Introduction to Bosonic String Theory

Eduard Alexis Larrañaga Rubio

Bogotá, D.C. November 2002
# Contents

## Introduction

## 1 Classical String Theory

1.1 World-Line of a Point Particle

1.1.1 Alternative Action for a point particle

1.2 World-Volume Action

1.3 Nambu-Goto Action

1.4 Polyakov Action

1.5 Symmetries of the Polyakov Action

1.6 Light-cone coordinates

1.7 Boundary Conditions

1.8 Oscillator expansions

1.9 Center of mass Position and Momentum

1.10 Classical Visaroro Algebra

## 2 Bosonic String Quantization

2.1 Covariant Canonical Quantization

2.1.1 Visaroro Operators

2.1.2 Bosonic String Spectrum

## Conclusion
Introduction

The development of fundamental physics in the past century arose from the identification and the overcoming of contradictions between the existing ideas. For example, the incompatibility of Maxwell's equations and Galilean invariance led Einstein to create the Special Theory of Relativity, and the inconsistency of this ones with Newtonian gravity produced the General Theory of Relativity. Same thing happened with the conciliation between special relativity and quantum mechanics, which led to the development of Quantum Field Theory.

Now, there is another incompatibility: General Relativity and Quantum Field theory. The quantization of gravity seems to be a non-renormalizable theory.

In the past year, String Theory has been the leading candidate for a theory that unifies all fundamental forces in nature in a consistent scheme. The way that string theory does this is to give up one of the basic assumptions of quantum field theory, that elementary particles are mathematical points, and instead to develop a quantum field theory of one-dimensional extended objects, called strings.

This string theory is still in progress, and there is not yet a complete description of the standard model of elementary particles based on it. However, there are some important features that seems to be generic in every kind of string theory: The first, and most important, is that general relativity is already incorporated in the theory. Although ordinary quantum field theory does not allow gravity, string theory requires it. The second fact is that the Yang-Mills gauge theories of the sort that comprise the standard model naturally arise in string theory, but there is not yet a fully understanding of why should we prefer the specific $SU(3) \otimes SU(2) \otimes U(1)$ gauge theory.

Here is given a little introduction to the most basic string theory: th bosonic string. First there is given the basics of the classical string theory as a generalization of the concept of world-line of a particle. Then the bosonic string is constructed and the oscillation modes are described. The covariant canonical quantization procedure is used and the bosonic spectra is shown.
INTRODUCTION
Chapter 1

Classical String Theory

1.1 World-Line of a Point Particle

Classically, a point particle follows a trajectory in space time that is called 'world-line'. The world line can be expressed as functions $x^\mu (\tau)$ with a parameter $\tau$. The path is an extremal of the action, that for a point particle of mass $m$ is proportional to the length of the world line:

$$S = m \int ds = m \int \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\tau$$  \hspace{1cm} (1.1)

where $\eta_{\mu\nu}$ is the flat Minkowski space-time metric. The momentum conjugate to $x^\mu (\tau)$ is:

$$p_\mu = -\frac{\partial L}{\partial \dot{x}^\mu} = -\frac{\partial}{\partial \dot{x}^\mu} \left( m \sqrt{-\dot{x}^\mu \dot{x}_\mu} \right) = m \frac{\dot{x}_\mu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}}$$  \hspace{1cm} (1.2)

$$p_\mu = m \frac{\dot{x}_\mu}{\sqrt{-\dot{x}^2}}$$  \hspace{1cm} (1.3)

This equation gives the constrain of mass-shell:

$$p^2 = -m^2$$  \hspace{1cm} (1.4)

The Lagrange equation from varying $S$ with respect to $x(\tau)$ is:

$$\frac{\partial L}{\partial x^\mu} - \frac{\partial}{\partial \tau} \left( \frac{\partial L}{\partial \dot{x}^\mu} \right) = 0$$  \hspace{1cm} (1.5)
\[
\frac{\partial}{\partial \tau} \left( \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}} \right) = 0 \quad (1.6)
\]

The canonical Hamiltonian is:
\[
H_{\text{can}} = \frac{\partial L}{\partial \dot{x}_\mu} \dot{x}^\mu - L = \frac{m\dot{x}_\mu \dot{x}^\mu}{\sqrt{-\dot{x}^2 \dot{x}_\mu}} - m \sqrt{-\dot{x}^2} \dot{x}_\mu \quad (1.7)
\]
\[
H_{\text{can}} = 0 \quad (1.8)
\]

So, \( H_{\text{can}} \) vanishes identically. This means that the constrain given in (1.4) governs the dynamics of the system completely. This constrain can be added to the Hamiltonian by using a Lagrange multiplier. This gives:
\[
H = \frac{N}{2m} \left( p^2 + m^2 \right) \quad (1.9)
\]

The equation of motion in Poisson bracket notation is:
\[
\dot{x}^\mu = \{ x^\mu, H \} = \frac{\partial H}{\partial p_\mu} = \frac{N}{2m} \frac{\partial (p^2 + m^2)}{\partial p_\mu} \quad (1.10)
\]
\[
\dot{x}^\mu = \frac{N}{m} p^\mu = N \frac{\dot{x}^\mu}{\sqrt{-\dot{x}^2}} \quad (1.11)
\]

So, it gives:
\[
\dot{x}^2 = -N^2 \quad (1.12)
\]

i.e. we are describing time-like trajectories.

The action in (1.1) is invariant under local reparametrizations. This kind of gauge invariance is represented by a change \( \tau \rightarrow \tilde{\tau} \). This reparametrization invariance is a one-dimensional analog of the four dimensional general coordinate invariance. The choice \( N = 1 \) corresponds to the called "static gauge":
\[
x^0 = \tau \quad (1.13)
\]

And in this gauge the action becomes:
\[
S = m \int \sqrt{1 - v^2} dt \quad (1.14)
\]

where the parameter is renamed \( t \) and:
1.1. WORLD-LINE OF A POINT PARTICLE

\[ \vec{v} = \frac{d\vec{x}}{dt} \] (1.15)

The equation of motion becomes:

\[ \frac{d\vec{p}}{dt} = 0 \] (1.16)

with:

\[ \vec{p} = m \frac{\vec{v}}{\sqrt{1 - v^2}} \] (1.17)

1.1.1 Alternative Action for a point particle

An alternative action for the point particle is:

\[ S = -\frac{1}{2} \int \left[ \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu e^{-1}(\tau) - m^2 e(\tau) \right] d\tau \] (1.18)

where \( e(\tau) \) (called the ‘einbein’) is a new independent function. This action allows to make the generalization to the massless case, and doesn’t have square roots like in (1.1) that makes the treatment of the quantum theory complicated. In order to prove that the action are equivalent, let’s make the variation of \( e(\tau) \):

\[ \delta S = \frac{1}{2} \int \left[ \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu e^{-2}(\tau) + m^2 \right] \delta e(\tau) d\tau \] (1.19)

Then, setting \( \delta S = 0 \) we obtain:

\[ \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu e^{-2}(\tau) + m^2 = 0 \] (1.20)

\[ e(\tau) = \frac{1}{m} \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \] (1.21)

If we substitute (1.21) in (1.18) we get:

\[ S = -\frac{1}{2} \int \left[ \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \left( \frac{1}{m} \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \right)^{-1} - m^2 \left( \frac{1}{m} \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \right) \right] d\tau \] (1.22)

\[ S = -\frac{1}{2} \int \left[ -m \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} - m \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \right] d\tau \] (1.23)
\[ S = \int m \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \, d\tau \]  

(1.24)

Hence, the actions are equivalent. Varying \( x^\mu \) in (1.18) we get:

\[ \delta S = -\frac{1}{2} \int [2 \dot{x}^\mu e^{-1}(\tau)] \delta (\dot{x}^\mu) \, d\tau \]  

(1.25)

\[ \delta S = - \int [\dot{x}^\mu e^{-1}(\tau)] \partial_\tau \delta x^\mu \, d\tau \]  

(1.26)

Integrating by parts we obtain:

\[ \delta S = - [\dot{x}^\mu e^{-1}(\tau)] \delta x^\mu + \int \partial_\tau [\dot{x}^\mu e^{-1}(\tau)] \delta x^\mu \, d\tau \]  

(1.27)

since at the extremes the variation is zero, then:

\[ \delta S = \int \partial_\tau [\dot{x}^\mu e^{-1}(\tau)] \delta x^\mu \, d\tau \]  

(1.28)

Then, setting \( \delta S = 0 \) gives:

\[ \partial_\tau [\dot{x}^\mu e^{-1}(\tau)] = 0 \]  

(1.29)

Substituting (1.21) we get:

\[ \partial_\tau \left[ \dot{x}^\mu \left( \frac{1}{m} \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \right)^{-1} \right] = 0 \]  

(1.30)

\[ \partial_\tau \left[ \frac{m \dot{x}^\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \right] = 0 \]  

(1.31)

and this is the same equation of motion as (1.6).
1.2 World-Volume Action

As we have seen, the action for a point particle was proportional to the length of its world-line. For strings, the action will be proportional to the surface area of its world-sheet, and in general for a \( p \)-brane the action involves the \((p+1)\)-dimensional volume:

\[
S = -T_p \int d\mu_{p+1}
\]

(1.32)

where the constant \( T_p \) makes the action dimensionless, so it has dimensions of \([\text{mass}]^{(p+1)} \) or \([\text{length}]^{-(p+1)} \). This factor is associated with the tension of the \( p \)-brane. For a 0-brane, (i.e. a point particle) it is just the mass.

Suppose that \( \xi^\alpha (\alpha = 0, 1, 2, ..., p+1) \) are the coordinates in the world-volume of the \( p \)-brane, and \( g_{\mu\nu} (\mu, \nu = 0, 1, 2, ..., d-1) \) is the metric of the \( d \)-dimensional space-time in which the \( p \)-brane propagates. Then, \( g_{\mu\nu} \) induces a metric in the world-volume given by:

\[
ds^2 = -g_{\mu\nu} dx^\mu dx^\nu = -g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta} d\xi^\alpha d\xi^\beta = G_{\alpha\beta} d\xi^\alpha d\xi^\beta
\]

(1.33)

where the induced metric is:

\[
G_{\alpha\beta} = -g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta}
\]

(1.34)

Then, the invariant volume element is given by:

\[
d\mu_{p+1} = \sqrt{-\det G_{\alpha\beta}} d^{p+1} \xi
\]

(1.35)

1.3 Nambu-Goto Action

In the case of 1-branes, i.e. strings, the world-volume action (1.32) becomes:

\[
S = -T \int \sqrt{-\det G_{\alpha\beta}} d^2 \xi
\]

(1.36)

If the space-time is flat Minkowski \( g_{\mu\nu} = \eta_{\mu\nu} \) we have:

\[
\sqrt{-\det G_{\alpha\beta}} = \sqrt{-\det \left( -\eta_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta} \right)}
\]

(1.37)

Defining \( \xi^0 = \tau \) and \( \xi^1 = \sigma \), then:
\[ \sqrt{-\det G_{\alpha\beta}} = \sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2} \]  
(1.38)

where:

\[ \dot{x} = \frac{\partial x}{\partial \tau} \quad x' = \frac{\partial x}{\partial \sigma} \]  
(1.39)

and the action for the string, called Nambu-Goto action becomes:

\[ S_{NG} = -T \int \sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2} d\sigma d\tau \]  
(1.40)

### 1.4 Polyakov Action

Again, as in the point particle case, the square root in the Nambu-Goto action make the quantum treatment complicated. So, we introduce an equivalent action:

\[ S_P = -\frac{T}{2} \int \sqrt{-\det h_{\alpha\beta}} \eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu d^2\xi \]  
(1.41)

where \( h_{\alpha\beta} (\sigma, \tau) \) is the world-sheet metric and \( \partial_\alpha x^\mu = \frac{\partial x^\mu}{\partial \xi_\alpha} \). The stress-tensor is defined as the variation of the action with respect to the metric:

\[ T_{\alpha\beta} \equiv -\frac{2}{T} \frac{1}{\sqrt{-\det h_{\alpha\beta}}} \frac{\delta S}{\delta h_{\alpha\beta}} \]  
(1.42)

\[ T_{\alpha\beta} = \eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \eta_{\mu\nu} \partial_\gamma x^\mu \partial_\delta x^\nu \]  
(1.43)

Hence, the Euler-Lagrange equation is:

\[ \partial_\alpha x \cdot \partial_\beta x - \frac{1}{2} h_{\alpha\beta} h_{\gamma\delta} \partial_\gamma x \cdot \partial_\delta x = 0 \]  
(1.44)

Solving for \( h_{\alpha\beta} \) we obtain:

\[ h_{\alpha\beta} = \partial_\alpha x \cdot \partial_\beta x = \eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \]  
(1.45)

Substituting this in (1.44) gives:

\[ h_{\alpha\beta} - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} h_{\gamma\delta} = h_{\alpha\beta} - \frac{1}{2} h_{\alpha\beta} \delta_\gamma^\gamma = h_{\alpha\beta} - h_{\alpha\beta} = 0 \]  
(1.46)
1.5. SYMMETRIES OF THE POLYAKOV ACTION

So, (1.45) means that the metric in the world-sheet $h_{\alpha\beta}$ is equal (at least classically) to the induced metric. Substituting this in the Polyakov action we obtain:

$$S_P = -\frac{T}{2} \int \sqrt{-\det (\eta_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu)} h_{\alpha\beta} h_{\alpha\beta} d^2 \xi$$  \hspace{1cm} (1.47)

$$S_P = -\frac{T}{2} \int \sqrt{-\det (\eta_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu)\delta_{\alpha}^\alpha d^2 \xi}$$  \hspace{1cm} (1.48)

$$S_P = -T \int \sqrt{-\det (\eta_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu) d^2 \xi} = S_{NG}$$  \hspace{1cm} (1.49)

Hence the Polyakov and the Nambu-Goto actions are equivalent, at least classically, since in the quantum treatment it is not in general.

1.5 Symmetries of the Polyakov Action

Polyakov action has the symmetries:

1. Poincaré invariance:

$$x^\mu \mapsto \omega^\mu_{\nu} x^\nu + a^\mu \hspace{1cm}, \hspace{1cm} h_{\alpha\beta} \mapsto h_{\alpha\beta} \hspace{1cm} (1.50)$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

2. Local 2-dimensional reparametrization invariance

$$\xi^\alpha \mapsto \xi'^\alpha (\xi^\beta) \hspace{1cm} (1.51)$$

3. Conformal or Weyl invariance

$$h_{\alpha\beta} \mapsto \Lambda (\xi^\gamma) h_{\alpha\beta} \hspace{1cm}, \hspace{1cm} x^\mu = x'^\mu \hspace{1cm} (1.52)$$

The two reparametrization invariance symmetries of $S$ allow us to choose a gauge in which the three independent components of $h_{\alpha\beta}$ are expressed with just one function. The usual choose is the conformal flat gauge:

$$h_{\alpha\beta} = e^{\Lambda(\xi)} \eta_{\alpha\beta} \hspace{1cm} (1.53)$$

where $\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2-dimensional Minkowski metric for a flat world-sheet. Substituting this in the action we obtain:
As can be seen, the gauge fixed action is quadratic in the $x$'s. Thus, mathematically it is the same as a theory of $d$ free scalar fields in two dimensions. Varying $x^\mu$ we obtain the equation of motion:

$$\ddot{x}^\mu - x''^\mu = 0 \quad (1.57)$$

This is simply a free two dimensional wave equation. Also, we have to take account of the constraints $T_{\alpha\beta} = 0$. In the conformal gauge this constrain is:

$$T_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\eta_{\mu\nu}\partial_\gamma x^\mu \partial_\delta x^\nu = 0 \quad (1.58)$$

This can be written as:

$$T_{00} = \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \frac{1}{2}\eta^{\gamma\delta}\eta_{\mu\nu}\partial_\gamma x^\mu \partial_\delta x^\nu = 0 \quad (1.59)$$

$$T_{00} = \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - \frac{1}{2}\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0 \quad (1.60)$$

$$T_{00} = T_{11} = \frac{1}{2}(\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu) = 0 \quad (1.61)$$

$$T_{01} = T_{10} = \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0 \quad (1.62)$$

Adding and substracting (1.61) and (1.62) gives the condition:

$$(\dot{x} \pm \dot{x'})^2 = 0 \quad (1.63)$$

Known as the Visaroro constraints, and they are the analog of the Gauss law in the string case.
1.6 Light-cone coordinates

Define the light-cone coordinates as:

\[ \xi^\pm = \tau \pm \sigma \]  

(1.64)

The flat metric becomes:

\[ ds^2 = -d\tau^2 + d\sigma^2 = -(d\tau + d\sigma)(d\tau - d\sigma) = -d\xi_+ d\xi_- \]  

(1.65)

Thus, the metric components are:

\[ g_{++} = g_{--} = 0 \quad , \quad g_{+-} = g_{-+} = -\frac{1}{2} \]  

(1.66)

and we have:

\[ \partial_\pm = \frac{1}{2} (\partial_\tau \pm \partial_\sigma) \]  

(1.67)

Then, the Polyakov action in the conformal gauge (1.56) may be written as:

\[ S = 2T \int \eta_{\mu\nu} \partial_+ x^\mu \partial_- x^\nu d^2 \xi \]  

(1.68)

The equations of motion from the Polyakov action (1.57) becomes:

\[ \ddot{x}^\mu - x''^\mu = \partial_\tau \partial_\tau x^\mu - \partial_\sigma \partial_\sigma x^\mu = 0 \]  

(1.69)

\[ (\partial_\tau + \partial_\sigma) (\partial_\tau x^\mu - \partial_\sigma x^\mu) = 0 \]  

(1.70)

\[ (\partial_\tau + \partial_\sigma) (\partial_\tau - \partial_\sigma) x^\mu = 0 \]  

(1.71)

\[ \partial_+ \partial_- x^\mu = 0 \]  

(1.72)

The stress-tensor in light-coordinates is:

\[ T_{++} = \frac{1}{2} \eta_{\mu\nu} \partial_+ x^\mu \partial_+ x^\nu = \frac{1}{2} \eta_{\mu\nu} (\dot{x}^\mu + x'^\mu) (\dot{x}^\nu + x'^\nu) = T_{00} + 2T_{10} \]  

(1.73)

\[ T_{--} = \frac{1}{2} \eta_{\mu\nu} \partial_- x^\mu \partial_- x^\nu = \frac{1}{2} \eta_{\mu\nu} (\dot{x}^\mu - x'^\mu) (\dot{x}^\nu - x'^\nu) = T_{00} - 2T_{10} \]  

(1.74)


\[ T_{+-} = T_{-+} = 0 \quad (1.75) \]

In these coordinates, the Energy-momentum conservation is given by:

\[ \nabla^\alpha T_{\alpha\beta} = -\frac{1}{2} (\partial_- T_{++} + \partial_+ T_{--}) = -\frac{1}{2} (\partial_+ T_{--} + \partial_- T_{++}) = 0 \quad (1.76) \]

but using (1.75) we obtain:

\[ \partial_- T_{++} = \partial_+ T_{--} = 0 \quad (1.77) \]

And finally, the Visaroro constraints (1.63) can be written as:

\[ T_{--} = \frac{1}{2} \eta_{\mu\nu} \partial_\mu x^\nu \partial_- x^\nu = 0, \quad T_{++} = \frac{1}{2} \eta_{\mu\nu} \partial_+ x^\mu \partial_+ x^\nu = 0 \quad (1.78) \]

### 1.7 Boundary Conditions

In string theory there are three important types:

In the case of **closed strings** the world-sheet is a tube, and one should impose periodicity in the spatial parameter \( \sigma \). Usually the period is chosen to be \( \pi \):

\[ x^\mu (\sigma + \pi, \tau) = x^\mu (\sigma, \tau) \quad (1.79) \]

For **open strings** the world-sheet is a strip. The conditions used are:

For **Neumann**:

\[ \frac{\partial x^\mu}{\partial \sigma} = 0 \quad \text{at} \quad \sigma = 0 \text{ or } \pi \quad (1.80) \]

For **Dirichlet**:

\[ \frac{\partial x^\mu}{\partial \tau} = 0 \quad \text{at} \quad \sigma = 0 \text{ or } \pi \quad (1.81) \]

Neumann condition imply that no momentum flows off the ends of the string. Dirichlet condition implies that the end points of the string are fixed in space-time, and that they ends on a physical object called \( D \)-brane ( \( D \) stands for Dirichlet).
1.8 Oscillator expansions

As we have seen the equation of motion for the string is (1.57):

\[ x^\mu - x'^{\mu} = 0 \]  \hspace{1cm} (1.82)

but we must treat each case of boundary condition separately:

1. Closed Strings

For closed strings the general solution of the two-dimensional wave equation is a sum of 'right' movers and 'left' movers:

\[ x^\mu (\sigma, \tau) = x^\mu_R (\tau - \sigma) + x^\mu_L (\tau + \sigma) \]  \hspace{1cm} (1.83)

This solution must satisfy the conditions:

- \( x^\mu (\sigma, \tau) \) is real
- \( x^\mu (\sigma + \pi, \tau) = x^\mu (\sigma, \tau) \)

This condition can be solved in terms of Fourier series as:

\[ x^\mu_R (\tau - \sigma) = \frac{1}{2} x^\mu_o + \frac{1}{2\pi T} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in(\tau - \sigma)} \]  \hspace{1cm} (1.84)

\[ x^\mu_L (\tau + \sigma) = \frac{1}{2} x^\mu_o + \frac{1}{2\pi T} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in(\tau + \sigma)} \]  \hspace{1cm} (1.85)

where \( \alpha_n^\mu \) and \( \tilde{\alpha}_n^\mu \) are Fourier modes that must satisfy (in order to \( x^\mu (\sigma, \tau) \) to be real):

\[ (\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu \quad \text{and} \quad (\tilde{\alpha}_n^\mu)^\dagger = \tilde{\alpha}_{-n}^\mu \]  \hspace{1cm} (1.86)

The center of mass coordinate \( x^\mu \) and the momentum \( p^\mu \) are real too, and the factor :

\[ \ell_s = \frac{1}{\sqrt{2\pi T}} \]  \hspace{1cm} (1.87)

is called the fundamental string length scale. So, the solution is:

\[ x^\mu (\sigma, \tau) = x^\mu_o + \frac{1}{\pi T} p^\mu \tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} \left( \alpha_n^\mu e^{in\sigma} + \tilde{\alpha}_n^\mu e^{-in\sigma} \right) e^{-i n \tau} \]  \hspace{1cm} (1.88)
CHAPTER 1. CLASSICAL STRING THEORY

Now, defining
\[ \alpha_0 = \tilde{\alpha}_0 = \frac{1}{\sqrt{\pi T}} \]
we can write in light-cone coordinates:
\[ \partial_{-} x^\mu (\sigma, \tau) = \partial_{-} x^\mu_R = \frac{1}{\sqrt{4\pi T}} \sum_n \alpha_n^\mu e^{-in(\tau - \sigma)} \]  
\[ \partial_{+} x^\mu (\sigma, \tau) = \partial_{+} x^\mu_L = \frac{1}{\sqrt{4\pi T}} \sum_n \tilde{\alpha}_n^\mu e^{-in(\tau + \sigma)} \]  

2. Open Strings

In the case of open strings we will use the Neumann boundary conditions:
\[ \frac{\partial x^\mu}{\partial \sigma} = 0 \quad \text{at} \; \sigma = 0 \; \text{or} \; \pi \]

Using the solution (1.83) with the expression for right and left movers (1.84) and (1.85) and substituting in the Neumann condition we get:
\[ \frac{\partial x^\mu}{\partial \sigma} \bigg|_{\sigma=0} = \frac{\partial x^\mu_R}{\partial \sigma} \bigg|_{\sigma=0} + \frac{\partial x^\mu_L}{\partial \sigma} \bigg|_{\sigma=0} = 0 \]
\[ -\frac{1}{2\pi T} p^\mu - \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} \alpha_n^\mu e^{-in\tau} + \frac{1}{2\pi T} \bar{p}^\mu + \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} \tilde{\alpha}_n^\mu e^{-in\tau} = 0 \]
\[ \frac{1}{2\pi T} (\bar{p}^\mu - p^\mu) + \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} (\tilde{\alpha}_n^\mu - \alpha_n^\mu) e^{-in\tau} = 0 \]

Then we obtain:
\[ \bar{p}^\mu = p^\mu \]
\[ \tilde{\alpha}_n^\mu = \alpha_n^\mu \]

This means that the right and left movers get mixed in the solution for the open string. Now, using the boundary condition at \( \sigma = \pi \) we get:
\[ \frac{\partial x^\mu}{\partial \sigma} \bigg|_{\sigma=\pi} = \frac{\partial x^\mu_R}{\partial \sigma} \bigg|_{\sigma=\pi} + \frac{\partial x^\mu_L}{\partial \sigma} \bigg|_{\sigma=\pi} = 0 \]
\[ -\frac{1}{2\pi T} p^\mu \pi - \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} \alpha_n^\mu e^{-in(\tau - \pi)} + \frac{1}{2\pi T} p^\mu \pi + \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} \alpha_n^\mu e^{-in(\tau + \pi)} = 0 \]
1.9. CENTER OF MASS POSITION AND MOMENTUM

\[ -\frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} \alpha_n^\mu e^{-in(\tau - \pi)} + \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} \alpha_n^\mu e^{-in(\tau + \pi)} = 0 \] (1.100)

\[ \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} \alpha_n^\mu (e^{-i\pi} - e^{i\pi}) = 0 \] (1.101)

Then, we get the condition:

\[ e^{-i\pi} - e^{i\pi} = 0 \] (1.102)

\[ -2i \sin (n\pi) = 0 \] (1.103)

So, \( n \) must be an integer. The solution now becomes:

\[ x^\mu (\tau, \sigma) = x_0^\mu + \frac{p_\mu}{\pi T} \tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \left( e^{in\sigma} + e^{-in\sigma} \right) \] (1.104)

\[ x^\mu (\tau, \sigma) = x_0^\mu + \frac{p_\mu}{\pi T} \tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} 2\cos (n\sigma) \] (1.105)

\[ x^\mu (\tau, \sigma) = x_0^\mu + \frac{p_\mu}{\pi T} \tau + \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos (n\sigma) \] (1.106)

Defining again \( \alpha_0^\mu = \frac{1}{\sqrt{\pi T}} p_\mu \) we get:

\[ \partial_\pm x^\mu (\tau, \sigma) = \frac{1}{2\pi T} p^\mu + \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} \alpha_n^\mu e^{-in(\tau \pm \sigma)} \] (1.107)

\[ \partial_\pm x^\mu (\tau, \sigma) = \frac{1}{\sqrt{4\pi T}} \sum_{n} \alpha_n^\mu e^{-in(\tau \pm \sigma)} \] (1.108)

1.9 Center of mass Position and Momentum

The position and momentum of the center of mass for the open and closed strings can be calculated, and they are given by:

\[ X^\mu_{\text{CM}} = \frac{1}{\pi} \int_{0}^{\pi} x^\mu (\tau, \sigma) d\sigma \] (1.109)

\[ X^\mu_{\text{CM}} = \frac{1}{\pi} \int_{0}^{\pi} x_0^\mu + \frac{p_\mu}{\pi T} \tau + \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos (n\sigma) d\sigma \] (1.110)

\[ X^\mu_{\text{CM}} = \frac{1}{\pi} \left[ x_0^\mu + \frac{p_\mu}{\pi T} \tau + \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \sin (n\sigma) \right] \] (1.111)


\[ X^\mu_{CM} = x^\mu_o + \frac{p^\mu}{\pi T} \tau \quad (1.112) \]

\[ p^\mu_{CM} = \int_0^\pi \Pi^\mu d\sigma = T \int_0^\pi \dot{x}^\mu d\sigma \quad (1.113) \]

\[ p^\mu_{CM} = T \int_0^\pi \left[ \frac{1}{\pi T} b^\mu + \frac{1}{\sqrt{\pi T}} \sum_{n \neq 0} \alpha_n^\mu e^{-in\tau} \cos (n\sigma) \right] d\sigma \quad (1.114) \]

\[ p^\mu_{CM} = p^\mu \quad (1.115) \]

We can see that \( x^\mu_o \) is the position of the center of mass at \( \tau = 0 \) and it moves as a free particle. The variables that describe the motion of the string are \( x^\mu_o \) and \( p^\mu \) plus the collection of \( \alpha_n^\mu \). This means that the string moves as a whole (center of mass) and vibrate in various modes.

### 1.10 Classical Visaroro Algebra

The equal-\( \tau \) Poison Brackets for the \( x^\mu \) and their conjugate momenta \( \Pi^\mu = T\dot{x}^\mu \) are:

\[ \{ x^\mu (\sigma, \tau), \dot{x}'^\nu (\sigma', \tau) \}_{PB} = \frac{1}{T} \eta^\mu^\nu \delta (\sigma - \sigma') \quad (1.116) \]

\[ \{ x^\mu (\sigma, \tau), x'^\nu (\sigma', \tau) \}_{PB} = 0 \quad (1.117) \]

\[ \{ \dot{x}^\mu (\sigma, \tau), \dot{x}'^\nu (\sigma', \tau) \}_{PB} = 0 \quad (1.118) \]

Using the expression for \( x^\mu (\sigma, \tau) \) we get:

\[ \{ \alpha_m^\mu, \alpha_n^\nu \} = \{ \bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu \} = -im\eta^\mu^\nu \delta_{m+n,0} \quad (1.119) \]

\[ \{ \bar{\alpha}_m^\mu, \alpha_n^\nu \} = 0 \quad (1.120) \]

\[ \{ \dot{x}_0^\mu, p^\nu \} = \eta^\mu^\nu \quad (1.121) \]

The Hamiltonian is given by:

\[ H = \int (\dot{x}^\mu \Pi^\mu - L) d\sigma = T \int (\dot{x}^\mu \dot{x}_\mu - L) d\sigma \quad (1.122) \]

\[ H = \int \left( T \dot{x}^\mu \dot{x}_\mu + \frac{T}{2} \eta^\alpha^\beta \eta_{\mu \nu} \partial_\alpha x^\mu \partial_\beta x^\nu \right) d\sigma \quad (1.123) \]
1.10. CLASSICAL VISARORO ALGEBRA

\[ H = \int \left[ T \dot{x}^\mu \dot{x}_\mu + \frac{T}{2} (-\dot{x}^\mu \dot{x}_\mu + x'^\mu x'_\mu) \right] d\sigma \quad (1.124) \]

\[ H = \frac{T}{2} \int [x^\mu \dot{x}_\mu + x'^\mu x'_\mu] d\sigma \quad (1.125) \]

This Hamiltonian can be expressed in terms of the oscillators in the closed string case, using \( \alpha^\mu_0 = \frac{1}{\sqrt{\pi T}} p^\mu \), as:

\[ H = \frac{T}{2} \int \frac{1}{4\pi T} \left( \sum_{m} \left( \left( \alpha^\mu_m \right)^{\dagger} e^{-im\sigma} + \left( \bar{\alpha}^\mu_m \right)^{\dagger} e^{im\sigma} \right) e^{im\tau} \right) \left( \sum_{n} \left( \left( \alpha_n \right)^\mu e^{in\sigma} + \left( \bar{\alpha}_n \right)^\mu e^{-in\sigma} \right) e^{-in\tau} \right) d\sigma \]

Since:

\[ \int e^{-im\sigma} e^{im\sigma} d\sigma = 2\pi \delta (m - n) \quad (1.126) \]

\[ \int e^{im\sigma} e^{in\sigma} d\sigma = 2\pi \delta (m + n) \quad (1.127) \]

we obtain:

\[ H = \frac{T}{2} \frac{1}{4\pi T} \left[ 2\pi \sum_{n} \left( \alpha_{-n} \cdot \alpha_n + \bar{\alpha}_n \cdot \alpha_{-n} e^{-2in\tau} + \alpha_n \cdot \bar{\alpha}_{-n} e^{-2in\tau} + \bar{\alpha}_{-n} \cdot \bar{\alpha}_n \right) \right. \]

\[ + 2\pi \sum_{n \neq 0} \left( \alpha_{-n} \cdot \alpha_n - \bar{\alpha}_n \cdot \alpha_{-n} e^{-2in\tau} - \alpha_n \cdot \bar{\alpha}_{-n} e^{-2in\tau} + \bar{\alpha}_{-n} \cdot \bar{\alpha}_n \right) \] \quad (1.128)

\[ H = \frac{T}{2} \frac{2\pi}{4\pi T} \left[ \alpha_0 \cdot \bar{\alpha}_0 + \bar{\alpha}_0 \cdot \alpha_0 + \sum_{n} \left( \alpha_{-n} \cdot \alpha_n + \bar{\alpha}_n \cdot \bar{\alpha}_{-n} \right) + \sum_{n \neq 0} \left( \alpha_{-n} \cdot \alpha_n + \bar{\alpha}_{-n} \cdot \bar{\alpha}_n \right) \right] \] \quad (1.129)
\[ H = \frac{1}{4} \left[ 2 (\alpha_0 \cdot \alpha_0 + \tilde{\alpha}_0 \cdot \tilde{\alpha}_0) + \sum_{n \neq 0} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) + \sum_{n \neq 0} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \right] \]  

(1.130)

\[ H = \frac{1}{4} \left[ 2 (\alpha_0 \cdot \alpha_0 + \tilde{\alpha}_0 \cdot \tilde{\alpha}_0) + \sum_{n \neq 0} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) + \sum_{n \neq 0} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \right] \]  

(1.131)

\[ H = \frac{1}{4} \left[ 2 (\alpha_0 \cdot \alpha_0 + \tilde{\alpha}_0 \cdot \tilde{\alpha}_0) + 2 \sum_{n \neq 0} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \right] \]  

(1.132)

\[ H = \frac{1}{2} \left[ (\alpha_0 \cdot \alpha_0 + \tilde{\alpha}_0 \cdot \tilde{\alpha}_0) + \sum_{n \neq 0} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \right] \]  

(1.133)

\[ H = \frac{1}{2} \sum_n (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \]  

(1.134)

For the open string using again \( \alpha_0^\mu = \frac{p^\mu}{\sqrt{\pi T}} \), we have:

\[ H = \frac{T}{2} \int \frac{1}{4\pi T} \left( \sum_m (\alpha_m^\mu) \left( e^{-in\sigma} + e^{i\sigma\sigma} \right) e^{im\tau} \right) \left( \sum_n (\alpha_n^\mu) \left( e^{i\sigma\sigma} + e^{-in\sigma} \right) e^{-in\tau} \right) \right) \]  

\[ + \frac{1}{4\pi T} \left( \sum_{m \neq 0} (\alpha_m^\mu) \left( -e^{-in\sigma} + e^{i\sigma\sigma} \right) e^{im\tau} \right) \left( \sum_{n \neq 0} (\alpha_n^\mu) \left( -e^{i\sigma\sigma} + e^{-in\sigma} \right) e^{-in\tau} \right) d\sigma \]  

\[ H = \frac{T}{2} \int \frac{1}{4\pi T} \left( \sum_m \alpha_{-m}^\mu \left( e^{-im\sigma} + e^{i\sigma\sigma} \right) e^{im\tau} \right) \left( \sum_n \alpha_n^\mu \left( e^{i\sigma\sigma} + e^{-in\sigma} \right) e^{-in\tau} \right) \right) \]  

\[ + \frac{1}{4\pi T} \left( \sum_{m \neq 0} \alpha_{-m}^\mu \left( -e^{-in\sigma} + e^{i\sigma\sigma} \right) e^{im\tau} \right) \left( \sum_{n \neq 0} \alpha_n^\mu \left( -e^{i\sigma\sigma} + e^{-in\sigma} \right) e^{-in\tau} \right) d\sigma \]
\[ H = \frac{T}{2} \frac{1}{4\pi T} \left[ \pi \sum_n (\alpha_{-n} \cdot \alpha_n + \alpha_n \cdot \alpha_n e^{-2in\tau} + \alpha_n \cdot \alpha_n e^{-2in\tau} + \alpha_{-n} \cdot \alpha_n) \right] \]

\[ + \pi \sum_{n \neq 0} (\alpha_{-n} \cdot \alpha_n - \alpha_n \cdot \alpha_n e^{-2in\tau} - \alpha_n \cdot \alpha_n e^{-2in\tau} + \alpha_{-n} \cdot \alpha_n) \right] \] (1.135)

\[ H = \frac{T}{2} \frac{\pi}{4\pi T} \left[ \alpha_0 \cdot \alpha_0 + \sum_n (\alpha_{-n} \cdot \alpha_n + \alpha_n \cdot \alpha_n) + \sum_{n \neq 0} (\alpha_{-n} \cdot \alpha_n + \alpha_{-n} \cdot \alpha_n) \right] \] (1.136)

\[ H = \frac{1}{8} \left[ 2(\alpha_0 \cdot \alpha_0 + \alpha_0 \cdot \alpha_0) + \sum_{n \neq 0} 2\alpha_{-n} \cdot \alpha_n + \sum_{n \neq 0} 2\alpha_{-n} \cdot \alpha_n \right] \] (1.137)

\[ H = \frac{1}{8} \left[ 4(\alpha_0 \cdot \alpha_0) + 4\sum_{n \neq 0} \alpha_{-n} \cdot \alpha_n \right] \] (1.138)

\[ H = \frac{1}{2} \left[ (\alpha_0 \cdot \alpha_0) + \sum_{n \neq 0} \alpha_{-n} \cdot \alpha_n \right] \] (1.139)

\[ H = \frac{1}{2} \sum_{n} \alpha_{-n} \cdot \alpha_n \] (1.140)

The Visaroro operators are defined as the Fourier modes of the Stress tensor. For the closed string they are:

\[ L_m = \frac{1}{2} \int_0^{2\pi} T_- e^{im(\tau - \sigma)} d\sigma \quad , \quad \bar{L}_m = \frac{1}{2} \int_0^{2\pi} T_+ e^{im(\tau + \sigma)} d\sigma \] (1.141)

Expressed in oscillators these operators are:

\[ L_m = \frac{1}{2} \sum_n \alpha_{m-n} \cdot \alpha_n \quad , \quad \bar{L}_m = \frac{1}{2} \sum_n \bar{\alpha}_{m-n} \cdot \bar{\alpha}_n \] (1.142)

Because of the properties of the \( \alpha \)'s, the operators satisfy:
In terms of the Virasoro operators, the Hamiltonian (1.134) can be expressed as:

\[ H = L_0 + \tilde{L}_0 \] (1.145)

For the open string the \( \tilde{\alpha}'s \) are equal to the \( \alpha's \), then the Virasoro operators are defined as:

\[ L_m = \frac{1}{2} \int_0^\pi \left[ T_- e^{im(\tau - \sigma)} + T_+ e^{im(\tau + \sigma)} \right] d\sigma \] (1.146)

or in terms of the oscillators:

\[ L_m = \frac{1}{2} \sum_n \alpha_{m-n} \cdot \alpha_n \] (1.147)

In this case, the Hamiltonian (1.140) can be written as:

\[ H = L_0 \] (1.148)

Using (1.119), (1.120) and (1.121) is easy to derive the Poisson brackets for the Virasoro operators:

\[ \{ L_m, L_n \}_{PB} = -i (m - n) L_{m+n} \] (1.149)

\[ \{ \tilde{L}_m, \tilde{L}_n \}_{PB} = -i (m - n) \tilde{L}_{m+n} \] (1.150)

\[ \{ L_m, \tilde{L}_n \}_{PB} = 0 \] (1.151)

For the open string the \( \tilde{L}'s \) operators are absent.
Chapter 2

Bosonic String Quantization

There exist three approaches to quantize classical strings:

1. **Canonical Quantization:**
   In this approach the classical variables become operators. Because of the constraints there are two options:
   
   (a) **Covariant Quantization**
   This procedure quantizes first and then imposes the constraints as conditions on states in Hilbert space. It is named covariant because it preserves the Lorentz invariance.
   
   (b) **Light-Cone Quantization**
   Here, the constraints are solved in the classical system, so it leaves a smaller number of classical variables. Then they are quantized. In this approach the manifest Lorentz invariance is lost.

2. **Path Integral Quantization**
   This way preserves the manifest Lorentz invariance, but contains ghost fields.

2.1 **Covariant Canonical Quantization**

In the canonical quantization the fields are replaced by operators, and the Poisson brackets are replaced by commutators following the rule:

\[
\{ \, \, \}^P_B \rightarrow -i \{ \, \, \} \quad (2.1)
\]

Following this rule, the relations \(\{1.1.6\} - \{1.1.8\}\) become:
\[ x^\mu (\sigma, \tau), \Pi^\nu (\sigma', \tau) = i\eta^{\mu\nu} \delta (\sigma - \sigma') \quad (2.2) \]
\[ [x^\mu (\sigma, \tau), x^\nu (\sigma', \tau)] = 0 \quad (2.3) \]
\[ \left[ \Pi^\mu (\sigma, \tau), \Pi^\nu (\sigma', \tau) \right] = 0 \quad (2.4) \]

Or in terms of the oscillator we can write:
\[ [\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m \eta^{\mu\nu} \delta_{m+n,0} \quad (2.5) \]
\[ [\tilde{\alpha}^\mu_m, \alpha^\nu_n] = 0 \quad (2.6) \]
\[ [x^\mu_0, p^\nu] = i\eta^{\mu\nu} \quad (2.7) \]

The first of this conditions can be written, absorbing the factor \( m \) as:
\[ \left[ \alpha^\mu_m, (\alpha^\nu_n)^\dagger \right] = \left[ \tilde{\alpha}^\mu_m, (\tilde{\alpha}^\nu_n)^\dagger \right] = \eta^{\mu\nu} \delta_{m,n} \quad (2.8) \]

In Quantum Mechanics the harmonic oscillator can be described using raising and lowering operators that satisfy the condition:
\[ [\alpha, \alpha^\dagger] = 1 \quad (2.9) \]

Then, we see that the expansion coefficients \( \alpha^\mu_n = (\alpha_n^\mu)^\dagger \) and \( \alpha^\mu_n \) are raising and lowering operators respectively. But there are one problem. Because of the metric tensor sign: \( \eta^{00} = -1 \) the time component of the oscillators has a minus sign. this means that they create states of negative norm, and it makes the quantum theory inconsistent. However, as we will see the classical condition \( T_{\alpha\beta} = 0 \) will eliminate the negative-norm states from physical spectrum.

Since the system is now an infinite set of harmonic oscillators, we may define the Hilbert space in a simple way. The ground-state is the one that is annihiliated by all the lowering operators, but since this state is not completly determined by this, we use the center-of-mass momentum operator. Diagonalizing \( p^\mu \) the states are caracterized by its momentum. So, the ground state may be written as:
\[ \alpha_m |0, p\rangle = 0 \quad \forall m > 0 \quad (2.10) \]

Other states are obtained by aplying the raising operators:
\[ \alpha^\mu_{-1} |0, p\rangle \ , \ \alpha^\mu_{-2} |p\rangle \ , \ \prod_m \alpha^\mu_m |0, p\rangle \quad (2.11) \]
2.1. COVARIANT CANONICAL QUANTIZATION

2.1.1 Visaroro Operators

In order to define the Visaroro operators in the quantum system we have to introduce the normal ordering operation that puts all positive frequency modes (lowering operators) to the right of the negative frequency modes (raising operators).

With this definition the Visaroro operators are defined as:

\[ L_m = \frac{1}{2} \sum_n : \alpha_{m-n} : \alpha_n : \]  \hspace{1cm} (2.12)

and similarly for the closed string:

\[ \tilde{L}_m = \frac{1}{2} \sum_n : \tilde{\alpha}_{m-n} : \tilde{\alpha}_n : \]  \hspace{1cm} (2.13)

From now on, for the closed string there are similar relations for the operators \( \tilde{L}_m \).

The only operator affected by the normal ordering is \( L_0 \):

\[ L_0 = \frac{1}{2} \alpha_0 \cdot \alpha_0 + \frac{1}{2} \sum_n : \alpha_{-n} : \alpha_n : \]  \hspace{1cm} (2.14)

\[ L_0 = \frac{1}{2} \alpha_0 \cdot \alpha_0 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \]  \hspace{1cm} (2.15)

Now, because of the arbitrariness in defining the normal ordering we have to include a normal-ordering constant \( q \) in the expression for \( L_0 \):

\[ L_0 \rightarrow L_0 - q \]  \hspace{1cm} (2.16)

The visaroro algebra can be calculated using the definition of the operators, and it gives (for a detailed calculation see [4]):

\[ [L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} m (m^2 - 1) \delta_{m+n,0} \]  \hspace{1cm} (2.17)

It’s important to note that there exist a difference between the operator and the Poisson algebra. This difference is a c-number which depends on the dimension of space-time, since in the case we have \( c = d \). This term is called the "conformal anomaly term" and the constant \( c \) is in general called the "central charge". This term is an inescapable consequence of the quantization, and is because of this that the string theory depends so much on the dimension of space-time.

The classical Visaroro constraints (1.63) cannot be imposed as operators constraints \( L_m |\psi\rangle = 0 \), because (2.17) gives:
\[ \langle \psi | [L_m, L_{-m}] | \psi \rangle = 2m \langle \psi | L_0 | \psi \rangle + \frac{d}{12} m (m^2 - 1) \langle \psi | \psi \rangle \neq 0 \quad (2.18) \]

Thus, the constraints are imposed as:

\[ (L_0 - q) | \phi \rangle = 0 \quad (2.19) \]

\[ L_m | \phi \rangle = 0 \quad m > 0 \quad (2.20) \]

where \( | \phi \rangle \) denotes a physical state. This is consistent with the classical constraints because:

\[ \langle \phi' | (L_m - q \delta_{m,0}) | \phi \rangle = 0 \quad \text{for all} \ m \quad (2.21) \]

As we will see, the constraint for \( L_0 \) is a generalization of the Klein-Gordon equation since it contains a term with \( p^2 = -\partial \cdot \partial \) and a term that determines the mass of the state.

### 2.1.2 Bosonic String Spectrum

**Open String.**

Using the form of \( L_0 \) given by (2.15), the constraint (2.19) gives:

\[ \frac{1}{2} \alpha_0 \cdot \alpha_0 + \sum_n \alpha_{-n} \cdot \alpha_n = q \quad (2.22) \]

\[ \frac{1}{2} \alpha_0 \cdot \alpha_0 = - \sum_n \alpha_{-n} \cdot \alpha_n + q \quad (2.23) \]

\[ \frac{1}{2\pi T} p^\mu p_\mu = - \sum_n \alpha_{-n} \cdot \alpha_n + q \quad (2.24) \]

And using the mass-shell condition \( p^\mu p_\mu = -m^2 \), we have:

\[ \frac{1}{2\pi T} m^2 = N - q \quad (2.25) \]

where is defined the level-number operator as:

\[ N = \sum_n \alpha_{-n} \cdot \alpha_n \quad (2.26) \]

Now, we will set \( q = 1 \) since this value is necessary for a consistent theory (as we will see later). Since each \( \alpha_{-n} \cdot \alpha_n \) has eigenvalues 0, 1, 2, 3, ... etc. then
2.1. COVARIANT CANONICAL QUANTIZATION

The number operator also has values 0, 1, 2, ... The ground state, $|0, p\rangle$ is given by $N = 0$. So we have:

$$\frac{1}{2\pi T}m^2 = -1$$  \hspace{1cm} (2.27)

This state corresponds to a Tachyon (since $p^\mu$ is space-like). This particle may cause a vacuum unstability, then it is not possible in a consistent quantum theory. Anyway, we will not worry about it now and will continue with the spectrum analysis.

The first excited state has $N = 1$ and (2.25) gives $m = 0$. To obtain this state we need:

$$|\phi\rangle = \xi_\mu \alpha^-_1 |0, p\rangle$$  \hspace{1cm} (2.28)

where $\xi^\mu$ is the polarization vector for a massless spin-1 particle. Now, the Visaroro constraint gives:

$$L_1 |\phi\rangle = 0$$  \hspace{1cm} (2.29)

$$\xi_\mu L_1 \alpha^-_1 |0, p\rangle = 0$$  \hspace{1cm} (2.30)

Since the operators $\alpha_m$ ($m > 0$) anihilates the ground-state, $L_1$ has just the contributions:

$$\xi_\mu \left( \frac{1}{2} \sum_n : \alpha_{1-n} \cdot \alpha_n : \right) \alpha^-_1 |0, p\rangle = 0$$  \hspace{1cm} (2.31)

$$\frac{1}{2} \xi_\mu (\alpha_0 \cdot \alpha_1 + \alpha_1 \cdot \alpha_0) \alpha^-_1 |0, p\rangle = 0$$  \hspace{1cm} (2.32)

$$\xi_\mu (\alpha_0)_\nu (\alpha^\mu_{\nu1} \alpha^-_1 |0, p\rangle = 0$$  \hspace{1cm} (2.33)

$$\xi_\mu (\alpha_0)_\nu \left( \alpha^\mu_{\nu1} \alpha^-_1 + \eta^{\mu\nu} \right) |0, p\rangle = 0$$  \hspace{1cm} (2.34)

$$\xi_\mu (\alpha_0)_\nu \alpha^\mu_{\nu1} |0, p\rangle + \xi_\mu (\alpha_0)_\nu \eta^{\mu\nu} |0, p\rangle = 0$$  \hspace{1cm} (2.35)

$$\xi_\mu (\alpha_0)^\mu |0, p\rangle = 0$$  \hspace{1cm} (2.36)

$$\xi_\mu \frac{1}{\sqrt{\pi T}} p^\mu |0, p\rangle = 0$$  \hspace{1cm} (2.37)

$$\xi_\mu p^\mu |0, p\rangle = 0$$  \hspace{1cm} (2.38)
Then, the Visaroro constraint implies that $\xi_\mu$ must satisfy:

$$p^\mu \xi_\mu = 0 \quad (2.39)$$

This condition ensures that the spin is transversely polarized, so there are just $d-2$ independent polarization states. The second excited state, with $N = 2$, has:

$$\frac{1}{2\pi T} m^2 = 1 \quad (2.40)$$

The most general state is a superposition:

$$|\phi\rangle = (\xi_\mu \alpha_2^\mu + \lambda_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu) |0, p\rangle \quad (2.41)$$

And again, the Visaroro constraints

$$L_1 |\phi\rangle = L_2 |\phi\rangle = 0 \quad (2.42)$$

restrict the values of $\xi_\mu$ and $\lambda_{\mu\nu}$.

**Closed String.**

For the closed string we have the conditions:

$$(L_0 - q) |\phi\rangle = 0 \quad (2.43)$$

$$\left(\tilde{L}_0 - q\right) |\phi\rangle = 0 \quad (2.44)$$

the sum of this conditions gives:

$$\left( L_0 + \tilde{L}_0 - 2q \right) |\phi\rangle = 0 \quad (2.45)$$

So:

$$\frac{1}{2} \alpha_0 \cdot \alpha_0 + \sum_n \alpha_{-n} \cdot \alpha_n + \frac{1}{2} \tilde{\alpha}_0 \cdot \tilde{\alpha}_0 + \sum_n \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = 2q \quad (2.46)$$

$$\alpha_0 \cdot \alpha_0 = -\sum_n \alpha_{-n} \cdot \alpha_n - \sum_n \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + 2q \quad (2.47)$$

$$\frac{1}{\pi T} p^\mu p_\mu = -m^2 \quad (2.48)$$

Using the mass-shell condition $p^\mu p_\mu = -m^2$, we have:

$$\frac{1}{\pi T} m^2 = N + \tilde{N} - 2q \quad (2.49)$$
where the level-number operators are:

\[ N = \sum_n \alpha_{-n} \cdot \alpha_n \] (2.50)

\[ \tilde{N} = \sum_n \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \] (2.51)

The difference between (2.43) and (2.44) gives:

\[ (L_0 - \tilde{L}_0) |\phi\rangle = 0 \] (2.52)

Then we have the level-matching condition:

\[ L_0 = \tilde{L}_0 \] (2.53)

\[ \frac{1}{2} \alpha_0 \cdot \alpha_0 + \sum_n \alpha_{-n} \cdot \alpha_n = \frac{1}{2} \tilde{\alpha}_0 \cdot \tilde{\alpha}_0 + \sum_n \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \] (2.54)

\[ \sum_n \alpha_{-n} \cdot \alpha_n = \sum_n \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \] (2.55)

\[ N = \tilde{N} \] (2.56)

The closed string states are then the tensor product of left moving and right moving states subject to the level matching condition. Then, the ground state for the closed string is (without displaying the momentum of the state):

\[ |0\rangle \otimes |0\rangle \] (2.57)

Since in the ground state has \( N = \tilde{N} = 0 \), then, setting \( q = 1 \) we obtain:

\[ \frac{1}{2\pi T} m^2 = -2 \] (2.58)

Again we obtain a spin 0 tachyon that makes the theory unstable. The first excited state is:

\[ |\phi\rangle = \xi_{\mu\nu} \left( \alpha_{\mu-1}^\nu |0\rangle \otimes \tilde{\alpha}_{-1}^\nu |0\rangle \right) \] (2.59)

This state has \( m^2 = 0 \). Again the Visaroro constraints

\[ L_1 |\phi\rangle = \tilde{L}_1 |\phi\rangle = 0 \] (2.60)

restricts the possible values of the polarization to:

\[ p^\mu \xi_{\mu\nu} = 0 \] (2.61)
This kind of tensor gives three distinct spin states: the symmetric part encodes a massless spin-2 particle \( g_{\mu\nu} \), the antisymmetric part is a massless antisymmetric tensor gauge field \( B_{\mu\nu} = -B_{\nu\mu} \); and finally there is encoded also an scalar field \( \Phi \).
Conclusion

We have described some of the basic ideas of the 26-dimensional bosonic string theory. There exist different kinds of bosonic strings, but all of them are sick, because in each case the spectra contains a tachyon. This means that the vacuum state is unstable, and it is still an open problem. The most remarkable feature of string theories is their dimensionality. Here we have seen that 26 dimensions are needed and there is no consistent explication of why this number is preferred.

Anyway the study of this toy model gives a better understanding of the different processes and features that the final theory should or should not have, and it is a necessary step for the study of more elaborated models, such as the Superstrings.
Bibliography

[1] Berkovits, N. *An Introduction to Superstring Theory and its Duality Symmetries*. Proceedings for the First School on Field Theory and Gravitation. Vitória, Brasil. 1997.

[2] Green, M. Schwarz J. and Witten, E. *Superstring Theory*. (2 Volumes) Cambridge University Press. U.K. 1987.

[3] Kiritsis, E. *Introduction to Superstring Theory*. Lectures presented at the Catholic University of Leuven and at the University of Padova. 1996-1997.

[4] Scherk, J. *An Introduction to the Theory of Dual Models and Strings*. Rev. Mod. Phys. 47-1. (1975)

[5] Schwarz, J. *Introduction to Superstring Theory*. Lectures presented at the NATO Advanced Study Institute on Techniques and Concepts of High Energy Physics. St. Croix, Virgin Islands. June 2000.