The Cauchy problem for the Pavlov equation9

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Abstract
Commutation of multidimensional vector fields leads to integrable nonlinear dispersionless PDEs that arise in various problems of mathematical physics and have been intensively studied in recent literature. This report aims to solve the scattering and inverse scattering problem for integrable dispersionless PDEs, recently introduced just at a formal level, concentrating on the prototypical example of the Pavlov equation, and to justify an existence theorem for global bounded solutions of the associated Cauchy problem with small data.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Integrable soliton equations, like the Korteweg–de Vries (KdV) [42], the nonlinear Schrödinger (NLS) [83] equations and their integrable (2 + 1)-dimensional generalizations, the Kadomtsev–Petviashvili [36] and Davey–Stewartson [13] equations respectively, play a key role in the study of waves propagating in weakly nonlinear and dispersive media. The inverse spectral transform (IST) method, introduced by Gardner et al [28], is the spectral method allowing one to solve the Cauchy problem for such PDEs, predicting that a localized disturbance evolves into a number of soliton pulses + radiation, and solitons arise as an exact balance between nonlinearity and dispersion [2, 3, 12, 81]. There is another important class of integrable PDEs, the so-called dispersionless PDEs (dPDEs), or PDEs of hydrodynamic type, arising in various problems of Mathematical Physics and intensively studied in the recent literature (see, f.i., in the multidimensional context, [10, 15, 18–20, 22, 24, 32, 38–41, 43–45, 63, 65, 66, 70–73, 78, 79, 82]). The class of integrable dPDEs includes relevant examples, like the dispersionless Kadomtsev–Petviashvili (dKP) equation [49, 74, 84], describing the evolution of weakly nonlinear, nearly one-dimensional waves in Nature, in the absence of dispersion and dissipation [49, 59, 74, 84], the first and second heavenly equations of Plebanski [68], relevant in complex gravity, and the dispersionless 2D Toda (or Boyer–Finley) equation [11, 25], whose elliptic and hyperbolic versions are relevant in twistor theory [11, 29] as integrable Einstein–Weyl geometries [33, 35, 76], and in the ideal Hele-Shaw problem [46, 48, 62, 64, 77].

Since integrable dPDEs arise from the condition of commutation \([L, M] = 0\) of pairs of one-parameter families of vector fields, implying the existence of common zero energy eigenfunctions (elements of the common kernel):

\[
[L, M] = 0 \Rightarrow L\psi = M\psi = 0,
\]

they can be in an arbitrary number of dimensions [82], unlike the soliton PDEs. In addition, due to the lack of dispersion, these multidimensional PDEs may or may not exhibit a gradient catastrophe at finite time. To investigate integrable dPDEs, a novel IST for vector fields, significantly different from that of soliton PDEs, has been recently constructed in [51–53], just at a formal level, (i) to solve their Cauchy problem, (ii) obtain the longtime behavior of solutions, (iii) construct distinguished classes of exact implicit solutions, (iv) establish if, due to the lack of dispersion, the nonlinearity of the dPDE is ‘strong enough’ to cause the gradient catastrophe of localized multidimensional disturbances, and (v) to study analytically the breaking mechanism [51–61].

It is important to remark that this novel IST is based on some critical assumptions, like existence of analytic eigenfunctions. In soliton theory we know that, in contrast with \(1 + 1\) systems, the relevant eigenfunctions for many \(2 + 1\) PDEs (like KPII) are not analytic [1], and the inverse problem is formulated as a \(\bar{\partial}\)-problem. But the methods used in soliton theory for proving the existence of the relevant eigenfunctions fail in the dispersionless case, since the corresponding operators are unbounded. In addition, since the Lax operators are vector fields, the kernel space is a ring, and the inverse problem is intrinsically nonlinear. At last, the dispersionless theory lacks of explicit regular localized solutions (solitons or lumps do not exist), and gradient catastrophes of different nature may occur at finite time.
For all these reasons, it is clearly important to make the IST for vector fields rigorous (even more important than for the case of soliton PDEs), and this is the main goal of this work. To do that, we choose, as illustrative example, the simplest integrable nonlinear dPDE available in the literature, the so-called Pavlov equation [15, 24, 66]

\[ v_{xt} + v_{xy} + v_{x}v_{xy} - v_{y}v_{xx} = 0, \quad v = v(x,y,t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R}, \]

arising in the study of integrable hydrodynamic chains [66], and in Differential Geometry as a particular example of Einstein–Weyl metric [15]. It was first derived in [14] as a conformal symmetry of the second heavenly equation.

If one assumes that the function \( v(x, y, t) \) depends on the variable \( x \) in a very special way, and the integration constant in the operator \( \partial_{x}^{-1}\partial_{y} \), appearing in the evolutionary form of (2), is chosen properly, then equation (2) reduces either to the NLS or to the KdV equation [21]. Therefore regular solutions of KdV and NLS can be used to generate regular solutions of (2), but localization can be achieved only in one of the two space variables.

As it was pointed out to the authors [80], the terms \( v_{xt} + v_{x}v_{xy} - v_{y}v_{xx} \) in equation (2) are in common (up to the interchange of \( x \) and \( y \)) with the zero pressure Prandtl equation for the potential \( \Phi [23] \):

\[ \Phi_{yt} - \Phi_{yy} + \Phi_{y} \Phi_{xy} - \Phi_{x} \Phi_{yy} = 0. \]

The main difference between these two equations is that the friction term of the Prandtl equation is replaced by the diffraction term of the Pavlov equation. While the zero-pressure Prandtl equation with suitable boundary conditions gives rise to blow-up at finite time [23], we prove in this paper that localized and sufficiently small initial data for Pavlov equation remain smooth at all times.

We remark that the inviscid Prandtl equation

\[ \Phi_{yt} + \Phi_{y} \Phi_{xy} - \Phi_{x} \Phi_{yy} = 0 \]

can be linearized using some partial Legendre transformation, and it also shows formation of singularities at finite time (unpublished result by Kuznetsov [47]).

Equation (2) arises as the commutativity condition (1) of the following pair of vector fields [15]

\[ L \equiv \partial_{y}(\lambda + v_{y})\partial_{t}, \]
\[ M \equiv \partial_{t}(\lambda^{2} + \lambda v_{x} - v_{y})\partial_{x}, \]

and is the \( u = 0 \) reduction of the following integrable system of dispersionless PDEs [53]

\[ u_{xt} + u_{xy} + (uu_{x})_{x} + v_{x}u_{y} - v_{y}u_{xx} = 0, \]
\[ v_{xt} + v_{xy} + u_{x}v_{y} + v_{x}v_{y} + v_{y}v_{xx} - v_{y}v_{xx} = 0, \]

describing the most general integrable Einstein–Weyl metric [16, 17]. This system reduces instead, for \( v = 0 \), to the celebrated dKP equation:

\[ u_{xt} + u_{xy} + (uu_{x})_{x} = 0, \quad u = u(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R}, \]

the simplest prototype integrable model for the study of wave breaking in multidimensions [55, 60].

Let us point out that, although the linearized versions of the Pavlov and dKP equations coincide, the formal IST predicts, at least for small data, a regular dynamics for the Pavlov equation, and the gradient catastrophe at finite time for the dKP equation.

In our paper we prove the following result:
Theorem 1.1. Suppose that $v_0(x, y)$ is a Schwartz function with compact support and satisfies a small norm condition (see definition 3.1). Then the IST method provides us with a real function $v(x, y, t)$ such that $v(x, y, 0) = v_0(x, y)$, the functions $\partial_x v(x, y, t)$, $\partial_y v(x, y, t)$, $\partial_{xx}^2 v(x, y, t)$, $\partial_{xy} v(x, y, t)$, $\partial_{yx} v(x, y, t)$, $\partial_{yy} v(x, y, t)$ lie in $C(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ and $C^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$ and satisfy the Pavlov equation (2).

Remark 1.1. The behavior of $\partial_t v(x, y, t)$ at $t = 0$ requires an extra investigation.

Remark 1.2. By introducing new variables $\xi = x + t$, $\tau = x - t$, we transform equation (2) to the form of a nonlinear wave equation. An important class of nonlinear wave equations is the class of equations satisfying the Klainermann’s null-condition [37], see also the textbook [34], first introduced for $n + 1$ equations, $n \geq 3$. The case of 2 spatial variables requires a special treatment [4, 5]. It is easy to check that, in the new variables, the Pavlov equation satisfies the null-conditions from [4], therefore, for sufficiently small Cauchy data, the solution is well-defined and regular for all positive times.

Let us point out that in [4] the non-characteristic Cauchy problem is studied, and the Cauchy data consist of the solution and its $\tau$-derivative at $\tau = 0$. Using the IST method, we study the characteristic Cauchy problem, and our initial data is the function $u(x, y, 0)$.

As in the soliton case, our IST formalism allows one to construct, in principle, also the long-time behavior of the solutions. We plan to do it in a forthcoming publication.

Since the realization of the scheme described above requires a rather big amount of technical work, including estimates on the behavior of the integral equations kernels, to make our text more transparent, we moved the proofs of the analytic estimates to the last section of our paper.

The authors would like to dedicate this paper to the memory of Manakov, who successfully devoted the last period of his life to the construction of the IST method for vector fields, and to its applications to the theory of integrable dispersionless PDEs in multidimensions.

2. The inverse scattering transform: a short summary

We find it convenient to summarize here the basic formal steps associated with this novel IST for the vector field $L$ in (5), allowing one to solve the Cauchy problem for the Pavlov equation [51, 52, 54], whose rigorous aspects will be investigated in the following sections.

The Direct Problem In our paper we always assume that $v(x, y)$ is a real-valued function. In analogy with the IST for KPI equation (whose Lax operator is the non-stationary Schrödinger operator, see [26, 50]), we make essential use of two sets of eigenfunctions—the real Jost eigenfunctions $\varphi_l(x, y, \lambda)$, $\lambda \in \mathbb{R}$, and the complex-analytic in $\lambda$ ones: $\Phi_l(x, y, \lambda)$, $\text{Im} \lambda \geq 0$; $\Phi_l^\pm(x, y, \lambda)$, $\text{Im} \lambda \leq 0$

$$L \varphi_l(x, y, \lambda) = 0, \quad L \Phi_l^\pm(x, y, \lambda) = 0,$$

$$\varphi_l(x, y, \lambda) \to x - \lambda y \quad \text{as} \quad y \to \pm \infty.$$  \hspace{1cm} (8)

The direct spectral transform consists of two steps

- Using the real Jost eigenfunctions, we construct the scattering data $\sigma(\xi, \lambda)$.
- Using the complex-analytic eigenfunctions, we construct the spectral data $\chi(\xi, \lambda)$ through the scattering data.

Step 1. For real $\lambda$, all eigenfunctions of $L$ have the following property: they are constant on the trajectories of the following ODE:
\[
\frac{dx}{dy} = \lambda + v(x, y) \tag{10}
\]
defining the characteristics of \( L \). Indeed, if the potential \( v \) is sufficiently regular and well-localized, the solution of the Cauchy problem \( x(y_0) = x_0 \) for the ODE (10) exists unique globally in the (time) variable \( y \), with the following free particle asymptotic behavior
\[
x(y) \to \lambda y + x_\pm(x_0, y_0, \lambda), \quad y \to \pm \infty. \tag{11}
\]
The asymptotic positions \( x_\pm(x_0, y_0, \lambda) \) are obviously constant when the point \( (x_0, y_0) \) moves along trajectories. Therefore \( x_\pm(x_0, y_0, \lambda) \) are solutions of the vector field equation
\[
[\partial_y + (\lambda + v_\pm(x_0, y_0))]\partial_{x_\pm}(x_0, y_0, \lambda) = 0.
\]
Due to (11) we have
\[
\phi_\pm(x_0, y_0, \lambda) \to x_0 - \lambda y_0 \quad \text{as} \quad y_0 \to \pm \infty,
\]
therefore they coincide with the real Jost eigenfunctions
\[
\phi_\pm(x_0, y_0, \lambda) = x_\pm(x_0, y_0, \lambda). \tag{12}
\]

**Definition 2.1.** Denote by \( \sigma(\xi, \lambda) \) the classical time-scattering datum, connecting the asymptotic behavior of the solutions at \( y \to +\infty \) and at \( y \to -\infty \)
\[
x_+/(x_0, y_0, \lambda) = x_-(x_0, y_0, \lambda) + \sigma(x_-(x_0, y_0, \lambda), \lambda),
\]
therefore
\[
\phi_\pm(x, y, \lambda) \to x - \lambda y + \sigma(x - \lambda y, \lambda) \quad \text{as} \quad y \to -\infty. \tag{13}
\]

**Step 2.** The problem of existence for complex (analytic) eigenfunctions \( \Phi^\pm \) of a vector field is usually highly nontrivial, and in all previous works by Manakov and Santini was only postulated and motivated by the analyticity properties of the Green’s functions of the undressed vector fields. In our paper we present a proof based on the following observation:

For \( \lambda \in \mathbb{C}/\mathbb{R} \), by the change of variables \( z = x - \lambda y, \quad z = x - \lambda y \), the Lax equation \( L\phi(x, y, \lambda) = 0 \) can be transformed into a linear Beltrami equation and can be solved. Moreover, we do not have to assume, at this stage, that the potential \( v(x, y) \) has small norm.

We show below that the limiting functions \( \Phi^\pm(x, y, \lambda) = \Phi(x, y, \lambda \pm i0), \lambda \in \mathbb{R} \) are also well-defined. Both real Jost eigenfunctions \( \phi_\pm(x, y, \lambda) \) enumerate the trajectories of our vector field, therefore any eigenfunction of \( L \) for \( \lambda \in \mathbb{R} \) can be represented as a function either of \( \phi_+(x, y, \lambda) \) or \( \phi_-(x, y, \lambda) \), and we have:
\[
\Phi^+(x, y, \lambda) = \phi_+(x, y, \lambda) + \chi_-(\phi_+(x, y, \lambda), \lambda) = \phi_+(x, y, \lambda) + \chi_-(\phi_+(x, y, \lambda), \lambda)
\]
\[
\Phi^-(x, y, \lambda) = \overline{\Phi^+(x, y, \lambda)}.
\]
defining the spectral data \( \chi_\pm(\xi, \lambda) \).

Assuming that the small \( \lambda_I = \text{Im} \lambda \) behavior be sufficiently good, we see that, for \( \lambda_I \to 0 \), the eigenfunction \( \Phi(x, y, \lambda) \) is almost constant on the trajectories of the vector field \( \hat{L} \equiv \hat{\partial}_y + (\lambda_R + v_\pm)(\hat{\partial}_x) \); these trajectories are straight lines \( \text{Re} z = \text{const} \) outside the support of \( v(x, y) \) and connect the lines \( \text{Re} z = \xi \) and \( \text{Re} z = \xi + \sigma(\xi, \lambda) \) as they go from \( -\infty \) to \( +\infty \) (see figure 1).

Assume now that \( \lambda_I < 0, |\lambda_I| \ll 1 \); then \( \Phi^+(x, y, \lambda) \) is holomorphic in \( z \) outside a small neighborhood of \( \mathbb{R} \): \( \Phi^+(x, y, \lambda) = \hat{\Phi}(z, \lambda) \) and, due to the almost constant behavior on the trajectories:
In the limit $\lambda \to 0$ we have

$$\hat{\phi}(\xi, \lambda) \sim \hat{\phi}(\xi + \sigma(\xi, \lambda) + i\epsilon, \lambda).$$

(15)

In the limit $\lambda \to 0$ we have

$$\hat{\phi}(\xi - i\epsilon, \lambda) = \Phi^-(x, y, \lambda), \quad y < -D_y,$$

(16)

$$\hat{\phi}(\xi + i\epsilon, \lambda) = \Phi^-(x, y, \lambda), \quad y > D_y,$$

(17)

therefore equation (14) implies

$$\hat{\phi}(\xi - i0, \lambda) = \xi + \chi_-(\xi, \lambda), \quad \hat{\phi}(\xi + i0, \lambda) = \xi + \chi_+(\xi, \lambda).$$

Hence the spectral data $\chi_{\pm}(\xi, \lambda)$ of the Pavlov equation satisfy the shifted Riemann–Hilbert (RH) problem

$$\sigma(\xi, \lambda) + \chi_-(\xi, \lambda), \lambda) - \chi_-(\xi, \lambda) = 0, \quad \xi \in \mathbb{R},$$

$$\partial_\xi \chi = 0 \quad \text{for} \quad \xi \in \mathbb{C}^\pm,$$

$$\chi \to 0 \quad \text{as} \quad |\xi| \to \infty.$$

(18)

Equation (18) defines the spectral data $\chi_{\pm}(\xi, \lambda)$ in terms of the scattering data $\sigma(\xi, \lambda)$. No small norm assumption is required also at this step.
Evolution of the spectral data. The evolution of the scattering and spectral data, following from the asymptotics (9) and (13), is given by the explicit formula [53, 54]:

\[
\sigma(\xi, \lambda, t) = \sigma(\xi - \lambda^2 t, \lambda, 0), \quad \chi_{\pm}(\xi, \lambda, t) = \chi_{\pm}(\xi - \lambda^2 t, \lambda, 0),
\]

implying that, from the eigenfunctions \( \varphi_{\pm}, \Phi^{\pm} \) of \( L \), one can constructs the common eigenfunctions \( \psi_{\pm}, \Psi^{\pm} \) of \( L \) and \( M \) through the formulas

\[
\psi_{\pm}(x, y, t, \lambda) = \varphi_{\pm}(x, y, t, \lambda) - \lambda^2 t, \quad \Psi^{\pm}(x, y, t, \lambda) = \Phi^{\pm}(x, y, t, \lambda) - \lambda^2 t,
\]

connected through equations

\[
\Psi^{-}(x, y, t, \lambda) = \psi_{-}(x, y, t, \lambda) + \chi_{-}(\psi_{-}(x, y, t, \lambda), \lambda) = \psi_{-}(x, y, \lambda) + \chi_{-}(\psi_{-}(x, y, \lambda), \lambda)
\]

\[
\Psi^{+}(x, y, t, \lambda) = \Psi^{-}(x, y, t, \lambda).
\]

The inverse problem. The reconstruction of the real eigenfunction \( \psi_{-} \) at time \( t \) from the spectral data \( \chi_{-} \) is provided by the solution of the nonlinear integral equation

\[
\psi_{-}(x, y, t, \lambda) = H_{\chi_{-}}(\psi_{-}(x, y, t, \lambda), \lambda) + \chi_{-R}(\psi_{-}(x, y, t, \lambda), \lambda) = x - \lambda y - \lambda^2 t,
\]

where \( \chi_{-R} \) and \( \chi_{-I} \) are the real and imaginary parts of \( \chi_{-} \) and \( H_{\lambda} \) is the Hilbert transform operator w.r.t \( \lambda \)

\[
H_{\lambda} f(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\lambda')}{\lambda - \lambda'} d\lambda'.
\]

We remark that, since \( \chi_{-}(\xi, \lambda) \) is analytic w.r.t \( \xi \) in the lower half-plane, its real and imaginary parts satisfy the relation \( \chi_{-R} - H_{\chi_{-}} = 0 \). Equation (22) expresses the fact that the rhs of (21) for \( \Psi^{-} \) is the boundary value of a function analytic in \( \lambda \) in the lower half-plane.

Once \( \psi_{-} \) is reconstructed from \( \chi_{-} \) solving the nonlinear integral equation (22), equation (21) gives \( \Psi^{\pm} \), and \( v \) is finally reconstructed from:

\[
v(x, y) = - \lim_{\lambda \to \infty} (\lambda[\Psi^{-}(x, y, \lambda) - (x - \lambda y - \lambda^2 t)]),
\]

or, better, as we shall see, from

\[
v(x, y, t) = - \frac{1}{\pi} \int_{\mathbb{R}} \chi_{-}(\psi_{-}(x, y, t, \zeta)) d\zeta.
\]

Remark 2.1. The main difficulty associated with the direct problem is in the proof of the existence of the analytic eigenfunction and of its limits on the real \( \lambda \) axis from above and below. While such a proof can be made in the Pavlov case, see section 3.2, in the dKP case the existence of the analytic eigenfunctions is proven, at the moment, only sufficiently far from the real \( \lambda \) axis [31]. We also remark that, soon after the formulation of the direct problem through the RH problem (15) [51], an alternative integral equation, obtained taking the Fourier transform of (15), was also suggested [52, 53]. It turns out that, while the construction of the spectral data from the scattering data through the RH problem with shift (15) does not present difficulties, the construction that makes use of the integral equation in Fourier space requires additional effort, due to the bad behavior of its kernel, and will not be considered in this paper.

Remark 2.2. A second inverse problem, a nonlinear RH (NRH) problem on the real line, was also introduced at a formal level [51–53], and intensively used (i) to study the longtime
behavior of the solutions of the target dPDE [55–57]; (ii) to detect if a localized initial disturbance evolving according to such a PDE goes through a gradient catastrophe at finite time (i.e. no gradient catastrophe for the second heavenly equation [52, 56] and for the Pavlov equation [54] was found, while a gradient catastrophe was indeed found for the dKP [57] and for the dispersionless 2D Toda [57] equations); (iii) to investigate analytically the wave breaking mechanism of such multidimensional waves [55, 57]; (iv) to construct classes of RH data giving rise to exactly solvable NRH problems, and to distinguished exact implicit solutions of the dispersionless PDEs through an algorithmic approach [9, 55–58]; (v) to detect integrable differential reductions of the associated hierarchy of PDEs [7, 8], like the Dunajski interpolating equation

\[v_t + v_{xy} + c v_x v_{xx} + v_x v_y - v_y v_{xx} = 0\]  

[16], an integrable PDE interpolating between the dKP and the Pavlov equations, corresponding to the reduction \(u = c v_x\) of system (6). The rigorous aspects of such a NRH inverse problem, as well as the connections with the above inverse problem, will also be investigated in a subsequent paper.

3. Direct spectral transform

3.1. The real eigenfunction

Throughout this paper, for all positive \(\mu\), these norms are equivalent, but in some situations it is necessary to choose an appropriate \(\mu\) to guarantee the contraction property for our integral operators.

In our paper we assume that the potential \(v(x, y)\) has compact support in \(x, y\). We expect that these constraints are not critical and can be weakened (for example, it should be enough to assume that the potential decays sufficiently fast as \(+\to\infty\)), but it may require a serious additional analytic work. To be more precise, let \(D_x, D_y\) be a pair of positive numbers, \(n > D_y\), and \(G \in v_{xy}\), such that

\[v(x, y) = 0 \text{ for } |x| > D_x \text{ or } |y| > D_y.\]  

The real eigenfunctions \(\varphi_\pm(x, y, \lambda)\) for the Pavlov equation are defined by the solution of the boundary value problem: for each fixed \(\lambda \in \mathbb{R}\), [56]

\[\partial_y \varphi_\pm + (\lambda + v_x)\partial_x \varphi_\pm = 0, \quad \text{for } x, y \in \mathbb{R},\]  

(27)

\[\varphi_\pm \to 0, \quad \text{as } y \to \pm\infty,\]  

(28)

where

\[\xi = x - \lambda y.\]  

(29)
Lemma 3.1. Suppose $v \in \mathcal{S}_{x,y}$ satisfying (26). The real eigenfunctions $\varphi_\pm(x - \lambda y)$ are smooth bounded functions.

Proof. The solvability and uniqueness of the boundary value problem of the first order partial differential equations (27) and (28) can be derived by solving the ordinary differential equation

$$\frac{dx}{dy} = \lambda + v(x, y), \quad x = x(y; x_0, y_0, \lambda), \quad x(y_0; x_0, y_0, \lambda) = x_0, \quad (30)$$

or, equivalently,

$$\frac{dh}{dy} = v(y + \lambda y, y), \quad h = h(y; \xi_0, y_0, \lambda), \quad h(y_0; \xi_0, y_0, \lambda) = \xi_0 = x_0 - \lambda y_0, \quad (31)$$

where

$$h(y) = x(y) - \lambda y.$$

Using the Picard iteration method on the integral equation defining the solution (see, for example, [6])

$$h(y; \xi_0, y_0, \lambda) = h_0 + \int_{y_0}^{y} v_i(h(y'; \xi_0, y_0, \lambda) + \lambda y', y') dy' \quad (32)$$

one shows that $x_{\pm}(x_0, y_0, \lambda) = h(\pm n; x_0 - \lambda y_0, y_0, \lambda)$ are smooth functions, $h(\pm n; x_0 - \lambda y_0, y_0, \lambda) = h_0$ are also bounded. Here we used the fact that $h(y; x_0 - \lambda y_0, y_0, \lambda)$ are constant in $y$ in the regions $y \geq n, y \leq -n$ due to the compact support of $v(x, y)$.

We see that

$$\sigma(\xi_0, \lambda) = h(n; \xi_0, -n, \lambda) - \xi_0 = \int_{-\infty}^{\infty} v_i(h(y'; -\infty, \lambda) + \lambda y', y') dy'$$

is also a regular function and the map $\xi \mapsto \xi + \sigma(\xi, \lambda)$ is regularly invertible for all $\lambda$. We do not require the small norm assumption at this step. □

For simplicity and convenience, we will use the following agreement: $C$ denotes a constant, possibly dependent of $\|v\|_{[\nu, \nu]}$, but independent of $x, y, t,$ and $\lambda$ throughout this paper. To construct the spectral data from the scattering data by solving the shifted Riemann–Hilbert problem, it is necessary to control the behavior of the scattering data and its derivatives for large $\lambda$. For solving the inverse problem we also need some estimates for large $\lambda$ and $\xi \sim \lambda^2$.

Proposition 3.1. Suppose $v \in \mathcal{S}_{x,y}$ such that $v(x, y) \equiv 0$ for $|y| > D$. Let us define the following constants $B_k = b_k[v]$ $k = 0, 1, 2, 3, \tilde{B}_k = \tilde{b}_k[v]$ $k = 0, 1$:

$$B_0 = \int_{-\infty}^{\infty} \max_{x \in \mathbb{R}} |v_1(x, y)| dy, \quad (33)$$

$$B_1 = \exp\left[\int_{-\infty}^{\infty} \max_{x \in \mathbb{R}} |v_{1x}(x, y)| dy\right] - 1, \quad (34)$$

$$B_2 = \left[\int_{-\infty}^{\infty} \max_{x \in \mathbb{R}} |v_{1xx}(x, y)| dy\right] (1 + B_1)^3, \quad (35)$$

3717
\[
B_3 = \left[ \int_{-\infty}^{+\infty} \left( \max_{x \in \mathbb{R}} |v_{xx}(x, y)| \right) dy \right] (1 + B_2)^2 B_2
\]
\[ \frac{1}{1 - B_1}, \]
\[ \frac{1}{1 - B_1}. \]

Then we have the following estimates on the scattering data:
\[ |\sigma(\xi, \lambda)| \leq B_0, \quad |\sigma_2(\xi, \lambda)| \leq B_1, \quad |\sigma_{22}(\xi, \lambda)| \leq B_2, \quad |\sigma_{222}(\xi, \lambda)| \leq B_3. \] (40)

Moreover, if \( B_1 < 1 \),
\[ \|\sigma(\xi, \lambda)\|_{L^2(\mathbb{R})} \leq \tilde{B}_0, \quad \|\sigma_2(\xi, \lambda)\|_{L^2(\mathbb{R})} \leq \tilde{B}_1. \]

The proof of proposition 3.1 is moved to the last section. It is rather straightforward and is based on some standard estimates from the ODE theory.

**Definition 3.1.** A potential \( v(x, y) \) satisfies the **small norm condition** if the following inequalities are fulfilled:

1. \( B_0 \leq \frac{\pi}{4} \)
2. \( B_1 \leq 1 \)
3. \( 8B_0 + 4B_2 + 2\sqrt{2} \tilde{B}_0 < \pi \)
4. \( 2B_1 + \frac{2}{\pi}(64B_1 + 16\tilde{B}_0) + \frac{4}{\pi}(8B_1 + 16B_2 + 56B_1 + 16B_1^2)(B_0 + \frac{2}{\pi}[2B_0 + \tilde{B}_0]) < \tan\left(\frac{\pi}{2}\right) \).

The meaning of the combinations of constants arising in this definition will be explained later.

**Proposition 3.2.** Suppose \( v \in \mathcal{S}_{x,y} \) satisfying (26) and \( |\lambda| \) is sufficiently large. Let us introduce new variables
\[ \lambda = \frac{1}{\lambda}, \quad \xi = \frac{\xi}{\lambda}. \]

Then, for sufficiently small \( \lambda \), the function
\[ \hat{\sigma}(\xi, \lambda) = \sigma(\xi/\lambda, 1/\lambda) \]
has the following properties:

1. It vanishes outside the interval \( |\hat{\xi}| \leq D_1 + |\hat{\lambda}|D_1 \) (see figure 2).
2. It is smooth in both variables \( \hat{\xi}, \hat{\lambda} \).

As a corollary we obtain that there exists a collection of positive constants \( C^{(\mu,k)} \) such that
\[ \| \partial^k_x \partial^\mu_\lambda \sigma(\xi, \lambda) \|_{L^\infty} < \frac{c^{(p,k)}}{1 + |\lambda|^{2 + p + k}}, \quad \mu \geq 0, \quad k \geq 0, \]  
\[ \| \partial_x^\mu \partial^\nu_\lambda \sigma(\xi, \lambda) \|_{L^2(\mathbb{C})} < \frac{c^{(p,k)}}{1 + |\lambda|^{2 + p + \nu}}, \quad \mu \geq 0, \quad k \geq 0. \]

As usual, we move the proof to the last section of our paper.

### 3.2. The complex eigenfunction

In this section, we prove that there exists a unique eigenfunction \( \Phi(x, y, \lambda) \) for each \( \lambda \in \mathbb{C}^\pm \). Moreover, \( \Phi(x, y, \lambda) \) is holomorphic in \( \lambda \in \mathbb{C}^\pm \), its boundary values on \( \mathbb{R} \), denoted as \( \Phi^\pm(x, y, \lambda) \), are well-defined and can be characterized by the shifted Riemann–Hilbert problem (61).

For \( \lambda \in \mathbb{C}^\pm \), we introduce the following complex notations:

\[
\begin{align*}
    z &= x - \lambda y, & \z &= x - \overline{\lambda} y, \\
    x &= \frac{1}{\lambda - \overline{\lambda}} (\overline{\lambda} z - \lambda \z), & y &= \frac{1}{\lambda - \overline{\lambda}} (z - \z), \\
    \partial_z &= -\frac{1}{\lambda - \overline{\lambda}} (\partial_z + \lambda \partial_{\z}), & \partial_{\z} &= \frac{1}{\lambda - \overline{\lambda}} (\partial_z + \overline{\lambda} \partial_{\z}), \\
    \partial_x &= \partial_z + \partial_{\z}, & \partial_{\z} &= - (\overline{\lambda} \partial_z + \lambda \partial_{\z}).
\end{align*}
\]

So \( W^{2,p}(dx\,dy) = W^{2,p}(dz\,d\z) = W^{2,p} \) for each \( \lambda \in \mathbb{C}^\pm \).

**Theorem 3.1.** For \( v \in \mathbb{S}_{x,y} \) and \( \lambda \in \mathbb{C}^\pm \), there exists a unique continuous eigenfunction \( \Phi(x, y, \lambda) \) and a positive function \( \epsilon(\lambda) \) such that

\[ \Phi = z + \partial_z^{-1} \alpha(z, \lambda), \quad z = x - \lambda y, \quad \alpha \in W^{2,p}(dz\,d\z), \quad \text{where} \quad |p - 2| < \epsilon(\lambda) \]

and

\[ \partial_x \Phi + (\lambda + v) \partial_{\z} \Phi = 0, \quad \text{for} \quad x, y \in \mathbb{R}, \]  
\[ \Phi(x, y, \lambda) - (x - \lambda y) \to 0, \quad \text{as} \quad x^2 + y^2 \to \infty. \]

Moreover, \( \Phi(x, y, \cdot) \) is holomorphic for \( \lambda \in \mathbb{C}^\pm \), and

\[ \Phi(x, y, \lambda) = \overline{\Phi(x, y, \overline{\lambda})}. \]  

If \( \lambda \to \pm \infty \) we have

\[ \Phi(x, y, \lambda) = x - \lambda y - \frac{1}{\lambda} v(x, y) + o \left( \frac{1}{\lambda} \right). \]

**Proof.** Equation (44) takes the following form:

\[
\left[ \partial_z + \frac{1}{\lambda - \overline{\lambda}} v(z, \z) (\partial_z + \partial_{\z}) \right] \Phi(z, \z, \lambda) = 0,
\]

or equivalently
\[
[\partial_z + b(z, \z, \lambda)\partial_z]\Phi(z, \z, \lambda) = 0, \quad (49)
\]

where

\[
b(z, \z, \lambda) = \frac{v_i(z, \z)}{2i\lambda + v_i(z, \z)}. \quad (50)
\]

The function \(v_i(z, \z)\) is real-valued, therefore

\[
lb(z, \z, \lambda) < 1. \quad (51)
\]

Using the representation

\[
\partial_z \partial_{\z}^{-1} f = \frac{1}{2\pi i} \int \int f(\z, \zeta) \frac{d\zeta \wedge d\zeta}{(\z - \zeta)^2},
\]

and the Zygmund–Calderon operator theory, it is easy to show that for any fixed \(\lambda \in \mathbb{C}^+\) there exist \(\varepsilon_2 > 0\) and \(\mu > 0\) such that for \(|p - 2| < \varepsilon_2\) the norm of the operator

\[
f \in W^{2,p}(\mu) \rightarrow b(z, \z, \lambda)\partial_z \partial_{\z}^{-1} f \in W^{2,p}(\mu)
\]

is smaller than 1 \([69]\). Then we can write \([75]\):

\[
\Phi(z, \z, \lambda) = z + \partial_z^{-1}\alpha(z, \z, \lambda) \quad (53)
\]

where \(\alpha(z, \z, \lambda)\) satisfies the following equation:

\[
[1 + b(z, \z, \lambda)\partial_z \partial_{\z}^{-1}\alpha(z, \z, \lambda) + b(z, \z, \lambda) = 0. \quad (54)
\]

This equation is uniquely solvable in the spaces \(W^{2,p}, |p - 2| < \varepsilon_2\). Therefore \(\partial_z^{-1}\alpha(z, \z, \lambda)\) is a decaying at infinity continuous function by Sobolev’s theorem, and

\[
\|\partial_z^{-1}\alpha\|_{L^\infty(\mu)} \leq C(\varepsilon_2)\|\alpha\|_{L^{2+\mu}(\mu)} + C_2(\varepsilon_2)\|\alpha\|_{L^{2+\mu}(\mu)}.
\]

We also have:

\[
|\det \begin{vmatrix} \partial_z \Phi & \partial_{\z} \Phi \\ \partial_z \bar{\Phi} & \partial_{\z} \bar{\Phi} \end{vmatrix}| = (1 - \lb(z, \z, \lambda)|^2)}|\partial_{\z} \Phi|^2 \geq 0, \quad (55)
\]

therefore for all regular points of the map \((z, \z) \rightarrow (\Phi, \bar{\Phi})\) the Jacobian is positive, and the number of preimages is the same for all regular points. It means that the number of preimages is the same for all regular points. This map is one-to-one at infinity, therefore it is invertible and we can use \(w = \Phi\) as a global coordinate on the \(z\)-plane. In this coordinate all solutions of (44) are functions holomorphic in \(w\) (see chapter II in [75]). So Liouville’s theorem implies that asymptotics (45) fixes the solution uniquely.

Let us show that \(\Phi(x, y, \lambda)\) is holomorphic in \(\lambda\) outside the real line. Differentiating (44) by \(\tilde{\lambda}\) we obtain

\[
L \partial_{\tilde{\lambda}} \Phi(x, y, \lambda) = 0, \quad (56)
\]

and
\[ \partial_\tau \Phi(x, y, \lambda) = o(1) \quad \text{as} \quad x^2 + y^2 \to \infty. \]  

(57)

Therefore \( \partial_\tau \Phi(x, y, \lambda) \) is a regular holomorphic function in \( w \) decaying at infinity, and by Liouville’s theorem \( \partial_\tau \Phi(x, y, \lambda) \equiv 0 \).

The reality condition (46) follows from applying Liouville’s theorem and the reality conditions \( v(x, y) = \overline{v(x, y)} \).

Let \( |\lambda| \gg 1 \). Taking into account, that \( dz \wedge d\sigma = 2i/\lambda \) dx \wedge dy we see, that

\[ \alpha(z, \bar{\sigma}, \lambda) = -b(z, \bar{\sigma}, \lambda) + \alpha(z, \bar{\sigma}, \lambda), \]

\[ \|\alpha(z, \bar{\sigma}, \lambda)\| \leq \frac{\max_z |b(z, \bar{\sigma}, \lambda)|}{1 - \max_z |b(z, \bar{\sigma}, \lambda)|} \|b(z, \bar{\sigma}, \lambda)\| = O\left(\frac{\sqrt{|\lambda|}}{\lambda^2}\right). \]

and

\[ \Phi(z, \bar{\sigma}, \lambda) = - \frac{1}{2l/\lambda} \partial_z^{-1} v(z, \bar{\sigma}) + o\left(\frac{1}{\lambda}\right) = z - \frac{1}{2l/\lambda} \partial_z^{-1}(\partial_z + \bar{\partial}_\sigma)v(z, \bar{\sigma}) + o\left(\frac{1}{\lambda}\right), \]

but

\[ \partial_z = - \frac{\bar{\sigma}}{\lambda} \partial_x - \frac{1}{\lambda} \partial_y, \]

therefore

\[ \Phi(z, \bar{\sigma}, \lambda) = z - \frac{1}{2l/\lambda} \partial_z^{-1}(\partial_z - \frac{\bar{\sigma}}{\lambda} \partial_x)v(z, \bar{\sigma}) + o\left(\frac{1}{\lambda}\right) = z - \frac{v(z, \bar{\sigma})}{\lambda} + o\left(\frac{1}{\lambda}\right). \]  

Starting from this point we will work with the Jost eigenfunction \( \varphi \) only; therefore we shall denote it simply by \( \varphi \), omitting the subscript:

\[ \varphi(x, y, \lambda) = \varphi(x, y, \lambda). \]  

(58)

**Theorem 3.2.** Suppose \( v \in \mathcal{S}_2 \) satisfying (26). The complex eigenfunction \( \Phi(x, y, \lambda) \) has continuous extensions on \( \mathbb{C} \cup \mathbb{R} \). Moreover, denote the limits on both sides of \( \mathbb{R} \) as \( \Phi^\pm \), then \( \partial_\lambda^\pm(\Phi^\pm - x + \lambda y) \in W^{1,2}(\mathbb{R}, d\lambda) \).

\[ \Phi(x, y, \lambda) = \varphi(x, y, \lambda) + \chi_- (\varphi(x, y, \lambda), \lambda) \]  

(59)

\[ \Phi^i(x, y, \lambda) = \Phi\left(x, y, \frac{\lambda}{\lambda}\right), \]  

(60)

where \( \chi_- (\xi, \lambda) \) is characterized by the Riemann–Hilbert problem with the shift function \( \sigma(\xi, \lambda) \):

\[ \sigma(\xi, \lambda) + \chi_-(\xi + \sigma(\xi, \lambda), \lambda) - \chi_-(\xi, \lambda) = 0, \quad \xi \in \mathbb{R}, \]

\[ \partial_\xi \chi = 0, \quad \xi \in \mathbb{C}, \]

\[ \chi \to 0, \quad |\xi| \to \infty. \]

(61)
As before, we move the proof to the last section.

Theorem 3.2 implies
\[ \Phi^+(x, y, \lambda) - \Phi^-(x, y, \lambda) = -2i\chi_-(\varphi(x, y, \lambda), \lambda), \quad \lambda \in \mathbb{R}, \]
and, due to (47)
\[ \Phi(x, y, \lambda) = x - \lambda y - \frac{1}{\pi} \int_{\mathbb{R}} \frac{\chi_-(\varphi(x, y, \zeta), \zeta)}{\zeta - \lambda} \, d\zeta, \quad \lambda \in \mathbb{C}^+, \]
\[ v(x, y) = -\frac{1}{\pi} \int_{\mathbb{R}} \chi_-(\varphi(x, y, \zeta), \zeta) \, d\zeta. \]

### 3.3. The shifted Riemann–Hilbert problem

In section 3.2 the following characterization for the boundary value of the complex eigenfunction
\[ \Phi^-(x, y, \lambda) = \varphi(x, y, \lambda) + \chi_-(\varphi(x, y, \lambda), \lambda), \quad \text{for} \ \lambda \in \mathbb{R}, \]
was justified. Here \( \chi_-(\xi, \lambda) \) satisfies the shifted Riemann–Hilbert problem (61).

The problem (61) can be converted into the following linear equation
\[ \chi_-(\xi, \lambda) = -\frac{1}{2\pi i} \int_{\mathbb{R}} f(\xi, \xi', \lambda) \chi_-(\xi', \lambda) \, d\xi' + g(\xi, \lambda) = 0, \]
where
\[ f(\xi, \xi', \lambda) = \frac{\partial \sigma(\xi', \lambda)}{\sigma(\xi', \lambda) - s(\xi, \lambda)} - \frac{1}{\xi' - \xi}, \]
\[ g(\xi, \lambda) = -\frac{1}{2} \sigma(\xi, \lambda) + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial \sigma(\xi', \lambda)}{\sigma(\xi', \lambda) - s(\xi, \lambda)} \sigma(\xi', \lambda) \, d\xi', \]
\[ s(\xi, \lambda) = \xi + \sigma(\xi, \lambda). \]
Under the assumptions that the mapping \( \xi \to \xi + \sigma(\xi, \lambda) \) is invertible for all \( \lambda \), that \( \sigma(\xi, \lambda) \) decay sufficiently fast for any fixed \( \lambda \) and be Hölder continuous, the unique solvability of \( \chi \) is proven in [27] by showing a Fredholm alternative for (66). Also this step does not require the small norm assumption.

Our goal in this section is to obtain some analytic estimates on the spectral data \( \chi_\pm \), including the large \( \lambda \)-asymptotic estimates, which are important in characterizing the complex eigenfunction and are indispensable for solving the inverse problem.

To simplify the calculations we shall use the following agreement in lemmas 3.2–3.3: we omit the \( \lambda \)-dependence in all formulas. It is convenient to denote:
\[ K\psi = \frac{1}{2\pi i} \int_{\mathbb{R}} f(\xi, \xi', \lambda) \psi(\xi') \, d\xi'. \]
It is natural to solve the integral equation (66) iteratively. Therefore we have to estimate the norm of \( K, \partial K \).

**Lemma 3.2.** Assume that the scattering data \( \sigma(\xi), \xi \in \mathbb{R} \) satisfy the following estimates:

1. \( \sigma(\xi) \) is 2 times continuously differentiable in \( \xi \).
2. \( |\sigma(\xi)| \leq C_0 \leq \frac{1}{2} \).
3. \(|\sigma'(\xi)| \leq C_1 \leq \frac{1}{7}\).
4. \(|\sigma''(\xi)| \leq C_2\).
5. \(||\sigma''(\xi)||_{L^2(\mathbb{C})} \leq \hat{C}_c||\).

Then we have the following estimate

\[
||K||_{L^\infty} \leq \frac{1}{\pi}[4C_0 + 2C_2 + \sqrt{2}\hat{C}_c].
\]

Assume that, in addition, the scattering data \(\sigma(\xi), \xi \in \mathbb{R}\) satisfy the following extra estimates:

1. \(\sigma(\xi)\) is 3 times continuously differentiable in \(\xi\).
2. \(C_2 \leq \frac{1}{7}\).
3. \(|\sigma'''(\xi)| \leq C_3\).

Then \(K\) maps the space \(L^\infty(d\xi)\) into the space \(C^1(\xi)\). Moreover, if \(h_2(\xi) = (Kh_1)(\xi)\), then

\[
|h_2(\xi)| \leq \frac{1}{\pi}(2C_3 + 4C_2^3 + 14C_1 + 4C_1^3) \cdot ||h(\xi)||_{L^\infty(\mathbb{C})}.
\]

The proof of this lemma is moved to the last section.

We also require some estimates on the function \(g(\xi)\).

**Lemma 3.3.** Assume that the scattering data satisfy the same estimates as in lemma 3.2 and

1. \(||\sigma(\xi)||_{L^2(\mathbb{C})} \leq \hat{C}_b||\).
2. \(||\sigma''(\xi)||_{L^2(\mathbb{C})} \leq \hat{C}_c||\).

Then we have:

1. \(|g(\xi)| \leq \frac{\hat{C}_b}{2} + \frac{1}{\pi}[2C_1 + \hat{C}_b]||\).
2. \(|g''(\xi)| \leq \frac{\hat{C}_c}{2} + \frac{1}{\pi}[16C_2 + 4\hat{C}_c]||\).

Moreover, if \(\sigma(\xi)\) has compact support: \(\sigma(\xi) = 0\) for \(|\xi| \geq R\), then

\[
|g(\xi)| \leq \frac{C_0}{2} + \frac{6R}{\pi}C_1 \leq \frac{8R}{\pi}C_1,
\]

\[
|g''(\xi)| \leq \frac{C_1}{2} + \frac{24R}{\pi}C_2 \leq \frac{26R}{\pi}C_2.
\]

The proof of this lemma is moved to the last section.

Combining the estimates from lemmas 3.2 and 3.3 we obtain the following:

**Proposition 3.3.** Assume that the potential \(v(x, y)\) satisfy the small norm constraints formulated in the definition 3.1. Then we have

\[
|\chi_\lambda(\xi, \lambda)| \leq \frac{1}{4} \tan\left(\frac{\pi}{8}\right).
\]

We show below, that this property guaranties the unique solvability of the inverse problem.

**Proof.** Equation (66) can be written in the short form:

\[
\chi = K\chi - g.
\]
If \( \|K\| < 1 \), it can be solved iteratively and
\[
\|\chi_-(\xi, \lambda)\| \leq \frac{1}{1 - \|K\|} \|g\|.
\]

By lemma 3.2, condition 3, the small norm conditions list means exactly that
\[
\|K\|_{L^\infty(d\xi)} \leq \frac{1}{2}.
\]

Therefore
\[
\|\chi_\varepsilon\|_{L^\infty(d\xi)} \leq 2\|g\|_{L^\infty(d\xi)} \leq B_0 + \frac{2}{\pi}[2B_1 + \hat{B}_0].
\]

By differentiating equation (71) with respect to \( \xi \), we obtain:
\[
\chi_\varepsilon = (K\chi_\varepsilon)_\xi - g_\varepsilon,
\]
and
\[
\|\chi_\varepsilon\|_{L^\infty(d\xi)} \leq \|(K\chi_\varepsilon)_\xi\|_{L^\infty(d\xi)} + \|g_\varepsilon\|_{L^\infty(d\xi)}.
\]

By lemma 3.2, in the small norm case
\[
\|(K\chi_\varepsilon)_\xi\|_{L^\infty(d\xi)} \leq \frac{1}{\pi}(2B_3 + 4B_2^2 + 14B_1^2 + 4B_1^2) \cdot \|\chi_\varepsilon\|_{L^\infty(d\xi)}.
\]

By lemma 3.3
\[
\|g_\varepsilon\|_{L^\infty(d\xi)} \leq \frac{B_1}{2} + \frac{1}{\pi}[16B_2 + 4\hat{B}_1].
\]

Therefore
\[
\|\chi_\varepsilon\|_{L^\infty(d\xi)} \leq \frac{1}{\pi}(2B_3 + 4B_2^2 + 14B_1^2 + 4B_1^2) \cdot \left( B_0 + \frac{2}{\pi}[2B_1 + \hat{B}_0]\right) + \frac{B_1}{2} + \frac{1}{\pi}[16B_2 + 4\hat{B}_1]
\]
\[
\leq \frac{1}{4} \tan \left( \frac{\pi}{8} \right) < \frac{1}{4}.
\]

The solution of the inverse problem also requires some estimates on \( \chi(\xi, \lambda) \) and its derivatives at \( \lambda \to \infty \). Let us show that, at large \( \lambda \), the leading term of the asymptotic behavior is determined by the linear part of (67).

More precisely,

**Lemma 3.4.** If \( v \in \mathcal{S}_{x,y} \) and \( v(x, y) = 0 \) for \( |y| \geq D_y \), then, for \( \lambda \to \pm \infty \), we have the following estimates:

1. \( \|K(\lambda)\|_{L^\infty(d\xi)} = O\left( \frac{1}{\lambda} \right) \)

2. For every sufficiently large \( \lambda \), the operator \( K(\lambda) \) maps the space \( L^\infty(d\xi) \) into the space \( C^\infty(\xi) \) and there exists a constant \( C_0(\lambda) \) such that
\[
\|\left( K(\lambda)f \right)_\xi \|_{L^\infty(d\xi)} \leq C_0(\lambda) \cdot \|f\|_{L^\infty(d\xi)}, \quad C_0(\lambda) = O\left( \frac{1}{\lambda^3} \right).
\]
Proof. To prove this lemma it is sufficient to compare formulas (41)–(42) with the estimates from lemma 3.2. □

Remark 3.1. Using the same approach, it is possible to prove analogous estimates for all derivatives; in particular, there exists a constant $C_2(\lambda)$ such that

$$
\| (K(\lambda)f)_{\xi} \|_{L^\infty(d\xi)} \leq C_2(\lambda) \cdot \| f \|_{L^\infty(d\xi)},
\quad C_2(\lambda) = O\left( \frac{1}{\lambda^2} \right)
$$

Proposition 3.4. Assume that $v \in \mathcal{S}_{x,y}$ and $v(x, y) \equiv 0$ for $|y| > D_x$. Then, for $\lambda \to \infty$, we have the following estimates

$$
\chi(\xi, \lambda) = -g(\xi, \lambda) + O\left( \frac{1}{\lambda^4} \right),
\quad \chi_\xi(\xi, \lambda) = -g_\xi(\xi, \lambda) + O\left( \frac{1}{\lambda^4} \right).
$$

If, in addition, $v(x, y)$ satisfies the compact support condition (26), i.e. $v(x, y) \equiv 0$ for $|x| \geq D_x$, then

$$
\| \chi(\xi, \lambda) \|_{L^\infty(d\xi)} = O\left( \frac{1}{\lambda^2} \right),
\quad \| \chi_\xi(\xi, \lambda) \|_{L^\infty(d\xi)} = O\left( \frac{1}{\lambda^2} \right)
$$

(73)

Remark 3.2. Using the same approach, it is possible to prove that in the compact support case

$$
\| \chi_\xi(\xi, \lambda) \|_{L^\infty(d\xi)} = O\left( \frac{1}{\lambda^2} \right).
$$

Proof of proposition 3.4.

From (67), proposition 3.2, lemmas 3.3 and 3.4 and the formula $R(\lambda) = D_x + \lambda \lambda D_y$, it follows immediately, that

$$
\| g(\xi, \lambda) \|_{L^\infty(d\xi)} = O\left( \frac{1}{\lambda^2} \right),
\quad \| g_\xi(\xi, \lambda) \|_{L^\infty(d\xi)} = O\left( \frac{1}{\lambda^2} \right),
$$

$$
\| \chi(\xi, \lambda) + g(\xi, \lambda) \|_{L^\infty(d\xi)} \leq \frac{\| K(\lambda) \|_{L^\infty(d\xi)}}{1 - \| K(\lambda) \|_{L^\infty(d\xi)}} \cdot \| g(\xi, \lambda) \|_{L^\infty(d\xi)} = O\left( \frac{1}{\lambda^4} \right),
\quad \| \chi_\xi(\xi, \lambda) + g_\xi(\xi, \lambda) \|_{L^\infty(d\xi)} \leq C_3(\lambda) \| \chi(\xi, \lambda) \|_{L^\infty(d\xi)} = O\left( \frac{1}{\lambda^4} \right).
$$

Proposition 3.5. Suppose $v \in \mathcal{S}_{x,y}$ with compact support and $v$ is small. Consider a curve in the $(\xi, \lambda)$-plane:

$$
\xi(\lambda) = x - \lambda y - \lambda^2 t + \omega(\lambda).
$$
Then, for fixed $t > 0$ and $\omega(\lambda) = O(1)$, we have, as $\lambda \to \pm \infty$,

$$l \partial^p_x \chi_-(\omega(\lambda) + x - \lambda y - \lambda^2 t, \lambda) = O\left(\frac{1}{|\lambda|^{1+2p}}\right)$$

(74)

**Proof.** Suppose the support of $v$ is contained in $\{|\xi| \leq D_x, |y| \leq D_y\}$. Therefore the support of $\sigma(\xi, \lambda)$ lies in the area $\{|\xi| \leq D_x + |\lambda| D_y, \xi \in \mathbb{R}\}$ (see Figure 2).

Outside this area $\sigma(\xi, \lambda) \equiv 0$, hence $\chi(\xi, \lambda)$ is holomorphic in $\xi$ in the complex plane outside the cut $[-D_x - |\lambda| D_y, D_x + |\lambda| D_y]$ on the real line. Therefore,

$$\chi(\xi, \lambda) = \frac{1}{2\pi i} \int_{-D_x - |\lambda| D_y}^{D_x + |\lambda| D_y} \frac{(\chi_+(\tau, \lambda) - \chi_-(\tau, \lambda))}{\tau - \xi} d\tau,$$

and

$$\partial^p_x \chi(\xi, \lambda) = \frac{1}{2\pi i} \int_{-D_x - |\lambda| D_y}^{D_x + |\lambda| D_y} \frac{(\chi_+(\tau, \lambda) - \chi_-(\tau, \lambda))}{(\tau - \xi)^2} d\tau.$$ 

It follows that, for $|\xi| > D_x + |\lambda| D_y$,

$$l \partial^p_x \chi_-(\xi, \lambda)$$

$$\leq \frac{1}{\pi} \cdot \left[\|\chi_-(\xi, \lambda)\|_{L^\infty} + \|\chi_+(\xi, \lambda)\|_{L^\infty}\right] \cdot (D_x + |\lambda| D_y) \cdot \sup_{\tau \in [-D_x - |\lambda| D_y, D_x + |\lambda| D_y]} \partial^p_x \left[\frac{1}{(\tau - \xi)}\right]$$

$$\leq C \frac{\mu!}{(1 + |\lambda|)(|\xi| - (D_x + |\lambda| D_y))^{p+1}}$$

Therefore (74) follows if $t > 0$. $\square$

4. The inverse problem

4.1. The reconstruction of the real eigenfunction

Assume that the **spectral data** $\chi_-(\xi, \lambda) = \chi_{-R}(\xi, \lambda) + i \chi_{-I}(\xi, \lambda)$ are given, where $\xi, \lambda \in \mathbb{R}$. Let us recall that $\chi_-(\xi, \lambda)$ is assumed to be analytic in $\xi$ the lower half-plane, or equivalently

$$\chi_{-R} - H_\xi \chi_{-I} = 0,$$  

(75)

where $H_\xi$ denotes the Hilbert transform w.r.t the variable $\xi$.

Our current aim is to construct the common eigenfunctions of the Lax pair for the Pavlov equation starting from the spectral data. By the Plemelj (Sokhotski) formula [27], Theorems 3.1 and 3.2, we have

$$\varphi(x, y, \lambda) + \chi_{-R}(\varphi(x, y, \lambda), \lambda) = x - \lambda y - \frac{1}{\pi} \int_{\mathbb{R}} \frac{\chi_{-I}(\varphi(x, y, \zeta), \zeta)}{\zeta - \lambda} d\zeta.$$  

(76)

Therefore, keeping in mind the time evolution (19) of the spectral data and the definition (20) of the common eigenfunctions of the vector field Lax pair, the nonlinear integral equation of the inverse problem reads:

$$\psi(x, y, t, \lambda) + \chi_{-R}(\psi(x, y, t, \lambda), \lambda) = x - \lambda y - \lambda^2 t - \frac{1}{\pi} \int_{\mathbb{R}} \frac{\chi_{-I}(\psi(x, y, t, \zeta), \zeta)}{\zeta - \lambda} d\zeta.$$  

(77)
In formulas (76) and (77) and below we write $\phi_\cdot, \psi_\cdot$ instead of $\phi -, \psi -$ to simplify our notations.

- We show that, if some appropriate constraints are imposed on the spectral data, the nonlinear integral equation (77) has a unique solution $\psi(x, y, t, \lambda)$.
- We show that the function $\psi(x, y, t, \lambda)$ is the real Jost eigenfunction for the Pavlov Lax Pair with the proper behavior at $y \to -\infty$, where the potential $\nu(x, y, t)$ is defined by formula (25).

**Theorem 4.1 (Global solvability for the IST equation (77)—part 1)** Suppose that the spectral data $\chi_\cdot(\xi, \lambda)$ satisfy the following constraints

1. $\chi_\cdot(\xi, \lambda), \partial_\xi \chi_\cdot(\xi, \lambda)$ are well-defined continuous functions.
2. $|\chi_\cdot(\xi, \lambda)| \leq \frac{C}{1 + |\lambda|}$.

Then, for all $x, y, t \in \mathbb{R}$, $t \geq 0$, equation (77) has a unique solution $\psi(x, y, t, \lambda)$ such that $\psi(x, y, t, \lambda) = x - \lambda y - \lambda^2 t + \omega(x, y, t, \lambda)$, where $\omega(x, y, t, \lambda) \in L^2(d\lambda)$.

**Proof.** The proof is based on the standard iteration procedure for contracting nonlinear maps. Equation (77) is equivalent to

$$R[\psi(f)](\lambda) = -\chi_\cdot(f(\lambda), \lambda) - \frac{1}{\pi} \int_{\mathbb{R}} \frac{\chi_\cdot(f(\zeta)), \zeta)}{\zeta - \lambda} d\zeta,$$

or equivalently,

$$R[\psi(f)](\lambda) = H_\cdot \circ \chi_\cdot(f(\lambda), \lambda) - \chi_\cdot(f(\lambda), \lambda).$$

From the constraints on the spectral data it immediately follows that the maps

$$f(\lambda) \to \chi_\cdot(f(\lambda), \lambda), \quad f(\lambda) \to \chi_\cdot(f(\lambda), \lambda)$$

map all measurable functions of $\lambda$ into the space $L^2(d\lambda)$; moreover the image of the map is located inside the ball of radius $R_0 = \sqrt{2} C$. $H_\cdot$ is a unitary operator in the space $L^2(\lambda)$; therefore, for any measurable function $f(\lambda)$, we know that $R[f(\lambda)] \in L^2(d\lambda)$, and $\|R[f(\lambda)]\|_{L^2(d\lambda)} \leq 2\sqrt{2} C$.

Let us check that operator $R$ is a contraction. Let $f(\lambda)$ be a measurable function, $g(\lambda) \in L^2(d\lambda)$. We have

$$\|R[f + g] - R[f]\|_{L^2(d\lambda)} \leq \|H_\cdot [\chi_\cdot(f(\lambda) + g(\lambda), \lambda) - \chi_\cdot(f(\lambda), \lambda)]\|_{L^2(d\lambda)} + \|\chi_\cdot(f(\lambda) + g(\lambda), \lambda) - \chi_\cdot(f(\lambda), \lambda)\|_{L^2(d\lambda)}$$

$$= \|\chi_\cdot(f(\lambda) + g(\lambda), \lambda) - \chi_\cdot(f(\lambda), \lambda)\|_{L^2(d\lambda)} + \|\chi_\cdot(f(\lambda) + g(\lambda), \lambda) - \chi_\cdot(f(\lambda), \lambda)\|_{L^2(d\lambda)}$$

3727
We know, that
\[
|\chi_- f(\lambda) + g(\lambda), \lambda) - \chi_- f(\lambda), \lambda)| \leq \max_{\xi, \lambda} |\partial_\xi \chi_- f(\xi, \lambda)| \cdot |g(\lambda)| \leq \frac{1}{4} |g(\lambda)|,
\]
\[
|\chi_- R(f(\lambda) + g(\lambda), \lambda) - \chi_- R(f(\lambda), \lambda)| \leq \max_{\xi, \lambda} |\partial_\xi \chi_- R(\xi, \lambda)| \cdot |g(\lambda)| \leq \frac{1}{4} |g(\lambda)|,
\]
therefore
\[
\|\mathcal{R}[f + g] - \mathcal{R}[f]\|_{L^2(d\lambda)} \leq \frac{1}{2} \|g\|_{L^2(d\lambda)}.
\]

Hence the iteration procedure:
\[
\omega_0(x, y, t, \lambda) = 0
\]
\[
\omega_{n+1}(x, y, t, \lambda) = \mathcal{R}[\omega_n(x, y, t, \lambda) + x - \lambda y - \lambda^2 t],
\]
perfectly converges in \(L^2(d\lambda)\).

Let us check now that the functions constructed above have the Jost property. Namely:

**Theorem 4.2.** Assume that the spectral data \(\chi_-(\xi, \lambda)\) satisfy the same constraints as in theorem 4.1, and

1. For each \(\lambda \in \mathbb{R}\) the function \(\chi_-(\xi, \lambda)\) is holomorphic in \(\xi\) in the lower half-plane.
2. \(\partial_\lambda \chi_-(\xi, \lambda), \partial_\xi \partial_\lambda \chi_-(\xi, \lambda), \partial_\xi^2 \chi_-(\xi, \lambda)\) are well-defined continuous functions.
3. There exists a positive constant \(C\) such, that
   \[
   |\chi_-(\xi, \lambda)| \leq \frac{C}{1 + |\lambda|^2},
   \]
   \[
   |\chi_-^2(\xi, \lambda)| \leq \frac{C}{1 + |\lambda|^2},
   \]
   \[
   |\partial_\xi \chi_-(\xi, \lambda)| \leq \frac{C}{1 + |\lambda|^2},
   \]
   \[
   |\partial_\xi^2 \chi_-(\xi, \lambda)| \leq \frac{C}{1 + |\lambda|^2},
   \]
   \[
   |\partial_\lambda \chi_-(\xi, \lambda)| \leq \frac{C}{1 + |\lambda|^2},
   \]
   \[
   |\partial_\lambda \chi_-(\xi, \lambda)| \leq \frac{C}{1 + |\lambda|^2},
   \]
4. For any \(\mathcal{D} > 0\), there exists a positive constant \(C(\mathcal{D})\) such that, for all \(\lambda\) such that \(|\lambda| \leq 4\mathcal{D}\),
   \[
   |\partial_\lambda \chi_-(\xi, \lambda)| \leq \frac{C(\mathcal{D})}{1 + |\xi|^2},
   \]
\[ |\partial_{\lambda} \chi_{-}(\xi, \lambda)| \leq \frac{C(D(\lambda))}{1 + |\xi|^2}, \tag{89} \]
\[ |\partial_{\xi} \partial_{\lambda} \chi_{-}(\xi, \lambda)| \leq \frac{C(D(\lambda))}{1 + |\xi|^2}, \tag{90} \]
\[ |\partial_{\xi}^2 \chi_{-}(\xi, \lambda)| \leq \frac{C(D(\lambda))}{1 + |\xi|^2}. \tag{91} \]

Then, for the functions constructed in theorem 4.1 with fixed \( \tau, t, \lambda_0 \in \mathbb{R}, t \geq 0 \), we have
\[ \omega(\tau + \lambda_0 t, y, t, \lambda_0) \to 0 \text{ for } y \to -\infty. \tag{92} \]

**Remark 4.1.** Let us point out that all conditions from theorem 4.2 hold for the spectral data constructed in the framework of the direct spectral transform (we assume again that our Cauchy data \( v_0(x, y) \) have compact support). Almost all of them were proved above, and the proof of the remaining ones are rather standard. Let us check, for example, (83).

Let \( |\xi| \) be sufficiently large. We have 2 regions.
1. Let \( |\lambda|^2 \geq |\xi| \). Then the second condition immediately follows from (73).
2. Let \( |\lambda|^2 < |\xi| \). From proposition 3.3 and (73) we obtain that there exists a constant \( C_0 \) such that
\[ \int_{-\infty}^{\infty} |\chi_{+}(\tau, \lambda) - \chi_{-}(\tau, \lambda)| \, d\tau < C_0 \text{ for all } \lambda. \tag{93} \]
If \( |\xi| > 2|D_2 + \sqrt{|\xi| D_1}| \), then we can use the same estimates as in proposition 3.5, and
\[ |\chi(\xi, \lambda)| \leq \frac{C_0}{\pi |\xi|}. \tag{94} \]
It completes the proof.

**Remark 4.2.** One can consider equation (77) without assuming that the spectral data \( \chi_{-}(\xi, \lambda) \) is holomorphic in \( \xi \) in the lower half-plane (or, equivalently, we do not assume that equation (75) is fulfilled). In this situation the function \( \psi(x, y, t, \lambda) \) will be also an eigenfunction for the Pavlov Lax operators \( L, M \) for some \( v(x, y, t) \) (see theorem 4.4), but the normalization of this eigenfunction will be different from (92).

We also require to study the linearized version of equation (77).

**Lemma 4.1.** Suppose that the scattering data \( \chi_{-}(\xi, \lambda) \) satisfy the same constraints as in theorem 4.1 (which are fulfilled if \( \chi_{-}(\xi, \lambda) \) was constructed through the small norm Cauchy data \( v(x, y) \)). Then, for \( \forall g \in L^p(\mathbb{R}, d\lambda), p = 2, 4 \) the integral equation
\[ f(x, y, t, \lambda) = \frac{1}{\pi} \int_{\mathbb{R}} \left[ \frac{\partial_{\xi} \chi_{-}(\psi(x, y, t, \zeta) - \lambda, \zeta) - \partial_{\lambda} \chi_{-}(\psi(x, y, t, \lambda) - \zeta)}{\lambda - \zeta} f(x, y, t, \lambda) + g(\lambda) \right] \, d\zeta \tag{95} \]
admits a unique solution \( f \) such that
Proof. The Hilbert transform is a unitary operator in $L^2(\mathbb{R}, d\lambda)$ and the norm of the Hilbert transform in $L^p(\mathbb{R}, d\lambda), 2 \leq p < \infty$ is equal to $\text{cot}\left(\frac{\pi}{2p}\right)$ (see [30, 67]); therefore for $p = 2$ or $p = 4$ one has

$$
\|f(x, y, t, \lambda)\|_{L^2(\mathbb{R}, d\lambda)} \leq 2 \|g\|_{L^2(\mathbb{R}, d\lambda)}. \tag{96}
$$

Therefore $[1 - \partial R/\partial \xi]_0$ is an invertible map on $L^p(\mathbb{R}, d\lambda)$ and the norm of the inverse operator in $L^p(\mathbb{R}, d\lambda)$ is not greater than 2:

$$
f(x, y, t, \lambda) = (1 - \partial R/\partial \xi)_0^{-1}g \in L^p(\mathbb{R}, d\lambda)
$$

and the estimate (96) follows.

Below we use the following simple corollary of the Sobolev embedding theorem:

**Lemma 4.2.** Let $f(\lambda)$ be an element of $H^1(d\lambda), \lambda \in \mathbb{R}$. Then $f(\lambda)$ is a continuous function and

$$
|f(\lambda)| \leq \sqrt{\|f\|_{L^2(d\lambda)}^2 + \|f_\lambda\|_{L^2(d\lambda)}} \tag{98}
$$

**Theorem 4.3 (Global solvability for the IST equation (77)—part 2)** Suppose that $\chi(\xi, \lambda)$ satisfies the same constraints as in theorem 4.1 and, in addition,

$$
\|\partial_\lambda^n \chi(\xi, \lambda)\|_{L^\infty(d\xi)} = O\left(\frac{1}{\lambda^{1+n}}\right), \quad n = 0, 1, 2, 3,
$$

$$
\|\partial_\lambda^n \partial_\lambda \chi(\xi, \lambda)\|_{L^\infty(d\xi)} = O\left(\frac{1}{\lambda^{3+n}}\right), \quad n = 0, 1. \tag{99}
$$

Let us denote:

$$
\omega = \omega(x, y, t, \lambda) = \psi(x, y, t, \lambda) - (x - \lambda y - \lambda^2 t)
$$

Then:

1. For all $x, y \in \mathbb{R}, t \geq 0$ the function $\omega(x, y, t, \lambda)$ lies in the space $H^1(d\lambda)$ and continuously depends on $x, y, t$ as an element of $L^2(d\lambda) \cap L^\infty(d\lambda)$. The norm of $\omega$ in the space $L^2(d\lambda)$ is uniformly bounded in $x, y, t$ (but the $H^1$-norm may be unbounded).

2. For all $x, y \in \mathbb{R}, t \geq 0$ the following derivatives of $\omega$:

$$
\partial_\omega, \partial_\omega \partial_\omega, \partial_\omega \partial_\omega, \partial_\omega^2 \omega, \partial_\omega \partial_\omega, \partial_\omega \partial_\omega, \partial_\omega \partial_\omega,
$$

are well-defined as elements of the space $L^2(\mathbb{R}, d\lambda)$, they continuously depend on $x, y, t$ and are uniformly bounded in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$, and $\psi(x, y, 0, \lambda) = \varphi(x, y, \lambda)$. 

3730
Proof.
1. To construct $\omega$, it is convenient to run the iteration procedure (77), simultaneously for $\omega$ and $\omega_\lambda$:

$$
\omega^{(n+1)}(x, y, t) = -\chi_{-R}(x - \lambda y - \lambda^2 t + \omega^{(n)}, \lambda) + H_\lambda[\chi_{-R}(x - \lambda y - \lambda^2 t + \omega^{(n)}, \lambda)]
$$

$$
\omega^{(n+1)}_\lambda(x, y, t) = g^{(n)}_\lambda - \partial_\lambda \chi_{-R}(x - \lambda y - \lambda^2 t + \omega^{(n)}, \lambda) \omega^{(n)}_\lambda
+ H_\lambda[\partial_\lambda \chi_{-R}(x - \lambda y - \lambda^2 t + \omega^{(n)}, \lambda) \omega^{(n)}_\lambda],
$$

where $H_\lambda$ is the Hilbert transform with respect to $\lambda$,

$$
g^{(n)}_\lambda = -\partial_\lambda \chi_{-R}(x - \lambda y - \lambda^2 t + \omega^{(n)}, \lambda)
+ H_\lambda[\partial_\lambda \chi_{-R}(x - \lambda y - \lambda^2 t + \omega^{(n)}, \lambda)]
+ \partial_\lambda \chi_{-R}(x - \lambda y - \lambda^2 t + \omega^{(n)}, \lambda) \cdot (y + 2\lambda t)
- H_\lambda[\partial_\lambda \chi_{-R}(x - \lambda y - \lambda^2 t + \omega^{(n)}, \lambda) \cdot (y + 2\lambda t)]
$$

In any compact area in the $x, y, t$ space the function $g^{(n)}_\lambda$ is bounded in $L^2(\lambda)$ uniformly in $\omega$. If $\|g^{(n)}_\lambda\|_{L^2(\lambda)} < F$, then for all $n$, $\|\omega^{(n)}_\lambda\|_{L^2(\lambda)} < 2F$. Therefore by lemma 4.2 the $L^2(\lambda)$ convergence of $\omega^{(n)}$ implies the $L^\infty(\lambda)$ convergence of $\omega^{(n)}$ and the convergence of $\omega^{(n)}_\lambda$ in $L^2(\lambda)$.

2. By taking derivatives of both sides of (77), we obtain the linearized integral equation by:

$$
\psi + \partial_\lambda \chi_{-R}(\psi, \lambda) \cdot \psi = 1 + H_\lambda[\partial_\lambda \chi_{-R}(\psi, \lambda) \psi(x, y, t, \lambda)],
$$

$$
\psi + \partial_\lambda \chi_{-R}(\psi, \lambda) \cdot \psi = -\lambda - H_\lambda[\partial_\lambda \chi_{-R}(\psi, \lambda) \psi(x, y, t, \lambda)],
$$

$$
\psi + \partial_\lambda \chi_{-R}(\psi, \lambda) \cdot \psi = -\lambda^2 - H_\lambda[\partial_\lambda \chi_{-R}(\psi, \lambda) \psi(x, y, t, \lambda)].
$$

In terms of $\omega(x, y, t, \lambda)$, equations (102)–(104) take the form:

$$
\omega_\lambda(x, y, t, \lambda) = g_\alpha + H_\lambda[\partial_\lambda \chi_{-R}(\psi, \lambda) \omega_\lambda] - \partial_\lambda \chi_{-R}(\psi, \lambda) \omega_\lambda_\alpha
$$

where $\alpha \in \{x, y, t\}$, and

$$
g(x, y, t, \lambda) = H_\lambda[\partial_\lambda \chi_{-R}(\psi, \lambda)] - \partial_\lambda \chi_{-R}(\psi, \lambda),
$$

$$
g(x, y, t, \lambda) = -H_\lambda[\partial_\lambda \chi_{-R}(\psi, \lambda) \lambda] + \partial_\lambda \chi_{-R}(\psi, \lambda) \lambda,
$$

$$
g(x, y, t, \lambda) = -H_\lambda[\partial_\lambda \chi_{-R}(\psi, \lambda) \lambda^2] + \partial_\lambda \chi_{-R}(\psi, \lambda) \lambda^2.
$$

From (99) it follows that $g, g_x, g_y \in L^2(\mathbb{R}, d\lambda) \cap L^4(\mathbb{R}, d\lambda)$. Therefore the existence of $\psi_x, \psi_y, \psi_t$ such that $\partial_\lambda(\psi - (x - \lambda y - \lambda^2 t))$, $\partial_\lambda(\psi - (x - \lambda y - \lambda^2 t))$, $\partial_\lambda(\psi - (x - \lambda y - \lambda^2 t)) \in L^2(\mathbb{R}, d\lambda) \cap L^4(\mathbb{R}, d\lambda)$ follows from lemma 4.1.

For the second derivatives of the wave function we have:

$$
\psi_{\alpha, \beta} = -\partial_\lambda \chi_{-R}(\psi, \lambda) \psi_\alpha \psi_\beta - \partial_\lambda \chi_{-R}(\psi, \lambda) \psi_{\alpha, \beta} + H_\lambda[\partial_\lambda \chi_{-R}(\psi, \lambda) \psi_\alpha \psi_\beta] + H_\lambda[\partial_\lambda \chi_{-R}(\psi, \lambda) \psi_{\alpha, \beta}],
$$

where
\[ g_{\alpha,\beta} = -\partial_x^2 \chi_{\alpha,\beta}(\psi, \lambda) \psi_{\alpha,\beta}^2 + H_0 [\partial_x^2 \chi_{\alpha,\beta}(\psi, \lambda) \psi_{\alpha,\beta}]. \]

From (99) and the properties of the first derivatives we obtain that \( g_{xx}, g_{xy}, g_{xt}, g_{xy} \) belong to \( L^2(\mathbb{R}, d\lambda) \); therefore equations (107) are uniquely solvable in \( L^\infty(\mathbb{R}, d\lambda) \).

Taking into account that \( \omega \) is continuous in \( x, y, t \) as an element of \( L^\infty(\mathbb{R}) \), we obtain that all coefficients of the linear equations are continuous in \( L^\infty(\mathbb{R}, d\lambda) \). This implies that the solutions are also continuous.

\[ \square \]

4.2. Eigenfunctions of the Lax equation and the Cauchy problem

**Theorem 4.4 (Global solvability for small initial data)** Suppose \( v_0(x, y) \in \mathcal{S}_{x,y} \) satisfying (26) and the sufficiently small condition from definition 3.1. Let \( \psi(x, y, t, \lambda) \) be the solution of the nonlinear inverse problem (77) obtained in theorem 4.3 with the data \( \chi(\xi, \lambda) \) constructed from \( v_0(x,y) \) through the direct problem. Define

\[ v(x,y,t) = -\frac{1}{\pi} \int_{\mathbb{R}} \chi_{-}(\psi(x,y,t,\lambda),\lambda) d\lambda, \quad (108) \]

Then

1. \( v, v_x, v_y, v_{xx}, v_{yy}, v_{xy} \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+) \cap L^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+) \),

2. Assume, in addition, that for \( \chi(\xi, \lambda) \) we have estimates from proposition 3.5. Then, for all \( t > 0 \), the function \( v(x, y, t) \) is well-defined and continuous in all variables.

3. \( \psi(x, y, t, \lambda) \) satisfies the Lax equations in the space \( L^2(d\lambda) \). More precisely, for each \( x, y, t \), functions \( L\psi, M\psi \) are well-defined elements of \( L^2(d\lambda) \) and

\[ L\psi = \partial_x \psi + (\lambda + v_x) \partial_x \psi = 0, \quad (111) \]

\[ M\psi = \partial_y \psi + (\lambda^2 + \lambda v_x - v_y) \partial_y \psi = 0 \quad (112) \]

for almost all \( \lambda \in \mathbb{R} \).

4. Let us define a pair of functions \( \Psi(\psi(x,y,t,\lambda),\lambda) \in \mathbb{R} \) by

\[ \Psi^- (x,y,t,\lambda) = \psi(x,y,t,\lambda) + \chi_{-}(\psi(x,y,t,\lambda),\lambda), \]

\[ \Psi^+ (x,y,t,\lambda) = \psi(x,y,t,\lambda) + \chi_{+}(\psi(x,y,t,\lambda),\lambda). \quad (113) \]

Then, for each \( x, y \in \mathbb{R}, t \geq 0 \), these functions admit natural analytic continuation in \( \lambda \) to the lower half-plane \( \mathbb{C}^- \) and the upper half-plane \( \mathbb{C}^+ \) respectively.

5. Denote by \( \Psi(x,y,t,\lambda) \lambda \in \mathbb{C} \setminus \mathbb{R} \) the function, coinciding with the analytic continuation of \( \Psi^-(x,y,t,\lambda) \) for \( \text{Im } \lambda > 0 \) and with the analytic continuation of \( \Psi^+(x,y,t,\lambda) \) for \( \text{Im } \lambda < 0 \). Then we have the following integral representation:

\[ \Psi(x,y,t,\lambda) = x - \lambda y - \lambda^2 t - \frac{1}{\pi} \int_{\mathbb{R}} \frac{\chi_{-}(\psi(x,y,\zeta,\lambda),\zeta)}{\zeta - \lambda} d\zeta, \quad \lambda \in \mathbb{C}^\times. \quad (114) \]

Denote by \( \tilde{\omega} = \tilde{\omega}(x,y,t,\lambda) \) the regular part of the wave function: \( \tilde{\omega} = \Psi - (x - \lambda y - \lambda^2 t) \)
Then for each fixed $\lambda \in \mathbb{C}^+ \setminus \mathbb{R}$ we have:

$$\omega, \omega_x, \omega_y, \omega_{xx}, \omega_{yy}, \omega_{xy} \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+) \cap L^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+),$$

(115)

and for any $\lambda \in \mathbb{C}^+ \setminus \mathbb{R}$ the analytic wave function $\Psi(x, y, t, \lambda)$ satisfies the Lax pair

$$L \Psi = \partial_t \Psi + (\lambda + v_t) \partial_x \Psi = 0,$$

(116)

$$M \Psi = \partial_t \Psi + (\lambda^2 + \lambda v_x - v_t) \partial_x \Psi = 0,$$

(117)

identically in $x, y, t$.

6. For $t = 0$ the function $v(x, y, t)$, constructed in terms of the inverse spectral transforms via (108), coincides with the Cauchy data $v_0(x, y)$ for the direct spectral transform:

$$v(x, y, 0) = v_0(x, y),$$

(118)

Proof.

1. The reality condition (109) follows from the fact that the inverse scattering equation (77) is real for real $\lambda$, and (108) has real coefficients. By differentiating (108) we obtain

$$v_0(x, y, t) = -\frac{1}{\pi} \int_\mathbb{R} \partial_x \chi_{x,f}(\psi, \lambda) \psi_0 d\lambda,$$

(119)

$$v_{0,\alpha}(x, y, t) = -\frac{1}{\pi} \int_\mathbb{R} [\partial_{\xi} \chi_{x,f}(\psi, \lambda) \psi_{0,\alpha} + \partial_{\xi} \chi_{x,f}(\psi, \lambda) \psi_{0,\beta}] d\lambda,$$

(120)

$\alpha, \beta \in \{x, y, t\}$. Using the properties (99), it follows that the only integral requiring regularization is the integral for $v_t$. This means that, for $t = 0$, the function $v_t$ may be discontinuous.

2. Let $t > 0$. We have

$$v_t(x, y, t) = -\frac{1}{\pi} \int_\mathbb{R} \partial_x \chi_{x,f}(x - \lambda y - \lambda^2 t + \omega(x, y, t, \lambda), \lambda) \psi_0 d\lambda,$$

(121)

and $\omega$ is a bounded function of $\lambda$, therefore the convergence of integral immediately follows from proposition 3.5.

3. To calculate $L \psi, M \psi$ we use the following simple formula. Let $f(\lambda)$ be a function such that $f(\lambda) \in L^p(\mathbb{R}), \lambda f(\lambda) \in L^p(\mathbb{R}), 1 < p < \infty$. Then

$$\lambda H[f(\lambda)] = H[\lambda f(\lambda)] + \frac{1}{\pi} \int_\mathbb{R} f(\lambda) d\lambda.$$

(122)

Applying $L$ to (77) we obtain:

$$L \psi = v_x - L(\chi_{x,f}(\psi, \lambda)) + (\partial_x + \lambda \partial_x + v_t \partial_x) H[(\chi_{x,f}(\psi, \lambda))]$$

$$= v_x - \partial_x \chi_{x,f}(\psi, \lambda) L \psi + H[(\chi_{x,f}(\psi, \lambda))] + \frac{1}{\pi} \int_\mathbb{R} (\partial_x \chi_{x,f}(\psi, \lambda)) d\lambda$$

$$= v_x - \partial_x \chi_{x,f}(\psi, \lambda) L \psi + H[(\partial_x \chi_{x,f}(\psi, \lambda)) L \psi] - v_t.$$

(123)

We obtain that $L \Psi \in L^2(\mathbb{R})$ and solves the homogeneous equation; therefore, by lemma 4.1, it is a zero element of $L^2(\mathbb{R})$. Analogously,
\[ M \psi = \lambda \psi - v - \partial \chi_{x,y} \psi \lambda M \psi + H[\partial \chi_{x,y} \psi \lambda M \psi] + \lambda \frac{1}{\pi} \int_{\mathbb{R}} (\partial \chi_{x,y} \psi \lambda M \psi) d\lambda + \frac{1}{\pi} \int_{\mathbb{R}} (\lambda \partial \chi_{x,y} \psi \lambda M \psi) d\lambda + v \psi \frac{1}{\pi} \int_{\mathbb{R}} (\partial \chi_{x,y} \psi \lambda M \psi) d\lambda \]

Taking into account that
\[
\lambda \partial \chi_{x,y} \psi \lambda = -\partial \chi_{x,y} \psi \lambda - v \partial \chi_{x,y} \psi \lambda,
\]
we obtain that
\[ M \psi \in L^2(d\lambda) \]
and
\[ M \psi = \partial \chi_{x,y} \psi \lambda M \psi + H[\partial \chi_{x,y} \psi \lambda M \psi]; \]

therefore
\[ M \psi = 0. \]

4. This property is exactly equivalent to the inverse problem equation (77).
5. From (113) it follows that
\[ \lambda \partial \chi_{x,y} \psi \lambda = -\partial \chi_{x,y} \psi \lambda - v \partial \chi_{x,y} \psi \lambda, \]
in \[ L^2(\mathbb{R}, d\lambda). \] The standard solution of the Riemann factorization problem in terms of the Cauchy integral immediately gives us (114). Combining theorem 4.3 and the Hölder inequality, we obtain (115) for fixed \[ \lambda \in \mathbb{C} \setminus \mathbb{R}. \]
6. Finally, restricting (114) to \[ t = 0, \] (127) yields (62) and (76). So \[ \psi(x, y, 0, \lambda) = \varphi(x, y, \lambda), \]
\[ \Psi(x, y, 0, \lambda) = \Phi(x, y, \lambda). \] Comparing (63), (64) with (114), (108) we obtain (118).

**Theorem 4.5.** Suppose \[ v_0(x, y) \in \mathcal{S}_{x,y} \] with compact support and satisfies the sufficiently small condition from definition 3.1. Then the Cauchy problem of the Pavlov equation
\[ v_{x} + v_{y} = v_{x} v_{x} - v_{x} v_{x}, \quad \forall x, y \in \mathbb{R}, t \in \mathbb{R}^+, \]
\[ v(x, y, 0) = v_0(x, y) \]
admits a real solution \[ v = v(x, y, t) \] such that \[ v, v_x, v_y, v_{xx}, v_{yy}, v_{xy}, v_{y} \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+) \cap L^\infty (\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+). \]

**Proof.** Applying proposition 3.5 and computing the compatibility of the Lax pair (111) and (112), we obtain
\[ (v_{x} + v_{y} - v_{x} v_{x} + v_{x} v_{y}) \partial \chi \psi \equiv 0. \]
Hence we obtain (128) by (115).

5. Summary of the results and concluding remarks

We have shown that the direct problem consists of the following steps:

- From the potential \[ v(x, y) \] we construct the scattering data \[ \sigma(\xi, \lambda) \], solving the ODE (10).
- From the scattering data \[ \sigma(\xi, \lambda) \] we construct the spectral data \[ \chi(\xi, \lambda) \] solving the shifted Riemann problem (18).
These two steps do not require small norm assumptions.

The inverse problem consists of the following two steps:

- From the spectral data $\chi(\xi, \lambda)$ we construct the real Jost eigenfunctions solving the nonlinear integral equation (22), under the small norm assumption.
- From the real eigenfunctions we construct the potential $v(x, y, t)$ using formula (108).

The following remark is important.

**Remark 5.1.** A careful reader may notice that the above basic steps do not involve explicitly the analytic eigenfunctions; therefore, strictly speaking, the Cauchy problem for the Pavlov equation can be solved without introducing them. However, their existence pervades the whole IST. Indeed, not only it is crucial in motivating the shifted Riemann problem (18) of the direct problem, but it is also equivalent to the nonlinear integral equation (22) of the inverse problem.

6. The analytic estimates

In this section we present the proofs of some of the analytical estimates we use in our paper.

**Proof of proposition 3.1.** The main tool for proving these estimates in the Gronwall’s inequality. By definition,

$$\sigma(\tau, \lambda) = \lim_{y \to +\infty} h(y, \tau, \lambda) - \tau,$$

where $h = h(y, \tau, \lambda)$ denotes the solution of the vector field ODE:

$$\frac{dh}{dy} = v_x(h + \lambda y, y, \lambda),$$ (129)

with the boundary condition:

$$\lim_{y \to -\infty} h(y, \tau, \lambda) = \tau.$$

Therefore:

$$\sigma(\tau, \lambda) = \int_{-\infty}^{+\infty} v_x(h(y, \tau, \lambda) + \lambda y, y)dy,$$

and

$$\|\sigma(\tau, \lambda)\| \leq \int_{-\infty}^{+\infty} \max_{x \in \mathbb{R}} |v_x(x, y)| dy = B_0.$$

The function $h_\tau$ satisfies the linearized equation:

$$\frac{dh_\tau}{dy} = v_x h_\tau,$$ (130)

with the boundary value

$$\lim_{y \to -\infty} h_\tau(y, \tau, \lambda) = 1.$$

Equation (130) can be written as:
\[
\frac{d}{dy} \log(h_\tau) = \nu_\delta(h(y, \tau, \lambda) + \lambda y, y),
\]
therefore
\[
|\log(h_\tau(y, \tau, \lambda))| \leq \int_{-\infty}^{+\infty} \left[ \max_{x \in \mathbb{R}} |v_\delta(x, y)| \right] dy,
\]
\[
\exp \left( -\int_{-\infty}^{+\infty} \left[ \max_{x \in \mathbb{R}} |v_\delta(x, y)| \right] dy \right) - 1 \leq h_\tau(y, \tau, \lambda) - 1 \leq \exp \left( \int_{-\infty}^{+\infty} \left[ \max_{x \in \mathbb{R}} |v_\delta(x, y)| \right] dy \right) - 1 = B_0,
\]
and
\[
|h_\tau(y, \tau, \lambda)| \leq B_1, \quad |h_\tau(y, \tau, \lambda)| \leq B_1 + 1,
\]
which automatically implies the necessary estimate on \(|\sigma_\tau(\tau, \lambda)|\).

The next step is to estimate the solutions of the equation for \(h_{\tau\tau}\)
\[
\frac{dh_{\tau\tau}}{dy} = \nu_\delta h_\tau^2 + v_\delta h_{\tau\tau}
\]  \quad (131)
with the boundary condition:
\[
\lim_{y \to -\infty} h_{\tau\tau}(y, \tau, \lambda) = 0.
\]
We have an inhomogeneous linear equation; therefore we can use the standard estimate:
\[
|h_{\tau\tau\tau}(y, \tau, \lambda)| \leq \left( \int_{-\infty}^{+\infty} \left[ \max_{x \in \mathbb{R}} |v_{\delta\delta}(x, y)| \right] dy \right) \cdot \left\{ \max_{x, y, \tau, \lambda} |h_\tau(y, \tau, \lambda)|^2 \right\}
\]
\[
\cdot \exp \left( \int_{-\infty}^{+\infty} \left[ \max_{x \in \mathbb{R}} |v_\delta(x, y)| \right] dy \right),
\]
which implies the estimate on \(|\sigma_{\tau\tau}(\tau, \lambda)|\). Equation for \(h_{\tau\tau\tau}\) has the form
\[
\frac{dh_{\tau\tau\tau}}{dy} = \nu_\delta h_\tau^2 + 3v_\delta h_\tau h_{\tau\tau} + v_\delta h_{\tau\tau\tau}
\]  \quad (132)
with the boundary condition:
\[
\lim_{y \to -\infty} h_{\tau\tau\tau}(y, \tau, \lambda) = 0.
\]
Again we can estimate the function \(|\sigma_{\tau\tau\tau}(\tau, \lambda)|\) as product of the integral of the modulus of the inhomogeneous term times the exponent of the modulus of the homogeneous coefficient:
\[
|h_{\tau\tau\tau\tau}(y, \tau, \lambda)| \leq \left[ \left( \int_{-\infty}^{+\infty} \left[ \max_{x \in \mathbb{R}} |v_{\delta\delta\delta}(x, y)| \right] dy \right) \cdot \left\{ \max_{x, y, \tau, \lambda} |h_\tau(y, \tau, \lambda)|^3 \right\} \right]
\]
\[
+ 3 \left[ \int_{-\infty}^{+\infty} \left[ \max_{x \in \mathbb{R}} |v_{\delta\delta}(x, y)| \right] dy \right] \cdot \left\{ \max_{x, y, \tau, \lambda} |h_\tau(y, \tau, \lambda)| \right\} \left( \int_{-\infty}^{+\infty} \left[ \max_{x \in \mathbb{R}} |v_\delta(x, y)| \right] dy \right) \cdot \exp \left( \int_{-\infty}^{+\infty} \left[ \max_{x \in \mathbb{R}} |v_\delta(x, y)| \right] dy \right).
\]
which implies the estimate on $|\sigma_{\tau\tau'}(\tau, \lambda)|$.

Let us denote $h(y, \tau, \lambda) = \tau + \tilde{h}(y, \tau, \lambda)$. Equations (129) and (130) can be interpreted as ODEs for the functions $\tilde{h}(y, \tau, \lambda), \tilde{h}_\tau(y, \tau, \lambda)$ in the Hilbert space $L^2(\mathrm{d}\tau)$. We obtain:

$$\frac{\mathrm{d}\tilde{h}}{\mathrm{d}y} = v_\tau(\tilde{h} + \tau + \lambda y, y),$$

(133)

$$\frac{\mathrm{d}\tilde{h}_\tau}{\mathrm{d}y} = v_{\tau\tau}(\tilde{h} + \tau + \lambda y, y) + v_{\tau}(\tilde{h} + \tau + \lambda y, y)\tilde{h}_\tau$$

(134)

We see that

$$\|\tilde{h}(y, \tau, \lambda)\|_{L^2(\mathrm{d}\tau)} \leq \int_{-\infty}^{+\infty} \|v_\tau(\tilde{h} + \tau + \lambda y, y)\|_{L^2(\mathrm{d}y)} \mathrm{dy},$$

$$\|\tilde{h}_\tau(y, \tau, \lambda)\|_{L^2(\mathrm{d}\tau)} \leq \left( \int_{-\infty}^{+\infty} \|v_{\tau\tau}(\tilde{h} + \tau + \lambda y, y)\|_{L^2(\mathrm{d}y)} \mathrm{dy} \right)^{1/2} \cdot \exp\left( \int_{-\infty}^{+\infty} \frac{1}{\max_{x \in \mathbb{R}}|v_\tau(x, \tilde{y})|} \mathrm{d}\tilde{y} \right).$$

We assume now that $B_1 < 1$. We know that

$$\|\cdots\|_{L^2(\mathrm{d}x)} \leq \left( \int_{-\infty}^{+\infty} \frac{\max_{x \in \mathbb{R}}|v_\tau(x, \tilde{y})|}{\min_{x \in \mathbb{R}}|v_\tau(x, \tilde{y})|} \mathrm{d}\tilde{y} \right)^{1/2} \cdot \|\cdots\|_{L^2(\mathrm{d}x)},$$

for a fixed $y, \lambda$, but $\frac{\mathrm{d}x}{\mathrm{d}\tau} = h_\tau(y, \tau, \lambda)$; therefore

$$\|v_\tau(\tilde{h} + \tau + \lambda y, y)\|_{L^2(\mathrm{d}\tau)} \leq \frac{1}{\sqrt{1 - B_1}} \cdot \|v_\tau(x, y)\|_{L^2(\mathrm{d}x)},$$

$$\|v_{\tau\tau}(\tilde{h} + \tau + \lambda y, y)\|_{L^2(\mathrm{d}\tau)} \leq \frac{1}{\sqrt{1 - B_1}} \cdot \|v_\tau(x, y)\|_{L^2(\mathrm{d}x)},$$

which completes the proof. $\square$

**Proof of proposition 3.2.** Due to definition 2.1, it is sufficient to prove the lemma for $|\lambda| \gg 1$. Thus we always assume $|\lambda| \gg 1$ in the following proof. The cases $\lambda \to +\infty$ and $\lambda \to -\infty$ are completely analogous, therefore we assume now that $\lambda \to +\infty$.

Let us rewrite the definition of the scattering data using $\xi$ and $x$ as new coordinates on the $(x, y)$-plane. The $x$-coordinate is expressed through $\xi, y$ using the following formulas:

$$x = h(y; \xi, -\infty, \lambda) + \lambda y = \xi + \lambda y + \int_{-\infty}^{y'} v_\tau(\lambda y' + h(y'; \xi, -\infty, \lambda), y')\mathrm{d}y'.$$

(135)

From the implicit function theorem, this map can be inverted with respect to $y$:

$$y = H(\xi, x, \lambda) = \frac{x - \xi}{\lambda} + \frac{H_\xi(\xi, x, \lambda)}{\lambda^2}, \quad H_1 = O(1)$$

where
\[
\frac{\partial y}{\partial x} = \frac{\partial H(\xi, x, \lambda)}{\partial x} = \frac{1}{\lambda + v(x, H(\xi, x, \lambda))},
\tag{136}
\]

or, equivalently
\[
\frac{\partial H_1(\xi, x, \lambda)}{\partial x} = \frac{v(x, \frac{x - \xi}{\lambda} + \frac{H_1(\xi, x, \lambda)}{\lambda})}{1 + v(x, \frac{x - \xi}{\lambda} + \frac{H_1(\xi, x, \lambda)}{\lambda})},
\tag{137}
\]

\[H_1(\xi, -\infty, \lambda) = 0.\]

We see, that
\[\sigma(\xi, \lambda) = -\frac{H_1(\xi, +\infty, \lambda)}{\lambda}.
\]

Let us denote
\[\hat{H}_1(\xi, x, \lambda) = H_1(\xi/x, x, 1/\lambda).
\]

Taking into account that
\[\hat{\lambda} = \frac{1}{\lambda}, \quad \hat{\xi} = \frac{\xi}{\lambda},\]

we obtain
\[
\frac{\partial \hat{H}_1(\xi, x, \lambda)}{\partial x} = \frac{v(x, -\hat{\xi} + \hat{\lambda}x + \hat{\lambda}^2 \hat{H}_1(\xi, x, \hat{\lambda}))}{1 + \hat{\lambda} v(x, -\hat{\xi} + \hat{\lambda}x + \hat{\lambda}^2 \hat{H}_1(\xi, x, \hat{\lambda}))}.
\tag{138}
\]

For \(|\hat{\lambda}| < \frac{1}{2\max|v_{\max}|x^{1/\lambda}}\) the right-hand side of (138) is smooth in \(\hat{\xi}, \hat{\lambda}\). We solve this equation in the finite interval \(-D_1 \leq x \leq D_1\); therefore \(\hat{H}_1(\xi, +\infty, \hat{\lambda}) = \hat{H}_1(\xi, D_1, \hat{\lambda})\) smoothly depends on the parameters. It is easy to check that, for \(|-\hat{\xi}| > D_1 + |\hat{\lambda}|D_2\) the right-hand side of (138) is identical to 0, therefore \(\hat{H}_1(\xi, \hat{\lambda}) \equiv 0\) in the region \(|\hat{\xi}| > D_1 + |\hat{\lambda}|D_2\) (see figure 2).

Expanding (138) at \(\hat{\lambda} = 0\) we obtain:
\[
\frac{\partial \hat{H}_1(\xi, x, \lambda)}{\partial x} = -v(x, -\hat{\xi}) + O(\lambda); \tag{139}
\]

therefore
\[
\hat{H}_1(\xi, +\infty, \hat{\lambda}) = \int_{-D_1}^{D_1} v(x, -\hat{\xi}) dx + O(\lambda) = v(D_1, -\hat{\xi}) - v(-D_1, -\hat{\xi}) + O(\lambda) = O(\lambda).
\]

From the Hadamard’s lemma it follows that
\[
\frac{\sigma(\xi/x, 1/\lambda)}{\lambda^2} = -\frac{\hat{H}_1(\xi, +\infty, \hat{\lambda})}{\hat{\lambda}}.
\]
is a regular function of $\xi, \lambda$ for sufficiently small $\lambda$. We proved the first part. To prove the corollary, let us point out that, in the new variables, 

$$\partial_\lambda = -\lambda^2 \partial_\lambda - \lambda \xi \partial_\xi, \quad \partial_\xi = \lambda \partial_\xi.$$ 

Therefore any differentiation of the scattering data with respect to $\lambda, \xi$ increases the order of zero with respect to $\lambda$ at the point $\lambda = 0$ by one. Taking into account that

$$\|L(\partial_\lambda^2)\| = \frac{\|L(\partial_\xi^2)\|}{\sqrt{\lambda^*(\lambda)}}$$

we finish the proof. □

**Proof of theorem 3.2.** To prove the theorem, let us make an appropriate change of variables. It will be done in 5 steps.

**Step 1:** Consider a point $(x, y) \in \mathbb{R}^2$. Denote by $\hat{h}(y'; x, y, \lambda_R)$ the solution of the ordinary differential equation

$$\frac{d\hat{h}}{dy'} = \lambda_R + v_1(\hat{h}, y')$$

with the boundary condition

$$\hat{h}(y; x, y, \lambda_R) = x.$$ 

The first change of variables $F_1 : (x, y) \rightarrow (x_1, y_1)$ is defined by:

$$\begin{cases}
    x_1 = \lim_{y' \rightarrow -\infty} \hat{h}(y'; x, y, \lambda_R) - \lambda_R y' = \varphi(x, y, \lambda_R), & y < 0 \\
    x_1 = \lim_{y' \rightarrow +\infty} \hat{h}(y'; x, y, \lambda_R) - \lambda_R y' = \varphi_0(x, y, \lambda_R), & y > 0 \\
    y_1 = y
\end{cases}$$

(142)

Of course the map is discontinuous on the line $y = 0$, and 

$$(\varphi_0)_y + (\lambda_R + v_1)(\varphi_0)_x = 0.$$  

(143)

In the new variables we have 

$$L = \partial_\lambda + (\lambda + v_1)\partial_\lambda = \partial_{x_1} + i\lambda \kappa(x_1, y_1) \partial_{y_1},$$

(144)

where

$$\kappa(x_1, y_1) = \frac{\partial \varphi_0}{\partial x}(x, y, \lambda_R) \bigg|_{(x, y) = F_1(x_1, y_1)}.$$  

(145)

Moreover, there exists a pair of positive constants $C_1, C_2$ such that:

$$0 < C_1 \leq \kappa(x_1, y_1) \leq C_2.$$  

(146)
Step 2: To investigate the boundary behaviors of the complex eigenfunction, we observe that, for \( \lambda = \lambda_R + i \lambda_I \), \( |\lambda| \ll 1 \), it is natural to conjecture that \( \Phi(x, y, \lambda) \) is almost constant on the trajectories of the vector field
\[
\hat{L} \equiv \partial_y + \lambda_R \partial_x + v_x \partial_z.
\]
These trajectories are defined by (140) and (141). Hence, if
\[
\hat{h}(x, y, y', \lambda_R) = \xi + \lambda_R y' \quad \text{as} \quad y' \to -\infty,
\]
then
\[
\hat{h}(x, y, y', \lambda_R) = \xi + \sigma(\xi, \lambda_R) + \lambda_R y' \quad \text{as} \quad y' \to +\infty,
\]
where \( \sigma(\xi, \lambda_R) \) is defined by definition 2.1 (see the proof of lemma 3.1).

Recall that \( z = x - \lambda_R \). Assume that the support of \( \Phi(z, \lambda) \) is located inside the strip \( |z| < \varepsilon \), \( \varepsilon \ll 1 \). Then \( \Phi(z, \xi, \lambda) \) is holomorphic in \( z \) outside a small neighbourhood of the real line and we have
\[
\Phi(\xi + \sigma(\xi, \lambda) + i\varepsilon, \lambda) \sim \Phi(\xi - i\varepsilon, \lambda) \quad \text{for} \quad \lambda_I < 0, \quad (150)
\]
\[
\Phi(\xi + \sigma(\xi, \lambda) - i\varepsilon, \lambda) \sim \Phi(\xi + i\varepsilon, \lambda) \quad \text{for} \quad \lambda_I > 0. \quad (151)
\]

Consider the Riemann–Hilbert problem with shift (61), or, via function
\[
w(\xi, \lambda_R) = \xi + \chi(\xi, \lambda_R),
\]
\[
w(\xi + \sigma(\xi) + i0, \lambda_R) = w(\xi - i0, \lambda_R), \quad \xi \in \mathbb{R},
\]
\[
w(z) = z + o(1) \quad \text{as} \quad z \to \infty. \quad (153)
\]

Then the hypothetical formulas for \( \Phi^-(x, y, \lambda_R) \), \( \Phi^+(x, y, \lambda_R) \) read:
\[
\Phi^-(x, y, \lambda_R) = w(\varphi^{-}(x, y, \lambda_R) - i0, \lambda_R) = w(\varphi^{-}(x, y, \lambda_R) + i0, \lambda_R)
\]
\[
\Phi^+(x, y, \lambda_R) = \overline{\Phi}(x, y, \lambda_R). \quad (154)
\]

Step 3: Assume \( \lambda_I < 0 \) from now on. Let us use the following rescaling: \( F_2: (x_1, y_1) \to (x_2, y_2) \)
\[
\begin{aligned}
x_2 &= x_1 \\
y_2 &= \lambda_N y_1 \\
z_2 &= x_2 - i y_2.
\end{aligned} \quad (155)
\]

In the new variables
\[
L = \lambda_I \left( \partial_{y_2} + i \frac{y_2}{\lambda_I} \partial_{x_2} \right). \quad (156)
\]

Step 4: Let us define a new complex variable \( z_3 \), \( F_3: (x_2, y_2) \to z_3 \) by
\[
z_3 = x_2 - i y_2 + \chi(x_2 - i y_2, \lambda_R), \quad (157)
\]
where \( \chi(\xi, \lambda) \) is the solution of the shifted Riemann–Hilbert problem (61) (existence of the solution is proved in [27]). Note that the composition \( F_3 \circ F_2 \circ F_1 \) is continuous by the property:
\[
\text{If} \quad F_1(x, -0) = (\xi, 0) \quad \text{then} \quad F_1(x, +0) = (\xi + \sigma(\xi, \lambda_R), 0). \quad (158)
\]
Consequently, (44) takes the form
\[ [\partial_z + q(z_3, z_3, \lambda)\partial_z] \Phi = 0 \]  \hspace{1cm} (159)

where
\[ lq(z_3, z_3, \lambda) < C_1 |\lambda_0| < 1, \]  \hspace{1cm} (160)

the support of \( q(z, z, \lambda) \) has area of order \( O(\lambda_0) \).

It is natural to consider Beltrami equation (159) in the space \( L^{2+\varepsilon}(d\zeta dz) \cap L^{2-\varepsilon}(d\zeta dz) \) where \( \varepsilon \) is sufficiently small. Again we can write
\[ \Phi(z_3, z_3, \lambda) = z_3 + \partial_z^{-1} \alpha(z_3, z_3, \lambda) \]  \hspace{1cm} (161)

where
\[ [1 + q(z_3, z_3, \lambda)\partial_z, \partial_z^{-1}] \alpha(z_3, z_3, \lambda) + q(z_3, z_3, \lambda) = 0. \]  \hspace{1cm} (162)

Taking into account (160) we see that
\[ |\alpha(z_3, z_3, \lambda)|_{L^p} = O(\lambda_1), \]  \hspace{1cm} (163)

Using the estimates from [75] we see that
\[ \|\Phi(z_3, z_3, \lambda) - z_3\|_{L^2(\partial_\Omega, d\zeta)} = O(\lambda_1), \]  \hspace{1cm} and \( \Phi(z_3, z_3, \lambda) \) uniformly converges to \( z_3 \).

Step 5: Consider the function \( \Phi(x, y, \lambda) \) on the line \( y = y_0 < -D_y \). We see that
\[ z_3 = \xi + i\lambda_0 y_0, \]  \hspace{1cm} where \( \xi = x - \lambda_0 y_0 \).

therefore
\[ \Phi(x, y_0, \lambda - i0) = z_3(z_3) \bigg|_{z_3 = \xi - i0} = \xi + \chi_-(\xi, \lambda). \]

On this line
\[ \phi(x, y, \lambda) = \xi, \]

therefore
\[ \Phi(x, y, \lambda - i0) = \phi(x, y, \lambda) + \chi_-(\phi(x, y, \lambda), \lambda). \]

The proof is completed. \( \square \)

**Proof of lemma 3.2.** To start with, let us point out that
\[ f(t, \tau) = \partial_\tau \log \delta(t, \tau), \]

where
\[ \delta(t, \tau) = \frac{s(t) - s(\tau)}{t - \tau} = \int_0^1 s'(\alpha t + [1 - \alpha] \tau) d\alpha, \]

\[ \partial_\tau^k \partial_t^l \delta(t, \tau) = \int_0^1 \alpha^k [1 - \alpha]^{l+1} s^{(k+l+1)}(\alpha t + [1 - \alpha] \tau) d\alpha. \]

Therefore
\[ |\partial_\tau^k \partial_t^l \delta(t, \tau)| \leq \max_{\xi} |s^{(k+l+1)}(\xi)|. \]
We see that, if the corresponding derivatives exist,

\[
\begin{align*}
    f &= \frac{\ddot{s}}{s}, \quad f' = \frac{\ddot{s} + \frac{2}{s^2} \dddot{s} - \frac{2}{s^2} \dddot{s}}{s} + \frac{2}{s^2} \dddot{s}, \\
    f'' &= \frac{\dddot{s}}{s}, \quad f''' = \frac{\ddot{s}}{s} - 2 \frac{\dddot{s}}{s^2} - \frac{\dddot{s}}{s^2} + 2 \frac{\dddot{s}}{s^2}.
\end{align*}
\]

and

\[
\begin{align*}
|f(t, \tau)| &\leq \frac{\max s^1}{\min s^1} + \frac{\max s^2}{\min s^1}, \\
|f'_s(t, \tau)| &\leq 3 \frac{\max s^1}{\min s^1} \frac{\max s^2}{\min s^2} + 2 \frac{\max s^1}{\min s^2} \frac{\max s^2}{\min s^2}, \quad (164)
\end{align*}
\]

We know that

\[
\begin{align*}
\|K\|_{L^\infty} &= \frac{1}{2\pi} \max_{t \in \mathbb{R}} \int |f(t, \tau)|d\tau = \frac{1}{2\pi} [I_1(t) + I_2(t)], \\
I_1(t) &= \int_{|\tau - t| \leq 1} \left| \frac{s'(\tau)}{s(\tau) - s(t)} - \frac{1}{\tau - t} \right|d\tau, \\
I_2(t) &= \int_{|\tau - t| > 1} \left| \frac{s'(\tau)}{s(\tau) - s(t)} - \frac{1}{\tau - t} \right|d\tau \leq I_{21} + I_{22}, \\
I_3(t) &= \int_{|\tau - t| > 1} \left| \frac{\sigma'(\tau)}{\sigma(\tau) - \sigma(t)} - \frac{1}{\tau - t} \right|d\tau, \\
I_{22}(t) &= \int_{|\tau - t| > 1} \left| \frac{\sigma'(\tau)}{\sigma(\tau) - \sigma(t)} - \frac{1}{\tau - t} \right|d\tau,
\end{align*}
\]

From (164) we see that

\[
|f(t, \tau)| \leq \frac{C_2}{1 - C_1} \leq 2C_2,
\]

and

\[
I_i \leq \int_{|\tau - t| \leq 1} 2C_2d\tau = 4C_2.
\]

Let us estimate now \(I_{21}\). We have \(s(\tau) - s(t) = \tau - t + \sigma(\tau) - \sigma(t)\). We assumed that \(\sigma(\tau) - \sigma(t) \leq 2C_0 \leq 1/2\); therefore

\[
\left| \frac{1}{s(\tau) - s(t)} - \frac{1}{\tau - t} \right| = \left| \frac{1}{\tau - t} \right| \left| \frac{1 + \frac{\sigma(\tau) - \sigma(t)}{\tau - t}}{\tau - t} - 1 \right| \leq \frac{4C_0}{(\tau - t)^2},
\]

and

\[
I_{21} \leq 8C_0.
\]

To estimate \(I_{22}\), we use the Hölder inequality.
\[ I_{22}(t) \leq \sqrt{\int_{|\tau - t| \leq 1} |\sigma'(\tau)|^2 \, d\tau} \cdot \sqrt{\int_{|\tau - t| \geq 1} \frac{1}{(s(\tau) - s(t))^2} \, d\tau} \]
\[ \leq \|\sigma'(\tau)\|_{L^2(d\tau)} \cdot \sqrt{\int_{|\tau - t| \geq 1} \frac{4}{(\tau - t)^2} \, d\tau} = \sqrt{8} \mathcal{C}_1. \]

Combining estimates for \( I_1, I_{21}, I_{22} \) we complete the proof of the first part.

To prove the second part, we use the standard estimate:
\[ |h_2(t)| \leq \frac{1}{2\pi} \left[ \max_{t \in \mathbb{R}} \int_{\tau \in \mathbb{R}} |\partial^2 f(t, \tau)| \, d\tau \right] \cdot \|h_1(t)\|_{L^\infty(d\tau)}. \]

We have:
\[ \int_{\tau \in \mathbb{R}} |\partial^2 f(t, \tau)| \, d\tau \leq I_1 + I_2, \]
where
\[ I_1 = \int_{|\tau - t| \leq 1} |\partial^2 f(t, \tau)| \, d\tau, \]
\[ I_2 = \int_{|\tau - t| > 1} |\partial^2 f(t, \tau)| \, d\tau. \]

From (164), we see that
\[ I_1 \leq 2 \left[ \frac{C_3}{1 - C_1} + \frac{C_2}{(1 - C_1)^2} \right] \leq 4C_1 + 8C_2^2. \]

Let us introduce the following notation:
\[ y = \hat{O}(x), \text{ if } |y| \leq |x|. \]

Let us estimate \( I_2 \). We have:
\[ \partial^2 f(t, \tau) = \frac{s'(\tau)s'(t)}{(s(\tau) - s(t))^2} - \frac{1}{(\tau - t)^2} \]
\[ = \frac{1}{(s(\tau) - s(t))^2} + \frac{\sigma'(\tau) + \sigma'(t)}{(s(\tau) - s(t))^2} + \frac{\sigma'(\tau)\sigma'(t)}{(s(\tau) - s(t))^2} - \frac{1}{(\tau - t)^2}. \]

By definition,
\[ s(\tau) - s(t) = \tau - t + \sigma(\tau) - \sigma(t) = (\tau - t) [1 + \hat{O}(C_1)] \]
\[ \frac{1}{(s(\tau) - s(t))^2} = \frac{1}{(\tau - t)^2} [1 + \hat{O}(6C_1)] = \hat{O}\left(\frac{4}{(\tau - t)^2}\right). \]

Therefore
\[ |I_2| \leq \int_{|\tau - t| > 1} \left[ \frac{6C_1}{(\tau - t)^2} + \frac{8C_1}{(\tau - t)^2} + \frac{4C_2^2}{(\tau - t)^2} \right] \, d\tau = 28C_1 + 8C_2^2. \]
Finally we obtain
\[
\int_{\tau \in \mathbb{R}} |\delta f(t, \tau)|\,dt \leq 4C_3 + 8C_2^2 + 28C_1 + 8C_1^2.
\]
\[
\square
\]

**Proof of lemma 3.3.** We have
\[
g(\xi) = g_1(\xi) + g_2(\xi),
\]
where
\[
g_1(\xi) = -\frac{1}{2} \sigma(\xi),
\]
\[
g_2(\xi) = \frac{1}{2\pi i} \int_{\mathbb{R}} \sigma(\xi' \circ s(\xi)) \,d\xi',
\]
or, equivalently,
\[
g_2(s^{-1}(\eta)) = \frac{1}{2\pi i} \int_{\mathbb{R}} \sigma(\xi' \circ \xi'') \,d\eta',
\]
where \(s^{-1}(\eta) = s^{-1}(\eta, \lambda)\) denotes the inversion of the function \(s(\xi, \lambda)\) with respect to \(\xi\):
\[
s(\xi, \lambda) = \eta.
\]

Let us denote:
\[
g_2(\eta) = g_2(s^{-1}(\eta)), \quad \sigma(\eta') = \sigma(s^{-1}(\eta')).
\]

We have:
\[
\hat{g}_2(\eta) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\hat{\sigma}(\eta'')}{\eta'' - \eta} \,d\eta',
\]
\[
\hat{g}_{2,\sigma}(\eta) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\hat{\sigma}_\eta(\eta''\eta)\,d\eta''}{\eta'' - \eta},
\]
\[
2\pi|\hat{g}_2(\eta)| = \left| \int_{|\eta'' - \eta| < 1} \frac{1}{\eta'' - \eta} \hat{\sigma}(\eta'') \,d\eta'' + \int_{|\eta'' - \eta| > 1} \frac{1}{\eta'' - \eta} \hat{\sigma}(\eta'') \,d\eta'' \right| \leq I_1 + I_2,
\]
where
\[
I_1 = \left| \int_{|\eta'' - \eta| < 1} \frac{\hat{\sigma}(\eta'') - \hat{\sigma}(\eta)}{\eta'' - \eta} \,d\eta'' \right| \leq 2\max|\hat{\sigma}_\eta(\eta)|,
\]
and \(I_2\) can be estimated using the Hölder inequality
\[
I_2 = \left| \int_{|\eta'' - \eta| > 1} \frac{1}{\eta'' - \eta} \,d\eta'' \right| \leq \|\hat{\sigma}\|_{L^2(\mathbb{R})} \cdot \left( \int_{|\eta'' - \eta| > 1} \frac{1}{(\eta'' - \eta)^2} \,d\eta'' \right)^{1/2} = \sqrt{2} \|\hat{\sigma}\|_{L^2(\mathbb{R})}.
\]

We obtained:
\[
|\hat{g}_2(\eta)| \leq \frac{1}{2\pi} \cdot [2\max|\hat{\sigma}_\eta(\eta)| + \sqrt{2} \|\hat{\sigma}\|_{L^2(\mathbb{R})}] .
\]
Similarly:

$$|\hat{g}_{2,\eta}(\eta)| \leq \frac{1}{2\pi} \cdot [2 \max|\partial_\eta \sigma(\eta)| + \sqrt{2} \|\hat{\sigma}\|_{L^2(\eta)}].$$

We have

$$\hat{\sigma}(\eta) = \frac{d\xi}{d\eta} \cdot \sigma(s^{-1}(\eta)), \quad \hat{\sigma}(\eta) = \left(\frac{d\xi}{d\eta}\right)^2 \cdot \sigma(s^{-1}(\eta)) + \frac{d^2\xi}{d\eta^2} \cdot \sigma(s^{-1}(\eta)),$$

$$\frac{d\xi}{d\eta} = \left(\frac{d\eta}{d\xi}\right)^{-1}, \quad \frac{d^2\xi}{d\eta^2} = -\frac{d^2\eta}{d\xi^2} \cdot \left(\frac{d\eta}{d\xi}\right)^{-3}, \quad d\eta = \frac{d\eta}{d\xi} \cdot d\xi.$$

We assumed that $B_1 < \frac{1}{2}$; therefore

$$\frac{1}{2} \leq \frac{d\xi}{d\eta} \leq 2, \quad \frac{1}{2} \leq \frac{d\eta}{d\xi} \leq 2, \quad \left|\frac{d^2\eta}{d\xi^2}\right| \leq 8|\sigma_\xi|,$$

$$|\hat{\sigma}(\eta)| \leq 2|\sigma(s^{-1}(\eta))|, \quad |\hat{\sigma}(\eta)| \leq 8|\sigma_\xi(s^{-1}(\eta))|, \quad |\hat{g}_{2,\eta}(\xi)| \leq 2|\hat{g}_{2,\eta}(\xi)|, \quad \|\ldots||_{L^2(\eta)} \leq \sqrt{2} \|\ldots||_{L^2(\eta)}.$$

Therefore

$$|g_{2}(\xi)| \leq \frac{1}{2\pi} \cdot [4 \max|\sigma_\xi(\xi)| + 2\|\sigma\|_{L^2(\xi)}].$$

Similarly:

$$|g_{2,\xi}(\xi)| \leq \frac{1}{\pi} \cdot [2 \cdot 8 \max|\sigma_\xi(\xi)| + 2\|\hat{\sigma}\|_{L^2(\xi)}].$$

Let us prove the second part.

Assume that $|\eta| \leq 2R$. Then

$$\hat{g}_{2}(\eta) = \frac{1}{2\pi i} \int_{|\eta'' - \eta'\xi| \leq 3R} \hat{\sigma}(\eta') d\eta'' = \frac{1}{2\pi i} \int_{|\eta'' - \eta'\xi| \leq 3R} \frac{\hat{\sigma}(\eta') - \hat{\sigma}(\eta)}{\eta'' - \eta} d\eta''.$$

We have:

$$|\hat{g}_{2}(\eta)| \leq \frac{1}{2\pi} \cdot 6R \cdot \max|\partial_\eta \sigma(\eta')| \leq \frac{6R}{\pi} \cdot C_i.$$

Consider now the case $|\eta| \geq 2R$. Then

$$\hat{g}_{2}(\eta) = \frac{1}{2\pi i} \int_{|\eta''| \leq R} \frac{\hat{\sigma}(\eta'')}{\eta'' - \eta} d\eta''.$$
\[ |\delta_\lambda(\eta)| \leq \frac{1}{2\pi} \cdot \|\delta(\eta)\|_{L^\infty([0,\pi])} \cdot \left| \int_{|\eta| \leq R} \frac{1}{|\eta^2 - \eta|} \, d\eta \right| \]
\[ \leq \frac{C_0}{2\pi} \cdot \log\left( \frac{|\eta| + R}{|\eta| - R} \right) \leq \frac{C_0}{2\pi} \cdot \log(3) \leq \frac{C_0}{\pi} \]

For a finite support function
\[ |\sigma(\xi)| \leq R \cdot \|\sigma(\xi)\|_{L^\infty([0,\pi])}, \text{ i.e. } C_0 \leq RC \]

The proof of the second formula is absolutely the same, but we take into account (165). \[ \square \]

**Proof of theorem 4.2.** In this part we always assume that \( t > 0 \) is fixed, \( \mathcal{D} \) is an arbitrary fixed positive constant, \( y < 0 \), \( |y| \) is sufficiently large (more precisely, \( |y| > 64\mathcal{D}t \)), \( t |y| \leq \mathcal{D} |y| \).

This proof consists of 3 steps:

1. We show that it is sufficient to obtain some \( L^2(\mathcal{D}\lambda) \) estimates on \( \omega \) and \( \omega_\lambda \).
2. We show that it is sufficient to estimate the first iteration of \( \omega \) and \( \omega_\lambda \) in \( L^2(\mathcal{D}\lambda) \).
3. We estimate the first iteration of \( \omega \) and \( \omega_\lambda \) in \( L^2(\mathcal{D}\lambda) \) for \( y \to -\infty \).

**Step 1.**

From lemma 4.2 it follows that it is sufficient to prove the following:
\[ \|\omega(x, y, t, \lambda)\|_{L^2(\mathcal{D}\lambda)} \cdot \|\omega_\lambda(x, y, t, \lambda)\|_{L^2(\mathcal{D}\lambda)} \to 0 \text{ as } y \to -\infty, \]
uniformly in \( x \) in the interval \( |x| \leq \mathcal{D} |y| \).

**Step 2.**

Let us recall that we use the following iteration procedure:
\[ \omega_{n+1}(x, y, t, \lambda) = -\chi_{-R}(x - \lambda y - \hat{\lambda}t + \omega_\lambda(x, y, t, \lambda), \lambda) \]
\[ + H_n[\chi_{-R}(x - \lambda y - \hat{\lambda}t + \omega_\lambda(x, y, t, \lambda), \lambda)], \]
\[ \omega_{n+1,\lambda}(x, y, t, \lambda) = I_1 + I_2 + y \cdot I_3, \]
where
\[ I_1 = -\chi_{-R,\lambda}(x - \lambda y - \hat{\lambda}t + \omega_\lambda(x, y, t, \lambda), \lambda) \]
\[ + H_n[\chi_{-R,\lambda}(x - \lambda y - \hat{\lambda}t + \omega_\lambda(x, y, t, \lambda), \lambda)] \]
\[ + 2t \chi_{-R,\lambda}(x - \lambda y - \hat{\lambda}t + \omega_\lambda(x, y, t, \lambda), \lambda) \cdot \lambda \]
\[ - 2t H_n[\chi_{-R,\lambda}(x - \lambda y - \hat{\lambda}t + \omega_\lambda(x, y, t, \lambda), \lambda) \cdot \lambda], \]
\[ I_2 = -\chi_{-R,\lambda}(x - \lambda y - \hat{\lambda}t + \omega_\lambda(x, y, t, \lambda), \lambda) \cdot \omega_{n,\lambda}(x, y, t, \lambda) \]
\[ + H_n[\chi_{-R,\lambda}(x - \lambda y - \hat{\lambda}t + \omega_\lambda(x, y, t, \lambda), \lambda) \cdot \omega_{n,\lambda}(x, y, t, \lambda)], \]
\[ I_3 = \chi_{-R,\lambda}(x - \lambda y - \hat{\lambda}t + \omega_\lambda(x, y, t, \lambda), \lambda) \]
\[ - H_n[\chi_{-R,\lambda}(x - \lambda y - \hat{\lambda}t + \omega_\lambda(x, y, t, \lambda)], \]
It is convenient to write:
\[ I_3 = I_{31} + I_{32} \]

\[ I_{31} = \chi_{-R, \xi}(x - \lambda y - \lambda^2 t + \omega_1(x, y, t, \lambda, \lambda) - \chi_{-R, \xi}(x - \lambda y - \lambda^2 t, \lambda) - H_R [\chi_{-L, \xi}(x - \lambda y - \lambda^2 t, \lambda)] \] (171)

\[ I_{32} = I_2(x, y, t, \lambda) = \chi_{-R, \xi}(x - \lambda y - \lambda^2 t, \lambda) - H_R [\chi_{-L, \xi}(x - \lambda y - \lambda^2 t, \lambda)] \] .

From (87) and (84) we immediately obtain that there exists a constant \( C_1 > 0 \) such that

\[ \|I_1\|_{L^2(d\lambda)} < C_1 \text{ for any } \omega(\lambda). \] (172)

Using the same arguments as in theorem 4.1 we immediately obtain

\[ \|I_2\|_{L^2(d\lambda)} < \frac{1}{2} \|\omega_n, 1\|_{L^2(d\lambda)}. \] (173)

From (86) we immediately obtain that there exists \( C_2 > 0 \) such that

\[ \|I_{31}\|_{L^2(d\lambda)} < C_2 \|\omega_n\|_{L^2(d\lambda)}. \] (174)

We also know that

\[ \|\omega_n\|_{L^2(d\lambda)} \leq 2 \|\omega_1\|_{L^2(d\lambda)}. \] (175)

Combining all these estimates we obtain:

\[ \|\omega_{n+1, \lambda}(x, y, t, \lambda)\|_{L^2(d\lambda)} < C_1 + \frac{1}{2} \|\omega_n, 1\|_{L^2(d\lambda)} + 2 |y| \cdot C_2 \|\omega_1\|_{L^2(d\lambda)} + |y| \cdot \|I_2\|_{L^2(d\lambda)}. \] (176)

Therefore, to prove theorem 4.2 it is sufficient to show that

\[ \|\omega_1\|_{L^2(d\lambda)} = o \left( \frac{1}{\sqrt{|y|}} \right), \quad \|I_2\|_{L^2(d\lambda)} = o \left( \frac{1}{\sqrt{|y|}} \right) \text{ as } y \to -\infty, \] (177)

uniformly in \( x \) for \( |y| \leq \mathcal{D}|y| \), where \( \mathcal{D} \) is an arbitrary positive constant.

**Step 3.**

The proof of both estimates are absolutely similar; moreover the second one is a little easier from a technical point of view. Let us estimate \( \omega_1 \):

\[ \omega_1(x, y, t, \lambda) = H_R [\chi_{-R}(x - \lambda y - \lambda^2 t, \lambda)] - \chi_{-K}(\tau - \lambda y - \lambda^2 t, \lambda). \]

It is convenient to represent \( \chi_{-}(\xi, \lambda) \) as a sum of three functions:

\[ \chi_{-}(\xi, \lambda) = \chi_{-}^{(1)}(\xi, \lambda) + \chi_{-}^{(2)}(\xi, \lambda) + \chi_{-}^{(3)}(\xi, \lambda) \] (178)

\[ \chi_{-}^{(1)}(\xi, \lambda) = \begin{cases} \chi_{-}(\xi, \lambda), & |\lambda| \leq 4\mathcal{D} \\ 0, & |\lambda| > 4\mathcal{D}, \end{cases} \] (179)

\[ \chi_{-}^{(2)}(\xi, \lambda) = \begin{cases} \chi_{-}(\xi, \lambda), & 4\mathcal{D} < |\lambda| \leq |y|/4t \\ 0, & \text{otherwise}, \end{cases} \] (180)
\[ \chi_-^{(3)}(\xi, \lambda) = \begin{cases} \chi_-^{(1)}(\xi, \lambda), & |\lambda| > |y|/4t \\ 0, & |\lambda| \leq |y|/4t. \end{cases} \] (181)

From (82) it follows immediately that there exists a constant \( C_3 > 0 \) such that

\[ \|\chi_-^{(3)}\|_{L^2(\lambda)} \leq \frac{C_3}{|y|^{3/2}}. \] (182)

If \( 4D < |\lambda| \leq |y|/4t \), then \( |\chi_-^{(3)}(\xi, \lambda)| \leq \frac{2C_3}{|\lambda|} \frac{|\lambda|}{|y|} \), and

\[ \|\chi_-^{(3)}\|_{L^2(\lambda)} \leq \frac{4C_3}{|y|^{1/2}} \sqrt{\frac{d\lambda}{\lambda^2}}. \] (183)

Let us denote by \( \omega^{(1)} \) the function:

\[ \omega^{(1)}(x, y, t, \lambda) = H_k[\chi_-^{(1)}(x - \lambda y - \lambda^2 t, \lambda)] - \chi_-^{(1)}(x - \lambda y - \lambda^2 t, \lambda). \]

We have shown that

\[ \|\omega^{(1)} - \omega^{(1)}\|_{L^2(\lambda)} = O\left(\frac{1}{|y|}\right), \]

therefore it is sufficient to estimate \( \|\omega^{(1)}\|_{L^2(\lambda)} \). We have

\[ \|\omega^{(1)}\|_{L^2(\lambda)} \leq \|\omega^{(1)}\|_{L^2(\lambda), |\lambda| \in [-2D, 2D]} + \|\chi_-^{(1)}\|_{L^2(\lambda), |\lambda| \in [2D, 4D]} + \|H_k[\chi_-^{(1)}]\|_{L^2(\lambda), |\lambda| \in [2D, 4D)}. \] (185)

For sufficiently large \( |y| \) and \( \frac{3}{2}D \leq |\lambda| \leq 4D \) we have

\[ |\chi_-^{(1)}(x - \lambda y - \lambda^2 t, \lambda)| \leq \frac{4C_3}{|y|}, \] (186)

and

\[ \|\chi_-^{(1)}\|_{L^2(\lambda), |\lambda| \in [2D, 4D]} \leq \frac{16C_3}{|y|}. \] (187)

Let us estimate the \( L^2 \)-norm of \( H_k[\chi_-^{(1)}(x - \lambda y - \lambda^2 t, \lambda)] \) on the interval \( |\lambda| > 2D \). We have

\[ H_k[\chi_-^{(1)}(x - \lambda y - \lambda^2 t, \lambda)] = \frac{1}{\pi} \int_{-4D}^{4D} \frac{\chi_-^{(1)}(x - \mu y - \mu^2 t, \mu) d\mu}{\lambda - \mu} = I(\lambda) + I_2(\lambda), \] (188)

where

\[ I(\lambda) = \frac{1}{\pi} \int_{-\frac{3}{2}D}^{\frac{3}{2}D} \frac{\chi_-^{(1)}(x - \mu y - \mu^2 t, \mu) d\mu}{\lambda - \mu}, \] (189)
From (186) it follows that
\[
\|I_2(\lambda)\|_{L^2(\mathbb{R}^4)} \leq \|\chi_{-}(x - \mu y - \mu^2 t, \mu)\|_{L^2(\mathbb{R}^4)} \leq \frac{20C}{|y|}. \tag{191}
\]
For \(|\lambda| > 2D\) we have
\[
\|I_2(\lambda)\|_{L^2(\mathbb{R}^4)} \leq \|\chi_{-}(x - \mu y - \mu^2 t, \mu)\|_{L^2(\mathbb{R}^4)} \leq \frac{C_4 + C_5 \log |y|}{|y|}. \tag{192}
\]
\[
\|I_2(\lambda)\|_{L^2(\mathbb{R}^4)} \leq \frac{C_4 + C_5 \log |y|}{|y|}. \tag{193}
\]
To complete the proof, we have to estimate \(\omega^{(l)}(x, y, t, \lambda)\) in the interval \(-2D \leq \lambda \leq 2D\).

For \(y < 0\) the function \(\chi_{-}(x - \mu y - \mu^2 t, \lambda)\) is holomorphic in \(\mu\) in the lower half-plane; therefore
\[
\chi_{-}^{(l)}(x - \mu y - \mu^2 t, \lambda) = H_0[\chi_{-}^{(l)}(x - \mu y - \mu^2 t, \lambda)]_{\mu=\lambda}, \tag{195}
\]
and
\[
\omega^{(l)}(x, y, t, \lambda) \nonumber = -\chi_{-}^{(l)}(x - \mu y - \mu^2 t, \lambda) + H_0[\chi_{-}^{(l)}(x - \mu y - \mu^2 t, \lambda)] \nonumber \\
= \frac{1}{\pi} \int_{-4D}^{4D} \frac{\chi_{-}^{(l)}(x - \mu y - \mu^2 t, \mu) d\mu}{\lambda - \mu} - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\chi_{-}^{(l)}(x - \mu y - \mu^2 t, \lambda) d\mu}{\lambda - \mu} \nonumber \nonumber \\
= \frac{1}{\pi} \int_{-4D}^{4D} \frac{\chi_{-}^{(l)}(x - \mu y - \mu^2 t, \mu) - \chi_{-}^{(l)}(x - \mu y - \mu^2 t, \lambda) d\mu}{\lambda - \mu} + I_3. \tag{196}
\]
where
\[
I_3 = -\frac{1}{\pi} \int_{|\mu| > 2D} \frac{\chi_{-}^{(l)}(x - \mu y - \mu^2 t, \lambda) d\mu}{\lambda - \mu}. \tag{197}
\]
If \(|y| > 4Dt\), then
\[
\left|I_3\right| \leq \frac{1}{\pi} \int_{|\mu| > 2D} \frac{4C}{|y||\mu|^2} d\mu \leq \frac{1}{\pi} \frac{2C}{|y|} \tag{198}
\]
\[
\frac{1}{\pi} \int_{-4D}^{4D} \frac{\chi_{-}^{(l)}(x - \mu y - \mu^2 t, \mu) - \chi_{-}^{(l)}(x - \mu y - \mu^2 t, \lambda) d\mu}{\lambda - \mu} = I_4 + I_5. \tag{199}
\]
where

\[ I_4 = \frac{1}{\pi} \int_{-4D}^{4D} \chi^{(1)}_{\lambda}(x - \mu y - \mu^2 t, \lambda) - \chi^{(1)}_{\lambda}(x - \mu y - \lambda^2 t, \lambda) \, d\mu, \]  

(200)

\[ I_5 = \frac{1}{\pi} \int_{-4D}^{4D} \chi^{(1)}_{\lambda}(x - \mu y - \mu^2 t, \mu) - \chi^{(1)}_{\lambda}(x - \mu y - \lambda^2 t, \mu) \, d\mu. \]  

(201)

We see that

\[ I_4 = \frac{1}{\pi} \int_{-4D}^{4D} t(\lambda + \mu) \chi^{(1)}_{\lambda}(\xi, \lambda) d\mu, \]  

where \( \xi \in [x - \mu y - \lambda^2 t, x - \mu y - \mu^2 t] \),

(202)

Denote:

\[ I_4 = I_{41} + I_{42}, \]  

(203)

where

\[ I_{41} = \frac{1}{\pi} \int_{|\mu| \leq 4D; |x - \mu y| > 64D^2} t(\lambda + \mu) \chi^{(1)}_{\lambda}(\xi, \lambda) d\mu, \]  

(204)

\[ I_{42} = \frac{1}{\pi} \int_{|\mu| \leq 4D; |x - \mu y| \leq 64D^2} t(\lambda + \mu) \chi^{(1)}_{\lambda}(\xi, \lambda) d\mu, \]  

\[ |I_{42}| \leq \frac{1024 CD^2 \lambda^2}{\pi \lambda y}, \]  

(205)

\[ |I_{41}| \leq \frac{8 D t}{\pi} \int_{|\mu| \leq 4D} \frac{C}{1 + \frac{|x - \mu y|}{4}} d\mu \leq \frac{16 C D t}{\pi \lambda y} \int_{-\infty}^{\infty} \frac{1}{1 + \beta^2} d\beta \leq \frac{16 C D t}{\lambda y}. \]  

(206)

Analogously,

\[ I_5 = \frac{1}{\pi} \int_{-4D}^{4D} \chi^{(1)}_{\lambda}(x - \mu y - \mu^2 t, \beta) d\beta, \]  

where \( |\beta| \leq 4D \),

(207)

\[ I_5 = I_{51} + I_{52}, \]  

(208)

\[ I_{51} = \frac{1}{\pi} \int_{|\mu| \leq 4D; |x - \mu y| > 64D^2} \chi^{(1)}_{\lambda}(x - \mu y - \mu^2 t, \beta) d\mu, \]  

\[ I_{52} = \frac{1}{\pi} \int_{|\mu| \leq 4D; |x - \mu y| \leq 64D^2} \chi^{(1)}_{\lambda}(x - \mu y - \mu^2 t, \beta) d\mu, \]  

\[ |I_{52}| \leq \frac{128 C D^2 \lambda}{\pi \lambda y}, \]  

(209)

\[ |I_{51}| \leq \frac{1}{\pi} \int_{|\mu| \leq 4D} \frac{C}{1 + \frac{|x - \mu y|}{2}} d\mu \leq \frac{4 C}{\pi \lambda y} \int_{0}^{\frac{5 D y}{2}} \frac{1}{1 + \beta} d\beta = \frac{4 C}{\pi \lambda y} \log \left( \frac{5 D y}{2} \right). \]  

(210)

3750
We have shown that there exist positive constants $C_6 = C_6(\mathcal{D})$, $C_7 = C_7(\mathcal{D})$ such that
\[ \|\omega^{(l)}\|_{L^2(\mathbb{R}^8)} \leq \frac{C_6 + C_7 \log |l|}{|l|^2}. \] (212)

Analogously, there exists a constant $C_8 = C_8(\mathcal{D})$ such that
\[ \|I_{22}(x, y, \tau, \lambda)\|_{L^2(\mathbb{R}^8)} \leq \frac{C_8}{|l|^2} \] (213)
\[ (\chi_{-, \xi}(\xi, \lambda), \chi_{-, \xi}(\xi, \lambda)) \text{ decay at } |\xi| \to \infty \text{ as } 1/|\xi|^2 \text{, therefore we have no logarithmic terms}. \]
The proof is completed.

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