Solution manifolds of differential systems with discrete state-dependent delays are almost graphs

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Abstract

We show that for a system

\[ x'(t) = g(x(t - d_1(Lx_t)), \ldots, x(t - d_k(Lx_t))) \]

of \( n \) differential equations with \( k \) discrete state-dependent delays the solution manifold, on which solution operators are differentiable, is nearly as simple as a graph over a closed subspace in \( C^1([-r, 0]; \mathbb{R}^n) \). The map \( L \) is continuous and linear from \( C([-r, 0]; \mathbb{R}^n) \) onto a finite-dimensional vectorspace, and \( g \) as well as the delay functions \( d_\kappa \) are assumed to be continuously differentiable.

Key words: Delay differential equation, state-dependent delay, solution manifold, almost graph

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1 Introduction

For an integer \( n > 0 \) and a real number \( r > 0 \) let \( C_n = C([-r, 0]; \mathbb{R}^n) \) and \( C^1_n = C^1([-r, 0]; \mathbb{R}^n) \) denote the Banach spaces of continuous and continuously

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differentiable maps \([-r, 0] \to \mathbb{R}^n\), respectively, with the norms given by

\[ |\phi|_C = \max_{-r \leq t \leq 0} |\phi(t)| \quad \text{and} \quad |\phi| = |\phi|_C + |\phi'|_C, \]

for a chosen norm on \(\mathbb{R}^n\). If a differential equation with state-dependent delay, like for example the equation

\[ x'(t) = g(x(t - \Delta)), \quad \Delta = d(x(t)) \tag{1} \]

with real functions \(g : \mathbb{R} \to \mathbb{R}\) and \(d : \mathbb{R} \to [0, r]\), is written in the general form

\[ x'(t) = f(x_t) \tag{2} \]

of an autonomous delay differential equation, with a map \(f : U \to \mathbb{R}^n\) defined on an open subset of \(C^1_n\), and \(x_t(s) = x(t + s), -r \leq s \leq 0\), then the associated solution manifold is the set

\[ X_f = \{ \phi \in U : \phi'(0) = f(\phi) \}. \]

If \(f\) is continuously differentiable and satisfies the extension property that

(e) each derivative \(Df(\phi) : C^1_n \to \mathbb{R}^n, \phi \in U\), continues to a linear map \(D_e f(\phi) : \mathbb{R}^n \to \mathbb{R}^n\) and the map

\[ U \times C_n \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n \]

is continuous

and if \(X_f\) is non-empty then \(X_f\) is a continuously differentiable submanifold of codimension \(n\) in \(C^1_n\), and the initial value problem for initial data in \(X_f\) is well-posed, with each solution operator continuously differentiable \([10, 14]\).

The extension property (e) is a version of the notion of being almost Fréchet differentiable which was introduced for maps on domains \(U \subset C_n\) by Mallet-Paret, Nussbaum, and Paraskevopoulos \([9]\).

Let us recall that in contrast to theory for differential equations with constant time lags \([3, 2]\) the initial value problem for equations with state-dependent delays is in general not well-posed for initial data in open subsets of the space \(C_n\).

The present paper about solution manifolds continues work which started with \([5, 13]\). Under a boundedness condition on the extended derivatives \(D_e f\), or under a condition on \(f\) which generalizes delays being bounded away from zero, solution manifolds are graphs which can be written as

\[ X_f = \{ \chi + \alpha(\chi) : \chi \in \text{dom} \} \]

with a continuously differentiable map \(\alpha\) from an open subset \(\text{dom}\) of the closed subspace

\[ X_0 = \{ \phi \in C^1_n : \phi'(0) = 0 \} \]

}\[ 2 \]
into a complementary space $Q \subset C^1_n$, see the proof of [5, Lemma 1] and [13, Theorem 2.4], respectively.

An example of the form (1) with $d : \mathbb{R} \to (0, r]$ in [13, Section 3] shows that in general solution manifolds do not admit any graph representation. However, [13, Theorem 5.1] says that for a class of systems with discrete state-dependent delays all of which are strictly positive the solution manifolds are nearly as simple as a graph, namely, they are almost graphs over $X_0$ in the sense of the following definition from [14, Section 1]:

A continuously differentiable submanifold $X$ of a Banach space $E$ is called an almost graph over a closed subspace $H \subset E$ if $H$ has a closed complementary subspace in $E$ and if there is a continuously differentiable map $\alpha : H \supset \text{dom} \to E$, $\text{dom}$ an open subset of $H$, such that

$$X = \{ \zeta + \alpha(\zeta) \in E : \zeta \in \text{dom} \},$$

$$\alpha(\zeta) = 0 \text{ on } \text{dom} \cap X,$$

$$\alpha(\zeta) \in E \setminus H \text{ on } \text{dom} \setminus X,$$

and the map $H \supset \text{dom} \ni \zeta \mapsto \zeta + \alpha(\zeta) \in X$ is a diffeomorphism onto $X$.

An example of an almost graph in the plane is the unit circle without $(0, r)$, for $H = \mathbb{R} \times \{0\}$ and $\alpha$ the inverse of the stereographic projection [14, Section 1].

A property stronger than being an almost graph is existence of an almost graph diffeomorphism as introduced in [14, Section 1]: For $E, X, H$ as above, and for an open set $O \subset E$,

a diffeomorphism $A : O \to E$ onto an open subset of $E$ is called an almost graph diffeomorphism with respect to $X, H,$ and $O$, if

$$A(X \cap O) = H \cap A(O)$$

and if $A$ leaves the points of $(X \cap O) \cap H$ fixed.

In [14, Section 1] it is shown that the existence of an almost graph diffeomorphism with respect to $X, H, O$ implies that $X \cap O$ is an almost graph over $H$.

The main results of [14], Theorems 3.5 and 4.8, imply that for a class of systems more general than those studied in [13] the associated solution manifolds carry a finite atlas of manifold charts whose domains are almost graphs over $X_0$. The size of the atlas is determined precisely by the zerosets of the delays. If for example the delay function $d$ in Eq. (1) is non-constant and has zeros then the atlas found in [14] consists of exactly 2 manifold charts.

Let us now recall the systems

$$x'(t) = g(x(t - d_1(Lx_1)), \ldots, x(t - d_k(Lx_1)))$$

as in the definition of a continuously differentiable submanifold.
introduced in [14]. In Eq. (3) the delays are given by compositions of a continuous linear map $L : C_n \to F$ onto a finite-dimensional normed vectorspace over the field $\mathbb{R}$, with continuously differentiable delay functions $d_\kappa : W \to [0, r]$, $W \subset F$ open and $\kappa \in \{1, \ldots, k\}$. The nonlinearity $g$ is a continuously differentiable map from an open subset $V \subset \mathbb{R}^{nk}$ into $\mathbb{R}^n$.

The notation for the argument of $g$ in Eq. (3) is an abbreviation for the column vector $y \in \mathbb{R}^{nk}$ with components

$$y_\nu = x_\nu(t - d_\kappa(Lx_t))$$

for $\nu = (\kappa - 1)n + \nu$ with $\kappa \in \{1, \ldots, k\}$ and $\nu \in \{1, \ldots, n\}$.

With regard to the form of the delays in Eq. (3) one may think of $L\phi$ as an approximation of $\phi \in C_n$ in the subspace $F \subset C_n$, and view $d_\kappa(L\cdot)$ as a substitute for a more general delay functional defined on an open subset of $C_n$.

In order that Eq. (3) makes sense it is assumed that

(V) there exist $\phi \in C^1_n$ with $L\phi \in W$ and $(\phi(-d_1(L\phi)), \ldots, \phi(-d_k(L\phi))) \in V$.

(in notation as described above for $\phi = x_t$).

With

$$\hat{\phi} = (\phi(-d_1(L\phi)), \ldots, \phi(-d_k(L\phi)))$$

we get that

$$U = \{ \phi \in C^1_n : L\phi \in W \text{ and } \hat{\phi} \in V \}$$

is non-empty, and Eq. (3) takes the form of Eq. (2) with $f : U \to \mathbb{R}^n$ given by

$$f(\phi) = g(\hat{\phi}).$$

According to [14] Propositions 2.1 and 2.3 the set $U$ is open, $f$ is continuously differentiable with property (e), and $X_f \neq \emptyset$, so that $X_f$ is a continuously differentiable submanifold of codimension $n$ in $C^1_n$.

In the present paper we consider system (3) under the above conditions and construct an almost graph diffeomorphism with respect to the whole solution manifold $X_f$, to the subspace $X_0$, and to the open neighbourhood $\mathcal{O} = U$ of $X_f$ in $C^1_n$. The result is stated as Theorem 3.5 below.

Let us emphasize that the finite atlas result from [14] holds true under further hypotheses on $g$ or on the delay functions $d_\kappa$ whereas the construction in Sections 2-3 below requires nothing beyond smoothness as stated above. This discrepancy reflects the fact that in the proof of [14] Theorem 5.1, as well as in the proofs of [13] Theorems 2.4 and 5.1, the invariance property

$$\overline{A}(\hat{\phi}) = \hat{\phi}$$

of a diffeomorphism $A$ from $X_f$ onto an open subset of $X_0$ is established and exploited. The approach in the present paper, which originated in the case study [15], proceeds without recourse to the said invariance property.
The Introduction of \[14\] lists a wide variety of special cases of system \[3\]. Therefore, by Theorem 3.5, the solution manifolds associated with many familiar delay differential systems with state-dependent delays are almost graphs over a closed subspace of \(C^1_n\). It is an open problem whether this is true for the solution manifold of the general equation \[2\] with a continuously differentiable \(f: U \to \mathbb{R}^n\), defined on an open \(U \subset C^1_n\), satisfying the extension property \((e)\).

There are differential equations with discrete delays so that the delay functions are not of the form \(\delta_\kappa(L\phi)\) with a continuous linear map \(L: \mathbb{C}^n \to \mathbb{F}\) into a finite dimensional vectorspace \(\mathbb{F}\). For example, threshold delays and transmission delays are implicitly defined (see e.g. \[1, 7, 11, 12, 4, 6\]), and the corresponding delay functions \(\sigma(\phi)\) are defined on an open subset of \(C^1_n\) in order to write the system in the form \[2\] with the required properties. It would be interesting to describe the solution manifolds for systems with threshold and transmission delays as well.

**Notation, conventions, preliminaries.** For subsets \(A \subset B\) of a topological space \(T\) we say \(A\) is open in \(B\) if \(A\) is open with respect to the relative topology on \(B\). Analogously for \(A\) closed in \(B\). The relation \(A \subset⊂ B\) for open subsets of \(T\) means that the closure \(\overline{A}\) of \(A\) is compact and contained in \(B\).

Finite-dimensional vectorspaces are always equipped with the canonical topology which makes them topological vectorspaces.

On \(\mathbb{R}^{nk}\) we use a norm which satisfies

\[
|y_j| \leq |y| \leq \sum_{i=1}^{nk} |y_i| \quad \text{for all} \quad y \in \mathbb{R}^{nk}, \ j \in \{1, \ldots, nk\}.
\]

In case \(V \neq \mathbb{R}^{nk}\) the expression \(\text{dist}(v, \mathbb{R}^{nk} \setminus V) = \min_{y \in \mathbb{R}^{nk} \setminus V} |y - v|\) defines a continuous function \(V \to (0, \infty)\).

An upper index as in \((x_1, \ldots, x_N)^{tr} \in \mathbb{R}^N\) denotes the transpose of the row vector \((x_1, \ldots, x_N)\). Vectors in \(\mathbb{R}^N\) which occur as argument of a map are always written as row vectors. The vectors of the canonical basis of \(\mathbb{R}^N\) are denoted by \(e_\nu, \nu \in \{1, \ldots, N\}\); \(e_\nu = 1\) for \(\nu = \mu\) and \(e_\nu = 0\) for \(\nu \neq \mu\), \(\nu\) and \(\mu\) in \(\{1, \ldots, N\}\).

Derivatives and partial derivatives of a map at a given argument are continuous linear maps, indicated by a capital \(D\). In case of real functions on domains in \(\mathbb{R}\) and in \(\mathbb{R}^N\), \(\phi'(t) = D\phi(t)1\) and \(\partial_\nu g(x) = D_\nu g(x)1\), respectively.

For \(n = 1\) we abbreviate \(C = C_1\) and \(C^1 = C^1_1\).

We define continuous bilinear products \(C \times \mathbb{R}^n \to \mathbb{C}^n\) and \(\mathbb{R}^n \times C_n \to C_n\) by

\[(\phi \cdot q)_\nu = q_\nu \phi \in C\]

and

\[(q \cdot \phi)_\nu = q_\nu \phi_\nu \in C\]
for \( \nu = 1, \ldots, n \). Obviously

\[
q \cdot \phi = \sum_{\nu=1}^{n} q_{\nu}(\phi_{\nu} \cdot e_{\nu})
\]

for all \( q \in \mathbb{R}^{n}, \phi \in C_{n} \).

The maps

\[
C_{n}^{1} \times W \ni (\psi, \eta) \mapsto \psi_{\nu}(-d_{\kappa}(\eta)) \in \mathbb{R}, \quad \nu \in \{1, \ldots, n\}, \quad \kappa \in \{1, \ldots, k\},
\]

are continuously differentiable, compare Part 2.1 of the proof of [14, Proposition 2.1]. It follows that also the map

\[
U \ni \phi \mapsto \hat{\phi} \in V \subset \mathbb{R}^{nk}
\]

is continuously differentiable.

The inclusion \( C_{n}^{1} \ni \phi \mapsto \phi \in C_{n} \), differentiation \( \partial : C_{n}^{1} \ni \phi \mapsto \phi' \in C_{n} \), and evaluation \( ev_{t} : C \ni \phi \mapsto \phi(t) \in \mathbb{R} \) at \( t \in [-r, 0] \) are continuous linear maps.

In the sequel a diffeomorphism is a continuously differentiable injective map with open image whose inverse is continuously differentiable.

## 2 Preparations

The following lemma is a version of [14, Proposition 4.1] which includes smallness in \( C \) of the function found.

**Lemma 2.1** Let \( \lambda : C \to \mathcal{F} \) be a continuous linear map into a finite-dimensional real vectorspace \( \mathcal{F} \), and let \( \epsilon > 0 \) be given. There exists \( \psi \in C^{1} \) with \( \lambda \psi = 0 \), \( \psi'(0) = 1 \), and \( |\psi|_{C} < \epsilon \).

**Proof.** 1. Proof that there exists a complementary space \( K \subset C^{1} \) for \( \lambda^{-1}(0) \) in \( C \). We have

\[
\lambda C^{1} = \overline{\lambda C^{1}} \quad \text{(with \quad \dim \lambda C^{1} \leq \dim \lambda C < \infty)} \ni \lambda C \quad \text{(with \quad C^{1} \text{ dense in } C)} \ni \lambda C^{1},
\]

hence \( \lambda C^{1} = \lambda C \). Set \( K = \sum_{j=1}^{k} \mathbb{R} \psi_{j} \) with preimages \( \psi_{j} \in C^{1} \) of a basis of \( \lambda C \) and verify \( C = \lambda^{-1}(0) \oplus K \).

2. The projection \( P : C \to C \) along \( K \subset C^{1} \) onto \( \lambda^{-1}(0) \) maps \( C^{1} \) into \( \lambda^{-1}(0) \cap C^{1} \). Choose a sequence \( (\phi_{m})_{m=1}^{\infty} \) in \( C^{1} \) with \( \phi_{m}'(0) = 1 \) for all \( m \in \mathbb{N} \) and \( |\phi_{m}|_{C} \to 0 \) as \( m \to \infty \). Then \( |(id - P)\phi_{m}|_{C} \to 0 \) as \( m \to \infty \). As \( K \) is
finite-dimensional we also get \(|(id - P)\phi_m| \to 0\) as \(m \to \infty\). It follows that 
\[ev_0 \partial (id - P)\phi_m \to 0\] as \(m \to \infty\). Using this and \(P\phi_m \in C^1\) we get 
\[1 = \phi'_m(0) = ev_0 \partial \phi_m = ev_0 \partial (P\phi_m + (id - P)\phi_m) = ev_0 \partial P\phi_m + ev_0 \partial (id - P)\phi_m\] for all \(m \in \mathbb{N}\), which yields \(ev_0 \partial P\phi_m \to 1\) as \(m \to \infty\). For \(m\) so large that 
\[ev_0 \partial P\phi_m \neq 0\] the functions \(\psi_m = \frac{1}{ev_0 \partial P\phi_m}P\phi_m\) belong to \(C^1\) and satisfy \(\lambda \psi_m = 0\) and \(\psi'_m(0) = ev_0 \partial \psi_m = 1\). For \(m\) sufficiently large we also obtain \(|\psi_m|_C < \epsilon\). □

The next proposition provides functions in \(C^1\) which after multiplication with the unit vectors \(e_v \in \mathbb{R}^n\) yield bases of subspaces which are complementary for \(X_0\) and depend on \(\phi \in U\) via \(v = \hat{\phi} \in V \subset \mathbb{R}^{nk}\). For \(\phi \in X_f\) the subspace is complementary also for the tangent space \(T_{\phi}X_f\). We omit proofs of these facts as they will not be used in the sequel. The almost graph diffeomorphism associated with \(X_f, X_0,\) and \(U\) which will be constructed in the next section is composed of translations by vectors in the complementary spaces just mentioned.

Recall the map \(L : C_n \to F\) from Eq. (3).

**Proposition 2.2** Let a continuous function \(h : V \to (0, \infty)\) and \(\nu \in \{1, \ldots, n\}\) be given. Then there exists a continuously differentiable map 
\[H_\nu : V \to C^1\] so that for all \(v \in V\), 
\[L(H_\nu(v) \cdot e_v) = 0, \quad (H_\nu(v))'(0) = 1, \quad |H_\nu(v)|_C \leq h(v),\] and for each \(\mu \in \{1, \ldots, nk\}\), 
\[|D_\mu H_\nu(v)1|_C \leq h(v).\]

**Proof.** 1. There is a sequence of non-empty open subsets \(V_{j1}, V_{j2}, V_j\) of \(V\), with \(j \in \mathbb{N}\), such that 
\[\bigcup_{j=1}^{\infty} V_j = V,\] and for every \(j \in \mathbb{N}\), 
\[V_j \subset \subset V_{j1} \subset \subset V_{j2} \subset \subset V_j \quad \text{and} \quad V_j \subset \subset V_{j+1,1}.\]
With \(\overline{V_{j2}} = \emptyset\) we have that for each integer \(j \geq 1\), 
\[(V_{j+1} \setminus \overline{V_{j2}}) \cap (V_j \setminus \overline{V_{j-1,2}}) = V_j \setminus \overline{V_{j2}}\]
while for integers \( j \geq 1 \) and \( k \geq j + 2 \),
\[
(V_k \setminus V_{k-1,2}) \cap (V_j \setminus V_{j-1,2}) = V_j \setminus V_{k-1,2} \subset V_j \setminus V_{j+1,2} = \emptyset.
\]

2. For every \( j \in \mathbb{N} \) choose a continuously differentiable function
\[
a_j : \mathbb{R}^{nk} \to [0,1]
\]
with
\[
a_j(v) = 1 \text{ on } V_j, \quad a_j(v) = 0 \text{ on } \mathbb{R}^{nk} \setminus V_j.
\]
For every \( j \in \mathbb{N} \) choose an upper bound
\[
A_j > 1 + \sum_{\mu=1}^{nk} \max_{v \in \mathbb{R}^{nk}} |D_{\mu}a_j(v)1| = \max_{v \in V_j} |a_j(v)| + \sum_{\mu=1}^{nk} \max_{v \in V_j} |D_{\mu}a_j(v)1|
\]
so that the sequence \((A_j)^\infty_{j=1}\) in \([1,\infty)\) is increasing.

The sequence \((h_j)^\infty_{j=1}\) given by \( h_j = \min_{v \in V_j} h(v) > 0 \) is nonincreasing. We have
\[
\frac{h_j}{2A_j} \leq h_j \text{ for all } j \in \mathbb{N},
\]
and the sequence \(((h_j/2A_j)^\infty_{j=1})\) is decreasing.

3. For each \( j \in \mathbb{N} \) apply Lemma 2.1 to \( \lambda : C \to F \) given by \( \lambda \phi = L(\phi \cdot e_v) \), and to \( \epsilon = h_j/2A_j \). This yields a sequence of functions \( \psi_{\nu,j} \in C^1, \ j \in \mathbb{N} \), which satisfy
\[
L(\psi_{\nu,j} \cdot e_v) = 0, \ (\psi_{\nu,j})'(0) = 1, \ |\psi_{\nu,j}|_C < \frac{h_j}{2A_j}.
\]
The maps
\[
H_{\nu,j} : V_j \setminus V_{j-1,2} \to C^1, \ j \in \mathbb{N},
\]
given by
\[
H_{\nu,j}(v) = a_j(v)\psi_{\nu,j} + (1 - a_j(v))\psi_{\nu,j+1}
\]
are continuously differentiable and satisfy
\[
L(H_{\nu,j}(v) \cdot e_v) = a_j(v)L(\psi_{\nu,j} \cdot e_v) + (1 - a_j(v))L(\psi_{\nu,j+1} \cdot e_v) = 0,
\]
\[
(H_{\nu,j}(v))'(0) = a_j(v)\psi_{\nu,j}'(0) + (1 - a_j(v))\psi_{\nu,j+1}'(0) = 1,
\]
\[
|H_{\nu,j}(v)|_C \leq a_j(v)|\psi_{\nu,j}|_C + (1 - a_j(v))|\psi_{\nu,j+1}|_C
\]
\[
\leq a_j(v)h_j + (1 - a_j(v))h_{j+1}
\]
\[
\leq a_j(v)h_j + (1 - a_j(v))h_j = h_j \leq h(v)
\]
for every \( j \in \mathbb{N} \) and all \( v \in V_j \setminus V_{j-1,2} \). Moreover, for such \( j \) and \( v \), and for every \( \mu \in \{1,\ldots, nk\} \),
\[
D_\mu H_{\nu,j}(v)1 = (D_\mu a_j(v)1)\psi_{\nu,j} - (D_\mu a_j(v)1)\psi_{\nu,j+1},
\]

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hence

\[ |D_\mu H_{\nu,j}(v)1|_C \leq |D_\mu a_j(v)1(\|\psi_{\nu,j}|_C + |\psi_{\nu,j+1}|_C) \]

\[ \leq A_j \left( \frac{h_j}{2A_j} + \frac{h_{j+1}}{2A_{j+1}} \right) \leq A_j \left( \frac{2h_j}{2A_j} \right) = h_j \leq h(v). \]

4. It remains to show that the maps \( H_{\nu,j}, j \in \mathbb{N}, \) define a map \( H_\nu : V \to C^1. \) This follows from Part 1 of the proof provided \( H_{\nu,j+1} \) and \( H_{\nu,j} \) coincide on the intersection \( V_j \setminus V_j^2 \) of their domains, for every \( j \in \mathbb{N}. \) For \( j \in \mathbb{N} \) and \( v \in V_j \setminus V_j^2 \) we have

\[ H_{\nu,j+1}(v) = a_{j+1}(v)\psi_{\nu,j+1} + (1 - a_{j+1}(v))\psi_{\nu,j+2} = \psi_{\nu,j+1} \]

due to \( a_{j+1}(v) = 1 \) on \( V_j, V_j \supset V_j \setminus V_j^2, \) and

\[ H_{\nu,j}(v) = a_j(v)\psi_{\nu,j} + (1 - a_j(v))\psi_{\nu,j+1} = \psi_{\nu,j+1} \]

due to \( a_j(v) = 0 \) on \( \mathbb{R}^n \setminus V_j \supset V_j \setminus V_j^2. \)

### 3 The almost graph diffeomorphism

The function \( h_\nu : V \to (0, \infty) \) given by

\[ h(v) = \min \{ 1, \text{dist}(v, \mathbb{R}^n \setminus V) \} \]

\[ \frac{1}{2(nk)^2(1 + \max_{i=1,\ldots,nk;\nu = 1,\ldots,n} |\partial_i g_\nu(v)| + \max_{\nu = 1,\ldots,n} |g_\nu(v)|), \]

in case \( V \neq \mathbb{R}^n \) and

\[ h(v) = \frac{1}{2(nk)^2(1 + \max_{i=1,\ldots,nk;\nu = 1,\ldots,n} |\partial_i g_\nu(v)| + \max_{\nu = 1,\ldots,n} |g_\nu(v)|) \]

for \( V = \mathbb{R}^n \) is continuous.

For \( h = h_\nu \) choose functions \( H_\nu : V \to C^1, \nu \in \{1,\ldots,n\}, \) according to Proposition 2.2 and define \( H : V \to C^1_n \) by \( H(v) = \sum_{\nu = 1}^n H_\nu(v) \cdot e_\nu, \) or equivalently, \( (H(v))_\nu = H_\nu(v) \) for \( \nu = 1,\ldots,n. \) The map \( A : U \to C^1_n \) given by

\[ A(\phi) = \phi - g(v) \cdot H(v) = \phi - \sum_{\nu = 1}^n g_\nu(v)(H_\nu(v) \cdot e_\nu) \quad \text{with} \quad v = \hat{\phi} \]

is continuously differentiable and satisfies \( A(X_f) \subset X_0 \) as for every \( \phi \in X_f \) we have

\[ (A(\phi))'(0) = \phi'(0) - (g(v) \cdot H(v))'(0) \quad \text{(with} \quad v = \hat{\phi}) \]

\[ = \phi'(0) - \left( \sum_{\nu = 1}^n g_\nu(v)(H_\nu(v) \cdot e_\nu) \right)'(0) \]
\[
= \phi'(0) - \sum_{\nu=1}^{n} g_{\nu}(v) (H_{\nu}(v))'(0) e_{\nu} \\
= \phi'(0) - g(v) \\
= \phi'(0) - g(\hat{\phi}) = 0 \quad \text{(with } \phi \in X_f),
\]
which means \(A(\phi) \in X_0\). From the above lines it is also obtained that if \(\phi \in U\) and \(A(\phi) \in X_0\), then \(0 = (A(\phi))'(0) = \phi'(0) - g(\hat{\phi})\), that is \(\phi \in X_f\).

We also have \(A(\phi) = \phi\) on \(X_0 \cap X_f\) since \(\phi \in X_0 \cap X_f\) yields \(0 = \phi'(0) = g(\hat{\phi})\), hence \(A(\phi) = \phi - g(\hat{\phi}) \cdot H(\hat{\phi}) = \phi\).

For \(A\) to be an almost graph diffeomorphism associated with the submanifold \(X_f \subset C_{1}^{n}\), with the open set \(U \subset C_{1}^{n}\), and with the closed subspace \(X_0 \subset C_{1}^{n}\), it remains to prove that \(A\) is a diffeomorphism onto an open subset of \(C_{1}^{n}\).

Observe that due to \(L(H_{\nu}(v) \cdot e_{\nu}) = 0\) we have
\[
LA(\phi) = L\phi - L \left( \sum_{\nu=1}^{n} g_{\nu}(v)(H_{\nu}(v) \cdot e_{\nu}) \right) = L\phi \quad \text{(4)}
\]
for every \(\phi \in U\).

Next we examine the relation between \(v = \hat{\chi}, \phi \in U\), and \(y = \hat{\chi}^{'\ast}\) for \(\chi = A(\phi)\).

Let \(\eta = L\chi = L\phi \in W\). For \(\iota = (\kappa - 1)n + \nu\) with \(\kappa \in \{1, \ldots, k\}\) and \(\nu \in \{1, \ldots, n\}\),
\[
y_{\iota} = \hat{\chi}_{\iota} = A(\hat{\phi})_{\iota} = [\phi_{\nu} - (g(v) \cdot H(v))_{\nu}(-d_{\kappa}(\eta))] \\
= \hat{\phi}_{\iota} - g_{\nu}(v)(H_{\nu}(v))(-d_{\kappa}(\eta)) \\
= v_{\iota} - g_{\nu}(v)(H_{\nu}(v))(-d_{\kappa}(\eta)) \\
= S(\eta, v)
\]
with the continuously differentiable map \(S : W \times V \to \mathbb{R}^{nk}\) given by
\[
S(\eta, v) = v - R(\eta, v)
\]
and
\[
R(\eta, v) = g_{\nu}(v)(H_{\nu}(v))(-d_{\kappa}(\eta)) \\
= ev_{\iota} - d_{\kappa}(\eta)(g_{\nu}(v)H_{\nu}(v))
\]
for \(\iota = (\kappa - 1)n + \nu\) with \(\kappa \in \{1, \ldots, k\}\) and \(\nu \in \{1, \ldots, n\}\).

**Proposition 3.1** (i) For all \((\eta, v) \in W \times V\),
\[
|D_{2}R(\eta, v)|_{L_{c}(\mathbb{R}^{nk}, \mathbb{R}^{nk})} \leq \frac{1}{2}.
\]
(ii) In case \( V \neq \mathbb{R}^{nk} \),

\[
|R(\eta, v)| < \frac{\text{dist}(v, \mathbb{R}^{nk} \setminus V)}{2}
\]

for all \((\eta, v) \in W \times V\).

**Proof.** 1. On assertion (i). Let \( \eta \in W \) be given and define \( R_\eta : V \to \mathbb{R}^{nk} \) by \( R_\eta(v) = R(\eta, v) \). For every \( v \in V \) and for all \( y \in \mathbb{R}^{nk} \) with \(|y| \leq 1\) we get

\[
|D_2 R(\eta, v)y| = |D_2 R_\eta(v)y| \leq \sum_{i=1}^{nk} \sum_{j=1}^{nk} |\partial_j R_{\eta,i}(v)\cdot y_j| \leq \sum_{i=1}^{nk} \sum_{j=1}^{nk} |\partial_j R_{\eta,i}(v)|,
\]

and for \( j \in \{1, \ldots, nk\} \) and \( \iota = (\kappa - 1)n + \nu \) with \( \kappa \in \{1, \ldots, k\} \) and \( \nu \in \{1, \ldots, n\} \), by the chain rule,

\[
|\partial_j R_{\eta,i}(v)| = |D_j R_{\eta,i}(v)|1 = |D_j g_\nu(v)(1) \cdot H_\nu(v) + g_\nu(v) \cdot D_j H_\nu(v)1| = |D_j g_\nu(v)(1) \cdot (H_\nu(v))(-d_\kappa(\eta)) + g_\nu(v) \cdot (D_j H_\nu(v)(1))(-d_\kappa(\eta))| \leq (|\partial_j g_\nu(v)| + |g_\nu(v)|)\delta(v) \quad \text{(with Proposition 2.2)}.
\]

It follows that

\[
|D_2 R(\eta, v)|_{L_\infty(\mathbb{R}^{nk}, \mathbb{R}^{nk})} = \sup_{|y| \leq 1} |D_2 R(\eta, v)y| \leq (nk)^2 \max_{j=1, \ldots, nk; \nu=1, \ldots, n} |\partial_j g_\nu(v)| + \max_{\nu=1, \ldots, n} |g_\nu(v)|\delta(v) \leq \frac{1}{2} \quad \text{(see the choice of } \delta).\]

2. On assertion (ii). Assume \( V \neq \mathbb{R}^{nk} \). For all \((\eta, v) \in W \times V \) and for each \( \iota = (\kappa - 1)n + \nu \) with \( \kappa \in \{1, \ldots, k\} \) and \( \nu \in \{1, \ldots, n\} \) we have

\[
|R_\iota(\eta, v)| = |g_\nu(v)(H_\nu(v))(-d_\kappa(\eta))| \leq |g_\nu(v)||H_\nu(v)| \leq |g_\nu(v)|\delta(v).
\]

Hence

\[
|R(\eta, v)| \leq \sum_{\iota=1}^{nk} |R_\iota(\eta, v)| \leq h(v) \sum_{\nu=1}^{nk} \max_{\nu=1, \ldots, n} |g_\nu(v)| \leq h(v) nk \max_{\nu=1, \ldots, n} |g_\nu(v)| < \frac{\text{dist}(v, \mathbb{R}^{nk} \setminus V)}{2}. \quad \square
\]

It is convenient to introduce the continuously differentiable maps \( S_\eta : V \ni v \mapsto S(\eta, v) \in \mathbb{R}^{nk}, \eta \in W \).
Proposition 3.2  

(i) The set \( \bigcup_{\eta \in W} \{ \eta \} \times S_\eta(V) \subset F \times \mathbb{R}^{nk} \) is open.

(ii) Each map \( S_\eta : V \rightarrow \mathbb{R}^{nk}, \eta \in W, \) is a diffeomorphism onto the open set \( S_\eta(V) = S(\{ \eta \} \times V) \subset \mathbb{R}^{nk}. \)

(iii) The map \( \bigcup_{\eta \in W} \{ \eta \} \times S_\eta(V) \ni (\eta, y) \mapsto S_\eta^{-1}(y) \in V \) is continuously differentiable.

Proof. 1. On assertion (i). Let \( \eta_0 \in W \) and \( y_0 = S_{\eta_0}(v_0) = S(\eta_0, v_0) \) with \( v_0 \in V \) be given. Choose a closed ball \( V_0 \subset V \) with center \( v_0 \) and radius \( \epsilon > 0 \), and an open neighbourhood \( W_0 \) of \( \eta_0 \) in \( W \) such that for all \( \eta \in W_0, \)

\[
|R(\eta, v_0) - R(\eta_0, v_0)| < \frac{\epsilon}{8}.
\]

For \( \eta \in W_0 \) and \( v, v_1 \) in \( V_0 \) the Mean Value Theorem in combination with Proposition 3.1 yield

\[
|R(\eta, v) - R(\eta, v_1)| \leq \frac{1}{2}|v - v_1|.
\]

Let \( Y \subset \mathbb{R}^{nk} \) denote the open ball with center \( y_0 \) and radius \( \epsilon/8 \). Let \( \eta \in W_0 \) and \( y \in Y \) be given and consider the map

\[
V_0 \ni v \mapsto y + R(\eta, v) \in \mathbb{R}^{nk},
\]

which is a contraction. For each \( v \in V_0, \)

\[
|y + R(\eta, v) - v_0| = |y + R(\eta, v) - (y_0 + R(\eta_0, v_0))| \leq |y - y_0| + |R(\eta, v) - R(\eta, v_0)| + |R(\eta, v_0) - R(\eta_0, v_0)| \leq \frac{\epsilon}{8} + \frac{1}{2}|v - v_0| + \frac{\epsilon}{8} \leq \frac{3\epsilon}{4} \leq \epsilon,
\]

and we see that the previous contraction has range in \( V_0 \). Consequently there is a fixed point

\[
v = y + R(\eta, v) \in V_0 \subset V.
\]

Hence \( y = v - R(\eta, v) = S_\eta(v) \). It follows that

\[
W_0 \times Y \subset \bigcup_{\eta \in W} \{ \eta \} \times S_\eta(V),
\]

which yields the assertion.

2. On assertion (ii). Let \( \eta \in W \) be given. It follows from assertion (i) that the set \( S_\eta(V) \subset \mathbb{R}^{nk} \) is open.

2.1. Proof that \( S_\eta \) is injective in case \( V \neq \mathbb{R}^{nk} \). Let \( v, \tilde{v} \) in \( V \) be given with \( S_\eta(v) = S_\eta(\tilde{v}) \). Then

\[
v - \tilde{v} = R(\eta, v) - R(\eta, \tilde{v}).
\]
Without loss of generality, \(\text{dist}(\tilde{v}, \mathbb{R}^{nk}\setminus V) \leq \text{dist}(v, \mathbb{R}^{nk}\setminus V)\). Using Proposition 3.1 (ii) we infer

\[
|\tilde{v} - v| = |R(\eta, \tilde{v}) - R(\eta, v)| < \frac{1}{2}(\text{dist}(\tilde{v}, \mathbb{R}^{nk}\setminus V) + \text{dist}(v, \mathbb{R}^{nk}\setminus V)) \leq \text{dist}(v, \mathbb{R}^{nk}\setminus V).
\]

It follows that the line segment \(v + [0, 1](\tilde{v} - v)\) belongs to \(V\), and the Mean Value Theorem in combination with Proposition 3.1 (i) yields

\[
|v - \tilde{v}| = |R(\eta, v) - R(\eta, \tilde{v})| \leq \frac{1}{2}|v - \tilde{v}|,
\]

which gives us \(v = \tilde{v}\).

2.2. The proof of injectivity of \(S_\eta\) for \(V = \mathbb{R}^{nk}\) is simpler due to convexity.

2.3. From the estimates

\[
|\text{id} - DS_\eta(v)|_{L_c(\mathbb{R}^{nk}, \mathbb{R}^{nk})} = |D_2 R(\eta, v)|_{L_c(\mathbb{R}^{nk}, \mathbb{R}^{nk})} \leq \frac{1}{2} < 1 \quad \text{for} \quad v \in V
\]

we infer that each \(DS_\eta(v), v \in V\), is an isomorphism. Therefore the Inverse Mapping Theorem applies and yields that \(S_\eta^{-1}\) is given by continuously differentiable maps on neighbourhoods of the values \(y \in S_\eta(V)\).

3. On assertion (iii). For every \(\eta \in W\) and \(y = S_\eta(v)\) with \(v \in V\) we have that \(v = S_\eta^{-1}(y)\) satisfies

\[
y - (v - R(\eta, v)) = 0,
\]

or equivalently,

\[
F(\eta, y, v) = 0
\]

for the continuously differentiable map \(F : (\cup_{\eta \in W} \{\eta\} \times S_\eta(V)) \times V \to \mathbb{R}^{nk}\) given by

\[
F(\eta, y, v) = y - (v - R(\eta, v)).
\]

Because of the estimates

\[
|D_3 F(\eta, y, v) - \text{id}|_{L_c(\mathbb{R}^{nk}, \mathbb{R}^{nk})} = |D_2 R(\eta, v)|_{L_c(\mathbb{R}^{nk}, \mathbb{R}^{nk})} \leq \frac{1}{2} < 1
\]

for \(\eta \in W, y \in S_\eta(V)\), and \(v \in V\), each map \(D_3 F(\eta, y, v)\) is an isomorphism. The Implicit Function Theorem applies and yields that the map

\[
\cup_{\eta \in W} \{\eta\} \times S_\eta(V) \ni (\eta, y) \mapsto S_\eta^{-1}(y) \in \mathbb{R}^{nk}
\]

is locally given by continuously differentiable maps. \(\square\)

Using Proposition 3.2 (i) and continuity we obtain that the set

\[
\mathcal{O} = \{\chi \in C^1_n : L\chi \in W \text{ and } \hat{\chi} \in S_\eta(V) \text{ for } \eta = L\chi\}
\]

is open. The map

\[
B : \mathcal{O} \to C^1_n
\]

is the map...
Proof. 1. In order to obtain
\[ B(\chi) = \chi + g(v) \cdot H(v) \quad \text{for} \quad v = S_n^{-1}(y), \quad y = \chi, \quad \eta = L\chi \]
is continuously differentiable. Analogously to Eq. (4) we have
\[ LB(\chi) = L\chi \quad \text{for every} \quad \chi \in \mathcal{O}. \]

**Proposition 3.3** \( A(U) \subset \mathcal{O} \) and \( B(A(\phi)) = \phi \) for all \( \phi \in U \).

**Proof.** Let \( \phi \in U \) be given and set \( \chi = A(\phi) \). Then \( L\chi = L\phi \in \mathcal{W} \), and
\[ y = \chi \]
satisfies \( y = S_\eta(v) \) for \( v = \phi \) and \( \eta = L\phi \). Hence \( \chi \in S_\eta(V) \). It follows
that \( A(\phi) = \chi \in \mathcal{O} \), and we obtain \( A(U) \subset \mathcal{O} \). With \( \phi, \chi, y, v, \eta \) as before, \( v = S_n^{-1}(y) \), and thereby,
\[ B(A(\phi)) = B(\chi) = \chi + g(v) \cdot H(v) = [\phi - g(v) \cdot H(v)] + g(v) \cdot H(v) = \phi. \quad \square \]

**Proposition 3.4** \( B(\mathcal{O}) \subset U \) and \( A(B(\chi)) = \chi \) for all \( \chi \in \mathcal{O} \).

**Proof.** 1. In order to obtain \( B(\mathcal{O}) \subset U \) let \( \chi \in \mathcal{O} \) be given and set \( \phi = B(\chi) \), \( \eta = L\chi, y = \chi \). As \( L\phi = L\chi \in \mathcal{W} \) we may consider \( \phi \in \mathbb{R}^{nk} \). In order to show \( \phi \in U \) we need to verify \( \phi \in V \). From \( \chi \in \mathcal{O} \) we have \( y = S_\eta(v) \) for some \( v \in V \).
For every \( \nu = (k - 1)n + \nu \) with \( \kappa \in \{1, \ldots, k\} \) and \( v \in \{1, \ldots, n\} \),
\[
\hat{v}_\nu = \phi_\nu(-d_\kappa(L\phi)) = \chi_\nu(-d_\kappa(L\chi)) + g_\nu(S_\eta^{-1}(y))(H_\nu(S_\eta^{-1}(y)))(-d_\kappa(L\phi))
\]
\[ = \chi_\nu(-d_\kappa(L\chi)) + g_\nu(v)(H_\nu(v))(-d_\kappa(L\chi))
\]
\[ = y_\nu + g_\nu(v)(H_\nu(v))(-d_\kappa(\eta))
\]
\[ = y_\nu + R_\nu(\eta, v)
\]
\[ = S_\eta(\eta, v) + R_\nu(\eta, v) = v, \]
which yields \( \phi = v \in V \).
2. For \( \chi, \phi, \eta, y, v \) as in Part 1 of the proof we saw that \( \phi = v \). It follows that
\[
A(B(\chi)) = A(\phi) = \phi - g(\phi) \cdot H(\phi)
\]
\[ = [\chi + g(S_\eta^{-1}(y)) \cdot H(S_\eta^{-1}(y))] - g(v) \cdot H(v)
\]
\[ = \chi \quad \text{(with} \quad v = S_\eta^{-1}(y)). \quad \square \]

**Theorem 3.5** The map \( A \) is an almost graph diffeomorphism with respect to the continuously differentiable submanifold \( X_f \), to the closed subspace \( X_0 \) of
codimension \( n \) in \( C_n^1 \), and to the open subset \( U \supset X_f \) of \( C_n^1 \), and the solution
manifold \( X_f \) is an almost graph over \( X_0 \).
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