A Bayesian nonparametric chi-squared goodness-of-fit test

June 20, 2016

Reyhaneh Hosseini and Mahmoud Zarepour

Department of Mathematics and Statistics
University of Ottawa

Abstract

The Bayesian nonparametric inference and Dirichlet process are popular tools in Bayesian statistical methodologies. In this paper, we employ the Dirichlet process in a hypothesis testing to propose a Bayesian nonparametric chi-squared goodness-of-fit test. In our new Bayesian nonparametric approach, we consider the Dirichlet process as the prior for the distribution of the data and carry out the test based on the Kullback-Leibler distance between the updated Dirichlet process and the hypothesized distribution. We prove that this distance asymptotically converges to the same chi-squared
distribution as the classical frequentist’s chi-squared test does. Moreover, a Bayesian nonparametric chi-squared test of independence for a contingency table is described. In addition, by computing the Kullback-Leibler distance between the Dirichlet process and the hypothesized distribution, a method to obtain an appropriate concentration parameter for the Dirichlet process is presented.

**Keywords:** Bayesian nonparametric inference, Dirichlet process, Pearson’s chi-squared test, chi-squared test of independence, goodness-of-fit test, Brownian bridge, Kullback-Leibler distance.

**MSC 2010:** Primary 62G20; secondary 62G10.

1 **Introduction**

The Bayesian nonparametric plays a crucial role in statistical inference. The Dirichlet process perhaps is the most popular prior in Bayesian nonparametric statistics and it has been applied in many different areas of statistical inference. The most common applications of Dirichlet process are in density estimation and clustering via mixture models. See for instance, Neal [31], Lo [27] and Escobar and West [12]. In this paper, we suggest a Bayesian nonparametric chi-squared goodness-of-fit test based on the Kullback-Leibler distance between the posterior Dirichlet process and the hypothesized distribution.

There are many one-sample and two-sample parametric goodness-of-fit tests in the literature. See for example, D’Agostino [10] for a review. The chi-squared test examines whether the data has a specified distribution $F_0$, i.e., the null hypothesis is given as $H_0 : F = F_0$ where $F_0$ is the true distribution for the observed data. Some extensions of chi-squared goodness-of-fit test to Bayesian model assessment where the test statistic is based on the
posterior distribution, are described by Johnson [23] and Johnson [24].

In Bayesian nonparametric inference, there are two strategies of goodness-of-fit test. The first strategy considers a prior for the true distribution of data and constructs the test based on the distance between the posterior distribution and the proposed one. For example, Muliere and Tardella [30], Swartz [33], Al Labadi and Zarepour [3, 4] considered the Dirichlet process prior and the Kolmogorov distance. Al Labadi and Zarepour [4] and Al Labadi et al. [2] carried out a goodness-of-fit test and a two-sample goodness-of-fit test, respectively by considering the Dirichlet process as a prior and the test statistic based on the Kolmogorov distance. Viele [36] used the Dirichlet process and the Kullback-Leibler distance for testing the discrete distributions. Hsieh [20] considered the Polya tree model as the prior and measured the Kullback-Leibler distance for testing the continuous distributions.

The second strategy is conducted by embedding the hypothesized model $H_0$ in an alternative model $H_1$ and placing a prior on that. To examine the hypothesized model, the Bayes factor is used as a measure of evidence against the hypothesized model. For example, Carota and Parmigiani [9] and Florens et al. [15] used a Dirichlet process prior for the alternative model. Tokdar and Martin [34] carried out a Bayesian test for normality by considering a Dirichlet process mixture for the alternative model. Some authors used other Bayesian nonparametric priors. For instance, Holmes et al. [19] described a Bayesian nonparametric two sample hypothesis testing based on a Polya tree prior. In order to test for the normal distribution, Berger and Guglielmi [5] considered a mixture of Polya trees for the alternative model distribution, while Verdinelli and Wasserman [35] suggested a mixture of Gaussian processes.

Our new proposed chi-squared goodness of fit test is based on the first approach discussed above. We consider a Dirichlet process prior for the distribution of the observed data
and define the chi-squared test statistic based on the Kullback-Leibler distance between the Dirichlet process posterior and the hypothesized distribution. In fact, in our Bayesian nonparametric approach, the test proceeds by constructing the chi-squared test statistic based on the distance between the observed probabilities obtained by the Dirichlet process posterior and the expected probabilities. Indeed, instead of counting the observed frequencies in each bin, we place a prior on the distribution of the data. The probability of each bin is obtained by the exact posterior probability of that bin. Then, our new test statistic compares the posterior probabilities with the probabilities under the null hypothesis. In this procedure, based on the suggested Dirichlet prior, we know the exact distribution of the test statistic. Using a similar approach, we also determine an appropriate concentration parameter for the Dirichlet process which is required to decide on an appropriate prior.

The outline of the paper is organized as follows. In Section 2, we give an essential background on Dirichlet process and its properties. In Section 3, we briefly review the definition of the Kullback-Leibler divergence. Following this, we obtain the Kullback-Leibler distance between the Dirichlet process and a continuous distribution and compute its mean and variance. Section 4 discusses a Bayesian nonparametric chi-squared goodness-of-fit test based on the Kullback-Leibler distance between the Dirichlet process posterior and the hypothesized distribution. In Section 5, we extend our suggested chi-squared test to present a Bayesian nonparametric chi-squared test of independence of two random variables. We also describe a method to obtain an appropriate concentration parameter based on the Kullback-Leibler distance between the Dirichlet process and the proposed distribution. Simulation studies of the tests with a data illustration appear in Section 6. In the final section, we conclude with a brief discussion and the Appendix contains the theoretical results.
2 Dirichlet Process

In this section, we review the construction, various properties and some series representations of the Dirichlet process. The Dirichlet process was initially formalized by Ferguson [13] for general Bayesian statistical modeling as a distribution over probability distributions.

**Definition 2.1.** (Ferguson [13]) Let $\mathcal{X}$ be a set, $\mathcal{A}$ be a $\sigma-$field of subsets of $\mathcal{X}$, $H$ be a probability measure on $(\mathcal{X}, \mathcal{A})$ and $\alpha > 0$. A random probability measure $P$ with parameters $\alpha$ and $H$ is called a Dirichlet process (denoted by $P \sim DP(\alpha H)$ ) on $(\mathcal{X}, \mathcal{A})$ if for any finite measurable partition $\{A_1, \ldots, A_k\}$ of $\mathcal{X}$, the joint distribution of the random variables $P(A_1), \ldots, P(A_k)$ is a $k$-dimensional Dirichlet distribution with parameters $\alpha H(A_1), \ldots, \alpha H(A_k)$, where $k \geq 2$.

We assume that if $H(A_k) = 0$, then $P(A_k) = 0$ with probability one. Then, a Dirichlet process is parameterized by $\alpha$ and $H$ which are called the concentration parameter and the base distribution, respectively. The base distribution is also the mean of the Dirichlet process, i.e., for any measurable set $A \subset \mathcal{X}$, $E(P(A)) = H(A)$. One of the most remarkable properties of the Dirichlet process is that it satisfies the conjugacy property. Let $X_1, \ldots, X_m$ be an i.i.d. sample from $P \sim DP(\alpha H)$. The posterior distribution of $P$ given $X_1, \ldots, X_m$ is a Dirichlet process with parameters

$$
\alpha_m^* = \alpha + m \quad \text{and} \quad H_m^* = \frac{\alpha}{\alpha + m} H + \frac{m}{\alpha + m} \sum_{i=1}^{m} \delta_{X_i}.
$$

(2.1)

and denoted by $P_m^* = (P \mid X_1, \ldots, X_m) \sim DP(\alpha_m^* H_m^*)$, where $\delta_X(\cdot)$ is the Dirac measure, i.e., $\delta_X(A) = 1$ if $X \in A$ and 0 otherwise.

As it is seen in (2.1), the posterior base distribution $H_m^*$ is a weighted average of $H$ and
the empirical distribution \( F_m = \sum_{i=1}^{m} \frac{1}{m} \delta_{X_i} \). Thus, for large values of \( \alpha \), \( H_{m}^\alpha \overset{a.s.}{\to} H \). On the other hand, as \( \alpha \to 0 \) or as the number of observations \( m \) grows large, \( H_{m}^\alpha \) becomes non-informative in the sense that \( H_{m}^\alpha \) is just given by the empirical distribution and is a close approximation of the true underlying distribution of \( X_i, i = 1, \ldots, m \). This confirms the consistency property of the Dirichlet process, i.e., the posterior Dirichlet process approaches the true underlying distribution. For a discussion about the consistency property of Dirichlet process, see Ghosal [16] and James [22].

A sum representation of Dirichlet process is presented by Ferguson [13] based on the work of Ferguson and Klass [14]. Specifically, let \((\theta_i)_{i \geq 1}\) be a sequence of i.i.d. random variables with common distribution \( H \) and \((E_k)_{k \geq 1}\) be a sequence of i.i.d. random variables from the exponential distribution with mean 1. If \( \Gamma_i = E_1 + \cdots + E_i \) and \((\Gamma_i)_{i \geq 1}\) are independent from \((\theta_i)_{i \geq 1}\), then,

\[
P = \sum_{i=1}^{\infty} \frac{L^{-1}(\Gamma_i)}{\sum_{i=1}^{\infty} L^{-1}(\Gamma_i)} \delta_{\theta_i} = \sum_{i=1}^{\infty} p_i \delta_{\theta_i} \tag{2.2}
\]

is a Dirichlet process with parameters \( \alpha \) and \( H \) where \( L(x) = \alpha \int_{x}^{\infty} t^{-1} e^{-t} dt, \ x > 0 \) and \( L^{-1}(y) = \inf \{ x > 0 : L(x) \geq y \} \). Ishwaran and Zarepour [21] introduced a finite sum approximation for the Dirichlet process which is easier to work with. Let \( p = (p_{1,n}, \ldots, p_{n,n}) \) has a Dirichlet distribution with parameters \((\alpha/n, \ldots, \alpha/n)\) denoted by \( \text{Dir}(\alpha/n, \ldots, \alpha/n) \) and \((\theta_i)_{1 \leq i \leq n}\) be a sequence of i.i.d. random variables with distribution \( H \) and independent of \((p_{i,n})_{1 \leq i \leq n}\). Also, let \((G_{i,n})_{1 \leq i \leq n}\) be i.i.d. random variables from \( \text{Gamma}(\alpha/n, 1) \) distribution and \( p_{i,n} = G_{i,n}/G_n \), where \( G_n = G_{1,n} + \cdots + G_{n,n} \). Then,

\[
P_n = \sum_{i=1}^{n} p_{i,n} \delta_{\theta_i} = \sum_{i=1}^{n} \frac{G_{i,n}}{G_n} \delta_{\theta_i} \tag{2.3}
\]
is called a finite-dimensional Dirichlet process and approximates the Ferguson’s Dirichlet process weakly. Another finite sum representation of the Dirichlet process with monotonically decreasing weights is presented in Zarepour and Al Labadi [39]. Specifically, let $\theta_i$ be a sequence of i.i.d. random variables with values in $\mathcal{X}$ and common distribution $H$ and independent of $(\Gamma_i)_{1 \leq i \leq n+1}$. Let $X_n \sim \text{Gamma}(\alpha/n, 1)$ and define

$$G_n(x) = \Pr(X_n > x) = \frac{1}{\Gamma(\alpha/n)} \int_x^\infty t^{(\alpha/n)-1}e^{-t}dt$$

and

$$G_n^{-1}(y) = \inf\{x: G_n(x) \geq y\}.$$ 

Then, as $n \to \infty$,

$$P_n = \sum_{i=1}^{n} \frac{G_n^{-1}(\frac{\Gamma_i}{\Gamma_{n+1}})}{\sum_{i=1}^{n} G_n^{-1}(\frac{\Gamma_i}{\Gamma_{n+1}})} \delta_{\theta_i} \xrightarrow{a.s.} P = \sum_{i=1}^{\infty} \frac{L^{-1}(\Gamma_i)}{\sum_{i=1}^{\infty} L^{-1}(\Gamma_i)} \delta_{\theta_i}. \quad (2.4)$$

If we define

$$p_{i,n} = \frac{G_n^{-1}(\frac{\Gamma_i}{\Gamma_{n+1}})}{\sum_{i=1}^{n} G_n^{-1}(\frac{\Gamma_i}{\Gamma_{n+1}})}, \quad (2.5)$$

then, $P_n$ can be written as

$$P_n = \sum_{i=1}^{n} p_{i,n} \delta_{\theta_i}. \quad (2.6)$$

This finite sum representation converges almost surely to Ferguson’s representation and empirically converges faster than the other representations. For other sum representations of Dirichlet process, see for example, [?] and Bondesson [S]. In the next section, we will discuss computing the Kullback-Leibler distance between the Dirichlet process and a continuous
distribution and its mean and variance.

3 Kullback-Leibler distance between the Dirichlet process and a continuous distribution

The Kullback-Leibler distance that measures the distance between two distributions introduced by Kullback and Leibler [26]. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two probability measures for discrete random variables on a measurable space $(\Omega, \mathcal{F})$. The Kullback-Leibler divergence between $\mathcal{P}$ and $\mathcal{Q}$ is defined as

$$D_{KL}(\mathcal{P} \parallel \mathcal{Q}) = \sum_i \mathcal{P}(i) \log \left( \frac{\mathcal{P}(i)}{\mathcal{Q}(i)} \right).$$

(3.1)

For continuous probability measures $\mathcal{P}$ and $\mathcal{Q}$ with $\mathcal{P}$ absolutely continuous with respect to $\mathcal{Q}$, the Kullback-Leibler distance is written as

$$D_{KL}(\mathcal{P} \parallel \mathcal{Q}) = \int \log \left( \frac{d\mathcal{P}}{d\mathcal{Q}} \right) d\mathcal{P}$$

where $\frac{d\mathcal{P}}{d\mathcal{Q}}$ is the Radon-Nikodym derivative of $\mathcal{P}$ with respect to $\mathcal{Q}$. Let $\mathcal{P} \ll \lambda$ and $\mathcal{Q} \ll \lambda$ where $\lambda$ is the Lebesgue measure. If the densities of $\mathcal{P}$ and $\mathcal{Q}$ with respect to Lebesgue measure are denoted by $p(x)$ and $q(x)$, respectively, then the Kullback-Leibler distance is written as

$$D_{KL}(\mathcal{P} \parallel \mathcal{Q}) = \int_{\mathbb{R}} p(x) \log \left( \frac{p(x)}{q(x)} \right) dx.$$  

(3.2)

We compute the distance between the random distribution $P$ from a Dirichlet process $DP(\alpha H)$ and a continuous distribution $F$ with density $f(x)$. Since $P$ is a discrete mea-
sure and $F$ is continuous, we estimate the density $f(x)$ by its histogram estimator on a partitioned space. Also, since the Kullback-Leibler distance is not symmetric, we compute both distances $D_{KL}(P \parallel F)$ and $D_{KL}(F \parallel P)$.

**Lemma 3.1.** Let $H$ and $F$ be two distributions defined on the same space $X$ and $P_n = \sum_{i=1}^{n} p_{i,n}\delta_{\theta_i}$ be a random distribution as defined in (2.3), i.e., $\theta_1, \ldots, \theta_n$ are i.i.d. generated from $H$ with corresponding order statistics $\theta_{(1)}, \ldots, \theta_{(n)}$. We have

$$D_{KL}(P_n \parallel F) = -\mathcal{H}(p) - \sum_{i=1}^{n} p_{i,n} \log(q_i)$$

(3.3)

and

$$D_{KL}(F \parallel P_n) = -\mathcal{H}(q) - \sum_{i=1}^{n} q_i \log(p_{i,n})$$

(3.4)

where $\mathcal{H}(p) = -\sum_{i=1}^{n} p_{i,n} \log(p_{i,n})$ is the entropy of $P_n$ and $\mathcal{H}(q) = -\sum_{i=1}^{n} q_i \log(q_i)$ with $q_i = \frac{\Delta F(x_i)}{\Delta x_i}$.

**Proof.** See the Appendix.

The mean and the variance of the Kullback-Leibler divergences (3.3) and (3.4) are given in the following Proposition and Remark.

**Proposition 3.1.** Let $H$ and $F$ be distributions defined on the same space $X$ and $P_n = \sum_{i=1}^{n} p_{i,n}\delta_{\theta_i}$ be a random distribution as defined in (2.3), i.e., $\theta_1, \ldots, \theta_n$ are i.i.d. generated from $H$ with corresponding order statistics $\theta_{(1)}, \ldots, \theta_{(n)}$. Then, the mean and the variance of the Kullback-Leibler divergence (3.3) are given as

$$E(D_{KL}(P_n \parallel F)) = n \left( \psi \left( \frac{\alpha}{n} + 1 \right) - \psi(\alpha + 1) \right) - \frac{1}{n} \sum_{i=1}^{n} \log(q_i)$$

(3.5)

9
and

\[
Var(D_{KL}(P_n \mid F)) = \sum_{i=1}^{n} \left\{ \Var(p_{i,n} \log(p_{i,n})) + (\log(q_i))^2 \Var(p_{i,n}) \right\} \\
-2 \sum_{i=1}^{n} \{ \log(q_i) \Cov(p_{i,n} \log(p_{i,n}), p_{i,n}) \} \\
+2 \sum_{i<j} \{ \Cov(p_{i,n} \log(p_{i,n}), p_{j,n} \log(p_{j,n})) + \log(q_i) \log(q_j) \Cov(p_{i,n}, p_{j,n}) \} \\
-4 \sum_{i<j} \{ \log(q_i) \Cov(p_{i,n} \log(p_{i,n}), p_{j,n} \log(p_{j,n})) \}, \tag{3.6}
\]

respectively, where

\[
\Var(p_{i,n}) = \frac{n-1}{n^2(\alpha+1)},
\]
\[
\Cov(p_{i,n}, p_{j,n}) = \frac{-1}{n^2(\alpha+1)},
\]
\[
\Var(p_{i,n} \log(p_{i,n})) = \frac{(\alpha/n + 1)}{n(\alpha + 1)} \left( \psi_1 \left( \frac{\alpha}{n} + 2 \right) - \psi_1(\alpha + 2) + \left[ \psi \left( \frac{\alpha}{n} + 2 \right) - \psi(\alpha + 2) \right]^2 \right) \\
- \left( \psi \left( \frac{\alpha}{n} + 1 \right) - \psi(\alpha + 1) \right)^2,
\]
\[
\Cov(p_{i,n} \log(p_{i,n}), p_{i,n}) = \frac{(\alpha/n + 1)}{n(\alpha + 1)} \left( \psi \left( \frac{\alpha}{n} + 2 \right) - \psi(\alpha + 2) \right) - \frac{1}{n} \left( \psi \left( \frac{\alpha}{n} + 1 \right) - \psi(\alpha + 1) \right),
\]
\[
\Cov(p_{i,n} \log(p_{i,n}), p_{j,n}) = \frac{\alpha}{n^2(\alpha + 1)} \left( \psi \left( \frac{\alpha}{n} + 1 \right) - \psi(\alpha + 2) \right) - \frac{1}{n} \left( \psi \left( \frac{\alpha}{n} + 1 \right) - \psi(\alpha + 1) \right),
\]
\[
\Cov(p_{i,n} \log(p_{i,n}), p_{j,n} \log(p_{j,n})) = \frac{\alpha}{n^2(\alpha + 1)} \left\{ \left( \psi \left( \frac{\alpha}{n} + 1 \right) - \psi(\alpha + 2) \right)^2 - \frac{\alpha \psi_1(\alpha + 2)}{n^2(\alpha + 1)} \right\} \\
- \left( \psi \left( \frac{\alpha}{n} + 1 \right) - \psi(\alpha + 1) \right)^2
\]

and \( \psi(\alpha) = \frac{d \ln(\Gamma(\alpha))}{d\alpha} \) and \( \psi_1(\alpha) = \frac{d^2 \ln(\Gamma(\alpha))}{d\alpha^2} = \frac{d \psi(\alpha)}{d\alpha} \) are called digamma and trigamma functions, respectively.

**Proof.** The proof is given in Appendix. \( \square \)

**Remark 3.1.** Let \( H \) and \( F \) be two distributions defined on the same space \( \mathcal{X} \) and \( P_n = \)
\[ \sum_{i=1}^{n} p_{i,n} \delta_{\theta_i} \] be the finite dimensional distribution as defined in (2.3), in which \( \theta_1, \ldots, \theta_n \) are i.i.d. generated from \( H \) with corresponding order statistics \( \theta_{(1)}, \ldots, \theta_{(n)} \). The mean and the variance of the Kullback-Leibler divergence (3.4) can be obtained as

\[
E(D_{KL}(F \parallel P_n)) = -\mathcal{H}(q) - \left( \psi\left(\frac{\alpha}{n}\right) - \psi(\alpha) \right)
\]

(3.7)

and

\[
Var(D_{KL}(F \parallel P_n)) = \sum_{i=1}^{n} q_i^2 \psi_1 \left( \frac{\alpha}{n} \right) - \psi_1(\alpha),
\]

(3.8)

respectively.

**Proof.** The proof is given in Appendix.

\[ \square \]

## 4 Bayesian nonparametric chi-squared goodness-of-fit test

The null hypothesis of the goodness-of-fit test is given as \( H_0 : F = F_0 \) where \( F \) is the true underlying distribution of the observed data and \( F_0 \) is some specified distribution. Pearson’s chi-squared goodness of fit test proceeds by partitioning the sample space into \( k \) non-overlapping bins and comparing the observed counts with the expected counts under the null hypothesis for each bin. Suppose \( X_1, \ldots, X_m \) is a sample of size \( m \) from the distribution \( F \). Let \( O_i \) and \( E_i, \ i = 1, \ldots, k \) denote the observed counts and the expected counts under the hypothesized distribution \( F_0 \) for bin \( k \), respectively. The Pearson’s goodness-of-fit test statistic is defined as
\[
X^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}
\]  \hspace{1cm} (4.1)

and \(X^2\) asymptotically converges to a chi-squared distribution with \(k - 1\) degrees of freedom.

To derive a counterpart Bayesian nonparametric test statistic similar to \(X^2\), we consider a Dirichlet process with parameters \(\alpha\) and \(H = F_0\) as a prior for the true distribution of data, i.e., \(X_1, \ldots, X_m \sim P\) where \(P \sim DP(\alpha H)\). Then, given \(X_1, \ldots, X_m\), the posterior distribution of \(P\) is a Dirichlet process \(P^*_m = (P \mid X_1, \ldots, X_m) \sim DP(\alpha^*_m H^*_m)\) where \(\alpha^*_m\) and \(H^*_m\) are as given in (2.1). We carry out the test based on the chi-squared distance between the posterior Dirichlet process \(P^*_m\) and the hypothesized distribution \(F_0\). Note that for the large sample size, both the Pearson’s goodness-of-fit test and the likelihood ratio test (the Kullback-Leibler distance) are asymptotically equivalent. For simplicity, we only consider Pearson’s goodness-of-fit test. Theorem 4.1 describes this connection and the asymptotic distribution for the law of the posterior distance for large sample size which is equivalent to the frequentist’s chi-squared test. This result follows from Al Labadi [11] and Lo [28], but we include a simple calculation to show the asymptotic distribution of

\[
D_{\alpha_m}(A) = \sqrt{m}(P^*_m(A) - H^*_m(A)) \quad \text{where } A \in \mathcal{X}.
\]

Notice that by having the partition \(\{A, A^c\}\) and the definition of Dirichlet process,

\[
P^*_m(A) \sim Beta(\alpha^*_m H^*_m(A), \alpha^*_m H^*_m(A^c)).
\]

Set \(Y = P^*_m(A)\) and \(v = H^*_m(A)\) where \(P^*_m\) and \(H^*_m\) are defined in (2.1). Then, for \(0 < y < 1\), the random variable \(Y\) has the probability density function

\[
f(y) = \frac{\Gamma(m)}{\Gamma(mv)\Gamma(m(1-v))} y^{\alpha^*_m v - 1}(1 - y)^{\alpha^*_m (1-v) - 1}.
\]
Thus, the probability density function of $Z = \sqrt{m}(Y - v)$ in its support is

$$f_Z(z) = \frac{\Gamma(m)}{\Gamma(mv)\Gamma(m(1-v))} \left( \frac{z}{\sqrt{m}} + v \right)^{\alpha_m v - 1} \left( 1 - \frac{z}{\sqrt{m}} - v \right)^{\alpha_m (1-v) - 1}.$$

(4.2)

By Scheffé’s theorem (Billingsley [7], page 29), we need to show that

$$f_Z(z) \to \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{z^2}{2\sigma^2} \right\},$$

where $\sigma^2 = F(A)(1 - F(A))$. By Stirling’s formula, we have

$$\Gamma(x) \approx \sqrt{2\pi x} x^{x-\frac{1}{2}} e^{-x} \text{ as } x \to \infty,$$

where we use the notation $f(x) \approx g(x)$ as $x \to \infty$ if $\lim_{x \to \infty} f(x)/g(x) = 1$. From (2.1), as $m \to \infty$, $H_m^{\alpha,\xi} \Rightarrow F$ and $\alpha_m = \alpha + m \approx m$. Then, the equation (4.2) can be rewritten as

$$f_Z(z) = \frac{\Gamma(m)}{\Gamma(mv)\Gamma(m(1-v))} \left( \frac{z}{\sqrt{m}} + v \right)^{mv - 1} \left( 1 - \frac{z}{\sqrt{m}} - v \right)^{m(1-v) - 1},$$

where $v = F(A)$. Then,
\[
\lim_{m \to \infty} f_Z(z) = \frac{1}{\sqrt{2\pi}} \lim_{m \to \infty} \left\{ \left( \frac{z}{\sqrt{m}} + v \right)^{m-1} \left( 1 - \frac{z}{\sqrt{m}} - v \right)^{m(1-v)-1} \right\}
\]

\[
= \frac{1}{\sqrt{2\pi v(1-v)}} \lim_{m \to \infty} \left\{ \left( \frac{z}{\sqrt{m}} + v \right)^{m-1} \left( 1 - \frac{z}{\sqrt{m}} - v \right)^{m(1-v)-1} \right\}
\]

\[
= \frac{1}{\sqrt{2\pi v(1-v)}} \lim_{m \to \infty} \left\{ \left( 1 + \frac{z}{\sqrt{mv}} \right)^{m-1} \left( 1 - \frac{z}{\sqrt{m(1-v)}} \right)^{m(1-v)-1} \right\}
\]

\[
= \frac{1}{\sqrt{2\pi v(1-v)}} \exp \left\{ \lim_{m \to \infty} m \ln (\eta_m) \right\}, \quad (4.3)
\]

where

\[
\eta_m = \left( 1 + \frac{z}{\sqrt{mv}} \right)^{v} \left( 1 - \frac{z}{\sqrt{m(1-v)}} \right)^{1-v}.
\]

Therefore,

\[
\lim_{m \to \infty} m \ln (\eta_m) = \lim_{m \to \infty} \frac{1}{1/m} \left\{ v \ln \left( 1 + \frac{z}{\sqrt{mv}} \right) + (1-v) \ln \left( 1 - \frac{z}{\sqrt{m(1-v)}} \right) \right\}.
\]

By applying the L’Hospital’s rule, we obtain
\[
\lim_{m \to \infty} m \ln (\eta_m) = \lim_{m \to \infty} \left( -m^2 \left\{ \frac{-vz}{2em^{3/2}} + \frac{(1-v)z}{2(1-v)m^{3/2}} \right\} \right)
= \lim_{m \to \infty} \frac{m}{2} \left\{ \frac{vz}{\sqrt{mv} + z} - \frac{(1-v)z}{\sqrt{m(1-v)} - z} \right\}
= \lim_{m \to \infty} \frac{m}{2} \left\{ \frac{-z^2}{(\sqrt{mv} + z)(\sqrt{m(1-v)} - z)} \right\}
= \frac{-z^2}{2v(1-v)}. \tag{4.4}
\]

Substituting (4.4) in (4.3) completes the proof of normality of \( D_{am}(A) = \sqrt{m}(P^*_m(A) - \mu^*_m(A)) \). A similar method proves that as \( m \to \infty \), for any partition \( \{A_1, \ldots, A_k\} \) of the space \( \mathcal{X} \),

\[
(D_{am}(A_1), D_{am}(A_2), \ldots, D_{am}(A_k)) \overset{d}{\to} (B_F(A_1), B_F(A_2), \ldots, B_F(A_k)),
\]

where \( B_F \) is the Brownian bridge.

Remark 4.1. A Gaussian process \( \{B_F(A), A \in \mathcal{X}\} \) is called a Brownian bridge if \( E(B_F(A)) = 0 \) and \( \text{Cov}(B_F(A_i), B_F(A_j)) = F(A_i \cap A_j) - F(A_i) \cap F(A_j) \), where \( A_i, A_j \in \mathcal{X} \). Now we can imply the following Lemma.

Lemma 4.1. Let \( X_1, \ldots, X_m \) be a random sample from the distribution \( H \). If \( P^*_m \) is the Dirichlet process posterior given \( X_1, \ldots, X_m \). Then, as \( m \to \infty \),

\[
D_{am}(\cdot) = \sqrt{m}(P^*_m(\cdot) - \mu^*_m(\cdot)) \overset{d}{\to} B_F(\cdot).
\]

For a detailed proof similar to what we presented here, see Al Labadi [1]. Also, see James
Al Labadi \cite{labadi} proved that as $\alpha \to \infty$, $D_\alpha(\cdot) = \sqrt{\alpha}(P(\cdot) - H(\cdot)) \overset{d}{\to} B_H(\cdot)$. Theorem \ref{thm:asymptotic_dist} describes the asymptotic distribution of the posterior distance for a large sample size.

**Theorem 4.1.** Suppose $X_1, \ldots, X_m$ is a random sample from a distribution $F$ on sample space $\mathcal{X}$. Let $P \sim DP(\alpha H)$ and $P_m^* = (P \mid X_1, \ldots, X_m) \sim DP(\alpha_m^* H_m^*)$, where $\alpha_m^* = \alpha + m$ and $H_m^* = \frac{\alpha}{\alpha + m} H + \frac{m}{\alpha + m} \sum_{i=1}^{m} \delta_{X_i}$. Let $D_{KL}(P_m^* \parallel H_m^*)$ denotes the Kullback-Leibler distance between $P_m^*$ and $H_m^*$. For any finite partition $\{A_1, \ldots, A_k\}$ of $\mathcal{X}$, define

$$\mathcal{D}(P_m^*, H_m^*) := \alpha_m^* \sum_{i=1}^{k} \frac{(P_m^*(A_i) - H_m^*(A_i))^2}{H_m^*(A_i)}.$$ \hfill (4.5)

Then, as $m \to \infty$, we have

$$2\alpha_m^* D_{KL}(P_m^* \parallel H_m^*) \sim \mathcal{D}(P_m^*, H_m^*) \overset{d}{\to} \chi_{(k-1)}^2.$$

**Proof.** See the Appendix. \hfill $\square$

Note that as the sample size $m$ increases, $H_m^* \overset{a.s.}{\to} F$ and therefore the posterior Dirichlet process $P_m^*$ converges to the true underlying distribution $F$ of the observed data $X_1, \ldots, X_m$. In our methodology, we compute the observed probability for bin $A_i$, $i = 1, \ldots, k$ of the partition $\{A_1, \ldots, A_k\}$ by calculating the posterior probability $P_m^*(A_i)$, $i = 1, \ldots, k$. Notice that in our Bayesian paradigm, we need to embed our prior information in our test statistic. In other words, the base distribution and the concentration parameter plays the role of the prior knowledge. Moreover, we do not count the observed frequencies in each bin. Instead, we calculate the exact posterior probability for each bin. Then, the $X^2$ distance in (4.5) compares the posterior probabilities with the hypothesized ones. Additionally, there is no
need to apply the asymptotic distribution as we know the exact distribution of the $X^2$
distance via a Monte Carlo simulation. Also, There are many discussions for choosing the
number of bins in the literature and different criterion are suggested by various authors.
See, for example, Koehler and Gan [25], Mann and Wald [29], Williams Jr [38], Watson [37], Hamdan [18], Dahiya and Gurland [11], Best and Rayner [6], Quine and Robinson [32] and Johnson [23]. In the following subsections, we first use the distance (4.5) to find
an appropriate concentration parameter for the Dirichlet process. Then, we carry out a
Bayesian nonparametric chi-squared goodness-of-fit test. We also extend our method to
present a Bayesian nonparametric test of independence. The described methods will be
illustrated by some examples in Section 6.

4.1 Selection of the concentration parameter of Dirichlet process

A challenging question in Bayesian nonparametric is to determine $\alpha$, the concentration pa-
rameter of the prior. To suggest an appropriate concentration parameter $\alpha$, fix $c$ and $q$ such
that

$$Pr(\mathcal{D}(P, F_0) \leq c) = q,$$ \hspace{1cm} (4.6)

where

$$\mathcal{D} = \mathcal{D}(P, F_0) = \alpha \sum_{i=1}^{k} \frac{(P(A_i) - F_0(A_i))^2}{F_0(A_i)}.$$

Throughout this paper, $\mathcal{D} = \mathcal{D}(P, F_0)$ denotes the prior distance. Also, let $\mathcal{D}^* = \mathcal{D}(P_m, F_0)$
stands for the posterior distance as given in (4.5), replacing $H^*_m$ by $F_0$. We can approximate
the distribution of the prior distance $\mathcal{D} = \mathcal{D}(P, F_0)$ by the empirical distribution of $N$
randomly generated values from $\mathcal{D}$. Thus, (4.6) can be approximated by the proportion of
\( \mathcal{D} \) values that are less than or equal to \( c \). We start with an initial value of \( \alpha \) and then we compute the probability (4.6). If the probability is close to the value of \( q \), we choose \( \alpha \), otherwise, we repeat this procedure by increasing or decreasing the value of \( \alpha \) to reach the value of \( q \). The results of a simulation study for an illustrated example are summarized in Table 1 in Section 6.

4.2 Goodness-of-fit test

Suppose \( X_1, \ldots, X_m \) is a random sample from a distribution \( F \). In order to test the null hypothesis \( H_0 : F = F_0 \), we place the Dirichlet process prior with parameters \( \alpha \) and \( F_0 \) on \( F \). Then, since under the null hypothesis, the true distribution of data is \( F_0 \), we calculate the distance between the Dirichlet process prior and \( F_0 \). The appropriate concentration parameter \( \alpha \) of the Dirichlet process can be calculated by the method explained in Subsection 4.1. We follow the approach of Swartz [33]. That is, for a fixed value of \( q \) and \( c \), we obtain \( \alpha \) by (4.6). Having \( \alpha \), we generate a random sample of size \( N \) from the Dirichlet process posterior with parameters \( \alpha^*_m \) and \( H^*_m \) as given earlier to get \( N \) random samples of \( \mathcal{D}^* = \mathcal{D}(P^*_m, F_0) \) as given in Theorem 4.1. The distribution of \( \mathcal{D}^* \) can be estimated by the empirical distribution of \( \mathcal{D}^* \) values. Hence, the posterior probability \( Pr(\mathcal{D}(P^*_m, F_0) \leq c) \) can be estimated by the proportion of \( \mathcal{D}^* \) which are less than or equal to \( c \). Here, our decision making is based on the comparison of the posterior probability and the prior probability \( q \), where \( q \) represents the prior belief that the underlying distribution \( F \) is practically equivalent to \( F_0 \). Usually \( q = 0.5 \) is considered. If the empirical posterior probability \( Pr(\mathcal{D}(P^*_m, F_0) \leq c) \) is less than \( q \), we reject the null hypothesis, otherwise there is no evidence to reject the null hypothesis.

Similar to the frequentist’s chi-squared goodness-of-fit test, we can also generalized the test to a family of distributions. Now, consider the null hypothesis \( H_0 : F = F_0 \) for some
\( \theta \in \Theta \). Therefore, the true underlying distribution \( F \) is a member of a family of distributions indexed by the parameter \( \theta \). Our approach for this case is similar to the simple hypothesis with the addition of a prior distribution \( \pi(\theta) \) on \( \theta \). Thus, the distance \( D(P_m^*, F_\theta) \) depends on the unknown parameter \( \theta \). In order to conduct the test, we first generate a random sample from the posterior distribution of \( \theta \) given \( X_1, \ldots, X_m \) that is given as

\[
g(\theta | X_1, \ldots, X_m) \propto \left( \prod_{i=1}^{m} f_\theta(x_i) \right) \pi(\theta), \tag{4.7}
\]

where \( f_\theta(x) \) is the density function corresponding to \( F_\theta \). By having a specified \( c \) and \( q \), we find the parameter \( \alpha \) such that \( \Pr(D(P, F_\bar{\theta}) \leq c) = q \), where \( \bar{\theta} = E(\theta) \). Then, we generate a random sample \( \theta_i^*, i = 1, \ldots, M \) from the posterior distribution \( g(\theta | X_1, \ldots, X_m) \). We obtain \( \theta_{Min} = \arg \min_{\theta_i^*} D(P_m^*, F_{\theta_i^*}) \), \( i = 1, \ldots, M \), where \( P_m^* \) is the posterior Dirichlet process with the base distribution \( H_{\theta_i^*}^* \) as given in (2.1) with \( H \) replaced by \( H_{\theta_i^*} \). We then generate a sample of size \( N \) from \( D(P_m^*, F_{\theta_{Min}}) \). Similar to the case of testing for the simple hypothesis, the decision is made by comparing the posterior probability \( \Pr(D(P_m^*, F_{\theta_{Min}}) \leq c) \) and \( q \).

Note that in the case of a non-standard distribution in (4.7), in order to sample from the posterior distribution, we need to apply some specialized techniques such as Metropolis-Hastings algorithm. In Section 6, some examples with simulation study are illustrated for the simple hypothesis \( H_0 : F = N(0,1) \) and the null hypothesis \( H_0 : F = \exp(\theta) \) with a Gamma (1.7, 2550) prior distribution for \( \theta \).
5 Bayesian nonparametric chi-squared test of independence

Here, we describe a Bayesian nonparametric chi-squared test of independence of two random variables. The null hypothesis of the chi-squared test of independence is given as \( H_0 : F_{X,Y}(x,y) = F_X(x)F_Y(y) \) against the alternative \( H_0 : F_{X,Y}(x,y) \neq F_X(x)F_Y(y) \) and hence it examines whether there is a significant relationship between two random variables \( X \) and \( Y \). Suppose \( \{A_j\}_{j=1}^r \) is a partition of the space \( \mathcal{X} \) of the random variable \( X \) and \( \{B_k\}_{k=1}^s \) is a partition of the space \( \mathcal{Y} \) of the random variable \( Y \), i.e., \( \mathcal{X} = \bigcup_{j=1}^r A_j \) and \( \mathcal{Y} = \bigcup_{k=1}^s B_k \). Let \( (X_l,Y_l) \overset{i.i.d.}{\sim} F(x,y), l = 1, \ldots, m \) be the sample data and \( H \) be a bivariate distribution. Then, the Dirichlet process posterior with parameters \( H_m^* \) and \( \alpha_m^* \) is written as

\[
P_m^* = \sum_{i=1}^\infty p_i^{(m)} \delta_{(X_i^*,Y_i^*)},
\]

where \( p_i^{(m)} \) is as given in (2.2), \( \alpha \) is replaced by \( \alpha_m^* \) and \( (X_i^*,Y_i^*) \), \( i = 1, \ldots, n \) are generated from \( H_m^* = \frac{\alpha}{\alpha+m} H + \frac{m}{\alpha+m} \sum_{i=1}^m \delta_{(X_i,Y_i)} \). In our new approach, we compute the observed probability at level \( j \) of the random variable \( X \) and at level \( k \) of the random variable \( Y \) by \( P_m^*(A_j \times B_k) \) and the corresponding expected probability is computed as \( P_m^*(A_j \times \mathcal{Y})P_m^*(\mathcal{X} \times B_k) \), where

\[
P_m^*(A_j \times B_k) = \sum_{i=1}^\infty p_i^{(m)} \delta_{(X_i^*,Y_i^*)}(A_j \times B_k)
\]

and

\[
P_m^*(A_j \times \mathcal{Y}) = \sum_{i=1}^\infty p_i^{(m)} \delta_{(X_i^*,Y_i^*)}(A_j \times \mathcal{Y}) = \sum_{i=1}^\infty p_i^{(m)} \delta_{X_i^*}(A_j)
\]

\[
P_m^*(\mathcal{X} \times B_k) = \sum_{i=1}^\infty p_i^{(m)} \delta_{(X_i^*,Y_i^*)}(\mathcal{X} \times B_k) = \sum_{i=1}^\infty p_i^{(m)} \delta_{Y_i^*}(B_k).
\]
Then, test statistic is given as

\[ D^* = \alpha_m^* \sum_{k=1}^{s} \sum_{j=1}^{r} \left( \frac{(P_m^*(A_j \times B_k) - P_m^*(A_j \times Y)P_m^*(X \times B_k))^2}{P_m^*(A_j \times Y)P_m^*(X \times B_k)} \right) \]  

(5.3)

which asymptotically converges to \( \chi^2_{(r-1) \times (s-1)} \). In order to carry out the test, we proceed a similar process as explained in Section 4 for the goodness-of-fit test. We generate a random sample of size \( N \) from the prior distance \( \mathcal{D} \), where \( \mathcal{D} \) is computed by (5.3) replacing \( \alpha_m^* \) by \( \alpha \) and the Dirichlet process posterior \( P_m^* \) by the Dirichlet process prior \( P \). By having a fixed value \( c \) and a fixed probability \( q \), an appropriate concentration parameter \( \alpha \) is obtained by the equation \( Pr(\mathcal{D} \leq c) = q \). Then, by generating a sample of size \( N \) from \( \mathcal{D}^* \), we can approximate the distribution of \( \mathcal{D}^* \) by the empirical distribution of \( \mathcal{D}^* \) values. Our decision is made by comparing the probabilities \( Pr(\mathcal{D}^* \leq c) \) and \( q \) and we reject the null hypothesis if \( Pr(\mathcal{D}^* \leq c) \) is less than \( q \). An illustrative example with a simulation study is discussed in Section 6.

### 6 Simulation study

This section provides some examples with simulation studies for the Bayesian nonparametric tests described in Section 4 and 5. For all the simulations, we use the finite sum representation to approximate the Dirichlet process as given in (2.6).

**Example 6.1.** We consider a Dirichlet process with the base distribution \( H = N(0, 1) \) and \( n = 2000 \) terms in the finite sum representation (2.6). We partition the space into \( k = 7 \) bins. Table 1 represents the probability (4.6) when \( F_0 = N(0, 1) \). The probabilities are computed for various values of \( \alpha \) and \( c \) and for a simulation of size \( N = 2000 \). As the Table 1 shows,
for example, if we set $q = 0.48$ and $c = 3$, $\alpha = 10$ is an appropriate concentration parameter.

| $\alpha$ | $c = 1$ | $c = 2$ | $c = 3$ | $c = 4$ | $c = 5$ | $c = 6$ |
|----------|---------|---------|---------|---------|---------|---------|
| 1        | 0.298   | 0.745   | 0.812   | 0.857   | 0.893   | 0.933   |
| 10       | 0.068   | 0.273   | 0.480   | 0.624   | 0.717   | 0.781   |
| 50       | 0.029   | 0.143   | 0.311   | 0.474   | 0.612   | 0.696   |
| 100      | 0.027   | 0.116   | 0.258   | 0.409   | 0.540   | 0.648   |
| 200      | 0.020   | 0.094   | 0.219   | 0.353   | 0.492   | 0.595   |
| 300      | 0.011   | 0.073   | 0.179   | 0.297   | 0.432   | 0.542   |
| 500      | 0.009   | 0.057   | 0.150   | 0.263   | 0.368   | 0.484   |

Table 1: The computed the probability $Pr(\mathcal{D}(P, F_0) < c)$ for different choices of $\alpha$ and $c$ in Example 6.1.

**Example 6.2.** Suppose $X_1, \ldots, X_{150}$ is a random sample from a standard Cauchy distribution. We want to test the null hypothesis $H_0: F = N(0, 1)$. We divide the sample space into $k = 7$ bins $A_i$, $i = 1, \ldots, 7$ as given in Table 2 and $Pr(A_i)$ shows the observed probability of each bin. We consider $H = N(0, 1)$ as the base measure and $n = 2000$ terms in the finite sum representation of Dirichlet process as given in (2.6). Then, an appropriate concentration parameter $\alpha = 100$ is obtained when $q = 0.54$ and $c = 5$. By sampling $N = 2000$ times from the Dirichlet process posterior $P_m^*$ and then $N = 2000$ realizations of $\mathcal{D}^*$, we obtain $Pr(\mathcal{D}(P_m^*, F_0) \leq c) = 0$. Thus, we reject the normality hypothesis of the data. Our decision is consistent with the classical chi-squared test which gives a p-value of $2.2 \times 10^{-16}$. Also, our decision is consistent with other choices of the base measure $H$, since the Dirichlet process posterior converges to the true underlying distribution as the data size increases. Table 2 illustrates the observed probabilities obtained by counting the data points in each bin and the corresponding probabilities computed by the Dirichlet process posterior.
$A_1 = (-\infty, -2]$  $A_2 = (-2, -1]$  $A_3 = (-1, 0]$  $A_4 = (0, 1]$  $A_5 = (1, 2]$  $A_6 = (2, 3]$  $A_7 = (3, \infty)$

|       | $P_{r}(A_i)$ | $P_{m}^{*}(A_i)$ | $F_0(A_i)$ |
|-------|--------------|-----------------|------------|
| $A_1$ | 0.133        | 0.072           | 0.023      |
| $A_2$ | 0.100        | 0.131           | 0.136      |
| $A_3$ | 0.313        | 0.342           | 0.341      |
| $A_4$ | 0.240        | 0.310           | 0.341      |
| $A_5$ | 0.060        | 0.069           | 0.136      |
| $A_6$ | 0.067        | 0.030           | 0.022      |
| $A_7$ | 0.087        | 0.046           | 0.001      |

Table 2: The computed probabilities $P_{r}(A_i)$, $P_{m}^{*}(A_i)$ and $F_0(A_i)$ where $P_{r}(A_i)$ is the observed probability obtained by counting the data points in $i$th bin, $P_{m}^{*}(A_i)$ is the corresponding probability computed by the Dirichlet process posterior for one simulation and $F_0(A_i)$ shows the corresponding expected probability under the null hypothesis.

Example 6.3. (Example 3.6. Hamada et al. [17]) Suppose we have an observed data of size $m = 31$ for the lifetime of the liquid crystal display (LCD) projector lamps. We want to test if the lifetime distribution of the liquid crystal display (LCD) projector lamps is an Exponential distribution with parameter $\theta > 0$. That is, we want to test the null hypothesis $H_0 : F_{\theta} = Exp(\theta)$, where $\theta$ has a Gamma (1.7, 2550) prior distribution. Hence, the posterior distribution of $\theta$ given data is a Gamma (32.7, 20457) distribution. We consider $k = 4$ bins. By specifying the values $q = 0.51$ and $c = 3$, the appropriate $\alpha = 100$ is obtained. We obtain $\theta_1^*, \ldots, \theta_M^*$ as realizations from the distribution of $(\theta | X_1, \ldots, X_{31})$ and we get $\theta_{Min} = 0.00136$. By generating $N = 2000$ times from $\mathcal{D}^* = \mathcal{D}(P_{m}^{*}, F_{\theta_{Min}})$, we obtain $Pr(\mathcal{D}(P_{m}^{*}, F_{\theta_{Min}}) \leq c) = 0.71$. Hence, there is no evidence to reject the null hypothesis.

Example 6.4. Suppose we have a random sample $(X_i, Y_i), i = 1, \ldots, 150$ from a bivariate
normal distribution \( F = N_2(\mu, \Sigma) \) where \( \mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) and \( \Sigma = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix} \). We consider five levels of variable \( X \) and four levels of variable \( Y \) as given in Table 3. We want to test the null hypothesis of independence as given in Section 5. Consider a Dirichlet process prior with base distribution \( H = N(\mu_1, \Sigma_1) \) where \( \mu_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) and \( \Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). For \( q = 0.5 \) and \( c = 20 \), by generating \( N = 2000 \) times from \( D \) and solving the equation \( \Pr(D < c) = q \), we obtain an appropriate concentration parameter \( \alpha = 100 \). By generating a sample of size \( N = 2000 \) from the posterior distance \( D^* \), we have \( \Pr(D^* < c) = 0 \). Therefore, we reject the null hypothesis of independence. The p-value of \( 8.34 \times 10^{-6} \) obtained by the classical chi-squared test of independence results in the same conclusion. Table 3 represents the probability of each category calculated by the Dirichlet process posterior.

| \( Y \) | \( B_1 = (\infty, -1] \) | \( B_2 = (-1, 0] \) | \( B_3 = (0, 1] \) | \( B_4 = (1, \infty) \) |
|---|---|---|---|---|
| \( A_1 = (-\infty, -1] \) | 0.076 | 0.069 | 0.005 | 0.066 | 0.000 |
| \( A_2 = (-1, 0] \) | 0.075 | 0.031 | 0.086 | 0.063 | 0.006 |
| \( A_3 = (0, 1] \) | 0.072 | 0.047 | 0.045 | 0.048 | 0.043 |
| \( A_4 = (1, \infty) \) | 0.014 | 0.061 | 0.044 | 0.025 | 0.125 |

Table 3: A sample table of probabilities computed by the Dirichlet process posterior in Example 6.4.

7 Discussion

In this paper, we proposed a Bayesian nonparametric chi-squared goodness of fit test based on the Kullback-Leibler distance between the Dirichlet process posterior and the hypothesized distribution. Our method proceeds by placing a Dirichlet process prior on the distribution of observed data and computing the probability of each bin of the partition from the Dirichlet
process posterior. The suggested method is in contrast with the frequentist’s Pearson’s chi-squared goodness of fit test which is based on counting the observations in each bin of the partition. We also extended our method to present a Bayesian nonparametric test of independence. Like the classical chi-squared test, we can generalize our goodness-of-fit test to several variables. For categorical observations with finite many categories, placing a Dirichlet distribution prior on the probabilities of categories and deriving the posterior Dirichlet distribution can establish similar tests. For example, the test of independence and conditional independence of qualitative observations follow easily.

Acknowledgments

This research was supported by grant funds from the Natural Science and Engineering Research Council of Canada.

References

[1] Al Labadi, L., 2012. On new constructive tools in bayesian nonparametric inference. Ph.D. thesis, Université d’Ottawa/University of Ottawa.

[2] Al Labadi, L., Masuadi, E., Zarepour, M., 2014. Two-sample bayesian nonparametric goodness-of-fit test. arXiv:1411.3427.
[3] Al Labadi, L., Zarepour, M., 2013. A bayesian nonparametric goodness of fit test for right censored data based on approximate samples from the beta-stacy process. Canadian Journal of Statistics 41 (3), 466–487.

[4] Al Labadi, L., Zarepour, M., 2014. Goodness-of-fit tests based on the distance between the dirichlet process and its base measure. Journal of Nonparametric Statistics 26 (2), 341–357.

[5] Berger, J. O., Guglielmi, A., 2001. Bayesian and conditional frequentist testing of a parametric model versus nonparametric alternatives. Journal of the American Statistical Association 96 (453), 174–184.

[6] Best, D., Rayner, J., 1981. Are two classes enough for the $\chi^2$ goodness-of-fit test? Statistica Neerlandica 35 (3), 157–163.

[7] Billingsley, P., 2013. Convergence of Probability Measures. John Wiley & Sons.

[8] Bondesson, L., 1982. On simulation from infinitely divisible distributions. Advances in Applied Probability 14 (4), 855–869.

[9] Carota, C., Parmigiani, G., 1994. On Bayes factors for nonparametric alternatives. Institute of Statistics and Decision Sciences, Duke University.

[10] D’Agostino, R. B., 1986. Goodness-of-Fit Techniques. Marcel Dekker, New York.

[11] Dahiya, R. C., Gurland, J., 1973. How many classes in the pearson chi-square test? Journal of the American Statistical Association 68 (343), 707–712.

[12] Escobar, M. D., West, M., 1995. Bayesian density estimation and inference using mixtures. Journal of the American Statistical Association 90 (430), 577–588.
[13] Ferguson, T. S., 1973. A bayesian analysis of some nonparametric problems. Annals of Statistics 1 (2), 209–230.

[14] Ferguson, T. S., Klass, M. J., 1972. A representation of independent increment processes without gaussian components. Annals of Mathematical Statistics 43 (5), 1634–1643.

[15] Florens, J.-P., Richard, J.-F., Rolin, J. M., 1996. Bayesian encompassing specification tests of a parametric model against a nonparametric alternative. Tech. Rep. 9608, Université catholique de Louvain, Institut de Statistique.

[16] Ghosal, S., 2010. The dirichlet process, related priors and posterior asymptotics. In: Bayesian Nonparametrics. Cambridge University Press, Cambridge, pp. 35–79.

[17] Hamada, M. S., Wilson, A., Reese, C. S., Martz, H., 2008. Bayesian Reliability. Springer, New York.

[18] Hamdan, M., 1963. The number and width of classes in the chi-square test. Journal of the American Statistical Association 58 (303), 678–689.

[19] Holmes, C. C., Caron, F., Griffin, J. E., Stephens, D. A., 2015. Two-sample bayesian nonparametric hypothesis testing. Bayesian Analysis 10 (2), 297–320.

[20] Hsieh, P.-H., 2013. A nonparametric assessment of model adequacy based on kullback-leibler divergence. Statistics and Computing 23 (2), 149–162.

[21] Ishwaran, H., Zarepour, M., 2002. Exact and approximate sum representations for the dirichlet process. Canadian Journal of Statistics 30 (2), 269–283.

[22] James, L. F., 2008. Large sample asymptotics for the two-parameter poisson–dirichlet process. Vol. 3. Institute of Mathematical Statistics, pp. 187–199.
[23] Johnson, V. E., 2004. A bayesian $\chi^2$ test for goodness-of-fit. Annals of Statistics 32 (6), 2361–2384.

[24] Johnson, V. E., 2007. Bayesian model assessment using pivotal quantities. Bayesian Analysis 2 (4), 719–733.

[25] Koehler, K. J., Gan, F., 1990. Chi-squared goodness-of-fit tests: Cell selection and power. Communications in Statistics-Simulation and Computation 19 (4), 1265–1278.

[26] Kullback, S., Leibler, R. A., 1951. On information and sufficiency. Annals of Mathematical Statistics 22 (1), 79–86.

[27] Lo, A. Y., 1984. On a class of bayesian nonparametric estimates: I. density estimates. Annals of Statistics 12 (1), 351–357.

[28] Lo, A. Y., 1987. A large sample study of the bayesian bootstrap. Annals of Statistics 15 (1), 360–375.

[29] Mann, H., Wald, A., 1942. On the choice of the number of class intervals in the application of the chi-square test. Annals of Mathematical Statistics 13 (3), 306–317.

[30] Muliere, P., Tardella, L., 1998. Approximating distributions of random functionals of ferguson-dirichlet priors. Canadian Journal of Statistics 26 (2), 283–297.

[31] Neal, R. M., 1992. Bayesian mixture modeling. In: Maximum Entropy and Bayesian Methods. Springer, pp. 197–211.

[32] Quine, M., Robinson, J., 1985. Efficiencies of chi-square and likelihood ratio goodness-of-fit tests. Annals of Statistics 13 (2), 727–742.
[33] Swartz, T., 1999. Nonparametric goodness-of-fit. Communications in Statistics-Theory and Methods 28 (12), 2821–2841.

[34] Tokdar, S. T., Martin, R., 2011. Bayesian test of normality versus a dirichlet process mixture alternative. arXiv:1108.2883.

[35] Verdinelli, I., Wasserman, L., 1998. Bayesian goodness-of-fit testing using infinite-dimensional exponential families. Annals of Statistics 26 (4), 1215–1241.

[36] Viele, K., 2000. Evaluating fit using dirichlet processes. Tech. Rep. 384, Department of Statistics, University of Kentucky.

[37] Watson, G., 1957. The $\chi^2$ goodness-of-fit test for normal distributions. Biometrika, 336–348.

[38] Williams Jr, C. A., 1950. The choice of the number and width of classes for the chi-square test of goodness-of-fit. Journal of the American Statistical Association 45 (249), 77–86.

[39] Zarepour, M., Al Labadi, L., 2012. On a rapid simulation of the dirichlet process. Statistics & Probability Letters 82 (5), 916–924.

**Appendix - Proofs of Theoretical Results**

Proof of Lemma 3.1

Suppose that the sample space is partitioned as $x_{(1)} < \cdots < x_{(n+1)}$ such that $x_{(i)} < \theta_{(i)} < x_{(i+1)}$, $i = 1, \ldots, n$. By definition of the Kullback-Leibler distance, we have
\[ D_{KL}(P_n \parallel F) = \sum_{i=1}^{n} \Delta P_n(x_i) \log \left( \frac{\Delta P_n(x_i)}{\Delta F(x_i)/\Delta x_i} \right) \]

\[ = \sum_{i=1}^{n} \Delta P_n(x_i) \log(\Delta P_n(x_i)) - \sum_{i=1}^{n} \Delta P_n(x_i) \log \left( \frac{\Delta F(x_i)}{\Delta x_i} \right) \]

\[ = \sum_{i=1}^{n} p_{i,n} \log(p_{i,n}) - \sum_{i=1}^{n} p_{i,n} \log \left( \frac{\Delta F(x_i)}{\Delta x_i} \right), \quad (7.1) \]

where \( \Delta F(x_i) = F(x_{i+1}) - F(x_i) \), \( \Delta x_i = x_{i+1} - x_i \),

\( p_{i,n} = P_n(x_{i+1}) - P_n(x_i) = P_n(\theta(i)) \) and \( H(p) = -\sum_{i=1}^{n} p_{i,n} \log(p_{i,n}) \) is the entropy of \( P_n \).

Similarly, we get

\[ D_{KL}(F \parallel P_n) = -H(q) - \sum_{i=1}^{n} q_i \log(p_{i,n}), \quad (7.2) \]

where \( q_i = \frac{\Delta F(x_i)}{\Delta x_i} \) and \( H(q) = -\sum_{i=1}^{n} q_i \log q_i \).

Proof of Theorem 3.1 and Remark 3.1

We have \( (p_1,n, \ldots, p_n,n) \sim \text{Dir}(\alpha/n, \ldots, \alpha/n). \) Thus, \( p_{i,n} \sim \text{Beta}(\frac{\alpha}{n}, \alpha(1-\frac{1}{n})) \), \( i = 1, \ldots, n \) and all computations for the mean and variance simply follow.

Proof of Theorem 4.1

We basically mimic the proof for the asymptotic frequentist’s chi-squared goodness-of-fit test. Define
\[ D^* = (\alpha + m) \sum_{i=1}^{k} \frac{(P_m^*(A_i) - H_m^*(A_i))^2}{H_m^*(A_i)}. \]  

(7.3)

Let \( Y_m^T = (Y_{1,m}, \ldots, Y_{k,m}) = (P_m^*(A_1), \ldots, P_m^*(A_k)) \) and \( v_m^T = (v_{1,m}, \ldots, v_{k,m}) = (H_m^*(A_1), \ldots, H_m^*(A_k)) \). By Lemma 4.1 as \( m \to \infty \),

\[ \sqrt{\alpha + m}(Y_m - v_m)^T \xrightarrow{d} N_k(0, \Sigma). \]  

(7.4)

In here, \( \Sigma = (\sigma_{ij})_{k \times k} \) is the covariance matrix with \( \sigma_{ii}^2 = \text{var}(Y_{i,m}) = F(A_i)(1 - F(A_i)), i = 1, \ldots, k \) and \( \sigma_{ij} = \text{cov}(Y_{i,m}, Y_{j,m}) = -F(A_i)F(A_j) \). Then, (7.3) can be written as

\[ D^* = (\alpha + m)(Y_m - v_m)^T \Sigma^{-1}(Y_m - v_m). \]  

(7.5)

Note that the sum of the \( j \)th column of \( \Sigma \) is \( F(A_j) - F(A_j)(F(A_1) + \cdots + F(A_k)) = 0 \), that implies the sum of the rows of \( \Sigma \) is the zero vector, therefore \( \Sigma \) is not invertible. To avoid dealing with this singular matrix, we define \( Y_m^{*T} = (Y_{1,m}, \ldots, Y_{k-1,m}) \). Let \( Y_m^* \) be the vector consisting of the first \( k-1 \) components of \( Y_m \). Then, the covariance matrix of \( Y_m^* \) is the upper-left \((k-1) \times (k-1)\) sub-matrix of \( \Sigma \) which is denoted by \( \Sigma^* \). Similarly, let \( v_m^{*T} \) denotes the vector \( v_m^* = (v_{1,m}, \ldots, v_{k-1,m}) \). It can be verified simply that \( \Sigma^* \) is invertible. Furthermore, (7.5) can be rewritten as

\[ D^* = (\alpha + m)(Y_m^* - v_m^*)^T (\Sigma^*)^{-1}(Y_m^* - v_m^*). \]  

(7.6)

Define

\[ Z_m^T = \sqrt{\alpha + m}(\Sigma^*)^{-1/2}(Y_m^* - v_m^*). \]
The central limit theorem implies $Z_m^T \overset{d}{\to} N_{k-1}(0, I)$. By definition, the $\chi^2_{(k-1)}$ distribution is the distribution of the sum of the squares of $k-1$ independent standard normal random variables. Therefore,

$$D^* = Z_m^T Z_m \overset{d}{\to} \chi^2_{(k-1)}.$$
Figure 7.1: (Left) The Q-Q plot of $N = 2000$ realizations of $\mathcal{D} = \mathcal{D}(P, H)$ with $\alpha = 100$, $H = N(0, 1)$, $k = 5$ and $n = 3000$ versus a $\chi^2_4$ distribution. (Middle) The empirical distribution function of $\mathcal{D}$ values and the cdf of a $\chi^2_4$ distribution. (Right) The histogram of $\mathcal{D}$ values and the pdf of a $\chi^2_4$ distribution.