On using angular cross-correlations to determine source redshift distributions

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ABSTRACT

We investigate how well the redshift distribution of a population of extragalactic objects can be reconstructed using angular cross-correlations with a sample whose redshifts are known. We derive the minimum variance quadratic estimator, which has simple analytic representations in very applicable limits and is significantly more sensitive than earlier proposed estimation procedures. This estimator is straightforward to apply to observations, it robustly finds the likelihood maximum and it conveniently selects angular scales at which fluctuations are well approximated as independent between redshift bins and at which linear theory applies. We find that the linear bias times number of objects in a redshift bin generally can be constrained with cross-correlations to fractional error $\approx \sqrt{10^2 N_{\text{bias}}/N}$, where $N$ is the total number of spectra per $dz$ and $N_{\text{bias}}$ is the number of redshift bins spanned by the bulk of the unknown population. The error is often independent of the sky area and sampling fraction. Furthermore, we find that sub-per cent measurements of the angular source density per unit redshift, $dN/dz$, are in principle possible, although cosmic magnification needs to be accounted for at fractional errors of $\lesssim 10$ per cent. We discuss how the sensitivity to $dN/dz$ changes as a function of photometric and spectroscopic depth and how to optimize the survey strategy to constrain $dN/dz$. We also quantify how well cross-correlations of photometric redshift bins can be used to self-calibrate a photometric redshift sample. Simple formulae that can be quickly applied to gauge the utility of cross-correlating different samples are given.

Key words: galaxies: evolution – cosmology: theory – dark energy – large-scale structure of Universe.

1 INTRODUCTION

In many spectral bands, the redshift distribution of a source population is difficult to determine (e.g. the radio, microwave, infrared and X-ray). Even in the optical, where photometric techniques are widely applied to estimate source redshifts, these techniques work better for certain galaxy types than for others. However, extragalactic objects that are close together on the sky are also likely to be close in redshift. Thus, angular cross-correlations between populations with poorly known redshifts and those with better known redshifts can be used to improve the determination of the former’s redshift distribution. Such reconstruction has a wide range of applications, from ascertaining the redshift distribution of diffuse backgrounds to calibrating photometric redshifts for the next generation of large-scale structure surveys.

Several previous studies have attempted to measure a population’s redshift distribution, $dN/dz$, by using its constituents’ proximity on the sky to sources with known redshifts, i.e. by computing angular cross-correlation statistics between the two populations (Seldner & Peebles 1979; Phillipps & Shanks 1987; Ho et al. 2008; Erben et al. 2009). Similar techniques have been used to search for contamination in photometrically selected redshift slices or to bound the median redshift of a sample (Padmanabhan et al. 2007; Erben et al. 2009; Benjamin et al. 2010, 2013). Different $dN/dz$ cross-correlation estimators have also been studied theoretically (Phillipps 1985; Newman 2008; Matthews & Newman 2010, 2012; Schulz 2010). However, it is unknown how close any of these estimators are to being optimal. It is also unclear which survey specifications (depth, area, sampling fraction, etc.) are best for reconstructing the redshift distribution of an unknown population.

This paper attempts to answer these questions. We write down the optimal $dN/dz$ estimator and show that in very applicable limits, intuitive formulae describe how well the redshifts of a given source population can be constrained from a population whose redshift distribution is better known. In the limit of a dense spectroscopic survey, we show that the fractional error in the number of galaxies in the unknown population that fall in spectroscopic redshift bin $z$ can be estimated to the precision

$$
\frac{\delta N(z)}{N(z)} \sim 0.1 \left( \frac{\beta(z)}{0.1} \frac{f_{\text{sky}}}{10^{-3}} \right)^{-1/2} \left( \frac{\epsilon_0}{10^3} \right)^{-1},
$$

where $\beta(z)$ is the number of objects in the redshift bin, $f_{\text{sky}}$ is the fraction of sky covered, and $\epsilon_0$ is the fractional depth of the survey.
where $f_{\delta z}$ is the sky coverage of the survey, $\ell_0$ is the multipole at which shot noise becomes equal to intrinsic clustering in either sample and $\beta(z)$ is the fraction of the unknown autopower (at multipoles less than $\ell_0$) that arises from redshift bin $z$. However, the result is even simpler in the limit of a sparse spectroscopic sample, having fewer than a thousand objects per degree square per $\Delta z$:

$$
\frac{\delta N(z)}{N(z)} \sim \left( \frac{N(z)}{10^3} \right)^{-1/2} \left( \frac{\beta(z)}{0.1} \right)^{-1/2},
$$

where $N(z)$ is the total number of spectra per unit redshift. In this ‘rare spectroscopic sample’ limit, the fractional error on $N(z)$ depends on the total number of spectra but not separately on the density of spectra, the sky area or the fraction of objects with spectra.

Angular cross-correlations to determine redshifts have applications beyond estimating $\delta N/dz$. For example, they could be used to measure the redshifts of unresolved cosmic infrared background (CIB) anisotropies (as was done in Kashlinsky et al. 2007) or to isolate foregrounds in cosmic microwave background (CMB) and high-redshift 21 cm maps. Angular cross-correlations can additionally be used to reconstruct three-dimensional correlations from angular clustering measurements (Seljak 1998; Padmanabhan et al. 2007). Furthermore, such cross-correlations are able to calibrate photometric redshift errors even when the spectroscopic population is not intrinsically identical to the unknown population. Applications that are not in the vein of precision cosmology likely need no better than a 10 per cent fractional constraint on $\delta N/dz$. However, per-cent-level or even better calibration of photometric redshifts is required to prevent redshift errors from being the limiting factor for cosmological parameter estimates with the next generation of weak lensing surveys (Huterer et al. 2006; Schneider et al. 2006; Bernstein & Huterer 2010; Zhang, Pen & Bernstein 2010; Cunha et al. 2012).

There are a wide range of surveys to which cross-correlation techniques could be applied. Recent spectroscopic surveys have gone wide over hundreds (Driver et al. 2011) or thousands of square degrees (Colless et al. 2001; Eisenstein et al. 2001; Drinkwater et al. 2010; Ahn et al. 2012) or deep over $\sim$1 square degree patches (Le Fèvre et al. 2005; Newman et al. 2012). Some are complete to a magnitude limit, whereas others more sparsely sample the sources (Lawrence et al. 1999; Eisenstein et al. 2001; Kochanek et al. 2012). The large spectroscopic data sets that should be available in the next decade include:

(i) the Baryon Oscillation Spectroscopic Survey (BOSS) galaxy sample, covering 10 000 deg$^2$ with 1.5 million redshifts of massive galaxies extending to $z \gtrsim 0.7$ (Dawson et al. 2013), and the WiggleZ survey with 240 000 redshifts over $0.2 < z < 1$ (Drinkwater et al. 2010).

(ii) the Sloan Digital Sky Survey (SDSS)-BOSS quasar sample, covering 10 000 deg$^2$ with $2 \times 10^5$ redshifts (Schneider et al. 2010; Shen et al. 2011; Ahn et al. 2012),

(iii) the Galaxy and Mass Assembly (GAMA) survey, covering 310 deg$^2$ with redshifts for $3.4 \times 10^5$ galaxies to a $z$-band magnitude limit of 19.8 (Driver et al. 2011),

(iv) DEEP2 (Newman et al. 2012), the VIMOS Very Large Telescope Deep Survey (Le Fèvre et al. 2005), the z-Cosmology Evolution Survey (zCOSMOS; Lilly et al. 2007) and, while not technically spectroscopic, COMBO-17 (Wolf et al. 2003), each with $\sim 10^4 - 10^5$ redshifts in $\sim 1$ deg$^2$ fields.

(v) the Hobby-EBERLY Telescope Dark Energy Experiment (HETDEX) survey gathering $10^6$ Ly$\alpha$ emitting galaxies over 200 deg$^2$ at $1.8 < z < 3.8$ (Hill et al. 2008),

(vi) 21 cm emission line surveys over wide fields with e.g. the Australian Square Kilometre Array Pathfinder (ASKAP; Johnston et al. 2008), which aims for $\sim 10^5$ galaxies to $z \lesssim 0.43$ (Duffy et al. 2012).

The proposed projects eBOSS and BigBOSS would increase the number of spectroscopically identified galaxies and quasars by an order of magnitude over the existing SDSS+BOSS samples (Schlegel et al. 2011). Ultimately, the Square Kilometre Array (SKA; projected for 2020) aims to capture a billion galaxies over half of the sky (Rawlings et al. 2004).

In addition, we are entering a new age of optical photometric surveys, with the Kilo Degree Survey (KIDS; 1500 deg$^2$ reaching an $i$-band magnitude limit of $i = 23$), the Dark Energy Survey (DES; 5000 deg$^2$ to $i = 25$) and the HyperSuprimeCam Project (HSC; 2000 deg$^2$ to $i = 26.2$) all currently gathering data. These surveys$^4$ will be followed in the next decade by Large Synoptic Sky Telescope (LSST), which aims to constrain the cosmological model using a ‘gold sample’ of galaxies with $i < 25.3$ over half of the sky, and Euclid, which will provide high-resolution images of galaxies out to $z \sim 2$ over 15 000 deg$^2$. While we do not model in detail any particular survey, we use the above to guide our discussion.

Fig. 1 shows characteristic number densities with redshift for some of the aforementioned spectroscopic surveys as well as for complete surveys to the specified $i$-band limiting magnitude. For these and ensuing calculations, we have parametrized the galaxy redshift probability distribution for an $i$-band magnitude limited sample as

$$
p(z | i) \sim \frac{1}{2z_0} \left( \frac{z}{z_0} \right)^2 \exp \left[ -\frac{z}{z_0} \right],
$$

$z_0 = 0.0417 i - 0.74,$

with a total angular number density of $1.7 \times 10^5 + 0.3(i - 25) \deg^{-2}$ (Coil et al. 2004; Hoekstra et al. 2006; Abell et al. 2009, ‘calibrated’ over the range $20.5 < i < 25.5$, although the deepest data can only constrain $i < 23$ and the behaviour above this threshold is inferred from mocks of semi-analytic galaxy-formation models applied to the Millennium Simulation; see also Efstathiou et al. 1991; Brainerd, Blandford & Smail 1996; Benjamin et al. 2010; Hildebrandt et al. 2012).

Cross-correlation techniques can also be applied to maps in the X-ray such as those made with the X-ray Multi-Mirror Mission (XMM–Newton), in the ultraviolet such as with the Galaxy Evolution Explorer (GALEX), and in the infrared such as with the Wide-field

1 While photometric redshifts are object specific, in practice weak lensing studies will likely use the statistical distribution from photometric redshifts owing to catastrophic errors (Mandelbaum et al. 2008; Cunha et al. 2009). In contrast, cross-correlations are not able to measure the redshifts of individual objects, but they are another way to measure this statistical distribution.

2 http://www.sdss3.org, http://www.gama-survey.org, http://deep.ps.uci.edu, http://cesam.oamp.fr/rdvsproject/, http://archive.eso.org/archive/adp/zCOSMOS/VIMOS_spectroscopy_v1.0/

3 http://www.sdss3.org/future/eboss.php, http://bigboss.lbl.gov

4 http://kids.strw.leidenuniv.nl/, http://www.darkenergysurvey.org, http://www.naoj.org/Projects/HSC/HSCProject.html, http://www.lsst.org/lsst/, http://sci.esa.int/euclid.
2 BASIC FORMALISM

We begin by introducing our notation and physical model, before deriving the most general form for our $dN/dz$ estimator and applying it to idealized, illustrative examples. Useful limits of our expressions are taken in Section 3, where we also build intuition for the mechanics of the estimator.

2.1 Model and notation

Initially we will discuss galaxy clustering in the spherical harmonic basis as our covariance matrix is maximally sparse in this space. We shall write expressions as if the galaxy samples cover the full sky, but often finite sky coverage can be included by simply multiplying by the sky covering fraction $(f_{\text{sky}})$. Section 5.1 generalizes our estimation methods to configuration space, while Section 5.3 discusses the generalization to finite sky coverage.

We denote the multipole moments of a `photometric’ population of objects with unknown redshifts and a `spectroscopic’ sample in which the redshifts are perfectly known in this study take a flat cold dark matter (CDM) cosmological model with $\Omega_m = 0.27$, $\Omega_{\Lambda} = 0.73$, $h = 0.71$, $\sigma_8 = 0.82$, $n_s = 0.96$ and $\Omega_0 = 0.046$, consistent with recent measurements (Larson et al. 2011). We treat the background cosmology as perfectly known in all calculations. Roman indices $(i,j,k)$ run from 1 to some maximum integer whilst Greek indices start from 0, and repeated indices that do not appear in the same quantity are summed. Table 1 provides definitions of some commonly appearing symbols.

5 http://xmm.esac.esa.int, http://www.galex.caltech.edu, http://wise.ssl.berkeley.edu, http://sci.esa.int/herschel/, http://www.princeton.edu/act/, http://pole.uchicago.edu, http://www.atnf.csiro.au/projects/mira/
Table 1. Definitions of commonly appearing symbols. The arguments are often dropped in the text, and hats on any symbol indicate an estimated value.

| Symbol | Description |
|--------|-------------|
| $a^{(i)}_x$ | the faint-end power-law index of the cumulative source number counts of population $x$ |
| $A(\ell, m)$ | covariance matrix of $p(\ell)$ with $s(\ell)$ with index 0 referring to $p$ |
| $b^{(x)}_{ij}$ | linear bias of population $x$ in redshift bin $i$ |
| $\beta_i(\ell)$ | fraction of the total angular power contributed by redshift bin $i$ (equation 43) |
| $C_{ij}(\ell)$ | matter density angular cross-power spectrum between redshift bins $i$ and $j$ |
| $\chi$ | the conformal distance; $d\chi = c(1+z)dt$ |
| $\delta^{(i)}(x, m)$ | overdensity in population $x$ |
| $\delta(k)$ | Kronecker delta |
| $D(z)$ | growth factor such that $D(0) = 1$; $D_i \equiv D(z_i)$ |
| $dN^{(i)}_{x}/dz$ | equal to $N^{(i)}_{x}/\Delta z_i$, where the subscript $i$ is dropped if redshift independent |
| $f^{(s)}_i$ | $i$-band limiting magnitude of sample $i$ (assumed complete unless otherwise specified) |
| $F$ | Fisher matrix; generally $\left[F^{-1}\right]_{ij}$ gives error in $z$ bin $i$ |
| $F_s$ | Fisher matrix in the Schur–Limber limit (Section 3.2) |
| $n$ | local power-law index of the density power spectrum such that $P(k) \sim k^n$ |
| $N_{\text{bin}}^{(i)}$ | number of redshift bins used in analysis |
| $N^{(i)}_{x}$ | average sky density in population $x$ in redshift bin $i$; $N^{(i)}_{x} = \sum_{i=1}^{N_{\text{bin}}} N^{(i)}_{x}$ |
| $N_{\text{tot}}^{(i)}$ | total number of spectroscopic galaxies per unit redshift in redshift bin $i$ |
| $\ell_{\text{NL}}$ | multipole where shot noise is equal to cosmic variance |
| $\ell_{\text{PLX}}$ | multipole where the logarithmic errors are at a factor of 2 (equation 34) |
| $p(\ell, m)$ | multipole moment of photometric population |
| $P(k(z))$ | the $z = 0$ linear-theory matter overdensity power spectrum |
| $s(x, m)$ | vector of multipole moments of spectroscopic $z$ bins ($s_j$ is component in redshift bin $i$) |
| $S(\ell)$ | the ‘Schur parameter’ (equation 29); $S \geq 1$, with equality holding in the rare limit |
| $w^{(i)}_{ps}$ | stochastic component of the cross-power between samples $x$ and $y$ in bin $i$; $w^{(i)}_{xy} \equiv w^{(i)}_{x,y}$ |
| $W_i(\chi)$ | the window function for redshift bin $i$; typically assumed to be a top-hat |

where $n_{\text{g}}(m_{\text{h}})$ is the halo mass function and $(n^{(i)}_{x}^p n^{(i)}_{y}^p m_{\text{h}})$ is the number of galaxies of type $x$ in a halo of mass $m_{\text{h}}$ times that in type $y$ and averaged over all haloes at fixed mass. This large-scale limit is a good approximation at the angular scales we consider. We will also adopt the simplifying notation $w^{(i)}_{ij} \equiv w^{(i)}_{x,y}$. We note that a measurement of the $N^{(p)}_{x}$ is not limited by sample variance, and it can be perfectly measured in the limit that the stochastic component is zero.

The cross-power spectrum of $s(\ell)$ and $p(\ell)$ is

$$\langle p s_j(\ell) \rangle = N^{(i)}_{x} b^{(i)}_{ij} \sum_{j=1}^{N_{\text{bin}}} N^{(p)}_{x} b^{(p)}_{ij} C_{ij}(\ell) + w^{(p)}_{ij}. \quad (8)$$

Finally,

$$\langle p^2(\ell) \rangle = \sum_{i=1}^{N_{\text{bin}}} \sum_{j=1}^{N_{\text{bin}}} \left[ N^{(p)}_{x} b^{(p)}_{ij} N^{(p)}_{x} b^{(p)}_{ij} C_{ij}(\ell) + w^{(p)}_{ij} \delta(k) \right]. \quad (9)$$

We will add to equations (6), (8) and (9) the generally smaller terms that owe to cosmic magnification later.

While our formalism is completely general, subsequent calculations (and the figures we present) assume

$$b^{(s)}_{ij} = D(z_i)^{-1}, \quad (10)$$

where $D(z)$ is the linear growth factor normalized so that $D(0) = 1$, and we will interchangeably use $\chi$ and $z$ for its argument. This choice leads to redshift-independent clustering, appropriate for several cosmological populations, especially if they are rare objects. In many instances this assumption will be benign, and our results can be simply rescaled by fixing $N^{(i)}_{x} b^{(s)}_{ij}$. We also assume

$$w^{(i)}_{ij} = \left( \frac{1 + 3 f^{(s)}_{\text{sat}}}{1 + f^{(s)}_{\text{sat}}} \right) N^{(i)}_{x}, \quad (11)$$

for the stochastic component of the power. We take the ‘overlap fraction’ to be $f^{(s)}_{\text{sat}} = 1$ unless stated otherwise (which means that the rarest min[$N^{(i)}_{x}$, $N^{(i)}_{y}$] sources are the same in both samples). In addition, we take a satellite fraction of $f^{(s)}_{\text{sat}} = 0$. Increasing $f^{(s)}_{\text{sat}}$ to 25 per cent—the largest fraction found for the relevant galaxies in Wetzel & White (2010, see their figs 8 and 12)—does not change our results appreciably.

The cross-power in the matter overdensity is

$$C_{ij}(\ell) = \int_0^\infty \frac{2 k^2 \delta k}{\pi} \alpha_i(k, z_i) \alpha_j(k, z_j) P(k), \quad (13)$$

$$\alpha_i(k, z_i) = \int_0^\infty \delta k D(\chi) W_i(\chi) j_i(k \chi), \quad (14)$$

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6 The normalization of the stochastic component can potentially be reduced for dense samples by differently weighting sources (Seljak, Hamaus & Desjacques 2009; Hamaus et al. 2010) instead of the galaxy number weighting used here.

7 The total linear bias of the photometric sample is $b^{(p)} = \sum_{i=1}^{N_{\text{bin}}} N^{(p)}_{x} b^{(p)}_{ij} / N^{(p)}_{x}$. where $D(z)$ is the linear growth factor normalized so that $D(0) = 1$, and we will interchangeably use $\chi$ and $z$ for its argument. This choice leads to redshift-independent clustering, appropriate for several cosmological populations, especially if they are rare objects. In many instances this assumption will be benign, and our results can be simply rescaled by fixing $N^{(i)}_{x} b^{(s)}_{ij}$. We also assume

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8 In the case of $f^{(s)}_{\text{sat}} = 1$ and equal numbers in both the $s$ and $p$ samples, both populations trace the same large-scale cosmological plus stochastic perturbations and the $N^{(p)}_{x}$ can be perfectly estimated.
where, in our top-hat $N^{(p)}_i$ bias, $W_i = \Delta x_i^{-1}$ for redshifts that fall in the range $z_{i-1} \leq z_i$ and zero otherwise. (For a discussion of how to evaluate $j_i$ and these highly oscillatory integrals over $j_i$ numerically, see Appendix D.) While not required, we have assumed linear theory such that $P(k)$ is the $z = 0$ linear-theory matter overdensity power spectrum. Equation (13) ignores redshift-space distortions (RSDs). RSDs contribute a small fraction to the angular fluctuations on relevant angular scales, with a larger impact on the fluctuations in the spectroscopic sample compared to the photometric (Appendix B).

We note that linear scales can only be used to reconstruct the product of the large-scale bias, $b^{(p)}$, and the number density, $N^{(p)}$, at any redshift (Newman 2008; Bernstein & Huterer 2010; Schulz 2010) as they always appear in combination. This product is sometimes the desired quantity (e.g. when cleaning a map of diffuse backgrounds), but for many applications it is $N^{(p)}$ itself that is desired. We discuss methods for breaking this degeneracy in Section 9. We will often write our constraints as on $N^{(p)}$ for notational simplicity, but please note that the constraints we quote are always on the combination $b^{(p)} N^{(p)}$.

Recently, Ménard et al. (2013) advocated using non-linear scales ($\lesssim 1$ proper Mpc) to constrain the $N^{(p)}$. In fact, most of the constraint from the Ménard et al. (2013) method appears to derive from $<300$ proper kpc (Schmidt et al. 2013), scales that are likely to reside within haloes. While small-scale measurements have the advantage that they can be applied to data sets even if there are significant calibration problems (Ménard et al. 2013), on non-linear scales it is less clear how to map cross-correlation amplitude to the redshift distribution of a population. This is especially true on intrahalo scales, as the correlations depend on how the two samples inhabit the same haloes. We shall not use non-linear scales for our estimator.

2.2 Estimator

To simplify notation, we define the combined covariance matrix of the photometric survey and the redshift slices of the spectroscopic survey:

$$A(\ell, m) \equiv \left( \begin{array}{c} \hat{p}(\ell, m) \\ \hat{s}(\ell, m) \end{array} \right),$$

(15)

where $\hat{s}^T = (\hat{s}_1, \ldots, \hat{s}_n)$ and note that $A = \hat{A}$. The argument $(\ell, m)$ will typically be dropped in subsequent expressions. The minimum variance estimator for $N^{(p)}_i$ that maximizes the likelihood function if it is Gaussian in this parameter near the maximum (as is likely if many modes are included in the estimate) is

$$\hat{N}^{(p)}_i = \frac{\hat{N}^{(p)}_{\text{last}}}{2} + \frac{1}{2} \left[ F^{-1} \right]_{ij} \sum_{\ell, m} \left( \frac{\hat{p}}{\hat{s}} \right) Q_{ij} \left( \frac{\hat{p}}{\hat{s}} \right),$$

(16)

where $Q_{ij} \equiv \sum_{\ell, m} A^{-1}_{ij} A^{-1}$.

$$Q_{ij} \equiv \sum_{\ell, m} A^{-1}_{ij} A^{-1}$$

(17)

If there is significant evolution in the overlap of the samples with redshift (or the size of haloes), this method will lead to artificial trends in the $N^{(p)}_i$ inferences. There may also be pathological cases where two populations do not significantly overlap [such as in the early- and late-type galaxies models considered in Ross & Brunner (2009)], which would greatly impact small-scale measurements while having minimal impact on large scales.

(e.g. Bond, Jaffe & Knox 1998; Tegmark et al. 1998; Dodelson 2003), where all repeated indices are summed and subscript, ‘$i$’ indicates a derivative with respect to the $i$th parameter, which for most of our discussion is the parameter $N^{(p)}_i$. The parameter $[\hat{N}^{(p)}_{\text{last}}]$ is initially a guess and, for subsequent iterations, the previous estimate. In addition, the $[\hat{N}^{(p)}_{\text{last}}]$ appear in the $A$ in the next iteration. Despite this we do not include hats on the $A$ (a slight notational inconsistency). One can also trivially recast the estimated quantity in equation (16) to be $b^{(p)} N^{(p)}$ rather than $N^{(p)}$, since $b^{(p)} N^{(p)}$ is what is truly constrained. Appendix A2 derives equations (16) and (17) and shows how they generalize to the case with priors on the $N^{(p)}_i$.

In the limit that many modes are included in the estimate (which is appropriate; Appendix A1),

$$F_{ij} = \frac{1}{2} \sum_{\ell, m} \text{Tr} \left[ A^{-1} A^{-1} A^{-1} A_{ij} \right]$$

(18)

and $F$ is the Fisher matrix. The estimator in this limit is the minimum variance quadratic estimator, and the variance of this estimator is $[\mathbf{F}^{-1}]_{ij}$ (e.g. Tegmark, Taylor & Heavens 1997). We will use equation (18) in our subsequent calculations.

Schulz (2010) and Matthews & Newman (2012) considered a maximum likelihood estimator approach to constrain the $N^{(p)}$, at least for their most general expressions. This approach should yield similar estimates to ours as the Fisher matrix, which sets our variance and saturates the Rao–Cramer bound (and so is optimal). In fact, quadratic estimators are prone to find local extrema and so a Markov chain Monte Carlo approach to find the maximum likelihood may yield more robust estimates (e.g. Christensens et al. 2001). However, the linearity of our estimator reduces the severity of this problem, and we show in Section 8 that it robustly finds the true minimum even when the initial guess for the $N^{(p)}$ is off by orders of magnitude.

It is worth noting two subtleties in our approach. First, we do not consider estimators for the $N^{(p)}$ that simultaneously estimate the $w^{(p)}$, although this would be a small generalization of equation (16). Instead, we assume that the $w^{(p)}$ can be measured independently from the $N^{(p)}$, which should hold because of the much different scaling of the cosmological and stochastic components in the $(\hat{s}, \hat{s})$. Larger $\ell$ can also be utilized for the $w^{(p)}$ estimate than are useful for constraining the $N^{(p)}$. Second, our expressions do not consider the case in which the true value for $N^{(p)}$ differs from the measured number density owing to large-scale modes on the scale of the survey. Such an error will be most important in narrow fields. One can take this effect into account by using the measured number in a prior on the field to field fluctuations and then marginalizing over the $N^{(p)}$ (Appendix A2).

2.3 Idealized application

Equation (18) allows us to estimate the sensitivity of a hypothetical survey. The solid curves in Fig. 2 show these estimates for an idealized case in which the $N^{(p)}$ are equal, have redshift-independent clustering (see equation 10) and span the redshift range 0–1 with 10 redshift bins. The curves represent contours of constant sensitivity on the parameter $b^{(p)} N^{(p)}$, where $i = N_{\text{bin}}/2$ (i.e. the fractional error on the bias times the angular number density of photometric objects in the fifth redshift bin) as a function of $dN^{(p)}/dz$ and $dN^{(p)}/dz$ used in the cross-correlations. The labels on the black solid curves are $\log_{10}$ of the fractional error. The solid curves in the right-hand panel of Fig. 3 are the same except assuming a survey in which...
Figure 2. The fractional error on the photometric number density for different spectroscopic and photometric samples. The contours represent the log_10 of the fractional error on N_p^i with i = N_\text{spec}/2. They consider an idealized survey in which the N_p^i are equal and span z = 0–1 with 10 redshift bins of the same width, covering 1 per cent of the sky (400 deg^2). Contours are labelled for the solid curves, and the corresponding contour for the other curves is the adjacent curve at higher number densities. The calculations assume our fiducial parameters except f_\text{frac} = 0. (For f_\text{frac} = 1, the curves buckle upwards when the number densities become equal.) The thick solid curves correspond to the sensitivity of the optimal estimator. The purple dotted curves show the approximation that sets to zero terms in F in which the derivatives hit A_\infty. The short-dashed green curves show the diagonal approximation to the remaining Fisher matrix, a limit that also works excellently. The long-dashed blue curves show the error on the estimator in the Schur–Limber limit (Section 3.2 and equation 35).

3 APPROXIMATIONS AND SPECIAL CASES

In this section, we provide an understanding of the shape of the contours in Figs 2 and 3, we discuss which scales contribute the N_p estimate and we provide intuitive formulae that can be quickly applied to gauge the utility of cross-correlating different samples.

3.1 The Limber approximation

If the theoretical power spectrum is smooth and our signal is coming primarily from scales which are small compared to the width of each redshift shell, then the Limber approximation applies (Limber 1953, 1954) and our expressions simplify significantly. The Limber approximation assumes that P(k_\perp, k_\parallel) varies slowly as a function of k_\perp compared to j_\parallel(k_\parallel) – which should hold when \chi \gg \Delta \chi.

Making use of the identity

\int k^2 dk j_\ell(k\chi) j_i(k\chi') = \frac{\pi}{2\chi^2} \delta^p(\chi - \chi'),

where \delta^p is the Dirac delta function, and the Limber approximation, C_\ell(\ell) – equation (13) – becomes diagonal (Kaiser 1992; White & Hu 2000):

C_\ell(\ell) = \delta_\ell^p \int_0^\infty d\chi D^2(\chi) W_i^2(\chi) \frac{P(\ell/\chi)}{\chi^2},

\approx \delta_\ell^p D^2(\chi) \frac{P(\ell/\chi)}{\chi^2 \Delta \chi},

where \delta_\ell^p is the Kronecker delta. We discuss how the Limber limit is approached and compute the corrections owing to RSDs in Appendix B [where we show that RSDs enter at O(\ell^{2}\Delta \chi/\chi^2) in the photometric sample, which means they contribute negligibly on scales where the Limber approximation applies].

The majority of past studies (Schneider et al. 2006; Newman 2008; Matthews & Newman 2010) have used the Limber approximation. Fig. 3 shows that this approximation provides a good estimate for the variance of our N_p estimate, with only a small error in the case of \Delta z = 0.1 (left-hand panel) and the error starting to become significant for \Delta z = 0.01 (right-hand panel). In both panels, compare the solid contours, which assume Limber, with the dashed contours, which do not. The Limber approximation is accurate because, as we will show, much of the estimator’s constraint derives from \ell where it should hold. (The per cent-level bias introduced by this approximation is quantified in Section 6.)

The covariance matrix of the photometric and spectroscopic surveys simplifies considerably in the Limber approximation, with only the A_0 terms and the diagonal components of A_ij being non-zero, namely

A_{00} = \sum_{i=1}^{N_\text{spec}} \left( b_i^p N_p^i \right)^2 C_{ij} + w_i^p,

(22)
\[ A_{00} = b_i^{(p)} N_i^{(p)} b_i^{(p)} N_i^{(p)} C_{ij} + w_i^{(p)}, \]
\[ A_{ij} = \delta_{ij} \left[ (b_i^{(p)} N_i^{(p)})^2 C_{ii} + w_i^{(p)} \right], \]
\[ [A_{00}] = b_i^{(p)} b_i^{(p)} N_i^{(p)} C_{ij}. \]

Furthermore, this \( \mathbf{A}(\ell, m) \) can be inverted analytically, yielding
\[ \mathbf{A}^{-1}_{00} = S \frac{A_{00}}{A_{00}}, \]
\[ \mathbf{A}^{-1}_{0i} = -S \frac{A_{0i}}{A_{00}} A_{ii} + S \frac{r^{2}_{ij}}{A_{ii}}, \]
\[ \mathbf{A}^{-1} = \frac{\delta_{ij}}{A_{ii}} + S \frac{A_{0i} A_{ij}}{A_{ii}} = \frac{\delta_{ij}}{A_{ii}} + S \sqrt{\frac{r^{2}_{ij}}{A_{ii} A_{jj}}}, \]

with
\[ S = A_{00} \left( A_{00} - \sum_{i=1}^{N_{\text{min}}} A_{ii} \right)^{-1} = \left( 1 - \sum_{i=1}^{N_{\text{min}}} r^{2}_{ij} \right)^{-1}, \]

where \( r_{ij}(\ell) \equiv A_{0i}/(A_{00} A_{0j})^{1/2} \) is the cross-correlation coefficient between \( p \) and \( s_i \), and again we are using the convention \( i, j \in 1 - N_{\text{min}} \). The above inverse can be derived using the Schur complement matrix identity and the Woodbury formula (e.g. Petersen & Pedersen 2008).

The ‘Schur parameter’, \( S \), is greater than or equal to unity and quantifies the extent of correlation between the spectroscopic and photometric samples. In the case of complete redshift overlap of the spectroscopic sample and in the absence of shot noise, \( S \rightarrow \infty \) and the \( N_i^{(p)} \) are perfectly constrained. If the unknown sample is limited by shot noise, or if the two samples cover different redshift ranges, \( S \rightarrow 1^{+} \). The implication is that even a small amount of noise diminishes considerably the constraining power of a mode.

In the analytic derivations that follow, we ignore derivatives in the \( A_{00} \) in equations (16) and (18), as this element provides only an integral-like constraint on the \( N_i^{(p)} \). For all relevant limits, the approximation of ignoring the \( A_{00} \)-derivatives is excellent. Fig. 2 compares the solid black error contours, which include the \( A_{00} \)-derivatives, with the nearly overlapping dotted purple contours, which do not. With this additional simplification, the Limber-approximation Fisher matrix (equation 18) is
\[ F_{ij} \approx \sum_{\ell, m} (\mathbf{A}^{-1}_{00} + [\mathbf{A}^{-1}_{00}]_{00} [\mathbf{A}^{-1}_{00}]_{00}) [A_{0i}, A_{0j}], \]
\[ = \sum_{\ell, m} S \frac{\delta_{ij}}{A_{ii}} + 2 S \sqrt{\frac{r^{2}_{ij}}{A_{ii} A_{jj}}} [A_{0i}, A_{0j}]. \]

Furthermore, the minimum variance quadratic estimator becomes
\[ \hat{N}_i^{(p)} = \left[ \hat{N}_i^{(p)} \right]_{\text{last}} + [\mathbf{F}^{-1}]_{00} \sum_{\ell, m} S [A_{0i}, A_{0j}] \times \left( \delta_{ij} + 2 S \sqrt{\frac{r^{2}_{ij} A_{jj}}{A_{ii} A_{jj}}} \hat{\beta}_{ij} - A_{00} \right), \]

where repeated indices that do not appear in the same quantity are summed. (The complete Limber estimator, where \( [A_{00}] \), terms are maintained, is given in Appendix A1. The complete estimator also involves autocorrelation terms, which we show in Section 7 can be important for photo-z calibration).\(^{10}\)

\(^{10}\) Our Limber ‘Fisher matrix’ that drops the off-diagonal terms can violate the Rao–Cramer bound, as can be noted in Fig. 2. The purple dotted contours are not above the black solid contours (which saturate the Rao–Cramer bound for our problem) at all number densities, falling just slightly below at the largest \( dN/(dz) \). This is not an issue for our purposes.
where the density power spectrum has power-law indices $-2$ and $-1$, $\ell_{Pk-2}$ and $\ell_{Pk-1}$, respectively. As we shall discuss further, correlations between two rare samples (where rare is defined as having $\ell_0 \lesssim \ell_{Pk-1}$) constrain $N_{\ell}^{(p)}$ primarily from multipoles with $\ell \sim \ell_{Pk-1}$. Rare and abundant samples use multipoles with $\ell \sim \ell_{Pk-2}$, which also holds in the case in which both samples are extremely abundant. It is also possible in less extreme examples (in which both samples are relatively abundant) for the information to derive primarily from the scale $\ell_0$.

To orient the reader, Fig. 5 shows estimates for the $C_\ell$ at $z = 1$ and for $\Delta z = 0.1$ that use linear theory, the Limber approximation and the Peacock & Dodds (1996) non-linear power spectrum. The vertical lines show $\ell_{Pk-2}$ and $\ell_{Pk-1}$. $\ell_0$ is the scale at which the (horizontal) stochastic power becomes equal to the $C_\ell$, i.e. where the red dotted lines intersect the black solid curve. We show the stochastic terms for two illustrative number densities. In particular, the upper horizontal line in Fig. 5 denotes the lowest number density at which $w_{\ell}^{(i)} > [b_i^{(i)} N_{\ell}^{(i)}]^2 \tilde C_{\ell}$ is satisfied at all $\ell$, which we denote as $\left[\frac{dN}{dz}\right]_{\ell}^{\text{crit}}$, where

$$\left[\frac{dN}{dz}\right]_{\ell}^{\text{crit}} \simeq 300 b^{-2} \left(1 + \frac{z}{2}\right)^{1.8} \text{deg}^{-2}. \quad (32)$$

Equation (32) uses the Limber approximation, takes $f_{\text{lim}}^{(0)} = 0$ and approximates the redshift dependence as a power law evaluated at $z = 1$. In addition, the lower horizontal line denotes the number density at which $w_{\ell}^{(i)} = [b_i^{(i)} N_{\ell}^{(i)}]^2 \tilde C_{\ell_i}(\ell_{Pk-2})$ or

$$\left[\frac{dN}{dz}\right]_{\ell}^{\text{crit}} \simeq 8000 b^{-2} \left(1 + \frac{z}{2}\right)^{1.8} \text{deg}^{-2}. \quad (33)$$

Both critical number densities are shown in Fig. 5 for our fiducial bias model. We return to the significance of these numbers in future sections.

We often will approximate the scale at which linear theory no longer holds as 

$$k_{\text{NL}} \simeq 0.25 (1 + z) \text{Mpc}^{-1}, \quad (34)$$

which we find is close to the scale in which the Peacock & Dodds (1996) non-linear density power spectrum overshoots linear theory.
Cross-correlations for source redshifts

3.2 The Schur–Limber limit

We now investigate the above Limber-approximation estimator in the limit $S(\ell) \rightarrow 1^+$ and show that a small tweak to this limit captures almost all of the information in the general case. We refer to the $S \rightarrow 1^+$ limit as the ‘Schur limit’ henceforth. In this limit the information originates from modes where $\sum r_i^2 \ll 1$, either because of incomplete overlap of the spectroscopic survey or because shot noise is important. In many interesting cases, this limit at least marginally holds. Importantly, both A and F are diagonal in the Schur limit, namely

$$F_{ij}^S \approx \sum_{\ell,m} [A_{\ell m}]^2 \delta_{ij}^\ell,$$

where the superscript $S$ denotes the Schur limit. Furthermore, the estimator becomes

$$\tilde{N}_{ij}^{(p)} = [\tilde{N}_{ij}^{(p)}]_{\text{last}} + \frac{1}{F_{ii}^S} \sum_{\ell,m} [A_{\ell m}] [A_{00}] \{ \tilde{p} \delta_i - A_{ii} \},$$

such that the number density in each bin is now estimated independently and is proportional to the cross-power, $\tilde{p} \delta_i$, minus a constant. The Schur–Limber approximation yields the long-dashed blue curves for the errors on the $N_{ij}^{(p)}$ shown in Fig. 2. These trace the contours in the full calculation (compare with the solid contours) at $dV/dz \lesssim 10^5 \text{deg}^{-2}$, but deviate if both samples have higher number densities, as is expected.

Three notes in passing: (1) the structure of $F^S$ is reminiscent of the optimal weight in the Feldman, Kaiser & Peacock (1994) definition of the effective volume. While our expression is in harmonic space, the structure has the form $[n P/(1 + n P)]^2$ just as in Feldman et al. (1994). This is not surprising as our estimator is asking a similar question to ‘What is the significance that the cross-power can be detected?’ (2) It is simple to show that the Schur–Limber estimator has the same error as fitting the amplitude of the cross-power as done in Ho et al. (2008) to constrain the redshift distribution of the NRAO VLA Sky Survey (NVSS) catalogue. (3) The Schur–Limber estimator is exact in the limits where Limber holds and $S = 1$, and does not require dropping certain derivative terms as was required to derive equation (31).

To see how the Schur–Limber estimator works, we take the case in which a single angle, $\ell$, mode contributes to the estimate such that

$$\tilde{N}_{ij}^{(p)} = [\tilde{N}_{ij}^{(p)}]_{\text{last}} + \hat{\delta} \tilde{p} \delta_i - A_{ii}.$$ 

If the true $N_{ij}^{(p)}$ differs from the fiducial model, $[N_{ij}^{(p)}]_{\text{last}}$, by $\delta N_{ij}^{(p)}$, we have the relations

$$\hat{\delta} \tilde{p} \delta_i = \left( [N_{ij}^{(p)}]_{\text{last}} + \delta N_{ij}^{(p)} \right) N_{ij}^{(s)} b_i^{(s)} b_j^{(s)} C_{ii} \bar{u}_{l}^{(ps)},$$

where $C_{ii}^{(data)}$ is the actual density power in this harmonic and

$$A_{ii} = [N_{ij}^{(p)}]_{\text{last}} \times N_{ij}^{(s)} b_i^{(s)} b_j^{(s)} C_{ii} \bar{u}_{l}^{(ps)}.$$ 

Plugging these into equation (37) yields

$$\left( \tilde{N}_{ij}^{(p)} \right) = [N_{ij}^{(p)}]_{\text{last}} + \delta N_{ij}^{(p)} = N_{ij}^{(p)},$$

noting that $(C_{ii}^{(data)}(\ell, m)) = C_{ii}$. Thus, the iteration converges in a single step, and the estimate is unchanged with subsequent iterations. The former is no longer the case when multiple $\ell$ are used in the estimate, but we show in Section 8 that the estimator still converges in just a few iterations.

The structure of the formula for the Fisher matrix in this Schur limit (equation 35) is also quite simple, and is most easily brought

by factor of 2 for the redshifts of interest. We define $\ell_{NL} \equiv \chi k_{NL}$, which is plotted in Figs 4 and 6 and throughout as the limit of validity of our assumptions. Fig. 6 shows that $\ell_0$ falls in the range in which both linear theory and the Limber approximation more or less apply across all relevant redshifts and number densities. Linear theory also applies for $\ell_{NL - 1}$ and (more approximately) $\ell_{NL - 2}$. We note that $\ell_{NL - 1}$ ($\ell_{NL - 2}$) corresponds to a transverse physical scale of $k \simeq 0.03 \text{Mpc}^{-1}$ ($k = 0.2 \text{Mpc}^{-1}$) (Table 2).

Table 2. The instantaneous power-law slope of the $\Lambda$CDM linear-theory power spectrum as a function of wavenumber, k, in Mpc$^{-1}$ ($n_{eff} \equiv d \log P/d \log k$). The values were computed using the Eisenstein & Hu (1998) matter transfer function without baryon acoustic features and for the fiducial cosmological parameters.

| $k$  | $n_{eff}$ | $k$  | $n_{eff}$ | $k$  | $n_{eff}$ |
|------|-----------|------|-----------|------|-----------|
| 0.01 | 0.05      | 0.1  | 1.7       | 1    | 2.4       |
| 0.02 | 0.7       | 0.2  | 2.0       | 2    | 2.5       |
| 0.05 | 1.3       | 0.5  | 2.3       | 5    | 2.7       |
out by considering the case where the underlying power spectrum is a power law, $C_\ell = c_\ell \ell^\beta$:

$$F_{ij} = \left[ N_{i}^{(p)} \right]^{-2} \sum_{\ell,m} \left[ C_{ij}^{(p)} \left( \ell^\beta + w^{(p)} \right) \left( c_{ij}^{(p)} \ell^\beta + w^{(p)} \right) \right],$$

where we have written $c_{ij}^{(p)} = [N_{i}^{(p)} b_{ij}^{(p)}] c_i$ and $c_{ij}^{(p)} = \sum_{i} c_i^{(p)}$. The CDM case can often locally be thought of a power law where the spectrum has a power-law index which becomes increasingly negative towards smaller scales (see Table 2). Equation (41) – which we remind the reader is valid in the Schur–Limber limit – provides intuition into the shape of the contours in Fig. 2. In particular, we now focus on three sublimits that bracket different regimes for the densities of galaxies being correlated.

### 3.3 Abundant galaxy limit

At $\ell$ where neither the photometric nor the spectroscopic survey is limited by shot noise, all $C_\ell$ contribute equally and the argument in the sum in equation (41) is roughly constant in $\ell$. However, once shot noise becomes appreciable for either survey ($\ell > \ell_0$), the argument in the sum scales as $\ell^\beta$. At scales where $n < -2$, which becomes increasingly satisfied at smaller scales with CDM spectra (see Table 2), this scaling cuts off the sum as shells of increasing $\ell$ contribute progressively less to $F$. If $n > -2$, this is not true, and there is information until scales where both surveys are limited by shot noise (or $n$ has steepened). This explanation is reflected by the contours in Fig. 2. For number densities where $\ell_0$ occurs at scales at which $n < -2$ (dN/dz $> 8000 b^{-2} \text{deg}^{-2}$), information is gained all the way until $\ell \sim \ell_0$. In this case, the contours are very boxy and equation (35) can be approximated as being clustering dominated at $\ell < \ell_0$ and being $0$ at $\ell > \ell_0$:

$$\frac{\delta N_{i}^{(p)}}{N_{i}^{(p)}} \approx \sqrt{\left[ F_{i}^{(p)} \right]_{i}} \ell_0 \left( f_{\text{sky}} \left[ \ell_0^2 - \ell_{\text{min}}^2 \right] \right)^{-1/2},$$

where $\ell_{\text{min}}$ is the minimum wavenumber used and $(\beta_i)$ is the $\ell$-averaged fraction of the angular power in the photometric sample that comes from $\ell$ bin $i$:

$$\beta_i = \sum_{j} \frac{\left[ N_{i}^{(p)} b_{ij}^{(p)} \right]^{2} C_{ij}(\ell, m)}{\sum_{j} \left[ N_{i}^{(p)} b_{ij}^{(p)} \right]^{2} C_{ij}(\ell, m)}.$$

For the simple case of slices of fixed number and distant observers (i.e. $\chi$ not changing appreciably across the sample), $(\beta) \sim N_{\text{bin}}^{-1}$. The left-hand panel of Fig. 7 shows how the sensitivity is increased with increasing dN/dz, fixing the photometric population (here a survey complete to $i = 23$) and the survey area. It shows that the prediction of equation (43) of a number density-independent error enters full effect at dN/dz $> 10^{5} \text{deg}^{-2}$, which is on par with the maximum number densities for medium-future experiments (see Fig. 1). Values of the Schur parameter greater than unity (equation 42 sets $S = 1$) result in some number density dependence even at high dN/dz.11 Also, evaluating equation (42) for parameters that match the case given in the left-hand panel of Fig. 7 – $\ell_0 = 2000$ (see Fig. 6), $\beta = 0.1$ and 100 deg$^{-2}$ – yields $\delta N/N = 0.03$, which is comparable to the values for the largest dN/dz in this plot.

We have used linear theory in our computations, but scales with $\ell > \ell_0$ should not be used in our formalism. Hence, a large enough patch of sky must be chosen to sample $\ell < \ell_0$ such that cross-correlations are fruitful. Evaluating equation (42) with $\ell_0 \rightarrow \ell_0 \sim 10^3$ implies that a square degree is required for cross-correlations to provide an O(1) constraint on dN/dz with our method.

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11 In fact, equation (42) should be regarded as an upper bound on the error since we set $S = 1$. When $S$ is large (and here we take $w_{ij}^{(p)} > w_{ij}^{(p)}$ and $w_{ij}^{(p)} > w_{ij}^{(p)}$, although similar conclusions apply regardless), $S \propto \sum_{i} [N_{i}^{(p)} b_{ij}^{(p)}]^{2}$. Including $S$ in the summation in equation (41) makes the kernel peak at $\ell_{P_k} - 2$ for high number densities rather than $\ell_0$. This results in the many–many case peaking at $\ell_{P_k - 2}$ in Fig. 4. However, the constraint on dN/dz only improves by a factor of $\sim 2$ for physically realizable number densities when accounting for $S \neq 1$ (as can be gleaned by comparing the Schur estimator’s error – the long-dashed blue curve – to the full estimator’s error – the solid black curve – at high densities in Fig. 2).
3.4 Rare spectroscopic sample

Another relevant limit of the Schur–Limber estimator is when the spectroscopic sample is sparse enough that it is dominated by shot noise. In this limit, the Schur approximation (\(S \approx 1\)) is always justified, and our equations simplify further so that the Fisher matrix becomes

\[
F_{ij} = N_i^{(s)} \sum_{b \neq i} \frac{\langle b_i^{(s)} b_j^{(s)} \rangle}{N_k^{(s)}} C_{ii} f_{\delta b} N_i^{(s)} f_{\delta b} \propto N_i^{(s)} f_{\delta b}
\]

for \(f_{\delta b} = 0\). Thus, in this limit the error on the \(N_i^{(s)}\) scales as the total number of spectra – it does not depend on the density of spectroscopic sources. It turns out that in many relevant cases cross-correlations will be in this regime (as discussed in Section 4).

The derivative of the photometric survey can be estimated by interpolating between them. What \(dN_i^{(s)}/dz\) are required to be in the rare limit? If \(dN_i^{(s)}/dz < \frac{(dN_i^{(s)}/dz)_{\Delta z}}{\Delta z}\), or roughly a hundred per square degree (equation 32), the sparse tracer limit certainly holds as the shot component always dominates by \(\delta N_i^{(s)}\) of \(N_i^{(s)}\). In this limit, the Schur approximation (\(S \approx 1\)) is approximately diagonal. However, empirically we find that the inverse of the full Fisher matrix of the minimum variance quadratic estimator is quite diagonal. However, empirically we find that the inverse of the full Fisher matrix of the minimum variance quadratic estimator is quite diagonal and is well approximated by the inverse of \(\sum_{\ell,m} S^{F}(\ell)\) (i.e. to ignoring the off-diagonal elements in \(S\)). This is illustrated by the dashed green contours in Fig. 2, which show the variance calculated with this expression for \(F^{-1}\).

\[
\delta \frac{N_i^{(s)}}{N_i^{(s)}} \approx \frac{0.6}{b_i^{(s)} D_1} \frac{N_i^{(s)} \langle \beta_i \rangle_c}{10^5 \ 0.1} \left( \frac{1 + z}{2} \right)^{-0.5},
\]

where we have assumed bins of fixed \(\Delta z\), \(\langle \beta_i \rangle_c\) is defined analogously to \(\langle \beta_i \rangle\) but weighted by \(C_{ii}\), and the redshift factor owes to how lengths map to angles and redshift intervals with \(z\) (which we evaluated at \(z = 1\), but this formula holds to 20 per cent for \(0.1 < z < 3\)).

3.5 Rare–rare limit

The final limit we consider is when the fluctuations in both samples are dominated by shot noise. In this limit, \(d F_{\delta}^{(s)}/d \log \ell \propto \ell^{2+\nu_2}\) such that the contribution to \(F_{\delta}^{(s)}\) decreases in bins of \(\ell \propto n^{-1} - 1\). As with the abundant–rare limit previously considered, we can also evaluate equation (35) in the rare–rare limit, which yields

\[
\delta \frac{N_i^{(s)}}{N_i^{(s)}} \approx 1.7 \frac{(dN_i^{(s)}/dz)_{\Delta z}}{10^5 \ 0.1} \left( \frac{1 + z}{2} \right)^{0.4},
\]

where \(f_i\) is the fraction of the photometric galaxies in redshift bin \(i\) (and equals the distant observer \(\beta_i\) in the case of redshift-independent clustering). This expression shows that at a minimum \(N_i^{(s)} \times dN_i^{(s)}/dz \gtrsim 10^6 \text{deg}^{-2}\) (47) is required for cross-correlations to be fruitful. The right-hand panel of Fig. 7 shows the constraints on the \(N_i^{(s)}\), again with the specifications \(\rho^{(s)} = 23\) and \(10^5\) total spectroscopic galaxies, but taking \(dN_i^{(s)}/dz = 10\text{deg}^{-2}\) for all the curves and assuming that only a fraction, \(f_i\), of photometric galaxies are used in the cross-correlations. When both the photometric and spectroscopic galaxies are in the rare limit, equation (46) shows that the sensitivity scales as \(\ell^{-1/2}\). We note that the peak of \(dN_i^{(s)}/dz\) for a survey complete to \(i = 23\) equals \(5 \times 10^5\text{deg}^{-2}\), so the \(f_i^{<0.01}\) curves should be in this limit, and we indeed find this scaling in this regime. This panel illustrates that cross-correlations can be used to constrain the redshift distribution of peculiar objects, comprising a part in \(10^5\) of the photometric sample in the case shown, and not just of the full sample.

The derivations that led to equation (46) implicitly assumed that the bias of the spectroscopic sample is known from autocorrelation function measurements. However, in the limit of a rare spectroscopic sample, the autocorrelations can be much noisier than the cross-correlations, calling into question this assumption. We show in Appendix A2 that in this case the fractional variance of the \(N_i^{(s)}\) is simply the fractional variance quoted in this section added to the fractional variance in the bias measurement.

Because the two limits given by equations (45) and (46) yield similar \(\delta N(z)/N(z)\) at the transition between the two regimes (at \(dN_i^{(s)}/dz \approx 0.1\) \(dN_i^{(s)}/dz \approx 10^5\text{deg}^{-2}\)), the sensitivity of an arbitrary photometric survey can be estimated by interpolating between them.

3.6 Generalizing the Schur limit

We showed that in the Schur–Limber limit, the Fisher matrix is diagonal. However, empirically we find that the inverse of the full Fisher matrix of the minimum variance quadratic estimator is quite diagonal and is well approximated by the inverse of \(\sum_{\ell,m} S^{F}(\ell)\) (i.e. ignoring the off-diagonal elements in \(S\)). This is illustrated by the dashed green contours in Fig. 2, which show the variance calculated with this expression for \(F^{-1}\).

The approximation of ignoring off-diagonals when computing the estimator variance from \(S\) is equivalent to not marginalizing over parameters other than \(N_i^{(s)}\). That \(F^{-1}\) is approximately diagonal thus means that one does not have to simultaneously estimate each of the \(N_i^{(s)}\) and rather can estimate each parameter independently for \(N_i^{(s)}\) near the peak of the likelihood.

4 APPLICATIONS

The previous section built intuition for the behaviour of the estimator. To bring out the appropriate limits, we considered simple \(dN/dz\) distributions, such as constants. This section considers more physically motivated parametrizations for the extragalactic populations. Fig. 8 is analogous to Fig. 4 but quantifies the scales...
that contribute to the constraint on the $N_i^{(p)}$ for realistic source models, plotting $d[1/F_i^{(-1)}]/d \log \ell$. In particular, Fig. 8 considers the following models:

- top panel: $i^{(o)} = 23$ over $40 \text{deg}^2$ and $i^{(p)} = 25.3$ – characteristic of the LSST gold sample,
- bottom panel: $dN_i^{(o)}/dz = 10 \text{deg}^2$ over $10^4 \text{deg}^2$ and $0 < z < 2.5$ – characteristic of SDSS quasars – and again $i^{(p)} = 25.3$.

In the model in the bottom panel, the kernel peaks near the scale $\ell_{PK} - z$, which corresponds to $\ell = 400$, 700 and 900 at $z = 0.5$, 1 and 1.5. This is as expected when at least one sample is abundant. In the model in the top panel, the information has a broad peak that falls between $\ell_{PK} - z$ and $\ell_0$, where $\ell_0 = 800$, 2000 and 3000 for the three redshifts considered. This is consistent with our arguments for the case of two abundant samples. In both of the models considered in Fig. 8, the majority of the information arises from linear scales [scales which fall leftwards of the filled dot on each curve, representing $F_{NL}(z)$]. We find that similar conclusions apply for a range of models.

Fig. 9 investigates the tradeoffs of depth versus area for attempts to constrain the $N_i^{(p)}$ in 50 redshift bins with $\Delta z = 0.05$ and spanning $0 < z < 2.5$. The top panel shows the fractional error on $b_i^{(p)} N_i^{(p)}$ for a photometric sample with the specifications of the LSST gold sample (which has $dN_i^{(o)}/dz > 10^4 \text{deg}^2$ over the entire redshift range) and for three spectroscopic samples that could be obtained with the same total time on a telescope. (More correctly, the limiting flux squared divided by the survey area is held constant.) We assume that the spectroscopic follow-up covers $40 \text{deg}^2$ at $i^{(p)} = 23$. Hence, it covers $1600 \text{deg}^2$ at $i^{(p)} = 21$ and $1.0 \text{deg}^2$ at $i^{(p)} = 25$. This panel illustrates that deeper is not necessarily better (compare only for the $i = 23$ case) in the top and bottom panels denote the variance of the Newman analogue estimator discussed in Section 5.2 without any cutoff at non-linear scales.
shown, and hence its errors blow up there. By contrast, while the $f(z) = 25$ sample is the least sensitive to $dN/vdz$ at intermediate redshifts (owing to its small $f_{BL}$), it is the most able to determine the distribution at the highest redshifts.

The middle panel of Fig. 9 is similar to the top panel but assumes that a random fraction, $f_r$, of all galaxies with $f(z) = 23$ are observed over a region of $40 f_r^{-1}$ deg$^2$ such that the total number of galaxies is fixed. This panel reinforces our result that the constraint on the $N_{\ell}^{(p)}$ depends primarily on the total number of spectroscopic galaxies and not their angular density, even though the case with $f_r = 1$ is in our abundant limit in which we no longer expect this scaling to hold exactly. We still find that this result approximately holds.

The bottom panel of Fig. 9 shows the case of a spectroscopic sample with the specifications of BigBOSS (whose $dN/dz$ is shown in Fig. 1) and the specified limiting photometric magnitudes.$^{12}$ This panel assumes that the surveys’ overlap is $10^2$ deg$^2$, but the error scales as the square root of the overlapping area. Despite the lower number densities of galaxies in the BigBOSS case compared to those in the top panel, BigBOSS has a total number of galaxies that exceeds the other cases by more than an order of magnitude and, thus, is the most sensitive of all the cross-correlation examples considered in Fig. 9. We note that to reach the $10^{-2}$ sensitivity quoted here, BigBOSS would likely need to correct for magnification bias (which is discussed in Section 6).

Omitting non-linear scales or introducing a redshift cutoff in the spectroscopic coverage has little impact on our results. The dashed curves in Fig. 9 include information from $\ell > \ell_{NL}$, whereas the solid curves do not. Excluding non-linear modes in the analysis only has a modest impact on the estimator, except in the $f(z) = 25$ case in the top panel, where the constraint is reduced by a factor of 3. This case is most impacted because (1) its $\ell_0$ falls at the most non-linear scales of the cases plotted and (2) the small $\ell$ field assumed in this case has already limited the scales that can contribute. Similar losses for each of the plotted cases also occur for a factor of 2 smaller $\ell_{NL}$.

In addition, we have assumed that the spectroscopic sample spans the entire redshift range of the photometric sample. A cutoff in the coverage of a spectroscopic sample, as could occur if an emission line falls out of the spectroscopic band of a survey, has little impact on our results below that cutoff. It has no impact to the extent that $S = 1$. When the additional condition $dN/vdz = 0$ was imposed for $z > 1.5$, which forces $S$ to be small, we found no change to the $f(z) = 25$ case in the top panel of Fig. 9, but a factor of 2.5 shift upwards for $f(z) = 25$ in that panel.

The photometric sample can often be divided into magnitude bins or into photometric redshift bins. For magnitude cuts, extra sensitivity is often gained by dividing the primary photometric sample because galaxies in different magnitude bins are more likely to also be at different redshifts. In particular, in the rare spectroscopic galaxy limit but where the photometric galaxies are more abundant than $dN/vdz$,$^{13}$ the signal scales inversely with the redshift extent of the photometric sample and does not depend on the amplitude of $dN/vdz$ (equation 45). Thus, the sensitivity is not improved by going deeper. The redshift distribution of galaxies given by our parametrization for $P(z|\ell)$ (equation 3) has mean $3z_0$ and variance $3z_0^2$. Because the variance of $P(z|\ell)$ increases with depth, deeper surveys will be somewhat less sensitive at the peak of $P(z|\ell)$ unless the sample is partitioned.$^{13}$ A partitioned sample can be easily accommodated in the quadratic estimator formalism. In Section 7, we discuss the gains from dividing by photometric redshift.

5 CONFIGURATION SPACE

The previous derivations were done in spherical harmonic space as this is the simplest basis for calculating the minimum variance estimator. However, when dealing with actual data, it can be more difficult to work with spherical harmonics as the survey window function enters non-trivially in convolution. Hence, many galaxy clustering analyses are done in configuration space. In this section, we show that the minimum variance estimator can be easily applied in this dual space (Section 5.1), compare with previous configuration-space $dN/dz$ estimators (Section 5.2) and finally discuss the impact of finite sky coverage (Section 5.3).

5.1 Configuration-space estimator

The harmonic space quadratic estimator can be written in the form

$$\sum_{\ell,m} v(\ell) p(\ell, m) s(\ell, m),$$

(48)

for some $v(\ell)$, plus analogous terms proportional to the autocorrelations. Writing $\tilde{p} s(\ell, m) = \int d\mathbf{n} \tilde{p} s(\mathbf{n}) Y^\ell_m(\mathbf{n})$, equation (48) becomes

$$\int d\mathbf{n} d\mathbf{n'} \tilde{p}(\mathbf{n}) v(\mathbf{n} \cdot \mathbf{n'}) s(\mathbf{n}),$$

(49)

where we have used the addition theorem for spherical harmonics (Abramowitz & Stegun 1964), $P_\ell$ is the Legendre polynomial of order $\ell$ and

$$v(\mathbf{x}) = \sum_{\ell} \frac{2\ell + 1}{4\pi} v(\ell) P_\ell(\mathbf{x}).$$

(50)

If we define $\hat{\omega}_{ps} (\mathbf{x}) \equiv \langle \tilde{p} s \rangle_x$, as the correlation function estimate where $x = \mathbf{n} \cdot \mathbf{n'}$ and $(\ldots)$, represents an average over all separation angles $x$ in the survey, equation (49) can be re-expressed as

$$8\pi^2 \int dx \ v(\mathbf{x}) \hat{\omega}_{ps} (\mathbf{x}).$$

(51)

Thus, the configuration-space estimator in the Schur–Lümmer limit is

$$\hat{N}_{ps}^{(p)}[\Sigma] = \left[ \hat{N}_{ps}^{(p)}[\Sigma] \right]_{\text{last}} + \frac{8\pi^2}{F_{\ell \ell}} \sum_u \Delta x_u v(x_u) \times \left\{ \hat{\omega}_{ps} (x_u) - \omega_{ps} (x_u) \right\},$$

(52)

where $\omega$ runs over the bins in (cosine of the) angle. A similar configuration-space estimator can be written for the full minimum variance quadratic estimator (equation 16).

For $\theta \ll 1$ rad (the scales that we will show are of primary interest), the result can be further simplified by making the flat sky approximation. Then, the Parseval identity,
The bottom panel of Fig. 10 shows $\theta \, v_i(\theta) \times \omega_{ps}$, which better represents the $\theta$ that contribute to the final estimate. Since measured correlations are weaker on large scales than small, the $\theta > 1^\circ$ behaviour of $v_i(\theta)$ is down weighted and really only sub-degree scales contribute significantly.

In practice, whether weights are applied during or after the computation of the correlation function depends on the survey to which cross-correlations are applied. In the case where the survey’s contiguous area is much larger than the kernel of $v_i(\theta) \gg 0.1-1^\circ$, the exact details of the survey window are irrelevant. The $\omega_{ps}(\theta)$ can be estimated with standard techniques (e.g. Hamilton 1993; Landy & Szalay 1993; Bernstein 1994) and then multiplied by the approximate $v_i$. This is the regime of most of the large-scale photometric and spectroscopic surveys, such as SDSS, WiggleZ, BOSS, GAMA, DES and LSST. The second regime, where the survey area is comparable to or smaller than the weighting kernel [e.g. with DEEP or Hubble Space Telescope (HST) fields], is more complex. Section 5.3 discusses this case.

5.2 Comparison to earlier work

Using cross-correlations to estimate redshift distributions has been championed by Newman (2008). The configuration-space expression for the optimal quadratic estimator (cf., equation 52) allows us to compare explicitly with the Newman (2008) method. Though the Newman (2008) method is neither optimal nor unbiased, it has some similarities to our estimator as we shall see.

The estimator in Newman (2008, and also Matthews & Newman 2010) involves non-linear, power-law fits to correlation functions over a specified range of scales and with specified, diagonal (i.e. ignoring bin-to-bin correlations in $\theta$ and $z$) weights. The estimator is thus a non-linear functional of the measured two-point functions. However, since the power-law fit is used mainly to divide out trends and fit for an amplitude, we can write an analogous estimator to Newman (2008) that contains essentially the same information. Our analogue estimator becomes very similar to that of Newman (2008) for power-law models.

Our analogue of the Newman (2008) estimator is

$$\tilde{N}_i^{(p)} = \eta_i^{-1} \sum_{j} v_j^{\text{New}} \left( \hat{p} \, \delta_i - u_i^{(p)} \right),$$

where

$$\eta_i = \sum_{j} v_j^{\text{New}} b_i^{(p)} b_j^{(p)} N_i^{(s)} C_{ij}.$$

This estimator returns $N_i^{(p)}$ if the Limber approximation holds and the underlying power spectra and biases are correctly guessed. When the sum in equation (55) is over configuration-space pixels (as in Newman 2008), the weighting is

$$v_j^{\text{New}}(r) = \begin{cases} 1 & r_{\text{min}} < r < r_{\text{max}} \\ 0 & \text{otherwise,} \end{cases}$$

where Newman (2008) chooses $r_{\text{min}} = 0$ and $r_{\text{max}} = 10 \, h^{-1} \text{Mpc}$. Fig. 10 compares the weights of our optimal estimator to that of our Newman analogue estimator. The thin green solid curve in the top panel corresponds to $\theta \, v_i^{\text{New}}(\theta)$ and the curve in the bottom panel corresponds to $\theta \, v_i^{\text{New}}(\theta) \times \omega_{ps}(\theta)$. The thick curves indicate the

Figure 10. The top panel shows $\theta \, v_i(\theta)$ for the illustrative cases considered in Fig. 4, again for the $i = 6$ redshift bin. The $\theta \times v_i(\theta)$ are the optimal estimator weights of the logarithmically binned cross-correlation function, $\omega_{ps}(\theta)$. The bottom panel is $\theta \times v_i(\theta) \times \omega_{ps}(\theta)$, which shows explicitly which angular scales the information derives. The thin solid green curve in each panel shows the weighting scheme used in our analogue of the Newman (2008) estimator, with $r_{\text{max}} = 10 \, h^{-1} \text{Mpc}$. All of the curves, aside from the Newman analogue ones, have down weighted non-linear modes by the factor $\exp[-\ell^2/r_{\text{NL}}^2]$. The curves in both panels are computed in the Limber and flat sky approximations.
same quantity for the optimal estimator for the same four extreme cases as considered earlier. The Newman analogue estimator uses similar scales to those selected by the optimal estimator, especially in the rare–rare case.

While the weights for the optimal quadratic and Newman analogue estimators are superficially similar, it becomes apparent that the estimators behave differently when examining the weights in more detail. The optimal estimator in the shot-noise-limited regime has configuration-space weights given by the density correlation function. However, the Newman analogue weights are simply a constant. The structure of the Newman analogue estimator is also much different in the signal-dominated regime. The optimal estimator has weight \( v_i(\theta) \propto \int \ell \, d\ell \, C_{\ell}^{-1} J_0(\hat{\ell}(\theta)), \) in the Schur–Limber approximation, in contrast to the constant configuration-space weights in our Newman analogue estimator.

The variance of these estimators also differs. The covariance of the minimum variance estimator is \( F^{-1} \), whereas the covariance of the Newman analogue estimator (in the Limber approximation) is

\[
\text{cov} \left( \tilde{N}_i^{(p)}, \tilde{N}_j^{(p)} \right) = \eta_i^{-1} \eta_j^{-1} \sum_{\ell,m} v_i^{\text{New}}(\ell) v_j^{\text{New}}(\ell) \times \left[ A_{0i}(\ell)A_{0j}(\ell) + A_{0i}(\ell)A_{1j}(\ell) \delta_{ij}^W \right],
\]

where the Fourier space (flat sky) Newman weights are the Hankle transform of equation (57):

\[
v_i^{\text{New}}(\ell) = \frac{X_i}{\ell} \left( \frac{J_1(\ell r_{\max}/X_i)}{r_{\max}} - \frac{J_1(\ell r_{\min}/X_i)}{r_{\min}} \right).
\]

The rapid oscillations at higher \( \ell \) damp the contribution of these modes. The dot–dashed curves in Fig. 9 (shown only for the \( i = 23 \) case) in the top and bottom panels correspond to the variance of the Newman analogue estimator without any non-linear cutoff in \( \ell \). The Newman analogue estimator performs substantially worse than the optimal estimator: a factor of 3–10, with the factor of 10 applying to the abundant galaxy case [which is most similar to the cases investigated in Newman (2008) and Matthews & Newman (2010)].

5.3 Finite sky coverage

Until now many of our expressions have implicitly assumed that the surveys cover the full sky, which is unlikely to be the case in practice. For surveys whose narrowest dimension is much larger than the scales where our estimator peaks, the correction for finite sky coverage is benign: we simply have a factor of \( f_{\text{sky}} \) to correct the number of modes in our Fisher matrix (e.g. Scott, Srednicki & White 1994; Jungman et al. 1996; Tegmark 1996; Knox 1997), as we have assumed in our prior example calculations. The effects of finite sky coverage have been studied extensively in the CMB (e.g. Hansen, Górski & Hivon 2002; Hivon et al. 2002; Efstathiou 2004) and large-scale structure literature (e.g. Peacock & Nicholson 1991; Feldman et al. 1994; Park et al. 1994; Tegmark et al. 1998).

The case of a general survey window function can be complex, but, if the width and height of the window are comparable, the effects of windowing are easily understood. Due to the convolution with the window function, \( \ell \)-modes which are separated by less than \( 2\pi/\Theta \) (where \( \Theta \) is the angular extent of the window function and for simplicity we are working in the flat sky approximation) are almost completely correlated and, thus, contain largely redundant information. In contrast, for modes separated by much more than \( 2\pi/\Theta \), the effects of the window function can be largely ignored.

Thus, the effects of finite sky coverage can be taken into account by replacing our sums over \( \ell \) with sums over \( L \) values which are integer multiples of \( 2\pi/\Theta \) and defining the \( C_L \) as bin averages of the \( C_\ell \). A simpler approximation, valid if the theoretical spectra are smooth, is to simply integrate from \( 2\pi/\Theta \) to infinity rather than zero to infinity in equation (53). If in computing the correlation function or power spectrum, we estimate the mean density from the survey itself, then the power is suppressed on large scales (often known as the integral constraint; Peebles 1980). An approximation to this suppression is to multiply \( C_L \) by \( |1 - W(\ell)|^2 \), where \( W(\ell) \) is the window function normalized so that \( W \to 1 \) as \( \ell \to 0 \).

6 BIAS OF APPROXIMATE ESTIMATORS

The minimum variance quadratic estimator under the approximation that off-diagonal terms in the Fisher matrix are zero is unbiased as long as the diagonal entries are appropriately calculated. In addition, dropping derivative terms in the quadratic estimator is unbiased since each derivative explores separate dependences. However, there are a few approximations that could incur bias: the Limber approximation, ignoring RSDs, including non-linear scales, cosmic magnification and assuming the incorrect cosmology. We do not consider the latter because it should be reduced to the per cent level with the coming generation of cosmological probes, but we consider the others. We can compute the bias of these approximations by substituting the full \( \langle \hat{p} \hat{r} \rangle \times \langle \hat{p} \hat{r} \rangle \) that includes the ignored terms into the approximate estimator and evaluating both near the input \( N_i^{(p)} \). Using this formalism, we address these biases here.

6.1 Limber approximation and RSDs

In the Limber approximation, which has been assumed by most previous investigations of \( dV/\partial z \) estimation from cross-correlations, the diagonals are accurately estimated in the limit \( \ell \Delta \chi \gg \chi \) (although, in practice, this condition has to be just weakly satisfied). Fig. 6 suggests that most scales that contribute to our estimate are safely in the Limber regime for \( \Delta \chi \sim 0.1 \). This will be less true for smaller \( \Delta \chi \). On angular scales favoured by our estimator, at which the matter power spectrum is decreasing with increasing \( \ell \), the Limber approximation results in an overprediction of the \( C_\ell \). Hence, our Schur–Limber estimator will result in an underprediction. However, setting to zero the \( \langle p s \rangle \) for \( i \neq j \) in the Limber approximation has the opposite effect. We find that the former effect is larger such that Limber results in an underprediction, with a fractional error of \( -(2-3) \times 10^{-3} \) for \( \Delta \chi = 0.01 \) and \( 0 < z < 1 \) for the cases where most of the information derives from \( \ell_{PK} - 2 \) (i.e. where one of the populations is abundant) and \( -(0.3-1) \times 10^{-2} \) for the cases where most of the information derives from \( \ell_{PK} - 1 \). For \( \Delta \chi = 0.1 \), the biases are of course significantly smaller than for \( \Delta \chi = 0.01 \). Thus, the Limber approximation will likely result in a bias that is smaller than the estimator’s variance even for applications with very large source populations.

The fact that the Limber approximation is as successful as it is suggests that RSDs will also induce a small bias (as RSDs are negligible on scales at which the Limber approximation holds; Appendix B). However, for reasons discussed in Appendix B, including RSDs is difficult in our current formalism as it requires a basis.

15 If \( dV/\partial z \) is being estimated as part of a programme aimed at constraining the cosmology, e.g. with gravitational lensing, the cosmology and \( dV/\partial z \) will have to be simultaneously varied.

16 We speculate that the surprising smallness of the biases in Limber results because of a near-cancellation of the two competing effects.
switch from our choice of top-hat redshift bins, which spuriously magnify the impact of RSDs. Thus, we do not quantify the magnitude of their small bias on the estimator. RSDs could be more important for calculating the \( \langle s_i^2 \rangle \), terms that do not appear in the Schur–Limber estimator (Appendix B).

6.2 Non-linear scales and the one halo term

Using scales that are non-linear can bias the estimator. The Schur–Limber estimator for \( N_s^{(p)} \) is biased by non-linear effects that occur at the redshift of the estimate, \( z_i \), and (fortunately) not by non-linearities at other redshifts. This is not the case for the minimum variance quadratic estimator (a fact that we have ignored). In our estimates in Section 4 and Fig. 9, we masked non-linear wavenumbers at \( z_i \) that met the criterion \( k > k_{NL}(z_i) \) (defined in equation 34) and found that this operation does not have a large impact on the sensitivity, except for the densest samples that were considered. This result owes to the broad range in \( \ell \) that contributes the information, which generally peaks at \( \ell < \ell_{NL} \) (Fig. 4). We find that if we reduce \( k_{NL} \) by an additional factor of 2, which corresponds to a wavenumber where the non-linear density power spectrum deviates from linear theory by just 10 per cent, the constraints are additionally degraded by a similarly small factor.

As long as they are modelled properly, non-linearities that trace the density field do not necessarily bias a measurement of \( N_s^{(p)} \) as the galaxies still trace the same large-scale density fluctuations. A bias will arise if intrahalo correlations contribute at scales where they are not in the white noise regime (as we have assumed). Fortunately, deviations from the large-scale limit generally occur at wavenumbers that are larger than \( k_{NL} \), especially if clusters and large, low-redshift groups are excluded from the cross-correlation analysis (see plots in Cooray & Sheth 2002).

6.3 Magnification bias

Magnification bias is the most significant of the biases we considered. Cosmic magnification results in additional off-diagonal terms in \( C \) that were zero in the Limber approximation. These terms are suppressed relative to the \( j \rightarrow j \), diagonal Limber term (equation 21) by the factor

\[
R_{ij}^{(s)} = \frac{\alpha_{ij}^{(s)} + \beta_{ij}^{(s)}}{\beta_{ij}^{(s)}} \left[ \frac{(1 + z_i) X_i}{2 \times 10^5 \text{Mpc}^2} \right] \left[ \frac{X_j}{X_i} \right]
\]

for \( i > j \), where \( \alpha_{ij}^{(s)} \) is the power-law index of the cumulative in decreasing flux source counts in bin \( i \) above a certain flux threshold (see Appendix C). Equation (60) ignores magnification–magnification correlations, which are smaller except perhaps for surveys at \( z \gg 1 \) (e.g. Heavens & Joachimi 2011).

For our simple Schur–Limber estimator, it is easy to compute the \( N_s^{(p)} \) estimator bias, being

\[
\text{frac.bias from mag} = \sum_{k, k<i} N_{ik}^{(p)} R_{ik}^{(s)} + \sum_{k, k<i} N_{ik}^{(p)} C_{kk} R_{kk}^{(s)},
\]

where \( C_{kk} \) is defined in equation (21). Thus, this estimator results in an overestimate when \( -\alpha_{ij}^{(s)} > 1 > 0 \). Evaluating this for our toy case of constant \( dN/dz \) from 0 < \( z < 1 \), one finds an \( \approx -0.5 \) per cent bias that is roughly constant with \( z_i \). In addition, equation (61) shows that if \( N_s^{(p)} \) is well below the peak in \( i \), this bias can be particularly severe. Fig. 11 illustrates the importance of magnification bias for a case in which the photometric sample consists of all galaxies with \( i^{(p)} < 25.3 \) and different spectroscopic samples, all covering \( 0 < z < 2.5 \) (Lower redshift samples would be less biased by magnification.) For simplicity, we take \( \alpha_{ij}^{(s)} = -2 \) for all populations, which emphasizes the effect (being characteristic of the bright end of quasar counts; fainter quasars have a slope \( \alpha \sim -0.5 \); Bartelmann & Schneider 2001; Scranton et al. 2005, and the faint-end slope for galaxies is \( \sim (0.5 - 1) \); Bouwens et al. 2012). The thick blue curves represent BigBOSS and 10\(^4\) deg\(^2\), the black curves a survey with \( dN^{(i)}/dz = 10 \text{deg}^{-2} \) over 10\(^4\) deg\(^2\), and the red curves a survey with \( i^{(s)} = 23 \) and 40 deg\(^2\). The solid (dashed) curves indicate that the bias results in an overestimate (underestimate). The top panel shows the bias relative to \( N_s^{(p)} \) and the bottom panel shows this relative to the fractional error. All curves simultaneously assume that the flux number counts of both populations have the \textit{rather steep power-law index of} \( a^{(s)} = -2 \), to emphasize the effect. The labelled thin blue curve in the top panel is the BigBOSS case with just the diagonal Schur–Limber estimator.
In all cases, magnification bias can be computed given an estimate for the $\alpha_i^{(s)}$ and removed. The main issue is uncertainty in the $\alpha_i^{(s)}$. It should be reasonably straightforward to remove the bias at redshifts greater than the peak in $\mathrm{dN/\mathrm{dz}}$ (where it is most severe) as the spectroscopic galaxies act as the sources and their $\alpha_i^{(s)}$ is easily measured. However, uncertainty in $\alpha_i^{(s)}$ could be the limiting factor in $N_i^{(p)}$ constraints at redshifts where the photometric galaxies act as the source, particularly in surveys that can place per cent-level errors on the $N_i^{(p)}$ and that extend to high redshifts. In such cases, the error will be approximately set by the fractional bias of $N_i^{(p)}$ owing to magnification (what is plotted in Fig. 11) times the fractional uncertainty in $\alpha_i^{(s)}$. Knowledge of $\alpha_i^{(s)}$ to 10 $\alpha_i^{(s)} + 1$ per cent precision is required for this not to be the limiting factor for the BigBOSS case considered above. Since magnification only depends on the sources’ $\alpha_i$ and not their $b_i$, the significant bias of BigBOSS also suggests that it can use magnification to break this degeneracy and separately estimate the $b_i^{(p)}$ to 10 $\alpha_i^{(s)} + 1$ per cent precision.

We revisit the impact of magnification in Section 7, showing that it is less onerous in the cases of (1) photo-z calibration and (2) estimating the redshift distribution of diffuse backgrounds.

Analogous to magnification, intervening dust can also correlate background galaxies with foreground ones for surveys in the optical and bluer wavelengths (Ménard et al. 2010). At linear scales, this effect will induce correlations that are a biased tracer of the projected density. The magnitude of this effect with redshift could be determined with multiband photometry using a population with uniform spectra, e.g. quasars, and this information would allow it to be corrected for in cross-correlation studies again to the extent that the $\alpha_i^{(s)}$ are known.

## 7 CALIBRATING PHOTO METRIC REDSHIFTS AND CLEANING CORRELATED ANISOTROPIES FROM MAPS

Our previous results can be generalized to spectroscopically calibrate the $\mathrm{dN/\mathrm{dz}}$ of a photometric population that is partitioned by photometric redshift, an application which is relevant for large-scale clustering and weak lensing analyses on photometric populations. When the catastrophic failure rate of the photometric redshift estimate is small, then it may be fruitful to self-calibrate by internal cross-correlations between different photometric redshift bins. However, if the catastrophic failure rate is large, there can be degeneracies in the reconstruction from self-calibrations, and it may be more robust to calibrate photometric redshifts with a spectroscopic sample. In Section 7.1, we discuss the latter, and Section 7.2 discusses the former. This section also addresses the more general problem of estimating the redshift distribution of a photometric sample in which other constraints exist for the sample’s redshift distribution. Finally, in Section 7.3 we discuss how our results can be used to statistically clean diffuse background maps.

### 7.1 Spectroscopic calibration

Considering binning the photometric sample by some property that we refer to as its ‘photo-z’, and we denote the sample in photometric redshift bin ‘m’ as ‘pm’. One can think of $m$ as, for example, indexing a probability distribution of the sample’s redshift as estimated from photometry. The goal is to use cross-correlations with a spectroscopic sample to constrain this probability distribution. The primary difference with the calculations in prior sections and this calculation is that the fluctuations from each photometric redshift bin are more likely localized in redshift than the full photometric sample. (We defer discussion of internal correlations between different photo-z bins to Section 7.2.)

If this is the case, our approximate formulae for the sensitivities in different limits (equations 44–46) are altered so that $f_i \approx N_i^{(pm)} / N_{\text{tot}}^{(pm)}$ and $\beta_i \approx [T_i^{(pm)}]^{1/2} / [T_{\text{tot}}^{(pm)}]^{1/2}$, where

\[ T_i^{(pm)} = D_i b_i^{(pm)} N_i^{(pm)} \]

and $N_i^{(pm)} [b_i^{(pm)}]$ is the sky density [linear bias] of the photometric galaxies in redshift bin $m$ that are actually at redshift $i$. Also, $N_{\text{tot}}^{(pm)} \equiv \sum_i N_i^{(pm)}$ and $T_{\text{tot}}^{(pm)} \equiv \sum_i [T_i^{(pm)}]^{1/2}$. These relations for $f_i$ and $\beta_i$ are exact in the distant observer approximation. With these replacements, we can recast our formulae in the rare and abundant limits for the case of photo-z calibration.

If the spectroscopic sample is in the rare limit, the potential constraint on the population in photo-z bin $m$ that is actually in redshift bin $i$ follows from equation (45) and is

\[ \frac{\delta T_i^{(pm)}}{T_{\text{tot}}^{(pm)}} \approx \frac{0.06}{b_i^{(pm)} D_i} \left( \frac{N_i^{(pm)}}{T_{\text{tot}}^{(pm)}} \right)^{-1/2} \left( \frac{1 + z_i}{2} \right)^{-0.5} \]

\[ \frac{\delta T_i^{(pm)}}{T_{\text{tot}}^{(pm)}} \approx 0.03 \left( \frac{f_{\text{sky}}}{0.001} \right)^{-1/2} \left( \frac{\ell_o}{10} \right)^{-1} \]

Equations (63) and (64) demonstrate that cross-correlations can be used to constrain the fractional number (times bias) from $pm$ in $i$ at the part in a hundred level with $10^{-3} - 10^{-9}$ spectra per unit redshift (for rare spectra) or $f_{\text{sky}} = 10^{-3}$ (for high spectral densities).

Fig. 12 presents estimates for how well the redshift distribution of a photo-z bin can be reconstructed in bins of size $\Delta z = 0.05$ with cross-correlations for the $z_m = 1.45$ photo-z bin, assuming that the ‘outlier’ photo-z’s that are not actually at the redshift $z_m$ are distributed uniformly in the range $0 < z < 2.5$. The solid curves assume that half of the galaxies in this photo-z bin reside outside of it, uniformly distributed so that $N_i^{(pm)} / N_{\text{tot}}^{(pm)} = 10^{-2}$ for $i \neq m$. The dashed curves are the same but for an outlier fraction of $N_i^{(pm)} / N_{\text{tot}}^{(pm)} = 10^{-3}$ so that only 5 per cent of galaxies reside outside the photo-z bin $z_m$. Despite these rather artificial outlier distributions, their comparison is useful for diagnosing how sensitive our results are to the details of the true outlier distribution.

The top panel of Fig. 12 shows the constraints from different spectroscopic samples with the specified $\mathrm{dN/\mathrm{dz}}$, which is held constant over $0 < z < 2.5$ and for fixed total number of spectra. This panel shows that equation (63) is in qualitative agreement with these estimates, noting that here $N_i^{(pm)} = 4 \times 10^4$. (We discuss the dip at $z_m = 1.45$ below.) Especially for the two lower number densities, the constraint depends weakly on the density of spectra as equation (63) predicts. The cases in this panel appear to depend modestly on the magnitude of this effect with redshift could be determined with multiband photometry using a population with uniform spectra, e.g. quasars, and this information would allow it to be corrected for in cross-correlation studies again to the extent that the $\alpha_i^{(s)}$ are known.

The middle panel of Fig. 12 is for a photometric sample with the specifications of the LSST gold sample ($\ell^0(p) = 25.3$) and for different...
Figure 12. Estimates for how well the redshift distribution of sources (times their bias) in the photo-z bin pm can be reconstructed with cross-correlations. Shown is the error in redshift bin $i$ divided by the total number of galaxies in bin photo-z pm (i.e. $\delta T_{i}^{(pm)}/T_{tot}^{(pm)}$, equation 62), assuming redshift bins of size $\Delta z = 0.05$. Our calculations assume that much of $N_{i}^{(pm)}$ resides in the $z_{m} = 1.45$ bin, with ‘outlier’ galaxies distributed uniformly in the range $0 < z < 2.5$, and that the number density at $z = 0$ is that of a survey complete to $\beta^{(0)} = 25.3$ unless specified otherwise. The solid curves take half of the galaxies in this photo-z bin to reside outside of $z_{m}$, uniformly distributed so that $N_{i}^{(pm)}/N_{tot}^{(pm)} = 10^{-2}$ for $i \neq m$. The dashed curves are the same but for $N_{i}^{(pm)}/N_{tot}^{(pm)} = 10^{-3}$ (so that most galaxies reside at $z = 0$). The top panel shows the constraints from different spectroscopic samples with the specified constant $\delta N^{(p)}/dz$ over $0 < z < 2.5$ and with $f_{sky}$ adjusted so that there are $10^{5}$ total spectra. The middle panel shows three different spectroscopic samples that could be obtained for the same total telescope time (with the same specifications as in the top panel of Fig. 9). The bottom panel is for a spectroscopic sample with the specifications of BigBOSS and the specified limiting photometric magnitudes. All curves truncate the summation over $\ell$ at $\ell_{SL}$. spectroscopic samples that could be obtained for the same total telescope time (with the same specifications as in the top panel of Fig. 9). In this case, both the photometric and spectroscopic galaxies are at least marginally in the dense limit such that equation (64) applies and the sensitivity scales roughly as $f_{sky}^{-1}$. In the three cases plotted, $f_{sky}$ equals $2.5 \times 10^{-4}, 10^{-3}$ and $4 \times 10^{-2}$. The predictions in this panel depend weakly on the outlier fraction (compare the solid and dashed curves, which in two of the cases lie on top of each other). The sensitivity of the follow-up to $i^{(0)} = 21$ also falls off substantially with increasing redshift, which reflects that the spectroscopic galaxies are entering the rare regime.

The bottom panel shows the cases of a spectroscopic sample with the specifications of BigBOSS, the specified limiting photometric magnitudes and where the surveys’ overlap is $10^{5} \text{deg}^{2}$. These cases depends negligibly on the outlier fraction. The BigBOSS sample is on the borderline of the rare limit (especially at the lowest and highest $z$) such that this panel is most difficult to relate to our predictions. The rare–abundant limit given by equation (63) appears to be most applicable for the case of $i^{(0)} = 25$ – BigBOSS has $N_{i}^{(pm)} \sim 10^{3}$ at $z \sim 1$. However, this limit does not appear to describe the error for the $i^{(0)} = 23$ case as this is considerably less sensitive: $i^{(0)} = 23$ is on the borderline of being in the rare limit with $\delta N^{(pm)}/dz = 5000 \text{deg}^{-2}$ at $z_{m}$.

Autocorrelations [which were dropped in the derivations that led to equations (63) and (64)] add additional information. We find that autocorrelation estimates do not improve the sensitivity for redshift bins that contain only a small fraction of $pm$ galaxies. However, for the redshifts that contain the bulk of $pm$, they can improve the constraint on $\delta T_{i}^{(pm)}/T_{tot}^{(pm)}$ by an order of magnitude. This can be seen by focusing on the dip at $z_{in} = 1.45$ in Fig. 12, which corresponds to the redshift that contains half or more of the galaxies. Equations (63) and (64) do not predict a dip. Especially with a rare spectroscopic sample as investigated in the top panel (where the cross-correlations can be quite noisy) and a low outlier fraction, much of the constraint on the number at $z_{m}$ owes to the large value of $\delta m^{2}$, which indicates that many galaxies are concentrated in a narrow range in redshift.

Bernstein & Huterer (2010) found that a 0.0015 error on the fractional number on ‘all outlying peaks’ in the photo-z distribution is required for uncertainty in the redshift distribution of the lenses not to be the limiting factor for the next generation of photometric weak lensing surveys. Equations (63) and (64) (and Fig. 12) show that such an error in the true redshift distribution of $pm$ would be difficult to achieve with spectroscopic cross-correlations [even ignoring that the $b_{i}^{(p)}$ also need to be constrained to $O(10^{-3} f_{i}^{-1})$, where $f_{i}$ is the contamination fraction]. The case of BigBOSS cross-correlations with a photometric sample complete to $i^{(0)} = 25$ over $10^{5} \text{deg}^{2}$ (green curves in the bottom panel of Fig. 12) achieves the smallest error of the cases considered. However, its error on $\delta T_{i}^{(pm)}/T_{tot}^{(pm)}$ in redshift bin $i$ with $\Delta z = 0.05$ is still only $\sim 0.003$. If, for example, an outlying peak in the photo-z distribution spanned a redshift range of 0.2, this would require four redshift bins and make the fractional error on the total number $\sim 0.006$. While this does not appear sufficient to satisfy the Bernstein & Huterer (2010) requirement, it is possible that the calibration requirements are less severe owing to cancelling effects (Cunha et al. 2012, who found that an $\sim 0.01$ outlier fraction may be tolerable). Quantitatively answering the question of whether a BigBOSS-like survey is sufficient for futuristic weak lensing surveys requires an analysis of the bias on cosmological parameters induced by the pattern of uncertainties we find.

Thus far we have ignored prior information on the redshift distribution of the photo-z subsample pm. Often it is the case that we have prior information on the distribution of $N_{i}^{(pm)}$, e.g. from the photometric redshift PDF per galaxy (Lima et al. 2008; Freeman et al. 2009; Sheth & Rossi 2010). In this case our formalism has only minor modifications. Appendix A2 reviews how the quadratic estimator formalism generalizes to include prior information. For a Gaussian prior on the $N_{i}$ (dropping $pm$ superscripts for simplicity), the estimator with a prior becomes

$$
\hat{N}_{i} = [\hat{N}_{i}]_{\text{lat}} + [F + F_{p}]_{i} \left\{ \sum_{\ell,m} \delta(p \cdot \hat{s}) Q_{\ell} \left( \frac{\hat{p}}{\hat{s}} \right) - \text{Tr}[A^{-1}A_{i}] \right\} + [F_{p}]_{i,\ell, m} \left( N_{F_{p}} - [\hat{N}_{i}]_{\text{lat}} \right),
$$

(65)
where \( \mathbf{F}_p \) and \( N_{D,i} \) are, respectively, the inverse covariance matrix and mean of the prior. The prior pulls the estimated quantity towards \( N_{D,i} \), and this pull dominates if the prior is more peaked than the likelihood of the data.

The final subtlety we address with regard to photo-z calibration is cosmic magnification. Section 6 showed that cosmic magnification can be a significant bias if unaccounted for redshift estimation of the entire photometric sample. Magnification may be less onerous for photo-z calibration to the extent that the redshifts of the photo-z samples are well localized because the locations of sources and lenses are more constrained. However, it is also true that the \( \alpha_\lambda^\prime \) may be less constrained in line photo-z bins than less restricted populations. Appendix C1 addresses how magnification can be accounted for in the case of photo-z’s.

### 7.2 Self-calibration of the photometric sample

Self-calibration of redshifts by cross-correlating different photo-z bins within a photometric sample has the potential to achieve a tighter constraint on the \( N_{i}^{(pm)} \) than calibration using correlations with spectroscopically identified galaxies, since spectroscopic samples are likely to be either sparser in number or distributed over narrower fields than photometric ones. Self-calibration of a photometric survey with cross-correlations has been investigated in several studies (Huterer et al. 2006; Schneider et al. 2006; Benjamin et al. 2010). Here we show that the maximum sensitivity to \( dN_{i}^{(pm)}/dz \) that can be achieved with photometric self-calibrations is strikingly similar to the previously considered case of abundant spectroscopic and photometric samples.

For self-calibration to be successful, the redshift distribution of the photometric sample \( pm \) needs to be much better known than in the case of calibration with spectroscopic cross-correlations. This is because the redshift of \( pn \) for all \( n \) is the only knowledge one has in measuring the redshift of \( pm \). If \( pn \) is not centred around a single redshift, it is unclear how finite (\( \bar{p}n/\bar{p}n \)) translates into the redshift distribution of sample \( pm \). To avoid this difficulty, we assume that most of sample \( pm \) falls into redshift bin \( z_{mn} \). This assumption is the best case scenario, and will allow us to put a lower bound on the constraint from self-calibrations.\(^{17}\) Thus, the covariance matrix of the different photo-z bins is

\[
\mathbf{B}_{mn} = \langle \mathbf{p}_{i}^{(pm)} \mathbf{p}_{j}^{(pn)} \rangle = \sum_{i,j} T_{i}^{(pm)} T_{j}^{(pn)} C_{ij} + w_{ij}^{(pm, pn)} \delta_{ij},
\]

and we have assumed the same discretization in redshift to specify both the photometric and actual redshift bins. In the second line, the sum is evaluated at only one value of \( i \) if \( m = n \) (i.e. the auto-correlation). The approximate equality in the last line follows from assuming that \( C_{ij} \) is diagonal (as holds in the Limber approximation), that \( T_{mn}^{(pm)} = D_{nm} b_{i}^{(pm)} N_{mn}^{(pm)} \gg \sum_{i} T_{i}^{(pm)} \), and from keeping terms that are \( \mathcal{O}(T_{i}^{(pm)} / T_{j}^{(pm)}) \) or larger. This is the limit in which the fraction of catastrophic photo-z’s is small and where the covariance matrix \( \mathbf{B}_{mn} \) is diagonally dominated. In this limit, and to

\[
\text{lowest order in } \alpha_{mn,j} = T_{i}^{(pm)/T_{j}^{(pn)}} \text{, the Fisher matrix with respect to the } T_{i}^{(pm)}/T_{j}^{(pn)} \text{ is}
\]

\[
\mathbf{F}_{T_{i}^{(pm)/T_{j}^{(pn)}}} \approx \sum_{l,m} \frac{T_{i}^{(pm)} C_{mn} T_{m}^{(pn)} C_{ln}}{B_{mn} B_{pn}},
\]

where \( B_{mn} \approx [T_{i}^{(pm)}]^{2} C_{mn} + w_{mn}^{(pm, pn)} \), and the matrix is zero between other combinations of parameters. The quadratic estimator for \( T_{i}^{(pm)}/T_{j}^{(pn)} \) in this limit can also easily be written as it only involves correlations between the photometric samples \( m \) and \( n \). Thus, in the diagonally dominated limit, the parameter \( T_{i}^{(pm)/T_{j}^{(pn)}} \) only correlates with \( T_{i}^{(pm)} \), and there is a perfect degeneracy that must be broken by adding a prior (often catastrophic errors occur in one redshift direction) or going to higher order terms that are suppressed by another factor of \( \alpha_{mn,j} \). [Including cosmic shear would also break this degeneracy (Zhang et al. 2010).] In the case of the prior that constraints \( T_{i}^{(pm)}/T_{j}^{(pn)} \) to be zero, many of our previous results hold as equation (67) is the same as equation (35) [and its subsequent incarnation in equation (42)] with the replacement \( \beta(z) = 1 \) and a slightly different number dependence. (In fact, we do not need the additional approximation of \( S = 1 \), as was made there.) Thus, if \( T_{i}^{(pm)} \gg 10^{4} b^{-2} \Delta z^{-2} \), so that the abundant limit holds,

\[
\frac{\delta T_{i}^{(pm)}}{T_{i}^{(pm)}} \approx 10^{-3} f_{sky}^{-1/2} \left( \frac{\bar{z}}{10^{2}} \right)^{-1}.
\]

Photometric self-calibration over a significant fraction of the sky is capable of part in \( 10^{3} \) accuracy required by the next generation of weak lensing surveys (e.g. Bernstein & Huterer 2010), but with the same caveats as noted in the previous subsection that (1) this method does not break the degeneracy between number and linear bias and (2) we have not calculated the bias on cosmological parameters as is necessary to truly quantify the potential of this method. In addition, this error only applies to the case of a single catastrophic error direction. If the latter does not hold, the constraint is likely to be weakened by the factor \( \sqrt{\alpha_{mn,j}} \).

More generally, the full covariance matrix of the photo-z bins, \( \mathbf{B}_{mn} \) (plus overlapping spectroscopic populations), can be used as the covariance matrix in the minimum variance quadratic estimator. This self-calibration estimator is likely to be more sensitive than the algorithm discussed in Benjamin et al. (2010), the only self-calibration method that we are aware of, as that algorithm uses linear combinations of the \( A_{nm} \) that encapsulate a subset of the full covariance and does not weight scales optimally.

### 7.3 Cleaning correlated anisotropies from a map

Our estimator is optimal for statistically estimating the level of (and, hence, cleaning) correlated anisotropies from angular cross-correlations between diffuse background/foreground maps and spectroscopic galaxies. The fractional errors we quote on number are equivalent to the error with which anisotropies can be statistically removed. Thus, the survey optimizations for this application are equivalent to those discussed for \( N_{i}^{(pm)} \) estimates. Our previous calculations suggest that correlating anisotropies can be cleaned statistically to the 1 per cent level. For wide-field observations of diffuse redshifted 21 cm emission, this factor of 100 could be helpful if extragalactic sources are found to be a limiting factor. For CMB analyses, cross-correlations could also be interesting for studying the redshift distribution and for expunging foregrounds. For example, it could better enable the separation of the CIB from CMB anisotropies generated at higher redshift. [CIB contamination is currently the limiting factor in measurements of the kinetic

\(^{17}\) This assumption requires a highly artificial top-hat photo-z distribution at \( z_{mn} \) for consistency. However, we expect that our result is more general than this choice.
Sunyaev–Zeldovich effect, which conveniently does not correlate with the \( s_i \), Kashlinsky et al. (2007) investigated correlations on \( \sim 10 \) arcmin scales between diffuse anisotropies in Spitzer and HST deep fields. Our results suggest that the sensitivity to the clustering component would be increased with wider fields (perhaps using shallower ground-based observations rather than HST, since we found that the extremely high number density in the HST fields is not useful).

For diffuse anisotropies, gravitational lensing enters at second order because lensing preserves surface brightness. Thus, at large scales its impact on correlating the anisotropies in a map with the spectroscopic sample is small. If the ‘spectroscopic’ sample is measured at sufficiently high redshifts that the magnification–magnification term becomes important, only then can magnification result in a linear order diffuse foreground–spectroscopic population cross-correlation signal. Magnification also has the effect of correlating the \( \hat{s}_i \), which can bias the estimate. However, both magnification effects are correctable as the \( \hat{q}_i^{(s)} \) can be measured.

Finally, the goal is sometimes to invert a measured 2D clustering signal to 3D clustering of a population using knowledge of \( dN/dz \). In the cases where the accuracy requirements are not stringent, knowledge of the mean redshift and the redshift width suffices to make this conversion. These quantities are typically easier to constrain than the full \( dN/dz \), and so far fewer spectra are required for the cross-correlation. Assuming a \( z \)-independent, power-law power spectrum and \( dN/dz \) that can be parametrized by a power of distance times an exponential of a power of distance, we found knowing just the mean and variance of \( dN/dz \) sufficed to invert the 2D clustering to 3D at the 10 per cent level.

8 MOCK SURVEYS

We are interested in understanding the robustness with which the proposed estimator converges to the input \( N^{(p)}_i \). To investigate its convergence, mock surveys are generated by decomposing the covariance matrix \( A \) into its eigenvectors \( e_\alpha \) and eigenvalues \( \lambda_\alpha \) for \( \alpha \in [0, N_{\mathrm{max}}] \). Then, a realization of the galaxy field at that multipole \( \ell \) that has this covariance matrix is given by

\[
g_\beta(\ell, m) = \sum_{\alpha=0}^{N_{\mathrm{max}}} r_\alpha \lambda_\alpha (\ell)^{1/2} [e_\alpha(\ell)]_\beta, \tag{69}
\]

where \( r_\alpha \) is a Gaussian deviate with unit variance. Here, \( g_\beta \) corresponds to the overdensity in redshift bin \( i \) of the spectroscopic survey and \( g_0 \) is the overdensity in the photometric sample. Our mocks assume that we are operating in a small enough patch such that there is a one-to-one mapping between wavevectors and spherical harmonics. In addition, our mocks assume linear theory and the Limber approximation. These approximations should not impact the conclusions per our previous results.\(^{15}\)

We generate 1000 mocks for two contrasting cases to illustrate the estimator’s performance:

(i) \( 10 \times 10 \text{ deg}^2 \) field with \( dN^{(s)}/dz = 10^3 \text{ deg}^{-2} \), \( dN^{(p)}/dz = 10^4 \text{ deg}^{-2} \), and 10 redshift bins spanning \( 0 < z < 1 \), each with 1000\(^2 \) angular pixels, specifications which result in \( \sim 10 \) per cent errors on the \( N^{(p)}_i \).

(ii) \( 30 \times 30 \text{ deg}^2 \) field with \( dN^{(s)}/dz = 10^4 \text{ deg}^{-2} \) and photometry up to \( \ell^{(p)} = 25.3 \), spanning \( 0 < z < 2.5 \) with 50 bins and 300\(^2 \) angular pixels, which result in \( \sim 1 \) per cent errors on the \( \hat{N}^{(p)}_i \).

The resolution of each mock is sufficient to resolve the scales that contain the bulk of the information (Section 5.3).

Next, we apply the estimator to the harmonic space realization of these mocks. (It would be equivalent to apply our estimator in real space using the results of Section 5.) Fig. 13 demonstrates that the minimum variance quadratic estimator converges to the expected Gaussian distribution of errors. This holds despite starting with initial estimates for the \( \hat{N}^{(i)}_i \) that are an order of magnitude smaller than their actual value. The top panel shows the case of a \( 10 \times 10 \text{ deg}^2 \) field with the specified populations and 10 bins spanning \( 0 < z < 1 \) (resulting in \( \sim 10 \) per cent errors). The bottom panel shows a \( 30 \times 30 \text{ deg}^2 \) field with a photometric sample complete to \( \ell^{(p)} = 25.3 \) and 50 bins spanning \( 0 < z < 2.5 \) (resulting in \( \sim 1 \) per cent errors). We find that the estimator robustly converges to its minimum, even when it starts far from it, and that in both cases there are zero outliers at \( > 5\sigma \) in the 1000 mocks.

Fig. 14 shows the walk of the \( \hat{N}^{(p)}_{\mathrm{med},2} \) estimate as a function of iteration number for the middle-redshift bin in the two cross-correlation cases. The solid curves denote the full minimum variance estimator and the dashed curves show the Schur–Limber estimator (which

\(^{15}\) These mocks have one significant advantage over a real survey: they are periodic. Hence, we do not have to worry about the survey window functions, and different modes on the lattice are truly independent. We discussed how to deal with these real-world complications in Section 5.3.
surveys capable of per cent-level $N_i^{(p)}$ determinations may be able to constrain the bias to 10 per cent.

Other possibilities for breaking this degeneracy require using additional scales or constraints not included in our earlier estimates. Such methods to break this degeneracy include modelling of the one-halo term in $\langle ps_i \rangle$; abundance matching or other modelling methods to map galaxy number to bias (e.g. Conroy, Wechsler & Kravtsov 2006, as $b_i^{(p)}$ is a weak function of mass for abundant haloes); galaxy–galaxy lensing with the photometric galaxies as both sources and lenses (using the $b_i^{(p)}N_i^{(p)}$ from cross-correlation measurements – the quantity needed for the lenses – to constrain $dN_i^{(p)}/dz_i$ of the sources); breaking up the photometric sample into subsamples and using that the autocorrelation of each subsample provides an integral constraint on its bias; and measurements of the second-order bias, either in the two-point function or higher order statistics. While several of these avenues appear promising, we shall not pursue them here.

10 CONCLUSIONS

Determining the redshift distribution of a particular population of astronomical objects is often quite difficult. However, since most cosmological objects are clustered (i.e. they trace the same matter field on large scales), objects that are close together on the sky are also likely to be close together in redshift. Thus, the redshift distribution of a population of objects can be determined by cross-correlating it in angle with a population whose redshift distribution is better known. This paper presented a new, optimal estimator for the redshift distribution of a given population in terms of cross-correlations. We found that this estimator (1) is quite intuitive in a number of limits, (2) is straightforward to apply to observations, (3) robustly finds the posterior maximum, and (4) conveniently selects angular scales at which the fluctuations are well approximated as independent between redshift bins and at which linear theory applies. In addition, we provided analytic formulae that can be used to quickly estimate the sensitivity of cross-correlations between overlapping surveys to $b_i dN_i/dz_i$ – the linear bias times angular number density per redshift. We compared our estimator to others suggested in the literature, showing that it produces considerably smaller errors than the familiar estimator of Newman (2008).

The optimal estimator’s fractional error on the number of objects (times their bias) in a redshift bin is $\approx \sqrt{10^4 N_{bin}/N^{(i)}}$ if the spectroscopic sample has a mean angular density of less than a few thousand and the unknown sample has a mean density larger than this value. Here, $N^{(i)}$ is the total number of spectra per unit redshift and $N_{bin}$ is the number of redshift bins spanned by the bulk of the unknown population. Thus, it is not necessarily better to use a narrow, deep spectroscopic survey covering tens of degrees than a wide, shallow one. Once the spectroscopic and unknown populations have $dN_i/dz_i \gg 10^4 b_i^{-2}$ deg$^{-2}$, the sensitivity scales simply with the fraction of sky covered (again with an intuitive formula) and no longer depends on just the total number of spectra. We found that upcoming spectroscopic surveys that aim for millions of spectra can potentially achieve per cent-level constraints on the $b_i dN_i/dz_i$ of

9 BREAKING THE BIAS – NUMBER DEGENERACY

Much of our discussion has ignored that cross-correlations do not constrain number alone but instead bias times number. The bias often can be parametrized as a smoothly and slowly varying function with redshift. An exception is samples with hard colour cuts, where the underlying galaxy population, and hence the large-scale bias, can change relatively quickly with $z$ at points where spectral features transition in and out of filters. In such cases, knowledge of $b_i^{(p)} N_i^{(p)}$ is more difficult to translate into knowledge about $N_i^{(p)}$.

For many applications, bias times number is in fact the quantity of interest, including attempts to measure 3D correlations with angular correlations or attempts to subtract correlated anisotropies from a map of diffuse backgrounds. However, knowing the bias is particularly important to the application of calibrating the lens redshifts for weak lensing surveys. RSDs as well as lensing magnification formally provide terms that break the bias–number degeneracy. However, we argued that breaking this degeneracy is unlikely with RSDs. Cosmic magnification is more promising. We argued that

Figure 14. Walk of the estimated $N_i^{(p)}$ for $i = N_{bin}/2$ as a function of iteration number for the two cross-correlation examples described in Fig. 13 and the text. The solid curves show the full minimum variance estimator and the dashed curves show the Schur–Limber estimator (which converges more quickly). The curves terminate after the last iteration changed the estimated $N_i^{(p)}$ by less than a part in 10$^4$ when averaged over all $i$. The initial guesses for the $N_i^{(p)}$ are taken to be an order of magnitude too small. The asymptotic value of each $N_i^{(p)}$ shown in this figure is within 2σ of the input $N_i^{(p)}$.
an unconstrained population. Furthermore, we showed that our estimates for the constraints on $b \, dN/dz$ also apply to spectroscopically calibrating samples binned by their photometric redshift, and we also commented on the sensitivity of photometric self-calibration. We investigated a number of approximations and how they bias the estimator. In the Limber approximation – which we found to be excellent for relevant redshift slice widths – the covariance matrix for this problem can be analytically inverted, allowing simple expressions for the estimator. We showed that the nearly optimal, Limber-approximation estimator can be expressed as an iteration of

$$\hat{N}_i = [\hat{N}]_{\text{last}} + \sum \phi_i \left( \hat{\rho}_s \hat{s}_i - \langle \hat{\rho} \hat{s}_i \rangle \right) / \sum \phi_i \frac{d\langle \hat{\rho} \hat{s}_i \rangle}{dN_i},$$

(70)

where the $\phi_i$ are weights comprised of intuitive combinations of the covariance matrix (equation 54) and $\hat{\rho} \hat{s}_i$ is the cross-correlation between the unknown sample and the spectroscopic sample in bin $z_i$. The summations are either evaluated over bins in angular separation or spherical harmonic indices depending on whether $\hat{\rho} \hat{s}_i$ is measured in configuration or harmonic space. In many limits, this estimator has the same error as the maximum likelihood estimate for the cross-power amplitude. Furthermore, we found that the bias from assuming the Limber approximation was minute and also argued that the same holds for RSDs. We found that cosmic variance is measured in configuration or harmonic space. In many limits, the techniques developed in this paper can be applied to a wide range of existing and upcoming surveys from DES, GAMA and WISE, to LSST, Euclid and the SKA. We intend to apply this estimator to observational data in a future paper.

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REFERENCES

Abell P. A. et al. (LSST Science Collaboration), 2009, preprint (arXiv:0912.0201)
Abramowizt M., Stegun I. A., 1964, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover, New York
Ahn C. P. et al., 2012, ApJS, 203, 21
Bartelmann M., Schneider P., 2001, Phys. Rep., 340, 291
Benjamin J., van Waerbeke L., Ménard B., Kilbinger M., 2010, MNRAS, 408, 1168
Benjamin J. et al., 2013, MNRAS, 431, 1547
Bernstein G. M., 1994, ApJ, 424, 569
Bernstein G., Huterer D., 2010, MNRAS, 401, 1399
Bond J. R., Jaffe A. H., Knox L., 1998, Phys. Rev. D, 57, 2117
Bouwens R. J. et al., 2012, ApJ, 752, L5
Brainder T. G., Blandford R. D., Smail I., 1996, ApJ, 466, 623
Christensen N., Meyer R., Knox L., Lucy B., 2001, Class. Quantum Grav., 18, 2677
Coil A. L. et al., 2004, ApJ, 609, 525
Colless M., Dalton G., Maddox S., Sutherland W., Norberg P., Taylor K., 2001, MNRAS, 328, 1039
Conroy C., Wechsler R. H., Kravtsov A. V., 2006, ApJ, 647, 201
Cooray A., Sheth R., 2002, Phys. Rep., 372, 1
Corbató F. J., Uretsky J. L., 1959, J. ACM, 6, 366
Cunha C. E., Lima M., Oyaizu H., Frieman J., Lin H., 2009, MNRAS, 396, 2379
Cunha C. E., Huterer D., Lin H., Busha M. T., Wechsler R. H., 2012, preprint (arXiv:1207.3347)
Dawson K. S. et al., 2013, AJ, 145, 10
Dodelson S., 2003, Modern Cosmology. Academic Press, Amsterdam
Drinkwater M. J. et al., 2010, MNRAS, 401, 1429
Driver S. P. et al., 2011, MNRAS, 413, 971
Duffy A. R., Meyer M. J., Staveley-Smith L., Berny M., Croton D. J., Korhalaiki B. S., Gerstmann D., Westerlund S., 2012, MNRAS, 426, 3385
Efstathiou G., 2004, MNRAS, 349, 603
Efstathiou G., Bernstein G., Tyson J. A., Katz N., Gauthakurta P., 1991, ApJ, 380, L47
Eisenstein D. J., Hu W., 1998, ApJ, 496, 605
Eisenstein D. J. et al., 2001, AJ, 122, 2267
Erben T. et al., 2009, A&A, 493, 1197
Feldman H. A., Kaiser N., Peacock J. A., 1994, ApJ, 426, 23
Freeman P. E., Newman J. A., Lee A. B., Richards J. W., Schafer C. M., 2009, MNRAS, 398, 2012
Fugmann W., 1988, A&A, 204, 73
Gillman E., Fiebig H. R., 1988, Comput. Phys., 2, 62
Hamaus N., Seljak U., Desjacques V., Smith R. E., Baldauf T., 2010, Phys. Rev. D, 82, 043515
Hamilton A. J. S., 1992, ApJ, 385, L5
Hamilton A. J. S., 1993, ApJ, 417, 19
Hansen F. K., Górski K. M., Hivon E., 2002, MNRAS, 336, 1304
Heavens A. F., Joachimi B., 2011, MNRAS, 415, 1681
Hildebrandt H. et al., 2012, MNRAS, 421, 2355
Hill G. J. et al., 2008, in Kodama T., Yamada T., Aoki K., eds, ASP Conf. Ser. Vol. 399, Panoramic Views of Galaxy Formation and Evolution. Astron. Soc. Pac., San Francisco, p. 115
Hivon E., Górski K. M., Netterfield C. B., Crill B. P., Brunet S., Hansen F., 2002, ApJ, 567, 2
Ho S., Hirata C., Padmanabhan N., Seljak U., Bahcall N., 2008, Phys. Rev. D, 78, 043519
Hockstra H. et al., 2006, ApJ, 647, 116
Hui L., Gaztañaga E., Loverde M., 2007, Phys. Rev. D, 76, 103502
Hui L., Gaztañaga E., Loverde M., 2008, Phys. Rev. D, 77, 063526
Huterer D., Takada M., Bernstein G., Jain B., 2006, MNRAS, 366, 101
Johnston S. et al., 2008, Exp. Astron., 22, 151
Jungman G., Kamionkowski M., Kosowsky A., Spergel D. N., 1996, Phys. Rev. D, 54, 1332
Kaiser N., 1987, MNRAS, 227, 1
Kaiser N., 1992, ApJ, 388, 272
Kashlinsky A., Arendt R. G., Mather J. E., Moseley S. H., 2007, ApJ, 666, L1
Knox L., 1997, ApJ, 480, 72
Kochanek C. S. et al., 2012, ApJS, 200, 8
Landy S. D., Szalay A. S., 1993, ApJ, 412, 64
Larson D. et al., 2011, ApJS, 192, 16
Lawrence A. et al., 1999, MNRAS, 308, 897
Le Fèvre O. et al., 2005, A&A, 439, 845
Lilly S. J. et al., 2007, ApJ, 666, 2
Lima M., Cunha C. E., Oyaizu H., Frieman J., Lin H., Sheldon E. S., 2008, MNRAS, 390, 118
Limber D. N., 1953, ApJ, 117, 134
Limber D. N., 1954, ApJ, 119, 655
Lucas S., 1995, J. Comput. Appl. Math., 64, 269
Mandelbaum R. et al., 2008, MNRAS, 386, 781
Matthews D. J., Newman J. A., 2010, ApJ, 721, 456
Matthews D. J., Newman J. A., 2012, ApJ, 745, 180
Ménard B., Scranton R., Fukugita M., Richards G., 2010, MNRAS, 405, 1025

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APPENDIX A: ESTIMATOR DETAILS

This appendix gives two generalizations of the minimum variance quadratic estimator (Appendix A1), then shows how a prior would impact the estimator (Appendix A2), and finally considers how the estimator and variance change with different basis choices to represent dN(θ)/dζ (Appendix A3).

A1 Full estimator

Here we write two more complete expressions for the estimator than were given in the text.

First, the estimator given by equation (16) is biased by different cosmic realizations except in the limit in which a large number of modes are used with comparable weight. The full, unbiased estimator replaces equation (16) with (Bond et al. 1998, for more on derivation see ensuing appendix)

\[
F_{ij}^{\text{full}} = F_{ij} + \sum_{f,m} \text{Tr} \left[ \left( \hat{p} \hat{s} \right) - A \right] \times \left[ A^{-1}A_{ij}A_{ij}^{-1} - \frac{1}{2} A^{-1}A_{ij}A_{ij}^{-1} \right].
\]  

This expression shows that the estimator is biased by using \( F_{ij} \) rather than \( F_{ij}^{\text{full}} \) at the level of \( N_{i,j}^{-1/2} \), where \( N_{i,j} \) is the number of modes that contribute. There are \( N_{i,j} = \ell^2 \sim 10^5 \ f_{\text{sky}} \) total modes that generally contribute to the estimator (at least when one sample is abundant). Thus, this error will impact the estimator at the \( 10^{-3} f_{\text{sky}}^{-1/2} \) level. This additional sample variance noise should typically be below the statistical error. We saw no evidence for this bias in the estimates from mock surveys in Section 8.

Secondly, we dropped terms that came from the dependence of \( A_{00} \) on the parameter being varied in the Limber-approximation estimator presented in the text (equation 31). The full estimator in the Limber regime is

\[
\hat{N}_k^{(p)} = [\hat{N}_k^{(p)}]_{\text{last}} + \left[ F_{ij}^{\text{full}} - F_{ij} \right] + \sum_{f,m} \text{Tr} \left[ \left( \hat{p} \hat{s} \right) - A \right] \times \left[ A^{-1}A_{ij}A_{ij}^{-1} - \frac{1}{2} A^{-1}A_{ij}A_{ij}^{-1} \right].
\]  

The autocorrelation terms that were (as a result of our approximation) omitted in equation (31) become important when \( \sum A_{00}/[A_{00} A_{00}] \sim 1 \). We found that their effect was most evident when considering photo-z calibration. In addition, there is a term that arises from the derivatives hitting \( A_{00} \) that is half the size of the second term that multiplies \( \hat{p} \hat{s} \) in equation (31).

All of our estimators can be written as sums over \( \ell \) or \( m \) and do not require keeping angular information. This may come as a surprise because each individual \( \ell, m \) mode contributes independent information and so it may seem suboptimal to combine them in annuli. However, one can note that this is also a symmetry of the likelihood function as \( \mathcal{L} \) can be written so that the argument in the exponent is proportional to \( \sum_{\ell,m} \text{Tr}[\mathbf{A}(\ell) \mathbf{A}^{-1}(\ell)] \), where \( \mathbf{A}(\ell) \) is the estimated covariance matrix (e.g. \( \hat{A}_{00}(\ell) \equiv (2\ell + 1)^{-1} \sum_{m} |p(\ell, m)|^2 \)).

A2 Impact of prior

The estimator given in equations (16) and (A4) follows from using the multidimensional Newton’s method to find the zeros of the derivative of the log of the data likelihood function, log \( \mathcal{L} \) (Bond et al. 1998).

\[
\hat{N}_j = [\hat{N}_j]_{\text{last}} - ([\text{log } \mathcal{L}])_j^{-1} [\text{log } \mathcal{L}]_j,
\]  

Newton’s method is applied to the log of the likelihood rather than the likelihood itself because Newton’s method provides exact estimates for the extrema of a quadratic function.
where $[\log \mathcal{L}]_{i}$ is the Hessian of $\log \mathcal{L}$, which upon ensemble average is the negative of the Fisher matrix. For a Gaussian likelihood with covariance matrix $\mathbf{C}$ and data vector $\Delta$, $[\log \mathcal{L}]_{i} = \Delta^{\mathcal{C}}$, $\mathbf{C}^{-1}\Delta/2$.

With this derivation in mind, it is straightforward to generalize equation (A3) to include a prior:

$$
\tilde{N}_{i} = [\tilde{N}_{i}]_{\text{last}} - \left([(\log \mathcal{L})_{i} + [\log \mathcal{L}_{p}]_{i}]^{-1} ([\log \mathcal{L}]_{i} + [\log \mathcal{L}_{p}]_{i}) \right),
$$

(A4)

where $\mathcal{L}_{p}$ is the prior likelihood function. The case of a Gaussian prior on the $N_{i}$ is given by equation (65).

As an application of the above, let us consider the case of our $N_{i}^{\mathrm{p}}$ estimator in which the $b_{i}^{\mathrm{p}}$ are imperfectly known and instead are constrained by prior information. Remember that since the $N_{i}^{\mathrm{p}}$ are estimated from large-scale cross-correlations, they are degenerate (ignoring e.g. magnification) with $b_{i}^{\mathrm{p}}$ and can only be separated with a prior from the autocorrelation measurements. In this case, the Fisher matrix of the parameters $N_{i}^{\mathrm{p}}$ and $b_{i}^{\mathrm{p}}$ plus a prior on $b_{i}^{\mathrm{p}}$ yields the new error matrix:

$$
\mathbf{F}^{\mathrm{p}} = \frac{F_{ii}^{2}}{[N_{i}^{\mathrm{p}}]_{i}^{-2}} \left( \begin{bmatrix} [N_{i}^{\mathrm{p}}]_{i}^{-2} & N_{i}^{\mathrm{p}} [b_{i}^{\mathrm{p}}]_{i}^{-2} \\ N_{i}^{\mathrm{p}} [b_{i}^{\mathrm{p}}]_{i}^{-2} & [b_{i}^{\mathrm{p}}]_{i}^{-2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{b_{i}}^{-2} \end{bmatrix} \right),
$$

(A5)

where $\sigma_{b_{i}}$ is the standard deviation of the Gaussian prior on $b_{i}^{\mathrm{p}}$ centred on $[b_{i}^{\mathrm{p}}]_{\text{prior}}$. Our previous results correspond to $\sigma_{b_{i}} \to 0$. (We are ignoring redshift-bin correlations in the prior for simplicity, but such correlations can be easily incorporated.) The fractional variance on a measurement of $N_{i}^{\mathrm{p}}$ for the case that $b_{i}^{\mathrm{p}}$ is held fixed.

The estimator in this limit is

$$
\tilde{N}_{i}^{\mathrm{p}} = [\tilde{N}_{i}]_{\text{last}} + \frac{1}{F_{ii}} \sum_{i} \left\{ \hat{A}_{i} \right\} \left\{ \tilde{p} \tilde{A}_{i} - A_{i} \right\} + N_{\mathrm{p}}^{\prime} \left( \frac{[b_{i}^{\mathrm{p}}]_{\text{prior}}}{[b_{i}^{\mathrm{p}}]_{i}^{-2} \text{last} - 1} \right),
$$

(A7)

with the complementary estimator for the bias being trivially

$$
\tilde{b}_{i}^{\mathrm{p}} = [b_{i}^{\mathrm{p}}]_{\text{prior}}.
$$

For the case of SDSS or BOSS quasars (where $N_{i}^{\mathrm{p}} \sim 10^{5}$), the variance in the measured bias is $\sigma_{b_{i}} \sim 0.1$ (Ross et al. 2009; White et al. 2012), which is comparable to the redshift error expected from cross-correlations (Fig. 7). However, for rare samples with fewer spectra than SDSS quasars, the uncertainty in $b_{i}^{\mathrm{p}}$ will dominate the error in the $N_{i}^{\mathrm{p}}$ that ignores the bias uncertainty.

### A3 Estimator and constraints in other bases

We have chosen a top-hat basis set for convenience, which also leads to an estimator that converges robustly to the likelihood peak. Other choices are clearly possible, and they may be preferred in some situations. For example, instead of $N_{i}^{\mathrm{p}}$, we could attempt the parameters of a particular functional form. Or we could expand $\Delta N/N$ as a sum of overlapping Gaussians or (orthogonal) polynomials times basis functions (e.g. a power law times an exponential).

While the quadratic estimator formalism is completely general, it is not trivial to recast the estimator in terms of an arbitrary basis set as $A$ needs to be recast in terms of the new parameter set. In many cases, this is not analytically expressible (with an exception being the linear case discussed below). However, it is trivial to translate our results for the error on a parameter into another basis set. The new Fisher matrix is given by the chain rule

$$
\mathbf{F} = \mathbf{W}^{T} \mathbf{F} \mathbf{W},
$$

(A8)

where $\mathbf{W}$ is the Jacobian matrix between the $N_{i}^{\mathrm{p}}$ and the new parameter set $\lambda_{i}$. We showed that the Fisher matrix is often well approximated as diagonal, such as in the Schur–Limiter limit. In this case

$$
F_{ij} \approx \frac{N_{\mathrm{bin}}}{F_{kk}} \sum_{k=1}^{N_{\mathrm{bin}}} \frac{dN_{i}^{\mathrm{p}}}{d\lambda_{k}} \frac{dN_{j}^{\mathrm{p}}}{d\lambda_{k}}.
$$

(A9)

Once the $N_{i}^{\mathrm{p}}$ are estimated with our technique, they can be combined to estimate the $\lambda_{i}$ with error given by $\mathbf{F}^{-1}$.

Fig. A1 shows an example using equation (A8) in which we changed basis to one in which $dN/N$ is constrained to have the smooth functional form specified in the key (a generalization of equation (3) for $P(z, i)$). This figure investigates the case of a photometric population with $\lambda_{i} = 23$ and with a low density of spectroscopic objects given by $\Delta N/N dz = 10 \text{ deg}^{-2}$, overlapping over a sky area of 1000 deg$^{2}$ (although the total number of spectra, here 10$^{4}$, is the essential quantity). It shows that the constraints are substantially improved even if a fairly general functional form is assumed (varying two parameters for the dotted curves and four for the dot–dashed). One advantage of parametrizing $dN/N dz$ with a smooth functional form is that the constraints do not depend on the choice of $\Delta z$.

Finally, we note that the formalism this paper developed for estimating the $N_{i}^{\mathrm{p}}$ can be trivially recast for models in which one instead aims to constrain some set of basis functions $\phi_{i}$ for which $dN/N dz = \sum c_{i} \phi_{i}(z)$, where $c_{i}$ are a set of coefficients. In this case, the primarily difference is that for the $\alpha_{i}(k, z_i)$ that went into...
calculating $C(\ell)$, the index $i$ no longer indexes the redshift bin but rather the basis function.

**APPENDIX B: EXTENDED LIMBER APPROXIMATION**

The Limber approximation is most applicable on small angular scales, where we may approximate the sky as flat and the spherical harmonic transform as a Fourier transform (e.g. White et al. 1999; Pápai & Szapudi 2008). With these approximations, the angular correlation function can be written as

$$w(\theta) = \int d\chi \frac{W(\chi)}{W(\chi_1)} W(\chi_2) \times \int \frac{d^2k}{(2\pi)^3} P(k) e^{i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)}, \quad (B1)$$

$$\approx \int \frac{d^2k}{(2\pi)^3} P(k, k_1) \int d\tilde{\chi} W^2(\tilde{\chi}) e^{i\mathbf{k} \cdot \mathbf{x}_1} \times \int dZ e^{i\mathbf{k} \cdot \mathbf{Z}}, \quad (B2)$$

$$= \frac{K_\perp}{2\pi} \int d\tilde{\chi} W^2(\tilde{\chi}) J_0(k_\perp, \tilde{\chi} \theta), \quad (B3)$$

where in the second line we have changed variables from $\chi_1$ to centre of mass and relative coordinates, $\tilde{\chi} = (\chi_1 + \chi_2)/2$ and $Z = \chi_1 - \chi_2$, and assumed that $W$ is so broad that $W(\tilde{\chi} \pm Z/2) \approx W(\tilde{\chi})$ (which is not always the case for the $W$ considered in the text). Writing $\ell = k_\perp \tilde{\chi}$ and using $J_0(\ell \theta) \approx P_\ell(\cos \theta)$ for $\theta \ll 1$ and $\ell \gg 1$, the angular power spectrum, $C_\ell$, is thus

$$C_\ell = \int d\chi \frac{W^2(\chi)}{\chi^2} P(k_\perp = \ell/\chi, k_\parallel = 0). \quad (B4)$$

The Limber approximation further results in correlations between non-overlapping redshift slices being zero.

One can compare the Limber approximation to the analytic solution for certain cases to see when and how well these approximations work. Let us assume $W(\chi)$ is a top-hat in $\chi$ in slices of width $\Delta \chi$ (as in the main body of this paper). Then, the cross-spectrum is

$$\ell^2 C_{ij} = k_\perp^2 \int \frac{dk_\perp}{2\pi} e^{i\mathbf{k} \cdot (\chi_1 - \chi_2)} \frac{k_\perp^2 \Delta \chi}{2} P(k_\perp, k_\parallel). \quad (B5)$$

Using the method of steepest descents (or approximating the power spectrum as a power law and using the asymptotic behaviour of the resulting Bessel functions), it can be shown that for $k_\perp |\chi_1 - \chi_2| \gg 1$

$$\ell^2 C_{ij} \rightarrow \ell^2 C_{ij}^{\text{asymp}} \equiv \frac{k_\perp^2 \Delta \chi}{2} P(k_\perp) e^{-k_\perp |\chi_1 - \chi_2|}. \quad (B6)$$

We can make further progress by assuming that $P(k)$ is a power law. In particular, if $P(k)$ is a power law with index $-2$, roughly the index on galaxy scales in our Universe, the integral in equation (B5) has simple poles that make the evaluation trivial:

$$\ell^2 C_{ij}^{\text{asymp}} \left\{ \begin{array}{ll} k_\perp \Delta \chi + \exp[-k_\perp \Delta \chi] - 1 & i = j; \\ \cosh[k_\perp \Delta \chi] - 1 & i \neq j. \end{array} \right. \quad (B7)$$

Note that when $i = j$ and $k_\perp \Delta \chi \gg 1$, we recover the Limber result $\ell^2 C_{ii} \approx (k_\perp^2 / \Delta \chi) P(k_\perp)$. In addition, at $k_\perp \Delta \chi = 2$ (the boundary of applicability used in Fig. 6), equation (B7) undershoots Limber by 40 per cent with this percentage decreasing roughly linearly with increasing $k_\perp \Delta \chi$. The errors from Limber will be smaller when $P(k)$ has a flatter power law, as is the case at $k_\perp \Delta \chi \sim 1$ for the $\Delta \chi$ considered in the text. That the Limber approximation works so well once $k_\perp \Delta \chi$ moderately exceeds unity helps explain why in the text we find it to be such a good approximation for our problem.

Next, consider the impact of RSDs in the Limber approximation, which have been neglected in all of our prior discussion. RSDs could be interesting for our purposes because they break the $b^{(i)} - N^{(i)}$ degeneracy. On linear scales the lowest order correction owing to RSDs is to multiply the power spectrum by $1 + 2\beta_\mu \mu^2$, where $\mu = k_\perp/k$ and $\beta_\mu \simeq \Omega_m h^2 / b^{(i)}$, with the redefinition of $\mu$ and $k$ to be the analogous redshift-space quantities (Kaiser 1987; Hamilton 1992). In the Limber approximation, $|k| \lesssim \Delta \chi^{-1}$ and so we expect $|\mu| \ll 1$ and the correction to be small. However, how quickly this falls off depends on $W(\chi)$. In the case of our top-hat window function and with the replacement $P(k_\perp, k) \rightarrow P(k_\perp)(1 + 2\beta_\mu \mu^2)$ – which is analogous to the Limber approximation – equation (B5) can be integrated analytically yielding

$$\ell^2 C_{ij} \approx \frac{k_\perp^2}{\Delta \chi} P(k_\perp) \left( 1 + \frac{2\beta_\mu \mu^2}{k_\perp \Delta \chi} \right), \quad (B8)$$

with the off-diagonals being zero. Thus, the RSD correction falls off slowly as $(k_\perp \Delta \chi)^{-1}$ in the case of top-hat $W$. A curiosity is that if we had approximated $\mu$ as $k_\perp/k_\parallel$, the integral would have diverged. Thus, in the case of a top-hat $W$, the RSD term arises from modes with $\mu \sim 1$.

However, smoother $W(\chi)$ result in RSDs having a weaker scaling in the Limber regime. Consider the case in which $W(\chi)$ is a Gaussian with standard deviation $\sigma$. The analogous equation to equation (B5) for this case is

$$\ell^2 C_{ij} \approx \frac{k_\perp^2}{\Delta \chi} \int \frac{dk_\perp}{2\pi} e^{i\mathbf{k} \cdot (\chi_1 - \chi_2)} \exp[-k_\perp^2 \sigma^2] P(k_\perp, k_\parallel). \quad (B9)$$

For large $\sigma$ the integral is dominated by small $k_\perp$, and we can Taylor series expand about $k_\perp = 0$ as above. In this case, the correction due to RSDs enters at order $O(k_\parallel \sigma^{-2})$. The RSD term is similar (merely increasing by a factor of 2) if one of the two window functions were much narrower than $\sigma$. In addition, exponential or triangle window functions also have RSDs entering at $O(k_\parallel \sigma^{-2})^2\!2!$

It is important for our calculations if the RSDs in Limber – an approximation that we showed holds excellently at angles that contribute to the estimator – contribute at $O(k_\parallel \sigma^{-2})$ rather than $O(k_\parallel \sigma^{-2})$, where $\sigma$ is the width of our window function. RSDs would be a promising signal to break the $b^{(i)} - N^{(i)}$ degeneracy if the former scaling holds, but are not in the case of the latter. It may appear with the formalism in the text, which uses top-hat $W$, that the $O(k_\parallel \sigma^{-2})$ scaling would apply. However, for the case of interest where the $dN/dz$ is a smooth function that is not known, we posit that one is always in the regime where the RSD term falls off as $O(k_\parallel \sigma^{-2})$. Basis functions can always be chosen that have smooth $W(\chi)$ and where the RSD terms contribute at $O(k_\parallel \sigma^{-2})$. That they contribute at $O(k_\parallel \sigma^{-2})$ for top-hat windows is a pathological result of our basis choice that implicitly assumes that the distribution of $dN/dz$ is a histogram with sharp breaks between redshift steps.

\footnote{This result that RSDs depend on the smoothness of $W(\chi)$ is analogous to the finding in Nock, Percival & Ross (2010). There, the impact of RSDs on the correlation function measured in a top-hat projection over $\sim 100$ Mpc was shown to be much more significant than when the effective window was smoothed with a pair-averaging scheme.}
APPENDIX C: MAGNIFICATION BIAS

The spatial density of observed galaxies is modulated by an additional factor that we have ignored so far of \((1 + \delta_i)\) owing to lensing magnification (Turner, Ostriker \& Gott 1984; Fugmann 1988; Narayan 1989; Hui, Gaztañaga \& Loverde 2007, 2008). In the weak lensing regime,

\[
\delta_i(\mathbf{n}, z_i) \equiv 2 \left( -a_i^{(s)} - 1 \right) \int_0^\infty d\chi \frac{\chi_i - \chi}{\chi_i} \nabla^2 \Phi(\chi, \mathbf{n}),
\]

where \(\nabla^2\) is the comoving Laplacian in the plane perpendicular to the radial direction and \(a_i^{(s)}\) is the power-law slope of the cumulative number of sources at the survey flux threshold and redshift \(z_i\). (Note that \(a_i^{(s)}\) is defined to be a negative number as long as the cumulative number decreases with increasing flux.) Thus, magnification generates additional correlations such that

\[
C_{ij} \rightarrow C_{ij} + C_{ij}^{\mu},
\]

where \(C_{ij}^{\mu}\) is the cross-correlation function between the galaxy overdensity field in redshift slice \(i\) and \(\delta_j\), and we are dropping the smaller \(C_{ij}^{\mu}\) term in the Limber regime, the expression for the new terms in equation (C2) is (Bartelmann \& Schneider 2001, their equation 7.9)

\[
C_{ij}^{\mu} = - \left( a_i^{(s)} + 1 \right) \frac{3H^2_{\Omega_0}}{c^2} \int d\chi \frac{W_i(\chi)Y_i(\chi)D^2(\chi)}{\chi} P\left( \frac{\ell}{\chi} \right),
\]

for \(i \neq j\). Otherwise, \(C_{ij}^{\mu} = 0\) (we ignore the contribution of magnification to the \(i = j\) elements), and we denote the source population in question by \(x\) and lens by \(y\) as it could be either the photometric or spectroscopic sample. Here,

\[
Y_i(\chi) = \int_\chi^\infty d\chi W_i(\chi) \left( \frac{\chi_i - \chi}{\chi} \right).
\]

Magnification depends only on the bias of the lens and not the source and so can break the degeneracy between bias and number. (This dependence may be opaque in our notation as the \(C_{ij}^{\mu}\) enter \(\mathbf{A}\) multiplied by factors of the bias.)

Noting that \(c^2/(3H^2_{\Omega_0}) = 2 \times 10^{7} \text{ Mpc}^2\), a back-of-the-envelope estimate for \(C_{ij}^{\mu}\) is

\[
C_{ij}^{\mu} \approx - \left( a_i^{(s)} + 1 \right) \frac{1}{b_i^{(s)}} \frac{(1 + z_j) D^2(\chi_j)}{(2 \times 10^7 \text{ Mpc}^2)} \left( \frac{\chi_j}{\chi_i} - \frac{1}{\chi_j} \right)
\]

when \(i \neq j\), and we have approximated \(W_i\) and \(W_j\) as sharply peaked around their respective redshifts. This is similar to the \(C_{ij}\) term without lensing (equation 21), differing most importantly by the factor \(\chi_i (1 + z_j) \chi_j / 2 \times 10^7 \text{ Mpc}^2\). This factor is \(O(10^{-2})\) for populations at \(z \sim 1\) and \(N_{\text{gal}} \sim 50\), but could be larger for higher redshift populations. Thus, magnification will add off-diagonal terms that are \(O(10^{-2})\) of the diagonal terms in \(\mathbf{C}\) that were zero in much of our treatment in the text. The new magnification terms have a larger impact on the components in \(\mathbf{A}\) involving \(p\), as these terms sum over \(i\) and \(j\) in \(C_{ij}\).

C1 Photo-z calibration with magnification

Here we discuss how magnification could potentially be corrected in the application of photo-z calibration investigated in Section 7.1 (and we use the same notation as introduced there). We consider a simplified problem in which most of the \(pm\) photo-z sample is concentrated at redshift \(z_m\). Then, there is a significant bias if the error on \(T_{ij}^{(pm)} / T_{ij}^{(s)}\) is comparable to \(C_{ij}^{\mu}/C_{ij}\), which we just showed is \(\mathcal{O}(\sqrt{N_{\text{gal}}})^{-1})\) for \(z_i \sim 1\).

The minimum variance estimator with a prior on the \(a_i^{(s)}\) (which enters analogously to the number prior in equation 65) can also be written for this simplified problem. First, the covariance matrix at some \(\ell\) and in the Limber approximation is

\[
D_{00} \approx |T_i^{(pm)}|^2 C_{mm} + w^{(pm)} + \mathcal{M},
\]

\[
D_{01} \approx |T_i^{(pm)}| T_j^{(s)} C_{ij} + w^{(pm)} + w^{(pm)},
\]

\[
D_{11} \approx |T_j^{(s)}|^2 C_{jj} + w^{(pm)},
\]

where \(\mathcal{M}\) encompasses the impact of photometric self-magnification, and we have dropped terms that do not contain \(T_{ij}^{(pm)}\) except the off-diagonal \(T_{ij}^{(pm)}\) terms for which the estimator’s sensitivity to \(T_{ij}^{(s)}\) derives. For the specified \(\mathbf{D}\) and a prior on \(a_i^{(s)}\) with variance \(\sigma_{a_i}\), the minimum variance quadratic estimator is

\[
\tilde{T}_{ij}^{(pm)} = \left[ T_{ij}^{(pm)} \right]_{\text{last}} + \left[ F^{-1} \right]_{11} \sum_{\ell, m} S T_{ij}^{(s)} C_{ij} \left[ p_{\alpha} - D_{01} \right],
\]

where \(S' = D_{00} D_{11} \left( D_{00} D_{11} + D_{01}^2 \right)^{-1} \text{det}[\mathbf{D}]^2\) and \(\alpha^{(s)}\) is set by the prior; we have assumed that \(T_{ij}^{(pm)}\) is well constrained by other cross (and auto) correlations (which is quite likely), and \(\mathbf{F}\) also has a simple analytic representation. This estimator is quite analogous to our previous estimator.

It is instructive to look at the variance on a measurement of \(T_{ij}^{(pm)}\) in a single mode:

\[
\left[ F^{-1} \right]_{11} = \frac{D_{00} D_{11} + S \left( T_{ij}^{(pm)} T_{ij}^{(s)} C_{ij}^{\mu}/(\alpha + 1)^2 \right) \sigma_{a_i}^2}{(S T_{ij}^{(s)} C_{ij})^2}.
\]

This equation shows that error on the magnification bias times \(S'\) (the latter term in the numerator) has to be comparable to the autopower terms (the former term) in order to change our previously quoted errors in Section 7.1. It also suggests that it may be desirable to down weight large-angle modes where \(S'\) is largest (that have the smallest noise) and, hence, where the fog from lensing is most disruptive.

APPENDIX D: RECURRENCE RELATIONS FOR (AND THE EVALUATION OF INTEGRALS OVER) SPHERICAL BESSEL FUNCTIONS

Our most general expressions for the auto- and cross-power spectra, equations (13) and (14), involved integrals over spherical Bessel functions. Numerical methods for evaluating spherical Bessel functions and integrating over them are well advanced, but do not seem to be widely known. This appendix gives the details of the algorithms used in this study. Further details can be
found in Miller (1952), Corbató & Uretsky (1959), Gillman & Fiebig (1988) and Poularikas (2000) or at http://www.utdallas.edu/cantrell/ee6481/lectures/bessres1.pdf.

First we address the evaluation of the \( j_i \). For small values of the argument, we use a series expansion of \( j_i(x) \). For larger values, we evaluate the \( j_i \) using a downwardly stable recurrence relation for \( r_\ell = j_i(j_{\ell-1}) \). Specifically we first initialize \( r_\ell \) by setting \( j_0(x) = 0 \) for \( L \) much larger than any \( \ell \) of interest (and \( x \)). Then the relation

\[
   r_{\ell-1} = \frac{1}{(2\ell - 1)x - r_\ell}
\]

is downwardly stable and can be used to find \( r_\ell \) for \( 0 < \ell < L \). The \( j_i \) can then be evaluated by moving up the hierarchy after initializing \( j_0(x) = \sin(x)/x \).

Equations (13) and (14) are difficult integrals to evaluate owing to the oscillatory nature of the \( j_i \). We experimented with using the scheme suggested in Lucas (1995) of decomposing the product of \( j_i \) into a sum of functions that each have a single oscillatory period at large arguments and then using the transformations discussed therein on a series where the \( n \)th member is our \( k \)-integral evaluated from 0 out to the \( n \)th zero of \( \pi \). This operation removes oscillatory behaviour in this slowly converging series so that it converges more quickly to the \( n \to \infty \) limit, and the integral converges for \( n \sim 10 \) (Lucas 1995). Experiments with some of the integral terms indicated that the Lucas (1995) method was much faster than a brute-force integration, but we were able to find a simpler implementation which was sufficiently fast and accurate. In particular, we ended up evaluating these integrals by brute force, integrating typically out to the 1000th zero of the \( \alpha(k, z) \) (which were pre-computed and stored in a table). A slight improvement in the convergence of the integral was obtained by applying a Gaussian damping to the integrand – based on the fact that \( k_\ell \gg \ell/\chi \) should not contribute much to the integral. The details of this damping did not affect our results.

**APPENDIX E: THE POWER-LAW CASE**

The main body of this paper used power-law approximations to the power spectrum and correlation function to understand the mechanics of the Schur–Limber estimator. To aid this discussion, here we work through expressions for the angular power spectrum and correlation function (and their relation) under these approximations.

Recall that within the Limber approximation (Section 3.1)

\[
   C_\ell = \int dx \frac{P(k) W^2(\chi)}{\chi^2},
\]

(E1)

where \( W(\chi) \) is the projection kernel that defines the 2D (projected) overdensity in terms of the 3D, and it integrates to unity against \( d\chi \).

We shall assume that \( W(\chi) \) is peaked at \( \chi_0 \) and of width \( \Delta \chi \) such that \( k\chi_0 \gg k\Delta \chi \gg 1 \) for scales, \( k \), which contribute significantly.

Assuming a power-law power spectrum of the form \( \Delta^2(k) = k^3 P(k)/2\pi^2 = (k/k_\star)^{3+n} \), with \( -2 < n < -1 \), the real-space 3D correlation function is

\[
   \xi(r) = \left( \frac{r_0}{r} \right)^\gamma = \int \frac{dk}{k} \Delta^2(k) j_0(kr) = B_n (kr)^{-3-n},
\]

(E2)

where \( B_n = -\sin(n\pi/2) \Gamma(2+n, 0) \), which, respectively, equals 1.25 and 1 for \( n = -3/2 \) and \(-1 \) (\( B_n \) diverges as \( n \to -3 \)). It follows from equation (E2) that \( \gamma = n+3 \) and \( r_0 = B_n^{1/\gamma}/k_\star \).

In the Limber approximation,

\[
   C_\ell = \frac{2\pi^2}{k_\star^3 \psi} \left( \frac{\ell}{k \chi_0} \right)^n,
\]

(E3)

where \( \psi = \chi_0^2 \Delta \chi \) is the volume per steradian. Using analogous relations to equation (E2), the 2D or projected correlation function is

\[
   w(\theta) = \left( \frac{\theta_\star}{\theta} \right)^{n+2} = \frac{\pi A_n}{k_\star^3 \psi} (k \chi_0)^n \theta^{-n-2},
\]

(E4)

where \( A_n = 2^{n+1} \Gamma(1+n/2)/(n/2+1) \approx 2.1 \) and 1 for \( n = -3/2 \) and \(-1 \) (\( A_n \) diverges as \( n \to -2^+ \)).

Particularly simple expressions hold in the case \( n = -1 \) for which \( A_n = B_n = 1 \), so \( \Delta^2 = (k/k_\star)^2 \).

\[
   \xi(r) = \left( \frac{r_0}{r} \right)^2 \quad \text{where} \quad r_0 = k_\star^{-1},
\]

(E5)

and

\[
   w(\theta) = \left( \frac{\theta_\star}{\theta} \right)^2 = \pi \left( \frac{r_0}{\chi_0} \right)^2 \left( \frac{\chi_0}{\Delta \chi} \right) \theta^{-1}.
\]

(E6)

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