Non-Commutative Integrability of the Grassmann Pentagram Map
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Abstract
The pentagram map is a discrete integrable system first introduced by Schwartz in 1992. It was proved to be integrable by Schwartz, Ovsienko, and Tabachnikov in 2010. Gekhtman, Shapiro, and Vainshtein studied Poisson geometry associated to certain networks embedded in a disc or annulus, and its relation to cluster algebras. Later, Gekhtman et al. and Tabachnikov reinterpreted the pentagram map in terms of these networks, and used the associated Poisson structures to give a new proof of integrability. In 2011, Mari Beffa and Felipe introduced a generalization of the pentagram map to certain Grassmannians, and proved it was integrable. We reinterpret this Grassmann pentagram map in terms of noncommutative algebra, in particular the double brackets of Van den bergh, and generalize the approach of Gekhtman et al. to establish a noncommutative version of integrability.

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0 Introduction

0.1 Background

The pentagram map was first introduced by Schwartz [Sch92] as a transformation of the moduli space of labeled polygons in $\mathbb{P}^2$. After labeling the vertices of an $n$-gon with the numbers $1, \ldots, n$, the diagonals are drawn which connects each $i$ to $i + 2$ (see Figure 1). The intersection points of these diagonals are taken to be the vertices of a new $n$-gon. Later, Schwartz, along with Ovsienko and Tabachnikov, generalized the map to the space of “twisted” $n$-gons in $\mathbb{P}^2$, and proved that this generalized pentagram map is completely integrable in the Liouville sense [OST10]. In 2011, Max Glick interpreted the pentagram map in terms of “$Y$-mutations” of a certain cluster algebra [Gli11]. In the literature, there are other names for $Y$-mutations or $Y$-dynamics. Fomin and Zelevinski have called them “coefficient dynamics” [FZ02] [FZ07]. Gekhtman, Shapiro, and Vainshtein refer to them as “$\tau$-coordinate mutations” [GSV10a]. Fock and Goncharov refer to them as “$X$-variable mutations” [FG06].

In 2016, Gekhtman, Shapiro, Vainshtein, and Tabachnikov, building on the work of both Glick and Postnikov [Pos06], interpreted a certain set of coordinates on the space of twisted polygons as weights on some directed graph, and the pentagram map in these coordinates as a certain sequence of “Postnikov moves” applied to this graph [GSTV16].
They used their previous work on the Poisson geometry of the space of edge weights of such graphs \([\text{GSV09}]\) \([\text{GSV10b}]\) to give a new proof of the integrability of the pentagram map.

Recently, Mari Beffa and Felipe considered a generalization of the pentagram map, where the twisted polygons were taken to be in the Grassmannian instead of projective space \([\text{FB15}]\), and they demonstrated a Lax representation for this version of the pentagram map, establishing integrability for this generalized version. The purpose of the present paper is to interpret this Grassmannian pentagram map as a transformation of a set of matrix-valued variables (and more generally as a formal noncommutative rational transformation), and use ideas from non-commutative Poisson geometry – namely the “double brackets” of Van den Bergh \([\text{Ber08}]\) and “\(H_0\)-Poisson structures” of Crawley-Boevey\([\text{CB11}]\) – to formulate a non-commutative version of integrability, generalizing the approach of Gekhtman, Shapiro, Vainshtein, and Tabachnikov, using weighted directed graphs.

### 0.2 Structure of the Paper

The structure of the paper is as follows. The first three sections are a condensed review of the basics of the pentagram map. We mainly follow the notations and conventions of Gekhtman, Shapiro, Vainshtein, and Tabachnikov \([\text{GSV09}]\) \([\text{GSV10b}]\) \([\text{GSTV16}]\). The later sections are an introduction to the Grassmannian pentagram map, and the development of the new theory. We now give a more detailed description of the sections.

In section 1, we review the definition of “twisted” polygons and the pentagram map. We also construct a system of coordinates on the moduli space of twisted polygons (following \([\text{GSTV16}]\)), and express the pentagram map in these coordinates.

In section 2, we define the weighted directed graphs that we will be using, and review the associated Poisson structure on the space of edge weights introduced in \([\text{GSV10b}]\). We also review the notion of “boundary measurements” for such graphs, and the “Postnikov moves” and “gauge transformations” which leave the boundary measurements invariant \([\text{Pos06}]\).

In section 3, we review the observation from \([\text{GSTV16}]\) that after identifying the coordinates from section 1 with certain weights on some graph, the pentagram map can be interpreted as a sequence of Postnikov moves and gauge transformations. This observation allows us to find invariants of the pentagram map, defined in terms of the boundary measurements. These invariants also turn out to be in involution with respect to the Poisson structure, giving the integrability of the pentagram map.

In section 4, we introduce the notion of twisted polygons in the Grassmannian, and define the pentagram map for these twisted Grassman polygons, following the presentation of Mari Beffa and Felipe \([\text{FB15}]\). Next, rather than putting real coordinates on the space of twisted Grassman polygons, we instead describe the pentagram map as a transformation of a set of matrix-valued variables, in an attempt to mimic, formally, the approach of \([\text{GSTV16}]\). We show that in these (non-commutative) variables, the pentagram map is a noncommutative version of the formula from section 1.

In section 5, we review the definitions and properties of “double brackets” on non-commutative algebras (due to Van den Bergh \([\text{Ber08}]\)). This is the formalism we will use in place of a usual Poisson structure in this non-commutative setting.

In section 6, we consider the same directed graph from section 2, but we consider non-commutative weights. We also consider the analogues of the boundary measurements, Postnikov moves, and gauge transformations in this
non-commutative setting. To make rigorous sense of the Postnikov moves, we introduce the free skew field in a set of formal non-commuting indeterminates. We then define a double bracket on the free skew field generated by the edges. This algebra acts as a sort of "space of non-commutative edge weights". The relationship with Goldman’s bracket on character varieties [Gol86] is discussed, giving the double bracket (and its induced brackets) a geometric interpretation in terms of intersection numbers on a certain surface. Finally, we find a set of elements/variables in the free skew field which are non-commutative analogues to the coordinates from section 1. The same sequence of Postnikov moves as in the classical case transforms these variables precisely by the formula from section 4 for the Grassman pentagram map.

In section 7, analogous to the classical case, we find noncommutative "invariants" of the map which are defined in terms of the boundary measurements. They are in fact invariant only modulo commutators. We also show that these invariants are, in some sense, involutive with respect to the non-commutative Poisson structure, giving some non-commutative version of integrability.

Finally, in section 8, we pose some lingering questions and directions for further research. Specifically, we ask how to extend the results of the present paper to more closely resemble the conditions of classical Liouville integrability (in the commutative case).

1 The “Classical” Pentagram Map

1.1 Twisted Polygons and the Pentagram Map

We refer to the pentagram map as the “classical” pentagram map to distinguish it from the Grassmann pentagram map (the subject of the later sections of the paper). First we give the basic idea, and then extend to so-called "twisted" polygons. The map was originally introduced by Schwartz [Sch92], but here we mainly follow the notations and conventions from [GSTV16]. Consider an $n$-gon in $\mathbb{P}^2 = \mathbb{R}\mathbb{P}^2$, with vertices labeled $p_1$ through $p_n$. Draw the diagonals of the $n$-gon which connect $p_i$ to $p_{i+2}$ (with indices read cyclically). Label the intersection of the lines $p_ip_{i+2}$ and $p_{i+1}p_{i+3}$ as $q_i$. Then the pentagram map, which we denote by $T$, sends the first labeled polygon $P$ to the labeled polygon $Q = T(P)$ whose vertices are $q_i$. An example for $n = 6$ is shown below in Figure 1.

Figure 1: The pentagram map for a hexagon
More generally, define a twisted \( n \)-gon to be a bi-infinite sequence \( (p_i)_{i \in \mathbb{Z}} \) of points in \( \mathbb{P}^2 \) such that \( p_{i+n} = Mp_i \) for all \( i \), where \( M \) is some projective transformation in \( \text{SL}_3(\mathbb{R}) \), referred to as the monodromy of the twisted \( n \)-gon. The usual notion of an \( n \)-gon can be thought of as the case where \( M = \text{Id} \), and the sequence is periodic. In this case we call the polygon a closed \( n \)-gon. The pentagram map, \( T \), can be defined analogously for twisted \( n \)-gons. We always assume twisted \( n \)-gons are “generic” in the sense that no three consecutive points \( p_i, p_{i+1}, p_{i+2} \) are collinear.

Call two twisted \( n \)-gons \( (p_i) \) and \( (q_i) \) projectively equivalent if there is some \( G \in \text{SL}_3(\mathbb{R}) \) so that \( q_i = Gp_i \) for every \( i \). We will denote by \( \mathcal{P}_n \) the moduli space of twisted \( n \)-gons under projective equivalence. The pentagram map commutes with projective transformations in the sense that for \( G \in \text{SL}_3(\mathbb{R}) \), the polygons \( T(P) \) and \( T(G \cdot P) \) are projectively equivalent. Thus it induces a well-defined map \( T: \mathcal{P}_n \to \mathcal{P}_n \).

**Theorem 1.** [OST10] The pentagram map \( T: \mathcal{P}_n \to \mathcal{P}_n \) is completely integrable.

### 1.2 Corrugated Polygons and Higher Pentagram Maps

In [GSTV16], the authors define generalized higher pentagram maps, which we define and discuss now. Instead of working in \( \mathbb{P}^2 \), we generalize to \( \mathbb{P}^{k-1} \), with the usual pentagram map being the specialization to \( k = 3 \). In the same way as before, we define twisted polygons in \( \mathbb{P}^{k-1} \) to be bi-infinite sequences of points \( (p_i)_{i \in \mathbb{Z}} \) in \( \mathbb{P}^{k-1} \) with monodromy \( M \in \text{SL}_k \) so that \( p_{i+n} = Mp_i \) for all \( i \). Define \( \mathcal{P}_{k,n} \) to be the set of all projective equivalence classes of twisted \( n \)-gons in \( \mathbb{P}^{k-1} \) with the genericity condition that any consecutive \( k \) points do not lie in a proper projective subspace. Keeping with the previous notation, we define \( \mathcal{P}_n := \mathcal{P}_{3,n} \).

Furthermore, define a corrugated polygon to be a twisted \( n \)-gon in \( \mathbb{P}^{k-1} \) with the additional property that \( p_i, p_{i+1}, p_{i+k-1}, p_{i+k} \) span a projective plane for all \( i \). In other words, for any lift \( v_i \) of \( p_i \) to \( \mathbb{R}^k \), the four vectors \( v_i, v_{i+1}, v_{i+k-1}, v_{i+k} \) span a 3-dimensional subspace. In particular, when \( k = 3 \), any twisted polygon is automatically corrugated.

Define \( \mathcal{P}^0_{k,n} \) to be the subset of corrugated polygons with the property that for each \( i \), any 3 of the 4 points \( p_i, p_{i+1}, p_{i+k-1}, p_{i+k} \) are not collinear. In other words, for any lift \( v_i \) and any \( i \), any 3 of the 4 vectors \( v_i, v_{i+1}, v_{i+k-1}, v_{i+k} \) are linearly independent. It is shown in [GSTV16] that \( \dim \mathcal{P}^0_{k,n} = 2n \).

Since \( p_i, p_{i+1}, p_{i+k-1}, p_{i+k} \) span a projective plane in \( \mathbb{P}^{k-1} \), the lines \( L^+_i = \overline{p_i, p_{i+k-1}} \) and \( L^-_i = \overline{p_{i+1}, p_{i+k}} \) must intersect. We call the intersection point \( q_i \). We can then define a pentagram map \( T: \mathcal{P}^0_{k,n} \to \mathcal{P}_{k,n} \) which sends \( (p_i) \) to \( (q_i) \). Note the codomain is not \( \mathcal{P}^0_{k,n} \), since the image my be degenerate.

### 1.3 Coordinates on the Moduli Space

Next we will construct a system of coordinates on \( \mathcal{P}_n \), following the presentation in [GSTV16]. The following construction will also give a system of coordinates on \( \mathcal{P}^0_{k,n} \) for the generalized higher pentagram maps, but we present only the case \( k = 3 \), for the sake of simplicity. Given a twisted \( n \)-gon \( (p_i) \), we lift it to a bi-infinite sequence of vectors \( (v_i) \) in \( \mathbb{R}^3 \). That is, \( v_i \) projects to \( p_i \) under the canonical map \( \mathbb{R}^3 \setminus \{0\} \to \mathbb{P}^2 \). The genericity assumption guarantees that for each \( i \), \( \{v_i, v_{i+1}, v_{i+2}\} \) is a basis for \( \mathbb{R}^3 \). Thus we have for each \( i \) a linear dependence relation:

\[
    v_{i+3} = a_i v_i + b_i v_{i+1} + c_i v_{i+2}
\]

(1)

Note that the sequences \( a_i, b_i, \) and \( c_i \) are \( n \)-periodic, and that none of the \( a_i, b_i, \) or \( c_i \) are zero. The following result is proved in more generality in [GSTV16], but we prove it here for the sake of presentation.
Proposition 1. [GSTV16] The lift \((v_i)\) can be chosen so that \(c_i = 1\) for all \(i\).

Proof. Given any lift \((v_i)\), any other lift \((\hat{v_i})\) differs by rescaling. That is, there are non-zero constants \(\lambda_i\) so that \(\hat{v_i} = \lambda_i v_i\). Then Equation 1 becomes:

\[
\hat{v}_{i+3} = \left( a_i \frac{\lambda_{i+3}}{\lambda_i} \right) \hat{v}_i + \left( b_i \frac{\lambda_{i+3}}{\lambda_{i+1}} \right) \hat{v}_{i+1} + \left( c_i \frac{\lambda_{i+3}}{\lambda_{i+2}} \right) \hat{v}_{i+2}
\]

We want to choose \(\lambda_i\) so that \(c_i \frac{\lambda_{i+3}}{\lambda_{i+2}} = 1\) for all \(i\). Re-arranging this equation, and re-indexing gives the recurrence \(\lambda_{i+1} = \frac{\lambda_i}{c_i - 2}\). We may therefore set \(\lambda_0 = 1\), and then define the rest by this recurrence. □

With the previous result in mind, we change notation and let \(x_i := b_i\) and \(y_i := a_i\), so that Equation 1 becomes

\[
v_{i+3} = y_iv_i + x_i v_{i+1} + v_{i+2}
\]

(2)

In addition to the above result, the numbers \(x_i, y_i\) are invariants of the projective equivalence class of a twisted polygon, and furthermore these numbers determine the twisted polygon uniquely, up to projective equivalence. Thus we see that \(\dim \mathcal{P}_n = 2n\), and that \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) gives a system of local coordinates on \(\mathcal{P}_n\), generically.

Next we will write the pentagram map in these coordinates, and see that it is a rational map. Since we consider the pentagram map acting on labeled polygons, we will abuse notation slightly and write \(T(p_i) = q_i\) for the individual vertices of the polygon. Taking the abuse a step further, if we have lifts \((v_i)\) of \((p_i)\) and \((w_i)\) of \((q_i) = (T(p_i))\), we will also write \(T(v_i) = w_i\). To see how the pentagram map transforms the \(x_i\) and \(y_i\) coordinates, we will look at how the pentagram map acts on a lift. First let’s introduce some notation. Let \(\mathcal{L}_i^+\) be the line between \(p_i\) and \(p_{i+2}\) and \(\mathcal{L}_i^-\) the line between \(p_{i+1}\) and \(p_{i+3}\). Recall that the pentagram map is given by \(q_i = T(p_i) = \mathcal{L}_i^+ \cap \mathcal{L}_i^-\). Let \(\mathcal{P}_i^+\) be the plane in \(\mathbb{R}^3\) which projects onto \(\mathcal{L}_i^+\), and similarly for \(\mathcal{P}_i^-\). Then any vector in the intersection \(\mathcal{P}_i^+ \cap \mathcal{P}_i^-\) can be taken to be the lift of the image \(w_i = T(v_i)\). In particular, re-arranging Equation 2 gives a candidate which can be written in two ways:

\[
v_{i+3} - x_{i+1}v_{i+1} = y_i v_i + v_{i+2} \in \mathcal{P}_i^- \cap \mathcal{P}_i^+
\]

(3)

Proposition 2. [GSTV16] The pentagram map \(T:\mathcal{P}_n \to \mathcal{P}_n\) is given in the \(x_i, y_i\) coordinates by

\[
\begin{align*}
x_i &\mapsto x_i \frac{x_{i+2} + y_{i+3}}{x_i + y_{i+1}} \\
y_i &\mapsto y_{i+1} \frac{x_{i+2} + y_{i+3}}{x_i + y_{i+1}}
\end{align*}
\]

Proof. The lifts of the image polygon's vertices will also satisfy the linear dependence relation as in Equation 1:

\[
T(v_{i+3}) = Y_i T(v_i) + X_i T(v_{i+1}) + Z_i T(v_{i+2})
\]

In the above equation, substitute for \(T(v_i)\) and \(T(v_{i+2})\) the expressions from Equation 3 belonging to \(\mathcal{P}_i^-\), and substitute for \(T(v_{i+1})\) and \(T(v_{i+3})\) the corresponding expressions belonging to \(\mathcal{P}_i^+\). Doing so, one obtains the equation

\[
v_{i+5} + y_{i+3} v_{i+3} = (X_i y_{i+1} - Y_i x_i) v_{i+1} + (X_i + Y_i - Z_i x_{i+2}) v_{i+3} + Z_i v_{i+5}
\]

Comparing coefficients on both sides, we can solve to get that

\[
\begin{align*}
Z_i &= 1 \\
X_i &= x_i \frac{x_{i+2} + y_{i+3}}{x_i + y_{i+1}} \\
Y_i &= y_{i+1} \frac{x_{i+2} + y_{i+3}}{x_i + y_{i+1}}
\end{align*}
\]

□
2 Networks and Poisson Structures

In this section, we review the relevant definitions and constructions needed to formulate the pentagram map in terms of edge-weighted directed graphs. First, we review the types of weighted directed graphs we will be working with, and an important class of transformations of such graphs. Then we review the Poisson structures introduced by Gekhtman, Shapiro, and Vainshtein on the space of weights of such graphs.

2.1 Weighted Directed Fat Graphs

A fat graph is a graph, together with a prescribed cyclic ordering of the half-edges incident to each vertex. In particular, any graph embedded onto a two-dimensional oriented surface is naturally a fat graph, with the orientation induced from that of the surface. Later on, we will only be considering such graphs which are embedded on a surface, so all graphs will assumed to be fat graphs, even if not explicitly stated.

A quiver (or directed graph) is a tuple $Q = (Q_0, Q_1, s, t)$ where $Q_0$ is the set of vertices of the underlying graph, $Q_1$ the set of edges, and $s, t: Q_1 \rightarrow Q_0$ are the "source" and "target" maps, indicating the direction of the arrows. We will be interested in weighted directed graphs, which additionally have the data of an assignment $Q_1 \rightarrow \mathbb{R}$ of a real number to each edge. Throughout, we will use the term "network" to mean a weighted directed fat graph.

Postnikov considered what he called perfect planar networks [Pos06], which are networks embedded in a disk such that

1. All vertices on the boundary are univalent and either sources or sinks
2. All internal vertices are trivalent and are neither sources nor sinks
3. All edge weights are positive real numbers

Note that the second condition, that internal vertices are trivalent, and are not sources or sinks, implies that they are one of two types: either a vertex has one incoming and two outgoing edges, or it has one outgoing and two incoming edges. When we draw networks, we will picture the former as white vertices, and the latter as black. This is pictured in Figure 2.

![Figure 2: Types of vertices in a perfect network](image)

For each such perfect network $N$, let $a_1, \ldots, a_k$ denote the sources on the boundary, and let $b_1, \ldots, b_l$ be the sinks on the boundary. Then, assuming $N$ is acyclic, the boundary measurement $M_{ij}$ is defined to be the sum of the weights of all paths from $a_i$ to $b_j$, where the weight of a path is the product of the weights of its edges. Later on, we will be interested in a particular acyclic network, so we will not discuss the more general definition for when $N$ has directed cycles. We arrange the boundary measurements into a matrix $B(N)$, called the boundary measurement matrix of the perfect network $N$. 

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In [Pos06], Postnikov describes several local “moves”, which change one network into another by changing just some small part and leaving the rest unchanged, with the important property that the boundary measurements are invariant under these moves. The moves are pictured in Figure 3, and are labeled with the corresponding edge weights.

We refer to the type I move as the “square move”, the type II move as “white-swapping”, and the type III move as “black-swapping”. It is an easy exercise to check that these three moves do not change the boundary measurements from a source to a sink in these pictures.

The last type of local move is called a gauge transformation. Let $t$ be any monomial in the coordinate variables (corresponding to the edge weights). Then at any vertex we may multiply all incoming weights by $t$ and all outgoing weights by $t^{-1}$. This obviously does not change the boundary measurements. The group generated by all gauge transformations is called the gauge group.

From now on, we will only consider networks where the sources and sinks are not interlaced, so that all sources can be pictured on the left side, and all sinks on the right. The reason for restricting to these networks is that they have the following nice property. One may consider “concatenating” two such networks in a disk, by gluing segments of their boundaries in a way that each sink on one is identified with a source of the other. The resulting edge after

Figure 3: Local Postnikov moves
identification is weighted by the product of the two edges. If the sources and sinks are not interlaced, as mentioned above, then the boundary measurement matrix of the resulting network after concatenation is the product of the two boundary measurement matrices.

We may also consider weighted directed networks embedded on an annulus [GSV10b]. Later, when we talk about the pentagram map, it will be these annular networks that we will consider. For simplicity, we will again only consider acyclic networks. Similar to the disk case, we assume any internal vertices are trivalent (and not sources or sinks), and that all boundary vertices are univalent. We also assume all vertices on the inner boundary component are sources, and all vertices on the outer boundary component are sinks. We also choose an oriented curve \( \rho \), which we call the cut, which connects the inner and outer boundary components of the annulus. Then in a similar way as before, we define a boundary measurement matrix, whose entries are the (signed) sums of path weights from a given source to a given sink. The sign is determined by the rotation number of the loop formed by the path, the cut, and segments of the boundaries (see [GSV10b]). Since the Postnikov moves are local, and do not take into account the global topology of the surface on which the network is embedded, they may also be applied in this case, and of course they still do not change the boundary measurements.

As in the disk case, we may consider concatenation of two annular networks, by identifying the outer boundary circle of one with the inner boundary circle of the other, in a way that each sink is identified with a source. If we assume, as mentioned above, that all sources are on the inner boundary circle and all sinks on the outer, then the resulting boundary measurement matrix is the product of the two boundary measurement matrices.

If there are the same number of sources and sinks, then it will sometimes be convenient to glue the inner and outer boundary circles together so that pairs of sources and sinks are identified, to obtain a network on the torus. We may then choose a new non-trivial loop to cut the torus along, obtaining a different network in an annulus, creating new pairs of sources and sinks. This is equivalent to first cutting the annulus, and then "swapping" the inner and outer parts. The new boundary measurement matrix will be conjugate to the original, since this amounts to changing a matrix \( AB \) to the matrix \( BA \).

Lastly, we define the modified edge weights, which are elements of the Laurent polynomial ring \( \mathbb{R}[\lambda^\pm] \) in the indeterminate \( \lambda \), defined as follows. Suppose an oriented edge \( \alpha \in Q_1 \) has weight \( x_\alpha \in \mathbb{R} \). It may happen that \( \alpha \) intersects the cut (it may even happen multiple times). If \( i \) is an intersection point, define \( \varepsilon_i = 1 \) if \( \alpha \) and the tangent vector to \( \rho \) at \( i \) form an oriented basis of the tangent space to the annulus, and define \( \varepsilon_i = -1 \) if they have the opposite orientation. Then the modified edge weight of \( \alpha \) is

\[
x_\alpha \prod_i \lambda^{\varepsilon_i}
\]

If we use the modified edge weights, then the boundary measurements become Laurent polynomials in \( \lambda \).

### 2.2 Poisson Structures on the Space of Edge Weights

Given a perfect network \( N \) in an annulus, as in the last section, we may forget the edge weights and consider the underlying quiver \( Q = (Q_0, Q_1) \). The space of edge weights of the quiver, \( \mathcal{E}_Q := (\mathbb{R}^*)^{Q_1} \), is the space of all possible choices of non-zero weights. Gekhtman, Shapiro, and Vainshtein defined a family of Poisson structures on \( \mathcal{E}_Q \) [GSV09], which we will review now.

The Poisson structures are defined locally at each vertex, and then the local Poisson brackets are shown to be "compatible" with concatenation (this will be made more precise later), and so they can be combined to give a global
Poisson bracket. This is outlined in [GSV09] and [GSV10b], but we present the construction here in detail, since we will mimic it very closely later on when we define a non-commutative analogue. Recall that for networks on an annulus, we require internal vertices are trivalent, and that they are neither sources nor sinks, and so they are either white or black, as mentioned before:

Let \( E^\circ = (\mathbb{R}^*)^3 \) with coordinates \( x, y, z \). We will think of the variables \( x, y, z \) as representing the edge weights at a “white” vertex picture above. Similarly, let \( E^\bullet = (\mathbb{R}^*)^3 \) with coordinates \( a, b, c \) correspond to a “black” vertex. Of course they are diffeomorphic, but we will define different Poisson brackets on them. Choose any \( \alpha, \beta, \gamma \in \mathbb{R} \), and define a “log-canonical” Poisson bracket \{−, −\} on \( E^\circ \) by

\[
\{x, y\}^\circ = \alpha xy, \quad \{x, z\}^\circ = \beta xz, \quad \{y, z\}^\circ = \gamma yz
\]

Similarly, for scalars \( \delta, \varepsilon, \zeta \in \mathbb{R} \), define \{−, −\} on \( E^\bullet \) by

\[
\{a, b\}^\bullet = \delta ab, \quad \{a, c\}^\bullet = \varepsilon ac, \quad \{b, c\}^\bullet = \zeta bc
\]

Next we will consider the operation of glueing/concatenating these local pictures together, as described before. This means identifying part of the boundary of one picture with part of another, in such a way that a sink is identified with a source. We then erase the common glued boundary, and remove the common identified vertex, identifying the two incident edges (which have a consistent orientation by construction). The weight of the new edge is associated with the product of the weights of the edges being glued. **Figure 4** illustrates glueing a black and white local picture:

The definitions of \{−, −\} \( E^\circ \) and \{−, −\} \( E^\bullet \) are “compatible” with this glueing procedure in a way that we now make precise. Recall that \( E_Q = (\mathbb{R}^*)^{Q_1} \) is the space of edge weights. We can associate to it the algebra \( E_Q := \mathbb{R}(Q_1) \), which is the field of rational functions with indeterminates corresponding to the arrows of the quiver. We also define \( \mathcal{H}_Q := (\mathbb{R}^*)^{Q_1} \cup \mathcal{H}_Q \) to be the *space of half-edge weights*. This is an assignment of a non-zero number to each half-edge. Each half edge is associated to the vertex which it is incident to. For an arrow \( \alpha \in Q_1 \), we call the corresponding half edges \( \alpha_s \) and \( \alpha_t \), for the source and target ends of the arrow. The algebra of functions \( H_Q \) corresponding to \( \mathcal{H}_Q \) is then the field of rational functions in twice as many variables, corresponding to \( \alpha_s \) and \( \alpha_t \) for \( \alpha \in Q_1 \).

Recall that \( Q_0 \) denotes the set of vertices of the quiver \( Q \). We can partition this into \( Q_0 = V^\circ \cup V^\bullet \cup V_\partial = V_i \cup V_\partial \), where \( V^\circ \) is the set of white vertices, \( V^\bullet \) the black vertices, and \( V_\partial \) the boundary vertices, and \( V_i = V^\circ \cup V^\bullet \) is the set of internal vertices. Note that since all internal vertices are trivalent, and all boundary vertices are univalent, the
set of half-edges is in bijection with \( V \coprod V \coprod V \coprod V \). Recall we defined the spaces \( E_\circ \cong E_\bullet \cong (\mathbb{R}^*)^3 \). We now additionally define \( E_\partial \cong \mathbb{R}^* \) with trivial Poisson bracket, corresponding to boundary vertices. Then the space of half-edges can be thought of as

\[
\mathcal{H}_Q \cong \prod_{s \in V_\partial} E_\partial \times \prod_{o \in V_\circ} E_\circ \times \prod_{i \in V_\bullet} E_\bullet.
\]

Under this identification, we realize the algebra \( H_Q \) as the tensor product

\[
H_Q \cong \bigotimes_{s \in V_\partial} E_\partial \otimes \bigotimes_{o \in V_\circ} E_\circ \otimes \bigotimes_{i \in V_\bullet} E_\bullet.
\]

From the Poisson brackets defined above on the algebras \( E_\circ, E_\bullet, \) and \( E_\partial \), we get a natural induced Poisson bracket on \( H_Q \), where \( \{\alpha, \beta\} = 0 \) if \( \alpha \) and \( \beta \) are half-edges which are not incident to a common vertex, and the boundary half-edges are casimirs.

We define a “glueing” map \( g: \mathcal{H}_Q \to E_Q \), represented by Figure 4, given by \( (\alpha_s, \alpha_t)_{\alpha \in Q_1} \mapsto (\alpha_s \alpha_t)_{\alpha \in Q_1} \). The earlier claim that the local brackets on \( E_\circ \) and \( E_\bullet \) are compatible with glueing is made precise by the following

**Proposition 3.** There is a unique Poisson bracket on \( E_Q \) such that the glueing map \( g: \mathcal{H}_Q \to E_Q \) is a Poisson morphism.

**Proof.** The statement that \( g \) is a Poisson map is equivalent to the pull-back map \( g^*: E_Q \to H_Q \) being a homomorphism of Poisson algebras. Note that for \( \alpha \in Q_1 \), the map \( g^* \) is given by \( g^*(\alpha) = \alpha_s \alpha_t \). It is obviously an associative algebra homomorphism. To be a Lie algebra homomorphism would mean that for any \( \alpha, \beta \in Q_1 \),

\[
\{g^*(\alpha), g^*(\beta)\}_{H_Q} = g^*(\{\alpha, \beta\}_{E_Q}).
\]

We claim that \( \{\alpha, \beta\}_{E_Q} \) is determined by this property. If \( \alpha \) and \( \beta \) do not share a common vertex, then obviously \( \{\alpha, \beta\} = 0 \). There are still many cases to consider, depending on which end (source or target) of each \( \alpha \) and \( \beta \) meet at the common vertex, and where in the cyclic ordering those half-edges are at that vertex. For example, let’s consider the picture in Figure 4, and try to define \( \{w, y\}_{E_Q} \). We only see one end of \( y \) in the figure, so let’s say \( g^*(y) = y_s y_t \), where \( y_s \) is the end we see at the white vertex. Then the condition that \( g^* \) be Poisson means we must have

\[
g^* \left( \{w, y\}_{E_Q} \right) = \{g^*(w), g^*(y)\}_{H_Q} \]

\[
= \{\alpha x, y_s y_t\}_{H_Q} \]

\[
= \alpha y_t \{x, y_s\}_{H_Q} \]

\[
= \alpha ax y_s y_t \]

\[
= \alpha g^*(w) g^*(y) \]

\[
= g^* (\alpha wy).
\]

Since \( g^* \) is injective, this uniquely defines \( \{w, y\}_{E_Q} \). The calculations for all other cases are similarly simple. We see that for \( \alpha, \beta \in Q_1 \) with a common vertex, the bracket \( \{\alpha, \beta\}_{E_Q} \) is given by the same expression as the bracket of the corresponding half-edges in \( H_Q \).

We will also want to consider the doubled quiver \( \overline{Q} \), which has the same vertex set as \( Q \), includes all the same arrows as \( Q \), plus additionally for each arrow \( \alpha \in Q_1 \), there is an “opposite” arrow \( \alpha^* \) with \( s(\alpha^*) = t(\alpha) \) and \( t(\alpha^*) = s(\alpha) \). Since we consider non-zero edge weights, we associate \( \alpha^* \) with the function \( \alpha^{-1} \) on \( E_Q \). Then a path in \( \overline{Q} \) may be represented as a Laurent monomial, which is the product of the edge weights along the path (allowing inverses).
Since Poisson brackets extend uniquely to localizations, we may extend the Poisson bracket defined above to all Laurent polynomials (and indeed any rational functions) on $\mathcal{E}_Q$.

We will now be interested in describing the Poisson bracket of two paths, thought of as rational functions on $\mathcal{E}_Q$. To do so, it will be convenient to introduce the constants

$$A = \delta - \varepsilon - \zeta$$
$$B = \alpha - \beta - \gamma$$

Consider two paths $f$ and $g$. Whenever the paths meet (share at least one edge in common), then there is a corresponding maximal subpath which $f$ and $g$ share. If the paths go in the same direction on the common subpath, we will say they are "parallel" on that common subpath. If $f$ and $g$ are parallel on a subpath, which is a proper subpath of both $f$ and $g$, then there are two possibilities, which are depicted in Figure 5. In the figure, $f$ is the blue path, $g$ is the red path. In the first situation (the left image in the figure), the paths are said to "touch", and in the second situation, they are said to "cross". The other possibility is that one of the paths (either $f$ or $g$) is a subpath of the other. In this case, if $f$ is a subpath of $g$, then we may write $g = fg'$, where $g'$ is the rest of $g$. We extend the notions of "touching" and "crossing" to the situation where $g = fg'$ by saying $f$ and $g$ "touch" on the subpath $f$ if $g'$ and $f$ touch, and similarly for "crossing".

![Figure 5: Two parallel paths sharing a common subpath](image)

We will denote by $f \cap g$ the set of all maximal common subpaths of $f$ and $g$. We will define a function $\epsilon(f, g)$ on the set $f \cap g$, and for $x \in f \cap g$, we denote the value by $\epsilon_x(f, g)$. It depends on whether $f$ and $g$ touch or cross, and on the colors of the vertices at the endpoints of the common subpath. The values are given in Table 1.

| type  | left endpt | right endpt | $\epsilon_x(f, g)$ |
|-------|------------|-------------|--------------------|
| touch | $\bullet$  | $\bullet$   | 0                  |
| touch | $\circ$    | $\circ$    | 0                  |
| touch | $\circ$    | $\bullet$  | $A + B$            |
| touch | $\bullet$  | $\circ$    | $-(A + B)$         |
| cross | $\bullet$  | $\bullet$  | $-2A$              |
| cross | $\circ$    | $\circ$    | $2B$               |
| cross | $\circ$    | $\bullet$  | $B - A$            |
| cross | $\bullet$  | $\circ$    | $B - A$            |

We define $\epsilon(f, g)$ to be skew-symmetric, so that $\epsilon(g, f) = -\epsilon(f, g)$ if the roles are switched. Also, there are other cases of intersecting paths which we have not considered. If we allow paths in $\mathcal{Q}$, then two paths can meet with opposing orientations on a common subpath. This can be obtained from the local pictures above by changing one of the paths to its inverse. Since Poisson brackets extend uniquely to localizations, this will not be an issue.
We then have the following formula:

**Proposition 4.** Suppose $f$ and $g$ are two paths. Then

\[
\{f, g\} = \sum_{x \in \text{fr} \cap \text{fr}^g} \varepsilon_x(f, g) fg
\]

*Proof.* Since $f$ and $g$ are monomials in the coordinate variables, it is obvious that $\{f, g\} = cfg$ for some constant $c$. We must prove that $c = \sum \varepsilon_x(f, g)$. Each time $f$ and $g$ share a proper subpath on which they are parallel, it will look like one of the two pictures in Figure 5. We can then write the paths as $f = f_0axwyb$ and $g = g_0cxywa$, where $xwy$ is the common subpath, $a, x, c$ are the edges incident to the "left" vertex (where the paths come together), and $y, b,$ and $d$ are the edges incident to the "right" vertex (where the paths diverge at the end of the common subpath). Then after expanding $\{f, g\}$ by the Leibniz rule, we get contributions from $\{a, c\}, \{a, x\}$, and $\{x, c\}$ at the left vertex, and $\{y, d\}, \{b, y\}$, and $\{b, d\}$ from the right vertex. Obviously the contributions from the common subpath cancel, since if $p$ and $q$ are two consecutive edges in the common subpath, we will get a contribution from both $\{p, q\}$ and $\{q, p\}$. There are of course other terms in the expansion coming from $f_0$ and $g_0$, but they will correspond to other common subpaths in the same way. It is therefore enough to consider the six terms mentioned above (3 each for the beginning and end vertices) for each common subpath.

In general, we must consider 36 different cases, since each endpoint could be either white or black, and there are three orientations for the arrows incident to each of the endpoint vertices. A simple calculation shows that the orientations at the vertices are in fact irrelevant. Therefore we only have 4 cases, depending on the colors of the vertices. As mentioned above, by the Leibniz rule, we can add the contributions from the endpoint vertices separately.

First consider the left endpoint. For all three choices of orientations, a black left endpoint will give a contribution of $-A$ to the coefficient, and a white left endpoint will give a contribution of $B$. For the right endpoint, a black vertex gives $A$, and a white gives $-B$. Combining all the possibilities together gives the list of values in the table for $\varepsilon_x(f, g)$.

We assumed so far that the common subpaths were parallel. If the paths are oriented in opposite directions on a common subpath, then it corresponds to reversing one of the paths in Figure 5. Then we can extend the table for $\varepsilon_x(f, g)$ by the rule $\{f, g^{-1}\} = -g^{-1}\{f, g\}$.

We also assumed so far that the subpaths were proper. It remains to consider when one of the paths is a subpath of the other. For instance, let's say $f$ is contained in $g$. Then we can write $g$ as $g = fg'$. Then by the Leibniz rule, $\{f, g\} = \{f, fg'\} = f\{f, g'\}$. It must be (since all vertices are trivalent) that the beginning of $g'$ is a common subpath with $f$, and so this is the term corresponding to the common subpath of $f$ and $g$ which is all of $f$. The rest of the common subpaths of $f$ and $g'$ are just usual proper common subpaths of $f$ and $g$, as discussed above. 

**Remark.** Note that this result depends only on the parameters $A$ and $B$, and not on the actual values of $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$. In particular, we could choose $\alpha = \beta = \delta = \varepsilon = 0$, and $\gamma = -B, \zeta = -A$, and the formula from the proposition would be the same. From now on, we will assume this is the case (that all but $\gamma$ and $\zeta$ are zero). In particular, we will mainly be interested in the case where $A = B = -\frac{1}{2}$, so that $\varepsilon_{x}(f, g) = 1$ when the paths touch, and $\varepsilon_{x}(f, g) = 0$ when the paths cross.
3 Interpreting the Pentagram Map in Terms of Networks

We now use the constructions from the previous section to realize the pentagram map as a sequence of Postnikov moves and gauge transformations of a particular network in an annulus. The properties of the boundary measurements will give us invariants of the pentagram map, which also turn out to be involutive with respect to the Poisson structures described in the previous section. This section summarizes the main points of [GSTV16].

3.1 The Quiver and Poisson Bracket

We now look at a very specific example of the quivers and Poisson structures discussed previously. The quiver is embedded in an annulus, and the case for $n = 5$ is shown below in Figure 6. In general, the number of square faces is $n$, and they are connected as in the figure. More specifically, the bottom right of the $i$th square face connects to the top left of the $(i+1)$th, and the top right of the $i$th square face is connected to the bottom left of the $(i+2)$th. The top and bottom edges of the boundary rectangle are identified, giving an annulus. We take the cut to be the top and bottom edges which we identify. The sources and sinks are labeled on the left and right edges, which become the inner and outer boundary circles, respectively. As was mentioned in the previous sections, we will sometimes consider the network on a torus by also identifying the left and right edges (up to a twist, indicated by the labels).

We consider the specific bracket mentioned at the end of the previous section, with $A = B = -\frac{1}{2}$. This means that for $f$ and $g$ paths in $Q$, that $\varepsilon_x(f, g) = 1$ when $f$ and $g$ touch, and $\varepsilon_x(f, g) = 0$ when $f$ and $g$ cross.

3.2 The $x, y$ Coordinates

Next, we will apply gauge transformations to this network and consider some other functions on $Q$. We start by defining our notation for the edge weights. We will consider the toric network obtained by glueing the boundary components together. So the $n$th square connects to the 1st square. We label the edge weights around each square face as follows:
Because of our choice of coefficients $A = B = -\frac{1}{2}$, the brackets between these coordinates are given by

\[
\{\beta_i, \alpha_i\} = \frac{1}{2} \beta_i \alpha_i \quad \{\beta_i, \gamma_i\} = \frac{1}{2} \gamma_i \beta_i \\
\{\alpha_i, \varepsilon_{i-1}\} = \frac{1}{2} \varepsilon_{i-1} \alpha_i \quad \{\gamma_i, \varepsilon_i\} = \frac{1}{2} \gamma_i \varepsilon_i
\]

After applying gauge transformations at the corners of each square face, we can obtain the following edge weights:

\[
{\beta_i, \varepsilon_i}_{i-1} = \frac{1}{2} \beta_i \varepsilon_{i-1} \quad {\gamma_i, \varepsilon_{i-1}} = \frac{1}{2} \gamma_i \varepsilon_{i-1}
\]

The effect of these gauge transformations is that all the edges other than those bounding the square faces have been set to 1. These new weights, which are monomials in the original weights, we will call by $a_i, b_i, c_i, d_i$:

\[
a_i = a_i \varepsilon_{i-1}^{-1} \quad b_i = \beta_i \\
c_i = \varepsilon_i^{-1} \gamma_i \quad d_i = \varepsilon_{i-1} \delta_i \varepsilon_i
\]

It is easily checked that the (non-zero) Poisson brackets of these monomials are given by

\[
\{b_i, a_i\} = \frac{1}{2} b_i a_i \quad \{b_i, c_i\} = \frac{1}{2} b_i c_i \\
\{a_i, d_i\} = \frac{1}{2} a_i d_i \quad \{c_i, d_i\} = \frac{1}{2} c_i d_i
\]

We will now apply further gauge transformations, in order to set as many edge weights as possible equal to 1. It is possible, after further gauge transformations, to set most of the weights equal to 1, except the bottom and left edges of each square face, and a few other edges. The result (again for $n = 5$) is pictured in Figure 7.

The weights in the new quiver after gauge transformations are given as follows

\[
x_i = \frac{a_i}{c_i-1 d_i-1 c_i-2} \\
y_i = \frac{b_i}{c_i d_i c_{i-1} d_i-1 c_i-2} \\
z = \prod_{k=1}^{n} d_k c_k
\]
Thinking of the quiver as being on a torus, all of these monomials represent loops. They are depicted in Figure 8. The blue loop is $x_4$, the red loop is $y_4$, and the green loop is $z$.

It is an easy calculation to see that $z$ is a Casimir of the Poisson bracket, and so the bracket descends to the level surface in $\mathcal{E}_Q$ defined by $z = 1$. We will now consider just edge weights lying on this hypersurface. In this case, we have that all edge weights except those labeled by $x_i$ and $y_i$ are equal to 1. Therefore, this level surface has coordinates given by $x_i$ and $y_i$, and so it is $2n$-dimensional. We will associate these coordinates with the $x_i, y_i$ coordinates on $\mathcal{P}_n$ introduced in section 1. Their brackets are given by

$$\{x_{i+1}, x_i\} = x_{i+1}x_i \quad \{y_i, x_i\} = y_ix_i$$
$$\{y_{i+1}, y_i\} = y_{i+1}y_i \quad \{y_i, x_{i+1}\} = y_ix_{i+1}$$
$$\{y_{i+2}, y_i\} = y_{i+2}y_i \quad \{x_i, y_{i+1}\} = x_iy_{i+1}$$
$$\{x_i, y_{i+1}\} = x_iy_{i+1}$$

### 3.3 The $p, q$ Coordinates

It will be convenient to also consider a different set of functions on $\mathcal{E}_Q$. We return to the $a, b, c, d$ coordinates before the gauge transformations which gave the $x, y$ coordinates. We define the face weights $p_i$ and $q_i$ as the counterclockwise paths around the faces of the quiver, taking inverses when the orientation disagrees:
\[ p_i = \frac{b_i}{a_i c_i d_i} \quad q_i = \frac{c_i - 2d_i - 1 a_{i+1}}{b_i} \]

These are related to the \( x, y \) coordinates by

\[ p_i = \frac{y_i}{x_i} \quad q_i = \frac{x_{i+1}}{y_i} \]

Define the quiver \( Q' \), dual to \( Q \), to have vertices corresponding to the faces of \( Q \), and arrows corresponding to arrows of \( Q \) connecting vertices of different colors, directed so that white vertices are on the left and black vertices on the right. The vertices of \( Q' \) are labeled by \( p_i \) and \( q_i \), with arrows \( p_i \rightarrow q_{i-1} \), \( p_i \rightarrow q_{i+2} \), \( q_i \rightarrow p_i \), and \( q_i+1 \rightarrow p_i \). This is pictured in Figure 9, with the dual quiver drawn in blue.

\[ \{ q_i, p_i \} = q_i p_i \quad \{ q_{i+1}, p_i \} = q_{i+1} p_i \]

\[ \{ p_i, q_{i-1} \} = p_i q_{i-1} \quad \{ p_i, q_{i+2} \} = p_i q_{i+2} \]

3.4 The Postnikov Moves and the Invariants

Now we will describe a sequence of Postnikov moves which will transform this quiver (considered as being on the torus) into an isomorphic quiver. The new edge weights obtained after this sequence will be the expressions for the pentagram map on \( P_n \) in the \( x, y \) coordinates given in section 1. We start with the quiver as in Figure 7, where all weights are 1 except the \( x_i \) and \( y_i \) weights on the bottom and sides of the square faces. We then apply the following moves, in order:

1. Perform the “square move” at each of the \( n \) square faces
2. Perform the “white-swap” at each white-white edge
3. Perform the “black-swap” at each black-black edge
After this sequence of moves, the underlying directed graph is isomorphic to the one we started with. However, the edge weights will not be of the same form. It remains to perform gauge transformations (as we did in the previous section) so that again all weights are 1 except the bottom and left of each square face. After these gauge transformations, and after choosing a particular choice of graph isomorphism, the $x, y$ weights transform as

$$
x_i \mapsto x_i \frac{x_{i+2} + y_{i+2}}{x_i + y_i}
y_i \mapsto y_{i+1} \frac{x_{i+1} + y_{i+1}}{x_{i+3} + y_{i+3}}
$$

This is almost (but not quite) the same as the expression for the pentagram map on $P_n$ given in section 1. But after making the change of variables $y_i \mapsto y_{i-1}$, the formulas agree. So it only differs by a shift of indices in the $y$-variables.

As discussed before, the Postnikov moves and gauge transformations performed above do not change the boundary measurements. However, since the quiver was considered on a torus, we may have had some of the vertices or edges “wrap around” from the right side to the left side when doing the “white-swap” and “black-swap” moves. This corresponds to cutting a piece off the right side of the picture, and gluing it back onto the left side. This changes the boundary measurement matrix up to conjugation. To see this, suppose that $B_1$ and $B_2$ are the boundary measurement matrices of the left and right segments where we “cut” the picture. Then the original boundary measurement matrix is $B = B_1 B_2$, and the new boundary measurement matrix is $B_2 B_1$. So in summary, the boundary measurement matrix of the resulting quiver is not necessarily the same as before, but is conjugate to the original.

Thus the components of the characteristic polynomial $\chi(t) = \det(\text{Id} + tB)$ are unchanged by the sequence of moves described above. Recall that we consider the elements of the boundary measurement matrix $B$ to be the modified edge weights, which are Laurent polynomials in the variable $\lambda$. So $\chi$ is a function of both $\lambda$ and $t$, which is polynomial in $t$, and Laurent in $\lambda$. We denote the coefficients (which are functions of the edge weights of the quiver) by $I_{ij}$:

$$
\chi(\lambda, t) = \sum_{i,j} I_{ij} \lambda^i t^j
$$

The discussion above implies that $I_{ij}$ are invariants of the pentagram map. Furthermore, we have:

**Theorem 2.** [GSTV16] Let $I_{ij}$ be as defined above. Then

(a) $I_{ij}$ are invariant under the pentagram map

(b) $\{I_{ij}, I_{k\ell}\} = 0$ for all $i, j, k, \ell$

(c) The pentagram map is completely integrable

We will also consider the dynamics in the $p, q$ coordinates described before. A simple calculation using the relations between $p, q$ and $x, y$ shows that the pentagram map transforms the $p, q$ coordinates by

$$
q_i \mapsto \frac{1}{p_{i+1}}
p_i \mapsto q_{i+2} \frac{(1 + p_i)(1 + p_{i+3})}{(1 + p_{i+1})(1 + p_{i+2})}
$$
In these coordinates we can interpret the Postnikov moves as cluster transformations. To see this, consider the dual quiver $Q'$ described before. Interpret the $p,q$ variables as the initial cluster of a cluster algebra whose exchange matrix $B = (b_{ij})$ is defined by the dual quiver $Q'$. We use the mutation formula

$$
\mu_k(x_i) = \begin{cases} 
\frac{1}{x_i} & \text{if } i = k \\
x_i(1 + x_k)^{b_{ik}} & \text{if } b_{ik} > 0 \\
x_i(1 + x_k^{-1})^{b_{ik}} & \text{if } b_{ik} < 0 
\end{cases}
$$

These are referred to as cluster $X$-variables (as opposed to cluster $A$-variables) by Fock and Goncharov [FG06]. They are also called “$\tau$-coordinates” by Gekhtman, Shapiro, and Vainshtein [GSV10a]. This type of cluster algebra is called a cluster Poisson algebra, since it inherits a natural Poisson bracket which is “compatib le” with mutation (see [GSV10a]), given by

$$\{ x_i, x_j \} = b_{ij} x_i x_j$$

Using the dual quiver $Q'$ for the exchange matrix $B$, this cluster Poisson bracket in the $p,q$ variables coincides with the bracket on $E_{Q'}$. Additionally, the square moves in the sequence giving the pentagram map coincide with mutations at the $p$-vertices. The sequence of mutations, $\mu$, that mutates at each $p_i$ once gives

$$\tilde{p}_i = \mu(p_i) = \frac{1}{p_i}, \quad \tilde{q}_i = \mu(q_i) = q_i \frac{(1 + p_{i+1})(1 + p_{i-2})}{(1 + p_i^{-1})(1 + p_{i-1}^{-1})}$$

The white-swap and black-swap Postnikov moves give a graph isomorphic to the original, which exchanges the $\tilde{p}$ and $\tilde{q}$ face weights. That is, the new square face weights $p^*_i$ and the new octagonal face weights $q^*_i$ are

$$p^*_i = \tilde{q}_i = q_i \frac{(1 + p_{i+1})(1 + p_{i-2})}{(1 + p_i^{-1})(1 + p_{i-1}^{-1})}$$

$$q^*_i = \tilde{p}_i = \frac{1}{p_i}$$

The pentagram map coincides with this formula up to the permutation of the variables which shifts the indices by $p_i^* \mapsto p_{i+2}^*$ and $q_i^* \mapsto q_{i+1}^*$. More technically, we have

$$T(p_i) = \tilde{q}_{i+1}, \quad T(q_i) = \tilde{p}_{i+2}$$

It is a simple calculation to verify that the brackets $\{ T(p_i), T(q_j) \}$ have exactly the same form as $\{ p_i, q_j \}$.

4 The Grassmann Pentagram Map

4.1 Twisted Grassmann Polygons and the Pentagram Map

Mari Beffa and Felipe studied a generalization of the pentagram map [FB15] to the Grassmann manifold. The exposition and notation in this section is largely borrowed from that paper, with minor variations. The projective plane $\mathbb{P}^2$ coincides with the Grassmannian $\text{Gr}(1, 3)$ of 1-dimensional subspaces of $\mathbb{R}^3$. A natural generalization would be
to consider $\text{Gr}(n, 3n)$, in which the previous case is just when $n = 1$. In actuality, Mari Beffa and Felipe considered more generally $\text{Gr}(n, mn)$, which generalizes $\mathbb{P}^{m-1}$, but we will focus here on the case $m = 3$.

Consider the set $\text{Mat}_{3n \times n}$ of $3n$-by-$n$ real matrices. There are two natural multiplication actions: on the left by $\text{GL}_{3n}$, and on the right by $\text{GL}_n$. If we call $\mathcal{M}_n \subset \text{Mat}_{3n \times n}$ the subset of rank-$n$ matrices, then we will identify $\text{Gr}(n, 3n)$ with the orbit space $\mathcal{M}_n/\text{GL}_n$ by this right action, since two matrices will be equivalent if they have the same column-span. Then $\text{Gr}(n, 3n)$ carries a natural left action by $\text{GL}_{3n}$, induced by the action on $\text{Mat}_{3n \times n}$.

Analogously to the $\mathbb{P}^2$ case, we define a twisted Grassmann $n$-gon (or just “polygon” if $n$ is understood) to be a bi-infinite sequence $(p_i)_{i \in \mathbb{Z}}$ of points in $\text{Gr}(n, 3n)$, together with an element $M \in \text{SL}_{3n}$ called the monodromy, such that $p_{i+n} = Mp_i$ for all $i$. As in the classical case, we will only consider twisted polygons with some nondegeneracy condition. More specifically, choose a lift $(P_i)$ of $(p_i)$ to $\mathcal{M}_n$. We require for any $i$ that the matrix $(P_i P_{i+1} P_{i+2})$ is nonsingular, or equivalently, that the combined columns of $P_i, P_{i+1}, P_{i+2}$ form a basis of $\mathbb{R}^{3n}$. If this nondegeneracy condition is satisfied, then we obtain something analogous to the linear dependence relations given in Equation 1.

For each $i$, there are matrices $A_i, B_i, C_i \in \text{GL}_n$ so that

$$P_{i+3} = P_i A_i + P_{i+1} B_i + P_{i+2} C_i$$

(4)

For the remainder of the paper, we will use the abbreviated phrase “twisted polygon” to mean “twisted Grassmann polygon”. We will use the notation $\mathcal{GP}_n$ to denote the moduli space of twisted Grassmann polygons in $\text{Gr}(n, 3n)$, up to the action of $\text{SL}_{3n}$. We will use the adjective “classical” when we wish to distinguish the $\mathbb{P}^2$ case. Next we will describe the pentagram map on Grassmann polygons.

Let $(p_i)$ be a twisted polygon in $\text{Gr}(n, 3n)$. Since $p_i$ and $p_{i+2}$ are $n$-dimensional subspaces, they span a $2n$-dimensional (codimension $n$) subspace $\mathcal{L}_i^+$. Similarly, let $\mathcal{L}_i^-$ be the span of $p_{i+1}$ and $p_{i+3}$. Then their intersection $\mathcal{L}_i^+ \cap \mathcal{L}_i^-$ is a codimension $2n$ (dimension $n$) subspace, which is again an element of $\text{Gr}(n, 3n)$. We define the pentagram map to be the map that sends $(p_i)$ to $(q_i)$, where $q_i = \mathcal{L}_i^+ \cap \mathcal{L}_i^-$. We will also refer to the pentagram map as $T: \mathcal{GP}_n \to \mathcal{GP}_n$. Abusing notation as in the classical case, we also write the map as if it were defined on individual vertices $T(p_i) = q_i$, and also write the map as if it is defined on lifts $T(P_i) = Q_i$ if $(P_i)$ and $(Q_i)$ are lifts of $(p_i)$ and $(q_i)$.

### 4.2 Corrugated Grassmann Polygons and Higher Pentagram Maps

Analogous to the classical case, we can also define generalized higher pentagram maps in the Grassmann case. This is a generalization from $\text{Gr}(n, 3n)$ to $\text{Gr}(n, kn)$. A twisted Grassmann polygon in $\text{Gr}(n, kn)$ is a sequence $(p_i)$ with monodromy $M \in \text{SL}_{kn}$ so that $p_{i+n} = Mp_i$ for all $i$. We denote the set of all equivalence classes of twisted polygons (up to the action of $\text{SL}_{kn}$) by $\mathcal{GP}_{k,n}$. In keeping with the notation of the previous section, we write just $\mathcal{GP}_n$ for $k = 3$.

We say that a twisted Grassmann polygon is corrugated if $p_i, p_{i+1}, p_{i+k-1}, p_{i+k}$ span a $3n$-dimensional subspace of $\mathbb{R}^{kn}$ for all $i$. Let $\mathcal{GP}_{k,n}^0$ denote the set of classes of corrugated polygons with the additional property that any 3 of the 4 subspaces $p_i, p_{i+1}, p_{i+k-1}, p_{i+k}$ span a $3n$-dimensional subspace.

We define the generalized higher pentagram map $T: \mathcal{GP}_{k,n}^0 \to \mathcal{GP}_{k,n}$ as follows. Let $\mathcal{L}_i^+$ be the span of $p_i$ and $p_{i+k-1}$, which is a $2n$-dimensional subspace of $\mathbb{R}^{kn}$, and similarly $\mathcal{L}_i^-$ is the span of $p_{i+1}$ and $p_{i+k}$. The “corrugated” property means that $\mathcal{L}_i^+$ and $\mathcal{L}_i^-$ span a $3n$-dimensional subspace. So we have

$$\dim(\mathcal{L}_i^+ \cap \mathcal{L}_i^-) = \dim \mathcal{L}_i^+ + \dim \mathcal{L}_i^- - \dim(\mathcal{L}_i^+ + \mathcal{L}_i^-) = 2n + 2n - 3n = n$$

So $q_i := \mathcal{L}_i^+ \cap \mathcal{L}_i^-$ is again an element of $\text{Gr}(n, kn)$, and we define the pentagram map to be $(p_i) \mapsto (q_i)$. 

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4.3 Expression for the Pentagram Map

In [FB15], Mari Beffa and Felipe find coordinates on \( \mathcal{GP}_{k,n} \), and find a Lax representation for the pentagram map. We take a different approach now. For simplicity, we will restrict to the case \( k = 3 \) for the rest of the paper. First, we give an analogue of Proposition 1.

**Proposition 5.** The lift \((\hat{P}_i)\) of \((p_i)\) can be chosen so that \( C_i = \text{Id}_n \) for all \( i \).

**Proof.** Any other lift \( \hat{P}_i \) differs from \( P_i \) by a change of basis. That is, there are \( G_i \in \text{GL}_n \) so that \( \hat{P}_i = P_i G_i \). In other words, \( P_i = \hat{P}_i G_i^{-1} \). Substituting this into Equation 4, we obtain

\[
\hat{P}_{i+3} = \hat{P}_i \left( G_i^{-1} A_i G_{i+3} \right) + \hat{P}_{i+1} \left( G_{i+1}^{-1} B_i G_{i+3} \right) + \hat{P}_{i+2} \left( G_{i+2}^{-1} C_i G_{i+3} \right)
\]

We want to prove that we can always ensure that \( G_i^{-1} C_i G_{i+3} = \text{Id} \). In other words, we desire \( G_{i+3} = G_i^{-1} G_{i+2} \). Re-indexing gives \( G_{i+1} = C_{i+1} G_i \). Now we may choose for instance \( G_0 = \text{Id} \), and determine the rest by this recurrence.

Just as in Equation 2, we change notation and define \( Y_i := A_i \) and \( X_i := B_i \), so that Equation 4 becomes

\[
P_{i+3} = P_i Y_i + P_{i+1} X_i + P_{i+2}
\]

By re-arranging this equation, we see that for a lift of the image, we may take for \( T(P_i) \) the subspace spanned by the columns of

\[
P_{i+3} - P_{i+1} X_i = P_i Y_i + P_{i+2}
\]

As mentioned above, we take a different approach than Mari Beffa and Felipe. Instead of putting real coordinates on \( \mathcal{GP}_n \) and describing the pentagram map in coordinates, we instead see how the map transforms these matrices \( X_i \) and \( Y_i \). The map is only defined generically, since we need to assume certain matrices are invertible.

**Proposition 6.** The pentagram map transforms the \( X_i \) and \( Y_i \) by

\[
X_i \mapsto (X_i + Y_{i+1})^{-1} X_i (X_{i+2} + Y_{i+3})
\]

\[
Y_i \mapsto (X_i + Y_{i+1})^{-1} Y_i (X_{i+2} + Y_{i+3})
\]

**Proof.** The proof is essentially the same as that of Proposition 2. Since the lift of the image under the pentagram map is again a twisted polygon, it satisfies a relation like in Equation 4:

\[
T(P_{i+3}) = T(P_i) A_i + T(P_{i+1}) B_i + T(P_{i+2}) C_i
\]

We substitute the expressions from Equation 6 into the equation above, using either the left-hand or right-hand side, according to the same convention used in the proof of Proposition 2. Doing so, we obtain

\[
P_{i+5} + P_{i+3} Y_{i+3} = P_{i+1} (Y_{i+1} B_i - X_i A_i) + P_{i+3} (A_i + B_i - X_{i+2} C_i) + P_{i+5} C_i
\]

Comparing coefficients of each \( P_i \), we conclude that \( C_i = \text{Id} \), and that

\[
X_i A_i = Y_{i+1} B_i
\]

\[
Y_{i+3} + X_{i+2} = A_i + B_i
\]

Assuming that each \( X_i + Y_{i+1} \) is invertible, these equations can be solved to give the desired result.

Notice that the expressions in the proposition are a non-commutative version of the expressions obtained in the classical case. For the remainder of the paper, we attempt to generalize much of what was done in the previous expository sections to noncommutative variables, using elements of a noncommutative ring in place of the commutative coordinates on \( \delta_Q \), and we introduce a noncommutative Poisson structure which mimics the classical counterpart in [GSTV16].
5 Non-Commutative Poisson Structures

5.1 Double Brackets

In this section, we discuss the basic constructions and definitions introduced by Van den Bergh [Ber08]. Throughout, \( \mathbb{K} \) is a field, and \( A \) is an associative \( \mathbb{K} \)-algebra. Unadorned tensor products are assumed to be over \( \mathbb{K} \). We will assume the characteristic of \( \mathbb{K} \) is zero, but we do not necessarily assume it is algebraically closed.

If \( A \) is an associative algebra, then \( A \otimes^n \) is an \( A \)-bimodule in the obvious way:

\[
x \cdot (a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n) \cdot y = xa_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n y
\]

We refer to this as the outer bimodule structure on \( A \otimes^n \).

**Definition 1.** A double bracket on \( A \) is a \( \mathbb{K} \)-bilinear map

\[
\{\cdot, \cdot\} : A \times A \to A \otimes A
\]

which satisfies:

1. \( \{\cdot, \cdot\} \) is a derivation in the second argument with respect to the outer bimodule structure:

\[
\{a, bc\} = \{a, b\} (1 \otimes c) + (b \otimes 1) \{a, c\}
\]

2. \( \{b, a\} = -\{a, b\}^\tau \), where \( (x \otimes y)^\tau := y \otimes x \)

Note that these properties imply that \( \{\cdot, \cdot\} \) also satisfies

\[
\{ab, c\} = (1 \otimes a) \{b, c\} + \{a, c\} (b \otimes 1)
\]

A double bracket \( \{\cdot, \cdot\} \) is called a double Poisson bracket if it additionally satisfies a version of the Jacobi identity:

\[
0 = \sum_{k=0}^{2} \sigma^k \circ (\{\cdot, \cdot\} \otimes \text{Id}) \circ (\text{Id} \otimes \{\cdot, \cdot\}) \circ \sigma^{-k}
\]

The right-hand side is an operator on \( A \otimes A \otimes A \), and \( \sigma \) is the permutation operator which sends \( x \otimes y \otimes z \) to \( z \otimes x \otimes y \).

For an algebra \( A \), let \( \mu : A \otimes A \to A \) be the multiplication map. If \( A \) has a double bracket \( \{\cdot, \cdot\} \), then we define another operation \( \{-, -\} : A \times A \to A \) by composing with \( \mu \):

\[
\{a, b\} := \mu(\{a, b\})
\]

**Proposition 7.** [Ber08] Suppose \( \{\cdot, \cdot\} \) is a double bracket on \( A \) (not necessarily a double Poisson bracket). The induced bracket \( \{-, -\} \) has the following properties

1. \( \{a, bc\} = \{a, b\} c + b \{a, c\} \) (Leibniz in the 2nd argument)
2. $\{ab, c\} = \{ba, c\}$ \hspace{1cm} (cyclic in the $1^{st}$ argument)

3. $\{a, b\} \equiv -\{b, a\} \mod [A, A]$ \hspace{1cm} (skew mod commutators)

Proof. Suppose that $\{a, b\} = \sum_i \omega_i \otimes \bar{\omega}_i$ and $\{a, c\} = \sum_i \eta_i \otimes \bar{\eta}_i$. Then using the Leibniz rule for $\{-, -\}$ in the second argument, and composing with multiplication, we get

$$\{a, bc\} = \sum_i \omega_i \bar{\omega}_i c + \sum_i \eta_i \bar{\eta}_i = \{a, b\}c + b\{a, c\}$$

This proves the first identity. The third identity follows from the fact that $\{a, b\} + \{b, a\} = \sum_i [\omega_i, \bar{\omega}_i]$.

For the second identity, suppose that $\{b, c\} = \sum_i \zeta_i \otimes \bar{\zeta}_i$. Then on the one hand, we have

$$\{ab, c\} = \sum_i \eta_i b \otimes \bar{\eta}_i + \sum_i \zeta_i \otimes a \bar{\zeta}_i$$

On the other hand, we have

$$\{ba, c\} = \sum_i \eta_i \bar{\eta}_i + \sum_i \zeta_i a \otimes \bar{\zeta}_i$$

Obviously both will be the same after composing with the multiplication map. \hfill \Box

**Definition 2.** For an associative algebra $A$, define the cyclic space, denoted $A^\sharp$, to be the vector space quotient $A/[A, A]$, by the linear span of all commutators.

Note that since $[A, A]$ is not in general an ideal, this is not necessarily an algebra. For a vector subspace $V \leq A$, we let $V^\sharp$ denote its image under the projection. Similarly, for an element $a \in A$, we use the notations $a^\sharp$ and $\overline{a}$ to denote its image under the projection. Note that conjugate elements are equivalent, since $xyx^{-1} - y = [xy, x^{-1}]$.

From this it follows that in $A^\sharp$, monomials are equivalent up to cyclic permutation, since for any $x_1, \ldots, x_n \in A$, we have $x_n x_1 \cdots x_{n-1} = x_n (x_1 \cdots x_n) x_n^{-1}$.

The following follows easily from the properties above.

**Proposition 8.** Suppose $\{\cdot, \cdot\}$ is a double bracket on $A$. Then there is a well-defined bilinear skew-symmetric map $\langle\cdot, \cdot\rangle: A^\sharp \times A^\sharp \to A^\sharp$, given by

$$\langle a^\sharp, b^\sharp \rangle = \{a, b\}^\sharp$$

The next few results indicate the extra structure which $\{-, -\}$ and $\langle\cdot, \cdot\rangle$ inherit if $\{\cdot, \cdot\}$ additionally satisfies the double Jacobi identity.

**Proposition 9.** Suppose that $\{\cdot, \cdot\}$ is a double Poisson bracket. Then

1. $\{-, -\}$ is a “Loday bracket” on $A$. That is, it satisfies a version of the Jacobi identity:

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + \{b, \{a, c\}\}$$
2. \( \langle - , - \rangle \) is a Lie bracket on \( A^\# \).

A more general notion is given by what William Crawley-Boevey calls an \( H_0 \)-Poisson structure:

**Definition 3. [CB11]** An \( H_0 \)-Poisson structure on an associative algebra \( A \) is a Lie bracket \( [ - , - ] \) on \( A^\# \) such that each \( [a, -] \) is induced by a derivation of \( A \).

In particular, any double Poisson bracket \( \{ - , - \} \) is an \( H_0 \)-Poisson structure by Proposition 7 and Proposition 9. However, there are \( H_0 \)-Poisson structures which do not arise from double Poisson brackets. Later, we will define a double bracket which is not double Poisson, but still has the property that the induced bracket \( \langle - , - \rangle \) on \( A^\# \) is a Lie bracket. This will be an example of an \( H_0 \)-Poisson structure which comes from a double bracket, but not a double Poisson bracket.

### 6 Non-Commutative Networks and Double Brackets

#### 6.1 Weighted Directed Fat Graphs (Revisited)

Let \( Q = (Q_0, Q_1) \) be a quiver as in section 2, which is embedded in an annulus, with all sources on the inner boundary circle, all sinks on the outer boundary circle, and all internal vertices trivalent. Define the algebra \( A = A_Q = \mathbb{Q}\langle \alpha \mid \alpha \in Q_1 \rangle \) to be the free associative algebra generated by the arrows of the quiver. Assuming, as before, that \( Q \) is acyclic, we can define the boundary measurement matrix \( B(Q) = (b_{ij}) \), where \( b_{ij} \) is the sum of the weights of all paths from source \( i \) to sink \( j \). Here, the weight of a path is the product of the weights, in order from left to right.

We again assign an indeterminate \( \lambda \) to the cut, and consider the boundary measurements to be Laurent polynomials in \( A[\lambda^\pm] \), where \( \lambda \) commutes with elements in \( A \).

We can define non-commutative gauge transformations. For any Laurent monomial \( t \) in the original edge weights (corresponding to the arrows of the quiver), we can multiply all incoming arrows at one vertex by \( t \) on the right, and multiply all outgoing arrows at that vertex by \( t^{-1} \) on the left. This will obviously not change the boundary measurements.

We can also define non-commutative versions of the the Postnikov moves, which preserve the boundary measurements. They are pictured in Figure 10. As in the commutative case, \( \Delta := b + adc \). We must take special care at this point to say what we mean by expressions such as \( (b + acd)^{-1} \). We do so now.

#### 6.2 The Free Skew Field and Mal’cev Neumann Series

In order to make sense of expressions of the form \( (x + y)^{-1} \), we need an appropriate notion of “noncommutative rational functions”. This will be the free skew field, which we will define below. Then, we will discuss how the free skew field can be identified with a certain subset of noncommutative formal power series.

For a set \( X = (x_1, \ldots, x_n) \) of formal noncommuting variables, the free skew field or universal field of fractions, which we will denote by \( \mathcal{F}_Q(X) \), or just \( \mathcal{F}(X) \), is a division algebra over \( \mathbb{Q} \) characterized by the following universal property:
There is an injective homomorphism $i: \mathbb{Q}\langle X \rangle \to \mathcal{F}_\mathbb{Q}(X)$ from the free associative algebra on $X$ into $\mathcal{F}_\mathbb{Q}(X)$ such that for any homomorphism $\varphi: \mathbb{Q}\langle X \rangle \to D$ into a division ring $D$, there is a unique subring $\mathbb{Q}\langle X \rangle \subset R_\varphi \subset \mathcal{F}_\mathbb{Q}(X)$ and a homomorphism $\psi: R \to D$ with $\varphi = \psi \circ i$, and such that if $0 \neq a \in R$ and $\psi(a) \neq 0$, then $a^{-1} \in R$.

The explicit construction of $\mathcal{F}(X)$ is a bit complicated (see [GGRW02] or [Coh77] for details), but informally it consists of noncommutative rational expressions in the variables $x_1, \ldots, x_n$, under some suitable notion of equivalence. For example, $\mathcal{F}(X)$ contains expressions such as $(1 + x)^{-1}$ and $w(x + y)^{-1}z$.

Another way to embed the free algebra into a division ring is by so-called Mal’cev Neumann series (also called Hahn-Mal’cev-Neumann series). Suppose that the free group generated by $X$ is given some order relation compatible with multiplication. This means that if $f \leq g$, then $hf \leq hg$ for any $h$. Given an ordering of the free group, the ring of Mal’cev Neumann series is the subset of all formal series over the free group which have well-ordered support. This is a division ring which contains $\mathbb{Q}\langle X \rangle$. Jaques Lewin proved [Lew74] that the free skew field $\mathcal{F}(X)$ is isomorphic to the subfield of the ring of Mal’cev Neumann series generated by the variables $x_1, \ldots, x_n$. In the next section, we show an example of expanding an element of the free skew field as a series.

Since $A = A_\mathbb{Q} = \mathbb{Q}\langle Q_1 \rangle$ is a free algebra, it is a subalgebra of $\mathcal{F}(Q_1)$. So, when we consider the noncommutative Postnikov “square move”, we interpret the expressions $(b + ade)^{-1}$ as elements of the free skew field $\mathcal{F}(Q_1)$. If we choose an order relation compatible with multiplication, then we may also view these expressions as noncommutative Mal’cev Neumann series. Thus, from now on, given a quiver, we will work more generally with the skew field.
\( \mathcal{F}(Q_1) \) rather than the free algebra \( \mathbb{Q} \langle Q_1 \rangle \).

### 6.3 An Example of a Series Expansion

Consider the free skew field on two variables \( x \) and \( y \). We will write the element \((x + y)^{-1}\) as a noncommutative power series. As mentioned above, we need to put an order on the free group generated by \( x \) and \( y \) to determine the ring of Mal’cev Neumann series. We will use the order induced by the Magnus embedding of the free group into the ring \( \mathbb{Z}[[x, y]] \) of formal power series in non-commuting variables \( x \) and \( y \) [MKS76].

More specifically, we embed the free group into \( \mathbb{Z}[[x, y]] \) by the map \( \alpha \mapsto 1 + \alpha \) and \( \alpha^{-1} \mapsto \sum_{i \geq 0} (-1)^i \alpha^i \) where \( \alpha \) is either \( x \) or \( y \). This embeds the free group as a subgroup of the multiplicative group of all power series with constant term 1. Choosing an order of the variables, say \( x < y \), this determines a graded lexicographic order on monomials/words in \( \mathbb{Z}[[x, y]] \). Then we define an order on \( \mathbb{Z}[[x, y]] \) where \( f < g \) if the coefficient of \( f \) at the first place they disagree is smaller. This induces an order on the free group. We then define the ring of Mal’cev Neumann series to be those series which have well-ordered support with respect to this ordering of the free group.

There are a couple different ways we could try to expand \((x + y)^{-1}\) as a series. We could factor out either of the variables, and then expand as a geometric series. For example, we could first factor out \( x \) to get

\[
(x + y)^{-1} = (x(1 + x^{-1}y))^{-1} = (1 + x^{-1}y)^{-1}x^{-1}
\]

Then we expand \((1 + x^{-1}y)^{-1}\) as a geometric series to get

\[
(x + y)^{-1} = (1 - x^{-1}y + x^{-2}y^2 - x^{-3}y^3 + \cdots)x^{-1} = x^{-1} - x^{-1}yx^{-1} + x^{-1}yx^{-1}y^{-1} + \cdots
\]

To see if the support is well-ordered, we need to go through the above construction. Under the Magnus embedding, we have

\[
x^{-1} \mapsto 1 - x + x^2 - x^3 + \cdots
\]
\[
x^{-1}yx^{-1} \mapsto 1 - 2x + y + 3x^2 - xy - yx + \cdots
\]
\[
x^{-1}yx^{-1}yx^{-1} \mapsto 1 - 3x + 2y + 6x^2 + y^2 - 3xy - 3yx + \cdots
\]
\[
\vdots
\]
\[
(x^{-1}y)^nx^{-1} \mapsto 1 - (n + 1)x + ny + \cdots
\]

The sequence of terms in our series expansion for \((x + y)^{-1}\) all differ at the coefficient for \( x \). We see that the coefficients are decreasing in the sequence \(-1, -2, -3, \ldots, -(n + 1), \ldots\). There is thus no lowest term in this sequence, and so it is not well-ordered. This expansion is then not a Mal’cev Neumann series for our chosen ordering.

If, however, we factor out \( y \) instead of \( x \), things will work out. Then we get

\[
(x + y)^{-1} = y^{-1}(1 + xy^{-1})^{-1} = y^{-1} - y^{-1}xy^{-1} + y^{-1}yx^{-1}y^{-1} + \cdots
\]

Then the general term, under the Magnus embedding, will be

\[
y^{-1}(xy^{-1})^n \mapsto 1 + nx - (n + 1)y + \cdots
\]

Now the sequence of \( x \) coefficients is the increasing sequence \( 0, 1, 2, 3, \ldots, n, \ldots \). Therefore the first term \( y^{-1} \) is the lowest in the sequence, and so the support is well-ordered. This expansion is thus a Mal’cev Neumann series.
6.4 Double Brackets Associated to a Quiver

We now define a family of double brackets on the skew field \( A_q := \mathcal{F}(Q_1) \) which generalize the Poisson structures on the space of edge weights \( E_q \) described in section 2. We again define it locally, and then describe the concatenation procedure by which they can be glued together. We start with the two local pictures of white and black vertices:

![Diagram of white and black vertex pictures](image)

Let \( A_o = \mathcal{F}(x, y, z) \) be the free skew field (over \( \mathbb{Q} \)) on 3 generators. We will think of the variables \( x, y, z \) as representing the edge weights on the “white” vertex picture above. Similarly, let \( A_\bullet = \mathcal{F}(a, b, c) \) correspond to the “black” vertex. Of course they are isomorphic as associative \( \mathbb{Q} \)-algebras, but we will define different brackets on them. Choose any \( \alpha, \beta, \gamma \in \mathbb{Q} \), and define a double bracket \( \{\{ - , - \}_o \} \) on \( A_o \) by

\[
\{\{ x, y \}_o \} = \alpha (1 \otimes xy) \\
\{\{ x, z \}_o \} = \beta (1 \otimes xz) \\
\{\{ y, z \}_o \} = \gamma (y \otimes z)
\]

Similarly, for scalars \( \delta, \varepsilon, \zeta \in \mathbb{Q} \), define \( \{\{ - , - \}_\bullet \} \) on \( A_\bullet \) by

\[
\{\{ a, b \}_\bullet \} = \delta (ba \otimes 1) \\
\{\{ a, c \}_\bullet \} = \varepsilon (ca \otimes 1) \\
\{\{ b, c \}_\bullet \} = \zeta (c \otimes b)
\]

Also define \( A_\partial = \mathcal{F}(x) = \mathbb{Q}(x) \) to be the field of rational functions in one variable, corresponding to a univalent boundary vertex, with trivial double bracket.

These double brackets are again “compatible” with concatenating/glueing in a similar sense as in the commutative case. Let \( H_q \) be the free skew field generated by the half-edges of \( Q \), denoted again by \( \alpha_s \) and \( \alpha_t \) for the source and target ends of \( \alpha \in Q_1 \). Then \( H_q \) is the free product of the algebras defined above:

\[
H_q \cong \left( \bigodot_{\alpha \in V_o} A_o \right) \ast \left( \bigodot_{\bullet \in V_\bullet} A_\bullet \right) \ast \left( \bigodot_{\partial \in V_\partial} A_\partial \right)
\]

As is mentioned in [Ber08], for algebras \( A \) and \( B \) with double brackets, there is a uniquely defined double bracket on the free product \( A \ast B \) such that \( \{\{ a, b \}\} = 0 \) for \( a \in A \) and \( b \in B \). Since the free product is the coproduct in the category of associative algebras (analogously the tensor product is the coproduct for commutative algebras), this is analogous to the product bracket on \( H_q \) in the commutative case.

Again we define a glueing map \( g^*: A_q \rightarrow H_q \) by \( g^*(\alpha) = \alpha_s \alpha_t \) for \( \alpha \in Q_1 \). Since the algebras are non-commutative, this is not actually the pull-back of a geometric/topological map, but we keep the notation for the sake of analogy. We have the following result, which mimics the commutative case:
Proposition 10. There is a unique double bracket on $A_Q$ so that the glueing homomorphism $g^*: A_Q \to H_Q$ satisfies

$$\{g^*(\alpha), g^*(\beta)\}_{H_Q} = (g^* \otimes g^*) \left( \{\alpha, \beta\}_{A_Q} \right)$$

Proof. As in the commutative case, there are many cases to consider, but all of them are similar and are simple calculations. We show one as an example to illustrate the idea. We again look at Figure 4, and try to define $\{w, y\}_{A_Q}$.

If it is to satisfy the desired property, we must have

$$\{g^*(w), g^*(y)\}_{H_Q} = \{\alpha, \beta\}_{A_Q}$$

This suggests that $\{w, y\}_{A_Q}$ must be defined to be $\alpha(1 \otimes wy)$. As mentioned above, all other cases are similar. Just as in the commutative case, the double bracket on $A_Q$ is given by the same expressions as in $H_Q$, treating edges as the corresponding half-edges which meet at a common vertex. 

We may also consider the doubled quiver $\overline{Q}$, as before. We then associate to each opposite arrow $\alpha^*$ the element $\alpha^{-1}$ in the free skew field. The double bracket defined above on generators extends in a unique way to the free skew field by the formulas

$$\{\beta, \alpha^{-1}\} = -(\alpha^{-1} \otimes 1) \{\beta, \alpha\} \ (1 \otimes \alpha^{-1})$$
$$\{\alpha^{-1}, \beta\} = -(1 \otimes \alpha^{-1}) \{\alpha, \beta\} \ (\alpha^{-1} \otimes 1)$$

We may then interpret any path in $\overline{Q}$ as a non-commutative Laurent monomial in $A_Q$.

6.5 A Formula for the Bracket

We will primarily be concerned with paths that are closed loops in $\overline{Q}$. Let $L \subset A_Q$ be the vector subspace spanned by all monomials which represent closed loops, and let $f, g \in L$ be two elements.

As before, we let $f \cap g$ denote the set of all maximal common subpaths. We will prove that, as in the commutative case, we only need to consider the ends of common subpaths in order to compute $\langle f, g \rangle$. First we give a result about the local structure of the double bracket.

Lemma 1. Let $x$ and $y$ be edges in $\overline{Q}$. That is, $x$ and $y$ can either be arrows in the quiver, or “reverse” arrows. Then

(a) If $y$ follows $x$ (i.e. $s(y) = t(x)$), then $\{x, y\} = \lambda(1 \otimes xy)$ for some $\lambda \in Q$.

(b) If $x$ and $y$ have the same source, then $\{x, y\} = \lambda(x \otimes y)$ for some $\lambda \in Q$.

(c) If $x$ and $y$ have the same target, then $\{x, y\} = \lambda(y \otimes x)$ for some $\lambda \in Q$. 

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Proof. The pictures below show the possibilities at a white vertex. The red are for part (a), the blue for part (b), and green for part (c). Note that there are 3 additional possibilities for part (a), given by the reversal/inverse of the pairs shown in red.

As in the picture from the previous section, call the unique incoming arrow $x$, and the outgoing arrows $y$ and $z$, in counter-clockwise order from $x$. Then the three pairs of edges in the picture above for (a) give

\[
\begin{align*}
\{(x, z)\} &= \beta(1 \otimes xz) \\
\{z^{-1}, y\} &= \gamma(1 \otimes z^{-1})(z \otimes y)(z^{-1} \otimes 1) = \gamma(1 \otimes z^{-1}y) \\
\{y^{-1}, x^{-1}\} &= -\alpha(x^{-1} \otimes y^{-1})(xy \otimes 1)(y^{-1} \otimes x^{-1}) = -\alpha(1 \otimes y^{-1}x^{-1})
\end{align*}
\]

For part (b), the pictured pairs give

\[
\begin{align*}
\{y, z\} &= \gamma(y \otimes z) \\
\{x^{-1}, y\} &= -\alpha(1 \otimes x^{-1})(1 \otimes xy)(x^{-1} \otimes 1) = -\alpha(x^{-1} \otimes y) \\
\{x^{-1}, z\} &= -\beta(1 \otimes x^{-1})(1 \otimes xz)(x^{-1} \otimes 1) = -\beta(x^{-1} \otimes z)
\end{align*}
\]

For part (c), the pairs pictured above give

\[
\begin{align*}
\{x, y^{-1}\} &= -\alpha(y^{-1} \otimes 1)(1 \otimes xy)(1 \otimes y^{-1}) = -\alpha(y^{-1} \otimes x) \\
\{x, z^{-1}\} &= -\beta(z^{-1} \otimes 1)(1 \otimes xz)(1 \otimes z^{-1}) = -\beta(z^{-1} \otimes x) \\
\{y^{-1}, z^{-1}\} &= \gamma(z^{-1} \otimes y^{-1})(y \otimes z)(y^{-1} \otimes z^{-1}) = \gamma(z^{-1} \otimes y^{-1})
\end{align*}
\]

All other possibilities at a black vertex are verified by similar calculations.

Lemma 2. Let $f, g \in \mathcal{L}$. The induced bracket $\langle f, g \rangle$ depends only on the endpoints of maximal common subpaths of $f$ and $g$.

Proof. The result can be stated more technically as follows. Suppose that $p$ is an edge in $f$ and $q$ is an edge in $g$. Expanding $\langle f, g \rangle$ with the Leibniz rule, there will be a term involving $\langle p, q \rangle$. The result says that unless $p$ and $q$ are incident to a common vertex which is an endpoint of a maximal common subpath, then this term is either zero, or it cancels with another term in the Leibniz expansion.

First of all, if $p$ and $q$ are not incident to the same vertex, then $\langle p, q \rangle = 0$. Since each vertex is trivalent, if two paths go through a common vertex, then they must have at least one edge in common at that vertex. So every non-zero contribution $\langle p, q \rangle$ comes from when $p$ and $q$ belong to a common subpath of $f$ and $g$. We now argue that even in this case, $\langle p, q \rangle$ is either zero or cancels unless $p$ and $q$ occur at the ends of a common subpath. For the remainder of the proof, let $w$ be a maximal common subpath of $f$ and $g$.

We first consider the simplest case, which is when $p$ and $q$ are consecutive edges in $w$. So suppose that $q$ immediately follows $p$ in $w$. Then we can write $w = w_1pqw_2$, and $f$ and $g$ can be written as $f = f_0w_1pqw_2$ and $g = g_0w_1pqw_2$. 

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Then after expanding using the Leibniz rules, we get two terms in $\langle f, g \rangle$ involving $\langle p, q \rangle$ and $\langle q, p \rangle$. By the previous lemma, $\langle p, q \rangle = \lambda (1 \otimes pq)$ for some $\lambda$, and $\langle q, p \rangle = -\lambda (pq \otimes 1)$. Then these two terms are given by

$$\lambda(g_0 w_1 p \otimes f_0 w_1)(1 \otimes pq)(qw_2 \otimes w_2) = \lambda g_0 w_1 pq w_2 \otimes f_0 w_1 pq w_2$$

and

$$-\lambda(g_0 w_1 \otimes f_0 w_1 p)(pq \otimes 1)(w_2 \otimes qw_2) = -\lambda g_0 w_1 pq w_2 \otimes f_0 w_1 pq w_2$$

Obviously, these terms cancel.

There is also the case that the path $w$ intersects itself, so it may be possible that $p$ and $q$ share a common vertex, but do not occur consecutively in the path. There are three cases, corresponding to parts $(a)$, $(b)$, and $(c)$ in the previous lemma.

First we consider part $(a)$ of the previous lemma. So suppose $p$ and $q$ are consecutive arrows in the quiver, but they occur non-consecutively in the path. Let $r$ be the third edge incident to the common vertex of $p$ and $q$, oriented outward. There are three cases: c orresponding to parts $(a)$, $(b)$, and $(c)$ in the previous lemma.

1. $p$ and $q$ occur consecutively twice, so $w = w_1 pq w_2 pq w_3$.
2. The path follows $r$ after $p$, and $p$ before $q$, so $w = w_1 pr w_2 pq w_3$.
3. The path follows $r$ after $p$, and $r^{-1}$ before $q$, so $w = w_1 pr w_2 r^{-1} q w_3$.

In cases $(2)$ and $(3)$, the vertex in question is the end of another common subpath, and so we don’t consider these cases. In case $(1)$, the brackets of the first $p$ and first $q$ cancel by the previous discussion, since they are consecutive edges in the path. The terms coming from the bracket of the first $p$ with the second $q$ and vice versa are given by

$$\lambda(g_0 w_1 pq w_2 pq w_3 \otimes f_0 w_1 pq w_3) \text{ and } -\lambda(g_0 w_1 pq w_2 pq w_3 \otimes f_0 w_1 pq w_3)$$

All other cases correspond to cases $(b)$ and $(c)$ from the previous lemma. All these cases imply that the vertex which $p$ and $q$ share is the end of some other common subpath, and so we don’t need to consider these. We have thus covered all cases, and the lemma is proved.

Write $f$ and $g$ as products of edges: $f = f_1 \cdots f_n$ and $g = g_1 \cdots g_m$. Then by the Leibniz rule:

$$\langle f, g \rangle = \sum_{i,j} (g_1 \cdots g_{j-1} \otimes f_1 \cdots f_{i-1}) \langle f_i, g_j \rangle (f_{i+1} \cdots f_n \otimes g_{j+1} \cdots g_m)$$

The result of the preceding lemma says that this sum is only over the pairs $f_i, g_j$ which are incident to a common vertex which is the end of a maximal common subpath of $f$ and $g$. So we will now compute what happens at these vertices.

As in the commutative case, there appear to be 36 cases to consider, since the two endpoints of a common subpath could each have one of two colors and one of three orientations. It happens again, however, that the orientations do not affect the outcome, which we formulate now as a lemma.

**Lemma 3.** The contribution to $\langle f, g \rangle$ coming from a common subpath $w \in f \cap g$ depends only on the colors of the endpoint vertices, and not on the orientations.
Proof. Let us first consider the beginning of a common subpath \(w\), where the paths \(f\) and \(g\) first come together. Then we may write \(f = f_1 \cdots f_i w_1 \cdots f_n\) and \(g = g_1 \cdots g_j w_1 \cdots g_m\), where \(w_1 = f_{i+1} = g_{j+1}\) is the first edge in the common subpath \(w\), and \(f_i\) and \(g_j\) are incident to the same vertex as \(w_1\). There are three terms in the expansion of \(\langle f, g \rangle\) coming from this vertex, corresponding to \(w\), given by

\[
\begin{align*}
(g_1 \cdots g_{j-1} \otimes f_1 \cdots f_{i-1}) \langle f_i, g_j \rangle (w_1 f_{i+2} \cdots f_n \otimes w_1 g_{j+2} \cdots g_m) \\
(g_1 \cdots g_j \otimes f_1 \cdots f_{i-1}) \langle f_i, w_1 \rangle (w_1 f_{i+2} \cdots f_n \otimes j g_{j+2} \cdots g_m) \\
(g_1 \cdots g_{j-1} \otimes f_1 \cdots f_i) \langle w_1, g_j \rangle (f_{i+2} \cdots f_n \otimes w_1 g_{j+2} \cdots g_m)
\end{align*}
\]

By Lemma 1, all three of these terms give the simple tensor

\[
g_1 \cdots g_j w_1 f_{i+2} \cdots f_n \otimes f_1 \cdots f_i w_1 g_{j+2} \cdots g_m,
\]

but with different coefficients. Which coefficient goes with which term depends on the orientation of the vertex. The following picture illustrates the three possibilities at a black vertex, with \(f_i w_1\) in blue and \(g_j w_1\) in red.

![Diagram](image)

The coefficients for the three pictures above are given in the following table:

| Case | \(\langle f_i, g_j \rangle\) | \(\langle f_i, w_1 \rangle\) | \(\langle w_1, g_j \rangle\) |
|------|----------------|----------------|----------------|
| (i)  | \(\zeta(g_j \otimes f_i)\) | \(\delta(1 \otimes f_i w_1)\) | \(\epsilon(g_j w_1 \otimes 1)\) |
| (ii) | \(\epsilon(g_j \otimes f_i)\) | \(\zeta(1 \otimes f_i w_1)\) | \(\delta(g_j w_1 \otimes 1)\) |
| (iii)| \(\delta(g_j \otimes f_i)\) | \(\epsilon(1 \otimes f_i w_1)\) | \(\zeta(g_j w_1 \otimes 1)\) |

In all cases, the three terms are the same simple tensor, so the coefficients add up. As we can see from the table, the combined coefficient is always \(-A = \epsilon + \zeta - \delta\). Similarly, for all three orientations at a white vertex, we will get the same simple tensor with a coefficient of \(B = \alpha - \beta - \gamma\). If we consider the endpoint vertex of the common subpath, we will again get the same simple tensor for all three terms. If the vertex is black, we will get a coefficient of \(A\), and if it is white, we will get \(-B\). These are exactly the same as the values for \(\epsilon_w(f, g)\) in the commutative case.

Let \(f_w g_w\) denote the loop which follows \(f\) starting with the path \(w\), followed by \(g\) starting at \(w\). Then putting the previous lemmas together gives

**Theorem 3.** Let \(f, g \in \mathcal{L}\). Then the induced bracket in \(A^1_Q\) is given by

\[
\langle f, g \rangle = \sum_{x : f \cap g} \epsilon_x(f, g) f_x g_x
\]

In particular, the subspace \(\mathcal{L}^2 \subset A^1_Q\) is closed under \(\langle -, - \rangle\).
Moreover, we have the following important observation

**Theorem 4.** The induced bracket \(\langle -, - \rangle\) makes \(\mathcal{L}^2\) a Lie algebra.

**Proof.** We only need to verify the Jacobi identity. So let \(f, g, h \in \mathcal{L}\). Then

\[
\langle \langle f, g \rangle, h \rangle = \sum_{i \in g \cap h} \varepsilon_i(g, h) \langle f, g_i h_i \rangle = \sum_{i \in g \cap h} \varepsilon_i(g, h) \left( \sum_{j \in f \cap g} \varepsilon_j(f, g) \langle f_j (g_i h_i)_j \rangle + \sum_{k \in f \cap h} \varepsilon_k(f, h) f_k (g_i h_i)_k \right)
\]

\[
\langle f, \langle g, h \rangle \rangle = \sum_{j \in f \cap g} \varepsilon_j(f, g) \langle f_j g_j, h \rangle = \sum_{j \in f \cap g} \varepsilon_j(f, g) \left( \sum_{k \in f \cap h} \varepsilon_k(f, h) (f_j g_k)_k h_k + \sum_{i \in g \cap h} \varepsilon_i(g, h) (f_j g_j)_i h_j \right)
\]

\[
\langle g, \langle f, h \rangle \rangle = \sum_{k \in f \cap h} \varepsilon_k(f, h) \langle g, f_k h_k \rangle = \sum_{k \in f \cap h} \varepsilon_k(f, h) \left( \sum_{i \in g \cap h} \varepsilon_i(g, h) (f_k h_k)_i - \sum_{j \in f \cap g} \varepsilon_j(f, g) (f_k h_k)_j \right)
\]

The Jacobi identity will hold if the first equation is equal to the sum of the second and third. Comparing the right-hand side of the first with the sum of the right-hand sides of the second and third, we see that the Jacobi identity will hold if the following identities are true:

\[
f_j (g_i h_i)_j = (f_j g_j)_i h_i = g_i (f_j h_j)_i
\]

It is an easy check that these identities are indeed true for all cycles. \(\square\)

**Remark.** This double bracket is not a double Poisson bracket (it does not satisfy the double Jacobi identity), so it does not immediately guarantee the Jacobi identity for \(\langle -, - \rangle\). Since we do have the Jacobi identity, though, for \(\langle - , - \rangle\), this makes it into an \(H_0\)-Poisson structure as defined by Crawley-Boevey [CB11].

### 6.6 Goldman’s Bracket and the Twisted Ribbon Surface

In this section, we give a geometric interpretation of the bracket just described. We first recall some preliminaries.

Let \(G\) be a connected Lie group and \(S\) a smooth oriented surface with fundamental group \(\pi := \pi_1(S)\). We consider the space of representations of \(\pi\) in \(G\), modulo conjugations, which we call \(\text{Rep}_G(\pi)\):

\[\text{Rep}_G(\pi) := \text{Hom}(\pi, G)/G\]

Let \(f : G \to \mathbb{R}\) be any invariant function on \(G\) (invariant under conjugation). Then for \(\alpha \in \pi\), we can define the function \(f_\alpha : \text{Rep}_G(\pi) \to \mathbb{R}\) by the formula \(f_\alpha([\varphi]) := f(\varphi(\alpha))\). In particular, if \(G\) is a group of matrices, we may take \(f = \text{tr}\). In this case we will write \(\text{tr}(\alpha)\) for \(f_\alpha\). Also, if \(G = \text{GL}_n(\mathbb{R})\), we will write \(\text{Rep}_n(\pi)\) instead of \(\text{Rep}_{\text{GL}_n(\mathbb{R})}(\pi)\).

In 1984, William Goldman described a symplectic structure on \(\text{Rep}_G(\pi)\) which generalizes the Weil-Petersson symplectic structure on Teichmüller space in the case that \(G = \text{PSL}_2(\mathbb{R})\) [Gol84]. Then, in 1986, he studied the Poisson bracket induced by this symplectic structure, in terms of the functions \(f_\alpha\) [Gol86]. Goldman gives explicit formulas for \(\{f_\alpha, f_\beta\}\) for various choices of the group \(G\), in terms of the topology of the surface and the intersection of curves representing \(\alpha\) and \(\beta\). In particular, when \(G = \text{GL}_n(\mathbb{R})\) and \(f = \text{tr}\), we have...
Theorem 5. [Gol86] The Poisson bracket of the functions $\text{tr}(\alpha)$ on $\text{Rep}_n(\pi)$ is given by

$$\{\text{tr}(\alpha), \text{tr}(\beta)\} = \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta) \text{tr}(\alpha_p \beta_p)$$

Here, $\alpha_p \beta_p$ denotes the loop which traverses first $\alpha$, and then $\beta$, both based at the point $p$, and $\varepsilon_p(\alpha, \beta)$ is the oriented intersection number of the curves at $p$.

Note the obvious similarity with our formula from Theorem 3. We will now formulate a geometric interpretation of our double bracket from a quiver so that a special choice of constants $A$ and $B$ realizes this Goldman bracket. First we define $\hat{\pi}$ to be the set of conjugacy classes in $\pi$. Goldman observes that the bracket above induces a Lie bracket on the vector space spanned by $\hat{\pi}$, the conjugacy classes in $\pi$ (the free homotopy classes of loops). Formally, we simply remove the “$\text{tr}$” in the formula above. So for $\alpha, \beta \in \hat{\pi}$, we have

$$[\alpha, \beta] = \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta) \alpha_p \beta_p$$

With this setup, we have the following

Theorem 6. [Gol86] $\mathbb{Z}\hat{\pi}$ is a Lie algebra with the bracket shown above, and the map $\text{tr}: \mathbb{Z}\hat{\pi} \to C^\infty(\text{Rep}_n(\pi))$ given by $\alpha \mapsto \text{tr}(\alpha)$ is a Lie algebra homomorphism.

In fact, our proof of Theorem 4 is essentially the same as Goldman’s original proof ([Gol86], Theorem 5.3).

In the literature, fat graphs are also commonly called “ribbon graphs”. From a fat graph $\Gamma$, one can construct an oriented surface with boundary, $S_\Gamma$, by replacing the edges with rectangular strips (ribbons) and the vertices with discs, where the ribbons are glued to the discs according to the cyclic ordering prescribed by the fat graph structure. We call this surface the ribbon surface associated to the fat graph. It is clear that the original graph $\Gamma$ is a deformation retract of $S_\Gamma$. We say that $\Gamma$ is a spine of the surface $S = S_\Gamma$.

Given a quiver $Q$ (oriented fat graph) with underlying unoriented graph $\Gamma$, we consider, as before, the subspace $\mathcal{L} \subset A$ of loops. Then in a natural way we identify the cyclic space $\mathcal{L}^\mathbb{Z}$ with $\mathbb{Q}\hat{\pi}$, the space generated by free homotopy classes of loops on $S_\Gamma$. In the commutative case, we considered the case $A = B = -\frac{1}{2}$, which gives $\varepsilon_p(f, g) = 1$ when $f$ and $g$ touch and $\varepsilon_p(f, g) = 0$ when $f$ and $g$ cross. We will again be primarily concerned with this specific choice of coefficients in the non-commutative case. This is opposite from what Goldman’s bracket, however. This is because if paths $f$ and $g$ touch in the quiver, then on $S_\Gamma$, they are homologous to paths which do not touch at all, and so their Goldman bracket should be zero. The way around this is to consider the “dual” surface.

We define the dual surface to $S_\Gamma$, which we denote $\widetilde{S}_\Gamma$, to be glued out of ribbons like $S_\Gamma$, except that whenever an edge joins vertices of different colors, the corresponding ribbon is given a half-twist. In this way, we think of the quiver as a planar projection picture of the ribbon surface, where one of the colors is the “top” of the ribbon, and the other color is the “bottom”. If we were to “untwist” the surface $\widetilde{S}_\Gamma$, and try to view it without twists, we would see that paths which touch in the quiver end up crossing in $\widetilde{S}_\Gamma$, and paths which cross in the quiver end up not touching in $\widetilde{S}_\Gamma$. Now that touching and crossing have been interchanged, we see that the Goldman bracket on $\widetilde{S}_\Gamma$ coincides with the induced bracket $\langle -, - \rangle$ on $\mathcal{L}^\mathbb{Z}$ when $A = B = -\frac{1}{2}$.

We will sometimes want to choose a vertex in the quiver $Q$, and only consider loops in the fundamental group of $S_\Gamma$ based at that point. After choosing a vertex, we let $\mathcal{L}^\bullet$ denote the subspace of loops which start and end at this point. Then $\mathcal{L}^\bullet$ is naturally identified with the group algebra of $\pi = \pi_1(S_\Gamma, \bullet)$. The induced bracket is independent of this choice of basepoint, since conjugate elements are equivalent in $\mathcal{L}^\bullet$. 

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We now point out some similar and related work which was brought to the author’s attention while working on the paper.

Remark. In \([MT12]\), Turaev and Massuyeau construct a quasi-Poisson bracket on the character variety \(\text{Rep}_{\pi}(\pi)\), which is induced by a double quasi-Poisson bracket on the group algebra of \(\pi\). As was mentioned above, we may identify the group algebra of \(\pi\) with \(L\), and so they are both double brackets on \(L\). However, the double bracket considered in this paper is not a double quasi-Poisson bracket, and so the two constructions are not exactly the same. However, the bracket of Turaev and Massuyeau also projects to the Goldman bracket in \(L^\natural\). Also, Semeon Artamonov, in his recent thesis \([Art18]\), constructed a much more abstract and categorical version of the quasi-Poisson structure of Turaev and Massuyeau.

6.7 The \(X, Y\) Variables

Just as in the commutative case, we will use gauge transformations to obtain new weights on the graph. Start with the quiver just as in Figure 6. Give the edge weights the same names as in the commutative case, but now they are formal noncommutative variables in \(F(Q_1)\). Perform gauge transformations to obtain variables \(a_i, b_i, c_i, d_i\) just as before. Note that the pictures in Section 3.2 are actually noncommutative gauge transformations, as they take into account whether multiplication happens on the left or right. In this case, the double bracket induced on the \(a, b, c, d\) variables is given by:

\[
\{ a_i, a_i \} = \frac{1}{2} b_i \otimes a_i \quad \{ b_i, c_i \} = \frac{1}{2} c_i \otimes b_i \\
\{ a_i, d_i \} = \frac{1}{2} 1 \otimes a_i d_i \quad \{ c_i, d_i \} = \frac{1}{2} d_i c_i \otimes 1
\]

Define the following monomials in the \(a, b, c, d\) variables:

\[
x_i = a_i c_{i-1}^{-1} d_{i-1}^{-1} c_{i-2}^{-1} \\
y_i = b_i c_{i-1}^{-1} d_{i-1}^{-1} c_{i-2}^{-1} \\
z_k = d_1 c_1 \cdots d_{k-1} c_{k-1} d_k
\]

These \(x_i\) and \(y_i\) are noncommutative versions of the same monomials given in the commutative case, and \(z_k\) are paths connecting the upper-left corner of the first square face to the upper-right corner of the \(k^{th}\) square face. We may again perform the exact same sequence of gauge transformations as in the commutative case to arrive at the weights depicted in Figure 7. However, these weights will now be noncommutative versions of the same monomials. In terms of the monomials defined above, the noncommutative weights we obtain in Figure 7 are given by

\[
X_i = z_{i-2} x_i z_{i-2}^{-1} \\
Y_i = z_{i-2} y_i z_{i-2}^{-1} \\
Z = z_n c_n
\]

Just as in the commutative case, these can be interpreted as closed loops in the quiver. The difference now in the non-commutative case is that all these loops share a common basepoint. This common basepoint is the upper-left
corner of the first square face. As in the previous section, let \( \mathcal{L} \subset \mathcal{L} \) denote the subspace of loops based at this point. Obviously \( \mathcal{L} \) is closed under multiplication, and so \( \mathcal{L} \) is an algebra. In fact, it is the group algebra of the fundamental group of the ribbon surface of the graph, and the fundamental group is generated by the \( X_i, Y_i, \) and \( Z \).

The following result says that \( \mathcal{L} \) is a subalgebra under the double bracket, which is not true of \( \mathcal{L}' \).

**Proposition 11.** The associative subalgebra \( \mathcal{L} \subset \mathcal{L} \) is closed under the double bracket. That is, \( \{ \mathcal{L}, \mathcal{L} \} \subseteq \mathcal{L} \otimes \mathcal{L} \). Furthermore, \( Z \) is a casimir of this bracket.

**Proof.** Let \( f, g \in \mathcal{L} \). Write \( f \) and \( g \) as monomials, \( f = f_1 \cdots f_k \) and \( g = g_1 \cdots g_\ell \), where each \( f_i \) and \( g_j \) are arrows in the quiver. Note that since they are both based at the point \( \bullet \), we have \( s(f_1) = s(g_1) = t(f_k) = t(g_\ell) = \bullet \). Now, using the Leibniz rule:

\[
\{ f, g \} = \sum_{i,j} (g_1 \cdots g_{j-1} \otimes f_1 \cdots f_{i-1}) \{ f_i, g_j \} (f_{i+1} \cdots f_k \otimes g_{j+1} \cdots g_\ell)
\]

Using the formulas from Lemma 1, and examining the three cases, we see that each term in the sum above is of the form \( \alpha \otimes \beta \), where \( \alpha \) and \( \beta \) are both in \( \mathcal{L} \).

A simple calculation shows that \( \{ Z, z_i \} = \{ Z, x_i \} = \{ Z, y_i \} = 0 \), from which is follows that \( \{ Z, X_i \} = \{ Z, Y_i \} = 0 \).

The bracket is then well-defined on the quotient \( \mathcal{L}^{(1)} := \mathcal{L} / (Z - 1) \), where \( Z \) is set to 1. The induced brackets in \( \mathcal{L}^{(1)} \) are given by:

\[
\langle X_{i+1}, X_i \rangle = X_{i+1}X_i \\
\langle Y_{i+2}, Y_i \rangle = Y_{i+2}Y_i \\
\langle Y_{i+1}, X_i \rangle = Y_{i+1}X_i \\
\langle X_{i+2}, Y_i \rangle = X_{i+2}Y_i
\]

If we perform the same sequence of Postnikov moves as in the commutative case, followed by gauge transformations, we get a graph isomorphism, with the same edges having weight 1 as before. This gives the transformation

\[
X_i \mapsto (X_i + Y_i)^{-1} X_i(X_{i+2} + Y_{i+2}) \\
Y_i \mapsto (X_{i+1} + Y_{i+1})^{-1} Y_{i+1}(X_{i+3} + Y_{i+3})
\]

Just as in the commutative case, this is almost the expression derived earlier for the pentagram map. It differs only by a shift in the \( Y \)-indices.

**6.8 The \( P, Q \) Variables**

We now define a non-commutative version of the \( p, q \) variables from the classical case. They are given by

\[
p_i = b_i c_i^{-1} d_i^{-1} a_i^{-1} \\
q_i = c_{i-2} d_{i-1} a_{i+1} b_{i-1}
\]

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Conjugating by the same $z_k$ paths as for the $X, Y$ variables, we obtain based versions:

$$P_i = z_{i-2} p_i z_{i-2}^{-1} \quad Q_i = z_{i-2} q_i z_{i-2}^{-1}$$

Similar to the commutative case, we have the following relation with the $X, Y$ variables:

$$P_i = Y_i X_i^{-1} \quad Q_i = X_{i+1} Y_i^{-1}$$

Their induced brackets are given by

$$\langle Q_i, P_i \rangle = Q_i P_i \quad \langle P_i, Q_{i-1} \rangle = P_i Q_{i-1}$$

$$\langle Q_{i+1}, P_i \rangle = Q_{i+1} Y_i^{-1} P_i Y_i \quad \langle P_i, Q_{i+2} \rangle = Y_i^{-1} P_i Q_{i+2}$$

From the formulas given in the previous section for the pentagram map in the $X, Y$ variables, we get that the pentagram map transforms the $P, Q$ variables by

$$P_i \mapsto X_{i+1}^{-1}(1 + P_{i+1}^{-1})^{-1} X_{i+1} \cdot (1 + P_{i+3}) Q_{i+2}(1 + P_{i+2}) \cdot Y_i^{-1}(1 + P_i) Y_i$$

$$Q_i \mapsto X_{i+1}^{-1} P_{i+1}^{-1} X_{i+1}$$

### 7 Invariance, Invariants, and Integrability

#### 7.1 Invariance of the Induced Bracket

We will show in this section that the induced bracket $\langle - , - \rangle$ on $(\mathcal{L}_n^{(1)})^g$ is invariant under the pentagram map. To do so, we will consider step-by-step how the weights and double bracket change under the Postnikov moves.

Recall that, starting with the $X, Y$ variables, the sequence of Postnikov moves which gives the pentagram map is as follows:

1. Perform the square move at each square face
2. Perform the white-swap move at each edge connecting two white vertices
3. Perform the black-swap move at each edge connecting two black vertices
4. Perform gauge transformations so all weights are 1 except the bottom and left of each square face

We will actually start with the $a, b, c, d$ weights instead of the $X, Y$ weights. The result will be the same, as we will show below. Application of a square move gives
The new edge weights are given by

\[
\begin{align*}
\tilde{b} &= b + adc = (1 + p^{-1})b \\
\tilde{c} &= \tilde{b}^{-1} ad = c^{-1} \cdot d^{-1} a^{-1} (1 + p)^{-1} ad \\
\tilde{a} &= dc\tilde{b}^{-1} = a^{-1} (1 + p)^{-1}
\end{align*}
\]

The double brackets of these new weights are

\[
\begin{align*}
\{\{\tilde{b}, \tilde{a}\}\} &= \frac{1}{2} (\tilde{a}\tilde{b} \otimes 1) \\
\{\{\tilde{b}, \tilde{c}\}\} &= \frac{1}{2} (1 \otimes \tilde{b}\tilde{c}) \\
\{\{\tilde{c}, \tilde{a}\}\} &= \frac{1}{2} (\tilde{d} \otimes \tilde{c})
\end{align*}
\]

Next, applying the white-swap and black-swap moves interchanges the square and octagonal faces. The resulting weights around the square faces is pictured in Figure 7.1.

Finally, we perform gauge transformations, as we did to get the original \(X, Y\) weights, so that all weights become 1 except those on the bottom and left edges of the square faces. We call these resulting weights \(\tilde{X}\) and \(\tilde{Y}\). If we define the “staircase” monomials \(\xi_k = \tilde{a}_1\tilde{b}_1 \cdots \tilde{a}_k\tilde{b}_k\), then we can express the new weights as

\[
\begin{align*}
\tilde{X}_i &= \xi_i \tilde{a}_{i+2}^{-1} \tilde{b}_{i+1}^{-1} \tilde{a}_{i+1}^{-1} \xi_i^{-1} \\
\tilde{Y}_i &= \xi_i \tilde{d}_{i+1}^{-1} \tilde{a}_{i+3}^{-1} \tilde{b}_{i+2}^{-1} \tilde{a}_{i+2}^{-1} \tilde{b}_{i+1}^{-1} \tilde{a}_{i+1}^{-1} \xi_i^{-1}
\end{align*}
\]

These weights are the images of \(X_i\) and \(Y_i\) under the pentagram map. That is, \(\tilde{X}_i = T(X_i)\) and \(\tilde{Y}_i = T(Y_i)\). As noted in the previous sections, these are given in terms of the original \(X, Y\) variables as \(\tilde{X}_i = \xi^{-1} X_i \Sigma_{i+2}\) and \(\tilde{Y}_i = \Sigma_{i+1}^{-1} Y_{i+1} \Sigma_{i+3}\), where \(\Sigma_i = X_i + Y_i\). We are now ready to prove the main theorem of this section:

**Theorem 7.** The induced bracket \(\langle -, - \rangle\) is invariant under the pentagram map.

**Proof.** We want to show that the induced brackets of \(\tilde{X}_i\) and \(\tilde{Y}_i\) have exactly the same form as the induced brackets of the \(X_i\) and \(Y_i\). Note that for \(a, b, c \in \mathbb{C}^\ast\), we have \(\langle a, c \rangle = \langle bab^{-1}, c \rangle\), since \(a^2 = (bab^{-1})^2\). So we will instead compute the brackets of the conjugate, but equivalent, elements:

\[
\begin{align*}
\bar{x}_i &= \xi^{-1}_i \bar{X}_i \xi_i = \bar{c}_i \bar{a}_{i+2}^{-1} \bar{b}_{i+1}^{-1} \bar{a}_{i+1}^{-1} \\
\bar{y}_i &= \xi^{-1}_i \bar{Y}_i \xi_i = \bar{d}_{i+1} \bar{a}_{i+3}^{-1} \bar{b}_{i+2}^{-1} \bar{a}_{i+2}^{-1} \bar{b}_{i+1}^{-1} \bar{a}_{i+1}^{-1}
\end{align*}
\]
We will compute the three possible combinations $\{x_i, x_j\}$, $\{y_i, y_j\}$, and $\{x_i, y_j\}$. For $\{x_i, x_j\}$, a simple calculation using the Leibniz rules for double brackets gives:

$$\{x_i, x_j\} = \frac{1}{2} \delta_{i,j+1} (c_j^{-1} a_{j+2}^{-1} x_i \otimes a_{j+2} c_j^{-1} x_j + x_i a_j^{-1} \otimes x_i a_j^{-1} a_i^{-1})$$

$$- \frac{1}{2} \delta_{i,j-1} (x_i a_j^{-1} a_j^{-1} \otimes x_i a_j^{-1} b_j + b_j^{-1} a_j^{-1} \otimes x_i a_j^{-1} x_j)$$

Therefore their induced brackets are

$$\langle \tilde{x}_i, \tilde{x}_j \rangle = \delta_{i,j+1} (c_j a_{j+2}^{-1} \tilde{x}_i (a_{j+2} c_j^{-1}) \tilde{x}_j - \delta_{i,j-1} (\tilde{b}_j^{-1} a_j^{-1}) \tilde{x}_i (a_j \tilde{b}_j) \tilde{x}_j$$

Conjugating the right hand side by $\xi_j$ gives the equivalent formula in terms of the $\tilde{X}_i$:

$$\langle \tilde{X}_i, \tilde{X}_j \rangle = (\delta_{i,j+1} - \delta_{i,j-1}) \tilde{X}_i \tilde{X}_j$$

This is exactly the same as the bracket formula for the original $X_i$. The calculations for $\langle \tilde{Y}_i, \tilde{Y}_j \rangle$ and $\langle \tilde{X}_i, \tilde{Y}_j \rangle$ are similar.

7.2 Invariance for the $P, Q$ Variables

Now define new face weights, after the square move, by

$$\tilde{p}_i = a_i b_i a_i^{-1}$$

$$\tilde{q}_i = d_i a_i^{-1} b_i^{-1} c_i^{-1}$$

We also define versions with a common base point, using the paths $\xi_k = a_1 b_1 \cdots a_k b_k$:

$$\tilde{P}_i = \xi_{i-1} \tilde{p}_i \xi_{i-1}^{-1}$$

$$\tilde{Q}_i = \xi_{i-2} \tilde{q}_i \xi_{i-2}^{-1}$$

The white-swap and black-swap Postnikov moves interchange the face weights: the new square faces have weight $\tilde{q}_i$ and the octagonal faces have weight $\tilde{p}_i$.

At this point, the graph is isomorphic to the original quiver. We can apply gauge transformations, as before, so that the only weights which are not equal to 1 are on the left and bottom edges of the square faces. These "post-pentagram" versions of $X, Y$ we call $\tilde{X}_i$ and $\tilde{Y}_i$:

$$\tilde{X}_i = \xi_i c_i b_i c_i^{-1} \xi_{i+2}^{-1}$$

$$\tilde{Y}_i = \xi_i a_i a_i^{-1} c_i^{-1} \xi_{i+2}^{-1}$$

They are related to the $\tilde{P}$’s and $\tilde{Q}$’s in a similar way as before:

$$\tilde{Q}_{i+2} = \tilde{Y}_i \tilde{X}_i^{-1}$$

$$\tilde{P}_{i+1} = \tilde{X}_{i+1} \tilde{Y}_i^{-1}$$

Then the face weights of the quiver after the entire sequence of Postnikov moves, which we denote $P^*$ and $Q^*$, are
\[Q_i^* = \bar{P}_i = X_i^{-1}P_i^{-1}X_i\]
\[P_i^* = \bar{Q}_i = Y_i^{-1}(1 + P_i^{-1})^{-1}Y_{i-1} \cdot (1 + P_{i+1})Q_i (1 + P_i^{-1})^{-1} \cdot X_{i-2}^{-1}(1 + P_{i-2})X_{i-2}\]

The induced brackets for \(P^*, Q^*\) are

\[\langle P_i^*, Q_j^* \rangle = (\delta_{i,j-1} - \delta_{i,j-2})P_i^*Q_j^* + (\delta_{ij} - \delta_{i,j+1})Y_{i-2}^{-1}Q_i^*Y_{i-2}P_j^*\]

After the following change of variables, the form of the bracket is the same as the original \(P, Q\) variables:

\[Q_i^* := Q_{i+1}^* = X_i^{-1}P_{i+2}^{-1}X_{i+2}\]
\[P_i^* := P_{i+2}^* = Y_i^{-1}(1 + P_i^{-1})^{-1}Y_i \cdot (1 + P_{i+2})Q_{i+1} (1 + P_i^{-1})^{-1} \cdot X_{i-1}^{-1}(1 + P_{i-1})X_{i-1}\]

Indeed, the pentagram map is given by \(T(P_i) = P_i^*\) and \(T(Q_i) = Q_i^*\).

### 7.3 The Monodromy Invariants

Let \(Q = Q_{k,n}\) be the quiver/network for the pentagram map, with the \(X_i\) and \(Y_i\) weights as before, and \(B = B(Q)\) its boundary measurement matrix. We restrict to choices of weights such that \(Z = 1\), so that the entries of \(B\) are elements of the Laurent polynomial ring \(\mathbb{Z}^{(1)}[\lambda^\pm]\). For each \(i \geq 1\), denote the coefficients of \(\lambda^k\) in \(tr(B^i)\) by \(t_{ik}\):

\[tr(B^i) = \sum_k t_{ik} \lambda^k\]

We will spend the remainder of this section proving the following theorem, which says that \(t_{ik}\) are invariants of the pentagram map.

**Theorem 8.** Let \(\tilde{Q}\) be the weighted quiver obtained from \(Q\) by applying the Postnikov moves and gauge transformations as described in the previous sections, and \(\tilde{B} = B(\tilde{Q})\) its boundary measurement matrix. Also let \(t_{ik}\) denote the coefficient of \(\lambda^k\) in \(tr(B^i)\). Then \(\tilde{t}_{ik} = t_{ik}\).

To begin proving this, we first make the following simple observation:

**Lemma 4.** Let \(R\) be an associative ring, and \(p, q \in R[\lambda^\pm]\). Then every coefficient of \([p, q]\) is in \([R, R]\).

**Proof.** Let \(p = \sum_i p_i \lambda^i\) and \(q = \sum_j q_j \lambda^j\). Then \([p, q] = \sum_{i,j} [p_i, q_j] \lambda^{i+j} \).

This implies the following

**Corollary 1.** Let \(f = \sum_i f_i \lambda^i\) and \(g = \sum_j g_j \lambda^j\) be Laurent polynomials in \(A := R[\lambda^\pm]\). If \(f \equiv g \mod [A, A]\), then \(f_i \equiv g_i \mod [R, R]\) for each \(i\).

We now consider traces of powers of matrices over general rings:
**Lemma 5.** Let $R$ be an associative ring, and $A, B \in \text{Mat}_n(R)$. Then $\text{tr}((AB)^k) \equiv \text{tr}((BA)^k) \mod [R, R]$ for all $k$.

**Proof.** If the matrices are given by $A = (a_{ij})$ and $B = (b_{ij})$ then

$$
\text{tr}(AB)^k = \sum_{i,j_1,\ldots,j_{k-1}} (AB)_{i,j_1} (AB)_{j_1,j_2} \cdots (AB)_{j_{k-1},i}
= \sum_{i,j_1,\ldots,j_{k-1}} \left( \sum_{\ell_1} a_{i\ell_1} b_{\ell_1,j_1} \right) \cdots \left( \sum_{\ell_k} a_{j_{k-1}\ell_k} b_{\ell_k,i} \right)
= \sum_{i,j_1,\ldots,j_{k-1}} \sum_{\ell_1,\ldots,\ell_k} a_{i\ell_1} b_{\ell_1,j_1} \cdots a_{j_{k-1}\ell_k} b_{\ell_k,i}
$$

Similarly, using $BA$ instead, we get

$$
\text{tr}(BA)^k = \sum_{i,j_1,\ldots,j_{k-1}} \sum_{\ell_1,\ldots,\ell_k} b_{i\ell_1} a_{\ell_1,j_1} \cdots b_{j_{k-1}\ell_k} a_{\ell_k,i}
$$

After re-indexing by $\ell_t \mapsto j_t$ (for $1 \leq t \leq k-1$), $\ell_k \mapsto i$, $i \mapsto \ell_1$, $j_t \mapsto \ell_{t+1}$, we get

$$
\text{tr}(BA)^k = \sum_{i,j_1,\ldots,j_{k-1}} \sum_{\ell_1,\ldots,\ell_k} b_{\ell_1,j_1} a_{j_1,\ell_2} \cdots b_{\ell_k,i} a_{i\ell_1}
$$

Clearly, this is the same as the expression for $\text{tr}((AB)^k) \mod [R, R]$, since each term with corresponding indices is the same after cyclicly shifting the last $a_{i\ell_1}$ to the beginning. 

Now we may prove **Theorem 8:**

**Proof.** The square move and gauge transformations do not change the boundary measurement matrix at all, and the “white-swap” and “black-swap” moves only change the boundary measurement matrix up to conjugation, since they may possibly require cutting a portion of the network on the right end off and gluing it back on to the left side. This transforms the boundary measurement matrix from one of the form $AB$ to another of the form $BA$. If we let $R = \mathcal{L}^{(1)}_*$ and $A = \mathcal{L}^{(1)}_*[\lambda^k]$, then by **Corollary 1** and **Lemma 5**, $\text{tr}(\tilde{B}^k)$ and $\text{tr}(B^k)$ differ by an element of $[A, A]$, and so $t_{ij} \equiv \tilde{t}_{ij} \mod [\mathcal{L}_*^{(1)}, \mathcal{L}_*^{(1)}]$. 

### 7.4 Involutivity of the Invariants

In this section, we will prove that the invariants from the previous section are an involutive family with respect to the induced bracket. More specifically:

**Theorem 9.** Let $Q$ be the network for the pentagram map on $\mathcal{GP}_{m,n}$, and $B = B(Q)$ its boundary measurement matrix, with $t_{ik}$ the homogeneous components of $\text{tr}(B^i)$ as defined before. Then for all $i, j, k, \ell$:

$$
\langle t_{ik}, t_{j\ell} \rangle = 0
$$
Recall that we use as the cut the identified top/bottom edge of the rectangle on which we draw the quiver. So the element \( t_{ik} \in \mathcal{L} \) is the sum over all loops in \( Q \) which are homologous to \((i, k)\) cycles on the torus. That is, if we lift these paths to the universal cover, then they cross fundamental domains \( i \) times horizontally and \( k \) times vertically (with sign). Let \( A_{ik} \) be the set of all such paths, so that

\[
t_{ik} = \sum_{p \in A_{ik}} p
\]

Then by bilinearity and the formula from **Theorem 3**, we have:

\[
\langle t_{ik}, t_{jk} \rangle = \sum_{f \in A_{ik}} \sum_{g \in A_{jk}} \langle f, g \rangle = \sum_{f \in A_{ik}} \sum_{g \in A_{jk}} \varepsilon(f, g) f^* g^*.
\]

We want to prove that this expression is zero. To do so, we will define a sign-reversing, fixed-point-free permutation on the set of terms appearing in the sum. That is, each term which appears can be paired with another term with opposite coefficient and the same monomial in \( A^2 \).

We start by choosing an arbitrary non-zero term in the sum, of the form \( \varepsilon(f, g) f^* g^* \). To define a permutation as suggested above, we need to know what are the other terms in the sum that are equivalent to \( f^* g^* \) in \( A^2 \). The answer is given by the following lemmas:

**Lemma 6.** Suppose \( f^* g^* \equiv f^* g^* \mod [\mathcal{L}^{(1)} \times \mathcal{L}^{(1)}] \), and that \( \varepsilon(f', g') \neq 0 \). Then \( * \) is in either \( f \cap g \), \( f \cap f' \), or \( g \cap g' \).

**Proof.** Certainly \( f' \) and \( g' \) together have the same combined set of edges as \( f \) and \( g \) since \( f^* g^* = f^* g^* \) cyclically. Then since two paths come together at \( * \) (since \( \varepsilon(f', g') \neq 0 \)), it must be that either \( f \) and \( g \) meet at \( * \), or \( f \) or \( g \) meets itself. \( \square \)

**Lemma 7.** Let \( \circ \) and \( * \) be as in the previous lemma, with \( f^* g^* = f^* g^* \). Additionally assume that \( (f, g) \) and \( (f', g') \) are both in \( A_{ik} \times A_{jk} \). Then:

(a) If \( * \in f \cap g \), then \( (f', g') \) are obtained from \( (f, g) \) by swapping the segments of \( f \) and \( g \) between \( \circ \) and \( * \).

(b) If \( * \in f \cap f' \), then \( (f', g') \) are obtained from \( (f, g) \) by swapping the entire loop \( g \) with the subloop of \( f \) based at \( * \).

**Proof.** (a) Denote by \( a \) and \( b \) the segments of \( f \) between \( \circ \) and \( * \), and similarly let \( x \) and \( y \) be the segments of \( g \) between \( \circ \) and \( * \), so we may write \( f = \circ a \cdot b \cdot x \cdot y \) and \( g = \circ x \cdot y \cdot a \). Then

\[
f^* g^* = \circ a \cdot b \cdot x \cdot y
\]

Since we assumed that \( f^* g^* = f^* g^* \) up to cyclic permutation, then we can write it starting at \( * \) as

\[
f^* g^* = \circ b \cdot x \cdot y \cdot a
\]

This means \( f' = \circ b \cdot x \) and \( g' = \circ y \cdot a \), or the other way around: \( f' = \circ y \cdot a \) and \( g' = \circ b \cdot x \). The two possibilities differ by switching the roles of \( f' \) and \( g' \). But since we assume \( (f', g') \in A_{ik} \times A_{jk} \), and since \( A_{ik} \neq A_{jk} \), only one of the possibilities will be correct.

After cyclically permuting, we have \( f' = \circ x \cdot b \) and \( g' = \circ a \cdot y \) (or the other way around). Thus we see that \( f' \) and \( g' \) are obtained from \( f \) and \( g \) by swapping either \( x \) with \( a \) or \( y \) with \( b \), which are the portions of the paths in between their common subpaths \( \circ \) and \( * \). This is illustrated in **Figure 12**. Note that since \( (f', g') \in A_{ik} \times A_{jk} \), the swapped segments must have the same intersection index with the cut.
Figure 12: A “type I” swap

Now suppose \( \ast \in f \cap f \). Let \( a, b, c \) be the segments of \( f \) between \( \bullet \) and \( \ast \), so that \( f = \bullet a b c \). Also let \( x \) be the rest of \( g \) after \( \bullet \), so that \( g = \bullet x \). Then

\[
f_{\bullet} g_{\bullet} = \bullet a b c x
\]

Cyclically permuting, and using the assumption that \( f_{\bullet} g_{\bullet} = f'_{\bullet} g'_{\bullet} \), we get

\[
f'_{\bullet} g'_{\bullet} = \ast c x a b
\]

Again there are two possibilities. Either \( f' = \ast c x \bullet a \) and \( g' = \ast b \), or \( f' \) and \( g' \) are flipped. But the condition that \( f' \in A_{ik} \) and \( g' \in A_{j\ell} \) ensures that only one of the two choices is correct. Suppose it is the first case, and \( f'_{\bullet} g'_{\bullet} = f_{\bullet} g_{\bullet} \). Then \( f'_{\bullet} \) is obtained from \( f_{\bullet} \) by removing the loop based at \( \ast \) and adding the loop \( g \), and \( g'_{\bullet} \) is simply the subloop of \( f_{\bullet} \) based at \( \ast \). This is illustrated in Figure 13. Note that since \( (f', g') \in A_{ik} \times A_{j\ell} \), the loops \( \bullet x \) and \( \ast b \) must cross the cut and rim the same number of times.

Figure 13: A “type II” swap

The next lemma is a converse to the previous one, so in fact the conditions given above completely characterize the set of terms in the expansion of \( \langle t_{ik}, t_{j\ell} \rangle \) with a given term \( f_{\bullet} g_{\bullet} \).

Lemma 8. Let \( (f, g) \in A_{ik} \times A_{j\ell} \) and \( \bullet \in f \cap g \) such that \( \varepsilon_{\bullet}(f, g) \neq 0 \).

\( (a) \) If there exists an \( \ast \in f \cap g \) such that the segments of \( f \) and \( g \) between \( \bullet \) and \( \ast \) (or between \( \ast \) and \( \bullet \)) cross the cut and rim the same number of times, then swapping those segments gives \( (f', g') \in A_{ik} \times A_{j\ell} \) such that \( f'_{\bullet} g'_{\bullet} = f_{\bullet} g_{\bullet} \).

\( (b) \) If there exists an \( \ast \in f \cap f \) such that \( g \) and the subloop of \( f \) based at \( \ast \) cross the cut and rim the same number of times, then swapping \( g \) and this subloop gives \( (f', g') \in A_{ik} \times A_{j\ell} \) such that \( f'_{\bullet} g'_{\bullet} = f_{\bullet} g_{\bullet} \).

Proof.
We will call the swaps from part (a) of the previous two lemmas “type I” swaps, and the swaps from part (b) will be called “type II” swaps. We are now ready to define our sign-reversing map, which we will denote $\sigma$, which acts on the set of terms appearing in the sum.

**Lemma 9.** There exists a fixed-point-free permutation $\sigma$ of the non-zero terms appearing in the expansion of $(t_{ik}, t_{j\ell})$ such that $\sigma(x) = -x$ for each term $x$.

**Proof.** Recall that $f$ is a term in $t_{ik}$ and $g$ a term in $t_{j\ell}$. Assume that $i \geq j$, so that $f$ goes around the torus more times “horizontally”. Starting at •, walk along $f$ until we reach the first admissible swap. If the first admissible swap is of type I, and occurs at $\ast$, then we will define the image of the term $\varepsilon_{\ast}(f, g) f_{\ast} g_{\ast}$ under $\sigma$ to be the term $\varepsilon_{\ast}(f', g') f'_{\ast} g'_{\ast}$, where $f'$ and $g'$ are obtained by performing the type I swap between $\bullet$ and $\ast$. In order for $\varepsilon_{\ast}(f', g') \neq 0$, it must be that $f$ and $g$ cross at $\ast$ (rather than touch). This guarantees that $\varepsilon_{\ast}(f', g') = -\varepsilon_{\ast}(f, g)$, note that by the preceding lemmas, the segments of $f$ and $g$ from $\bullet$ to $\ast$ must cross the cut and rim the same number of times. This means that on the universal cover, the segments of $f$ and $g$ between $\bullet$ and $\ast$ bound a contractible disc. Thus after performing the swap, $f'$ and $g'$ touch with opposite orientation — i.e. if $f$ was to the left, and $g$ to the right, then $f'$ is on the right, and $g'$ on the left. This can be seen in Figure 12.

If, on the other hand, the first admissible swap is of type II, we consider two separate cases. Recall that a type II swap means $f$ intersects itself at $\ast$, so we may write $f_{\ast} = f'_{\ast} f''_{\ast}$, and that $f''_{\ast} \in A_{ij}$ (the same as $g$). First we consider the case that $f''_{\ast}$ does not intersect $g$. In this case, we define the action of $\sigma$ on $\varepsilon_{\ast}(f, g) f_{\ast} g_{\ast}$ to be the term corresponding to performing the type II swap. Now, consider the second case, in which $f''_{\ast}$ does intersect $g$. In this case, let $\ast$ be the first intersection point after $\ast$. Then we define the action of $\sigma$ on $\varepsilon_{\ast}(f, g) f_{\ast} g_{\ast}$ to be the term corresponding to the type I swap from $\bullet$ to $\ast$. \hfill $\square$

**Theorem 9** now follows immediately as a corollary, since the lemma implies that there are an even number of terms equivalent to any given $f_{\ast} g_{\ast}$, half with coefficient $+1$ and half with coefficient $-1$.

## 8 Conclusions and Suggestions for Future Work

We have shown in **Theorem 8** and **Theorem 9** that the coefficients $t_{ij}$ of the traces of powers of the boundary measurement matrix are non-commutative invariants of the pentagram map, and that they form an involutive family under the induced Lie bracket $\langle - , - \rangle$ on $\mathbb{R}^2$. In this sense, we have established a form of non-commutative integrability for the Grassmann pentagram map, in terms of the $X_i, Y_i$ variables/matrices.

However, in the usual commutative setting, the definition of Liouville integrability requires not only a Poisson-commuting family of invariants, but also that these invariants are independent, and that they are a maximal family of such independent commuting invariants. These last two aspects — independence and maximality — were not addressed in the present paper.

An interesting question that remains is how to more fully complete this analogy with commutative integrability. In other words, what are the “right” notions of independence and maximality that would apply to this non-commutative situation?

The proof of **Theorem 9** is very combinatorial. It would be interesting to give an alternate proof using $R$-matrices, which would more closely resemble the method of proof in the classical case [GSTV16]. This would involve formulating an $R$-matrix double bracket on the noncommutative space of matrices, and formulating some result saying that the spectral invariants are in involution with respect to the induced $H_0$-Poisson structure.
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