INVARIENTS, EXPOUNENTS AND FORMAL GROUP LAWS

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Abstract. Let $W$ be the Weyl group of a crystallographic root system acting on the associated weight lattice by means of reflections. In the present notes we extend the notion of exponent of the $W$-action to the context of an arbitrary algebraic oriented cohomology theory of Levine-Morel and Panin-Smirnov and the associated formal group law. From this point of view the classical Dynkin index of the associated Lie algebra will be the second exponent of the deformation map from the multiplicative to the additive formal group law. We apply this generalized exponent to study the torsion part of an arbitrary oriented cohomology theory of a twisted flag variety.

1. Introduction

Let $W$ be the Weyl group of a crystallographic root system which acts by means of simple reflections on the respective weight lattice $\Lambda$. Consider the induced actions of $W$ on the polynomial ring $S^*(\Lambda)$ and the integral group ring $\mathbb{Z}[\Lambda]$ of $\Lambda$. The ring of invariants $\mathbb{Z}[\Lambda]^W$ can be identified with the representation ring of the associated linear algebraic group $G$ and according to the celebrated Chevalley theorem is a polynomial ring in classes of fundamental representations. On the other hand $S^*(\Lambda)^W \otimes \mathbb{Q}$ is known to be a polynomial ring as well with generators given by basic polynomial invariants.

The main goal of paper [1] was to establish the relationship between these two sets of invariants by means of the Chern class map. This was done by introducing the notion of an exponent – an integer $\tau_d$ which measures the difference between these invariants. In particular, it was proven that $\tau_2$ coincides with the Dynkin index of the associated Lie algebra (see Theorem 4.4 loc.cit.). Using the Riemann-Roch theorem and the Grothendieck $\gamma$-filtration it was also shown [1, Corollary 6.8] that the exponent $\tau_d$ bounds the annihilator of the torsion part of the Chow groups $\text{CH}^d$ for $d = 2, 3, 4$ of some twisted flag varieties, hence, providing new estimates for the torsion in small codimensions.

The next step was done in [2] where it was shown that $\tau_d$ divides the Dynkin index $\tau_2$ for all $d \geq 2$. This fact was together with Demazure’s description of the kernel of the characteristic map allowed to obtain a uniform bound for the annihilator of the torsion of $\text{CH}^d$ of strongly inner forms of flag varieties for all $d$, hence, pushing geometric applications of the exponent even further.

The goal of the present paper is to extend the notion of an exponent to the context of arbitrary algebraic oriented cohomology theories (o.c.t.) and the associated formal group laws (f.g.l.). From this point of view, the Dynkin index of a root system will be just the second exponent of the deformation from the multiplicative.

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In the present section we recall basic definitions and results of [1].

Let $\Lambda$ be a free abelian group of finite rank. Consider the integral group ring $\mathbb{Z}[\Lambda]$ of $\Lambda$. Its elements are finite linear combinations $\sum_j a_j e^{\lambda_j}$, $a_j \in \mathbb{Z}$, $\lambda_j \in \Lambda$. Consider the augmentation map $\epsilon_m : \mathbb{Z}[\Lambda] \to \mathbb{Z}$ given by $e^{\lambda} \mapsto 1$. Its kernel $I_m$ is an ideal in $\mathbb{Z}[\Lambda]$ generated by the differences $(1 - e^{-\lambda})$, $\lambda \in \Lambda$. Consider the $I_m$-adic
filtration on $\mathbb{Z}[\Lambda]$

$$\mathbb{Z}[\Lambda] = I_m^0 \supseteq I_m \supseteq I_m^2 \supseteq \ldots$$

Similarly, consider the symmetric algebra $S^*(\Lambda)$ of $\Lambda$ and the augmentation map $\epsilon: S^*(\Lambda) \to \mathbb{Z}$ which sends any polynomial to its constant term. Its kernel $I_a$ is an ideal generated by $\Lambda = S^1(\Lambda)$. Consider the $I_a$-adic filtration on $S^*(\Lambda)$

$$S^*(\Lambda) = I_a^0 \supseteq I_a \supseteq I_a^2 \supseteq \ldots.$$ 

The rings $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$ are non-isomorphic. However, they become isomorphic after truncation. Indeed, let $\{\omega_1, \ldots, \omega_n\}$ be a $\mathbb{Z}$-basis of $\Lambda$, then for each $d \geq 0$ we have two reverse isomorphisms defined by (cf. [1, Lem. 1.2] and [7, Def. 2.1])

$$\psi: S^*(\Lambda)/I_a^{d+1} \cong \mathbb{Z}[\Lambda]/I_m^{d+1}: \phi$$

$$\psi(\omega_j) = (1 - e^{-\omega_j}) \text{ and } \phi(e^{\sum_{i=1}^n a_i \omega_i}) = \prod_{j=1}^n (1 - \omega_j)^{-a_j}.$$ 

The morphisms $\phi$ and $\psi$ preserve the $I_a$- and $I_m$-adic filtrations. Set $I_a^{[d]} = I_a^d/I_a^{d+1}$ and $I_m^{[d]} = I_m^d/I_m^{d+1} = S^d(\Lambda)$. Restricting $\phi$ and $\psi$ to the subsequent quotients we obtain isomorphisms [11 Lem. 1.3]

$$\psi_d: I_a^{[d]} \cong I_m^{[d]}: \phi_d, \text{ where } \psi_d \left( \prod_{j=1}^d \lambda_j \right) = \prod_{j=1}^d (1 - e^{-\lambda_j}), \lambda_j \in \Lambda.$$ 

Observe that the isomorphisms $\phi$, $\psi$ depend on the choice of a basis of $\Lambda$ and the isomorphisms $\phi_d$, $\psi_d$ do not.

Let $W$ be a finite group which acts on $\Lambda$ by $\mathbb{Z}$-linear automorphisms. Consider the subrings of invariants $S^*(\Lambda)^W$ and $\mathbb{Z}[\Lambda]^W$. Let $I_a^W$ and $I_m^W$ denote the ideals generated by elements of $S^*(\Lambda)^W \cap I_a$ and $\mathbb{Z}[\Lambda]^W \cap I_m$ respectively.

The $W$-action is compatible with the $I_a$- and $I_m$-adic filtrations, i.e. $W(I_a^d) \subseteq I_a^d$ and $W(I_m^d) \subseteq I_m^d$ for all $d \geq 0$. Observe that the isomorphisms $\phi_d$ and $\psi_d$ are $W$-equivariant (in particular, we have $(I_a^{[d]})^W \simeq (I_m^{[d]})^W$) but the isomorphisms $\phi$ and $\psi$ are not.

Let $(I_a^W)^{(d)} \subseteq S^d(\Lambda)$ denote the $d$-th homogeneous component of the ideal $I_a^W$ and let $(I_m^W)^{(d)} = (I_m^W) \cap (I_m^W)/(I_m^W \cap I_m^{d+1})$ denote the $d$-th homogeneous component of the ideal $I_m^W$. Since the isomorphisms $\phi$ and $\phi_d$ preserve the filtrations, we have

$$\phi(I_m^W/I_m^{d+1}) \cap I_a^{[d]} = \phi_d \left( (I_m^W)^{(d)} \right).$$

2.1 Definition. We say that an action of $W$ on $\Lambda$ has a finite exponent in degree $d$ (see [11 Def. 2.1]) if there exists a non-zero integer $N_d$ such that

$$N_d \cdot (I_a^W)^{(d)} \subseteq \phi_d \left( (I_m^W)^{(d)} \right).$$ 

In this case the g.c.d. of all such $N_d$-s is called the $d$-th exponent of the $W$-action and is denoted by $\tau_d$.

Observe that if $\phi_d \left( (I_m^W)^{(d)} \right)$ is a subgroup of finite index in $(I_a^W)^{(d)}$, then $\tau_d$ is simply its exponent.
2.2 Remark. The main result of [1] says that in the case of a crystallographic root system an action of the Weyl group $W$ on the weight lattice $Λ$ has finite exponents in each degree (Cor. 6.3 loc.cit.) and $τ_2$ coincides with the Dynkin index of the associated Lie algebra (Thm. 4.4 loc.cit.). Moreover, according to [2, Prop. 5.6], if the root system is of type $B_n$ or $D_n$, then we have $τ_d | τ_2$ for all $d ≥ 3$.

3. Formal group algebras

In the present section we recall definition and basic properties of f.g.l. [13, Ch. IV] and formal group algebras [3, §2].

By $F$ we always denote a commutative one-dimensional f.g.l. over a commutative ring $R$ called the coefficient ring of $F$, i.e. $F$ is a power series in two variables

$$ F(u, v) = u + v + \sum_{i,j≥1} a_{ij} u^i v^j, \quad a_{ij} = a_{ji}, \quad a_{ij} ∈ R $$

which satisfies axioms of a group law.

A morphism of f.g.l. $f: F → F'$ over $R$ is a power-series $f(u) = u + O(2) ∈ R[[u]]$ such that $f(F(u, v)) = F'(f(u), f(v))$. For any f.g.l. $F$ there is a formal inverse (F-inverse) $i_F(u) ∈ R[[u]]$ which is defined by the identity $F(u, i_F(u)) = 0$.

We will use the following notation

$$ u + F v = F(u, v), \quad -F u = i_F(u) \quad \text{and} \quad a · F u = u + F u + F^2 u + ... + F^a u, \quad a ≥ 1. $$

3.1 Example. (a) The additive f.g.l. is given by $F_a(u, v) = u + v$.
(b) The multiplicative f.g.l. is given by $F_m(u, v) = u + v − uv$.
(c) The Lorentz f.g.l. is given by

$$ F_l(u, v) = \frac{u + v}{1 + uv} = u + v + \sum_{i≥1} (-1)^i \left( u^i v^{i+1} + u^{i+1} v^i \right). $$

(d) Let $E$ be an elliptic curve defined by (see [13, IV.1])

$$ E: \quad v = u^3 + a_1 uv + a_2 u^2 v + a_3 v^2 + a_4 uv^2 + a_6 v^3, \quad a_i ∈ \mathbb{Z}. $$

The group law on $E$ induces an elliptic f.g.l.

$$ F_E(u, v) = u + v - a_1 uv - a_2 (u^2 v + v^2 u) + 2a_3 (u^3 v + uv^3) + (a_1 a_2 - 3a_3) a_2^3 v^3 + O(5). $$

(e) There is a universal f.g.l. $F_u$. Its coefficient ring is called the Lazard ring $\mathbb{L}$. Any commutative one-dimensional f.g.l. over a ring $R$ corresponds to a ring homomorphism from $\mathbb{L}$ to $R$.

3.2 Definition. (cf. [3, Def. 2.4]) Let $F$ be a f.g.l. over a coefficient ring $R$, and let $Λ$ be an Abelian group. Consider the polynomial ring $R[x_Λ]$ in variables $x_λ, \ λ ∈ Λ$, and let $ε: R[x_Λ] → R, \ x_λ ↦ 0$, be the augmentation map. Let $R[[x_Λ]]$ be the ker $ε$-adic completion of $R[x_Λ]$. Let $\mathcal{J}_F$ be the closure of the ideal of $R[[x_Λ]]$ generated by $x_0$ and elements of the form $x_{λ_1 + λ_2} - (x_{λ_1} + F x_{λ_2})$ for all $λ_1, λ_2 ∈ Λ$. The formal group algebra (ring) $R[[Λ]]_F$ is defined to be the quotient

$$ R[[Λ]]_F := R[[x_Λ]]/\mathcal{J}_F. $$

The class of $x_λ$ in $R[[Λ]]_F$ will be denoted by the same letter. Notice that we have $x_{-λ} = ε(x_λ) = -F x_λ$ in $R[[Λ]]_F$. 


3.3 Example. Let $F_a(u, v) = u + v$ be the additive f.g.l. over $\mathbb{Z}$. The induced formal group ring over $\Lambda$ will be denoted as $\mathbb{Z}[[\Lambda]]_a$. There is a ring isomorphism
\[
\mathbb{Z}[[\Lambda]]_a \xrightarrow{\sim} \prod_{d \geq 0} S^d(\Lambda), \quad x_\lambda \mapsto \lambda.
\]

Let $F_m(u, v) = u + v - uv$ be the multiplicative f.g.l. over $\mathbb{Z}$. The induced formal group ring will be denoted as $\mathbb{Z}[[\Lambda]]_m$. There is a ring isomorphism
\[
\mathbb{Z}[[\Lambda]]_m \xrightarrow{\sim} \mathbb{Z}[\Lambda]^\wedge, \quad x_\lambda \mapsto (1 - e^{-\lambda}),
\]
where $\mathbb{Z}[\Lambda]^\wedge$ denotes the completion of the usual group ring $\mathbb{Z}[\Lambda]$ along the augmentation ideal.

4. Deformation maps

In the present section we introduce the completed analogues (called the deformation maps) of the maps $\phi, \psi, \phi_d, \psi_d$ of Section 2.

4.1 Definition. Given two f.g.l. $F$ and $F'$ over $R$ we define an $R$-algebra homomorphism $\Phi^{F \to F'}: R[[\Lambda]]_F \to R[[\Lambda]]_{F'}$ as follows.

First, we set $\Phi^{F \to F'}(x_0) := 0$ and $\Phi^{F \to F'}(x_{\omega_i}) := x_{\omega_i}$ for each basis element $\omega_i$ of $\Lambda$. Then for every $\lambda = \sum_{i=1}^n a_i \omega_i$ with $a_i \in \mathbb{Z}$ we define
\[
\Phi^{F \to F'}(x_\lambda) := (a_1 \cdot_F x_{\omega_1}) + (a_2 \cdot_F x_{\omega_2}) + \ldots + (a_n \cdot_F x_{\omega_n}).
\]

Finally, we set
\[
\Phi^{F \to F'}(x_\lambda + x_{\lambda'}) := \Phi^{F \to F'}(x_\lambda) + \Phi^{F \to F'}(x_{\lambda'})
\]
and
\[
\Phi^{F \to F'}(x_\lambda \cdot x_{\lambda'}) := \Phi^{F \to F'}(x_\lambda) \cdot \Phi^{F \to F'}(x_{\lambda'}).
\]

The map $\Phi^{F \to F'}$ will be called a deformation map from the formal group algebra $R[[\Lambda]]_F$ to $R[[\Lambda]]_{F'}$.

By definition the composite $\Phi^{F' \to F} \circ \Phi^{F \to F'}$ is the identity map. Therefore, $\Phi^{F \to F'}$ is an isomorphism with $\Phi^{F' \to F}$ being the inverse. Observe that all formal group algebras are (non-canonically) isomorphic to $R[[\omega_1, \ldots, \omega_n]]$ via the deformation map $\Phi^{F \to F_a}$ (cf. [3, Cor. 2.12]).

4.2 Example. Consider the deformation maps $\Phi^{F_0 \to F}$ and $\Phi^{F_m \to F}$ from the formal group algebras corresponding to the additive and multiplicative f.g.l. respectively. Let $\lambda = \sum_{i=1}^n a_i \omega_i$.

Since $a \cdot_{F_m} u = 1 - (1 - u)^a$ for $a \in \mathbb{Z}$, we obtain
\[
\Phi^{F_m \to F}(x_\lambda) = (1 - (1 - x_{\omega_1})^{a_1}) + \ldots + F_m (1 - (1 - x_{\omega_n})^{a_n}) = 1 - (1 - x_{\omega_1})^{a_1} \cdots (1 - x_{\omega_n})^{a_n}.
\]

Since $a \cdot_{F_0} u = a \cdot u$, we obtain
\[
\Phi^{F_a \to F}(x_\lambda) = a_1 x_{\omega_1} + a_2 x_{\omega_2} + \ldots + a_n x_{\omega_n}.
\]

Let $I_{F}$ denote the kernel of the augmentation map $\epsilon_F: R[[\Lambda]]_F \to R$, given by $x_\lambda \mapsto 0$. Consider the $I_{F}$-adic filtration
\[
R[[\Lambda]]_F = I_{F}^0 \supseteq I_{F} \supseteq I_{F}^2 \supseteq \cdots
\]
In other words, the deformation maps between f.g.l. Hence, it induces ring isomorphisms on quotients
\begin{equation}
\Phi^{F \to F'}(\mathcal{I}_F^d) = \mathcal{I}_{F'}^d, \quad d \geq 1,
\end{equation}

(2) \( \Phi^{F \to F'} : R[[\Lambda]]_F / \mathcal{I}_F^{d+1} \xrightarrow{\sim} R[[\Lambda]]_{F'} / \mathcal{I}_{F'}^{d+1} \) and \( \Phi_d^{F \to F'} : \mathcal{I}_F^d \xrightarrow{\sim} \mathcal{I}_{F'}^d \).

Observe that contrary to \( \Phi^{F \to F'} \) the isomorphism \( \Phi_d^{F \to F'} \) does not depend on the choice of a basis of \( \Lambda \). Indeed, \( \mathcal{I}_F^d \) is \( R \)-linearly generated by \( \prod_{i=1}^d x_{\lambda_i}, \lambda_i \in \Lambda \), and
\[ \Phi_d^{F \to F'} \left( \prod_{i=1}^d x_{\lambda_i} \right) = \prod_{i=1}^d x_{\lambda_i} . \]

### 4.3 Example
For \( F = F_a \) the isomorphism \( \Phi_d^{F \to F'} \) coincides with the isomorphism of \( \mathbb{K} \) Lemma 4.2).

Note that the symmetric algebra \( S^*(\Lambda) \) can be identified with the image of
\[ \varepsilon_a : S^*(\Lambda) \to \mathbb{Z}[[\Lambda]]_a, \quad \omega_i \mapsto x_{\omega_i} , \]
and the integral group ring \( \mathbb{Z}[\Lambda] \) can be identified with the image of
\[ \varepsilon_m : \mathbb{Z}[\Lambda] \to \mathbb{Z}[[\Lambda]]_m, \quad e^\lambda \mapsto (1 - x_{-\lambda}) . \]

The maps \( \varepsilon_a \) and \( \varepsilon_m \) preserve the filtrations, i.e. \( \varepsilon_a (I_a) \subseteq \mathcal{I}_a \) and \( \varepsilon_m (I_m) \subseteq \mathcal{I}_m \), where \( \mathcal{I}_a = \mathcal{I}_{F_a} \) and \( \mathcal{I}_m = \mathcal{I}_{F_m} \). Therefore, they induce isomorphisms on quotients
\begin{equation}
\varepsilon_a : S^*(\Lambda) / I_a^{d+1} \xrightarrow{\sim} \mathbb{Z}[[\Lambda]]_a / \mathcal{I}_a^{d+1} \quad \text{and} \quad \varepsilon_m : \mathbb{Z}[\Lambda] / I_m^{d+1} \xrightarrow{\sim} \mathbb{Z}[[\Lambda]]_m / \mathcal{I}_m^{d+1} .
\end{equation}

There are commutative diagrams
\[ S^*(\Lambda) / I_a^{d+1} \xrightarrow{\varepsilon_a} \mathbb{Z}[[\Lambda]]_a / \mathcal{I}_a^{d+1} \quad \text{and} \quad \mathbb{Z}[\Lambda] / I_m^{d+1} \xrightarrow{\varepsilon_m} \mathbb{Z}[[\Lambda]]_m / \mathcal{I}_m^{d+1} . \]

In other words, the deformation maps between \( F_m \) and \( F_a \) can be viewed as completions of the maps \( \phi, \phi_d \) and \( \psi, \psi_d \).

### 5. Exponents between formal group laws

In the present section we introduce the notion of an exponent between f.g.l. (see 5.1) and prove its existence (see 5.3).

As in Section 2 let \( W \) be a finite group which acts on \( \Lambda \) by \( \mathbb{Z} \)-linear automorphisms. It induces an action on the associated formal group algebra \( R[[\Lambda]]_F \) (see 3.2). Let \( R[[\Lambda]]_F^W \) denote the subring of invariants and let \( \mathcal{I}_F^W \) denote the ideal generated by \( R[[\Lambda]]_F^W \cap \mathcal{I}_F \).

As an immediate consequence of the definition the set \( R[[\Lambda]]_F^W \cap \mathcal{I}_F \) consists of formal sums of elements of the form
\[ \rho(x_{\lambda_1}, \ldots, x_{\lambda_s}) = \sum_{x_{\lambda_1} \cdots x_{\lambda_s} \in W(x_{\lambda_1}, \ldots, x_{\lambda_s})} x_{\lambda_1} \cdots x_{\lambda_s}, \quad s \geq 1. \]

Let \( \mathcal{I}_F^{(d)} = (\mathcal{I}_F^W \cap \mathcal{I}_F^W)/(\mathcal{I}_F^W \cap \mathcal{I}_F^{d+1}) \) denote the \( d \)-th homogeneous component of the ideal \( \mathcal{I}_F^W \) with respect to the \( \mathcal{I}_F \)-adic filtration.
5.1 Definition. Let $F$ and $F'$ be f.g.l. over a ring $R$. We say that an action of $W$ on $\Lambda$ has a finite exponent from $F$ to $F'$ in degree $d$ if there exists a positive integer $N^F_d$ of $F$-finite exponent in the sense of Definition 2.1. Moreover, the exponent $\tau^F_d$ coincides.

5.2 Example. We have $\tau^F_d = 1$ for any f.g.l. $F$ and $F'$. Indeed, considering a monomial $x_{\lambda_1} \cdots x_{\lambda_s}$ as a power series in the $x_{\lambda_i}$s, we have that $\deg(x_{\lambda_1} \cdots x_{\lambda_s}) \geq s$. Thus, we only need to consider the $W$-invariants $\rho(x_{\lambda})$, $\lambda \in \Lambda$. Since $F = F_a + O(2)$ for any f.g.l. $F$, the homogeneous components of degree one of $\rho(x_{\lambda})$ in $\mathbb{Z}[[\lambda]]_F$ and $\mathbb{Z}[[\lambda]]_a$ are the same. Since $I^a_w / I^a_a = 0$ (see [13, §3]), the statement follows.

5.3 Example. If $F = F_m$, i.e. $F_m(u,v) = u + v - u v$, then

$$\log_{F_m}(u) = \log(1 - u) = u + \frac{u^2}{2} + \frac{u^3}{3} + \ldots$$

For a general f.g.l. $F(u,v) = u + v + a_{11} uv + O(3)$ we have

$$\log_{F}(u) = u - \frac{a_{11} u^2}{2} + O(3).$$

A morphism of f.g.l. $f: F \to F'$ over $R$ induces a homomorphism of formal group rings $f^*: R[[\lambda]]_{F'} \to R[[\lambda]]_F$, $x_\lambda \mapsto f(x_\lambda)$ (see [3, §2.5]). By definition $f^*$ is $W$-equivariant and preserves the $I_F$-adic filtration. Therefore, we have induced maps

$$f^*: R[[\lambda]]_F^W \to R[[\lambda]]_{F'}^W$$

and

$$f^*: (I^W_F)^{(d)} \to (I^W_{F'})^{(d)}.$$
where the last map is defined by $\lambda_1 \cdots \lambda_d \mapsto x_{\lambda_1} \cdots x_{\lambda_d}$. Observe that by definition $\log^*_F$ coincides with the deformation map $\Phi_{d \rightarrow F}$ restricting on $(\mathcal{I}_d^{(d)})_\mathbb{Q}$.

The following theorem generalizes [1] Cor. 6.3

5.4 **Theorem.** Let $F$, $F'$ be f.g.l. defined over a ring of characteristic 0. Then a $W$-action on $A$ has a finite exponent from $F$ to $F'$.

**Proof.** The composite $\log^*_{F'} \circ (\log^*_F)^{-1}: (\mathcal{I}_d^{(d)})_\mathbb{Q} \rightarrow (\mathcal{I}_{d'}^{(d')})_\mathbb{Q}$ coincides with $\Phi_{d \rightarrow F'}$ and its restriction gives an isomorphism $(\mathcal{I}_d^{(d)})_\mathbb{Q} \cong (\mathcal{I}_{d'}^{(d')})_\mathbb{Q} \cong (\mathcal{I}_d^{(d)})_\mathbb{Q}$. Finally, observe that $(\mathcal{I}_d^{(d)})$ is finitely generated, so the exponents $\tau_{d \rightarrow F'}$ exist. \hfill $\square$

6. **Oriented cohomology and characteristic classes**

In the present section we recall several auxiliary facts concerning o.c.t. We refer to [10] and [12] for details and examples.

An o.c.t. is a contravariant functor $h$ from the category of smooth projective varieties over a field $k$ to the category of commutative unital rings which satisfies certain properties [10] §1.1. Given a morphism $f: X \rightarrow Y$ the functorial map $h(f)$ will be denoted by $f^*$ and called the pull-back. One of the characterizing properties of $h$ says that for any proper map $f: X \rightarrow Y$ there is an induced map $f_*: h(X) \rightarrow h(Y)$ of $h(Y)$-modules called the push-forward (here $h(X)$ is an $h(Y)$-module via $f^*$). A morphism of o.c.t. is a natural transformation of functors that also commutes with the push-forwards.

Another characterizing property of an o.c.t. is the existence of characteristic classes. The latter is a collection of maps $c^h_i: K_0(X) \rightarrow h(X)$, $i \geq 1$ that satisfy the following properties:

Let $c^h(x) = 1 + c^h_1(x)t + c^h_2(x)t^2 + \ldots \in h(X)[[t]]$ denote the total characteristic class. Then

1. $c^h(E) = 1$ for a trivial bundle $E$ over $X$,
2. $c^h_i(E) = 0$ for a bundle $E$ with $i > rk(E)$,
3. $c^h(E \oplus F) = c^h(E) \cdot c^h(F)$ for any two bundles $E$ and $F$ over $X$.

Given two line bundles $L_1$ and $L_2$ over $X$, we have [10] Lem. 1.1.3

\[ c^h_1(L_1 \otimes L_2) = c^h_1(L_1) + c^h_1(L_2), \]

where $F$ is a one-dimensional commutative f.g.l. over the coefficient ring $R = h(Spec(k))$ associated to $h$.

There is an o.c.t. $\Omega$ defined over a field of characteristic zero [10] §1.2, called algebraic cobordism, that is universal in the following sense: Given any o.c.t. $h$ there is a unique morphism $\Theta_h: \Omega \rightarrow h$ of o.c.t. Observe that the f.g.l. associated to $\Omega$ is the universal f.g.l. $F_a$.

6.1 **Lemma.** (cf. [6] Remark 3.2.3) Let $h$ be an o.c.t. and let $F$ be the respective f.g.l. Let $E$ and $E'$ be bundles over $X$ with Chern roots $\alpha_1, \ldots, \alpha_r$ and $\alpha'_1, \ldots, \alpha'_r$ respectively. Let $E'$ denote the dual of $E$. Then

1. $c^h(E') = \prod_{i=1}^r (1 + s(\alpha_i)t)$.
2. $c^h(E \otimes E') = \prod_{i,j} (1 + (\alpha_i + F \alpha'_j)t)$.
3. $c^h((^\Lambda^l E) = \prod_{1 \leq i_1 < \ldots < i_l \leq r} (1 + (\alpha_{i_1} + F \cdots + F \alpha_{i_l}))t)$ for $1 \leq l \leq r$. 

Proof. (i) and (ii) follow from \([5]\). To show (iii) assume that for \(E = \oplus_{l=1}^r L_l\) the statement holds. Now consider the vector bundle \(E \oplus L\) where \(L\) is a line bundle. By \([8\, \S 4.13]\) we have a short exact sequence for all \(1 \leq l \leq r + 1\)

\[
0 \to \bigwedge^{l-1}(E) \otimes L \to \bigwedge^l(E \oplus L) \to \bigwedge^l(E) \to 0.
\]

Set \(c_1^0(L) = \alpha_{r+1}\). By (ii) and induction we obtain for \(l \leq r\)

\[
c^h\left(\bigwedge^l(E \oplus L)\right) = c^h\left(\bigwedge^{l-1}(E) \otimes L\right) \cdot c^h\left(\bigwedge^l E\right)
\]

\[
= \prod_{1 \leq i_1 < \cdots < i_{r-1} \leq r} (1 + (\alpha_i + F \cdots + F \alpha_r + \alpha_{r+1})t)
\]

\[
= \prod_{1 \leq i_1 < \cdots < i_r \leq r+1} (1 + (\alpha_i + F \cdots + F \alpha_l)t).
\]

For \(l = r + 1\), since \(\bigwedge^l E = 0\) for all \(i > rk(E)\), by (ii) of the lemma and induction we conclude

\[
c^h\left(\bigwedge^{r+1}(E \oplus L)\right) = c^h\left(\bigwedge^r(E) \otimes L\right) = 1 + (\alpha_1 + F \cdots + F \alpha_{r+1}) t.
\]

Consider the following filtration on \(\mathfrak{h}(X)\) by subgroups \((R\text{-linear})\) generated by products of characteristic classes

\[
\gamma^d\mathfrak{h}(X) := (c_1^h(L_1) \cdot \cdots \cdot c_1^h(L_r)) \mid l \geq d, \ L_1, \ldots, L_r \text{ are line bundles}.
\]

6.2 Lemma. Let \(\mathfrak{h}\) be an o.c.t. and let \(F\) be the respective f.g.l. Let \(L_1, \ldots, L_r\) be line bundles over \(X\). Then

\[
c^h\left(\prod_{l=1}^r (1 - L_l^\gamma)\right) = (-1)^{r-1}(r - 1)! \cdot \prod_{l=1}^r c^h(L_l) \mod \gamma^{r+1}\mathfrak{h}(X).
\]

Proof. Denote \(\alpha_l = c_1^h(L_l), 1 \leq l \leq r\) and \(E = \oplus_{l=1}^r L_l\). We have

\[
c^h\left(\prod_{l=1}^r (1 - L_l^\gamma)\right) = c^h\left(\sum_{l=0}^r (-1)^l \bigwedge^l E^\gamma\right) = \prod_{l=1}^r c^h\left(\bigwedge^l E^\gamma\right)^{(-1)^l}
\]

\[
= \prod_{l=1}^r \prod_{1 \leq i_1 < \cdots < i_l \leq r} (1 + (\alpha_i + F \cdots + F \alpha_l)t)^{(-1)^l},
\]

where the second equality holds by the property (c) of characteristic classes and the last equality follows from Lemma \([6\, \S 4.13]\)(iii).

Since \(i(\alpha_l) = -\alpha_l + O(2)\), the \(r\)th characteristic class \(c^h\) modulo \(\gamma^{r+1}\mathfrak{h}(X)\) is given by the coefficient at \(t^r\) of the following polynomial

\[
\prod_{l=1}^r \prod_{1 \leq i_1 < \cdots < i_l \leq r} (1 - (\alpha_i + \cdots + \alpha_l)t)^{(-1)^l}
\]

and the desired formula then follows by \([6\, \text{Lem. 15.3]}\).
7. Characteristic classes and deformation maps

In the present section we establish a connection between $F$-exponents, characteristic classes and maps involving flag varieties (see \[7.2\]).

Let $G$ be a split simple simply connected linear algebraic group of rank $n$ over a field $k$. Fix a split maximal torus $T$ and a Borel subgroup $B$ so that $T \subset B \subset G$. Let $\mathfrak{B} = G/B$ be the variety of Borel subgroups of $G$ and let $\Lambda$ be the group of characters of $T$. Let $W$ be the Weyl group of the root system associated to $G$. It acts by linear automorphisms on $\Lambda$ via reflections. Let $t$ be the torsion index of $G$. Recall that $t = 1$ if $G$ is of type $A_n$ or $C_n$, and $t$ is a power of 2 if $G$ is of type $B_n$, $D_n$ or $G_2$.

Let $F$ be a f.g.l. over $R$. Consider the formal group ring $R[[\Lambda]]_F$ and the characteristic map defined in \[8 \S 6\]:

$$\epsilon_F: R[[\Lambda]]_F \longrightarrow \mathcal{H}(\Lambda)_F.$$  

If $F$ corresponds to an o.c.t. $\mathfrak{h}$ from \[8 \text{Thm. 13.12}\], then $\mathcal{H}(\Lambda)_F \simeq \mathfrak{h}(\mathfrak{B})$ and the map $\epsilon_F$ is given by $x_\lambda \mapsto c^h_1(L(\lambda))$, where $L(\lambda)$ is the associated line bundle.

7.1 Example. Characteristic map for the Chow theory $\text{CH}$, i.e. corresponding to the additive f.g.l., is given by

$$\epsilon_a: \mathbb{Z}[[\Lambda]]_{F_m} \longrightarrow \text{CH}(\mathfrak{B}), \quad x_\lambda \mapsto c_1(L(\lambda)).$$

Hence, restricting to $S^*(\Lambda)$ via $\epsilon_a$ we recover the usual characteristic map for Chow groups \[5 \S 1.5\].

Characteristic map for the Grothendieck $K_0$, i.e. corresponding to the multiplicative f.g.l., is given by

$$\epsilon_m: \mathbb{Z}[[\Lambda]]_{F_m} \longrightarrow K_0(\mathfrak{B}), \quad x_\lambda \mapsto 1 - [L(\lambda)^\vee].$$

Restricting to the integral group ring $\mathbb{Z}[\Lambda]$ via $\epsilon_m$ we recover the usual characteristic map for $K_0$ \[5 \S 1.6\] which maps $e^h$ to $[L(\lambda)]$.

Observe that the algebraic cobordism $\Omega$ defined over a field of characteristic zero satisfies the properties of \[8 \text{Thm. 13.12}\], therefore, we have the characteristic map $\epsilon_U: \mathbb{Z}[\Lambda]_u \rightarrow \Omega(\mathfrak{B})$ defined by $x_\lambda \mapsto c^U_1(L(\lambda))$.

7.2 Theorem. Let $\mathfrak{h}$ be an o.c.t. as considered in \[11 \text{Thm. 13.12}\]. Assume that the coefficient ring $R$ of $\mathfrak{h}$ has characteristic 0. Let $F$ be the respective f.g.l. over $R$. Then there is a commutative diagram of $R$-modules

\[
\begin{array}{ccc}
\mathcal{I}^{(d)}_m \downarrow (-1)^{d-1}(d-1)! \cdot \Phi^{F_m \rightarrow F} & \overset{\epsilon^{(d)}_m}{\longrightarrow} & \gamma^{(d)}K_0(\mathfrak{B}) \\
\downarrow \downarrow \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \downarrow \downarrow \\
\mathcal{I}^{(d)}_F & \overset{\epsilon^{(d)}_F}{\longrightarrow} & \gamma^{(d)}h(\mathfrak{B})
\end{array}
\]

where $\Phi^{F_m \rightarrow F}$ is the deformation map, $\epsilon^{(d)}_F$ is the $d$-th characteristic class and $\gamma^{(d)}h(X) = \gamma^d h(X)/\gamma^{d-1} h(X)$ is the $d$-th subsequent quotient.

If $\frac{1}{t} \in R$, then we have $\ker \epsilon^{(d)}_F = (\mathcal{I}^W_F)^{(d)}$.

Proof. The commutativity of the diagram follows from Lemma \[6.2\]. Now, suppose that $\frac{1}{t} \in R$, by \[8 \text{Thm. 6.9}\] there is an exact sequence of $R$-modules

$$0 \longrightarrow (\mathcal{I}^W_F) \longrightarrow (R[[\Lambda]]_F) \overset{\epsilon_F}{\longrightarrow} h(\mathfrak{B}) \longrightarrow 0.$$
Passing to the subsequent quotients of the $\mathcal{I}_F$-adic filtration we obtain the exact sequence

$$0 \longrightarrow (\mathcal{I}_F^W)^{(d)} \longrightarrow \mathcal{I}_F^{(d)} \xrightarrow{\iota_F^{(d)}} \gamma^{(d)}h(\mathfrak{B}) \longrightarrow 0 \ .$$

7.3 **Corollary.** For every $d \geq 0$ there is a group isomorphism

$$\text{CH}^d(\mathfrak{B}) \otimes R_Q \simeq \gamma^{(d)}h(\mathfrak{B}) \otimes \mathbb{Q}.$$

**Proof.** Follows from the isomorphism $\log_F^*: (\mathcal{I}_n^W)^{(d)} \simeq (\mathcal{I}_n^W)^{(d)}$ on the kernels, where $F$ is the respective f.g.l. \hfill \Box

8. **Exponents of root systems. Applications.**

In this section we describe some possible generators of $\mathcal{I}_F^W$ and the exponents $\tau_{F^{\to F'}}$ of the action of the Weyl group $W$ on the weight lattice $\Lambda$ for root systems of types $A_n$ ($n \geq 1$), $C_n$ ($n \geq 2$), $B_n$ ($n \geq 3$), $D_n$ ($n \geq 4$) and $G_2$. In the present section $R$ is always a ring of characteristic different from 2. Recall that the torsion index $t$ of $\mathfrak{O}$ is 1 for types $A_n$ and $C_n$ and a power of 2 for types $B_n$, $D_n$ and $G_2$.

8.1 **Definition.** Given a root system of classical Dynkin type $\mathfrak{D}$ of rank $n$ we define the $W$-invariant elements $\Theta^d_\mathfrak{D} \in R[[\Lambda]]^W \cap \mathcal{I}_F$ as follows.

For the type $A_n$ we set $\Theta^1_\mathfrak{A}$ to be the $(d+1)$-th elementary symmetric polynomial in $\{x_i\}$, where $\{e_{i,j}\}_{j=1}^{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$ on which the Weyl group $W$ acts by permutations [9, p.5].

For the type $B_n$ or $C_n$ we set $\Theta^2_\mathfrak{B}$ to be the $d$-th elementary symmetric polynomial in $\{x_i, x_{-i}\}$, where $\{e_{i,j}\}_{j=1}^{n}$ is the standard basis of $\mathbb{R}^n$ on which the Weyl group acts by permutations and sign changes [9, p.5].

For the type $D_n$ we set $\Theta^1_\mathfrak{D} := \Theta^d_\mathfrak{D}$ for $d = 1 \ldots n - 1$ and $\Theta^2_\mathfrak{D} := \prod_{i=1}^{n-1} (x_i - x_{-i})$, where $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbb{R}^n$ on which the Weyl group acts by permutations and sign changes which involve an even number of signs [9, p.5].

We will simply write $\Theta_i$, if the Dynkin type is clear for the context.

If $f \in \mathcal{I}_F^d \setminus \mathcal{I}_F^{d+1}$, we say that $\deg f = d$. For instance, reducing to the additive case via the isomorphism [2] we obtain that $\deg \Theta^d_\mathfrak{A} = d+1$ for $d = 1 \ldots n$, $\deg \Theta^2_\mathfrak{B} = 2d$ for $d = 1 \ldots n - 1$ and $\deg \Theta^2_\mathfrak{D} = n$. Moreover, if $F = F_n$, the g.c.d. of the coefficients of $\Theta^d_\mathfrak{B}$ and $\Theta^2_\mathfrak{D}$ in $S^*(\Lambda)$ is 1 if $2 \leq i \leq n - 1$, and is 2 if $i = 1$, and that of $\Theta^1_\mathfrak{D}$ (resp. $\Theta^2_\mathfrak{D}$) is 1 (resp. 2$^n$).

Let $(R[[\Lambda]]^W)^{(d)} = (R[[\Lambda]]^W \cap \mathcal{I}_F^d)/(R[[\Lambda]]^W \cap \mathcal{I}_F^{d+1})$ denote the $d$-th homogeneous component of $R[[\Lambda]]^W$. Given a $n$-tuple of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ we denote $\Theta(\alpha) := \prod_{i=1}^{n} \Theta_i^{\alpha_i}$, $|\alpha| := \sum_{i=1}^{n} \alpha_i \deg \Theta_i$, where $\deg \Theta_i$ is the degree of $\Theta_i$. Observe that $\deg \Theta(\alpha) = |\alpha|$. Let $(\Theta(\alpha))|_{|\alpha|=d}$ denote the $R$-linear span of all $\Theta(\alpha)$ such that $|\alpha| = d$.

8.2 **Lemma.** For classical Dynkin types, we have

$$S^*(\Lambda)^W \otimes \mathbb{Z}[1/t] = \mathbb{Z}[1/t][\Theta_1, \ldots, \Theta_n].$$

The set $\{\Theta_1, \ldots, \Theta_n\}$ is called a set of basic invariants.

**Proof.** By [4, §6], we know that the left hand side is a polynomial ring in $n$ generators $\{h_1, \ldots, h_n\}$. The degrees of $h_i$ are uniquely determined by the Dynkin type.

If $\mathfrak{D}$ is of type $A_n$, then by [11 Ch.I (2.4)], the conclusion follows.
If $\mathcal{D}$ is of type $C_n$, since the Weyl group acts on $\{e_i\}$ by sign changes and permutations, so each $W$-invariant element belongs to $\mathbb{Z}[e_1^2, \ldots, e_n^2]$ and is symmetric under permutations, so similar as the type $A_n$ case, we get the conclusion.

The proof for the type $B_n$ is similar to that of the $C_n$ after inverting 2, noticing that $\mathbb{Z}[1/2][e_1, \ldots, e_n] = \mathbb{Z}[1/2][\omega_1, \ldots, \omega_n]$.

Now consider the type $D_n$. Let $q_{2i} := \sum x_{e_i}^{2i}$ be the power sum of degree $2i$, and denote $Q = \{q_2, \ldots, q_{2n-2}, p_n\}$, $\mathcal{T} = \{\Theta_1, \ldots, \Theta_{n-1}, p_n\}$, $\mathcal{H} = \{h_1, \ldots, h_n\}$ and $\mathcal{E} = \{e_1, \ldots, e_n\}$. Denote by $|\partial A/\partial B|$ the Jacobian of $A$ with respect to $B$. By [9, 3.12], $Q$ is a set of basic invariants over $\mathbb{Q}$, and

$$|\partial Q/\partial \mathcal{E}| = (-2)^{n-1}(n-1)! \prod_{i<j} (e_i^2 - e_j^2).$$

From the Newton’s identities we see that $|\partial Q/\partial \mathcal{T}| = (-1)^{n-1}(n-1)!$, so $|\partial \mathcal{T}/\partial \mathcal{E}| = 2^n \prod_{i<j} (e_i^2 - e_j^2)$, and $\mathcal{T}$ is a set of basic invariants over $\mathbb{Q}$. Hence, $|\partial \mathcal{T}/\partial \mathcal{H}|$ is a rational number. Moreover, it divides $|\partial \mathcal{T}/\partial \mathcal{E}|$ in $\mathbb{Z}[1/2][h_1, \ldots, h_n]$. So $|\partial \mathcal{T}/\partial \mathcal{H}|$ is a power of 2. Therefore, $\mathcal{T}$ is a set of basic invariants over $\mathbb{Z}[1/2]$.

8.3 Lemma. Let $\mathcal{D}$ be of type $B_n$ (resp. type $D_n$), and $d \geq 1$ (resp. $n > d \geq 1$). Let $F = F_{\alpha}$, and view $\Theta$, as in $\mathbb{Z}[e_1, \ldots, e_n] \subseteq \mathbb{Z}[[\Lambda]]_{F_{\alpha}}$, then the set $\{\Theta(\alpha)\}_{|\alpha| = d}$ is linearly independent modulo any positive integer $M$.

Proof. Firstly, let $s_1, \ldots, s_n$ be elementary symmetric polynomials in $\mathbb{Z}[e_1, \ldots, e_n]$. Then its Jacobian is $\prod_{i<j} (e_i - e_j)$, so $s_1, \ldots, s_n$ are algebraically independent over $\mathbb{Z}/M\mathbb{Z}$. In particular, it shows that the set $\{f = s_1^i \cdots s_n^i | \deg f = d\}$ is linearly independent over $\mathbb{Z}/M\mathbb{Z}$.

If $\mathcal{D}$ is of type $B_n$, we identify the set $\{\Theta_i\}_{i=1}^n \subseteq \mathbb{Z}[e_1, \ldots, e_n]$ as the set of elementary symmetric polynomials in $\{e_i^2\}_{i=1}^n$, so the conclusion follows.

For the type $D_n$, since $d < n$, the elements involved are $\Theta_1, \ldots, \Theta_{n-1}$, so the conclusion follows by the same arguments.

8.4 Remark. In Lemma 8.3 the set $\{\Theta(\alpha)\}_{|\alpha| = d}$ may not be linearly independent in $\mathbb{Z}[\omega_1, \ldots, \omega_n]$ modulo certain $M$. For example, $\Theta_1/2 \in \mathbb{Z}[\omega_1, \ldots, \omega_n]$ but $\Theta_1/2 \notin \mathbb{Z}[e_1, \ldots, e_n]$, so $\Theta_1$ is linearly dependent in $\mathbb{Z}/2[\omega_1, \ldots, \omega_n]$ and linearly independent in $\mathbb{Z}/2[e_1, \ldots, e_n]$.

8.5 Lemma. Suppose $\mathcal{D}$ is of type $B_n$ (resp. $D_n$), and let $d \geq 2$ (resp. $d > n \geq 2$). Then

1. $2^d \cdot (R[\Lambda]^W)_{d}^{(d)} \subseteq \{\Theta(\alpha)\}_{|\alpha| = d}$.
2. $2^d \cdot (T^W_{\alpha})_{d} \subseteq \{\sum_{\deg \Theta_i \leq d} g_i \Theta_i | g_i \in T^{d - \deg \Theta_i}_{\alpha}\}$.

If $1/2 \in R$, then the “$\subseteq$” in (1) and (2) can be replaced by “$= $” after removing $2^d$.

Proof. The proof for the type $D_n$ with $d < n$ is similar to that for the type $B_n$, so we only consider the latter case.

1. Suppose $\mathcal{D}$ is of type $B_n$. We first prove it for $F = F_{\alpha}$. It is equivalent to show that

$$2^d \cdot S^d_{R}(\Lambda)^W \subseteq \{\Theta(\alpha)\}_{|\alpha| = d}.$$
For any \( f \in S^d_R(\Lambda)^W \), expressing each \( \omega_i \) in terms of \( e_j \), we see that \( f = \tilde{f} / 2^d \) for some \( \tilde{f} \in R[e_1, \ldots, e_n]^W \) of degree \( d \). By Lemma 8.2
\[
\tilde{f} = \sum_{|\alpha| = d} \frac{a_{\alpha}}{b_{\alpha}} \Theta(\alpha)
\]
for some \( b_{\alpha} \) is some power of 2. Clearing the denominators, we see that \( 2^r \cdot \tilde{f} = \sum a'_{\alpha} \Theta(\alpha) \) for some \( r, a'_{\alpha} \in \mathbb{Z} \). Since \( 2^r \) divides the left hand side, by Lemma 8.3 \( 2^r \) divides \( a'_{\alpha} \) for all \( \alpha \). Therefore, \( r = 0 \) and we obtain \( 2^d \cdot f = \sum a'_{\alpha} \Theta(\alpha) \).

For a general f.g.l. \( F \) we have (see (2))
\[
g \in (R[[\Lambda]]^W)(d) \subset (\mathcal{I}_F^d)^W \xrightarrow{\Phi_{F^d \to F}} (\mathcal{I}_F^d)^W = S^d_R(\Lambda)^W.
\]
Hence
\[
2^d \cdot \Phi_{F^d \to F}(g) = \sum_{|\alpha| = d} c_{\alpha} \Theta(\alpha) \in \mathcal{I}_F^d
\]
for some \( c_{\alpha} \in R \).

Applying the inverse \( \Phi_{F \to F^d} \) to both sides we obtain
\[
2^d \cdot g = \sum_{|\alpha| = d} c_{\alpha} \Theta(\alpha).
\]

(2) The group \( (\mathcal{I}_F^W)^{(d)} \) consists of \( R \)-linear combinations of products \( h = g \tilde{f} \) with \( \tilde{f} \in R[[\Lambda]]^W \cap \mathcal{I}_F \) of degree \( k \) \( (2 \leq k \leq d) \) and \( g \in R[[\Lambda]]^W \) of degree \( g = d - k \). So by (1),
\[
2^d h = 2^d g \tilde{f} = g \sum_{|\alpha| = k} b_{\alpha} \Theta(\alpha) = \sum_{\deg \Theta \leq d} g_{\Theta} \Theta.
\]

If \( 1/2 \in R \), the conclusion follows immediately. \( \square \)

8.6 Remark. For the type \( D_n \), if \( 1/2 \in R \), then the conclusion of the last part in Lemma 8.5 holds for \( d \geq n \) as well.

We now consider the \( G_2 \) type. Let \( \{e_1, e_2, e_3\} \) be the standard basis of \( \mathbb{R}^3 \). The fundamental weights are \( \omega_1 = e_3 - e_1 \) and \( \omega_2 = 2e_3 - e_1 - e_2 \), and the Weyl group associated to \( G_2 \) is the dihedral group \( D_6 \). Define
\[
\Theta_1^{G_2} = x_{\omega_1}x_{-\omega_1} + x_{\omega_2}x_{-\omega_2} + x_{\omega_1-\omega_2}x_{2\omega_1-2\omega_2} + x_{2\omega_1-2\omega_2}x_{\omega_1-\omega_2}
\]
and
\[
\Theta_2^{G_2} = x_{\omega_1}x_{-\omega_1}x_{\omega_2}x_{-\omega_2} + x_{\omega_1}x_{-\omega_2}x_{\omega_2}x_{-\omega_1} + x_{\omega_1-\omega_2}x_{2\omega_1-2\omega_2} + x_{2\omega_1-2\omega_2}x_{\omega_1-\omega_2}.
\]
A direct computation shows that \( \Theta_1^{G_2}, \Theta_2^{G_2} \) are both \( W \)-invariant, and \( \deg \Theta_1^{G_2} = 2, \deg \Theta_2^{G_2} = 6 \).

8.7 Lemma. For the \( G_2 \) type, we have \( S^*_R(\Lambda)^W = \mathbb{Z}[1/2][\Theta_1^{G_2}, \Theta_2^{G_2}] \).

Proof. By [4, §6], \( S^*_R(\Lambda)^W \) is a polynomial ring generated by two homogeneous polynomials of degree 2 and 6, respectively. Suppose \( \{h_1, h_2\} \) is a pair of generators. Note that the Jacobian
\[
\frac{\partial \Theta_1^{G_2}}{\partial \omega_j} = -4\omega_1\omega_2(\omega_1 - \omega_2)(2\omega_1 - \omega_2)(3\omega_1 - 2\omega_2)(3\omega_1 - \omega_2).
\]
So \( \Theta_1^{G_2} \) and \( \Theta_2^{G_2} \) are algebraically independent. So by [4, §3.11], \( \{\Theta_1^{G_2}, \Theta_2^{G_2}\} \) is a set of generators of the invariants over \( \mathbb{Q} \). Therefore, the Jacobian \( \frac{\partial \Theta_1^{G_2}}{\partial h_1} \) is a rational number dividing \( \frac{\partial \Theta_2^{G_2}}{\partial h_1} \), so it is a unit in \( \mathbb{Z}[1/2] \). The proof is finished. \( \square \)
8.8 Lemma. For each even integer \( d \geq 1 \), let \( r_d \) to be the remainder of \( d \) modulo 6, and let \( \zeta_d \equiv \frac{3d/6((d/6)+1)}{2} + \frac{r_d}{2} \). If \( \mathcal{O} \) is of type \( G_2 \), then

1. \( 2^d \cdot (R[[\Lambda]]^W_r)^{(d)} \subset \langle \Theta(\alpha) \rangle_{|\alpha|=d} \subset R[[\Lambda]]_r^W \).
2. \( 2^d \cdot (I^W_r)^{(d)} \subset \{ \sum_{\deg \Theta_i \leq d} g_i \Theta_i \mid g_i \in W^{d-\deg \Theta_i} \} \).

If \( 1/2 \in R \), then the \( \subset \) can be replaced by \( \subset \) after removing \( 2^d \).

Proof. (1) Suppose \( F = F_a \). For any \( f \in S_{R}(\Lambda)^W \), then \( f \in R[1/2][\Theta_1, \Theta_2] \), so by Lemma 8.7 we can write \( 2^r \cdot f = \sum_{i=0}^{[d/6]} a_i \Theta_1^{2i/3} \) with \( a_i \in R \). To prove the lemma, it suffices to show that the smallest \( r \) in the equation is less than or equal to \( \zeta_d \), i.e., to show that if \( r > \zeta_d \), then \( 2|a_i \) for all \( i \). Notice that \( \frac{a_i}{2} \in S_{R}^*(\Lambda) \), and \( \frac{a_i}{2} \not\in S_{R}^*(\Lambda) \).

We prove it for \( d = 6k \) first, in which case \( \zeta_d = \frac{3k(k+1)}{2} \). We proceed by induction on \( k \). If \( k = 1 \), then \( r > \zeta_6 = 3 \) and \( 2^r f = a \Theta_2 + b \Theta_1^3 \). Letting \( \omega_1 = \omega_2 = 1 \), we have \( \Theta_2(1,1) = 0 \) and \( \Theta_1(1,1) = 2 \), so \( 2^r | b \cdot 8 \), hence, \( 2^{r-3} | b \). So \( 2^{r-3} \) a since \( \Theta_2/2 \not\in S_{R}^*(\Lambda) \).

Assume the conclusion for \( k_0 - 1 \), then for \( k_0 \), note that \( \zeta_{6k_0} = \zeta_{6k_0-6} + 3k_0 \).

The equation becomes

\[
2^r f = \sum_{i=0}^{k_0} a_i \Theta_1^i \Theta_1^{3k_0-3i},
\]

and \( r > \zeta_{6k_0} \). Letting \( \omega_1 = \omega_2 = 1 \), we see that \( 2^r | a_0 2^{3k_0} \), so \( 2^{r-3k_0} | a_0 \). Therefore,

\[
2^{r-3k_0} \sum_{i=1}^{k_0} a_i \Theta_1^i \Theta_1^{3k_0-3i} = \Theta_2 \sum_{j=0}^{k_0-1} a_j \Theta_1^{j+1} \Theta_1^{3k_0-3-3j}.
\]

So \( 2^{6k_0-6} | 2^{r-3k_0} | \sum_{j=0}^{k_0-1} a_j \Theta_1^{j+1} \Theta_1^{3k_0-3-3j} \). By induction, it implies that \( 2 | a_j \) for \( j = 1, \ldots, k_0 \).

We then assume that \( d = 6k + r_d \) with \( r_d > 0 \) the reminder of \( d \) modulo 6. Then \( r > \zeta_d = \zeta_{6k} + r_d/2 \), and we have

\[
2^r f = \Theta_1^{r_d/2} \sum_{i=0}^{k} a_i \Theta_1^i \Theta_1^{3k-3i}.
\]

So \( 2^{r_d/2} | \sum_{i=0}^{k} a_i \Theta_1^i \Theta_1^{3k-3i} \). The conclusion then follows from the first part.

The proof of the rest is similar to that of 8.35. \( \square \)

8.9 Example. If \( \mathcal{O} \) is of type \( A_n \) or \( C_n \), then the torsion index is 1, and \( \{ e_i \}_{i=1}^n \) is a basis of \( \Lambda \). Following from the same idea of Lemma 8.5 we have

\[
(R[[\Lambda]]^W_r)^{(d)} = \langle \Theta(\alpha) \rangle_{|\alpha|=d}
\]

and

\[
(I^W_r)^{(d)} = \{ \sum_{\deg \Theta_i \leq d} g_i \Theta_i \mid g_i \in W^{d-\deg \Theta_i} \}.
\]

We are ready now to prove the main result of this section.

8.10 Theorem. Let \( F \) and \( F' \) be formal group laws over a ring \( R \) of characteristic different from 2. For the type \( B_n \) with \( n \geq 3 \) and \( d \geq 1 \) (resp. \( D_n \) with \( n \geq 4 \) and \( n > d \geq 1 \)), we have \( \tau_d^{F \rightarrow F'} \mid 2^d \). For the type \( G_2 \), \( \tau_{d \rightarrow F'} \mid 2^{2d} \), where \( \zeta_d \) was defined.
Proof. Suppose that $\mathcal{O}$ is of type $B_n$ or $D_n$. By Lemma 8.13 we have

$$2^d \cdot (\mathcal{I}_F^W)^{(d)} \subseteq \left\{ \sum_{\deg \Theta_i \leq d} g_i(\Theta_i, F') \mid g_i \in \mathcal{T}_{F'}(d-\deg \Theta_i) \right\} \subseteq \Phi_d^{F \to F'}((\mathcal{I}_F^W)^{(d)}),$$

as $\Phi_d^{F \to F'}(g_i(\Theta_i, F)) = g_i(\Theta_i, F')$, where $\Theta_i$ and $\Theta_i, F'$ are the respective $W$-invariant elements for the f.g.l. $F$ and $F'$. So $\tau_d^{F \to F'} \mid 2^d$ for any $F$ and $F'$.

The part for $G_2$ follows similarly from Lemma 8.8. The third statement follows immediately, after using Example 8.9 if $\mathcal{O}$ is of type $A_n$ or $C_n$. □

Let $h$ be an oriented cohomology theory satisfying the condition in [3, Theorem 13.12] with the corresponding formal group law $F$ over the coefficient ring $R$ of characteristic zero. Let $\mathfrak{B}$ be the variety of Borel subgroups and let $c^{(d)}_F : \mathcal{I}_F^{(d)} \to \gamma^{(d)}(\mathfrak{B})$ denote the characteristic map. Identifying the $\gamma^{(d)}(\mathfrak{B})$-module with $\mathfrak{B}$ and assuming that there are exist exponents $\deg \Theta$ for $\mathfrak{B}$, and the deformation map $\Phi_d^{F \to F'}$ induces an isomorphism ker $\Phi_d^{F \to F'}$. Let $G$ be a split, simple simply connected linear algebraic group of type $A_n, B_n, C_n, D_n$ or $G_2$.

If $\frac{R}{n} \in R$, then $\gamma^{(d)}(\mathfrak{B})$ is torsion free for every $d \geq 1$ and the deformation map $\Phi_d^{F \to F_a}$ induces an isomorphism $\gamma^{(d)}(\mathfrak{B}) \simeq \gamma^{(d)}(\mathfrak{B}; R)$. □

Proof. To see that $\gamma^{(d)}(\mathfrak{B})$ is torsion free, take $h'(-) = \text{CH}(-; R)$ and apply Theorem 8.10 and Corollary 8.12

Observe that $\ker c^{(d)}_a$ is finitely generated over $R$ so that the exponent $\tau'$ exists.

Let $z \in \ker c^{(d)}_a$. By Lemma 8.3, $z = \sum g_i(\Theta_i)$ for some $g_i \in \mathcal{I}_F^{(d-\deg \Theta_i)}$. Since $\Phi_d^{F \to F_a}$ maps $\prod_{i=1}^d x_i \in \mathcal{I}_F^{(d)}$ to $\prod_{i=1}^d x_i \in \mathcal{I}_a^{(d)}$, we obtain

$$\Phi_d^{F \to F_a}(z) = \sum g_i(\Theta_i) \in (\mathcal{I}_a^W)^{(d)} = \ker c^{(d)}_a.$$ 

Therefore, $\Phi_d^{F \to F_a}$ induces an isomorphism $\ker c^{(d)}_a \cong \ker c^{(d)}_a$, and hence an isomorphism on the cokernels $\gamma^{(d)}(\mathfrak{B}) \cong \gamma^{(d)}(\mathfrak{B}; X)$. □
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