Parallel submanifolds with an intrinsic product structure

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Abstract

Let $M$ and $N$ be Riemannian symmetric spaces and $f : M \to N$ be a parallel isometric immersion. We additionally assume that there exist simply connected, irreducible Riemannian symmetric spaces $M_i$ with $\dim(M_i) \geq 2$ for $i = 1, \ldots, r$ such that $M \cong M_1 \times \cdots \times M_r$. As a starting point, we describe how the intrinsic product structure of $M$ is reflected by a distinguished, fiberwise orthogonal direct sum decomposition of the corresponding first normal bundle. Then we consider the (second) osculating bundle $O_f$, which is a $\nabla^N$-parallel vector subbundle of the pullback bundle $f^*T_N$, and establish the existence of $r$ distinguished, pairwise commuting, $\nabla^N$-parallel vector bundle involutions on $O_f$. Consequently, the "extrinsic holonomy Lie algebra" of $O_f$ bears naturally the structure of a graded Lie algebra over the Abelian group which is given by the direct sum of $r$ copies of $\mathbb{Z}/2\mathbb{Z}$. Our main result is the following: Provided that $N$ is of compact or non-compact type, that $\dim(M_i) \geq 3$ for $i = 1, \ldots, r$ and that none of the product slices through one point of $M$ gets mapped into any flat of $N$, we can show that $f(M)$ is a homogeneous submanifold of $N$.

1 Introduction

Given a Riemannian symmetric space $N$ (briefly called "symmetric space") and an isometric immersion $f : M \to N$, we let $T M, T N, \perp f, h : T M \times T M \to \perp f$ and $S : T M \times \perp f \to T M$ denote the tangent bundles of $M$ and $N$, the normal bundle, the second fundamental form and the shape operator of $f$, respectively. Then $T M$ and $T N$ are equipped with the Levi Civita connections $\nabla^M$ and $\nabla^N$, respectively, whereas on $\perp f$ there is the usual normal connection $\nabla^\perp$ (obtained through projection). The equations of Gauß and Weingarten state that

$$\nabla^N_X T f Y = T f(\nabla^M_X Y) + h(X, Y) \quad \text{and} \quad \nabla^N_X \xi = -T f(S_\xi(X)) + \nabla^\perp_X \xi$$

for all $X, Y \in \Gamma(T M), \xi \in \Gamma(\perp f)$. On the vector bundle $L^2(T M, \perp f)$ there is a connection induced by $\nabla^M$ and $\nabla^\perp$ in a natural way, often called “Van der Waerden-Bortolotti connection”, characterized as follows: For every curve $c : \mathbb{R} \to M$, every parallel section $b : \mathbb{R} \to L^2(T M, \perp f)$ along $c$ and any two parallel sections $X, Y : \mathbb{R} \to T M$ along $c$, the curve $t \mapsto b(X(t), Y(t))$ is a parallel section of $\perp f$ along $c$.

Definition 1.1. $f$ is called parallel if its second fundamental form $h$ is a parallel section of the vector bundle $L^2(T M, \perp M)$.

Example 1.2 ("Extrinsic Circles", see [11]). A unit speed curve $c : J \to N$ is parallel if and only if it satisfies the equation

$$\nabla^N_\partial \nabla^N_\partial \dot{c} = -\kappa^2 \dot{c}$$

for some constant $\kappa \in \mathbb{R}$.

Surprisingly, not much is known so far about parallel isometric immersions in general. Only the special case of the "symmetric submanifolds" (in the sense of [2], [10] and [12]) of the symmetric spaces is perfectly well understood (cf. [11, Ch. 9.3]).

At a first glance, one could think that parallel submanifolds are simply the extrinsic analogue of symmetric spaces; remember that the latter ones are characterized by the condition $\nabla R = 0$. However,
this comparison is a bit shortcoming; for example, not every complete parallel submanifold is a symmetric submanifold unless \( N \) is a space form (cf. [3] Prop. 1). In fact, the discrepancy between the intrinsic and the extrinsic situation is even wider: Whereas every symmetric space is intrinsically a homogenous space, complete parallel submanifolds of \( N \) are not necessarily homogeneous submanifolds (see Section 2.6) unless \( N \) is Euclidian or a rank-1 symmetric space.

**Definition 1.3** ([1] A 1). Let \( M \) be a simply connected symmetric space.

(a) \( M \) is called “reducible” if it is the Riemannian product of two Riemannian spaces of dimension at least 1, respectively; otherwise \( M \) is called “irreducible”.

(b) There exists a Euclidian space \( M_0 \) and simply connected, irreducible symmetric spaces \( M_i \) with \( \dim(M_i) \geq 2 \) for \( i = 1, \ldots, r \) such that \( M \) is isometric to the Riemannian product \( M_0 \times \cdots \times M_r \).

In this case, the “factors” \( M_i \) are uniquely determined (up to isometry, respectively, and up to a permutation of \( \{M_1, \ldots, M_r\} \)) by means of the “de Rham Decomposition Theorem”, and \( M_0 \times \cdots \times M_r \) is called the “de Rham decomposition” of \( M \).

(c) In the situation of (b), we say that \( M \) has no Euclidian factor if \( M_0 \) is trivial; then \( M_1 \times \cdots \times M_r \) is called the “de Rham decomposition” of \( M \).

Let a parallel isometric immersion \( f : M \to N \) be given. It is well known that then \( M \) is necessarily a locally symmetric space (cf. [3] Prop. 4); in fact, we can even assume that \( M \) is a simply connected (globally) symmetric space. Here we consider the following more specific situation: Throughout this article, we assume that \( M \) and \( N \) both are symmetric spaces, that \( M \) is additionally simply connected and without Euclidian factor and that \( f : M \to N \) is a parallel isometric immersion.

**Example 1.4.** Suppose that there exist symmetric spaces \( N_i \) and parallel isometric immersions \( f_i : M_i \to N_i \) from simply connected, irreducible symmetric spaces \( M_i \) with \( \dim(M_i) \geq 2 \) for \( i = 1, \ldots, r \). Then \( M := M_1 \times \cdots \times M_r \) and \( N := N_1 \times \cdots \times N_r \) both are symmetric spaces, \( M \) has no Euclidian factor and the direct product map \( f := f_1 \times \cdots \times f_r : M \to N \) is a parallel isometric immersion, too.

But there are also examples which do not fit into the scheme of Example 1.4.

**Example 1.5** (The Segre embedding). Let positive integers \( l, m, n \) with \( n + 1 = (l + 1)(m + 1) \) be given and consider the parallel isometric immersion \( f : \mathbb{CP}^l \times \mathbb{CP}^m \to \mathbb{CP}^n \),

\[(|z_0 : \cdots : z_l|, [w_0 : \cdots : w_m]) \mapsto [z_0 w_0 : z_0 w_1 : \cdots : z_1 w_m] \text{ (all possible combinations)}, \]

also known as the “Segre embedding” (cf. [1] p. 260). Note that \( \mathbb{CP}^n \) is an irreducible symmetric space.

We are especially interested in the question how the (purely intrinsic) product structure of \( M \) influences the extrinsic geometry of \( f \).

This article is organized as follows:

The precise statement of our results and the required notation is given in Section 2; the main result of this article is Thm. 2.16 from Section 2.6. The corresponding proofs are given in the subsequent sections. The appendix provides some results on representations of Lie groups and Lie algebras; in particular, there we discuss the isotropy representations of symmetric spaces and their “extrinsic tensor products”.

2 Overview

Throughout this section, we always assume that \( M \) and \( N \) both are symmetric spaces, that \( M \) is additionally simply connected and without Euclidian factor and that \( f : M \to N \) is a parallel isometric immersion.

In the following, we implicitly identify \( M \) with its de Rham decomposition \( M_1 \times \cdots \times M_r \) (by means of

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1 According to [3] Thm. 7, for every (not necessarily complete) parallel submanifold \( M_{loc} \subset N \) there exists a simply connected Riemannian symmetric space \( M \), a parallel isometric immersion \( f : M \to N \) and an open subset \( U \subset M \), such that \( f|U : U \to M_{loc} \) is covering.
some fixed isometry $M \to M_1 \times \cdots \times M_r$. Let $L_i \subset M$ be the canonical foliation (whose leaves are the various product slices $L_i(p) := \{p_1 \times \cdots \times M_i \times \cdots \times \{p_r\}$ through $p = (p_1, \ldots, p_r) \in M_1 \times \cdots \times M_r$) and $D^i := TL_i$ for $i = 1, \ldots, r$ be the corresponding distribution of $M$. Then all product slices of $M$ are simply connected, irreducible symmetric spaces.

2.1 The canonical decomposition of the first normal bundle

We introduce the vector subbundles of $\perp f$ which are given by

$$F := \bigcup_{p \in M} \{ h(x, y) \mid x, y \in T_pM \} \in R,$$  \hspace{1cm} (3)

$$F^i := \bigcup_{p \in M} \{ h(x, y) \mid (x, y) \in D^i_p \times D^i_p \} \in R \text{ for } i = 1, \ldots, r,$$  \hspace{1cm} (4)

$$\overline{F} := \sum_{i=1}^r F^i.$$  \hspace{1cm} (5)

$F$ is usually called the “first normal bundle” of $f$. Obviously, Eq. 3-5 define parallel vector subbundles of $\perp f$; in particular, $F$ is equipped with $\nabla^\perp$ (through restriction).

Furthermore, let $F_p^i$ denote the maximal flat subbundle of $F$ and $F_p^i \subset F_p^i$ denote the orthogonal complement of $F_p^i \cap F_p^i$ in $F_p^i$ for each $p \in M$.

**Theorem 2.1.** (a) The linear spaces $F_p^i$ and $F_p^i$ are pairwise orthogonal for each $p \in M$ and $i, j = 1, \ldots, r$ with $i \neq j$, $\{i, j\} \neq \{k, l\}$, and

$$F = \overline{F} \oplus \bigoplus_{1 \leq i < j \leq k} F_{ij}$$  \hspace{1cm} (6)

is a fiberwise orthogonal decomposition into $\nabla^\perp$-parallel vector subbundles.

(b) The linear spaces $F_p^i$ and $F_p^i$ are pairwise orthogonal for each $p \in M$ and $i = 1, \ldots, r$. The same is true for $F_p^i$ and $F_p^i$ with $i \neq j$. Furthermore, $F_p^i \subset \overline{F}_p$ for each $p \in M$ and

$$\overline{F} = F^{1,2} \oplus \bigoplus_{i=1}^r F^i$$

is a fiberwise orthogonal decomposition into $\nabla^\perp$-parallel vector subbundles, too.

A proof of this theorem is given in Section 3.

**Corollary 2.2.** $f|_{L_i(p)} : L_i(p) \to N$ is a parallel isometric immersion for each $p \in M$ and $i = 1, \ldots, r$, too.

For a proof of this corollary see Section 3.

**Example 2.3** (Continuation of Example 2.1). Let $f : \mathbb{CP}^1 \times \mathbb{CP}^n \to \mathbb{CP}^n$ be the “Segre embedding”. Here all the product slices are totally geodesic submanifolds of $\mathbb{CP}^n$, hence $\overline{F}_p = 0$ and thus $F = F^{1,2}$. Furthermore, $f$ is “1-full”, i.e. $F = \perp f$ and therefore dim($F_p$) = 2 for each $p \in \mathbb{CP}^1 \times \mathbb{CP}^n$.

2.2 The $A$-gradation on $\mathfrak{so}(O_{p,f})$

In the following, $T_pM$ is seen as a linear subspace of $T_{f(p)}N$ by means of the injective map $T_pf : T_pM \to T_{f(p)}$ for each $p \in M$. Then $V := T_pM \oplus F_p$ is also a subspace of $T_{f(p)}N$, usually called the (second) “osculating space” at $p$. For each subspace $W \subset V$ we let $\sigma^W : V \to V$ denote the reflection in $W^\perp$, i.e. $\sigma^W|W = \text{Id}$, $\sigma^W|W^\perp = -\text{Id}$. Then the induced map $\text{Ad}((\sigma^W) : \mathfrak{so}(V) \to \mathfrak{so}(V), A \mapsto \sigma^W \circ A \circ \sigma^W$.

\footnote{Note that both $L_i$ and $D^i$ are in fact “intrinsically defined”, i.e. they do not depend on the chosen isometry $M \to M_1 \times \cdots \times M_r$.}
is a linear involution on $\mathfrak{so}(V)$. More specifically, let $i \in \{1, \ldots, r\}$ be given, consider the subspace of $V$ which is given by

$$D^i_p \oplus \bigoplus_{j \neq i} F_p^{ij}$$

and let $\sigma^i \in O(V)$ denote the corresponding reflection. Then, according to Thm. 2.1

$$\forall x \in D^i_p: \sigma^i(x) = -x, \tag{7}$$

$$\forall j \neq i, x \in D^j_p: \sigma^i(x) = x, \tag{8}$$

$$\forall j \neq i, \xi \in F_p^{ij}: \sigma^i(\xi) = -\xi, \tag{9}$$

$$\forall \xi \in F_p^{il}: \sigma^i(\xi) = \xi, \tag{10}$$

$$\forall k \neq i, l \neq i, \xi \in F_p^{kl}: \sigma^i(\xi) = \xi. \tag{11}$$

Therefore, we have $\sigma^i(T_p M) = T_p M$, $\sigma^i(F_p) = F_p$ and

$$\forall x, y \in T_p M, i, j = 1, \ldots, r: h(\sigma^i x, \sigma^j y) = \sigma^i h(x, y). \tag{12}$$

Furthermore, $\sigma^i \circ \sigma^j | T_p M = \sigma^j \circ \sigma^i | T_p M$, hence, by the above, $\sigma^i \circ \sigma^j h(x, y) = \sigma^j \circ \sigma^i h(x, y)$ for all $x, y \in T_p M$, thus $\sigma^i \circ \sigma^j | F_p = \sigma^j \circ \sigma^i | F_p$ also holds, and therefore we even have

$$\forall i, j = 1, \ldots, r: \sigma^i \circ \sigma^j = \sigma^j \circ \sigma^i. \tag{13}$$

**Definition 2.4.** Let $A$ denote the Abelian group whose elements are the functions $\delta : \{1, \ldots, r\} \to \{0, 1\}$ and whose group structure is given by $(\delta + \epsilon)(i) := \delta(i) + \epsilon(i) \mod 2 \mathbb{Z}$ for all $\delta, \epsilon \in A$ and $i = 1, \ldots, r$. Clearly,

$$A \cong \bigoplus_{i=1}^r \mathbb{Z}/2\mathbb{Z}.$$ 

Let $\text{Ad} : O(V) \to \text{Gl}(\mathfrak{so}(V))$ denote the adjoint representation and $\text{Eig}(\text{Ad}(\sigma^i), \lambda)$ denote the corresponding Eigenspace for each $\lambda \in \{-1, 1\}$ and $i = 1, \ldots, r$. By the above, $\{\text{Ad}(\sigma_i)\}_{i=1,\ldots,r}$ is a commuting family of linear involutions on $\mathfrak{so}(V)$, hence $\mathfrak{so}(V)_\delta := \bigcap_{i=1,\ldots,r} \text{Eig}(\text{Ad}(\sigma^i), (-1)^{\delta(i)})$ is a common Eigenspace of these involutions for each $\delta \in A$ and there is the splitting

$$\mathfrak{so}(V) = \bigoplus_{\delta \in A} \mathfrak{so}(V)_\delta. \tag{14}$$

Then

$$[\mathfrak{so}(V)_\delta, \mathfrak{so}(V)_\epsilon] \subset \mathfrak{so}(V)_{\delta + \epsilon}$$

for all $\delta, \epsilon \in A$. \tag{15}

In other words, $\mathfrak{so}(V)$ carries the structure of an $A$-graded Lie algebra. If $A \in \mathfrak{so}(V)_\delta$, then $A$ is called “homogeneous” and $\delta$ is called its “degree”.

**Remark 2.5.** Let $\sigma : V \to V$ denote the reflection in the first normal space $F_p$ and $\mathfrak{so}(V)_\pm$ denote the $\pm 1$-eigenspace of $\text{Ad}(\sigma)$; hence $\mathfrak{so}(V) = \mathfrak{so}(V)_+ \oplus \mathfrak{so}(V)_-$. In this way, $\mathfrak{so}(V)$ may be also seen as a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra. As a consequence of Eq. (14)-(15),

$$\sigma = \sigma^1 \circ \cdots \circ \sigma^r, \tag{16}$$

thus

$$A \to \{0, 1\}, \delta \mapsto |\delta| := \sum_{i=1}^r \delta(i) \mod 2 \mathbb{Z} \tag{17}$$
is a group homomorphism. Furthermore,
\[
\mathfrak{so}(V)_+ = \bigoplus_{|\delta|=0} \mathfrak{so}(V)_\delta ,
\]
\[
\mathfrak{so}(V)_- = \bigoplus_{|\delta|=1} \mathfrak{so}(V)_\delta .
\]

Hence, the \(A\)-gradation is “finer” than the \(\mathbb{Z}/2\mathbb{Z}\)-gradation.

2.3 Curvature invariance of the linear spaces \(\tilde{F}_p\) and \(F^i_p\) with \(i \neq j\)

According to [4 Prop. 7], for each \(p \in M\) and all \(\xi, \eta \in F_p\) the curvature endomorphism \(R^N(\xi, \eta) : T_{f(p)}N \to T_{f(p)}N, v \mapsto R^N(\xi, \eta)v\) has the following properties:
\[
R^N(\xi, \eta)(V) \subset V ,
\]
\[
R^N(\xi, \eta)V \in \mathfrak{so}(V)_+ .
\]

Here we additionally have:

**Theorem 2.6.** If \(\xi, \eta \in F^i_p\) or \(\xi, \eta \in F^j_p\), then, besides Eq. (20),
\[
R^N(\xi, \eta)V \in \mathfrak{so}(V)_0 .
\]

**Corollary 2.7.** \(\tilde{F}_p\) and \(F^i_p\) are curvature invariant subspace of \(T_{f(p)}N\) for each \(p \in M\) and \(i, j = 1, \ldots, r\) with \(i \neq j\).

A proof of Thm. 2.6 and Cor. 2.7 is given in Section 4.

2.4 Geometry of the second osculating bundle

In the following, we suppress the injective vector bundle homomorphism \(Tf : TM \to f^*TN\); for convenience, the reader may simply assume that \(M\) is a submanifold of \(N\) and \(f = \iota^M\). Then the “pull-back” \(f^*TN\) is the vector bundle over \(M\) which is given by \(TM \oplus \perp f\).

**Definition 2.8.**
(a) The split connection is the linear connection \(\nabla^\text{sp} := \nabla^M \oplus \nabla^\perp\) on \(f^*TN\).
(b) For each \(p \in M\) let \(h : T_pM \to \mathfrak{so}(T_{f(p)}N)\) be the linear map defined by
\[
\forall x, y \in T_pM, \xi \in \perp_pM : h(x)(y + \xi) := -S\xi x + h(x, y) .
\]
(c) Let \(L\) be a second Riemannian space and \(g : L \to M\) be any map. Sections of \(f^*TN\) along \(g\) which are parallel with respect to \(\nabla^N\) or \(\nabla^\text{sp}\) are called \(\nabla^N\)-parallel and “split-parallel”, respectively.

Now the equations of Gauß and Weingarten can be formally combined to
\[
\forall X \in \Gamma(TM), S \in \Gamma(f^*(TN)) : \nabla^N_X S = \nabla^\text{sp}_X S + h(X)S .
\]

In the same way, the combined Equations of Gauß, Codazzi and Ricci for the curvature are given by
\[
\forall x, y \in T_pM : R^N(x, y) = R^\text{sp}(x, y) + [h(x), h(y)] ,
\]
where \(R^\text{sp}\) denotes the curvature tensor of \(\nabla^\text{sp}\). Furthermore, the second osculating bundle
\[
O_f := TM \oplus \mathcal{F}
\]
is a parallel vector subbundle of \(f^*TN\) with respect to both \(\nabla^N\) and \(\nabla^\text{sp}\) (see [4 Prop. 6]). In particular,
\[
\forall p \in M, x, y \in T_pM : R^N(x, y)(O_p f) \subset O_p f .
\]

A linear subspace \(U \subset T_qN\) is called curvature invariant if \(R^N(U \times U \times U) \subset U\) holds.

\[3\]
Definition 2.9. (a) Let $\mathcal{E}$ be a vector bundle over $M$ and $\text{Hom}(\mathcal{E})$ denote the linear space of vector bundle maps on $\mathcal{E}$. More precisely, a map $F : \mathcal{E} \to \mathcal{E}$ belongs to $\text{Hom}(\mathcal{E})$ if and only if there exists a “base map” $F : M \to M$ such that the following diagram is commutative

$$
\begin{array}{ccc}
E & \xrightarrow{F} & E \\
\downarrow & & \downarrow \\
M & \xrightarrow{F} & M
\end{array}
$$

and $F|_{\mathcal{E}_p} : \mathcal{E}_p \to \mathcal{E}_{F(p)}$ is a linear map for each $p \in M$. Then we also say that $F$ is a vector bundle map along $F$.

(b) An invertible vector bundle map is also called a vector bundle isomorphism. If $F \in \text{Hom}(\mathcal{E})$ even satisfies $F \circ F = \text{Id}$, then $F$ will be called a vector bundle involution.

(c) Now assume that $\mathcal{E}$ is equipped with a linear connection $\nabla^\mathcal{E}$. A vector bundle map $F : \mathcal{E} \to \mathcal{E}$ along $F$ is called parallel if $F \circ S$ is a parallel section along the curve $F \circ c$ for every curve $c : \mathbb{R} \to M$ and every parallel section $S : \mathbb{R} \to \mathcal{E}$ along $c$.

Equipping $\mathcal{O}f$ with $\nabla^N$ or $\nabla^N_{\text{sp}}$, we thus obtain the notion of $\nabla^N$-parallel and split-parallel vector bundle maps on $\mathcal{O}f$, respectively. Then it is clear what is meant by a $\nabla^N$-parallel vector bundle involution on $\mathcal{O}f$.

For each $p \in M$ let $(p_1, \ldots, p_r) \in M_1 \times \cdots \times M_r$, let $\sigma_p^i$ be the direct product map on $M_1 \times \cdots \times M_r$ which is given by

$$
\sigma_p^i := \text{Id}_{M_1} \times \cdots \times \text{Id}_{M_{i-1}} \times \sigma^M_{p_i} \times \text{Id}_{M_{i+1}} \times \cdots \times \text{Id}_{M_r},
$$

where $\sigma^M_{p_i}$ denotes the corresponding geodesic symmetry for $i = 1, \ldots, r$.

Theorem 2.10. For each $p \in M$ there exists a family $\{\Sigma_p^i\}_{i=1,\ldots,r}$ of pairwise commuting, $\nabla^N$-parallel vector bundle involutions on $\mathcal{O}f$, characterized as follows:

- The base map of $\Sigma_p^i$ is given by Eq. (28),
- $\sigma_p^i(p) = p$ holds and $\Sigma_p^i|_{\mathcal{O}f}f$ is the reflection $\sigma^i$ described by Eq. (11)–(11).

A proof of this theorem is given in Section 5.

Remark 2.11. Put $\Sigma_p := \Sigma_p^1 \circ \cdots \circ \Sigma_p^r$ for some $p \in M$. Then the base map of $\sigma_p$ is the geodesic symmetry of $M$; furthermore, according to Eq. (11), $\Sigma_p|_{\mathcal{O}f}f$ is the reflection in $\perp_p f$. Therefore, $\Sigma_p$ is the “weak extrinsic symmetry” at $p$ whose existence was already proved in [4, Thm. 9]. Now we see how the intrinsic product structure of $M$ induces a distinguished factorization of $\Sigma_p$.

2.5 The “extrinsic holonomy Lie algebra” of $\mathcal{O}f$

For each differentiable curve $c : [0, 1] \to N$ let $(c)^N$ denote the parallel displacement in $TN$ along $c$, i.e.

$$
(1)\begin{array}{cc}
\frac{1}{0} & \frac{1}{0} \\
\frac{1}{0} & S(0) = S(1)
\end{array}
$$

for all parallel sections $S : [0, 1] \to TN$ along $c$ (cf. Def. 2.8). In the following, we equip $\mathcal{O}f$ with the linear connection $\nabla^N$ (see Sec. 2.4). Given a “base point” $p \in M$, we put $V := \mathcal{O}f \circ P$; then

$$
\text{Hol}(\mathcal{O}f) := \left\{ (\frac{1}{0} c)^N|V : V \to V | c : [0, 1] \to M \text{ is a loop with } c(0) = p \right\} \subset O(V). \tag{29}
$$

is the holonomy group of $\mathcal{O}f$ (see [8, Ch. II and III]).

Definition 2.12 ([3]). Let $\mathfrak{h}$ denote the Lie algebra of $\text{Hol}(\mathcal{O}f)$; then $\mathfrak{h}(\mathcal{O}f)$ is a subalgebra of $\mathfrak{so}(V)$, called the “extrinsic holonomy Lie algebra of $\mathcal{O}f$.”

6
Non surprisingly, the geometric structure of $O_f$ described by Thm. 2.10 strongly influences the structure of $\mathfrak{h}$. In [3, Thm. 3], we have already established the splitting $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$ with $\mathfrak{h}_\pm := \mathfrak{h} \cap \mathfrak{so}(V)_\pm$. Again, here we obtain a “finer” result (in the sense of Remark 2.5):

**Theorem 2.13.** 
(a) $\mathfrak{h}$ is an $\mathcal{A}$-graded subalgebra of $\mathfrak{so}(V)$, i.e. there is the splitting

$$\mathfrak{h} = \bigoplus_{\delta \in \mathcal{A}} \mathfrak{h}_\delta,$$

with $\mathfrak{h}_\delta := \mathfrak{h} \cap \mathfrak{so}(V)_\delta$ for each $\delta \in \mathcal{A}$.

(b) Put $\mathfrak{h}_0 := \mathfrak{h}_0$, (see Lemma 4.1) and $\mathfrak{h}_i := \mathfrak{h}_0 \oplus \mathfrak{h}_i$ for $i = 1, \ldots, r$. Then $\mathfrak{h}_i$ is an $\mathcal{A}$-graded subalgebra of $\mathfrak{h}$. Furthermore, we have

$$[\mathfrak{h}(x), \mathfrak{h}(y)] \subset \mathfrak{h}(x) \mathfrak{h}(y)$$

for each $x \in D_p^i$ and $i = 1, \ldots, r$.

(c) We have $R^N(x, y)V \subset V$ and $R^N(\xi, \eta)V \subset V$ for all $x, y \in D_p^i$ and $\xi, \eta \in \mathbb{R}^k$ or $\xi, \eta \in \mathbb{R}^k$ $(k, l = 1, \ldots, r)$; furthermore,

$$\mathfrak{h}_0 = \sum_{j=1}^r \left\{ R^N(x, y)|V | x, y \in D_p^j \right\}_{\mathbb{R}} + \left\{ R^N(\xi, \eta)|V | \xi, \eta \in \mathbb{R}^k \right\}_{\mathbb{R}} + \sum_{k, l=1}^r \left\{ R^N(\xi, \eta)|V | \xi, \eta \in \mathbb{R}^{kl} \right\}_{\mathbb{R}}.$$

A proof of this theorem is given in Section 5.

### 2.6 Homogeneity of parallel submanifolds

Let $I(N)$ denote the isometry group of $N$ (which is actually a Lie group, according to [3, Ch. IV]). Given a subset $M \subset N$, suppose that there exists a connected Lie subgroup $G \subset I(N)$ and some $p \in M$ such that $M$ is equal to the orbit $Gp$. Then $M$ is actually a submanifold of $N$, called a homogeneous submanifold. In this case, a standard argument shows that $M$ is even a complete Riemannian manifold. Hence, if $f : M \to N$ is a parallel isometric immersion from a simply connected symmetric space and $f(M)$ is a homogeneous submanifold of $N$, then $f$ is necessarily a Riemannian covering onto $f(M)$. The following stronger concept of extrinsic homogeneity was already used in [3]:

**Definition 2.14.** Let $M$ be a submanifold of $N$. We say that $M$ has extrinsically homogeneous tangent holonomy bundle if there exists a connected Lie subgroup $G \subset I(N)$ with the following properties:

- $g(M) = M$ for all $g \in G$.
- For each $p \in M$ and every curve $c : [0, 1] \to M$ with $c(0) = p$ there exists some $g \in G$ such that $g(p) = c(1)$ and that the parallel displacement along $c$ is given by

$$\begin{aligned}
\frac{1}{0} c' = T_{pg}|T_pM : T_pM \to T_{c(1)}M.
\end{aligned}$$

**Definition 2.15.** 
(a) An (intrinsically) flat totally geodesic submanifold of $N$ is briefly called a flat of $N$.

(b) According to [3, Ch. V, § 6], the rank of $N$ is the maximal dimension of a flat of $N$.

(c) According to [3, Ch. V, § 1], $N$ is called “of compact type” or “of non-compact type” if the Killing form of $I(N)$ restricted to $p$ is strictly negative or strictly positive, respectively.\(^4\)

\(^4\)Actually, $\mathfrak{h}$ is merely a $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebra.

\(^5\)However, $M$ is not necessarily “embedded”, i.e. its topology may be strictly finer than the “subset topology” (cf. [1, p. 7])

\(^6\)N is of compact (or non-compact) type if and only if the universal covering space of $N$ is compact (or non-compact) (cf. [3, Ch.V-VII]).
Let \( p \) be a fixed point of \( M \); then \( T_p M \) is seen as a subspace of \( T_{f(p)} N \), and so is \( D^i_p \) for \( i = 1, \ldots, r \). Let \( \exp^N \) denote the exponential spray of \( N \). Since \( f|L_i(p) \) is also a parallel isometric immersion according to Corollary 2.2, we can apply the Codazzi Equation to deduce that \( D^i_p \) is even a curvature invariant subspace of \( T_{f(p)} N \) (cf. Fn. 3\(^7\)). Hence, by virtue of a result due to E. Cartan,
\[
\bar{M}_i := \exp^N(D^i_p) \subset N
\]
(34)
is a totally geodesically embedded symmetric space. Furthermore, let \( \mathfrak{h} \) denote the Lie algebra from Def. 2.12.

**Theorem 2.16 (Main Theorem).** Besides the conventions made at the beginning of Section 3, we also assume that \( N \) is of compact or non-compact type and that \( \dim(M_i) \geq 3 \) holds for \( i = 1, \ldots, r \). Then the following assertions are equivalent:

(a) \( f(L_i(p)) \) is not contained in any flat of \( N \) for \( i = 1, \ldots, r \).
(b) \( f(M) \) is a homogeneous submanifold.
(c) \( f(M) \) is a submanifold with extrinsically homogeneous tangent holonomy bundle.
(d) We have
\[
\mathfrak{h}(T_p M) \subset \mathfrak{h}.
\]
(35)
(e) The symmetric space \( \bar{M}_i \) defined by Eq. (34) is irreducible if \( i = 1, \ldots, r \).

The direction “(a) \( \Rightarrow \) (c)” should be seen as the main result of this article; in case \( M \) is even irreducible, this implication already follows from [5, Thm. 5]. A proof of the Main Theorem can be found in Section 6.

The following is an immediate consequence of Theorem 2.16:

**Corollary 2.17.** In the situation of Theorem 2.16, if all factors of \( M \) are of dimension larger than the rank of \( N \), then \( f(M) \) is a homogeneous submanifold of \( N \).

### 3 Proof of Theorem 2.1 and Corollary 2.2

Let \( M \) be a simply connected Riemannian submanifold without Euclidian factor and \( M_0 \times \cdots \times M_r \) denote its de Rham decomposition (see Def. 1.3). Let \( I(M)^0 \) and \( I(M_i)^0 \) denote the connected components of the corresponding isometry groups, respectively. We keep \( p = (p_1, \ldots, p_r) \in M \) fixed and let \( K \) and \( K_i \) denote the isotropy subgroups of \( I(M)^0 \) and \( I(M_i)^0 \) at \( p \) and \( p_i \), respectively, for \( i = 1, \ldots, r \). Then, by means of the uniqueness of the de Rham decomposition, we have
\[
I(M)^0 \cong I(M_1)^0 \times \cdots \times I(M_r)^0,
\]
(36)
hence
\[
K \cong K_1 \times \cdots \times K_r.
\]
(37)
Furthermore, \( K \) and \( K_i \) both are connected (since \( M \) and its irreducible factors are simply connected).

**Definition 3.1.** A homogeneous vector bundle over \( M \) is a pair \((E, \alpha)\) where \( E \rightarrow M \) is a vector bundle and \( \alpha : I(M)^0 \times E \rightarrow E \) is an action through vector bundle isomorphisms (cf. Def. 2.4 (b)) such that the bundle projection of \( E \) is equivariant.

In the situation of the last definition,
\[
I(M)^0 \rightarrow M, \quad g \mapsto g(p)
\]
(38)
is a principal fiber bundle with structure group \( K \) and \( E \) is a vector bundle associated therewith via
\[
I(M)^0 \times E_p \rightarrow E, \quad (g, v) \mapsto \alpha(g, v).
\]
(39)
Therefore, one briefly writes “\( E \cong I(M)^0 \times_{K} E_p \)” or “\( E \cong (I(M)^0 \times E_p)/K \)” (cf. [5, Ch. 2.1]).

Now let \( f : M \rightarrow N \) be a parallel isometric immersion and \( F \) denote the first normal bundle of \( f \) (see Eq. 3\(^8\)), equipped with the linear connection \( \nabla \).

---

\(^7\) This condition simply means that \( M \) does not split off a factor whose dimension is 1 or 2 (cf. Def. 1.3).

\(^8\) An arbitrary symmetric space is called “irreducible” if its universal covering space is an irreducible symmetric space in the sense of Def. 1.3.
Proposition 3.2 ([4] Prop. 10]). There exists an action \( \alpha : I(M)^0 \times \mathbb{F} \to \mathbb{F} \) through isometric, \( \nabla^\bot \)-parallel vector bundle isomorphisms characterized by

\[
\alpha(g, h(x, y)) = h(T_p g x, T_p g y)
\]

(40)

for all \( x, y \in T_p M \) and \( g \in I(M)^0 \). Hence \( (\mathbb{F}, \alpha) \) is a homogeneous vector bundle over \( M \).

Let \( (\mathcal{E}, \alpha) \) be a homogeneous vector bundle over \( M \) (compare Def. 3.1 and \( I(M) \) denote the Lie algebra of \( I(M) \)). Following [5] Ch. 2.1, the Cartan decomposition \( I(M) = \mathfrak{t} \oplus \mathfrak{p} \) induces a distinguished connection \( \nabla^\mathcal{E} \), called the canonical connection. It can be obtained as follows: On the principal fiber bundle \( \mathbb{F} \), there is a \( I(M)^0 \)-invariant connection \( \mathcal{H} \) defined by

\[
\mathcal{H}_g := \{ X_g \mid X \in \mathfrak{p} \}
\]

(41)

for all \( g \in I(M)^0 \) where the elements of \( \mathfrak{p} \) are also considered as left-invariant vector fields on \( I(M)^0 \) (see [5] Vol. 1, p. 239). Since \( \mathcal{E} \) is an associated vector bundle via Eq. (39), the connection \( \mathcal{H} \) induces a linear connection \( \nabla^\mathcal{E} \). In order to relate the parallel displacement induced by \( \nabla^\mathcal{E} \) to the horizontal structure \( \mathcal{H} \), let a curve \( c : \mathbb{R} \to M \) with \( c(0) = p \) be given; then

\[
\forall v \in \mathcal{E}_p : \left(\begin{array}{c}
\frac{1}{|v|} v \\
0
\end{array}\right) = \alpha(\hat{c}(1), v),
\]

(42)

where \( \hat{c} : [0, 1] \to I(M)^0 \) denotes the \( \mathcal{H} \)-lift of \( c \) with \( \hat{c}(0) = \text{Id} \). One can also show that the canonical connection does not depend on the special choice of the base point \( p \).

Let \( \rho : K \to O(T_p M) \) denote the isotropy representation, which equips \( TM \) with the structure of a homogeneous vector bundle over \( M \), too. Note that \( \rho \) is a faithful representation (since \( K \) acts through isometries on \( M \)). Furthermore, let \( \text{Hol}(M) \subset O(T_p M) \) and \( \text{Hol}(\mathbb{F}) \subset O(\mathfrak{p}) \) denote the corresponding holonomy groups, respectively. Then \( \text{Hol}(M) \) and \( \text{Hol}(\mathbb{F}) \) both are connected (since \( M \) is simply connected).

**Lemma 3.3.**

(a) \( \nabla^M \) is the canonical connection of \( TM \).

(b) \( \rho \) maps \( K \) isomorphically onto \( \text{Hol}(M) \) such that \( K_i \) corresponds to \( \text{Hol}(M_i) \). Therefore, \( K_i \) acts irreducibly and non-trivially on \( D^i_p \) for \( i = 1, \ldots, r \).

(c) \( \nabla^\perp \) is the canonical connection of \( \mathbb{F} \) and the corresponding holonomy group \( \text{Hol}(\mathbb{F}) \) is the homomorphic image of \( K \) which is given by

\[
\{ \alpha(g, \cdot)\mathbb{F}_p \mid g \in K \}.
\]

(43)

**Proof.** For a): According to [8] Ch. X.2], the canonical connection \( \nabla^{TM} \) is a metric and torsion-free connection, hence \( \nabla^{TM} \) is the Levi Civita connection \( \nabla^M \).

For (b): We have \( \text{Hol}(M) \cong \text{Hol}(M_1) \times \cdots \times \text{Hol}(M_r) \) (since \( M \) the Riemannian product of the \( M_i \)'s) and \( K \cong K_1 \times \cdots \times K_r \) (see [37]). Thus we can assume that \( M \) is irreducible; then [4] Prop. 11] implies that the first statement of (b) holds on the level of Lie algebras. Thus also \( \text{Hol}(M) = \rho(K) \) holds, since the involved Lie groups are connected. Therefore, \( K_i \) acts irreducibly on \( D^i_p \) as a consequence of the de Rham Decomposition Theorem. In particular, \( K_i \) acts non-trivially on \( D^i_p \), since \( \dim(M_i) \geq 2 \).

For c): Let a point \( p \in M \), a curve \( c : [0, 1] \to M \) with \( c(0) = p \) and some \( g \in I(M)^0 \) with \( g(p) = p \) be given. Let \( \hat{c} \) denote the horizontal lift of \( c \) with \( \hat{c}(0) = \text{Id} \). Then it suffices to show that the parallel displacement described in Eq. (12) (with \( \mathbb{E} := \mathbb{F} \)) is equal to the parallel displacement with respect to \( \nabla^\perp \).

This for: Let \( \xi \in \mathbb{F}_p \) be given; thereby, we will assume that \( \xi = h(x, y) \) for certain \( x, y \in T_p M \). Then, in accordance with Part (a) and Eq. (12), the parallel displacement of \( x \) and \( y \) in \( TM \) along \( c \) is given by \( T\hat{c}(1) x \) and \( T\hat{c}(1) y \), respectively. Then

\[
\alpha(\hat{c}(1), \xi) = h(T\hat{c}(1) x, T\hat{c}(1) y)
\]

is the parallel displacement of \( \xi \) along \( c \) with respect to \( \nabla^\perp \) (since \( h \) is a parallel section of \( L^2(TM, \perp f) \)).

The previous arguments also show that \( \text{Hol}(\mathbb{F}) \) is given by Eq. (43).
The proof of the following lemma is “straight forward”:

**Lemma 3.4.** Let $(E, \alpha)$ be a homogeneous vector bundle over $M$ and $\nabla^E$ denote the corresponding canonical connection. Let $p \in M$ be a fixed point and $K$ be the corresponding isotropy subgroup of $1(M)^0$. Assigning to each $\nabla^E$-parallel section $s$ its value $s(p) \in E_p$ induces a one-one correspondence

$$
\text{parallel sections of } E \leftrightarrow K\text{-invariant elements of } E_p.
$$

(44)

**Proof of Thm. 2.1**

Proof. For (a): Clearly,

$$
F = \tilde{F} + \sum_{1 \leq i < j \leq r} F^{ij}.
$$

(45)

Now let $i, j, k, l$ with $i \neq j$ and $\{i, j\} \neq \{k, l\}$ be given; hence, it remains to show that $F^{ij} \perp F^{kl}$ holds.

For this: Without loss of generality, we will assume that $i$ is different from both $k$ and $l$. Furthermore, we note that $K_i$ is a Lie subgroup of $K$ according to Eq. (34). Hence, on the one hand, we have the action of $K_i$ on $E_p$ through orthogonal transformations (by means of $\alpha$); then $F^{kl} \perp F^{ij}$ both are invariant subspaces of $F_p$.

On the other hand, $K_i$ acts orthogonally on $T_p M$ by means of the isotropy representation. Then $D^p_i$ is an irreducible invariant subspace of $T_p M$ (according to Lemma 3.3 (b)) and $D^p_i$ is an invariant subspace on which $K_i$ acts trivially (since $i \neq j$). Therefore, the induced action of $K_i$ on $D^p_i \otimes D^p_j$ (as described in Def. 11) is isomorphic to the direct sum of $m_j$ copies of $D^p_i$ (where $m_j$ denotes the dimension of $M_j$), whereby the action of $K_i$ on each copy of $D^p_i$ is irreducible and non-trivial.

Therefore, and since $h: D^p_i \otimes D^p_j \rightarrow F_p^j$ is a surjective homomorphism in accordance with Eq. (40), $F^{ij}$ also decomposes into a direct sum of non-trivial and irreducible subspaces, by virtue of Lemma 3.3 (d) (here and in the following, we use the “Lie group version” of this Lemma, cf. Rem. A.3). On the other hand, since $i$ is different from both $k$ and $l$, $K_i$ acts trivially on $D^p_k \otimes D^p_l$. Thus $K_i$ also acts trivially on $F^{kl}$, because of Eq. (10). Thus $F^{ij} \perp F^{kl}$, according to Lemma 3.3 (a).

For (b): It suffices to show that

$$
F^2_p \subset \tilde{F}_p,
$$

(46)

$$
F^2_p \perp F^i_p \quad \text{for } i = 1, \ldots, r,
$$

(47)

$$
F^p_j \perp F^j_p \quad \text{for } i \neq j.
$$

(48)

For this: Since $M$ is simply connected, $F^2$ is pointwise spanned by the parallel sections of $F$. Hence sections of $F^2$ correspond uniquely to $K$-invariant elements of $F_p$, according to Lemma 3.3 (a) in combination with Lemma 3.4. Therefore, $F^2_p$ is the maximal subspace of $F_p$ on which $K$ acts trivially. Since we have already seen in the proof of Part (a) that $F^2_p$ is isomorphic to a direct sum of non-trivial and irreducible $K$-modules, it follows by arguments given before that $F^2_p \perp F^i_p$ for all $i \neq j$; now Part (a) implies that Eq. (49) holds.

Let $F^2_p = W_0 \oplus W_1 \oplus \cdots \oplus W_l$ be a decomposition into invariant subspaces such that $W_0$ is a trivial $K_i$-module and that $W_k$ is an irreducible and non-trivial $K_i$-module for $k = 1, \ldots, l$. Then $W_0 = \tilde{F}_p \cap \tilde{F}^2_p$ holds, since $K_j$ acts trivially on $\tilde{F}^2_p$ for $j \neq i$ anyway, according to Eq. (40). Therefore, $F^2_p$ is given by $W_1 \oplus \cdots \oplus W_l$. From Lemma 3.3 (a) we now conclude that Eq. (49) is valid. Finally, since $K_i$ acts trivially on $\tilde{F}^2_p$ for $i \neq j$, we obtain from a similar argument that Eq. (48) also holds.

**Lemma 3.5.** We have

$$
\forall x_i \in D^p_i, x_j \in D^p_j, \xi \in F^2_p:\quad S_{\xi} x_i \in D^p_j,
$$

(49)

$$
\forall x_j \in D^p_j, \xi \in F^2_p:\quad S_{\xi} x_j \in D^p_i,
$$

(50)

$$
\forall x_i, y_i \in D^p_i: \quad R^N(x_i, y_i)(D^p_i \oplus F^2_p) \subset D^p_i \oplus F^2_p
$$

(51)

for all $i, j = 1, \ldots, r$. 

10
Proof. For Eq. (40): For each \( x_k \in D^k_p \) with \( k \neq j \) we have
\[
\langle S_k x_i, x_k \rangle = -\langle h(x_i, x_k), \xi \rangle = 0 ,
\]
according to Thm. 2.1. Hence \( S_k x_j \in D^j_p \).

For Eq. (50): For each \( x_k \in D^k_p \) with \( k \neq j \) we have
\[
\langle S_k x_j, x_k \rangle = -\langle h(x_j, x_k), \xi \rangle = 0 ,
\]
by means of Thm. 2.1. Hence \( S_k x_j \in D^j_p \).

For Eq. (51): We have \([h(x_i), h(y_i)] F^i_p \subset \mathbb{R}^j_p \) and \([h(x_i), h(y_i)] D^j_p \subset D^i_p \) for all \( x_i, y_i \in D^i_p \), according to Eq. (49). Moreover, \( R^M(x_i, y_i) D^j_p \subset D^j_p \) (since \( D^j \) is a parallel vector subbundle of \( TM \)) and \( R^L(x_i, y_i) F^i_p \subset F \) (since \( F \) is parallel vector subbundle of \( F \)). Now the result follows from Eq. (25).

**Proof of Cor. 2.2**

Proof. Put \( f_i := f|L_i(p) : L_i(p) \rightarrow N \); then
\[
T_q L_i(p) = D_q^i \text{ and } \perp_q f_i = \bigoplus_{j \neq i} D_q^j \oplus \perp_q f ,
\]
holds for each \( q \in M \). Furthermore, since \( L_i(p) \subset M \) is totally geodesic, the second fundamental form of \( f_i \) is given by
\[
h_i(D^i_l \times D^i_q)
\]
for each \( q \in L_i(p) \); hence the first normal bundle of \( f_i \) is given by \( \mathbb{R}^i|L_i(p) \) (pullback of \( F \) to \( L_i(p) \)). I claim that \( \nabla^q \) coincides on \( \mathbb{R}^i|L_i(p) \) with the usual normal connection of \( f_i \).

For this: Let \( \xi \) be a section of \( \mathbb{R}^i \) along \( L_i(p) \). Then we have \( \nabla^q \xi = \nabla^q_X \xi - S_i(x) \), where \( \nabla^q_X \xi \) again is a section of \( \mathbb{R}^i \) along \( L_i(p) \) (because \( \mathbb{R}^i \) is a parallel vector subbundle of \( F \)) and \( S_i(x) \) again is a section of \( D^i \) along \( L_i(p) \), according to Lemma 3.5 (b). Therefore, by means of the Weingarten equation, the covariant derivative of \( \xi \) with respect to the normal connection of \( \perp f_i \) is given by \( \nabla^q_X \xi \).

Now let \( c : \mathbb{R} \rightarrow L_i(p) \) be a curve and \( X, Y \) be parallel sections of \( TL_i(p) \) along \( c \). Then \( c \) is a curve into \( M \) and \( X, Y \) are parallel sections of \( TM \) along \( c \), too. Hence \( t \mapsto h_X(t) := h(X(t), Y(t)) \) defines a parallel section of \( \mathbb{R}^i \) along \( c \), and hence, by the previous, \( \xi \) is also a parallel section of \( \perp f_i \) with respect to the usual normal connection of \( f_i \). Therefore, \( f_i \) is a parallel isometric immersion.

4 Proof of Theorem 2.6 and Corollary 2.7

We continue with the notation from Section 2.2. For \( i \in \{1, \ldots, r\} \) let \( \delta_i \) denote the characteristic function of \( \{i\} \), i.e.
\[
\delta_i(i) = 1 \text{ and } \delta_i(j) = 0 \text{ for } j \neq i .
\]

**Lemma 4.1.** (a) Let 0 denote the zero-function on \( \{1, \ldots, r\} \). Then
\[
so(V)_0 = \{ A \in so(V)_+ \ | A(D^i_p) \subset D^i_p \text{ and } A(F^i_p) \subset F^i_p \text{ for all } i, j = 1, \ldots, r \text{ with } j \neq i \} .
\]

(b) Put \( so(V)_i := so(V)_{\delta_i} \text{ for } i = 1, \ldots, r \). Then we have
\[
so(V)_i = \{ A \in so(V)_+ \ | A(D^i_p) \subset D^i_p \text{ and } A(D^i_p) \subset F^i_p \text{ for all } j \neq i \} .
\]

Therefore,
\[
h(x) \in so(V) , \text{ for all } x \in D^i_p \text{ and } i = 1, \ldots, r .
\]

(c) Put \( so(V)_{ij} := so(V)_{\delta_i + \delta_j} \text{ for } i, j = 1, \ldots, r \). Then, in addition to Eq. (27), we have
\[
\forall(x, y) \in D^i_p \times D^i_p : R^N(x, y)|V \in so(V)_{ij} .
\]

In particular (since \( \delta_i + \delta_i = 0 \) holds),
\[
\forall x, y \in D^i_p : R^N(x, y)|V \in so(V)_0 .
\]
Proof. Using Eq. (1)(1), the proof of Eq. (50) and (51) is straightforward. Then Eq. (53) immediately follows from Eq. (29) and (54).

For Eq. (56) and (57): Note that the curvature tensor for the split-connection of \( f^*TN \) (see Def. 2.8) is given by \( R^P(x, y) = R^N(x, y) \oplus R^N(x, y) \) for all \( x, y \in T_pM \), hence \( R^P(x, y)(V) \subseteq V \) and \( R^P(x, y)V \subseteq \mathfrak{so}(V)_0 \) for all \( x, y \in T_pM \) as a consequence of Eq. (53) and since Eq. (1)-(5) are parallel vector subbundles of \( f^*Tf \). Therefore, and by means of Eq. (25) and (55), Eq. (57) now follows. Moreover, in case \( i \neq j \), \( R^M(x, y) = 0 \) for all \( (x, y) \in D^i_p \times D^j_p \), hence, according to [11] Prop. 4 (d),

\[
R^L(x_1, y_1)h(x_2, y_2) = h(R^M(x_1, y_1) x_2, y_2) + h(x_2, R^M(x_1, y_1) y_2) = 0
\]

for all \( (x_1, y_1) \in D^i_p \times D^j_p, (x_2, y_2) \in T_pM \times T_pM \), i.e. \( R^L(x_1, y_1) \| T_p = 0 \). Consequently, \( R^N(x, y)\|V = [h(x), h(y)] \) for all \( (x, y) \in D^i_p \times D^j_p \), according to Eq. (29); therefore, and by means of arguments given before, now Eq. (56) also follows for \( i \neq j \).

Proof of Thm. 2.6

Proof. According to [11] Lemma 5] (see in particular Ed. (60) there), for each \( p \in M \) the following Equation holds on \( V \) for all \( x_1, \ldots, x_4 \in T_pM \):

\[
[h(x_1), [h(x_2), R^N(x_3, x_4)]] = R^N(h(x_1)h(x_2) x_3, x_4) + R^N(x_3, h(x_1)h(x_2) x_4) + R^N(h(x_1) x_3, h(x_2) x_4) + R^N(h(x_2) x_3, h(x_1) x_4). \tag{58}
\]

Furthermore, Eq. (29), (49) and (50) imply that

\[
\begin{align*}
\hbar(D^i_p) h(D^j_p)(D^k_p) &= \hbar(D^i_p) h(D^j_p)(D^k_p) \subset D^k_p, \tag{59} \\
\hbar(D^i_p) h(D^j_p)(D^k_p) \subset D^k_p &\tag{60}
\end{align*}
\]

for all \( i, j = 1, \ldots, r \). Using that \( R^N(h(x, x), h(x, x)) = 0 \) (because of the symmetry of \( h_p \)), Eq. (58) implies that for all \( x_i, y_i \in D^i_p \) and \( x_j, y_j \in D^j_p \) the following equations hold on \( V \):

\[
\begin{align*}
R^N(h(x_i, x_i), h(y_j, y_j)) &= [h(x_i), [h(y_j), R^N(x_i, y_i)]] - R^N(h(x_i) h(y_j) x_i, y_i) - R^N(x_i, h(x_i) h(y_j) y_j), \tag{61} \\
R^N(h(x_i, x_j), h(y_i, y_j)) &= [h(x_i), [h(y_i), R^N(x_i, y_i)]] - R^N(h(x_i) h(y_i) x_j, y_i) - R^N(x_j, h(x_i) h(y_j) y_i) - R^N(h(x_i, y_i), h(x_j, y_j)). \tag{62}
\end{align*}
\]

In order to establish Eq. (22), first assume that \( \xi, \eta \in \mathfrak{R}_p \). Furthermore, without loss of generality we can assume that there exists \( (x, y) \in D^i_p \times D^j_p \) such that \( \xi = h(x, x), \eta = h(y, y) \) (since \( h_p \) is symmetric). Then Eq. (22) is an immediate consequence of Eq. (15), (55), (50), (51) and (61).

Now assume that \( \xi, \eta \in \mathfrak{R}_p^y \). Using Eq. (62) in combination with the previous, Eq. (22) now follows by similar arguments as above.

Proof of Cor. 2.7

Proof. Use Eq. (58) in combination with Thm. 2.6

5 Proof of Theorem 2.10 and Theorem 2.13

The proof of the following basic lemma is left to the reader:

Lemma 5.1 (Continuation of Lemma 5.4). Let \((E, \alpha)\) be a homogeneous vector bundle over \( M \) and \( \nabla^E \) denote the corresponding canonical connection. Let \( p \in M \) be a fixed point and \( K \) be the corresponding isotropy subgroup of \( I(M)^0 \).
(a) For each $\sigma \in I(M)$, the “pull back bundle” $\sigma^*E \to M$ (whose total space is the fiber product $M \times_g E$) is a homogeneous vector bundle over $M$, too, by means of the action

$$I(M)^0 \times \sigma^*E \to \sigma^*E, (g, v) \mapsto \alpha(\sigma \circ g \circ \sigma^{-1}, v).$$

Moreover, the corresponding canonical connection is the one which is induced by $\nabla^E$ in the usual way.

(b) If $(\tilde{E}, \tilde{a})$ is a second homogeneous vector bundle over $M$, then the induced vector bundle $L^1(\tilde{E}, \tilde{E})$ is a homogeneous vector bundle over $M$, too, by means of the action

$$I(M)^0 \times L^1(\tilde{E}, \tilde{E}) \to L^1(\tilde{E}, \tilde{E}), (g, \ell : E_p \to \tilde{E}_p) \mapsto \tilde{a}_g|\tilde{E}_p \circ \ell \circ \alpha_g^{-1}|_{E_g(p)}.$$

Moreover, if $\nabla^E$ denotes the canonical connection of $\tilde{E}$, then the canonical connection on $L^1(\tilde{E}, \tilde{E})$ is the one which is induced by $\nabla^E$ and $\nabla^\tilde{E}$ in the usual way.

The proof of the following lemma is also straightforward:

**Lemma 5.2.** Let $V'$ be a Euclidian vector space and $W \subset V'$ be a subspace. Let $\sigma V' \to V'$ denote the reflection in $W^\perp$ and $\mathfrak{so}(V') = \mathfrak{so}(V')_+ \oplus \mathfrak{so}(V')_-$ be the decomposition into the eigenspaces of $\text{Ad}(\sigma)$ (see Remark 2.22). Then the natural map $\mathfrak{so}(V')_- \to \text{L}(W; W^\perp), A \mapsto A|W$ is a linear isomorphism.

**Proof of Thm. 2.10**

**Proof.** 1. **Step:** Let $p$ be a fixed point and let $\sigma^i_p$ be defined according to Eq. (28); then $\sigma^i_p(p) = p$ holds. We will show that $F$ admits a parallel vector bundle isomorphism $I^i_p$ along the base map $\sigma^i_p$ such that

$$I^i_p|F_p = \sigma^i|F_p, \quad (63)$$

$$\forall q \in M, x, y \in T_qM : I^i_p h(x, y) = h(T\sigma^i_p x, T\sigma^i_p y) \quad (64)$$

holds for $i = 1, \ldots, r$.

For this: As a consequence of Lemma 3.3 (c), Lemma 3.4 and Lemma 5.1, every parallel vector bundle homomorphism of $F$ along $\sigma^i_p$ uniquely corresponds to a parallel section of $E := L^1(F, \sigma^i_p F)$, where the latter is seen as a homogeneous vector bundle equipped with the corresponding canonical connection as described in Lemma 5.1. We let $K$ be the isotropy subgroup of $I(M)$ at $p$; then $E^2_p$ and $E^1_p$ are subspaces of $E_p$ which are invariant under the action of $K$; hence the linear map $\sigma^i|E_p$ (defined by Eq. (28)-(11)) is a $K$-invariant element of $E_p$. Therefore, according to Eq. (44), $\sigma^i|F_p$ uniquely extends to a parallel section $I^i_p$ of $E$, again by means of Lemma 3.3. Then the base map $\sigma^i_p$ is an involution on $M$ and $I^i_p|E_p = \sigma^i|E_p$ is a reflection of $F_p$, hence the parallelity of $I^i_p$ implies that $I^i_p|F_p$ is a vector bundle involution. It remains to establish Eq. (64).

Let $c : [0, 1] \to M$ be a curve with $c(0) = p$ and $c(1) = q$ and let $X, Y$ be parallel sections of $TM$ along $c$ with $X(1) = x$ and $Y(1) = y$. Consider the two sections $S_1$ and $S_2$ of $TM$ along the curve $\sigma^i_p \circ c$ defined by $S_1(t) := I^i_p(h(X(t), Y(t)))$ and $S_2(t) := h(T\sigma^i_p X(t), T\sigma^i_p Y(t))$. Using the parallelity of $f$ and the fact that $\sigma^i_p$ is an isometry of $M$, we see that both $S_1$ and $S_2$ are parallel sections. Furthermore $S_1(0) = S_2(0)$ holds, in accordance with Eq. (12); therefore $S_1 = S_2$, in particular Eq. (64) holds.

2. **Step:** Put $\Sigma^i_p := T\sigma^i_p \oplus I^i_p$. Then $\Sigma^i_p$ is a split-parallel vector bundle involution along $\sigma^i_p$, and $\Sigma^i_p|V$ is the reflection $\sigma^i$ described by Eq. (77)-(111). I claim that $\Sigma^i_p$ is also $\nabla^N$-parallel: In combination with Lemma 5.2 implies that

$$\forall q \in M, x \in T_qM, v \in O_q f : \Sigma^i_p(h(x) v) = h(T\sigma^i_p x)(\Sigma^i_p v). \quad (65)$$

Since $\Sigma^i_p$ is split-parallel, Eq. (65) combined with the Gauß-Weingarten equation Eq. (24) implies that $\Sigma^i_p$ is $\nabla^N$-parallel, too. The result follows.

The last assertion of the theorem follows from Eq. (13) in combination with the parallelity of $\Sigma^i_p$. ∎
Proof of Thm. 2.13

Proof. For (a): Let \( p \) be a fixed point, put \( V := \mathcal{O}_p f \) and let \( \sigma^i \) denote the reflection of \( V \) defined by Eq. (11). Then it suffices to show that

\[
\text{Ad}(\sigma^i)(h) = h.
\] (66)

Let \( \Sigma^i_p \) denote the symmetry of \( \mathcal{O}f \) described in Thm. 2.10 and let \( c : [0, 1] \to M \) be a loop with \( c(0) = p \). Remember that \( \Sigma^i_p \) is a \( \nabla^N \)-parallel vector bundle isomorphism of \( \mathcal{O}f \) along \( \sigma^i_p \) (see Eq. (28)) with \( \Sigma^i_p|\mathcal{O}_p f = \sigma^i \), in accordance with Thm. 2.10 hence

\[
\sigma^i \circ ([c])^N V = ([\sigma^i_p \circ c])^N \circ \sigma^i.
\] (67)

From the last line in combination with Eq. (20) we conclude that \( \text{Hol}(\mathcal{O}f) \) is invariant under group conjugation with \( \sigma^i \); thus Eq. (66) holds.

For (b): In accordance with [4, Thm. 3]

\[
[h(x), h] \subset h
\] (68)

holds for each \( x \in T_p M \). Now Eq. (31) follows as an immediate consequence of Part (a) in combination with Eq. (55) and the rules for graded Lie algebras (see Eq. (15)).

For (c): First, let us see that r.h.s. of (32) is contained in \( h_0 \).

For this: We have \( R^N(x, y)(V) \subset V \) and \( R^N(x, y)\text{'}V \in h_0 \) for all \( x, y \in D^j_p \) and \( j = 1, \ldots, r \), because of Eq. (57) and the Theorem of Ambrose/Singer (cf. [4, Proof of Thm. 3]). Now let \( \xi \in \mathbb{P}^j, \eta \in \mathbb{P}^j \) be given; thereby, we will assume that there exist \((x_j, x_k) \in D^j_p \times D^k_p\) such that \( \xi = h(x_j, x_j) \) and \( \eta = h(x_k, x_k) \). Then, in accordance with Eq. (20) and (22) we have \( R^N(\xi, \eta)(V) \subset V \) and \( R^N(\xi, \eta)|V \in so(V)_0 \); moreover, as a consequence of Eq. (61) combined with Eq. (68), even \( R^N(\xi, \eta)|V \in h_0 \) holds. Additionally using Eq. (62) we obtain the same result for all \( \xi, \eta \in \mathbb{P}^j \) and \( j, l = 1, \ldots, r \), which proves our claim.

In order to finally establish Eq. (32), we introduce the following linear subspaces of \( so(V) \),

\[
j_0 := \{ R^N(y_1, y_2)V \mid y_1, y_2 \in T_p M \}_R \quad \text{and} \quad j_2 := \{ [h(x_1), [h(x_2), [R^N(y_1, y_2)V]]]\mid x_1, x_2, y_1, y_2 \in T_p M \}_R.
\]

Then, according to [4, Proof of Thm. 3], we have \( h_0 \subset h_+ \) and, furthermore, using Eq. (35)-50, \( R^N(x_j, x_k)V \text{ and } [h(x_j), [h(x_k), [R^N(y_j, y_k)V]]] \) are homogeneous elements of degree \( \delta_j + \delta_j \) and \( \delta_j + \delta_j + \delta_j + \delta_j \), respectively, for all \((x_j, x_k, x_l, x_m) \in D^j_p \times D^j_p \times D^j_p \times D^j_p \) and \( j, k, l, m = 1, \ldots, r \). Thus \( h_0 \) is necessarily generated (as a vector space) by the following sets,

\[
S_{ij} := \{ R^N(x_j, y_j)V \mid (x_j, y_j) \in D^j_p \times D^j_p \}
\] (69)

\[
S_{jkl} := \{ [h(x_j), [h(x_k), [R^N(y_j, y_k)V]]]\mid (x_j, y_j), (y_j, y_k) \in D^j_p \times D^k_p \},
\] (70)

\[
S_{jkl} := \{ [h(x_j), [h(y_j), [R^N(x_k, y_k)V]]]\mid (x_j, y_j) \in D^j_p \times D^k_p \times D^k_p \}
\] (71)

with \( j, k = 1, \ldots, r \).

Clearly, \( S_{ij} \) is contained in to r.h.s. of (32) for all \( j = 1, \ldots, r \). Moreover, according to Eq. (58),

\[
[h(x_j), [h(x_k), R^N(y_j, y_k)] = -R^N(h(x_j), h(x_k), y_j, y_k) + R^N(y_j, h(x_j), h(x_k), y_k)
\]

\[
+ R^N(h(x_j, y_j), h(x_k, y_k) + R^N(h(x_k, y_j), h(x_j, y_k)) \quad \text{(on } V \text{)}
\]

\[
[h(x_j), [h(y_j), R^N(x_k, y_k)] = -R^N(h(x_j), h(y_j), x_k, y_k) + R^N(x_k, h(x_j), h(y_j), y_k)
\]

\[
+ R^N(h(x_j, x_k), h(y_j, y_k) + R^N(h(x_j, y_k), h(y_j, x_k)) \quad \text{(on } V \text{)}
\]

for all \((x_j, y_j) \in D^j_p \times D^j_p \text{ and } (x_k, y_k) \in D^k_p \times D^k_p \); hence, also using Eq. (59) and (60), both \( S_{jkl} \) and \( S_{jkl} \) are contained in r.h.s. of (32). This finishes the proof of Eq. (32). \qed
6 Proof of Theorem 2.16

Throughout this section, we assume that \( N \) is a symmetric space which is of compact or non-compact type, that \( M \) is a simply connected symmetric space without Euclidian factor whose de Rham decomposition is given by \( M_1 \times \cdots \times M_r \) and that \( f : M \rightarrow N \) is a parallel isometric immersion. As before, we let \( L_i \) denote the canonical foliation of \( M \) and \( D^i \) := \( TL_i \) be the corresponding distribution for \( i = 1, \ldots, r \). Keeping some \( p \in M \) fixed, \( T_pM \) and \( D_i(p) \) both are seen as subspaces of \( T_{f(p)}N \) by means of \( T_{f(p)}f \).

Because the curvature tensor of \( M \) is parallel and since \( L_i(p) \) is totally geodesic in \( M \), the Theorem of Ambrose/Singer implies that the holonomy Lie algebra of \( L_i(p) \) is the subalgebra of \( \mathfrak{so}(D_p) \) which is given by

\[
\mathfrak{hol}(L_i(p)) := \left\{ R^M(x,y)|D_p^i \mid x, y \in D_p^i \right\}_R
\]

for \( i = 1, \ldots, r \). Furthermore, since \( \text{Hol}(L_i(p)) \) is connected, Lemma 3.3 (b) in combination with Remark 3.3 yields:

**Lemma 6.1.** \( W_i \) is an irreducible \( \mathfrak{hol}(L_i(p)) \)-module for \( i = 1, \ldots, r \).

We put \( V := \mathcal{O}_pf \) and \( \mathfrak{so}(V)' := \mathfrak{so}(V)_0 \oplus \mathfrak{so}(V)_i \) (see Eq. (53) and (54)); then \( \mathfrak{so}(V)' \) is a \( \mathbb{Z}/2\mathbb{Z} \)-graded Lie algebra. Furthermore, let \( \mathfrak{h} \) be the Lie algebra from Def. 2.12 and \( \mathfrak{h}' \) be its subalgebra which was introduced in Thm. 2.13. Then there is the splitting \( \mathfrak{h}' = \mathfrak{h}_0 \oplus \mathfrak{h}_i \) turning \( \mathfrak{h}' \) into a graded subalgebra of \( \mathfrak{so}(V)' \).

We consider the usual positive definite scalar product on \( \mathfrak{so}(V) \),

\[
\langle A, B \rangle := -\text{trace}(A \circ B),
\]

and let \( P_i : \mathfrak{so}(V) \rightarrow \mathfrak{h}' \) denote the orthogonal projection onto \( \mathfrak{h}' \).

In accordance with Eq. (53), we introduce the linear map \( A' : D_p^i \rightarrow \mathfrak{so}(V)' \) given by

\[
A'(x) := (h(x) - P_i(h(x))
\]

for each \( x \in D_p^i \) and \( i = 1, \ldots, r \).

**Definition 6.2.** Let \( V' \) be a Euclidian vector space. For each subset \( X \subset \mathfrak{so}(V') \) we have the corresponding centralizer in \( \mathfrak{so}(V') \),

\[
c(X) := \left\{ A \in \mathfrak{so}(V') \mid \forall B \in X : A \circ B = B \circ A \right\}.
\]

**Lemma 6.3.** We have

\[
A'(x) \in c(\mathfrak{h}') \cap \mathfrak{so}(V)_i
\]

for each \( x \in D_p^i \) and \( i = 1, \ldots, r \). Furthermore, the following is true: Either \( A' \) is an injective map or \( h(x) \in \mathfrak{h}_i \) holds for each \( x \in D_p^i \).

**Proof.** For Eq. (76): By means of the splitting \( \mathfrak{h}' = \mathfrak{h}_0 \oplus \mathfrak{h}_i \) and using Eq. (53), (54) and (55), we can use analogous arguments as in [5] Proof of Prop. 11.

For the last assertion, we use Lemma 6.1 in combination with Eq. (53) and (52) in order to apply similar arguments as in [5] Proof of Prop. 11.

**Proposition 6.4.** Suppose that \( \dim(M_i) \geq 3 \) and that the symmetric space \( M_i \) defined by Eq. (53) is irreducible (cf. Fn. 3) for \( i = 1, \ldots, r \). Then the following estimate is valid,

\[
\dim(c(\mathfrak{h}') \cap \mathfrak{so}(V)_i) \leq 2.
\]

A proof of this Proposition will be given in Section 6.1.
Proof of Thm. 2.16

Proof. Let \( f: M \to N \) be a parallel isometric immersion from the simply connected Riemannian product space \( M \cong M_1 \times \cdots \times M_r \), where \( M_i \) is some irreducible symmetric space of dimension at least 3 for \( i = 1, \ldots, r \).

For \( "(c) \Rightarrow (d)" \): Here we have \( \dim(\mathfrak{h}(\bar{f}) \cap \mathfrak{so}(V)_p) < \dim(M_i) \), as a consequence of Prop. 6.4 hence Eq. (76) implies that the linear map \( \mathfrak{A}' \) defined by Eq. (74) is not injective. But then already \( \mathfrak{A}' = 0 \) according to Lemma 6.3 i.e. the availability of the relation

\[
\forall x \in D_p^i : h(x) \in \mathfrak{h}_i
\]  

(78)
is ensured for \( i = 1, \ldots, r \). This implies that Eq. (55) holds.

\( "(d) \Rightarrow (c)" \) follows from [5, Thm. 2 and Lemma 15].

\( "(c) \Rightarrow (b)" \) is trivial.

For \( "(b) \Rightarrow (a)" \): Let \( M \) be a symmetric space and \( f: M \to N \) be an isometric immersion such that there exists a connected Lie subgroup \( G \subseteq I(N) \) which acts transitively on \( f(M) \); in this situation, one can show that \( f: M \to f(M) \) is a covering map and there exists an equivariant Lie group homomorphism \( \tau: G \to I(M)^0 \) such that \( \tau(G) \) acts transitively on \( M \) (the proof for this fact is standard and left to the reader). We now see from Eq. (57) that \( \tau(G) \) acts transitively on \( L_i(p) \) and therefore \( f(L_i(p)) \) is a homogeneous submanifold of \( N \), too.

By contradiction, now we additionally assume that \( f(L_i(p)) \) is contained in some flat of \( N \). Since \( N \) is of compact or of non-compact type, every homogenous submanifold of \( N \) which is contained in some flat of \( N \), is intrinsically flat, too (cf. [5, Prop. 1]). Hence the previous implies that \( L_i(p) \) is a Euclidian space. Therefore, \( M_i \) is also a Euclidian space, which is contrary to our assumptions.

For \( "(a) \Rightarrow (c)" \): Since \( f(L_i(p)) \) is a parallel isometric immersion defined on the simply connected, irreducible symmetric space \( L_i(p) \) (in accordance with Cor. 2.22 and, furthermore, \( N \) is of compact or non-compact type, this direction follows from [5, Thm. 5]). \( \square \)

6.1 Proof of Proposition 6.4

Besides the conventions made at the beginning of Section 6 from now on we also assume that \( m_i := \dim(M_i) \) is at least 3 and that the symmetric space \( M_i \) defined by Eq. 34 is irreducible (in the sense of Fn. 8) for \( i = 1, \ldots, r \). In the following, we keep \( p \in M \) fixed and abbreviate \( W := T_pM \) and \( W_i := D_p^i \); then \( W \) and \( W_i \) both are seen as subspaces of \( T_{f(p)}N \).

In this situation, since \( M_i \) is totally geodesically embedded, \( W_i \) is a curvature invariant subspace of \( T_{f(p)}N \) and

\[
W_i \times W_i \times W_i \to W_i , (x, y, z) \mapsto R^N(x, y) z
\]
is the curvature tensor of \( \bar{M}_i \) at \( p \) (since \( \bar{M}_i \) is totally geodesic for \( i = 1, \ldots, r \)). Analogous to Eq. (72) and Lemma 6.1 we have:

Lemma 6.5. The holonomy Lie algebra of \( \bar{M}_i \) is the subalgebra of \( \mathfrak{so}(W_i) \) which is given by

\[
\mathfrak{hol}(M_i) := \left\{ R^N(x, y)|W_i \mid x, y \in W_i \right\}_\mathbb{R}.
\]  

(79)

Furthermore, \( W_i \) is an irreducible \( \mathfrak{hol}(M_i) \)-module for \( i = 1, \ldots, r \).

In accordance with Eq. (3) and (5), we also set \( U := \mathcal{F}_p, \ U_{ij} := \mathcal{F}_{ij}^p \) for \( i \neq j \) and \( \tilde{U} := \tilde{\mathcal{F}}_p \); then \( V := \mathcal{O}_p f = W \oplus U \) is the second osculating space of \( f \) at \( p \). Now we introduce the following linear spaces \( V_{ij} \) for \( i, j = 1, \ldots, r \),

\[
V_{ii} := W_i \oplus \tilde{U} , \quad V_{ij} := W_i \oplus U_{ij} \quad \text{for } i \neq j .
\]  

(80) 

(81)

Then we have \( A(V_{ij}) \subset (V_{ij}) \) for each \( A \in \mathfrak{h}^i \) and \( i, j = 1, \ldots, r \). Hence we can consider the corresponding centralizers \( \mathfrak{c}(\mathfrak{h}^i|V_{ij}) \subset \mathfrak{so}(V_{ij}) \) (cf. Def. 6.2).
In order to prove Eq. \((\ref{eq:77})\), we will additionally need the following two estimates for \(i = 1, \ldots, r\),
\[
\begin{align*}
\mathfrak{c}(\mathfrak{h}^i|V_{ij}) \cap \mathfrak{so}(V_{ij})_+ &= \{0\} \quad \text{for} \ i \neq j, \quad (82) \\
\dim (\mathfrak{c}(\mathfrak{h}^i|V_{ii}) \cap \mathfrak{so}(V_{ii})_-) &\leq 2. \quad (83)
\end{align*}
\]
A proof of these equations is given in Section \ref{sec:6.2} and \ref{sec:6.3}.

**Proof of Proposition \ref{prop:6.4}**

\textit{Proof.} Recall that
\[
U = U' \oplus \bigoplus_{i \neq j} U_{ij}
\]
is an orthogonal sum decomposition, according to Thm. \ref{thm:2.1}. Therefore, and because of Eq. \eqref{eq:56} and \eqref{eq:54}, we have
\[
\bigoplus_{j=1, \ldots, r} V_{ij} \subset V \quad \text{(as an orthogonal sum) and} \quad A(V_{ij}) \subset V_{ij} \quad \text{for each} \ A \in \mathfrak{so}(V)^+. \quad (85)
\]
Moreover, the splitting of vector spaces \eqref{eq:80}, \eqref{eq:81} induces the splitting \(\mathfrak{so}(V_{ij}) = \mathfrak{so}(V_{ij})_+ \oplus \mathfrak{so}(V_{ij})_- \) (as described in Remark \ref{rem:2.5}) such that, as a consequence of Eq. \eqref{eq:54},
\[
\mathfrak{so}(V)_i \to \bigoplus_{j=1, \ldots, r} \mathfrak{so}(V_{ij})_-, \ A \to \bigoplus_{j=1, \ldots, r} A|V_{ij} \quad \text{is an isomorphism}. \quad (86)
\]
Then Eq. \eqref{eq:80} and \eqref{eq:81} imply that
\[
\mathfrak{c}(\mathfrak{h}^i) \cap \mathfrak{so}(V)_i \cong \bigoplus_{j=1}^r \mathfrak{c}(\mathfrak{h}^j|V_{ij}) \cap \mathfrak{so}(V_{ij})_- ;
\]
now Eq. \eqref{eq:77} follows from Eq. \eqref{eq:82} and \eqref{eq:83}. \hfill \Box

\subsection*{6.2 Proof of Equation \eqref{eq:82}}

For this, we keep a pair \((i, j)\) with \(i \neq j\) fixed, let \(\mathbb{F}^{ij}\) be the vector bundle defined by Eq. \eqref{eq:41} and set
\[
U := \mathbb{F}^{ij}, \ W_i := D_i^{ij}, \ W_j := D_j^{ij} \quad \text{and} \quad U_{ij} := \mathbb{F}^{ij}. \quad (84)
\]
Then \(V_{ij} := W_i \oplus U_{ij}\) is the linear space defined by Eq. \eqref{eq:51}. We let \(K\) denote the isotropy subgroup of \(I(N)^0\) at \(p\), \(\rho : K \to T_pM\) be the corresponding isotropy representation and \(\mathfrak{a} : K \times \mathbb{F} \to \mathbb{F}\) be the action described by Eq. \eqref{eq:40}.

Let \(K \cong K_1 \times \cdots \times K_r\) be the induced product structure, see Eq. \eqref{eq:77}. Then \(K_1 \times K_j\) acts orthogonally on \(U_{ij}\) via \(\alpha\) (see again Eq. \eqref{eq:1} and Eq. \eqref{eq:40}).

**Proposition 6.6.** \textit{Only one of the following three cases can occur:}

- \(U_{ij} = \{0\}\).
- The dimension of \(U_{ij}\) is at least \(1/2 m_i m_j\) and \(K_1 \times K_j\) acts irreducibly on \(U_{ij}\).
- There is the splitting \(U_{ij} = U' \oplus U''\) into two irreducible invariant subspaces of the same dimension \(1/2 m_i m_j\).

\textit{Proof.} According to Lemma \ref{lem:A.11} the induced action of \(K_1 \times K_j\) on \(W_i \otimes W_j\) is either irreducible or \(W_i \otimes W_j\) is the direct sum of two irreducible invariant subspaces of the same dimension \(1/2 m_i m_j\). Furthermore, \(h_p : W_i \otimes W_j \to U_{ij}\) is a surjective homomorphism. Now the “Lie group version” of Lemma \ref{lem:A.3} \(d)\) (cf. Remark \ref{rem:A.4}) implies that either \(U_{ij}\) is also the direct sum of two irreducible invariant subspaces of the same dimension \(1/2 m_i m_j\) or \(U_{ij}\) is irreducible and of dimension at least \(1/2 m_i m_j\). \hfill \Box

We recall that \(\mathbb{F}^{ij}_q\) is a curvature invariant subspace of \(T_{f(q)}M\) for each \(q \in M\) (Cor. \ref{cor:2.7}). Hence, on the one hand, as \(q\) varies over \(M\),
\[
T : \mathbb{F}^{ij}_q \times \mathbb{F}^{ij}_q \times \mathbb{F}^{ij}_q \to \mathbb{F}^{ij}_q, (\xi_1, \xi_2, \xi_3) \mapsto R^N(\xi_1, \xi_2, \xi_3) \ \xi_3 \quad (87)
\]
defines a section of the induced vector bundle \(L^3(\mathbb{F}^{ij}; \mathbb{F}^{ij})\). Let the latter be equipped with the connection which is canonically induced by \(\nabla^\perp\).
Lemma 6.7. $T$ is a parallel section.

Proof. For any differentiable curve $c : \mathbb{R} \to M$ and all sections $\xi_1, \ldots, \xi_4$ of $\mathbb{F}^T$ along $c$, the function $f(t) := \langle R^N(\xi_1(t), \xi_2(t)) \xi_3(t), \xi_4(t) \rangle$ is constant according to [4, Prop. 8]. The result follows. 

On the other hand, using arguments given already in Sec. 2.6, we now see that

$$\tilde{M} := \exp^N(U_{ij}) \subset N$$

is also a totally geodesically embedded symmetric space and that $T_p$ is the curvature tensor of $\tilde{M}$ at $p$. Then, analogous to Eq. (79), the holonomy Lie algebra of $\tilde{M}$ is the subalgebra of $\mathfrak{so}(U_{ij})$ which is given by

$$\mathfrak{hol}(\tilde{M}) := \{ R^N(\xi_1, \xi_2) | U_{ij} \mid \xi_1, \xi_2 \in U_{ij} \}_\mathbb{R}.$$  

(88)

Let $U_{ij} = U_0 \oplus \cdots \oplus U_k$ be an orthogonal decomposition such that $U_0$ is the largest vector subspace of $U_{ij}$ on which $\mathfrak{hol}(\tilde{M})$ acts trivially and that $U_l$ is an irreducible $\mathfrak{hol}(\tilde{M})$-module for $l = 1, \ldots, k$. By virtue of the de Rham Decomposition Theorem, there exists a Euclidean space $\tilde{M}_0$ and irreducible symmetric spaces $\tilde{M}_l$ $(l = 1, \ldots, k)$ such that the universal covering space of $\tilde{M}$ is isometric to the Riemannian product of $\tilde{M}_0 \times \cdots \times \tilde{M}_k$ and that $U_l \cong T_p \tilde{M}_l$ for $l = 0, \ldots, k$. Moreover, this de Rham decomposition of $\tilde{M}$ (and hence the linear spaces $U_l$ are also unique (up to isometry, respectively, and a permutation of $\{\tilde{M}_1, \ldots, \tilde{M}_k\}$).

Corollary 6.8. (a) $U_l$ is invariant under the action of $K_i \times K_j$ on $U_{ij}$ for $l = 0, \ldots, k$.

(b) There are no more than the following five possibilities:

- $U_{ij}$ is trivial.
- $\tilde{M}$ is an irreducible symmetric space of dimension at least $1/2 m_i m_j$.
- $\tilde{M}$ is a flat of $N$.
- The universal covering space of $\tilde{M}$ is isometric to the Riemannian product $\tilde{M}_0 \times \tilde{M}_1$ where $\tilde{M}_0$ is Euclidean and $\tilde{M}_1$ is an irreducible symmetric space such that $\dim(\tilde{M}_0) = \dim(\tilde{M}_1) = 1/2 m_i m_j$.
- The universal covering space of $\tilde{M}$ is isometric to the Riemannian product $\tilde{M}_1 \times \tilde{M}_2$ of two irreducible symmetric spaces satisfying $\dim(\tilde{M}_1) = \dim(\tilde{M}_2) = 1/2 m_i m_j$.

Proof. For (a): Combining the two facts that the $\nabla^\perp$-holonomy group of $\mathbb{F}$ is given by Eq. (43) (see Lemma 4.3 (c)) and that $T$ is a $\nabla^\perp$-parallel section (see Lemma 6.7), we get that $T(g \xi_1, g \xi_2) g \xi_3 = g T(\xi_1, \xi_2) \xi_3$ for all $\xi_1, \xi_2, \xi_3 \in U_{ij}$ and $g \in K$. Hence $g(U_0)$ is also a trivial $\mathfrak{hol}(\tilde{M})$-module and $\mathfrak{hol}(\tilde{M})$ acts irreducibly on $g(U_l)$ for each $g \in K$ and $l = 1, \ldots, k$, too, in accordance with Eq. (59) Also using the uniqueness of the de Rham decomposition, a continuity argument now shows that $g(U_l) = U_l$ for each $g \in K$ and $l = 0, \ldots, k$. Our result follows.

(b) is now an immediate consequence of Prop. 6.9 and the uniqueness assertion of Lemma 4.3 (c) (again, we use its “Lie group version”).

In the following, we let $\mathfrak{h}_0$ be the Lie algebra described in Thm. 2.13 and $\mathfrak{g}$ be any subalgebra of $\mathfrak{h}_0$. Then Eq. (59) implies that there are induced representations,

$$\rho_j : \mathfrak{g} \to \mathfrak{so}(W_j), A \mapsto A|W_j : W_j \to W_j,$$

(90)

$$\rho_j : \mathfrak{g} \to \mathfrak{so}(U_{ij}), A \mapsto A|U_{ij} : U_{ij} \to U_{ij}.$$ 

(91)

for each $j \neq i$. Hence the linear space $\text{Hom}_{\mathfrak{g}}(W_j, U_{ij})$ is defined in accordance with Def. 4.1

Lemma 6.9. The natural isomorphism $\mathfrak{so}(V_{ij}) \to L(W_j, U_{ij})$ provided by Lemma 6.2 induces the inclusion

$$c(\mathfrak{h}^i|V_{ij}) \cap \mathfrak{so}(V_{ij}) \hookrightarrow \text{Hom}_{\mathfrak{g}}(W_j, U_{ij}).$$

(92)

Proof. Since $\mathfrak{g} \subset \mathfrak{h}^i$ is a subalgebra, we have $[A, \mathfrak{g}] = \{0\}$ for each $A \in c(\mathfrak{h}^i)$. Now it is straightforward to show that $A|W_j : W_j \to U_{ij}$ belongs to $\text{Hom}_{\mathfrak{g}}(W_j, U_{ij})$ for each $A \in c(\mathfrak{h}^i) \cap \mathfrak{so}(V_i)$. 

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We also recall the following result (cf. [5] Prop. 7),

**Lemma 6.10.** Let a linear subspace $V' \subset T_p \mathcal{N}$ be given. Then the following assertions are equivalent:

(a) $V'$ is a curvature isotropic subspace of $T_p \mathcal{N}$ (see Def. 2.15).

(b) $\exp^N(V')$ is a flat of $\mathcal{N}$.

(c) The sectional curvature of $\mathcal{N}$ vanishes on every 2-plane of $V'$, i.e. $⟨R^N(v, w)w, v⟩ = 0$ for all $v, w ∈ V'$.

**Proof of Eq. (82)**

**Proof.** According to Lemma 6.10, it suffices to show that $\text{Hom}_g(W_j, U_{ij}) = \{0\}$ for some subalgebra $g ⊂ h_0$. By means Cor. 2.28 it suffices to distinguish the following five cases:

**Case 1:** $\tilde{M}$ is trivial. Here we have $U_{ij} = \{0\}$, hence $\mathfrak{so}(V_{ij}) = \{0\}$; then our result is obvious.

**Case 2:** $\tilde{M}$ is an irreducible symmetric space of dimension at least $1/2m_i m_j$. Here we consider the subalgebra of $h_0$ which is given by

$$g := \{ R^N(\xi, \eta)|V | \xi, \eta ∈ U_{ij} \}_R,$$

see Cor. 2.27 and Thm. 2.13.

First, we note that $\mathfrak{so}(\tilde{M})$ acts irreducibly on $U_{ij}$, as a consequence of the deRham Decomposition Theorem. This together with Eq. (58) implies that the action of $g$ on $U_{ij}$ is irreducible. Therefore, and since, by assumption, $1/2m_i m_j ≥ 3/2 m_j > m_i$, Lemma A.2 implies that $\text{Hom}_g(U_{ij}, W_j) = \{0\}$ holds. Thus $\text{Hom}_g(W_j, U_{ij})$ is also trivial, according to Lemma A.3(e).

**Case 3:** $\tilde{M}$ is a Euclidean space. Here we consider the subalgebra of $h_0$ which is given by

$$g := \{ R^N(x, y)|V | x, y ∈ W_j \}_R,$$

see Cor. 2.27 and Thm. 2.13.

One the one hand, $U_{ij}$ is even a curvature isotropic subspace of $T_{\tilde{M}(p)} \mathcal{N}$, as a consequence of Lemma 6.10 in combination with Eq. (58). Therefore, we have $⟨R^N(x, y)\xi, \eta⟩ = ⟨R^N(\xi, \eta)x, y⟩ = 0$ for all $x, y ∈ W_j$ and $\xi, \eta ∈ U_{ij}$, i.e. $g$ acts trivially on $U_{ij}$. On the other hand, $g$ acts non-trivially and irreducibly on $W_j$, according to Lemma 6.10. Using now Lemma A.2 we now conclude that $\text{Hom}_g(W_j, U_{ij}) = \{0\}$.

**Case 4:** The universal covering space of $\tilde{M}$ is isometric to the Riemannian product $\tilde{M}_0 × \tilde{M}_1$ where $\tilde{M}_0$ is Euclidian and $\tilde{M}_1$ is an irreducible symmetric space such that $\dim(\tilde{M}_0) = \dim(\tilde{M}_1) = 1/2m_i m_j$.

Here we set $g := h_0$. Let $\lambda ∈$ $\text{Hom}_g(W_j, U_{ij})$ be given. I claim that $\lambda = 0$.

For this: Put $U_0 := T_p \tilde{M}_0$ and $U_1 := T_p \tilde{M}_1$. Then, following the arguments from Case 2, the linear space

$$\bar{g} := \{ R^N(\xi, \eta)|V | \xi, \eta ∈ U_1 \}_R$$

is actually a subalgebra of $g$ and $U_1$ is a $\bar{g}$-invariant subspace of $U_1$ such that $\text{Hom}_\bar{g}(W_j, U_1) = \{0\}$. Hence Lemma A.3(f) implies that $\lambda(W_j) ⊂ U_0$.

Then, repeating the arguments given in Case 3, we see that $g := \{ R^N(x, y)|V | x, y ∈ W_j \}_R$ is a subalgebra of $h_0$ and $\lambda(W_j)$ is a $g$-invariant subspace of $U_{ij}$ such that $\text{Hom}_g(W_j, \lambda(W_j)) = \{0\}$. We thus conclude that $\lambda = 0$ and therefore $\text{Hom}_g(W_j, U_{ij}) = \{0\}$.

**Case 5:** The universal covering space of $\tilde{M}$ is isometric to the Riemannian product $\tilde{M}_1 × \tilde{M}_2$ of two irreducible symmetric spaces satisfying $\dim(\tilde{M}_1) = \dim(\tilde{M}_2) = 1/2m_i m_j$. Again we put $g := h_0$. Let $\lambda ∈ \text{Hom}_g(W_j, U_{ij})$ be given. I claim that $\lambda = 0$.

For this: Put $U_1 := T_p \tilde{M}_1$ and $U_2 := T_p \tilde{M}_2$. Then, repeating an argument from Case 4, we conclude that $\lambda(W_j) ⊂ U_2$. Vice versa, we have $\lambda(W_j) ⊂ U_1$; therefore, $\lambda = 0$.

**6.3 Proof of Equation (83)**

For this, we keep $i ∈ \{1, \ldots, r\}$ fixed, put $f_i := f|L_i(p) : L_i(p) → N$ and let $\mathcal{O}f|L_i(p)$ and $f_i^*TN$ denote the corresponding pullback bundles; then $\mathcal{O}f|L_i(p)$ and $f_i^*TN$ both are vector bundles over $L_i(p)$.

Moreover, $\mathcal{O}f|L_i(p) ⊂ f_i^*TN$ is a parallel subbundle; therefore, we can repeat the construction from Section 2.5 to obtain the corresponding Holonomy group $\text{Hol}(\mathcal{O}f|L_i(p))$ with respect to the connection induced by $\nabla^N$. 19
**Lemma 6.15.** The pullback bundle \( V := O_f \) and let \( \mathfrak{h} \) be the subalgebra of \( \mathfrak{so}(V) \) introduced in Def. 2.12. Clearly, \( \tilde{\mathfrak{h}} \) is a subalgebra of \( \mathfrak{h} \), hence \( \tilde{\mathfrak{h}} \subset \mathfrak{h} \subset \mathfrak{so}(V) \) is a sequence of subalgebras.

**Proposition 6.12.** \( \tilde{\mathfrak{h}} \) is a graded subalgebra of \( \mathfrak{h} \), i.e.

\[
\tilde{\mathfrak{h}} = \bigoplus_{\delta \in \Lambda} \tilde{\mathfrak{h}}_{\delta} \cap \mathfrak{so}(V)_{\delta} .
\] (93)

Hence \( \tilde{\mathfrak{h}} \subset \mathfrak{h} \subset \mathfrak{so}(V) \) is actually a sequence of \( \mathbb{A} \)-graded subalgebras.

**Proof.** Let \( \Sigma_{\delta} \) denote the symmetries of \( O_f \) described in Thm. 2.10 for \( j = 1, \ldots, r \). Then the base map of \( \Sigma_{\delta} \) is the map \( \sigma_{\delta} \) described by Eq. (23); in particular, \( \sigma_{\delta}(L_i(p)) = L_i(p) \) holds for all \( j \). Therefore, \( \Sigma_{\delta} \) induces a parallel vector bundle involution on \( O_f \). We have directly follows from Eq. (93).

**Lemma 6.13.** (a) The pullback bundle \( D^j \oplus \mathbb{R}^j \) is a parallel subbundle of \( O_f \). The same is true for \( D^j \oplus \mathbb{R}^j \).

(b) \( \tilde{\mathfrak{h}} \) is already a subalgebra of \( \mathfrak{h} \). Hence the splitting Eq. (93) gets actually reduced to

\[
\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}} \cap \mathfrak{so}(V)_{0} \oplus \tilde{\mathfrak{h}} \cap \mathfrak{so}(V)_{i} ,
\] (94)

and \( \tilde{\mathfrak{h}} \subset \mathfrak{h} \subset \mathfrak{so}(V)^i \) is a sequence of \( \mathbb{Z}/2\mathbb{Z} \)-graded subalgebras.

**Proof.** For (a): Use the equations of Gauss and Weingarten in combination with Lemma 3.5.

For (b): Because of Part (a) combined with Eq. (54) and (64), we have \( \tilde{\mathfrak{h}} \subset \mathfrak{so}(V)^i \). The result immediately follows from Eq. (93).

We set \( W_i := D^i \) and \( \tilde{U} := \mathbb{R}^p \); then \( W_i \oplus \tilde{U} \) is the linear space \( V_{ii} \) defined by Eq. (80). By means of the previous lemma, we have \( A(V_{ii}) \subset V_{ii} \) for each \( A \in \tilde{\mathfrak{h}} \) and hence we can introduce the corresponding centralizer \( \mathfrak{c}(\tilde{\mathfrak{h}}| V_{ii}) \subset \mathfrak{so}(V_{ii}) \) (cf. Def. 6.2). In particular, Part (b) of Lemma 6.13 yields:

**Corollary 6.14.** We have

\[
\mathfrak{c}(\tilde{\mathfrak{h}}| V_{ii}) \subset \mathfrak{c}(\tilde{\mathfrak{h}}| V_{ii}) .
\] (95)

As mentioned already before, the vector space

\[
\mathfrak{g} := \{ \mathbb{R}^N(x, y)|V \mid x, y \in W_i \}
\]

is a subalgebra of \( \mathfrak{so}(V)_{0} \). Thus Eq. (53) also implies that there are induced representations,

\[
\rho_i : \mathfrak{g} \rightarrow \mathfrak{so}(W_i), \ A \mapsto A|W_i : W_i \rightarrow W_i ,
\]

\[
\tilde{\rho} : \mathfrak{g} \rightarrow \mathfrak{so}(\tilde{U}), \ A \mapsto A|\tilde{U} : \tilde{U} \rightarrow \tilde{U} .
\] (97)

We introduce the vector space \( \operatorname{Hom}_{\mathfrak{g}}(W_i, \tilde{U}) \) according to Def. 1.11 and consider the splitting \( \mathfrak{so}(V_{ii}) = \mathfrak{so}(V_{ii})_{+} \oplus \mathfrak{so}(V_{ii})_{-} \) induced by the splitting \( V_{ii} = W_i \oplus \tilde{U} \). By means of the Theorem of Ambrose/Singer, \( \mathfrak{g} \subset \tilde{\mathfrak{h}} \); hence we have (analogous to Lemma 6.9)

**Lemma 6.15.** The natural isomorphism \( \mathfrak{so}(V_{ii})_{-} \rightarrow L(W_i, \tilde{U}) \) provided by Lemma 5.2 induces the inclusion

\[
\mathfrak{c}(\tilde{\mathfrak{h}}| V_{ii}) \cap \mathfrak{so}(V_{ii})_{-} \hookrightarrow \operatorname{Hom}_{\mathfrak{g}}(W_i, \tilde{U}) .
\] (98)

Set \( U_{ii} := \mathbb{R}^p \); then \( U_{ii} \) is a subspace of \( \tilde{U} \).
Lemma 6.16. Let $U_{ii}^\perp$ denote the orthogonal complement of $U_{ii}$ in $\tilde{U}$.

(a) $\mathfrak{g}$ acts trivially on $U_{ii}^\perp$.
(b) We have $\lambda(W_i) \subset U_{ii}$ for each $\lambda \in \text{Hom}_g(W_i, \tilde{U})$.

Proof. For (a): Let $\xi \in U_{ii}^\perp$ be given. Using Lemma 6.15, we immediately obtain that $S_\xi x = 0$ for all $x \in W_i$. Hence

$$\forall x, y \in W_i : [h(x), h(y)] \xi = 0 .$$

(99)

Because of Eq. 99 and the Equations of Gauß, Codazzi and Ricci 25, it thus remains to show that

$$R^\perp(W_i \times W_i) \xi = \{0\} .$$

(100)

For this: Let $F^\sharp$ denote the maximal flat subbundle of $F$ and set $U^\sharp := F^\sharp_p$. As a consequence of Thm. 2.1, $\xi$ belongs to the linear space

$$U^\sharp \oplus \sum_{j \neq i} U_{jj} .$$

(101)

In case $\xi \in U^\sharp$, Eq. (100) is trivial. In case $\xi \in U_{jj}$ with $j \neq i$, we may assume that there exist $x_j, y_j \in W_j$ with $\xi = h(x_j, y_j)$. Then, according to [4, Prop. 4 (d)], and since $M$ is the Riemannian product of the $M_j$’s,

$$R^\perp(W_i \times W_i) \xi = h(R^M(W_i \times W_i) x_j, y_j) + h(x_j, R^M(W_i \times W_i) y_j) = \{0\} .$$

Eq. (100) follows.

(b) is now a consequence of Part (a) and arguments given already in the proof of Eq. (82).

Now recall that $f_i$ is a parallel isometric immersion, according to Cor. 2.2, then $U_{ii}$ is the first normal space and $V_i := W_i \oplus U_{ii}$ is the second osculating space of $f_i$ at $p$; note that $V_i$ is a subspace of $\tilde{V}_i$. Let $\mathcal{O}f_i$ denote the second osculating bundle of $f_i$ and let us apply Def. 2.12 to define its extrinsic holonomy Lie algebra $\mathfrak{h}_i$; then $\mathfrak{h}_i$ is a subalgebra of $\mathfrak{so}(V_i)$. Furthermore, there is the corresponding centralizer $c(\mathfrak{h}_i) \subset \mathfrak{so}(V_i)$ (cf. Def. (82)).

Corollary 6.17. There exists an injective map

$$c(\mathfrak{h}_i|V_i) \cap \mathfrak{so}(V_i)_- \hookrightarrow c(\mathfrak{h}_i) \cap \mathfrak{so}(V_i)_- .$$

(102)

Proof. Let $A \in c(\mathfrak{h}_i|V_i) \cap \mathfrak{so}(V_i)_-$ be given and $U_{ii}^\perp$ be defined as in Lemma 6.16. We will show that

$$A(W_i) \subset U_{ii} \text{ and } A(U_{ii}) \subset W_i ,$$

(103)

$$A|\tilde{U}_{ii} = 0 ,$$

(104)

$$[A|V_i, \mathfrak{h}_i] = \{0\} ;$$

(105)

hence $A \mapsto A|V_i$ gives the desired map (102).

For Eq. (103) and (104): On the one hand, we have $A(\tilde{U}) \subset W_i$ (since $A \in \mathfrak{so}(V_i)_-$). On the other hand, $A(W_i) \in \text{Hom}_g(W_i, \tilde{U})$ according to Lemma 6.16, hence $A(W_i) \subset U_{ii}$ because of Lemma 6.16. Now Eq. (103) follows and, moreover, $\langle A u, w \rangle = -\langle u, A w \rangle = 0$ for all $u \in U_{ii}^\perp$, $w \in W_i$. Thus Eq. (104) also holds.

For Eq. (105): Because $V_i$ is the second osculating space of $f_i$ at $p$ and since, furthermore, $\mathcal{O}f_i \subset \mathcal{O}f_i L_i(p)$ is a parallel subbundle as mentioned already before, $B(V_i) \subset V_i$ for each $B \in \mathfrak{h}_i$ and, moreover, $\mathfrak{h}_i \hookrightarrow \mathfrak{h}_i$, $B \mapsto B|V_i$ is a surjective map. Furthermore, $[A, B] = 0$ for all $B \in \mathfrak{h}_i$. The previous together with Eq. (103) obviously implies Eq. (105).
Proof of Eq. (102)
Proof. On the one hand, by means of Corollary 6.14 and 6.17 there exists a sequence of inclusions,

\[ c(h'|V'i) \cap \mathfrak{so}(V'i) \xrightarrow{\delta} c(h'|V'i) \cap \mathfrak{so}(V'i) \xrightarrow{102} c(h') \cap \mathfrak{so}(V) \xrightarrow{102} \]

Hence \( \dim(c(h'|V'i) \cap \mathfrak{so}(V'i)) \leq \dim(c(h') \cap \mathfrak{so}(V)) \).

On the other hand, by assumption, \( f_i \) is a parallel isometric immersion which is defined on the simply connected, irreducible symmetric space \( L_i(p) \) such that \( f_i(L_i(p)) \) is not contained in any flat of \( N \) and that \( \dim(L_i(p)) \geq 3 \). This situation was already investigated in [5] Sec. 3.2: According to Prop. 12 there, we have \( \dim(c(h') \cap \mathfrak{so}(V)) \leq 2 \). Now Eq. (102) follows. \( \square \)

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Appendix

A Representation theory

Let \( \mathfrak{k} \) be a Lie algebra over \( \mathbb{R} \), \( V \) be a vector space over a field \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \) and \( \rho: \mathfrak{k} \to \mathfrak{gl}_\mathbb{K}(V) \) be a representation. Then we also say that “\( V \) is a \( \mathfrak{k} \)-module”. \( V \) is called “irreducible” if \( \{0\} \) and \( V \) are the only invariant subspaces of \( V \); otherwise, \( V \) is called “reducible”.

Definition A.1. Let a second linear space \( V' \) over \( \mathbb{K} \) and a representation \( \rho': \mathfrak{k} \to \mathfrak{gl}_\mathbb{K}(V') \) be given. A linear map \( \lambda: V \to V' \) satisfying \( \lambda(\rho(A)v) = \rho'(A)\lambda(v) \) for all \( A \in \mathfrak{k} \) and \( v \in V \) will be briefly called a “homomorphism”. Then set of homomorphisms, denoted by \( \text{Hom}_\mathfrak{k}(V, V') \), is a vector space over \( \mathbb{K} \), too.

The next lemma is “standard”.

Lemma A.2 (Schur’s Lemma). If \( \lambda: V \to V' \) is a homomorphism, then both the kernel and the image of \( \lambda \) are invariant subspaces of \( V \) and \( V' \), respectively. In particular, if \( V \) is an irreducible \( \mathfrak{k} \)-module and \( \lambda \neq 0 \), then \( \lambda \) is injective.

In the following, we will always assume that \( V \) is a Euclidian space. The proof of the following lemma is left to the reader.

Lemma A.3. (a) Let invariant subspaces \( U \) and \( W \) of \( V \) be given. If \( U \) and \( W \) both are irreducible, then we have \( U \perp W \) unless \( U \cong V \) (as \( \mathfrak{k} \)-modules). Therefore, if \( U \) is isomorphic to a direct sum of non-trivial and irreducible \( \mathfrak{k} \)-modules and \( W \) is a trivial \( \mathfrak{k} \)-module, then \( U \perp W \).

(b) We can always find a decomposition \( V = W_1 \oplus \cdots \oplus W_k \) into pairwise orthogonal, invariant subspaces \( W_i \) with the following property: There exists an irreducible \( \mathfrak{k} \)-module \( W'_i \) and an integer \( m_i \) such that \( W_i \) is isomorphic to the direct sum of \( m_i \) copies of \( W'_i \) for \( i = 1, \ldots, k \), and such that \( \{W'_1, \ldots, W'_k\} \) are pairwise non-isomorphic (as \( \mathfrak{k} \)-modules).

(c) The subspaces \( W_i \), the “multiplicities” \( m_i \) and the modules \( W'_i \) are uniquely determined (the latter ones only up to isomorphy) for \( i = 1, \ldots, k \) (cf. [3] Ch. XVIII, Prop. 1.2]).

Let a second Euclidian space \( V' \) over \( \mathbb{K} \) and a representation \( \tilde{\rho}: \mathfrak{k} \to \mathfrak{gl}_\mathbb{K}(V') \) be given. Furthermore, let \( V \cong \bigoplus_{j \in J} W_j \) be any decomposition into irreducible submodules \( W_j \).

(d) If \( \lambda: V \to V' \) is a surjective homomorphism, then there exists a subset \( J' \subset J \) such that \( \lambda|_{\bigoplus_{j \in J'} W_j} \) induces an isomorphism onto \( V' \).

(e) For every \( \lambda \in \text{Hom}_\mathfrak{k}(V, V') \) the adjoint map \( \lambda^*: V' \to V \) belongs to \( \text{Hom}_\mathfrak{k}(V, V) \), and \( \lambda \mapsto \lambda^* \) induces \( \text{Hom}_\mathfrak{k}(V, V') \cong \text{Hom}_\mathfrak{k}(V', V) \).

(f) Suppose that \( V' = W \oplus U \) is the orthogonal sum of two invariant subspaces. If \( \text{Hom}_\mathfrak{k}(V, W) = \{0\} \), then \( \lambda(V) \subset U \) for each \( \lambda \in \text{Hom}_\mathfrak{k}(V, V') \).
Remark A.4. Let a Lie group \( K \), a vector space \( V \) and a representation \( \rho : K \to \text{GL}_\mathbb{C}(V) \) be given. Then we also say that “\( K \) acts linearly on \( V \)”.

Theorem of the highest weight” in combination with “Weyl’s dimension formula” (see [7, Ch. 5]), we have corresponding “Lie group versions” of Lemma A.3 and (in case \( V \) is Euclidian and \( K \) acts orthogonally on \( V \)) Lemma A.3.

Given a Lie algebra \( \mathfrak{t} \) over \( \mathbb{R} \), a Euclidian space \( V \) and a representation \( \rho : \mathfrak{t} \to \mathfrak{so}(V) \), let \( V^\mathbb{C} := V \oplus iV \) denote the corresponding “complexification” and \( \rho^\mathbb{C} : \mathfrak{t} \to \mathfrak{gl}_\mathbb{C}(V^\mathbb{C}) \) be the canonically induced representation.

Lemma A.5. (a) Suppose that there exists \( J \in \mathfrak{so}(V) \) with \( J^2 = -\text{Id} \) (equipping already \( V \) with the structure of a complex space) such that \( \rho(\mathfrak{k}) \subset \mathfrak{u}(V) \). Then

\[
V_{\pm i} := \left\{ v \mp iJv \mid v \in V \right\}.
\]

induces the decomposition \( V^\mathbb{C} = V_{+i} \oplus V_{-i} \) into invariant subspaces such that

\[
V \to V_{+i}, \quad v \mapsto 1/2(v - iJv)
\]

is a complex linear isomorphism of \( \mathfrak{t} \)-modules.

(b) Conversely, if \( V \) is irreducible, but \( V^\mathbb{C} \) is reducible, then there exists necessarily some \( J \in \mathfrak{so}(V) \) with \( J^2 = -\text{Id} \) such that \( \rho(\mathfrak{k}) \subset \mathfrak{u}(V) \).

Definition A.6. We set \( \mathfrak{t} := \mathfrak{t}_1 \oplus \mathfrak{t}_2 \) and \( V := V_1 \oplus \mathbb{C} V_2 \); then \( \mathfrak{t} \) is a Lie algebra over \( \mathbb{R} \) and \( V \) is a vector space over \( \mathbb{K} \). Moreover, there is a natural representation \( \rho_1 \otimes \rho_2 : \mathfrak{t} \to \mathfrak{gl}_\mathbb{K}(V) \), given by

\[
(\rho_1 \otimes \rho_2)(X_1 + X_2))v_1 \otimes v_2 := \rho_1(X_1)v_1 \otimes v_2 + v_1 \otimes \rho_2(X_2)v_2
\]

for all \( X_1 + X_2 \in \mathfrak{t}_1 \otimes \mathfrak{t}_2 \) and \( (v_1, v_2) \in V_1 \times V_2 \).

Lemma A.7. Let \( \mathfrak{t}_i \) be a semisimple Lie algebra (cf. [4, Ch. 1]), \( V_i \) be a Hermitian vector space and \( \rho_i : \mathfrak{t}_i \to \mathfrak{u}(V_i) \) be a complex-irreducible representation for \( i = 1, 2 \). Then \( \rho_1 \otimes \rho_2 \) is a complex-irreducible representation, too.

Proof. Set \( \mathfrak{t} := \mathfrak{t}_1 \oplus \mathfrak{t}_2 \) and \( \rho := \rho_1 \otimes \rho_2 \). Let \( g_i := \mathfrak{t}_i^\mathbb{C} \) denote the complexified Lie algebra, which is a complex semisimple Lie algebra. Then \( \rho_i \) induces an irreducible representation \( \rho_i : g_i \to \mathfrak{gl}(V_i) \). Moreover, if \( a_i \) is a maximal Abelian subspace of \( \mathfrak{t}_i \), then \( a_i \oplus i a_i \) is a “Cartan subalgebra” of \( g_i \) (see [4, Ch. II]).

Let \( \Delta_i \) denote the corresponding set of “roots”. Then the roots take real values on \( a_i \), and hence we may choose a “total ordering” on the dual space of \( a_i \) to obtain the corresponding set \( \Delta_i^+ \) of “positive roots” and the corresponding “highest weight” \( \lambda_i \) for \( i = 1, 2 \). Furthermore, \( \mathfrak{g} := \mathfrak{g}_1 \oplus \mathfrak{g}_2 \) is also a complex semisimple Lie algebra, \( (a_1 \oplus a_2) + i(a_1 \oplus a_2) \) is a Cartan subalgebra of \( \mathfrak{g} \) and \( \Delta^+ := \Delta_1^+ \cup \Delta_2^+ \) is the corresponding set of positive roots. Moreover, \( \rho \) induces a representation of \( \mathfrak{g} \) on \( V_1 \otimes \mathbb{C} V_2 \) whose weights are given by \( \mu_1 \otimes \mu_2 \), where \( \mu_1 \) and \( \mu_2 \) range over the weights of \( \rho_1 \) and \( \rho_2 \), respectively. In particular, \( \lambda := \lambda_1 + \lambda_2 \) is the highest weight of \( \rho \). Let \( V_{\lambda} \) be an irreducible \( \mathfrak{g} \)-submodule of \( V_1 \otimes \mathbb{C} V_2 \) such that the corresponding Eigenspace \( E_{\lambda} \subset V_{\lambda} \) is non-trivial (such a module clearly exists). Then, according to the “Theorem of the highest weight” in combination with “Weyl’s dimension formula” (see [7, Ch. 5]), we have \( \dim(V_{\lambda}) = \dim(V_1) \cdot \dim(V_2) = \dim(V) \), hence \( V_{\lambda} = V_1 \otimes \mathbb{C} V_2 \) and therefore already \( V_1 \otimes \mathbb{C} V_2 \) is an irreducible \( \mathfrak{t} \)-module.

Remark A.8. Let \( V \) be a Euclidian space, \( K \) be a connected Lie group and \( \rho : K \to \text{O}(V) \) be a representation. Let \( \mathfrak{t} \) be the Lie algebra of \( K \) and \( \rho : \mathfrak{t} \to \mathfrak{so}(V) \) be the induced representation. Since \( K \) is connected, a subspace \( W \subset V \) is \( K \)-invariant if and only if it is \( \mathfrak{k} \)-invariant. Hence \( K \) acts irreducible on \( V \) if and only if \( V \) is an irreducible \( \mathfrak{k} \)-module.

Let a simply connected symmetric space \( M \) be given. We let \( K \) be the isotropy subgroup of \( \text{I}(M)^0 \) at the origin \( p \) and \( \rho : K \to \text{O}(V) \) be the isotropy representation on \( V := T_pM \). Furthermore, let \( \mathfrak{t} \) be the Lie algebra of \( K \) and \( \rho : \mathfrak{t} \to \mathfrak{so}(V) \) be the induced representation.
Lemma A.9. Suppose that $M$ is an irreducible symmetric space. Then:

(a) $V$ is an irreducible $\mathfrak{k}$-module and and $\rho(\mathfrak{k})$ is the holonomy Lie algebra of $M$.

(b) $M$ is a Hermitian symmetric space if and only if there exists some $J \in O(V)$ with $J^2 = -\operatorname{Id}$ (equipping $V$ with the structure of a unitary space) such that $\rho(\mathfrak{k}) \subset u(V)$.

(c) We always have $\mathfrak{k} = \mathfrak{k} \oplus \mathfrak{c}$, where $\mathfrak{c}$ denotes the center of $\mathfrak{k}$. Furthermore, the commutator ideal $[\mathfrak{k}, \mathfrak{k}]$ is semisimple.

(d) In case $M$ is Hermitian, $\mathfrak{c}$ is 1-dimensional and $\rho(\mathfrak{c}) = \mathbb{R} J$ holds (see (b)). Otherwise, $\mathfrak{c} = \{0\}$ and hence $\mathfrak{c}$ is semisimple.

Proof. For (a) and (b): Since $K$ is connected (cf. Sec 8), the result follows from Lemma 3.3 (b) in combination with Remark A.8.

For (c): Since $\rho$ is a faithful representation, $\mathfrak{k}$ can be seen as a subalgebra of $\mathfrak{so}(T_p M)$. Then for any ideal $\mathfrak{a} \subset \mathfrak{k}$ its orthogonal complement $\mathfrak{a}^\perp$ (see (73)) is an ideal of $\mathfrak{k}$, too. Hence there exists a decomposition of $\mathfrak{k}$ into an Abelian subspace and simple ideals. Now (c) is obvious.

For (d): According to [3, Ch. VIII, § 7], $K$ has non-discrete center $Z_K$ if and only if $M$ is Hermitian and, in the Hermitian case, $\rho$ maps $Z_K$ isomorphically onto the circle subgroup $S^1 \subset U(V)$. Now (d) follows from (c).

Let $K_i$ be a Lie group, $V_i$ be a Euclidean space and $\rho_i : K_i \rightarrow O(V_i)$ be a representation for $i = 1, 2$.

Definition A.10. We let $K := K_1 \times K_2$ denote the product Lie group and set $V := V_1 \oplus V_2$. Then there is a natural representation $\rho_1 \otimes \rho_2 : K \rightarrow \text{GL}(V)$, given by

$$
\left( \rho_1 \otimes \rho_2 (g_1, g_2) \right) v_1 \otimes v_2 = \rho_1(g_1) v_1 \otimes \rho_2(g_2) v_2
$$

for all $(g_1, g_2) \in K_1 \times K_2$ and $(v_1, v_2) \in V_1 \times V_2$.

Let simply connected symmetric spaces $M_1$ and $M_2$ be given. Let $p_i$ be an origin of $M_i$, $K_i$ be the corresponding isotropy subgroup of $I(M_i)_0$ and $\rho_i : K_i \rightarrow O(V_i)$ be the isotropy representation on $V_i := T_{p_i} M_i$ for $i = 1, 2$.

Lemma A.11. Suppose that $M_1$ and $M_2$ both are irreducible symmetric spaces. Let $\rho_1 \otimes \rho_2$ be defined according to Def. A.10. Then:

(a) In case neither $M_1$ nor $M_2$ is a Hermitian symmetric space, $\rho_1 \otimes \rho_2$ is irreducible.

(b) The same is true if exactly one of $M_1$ or $M_2$ is a Hermitian symmetric space.

(c) In case $M_1$ and $M_2$ both are Hermitian symmetric spaces, we let $J_i$ denote the corresponding complex structure of $V_i$ for $i = 1, 2$. Then

$$
V_\pm := \left\{ v_1 \otimes v_2 \mid J_i v_1 \otimes J_i v_2 \pm \{v_1, v_2\} \in V_1 \times V_2 \right\}
$$

(110)

gives the decomposition $V_1 \otimes V_2 = V_+ \oplus V_-$ into two irreducible invariant subspaces of the same dimension.

Proof. Set $K := K_1 \times K_2$, $V := V_1 \oplus V_2$ and $\rho := \rho_1 \otimes \rho_2$. Since $K_i$ is connected, we may switch to the corresponding Lie algebras $\mathfrak{k}_i$ and their induced representations $\rho_i : \mathfrak{k}_i \rightarrow \mathfrak{so}(V_i)$ according to Remark A.8. Then $\rho_1 \otimes \rho_2$ is given by Eq. (108) (because of the “chain rule”).

For (a): As a consequence of Lemma A.5 in combination with Lemma A.9 in this case $\mathfrak{k}_i$ acts irreducibly on $V_i^\mathbb{C}$ for $i = 1, 2$, too. Hence $V^\mathbb{C}$ is a complex-irreducible $\mathfrak{k}$-module, too, according to Lemma A.7. Thus already $V$ is necessarily an irreducible $\mathfrak{k}$-module.

For (b): Without loss of generality, we can assume that $M_1$ is Hermitian but not $M_2$; let $J_1$ denote the complex structure of $V_1$. Then, according to Lemma A.5 (b), $\mathfrak{t}_2$ acts irreducibly on $V_2^\mathbb{C}$, whereas Eq. (106) gives the decomposition $V_2^\mathbb{C} = (V_2)^{+1} \oplus (V_2)^{-1}$ into two invariant subspaces such that $(V_2)^{+1}$ is an irreducible $\mathfrak{t}_1$-modules (over $\mathbb{C}$). Then $\mathfrak{t}_2$ and $\mathfrak{t}_1 := \{\mathfrak{t}_1, \mathfrak{t}_1\}$ both are semisimple Lie algebras and $\mathfrak{t}_1 = \mathfrak{c}_1 \oplus \mathfrak{t}_1$ holds according to Lemma A.9 (c), where $\mathfrak{c}_1$ denotes the center of $\mathfrak{t}_1$. Furthermore, we have $\rho_1(\mathfrak{c}_1) = \mathbb{R} J_1$ because of Lemma A.9 (d). In particular, $(V_1)^{+1}$ is even a complex-irreducible $\mathfrak{t}_1$-module.
(since $\epsilon_1$ acts through scalars on $(V_1)_+^i$). Therefore, we can apply Lemma A.7 to conclude that $(V_1)_+^i \otimes V_2^C$ is a complex-irreducible $\mathfrak{k}$-module. Furthermore, in accordance with Eq. (107),

$$V \rightarrow (V_1)_+^i \otimes V_2^C, v_1 \otimes v_2 \mapsto 1/2(v_1 - iJ_1 v_1) \otimes v_2$$

is an isomorphism of $\mathfrak{k}$-modules over $C$ (where $V$ is seen as a complex space by means of $J_1$). Therefore, $V$ is also irreducible over $C$. But $\rho_1(\epsilon_1) = R J_1$ holds, hence $V$ is irreducible over $R$, too. This finishes the proof for (b).

For (c): Here we obtain the decomposition $V_i^C = (V_i)_+^i \oplus (V_i)_-^i$ into $\mathfrak{k}$-invariant subspaces such that $\mathfrak{k}$ acts irreducibly on both $(V_i)_+^i$ and $(V_i)_-^i$ (over $C$) for $i = 1, 2$. Furthermore, using once again Eq. (107),

$$V_+ \rightarrow (V_1)_+^i \otimes (V_2)_+^i, (v_1 \otimes v_2 - J_1 v_1 \otimes J_2 v_2) \mapsto 1/2((v_1 - iJ_1 v_1) \otimes (v_2 - iJ_2 v_2))$$

is an isomorphism of $\mathfrak{k}$-modules over $C$ (where $V_+$ is seen as a complex space similar as before). Using similar arguments as before, we now conclude that $V_+$ is an irreducible $\mathfrak{k}$-module. Analogously, we can show that $V_-$ is an irreducible $\mathfrak{k}$-module, too. Furthermore,

$$V_+ \rightarrow V_-, v_1 \otimes v_2 - J_1 v_1 \otimes J_2 v_2 \mapsto v_1 \otimes v_2 + J_1 v_1 \otimes J_2 v_2$$

is a linear isomorphism. This finishes the proof for (c).

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