Testing order restrictions in contingency tables

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Abstract Though several interesting models for contingency tables are defined by a system of inequality constraints on a suitable set of marginal log-linear parameters, the specific features of the corresponding testing problems and the related procedures are not widely well known. After reviewing the most common difficulties which are intrinsic to inequality restricted testing problems, the paper concentrates on the problem of testing a set of equalities against the hypothesis that these are violated in the positive direction and also on testing the corresponding inequalities against the saturated model; we argue that valid procedures should consider these two testing problems simultaneously. By reformulating and adapting procedures appeared in the econometric literature, we propose a likelihood ratio and a multiple comparison procedure which are both based on the joint distribution of two relevant statistics; these statistics are used to divide the sample space into three regions: acceptance of the assumed equality constraints, rejection towards inequalities in the positive direction and rejection towards the unrestricted model. A simulation study indicates that the likelihood ratio based procedure perform substantially better. Our procedures are applied to the analysis of two real data sets to clarify how they work in practice.

Keywords Stochastic orderings · Chi-bar squared distribution · Positive association

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Models for contingency tables involving order restrictions may arise in several contexts. For two-way tables, when both variables have ordered categories, Dardanoni and Forcina (1998) extended the approach of Dykstra et al. (1995) for testing the set of inequalities implied by the assumption that the conditional distributions by row satisfy suitable stochastic orderings; Cazzaro and Colombi (2006) considered testing stochastic orderings by row and columns simultaneously. More generally, inequality constraints arise when we assume that a pair of ordered categorical variables are positively associated (Bartolucci et al. 2007) or that the strength of the association increases with respect to a third variable (Colombi and Forcina 2001).

Inequality constraints are also implied by certain graphical models containing latent variables when we restrict attention to the marginal distribution of the observed variables. In the very special case of item response models, Bartolucci and Forcina (2000) considered testing the order relation known as $M_{TP2}$ which is implied by the assumptions of conditional independence and monotonicity. For more recent research in the area, see Zwermik and Smith (2011) and Wermuth and Marchetti (2014).

Any of the models mentioned above may be formulated by constraining the elements of a linear transformation of a vector of marginal log-linear parameters to be non-negative or, equivalently, to be contained into a convex cone, rather than into a linear subspace. Due to the nature of such constraints, there are at least three different testing problems which may be of interest:

1. With models like $M_{TP2}$, it is usually of interest to test the non-negativity constraints against the saturated model to ascertain that the assumed restrictions are not in conflict with the data. For testing problems of this type see, for instance, Shapiro (1985) and Silvapulle and Sen (2005), Sect. 4.8.
2. When testing, for example, whether a treatment is better than a placebo on the basis of an ordered categorical response as in the first example of Sect. 5 below or when studying social mobility (see, for instance Dardanoni et al. 2012), one is more often interested in testing the equality constraints that all log-odds ratios of a given type are null against the alternative that they are all non-negative (inequality constraints of positive association). For this kind of inferential problems see for example Silvapulle and Sen (2005), Sect. 4.3.
3. In certain specific contexts, like for instance, the binary star graph models considered by Wermuth and Marchetti (2014), Sect. 4, or when one wants to test that a given treatment is the best within a set of competitors, one could be interested in testing that at least one inequality constraint is not satisfied as a strict inequality against the alternative that all the quantities of interest are strictly positive. For problems of this type see Sasabuchi (1980) and Perlman and Wu (2002).
threshold. The price to pay is that the procedure may have very low power against certain alternatives (see Dardanoni and Forcina 1998, 4.6). Moreover, especially in the context of contingency tables, the hypothesis, that non-negativity constraints hold as equalities, is of interest on its own, a feature which is not acknowledged by the Sasabuchi and Perlman approach.

Different solutions to this problem, proposed in the econometric literature, are based on combining the first two testing problems and consider the null hypothesis, defined by a parameter vector constrained to zero, against violations which are either contained into the positive orthant or are unrestricted. For instance, in the context of testing Lorenz curve orderings, Dardanoni and Forcina (1999) designed a procedure based on the joint distribution of two likelihood ratio statistics where the protection against accepting the alternative hypothesis that the inequalities hold (when they are violated) may be tuned. Starting from similar concerns, Bennet (2013) has proposed a procedure based on a multiple comparisons approach in the context of comparing two welfare distributions.

In this paper we concentrate on testing problems 1 and 2 simultaneously and propose two procedures, one obtained by reformulating the Dardanoni and Forcina’s procedure and the other obtained by an extension of the Bennet’s procedure to the context of testing inequality constraints in contingency tables. Both procedures proposed in this paper have the following features:

– they are designed to test the null hypothesis, that a system of equalities hold against either the alternative that one or more equalities are violated in the positive direction, or that violations are unrestricted;
– they may lead to one of the following decisions: accept the null hypothesis, reject the null in the direction of the inequality constraints, reject in the direction that the inequality constraints are also violated;
– rejection regions are determined by the joint distribution of two statistics;
– in addition to giving the user control on how the total error rate is split between the two alternatives, they depend on an additional parameter which affects the probability of detecting violations of the inequality constraints without changing the error rates.

The paper also discusses computational aspects and outline some improvements in the computation of probability weights for the chi-bar squared distributions. These probabilities, which are needed to apply the likelihood ratio procedures, are often considered as the main obstacle in applying likelihood ratio tests in the context of inequality constraints; we argue, instead, that recent advances in computational technology make such computations affordable in most cases.

In Sect. 2 we introduce the notation, present a formal statement of the testing problem, discuss its basic features and recall useful results on the chi-bar squared distribution. In Sect. 3 we present two new testing procedures, one based on a decomposition of the likelihood ratio statistics and the other on Multiple Comparisons approaches. The results of a simulation study for comparing the two procedures are discussed in Sect. 4; in principle, the likelihood ratios procedures, should make a more efficient use of the data: this conjecture seems to be supported by the results of the study. Finally, two real data sets are analyzed in Sect. 5 to illustrate the procedures.
2 Notation and preliminary results

Consider a contingency table determined by the joint distribution of $d$ discrete random variables and let $i$ be a vector of $d$ indices; let $p_i$ denote the probability that an observation falls in cell $i$ and let $p$ be the vector whose elements $p_i$ are arranged in lexicographic order relative to $i$ and assume that its elements are strictly positive. Let $\eta$ be a vector of marginal log-linear parameters as defined in Bartolucci et al. (2007) or Bergsma and Rudas (2002) and assume that the mapping between $\eta$ and $p$ is a diffeomorphism. Any such vector may be defined as

$$\eta = C \log(M \, p)$$

where $M$ is a matrix of 0’s and 1’s which produce the appropriate marginal or aggregated probabilities and $C$ is a matrix of row contrasts. This formulation allows to consider ordinary log-linear parameters as well as logits and higher order interactions of type global or continuation. Assume that $\eta$ is contained in an open subset of $\mathbb{R}^{t-1}$, where $t$ is the number of cells of the table.

Let $\mathcal{C}$ denote a closed convex cone and assume that $\mathcal{L}_0$, $\mathcal{L}_1$ are, respectively, the linear space of largest and smallest dimensions such that $\mathcal{L}_0 \subset \mathcal{C} \subset \mathcal{L}_1$. Let $\dim(\mathcal{L}_0) = q$, $\dim(\mathcal{L}_1) = r$; finally let $S \supseteq \mathcal{L}_1$ be the parameter space of the saturated model. Define

$$H_0 : \eta \in \mathcal{L}_0, \quad H_1 : \eta \in \mathcal{C}, \quad H_2 : \eta \in S$$

and note that $H_1$ will usually be defined by a set of equality and inequality constraints while $H_0$ is obtained by turning all inequalities into strict equalities.

2.1 The likelihood ratio statistics and the Chi-bar squared distribution

Let $L_{01}$ be the log-likelihood ratio for testing $H_0$ against $H_1$ and $L_{12}$ the log-likelihood ratio for testing $H_1$ against $H_2$. It is well known that the asymptotic distribution of $L_{02} = L_{01} + L_{12}$ is $\chi^2_s$, where $s = t - q - 1$. Let $F_0$ be the expected information matrix of $\eta$ under $H_0$ and $V_0 = F_0^{-1}$. It can also be shown (see for example Silvapulle and Sen 2005, 4.3) that

$$Pr(L_{01} > c \mid H_0) = \sum_{q} w_j(V_0, \mathcal{C}) Pr(\chi^2_{j-q} > c),$$

where $\chi^2_j$ denotes a chi square random variable with $j$ degrees of freedom. The above distribution, known as chi-bar squared, depends on the probability weights $w_j(V_0, \mathcal{C})$ whose definition and computation we discuss below. It can be shown that, under $H_0$, $L_{12}$ is asymptotically distributed like the sum of a $\chi^2_{t-1-r}$ and a chi-bar squared with weights $w_j(V_0, \mathcal{C})$ used in the reverse order (see for instance Silvapulle and Sen 2005, 4.8.5). Let $\hat{\eta}$ denote the unrestricted maximum likelihood estimate; an interesting geometric interpretation is that $L_{01}$ and $L_{12}$ are asymptotically equivalent to the squared norm of the projection of $\hat{\eta}$ onto, respectively, the convex cone defined

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by \( H_1 \) and onto its dual, in the metric defined by \( F_0^{-1} \). From this, the following simple expression for the asymptotic joint distribution of \( L_{01} \) and \( L_{12} \) can be derived (see for example Dardanoni and Forcina 1999, Lemma 2):

\[
Pr(L_{01} \leq c_1, L_{12} \leq c_2 \mid H_0) = \sum_{q} w_j(V_0, C) Pr(\chi_j^2 \leq c_1) Pr(\chi_{t-j-1}^2 \leq c_2)
\]

The previous joint distribution will play a key role in the next section where we consider testing procedures which use the statistics \( L_{01} \) and \( L_{12} \) simultaneously.

2.1.1 Computation of the probability weights

Fast and accurate computation of the weights \( w_j(V_0, C) \) is crucial in order to apply the testing procedures of Sect. 3.1 below, so we discuss two alternative methods that can be used to evaluate these probabilities. For simplicity, we restrict to the context where \( r = t - 1 \), meaning that \( H_1 \) is defined only by inequality constraints. The more general case where equality constraints are also present, so that \( L_1 \subset S \), can be reduced by replacing \( y \) with its projection onto \( L_1 \) and \( t - 1 \) with \( r \), see Shapiro (1988) for a detailed treatment.

**Lemma 1** Let \( x \) be distributed as multivariate normal \( N(\mathbf{0}, V_0) \), then \( w_j(V_0, C) \) is the probability that the projection of \( x \) onto \( C \) falls on a face which spans a linear space of dimension \( j \).

**Proof** See, for example, Silvapulle and Sen (2005), Prop. 3.6.1. \( \Box \)

Recent results (Genz and Bretz 2009) on multivariate normal integrals, make computation of weights fast and accurate for moderate values of \( r \). An algorithm for computing the probability weights derived from Kudo (Kudo 1963, pp. 409–416) is implemented in the R-packages ic-infer Grömping (2010) and hmmm (Colombi et al. 2014); a slightly different implementation is outlined in the “Appendix 1”.

When the number of inequalities is, say, larger than 15, exact computation becomes very hard. In such cases, one can use the procedure proposed by Dardanoni and Forcina (1998). According to this procedure sample points from the appropriate normal distribution are projected onto \( C \) and then the weights \( w_j(V_0, C) \) are estimated by the proportion of sample points falling on any face of \( C \) of dimension \( j \). In order to determine the minimum number of sample points required for accurate estimation of probability weights (see Dardanoni and Forcina 1998).

In general (see Silvapulle and Sen 2005, 4.3.1) nuisance parameters are going to affect \( V_0 \) and thus the probability weights. The formally correct procedure would be to search for the least favorable null distribution, a task which, however, may be very hard. In addition, it often turns out that the values of the nuisance parameters that produce the least favorable distribution are very extreme and substantially different from any plausible estimate obtained from the data. An alternative procedure would be to compute the weights after replacing the nuisance parameters with their maximum likelihood estimate; a simulation study (Dardanoni and Forcina 1998, 4.5) indicates
that the \( p \)-value computed in this way is sufficiently close to the one computed at the true value of the nuisance parameters when the sample size is moderately large.

In certain contexts one could remove the dependence on nuisance parameters by conditioning; for instance, if we are interested in the dependence structure of a two way table, we might condition to the row and columns totals, an approach explored by Bartolucci et al. (2001). The resulting null distributions are, however, hard to handle even with the power of modern computers and can be applied only with small sample sizes and in specific contexts.

3 Testing procedures

In this section we present two procedures for testing \( H_0 \), simultaneously against \( H_1 \) and \( H_2 \); they are designed to overcome the limitations of the basic procedure based on the \( L_{01} \) statistics and retain the central role of \( H_0 \).

Both procedures may lead to one of the following decisions: (1) accept \( H_0 \), (2) reject \( H_0 \) in the direction of \( H_1 \), meaning that there is strong evidence to support the assumed inequality constraints, (3) reject \( H_0 \) in the direction of \( H_2 \), if there is convincing evidence that the inequality constraints are violated.

3.1 Likelihood ratios

As emphasized, for instance, by Agresti and Coull (2002), the evidence in favour of \( H_1 \) provided by \( L_{01} \) may be highly misleading because a large value of this statistic, which, if considered in its own, would lead to reject \( H_0 \) in favour of \( H_1 \), is compatible with substantial violations of \( H_1 \). A geometric explanation is that this event will happen whenever \( \hat{\eta} \) is far away from \( L_0 \) and is not contained neither in \( C \), nor into its dual. Thus, before deciding that the assumed set of inequality constraints are satisfied, one should also examine \( L_{12} \); a large value of this statistic provides evidence that \( H_1 \) is violated in the direction of \( H_2 \).

We now describe a procedure for testing \( H_0 \) against both \( H_1 \) and \( H_2 \) which depends on a parameter that can be tuned in order to provide the desired amount of protection against rejecting \( H_0 \) in favour of \( H_1 \) when the assumed inequality constraints are violated in the sample. The procedure is derived from a similar one proposed by Dar-danoni and Forcina (1999) by improving the algorithm for determining the thresholds and adapting it to the context of contingency tables.

**Definition 1** The tunable LR procedure. Given the error rates \( \alpha_1 \), \( \alpha_2 \) and \( \alpha_{12} \leq \alpha_2 \) and the thresholds \( c_1 \), \( c_2 \) and \( c_{12} \), determined by the following equations:

\[
Pr(L_{12} > c_2 \mid H_0) = \alpha_2 - \alpha_{12}, \\
Pr(L_{01} \leq c_1, L_{12} \leq c_2 \mid H_0) = 1 - \alpha_1 - \alpha_2, \\
Pr(L_{01} > c_1, L_{12} \leq c_{12} \mid H_0) = \alpha_1,
\]

the procedure will:
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1. accept $H_0$ if $L_{12} \leq c_2$ and $L_{01} \leq c_1$.
2. reject $H_0$ in favour of $H_1$ if $L_{01} > c_1$ and $L_{12} \leq c_{12}$.
3. otherwise reject $H_0$ in favour of $H_2$.

The procedure requires the user to set the two error rates $\alpha_1, \alpha_2$ of rejecting towards $H_1$ and $H_2$ respectively, and to partition $\alpha_2$ into the tunable parameter $\alpha_{12}$ and $\alpha_2 - \alpha_{12}$.

A special case of the tunable procedure is when $\alpha_{12} = 0$ which implies $c_{12} = c_2$; this setting may be considered the most rough version of the procedure where the null is rejected towards $H_2$ if $L_{12} > c_2$, otherwise one rejects towards $H_1$ if $L_{01} > c_1$.

The other special case is when $\alpha_{12} = \alpha_2$ which implies that $c_2 = +\infty$. Figure 1 shows the location of the rejection regions for the tunable procedure and indicates that, when we increase $\alpha_{12}$, $c_2$ also increases, while $c_1$ and $c_{12}$ both decrease. It follows that, when $L_{01}$ is large, the evidence in favour $H_1$ is submitted to a stricter scrutiny. This suggests that large values of $\alpha_{12}$ are needed to detect smaller violations of $H_1$ in the direction of $H_2$. The drawback in using large values of $\alpha_{12}$ is that the procedure will become more conservative in rejecting $H_0$ in the direction of $H_2$.

3.2 Multiple comparison procedures

There is a collection of procedures which, due to their computational simplicity, have received much attention in Econometric applications aimed at comparing income inequalities or well-being, see for instance Bishop et al. (1991). Recently, Bennet (2013) has proposed a procedure which, in addition to being more flexible than Bishop’s procedure, should allow more protection against accepting a given order restriction when violated in the data. More precisely, Bennet is concerned with comparing two continuous distributions of well being and considered three alternatives to the null (the two distributions are equal): (1) the first distribution dominates the second, (2) the second distribution dominates the first and (3) no dominance relation hold.

Fig. 1 Acceptance region, $H_0$, and rejection regions, $H_1$, $H_2$, for $\alpha_{12} = 0$ on the left and $\alpha_{12}$ positive on the right.
To adapt an extension of Bennet’s approach to the context of contingency tables, we restrict attention to the case where $H_1$ is defined by $D \eta \geq 0$, where the $k \times (t - 1)$ matrix $D$ is of full row rank. Let $\tau = D \eta$ and $\hat{\tau}$ denote the unrestricted maximum likelihood estimator. Let $n$ denote the sample size; under $H_0$ and the usual regularity conditions, it follows that the asymptotic distribution of $z = \sqrt{n} \hat{\tau}$ is multivariate normal with covariance matrix $\Psi_0 = DV_0D'$. Let $\max(z)$ and $\min(z)$ denote the greatest and the lowest component respectively of $z$.

As in the likelihood ratio procedure described above, the user needs to set the significance levels $\alpha_1$, $\alpha_2$ and split $\alpha_2$ into $\alpha_{12}$ and $\alpha_2 - \alpha_{12}$.

**Definition 2 The tunable Multiple Comparisons (MC) procedure.** Given the error rates $\alpha_1$, $\alpha_2$ and $\alpha_{12} \leq \alpha_2$ and the thresholds $c_1$, $c_2$ and $c_{12}$, which satisfy the equations:

$$
Pr(\min(z) < -c_2 \mid H_0) = \alpha_2 - \alpha_{12},
$$

$$
Pr(\max(z) \leq c_1, \min(z) \geq -c_2 \mid H_0) = 1 - \alpha_1 - \alpha_2,
$$

$$
Pr(\max(z) > c_1, \min(z) \geq -c_{12} \mid H_0) = \alpha_1,
$$

the procedure will:

1. accept $H_0$ if $\min(z) \geq -c_2$ and $\max(z) \leq c_1$,
2. reject $H_0$ in favour of $H_1$ if $\max(z) > c_1$ and $\min(z) \geq -c_{12}$,
3. otherwise reject $H_0$ for $H_2$.

The shape of acceptance and rejection regions for $\alpha_{12} = 0$ and $\alpha_{12} > 0$ are displayed in Fig. 2. The extreme case when $\alpha_{12} = 0$ resembles the procedure proposed by Bishop et al. (1991) which, however, is determined by a single threshold and does not allows to control for the error rate in the direction of each alternative. When $\alpha_{12}$ increases, $c_2$ also increases while $c_1$, $c_{12}$ decrease; as a consequence, conditionally on $\max(z) \leq c_1$, the probability of rejecting $H_0$ in the direction of $H_2$ decreases; on the other hand,
conditionally on \( \max(z) > c_1 \), the probability of rejecting \( H_0 \) towards \( H_2 \), rather than \( H_1 \), increases. In this way, when \( \max(z) \) is sufficiently large, the evidence in favour of \( H_1 \) is submitted to a stricter scrutiny.

Multiple comparison procedures are computationally simpler for two reasons: (1) they can be applied without the need to fit the inequality constrained model and (2) they do not require computation of the probability weights of the chi bar squared distribution and exploit modern advances in the computation of probability integrals for the multivariate normal distribution. Critical values for the previous procedures rely on the computation of the probability \( \Phi(a, b, \Psi_0) \) that a multivariate normal \( N(0, \Psi_0) \) lies in the rectangle \([a, b]\). For a survey of this problem in the context of multiple comparison procedures see (Bretz et al. 2011, chapter 3) and (Genz and Bretz 2009, chapter 6). Because \( \alpha_1, \alpha_2 \) and \( \alpha_{12} \) are probabilities defined under the null hypothesis, in order to compute the critical values, the unknown \( \Psi_0 \) must be replaced by its estimate \( \hat{\Psi}_0 \) under \( H_0 \). Alternatively the elements of \( z \) can be divided by the standard error estimated under \( H_0 \) and the critical values computed by evaluating the previous multi-normal integrals using the correlation matrix corresponding to \( \hat{\Psi}_0 \).

### 3.3 R Software

The LR procedures are implemented in the R-packages **hmmm**, version 1.0-3, (Colombi et al. 2014) which computes the weights of the chi-bar squared distribution by the method described in the “Appendix 1” (see the help of the function *hmmm.chibar*). The multiple comparison procedures can be implemented by the R procedures **hmmm.MP** and **hmmm.MPE** available from the authors with their documentation. All the previous procedures to compute multivariate normal integrals, use the Genz and Bretz (2009) algorithm implemented in the R package **mvtnorm** (Genz et al. 2013). In some cases it could be convenient to replace the Genz and Brez’s algorithm with the one introduced by Craig (2008) and implemented in the R package **orthants** (Craig 2012). More numerical experiments are needed to assess the relative merits of the two integration algorithms; results that surely are bound to improve the applicability of the LR and MP procedures here presented.

### 4 Simulation study

To evaluate the performance of the various procedures, we used a targeted set of simulations concerning the problem of testing independence against the assumption that all the log-odds ratios of type local-local in \( 3 \times 3 \) and \( 3 \times 4 \) contingency tables are non negative. The sample size \( n \) and the number of replications \( N \) were fixed to 10,000. To keep the context simple, in all the simulations we set \( \alpha_1 = 0.02 \) and \( \alpha_2 = 0.03 \) combined with a range of values for the tuning parameter \( \alpha_{12} \). Initial simulations were used to check that all procedures achieved, with high accuracy, the correct size under \( H_0 \), the hypothesis of independence; then we considered various versions of \( H_1 \) and \( H_2 \) by selecting specific sets of local odds ratios and constructed the corresponding bivariate distributions having uniform marginals.
On the whole, the results of the simulation experiments indicate that reasonably large values of $\alpha_{12}$ give strong protection against rejecting $H_0$ toward $H_1$ when $H_2$ is true even for small violation of $H_1$. However the tuning parameter should be used with care because when $\alpha_{12} = \alpha_2$, both procedures tend to accept $H_0$ with probability close to 1 even if this is substantially violated in the direction of $H_2$. Results indicate also that the LR procedures perform better than the MC procedures; this, however, is no surprise because the MC procedures are based on a rectangular shape of the acceptance region and make a less efficient use of the data.

4.1 Power under $H_1$

All LR procedures seem to have high power even for moderate violations of $H_0$ in the direction of $H_1$, like in Tables 3a and 4a; with larger violations power gets very close to 1 as in Tables 3b and 4b. Though we explored only the case where all the log-odds ratios were equal, it seems reasonable to expect that the performance is determined by the smallest positive value. The power decreases slightly when $\alpha_{12}$ increases, this is consistent with the fact that larger values of $\alpha_{12}$ provide more protection against rejecting towards $H_1$ when $H_2$ is true. The power of MC procedures is considerably smaller with a relatively large error rate in the direction of $H_0$.

4.2 Power under $H_2$

Here the situation is much more complex because violations of $H_1$ can arise in many different directions and it is unlikely that a procedure can perform best under all possible $H_2$ alternatives. In the simulations we explored a limited range of possibilities which, however, seem to suggest some general conclusions.

The first result, which emerges clearly, is that the LR procedure with $\alpha_{12} = \alpha_2$ cannot be recommended because there are relevant versions of $H_2$ under which this procedure accepts $H_0$ with probability close to 1. This happens, for instance, when all log-odds ratios are negative, which means that all inequalities are violated, like in Table 3g,h. The procedure with $\alpha_{12} = \alpha_2$ has also a rather poor performance when the negative log-odds ratios dominate in number or in absolute value, like in Tables 3i and 4c. In all such cases, the procedure with $\alpha_{12} = \alpha_2$ tends to be terribly conservative. It is true that under few violations of $H_1$, like in Tables 3c,d,j and 4d,e,f,g the procedure with $\alpha_{12} = \alpha_2$ is the best; however, in these cases, most of the other tuned procedures have high power and the improvement produced by the procedure with $\alpha_{12} = \alpha_2$ is rather modest and certainly cannot compensates the very bad performance described above.

The performance of the tuned LR procedures seem to follow this general pattern: when violations of $H_1$ in the direction of $H_2$ are few or the negative log-odds ratios are sufficiently small in absolute value relatively to the positive ones, the procedure with $\alpha_{12} = 0$ tends to reject in the direction of $H_1$ with a relatively large error rate which, however, can be reduced dramatically by increasing $\alpha_{12}$, see for instance Tables 3c,d,j and 4d,e,f,g. Instead, when there is some kind of balance between negative and
positive log-odds ratios, like in Tables 3e,k and 4i, the LR procedures have a relatively large error rate in the direction of $H_0$ which increases with $\alpha_{12}$.

On the whole, MC procedures perform substantially worst than the corresponding LR procedure, with a few exceptions, typically when the negative and positive log-odds ratios are in some kind of balance: compare, for instance, the corresponding entries in Tables 5e,k and 3e,k. MC procedures also do better when the negative log-odds ratios are small or few in number but only for $\alpha_{12} = 0$; however, already at $\alpha_{12} = 0.015$ the LR procedure perform much better, like in Tables 3 and 5c,d. Usually, performance of MC procedures improve with $\alpha_{12}$, though they remain too much inferior relative to the corresponding LR procedures. The MC procedures are again substantially inferior when all log-odds ratios are negative, like in Table 5g,h.

### 5 Examples

Table 1 was used by Kateri and Agresti (2013) as an instance of a context where the statistic $L_{01}$ can give misleading evidence in favour of $H_1$ when we test $H_0$ (independence) against $H_1$ (all local log-odds ratios are non negative).

Here we have $L_{01} = 7.89$ and $L_{12} = 1.75$, so, when we compare these statistics with the critical values of several tunable LR procedures displayed in Table 2 for $\alpha_1 = 0.02$ and $\alpha_2 = 0.03$, we see that all procedures reject $H_0$ in favour of $H_2$, except when the tuning parameter is $\alpha_{12} = 0$. Thus, in this case, a procedure with $\alpha_{12} = 0.015$ seems to be a reasonable choice. For comparison, we also apply a set of multiple comparisons procedures.

The minimum and maximum unconstrained estimates of the log-odds ratios (studentized using the standard errors estimated under $H_0$) are equal to $-1.168$ and $2.186$ respectively. These statistics are assessed against the critical values of several tunable MC procedures displayed in Table 2. All the procedures accept $H_0$, a result which shows that the MC procedures tend to be more conservative.
As a second example, we analyze the data in Table 3 from Bartolucci et al. (2001); this is a $2 \times 5 \times 5$ contingency table where a sample of 2904 males were classified according to two age classes and 5 ordered categories of their own occupational prestige (OP) and that of their fathers. The first issue is whether association between father’s and son’s OP, measured by log-odds ratio of a suitable type, is stronger when sons are older; so, this is equivalent to assume that log-odds ratios increase with son age. If we use the global log-odds ratios, we have $L_{01} = 29.02$ and $L_{12} = 0.29$ and the conclusion is that $H_0$ is rejected in favour of $H_1$ by any LR tunable procedure with $\alpha_1 = 0.02$ and $\alpha_2 = 0.03$. For instance, using again exact weights, with $\alpha_{12} = 0$ the critical values are $c_1 = 25.197$, $c_2 = c_{12} = 11.120$, while with $\alpha_{12} = 0.029$ they are $c_1 = 22.117$, $c_2 = 19.731$, $c_{12} = 1.394$; in both cases $c_{12}$ is much larger than the observed value of $L_{12}$. Instead, if we measure the strength of association by the local log-odds ratios we have $L_{01} = 14.70$ and $L_{12} = 12.55$ and, though we reject again $H_0$ towards $H_1$ with $\alpha_{12} = 0$ because $c_1 = 12.214$, $c_2 = c_{12} = 23.820$, we reject towards $H_2$ if we set $\alpha_{12} = 0.015$ because $c_1 = 10.746$, $c_2 = 26.161$, $c_{12} = 10.785$. This example again shows that the LR tunable procedures allow us to submit $H_1$ to a severe scrutiny, while the standard procedure would often be too liberal relative to accepting the assumed ordering.

For the same data, it might be of interest to consider also the assumption that, when sons are younger, their OP is stochastically smaller than that of their fathers while the situation reverses when they are older; the idea behind is that, due to welfare improvement, sons will, on the whole, be better off than their fathers, but also that OP improves with age. This assumption compares the marginal distributions of fathers and sons conditionally on the age group of the sons, using logits of type global. Here $L_{01} = 130.00$ and $L_{12} = 5.12$ and, with $\alpha_1 = 0.02$, $\alpha_2 = 0.03$, $\alpha_{12} = 0$, the critical values are $c_2 = c_{12} = 9.606$ and $c_1 = 14.211$, so $H_0$ is rejected in favour of $H_1$ though there are substantial violations in the observed data. Instead, if we set $\alpha_{12} = 0.015$, with the critical values $c_1 = 12.719$, $c_2 = 11.292$, $c_{12} = 1.542$ $H_0$ must be rejected in favour of $H_2$. This example shows that, for large values of $L_{01}$ and small values of $L_{12}$, the LR with $\alpha_{12} = 0$ procedure can give false evidence in favour of $H_1$ when $H_2$ is true and that this drawback is eliminated by the introduction of the tuning probability $\alpha_{12}$.

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**Appendix 1: Computation of probability weights**

In order to compute the weights $w_i(V, C)$, it may be useful to summarize the geometry of the projection of a random vector $y \sim \mathcal{N}(0, V)$ onto a convex cone $C = \{ \eta : D\eta \preceq 0 \}$, where $D$ is a $k \times (t - 1)$ matrix of rank $k$. Let $H$ be the left component of the Cholesky decomposition of the positive definite matrix $\Psi = DV D'$. then $z = H^{-1} D y \sim \mathcal{N}(0, I_k)$, the transformation $\lambda = H^{-1} D \eta$ defines the cone $C^* = \{ \lambda : H\lambda \preceq 0, C^* \in \mathbb{R}^k \}$, it follows that: $\min_{\eta \geq 0}(y - \eta)'V^{-1}(y - \eta) = \min_{H\lambda \geq 0}(z - \lambda)'(z - \lambda)$. The cone $C^*$ may also be defined by its generating vec-
tors which are the columns of \( U = H^{-1} \): a vector \( z \) belongs to \( C^* \) if \( z = Uu \) where \( u \geq 0 \). In a similar way the dual cone \( C^{*0} \) is generated by the columns of \( W = -H' \) and note that \( U'W = -I \).

Within the euclidian metric, \( \Re^k \) can be partitioned into \( 2^k \) convex cones as follows: let \( J \) be the collection of all possible subsets of \( (1, \ldots, k) \), including the empty set and the whole set. For any pair \( i, j \in J \), \( i \cup j = (1, \ldots, k) \), let \( (U_i, W_j) \), be the matrix whose columns are, respectively, the columns of \( U \) with index in \( i \) and the columns of \( W \) with index in \( j \); the columns of this matrix generate the convex cone \( C^*(i) \) whose elements, when projected onto \( C^* \), belong to the face generated by the columns of \( U_i \), this face is itself a convex cone of dimension equal to the cardinality \( |i| \) of \( i \). Thus

\[
w_{i+q}(V, C) = w_i(I, C^*) = \sum_{|i| = i} P[z \in C^*(i)], \quad i = 0, 1, \ldots, k
\]

where \( q \) is the dimension of \( L_0 \).

To compute \( P[z \in C^*(i)] \) note that \( z \in C^*(i) \) if and only if \( t = (U_i, W_j)^{-1} z \geq 0 \), in other words, the linear transformation above reduces \( C^*(i) \) into the positive orthant for the multivariate normal random variable \( t \); thus, to compute \( P[t \in \Re^{k+}] \), the only quantity we need is \( \text{Var}(t) = \Omega \). Let \( \Psi = WW' \) and \( \Phi = (U'U) \) and note that \( \Psi = DV D' = \Phi^{-1} \). It can be shown that \( \Omega \) is block diagonal with elements given by \( (\Phi_{ii})^{-1} \) and \( (\Psi_{jj})^{-1} \), which are related by the well known formulas for the inverse of a partitioned matrix:

\[
(\Phi_{ii})^{-1} = \Psi_{ii} - \Psi_{ij}(\Psi_{jj})^{-1}\Psi_{ji}
\]
\[
(\Psi_{jj})^{-1} = \Phi_{jj} - \Phi_{ji}(\Phi_{ii})^{-1}\Phi_{ij}.
\]

So, if \( |i| \leq |j| \), it is convenient to compute \( (\Phi_{ii})^{-1} \) directly and \( (\Psi_{jj})^{-1} \) from the second expression above, instead, when \( |i| > |j| \), compute \( (\Psi_{jj})^{-1} \) directly and \( (\Phi_{ii})^{-1} \) from the first expression above. In any case, because \( \Omega \) is block diagonal, \( P[t \in \Re^{k+}] \) factorizes into the product of two lower dimensional integrals.

Because Proposition 3.6.1(3) in Silvapulle and Sen (2005) says that the weights with index \( j \) even or odd sum to 0.5, we may avoid computing the two weight which correspond to the largest number of side cones; these correspond to \((k/2 - 1, k/2)\) when \( k \) is even and to \(((k - 1)/2, (k + 1)/2)\) when \( k \) is odd.

**Appendix 2: Simulations results**

See Tables 3, 4 and 5.
Table 3  Likelihood ratio procedures in 3 × 3 tables

| $\alpha_{12}$ | 0.000 | 0.015 | 0.020 | 0.025 | 0.028 | 0.030 |
|----------------|-------|-------|-------|-------|-------|-------|
| (a) $H_1$: 0.08, 0.08, 0.08, 0.08 |
| H0 | 0.002 | 0.001 | 0.001 | 0.000 | 0.000 | 0.000 |
| H1 | 0.998 | 0.975 | 0.965 | 0.952 | 0.945 | 0.941 |
| H2 | 0.000 | 0.025 | 0.035 | 0.048 | 0.055 | 0.059 |
| (b) $H_1$: 0.15, 0.15, 0.15, 0.15 |
| H0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| H1 | 1.000 | 1.000 | 0.999 | 0.999 | 0.999 | 0.999 |
| H2 | 0.000 | 0.000 | 0.001 | 0.001 | 0.001 | 0.001 |
| (c) $H_2$: 0.08, 0.08, −0.08, 0.08 |
| H0 | 0.172 | 0.121 | 0.111 | 0.104 | 0.100 | 0.096 |
| H1 | 0.769 | 0.351 | 0.305 | 0.271 | 0.253 | 0.245 |
| H2 | 0.059 | 0.528 | 0.584 | 0.626 | 0.647 | 0.658 |
| (d) $H_2$: 0.15, 0.15, −0.15, 0.15 |
| H0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| H1 | 0.690 | 0.092 | 0.072 | 0.060 | 0.054 | 0.051 |
| H2 | 0.310 | 0.908 | 0.928 | 0.940 | 0.946 | 0.949 |
| (e) $H_2$: −0.08, 0.08, 0.08, −0.08 |
| H0 | 0.785 | 0.828 | 0.845 | 0.867 | 0.889 | 0.919 |
| H1 | 0.034 | 0.010 | 0.009 | 0.009 | 0.008 | 0.008 |
| H2 | 0.181 | 0.162 | 0.146 | 0.124 | 0.103 | 0.074 |
| (f) $H_2$: −0.15, 0.15, 0.15, −0.15 |
| H0 | 0.305 | 0.380 | 0.424 | 0.495 | 0.581 | 0.833 |
| H1 | 0.031 | 0.002 | 0.001 | 0.001 | 0.001 | 0.001 |
| H2 | 0.664 | 0.619 | 0.575 | 0.503 | 0.419 | 0.167 |
| (g) $H_2$: −0.08, −0.08, −0.08, −0.08 |
| H0 | 0.003 | 0.007 | 0.009 | 0.016 | 0.033 | 1.000 |
| H1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| H2 | 0.997 | 0.993 | 0.991 | 0.984 | 0.967 | 0.000 |
| (h) $H_2$: −0.15, −0.15, −0.15, −0.15 |
| H0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 |
| H1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| H2 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.000 |
| (i) $H_2$: −0.15, 0.04, 0.04, −0.15 |
| H0 | 0.047 | 0.080 | 0.105 | 0.154 | 0.230 | 1.000 |
| H1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| H2 | 0.953 | 0.920 | 0.895 | 0.846 | 0.770 | 0.000 |
Testing order restrictions...

Table 3 continued

| $\alpha_{12}$ | 0.000 | 0.015 | 0.020 | 0.025 | 0.028 | 0.030 |
|---------------|-------|-------|-------|-------|-------|-------|
| (j) $H_2$: $-0.04, 0.15, 0.15, -0.04$ | | | | | | |
| $H_0$ | 0.044 | 0.028 | 0.024 | 0.022 | 0.020 | 0.020 |
| $H_1$ | 0.925 | 0.468 | 0.412 | 0.373 | 0.353 | 0.340 |
| $H_2$ | 0.031 | 0.504 | 0.563 | 0.605 | 0.627 | 0.641 |
| (k) $H_2$: $-0.12, 0.08, 0.08, -0.12$ | | | | | | |
| 0 | 0.469 | 0.574 | 0.626 | 0.708 | 0.795 | 0.993 |
| $H_1$ | 0.001 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $H_2$ | 0.530 | 0.426 | 0.374 | 0.292 | 0.205 | 0.007 |

Each column corresponds to a different testing procedure identified by the value of the tuning parameter $\alpha_{12}$. Each sub-table corresponds to a different experiment, identified by the set of four log-odds ratios used to generate the data, and has three rows giving the relative frequencies of: (1) acceptance of $H_0$, (2) rejections in favour of $H_1$ and (3) rejections in favour of $H_2$.

Table 4 Likelihood ratio procedures in $3 \times 4$ tables

| $\alpha_{12}$ | 0.000 | 0.015 | 0.020 | 0.025 | 0.028 | 0.030 |
|---------------|-------|-------|-------|-------|-------|-------|
| (a) $H_2$: $0.06, 0.06, 0.06, 0.06, 0.06, 0.06$ | | | | | | |
| $H_0$ | 0.001 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $H_1$ | 0.999 | 0.960 | 0.946 | 0.930 | 0.921 | 0.916 |
| $H_2$ | 0.000 | 0.040 | 0.054 | 0.069 | 0.078 | 0.084 |
| (b) $H_2$: $0.12, 0.12, 0.12, 0.12, 0.12, 0.12$ | | | | | | |
| 0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $H_1$ | 1.000 | 0.998 | 0.997 | 0.995 | 0.994 | 0.993 |
| $H_2$ | 0.000 | 0.002 | 0.003 | 0.005 | 0.006 | 0.007 |
| (c) $H_2$: $-0.17, 0.15, 0.15, -0.17, -0.17, 0.15$ | | | | | | |
| $H_0$ | 0.172 | 0.248 | 0.292 | 0.371 | 0.474 | 0.942 |
| $H_1$ | 0.005 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $H_2$ | 0.823 | 0.751 | 0.708 | 0.629 | 0.526 | 0.058 |
| (d) $H_2$: $0.12, 0.12, -0.12, 0.12, 0.12, 0.12$ | | | | | | |
| $H_0$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $H_1$ | 0.952 | 0.483 | 0.428 | 0.390 | 0.367 | 0.356 |
| $H_2$ | 0.048 | 0.517 | 0.572 | 0.610 | 0.633 | 0.644 |
| (e) $H_2$: $0.15, 0.15, -0.15, 0.15, 0.15, 0.15$ | | | | | | |
| $H_0$ | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| $H_1$ | 0.902 | 0.331 | 0.285 | 0.251 | 0.235 | 0.227 |
| $H_2$ | 0.098 | 0.669 | 0.715 | 0.749 | 0.765 | 0.773 |
Table 4 continued

| $\alpha_{12}$ | 0.000 | 0.015 | 0.020 | 0.025 | 0.028 | 0.030 |
|---------------|-------|-------|-------|-------|-------|-------|
| (f) $H_2$: $-0.6, 0.15, -0.6, 0.15$ |       |       |       |       |       |       |
| H0            | 0.021 | 0.012 | 0.010 | 0.009 | 0.008 | 0.008 |
| H1            | 0.932 | 0.433 | 0.377 | 0.337 | 0.316 | 0.306 |
| H2            | 0.047 | 0.555 | 0.613 | 0.654 | 0.676 | 0.685 |
| (g) $H_2$: $0.15, 0.15, -0.15, 0.15, 0.15, 0.15$ |       |       |       |       |       |       |
| H0            | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| H1            | 0.902 | 0.331 | 0.285 | 0.251 | 0.235 | 0.227 |
| H2            | 0.098 | 0.669 | 0.715 | 0.749 | 0.765 | 0.773 |
| (i) $H_2$: $-0.15, 0.15, 0.15, -0.15, 0.15$ |       |       |       |       |       |       |
| H0            | 0.316 | 0.386 | 0.432 | 0.503 | 0.579 | 0.814 |
| H1            | 0.037 | 0.003 | 0.002 | 0.002 | 0.001 | 0.001 |
| H2            | 0.647 | 0.611 | 0.566 | 0.496 | 0.420 | 0.184 |

Each column corresponds to a different testing procedure identified by the value of the tuning parameter $\alpha_{12}$. Each sub-table corresponds to a different experiment, identified by the set of four log-odds ratios used to generate the data, and has three rows giving the relative frequencies of: (i) acceptance of $H_0$, (ii) rejections in favour of $H_1$ and (iii) rejections in favour of $H_2$.

Table 5 Multiple comparisons procedures in $3 \times 3$ tables

| $\alpha_{12}$ | 0.000 | 0.015 | 0.020 | 0.025 | 0.028 | 0.030 |
|---------------|-------|-------|-------|-------|-------|-------|
| (a) $H_1$: $0.08, 0.08, 0.08, 0.08$ |       |       |       |       |       |       |
| H0            | 0.573 | 0.464 | 0.438 | 0.417 | 0.406 | 0.401 |
| H1            | 0.427 | 0.533 | 0.558 | 0.578 | 0.589 | 0.593 |
| H2            | 0.001 | 0.004 | 0.004 | 0.005 | 0.005 | 0.006 |
| (b) $H_1$: $0.15, 0.15, 0.15, 0.15$ |       |       |       |       |       |       |
| H0            | 0.010 | 0.003 | 0.002 | 0.001 | 0.001 | 0.001 |
| H1            | 0.990 | 0.997 | 0.999 | 0.999 | 0.999 | 0.999 |
| H2            | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| (c) $H_2$: $0.08, 0.08, -0.08, 0.08$ |       |       |       |       |       |       |
| H0            | 0.608 | 0.547 | 0.533 | 0.520 | 0.516 | 0.514 |
| H1            | 0.255 | 0.234 | 0.230 | 0.228 | 0.228 | 0.229 |
| H2            | 0.137 | 0.219 | 0.238 | 0.251 | 0.256 | 0.257 |
| (d) $H_2$: $0.15, 0.15, -0.15, 0.15$ |       |       |       |       |       |       |
| H0            | 0.047 | 0.025 | 0.021 | 0.019 | 0.018 | 0.017 |
| H1            | 0.424 | 0.229 | 0.201 | 0.185 | 0.177 | 0.172 |
| H2            | 0.529 | 0.746 | 0.778 | 0.797 | 0.805 | 0.810 |
| (e) $H_2$: $-0.08, 0.08, 0.08, -0.08$ |       |       |       |       |       |       |
| H0            | 0.658 | 0.664 | 0.670 | 0.676 | 0.682 | 0.685 |
| H1            | 0.097 | 0.048 | 0.041 | 0.037 | 0.035 | 0.034 |
| H2            | 0.245 | 0.287 | 0.289 | 0.287 | 0.283 | 0.281 |
Table 5  continued

| $\alpha_{12}$ | 0.000 | 0.015 | 0.020 | 0.025 | 0.028 | 0.030 |
|---------------|-------|-------|-------|-------|-------|-------|
| (f) $H_2$: $-0.15, 0.15, 0.15, -0.15$ |       |       |       |       |       |       |
| H0            | 0.164 | 0.168 | 0.174 | 0.182 | 0.189 | 0.195 |
| H1            | 0.100 | 0.022 | 0.016 | 0.012 | 0.011 | 0.010 |
| H2            | 0.737 | 0.810 | 0.810 | 0.806 | 0.800 | 0.794 |
| (g) $H_2$: $-0.08, -0.08, -0.08, -0.08$ |       |       |       |       |       |       |
| H0            | 0.524 | 0.675 | 0.742 | 0.827 | 0.902 | 0.999 |
| H1            | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| H2            | 0.476 | 0.325 | 0.258 | 0.173 | 0.098 | 0.001 |
| (h) $H_2$: $-0.15, -0.15, -0.15, -0.15$ |       |       |       |       |       |       |
| H0            | 0.008 | 0.038 | 0.067 | 0.153 | 0.313 | 1.000 |
| H1            | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| H2            | 0.992 | 0.962 | 0.933 | 0.847 | 0.687 | 0.000 |
| (i) $H_2$: $-0.15, 0.04, 0.04, -0.15$ |       |       |       |       |       |       |
| H0            | 0.262 | 0.360 | 0.416 | 0.515 | 0.621 | 0.891 |
| H1            | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| H2            | 0.737 | 0.640 | 0.584 | 0.485 | 0.379 | 0.109 |
| (j) $H_2$: $-0.04, 0.15, 0.15, -0.04$ |       |       |       |       |       |       |
| H0            | 0.277 | 0.219 | 0.206 | 0.195 | 0.189 | 0.186 |
| H1            | 0.649 | 0.559 | 0.536 | 0.520 | 0.514 | 0.510 |
| H2            | 0.074 | 0.222 | 0.258 | 0.285 | 0.297 | 0.304 |
| (k) $H_2$: $-0.12, 0.08, 0.08, -0.12$ |       |       |       |       |       |       |
| H0            | 0.453 | 0.520 | 0.554 | 0.600 | 0.641 | 0.695 |
| H1            | 0.028 | 0.006 | 0.005 | 0.005 | 0.004 | 0.004 |
| H2            | 0.520 | 0.473 | 0.441 | 0.396 | 0.355 | 0.302 |

Each column corresponds to a different testing procedure identified by the value of the tuning parameter $\alpha_{12}$. Each sub-table corresponds to a different experiment, identified by the set of four log-odds ratios used to generate the data, and has three rows giving the relative frequencies of: (i) acceptance of $H_0$, (ii) rejections in favour of $H_1$ and (iii) rejections in favour of $H_2$.

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