On algebras of harmonic quaternion fields in $\mathbb{R}^3$

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Abstract

Let $\mathcal{A}(D)$ be an algebra of functions continuous in the disk $D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ and holomorphic into $D$. The well-known fact is that the set $\mathcal{M}$ of its characters (homomorphisms $\mathcal{A}(D) \to \mathbb{C}$) is exhausted by the Dirac measures $\{ \delta_{z_0} \mid z_0 \in D \}$ and a homeomorphism $\mathcal{M} \cong D$ holds. We present a 3d analog of this classical result as follows.

Let $B = \{ x \in \mathbb{R}^3 \mid |x| \leq 1 \}$. A quaternion field is a pair $p = \{ \alpha, u \}$ of a function $\alpha$ and vector field $u$ in the ball $B$. A field $p$ is harmonic if $\alpha, u$ are continuous in $B$ and $\nabla \alpha = \text{rot} u$, $\text{div} u = 0$ holds into $B$. The space $\mathcal{Q}(B)$ of such fields is not an algebra w.r.t. the relevant (point-wise quaternion) multiplication. However, it contains the commutative algebras $\mathcal{A}_\omega(B) = \{ p \in \mathcal{Q}(B) \mid \nabla_\omega \alpha = 0, \nabla_\omega u = 0 \} \ (\omega \in S^2)$, each $\mathcal{A}_\omega(B)$ being isometrically isomorphic to $\mathcal{A}(D)$. This enables one to introduce a set $\mathcal{M}^\mathbb{H}$ of the $\mathbb{H}$-valued linear functionals on $\mathcal{Q}(B)$ ($\mathbb{H}$-characters), which are multiplicative on each $\mathcal{A}_\omega(B)$, and prove that $\mathcal{M}^\mathbb{H} = \{ \delta_{x_0}^\mathbb{H} \mid x_0 \in B \} \cong B$, where $\delta_{x_0}^\mathbb{H}(p) = p(x_0)$.

Key words: 3d quaternion harmonic fields, real uniform Banach algebras, characters.

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0 Introduction

- The result, which our paper is devoted to, is announced in [6]. In a sense, it is a ‘by-product’ of activity in the framework of the algebraic approach to tomography problems on manifolds [2]–[9]. However, we hope that this result is of certain independent interest for the real uniform Banach algebras theory [1, 10]. Namely, we propose a 3d generalization of the well-known theorem on the characters of the disk-algebra of holomorphic functions. Perhaps, the most curious point is that such a generalization does exist although the relevant 3d analog of the disc-algebra is not an algebra.

- A subject to be generalized is the following well-known result.

Let $\mathcal{A}(D)$ be a commutative Banach algebra of functions continuous in the disk $D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ and holomorphic into $D$. The well-known fact (see, e.g., [12]) is that the set $\mathcal{M}$ of its characters (homeomorphisms $\mathcal{A}(D) \to \mathbb{C}$) endowed with the Gelfand topology, is exhausted by the Dirac measures $\{ \delta_{z_0} \mid z_0 \in D \}$ and the homeomorphism $\mathcal{M} \cong D$ holds.

Shortly, a 3d generalization, which is our main result, looks as follows.

Let $B = \{ x \in \mathbb{R}^3 \mid |x| \leq 1 \}$. A quaternion field is a pair $p = \{ \alpha, u \}$ of a function $\alpha$ and vector field $u$ in the ball $B$. Such fields are identified with the $\mathbb{H}$-valued functions and, hence, can be multiplied point-wise as quaternions. We say a field $p$ to be harmonic if $\alpha, u$ are continuous in $B$ and $\nabla \alpha = \text{rot} \, u$, $\text{div} \, u = 0$ holds into $B$. A space $\mathcal{Q}(B)$ of such fields is not an algebra w.r.t. the above mentioned multiplication. However, it contains the commutative algebras $\mathcal{A}_\omega(B) = \{ p \in \mathcal{Q}(B) \mid \nabla_\omega \alpha = 0, \nabla_\omega u = 0 \} \ (\omega \in S^2)$, which we call the axial algebras ($\omega$ is an axis). Each $\mathcal{A}_\omega(B)$ is isometrically isomorphic to $\mathcal{A}(D)$. This enables one to introduce the set $\mathcal{M}_\mathbb{H}$ of the $\mathbb{H}$-valued linear functionals on $\mathcal{Q}(B)$ ($\mathbb{H}$-characters), which are multiplicative on all $\mathcal{A}_\omega(B)$, and prove that $\mathcal{M}_\mathbb{H} = \{ \delta^\mathbb{H}_{x_0} \mid x_0 \in B \} \cong B$, where $\delta^\mathbb{H}_{x_0}(p) = p(x_0)$.

Sections 1,2 contain the definitions, statements of the results and comments on them. The basic proofs are placed in 3.

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\footnote{Unfortunately, with some inaccuracies in the formulations}
1 The 2d classics

We begin with the result, which is planned to be generalized.

Algebras

• Let $D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ be the disc on the complex plane. The continuous function algebra

$$C^\mathbb{C}(D) := \{ f = \varphi + i\psi \mid \varphi, \psi \in C^\mathbb{R}(D) \}; \quad \| f \| = \sup_D |f|,$$

$$\|fg\| \leq \|f\|\|g\|, \quad fg = gf, \quad \|f^2\| = \|f\|^2$$

is a Banach commutative uniform algebra.

• A function $f = \varphi + i\psi$ is holomorphic if the Cauchy-Riemann conditions

$$d\psi = \ast d\varphi \quad (\delta\psi = \delta\varphi = 0) \quad (1)$$

hold in the inner points of Dom $f \subset \mathbb{C}$. Here $\varphi$ and $\psi$ are regarded as the 0-forms, $\ast$ is the Hodge operator corresponding to the standard orientation of $\mathbb{C}$, $d$ and $\delta$ are the differential and codifferential respectively. The conditions in the brackets are fulfilled just by the well-known definitions. It is a form of writing the CR-conditions, which is most relevant to the forthcoming generalization.

• The (sub)algebra

$$\mathcal{A}(D) := \{ f \in C^\mathbb{C}(D) \mid f \text{ is holomorphic into } D \}$$

is also a commutative uniform Banach algebra.

Characters

By $\mathcal{L}(F, G)$ we denote the normed space of the linear continuous operators from a Banach space $F$ to a Banach space $G$.

Let $\mathcal{A}''(D) := \mathcal{L}(\mathcal{A}(D), \mathbb{C})$ be the dual space. Characters (multiplicative functionals) are defined as elements of the set

$$\mathcal{M} := \{ \mu \in \mathcal{A}'(D) \mid \mu(fg) = \mu(f)\mu(g), \quad f, g \in \mathcal{A}(D) \}$$
endowed with the Gelfand (*-weak) topology, which is determined by the convergence

\[ \{ \mu_j \to \mu \} \Leftrightarrow \{ \mu_j(f) \underset{\text{C}}{\rightarrow} \mu(f) \colon f \in \mathcal{A}(D) \} . \]

The set \( \mathcal{M} \) is also called a spectrum of the algebra \( \mathcal{A}(D) \).

An example of characters, which turns out to be universal, is provided by the Dirac measures \( \delta_{z_0} \in \mathcal{M} \):

\[ \delta_{z_0}(f) := f(z_0), \quad f \in \mathcal{A}(D) \quad (z_0 \in D). \]

Basic fact

For topological spaces, we write \( S \cong T \) if \( S \) and \( T \) are homeomorphic. For algebras, \( \mathcal{A} \cong \mathcal{B} \) means that \( \mathcal{A} \) and \( \mathcal{B} \) are isometrically isomorphic.

Our goal is to provide a 3d analog of the following classical result (see, e.g., [12]: Chapter III, paragraph 11, item 3).

**Theorem 1.** For any \( \mu \in \mathcal{M} \) there is a point \( z^\mu \in D \) such that \( \mu = \delta_{z^\mu} \). The map

\[ \gamma : \mathcal{M} \to D, \quad \mu \mapsto z^\mu \]

is a homeomorphism, so that \( \mathcal{M} \cong D \) holds. The Gelfand transform

\[ \Gamma : \mathcal{A}(D) \to C^\mathcal{C}(\mathcal{M}), \quad (\Gamma f)(\mu) = \mu(f), \quad \mu \in \mathcal{M} \]

is an isometric isomorphism onto its image, so that \( \mathcal{A}(D) \cong \Gamma(\mathcal{A}(D)) \) holds.

Thus, the spectrum of \( \mathcal{A}(D) \) is exhausted by Dirac measures: \( \mathcal{M} = \{ \delta_{z_0} \mid z_0 \in D \} \). Also, since \( (\Gamma f)(\mu) = \mu(f) = \delta_{z^\mu}(f) = f(z^\mu) \), the Gelfand transform just transfers functions from \( D \) to \( \mathcal{M} \) along the map \( \gamma \).

Notice in addition that the same set of the Dirac measures exhausts the spectrum of the 'big' algebra \( C^\mathcal{C}(D) \) [12].

## 2 The 3d analogs

**Quaternions**

- Recall that \( \mathbb{H} \) is a real algebra of the collections (quaternions)

\[ h = \alpha + u_1i + u_2j + u_3k, \quad \alpha, u_i \in \mathbb{R} \]
endowed with the component-wise linear operations and the norm (module) \( |\mathbf{h}| = |\alpha^2 + u_1^2 + u_2^2 + u_3^2|^{\frac{1}{2}} \). A multiplication is determined by the table
\[
\begin{align*}
i i &= j j = k k = -1; & ij &= k, & j k &= i, & k i &= j
\end{align*}
\]
and extended to \( \mathbb{H} \) by linearity and distributivity. The multiplication is associative but not commutative. The module obeys \( |g h| = |g| |h| \).

- A geometric quaternion is a pair of a scalar and 3d vector. The set of pairs
\[
\mathbb{H}_g = \{ p = \{\alpha, u\} | \alpha \in \mathbb{R}, \ u \in \mathbb{R}^3 \}
\]
is a real algebra w.r.t. the component-wise linear operations, the module \( |p| = |\alpha^2 + |u|^2|^{\frac{1}{2}} \), and multiplication
\[
 p q := \{ \alpha \beta - u \cdot v, \ \alpha v + \beta u + u \wedge v \} \quad \text{for} \quad p = \{\alpha, u\}, \ q = \{\beta, v\},
\]
where \( \cdot \) and \( \wedge \) are the standard inner and vector products in \( \mathbb{R}^3 \). The multiplication is noncommutative. The module obeys \( |p q| = |p||q| \).

The correspondence
\[
\mathbb{H} \ni \mathbf{h} = \alpha + u_1 i + u_2 j + u_3 k \leftrightarrow h = \{ \alpha, \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \} \in \mathbb{H}_g \quad (2)
\]
determines an isometric isomorphism of algebras: \( \mathbb{H} \cong \mathbb{H}_g \). As a consequence, \( \mathbb{H}_g \) is the left \( \mathbb{H} \)-module with the action
\[
\mathbf{h} p = h p, \quad \mathbf{h} \in \mathbb{H}, \ p \in \mathbb{H}_g. \quad (3)
\]

- A quaternion field is an \( \mathbb{H}_g \)-valued function on a domain in \( \mathbb{R}^3 \), i.e., a pair \( p = \{\alpha, u\} \), where \( \alpha \) is a function and \( u \) is a vector field defined on a domain in \( \mathbb{R}^3 \). The set of such fields is a real algebra and left \( \mathbb{H} \)-module w.r.t. the relevant point-wise operations.

Algebras and spaces
- Let \( B = \{ x \in \mathbb{R}^3 \mid |x| \leq 1 \} \) be a ball. The space of continuous fields
\[
C^\mathbb{H}(B) := \{ p = \{\alpha, u\} | \alpha \in C^\mathbb{R}(B), \ u \in C(B; \mathbb{R}^3) \}
\]
with the norm $\|p\| = \sup_B |p|$ obeying $\|pq\| \leq \|p\| \|q\|$ and $\|p^2\| = \|p\|^2$ is a (noncommutative) Banach uniform algebra and a left $\mathbb{H}$-module.

- A field $p = \{\alpha, u\}$ is said to be harmonic if the Cauchy-Riemann conditions
\[ d\alpha = \ast du', \quad \delta u' = 0 \quad (\delta \alpha = 0) \] hold in the inner points of $\text{Dom} \, p \subset \mathbb{R}^3$. Here $\alpha$ is regarded as a 0-form, $\ast$ is the Hodge operator corresponding to the standard orientation of $\mathbb{R}^3$, $d$ and $\delta$ are a differential and codifferential respectively, $u'$ is a 1-form dual to $u$, i.e., $u'(b) = b \cdot u$ on vector-fields $b$. The condition in the brackets is fulfilled automatically, whereas the 'gauge condition' $\delta u' = 0$ is now not trivial. In terms of the vector analysis operations, (4) is equivalent to
\[ \nabla \alpha = \text{rot} \, u, \quad \text{div} \, u = 0 \quad \text{into Dom} \, p. \] (5)

- The key object of the paper is the (sub)space of harmonic fields
\[ \mathcal{D}(B) = \{ p \in C^H(B) \mid p \text{ is harmonic into } B \}. \]

A simple calculation with regard to the definitions (3) and (5) enables one to check the following property (see [6] for detail).

**Proposition 1.** The space $\mathcal{D}(B)$ is a left $\mathbb{H}$-module: $h \in \mathbb{H}$ and $p \in \mathcal{D}(B)$ imply $hp \in \mathcal{D}(B)$.

In the mean time, $\mathcal{D}(B)$ is not a (sub)algebra: generically, $p, q \in \mathcal{D}(B)$ doesn’t imply $pq \in \mathcal{D}(B)$. It is the fact, which was perceived as an obstacle for existence of a 3d version of Theorem 1. However, we’ll see that such a version does exist, whereas the space $\mathcal{D}(B)$ turns out to be a relevant analog of the algebra $\mathcal{A}(D)$.

A constant quaternion field, which is equal to $h \in \mathbb{H}_g$ identically, will be denoted by the same symbol $h$. Such fields belong to $\mathcal{D}(B)$.

- Let $\omega \in S^2$ be a unit vector.

  For a function $\alpha$, we denote $\nabla_{\omega} \alpha = \omega \cdot \nabla \alpha$ and say $\alpha$ to be $\omega$-axial if $\nabla_{\omega} \alpha = 0$. For a vector field $u$, we denote by $\nabla_{\omega} u$ its covariant derivative in $\mathbb{R}^3$ and say $u$ to be $\omega$-axial if $\nabla_{\omega} u = 0$. A quaternion field $p = \{\alpha, u\}$ is $\omega$-axial if its components are $\omega$-axial. To be $\omega$-axial just means to be constant on the straight lines $x = x_0 + t\omega$ ($x_0 \in \mathbb{R}^3$, $t \in \mathbb{R}$).

- The following is proven in [6].
Proposition 2. An axial field \( p = \{ \varphi, \psi \omega \} \) is harmonic if and only if the functions \( \varphi, \psi \) obey

\[ \nabla \psi = \omega \wedge \nabla \varphi \quad \text{into Dom} \, p. \]

In this case, \( \varphi \) and \( \psi \) are \( \omega \)-axial and harmonic: \( \Delta \varphi = \Delta \psi = 0 \) holds into \( \text{Dom} \, p \).

If \( q = \{ \lambda, \rho \omega \} \), then \( pq = \{ \varphi \lambda - \psi \rho, [\varphi \rho + \psi \lambda] \omega \} \), i.e., \( pq \) is also \( \omega \)-axial and \( pq = qp \). Thus, one can multiply coaxial fields, the multiplication being commutative.

- By the aforesaid, the subspace \( A_\omega(B) = \{ p \in \mathcal{D}(B) \mid p \text{ is } \omega \text{-axial} \} \subset \mathcal{D}(B) \) (\( \omega \in S^2 \)) is a Banach commutative uniform algebra. Moreover, each \( A_\omega(B) \) is isometric to \( \mathcal{A}(D) \) via the map \( p \mapsto f_p = \tilde{\varphi} + i \tilde{\psi} \), where \( \tilde{\varphi} = \varphi|_{D_\omega}, \tilde{\psi} = \psi|_{D_\omega} \), and \( D_\omega = \{ x \in B \mid x \cdot \omega = 0 \} \) is the disc properly oriented and identified with \( D \subset \mathbb{C} \).

Thus, being not an algebra, the harmonic space contains algebras. A reserve of these algebras is rich enough: the following fact will be established later in sec 3.

Lemma 1. The relation

\[ \text{span} \{ A_\omega(B) \mid \omega \in S^2 \} = \mathcal{D}(B) \] (6)

is valid (the closure in \( C^\mathbb{H}(B) \)).

\( \mathbb{H} \)-characters

Recall that \( \mathcal{L}(F,G) \) is the space of linear operators from \( F \) to \( G \).

- In the 3d case, a role of the dual space \( \mathcal{A}'(D) \) is played by the space

\[ \mathcal{D}^\mathbb{H}(B) = \{ l \in \mathcal{L}(\mathcal{D}(B), \mathbb{H}) \mid l(hp) = \mathfrak{h} l(p), \forall p \in \mathcal{D}(B), \mathfrak{h} \in \mathbb{H} \}, \]

which we call an \( \mathbb{H} \)-dual to \( \mathcal{D}(B) \); its elements are named by \( \mathbb{H} \)-functionals.

By this definition, one has \( \mathfrak{h} l(p) = l(h \mathfrak{p}) = l(h) l(p) \), which implies

\[ l(h) = \mathfrak{h}, \quad \mathfrak{h} \in \mathbb{H} \] (7)
for all constant fields $h$.

In addition, note that $\mathcal{D}^\times(B)$ can be endowed with a left $\mathbb{H}$-module structure $[\mathbb{H}]$.

- In the capacity of a 3d-analog of the Gelfand spectrum $\mathcal{M}$ of algebra $\mathcal{A}(D)$, we propose the set

$$\mathcal{M}^\mathbb{H} := \{ \mu \in \mathcal{D}^\times(B) \mid \mu(pq) = \mu(p)\mu(q), \ \forall p, q \in \mathcal{A}_\omega(B), \ \omega \in S^2 \}$$

endowed with $\ast$-weak topology determined by the convergence

$$\{ \mu_j \to \mu \} \Leftrightarrow \{ \mu_j(f) \overset{\mathbb{H}}{\to} \mu(f), \ \forall f \in \mathcal{D}(B) \}.$$

It looks reasonable to call $\mathcal{M}^\mathbb{H}$ an $\mathbb{H}$-spectrum of the harmonic space $\mathcal{D}(B)$ and name its elements by $\mathbb{H}$-characters.

An example of $\mathbb{H}$-characters is provided by the ‘quaternion Dirac measures’

$$\delta^\mathbb{H}_{x_0}(p) = p(x_0), \quad p \in \mathcal{D}(B) \quad (x_0 \in B).$$

Note that $\delta^\mathbb{H}_{x_0}$ is well defined and multiplicative on the ‘big’ algebra $\mathcal{C}^\mathbb{H}(B)$.

**Main result**

The above mentioned example turns out to be universal: as will be proven later, the $\mathbb{H}$-spectrum is exhausted by the quaternion Dirac measures.

**Theorem 2.** For any $\mu \in \mathcal{M}^\mathbb{H}$, there is a point $x^\mu \in B$ such that $\mu = \delta^\mathbb{H}_{x^\mu}$. The map

$$\gamma : \mathcal{M}^\mathbb{H} \to B, \quad \mu \mapsto x^\mu$$

is a homeomorphism, so that $\mathcal{M}^\mathbb{H} \cong B$ holds. The Gelfand transform

$$\Gamma : \mathcal{D}(B) \to C(\mathcal{M}^\mathbb{H}; \mathbb{H}), \quad (\Gamma f)(\mu) := \mu(f), \ \mu \in \mathcal{M}^\mathbb{H}$$

is an isometry onto its image, so that $\mathcal{D}(B) \cong \Gamma(\mathcal{D}(B))$ holds.

Thus, $\Gamma$ just transfers the fields from $B$ to $\mathcal{M}^\mathbb{H}$ along the map $\gamma$.

Notice in addition that Dirac measures exhaust the set of $\mathbb{H}$-linear functionals of the ‘big’ algebra $\mathcal{C}^\mathbb{H}(B)$ $[\mathbb{H}, \mathbb{H}]$. 

8
Summary

For reader’s convenience, we present a correspondence table between the classical objects and their 3d-analogs.

| Classical Object | 3d-Analog |
|-----------------|-----------|
| disk $D \subset \mathbb{C}$ | ball $B \subset \mathbb{R}^3$ |
| algebra $C^c(D)$ | algebra $C^{ai}(B)$ |
| $f = \varphi + i\psi$: $d\psi = \ast d\varphi$ | $p = \{\alpha, u\}$: $du' = \ast d\alpha$, $\delta u' = 0$ |
| holom. func. algebra $\mathcal{A}(D)$ | harm. quat. field space $\mathcal{D}(B)$ |
| $\mathcal{A}(D)$ | axial algebras $\mathcal{A}_{\omega}(B) \subset \mathcal{D}(B)$ |
| $\mathcal{A}'(D)$ | $\mathbb{H}$-linear functionals $\mathcal{D}^\times(B)$ |
| $\mathcal{M} \cong D$ | $\mathcal{M}^\mathbb{H} \cong B$ |

Comments

- As is mentioned in Introduction, our results are obtained in the framework of algebraic version of the so-called BC-method, which is an approach to inverse problems of mathematical physics [2], [4]–[9]. More precisely, the impact comes from the impedance tomography of 3d Riemannian manifolds [3, 7]. To answer the following questions would be helpful for the progress in this application. The questions are put not for a ball $B$ but a domain $\Omega \subset \mathbb{R}^3$ that should not lead to confusion since the proper generalizations are evident. However, we don’t know the answers even for a ball.

1. For an algebra $\mathcal{A}$ and a set $S \subset \mathcal{A}$, by $\vee S$ we denote a minimal (sub)algebra in $\mathcal{A}$, which contains $S$. Does

$$\vee \mathcal{D}(\Omega) = C^\mathbb{H}(\Omega)$$

hold at least for a class of $\Omega$’s? Presumably, the answer may be got by the proper application of Corollary 1 from [10].

2. For $p \in C^\mathbb{H}(\Omega)$, we denote $p^\partial = p|_{\partial\Omega}$. A simple fact is that a harmonic quaternion field is determined by its boundary values [2]. Therefore, the algebras $\vee \mathcal{D}(\Omega) \subset C^\mathbb{H}(\Omega)$ and $\vee\{p^\partial | p \in \mathcal{D}(\Omega)\} \subset C^\mathbb{H}(\partial\Omega)$ turn out to be isomorphic (but not isometric!).

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2Moreover, owing to the relevant maximal principle, the map $\mathcal{D}(\Omega) \ni p \mapsto p^\partial \in C^\mathbb{H}(\partial\Omega)$ preserves the norms [6].
A boundary algebra

\[ \mathcal{B} = \overline{\{p^a \mid p \in \mathcal{D}(\Omega)\}} \]

(the closure in \( C^H(\partial \Omega) \)) is a noncommutative Banach uniform algebra. What is the relation between \( \mathcal{B} \) and \( C^H(\Omega) \)? Let \( \mathcal{M}(\mathcal{B}) \) be a structure space of \( \mathcal{B} \); is there a chance for \( \mathcal{M}(\mathcal{B}) \cong \Omega \)?

In fact, all of these questions are of auxiliary character. They become full-valued and important if \( \Omega \) is a Riemannian manifold with boundary. In such a case, the algebra \( \mathcal{B} \) is also well defined and seems to be a most promising device for solving 3d impedance tomography problem: to recover \( \Omega \) via its Dirichlet-to-Neumann operator [9, 7]. Any informative results on \( \mathcal{B} \) are welcomed.

- Theorem 1 is valid not only for a disc but a much wider class of domains on \( \mathbb{C} \) (see, e.g., [13]). In the mean time, one can show that Theorem 2 remains true for a convex bounded \( \Omega \subset \mathbb{R}^3 \). Also, it is valid for a torus \( \{x \in \mathbb{R}^3 \mid \text{dist}(x, L) \leq \kappa R\} \), where \( L \subset \mathbb{R}^3 \) is a circle of radius \( R \) and \( 0 < \kappa < 1 \). However, the class of available \( \Omega \)'s is not properly specified yet.

- Our considerations enable one to set up a ‘3d corona problem’ for the space of harmonic quaternion fields bounded in \( \mathcal{B} \) (by analogy with \( H^\infty(D) \)).

It would be interesting to extend Theorem 2 to \( \Omega \in \mathbb{R}^n \). Presumably, such an extension has to deal with the spaces and algebras of harmonic differential forms [9].

### 3 Proofs

In what follows, identifying the quaternions \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) with elements of the standard basis in \( \mathbb{R}^3 \) (see [2]), we write \( x = x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k} \in \mathbb{R}^3 \).

Recall that for the vector fields \( u = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} \) one defines the Laplacian by \( \Delta u = \Delta u_1 \mathbf{i} + \Delta u_2 \mathbf{j} + \Delta u_3 \mathbf{k} \). The harmonic vector fields are the ones satisfying \( \Delta u = 0 \).

#### Polynomials

Let

\[ \text{3 The 2d version of algebra } \mathcal{B} \text{ solves the 2d tomography problem on manifolds: see [2], [4], [5].} \]
Π be a linear space of the (scalar) polynomials of the variables \(x_1, x_2, x_3\); \(\Pi_n \subset \Pi\) the polynomials of degree \(n \geq 0\); \(\bar{\Pi}_n \subset \Pi_n\) the homogeneous polynomials, i.e., a linear span of monomials \(x_1^{r_1}x_2^{r_2}x_3^{r_3}\) with \(r_1 + r_2 + r_3 = n\);

\[ P = \{\alpha \in \Pi \mid \Delta \alpha = 0\}, \quad P_n = \{\alpha \in \Pi_n \mid \Delta \alpha = 0\}, \quad P_n = \{\alpha \in \Pi_n \mid \Delta \alpha = 0\} \]

the harmonic polynomials;

\[ P = \{\alpha \in \Pi \mid \Delta \alpha = 0\}, \quad P_n = \{\alpha \in \Pi_n \mid \Delta \alpha = 0\}, \quad P_n = \{\alpha \in \Pi_n \mid \Delta \alpha = 0\} \]

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the homogeneous polynomials;

\[ \mathcal{P} = \{p = \{\alpha, u\} \mid u = u_1i + u_2j + u_3k : \alpha, u_k \in \Pi; \ \nabla \alpha = \text{rot} u, \ \text{div} u = 0\} \]

the harmonic quaternion polynomials; the harmonicity implies \(\alpha, u_k \in \Pi\);

\[ \hat{\mathcal{P}}_n = \{\alpha, u \in \mathcal{P} \mid u = u_1i + u_2j + u_3k : \alpha, u_k \in \mathcal{P}_n\} \]

the homogeneous harmonic quaternion polynomials of degree \(n\);

The notations \(\{\ldots\}^\omega\) mean that a set \(\{\ldots\}\) consists of the objects (functions, vector fields, quaternion fields, etc), which are \(\omega\)-axial, i.e., take constant values on the straight lines \(\{x = x_0 + t\omega \mid x_0 \in \mathbb{R}^3, t \in \mathbb{R}\}\). So, \(P^\omega, \Pi^\omega, \mathcal{P}^\omega, \ldots\) are of clear meaning.

**Lemma 2.** The relation

\[ \mathcal{P} = \text{span} \{\mathcal{P}^\omega \mid \omega \in S^2\} \]

holds.

**Proof.**

- The relation \((x_1 + x_2i)^n = R_n(x_1, x_2) + I_n(x_1, x_2)i\) determines the \(k\)-axial polynomials \(R_n, I_n \in \hat{P}_n\) satisfying \(\nabla I_n = k \wedge \nabla R_n\). Any harmonic homogeneous polynomial of variables \(x_1, x_2\) of degree \(n\) is \(aR_n(x_1, x_2) + bI_n(x_1, x_2)\) with \(a, b \in \mathbb{R}\). Hence, \(\dim \hat{P}_n^k = 2\). Quite analogously, one has

\[ \dim \hat{P}_n^\omega = 2 \quad \text{for any } \omega \in S^2. \]

- We omit the proof of the following result, which is simply derived by induction.

**Proposition 3.** Let \(\omega_1, \ldots, \omega_r \in S^2\) be the pair-wise different vectors: \(\omega_i \neq \pm \omega_j\). The relation

\[ \dim \text{span} \left\{ \hat{P}_n^{\omega_k} \mid k = 1, \ldots, r\right\} = \begin{cases} 2r, & r \leq n \\ 2n + 1, & r > n \end{cases} \]

holds.
• Let \( \omega_1, \ldots, \omega_{n+1} \in S^2, \omega_i \neq \pm \omega_j \). By relations (9) and (10), the subspaces \( \dot{P} \omega_1, \ldots, \dot{P} \omega_n \subset \dot{P}_n \) are linearly independent, whereas \( \dot{P} \omega_1, \ldots, \dot{P} \omega_{n+1} \) are not independent, and \( \dim \dot{P} \omega_{n+1} \cap \text{span} \{ \dot{P} \omega_1, \ldots, \dot{P} \omega_n \} = 1 \). Therefore, choosing a nonzero \( \varphi_{n+1} \in \dot{P} \omega_{n+1} \cap \text{span} \{ \dot{P} \omega_1, \ldots, \dot{P} \omega_n \} \), one determines the nonzero \( \varphi_k \in \dot{P} \omega_k \) such that

\[
\varphi_1 + \cdots + \varphi_{n+1} = 0 \tag{11}
\]

holds. Moreover, \( \varphi_k \) are unique up to a constant multiplier: if \( \varphi'_k \in \dot{P} \omega_k \) and \( \varphi'_1 + \cdots + \varphi'_{n+1} = 0 \) then \( \varphi'_k = \lambda \varphi_k \) with some \( \lambda \in \mathbb{R} \).

Let \( \psi_k \in \dot{P} \omega_k \) be dual to \( \varphi_k \), i.e., \( \nabla \psi_k = \omega_k \wedge \nabla \varphi_k \). The element

\[
u^* = \psi_1 \omega_1 + \cdots + \psi_{n+1} \omega_{n+1}\]

is a nonzero vector field. Indeed, assuming \( \nu^* = 0 \), we have

\[
0 = \nu^* \cdot \omega_{n+1} = a_1 \psi_1 + \cdots + \psi_{n+1}
\]

with \( a_k = \omega_1 \cdot \omega_{n+1} \) and \( \psi_k \in \dot{P} \omega_k \). By the above mentioned uniqueness of the summands in (11) we get \( \psi_{n+1} = \lambda \varphi_{n+1} \) that contradicts to \( \nabla \psi_{n+1} = \omega_{n+1} \wedge \nabla \varphi_{n+1} \).

As a consequence of (11), (12), we get a nonzero quaternion field

\[
\{0, \nu^*\} = \{\varphi_1, \psi_1 \omega_1\} + \cdots + \{\varphi_{n+1}, \psi_{n+1} \omega_{n+1}\} \tag{13}
\]

which belongs to \( \mathcal{P}_n \), has the zero scalar component, and is expanded over the axial fields \( \{\varphi_k, \psi_k \omega_k\} \in \dot{P}_n \).

Since \( \{0, \nu^*\} \in \mathcal{P} \), one has \( \text{rot} \, \nu^* = \nabla 0 = 0 \). Hence, \( \nu^* \) is a potential field and one can represent

\[
u^* = \nabla \beta^* \quad \text{with} \quad \beta^* \in \dot{P}_{n+1}. \tag{14}
\]

• Let \( \mathcal{P}_0 \) be the subspace in \( \mathcal{P}_n \) of elements with zero scalar component, so that \( \{0, \nu^*\} \in \mathcal{P}_0 \). In the mean time, all elements of \( \mathcal{P}_0 \) are of the form (14), i.e.,

\[
\mathcal{P}_0 = \{0, \nabla \beta\} \mid \beta \in \dot{P}_{n+1}\}
\]

By \( O_3 \) we denote the group of rotations of \( \mathbb{R}^3 \). As is easy to check, \( \mathcal{P}_0 \) provides a finite-dimensional representation of the rotation group with the action

\[
R : \{0, \nabla \beta(x)\} \mapsto \{0, R[\nabla \beta(R^{-1}x)]\}, \quad x \in \mathbb{R}^3 \ (R \in O_3).
\]

12
As is obvious, such a representation is in fact identical to the standard representation of $O_3$ in $\hat{P}_{n+1}$. By the latter, it is irreducible.

In the meantime, the subspace $\text{span}\{R\{0,u^*\} \mid R \in O_3\} \subset \hat{P}_n$ is invariant w.r.t. the group action. Hence, the irreducibility yields

$$\text{span}\{R\{0,u^*\} \mid R \in O_3\} = \hat{P}_n.$$  \hspace{1cm} (15)

As it easily follows from (13), the fields $R\{0,u^*\}$ are also sums of the axial fields. Hence, each element of the left hand side in (15) is a finite sum of axial polynomial harmonic fields:

$$\hat{P}_n = \text{span}\{\hat{P}_n^\omega \mid \omega \in S^2\}.$$  \hspace{1cm} (16)

- Show that

$$\hat{P}_n = \text{span}\{\hat{P}_n^\omega \mid \omega \in S^2\}.$$  \hspace{1cm} (17)

Take $p = \{\alpha, u\} \in \hat{P}_n$. Represent $\alpha = \phi_1 + \cdots + \phi_l$ with $\phi_k \in P_n^\omega$. Let $\eta_k \in P_n^\omega$ satisfy $\nabla \eta_k = \omega_k \wedge \nabla \phi_k$, so that $p_k' = \{\phi_k, \eta_k \omega_k\} \in \hat{P}_n^\omega$. The quaternion polynomial $p' = p_1' + \cdots + p_l'$ belongs to the r.h.s. of (17). Representing $p = p' + p_0$, one has $p_0 = \{0, u_0\} \in \hat{P}_0$ by construction. In the meantime, (16) yields $p_0$ to be a sum of axial fields. Hence, eventually, $p$ is also a finite sum of axial polynomial harmonic fields, i.e., (17) does hold.

- Notice that the elements of $\mathcal{P}_0$ (constant fields) are axial. Then, representing

$$\mathcal{P} = \text{span}\{\hat{P}_n \mid n \geq 0\} \overset{17}{=} \text{span}\{\hat{P}_n^\omega \mid n \geq 0, \omega \in S^2\} = \text{span}\{\hat{P}_n^\omega \mid \omega \in S^2\},$$

we get (8) and prove Lemma 2.

**Density lemma**

Here we prove Lemma 1.

Take a field $p = \{\alpha, u\} \in \mathcal{P}(B)$ and show that it can be approximated by elements of $\mathcal{P}$.

\[\text{such a representation is not unique but any is available}\]
• We say $p = \{\alpha, u\}$ to be smooth and write $p \in \mathcal{S}$ if $\alpha \in C^2(B)$ and $u \in C^2(B; \mathbb{R}^3)$. As is well known, the lineal $\mathcal{Q}(B) \cap \mathcal{S}$ is dense in $\mathcal{Q}(B)$.

Let $p = \{\alpha, u\}$ be smooth. Fix a (small) $\varepsilon > 0$. For the harmonic divergence-free field $u$ one can find a harmonic polynomial vector field $v$ such that $\|u - v\|_{C^2(B; \mathbb{R}^3)} < \varepsilon$. Since $\text{div} u = 0$, the latter inequality yields

$$\|\text{div} v\|_{C^1(B)} < \text{const} \varepsilon.$$  \hfill (18)

Also, we have

$$\Delta \text{div} v = \text{div} \Delta v = 0 \quad \text{in } B, \quad$$ \hfill (19)

so that $\text{div} v$ is a ‘small’ scalar harmonic polynomial in the ball.

• By $\partial_{x_k}$ we denote the partial derivative w.r.t. $x_k$.

Let $q = q(x_1, x_2)$ be a polynomial satisfying

$$\Delta q(x_1, x_2) = -\partial_{x_3} [\text{div} v](x_1, x_2, 0).$$ \hfill (20)

The function

$$\tilde{\eta}(x_1, x_2, x_3) = q(x_1, x_2) + \int_0^{x_3} [\text{div} v](x_1, x_2, t) \, dt$$

is a scalar harmonic polynomial. Indeed,

\begin{align*}
\Delta \tilde{\eta}(x_1, x_2, x_3) &= \Delta q(x_1, x_2) + \int_0^{x_3} \left[ \partial_{x_1}^2 \text{div} v + \partial_{x_2}^2 \text{div} v \right] (x_1, x_2, t) \, dt + \\
&\quad + \partial_{x_3} [\text{div} v](x_1, x_2, x_3) \quad \overset{19}{=} \\
&= \Delta q(x_1, x_2) - \int_0^{x_3} \partial_{x_3} [\text{div} v](x_1, x_2, t) \, dt + \partial_{x_3} [\text{div} v](x_1, x_2, x_3) \quad \overset{20}{=} \\
&= -\partial_{x_3} [\text{div} v](x_1, x_2, 0) - \{\partial_{x_3} [\text{div} v](x_1, x_2, x_3) - \partial_{x_3} [\text{div} v](x_1, x_2, 0)\} + \\
&\quad + \partial_{x_3} [\text{div} v](x_1, x_2, x_3) = 0.
\end{align*}

Next, let $r = r(x_1, x_2)$ be a harmonic polynomial satisfying

$$\Delta r = 0, \quad r = q \quad \text{as } x_1^2 + x_2^2 = 1.$$

By the choice, one has

$$\Delta (q - r) \overset{20}{=} -\partial_{x_3} [\text{div} v](\cdot, \cdot, 0), \quad q - r = 0 \quad \text{as } x_1^2 + x_2^2 = 1.$$ \hfill (21)
Owing to estimate (18), the well-known properties of the elliptic Dirichlet problem (21) provide
\[ \| q - r \|_{C^2(B)} \leq \text{const} \varepsilon. \] (22)

Estimating the integral with regard to (18) and taking into account (22), we conclude that the function
\[ \eta = \tilde{\eta} - r, \]
\[ \eta(x_1, x_2, x_3) = q(x_1, x_2) - r(x_1, x_2) + \int_0^{x_3} \| \text{div} v \| (x_1, x_2, t) \, dt \]
is a harmonic polynomial, which satisfies
\[ \partial_{x_3} \eta = \text{div} v, \quad \| \eta \|_{C^1(B)} \leq \text{const} \varepsilon. \]

- By the latter, \( \eta k \) is a harmonic polynomial vector field, which satisfies \( \text{div} \eta k = \text{div} v \) and \( \| \eta k \|_{C^1(B;\mathbb{R}^3)} < \text{const} \varepsilon \). As a consequence, \( \tilde{u} = v - \eta k \) is a harmonic polynomial vector field satisfying \( \text{div} \tilde{u} = 0 \).

Representing \( u - \tilde{u} = u - v - \eta k \), we have
\[ \| u - \tilde{u} \|_{C^1(B;\mathbb{R}^3)} \leq \| u - v \|_{C^2(B;\mathbb{R}^3)} + \| \eta k \|_{C^1(B;\mathbb{R}^3)} \leq \text{const} \varepsilon. \] (23)
So, the vector component \( u \) of the smooth harmonic quaternion field \( p \) is approximated by the harmonic polynomial divergence-free vector field \( \tilde{u} \).

In the mean time, \( \text{rot} \tilde{u} \) is also a harmonic polynomial divergence-free vector field and the estimate
\[ \| \text{rot} u - \text{rot} \tilde{u} \|_{C(B;\mathbb{R}^3)} \leq \text{const} \varepsilon \] (24)

obviously follows from (23).

- By the construction of \( \tilde{u} \), one has \( \text{rot} \text{rot} \tilde{u} = \nabla \text{div} \tilde{u} - \Delta \tilde{u} = 0 \). Hence, the field \( \text{rot} \tilde{u} \) has a scalar potential in \( B \). Therefore, integrating over the proper paths in \( B \), one can find a scalar harmonic polynomial \( \tilde{\alpha} \) satisfying \( \nabla \tilde{\alpha} = \text{rot} \tilde{u} \) and \( \tilde{\alpha}(0, 0, 0) = \alpha(0, 0, 0) \).

The potential \( \tilde{\alpha} \) and the vector field \( \tilde{u} \) determine a harmonic polynomial quaternion field \( \tilde{p} = \{ \tilde{\alpha}, \tilde{u} \} \in \mathcal{P} \). The estimate
\[ \| \nabla \alpha - \nabla \tilde{\alpha} \|_{C(B;\mathbb{R}^3)} = \| \text{rot} u - \text{rot} \tilde{u} \|_{C(B;\mathbb{R}^3)} \leq \text{const} \varepsilon \]
(24)
easily implies \( \| \alpha - \tilde{\alpha} \|_{C(B)} \leq \text{const} \varepsilon \).
Summarizing, we arrive at
\[ \| p - \tilde{p} \|_{C^h(B)} = \left\{ \| \alpha - \tilde{\alpha} \|_{C(B)}^2 + \| u - \tilde{u} \|_{C(B,\mathbb{R}^3)}^2 \right\}^{\frac{1}{2}} \leq \text{const } \varepsilon \]
and conclude that \( \mathcal{P} \) is dense in the lineal of smooth fields. Since this lineal is dense in \( \mathcal{Q}(B) \), one obtains
\[ \mathcal{P} = \mathcal{Q}(B). \]  
(25)

- With regard to \( \mathcal{P}^\omega \subset \mathcal{A}_\omega(B) \), we have
\[ \mathcal{P} = \text{span} \{ \mathcal{P}^\omega \mid \omega \in S^2 \} \subset \text{span} \{ \mathcal{A}_\omega(B) \mid \omega \in S^2 \}, \]
whereas (25) follows to (6) and proves Lemma 1.

**Correspondence** \( \mu \mapsto x^\mu \)

- The quaternion fields of the form \( \pi(x) = \{ x \cdot a, Ax \} \), where \( a \in \mathbb{R}^3 \) is a vector and \( A \) is a constant 3×3-matrix, are called linear. Note that \( \nabla[x \cdot a] = a, \) \( \text{rot} \ Ax \) is a constant vector field, and \( \text{div} \ Ax = \text{tr} \ A \) holds. Therefore, a linear field is harmonic if and only if \( a = \text{rot} \ Ax \) and \( \text{tr} \ A = 0 \).

By \( \mathcal{L}(B) \subset \mathcal{Q}(B) \) we denote the subspace of the linear harmonic fields reduced to the ball.

- Let \( \omega_1, \omega_2, \omega_3 \) be a basis in \( \mathbb{R}^3 \) normalized by
\[ \omega_k \cdot \omega_l = \delta_{kl}; \quad \omega_1 \wedge \omega_2 = \omega_3, \quad \omega_2 \wedge \omega_3 = \omega_1, \quad \omega_3 \wedge \omega_1 = \omega_2. \]
The axial linear fields
\[ \pi_1(x) = \{ x \cdot \omega_1, [x \cdot \omega_2] \omega_3 \}, \quad \pi_2(x) = \{ x \cdot \omega_2, [x \cdot \omega_3] \omega_1 \}, \quad \pi_3(x) = \{ x \cdot \omega_3, [x \cdot \omega_1] \omega_2 \} \]
are called the *coordinate fields*. Since
\[ \nabla[x \cdot \omega_1] = \omega_1 = \omega_2 \wedge \omega_3 = \nabla[x \cdot \omega_2] \wedge \omega_3 = \text{rot} [x \cdot \omega_2] \omega_3, \quad \text{div} [x \cdot \omega_2] \omega_3 = 0, \]
the field \( \pi_1 \) is harmonic. Analogously, \( \pi_2, \pi_3 \) are harmonic.

We omit the proof of the following simple fact: harmonic linear fields are expanded over the coordinate fields. As example, any such \( \pi \) is uniquely represented in the form
\[ \pi = g \pi_2 + h \pi_3 \quad \text{with } g, h \in \mathbb{H}. \]  
(26)
In particular, the equality
\[ \pi_1 = o_3\pi_2 - o_2\pi_3 \] (27)
holds with \( o_k = \{0,\omega_k\} \in \mathbb{H} \) and can be verified by simple calculations.

- Take \( \mu \in \mathcal{M}^H \) and fix \( \omega \in S^2 \). Since \( \mu \) is multiplicative on \( \mathcal{A}_\omega(B) \), the image \( \mathcal{A}_\omega = \mu(\mathcal{A}_\omega(B)) \) is a commutative subalgebra of \( \mathbb{H} \). By (7), one has \( \{0,\omega\} \in \mathcal{A}_\omega \). Hence, \( \mathcal{A}_\omega = \{\{a,b\omega\} \mid a,b \in \mathbb{R}^3\} \) holds.

By the aforesaid, we have
\[ \mu(\pi_1) = \{a_{12},b_{12}\omega_3\}, \quad \mu(\pi_2) = \{a_{23},b_{23}\omega_1\}, \quad \mu(\pi_3) = \{a_{31},b_{31}\omega_2\} \]
with some \( a_{kl}, b_{kl} \in \mathbb{R} \).

Applying \( \mu \) to (27), we get
\[ \mu(\pi_1) = o_3\mu(\pi_2) - o_2\mu(\pi_3) \]
or, in detail,
\[ \{a_{12},b_{12}\omega_3\} = \{0,\omega_3\}\{a_{23},b_{23}\omega_1\} - \{0,\omega_2\}\{a_{31},b_{31}\omega_2\} = \]
\[ = \{0,a_{23}\omega_3 + b_{23}\omega_2\} - \{-b_{31},a_{31}\omega_2\} = \{b_{31},a_{23}\omega_3 + [b_{23} - a_{31}]\omega_2\}. \]
Comparing by components, we get \( a_{12} = b_{31}, \quad b_{12} = a_{23}, \quad b_{23} = a_{31} \).

- Denote \( a = a_{12} = b_{31}, \quad b = b_{12} = a_{23}, \quad c = b_{23} = a_{31} \) and define a vector
\[ x^\mu = a\omega_1 + b\omega_2 + c\omega_3 \in \mathbb{R}^3. \]

Summarizing the previous calculations, we easily conclude that
\[ \mu(\pi_1) = \pi_1(x^\mu), \quad \mu(\pi_2) = \pi_2(x^\mu), \quad \mu(\pi_3) = \pi_3(x^\mu). \] (28)
So, the functional \( \mu \), which is \( \mathbb{H} \)-linear and multiplicative on the axial algebras, determines a point \( x^\mu \in \mathbb{R}^3 \) associated with it via (28).

---

5Here and in what follows, we regard the linear fields to be reduced on \( B \), i.e., regard them as elements of \( \mathcal{L}(B) \).
Completing proof of Theorem 2
• Applying $\mu$ to (26), we have
\[
\mu(\pi) = \mu(g\pi_2 + h\pi_3) = g\mu(\pi_2) + h\mu(\pi_3) = g\pi_2(x^\mu) + h\pi_3(x^\mu) = [g\pi_2 + h\pi_3](x^\mu) = \pi(x^\mu) \tag{28}
\]
that extends (28) to $\mathcal{L}(B)$.

• As is well known, the disk algebra $\mathcal{A}(D)$ is generated by the functions 1, $z$. By perfect analogy with this fact, any axial algebra $\mathcal{A}_\omega(B) \cong \mathcal{A}(D)$ is generated by two fields $\{1, 0\}$ and $\{x \cdot \eta, [x \cdot \eta \wedge \omega] \omega \}$, where $\eta \in S^2$ is arbitrary provided $\eta \cdot \omega = 0$. Therefore, (29) is extended to $\mathcal{A}_\omega(B)$ by continuity and yields
\[
\mu(p) = p(x^\mu), \quad p \in \mathcal{A}_\omega(B) \quad (\omega \in S^2).
\]
Here we see that the point $x^\mu$ must belong to the ball $B$. Otherwise, one can chose (for instance, the polynomials) $p_j \in \mathcal{A}_\omega(B)$ such that $\|p_j\|_{\mathcal{A}(B)} \leq \text{const}$ and $|p_j(x^\mu)| \to \infty$ in contradiction to the continuity of $\mu$.

Then, in accordance with Lemma 1 one extends $\mu$ to $\mathcal{C}(B)$ an obtains $\mu(p) = p(x^\mu)$ for all $p \in \mathcal{A}(B)$. Hence, we conclude that $\mu = \delta_{x^\mu}$ is valid.

• The latter equality easily implies that the bijection $\mathcal{M} \ni \mu \mapsto x^\mu \in B$ is a homeomorphism of topological spaces. To show this one can just check that $\mu_j \to \mu$ in $\mathcal{M}$ is equivalent to $x^{\mu_j} \to x^\mu$ in $B$.

The proof of Theorem 2 is completed.

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