ON THE FINITE INVERSE PROBLEM IN ITERATIVE DIFFERENTIAL GALOIS THEORY

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ABSTRACT. In positive characteristic, nearly all Picard-Vessiot extensions are inseparable over some intermediate iterative differential extensions. In the Galois correspondence, these intermediate fields correspond to nonreduced subgroup schemes of the Galois group scheme. Moreover, the Galois group scheme itself may be nonreduced, or even infinitesimal. In this article, we investigate which finite group schemes occur as iterative differential Galois group schemes over a given ID-field. For a large class of ID-fields, we give a description of all occurring finite group schemes.

1. INTRODUCTION

Picard-Vessiot theory for iterative differential extensions in positive characteristic as conceived by Matzat and van der Put in [1] was restricted to separable extensions and algebraically closed fields of constants. This restriction was necessary, since the Galois group was given as (the rational points of) a linear algebraic group. Furthermore, intermediate fields over which the Picard-Vessiot ring is inseparable are not taken account in their Galois correspondence. In [5], the Picard-Vessiot theory was extended to perfect fields of constants and to inseparable extensions. This was made possible by constructing the Galois group as an affine group scheme. In the case of a separable PV-extension over an algebraically closed field of constants $C$, the $C$-rational points of this group scheme are exactly the original Galois group as defined by Matzat and van der Put.

In this article, we solve the inverse problem for finite group schemes in iterative Picard-Vessiot theory. That is, we give necessary and sufficient conditions for a finite group scheme to be a Galois group scheme over a given ID-field $F$. The general result which still depends on the inverse problem in classical Galois theory over $F$ is given in Theorem 5.1. In the case where $F$ itself is a PV-extension of an algebraic function field over algebraically closed constants, this classical inverse problem is solved. Therefore, in that case, we can explicitly describe the occurring Galois group schemes (cf. Corollary 5.2).

This article is structured as follows. In Section 2, we describe the notation and results regarding Picard-Vessiot theory in positive characteristic which are required later. Section 3 is dedicated to the case of infinitesimal Galois group schemes (i.e. purely inseparable PV-extensions), whereas results for finite reduced Galois group schemes are given in Section 4. Results from both sections are used in Section 5 to solve the inverse problem for finite group schemes.

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2. Basic notation

All rings are assumed to be commutative with unit. We will use the following notation (see also [6]). An iterative derivation on a ring $R$ is a homomorphism of rings $\theta : R \to R[[T]]$, such that $\theta(0) = \text{id}_R$ and for all $i, j \geq 0$, $\theta^{(i)} \circ \theta^{(j)} = (i+j)\theta^{(i+j)}$, where the maps $\theta^{(i)} : R \to R$ are defined by $\theta(r) = \sum_{i=0}^{\infty} \theta^{(i)}(r)T^i$. The pair $(R, \theta)$ is then called an ID-ring and $C_R := \{r \in R \mid \theta(r) = r\}$ is called the ring of constants of $(R, \theta)$. An iterative derivation $\theta$ resp. the ID-ring $(R, \theta)$ is called nontrivial, if $C_R \neq R$. An ideal $I \subseteq R$ is called an ID-ideal if $\theta(I) \subseteq I[[T]]$ and $R$ is ID-simple if $R$ has no proper nontrivial ID-ideals. An ID-ring which is a field is called an ID-field. Iterative derivations are extended to localisations by $\theta\left(\frac{s}{x}\right) := \theta(r)\theta(s)^{-1}$ and to tensor products by $\theta^{(k)}(r \otimes s) = \sum_{i+j=k} \theta^{(i)}(r) \otimes \theta^{(j)}(s)$

for all $k \geq 0$.

If $R$ is an integral domain of positive characteristic $p$, the iterative derivation induces a family of ID-subrings given by

$$R_\ell := \bigcap_{0 < j < p^\ell} \text{Ker}(\theta^{(j)}),$$

($\ell \in \mathbb{N}$) and also a family of ID-overrings $R_{[\ell]} := (R_\ell)^{p^{-\ell}}$ ($\ell \in \mathbb{N}$) in some inseparable closure with iterative derivation given by

$$\theta_{R_{[\ell]}}(x) := \left(\theta_R(x^{p^\ell})\right)^{p^{-\ell}}.$$

Notation From now on, $(F, \theta)$ denotes an ID-field of positive characteristic $p$, and $C = C_F$, its field of constants. We assume that $C$ is a perfect field, and that $\theta$ is non-degenerate, i.e., that $\theta^{(1)} \neq 0$.

Remark 2.1. In this setting, $F_\ell$ is an ID-subfield with $C_{F_\ell} = C$, and since $\theta(p^\ell)$ is a nilpotent $F_{\ell+1}$-linear endomorphism of $F_\ell$ and $\theta(p^\ell)$ is of nilpotence order $p$ with 1-dimensional kernel, one has $[F_\ell : F_{\ell+1}] = p$. Therefore, one obtains $[F : F_\ell] = p^\ell$

for all $\ell$.

Furthermore, $F_{[\ell]}$ is an ID-extension of $F$ with same constants (since $C$ is perfect), and $[F_{[\ell]} : F] = p^{\ell(m-1)}$, where $m$ denotes the degree of imperfection of $F$ (possibly infinite).

In [6], Prop. 4.1, it is shown that $F_{[\ell]}/F$ is the maximal purely inseparable ID-extension of $F$ of exponent $\leq \ell$.

We now recall some definitions from Picard-Vessiot theory:

Definition 2.2. Let $A = \sum_{k=0}^{\infty} A_k T^k \in \text{GL}_n(F[[T]])$ be a matrix with $A_0 = \mathbb{I}_n$ and for all $k, l \in \mathbb{N}$, $\binom{k}{l} A_{k+l} = \sum_{i+j=l} \theta^{(i)}(A_k) \cdot A_j$. An equation

$$\theta(y) = Ay,$$
where \( y \) is a vector of indeterminants, is called an \textbf{iterative differential equation} (IDE).

**Remark 2.3.** The condition on the \( A_k \) is equivalent to the condition that \( \theta^{(k)}(\theta^{(l)}(Y_{ij})) = (k+l)\theta^{(k+l)}(Y_{ij}) \) holds for a matrix \( Y = (Y_{ij})_{1 \leq i,j \leq n} \in \text{GL}_n(E) \) satisfying \( \theta(Y) = AY \), where \( E \) is some ID-extension of \( F \). (Such a \( Y \) is called a \textbf{fundamental solution matrix} ). The condition \( A_0 = 1_n \) is equivalent to \( \theta(Y_{ij}) = Y_{ij} \), and already implies that the matrix \( A \) is invertible.

**Definition 2.4.** An ID-ring \((R, \theta_R) \geq (F, \theta)\) is called a \textbf{Picard-Vessiot ring} (PV-ring) for the IDE \( \theta(y) = Ay \), if the following holds:

1. \( R \) is an ID-simple ring.
2. There is a fundamental solution matrix \( Y \in \text{GL}_n(R) \), i.e., an invertible matrix satisfying \( \theta(Y) = AY \).
3. As an \( F \)-algebra, \( R \) is generated by the coefficients of \( Y \) and by \( \det(Y)^{-1} \).
4. \( C_R = C_F = C \).

The quotient field \( E = \text{Quot}(R) \) (which exists, since such a PV-ring is always an integral domain) is called a \textbf{Picard-Vessiot field} (PV-field) for the IDE \( \theta(y) = Ay \).

For a PV-ring \( R/F \) one defines the functor

\[
\text{Aut}^{ID}(R/F) : \text{(Algebras}/C) \rightarrow \text{(Groups)}, D \mapsto \text{Aut}^{ID}(R \otimes_C D/F \otimes_C D)
\]

where \( D \) is equipped with the trivial iterative derivation. In [5], Sect. 10, it is shown that this functor is representable by a \( C \)-algebra of finite type, and hence, is an affine group scheme of finite type over \( C \). This group scheme is called the (iterative differential) \textbf{Galois group scheme} of the extension \( R \) over \( F \) – denoted by \( \text{Gal}(R/F) \), or also, the Galois group scheme of the extension \( E \) over \( F \), \( \text{Gal}(E/F) \), where \( E = \text{Quot}(R) \) is the corresponding PV-field.

Furthermore, \( \text{Spec}(R) \) is a \( (\text{Gal}(R/F) \times C F) \)-torsor and the corresponding isomorphism of rings

\[
\gamma : R \otimes_F R \rightarrow R \otimes_C C[\text{Gal}(R/F)]
\]

is an \( R \)-linear ID-isomorphism. Again, \( C[\text{Gal}(R/F)] \) is equipped with the trivial iterative derivation.

This torsor isomorphism \( \gamma \) is the key tool to establish the Galois correspondence between the closed subgroup schemes of \( \mathcal{G} = \text{Gal}(R/F) \) and the intermediate ID-fields of the extension \( E/F \), in more detail:

**Theorem 2.5.** (Galois correspondence) Let \( E/F \) be a PV-extension with PV-ring \( R \) and Galois group scheme \( \mathcal{G} \).

Then there is an inclusion reversing bijection between

\[ \mathcal{H} := \{ \mathcal{H} \mid \mathcal{H} \leq \mathcal{G} \text{ closed subgroup scheme of } \mathcal{G} \} \]

and

\[ \mathcal{M} := \{ M \mid F \leq M \leq E \text{ intermediate ID-field} \} \]

given by \( \Psi : \mathcal{H} \rightarrow \mathcal{M}, \mathcal{H} \mapsto E^H \) and \( \Phi : \mathcal{M} \rightarrow \mathcal{H}, M \mapsto \text{Gal}(E/M) \).

With respect to this bijection, \( \mathcal{H} \in \mathcal{H} \) is a normal subgroup of \( \mathcal{G} \), if and only if \( E^H \) is

\[ 1 \text{The PV-rings and PV-fields defined here were called pseudo Picard-Vessiot rings (resp. pseudo Picard-Vessiot fields) in [5] and [6]. This definition, however, is the most natural generalisation of the original definition of PV-rings and PV-fields to non algebraically closed fields of constants.} \]
a PV-field over $F$. In this case the Galois group scheme $\text{Gal}(E^N/F)$ is isomorphic to $\mathcal{G}/\mathcal{H}$.

(See [5], Thm. 11.5, resp. [6], Prop. 3.4 and Thm. 3.5 for the proof of this theorem.) The invariants $E^N$ are defined to be all elements $e = \frac{r}{s} \in E$, such that, for all $C$-algebras $D$, and all $h \in \mathcal{H}(D)$,

$$\frac{h(r \otimes 1)}{h(s \otimes 1)} = e \otimes 1 \in \text{Quot}(E \otimes_C D),$$

where $\text{Quot}(E \otimes_C D)$ denotes the localisation of $E \otimes_C D$ by all nonzerodivisors.

The following table shows some properties of the Galois group scheme and the corresponding properties of the PV-extension $E/F$:

| Property of $\mathcal{G}$ | Property of $E/F$ |
|---------------------------|-------------------|
| finite scheme with $\dim_{\text{Spec}(C)}(\mathcal{G}) = m$ | finite extension with $[E : F] = m$. |
| reduced scheme            | separable extension |
| infinitesimal scheme of height $n$ | purely inseparable extension of exponent $n$. |

The first correspondence is obtained by comparing dimensions in the torsor-isomorphism, since for finite extensions the PV-ring already is a field. The other two are proved in [5] and [6], respectively.

### 3. Purely inseparable PV-extensions

In this section, we are interested in PV-extensions with infinitesimal Galois group schemes, i.e., finite purely inseparable PV-extensions. Since the results presented here already appear in [6], most proofs are only sketched.

**Theorem 3.1.** [6 Thm. 4.2] Let $E$ be a PV-extension of $F$, and let $\ell \in \mathbb{N}$. Then $E[\ell]/F[\ell]$ is a PV-extension, and its Galois group scheme is related to $\text{Gal}(E/F)$ by $(\text{Frob}_\ell)^* (\text{Gal}(E[\ell]/F[\ell])) \cong \text{Gal}(E/F)$, where $\text{Frob}$ denotes the Frobenius morphism on $\text{Spec}(C)$.

**Proof.** We only sketch the proof, see [6] for details.

Let $R \subseteq E$ be the corresponding PV-ring. Since the iterative derivation is non-degenerate on $F$, one has $[F : F[\ell]] = p^\ell = [E : E[\ell]]$. This implies that there exists a fundamental solution matrix $Y \in \text{GL}_n(E_\ell)$ for some IDE $\theta(y) = Ay$ defining the PV-extension. One then shows that $A \in \text{GL}_n(F_\ell[[T^{p^\ell}]])$, and hence, $E_\ell/F_\ell$ is a PV-extension with Galois group scheme $\text{Gal}(E_\ell/F_\ell) \cong \text{Gal}(R/F)$. By taking $p^\ell$-th roots, one obtains that $R[\ell]$ is a PV-ring over $F[\ell]$ with fundamental solution matrix $\left( (Y_{i,j})^{p^\ell} \right)_{i,j}$. This gives the desired property:

$$(\text{Frob}_\ell)^* (\text{Gal}(E[\ell]/F[\ell])) \cong \text{Gal}(E_\ell/F_\ell) \cong \text{Gal}(E/F). \quad \Box$$

From this theorem we obtain a criterion for $F[\ell]/F$ being a PV-extension.

**Corollary 3.2.** [6 Cor. 4.3] Let $F$ be an ID-field which is a PV-extension of some nontrivial ID-field $L$ satisfying $L_1 = L^p$. Then $F[\ell]$ is a PV-extension of $F$, for all $\ell \in \mathbb{N}$. 


Proof. We first remark, that the iterative derivation on $L$ is non-degenerate, since otherwise $L^\ell = L_1 = L$ and hence $L = L^{\ell^k} \subseteq \ker(\theta^{\ell^k})$ for all $\ell$. This would imply that $\theta$ is trivial.

By Remark 2.1, the condition $L_1 = L^\ell$ implies that $L_{[\ell]} = L$ for all $\ell$. Hence, by the previous theorem, $F_{[\ell]}/L$ is a PV-extension, and therefore, $F_{[\ell]}/F$ is a PV-extension.

\begin{theorem}
Let $F$ be an ID-field with $C_F = C$ perfect, and suppose that $F$ has finite degree of imperfection.

Let $\tilde{C}_\ell$ denote the maximal subalgebra of $C_{F[\ell]} \otimes_{F[\ell]} C$ which is a Hopf algebra with respect to the comultiplication induced by

$$F_{[\ell]} \otimes_F F_{[\ell]} \longrightarrow (F_{[\ell]} \otimes_F F_{[\ell]}) \otimes_{F[\ell]} (F_{[\ell]} \otimes_F F_{[\ell]}), a \otimes b \mapsto (a \otimes 1) \otimes (1 \otimes b).$$

Then an infinitesimal group scheme of height $\leq \ell$ is realisable as a Galois group scheme over $F$, if and only if it is a factor group of $\text{Spec}(\tilde{C}_\ell)$.

In particular, if $F_{[\ell]}/F$ is a PV-extension, then $\tilde{C}_\ell = C_{F[\ell]} \otimes_{F[\ell]} C$, and $\text{Spec}(\tilde{C}_\ell)$ is isomorphic to $\Gal(F_{[\ell]}/F)$.

\end{theorem}

Proof. The first statement is proved in [6], Thm. 4.5. In the case where $F_{[\ell]}/F$ is a PV-extension, the torsor isomorphism [1] implies that $C[\Gal(F_{[\ell]}/F)] \cong C_{F[\ell]} \otimes_{F[\ell]} C$ is a Hopf algebra. Hence, $\tilde{C}_\ell = C_{F[\ell]} \otimes_{F[\ell]} C$, and $\text{Spec}(\tilde{C}_\ell) \cong \Gal(F_{[\ell]}/F)$.

\begin{corollary}
Cor. 4.6 \ Let $F$ be an ID-field and suppose that $F$ is a PV-extension of some nontrivial ID-field $L$ satisfying $L_1 = L^\ell$. An infinitesimal group scheme of height $\leq \ell$ is realisable as ID-Galois group scheme over $F$, if and only if it is a factor group of $\Gal(F_{[\ell]}/F)$. In particular, if $F_{[\ell]}/F$ is a PV-extension, then $\tilde{C}_\ell = C_{F[\ell]} \otimes_{F[\ell]} C$, and $\text{Spec}(\tilde{C}_\ell) \cong \Gal(F_{[\ell]}/F)$. $\square$

\end{corollary}

4. Finite separable PV-extensions

In this section we consider PV-extensions with finite reduced Galois group schemes, i.e., finite separable PV-extensions. Since iterative derivations extend uniquely to finite separable field extensions (see [3], 2.1, (5)), we obtain a close relationship to classical Galois extensions.

\begin{lemma}
Let $E$ be a finite (classical) Galois extension of $F$ which is geometric over $C$, and let $G$ be its Galois group. Let $E$ be equipped with the unique iterative derivation extending the iterative derivation on $F$. Then the extension $E/F$ is a PV-extension with Galois group scheme $\Gal(E/F) = \text{Spec}(C[G])$, the constant group scheme corresponding to $G$.

\end{lemma}

Proof. Let $\{x_1, \ldots, x_n\}$ be an $F$-basis of $E$, and let $a_{ij} \in F[[T]]$ such that $\theta(x_i) = \sum_{j=1}^n a_{ij}x_j$ for all $i = 1, \ldots, n$. Furthermore, let $G = \{\sigma_1, \ldots, \sigma_n\}$, and $Y = (\sigma_k(x_i))_{1 \leq i, k \leq n} \in \text{GL}_n(E)$. ($Y$ is invertible by Dedekind’s lemma on the independence of automorphisms.)

By definition, one has $\theta\left(\begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix}\right) = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, where $A = (a_{ij}) \in \text{Mat}_{n \times n}(F[[T]])$.

Since the extension of the iterative derivation of $F$ to $E$ is unique, all automorphisms are indeed ID-automorphisms, and therefore $\theta(Y) = AY$. 

\begin{flushright}
$\square$
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Therefore by Remark 4.1, \( \theta(y) = Ay \) is an IDE and \( Y \) is a fundamental solution matrix for this IDE. Furthermore, \( E \) is generated by the entries of \( Y \), and \( C_E = C \), since \( E/F \) is geometric. Hence, \( E \) is a PV-field for the IDE \( \theta(y) = Ay \).

Finally, \( \text{Spec}(C[G]) \) is a subgroup scheme of \( \text{Gal}(E/F) \), since \( G \) acts by ID-automorphisms, and \( \dim(C[G]) = n = [E : F] = \dim(C[\text{Gal}(E/F)]) \). Hence, \( \text{Spec}(C[G]) = \text{Gal}(E/F) \).

**Remark 4.2.** In the case that \( C \) is algebraically closed, the statement also follows easily from [4], 4.1.: in this case, Matzat and van der Put proved that the \( C \)-rational points of \( \text{Gal}(E/F) \) equal \( G \). Since for an algebraically closed field \( C \), a reduced group scheme is determined by its \( C \)-rational points, the claim follows.

**Lemma 4.3.** Let \( E/F \) be a finite separable PV-extension with Galois group scheme \( G \). Let \( E_C = E \otimes_C \bar{C} \) and \( F_C = F \otimes_C \bar{C} \) be the extensions of constants, where \( \bar{C} \) denotes an algebraic closure of \( C \). Then \( E_C/F_C \) is a finite (classical) Galois extension with Galois group \( \bar{G}(\bar{C}) \).

**Proof.** By definition of the Galois group scheme \( G \), the group \( \bar{G}(\bar{C}) \) is equal to \( \text{Aut}^{ID}(E_C/F_C) \). (Recall that in the finite case the PV-ring and the PV-field are equal.) Since \( E_C/F_C \) is separable, \( G \) is reduced and hence, \( E^\bar{G}(\bar{C}) = E^\bar{G} = F \).

Therefore, \( (E_C)^\bar{G}(\bar{C}) = F_C \), which implies that \( E_C \) is Galois over \( F_C \) with Galois group \( \bar{G}(\bar{C}) \).

**Remark 4.4.** The previous lemma tells us that all finite separable PV-extensions become classical Galois extensions after an algebraic extension of the constants. Actually, this extension of constants can be chosen to be finite Galois. Hence finite separable PV-extensions are almost classical Galois extensions in the sense of Greither and Pareigis (cf. [2], Def. 4.2).

## 5. Finite PV-extensions

We now consider the case of arbitrary finite PV-extensions, i.e., PV-extensions with finite Galois group schemes. By [1], Ch. II, §5, Cor. 2.4, every finite group scheme is the semi-direct product of an infinitesimal group scheme and a finite reduced group scheme. Hence, the results of the previous sections also give us information in this case.

**Theorem 5.1.** Let \( G \) be a finite group scheme over \( C \), \( G^0 \trianglelefteq G \) the connected component of \( G \) (an infinitesimal group scheme), and \( \mathcal{H} \trianglelefteq G \) the induced reduced group scheme. Assume that there is \( \ell \geq \text{ht}(G^0) \) such that \( F_{[\ell]}/F \) is a PV-extension. Then \( G \) is realisable over \( F \), if and only if \( G \cong G^0 \times \mathcal{H} \), \( G^0 \) is a factor group of \( \text{Gal}(F_{[\ell]}/F) \) and \( \mathcal{H} \) is realisable over \( F \).

**Proof.** Let \( G \cong G^0 \times \mathcal{H} \), such that \( G^0 \) is a factor group of \( \text{Gal}(F_{[\ell]}/F) \) and \( \mathcal{H} \) is realisable over \( F \) as \( \mathcal{H} \cong \text{Gal}(E''/F) \). By Theorem 3.3, \( G^0 \) is the Galois group scheme of some intermediate PV-field \( F \leq E' \leq F_{[\ell]} \). Since \( E''/F \) is separable and \( E'/F \) is purely inseparable, \( E' \) and \( E'' \) are linearly disjoint over \( F \), and so \( E' \otimes_F E'' \) is a PV-extension of \( F \) with Galois group scheme \( \text{Gal}(E' \otimes_F E''/F) \cong G^0 \times \mathcal{H} \). Hence, \( G \) is realisable over \( F \).

On the other hand, let \( G \) be realised over \( F \) as \( G \cong \text{Gal}(E/F) \). By [1], Ch. II, §5, Cor. 2.4, \( G \) is a semi-direct product \( G \cong G^0 \times \mathcal{H} \), and therefore \( \mathcal{H} \cong G/G^0 \cong \text{Gal}(E^0/F) \), i.e., \( \mathcal{H} \) is realisable over \( F \). Furthermore, \( E^\mathcal{H} \) is a purely inseparable
ID-extension of $F$ of height $\leq \text{ht}(\mathcal{G}^0)$. By assumption, there is $\ell \geq \text{ht}(\mathcal{G}^0)$ such that $F[\ell]/F$ is a PV-extension and therefore $F[\ell]$ is a PV-extension containing $E^H$. As in the first part of the proof, $\tilde{E} := F[\ell] \otimes_F E^G_0$ is a PV-extension of $F$ with Galois group $\text{Gal}(\tilde{E}/F) \cong \text{Gal}(F[\ell]/F) \times \text{Gal}(E^G_0/F) \cong \text{Gal}(F[\ell]/F) \times \mathcal{H}$. Since $E^H$ and $E^G_0$ are subfields of $\tilde{E}$, $E$ is also a subfield of $\tilde{E}$. Therefore, $\text{Gal}(E/F) \cong \mathcal{G}^0 \rtimes \mathcal{H}$ is a factor group of $\text{Gal}(\tilde{E}/F) \cong \text{Gal}(F[\ell]/F) \times \mathcal{H}$ which implies that $\mathcal{H}$ acts also trivially on $\mathcal{G}^0$, i.e., the semi-direct product $\mathcal{G}^0 \rtimes \mathcal{H}$ is in fact a direct product. Finally, we obtain that $E^H$ is a PV-extension of $F$ (since $\mathcal{H} \leq \mathcal{G}$ is a normal subgroup) with Galois group $\mathcal{G}^0$, and hence $\mathcal{G}^0$ is a factor group of $\text{Gal}(F[\ell]/F)$. \hfill \Box

**Corollary 5.2.** Let $C$ be algebraically closed, and let $F$ be a PV-extension of some function field $L/C$ in one variable with non-degenerate iterative derivation. Then the finite group schemes which occur as Galois group scheme over $F$ are exactly the direct products $\mathcal{G}^0 \rtimes \mathcal{H}$, where $\mathcal{H}$ is a constant group scheme (i.e., a reduced finite group scheme) and $\mathcal{G}^0$ is a factor group of some $\text{Gal}(F[\ell]/F)$.

**Proof.** By Corollary 5.2, $F[\ell]$ is a PV-extension of $F$ for all $\ell$. So by Theorem 5.1 we only have to show that every finite reduced group scheme $\mathcal{H}$ is realisable. Since $C$ is algebraically closed, the PV-extensions $E$ of $F$ with Galois group $\mathcal{H}$ are the (classical) Galois extensions with Galois group $\mathcal{H}(C)$. By [3], Thm. 4.4, the absolute Galois group of $L$, $\text{Gal}(L^{\text{sep}}/L)$, is a free group on infinitely many generators. Hence, there is an epimorphism $\phi : \text{Gal}(L^{\text{sep}}/L) \to \mathcal{H}(C) \times \text{Gal}(F \cap L^{\text{sep}}/L)$ such that the composition of $\phi$ and the projection $\text{pr}_2$ onto the second factor is the restriction map $\text{Gal}(L^{\text{sep}}/L) \to \text{Gal}(F \cap L^{\text{sep}}/L)$.

But this means that $\text{pr}_2 \circ \phi : \text{Gal}(L^{\text{sep}}/L) \to \mathcal{H}(C)$ corresponds to a Galois extension $\tilde{L}$ of $L$ with group $\mathcal{H}(C)$ which is linearly disjoint to $F$. Hence $\tilde{L} \otimes_F L$ is a Galois extension of $F$ with Galois group $\mathcal{H}(C)$. \hfill \Box

**Example 5.3.** Let $C$ be an algebraically closed field of positive characteristic $p$. We want $L/C$ to be a function field in one variable over $C$ with a non-degenerate iterative derivation $\theta$, and $F$ to be a PV-extension of $L$ with Galois group scheme $\mathbb{G}_m$. For example, we may take $L = C(t)$ with $\theta = \theta_1$ the iterative derivation with respect to $t$, given by $\theta_1(t) = t + 1 \cdot T \in L[T]$, and $F = L(t^\alpha)$, with $\alpha \in \mathbb{Z}_p \setminus \mathbb{Q}$ and the iterative derivation given by $\theta(t^\alpha) = (\alpha t^\alpha)/t^\alpha$.

By Theorem 5.3 for all $\ell \geq 0$, $F[\ell]/L$ is a PV-extension with $\text{Gal}(F[\ell]/L) \cong \mathbb{G}_m$, and the “restriction map” $\text{Gal}(F[\ell]/L) \cong \mathbb{G}_m \to \text{Gal}(F/L) \cong \mathbb{G}_m$ is given by the Frobenius map $x \mapsto x^{p^\ell}$. Hence, $\text{Gal}(F[\ell]/F) \cong \mu_{p^\ell}$, the “group of $p^\ell$th roots of unity.” The only factor groups of $\mu_{p^\ell}$ are $\mu_{p^k}$ where $k \leq \ell$. Hence by Theorem 5.1 the finite Galois group schemes over $F$ are exactly the group schemes of the form $\mu_{p^\ell} \times H$, where $\ell \geq 0$ and $H$ is finite reduced.

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