A FORMULA FOR SYMBOLIC POWERS

PAOLO MANTERO, CLETO B. MIRANDA-NETO, UWE NAGEL

ABSTRACT. Let $S$ be a Cohen-Macaulay ring which is local or standard graded over a field, and let $I$ be an unmixed ideal that is generically a complete intersection. Our goal in this paper is multi-fold. First, we give a multiplicity-based characterization of when an unmixed subideal $J \subseteq I^{(m)}$ equals the $m$-th symbolic power $I^{(m)}$ of $I$. Second, we provide a saturation-type formula to compute $I^{(m)}$ and employ it to deduce a theoretical criterion for when $I^{(m)} = I^m$. Third, we establish an explicit linear bound on the exponent that makes the saturation formula effective, and use it to obtain lower bounds for the initial degree of $I^{(m)}$. Along the way, we prove a generalized version of a conjecture raised by Eisenbud and Mazur about $\text{ann}_S(I^{(m)}/I^m)$, and we propose a conjecture connecting the symbolic defect of an ideal to Jacobian ideals.

1. INTRODUCTION

Symbolic powers of ideals have a relevant role in a number of results and open problems in the literature. Part of the reason is their geometric significance, which appears in a celebrated theorem proved by Zariski and Nagata (later generalized by Hochster and Eisenbud [9]) stating that the $m$-th symbolic power $I^{(m)}$ of a given radical ideal $I$ in a polynomial ring $R$ over a perfect field consists of all the hypersurfaces passing at least $m$ times through each point of the variety $V(I)$. (Recall that $I^{(m)} = I^m R_W \cap R$, where $W \subset R$ is the multiplicative set of all $R/I$-regular elements.) For further information, including the well-justified relevance of symbolic powers in the literature, we refer to the survey [4] and references therein, such as, e.g., [6], [15] and [17].

In this note we are concerned with the symbolic powers $I^{(m)}$ of an unmixed ideal $I$ that is generically a complete intersection in a Cohen-Macaulay ring $S$ which is either local or standard graded over a field.

As an illustration of our work, we observe that a main source of difficulties when investigating symbolic powers $I^{(m)}$ of an unmixed ideal $I$ is the lack of explicit formulas to compute $I^{(m)}$, even in the polynomial ring $k[x,y,z]$. So far, the only general formula
in the literature is the following saturation formula
\[ I^{(m)} = I^m : J^\infty, \quad \text{where} \quad J := \bigcap_{p \in \text{Ass}^*(I) - \text{Ass}(S/I)} p \]
and \( \text{Ass}^*(I) := \bigcup_{m \geq 1} \text{Ass}(S/I^m) \), (e.g. see Eisenbud [7, Prop 3.13], Herzog, Hibi and Trung [16, Section 3]). Since, in general, the set \( \text{Ass}^*(I) \) is not well–understood, the above is a theoretical formula and researchers (e.g. [13, Comment after Exercise 1.55]) have lamented the need for an explicit ideal \( H \), defined only in terms of \( I \) (and not depending on its powers \( I^m \) or \( m \)) such that
\[ I^{(m)} = I^m : H^\infty. \]
In one of our main results, Theorem 4.3, we fill this gap and provide an explicit description of an ideal \( H \), easily computed from the presentation of \( I \), satisfying the above formula.

We now outline the main results of this paper. In what follows, \( c \) denotes the height of the ideal \( I \).

- **(Theorem 3.2)** Let \( m \in \mathbb{Z}_+ \) and let \( J \subseteq I^{(m)} \) be an unmixed ideal with the same height \( c \). Then, \( J = I^{(m)} \) if and only if
  \[ e(S/J) = e(S/I) \left( \frac{c + m - 1}{c} \right). \]

- **(Theorem 4.3)** Let \( F_c(I) \) be the \( c \)-th Fitting ideal of \( I \). Then, for any \( m \in \mathbb{Z}_+ \),
  \[ I^{(m)} = I^m : F_c(I)^\infty. \]

  As a consequence, we obtain the following theoretical characterization for the equality between ordinary and symbolic powers: \( I^{(m)} = I^m \) if and only if the ideal \( F_c(I) \) contains an \( S/I^m \)-regular element (Corollary 4.6).

- **(Theorem 5.9)** We prove that, in general, we can replace the saturation with a fairly small power of \( F_c(I) \); more precisely, we have the formula
  \[ I^{(m)} = I^m : F_c(I)^{m-1}. \]

- **(Corollary 5.11)** We prove an explicit lower bound for the initial degree of \( I^{(m)} \) when \( S \) is a positively graded local domain (not necessarily a polynomial ring), and observe that it is sharp in some cases.

- **(Theorem 5.5)** We prove a generalized version of a conjecture of Eisenbud and Mazur in [10] concerning the annihilator of \( I^{(m)}/I^m \) for some determinantal ideals \( I \).

Furthermore, we propose the following conjecture over a polynomial ring \( S \) with coefficients in a perfect field \( k \).

**(Conjecture 6.5)** Let \( I \) be a radical unmixed ideal of \( S \). For any given \( g \in S \), let \( J(g) \) denote the ideal generated by the partial derivatives of \( g \) (possibly
computed via divided powers if char(k) = p > 0). Given m ≥ 2, assume $I^{(m)} = (I^m, g_1, \ldots, g_s)$. Then, the $g_i$'s can be taken so that they satisfy

$$\text{rad} \left( \sum_{1 \leq j \leq s} J(g_j) \right) = I.$$

Note that one containment holds easily since $J(g_j) \subseteq I$ for all $j$, by the classical Zariski–Nagata theorem.

This conjecture is supported by numerous examples, most of which have been computed via the computer algebra software Macaulay2 [14].

2. Preliminaries

In this section we establish a few conventions and basic facts used throughout the paper.

By ring we tacitly mean Noetherian, commutative ring with identity. The multiplicity of a module has two slightly different definitions depending on the assumptions on the ring.

- If $(S, \mathfrak{m})$ is a local ring with residue field $k$, dim$(S) = d$, and if $M$ is a finitely generated $S$-module, then the (Hilbert-Samuel) multiplicity of $M$ over $S$ is

$$e(\mathfrak{m}, M) = \lim_{t \to \infty} \frac{d!}{t^d} \dim_k \left( \frac{M}{\mathfrak{m}^t M} \right).$$

- If $S$ is a standard graded algebra over a field $k$ with homogeneous maximal ideal $\mathfrak{m}$, the multiplicity is defined in the same way except that the fraction $d!/t^d$ must be replaced with $(d-1)!/t^{d-1}$.

If $M = S$ one simply writes $e(\mathfrak{m}, S) = e(S)$ for the multiplicity of $S$.

For a finitely generated $S$-module $M$, the $j$-th Fitting ideal of $M$ over $S$ is defined from any given $S$-free presentation $S^{\mu} \xrightarrow{\varphi} S^\nu \to M \to 0$ as the ideal

$$F_j(M) = I_{\mu-j}(\varphi)$$

generated by the $(\mu - j)$-minors of $\varphi$, where $\varphi$ is regarded as a presentation matrix of $M$ over $S$ (in case $j \geq \mu$ we set $F_j(M) = S$). It is well-known that the Fitting ideals are independent of the choice of free presentation.

As usual, the height of an ideal $L$ is denoted $ht(L)$. Recall, for ideals $I, J$ of $S$, that:

- $I$ is unmixed of height $c$ if $ht(p) = c$ for every $p \in \text{Ass}_S(S/I)$;
- The unmixed part of $I$, denoted $I^{\text{unm}}$, is the ideal containing $I$ obtained by intersecting the minimal components of $I$ having minimal height (so $I$ is unmixed if and only if $I = I^{\text{unm}}$);
- $I$ is generically a complete intersection if the localization $I_p$ can be generated by an $S_p$-sequence for every $p \in \text{Ass}_S(S/I)$;
- The saturation of $I$ by $J$ is the ideal

$$I : J^\infty = \bigcup_{t \geq 1} I : J^t.$$
By Noetherianess, we have an equality $I : J^\infty = I : J^t$ for all $t \gg 0$.

Moreover, we use $\mu(M)$ to denote the minimal cardinality of a generating set of a finitely generated module $M$ (in either local or graded setting).

The key object of interest of the present paper is given in the following definition 2.1. **Definition.** Let $S$ be a ring, $I$ a proper ideal of $S$, $m \geq 2$ an integer. The $m$-th symbolic power of $I$ is the ideal

$$I^{(m)} = \bigcap_{p \in \text{Ass}(S/I)} I_p^m \cap S.$$  

2.2. **Remark.** Questions about symbolic powers of $I$ may change drastically depending on the presence, or lack of, embedded primes in $\text{Ass}(S/I)$. For instance, if $(S, m)$ is local and $m \in \text{Ass}(S/I)$ then the above definition gives $I^{(t)} = I^t$ for every $t \geq 1$. To avoid these trivializations, the vast majority of problems and papers about symbolic powers require $I$ to be (at least) radical or unmixed. We will make no exception and assume $I$ is unmixed throughout the paper. One should be aware that our results may fail for radical ideals having minimal primes of different heights; see, for instance, Example 4.5.

### 3. A NUMERICAL CHARACTERIZATION OF SYMBOLIC POWERS

While the inclusion $I^m \subseteq I^{(m)}$ always holds, in general equality occurs in relatively few cases (e.g., if $I$ is a complete intersection in a Cohen-Macaulay ring), and if the inclusion is strict it is often an extremely hard task to determine the minimal generators of $I^{(m)}$, even when $S$ is a polynomial ring in 3 variables over a field (in fact, even determining the degrees of some of these minimal generators!).

For special ideals (or classes of ideals) one may know some generators of $I^{(m)}$, and, for some reason, expect them to fully generate $I^{(m)}$. In this case one has the following question:

3.1. **Question.** Let $S$ be a Cohen-Macaulay ring, $I$ an unmixed $S$-ideal with $\text{ht}(I) = c$ and let $m \in \mathbb{Z}_+$. If $J \subseteq I^{(m)}$ is an unmixed ideal with $\text{ht}(J) = \text{ht}(I)$, when does the equality $J = I^{(m)}$ hold?

Theorem 3.2 answers Question 3.1 under mild assumptions. It provides a numerical criterion depending solely on $m$, $c$ and the multiplicities of $S/J$ and $S/I$. It can be used to prove that a “candidate” ideal is indeed equal to $I^{(m)}$.

3.2. **Theorem.** Let $S$ be a Cohen-Macaulay ring that is either local or standard graded over a field, and let $m \in \mathbb{Z}_+$. Let $I$ be an unmixed, generically complete intersection ideal and let $J \subseteq I^{(m)}$ be an ideal with $\text{ht}(J) = \text{ht}(I) = c$. Then the following assertions are equivalent:

(i) $I^{(m)} = J^{\text{unm}}$,

(ii) $e(S/J) = e(S/I)(c^m - 1)$. 

**Proof.** Since $I$ is unmixed and the ideals $J \subseteq I^{(m)}$ have the same height, the ideals $J^{\text{unm}}, I^{(m)}$ are unmixed of the same height. The assumptions on $S$ allow us to employ
the associativity formula (see e.g. [21, Theorem 14.7]) from which it follows that the equality \( I^{unm} = I^{(m)} \) holds if and only if the two ideals have the same multiplicity. Therefore, to prove the theorem it suffices to show that \( e(S/I^{(m)}) = e(S/I)^{(c+m-1) \choose c} \).

From the associativity formula and \( \text{Ass}_S(S/I^{(m)}) = \text{Ass}_S(S/I) \), we have

\[
e(S/I^{(m)}) = \sum_{p \in \text{Ass}_S(S/I)} e(S/p) \lambda(S/p/I_p^{m}),
\]

where \( \lambda(\cdot) \) denotes the length of a module. We now compute \( \lambda(S_p/I_p^{m}) \). By assumption, \( L := I_p \) is a complete intersection ideal in the local ring \( R := S_p \). Then

\[
gr_L(R) \cong (R/L)[x_1, \ldots, x_c]
\]

where \( x_1, \ldots, x_c \) are indeterminates over \( R/L \), and hence, for each \( t \geq 0 \), the \( R/L \)-module \( L^t/L^{t+1} \) is free of rank \( (c+t-1) \choose c-1 \). Using the short exact sequence

\[
0 \longrightarrow L^{m-1}/L^m \longrightarrow R/L^m \longrightarrow R/L^{m-1} \longrightarrow 0
\]

and induction on \( m \), we find

\[
\lambda(R/L^m) = \sum_{t=0}^{m-1} \lambda(L^t/L^{t+1}) = \sum_{t=0}^{m-1} \lambda(R/L) \left( \begin{array}{c} c+t-1 \\ c-1 \end{array} \right) = \lambda(R/L) \left( \begin{array}{c} c+m-1 \\ c \end{array} \right).
\]

Finally,

\[
e(S/I^{(m)}) = \sum_{p \in \text{Ass}_S(S/I)} e(S/p) \lambda(S_p/I_p^{m})
\]

\[
= \sum_{p \in \text{Ass}_S(S/I)} e(S/p) \lambda(S_p/I_p) \left( \begin{array}{c} c+m-1 \\ c \end{array} \right)
\]

\[
= e(S/I) \left( \begin{array}{c} c+m-1 \\ c \end{array} \right).
\]

3.3. Example. In the standard graded polynomial ring \( S = k[x, y, z, w, t] \), with \( k \) a perfect field, consider the following perfect radical ideal of height \( c = 3 \),

\[ I = (xz, xw, yw, yt, zt). \]

First, we note that \( S/I^2 \) is a Cohen-Macaulay ring, \( e(S/I) = 5 \) and \( e(S/I^2) = 5 \cdot 4 = 20 \). Hence, Theorem 3.2 yields

\[ I^{(2)} = I^2. \]

Second, we analyze the third symbolic power of \( I \). Since \( (c+m-1) \choose c = (5) \choose 3 = 10 \), Theorem 3.2 gives, once we have found a “candidate” unmixed ideal \( J \subseteq I^{(3)} \), that we just need \( e(S/J) = 5 \cdot 10 = 50 \).

We observe that \( g := xyzwt \in I^{(3)} \) (either by looking at the minimal primes of \( I \), or via the Zariski–Nagata theorem, which can be applied as \( I \) is radical), and \( g \notin I^3 \) for degree reasons. Thus, \( I^{(3)} \neq I^3 \) and \( (I^3, g) \subseteq I^{(3)} \). One can verify that \((I^3, g)\) is unmixed (in fact, perfect) and that \( e(S/(I^3, g)) = 50 \). By Theorem 3.2, we conclude that

\[ I^{(3)} = (I^3, xyzwt). \]
4. A SATURATION FORMULA FOR SYMBOLIC POWERS

We first specify the set-up for this section.

4.1. Assumptions. Let $S$ be a Cohen-Macaulay ring, $I$ an unmixed $S$-ideal that is generically a complete intersection with $\text{ht}(I) = c$.

For instance, in this setting $I^m = I^m$ if and only if $\text{Ass}_S(S/I^m) = \text{Ass}_S(S/I)$.

Our main motivation for this section is the following general question, which is wide-open because of the considerable challenges in computing the symbolic powers of ideals (even perfect ideals of height 2 in $k[x, y, z]$).

4.2. Question. Let $S, I$ be as in Assumptions 4.1 and let $m \in \mathbb{Z}_+$. How can $I^m$ be described effectively?

4.1. A saturation formula. Our central result of this section is the following formula for computing $I^m$ in terms of $I^m$ and a suitable ideal associated to $I$ that does not depend on $m$.

4.3. Theorem. Let $S, I$ be as in Assumptions 4.1. For any $m \in \mathbb{Z}_+$, one has

$$I^m = I^m : F_c(I)^\infty.$$ 

Proof. Let us first recall that

$$V(F_c(I)) = \{ p \in \text{Spec } S \mid \mu(I_p) > c \}.$$

Since $\text{ht}(I) = c$ and $I$ is generically a complete intersection it follows that $\text{ht}(F_c(I)) \geq c + 1$ and thus $I^{(m)} : F_c(I)^t = I^m$ for every $m \geq 1$ and $t \in \mathbb{Z}_+$.

Now, if $\text{Ass}_S(S/I^m) = \text{Ass}_S(S/I)$ then $I^{(m)} = I^m$, and by the above $I^m = I^m : F_c(I)^\infty$, proving the statement in this case.

We may then assume $\Gamma := \text{Ass}_S(S/I^m) - \text{Ass}_S(S/I)$ is non-empty, and write $\Gamma = \{ p_1, \ldots, p_r \}$. We now claim that $\Gamma \subseteq V(F_c(I))$.

Indeed, since $\text{Min}(I^m) = \text{Min}(I)$, any $p \in \Gamma$ is an embedded prime of $S/I^m$, and so $S_p/I_p^m$ has an embedded prime. In particular this implies that $I_p^m$ is not an unmixed ideal, and therefore $I_p$ is not a complete intersection ideal, i.e. $\mu(I_p) > c$. By the above observation, it follows that $p \in V(F_c(I))$, which proves the claim.

The claim yields that $Q : F_c(I)^\infty = S$ for every $p$-primary ideal $Q$ with $p \in \Gamma$. Writing $I^m = I^{(m)} \cap \left( \bigcap_{j=1}^{r} Q_j \right)$, where each $Q_j$ is a $p_j$-primary ideal, we then obtain

$$I^m : F_c(I)^\infty = \left( I^{(m)} : F_c(I)^\infty \right) \cap \left( \bigcap_{j=1}^{r} Q_j : F_c(I)^\infty \right) = I^{(m)} : F_c(I)^\infty = I^m.$$

4.2. Some examples, and equality between symbolic and ordinary powers. Below we illustrate that, in Theorem 4.3, neither of the two standing hypotheses on $I$ can be dropped, even in the presence of the other. We let $k$ denote a perfect field.
4.4. Example. Consider the height 3 ideal $I = (x, y, z)^2 \cap (y, z, w)$ in the polynomial ring $S = k[x, y, z, w]$. Note that $I$ is unmixed but not generically a complete intersection. Now, clearly

$$I^{(m)} = (x, y, z)^{2m} \cap (y, z, w)^m \quad \text{for any } m \geq 2.$$ 

A calculation shows that $I^2 = (x, y, z)^4 \cap (y, z, w)^2$, i.e., $I^2 = I^{(2)}$, but $I^2 \nsubseteq I^3 : \mathbf{F}_3(I)$. Hence $I^2 \neq I^3 : \mathbf{F}_3(I)^\infty$. An analogous behavior occurs for the third symbolic power. First we notice that $I^3 : \mathbf{F}_3(I)^\infty \neq I^3$ and $I^3 = (x, y, z)^6 \cap (y, z, w)^3 = I^{(3)}$, and then we get

$$I^{(3)} \neq I^3 : \mathbf{F}_3(I)^\infty.$$ 

4.5. Example. Consider the height 2 ideal $I = (xyz, xtu, zw, yw)$ in the polynomial ring $S = k[x, y, z, w, t, u]$. Note that $I$ is generically a complete intersection (even radical), but not unmixed because $(x, w)$ and $(x, y, z)$ are minimal prime divisors of $I$ (for the latter, notice that $I : (w) = (x, y, z)$). Now, according to [5] Remark 5.4, we have, for any $m \geq 2$,

$$I^{(m)} = I^m + gI^{m-2} \quad \text{with} \quad g = xyzw.$$ 

In particular, $I^{(2)} = (I^2, g)$. However, a computation with Macaulay2 [14] shows that, for example, the monomial $x^2yz^2t$ lies in $I^2 : \mathbf{F}_2(I)$ but not in $(I^2, g)$. Hence $I^{(2)} \neq I^2 : \mathbf{F}_2(I)^\infty$. To study the third symbolic power, by the above formula we have $I^{(3)} = I^3 + gI$. Suppose by way of contradiction that this ideal equals $I^{(3)} : \mathbf{F}_2(I)^\infty$. Then we would have $I^{(3)} : \mathbf{F}_2(I) \subseteq I^3 + gI$. But this is a contradiction as a calculation shows that the ideal $I^{(3)} : \mathbf{F}_2(I)$ contains, for instance, the monomial $x^3y^2z^2t \notin I^3 + gI$.

Next we give another consequence of the formula given in Theorem 4.3. First, we recall that it is an open problem to find sufficient (and possibly also necessary) conditions for the equality $I^{(m)} = I^m$ to occur (see, e.g., [4] Section 4.2). For instance, a celebrated conjecture in the context of Combinatorial Optimization raised by Conforti and Cornuéjols about the Max-Flow-Min-Cut property of clutters was translated by Gitler, Valencia and Villarreal into the following equivalent conjecture: for any square-free monomial ideal $I$, one has $I^{(m)} = I^m$ for all $m \geq 1$ if and only if $I$ has the so-called packing property; see [3] Conjecture 1.6, [12] Corollary 3.14.

In this context, our next application provides a potentially useful characterization of the equality between symbolic and ordinary powers.

4.6. Corollary. Let $I$ be as in Assumption 4.1 and let $m \in \mathbb{Z}_+$. Then,

$$I^{(m)} = I^m$$

if and only if $\mathbf{F}_c(I)$ contains an $S/I^m$-regular element.

Proof. We have $I^m : \mathbf{F}_c(I)^\infty = I^m : \mathbf{F}_c(I)^t$ for $t$ sufficiently large. Now if $g \in \mathbf{F}_c(I)$ is an $S/I^m$-regular element then so is $g^t \in \mathbf{F}_c(I)^t$, and hence $I^m : \mathbf{F}_c(I)^t = I^m$, which by Theorem 4.3 gives $I^{(m)} = I^m$. The converse is clear. □
4.7. **Example.** Let $S$ be either a polynomial ring (standard graded or localized at the ideal of the origin) or a power series ring, in 5 indeterminates $x, y, z, w, t$ over a field of characteristic zero. Let $I$ be the ideal generated by the maximal minors of the $3 \times 4$ matrix

$$
\begin{pmatrix}
x & w & y & t \\
y & t & z & x \\
z & x & w & y
\end{pmatrix}.
$$

Then $I$ is a radical unmixed ideal with $\text{ht}(I) = 2$. Now, by computations we find that $f := xy - zt \in F_c(I)$ satisfies $I^m : (f) = I^m$ at least if $m \leq 7$. By Corollary 4.6, we have

$$I^{(m)} = I^m \quad \text{for} \quad 2 \leq m \leq 7.$$ 

In fact, by [2, Theorem 1.1], one has $I^{(m)} = I^m$ for any $m \geq 1$.

5. **Effective Bounds and Proof of a Conjecture of Eisenbud and Mazur**

Theorem 4.3 implies that $I^{(m)} = I^m : F_c(I)^t$ for $t \gg 0$. So, for both theoretical and computational purposes one may be interested in determining values of $t$ which can be used. Our main question in this direction is the following.

5.1. **Question.** Let $S, I$ be as in Assumptions 4.1, and let $m \in \mathbb{Z}_+$. For which values of $t_0$ do we have

$$I^{(m)} = I^m : F_c(I)^t \quad \text{for all} \quad t \geq t_0?$$

Equivalently, for which values of $t_0$ is $I^{(m)} = I^m : F_c(I)^{t_0}$?

5.2. **Remark.** Let $S, I$ be as in Assumptions 4.1.

(1) Since $\text{ht}(F_c(I)) > \text{ht}(I)$, if one has $I^{(m)} \subseteq I^m : F_c(I)^{t_0}$ for some $t_0$, then equality holds. Thus, to show that $I^{(m)} = I^m : F_c(I)^{t_0}$, it suffices to prove

$$F_c(I)^{t_0} \subseteq I^m : I^{(m)} = \text{ann}_S(I^{(m)}/I^m).$$

(2) Part (1) provides an additional reason for our interest in Question 5.1: an answer to it automatically gives an estimate on the size of the ideal $I^m : I^{(m)}$, because it shows that $F_c(I)^{t_0} \subseteq \text{ann}_S(I^{(m)}/I^m)$.

5.3. **Remark.** Additional interest in Question 5.1 is provided by the observation that a tight bound on $t_0$ can be used to provide lower bounds for the initial degree of $I^{(m)}$. An instance of this application is exhibited in Corollary 5.11.

As an illustration of Remark 5.2(2) and the interest in $\text{ann}_S(I^{(m)}/I^m)$, we recall the following conjecture raised by Eisenbud and Mazur [10, Conjecture, p. 197] (notice, there is a typo: the ideal $F_1(I) = I$ should clearly be replaced with $F_2(I)$, which in this case is the maximal ideal $m$ generated by the 3 variables).

5.4. **Conjecture.** (Eisenbud and Mazur) If $I$ is the ideal of 2-minors of a $2 \times 3$ matrix of general linear forms in a polynomial ring in 3 variables, then

$$\text{ann}(I^{(m)}/I^m) = F_2(I)^{\lfloor \frac{m}{2} \rfloor}.$$
Conjecture 5.4 implies the following optimal answer to Question 5.1 for the ideal $I$ of the conjecture: $\left\lceil \frac{m}{N-1} \right\rceil$ is the smallest value we can take for $t_0$.

Conjecture 5.4 is proved in [23, Section 2] using tools from birational geometry. Here, we prove a stronger statement which implies Conjecture 5.4. Our result removes the assumption on the number of variables of the ring and it applies, for instance, to ideals of 2-minors of $2 \times N$ matrices of general linear forms, see Remark 5.6.

5.5. Theorem. Let $k$ be a perfect field, $S = k[x_0, \ldots, x_{N-1}]$ a standard graded polynomial ring in $N \geq 3$ variables, and $m = (x_0, \ldots, x_{N-1})$. Consider a radical homogeneous ideal $I$ of $S$ with $\text{ht}(I) = N - 1$ and minimally generated by \binom{N}{2} quadrics. Then, one has

$$\text{ann}_S(I^{(m)}/I^m) = m^{\left\lceil \frac{m(N-2)}{N-1} \right\rceil}.$$ 

Proof. Since $\text{ht}(I) = N - 1$ and $I$ is a radical ideal, the ring $S/I$ is Cohen–Macaulay. Now, let $\overline{k}$ denote the algebraic closure of $k$, and $T := S \otimes_k \overline{k} \cong \overline{k}[x_0, \ldots, x_{N-1}]$. Since $I$ is radical and $k$ is perfect, then $IT$ is radical in $T$. Then, by the purity of faithful flat extensions, it suffices to prove the statement in $T$, i.e., we may assume $k$ is algebraically closed.

Since $S/I$ is a reduced Cohen-Macaulay ring of dimension one and $k$ is algebraically closed, the ideal $I = I_X$ is the defining ideal of a set $X$ of points in $\mathbb{P}^N_k$. By assumption, $I$ is generated by \binom{N}{2} quadrics. It follows that $|X| = N$ and the points of $X$ span $\mathbb{P}^{N-1}_k$, therefore, $X$ is a star configuration in $\mathbb{P}^{N-1}_k$.

Let us denote by $\alpha(-)$ the initial degree of a homogeneous ideal. One has $\alpha(I^{(m)}_X) = m + \left\lceil \frac{m}{N-1} \right\rceil$ (e.g. [20, Theorem 4.12(1)]) and $\text{reg}(I^{(m)}_X) = 2m$ (e.g. [20, Corollary 7.3], or [1, Corollary 4.4]). In particular, this implies $[I^m_X]_d = \left[I^{(m)}_X\right]_d$ for every $d \geq 2m$. It follows that

$$m^eI^{(m)}_X \subseteq I^m_X \quad \text{for} \quad e = 2m - \left(m + \left\lceil \frac{m}{N-1} \right\rceil \right) = \left\lfloor \frac{(N-2)m}{(N-1)} \right\rfloor,$$

and so $m^e \subseteq \text{ann}_S(I^{(m)}_X/I^m_X)$. To prove equality we show that no form of degree $< e$ lies in $\text{ann}_S(I^{(m)}_X/I^m_X)$. In fact, consider a form $0 \neq g \in I^{(m)}_X$ of minimum degree. If there exists a form $f \in \text{ann}_S(I^{(m)}_X/I^m_X)$ of degree $< e$, then $fg \in I^m_X$ and $\deg(fg) < e + \alpha(I^{(m)}_X) = 2m$, contradicting the fact that $I^m_X$ is generated in degree $2m$. □

5.6. Remark. The assumptions of Theorem 5.5 are satisfied if $I \subset S$ is the ideal generated by the 2-minors of a $2 \times N$ matrix whose entries are general linear forms. So, Conjecture 5.4 follows from the special case $N = 3$.

We now return to discussing bounds on the integer $t_0$ of Question 5.1 in our general setting. We begin by deriving a non-explicit estimate for $t_0$. To this end, recall that the index of nilpotency of an ideal $L \neq S$ is

$$\text{nil} L = \min\{r \geq 1 \mid (\sqrt{L})^r \subseteq L\}.$$
Now, with notation as in the proof of Theorem 4.3, fix \( m \) and, for each \( j = 1, \ldots, r \), let
\[
\eta_j = \max\{\text{nil } Q_j\}
\]
Then an easy adaption of the proof of Theorem 4.3 gives the following estimate.

5.7. Corollary. Setting \( t_0 = \max\{\eta_1, \ldots, \eta_r\} \), one has
\[
I^{(m)} = I^m : F_c(I)^{t_0}
\]

The difficulty in applying this observation lies is the computation of \( \eta_i \).

So, we are led to look for explicit bounds for \( t_0 \). For instance, if \( m = 2 \) then it follows by work of Eisenbud and Mazur [10] that the smallest possible value of \( t_0 \) (i.e., \( t_0 = 1 \)) works. In fact we have the following theorem, which has evident potential computational applications.

5.8. Theorem. Let \( S, I \) be as in Assumptions 4.7. Then one has
\[
I^{(2)} = I^2 : F_c(I)^t
\]
for every \( t \geq 1 \).

Proof. The statement follows by the chain of inclusions
\[
I^{(2)} \subseteq I^2 : F_c(I) \subseteq I^2 : F_c(I)_{\infty} = I^{(2)},
\]
where the equality holds by Theorem 4.3 and the leftmost inclusion follows by [10 Theorem 6].

Our main result in this section generalizes Theorem 5.8. Its proof is inspired by [18 Theorem 2.1] and employs linkage. For basic terminology about linkage we refer the reader to [19] or [22].

5.9. Theorem. Let \( S, I \) be as in Assumptions 4.7. Then, for every \( m \geq 2 \), one has
\[
I^{(m)} = I^m : F_c(I)^{m-1}
\]

Proof. Without loss of generality we may assume that the residue field is infinite. Thus, we can take a system of generators of \( I \) so that any \( c \) of them form a regular sequence. Let \( \Delta \in F_c(I) \) be any \((\mu(I) - c)\)-th minor of the presentation matrix of this system of generators. It follows by Cramer’s rule that \( \Delta \) lies in an ideal geometrically linked to \( I \) (see [18 Proof of Corollary 2.4]). We now prove the following claim.

Claim. If \( L \) is any ideal geometrically linked to \( I \), then \( LI^{(t)} \subseteq CI^{(t-1)} \) for every \( t \geq 2 \), where \( C = L \cap I \) is the complete intersection defining the link.

Since \( C/C^t \) is a free \( S/C^t \)-module, the module
\[
C/CI^{(t-1)} \cong (C/C^t)/(I^{(t-1)}/C^{t-1})(C/C^t) \cong C/C^t \otimes_{S/C^{t-1}} S/I^{(t-1)}
\]
is isomorphic to a direct sum of copies of \( S/I^{(t-1)} \) as an \( S/C^t \)-module, and thus as an \( S \)-module. It follows that \( \text{Ass}_S(C/CI^{(t-1)}) = \text{Ass}_S(S/I^{(t-1)}) \subseteq \text{Ass}_S(S/I) \). Now, from the short exact sequence
\[
0 \to C/CI^{(t-1)} \to S/CI^{(t-1)} \to S/C \to 0
\]
we deduce that \( \text{Ass}_S(S/CI^{(t-1)}) \subseteq \text{Ass}_S(S/I) \cup \text{Ass}_S(S/C) = \text{Ass}_S(S/C) \). Therefore \( CI^{(t-1)} \) is an unmixed ideal of height \( c = \text{ht} I \).

We show \( LI^{(t)} \subseteq CI^{(t-1)} \) by proving the inclusion locally at each \( p \in \text{Ass}_S(S/CI^{(t-1)}) \). If \( I \subseteq p \), then \( p \in \text{Min}(I) \). Since \( C = L \cap I \) and \( \text{Ass}_S(S/L) \cap \text{Ass}_S(S/I) \) is empty, we obtain \( I_p = C_p \) and \( L_p = S_p \), so \( (LI^{(t)})_p = C^t_p = (CI^{(t-1)})_p \). If \( p \) does not contain \( I \), then \( I_p = S_p \) and \( L_p = C_p \), and in this case \( (LI^{(t)})_p = C_p = (CI^{(t-1)})_p \). This proves the claim.

Now, if \( \Delta_1, \ldots, \Delta_{m-1} \) are any of the minors generating \( F_c(I) \), then, by the above, there exist complete intersections \( C_1, \ldots, C_{m-1} \) in \( I \) such that \( \Delta_i I^{(t)} \subseteq C_i I^{(t-1)} \) for every \( t \geq 2 \). Note we can assume \( m \geq 3 \) (by Theorem 5.8). Then,

\[
\Delta_1 \cdots \Delta_{m-1} I^{(m)} \subseteq \Delta_1 \cdots \Delta_{m-2} C_{m-1} I^{(m-1)} \subseteq \ldots \subseteq C_1 C_2 \cdots C_{m-1} I \subseteq I^m.
\]

Therefore \( I^{(m)} F_c(I)^{m-1} \subseteq I^m \), and by Remark 5.2(1) this concludes the proof.

**5.10. Remark.** While it is now obvious that for \( m = 2 \) the above result provides an optimal exponent \( t_0 \) for Question 5.1 one may ask how tight the bound \( t_0 \leq m - 1 \) is in case \( m \geq 3 \). There are some examples for which the bound is sharp, and examples where actually lower bounds apply. Sharpness can be seen, e.g., in parts (ii) and (iii) of Example 6.3 (to be given in the next section), corresponding respectively to the cases \( m = 3 \) and \( m = 4 \). Finally, to illustrate the opposite situation, just pick \( I \subset k[x, y, z] \) as the ideal of 3 points in \( \mathbb{P}^2 \), then \( F_2(I) = m \) and by Theorem 5.5 it follows that the optimal \( t_0 \) is \( \lfloor \frac{m}{2} \rfloor \) which is strictly smaller than \( m - 1 \) for \( m \geq 3 \).

We now illustrate a couple of applications of Theorem 5.9. The first one, Corollary 5.11 provides a lower bound on the initial degree of \( I^{(m)} \) (in a fairly general setting), and the second one, Corollary 5.13 provides an approximation of \( I^{(m)} \) via the annihilator of the \( (c + 1) \)-th exterior power of \( I \).

Recall that for a homogeneous ideal \( I \) in a graded reduced ring \( S \), we clearly have \( \alpha(I^{(m)}) \leq \alpha(I^m) = m \alpha(I) \). Finding lower bounds on \( \alpha(I^{(m)}) \) is in general a much harder task. As a first consequence of Theorem 5.9 we obtain an explicit lower bound for any unmixed, generically complete intersection ideal \( I \). We remark that it applies not just to polynomial rings, but actually to any positively graded Cohen–Macaulay domain.

**5.11. Corollary.** In addition to Assumptions 4.7 assume \( S \) is a domain positively graded over a field and \( I \) is a homogeneous ideal. If \( I^{(m)} = I^m : F_c(I)^{t_0} \), then one has

\[
\alpha(I^{(m)}) \geq m \alpha(I) - t_0 \alpha(F_c(I)).
\]

In particular, \( \alpha(I^{(m)}) \geq m(\alpha(I) - \alpha(F_c(I))) + \alpha(F_c(I)) \).

**Proof.** By assumption \( F_c(I)^{t_0} I^{(m)} \subseteq I^m \), so \( \alpha(I^m) \leq \alpha(F_c(I)^{t_0} I^{(m)}) \). Since \( S \) is a domain, \( \alpha(I^m) = m \alpha(I) \) and \( \alpha(F_c(I)^{t_0} I^{(m)}) = \alpha(F_c(I)^{t_0}) + \alpha(I^{(m)}) = t_0 \alpha(F_c(I)) + \alpha(I^{(m)}) \). This proves the first inequality. The second one follows with \( t_0 = m - 1 \) by Theorem 5.9.

If \( t_0 \) is chosen optimally, one obtains equalities in some cases.
5.12. Example. In \( k[x, y, z] \) let \( I \) be any one of the following two ideals:

(i) \( I = (xy, xz, yz) \) (the defining ideal of the three coordinate points in \( P^2 \)),
(ii) \( I \) is the ideal of 2-minors of any \( 2 \times 3 \) matrix of general linear forms.

In either case by Theorem 5.5 one has \( I^{(m)} = I^m : F_2(I)^{\lceil \frac{m}{2} \rceil} \). Thus, Corollary 5.11 gives
\[
\alpha(I^{(m)}) \geq \alpha(I) - \left\lfloor \frac{m}{2} \right\rfloor = 2m - \left\lfloor \frac{m}{2} \right\rfloor = m + \left\lceil \frac{m}{2} \right\rceil.
\]

In the proof of Theorem 5.5 we have already seen that \( m + \left\lceil \frac{m}{2} \right\rceil = \alpha(I^{(m)}) \). Hence, in this example the lower bound in Corollary 5.11 is actually attained.

In general, however, the bound of Corollary 5.11 is not sharp. For example, in \( R = k[x, y, z] \), let \( I = (x(y^2 - z^2), y(z^2 - x^2), z(x^2 - y^2)) \) be the defining ideal of a set of 7 simple points in \( P^2 \) (this is the first of the so-called Fermat ideals). One has \( F_2(I) = m \) and, if \( m = 2 \), of course the smallest \( t_0 \) one can take is \( t_0 = m - 1 = 1 \); in this case \( \alpha(I^{(2)}) = 6 > 5 = 2 \cdot 3 - 1 \).

Our second application of Theorem 5.9 is an “approximation” result. It suggests that the annihilator of the \((c + 1)\)-th exterior power of \( I \), which is known to contain \( I \) if \( \text{char}(S) = 0 \), may be employed to study symbolic powers. We thank M. Mastroeni for pointing out that the rightmost inclusion in the statement below is actually an equality.

5.13. Corollary. Let \( S, I \) be as in Assumption 4.1 and let \( m \in \mathbb{Z}_+ \). If \( \text{char}(S) = 0 \) then
\[
I^m : (\text{ann} \wedge^{c+1} I)^{m-1} \subseteq I^{(m)} = I^m : (\text{ann} \wedge^{c+1} I)^{m-1} F_{c+1}(I)^{m-1}.
\]

Proof. Applying [8] Lemma 2.1], one gets inclusions
\[
(\text{ann} \wedge^{c+1} I) F_{c+1}(I) \subseteq F_c(I) \subseteq \text{ann} \wedge^{c+1} I.
\]

This yields
\[
I^m : (\text{ann} \wedge^{c+1} I)^{m-1} \subseteq I^m : F_c(I)^{m-1} \subseteq I^m : ((\text{ann} \wedge^{c+1} I) F_{c+1}(I))^{m-1},
\]
where the ideal in the middle is \( I^{(m)} \) by Theorem 5.9. Finally, both \( F_{c+1}(I) \) and \( \text{ann} \wedge^{c+1} I \) contain \( F_c(I) \), so in particular, \( (\text{ann} \wedge^{c+1} I) F_{c+1}(I) \) has height \( > c \), thus Remark 5.2 implies that the right-most inclusion is actually an equality.

6. A conjecture connecting symbolic defects and Jacobian ideals

We conclude the paper with a conjecture supported by a number of computations (performed with the computer algebra system Macaulay2 [14]) related to the main results of this paper.

For this section, we let \( S = k[x_1, \ldots, x_n] \) be a standard graded polynomial ring over a perfect field \( k \), with homogeneous maximal ideal \( m \). For a suitable ideal \( I \) of \( S \), we propose a conjecture connecting the module \( I^{(m)} / I^m \) to Jacobian ideals. In what follows, we denote the radical of \( I \) by \( \text{rad} \ I \). We first recall the following concept, introduced in [11].
6.1. **Definition.** Let $I$ be an unmixed or radical ideal $I$ of $S$ and $m \in \mathbb{Z}_+$. The $m$-th symbolic defect of $I$ is

$$s\text{defect}(I, m) = \mu(I^{(m)}/I^m),$$

i.e., the minimal number of generators of $I^{(m)}/I^m$.

Trivially, $s\text{defect}(I, 1) = 0$. Moreover, it is well-known that if $I$ can be generated by a regular sequence (i.e., a complete intersection ideal) then $s\text{defect}(I, m) = 0$ for every $m$; see [24, Appendix 6, Lemma 5].

6.2. **Notation.** Given a polynomial $g \in S$, we recall that the Jacobian (or gradient) ideal of $g$ is the ideal

$$J(g) = \left( \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n} \right).$$

(If char$(k) = p > 0$ one uses divided powers to compute “partial derivatives”.) Since $k$ is perfect, if $I$ is a radical ideal of $S$ then for any $g \in I^{(2)}$ we have $J(g) \subseteq I$ by virtue of the Zariski-Nagata theorem (see [4, subsection 2.1]), and hence $\text{rad} J(g) \subseteq I$.

It is then natural to ask when equality holds. We analyze some examples below, where we also compute symbolic defects.

6.3. **Example.** (char$(k) = 0$.) Let $I$ be the ideal of $S = k[x, y, z]$ generated by the maximal minors of the $2 \times 3$ matrix

$$
\begin{pmatrix}
  x + y + z & x + y & y + z \\
  x & y & z
\end{pmatrix}.
$$

Then $I$ is a radical unmixed ideal with $\text{ht}(I) = 2$ and $F_2(I) = m$.

(i) We have $s\text{defect}(I, 2) = 1$. Indeed, Theorem 5.8 yields $I^{(2)} = I^2$: $m$, and one obtains

$$I^{(2)} = (I^2, g) \quad \text{with } g = x^3 - xy^2 + y^3 - x^2z - 3xyz + y^2z + 2y^2 + z^3.$$

Here, $J(g) = I$. In particular, $J(g)$ is radical.

(ii) We have $s\text{defect}(I, 3) = 3$. Explicitly, applying Theorem 5.9,

$$I^{(3)} = I^3: F_2(I)^2 = I^3: m^2 = (I^3, g_1, g_2, g_3), \quad \text{where}$$

$$g_1 = x^5 - 2x^3y^2 + x^2y^3 + xy^4 - y^5 - x^4z - 4x^3yz + 2x^2y^2z + 4xy^2z - 3y^3z^2 + x^2z^3 - yz^4,$$

$$g_2 = x^4y + 3x^3y^2 - x^2y^3 - 2xy^4 + 3y^5 - 3x^4z - 2x^3yz - 3x^2y^2z - 10xyz + 2y^4z + 2x^3z^2 + 10x^2yz^2 + 3xyz^2 + 4y^3z^2 + 3y^2z^3 - 2xyz^3 - 3xz^4 - 3yz^4 - z^5,$$

$$g_3 = x^3y^2 - xy^4 + y^5 - x^4z - 4xy^3z + y^4z + x^3z^2 + 3x^2yz^2 - y^2z^2 + 2y^3z^2 - 2xyz^3 + y^2z^3 - xz^4.$$

By a computation we find that $\text{rad} (J(g_1) + J(g_2) + J(g_3)) = I$. 

A FORMULA FOR SYMBOLIC POWERS
(iii) We have $\text{sdefect}(I, 4) = 4$. By Theorem 5.9,
\[ I^{(4)} = I^4: F_2(I)^3 = I^4: m^3 = (I^4, g_1, g_2, g_3, g_4) \]
for suitable polynomials $g_1, g_2, g_3, g_4 \in S$ of large monomial support (which we will not write down explicitly). Their degrees are $6, 7, 7, 7$. Once again it can be confirmed that $\text{rad} (J(g_1) + J(g_2) + J(g_3) + J(g_4)) = I$.

6.4. Example. (char($k$) = 0.) Let $I$ be the ideal of $S = k[x, y, z, w, t]$ generated by the 2-minors of the $3 \times 3$ generic Hankel matrix
\[ \begin{pmatrix} x & y & z \\ y & z & w \\ z & w & t \end{pmatrix}. \]
Then $I$ is a radical unmixed ideal with $\text{ht}(I) = 3$ and $F_5(I) = m^3$.

(i) We have $\text{sdefect}(I, 2) = 1$. More precisely, applying Theorem 5.9, one computes
\[ I^{(2)} = I^2: m^3 = (I^2, g) \]
where $g = z^3 - 2yzw + xw^2 + y^2t - xzt$.
In this case, we have $\text{rad} J(g) = I$, and $J(g)$ is not radical.

(ii) We have $\text{sdefect}(I, 3) = 6$. Indeed, using Theorem 5.9
\[ I^{(3)} = I^3: F_3(I)^2 = I^3: m^3 = (I^3, g_1, \ldots, g_6), \]
where
\[
\begin{align*}
g_1 &= y^3z^3 - xz^4 - 2yzw^2 + 2xyz^2 - 6x^2z + 2x^2t, \\
g_2 &= yz^4 - 4y^2z^2w - xz^3w + 3xzyw^2 - x^2w^2 - 6xyzt + x^2zt, \\
g_3 &= yz^3w - 2y^2zw^2 + xyw^2 - xz^2 + 3xzt - 2xyzt + x^2zt, \\
g_4 &= z^4w - 2yzw^2 - xz^3 + 3y^2zt - x^2w^2 - y^3t + 2xzt, \\
g_5 &= z^5 - 2y^2zw^2 + xzw^2 + 2xyw^2 + y^2zt - 4x^2t + 2xyzt - 3x^2w^2 - 3xy^2t^2 + 3x^2zt^2, \\
g_6 &= z^6 - 2y^2zw^2 + xw^2 + z^2t + 2yzw^2 + y^2w^2 - 2xzw^2 - y^3w^2 + z^2t^2.
\end{align*}
\]
A computation shows that $\text{rad} (J(g_1) + \ldots + J(g_6)) = I$.

These examples are just a few among a much larger set of experiments we computed, for various values of $n$ and $m$. All of them support the following conjecture.

6.5. Conjecture. (char($k$) = 0.) Let $I$ be a radical unmixed ideal of $S$, and $m \geq 2$. If $s := \text{sdefect}(I, m) \geq 1$, then we can write $I^{(m)} = (I^m, g_1, \ldots, g_s)$ for some $g_1, \ldots, g_s \in S$ satisfying
\[ \text{rad} (J(g_1) + \ldots + J(g_s)) = I. \]

6.6. Remark. It is worth observing that:

(a) The inclusion $\text{rad} (J(g_1) + \ldots + J(g_s)) \subseteq I$ holds by the Zariski–Nagata Theorem;
The $g_i$’s must be chosen carefully – there are examples of specific $g_i$’s such that $I^{(m)} = (I^m, g_1, \ldots, g_s)$, but $g_1, \ldots, g_s$ do not satisfy Conjecture 6.5. To illustrate it, we now provide two examples where $S, I$ are graded but the $g_i$’s cannot be homogeneous – see Examples 6.7 and 6.8 below. Note that these symbolic powers even have symbolic defect 1.

The first example is a squarefree monomial ideal.

6.7. Example. (char $(k) \neq 2$.) Consider the ideal $I = (xy, xz, yz, tu) \subset k[x, y, z, t, u]$. Then $I$ is a radical ideal having precisely 6 associated primes of height 3. One can check that $I^{(2)} = (I^2, xyz)$, so sdefect$(I, 2) = 1$. Since $I^2$ is generated in degree 4, $g := xyz$ is the only homogeneous element $h \in S$ with the property that $I^{(2)} = (I^2, h)$. However, $J(g) = (xy, xz, yz)$, and thus we observe that $\text{rad } J(g) \neq I$.

On the other hand, we can also write $I^{(2)} = (I^2, g_1)$, $g_1 := g + t^2u^2 = xyz + t^2u^2$ and we clearly have $\text{rad } J(g_1) = \text{rad } (xy, xz, yz, t^2u, tu^2) = I$. So, for the conjecture to be true in this case one needs to take an element $g_1$ which is not homogeneous.

The second example is more geometric in nature.

6.8. Example. Let $I$ be the ideal of 5 general points in $\mathbb{P}^2$, then $I$ is generated by a quadric and two cubics, and $I^{(2)}/I^2$ is generated by a single quintic, so sdefect$(I, 2) = 1$. Macaulay2 [14] computations suggest that no homogeneous element $g_1$ of degree 5 in $I^{(2)}$ satisfies Conjecture 6.5.

REFERENCES

[1] J. Biermann, H. de Alba, F. Galetto, S. Murai, U. Nagel, A. O’Keefe, T. Römer, A. Seceleanu, Betti numbers of symmetric shifted ideals, J. Algebra 560 (2020), 312–342.
[2] S. Cooper, G. Fatabbi, E. Guardo, A. Lorenzini, J. Migliore, U. Nagel, A. Seceleanu, J. Szpond, A. Van Tuyl, Symbolic powers of codimension two Cohen-Macaulay ideals, Comm. Algebra 48 (2020), 4663–4680.
[3] G. Cornuéjols, M. Conforti, A decomposition theorem for balanced matrices, Integer Programming and Combinatorial Optimization 74 (1990), 147–169.
[4] H. Dao, A. De Stefani, E. Grifo, C. Huneke, L. Núñez-Betancourt, Symbolic Powers of Ideals, In: Singularities and Foliations. Geometry, Topology and Applications, pp. 387–432. Springer Proceedings in Mathematics & Statistics 222. Springer, Cham. (2018).
[5] M. DiPasquale, B. Drabkin, On resurgence via asymptotic resurgence, J. Algebra 587 (2021), 64–84.
[6] L. Ein, R. Lazarsfeld, K. E. Smith, Uniform bounds and symbolic powers on smooth varieties, Invent. Math. 144 (2001), 241–252.
[7] D. Eisenbud, Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics 150. Springer-Verlag, New York, 1995.
[8] D. Eisenbud, M. L. Green, Ideals of minors in free resolutions, Duke Math. J. 75 (1994), 339–352.
[9] D. Eisenbud, M. Hochster, A Nullstellensatz with nilpotents and Zariski’s main lemma on holomorphic functions, J. Algebra, 58(1) (1979), 157–161.
[10] D. Eisenbud, B. Mazur, Evolutions, symbolic squares, and Fitting ideals, J. Reine Angew. Math. 488 (1997), 189–201.
[11] F. Galetto, A. V. Geramita, Y.-S. Shin, A. Van Tuyl, The symbolic defect of an ideal, J. Pure Appl. Algebra 223 (2019), 2709–2731.
[12] I. Gitler, C. E. Valencia, R. Villarreal, A note on Rees algebras and the MFMC property, Beiträge Algebra Geom. 48 (2007), 141–150.
[13] E. Grifo, Topics in commutative algebra: Symbolic Powers. https://www.math.utah.edu/agtrtg/commutative-algebra/Grifo-symbolic-powers.pdf
[14] D. R. Grayson, M. E. Stillman, Macaulay2, a software system for research in algebraic geometry. Available at [http://www.math.uiuc.edu/Macaulay2/]
[15] B. Harbourne, C. Huneke, Are symbolic powers highly evolved?, J. Ramanujan Math. Soc. 28 (2013), 311–330.
[16] J. Herzog, T. Hibi, and N. V. Trung, Symbolic powers of monomial ideals and vertex cover algebras, Adv. Math. 210 (2007), 304–322.
[17] M. Hochster, C. Huneke, Comparison of symbolic and ordinary powers of ideals, Invent. Math. 147 (2002), 349–369.
[18] C. Huneke, J. Ribbe, Symbolic squares in regular local rings, Math. Z. 229 (1998), 31–44.
[19] C. Huneke, B. Ulrich, The structure of linkage, Ann. of Math. 126 (1987), 277–334.
[20] P. Mantero, The structure and free resolutions of the symbolic powers of star configurations of hypersurfaces, Trans. Amer. Math. Soc. 373 (2020), 8785–8835.
[21] H. Matsumura, Commutative Ring Theory, 2nd ed. Cambridge Stud. Adv. Math. 8, Cambridge Univ. Press, 1989. Translated from the Japanese by M. Reid.
[22] J. Migliore, Introduction to Liaison theory and Deficiency modules, Progress in Math. 165, Birkhäuser, 1998.
[23] Z. Ramos, A. Simis, Symbolic powers of perfect ideals of codimension 2 and birational maps, J. Algebra 413 (2014), 153–197.
[24] O. Zariski, P. Samuel, Commutative Algebra, vol. II, Springer, 1960.