Anisotropic Fermi superfluid via p-wave Feshbach resonance

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We investigate theoretically Fermionic superfluidity induced by Feshbach resonance in the orbital p-wave channel. We show that, due to the dipole interaction, the pairing is extremely anisotropic. When this dipole interaction is relatively strong, the pairing has symmetry $k_z$. When it is relatively weak, it is of symmetry $k_x + i\beta k_y$ (up to a rotation about $\hat{z}$, here $\beta < 1$). A phase transition between these two states can occur under a change in the magnetic field or the density of the gas.

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In the earlier investigations of trapped Bose and Fermi gases, the interactions between particles are in general weak since the gases are typically dilute, with the interparticle distances much larger than the scattering $a$ between the particles (see, e.g., [1, 2] and reference therein). Recently however, it has been recognized that the interaction among particles can be tuned with the use of Feshbach resonances [3]. In the case of two Fermion species and a Feshbach resonance between them, one can tune, by varying the magnetic field and hence the resonance level energy, the interaction from (its "background" value through) weakly to strongly attractive. The ground state of the system is expected to evolve from a Bardeen-Cooper-Schrieffer (BCS) superfluid with long-range (compared with interparticle distances) Cooper pairing to a Bose-Einstein condensate (BEC) of tightly bound molecules. This cross-over has already been a subject of intensive and experimental investigations.

Practically all the above investigations are for the case where the resonance involves an s-wave bound state in the "closed" channel. Thus the aforementioned states involve pairing only in the s-wave state. More recently, it has been demonstrated experimentally also that p-wave Feshbach resonances exist. This thus raises the possibility of superfluids with p-wave Cooper pairing or BEC of p-wave molecules.

Consider now therefore a single Fermion species with an effective interaction induced by the p-wave Feshbach resonance. This brings in mind the well-known system of superfluid $^3$He. $^3$He has (nuclear) spin 1/2, in general not spin-polarized and basically spatially isotropic. Back in the 60’s, Anderson and Morel [12] investigated theoretically the superfluid state for this system by assuming that the pairing exists only between the same (say $\uparrow$) species. They found that the ground state of this system corresponds to Cooper pairing in the $l = 1$, $m = 1$ channel, i.e., the pair-wavefunction has the symmetry of the spherical harmonics $Y^1_1(k) \propto (k_x + ik_y)$ (or its spatial rotation). This state is realized in the $^3$He A-phase and is known as the "axial" phase [11]. One can thus conclude immediately that for a spin-polarized but otherwise spatially isotropic system, the pairing is again expected to be in the $Y^1_1(k)$ state. This argument is in agreement with the findings of Ref [10].

However, in our atomic system of interest, the interaction is far from isotropic. In particular, as demonstrated and explained by Ticknor et al [3] for the case of $^{40}$K, due to the magnetic dipole of the alkali atoms, the $l = 1$, $m = 0$ resonance occurs at a higher magnetic field than the $m = \pm 1$ ones. For a given magnetic field, the induced effective interaction is actually more attractive in the $m = 0$ channel than the $m = \pm 1$ ones. It is now thus non-trivial what the symmetry of ground state should be. This is the specific question on which we would concentrate in this paper.

Our findings are basically as follows. For a sufficiently dilute Fermi gas or for the case where the $m = 0$ and $m = \pm 1$ resonances are sufficiently far apart, the pairing occurs only in the $m = 0$ channel. That is, the ground state is a BCS state with Cooper pairing of symmetry $Y^0_0(k) \propto k_z$ for "large" magnetic fields and a BEC state of $m = 0$ Bosons for magnetic fields sufficiently below the resonance(s). There is no pairing or Boson in the $m = \pm 1$ channel. The orbital symmetry of this pairing is the same as the "polar" phase in the $^3$He literature [11]. For a sufficiently dense gas or for the case where the resonances are sufficiently closed to each other, the BCS case at high magnetic field corresponds to a state with Cooper pairing $\propto (k \cdot \hat{z} + i\beta (k \cdot \hat{a}))$. Here $\hat{z}$ is along the magnetic field direction and $\hat{a}$ is a vector in the plane perpendicular to $z$, and $\beta$ is a number less than unity and is field dependent. This state is thus intermediate between the polar and axial phases. At lower fields, $\beta$ may vanish and again the BCS pairing or the BEC condensation is again entirely in the $m = 0$ channel. These results are summarized in Figs. 1 and 2.

We begin with the Hamiltonian $H = H_f + H_b + H_\alpha$ where

$$H_f = \sum_k \left( \frac{\hbar^2 k^2}{2M} - \mu \right) a_k^\dagger a_k$$  \hfill (1)

$$H_b = \sum_{\vec{q}} \left( -2\mu + \delta_m + \frac{\hbar^2 q^2}{4M} \right) b_{\vec{q},m}^\dagger b_{\vec{q},m}$$  \hfill (2)
This Hamiltonian is a generalization of the ones already commonly employed for s-wave Feshbach resonances to the case of several \( l = 1 \), \( m = 0, \pm 1 \) closed channels. \( H_f \) and \( H_b \) are the Hamiltonians for the free Fermions and Bosons (particle pairs in the “closed” channels) respectively, and \( H_{\alpha} \) represents the Feshbach coupling. \( \alpha_k \) is the annihilation operator for a Fermion with momentum \( \tilde{k} \), \( b_{q} \) the corresponding operator for a Boson with angular momentum \( l = 1 \) and \( z \)-axis projection \( m \) with momentum \( \tilde{q} \), \( \mu \) and \( M \) are the chemical potential and mass of the Fermions. In \( H_{\alpha} \), the factor \( Y^{\alpha}_{m}(\tilde{k}) \) reflects the symmetry of the \( l, m \) bound state and the linear factor in \( k \) arises from the small momentum approximation for the coupling. \( \delta_{m} \) is the (bare) detuning of the energy of the closed channel with angular momentum projection \( m \), and \( \alpha_{m} \) the corresponding coupling constant. For low energy scattering between a pair of particles of momentum \( \pm \tilde{k} \), the scattering amplitude in the \( l = 1, m \) partial wave can be parameterized as, using the same notations as ref \([8]\),

\[
H_{\alpha} = \frac{1}{L^{3/2}} \sum_{m, \tilde{q}, \tilde{k}} \left\{ b_{\tilde{q}, m} \left( i \sqrt{4 \pi} k Y^{m*}_{\alpha}(\tilde{k}) \right) a_{-\tilde{k} + \tilde{q}/2} a_{\tilde{k} + \tilde{q}/2} + h.c. \right\}
\]  

(3)

\( v_m \) has the dimension of a volume and \( c_m \) an inverse length. The magnetic field dependent parameters \( v_m \), \( c_m \) are in principle available experimentally and have already been measured for 40K for atoms in the \( |f, m_f = 9/2, -7/2 \rangle \) hyperfine states. \([8]\) The bare parameters \( \delta_{m} \) and \( \alpha_{m} \) can be related to the physical parameters \( v_m \), \( c_m \) by considering scattering between two Fermions using the Hamiltonian given in eqs \([4, 8]\), \([10]\). These relations are:

\[
f_{m}(k) = \frac{k^2}{-\frac{1}{v_m} + c_m k^2 - i\epsilon_k^3} \]  

(4)

\[
v_m = \frac{4 \pi a^2}{M} \left( \frac{\delta_{m}}{|\alpha_{m}|^2} - \frac{1}{L^3} \sum_{\tilde{k}} \frac{M}{\hbar^2} \right) \]  

(5)

\[
c_m = -\frac{4 \pi a^2}{M} \left( \frac{\epsilon_k^2}{M|\alpha_{m}|^2} + \frac{1}{L^3} \sum_{\tilde{k}} \frac{1}{2\epsilon_k} \right) \]  

(6)

Here \( \epsilon_k \equiv \frac{k^2}{2M} \). The divergent sums over \( \tilde{k} \) on the right-hand-sides of the above two equations can be regulated either by introducing a cut-off or invoking the fact that the coupling \( \alpha_{m} \) must actually decay to zero at large momenta. Below we shall express all physical quantities in \( v_m \) and \( c_m \) in the final expressions and omit these explicit cutoffs.

With Feshbach resonance for the sub-channel \( m \), \( 1/v_m \) is field dependent, vanishing at a field \( B_{m}^* \). In contrast, \( c_m \) has a definite sign. For the ease of discussions, we shall assume that \( c_m < 0 \) and field independent, \(-1/v_m \) is an increasing function of field \(-1/v_m > (\sim) 0 \) for \( B > (\sim) B_{m}^* \), as in the case of 40K. (This corresponds to the case where \( \delta_{m} \) is an increasing function of field and \( \alpha_{m} \) weakly field dependent, c.f. eqs \([4, 8]\) and \([10]\)). For \( B < B_{m}^* \), a bound state appears. The energy of this bound state is given by \(-\epsilon_{b,m} = -\frac{1}{2} \kappa_{m}^2 / M \) with \( k = i\kappa_{m} \) being a pole for \( f_{m}(k) \). For small detuning below the resonance, \( \kappa_{m}^2 = 1/(-c_{m}(v_m)) \). Since \( 1/v_m \) should be roughly linear in \( B \) near the resonance, \( \epsilon_{b,m} \) increases linearly with \( (B_{m} - B) \) (in contrast to s-wave, where it is quadratic).

Moreover, as explained in \([8]\), due to the dipole interaction, \( B_0^* > B_1^* = B_{m}^* - 1 \). Thus in the field range of interest, \(-1/v_1 = -1/v_{-1} > -1/v_0 \). We can say that, at a given field, the effective interaction between the Fermions is less attractive for relative angular momentum projections \( m = \pm 1 \) than \( m = 0 \).

Now we proceed to find the ground state for the many-body problem. We assume a mean-field theory and replace \( b_{\tilde{q}, m} \) by c-numbers. Only its \( \tilde{q} = 0 \) value is non-vanishing. It is convenient to introduce the symbols

\[
D_m = -i \sqrt{4 \pi |\alpha_{m}| a_{0,m}} / L^{3/2} \]  

(7)

\[
\Delta_{k} = \sum_{m} D_m k Y^{m}_{1}(\tilde{k}) \]  

(8)

so that \( H_{\alpha} \) becomes \( \Delta_k^2 a_{-\tilde{k}} a_{\tilde{k}} + h.c. \). The Fermionic part of the Hamiltonian can be solved by Bogoliubov transformation. The value for \( b_{0,m} \) can also be easily found since the Hamiltonian is quadratic in this variable. In terms of \( D_m \), we get

\[
(-2\mu + \delta_{m}) D_m = \frac{4 \pi a^2}{L} \sum_{\tilde{k}} |\alpha_{m}|^2 k Y^{m*}_{1}(\tilde{k}) \frac{\Delta_{k}^2}{2[\epsilon_{k} - \mu + |\Delta_k^2|^{1/2}]} \]  

(9)

Using eq \([5]\) and \([6]\) we obtain
and a corresponding equation with $m = 1 \leftrightarrow m = -1$. Here

$$h(\vec{k}) = \frac{k^2}{[(\epsilon_k - \mu)^2 + (\Delta_\vec{k})^2]^{1/2}} - \frac{k^2(1 + \mu/\epsilon_k) - |\Delta_\vec{k}|^2}{2\epsilon_k}. \tag{12}$$

These equations are to be solved together with the number equation

$$n = \frac{1}{L^3} \left( \sum_k \langle a^+_k a_k \rangle + 2 \sum_m |b_{0,m}|^2 \right) \tag{13}$$

$\langle a^+_k a_k \rangle$ is given by $v^2_k = \frac{1}{2}(1 - \frac{\epsilon_k - \mu}{(\epsilon_k - \mu)^2 + (\Delta_\vec{k})^2}$. Since $\Delta_\vec{k}$ from eq 8 is linear in $k$, the sum in eq 13 over $\vec{k}$ is formally divergent due to the large $\vec{k}$ contributions. However, we can regularize it by employing again eq 8. We have finally

$$n = \frac{1}{L^3} \sum_k \left( \frac{v^2_k - |\Delta_\vec{k}|^2}{4\epsilon_k^2} + \frac{M^2}{2(4\pi\hbar)^2} \sum_m (-c_m)|D_m|^2 \right) \tag{14}$$

Eqs 10, 11, 13 are our principal equations, with parameters characterizing the Feshbach resonances expressed entirely in $v_m$ and $c_m$. These equations determine the order parameters $D_m$ and chemical potential $\mu$ for given density $n$ and "interaction parameters" $v_m$ and $c_m$. For simplicity, in writing these equations we have already dropped the terms with explicit $1/|\tilde{\alpha}_m|^2$ factors. These terms are small under the "wide-resonance" regime $R(b)$.  

Since the interaction is less attractive for angular momentum projections $m = \pm 1$, for sufficiently large difference between $-1/\tilde{\nu}_0$ and $-1/v_{\pm 1}$ we expect (and verify below) that the pairing is entirely in the $m = 0$ partial wave. We thus first begin our analysis by assuming that only $D_0$ is non-vanishing. Eqs 10 and 11 can be solved simultaneously similar to the s-wave case. It is convenient to express the results in dimensionless form. We define $\tilde{\mu} \equiv \mu/\epsilon_F$, $D_{m} \equiv D_m/v_F$, $\tilde{c}_m \equiv n^{-1/3} c_m$, and $\tilde{v}_m \equiv n v_m$ where $\epsilon_F \equiv \hbar^2 k^2/2M$, $v_F \equiv \epsilon_k/M$, and $k_F \equiv 6\pi^2 n$. The results are as shown in Fig. 3 (for the case $\tilde{c}_0 = \tilde{c}_1 = -100$, see below for the reason of this choice). In the BCS regime ($\epsilon_{\tilde{c}_0} > 1$ or $B - B_0^*$, not shown explicitly), $\tilde{\mu} \rightarrow 1$, and $\tilde{D}_0$, being proportional to the magnitude of the BCS gap, is $\ll 1$. In the BEC ($-1/\tilde{\nu}_0 \ll -1$ or large and negative $B - B^*_0$), $\tilde{\mu}$ is approximately $-\epsilon_0/2$ and $D_0$ approaches a constant. This latter value can be obtained from eq 11 as $(32\pi/3)^{1/3}(\tilde{c}_0)^{-1/2}$.

The "cross-over" behavior in Fig 3 is analogous to the s-wave case, where the corresponding x-axis is $x = -1/(n^{1/3}a)$ where $a$ is the s-wave scattering length. Note here $D_0$ has the dimension of $(\text{energy} \times \text{inverse length})$ and behaves differently from the s-wave $\Delta$ in the BEC limit. We have also performed calculations for other values of $\tilde{c}_0$. The size of the crossover region is roughly proportional to the value of $\tilde{c}_0$. For example, for $\tilde{c}_0 = -200$, the corresponding results can be captured well by replacing the x-axis by $-1/2\tilde{\nu}_0$ and dividing $\tilde{D}_0$ by $1/\sqrt{2}$ in Fig 3. The result in Fig 3 is similar to that in Ref 11, even though the latter actually studied a different ($D_1 \neq 0$, $D_0 = D_{-1} = 0$) state.

The above behavior applies only to sufficiently large $-1/v_{\pm 1} - (-1/\tilde{\nu}_0) > 0$. When this difference is sufficiently small, $D_{\pm 1}$ will become finite. The critical value for $1/v_{\pm 1}$, denoted by $1/v^*_{\pm 1}$, can be found by putting $D_{\pm 1} = 0$ in eq 11 and linearizing eq 11 in $D_{\pm 1}$. We obtain, for $c_1 = c_0$,

$$-\frac{1}{\tilde{v}_1} + \frac{1}{\tilde{v}_0} = \frac{3(6\pi)^{3/2}}{5\pi} \tilde{D}_0^2 (\tilde{c}_0)$$

$$+9\pi \int_0^\infty dx \int_{-1}^1 dy \frac{x^4(1-3y^2)}{(ax^2 - \tilde{\mu})^2 + (2\pi D_0^2 x^2 y^2)^{1/2}} \tag{15}$$

In the BCS limit ($-1/\tilde{\nu}_0 \gg 1$), the first term in eq 15 is negligible whereas the second term becomes a constant independent of $D_0$. From this we get $-1/\tilde{v}_1^* + 1/\tilde{\nu}_0 \rightarrow 12\pi \approx 37.7$. In BEC limit ($-1/\tilde{\nu}_0 \ll -1$), the main contribution comes from the first term in eq 15. Using the aforementioned asymptotic values of $D_0$ we get $-1/\tilde{v}_1^* + 1/\tilde{\nu}_0 \rightarrow 48\pi/5 \approx 30.2$. $-1/\tilde{v}_1^* + 1/\tilde{\nu}_0$ is shown as the thick black line ($\tilde{D}_0 = 0$) in Fig 3.

For $-1/\tilde{v}_1^* + 1/\tilde{\nu}_0$ less than this critical value, $D_{\pm 1}$ are finite. Eqs 10, 11 involve three complex unknowns $D_m$. By gauge invariance we can always choose $D_0$ to be real. Under this choice, the solutions we found belong to the class $D_1 = D_{-1}$. Writing $D_1 = |D_1|e^{i\chi}, \Delta_k$
then has the angular dependence \( \propto D_0 \hat{k}_z + i \sqrt{2} D_1 |\hat{k}\cdot\hat{a}\) \( \propto (\hat{k}_z + i\hat{\beta}\cdot\hat{a}) \) where \( \hat{a} = (\cos \chi) \hat{y} + (\sin \chi) \hat{x} \) is a unit vector perpendicular to \( \hat{z} \) and \( \hat{\beta} = \sqrt{2} D_1 / D_0 \). A particular solution is given by the case where \( D_1 \) and \( D_\perp \) are both real where \( \hat{a} = \hat{y} \). The other solutions are simply related to this one by a rotation about \( \hat{z} \). Without loss of generality we shall therefore pretend that \( D_m \)'s are all real.

The contours of the order parameters \( D_{\pm1} \) are also shown in Fig. 2. In the \( D_{\pm1} = 0 \) phase, the state is rotationally invariant about \( \hat{z} \), whereas this symmetry is broken in the \( D_{\pm1} \neq 0 \) phase. There is a (quantum) phase transition between these two phases when one crosses the line \(-1/\tilde{v}_1^2 + 1/\tilde{v}_0 \).

For the \( ^{40}\text{K} \) case studied in \( ^{4} \), the Feshbach resonances are at \( B_0 \approx 198.8G \) and \( B_1 \approx 198.4G \). There, \( c_1 \) and \( c_0 \) are both only weakly field dependent and are approximately given by \(-0.02 a_0^{-3} \). Our choice of \( \tilde{c} = -100 \) above corresponds to a density of roughly \( 6 \times 10^{13} \text{cm}^{-3} \). Near the resonant fields, \(-1/\tilde{v}_1 + 1/\tilde{v}_0 \approx 2.1 \times 10^{-8} \tilde{a}_0^{-3} \) and is roughly field independent. Thus the density determines the values for both \( \tilde{c}_0 \) and \(-1/\tilde{v}_1 + 1/\tilde{v}_0 \) while varying the magnetic field corresponds roughly to moving along a horizontal line on our phase diagram of Fig. 2 (with increasing field towards the right and the distance of the line from the x-axis proportional to \( n^{-1} \)).

While preparing this manuscript, we become aware of Ref. [17] which studies essentially the same problem. Whereas our results agree with [15] for very large and very small splitting between the resonances, the conclusions differ in the intermediate splitting regime. Our prediction is that the state is \( \sim \hat{k}_z + i\beta \hat{k}_y \) on the BCS side whereas it should be \( \hat{k}_x \) on the BEC side. Their conclusion is the opposite. The reason for this disagreement is not yet understood. We believe that our results are more reasonable. For large positive detuning, the splitting should be less relevant and the pairing state should resemble more that of the isotropic system. On the BEC side, the system should be closer to a Bose condensate of lowest energy (\( \hat{k}_z \)) molecules.

In conclusion, we have shown that p-wave Feshbach resonance in general leads to anisotropic Fermi superfluids. The symmetry of the ground state depends on both the density and magnetic fields.

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**FIG. 1:** The dimensionless parameters \( \tilde{D}_0 \) and \( \tilde{\mu} \) as functions of \(-1/\tilde{v}_0 \). \( \tilde{c}_0 = c_1 = -100 \) and \( D_{\pm1} = 0 \) in this case. The dashed line represents \(-e_b/2\epsilon_F\).
FIG. 2: Contour plot of $\tilde{D}_{\pm 1}$ as a function of $-1/\tilde{v}_1 + 1/\tilde{v}_0$ and $-1/\tilde{v}_0$ for $\tilde{c}_0 = \tilde{c}_1 = -100$. The line for $\tilde{D}_{\pm 1} = 0$ corresponds to the critical value $-1/\tilde{v}_1^* + 1/\tilde{v}_0$. 