Recurrent Variational Approach to the Two-Leg Hubbard Ladder

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We applied the Recurrent Variational Approach to the two-leg Hubbard ladder. At half-filling, our variational Ansatz was a generalization of the resonating valence bond state. At finite doping, hole pairs were allowed to move in the resonating valence bond background. The results obtained by the Recurrent Variational Approach were compared with results from Density Matrix Renormalization Group.

I. INTRODUCTION

In the hope to get a better understanding of strongly interacting systems, there has been considerable interest in ladder systems. These ladder systems have proven to be a theoretical wonderland, both analytically and numerically. However, much of the analytic work done on ladder systems has been in weak coupling (or perturbatively in some parameter), namely because there are very few analytic methods at strong coupling. Exact diagonalization, Monte Carlo, and Density Matrix Renormalization Group methods have been the primary tools for studying these systems at strong coupling. Each of these methods has both strengths and weaknesses when considering the lattice sizes, temperatures, and couplings on can consider.

With the ability to fabricate these materials, ladders are not just a theoretical playground. For example, in $(VO)_2P_2O_7$, there are well separated two-leg ladders composed of $VO_2$. Also the cuprate-like material $SrCu_2O_3$ consists of weakly coupled $CuO_2$ two-leg ladders, and the material $Sr_2Cu_3O_5$ consists of weakly coupled $CuO_2$ three-leg ladders.

Recently, a powerful analytic method was developed to deal with strongly coupled quasi-one dimensional systems — the Recurrent Variational Approach (RVA). This method is similar in spirit to Wilson’s Numerical Renormalization Group and White’s Density Matrix Renormalization Group (DMRG). The key idea in all of these methods is to build up the system by adding on sites at the boundary. However, the real power of the RVA is that, though analytic, the physics of the problem is taken into account in an unbiased way and elucidated quite clearly. For example, the importance of different configurations in the ground state wave function is determined without any outside assumptions, and the physics of these configurations is made clear.

In this work, we apply the RVA to the two-leg Hubbard ladder at strong coupling and small dopings. Though a considerable amount of work has been done on the Hubbard ladder and many of its properties are known, we are unaware of any work which has put this information together and constructed a ground state wave function. Our goal in this work is to provide a simple physical picture of what the ground state might look like and to go ahead and construct a ground state wave function.

The Hamiltonian of the two-leg Hubbard ladder is given by

$$H = -t \sum_{i,s} \left( c_{i,s}^\dagger c_{i+\hat{x},s} + h.c. \right) - t_\perp \sum_{i,s} \left( c_{i,s}^\dagger c_{i+\hat{y},s} + h.c. \right) + U \sum_i n_{i,\uparrow} n_{i,\downarrow},$$

where $c_{i,s}^\dagger$ creates a fermion at site $i$ with spin $s$, $n_{i,s} = c_{i,s}^\dagger c_{i,s}$, $t$ is the hopping matrix element along the chain, $t_\perp$ is the hopping matrix element perpendicular to the chain (i.e., along the rung), and $U$ is the on-site Coulomb repulsion. Site $i$ has coordinates $(x,y)$ with $1 \leq x \leq N$ and $y = 1, 2$. It will also be convenient to introduce the following two operators:

$$\Delta_{ij}^\dagger = c_{i,\uparrow}^\dagger c_{j,\downarrow}^\dagger + c_{j,\downarrow}^\dagger c_{i,\uparrow}^\dagger,$$

which creates a singlet across sites $i$ and $j$, and

$$D_i = c_{i,\uparrow}^\dagger c_{i,\downarrow}^\dagger,$$

which creates a doubly occupied site.

The rest of the paper is organized as follows. In Sec. II we consider the half-filled case. In Sec. III we consider the Hubbard ladder at small dopings. In Sec. IV we present our results for the ground state energies and compare with DMRG. Finally, in Sec. V we summarize our results and present some concluding remarks.

II. HALF-FILLED HUBBARD LADDER

We begin with the half-filled ladder. What are the ingredients necessary to construct a wave function which captures the physics of the half-filled Hubbard ladder? A trail of clues has been laid by previous works. First of all, we know that at strong coupling the half-filled...
We will be mainly interested in the case \( t = t_\perp = 1 \).

In Table I we list the values of the parameters for several values of \( U \) (with \( t = t_\perp = 1 \)).

Notice that the solution to the 2x2 plaquette (with \( t = t_\perp = 1 \)) has \( D_4 \) symmetry. \( (D_4 \) is the symmetry group of the square.) However, the ground state does not transform in the scalar representation of \( D_4 \); it transforms in the \( B_2 \) representation of \( D_4 \). \( (B_2 \) is the one-dimensional representation which changes sign upon rotation by 90° and reflection about the diagonals.) \( B_2 \) coincides with the standard \( d_{x^2-y^2} \) symmetry. The \( B_2 \) representation is what forbids some configurations, as those shown in Fig. 3 from appearing in the ground state wave function.

TABLE I. Values of the parameters for the half-filled 2x2 plaquette (with \( t = t_\perp = 1 \)) which give the exact groundstate.

\[
\begin{array}{cccccc}
U & a_1 & a_3 & a_5 & a_2 & a_4 \\
8 & -1.0 & -0.0762 & -0.3306 & -a_3 & -a_5 \\
16 & -1.0 & -0.0221 & -0.1807 & -a_3 & -a_5 \\
24 & -1.0 & -0.0101 & -0.1229 & -a_3 & -a_5 \\
\end{array}
\]

Using the configurations of the 2x2 plaquette as the basis of our Ansatz for the ladder, a typical configuration for the ladder is shown in Fig. 3. The ground state will be a superposition of all possible configurations of the type shown in Fig. 3 Therefore, it seems like working with this kind of state will be a formidable task. Fortunately, the RVA gives us a straightforward way of dealing with such a state -- generate it recursively. Specifically, the RVA builds the ground state of a ladder with \( N + \nu \) rungs using the knowledge of the ground states of a ladder with \( N, N + 1, \ldots, N + \nu - 1 \) rungs. This is achieved by recursion relations which express the ground state \( | N + \nu \rangle \) in terms of the ground states \( \{ | N + i \rangle \} \) with \( i = 0, \ldots, \nu - 1 \).

Using these ingredients, we consider the following ansatz for the half-filled Hubbard ladder which is shown in Fig. 3.
\[ |N + 2\rangle = |\phi_0\rangle_{N+2} | N + 1\rangle + \alpha |\phi_1\rangle_{N+2} | N + 1\rangle
+ \beta | \phi_2\rangle_{N+1,N+2} | N\rangle + \gamma | \phi_3\rangle_{N+1,N+2} | N\rangle
+ \xi | \phi_4\rangle_{N+1,N+2} | N\rangle + \eta | \phi_5\rangle_{N+1,N+2} | N\rangle
+ \delta | \phi_6\rangle_{N+1,N+2} | N - 1\rangle
+ \varepsilon | \phi_7\rangle_{N,N+1,N+2} | N - 1\rangle \]

(6)

schematically in Fig. 4. For completeness, the states in terms of the operators in Eqs. (3) and (4) are given in Appendix A.

A few words are in order about the configurations in our Ansatz. i) \(\beta\), the weight of the horizontal bond (relative to the vertical bond), is the “RVB parameter”. For a true RVB state, we would have \(\beta = -1\). Although we do not expect \(\beta = -1\), from work on the Heisenberg model the variational parameters were shown to evolve smoothly with system size; we expect \(|\beta| = O(1)\). ii) Suppose we iterate the recursion relations once. Then \(|\phi_0\rangle\) and \(|\phi_1\rangle\) (and also \(|\phi_5\rangle\)) generate the terms shown in Fig. 3. In the ground state of the 2x2 plaquette, the states in Fig. 4(b) have weight \(O(\alpha)\), the states in Fig. 4(c) have weight \(O(\alpha^2)\), and the states in Fig. 4(d) do not appear. Since we expect the parameters to evolve smoothly, we expect \(\eta \approx -\alpha^2\). (\(\eta\) is behaving as a “counter-term”; its job is to subtract off the \(\alpha^2\) contribution from \(|\phi_1\rangle\).) iii) Even though we no longer have \(D_4\) symmetry, initial calculations showed that the states in Fig. 4(b) do not appear in the ground state of the Hubbard ladder. (This is another indication that the parameters evolve smoothly from the 2x2 case to the ladder.) Therefore, we do not consider them in what follows. iv) Since the configurations \(|\phi_6\rangle_{N,N+1,N+2}\) and \(|\phi_7\rangle_{N,N+1,N+2}\) appear as intermediate states for the resonances shown in Fig. 6, it is necessary to include these states in the Ansatz to give us an RVB state. This can easily be seen by comparing the weights of the coefficients as shown in Table 1 (see below).

In order to compute the values of the coefficients, we treat them as variational parameters and minimize the ground state energy with respect to these parameters. The ground state energy and other quantities appear as recursion relations. It will be useful to define

\[ E_N = \langle N | H_N | N \rangle, \]
\[ D_N = \langle N - 1 | N \langle \phi_0 | H_N | N \rangle, \]
\[ C_N = \langle N - 1 | N \langle \phi_1 | H_N | N \rangle, \]
\[ Z_N = \langle N | N \rangle, \]
\[ Y_N = \langle N - 1 | N \langle \phi_0 | N \rangle, \]
\[ X_N = \langle N - 1 | N \langle \phi_1 | N \rangle. \]

(7)

They are supplemented by the initial conditions

\[ Z_0 = 1, Y_0 = 0, X_0 = 0, \]
\[ E_0 = 0, D_0 = 0, C_0 = 0. \]

(8)

To determine the values of the variational parameters for a given (finite) value of \(N\), we iterate the recursion relations and minimize the quantity \(E_N/Z_N\) numerically. The actual recursion relations are quite unwieldy; we have relegated them, as well as their derivation, to Appendix A.

The values of the variational parameters for various values of \(U\) (with \(t = t_\perp = 1\)) are shown in Table 1. The results were obtained on a 2 x 32 ladder. Notice that i) \(|\beta| = O(1)\), and we have produced an RVB state. ii) \(|\eta| = O(\alpha^2)\), and \(\eta\) is indeed behaving as a counterterm. iii) \(|\delta|, |\varepsilon| \approx \alpha/3\). Therefore, these configurations are non-negligible, suggesting that the resonances shown in Fig. 6 are important to the RVB picture.

### III. THE DOPED HUBBARD LADDER

Now, we consider the doped Hubbard ladder. In Ref. 9 it was shown that hole pairs moving through an RVB

| \(U\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\xi\) | \(\eta\) | \(\delta\) | \(\varepsilon\) |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 8   | .3296 | -.8710 | -.0782 | -.2877 | -.0938 | .1031 | .1031 |
| 16  | .1848 | -.8800 | -.0243 | -.1606 | -.0299 | .0639 | .0639 |
| 24  | .1265 | -.8817 | -.0113 | -.1097 | -.0142 | .0451 | .0451 |

### TABLE II. Values of the variational parameters for a 2 x 32 half-filled ladder with \(t = t_\perp = 1\).
background captures the essential physics of the $t-J$ ladder at small dopings. Since the $t-J$ model is the large U limit of the Hubbard model, we expect this picture to hold for the Hubbard model at large $U$ and small dopings. Therefore, we consider an Ansatz of hole pairs moving through our generalized RVB background for the Hubbard ladder at small dopings.

Since the structure of the hole pairs is based on the exact solution to the 2x2 plaquette with 2 holes, we consider the 2x2 case in detail below.

A. The 2x2 Plaquette with Two Holes

The ground state of the 2x2 plaquette with 2 holes is given by
\[
|\psi^h_{2}\rangle = |\varphi^h_{0}\rangle + b_1 |\varphi^h_{1}\rangle + b_2 |\varphi^h_{2}\rangle + b_3 |\varphi^h_{3}\rangle
\]
where
\[
|\varphi^h_{0}\rangle = \Delta^\dagger_{1,(1,1,2)} |0\rangle + \Delta^\dagger_{2,(2,1,2)} |0\rangle,
|\varphi^h_{1}\rangle = \Delta^\dagger_{1,(1,2,1)} |0\rangle + \Delta^\dagger_{1,(2,1,2)} |0\rangle,
|\varphi^h_{2}\rangle = \Delta^\dagger_{1,(1,2,2)} |0\rangle + \Delta^\dagger_{1,(2,1,2)} |0\rangle,
|\varphi^h_{3}\rangle = D^\dagger_{1,(1,1,2)} |0\rangle + D^\dagger_{1,(2,1,2)} |0\rangle
\]
\[
+ D^\dagger_{1,(2,2,1)} |0\rangle + D^\dagger_{1,(2,2,2)} |0\rangle.
\]

\[
|\psi^h_{2}\rangle
\]
is shown schematically in Fig. 7.

In Table III we list the values of the parameters for several values of $U$ (with $t = t_\perp = 1$). Notice that the solution to the 2x2 case (with $t = t_\perp$) has $D_4$ symmetry. However, now the ground state transforms in the scalar representation of $D_4$. Recalling that the ground state of the half-filled ladder transforms in the $B_2$ representation, we see that the operator that creates a hole pair out of the undoped system has $d_{x^2+y^2}$ symmetry. These facts continue to be true for low doping in larger ladders. Also, notice that $b_3 = O(\alpha)$ where $\alpha$ is from the half-filled ladder. This will play a role in writing our Ansatz for the doped ladder.

B. The Ladder

Using the configurations of the generalized RVB state as well as the hole pair configurations, a typical configuration for the doped ladder is shown in Fig. 8.

The ground state will be a superposition of all such configurations shown in Fig. 8. Fortunately, we can generate such a state recursively. Specifically, we build the ground state of a ladder with $N + \nu$ rungs and $P + \mu$ holes using the knowledge of the ground states of a ladder with $N, N + 1, \ldots, N + \nu - 1$ rungs and $P, P + 1, \ldots, P + \mu$ holes. This is achieved by recursion relations which express the ground state $| N + \nu, P + \mu\rangle$ in terms of the ground states $\{| N + i, P + j\rangle\}$ with $i = 0, \ldots, \nu - 1$ and $j = 0, \ldots, \mu$.

Using the above ingredients, we consider the following Ansatz for the doped Hubbard ladder

\[
| N + 2 P + 1\rangle = |\phi_0\rangle_{N+2} | N+1 P+1\rangle + \alpha |\phi_1\rangle_{N+2} | N+1 P+1\rangle + \beta |\phi_2\rangle_{N+1,N+2} | N P+1\rangle
\]
\[
+ |\phi_3\rangle_{N+1,N+2} | N P+1\rangle + |\phi_4\rangle_{N+1,N+2} | N P+1\rangle + \xi |\phi_5\rangle_{N+1,N+2} | N P+1\rangle
\]
\[
+ \delta |\phi_6\rangle_{N,N+1,N+2} | N-1 P+1\rangle + \epsilon |\phi_7\rangle_{N,N+1,N+2} | N-1 P+1\rangle
\]
\[
+ |\phi_8\rangle_{N+1,N+2} | N+1 P\rangle + \lambda |\phi_9\rangle_{N+1,N+2} | N P\rangle + \zeta |\phi_{10}\rangle_{N+1,N+2} | N P\rangle
\]
\[
+ \mu |\phi_{11}\rangle_{N,N+1,N+2} | N-1 P\rangle + \nu |\phi_{12}\rangle_{N,N+1,N+2} | N-1 P\rangle
\]

(11)

| $U$ | $b_1$ | $b_2$ | $b_3$
|---|---|---|---|
| 8 | 1.0 | 1.2470 | 0.3569 |
| 16 | 1.0 | 1.3131 | 0.2100 |
| 24 | 1.0 | 1.3420 | 0.1483 |
They are supplemented by the initial conditions

\begin{align*}
Z_{N,P=0} &= 1, Y_{N,P=0} = 0, X_{N,P=0} = 0, \\
E_{N,P=0} &= 0, D_{N,P=0} = 0, C_{N,P=0} = 0, \\
F_{N<P,P} &= 0 \text{ for } F = Z, Y, X, E, D, C.
\end{align*}

To determine the values for the variational parameters for given (finite) values of \( N \) and \( P \), we iterate the recursion relations and we minimize the quantity \( E_{N,P}/Z_{N,P} \). The actual recursion relations are quite unwieldy; we have relegated them, as well as their derivation, to Appendix B.

What is the nature of the state we have constructed? In order to answer this question, we plot \( \beta, \lambda, \) and \( \zeta \) vs. doping. These parameters contain most of the physics of our Ansatz. \( \beta \) is the “RVB parameter”; \( \lambda \) and \( \zeta \) are the weights of the hole pair configurations. The results were obtained on a \( 2 \times 32 \) ladder.

First, consider Fig. 10(a). \( \beta \) begins at \( O(-1) \) and increases (i.e., becomes less negative) with doping until a critical doping, \( x_c \), where it vanishes. Beyond this doping, \( \beta \) is positive. This has also been found for the \( t-J \) ladder. Upon doping, the hole pairs cause destructive interference which degrades the RVB mechanism. For \( x > x_c \), this destructive interference has driven \( \beta \) positive, and it is no longer appropriate to think of our state as describing hole pairs moving through an RVB background.

Similar to the \( t-J \) ladder, the difference between \( x < x_c \) and \( x > x_c \) can be attributed to two different internal structures of the hole pairs. For \( x < x_c \), the hole pairs have a \( d_{x^2-y^2} \) structure relative to the RVB background. For \( x > x_c \), the hole pairs have an s-wave like symmetry relative to their background.

Now consider Figs. 10(b) and 10(c). First of all, notice that \( \lambda > \zeta \). This shows the importance of the diagonal frustrating bonds for all dopings. Also, notice that \( \lambda \) and \( \zeta \) both reach their maximum at \( x = 1/2 \). At \( x = 1/2 \) the system is essentially a large scale reproduction of the \( 2 \times 2 \) plaquette. Indeed, the values of \( \lambda \) and \( \zeta \) at \( x = 1/2 \) are similar to their values for the \( 2 \times 2 \) plaquette.
IV. GROUND STATE ENERGIES

First, we show results for Energy per site vs. $U$ in Fig. 11. For comparison, DMRG results are presented for the same set of parameters. At half-filling, as we would expect, our Ansatz is most accurate for large $U$ and large $t_{\perp}$. The ground state energy per site for a 2 x 32 half-filled ladder as a function of $U$ for various $t_{\perp}$ is shown in Fig. 11(a). For $U = 8$ and $t_{\perp} = 1$, the energy from the RVA agrees with DMRG to within 90% and improves as $U$ or $t_{\perp}$ is increased. Up to about $U = 10$, longer bonds (extending over at least 3 rungs) are coming into play. These states should be included in the Ansatz to further improve the overlap with the ground state. At $U = 16$ and $t_{\perp} = 1$, our ansatz gives a ground state energy within 94% of the DMRG result.

It should be noted that for the Heisenberg ladder, the RVB state gives a ground state energy within 94% of true ground state energy, obtained from DMRG.

A recent
DMRG study of different ladder models found that the Hubbard model and Heisenberg model begin to agree only for rather large $U$ ($U \approx 16$). Therefore, it is not surprising that the RVB picture becomes as good for the Hubbard model as it is for the Heisenberg model at $U \approx 16$.

Fig. 13(b) shows the ground state energy of the RVA Ansatz as a function of $U$ for various $t_\perp$ for a doping of $x = 1/8$ on a $2 \times 32$ ladder. Again, we show energies obtained from DMRG for the same set of parameters. For $U = 16$ and $t_\perp = 1$, the two energies agree to only within 77% and improves slightly as $t_\perp$ is increased. For example, at $U = 16$ and $t_\perp = 2$, the overlap of energies increases to 87%. Further discrepancies occur when the doping is increased to $x = 1/4$ (see Fig. 13(c)).

The differences in ground state energies occur due to the importance of “pair-breaking” configurations, like those shown in Fig. 12. The weights of these types of states increase as we move away from half-filling. Consequently, to further improve the RVA Ansatz, such states must be included in the wave function. Note also that our Ansatz would be essentially exact for the case where hole pairs are well localized on a rung. For the $t-J$ model with $J_{\text{rung}} \gg J_{\text{chain}}, t$, pairs are well localized along a rung, and the ground state is essentially a product of rung singlets and rung hole pairs. However, for the Hubbard model at strong coupling (i.e., $U \gg t_\perp, t$), this is not the case. Holes would always rather occupy adjacent rungs, even for $t_\perp \gg t$, since this minimizes the Coulomb energy from doubly occupied sites. To see this consider a $2 \times 2$ plaquette with 2 holes; let $t \ll t_\perp$ and $t, t_\perp \ll U$. With 1 particle on each rung (i.e., one hole on each hole rung), the ground state energy is approximately $-2t_\perp$; with both particles on the same rung (i.e., both holes on the same rung), the ground state energy is approximately $-J = -4t_\perp^2/U$. Therefore, at large $U$, the particles would rather occupy adjacent rungs. (See Fig. 13)

The situation we have with the doped ladder is similar to what we had for the half-filled ladder in the early stages of this work. We found that without the states $| \phi_0 \rangle$ and $| \phi_7 \rangle$ which extend over three rungs (see Fig. 4), the RVA did not accurately reproduce the ground state even at extremely large $U$. However, once we included $| \phi_0 \rangle$ and $| \phi_7 \rangle$, the results from the RVA improved drastically. Based on these results, we expect the RVA to greatly improve by including the states shown in Fig. 12.

In Fig. 14, we plot Energy per site vs. Doping for a $2 \times 32$ ladder for $U = 8$ and $U = 16$ (with $t_\perp = 1$) in order to better understand the region of validity of our RVA ansatz. Again, we see good agreement with DMRG results at half-filling. However, as soon as we dope, configurations like those shown in Fig. 12 are also important.

It is interesting to note that the idea of hole pairs moving through the RVB background seems to more accurately represent the ground state of the $t-J$ model than the Hubbard model. Using the well known relation at strong coupling, $J \approx 4t_\perp^2$, the RVA agrees to within 92% of the true ground state energy of the $t-J$ ladder for $J = 0.5$ ($U = 8$) at a doping of $x = 1/8$. There are two ways to interpret this; either the t-J model supports pairing better than the Hubbard model, or we must view the hole pairs in the Hubbard model as having a larger size (i.e. larger coherence length.)

V. CONCLUDING REMARKS

To summarize, we applied the Recurrent Variational Approach to the two-leg Hubbard ladder. Our results were in qualitative agreement with previous results on the Heisenberg and t-J ladders. For the half-filled ladder, the generalized RVB state became more accurate in the parameter regime where the Hubbard and Heisenberg ladders were shown to coincide. However, comparison of the RVA with DMRG for the doped ladder indicate that hole-pairs moving through an RVB background
is incomplete; it does not capture the essential physics. “Pair-breaking” configurations are also necessary to capture the essential physics.

As we saw, the strength of the RVA is the ease in which we could extract the physics. We were able to see the importance of the configurations in our Ansatz quite easily. Furthermore, the RVA has a natural way in which to include longer bonds in the Ansatz to more accurately represent the groundstate wavefunction. The importance of such additional states to the physics of the ladder is not easily probed with other techniques.

Generalized RVB states similar to ours have been considered previously for the half-filled Hubbard ladder. Fano et al. were even able to produce an Ansatz coming within 98% of the true groundstate energy for a (half-filled) 2x4 ladder at $U = 16$. (Their Ansatz included diagonal bonds of length $\sqrt{3}$.) However, none of these works considered the doped case. Using the approach in Ref. 17 (in terms of dimer coverings), it appears to be a formidable task to consider doping. This is one of the strengths of the RVA: doping is handled rather easily. Even though our results for the doped ladder showed that hole pairs moving through an RVB background is incomplete, the RVA offers a straightforward way to improve the situation, namely include “pair-breaking” configurations (shown in Fig. 12) in the Ansatz.

Another (and probably better) way to improve the situation for the doped ladder is to consider a Matrix Product Ansatz. A Matrix Product Ansatz can be generated by first order recursion relations. In the RVA, the size of the hole-pairs are fixed. (In our case, the hole pairs had a size of one lattice spacing.) However, by construction, the Matrix Product Ansatz takes into account hole pairs of arbitrary size. We leave this (and other possibilities) for future work.

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APPENDIX A: THE HALF-FILLED LADDER

The states in the Ansatz for the half-filled ladder of Eq. (3) are given by

$$
|\phi_0\rangle_{N+2} = \Delta^\dagger_{(N+2,1),(N+2,2)} |0\rangle_{N+2},
$$

$$
|\phi_1\rangle_{N+2} = D^\dagger_{(N+2,1)} |0\rangle_{N+2} + D^\dagger_{(N+2,2)} |0\rangle_{N+2},
$$

$$
|\phi_2\rangle_{N+1,N+2} = \Delta^\dagger_{(N+1,1),(N+1,2)} \Delta^\dagger_{(N+2,1),(N+2,2)} |0\rangle_{N+1,N+2},
$$

$$
|\phi_3\rangle_{N+1,N+2} = D^\dagger_{(N+1,1)} D^\dagger_{(N+1,2)} |0\rangle_{N+1,N+2} + D^\dagger_{(N+2,1)} D^\dagger_{(N+2,2)} |0\rangle_{N+1,N+2},
$$

$$
|\phi_4\rangle_{N+1,N+2} = \Delta^\dagger_{(N+1,1),(N+2,1)} D^\dagger_{(N+1,2)} |0\rangle_{N+1,N+2} + \Delta^\dagger_{(N+1,1),(N+2,2)} D^\dagger_{(N+2,2)} |0\rangle_{N+1,N+2}
+ \Delta^\dagger_{(N+1,2),(N+1,2)} D^\dagger_{(N+1,1)} |0\rangle_{N+1,N+2} + \Delta^\dagger_{(N+1,2),(N+2,2)} D^\dagger_{(N+2,1)} |0\rangle_{N+1,N+2},
$$

$$
|\phi_5\rangle_{N+1,N+2} = D^\dagger_{(N+1,1)} D^\dagger_{(N+2,2)} |0\rangle_{N+1,N+2} + D^\dagger_{(N+1,2)} D^\dagger_{(N+2,1)} |0\rangle_{N+1,N+2},
$$

$$
|\phi_6\rangle_{N+1,N+2} = \Delta^\dagger_{(N+2,1),(N+2,2)} \Delta^\dagger_{(N+1,1),(N+2,1)} D^\dagger_{(N+2,2)} |0\rangle_{N+1,N+2}
+ \Delta^\dagger_{(N+2,2),(N+1,2)} \Delta^\dagger_{(N+1,1),(N+2,1)} D^\dagger_{(N+1,2)} |0\rangle_{N+1,N+2}
+ \Delta^\dagger_{(N+2,2),(N+2,2)} \Delta^\dagger_{(N+1,2),(N+2,2)} D^\dagger_{(N+1,1)} |0\rangle_{N+1,N+2}
+ \Delta^\dagger_{(N+2,2),(N+2,2)} \Delta^\dagger_{(N+1,2),(N+2,2)} D^\dagger_{(N+1,1)} |0\rangle_{N+1,N+2},
$$

$$
|\phi_7\rangle_{N+1,N+2} = \Delta^\dagger_{(N+1,1),(N+2,2)} \Delta^\dagger_{(N+2,1),(N+2,2)} D^\dagger_{(N+1,2)} |0\rangle_{N+1,N+2}
+ \Delta^\dagger_{(N+1,1),(N+2,2)} \Delta^\dagger_{(N+2,1),(N+2,2)} D^\dagger_{(N+2,2)} |0\rangle_{N+1,N+2}
+ \Delta^\dagger_{(N+2,2),(N+2,1)} \Delta^\dagger_{(N+1,1),(N+2,2)} D^\dagger_{(N+1,2)} |0\rangle_{N+1,N+2}
+ \Delta^\dagger_{(N+2,2),(N+2,1)} \Delta^\dagger_{(N+1,1),(N+2,2)} D^\dagger_{(N+1,2)} |0\rangle_{N+1,N+2}.
$$

To derive the recursion relations, the following inner products are necessary:
Using these inner products, a straightforward calculation gives the following (coupled) recursion relations:

\[
Z_{N+2} = (2 + 2a^2)Z_{N+1} - 2B_{N+1} + 2\eta X_{N+1} + (4\beta^2 + 2\gamma^2 + 8\xi^2 + 2\eta^2)Z_N \\
+ (16\delta^2 + 16\epsilon^2 - 16\delta\epsilon)Z_{N-1} - 8\xi\epsilon Y_N + 8\delta\epsilon Y_{N-1},
\]

\[
Y_{N+2} = 2Z_{N+1} - 8\delta\epsilon Y_{N+1} - 4\delta\epsilon Y_N - 8\delta\epsilon Y_{N-1},
\]

\[
X_{N+2} = 2\alpha Z_{N+1} + \eta X_{N+1},
\]

\[
E_{N+2} = (2 + 2a^2)E_{N+1} + (-8t_\perp\alpha + 2U\alpha^2)Z_{N+1} - 2\beta D_{N+1} + 2\eta C_{N+1} \\
+ (8t_\perp + 4t_\perp\alpha\beta)Y_{N+1} + (4\alpha^2 + 2\gamma^2 + 8\xi^2 + 2\eta^2)E_N \\
+ (4\beta^2 + 2\gamma^2 + 8\xi^2 + 2\eta^2)E_N + (32t_\perp\beta + 4U\gamma^2 - 16t_\perp\gamma\xi + 8U\xi^2 - 16t_\perp\xi^2 + 8U\xi^2 + 4U\eta^2)Z_N \\
+ (16\beta\delta - 8t_\perp + 8t_\perp\epsilon + 16t_\perp\beta + 8t_\perp\gamma + 8U\xi + 8t_\perp\xi - 8t_\perp\delta)Y_N + 8t_\perp\delta N_X_N \\
+ (32t_\perp\xi + 16U\beta^2 + 16U\gamma^2 + 32t_\perp\delta + 16t_\perp\delta - 16U\delta^2 + 16t_\perp\delta^2 + 32t_\perp\delta N_X_N - 8\xi\epsilon D_N \\
+ (16\alpha^2 + 16\epsilon^2 - 16\delta\epsilon)E_{N-1} + 8\delta\epsilon D_{N-1} + (8U\delta^2 - 16\delta\epsilon)Y_{N-1},
\]

\[
D_{N+2} = 2E_{N+1} - 4t_\perp\alpha Z_{N+1} - \beta D_{N+1} + 4t_\perp\eta Y_{N+1} + 2t_\perp\eta X_{N+1} + (-4\epsilon - 4\delta)Y_N + 4t_\perp\delta X_N \\
+ (16\beta\delta + 8t_\perp\epsilon + 8U\xi + 8t_\perp\delta + 16t_\perp\delta Z_{N-1} - 8\delta\epsilon E_{N-1} + 4\delta\epsilon D_{N-1} + 4U\delta Y_{N-1},
\]

\[
C_{N+2} = 2\alpha E_{N+1} + (-2t_\perp + 2U\alpha)Z_{N+1} + 2t_\perp\beta Y_{N+1} + (-4t_\perp + U\eta)X_{N+1} + \eta C_{N+1} \\
+ 16t_\perp\xi\delta Z_{N-1} - 8t_\perp\delta\epsilon Y_{N-1} + 4t_\perp\alpha\epsilon Y_N.
\]

(2)

**APPENDIX II: THE DOPED LADDER**

For the doped ladder (see Eq. ([1], | φ_0, N+1, N+2 | φ_1, N+1, N+2 | φ_2, N+1, N+2 | φ_3, N+1, N+2 | φ_4, N+1, N+2 | φ_5, N+1, N+2, and | φ_7, N+1, N+2 are the same as the half-filled case, and

\[
| φ_0, N+2 = | 0, N+2, \\
| φ_1, N+1, N+2 = \Delta^+_N|0, N+1, N+2 + \Delta^+_N|N+1, 1, N+2 | 0, N+1, N+2, \\
| φ_2, N+1, N+2 = \Delta^+_N|0, N+1, N+2 + \Delta^+_N|N+2, 1, N+2 | 0, N+1, N+2, \\
| φ_3, N+1, N+2 = \Delta^+_N|N+1, 1, N+1, N+2 + \Delta^+_N|N+2, 1, N+2 | 0, N+1, N+2, \\
| φ_4, N+1, N+2 = \Delta^+_N|N+1, 1, N+2 + \Delta^+_N|N+2, 1, N+1 | 0, N+1, N+2.
\]

(1)

To derive the recursion relations, we use the inner products from the half-filled case as well as the following recursion relations:

\[
N+2|φ^h_0, φ^h_0, N+2 = 1, \\
N+1, N+2|φ^h_0, φ^h_0, N+1, N+2 = 4, \\
N+1, N+2|φ^h_0, φ^h_0, N+1, N+2 = 4, \\
N+1, N+2|φ^h_0, φ^h_0, N+1, N+2 = 8, \\
N+1, N+2|φ^h_0, φ^h_0, N+1, N+2 = 8, \\
\langle N + 1P + 1 | N+2, φ_0, φ_0, N+1, N+2 | N - 1P \rangle = (-1)^\langle N + 1P + 1 | φ_0, φ_0, N+1 | N - 1P \rangle, \\
\langle NP | N+1, N+2, φ^h_0, φ^h_0, N+1, N+2 | N - 1P \rangle = (-2)^\langle NP | φ_0, φ_0 | N - 1P \rangle.
\]

(2)

Using these inner products, a straightforward calculation gives the following (coupled) recursion relations:

\[
Z_{N+2, p+1} = (2 + 2a^2)Z_{N+1, p+1} - 2\beta Y_{N+1, p+1} + 2\eta X_{N+1, p+1} + (4\beta^2 + 2\gamma^2 + 8\xi^2 + 2\eta^2)Z_{N, p+1} \\
+ (16\delta^2 + 16\epsilon^2 - 16\delta\epsilon)Z_{N-1, p+1} - 8\xi\epsilon Y_{N+1, p+1} + 8\delta\epsilon Y_{N-1, p+1} + \omega^2 Z_{N+1, p} + (4\lambda^2 + 4\xi^2)Z_{N, p} \\
- 4\lambda\epsilon Y_{N, p} + (8\beta^2 + 8\gamma^2 - 8\lambda\mu)Z_{N-1, p} + 4\mu Y_{N-1, p},
\]

\[
Y_{N+2, p+1} = 2Z_{N+1, p+1} - \beta Y_{N+1, p+1} - 8\delta\epsilon Y_{N-1, p+1} + 4\delta\epsilon Y_{N-1, p+1} - 4\lambda\epsilon Z_{N-1, p} + 2\mu Y_{N-1, p},
\]

9
\[ X_{N+2,p+1} = 2\alpha Z_{N-1,p+1} + \eta X_{N+1,p+1}, \]
\[ E_{N+2,p+1} = (2 + 2\alpha^2)E_{N+1,p+1} + (-8t_1\alpha + 2U\alpha^2)Z_{N+1,p+1} - 2\beta D_{N+1,p+1} + 2\alpha\eta C_{N+1,p+1} \]
\[ + (8\xi + 4t_1\alpha\beta)Y_{N+1,p+1} + (-4t_1\eta - 8t_0\xi + 2U\alpha\eta)X_{N+1,p+1} \]
\[ + (4\beta^2 + 2\gamma^2 + 8\xi^2 + 2\eta^2)E_{N+1,p+1} + (-32t_1\beta\xi + 4U\gamma^2 - 16\eta\xi + 8U\xi^2 - 16t_1\xi\eta + 4U\eta^2)Z_{N+1,p+1} \]
\[ + (16t_1\beta\delta - 8t_0\xi + 8t_0\tau \tau - 16t_1\xi\delta + 16U\xi\delta - 8t_0\tau \tau - 8t_0\tau \tau - 8t_0\tau \tau)Y_{N+1,p+1} + 8t_0\alpha\xi X_{N+1,p+1} \]
\[ + (32t_1\beta\xi + 16U\gamma^2 + 16U\xi^2 + 32t_1\alpha\xi)X_{N-1,p+1} - 8\xi\xi D_{N+1,p+1} \]
\[ + (16\delta^2 - 16\xi\delta)E_{N-1,p+1} + 8\xi\xi D_{N-1,p+1} + (8U\xi - 16t_1\alpha\delta)Y_{N-1,p+1} \]
\[ + E_{N+1,p+1} + (4\lambda^2 + 4\xi^2)E_{N,p+1} + (8\mu^2 + 8\nu^2 - 8\lambda\mu)E_{N-1,p} - 4t_1\lambda Y_{N+1,p} - 4t_1\xi X_{N+1,p} \]
\[ + (-16t_1\alpha\xi - 8t_1\lambda - 8t_0\alpha\xi)Z_{N,p} + (4t_1\nu + 8t_1\xi\nu + 4t_1\mu + 4t_1\nu)Y_{N,p} \]
\[ + (-16t_1\beta\nu + 8t_1\nu + 16t_1\alpha\lambda\mu - 16t_1\beta\mu)Z_{N-1,p} - 4t_1\nu D_{N,p} + 4t_1\nu D_{N-1,p} \]
\[ - 8t_1\lambda\mu Y_{N-1,p}, \]
\[ D_{N+2,p+1} = 2E_{N+1,p+1} - 4t_1\alpha Z_{N+1,p+1} - \beta D_{N+1,p+1} + 4t_1\xi Y_{N+1,p+1} + 2t_1\eta X_{N+1,p+1} \]
\[ + (-4t_0\xi - 4t_1\eta)Y_{N+1,p+1} + 4t_1\alpha\xi X_{N+1,p+1} + (16t_1\beta + 8t_1\gamma\delta - 8t_1\xi\delta + 8t_1\eta\delta + 16t_1\beta\xi Z_{N-1,p+1} \]
\[ - 8\xi\xi E_{N-1,p+1} + 4t_1\delta D_{N-1,p+1} + 4t_1\delta Y_{N-1,p+1} - 4t_1\lambda Z_{N,p} + (2t_1\nu + 2t_1\mu)Y_{N,p} \]
\[ - 4t_1\lambda E_{N-1,p} + 2t_1\nu D_{N-1,p} + (4t_1\mu + 8t_1\xi\mu)Z_{N-1,p}, \]
\[ C_{N+2,p+1} = 2\alpha E_{N+1,p+1} + (-4t_1 + 2U\alpha)Z_{N+1,p+1} + 2t_1\beta Y_{N+1,p+1} + (-4t_1 + U\eta)X_{N+1,p+1} \]
\[ + \eta C_{N-1,p+1} + 16t_1\lambda\delta Z_{N-1,p+1} - 8t_0\xi\xi Y_{N-1,p+1} + 4t_0\alpha Y_{N+1,p+1} - 4t_0\etd Z_{N,p} \]
\[ + 8t_0\lambda\mu Z_{N-1,p} - 4t_0\mu\nu Y_{N-1,p}. \]

(3)

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