Y(\(nS\)) \(\rightarrow\) \(B_c^*D\) decays with perturbative QCD approach

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Abstract

The Y(\(nS\)) \(\rightarrow\) \(B_c^*D\) weak decays \((n = 1, 2, 3)\) are investigated with perturbative QCD approach. It is found that the CKM-favored Y(\(nS\)) \(\rightarrow\) \(B_c^*D_s\) decays have branching ratio of \(\mathcal{O}(10^{-10})\), which might be potentially accessible to the future LHC and SuperKEKB experiments.
I. INTRODUCTION

ϒ(1S), ϒ(2S) and ϒ(3S) are spin-triplet S-wave $b\bar{b}$ bound states carrying with quantum number of $I^GJ^{PC} = 0^{-+} - - [1]$. They all lie below the open bottom threshold. They must strongly decay into two light hadrons via $b\bar{b}$ annihilation into at least three gluons. So their decay width is very narrow, only dozens of keV. [Hereinafter, for simplicity sake, we will use a notation ϒ(nS) to represent ϒ(1S), ϒ(2S), and ϒ(3S) mesons.] Since their discovery in 1977 [2, 3], ϒ(nS) has been attracting much attention from experimentalists and theorists. Thanks to the excellent performance from experimental groups of CLEO, BaBar, Belle, CDF, D0, LHCb, ATLAS and so on, remarkable achievements have been made in understanding of the nature of upsilon [1, 4]. The strong and electromagnetic ϒ(nS) decay modes have been carefully investigated. With accumulation of ϒ(nS) data samples, it might be possible to search for ϒ(nS) weak decay at future LHC and SuperKEKB experiments.

Theoretically, both valence quarks of ϒ(nS) can decay individually via the weak interaction. The $b \to c$ transition is particularly favored by a hierarchy of the Cabibbo-Kabayashi-Maskawa (CKM) matrix elements. So ϒ(nS) decay into final states containing a $B_c^{(*)}$ meson should, in principle, have a relatively large branching fraction among its weak decay modes. Recently, some phenomenological QCD-inspired methods have been vividly developed to deal with heavy quark weak decay, such as perturbative QCD (pQCD) approach [5–7], QCD factorization (QCDF) [8–12] and soft and collinear effective theory [13–16]. The ϒ(nS) → $B_c^{(*)}D$ decays offer a good plaza to ulteriorly test various phenomenological models and to further explore the underlying dynamical mechanism of heavy quarkonium weak decay. In addition, as far as we know, there is no experimental measurement report and few theoretical work related to ϒ(nS) → $B_c^{(*)}D$ decay for the moment. Herein, we will study the bottom-and charm-changing ϒ(nS) → $B_c^{(*)}D$ weak decays with pQCD approach to provide future experimental exploration with a useful reference.

This paper is organized as follows. The section II devotes to theoretical framework and amplitudes for ϒ(nS) → $B_c^{(*)}D$ decay. The numerical results and discussion are presented in section III. We summarize in the last section.
II. THEORETICAL FRAMEWORK

A. The effective Hamiltonian

The effective weak Hamiltonian describing nonleptonic $\Upsilon(nS) \to B^*_c D$ decays is written as \[17\]
\[
\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \left\{ V_{cb} V_{cq}^* \sum_{i=1}^{2} C_i(\mu) Q_i(\mu) - V_{tb} V_{tq}^* \sum_{j=3}^{10} C_j(\mu) Q_j(\mu) \right\} + \text{h.c.},
\]
where $G_F \simeq 1.166 \times 10^{-5}$ GeV$^{-2}$ is the Fermi constant \[1\]; $q = d$ and $s$; the CKM factors can be expressed as
\[
V_{cb} V_{cs}^* = +A\lambda^2 - \frac{1}{2} A\lambda^4 - \frac{1}{8} A\lambda^6(1 + 4A^2) + \mathcal{O}(\lambda^7),
\]
for $\Upsilon(nS) \to B^*_c D_s$ decays; and
\[
V_{cb} V_{cd}^* = -A\lambda^3 + \mathcal{O}(\lambda^7),
\]
for $\Upsilon(nS) \to B^{(*)}_c D_d$ decays; $A$, $\lambda$, $\rho$ and $\eta$ are Wolfenstein parameters \[1, 18\].

The local tree operators $Q_{1,2}$, QCD penguin operators $Q_{3,...,6}$, and electroweak operators $Q_{7,...,10}$ are defined below.

\[
Q_1 = [\bar{c}_\alpha \gamma_\mu(1 - \gamma_5) b_\alpha][\bar{q}_\beta \gamma^\mu(1 - \gamma_5)c_\beta],
\]
\[
Q_2 = [\bar{c}_\alpha \gamma_\mu(1 - \gamma_5) b_\alpha][\bar{q}_\beta \gamma^\mu(1 - \gamma_5)c_\alpha],
\]
\[
Q_3 = \sum_q [\bar{q}_\alpha \gamma_\mu(1 - \gamma_5) b_\alpha][\bar{q}_\beta \gamma^\mu(1 - \gamma_5)q_\beta],
\]
\[
Q_4 = \sum_q [\bar{q}_\alpha \gamma_\mu(1 - \gamma_5) b_\beta][\bar{q}_\beta \gamma^\mu(1 - \gamma_5)q_\alpha],
\]
\[
Q_5 = \sum_q [\bar{q}_\alpha \gamma_\mu(1 - \gamma_5) b_\alpha][\bar{q}_\beta \gamma^\mu(1 + \gamma_5)q_\beta],
\]
\[
Q_6 = \sum_q [\bar{q}_\alpha \gamma_\mu(1 - \gamma_5) b_\beta][\bar{q}_\beta \gamma^\mu(1 + \gamma_5)q_\alpha],
\]
\[
Q_7 = \sum_q \frac{3}{2} e_q [\bar{q}_\alpha \gamma_\mu(1 - \gamma_5) b_\alpha][\bar{q}_\beta \gamma^\mu(1 + \gamma_5)q_\beta],
\]
\[ Q_8 = \sum_{q'} \frac{3}{2} e_{q'} \left[ \bar{q}_a \gamma_\mu (1 - \gamma_5)b_\beta \right] \left[ q'_\beta \gamma^\mu (1 + \gamma_5) q'_\alpha \right], \] (13)

\[ Q_9 = \sum_{q'} \frac{3}{2} e_{q'} \left[ \bar{q}_a \gamma_\mu (1 - \gamma_5)b_\alpha \right] \left[ q'_\beta \gamma^\mu (1 - \gamma_5) q'_\beta \right], \] (14)

\[ Q_{10} = \sum_{q'} \frac{3}{2} e_{q'} \left[ \bar{q}_a \gamma_\mu (1 - \gamma_5)b_\beta \right] \left[ q'_\beta \gamma^\mu (1 - \gamma_5) q'_\alpha \right], \] (15)

where \( \alpha \) and \( \beta \) are color indices; \( q' = u, d, s, c, b \) has an electric charge \( e_{q'} \) in the unit of \( |e| \).

The scale \( \mu \) separates physical contributions into two components. The Wilson coefficients \( C_i(\mu) \) summarize the physical contributions above \( \mu \), and has been reliably computed to the next-to-leading order with perturbation theory \[17\]. The hadronic matrix elements (HME), where the local operators are sandwiched between initial and final hadron states, contain the physical contributions below \( \mu \). Due to the incorporation of long distance contributions and the entanglement of perturbative and nonperturbative effects, HME is not yet fully understood until now. However, in order to evaluate the amplitudes, one has to face directly the HME’s calculation based on some approximation and assumptions, which leads to large theoretical uncertainties.

**B. Hadronic matrix elements**

Phenomenologically, combining factorization hypothesis \[19–21\] and hard-scattering approach \[22–26\], HME could be written as the convolution of hard scattering kernel function \( \mathcal{T} \) and distribution amplitudes (DAs) of participating hadrons. DAs are nonperturbative but universal inputs, which can be obtained from nonperturbative methods or fitted from experimental data. In order to eliminate the endpoint singularities accompanying with spectator rescattering and annihilation contributions based on a collinear approximation \[10–12\], and in the meantime to provide an effective cutoff on nonperturbative contributions, the transverse momentum of valence quarks is kept explicitly and a Sudakov factor for each of DAs is introduced compulsorily with pQCD approach \[5–7\]. A general pQCD amplitude is made up of three parts: the Wilson coefficients \( C_i \) absorbing physical contributions above a typical scale of \( t \), hard scattering kernel function \( \mathcal{T} \) accounting for heavy quark weak decay, and wave functions \( \Phi \), i.e.,

\[
\int dk \, C_i(t) \, \mathcal{T}(t, k) \prod_j \Phi_j(k) \, e^{-S_j},
\] (16)

where \( k \) is the momentum of valence quarks, and \( e^{-S_j} \) is a Sudakov factor.
C. Kinematic variables

In the $\Upsilon(nS)$ rest frame, the light cone kinematic variables are defined as follows.

\[ p_{\Upsilon} = p_1 = \frac{m_1}{\sqrt{2}}(1, 1, 0), \]  
\[ p_{B_c^*} = p_2 = (p^+_2, p^-_2, 0), \]  
\[ p_D = p_3 = (p^-_3, p^+_3, 0), \]  
\[ p^\pm_i = \frac{E_i \pm p_i}{\sqrt{2}}, \]  
\[ k_i = x_i p_i + (0, 0, \vec{k}_iT), \]  
\[ \epsilon^\parallel_1 = \frac{p_1}{m_1} - \frac{m_1}{p_1 \cdot n_+} n_+, \]  
\[ \epsilon^\parallel_2 = \frac{p_2}{m_2} - \frac{m_2}{p_2 \cdot n_-} n_-, \]  
\[ \epsilon^{\perp}_{1,2} = (0, 0, \vec{\epsilon}_{iT}), \]  
\[ n_+ = (1, 0, 0), \]  
\[ n_- = (0, 1, 0), \]  
\[ s = 2p_2 \cdot p_3 = m_1^2 - m_2^2 - m_3^2, \]  
\[ t = 2p_1 \cdot p_2 = m_1^2 + m_2^2 - m_3^2 = 2m_1 E_2, \]  
\[ u = 2p_1 \cdot p_3 = m_1^2 - m_2^2 + m_3^2 = 2m_1 E_3, \]  
\[ u + t - s = m_1^2 + m_2^2 + m_3^2, \]  
\[ st + su - tu - 4m_1^2 p^2 = 0, \]

where $x_i$ and $k_{iT}$ are the longitudinal momentum fraction and transverse momentum of valence quark, respectively; $\epsilon^\parallel_i$ and $\epsilon^{\perp}_i$ are the longitudinal and transverse polarization vectors, respectively, satisfying relations $\epsilon^2_i = -1$ and $\epsilon_i \cdot p_i = 0$; the subscript $i = 1, 2, 3$ on variables $(E_i, p_i, m_i, \epsilon_i)$ corresponds to $\Upsilon(nS)$, $B_c^*$, and $D$ mesons, respectively; $n_+$ and $n_-$ are the positive and negative null vectors, respectively; $s$, $t$ and $u$ are Lorentz-invariant variables. These kinematic variables are showed in Fig.2(a).
D. Wave functions

The definitions of wave functions are [27, 28],

\[ \langle 0| b_i(z) \bar{b}_j(0)| Y(p_1, \epsilon_1^\perp) \rangle = \frac{f_T}{4} \int dk_1 e^{-i k_1 \cdot z} \left\{ f_1^\perp \left[ m_1 \phi^T_\perp(k_1) - \not{p}_1 \phi^L_\perp(k_1) \right] \right\}_{ji}, \]

\[ \langle 0| b_i(z) \bar{b}_j(0)| Y(p_1, \epsilon_1^\perp) \rangle = \frac{f_T}{4} \int dk_1 e^{-i k_1 \cdot z} \left\{ f_1^\perp \left[ m_1 \phi^V_\perp(k_1) - \not{p}_1 \phi^T_\perp(k_1) \right] \right\}_{ji}, \]

\[ \langle B_c^*(p_2, \epsilon_2^\perp)| \bar{c}_j(z) b_j(0)|0 \rangle = \frac{f_{B_c^*}}{4} \int_0^1 dk_2 e^{i k_2 \cdot z} \left\{ f_2^\perp \left[ m_2 \phi^V_{B_c^*}(k_2) + \not{p}_2 \phi^T_{B_c^*}(k_2) \right] \right\}_{ji}, \]

\[ \langle B_c^*(p_2, \epsilon_2^\perp)| \bar{c}_j(z) b_j(0)|0 \rangle = \frac{f_{B_c^*}}{4} \int_0^1 dk_2 e^{i k_2 \cdot z} \left\{ f_2^\perp \left[ m_2 \phi^V_{B_c^*}(k_2) + \not{p}_2 \phi^T_{B_c^*}(k_2) \right] \right\}_{ji}, \]

\[ \langle D(p_3)| c_i(0) \bar{q}_j(z)|0 \rangle = \frac{i f_{D}}{4} \int dk_3 e^{i k_3 \cdot z} \left\{ \gamma_5 \left[ \not{p}_3 \phi^D(k_3) + m_3 \phi^P_D(k_3) \right] \right\}_{ji}, \]

where \( f_T, f_{B_c^*}, f_D \) are decay constants; wave functions \( \Phi^{v,T}_{T,B_c^*} \) and \( \Phi^D \) are twist-2; \( \Phi^{V,T}_{T,B_c^*} \) and \( \Phi^P_D \) are twist-3. The explicit expressions of DAs are [29]

\[ \phi^v_T(x) = \phi^T_T(x) = A_1 x \bar{x} \exp \left\{ - \frac{m_b^2}{8 \omega_T^2 x \bar{x}} \right\}, \]

\[ \phi^l_T(x) = A_2 (\bar{x} - x)^2 \exp \left\{ - \frac{m_b^2}{8 \omega_T^2 x \bar{x}} \right\}, \]

\[ \phi^V_T(x) = A_3 \left\{ 1 + (\bar{x} - x)^2 \right\} \exp \left\{ - \frac{m_b^2}{8 \omega_T^2 x \bar{x}} \right\}, \]

\[ \phi^{v,t,V}_{T(3S)}(x) = B_t \phi^{v,t,V}_{T(1S)}(x) \left\{ 1 + \frac{m_b^2}{2 \omega_T^2 x \bar{x}} \right\}, \]

\[ \phi^{v,t,V}_{T(3S)}(x) = C_t \phi^{v,t,V}_{T(1S)}(x) \left\{ \left( 1 - \frac{m_b^2}{2 \omega_T^2 x \bar{x}} \right)^2 + 6 \right\}, \]

\[ \phi^v_{B_c^*}(x) = \phi^T_{B_c^*}(x) = D_1 x \bar{x} \exp \left\{ - \frac{\bar{x} m_c^2 + x m_b^2}{8 \omega_c^2 x \bar{x}} \right\}, \]

\[ \phi^l_{B_c^*}(x) = D_2 (\bar{x} - x)^2 \exp \left\{ - \frac{\bar{x} m_c^2 + x m_b^2}{8 \omega_c^2 x \bar{x}} \right\}, \]

\[ \phi^V_{B_c^*}(x) = D_3 \left\{ 1 + (\bar{x} - x)^2 \right\} \exp \left\{ - \frac{\bar{x} m_c^2 + x m_b^2}{8 \omega_c^2 x \bar{x}} \right\}, \]

\[ \phi^a_D(x) = E_1 x \bar{x} \exp \left\{ - \frac{\bar{x} m_q^2 + x m_c^2}{8 \omega_q^2 x \bar{x}} \right\}, \]

\[ \phi^P_D(x) = E_2 \exp \left\{ - \frac{\bar{x} m_q^2 + x m_c^2}{8 \omega_q^2 x \bar{x}} \right\}, \]
where \( \bar{x} = 1 - x \); parameter \( \omega_i = m_i \alpha_s(m_i) \) determines the average transverse quark momentum according to nonrelativistic quantum chromodynamics (NRQCD) power counting rules \[30\]; parameters \( A_i, B_i, C_i, D_i, E_i \) are normalization coefficients,

\[
\int_0^1 dx \, \phi_i^x(x) = 1, \quad \text{for } i = v, t, V, T, \tag{47}
\]

\[
\int_0^1 dx \, \phi_i^{B_c}(x) = 1, \quad \text{for } i = v, t, V, T, \tag{48}
\]

\[
\int_0^1 dx \, \phi_i^D(x) = 1, \quad \text{for } i = a, p. \tag{49}
\]

FIG. 1: The normalized distribution amplitudes for \( \Upsilon(nS), B_c^*, D \) mesons.

The shape lines of DAs for \( \Upsilon(nS), B_c^*, D \) mesons are displayed in Fig. 1. It is clearly seen that DAs fall quickly down to zero at endpoint \( x, \bar{x} \to 0 \) due to suppression from exponential functions, which offer a natural cutoff for soft contributions.

E. Decay amplitudes

The Feynman diagrams for \( \Upsilon(nS) \to B_c^* D_s \) decay are showed in Fig. 2. There are two types. One is emission topology, and the other is annihilation topology. Each type is further subdivided into factorizable and nonfactorizable diagrams.

After a detail calculation, amplitude for \( \Upsilon(nS) \to B_c^* D \) decay is written as

\[
\mathcal{A}(\Upsilon \to B_c^* D) = \mathcal{A}_L(\epsilon_1, \epsilon_2) + \mathcal{A}_N(\epsilon_1^\perp \cdot \epsilon_2^\perp) + i \mathcal{A}_T \varepsilon_{\mu \rho \alpha \beta} \epsilon_1^\mu \epsilon_2^\rho \epsilon_1^\alpha p_1^\beta p_2^\beta, \tag{50}
\]
which is also written as the helicity amplitudes,

\[ M_0 = -\mathcal{C} A_L(\epsilon_1^\parallel, \epsilon_2^\parallel), \]  
\[ M_{\parallel} = \sqrt{2} \mathcal{C} A_N, \]  
\[ M_{\perp} = \sqrt{2} \mathcal{C} m_1 p A_T, \]

\[ \mathcal{C} = i G_F C_F \sqrt{\frac{2}{3}} \frac{N_c}{f_{B_s} f_D}, \]

where \( C_F = 4/3 \) and the color number \( N_c = 3 \).

The expression of polarization amplitude \( A_j \) is

\[ A_j = V_{cb} V_{cq}^* \left\{ (A_{a,j}^{LL} + A_{b,j}^{LL}) a_1 + (A_{c,j}^{LL} + A_{d,j}^{LL}) C_2 \right\} 
- V_{tb} V_{tq}^* \left\{ (A_{a,j}^{LL} + A_{b,j}^{LL}) (a_4 + a_{10}) + (A_{e,j}^{LL} + A_{d,j}^{LL}) (C_3 + C_9) 
+ (A_{e,j}^{LL} + A_{f,j}^{LL}) (C_3 + C_4 - \frac{1}{2} C_9 - \frac{1}{2} C_{10}) 
+ (A_{g,j}^{LL} + A_{h,j}^{LL}) (a_3 + a_4 - \frac{1}{2} a_9 - \frac{1}{2} a_{10}) 
+ (A_{e,j}^{LR} + A_{f,j}^{LR}) (C_6 - \frac{1}{2} C_8) + (A_{g,j}^{LR} + A_{h,j}^{LR}) (a_5 - \frac{1}{2} a_7) 
+ (A_{e,j}^{SP} + A_{d,j}^{SP}) (C_5 + C_7) + (A_{e,j}^{SP} + A_{f,j}^{SP}) (C_5 - \frac{1}{2} C_7) \right\}, \]  

where \( M_0, M_{\parallel}, M_{\perp}, \mathcal{C} \), and the color number \( N_c = 3 \).
where the subscript $j = L, N, T$ denotes to three different helicity amplitudes; the expressions of building blocks $A_{i,j}^k$ are collected in Appendix; $C_i$ is Wilson coefficient, parameter $a_i$ is defined as

$$a_i = \begin{cases} 
C_i + C_{i+1}/N_c, & \text{for odd } i; \\
C_i + C_{i-1}/N_c, & \text{for even } i.
\end{cases}$$

(56)

III. NUMERICAL RESULTS AND DISCUSSION

In the $\Upsilon(nS)$ rest frame, branching ratio for $\Upsilon(nS) \to B_c^+D$ decay is defined as

$$\mathcal{B}r = \frac{1}{12\pi} \frac{p}{m_\Upsilon^2 \Gamma_\Upsilon} \left\{ |\mathcal{M}_0|^2 + |\mathcal{M}_\||^2 + |\mathcal{M}_\perp|^2 \right\},$$

(57)

where $p$ is the center-of-mass momentum of final states; $\Gamma_\Upsilon$ is a total decay width.

The input parameters are listed in Table I. If it is not stated explicitly, their central values will be used as the default inputs. Our numerical results are collected in Table. III where theoretical uncertainties come from scale $(1\pm0.1)t$, mass $m_b$ and $m_c$, and CKM parameters, respectively. The following is some comments.

| TABLE I: The numerical values of input parameters. |
|-----------------------------------------------|
| **Wolfenstein parameters$^a$ [1]** |
| $A = 0.814^{+0.023}_{-0.024}$, $\lambda = 0.22537\pm0.00061$, $\bar{\rho} = 0.117\pm0.021$, $\bar{\eta} = 0.353\pm0.013$ |
| mass, width and decay constant |
| $m_{\Upsilon(1S)} = 9460.30\pm0.26$ MeV [1], $\Gamma_{\Upsilon(1S)} = 54.02\pm1.25$ keV [1], $f_{\Upsilon(1S)} = 676.4\pm10.7$ MeV [31], |
| $m_{\Upsilon(2S)} = 10023.26\pm0.31$ MeV [1], $\Gamma_{\Upsilon(2S)} = 31.98\pm2.63$ keV [1], $f_{\Upsilon(2S)} = 473.0\pm23.7$ MeV [31], |
| $m_{\Upsilon(3S)} = 10355.2\pm0.5$ MeV [1], $\Gamma_{\Upsilon(3S)} = 20.32\pm1.85$ keV [1], $f_{\Upsilon(3S)} = 409.5\pm29.4$ MeV [31], |
| $m_{B_c^+} = 6332\pm9$ MeV [32], $f_{B_c^+} = 422\pm13$ MeV [33], $m_b = 4.78\pm0.06$ GeV [1], |
| $m_{D_s} = 1968.30\pm0.11$ MeV [1], $f_{D_s} = 257.5\pm4.6$ MeV [1], $m_c = 1.67\pm0.07$ GeV [1], |
| $m_{D_d} = 1869.61\pm0.10$ MeV [1], $f_{D_d} = 204.6\pm5.0$ MeV [1], $m_s \simeq 0.51$ GeV [34], |
| $\Lambda_{QCD}^{(5)} = 214\pm7$ MeV [1], $\Lambda_{QCD}^{(4)} = 297\pm8$ MeV [1], $m_d \simeq 0.31$ GeV [34]. |

$^a$The relation between parameters $(\rho, \eta)$ and $(\bar{\rho}, \bar{\eta})$ is [1]: $(\rho + i\eta) = \frac{\sqrt{1 - A^2\lambda^4(\bar{\rho} + i\bar{\eta})}}{\sqrt{1 - \lambda^2[1 - A^2\lambda^4(\bar{\rho} + i\bar{\eta})]}}$.

(1) By and large, due to the hierarchical structure of CKM factors $|V_{cb}V_{cs}^\ast| > |V_{cd}V_{cd}^\ast|$, there is a general hierarchical relationship among branching ratios $\mathcal{B}r(\Upsilon(nS) \to B_c^+D_s) > \mathcal{B}r(\Upsilon(nS) \to B_c^+D_d)$. 

9
(2) In principle, it is expected to have relations $Br(\Upsilon(3S)\rightarrow B_c^+D_s) > Br(\Upsilon(2S)\rightarrow B_c^+D_s) > Br(\Upsilon(1S)\rightarrow B_c^+D_s)$ for the same $D$ meson, due to the fact $\Gamma_{\Upsilon(3S)} < \Gamma_{\Upsilon(2S)} < \Gamma_{\Upsilon(1S)}$. However, the numbers in Table II are beyond our expectation. Why is it that? Besides convolution integral of DAs resulting in different $\Upsilon(nS)\rightarrow B_c^+$ transition form factors, one of the possible essential causation is that branching ratio is proportional to factor $f_T^2/m_T^2 \Gamma_\Upsilon$ and

$$\frac{f_{T(1S)}^2}{m_{T(1S)}^2 \Gamma_{\Upsilon(1S)}} : \frac{f_{T(2S)}^2}{m_{T(2S)}^2 \Gamma_{\Upsilon(2S)}} : \frac{f_{T(3S)}^2}{m_{T(3S)}^2 \Gamma_{\Upsilon(3S)}} \approx 1.2 : 0.9 : 1.0. \quad (58)$$

(3) Branching ratios for $\Upsilon(nS)\rightarrow B_c^+D_s$ decays can reach up to $\mathcal{O}(10^{-10})$. In the center-of-mass frame of $\Upsilon(nS)$, the final states are back-to-back, and have opposite electric charges. In addition to abundant $\Upsilon(nS)$ data samples in the future experiment, the “charge tag” and “flavor tag” technique can be used to effectively reconstruct events and reduce background. So $\Upsilon(nS)\rightarrow B_c^+D_s$ decay might be measurable at the running LHC and forthcoming SuperKEKB. For example, the $\Upsilon(nS)$ production cross section in p-Pb collision is about a few $\mu$b at LHCb [35] and ALICE [36]. More than $10^{11}$ $\Upsilon(nS)$ data samples per $ab^{-1}$ data collected at LHCb and ALICE are in principle available, corresponding to dozens of $\Upsilon(nS)\rightarrow B_c^+D_s$ events.

(4) The momentum transition in $\Upsilon(nS)\rightarrow B_c^+D$ decay may be not large enough, because of $m_{B_c} + m_D > 8$ GeV. It is natural to question the validity of perturbative calculation with pQCD approach. Therefore, it is very necessary to check what percentage of contributions comes from the perturbative region. In Fig.3 contributions to branching ratio from different regions of $\alpha_s/\pi$ are plotted. It is clearly seen that more than 80% (90%) contributions come from $\alpha_s/\pi \leq 0.2$ (0.3) regions, implying that pQCD approach is applicable to the concerned processes, and many combined factors (such as the choice of scale $t$, Sudakov factor, wave function models, and so on) ensure a reliable perturbative calculation. Compared with the second bin contribution where $\alpha_s/\pi = 0.2$, the first bin contribution where $\alpha_s/\pi = 0.1$ is relatively small. One of crucial reasons might be that the absolute values of parameter $a_1$
FIG. 3: Contributions to branching ratio from different regions of $\alpha_s/\pi$ (horizontal axis), where the numbers over histogram denote the percentage of the corresponding contributions.

and coupling $\alpha_s$ decrease along with the increase of renormalization scale.

(5) Besides the uncertainties listed in Table II, decay constants $f_\Upsilon$ and $f_{B_c^*}$ (decay width $\Gamma_\Upsilon$) can bring about 7% (2%), 12% (8%), 16% (9%) uncertainties for $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$ decays, respectively, mainly from $f_{\Upsilon(2S,3S)}$ and $\Gamma_{\Upsilon(2S,3S)}$. These are at least two ways to reduce theoretical uncertainty. One way is to construct some relative ratios, for example, $\mathcal{B}r(\Upsilon(nS)\to B_{c}^*D_d)/\mathcal{B}r(\Upsilon(nS)\to B_{c}^*D_s)$ and $\mathcal{B}r(\Upsilon(mS)\to B_{c}^*D)/\mathcal{B}r(\Upsilon(nS)\to B_{c}^*D)$. The
other is to consider higher order corrections to HME, more realistic DAs models, and so on. Our results are just an order of magnitude estimation on branching ratio.

IV. SUMMARY

With anticipation of the potential prospects of $\Upsilon(nS)$ physics at high-luminosity heavy-flavor factories, search for $\Upsilon(nS)$ weak decay seems to be experimentally feasible. A theoretical study of $\Upsilon(nS)$ weak decay is seasonable and necessary. In this paper, we investigated the bottom- and charm-changing $\Upsilon(nS) \rightarrow B_{c}^{*}D_{s,d}$ decays with phenomenological pQCD approach. It is expected that branching ratio for $\Upsilon(nS) \rightarrow B_{c}^{*}D_{s}$ decay could be up to $O(10^{-10})$, which might be measurable at the future LHC and SuperKEKB experiments.

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Appendix A: Building blocks of decay amplitudes

For the sake of simplicity, we decompose the amplitude Eq. (55) into some building blocks $A_{i,j}^{k}$, where the subscript $i$ corresponds to the indices of Fig. 2, the subscript $j = L, N, T$ relates with different helicity amplitudes; the superscript $k$ refers to one of the three possible Dirac structures $\Gamma_{1} \otimes \Gamma_{2}$ of the four-quark operator $(\bar{q}_{1}\Gamma_{1}q_{2})(\bar{q}_{1}\Gamma_{2}q_{2})$, namely $k = LL$ for $(V-A) \otimes (V-A)$, $k = LR$ for $(V-A) \otimes (V+A)$, and $k = SP$ for $-2(S-P) \otimes (S+P)$. The explicit expressions of $A_{i,j}^{k}$ are written as follows.

$$A_{a,L}^{LL} = \mathcal{I}_{a} \phi_{\Gamma}^{V}(x_{1}) \left\{ \phi_{B_{c}^{*}}^{V}(x_{2}) \left[ m_{1}^{2} s - (4 m_{1}^{2} p^{2} + m_{2}^{2} u) \bar{x}_{2} \right] + \phi_{B_{c}^{*}}^{T}(x_{2}) m_{2} m_{b} u \right\},$$

(A1)

$$A_{a,N}^{LL} = m_{1} \mathcal{I}_{a} \phi_{\Gamma}^{V}(x_{1}) \left\{ \phi_{B_{c}^{*}}^{V}(x_{2}) m_{2} (u - s \bar{x}_{2}) + \phi_{B_{c}^{*}}^{T}(x_{2}) m_{b} s \right\},$$

(A2)

$$A_{a,T}^{LL} = -2 m_{1} \mathcal{I}_{a} \phi_{\Gamma}^{V}(x_{1}) \left\{ \phi_{B_{c}^{*}}^{V}(x_{2}) m_{2} x_{2} + \phi_{B_{c}^{*}}^{T}(x_{2}) m_{b} \right\},$$

(A3)

$$A_{a,L}^{SP} = 2 m_{3} \mathcal{I}_{a} \phi_{\Gamma}^{V}(x_{1}) \left\{ \phi_{B_{c}^{*}}^{V}(x_{2}) m_{b} t + \phi_{B_{c}^{*}}^{T}(x_{2}) m_{2} (2 m_{1}^{2} - t \bar{x}_{2}) \right\},$$

(A4)

$$A_{a,N}^{SP} = 2 m_{1} m_{3} \mathcal{I}_{a} \phi_{\Gamma}^{V}(x_{1}) \left\{ \phi_{B_{c}^{*}}^{V}(x_{2}) 2 m_{2} m_{b} + \phi_{B_{c}^{*}}^{T}(x_{2}) (t - 2 m_{1}^{2} \bar{x}_{2}) \right\},$$

(A5)
\[\mathcal{A}_{a,T}^{SP} = 4 m_1 m_3 \mathcal{I}_b \phi_T^v (x_1) \phi_{B_z}^T (x_2),\]  
\[\mathcal{A}_{b,L}^{LL} = \mathcal{I}_b \phi_{B_z}^v (x_2) \left\{ \phi_T^v (x_1) \left[ m_2^2 u - m_2^2 (s - 4 p^2) \bar{x}_1 \right] + \phi_T^v (x_1) m_1 m_c s \right\},\]  
\[\mathcal{A}_{b,N}^{LL} = m_2 \mathcal{I}_b \phi_{B_z}^v (x_2) \left\{ \phi_T^v (x_1) m_1 (s - u \bar{x}_1) + \phi_T^v (x_1) m_c u \right\},\]  
\[\mathcal{A}_{b,T}^{LL} = -2 m_2 \mathcal{I}_b \phi_{B_z}^v (x_2) \left\{ \phi_T^v (x_1) m_1 x_1 + \phi_T^v (x_1) m_c \right\},\]  
\[\mathcal{A}_{b,L}^{SP} = 2 m_3 \mathcal{I}_b \phi_{B_z}^v (x_2) \left\{ \phi_T^v (x_1) m_c t + \phi_T^v (x_1) m_1 (2 m_2^2 - t \bar{x}_1) \right\},\]  
\[\mathcal{A}_{b,N}^{SP} = 2 m_2 m_3 \mathcal{I}_b \phi_{B_z}^v (x_2) \left\{ \phi_T^v (x_1) 2 m_1 m_c + \phi_T^v (x_1) (t - 2 m_1^2 \bar{x}_1) \right\},\]  
\[\mathcal{A}_{b,T}^{SP} = 4 m_2 m_3 \mathcal{I}_b \phi_T^v (x_1) \phi_{B_z}^V (x_2),\]  
\[\mathcal{A}_{c,L}^{LL} = \mathcal{I}_c \phi_D^v (x_3) \left\{ \phi_T^v (x_1) \phi_{B_z}^v (x_2) 4 m_1^2 p^2 (x_1 - \bar{x}_3) \right.\]  
\[+ \phi_T^v (x_1) \phi_{B_z}^v (x_2) m_1 m_2 (u x_1 - s x_2 - 2 m_3 \bar{x}_3) \},\]  
\[\mathcal{A}_{c,N}^{LL} = \mathcal{I}_c \phi_T^v (x_1) \phi_{B_z}^v (x_2) \phi_D^v (x_3) \left\{ m_2^2 s (x_1 - \bar{x}_3) + m_2^2 u (\bar{x}_3 - x_2) \right\},\]  
\[\mathcal{A}_{c,T}^{LL} = 2 \mathcal{I}_c \phi_T^v (x_1) \phi_{B_z}^v (x_2) \phi_D^v (x_3) \left\{ m_1^2 (\bar{x}_3 - x_1) + m_2^2 (x_2 - \bar{x}_3) \right\},\]  
\[\mathcal{A}_{c,L}^{SP} = \mathcal{I}_c \phi_D^v (x_3) \left\{ \phi_T^v (x_1) \phi_{B_z}^v (x_2) m_2 m_3 (2 m_1^2 x_1 - t x_2 - u \bar{x}_3) \right.\]  
\[+ \phi_T^v (x_1) \phi_{B_z}^v (x_2) m_1 m_3 (t x_1 - 2 m_2^2 x_2 - s \bar{x}_3) \},\]  
\[\mathcal{A}_{c,N}^{SP} = \mathcal{I}_c \phi_D^v (x_3) \left\{ \phi_T^v (x_1) \phi_{B_z}^v (x_2) m_1 m_3 (t x_1 - 2 m_2^2 x_2 - s \bar{x}_3) \right.\]  
\[+ \phi_T^v (x_1) \phi_{B_z}^v (x_2) m_2 m_3 (2 m_1^2 x_1 - t x_2 - u \bar{x}_3) \},\]  
\[\mathcal{A}_{c,T}^{SP} = 2 m_3 \mathcal{I}_c \phi_D^v (x_3) \left\{ \phi_T^v (x_1) \phi_{B_z}^v (x_2) m_1 (x_1 - \bar{x}_3) + \phi_T^v (x_1) \phi_{B_z}^v (x_2) m_2 (\bar{x}_3 - x_2) \right\},\]  
\[\mathcal{A}_{d,L}^{LL} = \mathcal{I}_d \left\{ \phi_T^v (x_1) \phi_{B_z}^v (x_2) \phi_D^v (x_3) \right.\]  
\[+ \phi_T^v (x_1) \phi_{B_z}^v (x_2) \left[ \phi_D^v (x_3) 4 m_1^2 p^2 (x_3 - x_2) - \phi_D^v (x_3) m_3 m_c t \right],\]  
\[\mathcal{A}_{d,N}^{LL} = \mathcal{I}_d \left\{ \phi_T^v (x_1) \phi_{B_z}^v (x_2) \phi_D^v (x_3) \right.\]  
\[+ \phi_T^v (x_1) \phi_{B_z}^v (x_2) \phi_D^v (x_3) 2 m_1 m_2 m_3 m_c \},\]  
\[\mathcal{A}_{d,T}^{LL} = 2 \mathcal{I}_d \phi_T^v (x_1) \phi_{B_z}^v (x_2) \phi_D^v (x_3) \left\{ m_1^2 (x_1 - \bar{x}_3) - m_2^2 (x_2 - \bar{x}_3) \right\},\]  
\[\mathcal{A}_{d,L}^{SP} = \mathcal{I}_d \left\{ \phi_T^v (x_1) \phi_{B_z}^v (x_2) m_2 \left[ \phi_D^v (x_3) m_2 (t x_2 + u x_3 - 2 m_1^2 x_1) - \phi_D^v (x_3) m_c u \right] \right.\]  
\[+ \phi_T^v (x_1) \phi_{B_z}^v (x_2) m_1 \left[ \phi_D^v (x_3) m_3 (2 m_2^2 x_2 + s x_3 - t x_1) - \phi_D^v (x_3) m_c s \right],\]
\[
\mathcal{A}_{d,N}^{SP} = \mathcal{I}_d \left\{ \phi_T^V(x_1) \phi_{B_2}^T(x_2) m_1 \left[ \phi_D^p(x_3) m_3 (2 m_2^2 x_2 + s x_3 - t x_1) - \phi_D^p(x_3) m_c s \right] \\
+ \phi_T^T(x_1) \phi_{B_2}^V(x_2) m_2 \left[ \phi_D^p(x_3) m_3 (t x_2 + u x_3 - 2 m_1^2 x_1) - \phi_D^p(x_3) m_c u \right] \right\}, \quad (A23)
\]

\[
\mathcal{A}_{d,T}^{SP} = 2 \mathcal{I}_d \left\{ \phi_T^V(x_1) \phi_{B_2}^T(x_2) m_1 \left[ \phi_D^p(x_3) m_3 (x_3 - x_1) - \phi_D^p(x_3) m_c \right] \\
+ \phi_T^T(x_1) \phi_{B_2}^V(x_2) m_2 \left[ \phi_D^p(x_3) m_c + \phi_D^p(x_3) m_3 (x_2 - x_3) \right] \right\}, \quad (A24)
\]

\[
\mathcal{A}_{e,L}^{LL} = \mathcal{I}_e \left\{ \phi_{B_2}^V(x_2) \phi_D^a(x_3) \left[ \phi_T^V(x_1) 4 m_1^2 p^2 (x_1 - \bar{x}_3) - \phi_T^V(x_1) m_1 b s \right] \\
+ \phi_T^V(x_1) \phi_{B_2}^T(x_2) \phi_D^p(x_3) m_2 m_3 (2 m_2^2 x_2 - t x_2 - u \bar{x}_3) \right\}, \quad (A25)
\]

\[
\mathcal{A}_{e,N}^{LL} = \mathcal{I}_e \left\{ \phi_T^V(x_1) \phi_{B_2}^T(x_2) \phi_D^p(x_3) m_1 m_3 (t x_1 - 2 m_2^2 x_2 - s \bar{x}_3) \\
- \phi_T^T(x_1) \phi_{B_2}^V(x_2) \phi_D^a(x_3) m_2 m_b u \right\}, \quad (A26)
\]

\[
\mathcal{A}_{e,T}^{LL} = 2 \mathcal{I}_e \left\{ \phi_T^V(x_1) \phi_{B_2}^T(x_2) \phi_D^p(x_3) m_1 m_3 (x_1 - \bar{x}_3) - \phi_T^T(x_1) \phi_{B_2}^V(x_2) \phi_D^p(x_3) m_2 m_b \right\}, \quad (A27)
\]

\[
\mathcal{A}_{e,L}^{LR} = \mathcal{I}_e \left\{ \phi_{B_2}^V(x_2) \phi_D^a(x_3) \left[ \phi_T^V(x_1) t (s x_2 + 2 m_3^2 \bar{x}_3 - u x_1) + \phi_T^V(x_1) m_1 m_b s \right] \\
+ \phi_{B_2}^T(x_2) \phi_D^p(x_3) m_2 m_3 \left[ \phi_T^V(x_1) (t x_2 + u \bar{x}_3 - 2 m_1^2 x_1) + \phi_T^T(x_1) 4 m_1 m_b \right] \right\}, \quad (A28)
\]

\[
\mathcal{A}_{e,N}^{LR} = \mathcal{I}_e \left\{ \phi_{B_2}^V(x_2) \phi_D^a(x_3) m_2 \left[ \phi_T^V(x_1) 2 m_1 (s x_2 + 2 m_3^2 \bar{x}_3 - u x_1) + \phi_T^T(x_1) m_b u \right] \\
+ \phi_{B_2}^T(x_2) \phi_D^p(x_3) m_3 \left[ \phi_T^V(x_1) m_1 (2 m_2^2 x_2 + s \bar{x}_3 - t x_1) + \phi_T^T(x_1) 2 m_b t \right] \right\}, \quad (A29)
\]

\[
\mathcal{A}_{e,T}^{LR} = 2 \mathcal{I}_e \left\{ \phi_{B_2}^T(x_2) \phi_D^p(x_3) m_3 \left[ \phi_T^V(x_1) m_1 (\bar{x}_3 - x_1) + \phi_T^T(x_1) 2 m_b \right] \\
- \phi_T^T(x_1) \phi_{B_2}^V(x_2) \phi_D^a(x_3) m_2 m_b \right\}, \quad (A30)
\]

\[
\mathcal{A}_{e,L}^{SP} = \mathcal{I}_e \left\{ \phi_{B_2}^V(x_2) \phi_D^a(x_3) m_2 \left[ \phi_T^V(x_1) m_b u + \phi_T^V(x_1) m_1 (s x_2 + 2 m_3^2 \bar{x}_3 - u x_1) \right] \\
+ \phi_{B_2}^T(x_2) \phi_D^p(x_3) m_3 \left[ \phi_T^V(x_1) m_b t + \phi_T^V(x_1) m_1 (2 m_2^2 x_2 + s \bar{x}_3 - t x_1) \right] \right\}, \quad (A31)
\]

\[
\mathcal{A}_{e,N}^{SP} = \mathcal{I}_e \left\{ \phi_{B_2}^T(x_2) \phi_D^p(x_3) \left[ \phi_T^V(x_1) m_1 m_b s + \phi_T^V(x_1) \{ m_1^2 s (\bar{x}_3 - x_1) + m_2^2 u (x_2 - \bar{x}_3) \} \right] \\
+ \phi_{B_2}^V(x_2) \phi_D^p(x_3) m_2 m_3 \left[ \phi_T^V(x_1) 2 m_1 m_b + \phi_T^T(x_1) (t x_2 + u \bar{x}_3 - 2 m_1^2 x_1) \right] \right\}, \quad (A32)
\]

\[
\mathcal{A}_{e,T}^{SP} = 2 \mathcal{I}_e \left\{ \phi_{B_2}^T(x_2) \phi_D^p(x_3) \left[ \phi_T^V(x_1) m_1 m_b + \phi_T^T(x_1) \{ m_1^2 (\bar{x}_3 - x_1) + m_2^2 (x_2 - \bar{x}_3) \} \right] \\
+ \phi_T^T(x_1) \phi_{B_2}^V(x_2) \phi_D^a(x_3) m_2 m_3 (\bar{x}_3 - x_2) \right\}, \quad (A33)
\]
\[ A_{f,L}^{LL} = \mathcal{I}_f \left\{ \phi_{B_2}^v(x_2) \phi_D^v(x_3) \left[ \phi_T^v(x_1) t (u \bar{x}_1 - s x_2 - 2 m_2^2 \bar{x}_3) - \phi_T^v(x_1) m_1 m_b s \right] + \phi_{B_2}^v(x_2) \phi_D^p(x_3) m_2 m_3 \left[ \phi_T^v(x_1) (2 m_1^2 \bar{x}_1 - t x_2 - u \bar{x}_3) - \phi_T^v(x_1) 4 m_1 m_b \right] \right\} \] (A34)

\[ A_{f,N}^{LL} = \mathcal{I}_f \left\{ \phi_{B_2}^v(x_2) \phi_D^v(x_3) m_2 \left[ \phi_T^v(x_1) 2 m_1 (u \bar{x}_1 - s x_2 - 2 m_2^2 \bar{x}_3) - \phi_T^v(x_1) m_b u \right] + \phi_{B_2}^v(x_2) \phi_D^p(x_3) m_3 \left[ \phi_T^v(x_1) m_1 (t \bar{x}_1 - 2 m_2^2 x_2 - s \bar{x}_3) - \phi_T^v(x_1) 2 m_b t \right] \right\}, \] (A35)

\[ A_{f,T}^{LL} = 2 \mathcal{I}_f \left\{ \phi_{B_2}^T(x_2) \phi_D^p(x_3) m_3 \left[ \phi_T^v(x_1) m_1 (\bar{x}_1 - \bar{x}_3) - \phi_T^v(x_1) 2 m_b \right] + \phi_{B_2}^T(x_2) \phi_D^p(x_3) m_2 m_b \right\}, \] (A36)

\[ A_{f,L}^{LR} = \mathcal{I}_f \left\{ \phi_{B_2}^v(x_2) \phi_D^o(x_3) \left[ \phi_T^v(x_1) 4 m_1^2 p^2 (\bar{x}_3 - \bar{x}_1) + \phi_T^v(x_1) m_1 m_b s \right] + \phi_{B_2}^v(x_1) \phi_{B_2}^o(x_2) \phi_D^o(x_3) m_2 m_3 (t x_2 + u \bar{x}_3 - 2 m_2^2 \bar{x}_1) \right\}, \] (A37)

\[ A_{f,N}^{LR} = \mathcal{I}_f \left\{ \phi_T^v(x_1) \phi_{B_2}^T(x_2) \phi_D^o(x_3) m_1 m_3 (2 m_2^2 x_2 + s \bar{x}_3 - t \bar{x}_1) + \phi_T^v(x_1) \phi_{B_2}^o(x_2) \phi_D^o(x_3) m_2 m_b u \right\}, \] (A38)

\[ A_{f,T}^{LR} = 2 \mathcal{I}_f \left\{ \phi_{B_2}^T(x_1) \phi_{B_2}^v(x_2) \phi_D^o(x_3) m_2 m_b + \phi_T^v(x_1) \phi_{B_2}^o(x_2) \phi_D^o(x_3) m_1 m_3 (\bar{x}_3 - \bar{x}_1) \right\}, \] (A39)
\[ A_{h,N}^{LL,LR} = m_1 m_2 I_h \phi_{B_2}^V(x_2) \left\{ \phi_{D}^V(x_3) (s + 2 m_3^2 \bar{x}_3) - \phi_{D}^V(x_3) 2 m_3 m_b \right\}, \]  
(A47)

\[ A_{h,T}^{LL,LR} = 2 m_1 m_2 I_h \phi_{B_2}^V(x_2) \phi_{D}^V(x_3), \]  
(A48)

\[ I_a = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 H_a(\alpha_e, \beta_a, b_1, b_2) E_a(t_a) \alpha_s(t_a), \]  
(A49)

\[ I_b = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 H_b(\alpha_e, \beta_b, b_1, b_2) E_b(t_b) \alpha_s(t_b), \]  
(A50)

\[ I_c = \frac{1}{N_c^2} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 
\times H_c(\alpha_e, \beta_c, b_2, b_3) E_c(t_c) \alpha_s(t_c) \delta(b_1 - b_2), \]  
(A51)

\[ I_d = \frac{1}{N_c^2} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 
\times H_d(\alpha_e, \beta_d, b_2, b_3) E_d(t_d) \alpha_s(t_d) \delta(b_1 - b_2), \]  
(A52)

\[ I_e = \frac{1}{N_c^2} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 
\times H_e(\alpha_o, \beta_e, b_1, b_2) E_e(t_e) \alpha_s(t_e) \delta(b_2 - b_3), \]  
(A53)

\[ I_f = \frac{1}{N_c^2} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 
\times H_f(\alpha_o, \beta_f, b_1, b_2) E_f(t_f) \alpha_s(t_f) \delta(b_2 - b_3), \]  
(A54)

\[ I_g = \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 H_g(\alpha_o, \beta_g, b_2, b_3) E_g(t_g) \alpha_s(t_g), \]  
(A55)

\[ I_h = \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 H_h(\alpha_o, \beta_h, b_2, b_3) E_h(t_h) \alpha_s(t_h), \]  
(A56)

where $\bar{x}_i = 1 - x_i$ and $x_i$ are longitudinal momentum fraction of valence quarks; $b_i$ is the conjugate variable of the transverse momentum $k_{iT}$; Sudakov factors $E_i$ are defined as

\[ E_i(t) = \begin{cases} 
\exp\{-S_Y(t) - S_{B_2}(t)\}, & i = a, b \\
\exp\{-S_Y(t) - S_{B_2}(t) - S_D(t)\}, & i = c, d, e, f \\
\exp\{-S_{B_2}(t) - S_D(t)\}, & i = g, h
\end{cases}, \]  
(A57)

\[ S_Y(t) = s(x_1, p_1^+, 1/b_1) + 2 \int_{1/b_1}^t \frac{d\mu}{\mu} \gamma_q, \]  
(A58)

\[ S_{B_2}(t) = s(x_2, p_2^+, 1/b_2) + 2 \int_{1/b_2}^t \frac{d\mu}{\mu} \gamma_q, \]  
(A59)

\[ S_D(t) = s(x_3, p_3^+, 1/b_3) + 2 \int_{1/b_3}^t \frac{d\mu}{\mu} \gamma_q, \]  
(A60)
The definition of functions $H_i$ and scale $t_i$ are the same as that of Ref. [29].

[1] K. Olive et al. (Particle Data Group), Chin. Phys. C 38, 090001 (2014).
[2] S. Herb et al., Phys. Rev. Lett. 39, 252 (1977).
[3] W. Innes et al., Phys. Rev. Lett. 39, 1240 (1977).
[4] C. Patrignani, T. Pedlar and J. Rosner, Annu. Rev. Nucl. Part. Sci. 63, 21 (2013).
[5] H. Li, Phys. Rev. D 52, 3958 (1995).
[6] C. Chang, H. Li, Phys. Rev. D 55, 5577 (1997).
[7] T. Yeh, H. Li, Phys. Rev. D 56, 1615 (1997).
[8] M. Beneke et al., Phys. Rev. Lett. 83, 1914 (1999).
[9] M. Beneke et al., Nucl. Phys. B 591, 313 (2000).
[10] M. Beneke et al., Nucl. Phys. B 606, 245 (2001).
[11] D. Du, D. Yang and G. Zhu, Phys. Lett. B 488, 46 (2000).
[12] D. Du, D. Yang and G. Zhu, Phys. Rev. D 64, 014036 (2000).
[13] C. Bauer et al., Phys. Rev. D 63, 114020 (2001).
[14] C. Bauer, D. Pirjol, I. Stewart, Phys. Rev. D 65, 054022 (2002).
[15] C. Bauer et al., Phys. Rev. D 66, 014017 (2002).
[16] M. Beneke et al., Nucl. Phys. B 643, 431 (2002).
[17] G. Buchalla, A. Buras, M. Lautenbacher, Rev. Mod. Phys. 68, 1125, (1996).
[18] L. Wolfenstein, Phys. Rev. Lett. 51, 1945 (1983).
[19] D. Fakirov and B. Stech, Nucl. Phys. B 133, 315 (1978).
[20] M. Wirbel, B. Stech, and M. Bauer, Z. Phys. C 29, 637 (1985).
[21] J. Bjorken, Nucl. Phys. B (Proc. Suppl.) 11, 325 (1989).
[22] G. Lepage and S. Brodsky, Phys. Lett. B 87, 359 (1979).
[23] G. Lepage and S. Brodsky, Phys. Rev. D 22, 2157 (1980).
[24] A. Duncan and A. Mueller, Phys. Lett. B 90, 159 (1980).
[25] A. Duncan and A. Mueller, Phys. Rev. D 21, 1636 (1980).
[26] A. Efremov and A. Radyushkin, Phys. Lett. B 94, 245 (1980).
[27] T. Kurimoto, H. Li, A. Sanda, Phys. Rev. D 65, 014007 (2001).
[28] P. Ball, and G. Jones, JHEP 0703, 069 (2007).

17
[29] J. Sun et al., Phys. Lett. B 752, 322 (2016).
[30] G. Lepage et al., Phys. Rev. D 46, 4052 (1992).
[31] Y. Yang et al., Phys. Lett. B 751, 171 (2015).
[32] R. Dowdall et al. (HPQCD Collaboration), Phys. Rev. D 86, 094510 (2012).
[33] B. Colquhoun et al. (HPQCD Collaboration), Phys. Rev. D 91, 114509 (2015).
[34] A. Kamal, Particle physics, Springer, p.298 (2014).
[35] R. Aaij et al. (LHCb Collaboration), JHEP 1407, 094 (2014).
[36] B. Abelev et al. (ALICE Collaboration), Phys. Lett. B 740, 105 (2015).