HAUSDORFF COMPACTIFICATIONS

MATT INSALL, PETER A. LOEB, AND MALGORZATA ANETA MARCINIAK

Abstract. Previously, the authors used the insights of Robinson’s nonstandard analysis as a powerful tool to extend and simplify the construction of some compactifications of regular spaces. They now show that any Hausdorff compactification is obtainable with their method.

1. Introduction

In [3], we extended to more general compactifications the work on topological ends of Insall and Marciniak in [2]. Fix an appropriately saturated nonstandard extension of a regular, noncompact topological space \((X, T)\). In [3], we showed that points needed to form a compactification can be formed from equivalence classes of points not in the monad of any standard point. Any equivalence relation on such points works, but not every compactification of \(X\) can be obtained this way. We show here that any Hausdorff compactification of \(X\) can be obtained using a natural equivalence relation. We conclude with brief discussions of an application to a moduli space of triangles and to the Martin boundary in potential theory and probability.

2. General Compactifications

First we review the construction in [3]. Also for background, see [8], [9], and [7]. Let \((X, T)\) be a regular, noncompact topological space.

2.1. Definition. By a compactification of \((X, T)\), we mean a compact space \((Y, T_Y)\) such that \(X\) is a dense subset of \((Y, T_Y)\). We require here that the identity mapping of \(X\) into \(Y\) is a homeomorphism from \((X, T)\) to \(X\) with the relative \(T_Y\)-topology.

2.2. Example. A simple example where the latter requirement fails is given by the set \(X = \{1/n : n \in \mathbb{N}\} \cup \{0\}\) with the discrete topology. We set \(Y = X\), but now the singleton \(\{0\}\) is no longer an open set. Its neighborhoods consist of all intervals in \(X\) from 0 to \(1/n\), \(n \in \mathbb{N}\). Clearly, the set \(X\) is a dense and continuous image in \(Y\), but \(Y\) is not a compactification of \(X\) in the sense of Definition 2.1.
We now fix a \( \kappa \)-saturated nonstandard extension of \( (X, T) \), where \( \kappa \) is greater than the cardinality of the topology \( T \) on \( X \).

### 2.3. Definition

We call a point \( x \in {}^*X \) **remote** if \( x \) is not near-standard, i.e., not in the monad of any standard point of \( X \). Given an equivalence relation on the set of remote points of \( {}^*X \), we write \( x \sim y \) if \( x \) and \( y \) are remote and equivalent.

Let \( Y \) be the point set consisting of points of \( X \), called **s-points**, together with all equivalence classes of remote points, where each such equivalence class is treated as a single point. We call each such point of \( Y \) an **r-point**. We supply \( X \) with the S-topology, that is the topology generated by the nonstandard extensions of standard open subsets of \( X \). Let \( \varphi \) be the mapping from \( {}^*X \) onto \( Y \) that sends near-standard points to their standard parts and remote points to their respective r-points. The **neighborhood filter base** \( B(y) \) at an r-point \( y \) in \( Y \) consists of all sets of the form \( \varphi({}^*U) \), where \( U \) is a standard open subset of \( X \) with nonstandard extension containing the entire equivalence class corresponding to \( y \) (whence \( U \neq \emptyset \)). The **neighborhood filter base** \( B(x) \) at an s-point \( x \) in \( Y \) consists of all sets of the form \( \varphi({}^*U) \), where \( U \) is a standard open subset of \( X \) with \( x \in U \). For each point \( p \in Y \), \( B(p) \) is in fact a filter base (see [3].) As usual, a set \( O \subseteq Y \) is called “open” if for each point \( p \in O \), there is an element \( \varphi({}^*U) \in B(p) \) with \( \varphi({}^*U) \subseteq O \). The collection of open sets forms a topology on \( Y \). Note that we have not made any claim about the interior with respect to \( Y \) of any member of any neighborhood filter base. We let \( T_Y \) denote the topology on \( Y \), i.e., the collection of open sets, generated by the neighborhood filter bases. The following result is established in [3] using the fact (see [10] and [11]) that \( {}^*X \) with the S-topology is compact.

### 2.4. Theorem

The map \( \varphi \) is a continuous surjection from \( {}^*X \) onto \( Y \), whence, \( Y \) is compact. Moreover, the point set \( X \) is dense in \( Y \) supplied with the \( T_Y \)-topology, and \( T \) is stronger than, or equal to, the relative \( T_Y \)-topology on \( X \).

Recall that all sets in \( T \) are themselves in \( T_Y \) if \( T \) is a locally compact topology on \( X \). The following is an example where the \( T \)-topology on \( X \) is strictly stronger than the relative \( T_Y \)-topology.

### 2.5. Example

Let \( X \) be the rational numbers in the interval \( [0, 1] \). A point \( x \) in \( {}^*X \) is remote if and only if \( x \) is in the monad of an irrational point in \( [0, 1] \). Put all remote points in the same equivalence class, so there is only one r-point, denoted by \( \alpha \), in \( Y \). If \( U \) is a nonempty open set in \( X \), then \( {}^*U \) contains remote points. In order for there to be a set \( V \) in \( T_Y \) for which the restriction to \( X \) is \( U \), it is necessary that \( V \) contains an element of the neighborhood filter base of \( \alpha \). That is, \( V \) must contain a set \( \varphi[{}^*W] \) where \( W \) is in \( T \) and \( {}^*W \) contains every remote point in \( {}^*X \). Such a subset \( W \) of \( X \) must contain every point of \( X \) except perhaps those in a compact subset of \( X \). Therefore, the point set \( Y \) consists of the points of \( X \) together with...
the point \( \alpha \), and the nonempty elements of \( T_Y \) are the complements in \( Y \) of compact subsets of \( X \).

In the next section, we show that any Hausdorff compactification of a Hausdorff space \((X, T)\) can be produced with an equivalence relation on the remote points of \( *X \). First, however, we give an example showing that this may not be true if the compactification is not Hausdorff. It is also an example of a Hausdorff space \((X, T)\) that is not locally compact, but still forms an open set in a compactification.

2.6. Example. We modify Example 2 on Page 630 from [4]. Let \( X \) be the subset of the plane given by
\[
X = \{(x, 0) : x \text{ rational}, 0 \leq x \leq 1\}.
\]
To form the compactification \( Y \), we adjoin to \( X \) the subset in the plane given by
\[
\Delta = \{(x, 1) : 0 \leq x \leq 1\}.
\]
A typical neighborhood of a point \((x_0, 0)\) in \( X \) is given by the relative plane topology. That is, it is given by a constant \( \varepsilon > 0 \) and has the form
\[
\{(x, 0) \in X : |x - x_0| < \varepsilon\}.
\]
A typical neighborhood of a point \((x_1, 1)\) in \( \Delta \) is given by a constant \( \delta > 0 \) and has the form
\[
\{(x, 1) \in \Delta : |x - x_1| < \delta\} \cup \{(x, 0) \in X : |x - x_1| < \delta\}.
\]
Clearly, \( X \) is dense in \( Y = X \cup \Delta \), and \( Y \) is not Hausdorff. Moreover, \( Y \) is compact, since any net has a cluster point in \( \Delta \). The compactification \( Y \) cannot be obtained using an equivalence relation on the remote points of \( *X \) when the neighborhoods are formed as in [3]. To see this, suppose \((r, 0)\) is a remote point in \( *X \) that is in the equivalence class forming the point \((0, 1)\) in \( \Delta \). Then \( r \) is in the monad of a standard irrational \( s \) in \([0, 1]\). If the construction from [3] works here, then there must be a standard open set \( U \) in \( X \) such that \((r, 0)\) is in \( *U \), and every near-standard point in \( *U \) has \( x \)-coordinate less than \( s/2 \). But this is impossible.

3. Hausdorff Compactifications

Assume that \((X, T)\) is a Hausdorff space, and let \((Z, T_Z)\) be a Hausdorff compactification of \((X, T)\). That is, \( X \) is dense subset of the compact Hausdorff space \( Z \), and the mapping from \( X \) to \( Z \) as a subset of \( Z \) is a homeomorphism.

3.1. Definition. Remote points \( p \) and \( q \) in \( *X \) are equivalent, i.e., \( p \sim q \), if \( p \) and \( q \) are in the monad of a point \( z \in Z \setminus X \). Let \((Y, T_Y)\) be the compact space produced with this equivalence relation. Denote by \( F \) the mapping from \( Z \) to \( Y \) that is the identity mapping from \( X \) as a subset of \( Z \) to \( X \) as a subset of \( Y \), and maps each point \( z \in Z \setminus X \) to the \( r \)-point in \( Y \) formed from the equivalence class formed by those points in \( *X \) that are in the monad of \( z \).
3.2. **Theorem.** The map $F$ is a bijection, and indeed, a homeomorphism from $(Z, T_Z)$ onto $(Y, T_Y)$. It follows that any Hausdorff compactification of $X$ can be obtained from an equivalence relation on the remote points of $^*X$.

**Proof.** It is clear that $F$ is bijective. As is well known, it is sufficient to show that $F^{-1}$ is continuous. For then, the inverse image using $F$ of an open set in $Y$ will be the open complement in $Z$ of the compact image of a closed, therefore compact set in $Y$. Fix $z \in Z$, and an open neighborhood $U$ of $z$. By the regularity of $(Z, T_Z)$, there is an open neighborhood $V$ of $z$ for which the closure is contained in $U$. Let $O = V \cap X$. Since $(Z, T_Z)$ is a compactification of $(X, T)$, $O \in T$. Moreover, $^*O$ contains $F(z)$, and $\varphi (^*O)$ is a closed neighborhood of $F(z)$ that maps, using $F^{-1}$, into $U$. It follows that $F^{-1}$ is continuous. \qed

3.3. **Example.** As an application to moduli spaces, we compactify the space of similar triangles in $\mathbb{R}^2$ discussed by Madeline Brandt in [1]. Here, two triangles are equivalent with respect to this space of triangles if they are similar. Each such equivalence class is represented by the member with a leg of shortest length on the interval in the $y$-axis from vertex $(0, 0)$ to vertex $(0, 1)$, and with the third vertex in the parameter space

$$S = \left\{(x, y) \in \mathbb{R}^2 : x > 0, \ y \geq 1/2, \ x^2 + (y - 1)^2 \geq 1 \right\}.$$ 

Thinking of $S$ as a subset of the complex plane, the remote points are the points in the nonstandard extension of $S$ with argument in the monad of $\pi/2$ together with points with unlimited modulus and complex argument in the interval $^*(0, \pi/2)$. To form a compactification, we can call equivalent all remote points with argument in the monad of $\pi/2$, and represent that equivalence class with one point at $2i$. On the other hand, we can break up that equivalence class so there is just one point for members with unlimited modulus, and represent that set with the nonnegative $y$-axis. We can then call two remote points with infinitesimally close, limited moduli and argument in the monad of $\pi/2$ equivalent. That equivalence class is then represented by the line from the origin to the unique standard part of the members. Each remaining remote point is the third vertex of a nonstandard triangle. That triangle produces a “standard part” in $S$ consisting of the standard parts of those points of limited modulus in the triangle. We call two such remote points equivalent if their arguments have the same standard part $\theta < \pi/2$. The corresponding triangles have the same standard part in $S$. The corresponding equivalence class can be represented by that standard part, which is the figure formed by the line from $(0, 0)$ to $(0, 1)$ together with the infinite line segment with endpoint $(0, 0)$ forming an angle $\theta$ with the $x$-axis and the parallel infinite line segment with endpoint $(0, 1)$.

3.4. **Example.** The Martin compactification (see [5]) is an important construction in potential theory; it can be seen as an application of Theorem 3.14 in [3]. Martin’s boundary is also of great importance in probability.
theory. Motivated by our Theorem 3.2, the second author obtained a probability formulated equivalence relation in [6] that yields the Martin compactification for a number of illuminating examples. His approach, “looking inside” a domain, only makes sense when one can speak of points that are neither points of an existing boundary nor points in a compact subset of the domain.

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Department of Mathematics and Statistics, Missouri University of Science and Technology, 400 W. 12th St., Rolla, MO 65409-0020
E-mail address: insall@mst.edu

Department of Mathematics, University of Illinois, 1409 West Green St., Urbana, Ill. 61801, USA
E-mail address: PeterA3@AOL.com

Department of Mathematics, Engineering and Computer Science, LaGuardia Community College of the City University of New York, 31-10 Thomson Avenue, Long Island City, NY
E-mail address: goga@mimuw.edu.pl