$\sigma$-BIDERIVATIONS AND $\sigma$-COMMUTING MAPS OF TRIANGULAR ALGEBRAS

JOE REPKA$^1$, JUANA SÁNCHEZ-ORTEGA$^2$

Abstract. Let $A$ be an algebra and $\sigma$ an automorphism of $A$. A linear map $d$ of $A$ is called a $\sigma$-derivation of $A$ if $d(xy) = d(x)y + \sigma(x)d(y)$, for all $x, y \in A$. A bilinear map $D : A \times A \to A$ is said to be a $\sigma$-biderivation of $A$ if it is a $\sigma$-derivation in each component. An additive map $\Theta$ of $A$ is $\sigma$-commuting if it satisfies $\Theta(x)x - \sigma(x)\Theta(x) = 0$, for all $x \in A$. In this paper, we introduce the notions of inner and extremal $\sigma$-biderivations and of proper $\sigma$-commuting maps. One of our main results states that, under certain assumptions, every $\sigma$-biderivation of a triangular algebra is the sum of an extremal $\sigma$-biderivation and an inner $\sigma$-biderivation. Sufficient conditions are provided on a triangular algebra for all of its $\sigma$-biderivations (respectively, $\sigma$-commuting maps) to be inner (respectively, proper). A precise description of $\sigma$-commuting maps of triangular algebras is also given.

1. Introduction

Triangular algebras were introduced by Chase [7] in the early 1960s. He ended up with these structures in the course of his study of the asymmetric behavior of semi-hereditary rings; he provided an example of a left semi-hereditary ring which is not right semi-hereditary. Since their introduction, triangular algebras have played an important role in the development of ring theory. In 1966, Harada [19] characterized hereditary semi-primary rings by using triangular algebras; in his paper, triangular algebras are named generalized triangular matrix rings. Later on, Haghany and Varadarajan [17] studied many properties of triangular algebras; they used the terminology of formal triangular matrix rings.

Derivations are a fundamental notion in mathematics. They have been objects of study by many well-known mathematicians. In the middle/late 1990s, several authors undertook the study of derivations and related maps over some particular families of triangular algebras (see for e.g. [10, 11, 21, 30] and references therein). Motivated by those works Cheung [8] initiated, in his thesis, the study of linear maps of (abstract) triangular algebras. He described automorphisms, derivations, commuting maps and Lie derivations of triangular algebras. Cheung’s work has inspired several authors to investigate many distinct maps of triangular algebras.

The present paper is devoted to the study of the so-called $\sigma$-biderivations and $\sigma$-commuting maps of triangular algebras. Our motivation arises from recent developments of the theory of maps of triangular algebras; more concretely, we could say that our interest was sparked by the work of Cheung [9] on commuting maps of triangular algebras, the paper of Benkovič [11] on biderivations of triangular algebras.
and the study of $\sigma$-biderivations and $\sigma$-commuting maps of nest algebras due to Yu and Zhang [29].

The study of biderivations of rings and algebras has a rich history and nowadays it is still an active area of research. Also, biderivations have already been shown of utility in connection with distinct areas; for example, Skosyrskii [26] used them to investigate noncommutative Jordan algebras, while Farkas and Letzer [15] treated them in their study of Poisson algebras.

Let $A$ be an algebra over a unital commutative ring of scalars; a bilinear map $D : A \times A \to A$ is said to be a biderivation of $A$ if it is a derivation in each argument. The map $\Delta_{\lambda} : (x, y) \mapsto \lambda[x, y]$ is an example of a biderivation, provided that $\lambda$ lies in the center $Z(A)$ of $A$. Biderivations of this form are called inner biderivations; here, $[x, y]$ stands for the commutator $xy - yx$.

Notice that if $d$ is a derivation of a commutative algebra, then the map $(x, y) \mapsto d(x)d(y)$ is a biderivation. In the noncommutative case, it happens quite often that all biderivations are inner. The classical problem under study is to determine whether every biderivation of a noncommutative algebra is inner.

In 1993, Brešar et al. [6] proved that every biderivation of a noncommutative prime ring $R$ is inner. One year after, Brešar [3] extended the result above to semiprime rings. Those results have been shown to be very useful in the study of the so-called commuting maps that, as we will point out below, are closely related to biderivations.

Recently, motivated by Cheung’s work, Zhang et al. [31, Theorem 2.1] proved that, under some mild conditions, every biderivation of a nest algebra is inner. Later on, Zhao et al. [32] proved, using the results of [31], that every biderivation of an upper triangular matrix algebra is a sum of an inner biderivation and an element of a particular class of biderivations. Benkovič [1] was the first author to study biderivations of (abstract) triangular algebras. He determined the conditions which need to be imposed on a triangular algebra to ensure that all its biderivations are inner.

It turns out that biderivations are closely connected to the thoroughly studied commuting (additive) maps. A map $\Theta$ of an algebra $A$ into itself is said to be commuting if $\Theta(x)$ commutes with $x$, for every $x \in A$. The usual goal when dealing with a commuting map is to provide a precise description of its form. It is straightforward to check that the identity map and any central map (a map having its image in the center of the algebra) are examples of commuting maps. Moreover, the sum and the pointwise product of commuting maps produce commuting maps. For example, it is easy to check that the following map

$$\Theta(x) = \lambda x + \Omega(x), \quad \forall x \in A,$$

is commuting, for any $\lambda \in Z(A)$ and any choice of central map $\Omega$ of $A$. We will refer to commuting maps of this form as proper commuting maps.

The first important result on commuting maps dates back to 1957 and it is due to Posner [24]. Posner’s theorem states that the existence of a nonzero commuting derivation of a prime ring implies the commutativity of the ring. Notice that Posner’s theorem can be reformulated as follows:

**Theorem.** If $d$ is a commuting derivation of a noncommutative prime ring $R$, then $d = 0$. 

Posner’s theorem has been generalized by a number of authors in many different ways. We refer the reader to the well-written survey [5], where the development of the theory of commuting maps and related maps is presented; see also [5, Section 3] for applications of biderivations and commuting maps to other areas. Let us point out here that one of the main applications of commuting maps can be found in their connection with the several conjectures on Lie maps of associative rings, formulated by Herstein [20] in 1961.

Let us now come back to the relationship between biderivations and commuting maps; notice that if $\Theta$ is a commuting map of an algebra $A$, then the map $D_\Theta : A \times A \to A$ given by $D_\Theta(x,y) = [\Theta(x),y]$, for all $x,y \in A$; that is to say that $D_\Theta$ is an inner biderivation. Then it follows that $\Omega_\Theta(x) := \Theta(x) - \lambda x \in Z(A)$, for every $x \in A$; in other words, $\Theta$ is a proper commuting map. Therefore, in order to show that every commuting map is proper, it is enough to prove that every biderivation is inner.

Let $\sigma$ be an automorpshism of an algebra $A$. Recall that a linear map $d$ of $A$ is called a $\sigma$-derivation if $d(xy) = \sigma(x)d(y) + d(x)y$, for all $x,y \in A$. A $\sigma$-biderivation is a bilinear map which is a $\sigma$-derivation in each argument.

The structure of $\sigma$-biderivations of prime rings was investigated by Brešar [4]. As an application, he characterized additive maps $f$ of a prime ring satisfying $f(x) = \sigma(x)f(x)$, for all $x$ in the ring. We will refer to such maps as $\sigma$-commuting maps. In the context of triangular algebras, $\sigma$-biderivations and $\sigma$-commuting maps of nest algebras have been studied by Yu and Zhang [29] in 2007.

The paper is organized as follows: in Section 2 we gather together basic definitions and elementary properties needed throughout the paper. Section 3 deals with the study of $\sigma$-biderivations of triangular algebras, while $\sigma$-commuting maps are investigated in Section 4.

2. Preliminaries

Throughout the paper we consider unital associative algebras, and we tacitly assume that all of them are algebras over a fixed commutative unital ring of scalars $R$. In the subsequent subsections, we recall some definitions and basic results, and introduce some notation.

2.1. Triangular algebras. Let $A$ and $B$ be unital associative algebras and $M$ a nonzero $(A,B)$-bimodule. The following set becomes an associative algebra under the usual matrix operations:

$$\mathcal{T} = \text{Trian}(A,M,B) = \left\{ \begin{pmatrix} a & m \\ M & b \end{pmatrix} : a \in A, m \in M, b \in B \right\}.$$  

An algebra $\mathcal{T}$ is called a triangular algebra if there exist algebras $A$, $B$ and a nonzero $(A,B)$-bimodule $M$ such that $\mathcal{T}$ is isomorphic to Trian $(A,M,B)$.

Given $\mathcal{T} = \text{Trian}(A,M,B)$, a triangular algebra, let us denote by $\pi_A$, $\pi_B$ the two natural projections from $\mathcal{T}$ onto $A$, $B$, respectively defined as follows:

$$\pi_A : \begin{pmatrix} a & m \\ b \end{pmatrix} \mapsto a, \quad \pi_B : \begin{pmatrix} a & m \\ b \end{pmatrix} \mapsto b, \quad \text{for all} \quad \begin{pmatrix} a & m \\ b \end{pmatrix} \in \mathcal{T}.$$
The center $Z(T)$ of $T$ was computed in [8] (see also [9, Proposition 3]); it is the following set:
\[ Z(T) = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a \in T, \text{ for all } a \in M \right\}. \]
Moreover, it follows that $\pi_A(Z(T)) \subseteq Z(A)$ and $\pi_B(Z(T)) \subseteq Z(B)$, and there exists a unique algebra isomorphism $\tau : \pi_A(Z(T)) \rightarrow \pi_B(Z(T))$ such that $am = m\tau(a)$, for all $m \in M$.

The most important examples of triangular algebras are the following:

- **Upper triangular matrix algebras.** Let us denote by $M_{n \times m}(R)$ the algebra of all $n \times m$ matrices over $R$, and by $T_n(R)$ the algebra of all $n \times n$ upper triangular matrices over $R$. Given $n \geq 2$, the algebra $T_n(R)$ can be represented as a triangular algebra as follows
\[ T_n(R) = \begin{pmatrix} T_{\ell}(R) & M_{\ell \times (n-\ell)}(R) \\ & T_{n-\ell}(R) \end{pmatrix}, \]
where $\ell \in \{1, \ldots, n-1\}$.

- **Block upper triangular matrix algebras.** Let $\mathbb{N}$ be the set of all positive integers and $n \in \mathbb{N}$. For any $m \in \mathbb{N}$ such that $m \leq n$, we write $d_n$ to denote an element $(d_1, \ldots, d_m) \in \mathbb{N}^m$ which satisfies $n = d_1 + \cdots + d_m$. The block upper triangular matrix algebra $B_n^{d_n}(R)$ is the following subalgebra of $M_n(R)$:
\[ B_n^{d_n}(R) = \begin{pmatrix} M_{d_1}(R) & M_{d_1 \times d_2}(R) & \cdots & M_{d_1 \times d_m}(R) \\ 0 & M_{d_2}(R) & \cdots & M_{d_2 \times d_m}(R) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{d_m}(R) \end{pmatrix}. \]
Notice that the full matrix algebra $M_n(R)$ and the upper triangular matrix algebra $T_n(R)$ are two special cases of block upper triangular matrix algebras. Given $n \geq 2$, assume that $B_n^{d_n}(R) \neq M_n(R)$; then $B_n^{d_n}(R)$ can be seen as a triangular algebra of the form
\[ B_n^{d_n}(R) = \begin{pmatrix} B_\ell^{d_{\ell}}(R) & M_{\ell \times (n-\ell)}(R) \\ B_{n-\ell}^{d_{n-\ell}}(R) \end{pmatrix}, \]
where $k \in \{1, \ldots, m-1\}$, $\ell = d_1 + d_2 + \cdots + d_k$, $\bar{d}_\ell = (d_{k+1}, \ldots, d_k) \in \mathbb{N}^k$, and $d_{n-\ell} = (d_{k+1}, \ldots, d_m) \in \mathbb{N}^{m-k}$.

- **Triangular Banach algebras.** Let $(\mathcal{A}, \| \cdot \|_A)$ and $(\mathcal{B}, \| \cdot \|_B)$ be two Banach algebras, and $M$ a Banach $(\mathcal{A}, \mathcal{B})$-bimodule. Then the triangular algebra $T = \text{Trian}(\mathcal{A}, M, \mathcal{B})$ is a Banach algebra with respect to the following norm:
\[ \left\| \begin{pmatrix} a & m \\ b & 0 \end{pmatrix} \right\|_T = \|a\|_A + \|m\|_M + \|b\|_B, \quad \text{for all } \begin{pmatrix} a & m \\ b & 0 \end{pmatrix} \in T. \]
The algebra $T$ is called a **triangular Banach algebra**.

- **Nest algebras.** Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. A **nest** is a chain $\mathcal{N}$ of closed subspaces of $\mathcal{H}$, which contains $0$ and $\mathcal{H}$, and is closed under arbitrary intersections and closed linear span. A nest...
\( \mathcal{N} \) is said to be **trivial** if \( \mathcal{N} = \{0, \mathcal{H}\} \); otherwise, it is called a **nontrivial nest**. The **nest algebra** associated to \( \mathcal{N} \) is the set

\[
\mathfrak{T}(\mathcal{N}) = \{ T \in B(\mathcal{H}) \mid T(N) \subseteq N, \forall N \in \mathcal{N} \}.
\]

Any nest algebra \( \mathfrak{T}(\mathcal{N}) \) associated to a nontrivial nest \( \mathcal{N} \) can be seen as a triangular algebra. In fact, given \( N \in \mathcal{N} - \{0, \mathcal{H}\} \), denote by \( E \) the orthogonal projection onto \( N \). Then \( N_1 = E(\mathcal{N}) \) and \( N_2 = (1 - E)(\mathcal{N}) \) are nests of \( N \) and \( N^\perp \), respectively. Moreover, \( \mathfrak{T}(N_1) = E\mathfrak{T}(\mathcal{N})E \) and \( \mathfrak{T}(N_2) = (1 - E)\mathfrak{T}(\mathcal{N})(1 - E) \) are nest algebras and

\[
\mathfrak{T}(\mathcal{N}) = \begin{pmatrix} \mathfrak{T}(N_1) & E\mathfrak{T}(\mathcal{N})(1 - E) \\ E\mathfrak{T}(\mathcal{N})(1 - E) & \mathfrak{T}(N_2) \end{pmatrix}.
\]

At this point, it should be mentioned that finite dimensional nest algebras are isomorphic to complex block upper triangular matrix algebras. We refer the reader to [12] for the general theory of nest algebras.

**Notation 2.1.** Let \( \mathcal{T} = \text{Trian}(A, M, B) \) be a triangular algebra. We will write \( 1_A, 1_B \) to denote the units of the algebras \( A, B \), respectively. The unit of \( \mathcal{T} \) is the element:

\[
1 := \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}.
\]

It is straightforward to check that the following elements are orthogonal idempotents of \( \mathcal{T} \).

\[
p := \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}, \quad q := 1 - p = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix} \in \mathcal{T}.
\]

Hence, we can consider the Peirce decomposition of \( \mathcal{T} \) associated to the idempotent \( p \). In other words, we can write \( \mathcal{T} = p\mathcal{T}p \oplus p\mathcal{T} \oplus q\mathcal{T}q \). Note that \( p\mathcal{T}p, q\mathcal{T}q \) are subalgebras of \( \mathcal{T} \) isomorphic to \( A, B \), respectively; while \( p\mathcal{T}q \) is a \( (p\mathcal{T}p, q\mathcal{T}q) \)-bimodule isomorphic to \( M \). To ease the notation in what follows, we will identify \( p\mathcal{T}p, q\mathcal{T}q, p\mathcal{T}q \) with \( A, B, M \), respectively. Thus, any element of \( \mathcal{T} \) can be expressed as:

\[
x = \begin{pmatrix} a & m \\ b & \end{pmatrix} = pap + pmq + qbq = a + m + b,
\]

by using the identification above.

### 2.2. Derivations and related maps

In this subsection, we collect all the definitions of the maps that will be used along the paper. Although we have already introduced some of those maps in the introductory section, we will include their definition here for the sake of completeness.

**Definitions 2.2.** Let \( \mathcal{A} \) be an algebra and \( \sigma \) an automorphism of \( \mathcal{A} \). Let us denote by \( \text{Id}_\mathcal{A} \) the identity map on \( \mathcal{A} \).

- A linear map \( d : \mathcal{A} \to \mathcal{A} \) is called a **derivation of** \( \mathcal{A} \) if it satisfies

\[
d(xy) = d(x)y + xd(y), \quad \forall x, y \in \mathcal{A}.
\]

- A linear map \( d : \mathcal{A} \to \mathcal{A} \) is called a **\( \sigma \)-derivation of** \( \mathcal{A} \) if it verifies

\[
d(xy) = d(x)y + \sigma(x)d(y), \quad \forall x, y \in \mathcal{A}.
\]

Note that every derivation is an \( \text{Id}_\mathcal{A} \)-derivation. In the literature (see, for example, [LS 28]), some authors have used the terminology of skew-derivations for \( \sigma \)-derivations.
A bilinear map $D : A \times A \to A$ is said to be a biderivation of $A$ if it is a derivation in each argument; that is to say that for every $y \in A$, the maps $x \mapsto D(x, y)$ and $x \mapsto D(y, x)$ are derivations of $A$. In other words:

$$D(xy, z) = xD(y, z) + D(x, z)y, \quad D(x, yz) = yD(x, z) + D(x, y)z,$$

for all $x, y, z \in A$. The map $D$ is called a $\sigma$-biderivation of $A$ provided that $D$ is a $\sigma$-derivation in each argument.

Suppose that $A$ is noncommutative and take $\lambda \in \mathbb{Z}(A)$. The bilinear map $\Delta_{\lambda} : A \times A \to A$ given by

$$\Delta_{\lambda}(x, y) = \lambda [x, y], \quad \forall x, y \in A$$

is an example of a biderivation. Biderivations of the form above are called inner biderivations.

Given $x_0 \in A$ such that $x_0 \notin \mathbb{Z}(A)$; suppose that $x_0$ satisfies $[[x, y], x_0] = 0$ for all $x, y \in A$. It was proved in [1] that the bilinear map $\psi_{x_0} : A \times A \to A$ given by

$$\psi_{x_0}(x, y) = [x, [y, x_0]], \quad \forall x, y \in A$$

is a biderivation. Biderivations of this form appear first in [1]; they were named extremal biderivations.

In this paper, we introduce the notions of inner $\sigma$-biderivations and extremal $\sigma$-biderivations (see Definitions 3.2 and 3.10 below).

**Example 2.3.** Let $A$ be an algebra and $\sigma$ an automorphism of $A$. It is easy to see that the map $d := \text{Id}_{A} - \sigma$ is a $\sigma$-derivation of $A$, although $d$ is not, in general, a derivation of $A$. For example, let $K$ be a field of characteristic not 2, and take the algebra of polynomials $K[x]$ and $\sigma$ the automorphism of $K[x]$ which maps $x$ to $-x$. Then the $\sigma$-derivation $d$ of $K[x]$, defined as before, satisfies

$$d(x^2) = x^2 - \sigma(x^2) = x^2 - \sigma(x)\sigma(x) = x^2 - (-x)(-x) = 0,$$

$$d(x)x + xd(x) = 4x^2,$$

since $d(x) = x - \sigma(x) = 2x$. This shows that $d$ is not a derivation of $K[x]$. On the other hand, the map $D(x, y) = d(x)d(y)$ is a $\sigma$-biderivation, since $K[x]$ is a commutative algebra. But $D$ is not a biderivation, since $D(x^2, x) = 0$ and $D(x, x)x + xD(x, x) = 8x^3$.

Notice that these examples also work for the finite-dimensional algebra of polynomials of degree $\leq N$ (i.e., the polynomial algebra $K[x]$ modulo the ideal generated by $x^{N+1}$), since $\sigma$ preserves degrees.

**Definitions 2.4.** Let $A$ be an algebra, $\sigma$ an automorphism of $A$ and $\Theta$ a linear map $A \to A$.

- The map $\Theta$ is a commuting map $A \to A$ if $\Theta(x)$ commutes with $x$, for every $x \in A$, i.e., $[\Theta(x), x] = 0$, for all $x \in A$.
- $\Theta$ is called $\sigma$-commuting if it verifies $\Theta(x)x = \sigma(x)\Theta(x)$, for all $x \in A$. In particular, in the case where $\sigma = \text{Id}_{A}$, the notion of a commuting map is recovered. At this point, it is worth mentioning that the concept of a skew-commuting map has a different meaning (see, for example, [2]).
Example 2.5. Let $T_2(\mathbb{C})$ be the algebra of $2 \times 2$ upper triangular matrices over the complex numbers. The map $\sigma : T_2(\mathbb{C}) \to T_2(\mathbb{C})$ given by

$$\sigma \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) := \left( \begin{array}{cc} a & -b \\ c & d \end{array} \right),$$

for all $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in T_2(\mathbb{C})$,

is an automorphism of $T_2(\mathbb{C})$. It is straightforward to show that the linear map $\Theta$ of $T_2(\mathbb{C})$ given by

$$\Theta \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) := \left( \begin{array}{cc} a & b \\ c & -d \end{array} \right),$$

for all $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in T_2(\mathbb{C})$,

is a $\sigma$-commuting map of $T_2(\mathbb{C})$; however, it is not commuting since, for example, for $x = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$, we have that $[x, \Theta(x)] = \left( \begin{array}{cc} 0 & -2 \\ 0 & 0 \end{array} \right) \neq 0$.

2.3. Some basic results on triangular algebras. Let $\mathcal{T} = \text{Trian}(A, M, B)$ be a triangular algebra. In the sequel (for convenience) we will assume that the bimodule $M$ is faithful as a left $A$-module and also as a right $B$-module, although these assumptions might not always be necessary.

Automorphisms of triangular algebras were studied by Cheung [8]. Here, we are interested in the following description given by Khazal et al. [22].

Theorem 2.6. [22, Theorem 1] Let $\mathcal{T} = \text{Trian}(A, M, B)$ be a triangular algebra such that the algebras $A$ and $B$ have only trivial idempotents. Then every automorphism $\sigma$ of $\mathcal{T}$ is of the following form:

$$(1) \quad \sigma \left( \begin{array}{cc} a & m \\ b & d \end{array} \right) = \left( \begin{array}{cc} f_\sigma(a) & f_\sigma(a)m - m_\sigma g_\sigma(b) + \nu_\sigma(m) \\ g_\sigma(b) \end{array} \right),$$

where $f_\sigma, g_\sigma$ are automorphisms of $A, B$, respectively, $m_\sigma$ is an element of $M$ and $\nu_\sigma : M \to M$ is a linear bijective map which satisfies $\nu_\sigma(am) = f_\sigma(a)\nu_\sigma(m)$ and $\nu_\sigma(mb) = \nu_\sigma(m)g_\sigma(b)$, for all $a \in A, b \in B, m \in M$.

In what follows, we will use the theorem above without further mention. In other words, whenever $\sigma$ is an automorphism of a triangular algebra which satisfies the hypothesis of Theorem 2.6 we will assume that $\sigma$ is written as in (1).

Remark 2.7. Notice that in Theorem 2.6 an additional hypothesis has been imposed on a triangular algebra $\mathcal{T} = \text{Trian}(A, M, B)$, namely: the algebras $A$ and $B$ have only trivial idempotents. We refer the reader to [18, Section 2.7] for examples of triangular algebras with $A$ and $B$ having only trivial idempotents.

In our study of $\sigma$-biderivations and $\sigma$-commuting maps of a triangular algebra $\mathcal{T} = \text{Trian}(A, M, B)$, we will require that the algebras $A$ and $B$ have only trivial idempotents, since the description of the automorphism $\sigma$ of $\mathcal{T}$ provided in Theorem 2.6 will be crucial for our computations. However, it should be pointed out that the results obtained below do not really require that $A$ and $B$ have only trivial idempotents; they all hold for any automorphism $\sigma$ that can be written in the form (1). In particular, they always hold for $\sigma = \text{Id}_\mathcal{T}$. Accordingly, the results presented in this paper constitute a generalization of the results of Benkovič [1] on biderivations of triangular algebras, and of the results of Cheung [8] on commuting maps of triangular algebras.

The following concept will play an important role for our purposes.
Definition 2.8. [28] Definition 2.3] Let \( A \) be an algebra and \( \sigma \) an automorphism of \( A \). The \( \sigma \)-center of \( A \) is the set \( Z_\sigma(A) \) given by
\[
Z_\sigma(A) = \{ \lambda \in A : \sigma(x)\lambda = \lambda x, \text{ for all } x \in A \}.
\]
A linear map from \( A \) to \( Z_\sigma(A) \) will be called \( \sigma \)-central.

Lemma 2.9. [28] Lemma 2.5] Let \( A \) be an algebra and \( \sigma \) an automorphism of \( A \). The \( \sigma \)-center of \( A \) is a subspace of \( A \) which is invariant under \( \sigma \); in other words: \( \sigma(\lambda) \in Z_\sigma(A) \) for every \( \lambda \in Z_\sigma(A) \).

Given a triangular algebra \( T = \text{Trian}(A,M,B) \) and \( \sigma \) an automorphism of \( T \), in view of (1) it is natural to expect that \( Z_{f_n}(A) \) and \( Z_{g_n}(B) \) will be related to \( Z_\sigma(T) \). The description of \( Z_\sigma(T) \) and the relationship between \( Z_\sigma(T) \), \( Z_{f_n}(A) \), and \( Z_{g_n}(B) \) was investigated in [28], where the authors proved the following result:

Lemma 2.10. [28] Lemmas 2.6, 2.8 and 2.9] Let \( T = \text{Trian}(A,M,B) \) be a triangular algebra and \( \sigma \) an automorphism of \( T \). Assume that the algebras \( A \) and \( B \) have only trivial idempotents. Then
\[
Z_\sigma(T) = \left\{ \left( \begin{array}{c} a \\ -m \sigma b \\ b \end{array} \right) \in T, \text{ such that } am = \nu_\sigma(m)b, \text{ for all } m \in M \right\}.
\]
Moreover, \( \pi_A(Z_\sigma(T)) \subseteq Z_{f_n}(A) \), \( \pi_B(Z_\sigma(T)) \subseteq Z_{g_n}(B) \), and there exists a unique algebra isomorphism \( \eta : \pi_B(Z_\sigma(T)) \rightarrow \pi_A(Z_\sigma(T)) \) such that \( \eta(b)m = \nu_\sigma(m)b \), for all \( b \in Z_\sigma(T) \), \( m \in M \).

3. \( \sigma \)-biderivations of triangular algebras

This section is devoted to the study of \( \sigma \)-biderivations of triangular algebras. We start by investigating what should be the suitable generalization of an inner biderivation in the \( \sigma \)-maps setting.

Proposition 3.1. Let \( A \) be a noncommutative algebra, \( \sigma \) an automorphism of \( A \) and \( \lambda \in Z_\sigma(A) \). Then the map \( \Delta_\lambda : A \times A \rightarrow A \) given by
\[
\Delta_\lambda(x,y) = \lambda [x,y], \quad \forall x,y \in A
\]
is a \( \sigma \)-biderivation of \( A \).

Proof. Take \( x,y,z \in A \).
\[
\Delta_\lambda(x,y)\Delta_\lambda(y,z) = \lambda xzy - \lambda zxy + \sigma(x)\lambda yz - \sigma(x)\lambda zy = \\
= \lambda xzy - \lambda zxy + \lambda xyz - \lambda xzy = \lambda xyz - \lambda zxy,
\]
since \( \lambda \in Z_\sigma(A) \). On the other hand, we have that
\[
\Delta_\lambda(xy,z) = \lambda xyz - \lambda zxy.
\]
Therefore, \( \Delta_\lambda(xy,z) = \Delta_\lambda(x,y)\Delta_\lambda(y,z) \). Similarly, one can show that \( \Delta_\lambda(x,yz) = \Delta_\lambda(x,y)z + \sigma(y)\Delta_\lambda(x,z) \), concluding the proof. \( \square \)

The result above makes it possible to introduce the following concept.

Definition 3.2. Let \( A \) be a noncommutative algebra and \( \sigma \) an automorphism of \( A \). Maps of the form (2) will be called inner \( \sigma \)-biderivations.
One of our main results in this section is the following theorem. For a triangular algebra and the idempotent \( p \) defined in Notation (2.1), it provides sufficient conditions to ensure that the \( \sigma \)-biderivations vanishing at \((p, p)\) are indeed inner. At this point, a natural question arises:

**Question 3.3.** What can be said about the \( \sigma \)-biderivations of \( T \) which do not vanish at \((p, p)\)?

This question will be treated in Theorem 3.4 below.

**Theorem 3.4.** Let \( T = \text{Trian}(A,M,B) \) be a triangular algebra and \( \sigma \) an automorphism of \( T \). Assume that the algebras \( A \) and \( B \) have only trivial idempotents. Suppose that \( T \) satisfies the following conditions:

1. \( \pi_A(Z_{\sigma}(T)) = Z_{f_\sigma}(A) \) and \( \pi_B(Z_{\sigma}(T)) = Z_{g_\sigma}(B) \).
2. Either \( A \) or \( B \) is noncommutative.
3. If \( \lambda \in Z_{\sigma}(T) \) satisfies \( \lambda x = 0 \) for some nonzero element \( x \) in \( T \), then \( \lambda = 0 \).
4. Every linear map \( \xi : M \rightarrow M \) satisfying \( \xi(amb) = f_\sigma(a)\xi(m)b \) for all \( a \in A, m \in M, b \in B \), can be expressed as \( \xi(m) = \lambda_0 m + \nu_\sigma(m)\mu_0 \) for all \( m \in M \) and certain \( \lambda_0 \in Z_{f_\sigma}(A) \) and \( \mu_0 \in Z_{g_\sigma}(B) \).

Then every \( \sigma \)-biderivation \( D \) of \( T \) such that \( D(p, p) = 0 \) is an inner \( \sigma \)-biderivation.

**Remark 3.5.** The results of the present section generalize the results of [1] Section 4 (see Section 5 for details). Theorem 3.4 is the analogue of [1] Theorem 4.11 for \( \sigma \)-maps. (See also [1] Remark 4.12.) Note that Condition (iv) could be replaced by the following condition:

(iv)' Every \( \sigma \)-derivation of \( T \) is inner.

The \( \sigma \)-derivations of triangular algebras have been already studied by Han and Wei [18]; however, the innerness of \( \sigma \)-derivations has not been treated yet. It is not difficult to prove that a \( \sigma \)-derivation of \( T \) is inner if and only if its associated linear map \( \xi : M \rightarrow M \) (see [18] Theorem 3.12) for a precise description of \( \sigma \)-derivations)

is of the following form:

\[
\xi(m) = \lambda_0 m + \nu_\sigma(m)\mu_0, \quad \forall m \in M,
\]

for certain \( \lambda_0 \in Z_{f_\sigma}(A) \) and \( \mu_0 \in Z_{g_\sigma}(B) \). From this, one can easily derive that Condition (iv)' above implies Condition (iv) in Theorem 3.4.

In what follows, we will collect the appropriate results and develop the machinery needed to prove Theorem 3.4.

**Lemma 3.6.** (See [1] Lemma 2.3 and [11] Lemma 4.2) Let \( A \) be an algebra, \( \sigma \) an automorphism of \( A \), and \( D \) a \( \sigma \)-biderivation of \( A \). Then

1. \( D(x,y)[u,v] = \sigma([x,y])D(u,v) = [\sigma(x),\sigma(y)]D(u,v), \) for all \( x, y, u, v \in A \).
2. \( D(x,1) = D(1,x) = D(x,0) = D(0,x) = 0, \) for all \( x \in A \).
3. For an idempotent \( e \) of \( A \), it follows that \( D(e,e) = -D(e,1-e) = -D(1-e,e) = D(1-e,1-e) \).

**Lemma 3.7.** Let \( T = \text{Trian}(A,M,B) \) be a triangular algebra, \( \sigma \) an automorphism of \( T \), and \( D \) a \( \sigma \)-biderivation of \( T \). Assume that the algebras \( A \) and \( B \) have only trivial idempotents. If \( x,y \in T \) are such that \([x,y] = 0\), then \( D(x,y) = pD(x,y)q \), where \( p, q \) are as in Notation 2.1.
Proof. Take \( x, y \in \mathcal{T} \) with \([x, y] = 0\). Considering the Peirce decomposition of \( \mathcal{T} \) associated to the idempotent \( p \) of \( \mathcal{T} \), we can write \( D(x, y) \) as follows:

\[
D(x, y) = pD(x, y)p + pD(x, y)q + qD(x, y)q.
\]

For \( m \in M \), applying Lemma 3.6 (i) we get that

\[
0 = D(p, pmq)[x, y] = \sigma([p, pmq])D(x, y) = \sigma(pm)D(x, y) = \nu_\sigma(m)qD(x, y),
\]

which implies that \( pMqD(x, y) = 0 \), since \( \nu_\sigma \) is a bijection. From here, we can conclude that \( MqD(x, y)q = 0 \); which yields \( qD(x, y)q = 0 \), since \( M \) is a faithful right \( B \)-module. Given \( m \in M \), a second application of Lemma 3.6 (i) produces:

\[
D(x, y)pmq = D(x, y)[p, pmq] = \sigma([x, y])D(p, pmq) = 0.
\]

Thus, \( pD(x, y)pM = 0 \), which yields \( pD(x, y)p = 0 \). The result now follows from (3). \( \square \)

Motivated by the fact that extremal biderivations played an important role in the study of biderivations of triangular algebras; our next task will be to investigate what should be called an extremal \( \sigma \)-biderivation.

Let \( \mathcal{A} \) be an algebra and \( \sigma \) an automorphism of \( \mathcal{A} \). We introduce a new bilinear operation on \( \mathcal{A} \):

\[
[x, y]_\sigma = \sigma(x)y - yx, \quad \text{for all } x, y \in \mathcal{A}.
\]

We will call (4) the \( \sigma \)-commutator of \( \mathcal{A} \). Note that

\[
Z_\sigma(\mathcal{A}) = \{ \lambda \in \mathcal{A} : [x, \lambda]_\sigma = 0 \ \text{for all } x \in \mathcal{A} \}.
\]

The next result collects some elementary properties of the \( \sigma \)-commutator. We omit its proof since it consists of elementary calculations.

**Lemma 3.8.** Let \( \mathcal{A} \) be an algebra and \( \sigma \) an automorphism of \( \mathcal{A} \). For \( x, y, z \in \mathcal{A} \) it follows that

- (i) \([xy, z]_\sigma = [x, z]_\sigma y + \sigma(x)[y, z]_\sigma\),
- (ii) \([x, [y, z]]_\sigma = [[x, y], z]_\sigma + [y, [x, z]]_\sigma\).

**Proposition 3.9.** Let \( \mathcal{A} \) be an algebra and \( \sigma \) an automorphism of \( \mathcal{A} \). The symmetric bilinear map \( \psi_{x_0} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \) given by

\[
\psi_{x_0}(x, y) = [x, [y, x_0]]_\sigma, \quad \forall x, y \in \mathcal{A}
\]

is a \( \sigma \)-biderivation, provided that the element \( x_0 \in \mathcal{A} \) satisfies \( x_0 \notin Z_\sigma(\mathcal{A}) \) and \([\mathcal{A}, \mathcal{A}], x_0]_\sigma = 0\).

Proof. Let \( x_0 \in \mathcal{A} \) such that \( x_0 \notin Z_\sigma(\mathcal{A}) \) and \([\mathcal{A}, \mathcal{A}], x_0]_\sigma = 0\). Clearly, \( \psi_{x_0} \) is a bilinear map. To show its symmetry, apply Lemma 3.8 (ii) to get that

\[
\psi_{x_0}(x, y) = [x, [y, x_0]]_\sigma = [[x, y], x_0]_\sigma + [y, [x, x_0]]_\sigma = \psi_{x_0}(y, x),
\]

as desired. In order to prove that \( \psi_{x_0} \) is a biderivation, it is enough to check that it is a derivation in its first argument. Given \( x, y, z \in \mathcal{A} \), applying Lemma 3.8 (i) we obtain that

\[
\psi_{x_0}(xy, z) = [xy, [z, x_0]]_\sigma = [x, [z, x_0]]_\sigma y + \sigma(x)[y, [z, x_0]]_\sigma = \psi_{x_0}(x, z)y + \sigma(x)\psi_{x_0}(y, z),
\]

which concludes the proof. \( \square \)

In view of the previous result, we introduce the following terminology:
**Definition 3.10.** Let $A$ be an algebra and $\sigma$ an automorphism of $A$. An **extremal $\sigma$-biderivation** is a bilinear map of $A$ of the form $(\sigma)$. 

Now, we are in a position to answer Question \textbf{3.6.6.}

**Theorem 3.11.** Let $T$ be a triangular algebra, $\sigma$ an automorphism of $T$, and $D$ a $\sigma$-biderivation of $T$. Assume that $A$ and $B$ have only trivial idempotents and that $D(p, p) \neq 0$; then $D$ can be written as a sum of an extremal $\sigma$-biderivation and a biderivation vanishing at $(p, p)$. Moreover, $D = D(D(p, p), D(p, p))$ where $D(D(p, p), D(p, p))$ is the extremal $\sigma$-biderivation associated to the element $x_0 = D(p, p)$, and $D_0$ is a biderivation of $T$ satisfying $D_0(p, p) = 0$.

**Proof.** Let $D$ be a $\sigma$-biderivation of $T$ such that $D(p, p) \neq 0$. Note that from $[p, p] = 0$, we get that $D(p, p) = pD(p, p)q$, by an application of Lemma \textbf{3.7.} Lemma \textbf{2.10.} allows us to conclude that $D(p, p) \notin Z_\sigma(T)$. Next, we claim that $[[x, y], D(p, p)]_\sigma = 0$, for every $x, y \in T$. In fact, given $x, y \in T$, we have that

$$[[x, y], D(p, p)]_\sigma = \sigma([x, y])D(p, p) - D(p, p)[x, y] = \sigma(x)D(p, p) - D(p, p)x, y = 0,$$

by Lemma \textbf{3.6.} Therefore, it makes sense to consider the extremal $\sigma$-biderivation $D(D(p, p), D(p, p))$. Let $D_0 = D - D(D(p, p), D(p, p))$; it remains to check that $D_0(p, p) = 0$. We will show that $D(D(p, p), D(p, p)) = D(p, p)$.

$$D(D(p, p), D(p, p)) = [p, [p, D(p, p)]_\sigma]_\sigma = [p, \sigma(p)D(p, p), D(p, p)]_\sigma = \sigma(p)\sigma(p)D(p, p) - \sigma(p)D(p, p),$$

Taking into account that $D(p, p) = pD(p, p)q$ and that $\sigma(p) = \left(\begin{array}{cc} A & m_{\sigma} \\ 0 & 0 \end{array}\right)$, the right hand side of the equality above becomes $D(p, p)$; that is, $D(D(p, p), D(p, p)) = D(p, p)$, which finishes the proof. □

**Proof of Theorem \textbf{3.11.}** Let $D$ be a $\sigma$-biderivation of $T$ such that $D(p, p) = 0$. Take

$$x = \left(\begin{array}{cc} a_x & m_x \\ b_x & 0 \end{array}\right) = a_x + m_x + b_x, \quad y = \left(\begin{array}{cc} a_y & m_y \\ b_y & 0 \end{array}\right) = a_y + m_y + b_y,$$

two arbitrary elements of $T$. The bilinearity of $D$ implies that

$$(6) \quad D(x, y) = D(a_x, a_y) + D(a_x, m_y) + D(a_x, b_y) + D(m_x, a_y) + D(m_x, m_y) + D(m_x, b_y) + D(b_x, a_y) + D(b_x, m_y) + D(b_x, b_y).$$

Note that an application of Lemma \textbf{3.10.} yields that $D(p, q) = D(q, p) = D(q, q) = 0$. In what follows, we will distinguish several cases:

\textbullet Case 1. $D(a, b) = D(b, a) = 0$, for all $a \in A$ and $b \in B$.

Applying Lemma \textbf{3.7.} taking into account that $[a, b] = 0$, we get that $D(a, b) = pD(a, b)q$ and $D(b, a) = pD(b, a)q$. Thus:

$$D(a, b) = pD(a, b)q = pD(ap, qb)q = pD(a, qb)pq + p\sigma(a)D(p, qb)q = \sigma(a)D(p, qb)q$$

$$= \sigma(a)D(p, q)bq + \sigma(a)\sigma(q)D(p, b)q = \sigma(aq)D(p, b)q = 0.$$
It remains to check that $D(b, a) = 0$. Similar calculations give:

$$D(b, a) = pD(b, a)q = pD(qb, ap)q = pD(q, ap)bq + pσ(q)D(b, ap)q = pD(q, ae)b$$

$$-m_aD(b, ae)q = pD(q, a)pb + pσ(a)D(q, p)b - m_aD(b, a)pq$$

$$-m_aσ(a)D(b, e)q = 0,$$

which finishes the proof of Case 1.

\* Case 2. $D(p, b) = D(b, p) = D(p, a) = D(a, p) = 0$, for all $a \in A$ and $b \in B$.

By Case 1 we have that $D(p, b) = D(b, p)$, for all $b \in B$. Given $a \in A$, an application of Lemma 3.6 (ii) gives:

$$0 = D(1, a) = D(p, a) + D(q, a) \quad \text{Case 1}$$

$$0 = D(a, 1) = D(a, p) + D(a, q) \quad \text{Case 1}$$

concluding the proof of Case 2.

\* Case 3. $D(q, b) = D(b, q) = D(q, a) = D(a, q) = 0$, for all $a \in A$ and $b \in B$.

The proof is analogous to that of Case 2.

\* Case 4. $D(a, m)p = D(m, a)p = 0$, for all $a \in A$ and $m \in M$.

Given $a \in A$ and $m \in M$, we have that

$$D(a, m)p = D(a, mq)p = σ(m)D(a, q)p + D(a, m)q \quad \text{Case 3}$$

$$D(m, a)p = D(mq, a)p = σ(m)D(q, a)p + D(m, a)q \quad \text{Case 3}$$

finishing the proof of Case 4.

The next case can be proved in a similar way.

\* Case 5. $σ(q)D(b, m) = σ(q)(D(m, b) = 0$, for all $m \in M$ and $b \in B$.

\* Case 6. There exists $λ_0 \in Z_{f_a}(A)$ such that

$$D(a, m) = -D(m, a) = λ_0 am, \quad D(m, b) = -D(b, m) = λ_0 mb$$

for all $a \in A$, $m \in M$ and $b \in B$.

Consider the map $ξ : M \to M$ given by $ξ(m) = D(p, m)$, for all $m \in M$. Notice that $ξ$ is well-defined, i.e., $ξ(m) \in M$, for all $m \in M$. In fact, given $m \in M$, we have that

$$D(p, m) = D(p, pmq) = D(p, pm)q + σ(pm)D(p, q) = σ(p)D(p, m)q + D(p, pm)q$$

since $σ(p) = \left(\begin{array}{cc} 1_A & m_a \\ 0 & 0 \end{array}\right)$. Trivially, $ξ$ is an additive map. Moreover, $ξ$ satisfies

$$ξ(amb) = f_σ(a)ξ(m)b, \text{ for all } a \in A, m \in M, b \in B.$$ 

In fact, given $a \in A, m \in M, b \in B$, we have that

$$ξ(amb) = D(p, amb) = σ(a)D(p, mb) + D(p, a)mb \quad \text{Case 2}$$

$$+ σ(a)D(p, m)b \quad \text{Case 2}$$

$$σ(a)D(p, m)b = f_σ(a)ξ(m)b,$$

since $σ(a) = f_σ(a) = \left(\begin{array}{cc} f_σ(a) & f_σ(a)m_a \\ 0 & 0 \end{array}\right)$, and $D(p, m) \in M$. Thus Condition (iv) implies that there exist $λ_0 \in Z_{f_a}(A)$ and $μ_0 \in Z_{f_σ}(B)$ such that $ξ(m) = λ_0 m + ν_σ(m)μ_0$, for all $m \in M$. 
Lemma 2.11 and Condition (i) allow us to consider the element \( \tilde{\lambda}_0 := \lambda_0 + \eta(\mu_0) \in Z_{f_x}(A) \). The calculations above jointly with a second use of Lemma 2.11 imply that
\[
D(p, m) = \xi(m) = \lambda_0 m + \nu_\sigma(m) \mu_0 = (\lambda_0 + \eta(\mu_0)) m = \tilde{\lambda}_0 m, \quad \forall m \in M.
\]

Similarly, one can find \( \tilde{\mu}_0 \in Z_{f_x}(A) \) such that \( D(m, p) = \tilde{\mu}_0 m \), for all \( m \in M \). Next, we claim that \( \tilde{\lambda}_0 + \tilde{\mu}_0 = 0 \). From Condition (ii) we have that either \( A \) or \( B \) is noncommutative. Let us assume, for example, that \( A \) is noncommutative. Then, there exist \( a, a' \in A \) such that \([a, a'] \neq 0\). Applying Lemma 3.6 (i) we get that
\[
D(a, a')[p, m] = \sigma([a, a'])D(p, m) = f_\sigma([a, a'])\tilde{\lambda}_0 m = \tilde{\lambda}_0[a, a']m,
\]
\[
D(a, a')[m, p] = \sigma([a, a'])D(m, p) = f_\sigma([a, a'])\tilde{\mu}_0 m = \tilde{\mu}_0[a, a']m,
\]
for all \( m \in M \). Thus: \((\tilde{\lambda}_0 + \tilde{\mu}_0)[a, a']M = 0\), which implies that \((\tilde{\lambda}_0 + \tilde{\mu}_0)[a, a'] = 0\), since \( M \) is a faithful left \( A \)-module. From here, Condition (iii) applies to get that \( \tilde{\lambda}_0 + \tilde{\mu}_0 = 0 \), which finishes the proof of our claim. Notice that we have just proved that
\[
D(p, m) = -D(m, p) = \tilde{\lambda}_0 m, \quad \forall m \in M.
\]

From here, an application of Lemma 3.6 (ii) gives that
\[
D(m, q) = -D(q, m) = \tilde{\lambda}_0 m, \quad \forall m \in M.
\]

For \( a \in A \), it follows that
\[
D(a, m) = D(ap, m) = \sigma(a)D(p, m) + D(a, m)p \overset{\text{Case 4}}{=} f_\sigma(a)\tilde{\lambda}_0 m = \tilde{\lambda}_0 am,
\]
\[
D(m, a) = D(m, ap) = \sigma(a)D(m, p) + D(m, ap) \overset{\text{Case 4}}{=} -f_\sigma(a)\tilde{\lambda}_0 m = -\tilde{\lambda}_0 am,
\]
for all \( m \in M \). Proceeding as above, applying Case 5 one can prove that
\[
D(m, b) = -D(b, m) = \tilde{\lambda}_0 mb,
\]
for all \( m \in M \) and \( b \in B \), as desired.

\( \diamond \) Case 7. \( D(a, a') = \tilde{\lambda}_0[a, a'] \) for every \( a, a' \in A \).

Given \( a, a' \in A \), we start by showing that \( D(a, a') \in A \).

\[
D(a, a') = D(pap, a') = \sigma(p)D(ap, a') + D(p, a')ap \overset{\text{Case 2}}{=} \sigma(p)\sigma(a)D(p, a') + \sigma(p)D(a, a')p \overset{\text{Case 2}}{=} \sigma(p)D(a, a')p \in A.
\]

For \( m \in M \), apply Lemma 3.6 (i) to obtain that
\[
D(a, a')m = D(a, a')[p, m] = \sigma([a, a'])D(p, m) = f_\sigma([a, a'])\tilde{\lambda}_0 m = \tilde{\lambda}_0[a, a']m, \quad \forall m \in M.
\]

Thus: \((D(a, a') - \tilde{\lambda}_0[a, a'])M = 0\), which yields that \( D(a, a') = \tilde{\lambda}_0[a, a'] \).

\( \diamond \) Case 8. \( D(b, b') = \begin{pmatrix} 0 & -m_\sigma \eta^{-1}(\tilde{\lambda}_0)[b, b'] \\ -\eta^{-1}(\tilde{\lambda}_0)[b, b'] & 0 \end{pmatrix} \), for all \( b, b' \in B \).

Take \( b, b' \in B \) and write \( D(b, b') = \begin{pmatrix} a'' & m'' \\ b'' & 0 \end{pmatrix} \). On the other hand, we have that
\[
D(b, b') = D(qbq, b') = \sigma(q)D(bq, b') + D(q, b')bq = \sigma(q)\sigma(b)D(q, b') + \sigma(q)D(b, b')q \overset{\text{Case 3}}{=} \sigma(q)D(b, b')q = \begin{pmatrix} 0 & -m_\sigma b'' \\ b'' & 0 \end{pmatrix}.
\]
which implies that $a'' = 0$ and $m'' = -m_x b''$. Given $m \in M$, from Lemma 3.8 (i) we obtain that 
$$
\sigma(m)D(b, b') = \sigma([p, m])D(b, b') = D(p, m)[b, b'] = \tilde{\lambda}_0 m[b, b'] = \nu_\sigma(m)\eta^{-1}(\tilde{\lambda}_0)[b, b'].
$$
Taking into account the fact that $\sigma(m)D(b, b') = \nu_\sigma(m)b''$, we have that 
$$
\nu(m)(b'' - \eta^{-1}(\tilde{\lambda}_0)[b, b']) = 0, \quad \forall m \in M.
$$
Combining the bijectivity of $\nu$ with the fact that $M$ is faithful as a $B$-module, we obtain that $b'' = \eta^{-1}(\tilde{\lambda}_0)$, finishing the proof of Case 8.

$\diamond$ Case 9. $D(m, m') = 0$, for all $m, m' \in M$.

Given $m, m' \in M$, since $[m, m'] = 0$, Lemma 3.7 can be applied to get that $D(m, m') = pD(m, m')q \in \mathcal{M}$. Next, fix $m_0 \in M$ and define a map $\xi : M \to M$ by $\xi(m) = D(m, m_0)$, for all $m \in M$. The above calculation yields that $\xi$ is well-defined. Moreover, $\xi$ is an additive map satisfying $\xi(amb) = f(a)\xi(m)b$, for all $a \in A$, $m \in M$ and $b \in B$. In fact, let $a \in A, m \in M$, and $b \in B$. Then

$$
\xi(amb) = D(amb, m_0) = \sigma(a)D(mb, m_0) + D(a, m_0)mb \overset{\text{Case 6}}{=} \sigma(a)\sigma(m)D(b, m_0) + \sigma(a)D(m, m_0)b + \tilde{\lambda}_0 am_0 mb \overset{\text{Case 6}}{=} -\sigma(a)\sigma(m)\tilde{\lambda}_0 m_0 b + f(a)D(m, m_0)b
$$

since $\tilde{\lambda}_0 m_0 = \nu_\sigma(m_0)\eta^{-1}(\tilde{\lambda}_0)$ and $\nu_\sigma(m)\nu_\sigma(m_0) = 0$. Now apply Condition (iv) to find $\lambda'' \in Z_{f_s}(A)$ and $\mu'' \in Z_{g_s}(B)$ such that $\xi(m) = \lambda'' m + \nu_\sigma(m)\mu''$, for all $m \in M$. Set $\lambda_{m_0} := \lambda'' + \eta(\mu'') \in Z_{f_s}(A)$. Notice that $\xi(m) = \lambda'' m + \nu_\sigma(m)\mu'' = \lambda_{m_0} m$, for all $m \in M$. From here, proceeding as in the proof of Case 6, we get that $\lambda_{m_0} = 0$, which implies that $\xi(m) = D(m, m_0) = 0$, for all $m \in M$ and $m_0 \in M$.

Taking into account what has already been proved, (5) can be rewritten as:

$$
D(x, y) = \tilde{\lambda}_0[a_x, a_y] + \tilde{\lambda}_0 a_x m_y - \tilde{\lambda}_0 a_y m_x + \tilde{\lambda}_0 m_x b_y - \tilde{\lambda}_0 m_y b_x - m_x \eta^{-1}(\tilde{\lambda}_0)[b_x, b_y] + \eta^{-1}(\tilde{\lambda}_0)[b_x, b_y].
$$

To finish, we will show that $D = \Delta_\lambda$, where $\Delta_\lambda$ is the inner $\sigma$-biderivation associated to the element $\lambda = \begin{pmatrix} \tilde{\lambda}_0 & -m_x \eta^{-1}(\tilde{\lambda}_0) \\ \eta^{-1}(\tilde{\lambda}_0) & \tilde{\lambda}_0 \end{pmatrix}$. Notice that it makes sense to consider $\Delta_\lambda$ since $\lambda \in Z_{\sigma}(T)$. On the other hand, from

$$
[x, y] = \begin{pmatrix} a_x, a_y \\ a_x m_y + m_x b_y - a_y m_x - m_y b_x \\ [b_x, b_y] \end{pmatrix},
$$

it is easy to check that $\tilde{\lambda}_0[x, y] = D(x, y)$, which concludes the proof of the theorem.

4. $\sigma$-COMMUTING MAPS OF TRIANGULAR ALGEBRAS

Let $\mathcal{A}$ be an algebra and $\sigma$ an automorphism of $\mathcal{A}$. Notice that in terms of the $\sigma$-commutator, a linear map $\Theta$ of $\mathcal{A}$ is $\sigma$-commuting if it satisfies

$$
\sigma (\sigma - 1) \quad [x, \Theta(x)]_\sigma = 0, \quad \forall x \in \mathcal{A}.
$$

A linearization of (4.1) gives that

$$
[x, \Theta(y)]_\sigma = -[y, \Theta(x)]_\sigma, \quad \forall x, y \in \mathcal{A}.
$$
Given \( \lambda \in \mathbb{Z}_\sigma(A) \) and a \( \sigma \)-central map \( \Omega \) of \( A \), i.e., a linear map \( \Omega : A \to \mathbb{Z}_\sigma(A) \); it is straightforward to check that the map \( \Theta : A \to A \) defined by
\[
(7) \quad \Theta(x) = \lambda x + \Omega(x), \quad \forall x \in A
\]
is \( \sigma \)-commuting. Motivated by this fact and keeping in mind the concept of proper commuting maps, we introduce the following notion:

**Definition 4.1.** Let \( A \) be an algebra and \( \sigma \) an automorphism of \( A \). Maps of the form \( (7) \) will be called **proper \( \sigma \)-commuting maps**.

In this section, we will investigate when a \( \sigma \)-commuting map of a triangular algebra is proper. Sufficient conditions will be given on a triangular algebra for all \( \sigma \)-commuting maps to be proper.

Let us start by studying the structure of \( \sigma \)-commuting maps of triangular algebras.

**Theorem 4.2.** Let \( T = \text{Trian}(A, M, B) \) be a triangular algebra and \( \sigma \) an automorphism of \( T \). Assume that the algebras \( A \) and \( B \) have only trivial idempotents. Then every \( \sigma \)-commuting map \( \Theta \) of \( T \) is of the following form
\[
\text{(\( \sigma \)-cm)}
\]
\[
\Theta \left( \begin{array}{cc} a & m \\ b & b \end{array} \right) = \left( \begin{array}{ccc} \delta_1(a) + \delta_2(m) + \delta_3(b) & -m_\sigma(\mu_1(a) + \mu_2(m) + \mu_3(b)) + \delta_1(1_A)m - \nu_\sigma(\mu_1(1_A)) \\
\mu_1(a) + \mu_2(m) + \mu_3(b) & & \end{array} \right),
\]
where
\[
\delta_1 : A \to A, \quad \delta_2 : M \to Z_{f_\sigma}(A), \quad \delta_3 : B \to Z_{f_\sigma}(A),
\]
\[
\mu_1 : A \to Z_{g_\sigma}(B), \quad \mu_2 : M \to Z_{g_\sigma}(B), \quad \mu_3 : B \to B,
\]
are linear maps such that
(i) \( \delta_1 \) is an \( f_\sigma \)-commuting map of \( A \),
(ii) \( \mu_3 \) is a \( g_\sigma \)-commuting map of \( B \),
(iii) \( \delta_1(a)m - \nu_\sigma(m)\mu_1(a) = f_\sigma(\delta_1(1_A)m - \nu_\sigma(m)\mu_1(1_A)) \),
(iv) \( \nu_\sigma(m)\mu_3(b) - \delta_3(b)m = (\nu_\sigma(m)\mu_3(1_B) - \delta_3(1_B)m)b \),
(v) \( \delta_2(m)m = \nu_\sigma(m)\mu_2(m) \),
(vi) \( \delta_1(1_A)m - \nu_\sigma(m)\mu_1(1_A) = \nu_\sigma(m)\mu_3(1_B) - \delta_3(1_B)m \),
for all \( a \in A, m \in M \) and \( b \in B \).

**Proof.** Let \( \Theta \) be a \( \sigma \)-commuting map of \( T \); write:
\[
\Theta \left( \begin{array}{cc} a & m \\ b & b \end{array} \right) = \left( \begin{array}{ccc} \delta_1(a) + \delta_2(m) + \delta_3(b) & \tau_1(a) + \tau_2(m) + \tau_3(b) \\
\mu_1(a) + \mu_2(m) + \mu_3(b) & & \end{array} \right),
\]
for all \( \left( \begin{array}{cc} a & m \\ b & b \end{array} \right) \in T \). In what follows, we will apply equations (\( \sigma \)-c1) and (\( \sigma \)-c2) many times with appropriate elements of \( T \). Apply (\( \sigma \)-c1) with \( x = \left( \begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right) \) to get that
\[
(8) \quad [a, \delta_1(a)]_{f_\sigma} = 0, \quad f_\sigma(\delta_1(a) + m_\sigma\mu_1(a)) = 0,
\]
for all \( a \in A \). This shows (i) and implies that
\[
(9) \quad \tau_1(1_A) + m_\sigma\mu_1(1_A) = 0,
\]
by making \( a = 1_A \) in the second equation of \( (8) \). A second application of \( (c) \) with \( x = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \) proves (ii), and also that

\[
(10) \quad m_3(1_B) + \tau_3(1_B) = 0.
\]

Next, take \( x = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \) and \( y = q \) in \( (d) \) to obtain (using \( (10) \)) that

\[
(11) \quad [a, \delta_3(1_B)]_{f_\sigma} = 0, \quad [b, \mu_3(1_B)]_{g_\sigma} = 0,
\]

\[
(12) \quad -m_\sigma g_\sigma(b)\mu_3(1_B) - \tau_3(1_B)b - m_\sigma \mu_1(a) - \tau_1(a) - m_\sigma \mu_3(b) - \tau_3(b) = 0
\]

for all \( a \in A, m \in M \) and \( b \in B \). A use of the second equation of \( (11) \) allows us to write \( (12) \) as

\[
(13) \quad \tau_1(a) + m_\sigma \mu_1(a) + \tau_3(b) + m_\sigma \mu_3(b) = 0,
\]

for all \( a \in A, m \in M \) and \( b \in B \). Next, make \( a = 1_A \) (respectively, \( b = 1_B \)) in \( (13) \) and apply \( (c) \) (respectively, \( (10) \)) to obtain the following equations:

\[
(14) \quad \tau_1(a) + m_\sigma \mu_1(a) = 0, \quad \tau_3(b) + m_\sigma \mu_3(b) = 0,
\]

for all \( a \in A \) and \( b \in B \). On the other hand, (v) follows from \( (c) \) by considering

\[
 x = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}. \quad \text{Now letting } x = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \text{ in } (d), \text{ we produce the following equations:}
\]

\[
(15) \quad [a, \delta_2(m)]_{f_\sigma} = 0,
\]

\[
(16) \quad f_\sigma(a)(\tau_2(m) + m_\sigma \mu_2(m)) = \delta_1(a)m - \nu_\sigma(m)\mu_1(a),
\]

for all \( a \in A \) and \( m \in M \). Note that \( (15) \) says that \( \delta_2(M) \subseteq Z_{f_\sigma}(A) \). Let \( a = 1_A \) in \( (10) \) to obtain

\[
(17) \quad \tau_2(m) + m_\sigma \mu_2(m) = \delta_1(1_A)m - \nu_\sigma(m)\mu_1(1_A),
\]

for all \( m \in M \). Notice that (iii) follows from \( (10) \) and \( (17) \).

Now taking \( x = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \) and \( y = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \) in \( (d) \), we get that

\[
(18) \quad [b, \mu_2(m)]_{g_\sigma} = 0,
\]

\[
(19) \quad -\tau_2(m)b - m_\sigma g_\sigma(b)\mu_2(m) + \nu_\sigma(m)\mu_3(b) - \delta_3(b)m = 0,
\]

for all \( m \in M \) and \( b \in B \). Equation \( (13) \) gives \( \mu_2(M) \subseteq Z_{g_\sigma}(B) \). An application of \( (13) \) in \( (19) \) allows us to write that

\[
(20) \quad (\tau_2(m) + m_\sigma \mu_2(m))b = \nu_\sigma(m)\mu_3(b) - \delta_3(b)m,
\]

for all \( m \in M \). Making \( b = 1_B \) in \( (20) \) we obtain that

\[
(21) \quad \tau_2(m) + m_\sigma \mu_2(m) = \nu_\sigma(m)\mu_3(1_B) - \delta_3(1_B)m,
\]

for all \( m \in M \). From \( (20) \) and \( (21) \) we prove (iv). On the other hand, (vi) follows from \( (14) \) and \( (21) \). Next notice that an application of \( (14) \) and \( (17) \) gives:

\[
\tau_1(a) + \tau_2(m) + \tau_3(b) = -m_\sigma(\mu_1(a) + \mu_2(m) + \mu_3(b)) + \delta_1(1_A)m - \nu_\sigma(m)\mu_1(1_A),
\]

as desired. It remains to show that \( \mu_1(A) \subseteq Z_{g_\sigma}(B) \) and \( \delta_3(B) \subseteq Z_{f_\sigma}(A) \), which follows from an application of \( (d) \) with \( x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \) and \( y = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \), finishing the proof. \( \square \)
In what follows, we will use the theorem above without further mention. In other words, whenever we are given a \( \sigma \)-commuting map \( \Theta \) of a triangular algebra \( T \), which satisfies the hypotheses of Theorem 4.2, we will assume that \( \Theta \) is of the form (22).

Our next result provides a characterization of the properness of a \( \sigma \)-commuting map of \( T \) in terms of its behavior with the \( \sigma \)-center of \( T \).

**Theorem 4.3.** Let \( T = \text{Trian}(A,M,B) \) be a triangular algebra and \( \sigma \) an automorphism of \( T \). Assume that the algebras \( A \) and \( B \) have only trivial idempotents. Then for every \( \sigma \)-commuting map \( \Theta \) of \( T \), the following conditions are equivalent:

(i) \( \Theta \) is a proper \( \sigma \)-commuting map.

(ii) \( \mu_1(A) \subseteq \pi_B(Z_{\sigma}(T)), \delta_3(B) \subseteq \pi_A(Z_{\sigma}(T)) \) and

\[
\begin{pmatrix}
\delta_2(m) & -m_{\sigma}\mu_2(m) \\
\mu_2(m) & m_{\sigma}\mu_2(m)
\end{pmatrix} \in Z_{\sigma}(T), \quad \forall m \in M.
\]

(iii) \( \delta_1(1_A) \in \pi_A(Z_{\sigma}(T)), \mu_1(1_A) \in \pi_B(Z_{\sigma}(T)) \) and

\[
\begin{pmatrix}
\delta_2(m) & -m_{\sigma}\mu_2(m) \\
\mu_2(m) & m_{\sigma}\mu_2(m)
\end{pmatrix} \in Z_{\sigma}(T), \quad \forall m \in M.
\]

In order to prove the theorem above, we need a preliminary result.

**Lemma 4.4.** Let \( T = \text{Trian}(A,M,B) \) be a triangular algebra and \( \sigma \) an automorphism of \( T \). Assume that the algebras \( A \) and \( B \) have only trivial idempotents. Then for every \( \sigma \)-commuting map \( \Theta \) of \( T \), it follows that

\[
[A,A] \subseteq \mu_1^{-1}(\pi_B(Z_{\sigma}(T))) \triangleleft A, \quad [B,B] \subseteq \delta_3^{-1}(\pi_A(Z_{\sigma}(T))) \triangleleft B.
\]

**Proof.** Let us show, for example, that \( [A,A] \subseteq \mu_1^{-1}(\pi_B(Z_{\sigma}(T))) \triangleleft A \). Analogously, one can prove that \( [B,B] \subseteq \delta_3^{-1}(\pi_A(Z_{\sigma}(T))) \triangleleft B \). Set \( a,a' \in A \) and \( m \in M \).

Applications of Theorem 4.2 (iii) give the following:

\[
d_1(a'a)m - \nu_{\sigma}(m)\mu_1(a'a) = f_\sigma(a'a)(d_1(1_A)m - \nu_{\sigma}(m)\mu_1(1_A))
\]

\[
= f_\sigma(a')(d_1(a)m - \nu_{\sigma}(m)\mu_1(a))
\]

\[
d_1(aa')m - \nu_{\sigma}(m)\mu_1(aa') = f_\sigma(aa')(d_1(1_A)m - \nu_{\sigma}(m)\mu_1(1_A))
\]

\[
= f_\sigma(a)(f_\sigma(a')d_1(1_A)m - f_\sigma(a')\nu_{\sigma}(m)\mu_1(1_A))
\]

\[
= f_\sigma(a)(d_1(1_A)a'm - \nu_{\sigma}(a'm)\mu_1(1_A))
\]

\[
= d_1(a)a'm - \nu_{\sigma}(a'm)\mu_1(1_A).
\]

From (22) and (23) we get that

\[
d_1([a,a'])(m - \nu_{\sigma}(m)\mu_1([a,a']]) + f_\sigma(a')(d_1(a)m - d_1(a)a'm + \nu_{\sigma}(a'm)\mu_1(1_A))
\]

\[
- f_\sigma(a')\nu_{\sigma}(m)\mu_1(1_A) = 0.
\]

Taking into account that \( \nu_{\sigma}(a'm) = f_\sigma(a')\nu_{\sigma}(m) \) and that \( \delta_1(A) \subseteq Z_{f_\sigma}(A) \), the identity above becomes:

\[
d_1([a,a'])(m - \nu_{\sigma}(m)\mu_1([a,a'])) = 0,
\]

which allows us to conclude that

\[
\begin{pmatrix}
d_1([a,a']) & -m_{\sigma}\mu_1([a,a']) \\
\mu_1([a,a']) & m_{\sigma}\mu_1([a,a'])
\end{pmatrix} \in Z_{\sigma}(T),
\]
Therefore, (24) and (25) allow us to conclude that

This yields that

\[(\delta_1(a') - f_\sigma(a')\delta_1(a) + f_\sigma(a')\eta(\mu_1(a)))m = \nu_\sigma(m)\mu_1(a'a).\]

This proves that \(\mu_1(a'a)\in\pi_B(Z_\sigma(T))\), that is, \(a'a\in\mu_1^{-1}(\pi_B(Z_\sigma(T)))\). To show that \(\mu_1^{-1}(\pi_B(Z_\sigma(T)))\) is an ideal of \(A\), as desired.

\[\square\]

**Proof of Theorem 4.3** We are going to show that (i) \(\iff\) (iii) and (ii) \(\iff\) (iii).

Let \(\Theta\) be a \(\sigma\)-commuting map of \(T\).

(i) \(\Rightarrow\) (iii). Suppose that \(\Theta\) is proper. Then there exist \(\lambda \in Z_\sigma(T)\) and a linear map \(\Omega : T \to Z_\sigma(T)\) such that \(\Theta(x) = \lambda x + \Omega(x)\), for all \(x \in T\). From the proof of [28, Lemma 2.9], we can express \(\lambda\) as follows:

\[\lambda = \begin{pmatrix} a_\lambda & -m_\sigma \eta^{-1}(a_\lambda) \\ \eta^{-1}(a_\lambda) & \end{pmatrix},\]

for some \(a_\lambda \in \pi_A(Z_\sigma(T))\). Take \(m \in M\) and compute \(\Theta(x)\) for \(x = \begin{pmatrix} 0 \\ m \\ 0 \end{pmatrix}\); let us assume that \(\Omega(x) = \begin{pmatrix} a_m & -m_\sigma \eta^{-1}(a_m) \\ \eta^{-1}(a_m) \end{pmatrix}\), where \(a_m \in \pi_A(Z_\sigma(T))\). Then, we have that

\[\Theta(x) = \begin{pmatrix} a_m & a_\lambda m - m_\sigma \eta^{-1}(a_m) \\ \eta^{-1}(a_m) & \end{pmatrix}.\]

On the other hand, we can write that

\[\Theta(x) = \begin{pmatrix} \delta_2(m) & \delta_1(1_A)m - \nu_\sigma(m)\mu_1(1_A) - m_\sigma \mu_2(m) \\ \mu_2(m) & \end{pmatrix}\]

Therefore, (24) and (25) allow us to conclude that

\[a_m = \delta_2(m), \quad \mu_2(m) = \eta^{-1}(a_m),\]

\[a_\lambda m - m_\sigma \eta^{-1}(a_m) = \delta_1(1_A)m - \nu_\sigma(m)\mu_1(1_A) - m_\sigma \mu_2(m).\]

From (20) we get that

\[\begin{pmatrix} \delta_2(m) & -m_\sigma \mu_2(m) \\ \mu_2(m) & \end{pmatrix} \in Z_\sigma(T),\]
for all $m \in M$. Note that \((27)\) becomes
\begin{equation}
\label{eq:27}
a_\lambda m = \delta_1(1_A)m - \nu_\sigma(m)\mu_1(1_A), \quad \forall m \in M,
\end{equation}
by an application of \((26)\). Taking into account the fact that $a_\lambda \in Z_\sigma(T)$, \((28)\) can be rewritten as
\[
\delta_1(1_A)m = \nu_\sigma(m)(\mu_1(1_A) + \eta^{-1}(a_\lambda)), \quad \forall m \in M,
\]
which implies that
\[
\begin{pmatrix}
\delta_1(1_A) & -m_\sigma(\mu_1(1_A) + \eta^{-1}(a_\lambda)) \\
\mu_1(1_A) + \eta^{-1}(a_\lambda) & \\
\end{pmatrix}
\in Z_\sigma(T),
\]
for all $m \in M$. In particular, we have that $\delta_1(1_A) \in \pi_A(Z_\sigma(T))$. On the other hand, writing \((28)\) as
\[
(\delta_1(1_A) - a_\lambda)m = \nu_\sigma(m)\mu_1(1_A), \quad \forall m \in M,
\]
we obtain that
\[
\begin{pmatrix}
\delta_1(1_A) - a_\lambda & -m_\sigma\mu_1(1_A) \\
\mu_1(1_A) & \\
\end{pmatrix}
\in Z_\sigma(T),
\]
for all $m \in M$. Thus: $\mu_1(1_B) \in \pi_B(Z_\sigma(T))$, concluding the proof of (iii).

(iii) $\Rightarrow$ (i).

Let us start by noticing that $\delta_1(1_A) \in \pi_A(Z_\sigma(T))$ and $\mu_1(1_A) \in \pi_B(Z_\sigma(T))$ allow us to write $\eta(\mu_1(1_A))$ and $\eta^{-1}(\delta_1(1_A))$, respectively. Thus, it makes sense to consider the element
\[
\lambda := \begin{pmatrix}
\delta_1(1_A) - \eta(\mu_1(1_A)) & -m_\sigma(\eta^{-1}(\delta_1(1_A)) - \mu_1(1_A)) \\
\eta^{-1}(\delta_1(1_A)) - \mu_1(1_A) & \\
\end{pmatrix}
\in T.
\]
Note that $\lambda \in Z_\sigma(T)$ since
\[
(\delta_1(1_A) - \eta(\mu_1(1_A)))m = \nu_\sigma(m)(\eta^{-1}(\delta_1(1_A)) - \mu_1(1_A)), \quad \forall m \in M.
\]
Next, we claim that $\Omega(x) := \Theta(x) - \lambda x \in Z_\sigma(T)$, for all $x \in T$. Given $x = \begin{pmatrix} a_x & m_x \\ b_x & \end{pmatrix} \in T$, we have that
\[
\lambda x = \begin{pmatrix}
(\delta_1(1_A) - \eta(\mu_1(1_A)))a_x & (\delta_1(1_A) - \eta(\mu_1(1_A)))m_x \\
-m_\sigma(\eta^{-1}(\delta_1(1_A)) - \mu_1(1_A))b_x & \\
(\eta^{-1}(\delta_1(1_A)) - \mu_1(1_A))b_x & \\
\end{pmatrix},
\]
which implies that
\begin{equation}
\label{eq:29}
\Omega(x) = \begin{pmatrix}
\delta_1(a_x) - \delta_1(1_A)a_x + \eta(\mu_1(1_A))a_x & -m_\sigma\mu_1(a_x) \\
\mu_1(a_x) & \\
\delta_3(b_x) & -m_\sigma(\mu_3(b_x) - \eta^{-1}(\delta_1(1_A))b_x + \mu_1(1_A)b_x) \\
\mu_3(b_x) - \eta^{-1}(\delta_1(1_A))b_x + \mu_1(1_A)b_x & \\
\end{pmatrix}
\end{equation}
Thus:
\[
\begin{pmatrix}
\delta_2(m_x) & -m_\sigma \mu_2(m_x) \\
\mu_2(m_x)
\end{pmatrix}
\]

To finish, we will show that the three terms in (29) are indeed in \(Z_\sigma(T)\). Notice that the last term belongs to \(Z_\sigma(T)\), by hypothesis. Given \(m \in M\), it is enough to show that
\[
(\delta_1(a_x) - \delta_1(1_A)a_x + \eta(\mu_1(1_A))a_x)m - \nu_\sigma(m)\mu_1(a_x) = 0,
\]
\[
\delta_3(b_x)m - \nu_\sigma(m)(\mu_3(b_x) - \eta^{-1}(\delta_1(1_A))b_x + \mu_1(1_A)b_x) = 0.
\]

Let us start by proving the first identity above. Apply Theorem 4.2 (iv) and (vi) to obtain that
\[
(\delta_1(a_x) - \delta_1(1_A)a_x + \eta(\mu_1(1_A))a_x)m - \nu_\sigma(m)\mu_1(a_x) =
\]
\[
f_x(a_x)\delta_1(1_A)m - f_x(a_x)\nu_\sigma(m)\mu_1(1_A) - \delta_1(1_A)a_x m + \eta(\mu_1(1_A))a_x m =
\]
\[
[a_x, \delta_1(1_A)]f_x m + \eta(\mu_1(1_A))a_x m - \nu_\sigma(a_x m)\mu_1(1_A) = 0,
\]
since \(\delta_1(1_A) \in \pi_A(Z_\sigma(T)) \subseteq \pi_f(A)\) and \(\mu_1(1_A) \in \pi_B(Z_\sigma(T))\), by (iii).

It remains to prove the second of the two displayed identities above. Apply Theorem 4.2 (iv) and (vi) to obtain that
\[
\delta_3(b_x)m - \nu_\sigma(m)(\mu_3(b_x) - \eta^{-1}(\delta_1(1_A))b_x + \mu_1(1_A)b_x) =
\]
\[
(\delta_3(1_B)m - \nu_\sigma(m)\mu_3(1_B))b_x + \nu_\sigma(m)\eta^{-1}(\delta_1(1_A))b_x - \nu_\sigma(m)\mu_1(1_A)b_x =
\]
\[
(\nu_\sigma(m)\mu_1(1_A) - \delta_1(1_A)m)b_x + \eta_\sigma(m)\mu_1(1_A))b_x =
\]
\[
(\nu_\sigma(m)\eta^{-1}(\delta_1(1_A)) - \delta_1(1_A)m)b_x = 0,
\]
since \(\delta_1(1_A) \in \pi_A(Z_\sigma(T))\).

(ii) \(\Rightarrow\) (iii).

It remains to show that \(\delta_1(1_A) \in \pi_A(Z_\sigma(T))\). Given \(m \in M\), notice that \(\delta_3(1_B) \in \pi_A(Z_\sigma(T))\) allows us to write \(\delta_3(1)m = \nu_\sigma(m)\eta^{-1}(\delta_3(1_B))\). Keeping this fact in mind, an application of Theorem 4.2 (vi) gives the following:
\[
\delta_1(1_A)m = \nu_\sigma(m)(\mu_1(1_A) + \mu_3(1_B) - \eta^{-1}(\delta_3(1_B))),
\]
which implies that
\[
\begin{pmatrix}
\delta_1(1_A) & -m_\sigma(\mu_1(1_A) + \mu_3(1_B) - \eta^{-1}(\delta_3(1_B)) \\
\mu_1(1_A) + \mu_3(1_B) - \eta^{-1}(\delta_3(1_B))
\end{pmatrix}
\in \pi_A(Z_\sigma(T)).
\]

Thus: \(\delta_1(1_A) \in \pi_A(Z_\sigma(T))\), proving (iii).

(iii) \(\Rightarrow\) (ii).

From \(\mu_1(1_A) \in \pi_B(Z_\sigma(T))\), we get that \(1_A \in \mu_1^{-1}(\pi_B(Z_\sigma(T)))\), which is an ideal of \(A\) by Lemma 4.4. Hence: \(\mu_1^{-1}(\pi_B(Z_\sigma(T))) = A\), and therefore \(\mu_1(A) \subseteq \pi_B(Z_\sigma(T))\). In order to show that \(\delta_3(B) \subseteq \pi_A(Z_\sigma(T))\), we first need to prove that \(\delta_3(1_B) \in \pi_A(Z_\sigma(T))\). To this end, apply Theorem 4.2 (vi), taking into account the fact that \(\delta_1(1_A) \in \pi_A(Z_\sigma(T))\), to obtain that
\[
\delta_3(1_B)m = \nu_\sigma(m)(\mu_3(1_B) + \mu_1(1_A)) - \delta_1(1_A)m =
\]
\[
\nu_\sigma(m)(\mu_3(1_B) + \mu_1(1_A) - \eta^{-1}(\delta_1(1_A))), \quad \forall m \in M.
\]
This implies that
\[
\begin{pmatrix}
\delta_3(1_B) & -m_\sigma(\mu_1(1_A) + \mu_3(1_B) - \eta^{-1}(\delta_1(1_A))) \\
\mu_1(1_A) + \mu_3(1_B) - \eta^{-1}(\delta_1(1_A))
\end{pmatrix} \in \mathcal{Z}_\sigma(\mathcal{T}),
\]
and therefore \(\delta_3(1_B) \in \pi_A(\mathcal{Z}_\sigma(\mathcal{T}))\). Now take \(b \in B, m \in M\) and apply Theorem 4.2 (iv), taking into account the fact that \(\delta_3(1_B)m = \nu_\sigma(m)\eta^{-1}(\delta_3(1_B))\), to get that
\[
\delta_3(b)m = \nu_\sigma(m)\mu_3(b) - \delta_3(1_B)mb + \mu_\sigma(m)\mu_3(1_B)b
= \nu_\sigma(m)(\mu_3(b) - \eta^{-1}(\delta_3(1_B)))b + \mu_3(1_B)b,
\]
which says that
\[
\begin{pmatrix}
\delta_3(b) & -m_\sigma(\mu_3(b) + \mu_3(1_B)b - \eta^{-1}(\delta_3(1_B))b) \\
\mu_3(b) + \mu_3(1_B)b - \eta^{-1}(\delta_3(1_B))b
\end{pmatrix} \in \mathcal{Z}_\sigma(\mathcal{T}).
\]
Thus: \(\delta_3(b) \in \pi_A(\mathcal{Z}_\sigma(\mathcal{T}))\), concluding the proof of Theorem 4.5. \(\Box\)

Our last result states sufficient conditions on a triangular algebra \(\mathcal{T}\) to guarantee that all its \(\sigma\)-commuting maps are proper.

**Theorem 4.5.** Let \(\mathcal{T} = \text{Triang}(A, M, B)\) be a triangular algebra and \(\sigma\) an automorphism of \(\mathcal{T}\). Assume that the algebras \(A\) and \(B\) have only trivial idempotents. Suppose that \(\mathcal{T}\) satisfies the following conditions:

(i) Either \(Z_{f_\sigma}(A) = \pi_A(\mathcal{Z}_\sigma(\mathcal{T}))\) or \(B = [B, B]\).

(ii) Either \(Z_{g_\sigma}(B) = \pi_B(\mathcal{Z}_\sigma(\mathcal{T}))\) or \(A = [A, A]\).

(iii) There exists \(m_0 \in M\) such that
\[
Z_{\sigma}(\mathcal{T}) = \left\{ \begin{pmatrix} a & -m_\sigma b \\ b & \end{pmatrix} \in \mathcal{T}, \text{ such that } am_0 = \nu_\sigma(m_0)b \right\}.
\]

Then every \(\sigma\)-commuting map of \(\mathcal{T}\) is proper.

**Proof.** Let \(\Theta\) be a \(\sigma\)-commuting map of \(\mathcal{T}\). By Theorem 4.2 we have that \(\mu_1(A) \subseteq Z_{g_\sigma}(B)\). If \(Z_{g_\sigma}(B) = \pi_B(\mathcal{Z}_\sigma(\mathcal{T}))\), we can conclude that \(\mu_1(A) \subseteq \pi_B(Z_{\sigma}(\mathcal{T}))\). On the other hand, if \(A = [A, A]\), apply Lemma 4.3 to get that \(A \subseteq \mu_1^{-1}(\pi_B(Z_{\sigma}(\mathcal{T}))\)), that is, \(\mu_1(A) \subseteq \pi_B(Z_{\sigma}(\mathcal{T}))\). Reasoning as above, now using the fact that \(\delta_3(B) \subseteq Z_{f_\sigma}(A)\) and applying (i), we get that \(\delta_3(B) \subseteq \pi_A(Z_{\sigma}(\mathcal{T}))\).

In view of Theorem 4.3 to prove that \(\Theta\) is proper, it remains to show that
\[
(30) \begin{pmatrix}
\delta_2(m) & -m_\sigma\mu_2(m) \\
\mu_2(m)
\end{pmatrix} \in \mathcal{Z}_\sigma(\mathcal{T}), \quad \forall m \in M.
\]
From Theorem 4.2 we have that \(\delta_2(M) \subseteq Z_{f_\sigma}(A)\) and \(\mu_2(M) \subseteq Z_{g_\sigma}(B)\), which imply that
\[
(31) \begin{pmatrix}
\delta_2(m)m = \nu_\sigma(m)\mu_2(m), \quad \forall m \in M.
\end{pmatrix}
\]
In particular, for \(m = m_0\) we get that \(\delta_2(m_0)m_0 = \nu_\sigma(m_0)\mu_2(m_0)\), which gives
\[
(30) \begin{pmatrix}
\delta_2(m_0) & -m_\sigma\mu_2(m_0) \\
\mu_2(m_0)
\end{pmatrix} \in \mathcal{Z}_\sigma(\mathcal{T}),
\]
by an application of (iii). Therefore:

\[
\begin{pmatrix}
0 & m \\
0 & \delta_2(m_0) & -m_\sigma \mu_2(m_0) \\
\mu_2(m_0)
\end{pmatrix}
\] 

Expanding the above product, we get that

\[(32) \quad \delta_2(m_0)m = \nu_\sigma(m)\mu_1(m_0), \quad \forall m \in M.
\]

Given \(m \in M\), applying (31) with \(m + m_0\) and making use of (32), we get that

\[\delta_2(m)m_0 = \nu_\sigma(m_0)\mu_1(m),
\]

which, by an application of (iii), proves (30), concluding the proof of the theorem.

\[\square\]

**Remark 4.6.** Let \(\sigma\) be an automorphism of an algebra \(A\). Notice that, as happened with biderivations and commuting maps, the study of the properness of \(\sigma\)-commuting maps of \(A\) can be reduced to the study of the innerness of its \(\sigma\)-biderivations. Let \(\Theta\) be any \(\sigma\)-commuting map of \(A\); applying Lemma 3.8 (i) and (ii), it is not difficult to prove that the map \(D_\Theta : A \times A \to A\) given by

\[D_\Theta(x, y) = [x, \Theta(y)]\sigma,\]

for \(x, y \in A\), is a \(\sigma\)-biderivation of \(A\). Let us assume now that \(D_\Theta\) is inner, i.e., \(D_\Theta(x, y) = \lambda[x, y]\), for some \(\lambda \in \mathbb{Z}_\sigma(A)\). This implies that the map \(\Omega_\Theta(y) := \Theta(y) - \lambda y\) is \(\sigma\)-central; in fact:

\[
\begin{align*}
[x, \Omega_\Theta(y)]_\sigma &= [x, \Theta(y) - \lambda y]_\sigma = \sigma(x)\Theta(y) - \sigma(x)\lambda y - \Theta(y)x + \lambda yx \\
&= [x, \Theta(y)]_\sigma - \lambda xy + \lambda yx = [x, \Theta(y)]_\sigma - \lambda[x, y] \\
&= D_\Theta(x, y) - \lambda[x, y] = 0,
\end{align*}
\]

for all \(x \in A\). Therefore, we can conclude that \(\Theta\) is a proper \(\sigma\)-commuting map.

Suppose now that \(A\) is a triangular algebra and notice that \(D_\Theta(p, p) = 0\). In other words, \(D_\Theta\) is one of the \(\sigma\)-biderivations studied in Theorem 3.4. Taking into account the calculations above, under the assumptions of Theorem 3.4, we can guarantee that every \(\sigma\)-commuting map of \(A\) is proper. Nevertheless, Theorem 4.5 above gives us more precise information when dealing directly with \(\sigma\)-commuting maps; it is a generalization of [9, Theorem 2].

The next result shows that, under some mild conditions, the identity is the only commuting automorphism of a triangular algebra.

**Theorem 4.7.** Let \(T = \text{Trian}(A, M, B)\) be a triangular algebra such that the algebras \(A\) and \(B\) have only trivial idempotents. If \(\sigma\) is a commuting automorphism of \(T\), then \(\sigma = 1_{T}\).

**Proof.** Let \(\sigma\) be an automorphism of \(T\). By Theorem 2.6, we know that \(\sigma\) is of the following form:

\[
\sigma \begin{pmatrix} a & m \\ b & \end{pmatrix} = \begin{pmatrix} f_\sigma(a) & f_\sigma(a)m_\sigma - m_\sigma g_\sigma(b) + \nu_\sigma(m) \\ \nu_\sigma(b) & \end{pmatrix},
\]

where \(f_\sigma, g_\sigma\) are automorphisms of \(A, B\), respectively, \(m_\sigma\) is a fixed element of \(M\), and \(\nu_\sigma : M \to M\) is a linear bijective map satisfying

\[(33) \quad \nu_\sigma(am) = f_\sigma(a)\nu_\sigma(m), \quad \nu_\sigma(mb) = \nu_\sigma(m)g_\sigma(b),\]
for all \( a \in A, b \in B, m \in M \). Suppose that \( \sigma \) is a commuting map of \( T \) and let us show that \( \sigma = \text{Id}_T \). Consider the elements \( x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \) and \( y = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \); since \([x, \sigma(y)] + [y, \sigma(x)] = 0\), we get that \( \sigma(m) = f_\sigma(a)m = 0 \), for all \( a \in A \) and \( m \in M \). In particular, for \( a = 1_A \) we get that \( \sigma = \text{Id}_M \), which together with (33) implies that \((a - f_\sigma(a))M = 0, \quad M(b - g_\sigma(b)) = 0,\) for all \( a \in A, b \in B \). Therefore: \( f_\sigma = \text{Id}_A \), since \( M \) is a faithful \((A,B)\)-bimodule. To finish, to show that \( m_\sigma = 0 \), let \( x = p \) and apply the fact that \([p, \sigma(p)] = 0\). □

We close the section by showing that Posner’s theorem also holds for triangular algebras.

**Theorem 4.8.** Let \( T = \text{Trian}(A,M,B) \) be a triangular algebra and \( \sigma \) an automorphism of \( T \). Assume that the algebras \( A \) and \( B \) have only trivial idempotents. If a \( \sigma \)-derivation \( d \) of \( T \) is \( \sigma \)-commuting, then \( \sigma = 0 \).

**Proof.** Let \( d \) be a \( \sigma \)-derivation of \( T \) which is \( \sigma \)-commuting. By [18, Theorem 3.12], we know that \( d \) is of the following form:

\[
d \begin{pmatrix} a & m \\ b & c \end{pmatrix} = \begin{pmatrix} d_A(a) & f_\sigma(a)m_d - m_db - m_\sigma d_B(b) + \xi_d(m) \\ d_B(b) \end{pmatrix},
\]

where \( d_A \) is an \( f_\sigma \)-derivation of \( A \), \( d_B \) is a \( g_\sigma \)-derivation of \( B \), \( m_d \) is a fixed element of \( M \), and \( \xi_d : M \to M \) is a linear map which satisfies

\[
\xi_d(ab) = d_A(a)m + f_\sigma(a)\xi(m), \quad \xi(mb) = \xi(m)b + \nu_\sigma(m)d_B(b),
\]

for all \( a \in A, b \in B, m \in M \). An application of (32) with \( x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \) and \( y = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \) gives that \( f_\sigma(a)\xi(m) - d_A(a)m = 0 \), for all \( a \in A \) and \( m \in M \). In particular, for \( a = 1_A \) we get that \( \xi = 0 \). Then from (34) we obtain that \( d_A(a)M = 0, \quad \nu_\sigma(M)d_B(b) = 0, \) for all \( a \in A \) and \( b \in B \). Since \( \nu_\sigma \) is bijective and \( M \) is a faithful \((A,B)\)-bimodule, we see that \( d_A = d_B = 0 \). Applying (31) with \( x = p \), we get that \( m_d = 0 \), finishing the proof. □

**Corollary 4.9.** If \( d \) is a commuting derivation of a triangular algebra, then \( d = 0 \).

### 5. Some facts about \((\alpha,\beta)\)-biderivations, \((\alpha,\beta)\)-commuting maps and generalized matrix algebras.

In 2011, Xiao and Wei [27] introduced a generalization of \( \sigma \)-biderivations and \( \sigma \)-commuting maps that they named \((\alpha,\beta)\)-biderivations and \((\alpha,\beta)\)-commuting maps. In their paper, they studied \((\alpha,\beta)\)-biderivations and \((\alpha,\beta)\)-commuting maps of nest algebras. In the present paper, we have investigated \( \sigma \)-biderivations and \( \sigma \)-commuting maps since, as will be shown below, the study of \((\alpha,\beta)\)-biderivations (respectively, \((\alpha,\beta)\)-commuting maps) of any algebra can be reduced to the study of its \( \sigma \)-biderivations (respectively, \( \sigma \)-commuting maps).
Let $\mathcal{A}$ be an algebra and $\alpha, \beta, \sigma$ automorphisms of $\mathcal{A}$. Recall that a linear map $d : \mathcal{A} \to \mathcal{A}$ is called an $(\alpha, \beta)$-derivation of $\mathcal{A}$ if it satisfies
\[ d(xy) = d(x)\beta(y) + \alpha(x)d(y), \quad \forall x, y \in \mathcal{A}. \]
A bilinear map $D : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is said to be an $(\alpha, \beta)$-biderivation of $\mathcal{A}$ if it is an $(\alpha, \beta)$-derivation in each argument. An $(\alpha, \beta)$-commuting map of $\mathcal{A}$ is a linear map $\Theta$ satisfying that $\Theta(x)\alpha(x) = \beta(x)\Theta(x)$, for all $x \in \mathcal{A}$. Clearly, every $\sigma$-derivation can be seen as an $(\alpha, \beta)$-derivation with $\alpha = \text{Id}_A$ and $\beta = \sigma$. The same can be said for $\sigma$-biderivations and $\sigma$-commuting maps.

It is straightforward to check that every $(\alpha, \beta)$-biderivation $D$ (respectively, $(\alpha, \beta)$-commuting map $\Theta$) of $\mathcal{A}$ gives rise to an $\alpha^{-1}\beta$-biderivation (respectively, $\alpha^{-1}\beta$-commuting map) by considering $\alpha^{-1}D$ (respectively, $\alpha^{-1}\Theta$). Accordingly, it is sufficient to restrict attention to $\sigma$-biderivations (respectively, $\sigma$-commuting maps).

In the last few years, many results on maps of triangular algebras have been extended to the setting of generalized matrix algebras (GMAs); see, for example, [13, 14, 23, 27] and references therein. GMAs were introduced by Sands [25] in the early 1970s. He ended up with these structures during his study of radicals of rings in Morita contexts. Specifically, a Morita context
\[
(A, B, M, N, \Phi_{MN}, \Psi_{NM}),
\]
consists of two $R$-algebras $A$ and $B$, two bimodules $AM$ and $BN$, and two bimodule homomorphisms $\Phi_{MN} : M \otimes_B N \to A$ and $\Psi_{NM} : N \otimes_A M \to B$ such that the following diagrams are commutative.

\[
\begin{array}{ccc}
M \otimes_B N \otimes_A M & \xrightarrow{\Phi_{MN} \otimes I_M} & A \otimes_A M \\
\downarrow & & \downarrow \\
M \otimes_B B & \xrightarrow{\simeq} & M \\
\end{array}
= \begin{array}{ccc}
N \otimes_B N \otimes_A M & \xrightarrow{\Psi_{NM} \otimes I_N} & B \otimes_B N \\
\downarrow & & \downarrow \\
N \otimes_A A & \xrightarrow{\simeq} & N \\
\end{array}
\]

Let $(A, B, M, N, \Phi_{MN}, \Psi_{NM})$ be a Morita context, where at least one of the two bimodules is nonzero. The set
\[
\mathcal{G} = \left\{ \begin{pmatrix} a & M \\ N & b \end{pmatrix} : a \in A, m \in M, n \in N, b \in B \right\},
\]
can be endowed with an $R$-algebra structure under the usual matrix operations. We will call $\mathcal{G}$ a generalized matrix algebra. Note that any triangular algebra can be seen as a generalized matrix algebra with $N = 0$.

The extension of our results proceeds through the study of automorphisms of GMAs. However, the description of automorphisms of GMAs is still an open problem. (See [23 Question 4.4])

Acknowledgements

Both authors were supported by a grant from the Natural Sciences and Engineering Research Council (Canada). The second author was also supported by the Spanish MEC and Fondos FEDER jointly through project MTM2010-15223, and by the Junta de Andalucía (projects FQM-336 and FQM2467). She thanks Professor Nantel Bergeron, the Department of Mathematics at the University of Toronto and the Fields Institute for her visit from August 2012 to September 2013.
σ-BIDERIVATIONS AND σ-COMMUTING MAPS OF TRIANGULAR ALGEBRAS

References

[1] D. Benkovič: Biderivations of triangular algebras. Linear Algebra Appl. 431 (2009), 1587–1602.
[2] M. Brešar: On skew-commuting mappings of rings. Bull. Austral. Math. Soc. 47 (1993), 291–296.
[3] M. Brešar: On certain pairs of functions of semiprime rings. Proc. Amer. Math. Soc. 120 (1994), 709–713.
[4] M. Brešar: On Generalized Biderivations and Related Maps. J. Algebra 172 (1995), 764–786.
[5] M. Brešar: Commuting maps: a survey. Taiwanese J. Math. 8 (2004) 361–397.
[6] M. Brešar, W. S. Martindale 3rd, C. R. Miers: Centralizing maps in prime rings with involution. J. Algebra 161 (1993), 342–357.
[7] S. U. Chase: A generalization of the ring of triangular matrices. Nagoya Math. J. 18 (1961), 13–25.
[8] W. S. Cheung: Mappings on triangular algebras. Ph.D. Dissertation, University of Victoria, 2000.
[9] W. S. Cheung: Commuting maps of triangular algebras. J. London Math. Soc. 63 (2001), 117–127.
[10] E. Christensen: Derivations of nest algebras. Math. Ann. 229 (1977), 155–161.
[11] S. P. Coelho, C. P. Milies: Derivations of upper triangular matrix rings. Linear Algebra Appl. 187 (1993), 263–267.
[12] K. R. Davidson: Nest algebras in: Pitman Research Notes in Mathematical Series, vol. 191, Longman, London/ New York, 1988.
[13] Y. Du, Y. Wang: Lie derivations of generalized matrix algebras. Linear Algebra Appl. 437 (2012), 2719–2726.
[14] Y. Du, Y. Wang: Biderivations of generalized matrix algebras. Linear Algebra Appl. 438 (2013), 4483–4499.
[15] D. R. Farkas, G. Letzter: Ring theory from symplectic geometry. J. Pure Applied Algebra 125 (1998), 401–416.
[16] B. E. Forrest, L. W. Marcoux: Derivations of triangular Banach algebras. Indiana Univ. Math. J. 45 (1996), 441–462.
[17] A. Haghighi, K. Varadarajan: Study of formal triangular matrix rings. Comm. Algebra 27 (1999), 5507–5525.
[18] D. Han, F. Wei: Jordan (α, β)-derivations on triangular algebras and related mappings. Linear Algebra Appl. 434 (2011), 259–284.
[19] M. Harada: Hereditary semi-primary rings and triangular matrix rings. Nagoya Math. J. 27 (1966), 463–484.
[20] I. N. Herstein: Lie and Jordan structures in simple, associative rings. Bull. Amer. Math. Soc. 67 (1961), 517–531.
[21] S. Jøndrup: Automorphisms and derivations of upper triangular matrix rings. Linear Algebra Appl. 221 (1995), 205–218.
[22] R. Khazal, S. Dăscălescu, L. V. Wyk: Isomorphism of generalized triangular matrix rings and recovery of titles. Int. J. Math. Sci. 9 (2003), 553–538.
[23] Y. B. Li, F. Wei: Semi-centralizing maps of generalized matrix algebras. Linear Algebra Appl. 436 (2012), 1122–1153.
[24] E. C. Posner: Derivations in prime rings. Proc. Amer. Math. Soc. 8 (1957) 1093–1100.
[25] A. D. Sands: Radicals and Morita contexts. J. Algebra 24 (1973), 335–345.
[26] Skornyakov: Strongly prime noncommutative Jordan algebras. Trudy Inst. Mat. (Novosibirsk) 16 (1989), 131–164 (in Russian).
[27] Z. Xiao, F. Wei: Commuting mappings of generalized matrix algebras. Linear Algebra Appl. 433 (2010) 2178–2197.
[28] W. Yang, J. Zhu: Characterizations of additive (generalized) ξ-Lie (α, β)-derivations on triangular algebras. Linear and Multilinear A. 61 (2013) no. 6, 811–830.
[29] W.-Y. Yu, J.-H. Zhang: σ-biderivations and σ-commuting maps on nest algebras. Acta Math. Sin. (Chin. Ser.) 50 (2007) 1391–1396.
[30] J.-H. Zhang: Jordan derivations of nest algebras. Acta Math. Sin. (Chin. Ser.) 41 (1998) 205–212.
[31] J.-H. Zhang, S. Feng, H.-X. Li, R.-H. Wu: Generalized biderivations of nest algebras. *Linear Algebra Appl.* **418** (2006) 225–233.

[32] Y. Zhao, D. Wang, R. Yao: Biderivations of upper triangular matrix algebras over commutative rings. *Int. J. Math. Game Theory Algebra* **18** (2009) no. 6, 473–778.

(1) **University of Toronto**, Toronto, ON, Canada

(2) **Departamento de Álgebra, Geometría y Topología, Universidad de Málaga**, Málaga, Spain

*E-mail address:* repka@math.toronto.edu

*E-mail address:* jsanchez@uma.es