Simulating Interaction of Liquid Steel with Gate Wall at Harmonic Motion

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Abstract. The problem of determining the forces of interaction of a viscous fluid with the cylindrical pipe wall is considered. It is assumed that near the pipe wall, the fluid motion is completely determined by viscous forces. The pipe moves along the streamline. The annular fluid element motion law is a special case of the Navier–Stokes equation in a cylindrical coordinate system. The equation is solved by the Fourier method in Bessel functions. Considering the orthogonality of the eigenfunctions, an equation for the squared norm is found. As an example, the case is considered when the pipe is subjected to vibration. Equations have been obtained for the velocities and viscous friction forces in the laminar sublayer. It has been found that when the pipe moves harmonically, the velocities and shear stresses at the pipe wall do not reach their maximum synchronously. The distribution of velocities and stresses in the section of the steel-pouring ladle gate channel has been considered for three vibration modes. The solution provided can be, in particular, used to determine the fluid–pipe wall interaction forces when the pipe is technologically affected by vibration, impulse, etc., as well as study moving joints such as piston, plunger, etc.

1. Introduction

In the steel casting, there is a problem of the skull formation in the gate channel. A way to reduce the skull formation rate is vibration action on the gate, which has earlier been established by studies performed on the physical model of the steel-pouring ladle gate [1]. To estimate the efficiency of such an action, the dependence of the viscous friction forces between the channel wall and liquid steel on the vibration characteristics should be determined.

1.1. Relevance

Many studies, both experimental [2-4] and theoretical, using numerical [5, 6] and analytical [7-10] techniques, are devoted to the problems of viscous fluid mechanics. In [11, 12], the dynamics of a viscous fluid under circular motions of a cylinder is studied using a numerical and analytical approach to solve the problem; however, the examples considered in the papers do not give a universal solution but are only particular case studies. The motion of liquids relative to solids is a complex and multifactorial process. However, to consider various applied problems, provisions and hypotheses are widely used, simplifying the mathematical description of the process, which allows obtaining simpler engineering dependencies to determine the desired quantities.
1.2. Formulation of the problem

Consider the liquid metal motion along the gate channel as a viscous liquid flow in a smooth cylindrical pipe. According to one of the widespread models [2, 4, 13], we assume that the entire flow in the pipe is divided into two characteristic regions: 1) the flow core, where the flow is turbulent and the effect of viscosity is negligible, and 2) the near-wall layer (laminar sublayer), where the motion is entirely determined by the viscosity forces (Fig. 1)

\[ \tau = \eta \frac{\partial \mathbf{v}}{\partial r}, \]

where \( \eta \) is the dynamic viscosity. The conditional interlayer zone will be determined by the \( R^* \) size (Fig. 1). The law of the annular fluid element motion in a laminar sublayer at a distance \( r \) from the pipe center is a particular form of the Navier–Stokes equations in cylindrical coordinates [13]

\[ \frac{\nu}{\rho} \frac{\partial}{\partial r} \left( r \frac{\partial \mathbf{v}}{\partial r} \right) - r \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad (1) \]

where \( \nu = \eta / \rho \) is the kinematic viscosity, \( \rho \) is the fluid density.

For convenience, in equation (1), pass to the dimensionless quantities

\[ \frac{\partial}{\partial \xi} \left( \xi \frac{\partial \mathbf{v}}{\partial \xi} \right) - \xi \frac{\partial \mathbf{v}}{\partial \xi} = -\xi \frac{\partial \bar{p}}{\partial \xi}, \quad (2) \]

where \( \xi = r / R \) is the relative radius, \( \tilde{v} = vR / \nu \) is the dimensionless velocity, \( \tilde{t} = tv / R^2 \) is the dimensionless time, \( \chi = x / R \) is the relative longitudinal coordinate, \( \bar{p} = pR^2 / (\nu \eta) \) is the dimensionless pressure.

The below homogeneous equation corresponds to the non-homogeneous equation (2)

\[ \frac{\partial}{\partial \xi} \left( \xi \frac{\partial \mathbf{v}}{\partial \xi} \right) - \xi \frac{\partial \mathbf{v}}{\partial \xi} = 0. \quad (3) \]

\[\text{Figure 1. Diagram of the movement of liquid metal along the gate channel.}\]
We solve equation (3) by the Fourier method as a series of the product of two functions

\[ \bar{v}(\xi, \tau) = \sum_{n=1}^{\infty} W_n(\xi) W_n(\tau). \]  

Substituting series (4) into equation (3) and dividing the variables, we obtain

\[ \frac{\xi W_n'' + W_n'}{\xi W_n} = \frac{\dot{T}_n}{T_n} = -\lambda_n^2, \]

where \( \lambda_n^2 \) is the division constant, the prime and the dot denote the derivative with respect to \( \xi \) and \( \tau \), respectively.

Thus, equation (3) is divided into two homogeneous differential equations

\[ \dot{T}_n + \lambda_n^2 T_n = 0; \quad \xi W_n'' + W_n' + \lambda_n^2 \xi W_n = 0. \]  

The solution to equation (5) is known

\[ T_n(\tau) = A_n \exp\left(-\lambda_n^2 \tau\right). \]  

Equation (6) is solved in the form [14]

\[ W_n(\xi) = C_1 J_0(\lambda_n \xi) + C_2 Y_0(\lambda_n \xi). \]  

For the case of fluid flow in a cylindrical pipe considered, the boundary conditions have the form

\[ \nu(R^*, \tau) = 0; \quad \nu(R, \tau) = 0. \]  

For the convenience of solving the boundary value problem (1) with conditions (9), we rewrite equation (8) as follows

\[ W_n(\xi) = C_1 A(\lambda_n \xi) + C_2 B(\lambda_n \xi), \]

where

\[ A(\xi) = \frac{\pi}{2} \xi_0 (Y_0(\xi) J_1(\xi_0) - J_0(\xi) Y_1(\xi_0)); \]
\[ B(\xi) = \frac{\pi}{2} \xi_0 (Y_0(\xi) J_0(\xi_0) - J_0(\xi) Y_0(\xi_0)). \]

The derivative of the function (10) will be

\[ W_n'(\xi) = \lambda_n (C_1 C(\lambda_n \xi) + C_2 D(\lambda_n \xi)), \]

where
\[ C(\zeta) = \frac{\pi}{2} \zeta_0 (J_1(\zeta_0)Y_1(\zeta_0) - Y_1(\zeta_0)J_1(\zeta_0)); \]
\[ D(\zeta) = \frac{\pi}{2} \zeta_0 (J_1(\zeta_0)Y_0(\zeta_0) - Y_1(\zeta_0)J_0(\zeta_0)). \]

Here \( \zeta_0 = \lambda_n k \) is the argument value at the starting point, \( k = R^*/R \). Based on the recurrence relations for the Bessel functions \([15]\), these functions will have the following particular values:

\[ A(\zeta_0) = D(\zeta_0) = 1; \quad B(\zeta_0) = C(\zeta_0) = 0. \]

The approach applied is similar to the use of Krylov’s functions when considering boundary value problems of elastic rod statics and dynamics; similar functions have also been used in solving the problems of inhomogeneous rod dynamics \([16, 17]\).

From the first boundary condition (9), \( C_2 = 0 \), and from the second boundary condition, we obtain the equation for the eigenvalues \( \lambda_n \)

\[ A(\lambda_n) = 0. \quad (11) \]

Show that the eigenfunctions of the considered boundary value problem will be orthogonal with the function \( q(\xi) = \xi \). Consider equations (6) for two different forms \( n \) and \( m \):

\[ (\xi W'_n) = -\lambda_n^2 \xi W_n; \]
\[ (\xi W'_m) = -\lambda_m^2 \xi W_m. \]

Multiply the upper and lower equations by the functions \( W_m \) and \( W_n \), respectively. We subtract the second equation from the first one and integrate the resulting difference with respect to the variable \( \xi \)

\[ \left( \lambda_m^2 - \lambda_n^2 \right) \int_k^1 \xi W_n W_m d\xi = \frac{1}{k} \left( (\xi W'_n) W_m - (\xi W'_m) W_n \right) d\xi. \]

Taking the integrals by parts and considering (9), we obtain the dependence

\[ \left( \lambda_m^2 - \lambda_n^2 \right) \int_k^1 \xi W_n W_m d\xi = \left[ \xi W_n W_m - \xi W'_m W_n \right]_{n=1} = 0. \quad (12) \]

To determine the squared norm of the eigenfunctions in equation (12), we pass to the limit \( m \to n \), i.e. \( \lambda_m = \lambda_n + \delta \lambda \), and \( W_m = W_n + \frac{\partial W_n}{\partial \lambda} \delta \lambda \); neglecting the quantities of the second order of smallness, we obtain

\[ 2\lambda_n \delta \lambda \int_k^1 \xi W_n^2(\xi) d\xi = \left| \xi W'_n \frac{\partial W_n}{\partial \lambda} \delta \lambda - \xi \frac{\partial W'_n}{\partial \lambda} \delta \lambda W_n \right|_{n=1} = 0, \]

Having determined the partial derivatives, we find the final form of the squared norm equation
\[ \Delta_n^2 = \int k \xi W_n^2(\xi) d\xi = \frac{1}{2} \xi^2 W_n^2 + \frac{\xi^2}{\lambda_n^2} (W_n')^2. \] (13)

Consider the case when the gate is subjected to harmonic action, while the pipe velocity changes according to the law
\[ u(t) = A_u \theta \sin(\theta t), \] (14)

where \( A_u \) is the gate motion amplitude, \( \theta \) is the frequency of forced vibrations. The fluid motion will be considered a complex one, consisting of the relative fluid motion along the pipe and the frame motion of the pipe itself, the absolute velocity will be
\[ \nu_{\theta \theta} = \nu + u. \] (15)

Substituting equation (15) into equation (1) and considering that the frame speed \( u \) does not depend on the variable \( r \), the motion equation takes the form
\[ \frac{\partial}{\partial r} \left( r \frac{\partial \nu}{\partial r} \right) - \frac{1}{\nu} r \frac{\partial \nu}{\partial t} = \frac{1}{\nu} r \frac{\partial u}{\partial t} - \frac{1}{\rho} r \frac{\partial p}{\partial \xi}, \]

considering the previously adopted designations for dimensionless quantities, we obtain
\[ \frac{\partial}{\partial \xi} \left( \xi \frac{\partial \nu}{\partial \xi} \right) - \xi \frac{\partial \nu}{\partial \tau} = \xi \tilde{A}_u \tilde{\theta}^2 \cos(\tilde{\theta} \tau) - \xi \frac{\partial \tilde{p}}{\partial \xi}. \] (16)

where \( \tilde{A}_u = A_u / R, \tilde{\theta} = \theta R^2 / \nu. \)

1.3. Theoretical research
Consider only the oscillatory gate motion action on the fluid motion inside the channel \( (\rho = 0) \); to do this, we solve the equation (16) with the initial condition
\[ \nu(r,0) = 0. \] (17)

Substituting series (4) into equation (16) and considering equation (6), we obtain
\[ \sum_{n=1}^\infty \left( \frac{\tilde{t}_n}{\nu} + \lambda_n^2 T_n \right) \xi W_n = -\xi \tilde{A}_u \tilde{\theta}^2 \cos(\tilde{\theta} \tau). \] (18)

Multiplying both hands of equality (18) by \( W_n \) and integrating over the variable \( \xi \), we obtain an equation to determine the function \( T_n \)
\[ \tilde{t}_n + \lambda_n^2 T_n = \frac{\tilde{A}_u \tilde{\theta}^2 \cos(\tilde{\theta} \tau)}{\Delta_n^2 \lambda_n^2} \langle \xi W_n' \rangle_{\xi=1}. \] (19)
where \( \Delta_n^2 = \frac{1}{2} \left( C(\lambda_n)^2 - k^2 \right) \), \( (\xi \omega_n)^2 \xi = \lambda_n C(\lambda_n) \).

Considering (17), the solution to equation (19) has the form

\[ T_n(\tilde{t}) = \frac{2\tilde{A}_n \tilde{\theta}^2 C(\lambda_n)}{\left( \tilde{\theta}^2 + \lambda_n^4 \right) \lambda_n \left( C(\lambda_n)^2 - k^2 \right)} \left[ \tilde{\theta} \sin(\tilde{\theta} \tilde{t}) + \lambda_n^2 \left( \cos(\tilde{\theta} \tilde{t}) - \exp(-\lambda_n^2 \tilde{t}) \right) \right]. \]

Then the equation for the absolute velocity will be

\[ \tilde{v}_{\text{abs}}(\xi, \tilde{t}) = \tilde{A}_n \tilde{\theta} \sin(\tilde{\theta} \tilde{t}) + \sum_{n=1}^{\infty} \frac{2\tilde{A}_n \tilde{\theta}^2 C(\lambda_n)}{\left( \tilde{\theta}^2 + \lambda_n^4 \right) \lambda_n \left( C(\lambda_n)^2 - k^2 \right)} \lambda_n \xi \left[ \tilde{\theta} \sin(\tilde{\theta} \tilde{t}) + \lambda_n^2 \left( \cos(\tilde{\theta} \tilde{t}) - \exp(-\lambda_n^2 \tilde{t}) \right) \right]. \tag{20} \]

The equation for the shear stresses will be as follows

\[ \tau(\xi, \tilde{t}) = \sum_{n=1}^{\infty} \frac{2\tilde{A}_n \tilde{\theta}^2 C(\lambda_n)}{\left( \tilde{\theta}^2 + \lambda_n^4 \right) \lambda_n \left( C(\lambda_n)^2 - k^2 \right)} \lambda_n \xi \left[ \tilde{\theta} \sin(\tilde{\theta} \tilde{t}) + \lambda_n^2 \left( \cos(\tilde{\theta} \tilde{t}) - \exp(-\lambda_n^2 \tilde{t}) \right) \right], \tag{21} \]

where \( \tau = \tau R^2 / (\rho \eta) \) are the dimensionless stresses.

The plot in Fig. 2 shows the dependences of the relative velocity (20) and relative stresses (21) at the channel wall (\( \xi = 1 \)) for the parameter values: \( k = 0.01 \); \( \tilde{A}_n = 1 \); \( \tilde{\theta} = 100 \). As can be seen from the plot, stresses and velocity do not reach their maximum synchronously but have a phase shift, which remains constant for a wide range of changes in the model parameter values and is \( \varphi = \pi / 4 \).

Studying the action of the vibration frequency \( \tilde{\theta} \) on the distribution of velocities and stresses over the channel cross-section has shown that with an increase in \( \tilde{\theta} \), the velocities and stresses change significantly only in a limited area near the wall. Thus, at \( \tilde{\theta} > 2000 \), this area does not exceed a tenth of the radius (\( k > 0.9 \)).

**Figure 2.** Dependence of the fluid velocity and shear stresses at the channel wall (\( \xi = 1 \)) on the dimensionless time.

As an example, consider the distribution of maximum velocities and stresses in the near-wall zone of the gate channel, \( R = 35 \text{ mm} \), during steel pouring (\( \rho = 7200 \text{ kg/m}^3 \); \( \eta = 7070 \cdot 10^{-6} \text{ Pa\cdot s} \).
\( v = 1.01 \cdot 10^{-6} \text{ m}^2/\text{s} \). The parameters of the vibration action on the gate are taken from [1]: the vibration amplitude \( A_u = 1 \text{ mm} \) and three frequency modes: \( \theta_1 = 52 \text{ rad/s} \), \( \theta_2 = 104 \text{ rad/s} \), \( \theta_3 = 157 \text{ rad/s} \) (\( \tilde{A}_u = 0.029 \); \( \tilde{\theta}_1 = 6.487 \cdot 10^4 \); \( \tilde{\theta}_2 = 1.297 \cdot 10^5 \); \( \tilde{\theta}_3 = 1.904 \cdot 10^5 \)). The laminar sublayer thickness is assumed to be \( \delta_s = 1 \text{ mm} \) (\( R^s = 34 \text{ mm} \); \( k = 0.971 \)). The plots (Fig. 3) show the epures of the maximum velocities for the corresponding time instants \( T_i = \pi/(2\tilde{\theta}) \) and maximum shear stresses for the time instants \( T_i = 5\pi/(4\tilde{\theta}) \). A change in the parameter \( k \) at \( k < 0.99 \) has practically no effect on the calculation results.

**Figure 3.** Diagrams of velocities and stresses in the near-wall zone of the channel when pouring liquid steel for three modes of vibration action on the gate: \( \theta_1 = 52 \text{ rad/s} \); \( \theta_2 = 104 \text{ rad/s} \); \( \theta_3 = 157 \text{ rad/s} \).

In [4, 13], a semiempirical formula is given for the laminar sublayer thickness

\[
\delta_L = \frac{11.5 q}{\sqrt{\nu \tau_w}}
\]

(22)

where \( \tau_w \) is the shear stresses on the pipe wall. Substituting the maximum stresses according to formula (21) into equation (22), for the cases under consideration, we obtain \( \delta_L = 0.26 \div 0.58 \text{ mm} \), which is consistent with the size of the area, within which viscous forces significantly affect the fluid velocity (Fig. 3).

2. Practical Advice
The model can be useful to determine the forces of interaction of a fluid with a pipe wall (e.g., liquid steel with a refractory nozzle of a steel-pouring ladle) when the pipe is technologically affected by vibration, impulse, etc., as well as study moving joints such as piston, plunger, etc. However, it should be considered that the assumptions such as the lack of roughness, the pipe cylindricality, the presence of a laminar sublayer, which in the general case may be unsteady or even absent [4], may significantly affect the real motion and interaction of a viscous fluid with the vessel wall. Therefore, the results obtained can be considered preliminary, and laboratory studies are required to clarify them.

3. Conclusion
The paper provides a mathematical model of the axisymmetric fluid motion in a cylindrical pipe under the action of viscous forces. For the near-wall layer, a motion equation is obtained, the solution to
which is sought in analytical form by the Fourier method. In contrast to well-known studies, the adopted approach gives wider possibilities for solving various boundary value problems of fluid motion in a cylindrical pipe. The solutions provided have been used to study the interaction of liquid steel with the steel-pouring ladle gate channel wall. Numerical values are obtained for the shear stresses on the channel wall depending on the gate vibration amplitude and frequency. However, to determine the complex effect of vibrations on the mechanism of skull formation in the gate channel, further theoretical and experimental studies are required.

4. References

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