A redefinition of concurrence and its generalisation to bosonic subsystems of $N$ qubit systems

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We refine the notion of concurrence in this paper by a redefinition of the concept. The new definition is simpler, computationally straightforward, and allows the concurrence to be directly read off from the state. It has all the positive features of the definition given by Wootters over and above which it can discriminate between different systems to which the Wootters prescription would assign the same value. Finally, the definition leads to a natural extension of the notion to multiqubit concurrence, which we illustrate with examples from quantum error correction codes.

I. INTRODUCTION

Quantum entanglement (QE) is the central feature of quantum mechanics that distinguishes a quantum system from its classical counterpart. It is also the corner stone on which many of the novel applications of quantum mechanics - to quantum computation, quantum information theory, quantum cryptography and quantum teleportation - are based. Indeed, it is this promise that has led to a renewal of interest in QE in recent years.

The simplest example of QE is afforded by a bipartite system of two spin half (or pseudo spin half) particles, where the Hilbert space has the minimal dimension four. This system is also called a two qubit system (2QS) in the context of quantum computations. For a 2QS, there is essentially only one measure of entanglement; it may be given e.g., by the von-Neumann entropy $E_N$, or expressed in terms of the sum of bilinears in the eigenvalues as given by the quantity $1 - Tr\rho^{(r)^2} \equiv E_{tr}$, or in terms of $det\rho^{(r)} = E_d$. $\rho^{(r)}$ is the reduced density matrix obtained by taking a marginal trace over one of the spin degrees of freedom. States with a vanishing entanglement are separable (in the strong sense): they admit a factorisation $|1, 2 > = |1 > \otimes |2 >$. On the other hand, fully entangled states have the maximum correlation, and are collectively designated as Bell states.

Consider now a N-qubit system (NQS), realised as a multipartite system of $N$ spin halves. NQS upto $N = 4$ have been prepared experimentally with photons, for example [1]. It is easy to check that for an NQS, $E_N$, $E_{tr}$ and $E_d$ are not equivalent. Further, none of them is exhaustive. A natural question that arises is how one may identify a complete set of entanglement measures for any NQS. There have been several proposals [2] that attempt at answering this question. One note worthy proposal is from Sixia Yu et al [3] who posit a hierarchy of $N - 1$ classes of entanglement. The total number grows exponentially with increasing $N$.

An alternative approach is to look for physically interesting measures, apart from $E_N$ and $E_{tr}$, with (potential) applicability to quantum computation. This simpler approach would also give direction to the experimentalists to prepare NQS in specified states.

In this context, a useful question to ask is what the entanglement property of a 2QS is, when looked upon as a subsystem of an NQS, when $N \geq 3$ (There would be no new information when $N = 2$). There would be $NC_2$ such subentanglements, as analogues of two particle correlations. In defining the new measure, we seek to determine the 'proximity' of a given 2QS to the classic Bell states mentioned above.

Wootters [4] has developed a closely related concept called concurrence (hereafter denoted $C_W$) for 2QS, which is required to characterise the so called 'entanglement of formation' (EOF) [5]. Indeed, EOF is defined to be the minimum entropy carried by a mixed state, under all possible resolutions

$$\rho = \sum_i p_i |\psi_i > < \psi_i|$$

of a reduced density matrix of a 2QS: the set $\{|\psi_i > < \psi_i|\}$ does not necessarily form a basis - much less an orthonormal set. $C_W$, which measures this EOF is then defined as follows:

Definition 1 (Wootters:) The concurrence for a mixed 2QS is given by

$$C_W = max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

where the $\lambda_i$ are the eigenvalues, arranged in decreasing order of magnitude, of the operator $R = (\sqrt{\rho}, \sqrt{\rho})^{1/2}$: $R = \sigma_x \otimes \sigma_x \rho \sigma_y \otimes \sigma_y$.

The so called EOF is itself given by

Proposition 1 The entanglement of formation is given by

$$E(C_W) = h\left(\frac{1 + \sqrt{1 - C_W^2}}{2}\right): h(x) = -x\log_2 x - (1 - x)\log_2(1 - x).$$

The above definitions are attractive, and have been used to estimate bipartite entanglement in spin systems [6]. There are also reports that states with nonvanishing $C_W$ have been prepared experimentally [10]. Attempts have
been made to generalise the definition to higher spin sectors [11]. There is, thus, quite an interest in $C_W$.

The definition of $C_W$, rather involved in its form, crucially hinges on the concept of EOF. However, it is well known that there is no way of inferring how a system is prepared [8], no matter which property of the density matrix we look at, unless it corresponds to a pure state. Thus the physical interpretation of EOF is rather dubious, and does not seem to have any operational significance. Indeed, suppose the expansion is in the ordered basis $|12\rangle\equiv\sum_{m_1,m_2,m_3}\alpha_{m_1,m_2,m_3}|m_1,m_2,m_3\rangle$.

The computational simplicity of $C_R$ is evident, from its very form, if only we demonstrate its viability. A convenient method is to use $C_W$ as a benchmark to test the new definition against. For, the definition of $C_W$ is also motivated by the entanglement measure $\mathcal{E}_e$. Figure 1. gives a comparison of the two definitions, where the states are generated randomly. One sees an overall agreement.

### II. CONCURRENCE

As a warm up, and for motivation, consider a 2QS first. Let a 2QS be in the state

$$|1,2\rangle = \alpha|\uparrow\uparrow\rangle + \alpha_2|\uparrow\downarrow\rangle + \alpha_3|\downarrow\uparrow\rangle + \alpha_4|\downarrow\downarrow\rangle.$$  

The key step in defining concurrence is in recognising that $\mathcal{E}_e = 2|\alpha_1\alpha_4 - \alpha_2\alpha_3|$ is a measure of entanglement, equivalent to the standard measures listed above. $\mathcal{E}_e$ is much simpler to evaluate, though. Our notion of concurrence is inspired by, and is very close in its definition to, $\mathcal{E}_e$. Although $\mathcal{E}_e$ does not carry any new information, it does exhibit a symplectic structure which is not apparent in the other two measures. Further, as observed by Hill and Wooters [5], it can also be written in the form $|\langle12|1,2\rangle|$, the inner product of $|1,2\rangle$ with its conjugate (equivalently, time reversed) state which we define thus:

$$|\langle12|\rangle = \sum_{m_1,m_2}(-1)^{m_1+m_2}\alpha_{m_1,m_2}^{*}|m_1,m_2\rangle - |m_1,m_2\rangle.$$  

The main thrust of the paper is that this form of entanglement for a 2QS needs little modification in defining concurrence in NQS, $N \geq 3$. The definition will be given in steps, so as not to have a cluttered notation.

#### The 3QS: Consider the simplest case, of a 3QS, prepared in a pure state. Let

$$|1,2,3\rangle = \sum_{m_1,m_2,m_3}\alpha_{m_1,m_2,m_3}|m_1,m_2,m_3\rangle.$$  

We define:

**Definition 2** The concurrence between the first two qubits of a 3QS is given by

$$C_R^{(12)} = |\langle12|3|1,2,3\rangle|;$$

$$|\langle12|3\rangle = \sum (-1)^{m_1+m_2}\alpha_{m_1,m_2,m_3}^{*}|m_1,m_2,m_3\rangle - |m_1,m_2,m_3\rangle > .$$

which is obtained by a partial conjugation – on the sector (12). The definition of concurrences $C^{(13)}$ and $C^{(23)}$ follow similarly. By definition $C_R^{(ij)}$ take values in the range $[0,1]$. $C^{(12)}$ also has an in built symplectic structure; to see this, consider the quadruplet of vectors

$$\{V_1 = (\alpha_1,\alpha_2), V_2 = (\alpha_3,\alpha_4), V_3 = (\alpha_5,\alpha_6), V_4 = (\alpha_7,\alpha_8)\};$$

the expansion is in the ordered basis $\{|\uparrow\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle\}$. The concurrence is now simply written as

$$C_R^{(12)} = 2|V_1V_1 - \tilde{V}_3V_2|,$$

which naturally generalises the definition of entanglement in a 2QS in a manner appropriate to our purpose [9]. $\tilde{V}$ is the transpose of $V$. One can straight away construct quadruplets for $C_R^{(23)}$, $C_R^{(13)}$ by inspection. It is also straightforward to generalise the definition when the parent state is an NQS in a pure state. We write the expression for the (12) sector:

**Definition 3** The concurrence of the first two qubits of a NQS in a pure state $|1,2,\ldots,N\rangle = \sum_{m_1,m_2,\ldots,m_N}\alpha_{m_1,m_2,\ldots,m_N}|m_1,m_2,\ldots,m_N\rangle$ is given by $|\langle12|3,\ldots,N|1,2,\ldots,N\rangle|$, where the conjugate state is now defined to be $|\langle12|3,\ldots,N\rangle|$. It is also a straight forward to generalise the definition when the parent state is an NQS in a pure state. We write the expression for the (12) sector:

$$C_R^{(12)} = 2|\tilde{V}_1V_1 - \tilde{V}_3V_2|,$$

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that each of them has a preferential proximity to one of the Bell states which is more clear in Fig.2.

The computational advantage is of course evident from the form of $C_R$. For, if we start with a NQS, there are $NC_2$ such concurrences. Even in regions where $C_W$ has no interpretational problem, Definition 1 requires an evaluation of three reduced density matrices, a determination of their partner states, and an evaluation of the eigenvalues of $R$. The necessity of these computations gets obviated in the determination of $C_R$.

Indeed, let

$$|1, 2, \ldots, N> = \sum \alpha_{m_1, m_2, \ldots, m_N} |m_1, m_2, \ldots, m_N>$$

be the state in which an NQS is. We define the conjugate state

$$|\{12\cdots k\}k+1, \ldots, N> = \sum (-1)^{\sum_i m_i} \alpha^{*}_{m_1, m_2, \ldots, m_N} |−m_1, −m_2, \ldots, −m_k, m_{k+1}, \ldots, m_N>$$

Definition 4 The concurrence in the first $k$ qubits of an NQS is given by

$$C^{(12\cdots k)}_R = |<\{12\cdots k\}k+1, \ldots, N|1, 2, \ldots, N>|.$$  

The above definition is unfortunately restrictive; It works only when the subsystem of the NQS is bosonic; The fermionic states ($k$ odd) are always orthogonal to their conjugates. Apart from this, there is generality with respect to the choice of the $k$ qubits since any $k$ qubits may be chosen to be in the order given, by a permutation. The number of independent concurrence measures is given by $NC_k$.

We proceed to enlarge the definition of concurrence to higher spin sectors. Note that there is no analogue of $C_W$ here, since the criterion for EOF which was evolved in [4] is specific to a 2QS. The generalisation in our case is naturally suggested by the form for $C_R$ written above.

Figure 3 displays the 4Q concurrences for states generated from a 6QS. As an illustration, and as an example of the discerning capability of the higher order concurrences which we have defined here, the Shor code will be compared with the Steane code, both of which are used in quantum error correction [12]. The Shor code $|0_L> (= |1_L>)$ is a nine qubit state, written as a direct product of three 3Q states each of which have the
form $|000\rangle \pm |111\rangle$, in writing which we have employed the qubit notation: $0 \rightarrow \downarrow (\frac{1}{\sqrt{2}}), 1 \rightarrow \uparrow (\frac{1}{\sqrt{2}})$. This entangled state has vanishing concurrences in all orders. In contrast, the Steane code, constructed with seven qubits has a more involved structure; it is not separable the way the Shor state is, and is given by $|0_L\rangle = \frac{1}{\sqrt{8}}(|0000000\rangle + |1010101\rangle + |1100110\rangle + |0001111\rangle + |0110011\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle)$. As a manifestation, although the 2Q concurrences vanish, the 4Q concurrences survive in the sectors \{(1247), (1256), (1346), (1357), (2345), (2367), (4567)\}, and all of them attain the maximum allowed value 1. The 6Q concurrences vanish. These features highlight quantitatively the manner in which the two codes differ.

Finally, the question of handling mixed states still remain. Not getting into the general case, only the 2Q concurrence will be discussed. The general case only involves rewriting the argument with more indices. Indeed, given a 2QS in a mixed state $\rho$, the method is to look upon $\rho$ as having descended from a pure parent state of a higher dimension $N$. Significant, $N \leq 4$. The determination of parent state is done by simple inspection. It also follows that if $\rho$ has only real components, then $C_R = 2|\rho_{14} - \rho_{23}|$. It is not difficult to check that although the parent state is not unique, the concurrence determined will be so - reflecting the fact that it is a property of the system, and not of the parent - introduced purely as an intermediate step. This final remark completes the programme undertaken.

In conclusion, we have redefined in this note the concept of concurrence which has a much simpler form, and is computationally trivial - compared to $C_W$ and its generalisations thereof. It is not based on the notion of EOF. While concurrences do not completely characterise a system - which was not our aim - it appears that they do constitute an important subset of entangled states, especially in the two qubit sector. The higher order concurrences also have a significance, as illustrated by the concurrence properties of Shor and the Steane codes. The true import of entanglements of this kind would manifest with only a more geometric approach, which will be taken up in a subsequent publication.

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