A Jacobi algorithm for distributed model predictive control of dynamically coupled systems

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Abstract— In this paper we introduce an iterative Jacobi algorithm for solving distributed model predictive control (DMPC) problems, with linear coupled dynamics and convex coupled constraints. The algorithm guarantees stability and persistent feasibility, and we provide a localized procedure for constructing an initial feasible solution by constraint tightening. Moreover, we show that the solution of the iterative process converges to the centralized MPC solution. The proposed iterative approach involves solving local optimization problems consisting of only few subsystems, depending on the choice of the designer and the sparsity of dynamical and constraint couplings. The gain in the overall computational load compared to the centralized problem is balanced by the increased communication requirements. This makes our approach more applicable to situations where the number of subsystems is large, the coupling is sparse, and local communication is relatively fast and cheap. A numerical example illustrates the effects of the local problem size, and the number of iterations on convergence to the centralized solution.

I. INTRODUCTION

Model predictive control (MPC) is the most successful advanced control technology implemented in industry due to its ability to handle complex systems with hard input and state constraints [5], [8], [9]. The essence of MPC is to determine a control profile that optimizes a cost criterion over a prediction window and then to apply this control profile until new process measurements become available. Then the whole procedure is repeated and feedback is incorporated by using the measurements to update the optimization problem for the next step.

For the control problem of large-scale networked systems, centralized MPC is considered impractical, inflexible and unsuitable due to information exchange requirements and computational aspects. The subsystems in the network may belong to different authorities that prevent sending all necessary information to one processing center. Moreover, the optimization problem yielded by centralized MPC can be excessively large for real-time computation. In order to deal with these limitations, distributed MPC is proposed for control of large-scale systems, by decomposing the overall system into small subsystems. The subsystems employ distinct MPC controllers, use local information from neighboring subsystems, and collaborate to achieve globally attractive solutions.

Approaches to distributed MPC design differ from each other in the problem setup. In [3], Camponogara et al. studied stability of coordination-based distributed MPC with several information exchange conditions. In [4], Dunbar and Murray proposed a distributed MPC scheme for problems with coupled cost function, utilizing predicted trajectories of the neighbors in each subsystem’s optimization. Keviczky et al. proposed a distributed MPC scheme with a sufficient stability test for dynamically decoupled systems in [7], in which each subsystem optimizes also over the behaviors of its neighbors. Richards and How in [10] proposed a robust distributed MPC method for networks with coupled constraints, based on constraint tightening and a serial solution approach.

A distributed MPC scheme for dynamically coupled systems called feasible-cooperation MPC (FC-MPC) was proposed by Venkat et al. in [11], [12], based on a parallel synchronous approach for cooperative optimization. This scheme works only for input-coupled linear time-invariant (LTI) subsystem dynamics without state constraints, and is not applicable to problems with constraints between subsystems. In this paper, we propose an extension of this scheme in several ways in order to solve these issues.

The distributed MPC algorithm described in this paper is able to handle LTI dynamics with general dynamical couplings, and the presence of convex coupled constraints. Each local controller optimizes not only for itself, but also for its neighbors in order to gain better overall performance. Global feasibility and stability are achieved, whilst the algorithm can be implemented using local communications. The proposed algorithm is based on an MPC framework with zero terminal point constraint for increased clarity and simplicity. We highlight an open research question that needs to be addressed for a full treatment of the terminal cost based version of this MPC framework, which would allow reduced conservativeness. While other distributed MPC methods typically assume an initial feasible solution to be available, we incorporate a decentralized method to determine an initial feasible solution.

The problem formulation is described in Section II followed by two variations of the algorithm in Section III. It is shown that an algorithm using local communication exists and it is equivalent to one that is based on global communication. In Section IV we analyze the feasibility, stability and optimality of the algorithm. Different ways of customizing the proposed algorithm and a trade-off between communications and computational aspects are discussed in Section V. Finally, Section VI illustrates the algorithm in a numerical example and Section VII concludes the paper.
II. PROBLEM DESCRIPTION

A. Coupled subsystem model

Consider a plant consisting of $M$ subsystems. Each subsystem’s dynamics is assumed to be influenced directly by only a small number of other subsystems. Let each subsystem be represented by a discrete-time, linear time-invariant model of the form:

$$x_{t+1}^i = \sum_{j=1}^{M} (A_{ij} x_t^j + B_{ij} u_t^j), \quad (1)$$

where $x_t^i \in \mathbb{R}^{n_i}$ and $u_t^i \in \mathbb{R}^{m_i}$ are the states and control inputs of the $i$-th subsystem at time $t$, respectively.

Remark 2.1 This is a very general model class for describing dynamical coupling between subsystems and includes as a special case the combination of decentralized models and interaction models in [11].

We define the neighborhood of $i$, denoted by $N^i$, as the set of indices of subsystems that have either direct dynamical or convex constraint coupling with subsystem $i$. In Figure 1, we demonstrate this with an interaction map where each node stands for one subsystem, the dotted links show constraint couplings and the solid arrows represent dynamical couplings. The neighborhood $N^4$ of subsystem 4 is the set of $\{1, 4, 2, 5\}$. We will refer to vectors and sets related to nodes in $N^i$ with a superscript $+i$. The collection of all other nodes that are not included in $N^i$ will be referred to with a superscript $i$.

B. Convex coupled constraints

Each subsystem $i$ is assumed to have local convex coupled constraints involving only a small number of the others. If we fix the control inputs and the corresponding states of the nodes outside $N^i$, the state and input constraints involving the nodes in $N^i$ can be defined in the following way:

$$x_t^{+i} \in X^{+i}(x_t^i), \quad u_t^{+i} \in U^{+i}(u_t^i), \quad \forall i = 1, \ldots, M \quad (2)$$

where $X^{+i}(x_t^i)$ and $U^{+i}(u_t^i)$ are closed and convex sets parameterized by the states and control inputs of nodes outside $N^i$.

Remark 2.2 Note that the constraints involving nodes of $N^i$ in general do not depend on every other state and input outside $N^i$, only on the immediate neighbors of $N^i$. The notation in (2) is used for simplicity.

C. Centralized model

Let $x = \begin{bmatrix} x_1^T & \cdots & x_M^T \end{bmatrix}^T$, $u = \begin{bmatrix} u_1^T & \cdots & u_M^T \end{bmatrix}^T$ denote the aggregated states and inputs of the full plant, with dimensions $\mathbb{R}^{\sum_{i=1}^{M} n_i}$, and $\mathbb{R}^{\sum_{i=1}^{M} m_i}$, respectively. The matrices $A$ and $B$ will denote the aggregated subsystem dynamics matrices and are assumed to be stabilizable:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1M} \\ \vdots & \ddots & \vdots \\ A_{M1} & \cdots & A_{MM} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & \cdots & B_{1M} \\ \vdots & \ddots & \vdots \\ B_{M1} & \cdots & B_{MM} \end{bmatrix}.$$

$$x_{t+1} = Ax_t + Bu_t.$$

The full (centralized) plant model is thus represented as:

$$x_{t+1} = Ax_t + Bu_t,$$  \quad (3)

Remark 2.3 The centralized model defined in (3) is more general than the so-called composite model employed in [11]. In our approach, the centralized model can represent both couplings in states and inputs. In [11], the authors use an input-coupled composite model, which requires the subsystems’ states to be decoupled, allowing only couplings in inputs.

D. Centralized MPC problem

The centralized MPC problem is formulated based on a typical quadratic MPC framework [8] with prediction horizon $N$, and the following quadratic cost function at time step $t$:

$$V_t = \sum_{k=0}^{N-1} x_{k,t}^T Q x_{k,t} + u_{k,t}^T R u_{k,t}$$  \quad (4)

where $x_{k,t}$ denotes the centralized state vector at time $t+k$ obtained by starting from the state $x_{0,t} = x_t$ and applying to system (3) the input sequence $u_{0,t}, \ldots, u_{k-1,t}$. $Q = diag(Q_1, \ldots, Q_M)$, $R = diag(R_1, \ldots, R_M)$ with $diag(\cdot)$ function representing the block diagonal matrix. Matrices $Q_i$ are positive semidefinite and $R_i$ are positive definite.

Let $x_t = \begin{bmatrix} x_{1,t}^T, \ldots, x_{N,t}^T \end{bmatrix}^T$, $u_t = \begin{bmatrix} u_{0,t}^T, \ldots, u_{N-1,t}^T \end{bmatrix}^T$. The centralized MPC problem is then defined as:

$$V_t(x_t) = \min_{x_{0,t}, u_t} \sum_{k=0}^{N-1} x_{k,t}^T Q x_{k,t} + u_{k,t}^T R u_{k,t}$$  \quad (5)

s.t. $x_{k+1,t} = Ax_{k,t} + Bu_{k,t}, k = 0, \ldots, N - 1$,

$u_{k,t} \in \mathcal{U}, k = 0, \ldots, N - 1$,

$x_{N,t} = 0$,

$x_{0,t} = x_t$,

where $\mathcal{U}$ and $\mathcal{X}$ are defined as $\cap_{i=1}^{M} CE(U^{+i})$ and $\cap_{i=1}^{M} CE(X^{+i})$, respectively. The $CE(\cdot)$ operator denotes
cylindrical extension to the set of $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, respectively. In other words, if $\mathcal{X}^i \subset \mathbb{R}^d$ then $CE(\mathcal{X}^i) = \mathcal{X}^i \times \mathbb{R}^{m_1}$. The vector $x_t$ contains the measured states at time step $t$.

Let $u^*_t = [(u^*_{0,t})^T, \ldots, (u^*_{N-1,t})^T]^T$ denote the optimal control solution of (5) at time $t$. Then, the first sample of $u^*_t$ is applied to the overall system:

$$u_t = u^*_0,t. \tag{6}$$

The optimization (5) is repeated at time $t+1$, based on the new state $x_{t+1}$. In [6] it was shown that with prediction horizon $N$ long enough to allow a feasible solution to the optimization problem, the closed-loop system (3)-(6) is stable.

Before formulating the distributed MPC problems, we eliminate the state variables in the centralized MPC formulation. In the following we will also assume $t=0$ without loss of generality and drop subscript $t$ for simplicity. The set of dynamics equations allows us to write the predicted states as

$$x = \alpha u + \beta(x_0), \tag{7}$$

where

$$\alpha = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}, \quad \beta(x_0) = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} x_0.$$

Using the above equations, we can eliminate state variables in the centralized MPC leading to the following problem:

$$\min_{u} V(u, x_0) \tag{8}$$

$$= \min_{u} u^T (\alpha^T Q \alpha + R) u + 2(\alpha^T Q \beta)^T u + \beta^T Q \beta$$

s.t. $u \in \tilde{U}$,

$$\alpha u + \beta(x_0) \in \tilde{X},$$

$$F u + A^N x_0 = 0,$$

where $\tilde{U} = \prod_{k=1}^N U$ and $\tilde{X} = \prod_{k=1}^N X$. $F = [A^{N-1}B, A^{N-2}B, \ldots, B]$ is the last block row of $\alpha$. The matrices $Q$ and $R$ are block-diagonal, built from weighting matrices $Q$ and $R$. Therefore $Q$ is positive semidefinite and $R$ is positive definite, making the cost function $V(u, x_0)$ strictly convex with any given $x_0$.

E. Distributed MPC problem

We will solve problem (8) by dividing it into smaller, overlapping DMPC problems, with each DMPC assigned to one subsystem but optimizing also over neighboring subsystems at the same time. At each time step, by solving DMPCs and combining the local solutions in an iterative process, we will get an increasingly accurate approximate solution of the centralized MPC problem.

In the DMPC problem for subsystem $i$, the global cost function is optimized with respect to a reduced set of variables: control inputs of $i$ and its neighbors, denoted together by $u^{+i}$. Each DMPC will guarantee that all constraints of the centralized MPC problem are satisfied. The DMPC of subsystem $i$ can be recast into the following optimization problem:

$$\min_{u^{+i}} V(u, x_0) \tag{9}$$

s.t. $u \in \tilde{U}$,

$$\alpha u + \beta(x_0) \in \tilde{X},$$

$$F u + A^N x_0 = 0,$$

$$\tilde{u}^i = u^{+i}.$$

where $\tilde{u}^{i,a}$ denotes the assumed inputs of all non-neighbors of $i$. For now we assume that in the beginning of each step, each node $j$ transmits its assumed inputs for $u^j = [(u^j_0)^T, \ldots, (u^j_{N-1})^T]^T$ to the entire network, node $i$ receive these vectors from $\forall j \not\in N^i$ to form $u^{i,a}$.

Note that $u$ is the combination of $u^{+i}$ and $u^i$. With each $i$, we can construct two pairs of matrices $\alpha^{+i}, \alpha^i$ and $F^{+i}, F^i$ so that:

$$\alpha u = \alpha^{+i} u^{+i} + \alpha^i u^i \tag{10}$$

$$F u = F^{+i} u^{+i} + F^i u^i.$$

By eliminating the input constraints which involve only $u^i$, the DMPC problem (9) is equivalent to the following

$$\min_{u^{+i}} V(u, x_0) \tag{11}$$

s.t. $u^{+i} \in \tilde{U}^{+i}$,

$$\alpha^{+i} u^{+i} + \alpha^i u^i + \beta(x_0) \in \tilde{X},$$

$$F^{+i} u^{+i} + F^i u^i + A^N x_0 = 0,$$

$$\tilde{u}^i = u^{+i}.$$

in which $\tilde{U}^{+i} = \prod_{k=1}^N U^{+i}(u^i_k)$. The optimal solution of (11), (12) will be denoted by $u^{+i}$. For implementation, we introduce the notion of the $r$-step extended neighborhood for each subsystem $i$, denoted by $N^i_r$, which contains all nodes that can be reached from node $i$ in not more than $r$ links. $N^i_r$ is the union of subsystem indices in the neighborhoods of all subsystems in $N^i_{r-1}$:

$$N^i_r = \bigcup_{j \in N^i_{r-1}} N^j, \tag{12}$$

where $N^i_0 := N^i$. We see that in order to solve (11), subsystem $i$ only needs information from other subsystems in $N^i_{N+1}$, the initial states and assumed inputs of subsystems outside $N^i_{N+1}$ are redundant. Remark 2.4 The MPC formulation using terminal point constraint described above simplifies our exposition but it is rather conservative. This could be alleviated by using a dual-mode MPC formulation with a terminal cost function. However, in order for this to be a truly distributed approach, the terminal cost function associated with the terminal controllers should have a sparse structure. This would allow the construction of a centralized Lyapunov function in a
local way, using only local information. In [11], the authors try to bypass this obstacle by using additional restrictive assumptions: they employ zero terminal controllers and require all subsystems and interaction models (coupled via the inputs only) to be stable. These assumptions can actually be more conservative than using a terminal point constraint, preventing the application of the FC-MPC method in general dynamically coupled systems. Finding terminal controllers that lead to a structured terminal cost is an open problem and subject of our current research.

III. JACOBI-TYPE ALGORITHM

In this section we present an iterative procedure to approximate the centralized MPC solution by repeatedly calculating and combining the solutions the local DMPC problems described in the previous section. We will show two versions of our approach, which are based on Jacobi distributed optimization [2]. The proposed algorithms maintain feasibility of intermediate solutions and converge to the centralized MPC solution asymptotically. The first version uses global communication and can be considered as an extension of FC-MPC [11]. The second version relies on local communication and represents the main contribution of the paper. We will show that the two versions are equivalent to each other, which leads to simplified analysis in Section IV.

A. Globally and locally communicating algorithms

For each time step \( t \), we assume that a feasible input \( u^f \) is given for the entire system. (Section IV-A discusses a method of obtaining such a feasible initial control sequence in a distributed way, given a known initial condition.) In each step of the proposed DMPC scheme, the subsystems cooperate and perform a Jacobi algorithm, where each subsystem iteratively solves the optimization problem \((11)\) with regards to its local variables, and incorporates a convex combination of neighboring local solutions.

During every MPC sampling period, a distributed iterative loop is employed, and is indexed by \( p \). At each iteration \( p \), \( u^f \) is updated. We will refer to vectors obtained in these iterations with subscript \((p)\).

For \( p = 1 \), we initialize the iteration with \( u_{(0)} = u^f \). Let \( u^{s^i}_{(p)} \) denote the control sequence of the whole system stored in the memory of subsystem \( i \) at iteration \( p \). For making convex combinations, each subsystem \( i \) is assigned a weight \( \lambda^i \in (0, 1) \), satisfying \( \sum_{i=1}^{M} \lambda^i = 1 \). The choice of weights is arbitrary and could depend on the specific problem, the simplest choice will be equal weights \( \lambda^i = \frac{1}{M} \). We propose then the following iterative algorithm:

Algorithm 3.1 (Jacobi DMPC with global communication):

Given \( N, \ p_{\text{max}} > 0, \ \epsilon > 0 \):

1. \( p \leftarrow 1, \ u_{(0)} \leftarrow u^f \), \( \rho^i \leftarrow \) large number, \( \forall i = 1, \ldots, M \).

\( \text{while } \rho^i > \epsilon \text{ for some } i \text{ and } p \leq p_{\text{max}} \)

\( \text{a. for each } i = 1, \ldots, M \)

\( \text{Construct new } u^{s^i}_{(p)} \text{ from } u_{(p-1)}^{s^i} \).

\( \text{Solve problem } (9) \text{ to get } u_{(p)}^{s^i} \) Construct a global input vector \( u^{s^i}_{(p)} \) from \( u^{s^i}_{(p-1)} \) and \( u^{s^i} \). Transmit \( u^{s^i}_{(p)} \) to a central update location.

\( \text{end (for)} \)

\( \text{b. Merge local solutions according to the following convex combination:} \)

\[
        u_{(p)} = \sum_{i=1}^{M} \lambda^i u^{s^i}_{(p)}
\]

\( \text{(13)} \)

\( \text{c. Compute the progress and iterate:} \)

\( \rho^i = \| u_{(p)} - u_{(p-1)} \| \)

\( p \leftarrow p + 1 \)

\( \text{end (while)} \)

2. Each subsystem implements the first input value in \( u_{(p)}^f \):

\[
        u^i_t = u^i_{0,(p)}
\]

\( \text{(14)} \)

3. Shift the predicted input sequence one step to make a feasible solution for the following MPC update:

\[
        u^f = [u_{1,(p)}, \ldots, u_{N-1,(p)}, 0].
\]

4. \( t \leftarrow t + 1 \). Measure new initial states \( x_t \), go to step 1.

Algorithm 3.2 (Jacobi DMPC with local communication):

Given \( N, \ p_{\text{max}} > 0, \ \epsilon > 0 \), and assuming each subsystem \( i \) knows a feasible input \( u^i \) for all subsystems \( j \in N^i_{N_{i+1}} \).

1. \( p \leftarrow 1, \ \rho^i \leftarrow \) large number, \( \forall i = 1, \ldots, M \).

\( \text{while } \rho^i > \epsilon \text{ for some } i \text{ and } p \leq p_{\text{max}} \)

\( \text{for each } i = 1, \ldots, M \)

\( \text{a. Subsystem } i \text{ solves the local problem } (11) \text{, using } \{ u^{j,i}_t \left| \forall j \in N^i_{N_{i+1}} \setminus N^i \} \text{ as assumed inputs for subsystems outside } N^i \text{ but inside } N^i_{N_{i+1}}. \)

The solution is comprised of \( \{ u^{j,i}_t \left| j \in N^i \} \).

\( \text{b. Subsystem } i \text{ receives solutions for itself calculated by its neighbors } \{ u^{j,i}_t \left| j \in N^i \} \text{, then updates its solution for iterate } p \text{ according to:} \)

\[
        u^{i}_t(p) = \sum_{j \in N^i} \lambda^{j,i} u^{i,j}_t(p) + \left( 1 - \sum_{j \in N^i} \lambda^{j,i} \right) u^{i,(p-1)}
\]

\( \text{(15)} \)

\( \text{c. Calculate the progress:} \)

\( \rho^i = \| u^i_t - u^i_{(p-1)} \| \)

\( \text{d. } u^{i,f} \leftarrow u^{i,(p)}_t \text{, subsystem } i \text{ transmits new } u^{i,f} \text{ to all subsystems in } N^i_{N_{i+1}}. \)

\( \text{end (for)} \)
The local problem of the are transmitted to subsystem \( \{ p \} \).

2. Each subsystem \( i \) implements the first input value in \( u_{i_0}^{(p)} \):

\[
u_i^t = u_{i_0}^{(p)}. \tag{16}\]

3. Shift the predicted input sequence by one step to make a feasible solution for the following MPC update:

\[
u^t = [u_1^{(p)}, \ldots, u_N^{(p)}, 0], \quad i = 1, \ldots, M.\]

4. \( t \leftarrow t + 1 \). Measure new initial states \( x_i^t \), go to step 1.

The major difference between Algorithms 3.1 and 3.2 is at step 1b: in Algorithm 3.1 the convex combination is performed on the global control input vector, while in Algorithm 3.2 each local controller performs convex combination using its local control input vectors, therefore removing the need of a coordinator. In the sequel, we will show that the two algorithms are equivalent, thus allowing us to implement Algorithm 3.2 while using Algorithm 3.1 for analysis.

B. Equivalence of the two algorithms

The two crucial differences between Algorithm 3.2 and 3.1 are the communication requirement and the update method. We already mentioned that the optimization problem (11) is equivalent to (9), thus each subsystem only has to transmit its new results to the subsystems inside \( \mathcal{N}_{N+1}^i \). This leads to the local communications in Algorithm 3.2. Now we will show that the local update of Algorithm 3.2 is also equivalent to the global update of Algorithm 3.1.

Consider Algorithm 3.2 at the beginning of iteration \( p \), a local input vector \( u_{p-1}^i \) is given for each \( i \). Then each subsystem \( j \in \mathcal{N}^i \) computes \( u_{j}^{i_j} \) and sends these solutions to \( i \), which forms the final update for itself \( u_{p}^i \). Note that \( i \in \mathcal{N}^j \iff j \in \mathcal{N}^i \), so we have

\[
u_i^p = \sum_{j \in \mathcal{N}^i} \lambda^j \ u_{i_j}^{p_j} + \left( 1 - \sum_{j \in \mathcal{N}^i} \lambda^j \right) \ u_{i_0}^{(p-1)} \tag{17}\]

\[
= \sum_{j \in \mathcal{N}^i} \lambda^j \ u_{i_j}^{p_j} - \left( \sum_{j \in \mathcal{N}^i} \lambda^j \right) \ u_{i_0}^{(p-1)} + \ u_{i_0}^{(p-1)} \tag{18}\]

Comparing (17) and (18) we can see that the local update of Algorithm 3.2 and the global update of Algorithm 3.1 yield the same result. This implies that Algorithm 3.2 does exactly what Algorithm 3.1 does, except it only needs to use regional information (each subsystem \( i \) needs to communicate with subsystems in the region \( \mathcal{N}_{N+1}^i \)). If \( M \gg N \) and the interaction map is relatively sparse, this region will be much smaller than the whole network, thus DMPC problems can be considered local optimization problems.

The equivalence of the two proposed DMPC algorithms allows us to prove their feasibility, stability and optimality aspects by analyzing the globally communicating algorithm, which is more comprehensive than the locally communicating algorithm. In the next sections, we refer to Algorithm 3.1 for analysis.

IV. Feasibility, stability and optimality

A. Constructing initial feasible solutions locally

Although in current literature it is typically assumed that an initial centralized feasible solution exists and is available, in this section we give a simple but implementable way of actually constructing it in a distributed way assuming that the global initial state is available in advance.

The initial feasible prediction input \( u_0^t \) at time \( t = 0 \) can be calculated locally by using an inner approximation of the global feasible set, which we will denote with \( \Omega \) based on all the constraints appearing in (8) and the global initial state, which is assumed to be available. Consider an inner-hyperbox (or hyperrectangular) approximation \( \mathcal{B} \) of the feasible set \( \Omega \), which then takes the form of a Cartesian product:

\[
\mathcal{B} = \mathcal{B}^1 \times \cdots \times \mathcal{B}^M, \quad \mathcal{B} \subset \Omega. \tag{19}\]

This approximation essentially decomposes and decouples the constraints among subsystems by performing constraint tightening. Each subsystem \( i \) will thus have to include \( \mathcal{B}^i \) in their local problem setup. Since the Cartesian product of these local constraint sets are included in the globally feasible set \( \Omega \), any combination of local solutions within \( \mathcal{B}^i \) will be globally feasible as well. Needless to say that
the local constraint sets that arise from this inner-hyperbox approximation will be in general quite conservative, but at the same time will allow the construction of a centralized feasible solution locally to initialize Algorithm 3.1.

Calculation of the inner-hyperbox approximation can be performed a priori and the local $B^i$ constraints distributed to each subsystem. A polynomial-time procedure to compute a maximum volume inner box of $\Omega$ could follow the procedure described in [1]. Let us denote the dimension of the global input vector with $d = \sum_{i=1}^{M} m_i$. If we represent a box as $B(\bar{u}, \bar{u}) = \{ u \in \mathbb{R}^d \mid \bar{u} \leq u \leq \bar{u} \}$, then $B(\bar{u}^*, \bar{u}^* + v^*)$ is a maximum volume inner box of the full-dimensional polytope defined as $\Omega = \{ u \in \mathbb{R}^d \mid Au \leq b \}$, where $(\bar{u}^*, v^*)$ is an optimal solution of

$$\max_{v} \sum_{j=1}^{d} \ln v_j$$

s.t. $Au + A^+ v \leq b$. \hspace{1cm} (20)

The matrix $A^+$ is the positive part of $A$. Obtaining the local component-wise constraints $B^i$ is then straightforward. For time steps other than $t = 0$, we construct a feasible solution by performing step 3 of Algorithm 3.1.

**B. Maintaining feasibility throughout the iterations**

Observe that in step 1.a, we get $M$ feasible solutions $u_{(p)}^i$ for the centralized problem (8). In step 1.b, we construct the new control profile $u_{(p)}$ as a convex combination of these solutions. Since problem (8) is a convex constrained QP, any convex combination of $\{u_{(p)}^i\}_{i=1}^{M}$ also satisfies the convex constraint set. Therefore $u_{(p)}$ is a feasible solution of optimization problems (8), and (9) for all $i$.

**C. Stability analysis**

Showing stability of the closed-loop system (3)–(14) follows standard arguments for the most part [6], [9]. In the following, we describe only the most important part for brevity, which considers the nonincreasing property of the value function. The proof in this section is closely related to the stability proof of the FC-MPC method in [12], the main difference is due to the underlying MPC schemes: this method uses terminal point constraint MPC while FC-MPC uses dual-mode MPC.

Let $\bar{p}_t$ and $\bar{p}_{t+1}$ stand for the last iteration number of Algorithm 3.1 at step $t$ and $t + 1$, respectively. Let $V_t = V(u_{(p)}, x_t)$ and $V_{t+1} = V(u_{(p+1)}, x_{t+1})$ denote the cost values associated with the final solution at step $t$ and $t + 1$, respectively. At step $t + 1$, let $\Phi_{(p+1)} = V(u_{(p+1)}, x_t)$ denote the global cost associated with solution of subsystem $i$ at iterate $p + 1$, and $\Phi_{(p)} = V(u_{(p)}, x_t)$ the cost corresponding to the combined solution at iterate $p$.

The global cost function can be used as a Lyapunov function, and its nonincreasing property can be shown following the chain:

$$V_{t+1} \leq \cdots \leq \Phi_{(p+1)} \leq \Phi_{(p)} \leq \cdots \leq \Phi_{(1)} \leq V_t - x_t^T Q x_t - u_t^T R u_t$$

The two main components of the above inequality chain are shown in the following two subsections.

**Showing that $\Phi_{(p+1)} \leq \Phi_{(p)}$**

The cost $V(u, x_t)$ is a convex function of $u$, thus

$$V \left( \sum_{i=1}^{M} \lambda_i^p u_{(p+1)}^i, x_t \right) \leq \sum_{i=1}^{M} \lambda_i^p V \left( u_{(p+1)}^i, x_t \right)$$

Moreover, each $u_{(p+1)}^i$ is the optimizer of $i$-th local problem starting from $u_{(p)}$, therefore we have:

$$V \left( u_{(p+1)}^i, x_t \right) = \Phi_{(p+1)} \leq \Phi_{(p)}$$

Substituting (22) into (21) leads to:

$$\Phi_{(p+1)} = V \left( \sum_{i=1}^{M} \lambda_i^p u_{(p+1)}^i, x_t \right) \leq \sum_{i=1}^{M} \lambda_i^p \Phi_{(p)} = \Phi_{(p)}$$

Using the above inequality, we can trace back to $p = 1$:

$$V_{t+1} \leq \cdots \leq \Phi_{(p+1)} \leq \Phi_{(p)} \leq \cdots \leq \Phi_{(1)}$$

**Showing that $\Phi_{(1)} \leq V_t - x_t^T Q x_t - u_t^T R u_t$**

At step $t + 1$ and iteration $p = 1$, recall that the initial feasible solution $u^f$ of the centralized MPC is built by Algorithm 3.1 at the end of step $t$ in the following way:

$$u^f = [u_{1.}(\bar{p}_t), \ldots, u_{N-1.}(\bar{p}_t), 0]$$

The DMPC of each subsystem $i$ optimizes the cost with respect to $u^i$ starting from $u^f$, therefore $\forall i = 1, \ldots, M$:

$$V \left( u_{(1)}^i, x_t \right) \leq V \left( u^f, x_t \right)$$

$$\iff \Phi_{(1)} \leq \sum_{k=1}^{N-1} \left( x_{k.}^T \bar{p}_t \right) Q x_{k.} + u_{k.}^T \bar{p}_t R u_{k.}$$

$$\iff \Phi_{(1)} \leq \Phi(u_{(1)}) - x_{0.}^T \bar{p}_t Q x_{0.} - u_{0.}^T \bar{p}_t R u_{0.}$$

$$\iff \Phi_{(1)} \leq V_t - x_t^T Q x_t - u_t^T R u_t$$

Moreover, due to the convexity of $V(u, x_t)$ and the convex combination update $u_{(1)} = \sum_{i=1}^{M} \lambda_i u_i$, we obtain

$$\Phi_{(1)} = V \left( \sum_{i=1}^{M} \lambda_i u_{(1)}^i, x_t \right) \leq \sum_{i=1}^{M} \lambda_i^f V \left( u_{(1)}^i, x_t \right)$$

$$\iff \Phi_{(1)} \leq \sum_{i=1}^{M} \lambda_i^f \Phi_{(1)} \leq \sum_{i=1}^{M} \lambda_i^f \left[ V_t - x_t^T Q x_t - u_t^T R u_t \right]$$

$$\iff \Phi_{(1)} \leq V_t - x_t^T Q x_t - u_t^T R u_t$$

The above inequalities show that the value function decreases along closed-loop trajectories of the system. The rest of the proof for stability follows standard arguments found for instance in [6], [11].
D. Optimality analysis

Using the descent approach, we will show that the solution of Algorithm 3.1 approaches the solution of the centralized MPC in (8), as \( p \to \infty \). We characterize the optimality of the proposed iterative procedure by using the following results.

**Lemma 4.1:** A limit point of \( \{u(p)\} \) is guaranteed to exist.

*Proof:* The feasible set of (9) is compact. It is shown that every \( u(p) \) is feasible, therefore this sequence is bounded, thus converges. \( \square \)

**Lemma 4.2:** Every limit point of \( \{u(p)\} \) is an optimal solution of (8).

*Proof:* We will make use of the strict convexity of \( V(\cdot) \) and a technique, which is inspired by the proof of Gauss-Seidel distributed optimization algorithms in [2]. In our context however, we address also the overlapping variables that are present in the local optimization problems.

Let \( v = (v^1, ..., v^M) \) be a limit point of \( \{u(p)\} \). Assume \( (u(p')) \) is a subsequence of \( \{u(p)\} \) that converges to \( v \).

In the following, we drop parameter \( x \) in \( V(\cdot) \) for simplicity. Using the continuity of \( V(\cdot) \) and the convergence of \( \{u(p')\} \) to \( v \), we see that \( V(u(p')) \) converges to \( V(v) \). This implies the entire sequence \( \{V(u(p))\} \) converges to \( V(v) \). It now remains to show that \( V(\cdot) \) minimizes \( V(\cdot) \) over the feasible set of (8).

We first show that \( u^1_{p'+1} - u^1_p \) converges to zero. Recall that at iteration \( p \), \( u^1_{p'} \) and \( u^1_{p'} \) forms \( u^1 \), at iteration \( p + 1 \), \( u^1_{p+1} \) and \( u^1_{p+1} \) forms \( u^1_{p+1} \). Assume the contrary, or \( u^1_{p+1} - u^1_p \) does not converge to zero. There exists some \( \epsilon > 0 \) such that \( \|u^1_{p+1} - u^1_p\| \geq \epsilon \) for all \( p' \).

Let us fix some \( \gamma \in (0, 1) \) and define

\[
\bar{s}^1_{p'2} = \gamma u^1_{p'} + (1 - \gamma) u^1_{p'+1}, \tag{23}
\]

this means \( s^1_{p'} \) lies between \( u^1_{p'} \) and \( u^1_{p'+1} \), only differs from them in the values of \( u^1 \).

Notice that \( s^1_{p'} \) belongs to a compact set and therefore has a limit point, denoted \( s^1_{\infty} \). Since \( \gamma \neq 1 \) and \( u^1_{p'} \neq \bar{u}^1_{p'+1} \), \( \forall p' \), we have \( s^1_{\infty} \neq \bar{v} \).

Using convexity of \( V(\cdot) \), we obtain

\[
V(s^1_{p'}) \leq \max\{V(u^1_{p'}), V(s^1_{p'+1})\}. \tag{24}
\]

By definition, \( u^1_{p'+1} \) minimizes \( V(\cdot) \) over the subspace of \( u^1 \). So we have:

\[
V(u^1_{p'+1}) \leq V(s^1_{p'}) \leq V(u^1_{p'}). \tag{25}
\]

As \( V(u^1_{p'}) \) and \( V(u^1_{p'+1}) \) both converge to \( V(v) \), taking limit of (25), we conclude that \( V(v) = V(s^1_{\infty}) \), for any \( \gamma \in (0, 1) \). This contradicts the strict convexity of \( V(\cdot) \) in the subspace of \( u^1 \). The contradiction establishes that \( u^1_{p'+1} - u^1_{p'} \) converges to zero, leading to the convergence of \( u^1_{p'+1} \) to \( v^1 \).

We have, by definition

\[
V(u^1_{p'+1}) \leq V(u^1_{p'}, v^1). \tag{29}
\]

Taking the limit as \( p' \) tends to infinity, we obtain

\[
V(v) \leq V(u^1, v^1). \tag{30}
\]

or \( v \) is optimizer of \( V(\cdot) \) in the subspace of \( u^1 \). If we further consider \( V(\cdot) \) in a subspace corresponding to \( u^1 \), then \( V(v) \) is still a minimum. Thus, the necessary optimality condition gives \( \nabla_u V(v)^T(u^1 - v^1) \geq 0 \), \( \forall u^1 \in \Omega_1 \) where \( \Omega_1 \) is the feasible set of (9) with \( i = 1 \).

Repeating the procedure, we obtain

\[
\nabla_u V(v)^T(u^i - v^i) \geq 0, \tag{31}
\]

for all \( u^i \) such that \( u^i \) is a feasible solution of (9).

By summing up the system of equations in (31) for \( i = 1, \cdots, M \), we get:

\[
\nabla_u V(v)^T(u - v) \geq 0, \tag{32}
\]

for all \( u \) that is a feasible solution of (8).

This shows that \( v \) satisfies the optimality condition of problem (8). \( \square \)

Using strict convexity of \( V(\cdot) \), it follows that \( v \) is in fact the global optimizer of Algorithm 3.1.

V. Communications and computational aspects

In this section, we discuss the communications and computational aspects of our approach and illustrate the freedom that the designer has in choosing the appropriate trade-off and performance level in a certain application.

Although the overall computational load is reduced by employing the distributed Algorithm 3.2, its iterative nature implies that communication between neighboring systems increases in exchange. This trade-off is illustrated in Table I, which compares the communication requirements of the centralized and our distributed MPC approach. This overview suggests that our scheme is mostly applicable in situations where local communication is relatively fast and cheap.

**TABLE I**

| Communication | Centralized MPC | Distributed MPC |
|---------------|----------------|-----------------|
| **Global**    |                 |                 |
| **Local**     |                 |                 |
| Each subsystem communicates with | Central coordinator | Other subsystems in \((N + 1)\)-step extended neighborhood |
| Total number of messages sent in each time step | \(2 \times M\) | \(p_{max} \times 2 \times \sum_{i=1}^{M} |W_{N+1}|\) |
Distributed MPC

At oscillator operation of oscillator where problem setup consists of algorithm 3.2 to a problem involving coupled oscillators. The increase of is nevertheless clear: each subsystem will have more precise neighbors. Although we do not provide a formal proof of this, optimizing for its direct neighbors when solving the local optimization problems will need to be solved in one sampling period. In the proposed Algorithms 3.1, 3.2, each subsystem for masses of each oscillator. The setup is shown in Figure II. Table II shows the difference in size of the optimization problems solved by the distributed and the centralized method. Since $|\mathcal{N}| \ll M$, where $M$ is the total number of subsystems, the local optimization problems in DMPC are much smaller than the centralized one. Note that during one sampling period, the local DMPC optimization problems are solved at most $p_{max}$ times. Nevertheless, DMPC is in general more computationally efficient than the centralized MPC, with a proper choice of $p_{max}$.

Choosing an appropriately high $p_{max}$ value leads to better performance of the whole system. The trade-off is that the increase of $p_{max}$ will also lead to increased communications, and more local optimization problems will need to be solved in one sampling time.

Another way to customize the algorithm is to expand the size of the neighborhood that each subsystem optimizes for. In the proposed Algorithms 3.1, 3.2 each subsystem optimizes for its direct neighbors when solving the local optimization. We may have better performance when each subsystem optimizes also for its 2, 3, k-step expanded neighbors. Although we do not provide a formal proof of this, we will give an illustration in the following section through a numerical example. The intuition behind this phenomenon is nevertheless clear: each subsystem will have more precise predictions when it takes into account the behaviors of more neighboring subsystems.

**VI. NUMERICAL EXAMPLE**

In this section, we illustrate the application of Algorithm 3.2 to a problem involving coupled oscillators. The problem setup consists of $M$ oscillators that can move only along the vertical axis, and are coupled by springs that connect each oscillator with its two closest neighbors. An exogenous vertical force will be used as the control input for each oscillator. The setup is shown in Figure II.

Each oscillator is considered as one subsystem. The superscript $i$ denote the index of oscillators. The dynamics equation of oscillator $i$ is then defined as

$$ma^i = k_1 p_i^i - f_s v_i^i + k_2 (p_i^{i-1} - p_i^i) + k_3 (p_i^{i+1} - p_i^i) + F_i^i,$$  \hspace{1cm} (33)

where $p_i^i$, $v_i^i$ and $a_i^i$ denote the position, velocity and acceleration of oscillator $i$, respectively. The control force exerted at oscillator $i$ is $F_i^i$ and the parameters are defined as

- $k_1$: stiffness of vertical spring at each oscillator
- $k_2$: stiffness of springs that connect the oscillators
- $m$: mass of each oscillator
- $f_s$: friction coefficient of movements

From some nonzero initial state, the system needs to be stabilized subject to the constraints:

$$\left| p_i^i - \frac{v_i^{i-1} + p_i^{i+1}}{2} \right| \leq 4, \ \forall i = 2, ..., M - 1 \hspace{1cm} (34)$$

Based on dynamical couplings and constraint couplings, the neighborhood of each subsystem inside the chain is defined to contain itself and its two closest neighbors $\mathcal{N} = \{i - 1, i, i + 1\}$, $i = 2, ..., M - 1$, while for the two ends $\mathcal{N}^1 = \{1, 2\}$ and $\mathcal{N}^M = \{M, M - 1\}$. We define the state vector as $x_i^i = [p_i^i, v_i^i]^T$, and the input as $u_i^i = F_i^i$. The discretized dynamics with sampling time $T_s$ is represented by the following matrices:

$$A_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \forall j \notin N^i$$

$$A_{i,i-1} = \begin{bmatrix} 0 & 0 \\ T_s k_2 & 0 \end{bmatrix}, \forall i = 2, ..., M$$

$$A_{ii} = \begin{bmatrix} 1 & T_s \\ T_s (k_1 - 2 k_2) & 1 - T_s f_s \end{bmatrix}, \forall i = 1, ..., M$$

$$A_{i,i+1} = \begin{bmatrix} 0 & 0 \\ T_s k_2 & 0 \end{bmatrix}, \forall i = 1, ..., M - 1$$

$$B_{ij} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \forall j \neq i$$

$$B_{ii} = \begin{bmatrix} 0 \\ T_s \end{bmatrix}, \forall i = 1, ..., M$$

The following parameters were used in the simulation example:

$$k_1 = 0.4, \ \ k_2 = 0.3$$

$$f_s = 0.4, \ \ T_s = 0.05, \ \ m = 1$$

$$M = 40, \ \ N = 20$$

$$Q_i = \begin{bmatrix} 100 & 0 \\ 0 & 0 \end{bmatrix}, \ \ R_i = 10$$

Starting from the same feasible initial state, we apply Algorithm 3.2 with $p_{max} = 2, 20$ and 100. The results are compared to the solution obtained from the centralized MPC approach. The results indicate that all states of the 40 subsystems are stabilized. Figure IV shows the evolution of the overall cost achieved by DMPC compared to the cost of the centralized approach. We can see that the difference is reduced by choosing a larger $p_{max}$ value. The analysis guarantees in fact that the DMPC solution converges to the centralized one as $p$ tends to infinity.

As mentioned above, another way to customize the proposed distributed MPC algorithm is for each local problem to consider optimizing over the inputs of subsystems in a larger neighborhood. Figure II illustrates the effect of optimizing in each subproblem over an r-step extended neighborhood, with $r = \{1, 5, 10\}$. Fixing the number of maximum subiterations to $p_{max} = 2$, we can observe a steady improvement in performance until the increased neighborhood of each subsystem covers essentially all other subsystems and end up with a centralized problem.
VII. Conclusions

We presented a Jacobi algorithm for solving distributed model predictive control problems, which is able to deal with general linear coupled dynamics and convex coupled constraints. We incorporated neighboring subsystem models and constraints in the formulation of the local problems for enhanced performance. Global feasibility and stability were achieved, and a local implementation of the algorithm was given, which relies on information exchange from an extended set of “nearby” neighboring subsystems. It was shown that the distributed MPC solution converges to the centralized one through a localized iterative procedure. An a priori approximation procedure was proposed, which allows to construct an initial feasible solution locally by tightening constraints. We also discussed the trade-off between communications and computational aspects, the effect of increasing the maximum number of iterations ($p_{\text{max}}$) in one sampling period and the potential improvements that can be gained by incorporating several subsystems into a local optimization.

We are currently working on an extension of the algorithm, which allows the use of terminal costs in a dual-mode MPC formulation.

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