisms $\varphi_1^\pm$ are known. $H$ will be the quantum group $U_q sl(N)$ or $U_q so(N)$, and $A_1$ is the $U_q sl(N)$- or $U_q so(N)$-covariant Heisenberg algebra (Section 3.1), the $U_q so(N)$-covariant quantum space/sphere (Section 3.2). In Ref. \cite{9} we have treated also the $U_q so(3)$-covariant $q$-fuzzy sphere. As generators of H it will be convenient in either case to use the Faddeev-Reshetikhin-Takhtadjan (FRT) generators \cite{7} $L^+_a \in H^+$ and $L^-_a \in H^-$. They are related to $R$ by

$$L^+_a := R^{(1)}(g) \rho_a R^{(2)} \quad \text{and} \quad L^-_a := \rho_a R^{-1},$$

where $\rho_a(g)$ denote the matrix elements of $g \in U_q g$ in the fundamental $N$-dimensional representation $\rho$ of $U_q g$. In fact they provide, together with the square roots of the elements $L^\pm_i$, a (overcomplete) set of generators of $U_q g$.

#### 3.1. Unbraiding ‘chains’ of braided Heisenberg algebras

In this subsection we consider the braided tensor product of $M \geq 2$ copies of the $U_q g$-covariant deformed Heisenberg algebras $D_{r,g}$, $g = sl(N)$, $so(N)$. Such algebras have been introduced in Ref. \cite{13, 14, 2}. They are unital associative algebras generated by $x^i, \partial_j$ fulfilling the relations

$$P_{a,hk} x^h x^k = 0, \quad P_{a,hk} \partial_j \partial_l = 0, \quad \partial_i x^j = \delta_i^j + (q\gamma R)_{ih} x^h \partial_k, \quad \partial_i x^j = \delta_i^j + (q\gamma R)_{ij} x^i \partial_k,$$

(16)
where $\gamma = q^{\frac{1}{2}}, 1$ respectively for $\mathfrak{g} = sl(N), so(N)$, and the exponent $\epsilon$ can take either value $\epsilon = 1, -1$. \( \hat{R} \) denotes the braid matrix of $U_q \mathfrak{g}$ [given in formulae (27)], and the matrix $P_\alpha$ is the deformed antisymmetric projector appearing in the decomposition (28) of the latter. The coordinates $x^i$ transform according to the fundamental $N$-dimensional representation, whereas the ‘partial derivatives’ transform according to the contragredient representation,

$$x^i < g = \rho^i_j(g)x^j, \quad \partial_i < g = \partial_i \rho^h_i(S^{-1}g). \quad (17)$$

In our conventions the indices will take the values $i = 1, ..., N$ if $\mathfrak{g} = sl(N)$, whereas if $\mathfrak{g} = so(N)$ they will take the values $i = -n, \ldots, 1, 0, 1, \ldots n$ for $N$ odd, and $i = -n, \ldots, -1, 1, \ldots n$ for $N$ even; here $n := \frac{N}{2}$ denotes the rank of $so(N)$. We shall enumerate the different copies of $\mathcal{D}_\epsilon \mathfrak{g}$ by attaching to them an additional Greek index, e.g. $\alpha = 1, 2, ..., M$. The prescription (8) gives the following “cross” commutation relations between their respective generators ($\alpha < \beta$).

$$x^{\alpha,i} x^{\beta,j} = \hat{R}_{ij}^{\alpha,k} x^{\beta,k} x^{\alpha,i}, \quad \partial_{\alpha,i} x^{\beta,j} = \hat{R}_{ij}^{\alpha,k} \partial_{\alpha,k} x^{\beta,j}, \quad \partial_{\alpha,i} \partial_{\beta,j} = \hat{R}_{ij}^{k} \partial_{\alpha,k} \partial_{\beta,h}. \quad (18)$$

Algebra homomorphisms $\varphi_1 : \mathcal{A}_1 \triangleright H \rightarrow \mathcal{A}_1$, for $H = U_q \mathfrak{g}$ and $\mathcal{A}_1$ equal to (a suitable completion of) $\mathcal{D}_\epsilon \mathfrak{g}$ have been constructed in Ref. [8]. This is the $q$-analog of the well-known fact that the elements of $\mathfrak{g}$ can be realized as “vector fields” (first order differential operators) on the corresponding $\mathfrak{g}$-covariant (undeformed) space, e.g. $\varphi_1 (E^i_j) = x^i \partial_j - \frac{1}{N} \delta^i_j$ in the $\mathfrak{g} = sl(N)$ case. The maps $\varphi^\pm_1$ needed to apply Theorem 4 are simply the restrictions to $\mathcal{A}_1 \triangleright H^\pm$ of $\varphi_1$ of Ref. [8].

The unbraiding procedure is recursive. We just describe the first step, which consists of using the homomorphism $\varphi^\pm_1$ to unbraid the first copy from the others. According to the main theorem, if we set

$$y^{1,i} \equiv x^{1,i}, \quad \partial_{y,1,a} \equiv \partial_{1,a} \quad (19)$$

$$y^{\alpha,i} \equiv \chi^{-}(x^{\alpha,i}) = \varphi_1(R^{-1(2)}) \rho^\alpha_j(R^{-1(1)}) x^{\alpha,j} = \varphi_1(L^{-1}) x^{\alpha,j}, \quad (20)$$

$$\partial_{y,\alpha,a} \equiv \chi^{-}(\partial_{\alpha,a}) = \varphi_1(SR^{-1(2)}) \rho^\alpha_d(R^{-1(1)}) \partial_{\alpha,d} = \varphi_1(SL^{-1}) \partial_{\alpha,d} \quad (21)$$

with $\alpha > 1$. By Theorem 4 $y^{1,i} \equiv x^{1,i}$ and $\partial_{y,1,i} \equiv \partial_{1,i}$ will commute with $y^{2,i}, ..., y^{M,i}$ and $\partial_{y,2,i}, ..., \partial_{y,M,i}$. As we see, the FRT generators are special because they appear in the redefinitions (20-21). The explicit expression of
\( \varphi_1(L^{-i}) \) in terms of \( x^i, \partial_i \) for \( U_q sl(2), U_q so(3) \) has been given in Ref. [9]. For different values of \( N \) it can be found from the results of Ref. [8, 5] by passing from the generators adopted there to the FRT generators.

By completely analogous arguments one determines the alternative unbraiding procedure for the braided tensor product stemming from prescription [3]. \( A_1 \rtimes H \) is a \( * \)-algebra and the map \( \varphi_1 \) is a \( * \)-homomorphism both for \( q \) real and \( |q| = 1 \). But \( \varphi_{\pm} \) are \( * \)-homomorphisms only for \( |q| = 1 \). In the latter case the \( * \)-structure of \( A_1 \) is
\[
(x^i)^* = x^i, \quad (\partial_i)^* = -q^\pm N g^{kh} g_{ki} \partial_h
\]
(22)

Applying Proposition 1 in the latter case we find that \( * \) maps \( A_1 \) as well as each of the commuting subalgebras \( \tilde{A}_{\pm} \) into itself.

3.2. Unbraiding ‘chains’ of braided quantum Euclidean spaces or spheres

In this section we consider the braided tensor product of \( M \geq 2 \) copies of the quantum Euclidean space \( \mathbb{R}^N_q \) (the \( U_q so(N) \)-covariant quantum space), i.e. of the unital associative algebra generated by \( x^i \) fulfilling the relations (16), or of the quotient space of \( \mathbb{R}^N_q \) obtained by setting \( r^2 := x^i x_i = 1 \) [the quantum \((N-1)\)-dimensional sphere \( S^{N-1}_q \)]. (Thus, these will be subalgebras of the Heisenberg algebras \( D_{+, so(N)}, D_{+, so(N)} \) considered in the previous subsection). Again, the multiplet \((x^i)\) carries the fundamental \( N \)-dimensional representation \( \rho \) of \( U_q so(N) \). As before, we shall enumerate the different copies of the quantum Euclidean space or sphere by attaching an additional Greek index to them, e.g. \( \alpha = 1, 2, ..., M \). The prescription (18) gives the cross commutation relations (18)\( _1 \).

According to Ref. [8], to define \( \varphi_{1} \) (for \( q \neq 1 \)) one actually needs a slightly enlarged version of \( \mathbb{R}^N_q \) (or \( S^{N-1}_q \)). One has to introduce some new generators \( \sqrt{r} \), with \( 0 \leq a \leq N/2 \), together with their inverses \((\sqrt{r})^{-1}\), requiring that
\[
 r_a^2 = \sum_{h=-a}^{a} x^h x_h = \sum_{h=-a}^{a} g_{hk} x^h x^k
\]
(23)
(note that, having set \( n := \lfloor N/2 \rfloor \), \( r_0^2 \) coincides with \( r^2 \), whereas for odd \( N \), \( r_0^2 = (x^0)^2 \), so we are adding also \((x^0)^{-1}\) as a new generator). In fact, the commutation relations involving these new generators can be fixed consistently,
and turn out to be simply \( q \)-commutation relations. \( r \) plays the role of ‘deformed Euclidean distance’ of the generic ‘point’ \( (x^i) \) of \( \mathbb{R}^N_q \) from the ‘origin’; \( r_a \) is the ‘projection’ of \( r \) on the ‘subspace’ \( x^i = 0, |i| > a \). In the previous equation \( g_{hk} \) denotes the ‘metric matrix’ of \( SO_q(N) \), \( g_{ij} = q^{r_i - r_j} \), which is a \( SO_q(N) \)-isotropic tensor and a deformation of the ordinary Euclidean metric. Here, \( (\rho_i) := (n - \frac{1}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, \frac{1}{2} - n) \) for \( N \) odd, \( (\rho_i) := (n - 1, \ldots, 0, 0, \ldots, 1 - n) \) for \( N \) even. \( g_{ij} \) is related to the trace projector appearing in (28) by

\[
P^{ij}_{kl} = (g^{sm} g_{sm})^{-1} g^{ij} g_{kl}.
\]

The extension of the action of \( H \) to these extra generators is uniquely determined by the constraints the latter fulfil. In the case of even \( N \) one needs to include also the FRT generators \( \mathcal{L}^+ \), \( \mathcal{L}^- \) (which are generators of \( H \)) among the generators of \( A_1 \). In appendix \( 3.2 \) we recall the explicit form of \( \varphi_1^\pm \) in the present case.

Note that the maps \( \varphi_1^\pm \) have no analog in the “undeformed” case \( (q = 1) \), because \( A_1 \) is abelian, whereas \( H \) is not.

The unbraiding procedure is recursive. The first step consists of using the homomorphism \( \varphi_1^\pm \) found in Ref. [3] to unbraid the first copy from the others. Following Theorem 1, we perform the change of generators (19), (20) in \( A^- \). In view of formula (29) we thus find

\[
y_{\alpha,1} := x_{\alpha,1}, \quad y_{\alpha,0} := g^h [\mu_1^1, x^{1,k}] q g_{kj} x_{\alpha,j}, \quad \alpha > 1.
\]

(24)

The suffix 1 in \( \mu_1^1 \) means that the special elements \( \mu_a \) defined in (30) must be taken as elements of the first copy. In view of (30) we see that \( g^h [\mu_1^1, x^{1,k}] q g_{kj} \) are rather simple polynomials in \( x^i \) and \( r_a^{-1} \), homogeneous of total degree 1 in the coordinates \( x^i \) and \( r_a \). Using the results given in the appendix we give now the explicit expression of (24)2 for \( N = 3 \):

\[
y_{\alpha,-} = -q h \gamma_1 \frac{r}{x_0} x_{\alpha,-} \]
\[
y_{\alpha,0} = \sqrt{q} (q + 1) \frac{1}{x_0} x^+ x_{\alpha,0} + x_{\alpha,0} \]
\[
y_{\alpha,+} = \frac{\sqrt{q} (q + 1)}{h \gamma_1 r x^0} (x^+)^2 x_{\alpha,-} + \frac{q^{-1} + 1}{h \gamma_1 r} x^+ x_{\alpha,0} - \frac{1}{q h \gamma_1 r} x^0 x_{\alpha,+}
\]

(25)

for any \( \alpha = 2, \ldots, M \). Here we have set \( x^i \equiv x^{1,i} \), \( h \equiv \sqrt{q-1}/\sqrt{q} \), replaced for simplicity the values \(-1, 0, 1\) of the indices by the ones \(-, 0, +\) and denoted by \( \gamma_1 \in \mathbb{C} \) a free parameter. By Theorem \( 1 \) \( y_{1,i} \equiv x^{1,i} \) commutes with \( y_{2,i} \), ..., \( y_{M,i} \).
The alternative unbraiding procedure for the braided tensor product algebra stemming from prescription (5) arises by iterating the change of generators

\[ y^{M,i} := x^{M,i}, \quad y^{\alpha,i} := \varphi_M(L^+_{\alpha,j})x^{\alpha,j} = g^{ih}[\bar{\mu}^M_h, x^{M,k}]_{q^{-1}}g_{kj} x^{\alpha,j}, \]  

(26)

\( \alpha < M \). The special elements \( \bar{\mu}_a \) are defined in (30), and suffix \( M \) means that we must take \( \bar{\mu}_a \) as an element of the \( M \)-th copy of \( \mathbb{R}_q^N \) (or \( S_q^{N-1} \)).

\( y^{M,i} \equiv x^{M,i} \) commutes with \( y^{1,i}, \ldots, y^{M-1,i} \).

When \( |q| = 1 \), by a suitable choice (32) of \( \gamma_1, \bar{\gamma}_1 \), as well as of the other free parameters \( \gamma_a, \bar{\gamma}_a \) appearing in the definitions of \( \varphi^\pm \) for \( N > 3 \), one can make \( \varphi^\pm \) into \(*\)-homomorphisms. Applying Proposition 1 in the latter case we find that \(*\) maps \( A_1 \) as well as each of the commuting subalgebras \( \tilde{A}_i^\pm \) into itself.

The braid matrix \( \hat{R} \) is related to \( R \) by \( \hat{R}_{hh} \equiv R^i_{ij} := (\rho_h \otimes \rho_k)R \). With the indices’ convention described in sections 3.2, 3.1 \( \hat{R} \) is given by

\[ \hat{R} = q^{-1} \left[ q \sum_i e^i_i \otimes e^i_i + \sum_{i \neq j} e^i_i \otimes e^j_j + k \sum_{i<j} e^i_i \otimes e^j_j \right] \]  

(27)

\[ \hat{R} = q \sum_{i \neq 0} e^i_i \otimes e^i_i + \sum_{i \neq j, i+j=0} e^i_i \otimes e^j_j + q^{-1} \sum_{i \neq 0} e^{-i}_i \otimes e^{-i}_i \]

\[ \quad + k \sum_{i<j} (e^i_i \otimes e^j_j - q^{\rho_i^+ \rho_i} e^{-i}_i \otimes e^{-j}_j) \]

for \( g = sl(N), so(N) \) respectively. Here \( e^i_j \) is the \( N \times N \) matrix with all elements equal to zero except for a 1 in the \( i \)th column and \( j \)th row. The braid matrix of \( so(N) \) admits the orthogonal projector decomposition

\[ \hat{R} = qP_s - q^{-1}P_a + q^{1-N}P_t. \]  

(28)

\( P_a, P_t, P_s \) are the \( q \)-deformed antisymmetric, trace, trace-free symmetric projectors. There are just two projectors \( P_a, P_s \) in decomposition of the braid matrix of \( sl(N) \). The latter is obtained from (28) just by deleting the third term.

We now recall the explicit form of maps \( \varphi^\pm \) for the quantum Euclidean spaces or spheres, found in Ref. 3. These are algebra homomorphisms \( \varphi^\pm : \mathbb{R}_q^N \times U_q^+ so(N) \to \mathbb{R}_q^N \). We introduce the short-hand notation \([A, B]_x = \]
$AB - xBA$. The images of $\varphi^-$ (resp. $\varphi^+$) on the negative (resp. positive) FRT generators read

$$
\varphi^-(L_j^{-i}) = g^{ih}[\mu_h, x^k]_q g_{kj}, \quad \varphi^+(L_j^{+i}) = g^{ih}[\bar{\mu}_h, x^k]_{q^{-1}} g_{kj}, \quad (29)
$$

where

$$
\mu_0 = \gamma_0(x^0)^{-1} \quad \quad \bar{\mu}_0 = \bar{\gamma}_0(x^0)^{-1} \quad \text{for } N \text{ odd},
$$

$$
\mu_{\pm 1} = \gamma_{\pm 1}(x^{\pm 1})^{-1} \mathcal{L}^{\pm 1}_1 \quad \bar{\mu}_{\pm 1} = \bar{\gamma}_{\pm 1}(x^{\pm 1})^{-1} \mathcal{L}^{\pm 1}_1 \quad \text{for } N \text{ even},
$$

$$
\mu_a = \gamma_a r_{|a|}^r_{|a|-1} x^{-a} \quad \bar{\mu}_a = \bar{\gamma}_a r_{|a|}^r_{|a|-1} x^{-a} \quad \text{otherwise,}
$$

(30)

and $\gamma_a, \bar{\gamma}_a \in \mathbb{C}$ are normalization constants fulfilling the conditions

$$
\gamma_0 = -q^{-\frac{1}{2}}h^{-1} \quad \bar{\gamma}_0 = q^{\frac{1}{2}}h^{-1} \quad \text{for } N \text{ odd},
$$

$$
\gamma_1 \gamma_{-1} = \begin{cases} -q^{-1}h^{-2} & \text{for } N \text{ odd,} \\ k^{-2} & \text{for } N \text{ even,} \end{cases} \quad \bar{\gamma}_1 \bar{\gamma}_{-1} = \begin{cases} -qh^{-2} & \text{for } N \text{ odd,} \\ k^{-2} & \text{for } N \text{ even,} \end{cases}
$$

$$
\gamma_a \gamma_{-a} = -q^{-1}k^{-2} \omega_a \omega_{a-1} \quad \bar{\gamma}_a \bar{\gamma}_{-a} = -qk^{-2} \omega_a \omega_{a-1} \quad \text{for } a > 1.
$$

(31)

Here $k := q - 1/q$, $\omega_a := (q^{\rho_a} + q^{-\rho_a})$. Incidentally, for odd $N$ one can choose the free parameters $\gamma_a, \bar{\gamma}_a$ in such a way that $\varphi^+, \varphi^-$ can be ‘glued’ into an algebra homomorphism $\varphi : \mathbb{R}_q^N > \triangleright U_q so(N) \to \mathbb{R}_q^N$ [3]. When $|q| = 1$, the $\ast$-structure is given by $(x^i)^\ast = x^i$ [see (23)]. It turns out that $\varphi^\pm$ are $\ast$-homomorphisms if, in addition,

$$
\gamma_{\pm 1} = -\bar{\gamma}_{\pm 1} \quad \text{if } N \text{ even,}
$$

$$
\gamma_{a} = -\bar{\gamma}_{a} \begin{cases} 1 & \text{if } a < 0 \\ q^{-2} & \text{if } a > 0 \end{cases} \quad \text{otherwise.} \quad (32)
$$

References

[1] A. Van Daele and S. Van Keer, Compositio Mathematica 91, 201 (1994). A. Borowiec, W. Marcinek, “On crossed product of algebras”, math-ph/0007031 and references therein; “Hopf modules and their duals”, math.QA/0007151.

[2] U. Carow-Watamura, M. Schlieker, S. Watamura, Z. Physik C 49 (1991) 439.
[3] B. L. Cerchiai, G. Fiore, J. Madore, “Geometrical Tools for Quantum Euclidean Spaces”, to appear in Commun. Math. Phys., math.QA/0002007

[4] B.L. Cerchiai, J. Madore, S. Schraml, J. Wess, Eur.Phys.J. C16 (2000), 169-180.

[5] C.-S. Chu, B. Zumino, “Realization of vector fields from quantum groups as pseudodifferential operators on quantum spaces”, Proc. XX Int. Conf. on Group Theory Methods in Physics, Toyonaka (Japan), 1995, and q-alg/9502005.

[6] V. Drinfeld, “Quantum groups,” in I.C.M. Proceedings, Berkeley, (1986) p. 798.

[7] L.D. Faddeev, N.Y. Reshetikhin, L. Takhtadjan, Leningrad Math. J. 1 (1990), 193.

[8] G. Fiore, Commun. Math. Phys. 169 (1995), 475-500.

[9] G. Fiore, H. Steinacker, J. Wess “Unbraiding the braided tensor product”, Preprint 00-30 Dip. Matematica e Applicazioni, Università di Napoli, math/0007174.

[10] A. Joyal, R. Streat, Braided Monoidal Categories, Mathematics Reports 86008, Macquarie University, 1986.

[11] S. Majid, Int. J. Mod. Phys. A5, 1 (1990); J. Algebra 130, 17 (1990); Lett. Math. Phys. 22, 167 (1991); J. Algebra 163, 191 (1994). For a review: S. Majid, Foundations of Quantum Groups, Cambridge Univ. Press (1995); and references therein.

[12] O. Ogievetsky, Lett. Math. Phys. 24 (1992), 245.

[13] W. Pusz, S. L. Woronowicz, Rep. Math. Phys. 27 (1989), 231.

[14] J. Wess, B. Zumino, Nucl. Phys. (Proc. Suppl.) 18B (1990) 302.