KK-fibrations arising from Rieffel deformations

Amandip Sangha

Abstract

The bundle map \( \pi_h : \Gamma((A_t J)_{t \in [0,1]}) \to A_{hJ} \), for every \( h \in [0,1] \), of the continuous field \((A_t)_{t \in [0,1]}\) associated to the Rieffel deformation \( A_J \) of a C*-algebra \( A \) is shown to be a KK-equivalence by using a 2-cocycle twisting approach and RKK-fibrations.

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1 Introduction

In [15] M. A. Rieffel introduced a C*-algebraic framework for deformation quantization whereby a C*-algebra \( A \) equipped with an action of \( \mathbb{R}^n \) by automorphisms and further supplied with a skew-symmetric \( J \in M_n(\mathbb{R}) \), produces a C*-algebra \( A_J \) with multiplication \( \times_J \), often referred to as the Rieffel deformation of the original algebra. Several other well-known examples of C*-algebras can be shown to arise in this way. The K-theory of the deformed algebra was studied in [16], revealing that the deformed algebra \( A_J \) and the original algebra \( A \) have the same K-groups.

There, the key technique was to show that \( A_J \) was strongly Morita equivalent to a certain crossed product of (a stabilization and suspension of) \( A \) by \( \mathbb{R}^n \), followed by an application of the Connes-Thom result in K-theory, stability and Morita invariance of the K-functor.

Some operator algebraic approaches to deformation quantization use various notions of “twists”, utilizing an action (e.g. of a group) combined with a distinguished element satisfying some cocyclicity-condition (e.g. a group 2-cocycle) as ingredients towards deforming a given algebra equipped with said action. One such procedure is explored by Kasprzak in [7] where a locally compact abelian group \( G \) acts on a C*-algebra \( A \). Given a 2-cocycle \( \psi \) on the dual group \( \hat{G} \), there is a method for obtaining a deformed algebra \( A^\psi \). This procedure encompasses in particular Rieffel deformation as the case \( G = \mathbb{R}^n \) with a certain choice of 2-cocycle on \( \mathbb{R}^n \). Concerning K-theory, there is an isomorphism \( A^\psi \times G \cong A \times G \) of crossed products which, for the case
$G = \mathbb{R}^n$, when combined with the Connes-Thom result yields an identification of the K-groups of the deformed and undeformed algebras respectively.

The present paper discusses the continuous field over $[0,1]$ of the Rieffel deformation and shows that the evaluation map is a KK-equivalence. Namely, for a C*-algebra $A$ with an action of $\mathbb{R}^n$ and given a skew-symmetric matrix $J$, taking $t \in [0,1]$ and using $tJ$ as the skew-symmetric matrix gives the Rieffel deformation $A_{tJ}$. This will constitute a continuous field $(A_{tJ})_{t \in [0,1]}$ as was already explored in the original monograph [15]. We show that the evaluation map of the bundle algebra $\pi_h : \Gamma((A_{tJ})_{t \in [0,1]}) \to A_{hJ}$, for each $h \in [0,1]$, is a KK-equivalence. To accomplish this we shall employ the deformation approach of Kasprzak and consider a deformed bundle algebra $B^\psi$ which will be a $C([0,1])$-algebra equipped with a fibrewise action. As such, RKK-theory naturally enters and we show that $B^\psi$ is an RKK-fibration in the sense of [4] by appealing to the fibrewise action and the Connes-Thom result in RKK-theory. The important consequence of being an RKK-fibration here, is that the evaluation map of the $C([0,1])$-algebra becomes a KK-equivalence. Finally, the deformed bundle algebra $B^\psi$ will be shown to be $C([0,1])$-linearly *-isomorphic to the bundle algebra $\Gamma((A_{tJ})_{t \in [0,1]})$ of the continuous field of the Rieffel deformation, thus yielding the promised result.

We now give a more specific outline of the paper. Section 2 explains the approach to deformation taken in [7], where one starts with the action of a locally compact abelian group $G$ with a 2-cocycle $\psi$ on the Pontryagin dual $\hat{G}$. A certain subalgebra $A^\psi \subseteq M(A \rtimes G)$ is obtained as the Landstad algebra of the G-product $(A \rtimes G, \lambda, \tilde{\alpha})$. After presenting the basic preliminaries and some of the needed results, we specialize to $G = \mathbb{R}^n$ with our specific 2-cocycle $\psi_J$. Section 3 discusses the relevant bundle and collects a few needed ingredients from [4] on RKK-fibrations and their relation to KK-equivalences, and then proceeds to establish that the aforementioned bundle is an RKK-fibration. Section 4 recalls the main notions of Rieffel deformation, the associated continuous field and the relation to the 2-cocycle deformation. The main result regarding the evaluation map of the bundle algebra of the continuous field is then achieved as a consequence of the RKK-fibration laid forth in the preceding section. In section 5 we comment on the special case called theta deformation, in which the action is not by $\mathbb{R}^n$ but $\mathbb{T}^n$. There, a different bundle algebra is plausible. Namely, taking a fix-point algebra description of the deformed algebra, we use a strong Morita equivalence to a certain crossed product algebra by the integers and work with an integer crossed product bundle algebra. One is able to show that the related bundle evaluation map has a KK-contractible kernel by applying the Pimsner-Voiculescu six-term exact sequence, so the KK-equivalence follows. Finally we describe the invariance of the index pairing which we understand as a KK-product between elements of the K-group with (in particular) the Fredholm module coming from a spectral triple.

## 2 Twisting by a 2-cocycle

We recall the approach to deformation as in [7]. The idea is based on twisting a dual C*-dynamical system by a 2-cocycle of the dual group. First we recollect some preliminaries on C*-dynamical systems and G-products (cf. [13] §7.8).

**Definition 2.1.** Let $G$ be a locally compact abelian group and $\hat{G}$ its Pontryagin dual group. Let $B$ be a C*-algebra with a strict-continuous unitary-valued homomorphism $\lambda : G \to M(B)$, and let $\hat{\rho}$ be a strongly continuous action $\hat{\rho} : \hat{G} \to Aut(B)$ satisfying

$$\hat{\rho}_\chi(\lambda_\gamma) = \chi(\gamma)\lambda_\gamma$$

for all $\chi \in \hat{G}$ and $\gamma \in G$. The triple $(B, \lambda, \hat{\rho})$ is called a $G$-product. One also simply refers to $B$ as a $G$-product when the rest is implicitly understood.
Given a $G$-product $(B, \lambda, \tilde{\rho})$, one may extend the given unitary representation $\lambda$ to the $\ast$-homomorphism $\lambda : C^*(G) \to M(B)$. Using the Fourier transform to identify $C^*(G) \cong C_0(\hat{G})$ we write $\lambda : C_0(\hat{G}) \to M(B)$. This map is injective and we often omit $\lambda$ from the notation.

**Definition 2.2.** Let $(B, \lambda, \tilde{\rho})$ be a $G$-product and let $x \in M(B)$. The element $x$ satisfies the Landstad conditions if:

(i) $\hat{\rho}_\chi(x) = x$ for all $\chi \in \hat{G}$,
(ii) the map $G \ni \gamma \mapsto \lambda_\gamma x \lambda_\gamma^* \in M(B)$ is norm continuous,
(iii) $fxg \in B$ for all $f, g \in C_0(\hat{G})$.

The set of elements satisfying the Landstad conditions turns out to be a subalgebra in $M(B)$. We shall refer to this subalgebra as the Landstad algebra of the $G$-product.

The foremost example of a $G$-product is produced by the crossed product construction. Indeed, given an abelian $C^*$-dynamical system $(B, G, \alpha)$, the triple $(B \rtimes_\alpha G, \lambda, \tilde{\rho})$ is a $G$-product whose Landstad algebra is precisely $B$. The following result states that any $G$-product arises in this way.

**Theorem 2.3.** [13 Theorem 7.8.8] A $C^*$-algebra $B$ is a $G$-product $(B, \lambda, \tilde{\rho})$ if and only if there exists a $C^*$-dynamical system $(C, G, \beta)$ for which $B \cong C \rtimes_\beta G$. The $C^*$-dynamical system is unique up to covariant isomorphism, the $C^*$-algebra $C$ is just the associated Landstad algebra and $\beta = \text{Ad} \lambda$.

Recall that a 2-cocycle $\psi$ on the abelian group $\hat{G}$ is a continuous function

$$\psi : \hat{G} \times \hat{G} \to \mathbb{T}$$

satisfying

(i) $\psi(e, \chi) = \psi(\chi, e) = 1$ for all $\chi \in \hat{G}$,
(ii) $\psi(\chi_1, \chi_2 + \chi_3) = \psi(\chi_1 + \chi_2, \chi_3) = \psi(\chi_1, \chi_2)$ for all $\chi_1, \chi_2, \chi_3 \in \hat{G}$.

Given an element $\chi \in \hat{G}$, define the function $\psi_\chi \in C_b(\hat{G})$ by

$$\psi_\chi(\hat{\sigma}) = \psi(\chi, \hat{\sigma})$$

for $\hat{\sigma} \in \hat{G}$.

Observing that $C_\beta(\hat{G}) = M(C_0(\hat{G}))$, use the obvious extension $\lambda : C_b(\hat{G}) \to M(B)$ and obtain unitaries

$$U_\chi = \lambda(\psi_\chi) \in M(B).$$

The 2-cocycle condition for $\psi$ implies the following commutation rule for these unitaries

$$U_{\chi_1 + \chi_2} = \tilde{\psi}(\chi_1, \chi_2) U_{\chi_1} \hat{\rho}_{\chi_1}(U_{\chi_2}).$$

**Lemma 2.4.** [4 Theorem 3.1] Let $(B, \lambda, \tilde{\rho})$ be a $G$-product and $\psi$ a 2-cocycle on $\hat{G}$. Use the unitaries of (2.4) to define the strongly continuous action $\tilde{\rho}^{\psi} : \hat{G} \to \text{Aut}(B)$,

$$\tilde{\rho}^{\psi}_\chi(b) = U_\chi^* \hat{\rho}_\chi(b) U_\chi$$

for $\chi \in \hat{G}$ and $b \in B$. Then $(B, \lambda, \tilde{\rho}^{\psi})$ is a $G$-product.
Definition 2.5 (Kasprzak deformation). Let $A$ be a separable C*-algebra with strongly continuous action $\alpha : G \to Aut(A)$ of the locally compact abelian group $G$, and $\psi$ a 2-cocycle on $G$. The $G$-product $(A \rtimes_\alpha G, \lambda, \tilde{\alpha})$ gives rise to the $G$-product $(A \rtimes_\alpha G, \lambda, \tilde{\alpha}^\psi)$ by Lemma 2.4. The deformed algebra $A^\psi$ is by definition the Landstad algebra of the $G$-product $(A \rtimes_\alpha G, \lambda, \tilde{\alpha}^\psi)$.

An interesting result is obtained by considering the original action on the deformed algebra. Denote by $\alpha^\psi : G \to Aut(A^\psi)$ the action $\alpha^\psi_\beta(x) = \lambda_\beta x \lambda_\beta^*$, if we consider the crossed product of the C*-dynamical system $(A^\psi, G, \alpha^\psi)$ we get

Lemma 2.6. $A^\psi \rtimes_\alpha^\psi G \cong A \rtimes_\alpha G$.

Proof. The proof is a literal application of Theorem 2.3. Indeed, let $B = A \rtimes_\alpha G$ and consider the $G$-product $(B, \lambda, \tilde{\alpha}^\psi) = (A \rtimes_\alpha G, \lambda, \tilde{\alpha}^\psi)$. The Landstad algebra of this $G$-product is what we have called $A^\psi$ by definition, which is the algebra $C = A^\psi$ referred to in Theorem 2.3. Furthermore $\alpha^\psi = Ad \lambda$, which is the action $\beta$ in that theorem. In other words the C*-dynamical system is $(C, G, \beta) = (A^\psi, G, \alpha^\psi)$ and the theorem yields the isomorphism $B \cong C \rtimes_\beta G$, in our case $A \rtimes_\alpha G \cong A^\psi \rtimes_\alpha^\psi G$ as claimed.

Following [13] and [10], we may further describe the *-isomorphism $A^\psi \rtimes_\alpha^\psi G \to A \rtimes_\alpha G$ as mapping $y \otimes g \mapsto y \lambda_\beta g$, for $y \in A^\psi$ and $g \in C_c(G)$. \hfill \Box

Let our separable C*-algebra $A$ be equipped with a strongly continuous action $\sigma : \mathbb{R}^n \to Aut(A)$, and let $J \in M_n(\mathbb{R})$ be a skew-symmetric matrix. On $\mathbb{R}^n$ we consider the symmetric bicharacter

$$e : \mathbb{R}^n \times \mathbb{R}^n \to T$$

$$e(u, v) = e^{2\pi i u \cdot v}$$

which gives the group isomorphism $\mathbb{R}^n \cong \hat{\mathbb{R}^n}$ by $u \mapsto e_1^u$ where $e_1^u(v) = e(u, v)$. We use the 2-cocycle $\psi_J : \mathbb{R}^n \times \mathbb{R}^n \to T$,

$$\psi_J(e_1^u, e_1^v) = e_1^u(Jv)v = e(u, Jv) = e^{2\pi i u \cdot Jv}.$$  \hfill (2.2)

By Lemma 2.4 the $\mathbb{R}^n$-product $(A \rtimes_\sigma \mathbb{R}^n, \lambda, \hat{\sigma})$ combined with the 2-cocycle $\psi_J$ gives the $\mathbb{R}^n$-product $(A \rtimes_\sigma \mathbb{R}^n, \lambda, \hat{\sigma}^{\psi_J})$, and the deformed algebra $A^{\psi_J}$ is the corresponding Landstad algebra.

3 Bundle structure and RKK-fibration

Let $X$ be a locally compact Hausdorff space. A C*-algebra $B$ is called a $C_0(X)$-algebra (cf. [6, 1.5]) if there is a non-degenerate *-homomorphism $\Phi_B : C_0(X) \to ZM(B)$. One also writes $fb = \Phi_B(f)b$, for $f \in C_0(X)$ and $b \in B$. For each $x \in X$, let $I_x = \{f \in C_0(X) : f(x) = 0\}$ be the ideal of functions vanishing at $x$, then $I_x B \subseteq B$ is an ideal and the quotient $B_x = B/(I_x B)$ is called the fiber over $x$. The quotient map $q_x : B \to B_x$ is also referred to as evaluation at $x$.

Recall that we are considering a strongly continuous action $\sigma : \mathbb{R}^n \to Aut(A)$ on a separable C*-algebra $A$, and a real skew-symmetric matrix $J$. Let $B = C([0,1]) \otimes A = C([0,1], A)$ be equipped with the obvious $C([0,1])$-algebra structure $\Phi_B : C([0,1]) \to ZM(B)$, $\Phi_B(f)(g \otimes a) = fg \otimes a$. Define the action $\beta : \mathbb{R}^n \to Aut(B)$

$$\beta_x(y)(s) = \sigma_{\sqrt{x}}(y(s)),$$  \hfill (3.1)

for $x \in \mathbb{R}^n$, $y \in B$, $s \in [0,1]$. Let $\psi = \psi_J$ be the 2-cocycle from (2.2). Then the 2-cocycle deformation $B^\psi$ is by definition the Landstad algebra of the $\mathbb{R}^n$-product $(B \rtimes_\beta \mathbb{R}^n, \lambda, \hat{\beta}^\psi)$. Recall the action $\beta^\psi := \beta^\psi : \mathbb{R}^n \to Aut(B^\psi)$ from the remark preceding Lemma 2.6.
Lemma 3.1. The deformed algebra $B^\psi$ is a $C([0,1])$-algebra and the action $\beta^\psi : \mathbb{R}^n \to Aut(B^\psi)$ is fiberwise. There is a $C([0,1])$-linear *-isomorphism

$$B^\psi \rtimes_{\beta^\psi} \mathbb{R}^n \to B \rtimes_{\beta} \mathbb{R}^n.$$ 

Proof. Clearly the action $\beta$ on $B$ is $C([0,1])$-linear, i.e. for every $x \in \mathbb{R}^n$, $\beta_x(\Phi_B(f)y) = \Phi_B(f)\beta_x(y)$ for every $f \in C([0,1])$ and $y \in B$. This entails that $\Phi_{B \rtimes_{\beta} \mathbb{R}^n} : C([0,1]) \to ZM(B \rtimes_{\beta} \mathbb{R}^n)$ given by $(\Phi_{B \rtimes_{\beta} \mathbb{R}^n}(f)y) = \Phi_B(f)(y)$ gives a $C([0,1])$-algebra structure on $B \rtimes_{\beta} \mathbb{R}^n$.

Concerning the dual action $\hat{\beta} : \mathbb{R}^n \to Aut(B \rtimes_{\beta} \mathbb{R}^n)$, for each $w \in \mathbb{R}^n$ the canonically extended automorphism $\hat{\beta}_w : M(B \rtimes_{\beta} \mathbb{R}^n) \to M(B \rtimes_{\beta} \mathbb{R}^n)$ satisfies $\hat{\beta}_w(\Phi_{B \rtimes_{\beta} \mathbb{R}^n}(f)) = \Phi_{B \rtimes_{\beta} \mathbb{R}^n}(f)$ for every $f \in C([0,1])$. It then follows that for any $y \in M(B \rtimes_{\beta} \mathbb{R}^n)$,

$$\hat{\beta}_w^\psi(\Phi_{B \rtimes_{\beta} \mathbb{R}^n}(f)) = U_w^* \hat{\beta}_w(\Phi_{B \rtimes_{\beta} \mathbb{R}^n}(f))U_w = U_w^* \Phi_{B \rtimes_{\beta} \mathbb{R}^n}(f)U_w = \Phi_{B \rtimes_{\beta} \mathbb{R}^n}(f),$$

i.e. $\Phi_{B \rtimes_{\beta} \mathbb{R}^n}(C([0,1])) \subseteq M(B \rtimes_{\beta} \mathbb{R}^n)\hat{\beta}^\psi = B^\psi$. Combined with the fact that $\Phi_{B \rtimes_{\beta} \mathbb{R}^n}(C([0,1])) \subseteq ZM(B \rtimes_{\beta} \mathbb{R}^n)$, this entails that we may define $\Phi_{B^\psi} = \Phi_{B \rtimes_{\beta} \mathbb{R}^n}$ to obtain a $C([0,1])$-algebra structure on $B^\psi$.

The action $\beta^\psi : \mathbb{R}^n \to Aut(B^\psi)$ is $\beta^\psi(y) = \lambda_x y \lambda_x^*$, for $y \in B^\psi$, and so

$$\beta^\psi_x(\Phi_{B^\psi}(f)) = \beta_x^\psi(\Phi_{B \rtimes_{\beta} \mathbb{R}^n}(f)y) = \lambda_x(\Phi_{B \rtimes_{\beta} \mathbb{R}^n}(f)y) \lambda_x^* = \Phi_{B \rtimes_{\beta} \mathbb{R}^n}(f)\lambda_x y \lambda_x^* = \Phi_{B^\psi}(f)\lambda_x y \lambda_x^*,$$

i.e. the action $\beta^\psi$ is fiberwise and hence naturally makes the crossed product $B^\psi \rtimes_{\beta^\psi} \mathbb{R}^n$ a $C([0,1])$-algebra where $\Phi_{B^\psi \rtimes_{\beta^\psi} \mathbb{R}^n} : C([0,1]) \to ZM(B^\psi \rtimes_{\beta^\psi} \mathbb{R}^n)$ is given by the composition of $\Phi_{B^\psi}$ with the inclusion $M(B^\psi) \subseteq M(B^\psi \rtimes_{\beta^\psi} \mathbb{R}^n)$. By Lemma 2.4

$$B^\psi \rtimes_{\beta^\psi} \mathbb{R}^n \cong B \rtimes_{\beta} \mathbb{R}^n,$$  \hspace{1cm} (3.2)

and we claim this *-isomorphism to be $C([0,1])$-linear. Indeed, denote this *-isomorphism $S : B^\psi \rtimes_{\beta^\psi} \mathbb{R}^n \to B \rtimes_{\beta} \mathbb{R}^n$, which by Lemma 2.4 can be described as $S(y \otimes g) = y \lambda_y$ for $y \in B^\psi$ and $g \in C_c(\mathbb{R}^n)$, and it follows that

$$S(\Phi_{B^\psi \rtimes_{\beta^\psi} \mathbb{R}^n}(f)(y \otimes g)) = S(\Phi_{B^\psi}(f)y \otimes g) = \Phi_{B^\psi}(f)y \lambda_y = \Phi_{B \rtimes_{\beta} \mathbb{R}^n}(f)y \lambda_y = \Phi_{B \rtimes_{\beta} \mathbb{R}^n}(f)y \lambda_y = \Phi_{B \rtimes_{\beta} \mathbb{R}^n}(f)S(y \otimes g)$$

for any $f \in C([0,1])$, i.e. $S \circ \Phi_{B^\psi \rtimes_{\beta^\psi} \mathbb{R}^n} = \Phi_{B \rtimes_{\beta} \mathbb{R}^n} \circ S$. \hfill \Box

Let $f : Y \to X$ be a continuous map between locally compact spaces. The pullback construction gives a $C_0(X)$-algebra structure on $C_0(Y)$, since $f^* : C_0(X) \to C_0(Y)$ and $C_b(Y) = ZM(C_0(Y))$, we let $\Phi_{C_0(Y)} : C_0(X) \to ZM(C_0(Y))$, $\Phi_{C_0(Y)}(k) = f^*(k)$ be the pointwise multiplication operator by the pullback

$$\Phi_{C_0(Y)}(k)h = f^*(k)h$$

for $k \in C_0(X)$, $h \in C_0(Y)$.  


Given a $C_0(X)$-algebra $B$, a locally compact space $Y$ and $f : Y \to X$ a continuous map, the pullback $f^*(B)$ of $B$ along $f$ is the $C_0(Y)$-algebra

$$f^*(B) = C_0(Y) \otimes_{C_0(X)} B.$$  

(3.3)

The balanced tensor product in $\otimes$ is by definition the quotient of $C_0(Y) \otimes B$ by the ideal generated by

$$\{ \Phi_{C_0(Y)}(k) g \otimes b - g \otimes \Phi_B(k) b \mid g \in C_0(Y), b \in B, k \in C_0(X) \}.$$  

The $C_0(Y)$-algebra structure on $f^*(B)$ is pointwise multiplication on the left, $\Phi_{f^*(B)} : C_0(Y) \to ZM(f^*(B))$, $\Phi_{f^*(B)}(h)(g \otimes b) = hg \otimes b$, for $h, g \in C_0(Y)$ and $b \in B$. Note that the fiber $f^*(B)_y$ over $y \in Y$ is $B_{f(y)}$. Indeed, as in the balanced tensor product one has $I_y C_0(Y) \otimes_{C_0(X)} B = C_0(Y) \otimes_{C_0(X)} I_{f(y)} B$, then

$$f^*(B)_y = C_0(Y) \otimes_{C_0(X)} B/I_y C_0(Y) \otimes_{C_0(X)} B$$

$$= C_0(Y) \otimes_{C_0(X)} B/C_0(Y) \otimes_{C_0(X)} I_{f(y)} B$$

$$= B/I_{f(y)} B = B_{f(y)}.$$

Recall that given two graded, separable C*-algebras $A$ and $B$, the group $KK(A, B)$ is the set of Kasparov $A$-$B$-modules (also called Kasparov cycles) modulo an appropriate equivalence relation (e.g. homotopy equivalence). Briefly, a Kasparov is a countably generated right Hilbert $E$-module. Namely, for two $C$-algebras $B$ and $C$ where

$$\{ \Phi_{C_0(Y)}(k) g \otimes b - g \otimes \Phi_B(k) b \mid g \in C_0(Y), b \in B, k \in C_0(X) \}.$$  

The balanced tensor product in $\otimes$ is by definition the quotient of $C_0(Y) \otimes B$ by the ideal generated by

$$\{ \Phi_{C_0(Y)}(k) g \otimes b - g \otimes \Phi_B(k) b \mid g \in C_0(Y), b \in B, k \in C_0(X) \}.$$  

The $C_0(Y)$-algebra structure on $f^*(B)$ is pointwise multiplication on the left, $\Phi_{f^*(B)} : C_0(Y) \to ZM(f^*(B))$, $\Phi_{f^*(B)}(h)(g \otimes b) = hg \otimes b$, for $h, g \in C_0(Y)$ and $b \in B$. Note that the fiber $f^*(B)_y$ over $y \in Y$ is $B_{f(y)}$. Indeed, as in the balanced tensor product one has $I_y C_0(Y) \otimes_{C_0(X)} B = C_0(Y) \otimes_{C_0(X)} I_{f(y)} B$, then

$$f^*(B)_y = C_0(Y) \otimes_{C_0(X)} B/I_y C_0(Y) \otimes_{C_0(X)} B$$

$$= C_0(Y) \otimes_{C_0(X)} B/C_0(Y) \otimes_{C_0(X)} I_{f(y)} B$$

$$= B/I_{f(y)} B = B_{f(y)}.$$

The $KK$-product is a bilinear map

$$KK(A, D) \times KK(D, B) \to KK(A, B)$$

$$(x, y) \mapsto xy$$

where $A$, $B$ and $D$ are separable (and $D$ is $\sigma$-unital) C*-algebras. There is a multiplicatively neutral element $1_D = [(D, 1_0, 0)] \in KK(D, D)$ such that for any $x \in KK(A, D)$ and $y \in KK(D, B)$ one has $x1_D = x$ and $1_D y = y$.

An element $x \in KK(A, B)$ is called a $KK$-equivalence if it is invertible with respect to the $KK$-product, i.e. if there exists an element $y \in KK(B, A)$ such that $xy = 1_A \in KK(A, A)$ and $yx = 1_B \in KK(B, B)$.

Given a graded $*$-homomorphism $\phi : A \to B$, then $(B, \phi, 0)$ is the naturally associated Kasparov $A$-$B$-module. We say $\phi$ is a $KK$-equivalence if the corresponding element $[(B, \phi, 0)] \in KK(A, B)$ is a $KK$-equivalence.

Regarding $C_0(X)$-algebras there is a further refinement of the $KK$-groups called RKK-groups ($\mathbb{R}$). Namely, for two $C_0(X)$-algebras $A$ and $B$, the group $RKK(X; A, B)$ consists of Kasparov $A$-$B$-modules $(E, \phi, F)$ as before, only with the additional requirement

$$(fa) \cdot e \cdot b = a \cdot e \cdot (fb)$$

(3.4)

for any $f \in C_0(X)$, $a \in A$, $b \in B$ and $e \in E$.

The notions $RKK(X; \cdot, \cdot)$-product and $RKK(X; \cdot, \cdot)$-equivalence are similar to those of the $KK$-counterpart.

We let $\Delta^p \subseteq \mathbb{R}^{p+1}$ denote the standard $p$-simplex.

**Definition 3.2.** A $C_0(X)$-algebra $B$ is called a $KK$-fibration if for every positive integer $p$, every continuous map $f : \Delta^p \to X$ and every element $v \in \Delta^p$ the evaluation $q_v : f^*(B) \to B_{f(v)}$ is a $KK$-equivalence.
Definition 3.3. A $C_0(X)$-algebra $B$ is called an RKK-fibration if for every positive integer $p$, every continuous map $f : \Delta^p \to X$ and every element $v \in \Delta^p$, $f^*(B)$ is RKK$(\Delta^p; \cdot, \cdot)$-equivalent to $C(\Delta^p, B_{f(v)})$.

Remark 3.4. Given a C*-algebra $A$, the canonical $C_0(X)$-algebra $B = C_0(X) \otimes A$ is an RKK-fibration. Indeed, given $f : \Delta^p \to X$ and $v \in \Delta^p$, the pullback

$$f^*(B) = C(\Delta^p) \otimes C_0(X) \otimes A$$

is $C(\Delta^p)$-linearly *-isomorphic to $C(\Delta^p, B_{f(v)}) = C(\Delta^p) \otimes B_{f(v)} = C(\Delta^p) \otimes A$ by the map

$$h \otimes g \otimes a \mapsto \Phi_{C(\Delta^p)}(g)h \otimes a = f^*(g)h \otimes a,$$

where $h \in C(\Delta^p)$, $g \in C_0(X)$ and $a \in A$. This implies the required RKK$(\Delta^p; \cdot, \cdot)$-equivalence.

Note also that the property of being an RKK-fibration is preserved under RKK-equivalence. The following observation ([$4$, Remark 1.4]) will be useful.

Lemma 3.5. An RKK-fibration is a KK-fibration.

Proof. Suppose $B$ is an RKK-fibration, let $f : \Delta^p \to X$ and $v \in \Delta^p$. Concisely put, we get the following commutative diagram in the KK category in which all arrows but the right vertical arrow are already known to be isomorphisms

$$\begin{array}{ccc}
C(\Delta^p, B_{f(v)}) & \xrightarrow{r} & f^*(B) \\
\downarrow{ev_v} & & \downarrow{q_v} \\
B_{f(v)} & \xrightarrow{r(v)} & B_{f(v)}
\end{array}$$

so it follows that the right vertical arrow $q_v$ must be an isomorphism as well.

In details, by assumption there exists an invertible element

$$r \in RKK(\Delta^p; C(\Delta^p, B_{f(v)}), f^*(B)).$$

Here $C(\Delta^p, B_{f(v)}) = C(\Delta^p) \otimes B_{f(v)}$ is the canonical $C(\Delta^p)$-algebra with constant fiber $B_{f(v)}$ over each point of $\Delta^p$, its bundle projection map being just the evaluation $ev_v : C(\Delta^p, B_{f(v)}) \to B_{f(v)}$, $ev_v(f \otimes b) = f(w) b$, for any $w \in \Delta^p$, and it gives in particular the KK-equivalence $[ev_v] \in KK(C(\Delta^p, B_{f(v)}), B_{f(v)})$. Recall also that $f^*(B)$ has fiber $B_{f(v)}$ over the point $v \in \Delta^p$, denote this bundle projection map $q_v$. From the invertible element $r \in RKK(\Delta^p; C(\Delta^p, B_{f(v)}), f^*(B))$ we get an invertible element $r(v) \in KK(B_{f(v)}, B_{f(v)})$ which implements the KK-equivalence between the fibers. It follows from

$$[q_v] \cdot r = r(v)[ev_v]$$

that $[q_v] = r(v)[ev_v]r^{-1}$ is a KK-equivalence.

Recall the Connes-Thom isomorphism in K-theory $K_i(A \rtimes_\alpha \mathbb{R}) \cong K_{i-1}(A)$, $i = 0, 1$, where $\alpha \in Aut(A)$ is a continuous action. The analogous result in KK-theory establishes the existence of an invertible element $t_\alpha \in KK^1(A, A \rtimes_\alpha \mathbb{R}) = KK(SA, A \rtimes_\alpha \mathbb{R})$, the Thom element. In other words, $A$ and $A \rtimes_\alpha \mathbb{R}$ are KK-equivalent with dimension shift 1. The case of an $\mathbb{R}^n$-action is handled by repeated application of the above, yielding a KK-equivalence with total dimension shift $n (mod 2)$. In dealing with $C_0(X)$-algebras we shall make use of the following RKK-version of the Connes-Thom isomorphism (see [6 §4])
Theorem 3.6. \cite{3} \textbf{Theorem 3.5} Let $A$ be a $C_0(X)$-algebra and $\alpha : R^n \rightarrow Aut(A)$ a fibrewise action. There exists an invertible element
\[ t_\alpha \in RKK^n(X; A, A \rtimes_\alpha R^n). \]
Hence $A$ and $A \rtimes_\alpha R^n$ are RKK-equivalent with dimension shift $n \mod 2$.

Theorem 3.7. $B^\psi$ is an RKK-fibration.

\textbf{Proof.} It follows from Theorem 3.6 that $B^\psi$ is $RKK([0,1]; \cdot, \cdot)$-equivalent, with dimension shift $n \mod 2$, to $B^\psi \rtimes_\beta R^n$. By the isomorphism \cite{5} the latter algebra is $RKK([0,1]; \cdot, \cdot)$-equivalent to $B \rtimes_\beta R^n$, which by Theorem 3.6 again is $RKK([0,1]; \cdot, \cdot)$-equivalent, with another dimension shift $n \mod 2$, to $B = C([0,1]) \otimes A$. The total dimension shift thus far is $2n \mod 2 = 0$, i.e. the net effect being no dimension shift, so $B^\psi$ is plainly $RKK([0,1]; \cdot, \cdot)$-equivalent to $B$. Finally, the algebra $B = C([0,1]) \otimes A$ is clearly an $RKK$-fibration (Remark 3.4), thus proving the claim. \end{proof}

It follows from Theorem 3.6 and Lemma 3.6 that $B^\psi$ is a KK-fibration. Taking the identity function of the 1-simplex, $f : \Delta^1 = [0,1] \rightarrow [0,1]$, $f(s) = s$, we conclude that the evaluation map $q_s : B^\psi \rightarrow (B^\psi)_s$ is a KK-equivalence. Although maybe not completely transparent thus far, it will be made clear in section 3 that $B^\psi \cong \Gamma((A_t)_{t \in [0,1]})$ is the bundle algebra of the continuous field over $[0,1]$ of the Rieffel deformation and $(B^\psi)_s \cong A_{t,s}$ is the fiber over the point $s \in [0,1]$. 

4 \textbf{The continuous field of the Rieffel deformation}

We briefly recall some of the basic facts from \cite{15} concerning Rieffel deformation. Let $\sigma : R^n \rightarrow Aut(A)$ be a strongly continuous action on a separable C*-algebra $A$, and $J \in M_n(R)$ a skew-symmetric matrix. Let $\tau$ be the translation action on the Frechet space $C_b(R^n, A)$ and let $C_\sigma(R^n, A)$ be the largest subspace on which $\tau$ is strongly continuous. Denote by $B^A = B^A(R^n) \subseteq C_\sigma(R^n, A)$ the subalgebra of smooth elements for the action $\tau$. For any $F \in B^A(R^n \times R^n)$ the integral
\[
\iint F(u, v)e^{2\pi i u \cdot v} \, du \, dv
\]
exists, as shown in \cite{15} Chapter 1] by considerations of oscillatory integrals. For $f, g \in B^A(R^n)$, the function $(u, v) \mapsto \tau_J(u)(f)(x)\tau_J(v)(g)(x)$ is an element of $B^A(R^n \times R^n)$ for each $x \in R^n$, hence the following integral is well defined
\[
(f \times_J g)(x) = \iint \tau_J(u)(f)(x)\tau_J(v)(g)(x)e^{2\pi i (u \cdot v)}, \tag{4.1}
\]
and it turns out $\times_J$ defines an associative product on $B^A(R^n)$, and we denote by $B^A_J = (B^A(R^n), \times_J)$ this algebra structure. Let $S^A \subseteq B^A$ be the subspace of $A$-valued Schwartz functions. This is naturally a right Hilbert $A$-module for the $A$-valued inner product $(f, g)_A = \int f(x)^* g(x)$. Considering the product $\times_J$, it turns out $S^A_J$ is an ideal in $B^A_J$, this still being compatible with the Hilbert C*-module structure. In this way $S^A_J$ carries a representation $L = L^J$ of $B^A_J$ by adjointable operators
\[
L : B^A_J \rightarrow \mathcal{L}(S^A_J)
\]
\[
L_J(\xi) = f \times_J \xi, \quad f \in B^A_J, \xi \in S^A_J.
\]
Let $A^\infty \subseteq A$ denote the dense $*$-subalgebra of smooth elements for the action $\sigma$. For $a, b \in A^\infty$, the function $(u, v) \mapsto \sigma_{Ju} \sigma_{Jv}(b)$ is an element of $B^A(\mathbb{R}^n \times \mathbb{R}^n)$ and we may define

$$a \times_J b = \int \sigma_{Ju}(a) \sigma_{Jv}(b) e^{2\pi i u \cdot v} \, du \, dv.$$ 

The homomorphism $A \rightarrow C_0(\mathbb{R}^n, A)$, $a \mapsto \tilde{a}$, $\tilde{a}(x) = \sigma_x(a)$, is equivariant for the respective actions $\sigma$ and $\tau$, thus maps $A^\infty \rightarrow B^A$. Moreover, $a \times_J b = \tilde{a} \times_J \tilde{b}$, i.e. this is a homomorphism for the products $\times_J$. Thus $A^\infty$ is represented on $S^A_1$, and we define a new norm $|| \cdot ||_J$ on $A^\infty$, $||a||_J = ||L_\tilde{a}||$.

**Definition 4.1** (Rieffel deformation). Equip $A^\infty$ with the product $\times_J$ and the norm $|| \cdot ||_J$. This completion is denoted $A_J$ and is called the deformation of $A$ along $\sigma$ by $J$, or in short the Rieffel deformation of $A$.

Below we list some of the properties of the Rieffel deformation.

**Lemma 4.2** (Properties of the Rieffel deformation). Let $A$ be a separable $C^*$-algebra, $\sigma : \mathbb{R}^n \rightarrow \text{Aut}(A)$ a strongly continuous action and $J \in M_n(\mathbb{R})$ such that $J^t = -J$.

(i) $\times_J$ is associative and the involution $*$ for $A$ is also an involution for $A_J$, which thus becomes a $C^*$-algebra
(ii) $a \times_J b = ab$ for $J = 0$
(iii) For every fixed point $a \in A^\sigma$, $a \times_J b = ab$ and $b \times_J a = ba$ for every $b \in A$
(iv) $(A_J)_K = A_{J+K}$ for any skew-symmetric $K \in M_n(\mathbb{R})$
(v) The action $\sigma$ is also an action on $A_J$, $\sigma : \mathbb{R}^n \rightarrow \text{Aut}(A_J)$. Moreover $(A_J)^\infty = (A^\infty)_J$
(vi) The dense subalgebra $(A^\infty)_J \subseteq A_J$ is stable under holomorphic functional calculus
(vii) Given a $\sigma$-invariant ideal $I \subseteq A$, the equivariant short exact sequence

$$0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$$

implies a short exact sequence

$$0 \rightarrow I_J \rightarrow A_J \rightarrow (A/I)_J \rightarrow 0$$

(viii) For any $T \in M_n(\mathbb{R})$, define a new action $\sigma^T$ by $\sigma^T_x(a) = \sigma_{Tx}(a)$, for $x \in \mathbb{R}^n$, $a \in A$. Performing the deformation procedure for the action $\sigma^T$ and skew-symmetric matrix $J$, denote by $\times^T_J$ the deformed product so obtained. Then

$$\times^T_J = \times_{TJT^t}.$$ 

The equivalence between the Rieffel deformation $A_J$ and the 2-cocycle deformation $A^{\psi_J}$ is given by the $*$-isomorphism of the following lemma. Recall that one considers the $\mathbb{R}^n$-product $(A \ltimes_\sigma \mathbb{R}^n, \lambda, \tilde{\psi}_J)$, the 2-cocycle $\psi_J$ in (2.2) and $A^{\psi_J} \subseteq M(A \ltimes_\sigma \mathbb{R}^n)$ is the subalgebra satisfying the Landstad conditions.

**Lemma 4.3.** There is a $*$-isomorphism

$$T : A^{\psi_J} \rightarrow A_J.$$
Proof. We refer the reader to [5] for details, and give only the form of the *-isomorphism here. Let \( y \in C_c(\mathbb{R}^n, A) \subseteq A \rtimes \mathbb{R}^n \subseteq M(A \rtimes \mathbb{R}^n) \), and suppose \( y \in A^\psi_J \), which means that \( \tilde{\sigma}_y^J(x) = y \) for all \( x \in \mathbb{R}^n \). The isomorphism \( T \) is described on such elements by

\[
T(y) = \int_{\mathbb{R}^n} y(v) \, dv.
\]

We consider \( B = C([0,1]) \otimes A \) with the action \( \beta \) as in (3.1). Note that \( \beta_x(y)(s) = \sigma_x^1(y(s)) \) (see Lemma 4.2 (i)). For every \( x \in \mathbb{R}^n \), let \( \overline{b}_x \in \text{Aut}(M(B)) \) denote the canonical extension of \( \beta_x \) to the multiplier algebra, namely for \( L \in M(B) \), \( \overline{b}_x(L)(b) = \beta_x(L(\beta_x^{-1}(b))) \), for \( b \in B \). A quick calculation reveals that for every \( f \in C([0,1]) \), \( \overline{b}_x(\Phi_B(f)) = \Phi_B(f) \), i.e. \( \Phi_B(C([0,1])) \subseteq M(B)^\mathbb{R} \). It is also clear that \( \Phi_B(C([0,1])) \subseteq M(B)^\mathbb{R} \). From the inclusion \( B \subseteq M(B) \) as a \( \beta \)-invariant ideal we get \( B_J \subseteq M(B)_J \) by Lemma 4.2 (vii), and working inside \( M(B)_J \) get from Lemma 4.2 (iii)

\[
\Phi_B(f) \times_J y = \Phi_B(f)y = y\Phi_B(f) = y \times_J \Phi_B(f)
\]

for \( y \in B^\infty \) and \( f \in C([0,1]) \), as \( \Phi_B(f) \in M(B)^\mathbb{R} \). This yields a \( C([0,1]) \)-algebra structure on \( B_J \), denoted \( \Phi_{B_J} : C([0,1]) \rightarrow ZM(B_J) \) given by \( \Phi_{B_J}(f)y = \Phi_B(f) \times_J y = \Phi_B(f)y \). As such, \( B_J \) is an essential \( C([0,1]) \)-module, i.e. \( C([0,1])B_J = B_J \).

**Theorem 4.4.** \( (A_tJ)_{t \in [0,1]} \) is a continuous field of \( C^* \)-algebras, where we take as the algebra of sections \( \Gamma((A_tJ)_{t \in [0,1]}) \) to be the algebra \( B_J \).

**Proof.** For each \( s \in [0,1] \) let \( K^s = I_s \otimes A \) be the ideal consisting of elements of \( B = C([0,1], A) \) which vanish at the point \( s \). Clearly, \( B/K^s = A \). The short exact sequence

\[
0 \longrightarrow K^s \longrightarrow B \longrightarrow A \longrightarrow 0
\]

is equivariant for \( \beta \) acting on \( K^s \) and \( \sigma_1^\mathbb{R} \) acting on \( A \), so by Lemma 4.2 (vii) (cf. also Theorem 7.7 of [15]) we get a short exact sequence

\[
0 \longrightarrow K^s_J \longrightarrow B_J \longrightarrow A^\mathbb{R}_{\mathbb{R}_1} \longrightarrow 0
\]

The fiber \( (B_J)_s \) over \( s \in [0,1] \) of the \( C([0,1]) \)-algebra \( B_J \) is by definition the quotient \( (B_J)_s = B_J/(I_sB_J) \). It is shown in [15] that \( K^s_J = I_sB_J \), consequently \( (B_J)_s = B_J/(I_sB_J) = B_J/K^s_J = A^\mathbb{R}_{\mathbb{R}_1} \). Moreover, from Lemma 4.2 (viii) it follows that \( A^\mathbb{R}_{\mathbb{R}_1} = A_{\mathbb{R}_1} = A_{\mathbb{R}_1} \), thus the bundle projection is \( \pi_s : B_J \rightarrow A_{\mathbb{R}_1} \). Theorem 8.3 of [15] (see also Proposition 1.2 of [14]) establishes the continuity of the field \( (A_tJ)_{t \in [0,1]} \), for which \( B_J \) is a maximal algebra of cross sections, henceforth denoted \( \Gamma((A_tJ)_{t \in [0,1]}) \).

Considering the *-isomorphism of Lemma 4.3 at the level of bundles, we get

**Lemma 4.5.** The *-isomorphism

\[
T : B^\psi \longrightarrow B_J
\]

is \( C([0,1]) \)-equivariant, i.e. \( T \circ \Phi_B^\psi = \Phi_{B_J} \circ T \).

**Proof.** Let \( b \in C_c(\mathbb{R}^n, B^\infty) \subseteq B \rtimes \beta \mathbb{R}^n \subseteq M(B \rtimes \beta \mathbb{R}^n) \) be an element such that \( \tilde{\sigma}_y^J(b) = b \) for all \( x \in \mathbb{R}^n \), i.e. \( b \) is an element of \( B^\psi \). The *-isomorphism is described on such elements by

\[
T(b) = \int_{\mathbb{R}^n} b(v) \, dv.
\]
Furthermore
\[ T(\Phi_B^\sigma(f)b) = \int_{\mathbb{R}^n} (\Phi_B^\sigma(f)b)(v) \, dv = \int_{\mathbb{R}^n} (\Phi_B \circ (\Phi_{\xi,J})(\Phi_B^\sigma(f)b))(v) \, dv \]
\[ = \int_{\mathbb{R}^n} \Phi_B(f)(b(v)) \, dv = \Phi_B(f) \int_{\mathbb{R}^n} b(v) \, dv, \]
and since \( \Phi_{B,J} = \Phi_B \) as in (4.2), the claim follows. \( \square \)

**Theorem 4.6.** Let \( h \in [0,1] \). The evaluation map
\[ \pi_h : \Gamma((A_{i,J})_{i \in [0,1]}) \rightarrow A_{h,J} \]
is a KK-equivalence.

**Proof.** As \( \Gamma((A_{i,J})_{i \in [0,1]}) = B_J \) is \( C([0,1]) \)-linearly \(*\)-isomorphic to \( B^\psi \), and \( B^\psi \) is an RKK-fibration (Theorem 3.7), thus \( \Gamma((A_{i,J})_{i \in [0,1]}) \) is an RKK-fibration and hence a KK-fibration (Lemma 3.5). So for any \( f : \Delta^p \rightarrow [0,1] \) and \( v \in \Delta^p \), the quotient map \( q_v : f^* (\Gamma((A_{i,J}))) \rightarrow A_{f(v),J} \) is a KK-equivalence. We take the identity function of the 1-simplex, namely \( f : \Delta^1 = [0,1] \rightarrow [0,1] \), \( f(s) = s \). Then \( f^* (\Gamma((A_{i,J}))) = \Gamma((A_{i,J})) \), \( q_h = \pi_h \) and
\[ \pi_h : \Gamma((A_{i,J})) \rightarrow A_{h,J} \]
is a KK-equivalence, for every \( h \in [0,1] \). \( \square \)

## 5 Comments

### 5.1 Theta deformation

Here we discuss a special case of Rieffel deformation, namely *theta deformation* and one possible variation to the above approach to KK-equivalence by bundle methods. Theta deformation concerns a separable C*-algebra \( A \) on which there is a strongly continuous action of the \( n \)-torus, \( \sigma : \mathbb{T}^n \rightarrow \text{Aut}(A) \), with a given skew-symmetric matrix \( \theta \in M_n(\mathbb{R}) \). This is just a special case of Rieffel deformation in which the \( n \)-torus is regarded as the quotient \( \mathbb{T}^n = \mathbb{R}^n / 2\pi \mathbb{Z}^n \), and one obtains the deformed algebra \( A_{\theta} \). An alternative and perhaps more direct picture can be given by following [1]. First define \( C(\mathbb{T}^n_\theta) \) to be the unital C*-algebra generated by unitaries \( u_1, \ldots, u_n \) with relations
\[ u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j, \quad \text{for } j, k = 1, \ldots, n. \]
(Note that this is just the Rieffel deformation \( C(\mathbb{T}^n)_{\theta} \) of the commutative C*-algebra \( C(\mathbb{T}^n) \) with respect to the translation action of the \( n \)-torus; the notation \( C(\mathbb{T}^n) \) is suggestive of the terminology of "noncommutative manifolds" as in [1]). On \( C(\mathbb{T}^n_\theta) \) there is the action \( \tau : \mathbb{T}^n \rightarrow \text{Aut}(C(\mathbb{T}^n_\theta)) \), \( \tau_s(u_j) = e^{2\pi i s_j} u_j \), for \( s \in \mathbb{T}^n \). By considering the diagonal action \( \sigma \circ \tau^{-1} : \mathbb{T}^n \rightarrow \text{Aut}(A \otimes C(\mathbb{T}^n_\theta)) \) one defines the theta deformed algebra
\[ A_{\theta} = (A \otimes C(\mathbb{T}^n_\theta))^{\sigma \circ \tau^{-1}} \tag{5.1} \]
as the fixed-point C*-subalgebra for this diagonal action.

We shall define a continuous C*-bundle over \([0,1]\) whose fiber over \( t \in [0,1] \) will not be \( A_{\theta} \) per se, but will be strongly Morita equivalent to it. The benefit of this particular bundle will be that the evaluation map will easily be seen to yield a KK-equivalence element, and the remaining KK-equivalence is then given by the strong Morita equivalence. First we record the result we need regarding the strong Morita equivalence.
Lemma 5.1. \( A_\theta \sim_M A \rtimes_{\sigma} \mathbb{T}^n \rtimes_{\gamma_1} \mathbb{Z} \times \cdots \times_{\gamma_n} \mathbb{Z} \).

Proof. By results of [12] we get the strong Morita equivalence
\[
(A \otimes C(T^0))^{\sigma \otimes \tau^{-1}} \sim_M (A \otimes C(T^0)) \rtimes_{\sigma \otimes \tau^{-1}} \mathbb{T}^n.
\]
The latter crossed product algebra is *-isomorphic to the crossed product in the statement of the lemma, which we now define. Let \( \gamma_1 \in Aut(A \rtimes_{\sigma} \mathbb{T}^n) \) be \( \gamma_1(g)(s) = e^{2\pi is}g(s) \) for \( g \in A \rtimes_{\gamma_1} \mathbb{T}^n \) and let \( u_1 \) be the implementing unitary. Proceed inductively to define actions \( \gamma_2, \ldots, \gamma_n \) with implementing unitaries \( u_2, \ldots, u_n \) so that
\[
(5.2) \quad \gamma_j(u_k) = e^{2\pi i\theta_j,k}u_k, \quad j < k,
\]
so the covariance relation \( \gamma_j(u_k) = u_j u_k^* e^{2\pi i\theta_j,k} u_k \) means precisely \( u_j u_k = e^{2\pi i\theta_j,k} u_k u_j \).

The *-isomorphism \( (A \otimes C(T^0)) \rtimes_{\sigma \otimes \tau^{-1}} \mathbb{T}^n \to A \rtimes_{\gamma_1} \mathbb{T}^n \rtimes_{\gamma_2} \mathbb{Z} \times \cdots \times_{\gamma_n} \mathbb{Z} \) can be explicitly described on the dense *-subalgebra \( A \otimes C(T^0) \otimes C(\mathbb{T}^n) \) as \( a \otimes u_j \otimes h \mapsto u_j(ah) \) where one understands \( ah \in A \otimes C(\mathbb{T}^n) \subseteq A \rtimes_{\gamma_1} \mathbb{T}^n \).

Let \( B = C([0,1]) \otimes A \rtimes_{\gamma_1} \mathbb{T}^n = C([0,1], A \rtimes_{\sigma} \mathbb{T}^n) \). We may decompose \( \sigma \) into its coordinate actions \( \sigma_1, \ldots, \sigma_n \) where \( \sigma_j(z) = \sigma_1^{(j, \ldots, j, 0, \ldots, 0)} \) for \( z \in \mathbb{T} \). For \( j, k = 1, \ldots, n \) let \( h_{j,k} \in C([0,1]) \) be the function
\[
h_{j,k}(t) = e^{2\pi i\theta_j,k}.
\]
Define \( \alpha_1 \in Aut(B) \) by
\[
\alpha_1(f \otimes g) = f \otimes \hat{\sigma}_1(g), \quad f \in C([0,1]), g \in A \rtimes_{\gamma_1} \mathbb{T}^n
\]
and let \( v_1 \) be the unitary implementing \( \alpha_1 \) in \( B \rtimes_{\alpha_1} \mathbb{Z} \). Define \( \alpha_2 \in Aut(B \rtimes_{\alpha_1} \mathbb{Z}) \) by
\[
\alpha_2((f \otimes g)v_1^m) = (h_{1,2} f \otimes \hat{\sigma}_{2}^m(g))v_1^m, \quad m \in \mathbb{Z}.
\]
Proceeding inductively we thus obtain actions \( \alpha_1, \ldots, \alpha_n \) with respective implementing unitaries \( v_1, \ldots, v_n \),
\[
\alpha_k((f \otimes g)v_j^m) = v_k((f \otimes g)v_j^m)^* v_k = (h_{j,k} f \otimes \hat{\sigma}_{k}^m(g))v_j^m.
\]
Let \( \pi_1 : C([0,1]) \otimes A \rtimes_{\gamma_1} \mathbb{T}^n \to A \rtimes_{\gamma_1} \mathbb{T}^n \) be the evaluation map, \( \pi_1(f \otimes g) = f(t)g \). For each \( t \in [0,1] \), starting with \( A \rtimes_{\gamma_1} \mathbb{T}^n \) inductively define actions \( \gamma_1^t, \ldots, \gamma_n^t \) as in \( (5.2) \) with respective unitaries \( u_1, \ldots, u_n \) such that
\[
\gamma_j^t(u_k) = e^{2\pi i\theta_j,k}u_k.
\]
Note that the actions \( \gamma_j \) of \( (5.2) \) are just \( \gamma_j = \gamma_j^1 \) with \( t = 1 \). Furthermore, \( \pi_t \circ \alpha_1 = \gamma_1^t \circ \pi_t \), i.e. \( \pi_1 \) is a \( \mathbb{Z} \)-equivariant *-homomorphism between the \( \mathbb{C}^* \)-dynamical systems and so passes to a *-homomorphism between the crossed products
\[
\pi_t : (C([0,1]) \otimes A \rtimes_{\gamma_1} \mathbb{T}^n) \rtimes_{\alpha_1} \mathbb{Z} \to A \rtimes_{\gamma_1^t} \mathbb{T}^n \rtimes_{\gamma_1^t} \mathbb{Z}, \quad \quad (5.3)
\]
which is a continuous \( \mathbb{C}^* \)-bundle. Iterating this, one has \( \pi_t \circ \alpha_j = \gamma_j^t \circ \pi_t \) for each \( j = 1, \ldots, n \), where \( \pi_t \) is understood on the appropriate crossed product. Thus we get a continuous \( \mathbb{C}^* \)-bundle
\[
\pi_t : C([0,1], A \rtimes_{\gamma_1} \mathbb{T}^n) \rtimes_{\alpha_1} \mathbb{Z} \times \cdots \times_{\alpha_n} \mathbb{Z} \to A \rtimes_{\gamma_1^t} \mathbb{T}^n \rtimes_{\gamma_1^t} \mathbb{Z} \times \cdots \times_{\gamma_n^t} \mathbb{Z}, \quad \quad (5.4)
\]
For each \( t \in [0, 1] \) let \( I_t = \{ f \in C([0, 1]) \mid f(t) = 0 \} \) be the ideal of functions vanishing at the point \( t \). The ideal \( I_t \otimes A \rtimes_\sigma \mathbb{T}^n \subseteq C([0, 1]) \otimes A \rtimes_\sigma \mathbb{T}^n \) is \( \alpha_1 \)-invariant, so it follows that the kernel of the \(*\)-homomorphism \( \pi_t \) in (5.3) is
\[
ker \pi_t = (I_t \otimes A \rtimes_\sigma \mathbb{T}^n) \rtimes_{\alpha_1} \mathbb{Z}.
\]
By iteration, it follows that the kernel of the \(*\)-homomorphism \( \pi_t \) in (5.4) is
\[
k\ker \pi_t = (I_t \otimes A \rtimes_\sigma \mathbb{T}^n) \rtimes_{\alpha_1} \mathbb{Z} \rtimes \cdots \rtimes_{\alpha_n} \mathbb{Z}.
\]
Using a homeomorphism of \([0, 1]\) to itself, mapping \( t \) to 1, there is a \(*\)-isomorphism \( I_t \cong C_0([0, 1]) \). This means \( I_t \otimes A \rtimes_\sigma \mathbb{T}^n \cong C_0([0, 1]) \otimes A \rtimes_\sigma \mathbb{T}^n = Cone(A \rtimes_\sigma \mathbb{T}^n) \), hence
\[
k\ker \pi_t = Cone(A \rtimes_\sigma \mathbb{T}^n) \rtimes_{\alpha_1} \mathbb{Z} \rtimes \cdots \rtimes_{\alpha_n} \mathbb{Z}.
\]
(5.5)

We recall a few general facts which we will appeal to shortly, in particular contractibility of cones and the Pimsner-Voiculescu six-term exact sequence. First, a \( C^* \)-algebra \( B \) is called KK-contractible if \( KK(B, B) = 0 \). This also implies \( KK(B, D) = 0 = KK(D, B) \) for any other \( C^* \)-algebra \( D \).

Suppose there is an action \( \beta \in Aut(B) \). The Pimsner-Voiculescu six-term exact sequence in KK-theory is
\[
\begin{array}{cccccc}
KK(D, B) & \overset{1-\beta_*}{\longrightarrow} & KK(D, B) & \longrightarrow & KK(D, B \rtimes_\beta \mathbb{Z}) \\
\uparrow & & \downarrow & & \\
KK^1(D, B \rtimes_\beta \mathbb{Z}) & \leftarrow & KK^1(D, B) & \leftarrow & KK^1(D, B)
\end{array}
\]
Observe that if \( B \) is KK-contractible, then the six-term exact sequence reads
\[
\begin{array}{cccccc}
0 & \overset{1-\beta_*}{\longrightarrow} & 0 & \longrightarrow & KK(D, B \rtimes_\beta \mathbb{Z}) \\
\uparrow & & \downarrow & & \\
KK^1(D, B \rtimes_\beta \mathbb{Z}) & \leftarrow & 0 & \leftarrow & 0
\end{array}
\]
and using in particular \( D = B \rtimes_\beta \mathbb{Z} \) we deduce \( KK(B \rtimes_\beta \mathbb{Z}, B \rtimes_\beta \mathbb{Z}) = 0 \), i.e. \( B \rtimes_\beta \mathbb{Z} \) is KK-contractible.

Given any separable \( C^* \)-algebra \( D \), its cone \( Cone(D) = C_0([0, 1]) \otimes D \) is KK-contractible.

**Theorem 5.2.** For every \( t \in [0, 1] \) the bundle map
\[
\pi_t : C([0, 1], A \rtimes_\sigma \mathbb{T}^n) \rtimes_{\alpha_1} \mathbb{Z} \times \cdots \rtimes_{\alpha_n} \mathbb{Z} \rightarrow A \rtimes_\sigma \mathbb{T}^n \rtimes_{\gamma_1} \mathbb{Z} \times \cdots \rtimes_{\gamma_n} \mathbb{Z}
\]
gives a KK-equivalence.

**Proof.** From (5.5) \( ker \pi_t = Cone(A \rtimes_\sigma \mathbb{T}^n) \rtimes_{\alpha_1} \mathbb{Z} \times \cdots \rtimes_{\alpha_n} \mathbb{Z} \). Then the KK-contractibility of \( Cone(A \rtimes_\sigma \mathbb{T}^n) \) combined with a repeated Pimsner-Voiculescu six-term sequence argument as above establishes that \( ker \pi_t \) is KK-contractible. This implies that \( \pi_t \) gives a KK-equivalence element.
5.2 Invariance of the index

The index pairing is the pairing between K-theory and K-homology

\[ K_0(A) \times K^0(A) \longrightarrow \mathbb{Z} \]

\[ ([e], [(\mathcal{H}, F)]) = \text{index } (e(F^+ \otimes 1_k)e : e\mathcal{H}^k \longrightarrow e\mathcal{H}^k), \]

(5.6)

for a projection \( e \in M_k(A) \) and Fredholm module \((\mathcal{H}, F)\) for \( A \). This pairing is nothing but the KK-product

\[ KK(\mathbb{C}, A) \times KK(A, \mathbb{C}) \longrightarrow KK(\mathbb{C}, \mathbb{C}) \]

(5.7)

after the identifications \( K_0(A) = KK(\mathbb{C}, A), K^0(A) = KK(A, \mathbb{C}) \) and \( KK(\mathbb{C}, \mathbb{C}) = \mathbb{Z} \).

See also [17] for a discussion of theta deformation and the invariance of the index, and moreover a calculation of the Chern character map for the deformation.

Given an even spectral triple \((A, \mathcal{H}, D)\) there is the associated Fredholm module \((\mathcal{H}, F)\) with \( F = D|D|^{-1} \).

Our separable C*-algebra \( A \) is assumed equipped with an action \( \sigma : \mathbb{T}^n \longrightarrow \text{Aut}(A) \), and let \( \mathcal{A} \subseteq A \) be the dense *-subalgebra of smooth elements for the action. Suppose \((A, \mathcal{H}, D)\) is a spectral triple, with a *-representation \( \varphi : A \longrightarrow B(\mathcal{H}) \). Assume the action to be unitarily implemented by \( U : \mathbb{T}^n \longrightarrow B(\mathcal{H}), \varphi(\sigma_s(a)) = U_s^*\varphi(a)U_s \), and that \( U_sD = DU_s \) for each \( s \in \mathbb{T}^n \).

Theta deformation is an isospectral deformation, meaning that the same data \((\mathcal{H}, D)\) which describes a noncommutative geometry for \( A \), is also taken to serve a noncommutative geometry for \( A_\theta \). In order to study these aspects, it is useful to work with the following picture of the deformation. Any element \( a \in A \) decomposes into a norm convergent series \( a = \sum r \in 2^n a_r \) where each \( a_r \in \mathcal{A} \) satisfies \( \sigma_s(a_r) = e^{-2\pi ir \cdot s}a_r \), for \( s \in \mathbb{T}^n \). Given two elements \( a, b \in \mathcal{A} \) with decompositions \( a = \sum a_r \) and \( b = \sum b_p \), the product \( \times_\theta \) takes the form

\[ a_r \times_\theta b_p = e^{2\pi ir \cdot \theta p}a_r b_p \]

(5.8)

between two component elements \( a_r \) and \( b_p \). The product \( a \times_\theta b \) is then the linear extension of the componentwise product (5.8). The *-algebra \( \mathcal{A}_\theta \) is just \( \mathcal{A} \) equipped with this product. The correspondence with the definition in (5.1) is just

\[ a_r \mapsto a_r \otimes u_1^{r_1} \cdots u_n^{r_n} \in (A \otimes C(\mathbb{T}^n_\theta))^{\sigma \otimes r^{-1}}. \]

We have a representation \( \varphi_\theta \) of \( \mathcal{A}_\theta \) on the same Hilbert space, \( \varphi_\theta : \mathcal{A}_\theta \longrightarrow B(\mathcal{H}) \), by

\[ \varphi_\theta(a) = \sum_r \varphi(a_r)U_{q(\theta r)}, \]

\( A_\theta \) is then the norm closure and \((\mathcal{A}_\theta, \mathcal{H}, D_\theta)\) is the deformed spectral triple, where \( D_\theta = D \).

For the even spectral triple \((A, \mathcal{H}, D)\) we shall denote by \([D] = [(\mathcal{H}, \varphi, F)] \in K^0(A)\) the corresponding element of K-homology. Likewise we denote by \([D_\theta] = [(\mathcal{H}, \varphi_\theta, F)] \in K^0(A_\theta)\) the element associated to the spectral triple \((\mathcal{A}_\theta, \mathcal{H}, D_\theta)\).

**Corollary 5.3.** The KK-equivalence of Theorem 4.6 induces an isomorphism \( K^0(A) \cong K^0(A_\theta) \) mapping \([D] \mapsto [D_\theta]\).

**Proof.** Let \( \Gamma = \Gamma((A_\theta)_{\theta \in [0,1]}). \) From the bundle maps \( \pi_0 : \Gamma \longrightarrow A \) and \( \pi_1 : \Gamma \longrightarrow A_\theta \) we get by Theorem 4.6 the KK-equivalence elements \([\pi_0] \in KK(\Gamma, A)\) and \([\pi_1] \in KK(\Gamma, A_\theta)\). The relevant
mappings between KK-groups is described by the KK-products

\[
\begin{array}{c}
\xymatrix{
KK(\Gamma, C) \\
KK(A, C) \ar[r]_{\pi_0} \ar[rru]_{\pi_1} & & KK(A_\theta, C) \\
[p_0] & & [p_1]\n}
\end{array}
\]

where \([p_0] = [(A, \pi_0, 0)] \in KK(A, C)\) and \([p_1] = [(A_\theta, \pi_1, 0)] \in KK(\Gamma, A_\theta)\) are the KK-cycle descriptions.

The element \([D] = [(\mathcal{H}, \phi, F)] \in KK(A, C)\) is the element canonically associated to the given spectral triple \((A, \mathcal{H}, D)\) as explained above, and upon taking the KK-product we get

\[
[p_0] \cdot [D] = [(\mathcal{H}, \phi \circ \pi_0, F)] \in KK(\Gamma, C).
\] (5.9)

Likewise \([D_\theta] = [(\mathcal{H}, \phi_\theta, F)] \in KK(A_\theta, C)\) is the element associated to the deformed spectral triple \((A_\theta, \mathcal{H}, D_\theta)\), and the KK-product is then

\[
[p_1] \cdot [D_\theta] = [(\mathcal{H}, \phi \circ \pi_1, F)] \in KK(\Gamma, C).\] (5.10)

It will be enough to establish the equality \([p_0] \cdot [D] = [p_1] \cdot [D_\theta]\) in \(KK(\Gamma, C)\). This follows from homotopy of KK-cycles. Indeed, let \((E, \phi, F) \in KK(\Gamma, I C)\) be the element where \(E = C([0, 1], \mathcal{H}), \phi : \Gamma \to L_I C(E), (\phi(s)\xi)(t) = s(t)\xi(t)\), and \(I C = C([0, 1]) \otimes C = C([0, 1])\). Let \(e_{v_0}\) and \(e_{v_1}\) denote the respective evaluation morphisms \(E \to \mathcal{H}\). It is easy to check (using details explained in [2]) that \((E, \phi, F)\) provides a homotopy between the KK-cycles (5.9) and (5.10), i.e. isomorphisms of the KK-cycles with the pushouts of \(e_{v_0}\) and \(e_{v_1}\) respectively,

\[
(E_{v_0}, \phi_{e_{v_0}}, F_{e_{v_0}}) \cong [(\mathcal{H}, \phi \circ \pi_0, F)] \quad \text{and} \quad (E_{v_1}, \phi_{e_{v_1}}, F_{e_{v_1}}) \cong [(\mathcal{H}, \phi \circ \pi_1, F)].
\]

The KK-equivalence of Theorem 4.6 implies the isomorphisms

\[
K_0(A) = KK(C, A) \to KK(C, A_\theta) = K_0(A_\theta), \quad [e] \mapsto [e] \cdot [p_0]^{-1} \cdot [p_1],
\]

and

\[
K^0(A) = KK(A, C) \to KK(A_\theta, C) = K^0(A_\theta), \quad [(\mathcal{H}, F)] \mapsto [p_1]^{-1} \cdot [p_0] \cdot [(\mathcal{H}, F)],
\]

and regarding the index pairing (5.6) or equivalently the KK-product (5.7), we get

\[
\begin{array}{c}
K_0(A) \times K^0(A) \xrightarrow{\text{index}} \mathbb{Z} \\
\downarrow \quad \downarrow \\
K_0(A_\theta) \times K^0(A_\theta) \xrightarrow{\text{index}} \mathbb{Z}
\end{array}
\]

where \([e] \cdot [(\mathcal{H}, F)]\) is the top index pairing and

\[
[e] \cdot [p_0]^{-1} \cdot [p_1] \cdot [p_1]^{-1} \cdot [p_0] \cdot [(\mathcal{H}, F)] = [e] \cdot [(\mathcal{H}, F)]
\]

is the bottom index pairing after having followed the isomorphisms induced by the KK-equivalences.
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Amandip Sangha  
Department of Mathematics,  
University of Oslo,  
PO Box 1053 Blindern,  
N-0316 Oslo, Norway.  
amandip.s.s.sangha@gmail.com