Nordhaus-Gaddum-type theorem for conflict-free connection number of graphs

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Abstract

An edge-colored graph \(G\) is conflict-free connected if, between each pair of distinct vertices, there exists a path containing a color used on exactly one of its edges. The conflict-free connection number of a connected graph \(G\), denoted by \(cfc(G)\), is defined as the smallest number of colors that are needed in order to make \(G\) conflict-free connected. In this paper, we determine all trees \(T\) of order \(n\) for which \(cfc(T) = n - t\), where \(t \geq 1\) and \(n \geq 2t + 2\). Then we prove that \(1 \leq cfc(G) \leq n - 1\) for a connected graph \(G\), and characterize the graphs \(G\) with \(cfc(G) = 1, n - 4, n - 3, n - 2, n - 1\), respectively. Finally, we get the Nordhaus-Gaddum-type theorem for the conflict-free connection number of graphs, and prove that if \(G\) and \(\overline{G}\) are connected, then \(4 \leq cfc(G) + cfc(\overline{G}) \leq n\) and \(4 \leq cfc(G) \cdot cfc(\overline{G}) \leq 2(n - 2)\), and moreover, \(cfc(G) + cfc(\overline{G}) = n\) or \(cfc(G) \cdot cfc(\overline{G}) = 2(n - 2)\) if and only if one of \(G\) and \(\overline{G}\) is a tree with maximum degree \(n - 2\) or a \(P_5\), and the lower bounds are sharp.

Keywords: Edge-coloring; connectivity; conflict-free connection number; Nordhaus-Gaddum-type.

\(^*\)Supported by NSFC Nos. 11371205, 11531011, 11601254 and 11551001, and the SFQP Nos. 2016-ZJ-948Q and 2014-ZJ-907, and the project on the key lab of IOT of Qinghai province (No. 2017-Z-Y21).
AMS subject classification 2010: 05C15, 05C40, 05C75.

1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [4] for graph theoretical notation and terminology not described here. Let $G$ be a graph. We use $V(G)$, $E(G)$, $n(G)$, $m(G)$, and $\Delta(G)$ to denote the vertex-set, edge-set, number of vertices, number of edges, and maximum degree of $G$, respectively. For $v \in V(G)$, let $N(v)$ denote the neighborhood of $v$ in $G$, and let $d(v)$ denote the degree of $v$ in $G$, and $d_F(v)$ denote the degree of $v$ in a subgraph $F$ of $G$. Given two graphs $G$ and $H$, the union of $G$ and $H$, denoted by $G \cup H$, is the graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. The join of $G$ and $H$, denoted by $G + H$, is obtained from $G \cup H$ by joining each vertex of $G$ to every vertex of $H$.

Let $G$ be a nontrivial connected graph with an associated edge-coloring $c : E(G) \to \{1, 2, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may have the same color. If adjacent edges of $G$ are assigned different colors by $c$, then $c$ is a proper (edge-)coloring. For a graph $G$, the minimum number of colors needed in a proper coloring of $G$ is referred to as the edge-chromatic number of $G$ and denoted by $\chi'(G)$. A path of an edge-colored graph $G$ is said to be a rainbow path if no two edges on the path have the same color. The graph $G$ is called rainbow connected if every pair of distinct vertices of $G$ is connected by a rainbow path in $G$. An edge-coloring of a connected graph is a rainbow connection coloring if it makes the graph rainbow connected. This concept of rainbow connection of graphs was introduced by Chartrand et al. [7] in 2008. For a connected graph $G$, the rainbow connection number $rc(G)$ of $G$ is defined as the smallest number of colors that are needed in order to make $G$ rainbow connected. The reader who are interested in this topic can see [20, 21] for a survey.

Inspired by rainbow connection coloring and proper coloring in graphs, Andrews et al. [3] and Borozan et al. [5] introduced the concept of proper-path coloring. Let $G$ be a nontrivial connected graph with an edge-coloring. A path in $G$ is called a proper path if no two adjacent edges of the path receive the same color. An edge-coloring $c$ of a connected graph $G$ is a proper-path coloring if every pair of distinct vertices of $G$ are connected by a proper path in $G$. And if $k$ colors are used, then $c$ is called a proper-path $k$-coloring. An edge-colored graph $G$ is proper connected if any two vertices of $G$ are connected by a proper path. For a connected graph $G$, the minimum number of colors that are needed in order to make $G$ proper connected is called the proper connection number of $G$, denoted by $pc(G)$. Let $G$ be a nontrivial
connected graph of order $n$ and size $m$ (number of edges). Then we have that $1 \leq pc(G) \leq \min\{\chi'(G), rc(G)\} \leq m$. For more details, we refer to [13, 17, 22] and a dynamic survey [18].

A coloring of vertices of a hypergraph $H$ is called conflicted-free if each hyperedge $E$ of $H$ has a vertex of unique color that does not get repeated in $E$. The smallest number of colors required for such a coloring is called the conflicted-free chromatic number of $H$. This parameter was first introduced by Even et al. [12] in a geometric setting, in connection with frequency assignment problems for cellular networks. One can find many results on conflict-free coloring, see [9, 10, 25].

Recently, Czap et al. [8] introduced the concept of conflict-free connection of graphs. An edge-colored graph $G$ is called conflicted-free connected if each pair of distinct vertices is connected by a path which contains at least one color used on exactly one of its edges. This path is called a conflicted-free path, and this coloring is called a conflict-free connection coloring of $G$. The conflict-free connection number (or, cfc number, for short) of a connected graph $G$, denoted by $cfc(G)$, is the smallest number of colors needed to color the edges of $G$ so that $G$ is conflict-free connected. In [8], they showed that it is easy to get the conflict-free connection number for 2-connected graphs and very difficult for other connected graphs, including trees.

A Nordhaus-Gaddum-type result is a (tight) lower or upper bound on the sum or product of the values of a parameter for a graph and its complement. The name “Nordhaus-Gaddum-type” is given because Nordhaus and Gaddum [24] first established the following type of inequalities for chromatic numbers in 1956. They proved that if $G$ and $\overline{G}$ are complementary graphs on $n$ vertices whose chromatic numbers are $\chi(G)$ and $\chi(\overline{G})$, respectively, then

\[
2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1, \quad n \leq \chi(G) \cdot \chi(\overline{G}) \leq \left(\frac{n + 1}{2}\right)^2.
\]

Since then, the Nordhaus-Gaddum type relations have received wide attention: diameter [15], domination number [14, 29], connectivity [16], generalized edge-connectivity [19], rainbow connection number [6], list coloring [11], Wiener index [23] and some other chemical indices [27], and so on. For more results, we refer to a recent survey paper [1] by Aouchiche and Hansen.

Let us give an overview of the rest of this paper. In Section 2, we present some upper bounds for the conflict-free connection number. In Section 3, we determine all trees $T$ of order $n$ for which $cfc(T) = n - t$, where $t \geq 1$ and $n \geq 2t + 2$. In Section 4, graphs $G$ with $cfc(G) = 1, n - 4, n - 3, n - 2, n - 1$ are respectively characterized. In Section 5, we get the Nordhaus-Gaddum-type theorem for the conflict-free connection number of graphs, and prove that if $G$ and $\overline{G}$ are connected,
then $4 \leq cfc(G) + cfc(\overline{G}) \leq n$ and $4 \leq cfc(G) \cdot cfc(\overline{G}) \leq 2(n - 2)$, and moreover, $cfc(G) + cfc(\overline{G}) = n$ or $cfc(G) \cdot cfc(\overline{G}) = 2(n - 2)$ if and only if one of $G$ and $\overline{G}$ is a tree with maximum degree $n - 2$ or a $P_5$, and the lower bounds are sharp.

2 Preliminaries

At the very beginning, we state some fundamental results on the conflict-free connections of graphs, which will be used in the sequel.

**Lemma 2.1** [8] If $P_n$ is a path on $n$ edges, then $cfc(P) = \lceil \log_2(n + 1) \rceil$.

It is obvious that $cfc(K_{1,n-1}) = n - 1$ for $n \geq 2$. In [8] the authors obtained the upper and lower bounds of the conflict-free connection number for trees in terms of the maximum degree $\Delta$.

**Lemma 2.2** [8] If $T$ is a tree on $n$ vertices with maximum degree $\Delta(T) \geq 3$ and diameter $d(T)$, then

$$\max\{\Delta(T), \log_2(d(T))\} \leq cfc(T) \leq \frac{(\Delta(T) - 2) \log_2(n)}{\log_2(\Delta(T)) - 1}.$$ 

**Lemma 2.3** [8] If $G$ is a noncomplete 2-connected graph, then $cfc(G) = 2$.

A *block* of a graph $G$ is a maximal connected subgraph of $G$ that has no cut-vertex. If $G$ is connected and has no cut-vertex, then $G$ is a block. An edge is a block if and only if it is a cut-edge, this block is called trivial. Then any nontrivial block is 2-connected.

**Lemma 2.4** [8] Let $G$ be a connected graph. Then from its every nontrivial block an edge can be chosen so that the set of all such chosen edges forms a matching.

From Lemma 2.4 we can extend the result of Lemma 2.3 to 2-edge-connected graphs in the following.

**Corollary 2.5** Let $G$ be a noncomplete 2-edge-connected graph. Then $cfc(G) = 2$.

*Proof.* Since $G$ is not a complete graph, it follows that $cfc(G) \geq 2$. It suffices to show that $cfc(G) = 2$. From Lemma 2.4 we can choose an edge in each nontrivial block such that the set $S$ of such chosen edges forms a matching. Then we color the edges from $S$ with color 2 and color the remaining edges of $G$ with color 1. It
is easy to check that this coloring is a conflict-free connection coloring of $G$. Thus, $cfc(G) = 2$.

Let $C(G)$ be the subgraph of $G$ induced on the set of cut-edges of $G$, and let $h(G) = \max\{cfc(T) : T$ is a component of $C(G)\}$. The following theorem provides a sufficient condition for graphs $G$ with $cfc(G) = 2$.

**Lemma 2.6** [8] If $G$ is a connected graph and $C(G)$ is a linear forest whose each component has an order 2, then $cfc(G) = 2$.

**Lemma 2.7** [8] If $G$ is a connected graph, then

$$h(G) \leq cfc(G) \leq h(G) + 1.$$  

Moreover, the bounds are sharp.

Note that it is supposed to define $h(G) = 1$ for the 2-edge-connected graph $G$ in addition. Next, we give a sufficient condition such that the lower bound is sharp in Lemma 2.7 for $h(G) \geq 2$.

**Proposition 2.8** Let $G$ be a connected graph with $h(G) \geq 2$. If there exists a unique component $T$ of $C(G)$ such that $cfc(T) = h(G)$, then $cfc(G) = h(G)$.

**Proof.** Let $T_1, T_2, \ldots, T_s$ be the components of $C(G)$, and let $B_1, B_2, \ldots, B_r$ be the nontrivial blocks of $G$. Suppose that $T_1$ is the unique component $T$ of $C(G)$ with $cfc(T) = h(G)$. We provide an edge-coloring of $G$ as follows. We first color the edges of $T_1$ with $h(G)$ colors $\{1, \ldots, h(G)\}$ such that $T_1$ is conflict-free connected. Then color the edges of $T_i$ with at most $h(G) - 1$ colors $\{1, \ldots, h(G) - 1\}$ such that $T_i$ is conflict-free connected for $2 \leq i \leq s$. Next, we color the edges of $B_i$ for $1 \leq i \leq r$. By Lemma 2.4, we choose an edge in $B_i$ such that the set $S$ of such chosen edges forms a matching. We color the edges from $S$ with color $h(G)$ and color the remaining edges of $G$ with color 1. Note that this coloring is a conflict-free connection coloring of $G$. So, we have $cfc(G) = h(G)$.

Recall that the edge-connectivity of a connected graph $G$, denoted by $\lambda(G)$, is the minimum size of an edge-subset whose removal from $G$ results a disconnected graph.

The following result will be useful in our discussion.

**Lemma 2.9** Let $G$ be a connected graph of order $n$ with $\lambda(G) = 1$. Then $cfc(G) \leq n - 2r$, where $r$ is the number of nontrivial blocks of $G$. 
Proposition 2.12 If \( G \) is a nontrivial connected graph and \( H \) is a connected spanning subgraph of \( G \), then \( cfc(G) \leq cfc(H) \).

Proposition 2.12 If \( G \) is a connected graph with size \( m_G \) and \( H \) is a connected subgraph of \( G \) with size \( m_H \), then \( cfc(G) \leq cfc(H) + m_G - m_H \).

Proof. Suppose that \( a = cfc(H) \) and \( b = cfc(H) + m_G - m_H \). Let \( c_H \) be a conflict-free connection coloring of \( H \) using the colors \( 1, \ldots, a \). Then \( c_H \) can be extended to a conflict-free connection coloring \( c_G \) of \( G \) by assigning the \( m_G - m_H \) colors \( a, a + 1, \ldots, b \) to the \( m_G - m_H \) edges in \( E(G) - E(H) \), which implies that \( cfc(G) \leq cfc(H) + m_G - m_H \).

The following result is an immediate consequence of Lemma 2.9.

Corollary 2.10 Let \( G \) be a connected graph of order \( n \) with \( \lambda(G) = 1 \). If \( G \) has a unique nontrivial block \( B \), then \( cfc(G) \leq n + 1 - |V(B)| \).

Next, we give some upper bounds for \( cfc(G) \), which will be useful in our discussion.

It is clear that the addition of an edge to \( G \) can not increase \( cfc(G) \).

3 Trees with given cfc numbers

In the sequel, let \( K_n, K_{s,t}, P_n \), and \( C_n \) denote the complete bipartite graph of order \( s + t \), complete graph of order \( n \), path of order \( n \), cycle of order \( n \), respectively.
Clearly, a star of order $t + 1$ is exactly $K_{1,t}$. In addition, a double star is a tree with diameter 3.

The following theorem indicates that when the maximum degree of a tree is large, we can give the conflict-free connection number immediately by its maximum degree.

**Theorem 3.1** Let $T$ be a tree of order $n$, and let $t$ be a natural number such that $t \geq 1$ and $n \geq 2t + 2$. Then $cfc(T) = n - t$ if and only if $\Delta(T) = n - t$.

We proceed our proof by the following two lemmas.

**Lemma 3.2** Let $T$ be a tree of order $n$, and let $t$ be a natural number such that $t \geq 1$ and $n \geq 2t + 2$. If $cfc(T) = n - t$, then $\Delta(T) = n - t$.

**Proof.** Let $T$ be a tree of order $n$ such that $cfc(T) = n - t$ for $1 \leq t \leq 2n + 2$. It suffices to show that $\Delta(T) = n - t$. Let $T'$ be a sub tree of $T$ obtained from $T$ by deleting all pendant vertices. Set $E(T') = \{e_1, \ldots , e_r\}$. Consider any edge $e_i$ of $E(T')$. There exist two components $A_i$ and $B_i$ of $T - e_i$ for $1 \leq i \leq r$. Without loss of generality, assume that $e(A_i) \geq e(B_i) \geq 1$ for each $i$ ($1 \leq i \leq r$). Let $B_m$ ($1 \leq m \leq r$) be the maximum component in $\{B_1, \ldots , B_r\}$.

**Claim 1.** $e(B_m) \leq t - 1$.

**Proof of Claim 1:** Assume, to the contrary, that $e(B_m) \geq t$. We define an edge-coloring of $T$ as follows: color the edges of $A_i$ and $e_i$ with distinct colors, then color the edges of $B_i$ with distinct colors that assigned to the edges of $A_i$. Note that $A_i$ and $B_i$ are conflict-free connected, and the color assigned to $e_i$ is used only once. Thus, it is easy to see that this coloring is a conflict-free connection coloring, and hence $cfc(T) \leq n - 1 - t$, a contradiction.

**Claim 2.** For each pair of $B_p$ and $B_q$, $V(B_p) \cap V(B_q) = \emptyset$ or $V(B_p) \subseteq V(B_q)$ or $V(B_q) \subseteq V(B_p)$.

**Proof of Claim 2:** Note that $T - e_p - e_q$ has three components, say $X$, $Y$, and $Z$. If there exists one of $B_p$, $B_q$ such that it contains two components of $T - e_p - e_q$, then without loss of generality, we assume $B_q = X \cup e_p \cup Y$. Then $A_q = Z$ and $e(Z) \geq e(B_q)$, and hence $\{A_p, B_p\} = \{X, Y \cup e_q \cup Z\}$. Since $e(A_p) \geq e(B_p)$ and $e(Y \cup e_q \cup Z) > e(Z) \geq e(B_q) > e(X)$, it follows that $X = B_p$, and hence $V(B_p) \subseteq V(B_q)$. If both $B_p$ and $B_q$ have the property that each of them contains only one component of $T - e_p - e_q$, then $B_p \cap B_q = \emptyset$.

Let $H = \bigcup_{1 \leq i \leq r} B_i$ be a subgraph of $T$. It follows from Claim 2 that $B_m$ is also a maximum component of $H$.

Let $F$ be a subgraph obtained from $T$ by deleting the edges of $H$, and then deleting isolated vertices. Then we have the following claim.
Lemma 2.2, which is impossible. Therefore, \( e_k \notin E(T') \). Since \( T - e_k \) has two components \( A_k \) and \( B_k \), it follows that some edges of \( F \) are contained in \( B_k \), which is impossible. Thus, \( F \) is a star. 

Claim 4. \( e(H) = t - 1 \).

Proof of Claim 4: At first, we show that \( e(H) \leq t - 1 \). By contradiction, assume that \( e(H) \geq t \). Since \( e(H) \geq t \), one may take \( t \) edges of \( H \) such that the edges of \( B_m \) must be chosen. Let \( D \) be the subgraph of \( H \) induced on the set of these \( t \) edges, and let \( D_1, \ldots, D_d \) be the components of \( D \), in which \( D_1 = B_m \). Clearly, \( D_1 \) is a maximum component of \( D \). Note that for each \( D_i \) (\( 2 \leq i \leq d \)), there exists one \( B_{a_i} \) such that \( D_i \subseteq B_{a_i} \). Since \( t = e(D) = e(D_1) + \cdots + e(D_d) \) and \( e(D_i) \geq 1 \) for \( 1 \leq i \leq d \), it follows that \( e(B_{a_i}) = e(D_i) \leq t - (d - 1) = t - d + 1 \), and hence \( e(B_{a_i}) = e(D_i) \leq n - 1 - t - d \) for \( n \geq 2t + 2 \). Next, we will provide a conflict-free connection coloring of \( T \) with at most \( n - t - 1 \) colors.

Step 1. Color the edges of \( T \setminus \{ D \cup \{ e_{a_1} \cup \ldots \cup e_{a_d} \} \} \) with \( n - 1 - t - d \) distinct colors \( 1, \ldots, n - 1 - t - d \), and color the edges of \( \{ e_{a_1}, \ldots, e_{a_d} \} \) with \( d \) fresh colors \( n - t - d, \ldots, n - 1 - t \).

Step 2. Color the edges of \( D = D_1 \cup \ldots \cup D_d \) with used colors. Color each edge of \( D_1 \) with distinct colors that are assigned to the edges in \( T \setminus \{ D \cup \{ e_{a_1} \cup \ldots \cup e_{a_d} \} \} \). For any other component \( D_i \) (\( 2 \leq i \leq d \)) of \( D \), \( e(D_i) = e(B_{a_i}) - e(B_{a_i} \setminus D_i) \leq e(B_m) - e(B_{a_i} \setminus D_i) = n - 1 - t - d - e(B_{a_i} \setminus D_i) \). Next, color each edge of \( D_i \) with distinct colors that are assigned to the edges in \( T \setminus \{ D_1 \cup \{ e_{a_1} \cup \ldots \cup e_{a_d} \} \cup (B_{a_i} \setminus D_i) \} \).

In order to prove that this coloring is a conflict-free connection coloring, it suffices to show that for each pair of vertices \( x, y \), there exists a conflict-free path between them. Note that \( D_i \) is conflict-free connected, the edges of \( E(T) \setminus E(D) \) are colored with distinct colors, and the color assigned to each edge of \( \{ e_{a_1} \cup \ldots \cup e_{a_d} \} \) is used only once under this coloring. Thus, we only need to consider the case that \( x, y \) are in distinct components of \( D \). Without loss of generality, assume that \( x \in D_i \) and \( y \in D_j \), where \( 1 \leq i \neq j \leq d \). The edge assigned the unique color on the conflict-free path between them is \( e_{a_i} \) (or \( e_{a_j} \)). Thus, \( cfc(T) = n - t - 1 \), a contradiction. Thus, \( e(H) \leq t - 1 \). By Claim 2, we have \( F = K_{1, n - 1 - e(H)} \). If \( e(H) \leq t - 2 \), then \( \Delta(T) \geq \Delta(F) = n - 1 - e(H) \geq n - t + 1 \), and hence \( cfc(T) \geq \Delta(T) \geq n - t + 1 \) by Lemma 2.2, which is impossible. Therefore, \( e(H) = t - 1 \).

Let \( v \) be a non-leaf vertex of \( F \) with \( d_F(v) = n - 1 - e(H) = n - t \). Since \( n \geq 2t + 2 \), it follows that \( d_F(v) = n - t \geq t + 2 \geq 3 \). We claim that \( d(v) = n - t \). Assume, to the
contrary, that there exists an edge $f$ incident with $v$ such that $f \in E(H)$. Then there exists some $B'$ satisfying $f \in E(B')$, and hence the edges of $F$ are contained in $B'$, which is impossible. Thus, $d(v) = n - t \geq t + 2$. Consider any leaf $u$ of $F$. Since $u$ is not adjacent to the other leaves of $F$, we have $d(u) \leq n - 1 - (n - t - 1) = t < d(v)$. Consider any vertex $w$ of $V(T) \setminus V(F)$. It is adjacent to at most one vertex of $F$; otherwise, there exists a cycle in $T$, a contradiction. Thus, $d(u) \leq n - 1 - (n - t) = t - 1 < d(v)$. As a result, $\Delta(T) = d(v) = n - t$.

**Lemma 3.3** Let $T$ be a tree of order $n$, and let $t$ be a natural number such that $t \geq 1$ and $n \geq 2t + 2$. If $\Delta(T) = n - t$, then $cfc(T) = n - t$.

*Proof.* For convenience, we still use the notation in Lemma 3.2. Since $\Delta(T) = n - t$, then it follows from Lemma 2.2 that $cfc(T) \geq n - t$. Let $cfc(T) = n - t + u = n - (t - u)$ where $0 \leq u \leq t - 1$. It is sufficient to show that $u = 0$. By Lemma 3.2, we have that $e(H) = t - u - 1$ and $F = K_{1, n - t + u}$. Note that $\Delta(F) = n - t + u$ and so $\Delta(T) \geq \Delta(F)$, but $\Delta(T) = n - t$. Thus, $u = 0$. The proof is complete.

**4 Graphs with given small or large cfc numbers**

We first give sharp lower and upper bounds of $cfc(G)$ for a connected graph $G$.

**Proposition 4.1** Let $G$ be a connected graph of order $n$. Then

$$1 \leq cfc(G) \leq n - 1.$$ 

*Proof.* The lower bound is trivial. For the upper bound, we assign distinct colors to the edges of a given spanning tree of $G$, and color the remaining edges with one used colors. Since this coloring is a conflict-free connection coloring of $G$, it follows that $cfc(G) \leq n - 1$.

Graphs with $cfc(G) = 1$ can be easily characterized.

**Proposition 4.2** If $G$ is a connected graph of order $n$, then $cfc(G) = 1$ if and only if $G = K_n$.

*Proof.* It is obvious that $cfc(K_n) = 1$. Conversely, let $G$ be a connected graph with $cfc(G) = 1$. If $diam(G) \geq 2$, then we let $x, y$ be two vertices with $d(x, y) = diam(G)$. Since the conflict-free path between $x$ and $y$ needs at least two
colors, it follows that $cfc(G) \geq 2$, a contradiction. Thus, $diam(G) = 1$, which implies $G = K_n$. 

Next, we present a sufficient and necessary condition for a graph $G$ with $cfc(G) = 2$ under the case $diam(G) \geq 3$.

**Lemma 4.3** [27] Let $G$ be a connected graph with connected complement $\overline{G}$. Then

(i) if $diam(G) > 3$, then $diam(\overline{G}) = 2$,

(ii) if $diam(G) = 3$, then $\overline{G}$ has a spanning subgraph which is a double star.

**Theorem 4.4** Let $G$ be a connected noncomplete graph such that $diam(G) \geq 3$. Then $cfc(G) = 2$ if and only if there exist at most two cut-edges incident with any vertex of $G$.

**Proof.** Suppose $cfc(G) = 2$. Assume, to the contrary, that there exist at least three cut-edges incident with some vertex of $G$. In order to make $G$ conflict-free connected, these cut-edges need to be assigned three distinct colors. Thus, $cfc(G) \geq 3$, a contradiction. For the converse, if $G$ is 2-edge-connected, then $cfc(G) = 2$ by Corollary 2.5. Next, we only consider the case that $G$ has at least one cut-edge. If $n = 4, 5$, then it is easy to see that the result holds. Next, assume that $n \geq 6$. In order to complete our proof, we distinguish the following two cases.

**Case 1.** $diam(G) > 3$.

It follows from Lemma 4.3 that $diam(G) = 2$. Take a vertex $v$ of maximum degree in $G$, let $N_1(v) = \{u_1, \ldots, u_a\}$ denote the neighborhood of $v$, and $N_2(v) = V \setminus (N_1(v) \cup v) = \{w_1, \ldots, w_b\}$, where $a \geq 1$ and $a + b = n - 1$. Consider any vertex $w_i$ $(1 \leq i \leq b)$ in $N_2(v)$. Suppose $d(w_i) = 1$, and let $u_{p_i}$ be the unique neighbour vertex. Since $diam(G) = 2$, it follows that $u_{p_i}$ is adjacent to all other vertices in $N_1(v)$. Let $W = \{w_i | d(w_i) = 1\}$, and $U = \{u_{p_i} | u_{p_i}$ is the unique neighbor of $w_i\}$. Since $diam(G) = 2$, it follows that $|W| \leq 2$ and $|U| \leq 1$. Note that each vertex of $N_2(v) \setminus W$ is adjacent to at least two vertices of $N_1(v)$. If $W = \emptyset$, then it is easy to see that $cfc(G) = 2$. Suppose $W \neq \emptyset$. It follows that $G[V \setminus W]$ is 2-edge-connected. Let $c'$ be the conflict-free connection coloring of $G[V \setminus W]$ with two colors 1 and 2. Then $c'$ can be extended to a conflict-free connection coloring $c$ of $G$ by assigning distinct colors to cut-edges incident with the vertex in $U$ (if these edges exist). It is easy to see that $G$ is conflict-free connected under this coloring $c$, and thus $cfc(G) = 2$.

**Case 2.** $diam(G) = 3$.
It follows from Lemma 4.3 that $G$ has a spanning subgraph $T$ which is a double star. We first present the following claim.

**Claim** If $e$ is a cut-edge of $G$, then $e \in E(T)$.

**Proof of Claim:** Assume, to the contrary, that $e$ is a cut-edge of $G$ and $e \notin E(T)$. Since $T$ is a spanning tree of $G - e$, it follows that $G - e$ is connected, which is impossible.

Let $u, v$ be two non-leaf vertices of $T$, and let $A = \{u\} \cup (N_T(u) \setminus \{v\}) = \{u, u_1, \ldots, u_a\}$ and $B = \{v\} \cup (N_T(v) \setminus \{u\}) = \{v, v_1, \ldots, v_b\}$, where $a + b = n - 2$.

Let $H_1 = G[A]$ and $H_2 = G[B]$.

**Case 2.1.** $uv$ is a cut-edge of $G$.

It follows that each of $u, v$ is incident with at most one cut-edge in $H_1$, $H_2$, respectively. Note that $C(G)$ has a unique path whose order is at most 4, it follows from Lemma 2.6 and Proposition 2.8 that $cfc(G) = 2$.

**Case 2.2.** $uv$ is not a cut-edge of $G$.

It follows that there exists an edge $e$ other than $uv$ in $E[A, B]$. Since each of $u, v$ is incident with at most two cut-edges in $H_1$, $H_2$, respectively. Let $H'_i$ be the resulting graph obtained by deleting pendent vertices from $H_i$ (if these vertices exists) for $i = 1, 2$. Note that $H'_i$ is 2-edge-connected. Then $cfc(H_i) = h(H_i) = 2$ by Proposition 2.8. Let $c_i$ be the conflict-free connection coloring of $H_i$ with two colors 1 and 2 for $i = 1, 2$. Then $c_1$ and $c_2$ can be extended to a conflict-free connection coloring $c$ of $G$ by coloring the edge $uv$ with color 1, the edge $e$ with color 2. It is easy to see that $G$ is conflict-free connected under this coloring $c$, and thus $cfc(G) = 2$.

**Theorem 4.5** Let $G$ be a connected graph of order $n$ ($n \geq 2$). Then $cfc(G) = n - 1$ if and only if $G = K_{1,n-1}$.

**Proof.** The sufficiency is trivial, and so we only give the proof of the necessity. Suppose that $cfc(G) = n - 1$. We claim that $G$ is a tree. Assume, to the contrary, that $G$ is not a tree. Then $G$ contains a cycle, and so there exists a nontrivial block containing this cycle. By Lemma 2.9, we have $cfc(G) \leq n - 2$, which is impossible.

In order to complete our proof, it is sufficient to show that $diam(G) = 2$. If this is not the case, then $diam(G) \geq 3$. Let $P$ be the path of length $diam(G)$. From Lemma 2.11 $cfc(P) = \lceil \log_2(diam(G) + 1) \rceil \leq diam(G) - 1$. From Proposition 2.12 we have $cfc(G) \leq cfc(P) + n - 1 - diam(G) \leq n - 2$, a contradiction.

For a nontrivial graph $G$ for which $G + uv \cong G + xy$ for every two pairs $\{u, v\}$ and $\{x, y\}$ of nonadjacent vertices of $G$, the graph $G + e$ is obtained from $G$ by adding the edge $e$ joining two nonadjacent vertices of $G$. 11
Theorem 4.6 Let $G$ be a connected graph of order $n$ ($n \geq 3$). Then $cfc(G) = n - 2$ if and only if $G$ is a tree with $\Delta(G) = n - 2$ for $n \geq 4$, or $G \in \{K_3, K_{1,3} + e, K_{2,2}, K_{2,2} + e, P_5\}$.

Proof. If $G$ is a tree with $\Delta(G) = n - 2$, then it follows from Lemma 2.2 and Theorem 1.5 that $cfc(G) = n - 2$. From Lemmas 2.1 and 2.3 and Proposition 4.2 we have $cfc(G) = n - 2$ if $G \in \{K_3, K_{1,3} + e, K_{2,2}, K_{2,2} + e, P_5\}$. Thus, it remains to verify the converse. Let $G$ be a connected graph with $cfc(G) = n - 2$. If $3 \leq n \leq 5$, then it is easy to verify that $G$ is a tree of order 5 with $\Delta(G) = 3$, or $G \in \{K_3, P_4, K_{1,3} + e, K_{2,2}, K_{2,2} + e, P_5\}$. From now on, we assume $n \geq 6$. In order to prove our result, we present the following claim.

Claim. $G$ is a tree.

Proof of Claim: Assume, to the contrary, that $G$ is not a tree. Then $G$ contains a cycle, and so there exists a nontrivial block containing this cycle. If $G$ has at least two nontrivial blocks, then it follows from Lemma 2.9 that $cfc(G) \leq n - 4$, which is impossible. Suppose that there exists only one nontrivial block $B$. Let $T_1, T_2, \ldots, T_s$ be the components of $C(G)$. Note that $E(G) = E(B) \cup E(T_1) \cup E(T_2) \cup \cdots \cup E(T_s)$ and $n = |V(G)| = |V(B)| + |V(T_1)| + |V(T_2)| + \cdots + |V(T_s)| - (r + s - 1)$. If $|V(B)| \geq 4$, then it follows from Corollary 2.10 that $cfc(G) \leq n + 1 - |V(B)| \leq n - 3$, a contradiction. Suppose $|V(B)| = 3$. We assign $|V(T_1)| - 1 + |V(T_2)| - 1 + \cdots + |V(T_s)| - 1$ distinct colors $\{1, \ldots, |V(T_1)| - 1 + |V(T_2)| - 1 + \cdots + |V(T_s)| - 1\}$ to the edges of $T_1 \cup T_2 \cup \cdots \cup T_s$, and then color the edges of $B$ with three used colors $\{1, 2, 3\}$. It is easy to check out that this coloring is a conflict-free connection coloring of $G$, and so $cfc(G) \leq |V(B)| - 3 + |V(T_1)| - 1 + |V(T_2)| - 1 + \cdots + |V(T_s)| - 1 = n - 3$, which is impossible.

Since $n \geq 6$, it follows from Lemma 3.2 that $\Delta(G) = n - 2$. We complete the proof.

A graph is unicyclic if it is connected and contains exactly one cycle. Note that $K_{1,n-1} + e$ is unicyclic, whose cycle is a triangle. Next, we consider another class of unicyclic graphs, whose cycles are also a triangle. Let $S_{a,n-a}$ be a tree with diameter 3, such that the two non-leaf vertices have degree $a$ and $n - a$, the unicyclic graph $U_n$ is obtained from $S_{3,n-3}$ by adding an edge joining the two neighbouring leaves of the vertex of degree 3. Next, we study the conflict-free connection numbers of $K_{1,n-1} + e$ or $U_n$.

Lemma 4.7 If $G$ is $K_{1,n-1} + e$ or $U_n$ with $n \geq 5$, then $cfc(G) = n - 3$.

Proof. If $G$ is $K_{1,n-1} + e$ or $U_n$ with $n \geq 5$, then $G$ has the property that $h(G) \geq 2$.
and there exists a unique component $T$ of $C(G)$. Noticing that $T = K_{1, n-3}$, then $cfc(G) = h(G) = n - 3$ by Proposition 2.8 and Theorem 4.5.

**Theorem 4.8** Let $G$ be a connected graph of order $n \geq 4$. Then $cfc(G) = n - 3$ if and only if $G$ satisfies one of the following conditions.

(i) $G$ is a tree with $\Delta(G) = n - 3$, where $n \geq 6$,

(ii) $G = K_{1, n-1} + e$, where $n \geq 5$,

(iii) $G = U_n$, where $n \geq 5$,

(iv) $G$ is a 2-edge-connected and non-complete graph of order 5,

(v) $G \in \{K_4, P_6, G_1, G_2, G_3, G_4, G_5, G_6\}$, where $G_1, G_2, G_3, G_4, G_5, G_6$ are showed in Fig. 1.

**Proof.** If $G$ is a tree with $\Delta(G) = n - 3$ for $n \geq 6$, then $cfc(G) = n - 3$ by Lemma 2.2, Theorems 4.5 and 4.6. If $G = K_{1, n-1} + e$ or $U_n$ with $n \geq 5$, then $cfc(G) = n - 3$ by Lemma 4.7. If $G$ is a 2-edge-connected and non-complete graph of order 5, then it follows from Corollary 2.5 that $cfc(G) = n - 3 = 2$. Clearly, $cfc(K_4) = cfc(P_6) = n - 3$, and each graph $G_i$ in Fig. 1 satisfies $cfc(G_i) = n - 3$ for $1 \leq i \leq 6$. Conversely, let $G$ be a connected graph of order $n \geq 4$ such that $cfc(G) = n - 3$. If $n = 4, 5$, then $G$ is a 2-edge-connected and noncomplete graph of order 5 or $G \in \{K_4, K_{1, 4} + e, U_5, G_1, G_2, G_3, G_4, G_5\}$. From now on, we assume $n \geq 6$. We distinguish the following two cases to show this theorem.

**Case 1.** $G$ is a tree of order $n$ with $cfc(G) = n - 3$.

For $n = 6$, if $\Delta(G) = 2$, then $G = P_6$ by Lemma 2.1 if $\Delta(G) = 3$, the result holds trivially; if $\Delta(G) \geq 4$, then it follows from Lemma 2.2 that $cfc(G) \geq 4 > n - 3$, which is impossible. Suppose $n = 7$. If $\Delta(G) = 2$, then $G = P_7$, and so $cfc(G) = 3$ by Lemma 2.1, a contradiction. If $\Delta(G) = 3$, then $G$ is one of five trees in Fig. 2.
Figure 2: Five trees of order 7 with maximum degree 3

Note that the first tree in Fig. 2 has conflict-free connection number 4, and the others have conflict-free connection number 3. Thus, $G$ is the first tree in Fig. 2, which is exactly $G_6$ in Fig. 1. If $\Delta(G) = 4$, then the result always follows. If $\Delta(G) \geq 5$, then $cf(G) \geq 5 > n - 3$ by Lemma 2.10, which is again impossible. If $n \geq 8$, then $\Delta(G) = n - 3$ by Lemma 3.2.

**Case 2.** $G$ contains a cycle.

Since $G$ contains a cycle, it follows that there exists a nontrivial block containing this cycle. Let $T_1, T_2, \ldots, T_s$ be the components of $C(G)$ such that $h(G) = cf(T_1) \geq cf(T_2) \geq \cdots \geq cf(T_s)$. If $G$ has at least two nontrivial blocks, then it follows from Lemma 2.9 that $cf(G) \leq n - 4$, which is impossible. Thus, $G$ has only one nontrivial block $B$. If $|V(B)| \geq 5$, then $cf(G) \leq n + 1 - |V(B)| \leq n - 4$ by Corollary 2.10, a contradiction. Suppose $|V(B)| = 4$. It follows from the proof ofLemma 2.9 that $cf(G) \leq h(G) + 1 \leq |V(T_1)| \leq n - |V(B)| + (1 + s - 1) - 2(s - 1) \leq n - |V(B)| - s + 2 \leq n - 2 - s$. If $s \geq 2$, then $cf(G) \leq n - 4$, which is impossible. If $s = 1$, then $cf(T_1) \leq |V(T_1)| - 1 \leq n - 3 - 1 = n - 4$, and hence $cf(G) = h(G) = cf(T_1) \leq n - 4$ by Proposition 2.8, which is again impossible. Suppose $|V(B)| = 3$. It follows from the proof of Lemma 2.9 that $cf(G) \leq h(G) + 1 \leq |V(T_1)| \leq n - |V(B)| + (1 + s - 1) - 2(s - 1) \leq n - |V(B)| - s + 2 \leq n - 1 - s$. If $s \geq 3$, then $cf(G) \leq n - 4$, which is impossible. Suppose $s = 2$. If $|V(T_2)| \geq 3$, then $h(G) = cf(T_1) \leq |V(T_1)| - 1 \leq n - |V(B)| - |V(T_2)| + 2 - 1 = n - 5$, and so $cf(G) \leq h(G) + 1 \leq n - 4$ by Lemma 2.7, a contradiction. If $|V(T_2)| = 2$, then $|V(T_1)| \geq 3$ and so there exists only one component $T_1$ of $C(G)$ such that $h(G) = cf(T_1) \leq |V(T_1)| - 1 \leq n - 4$. Thus, $cf(G) = h(G) \leq n - 4$ by Proposition 2.8, a contradiction. If $s = 1$, then $|V(T_1)| = n - 2$. We claim that $\Delta(T_1) = n - 3$. Assume, to the contrary, that $\Delta(T_1) \leq n - 4$. We have $cf(G) = h(G) = cf(T_1) \leq n - 4$ by
Proposition 2.8, Theorems 4.5 and 4.6; a contradiction. Thus, \(\Delta(T_1) = n - 3\), which implies that \(G = K_{1,n-1} + e\) or \(G = U_n\). The proof is complete.

Let \(n\) be a natural number with \(n \geq 7\). We now define a sequence of graph classes, which will be used later.

- Let \(U_n^1\) be a graph obtained from \(U_{n-1}\) by adding a pendent edge to a vertex of degree 2 of \(U_{n-1}\).
- Let \(U_n^2\) be a graph obtained from \(U_{n-1}\) by adding a pendent edge to a vertex of degree 1 of \(U_{n-1}\).
- Let \(U_n^3\) be a graph obtained from \(K_{1,n-4}\) and \(K_3\) by joining a leaf vertex of \(K_{1,n-4}\) and a vertex of \(K_3\).
- Let \(U_n^4\) be a graph obtained from \(K_{1,n-2} + e\) by adding a pendent edge to a vertex of degree 2 of \(K_{1,n-2} + e\).
- Let \(U_n^5\) be a graph obtained from \(K_{1,n-2} + e\) by adding a pendent edge to a leaf vertex of \(K_{1,n-2} + e\).
- Let \(U_n^6\) be obtained from \(S_{4,n-4}\) by adding an edge joining the two neighboring leaves of the vertex of degree 4.
- Let \(W_n^1\) be a graph obtained from \(C_4\) and \(K_{1,n-4}\) by identifying a vertex of \(C_4\) and a leaf vertex of \(K_{1,n-4}\).
- Let \(W_n^2\) be a graph obtained from \(K_4\) and \(K_{1,n-4}\) by identifying a vertex of \(K_4\) and a leaf vertex of \(K_{1,n-4}\).
- Let \(W_n^3\) or \(W_n^4\) be a graph obtained from \(W_n^2\) by deleting an edge of \(K_4\).
- Let \(W_n^5\) be a graph obtained from \(C_4\) and \(K_{1,n-4}\) by identifying a vertex of \(C_4\) and the non-leaf vertex of \(K_{1,n-4}\).
- Let \(W_n^6\) be a graph obtained from \(K_4\) and \(K_{1,n-4}\) by identifying a vertex of \(K_4\) and the non-leaf vertex of \(K_{1,n-4}\).
- Let \(W_n^7\) or \(W_n^8\) be a graph obtained from \(W_n^6\) by deleting an edge of \(K_4\).

The following lemma is a preparation for the proof of Theorem 4.10.

**Lemma 4.9** If \(G \in \{U_n^1, \ldots, U_n^6, W_n^1, \ldots, W_n^8\}\) for \(n \geq 7\), then \(cfc(G) = n - 4\).
Proof. Note that each of these graph classes has the property that $h(G) \geq 2$ and there exists a unique component $T$ of $C(G)$ such that $cfc(T) = h(G)$. Since $T = K_{1,n-4}$, it follows that $cfc(G) = h(G) = n - 4$ by Proposition 2.8 and Theorem 4.5.

Theorem 4.10 Let $G$ be a connected graph of order $n \geq 5$. Then $cfc(G) = n - 4$ if and only if $G$ is one of the following cases.

(i) $G$ is a tree with $\Delta(G) = n - 4$ except for $G = G_6$ in Fig. 1, where $n \geq 7$.

(ii) $G$ is one of the 14 graph classes $\{U_n^1, \ldots, U_n^6, W_n^1, \ldots, W_n^8\}$ for $n \geq 7$.

(iii) $G$ is a connected non-complete graph of order 6 such that $G$ contains a cycle.

(iv) $G \in \{K_5, P_7, H_1, \ldots, H_5, H_{11}, H_{12}\}$, where $H_1, \ldots, H_5, H_{11}, H_{12}$ are shown in Fig. 3.

Proof. If $G$ is a tree with $\Delta(G) = n - 4$ except for $G = G_6$ in Fig. 1, then $cfc(G) = n - 4$ by Lemma 2.2. Theorems 4.3, 4.6 and 4.8 for $n \geq 7$. If $G \in \{U_n^1, \ldots, U_n^6, W_n^1, \ldots, W_n^8\}$ for $n \geq 7$, then $cfc(G) = n - 4$ by Lemma 4.9. Suppose that $G$ is a connected and non-complete graph of order 6. We first assume that $G$ is a tree. If $\Delta(G) = 2$, then $cfc(G) = 3 \neq n - 4$, a contradiction. If $\Delta(G) \geq 3$, then it follows from Lemma 2.2 that $cfc(G) \geq 3 > n - 4$. Next, we deal with the case that $G$ has a cycle. Let $C$ be the longest cycle. It is not hard to verify that $cfc(G) = 2$ in any case $|V(C)| = 3, 4, 5, \text{ or } 6$. It is clear that $cfc(K_5) = cfc(P_7) = n - 4$, and $cfc(H_i) = 4 = n - 4$ for $1 \leq i \leq 5$ and $cfc(H_{11}) = 5 = n - 4$ and $cfc(H_{12}) = cfc(H_{13}) = 3 = n - 4$ in Fig. 3. Conversely, we let $G$ be a connected graph of order $n \geq 5$ with $cfc(G) = n - 4$. If $n = 5$, then $G = K_5$. If $n = 6$, then we can obtain that $G$ is neither a complete graph nor a tree. Next, we assume $n \geq 7$. We distinguish the following two cases to show this theorem.

Case 1. $G$ is a tree of order $n$ with $cfc(G) = n - 4$.

If $n \geq 10$, then $\Delta(G) = n - 4$ by Lemma 3.2. We only need to consider the case $7 \leq n \leq 9$. If $\Delta(G) = 2$, then it follows from Lemma 2.1 that $G = P_7$. Suppose $\Delta(G) = 3$. If $n = 7$, then it is easy to check that $G$ is not $G_6$ shown in Fig. 1. If $n = 8$, then $G$ is one of the first 10 trees shown in Fig. 3. One can verify that $cfc(H_i) = 4$ for $1 \leq i \leq 5$ and $cfc(H_i) = 3$ for $6 \leq i \leq 10$. Thus, $G \in \{H_1, \ldots, H_5\}$ shown in Fig. 3 in this case. If $n = 9$, then $cfc(G) \leq 4$, a contradiction. Suppose $\Delta(G) = 4$. If $n = 7$, then it follows from Lemma 2.2 that $cfc(G) \geq \Delta(G) = 4 > n - 4$,
which is impossible. If \( n = 8 \), then \( G \) is one of 7 trees shown in Fig. 4, and hence the result clearly holds. If \( n = 9 \), then \( G = H_{11} \) shown in Fig. 3. Suppose \( \Delta(G) = 5 \). If \( 7 \leq n \leq 8 \), then \( \text{cfc}(G) \geq \Delta(G) = 5 > n - 4 \) by Lemma 2.2, which is again impossible. If \( n = 9 \), then it is not hard to see that the result follows. If \( \Delta(G) \geq 6 \), then it follows from Lemma 2.2 that \( \text{cfc}(G) \geq \Delta(G) = 6 > n - 4 \), a contradiction.

Case 2. \( G \) contains a cycle.
Since $G$ contains a cycle, it follows that there exists a nontrivial block containing this cycle. Let $T_1, T_2, \ldots, T_s$ be the components of $C(G)$ such that $h(G) = cfc(T_1) \geq cfc(T_2) \geq \cdots \geq cfc(T_s)$. If $G$ has at least three nontrivial blocks, then it follows from Lemma 2.9 that $cfc(G) \leq n - 6$, which is impossible. Suppose $G$ has two nontrivial blocks $B_1$ and $B_2$ with $|V(B_1)| \geq |V(B_2)| \geq 3$. It follows from the proof of Lemma 2.9 that $cfc(G) \leq h(G) + 1 \leq |V(T_1)| \leq n - |V(B_1)| - |V(B_2)| + (2 + s - 1) - 2(s - 1) \leq n - |V(B_1)| - |V(B_2)| - s + 3$. If $|V(B_1)| \geq 4$, then $cfc(G) \leq n + 2 - |V(B_1)| - |V(B_2)| \leq n - 5$, a contradiction. Suppose that $|V(B_1)| = |V(B_2)| = 3$. If $s \geq 2$, then $cfc(G) \leq n - |V(B_1)| - |V(B_2)| - s + 3 \leq n - 5$, a contradiction. If $s = 1$, then $cfc(G) = h(G) = cfc(T_1) \leq |V(T_1)| - 1 = n - 4 - 1 \leq n - 5$ by Proposition 2.8, a contradiction. Next we only need to consider that $G$ has only one nontrivial block $B$. If $|V(B)| \geq 6$, then $cfc(G) \leq n + 1 - |V(B)| \leq n - 5$ by Corollary 2.10, a contradiction. Suppose $|V(B)| = 5$. It follows from the proof of Lemma 2.9 that $cfc(G) \leq h(G) + 1 \leq |V(T_1)| \leq n - |V(B)| + (1 + s - 1) - 2(s - 1) \leq n - |V(B)| - s + 2 \leq n - 3 - s$. If $s \geq 2$, then $cfc(G) \leq n - 5$, which is impossible. If $s = 1$, then $cfc(T_1) \leq |V(T_1)| - 1 \leq n - 4 - 1 = n - 5$, and hence $cfc(G) = h(G) = cfc(T_1) \leq n - 5$ by Proposition 2.8, which is again impossible. Suppose $|V(B)| = 4$. It follows from the proof of Lemma 2.9 that $cfc(G) \leq h(G) + 1 \leq |V(T_1)| \leq n - |V(B)| + (1 + s - 1) - 2(s - 1) \leq n - |V(B)| - s + 2 \leq n - 2 - s$. If $s \geq 3$, then $cfc(G) \leq n - 5$, which is impossible. Suppose $s = 2$. If $|V(T_2)| \geq 3$, then $h(G) = cfc(T_1) \leq |V(T_1)| - 1 \leq n - |V(B)| - |V(T_2)| + 2 - 1 = n - 6$, and so $cfc(G) \leq h(G) + 1 = 5$, a contradiction. If $|V(T_2)| \geq 2$, then $|V(T_1)| \geq 3$ and so $cfc(G) = h(G) = cfc(T_1) \leq |V(T_1)| - 1 \leq n - 5$ by Proposition 2.8, a contradiction. If $s = 1$, then $|V(T_1)| = n - 3$. Note that $cfc(G) = h(G) = cfc(T_1) = n - 4$ by Proposition 2.8, and hence it follows from Theorem 4.5 that $\Delta(T_1) = n - 4$. As a result, $G$ is a graph class in $\{W_n^1, \ldots, W_n^3\}$. Suppose $|V(B)| = 3$. Note that $s \leq 3$. Suppose $s = 3$. As $n \geq 7$, we have $h(G) \geq 2$. If there exist two components $T_1, T_2$ such that $cfc(T_1) = cfc(T_2) = h(G) \geq 2$, then $cfc(G) \leq h(G) + 1 \leq |V(T_1)| = n - |V(B)| - |V(T_2)| - |V(T_3)| + (1 + 1) \leq n - 5$, a contradiction. If there exists only one component $T_1$ such that $cfc(T_1) = h(G) \geq 2$, then $cfc(G) = h(G) \leq |V(T_1)| - 1 \leq n - |V(B)| - |V(T_2)| - |V(T_3)| + (1 + 3 - 1) - 1 \leq n - 5$ by Proposition 2.8, a contradiction. Suppose $s = 2$. As $n \geq 7$, we have $h(G) \geq 2$. Without loss of generality, assume $cfc(T_1) \geq cfc(T_2)$. If $cfc(T_1) = cfc(T_2) = 3$, then $cfc(G) \leq h(G) + 1 \leq |V(T_1)| = n - |V(B)| - |V(T_2)| + (1 + 2 - 1) \leq n - 5$, a contradiction. If $cfc(T_1) = cfc(T_2) = 2$, then the unique graph $H_{12}$ satisfies $cfc(H_{12}) = n - 4$. If $cfc(T_1) > cfc(T_2) \geq 2$, then $cfc(G) = h(G) \leq |V(T_1)| - 1 = n - |V(B)| - |V(T_2)| + (1 + 2 - 1) - 1 \leq n - 5$ by Proposition 2.8, a contradiction. If $cfc(T_1) > cfc(T_2) = 1$, then $T_1$ is a star by
Theorem 4.5, and hence $G$ is $U_1^2$ or $U_4^4$. Suppose $s = 1$. Since $|V(T_1)| = n - 2$, and $cfc(T_1) = h(G) = cfc(G) = n - 4$ by Proposition 2.8, it follows from Theorem 4.6 that $T_1$ is a tree with $\Delta(T) = n - 4$ for $n \geq 6$ or $T = P_5$. Thus, $G \in \{U_2^2, U_3^3, U_5^5, U_6^6\}$ or $G = H_{13}$. We complete the proof.

5 Nordhaus-Gaddum-type theorem

Note that if $G$ is a connected graph of order $n$, then $m(G) \geq n - 1$. If both $G$ and $\overline{G}$ are connected, then $n \geq 4$, since

$$2(n - 1) \leq m(G) + m(\overline{G}) = m(K_n) \leq \frac{n(n - 1)}{2}.$$ 

In the sequel, we always assume that all graphs have at least 4 vertices, and both $G$ and $\overline{G}$ are connected.

**Theorem 5.1** Let $G$ and $\overline{G}$ be connected graphs of order $n$ ($n \geq 4$). Then $4 \leq cfc(G) + cfc(\overline{G}) \leq n$ and $4 \leq cfc(G) \cdot cfc(\overline{G}) \leq 2(n - 2)$. Moreover,

(i) $cfc(G) + cfc(\overline{G}) = n$ or $cfc(G) \cdot cfc(\overline{G}) = 2(n - 2)$ if and only if one of $G$ and $\overline{G}$ is a tree with maximum degree $n - 2$ or a $P_5$,

(ii) the lower bounds are sharp.

We proceed the proof of Theorem 5.1 by Proposition 5.2, Theorems 5.3 and 5.6.

First, we study Nordhaus-Gaddum-type problem for conflict-free connection number of graphs $G$ for the case both $G$ and $\overline{G}$ are 2-edge-connected. It is known that if $G$ is a 2-edge-connected graph of order 4, then $\overline{G}$ is disconnected. Thus, we need to assume that $n \geq 5$ for the case that $G$ is a 2-edge-connected in the following.

**Proposition 5.2** Let $G$ be a connected graph of order $n$ ($n \geq 5$). If both $G$ and $\overline{G}$ are 2-edge-connected graphs, then $cfc(G) + cfc(\overline{G}) = 4$ and $cfc(G) \cdot cfc(\overline{G}) = 4$.

**Proof.** Since both $G$ and $\overline{G}$ are connected, it follows that $G$ is neither an empty graph nor a complete graph. Thus, $G$ and $\overline{G}$ are 2-edge-connected and noncomplete graphs, which implies $cfc(G) = 2$ and $cfc(\overline{G}) = 2$ by Corollary 2.5. The results hold.

Secondly, we turn to investigate the Nordhaus-Gaddum-type problem for graphs $G$ such that $G$ is 2-edge-connected and $\lambda(\overline{G}) = 1$. 

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Theorem 5.3 Let $G$ be a 2-edge-connected graph of order $n$ ($n \geq 5$) and $\lambda(G) = 1$. Then $4 \leq cfc(G) + cfc(\overline{G}) \leq n$ and $4 \leq cfc(G) \cdot cfc(\overline{G}) \leq 2(n - 2)$. Moreover,

(i) $cfc(G) + cfc(\overline{G}) = n$ or $cfc(G) \cdot cfc(\overline{G}) = 2(n - 2)$ if and only if $\overline{G} = P_5$,

(ii) the lower bounds are sharp.

Proof. We proceed our proof by the following three claims.

Claim 1. $4 \leq cfc(G) + cfc(\overline{G}) \leq n$ and $4 \leq cfc(G) \cdot cfc(\overline{G}) \leq 2(n - 2)$.

Since both $G$ and $\overline{G}$ are connected, it follows that $G$ is neither an empty graph nor a complete graph. Thus, $G$ is a 2-edge-connected and noncomplete graph, which implies that $cfc(G) = 2$ and $cfc(\overline{G}) \geq 2$ by Corollary 2.5 Propositions 4.1 and 4.2. As a result, the lower bounds clearly hold. It remains to verify the upper bound. Firstly, we claim that $cfc(\overline{G}) \leq n - 2$. Assume, to the contrary, that $cfc(\overline{G}) = n - 1$. It follows that $\overline{G} = K_{1, n - 1}$ by Theorem 4.5, which implies that $G$ has an isolated vertex, a contradiction. Thus, $cfc(\overline{G}) \leq n - 2$, and hence $cfc(G) + cfc(\overline{G}) \leq n$ and $cfc(G) \cdot cfc(\overline{G}) \leq 2(n - 2)$.

Claim 2. $cfc(G) + cfc(\overline{G}) = n$ or $cfc(G) \cdot cfc(\overline{G}) = 2(n - 2)$ if and only if $\overline{G} = P_5$.

Proof of Claim 2: Since $cfc(G) = 2$, it follows that $cfc(G) + cfc(\overline{G}) = n$ or $cfc(G) \cdot cfc(\overline{G}) = 2(n - 2)$ if and only if $cfc(\overline{G}) = n - 2$. In order to complete our proof, it is sufficient to show that $cfc(\overline{G}) = n - 2$ if and only if $\overline{G} = P_5$. Let $\overline{G}$ be a connected graph such that $cfc(\overline{G}) = n - 2$. It follows from Theorem 4.6 that $\overline{G}$ is a tree with $\Delta(\overline{G}) = n - 2$ or $\overline{G} \in \{K_{1,3} + e, K_{2,2}, K_{2,2} + e, P_5\}$. Suppose that $\overline{G}$ is a tree with $\Delta(\overline{G}) = n - 2$. Let $v$ be a vertex of $\overline{G}$ with maximum degree $n - 2$, and $w$ be the unique vertex that is not adjacent to $v$ in $\overline{G}$. Note that $vw$ is a cut-edge of $G$, which contradicts to that $G$ is 2-edge-connected. If $\overline{G} \in \{K_{1,3} + e, K_{2,2}, K_{2,2} + e\}$, then it is easy to see that $G$ is disconnected, a contradiction. Next, we only need to consider the case $\overline{G} = P_5$. It is obtained that $G$ is a 2-edge-connected and noncomplete graph of order 5, and so $cfc(G) = 2$.

Claim 3. The lower bounds in Theorem 5.3 are sharp.

Proof of Claim 3: The following example shows that the lower bounds in Theorem 5.3 are best possible. Suppose that $H$ is a 2-edge-connected and noncomplete graph, and put one pair of nonadjacent vertices $x, y$ of $H$. Let $u, v$ be two new vertices with $u, v \notin V(H)$, and let $\overline{G}$ be a graph such that $V(\overline{G}) = V(H) \cup \{u, v\}$ and $E(\overline{G}) = E(H) \cup ux \cup uy \cup uv$. Clearly, $\lambda(\overline{G}) = 1$ and $G$ is 2-edge-connected.
follows from Corollary 2.5 and Lemma 2.6 that \(cf_c(G) = 2\) and \(cf_c(G) = 2\).

Finally, we discuss the Nordhaus-Gaddum-type problem for conflict-free connection number of graphs \(G\) such that \(\lambda(G) = \lambda(G) = 1\).

The following two lemmas are preparations for the proof of Theorem 5.6.

**Lemma 5.4** [2] A graph \(G\) with \(p\) points satisfies the condition \(\lambda(G) = \lambda(G) = 1\) if and only if \(G\) is a connected graph with a bridge and \(\Delta = p - 2\).

**Lemma 5.5** Let \(G\) be a connected graph of order \(n\) (\(n \geq 4\)). If \(\lambda(G) = 1\) and \(\lambda(G) = 1\), then at least one of \(G\) and \(\overline{G}\) has conflict-free connection number 2.

**Proof.** By contradiction. Suppose that \(cf_c(G) \geq 3\) and \(cf_c(G) \geq 3\). Since \(cf_c(G) \geq 3\), it follows from Lemma 2.7 that \(h(G) \geq 2\). At first, we present the follow claim.

**Claim 1.** \(G\) has at least three cut-edges.

**Proof of Claim 1:** Let \(T\) be the component of \(C(G)\) such that \(cf_c(T) = h(G) \geq 2\). It follows that \(m(T) \geq 2\). Suppose \(m(T) = 2\). It follows that \(h(G) = 2\) and \(cf_c(G) = 3\). By Propositon 2.8, there exists another component \(T'\) of \(C(G)\) such that \(cf_c(T') = h(G) = 2\). Clearly, \(m(T') \geq 2\). Note that each edge of both \(T\) and \(T'\) is a cut-edge. Then \(G\) has at least four cut-edges. Suppose \(m(T) \geq 3\). Since every edge of \(T\) is a cut-edge, it follows that \(G\) has at least three cut-edges.

It is obtained that \(G\) has a cut-edge and a vertex \(v\) of degree \(n - 2\) by Lemma 5.4. Let \(X = \{x_1, \ldots, x_{n-2}\}\) be the neighborhood of \(v\), and \(w\) be the unique vertex that is not adjacent to \(v\). Note that \(w\) is adjacent to at least one vertex in \(X\). Without loss of generality, assume that \(wx_1 \in E(G)\). Let \(T\) be a spanning tree induced on the set of edges \(\{vx_1, \ldots, vx_{n-2}, wx_1\}\) of \(G\). Next, we give another claim.

**Claim 2.** If \(e\) is a cut-edge of \(G\), then \(e \in E(T)\).

**Proof of Claim 2:** Assume, to the contrary, that \(e\) is a cut-edge of \(G\) and \(e \notin E(T)\). Since \(T\) is a spanning tree of \(G\)–\(e\), it follows that \(G\) – \(e\) is connected, which is impossible.

By Claims 1 and 2, there exist at least two cut-edges incident with \(v\) in \(G\), say \(vx_i\) and \(vx_j\) for \(1 \leq i < j \leq n - 2\). If \(i \geq 2\), then \(d(x_i) = d(x_i) = 1\) and \(n \geq 5\). Noticing that \(d_G(x_i) = d_G(x_i) = n - 2\), one can easily obtain that \(\overline{G}[V \setminus v]\) contains \(K_{2,n-3}\) as a subgraph \(H_1\). Let \(H'\) be a spanning subgraph of \(\overline{G}\) obtained from \(H_1\) by joining \(w\) to \(v\). Then \(cf_c(H') = 2\) by Lemma 2.3, which implies that \(cf_c(\overline{G}) \leq cf_c(H') = 2\) by Proposition 2.11, a contradiction. Suppose \(i = 1\). Note that \(wx_1\) is also a cut-edge, and \(d_{\overline{G}}(w) = d_{\overline{G}}(x_j) = n - 2\). If \(n = 4\), then \(G = \overline{G} = P_4\), thus \(cf_c(G) = cf_c(\overline{G}) = 2\).
a contradiction. If \( n = 5 \), then it is easy to see that \( cfc(\overline{G}) = 2 \), a contradiction. If \( n \geq 6 \), then \( \overline{G}[V \setminus \{v, x_1\}] \) contains \( K_{2,n-4} \) as a subgraph \( H_2 \). Let \( H'' \) be a spanning subgraph of \( \overline{G} \) obtained from \( H_2 \) by adding two edges \( uv \) and \( x_jx_1 \). Then \( cfc(H'') = 2 \) by Lemma 2.6 which means that \( cfc(\overline{G}) \leq cfc(H'') = 2 \) by Proposition 2.11 a contradiction.

**Theorem 5.6** Let \( G \) be a connected graph of order \( n \geq 4 \). If \( \lambda(G) = 1 \) and \( \lambda(\overline{G}) = 1 \), then \( 4 \leq cfc(G) + cfc(\overline{G}) \leq n \) and \( 4 \leq cfc(G) \cdot cfc(\overline{G}) \leq 2(n - 2) \). Moreover,

(i) \( cfc(G) + cfc(\overline{G}) = n \) or \( cfc(G) \cdot cfc(\overline{G}) = 2(n - 2) \) if and only if one of \( G \) and \( \overline{G} \) is a tree with maximum degree \( n - 2 \).

(ii) \( cfc(G) + cfc(\overline{G}) = 4 \) or \( cfc(G) \cdot cfc(\overline{G}) = 4 \) if and only if each of \( G \) and \( \overline{G} \) has the property that there exist at most two cut-edges incident with the vertex of maximum degree.

**Proof.** We proceed our proof by the following three claims.

**Claim 1.** \( 4 \leq cfc(G) + cfc(\overline{G}) \leq n \) and \( 4 \leq cfc(G) \cdot cfc(\overline{G}) \leq 2(n - 2) \).

**Proof of Claim 1:** Since both \( G \) and \( \overline{G} \) are connected, it follows that \( G \) is neither an empty graph nor a complete graph. Thus, \( cfc(G) \geq 2 \) and \( cfc(\overline{G}) \geq 2 \) by Propositions 1.1 and 1.2. As a result, the lower bounds trivially hold. For the upper bound, we first claim that \( cfc(G) \leq n - 2 \). Suppose, to the contrary, that \( cfc(G) = n - 1 \). Then \( G = K_{1,n-1} \) by Theorem 1.3. It follows that \( \overline{G} \) has an isolated vertex, a contradiction. With a similar argument, one can show that \( cfc(\overline{G}) \leq n - 2 \). Since at least one of \( G \) and \( \overline{G} \) has conflict-free connection number 2 by Lemma 5.5, then \( cfc(G) + cfc(\overline{G}) \leq n \). \( \square \)

**Claim 2.** \( cfc(G) + cfc(\overline{G}) = n \) or \( cfc(G) \cdot cfc(\overline{G}) = 2(n - 2) \) if and only if one of \( G \) and \( \overline{G} \) is a tree with maximum degree \( n - 2 \).

**Proof of Claim 2:** By Claim 1 and Lemma 5.5 we have that \( cfc(G) + cfc(\overline{G}) = n \) or \( cfc(G) \cdot cfc(\overline{G}) = 2(n - 2) \) if and only if \( cfc(G) = n - 2 \) and \( cfc(\overline{G}) = 2 \), or \( cfc(G) = 2 \) and \( cfc(\overline{G}) = n - 2 \). By symmetry, we only need to consider one case, say \( cfc(G) = n - 2 \) and \( cfc(\overline{G}) = 2 \). Suppose that \( cfc(G) = n - 2 \). It follows from Theorem 4.6 that \( G \) is a tree with \( \Delta(G) = n - 2 \), or \( G \in \{ K_{1,3} + e, K_{2,2}, K_{2,2} + e, P_5 \} \). If \( G \) is a tree with \( \Delta(G) = n - 2 \), then \( cfc(\overline{G}) = 2 \) by Lemma 5.5. If \( G \in \{ K_{1,3} + e, K_{2,2}, K_{2,2} + e \} \), then it is easy to see that \( \overline{G} \) is disconnected. Next, we only need to consider the case \( G = P_5 \). Note that \( \overline{G} \) is a 2-edge-connected and noncomplete graph of order 5, which contradicts to the fact that \( \lambda(\overline{G}) = 1 \). Thus, \( G \) is a tree with \( \Delta(G) = n - 2 \). \( \square \)

**Claim 3.** \( cfc(G) + cfc(\overline{G}) = 4 \) or \( cfc(G) \cdot cfc(\overline{G}) = 4 \) if and only if each of
$G$ and $\overline{G}$ has the property that there exist at most two cut-edges incident with the vertex of maximum degree.

**Proof of Claim 3:** At first, we show that $cfc(G) + cfc(\overline{G}) = 4$ if and only if each of $G$ and $\overline{G}$ has the property that there exist at most two cut-edges incident with the vertex of maximum degree. For the necessity. Let $G$ be a connected graph of order $n \geq 4$ such that $\lambda(G) = 1$ and $\lambda(\overline{G}) = 1$ and $cfc(G) + cfc(\overline{G}) = 4$. Suppose that one of $G$ and $\overline{G}$, say $G$, has the opposite property that there exist at least three cut-edges incident with the vertex of maximum degree. In order to make $G$ conflict-free connected, these cut-edges need to be assigned three distinct colors. Thus, $cfc(G) \geq 3$, a contradiction.

For the sufficiency. If each of $G$ and $\overline{G}$ has the property that there exist at most two cut-edges incident with the vertex of maximum degree. By symmetry, we only need to show that $cfc(G) = 2$. Note that $G$ has a cut-edge and a vertex $v$ of degree $n - 2$ by Lemma 5.1. Let $X = \{x_1, \ldots, x_{n-2}\}$ be the neighborhood of $v$, and $w$ be the unique vertex that is not adjacent to $v$. Note that $w$ is adjacent to at least one vertex in $X$. Without loss of generality, assume that $wx_1 \in E(G)$. Suppose that there exists only one cut-edge $vx_j$ incident with $v$. If $wx_1$ is not a cut-edge, then let $F_1 = G[V \setminus \{x_j\}]$ and $F' = F_1 + vx_j$. Noticing that $F_1$ is 2-edge-connected, it follows that $cfc(F') = 2$ by Lemma 2.6. Clearly, $F'$ is a spanning subgraph of $G$, and hence $cfc(G) \leq cfc(F') = 2$ by Proposition 2.11. Thus $cfc(G) = 2$. If $wx_1$ is a cut-edge, then let $H_1 = G[V \setminus \{x_j, w\}]$ and $H' = H_1 + vx_j + x_1w$. It is obvious that $H_1$ is 2-edge-connected. And it is known that $H_1$ is a cut-edge, then let $F_2 = G[V \setminus \{x_i, x_j, \}]$ and $F'' = F_2 + vx_i + vx_j$. Note that $F_2$ is 2-edge-connected, and $F''$ has only one component $T$ of $C(G)$. Thus, $cfc(F'') = h(F'') = cfc(T) = 2$ by Proposition 2.8. Clearly, $F''$ is a spanning subgraph of $G$, and hence $cfc(G) \leq cfc(F'') = 2$ by Proposition 2.11. Thus $cfc(G) = 2$. If $wx_1$ is a cut-edge, then let $H_2 = G[V \setminus \{x_i, x_j, w\}]$ and $H'' = H_2 + vx_i + vx_j + x_1w$. Clearly, $H_2$ is 2-edge-connected. And it is known that $H''$ has two components $T_1$ and $T_2$ of $C(G)$. Without loss of generality, assume that $T_1$ is the path $x_i vx_j$ (or $vx_iw$), and $T_2$ is the edge $x_1w$ (or $vx_j$). Thus, $cfc(H'') = h(H'') = cfc(T_1) = 2$ by Proposition 2.8. Clearly, $H''$ is a spanning subgraph of $G$, and hence $cfc(G) \leq cfc(H'') = 2$ by Proposition 2.11. Thus $cfc(G) = 2$.

With a similar argument, one can obtain that $cfc(G) \cdot cfc(\overline{G}) = 4$ if and only if each of $G$ and $\overline{G}$ has the property that there exist at most two cut-edges incident with the vertex of maximum degree. 


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