Hyperbolic Orbits for Restricted Three-body Problems with Fixed Energy*

Donglun Wu and Shiqing Zhang
Yangtze Center of Mathematics and College of Mathematics, Sichuan University, Chengdu 610064, People’s Republic of China

Abstract The existence of hyperbolic orbits is proved for a class of restricted three-body problems with a fixed energy by taking limit for a sequence of periodic solutions which are obtained by variational methods.

Keywords Hyperbolic Orbits; Variational Methods; Restricted Three-body Problems; Fixed Energy.
2000 MSC: 34C15, 34C25, 58F

1 Introduction and Main Results

In this paper, we consider the following second order Hamiltonian systems

\[ \ddot{u}(t) + \nabla V(u(t)) = 0 \quad (1) \]

with

\[ \frac{1}{2} |\dot{u}(t)|^2 + V(u(t)) = H. \quad (2) \]

where \( u \in C^2(R^1, R^N), V \in C^1(R^N, R^1). \) Subsequently, \( \nabla V(x) \) denotes the gradient with respect to the \( x \) variable, \( (\cdot, \cdot) : R^N \times R^N \rightarrow R \) denotes the standard Euclidean inner product in \( R^N \) and \(| \cdot |\) is the induced norm.

The restricted three-body problem is a reduced model for the Newtonian three-body problems. The existence of periodic orbits, hyperbolic orbits for this model has been studied by many mathematicians [1, 2, 4, 7, 8, 9, 15, 21, 23, 27] and the references therein. In this paper, an orbit of this problem is said to be hyperbolic if two of the three bodies remain bounded while the third goes to infinity with vanishing velocity. A special type of restricted three-body problem was considered by Sitninkov [20] and Moser [13]: Under Newton’s law of attraction, two mass points of equal mass \( m_1 = m_2 = \frac{1}{2} \) moving in the plane of their elliptic orbits such that the center of masses is at rest, the third mass point \( m \) which does not influence the motion of the first two moving on the line perpendicular to the plane containing the first two mass points and going through the center of mass. Let \( u \) be the coordinate describing the motion of \( m \) and the center of mass of the first two mass points is at the origin. The restricted three-body problem consists in determining \( u \) such that:

\[ -\ddot{u}(t) = \frac{u(t)}{(|u(t)|^2 + |r(t)|^2)^{3/2}}, \]

*Supported partially by NSF of China.
where \( r(t) = r(t + 2\pi) \) is the distance from the center of mass to one of the first two mass points. For a small \( \varepsilon > 0 \), the function \( r \) has the form (see Moser [13]):

\[
r(t) = \frac{1}{2}(1 - \varepsilon \cos t) + O(\varepsilon^2).
\]

Souissi [21] used variational minimax methods and approximations to prove the existence of at least one parabolic orbit of the circular restricted three-body problem with \( 0 < \alpha < 1 \) for

\[
\ddot{u}(t) + \frac{\alpha u(t)}{(|u(t)|^2 + \varepsilon |\dot{r}(t)|^2)^{\frac{\alpha+2}{2}}} = 0.
\] (3)

With \( 0 < \alpha < 2 \), Zhang [27] has proved

**Theorem 1.1 (See[27])**. For (3) with \( 0 < \alpha < 2 \), there exists one odd parabolic or hyperbolic orbit which minimizes the corresponding variational functional.

The above results are obtained for Newtonian weak force type potentials. For the two-body problems with charges, Wu and Zhang [26] have proved the existence of hyperbolic orbits for a class of singular Hamiltonian systems with fixed energy, they obtained the following theorem.

**Theorem 1.2 (See[26])** Suppose that \( V \in C^1(R^N \setminus \{0\}, R^1) \) satisfies

(A1) \( V(-x) = V(x), \forall x \in R^N \setminus \{0\} \),

(A2) there is a constant \( \alpha \in (0, 2) \) such that

\[
(x, \nabla V(x)) = -\alpha V(x) < 0 \quad \text{for any} \quad x \in R^N \setminus \{0\}.
\]

Then for any \( H > 0 \), there is at least one hyperbolic orbit for systems (1)-(2).

Similarly, in restricted three-body problems, we can also consider three bodies which are charged. Suppose \( e_1, e_2 \) and \( e \) represent the charges of \( m_1, m_2 \) and \( m \) with \( e_1 = e_2 \), then the effect force between the mass points not only obey the Newton’s but also Coulomb’s laws. When \( |e_1| = |e_2| \) are small enough, the first two bodies attract each other which implies that they can move in their elliptic orbits. The motion equation of the third mass point is

\[
m\ddot{u}(t) + \frac{\alpha(m + 2\varepsilon e_1)u(t)}{(|u(t)|^2 + |r(t)|^2)^{\frac{\alpha+2}{2}}} = 0.
\] (4)

In this model, the potential is not singular which is much different from the Newtonian type potentials. As to the existence of periodic orbits for non-singular Hamiltonian systems with fixed energy, there have been many works. In 1978, Rabinowitz [19] used variational methods for strongly indefinite functionals to study the existence of a periodic solution of a class of Hamiltonian systems on any given regular energy hyperface. He obtained the following result.

**Theorem 1.3 (See[19])** Let \( H \in C^1(R^{2N}, R^1) \). Suppose

(B1) for some \( b \neq 0 \), \( H^{-1}(b) \) is radially homeomorphic to \( S^{2N-1} \).

(B2) \( H_{\zeta} \neq 0, \forall \zeta \in H^{-1}(b) \).
Then the Hamiltonian system

\[
d\frac{dz}{dt} = JHz
\]  
(5)

possesses a periodic solution on \( H^{-1}(b) \), where \( z = (p, q) \in R^{2N} \), \( H = H(p, q) \), \( H_z = (\frac{\partial H}{\partial p}, \frac{\partial H}{\partial q}) \), \( J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \) \( 2N \times 2N \).

Since the pioneering work of Rabinowitz, there are many works on the existence of periodic solutions for (5) or second order Hamiltonian systems. As to the unbounded orbits for non-singular Hamiltonian systems with a fixed energy, there are only few papers relating to this topic. In 1994, E. Serra [22] has obtained the existence of Homoclinic orbits at infinity for a class of second order conservative systems. In his paper, he treated the systems with zero energy and he approximated the homoclinic orbits with a sequence of brake orbits which are obtained by variational methods. He obtained the following theorem.

**Theorem 1.4 (See[22])** Suppose that the potential \( V \in C^2(R^N, R^1) \) satisfies

(C1) \( V(x) < 0 \) for all \( x \in R^N \),

(C2) there exist \( R_0 > 0 \), \( \gamma > 2 \) such that

\[
V(x) = -\frac{1}{|x|\gamma} + W(x), \quad \forall \ |x| \geq R_0,
\]

(C3) \( \lim_{|x|\to+\infty} W(x)|x|\gamma = 0 \),

(C4) \( (x, \nabla W(x)) > 0 \), \( \forall |x| \geq R_0 \).

Then there exists at least one solution to the problem

\[
\begin{cases}
\ddot{u}(t) + \nabla V(u(t)) = 0, & \text{for all } t \in R, \\
\lim_{t\to\pm\infty} |u(t)| = +\infty, \\
\lim_{t\to\pm\infty} \dot{u}(t) = 0.
\end{cases}
\]  
(6)

**Definition 1.5 (See[4])** An orbit \( u(t) \) is called a parabolic orbit, if we have

\[
|u(t)| \to +\infty, \quad |\dot{u}(t)| \to 0 \quad \text{as} \quad |t| \to +\infty;
\]

An orbit \( u(t) \) is called a hyperbolic orbit, if we have

\[
|u(t)| \to +\infty, \quad |\dot{u}(t)| > 0 \quad \text{as} \quad |t| \to +\infty.
\]

Motivated by above papers, we have following theorems.

**Theorem 1.6** Suppose \( V \in C^1(R^N, R^1) \) satisfies

(V1) \( V(0) \geq V(-x) = V(x) > 0 \), for all \( x \in R^N \).

(V2) \( (x, \nabla V(x)) \to 0 \), as \( |x| \to +\infty \).

(V3) \( V(x) \to 0 \), as \( |x| \to +\infty \).
Then system (1)-(2) possesses at least one hyperbolic orbit for any given $H > V(0)$.

**Theorem 1.7** Suppose $V \in C^1(R^N, R^1)$ satisfies (V$_2$), (V$_3$) and

(V$_4$) $V(-x) = V(x) < 0$, for all $x \in R^N$.

Then systems (1)-(2) possesses at least one hyperbolic orbits for any $H > 0$.

**Remark 1** In this paper, we use the 1/2-antisymmetrical constraint to reduce the norm. Since the potential in this paper has no singularity, we can also reduce the norm on the odd-antisymmetry constrain space which is

$$E_R = \left\{ q \in H^1 \mid q(-t) = -q(t), \left| q \left( -\frac{1}{2} \right) \right| = \left| q \left( \frac{1}{2} \right) \right| = R \right\},$$

where $H^1 = W^{1, 2}([-\frac{1}{2}, \frac{1}{2}], R^N)$. And all the proofs are similar to this paper.

**Remark 2** Notice that in model (4), if $e$ has different sign with $e_1$, $e_2$ and $|e|$ is large enough, the parameter $\alpha(m + 2ee_1)$ is negative, which satisfies all conditions in Theorem 1.6. On the other hand, if $e$ has the same sign with $e_1$ and $e_2$, the parameter $\alpha(m + 2ee_1)$ is positive, which satisfies all conditions in Theorem 1.7.

2 Variational Settings

Let $L^\infty([0, 1], R^N)$ be a space of measurable functions from $[0, 1]$ into $R^N$ and essentially bounded under the following norm

$$\|q\|_{L^\infty([0, 1], R^N)} := \text{esssup}\{|q(t)| : t \in [0, 1]\}.$$  

Similar to A. Ambrosetti and V. Coti. Zelati in [1], we use the 1/2-antisymmetrical constraint to reduce the norm. The space where we define the functional is as follow.

$$M_R = \{ q \in H^1 \mid q(t + \frac{1}{2}) = -q(t), |q(0)| = |q(1)| = R \}.$$  

For any $q \in H^1$, we know that the following norms are equivalent to each other

$$\|q\|_{H^1} = \left( \int_0^1 |\dot{q}(t)|^2 dt \right)^{1/2},$$

$$\|q\|_{H^1} = \left( \int_0^1 |\dot{q}(t)|^2 dt \right)^{1/2} + \left( \int_0^1 |q(t)|^2 dt \right)^{1/2},$$

$$\|q\|_{H^1} = \left( \int_0^1 |\dot{q}(t)|^2 dt \right)^{1/2} + |q(0)|.$$  

If $q \in M_R$, we have $\int_0^1 q(t) dt = 0$, then by Poincaré-Wirtinger’s inequality, we obtain that the above norms are equivalent to

$$\|q\|_{H^1} = \left( \int_0^1 |\dot{q}(t)|^2 dt \right)^{1/2}.$$  

Moreover, let $f : M_R \to R^1$ be the functional defined by

$$f(q) = \frac{1}{2} \int_0^1 |\dot{q}(t)|^2 dt \int_0^1 (H - V(q(t))) dt = \frac{1}{2} \|q\|^2 \int_0^1 (H - V(q(t))) dt.$$ (7)
Then one can easily check that $f \in C^1(M_R; R^1)$ and

$$
\langle f'(q), q(t) \rangle = \|q\|^2 \int_0^1 \left( H - V(q(t)) - \frac{1}{2} \langle \nabla V(q(t)), q(t) \rangle \right) dt.
$$

(8)

Firstly, we prove Theorem 1.6. To prove this theorem, our way is to approach the hyperbolic orbits with a sequence of periodic orbits which are obtained by the minimizing theory. We need the following lemma which is proved by A. Ambrosetti and V. Coti. Zelati in [1].

**Lemma 2.1 (See[1])** Let $f(q) = \frac{1}{2} \int_0^1 |\dot{q}(t)|^2 dt \int_0^1 (H - V(q(t))) dt$ and $\tilde{q} \in H^1$ be such that $f'(\tilde{q}) = 0$, $f(\tilde{q}) > 0$. Set

$$
T^2 = \frac{\frac{1}{2} \int_0^1 |\dot{q}(t)|^2 dt}{\int_0^1 (H - V(\tilde{q}(t)) dt}.
$$

Then $\tilde{u}(t) = \tilde{q}(t/T)$ is a non-constant $T$-periodic solution for (1) and (2).

**Remark 3** In view of the proof of Lemma 2.3 in [1], we can see that the condition $f(\tilde{q}) > 0$ in Lemma 3.1 can be replaced by $\int_0^1 |\dot{q}(t)|^2 dt > 0$.

**Lemma 2.2 (Palais[24])** Let $\sigma$ be an orthogonal representation of a finite or compact group $G$ in the real Hilbert space $H$ such that for any $\sigma \in G$,

$$
f(\sigma \cdot x) = f(x),
$$

where $f \in C^1(H, R^1)$. Let $S = \{x \in H| \sigma x = x, \forall \sigma \in G\}$, then the critical point of $f$ in $S$ is also a critical point of $f$ in $H$.

**Lemma 2.3 (Translation Property[16])** Suppose that, in domain $D \subset R^N$, we have a solution $\phi(t)$ for the following differential equation

$$
x^{(n)} + F(x^{(n-1)}, \ldots, x) = 0,
$$

where $x^{(k)} = d^k x/dt^k$, $k = 0, 1, \ldots, n$, $x^{(0)} = x$. Then $\phi(t - t_0)$ with $t_0$ being a constant is also a solution.

3 The Proof of Theorem 1.6

Firstly, we prove the existence of the approximate solutions, then we study the limit procedure. In order to obtain the critical points of the functional and make some estimations, we need the following lemma.

**Lemma 3.1** Suppose the conditions of Theorem 1.6 hold, then for any $R > 0$, there exists at least one periodic solution on $M_R$ for the following systems

$$
\ddot{q}(t) + \nabla V(q(t)) = 0, \quad \forall \ t \in \left(-\frac{T_R}{2}, \frac{T_R}{2}\right)
$$

(9)

with

$$
\frac{1}{2} |\dot{q}(t)|^2 + V(q(t)) = H, \quad \forall \ t \in \left(-\frac{T_R}{2}, \frac{T_R}{2}\right),
$$

(10)
where $T_R$ is defined as

$$T_R^2 = \frac{1}{2} \int_0^1 |\dot{q}_R(t)|^2 dt \int_0^1 (H - V(q_R(t))) dt,$$

(11)

where $q_R(t)$ is the minimizer for the functional.

**Proof.** We notice that $H^1$ is a reflexive Banach space and $M_R$ is a weakly closed subset of $H^1$. By the definition of $f$, $(V_1)$ and $H > V(0)$, we obtain that $f$ is a functional bounded from below and

$$f(q) = \frac{1}{2} \|q\|^2 \int_0^1 (H - V(q(t))) dt \geq \frac{(H - V(0))}{2} \|q\|^2 \to +\infty \quad \text{as} \quad \|q\| \to +\infty.$$ 

Furthermore, it is easy to check that $f$ is weakly lower semi-continuous. Then, we can see that for every $R > 0$ there exists a minimizer $q_R \in M_R$ such that

$$f'(q_R) = 0, \quad f(q_R) = \inf_{q \in M_R} f(q) \geq 0. \quad (12)$$ 

It is easy to see that $\|q_R\|^2 = \int_0^1 |\dot{q}_R(t)|^2 dt > 0$, otherwise we deduce that $|q_R(t)| \equiv R > 0$, on the other hand, by the 1/2-antisymmetry of $q_R$, we have $q_R \equiv 0$, which is a contradiction. Then by Lemmas 2.1-2.3, $u_R(t) = q_R(\frac{t + T_R}{T_R}) : \left(-\frac{T_R}{2}, \frac{T_R}{2}\right) \to M_R$ is a non-constant $T_R$-periodic solution satisfying (9) and (10). The proof of this lemma is finished.

**Remark 4** In our model, the set $M_R$ is a closed set in set $H^1$. We minimize the functional on the set $M_R$, however, we can not show that $u_R(t)$ solve the equation at $\pm \frac{T_R}{2}$. But it is true that we do not need that $u_R(t)$ is a solution at these two moments. Furthermore, we know that $u_R(t)$ still has definition at $\pm \frac{T_R}{2}$ and $|u_R(\pm \frac{T_R}{2})| = R$.

Subsequently, we need to let $R \to +\infty$. But before doing this, we need to prove $u_R$ can not diverge to infinity uniformly as $R \to +\infty$, which is the following lemma.

**Lemma 3.2** Suppose that $u_R(t) : \left[-\frac{T_R}{2}, \frac{T_R}{2}\right] \to M_R$ is the solution obtained in Lemma 3.1, then $\min_{t \in \left[-\frac{T_R}{2}, \frac{T_R}{2}\right]} |u_R(t)|$ is bounded from above. More precisely, there is a constant $M > 0$ independent of $R$ such that

$$\min_{t \in \left[-\frac{T_R}{2}, \frac{T_R}{2}\right]} |u_R(t)| \leq M \quad \text{for all} \quad R > 0.$$ 

**Proof.** Since $q_R \in M_R$, it is easy to see that $u_R(t) = q_R(\frac{t + T_R}{T_R})$ satisfies $u_R(-\frac{T_R}{2}) = u_R(\frac{T_R}{2})$ and $\dot{u}_R(-\frac{T_R}{2}) = \dot{u}_R(\frac{T_R}{2})$, then we have that

$$\left(u_R \left(\frac{T_R}{2}\right), \dot{u}_R \left(\frac{T_R}{2}\right)\right) - \left(u_R \left(-\frac{T_R}{2}\right), \dot{u}_R \left(-\frac{T_R}{2}\right)\right) = \int_{-\frac{T_R}{2}}^{\frac{T_R}{2}} \frac{d}{dt} (u_R(t), \dot{u}_R(t)) dt$$

$$= \int_{-\frac{T_R}{2}}^{\frac{T_R}{2}} (u_R(t), \dot{u}_R(t)) dt$$
\[
\int_{T_R^2}^{T_R^2} (\dot{u}_R(t))^2 + (u_R(t), \ddot{u}_R(t))dt = \int_{T_R}^{T_R} 2(H - V(u_R(t))) - (\nabla V(u_R(t), u_R(t)))dt.
\]

Then we obtain that
\[
\int_{T_R^2}^{T_R^2} 2H - (2V(u_R(t)) + (\nabla V(u_R(t), u_R(t))) dt = 0.
\]

There are two cases needed to be discussed.

**Case 1.** \(2H - (2V(u_R(t)) + (\nabla V(u_R(t), u_R(t)))) \equiv 0\), which implies that
\[
2H = 2V(u_R(t)) + (\nabla V(u_R(t), u_R(t)), \quad \text{a.e. } t \in \left[-\frac{T_R}{2}, \frac{T_R}{2}\right].
\]

Hypotheses (V2), (V3) imply that there exists a constant \(M_1 > 0\) independent of \(R\) such that
\[
\min_{t \in \left[-\frac{T_R}{2}, \frac{T_R}{2}\right]} |u_R(t)| \leq M_1.
\]

**Case 2.** \(2(H - V(u_R(t)) - (\nabla V(u_R(t), u_R(t)))\) changes sign in \([-\frac{T_R}{2}, \frac{T_R}{2}]\). Then there exists \(t_0 \in \left[-\frac{T_R}{2}, \frac{T_R}{2}\right]\) such that
\[
2H - (2V(u_R(t_0)) + (\nabla V(u_R(t_0), u_R(t_0))) < 0,
\]

which implies that
\[
2H < 2V(u_R(t_0)) + (\nabla V(u_R(t_0), u_R(t_0))).
\]

It follows from \(H > 0\) and hypotheses (V2), (V3) that there exists a constant \(M_2 > 0\) independent of \(R\) such that
\[
\min_{t \in \left[-\frac{T_R}{2}, \frac{T_R}{2}\right]} |u_R(t)| \leq M_2.
\]

Then the proof is completed.

**Lemma 3.3** Suppose that \(R > M\) and \(u_R(t)\) is the solution for (9) – (10) obtained in Lemma 3.1, where \(M\) is from Lemma 3.2. Set
\[
t_+ = \sup \left\{ t \in \left[-\frac{2R}{2}, \frac{2R}{2}\right] \mid |u_R(t)| \leq L \right\}
\]
and
\[
t_- = \inf \left\{ t \in \left[-\frac{2R}{2}, \frac{2R}{2}\right] \mid |u_R(t)| \leq L \right\}
\]
where \(L\) is a constant independent of \(R\) such that \(M < L < R\). Then we have that
\[
\frac{T_R}{2} - t_+ \to +\infty, \quad t_+ + \frac{T_R}{2} \to +\infty \quad \text{as } R \to +\infty.
\]
**Proof.** By the definition of \( u_R(t) \) we have that
\[
|u_R\left(-\frac{T_R}{2}\right)| = |u_R\left(\frac{T_R}{2}\right)| = R.
\]
Then, by \((V_1)\) and the definitions of \( t_+ \), we have
\[
\int_{t_+}^{T_R} \sqrt{H - V(u_R(t))} \, |\dot{u}_R(t)| \, dt \geq \sqrt{2} \int_{t_+}^{T_R} \sqrt{H - V(0)} \, |\dot{u}_R(t)| \, dt
\]
\[
\geq \sqrt{H - V(0)} \int_{t_+}^{T_R} |\dot{u}_R(t)| \, dt
\]
\[
\geq \sqrt{H - V(0)} \left| \int_{t_+}^{T_R} \dot{u}_R(t) \, dt \right|
\]
\[
\geq \sqrt{H - V(0)} (R - L). \tag{13}
\]
It follows from Lemma 3.1 and \((V_1)\) that
\[
\int_{t_+}^{T_R} \sqrt{H - V(u_R(t))} \, |\dot{u}_R(t)| \, dt = \sqrt{2} \int_{t_+}^{T_R} \sqrt{H - V(u_R(t))} \sqrt{H - V(u_R(t))} \, dt
\]
\[
\leq \sqrt{2} H \left( \frac{T_R}{2} - t_+ \right)
\]
Combining (13) with the above estimation, we obtain that
\[
\sqrt{H - V(0)} (R - L) \leq \sqrt{2} H \left( \frac{T_R}{2} - t_+ \right).
\]
Then we have
\[
\frac{T_R}{2} - t_+ \to +\infty, \quad \text{as} \quad R \to +\infty.
\]
The limit for \( t_- + \frac{T_R}{2} \) can be obtained in the similar way. The proof is completed.

**The Limit Procedure.** Subsequently, we set that
\[
t^* = \inf \left\{ t \in \left[-\frac{T_R}{2}, \frac{T_R}{2}\right] \mid |u_R(t)| = M \right\}
\]
and
\[
u_R^*(t) = u_R(t - t^*)
\]
Since \( L > M \), we can deduce that \( t_+ \geq t^* \geq t_- \), which implies that
\[
-\frac{T_R}{2} + t^* \to -\infty, \quad \frac{T_R}{2} + t^* \to +\infty \quad \text{as} \quad R \to \infty.
\]
Then it follows from (13) that
\[
\frac{1}{2} \left| \dot{u}_R^*(t) \right|^2 + V(u_R^*(t)) = H, \quad \forall \ t \in \left(-\frac{T_R}{2} + t^*, \frac{T_R}{2} + t^*\right).
\]
which implies that
\[
\left| \dot{u}_R^*(t) \right|^2 = 2(H - V(u_R^*(t))), \quad \forall \ t \in \left(-\frac{T_R}{2} + t^*, \frac{T_R}{2} + t^*\right).
\]
By \((V_3)\) and \(V \in C^1(R^N, R^1)\), we can deduce that there exists a constant \(M_4 > 0\) independent of \(R\) such that
\[
|V(u_R^*(t))| \leq M_4 \quad \text{for all} \quad t \in \left(-\frac{T_R}{2} + t^*, \frac{T_R}{2} + t^*\right).
\]
Then there is a constant \(M_5\) independent of \(R\) such that
\[
|\dot{u}_R^*(t)| \leq M_5 \quad \text{for all} \quad t \in \left(-\frac{T_R}{2} + t^*, \frac{T_R}{2} + t^*\right),
\]
which implies that
\[
|u_R^*(t_1) - u_R^*(t_2)| \leq \int_{t_1}^{t_2} |\dot{u}_R^*(s)| ds \leq M_5|t_1 - t_2|.
\]
for each \(R > 0\) and \(t_1, t_2 \in \left(-\frac{T_R}{2} + t^*, \frac{T_R}{2} + t^*\right)\), which shows \(\{u_R^*\}\) is equicontinuous.

From the above lemmas, we have proved there is at least one hyperbolic solution for \((1) - (2)\) with \(H > 0\). We finish the proof of Theorem 1.6.

4 The Proof of Theorem 1.7

Since the potential in Theorem 1.7 is negative and of \(C^1\) class, the proof of this theorem is more simple. Similar to the proof of Theorem 1.6, we consider the functional \((7)\) on \(M_R\) which is \(f : M_R \to R\).

Lemma 4.1 Suppose the conditions of Theorem 1.5 hold, then for any \(R > 0\), there exists at least one periodic solution on \(M_R\) for the following systems
\[
\ddot{q}(t) + \nabla V(q(t)) = 0, \quad \forall \ t \in \left(-\frac{T_R}{2}, \frac{T_R}{2}\right)
\]
with
\[
\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = H, \quad \forall \ t \in \left(-\frac{T_R}{2}, \frac{T_R}{2}\right).
\]

Proof. We notice that \(H^1\) is a reflexive Banach space and \(M_R\) is a weakly closed subset of \(H^1\). Since \(H > 0\), we obtain that
\[
f(q) = \frac{1}{2}\|q\|^2 \int_0^1 (H - V(q(t)))dt \geq \frac{H}{2}\|q\|^2,
\]
which implies that \(f\) is a functional bounded from below, furthermore, it is easy to check that \(f\) is weakly lower semi-continuous and
\[
f(q) \to +\infty \quad \text{as} \quad \|q\| \to +\infty.
\]
Then, we conclude that for every $R > 0$ there exists a minimizer $Q_R \in M_R$ such that
\[ f'(Q_R) = 0, \quad f(Q_R) = \inf_{q \in M_R} f(q) > 0. \]

It is easy to see that $\|Q_R\|^2 = \int_0^1 |\dot{Q}_R(t)|^2 dt > 0$, otherwise we deduce that $|Q_R(t)| \equiv R$, on the other hand, by the 1/2-antisymmetry of $Q_R$, we have $Q_R \equiv 0$, which is a contradiction. This implies that $f(Q_R) > 0$. Then let
\[
T_R^2 = \frac{\frac{1}{2} \int_0^1 |\dot{Q}_R(t)|^2 dt}{\int_0^1 (H - V(Q_R(t))) dt},
\]
then by Lemmas 2.1-2.3, $U_R(t) = Q_R \left( \frac{t + T_R}{T_R^2} \right) : \left( -\frac{T_R}{2}, \frac{T_R}{2} \right) \to M$ is a non-constant $T_R$-periodic solution satisfying (14) and (15). The proof of this lemma is finished.

Subsequently, we need to show that $U_R(t)$ can not diverge to infinity uniformly as $R \to +\infty$. Moreover, we prove the following lemma.

**Lemma 4.2** Suppose that $U_R(t) : \left[ -\frac{T_R}{2}, \frac{T_R}{2} \right] \to M$ is the solution obtained in Lemma 4.1, then $\min_{t \in \left[ -\frac{T_R}{2}, \frac{T_R}{2} \right]} |U_R(t)|$ is bounded from above. More precisely, there is a constant $M' > 0$ independent of $R$ such that
\[
\min_{t \in \left[ -\frac{T_R}{2}, \frac{T_R}{2} \right]} |U_R(t)| \leq M' \text{ for all } R > 0.
\]
The proof of this lemma is same with that of Lemma 3.2.

**Lemma 4.3** Suppose that $R > M'$, where $M'$ is defined in Lemma 4.2 and $U_R(t)$ is the solution for (14) and (15) obtained in Lemma 4.1. Set
\[
t_+ = \sup \left\{ t \in \left[ -\frac{T_R}{2}, \frac{T_R}{2} \right] \left| |U_R(t)| \leq l \right. \right\}
\]
and
\[
t_- = \inf \left\{ t \in \left[ -\frac{T_R}{2}, \frac{T_R}{2} \right] \left| |U_R(t)| \leq l \right. \right\}
\]
where $l$ is a constant independent of $R$ such that $M' < l < R$. Then we have that
\[
\frac{T_R}{2} - t_+ \to +\infty, \quad t_- + \frac{T_R}{2} \to +\infty \quad \text{as } R \to +\infty.
\]

**Proof.** By the definition of $U_R(t)$ we have that
\[
\left| U_R \left( -\frac{T_R}{2} \right) \right| = \left| U_R \left( \frac{T_R}{2} \right) \right| = R.
\]
Then, by $(V_4)$ and the definitions of $t_+$ and $t_-$, we have
\[
\int_{t_+}^{T_R} \sqrt{H - V(U_R(t))} |\dot{U}_R(t)| dt \geq \sqrt{H} \int_{t_+}^{T_R} |\dot{U}_R(t)| dt \geq \sqrt{H} \left| \int_{t_+}^{T_R} \dot{U}_R(t) dt \right| \geq \sqrt{H} (R - l) \quad (17)
\]
and
\[ \int_{-\frac{T_R}{2}}^{t_-} \sqrt{H - V(U_R(t))} |\dot{U}_R(t)| dt \geq \sqrt{H} \int_{-\frac{T_R}{2}}^{t_-} |\dot{U}_R(t)| dt \geq \sqrt{H} \left[ \int_{-\frac{T_R}{2}}^{t_-} \dot{U}_R(t) dt \right] \geq \sqrt{H} (R - l). \] (18)

Since \( V \in C^1(R^N, \mathbb{R}^1) \), it follows from (V3), that there exists a constant \( M_6 > 0 \) independent of \( R \) such that
\[ |V(U_R(t))| \leq M_6 \quad \text{for all } t \in \left[-\frac{T_R}{2}, \frac{T_R}{2}\right], \]
which implies that
\[ \int_{t_+}^{T_R} \sqrt{H - V(U_R(t))} |\dot{U}_R(t)| dt = \sqrt{2} \int_{t_+}^{T_R} (H - V(U_R(t))) dt \leq \sqrt{2} (H + M_6) \left( \frac{T_R}{2} - t_+ \right). \]

Combining (17) with the above estimates, we obtain that
\[ \sqrt{H} (R - L) \leq \sqrt{2} (H + M_6) \left( \frac{T_R}{2} - t_+ \right). \]

Then we have
\[ \frac{T_R}{2} - t_+ \to +\infty, \quad \text{as } R \to +\infty. \]

The limit for \( t_- + \frac{T_R}{2} \) can be obtained in the similar way. The proof is completed.

The following limit procedure is similar to the proof in Theorem 1.6. □

References

[1] A. Ambrosetti and V. Coti Zelati, Closed orbits of fixed energy for singular Hamiltonian systems, Arch. Rat. Mech. Anal. 112(1990), 339-362.

[2] A. Ambrosetti and V. Coti Zelati, Periodic solutions for singular Lagrangian systems, Springer, 1993.

[3] V. Benci and P. H. Rabinowitz, Critical point theorems for indefinite functionals Inv. Math. 52(1979) 241-73

[4] V. Barutello, S. Terracini, and G. Verzini. Entire parabolic trajectories as minimal phase transitions. Preprint, arXiv:1105.3358v1 [math.DS], 2011.

[5] G. Cerami, Un criterio di esistenza per i punti critici su varietà illimitate, Rend. Acad. Sci. Lett. Ist. Lombardo 112(1978), 332-336.

[6] K. C. Chang, Infinite dimensional Morse theory and multiple solution problems, Birkhauser, 1993.

[7] M. Degiovanni and F. Giannoni, Periodic solutions of dynamical systems with Newtonian type potentials, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 15(1988), 467-494.

[8] P. Felmer, K. Tanaka, Hyperbolic-like solutions for singular Hamiltonian systems, Nonlinear Differ. Equ. Appl. 7 (2000) 43-65.
[9] W. Gordon, Conservative dynamical systems involving strong forces, Trans. Amer. Math. Soc. 204(1975), 113-135.

[10] E. W. C. Van Groesen, Analytical mini-max methods for Hamiltonian break orbits of prescribed energy, J. Math. Anal. Appl. 132 (1988), 1-12.

[11] Y. Lv, C. L. Tang, Existence of even homoclinic orbits for second-order Hamiltonian systems, Nonlinear Anal. 67 (2007), no. 7, 2189–2198.

[12] M. Izydorek, J. Janczewska, Homoclinic solutions for nonautonomous second-order Hamiltonian systems with a coercive potential, J. Math. Anal. Appl. 335 (2007), no. 2, 1119–1127.

[13] J. Moser, Stable and Random Motions in Dynamical Systems. Ann Math Studies 77. Princeton: Princeton University Press, 1973.

[14] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian systems, Appl. Math. Sci., vol. 74, Springer-Verlag, New York, 1989.

[15] E. Maderna, A. Venturelli, Globally minimizing parabolic motions in the Newtonian N-body problem, Arch. Ration. Mech. Anal. 194 (2009), 283-313.

[16] F. Verhulst, Nonlinear Differential Equations and Dynamical Systems, Springer, Berlin, Heidelberg, 1990.

[17] P. H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, Proc. Roy. Soc. Edinburgh Sect. A 114 (1990) 33-38.

[18] P. H. Rabinowitz, Periodic and Heteroclinic orbits for a periodic Hamiltonian system, Ann. Inst. H. Poincare Anal. Non Lineaire 6 (5) (1989) 331-346.

[19] P. H. Rabinowitz, Periodic solutions of Hamiltonian systems, Communications on Pure and Applied Mathematics, 31 (2) (1978) 157-184.

[20] K. Sitnikov, Existence of oscillating motion for the three-body problem. J Dokl Akad Nauk USSR, 133(1960) 303-306

[21] Souissi C. Existence of parabolic orbits for the restricted three-body problem, Annals of University of Craiova. Math Comp Sci Ser, 31(2004) 85-93

[22] E. Serra, Homoclinic orbits at infinity for second order conservative systems, Nonlinear Differ. Equ. Appl. 1 (1994), 249-266.

[23] E. Serra, S. Terracini, Noncollision solutions to some singular minimization problems with Keplerian-like potentials, Nonlinear Anal. TMA 22(1994), 45-62.

[24] R. Palais, The principle of symmetric criticality, CMP 69(1979), 19-30.

[25] D. L. Wu, X. P. Wu, C. L. Tang, Homoclinic solutions for a class of nonperiodic and noneven second-order Hamiltonian systems, J. Math. Anal. Appl. 367 (2010) 154-166.

[26] D. L. Wu, S. Q. Zhang, Hyperbolic Orbits for a Class of Singular Hamiltonian Systems with Repulsive Potentials. arXiv:1205.3891v1 [math.CA], 2012.

[27] S. Q. Zhang, Variational Minimizing Parabolic and Hyperbolic Orbits for the Restricted 3-Body Problems, Sci. China, doi: 10.1007/s11425-011-4311-9.

[28] S. Q. Zhang, Periodic Solutions for some second order Hamiltonian systems, Nonlinearity, 22 (2009) 2141-2150.