The massless XXZ chain
with a boundary

Takeo Kojima

Department Mathematics, College of Science and Technology,
Nihon University, Chiyoda-ku, Tokyo 101-0062, Japan

November 13, 2018

Abstract

We study the XXZ chain with a boundary at massless regime $-1 < \Delta < 1$. We give the free fields realizations of the boundary vacuum state and it’s dual. Using these realizations, we give the integral representations for the correlation functions.

1 Introduction

The one-dimentional massless spin $\frac{1}{2}$ XXZ chain with a boundary is a system described by the Hamiltonian,

$$
\mathcal{H}_B = -\frac{1}{2} \sum_{n=1}^{\infty} (\sigma_{n+1}^x \sigma_n^x + \sigma_{n+1}^y \sigma_n^y + \Delta \sigma_{n+1}^z \sigma_n^z) + h \sigma_1^z, \quad -1 < \Delta < 1.
$$

(1.1)

Here the $\sigma_n^x$, $\sigma_n^y$ and $\sigma_n^z$ stand for the Pauli matrices acting on the $n$-th site of the semi-infinite spin chain:

$$
\cdots \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2.
$$

(1.2)
In this paper we are interested in so-called massless regime:

\[-1 < \Delta = - \cos \left( \frac{\pi}{\xi + 1} \right) < 1, \xi: \text{generic},\]

where the spectrum of the Hamiltonian (1.1) is gapless. In recent works [1, 2, 3, 4] the various massless models “without boundary” were discussed, in the framework of the free field approach. In this paper we shall study the massless model “with a boundary”, in the framework of the free field approach.

In the earlier work [5] the massive XXZ chain with a boundary was considered. The diagonalization of the Hamiltonian was obtained and the integral representations of the correlation functions were derived, in the framework of the free field approach. It’s $U_q(\widehat{sl_n})$-generalization was achieved in [6]. In the work [7], Baxter’s Corner Transfer Matrix Method were extended to the boundary problem- the XYZ model with a boundary. The quantum Knizhnik-Zamolodchikov equation with boundary reflection, which governs the correlation functions, was derived. An infinite product formula of the one-point function was derived by solving the difference equations. The Corner Transfer Matrix Method can be applied to only massive models. Fortunately, the massless XXZ spin is a limiting case of the massive model- the XYZ model [1]. Therefore the correlation functions of the massless XXZ spin with a boundary were described by the following systems of the difference equations, which imply in particular the quantum Knizhnik-Zamolodchikov equation with reflections.

\[
G(\beta_1, \ldots, \beta_{2N}) = \sum_{\epsilon_j, \epsilon'_j=\pm} R(\beta_j - \beta_{j+1})_{\epsilon_j, \epsilon'_j+1} G(\beta_1, \ldots, \beta_j, \beta_{j+1}, \ldots, \beta_{2N}), \quad (1.3)
\]

and

\[
G(\beta_1, \ldots, \beta_{2N-1}, -\beta_{2N})_{\epsilon_1, \ldots, \epsilon_{2N}} = K(\beta_{2N})_{\epsilon_{2N}} G(\beta_1, \ldots, \beta_{2N-1}, \beta_{2N})_{\epsilon_1, \ldots, \epsilon_{2N}},
\]

\[
G(\beta_1 + \pi i, \ldots, \beta_{2N-1}, \beta_{2N})_{\epsilon_1, \ldots, \epsilon_{2N}} = K(-\beta_1 + \pi i, \beta_2, \ldots, \beta_{2N})_{\epsilon_1, \ldots, \epsilon_{2N}}.
\]

(1.4)

Here $G_N(\beta_1, \ldots, \beta_{2N})$ is a function with values in $\mathbb{C}^{\otimes 2N}$. The matrix $R(\beta)$ denotes the $R$-matrix defined in (2.1). The matrix $K(\beta)$ denotes the boundary $K$-matrix defined in (2.14). In this paper we construct the integral representations of the N-point correlation functions, which satisfy the above systems of difference equations.
In this connection we should mention about the works [8, 9]. They constructed the bosonizations of the boundary vacuum state of the type-II vertex operators, for the sine-Gordon model [8] and the SU(2)-invariant massive thirring model [9], by using Lukyanov’s bosonizations of the vertex operators with uv-cutoff [10]. Therefore their constructions started with bosons with uv-cutoff, and the form factors were derived after removing the cutoff parameter at final stage. In this paper we prefer to work directly with operators with the cutoff parameter removed [1], and construct the bosonizations of the boundary state associated with the type-I vertex operators.

Now a few words about the organization of the paper. In section 2 we formulate our problem. In section 3 we construct the bosonizations of the boundary vacuum state and its dual state. In section 4 we derived the integral representations for the correlation functions. In Appendix A we summarize the bosonizations of the vertex operators [1]. In Appendix B we summarize the Multi-Gamma functions.

2 Formulation

The purpose of this section is to formulate the problem.

Let us set the $R$-matrix as

$$R(\beta) = r(\beta) \begin{pmatrix} 1 & b(\beta) & c(\beta) \\ b(\beta) & c(\beta) & b(\beta) \\ c(\beta) & b(\beta) & 1 \end{pmatrix}, \quad (2.1)$$

where we set the components as

$$b(\beta) = -\frac{\text{sh} \left( \frac{\beta}{\xi + 1} \right)}{\text{sh} \left( \frac{\beta + \pi i}{\xi + 1} \right)}, \quad c(\beta) = \frac{\text{sh} \left( \frac{\pi i}{\xi + 1} \right)}{\text{sh} \left( \frac{\beta + \pi i}{\xi + 1} \right)}. \quad (2.2)$$

Here we set

$$r(\beta) = -\frac{S_2(i\beta|2\pi, \pi(\xi + 1))S_2(-i\beta + \pi|2\pi, \pi(\xi + 1))}{S_2(-i\beta|2\pi, \pi(\xi + 1))S_2(i\beta + \pi|2\pi, \pi(\xi + 1))}. \quad (2.3)$$

where $S_2(\beta|\omega_1\omega_2)$ is the double sine function defined in Appendix B.

Let $\{v_+, v_-\}$ denote the natural basis of $V = \mathbb{C}^2$. When viewed as an operator on $V \otimes V$,
the matrix elements of $R(\beta)$ are defined by
\[ R(\beta)v_{k_1} \otimes v_{k_2} = \sum_{j_1, j_2 = \pm} v_{j_1} \otimes v_{j_2} R(\beta)^{h_1 h_2}_{j_1 j_2}. \] (2.4)

The $R$-matrix satisfies the Yang-Baxter equation:
\[ R_{12}(\beta_1 - \beta_2)R_{13}(\beta_1 - \beta_3)R_{23}(\beta_2 - \beta_3) = R_{23}(\beta_2 - \beta_3)R_{13}(\beta_1 - \beta_3)R_{12}(\beta_1 - \beta_2). \] (2.5)

The normalization factor $r_0(\beta)$ is so chosen that the unitarity and crossing relations are
\[ R_{12}(\beta)R_{21}(-\beta) = id, \] (2.6)
\[ R(-\beta)^{k_1 k_2}_{j_1 j_2} = R(\beta - \pi i)^{-j_2 k_1}_{-k_2 j_1}. \] (2.7)

The commutation relation of the type-I vertex operator $\Phi_j(\beta)$ is given by
\[ \Phi_{j_1}(\beta_1)\Phi_{j_2}(\beta_2) = \sum_{k_1, k_2 = \pm} R(\beta_1 - \beta_2)^{k_1 k_2}_{j_1 j_2} \Phi_{k_2}(\beta_2)\Phi_{k_1}(\beta_1). \] (2.8)

The bulk scattering matrix is given by
\[ S(\beta) = s(\beta) \begin{pmatrix} 1 & b'(\beta) & c'(\beta) \\ b'(\beta) & c'(\beta) & 1 \end{pmatrix}, \] (2.9)
where we set the components as
\[ b'(\beta) = \frac{\text{sh} \left( \frac{\beta}{\xi} \right)}{\text{sh} \left( \frac{i \pi - \beta}{\xi} \right)}, \quad c'(\beta) = \frac{\text{sh} \left( \frac{\pi i}{\xi} \right)}{\text{sh} \left( \frac{i \pi - \beta}{\xi} \right)}. \] (2.10)

Here we set
\[ s(\beta) = \frac{S_2(-i\beta|2\pi, \pi \xi)S_2(\pi + i\beta|2\pi, \pi \xi)}{S_2(i\beta|2\pi, \pi \xi)S_2(\pi - i\beta|2\pi, \pi \xi)}. \] (2.11)

The commutation relation of the type-II vertex operator $\Psi_j(\beta)$ is given by
\[ \Psi_{j_1}(\beta_1)\Psi_{j_2}(\beta_2) = \sum_{k_1, k_2 = \pm} S(\beta_1 - \beta_2)^{k_1 k_2}_{j_1 j_2} \Psi_{k_2}(\beta_2)\Psi_{k_1}(\beta_1). \] (2.12)

The commutation relation of type-I and type-II vertex operators is given by
\[ \Psi_{j_1}(\beta_1)\Phi_{j_2}(\beta_2) = j_1 j_2 \tan \left( \frac{\pi}{4} + \frac{i\beta_1 - \beta_2}{2} \right) \Phi_{j_2}(\beta_2)\Psi_{j_1}(\beta_1). \] (2.13)
The free field realization of the vertex operators on the Fock space of boson was given in [1]. We summarized the bosonization of the vertex operators in Appendix A.

Let us set the boundary $K$-matrix by

$$K(\beta) = k(\beta) \begin{pmatrix} 1 & 0 \\ 0 & \text{sh} \left( \frac{\mu + \beta}{\xi + 1} \right) \end{pmatrix},$$

where the normalization factor is given by

$$k(\beta) = k_0(\beta)k_1(\beta),$$

where

$$k_0(\beta) = \frac{S_2(-2i\beta + 4\pi|4\pi, \pi(\xi + 1))S_2(2i\beta + 3\pi|4\pi, \pi(\xi + 1))}{S_2(2i\beta + 4\pi|4\pi, \pi(\xi + 1))S_2(-2i\beta + 3\pi|4\pi, \pi(\xi + 1))},$$

$$k_1(\beta) = \frac{S_2(-i\beta + i\mu + \pi|2\pi, \pi(\xi + 1))S_2(i\beta + i\mu + 2\pi|2\pi, \pi(\xi + 1))}{S_2(i\beta + i\mu + \pi|2\pi, \pi(\xi + 1))S_2(-i\beta + i\mu + 2\pi|2\pi, \pi(\xi + 1))}.$$  

The matrix elements $K(\beta)^k_j$ are defined by

$$K(\beta)v_k = \sum_{j=\pm} v_jK(\beta)^k_j.$$  

The $R$-matrix and the $K$-matrix satisfy the Boundary Yang-Baxter equation.

$$K_2(\beta_2)R_{21}(\beta_1 + \beta_2)K_1(\beta_1)R_{12}(\beta_1 - \beta_2) = R_{21}(\beta_1 - \beta_2)K_1(\beta_1)R_{12}(\beta_1 + \beta_2)K_2(\beta_2).$$

The normalization factor $k(\beta)$ is so chosen that the boundary unitarity and the boundary crossing relations are

$$K(\beta)K(-\beta) = id,$$

$$K \left( \beta + \frac{\pi i}{2} \right) = \sum_{k=\pm} R(2\beta)^{-j\bar{j}}_{k,\bar{k}}K \left( -\beta + \frac{\pi i}{2} \right)^k_{\bar{k}}.$$  

Let us set the renormalized transfer matrix

$$\mathcal{T}_B(\beta) = g^{-1}\sum_{j=\pm} \Phi^*_j(\beta) K(\beta)^j \Phi_j(\beta).$$

Here we set the dual vertex operator as

$$\Phi^*_j(\beta) = \Phi_{-j}(\beta + \pi i), \quad (j = \pm).$$
The constant factor is given by
\[ g = -\frac{1}{\pi i} e^{-(\gamma + \ln(\pi(\xi + 1)))} \sin \left( \frac{\pi}{\xi + 1} \right) \left( \pi(\xi + 1)\Gamma\left( \frac{1}{\xi + 1} \right) \right)^2 \lim_{\beta \to 0} \frac{g(-\beta - \pi i)}{\beta}, \] (2.24)
where \( g(\beta) \) is given in (3.18).

The renormalized transfer matrix has the following relations.

\[ [\mathcal{T}_B(\beta_1), \mathcal{T}_B(\beta_2)] = 0, \quad (\beta_1, \beta_2 \in \mathbb{R}), \] (2.25)
\[ \mathcal{T}_B(0) = id, \quad \mathcal{T}_B(\beta)\mathcal{T}_B(-\beta) = id, \] (2.26)
\[ \mathcal{T}_B(-\beta + \pi i) = \mathcal{T}_B(\beta). \] (2.27)

The Hamiltonian \( \mathcal{H}_B \) (1.1) and the renormalized transfer matrix \( \mathcal{T}_B(\beta) \) (2.22) are related through the formula, naively.

\[ \left( \frac{d}{d\beta} \mathcal{T}_B \right)(0) \sim \mathcal{H}_B. \] (2.28)

Inspite of constructing the eigenstate of the Hamiltonian (1.1), we solve the following eigenstate problem of the transfer matrix \( \mathcal{T}_B(\beta) \).

\[ \mathcal{T}_B(\beta)|B\rangle = |B\rangle, \] (2.29)
\[ \langle B|\mathcal{T}_B(\beta) = \langle B|. \] (2.30)

### 3 Boundary state

In this section we invoke the bosonization method to find the explicit formulae for the boundary state \(|B\rangle\), assuming uniqueness. The boundary state is determined by the following relation.

\[ \mathcal{T}_B(\beta)|B\rangle = |B\rangle. \] (3.1)

Acting the vertex opertaors \( \Phi_j(-\beta) \) from the left, we have the equivalent relation.

\[ K(\beta)^j \Phi_j(\beta)|B\rangle = \Phi_j(-\beta)|B\rangle, \quad (j = \pm). \] (3.2)

Here we have used the duality relation.

\[ \Phi_j(\beta)\Phi_k^*(\beta) = g \times \delta_{j,k}, \quad (j, k = \pm), \] (3.3)
where \( g \) is defined in (2.24).

We make the ansatz that the boundary state has the following form.

\[
|B\rangle = e^{F}|vac\rangle, \tag{3.4}
\]

where

\[
F = \frac{1}{2} \int_0^\infty \frac{A(t)}{[b(t), b(-t)]} b(-t)^2 dt + \int_0^\infty \frac{B(t)}{[b(t), b(-t)]} b(-t) dt. \tag{3.5}
\]

When we set the coefficients as \( A(t) = -1 \) and \( B(t) = \frac{1}{t} \sh \left( \frac{\pi t}{2} \right) \sh \left( \frac{i \mu - \pi \xi}{2} t \right) \)

\[
- \frac{1}{t} \sh \left( \frac{\pi t}{4} \right) \ch \left( \frac{\pi \xi t}{4} \right), \tag{3.6}
\]

the boundary state \( |B\rangle \) satisfy the characterizing relation (3.2).

Let us prove the relation (3.2). In what follows we use the abbreviations : \( \omega_1 = 2\pi, \omega_2 = \pi(\xi + 1) \), and

\[
U_+ (\beta) = \exp \left( -\int_0^\infty \frac{b(t)}{\sh \pi t} e^{i\beta t} dt \right), \quad U_- (\beta) = \exp \left( \int_0^\infty \frac{b(-t)}{\sh \pi t} e^{-i\beta t} dt \right), \tag{3.7}
\]

\[
\bar{U}_+ (\alpha) = \exp \left( \int_0^\infty \frac{b(t)}{\sh \frac{\pi t}{2}} e^{i\alpha t} dt \right), \quad \bar{U}_- (\alpha) = \exp \left( -\int_0^\infty \frac{b(-t)}{\sh \frac{\pi t}{2}} e^{-i\alpha t} dt \right). \tag{3.8}
\]

In what follows we omit non-essential constant factors.

At first we explain the formulas of the form

\[
X(\beta_1)Y(\beta_2) = C_{XY}(\beta_1 - \beta_2) : X(\beta_1)X(\beta_2) :, \tag{3.9}
\]

where \( X, Y = U_j \), and \( C_{XY}(\beta) \) is a meromorphic function on \( \mathbb{C} \). These formulae follow from the commutation relation of the free bosons. When we compute the contraction of the basic operators, we often encounter an integral

\[
\int_0^\infty F(t) dt, \tag{3.10}
\]

which is divergent at \( t = 0 \). Here we adopt the following prescription for regularization : it should be understood as the countour integral,

\[
\int_C F(t) \frac{\log(-t)}{2\pi i} dt, \tag{3.11}
\]
where the contour $C$ is given by

$$0 \quad \text{Contour } C$$

The action of the basic operator on the boundary state is given by

$$U_+ (\beta) |B\rangle = \text{Const.} h(\beta) U_- (-\beta) |B\rangle,$$

where

$$h(\beta) = \frac{\Gamma_2(-2i\beta + 4\pi|2\omega_1, \omega_2\rangle \Gamma_2(-2i\beta + \pi(\xi + 1)|2\omega_1, \omega_2\rangle)}{\Gamma_2(-2i\beta + 3\pi|2\omega_1, \omega_2\rangle \Gamma_2(-2i\beta + \pi(\xi + 1)|2\omega_1, \omega_2\rangle)} \times \frac{\Gamma_2(-i\beta + i\mu + \pi|\omega_1, \omega_2\rangle \Gamma_2(-i\beta - i\mu + \pi(\xi + 1)|\omega_1, \omega_2\rangle)}{\Gamma_2(-i\beta + i\mu + 2\pi|\omega_1, \omega_2\rangle \Gamma_2(-i\beta - i\mu + \pi(\xi + 1)|\omega_1, \omega_2\rangle)}.$$  

The function $h(\beta)$ satisfies

$$K(\beta) = k(\beta) = k_0(\beta) k_1(\beta) = \frac{h(-\beta)}{h(\beta)}.$$  

We have

$$h(-\beta) \Phi_+(\beta) |B\rangle = \text{Const.} h(-\beta) h(\beta) U_- (\beta) U_- (-\beta) |B\rangle.$$  

Now we have proved the “+”-part of characterizing relation of the boundary state (3.2).

$$K(\beta) \Phi_+(\beta) |B\rangle = \Phi_+(-\beta) |B\rangle.$$  

Next we shall prove the “−”-part of (3.2).

The commutation relation of the basic operator is given by

$$U_+ (\beta_1) U_- (\beta_2) = g(\beta_1 - \beta_2) U_- (\beta_2) U_+ (\beta_1),$$

where we set

$$g(\beta) = e^{\pi \mu \xi} \frac{\Gamma_2(-i\beta + 2\pi|\omega_1, \omega_2\rangle \Gamma_2(-i\beta + \pi(\xi + 1)|\omega_1, \omega_2\rangle)}{\Gamma_2(-i\beta + \pi|\omega_1, \omega_2\rangle \Gamma_2(-i\beta + \pi(\xi + 1)|\omega_1, \omega_2\rangle)}.$$  

8
The action of the basic operator on the boundary state is given by

$$\bar{U}_+(\alpha)|B\rangle = \text{Const}.I(\alpha)\bar{U}_-(-\alpha)|B\rangle,$$  
\hspace{1cm} (3.19)

where we set

$$I(\alpha) = -\frac{2i\alpha}{\pi(\xi + 1)} \frac{\Gamma\left(\frac{-i\mu - i\alpha}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)}\right)}{\Gamma\left(\frac{i\mu - i\alpha}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)}\right)}.$$  
\hspace{1cm} (3.20)

From direct calculation, we have

$$h(\beta)^{-1}\text{sh}\left(\frac{\mu + \beta}{\xi + 1}\right)\Phi_-(\beta)|B\rangle$$
$$= \text{Const.} \times \text{sh}\left(\frac{\mu + \beta}{\xi + 1}\right) \int_{-\infty}^{\infty} d\alpha \prod_{\epsilon_1, \epsilon_2 = \pm} \Gamma\left(\frac{i(\epsilon_1 \alpha + \epsilon_2 \beta)}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)}\right)$$
$$\times \text{sh}\left(\frac{\alpha + \beta}{\xi + 1} - \frac{\pi i}{2(\xi + 1)}\right) \times \alpha \frac{\Gamma\left(\frac{-i\mu - i\alpha}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)}\right)}{\Gamma\left(\frac{i\mu - i\alpha}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)}\right)}$$
$$\times U_-(\beta)\bar{U}_-(-\beta)\bar{U}_-(\alpha)\bar{U}_-(-\alpha)|B\rangle.$$  
\hspace{1cm} (3.21)

Note that the operator part $U_-(\beta)\bar{U}_-(-\beta)\bar{U}_-(\alpha)\bar{U}_-(-\alpha)$ is invariant under the change of variables $\alpha \leftrightarrow -\alpha, \beta \leftrightarrow -\beta$.

We get

$$h(\beta)^{-1}\text{sh}\left(\frac{\mu + \beta}{\xi + 1}\right)\Phi_-(\beta)|B\rangle - h(-\beta)^{-1}\text{sh}\left(\frac{\mu - \beta}{\xi + 1}\right)\Phi_-(\beta)|B\rangle$$
$$= \text{Const.} \times \text{sh}\left(\frac{2\beta}{\xi + 1}\right) \int_{-\infty}^{\infty} d\alpha \prod_{\epsilon_1, \epsilon_2 = \pm} \Gamma\left(\frac{i(\epsilon_1 \alpha + \epsilon_2 \beta)}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)}\right)$$
$$\times \prod_{\epsilon = \pm} \Gamma\left(\frac{i(\epsilon \mu + \epsilon \alpha)}{\pi(\xi + 1)} + 1 - \frac{1}{2(\xi + 1)}\right) \times \alpha \prod_{\epsilon = \pm} \text{sh}\left(\frac{\mu + \epsilon \alpha}{\xi + 1} - \frac{\pi i}{2(\xi + 1)}\right)$$
$$\times U_-(\beta)\bar{U}_-(-\beta)\bar{U}_-(\alpha)\bar{U}_-(-\alpha)|B\rangle.$$  
\hspace{1cm} (3.22)

The integrand of (RHS) of the above equation is anti-symmetric to a change of the variable $\alpha \leftrightarrow -\alpha$. It means the left-hand side becomes zero after taking integral. Therefore we arrive at the following.

$$h(-\beta)\text{sh}\left(\frac{\mu + \beta}{\xi + 1}\right)\Phi_-(\beta)|B\rangle = h(\beta)\text{sh}\left(\frac{\mu - \beta}{\xi + 1}\right)\Phi_-(\beta)|B\rangle.$$  
\hspace{1cm} (3.23)

Now we have proved the “-”-part of the characterizing relation of the boundary vacuum state (3.2).
The dual boundary state $\langle B|\rangle$ is determined by the following relation.

$$\langle B|\mathcal{T}_B(\beta) = \langle B|.$$  \hspace{1cm} (3.24)

Acting the dual vertex operator $\Phi_j^*(\beta)$ from right, we have the equivalent relation.

$$\langle B|\Phi_j(-\beta + \pi i)K(\beta)^{-j} = \langle B|\Phi_j(\beta + \pi i), \; (j = \pm).$$  \hspace{1cm} (3.25)

We make the ansatz that the boundary state has the following form.

$$\langle B| = \langle \text{vac}|e^G,$$  \hspace{1cm} (3.26)

where

$$G = \frac{1}{2} \int_0^\infty \frac{C(t)}{[b(t), b(-t)]}b(t)^2dt + \int_0^\infty \frac{D(t)}{[b(t), b(-t)]}b(t)dt.$$  \hspace{1cm} (3.27)

When we set the coefficients as $C(t) = -e^{-2\pi t}$ and

$$D(t) = e^{-\pi t} \frac{\text{sh} \left( \frac{\pi t}{2} \right) \text{sh} \left( (i\mu - \frac{\pi \xi}{2} - \pi) t \right)}{\text{sh} \left( \frac{\pi}{2}(\xi + 1)t \right)} + e^{-\pi t} \frac{\text{sh} \left( \frac{\pi t}{2} \right) \text{sh} \left( \frac{\pi t}{4} \right) \text{ch} \left( \frac{\pi \xi}{4} t \right)}{\text{sh} \left( \frac{\pi}{4}(\xi + 1)t \right)},$$  \hspace{1cm} (3.28)

the state $\langle B|$ satisfies the characterizing relation (3.25). It can be shown as the same manner as the case of the state $|B\rangle$. Here we omit details.

4 Correlation functions

In this section we calculate the vacuum expectation values of type-I vertex operators, and obtain them as integrals of meromorphic functions involving Multi-Gamma functions.

We shall consider the $2N$-point functions defined by

$$G_{\epsilon_1,\ldots,\epsilon_{2N}}(\beta_1, \cdots, \beta_{2N}) = \frac{\langle B|\Phi_{\epsilon_1}^*(\beta_1)\cdots\Phi_{\epsilon_{2N}}^*(\beta_{2N})|B\rangle}{\langle B|B\rangle}.$$  \hspace{1cm} (4.1)

From the commutation relation of the vertex operators (2.8), the vacuum expectation values (4.1) satisfy the $R$-matrix symmetry (1.3). From the reflection relations (3.2) and (3.25), the vacuum expectation values (4.1) satisfy the reflection conditions (1.4).

Specializing the spectral parameters, they give multi-point correlation functions of the local spin operators of the massless XXZ spin with a boundary :

$$G_{-\epsilon_1,\ldots,-\epsilon_N,\epsilon_N,\ldots,\epsilon_1}(\beta_1 + \pi i, \cdots, \beta_N + \pi i, \beta_N, \cdots, \beta_1).$$  \hspace{1cm} (4.2)
After specializing the spectral parameters as the above, our integral representations of the correlation functions (4.1) can be compared with the formulae for the correlation functions of the boundary XYZ spin, which can be derived by mapping to the boundary SOS model \[11\]. Y. Hara \[12\] considered the mapping method \[13\] of the boundary XYZ model, and derived the explicit formulae of the one-point functions, whose N-point generalization seems to be tedious but straightforward \[14\]. Comparison with two formulae is our future problem.

Now let us calculate the vacuum expectation value \(4.1\), explicitly. Fixing indexes \(\{\epsilon_1, \cdots, \epsilon_{2N}\}\), let us denote by \(A\) the index set

\[
A = \{a|\epsilon_a = -, 1 \leq a \leq 2N\}. \tag{4.3}
\]

In order to evaluate the expectation value (4.1), we invoke the bosonization formulae of the vertex operators and the boundary state. By normal-ordering the product of vertex operators, we have the following formula.

\[
\begin{align*}
G_{\epsilon_1 \cdots \epsilon_{2N}}(\beta_1, \cdots, \beta_{2N}) &= \prod_{1 \leq b_1 < b_2 \leq 2N} \frac{\Gamma_2(i(\beta_{b_2} - \beta_{b_1}) + 2\pi|\omega_1\omega_2| \Gamma_2(i(\beta_{b_2} - \beta_{b_1}) + \pi(\xi + 1)|\omega_1\omega_2)}{\Gamma_2(i(\beta_{b_2} - \beta_{b_1}) + \pi|\omega_1\omega_2| \Gamma_2(i(\beta_{b_2} - \beta_{b_1}) + \pi(\xi + 2)|\omega_1\omega_2)}) \\
& \times \prod_{a \in A} \int_{-\infty}^{\infty} d\alpha_a \prod_{a \in A} \Gamma \left( \frac{i(\alpha_a - \beta_a)}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)} \right) \Gamma \left( \frac{i(\beta_a - \alpha_a)}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)} \right) \\
& \times \prod_{a_1 < a_2, a_1, a_2 \in A} \left\{ \frac{\Gamma \left( \frac{i(\alpha_{a_2} - \alpha_{a_1})}{\pi(\xi + 1)} + \frac{1}{\xi + 1} \right)}{\Gamma \left( \frac{i(\alpha_{a_2} - \alpha_{a_1})}{\pi(\xi + 1)} + \frac{1}{\xi + 1} \right)} \right\} \\
& \times \prod_{a < b, a \in A, 1 \leq b \leq 2N} \frac{\Gamma \left( \frac{i(\beta_b - \alpha_a)}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)} \right)}{\Gamma \left( \frac{i(\beta_b - \alpha_a)}{\pi(\xi + 1)} + \frac{1}{\xi + 1} \right)} \prod_{b < a, a \in A, 1 \leq b \leq 2N} \frac{\Gamma \left( \frac{i(\alpha_a - \beta_b)}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)} \right)}{\Gamma \left( \frac{i(\alpha_a - \beta_b)}{\pi(\xi + 1)} + \frac{1}{\xi + 1} \right)} \\
& \times J(\{\beta_b\}_{b=1}^{2N}\{\alpha_a\}_{a \in A}) \tag{4.4}
\end{align*}
\]

Here we set

\[
J(\{\beta_b\}_{b=1}^{2N}\{\alpha_a\}_{a \in A}) = \frac{\langle B| \exp \left( \int_0^\infty X(t)b(-t)dt \right) \exp \left( \int_0^\infty Y(t)b(t)dt \right) |B \rangle}{\langle B|B \rangle}, \tag{4.5}
\]
where

\[
X(t) = \sum_{b=1}^{2N} e^{-i\beta_b t} \frac{\text{sh}(\pi t)}{\text{sh}(\frac{\pi t}{2})} - \sum_{a \in A} e^{-i\alpha_a t} \frac{\text{sh}(\pi t)}{\text{sh}(\frac{\pi t}{2})}. \quad (4.6)
\]

\[
Y(t) = -\sum_{b=1}^{2N} e^{i\beta_b t} \frac{\text{sh}(\pi t)}{\text{sh}(\frac{\pi t}{2})} + \sum_{a \in A} e^{i\alpha_a t} \frac{\text{sh}(\pi t)}{\text{sh}(\frac{\pi t}{2})} = -X^*(t). \quad (4.7)
\]

Next we evaluate the quantity \(J(\{\beta_b\}|\{\alpha_a\})\). Using the completeness relation of the coherent states \([5]\), and performing the integral calculations, we have

\[
J(\{\beta_b\}_{b=1}^{2N}|\{\alpha_a\}_{a \in A}) = \exp \left( \int_0^\infty \frac{1}{1 - A(t)C(t)} \frac{\text{sh}(\pi t) \text{sh}(\frac{\pi t}{2})}{\text{tsh}(\frac{\pi}{2}(\xi + 1)t)} \right.
\]

\[
\times \left( \frac{1}{2} C(t) X(t)^2 + A(t) C(t) X(t) Y(t) + \frac{1}{2} A(t) Y(t)^2 \right)
\]

\[
+ \int_0^\infty \frac{1}{1 - A(t)C(t)} \{ (D(t) + C(t)B(t))X(t) + (B(t) + A(t)D(t))Y(t) \} dt \right) \quad (4.8)
\]

Here \(A(t), B(t), C(t)\) and \(D(t)\) are the coefficient functions in the boundary state and its dual. It is evaluated as follows.

\[
J(\{\beta_b\}|\{\alpha_a\}) = J_\beta(\{\beta_b\}) J_\alpha(\{\alpha_a\}) J_{\beta\alpha}(\{\beta_b\}|\{\alpha_a\}). \quad (4.9)
\]

Here we set

\[
J_\beta(\{\beta_b\}) = \prod_{b=1}^{2N} S_2(i\mu + i\beta_b + \pi|\omega_1\omega_2) S_2(2i\beta_b + 3\pi|2\omega_1\omega_2) S_2(-2i\beta_b + 2\pi|2\omega_1\omega_2)
\]

\[
\times \prod_{b=1}^{2N} \frac{S_3(2i\beta_b + 3\pi|\omega_1\omega_2) S_3(-2i\beta_b + \pi|\omega_1\omega_2) S_3(2i\beta_b + 4\pi|\omega_1\omega_2) S_3(-2i\beta_b + 2\pi|\omega_1\omega_2)}{S_3(2i\beta_b + 3\pi|2\omega_1\omega_2) S_3(-2i\beta_b + 2\pi|2\omega_1\omega_2)}
\]

\[
\times \prod_{1 \leq b_1 < b_2 \leq 2N} \frac{S_3(i(\beta_{b_1} + \beta_{b_2}) + 3\pi|\omega_1\omega_2) S_3(-i(\beta_{b_1} + \beta_{b_2}) + \pi|\omega_1\omega_2) S_3(i(\beta_{b_1} + \beta_{b_2}) + 4\pi|\omega_1\omega_2) S_3(-i(\beta_{b_1} + \beta_{b_2}) + 2\pi|\omega_1\omega_2)}{S_3(i(\beta_{b_1} + \beta_{b_2}) + 3\pi|2\omega_1\omega_2) S_3(-i(\beta_{b_1} + \beta_{b_2}) + \pi|2\omega_1\omega_2) S_3(i(\beta_{b_1} + \beta_{b_2}) + 4\pi|2\omega_1\omega_2) S_3(-i(\beta_{b_1} + \beta_{b_2}) + 2\pi|2\omega_1\omega_2)}
\]

\[
\times \prod_{1 \leq b_1 < b_2 \leq 2N, \epsilon = \pm} \left\{ \frac{S_2(i\epsilon(\beta_{b_1} - \beta_{b_2}) + \pi|\omega_1\omega_2) S_3(i\epsilon(\beta_{b_1} - \beta_{b_2}) + \pi|\omega_1\omega_2)}{S_2(i\epsilon(\beta_{b_1} - \beta_{b_2}) + 4\pi|\omega_1\omega_2) S_3(i\epsilon(\beta_{b_1} - \beta_{b_2}) + 4\pi|\omega_1\omega_2)} \right\}. \quad (4.10)
\]
Here we omit an irrelevant constant.

In order to get the integral representations of the form factors of the local spin operators, we have to calculate the vacuum expectation value of both type-I and type-II vertex operators, and obtain them as integrals of meromorphic functions of Multi-Gamma functions.

\[ F_{j_1 \cdots j_N}^{k_1 \cdots k_M} (\gamma_1 \cdots \gamma_M | \beta_1 \cdots \beta_N) = \frac{\langle B | \Phi_{k_1} (\gamma_1) \cdots \Phi_{k_M} (\gamma_M) \Psi_{j_1} (\beta_1) \cdots \Psi_{j_N} (\beta_N) | B \rangle}{\langle B | B \rangle}. \]  

Calculation of the vacuum expectation values is tedious but straightforward. We can perform it as the same manner as the correlation functions.
Acknowledgements. This work was partly supported by Grant-in-Aid for Encouragements for Young Scientists (A) from Japan Society for the Promotion of Science. (11740099)

References

[1] M. Jimbo, H. Konno and T. Miwa: Massless XXZ model and degeneration of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl}_2)$, *Deformation theory and Symplectic Geometry*, Eds. D. Sternheimer, J. Rawnsley and S. Gutt, Math.Phys. Studies, Kluwer, 20, 117-138, 1997.

[2] M. Jimbo and T. Miwa: Quantum KZ equation with $|q| = 1$ and correlation functions of the XXZ model in the gapless regime, *J.Phys.*A29, 2923-2958, (1996).

[3] T. Kojima and S. Yamasita: The critical $A_{n-1}^{(1)}$ chain, [nlin-sl/0004013], (2000), submitted to publication.

[4] T. Kojima: The 19-Vertex Model at critical regime $|q| = 1$, [nlin-sl/0005023], (2000), submitted to publication.

[5] M. Jimbo, R. Kedem, T. Kojima, H. Konno and T. Miwa: XXZ chain with a boundary, *Nucl.Phys.*B441 [FS], 437-470, (1995).

[6] H. Furutsu and T. Kojima: The $U_q(\widehat{sl}_n)$-analogue of the XXZ chain with a boundary, [solv-int/9905009], to appear in *J.Math.Phys.* (2000).

[7] M. Jimbo, R. Kedem, H. Konno, T. Miwa and R. Weston: Difference Equations in Spin Chains with a Boundary, *Nucl.Phys.*B448, 429-456, (1995).

[8] B. Hou, K. Shi, Y. Wang and W. Yang: Bosonization of Quantum Sine-Gordon Field with Boundary, *Int.J.Mod.Phys.*A12, No.9, 1711-1741, (1997).

[9] H. Furutsu, T. Kojima, Y.-H. Quano: Form factors of the $SU(2)$-invariant Thirring model with boundary reflection, [solv-int/9910012], to appear in Int.J.Mod.Phys. A, (2000).
A Vertex Operators

Here we summarize the bosonizations of the vertex operators [1].

Let us set free bosons \( b(t)(t \in \mathbb{R}) \) which satisfy

\[
[b(t), b(t')] = \frac{\text{sh}(\frac{\pi t}{2})\text{sh}(\pi t)\text{sh} \frac{\pi t \xi}{2}}{t \text{sh} \frac{\pi t}{2}} \delta(t + t').
\]  

(A.1)

Let us set \( a(t) \) by

\[
b(t)\text{sh} \frac{\pi t(\xi + 1)}{2} = a(t)\text{sh} \frac{\pi t \xi}{2}.
\]  

(A.2)

Let us consider the Fock space \( \mathcal{H} \) generated by the vacuum \( |\text{vac}\rangle \) which satisfies

\[
b(t)|\text{vac}\rangle = 0 \quad \text{if} \quad t > 0.
\]  

(A.3)

The bosonization of the type-I vertex operators is given by

\[
\Phi_+(\beta) = U(\beta),
\]

(A.4)

\[
\Phi_-(\beta) = \int_{\mathcal{C}_t} d\alpha : U(\beta)\bar{U}(\alpha) :
\times \Gamma \left( \frac{i(\alpha - \beta)}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)} \right) \Gamma \left( -\frac{i(\alpha - \beta)}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)} \right),
\]  

(A.5)
where we have set
\[ U(\alpha) =: \exp \left( -\int_{-\infty}^{\infty} \frac{b(t)}{\sinh \pi t} e^{i\alpha t} dt \right) ; \quad \bar{U}(\alpha) =: \exp \left( \int_{-\infty}^{\infty} \frac{b(t)}{\sinh \frac{\pi}{2} t} e^{i\alpha t} dt \right) \quad . \] (A.6)

The bosonization of the type-II vertex operators is given by
\[ \Psi_+(\beta) = V(\beta), \quad \Psi_-(\beta) = \int_{C_{II}} d\alpha : V(\beta)\bar{V}(\alpha) : \times \left( \frac{i(\alpha - \beta)}{\pi \xi} - \frac{1}{2\xi} \right) \left( -\frac{i(\alpha - \beta)}{\pi \xi} + \frac{1}{2\xi} \right) , \] (A.7)

where we have set
\[ V(\alpha) =: \exp \left( \int_{-\infty}^{\infty} \frac{a(t)}{\sin \pi t} e^{i\alpha t} dt \right) ; \quad \bar{V}(\alpha) =: \exp \left( -\int_{-\infty}^{\infty} \frac{a(t)}{\sin \frac{\pi}{2} t} e^{i\alpha t} dt \right) \quad . \] (A.8)

Here the integration contours are chosen as follows. The contour \( C_I \) is \((-\infty, \infty)\). The poles
\[ \alpha - \beta = \frac{\pi i}{2} + n\pi(\xi + 1)i, \quad (n \in \mathbb{N}) \] (A.10)

of \( \Gamma \left( \frac{i(\alpha - \beta)}{\pi \xi} + \frac{1}{2(\xi + 1)} \right) \) are above \( C_I \) and the poles
\[ \alpha - \beta = -\frac{\pi i}{2} - n\pi(\xi + 1)i, \quad (n \in \mathbb{N}) \] (A.11)

of \( \Gamma \left( -\frac{i(\alpha - \beta)}{\pi \xi} + \frac{1}{2(\xi + 1)} \right) \) are below \( C_I \). The contour \( C_{II} \) is \((-\infty, \infty)\) except that the poles
\[ \alpha - \beta = -\frac{\pi i}{2} + n\pi \xi i, \quad (n \in \mathbb{N}) \] (A.12)

of \( \Gamma \left( \frac{i(\alpha - \beta)}{\pi \xi} - \frac{1}{2\xi} \right) \) are above \( C_{II} \) and the poles
\[ \alpha - \beta = \frac{\pi i}{2} - n\pi \xi i, \quad (n \in \mathbb{N}) \] (A.13)

of \( \Gamma \left( -\frac{i(\alpha - \beta)}{\pi \xi} - \frac{1}{2\xi} \right) \) are below \( C_{II} \).
B Multi Gamma functions

Here we summarize the multiple gamma and the multiple sine functions. Let us set the functions $\Gamma_1(x|\omega), \Gamma_2(x|\omega_1, \omega_2)$ and $\Gamma_3(x|\omega_1, \omega_2, \omega_3)$ by

$$\log \Gamma_1(x|\omega) + \gamma B_{11}(x|\omega) = \int_C \frac{dt}{2\pi it} e^{-xt} \log \left( \frac{-t}{1 - e^{-\omega t}} \right),$$

(B.1)

$$\log \Gamma_2(x|\omega_1, \omega_2) - \frac{\gamma}{2} B_{22}(x|\omega_1, \omega_2) = \int_C \frac{dt}{2\pi it} e^{-xt} \frac{\log(-t)}{(1 - e^{-\omega_1 t})(1 - e^{-\omega_2 t})},$$

(B.2)

$$\log \Gamma_3(x|\omega_1, \omega_2, \omega_3) + \frac{\gamma}{3!} B_{33}(x|\omega_1, \omega_2, \omega_3) = \int_C \frac{dt}{2\pi it} e^{-xt} \frac{\log(-t)}{(1 - e^{-\omega_1 t})(1 - e^{-\omega_2 t})(1 - e^{-\omega_3 t})},$$

(B.3)

where the functions $B_{jj}(x)$ are the multiple Bernoulli polynomials defined by

$$\frac{t^r e^{xt}}{\prod_{j=1}^r (e^{\omega_j t} - 1)} = \sum_{n=0}^\infty \frac{t^n}{n!} B_{r,n}(x|\omega_1 \cdots \omega_r),$$

(B.4)

more explicitly

$$B_{11}(x|\omega) = \frac{x}{\omega} - \frac{1}{2},$$

(B.5)

$$B_{22}(x|\omega) = \frac{x^2}{\omega_1 \omega_2} - \left( \frac{1}{\omega_1} + \frac{1}{\omega_2} \right) x + \frac{1}{2} + \frac{1}{6} \left( \frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} \right).$$

(B.6)

Here $\gamma$ is Euler’s constant, $\gamma = \lim_{n \to \infty} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n)$.

Here the contour of integral is given by

\[ \text{Contour C} \]
Let us set
\[ S_1(x|\omega) = \frac{1}{\Gamma_1(\omega - x|\omega)\Gamma_1(x|\omega)}, \]  
\[ S_2(x|\omega_1, \omega_2) = \frac{\Gamma_2(\omega_1 + \omega_2 - x|\omega_1, \omega_2)}{\Gamma_2(x|\omega_1, \omega_2)}, \]  
\[ S_3(x|\omega_1, \omega_2, \omega_3) = \frac{1}{\Gamma_3(\omega_1 + \omega_2 + \omega_3 - x|\omega_1, \omega_2, \omega_3)\Gamma_3(x|\omega_1, \omega_2, \omega_3)}. \]  
(B.7)  
(B.8)  
(B.9)

We have
\[ \Gamma_1(x|\omega) = e^{(\frac{x}{\omega} - \frac{1}{2})\log \frac{\Gamma(x/\omega)}{\sqrt{2\pi}}}, \quad S_1(x|\omega) = 2\sin(\pi x/\omega), \]  
(B.10)

\[ \frac{\Gamma_2(x + \omega_1|\omega_1, \omega_2)}{\Gamma_2(x|\omega_1, \omega_2)} = \frac{1}{\Gamma_1(x|\omega_2)}, \quad \frac{S_2(x + \omega_1|\omega_1, \omega_2)}{S_2(x|\omega_1, \omega_2)} = \frac{1}{S_1(x|\omega_2)}, \quad \frac{\Gamma_1(x + \omega|\omega)}{\Gamma_1(x|\omega)} = x. \]  
(B.11)

\[ \frac{\Gamma_3(x + \omega_1|\omega_1, \omega_2, \omega_3)}{\Gamma_3(x|\omega_1, \omega_2, \omega_3)} = \frac{1}{\Gamma_2(x|\omega_2, \omega_3)}, \quad \frac{S_3(x + \omega_1|\omega_1, \omega_2, \omega_3)}{S_3(x|\omega_1, \omega_2, \omega_3)} = \frac{1}{S_2(x|\omega_2, \omega_3)}. \]  
(B.12)

\[ \log S_2(x|\omega_1\omega_2) = \int_C \frac{\text{sh}(x - \frac{\omega_1 + \omega_2}{2})t}{2\text{sh}\frac{\omega_1 t}{2}\text{sh}\frac{\omega_2 t}{2}} \log(-t) \frac{dt}{2\pi it}, \quad (0 < \text{Re}x < \omega_1 + \omega_2). \]  
(B.13)

\[ S_2(x|\omega_1\omega_2) = \frac{2\pi}{\sqrt{\omega_1\omega_2}}x + O(x^2), \quad (x \to 0). \]  
(B.14)

\[ S_2(x|\omega_1\omega_2)S_2(-x|\omega_1\omega_2) = -4\sin\frac{\pi x}{\omega_1}\sin\frac{\pi x}{\omega_2}. \]  
(B.15)