NOTE ON A PROPOSED PROOF OF THE RIEMANN HYPOTHESIS BY JIN GYU LEE

JACQUES GÉLINAS

Abstract. This is a reformulation and refutation of a proposed proof of the Riemann hypothesis published in electronic form on the Internet in 2013 [5, 6]. Proceeding by contradiction, the author wants to prove that if \( \zeta(s) = 0 \) where \( 1/2 < \Re s < 1 \), then \( \zeta(2s) = 0 \), which is known to be impossible. We show that this version of the proof is incomplete.

1. On the right of the critical strip

Consider the \( \zeta(s) \) function of Riemann, represented by a Dirichlet series, an Euler product, or via the Dirichlet alternating zeta integral function \( \eta(s) \) as:

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (\Re s > 1),
\]

\[
= \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}, \quad (\Re s > 1),
\]

\[
= \frac{1}{\log 2} \eta(s), \quad (\Re s > 0, s \neq 1 + \frac{2k\pi i}{\log 2}, k \text{ integer}),
\]

\[
= \frac{\eta'(s)}{\log 2}, \quad (s = 1 + \frac{2k\pi i}{\log 2}, k \text{ nonzero integer}),
\]

\[
\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, \quad (\Re s > 0).
\]

In his second proof of the functional equation of the zeta function discovered by Euler [3], Riemann [9] has shown that \( \xi(s) := \pi^{s/2}\Gamma(1+s/2)(s-1)\zeta(s) \) is an integral function left unchanged by the mapping \( s \mapsto 1 - s \). The non trivial zeros of \( \zeta(s) \) are thus located symmetrically with respect to the line \( \Re s = 1/2 \), inside the critical strip \( 0 < \Re s < 1 \) because the Euler product (1.2) is not zero for \( \Re s > 1 \), nor is the limit for \( \Re s = 1 \) of an expression derived from it. Riemann has stated that all these non trivial zeros are very likely located on the critical line \( \Re s = 1/2 \) itself.

In order to prove this “Riemann Hypothesis”, it is sufficient to show that if \( \eta(s) \) had one zero in the right hand side \( 1/2 < \Re s < 1 \) of the critical strip, then \( \zeta(2s) \) would also vanish while \( 2s \) is outside the critical strip, contradicting (1.2). This is Mr Lee’s claim and the main goal of [6]:

\[
\frac{1}{2} < \Re s < 1 \& \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = 0 \implies \Re(2s) > 1 \& \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = 0.
\]

This work was done while the author was a retired mathematician.
2. Arithmetic functions

The following two functions are defined on the natural numbers in : first, with
the convention that $\Omega(1) = 0$, [6, definition 3.1]

$$
\Omega(m) := \sum_{k=0}^{K} r_k \quad \text{where, for distinct primes } p_k, \ m = \prod_{k=0}^{K} p_k^{r_k},
$$

and secondly [6, definition 3.3]

$$
\beta(n) := \sum_{m=1, m | n} (-1)^{l+1} (-1)^{\Omega(m)}; \quad (l = \frac{n}{m}).
$$

According to [6, theorem 3.7] :

$$
\beta(n) =
\begin{cases} 
1 & \text{if } n \text{ is the square of a natural number,} \\
-2 & \text{if } n \text{ is twice the square of a natural number,} \\
0 & \text{otherwise.}
\end{cases}
$$

The values of the two functions at the first integers can be generated by the following
PARI/GP [8] program as confirmed by the sample output :

```parigp
omeg(n=1) = local(f = factor(n)[,2]); sum(k=1,length(f),f[k])
beta(n=1) = if( issquare(n), 1, if(n%2==0 && issquare(n/2), -2, 0))
betd(n=1) = local(d = divisors(n));
           \sum(m=1,length(d), (-1)^{n/d[m] + 1 + \text{omeg}(d[m])})
vector(20,m,omeg(m)) \ [0,1,1,2,1,2,1,3,1,2,2,4,1,3,1,3,1,3,1,3]
vector(20,n,beta(n)) \ [1,-2,0,1,0,0,0,-2,1,0,0,0,0,0,1,0,-2,0,0]
sum(n=1,10^5, beta(n) == betd(n)) \ 100000
```

3. A new expression for $\zeta(2s)$

If $R > 1/2$, then $R(2s) > 1$ and the Dirichlet series representing $\zeta(2s)$ converges
absolutely (and also uniformly on compact sets). We can certainly write, replacing
the usual exponent $1 - 2s$ for $\eta(2s)$ in [13] by the exponent $1 - s$ as in [6] :

$$
(1 - 2^{1-s}) \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = \sum_{n=1}^{\infty} \left( \frac{1}{(n^2)^s} - \frac{2}{(2n^2)^s} \right),
$$

$$
= \left( \frac{1}{1^s} - \frac{2}{2^s} \right) + \left( \frac{1}{4^s} - \frac{2}{8^s} \right) + \left( \frac{1}{9^s} - \frac{2}{18^s} \right) + \left( \frac{1}{16^s} - \frac{2}{32^s} \right) + \cdots .
$$

After rearranging terms by absolute convergence and using (2.3), this becomes

$$
= \frac{1}{1^s} - \frac{2}{2^s} + \frac{1}{4^s} - \frac{2}{8^s} + \frac{1}{9^s} + \frac{1}{16^s} - \frac{2}{18^s} + \frac{1}{25^s} - \frac{2}{32^s} + \frac{1}{36^s} + \cdots ,
$$

and finally, inserting the definition (2.2) of $\beta(n)$, [6, eq. 3.4, 3.5]

$$
(1 - 2^{1-s})\zeta(2s) = \sum_{n=1}^{\infty} \sum_{m=1, m | n} \frac{(-1)^{l+1} (-1)^{\Omega(m)}}{m^s l^s}, \quad (l = \frac{n}{m}).
$$
The previous equations are illustrated by the following PARI/GP \cite{pari} program:

```pari
default(realprecision,100);
default(format,"g1.5");
xis(s=1/2) = Pi^(-s/2)*gamma(1+s/2)*(s-1)*zeta(s);
xis(s=1/2) = {
  if(s==0||s==1, return(1/2));
  if(real(s) == 1/2, return( real(xis(s)) ));
  if(real(s) < 1/2, return( xis(1-s) ));
  xis(s);
}

lhs(s) = (1-2^(1-s))*zeta(2*s);
rhs(s,N) = sum(n=1,10^N, beta(n)/n^s);
s1 = 1/2 + I*solve(t=14,15,xi(1/2+I*t)) \ 1/2 + 14.135*I
s2 = 1/2 + I*solve(t=20,22,xi(1/2+I*t)) \ 1/2 + 21.022*I
s3 = 1/2 + I*solve(t=24,26,xi(1/2+I*t)) \ 1/2 + 25.011*I

lhs(s1) \ 3.9207 - 2.4547*I
lhs(s2) \ 1.0729 + 1.3198*I
lhs(s3) \ 0.82614 - 0.35665*I

vector(5,N, abs( 1 - rhs(s1,N)/lhs(s1)) ) \ [0.26899, 0.011621, 0.0045613, 0.0038153, 0.0031909]
vector(5,N, abs( 1 - rhs(s2,N)/lhs(s2)) ) \ [0.40567, 0.084822, 0.011169, 0.0082009, 0.0058860]
vector(5,N, abs( 1 - rhs(s3,N)/lhs(s3)) ) \ [0.32487, 0.34041, 0.021146, 0.014346, 0.0094130]
```

Although it is known that the Dirichlet series for $\zeta(s)$ diverges when $\Re s = 1$, the series $\sum \beta(n)/n^s$ could still converge to a nonzero value when $\Re(2s) = 1$, according to these particular examples of the first non-trivial zeros of $\zeta(s)$.

4. Series of zero series

If $s$ is a zero of $\eta(s)$ such that $\Re s > 0$, then for any natural number $m$ \cite{lee} p. 10

\[ \frac{(-1)^{\Omega(m)}}{m^s} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l^s} \frac{(-1)^{\Omega(m)}}{m^s}(0) = 0, \]

which yields, obviously, \cite{lee} p. 11

\[ \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}(-1)^{\Omega(m)}}{m^s l^s} = 0. \] (4.1)

Both (3.1) and (4.1) are absolutely convergent if $\Re s > 1/2$, when viewed as a single series in $n$ or $m$ respectively; but if $\Re s < 1$ the two double series must be considered as conditionally convergent only.
5. Comparison of double series

In order to prove (1.4), Mr. Lee states in [6, theorem 3.10] that the double sums in (3.1) and (4.1) are equal because the product sets of integers \((n, l)\) in (3.1) and \((m, l)\) in (4.1) are identical. In fact, what needs to be justified is that the order of summation in the double series can be changed, so that:

\[
\sum_{n=1}^{\infty} \sum_{m=1,m|n}^{n} \frac{(-1)^{l+1}(-1)^{l(m)}}{l^s m^s}, \quad (l = \frac{n}{m})
\]

(5.1)

\[
= \sum_{m=1}^{\infty} \sum_{n=m,m|n}^{\infty} \frac{(-1)^{l+1}(-1)^{l(m)}}{l^s m^s}, \quad (l = \frac{n}{m})
\]

(5.2)

\[
= \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}(-1)^{l(m)}}{l^s m^s}
\]

\[
= \sum_{m=1}^{\infty} \frac{(-1)^{l(m)}}{m^s} \left( \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l^s} \right).
\]

Some justification is needed here for the inversion between (5.1) and (5.2) since according to Riemann’s rearrangement theorem for single series [4, p. 318],

“If a series converges, but not absolutely, its sum can be made to have any arbitrary value by a suitable derangement of the series; it can also be made divergent or oscillatory.” [1, p. 74]

Although the original and the reordered series have exactly the same terms, the second can diverge or converge conditionally to a different sum.

A double series can be summed by columns first (3.1), by rows first (4.1), or by expanding rectangles in the sense of Pringsheim [10, §2.5]. For positive terms, the three sums are equal if any one of them converges.

“When the terms of the double series are positive, its convergence implies the convergence of all the rows and columns, and its sum is equal to the sum of the two repeated series.” [1, p. 84]

“The terms being always positive, if either repeated series is convergent, so is the other and also the double series; and the three sums are the same.” [1, p. 84]

But the situation for general terms is not so simple. A striking example from Cesàro makes this clear [1, p. 89]: the sum of row \(m\) is \(1/2^m\), converging absolutely, and so the sum by rows first is 1. But the sum of column \(n\) is \((-1)^{n+1}\), so the sum by columns first is oscillating. Thus

“the sum of a non-absolutely convergent double series may have different values according to the mode of summation” [1, p. 89].

However, Pringsheim has proven the following result:

“If the rows and columns converge, and if the double series is convergent, then the repeated sums are equal” [1, p. 81].

For the case at hand, column \(n\) is finite and has for sum \(\beta(n)/n^s\), while each row \(m\) converges to 0 if \(\eta(s) = 0\). But it is not proven in [6] that the sums by expanding rectangles converge, and so Pringsheim’s theorem cannot be applied.
6. Conclusion

The simple proof of the “Riemann Hypothesis” proposed in [6], although interesting and original, is still incomplete: a crucial theorem presents conditionally convergent infinite series as sums over sets, without specifying the order of summation, and without providing any justification for disregarding this order.

After being made aware of this gap in his proof, the author of [5] and [6] has suspended [5] and proposed in [7] a justification based on the Moore theorem for the inversion of two limits, one of which is uniform [2, p. 28].

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Ottawa, Canada

E-mail address: jacquesg00@hotmail.com