Nested Bethe Ansatz for RTT-Algebra of $U_q(\text{sp}(4))$ Type

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Abstract—We study the highest weight representations of the RTT-algebras for the R-matrix of $U_q(\text{sp}(4))$ type by the nested algebraic Bethe ansatz. It is a generalization of our study for R-matrix of $U_q(\text{gl}(n))$.

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1. INTRODUCTION

The formulation of the quantum inverse scattering method, or algebraic Bethe ansatz, by the Leningrad school [1] provides eigenvectors and eigenvalues of the transfer matrix. The latter is the generating function of the conserved quantities of a large family of quantum integrable models. The transfer matrix eigenvectors are constructed from the representation theory of the RTT-algebras. In order to construct these eigenvectors, one should first prepare Bethe vectors depending on a set of complex variables. The first formulation of the Bethe vectors for the $q$-invariant models was given by P.P. Kulish and N.Yu. Reshetikhin in [2] where the nested algebraic Bethe ansatz was introduced. These vectors are given by recursion on the rank of the algebra. Our calculation is some $q$-generalization of the construction which we published in our recent a work [6] for the non-deformed case of $\text{sp}(4)$. Our construction of Bethe vectors used the new RTT-algebra $\tilde{\text{sp}}(4)$ which is defined in Section 2 and is not the RTT-subalgebra of $\text{sp}(4)$.

Recently, we dealt with the Nested–Bethe ansatz for the RTT-algebra of $U_q(\text{sp}(2n))$ type [9]. In this part, we briefly summarize the results of this work in the case of the RTT-algebra of $U_q(\text{sp}(4))$ type.

The R-matrix of $U_q(\text{sp}(4))$ type has the form

$$R(x) = \frac{1}{\alpha(x)} \left( \sum_{i,k} E_i^k \otimes E_k^i + f(x) \sum_i E_i^i \otimes E_i^i \right. + f(x^{-1}) \sum_i E_i^i \otimes E_i^{-i} + g(x) \sum_{i<k} E_i^k \otimes E_k^i - g(x^{-1}) \sum_{i<k} E_k^i \otimes E_i^k \right. \right) \right)$$

where the census indices take place $i, k = \pm 1, \pm 2$, $\epsilon_i = \text{sgn}(i)$ and

$$f(x) = \frac{x q - x^{-1}}{x - x^{-1}}, \quad g(x) = \frac{x(q - q^{-1})}{x - x^{-1}}$$

$$\alpha(x) = 1 + \frac{q - q^{-1}}{x - x^{-1}}$$

This R-matrix satisfies the Yang–Baxter equation

$$R_{1,2}(x)R_{1,3}(xy)R_{2,3}(y) = R_{2,3}(y)R_{1,3}(xy)R_{1,2}(x)$$

and is invertible. Therefore, by using the RTT-equation

$$R_{1,2}(xy^{-1})T_1(x)T_2(y) = T_2(y)T_1(x)R_{1,2}(xy^{-1})$$

where $T(x) = \sum_{i,k=-n} E_i^k \otimes T_k^i(x)$.
we define the RTT-algebra of $U_q(\text{sp}(4))$ type. From the invertibility of the R-matrix we have that the operator

$$H(x) = \text{Tr}(T(x)) = \sum_{i=2}^{2} T_i(x)$$

fulfills the equations $H(x)H(y) = H(y)H(x)$ for any $x$ and $y$.

We suppose that in the representation space $\mathcal{W}$ of the RTT-algebra $\mathcal{A}$ there exists a vacuum vector $\omega \in \mathcal{W}$, for which $\mathcal{W} = \mathcal{A}\omega$ and

$$T_i(x)\omega = 0 \quad \text{for} \quad i < k,$$

$$T_i(x)\omega = \lambda_i(x)\omega \quad \text{for} \quad i = \pm 1, \pm 2.$$

In the vector space $\mathcal{W} = \mathcal{A}\omega$, we will look for eigenvectors of $H(x)$.

In [9] we showed that if we restrict our considerations to the space $\mathcal{W}_0 = \mathcal{A}^{(+)\omega} \subset \mathcal{W} = \mathcal{A}\omega$, where RTT-subalgebras $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$ are generated by $T_i(x)$ and $T_i^-(x)$, where $i, k = 1, 2$, it is possible to write commutation relations between

$$T^{(+)i}(x) = \sum_{i,k=1}^{2} E_i \otimes T_k^i$$

and

$$T^{(-)i}(x) = \sum_{i,k=1}^{2} E_i^\dagger \otimes T_k^{-i}(x)$$

in the form of RTT-equations

$$R^{(\epsilon_1, \epsilon_2)}_{12}(xy^{-1})T_{12}^{(\epsilon_1)}(x)T_{12}^{(\epsilon_2)}(y) = T_{12}^{(\epsilon_1)}(y)T_{12}^{(\epsilon_2)}(x)R^{(\epsilon_1, \epsilon_2)}_{12}(xy^{-1}),$$

where $\epsilon_1, \epsilon_2 = \pm$ and

$$R^{(+)}(x) = \frac{1}{f(x)} \left[ \sum_{i,j=1 \leq i \neq j} E_i \otimes E_j + f(x) \sum_{i,j=1} E_i \otimes E_j \right. \left. + g(x)E_1^2 \otimes E_2^2 - g(x^{-1})E_1^2 \otimes E_2^2 \right],$$

$$R^{(-)}(x) = \frac{1}{f(x)} \left[ \sum_{i,j=1 \leq i \neq j} E_i \otimes E_j^\dagger + f(x) \sum_{i,j=1} E_i \otimes E_j^\dagger \right. \left. + g(x)E_1^\dagger \otimes E_2^{-1} - g(x^{-1})E_1^\dagger \otimes E_2^{-1} \right],$$

$$R^{(\cdot \cdot)}(x) = \frac{1}{f(x)} \left[ \sum_{i,j=1 \leq i \neq j} E_i \otimes E_j + f(x^{-1}) \sum_{i,j=1} E_i \otimes E_j \right. \left. + qg(x^{-1})E_1^\dagger \otimes E_2^{-1} - q^{-1}g(xq^{-1})E_1^\dagger \otimes E_2^{-1} \right. \left. + q^{-1}g(xq^{-1})E_2^{-1} \otimes E_1^\dagger - qg(x^{-1})E_2^{-1} \otimes E_1^\dagger \right].$$

The RTT-equation (2) can be written in the form of a single RTT-equation

$$\tilde{R}^{(+)}_{12}(xy^{-1})\tilde{T}_1(x)\tilde{T}_2(y) = \tilde{T}_2(y)\tilde{T}_1(x)\tilde{R}^{(+)}_{12}(xy^{-1}),$$

where

$$\tilde{R}(x) = R^{(+,+)}(x) + R^{(+,+)}(x) + R^{(+,-)}(x) + R^{(-,-)}(x),$$

$$\tilde{T}(x) = T^{(+)}(x) + T^{(-)}(x).$$

Since the R-matrix $\tilde{R}(x)$ satisfies the Yang–Baxter equation

$$\tilde{R}^{(+)}_{12}(x)\tilde{R}^{(+)}_{13}(y)\tilde{R}^{(+)}_{23}(z) = \tilde{R}^{(+)}_{23}(z)\tilde{R}^{(+)}_{13}(y)\tilde{R}^{(+)}_{12}(x),$$

and is invertible, we can define the RTT-algebra denoted by $\tilde{\mathcal{A}}$. If we want to point out that $T^{(+)}(x)$ is an element of the RTT–algebra $\tilde{\mathcal{A}}$, we will write

$$T^{(+)}(x) = \sum_{i,k=1}^{2} E_i \otimes T_k^i(x),$$

$$T^{(-)}(x) = \sum_{i,k=1}^{2} E_i^\dagger \otimes T_k^{-i}(x).$$

In the standard way by using (2) we obtain that in the RTT-algebra $\tilde{\mathcal{A}}$ the operators $\tilde{H}^{(+)}(x)$ and $\tilde{H}^{(-)}(y)$ commute with each other.

We look for Bethe vectors in the form

$$\mathcal{V}(u) = \langle B_{u_1 \ldots u_M}(u), \Phi \rangle,$$

where $u = (u_1, u_2, \ldots, u_M)$ are different complex numbers,

$$B_{u_1 \ldots u_M}(u) = \frac{1}{\prod_{i_1, \ldots, M} f^{i_1} \otimes \cdots \otimes f^{i_M} \otimes e_{-i_1} \otimes \cdots \otimes e_{-i_M}} \Phi \in \mathcal{W},$$

where $\Phi \in \mathcal{W}$ and $e_i$ is the basis of space $\mathcal{V}^+$, $e_{-i}$ is the basis of space $\mathcal{V}^-$, and $f_i$ and $f_{-i}$ are dual bases in spaces $\mathcal{V}^+$ and $\mathcal{V}^-$, respectively.
In [9] we introduced for any \( u \) the operators
\[
\hat{T}_{0,1,...,M}^{(+)}(x;u) = \hat{R}_{0,1,...,M}^{(+)}(xu^{-1}) \cdots \hat{R}_{0,M}^{(+)}(xu_M^{-1}),
\]
\[
\times \hat{T}_{0,1,...,M}^{(-)}(x) \hat{R}_{0,M}^{(-)}(xu_M^{-1}) \cdots \hat{R}_{0,1}^{(-)}(xu^{-1}),
\]
\[
\hat{T}_{0,1,...,M}^{(+)}(x;u) = \hat{R}_{0,1,...,M}^{(+)}(xu^{-1}) \cdots \hat{R}_{0,M}^{(+)}(xu_M^{-1}) \hat{T}_{0}^{(-)}(x)
\]
\[
\times \hat{R}_{0,M}^{(-)}(xu_M^{-1}) \cdots \hat{R}_{0,1}^{(-)}(xu^{-1}),
\]
where
\[
\hat{R}_{0,1}^{(+)}(x) = \frac{1}{f(x^{-1})} \left( \sum_{i,k=1}^{2} E_{i}^{+} \otimes F_{k}^{-} \otimes L_{-} \right.
\]
\[
+ f(x^{-1}) \sum_{i=1}^{2} E_{i}^{-} \otimes F_{i}^{+} \otimes L_{+} + g(x^{-1})E_{2}^{+} \otimes F_{1}^{-} \otimes L_{-},
\]
\[
\hat{R}_{0,1}^{(-)}(x) = \sum_{i,k=1}^{2} E_{i}^{-} \otimes F_{k}^{+} \otimes L_{-} + f(xq) \sum_{i=1}^{2} E_{i}^{-} \otimes F_{i}^{+} \otimes L_{+}
\]
\[
+ gg(xq)E_{2}^{+} \otimes F_{1}^{-} \otimes L_{-},
\]
\[
\hat{T}_{0,1,...,M}^{(+)}(x;u) = \sum_{i,k=1}^{2} E_{i}^{+} \otimes F_{k}^{-} \otimes E_{-}^{-} + f(x^{-1}) \sum_{i=1}^{2} E_{i}^{+} \otimes F_{i}^{-} \otimes E_{-}^{-}
\]
\[
+ g(x^{-1})E_{2}^{+} \otimes F_{1}^{-} \otimes E_{-}^{-},
\]
\[
\hat{T}_{0,1,...,M}^{(-)}(x;u) = \frac{1}{f(x)} \left( \sum_{i,k=1}^{2} E_{i}^{-} \otimes F_{k}^{+} \otimes E_{-}^{-}
\]
\[
+ f(x) \sum_{i=1}^{2} E_{i}^{-} \otimes F_{i}^{+} \otimes E_{-}^{-} + g(x)E_{2}^{-} \otimes F_{1}^{+} \otimes E_{-}^{-},
\]
and define the operators \( \hat{T}_{k}^{(+)}(x;u) \) and \( \hat{T}_{k}^{(-)}(x;u) \) by the relationships
\[
\hat{T}_{0,1,...,M}^{(+)}(x;u) = \sum_{i,k=1}^{2} E_{i}^{+} \otimes F_{k}^{-} \otimes L_{-},
\]
\[
\hat{T}_{0,1,...,M}^{(-)}(x;u) = \sum_{i,k=1}^{2} E_{i}^{-} \otimes F_{k}^{+} \otimes L_{-}.
\]

For organized \( M \)-tuples \( u = (u_1, ..., u_M) \) denote by \( \overline{u} \) the set \( \overline{u} = \{ u_1, ..., u_M \} \), define
\[
\overline{u}_k = \overline{u} \setminus \{ u_k \} = \{ u_1, ..., u_{k-1}, u_{k+1}, ..., u_M \},
\]
\[
F(x;\overline{u}^{-1}) = \prod_{k=1}^{M} f(x u_k^{-1}),
\]
\[
F(x^{-1},\overline{u}) = \prod_{k=1}^{M} f(x^{-1} u_k).
\]

One of the main results of [9] is

**Proposition 1.** Let \( \Phi \) be a common eigenvector of the operators
\[
\hat{H}_{0,1,...,M}^{(+)}(x;u) = \hat{T}_{0}^{(+)}(x;u) = \hat{T}_{0}^{(-)}(x;u) = \hat{T}_{1}^{(-)}(x;u) + \hat{T}_{2}^{(-)}(x;u),
\]
\[
\hat{H}_{0,1,...,M}^{(+)}(x;u) = \hat{T}_{0}^{(+)}(x;u) = \hat{T}_{1}^{(-)}(x;u) + \hat{T}_{2}^{(-)}(x;u)
\]
with eigenvalues \( \hat{E}_{0,1,...,M}^{(+)}(x;u) \) and \( \hat{E}_{0,1,...,M}^{(-)}(x;u) \). If the relations
\[
F(u_k^{-1},\overline{u}_k) \hat{E}_{0,1,...,M}^{(+)}(u_k;\overline{u}_k) = F(u_k,\overline{u}_k) \hat{E}_{0,1,...,M}^{(-)}(u_k;\overline{u}_k)
\]
are true for each \( u_k \in \overline{u} \), then \( \langle B_{0,1,...,M}(u), \Phi \rangle \) is the eigenvector of \( H(x) = H^{(+)}(x) + H^{(-)}(x) \) with eigenvalue
\[
\hat{E}_{0,1,...,M}(x;\overline{u}) = F(x^{-1};\overline{u}) \hat{E}_{0,1,...,M}^{(+)}(x;u)
\]
\[
+ F(x;\overline{u}) \hat{E}_{0,1,...,M}^{(-)}(x;u).
\]

So to find eigenvectors of the operator \( H(x) \), it is enough to find common eigenvectors of the operators \( \hat{H}_{0,1,...,M}^{(+)}(x;u) \) and \( \hat{H}_{0,1,...,M}^{(-)}(x;u) \).

Other important results of [9] are the RTT-equations
\[
\hat{R}_{0,1}^{(e,e^{'})}(x;u) \hat{R}_{0,1}^{(e,e^{'})}(x;u) = \hat{R}_{0,1}^{(e,e^{'})}(x;u) \hat{R}_{0,1}^{(e,e^{'})}(x;u),
\]
\[
\hat{R}_{0,1}^{(e,e^{'})}(x;u) \hat{R}_{0,1}^{(e,e^{'})}(x;u) = \hat{R}_{0,1}^{(e,e^{'})}(x;u) \hat{R}_{0,1}^{(e,e^{'})}(x;u),
\]
which hold for any \( e, e^{'} = \pm \) and for any \( u \). It means that the operators \( \hat{T}_{k}^{(+)}(x;u) \) and \( \hat{T}_{k}^{(-)}(x;u) \) generate the RTT-algebra \( \hat{A}_2 \) for any \( u \).

Finally, it is shown in [9] that for the vector
\[
\hat{\Omega} = \bigotimes_{M}^{M} t^{i} \otimes \bigotimes_{M}^{M} e_{-}^{i} \otimes \bigotimes_{M}^{M} e_{-}^{-} \otimes \omega
\]
we have
\[
\hat{T}_{k}^{(+)}(x;u) \hat{\Omega} = 0, \quad \hat{T}_{k}^{(-)}(x;u) \hat{\Omega} = \mu_{k}(x;u) \hat{\Omega}
\]
for \( k = 1, 2 \)
\[
\hat{T}_{k}^{(+)}(x;u) \hat{\Omega} = 0, \quad \hat{T}_{k}^{(-)}(x;u) \hat{\Omega} = \mu_{-k}(x;u) \hat{\Omega}
\]
for \( k = 1, 2 \),
where
\[
\mu_{1}(x;\overline{u}) = \lambda_{1}(x) F(x^{-1} q^{-1};\overline{u}),
\]
\[
\mu_{2}(x;\overline{u}) = \lambda_{2}(x) F(xq^{-1};\overline{u}),
\]
\[
\mu_{-1}(x;\overline{u}) = \lambda_{-1}(x) F(xq^{-1};\overline{u}),
\]
\[
\mu_{-2}(x;\overline{u}) = \lambda_{-2}(x) F(x^{-1} q^{-1};\overline{u}),
\]
i.e. \( \hat{\Omega} \) is a vacuum vector for the representation of the RTT-algebra \( \hat{A}_2 \).

So to find our own vectors of the operator \( H(x) \) for the RTT-algebra of \( U_q(sp(4)) \) type, just formulate the Bethe ansatz for the RTT-algebra \( \hat{A}_2 \).
3. COMMON EIGENVECTORS
OF THE OPERATORS \( \hat{H}^{(+)}(x) \) AND \( \hat{H}^{(-)}(x) \)
IN THE RTT-ALGEBRA \( \mathcal{A}_2 \)

It is possible from the commutation relations in the RTT-algebra \( \mathcal{A}_2 \) to prove that for each \( x \) and \( y \)

\[
\hat{T}_1^2(x)\hat{T}_1^2(y) = \hat{T}_1^2(y)\hat{T}_1^2(x),
\]

\[
\hat{T}_2^{-1}(x)\hat{T}_2^{-1}(y) = \hat{T}_2^{-1}(y)\hat{T}_2^{-1}(x),
\]

\[
\hat{T}_1^2(x)\hat{T}_2^{-1}(y) = \hat{T}_2^{-1}(y)\hat{T}_1^2(x),
\]

\[
\hat{T}_1^2(x)\hat{T}_1^2(y) = \hat{T}_2^{-1}(y)\hat{T}_1^2(x)
\]

hold.

Let \( \tilde{\omega} \) be a vacuum vector for the representation of the RTT-algebra \( \mathcal{A}_2 \), i.e. we have

\[
\hat{T}_1^2(x)\tilde{\omega} = \hat{T}_2^{-1}(x)\tilde{\omega} = 0,
\]

\[
\hat{T}_2^{-1}(x)\tilde{\omega} = \mu_2(x)\tilde{\omega}, \quad i = 1, 2.
\]

Common eigenvectors of the operators \( \hat{H}^{(+)}(x) \) and \( \hat{H}^{(-)}(x) \) will be searched for in the form

\[
| \varpi; \omega \rangle = \hat{T}_1^2(v_1)\hat{T}_1^2(v_2)\ldots \hat{T}_2^{-1}(v_p)\tilde{\omega} = \hat{T}_2^{-1}(w_1)\ldots \hat{T}_2^{-1}(w_q)\tilde{\omega},
\]

where \( \varpi \) and \( \omega \) are the sets \( \varpi = \{v_1, v_2, \ldots, v_p\} \) and \( \omega = \{w_1, w_2, \ldots, w_q\} \).

**Proposition 2.** For any \( x, \varpi \) and \( \omega \) we have

\[
\hat{T}_1^2(x)| \varpi; \omega \rangle = \mu_1(x)F(x; \varpi^{-1})F(xq^{-2}; \omega^{-1})| \varpi; \omega \rangle
\]

\[-\sum_{v_r \in \varpi} \mu_2(v_r)g(xv_r^{-1})F(v_r; \varpi_r^{-1})F(v_rq^{-2}; \omega)| x, \varpi; \omega \rangle
\]

\[\times \sum_{w_s \in \omega} \mu_2(w_s)g(xw_s^{-1}q^{-2})F(w_sq^{-2}; \varpi)| x, \varpi; \omega \rangle
\]

\[\hat{T}_2^{-1}(x)| \varpi; \omega \rangle = \mu_2(x)F(x^{-1}; \varpi)F(xq^{-2}; \omega)| \varpi; \omega \rangle
\]

\[\sum_{v_r \in \varpi} \mu_2(v_r)g(xv_r^{-1}q^{-2})F(v_r^{-1}; \varpi_r)F(v_rq^{-2}; \omega)| x, \varpi; \omega \rangle
\]

\[-\sum_{w_s \in \omega} \mu_1(w_s)g(xw_s^{-1})F(w_s^{-1}q^{-2})F(w^{-1}q^{-2}; \varpi)| \varpi; x, \omega \rangle
\]

\[\hat{T}_2^{-1}(x)| \varpi; \omega \rangle = \mu_1(x)F(x^{-1}; \varpi)F(xq^{-2}; \omega)| \varpi; \omega \rangle
\]

\[-\sum_{v_r \in \varpi} \mu_2(v_r)g(xv_r^{-1}q^{-2})F(v_r^{-1}; \varpi_r)F(v_rq^{-2}; \omega)| x, \varpi; \omega \rangle
\]

\[\hat{T}_1^2(x)| \varpi; \omega \rangle = \mu_2(x)F(x^{-1}; \varpi)F(xq^{-2}; \omega)| \varpi; \omega \rangle
\]

\[-\sum_{w_s \in \omega} \mu_2(w_s)g(xw_s^{-1}q^{-2})F(w^{-1}q^{-2}; \varpi)| x, \omega \rangle
\]

From this statement we obtain for the action of the operators \( \hat{H}^{(\pm)}(x) \)

\[
\hat{H}^{(+)}(x)| \varpi; \omega \rangle = \hat{T}_1^2(x)| \varpi; \omega \rangle + \hat{T}_2^{-1}(x)| \varpi; \omega \rangle
\]

\[= (\mu_1(x)F(x; \varpi^{-1})F(xq^{-2}; \omega^{-1})
\]

\[+ \mu_2(x)F(x^{-1}; \varpi)F(xq^{-2}; \omega^{-1})
\]

\[- \sum_{v_r \in \varpi} g(xv_r^{-1})F(v_r; \varpi_r^{-1})F(v_rq^{-2}; \omega)-\mu_2(v_r)F(v_r^{-1}; \varpi_r)F(v_rq^{-2}; \omega)| x, \varpi; \omega \rangle
\]

\[- \sum_{w_s \in \omega} g(xw_s^{-1}q^{-2})(\mu_1(w_s)F(w_s^{-1}q^{-2}; \varpi)| w_s^{-1}q^{-2}; \varpi \rangle
\]

\[\hat{H}^{(-)}(x)| \varpi; \omega \rangle = \hat{T}_2^{-1}(x)| \varpi; \omega \rangle
\]

\[= (\mu_1(x)F(x^{-1}; \varpi)F(xq^{-2}; \omega) + \mu_2(x)F(x^{-1}; \varpi)F(xq^{-2}; \omega)| \varpi; \omega \rangle
\]

\[\times F(x^{-1}; \varpi)| \varpi; \omega \rangle + \sum_{v_r \in \varpi} g(xv_r^{-1}q^{-2})(\mu_1(v_r)F(v_r^{-1}; \varpi)| x, \omega \rangle
\]

\[\times F(v_r^{-1}; \varpi)| x, \omega \rangle + \sum_{w_s \in \omega} g(xw_s^{-1}q^{-2})(\mu_1(w_s)F(w_s^{-1}q^{-2}; \varpi)| x, \omega \rangle
\]

\[\times F(w_s^{-1}q^{-2}; \varpi)| x, \omega \rangle - \mu_2(w_s)F(w_s^{-1}q^{-2}; \varpi)| \varpi; x, \omega \rangle
\]

and the following statement:

**Proposition 3.** If for each \( v_r \in \varpi \) and \( w_s \in \omega \) the Bethe conditions are fulfilled

\[
\mu_1(v_r)F(v_r; \varpi_r^{-1})F(v_rq^{-2}; \omega^{-1})
\]

\[= \mu_2(v_r)F(v_r^{-1}; \varpi_r)F(v_rq^{-2}; \omega),
\]

\[\mu_1(w_s)F(w_s^{-1}q^{-2}; \varpi)| w_s^{-1}q^{-2}; \varpi \rangle
\]

(4)

the vectors \( | \varpi; \omega \rangle = \hat{T}_1^2(\varpi)| \omega \rangle\hat{T}_2^{-1}(\omega)\tilde{\omega} \) are common eigenvectors of the operators \( \hat{H}^{(+)}(x) \) and \( \hat{H}^{(-)}(x) \) with eigenvalues

\[
\hat{E}^{(+)}(x; \varpi; \omega) = \mu_1(x)F(x; \varpi^{-1})F(xq^{-2}; \omega^{-1})
\]

\[+ \mu_2(x)F(x^{-1}; \varpi)F(xq^{-2}; \omega),
\]

\[
\hat{E}^{(-)}(x; \varpi; \omega) = \mu_1(x)F(x^{-1}; \varpi)F(xq^{-2}; \omega)
\]

\[+ \mu_2(x)F(xq^{-2}; \varpi)| x, \omega \rangle.
\]

4. BETHE CONDITIONS
AND BETHE EIGENVECTORS
FOR THE RTT-ALGEBRA OF \( U_q(sp(4)) \) TYPE

In Section 2, we have mentioned that the operators \( \hat{T}_1^2(x; \mu) \) and \( \hat{T}_2^{-1}(x; \mu) \) generate \( \mu \) the RTT-algebra \( \mathcal{A}_2 \) for each and the vector \( \tilde{\Omega} \) is the vacuum vector with weights
\[
\begin{align*}
\mu_i(x;\overline{u}) &= \lambda_i(x)F(x^{-1}q;\overline{u}), \\
\mu_2(x;\overline{u}) &= \lambda_2(x)F(xq^{-1};\overline{u}^{-1}), \\
\mu_{-1}(x;\overline{u}) &= \lambda_{-1}(x)F(xq;\overline{u}), \\
\mu_{-2}(x;\overline{u}) &= \lambda_{-2}(x)F(x^{-1}q^{-1};\overline{u}).
\end{align*}
\]

Proposition 4 says that if for each \( v, w \) the Bethe conditions are fulfilled

\[
\mu_i(v;\overline{v})F(v^{-1}q^{-2};\overline{v}^{-1}) = \mu_2(v;\overline{v})F(v^{-1}q^{-2};\overline{v}),
\]

then the vectors \( \Phi(u;\overline{v};\overline{w}) = \hat{T}^{(+)\,1}_{1}(u;\overline{v})\hat{T}^{(-)\,1}_{-1}(u;\overline{w})\tilde{\Omega} \) are common eigenvectors of the operators \( \hat{H}^{(+)}_{-1}(x;u) \) and \( \hat{H}^{(-)}_{1}(x;u) \) with eigenvalues

\[
\begin{align*}
\hat{E}^{(+)}_{-1}(x;u;\overline{v};\overline{w}) &= \mu_i(x;\overline{u})F(x^{-1}q;\overline{v})F(xq^{-1};\overline{w}^{-1}) \\
&+ \mu_2(x;\overline{u})F(x^{-1};\overline{v})F(x^{-1}q^{-2};\overline{v}^{-1}), \\
\hat{E}^{(-)}_{1}(x;u;\overline{v};\overline{w}) &= \mu_{-1}(x;\overline{u})F(x^{-1}q^{-2};\overline{v})F(x;\overline{v}^{-1}) \\
&+ \mu_{-2}(x;\overline{u})F(xq^{-2};\overline{v}^{-1})F(x;\overline{v}).
\end{align*}
\]

From relation (3) it follows that if for each \( u_k \in \overline{u} \) we have

\[
\begin{align*}
F(u_k^{-1};\overline{u}_k)\hat{E}^{(+)}_{-1}(u_k;u;\overline{v};\overline{w}) &= F(u_k;\overline{u}_k)\hat{E}^{(-)}_{1}(u_k;u;\overline{v};\overline{w})
\end{align*}
\]

then the vector

\[
\Psi(u;\overline{v};\overline{w}) = (B^{(-)}_{-1})(u,\Phi(u;\overline{v};\overline{w}))
\]

is the eigenvector of the operator \( H(x) \). From this we obtain the following theorem:

**Theorem.** Let the Bethe condition

\[
\begin{align*}
\lambda_i(u_i)F(u_i^{-1};\overline{u}_i)F(u_iq;\overline{u}_i)F(u_i;\overline{v})F(u_iq^{-2};\overline{w}^{-1}) \\
&= \lambda_{-1}(u_k)F(u_k;\overline{u}_k)F(u_k;\overline{u}_k)F(u_k^{-1}q^{-2};\overline{u}_k)F(u_k^{-1};\overline{v}), \\
\lambda_i(v_i)F(v_i^{-1}q^{-1};\overline{v})F(v_i^{-1};\overline{v}^{-1})F(v_iq^{-2};\overline{v}^{-1}) \\
&= \lambda_{-2}(v)F(vq^{-2};\overline{v})F(v^{-1};\overline{v})F(v^{-1}q^{-2};\overline{v}) \\
&= \lambda_{-1}(w_i)F(w_iq^{-1};\overline{u})F(w_iq^{-2};\overline{u}_i)F(w_i^{-1};\overline{v}) \\
&= \lambda_{-2}(w_i)F(w_i^{-1}q^{-1};\overline{u})F(w_iq^{-2};\overline{u}_i)F(w_i^{-1};\overline{v})
\end{align*}
\]

be fulfilled for any \( u_k \in \overline{u}, v_i \in \overline{v} \) and \( w_i \in \overline{w} \), then the vectors (5) are eigenvectors of \( H(x) \) with eigenvalues

\[
\begin{align*}
E(x;\overline{u};\overline{v};\overline{w}) &= \lambda_i(x)F(x^{-1};\overline{u})F(x^{-1}q;\overline{u})F(x;\overline{v}), \\
&+ F(xq^{-2};\overline{u})F(x^{-1}q^{-2};\overline{v})F(x;\overline{v}) + F(x;\overline{v})F(Fx^{-1}q^{-2};\overline{v})F(x;\overline{v}), \\
&+ \lambda_{-2}(x)F(x;\overline{u})F(x^{-1}q^{-2};\overline{v})F(xq^{-2};\overline{v})F(x;\overline{v}).
\end{align*}
\]

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