RG FLOWS OF NON-DIAGONAL MINIMAL MODELS
PERTURBED BY $\phi_{1,3}$

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Abstract

Studying perturbatively, for large $m$, the torus partition function of both $(A, A)$ and $(A, D)$ series of minimal models in the Cappelli, Itzykson, Zuber classification, deformed by the least relevant operator $\phi_{(1,3)}$, we disentangle the structure of $\phi_{1,3}$ flows. The results are conjectured on reasonable ground to be valid for all $m$. They show that $(A, A)$ models always flow to $(A, A)$ and $(A, D)$ ones to $(A, D)$. No hopping between the two series is possible. Also, we give arguments that there exist 3 isolated flows $(E, A) \rightarrow (A, E)$ that, together with the two series, should exhaust all the possible $\phi_{1,3}$ flows. Conservation (and symmetry breaking) of non-local currents along the flows is discussed and put in relation to the $A, D, E$ classification.

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1 - A well established result in Conformal Field Theory (CFT) is that, perturbing the $m$-th diagonal minimal model (where the level $m$ parametrizes the central charge as $c = c(m) = 1 - \frac{6}{m(m+1)}$, $m = 3, 4, 5...$) by its least relevant operator, namely the scalar primary field $\phi_{1,3}$, for positive values of the perturbing coupling constant one moves along a Renormalization Group (RG) flow that ends at the $(m - 1)$-th diagonal minimal model [1, 2], while for negative values of the coupling a massive theory is defined, whose scattering matrix is known [3].

However, all the investigations so far on this subject have not considered the non-diagonal minimal models, i.e. the models labelled by $(A, D)$ in the Cappelli, Itzykson, Zuber ADE classification [4]. $\phi_{1,3}$ perturbation of $(A, D)$ models are possible, as for all of them a $\phi_{1,3}$ scalar operator surely exists in the local sector and is unique [1]. It is not difficult to convince oneself (see below) that also in the $(A, D)$ series $\phi_{1,3}$ perturbations with positive coupling induce a flow from level $m$ to level $m - 1$ models. In general at level $m - 1$ there are two models: $(A, A)$ and $(A, D)$ [2]. The question arises: does $(A, D)$ at level $m$ flow to $(A, D)$ or to $(A, A)$ at level $m - 1$? and what is the role of $(A, E)$ models? is there any classification scheme for these flows? In the following we try to give an answer to these questions.

We consider a CFT (that we call “ultraviolet” (UV)), belonging to the set of minimal models, with action $S_{UV}$ and central charge $c_{UV} = c(m)$, and perturb it by its least relevant scalar operator $\phi_{1,3}(z, \bar{z})$. The (euclidean) action of the perturbed model is then

$$S = S_{UV} + g \int dzd\bar{z} \phi_{1,3}(z, \bar{z})$$

(1)

In lagrangian formalism the partition function is expressed in terms of a functional integral

$$Z(g) = \int D\phi e^{-S[\phi]}$$

(2)

where $\phi$ is a set of “fundamental” fields, whose nature is not important in the

1Except for the $(A_4, D_4)$ theory where there are two $\phi_{1,3}^\pm$ operators.

2At levels 11, 12, 17, 18, 29, 30 there is a third exceptional model $(A, E)$. 
following. This quantity, divergent on the sphere, is well defined on a toroidal
geometry $T$. Substituting (1) in (2) and developing to first order in the perturbing
parameter $g$

$$Z(g) = Z_{UV} - g \int_T d^2 w \langle \phi_{1,3}(w, \bar{w}) \rangle_T + O(g^2)$$ (3)

where $w, \bar{w}$ are coordinates on the torus $T$. Translation invariance ensures that one
point functions on the torus are independent of the position. $\int_T d^2 w$ measures the
area of the torus, thus leading to

$$Z(g) = Z_{UV} - g \tau_I \langle \phi_{1,3} \rangle_T + O(g^2)$$ (4)

Of course both $Z_{UV}$ and $\langle \phi_{1,3} \rangle_T$ have a dependance on the modular parameters
$\tau, \bar{\tau}$. Here $\tau_I = \text{Im} \tau = -\text{Im} \bar{\tau}$. A.Zamolodchikov [1], and Ludwig and Cardy [2] have
shown that the applicability of this perturbation expansion is suitable for $m$ large,
i.e. when the conformal dimension of the $\phi_{1,3}$ operator $\Delta_{1,3} = 1 - \epsilon$, with $\epsilon = \frac{2}{m+1}$
tends to 1 and $\phi_{1,3}$ becomes nearly marginal. $\epsilon$ can be then seen as the parameter
of an $\epsilon$-expansion around dimension 2. Indeed, in this regime the Callan-Symanzyk
$\beta$-function

$$\beta(g) = 2\epsilon g - (\pi C_{(13)(13)}^{(13)} + O(\epsilon))g^2 + O(g^3)$$ (5)

where the structure constant $C_{(13)(13)}^{(13)} = 4/\sqrt{3} + O(\epsilon)$ for both $(A, A)$ and $(A, D)$
(see below), shows a non-trivial zero, with infrared (IR) nature, at the point $g^* = \frac{\sqrt{3}}{2\pi} + O(\epsilon^2)$, which is still in the perturbative region. At the IR point conformal
invariance is restored. Central charge $c$ can only decrease on the RG flow from UV
to IR [8], and $c_{UV} - c_{IR}$ can be estimated, integrating the $\beta$-function [1, 2], to be
$12/m^3 + O(\epsilon^4)$ in agreement with $c_{IR} = c(m-1)$.

From the arguments given in [1, 8] it is easy to infer that if the UV theory
is in the $(A, A)$ series, the IR one must also belong to that series. This fact has
been recently confirmed by a calculation of the perturbed partition function at one
loop [3] that we are going to reproduce and extend for the non-diagonal case in
what follows. If however the UV model is in the $(A, D)$ series, from the previous
discussion and the results of [1, 8] it is not clear if it would flow to the IR $(A, D)$ or
to the \((A,A)\) model at \(m-1\). To disentangle between these two possibilities one could study wandering of non scalar operators, which can be cumbersome, or turn to the analysis of the torus partition function along the lines of a recent paper of Ghoshal and Sen \[3\]. Here we shall adopt this second strategy. As \(Z_{IR} = Z(g^*)\), the central object in our calculation is the difference \(\delta Z = Z_{UV} - Z_{IR}\), that is given, at one loop, by the expression

\[
\delta Z = \frac{\sqrt{3} \epsilon \tau I}{2 \pi} \langle \phi_{1,3} \rangle_T + O(\epsilon)
\]

(6)

where \(\langle \phi_{1,3} \rangle_T\) can contain terms proportional to \(1/\epsilon\).

2 - First of all, we have to compute the difference between the candidate UV and IR partition functions in order to compare them later with the one loop calculation. The most economic way to do this, as cleverly suggested in \[3\], is to use the formulas in \[6\] that map the minimal model partition functions on the torus into combinations of the gaussian ones \(Z(R)\), \(R\) being the compactification radius of the gaussian free boson. As we only need to analyze the large \(m\) limit, we can make use of the asymptotics \(Z(R)_{R \to \infty} \sim \sqrt{2} R Z_0\) where

\[
Z_0 = \frac{1}{\sqrt{\tau I} |\eta(\tau)|^2}
\]

(7)

The calculation for the \((A,A)\) series has already been performed in \[3\]. The results for the cases we need to analyze can be summarized as follows:

- for \((A,A)\) \(\to (A,A)\) \(\Rightarrow\) \(\delta Z = \frac{1}{2} Z_0\) (8)
- for \((A,D)\) \(\to (A,D)\) \(\Rightarrow\) \(\delta Z = \frac{1}{4} Z_0\) (9)
- for \((A,D)\) \(\to (A,A)\) \(\Rightarrow\) \(\delta Z = \frac{m}{2} Z_0\) (10)

Hence these three situations can be disentangled by the different coefficients of \(Z_0\) in \(\delta Z\). It is interesting to notice that in the last case \(\delta Z\) is divergent as \(m\) grows, thus signaling that perturbation analysis may not be applicable to this situation. This is not important, as the calculation that follows will show that this possibility is ruled out in favour of the second one.
The main object to compute in order to estimate $\delta Z$ is then the one point function on the torus of the operator $\phi_{1,3}$ at the UV point. Expressions for this function are known by various arguments \cite{7}, but they are in general quite cumbersome to use. Luckily when $\Delta_{1,3} \to 1$, Ghoshal and Sen \cite{5} have shown by a simple argument (that can be extended from $(A,A)$ to $(A,D)$ with no substantial modification) that the final expression of such a one point function simplifies to

$$\langle \phi_{1,3} \rangle_T = 4\pi^2 \sum_{F \in \mathcal{A}} C^F_{(13),F} \chi_h \chi_{\bar{h}}^* \tag{11}$$

where $\chi_h(\tau)$ are the characters of the Virasoro irreducible module of highest weight $h$ and the sum is performed over the set $\mathcal{A}$ of all primary fields $F = (h, \bar{h})$ in the modular invariant sector of the model. This set $\mathcal{A}$ (i.e. the fusion algebra) is known to have different characterizations for $(A,A)$ and $(A,D)$ series. Also, the structure constants $C^F_{(13),F}$ can take different values in the $(A,A)$ and $(A,D)$ cases respectively. For $(A,A)$ models, primary fields are all scalar and labelled by a couple of integers $(r, s)$ such that $1 \leq r \leq m - 1$ and $1 \leq s \leq m$, together with a doubling: $(r, s) = (m - r, m + 1 - s)$. Hence

$$\sum_{F \in \mathcal{A}} = \frac{1}{2} \sum_{r=1}^{m-1} \sum_{s=1}^{m} \tag{12}$$

For $(A,D)$ models \cite{9} the algebra $\mathcal{A}$ naturally acquires a $\mathbb{Z}_2$ grading and splits in two parts: $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ such that $\mathcal{A}_0 \times \mathcal{A}_0 = \mathcal{A}_1 \times \mathcal{A}_1 = \mathcal{A}_0$ and $\mathcal{A}_0 \times \mathcal{A}_1 = \mathcal{A}_1$. The $\mathcal{A}_0$ subalgebra is composed of scalar primary fields, while the $\mathcal{A}_1$ set contains scalars as well as bosonic fields of higher spin.

For $m$ odd, $(A_{m-1}, D_{\frac{m+3}{2}}) \Rightarrow \begin{cases} \mathcal{A}_0 = \{ \phi_{r,s} | \Delta_{rs} = \bar{\Delta}_{rs} \text{ and } s \text{ odd} \} \\ \mathcal{A}_1 = \{ \phi_F | \Delta_F = \Delta_{rs}, \bar{\Delta}_F = \Delta_{r,m+1-s} \text{ and } s = (m+1)/2 \text{ mod } 2 \} \end{cases} \tag{13}$

for $m$ even, $(D_{\frac{m+1}{2}}, A_m) \Rightarrow \begin{cases} \mathcal{A}_0 = \{ \phi_{r,s} | \Delta_{rs} = \bar{\Delta}_{rs} \text{ and } r \text{ odd} \} \\ \mathcal{A}_1 = \{ \phi_F | \Delta_F = \Delta_{rs}, \bar{\Delta}_F = \Delta_{m-r,s} \text{ and } r = m/2 \text{ mod } 2 \} \end{cases}$

This allows to specify the sums in eq.\cite{11} in the various cases.
• for \((A_{m-1}, A_m)\)

\[
\langle \phi_{1,3} \rangle_T = 2\pi^2 \sum_{r=1}^{m-1} \sum_{s=1}^{m} C^{(rs)}_{(13)(rs)} |\chi_{rs}|^2 + O(\epsilon)
\]  

(14)

• for \((A_{m-1}, D_{m+3}/2)\), \(m\) odd

\[
\langle \phi_{1,3} \rangle_T = 2\pi^2 \sum_{r=1}^{m-1} \sum_{s=1}^{m} C^{(rs)}_{(13)(rs)} |\chi_{rs}|^2
\]

\[
+ 2\pi^2 \sum_{r=1}^{m-1} \sum_{s=1}^{m} \text{C}^{F}_{(13),F} \chi_{rs} \chi^*_r \chi_{m+1-s} + O(\epsilon)
\]

(15)

• for \((D_{m+1}, A_m)\), \(m\) even

\[
\langle \phi_{1,3} \rangle_T = 2\pi^2 \sum_{r=1}^{m-1} \sum_{s=1}^{m} C^{(rs)}_{(13)(rs)} |\chi_{rs}|^2
\]

\[
+ 2\pi^2 \sum_{r=1}^{m-1} \sum_{s=1}^{m} \text{C}^{F}_{(13),F} \chi_{rs} \chi^*_r \chi_{m-s} + O(\epsilon)
\]

(16)

The structure constants \(C^{(rs)}_{(13)(rs)}\) for the \((A, A)\) series can be deduced from the general expressions of Dotsenko and Fateev [10]

\[
C^{(rs)}_{(13)(rs)} = -\frac{1}{2\sqrt{3}} \frac{\Gamma \left( \frac{m+\lambda}{m+1} \right) \Gamma \left( \frac{m-\lambda}{m+1} \right)}{\Gamma \left( \frac{1+\lambda}{m+1} \right) \Gamma \left( \frac{1-\lambda}{m+1} \right)}
\]

(17)

where \(\lambda = (m+1)r - ms\). For the \((A, D)\) series the structure constants have been given by Petkova [9] and, as far as only field in the \(A_0\) subalgebra are concerned, they coincide with those of the \((A, A)\) series. We shall see that we need indeed only this subset of \((A, D)\) structure constants; for the interested reader, a nice expression for the remaining ones can be found in [9].

4 - The other delicate point is the analysis of asymptotics of characters for large \(m\). Also this problem has been carried out in [3] for the \((A, A)\) case. The general expression for a Virasoro character is

\[
\chi_{rs}(\tau) = \frac{K_{r,s}(\tau) - K_{r,-s}(\tau)}{\eta(\tau)}
\]

(18)
where $\eta(\tau)$ is the Dedekind function and

\[
K_{r,s}(\tau) = \sum_{k=-\infty}^{+\infty} e^{2\pi i \tau \frac{2m(m+1)k + (m+1)r - ms^2}{4m(m+1)}}
\]  

(19)

For $m$ large, terms are exponentially suppressed in $K_{r,s}$ unless $k = 0$ and $r - s \sim O(1)$. The other function $K_{r,-s}$ is made of exponentially suppressed terms, unless $k = 0$ and both $r, s \sim O(1)$, or $k = -1$ and both $r, s \sim O(m)$. The set of characters that take contributions from $K_{r,-s}$ is then of measure zero (in the large $m$ limit) compared to the whole rectangle $1 \leq r \leq m-1, 1 \leq s \leq m$. Hence contributions to the sum over this rectangle from the second piece can be neglected in the large $m$ limit. This analysis is sufficient to carry out the calculation of $\langle \phi_{1,3} \rangle_T$ for the $(A,A)$ case \[5\], namely taking as an expression of the character for large $m$ the following

\[
\chi_{rs} = \frac{1}{\eta} e^{\pi i \tau \frac{\lambda^2}{2m(m+1)}}
\]  

(20)

What is surprising is that in the $(A,D)$ case, the contributions from the mixed terms are always suppressed. Indeed, in the off-diagonal terms the characters $\chi_{r,m+1-s}$ for $m$ odd or $\chi_{m-r,s}$ for $m$ even appear. Take $m$ odd to fix the ideas. Terms are exponentially suppressed in $K_{r,m+1-s}$ unless $k = 0$ and $r - s + m \sim O(1)$, and in $K_{r,-(m+1)+s}$ unless $k = 0$ and $r \sim O(1), s \sim O(m)$, or $k = -1$ and $r \sim O(m), s \sim O(1)$. Thus $K_{r,-(m+1)+s}$ gives contributions from a set of measure zero (for $m$ large) and can be neglected. The only term that may survive is the product $K_{r,s}K_{r,m+1-s}^*$, which may have contributions only from the intersection between the non suppressed $(r,s)$ region for $K_{r,s}$ and that for $K_{r,m+1-s}$. This intersection, however, in the large $m$ limit is of measure zero too, so that it can be discarded as negligible. The contribution from the structure constants in the non-diagonal terms can at maximum account for a polynomial divergence in $m$, that is anyway suppressed by the exponential dump of all the terms in the characters. The same argument can be carried out for the $m$ even case. In conclusion, for large $m$ there is no contribution to $\langle \phi_{1,3} \rangle_T$ from the terms non diagonal in the characters in the expressions \[13,14\].
This fact allows to finally compute $\delta Z$ for both $(A, A)$ and $(A, D)$ series. All what we have to do is to collect expressions (17,20) and use them into (14,15,16) to evaluate the one point function of $\phi_{1,3}$ and hence $\delta Z$. It is useful, instead of trying to sum on $r$ and $s$ separately, to express the sums in terms of $\lambda$, i.e.

$$\langle \phi_{1,3} \rangle_T = -\frac{\pi^2}{\sqrt{3} |\eta|^2} \sum_{\lambda} \frac{\Gamma\left(\frac{m+\lambda}{m+1}\right) \Gamma\left(\frac{m-\lambda}{m+1}\right)}{\Gamma\left(\frac{1+\lambda}{m+1}\right) \Gamma\left(\frac{1-\lambda}{m+1}\right)} e^{-\pi \tau I \frac{\lambda^2}{m(m+1)}} + O(\epsilon)$$

(21)

The set of $\lambda$ over which the sum has to be performed is the only difference between $(A, A)$ and $(A, D)$ cases:

- for $(A, A)$, $\lambda$ runs from $-m(m-1)$ to $m(m-1)$ with the exclusion of all values that divide $m$ or $m+1$,

- for $(A, D)$, $\lambda$ takes all odd values with the exclusion of those that divide $m$ or $m+1$. Indeed for $m$ odd, $s$ is always odd in (15), and $(m+1)r - ms = \text{even} - \text{odd} = \text{odd}$. Similarly for $m$ even $(m+1)r - ms = \text{odd} - \text{even} = \text{odd}$.

Next, for $(A, A)$ models, we can transform the sum into an integral by considering the variable $x = \lambda/(m+1)$, which, for large $m$, is dense in its integration interval $(-\infty, +\infty)$. The values of $\lambda$ dividing $m$ or $m+1$ become a set of measure zero in the variable $x$ and this constraint can be forgotten. Finally, remembering that $\Gamma(1+x) = x\Gamma(x)$, for $(A, A)$ we have the integral

$$\delta Z = \frac{\pi \epsilon}{4 |\eta|^2} \int_{-\infty}^{+\infty} dx x^2 e^{-\pi \tau I x^2} + O(\epsilon) = \frac{1}{2} Z_0 + O(\epsilon)$$

(22)

For $(A, D)$ as $\lambda$ takes odd values only, it is convenient to parametrize $\lambda = 2\rho - 1$ and then define $x = \rho/(m+1)$, therefore

$$\delta Z = \frac{\pi \epsilon}{4 |\eta|^2} \int_{-\infty}^{+\infty} dx 4x^2 e^{-4\pi \tau I x^2} + O(\epsilon) = \frac{1}{4} Z_0 + O(\epsilon)$$

(23)

Hence, comparing with the results of sect.2, we can state that for $m$ large, $\phi_{1,3}$ perturbations can only let $(A, A)$ flow to $(A, A)$ and $(A, D)$ flow to $(A, D)$. The picture for large $m$ then shows two series between which there is no possible bridge.
created by a $\phi_{1,3}$ flow. In our opinion, this result can be reasonably conjectured to be true for all $m$, as we can not see any reason why a particular value of $m$ might be selected for which this picture breaks. When the lowest $(A, D)$ model is reached, namely at $m = 5$ (3-state Potts model $(A_4, D_4)$), we could still imagine a flow along the “$(A, D)$” series to the $(D_3, A_4)$ model. The coincidence $D_3 = A_3$ tells us that this model has to be identified with that of the $(A, A)$ series, namely the tricritical Ising model $(A_3, A_4)$ at $m = 4$. We then conclude that at $m = 5$ the series of $(A, D) \phi_{1,3}$ flows finally converges into the $(A, A)$ one. This flow from 3-Potts to tricritical Ising, that should not be confused with the flow $(A_4, A_5) \to (A_3, A_4)$, has been recently described by Fateev and Al.Zamolodchikov [11].

6 - Assuming that the picture just described is valid in general, an intriguing observation comes to the eye. The $\phi_{1,3}$ flows among minimal models are such that one of the two Lie algebras labelling the UV model in the ADE classification of [4] is still present in the IR one. The other algebra has been substituted by a lower rank one. More precisely, if we order the two Lie algebras labelling a model such that the first one has lower dual Coxeter number, the rule is that the first Lie algebra of the UV model always coincides with the second algebra of the IR one. The IR model then remembers part of the structure of the partition function of the UV one, and this is a clear signal that there is some “structure” conserved along the flow. This structure can be easily identified with the algebra of non-local conserved currents described in [3]. Each minimal model has two (left) algebras of conserved currents, generated through operator product expansion by the non-local operators $\psi(z)$ and $J(z)$ of conformal dimension $(\Delta_{13}, 0)$ and $(\Delta_{31}, 0)$ respectively (corresponding right algebras are generated by $(0, \Delta_{13})$ and $(0, \Delta_{31})$). The two indices labelling primary fields in minimal models are related to the two different $S\hat{U}(2)$ structures appearing in the Goddard, Kent, Olive (GKO) [12] construction. The first (resp. second) of the two ADE algebras in the classification scheme is in relation with the first (resp. second) $S\hat{U}(2)$ structure, hence with the first (resp.
second) index of primary fields \( r \) (resp. \( s \)). The perturbing field \( \phi_{1,3} \) behaves like an identity w.r.t. the first \( SU(2) \) structure and breaks explicitly only the second one. This explains why the first ADE algebra is still present at the end of the flow. The non-local algebra associated to this first structure is the one generated by the current \( J(z) \) and indeed this is shown in [3] to be conserved off criticality too, at least by perturbative arguments. A well known result for \((A,A)\) that can trivially be extended to \((A,D)\) is that the fields with left conformal dimension \( \Delta_{31} \) at UV evolve to fields with left conformal dimension \( \Delta_{13} \) at IR along a \( \phi_{1,3} \) flow. This means that the \( J(z) \) current of the UV point evolves towards the \( \psi(z) \) field of the IR point. The algebra generated by \( \psi(z) \) is associated with the second index \( s \) of primary fields and with the second algebra in the ADE classification. This clarifies why the first algebra at UV becomes the second at IR.

To be more precise, instead of speaking of conservation of \( J(z) \) along the flow, it would be better to think in terms of spontaneous symmetry breaking. Indeed, as widely documented [13, 14] in the literature for the simplest cases of tricritical Ising flowing to Ising and tricritical 3-state Potts flowing to 3-state Potts, the UV conserved current \( J(z) \) ensures a symmetry of the theory which is no more present in the IR limit. For example the \( J(z) \) current for tricritical Ising is the fermionic partner of stress-energy tensor, guaranteeing supersymmetry of the UV point. The IR point (Ising model) is not supersymmetric, and the \( \psi(z) \) field (the Onsager fermion) has been interpreted as the goldstino of the spontaneously broken supersymmetry. We think that this picture can be extended to the general case: along the \( \phi_{1,3} \) flows there is spontaneous symmetry breaking of the symmetry encoded in the non-local algebra generated by the \( J(z) \) current. The corresponding goldstino evolves to the \( \psi(z) \) field as the IR limit is approached. Notice that the sum of the left conformal dimension of the spontaneously broken current at UV and of the goldstino field at IR always equals 2, and this intriguing observation seems to be a general feature of two dimensional spontaneous symmetry breaking to be better understood.

Tricritical Ising can also be seen as the lowest of a series of minimal models for
the superconformal algebra generated by stress-energy tensor and the \(J(z)\) current. It is known that “fractional” superconformal algebras can be generated by the stress-energy tensor plus \(J(z)\) currents in general \([13, 4, 16]\) and show series of minimal models that can be identified with \(SU(2) \otimes SU(2)/SU(2)\) GKO cosets. Flows between models in these series have been studied, at least for \((A, A)\) series \([17]\).

The lowest model of each series is a minimal model (in the usual Virasoro sense), and if we insist to perturb it by \(\phi_{1,3}\) we go to an IR limit that does not belong to the series of \(J(z)\) invariant models any more, thus forcing spontaneous symmetry breaking of the fractional supersymmetry.

Finally, the (spontaneously broken) conservation of the current \(J(z)\) allows us to complete the picture of \(\phi_{1,3}\) flows between minimal models, by considering the \((A, E)\) models too. Let us concentrate on \(E_6\) (the same arguments can be repeated for \(E_7\) and \(E_8\)). There are two \((A, E_6)\) models, namely \((E_6, A_{12})\) at \(m = 12\) and \((A_{10}, E_6)\) at \(m = 11\). Let us first consider the \((E_6, A_{12})\) model as an UV point. The \(\phi_{1,3}\) operator exists in the model and can be shown to generate a perturbation whose \(\beta\)-function is of the same form as for \((A, A)\) series. Hence we reasonably expect that it flows to an IR model at \(m = 11\). There are 3 models at \(m = 11\), but if we further assume that the first ADE algebra of UV must coincide with the second of IR, we uniquely select the \((A_{10}, E_6)\) model. Thus this argument strongly suggests that there is a flow \((E_6, A_{12}) \rightarrow (A_{10}, E_6)\). Next one can ask if the \((A_{10}, E_6)\) model can be considered as an UV point of a \(\phi_{1,3}\) flow. The answer is no, for the simple reason that there is no local scalar operator \(\phi_{1,3}\) in this model, as an inspection of the modular invariant partition function shows. Hence we cannot even speak of \(\phi_{1,3}\) perturbation of this model. The other question is if it exists a model at \(m > 12\) that can flow to \((E_6, A_{12})\). This model should contain \(A_{12}\) as its first Lie algebra, hence the only candidates are \((A_{12}, A_{13})\) or \((A_{12}, D_8)\) at \(m = 13\). Both are ruled out simply by the fact that the \(\phi_{1,3}\) flow defines a theory with no ambiguity, i.e. it must have a unique IR limit, which is \((A_{11}, A_{12})\) and \((D_6, A_{12})\) respectively, living no room for \((E_6, A_{12})\) as an IR limit of a \(\phi_{1,3}\) flow. Hence we conclude that,
besides the two series \((A, A) \to (A, A)\) and \((A, D) \to (A, D)\) there exist only 3 other isolated flows \((E, A) \to (A, E)\) and this should reasonably exhaust all the possible \(\phi_{1,3}\) flows between minimal models, as summarized in the following table:

- for all \(m\): \((A_{m-1}, A_m) \to (A_{m-2}, A_{m-1})\) conserving \(A_{m-1}\)
- for \(m\) even: \((D_{\frac{m+1}{2}}, A_m) \to (A_{m-2}, D_{\frac{m+1}{2}})\) conserving \(D_{\frac{m+1}{2}}\)
- for \(m\) odd: \((A_{m-1}, D_{\frac{m+3}{2}}) \to (D_{\frac{m+1}{2}}, A_{m-1})\) conserving \(A_{m-1}\)
- for \(m = 12\): \((E_6, A_{12}) \to (A_{10}, E_6)\) conserving \(E_6\)
- for \(m = 18\): \((E_7, A_{18}) \to (A_{16}, E_7)\) conserving \(E_7\)
- for \(m = 30\): \((E_8, A_{30}) \to (A_{28}, E_8)\) conserving \(E_8\)

Notice that the possible algebras of conserved currents (generated by \(J(z)\)) follow an ADE classification too. Also notice that this list is in agreement with similar results on the perturbation of minimal models coupled to gravity recently obtained using KdV techniques by Di Francesco and Kutasov [18].

Concluding, this paper only gives a perturbative argument and makes use of some observation concerning the conserved non-local currents to disentangle the structure of \(\phi_{1,3}\) flows between minimal models. We believe that a better understanding of the complete picture of integrable perturbations of minimal models, that should include for example knowledge of the scattering matrices for \(\phi_{1,3}\) perturbed \((A, D)\) and \((A, E)\) models in the negative coupling (massive) direction, as well as correlation functions for non-critical models, could be obtained by a deeper study of the non-local currents and the quantum group symmetries underlying them.

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