Hydrodynamic modes and pulse propagation in a cylindrical Bose gas above the Bose-Einstein transition

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Abstract

We study hydrodynamic oscillations of a cylindrical Bose gas above the Bose-Einstein transition temperature using the hydrodynamic equations derived by Griffin, Wu and Stringari. This extends recent studies of a cylindrical Bose-condensed gas at $T = 0$. Explicit normal mode solutions are obtained for non-propagating solutions. In the classical limit, the sound velocity is shown to be the same as a uniform classical gas. We use a variational formulation of the hydrodynamic equations to discuss the propagating modes in the degenerate Bose-gas limit and show there is little difference from the classical results.

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I. INTRODUCTION

Recently Andrews et al. [1] reported a measurement of a sound pulse propagation of Bose-condensed cloud in a highly asymmetric cigar-shaped trap. They have measured the sound velocity in a Bose-condensate as a function of the density. Zaremba [2] gave a detailed analysis of the collective excitations in a cylindrical Bose gas starting from the $T = 0$ quantum hydrodynamic equation of Stringari [3], which is based on the Thomas-Fermi approximation. The sound velocity of a condensate pulse obtained in Ref. [2] is in good agreement with the experimental observations [1]. More recently, several other theoretical studies have discussed the excitations and pulse propagation in a cigar-shaped trap at $T = 0$ [4–6].

In this paper, we consider the analogous modes in a hydrodynamic regime (where collisions ensure local thermal equilibrium) for a cylindrical Bose gas above the Bose-Einstein transition temperature $T_{\text{BEC}}$. Our starting point is the hydrodynamic equations derived by Griffin, Wu and Stringari [7].

II. HYDRODYNAMIC NORMAL MODE EQUATIONS FOR A CYLINDRICAL BOSE GAS

The linearized hydrodynamic equation for the velocity fluctuations $v(r,t)$ derived by Griffin, Wu and Stringari is [7]:

$$m \frac{\partial^2 v}{\partial t^2} = \frac{5P_0(r)}{3n_0(r)} \nabla (\nabla \cdot v) - \nabla [v \cdot \nabla U_0(r)] - \frac{2}{3}(\nabla \cdot v) \nabla U_0(r) - \frac{\partial}{\partial t} \nabla \delta U(r,t),$$  \hspace{1cm} (1)

where $U_0(r)$ is the static cylindrical trap potential and $\delta U(r,t)$ is a small time-dependent external perturbation. The equilibrium local density $n_0(r)$ and the equilibrium local kinetic pressure $P_0(r)$ in (1) are given by

$$n_0(r) = \frac{1}{\Lambda^3} g_{3/2}(z_0), \quad P_0(r) = \frac{k_B T}{\Lambda^3} g_{5/2}(z_0),$$

where $z_0(r) \equiv \frac{1}{k_B T}(\mu - U_0)$ is the local equilibrium fugacity, $\Lambda \equiv (2\pi \hbar^2 / mk_B T)^{1/2}$ is the thermal de Broglie wave-length and $g_n(z) = \sum_{l=1}^{\infty} z^l / l^n$ are the well-known Bose-Einstein functions.
Throughout this paper, we shall limit our discussion to a purely cylindrical harmonic trap potential

\[ U_0(r) = \frac{1}{2} m \omega_0^2 (x^2 + y^2). \]  

(3)

Inserting this into (1), we obtain coupled equations for the radial \( v_{\perp} \) and axial \( v_z \) velocity fluctuations

\[
m \frac{\partial^2 v_{\perp}}{\partial t^2} = \frac{5 P_0}{3 n_0} \nabla_\perp (\nabla_\perp \cdot v_{\perp}) - m \omega_0^2 \nabla_\perp (r_{\perp} \cdot v_{\perp}) - \frac{2}{3} m \omega_0^2 (\nabla_\perp \cdot v_{\perp}) r_{\perp} \\
+ \left( \frac{5 P_0}{3 n_0} \nabla_\perp - \frac{2}{3} m \omega_0^2 r_{\perp} \right) \frac{\partial v_z}{\partial r_{\perp}} ,
\]

(4a)

\[
m \frac{\partial^2 v_z}{\partial t^2} = \frac{5 P_0}{3 n_0} \frac{\partial^2 v_z}{\partial z^2} + \left( \frac{5 P_0}{3 n_0} \nabla_\perp - m \omega_0^2 r_{\perp} \right) \cdot \frac{\partial v_{\perp}}{\partial z} ,
\]

(4b)

where we have set \( \delta U(r, t) = 0 \) since we are interested in normal mode solutions (driven solutions will be discussed in Section VI). We use the convention \( v(r, t) = v_\omega(r)e^{-i \omega t} \).

Finally, we shall look for solutions of the kind

\[ v_\omega(r) = (x f(r_{\perp}), y f(r_{\perp}), h(r_{\perp}))e^{ikz}, \]

(5)

where \( r_{\perp} = \sqrt{x^2 + y^2} \). That is, we assume that the functions \( f \) and \( h \) do not depend on the radial azimuthal angle.

Using (1) in (4), one finds, after some algebra, a coupled set of equations for the radial function \( f \) and the axial function \( h \)

\[
\omega^2 f = -c_0^2(r_{\perp}) \left( \frac{\partial^2 f}{\partial r_{\perp}^2} + \frac{3}{r_{\perp}} \frac{\partial f}{\partial r_{\perp}} \right) + \omega_0^2 \left( \frac{10}{3} f + \frac{5}{3} r_{\perp} \frac{\partial f}{\partial r_{\perp}} \right) - ik \left[ c_0^2(r_{\perp}) \frac{1}{r_{\perp}} \frac{\partial h}{\partial r_{\perp}} - \frac{2}{3} \omega_0^2 h \right],
\]

(6a)

\[
\omega^2 h = c_0^2(r_{\perp}) k^2 h - ik \left[ c_0^2(r_{\perp}) \left( 2 f + r_{\perp} \frac{\partial f}{\partial r_{\perp}} \right) - \omega_0^2 r_{\perp}^2 f \right].
\]

(6b)

The position-dependent local "sound velocity" \( c_0(r_{\perp}) \) is defined by

\[
c_0^2(r_{\perp}) \equiv \frac{5 P_0(r_{\perp})}{3 m n_0(r_{\perp})} = \frac{5 k_B T}{3 m} B(z_0), \quad B(z_0) \equiv \frac{g_5/2(z_0)}{g_3/2(z_0)}.
\]

(7)
One can show that the normal mode solutions of (6a) and (6b) satisfy the following orthogonality

$$\int dr \, n_0(r) v^*_{\omega'}(r) \cdot v_{\omega}(r) = 0, \text{ if } \omega_{n'} \neq \omega_n,$$

or, more explicitly,

$$\int_0^\infty dr_\perp r_\perp n_0(r_\perp)[r^2_\perp f^*_{n'}(r_\perp) f_n(r_\perp) + h^*_{n'}(r_\perp) h_n(r_\perp)] = 0, \text{ if } n' \neq n.$$  \hspace{1cm} (8b)

Here the label \( n \) specifies the different normal mode solutions. We remark that while, it is not obvious, if we set the trap frequency \( \omega_0 \) to zero, (6a) and (6b) have solutions involving Bessel functions \( J(k_\perp r_\perp) \) with the expected phonon dispersion relation \( \omega^2 = c_0^2(k^2 + k_\perp^2) \).

It is convenient to introduce a dimensionless radial variable

$$s \equiv \frac{r^2_\perp}{R^2}, \quad R \equiv \left( \frac{2k_B T}{m\omega^2_0} \right)^{1/2}.$$  \hspace{1cm} (9)

We also introduce a dimensionless frequency and wavevector

$$\bar{\omega} \equiv \frac{\omega}{\omega_0}, \quad \bar{k} \equiv kR.$$  \hspace{1cm} (10)

We observe from (2) that \( R \) denotes the "size" of the radial density profile produced by the harmonic potential trap. In these units, we note \( z_0 = e^{\beta \mu_0} e^{-s} \) and the classical density profile \( n_0(r_\perp) \sim e^{-s} \). In terms of these new dimensionless variables, it is useful to introduce the new functions:

$$\bar{f}(s) \equiv -iRf(r_\perp), \quad \bar{h}(s) \equiv h(r_\perp).$$  \hspace{1cm} (11)

Using (6), the coupled equations for \( \bar{f} \) and \( \bar{h} \) are given by

$$\bar{\omega}^2 \bar{f} = \hat{L}[\bar{f}] - \bar{k} \left( \frac{5}{3} B(z_0) \frac{d\bar{h}}{ds} - \frac{2}{3} \bar{h} \right),$$

$$\bar{\omega}^2 \bar{h} = \frac{5}{6} B(z_0) \bar{k}^2 \bar{h} - \bar{k} \left[ \frac{5}{3} B(z_0) s \frac{d\bar{f}}{ds} + \left( \frac{5}{3} B(z_0) - s \right) \bar{f} \right],$$  \hspace{1cm} (12b)

where we have introduced the operator \( \hat{L} \)

$$\hat{L}[\bar{f}] \equiv - \frac{10}{3} \left[ B(z_0) s \frac{d^2\bar{f}}{ds^2} + (2B(z_0) - s) \frac{d\bar{f}}{ds} - \bar{f} \right].$$  \hspace{1cm} (13)
The rest of this paper is based on the equations in (12a) and (12b), which determine the normal mode velocity fluctuations using (11) and (5). The associated density fluctuations $\delta n(r,t)$ can be found by using the number conservation law

$$\frac{\partial \delta n}{\partial t} = - \nabla \cdot (n_0 \mathbf{v}).$$ \hspace{1cm} (14)

III. NON-PROPAGATING MODES

We first examine non-propagating solutions ($k = 0$) of (12a) and (12b), in which case they reduce to the two independent equations

$$\bar{\omega}^2 \bar{f} = \hat{L} \bar{f}, \quad \bar{\omega}^2 \bar{h} = 0. \hspace{1cm} (15)$$

There is a trivial zero frequency solution $\bar{f} = 0$, $\bar{h} \neq 0$, corresponding to $v_\perp = 0, v_z \neq 0$. In this case, the dependence of $\bar{h}$ on $s$ cannot be determined uniquely. Using (14), one finds that $\delta n = 0$ for this mode. The interesting solutions of (15) with $\bar{\omega} \neq 0$ are given by

$$\bar{h} = 0, \quad \bar{\omega}^2 \bar{f} = \hat{L} \bar{f}. \hspace{1cm} (16)$$

These correspond to oscillations only in the radial direction, which we shall now discuss.

In the classical gas limit, the operator $\hat{L}$ in (14) simplifies since $B(z_0) = 1$. In this case, we can obtain the complete set of normal mode solutions of (14), namely

$$\bar{f}_n^{(0)}(s) = \frac{1}{n! \sqrt{n}} \frac{d^n}{ds^n} L_n(s), \quad [\bar{\omega}_n^{(0)}]^2 = \frac{10n}{3}, \quad n = 1, 2, 3, \cdots. \hspace{1cm} (17)$$

Here $L_n(s)$ is the Laguerre polynomial defined by

$$L_n(s) \equiv e^s \frac{d^n}{ds^n} (s^n e^{-s}), \hspace{1cm} (18)$$

and the orthogonal functions $\bar{f}_n^{(0)}$ are normalized according to (see (8b))

$$\int_0^\infty ds \, se^{-s} \bar{f}_n^{(0)}(s) \bar{f}_{n'}^{(0)}(s) = \delta_{nn'}. \hspace{1cm} (19)$$

In ordinary variables, the dispersion relation of these $k = 0$ solutions corresponds to
\[ \omega_n^2 = n \frac{10}{3} \omega_0^2, \quad n = 1, 2, 3, \ldots \]  

(20)

and the associated density fluctuation is given by

\[ \delta n(r, t) \propto L_n(s = r^2 / R^2) \exp(-r^2 / R^2)e^{-i\omega_n t}. \]  

(21)

In the lowest \((n = 1)\) mode with \(\omega_1^2 = 10\omega_0^2 / 3\), \(\bar{f}^{(0)}(s)\) is independent of \(s\) and \(L_1(s) = 1 - s\). This is the two-dimensional radial breathing mode. We note that this particular mode corresponds to one of the coupled monopole-quadrupole modes found in Ref. [7] for an anisotropic trap in the limit that the axial trap frequency \((\omega_z)\) is set to zero. In this same limit \((\omega_z = 0)\), the other mode has zero frequency and a velocity fluctuation is given by \(v_\omega = \alpha(x, y, -5z)\). This kind of solution is not described by the form (5) which we are dealing with in this paper.

For a degenerate Bose gas, in which \(B(z_0)\) is now weakly dependent on \(s\) through \(z_0 = e^{\beta\mu_0 - s}\), one cannot solve (16) analytically. We discuss variational solutions in Section V. However one can check that \(\bar{f}(s) = \text{constant}\) is a solution, with frequency \(\omega^2 = \frac{10}{3} \omega_0^2\). This is the analogue of the \(n = 1\) normal mode in [7] for the classical gas (we recall that \(\bar{f}^{(0)}(0)\) is independent of \(s\)).

For comparison with (20), the analogous non-propagating normal modes in a cigar trap at \(T = 0\) have a spectrum given by [3]

\[ \omega_l^2 = 2l(l + 1)\omega_0^2, \quad l = 0, 1, 2, \ldots \]  

(22)

### IV. PROPAGATING MODES

In this section, we discuss the more interesting propagating solution \((\bar{k} \neq 0)\) of equations (12a) and (12b). In the classical limit \((B(z_0) = 1)\), one immediately finds a phonon mode solution

\[ \bar{h} = \exp(2s/5), \quad \bar{f} = 0, \quad \bar{\omega}^2 = \frac{5}{6} \bar{k}^2. \]  

(23)
This is a longitudinal sound wave with the dispersion relation \( \omega = c_0 k \), where the sound velocity \( c_0^2 = 5k_B T/3m \) is the same as for a uniform classical gas. There is no radial oscillation (i.e. \( \bar{f} = 0 \)) associated with this phonon mode in the classical limit. Using (14), the associated density fluctuation is found to be

\[
\delta n(r, t) \propto \exp(-3r_{\perp}^2/5R^2) \exp(ikz - ic_0 kt).
\] (24)

In the classical limit, one can show that the dispersion relation \( \omega = c_0 k \) is valid for any cylindrical trap potential \( U_0(r_\perp) \), i.e. it is not limited to parabolic potentials. In this more general case, the solution of the hydrodynamic equation corresponding to the phonon-like mode in the classical limit is given by \( h = \exp(2U_0/5k_B T) \) and \( f = 0 \), with the associated density fluctuation \( \delta n \propto \exp(-3U_0/5k_B T) \).

In order to find how the non-propagating normal mode solutions in (17) are modified when \( k \neq 0 \), we expand \( \bar{f} \) as follows:

\[
\bar{f}(s) = \sum_n a_n \bar{f}_n^{(0)}(s).
\] (25)

This follows the approach of Zaremba [2] for a Bose-condensed gas at \( T = 0 \) in a cigar-shaped trap. Substituting (25) into (12a) and (12b), we obtain the coupled linear equations for the coefficients \( a_n \):

\[
\left( \bar{\omega}^2 - \left[ \bar{\omega}_n^{(0)} \right]^2 - \frac{5}{6} \frac{\left[ \bar{\omega}_n^{(0)} \right]^2}{\bar{\omega}^2 - \frac{2}{3} k^2 \bar{\omega}} \right) a_n + \frac{2}{3} \frac{k^2}{\bar{\omega}^2 - \frac{2}{3} k^2} \sum_{n'} M_{nn'} a_{n'} = 0,
\] (26)

where the matrix elements \( M_{nn'} \) are defined by

\[
M_{nn'} \equiv \int_0^\infty ds \ s^2 e^{-s} \bar{f}_n^{(0)}(s) \bar{f}_{n'}^{(0)}(s).
\] (27)

Using the identity for Laguerre polynomials

\[
\int_0^\infty ds \ e^{-s} L_n(s) L_{n'}(s) = \delta_{nn'}(n!)^2,
\] (28)

we find

\[
M_{nn'} = 2n\delta_{nn'} - \sqrt{nn'}\delta_{n',n\pm 1}.
\] (29)
To lowest order in $\bar{k}^2$, one finds the eigenvalue $\bar{\omega}^2$ of (26) is given by ($M_{nn} = 2n$)

$$\bar{\omega}^2 \simeq \frac{10}{3} n + \left( \frac{5}{6} - \frac{M_{nn}}{5n} \right) \bar{k}^2 = \frac{10}{3} n + \frac{13}{30} \bar{k}^2. \quad (30)$$

In ordinary units, the excitation spectrum is given by

$$\omega^2_n(k) = n \frac{10}{3} \omega_0^2 + \frac{13}{15} k_B T m k^2, \quad n = 1, 2, 3, \ldots. \quad (31)$$

We note that the correction term in (31) is of order $(kR)^2$ relative to the first term, which is assumed to be large. Thus this spectrum for propagating modes for a classical gas in a cylindrical harmonic trap is only valid for $kR \ll 1$, where $R$ is the radial size of the trapped gas density profile.

**V. VARIATIONAL SOLUTIONS**

For a Bose gas above $T_{\text{BEC}}$, one cannot easily solve the coupled equations in (12a) and (12b) for $k \neq 0$. An alternative approach is to recast these equations into a variational form, following recent work [10] in solving the two-fluid hydrodynamic equations for a trapped Bose-condensed gas [11]. One finds that the functional

$$E[\bar{f}, \bar{h}] \equiv \text{Re} \int_0^\infty ds \frac{g_{3/2}(z_0)}{s} \left[ \bar{s} \bar{f}^* \hat{L}[\bar{f}] + \frac{5}{6} B(z_0)|\bar{h}|^2 \bar{k}^2 - 2s \bar{f}^* \left( \frac{5}{3} B(z_0) \frac{d\bar{h}}{ds} - \frac{2}{3} \bar{h} \right) \bar{k} \right] \int_0^\infty ds \frac{g_{3/2}(z_0)}{s} |\bar{f}|^2 + |\bar{h}|^2$$

has the property that conditions $\delta E/\delta \bar{f} = 0, \delta E/\delta \bar{h} = 0$ yields Eqs.(12a) and (12b). Thus the normal mode eigenvalues $\omega^2$ are given by the stationary value of this functional $E[\bar{f}, \bar{h}]$.

For $\bar{k} = 0$ with $\bar{\omega} \neq 0$, we can use (32) to estimate the normal mode frequencies using the classical solutions of (17), $\bar{f} = \bar{f}_n^{(0)}$ and $\bar{h} = 0$, as trial functions. The frequency so determined is given by

$$\bar{\omega}^2 = \frac{\int_0^\infty ds \frac{g_{3/2}(z_0)}{s} \bar{f}_n^{(0)} \hat{L}[\bar{f}_n^{(0)}]}{\int_0^\infty ds \frac{g_{3/2}(z_0)}{s} |\bar{f}_n^{(0)}|^2}
= \frac{10}{3} n + \frac{10}{3} \int_0^\infty ds \frac{[g_{3/2}(z_0) - g_{5/2}(z_0)]\bar{f}_n^{(0)}}{\int_0^\infty ds \frac{g_{3/2}(z_0)}{s} |\bar{f}_n^{(0)}|^2} \left( s^2 \frac{d^2 \bar{f}_n^{(0)}}{ds^2} + 2s \frac{d\bar{f}_n^{(0)}}{ds} \right). \quad (33)$$
The second term in (33) gives the quantum correction to the classical limit.

For \( k \neq 0 \), the most interesting propagating mode is the phonon mode with \( \omega \propto k \). For trial functions in (32), we take (see (23))

\[
\bar{f} = \bar{k} A_f, \quad \bar{h} = A_h \exp(2s/5),
\]

where the constants \( A_f \) and \( A_h \) are real and independent of \( s \). To first order in \( \bar{k} \), we find a phonon-like solution \( \bar{\omega} = \bar{c} \bar{k} \), with the (dimensionless) sound velocity \( \bar{c} \) given by

\[
\bar{c}^2 = \left( \frac{5}{6} \int_0^\infty ds \, g_{5/2}(z_0) e^{4s/5} - \frac{2}{15} \left\{ \int_0^\infty ds \, [g_{3/2}(z_0) - g_{5/2}(z_0)] e^{2s/5} \right\}^2 / \int_0^\infty ds \, g_{3/2}(z_0) e^{4s/5} \right) \int_0^\infty ds \, g_{3/2}(z_0) e^{4s/5}.
\]

(35)

The amplitudes in (34) which are associated with this phonon mode have the ratio

\[
\frac{A_f}{A_h} = - \frac{\int_0^\infty ds \, [g_{3/2}(z_0) - g_{5/2}(z_0)] e^{2s/5}}{5 \int_0^\infty ds \, g_{3/2}(z_0) s}.
\]

(36)

One can see that in a degenerate Bose gas, where \( g_{3/2}(z_0) \neq g_{5/2}(z_0) \), the radial oscillations \( (A_f) \) are coupled to the axial oscillations \( (A_h) \).

The normal mode frequencies given by the variational expressions in (33) and (35) can be numerically calculated. All the integrals can be evaluated analytically using the useful identity,

\[
\int_0^\infty ds \, g_n(z_0) s^m = m! g_{n+m+1}(\tilde{z}_0),
\]

where \( \tilde{z}_0 \equiv e^{\beta \mu_0} \) and \( g_n \) is the Bose-Einstein function, as defined below (2). It is useful to plot the temperature dependence relative to the \( T_{\text{BEC}} \) for an ideal gas in a cigar-shaped trap described by (3). For a trap of length \( L \) with \( N \) atoms, we have (using (2))

\[
N = \int d\mathbf{r} \, n_0(\mathbf{r}) = \frac{2\pi L}{\Lambda^3} \int_0^\infty dr \, r \, g_{3/2}(z_0) = \frac{\pi R^2 L}{\Lambda^3} \int_0^\infty ds \, g_{3/2}(\tilde{z}_0 e^{-s})
\]

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When $T = T_{\text{BEC}}$, we have $\mu_0 = 0$ and hence $\tilde{z}_0 = 1$. Thus (38) gives the Bose-Einstein transition temperature

$$k_B T_{\text{BEC}} = \hbar \omega_0 \left[ \frac{N}{L} \left( \frac{2\pi \hbar^2}{m \omega_0} \right)^{1/2} \frac{1}{\zeta(5/2)} \right]^{2/5}$$

for a cigar trap in the usual semiclassical approximation.

In Fig. 1, we show how the sound velocity given by (33) varies with temperature down to $T_{\text{BEC}}$, relative to the classical value $c_0 = \sqrt{5k_B T/3m}$. In Fig. 2, we show the temperature-dependent results for the frequencies of the non-propagating modes based on (33). We recall that since $f_1^{(0)}$ is constant, the correction term in (33) vanishes for the $n = 1$ mode. The variational calculations shown in Figs. 1 and 2 indicate that there is little change in the normal mode frequencies given by the classical limit for temperature down to $T = T_{\text{BEC}}$. As noted at the end of Ref. [7], these results are to be expected since the only place where the Bose nature of the gas enters is in the first term of Eq. (1). This involves the ratio $B(z_0) = g_{5/2}(z_0)/g_{3/2}(z_0)$, which is remarkably close to the classical value of unity, even at the center of the trap.

VI. PROPAGATION OF SOUND PULSES

In this section, following the approach of Zaremba [2], we discuss the propagation of sound pulses induced by a small external perturbation $\delta U(r, t)$. We assume that $\delta U(r, t)$ has no radial dependence and is switched on at $t = 0$, i.e. the external perturbation is of the form

$$\delta U(r, t) = \delta U(z) \theta(t).$$

The equation of motion (1) with the external perturbation $\delta U$ can be solved [2] by introducing a Fourier representation of the velocity fluctuations (compare with (3)) and the external perturbation

$$= L \left( \frac{m \omega_0}{2\pi \hbar} \right)^{1/2} \left( \frac{k_B T}{\hbar \omega_0} \right)^{5/2} g_{5/2}(\tilde{z}_0).$$

(38)
\[ \mathbf{v}(r, t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikz} (xf(k, r_\perp, t), yf(k, r_\perp, t), hf(k, r_\perp, t)), \]
\[ \delta U(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikz} \delta U(k). \] (41)

Taking the Fourier transform of (41) and using the radial variable \( s \) defined in (9), we obtain a coupled set of equations (compare with (12a) and (12b))
\[ \frac{\partial^2 \bar{f}}{\partial t^2} + \omega_n^2 \left[ \hat{\mathcal{L}}[\bar{f}] - \bar{k} \left( \frac{5}{3} B(z_0) \frac{\partial \bar{h}}{\partial s} - \frac{2}{3} \bar{h} \right) \right] = 0, \] (42a)
\[ \frac{\partial^2 \bar{h}}{\partial t^2} + \omega_n^2 \left\{ \frac{5}{6} B(z_0) \bar{k}^2 \bar{h}^2 - \bar{k} \left[ \frac{5}{3} B(z_0) s \frac{\partial \bar{f}}{\partial s} + \left( \frac{5}{3} B(z_0) - s \right) \bar{f} \right] \right\} = -i \frac{k}{m} \delta U(k) \delta(t). \] (42b)

Here we have used notation analogous to that in (11), but now \( \bar{f} \) and \( \bar{h} \) also depend on \( t \) and are for particular \( k \) component.

In order to solve these coupled equations, we expand \( \bar{f} \) and \( \bar{h} \) as follows,
\[ \bar{f}(k, s, t) = \sum_m b_m(k, t) \bar{f}_m(k, s), \quad \bar{h}(k, s, t) = \sum_m b_m(k, t) \bar{h}_m(k, s), \] (43)
where the basis functions \( \bar{f}_m \) and \( \bar{h}_m \) are the normal mode solutions of (12). These have eigenvalues \( \bar{\omega}_m^2(k) \) and form an orthonormal set (see (8b)),
\[ \int_0^\infty ds \ g_{3/2}(z_0) [s \bar{f}_m^* \bar{f}_n + \bar{h}_m^* \bar{h}_n] = \delta_{mn}. \] (44)

Substituting the expression (43) into (42) and using (44), we obtain
\[ \frac{\partial^2 b_n}{\partial t^2} + \omega_n^2(k) b_n = F_n(k) \delta(t), \] (45)
where \( \omega_n(k) \equiv \bar{\omega}_n(k) \omega_0 \) and
\[ F_n(k) \equiv -i k \frac{\delta U(k)}{m} \int_0^\infty ds \ g_{3/2}(z_0) \bar{h}_n^*(k, s). \] (46)

The solution of (43) with the boundary condition \( b_n(t < 0) = 0 \) is given by
\[ b_n(k, t) = \theta(t) \frac{F_n(k)}{\omega_n(k)} \sin[\omega_n(k)t], \] (47)
where the quantum number \( n \) refers to the radial degree of freedom.

In order to analyse the time evolution of the associated density fluctuations, it is convenient to work with the radially-averaged density.
\[ \overline{\delta n}(z, t) \equiv \int d\mathbf{r}_\perp \delta n(\mathbf{r}, t). \] (48)

Using (14), one finds

\[
\frac{\partial}{\partial t} \overline{\delta n} = - \int d\mathbf{r}_\perp \nabla \cdot (n_0 \mathbf{v}) = - \int d\mathbf{r}_\perp n_0 \frac{\partial v_z}{\partial z} \\
= -i \frac{2\pi}{\Lambda^3} \int_0^\infty r_\perp dr_\perp g_{3/2}(z_0) \int_{-\infty}^\infty \frac{dk}{2\pi} k e^{ikz} \tilde{h}(k, r_\perp, t) \\
= - \frac{\pi R^2}{\Lambda^3} \sum_n \int_{-\infty}^\infty \frac{dk}{2\pi} e^{ikz} k^2 \delta U(k) \frac{\delta}{\delta \omega_n(k)} \theta(t) \sin \omega_n(k)t \left| \int_0^\infty ds g_{3/2}(z_0) \tilde{h}_n(k, s) \right|^2. \tag{49}
\]

Integrating (49) over \( t \), one finds

\[
\overline{\delta n}(z, t) = - \frac{\pi R^2}{\Lambda^3} \sum_n \int_{-\infty}^\infty \frac{dk}{2\pi} e^{ikz} k^2 \delta U(k) \frac{\delta}{\delta \omega_n(k)} \theta(t) \left[ 1 - \cos \omega_n(k)t \right] \left| \int_0^\infty ds g_{3/2}(z_0) \tilde{h}_n(k, s) \right|^2. \tag{50}
\]

One can see that a low frequency phonon mode makes a large contribution to (50). To illustrate the contribution to (50) which is associated with the phonon density fluctuations \( \omega_n = ck \) in the classical limit, we use \( \tilde{h}_n(k, s) = A_n \exp(2s/5) \) (see (23)) where the normalization condition (44) gives \( A_n^2 = 1/5z_0 \). The contribution to (50) of this classical sound wave is given by (we use \( N_L = \pi R^2 \Lambda^3 z_0 \) appropriate to the classical limit)

\[
\overline{\delta n}(z, t) = - \frac{5}{9} \frac{N \theta(t)}{L mc^2} \left\{ \delta U(z) - \frac{1}{2} \left[ \delta U(z - ct) + \delta U(z + ct) \right] \right\}, \tag{51}
\]

which has the form of a propagating pulse moving with a speed \( \pm c \).

**VII. CONCLUDING REMARKS**

In this paper, we have given a detailed analysis of the hydrodynamic normal modes of a Bose gas in a cigar-shaped trap above \( T_{\text{BEC}} \). We have discussed the non-propagating and propagating modes, both in the classical limit as well as in the degenerate Bose limit just above \( T_{\text{BEC}} \). Our results complement the analogous studies [2,4–6] of such modes in the quantum hydrodynamic limit at \( T = 0 \). In contrast with the \( T = 0 \) analysis, which works with a single equation for the density fluctuations, we have to work with coupled equations.
for the velocity fluctuations. For simplicity, we have considered the limit of a uniform gas along the axial direction. We note that Stringari [4] has discussed the condensate modes at $T = 0$ in the limit of a very weak trap in the axial direction ($\omega_z \ll \omega_0$).

As in Ref. [7], we have ignored the Hartree-Fock mean field contribution to the hydrodynamic equations. Such terms are given in Eq. (6) of Ref. [11]. In place of (1), we obtain the linearized velocity equation

$$m \frac{\partial^2 v}{\partial t^2} = \frac{5P_0(r)}{3n_0(r)} \nabla(\nabla \cdot v) - \nabla[v \cdot \nabla U(r)] - \frac{2}{3}(\nabla \cdot v)\nabla U(r)$$
$$+ 2g \nabla(\nabla \cdot n_0 v) - \frac{\partial}{\partial t} \nabla \delta U(r,t).$$

(52)

This now involves the effective trap potential

$$U(r) = U_0(r) + 2gn_0(r),$$

(53)

which also appears in the equilibrium fugacity $z_0 = e^{\beta(\mu_0 - U)}$ in the expressions for $n_0(r)$ and $P_0(r)$. The usual $s$-wave scattering interaction is $g = 4\pi a\hbar^2/m$. The analysis given in this paper can be generalized [12] to include the effects of this HF mean field but it is much more complicated. We simply quote some final results for the classical limit. The $n = 1$ non-propagating mode has a frequency given by

$$\omega_1^2 = \frac{10}{3} \omega_0^2 \left(1 - \frac{gn_0(r = 0)}{2k_B T}\right),$$

(54)

where $n_0(r = 0)$ is the density at the center of the cylindrical trap. The sound velocity corresponding to (23) is given by

$$c^2 = \frac{5k_B T}{3m} + \frac{gn_0(r = 0)}{3m}.$$  

(55)

The two-fluid hydrodynamic equations for a trapped Bose-condensed gas ($T < T_{\text{BEC}}$) have been recently discussed by Zaremba and the authors [13]. These equations have been used to study first and second sound modes in a dilute uniform Bose gas [13] at finite temperatures. It is found that first sound corresponds mainly to an oscillation of the non-condensate, with a velocity given by
\[ u_1^2 = \frac{5}{3} \frac{k_B T g_{5/2}(z_0)}{m} + \frac{2g\tilde{n}_0}{m}. \]  
\( (56) \)

In contrast, the second sound mode mainly corresponds to an oscillation of the condensate, with a velocity given by

\[ u_2^2 = \frac{gn_c}{m}. \]  
\( (57) \)

Here \( n_c(\tilde{n}_0) \) is the equilibrium condensate (non-condensate) density. As discussed in Ref. [13], to a good approximation, one can use

\[ \tilde{n}_0 = \frac{1}{\Lambda^3} g_{3/2}(z_0), \]  
\( (58) \)

where the equilibrium fugacity is \( z_0 = e^{-\beta g n_{\text{eq}}} \).

In principle, we could use the equations in Ref. [11] to extend the analysis of the present paper and discuss the propagating first and second sound modes in a cigar-shaped trap. Here we limit ourselves to some qualitative remarks. One expects to find an expression similar to (51) for the propagation of a pulse, and there should be distinct first and second sound pulses moving with velocities quite close to \( u_1 \) and \( u_2 \) as given above. However, as the expression in (51) shows, the relative amplitude of these two modes is proportional to \( 1/u_1^2 \).

We conclude that if pulse experiments such as in Ref. [1] were done in the hydrodynamic region, most of the weight would be in the second sound pulse if \( u_2^2 \ll u_1^2 \). This mode, given by (57), is the natural hydrodynamic analogue of the Bogoliubov mode exhibited in the quantum hydrodynamic region at \( T = 0 \) [1,2]. At temperatures close to \( T_{\text{BEC}} \), the first sound pulse has a much faster speed and thus its intensity will be very weak. The observation of distinct first and second sound pulses in cigar-shaped traps would be very dramatic evidence for superfluid behavior in dilute Bose gases. The experiment would best be done at intermediate or lower temperatures, where \( u_1 \) and \( u_2 \) are more comparable in magnitude.
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FIGURE CAPTIONS

Fig.1: The sound velocity \( c \) as a function of temperature relative to \( T_{\text{BEC}} \). The values are normalized to the classical gas result \( c_0 = \sqrt{5k_B T/m} \).

Fig.2: The normal mode frequencies \( \omega_n \) for \( k = 0 \) as a function of temperature, as given by (33). The frequencies are normalized to the radial trap frequency \( \omega_0 \).
Figure 1

Sound Velocity

\[ \frac{c}{c_0} \]

\[ \frac{T}{T_{BEC}} \]
Figure 2

$\frac{\omega_n}{\omega_0}$ vs $\frac{T}{T_{\text{BEC}}}$

- $n=1$
- $n=2$
- $n=3$
- $n=4$