MIXING OPERATORS WITH NON-PERFECTLY SPANNING UNIMODULAR EIGENVECTORS

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Abstract. For arbitrary closed countable subsets Z of the unit circle examples of topologically mixing operators on Hilbert spaces are given which have a densely spanning set of unimodular eigenvectors with eigenvalues restricted to Z. In particular, these operators cannot be ergodic in the Gaussian sense.

1. Introduction and main result

The dynamical behaviour of linear operators acting on Fréchet spaces X has been investigated intensively in the last years. Recommended introductions are the textbooks [3] and [9], and also the recent article [12]. It turns out that the richness of eigenvectors corresponding to unimodular eigenvalues strongly influences the metric dynamical properties of linear operators. In particular, a linear operator on a Hilbert space admits a Gaussian invariant measure of full support if and only if it has spanning unimodular eigenvectors and is ergodic in the Gaussian sense (that is, ergodic with respect to a Gaussian measure of full support) if and only if it has perfectly spanning unimodular eigenvectors. For these and corresponding results we refer in particular to [1], [2], [4], [10] and again to [12].

Due to recent deep results of Menet ([14]) and Grivaux, Matheron and Menet ([12]), in the situation of Hilbert spaces X the picture has become quite complete for the case of chaotic operators, that is, for hypercyclic operators having eigenvectors corresponding to roots of unity (i.e. periodic vectors) which span a dense subspace of X. Less is known, however, in the case of absence of periodic or almost periodic vectors (cf. [12 Section 1.3]). In [11 Question 3] (see also [12 Question 7.7]) it is asked if a hypercyclic operator with a densely spanning set of eigenvectors corresponding to rationally independent eigenvalues is already ergodic. In this paper, we give examples of topologically mixing operators on Hilbert spaces which have a densely spanning set of eigenvectors with unimodular eigenvalues restricted to an arbitrary prescribed closed countable subset of the unit circle $T$. In particular, such operators cannot be ergodic in the Gaussian sense.

For an open set $\Omega$ in the extended plane $C_\infty$ with $0 \in \Omega$ we denote by $H(\Omega)$ the Fréchet space of functions holomorphic in $\Omega$ and vanishing at $\infty$ endowed with the topology of locally uniform convergence, where, as usual, via stereographic projection we identify $C_\infty$ and the sphere $S^2$ endowed with the spherical metric. We consider Bergman spaces on general open sets $\Omega \subset C_\infty$ with $0 \in \Omega$: For
$0 \leq p < \infty$ let $A^p(\Omega)$ be the space of all functions $f$ holomorphic in $\Omega$ that satisfy

$$\|f\|_p := \|f\|_{\Omega,p} := \left( \int_{\Omega} |f|^p \, dm_2 \right)^{1/p} < \infty,$$

where $m_2$ denotes the spherical measure on $\mathbb{C}_\infty$. Then $A^0(\Omega) = H(\Omega)$ and for $p \geq 1$ the spaces $(A^p(\Omega), \| \cdot \|_p)$ are Banach spaces. In the case $p = 2$, the norm is induced by the inner product $(f, g) \mapsto \int_{\Omega} f \overline{g} \, dm_2$.

For $\Omega$ being bounded in the plane or containing $\infty$, we define $T = T_{A^p(\Omega)} : A^p(\Omega) \to A^p(\Omega)$ by

$$T f(z) := (f(z) - f(0))/z \quad (z \neq 0), \quad T f(0) := f'(0).$$

If $f(z) = \sum_{\nu=0}^{\infty} a_{\nu} z^\nu$, then

$$T f(z) = \sum_{\nu=0}^{\infty} a_{\nu+1} z^\nu$$

for $|z|$ sufficiently small. We call $T$ the Taylor (backward) shift on $A^p(\Omega)$. If we write $S_n f(z) := \sum_{\nu=0}^{n} a_{\nu} z^\nu$ for the $n$-th partial sum of the Taylor expansion $\sum_{\nu=0}^{\infty} a_{\nu} z^\nu$ of $f$ about 0, then by induction it is easily seen that

$$T^{n+1} f(z) = (f - S_n f(z))/z^{n+1} \quad (z \neq 0), \quad T^{n+1} f(0) = a_{n+1},$$

for $n \in \mathbb{N}_0$. In [5] it is shown that for open sets $\Omega$ with $0 \in \Omega$ the Taylor shift $T$ is topologically mixing on $H(\Omega)$ if and only if each connected component of $\mathbb{C}_\infty \setminus \Omega$ meets $\mathbb{T}$. Results concerning topological and metric dynamics of the Taylor shift on Bergman spaces are proved in [6] and [10]. We write $M^* := (\mathbb{C}_\infty \setminus M)^{-1}$ for $M \subset \mathbb{C}_\infty$. Then $\Omega^*$ is a compact plane set (note that $0 \in \Omega^*$) and the spectrum of $T$ is contained in $\Omega^*$. In the case $1 \leq p < 2$ is easily seen that $f \in A^p(\Omega)$ is an eigenfunction for $T$ if and only if, for some $\alpha \in \Omega^*$, the function $f$ is a scalar multiple of $\gamma_\alpha$, where $\gamma_\alpha$ is defined by

$$\gamma_\alpha(z) = 1/(1 - \alpha z)$$

for $z \in \Omega$ (with $0 \cdot \infty := 0$). In this case, $\alpha$ is the corresponding eigenvalue and the spectrum as well as the point spectrum both equal $\Omega^*$. If $p \geq 2$, then the functions $\gamma_\alpha$ still belong to $A^p(\Omega)$ for all $\alpha$ in the interior of $\Omega^*$, but in general not for $\alpha$ belonging to the boundary of $\Omega^*$. If $\Omega$ has small spherical measure near a boundary point $1/\alpha$ of $\Omega$, it may, however, happen that $\gamma_\alpha$ is again an eigenfunction of $T$ (that is, $\gamma_\alpha$ belongs to $A^p(\Omega)$). For example, $\gamma_1 \in A^2(\Omega)$ in the case of the crescent-shaped region $\Omega = \mathbb{D} \setminus \{z : |z - 1/2| \leq 1/2\}$ where $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$. This opens up the possibility to place eigenvalues at certain points of the boundary of $\Omega^*$. A corresponding construction leads to our main result. We write $\mathcal{E}(T)$ for the set of unimodular eigenvalues of $T$, which for $T = T_{A^2(\Omega)}$ equals the set of $\lambda \in \mathbb{T}$ such that $\gamma_\lambda \in A^2(\Omega)$.

**Theorem 1.1.** Let $Z \subset \mathbb{T}$ be an infinite closed set. Then there is an open set $\Omega \subset \mathbb{D}$ so that the Taylor shift $T = T_{A^2(\Omega)}$ is topologically mixing, $\mathcal{E}(T) \subset Z$ and $\{\gamma_\lambda : \lambda \in \mathcal{E}(T)\}$ spans a dense subspace of $A^2(\Omega)$.

**Remark 1.2.** Since Hilbert spaces are of cotype 2, the main theorem from [4] implies that, in the situation of Theorem 1.1, for countable $Z$ the Taylor shift $T$ is not ergodic in the Gaussian sense. So Question 3 from [11] can be answered in the negative, at least in the weak form that ergodicity in the Gaussian sense does not always follow from the existence of a spanning set of eigenvectors corresponding.
to a rationally independent set of unimodular eigenvectors. We are left with the open question whether $T$ is ergodic with respect to some measure of full support or (upper) frequently hypercyclic.

If in the situation of Theorem 1.1 the set $Z$ consists of roots of unity, then the Taylor shift is chaotic with unimodular eigenvalues only in $Z$ and not ergodic in the Gaussian sense. So we have an alternative construction for a chaotic operator on a Hilbert space that is not ergodic in the Gaussian sense. The first construction of such an operator on $\ell_2(\mathbb{N})$ given by Menet (14) even leads to an operator which is not (upper) frequently hypercyclic. The approach via Taylor shift is, however, quite different and gives more flexibility in prescribing unimodular eigenvectors.

2. Proof of Theorem 1.1

As main tool for the proof of Theorem 1.1 we seek results on rational approximation in the mean.

Remark 2.1. Let $\Omega$ be an open and bounded set in $\mathbb{C}$ and suppose that each point on the boundary of $\Omega^*$ belongs to the boundary of some component of the interior $(\Omega^*)^c$ of $\Omega^*$. Then Theorem 4 from (13) implies that the span of $\{\gamma_\alpha : \alpha \in (\Omega^*)^c\}$ is dense in $A^2(\Omega)$. If $\Omega$ is open in the extended plane with $\infty \in \Omega$, this is also the case (see (16, Remark 2.6)).

As indicated in the introduction, the function $\gamma_\alpha$ belongs to $A^2(\Omega)$ if $\alpha$ belongs to the boundary of $\Omega^*$ and $\Omega$ has small spherical measure near $1/\alpha$. For $k \in \mathbb{N}_0$, $z \in \Omega$ and $\alpha \in \mathbb{C}$ we write

$$\gamma_{\alpha,k}(z) := z^k/(1 - \alpha z)^{k+1} = z^k \gamma_{\alpha,k+1}(z).$$

We show that in the case of sufficiently small measure near $1/\alpha$ all functions $\gamma_{\alpha,k}$ belong to $A^2(\Omega)$ and that under appropriate conditions they span a dense subspace of $A^2(\Omega)$. The approach is strongly influenced by the proof of a result on the completeness of polynomials in $A^2(\Omega)$ (see (15, Theorem 12.1), cf. also (8, Chapter I, Section 3)).

Let $\delta > 0$ and let the "cup" $C_\delta$ be defined as the interior of the convex hull of $\{t + is(t) : \delta \leq t \leq \delta\}$, where

$$s(t) := \exp(-\exp(1/|t|))$$

(with $s(0) = 0$). With that, we say that two components $A, B$ of an open set $U \subset S := \mathbb{R} + i(-\pi/2, \pi/2)$ are directly bridged at $w \in S$, if $\omega \in \mathbb{T}$ and $\delta > 0$ exist with $w + \omega C_\delta \subset A$ and $w - \omega C_\delta \subset B$. If $\varphi : S \to S^2$ is the standard parametrisation of $S^2 \setminus \{(0,0,1)\}$, that is,

$$\varphi(t + is) = (\cos(s) \cos(t), \cos(s) \sin(t), \sin(s))$$

for $t \in \mathbb{R}$ and $-\pi/2 < s < \pi/2$, we say that two components $C, D$ of an open set $V \subset C_\infty$ with $0, \infty \notin \partial V$ are directly bridged, if the corresponding inverse images under $\varphi$ in $S$ are bridged at some $w$. In this case $\zeta = \varphi(w)$ is said to be a bridge point for $(C, D)$. We say that $C, D$ are bridged if finitely many components $C_0, C_1, \ldots, C_m$ exist with $C_0 = C, C_m = D$ and so that $C_j, C_{j-1}$ are directly bridged. If $\mathcal{C}$ denotes the set of components of $V$, then bridging induces an equivalence relation $\sim$ on $\mathcal{C}$.

If a system $\mathcal{D} \subset \mathcal{C}$ is a complete system of representatives for $\sim$, we briefly say that the system is complete for $V$. 
Theorem 2.2. Let $\Omega$ be an open set in $\mathbb{C}_\infty$ which is bounded in $\mathbb{C}$ or contains $\infty$ and suppose that each point on the boundary of $\Omega^*$ belongs to the boundary of some component of the interior $(\Omega^*)^\circ$ of $\Omega^*$. Moreover, suppose $D$ to be complete for $(\Omega^*)^\circ$ and for $D \in D$ let $\alpha_D$ be either in $D$ or a bridge point of $(C, D)$ for some $C \in \mathcal{C}$. Then both $\{\gamma_\alpha : \alpha \in \bigcup_{D \in D} D\}$ and $\{\gamma_{\alpha,D,k} : D \in D, k \in \mathbb{N}_0\}$ have dense span in $A^2(\Omega)$.

Proof. Let $g \in A^2(\Omega) = A^2(\Omega^\prime)$. Then the Cauchy transform $V_g : (\Omega^*)^\circ \to \mathbb{C}$ of $g$, defined by

$$(V_g)(\alpha) = \int_{\Omega} \gamma_\alpha(z)\overline{g}(z) \, \text{dm}_2(z)$$

is holomorphic with

$$(V_g)^{(k)}(\alpha) = k! \int_{\Omega} \gamma_{\alpha,k}(z)\overline{g}(z) \, \text{dm}_2(z)
$$

for all $k \in \mathbb{N}_0$. According to the Hahn-Banach theorem and Remark 2.1 it suffices to show that $V_g = 0$ under each of the conditions stated in the theorem.

1. Suppose that $g \perp \gamma_\alpha$ for all $\alpha \in \bigcup_{D \in D} D$, that is $(V_g)|_D = 0$ for all $D \in D$. If $D = \mathcal{C}$ then $V_g = 0$. If $C \subset \mathcal{C} \setminus D$ then $C, D$ are bridged for some $D \in D$. We can assume that $C, D$ are directly bridged. Let $\zeta$ be a bridge point for $(C, D)$. Using the assumption on the flatness of $\Omega$ near $\zeta$ it can be shown by differentiation of the parameter integral that $(V_g)^{(k)}$ has a two-sided non-tangential limit at $\zeta$ for all $k \in \mathbb{N}_0$ and that

$$(V_g)^{(k)}(\zeta) = k! \int_{\Omega} \gamma_{\zeta,k}(z)\overline{g}(z) \, \text{dm}_2(z)$$

(cause proof of Theorem 12.1 in [15]). We show that $(V_g)^{(k)}(\zeta) = 0$ for all $k$ implies $(V_g)|_C = 0$ and $(V_g)|_D = 0$ (cause again the proof of Theorem 12.1 in [15]). Since $(V_g)|_D = 0$ we have $(V_g)^{(k)}(\zeta) = 0$ for all $k$, and then also $(V_g)|_C = 0$.

Without loss of generality we can assume that $\zeta = 1$ and that $C$ lies in $D$ and outside $\overline{D}$. Considering the fact that $\zeta = 1$ is a bridge point, we may fix $0 < r < 1$ in such a way that the corresponding "cup"-sets $\pm C_r$ satisfy $\varphi^{-1}(\Omega) \cap \pm C_r = \emptyset$. By the identity theorem for holomorphic functions, it suffices to prove that $V_g$ belongs to a quasi-analytic subclass of $C^\infty(I)$, where

$$I = \varphi([r, r/2])$$

(note that $I$ is a compact interval in $\mathbb{R}$ with 1 in its interior).

To this end, we estimate the derivatives of $V_g$ on $I$. By the Cauchy-Schwarz inequality we have

$$(2) \quad \|(V_g)^{(k)}(x)\| \leq k!\|g\|_2 \left(\int_{\Omega} \frac{\text{dm}_2(z)}{|1 - xz|^{2k+2}}\right)^{1/2}$$

for $k \in \mathbb{N}_0$. So it suffices to estimate the latter integrals. We define

$$W_A(k, x) := \int_A \frac{\text{dm}_2(z)}{|1 - xz|^{2k+2}}$$

for $k \in \mathbb{N}_0$, $x \in I$ and measurable $A \subset \Omega$. With $Q := [r, r] + i[-r,r]$, we have

$$\sup_{x \in I} W_{\Omega \setminus \varphi(Q)}(k, x) = \mathcal{O}(q^k)$$
for some positive $q$. To estimate $W_{\varphi(Q)|Q}\Omega$, we observe that the shape of $\Omega$ in $\varphi(Q)$ allows that with some constant $c > 0$ we have $|1 - xz| \geq c|1 - z|$ for all $x \in I$ and all $z \in \Omega \cap \varphi(Q)$. Thus, for $x \in I$ we obtain (by substituting $u = e^{1/t}$ in the last step)

$$W_{\Omega\cap \varphi(Q)}(k, x) \leq cW_{\Omega\cap \varphi(Q)}(k, 1) \leq c \int_r^r \int_{-s(t)}^{s(t)} \frac{\cos s}{t + k s^2} ds dt$$

$$\leq 4c \int_0^r \frac{s(t)}{t^{2k+2}} dt = 4c \int_{e^{1/r}}^\infty e^{-u} u^{-1} \log^{2k}(u) du.$$

For $k$ sufficiently large and $u \geq k^2$ we have $\log^{2k}(u) \leq e^{u/2}$. Hence, by splitting up the integral at $k^2$, one can see that

$$\int_{e^{1/r}}^\infty e^{-u} u^{-1} \log^{2k}(u) du = O(k^2 \log^{2k}(k^2)) = O(5^k \log^{2k}(k)).$$

Putting together we find that $\sup_{x \in I} W_{\Omega}(k, x) = O(5^k \log^{2k}(k))$. We now combine the latter with \textcircled{2} to obtain

$$\sup_{x \in I} (Vg)^{(k)}(x) = O(k! 5^{k/2} \log^k(k)).$$

The Denjoy-Carleman theorem now shows that $Vg$ belongs to a quasi-analytic subclass of $C^\infty(I)$.

2. Suppose that $g \perp \gamma_{\alpha, D, k}$, that is, $(Vg)(\alpha_D)^{(k)} = 0$, for all $D \in \mathcal{D}$ and $k \in \mathbb{N}_0$, and let $C \in \mathcal{C}$. If $C \in \mathcal{D}$ then $\alpha_C \in C$ or $\alpha_C$ is a bridge point of $C, E$ for some $E \in \mathcal{C}$. In both cases, $(Vg)|_C = 0$. If $C \notin \mathcal{D}$, then there is $D \in \mathcal{D}$ so that $C, D$ are bridged. Again, we can assume that $C, D$ are directly bridged. Let $\zeta$ be a bridge point for $(C, D)$. If $\alpha_D \in D$, then $(Vg)|_D = 0$. Hence $(Vg)^{(k)}(\zeta) = 0$ for all $k$ and then $(Vg)|_C = 0$. If $\alpha_D$ is a bridge point of $(D, E)$ for some $E \in \mathcal{C}$, then $(Vg)^{(k)}(\alpha_D) = 0$ for all $k$ and again $(Vg)|_D = 0$. As in the first case, $(Vg)|_C = 0$. \hfill \Box

Remark 2.3. If, for some component $D \in \mathcal{C}$, the single set system $\{D\}$ is complete and if $\alpha$ is a point in $D$ or a bridge point of $(C, D)$ for some component $C$, then the span of $\{\gamma_{\alpha, k} : k \in \mathbb{N}_0\}$ is dense in $A^2(\Omega)$. In particular, in case $\alpha = 0$ we conclude that the polynomials form a dense set in $A^2(\Omega)$ (cf. Theorem 12.1 in [13], where actually a weaker condition on the sharpness of $\Omega$ near $1/\alpha$ is proved to be sufficient).

From the Godefroy-Shapiro criterion (see e.g. [9] Theorem 3.1) and Theorem \textcircled{2} we obtain

Corollary 2.4. Let $\Omega$ be an open set in $\mathbb{C}_\infty$ which is bounded in $\mathbb{C}$ or contains $\infty$ and suppose that each point in the boundary of $\Omega^*$ belongs to the boundary of some component of $(\Omega^*)^o$. If complete systems $D$ and $D'$ exist with $D \subset D'$ for all $D \in \mathcal{D}$ and $D' \subset \mathbb{C}_\infty \setminus \mathbb{D}$ for all $D' \in \mathcal{D}'$, then $T_{A^2(\Omega)}$ is topologically mixing.

Example 2.5. Let $\Omega = \mathbb{C}_\infty \setminus (\overline{U \cup G}) = ((1/U) \cup (1/G))^*$, where $G, 1/U \subset \mathbb{D}$ are domains with $0 \notin \overline{G}$ and so that $U, G$ (or, equivalently, $1/G, 1/U$) are bridged. If $\zeta$ is a corresponding bridge point, then for each $\alpha \in (1/U) \cup (1/G) \cup \{1/\zeta\}$ the linear span of the rational functions $\{\gamma_{\alpha, k} : k \in \mathbb{N}\}$ forms a dense set in $A^2(\Omega)$ (Remark \textcircled{23} with $D = \{1/U\}$ or $D = \{1/G\}$). By Corollary \textcircled{24} the Taylor shift $T_{A^2(\Omega)}$ is mixing. If $0 \in 1/U$ (e.g. for $1/U = \mathbb{D}$, in which case $\Omega = \mathbb{D} \setminus \overline{G}$), the polynomials
are dense in $A^2(\Omega)$ and the Taylor shift on $A^2(\mathbb{D})$ is a quasi-factor of the Taylor shift on $A^2(\Omega)$.

**Theorem 2.6.** For each infinite closed set $Z \subset \mathbb{T}$ a domain $G \subset \mathbb{D}$ bridged to $C_\infty \setminus \overline{\mathbb{D}}$ exists with $\overline{G} \cap \mathbb{T} \subset Z$ and so that the span of $\{ \gamma_\zeta : \zeta \in \overline{G} \cap \mathbb{T} \}$ is dense in $A^2(\mathbb{D} \setminus \overline{G})$.

**Proof.** According to Example 2.5 with $1/U = \mathbb{D}$, for the proof of the denseness of the span of $\{ \gamma_\zeta : \zeta \in \overline{G} \cap \mathbb{T} \}$ it suffices to show that $\gamma_\zeta \in A^2(\mathbb{D} \setminus \overline{G})$ for $\zeta \in \overline{G} \cap \mathbb{T}$ and that, for some $\alpha \in \mathbb{D} \cup \{ 1 \}$, all $\gamma_{\alpha,k}$ belong to the closure of the span of $\{ \gamma_\zeta : \zeta \in \overline{G} \cap \mathbb{T} \}$.

We suppose $1$ to be an accumulation point of $Z$ and write
\[ U_r := \{ z \in \mathbb{C} : |z - 1| < r \} \cap \mathbb{D} \]
for $r > 0$. Moreover, we consider ”flat cups” $C_{\delta, \rho}$ defined as above with $s(t)$ replaced by $p(t)$, where $0 < \rho \leq 1$.

Let $0 < r < 1$ and $J \subset \mathbb{N}_0$ finite. According to Runge’s theorem, a finite set $F \subset Z \cap U_r$ and a family $(R_j)_{j \in J}$ in span$\{ \gamma_{\zeta} : \zeta \in F \}$ exist with
\[ \max_{z \in \mathbb{T} \setminus U_r} |\gamma_{\zeta,j}(z) - R_j(z)| < 1/n \]
for $j \in J$.

Let now $0 < r_0 < 1$. We choose a finite set $Z_0 \subset Z \cap U_{r_0}$ and a function $R_0^{(0)} \in \text{span}\{ \gamma_{\zeta} : \zeta \in Z_0 \}$ with
\[ \max_{z \in \mathbb{T} \setminus U_{r_0}} |\gamma_{\zeta,0}(z) - R_0^{(0)}(z)| < 1. \]

With conv denoting the convex hull with respect to the plane $\mathbb{C}$, we define
\[ G_1 := U_{r_0} \cap \bigcup_{\zeta \in Z_0} \text{conv}(\zeta \varphi(C_{\delta, \rho})) \]
where we choose $\rho > 0$ so small and $\delta > 0$ so large that $G_1$ is connected with
\[ \int_{U_{r_0} \setminus G_1} |R_0^{(0)}|^2 \, dm_2 < 1 \quad \text{and} \quad \int_{U_{r_0} \setminus G_1} |\gamma_{\zeta,0}|^2 \, dm_2 < 1. \]

For a second step we fix $0 < r_1 < 1/2$. Then we find a finite set $Z_1 \subset Z \cap U_{r_1}$ and $R_0^{(1)}, R_1^{(1)} \in \text{span}\{ \gamma_{\zeta} : \zeta \in Z_1 \}$ such that
\[ \max_{z \in \mathbb{D} \setminus U_{r_1}} |\gamma_{\zeta,j}(z) - R_1^{(1)}(z)| < 1/2. \]

for $j = 0, 1$. We define
\[ G_2 := G_1 \cup \left( U_{r_1} \cap \bigcup_{\zeta \in Z_1} \text{conv}(\zeta \varphi(C_{\delta, \rho})) \right) \]
where we choose $\delta, \rho$ in such a way that $G_2$ is connected with
\[ \int_{U_{r_1} \setminus G_2} |R_j^{(1)}|^2 \, dm_2 < 1/2 \quad \text{and} \quad \int_{U_{r_1} \setminus G_2} |\gamma_{1,j}|^2 \, dm_2 < 1/2 \]
for $j = 0, 1$. Successively, we obtain $0 < r_n < 1/n$, finite sets $Z_n \subset Z \cap U_{r_n}$, functions $R_j^{(n)} \in \text{span}\{ \gamma_{\zeta} : \zeta \in Z_n \}$ such that
\[ \max_{z \in \mathbb{D} \setminus U_{r_{n-1}}} |\gamma_{\zeta,j}(z) - R_j^{(n-1)}(z)| < 1/n \]
for \(j = 0, \ldots, n-1\) and increasing domains \(G_n \subset \mathbb{D} \cap U_n\) with \(\overline{G_n} \cap T = Z_0 \cup \cdots \cup Z_n\), bridged to \(C_\infty \setminus \mathbb{B}\) at all \(\zeta \in Z_0 \cup \cdots \cup Z_n\). and such that

\[
\int_{U_{r_{n-1}} \setminus G_n} |R_j^{(n)}|^2 \, dm_2 < 1/(n+1) \quad \text{and} \quad \int_{U_{r_{n-1}} \setminus G_n} |\gamma_{1,j}|^2 \, dm_2 < 1/(n+1)
\]

for \(j = 0, \ldots, n-1\).

The construction shows that \(G := \bigcup_{n \in \mathbb{N}} G_n\) satisfies all requirements. \(\Box\)

**Proof of Theorem 1.1.** Let \(\Omega = \mathbb{D} \setminus \overline{G}\) where \(G\) is as in Theorem 2.6. Then \(\overline{G} \cap T \subset \mathcal{E}(T)\). According to Example 2.5 and Theorem 2.6, \(T\) is topologically mixing. Since each point in \(T \setminus \overline{G}\) is an interior point of \(\mathbb{B}\), no point in \(T \setminus \overline{G}\) belongs to the point spectrum, that is \(\mathcal{E}(T) \subset \mathbb{Z}\). \(\Box\)

**Remark 2.7.**

1. By deleting sufficiently small parts from \(G\) it is possible to modify \(G\) to an open set \(W\) in such a way that \(\Omega = \mathbb{D} \setminus W\) is connected and the Taylor shift \(T_{A^2}(\Omega)\) satisfies the same conditions as in Theorem 1.1.

2. The statement (and proof) of Theorem 2.6 can be modified in such a way that \(\Omega^* \cap T \subset \mathbb{Z}\), i.e. the spectrum intersects \(T\) only in \(\mathbb{Z}\) (cf. [11, Question 3]).

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