Extreme value statistics for truncated Pareto-type distributions

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Abstract

Recently some authors have drawn attention to the fact that there might be practical problems with the use of unbounded Pareto distributions, for instance when there are natural upper bounds that truncate the probability tail. Aban, Meerschaert and Panorska (2006) derived the maximum likelihood estimator for the tail index of a truncated Pareto distribution with right truncation point $T$. The Hill (1975) estimator is then obtained from this maximum likelihood estimator letting $T \to \infty$. The problem of extreme value estimation under (right) truncation was also introduced in Nuyts (2010) who proposed a similar estimator for the tail index and considered trimming of the number of extreme order statistics. Given that in practice one does not always know if the distribution is truncated or not, we propose estimators for extreme quantiles and $T$ that are consistent both under truncated and non-truncated Pareto-type distributions. In this way we extend the classical extreme value methodology adding the truncated Pareto-type model with truncation point $T \to \infty$ as the sample size $n \to \infty$. Finally we present some practical examples, asymptotics and simulation results.

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1 Introduction

The Pareto distribution is a simple and very popular model for positive data with power law probability tail. Using the notation from Aban et al. (2006)

\[ P(W > w) = \tau^{\alpha} w^{-\alpha} \quad \text{for} \quad w \geq \tau > 0 \quad \text{and} \quad \alpha > 0 \]  

(1)

it is considered as the standard example in the max domain of attraction of the Fréchet distribution. For instance, losses in property and casualty insurance often have a heavy right tail behaviour making it appropriate for including large events in applications such as excess-of-loss pricing and Enterprise Risk Management (ERM). There might be some practical problems with the use of the Pareto distribution and its generalization to the Pareto-type model, because some probability mass can still be assigned to loss amounts that are unreasonable large or even physically impossible. In ERM this leads to the concept of maximum probable loss (MPL). In certain applications there is a natural upper bound that truncates the probability tail. Evidence of truncated power type laws can be found in insurance, finance, earthquake modeling among others. For references we refer to Clark (2013) and Aban et al. (2006). These authors considered the upper-truncated Pareto distribution

\[ P(X > x) = \frac{\tau^\alpha (x^{-\alpha} - T^{-\alpha})}{1 - (\tau/T)^\alpha} \]  

(2)

with density

\[ f_X(x) = \frac{\alpha \tau^\alpha x^{-\alpha-1}}{1 - (\tau/T)^\alpha} \]  

(3)

for \( 0 < \tau \leq x \leq T \leq \infty \), where \( \tau < T \).

Aban et al. (2006) considered the estimation of the parameters by obtaining the conditional maximum likelihood estimator (MLE) based on the \( k + 1 \) (\( 0 \leq k < n \)) largest order statistics representing only the portion of the tail where the truncated Pareto approximation holds. They showed that when \( X_{1,n} \leq \ldots \leq X_{n-k,n} \leq X_{n-k+1,n} \leq \ldots \leq X_{n,n} \), the MLE are

\[ \hat{T}_A = X_{n,n}, \quad \hat{\tau}_A = k^{1/\hat{\alpha}_A} X_{n-k,n} \left( n - (n - k)(X_{n-k,n}/X_{n,n})^{\hat{\alpha}_A} \right)^{-1/\hat{\alpha}_A} \]

while \( \hat{\alpha}_A \) solves the equation

\[ \frac{1}{k} \sum_{j=1}^{k} (\log X_{n-j+1,n} - \log X_{n-k,n}) = \frac{1}{\hat{\alpha}_A} + \frac{(X_{n-k,n}/X_{n,n})^{\hat{\alpha}_A} \log(X_{n-k,n}/X_{n,n})}{1 - (X_{n-k,n}/X_{n,n})^{\hat{\alpha}_A}}. \]  

(4)
This estimator $1/\hat{\alpha}_A$ is then considered as an extension of Hill’s (1975) estimator $H_{k,n} = \frac{1}{k} \sum_{j=1}^{k} (\log X_{n-j+1,n} - \log X_{n-k,n})$ to the case of a truncated Pareto distribution with $T < \infty$, while $H_{k,n}$ was introduced as an estimator of $\frac{1}{\alpha}$ when $T = \infty$.

Independently Nuyts (2010) considered an adaptation of the Hill (1975) estimator through the estimation of

$$\mathbb{E}(\log W | W \in [L, R]) = \frac{\int_{L}^{R} \log(w) f(w)dw}{\int_{L}^{R} f(w)dw} \tag{5}$$

for some positive numbers $0 < L < R$, taking $W$ to be strict Pareto (1).

Then, (1) and (5) lead to

$$\mathbb{E}(\log W | W \in (L, R]) = \frac{1}{\alpha} + \frac{L^{-\alpha} \log L - R^{-\alpha} \log R}{L^{-\alpha} - R^{-\alpha}}. \tag{6}$$

Considering $L = Q(1 - \frac{k}{n})$ and $R = Q(1 - \frac{r}{n})$, the $k/n$ and $r/n$ ($1 \leq r < k < n$) upper quantiles which are estimated by $X_{n-k+1,n}$ and $X_{n-r+1,n}$ respectively, the estimator of Nuyts (2010) of $1/\alpha$ is obtained from solving

$$\frac{1}{k_r} \sum_{j=r}^{k} \log(X_{n-j+1,n}) = \frac{1}{\alpha} + \frac{X_{n-k+1,n}^{-\alpha} \log X_{n-k+1,n} - X_{n-r+1,n}^{-\alpha} \log X_{n-r+1,n}}{X_{n-k+1,n}^{-\alpha} - X_{n-r+1,n}^{-\alpha}}, \tag{7}$$

with $k_r = k - r + 1$. After some algebra,

$$\frac{1}{k_r} \sum_{j=r}^{k} \log(X_{n-j+1,n}) - \log(X_{n-k,n}) = \frac{1}{\alpha} + \frac{(X_{n-k+1,n}/X_{n-r+1,n})^\alpha \log(X_{n-k,n}/X_{n-r+1,n})}{1 - (X_{n-k+1,n}/X_{n-r+1,n})^\alpha} + \log(X_{n-k+1,n}/X_{n-k,n}) \frac{1}{1 - (X_{n-k+1,n}/X_{n-r+1,n})^\alpha}.$$ 

The last term on the right hand side is of smaller order than the other terms as can be shown by asymptotic arguments as developed in the Appendix. Hence one is lead to delete the last term, and then, in case $r = 1$, this equation is only a minor adaptation of (4). We conclude that the estimators of Nuyts (2010) and Aban et al. (2006) are basically the same as it
has been observed in simulations too. Deleting the last term in the above expression and considering the trimming procedure from Nuyts (2010) we consider the estimator \( \hat{\alpha}_{r,k,n} \) defined from

\[
\frac{1}{k_r} \sum_{j=r}^{k} \log \left( \frac{X_{n-j+1,n}}{X_{n-k,n}} \right) = 1/\hat{\alpha}_{r,k,n} + \frac{(X_{n-k,n}/X_{n-r+1,n})^{\hat{\alpha}_{r,k,n}} \log(X_{n-k,n}/X_{n-r+1,n})}{1 - (X_{n-k,n}/X_{n-r+1,n})^{\hat{\alpha}_{r,k,n}}}
\]

for \( 1 \leq r < k < n \). In asymptotic settings we will consider \( r \) either fixed or intermediate, such that when \( k \) and \( n \) go to infinity, \( r/k \to \lambda \in [0, 1) \).

In what follows we use the notation

\[
H_{r,k,n} = \frac{1}{k_r} \sum_{j=r}^{k} \log X_{n-j+1,n} - \log X_{n-k,n}
\]

and

\[
R_{r,k,n} = \frac{X_{n-k,n}}{X_{n-r+1,n}}.
\]

Furthermore remark that \( H_{k,n} = H_{1,k,n} \). Hence the estimator \( \hat{\alpha}_{r,k,n} \) is defined as the solution of

\[
H_{r,k,n} = \frac{1}{\alpha} + \frac{R_{r,k,n}^{\alpha} \log R_{r,k,n}}{1 - R_{r,k,n}^{\alpha}}.
\] (8)

The solution of \( (8) \) can be approximated using the Newton-Raphson iteration on the function

\[
f \left( \frac{1}{\alpha} \right) = H_{r,k,n} - \frac{1}{\alpha} - \frac{R_{r,k,n}^{\alpha} \log R_{r,k,n}}{1 - R_{r,k,n}^{\alpha}}
\]

to get

\[
\frac{1}{\hat{\alpha}_{r,k,n}^{(l+1)}} = \frac{1}{\hat{\alpha}_{r,k,n}^{(l)}}
\]

\[
H_{r,k,n} - \left( \hat{\alpha}_{r,k,n}^{(l)} \right)^{-1} - \frac{\hat{\alpha}_{r,k,n}^{(l)} \log R_{r,k,n}}{1 - \hat{\alpha}_{r,k,n}^{(l)}} + \frac{\hat{\alpha}_{r,k,n}^{(l)} \log^2 R_{r,k,n}}{(1 - \hat{\alpha}_{r,k,n}^{(l)})^2}, \ l = 0, 1, \ldots
\] (9)
where for instance Hill’s estimator can serve as an initial value: \( \hat{\alpha}^{(0)} = 1/H_{k,n} \).

We will study the behaviour of \( \hat{\alpha}_{r,k,n} \) in case of both Pareto-type and truncated Pareto-type distributions. In practical applications, typically it is not known a priori if a truncated distribution is more appropriate in tail fitting, see the discussion in Aban et al. (2006). Hence preferably the proposed estimation methods should work for both settings.

We will make use of the survival function \( 1 - F(x) = \mathbb{P}[X > x] \) and of the tail function \( U(x) = Q(1 - \frac{1}{x}) (x > 1) \) where \( Q(p) := \inf\{x : F(x) \geq p\} \) denotes the quantile function.

Pareto-type distributions are defined by

\[
1 - F(x) = x^{-\alpha} \ell_F(x) \quad (10)
\]

where \( \ell_F \) is a slowly varying function at infinity, i.e. \( \lim_{t \to \infty} \ell_F(tx)/\ell_F(t) = 1 \) for every \( x > 0 \). In extreme value statistics the parameter \( \xi := 1/\alpha \) is referred to as the extreme value index (EVI). The EVI \( \xi \) is the shape parameter in the generalized extreme value distribution

\[
G_\xi(x) = \exp \left( -\left(1 + \xi x\right)^{-1/\xi} \right), \quad \text{for } 1 + \xi x > 0.
\]

This class of distributions is the set of the unique non-degenerate limit distributions of a sequence of maximum values, linearly normalized. In case \( \xi > 0 \) the class of distributions for which the maxima are attracted to \( G_\xi \) corresponds to the Pareto-type distributions (10).

We define the truncated version of a Pareto-type distribution by

\[
1 - F_T(x) = C_T \left( x^{-\alpha} \ell_F(x) - T^{-\alpha} \ell_F(T) \right) . \quad (11)
\]

The constant \( C_T = (\tau^{-\alpha} \ell_F(\tau) - T^{-\alpha} \ell_F(T))^{-1} \) is specified by the condition \( 1 - F_T(\tau) = 1 \), where \( \tau > 0 \) is the lower bound of the range of \( x \). Below we derive that the corresponding quantile function \( Q_T(1 - p) \) can be written as

\[
Q_T(1 - p) = T \left( 1 + \frac{p}{D_T} \right)^{-1/\alpha} \zeta_{1/p,T} \quad (12)
\]

where \( D_T = C_T (T \ell(T))^{-\alpha} \) and \( \zeta_{1/p,T} \to 1 \) if \( T \to \infty \) and \( p \to 0 \), assuming that the quantity \( p/D_T \) remains bounded away from \( \infty \) as \( p \to 0 \) and \( T \to \infty \).
\( \infty \). For instance in case of the simple truncated Pareto distribution (2), an expansion for \( p / D_T \to 0 \) of \( Q_T(1 - p) \) yields

\[
Q_T(1 - p) = T \left( 1 - \frac{1}{\alpha D_T} \frac{p}{1 + o_p(1)} \right).
\]

From (12) using for instance (2.11) in Beirlant et al. (2004), it follows that truncated Pareto-type distributions belong to the Weibull domain of attraction for maxima with EVI \( \xi = -1 \) when \( p / D_T \to 0 \).

Remark that the quantity \( D_T \) equals the odd’s ratio \( \frac{P(X > T)}{P(X \leq T)} \) of the truncated probability mass under the untruncated Pareto-type distribution (10). If the underlying \( X \) is truncated at a quantile level \( T = Q(1 - \pi) \), then \( D_T = \pi / (1 - \pi) \). Hence asymptotic conditions on \( p / D_T \) as \( p \to 0 \) and \( T \to \infty \), amount to conditions on the relative behaviour between the odd’s ratios \( \pi / (1 - \pi) \) and \( p / (1 - p) \), i.e. between the truncated probability and the tail probability \( p \) of interest. For instance, if \( p \) is negligible compared to \( D_T \), or \( p / D_T \to 0 \), then the truncation is significant with respect to the quantile estimation exercise.

As suggested in Clark (2013) \( \alpha \) could also be taken to be zero or negative. For instance formally setting \( \alpha = -1 \) in (2), one obtains the tail of a uniform type distribution. Finite tail distributions following (13) with negative value of \( \alpha \) show a fast rate of convergence to \( T \) when \( p \to 0 \) and \( T \) is a big number (due to the presence of \( D_T \) in (13)). In the applications we have in mind here, convergence to \( T \) is slow and hence a positive value of \( \alpha \) is appropriate. Moreover we allow the truncation point \( T \) to be large, expressed by \( T \to \infty \). In an asymptotic setting this means that we consider a sequence of models indexed by the truncation point. This approach is new and allows to bridge tail models with \( \xi = -1 \) and Pareto models. In this setup we improve upon the well-established extreme quantile estimation methods. Moreover in developing statistical methods we consider the cases \( k / (nD_T) \to 0 \), respectively \( k / (nD_T) \to k > 0 \) finite, and \( k / (nD_T) \to \infty \), corresponding to whether the truncated probability mass \( T^{-\alpha} \ell_F(T) \) is large, intermediate or small with respect to the proportion of data used in the extreme value estimation. The final case (with \( k / (nD_T) \to \infty \)) can then be considered adjacent to the (untruncated) Pareto-type models with \( \xi > 0 \) and appear in practice when \( T \) is so high corresponding to the data that no truncation effect is visible from the data.
In the next section we provide estimators for $T$ and for extreme quantiles. We also consider the problem of deciding between a Pareto-type (Pa) case (10) and a truncated Pareto-type (TPa) case (11). To this purpose we construct a truncated Pareto QQ-plot. This construction also allows an alternative motivation for the estimator $\hat{\alpha}_{r,k,n}$. In section 3 we discuss the asymptotic properties of $\hat{\alpha}_{r,k,n}$ and the extreme quantile estimators under (10), and (11). We also consider the effect of the trimming parameter $r$. Finally we conclude with simulation results and practical examples.

2 Statistical methods for truncated Pareto-type distributions

In case of the truncated strict Pareto model (see [2]) we have that

$$Q_T(1 - p) = C_T^{1/\alpha} (D_T + p)^{-1/\alpha} = T \left( 1 + \frac{p}{D_T} \right)^{-1/\alpha}.$$  \hspace{1cm} (14)

The estimation of $D_T$ hence is an intermediate step in important estimation problems following the estimation of $\alpha$, namely of extreme quantiles and of the endpoint $T$. From (14)

$$\left( \frac{Q_T(1 - \frac{k}{n})}{Q_T(1 - \frac{r}{n})} \right)^{\alpha} = \left( 1 + \frac{r}{nD_T} \right) \left( 1 + \frac{k}{nD_T} \right) = \frac{r}{k} \left( 1 + D_{T,n}^T \right) \left( 1 + D_{T,K}^n \right)$$

from which, estimating $\frac{Q_T(1 - \frac{k}{n})}{Q_T(1 - \frac{r}{n})}$ by $R_{r,k,n}$, we propose with $\lambda_{r,k} = r/(k + 1)$

$$\hat{D}_T := \hat{D}_{T,r,k,n} = \frac{k}{n} \frac{R_{r,k,n}^{\hat{\alpha}_{r,k,n}} - \lambda_{r,k}}{1 - R_{r,k,n}^{\hat{\alpha}_{r,k,n}}}$$ \hspace{1cm} (15)

as an estimation method for $D_T$ in case of truncated and non-truncated Pareto-type distributions. In practice we will make use of the admissible estimator

$$\hat{D}_T^{(0)} = \max \left\{ \hat{D}_T, 0 \right\}.$$  

In case $D_T > 0$, estimators of $T$ and extreme quantiles $q_p = Q(1 - p)$ can be constructed on the basis of (14), contrasting the expression for $Q_T(1 - p)$
with the corresponding expression at an anchor point $X_{n-k,n}$, which serves as the empirical estimator of $Q_T(1 - k/n)$:

$$
\log \hat{q}_{p,r,k,n} = \log X_{n-k,n} + \frac{1}{\hat{\alpha}_{r,k,n}} \log \left( \frac{\hat{D}_T + \frac{k}{n}}{\hat{D}_T + p} \right),
$$

(16)

$$
\hat{q}_{p,r,k,n} = X_{n-k,n} \left( \frac{k}{np} \right)^{1/\hat{\alpha}_{r,k,n}} \left( \frac{1 + \frac{n\hat{D}_T}{k}}{1 + \frac{p}{\hat{D}_T}} \right)^{1/\hat{\alpha}_{r,k,n}},
$$

(17)

$$
\log \hat{T}_{r,k,n} = \max \left\{ \log X_{n-k,n} + \frac{1}{\hat{\alpha}_{r,k,n}} \log \left( 1 + \frac{k}{n\hat{D}_T} \right), \log X_{n,n} \right\},
$$

(19)

where the expression for $\hat{T}_{r,k,n}$ follows from letting $p \to 0$ in the above expressions for $\hat{q}_{p,r,k,n}$. The maximum of the value following from (19) and $X_{n,n}$ is taken in order for this endpoint estimator to be admissible. It now follows that in case $\hat{D}_T > 0$

$$
\hat{q}_{p,r,k,n} = \hat{T}_{r,k,n} \left( 1 + \frac{p}{\hat{D}_T} \right)^{-1/\hat{\alpha}_{r,k,n}}
$$

which is consistent with (12). Expression (18) for $\hat{q}_{p,r,k,n}$ constitutes an adaptation of the Weissman (1978) estimator

$$
\hat{q}_{p,k,n}^{(W)} = X_{n-k,n} \left( \frac{k}{np} \right)^{H_{1,k,n}}
$$

(20)

under (10) to the truncated Pareto case, and is more adapted to the case $k/(nD_T) \to \infty$. Version (17) can be linked to the cases where $k/(nD_T)$ is bounded away from $\infty$. In practice we always use version (16) which can be applied in all cases. Remark that such alternative expressions do not exist for the estimation for the endpoint $T$ as in case $D_T = 0$ no finite endpoint exists.

Based on a chosen value $\hat{D}_{T,r,k^{*},n}$ for particular $k^{*}$, we propose the truncated Pareto (TPa) QQ-plot to verify the validity of (12):

$$
\left( \log X_{n-j+1,n}, \log \left( \hat{D}_{T,r,k^{*},n} + j/n \right) \right), \ j = 1, \ldots, n.
$$

(21)
Under (12) an ultimately linear pattern should be observed to the right of some anchor point, i.e. at the points with indices \(j = 1, \ldots, k\) for some \(1 < k < n\). From this, we propose to choose the value of \(k^*\) in practice as the value that maximizes the correlation between \(\log X_{n-j+1,n}\) and 
\[
\log \left( \hat{D}_{T,r,k^*,n} + j/n \right) \quad \text{for } j = 1, \ldots, k^* \quad \text{and } k > 10.
\]
Remark that when \(T = \infty\) or \(D_T = 0\) the TPa QQ-plot agrees with the classical Pareto QQ-plot
\[
(\log X_{n-j+1,n}, \log(j/n)) \quad j = 1, \ldots, n.
\]

We finally remark that in the spirit of the Hill estimator \(H_{1,k,n}\) which can be viewed as a slope estimator based on the Pareto QQ-plot (see for instance Beirlant et al. (1996) and Aban and Meerschaert (2004)), a simple estimator of \(\alpha\) can then be obtained as the ratio of the average increase in the vertical over the horizontal direction in the TPa QQ-plot to the right of an anchor point:
\[
\frac{H_{r,k,n}}{-\frac{1}{k_r} \sum r \log(1 + \frac{j}{nD_T}) + \log(1 + \frac{k+1}{nD_T})}.
\]
But as \(\hat{D}_r\) depends on \(\alpha\) an estimator of \(1/\alpha\) can then be found from the equation
\[
\frac{H_{r,k,n}}{-\frac{1}{k_r} \sum r \log(1 + \frac{j}{nD_{T,\alpha}}) + \log(1 + \frac{k+1}{nD_{T,\alpha}})} = \frac{1}{\alpha}.
\]
with \(D_{T,\alpha} = \frac{k}{n} \frac{R_{r,k,n}^\alpha}{1 - R_{r,k,n}^\alpha} \lambda_{r,k}\). In Appendix 2 we derive however that the estimator obtained from solving (23) and \(\hat{\alpha}_{r,k,n}\) are asymptotically equivalent. The estimator based on (23) can be seen to correspond to the least squares slope estimator when the regression line is forced to pass through the anchor point.

### 3 Asymptotic distributions of the estimators

In this section we derive the large sample distribution of \(\hat{\alpha}_{r,k,n}\) and \(\hat{q}_{p,r,k,n}\) defined in (8) and (16) for truncated Pareto-type distributions in case \(k/(nD_T)\) is bounded away from \(\infty\), or when \(k/(nD_T) \to \infty\). The case of Pareto-type distributions (10) are then be considered as a limit case of \(k/(nD_T) \to \infty\). The proofs are deferred to Appendix.

We first develop more precise expressions for the quantile function \(Q_T(1-p)\) for truncated Pareto-type distributions assuming that \(F_T\) is continuous.
To this end set $\ell^{-\alpha}(x) = \ell_F(x)$ and let $\ell^*$ denote the de Bruyn conjugate of $\ell$. Solving the equation $y^{1/\alpha} = x\ell(x)$ by $x = y^{1/\alpha}\ell^*(y^{1/\alpha})$ (see for instance Proposition 2.5 and page 80 in Beirlant et al. (2004)), we find that the tail function $U_T(y) = Q_T(1 - \frac{1}{y})$ corresponding to $1 - F_T$ is given by

$$U_T(y) = \left(\frac{C_T y}{1 + (T\ell(T))^{-\alpha}}\right)^{-1/\alpha} \ell^* \left(\frac{1}{1 - \frac{1}{DTy}}\right)^{-1/\alpha}.$$ (24)

In order to derive asymptotic results for our estimators we have to consider the different cases for the balance between $1/y$ and $DT$ as $y,T \to \infty$. These are specified in the following Proposition. In the asymptotic results we will apply this with $y = n/k$.

**Proposition 1.**

(a) If $yDT$ is bounded away from 0 as $y,T \to \infty$, then

$$U_T(y) = T \left(1 + \frac{1}{DTy}\right)^{-1/\alpha} \{\ell(T)\ell^*(T\ell(T))\} \frac{\ell^* \left((T\ell(T))[1 + \frac{1}{DTy}]^{-1/\alpha}\right)}{\ell^*(T\ell(T))}.$$ (25)

where $\ell(T)\ell^*(T\ell(T)) \to 1$ and $\frac{\ell^* \left(T\ell(T)[1 + \frac{1}{DTy}]^{-1/\alpha}\right)}{\ell^*(T\ell(T))} \to 1$.

(b) If $yDT \to 0$ as $y,T \to \infty$, then

$$U_T(y) = (yC_T)^{1/\alpha} \ell^* \left((yC_T)^{1/\alpha}\right) \left(1 + DTy\right)^{-1/\alpha} \frac{\ell^* \left((yC_T)^{1/\alpha} \left[1 + DTy\right]^{-1/\alpha}\right)}{\ell^* \left((yC_T)^{1/\alpha}\right)}.$$ (26)

where $\frac{\ell^* \left((yC_T)^{1/\alpha} \left[1 + DTy\right]^{-1/\alpha}\right)}{\ell^* \left((yC_T)^{1/\alpha}\right)} \to 1$.

Remark that in case $yDT \to 0$ as $y,T \to \infty$ the tail function $U_T$ is asymptotically equivalent to the Pareto type tail function $(yC_T)^{1/\alpha} \ell^* \left((yC_T)^{1/\alpha}\right)$ which is the model corresponding to the case $T = \infty$. Hence in case (b) we have that $\xi = 1/\alpha > 0$, compared to the case where $yDT$ is bounded away from 0 in which case $\xi = -1$.

In order to derive the asymptotic results for the estimators we will make use of a second order slow variation condition on $\ell^*$ specifying the rate of
convergence of $\frac{\ell^*(tx)}{\ell^*(x)}$ to 1 as $x \to \infty$, which is used typically in all asymptotic results in extreme value methods (see for instance Theorem 3.2.5 in de Haan and Ferreira (2006)):

$$
\lim_{x \to \infty} \frac{1}{b^*(x)} \log \frac{\ell^*(tx)}{\ell^*(x)} = h_{\rho^*}(t)
$$

with $\rho^* < 0$, $h_{\rho^*}(t) = (t^{\rho^*} - 1)/\rho^*$, and $b^*$ regularly varying with index $\rho^*$.

Throughout we will also that as $r,k \to \infty$ for some $\lambda \in [0,1)$

$$
r/k - \lambda = O\left(\frac{1}{k}\right).
$$

Using (28) guarantees that we can interchange $r/k$ and $\lambda$ in the asymptotic results and proofs.

**Theorem 1.** Let (27) and (28) hold and let $n,k = k_n \to \infty$, $k/n \to 0$. Then

(a) if $k/(nD_T) \to 0$ and $(nD_T)/k^{3/2} \to 0$,

$$
\frac{1}{\hat{\alpha}_{r,k,n}} - \frac{1}{\alpha} = \left(\frac{nD_T}{k^{3/2}} \frac{1}{\alpha(1-\lambda)} \mathcal{N}_\lambda^{(1)} + b^*(T\ell(T))\left(\frac{\alpha-1}{\alpha} - \frac{\rho^*}{\alpha^2}\right)\right) (1 + o_p(1)),
$$

with

$$
\mathcal{N}_\lambda^{(1)} = \left(\frac{W(1) + W(\lambda)}{2} - \frac{1}{1-\lambda} \int_{\lambda}^{1} W(u)du\right) \sim \mathcal{N}(0, (1-\lambda)/12),
$$

(b) if $k/(nD_T) \to \kappa < \infty$,

$$
\frac{1}{\hat{\alpha}_{r,k,n}} - \frac{1}{\alpha} = \left(\frac{1}{\delta_{\kappa,\lambda}} \frac{\alpha}{\sqrt{k}} \left(-\frac{1}{1-\lambda} \int_{\lambda}^{1} W(u)d\log(1+\kappa u)
\right.
\right.
\left.
\left.
+ W(1) \left\{1 - \frac{1 + \kappa\lambda}{\kappa(1-\lambda)} \log\left(\frac{1 + \kappa}{1 + \kappa\lambda}\right)\right\}
\right.
\right.
\left.
\left.
- W(\lambda) \left\{1 - \frac{1 + \kappa}{\kappa(1-\lambda)} \log\left(\frac{1 + \kappa}{1 + \kappa\lambda}\right)\right\}\right)

+ b^*(T\ell(T))\left(\frac{\beta_{\kappa,\lambda}}{\delta_{\kappa,\lambda}}\right) (1 + o_p(1)),
$$
with asymptotic variance $\sigma^2_{\kappa,\lambda}/(k\alpha^2)$ and

$$
\beta_{\kappa,\lambda} = A_{\kappa,\lambda} - B_{\kappa,\lambda}c_{\kappa,\lambda},
$$

$$
A_{\kappa,\lambda} = \frac{1}{1-\lambda} \int_{\lambda}^{1} h_{\rho^*}([1+\kappa u]^{-1/\alpha}) du - h_{\rho^*}([1+\kappa]^{-1/\alpha}),
$$

$$
B_{\kappa,\lambda} = h_{\rho^*}([1+\kappa]^{-1/\alpha}) - h_{\rho^*}([1+\kappa\lambda]^{-1/\alpha}),
$$

$$
c_{\kappa,\lambda} = \frac{1 + \kappa\lambda}{(1-\lambda)\kappa} + \frac{(1+\kappa\lambda)(1+\kappa)}{(1-\lambda)^2\kappa^2} \log \left( \frac{1 + \kappa \lambda}{1 + \kappa} \right),
$$

$$
\sigma^2_{\kappa,\lambda} = \frac{1}{(1-\lambda)\delta_{\kappa,\lambda}},
$$

$$
\delta_{\kappa,\lambda} = 1 - \frac{(1 + \kappa\lambda)(1 + \kappa)}{(1-\lambda)^2\kappa^2} \log^2 \left( \frac{1 + \kappa}{1 + \kappa\lambda} \right).
$$

(c) if $k/(nD_T) \to \infty$ and $D_T = o((n/k)^{-1+\rho^*/\alpha})$

$$
1/\hat{\alpha}_{r,k,n} - 1/\alpha = \left( \frac{\sigma^2(\lambda)}{\alpha\sqrt{k}} N^{(1)}_\lambda + b^*((C_T n/k)^{1/\alpha}) \beta(\lambda) \right) (1 + o_p(1)),
$$

with

$$
N^{(2)}_\lambda = - \int_{\lambda}^{1} \frac{W(u)}{u} du + \frac{W(1)}{1-\lambda} (1 + \frac{\lambda}{1-\lambda} \log \lambda) - \frac{W(\lambda)}{1-\lambda} (1 + \frac{1}{1-\lambda} \log \lambda)
$$

$$
\sim N(0, \sigma^{-2}(\lambda)),
$$

and

$$
\beta(\lambda) = \left( 1 - \frac{\lambda \log^2 \lambda}{(1-\lambda)^2} \right)^{-1}
$$

$$
\left( \frac{1}{\rho^*(1-\frac{\rho^*}{\alpha})} \frac{1 - \lambda^{1-\frac{\rho^*}{\alpha}}}{1-\lambda} - \frac{1}{\rho^*} + \frac{\lambda}{1-\lambda} h_{\rho^*}(1/\lambda) \left( \frac{\log(\lambda)}{1-\lambda} + 1 \right) \right);
$$

$$
\sigma^2(\lambda) = \left( (1-\lambda)(1 - \frac{\lambda \log^2(\lambda)}{(1-\lambda)^2} \right)^{-1}.
$$

Here $\beta(0) = (\alpha(1 - \frac{\rho^*}{\alpha}))^{-1}$ and $\sigma^2(0) = 1$.

**Remark 1.** Theorem 1(a) entails that in case $k/(nD_T) \to 0$, $k$ should grow with $n$ to infinity as $n^{1-\eta}$ where $\eta < \frac{1}{3}$ in order to obtain a reasonable...
estimation rate. This means in practice that in case of a truncated Pareto-type distribution the number of extremes \( k \) should be taken large. Also the presence of \( D_T \) in the standard deviation guarantees even faster convergence for large values of \( T \). Moreover there is a bias of order \( b^*(T \ell(T)) \) which is only negligible if \( T \) is a reasonably large value and \( -\rho^* \) is sufficiently large.

**Remark 2.** Robustness under Pareto-type models has received quite some attention in the literature (see for instance Hubert et al. (2013) and the references therein) while the classical estimators such as Hill’s (1975) estimator are known to be highly non robust against outliers. The estimator \( \hat{\alpha}_{r,k,n} \) provides a way to robustify the Hill estimator \( H_{1,k,n} \) using a trimming procedure (with \( r > 1 \)). Trimming of course makes the estimator more robust against outliers, but decreases the efficiency of the estimator. This is illustrated in Figure 13 of Appendix 4, plotting the functions \( \sigma^2(\lambda) \) and \( \beta(\lambda) \) for \( \lambda \in [0, 1/4] \). The robustness properties of the estimation procedures presented here will be studied elsewhere.

**Remark 3.** In case \( k/(nD_T) \to \infty \) and \( \lambda = 0 \) the asymptotic result for \( \hat{\alpha}_{r,k,n} \) is identical to that of the Hill estimator \( H_{1,k,n} \) as given for instance in Beirlant et al. (2004), section 4.2. To see this remark that the main slowly varying component of the tail function \( U_T \) equals \( \ell_U(x) := \ell^*((C_T x)^{1/\alpha}) \). Based on (27) we find that for every \( t > 0 \) and \( x \to \infty \)

\[
\log \frac{\ell_U(tx)}{\ell_U(x)} \to b^*((C_T x)^{1/\alpha})h_{\rho^*}(t^{1/\alpha}) = b_U(x)h_{\rho^*/\alpha}(t)
\]

where \( b_U(x) = b^*((C_T x)^{1/\alpha})/\alpha \) is regularly varying with index \( \rho^*/\alpha \). Hence the asymptotic bias in Theorem 1(c) when \( \lambda = 0 \) equals \( b_U(n/k)/(1 - \frac{\rho^*}{\alpha}) \) which is the form found in literature for the bias of the Hill estimator.

Concerning asymptotic results for the extreme quantile estimator \( \hat{q}_{p,r,k,n} \) we confine ourselves to the cases \( k/(nD_T) \to 0 \) and \( k/(nD_T) \to \infty \) due to the complexity of the intermediate case. A similar result when \( k/(nD_T) \to \kappa \in (0, \infty) \) can readily be obtained using similar techniques as in the cases presented here.

**Theorem 2.** Let (27) and (28) hold and let \( n, k = k_n \to \infty \), \( k/n \to 0 \), and \( np_n = o(k) \). Then
(a) if \( k/(nD_T) \to 0 \) and \( (nD_T)/k^{3/2} \to 0 \)

\[
\log \hat{q}_{p,r,k,n} - \log q_p = O((k/nD_T)^2) + o \left( \frac{k}{nD_T} b^*(T \ell(T)) \right) + o_p \left( \frac{1}{\sqrt{k}} \right);
\]

(b) if \( p_n/D_T \to \infty \), \( D_T = o((n/k)^{-1+\rho^*/\alpha}) \), \( \log(np_n) = o(\sqrt{k}) \), and in case \( \lambda > 0 \) assuming \( \frac{nD_T}{k} \left( \sqrt{k} + b^*((C_T n/k)^{1/\alpha}) \right) \to \infty \),

\[
\log \hat{q}_{p,r,k,n} - \log q_p
= \log \left( \frac{k}{np_n} \right) \left\{ \frac{\sigma^2(\lambda)}{\alpha \sqrt{k}} \mathcal{N}^{(1)}_\lambda + b^*((C_T n/k)^{1/\alpha}) \beta(\lambda) \right\} (1 + o_p(1))
\]

\[
- \frac{k}{n} (p + D_T)^{-1} \frac{\lambda}{\alpha (1 - \lambda)} \left\{ \frac{1}{\sqrt{k}} \mathcal{N}^{(3)}_\lambda - \frac{\log(\lambda)\sigma^2(\lambda)}{\sqrt{k}} \mathcal{N}^{(2)}_\lambda - ab^*((C_T n/k)^{1/\alpha}) \zeta(\lambda) \right\} (1 + o_p(1))
\]

where \( \mathcal{N}^{(3)}_\lambda = W(\lambda)/\lambda - W(1) \sim \mathcal{N}(0, (1 - \lambda)/\lambda) \) and \( \zeta(\lambda) = (\log(\lambda))\beta(\lambda) + h_{\rho^*/\lambda}(\lambda^{-1}) \).

**Remark 4.** In case \( k/(nD_T) \to 0 \) both the asymptotic bias and the stochastic part of \( \hat{q}_p \) are of lower order than the asymptotic bias of the estimator \( 1/\hat{\alpha}_{r,k,n} \). This is also confirmed by the simulation results in section 4 where the plots of the quantile estimators are found to be quite horizontal as a function of \( k \), compared to other estimators found in extreme value analysis. In case \( k/(nD_T) \to \infty \), the second term on the right hand side of the asymptotic expansion disappears in case of no trimming, i.e. when \( \lambda = 0 \), and hence the condition \( \frac{nD_T}{k} \left( \sqrt{k} + b^*((C_T n/k)^{1/\alpha}) \right) \to \infty \) can disappear from the statement of the theorem. In fact a different behaviour of the extreme quantile estimator is observed in the simulations (see Figures 11 and 12 when trimming the data. When taking \( \lambda = 0 \) the asymptotic distribution of \( \hat{q}_p \) is comparable to the asymptotic result obtained for the Weissman estimator (see for instance Beirlant et al. (2004), section 4.6).

## 4 Practical examples and simulations

For a first illustration we use the data set containing fatalities due to large earthquakes as published by the U.S. Geological Survey on http://earthquake.
usgs.gov/earthquakes/world/ which were also used in Clark (2013). It contains the estimated number of deaths for the 124 events between 1900 and 2011 with at least 1000 deaths.

In Figure 1 (left) the Pareto QQ plot (or log-log plot) is given. A curvature is appearing at the largest observations which indicates that the unbounded Pareto pattern could be violated in this example. On this plot the extrapolations using a Pareto distribution (linear pattern) and a truncated Pareto model using the truncated Pareto model are plotted based on the largest 21 data points as it was proposed in Clark (2013).

In Figure 2 the estimates and are plotted against . Here we have chosen as a typical value where both plots are horizontal in . Also the value used in Clark (2013) using the largest 21 observations is indicated. The truncated Pareto QQ-plot is given in Figure 1 (right), using the above mentioned value . Finally in Figure 3 the estimates of the extreme quantile using (16) and the endpoint using (19) are presented as a function of . They are contrasted with the values obtained by the classical method of moment estimates as introduced in Dekkers et al. (1989) illustrating the slow convergence of the classical extreme value methods in the truncated Pareto type model we study here. For any real EVI, the classical moment estimator is defined by

\[ \hat{\xi}_{n,k}^{\text{MOM}} := M_{n,k}^{(1)} + \hat{\xi}_{n,k}^{-}, \quad \hat{\xi}_{n,k}^{-} := 1 - \frac{1}{2} \left[ 1 - \left( \frac{M_{n,k}^{(1)}}{M_{n,k}^{(2)}} \right)^{1} \right], \]

with \[ M_{n,k}^{(j)} := \frac{1}{k} \sum_{i=0}^{k-1} \ln^{j} (X_{n-i,n}/X_{n-k,n}), \quad j = 1, 2, \]

which constitutes a consistent estimator for \( \xi \). The Hill estimator is \( M_{n,k}^{(1)} = H_{1,k,n} \). The MOM-estimators for high quantiles and right endpoint, based on the moments estimator \( \hat{\xi}_{n,k}^{\text{MOM}} \), are defined by (see de Haan and Ferreira (2006), §4.3.2, for details).

\[ q_{p}^{\text{MOM}} := X_{n-k,n} + X_{n-k,n}M_{n,k}^{(1)}(1 - \hat{\xi}_{n,k}^{-}) \left( \frac{\hat{\xi}_{n,k}^{-}}{\hat{\xi}_{n,k}^{\text{MOM}}} \right) - 1 \]

and

\[ \hat{T}^{\text{MOM}} := \max \left( \hat{T}^{(M)}, X_{n,n} \right), \quad \hat{T}^{(M)} := X_{n-k,n} - \frac{X_{n-k,n}M_{n,k}^{(1)}(1 - \hat{\xi}_{n,k}^{-})}{\hat{\xi}_{n,k}^{\text{MOM}}}. \]

Notice that in (31) \( \hat{T}^{\text{MOM}} \) corresponds to the admissible version of the moment endpoint estimator \( \hat{T}^{(M)} \), since the latter can return values below the
sample maximum. If we focus on Figure 3 (right) it is clear that the $\hat{T}_{MOM}$ does not improve the information given by the sample maximum, for a large range of thresholds $k$, contrasting with the behaviour of the proposed $\hat{T}_{1,k,n}$. Concerning the high quantile estimation, the chosen value $p = 0.01$ is directly related with the modest sample size here of $n = 124$. Similar to the endpoint estimation, for this data set the new quantile estimates $\hat{q}_{0.99,1,k}$ also reveals a stable pattern on $k$, in Figure 3 (left). Overall we can conclude that the truncated Pareto-type model with a truncation point $T$ around 400,000 deaths offers a convincing fit.

Another example where the truncated Pareto-type model is fitting well to the tail is found with the distribution of seismic moments of shallow earthquakes at depth less than 70 km, between 1977 and 2000, the data of which can be found in Pisarenko and Sornette (2003). The tails of these distributions were also considered in Section 6.3 in Beirlant et al. (2004) both for subduction and mid ocean ridge zones. Here we concentrate on the subduction zone data. In Beirlant et al. (2004), page 200, the use of $k = 1157$ is suggested in order to obtain a proper fit to the upper tail of the underlying distribution. The tail fit is revisited here in Figure 4 (left) using truncated and non-truncated Pareto models. For this data set, in Figure 4 (right), the truncated Pareto (TPa) QQ-plot (21), associated with the validity of (12), has been built on the chosen value $k^* = 3981$, which maximizes the correlation between $\log X_{n-j+1,n}$ and $\log (\hat{D}_T + j/n)$, $j = 1, \cdots, k$, for $10 < k < n$.

Figure 1: Log-log QQ-plot (left) of the earthquake fatalities with extrapolations anchored at $\log(X_{n-21,n})$ based on a non-truncated Pareto model (1) (dotted line) and a truncated Pareto model (21) (full line). TPa QQ-plot (21) (right) for the earthquake fatalities data set using $r = 1$ and $k^* = 100$. 
Figure 2: Plots of $\hat{\alpha}_{1,k,n}$ and $\hat{D}_{T,1,k}$ ($k = 1, \ldots, 124$) for the earthquake fatalities data set, marking the values at $k = 21, 100$.

Figure 3: Quantile estimates $\hat{q}_{0.01,1,k,n}$ (16) (left) and $\hat{T}_{1,k,n}$ (19) (right) for the earthquake fatalities data set, contrasted with the method of moments quantile and endpoint estimators, in (30) and (31), respectively.
Figure 4: Log-log QQ-plot (left) of the seismic moment with extrapolations based on a non-truncated Pareto model (1) (dotted line) and a truncated Pareto model (2) (full line). TPa QQ-plot (21) (right) for the seismic moments data set using $r = 1$ and the top $k^* = 3981$ data.

Figure 5: Plots of $\hat{\alpha}_{1,k,n}$ and $\hat{D}_{T,1,k}$ for the seismic moments data set.

Figure 6: Quantile estimates $\hat{q}_{0.0005,1,k}$ (left) and $\hat{T}_{1,k}$ (right) for the seismic moments data set, contrasted with the method of moments quantile and endpoint estimators, in (30) and (31), respectively.
The finite sample behaviour of the proposed estimators $\hat{\alpha}_{r,k,n}$ based on (8) and (9), $\hat{q}_{p,r,k,n}$ from (16), and $\hat{T}_{r,k,n}$ from (18) has been studied through an extensive Monte Carlo simulation procedure, both for truncated and non-truncated Pareto-type distributions.

Here we will only present results concerning Pareto and Burr distributions, with truncated and non-truncated versions:

1. **Non-truncated models**

   (a) **Pareto**($\alpha$), $\alpha = 1, 2$
   
   $F(x) = 1 - x^{-\alpha}, \ x > 1, \ \alpha > 0,$ \hspace{1cm} (32)

   (b) **Burr**($\alpha, \rho$), $\alpha = 1, 2, \ \rho = -1$
   
   $F(x) = 1 - (1 + x^{-\rho \alpha})^{1/\rho}, \ x > 0, \ \rho < 0, \ \alpha > 0.$ \hspace{1cm} (33)

2. **Truncated models**

   (a) **Truncated-Pareto**($\alpha, T$), $\alpha = 2$ and $T$ a high quantile from the corresponding Pareto model (32)
   
   $F(x) = \frac{1 - x^{-\alpha}}{1 - T^{-\alpha}}, \ 1 < x < T, \ \alpha > 0.$ \hspace{1cm} (34)

   Here we use $T = 3.1623$, respectively $T = 1.4142$, the 90 percentile, respectively the median, of the corresponding non-truncated Pareto model.

   (b) **Truncated-Burr**($\alpha, \rho, T$), $\alpha = 2, \ \rho = -1$ and $T$ a high quantile from the corresponding Burr model (33)
   
   $F(x) = \frac{1 - (1 + x^{-\rho \alpha})^{1/\rho}}{1 - (1 + T^{-\rho \alpha})^{1/\rho}}, \ 0 < x < T, \ \rho < 0, \ \alpha > 0.$ \hspace{1cm} (35)

   Here we use $T = 3$, being the 90 percentile of the corresponding non-truncated Burr distribution.

   Remark that in case of (35) $\ell^*(y) = 1 + \frac{1}{\rho} y^{\alpha \rho} (1 + o(1))$ when $y \to \infty$ and $\rho^* = \alpha \rho$. 

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For a particular data set from an unknown but apparently heavy-tailed distribution, the practitioner does not know if the distribution comes from a truncated or a non-truncated Pareto-type distribution and hence we have to study the behaviour of the proposed estimators under both cases, and compare them with the existing extreme value estimators. As mentioned before, truncated Pareto-type distributions belong to the Weibull domain of attraction for maxima with EVI $\xi = -1$ so that the moment estimator (29) almost surely converges to -1. Also, for these models, $1/H_{1,k,n}$ does not constitute a consistent estimator neither for $\alpha$ nor for $\xi$, since in case $\xi < 0$ the Hill estimator $H_{1,k,n}$ almost surely tends to zero when $k/n \to 0$ as $k,n \to \infty$. Only when $T = \infty$ we have that $\hat{\alpha}_{r,k,n}$ and $1/H_{1,k,n}$ estimate the same value $1/\xi$.

When estimating an extreme quantile the estimator (30) based on the moment estimator is designed both for truncated and non-truncated cases and is to be compared with the estimation procedure defined in (18). The same holds for endpoint estimators (19) and (31) in case of truncated models. Finally $\hat{q}_{p,r,k,n}$ and the Weissman (1978) extreme quantile estimator $\hat{q}_{p,k,n}^{(W)}$ from (20) are competitors in case of non-truncated Pareto-type distributions only.

In Figures 11-12, the “trimmed-Hill, not corrected” refers to $H_{r,k,n}$, which coincides with the Hill estimator for $r = 1$. The $\alpha$-estimator $\hat{\alpha}$ is the solution of (8), approximated using the Newton-Raphson iteration as in (9), with an initial value $\hat{\alpha}^{(0)} = 1/H_{r,k,n}$.

In Figures 8-10 we present the relative performance of the estimators with $r = 1$ and $r = 10$ for the truncated Pareto model (34) and the truncated Burr model (35), while in Figures 11-12 we consider the corresponding non-truncated models (32) and (33). Observe that $\hat{\alpha}_{r,k,n}$ appears to be not too
sensitive for small changes of \( r \). It appears that when the model is of Pareto-type, whether truncated or not, the estimators proposed here are performing well. Adding to this, in Figure 8 the EVI moment estimator systematically overestimates the true value of \( \xi = -1 \) for these upper tail truncated distributions as in Figure 7 (left). Only for a lower truncation point, for instance equal to the median of Pareto(\( \alpha = 2 \)) (see also Figure 7 (right)), the situation naturally improves for the MOM-estimators (Figure 9 (up)). This confirms that the methods proposed here are especially useful for truncated Pareto-type models with a truncation point \( T \) equal to a high quantile of the corresponding non-truncated Pareto-type distribution. On the other hand,
for these truncated Pareto type models, the convergence of the new quantile and endpoint estimators seems to be attained at lower thresholds (or high $k$) with high accuracy, contrasting with higher thresholds (or lower $k$) for MOM class estimators. With quantile estimation in Figures 8-10 an erratic behaviour appears for some smaller or larger values of $k$ which becomes more apparent when $T$ corresponds to lower quantiles of the underlying Pareto distribution. This is a consequence of the use of $\hat{D}_T^{(0)} = \max\{\hat{D}_T, 0\}$ rather than $\hat{D}_T$ in practice. If we assume that $T$ is finite then using simply $\hat{D}_T$ rather than $\hat{D}_T^{(0)}$ produces much smoother performance in extreme quantile estimation. On the other hand in case of non-truncated models the use of $\hat{D}_T$ instead of $\hat{D}_T^{(0)}$, leads to extreme quantile estimates that are quite sensitive with respect to the value of $D_T$. While the stable parts in the plots of quantile estimate are readily apparent anyway, we here use $\hat{D}_T^{(0)}$ in (16).

In case of non-truncated Pareto-type models (see Figures 11 and 12) concerning high quantile estimation, we also consider the Weissman (1978) estimator $\hat{q}_p^{(W)}$ defined in (20) besides the newly proposed estimator $\hat{q}_{p,r,k}$ and the MOM-estimator $\hat{q}_p^{\text{MOM}}$, defined in (16) and (30) respectively. Taking into account that $H_{1,k,n}$ and $\hat{q}_p^{(W)}$ are designed for this particular situation, we can conclude that the newly proposed estimators perform reasonably well here if we compare with the classical extreme value estimators $H_{1,k,n}$ and $\hat{\xi}_{n,k}$. However, on average $\hat{q}_{p,r,k}$ underestimates $Q(0.999)$, taking into account Figure 11 (left). Nevertheless this larger bias is balanced by a lower variance, which results in a MSE competitive with the one from MOM-class $\hat{q}_p^{\text{MOM}}$. Finally remark that trimming has a large influence on the estimation of extreme quantiles in non-truncated Pareto-type distributions. This corresponds to the asymptotic result in Theorem 2(b) where an additional term is present in case $\lambda > 0$. 


Figure 9: Truncated-Pareto(\(\alpha = 2, T = 1.4142\)): Estimation of \(\alpha\) using the
Newton-Raphson procedure with initial value \(\hat{\alpha}^{(0)} = 1/H_{r,k,n}\), \(r = 1, 10\) (up).
Estimation of the high quantile \(q_{0.001}\) (center) and right endpoint \(T\) (down)
using \(q_{0.001,r,k}\) defined in (16) and \(T_{r,k}\) as in (19), \(r = 1, 10\).
Figure 10: Truncated-Burr($\alpha = 2, \rho = -1, T = 3$): Estimation of $\alpha$ (up), high quantile $q_{0.001}$ (center) and right endpoint $T$ (down) using $\hat{\alpha}_{r,k,n}$, $\hat{q}_{0.001,r,k}$ defined in (16) and $\hat{T}_{r,k,n}$ as in (19), for $r = 1, 10$. 
Figure 11: Pareto($\alpha = 2$): Estimation of $\alpha$ using the Newton-Raphson procedure with initial value $\hat{\alpha}^{(0)} = 1/H_{r,k,n}$ (up), $r = 1, 10$. Estimation of high quantile $q_{0.001}$ using $\hat{q}_{0.001,1,k}$ defined in (16) (down).

Figure 12: Burr($\alpha = 2; \rho = -1$): Estimation of $\alpha$ using the Newton-Raphson procedure with initial value $\hat{\alpha}^{(0)} = 1/H_{r,k,n}$ (up), $r = 1, 10$. Estimation of high quantile $q_{0.001}$ using $\hat{q}_{0.001,1,k}$ defined in (16) (down).
5 Conclusion

We have proposed extreme value methods that can be used both for unbounded Pareto-type distributions and truncated Pareto-type distributions. These methods are especially designed for cases where the truncation point is rather high, which is mathematically expressed by letting \( T \) tend to \( \infty \) in the asymptotics. Several possible areas for new research appear from this work. For instance linking truncation with all domains of attraction for maxima, especially in case of the Gumbel domain of attraction with \( \xi = 0 \). Also bringing in covariate information in the model appears of importance. For instance modelling large earthquakes using geographical information is a problem of interest. Finally the robustness properties of the estimators proposed here should be studied further.

Appendix 1. Derivation of Proposition 1

Set \( \ell^{-\alpha}(x) = \ell_F(x) \) and let \( \ell^* \) denote the de Bruyn conjugate of \( \ell \). Solving the equation \( y^{1/\alpha} = x\ell(x) \) by \( x = y^{1/\alpha}\ell^*(y^{1/\alpha}) \) (see for instance Proposition 2.5 and page 80 in Beirlant et al. (2004)), we find that the tail function \( U_T(y) = Q_T(1 - \frac{1}{y}) \) corresponding to \( 1 - F_T \) is given by

\[
U_T(y) = ((C_T y)^{-1} + (T\ell(T))^{-\alpha})^{-1/\alpha}\ell^*((1/(C_T y) + (T\ell(T))^{-\alpha})^{-1/\alpha}).
\] (36)

First consider the case where \( yD_T \) is bounded away from 0 as \( y,T \to \infty \). Then apply \( [(C_T y)^{-1} + (T\ell(T))^{-\alpha}]^{-1/\alpha} = T\ell(T)\left[1 + \frac{1}{D_T y}\right]^{-1/\alpha} \) twice so that

\[
U_T(y) = T\ell(T)\left(1 + \frac{1}{D_T y}\right)^{-1/\alpha}\ell^*\left(T\ell(T)\left[1 + \frac{1}{D_T y}\right]^{-1/\alpha}\right),
\]

while multiplying by \( \ell^*(T\ell(T)) \) on both the top and bottom leads to

\[
U_T(y) = T\left(1 + \frac{1}{D_T y}\right)^{-1/\alpha}\{\ell(T)\ell^*(T\ell(T))\}^{\ell}\frac{T\ell(T)[1 + \frac{1}{D_T y}]^{-1/\alpha}}{\ell^*(T\ell(T))}.
\]

As \( T \to \infty \) we have \( \ell(T)\ell^*(T\ell(T)) \to 1 \) by the definition of the de Bruyn conjugate. Assuming that for some constants \( 0 < m < M < \infty \) we have \( m < D_T y < M \) as \( y,T \to \infty \), then it follows from the Uniform Convergence Theorem for regularly
varying functions (Seneta, 1976) that
\[
\frac{\ell^* \left( T \ell(T) \left[ 1 + \frac{1}{DTy} \right]^{-1/\alpha} \right)}{\ell^* (T \ell(T))} \to 1. \tag{37}
\]

(37) clearly also holds when \( yDT \) tends to \( \infty \).

Alternatively, when \( yDT \to 0 \) as \( y,T \to \infty \), use
\[
\frac{\ell^* \left( (yC_T)^{1/\alpha}[1+yDT]^{-1/\alpha} \right)}{\ell^* (yC_T)^{1/\alpha}}\to 1.
\]

(38)

Multiplying by \( \ell^* ((yC_T)^{1/\alpha}) \) on both the top and bottom leads to (26). Finally it follows that
\[
\ell^* \left( (yC_T)^{1/\alpha}[1+yDT]^{-1/\alpha} \right) \to 1 \text{ as } (yC_T)^{1/\alpha} \to \infty \text{ and } [1+yDT]^{-1/\alpha} \to 1.
\]

**Appendix 2. Outline of proof of Theorems 1 and 2**

The Mean Value Theorem implies that
\[
1/\hat{\alpha}_{r,k,n} - 1/\alpha = -f(1/\alpha)/f'(1/\hat{\alpha}) \text{ where } \hat{\alpha} = \tilde{\alpha}_{r,k,n} \text{ is between } \alpha \text{ and } \hat{\alpha}_{r,k,n}, \text{ where } f(1/\alpha) = H_{r,k,n} - 1/\alpha - \frac{R_{r,k,n}^\alpha}{1-R_{r,k,n}^\alpha} \log R_{r,k,n}.
\]

Then we obtain that the limit distribution of \( 1/\hat{\alpha}_{r,k,n} \) is found from the asymptotic distribution of
\[
(1 - \hat{\alpha}^2 \frac{R_{r,k,n}^\alpha \log^2 R_{r,k,n}}{(1 - R_{r,k,n}^\alpha)^2})^{-1} \left( H_{r,k,n} - 1/\alpha - \frac{R_{r,k,n}^\alpha \log R_{r,k,n}}{1 - R_{r,k,n}^\alpha} \right).
\]

Hence the asymptotic behaviour of \( H_{r,k,n} \) and \( \log R_{r,k,n} \) constitute essential building blocks in the derivation of the asymptotics for \( \hat{\alpha}_{r,k,n} \). We consider these in the following Propositions.

For this we make use of the result (see de Haan and Ferreira (2006, Theorem 7.2.2)) that for some standard Wiener process \( W \) (with \( E(W(s)W(t)) = \min(s,t) \)) we have uniformly over all \( j = 1, \ldots, k \), as \( k, n \to \infty \), \( k/n \to 0 \)
\[
\sqrt{k} \left( \frac{n}{k} U_{j,n} - \frac{j}{k} \right) - W(\frac{j}{k}) \to_p 0. \tag{38}
\]

**Proposition 2.** Let (27) and (28) hold and let \( n, k = k_n \to \infty \), \( k/n \to 0 \). Then
(a) if $k/(nD_T)$ is bounded away from $\infty$, 
\[
H_{r,k,n} = \frac{1}{\alpha} \left( 1 - \frac{1}{1 - \lambda_{r,k}} \frac{1 + (k/nD_T)\lambda_{r,k}}{k/(nD_T)} \log \left( \frac{1 + k/(nD_T)}{1 + k/(nD_T)\lambda_{r,k}} \right) \right) \\
+ \frac{1}{\alpha} \left( W(1) \frac{k/(nD_T)}{1 + k/(nD_T)} - \frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} W(u) d\log(1 + \frac{k}{nD_T}u) \right) (1 + o_p(1)) \\
+ b^*(T(T)) A_{r,k,n,T} (1 + o(1)).
\]

where 
\[
A_{r,k,n,T} = \frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} h_{r,k}^* \left( [1 + (k/(nD_T))u]^{-1/\alpha} \right) du - h_{r,k}^* \left( [1 + (k/(nD_T))]^{-1/\alpha} \right)
\]

(b) if $k/(nD_T) \to \infty$
\[
H_{r,k,n} = \frac{1}{\alpha} \left( 1 + \lambda_{r,k} \log \frac{\lambda_{r,k}}{1 - \lambda_{r,k}} \right) + \frac{1}{\alpha} \frac{1}{\sqrt{k}} \left( W(1) - \frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} \frac{W(u)}{u} du \right) (1 + o_p(1)) \\
+ \left( \frac{1}{2\alpha} \frac{nD_T}{k} (1 - \lambda_{r,k}) + b^* \left( C_T^2(n/k)^{1/2} \right) \right) \frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} h_{r,k}^* (u^{-1/\alpha}) du (1 + o(1)).
\]

**Proof** Let $j_r = j - r + 1$ and let $U_{1,n} \leq U_{2,n} \leq \ldots \leq U_{n,n}$ denote order statistics from an i.i.d. sample of size $n$ from the uniform $(0,1)$ distribution. Then using summation by parts and the fact that $X_{n-j+1,n} = d Q_T(1 - U_{j,n}) (j = 1, \ldots, n)$

\[
H_{r,k,n} = \frac{1}{k_r} \sum_{j=r}^{k} j_r (\log X_{n-j+1,n} - \log X_{n-j,n}) \\
= -\frac{1}{k_r} \sum_{j=r}^{k} j_r (\log Q_T(1 - U_{j+1,n}) - \log Q_T(1 - U_{j,n})) \\
= -\frac{k + 1}{1 - \lambda_r} \frac{1}{k + 1} \sum_{j=r}^{k} \left\{ \frac{j + 1}{k + 1} - \lambda_{r,k} \right\} \int_{U_{j,n}}^{U_{j+1,n}} d\log Q_T(1 - w).
\]

Using (38) $H_{r,k,n}$ can now be approximated as $k, n \to \infty$ by the integral

\[
-\frac{k}{1 - \lambda_r} \int_{\lambda_{r,k}}^{1} (u - \lambda_{r,k}) \left\{ \int_{u+W(u)/\sqrt{k}+1/k}^{u+W(u)/\sqrt{k}} d\log Q_T(1 - \frac{k}{n}w) \right\} du.
\]

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Using the mean value theorem on the inner integral between $u + W(u)/\sqrt{k}$ and $u + W(u)/\sqrt{k} + 1/k$, followed by an integration by parts, we obtain

$$-\frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} (u - \lambda_{r,k})d \log Q_T(1 - \frac{k}{n}(u + \frac{W(u)}{\sqrt{k}}))$$

$$= - \log Q_T(1 - \frac{k}{n}(1 + \frac{W(1)}{\sqrt{k}})) + \frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} \log Q_T(1 - \frac{k}{n}(u + \frac{W(u)}{\sqrt{k}}))du.$$

(39)

First, let $k/(nD_T)$ be bounded away from $\infty$. Then using Proposition 1(a) the approximation (39) of $H_{r,k,n}$ equals

$$\frac{1}{\alpha} \log(1 + \frac{k}{nD_T}(1 + \frac{W(1)}{\sqrt{k}})) - \log \ell^*(T\ell(T)[1 + \frac{k}{nD_T}(1 + \frac{W(1)}{\sqrt{k}})]^{-1/\alpha})$$

$$- \frac{1}{\alpha} \int_{\lambda_{r,k}}^{1} \log \left(1 + \frac{k}{nD_T}(u + \frac{W(u)}{\sqrt{k}})\right) du$$

$$+ \frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} \log \ell^*(T\ell(T)[1 + \frac{k}{nD_T}(u + \frac{W(u)}{\sqrt{k}})]^{-1/\alpha}) du.$$

Subtracting $\log \ell^*(T\ell(T))$ from both the second and fourth line, and using the approximations

$$\log \left(1 + \frac{k}{nD_T}(u + \frac{W(u)}{\sqrt{k}})\right) = \log(1 + \frac{k}{nD_T}u) + \frac{W(u)}{\sqrt{k}} \frac{k}{nD_T}u, 0 < u \leq 1,$$

with $u^*$ between $u$ and $u + W(u)/\sqrt{k}$, and

$$\frac{\ell^*(T\ell(T)[1 + \frac{k}{nD_T}(u + \frac{W(u)}{\sqrt{k}})]^{-1/\alpha})}{\ell^*(T\ell(T))} = b^*(T\ell(T))h_{\rho^*}\left(1 + \frac{k}{nD_T}u\right)^{-1/\alpha}(1 + o_p(1))$$

we finally obtain the stated result in (a).

Secondly, consider $k/(nD_T) \to \infty$. Then using Proposition 1(b) we obtain using
similar steps as in the preceeding case that approximation \(39\) of \(H_{r,k,n}\) equals

\[
\begin{align*}
\frac{1}{\alpha} \log(1 + \frac{W(1)}{\sqrt{k}}) \\
+ \frac{1}{\alpha} \log(1 + \frac{nD_T}{k} (1 + \frac{W(1)}{\sqrt{k}})^{-1}) \\
+ \frac{1}{\alpha} \frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} \log(u + \frac{W(u)}{\sqrt{k}})du \\
- \frac{1}{\alpha} \frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} \log(1 + \frac{nD_T}{k} (u + \frac{W(u)}{\sqrt{k}})^{-1})du \\
+ \frac{1}{\alpha} \frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} \log \left( \frac{\ell^*(C_T^{-1} (n/(ku))^{\frac{1}{2}} [1 + nD_T/(ku)]^{-1/\alpha})}{\ell^*(C_T^{-1} (n/k)^{\frac{1}{2}} [1 + nD_T/k]^{-1/\alpha})} \right) du.
\end{align*}
\]

This is now approximated by

\[
\begin{align*}
\frac{1}{\alpha} \left( 1 + \frac{\lambda \log \lambda}{1 - \lambda} \right) + \frac{1}{\alpha} \frac{1}{\sqrt{k}} \left( W(1) - \frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} \frac{W(u)}{u} du \right) \\
+ \frac{1}{2\alpha} \frac{nD_T}{k} (1 - \lambda_{r,k}) \\
+ b^* (C_T^{-1} (n/k)^{\frac{1}{2}}) \frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} k_{\rho^*} \left( u^{-1/\alpha} \left( \frac{1 + nD_T/(ku)}{1 + nD_T/k} \right)^{-1/\alpha} \right) du.
\end{align*}
\]

The result then follows by approximating \(\frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} k_{\rho^*} (u^{-1/\alpha} \left( \frac{1 + nD_T/(ku)}{1 + nD_T/k} \right)^{-1/\alpha}) du\) by \(b^* (C_T^{-1} (n/k)^{\frac{1}{2}}) \frac{1}{1 - \lambda_{r,k}} \int_{\lambda_{r,k}}^{1} k_{\rho^*} (u^{-1/\alpha})du\).

In a similar way we obtain an asymptotic result for \(\log R_{r,k,n}\).

**Proposition 3.** Let \((27)\) and \((28)\) hold and let \(n,k = k_n \to \infty, k/n \to 0\). Then

(a) if \(k/(nD_T)\) is bounded away from \(\infty\),

\[
\begin{align*}
\log R_{r,k,n} &= \frac{1}{\alpha} \log \left( 1 + \frac{k/(nD_T)}{1 + \frac{k}{nD_T}} \lambda_{r,k} \right) \\
&- \frac{1}{\alpha \sqrt{k}} \left( W(1) \frac{k/(nD_T)}{1 + k/(nD_T)} - W(\lambda_{r,k}) \frac{k/(nD_T)}{1 + \lambda_{r,k} k/(nD_T)} \right) (1 + o_p(1)) \\
&+ b^* (T\ell(T)) B_{r,k,n,T} (1 + o(1)).
\end{align*}
\]

where

\[
B_{r,k,n,T} = h_{\rho^*} ([1 + (k/(nD_T))]^{-1/\alpha}) - h_{\rho^*} ([1 + (k/(nD_T))\lambda_{r,k}]^{-1/\alpha})
\]
(b) if \( k/(nD_T) \to \infty \)

\[
\log R_{r,k,n} = \frac{1}{\alpha} \log \lambda_{r,k} + \frac{1}{\alpha \sqrt{k}} (W(\lambda_{r,k})/\lambda_{r,k} - W(1)) (1 + o_p(1))
+ \left(-b^* (C_T^{1/\alpha} (n/k)^{1/\alpha}) R_{r,k,n} + \frac{1}{\alpha} D_T (n/k)(\lambda_{r,k}^{-1} - 1) \right) (1 + o(1))
\]

**Proof of Theorem 1**  
First we derive the consistency of \( \hat{\alpha}_{r,k,n} \) under the conditions of Theorem 1, so that then \( \tilde{\alpha} \to p \alpha \). Aban et al. (2006, see A.4) showed that \( \tilde{f}(t) := \frac{1}{t} + \frac{R_{r,k,n} \log R_{r,k,n}}{1 - R_{r,k,n}} - H_{r,k,n} \) is a decreasing function in \( t \in (0, \infty) \). Moreover \( \lim_{t \to \infty} \tilde{f}(t) = -H_{r,k,n} < 0 \) and \( \lim_{t \to 0} \tilde{f}(t) = -(\log R_{r,k,n})/2 - H_{r,k,n} \). Showing that asymptotically under the conditions of the theorem \(-H_{r,k,n}/2 - H_{r,k,n} > 0 \) using Propositions 2 and 3 in both cases (a) and (b), we have then that there is a unique solution to the equation \( \tilde{f}(t) = 0 \). Remark with Propositions 2 and 3 that for the true value \( \alpha \) we have \( \tilde{f}(\alpha) = o_p(1) \), since \( H_{r,k,n} \) and \( \frac{1}{\alpha} + \frac{R_{r,k,n} \log R_{r,k,n}}{1 - R_{r,k,n}} \) asymptotically are equal, namely to \( \frac{1}{\alpha} \left(1 - \frac{1 + \frac{k}{nD_T} \lambda_{r,k}}{1 - \frac{k}{nD_T} \lambda_{r,k}} \right) \log \frac{1 + \frac{k}{nD_T} \lambda_{r,k}}{1 + \frac{k}{nD_T} \lambda_{r,k}} \) in case (a), and \( \frac{1}{\alpha} \left(1 + \frac{\lambda_{r,k} \log \lambda_{r,k}}{1 - \lambda_{r,k}} \right) \) in case (b). So the true value \( \alpha \) asymptotically is a solution from which the consistency follows.

Now using Propositions 2 and 3 we obtain that

\[
1 - \hat{\alpha}^2 R_{r,k,n} \log^2 R_{r,k,n} = \delta_{r,k,n,T} (1 + o_p(1)),
\]

where

\[
\delta_{r,k,n,T} = 1 - \left(1 + \frac{k}{nD_T} \lambda_{r,k}\right)\left(1 + \frac{k}{nD_T}\right)^2 \log^2 \left(1 + \frac{k}{nD_T} \lambda_{r,k}\right).
\]

Next, consider \( g(H_{r,k,n}, \log R_{r,k,n}) = H_{r,k,n} - 1/\alpha - \frac{R_{r,k,n} \log R_{r,k,n}}{1 - R_{r,k,n}} \) with

\[
g(x, y) = x - \frac{1}{\alpha} - y \frac{e^{\alpha y}}{1 - e^{\alpha y}}.
\]

The Taylor approximation of \( g(H_{r,k,n}, \log R_{r,k,n}) \) around the asymptotic expected value \( E_\infty H_{r,k,n} \) and \( E_\infty \log R_{r,k,n} \) yields

\[
\hat{g}(H_{r,k,n}, \log R_{r,k,n}) = g(E_\infty H_{r,k,n}, E_\infty \log R_{r,k,n}) + (H_{r,k,n} - E_\infty H_{r,k,n})
+ (\log R_{r,k,n} - E_\infty \log R_{r,k,n}) \frac{\partial g}{\partial y}(E_\infty H_{r,k,n}, E_\infty \log R_{r,k,n}),
\]

(41)
with, based on Proposition 3,
\[ \frac{\partial g}{\partial y}(E_{\infty}H_{r,k,n}, E_{\infty} \log R_{r,k,n}) = -c_{r,k,n,T}(1 + o(1)), \]  
(42)

where
\[ c_{r,k,n,T} = 1 + \frac{k}{nD_T} \lambda_r,k \] 
\[ (1 - \lambda_r,k) \frac{k}{nD_T} \frac{(1 + \frac{k}{nD_T} \lambda_r,k)(1 + \frac{k}{nD_T} \lambda_r,k)}{(1 - \lambda_r,k)^2(\frac{k}{nD_T})^2} \log \left( 1 + \frac{k}{nD_T} \lambda_r,k \right) \] 
\[ (1 + \frac{k}{nD_T} \lambda_r,k)(1 + \frac{k}{nD_T} \lambda_r,k) \] 
\[ (1 - \lambda_r,k)^2 \left( \frac{k}{nD_T} \right)^2 \log \left( 1 + \frac{k}{nD_T} \lambda_r,k \right) \] 
\[ (1 + \frac{k}{nD_T} \lambda_r,k) \] 
\[ (1 + \frac{k}{nD_T} \lambda_r,k)(1 + \frac{k}{nD_T} \lambda_r,k) \] 
\[ (1 - \lambda_r,k)^2 \left( \frac{k}{nD_T} \right)^2 \log \left( 1 + \frac{k}{nD_T} \lambda_r,k \right) \] 

From Propositions 2 and 3, (41), (42), and (40) we find that the stochastic part in the development of \( \hat{\alpha}_{r,k,n} - \alpha^{-1} \) is given by
\[ \frac{1}{\delta_{r,k,n,T} \alpha \sqrt{k}} \left( -\frac{1}{1 - \lambda_r,k} \int_{\lambda_r,k}^1 W(u) d \log \left( 1 + \frac{k}{nD_T} u \right) \right) \] 
\[ + \frac{W(1)}{1 - \lambda_r,k} \left\{ 1 - \frac{1 + \frac{k}{nD_T} \lambda_r,k}{\frac{k}{nD_T}(1 - \lambda_r,k)} \log \left( 1 + \frac{k}{nD_T} \lambda_r,k \right) \right\} \] 
\[ - \frac{W(\lambda_r,k)}{1 - \lambda_r,k} \left\{ 1 - \frac{1 + \frac{k}{nD_T} \lambda_r,k}{\frac{k}{nD_T}(1 - \lambda_r,k)} \log \left( 1 + \frac{k}{nD_T} \lambda_r,k \right) \right\} \right). \]

Developing for \( k/(nD_T) \to 0 \), respectively taking limits for \( k/(nD_T) \to \kappa \) and \( k/(nD_T) \to \infty \) leads to the stated asymptotic variances in the different cases (a), (b), respectively (c).

From (41), (42), and the asymptotic bias expressions in Propositions 2 and 3 one finds the asymptotic bias expressions of \( \hat{\alpha}_{r,k,n} - \alpha^{-1} \). For instance in case \( k/(nD_T) \) is bounded away from \( \infty \) we find that
\[ g(E_{\infty}H_{r,k,n}, E_{\infty} \log R_{r,k,n}) = b^*(T \ell(T))(A_{r,k,n,T} - B_{r,k,n,T}c_{r,k,n,T})(1 + o(1)). \]  
(43)

Condition \( D_T = o((n/k)^{-1+\frac{1}{\alpha}}) \) in Theorem 1(b) entails that the bias term due to the factor \( (1 + D_Ty)^{-1/\alpha} \) in Proposition 1(b) is negligible with respect to the bias term due to the last factor based on \( \ell^* \) in Proposition 2(b).

**Proof of Theorem 2.** First consider the case \( k/(nD_T) \to 0 \). Then observe that \( p/D_T = o(k/(nD_T)) \). Also, after some algebra, starting from (17), using
Using the mean value theorem we obtain that
\[ T_{r,k,n}^{(1)} = -\frac{1}{\alpha} \sqrt{k} \left( \frac{W(1)}{k} \right) (1 + o_p(1)). \]

Next, using the result from Theorem 1(a),
\[ T_{r,k,n}^{(2)} = \frac{1}{\alpha(1 - \lambda)^2} \frac{1}{\sqrt{k}} \mathcal{N}_{\lambda}^{(1)} (1 + o_p(1)) \]
\[ + b^*(T_\ell(T)) \frac{k}{nD_T} (\alpha^{-1} - \rho^*/\alpha^2)(1 + o_p(1)). \]

Furthermore using (27) and the fact that \( nU_{k+1,n}/k \to 1 \) as \( k, n \to \infty, k/n \to 0 \) and that \( p/D_T = o(k/(nD_T)) \)
\[ T_{r,k,n}^{(3)} = b^*(T_\ell(T)) \left[ k_p^* \left( \left[ 1 + \frac{k}{nD_T} \right]^{-1/\alpha} \right) - k_p^* \left( \left[ 1 + \frac{p}{D_T} \right]^{-1/\alpha} \right) \right] (1 + o_p(1)) \]
\[ = -\frac{1}{\alpha} b^*(T_\ell(T)) \frac{k}{nD_T} (1 + o_p(1)). \]

Remains the evaluation of \( T_{r,k,n}^{(3)} \). To this end remark that
\[ \log \left( 1 + \frac{k}{nD_T} \right) = \log \left( \frac{1 - \lambda_{r,k}}{R_{r,k,n}^{\phi_{r,k,n}} - \lambda_{r,k}} \right). \]
We hence need to develop an asymptotic expansion for $\hat{\alpha}_{r,k,n} \log R_{r,k,n}$. Using Theorem 1(a) and Proposition 2 leads to

$$
\hat{\alpha}_{r,k,n} \log R_{r,k,n} \\
\sim \left( \alpha - \frac{n D_T}{k^{3/2}} \frac{12 \alpha}{(1 - \lambda)^2} \mathcal{N}_\lambda^{(1)} - b^*(T \ell(T))(\alpha - \rho^*) \right) \\
\left( - \frac{1 - \lambda}{\alpha} \frac{k}{n D_T} + \frac{1}{2 \alpha} \left( \frac{k}{n D_T} \right)^2 (1 - \lambda^2) - b^*(T \ell(T)) \frac{k}{\alpha} \frac{1 - \lambda}{n D_T} \right. \\
\left. + O_p \left( \frac{k}{n D_T} \right) \right)
$$

$$
\sim - \frac{k}{n D_T} (1 - \lambda) \left\{ 1 + b^*(T \ell(T)) \frac{\rho^*}{\alpha} \right\} + \frac{1}{2} \left( \frac{k}{n D_T} \right)^2 (1 - \lambda^2) \\
+ \frac{12}{1 - \lambda} \frac{1}{\sqrt{k}} \mathcal{N}_\lambda^{(1)} (1 + o_P(1)).
$$

From this we obtain that

$$
R_{\hat{\alpha}_{r,k,n} - \lambda, r,k} \sim (1 - \frac{k}{n D_T}) + (\frac{k}{n D_T})^2 (1 - \lambda) - (\rho^*/\alpha)(\frac{k}{n D_T}) (1 - \lambda) b^*(T \ell(T)) \\
+ \frac{12}{1 - \lambda} \frac{1}{\sqrt{k}} \mathcal{N}_\lambda^{(1)} (1 + o_P(1)),
$$

from which

$$
\log \left( \frac{1 - \lambda_{r,k}}{R_{\hat{\alpha}_{r,k,n} - \lambda, r,k}} \right) \sim - \log \left( 1 - \frac{k}{n D_T} + (\frac{k}{n D_T})^2 - \frac{k}{n D_T} b^*(T \ell(T)) \frac{\rho^*}{\alpha} \right) \\
+ \frac{12}{(1 - \lambda)^2} \frac{1}{\sqrt{k}} \mathcal{N}_\lambda^{(1)} (1 + o_P(1))
$$

$$
\sim \frac{k}{n D_T} - (\frac{k}{n D_T})^2 + \frac{k}{n D_T} \frac{\rho^*}{\alpha} b^*(T \ell(T)) \\
- \frac{12}{(1 - \lambda)^2} \frac{1}{\sqrt{k}} \mathcal{N}_\lambda^{(1)} (1 + o_P(1))
$$

and

$$
\log \left( 1 + \frac{k}{n D_T} \right) - \log \left( 1 + \frac{k}{n D_T} \right) \sim - \frac{1}{2} \left( \frac{k}{n D_T} \right)^2 + \frac{k}{n D_T} \frac{\rho^*}{\alpha} b^*(T \ell(T)) \\
- \frac{12}{(1 - \lambda)^2} \frac{1}{\sqrt{k}} \mathcal{N}_\lambda^{(1)} (1 + o_P(1)).
$$

Furthermore

$$
\log \left( 1 + \frac{p}{D_T} \right) - \log \left( 1 + \frac{p}{D_T} \right)
$$

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is asymptotically equivalent to
\[
\frac{p/D_T}{1 + (p/D_T)}(D_T - 1) = o(D_T - 1)
\]
and hence asymptotically negligible with respect to \(\log \left(1 + \frac{k}{nD_T}\right)\log \left(1 + \frac{k}{nD_T}\right)\).

Using Theorem 1(a) then leads to
\[
T^{(3)}_{r,k,n} = -\frac{1}{2} \left(\frac{k}{nD_T}\right)^2(1 + o(1)) + \frac{k}{nD_T} \frac{\rho^*}{\alpha^2} b^*(T\ell'(T))(1 + o(1))
- \frac{12}{(1 - \lambda)^2} \frac{1}{\sqrt{\lambda}} \mathcal{N}^{(1)}(1 + o_p(1)).
\]

Combining the developments for \(T^{(i)}_{r,k,n}, i = 1, ..., 4\) leads to the stated result.

Next we consider the case \(k/(nD_T) \to \infty\). After some algebra starting from expression (18)
\[
\log \hat{q}_{p,r,k,n} - \log q_p
= \frac{1}{\alpha} \left\{ \log \left(1 + \frac{D_T}{1 + D_T} \right) - \log \left(1 + \frac{D_T}{1 + \frac{D_T}{p}} \right) \right\}
+ \frac{1}{\alpha} \left\{ \log \left(1 + \frac{\hat{D}_T}{1 + \frac{\hat{D}_T}{k}} \right) - \log \left(1 + \frac{D_T}{1 + \frac{D_T}{p}} \right) \right\}
+ \frac{1}{\alpha} \left\{ \log \left(1 + \frac{D_T}{k} \right) - \log \left(1 + \frac{D_T}{1 + \frac{D_T}{U_{k+1,n}}} \right) \right\}
+ \log \frac{\ell^*\left((CT/U_{k+1,n})^{1/\alpha}[1 + \frac{D_T}{U_{k+1,n}}]^{-1/\alpha}\right)}{\ell^*\left(C^{1/\alpha}\frac{p^{-1/\alpha}[1 + \frac{D_T}{p}]^{-1/\alpha}}{1 + \frac{D_T}{p}}\right)}
= \sum_{i=1}^{6} Y^{(i)}_{r,k,n}.
\]

Using (27) and the fact that \(nU_{k+1,n}/k \to 1\) as \(k, n \to \infty, k/n \to 0\), one obtains
\[
Y^{(6)}_{r,k,n} \sim \frac{1}{\rho^*} b^*\left((CTn/k)^{1/\alpha}\right).
\]
This can be derived using Lemma 4.3.5 in de Haan and Ferreira (2006) replacing $U(t)$ by $t^{1/\alpha} \ell^\alpha((C_T t)^{1/\alpha})$, $a(t)$ by $\frac{1}{\alpha} t^{1/\alpha} \ell^\alpha((C_T t)^{1/\alpha})$, $\rho$ by $\rho^*$, $A(t)$ by $b^*((C_T t)^{1/\alpha})$, and $x = x(t)$ by $(k/(np))^{1/\alpha}$.

Next using the mean value theorem

$$Y_{r,k,n}^{(5)} \sim \frac{1}{\alpha} \frac{n}{k} \left( U_{k+1,n} - \frac{k}{n} \right) \frac{nD_T}{k} \frac{1}{1 + \frac{nD_T}{k}}$$

so that using (38)

$$Y_{r,k,n}^{(1)} + Y_{r,k,n}^{(5)} \sim -\frac{1}{\alpha} \frac{W(1)}{\sqrt{k}} (1 + \frac{nD_T}{k})^{-1}.$$

Moreover using Theorem 1(c) and $nD_T/k \to 0$

$$Y_{r,k,n}^{(2)} + Y_{r,k,n}^{(3)} \sim \log \left( \frac{k}{np} (1 + (D_T/p))^{-1} \right)$$

$$\times \left( \frac{\sigma^2(\lambda)}{\alpha \sqrt{k}} \mathcal{N}_\lambda^{(2)} + b^*((C_T n/k)^{1/\alpha}) \beta(\lambda) \right).$$

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Remains $Y_{r,k,n}^{(4)}$. To this end remark that using Proposition 3(b)

\[ \frac{1}{\alpha} \log \left( \frac{1 + \frac{nD_T}{k}}{1 + \frac{D_T}{p}} \right) \]

\[ = -\frac{1}{\alpha} \log \left\{ \frac{(1 - \lambda \frac{k}{np}) - (1 - \frac{k}{np})R_{r,k,n}^{\hat{y}}} {1 - \lambda} \right\} \]

\[ \sim -\frac{1}{\alpha} \log \{1 - (1 - \frac{k}{np})\lambda \left[ 1 + \frac{1}{\sqrt{k}}(\lambda^{(3)} - \sigma^2(\lambda)\lambda^{(2)}) \right. \]

\[ \left. - \alpha b^*((C_Tn/k)^{1/\alpha})\zeta(\lambda) + \frac{nD_T}{k} \frac{1 - \lambda}{\lambda} \right\} \]

\[ + \frac{1}{\alpha} \log(1 - \lambda) \]

\[ \sim -\frac{1}{\alpha} \log \{1 - (1 - \frac{k}{np})\left[ \frac{\lambda}{1 - \sqrt{k}}(\lambda^{(3)} - \sigma^2(\lambda)\lambda^{(2)}) \right. \]

\[ \left. - \alpha b^*((C_Tn/k)^{1/\alpha})\zeta(\lambda) + \frac{nD_T}{k} \right\} \]

\[ = -\frac{1}{\alpha} \log \{1 - \frac{nD_T}{k} + \frac{D_T}{p} \}

\[ - \frac{\lambda}{1 - \lambda} (1 - \frac{k}{np}) \frac{1}{\sqrt{k}}(\lambda^{(3)} - \sigma^2(\lambda)\lambda^{(2)}) \]

\[ + \frac{\alpha \lambda}{1 - \lambda} (1 - \frac{k}{np})b^*((C_Tn/k)^{1/\alpha})\zeta(\lambda) \}

\[ \sim -\frac{1}{\alpha} \log \left\{ \frac{1 + \frac{D_T}{p}}{1 + \frac{nD_T}{k}} \right\} \]

\[ - \frac{\lambda}{1 - \lambda} (1 - \frac{k}{np}) \frac{1}{\sqrt{k}}(\lambda^{(3)} - \sigma^2(\lambda)\lambda^{(2)}) \]

\[ + \frac{\alpha \lambda}{1 - \lambda} (1 - \frac{k}{np})b^*((C_Tn/k)^{1/\alpha})\zeta(\lambda) \}. \]
Hence

\[ Y_{r,k,n}^{(4)} \sim -\frac{1}{\alpha} \log \left( 1 - \frac{\lambda}{1 - \lambda} \frac{1 + \frac{nD_T}{k}}{1 + \frac{D_T}{p}} (1 - \frac{k}{np}) \right) \]

\[ \times \left[ \frac{1}{\sqrt{k}} (N_\lambda^{(3)} - \sigma^2(\lambda)(\log \lambda)N_\lambda^{(2)}) \right] \]

\[ -\alpha b^* \left( (C_T n/k)^{1/\alpha} \zeta(\lambda) \right) \]

\[ \sim \frac{\lambda}{1 - \lambda} \frac{1 + \frac{nD_T}{k}}{1 + \frac{D_T}{p}} \left( 1 - \frac{k}{np} \right) \left\{ \frac{1}{\alpha \sqrt{k}} (N_\lambda^{(3)} - \sigma^2(\lambda)(\log \lambda)N_\lambda^{(2)}) \right\} \]

\[ -b^* \left( (C_T n/k)^{1/\alpha} \zeta(\lambda) \right) \]

where the last step with the development of the log is only valid when \( \frac{\sqrt{k}}{(nD_T)} \rightarrow 0 \) since then \( \frac{1}{1 + \frac{D_T}{p}} (1 - \frac{k}{np})/\sqrt{k} \rightarrow 0 \). Theorem 2(b) now follows from combining the different developments of the \( Y_{r,k,n}^{(i)} \) \( (i = 1, \ldots, 6) \).

\[ \square \]

**Appendix 3. Equivalence of \( \hat{\alpha}_{r,k,n} \) and the solution of (23)**

For \( k \rightarrow \infty \) the denominator in (23) is asymptotically equivalent to

\[ 1 - \frac{1}{1 - \lambda_{r,k}} (\lambda_{r,k} + (nD_{T,\alpha}/k)) \log \left( \frac{1 + \frac{k+1}{nD_{T,\alpha}}}{1 + \frac{k+1}{nD_{T,\alpha}} \lambda_{r,k}} \right) . \]

Indeed this equivalence follows when approximating \(-\frac{1}{\lambda_{r,k}} \sum_k \log(1 + \frac{k}{nD_{T,\alpha}})\) by the Riemann integral \(-\frac{1}{\lambda_{r,k}} \int_{\lambda_{r,k}}^1 \log(1 + \frac{k}{nD_{T,\alpha}} u) du\) and working out this integral.

Next under the conditions and with the method of proof of Theorem 1 it follows that

\[ \log \left( \frac{1 + \frac{k+1}{nD_{T,\alpha}}}{1 + \frac{k+1}{nD_{T,\alpha}} \lambda_{r,k}} \right) / \log R_{r,k,n} \]

converges in probability to \(-\alpha\). Furthermore remark that

\[ \frac{1}{1 - \lambda_{r,k}} \left( \frac{R_{r,k,n}^{\alpha}}{1 - R_{r,k,n}^{\alpha}} - \frac{\lambda_{r,k}}{1 - R_{r,k,n}^{\alpha}} \right) = \frac{R_{r,k,n}^{(6)}}{1 - R_{r,k,n}^{\alpha}} . \]
Hence (23) is asymptotically equivalent to the equation

\[
H_{r,k,n} / \left( 1 + \alpha \frac{R_{r,k,n}^\alpha}{1 - R_{r,k,n}^\alpha} \log R_{r,k,n} \right) = \frac{1}{\alpha},
\]

which is the defining equation for \( \hat{\alpha}_{r,k,n} \).

**Appendix 4. The effect of trimming on bias and variance of \( \hat{\alpha}_{r,k,n} \)**

Trimming the estimator \( \hat{\alpha}_{r,k,n} \) decreases its efficiency with respect to the case \( r = 1 \). This is illustrated in Figure 13, plotting the functions \( \sigma^2(\lambda) \) and \( \beta(\lambda) \) for \( \lambda \in [0, 1/4] \), from Theorem 1(c).

![Figure 13: Functions \( \sigma^2(\lambda) \) and \( \beta(\lambda) \) for \( \lambda \in [0, 1/4] \).](image)

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