A NOTE ON THE CARISTI FIXED POINT THEOREM

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Abstract. We generalise the Caristi Fixed Point Theorem to the mappings of the complete semi-metric spaces.

1. Main Theorems

1.1. Definitions and Assumptions. Let $X$ be a semi-metric space with collection of semi-metrics $\{d_\alpha(\cdot,\cdot)\}_{\alpha \in A}$, and $A$ is an arbitrary non-empty set. Recall that a semi-metric satisfy the following properties.

For any $x, y, z \in X$ one has

$$d_\alpha(x,y) = d_\alpha(y,x), \quad d_\alpha(y,x) \leq d_\alpha(y,z) + d_\alpha(z,x), \quad d_\alpha(x,y) \geq 0$$

and $d_\alpha(x,x) = 0$.

Assume also that the space $X$ is separated in the following sense. For any different points $x, y \in X$ there exists a semi-metric $d_\alpha$ such that $d_\alpha(x,y) > 0$.

Definition 1.1. Let $(Q, \prec)$ be a partially ordered set with the following extra condition. For every pair $n, m \in Q$ there is an element $l \in Q$ such that $m, n \prec l$.

Then a set $\{x_q\}_{q \in Q} \subset X$ is said to be a net.

Definition 1.2. We shall say that a net $\{x_q\}$ tends to $x$ iff for any $\varepsilon > 0$ and for any $\alpha \in A$ there is $c \in Q$ such that if $q \succ c$ then $d_\alpha(x_q, x) < \varepsilon$.

We shall denote this in the following way:

$$\lim_{q \in Q} x_q = x.$$

Definition 1.3. A net $\{x_q\}_{q \in Q}$ is a Cauchy net iff for any $\varepsilon > 0$ and for any $\alpha \in A$ there is $c \in Q$ such that if $q', q \succ c$ then $d_\alpha(x_{q'}, x_q) < \varepsilon$.

Suppose that the space $X$ is complete i.e. every Cauchy net $\{x_q\}_{q \in Q}$ tends to a point $x$.

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Definition 1.4. We shall say that a function \( g : X \to \mathbb{R} \) is lower semi-continuous at a point \( x' \) if for every net \( x_q \to x' \) one has
\[
\liminf_{x_q \to x'} g(x_q) \geq g(x').
\]
The last expression implies that \( a_q \to a \geq g(x') \), \( a_q = \inf_{q \prec u} g(x_u) \).

A function \( g : X \to \mathbb{R} \) is called lower semi-continuous provided it is lower semi-continuous at all the points \( x \in X \).

1.2. Generalised Caristi’s Theorems. Consider a mapping \( f : X \to X \).

Theorem 1.1. Suppose that for each \( \alpha \in A \) there exists a lower semi-continuous and bounded from below function \( \psi_\alpha : X \to \mathbb{R} \) such that
\[
d_\alpha(x, f(x)) \leq \psi_\alpha(x) - \psi_\alpha(f(x)).
\] (1.1)

Then the mapping \( f \) has a fixed point: \( f(\hat{x}) = \hat{x} \).

Remark 1. From Theorem 1.1 one can obtain a version of the contraction mapping principle for semi-metric spaces.

Suppose a mapping \( f : X \to X \) is continuous and has the following property.

For any \( \alpha \in A \) one can define an element \( \gamma \in A \) and a number \( c > 0 \) such that
\[
d_\gamma(f(x), f(y)) \leq d_\gamma(x, y) - cd_\alpha(x, y), \quad \forall x, y \in X.
\]

Then mapping \( f \) has a unique fixed point.

To prove this one should take \( \psi_\alpha(x) = d_\gamma(x, f(x))/c \).

If \( A \) consists of a single element this proposition becomes the standard contraction mapping theorem.

Theorem 1.2. If under the conditions of Theorem 1.1 at least one of the functions \( \{\psi_\alpha\} \) does not attain its minimum then the set of fixed points of the mapping \( f \) is not compact.

For the case of metric \( X \) theorems 1.1 1.2 have been obtained by Caristi in [1]. Since that time these results have been generalised by many authors in different directions.

2. Proof of the Theorems

2.1. Proof of Theorem 1.1. Introduce in the set \( X \) a partial order by the formula
\[
x \ll y \iff d_\alpha(x, y) \leq \psi_\alpha(x) - \psi_\alpha(y), \quad \forall \alpha \in A.
\]

It is easy to check that the relation \( \ll \) is a partial order indeed.

Formula (1.1) takes the form \( x \ll f(x) \) for all \( x \in X \). Thus to complete the proof it is enough to show that the set \( X \) has a maximal element.

This in its part immediately follows from the Zorn Lemma if only we check that any chain of \( X \) possess a supremum.
Let \( Z \subseteq X \) be a chain. Since elements of \( Z \) enumerate themselves the set \( Z \) can be considered as a net with \( Q = Z \). So we rewrite the set \( Z \) as \( \{z_q\}_{q \in Z} \).

Show that \( \{z_q\} \) is a Cauchy net.

Take any pair of elements from this net \( z_p, z_q \in Z, z_p \ll z_q \),

\[
\psi_\alpha(z_q) \leq \psi_\alpha(z_p) - d_\alpha(z_p, z_q) .
\] (2.1)

The last inequality implies that the function \( \psi_\alpha \) is decreased on the net \( \{z_q\} \).

So the following limit exists:

\[
\lim_{q \in Z} \psi_\alpha(z_q) = c_\alpha.
\]

Thus passing to the limit in formula (2.1) we get

\[
\lim_{p,q \in Z} d_\alpha(z_p, z_q) = 0.
\]

Consequently, \( \{z_q\} \) is a Cauchy net and \( z_q \to \hat{z} \).

To show that \( \hat{z} \) is a supremum for the chain \( Z \) we consider again formula (2.1):

\[
d_\alpha(z_p, \hat{z}) = \lim_{q \in Z} d_\alpha(z_p, z_q) \leq \psi_\alpha(z_p) - \psi_\alpha(z_q) \leq \psi_\alpha(z_p) - \psi_\alpha(\hat{z}).
\]

The Theorem is proved.

2.2. Proof of Theorem 1.2. Denote by \( P \) a set of the fixed points to the mapping \( f \). Assume that for some index \( \alpha = \alpha' \) the function \( \psi_{\alpha'} \) does not achieve its minimum in \( X \) and the set \( P \) is compact.

Then the function \( \psi_{\alpha'} \) attains its minimum in \( P \), say

\[
\min_{x \in P} \psi_{\alpha'}(x) = \psi_{\alpha'}(w) = c, \quad f(w) = w.
\]

Consider a set \( D_\sigma = \{x \in X \mid \psi_{\alpha'}(x) \leq c - \sigma\} \). By assumption there exists a small \( \sigma > 0 \) such that the set \( D_\sigma \) is not empty.

Being endowed with the same semi-metrics \( \{d_\alpha\} \), the set \( D_\sigma \) becomes a complete semi-metric space.

Indeed, show that \( D_\sigma \) is a closed subset of \( X \). Let \( x_q \to x, \quad \{x_q\} \subseteq D_\sigma \). Then

\[
\psi_{\alpha'}(x) \leq \liminf_{x_q \to x} \psi_{\alpha'}(x_q) \leq c - \sigma.
\]

Thus \( x \in D_\sigma \).

The inclusion \( f(D_\sigma) \subseteq D_\sigma \) is obvious:

\[
\psi_{\alpha'}(f(x)) \leq \psi_{\alpha'}(x) - d_{\alpha'}(x, f(x)) \leq c - \sigma.
\]

By Theorem 1.1 the mapping \( f \) has a fixed point \( f(\tilde{x}) = \tilde{x} \in D_\sigma \). This provides a contradiction since \( \tilde{x} \notin P \).

The Theorem is proved.
REFERENCES

[1] Caristi, J: Fixed point theorems for mappings satisfying inwardness conditions. Trans Am Math Soc. 215, 241-251 (1976)