MONOID EMBEDDINGS OF SYMMETRIC VARIETIES

BY

MAHIR BILEN CAN (New Orleans, LA), ROGER HOWE (New Haven, CT) and LEX RENNER (London, ON)

Abstract. We determine when an antiinvolution on an adjoint semisimple linear algebraic group extends to an antiinvolution on a J-irreducible monoid. Using this information, we study a special class of compactifications of symmetric varieties. Extending the work of Springer on involutions, we describe the parametrizing sets of Borel orbits in these special embeddings.

1. Introduction. Let $G$ be a complex reductive algebraic group and let $\theta : G \to G$ be an algebraic group automorphism such that $\theta^2 = \text{id}$. The fixed subgroup $H := \{ g \in G : \theta(g) = g \}$ is called the symmetric subgroup associated with $\theta$, and the corresponding quotient $G/H$ is called a symmetric variety. Let $B$ be a Borel subgroup of $G$. From the works of Richardson and Springer [16] and Helminck [7, 8], we know that there is a close relationship between the set of $B$-orbits in $G/H$ and the set of involutions in the Weyl group of $G$. In particular, we know that the number of $B$-orbits in $G/H$ is finite [10, 17].

A purpose of our paper is to show that the sets of $B$-orbits in certain “monoid embeddings” of the symmetric varieties are closely related to the sets of involutions in certain finite inverse semigroups. We proceed to explain what we mean by a monoid embedding.

A reductive monoid is a linear algebraic monoid whose group of units is a reductive algebraic group. Let $M$ be a reductive monoid and let $\theta_{\text{an}}$ be an antiinvolution, that is, $\theta_{\text{an}} : M \to M$ is an automorphism of $M$ such that $\theta_{\text{an}}^2 = \text{id}$ and for every $m_1, m_2 \in M$ we have $\theta_{\text{an}}(m_1m_2) = \theta_{\text{an}}(m_2)\theta_{\text{an}}(m_1)$. If $G$ denotes the group of units of $M$, then we denote the restriction of $\theta_{\text{an}}$ to $G$ by the same notation. Then the morphism $\theta : G \to G$ defined by $\theta(g) := \theta_{\text{an}}(g)^{-1} \ (g \in G)$ is an involutory algebraic group automorphism. As before, let us denote by $H$ the fixed subgroup of $\theta$. The morphic map

$$\tau : M \to M, \quad m \mapsto m\theta_{\text{an}}(m),$$

2010 Mathematics Subject Classification: 14M17, 20G05, 20M32.
Key words and phrases: Borel orbits, reductive monoids, symmetric varieties.
Received 18 June 2018; revised 31 July 2018.
Published online 15 February 2019.

DOI: 10.4064/cm7644-7-2018 [1] © Instytut Matematyczny PAN, 2019
restricts to give a morphism on $G$ and we denote this restriction by $\tau$ as well. The image of $\tau$ on $G$ is denoted by $P$. Note that $P$ is a closed subvariety of $G$, and furthermore it is isomorphic to $G/H$ as a variety (see [15, Lemma 2.4]).

The \textit{twisted (conjugation) action} of $G$ on $M$, denoted by $\ast$, is defined by

$$g \ast m = gm\theta_{\text{an}}(g) = gm\theta(g)^{-1} \quad (g \in G, m \in M).$$

It is easy to check that $P$ is stable under the twisted action. In fact, $P = G \ast 1_G$. It is not hard to see also that the set of $B\ast$-orbits in $P$ is in bijection with the set of $B$-orbits in $G/H$. We call the Zariski closure of $P$ in $M$ the \textit{monoid embedding} of $P$. Since $P$ is isomorphic to $G/H$ and since $\overline{P}$ is $G\ast$-stable, we view the monoid embedding of $P$ as an equivariant embedding of $G/H$ in the reductive monoid $M$.

An important auxiliary variety for our purposes is the fixed subvariety $Q$ defined by

$$Q := \{g \in G : \theta_{\text{an}}(g) = g\}.$$ 

It is easy to check that $Q$ is closed in $G$, $P \subseteq Q$, and that $Q$ is $G\ast$-stable. We know from Springer’s work [17] that if $B$ is a $\theta$-stable Borel subgroup of $G$, then $Q$ has only finitely many $B\ast$-orbits. As in the case of $P$, the parametrizing set of $B\ast$-orbits in $Q$ is closely related to the set of involutions in the Weyl group $W := N_G(T)/T$, where $T$ is a maximal torus contained in $B$ and $N_G(T)$ is the normalizer of $T$ in $G$. (Here, by an involution in $W$ we mean an element $\sigma \in W$ such that $\sigma^2 = \text{id}$.)

Let $M_Q$ denote the following (closed) subvariety of $M$:

$$M_Q := \{m \in M : \theta_{\text{an}}(m) = m\}.$$ 

Clearly, $\overline{P} \subseteq \overline{Q} \subseteq M_Q$ and $G$ acts on the sets $P, Q, \overline{P}, \overline{Q}$, and on $M_Q$ by the same formula $g \ast m = gm\theta_{\text{an}}(g)$. Our first main result is about the parametrizing sets of $B\ast$-orbits in the embeddings of $P$ and $Q$ in $M$.

\textbf{Theorem 1.1.} Let $M$ be a normal reductive monoid with unit group $G$, $\theta_{\text{an}}$ be an antiinvolution on $M$, and let $\theta$ denote the involutive automorphism on $G$ that is defined by $\theta(g) = \theta_{\text{an}}(g)^{-1}$ for $g \in G$. We fix a pair $(T, B)$ of a $\theta$-stable maximal torus and a Borel subgroup in $G$, and we let $\overline{N}$ denote the Zariski closure in $M$ of the normalizer of $T$ in $G$. In this case, the following sets are finite and they are in bijection with each other:

1. $B\ast$-orbits in $\overline{Q}$ (respectively, $B\ast$-orbits in $\overline{P}$),
2. $T \times H$-orbits in $\tau^{-1}(\overline{N} \cap \overline{Q})$ (respectively, $T \times H$-orbits in $\tau^{-1}(\overline{N} \cap \overline{P})$).

The \textit{Renner monoid} of $M$, defined by $R = \overline{N}/T$, is a generalization of the Weyl group of $G$ (see [13]). It is a finite inverse semigroup and $W$ is its group of invertible elements. As a consequence of Theorem 1.1, we will show that a certain subset of $R$ can be used for studying $B\ast$-orbits in $\overline{P}$. In
some special cases this subset of $\mathbb{R}$ gives a parametrization of the full set of $B^*$-orbits; see the examples in Section 4.

With hindsight, our first main result raises the question of finding antiinvolutions on reductive monoids. To answer this question, in our second main result, we focus on a particular subclass of reductive monoids. A semisimple monoid is a reductive monoid which is normal, has a one-dimensional center and a zero element. Interesting examples of such monoids include the cones over certain representations of semisimple groups.

Let $G_0$ be a semisimple algebraic group of adjoint type and let $\rho : G \to \text{GL}(V)$ be a finite-dimensional irreducible rational representation of $G_0$. Let $Z_V$ denote the cone over $\rho(G)$ in $\text{End}(V)$. Then $Z_V$ has the structure of a reductive monoid. Let us mention that $Z_V$ is known to be normal as an algebraic variety if the representation $(\rho, V)$ is a minuscule representation in the sense of [3, Theorem 3.1]. (We will review De Concini’s theorem in the preliminaries section.) In the following result we denote by $G$ the reductive group of units in $Z_V$.

**Theorem 1.2.** Let $G_0$ be a complex semisimple algebraic group of adjoint type and let $\theta_0 \in \text{Aut}(G_0)$ be an involutory algebraic group automorphism. If $(\rho, V)$ is a minuscule representation of $G_0$ with the highest weight $\omega$ such that $\theta_0^*\omega = -\omega$ and $Z_V$ is normal, then there exists a unique morphism $\theta_{\text{an}} : Z_V \to Z_V$ such that

1. $\theta_{\text{an}}(xy) = \theta_{\text{an}}(y)\theta_{\text{an}}(x)$ for all $x, y \in Z_V$;
2. $\theta_{\text{an}}^2$ is the identity map on $Z_V$;
3. $\theta_{\text{an}}(g) = \theta(g)^{-1}$ for all $g \in G$, where $\theta$ is the unique extension of $\theta_0$ to $G$.

We conclude our introduction by giving a brief overview of our article. In Section 2, we set our notation and review some facts from the theory of linear algebraic monoids and the representation theory of reductive algebraic groups. In Section 3, we prove our second main result, Theorem 1.2. In Section 4, we characterize the parametrizing sets of Borel orbits in $P$. In particular we prove our Theorem 1.1 in Section 4. Finally, we close our paper with some remarks in Section 5.

**2. Preliminaries.** Unless otherwise mentioned, all reductive groups are assumed to be connected and all semigroups are defined over $\mathbb{C}$. The representations we consider here are all rational and finite-dimensional.

The general linear group of invertible $n \times n$ matrices is denoted by $\text{GL}_n$, and the monoid of $n \times n$ matrices by $\text{Mat}_n$. The Lie algebra of a linear algebraic group $G$ is denoted by $\text{Lie}(G)$.

Let $G$ be a reductive algebraic group, $T$ a maximal torus in $G$, and $B$ a Borel subgroup containing $T$. We use $X(T)$ to denote the character group
of $T$, and $E$ to denote the real vector space $X(T) \otimes \mathbb{Z} \mathbb{R}$. In addition, we fix the following notation:

- $\Phi \subset E$: the set of weights of the adjoint representation;
- $\Delta \subset \Phi$: the set of simple roots determined by $(B,T)$;
- $\Phi^+ \subset \Phi$: the set of positive roots determined by $\Delta$;
- $\Lambda_r \subset X(T)$: the root lattice generated by $\Delta$.

If $\alpha$ is a root from $\Phi$, then the associated coroot, $2\alpha/(\alpha,\alpha)$, is denoted by $\check{\alpha}$. Suppose that $\alpha_1, \ldots, \alpha_n$ is the list of simple roots from $\Delta$. The set of fundamental weights, $\{\omega_1, \ldots, \omega_n\}$, is the dual of the coroot basis $\{\check{\alpha}_1, \ldots, \check{\alpha}_n\}$ for the dual vector space $\text{Lie}(T)^*$.

Irreducible representations of $G$ are parametrized by the semigroup of dominant weights (with respect to $T$). A dominant weight $\lambda$ is called minuscule if $\langle \lambda, \check{\alpha} \rangle \leq 1$ for all positive coroots $\check{\alpha}$.

The dominance partial order on weights is defined by $\mu \preceq \lambda$ if and only if $\lambda - \mu$ is a positive linear combination of positive roots.

If $\lambda$ is a dominant weight, then we denote by $\Sigma(\lambda)$ the set of dominant weights $\mu$ such that $\mu \preceq \lambda$. The set $\Sigma(\lambda)$ is finite and it is called the saturation of $\lambda$.

**2.1. Reductive monoids.** The purpose of this section is to introduce the notion of a reductive monoid. For details, see [14][11]. For a more recent exposition of the basic ideas behind algebraic monoids we recommend Brion’s article [11], and for the combinatorics of Renner monoids we recommend [12].

Let $M$ be a linear algebraic monoid with the group of invertible elements $G$. The set of idempotents in $M$ is denoted by $E(M)$. If $G$ is a reductive group and $M$ is an irreducible algebraic variety, then $M$ is called a reductive monoid. Note that there is no normality assumption on $M$.

Let $T$ be a maximal torus in $G$ and let $B$ be a Borel subgroup containing $T$. Clearly, $\overline{T}$ is a reductive and commutative submonoid of $M$. As before, we denote by $R$ (resp. by $W$), the Renner monoid $N_G(T)/T$ (resp. the Weyl group $N_G(T)/T$) of $M$ (resp. of $G$).

The “generalized” Bruhat–Chevalley order on $R$ is defined by

\begin{equation}
\sigma \leq \tau \quad \text{if and only if} \quad B\sigma B \subseteq B\tau B,
\end{equation}

where $\tau$ and $\sigma$ are from $R$ and the bar stands for the Zariski closure in $M$.

There is a canonical partial order $\leq$ on the set $E(\overline{T})$ of idempotents of $\overline{T}$ defined by

\begin{equation}
e \leq f \quad \text{if and only if} \quad ef = e = fe.
\end{equation}

Notice that $E(\overline{T})$ is invariant under the conjugation action of the Weyl group $W$. A subset $\Lambda \subseteq E(\overline{T})$ is called a cross-section lattice (or a Putcha
lattice) if $A$ is a set of representatives for the $W$-orbits on $E(T)$ and the bijection $A \to G \backslash M/G$ defined by $e \mapsto GeG$ is order preserving. There is a close relationship between cross-section lattices and Borel subgroups. The right centralizer of $A$ in $G$ is the subgroup

$$C_G^r(A) = \{g \in G : ge = ege \text{ for all } e \in A\}.$$  

If $M$ has a zero, then for all Borel subgroups of $G$ containing $T$ the set $A(B) = \{e \in E(T) : Be = eBe\}$ is a cross-section lattice with $B = C_G^r(A)$, and for any cross-section lattice $A$, the right centralizer $C_G^r(A)$ is a Borel subgroup containing $T$ with $A = A(C_G^r(A))$. See [11, Theorem 9.10].

The decomposition $M = \bigsqcup_{e \in A} GeG$ into $G \times G$ orbits has a finite counterpart: $R = \bigsqcup_{e \in A} WeW$. Moreover, the partial order (4) on $A$ agrees with the order induced from the Bruhat–Chevalley order (3).

$E(T)$ is a relatively complemented lattice, antiisomorphic to a face lattice of a convex polytope. For $T$ contained in a $J$-irreducible monoid, the associated polytope is described explicitly in Section 2.2. Let $A$ be a cross-section lattice in $E(T)$. The Weyl group of $T$ (relative to $B = C_G^r(A)$) acts on $E(T)$, and furthermore

$$E(T) = \bigsqcup_{w \in W} wAw^{-1}.$$  

Let $S$ be a semigroup and let $M = S^1$ be the monoid obtained from $S$ by adding a unit element if it is not already present. Let $a, b \in M$. The following are four of the five equivalence relations which are collectively known as Green’s relations, of utmost importance for semigroup theory:

- $a \mathcal{L} b$ if $Ma = Mb$;
- $a \mathcal{R} b$ if $aM = bM$;
- $a \mathcal{J} b$ if $MaM = MbM$;
- $a \mathcal{H} b$ if $a \mathcal{L} b$ and $a \mathcal{R} b$.

It turns out that the unit group $G$ of a reductive monoid $M$ is big in the sense that $a \mathcal{L} b$ if $G a = G b$, $a \mathcal{R} b$ if $a G = b G$, and $a \mathcal{J} b$ if $G a G = G b G$ (see [11, Proposition 6.1]). Furthermore, a cross-section lattice is a representative for the set of $\mathcal{J}$-classes in $M$. A reductive monoid $M$ is called $J$-irreducible if $M$ has a unique, nonzero, minimal $G \times G$-orbit.

We continue with the assumption that $M$ is a reductive group with unit group $G$. Among the important submonoids of $M$ are those of the form $e Me$ ($e \in E(T)$). Let $C_G(e)$ denote the centralizer of $e$ in $G$. If $e$ is from the cross-section lattice $A$, then $e C_G(e)$ is the unit group of $e Me$. We will also need the following fact.

**Lemma 2.1.** Let $e \in E(T)$ be an idempotent and let $H$ denote its $\mathcal{H}$-class, that is, $H = e C_G(e)$. Let $B$ be a Borel subgroup of $G$ containing $T$, 

hence $e \in E(\overline{B})$. In this case, $C_B(e)$ and $eB_0 = eC_B(e)$, respectively, are Borel subgroups of $C_G(e)$ and $H$.

For the proofs of the facts stated in the paragraph before the lemma as well as for the proof of the lemma, see [11, Corollary 7.2].

2.2. $J$-irreducible monoids. Let $G_0$ denote a semisimple linear algebraic group of adjoint type, and $T_0$ be a maximal torus in $G_0$. If $(\rho_0, V)$ is a representation of $G_0$, then the group $\mathbb{C}^* \cdot \rho_0(G_0)$, which we denote by $G$, is reductive. If $\rho_0$ is faithful, then up to isomorphism, $T_0$ and $\rho_0(T_0)$ differ by a finite set of central elements. In this case, when there is no danger of confusion, we will denote the image $\rho_0(T_0)$ by $T_0$.

Let $T \subseteq G$ be a maximal torus containing $T_0$, and let $T \subset \text{GL}(V)$ denote an $n$-dimensional maximal torus containing $T$. (Here, $n = \dim V$.) Accordingly, we have a nested sequence of Euclidean spaces:

$$E_0 = X(T_0) \otimes \mathbb{Z} \mathbb{R} \subset E = X(T) \otimes \mathbb{Z} \mathbb{R} \subset E = X(T) \otimes \mathbb{Z} \mathbb{R}.$$  

Note that $\dim E = \dim E_0 + 1$.

If $\varepsilon_i$ (for $i = 1, \ldots, n$) denotes the standard $i$th coordinate function on $T$, then $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is a basis for $E$, and $E$ is spanned by the restrictions $\varepsilon_i|_T$, $i = 1, \ldots, n$. Let $\chi \in X(T)$ denote the restriction of the character whose $n$th power is the determinant on $\text{GL}(V)$. Stated differently in additive notation, $\chi$ is the restriction to $T$ of the rational character $\frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n)$. We denote $\varepsilon_i|_T$ by $\chi_i$ and set

$$\tilde{\chi}_i := \chi_i - \chi \quad \text{for } i = 1, \ldots, n.$$  

If $K$ is an arbitrary group, then the center of $K$ is customarily denoted by $Z(K)$. In our case, since $Z(G) = \mathbb{C}^* \cdot Z(G_0)$, the character group of $Z(G)$ is generated by one element, which is $\chi$. Thus, $E = \mathbb{R}\chi \oplus E_0$. In fact, $\chi$ vanishes on $T_0$. It follows from these observations that

- $\{\tilde{\chi}_1, \ldots, \tilde{\chi}_{n-1}, \chi\}$ spans $E$;
- if $x \in T$ lies in $T_0 \subset T$, then $\tilde{\chi}_i|_{T_0}(x) = \chi_i(x)$;
- $\{\tilde{\chi}_1, \ldots, \tilde{\chi}_{n-1}\}$ spans $E_0$.

In this paper, we are interested in the $J$-irreducible monoids that come from a faithful representation.

**Definition 2.1.** Let $(\rho_0, V)$ be a faithful, irreducible representation of $G_0$. The $J$-irreducible monoid associated with $(\rho_0, V)$ is the affine variety $\mathbb{C}^* \cdot \rho_0(G_0)$ together with its monoid structure induced from $\text{End}(V)$.

**Remark 2.1.** The definition of a $J$-irreducible monoid which is given at the end of Section 2.1 agrees with Definition 2.1 (see [14, Lemma 7.8]).

The representation $(\rho_0, V)$ of $G_0$ gives a representation of $G$ by

$$zg \cdot v = z\rho_0(g)(v) \in V,$$

(5)
where \( z \in \mathbb{C}^* \), \( g \in G_0 \), and \( v \in V \). Another such simple but useful observation is that, since \( G_0 \cong (G,G) \), the Weyl group \( W \) of \((G_0,T_0)\) is isomorphic to that of the pair \((G,T)\).

**Lemma 2.2** ([12, Proposition 3.5]). Let \((\rho_0,V)\) be an irreducible representation of \(G_0\). If \(\lambda\) is the \(T_0\)-highest weight of \((\rho_0,V)\) and \(P\) denotes the convex hull of \(\{w \cdot (\chi + \lambda) : w \in W\} = \{\chi + w \cdot \lambda : w \in W\} \), then the set of weights of \(T\) with respect to (5) is contained in \(P\).

Next, we briefly review a result of De Concini on the normality of \(J\)-irreducible monoids. Let \(\lambda\) be a dominant weight for \(G_0\) and let \((\rho_0,V)\) denote the corresponding irreducible representation of \(G_0\). We define \((\eta,W_\lambda)\) as the following sum of irreducible representations of \(G_0\):

\[
W_\lambda := \bigoplus_{\mu \in \Sigma(\lambda)} V(\mu).
\]

Here \(V(\mu)\) denotes the irreducible representation of \(G_0\) with highest weight \(\mu\). Finally, we set

\[
Z_\lambda := Z_{V(\lambda)} \quad \text{and} \quad Z_\lambda := Z_{W_\lambda},
\]

where \(Z_{W_\lambda}\) is the cone over \(\eta(W_\lambda)\) in \(\text{End}(W_\lambda)\).

**Theorem 2.1** (De Concini [3, Theorem 3.1]).

1. \(Z_\lambda\) is a normal variety with rational singularities.
2. If \(V\) is a \(G_0\)-module of highest weight \(\lambda\), then \(Z_\lambda\) is the normalization of \(Z_\lambda\), and it is equal to \(Z_\lambda\) if and only if \(W_\lambda\) is a subrepresentation of \(V\). In particular, \(Z_\lambda = Z_\lambda\) if and only if \(\lambda\) is minuscule.

**3. A proof of Theorem 1.2** Let \(G_0\) be a semisimple algebraic group of adjoint type, and \(\theta_0\) be an involutory linear algebraic group automorphism of \(G_0\). Let \((T_0,B_0)\) be a \(\theta_0\)-stable pair of a maximal torus \(T_0\) and a Borel subgroup \(B\) such that \(T_0 \subset B_0\). The isotropic subtorus \(T'_0\) and the anisotropic subtorus \(T'_1\) are defined by

\[
T'_0 = \{ t \in T_0 : \theta(t) = t \}, \quad T'_1 = \{ t \in T_0 : \theta(t) = t^{-1} \}.
\]

The multiplication map \(T'_1 \times T'_0 \to T_0\) is an isogeny, that is, a surjective homomorphism with a finite kernel.

Among all \(\theta\)-stable maximal tori, we work with the one for which the dimension \(l := \dim T'_1\) is maximal. The integer \(l\) is called the rank of the symmetric variety \(G_0/G_0^\theta\). \((G_0^\theta\) is the fixed subgroup of \(\theta\).)

Let \(\Phi\) denote the set of roots of \(G_0\) relative to \(T_0\). Passing to the Lie algebra setting by differentiation, we view \(\Phi\) as a subset of the dual vector space \(\text{Lie}(T_0)^*\) of the Lie algebra of \(T_0\). Since \(\theta\) is an automorphism of \(T_0\), it...
induces a linear map

\[ \theta^* : \text{Lie}(T_0)^* \rightarrow \text{Lie}(T_0)^*, \]

which in turn induces an involution on \( \Phi \). Define

\[ \Phi_0 = \{ \alpha \in \Phi : \theta^*(\alpha) = \alpha \}, \quad \Phi_1 = \Phi - \Phi_0. \]

Lemma 3.1 \([11, \text{Lemma 1.2}]\). There exists a system \( \Phi^+ \subseteq \Phi \) of positive roots such that \( \theta^*(\alpha) \in \Phi - \Phi^+ \) for all \( \alpha \in \Phi^+ \cap \Phi_1 \).

We fix a set \( \Phi^+ \) of positive roots as in Lemma 3.1. Let \( \Delta \) denote the associated set of simple roots and let

\[ \Delta_0 = \Phi_0 \cap \Delta, \quad \Delta_1 = \Phi_1 \cap \Delta. \]

Observe that \( |\Delta_1| \geq \dim T_1 = l \). It turns out that there exists an ordering \( \{\alpha_1, \ldots, \alpha_j\} \) of the elements of \( \Delta_1 \) such that the differences \( \alpha_i - \theta^*(\alpha_i) \) are mutually distinct for \( i = 1, \ldots, l \), and for each \( i \in \{l + 1, \ldots, j\} \) there exists \( s \in \{1, \ldots, l\} \) such that \( \alpha_s - \theta^*(\alpha_s) = \alpha_s - \theta^*(\alpha_s) \) (see \([4, \text{Section 1.4}]\)).

A restricted simple root \( \overline{\alpha} \) is a weight of the form

\[ \overline{\alpha} = \frac{\alpha_i - \theta^*(\alpha_i)}{2} \quad \text{for some } i \in \{1, \ldots, l\}. \]

In this case, we denote \( \overline{\alpha} \) by \( \overline{\alpha}_i \), and denote by \( \overline{\Delta}_1 = \{\overline{\alpha}_1, \ldots, \overline{\alpha}_l\} \) the set of all restricted simple roots. Suppose now that \( \Delta_0 = \{\beta_1, \ldots, \beta_k\} \). In accordance with the partitioning \( \Delta = \Delta_0 \sqcup \Delta_1 \), we divide the set of fundamental weights of \( \Delta \) into two disjoint subsets \( \{\omega_1, \ldots, \omega_j\} \sqcup \{\zeta_1, \ldots, \zeta_k\} \) so that for each \( i \in \{1, \ldots, j\} \),

\[ (\omega_i, \beta_s^\vee) = 0 \quad \text{for } s = 1, \ldots, k, \] and \( (\omega_i, \alpha_r^\vee) = \delta_{i,r} \quad \text{for } r = 1, \ldots, j \),

and similarly for \( \zeta_i \)'s. As shown in \([4, \text{pp. 5, 6}]\), \( \theta^* \) induces an involution \( \tilde{\theta} \) on the indices \( \{1, \ldots, j\} \) such that \( \theta^*(\omega_i) = -\omega_{\tilde{\theta}(i)} \). Thus, we arrive at a crucial definition for our purposes:

**Definition 3.1.** A dominant weight \( \lambda \) of \( G_0 \) is said to be special (or \( \theta \)-special) if \( \theta^*(\lambda) = -\lambda \). If \( (\rho, V) \) is an irreducible representation with a \( \theta \)-special highest weight, then we call \( \rho \) a \( \theta \)-special representation of \( G_0 \).

Now, let \( \theta_0 : G_0 \rightarrow G_0 \) be an involutory automorphism on \( G_0 \). We choose a \( \theta_0 \)-stable maximal torus \( T_0 \) in \( G_0 \). Let \( \lambda \) be a special, dominant weight with the corresponding irreducible representation \( (\rho_0, V) \). Assume also that \( (\rho_0, V) \) is faithful. As before, we define the reductive group \( G \) by setting \( G = \mathbb{C}^* \cdot \rho_0(G_0) \subset GL(V) \). We claim that there exists an “extension” \( \theta : G \rightarrow G \) of \( \theta_0 \) to \( G \). To this end we define

\[ \theta(c\rho_0(g)) = c^{-1}\rho_0(\theta_0(g)), \quad g \in G_0, \ c \in \mathbb{C}^*. \tag{6} \]

To prove \( \theta \) is well defined, suppose \( g, g' \in G_0 \) and \( c, c' \in \mathbb{C}^* \) are such that \( c\rho_0(g) = c'\rho_0(g') \). Let \( \alpha \in \mathbb{C}^* \) denote \( cc'^{-1} \). Then \( \rho_0(g^{-1}g') = \alpha 1_{GL(V)} \in GL(V) \).
But $G_0$ is of adjoint type, $\rho_0$ is faithful, and $\alpha$ is a central element in $\GL(V)$. Therefore, $\alpha = \id$, hence $g = g'$ and $c = c'$. Finally, note that

$$\theta(\theta(c\rho_0(g))) = \theta(c^{-1}\rho_0(\theta_0(g))) = c\rho_0(\theta_0(\theta(g))) = c\rho_0(g) \quad \text{for all } c \in \mathbb{C}^* \text{ and } g \in G_0.$$  

The antiinvolution corresponding to $\theta$ is, by definition, the composition $\theta_{an} := \theta \circ \iota$ of $\theta$ with the “inverting” morphism $\iota : g \mapsto g^{-1}$. The map induced by $\theta_{an}$ on the character group $X(T)$ is denoted $\theta_{an}^*$. Then $\theta^*$ and $\theta_{an}^*$ are related by

$$\theta_{an}^*(\chi) = -\theta^*(\chi) \quad \text{for } \chi \in X(T).$$

In particular, if $\theta^*(\lambda) = -\lambda$, then $\theta_{an}^*(\lambda) = \lambda$.

We are ready to prove Theorem 1.2. Let us paraphrase it for completeness: If $M$ is a normal $J$-irreducible monoid that is obtained from a $\theta$-special minuscule representation of $G$, then there exists a unique morphism $\theta_{an} : M \to M$ such that

1. $\theta_{an}(xy) = \theta_{an}(y)\theta_{an}(x)$ for all $x, y \in M$;
2. $\theta_{an}^2$ is the identity map on $M$;
3. $\theta_{an}(g) = \theta(g)^{-1}$ for all $g \in G$, where $\theta$ is the involution that is extended from $\theta_0$ on $G_0$.

**Proof of Theorem 1.2.** Since $\theta_{an}$ agrees with $\theta$ (after composing with $\iota$, of course) on $G$, the uniqueness is clear. We are going to show that $\theta_{an}$ extends to the whole $J$-irreducible monoid $M := Z\chi$ associated to an irreducible representation $(\rho_0, V)$ of $G_0$ with the highest weight $\lambda$. Let $\rho$ denote the representation of $G$ as defined in [5].

First, we note that, by Theorem 2.1, $M$ is a normal reductive monoid. Let $T_0$ denote the maximal torus of $G_0$ such that $T = \mathbb{C}^* \cdot \rho_0(T_0)$, and let $\langle \Pi(\rho) \rangle$ denote the submonoid of $X(T)$ generated by the weights $\Pi(\rho)$ of $T$. The coordinate ring of the affine torus embedding $\bar{T}$ is equal to the monoid-ring $R = \mathbb{C}[\langle \Pi(\rho) \rangle]$ (see [12] Lemma 3.2). Therefore, $\bar{T} = \Spec(R)$. On the other hand, by Lemma 2.2, we know that $\Pi(\rho)$ is contained in the convex hull $\mathcal{P}$ of $W \cdot (\lambda + \chi)$, where $\chi$ is the $n$th root of the determinant on $\GL(V)$.

Since $\lambda$ is special, by [1] Lemma 1.6] there is a $G$-isomorphism $V^\theta \cong V^*$, hence $\theta^*(\Pi(\rho)) = \Pi(\rho^*) = -\Pi(\rho)$ and it follows that $\theta_{an}^*(\Pi(\rho)) = \Pi(\rho)$. In particular, it induces an antiinvolution $\theta_{an}$ on $\bar{T} = \Spec(R)$. Since $\theta \circ \iota = \theta_{an}$ on $T$, by the “extension principle” (see [12] Corollary 4.5]) there exists a unique morphism $\theta_{an} : M \to M$ which agrees with $\theta \circ \iota$ on $G$ and agrees with $\theta_{an}$ on $\bar{T}$. Since $\theta_{an}^2 = \id$ on $G$, and since $G$ is dense on $M$, we see that $\theta_{an}^2 = \id$ on $M$. Finally, since $(x, y) \mapsto \theta_{an}(xy)$ and $(x, y) \mapsto \theta_{an}(y)\theta_{an}(x)$ are morphisms from $M \times M$ into $M$ agreeing on the open dense set $G \times G$, they agree everywhere. ■
4. A proof of Theorem 1.1. We start with providing the details of some useful facts which we briefly mentioned earlier in Section 2.1.

Lemma 4.1 (Generalized Bruhat–Chevalley decomposition, [13]). Let $M$ be a reductive monoid with the group of invertible elements $G$, and let $T \subseteq B$ be a maximal torus contained in a Borel subgroup. Let $\overline{N}$ denote the closure in $M$ of the normalizer $N = N_G(T)$ of $T$. If $m \in M$, then there exist $b_1, b_2 \in B$ and $\overline{n} \in \overline{N}$ such that $m = b_1 \overline{n} b_2$. This leads to the Bruhat–Chevalley decomposition

$$M = \bigcup_{\overline{n} \in R} B\overline{n}B,$$

where the union is disjoint and $R = \overline{N}/T$ is the Renner monoid of $M$.

Fix $\overline{n} \in \overline{N}$, and let $V_{\overline{n}} \subseteq U$ denote the subgroup $\{u \in U : u\overline{n}B \subseteq \overline{n}B\}$. Then $V_{\overline{n}}$ is closed and $T$-stable under conjugation. Therefore, there exists a complementary subgroup

$$U_{\overline{n},1} = \prod_{u_\alpha \notin V_{\overline{n}}} U_\alpha.$$

Complementary in this context means that the product morphism $U_{\overline{n},1} \times V_{\overline{n}} \to U$ is an isomorphism of algebraic groups.

In a similar manner, let $Z_{\overline{n}} \subseteq U$ denote the closed subgroup $\{u \in U : \overline{n}Tu = \overline{n}T\}$. Also in this case, $Z_{\overline{n}}$ is $T$-stable under conjugation; let

$$U_{\overline{n},2} = \prod_{u_\alpha \notin Z_{\overline{n}}} U_\alpha$$

denote its complementary subgroup. The precise structure of the orbit $B\overline{n}B$ is exhibited in the next result:

Lemma 4.2 ([14, Lemma 13.1]). The product morphism $U_{\overline{n},1} \times \overline{n}T \times U_{\overline{n},2} \to B\overline{n}B$ is an isomorphism of varieties.

As a consequence of Lemmas 4.1 and 4.2, we have the following important observation:

Uniqueness Criterion. Given $m \in M$, there exist unique $u \in U_{\overline{n},1}$, $v \in U_{\overline{n},2}$, and $\overline{n} \in \overline{N}$ such that

$$m = u\overline{n}v.$$
Assume from now on that $T$ is a $\theta$-stable maximal torus of the $\theta$-stable Borel subgroup $B \subseteq G$. Notice in this case that the corresponding unipotent subgroup $U \subset B$ has to be $\theta$-stable as well. Moreover, since $T$ is $\theta$-stable, if $n$ is an element from the normalizer $N = N_G(T)$, then $\theta(n)t\theta(n)^{-1} = \theta(n)\theta(t')\theta(n^{-1}) = \theta(nt'n^{-1}) \in T$ for some $t' \in T$. In other words, $\theta(N) = N$. It follows that the Zariski closure $\overline{N}$ is $\theta_{an}$-stable.

**Proposition 4.1.** Any $B^*$-orbit in $M_Q$ contains an element of $\overline{N}$.

**Proof.** For $m \in M_Q$, we know from the Uniqueness Criterion that there exist unique $u \in U_{\pi,1}$, $v \in U_{\pi,2}$, and $\pi \in \overline{N}$ such that $m = u\pi v$. Then

$$u\pi v = m = \theta_{an}(m) = \theta_{an}(v)\theta_{an}(\pi)\theta_{an}(u) = \theta(v)^{-1}\theta_{an}(\pi)\theta(u)^{-1}.$$

Since the Bruhat–Chevalley decomposition (7) is a disjoint union, we see that $\theta_{an}(\pi) \in \pi T$. Let $t \in T$ be such that $\theta_{an}(\pi) = \pi t$.

Let $a$ denote $v\theta(u)$. It is clear that $\pi a$ lies in the $B^*$-orbit of $m$. Therefore, we have

$$\pi a = \theta_{an}(\pi a) = \theta_{an}(a)\theta_{an}(\pi) = \theta(a)^{-1}\pi t \quad \text{for some } t \in T,$$

or

$$\theta(a)\pi a = \pi t \quad \text{for some } t \in T. \quad (11)$$

Suppose $\pi = en$ for some $e \in E(\overline{T})$ and $n \in N$. By (11), we see that

$$\theta(a)e = ena^{-1}n^{-1}.$$

In particular, $\theta(a)e = e\theta(a)e$. Since $U$ is $\theta$-stable, we know that $\theta(a) \in U$, therefore $\theta(a)e$ is an element of $eUe$. Notice that $nt = t'n$ for some $t' \in T$, so $\theta(a)e = et'n a^{-1}n^{-1}$. At the same time, $na^{-1}n^{-1} \in U$. In other words, $\theta(a)e = et'u'$, where $t' \in T$, $u' := na^{-1}n^{-1} \in U$, and we know that $et'u' \in eUe$. Therefore, it is harmless to continue with $\theta(a)e = eu'$.

Since square roots exist in unipotent groups, we see that $(e\theta(a)e)^{1/2} = e\theta(a)^{1/2}e = (ena^{-1}n^{-1}e)^{1/2} = e(na^{-1}n^{-1})^{1/2}e = e(na^{-1/2}n^{-1})e$.

The unit group of $eMe$ is $eC_G(e)$, and $eC_B(e) = eBe$ is a Borel subgroup of $eC_G(e)$ (see Lemma 2.1). Since $eUe \subseteq eC_B(e)$, we find that $e\theta(a)^{1/2}e = \theta(a)^{1/2}e$ and that $e(na^{-1/2}n^{-1})e = ena^{-1/2}n^{-1}$. Now, on the one hand, we have $\theta(a)^{1/2}e = ena^{-1/2}n^{-1}$, or equivalently $\theta(a)^{1/2}na^{1/2} = \pi$. On the other hand, $\theta(a)^{1/2}\pi a^{1/2} = \theta(a)^{1/2}*(\pi a)$. Hence, $\pi$ is contained in the $B^*$-orbit of $m$. □

**Remark 4.1.** Recall that $\tau : M \to M$ is defined by $\tau(x) = x\theta_{an}(x)$. The image of $\tau$ is contained in $M_Q$; if $m \in M$, then

$$\theta_{an}(\tau(m)) = \theta_{an}(m\theta_{an}(m)) = \theta_{an}(\theta_{an}(m))\theta_{an}(m) = m\theta_{an}(m) = \tau(m).$$

**Lemma 4.3.** $T \times H$ acts on $\tau^{-1}(\overline{N})$ by $(t, h) \cdot m = tmh^{-1}$. 


Proof. It suffices to check that for all \( t \in T, \ h \in H, \) and \( m \in \tau^{-1}(\mathcal{N}) \), the image \( \tau(tmh^{-1}) \) is contained in \( \mathcal{N} \). But
\[
\tau(tmh^{-1}) = tm\theta_{an}(m)\theta(t)^{-1}.
\]
Since \( \tau(m) = m\theta_{an}(m) \in \mathcal{N} \) and since \( \mathcal{N} \) is \( T \)-stable, the proof is finished.

Remark 4.2. Let \((T,B)\) be a pair of a \( \theta \)-stable maximal torus and a Borel subgroup such that \( T \subseteq B \). Let \( \mathcal{V} \) denote the set of all \( g \in G \) such that \( \tau(g) \in N_G(T) \). It is easy to verify that \( \mathcal{V} \subset G \) is closed under the action of \( T \times H \),
\[
(t,h) \cdot g = tgh^{-1} \quad \text{for } t \in T, \ h \in H, \ g \in G.
\]
Let \( \mathcal{V} \) denote the set of \( T \times H \)-orbits in \( \mathcal{V} \). For \( v \in \mathcal{V} \), let \( x(v) \in \mathcal{V} \) denote a representative of the orbit \( v \). The inclusion \( \mathcal{V} \hookrightarrow G \) induces a bijection from \( \mathcal{V} \) onto the set of \( B \times H \)-orbits in \( G \). In particular, \( G \) is the disjoint union of the double cosets \( Bx(v)H, \ v \in \mathcal{V} \) (see [8]).

Theorem 4.1. The following sets are in bijection with each other:
1. \( B^\ast \)-orbits in \( M_Q \),
2. \( T \times H \)-orbits in \( \tau^{-1}(\mathcal{N}) \subset M \).

Proof. We start with an observation: under \( \tau \), the set of \( B \times H \)-orbits in \( M \) is surjectively mapped onto the set of \( B^\ast \)-orbits in \( M_Q \). To see this, first, we show that any \( B \times H \)-orbit in \( M \) is mapped by \( \tau \) onto a \( B^\ast \)-orbit in \( M_Q \). Let \( \mathcal{O}_a \) be the \( B \times H \)-orbit of an element \( a \) from \( M \). Since
\[
\tau(bah^{-1}) = bah^{-1}\theta_{an}(bah^{-1}) = ba\theta_{an}(a)\theta_{an}(b) = b \ast \tau(a),
\]
we see that \( \tau(\mathcal{O}_a) = B \ast \tau(a) \). Next, we will show that any \( B^\ast \)-orbit in \( M_Q \) comes from a \( B \times H \)-orbit in \( M \). Let \( x \in M_Q \). By Proposition 4.1, we know that any \( B^\ast \)-orbit in \( M_Q \) intersects \( \mathcal{N} \). In particular, \( B \ast x \cap \mathcal{N} \neq \emptyset \). Let \( \bar{\pi} \in \mathcal{N} \) be such that \( b \ast x = \bar{\pi} \) for some \( b \in B \). By Lemma 4.3, we know that \( T \times H \) acts on \( \tau^{-1}(\mathcal{N}) \). Let \( a \in \tau^{-1}(\mathcal{N}) \) be such that \( \tau(a) = \bar{\pi} \). Then the \( B \times H \)-orbit \( \mathcal{O}_a \) of \( a \) is mapped to \( B \ast \tau(a) = B \ast x \). Now we know that \( B \times H \)-orbits in \( M \) are mapped onto \( B^\ast \)-orbits in \( M_Q \).

Incidentally, the argument in the above paragraph shows the following: the assignment defined by
\[
(12) \quad f : (T \times H)a \mapsto T \ast \tau(a) \mapsto B \ast \tau(a)
\]
is a surjective map between the set of \( T \times H \)-orbits in \( \tau^{-1}(\mathcal{N}) \) and the set of \( B^\ast \)-orbits in \( M_Q \). We proceed to show that \( f \) is injective.

Let \( O \) be a \( B^\ast \)-orbit in \( M_Q \) and let \( \bar{\pi}_1, \bar{\pi}_2 \in \mathcal{N} \cap O \). Then there exists \( b \in B \) such that
\[
\bar{\pi}_1 = b \ast \bar{\pi}_2 = b\bar{\pi}_2\theta_{an}(b).
\]
Since the Bruhat–Chevalley decomposition \( M = \bigcup_{r \in R} BrB \) is a disjoint union, and \( B \) is \( \theta_{an} \)-stable, we see from the uniqueness criterion that \( b \in T \)
and \( \overline{\tau_2} \in \overline{\tau_1 T} \). In other words, there exists \( t \in T \) such that \( t \ast \overline{\tau_2} = \overline{\tau_1} \). Let \( a_1, a_2 \in \tau^{-1}(\overline{N}) \) be such that \( \tau(a_1) = \overline{\tau_1} \) and \( \tau(a_2) = \overline{\tau_2} \). Then \( t \ast \tau(a_2) = \tau(a_1) \). But \( t \ast \tau(a_2) = \tau(ta_2) \), or equivalently \( ta_2 \in \tau^{-1}(\overline{\tau_1}) \). Consequently, the intersection with \( \overline{N} \) of a \( B \ast \)-orbit \( O \) (\( = \tau(\overline{O_{a_1}}) \)) is covered by a single \( T \times H \)-orbit in \( \tau^{-1}(\overline{N}) \). In particular, the map (12) is one-to-one. ■

**Remark 4.3.** An important corollary of the proof of Theorem 4.1 is that the number of \( B \ast \)-orbits in \( MQ \) is finite. Indeed, any \( B \ast \)-orbit in \( MQ \) intersects \( \overline{N} \) along a \( T \ast \)-orbit and \( \overline{N} / T \) is a finite semigroup.

Now we are ready to prove our first main result, which states that the following sets are finite and they are in bijection with each other:

1. \( B \ast \)-orbits in \( \overline{Q} \) (respectively, \( B \ast \)-orbits in \( \overline{P} \)),
2. \( T \times H \)-orbits in \( \tau^{-1}(\overline{N} \cap \overline{Q}) \) (respectively, \( T \times H \)-orbits in \( \tau^{-1}(\overline{N} \cap \overline{P}) \)).

**Proof of Theorem 4.1.** Since \( MQ \) is closed, the inclusions \( P \subseteq Q \subseteq MQ \) imply that \( \overline{P} \subseteq \overline{Q} \subseteq \overline{MQ} \). Moreover, we know that \( P \) and \( Q \), and hence \( \overline{P} \) and \( \overline{Q} \), are \( B \ast \)-stable. By Remark 4.3 we know that \( MQ \) comprises finitely many \( B \ast \)-orbits. The rest of the proof follows from Theorem 4.1. ■

There is a well known classification, due to Cartan, of involutions on semisimple groups. For the classical groups, up to inner automorphisms there are seven types of involutions in total. For the exceptional groups there are in total nine involutions. See [6, Chapter X, Section 6] for a complete list.

We finish this section by presenting some examples.

**Example 4.1.** Let \( G_0 \) denote \( \text{PSL}_n \), the projective special linear group of \( n \times n \) matrices with determinant 1. Then \( \theta_0 : G_0 \rightarrow G_0 \) defined by \( \theta_0(g) = (g^{-1})^\top \) (\( g \in G_0 \)) is an involutory automorphism. Let \( T_0 \) denote the maximal torus of diagonal matrices in \( G_0 \) and let \( \omega_1 \) denote the first fundamental weight. Let \( (\rho_0, V) \cong (\text{id}, \mathbb{C}^n) \) denote the corresponding irreducible (minuscule) representation. Then the \( J \)-irreducible monoid associated with \( \omega_1 \) is nothing but the monoid of \( n \times n \) matrices,

\[
Z_{\omega_1} := \mathbb{C}^\ast \cdot \text{PSL}_n = \text{Mat}_n,
\]

which we denote by \( M \). Then the unit group of \( M \) is \( G = \text{GL}_n \). Clearly, \( \theta_0 \) extends to \( G \) by the same formula, \( \theta(g) = (g^{-1})^\top \) (\( g \in G \)). The \( G \ast \)-orbit of the identity is equal to the set of invertible symmetric \( n \times n \) matrices,

\[
G \ast 1_{\text{GL}_n} = P = \{ gg^\top : g \in \text{GL}_n \}.
\]

We observe that for our choices of \( \theta \) and \( G_0 \), the subvariety \( Q := \{ g \in G : \theta(g) = g^{-1} \} \) is equal to \( P \). Therefore, in \( M \), we have

\[
\overline{Q} = \overline{P} = \text{Sym}_n,
\]

the affine variety of symmetric \( n \times n \) matrices. Also, we notice that the unique antiinvolution on \( M \) that is extended from the involution \( \theta \) on \( G \) is given by

\[
\tau(g) = (g^{-1})^\top.
\]
\[ \theta_{\text{an}}(m) = m^\top \] for \( m \in M \). Therefore, \( M_Q \) is equal to \( \text{Sym}_n \) as well. Finally, we know from [18] that \( B^* \)-orbits in \( \text{Sym}_n \) are parametrized by the \( n \times n \) “partial involutions” in the “rook monoid” \( R_n \). Here, the rook monoid is the Renner monoid of \( \text{Mat}_n \); it is the finite monoid which consists of \( n \times n \) 0/1 matrices with at most one 1 in each row and column. A partial involution in \( R_n \) is an element \( x \in R_n \) such that \( x^\top = x \).

**Example 4.2.** Let \( (\rho_0, V) \) denote the second fundamental representation \( V = \bigwedge^2 \mathbb{C}^{2n} \) of \( G_0 := \text{PSL}_{2n} \). As before, let \( T_0 \) denote the maximal torus consisting of all diagonal matrices in \( G_0 \). We consider the involution \( \theta_0(g) = -J(g^{-1})^\top J \) (\( g \in G_0 \)), where \( J \) is the \( 2n \times 2n \) block diagonal matrix
\[ J = \text{diag}(J_2, \ldots, J_2) \quad \text{with} \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

More explicitly, \( V \) is equal to the space of \( 2n \times 2n \) skew-symmetric matrices, and the action of \( G_0 \) on \( V \) is given by
\[ g \cdot A = (g^{-1})^\top Ag^{-1}. \]

For simplicity, denote by \( \phi_g \) the operator \( \rho_0(g) \) on \( V \) (\( g \in G_0 \)). Note that for \( g = J \) we have \( \phi_J^2 = 1_{\text{GL}(V)} \). It is not difficult to show that \( \rho_0 \) is faithful, and that the extension of \( \theta_0 \) to \( G = \mathbb{C}^* \cdot \rho_0(G_0) \) is given by
\[ \theta(c\phi_g) = c\phi_J\phi(g^{-1})^\top \phi_J \quad \text{for all} \quad g \in G_0 \quad \text{and} \quad c \in \mathbb{C}^*. \]

Now, let \( y \in Q \). If \( y = c\phi_g \) for some \( g \in G_0 \), and \( c \in \mathbb{C}^* \), then
\[ c^{-1}\phi_g^{-1} = \theta(y) = c\phi_J\phi(g^{-1})^\top \phi_J, \quad \text{or equivalently} \quad 1_V = c^2\phi_J(g^{-1})^\top J. \]

Since \( \rho_0 \) is faithful, we have \( c = 1 \), and \( gJ(g^{-1})^\top J = 1_{G_0} \), or \( g^{-1} = J(g^{-1})^\top J \). In other words, \( Q \) is isomorphic to
\[ Q_0 := \{ g \in \text{PSL}_{2n} : \theta_0(g) = g^{-1} \}. \]

On the other hand, we know that \( Q_0 \) is equal to \( P_0 := \{ g\theta_0(g^{-1}) : g \in \text{PSL}_{2n} \} \) (see [5, Section 11.3.5]). Since the image of \( P_0 \) under \( \rho_0 \) equals \( P \), we see that \( P = Q \), so \( P \) is a closed subvariety of \( G \).

Let \( \theta_{\text{an}} \) be the unique antiinvolution extension of \( \theta \) to the monoid \( M \) of \( (\rho_0, V) \). Then
\[ \overline{Q} = M_Q = \{ y \in M : \theta_{\text{an}}(y) = y \}. \]

Next, we compute the parametrizing set of \( B^* \)-orbits in \( M_Q = M_P \). To this end, we determine the normalizer of \( T \) in \( G \). We claim that \( N_G(T) = \mathbb{C}^* \rho_0(N_{G_0}(T_0)) \). Indeed, let \( x = c\rho_0(g) \in G \) be an element from the normalizer of \( T \), and let \( t \in T \). Since \( t = d\rho_0(t') \) for some \( t' \in T_0 \) and \( d \in \mathbb{C}^* \), we have \( xtx^{-1} = d\rho_0(gt'g^{-1}) \in T \), or equivalently \( gt'g^{-1} \in T_0 \). Thus, \( t \) lies in \( \mathbb{C}^* \rho_0(N_{G_0}(T_0)) \). The converse inclusion is obvious.
Let us look at a typical element of $N_G(T)$. Assume that $g$ is a monomial matrix, that is, every row and every column have exactly one nonzero entry. We will prove that, once a basis is fixed, $\rho_0(g) = \phi_g$ is a monomial matrix as well. Towards this end, we choose the following basis:

$$F_{i,j} = E_{i,j} - E_{j,i}, \quad 1 \leq i < j \leq 2n,$$

where $E_{i,j}$’s are the elementary matrices. Suppose that the inverse of $g \in G_0$ is the matrix $g^{-1} = (g_{k,l})_{k,l=1}^{2n}$. Obviously, $g^{-1}$ is a monomial matrix as well. Since $\rho_0(g) \cdot E_{i,j} = (g^{-1})^\top E_{i,j} g^{-1} = (g_{i,k}g_{j,l})_{k,l=1}^{2n}$, we see that

$$(13) \quad g \cdot F_{i,j} = \rho_0(g) \cdot F_{i,j} = (g_{i,k}g_{j,l} - g_{j,k}g_{i,l})_{k,l=1}^{2n}.$$

We continue with a special case of our claim by assuming that $g$ is a diagonal matrix. Then the $(k,l)$th entry (with $k < l$) of $g \cdot F_{i,j}$ is nonzero if and only if $i = k$ and $j = l$. In this case, $g \cdot F_{i,j} = g_{i,i}g_{j,j}F_{i,j}$. Thus, the matrix representing $\rho_0(g)$ is the $n(n-1) \times n(n-1)$ diagonal matrix $\text{diag}(s_{1,2}, s_{1,3}, \ldots, s_{n-1,n})$ with $s_{i,j} = g_{i,i}g_{j,j}$. Now, more generally, assume that $g$ is a monomial matrix. Then the $(k,l)$th entry (with $k < l$) $g_{i,k}g_{j,l} - g_{j,k}g_{i,l}$ of $g \cdot F_{i,j}$ is nonzero if and only if one of the following is true:

(i) the entries of $g^{-1}$ at its $(i,k)$th and the $(j,l)$th positions are nonzero at the same time, or
(ii) the entries of $g^{-1}$ at its $(i,l)$th and the $(j,k)$th positions are nonzero at the same time.

Observe that (i) and (ii) do not hold true at the same time. Observe also that for each $i < j$ there exists a unique pair $(k,l)$ with $k < l$ such that either (i) is true, or (ii) is true. Therefore, if $g^{-1}$ is a monomial matrix, then

$$(14) \quad g \cdot F_{i,j} = \begin{cases} g_{i,k}g_{j,l}F_{k,l} & \text{if } g_{i,k}g_{j,l} \neq 0, \\ g_{i,l}g_{j,k}F_{k,l} & \text{if } g_{i,l}g_{j,k} \neq 0. \end{cases}$$

It follows that if $g$ is a monomial matrix, then so is the matrix of $\rho_0(g) = \phi_g$.

Now, let $x \in M$ be an element from $\overline{N_G(T)}$. Since the elements of $\overline{N_G(T)}$ are obtained from those of $N_G(T)$ by taking limits (in the algebraic sense), we see that $x \cdot F_{i,j}$ is either identically zero, or is a scalar multiple of $F_{k,l}$ for some $k,l$ as in (14). In other words, $x$ is obtained from the image of a monomial matrix in $G_0$ by replacing some of its entries by zeros.

It is well known that the invertible symmetric monomial matrices modulo the maximal torus of diagonal matrices represent the $BS^*$-orbits in $Q$, and furthermore the finite set of orbit representatives is in bijection with the fixed point free involutions of the symmetric group $S_{2n}$ (see [16]). Thus, in our case, the representing matrices are those that are obtained from the fixed point free monomial matrices by replacing some of the nonzero entries by...
zeros. These are precisely the “partial fixed point free involutions,” introduced in [2].

5. Final remarks. Given a reductive monoid $M$ with an antiinvolution $\theta_{\text{an}}$, we now have the notion of a symmetric submonoid

$$M_{\text{an}} := \{ m \in M : m\theta_{\text{an}}(m) = 1_M \}.$$ 

Observe that the identity element $1_M$ of $M$ is the identity element $1_G$ of $G$. Therefore, $\theta_{\text{an}}(1_M) = \theta_{\text{an}}(1_G) = \theta_{\text{an}}(1_G)\theta_{\text{an}}(1_G)$, hence $\theta_{\text{an}}(1_M) = 1_M$. In other words, $1_M \in M_{\text{an}}$. Also, if $m_1, m_2 \in M_{\text{an}}$, then

$$m_1m_2\theta_{\text{an}}(m_1m_2) = m_1m_2\theta_{\text{an}}(m_2)\theta_{\text{an}}(m_1) = m_1 \cdot 1_M \cdot \theta_{\text{an}}(m_1) = 1_M.$$ 

Therefore, $m_1m_2 \in M_{\text{an}}$. Note that if $g \in G$ lies in $M_{\text{an}}$, then $1_G = g^{-1}\theta_{\text{an}}(g^{-1}) = g^{-1}\theta(g)$, hence $\theta(g) = g$. In other words, the group of invertible elements of $M_{\text{an}}$ is the fixed subgroup $H = G^\theta$. The above argument provides us with an effective way of producing new linear algebraic monoids, one for each antiinvolution $\theta_{\text{an}}$ on $M$.

REFERENCES

[1] M. Brion, On algebraic semigroups and monoids, in: Algebraic Monoids, Group Embeddings, and Algebraic Combinatorics, Fields Inst. Commun. 71, Springer, New York, 2014, 1–54.
[2] Y. Cherniavsky, On involutions of the symmetric group and congruence $B$-orbits of anti-symmetric matrices, Int. J. Algebra Comput. 21 (2011), 841–856.
[3] C. De Concini, Normality and non normality of certain semigroups and orbit closures, in: Algebraic Transformation Groups and Algebraic Varieties, Encyclopaedia Math. Sci. 132, Springer, Berlin, 2004, 15–35.
[4] C. De Concini and C. Procesi, Complete symmetric varieties, in: Invariant Theory (Montecatini, 1982), Lecture Notes in Math. 996, Springer, Berlin, 1983, 1–44.
[5] R. Goodman and N. R. Wallach, Symmetry, Representations, and Invariants, Grad. Texts in Math. 255, Springer, Dordrecht, 2009.
[6] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Grad. Stud. Math. 34, Amer. Math. Soc., Providence, RI, 2001.
[7] A. G. Helminck, Algebraic groups with a commuting pair of involutions and semisimple symmetric spaces, Adv. Math. 71 (1988), 21–91.
[8] A. G. Helminck, Computing $B$-orbits on $G/H$, J. Symbolic Comput. 21 (1996), 169–209.
[9] Z. Li, Z. Li, and Y. Cao, Algebraic monoids and Renner Monoids, in: Algebraic Monoids, Group Embeddings, and Algebraic Combinatorics, Fields Inst. Commun. 71, Springer, New York, 2014, 141–187.
[10] T. Matsuki, The orbits of affine symmetric spaces under the action of minimal parabolic subgroups, J. Math. Soc. Japan 31 (1979), 331–357.
[11] M. S. Putcha, Linear Algebraic Monoids, London Math. Soc. Lecture Note Ser. 133, Cambridge Univ. Press, Cambridge, 1988.
[12] L. E. Renner, Classification of semisimple algebraic monoids, Trans. Amer. Math. Soc. 292 (1985), 193–223.
[13] L. E. Renner, *Analogue of the Bruhat decomposition for algebraic monoids*, J. Algebra 101 (1986), 303–338.

[14] L. E. Renner, *Linear Algebraic Monoids*, Encyclopaedia Math. Sci. 134, Springer, Berlin, 2005.

[15] R. W. Richardson, *Orbits, invariants, and representations associated to involutions of reductive groups*, Invent. Math. 66 (1982), 287–312.

[16] R. W. Richardson and T. A. Springer, *The Bruhat order on symmetric varieties*, Geom. Dedicata 35 (1990), 389–436.

[17] T. A. Springer, *Some results on algebraic groups with involutions*, in: Algebraic Groups and Related Topics (Kyoto/Nagoya, 1983), Adv. Stud. Pure Math. 6, North-Holland, Amsterdam, 1985, 525–543.

[18] F. Szechtman, *Equivalence and congruence of matrices under the action of standard parabolic subgroups*, Electron. J. Linear Algebra 16 (2007), 325–333.

Mahir Bilen Can
Department of Mathematics
Tulane University
6823 St. Charles Ave.
New Orleans, LA 70118, U.S.A.
E-mail: mahirbilencan@gmail.com

Lex Renner
Department of Mathematics
Middlesex College
Western University
London, ON N6A 5B7, Canada
E-mail: lex@uwo.edu

Roger Howe
Department of Mathematics
Yale University
442 Dunham Lab
10 Hillhouse Ave.
New Haven, CT 06511, U.S.A.
E-mail: roger.howe@yale.edu