Geometry and rigged strings in Bethe Ansatz.

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Abstract. The main purpose of this report is a thorough analysis of completeness of solutions of the one-dimensional Heisenberg Hamiltonian through the hypothesis of strings. A somehow astonishing conclusion emerges from studying of the structure of the classical configuration space of this system. Namely, all allowed information concerning quantum states, which are exact solutions of the Bethe equations, encoded in quantum numbers, are predictable via a bijection between the set of the magnetic configurations and the string configurations. This startling and beautiful observation constitutes the proof of the completeness of the eigenstates of the Heisenberg Hamiltonian, deduced in a purely combinatorial way. We interpret the set of all magnetic configurations with a fixed number \( r \) of spin deviations as the classical configuration space of a hypothetic system of \( r \) Bethe pseudoparticles, which move, in a stroboscopic manner, on the magnetic ring. The geometry of this configuration space, induced by the action of Heisenberg Hamiltonian and the translation symmetry group of the ring, implies the structure of a locally \( r \)-dimensional hypercubic lattice with well defined \( F \)-dimensional boundaries, \( 1 \leq F \leq r \). We demonstrate that rigged string configurations originate from these boundaries, depending upon the island structure of spin deviations. We show that a relatively simple combinatoric definition of rigged strings reproduces completely exact results of Bethe Ansatz. It is expressed in terms of a combined bijection: Robinson-Schensted with Kerov-Kirillov-Reshetikhin (RSKKR) which produces a geography of exact Bethe Ansatz solutions on the classical configuration space.

1. Introduction

The famous Bethe Ansatz (BA) [1] provides the exact solution of the eigenproblem for the Heisenberg Hamiltonian for a finite magnetic ring of \( N \) spins \( s = 1/2 \). Eigenfunctions are classified by combinatorial objects called rigged string configurations [2], which are a refined version of Takahashi integers [3]. Kerov, Kirillov and Reshetikhin (KKR)[2] introduced a bijection between the irreducible basis of the Weyl duality [4] (cf. also [5], [6]) and the set of exact BA eigenfunctions. Another bijection, known as the Robinson - Schensted (RS) algorithm [7], [8] (cf. also [9]-[11]), associates each magnetic configuration, treated as a word of the length \( N \) in the alphabet \{+,-\} of single-node spins, with the irreducible basis of the Weyl duality, that is a pair of standard Weyl and Young tableaux with the same shape.

The set of all magnetic configurations with a given number \( r \) of spin deviations from the magnetic saturation (all spins "+") can be equipped with a geometry imposed by the action of the Heisenberg Hamiltonian and the cyclic group \( C_N \) - the translational symmetry group of the magnet [12]. This set can be also interpreted along the Schrodinger picture of Quantum Mechanics, as the ”classical” configuration space for the system of \( r \) spin deviations (Bethe...
pseudoparticles), moving in a stroboscopic way on the magnetic ring. In the present paper we aim to consider some properties of a composed bijection, $KKR \circ RS$, which maps magnetic configurations immediately to exact BA solutions, omitting thus the basis of duality of Weyl. We aim to point out that strings of the length $l > 1$, interpreted usually as bound states of $l$ magnons, originate indeed from boundaries of the classical configuration space, i.e. from those magnetic configurations in which some Bethe pseudoparticles gather into islands. In this way, the $RSKKR$ bijection establishes a “geography” of distribution of rigged string configurations over the classical configuration space.

2. Preliminaries

Let $\tilde{N} = \{j = 1, 2, \ldots, N\}$ be the set of nodes of a magnetic ring, and $\tilde{2} = \{+, -\}$ - the set of single-node spin projections. Then the set of all magnetic configurations is

$$\tilde{2}^{\tilde{N}} = \{f : \tilde{N} \rightarrow \tilde{2}\},$$

and the space $\mathcal{H}$ of all quantum states of the magnetic ring is spanned on this set, i.e.

$$\mathcal{H} = l_{\mathbb{C}} \tilde{2}^{\tilde{N}}.$$  

Moreover, the set $\tilde{2}^{\tilde{N}}$ provides an orthonormal basis in $\mathcal{H}$. We recall briefly that the space $\mathcal{H}$ of all quantum states of the Heisenberg magnet, with $\dim \mathcal{H} = 2^N$, decomposes as

$$\mathcal{H} = \sum_{r=0}^{N} \mathcal{H}^r,$$

into subspaces $\mathcal{H}^r$, with $\dim \mathcal{H}^r = \binom{N}{r}$, with the fixed number $r$ of Bethe pseudoparticles (spin deviations). Stated otherwise, $\mathcal{H}^r$ is the space of all states of the magnet with the projection

$$M = N/2 - r$$

of the total spin, or of all states with the weight

$$\mu = \{N - r, r\}.$$  

Clearly, $\mathcal{H}^r$ and $\mathcal{H}^{N-r}$ are particle-hole counterparts, so we restrict in the sequel to $r \leq N/2$. Then, each space $\mathcal{H}^r$ decomposes as

$$\mathcal{H}^r = \sum_{S=N/2-r}^{N/2} \mathcal{H}^{rS}$$

into subspaces $\mathcal{H}^{rS}$ with the total spin $S$ (and $M = N/2 - r$). By the duality of Weyl [4] between the actions of the symmetric group $\Sigma_N$ and the unitary group $U(n)$, $n = 2s + 1 = 2$, in the space $\mathcal{H}$ (cf., e.g., [5], [6] for detail), it defines the partition $\lambda$ of the integer $N$, of the shape

$$\lambda = \{N - r', r'\},$$

such that $\lambda$ labels the irrep $\Delta^\lambda$ of $\Sigma_N$,

$$\dim \Delta^\lambda = \dim \mathcal{H}^{rS},$$
and

\[ S = \frac{N}{2} - r', \quad 0 \leq r' \leq r. \quad (9) \]

For \( r \leq N/2 \), orbits of the symmetric group \( \Sigma_N \) are classical configuration spaces \( Q^{(r)} \) for the system of \( r \) spin deviations, such that

\[ \mathcal{H}^r = l_{CC} \mathcal{O}_\mu \equiv l_{CC} Q^{(r)}, \quad (10) \]

and

\[ \dim \mathcal{H} = |Q^{(r)}| = \binom{N}{r}. \quad (11) \]

Let \( j = (j_1, j_2, \ldots, j_r) \), for some set \( 1 \leq j_1 < j_2 < \ldots < j_r \leq N \),

\[ \text{be the distribution of Bethe pseudoparticles which specifies a magnetic configuration } f \in Q^{(r)}, \]

and let

\[ t = (t_1, t_2, \ldots, t_r), \quad \sum_{\alpha \in \tilde{r}} t_{\alpha} = N. \quad (13) \]

be the relative distribution of Bethe pseudoparticles [13] within an orbit of the translational symmetry group of the ring \( C_N \) (\( C_N \)-orbit), being subgroup of the symmetric group \( \Sigma_N \). The vector \( t \) provides a unique label for classification of \( C_N \)-orbits in \( Q^{(r)} \), so that

\[ Q^{(r)} = \bigcup_{t \in T} \mathcal{O}_t, \quad (14) \]

where \( T = Q^{(r)}/C_N \) is the set of orbits, which we refer to, after [13], as a skeleton of the configuration space \( Q^{(r)} \) (cf. [13] for detail definitions).

Let

\[ f^t = (t, 1) = \boxed{- + + \ldots + - + \ldots + - \ldots - + \ldots +} \]

\[ t_1 \quad t_2 \quad t_r \]

be, by convention, the initial magnetic configuration in the orbit \( \mathcal{O}_t \). The distribution \( j^t \) of Bethe pseudoparticles for the initial magnetic configuration \( f^t \) is given by

\[ j^t_{\alpha} = 1 + \sum_{\alpha' < \alpha} t_{\alpha'}, \quad \alpha \in \tilde{r}. \quad (16) \]

An arbitrary configuration in this orbit has the form of composition of mappings

\[ (t, j) = (t, 1) \circ C_N^{1 - j}, \quad j \in \tilde{N}, \quad (17) \]

where the mapping \( C_N : \tilde{N} \rightarrow \tilde{N} \) is defined by

\[ C_N = \left( \left( \frac{j}{(j + 1) \mod N} \right) \in \Sigma_N, \quad j \in \tilde{N} \right. \quad (18) \]

(we use the same symbol for the group \( C_N \subset \Sigma_N \), and for its generating element \( C_N \)).
3. Structure of islands of spin deviations

We recall that the action of the Heisenberg Hamiltonian $\hat{H}$ in $\mathcal{H}^r$ is given by

$$2\hat{H}|j> = \sum_{j' \in Q^j_{(r)}} (|j'> - |j>), \ j \in Q^{(r)},$$

where $|j> \in \mathcal{H}^r$ is the quantum state of the model, corresponding to the magnetic configuration determined by the distribution $j \in Q^{(r)}$ of Bethe pseudoparticles, and $Q^j_{(r)} \subset Q^{(r)}$ is the set of all nearest neighbours of the magnetic configuration $j$ within $Q^{(r)}$.

In this way, the dynamics (19) of BA imposes, in a local manner, the structure of a hypercubic lattice in the classical configuration space $Q^{(r)}$: for a generic position $j \in Q^{(r)}$, its nearest neighbours $j' \in Q^j_{(r)}$ define a local coordinate system in a real $r$-dimensional linear space $\mathbb{R}^r$ into which $Q^j_{(r)}$ can be embedded, with the origin at $j$. Such a description reflects the $r$ degrees of freedom for stroboscopic movements of the system $\tilde{r}$ of Bethe pseudoparticles over the ring $\tilde{N}$: a pseudoparticle $\alpha \in \tilde{r}$ can jump forward or back by one step on the crystal $\tilde{N}$, independently on other pseudoparticles $\alpha' \neq \alpha$.

The classical configuration space $Q^{(r)}$, however, is not homogeneous with respect to the degree of freedom. A look at the form (15) of a magnetic configuration reveals that some pseudoparticles can be the nearest neighbours each of other, forming thus "islands" of spin deviations in the "sea" of nodes with the spin $+\frac{1}{2}$. Clearly, each island contributes to a single degree of freedom, since its leftmost pseudoparticle can only jump to the left, its rightmost - to the right, and all internal pseudoparticles of this island are kinematically frozen.

The structure of islands of spin deviations is defined as a sequence of lengths

$$\xi = (\xi_1, \xi_2, \ldots, \xi_F), \ \sum_{\gamma=1}^F \xi_\gamma = r.$$  

(20)

It thus imposes a decomposition

$$Q^{(r)} = \bigcup_{F=1}^r Q^{(r,F)}$$

(21)

of the classical configuration space into subsets

$$Q^{(r,F)} = \{|j \in Q^{(r)}| |Q^j_{(r)}| = 2F\},$$

(22)

consisting of $C_N$-orbits with exactly $F \leq r$ islands of spin deviations. The case $F = r$ is generic, whereas subsets $Q^{(r,F)}$ with $F < r$ correspond to boundaries of the classical configuration space with a lower dimension. Thus the integer $F$, defined in Eq. (20) as the number of islands of spin deviations for a magnetic configuration within an orbit $O_t$, plays also the role of local dimension for all elements of the boundary $Q^{(r,F)}$ (such that $O_t \subset Q^{(r,F)}$) of the classical configuration space $Q^{(r)}$. The local dimension $F$ varies within the range

$$1 \leq F \leq r.$$  

(23)

It will follow from the sequel (Sect. 5-6) that

$$r' \leq F,$$

(24)

where $r'$ determines the total spin $S = N/2 - r'$ of the RSKKR image of $j \in Q^{(r,F)}$. 


4. The combinatorics of strings.
A string configuration \([2], [14], [15]\) \(\nu\) is, by the definition, a partition of the integer \(r', \nu \vdash r'\). Each part of the partition \(\nu\) is referred to as a string, and its size (the number of boxes in the Young diagram of \(\nu\)) - the length of the string. Denoting by \(m_l\) the number of strings of the length \(l\) in \(\nu\), we have

\[
\sum_l l m_l = r'.
\]  

Each string \((lv), v \in \tilde{m}_l = \{1, 2, \ldots, m_l\}\), is equipped with a nonnegative integer \(L\), called its rigging. The rigging \(L\) varies within the range

\[
0 \leq L \leq P_l, \tag{26}
\]

where

\[
P_l = N - 2Q_l, \tag{27}
\]

with \(Q_l\) being the number of boxes in the first \(l\) columns of the Young diagram of the string configuration \(\nu\). The integer \(P_l\) is referred to as the number of holes for strings of the length \(l\). The pair \(\nu L\), where

\[
L = \{L_{lv} | l = 1, 2, \ldots; v \in \tilde{m}_l\}, \tag{28}
\]

is the set of all riggings of \(\nu\), is referred to as a rigged string configuration. All \(m_l\) strings of the length \(l\) are distinguished by their riggings only, so that they can be arranged in \(\nu L\) in the nondecreasing order (from the top to the bottom of the \(l \times m_l\) rectangle incorporated in the Young diagram \(\nu \vdash r'\)), that is

\[
0 \leq L_{lv} \leq L_{lv'} \leq P_l \quad \text{for } 1 \leq v < v' \leq m_l, \; l = 1, 2, \ldots \tag{29}
\]

Such a counting implies that the set \(z(\nu) = \{\nu L\}\) of all rigged string configurations corresponding to a given \(\nu \vdash r'\) has the cardinality

\[
|z(\nu)| = \prod_l \left(\frac{P_l + m_l}{m_l}\right), \tag{30}
\]

and, moreover,

\[
\dim \Delta^{\lambda} = \sum_{\nu \vdash r'} |z(\nu)|. \tag{31}
\]

Each rigged string configuration \(\nu L\) classifies an exact eigenstate of the Heisenberg Hamiltonian \(\hat{H}\) in each space \(\mathcal{H}'\), \(r' \leq r \leq N/2\) (cf. Eq. (19)). The eigenstate \(|\nu L\rangle\) is a superposition \([6]\) of magnetic configurations \(|f\rangle, f \in Q^{(r)}\), and can be obtained as a solution of highly non-linear Bethe equations. It is amazing that this state is fully characterised by the combinatoric data \(\nu L\), which can be simply interpreted in terms of strings. In this combinatoric interpretation, a string of the length \(l\) is a collection of \(l\) consecutive nodes \(j, j + 1, \ldots, j + l - 1\), each occupied by a spin deviation, followed by next \(l\) nodes \(j + l, j + l + 1, \ldots, j + 2l - 1\), occupied by a spin \(\pi + \pi\) \(\in 2\). In the case when there are no other strings, then the remaining \(P_l = N - 2l\) nodes of the crystal \(N\) play the role of the ”sea of holes” for the string configuration \(\nu = \{l\}\). The rigging of this \(l\)-string is \(L = j - 1\), which coincides with the number of holes to the left of the molecule representing this string \([16]\). All possible riggings of this \(l\)-string correspond to various locations of the molecule inside the open chain \(N\), with distinguished initial and final node. In other words, the left boundary node \(j\) of such a string should satisfy \(1 \leq j \leq N - 2l\). Clearly, the number of such locations is \(|z(\{l\})| = P_l + 1 = N - 2l + 1\), in accordance with Eq. (30). More complex string configurations require some combinatoric extension which we describe in the next paragraph.
5. The RSKKKR bijection

In fact, such an apparently simplified, but surprisingly adequate picture of an exact BA solution, emerges by virtue of existence of two bijections. The first is Robinson-Schensted correspondence [7], [8] between the set \( \tilde{2}^N \) of all magnetic configurations and the set \( b_{\text{irr}} \) of an irreducible basis of the duality of Weyl, and the second - the bijection of Kerov, Kirillov and Reshetikhin [2] between the set \( b_{\text{irr}} \) and appropriate rigged string configurations. The irreducible basis of the duality of Weyl, written in the form

\[
b_{\text{irr}} = \bigcup_{0 \leq \nu' \leq N/2} \text{SYT}(\{N - \nu', \nu\}) \times \text{WT}(\{N - \nu', \nu\}, \overline{2}),
\]

(32)

where \( \text{SYT}(\lambda) \) is the set of all standard Young tableaux of the shape \( \lambda \) in the alphabet \( \tilde{N} \) of nodes, and \( \text{WT}(\lambda, \overline{2}) \) is the set of all Weyl tableaux (that is, semistandard Young tableaux) of the same shape \( \lambda \) in the alphabet \( \overline{2} \) of spins, with \( \lambda = \{N - \nu', \nu\} \), mediates thus between the initial basis \( \tilde{2}^N \) for quantum computations, and the final basis of rigged string configurations which classify the exact BA eigenstates. Here we consider the composition of these two bijections, which runs immediately from magnetic configurations to rigged string configurations, omitting thus the pairs of Young and Weyl tableaux of the duality of Weyl. The classical configuration space \( \mathcal{Q}^{(r)} \) serves as the source of the bijection, and the target is the basis of those exact BA eigenstates which span the space \( \mathcal{H}^r = l_{\mathbb{C}} \mathcal{Q}^{(r)} \). We thus describe the composite RSKKKR bijection

\[
\rho : \mathcal{Q}^{(r)} \longrightarrow \text{RC}(N, r),
\]

(33)

where

\[
\text{RC}(N, r) = \bigcup_{r' = 0}^{r} \text{RC}(N, r, r'), \quad \text{and} \quad \text{RC}(N, r, r') = \bigcup_{\nu \vdash \nu'} z(\nu).
\]

(34)

Here \( \text{RC}(N, r) \), \( \text{RC}(N, r, r') \), and \( z(\nu) \) denotes the set of all rigged string configurations on the magnetic ring \( \tilde{N} \) which correspond to the system of \( r \) Bethe pseudoparticles, all those with the total spin \( S = N/2 - r' \) and its projection \( M = N/2 - r \), and all riggings of the string configuration \( \nu \vdash \nu' \), respectively. The completeness of exact BA solutions reads

\[
l_{\mathbb{C}} \text{RC}(N, r, r') = \mathcal{H}^{rS}, \quad \text{and} \quad l_{\mathbb{C}} \text{RC}(N, r) = \mathcal{H}^r, \quad S = N/2 - r'.
\]

(35)

We proceed to describe the RSKKKR bijection in detail. To this aim, we introduce the notion of a minimal compensated subword of a magnetic configuration \( f \in \mathcal{Q}^{(r)} \). Let

\[
v = f(j)f(j + 1) \ldots f(j + 2A - 1), \quad 2A \leq N - j + 1,
\]

(36)

be a subword of \( f \), consisting of \( 2A \) letters of consecutive nodes. Let \( F^\alpha \) be the subword of \( v \), consisting of its first \( \alpha \) letters, i.e.

\[
F^\alpha = f(j)f(j + 1) \ldots f(j + \alpha - 1), \quad \alpha = 1, 2, \ldots, 2A,
\]

(37)

and let \( w(F^\alpha) = \{w_1, w_2\} \) be the weight of the word \( F^\alpha \), so that \( w_i \) is the number of occurrences of the letter \( i \in \overline{2} \) in \( F^\alpha \), and thus \( w_1 + w_2 = \alpha \). By the definition, \( v \) is called a minimal compensated subword of \( f \) if

\[
w_1 \begin{cases} < w_2 & \text{for } \alpha < 2A, \\ = w_2 = A & \text{for } \alpha = 2A. \end{cases}
\]

(38)

A minimal compensated word of the length \( 2A \) corresponds to an ensemble of \( A \) pairs of nodes (in general not adjacent), each pair consisting of a Bethe pseudoparticle, followed by a compensating node occupied by the spin \"'\( +\)"\.


We proceed to consider some properties of the mapping $K$. Behaviour of RSKKR preimages of strings under translations of the second.

In fact, the first rule can be looked at as a special case configuration space by $v$ in other words, $K$ such that $v_1, v_2, \ldots, v_n$ are minimal compensated subwords, whereas $u_1, u_2, \ldots, u_{n+1}$ are formed consecutively from the remaining letters of $f$ (some $u_i$ might have zero length, when it happens that $v_{i-1}$ and $v_i$ are adjacent). The main algorithm which determines the RSKKR bijection can be expressed transparently in terms of paths. The path $[14] p(f)$, corresponding to a magnetic configuration $f$ in the RSKKR bijection is given by

$$p_j(f) = \begin{cases} \text{up} & \text{if } f(j) = "-", \\ \text{down} & \text{if } f(j) = "+" \text{ and } f(j) \in u(f), \\ \text{and } f(j) \in v(f), \end{cases}, j \in \tilde{N}. \quad (40)$$

Thus all Bethe pseudoparticles in a configuration $f$ correspond to steps up in the path $p(f)$, whereas the nodes with $f(j) = "+"$ correspond to the step either up or down, depending if they belong to the subword $u(f)$ or $v(f)$, respectively. Each straight line $u_a$ belongs to the "sea of holes" for all strings, whereas a multipyramid $v_\alpha$ constitutes a collection of overlapping strings. We proceed to interpret such a multipyramid.

Each of two or more pyramids can move within its range of riggings, obeying two simple rules of navigation [16]. Firstly, two $l$-strings are mutually inpenetrable, that is, they behave like hard-core objects. It is due to the fact that the strings cannot exchange their positions in the one-dimensional ring $\tilde{N}$ since the order of the queue of Bethe pseudoparticles is fixed. Secondly, for $l < l'$, the $l$-string, treated as a pyramid, can slide over both slopes of the $l'$-string until the heights of their tops coincide. In fact, the first rule can be looked at as a special case of the second.

As a result, each multipyramid $v(f)$ represents a collection of overlapping strings.

### 6. Behaviour of RSKKR preimages of strings under translations

We proceed to consider some properties of the mapping $K_N : RC(N, r) \rightarrow RC(N, r)$, defined by

$$K_N = \rho \circ A(C_N) \circ \rho^{-1}. \quad (41)$$

In other words, $K_N$ is the push-forward of the shift $A(C_N) : Q^{(r)} \rightarrow Q^{(r)}$ from the classical configuration space $Q^{(r)}$ to the corresponding set $RC(N, r)$ of exact BA solutions along the RSKKR bijection $\rho : Q^{(r)} \rightarrow RC(N, r)$.

The Fig. 1 presents the travel of the coupled system of a 1-string and a 3-string over the ring of $N = 10$ nodes along the orbit $t = (1216)$. The corresponding multipyramid

$$v = -- + - ++$$

has the size $2(1 + 3) = 8$, and can be thus located on the open chain of 10 nodes in three different positions. RSKKR preimages of these positions are given by magnetic configurations $(t, j)$ with $j \in \{1, 2, 3\}$. For the next three nodes, $j \in \{4, 5, 6\}$, the right slope of the 3-string disappears at the terminal node of the chain, so this string is shortened to 2-string ($j = 4$), 1-string ($j = 5$), and vanishes completely at $j = 6$. Each such shortening results in extension of the sea of unbounded nodes by 2: one "+" which arises at the beginning of the chain, and one "-", which is no longer bound to the shortened string. Then, at the nodes $j \in \{7, 8\}$ a string arises at the initial node (1-string and 2-string for $j = 7$ and 8, respectively). In the meantime, the 1-string reaches the terminal node for $j = 8$, then disappears at $j = 9$, and arises again at $j = 10$ as the neighbour of the 2-string. The next step, corresponding to $j = 1$, creates the multipyramid (42) at its leftmost position.
To describe this process in detail, one has to distinguish three cases: (i) $f(N) = +$ and $f(N) \in u(f) \equiv$ the last node is occupied by the spin \( + \), and belongs to the sea of holes; (ii) $f(N) = +$ and $f(N) \in v(f) \equiv$ the last node is \( + \) and belongs to the (last) minimal compensated subword $v_a(f) \subset v(f)$; (iii) $f(N) = -$. The total spin $S' = N^2 - r'$ changes in the cases described above according to the formula

$$S' = \begin{cases} S & \text{in the case (i)}, \\ S - 1 & \text{in the case (ii)}, \\ S + 1 & \text{in the case (iii)}. \end{cases}$$

(43)

It is worth to observe that the total number of strings,

$$q(f) = \sum_{l \geq 1} m_l,$$

(44)

can be deduced immediately from the shape of the path $p(f)$ - it is just equal to the number of cuspidal maxima inside the open chain $\tilde{N}$. Clearly the variation of $q(f)$ along an orbit $O_t$ is much smaller than that of $S$, namely

$$q(t, j) \in \{F, F - 1\} \text{ for } O_t \subset Q^{(r,F)}.$$

(45)

It follows from the observation that at most one island can approach the rightmost end of the open chain, and thus not to contribute to a string, at each step along the orbit $O_t$. 

Figure 1. The RSKKR bijection for the $C_{10}$-orbit $t = (1216)$ ($N = 10, r = 4$).
7. Final remarks and conclusions

The main purpose of this report was a thorough analysis of completeness of the exact solutions of the eigenproblem for the Heisenberg Hamiltonian for a finite magnetic ring of \( N \) spins \( s = 1/2 \). We have shown, that this result can be obtained in a purely combinatorial way, merely by studying the geometry of the classical configuration space of this system, by means of the RSKKRR bijection. This composite bijection maps magnetic configurations immediately to exact BA solutions, omitting thus the basis of the duality of Weyl. We also pointed to the fact that the number of uncompensated spin deviations in a magnetic configuration are responsible for variation of the total spin \( S \).

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