Distribution free goodness of fit tests for regularly varying tail distributions

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Abstract We discuss in this paper a possibility of constructing a whole class of asymptotic distribution-free tests for testing regularly varying tail distributions. The idea is that we treat the tails of distributions as members of a parametric family and using MLE to estimate the exponent. No matter what the exponent’s estimator is, we are able to transform the whole class into a specific distribution with a prefix exponent so that we are free from choosing any functional of the tail empirical process as a distribution-free test statistic. The asymptotic behaviour of some new tests, as examples from the whole class of new tests, are demonstrated as well.

Keywords Regularly varying · distribution free · goodness of fit test · unitary transformation.

1 Introduction

Suppose that we have a random sample $X_1, X_2, \cdots, X_n$ of a random variable $X$ following some unknown distribution $F$. No matter how $F$ behaves, we are mainly interested in the right tail behavior of $F$, i.e., in $F(x)$ when $F(x) \to 1$. Among the class of heavy-tail distributions, regularly varying tail distributions has been attracted most attention due to its various applications. Briefly, a distribution $F$ is said to be regularly varying in tail if

$$\lim_{x \to \infty} \frac{1-F(tx)}{1-F(x)} = t^{-\theta}$$

for all $t > 0$ where $\theta > 0$ is called the index or exponent of regular variation. By this definition, regularly varying tail distributions are often indicated as Pareto-type distributions.

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Regular variation of the tail of a distribution often appears as a natural condition in various theoretical results of probability theory. The most typical example is that it is the condition for the distribution of the partial maxima to belong to the domain of attraction of extreme value distributions. This could be found in various references, we refer to books of de Haan and Ferreira [8] and Resnick [27] among others. Moreover, regular variation naturally arises as a common phenomenon in numerous practical case studies such as finance, insurance, physics, geology, hydrology and engineering, etc. Therefore, detecting and testing regular variation plays an important role in probabilistic areas as well as in practical applications.

The problem of estimating the exponent $\theta$ has a very rich literature. Among various studies, the Hill estimator is the most common one, see Hill [18]. Recall that the Hill estimator of the exponent $\theta$ essentially takes on the following form

$$\hat{\theta}_H = \left( \frac{1}{k} \sum_{i=1}^{k} \ln X_{i,n} - \ln X_{k,n} \right)^{-1}$$

where $k = k(n)$ is a certain sample fraction, $k(n) \to \infty$ in an appropriate way and $X_{1,n} \leq \cdots \leq X_{n,n}$ denotes the upper order statistics of the sample $X_1, \cdots, X_n$. Many other estimators was proposed, most of them is also based on the upper order statistics and is not too difficult to compute. These include, to name a few, popular estimators introduced by Dekkers et. al. [9], de Haan and Resnick [21], Pickands [25], Teugels [29] among others. The rate of the convergence of an estimator was also discussed, for example, in Hall and Welsh [10].

In contrast to the numerous number of approaches to estimate the exponent $\theta$, using goodness of fit (gof) for testing regular variation has been addressed in a modest number of studies. Not long ago, Beirlant et al. [1] modified the Jackson statistic - which was originally proposed as a gof for testing exponentiality - for testing Pareto-type data. Koning and Peng [26] examined the Kolmogorov-Smirnov, Berk-Jones and the estimated score tests and compare them in terms of Bahadur efficiency. In that paper, the Berk-Jones and estimated score tests, which are not based on the empirical process, were shown to perform better than the Kolmogorov-Smirnov test.

Note that regularly varying tail distributions are members of the family of the generalized Pareto distribution (GPD), which is of the form

$$F(x; k, \sigma) = \begin{cases} 1 - (1 - \frac{k x}{\sigma})^{1/k}, & k \neq 0, \sigma > 0 \\ 1 - e^{-x/\sigma}, & k = 0, \sigma > 0 \end{cases}$$

where $k$ and $\sigma$ are shape and scale parameters and $x > 0$. To test the fit of data to a GPD, there has been several studies such as Davison and Smith [5], Choulakian and Stephens [3]. In these papers, the critical values for Cramer-von Mises statistic and Anderson-Darling statistic, which is a weighted Cramer-von Mises statistic, were given.
In this paper, we will introduce a wide class of asymptotically distribution free gof tests for testing regularly varying tail distributions. Our approach follows a new method introduced in Khmaladze [23]. In that paper, the author proposed a unitary transformation which enables us to create a class of gof tests for both simple and parametric hypothesis testing problems. The method for the latter will be adopted for our problem, which we will present in Section 2.2. Briefly speaking, we will use exactly the same transformation in [23] to derive a modification of the empirical process. Only this time, the assumption that \( F \) belongs to a parametric family of distributions is no longer available and the fact is that the right tail just partially describes the distribution \( F \). Hence, we can only consider the tail empirical process and transform it. The transformed tail empirical process, under the hypothesis of interest, possesses a limit in distribution asymptotically free from any underlying distribution \( F \) as well as the unknown index/exponent of the regular variation. Therefore, any appropriate functionals of the transformed process could be used as asymptotically distribution free test statistics. Section 2.1 will be devoted for the main literature of the empirical process based on the tail.

Some simulation results will be documented in Section 3. Namely, we will take the Kolmogorov-Smirnov (KS), Cramer-von Mises (\( \Omega^2 \)) and Anderson-Darling (\( A^2 \)) tests as examples from the new class of asymptotically distribution free gof tests for demonstration. The asymptotically distribution free property of these test statistics will be illustrated for different choices of the original distribution \( F \).

2 Main results

2.1 Overall review

Suppose that from a random sample \( X_1, \ldots, X_n \) we are only considering excesses over a certain threshold \( x_0 \) which is considerably large. Let us denote the subsample of all observations exceeding \( x_0 \) by \( \tilde{X}_1, \ldots, \tilde{X}_m \). Generally, the choice of \( x_0 \) as well as the sample fraction \( m/n \) may require some educated guessing. Hill [18] suggested \( m \) be chosen as an adaptive, data-analytic basis. Several methods for choosing \( m \) or \( x_0 \) based on survey data could be found in Drees and Kaufmann [11], Danielsson et al. [4] and Guillou and Hall [20]. Nevertheless, there has existed various research assuming that \( m \) is known such that \( m \to \infty \) as \( n \to \infty \) but \( m = o(n) \). With this assumption and some prior knowledge about the underlying distribution function \( F \), Haeusler and Teugels [17] derived a general condition which can be used to determine the optimal value \( m \) explicitly. Hall [15] also considered \( m \) having a deterministic value and a quite common estimate of the exponent was introduced. Throughout this paper, our approach will be based on the same such assumption, that also means \( x_0 \) fixed.

Our main aim is to create a class of gof test for testing the hypothesis \( H_0 : \) “\( F \) is a regularly varying tail distribution” against the alternative \( H_1 : \) “\( F \) is
not a regularly varying tail distribution”. It is sensible that the exponent $\theta$ needs to be estimated based on the right tail only. Specifically, we only consider observed values $\{\tilde{X}_1, \cdots, \tilde{X}_m\}$, which now should be looked at as a sample of a different random variable, let say, $\tilde{X}$.

Denote $\tilde{T} = \frac{\tilde{X}}{x_0}$ and $T = \tilde{T} - 1$. Under the hypothesis of interest and by the definition of the regular variation, the survival distribution of $\tilde{X}$ conditional on the specified large value $x_0$ is

$$\frac{P\{\tilde{X} \geq x_0 t\}}{P\{\tilde{X} \geq x_0\}} = \left(1 - \frac{F(x_0 t)}{1 - F(x_0)}\right) = t^{-\theta} + o(1), \quad t \geq 1, \theta \geq 0. \quad (4)$$

Then, the distribution of the positive continuous random variable $T$ under the null hypothesis is

$$H_\theta(t) = 1 - (1 + t)^{-\theta}, \quad t \geq 0. \quad (5)$$

Clearly, the density function is $h(t) = \theta(1 + t)^{-(\theta+1)}$. Viewing $T$ as a new random variable, we can define the tail empirical process in a similar way as of the standard empirical process, i.e., we have

$$H_m(t) = \frac{1}{m} \sum_{i=1}^{m} 1\{T_i \leq t\}. \quad (6)$$

The tail parametric empirical process $\hat{v}_m$ is

$$\hat{v}_m(t) = \sqrt{m}[H_m(t) - H_{\hat{\theta}_m}(t)]. \quad (7)$$

where $\hat{\theta}_m$ is an estimator of $\theta$ calculated from the subsample $\tilde{X}_1, \cdots, \tilde{X}_m$ (or $T_1, \cdots, T_m$). Assume that the true unknown exponent under the hypothesis $H_0$ is $\theta_0$. Let $\hat{\theta}_m$ be the maximum likelihood estimator (MLE), then it is the solution of the equation

$$\sum_{i=1}^{m} \frac{\partial \log h(T_i, \theta)}{\partial \theta} = 0. \quad (8)$$

That yields

$$\hat{\theta}_m = \frac{m}{\sum_{i=1}^{m} \log(T_i + 1)}, \quad (9)$$

which actually coincides with the Hill estimator $\hat{\theta}^H$. This estimator was proved to be consistent in the sense that

$$\hat{\theta}^H = \hat{\theta}_m \xrightarrow{P} \theta_0 \quad (10)$$

under the condition that $m \to \infty$ as $n \to \infty$ such that $m/n \to 0$. The proof of this convergence could be found in Mason [24]. Regarding the asymptotic
normality of the estimator, we refer to Haeusler and Teugels [17], Geluk et al. [19], de Haan and Resnick [7] among various others.

We will spend few more lines here to review the property of the limit in distribution of the process $\hat{v}_{mH}$ because it is essential for the method we present below. For all the concepts and terminologies, we follow and keep the same as in [23]. For the sake of lucidity, we extract and represent the method only for our particular problem.

Consider the space $L_2(H)$. Recall that a function $\phi$ is integrable with respect to $H$ if $\int_0^\infty \phi(s)H(ds) < \infty$ and square integrable if $\int_0^\infty \phi^2(s)H(ds) < \infty$. The space $L_2(H)$ consists of all square integrable functions with respect to $H$. The inner product and norm in $L_2(H)$ are defined as usual. That is, for any $\phi, \tilde{\phi} \in L_2(H)$ we have

$$\|\phi\|^2_H = \int_0^\infty \phi^2(s)H(ds),$$

$$\langle \phi, \tilde{\phi} \rangle_H = \int_0^\infty \phi(s)\tilde{\phi}(s)H(ds).$$

Denote by $w_H(\phi)$ a function-parametric $H$-Brownian motion where $\phi$ is a square integrable function in $L_2(H)$. That means, $w_H(\phi)$ for each $\phi$ is a Gaussian random variable with expected value 0 and variance $\mathbb{E}w_H^2(\phi) = \|\phi\|^2_H$. This also implies that the covariance between $w_H(\phi)$ and $w_H(\tilde{\phi})$ is

$$\mathbb{E}w_H(\phi)w_H(\tilde{\phi}) = \langle \phi, \tilde{\phi} \rangle_H.$$

If $\phi_t(t^*) = 1_{t^* \leq t}$ then $w_H(\phi_t) := w_H(t)$ is simply the Brownian motion in time $H(t)$. As usual, a linear transformation of $w_H(t)$ which is of the form $v_H(t) = w_H(t) - H(t)w_H(\infty)$ is the Brownian bridge in time $H(t)$.

Assuming that under the null hypothesis $H_0$, the true unknown exponent of the regular variation is $\theta_0$. Denote by $v_{mH}$ the usual tail empirical process, that is,

$$v_{mH}(t) = \sqrt{m}[H_m(t) - H_{\theta_0}(t)].$$

Let us stress that, since we assumed $x_0$ is known, the limit in distribution of $v_{mH}(t)$, as a conditional tail empirical process, is similar to that of the standard empirical process, which is the Brownian bridge $v_H(t)$. For a full description on the property of general tail empirical process, we refer to Einmahl [13,14]. It was proved in these papers that if such process is unconditional on $x_0$, the limit in distribution of $v_{mH}(t)$ is a Brownian motion in time $H(t)$. On a recent approach on distribution free gof test for testing tail copula by Can et al. [2], the tail empirical process - constructed on tail copula, was mapped to a standard Brownian motion. Their construction based on the innovative martingale method in Khmaladze [21] which is known as the Khmaladze transformation. Our approach here use a different Khmaladze transformation in [23], and so let us call it Khmaladze-2.
Considers the function-parametric version of the tail empirical process

\[ v_{m\theta}(\phi) = \int_{0}^{\infty} \phi(t)v_{m\theta}(\phi) \, dt = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \left[ \phi(T_i) - E \phi(T_i) \right] \]

(12)

where \( \phi \) is a function in \( L_2(H) \). In a similar way to achieve the limit of \( v_{m\theta}(\phi) \), the limit in distribution of \( v_{m\theta}(\phi) \) with some proper restriction on functions \( \phi \) is called a function-parametric \( H \)-Brownian bridge. Specifically,

\[ \nu_{H}(\phi) = w_{H}(\phi) = \langle \phi, 1 \rangle_{H} w_{H}(1), \]

(13)

where 1 stands for the function identically equals to 1.

We can also say more about the asymptotic behavior of the tail parametric empirical process where \( \hat{\theta}_m \) is the MLE which is also the Hill’s estimator. It is well-known since long ago that under some usual and mild constraints, MLE \( \hat{\theta}_m \) possesses the asymptotic property

\[ \sqrt{m} (\hat{\theta}_m - \theta_0) = \Gamma_{H}^{-1} \int_{0}^{\infty} \frac{\hat{h}_{\theta_0}(t)}{h_{\theta_0}(t)} v_{m\theta}(dt, \theta_0) + o_P(1), \quad m \to \infty \]

(14)

where \( h_{\theta_0} \) denotes the hypothetical density and \( \hat{h}_{\theta_0} \) its derivatives in \( \theta \). Denote by

\[ \Gamma_{H} = \int_{0}^{\infty} \frac{\hat{h}_{\theta_0}^2(t)}{h_{\theta_0}^2(t)} \, H(dt) \]

(15)

the Fisher information. Then it is easy to check that \( \Gamma_{H} = 1/\theta_0^2 \). As a consequence of (14), we can expand the process \( \hat{v}_{m\theta}(t) \) as

\[ \hat{v}_{m\theta}(t) = v_{m\theta}(t) - \sqrt{m}[H_{\theta_0}(t) - H_{\theta_0}(t)] \]

\[ = v_{m\theta}(t) - \int_{0}^{t} \frac{\hat{h}_{\theta_0}(t)}{h_{\theta_0}(t)} H_{\theta_0}(dt, \theta_0) \Gamma_{H}^{-1} \int_{0}^{\infty} \frac{\hat{h}_{\theta_0}(t)}{h_{\theta_0}(t)} v_{m\theta}(dt, \theta_0) + o_P(1) \]

\[ = v_{m\theta}(t) - \int_{0}^{t} \beta_{H}(t) H_{\theta_0}(dt) \int_{0}^{\infty} \beta_{H}(t) v_{m\theta}(dt, \theta_0) + o_P(1) \]

(16)

where

\[ \beta_{H}(t) = \Gamma_{H}^{-1/2} \frac{\hat{h}_{\theta_0}(t)}{h_{\theta_0}(t)} \]

(17)

denotes the normalized score function. This expression represents the limit in distribution \( \hat{v}_{m\theta}(t) \) of the process \( \hat{v}_{m\theta}(t) \), that is, an orthogonal projection of the process \( v_{H} \) parallel to the normalized score function \( \beta_{H} \). Note that \( \beta_{H} \) by its definition is of unit norm in the space \( L_2(H) \). Moreover, functions \( \beta_{H} \) and \( \beta_{H} \) are orthogonal. Therefore, from (16) we have the limit in distribution of the process \( \hat{v}_{m\theta}(\phi) \) is

\[ \hat{v}_{H}(\phi) = v_{H}(\phi) - \langle \phi, \beta_{H} \rangle_{H} v_{H}(\beta_{H}) \]

\[ \rightarrow \quad w_{H}(\phi) - \langle 1, \phi \rangle_{H} w_{H}(1) - \langle \beta_{H}, \phi \rangle_{H} w_{H}(\beta_{H}). \]

(18)
This implies that \( \hat{v}_H \) is an orthogonal projection of the function-parametric \( H \)-Brownian motion \( w_H \) parallel to the subspace generated by 2 functions \( \{1, \beta_H\} \). The process \( \hat{v}_H \) depends not only on the true unknown exponent \( \theta_0 \) but also the score function \( \beta_H \). In the terminology of [23], the process \( \hat{v}_H \) is called a \( \beta_H \)-projected \( H \)-Brownian motion.

2.2 Method

Our approach follows the method in Khmaladze [23] for parametric family of distributions. The main idea can be briefly explained as follows: Under the null hypothesis, the empirical process \( \hat{v}_{mH}(\phi) \) (see (7) and (12)) constructed on the tail of an unknown distribution \( F \), from a fixed threshold \( x_0 \), possesses an “unspecified” limit in distribution (see (18)). “Unspecified” here means that the limit depends on some unknown parameter. We will map the process \( \hat{v}_{mH} \) into another process \( \hat{v}_{mG} \) (see (22)) whose limit in distribution is specified. The mapping procedure is one-to-one to guarantee that no statistical information is lost.

We chose \( G \) to be the exponential distribution \( G(t) = 1 - e^{-t} \) simply by preference with some reasons. That is, both \( G \) and \( H \) are members of the GPD family (see (3)). The distribution \( G \) is the limiting distribution of the GPD(\( k, \sigma \)) as \( k \to 0 \) and scaled by \( \sigma = 1 \). A research by Davison and Smith [5] employed Kolmogorov-Smirnov and Anderson-Darling statistics to test the fit of the GPD to data using the critical value derived from an exponential distribution for the purpose. They discussed that the approach may be suspect since the exponential distribution is just a member of the whole family though.

It is obvious that distributions \( G \) and \( H \) are equivalent or in other words, mutually absolutely continuous. It is also easy to check that the Fisher information of \( G \) is \( \Gamma_G = 1 \). Put

\[
\ell(t) = \sqrt{\frac{dG}{dH}(t)} = \theta^{-\frac{1}{2}}(1 + t)^{\frac{\theta+1}{2}}e^{-\frac{1}{2}},
\]

then this function belongs to \( L_2(H) \). In addition, if \( \phi \in L_2(G) \) then \( \ell\phi \in L_2(H) \) and \( \|\phi\|_G = \|\ell\phi\|_H \). Note that when \( H \) and \( G \) are equivalent, it is known that a transformation from a \( H \)-Brownian motion into a \( G \)-Brownian motion is straightforward by multiplication. Namely, \( w_H(\ell\phi) = w_G(\phi) \) is a \( G \)-Brownian motion in \( L_2(G) \). However, mapping a Brownian bridge such as \( v_H \) or \( \hat{v}_H \) to another Brownian bridge is not that straightforward any more. The fact is that \( v_H(\ell\phi) \) depends on both \( H \) and \( G \) and so does \( \hat{v}_H(\ell\phi) \).

Consider a subspace \( \hat{L} \) of \( L_2(H) \) generated by four functions \( \{1, \beta_H, \ell, \ell\beta_G\} \). These functions are of unit norm in \( L_2(H) \). More explicitly, the score functions \( \beta_H \) and \( \beta_G \) are

\[
\beta_H(t) = 1 - \theta \log(1 + t), \quad \beta_G(t) = 1 - t.
\]
For any function $f$ and $g$ in $L_2(H)$, define the unitary operator $K_{f,g}$ as

$$K_{f,g} = I - \frac{1}{1 - \langle f, g \rangle_H} (g - f)(g - f, \cdot)_H$$

where $I$ is the identity function. This unitary operator will turn the function $f$ into $g$ and reversely $g$ to $f$ and any function which is orthogonal to $f$ and $g$ in the subspace $L_2(H)$ into itself.

Step into the method, we first consider the unitary operator $K_{1,\ell} = I - \frac{1}{1 - \langle 1, \ell \rangle_H} (\ell - \mathbb{1})(\ell - \mathbb{1}, \cdot)_H$. This operator will map $\ell$ to $1$ and $1$ to $\ell$. Next, consider the image of the function $\ell \beta_G$ via $K_{1,\ell}$, which is

$$\tilde{\ell} \beta_G = \ell \beta_G - \frac{1}{1 - \langle 1, \ell \rangle_H} (\ell - 1)(\ell - 1, \ell \beta_G)_H$$

$$= \ell \beta_G - \frac{1}{1 - \langle 1, \ell \rangle_H} \int_0^\infty (s - 1)(s - 1, \ell \beta_G)_H ds.$$

Then consider the operator $K_{\beta_H, \tilde{\ell} \beta_G}$ defined as

$$K_{\beta_H, \tilde{\ell} \beta_G} = I - \frac{1}{1 - \langle \beta_H, \tilde{\ell} \beta_G \rangle_H} (\tilde{\ell} \beta_G - \beta_H)(\tilde{\ell} \beta_G - \beta_H, \cdot)_H.$$

Set $\tilde{K} = K_{\beta_H, \tilde{\ell} \beta_G} K_{1,\ell}$, then this unitary operator will map $\ell$ to $\mathbb{1}$ and $\ell \beta_G$ to $\beta_H$. The non-uniqueness of such unitary operator like $\tilde{K}$ was discussed thoroughly in Khmaladze [23], Section 3.4. Nevertheless, we believe that this operator $\tilde{K}$ is simple enough for practical purpose, especially with only one parameter.

The main result for testing composite hypothesis was stated as Theorem 7 in [23], it is essential so we restate it here accordingly to our notations.

**Theorem 1** (Restatement of Theorem 7 in [23])

If $\hat{v}_H$ is a $\beta_H$-projected $H$-Brownian motion and $G$ is absolutely continuous with respect to $H$, then

$$\hat{v}_G(\phi) = \hat{v}_H(\tilde{K}(\ell \phi)) = \tilde{K}(\hat{v}_H(\ell \phi))$$

is a $\beta_G$-projected $G$-Brownian motion.

As a consequence, transform the function-parametric tail empirical process $\hat{v}_{mH}(\phi)$ by $\tilde{K}$, we obtain another process

$$\hat{v}_{mG}(\phi) = \hat{v}_{mH}(\tilde{K}(\ell \phi)) = \tilde{K}(\hat{v}_{mH}(\ell \phi)),$$

which has $\hat{v}_G(\phi)$ as a limit in distribution.
Let
\[ \phi_x(t) = 1_{\{t \leq x\}} \]
be series of indicator functions defined on \( L_2(H) \) depending on \( x \). Let \( x \) runs from 0 to \( \infty \). From Theorem 1, we know that the limit in distribution \( \hat{v}_m \) of the process \( \hat{v}_m(\hat{K}(\ell \phi_x)) \) is a \( \beta \)-projected G-Brownian motion in \( \phi_x \). Hence, any statistic as an appropriate function based on \( \hat{v}_m(\hat{K}(\ell \phi_x)) \) will be asymptotically distribution free. Denote by \( \tilde{\phi}_x \) the image of \( \ell \phi_x \) via the operator \( \hat{K} \), that is,
\[ \tilde{\phi}_x = \hat{K}(\ell \phi_x) = K_{\beta_H, \ell \beta_G} K_{L, \ell}(\ell \phi_x). \]

For practical purpose, the form of \( \tilde{\phi}_x \) is
\[ \tilde{\phi}_x = \ell \phi_x - \frac{1}{1 - \langle \ell, 1 \rangle_H} (\ell - 1)(\ell - 1, \ell \phi_x)_H - \frac{1}{1 - \langle \beta_H, \ell \beta_G \rangle_H} (\ell \beta_G - \beta_H)
\[ \times \left[ (\ell \beta_G - \beta_H, \ell \phi_x)_H - \frac{\langle \ell - 1, \ell \phi_x \rangle_H}{1 - \langle \ell, 1 \rangle_H} (\ell \beta_G - \beta_H, \ell - 1)_H \right]. \]

Applying the unitary operator \( \hat{K} \) on the process \( \hat{v}_m(\ell \phi_x) \) we have
\[ \hat{v}_m(\phi_x) = \hat{K}(\hat{v}_m(\ell \phi_x)) = \hat{v}_m(\tilde{\phi}_x) = \int_0^\infty \tilde{\phi}_x(t) \hat{v}_m(dt) \]
\[ = \frac{1}{\sqrt{m}} \sum_{i=1}^m [\tilde{\phi}_x(\hat{T}_i)] - E_{\hat{\theta}_n} \tilde{\phi}_x(\hat{T}_i)], \]
(24)
Here \( E_{\hat{\theta}_n} \) denotes the expected value with respect to the distribution \( H_{\hat{\theta}_n} \).
That means, \( E_{\hat{\theta}_n} f(\hat{T}) = \int_0^\infty f(s) h_{\hat{\theta}_n}(s) ds \) for any integrable function \( f \). As a result of the Theorem [3], the limit in distribution of process \( \hat{v}_m(\tilde{\phi}_x) = \hat{K}(\hat{v}_m(\ell \phi_x)) \) is \( \hat{v}_G(\phi_x) \) - a projected G-Brownian motion. We demonstrate in the next Section that any functionals of \( \hat{v}_m(\tilde{\phi}_x) \) is asymptotically distribution free.

3 Simulation results

The main purpose of this section is to show the asymptotically distribution free property of the new test statistics based on the transformed empirical process \( \hat{v}_m(\tilde{\phi}_x) \). To do so, we will take some prevalent gof tests as examples, namely the Kolmogorov-Smirnov (KS), the Cramer-von Mises and the Anderson-Darling tests. These test statistics are some specific functionals of the transformed process \( \hat{v}_m(\tilde{\phi}_x) \). Namely, an analogue version of the Kolmogorov-Smirnov test statistic is
\[ KS = \max_x \left| \hat{v}_m(\tilde{\phi}_x) \right|. \]
(25)
The Cramér-von Mises statistics should be of the form

$$\Omega^2 = \int_{0}^{\infty} \tilde{v}_{mH}(\tilde{\phi}_x) dG(x),$$

and its weighted version called Anderson-Darling statistic is

$$A^2 = \int_{0}^{\infty} \frac{\tilde{v}_{mH}^2(\tilde{\phi}_x)}{G(x)(1 - G(x))} dG(x).$$  \hspace{1cm} (26)
As shown in Section 2.2, the limit in distribution of the process \( \tilde{v}_{mH}(\tilde{\phi}_x) \) is a projected Brownian motion in time \( G(x) \). Hence, theoretically these test statistics are asymptotically distribution free. Especially, it makes sense that we integrate with respect to nothing else but \( G(x) \) in order to get the \( \Omega^2 \) and \( A^2 \) tests. In principle, \( x \) runs from 0 to infinity. However, we choose \( x \in \{0.1, 0.2, \cdots, 7.9, 8\} \) since \( G(8) = 0.9997 \approx 1 \) hence we practically just need \( x_{\text{max}} = 8 \) as the maximum value for \( x \). The \( \Omega^2 \) and \( A^2 \) tests can be approximated as follows:

\[
\Omega^2 \approx 0.1 \times \sum_{x=0.1}^{8} \tilde{v}_{mH}(\tilde{\phi}_x)e^{-x},
\]

\[
A^2 \approx 0.1 \times \sum_{x=0.1}^{8} \frac{\tilde{v}_{mH}(\tilde{\phi}_x)}{1-e^{-x}}.
\]

To create the curves of the cumulative distributions of the new tests, we choose Pareto and Cauchy distributions as the underlying distributions. It is known that the tail of the Cauchy distribution is regularly varying with the exponent \( \theta_0 = 1 \). For the Pareto distributions, the exponent is arbitrarily selected and positive. We did choose some \( \theta_0 \) ranging from 0.5 to 10. Sample size \( n \) for simulation needs to be large to guarantee that the tail is sufficiently big, namely, \( m \) is not less than 40. Hence, we chose \( n \) at least equals 1000. We also chose different threshold \( x_0 \), namely \( x_0 = 3, 5, 10 \). Usually, each curve is produced by 5000 iterations of simulation.

Figures 1, 2 and 3 show the plots of the cumulative functions for \( KS, \Omega^2, A^2 \) test statistics respectively with different choices of the threshold \( x_0 \), different sample sizes and different underlying distributions. As we can see in these figures, the two curves of the cumulative distribution functions in each plot are not distinguishable, which practically demonstrate the asymptotically distribution free property of the new test statistics. Moreover, as we notice, for different \( x_0 \), the difference between these curves is also very minor.
Regarding the time of the procedure, it took approximately 1 hour to create the cumulative distribution functions with $x_0 = 5$ and sample size $n = 5000$ by 5000 iterations for two different original distributions $F$. Therefore, we believe that the method is easy to implement and also quick.

4 Discussion

It is possible to extend this approach to testing multidimensional regularly varying tail distributions. As long as we are able to find a transparent expression for the class of regularly varying tail distributions with an explicit parametric family form, we are able to transform the whole family into a specific distribution, from that we can build up a whole class of asymptotic distribution-free test statistics.

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References

1. Beirlant, J., de Wet, T. and Goegebeur, Y., A goodness of fit statistic for Pareto-type behaviour, Journal of Computational and Applied Mathematics, 186, pp. 99-116, (2006)
2. Can, S. M, Einmahl, J. H. J, Khmaladze, E. V and Laeven, R. J. A., Asymptotically distribution-free goodness-of-fit testing for tail copulas, Ann. Statist., 43 (2), pp. 878-902, (2015)
3. Choulakian, V. and Stephens, M. A., Goodness of fit tests for the generalized Pareto distribution, Technometrics, 43, No. 4, pp. 478-484, (2001)
4. Danielsson, J., de Haan, L., Peng, L. and de Vries, C. G., Using a bootstrap method to choose the sample fraction in tail index estimation, Journal of Multivariate Analysis, 76, No. 2, pp. 236-248, (2001)
5. Davison, A. C., Smith, R. L., Models for exceedances over high threshold, J. Roy. Statist. Soc. B: Methodological 52 (3), pp. 393-442. (1990)
6. De Haan, L. and Resnick, S. I., A simple asymptotic estimate for the index of a stable distribution, J. R. Statist. Soc. B, 42, pp. 83-88, (1980)
7. De Haan, L. and Resnick, S. I., On asymptotic normality of the Hill estimator, Comm. Statist. Stochastic Models, 14(4), pp. 849-866, (1998)
8. De Haan, L. and Ferreira, A., Extreme value theory, Springer, (2006)
9. Dekkers, A. L. M., Einmahl, J. H. J. and de Haan, L., A moment estimator for the index of an extreme value distribution, Ann. Statist., 17, pp. 1833-1855, (1989)
10. Dietrich, D., de Haan, L. and Husler, J., Testing extreme value conditions, Extremes, 5, 71-85, (2002)
11. Drees, H., De Haan, E., and Kaufmann, E., Selecting the optimal sample fraction in univariate extreme value estimation, Stochastic Processes and their Applications, 75, No. 2, pp. 149-172, (1998)
12. Drees, H., De Haan, L., and D., Approximations to the tail empirical distribution function with application to testing extreme value conditions, J. Statist. Plann. Inference, 136, pp. 3498-3538, (2006)
13. Einmahl, J. H. J, The empirical distribution functions as a tail estimator, Statist. Neerlandica, 44, pp.79-82, (1990)
14. Einmahl, J. H. J, Limit theorems for tail processes and application to intermediate quantile estimation, J. Statistical Planning and Inference, 32, pp. 137-145, (1992)
15. Hall, P., On some simple estimates of an exponent of regular variation, J. R. Statist. Soc. B, Vol. 44, No. 1, pp. 37-42, (1984)
16. Hall, P. and Welsh, A. H., Best attainable rates of convergence for estimates of parameters of regular variation. Annals of Statistics, 12(3), pp. 1079-1084, (1984)
17. Haeusler, E. and Teugels, J. L., On asymptotic normality of Hill's estimator for the exponent of regular variation. Annals of Statistics, 13(2), pp. 743-756, (1985)
18. Hill, B. M., A simple general approach to inference about the tail of a distribution. Annals. Math. Statist., 3, pp. 1163-1174
19. Geluk, J., de Haan, L., Resnick, S. and Starica, C., Second-order regular variation, convolution and the central limit theorem, Stochastic Processes and their applications, 69(2), pp. 139-159, (1997)
20. Guillou, A. and Hall, P., A diagnostic for selecting the threshold in extreme value analysis. Journal of Royal Statistical Society, Series B: Methodology, 63, No. 2, pp. 293-305, (2001)
21. Khmaladze, E., Goodness of fit problem and scanning innovation martingales, Annals of Statistics, 21, pp. 798–829, (1993)
22. Khmaladze, E., Note on distribution free testing for discrete distribution, Annals of Statistics, 41, pp. 2979–2993, (2013)
23. Khmaladze, E., Unitary transformations, empirical processes and distribution free testing, Bernoulli, 22, pp. 563–588, (2016)
24. Mason, D., Law of large numbers for sums of extreme values, Ann. Prob., 10, pp. 754-764, (1982)
25. Pickands, III J., Statistical inference using extreme order statistics, Ann. Statist., 3, pp. 119–131, (1975)
26. Koning, A. J. and Peng, L., Goodness of fit tests for heavy tailed distribution, Journal of Statistical Planning and Inference, 138, pp. 3960-3981, (2008)
27. Resnick, S. I., Extreme values, regular variation and point process, Springer-Verlag, New York (1987)
28. Smith, R. L., Estimating tails of probability distributions, Annals of Statistics, 15, pp. 1174-1207, (1987)
29. Teugels, J. L., Limit theorems on order statistics, Annals. Prob., Vol. 9, No. 5, pp. 868-880, (1981)