An Accelerated Method for Derivative-Free Smooth Stochastic Convex Optimization

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Abstract
We consider an unconstrained problem of minimization of a smooth convex function which is only available through noisy observations of its values, the noise consisting of two parts. Similar to stochastic optimization problems, the first part is of a stochastic nature. On the opposite, the second part is an additive noise of an unknown nature, but bounded in the absolute value. In the two-point feedback setting, i.e. when pairs of function values are available, we propose an accelerated derivative-free algorithm together with its complexity analysis. The complexity bound of our derivative-free algorithm is only by a factor of $\sqrt{n}$ larger than the bound for accelerated gradient-based algorithms, where $n$ is the dimension of the decision variable. We also propose a non-accelerated derivative-free algorithm with a complexity bound similar to the stochastic-gradient-based algorithm, that is, our bound does not have any dimension-dependent factor. Interestingly, if the difference between the starting point and the solution is a sparse vector, for both our algorithms, we obtain better complexity bound if the algorithm uses a $\ell_1$-norm proximal setup, rather than the Euclidean proximal setup, which is a standard choice for unconstrained problems

Keywords: Derivative-Free Optimization, Zeroth-Order Optimization, Stochastic Convex Optimization, Smoothness, Acceleration

1. Introduction

Derivative-free or zeroth-order optimization Rosenbrock (1960); Brent (1973); Spall (2003) is one of the oldest areas in optimization, which constantly attracts attention of the learning community, mostly in connection to the online learning in the bandit setup Bubeck and Cesa-Bianchi (2012). We study stochastic derivative-free optimization problems in a two-point feedback situation, considered by Agarwal et al. (2010); Duchi et al. (2015); Shamir (2017) in the learning community and by Nesterov and Spokoiny (2017); Stich et al. (2011); Ghadimi and Lan (2013); Ghadimi et al. (2016); Gasnikov et al. (2016a) in the optimization community. Two-point setup allows to prove complexity bounds, which typically coincide with the complexity bounds for gradient-based algorithms up to a small-degree polynomial of $n$, where $n$ is the dimension of the decision variable. On the contrary, problems with one-point feedback are harder and complexity bounds for such problems either have worse dependence on $n$, or worse dependence on the desired accuracy of the solution, see Nemirovsky and Yudin (1983); Protasov (1996); Flaxman et al. (2005); Agarwal et al. (2011); Jamieson et al. (2012); Shamir (2013); Liang et al. (2014); Bach and Perchet (2016); Bubeck et al. (2017) and the references therein.

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More precisely, we consider the following optimization problem

$$\min_{x \in \mathbb{R}^n} \left\{ f(x) := \mathbb{E}_\xi[F(x, \xi)] = \int_{\mathcal{X}} F(x, \xi) dP(x) \right\}, \tag{1}$$

where $\xi$ is a random vector with probability distribution $P(\xi), \xi \in \mathcal{X}$, and the function $f(x)$ is closed and convex. Note that $F(x, \xi)$ can be non-convex at $x$ with positive probability. Moreover, we assume that, for $P$ almost every $\xi$, the function $F(x, \xi)$ has gradient $g(x, \xi)$, which is $L(\xi)$-Lipschitz continuous with respect to the Euclidean norm and $L_2 := \sqrt{\mathbb{E}_\xi L(\xi)^2} < +\infty$. Under these assumptions, $\mathbb{E}_\xi g(x, \xi) = \nabla f(x)$ and $f$ has $L_2$-Lipschitz continuous gradient with respect to the Euclidean norm. Also we assume that

$$\mathbb{E}_\xi[\|g(x, \xi) - \nabla f(x)\|_2^2] \leq \sigma^2,$$  

where $\| \cdot \|_2$ is the Euclidean norm. We emphasize that, unlike Duchi et al. (2015), we do not assume that $\mathbb{E}_\xi[\|g(x, \xi)\|_2^2]$ is bounded since it is not the case for many unconstrained optimization problems, e.g. for deterministic quadratic optimization problems.

Finally, we assume that an optimization procedure, given a pair of points $(x, y) \in \mathbb{R}^{2n}$, can obtain a pair of noisy stochastic realizations $(\hat{f}(x, \xi), \hat{f}(y, \xi))$ of the objective value $f$, which we refer to as oracle call. Here

$$\hat{f}(x, \xi) = F(x, \xi) + \eta(x, \xi), \quad |\eta(x, \xi)| \leq \Delta, \; \forall x \in \mathbb{R}^n,$$

and $\xi$ is independently drawn from $P$. This makes our problem more complicated than problems studied in the literature. Not only we have stochastic noise in the problem (1), but an additional noise $\eta(x, \xi)$, which can be adversarial (see Section 2.3 for the detailed explanation).

We notice that our model of two-point feedback oracle is pretty general and covers deterministic exact oracle and even specific types of one-point feedback oracle. For example, if the function $F(x, \xi)$ is separable, i.e. $F(x, \xi) = f(x) + h(\xi)$, where $\mathbb{E}_\xi[h(\xi)] = 0$, $|h(\xi)| \leq \frac{\Delta}{2}$ for all $\xi$ and the oracle gives us $F(x, \xi)$ in the given point $x$, then for all $\xi_1, \xi_2$ we can define $\hat{f}(x, \xi_1) = F(x, \xi_1)$ and $\hat{f}(y, \xi_2) = F(y, \xi_2) = F(y, \xi_1) + h(\xi_2) - h(\xi_1)$. Since $|h(\xi_2) - h(\xi_1)| \leq |h(\xi_2)| + |h(\xi_1)| \leq \Delta$ we can use representation (3) omitting dependence of $\eta(x, \xi_1)$ on $\xi_2$, because in our analysis we only rely on the fact that $|\eta(x, \xi)| \leq \Delta$ for all $x \in \mathbb{R}^n$ almost surely in $\xi$. Moreover, such oracle can be met in practice, since rounding errors can be put into this context, e.g. by artificial adding random bit modulo 2 to the last bit in machine’s number representation format (see Gasnikov et al. (2016b) for details).

As it is known Lan (2012); Devolder (2011); Dvurechensky and Gasnikov (2016), if the stochastic approximation $g(x, \xi)$ for the gradient of $f$ is available, an accelerated gradient method has oracle complexity bound (i.e. overall number of first-order oracle calls) $O\left(\max\left\{\sqrt{L_2 R_2^2 / \varepsilon}, \sigma^2 R_2^2 / \varepsilon^2\right\}\right)$, where $\varepsilon$ is the target optimization error, the goal being to find such $\hat{x}$ that $\mathbb{E} f(\hat{x}) - f^* \leq \varepsilon$. Here $f^*$ is the global optimal value of $f$. The question, to which we give a positive answer in this paper, is as follows.

**Is it possible to solve a stochastic optimization problem with the same $\varepsilon$-dependence in the iteration and sample complexity and only noisy observations of the objective value?**

Problem (1) is an unconstrained problem and the first choice of geometric setup is usually given by Euclidean norm. Surprisingly, as we show below, using a non-standard proximal setup given by $\| \cdot \|_1$-norm can give some benefits. So, we solve the problem (1) using two proximal setups Ben-Tal and Nemirovski (2015), characterized by the value $p \in \{1, 2\}$ and its conjugate $q \in \{2, \infty\}$, given by the identity $\frac{1}{p} + \frac{1}{q} = 1$. The case $p = 1$ corresponds to the choice of $\| \cdot \|_1$-norm in $\mathbb{R}^n$ and corresponding prox-function, which is strongly convex with respect to this norm (we provide the details below). The case $p = 2$ corresponds to the choice of the Euclidean $\| \cdot \|_2$-norm in $\mathbb{R}^n$ and squared Euclidean norm as the prox-function.
Table 1: Comparison of oracle complexity (total number of oracle calls) of different methods with two point feedback discussed in the paper for convex optimization problems. In the column "Assumptions" we use "bound. gr." to define $\mathbb{E} \left[ \|g(x, \xi)\|^2 \right] \leq M^2$ and "bound. var." to define $\mathbb{E}[\|g(x, \xi) - \nabla f(x)\|^2] \leq \sigma^2$. Column $p = 1$ corresponds to the support of non-Euclidean setup, column "Stoch." to the support of stochastic optimization methods, "Noise" corresponds to the support of additional noise of an unknown nature.

### 1.1. Related Work
Non-smooth deterministic and stochastic problems in the two-point derivative-free setting was considered by Nesterov and Spokoiny (2017).\(^1\) Non-smooth stochastic problems were considered by Shamir (2017) and independently by Bayandina et al. (2018), the latter paper considering also problems with additional noise of an unknown nature in the objective value. Duchi et al. (2015) consider the smooth stochastic optimization problems, yet under additional quite restrictive assumption $\mathbb{E}[\|g(x, \xi)\|^2] < +\infty$. Their bound was improved by Gasnikov et al. (2016a, 2017) for the problems with non-Euclidean proximal setup and noise in the objective value. Strongly convex problems with different smoothness assumptions were considered by Gasnikov et al. (2017); Bayandina et al. (2018). Smooth stochastic convex optimization problems, without the assumption that $\mathbb{E}[\|g(x, \xi)\|^2] < +\infty$, were studied by Ghadimi et al. (2016); Ghadimi and Lan (2013) for the Euclidean case. Accelerated and non-accelerated derivative-free method, but for deterministic problems was proposed in Nesterov and Spokoiny (2017) and extended in Bogolubsky et al. (2016); Dvurechensky et al. (2017) for the case of additional bounded noise in the function value. Table 1 presents a detailed comparison of our results and results in the literature on two-point feedback derivative-free optimization and assumptions, under which they are obtained.

We also mention the works Nemirovsky and Yudin (1983); Protasov (1996); Flaxman et al. (2005); Saha and Tewari (2011); Dekel et al. (2015); Gasnikov et al. (2017); Agarwal et al. (2011); Liang et al. (2014); Belloni et al. (2015); Bubeck et al. (2017); Shamir (2013); Jamieson et al. (2012); Hazan and Levy (2014); Bach and Perchet (2016); Jamieson et al. (2012) who study derivative-free optimization with one-point feedback in different settings, and works Nesterov (2005); Allen-Zhu and Orecchia (2014) on coupling

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\(^1\) We list the references in the order of the date of the first appearance, but not in the order of the date of official publication.
non-accelerated methods to obtain acceleration, which inspired our work. A more detailed description of the related work can be found in the Appendix.

1.2. Our Contributions

As our main contribution, we propose an accelerated method for smooth stochastic derivative-free optimization, which we call Accelerated Randomized Derivative-Free Directional Search (ARDFDS). Our method has the complexity bound

$$\tilde{O} \left( \max \left\{ n^{\frac{1}{2}} + \frac{1}{\epsilon} \sqrt{\frac{L_2 R_p^2}{\epsilon}}, \frac{n \sigma^2 R_p^2}{\epsilon^2} \right\} \right),$$

where $R_p$ characterizes the distance in $\| \cdot \|_p$-norm between the starting point of the algorithm and a solution to (1). In the Euclidean case $p = q = 2$, the first term in the above bound has better dependence on $n$, $\epsilon$, $L_2$, and $R_2$ than the bound in Ghadimi et al. (2016); Ghadimi and Lan (2013). Unlike these papers, our bound also covers the non-euclidean case $p = 1, q = \infty$ and due to that allows to obtain better complexity bounds.

To illustrate this, let us start method from a point $x_0$ and define the sparsity $s$ of the vector $x_0 - x^*$, i.e. $\|x_0 - x^*\|_1 \leq s \cdot \|x_0 - x^*\|_2$ and $1 \leq s \leq \sqrt{n}$. Then the complexity of our method for $p = 1, q = \infty$ is

$$\tilde{O} \left( \max \left\{ \frac{ns^2 L_2}{\epsilon}, \frac{s^2 \sigma^2}{\epsilon^2} \right\} \right),$$

which is always no worse than the complexity for $p = q = 2$

$$\tilde{O} \left( \max \left\{ \frac{n^2 L_2}{\epsilon}, \frac{n \sigma^2}{\epsilon^2} \right\} \right)$$

and allows to gain up to $\sqrt{n}$ if $s$ is close to 1. Notably, this is done automatically, without any prior knowledge of $s$.

Unlike other authors, we consider additional, possibly adversarial noise $\eta(x, \xi)$ in the objective value and analyze how this noise affects the convergence rate estimates. We emphasize that even if noise is uncontrolled, e.g. we do not know noise level $\Delta$ or we cannot reduce $\Delta$, we can run our algorithms and still expect the convergence (see Section 2.3 for the detailed explanation). This is important when the objective is given as a solution to some auxiliary problem which can’t be solved exactly, e.g. in bi-level optimization or reinforcement learning. It should also be mentioned that our assumption $\mathbb{E}_{\xi}[L(\xi)^2] < +\infty$ is weaker than the assumption $L(\xi) \leq L_2$ a.s. in $\xi$, which is used in Ghadimi et al. (2016); Ghadimi and Lan (2013).

As our second contribution, we propose a non-accelerated Randomized Derivative-Free Directional Search (RDFDS) method with the complexity bound

$$\tilde{O} \left( \max \left\{ \frac{n^2 L_2 R_p^2}{\epsilon}, \frac{n \sigma^2 R_p^2}{\epsilon^2} \right\} \right),$$

where unlike Ghadimi et al. (2016); Ghadimi and Lan (2013) a non-euclidean case $p = 1, q = \infty$ with the gain in the complexity up to the factor of $n$ is possible. Interestingly, in this case, we obtain a nearly dimension independent complexity bound despite we use only noisy function value observations.

1.2.1. Why it is important to improve the first term of the maximum?

1. Acceleration when $n$ is big. First of all, we notice that the first term dominates the second term when $\sigma^2 \leq \frac{\epsilon^2 n^{\frac{1}{2}}}{R_p}$ in the accelerated case and $\sigma^2 \leq \epsilon L_2$ in the non-accelerated case which could be met in practice if $\epsilon$, $L_2$ and $n$ are big enough in comparison with $R_p$. For ARDFDS with $p = 1, q = \infty$ it means that if we want to get $\epsilon$-solution with $\epsilon = 10^{-3}$ and $L_2 = 100$, $R_p = 10$, $n = 10000$ (or bigger), then the variance should satisfy $\sigma^2 \leq 10^{-1}$ in order to have accelerated rate for ARDFDS, which, actually, is not a very restrictive assumption.

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2. **Better dimension-dependence in the deterministic case.** We underline also that even in the deterministic case $\sigma = 0$ without additive noise both our non-accelerated and accelerated complexity bounds for $p < 2$ and, in particular, for $p = 1$ are new. Moreover, disregarding $\ln n$ factors, the existing bounds Nesterov and Spokoiny (2017) are $n/s^2$ and $n/s$ times worse than our new bounds respectively in non-accelerated and accelerated cases, where $s \in [1, \sqrt{n}]$. Importantly, in the non-accelerated case our bound is dimension-independent up to a $\ln n$ factor.

3. **Parallel computation of mini-batches makes acceleration reasonable when $\sigma^2$ is not small.** Imagine the situation when one can have an access to $k \geq m = n^{\frac{1}{q} - \frac{1}{p}} \frac{\sqrt{R^2_p}}{\varepsilon n L_2}$ processors (or workers). For example, if $\sigma^2 = 1$, which is not small, $n = 10000$, $R_p = 10$, $\varepsilon = 10^{-3}$ and $L_2 = 100$, then we need $k = 10^{2.5} \approx 316$ processors which is just nothing for modern supercomputers and clusters that often have $\sim 10^5 - 10^6$ processors. It means that one can call zeroth-order oracle on each processor, compute own stochastic approximation of the gradient which is described below and after that send it to one chosen processor (directly or via others processors) which just take an average of received data. In other words, one can compute mini-batches in parallel. Such technique gives us an opportunity to reduce real time of computing stochastic approximation of the gradient on each iteration and to make running time equal to $O(N)$ (i.e. iteration complexity). But $N$ is equal exactly to the first term in our bounds (see Table 2 for the details) which means that in such situation it is possible to make real running time of the method proportional to the first term in bounds we obtained. More rigorously, it means that if we have $k$ processors than due to parallel computing of mini-batches we can obtain the real working time equal to $	ilde{O} \left( \max \left\{ n^{\frac{1}{q} + \frac{1}{q}} \frac{\sqrt{R^2_p}}{\varepsilon}, \frac{1}{k}, \frac{n^{\frac{2}{q}} \sigma^2 R^2_p}{\varepsilon^2} \right\} \right)$ and $	ilde{O} \left( \max \left\{ n^{\frac{2}{q}} L_2 R^2_p, 1 \frac{n^{\frac{2}{q}} \sigma^2 R^2_p}{\varepsilon^2} \right\} \right)$ for ARDFDS and RDFDS respectively. We notice that $k$ could be even smaller that $m = n^{\frac{1}{q} - \frac{1}{p}} \frac{\sqrt{R^2_p}}{\varepsilon n L_2}$ to have that the first term of maximum is bigger than the second one (see the first reason in this list).

### 2. Algorithms for Stochastic Convex Optimization

#### 2.1. Preliminaries

**Proximal setup.** Let $p \in [1, 2]$ and $\|x\|_p$ be the $p$-norm in $\mathbb{R}^n$ defined as $\|x\|_p^n = \sum_{i=1}^{n} |x_i|^p$ and $\|\cdot\|_q$ be its dual, defined by $\|g\|_q = \max_{x} \left\{ \langle g, x \rangle, \|x\|_p \leq 1 \right\}$, where $q \in [2, \infty]$ is the conjugate number to $p$, given by $\frac{1}{p} + \frac{1}{q} = 1$, and, for $q = \infty$, we define $\|x\|_\infty = \max_{i=1,...,n} |x_i|$. We choose a prox-function $d(x)$, which is continuous, convex on $\mathbb{R}^n$ and is 1-strongly convex on $\mathbb{R}^n$ with respect to $\|\cdot\|_p$, i.e., for any $x, y \in \mathbb{R}^n$ $d(y) - d(x) - \langle \nabla d(x), y - x \rangle \geq \frac{1}{2} \|y - x\|_p^2$. Without loss of generality, we assume that $\min_{x \in \mathbb{R}^n} d(x) = 0$. We define also the corresponding Bregman divergence $V[z](x) = d(x) - d(z) - \langle \nabla d(z), x - z \rangle$, $x, z \in \mathbb{R}^n$. Note that, by the strong convexity of $d$, $V[z](x) \geq \frac{1}{2} \|x - z\|_p^2$, $x, z \in \mathbb{R}^n$. (6)

For the case $p = 1$, we choose the following prox function Ben-Tal and Nemirovski (2015) $d(x) = \frac{c_0(n-1)(2-\kappa)/\kappa \ln n}{2^2} \|x\|_k^2$, where $\kappa = 1 + \frac{1}{\ln n}$ and, for the case $p = 2$, we choose the prox-function to be proportional to the Euclidean norm: $d(x) = \frac{1}{2} \|x\|_2^2$. 

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Main technical lemma. In our proofs of complexity bounds, we rely on the following lemma. The proof is rather technical and is provided in the appendix.

**Lemma 1** Let $e \in RS_2(1)$, i.e., be a random vector uniformly distributed on the surface of the unit Euclidean sphere in $\mathbb{R}^n$. $p \in [1, 2]$ and $q$ be given by $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $n \geq 8$,

\[
\mathbb{E}_e \|e\|_q^2 \leq \rho_n, \quad (7)
\]

\[
\mathbb{E}_e (\langle s, e \rangle^2 \|e\|_q^2) \leq \frac{6\rho_n}{n} \|s\|_2^2, \quad \forall s \in \mathbb{R}^n. \quad (8)
\]

with $\rho_n \text{ def} = \min\{q - 1, 16 \ln n - 8\} n^{\frac{2}{n}-1}$.

**Stochastic approximation of the gradient.** Based on the noisy observations (3) of the objective value, we form the following stochastic approximation of $\nabla f(x)$

\[
\tilde{\nabla}^m f^t(x) = \frac{1}{m} \sum_{i=1}^{m} \tilde{f}(x + te, \xi_i) - \tilde{f}(x, \xi_i) e, \quad (9)
\]

where $e \in RS_2(1)$, $\xi_i, i = 1, ..., m$ are independent realizations of $\xi$, $m$ is the batch size, $t$ is some small positive parameter, which we call smoothing parameter.

2.2. Algorithms and Main Theorems

Our Accelerated Randomized Derivative-Free Directional Search (ARDFDS) method is listed as Algorithm 1.

**Algorithm 1** Accelerated Randomized Derivative-Free Directional Search (ARDFDS)

**Input:** $x_0$ — starting point; $N$ — number of iterations; $m \geq 1$ — batch size; $t > 0$ — smoothing parameter, $\{\alpha_k\}_{k=1}^N$ — stepsizes.

**Output:** point $y_N$.

1. $y_0 \leftarrow x_0$, $z_0 \leftarrow x_0$.
2. for $k = 0, \ldots, N-1$ do
3. $\tau_k \leftarrow \frac{2}{k+2}$
4. Generate $e_{k+1} \in RS_2(1)$ independently from previous iterations and $\xi_i, i = 1, ..., m$ independent realizations of $\xi$.
5. $x_{k+1} \leftarrow \tau_k z_k + (1 - \tau_k) y_k$.
6. Calculate $\tilde{\nabla}^m f^t(x_{k+1})$ given in (9).
7. $y_{k+1} \leftarrow x_{k+1} - \frac{1}{L_2} \tilde{\nabla}^m f^t(x_{k+1})$.
8. $z_{k+1} \leftarrow \arg\min_{z \in \mathbb{R}^n} \left\{ \alpha_{k+1} n \left( \tilde{\nabla}^m f^t(x_{k+1}), z - z_k \right) + V[z_k] (z) \right\}$
9. end for
10. return $y_N$

**Theorem 2** Let ARDFDS be applied to solve problem (1) with $\alpha_{k+1} = \frac{k+2}{90n^2\rho_n L_2}$ which implies $\tau_k = \frac{2}{k+2} = \frac{1}{48\alpha_{k+1} n^2 \rho_n L_2}$. If we set $\Theta_p = \frac{1}{2} \frac{M}{N} \left( \frac{nL_2}{mL_2} \right)$, then

\[
\mathbb{E}[f(y_N)] - f(x^*) \leq \frac{384\rho_n L_2 \Theta_p \Theta_e}{N^2} + \frac{384N^2 \rho_n L_2}{m} + \frac{12\sqrt{2\Theta_p}}{N^2} \left( \frac{L_2 t}{2} + \frac{2a}{t} \right)
\]

\[
+ \frac{6N}{L_2} \left( L_2^2 t^2 + \frac{16\Delta^2}{t^2} \right) + \frac{N^2}{24n\rho_n L_2} \left( L_2^2 t^2 + \frac{16\Delta^2}{t^2} \right), \quad \forall n \geq 8. \quad (10)
\]

6
Sketch of the proof of the Theorem 2 for the deterministic case with exact oracle. To introduce the main idea and intuition behind our proof of the Theorem 2 we consider here the most simple situation when \( \sigma^2 = 0 \) (i.e. deterministic problem) and \( \Delta = 0 \) (i.e. exact oracle). In this case technical details corresponding to the control of variance and noise do not disturb us to show what is really important to get better dimension-dependence in the oracle complexity.

First of all, we notice that in this special case we don’t need to do mini-batches of size \( m > 1 \), so, we can set \( m = 1 \) and rewrite \( \nabla^m f(x) \) as

\[
\nabla^m f(x) \equiv \nabla f(x) = \frac{f(x + te) - f(x)}{t} e = (\langle \nabla f(x), e \rangle + \theta(x, t, e)) e, \tag{11}
\]

where \( \theta(x, t, e) = \frac{f(x + te) - f(x)}{t} - \langle \nabla f(x), e \rangle = \frac{1}{t} \left(f(x + te) - f(x) - \langle \nabla f(x), te \rangle\right)\). Note that due to \( L_2 \)-smoothness we have

\[
|\theta(x, t, e)| \leq \frac{1}{t} \cdot \frac{L_2 t^2}{2} = \frac{L_2 t}{2}. \tag{12}
\]

The technique of linear coupling from Allen-Zhu and Orecchia (2014) inspired us to study how this method behaves when we use gradient approximation by finite differences like in (11) instead of true gradient in the method. In the linear coupling method one can choose any norm to define \( L \)-smoothness of the objective. However, we are also motivated to have clear comparison in the deterministic case with other methods like RSPGF from Ghadimi and Lan (2013); Ghadimi et al. (2016), RS from Nesterov and Spokoiny (2017); Bogolubsky et al. (2016) and AccRS from Nesterov and Spokoiny (2017); Dvurechensky et al. (2017), which use \( L_2 \)-smoothness with respect to Euclidean norm. Therefore, we decide to use \( L_2 \)-smoothness of the objective in the \( \ell_2 \)-norm and, as the consequence, it is natural for our purposes to consider gradient step in the linear coupling method with respect to Euclidean norm, since it is a well-known fact the gradient descent step with constant stepsize \( \frac{1}{L_2} \) could be obtained in view of minimization of the quadratic upper bound coming from Lipschitz-continuity of the gradient: \( \arg \min_{x \in \mathbb{R}^n} \{ f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{L_2}{2} \| x - x_k \|^2 \} = x_k - \frac{1}{L_2} \nabla f(x_k) \). Thus, in the line 7 of the ARDFDS we set \( y_{k+1} = x_{k+1} - \frac{1}{L_2} \nabla f(x) \). We substitute classic coefficient \( -\frac{1}{L_2} \) by \(-\frac{1}{2L_2}\) just because of some technical issues that we analyse accurately in the full prove which one can find in the appendix. But if we also chose mirror descent step with Bregman divergence corresponding to the \( \ell_2 \) proximal setup, we will obtain the same bound as for AccRS from Nesterov and Spokoiny (2017); Dvurechensky et al. (2017). Then we switch gears and come up with the following counter-intuitive idea. Let consider mirror descent step with respect to \( \ell_p \) proximal setup, where \( p \in [1, 2] \), but remain the gradient descent step the same as for choosing Lipschitz-continuity of the gradient with respect to \( \ell_2 \) norm. Next we emphasize the place in our proof where this idea naturally appears.

The first step of the proof is the standard analysis of the one iteration of mirror descent. We have

\[
\alpha_{k+1} n \langle \nabla f^t(x_{k+1}), z_k - x_* \rangle = \alpha_{k+1} n \langle \nabla f^t(x_{k+1}), z_k - z_{k+1} \rangle + \alpha_{k+1} n \langle \nabla f^t(x_{k+1}), z_{k+1} - x_* \rangle \tag{13}
\]

\[
\leq \alpha_{k+1} n \langle \nabla f^t(x_{k+1}), z_k - z_{k+1} \rangle + \langle -\nabla V[z_k](z_{k+1}), z_{k+1} - x_* \rangle \tag{14}
\leq \alpha_{k+1} n \langle \nabla f^t(x_{k+1}), z_k - z_{k+1} \rangle + \langle \alpha_{k+1} n \langle \nabla f^t(x_{k+1}), z_k - z_{k+1} \rangle - \frac{1}{2} \| z_k - z_{k+1} \|^2 \|_p \rangle + V[z_k](x_*) - V[z_{k+1}](x_*), \tag{15}
\]

where \( \tag{13} \) follows from the definition of \( z_{k+1} \), whence \( \langle \nabla V[z_k](z_{k+1}) + \alpha_{k+1} n \nabla f^t(x_{k+1}), u - z_{k+1} \rangle \geq 0 \) for all \( u \in \mathbb{R}^n \) and, as a consequence, for \( u = x_* \); \( \tag{14} \) follows from the ”magic identity” Fact 5.3.3 in Ben-Tal
and Nemirovski (2015) for the Bregman divergence; ③ follows from (6); and ④ follows from the Fenchel inequality $\zeta(s, z) - \frac{1}{2}\|z\|^2 \leq \frac{q^2}{4}\|s\|^2$. Letting $\theta_{k+1} = \theta(x_{k+1}, t, e_{k+1})$, we continue

$$\alpha_{k+1}n\langle \nabla f^t(x_{k+1}), z_k - x_s \rangle = \alpha_{k+1}n\langle \nabla f(x_{k+1}), e_{k+1} \rangle e_{k+1}, z_k - x_s \rangle + \alpha_{k+1}n\theta_{k+1}\langle e_{k+1}, z_k - x_s \rangle \\
\geq \alpha_{k+1}n\langle \nabla f(x_{k+1}), e_{k+1} \rangle e_{k+1}, z_k - x_s \rangle - \alpha_{k+1}n|\theta_{k+1}| \cdot \|e_{k+1}, z_k - x_s \| \\
\geq \alpha_{k+1}n\langle \nabla f(x_{k+1}), e_{k+1} \rangle e_{k+1}, z_k - x_s \rangle - \frac{\alpha_{k+1}nL_2^2}{2}\|e_{k+1}, z_k - x_s \|.
$$

Using triangle inequality and $\|a + b\|^2 \leq (\|a\| + \|b\|)^2 \leq 2\|a\|^2 + 2\|b\|^2$ we obtain $\|\nabla f(x_{k+1})\|^2 \leq 2\|\nabla f(x_{k+1}), e_{k+1}\|^2 + \frac{L_2^2\|e_{k+1}\|^2}{2}$. Putting all together we get

$$\alpha_{k+1}n\langle \nabla f(x_{k+1}), e_{k+1} \rangle e_{k+1}, z_k - x_s \rangle \leq \alpha_{k+1}n^2\|\nabla f(x_{k+1}), e_{k+1}\|^2 + V[z_k](x_s) - V[z_{k+1}](x_s) + W_k,$$

where $W_k = \frac{\alpha_{k+1}nL_2^2}{2}\|e_{k+1}, z_k - x_s\| + \frac{\alpha_{k+1}^2nL_2^2}{4}\|e_{k+1}\|^2$. If we take the expectation $\mathbb{E}_{e_{k+1}[\cdot]}$ with respect randomness coming from $e_{k+1}$ from both sides of the previous inequality, we will establish

$$\alpha_{k+1}\langle \nabla f(x_{k+1}), z_k - x_s \rangle \leq \alpha_{k+1}^2n^2\mathbb{E}_{e_{k+1}}[\|\nabla f(x_{k+1}), e_{k+1}\|^2] + V[z_k](x_s) - V[z_{k+1}](x_s) + \mathbb{E}_{e_{k+1}}[W_k],$$

since $\mathbb{E}_{e_{k+1}}[\langle \nabla f(x_{k+1}), e_{k+1} \rangle e_{k+1}] = \frac{\nabla f(x_{k+1})}{n}$ that can be obtained from $\mathbb{E}_{e_{k+1}}[(e_{k+1}^i)^2] = \frac{1}{n}$ and $\mathbb{E}_{e_{k+1}}[e_{k+1}^j e_{k+1}^j] = 0$ for $i \neq j$, where $e_{k+1}^i$ is the $i$-th component of the $e_{k+1}$. Note that before this moment our proof looks similar for all $p \in [1, 2]$. Now it is the right time to reveal the main trick in our proof.

The second step of the proof consists of upper bounding $\mathbb{E}_{e_{k+1}}[\|\nabla f(x_{k+1}), e_{k+1}\|^2]$ by $f(x_{k+1}) - \mathbb{E}_{e_{k+1}}[f(y_{k+1})]$ with some multiplicative and additive terms to connect progresses of mirror descent and gradient descent steps. But since we have $L_2$-smoothness defined with respect to Euclidean norm it is more convenient first to upper bound $\mathbb{E}_{e_{k+1}}[\|\nabla f(x_{k+1}), e_{k+1}\|^2]$ by $\|\nabla f(x_{k+1})\|^2$ with some multiplicative factor which is the most important step in the whole proof since it influences the dimension-dependence in the final bound and after that to upper bound $\|\nabla f(x_{k+1})\|^2$ by $f(x_{k+1}) - \mathbb{E}_{e_{k+1}}[f(y_{k+1})]$ with some multiplicative and additive terms using standard gradient descent analysis. Our main technical lemma (see Lemma 1) helps us here:

$$\mathbb{E}_{e_{k+1}}[\|\nabla f(x_{k+1}), e_{k+1}\|^2] \leq \frac{6\rho_n}{n}\|\nabla f(x_{k+1})\|^2, \quad \rho_n = \min \{q - 1, 16\ln n - 8\} n^{q - 1}, \forall n \geq 8.$$

It remains to estimate $\|\nabla f(x_{k+1})\|^2$. Using $L_2$-smoothness and invoking gradient step we have

$$f(y_{k+1}) \leq f(x_{k+1}) - \frac{1}{2L_2}\|\nabla f(x_{k+1})\|^2 + \frac{L_2}{2}\|\nabla f(x_{k+1})\|^2 \leq f(x_{k+1})$$

$$- \frac{1}{2L_2}\langle \nabla f(x_{k+1}), e_{k+1} \rangle^2 - \frac{1}{2L_2}\theta(x_{k+1}, t, e_{k+1}) \langle \nabla f(x_{k+1}), e_{k+1} \rangle \\
+ L_2\left(\frac{1}{2L_2}\langle \nabla f(x_{k+1}), e_{k+1} \rangle + \frac{1}{2L_2}\theta(x_{k+1}, t, e_{k+1})\right)^2 \leq f(x_{k+1}) - \frac{1}{4L_2}\|\nabla f(x_{k+1}), e_{k+1}\|^2 + \frac{L_2^2}{16}.$$
Using Lemma B.10 from Bogolubsky et al. (2016) and rearranging the terms we get \( \| \nabla f(x_{k+1}) \|_2^2 \leq 4nL_2 \left( f(x_{k+1}) - \mathbb{E}_{\epsilon_{k+1}} [f(y_{k+1})] \right) + \frac{L_2 n t^2}{4} \). If we multiply the right hand side of the previous inequality by 2 we will still obtain right inequality \( \| \nabla f(x_{k+1}) \|_2^2 \leq 8nL_2 \left( f(x_{k+1}) - \mathbb{E}_{\epsilon_{k+1}} [f(y_{k+1})] \right) + \frac{L_2 n t^2}{2} \). For the deterministic case it is no reason to do so, but we do it to have clear matching with the general case. Putting all together we get

\[
\alpha_{k+1} \langle \nabla f(x_{k+1}), z_k - x \rangle \leq 48\alpha_{k+1}^2 n^2 \rho_n L_2 \left( f(x_{k+1}) - \mathbb{E}_{\epsilon_{k+1}} [f(y_{k+1})] \right) + V[z_k](x) - \mathbb{E}_{\epsilon_{k+1}} [V[z_{k+1}](x)] + t\tilde{W}_k,
\]

(13)

where \( \tilde{W}_k = W_k + 3\alpha_{k+1}^2 n^2 \rho_n L_2 t \).

The remaining part of the proof is very similar to the original proof of linear coupling method, but still hides non-trivial technique how to estimate sum of \( \mathbb{E} \left[ \tilde{W}_k \right] \). However, for the simplicity in this sketch of the proof we omit the details about estimating terms connecting with \( \tilde{W}_k \) (see appendix for the detailed proof). Further,

\[
\alpha_{k+1} \langle f(x_{k+1}) - f(x) \rangle \leq \alpha_{k+1} \langle \nabla f(x_{k+1}), x_{k+1} - x \rangle = \alpha_{k+1} \langle \nabla f(x_{k+1}), x_{k+1} - z_k \rangle + \alpha_{k+1} \langle \nabla f(x_{k+1}), z_k - x \rangle \leq \frac{1 - \tau_k}{\tau_k} (1 - \tau_k) \alpha_{k+1} \langle f(y_{k+1}) - f(x_{k+1}) \rangle + \alpha_{k+1} \langle \nabla f(x_{k+1}), z_k - x \rangle \leq \frac{1 - \tau_k}{\tau_k} (1 - \tau_k) \alpha_{k+1} \langle f(y_{k+1}) - f(x_{k+1}) \rangle + V[z_k](x) - \mathbb{E}_{\epsilon_{k+1}} [V[z_{k+1}](x)] + t\tilde{W}_k.
\]

That is, ① is since \( x_{k+1} = \tau_k z_k + (1 - \tau_k) y_k \Leftrightarrow \tau_k(x_{k+1} - z_k) = (1 - \tau_k)(y_k - x_{k+1}) \), ② follows from the convexity of \( f \) and inequality \( 1 - \tau_k \geq 0 \) and ③ is since \( \tau_k = \frac{1}{192n^2 \rho_n L_2} \). Rearranging the terms we get \( 48n^2 \rho_n L_2 \alpha_{k+1}^2 \mathbb{E}[f(y_{k+1})] - (48n^2 \rho_n L_2 \alpha_{k+1}^2 - \alpha_{k+1} f(y_k) - V[z_k](x) + \mathbb{E}_{\epsilon_{k+1}} [V[z_{k+1}](x)] - t\tilde{W}_k \leq \alpha_{k+1} f(x_{k+1}) \). Taking the full expectation from the both sides of the previous inequality and summing the results for \( k = 0, 1, \ldots, N \) we get \( 48n^2 \rho_n L_2 \alpha_{k+1}^2 \mathbb{E}[f(y_N)] - \sum_{k=1}^{N-1} \frac{1}{192n^2 \rho_n L_2} \mathbb{E}[f(y_k)] - V[z_0](x_0) + \mathbb{E}[V[z_N](x_N)] - t \sum_{k=0}^{N-1} \mathbb{E} \left[ \tilde{W}_k \right] \leq \sum_{k=0}^{N-1} \alpha_{k+1} f(x_{k+1}) \), where we use the simple recursion \( 48n^2 \rho_n L_2 \alpha_{k+1}^2 - \alpha_{k+1} + \frac{1}{192n^2 \rho_n L_2} = 48n^2 \rho_n L_2 \alpha_{k+1}^2 \) which trivially follows from our choice of \( \alpha_{k+1} \). We define \( \Theta_p := V[z_0](x^*) \). Since \( \sum_{k=0}^{N-1} \alpha_{k+1} = \frac{N(N+3)}{192n^2 \rho_n L_2} \), \( V[z_N](x_N) \geq 0 \) and for all \( k = 1, \ldots, N \), \( f(y_k) \leq f(x^*) \), we obtain

\[
\frac{(N + 1)^2}{192n^2 \rho_n L_2} \mathbb{E}[f(y_N)] \leq f(x^*) \left( \frac{(N + 3)N}{192n^2 \rho_n L_2} - \frac{N - 1}{192n^2 \rho_n L_2} \right) + \Theta_p + t \sum_{k=0}^{N-1} \mathbb{E} \left[ \tilde{W}_k \right],
\]

and after rearranging the terms we get

\[
\mathbb{E}[f(y_N)] - f(x^*) \leq \frac{192n^2 \rho_n L_2 \Theta_p}{(N + 1)^2} + t \cdot \frac{192n^2 \rho_n L_2}{(N + 1)^2} \sum_{k=0}^{N-1} \mathbb{E} \left[ \tilde{W}_k \right].
\]

(14)
Moreover, if the dimension \( n \) Let RDFDS with Theorem 3 represents the total number of our oracle which means that the oracle complexity is also \( O(\varepsilon) \) it is clear why \( p = 1 \) is always better than \( p = 2 \). That is, for our choice of prox-functions described in the Section 2.1 we have \( \Theta_1 = O(\|x_0 - x_*\|^2) \) and \( \Theta_2 = \frac{1}{2}\|x_0 - x_*\|^2 = O(\|x_0 - x_*\|^2) \). Moreover, if the dimension \( n \) is big enough then \( \rho_\star = \min\{q - 1, 16 \ln n - 2\} \) is \( \tilde{O}(\frac{1}{n^2}) \) for the choice of \( p = 1 \) and \( \tilde{O}(\frac{1}{n}) \) for the choice of \( p = 2 \). What is more, due to Cauchy-Schwartz inequality \( \|x_0 - x_*\|_1 \leq \sqrt{n}\|x_0 - x_*\|_2 \). Taking it all into account we obtain that in general situation oracle complexity of ARDFDS with \( p = 1 \) is not worse then for ARDFDS with \( p = 2 \). But if we are lucky and set \( x_0 \) in such a way that \( x_0 - x_* \) is sparse (i.e. we can choose \( x_0 = 0 \) if we minimize the loss for over-parameterized models, since the solution \( x_* \) is sparse), we will have \( \|x_0 - x_*\|_1 = O(\|x_0 - x_*\|_2) \) and \( \tilde{O}\left(\frac{n\sqrt{L_2\Theta_2}}{\varepsilon}\right) \) oracle complexity for the choice of \( p = 1 \) which is \( \sqrt{n} \) better then \( \tilde{O}\left(\frac{n\sqrt{L_2\Theta_2}}{\varepsilon}\right) \).

Now it is clearly why it was so important to get a tight upper bound of \( \mathbb{E}_{\varepsilon_{k+1}}[\|\nabla f(x_{k+1}), \varepsilon_{k+1} x_{k+1}\|_2^2] \) by \( \|\nabla f(x_{k+1})\|_2^2 \), since it is the only place in our proof where the constant \( \rho_\star \) appears and \( \rho_\star \) is, actually, the only thing that improves the dimension-dependence in the oracle complexity.

In the Table 2 we give the appropriate choice of the ARDFDS parameters \( N, m, t \) and accuracy of the function values evaluation \( \Delta \) where \( \varepsilon_{k+1} = \frac{k+2}{8n^2\rho_\star L_2} \) (see Corollary 8 in the appendix). The last row represents the total number \( Nm \) of function evaluations, which was advertised in (4).

Our Randomized Derivative-Free Directional Search (RDFS) method is listed as Algorithm 2.

**Theorem 3** Let RDFDS with \( \alpha = \frac{1}{48n\rho_\star L_2} \) be applied to solve problem (1) and \( \Theta_\star = V[x_0](x^\star) \). Then

\[
\mathbb{E}[f(\bar{x}_N)] - f(x_\star) \leq 384n\rho_\star L_2 \Theta_\star + \frac{2n^2}{L_2 m} + \left( \frac{n}{6L_2} + \frac{N}{3L_2 \rho_\star} \right) \left( \frac{L_2^2 t^2}{2} + \frac{8\Delta^2}{t^2} \right) + \frac{8\sqrt{2n\rho_\star}}{N} \frac{L_2 t}{2} + \frac{2\Delta}{t},
\]

for all \( n \geq 8 \).

The proof of the theorem and appropriate choice of parameters are given in the Section F in the Appendix. The proof of this result is pretty similar to the proof of the Theorem 2.
Algorithm 2 Randomized Derivative-Free Directional Search (RDFDS)

**Input:** \(x_0\) — starting point; \(N\) — number of iterations; \(m \geq 1\) — batch size; \(t > 0\) — smoothing parameter, \(\alpha\) — stepsize.

**Output:** point \(\bar{x}_N\).

1: for \(k = 0, \ldots, N-1\). do
2: Generate \(e_{k+1} \in RS_2(1)\) independently from previous iterations and \(\xi_i, i = 1, \ldots, m\) – independent realizations of \(\xi\).
3: \(x_{k+1} \leftarrow \arg\min_{x \in \mathbb{R}^n} \left\{ \alpha n \left\langle \widetilde{\nabla}^m f^l(x_k), x - x_k \right\rangle + V[x_k](x) \right\} \).
4: Calculate \(\widetilde{\nabla}^m f^l(x_{k+1})\) given in (9).
5: end for
6: return \(\bar{x}_N \leftarrow \frac{1}{N} \sum_{k=0}^{N-1} x_k\)

2.3. Role of the parameters \(\Delta, t\) and \(\sigma^2\) in our algorithms

**Role of \(\Delta\) and \(t\).** We want to mention that there is no need to know the noise level \(\Delta\) to run our algorithms. As it can be seen from (10), the ARDFS method is robust (in the sense of Nemirovski et al. (2009)) to the choice of the smoothing parameter \(t\). Namely, if we under/overestimate \(t\) by a constant factor, the corresponding terms in the convergence rate will increase only by a constant factor.

Our Theorems 2 and 3 are applicable in two situations, the noise being a) controlled and b) uncontrolled.

a) Our assumptions on the noise level in Tables 2 and 4 can be met in practice. For example, in Bogolubsky et al. (2016), the objective function is defined by some auxiliary problem and its value can be calculated with accuracy \(\Delta\) at the cost proportional to \(\ln \frac{1}{\Delta}\), which would result in only a \(\ln \frac{1}{\varepsilon}\) factor in the total complexity of our methods in this paper.

b) One can minimize the r.h.s. of (10) and (15) in \(N\) and obtain the minimum possible accuracy of the solution. This minimum error can not be arbitrarily small. But, this is reasonable: one can not solve the problem with better accuracy than the accuracy of the available information. And here we come to the curious phenomenon: it can happen that accelerated algorithm works worse than a non-accelerated one. This is not an issue of our approach, but a general drawback of accelerated methods, as for the full gradient methods (see Devolder et al. (2014)) it was shown that accelerated gradient method accumulates the error and it was proved that it is impossible to have acceleration without error accumulation. We conjecture that, in our derivative-free setting, it is impossible to have slower rate of error accumulation than we have.

**Role of \(\sigma^2\).** Although, all the related works, which we are aware of, assume \(\sigma^2\) to be known, adaptivity to the variance \(\sigma^2\) is a very important direction of future work.

3. Experiments

We considered the following artificial problem in order to emphasize main theoretical contributions of our work. That is, we construct the objective function such that it reveals advantages of using \(\| \cdot \|_1\)-norm setup instead of classical Euclidean setup.
The objective function is Nesterov’s function $f(x) = \frac{L_2}{4} \left( \frac{1}{2} \sum_{i=1}^{n-1} (a_i - x_i^{i+1})^2 + (x^n)^2 \right) - x^1$ from Nesterov (2004), where $x^i$ is the $i$-th component of vector $x \in \mathbb{R}^n$. It is well-known and one can easily check that $f$ is convex, $L_2$-smooth w.r.t. $\|\cdot\|_2$-norm and the objective attains its minimal value $f_\ast = \frac{L_2}{8} \left( -1 + \frac{1}{n+1} \right)$ at the point $x_\ast = (x_1^\ast, \ldots, x_n^\ast)^\top$ such that $x_i^\ast = 1 - \frac{i}{n+1}$. We introduce stochastic noise in our function and consider $F(x, \xi) = f(x) + \xi(a, x)$, where $\xi$ is Gaussian random variable with mean $\mu = 0$ and variance $\sigma^2$ and $a \in \mathbb{R}^n$ is some vector from the unit Euclidean sphere ($\|a\|_2^2 = 1$). It implies that $f(x) = \mathbb{E}_\xi [F(x, \xi)]$ and $F(x, \xi)$ is $L_2$-smooth in $x$ w.r.t. $\|\cdot\|_2$-norm since $g(x, \xi) - g(y, \xi) = \nabla f(x) - \nabla f(y)$. Moreover, $\mathbb{E}_\xi [\|g(x, \xi) - \nabla f(x)\|_2^2] = \mathbb{E}_\xi [\|\xi a\|_2^2] = \|a\|_2^2 \mathbb{E}_\xi [\xi^2] = \sigma^2$ for all $x \in \mathbb{R}^n$. Also we introduce the additive noise $\eta(x) = \Delta \sin \|x\|_2$. It is clear that $|\eta(x)| \leq \Delta$ for all $x \in \mathbb{R}^n$. Thus, we consider the situation when oracle takes as an input a pair of points $(x, y)$ and returns noisy stochastic realizations $(\tilde{f}(x, \xi), \tilde{f}(y, \xi))$, where $\tilde{f}(x, \xi) = F(x, \xi) + \eta(x) = f(x) + \xi(a, x) + \Delta \sin \|x\|_2$. We compare our methods in Euclidean and $\|\cdot\|_1$-norm proximal setups relatively and RSPGF to define the corresponding method from Ghadimi and Lan (2013).

4. Conclusion

In this paper, we propose two new algorithms for smooth stochastic derivative-free optimization with two-point feedback inexact oracle. Our first algorithm is an accelerated one and the second one is a non-accelerated one. Interestingly, despite the traditional choice of $\ell_2$-norm proximal setup for unconstrained
optimization problems, our analysis shows that the method with $\ell_1$-norm proximal setup has better complexity.

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An Accelerated Method for Derivative-Free Smooth Stochastic Convex Optimization: Supplementary Material

Appendix A. Detailed Review of the Related Work

Complexity bounds for derivative-free optimization with exact function observations were obtained in Nemirovsky and Yudin (1983); Protasov (1996). The related work on stochastic problems can be divided in two large groups, namely, a group, considering one-point feedback, and a group, considering two-point feedback. A unified view on these two cases was presented in Hu et al. (2016).

One-point feedback. Strictly speaking, this setup allows to form an approximation for the gradient using two observations of the objective function, but these two observations correspond to two different realizations of the random vector $\xi$. Most of the authors in this group solve a more general problem of bandit convex optimization and obtain bounds on the so-called regret. It is well known Cesa-bianchi et al. (2002) that a bound on the regret can be converted to a bound on the expected optimization error $f(\hat{x}) - f^*$ in stochastic optimization, where $f^*$ is an optimal value of $f$. To compare the results in the literature, we compare complexity bounds, that is, the total number of oracle call to achieve the expected optimization error $\varepsilon$. Also we enumerate results for the case of one-point feedback focusing only on the dependence on dimension $n$ and accuracy $\varepsilon$ since our results obtained for the two-point feedback oracle and, therefore, these results do not conquer with results we obtained.

In the early work, Flaxman et al. (2005) obtained complexity $\tilde{O}(n^2/\varepsilon^4)$ for convex non-smooth problems and $\tilde{O}(n^2/\varepsilon^3)$ for strongly convex non-smooth problems (see also Saha and Tewari (2011); Dekel et al. (2015); Gasnikov et al. (2017) on some improvements on these sub $1/\varepsilon^2$ rates). Agarwal et al. (2011) provide an algorithm with complexity bound $\tilde{O}(n^{32}/\varepsilon^2)$, which was later improved by Liang et al. (2014) to $\tilde{O}(n^{14}/\varepsilon^2)$ for convex functions, and by Belloni et al. (2015) to $\tilde{O}(n^{6.5}/\varepsilon^2)$. Bubeck et al. (2017) conjecture that their algorithm has complexity $\tilde{O}(n^3/\varepsilon^2)$, while the lower bound of Shamir (2013) is $\Omega(n^2/\varepsilon^2)$ even for strongly convex functions.

Smoothness of the objective function allows to obtain better upper bounds. In this case, Jamieson et al. (2012); Hazan and Levy (2014) proved $\tilde{O}(n^3/\varepsilon^2)$ bound for strongly convex problems. Later, Gasnikov et al. (2017) obtained a bound $\tilde{O}(n^{5}/\varepsilon^3)$ for convex problems and $\tilde{O}(n^{2}/\varepsilon^2)$ for strongly convex problems. Bach and Perchet (2016) obtained a bound $\tilde{O}(n^{2}/\varepsilon^3)$ for convex problems and $\tilde{O}(n^{2}/\varepsilon^2)$ for strongly convex problems. For the smooth case and both convex and strongly convex problems, Jamieson et al. (2012) proved a lower bound $\Omega(n/\varepsilon^2)$ and Shamir (2013) obtained an $\Omega(n^2/\varepsilon^2)$ lower bound.

Two-point feedback. Non-smooth deterministic problem of this type was considered by Nesterov and Spokoiny (2017), who proved in the Euclidean setup an $O(nM_2^2 R_2^2/\varepsilon^2)$ complexity bound, where $M_2$ is a Lipschitz constant of the function $f$ in $\|\cdot\|_2$-norm and $R = \|x_0 - x^*\|_2$. For the smooth stochastic optimization problems similar bound $O(s(n)M^2 R^2/\varepsilon^2)$ was obtained by Duchi et al. (2015) in the non-Euclidean setup, where $s(n)$ is a special function of $n$ specified in the Assumption D of Duchi et al. (2015), $\mathbb{E}_{\xi} \|g(x, \xi)\|_2^2 \leq M^2$, which is very restrictive and does not hold for many unconstrained optimization problems, and $R$ is a radius of the feasible set that means that Duchi et al. (2015) additionally assumed $c > 0$.

2. $\tilde{O}$ hides polylogarithmic factors $(\ln n)^c$.
3. We list the references in the order of the date of the first appearance, but not in the order of the date of publication.
that the feasible set is bounded. However, the authors didn’t provide examples when this bound outperforms previous bound by Nesterov and Spokoiny (2017) and show that in the stochastic setting it recovers $O(nM_2^2R_z^2/\varepsilon^2)$ for the Euclidean setup. Next, these bound were improved by Gasnikov et al. (2016a, 2017) to $O(n^{2/3}M_2R_z^2/\varepsilon^2)$, where $p \in \{1, 2\}$, $\frac{1}{p} + \frac{1}{q} = 1$, $E_{x,\xi} \|g(x, \xi)\|_2^2 \leq M_2^2$ and $R_p$ is the radius of the feasible set in the $\| \cdot \|_p$ norm. Then, for non-smooth case Shamir (2017) obtained the same bound in $\| \cdot \|_1$-norm proximal setup $\tilde{O}(M_2^2R_z^2/\varepsilon^2)$, where $f$ is $M_2$-Lipschitz w.r.t. $\| \cdot \|_1$-norm and $R_1$ is the radius of the feasible set in the $\| \cdot \|_1$-norm. Using the different technique, authors of Bayandina et al. (2018) independently from Shamir (2017) obtained the similar bound $\tilde{O}(n^{2/3}M_2^2R_z^2/\varepsilon^2)$ but for the case when feasible set is not necessary bounded and with additional assumption that oracle gives values of the objective with small additive noise of unknown nature; $R_p$ is distance in $\| \cdot \|_p$-norm between starting point and the solution. For $\mu_2$-strongly convex w.r.t. to $\| \cdot \|_2$-norm problems, Gasnikov et al. (2017); Bayandina et al. (2018) proved a bound $\tilde{O}(n^{2/3}M_2^2/\mu_2\varepsilon))$ for the non-smooth objective function and Gasnikov et al. (2017) proved a bound $\tilde{O}(nM_2^2/\mu_2\varepsilon))$ for the fully smooth case. See also the comparison of these results in the Table 3.

In the fully smooth case, without the assumption that $E\|g(x, \xi)\|_2^2 < +\infty$, Ghadimi and Lan (2013) proposed an algorithm with the bound

$$\tilde{O}\left(\max\left\{\frac{nL_2R_z^2}{\varepsilon}, \frac{n\sigma^2R_z^2}{\varepsilon^2}\right\}\right)$$

for the Euclidean case.

**Deterministic problems.** Accelerated derivative-free method for deterministic problems was proposed in Nesterov and Spokoiny (2017) for the Euclidean case with the bound $O(n\sqrt{L_zR_z^2}/\varepsilon)$. A non-accelerated derivative-free method for deterministic problems with additional bounded noise in function values was proposed in Bogolubsky et al. (2016) together with $O(nL_zR_z^2/\varepsilon)$ bound and application to learning parameter of a parametric PageRank model. Deterministic problems with additional bounded noise in function values were also considered in Dvurechensky et al. (2017), where several accelerated derivative-free methods, including Derivative-Free Block-Coordinate Descent, were proposed and a bound $O(n\sqrt{LR_z^2}/\varepsilon)$ was proved, where $L$ depends on the method and, in some sense, characterizes the average over blocks of coordinates Lipschitz constant of the derivative in the block. All these results are not applicable in our setting since these algorithms are designed for deterministic problems.

**Appendix B. Experiments**

**B.1. How to perform mirror descent step efficiently**

The mirror descent step in the main algorithm requires to solve the following optimization problem:

$$z_{k+1} \leftarrow \arg\min_{z \in \mathbb{R}^n} \left\{ \alpha_{k+1} n \left\langle \tilde{\nabla}^m f^t(x_{k+1}), z - z_k \right\rangle + V[z_k](z) \right\}.$$ 

For general prox-functions it means that one needs to solve an auxiliary optimization problem. However, for the aforementioned prox-function one can to obtain an implicit formula for $z_{k+1}$. However, our choices of prox-structures give us an opportunity to obtain implicit formula for $z_{k+1}$.

In the Euclidean case our theory suggests to use $d(x) = \frac{1}{2}\|x\|_2^2$. It is easy to show that for this particular prox-function the corresponding Bregman divergence is equal to $V[z](y) = \frac{1}{2}\|x - y\|_2^2$ and we obtain well-known formula in proximal operators theory:

$$z_{k+1} = \arg\min_{z \in \mathbb{R}^n} \left\{ \alpha_{k+1} n \left\langle \tilde{\nabla}^m f^t(x_{k+1}), z - z_k \right\rangle + \frac{1}{2}\|z_k - z\|_2^2 \right\} = z_k - \alpha_{k+1}n\tilde{\nabla}^m f^t(x_{k+1}). \quad (16)$$
| Method       | Assumptions                                                                                                                                  | Oracle complexity                                                                      |
|--------------|----------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------|
| MD [Duchi et al. (2015)] | not only Euclidean setup, $\mathbb{E}_\xi [\|g(x, \xi)\|^2] \leq M_2^2$, bounded domain, smoothness of the objective, no noise    | $\tilde{O} \left( \frac{nM_2^2 R_p^2}{\varepsilon^2} \right)$                         |
| MD [Gasnikov et al. (2016a, 2017)] | not only Euclidean setup, $\mathbb{E}_\xi [\|g(x, \xi)\|^2] \leq M_2^2$, bounded domain, smoothness of the objective | $\tilde{O} \left( \frac{n^\frac{2}{3} M_2^2 R_p^2}{\varepsilon^2} \right)$            |
| MD [Shamir (2017)] | the objective could be non-smooth, not only Euclidean setup, $|f(x) - f(y)| \leq M_2 \|x - y\|_2$, bounded domain, no noise | $\tilde{O} \left( \frac{1}{\varepsilon^2} \right)$                                    |
| MD [Gasnikov et al. (2016b), Bayandina et al. (2018)] | the objective could be non-smooth, not only Euclidean setup, $\mathbb{E}_\xi [\|g(x, \xi)\|^2] \leq M_2^2$ | $\tilde{O} \left( \frac{n^\frac{2}{3} M_2^2 R_p^2}{\varepsilon^2} \right)$            |
| RSPGF [Ghadimi et al. (2016), Ghadimi and Lan (2013)] | (2), no noise, smoothness of the objective, only Euclidean setup | $\tilde{O} \left( \max \left\{ \frac{nL_2 R_p^2}{\varepsilon^2}, \frac{n^3\sigma^2 R_p^2}{\varepsilon^2} \right\} \right)$ |
| RDFDS (New!) [This paper] | (2), not only Euclidean setup, smoothness of the objective | $\tilde{O} \left( \max \left\{ \frac{n^2 L_2 R_p^2}{\varepsilon^2}, \frac{n^7\sigma^2 R_p^2}{\varepsilon^2} \right\} \right)$ |
| ARS [Nesterov and Spokoiny (2017)] | $\sigma^2 = 0$, no noise, smoothness of the objective, only Euclidean setup | $\tilde{O} \left( \frac{n \sqrt{L_2 R_p^2}}{\varepsilon^2} \right)$                       |
| ARDFDS (New!) [This paper] | (2), not only Euclidean setup, smoothness of the objective | $\tilde{O} \left( \max \left\{ \frac{n^2 \sigma^2}{\varepsilon^2}, \frac{n^7\sigma^2 R_p^2}{\varepsilon^2} \right\} \right)$ |

Table 3: Comparison of oracle complexity (total number of oracle calls) of different methods with two point feedback discussed in the paper for convex optimization problems. Unless otherwise stated, it is assumed that the feasible set is unbounded and the values of the functions are known with additive small noise of unknown nature.
For the case $p = 1$, we choose the following prox-function: Ben-Tal and Nemirovski (2015)

$$d(x) = \frac{en(\kappa-1)(2-\kappa)/\kappa \ln n}{2} \|x\|_2^2 \overset{\text{eqdef}}{=} A_n \|x\|_\kappa^2, \quad \kappa = 1 + \frac{1}{\ln n}.$$  

Let us show that the solution for the optimization problem

$$\langle s, z - y \rangle + V[y](z) \rightarrow \min_{z \in \mathbb{R}^n}, \quad s = \alpha_{k+1} n \bar{V}^m f^i(x_{k+1})$$

is defined by equations

$$z_i = \text{sign}(\hat{s}_i) \left( \frac{|\hat{s}_i|}{2} \right)^{\frac{1}{\kappa-1}} \left( \sum_{i=1}^{n} \left( \frac{|s_i|}{2} \right)^{\frac{\kappa}{\kappa-1}} \right)^{\frac{\kappa-2}{\kappa}}, \quad \hat{s}_i = -\frac{s}{A_n} + \nabla_z (\|z\|_\kappa^2) \bigg|_{z=y}, \quad i = 1, 2, \ldots, n$$

where $z_i$ is the $i$-th component of vector $z$. Taking into account the identity $V[y](z) = d(z) - d(y) - \langle \nabla d(y), z - y \rangle$ and $d(x) = A_n \|x\|_\kappa^2$ we can solve the equivalent optimization problem

$$-\langle \hat{s}, z \rangle + \|z\|_\kappa^2 \rightarrow \min_{z \in \mathbb{R}^n}.$$  

Firstly, function $f(z) = -\langle \hat{s}, z \rangle + \|z\|_\kappa^2$ is strongly convex on $\mathbb{R}^n$ and, therefore,

$$0 \in \nabla_z (\|z\|_\kappa^2) \bigg|_{z=z^*} - \hat{s},$$

where $\nabla_z (\|z\|_\kappa^2) \bigg|_{z=z^*}$ is subdifferential of $\|z\|_\kappa^2$ at $z = z^*$. From this we obtain

$$\frac{\partial}{\partial z_i} (\|z\|_\kappa^2) = \frac{2}{\kappa} \left( \sum_{i=1}^{n} |z_i|^{\kappa} \right)^{\frac{2-\kappa}{\kappa}} \kappa |z_i|^{\kappa-1} \text{sign}(z_i)$$

$$= 2 \left( \sum_{i=1}^{n} |z_i|^{\kappa} \right)^{\frac{2-\kappa}{\kappa}} \kappa |z_i|^{\kappa-1} \text{sign}(z_i)$$

and it should be equal to $\hat{s}_i = \text{sign}(\hat{s}_i) |s_i|$ if $z = z^*$. This gives us

$$\text{sign}(z_i) = \text{sign}(\hat{s}_i)$$

and

$$\left( \sum_{j=1}^{n} |z_j|^{\kappa} \right)^{\frac{2-\kappa}{\kappa-1}} |z_i|^{\kappa} = \left( \frac{|\hat{s}_i|}{2} \right)^{\frac{\kappa}{\kappa-1}}, \quad i = 1, 2, \ldots, n.$$  

Summing identities (17) for $i = 1, 2, \ldots, n$ we get

$$\left( \sum_{j=1}^{n} |z_j|^{\kappa} \right)^{\frac{1}{\kappa-1}} = \sum_{i=1}^{n} \left( \frac{|\hat{s}_i|}{2} \right)^{\frac{\kappa}{\kappa-1}} \implies \sum_{j=1}^{n} |z_j|^{\kappa} = \left( \sum_{i=1}^{n} \left( \frac{|\hat{s}_i|}{2} \right)^{\frac{\kappa}{\kappa-1}} \right)^{\kappa-1}.$$

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By plugging it into (17) we have

$$|z_i| = \left( \frac{|\hat{s}_i|}{2} \right)^{-\frac{1}{
}} \left( \sum_{i=1}^{n} \left( \frac{|\hat{s}_i|}{2} \right)^{-\frac{1}{
}} \right)^{\frac{n-2}{
}}, \quad i = 1, 2, \ldots, n$$

and finally

$$z_i = \text{sign}(\hat{s}_i) \left( \frac{|\hat{s}_i|}{2} \right)^{-\frac{1}{
}} \left( \sum_{i=1}^{n} \left( \frac{|\hat{s}_i|}{2} \right)^{-\frac{1}{
}} \right)^{\frac{n-2}{
}}, \quad i = 1, 2, \ldots, n.$$ 

This simple analysis shows us that to compute $z_{k+1}$ we need to perform $O(n)$ arithmetical operations.

### B.2. Detailed description of the numerical experiments

We set desired accuracy $\varepsilon = 10^{-3}$ and adjust parameters $\Delta$ and $t$ according to Tables 2 and 4. To emphasize the importance of improving the first term of the maximum in our complexity bounds we choose $\sigma^2$ such that $\sigma^2 \leq \frac{2^{3/2} \sqrt{n}}{\sqrt{L_2}} \cdot \sqrt{\frac{L_2}{2}}$ in order to make optimal batch-size $m = 1$ for the accelerated method with $p = 1$ and the starting point $x_0$ which have all coordinates equal to corresponding coordinates of $x_*$ except the first coordinate which is equal to $x_0^1 = x_*^1 + L_2$ and also we choose $L_2 = 10$. In this case $\|x_0 - x_*\|_1 = \|x_0 - x_*\|_2 = L_2$ and oracle complexity actually equal iteration complexity up to $O(1)$ factor. For such case our theory says that our methods with $p = 1$ should work much faster than in the Euclidean case for the big $n$.

As the theory predicts our methods with $p = 1$ work better and better when $n$ is growing (see Figure 1). In our series of experiments we run algorithms until they reach the desired functional accuracy $\varepsilon = 10^{-3}$. Simple calculations chows that for our choice of $x_0$ we have $f(x_0) - f(x_*) = 250$, therefore, to reach functional accuracy $\varepsilon = 10^{-3}$ we need to have relative accuracy $\frac{f(x_k) - f(x_*)}{f(x_0) - f(x_*)} = 10^{-3} \in [10^{-6}, 10^{-5}]$. We adjusted stepizes $\alpha_{k+1}$ for each of our methods (see the Section B.3 for the details). One see that for large $n$ the choice of $p = 1$ becomes more beneficial than the standard choice of $p = 2$. We note that all of our methods outperforms RSPGF method from Ghadimi and Lan (2013) and relate to each other in terms of convergence as our theory predicts.

### B.3. How to choose $\alpha_{k+1}$

We give all the analysis of our algorithms for the choice of $\alpha_{k+1} = \frac{k+2}{96n^2 \rho_n L_2}$ and $\alpha = \frac{1}{48n \rho_n L_2}$. However, our methods do not require to choose exactly such parameters in order to get the convergence. In practice, it is better to choose $\alpha_{k+1} = \gamma \cdot \frac{k+2}{96n^2 \rho_n L_2}$ and $\alpha = \gamma \cdot \frac{1}{48n \rho_n L_2}$ and tune $\gamma$ instead of choosing $\frac{k+2}{96n^2 \rho_n L_2}$ and $\frac{1}{48n \rho_n L_2}$ respectively. And here we come up with the question: what is the optimal $\gamma$ to choose in order to get better convergence?

We do not provide this type of analysis since the main goal of our paper is to improve dimensional dependence of the existing bounds for zeroth-order minimization problems. Nevertheless, we studied this question for the problem described in the main part of this paper in Section 3 for the dimension $n = 100$. We run our algorithms for different values of $\gamma \in [1, 48]$ (see Figure 2).

We notice that the choice $\gamma = 1$ is never optimal in our experiments, which means that it is better to set $\gamma > 1$. In contrast, if we take $\gamma$ too big (e.g. $\gamma = 48$), then ARDFDS for $p = 2$ could diverge. One can see from the provided graphs that optimal $\gamma$ for different proximal setups could be very different (e.g. for ARDFDS with $p = 2$ the optimal value of $\gamma$ is close to 8 and for $p = 1$ it is closer to 2000). In the experiments we use $\gamma = 8$ for ARDFDS with $p = 2$, $\gamma = 2000$ for ARDFDS with $p = 1$, $\gamma = 32$ for RDFDS with $p = 2$ and $\gamma = 1000$ for RDFDS with $p = 1$. 

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Figure 2: The $y$-axis corresponds to the relative accuracy $\frac{f(x_k) - f(x^*)}{f(x_0) - f(x^*)}$ and the $x$-axis shows number of iterations. We use _E and _NE in the graphs to define $\| \cdot \|_2$-norm and $\| \cdot \|_1$-norm proximal setups relatively. Numbers in the legend of each graphs correspond to the values of $\gamma$. Dimension of the problem $n = 100$. 
Appendix C. Inequalities for Gradient Approximation

The proof of the main theorem relies on the following technical result, which connects finite-difference approximation (9) of the stochastic gradient with the stochastic gradient itself and also with $\nabla f$.

**Lemma 4** For all $x, s \in \mathbb{R}^n$, we have

$$E_e\|\tilde{\nabla}^m f^t(x)\|_q \leq \frac{12\rho_m}{n} \|g^m(x, \xi_m)\|_2^2 + \frac{\rho_m \Delta^2}{m} \sum_{i=1}^m L(\xi_i)^2 + \frac{16\rho_m \Delta^2}{t^2},$$

(18)

$$E_e\|\tilde{\nabla}^m f^t(x)\|_q^2 \geq \frac{1}{2} \|g^m(x, \xi_m)\|_2^2 - \frac{t^2}{2m} \sum_{i=1}^m L(\xi_i)^2 - \frac{8\Delta^2}{t^2},$$

(19)

$$E_e(\tilde{\nabla}^m f^t(x), s) \geq \frac{1}{n} \langle g^m(x, \xi_m), s \rangle - \frac{\|s\|_p^2}{2mn} \sum_{i=1}^m L(\xi_i) - \frac{2\|s\|_p^2}{\sqrt{n}},$$

(20)

$$E_e\|\nabla f(x), e - \tilde{\nabla}^m f^t(x)\|_q^2 \leq \frac{2}{n} \|\nabla f(x) - g^m(x, \xi_m)\|_2^2 + \frac{t^2}{m} \sum_{i=1}^m L(\xi_i)^2 + \frac{16\Delta^2}{t^2},$$

(21)

where $g^m(x, \xi_m) := \frac{1}{m} \sum_{i=1}^m g(x, \xi_i)$, $\Delta$ is defined in (3), $L(\xi)$ is the Lipschitz constant of $g(\cdot, \xi)$, which is the gradient of $F(\cdot, \xi)$.

**Proof.** First of all, we rewrite $\tilde{\nabla}^m f^t(x)$ as follows

$$\tilde{\nabla}^m f^t(x) = \left(\langle g^m(x, \xi_m), e \rangle + \frac{1}{m} \sum_{i=1}^m \theta(x, \xi_i, t, e)\right) e,$$

where

$$\theta(x, \xi_i, t, e) = \frac{F(x + te, \xi_i) - F(x, \xi_i)}{t} - \langle g(x, \xi_i), e \rangle + \frac{\Delta(x + te, \xi_i) - \Delta(x, \xi_i)}{t}, \quad i = 1, \ldots, m.$$ 

By Lipshitz smoothness of $F(\cdot, \xi)$ and (3), we have

$$|\theta(x, \xi_i, t, e)| \leq \frac{L(\xi) t}{2} + \frac{2\Delta}{t}.$$ 

(22)

**Proof of (18).**

$$E_e\|\tilde{\nabla}^m f^t(x)\|_q^2 = E_e\left\| \left(\langle g^m(x, \xi_m), e \rangle + \frac{1}{m} \sum_{i=1}^m \theta(x, \xi_i, t, e)\right) e \right\|_q^2$$

$$\leq 2E_e\|\langle g^m(x, \xi_m), e \rangle\|_q^2 + 2 E_e\left\| \frac{1}{m} \sum_{i=1}^m \theta(x, \xi_i, t, e)e \right\|_q^2$$

$$\leq \frac{12\rho_m}{n} \|g^m(x, \xi_m)\|_2^2 + \frac{2\rho_m}{m} \sum_{i=1}^m \left(\frac{L(\xi_i) t}{2} + \frac{2\Delta}{t}\right)^2$$

$$\leq \frac{12\rho_m}{n} \|g^m(x, \xi_m)\|_2^2 + \frac{\rho_m \Delta^2}{m} \sum_{i=1}^m L(\xi_i)^2 + \frac{16\rho_m \Delta^2}{t^2},$$

where $\oplus$ holds since $\|x + y\|_q^2 \leq 2\|x\|_q^2 + 2\|y\|_q^2, \forall x, y \in \mathbb{R}^n$; $\oplus \oplus$ follows from inequalities (7), (8), (22) and the fact that, for any $a_1, a_2, \ldots, a_m > 0$, it holds that $\left(\sum_{i=1}^m a_i\right)^2 \leq m \sum_{i=1}^m a_i^2$. 

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Proof of (19).
\[
E_e\|\nabla^m f^t(x)\|^2_2 = E_e\left\| \left\langle g^m(x, \xi_m), e \right\rangle + \frac{1}{m} \sum_{i=1}^{m} \theta(x, \xi_i, t, e) \right\|^2_2 \\
\geq \frac{1}{2} E_e\|g^m(x, \xi_m), e\|^2_2 - \frac{1}{m} \sum_{i=1}^{m} \left( \frac{L(\xi_i) t}{2} + \frac{\Delta}{t} \right)^2 \\
\geq \frac{1}{2m} \|g^m(x, \xi_m)\|^2_2 - \frac{L^2}{2m} \sum_{i=1}^{m} L(\xi_i)^2 - \frac{8\Delta^2}{t},
\]

(24)

where \( \bigcirc \) follows from (22) and inequality \( \|x + y\|^2 \geq \frac{1}{2} \|x\|^2 + \frac{1}{2} \|y\|^2, \forall x, y \in \mathbb{R}^n \); \( \bigcirc \) follows from \( e \in RS_2(1) \) and Lemma B.10 in Bogolubsky et al. (2016), stating that, for any \( s \in \mathbb{R}^n \), \( E_e\langle s, e\rangle^2 = \frac{1}{n} \|s\|_2^2 \).

Proof of (20).
\[
E_e\langle \nabla^m f(x), s \rangle = E_e\langle \langle g^m(x, \xi_m), e \rangle, s \rangle + E_e\frac{1}{m} \sum_{i=1}^{m} \theta(x, \xi_i, t, e) \langle e, s \rangle \\
\geq \frac{1}{m} \|g^m(x, \xi_m), s\| - \frac{1}{m} \sum_{i=1}^{m} \left( \frac{L(\xi_i) t}{2} + \frac{\Delta}{t} \right) E_e\langle |\langle e, s \rangle| \rangle \\
\geq \frac{1}{m} \|g^m(x, \xi_m), s\| - \frac{t\|\|s\|_p}{2m\sqrt{n}} \sum_{i=1}^{m} L(\xi_i) - \frac{8\Delta^2\|s\|_p}{t}
\]

(25)

where \( \bigcirc \) follows from \( E_e[n(g, e)|e] = g, \forall g \in \mathbb{R}^n \) and (22); \( \bigcirc \) follows from Lemma B.10 in Bogolubsky et al. (2016), since \( E_e|\langle s, e\rangle| \leq \sqrt{E_e\langle s, e\rangle^2} \), and the fact that \( \|x\|_2 \leq \|x\|_p \) for \( p \leq 2 \).

Proof of (21).
\[
E_e\|\langle \nabla f(x), e \rangle - \nabla^m f^t(x)\|^2_2 = E_e\|\langle \nabla f(x), e \rangle - \langle g^m(x, \xi_m), e \rangle - \frac{1}{m} \sum_{i=1}^{m} \theta(x, \xi_i, t, e) \rangle\|^2_2 \\
\leq \frac{1}{m} \|\langle \nabla f(x) - g^m(x, \xi_m), e \rangle\|^2_2 \\
+ 2E_e\frac{1}{m} \sum_{i=1}^{m} \theta(x, \xi_i, t, e) \|e\|^2_2 \\
\leq \frac{2}{m} \|\nabla f(x) - g^m(x, \xi_m)\|^2_2 + \frac{L^2}{m} \sum_{i=1}^{m} L(\xi_i)^2 + \frac{16\Delta^2}{t^2},
\]

(26)

where \( \bigcirc \) holds since \( \|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2, \forall x, y \in \mathbb{R}^n \); \( \bigcirc \) follows from \( e \in S_2(1) \) and Lemma B.10 in Bogolubsky et al. (2016), and (22).

Appendix D. Estimates for the Progress of the Method

The following lemma estimates the progress in step 7 of ARDFDS, which is a gradient step.

Lemma 5 Assume that \( y = x - \frac{1}{2L_2} \nabla^m f^t(x) \). Then,
\[
\|g^m(x, \xi_m)\|^2_2 \leq 8nL_2(f(x) - E_e f(y)) + 8\|\nabla f(x) - g^m(x, \xi_m)\|^2_2 \\
+ 5m^2 \sum_{i=1}^{m} L(\xi_i)^2 + \frac{8nm\Delta^2}{t^2},
\]

(27)

where \( g^m(x, \xi_m) \) is defined in Lemma 4, \( \Delta \) is defined in (3), \( L(\xi) \) is the Lipschitz constant of \( g(\cdot, \xi) \), which is the gradient of \( F(\cdot, \xi) \).
**Proof.** Since \( \nabla^m f^t(x) \) is collinear to \( e \), we have that, for some \( \gamma \in \mathbb{R} \), \( y - x = \gamma e \). Then, since \( \|e\|_2 = 1 \),

\[
\langle \nabla f(x), y - x \rangle = \langle \nabla f(x), e \rangle \gamma = \langle \nabla f(x), e \rangle \langle e, y - x \rangle = \langle \nabla f(x), e \rangle e, y - x \rangle.
\]

From this and \( L_2 \)-smoothness of \( f \) we obtain

\[
f(y) \leq f(x) + \langle \nabla f(x), e \rangle e, y - x \rangle + \frac{L_2}{2} \|y - x\|^2
\]

\[
= f(x) + \langle \nabla^m f^t(x), y - x \rangle + L_2 \|y - x\|^2 + \langle \nabla f(x), e \rangle e - \nabla^m f^t(x), y - x \rangle - \frac{L_2}{2} \|y - x\|^2
\]

\[
\leq f(x) + \langle \nabla^m f^t(x), y - x \rangle + L_2 \|y - x\|^2 + \frac{1}{\sqrt{m}} \|\nabla f(x), e \rangle e - \nabla^m f^t(x)\|_2^2,
\]

where \( \od \) follows form the Fenchel inequality \( \langle s, z \rangle - \frac{1}{2} \|z\|^2 \leq \frac{1}{2 \alpha} \|s\|^2 \). Using \( y = x - \frac{1}{\sqrt{m}} \nabla^m f^t(x) \), we get

\[
\frac{1}{\sqrt{m}} \|\nabla^m f^t(x)\|_2^2 \leq f(x) - f(y) + \frac{1}{\sqrt{m}} \|\nabla f(x), e \rangle e - \nabla^m f^t(x)\|_2^2
\]

Taking the expectation in \( e \) we obtain

\[
\frac{1}{\sqrt{m}} \left( \frac{1}{\sqrt{m}} \|g^m(x, \xi_m)\|_2^2 - \frac{1}{2 \sqrt{m}} \sum_{i=1}^{m} L(\xi_i)^2 \right) \leq \frac{1}{\sqrt{m}} \mathbb{E}_e \|\nabla^m f^t(x)\|_2^2
\]

\[
\leq f(x) - \mathbb{E}_e f(y) + \frac{1}{\sqrt{m}} \mathbb{E}_e \|\nabla f(x), e \rangle e - \nabla^m f^t(x)\|_2^2
\]

\[
\leq f(x) - \mathbb{E}_e f(y) + \frac{1}{\sqrt{m}} \mathbb{E}_e \left( \frac{2}{\sqrt{m}} \|\nabla f(x) - g^m(x, \xi_m)\|_2^2 + \frac{1}{2 \sqrt{m}} \sum_{i=1}^{m} L(\xi_i)^2 + \frac{16 \Delta^2}{\sqrt{m}} \right),
\]

Rearranging the terms, we obtain the statement of the lemma.

**D.1. Progress of the Mirror Descent Step**

The following lemma estimates the progress in step 8 of ARDFDS, which is a Mirror Descent step.

**Lemma 6** Assume that \( z_+ = \arg \min_{u \in \mathbb{R}^n} \left\{ \alpha n \langle \nabla^m f^t(x) \rangle u - z \rangle + V[z](u) \right\} \). Then,

\[
\alpha \langle g^m(x, \xi_m), z - u \rangle \leq 6 \alpha^2 n \rho \|g^m(x, \xi_m)\|_2^2 + V[z](u) - \mathbb{E}_e [V[z_+](u)] + \frac{\alpha^2 n \rho \|z - u\|_p^2}{2} \left( \frac{1}{\sqrt{m}} \sum_{i=1}^{m} L(\xi_i)^2 + \frac{16 \Delta^2}{\sqrt{m}} \right)
\]

\[
+ \alpha \sqrt{n} \|z - u\|_p \left( \frac{1}{\sqrt{m}} \sum_{i=1}^{m} L(\xi_i) + \frac{2 \Delta}{\sqrt{m}} \right),
\]

where \( g^m(x, \xi_m) \) is defined in Lemma 4, \( \Delta \) is defined in (3), \( L(\xi) \) is the Lipschitz constant of \( g(\cdot, \xi) \), which is the gradient of \( F(\cdot, \xi) \).

**Proof.** For all \( u \in \mathbb{R}^n \), we have

\[
\alpha n \langle \nabla^m f^t(x) \rangle u - z \rangle = \alpha n \langle \nabla^m f^t(x) \rangle z - z_+ \rangle + \alpha n \langle \nabla^m f^t(x), z_+ - u \rangle
\]

\[
\leq \alpha n \langle \nabla^m f^t(x) \rangle z - z_+ \rangle + \langle -\nabla V[z](z_+), z_+ - u \rangle
\]

\[
= \alpha n \langle \nabla^m f^t(x) \rangle z - z_+ \rangle + V[z](u) - V[z_+](u) - V[z](z_+)
\]

\[
\leq \left( \alpha n \langle \nabla^m f^t(x) \rangle z - z_+ \rangle - \frac{1}{2} \|z - z_+\|_p^2 + V[z](u) - V[z_+](u)
\]

\[
\leq \frac{\alpha^2 n \rho \|z - u\|_p^2}{2} \leq \frac{16 \Delta^2}{\sqrt{m}} \left( \frac{1}{\sqrt{m}} \sum_{i=1}^{m} L(\xi_i) + \frac{2 \Delta}{\sqrt{m}} \right),
\]

where \( g^m(x, \xi_m) \) is defined in Lemma 4, \( \Delta \) is defined in (3), \( L(\xi) \) is the Lipschitz constant of \( g(\cdot, \xi) \), which is the gradient of \( F(\cdot, \xi) \).
where \( \Omega \) follows from the definition of \( z_+ \), whence \( \langle \nabla V[z](z_+) + \alpha n \tilde{V}^m f^t(x), u - z_+ \rangle \geq 0 \) for all \( u \in \mathbb{R}^n \); \( \Omega \) follows from the "magic identity" Fact 5.3.3 in Ben-Tal and Nemirovski (2015) for the Bregman divergence; \( \Omega \) follows from (6); and \( \Omega \) follows from the Fenchel inequality \( \zeta(s, z) - \frac{1}{2} \| z \|^2 \leq \frac{\zeta^2}{2} \| s \|^2 \).

Taking expectation in \( e \) and applying (20) with \( s = z - u, \) (18), we get

\[
\alpha n \left( \frac{1}{n} \langle g^m(x, \xi_m), z - u \rangle - \frac{t \| z - u \|_p}{2m \sqrt{n}} \sum_{i=1}^m L(\xi_i) - \frac{2\Delta \| z - u \|_p}{t \sqrt{n}} \right)
\leq \alpha n \mathbb{E}_e \langle \tilde{V}^m f^t(x), z - u \rangle \leq \frac{\alpha^2 \rho_n^2}{2} \| g^m(x, \xi_m) \|_p^2 + \frac{\rho_n^2}{m} \sum_{i=1}^m L(\xi_i)^2 + \frac{16 \rho_n \Delta^2}{\rho} + \mathbb{E}_e[V[z_+](u)]
\]

(30)

Rearranging the terms, we obtain the statement of the lemma.

Appendix E. Proof of Theorem 2

First, we prove the following lemma, which estimates the one-iteration progress of the whole algorithm.

Lemma 7. Let \( \{x_k, y_k, z_k, \alpha_k, \tau_k\}, k \geq 0 \) be generated by ARDFDS. Then, for all \( u \in \mathbb{R}^n \),

\[
48n \rho_n L_2 \alpha_{k+1}^2 \mathbb{E}_{e, \xi} \left[ f(y_{k+1}) \mid \mathcal{E}_k, \Xi_k \right] - (48n \rho_n L_2 \alpha_{k+1}^2 - \alpha_{k+1}) f(y_k)
\]

\[
- V[z_k](u) + \mathbb{E}_e[V[z_{k+1}](u) \mid \mathcal{E}_k, \Xi_k] - R_{k+1} \leq \alpha_{k+1} f(u),
\]

(31)

\[
R_{k+1} := 48a_k^2 n \rho_n \alpha_{k+1}^2 + \frac{61 \alpha_{k+1}^2 n^2 \rho_n^2}{2} \left( L_2^2 t^2 + \frac{16 \Delta^2}{t^2} \right) + \alpha_{k+1} \sqrt{n} \| z_k - u \|_p \left( \frac{L_2^2 t^2}{2} + \frac{2 \Delta^2}{t} \right).
\]

(32)

where \( \Delta \) is defined in (3), \( \mathcal{E}_k \) and \( \Xi_k \) denote the history of realizations of \( e_1, \ldots, e_k \) and \( \xi_{1,1}, \ldots, \xi_{k,m} \) respectively, up to the step \( k \).

Proof. Combining (27) and (28), we obtain

\[
\alpha \langle g^m(x_{k+1}, \xi_{m(k+1)}), z - u \rangle \leq 48n \rho_n L_2 \left( f(x_{k+1}) - \mathbb{E}_e f(y_{k+1}) \right) + V[z_k](u) + \mathbb{E}_e[V[z_{k+1}](u)]
\]

\[
+ 48 \alpha^2 n \rho_n \| \nabla f(x_{k+1}) - g^m(x_{k+1}, \xi_{m(k+1)}) \|_2^2
\]

\[
+ \frac{61 \alpha^2 n^2 \rho_n^2}{2} \left( \frac{t^2}{m} \sum_{i=1}^m L(\xi_i)^2 + \frac{16 \Delta^2}{t^2} \right)
\]

\[
+ \alpha \sqrt{n} \| z_k - u \|_p \left( \frac{L_2^2 t^2}{2m} \sum_{i=1}^m L(\xi_i)^2 + \frac{2 \Delta^2}{t} \right),
\]

(33)

where \( g^m(x, \xi_m) \) is defined in Lemma 4 and the expectation in \( e \) is conditional to \( \mathcal{E}_k \). By the definition of \( g^m(x, \xi_m) \) and (2), \( \mathbb{E}_{\xi} g^m(x, \xi_m) = \nabla f(x) \) and \( \mathbb{E}_{\xi} \| \nabla f(x_{k+1}) - g^m(x_{k+1}, \xi_{m(k+1)}) \|_2^2 \leq \frac{a^2}{m} \). Using these two facts and taking the expectation in \( \xi_{m(k+1)} \) conditional to \( \Xi_k \), we obtain

\[
\alpha_{k+1} \langle \nabla f(x_{k+1}), z_k - u \rangle \leq 48 \alpha_{k+1}^2 n \rho_n L_2 \left( f(x_{k+1}) - \mathbb{E}_e f(y_{k+1}) \right)
\]

\[
+ V[z_k](u) - \mathbb{E}_e[V[z_{k+1}](u) \mid \mathcal{E}_k, \Xi_k] + R_{k+1}.
\]

(34)
Further, 
\[
\alpha_{k+1}(f(x_{k+1}) - f(u)) \leq \alpha_{k+1}(\nabla f(x_{k+1}), x_{k+1} - u) \\
= \alpha_{k+1}(\nabla f(x_{k+1}), x_{k+1} - z_k) + \alpha_{k+1}(\nabla f(x_{k+1}), z_k - u) \\
\overset{(3)}{=} \left(1 - \frac{\tau_k}{\tau_k}\right)\alpha_{k+1}(\nabla f(x_{k+1}), y_k - x_{k+1}) + \alpha_{k+1}(\nabla f(x_{k+1}), z_k - u) \\
\overset{(2)}{=} \left(1 - \frac{\tau_k}{\tau_k}\right)\alpha_{k+1}(f(y_k) - f(x_{k+1})) + \alpha_{k+1}(\nabla f(x_{k+1}), z_k - u) \\
\overset{(34)}{\leq} \left(1 - \frac{\tau_k}{\tau_k}\right)\alpha_{k+1}(f(y_k) - f(x_{k+1})) \\
+ 48\alpha_{k+1}^2n^2\rho_nL_2\left(f(x_{k+1}) - \mathbb{E}_{\xi_k, \zeta_k}[f(y_{k+1})] \right) \\
+ V[z_k](u) - \mathbb{E}_{\xi_k, \zeta_k}[V[z_k+1](u)] + R_{k+1} \\
\overset{(2)}{=} \left(48\alpha_{k+1}^2n^2\rho_nL_2 - \alpha_{k+1}\right)(f(y_k) - f(x_{k+1})) + 48\alpha_{k+1}^2n^2\rho_nL_2\mathbb{E}_{\xi_k, \zeta_k}[f(y_{k+1})] \\
+ \alpha_{k+1}f(x_k) + V[z_k](u) - \mathbb{E}_{\xi_k, \zeta_k}[V[z_{k+1}](u)] + R_{k+1}.
\]
That is, (3) is since \(x_{k+1} = \tau_k u + (1 - \tau_k)y_k \iff \tau_k(x_{k+1} - z_k) = (1 - \tau_k)(y_k - x_{k+1})\), (2) follows from the convexity of \(f\) and inequality \(1 - \tau_k \geq 0\) and (3) is since \(\tau_k = \frac{1}{48\alpha_{k+1}^2n^2\rho_nL_2}\). Rearranging the terms, we obtain the statement of the lemma.

**Proof of Theorem 2.** Note that \(48n^2\rho_nL_2\alpha_{k+1}^2 - \alpha_{k+1} + \frac{1}{192n^2\rho_nL_2} = 48n^2\rho_nL_2\alpha_{k+1}^2\). That is, 
\[
48n^2\rho_nL_2\alpha_{k+1}^2 - \alpha_{k+1} + \frac{1}{192n^2\rho_nL_2} = \frac{(k+2)^2}{96n^2\rho_nL_2} - \frac{k+2}{96n^2\rho_nL_2} + \frac{1}{192n^2\rho_nL_2} = \frac{(k+1)^2}{192n^2\rho_nL_2} = 48n^2\rho_nL_2\alpha_{k+1}^2.
\]

Taking full expectation \(\mathbb{E}[\cdot] = \mathbb{E}_{\xi_1, \ldots, \xi_N, \zeta_1, \ldots, \zeta_{N,m}}[\cdot]\) from both sides of (31) for \(k = 0, \ldots, l - 1\), where \(l \leq N\) and telescoping the obtained inequalities\(^4\) we have 
\[
48n^2\rho_nL_2\alpha_{k+1}^2\mathbb{E}[f(y_l)] + \sum_{k=1}^{l-1} \frac{1}{192n^2\rho_nL_2} \mathbb{E}[f(y_k)] - V[z_0](u) \\
+ \mathbb{E}[V[z_l](u)] - \zeta_1 \sum_{k=0}^{l-1} \alpha_{k+1} \mathbb{E}[\|u - z_k\|_p] - \zeta_2 \sum_{k=0}^{l-1} \alpha_{k+1}^2 \leq \sum_{k=0}^{l-1} \alpha_{k+1} f(u),
\]
where we denoted 
\[
\zeta_1 := \sqrt{n} \left( \frac{L_2t}{2} + \frac{2\Delta}{t} \right), \quad \zeta_2 := 48n\rho_n \frac{\sigma^2}{m} + \frac{61n^2\rho_n}{2} \left( L_2^2t^2 + \frac{16\Delta^2}{t} \right).
\]
We set \(u = x^*\) in (35), where \(x^*\) is a solution to (1), and define \(\Theta_p := V[z_0](x^*), R_k := \mathbb{E}[\|x^* - z_k\|_p]\).

Also, from (6), we have that \(\zeta_1 \alpha_1 R_0 \leq \sqrt{\mathbb{E}[\cdot]} / 48n^2\rho_nL_2\). To simplify the notation, we define \(B_l := \zeta_2 \sum_{k=0}^{l-1} \alpha_{k+1}^2 + \Theta_p + \sqrt{\frac{24\rho_n\zeta_1}{48n^2\rho_nL_2}}\). Since \(\sum_{k=0}^{l-1} \alpha_{k+1} = \frac{l(l+1)}{192n^2\rho_nL_2}\) and, for all \(i = 1, \ldots, N\), \(f(y_i) \leq f(x^*)\), we obtain from (35) 
\[
\frac{(l+1)^2}{192n^2\rho_nL_2} \mathbb{E}[f(y_l)] \leq f(x^*) \left( \frac{(l+3)(l+4)}{192n^2\rho_nL_2} - \frac{l+1}{192n^2\rho_nL_2} \right) + B_l - \mathbb{E}[\mathbb{V}[z_l](x^*)] + \zeta_1 \sum_{k=1}^{l-1} \alpha_{k+1} R_k,
\]
\[
0 \leq \frac{(l+1)^2}{192n^2\rho_nL_2} \mathbb{E}[f(y_l)] - f(x^*) \leq B_l - \mathbb{E}[\mathbb{V}[z_l](x^*)] + \zeta_1 \sum_{k=1}^{l-1} \alpha_{k+1} R_k.
\]

\(^4\) Note that \(\alpha_1 = \frac{2}{96n^2\rho_nL_2} = \frac{1}{48n^2\rho_nL_2}\) and therefore \(48n^2\rho_nL_2 \alpha_1^2 - \alpha_1 = 0\).
which gives
\[ \mathbb{E}[V[z_l](x^*)] \leq B_l + \zeta_1 \sum_{k=1}^{l-1} \alpha_{k+1} R_k. \tag{38} \]

Moreover,
\[ \frac{1}{2} (\mathbb{E}[\|z_l - x^*\|_p])^2 \leq \frac{1}{2} \mathbb{E}[\|z_l - x^*\|_p] \leq \mathbb{E}[V[z_l](x^*)] \leq B_l + \zeta_1 \sum_{k=1}^{l-1} \alpha_{k+1} R_k, \tag{39} \]

whence,
\[ R_l \leq \sqrt{2} \cdot \sqrt{B_l + \zeta_1 \sum_{k=1}^{l-1} \alpha_{k+1} R_k}. \tag{40} \]

Applying Lemma 10 for \( a_0 = \zeta_2 \alpha_1^2 + \Theta_p + \sqrt{\frac{2 \Theta_p}{4n^2 \rho_n L_2}} \), \( a_k = \zeta_2 \alpha_{k+1}^2 \), \( b = \zeta_1 \) for \( k = 1, \ldots, N - 1 \), we obtain
\[ B_l + \zeta_1 \sum_{k=1}^{l-1} \alpha_{k+1} R_k \leq \left( \sqrt{B_l + \sqrt{2} \zeta_1 \cdot \frac{t^2}{96n^2 \rho_n L_2}} \right)^2, \quad l = 1, \ldots, N \tag{41} \]

Since \( V[z](x^*) \geq 0 \), by inequality (37) for \( l = N \) and the definition of \( B_l \), we have
\[ \frac{(N+1)^2}{192n^2 \rho_n L_2} \left( \mathbb{E}[f(y_N)] - f(x^*) \right) \leq \left( \sqrt{B_N + \sqrt{2} \zeta_1} \cdot \frac{N^2}{96n^2 \rho_n L_2} \right)^2 \leq 2B_N + 4\zeta_1 \cdot \frac{N^4}{(96n^2 \rho_n L_2)^2} \tag{42} \]
where \( \Box \) is due to the fact that \( \forall a, b \in \mathbb{R} \quad (a + b)^2 \leq 2a^2 + 2b^2 \) and \( \Box \) is because \( \sum_{k=0}^{N-1} \alpha_{k+1}^2 = \frac{N-1}{(96n^2 \rho_n L_2)^2} \sum_{k=2}^{N+1} k^2 \leq \frac{1}{(96n^2 \rho_n L_2)^2} \cdot \frac{(N+1)(N+2)(2N+3)}{6} \leq \frac{1}{(96n^2 \rho_n L_2)^2} \cdot \frac{(N+1)^2(N+2)(3N+1)}{6} = \frac{(N+1)^3}{(96n^2 \rho_n L_2)^2} \).

Dividing (42) by \( \frac{(N+1)^2}{192n^2 \rho_n L_2} \) and substituting \( \zeta_1, \zeta_2 \) from (36), we obtain
\[ \mathbb{E}[f(y_N)] - f(x^*) \leq \frac{384 \Theta_p n^2 \rho_n L_2}{(N+1)^2} + \frac{12 \sqrt{2} \Theta_p}{(N+1)^2} \zeta_1 + \frac{384(N+1) \zeta_2}{(96n^2 \rho_n L_2)^2} + \frac{12n^2 \rho_n L_2 (N+1)^2}{N^2} \]
\[ + \frac{6N}{L_2} \left( L_2^2 t^2 + \frac{16 \Delta^2}{t^2} \right) + \frac{N}{24n^2 \rho_n L_2} \left( L_2^2 t^2 + \frac{16 \Delta^2}{t^2} \right). \]

**Corollary 8** If \( N = O \left( \sqrt{\frac{n^2 \rho_n L_2 \Theta_p}{\varepsilon}} \right) \) then ARDFS with \( \alpha_{k+1} = \frac{k+2}{96n^2 \rho_n L_2} \) applied to solve problem (1) after \( N \) iterations produce such point \( y_N \) that \( \mathbb{E}[f(y_N)] - f(x^*) \leq \varepsilon \) if additionally
\[ m = O \left( \max \left\{ 1, \frac{\sigma^2}{\varepsilon^2} \cdot \sqrt{\frac{n^2 \rho_n \Theta_p}{L_2^2}} \right\} \right), \quad \Delta = O \left( \min \left\{ \frac{\varepsilon^2}{nL_2^2 \Theta_p}, \frac{\varepsilon^2}{\sqrt{n^2 \rho_n L_2^2 \Theta_p}} \right\} \right) \text{ and} \]
\[ t = O \left( \min \left\{ \frac{\varepsilon}{L_2 \sqrt{n^2 \rho_n \Theta_p}}, \frac{\varepsilon^3}{\sqrt{n^2 \rho_n L_2^2 \Theta_p}} \right\} \right), \]where smoothing parameter \( t \) is chosen as \( 2 \sqrt{\frac{\Delta}{L_2}} \) in order to minimize \( L_2^2 t^2 + \frac{16 \Delta^2}{t^2} \). In this case overall number of oracle calls is \( N m = O \left( \max \left\{ \sqrt{\frac{n^2 \rho_n L_2 \Theta_p}{\varepsilon}}, \frac{n^2 \rho_n \sigma^2 \Theta_p}{\varepsilon^2} \right\} \right). \)
Appendix F. Randomized Derivative-Free Directional Search

In this section we prove the convergence rate theorem for Randomized Derivative-Free Directional Search algorithm.

Proof of Theorem 3.

\[ \alpha n(\nabla^m f^t(x_k), x_k - x_s) = \alpha n(\nabla^m f^t(x_k), x_k - x_{k+1}) + \alpha n(\nabla^m f^t(x_k), x_{k+1} - x_s) \]

\[ \leq \alpha n(\nabla^m f^t(x_k), x_k - x_{k+1}) + \langle \nabla V[x_k](x_{k+1}), x_{k+1} - x_s \rangle \]

\[ \leq \alpha n(\nabla^m f^t(x_k), x_k - x_{k+1}) + V[x_k](x_s) - V[x_{k+1}](x_s) - V[x_k](x_{k+1}) \]

where \( \odot \) follows from \( x_{k+1} = \arg \min_{x \in \mathbb{R}^n} \{ V[x_k](x) + \alpha n(\nabla^m f^t(x_k), x) \} \), whence \( \langle \nabla V[x_k](x_{k+1}) + \alpha n(\nabla^m f^t(x_k), x - x_{k+1}) \rangle \geq 0 \) for all \( x \in \mathbb{R}^n \), \( \odot \) follows from triangle equality for Bregman divergence and \( \odot \) is due to \( V[x](y) \geq \frac{1}{2} \| x - y \|^2_\rho \). Taking conditional mathematical expectation \( \mathbb{E}_{e_{k+1}}[\cdot | \mathcal{E}_k] \) from both sides of (43) we get

\[ \alpha n \mathbb{E}_{e_{k+1}}[\langle \nabla^m f^t(x_k), x_k - x_s \rangle | \mathcal{E}_k] \leq \alpha n^2 \mathbb{E}_{e_{k+1}}[\| \nabla^m f^t(x_k) \|^2_\rho | \mathcal{E}_k] \]

\[ + V[x_k](x_s) - \mathbb{E}_{e_{k+1}}[V[x_{k+1}](x_s) | \mathcal{E}_k] \]

(44)

From (44), (18) and (20) for \( s = x_k - x_s \) we obtain

\[ \langle g^m(x_k, \xi^{(k+1)}_m), x_k - x_s \rangle \leq 24 \alpha^2 n^2 \rho_n \Lambda_2(f(x_k) - f(x_s)) \]

\[ + 12 \alpha^2 n^2 \rho_n \|
abla f(x_k) - g^m(x_k, \xi^{(k+1)}_m)\|^2_2 \]

\[ + \alpha^2 n^2 \rho_n \cdot \frac{\Delta}{2m} \sum_{i=1}^m L_2(\xi_{k+1,i})^2 + \frac{8 \alpha^2 n^2 \rho_n \Delta^2}{t^2} \]

\[ + \alpha \sqrt{m} \| x_k - x_s \|_p \cdot \frac{t}{2m} \sum_{i=1}^m L_2(\xi_{k+1,i}) + \frac{2 \alpha \sqrt{m} \| x_k - x_s \|_p}{t} \]

\[ + V[x_k](x_s) - \mathbb{E}_{e_{k+1}}[V[x_{k+1}](x_s) | \mathcal{E}_k] \]

Taking conditional mathematical expectation \( \mathbb{E}_{\xi_{k+1}}[\cdot | \mathcal{E}_k] = \mathbb{E}_{\xi_{k+1} \xi_{k+2} \cdots \xi_{k+m}}[\cdot | \xi_1, \xi_2, \ldots, \xi_{k,m}] \) from the both sides of previous inequality and using convexity of \( f \) and (2) we have

\[ \left( \alpha - 24 \alpha^2 n^2 \rho_n \Lambda_2 \right) (f(x_k) - f(x_s)) \leq 12 \alpha^2 n^2 \rho_n \frac{\Delta^2}{m} + \alpha^2 n^2 \rho_n \left( \frac{L_2 t^2}{2} + \frac{8 \Delta^2}{t^2} \right) \]

\[ + \alpha \sqrt{m} \| x_k - x_s \|_p \left( \frac{L_2 t}{2} + \frac{2 \Delta}{t} \right) \]

\[ + V[x_k](x_s) - \mathbb{E}_{e_{k+1}, \xi_{k+1}}[V[x_{k+1}](x_s) | \mathcal{E}_k, \Xi_k] \]

(45)

because \( \alpha = \frac{1}{48 n^2 \rho_n \Lambda_2} \). Denote

\[ \zeta_1 = \frac{L_2 t}{2} + \frac{2 \Delta}{t}, \quad \zeta_2 = \frac{L_2 t^2}{2} + \frac{8 \Delta^2}{t^2} \]

(46)

Note that

\[ \zeta_1^2 = \left( \frac{L_2 t}{2} + \frac{2 \Delta}{t} \right)^2 \leq 2 \cdot \frac{L_2 t^2}{4} + 2 \cdot \frac{4 \Delta^2}{t^2} = \zeta_2 \]

(47)
Taking mathematical expectation \(\mathbb{E}[\cdot] = \mathbb{E}_{x_1,\ldots,x_N,\xi_1,\ldots,\xi_N,\epsilon}[\cdot]\) from inequalities (45) for \(k = 0, \ldots, l - 1\), where \(l \leq N\), and summing them we get

\[
0 \leq \frac{N\alpha}{4} \left( \mathbb{E}[f(\bar{x})] - f(x^*) \right) \leq l \cdot 12\alpha^2 n \rho n \frac{\sigma^2}{m} + l \alpha^2 n^2 \rho \zeta_2 \\
+ \alpha \sqrt{n} \zeta_1 \sum_{k=0}^{l-1} \mathbb{E}[\|x_k - x^*\|_p] + V[x_0](x^*) - \mathbb{E}[V[x](x^*)],
\]

(48)

where \(\bar{x} \triangleq \frac{1}{l} \sum_{k=0}^{l-1} x_k\). From the previous inequality we get

\[
\frac{1}{2} \left( \mathbb{E}[\|x_l - x^*\|_p] \right)^2 \leq \frac{1}{2} \mathbb{E}[\|x_l - x^*\|^2_2] \leq \mathbb{E}[V[x](x^*)] \\
\leq \Theta_p + l \cdot 12\alpha^2 n \rho n \frac{\sigma^2}{m} + l \alpha^2 n^2 \rho \zeta_2 + \alpha \sqrt{n} \zeta_1 \sum_{k=0}^{l-1} \mathbb{E}[\|x_k - x^*\|_p],
\]

(49)

whence \(\forall l \leq N\) we obtain

\[
\mathbb{E}[\|x_l - x^*\|_p] \leq \sqrt{2} \sqrt{\Theta_p + l \cdot 12\alpha^2 n \rho n \frac{\sigma^2}{m} + l \alpha^2 n^2 \rho \zeta_2 + \alpha \sqrt{n} \zeta_1 \sum_{k=0}^{l-1} \mathbb{E}[\|x_k - x^*\|_p].}
\]

(50)

Denote \(R_k = \mathbb{E}[\|x^* - x_k\|_p]\) for \(k = 0, \ldots, N\). Applying Lemma 11 for \(a_0 = \Theta_p + \alpha \sqrt{n} \zeta_1 \mathbb{E}[\|x_0 - x^*\|_p] \leq \Theta_p + \alpha \sqrt{2n} \Theta_p \zeta_1, a_k = 12\alpha^2 n \rho n \frac{\sigma^2}{m} + \alpha^2 n^2 \rho \zeta_2, b = \sqrt{n} \zeta_1\) for \(k = 1, \ldots, N-1\) we have for \(l = N\)

\[
\begin{aligned}
0 \leq \left( \sqrt{\Theta_p + N \cdot 12\alpha^2 n \rho n \frac{\sigma^2}{m} + N \alpha^2 n^2 \rho \zeta_2 + \alpha \sqrt{2n} \Theta_p \zeta_1 + \sqrt{2n} \zeta_1 \alpha N} \right)^2 \\
\leq 2\Theta_p + 24N\alpha^2 n \rho n \frac{\sigma^2}{m} + 2N \alpha^2 n^2 \rho \zeta_2 + 2\alpha \sqrt{2n} \Theta_p \zeta_1 + 4n \zeta_1^2 \alpha^2 N^2,
\end{aligned}
\]

whence

\[
\mathbb{E}[f(\bar{x}_N)] - f(x^*) \leq \frac{384n \rho \sigma^2 L_2 \Theta_p}{N} + \frac{2\sigma^2 L_2}{m} + \frac{\alpha \sqrt{2n} \Theta_p \zeta_1}{N} + \frac{\zeta_2 N}{3L_2 \rho n} \\
+ \frac{8\sqrt{2n} \Theta_p \zeta_1}{N} + \frac{8\sqrt{2n} \Theta_p}{N} \left( \frac{L_2 \Delta^2}{2} + \frac{8\Delta^2}{t^2} \right) + \frac{8\sqrt{2n} \Theta_p}{N} \left( \frac{L_2 \Delta^2}{2} + \frac{8\Delta^2}{t} \right),
\]

(47)

(48)

because \(\alpha = \frac{1}{48n \rho \sigma^2 L_2}\).

**Corollary 9** If \(N = O\left(\frac{n \rho \sigma^2 L_2 \Theta_p}{\epsilon}\right)\) then RDFDS with \(\alpha = \frac{1}{48n \rho \sigma^2 L_2}\) applied to solve problem (1) after \(N\) iterations produce such point \(\bar{x}_N\) that \(\mathbb{E}[f(\bar{x}_N)] - f(x^*) \leq \epsilon\) if additionally \(m = O\left(\max \left\{1, \frac{\sigma^2}{L_2} \right\}\right)\), \(\Delta = O\left(\min \left\{\frac{\epsilon}{n}, \frac{\epsilon^2}{nL_2 \Theta_p}\right\}\right)\) and \(t = O\left(\min \left\{\sqrt{\frac{\epsilon}{nL_2 \Theta_p}}, \frac{\epsilon}{nL_2 \Theta_p}\right\}\right)\), where smoothing parameter \(t\) is chosen as \(2\sqrt{\frac{\Delta}{L_2}}\) in order to minimize \(\frac{L_2 \Delta^2}{2} + \frac{8\Delta^2}{t^2}\). In this case overall number of oracle calls is \(Nm = O\left(\max \left\{\frac{n \rho \sigma^2 L_2 \Theta_p}{\epsilon}, \frac{n \rho \sigma^2 L_2 \Theta_p}{\epsilon^2}\right\}\right)\).

We summarize the appropriate parameters of the algorithm in Table 4.
Here we prove that, for \( e \in RS_2(1) \)

\[
E[\|e\|_q^2] \leq \min\{q - 1, 16\ln n - 8\} n^{\frac{q}{2}-1},
\]

\[
E[\langle s, e \rangle^2 | e|_q^2] \leq 6\|s\|_2^2 \min\{q - 1, 16\ln n - 8\} n^{\frac{q}{2}-2}.
\]

We start with proving the following inequality, which could be rough for big \( q \):

\[
E[|e|_q^2] \leq (q - 1) n^{\frac{q}{2}-1}, \quad 2 \leq q < \infty.
\]

We have

\[
E[|e|_q^2] = E \left[ \left( \sum_{k=1}^n |e_k|^q \right)^{\frac{2}{q}} \right] \overset{\text{1}}{=\!\!\!\!\!\!\!}\left( E \left[ \sum_{k=1}^n |e_k|^q \right] \right)^{\frac{2}{q}} \overset{\text{2}}{=\!\!\!\!\!\!\!}\left( n E[|e_2|^q] \right)^{\frac{2}{q}},
\]

where \( \text{1} \) is due to probabilistic version of Jensen’s inequality (function \( \varphi(x) = x^{\frac{2}{q}} \) is concave, because \( q \geq 2 \)) and \( \text{2} \) is because mathematical expectation is linear and components of vector \( e \) are identically distributed.

Moreover, due to Poincare lemma, we have

\[
e = \frac{d}{\sqrt{\xi_1^2 + \cdots + \xi_n^2}},
\]

where \( \xi \) is Gaussian random vector which mathematical expectation is zero vector and covariance matrix is identical. Then

\[
E[|e_2|^q] = E \left[ \left( \frac{|\xi_2|^q}{\xi_1^2 + \cdots + \xi_n^2} \right)^{\frac{2}{q}} \right] = \int_{\mathbb{R}^n} |x_2|^q \left( \frac{n}{\xi_1^2 + \cdots + \xi_n^2} \right)^{\frac{q}{2}} \cdot \frac{1}{(2\pi)^{\frac{n}{2}}} \cdot \exp \left( -\frac{1}{2} \sum_{k=1}^n x_k^2 \right) dx_1 \ldots dx_n.
\]

Table 4: Algorithm 2 parameters for the cases \( p = 1 \) and \( p = 2 \).

| \( N \) | \( p = 1 \) | \( p = 2 \) |
|---|---|---|
| \( m \) | \( O \left( \max \left\{ \frac{1}{n^2}, \frac{n^2}{nL_2^2\Theta_1} \right\} \right) \) | \( O \left( \max \left\{ 1, \frac{n^2}{nL_2^2\Theta_2} \right\} \right) \) |
| \( \Delta \) | \( O \left( \min \left\{ \frac{\varepsilon}{n^2}, \frac{\varepsilon^2}{nL_2^{2\Theta_1}} \right\} \right) \) | \( O \left( \min \left\{ \frac{\varepsilon}{n^2}, \frac{\varepsilon^2}{nL_2^{2\Theta_2}} \right\} \right) \) |
| \( t \) | \( O \left( \min \left\{ \sqrt{\frac{\varepsilon}{nL_2^2}}, \frac{\varepsilon}{\sqrt{nL_2^{2\Theta_1}}} \right\} \right) \) | \( O \left( \min \left\{ \sqrt{\frac{\varepsilon}{nL_2^2}}, \frac{\varepsilon}{\sqrt{nL_2^{2\Theta_2}}} \right\} \right) \) |

Appendix G. Proof of Lemma 1

Here we prove that, for \( e \in RS_2(1) \)

We have

\[
E[|e_2|^q] = E \left[ \left( \frac{|\xi_2|^q}{\xi_1^2 + \cdots + \xi_n^2} \right)^{\frac{2}{q}} \right] = \int_{\mathbb{R}^n} |x_2|^q \left( \frac{n}{\xi_1^2 + \cdots + \xi_n^2} \right)^{\frac{q}{2}} \cdot \frac{1}{(2\pi)^{\frac{n}{2}}} \cdot \exp \left( -\frac{1}{2} \sum_{k=1}^n x_k^2 \right) dx_1 \ldots dx_n.
\]
Consider spherical coordinates:

\[
x_1 = r \cos \varphi \sin \theta_1 \ldots \sin \theta_{n-2},
\]

\[
x_2 = r \sin \varphi \sin \theta_1 \ldots \sin \theta_{n-2},
\]

\[
x_3 = r \cos \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2},
\]

\[
x_4 = r \cos \theta_2 \sin \theta_3 \ldots \sin \theta_{n-2},
\]

\[\ldots\]

\[
x_n = r \cos \theta_{n-2},
\]

\(r > 0, \ \varphi \in [0, 2\pi), \ \theta_i \in [0, \pi], \ i = 1, n-2.\)

The Jacobian of mapping is

\[
\det \left( \frac{\partial (x_1, \ldots, x_n)}{\partial (r, \varphi, \theta_1, \ldots, \theta_{n-2})} \right) = r^{n-1} \sin \theta_1 (\sin \theta_2)^2 \ldots (\sin \theta_{n-2})^{n-2}.
\]

Then mathematical expectation \(\mathbb{E}[|e_2|^q]\) could be rewritten in the following form:

\[
\mathbb{E}[|e_2|^q] = \int_{r>0, \varphi \in (0, 2\pi)} \int_{\theta \in [0, \pi], i = 1, n-2} r^{n-1} \sin \varphi |q| \sin \theta_1 |q+1| \sin \theta_2 |q+2| \ldots | \sin \theta_{n-2} |q+n-2| e^{-\frac{r^2}{2}} \sin \varphi \, d\varphi \, dr \, d\theta_{n-2}
\]

where

\[
I_r = \int_0^{+\infty} r^{n-1} e^{-\frac{r^2}{2}} \, dr,
\]

\[
I_{\varphi} = \int_0^{2\pi} |\sin \varphi|^q \, d\varphi = 2 \int_0^{\pi} |\sin \varphi|^q \, d\varphi,
\]

\[
I_{\theta_i} = \int_0^{\pi} |\sin \theta_i|^q \, d\theta_i, \ i = 1, n-2.
\]

Now we are going to compute these integrals. Start with \(I_r:\)

\[
I_r = \int_0^{+\infty} r^{n-1} e^{-\frac{r^2}{2}} \, dr = /r = \sqrt{2t} / = \int_0^{+\infty} (2t)^{\frac{n-1}{2}} e^{-t} \, dt = 2^{\frac{n-1}{2}} t^{-\frac{1}{2}} \Gamma\left(\frac{n}{2}\right).
\]

To compute other integrals it is useful to consider the following integral \((\alpha > 0):\)

\[
\int_0^{\pi} |\sin \varphi|^\alpha \, d\varphi = 2 \int_0^{\pi} |\sin \varphi|^\alpha \, d\varphi = 2 \int_0^{\pi} (\sin^2 \varphi)^{\frac{\alpha}{2}} \, d\varphi = /t = \sin^2 \varphi /\]

\[
= \int_0^t \left( \frac{\alpha+1}{2} \right) - \frac{1}{2} \right) = B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{\alpha+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2} + \frac{1}{2}\right)} = \sqrt{\pi} \frac{\Gamma\left(\frac{\alpha+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2} + \frac{1}{2}\right)}.
\]

From this we obtain

\[
\mathbb{E}[|e_2|^q] = \frac{1}{(2\pi)^{\frac{n}{2}}} I_r \cdot I_{\varphi} \cdot I_{\theta_1} \cdot I_{\theta_2} \ldots \cdot I_{\theta_{n-2}}
\]

\[
= \frac{1}{(2\pi)^{\frac{n}{2}}} \cdot 2^{\frac{n-1}{2}} t^{-\frac{1}{2}} \cdot \frac{\Gamma\left(\frac{\alpha+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2} + \frac{1}{2}\right)} \cdot \sqrt{\pi} \frac{\Gamma\left(\frac{\alpha+2}{2}\right)}{\Gamma\left(\frac{\alpha+3}{2}\right)} \cdot \sqrt{\pi} \frac{\Gamma\left(\frac{\alpha+3}{2}\right)}{\Gamma\left(\frac{\alpha+4}{2}\right)} \ldots \cdot \sqrt{\pi} \frac{\Gamma\left(\frac{\alpha+n-1}{2}\right)}{\Gamma\left(\frac{\alpha+n}{2}\right)}
\]

\(= \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha+n}{2}\right)} \).

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Now, we want to show that \( \forall q \geq 2 \)
\[
\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{q+1}{2} \right)}{\Gamma \left( \frac{q+n}{2} \right)} \leq \left( \frac{q-1}{n} \right)^{\frac{q}{2}}. \tag{57}
\]

At the beginning show that (57) holds for \( q = 2 \) (and arbitrary \( n \)):
\[
\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{3}{2} \right)}{\Gamma \left( \frac{q+n}{2} \right)} - \frac{1}{n} = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma \left( \frac{n}{2} \right) \cdot \frac{1}{2} \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{q}{2} \right)} - \frac{1}{n} = \frac{1}{n} - \frac{1}{n} = 0 \leq 0.
\]

Consider the function
\[
f_n(q) = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{q+1}{2} \right)}{\Gamma \left( \frac{q+n}{2} \right)} - \left( \frac{q-1}{n} \right)^{\frac{q}{2}}
\]
where \( q \geq 2 \). Also consider \( \psi(x) = \frac{d}{dx} \ln(\Gamma(x)) \) with \( x > 0 \) which is called (digamma function). For gamma function it holds
\[
\Gamma(x+1) = x\Gamma(x), \quad x > 0.
\]

Taking natural logarithm from it and taking derivative w.r.t. \( x \):
\[
\frac{\ln \Gamma(x+1)}{dx} = \ln \Gamma(x) + \ln x, \quad \frac{d}{dx} \ln(\Gamma(x+1)) = \frac{d}{dx} \ln(\Gamma(x)) + \frac{1}{x},
\]
which could be written in digamma-function-notation:
\[
\psi(x+1) = \psi(x) + \frac{1}{x}. \tag{58}
\]

One can show that digamma function is monotonically increases when \( x > 0 \). To prove this fact we are going to show that
\[
\left( \frac{\Gamma'(x)}{\Gamma(x)} \right)^2 < \Gamma(x) \Gamma''(x). \tag{59}
\]
That is,
\[
\left( \frac{\Gamma'(x)}{\Gamma(x)} \right)^2 = \left( \int_0^{+\infty} e^{-t} \ln t \cdot t^{x-1} dt \right)^2 < \int_0^{+\infty} \left( e^{-\frac{t}{2}} t^{x-1} \right)^2 dt \cdot \int_0^{+\infty} \left( e^{-\frac{t}{2}} t^{x-1} \ln t \right)^2 dt
\]
\[
= \int_0^{+\infty} e^{-t} t^{x-1} dt \cdot \int_0^{+\infty} e^{t} t^{x-1} \ln^2 t dt,
\]
where \( \circ \) follows from Cauchy-Schwartz inequality (the equality cannot occur because functions \( e^{-\frac{t}{2}} t^{x-1} \) and \( e^{-\frac{t}{2}} t^{x-1} \ln t \) are linearly independent). From (59) follows that
\[
\frac{d^2}{dx^2} \ln(\Gamma(x)) = \left( \frac{\Gamma'(x)}{\Gamma(x)} \right)' = \frac{\Gamma''(x)}{\Gamma(x)} - \frac{\left( \frac{\Gamma'(x)}{\Gamma(x)} \right)^2}{(\Gamma(x))^2} > 0,
\]
which shows that digamma function increases.
Now we show that \( f_n(q) \) decreases on the interval \([2, +\infty)\). To obtain it is sufficient to consider \( \ln(f(q)) \):

\[
\ln(f_n(q)) = \ln\left(\frac{\Gamma\left(\frac{q+n}{2}\right)}{\sqrt{n} \sqrt{\pi}}\right) + \ln\left(\Gamma\left(\frac{q+1}{2}\right)\right) - \ln\left(\Gamma\left(\frac{2n}{2}\right)\right) - \frac{q}{2} (\ln(q) - \ln n),
\]

\[
\frac{d(\ln(f_n(q)))}{dq} = \frac{1}{2} \psi\left(\frac{q+1}{2}\right) - \frac{1}{2} \psi\left(\frac{q+n}{2}\right) - \frac{1}{2} \ln(q - 1) - \frac{q}{2(q-1)} + \frac{1}{2} \ln n.
\]

We are going to show that \( \frac{d(\ln(f_n(q)))}{dq} < 0 \) for \( q \geq 2 \). Let \( k = \lfloor \frac{q}{2} \rfloor \) (the closest integer which is no greater than \( \frac{q}{2} \)). Then \( \psi\left(\frac{q+n}{2}\right) > \psi\left(k - 1 + \frac{q+1}{2}\right) \) and \( \ln n \leq \ln(2k+1) \), whence

\[
\frac{d(\ln(f_n(q)))}{dq} < \frac{1}{2} \left( \psi\left(\frac{q+1}{2}\right) - \psi\left(k - 1 + \frac{q+1}{2}\right) \right) - \frac{1}{2} \ln(q - 1) - \frac{q}{2(q-1)} + \frac{1}{2} \ln(2k+1)
\]

\[
= \frac{1}{2} \left( \psi\left(\frac{q+1}{2}\right) - \sum_{i=1}^{k-1} \frac{1}{i + \frac{k-1}{2}} - \psi\left(\frac{q+1}{2}\right) \right) - \frac{q}{2(q-1)} + \frac{1}{2} \ln(2k+1)
\]

\[
\leq - \frac{1}{2} \sum_{i=1}^{k-1} \frac{q+1}{2} - \frac{2}{q-1} + \frac{1}{2} \ln\left(\frac{2k+1}{q-1}\right)
\]

\[
= - \frac{1}{2} \left( \frac{2}{q-1} + \frac{2}{q+1} + \frac{2}{q+3} + \ldots + \frac{2}{q+2k-3} \right) + \frac{1}{2} \ln\left(\frac{2k+1}{q-1}\right)
\]

\[
< - \frac{1}{2} \ln\left(\frac{q+2k-1}{q-1}\right) + \frac{1}{2} \ln\left(\frac{2k+1}{q-1}\right) \leq - \frac{1}{2} \ln\left(\frac{2k+1}{q-1}\right) + \frac{1}{2} \ln\left(\frac{2k+1}{q-1}\right)
\]

\[
= 0,
\]

where \( \oplus \) and \( \ominus \) is because \( q \geq 2 \), \( \ominus \) is due to estimation of integral of \( \frac{1}{x^2} \) by integral of \( g(x) = \frac{1}{x - 1/2} \), \( x \in [q - 1 + 2i, q - 1 + 2i + 2], i = 0, 2k - 1 \) which is no less than \( f(x) \):

\[
\frac{2}{q-1} + \frac{2}{q+1} + \frac{2}{q+3} + \ldots + \frac{2}{q+2k-3} > \int_{q-1}^{1} \frac{1}{x} dx = \ln\left(\frac{q+2k-1}{q-1}\right).
\]

So, we shown that \( \frac{d(\ln(f_n(q)))}{dq} < 0 \) for \( q \geq 2 \) arbitrary natural number \( n \). Therefore for any fixed number \( n \) the function \( f_n(q) \) decreases as \( q \) increase, which means that \( f_n(q) \leq f_n(2) = 0 \), i.e., (57) holds. From this and (54),(56) we obtain that \( \forall \ q \geq 2 \)

\[
E[|e|^{2\frac{1}{2}}]^{(54)} \leq (nE[|e_2|^{q}]^{\frac{2}{q}})^{\frac{2}{q}} \leq (q - 1)^{\frac{2}{q}} - 1.
\]

However, inequality (60) is useless when \( q \) is big (with respect to \( n \)). Consider left hand side of (60) as function of \( q \) and find its minimum for \( q \geq 2 \). Consider \( h_n(q) = \ln(q - 1) + \left(\frac{2}{q - 1}\right) \ln n \) (it is logarithm of the right hand side of (60)). Derivative of \( h(q) \) is

\[
\frac{dh(q)}{dq} = \frac{1}{q-1} - \frac{2 \ln n}{q^2},
\]

\[
q^2 - 2q \ln n + 2 \ln n = 0.
\]

If \( n \geq 8 \), then the point where the function obtains its minimum on the set \([2, +\infty)\) is \( q_0 = \ln n \left(1 + \sqrt{1 - \frac{2}{\ln n}}\right)\) (for the case \( n \leq 7 \) it turns out that \( q_0 = 2 \); further without loss of generality we assume \( n \geq 8 \). Therefore
for all $q > q_0$ it is more useful to use the following estimation:

\[
E[||e||_q^4] \overset{\text{I}}{\leq} E[||e||_{q_0}^4] \overset{(60)}{\leq} (q_0 - 1)n^{\frac{2}{q_0} - 1} \overset{\text{II}}{\leq} (2\ln n - 1)n^{\frac{2}{n} - 1} \\
= (2\ln n - 1)\frac{2}{n} \leq (16\ln n - 8)\frac{1}{n} \\
\leq (16\ln n - 8)n^{\frac{2}{n} - 1},
\]

where $\text{I}$ is due to $||e||_q < ||e||_{q_0}$ for $q > q_0$, $\text{II}$ follows from $q_0 \leq 2\ln n$, $q_0 \geq \ln n$. Putting estimations (60) and (61) together we obtain (51).

Now we are going to prove (52). Firstly, we want to estimate $\sqrt{E[||e||_q^4]}$. Due to probabilistic Jensen’s inequality ($q \geq 2$)

\[
E[||e||_q^4] = E \left[ \left( \sum_{k=1}^{n} |e_k|^q \right)^2 \right] \leq \left( E \left[ \left( \sum_{k=1}^{n} |e_k|^q \right)^2 \right] \right)^{\frac{2}{q}} \\
\overset{\text{I}}{\leq} \left( E \left[ \left( \sum_{k=1}^{n} |e_k|^{2q} \right) \right] \right)^{\frac{2}{q}} \overset{\text{II}}{=} n^2 E[|e_1|^{2q}] \overset{(56),(57)}{=} n^\frac{4}{q} \left( \frac{2q-1}{n} \right)^{\frac{2}{q}} = (2q - 1)^{n} \frac{2}{n^{\frac{2}{q} - 2}},
\]

where $\text{I}$ is because $\left( \sum_{k=1}^{n} x_k \right)^2 \leq n \sum_{k=1}^{n} x_k^2$ for $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and $\text{II}$ follows from that mathematical expectation is linear and components of the random vector $e$ are identically distributed. From this we obtain

\[
\sqrt{E[||e||_q^4]} \leq (2q - 1)n^{\frac{2}{n} - 1}.
\]

Consider the right hand side of the inequality (62) as a function of $q$ and find its minimum for $q \geq 2$. Consider $h_n(q) = \ln(2q - 1) + \left( \frac{2}{q} - 1 \right) \ln n$ (logarithm of the right hand side (62)). Derivative of $h(q)$ is

\[
\frac{dh(q)}{dq} = \frac{2q - 1}{2q - 2} - \frac{2 \ln n}{q^2},
\]

$$q^2 - 2q \ln n + n \ln n = 0.$$

If $n \geq 3$, the the point where the function obtains its minimum on the set $[2, +\infty)$ is $q_0 = \ln n \left( 1 + \sqrt{1 - \frac{1}{\ln n}} \right)$ (for the case $n \leq 2$ it turns out that $q_0 = 2$; further without loss of generality we assume that $n \geq 3$). Therefore for all $q > q_0$:

\[
\sqrt{E[||e||_q^4]} \overset{\text{I}}{\leq} \sqrt{E[||e||_{q_0}^4]} \overset{(62)}{\leq} (2q_0 - 1)n^{\frac{2}{q_0} - 1} \overset{\text{II}}{\leq} (4 \ln n - 1)n^{\frac{2}{n} - 1} \\
= (4 \ln n - 1)\frac{2}{n} \leq (32 \ln n - 8)\frac{1}{n} \\
\leq (32 \ln n - 8)n^{\frac{2}{n} - 1},
\]

where $\text{I}$ is due to $||e||_q < ||e||_{q_0}$ for $q > q_0$, $\text{II}$ follows from $q_0 \leq 2\ln n$, $q_0 \geq \ln n$. Putting estimations (62) and (63) together we get inequality

\[
\sqrt{E[||e||_q^4]} \leq \min\{2q - 1, 32 \ln n - 8\}n^{\frac{2}{n} - 1}.
\]
Now we are going to find $\mathbb{E}[(s, e)^4]$, where $s \in \mathbb{R}^n$ is some vector. Let $S_n(r)$ be a surface area of $n$-dimensional Euclidean sphere with radius $r$ and $d\sigma(e)$ be unnormalized uniform measure on $n$-dimensional Euclidean sphere. From this it follows that $S_n(r) = S_n(1)r^{n-1}$, $S_{n-1}(1) = \frac{n-1}{n\sqrt{\pi}} \Gamma(\frac{n+1}{2})$. Besides, let $\varphi$ be the angle between $s$ and $e$. Then

$$
\mathbb{E}[(s, e)^4] = \frac{1}{S_n(1)} \int_S (s, e)^4 d\sigma(\varphi) = \frac{1}{S_n(1)} \int_0^\pi ||s||^2 \frac{1}{2} \cos \frac{1}{2} \varphi S_{n-1}(\sin \varphi) d\varphi
$$

$$
= ||s||^2 \frac{1}{2} \frac{S_{n-1}(1)}{S_n(1)} \int_0^\pi \cos^4 \varphi \sin^{n-2} \varphi d\varphi
$$

$$
= ||s||^2 \frac{1}{2} \cdot \frac{n-1}{n\sqrt{\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})} \frac{\pi}{2} \cos^4 \varphi \sin^{n-2} \varphi d\varphi. \tag{65}
$$

Compute the integral:

$$
\int_0^\pi \cos^4 \varphi \sin^{n-2} \varphi d\varphi = 2 \int_0^{\frac{\pi}{2}} \cos^4 \varphi \sin^{n-2} \varphi d\varphi = \int_0^{\frac{\pi}{2}} t^{n-3} (1 - t)^{\frac{3}{2}} dt = B(\frac{n-1}{2}, \frac{5}{2})
$$

$$
= \frac{\Gamma(\frac{n}{2})\Gamma(\frac{5-n}{2})}{\Gamma(\frac{n+1}{2})} = \frac{\frac{3}{2}\Gamma(\frac{n}{2})\Gamma(\frac{5-n}{2})}{\Gamma(\frac{n+1}{2})} = \frac{3}{n+2} \cdot \sqrt{\pi} \Gamma(\frac{n+1}{2})
$$

From this and (65) we obtain

$$
\mathbb{E}[(s, e)^4] = ||s||^2 \cdot \frac{n-1}{n\sqrt{\pi}} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})} \cdot \frac{3}{n+2} \cdot \sqrt{\pi} \Gamma(\frac{n+1}{2})
$$

$$
= ||s||^2 \cdot \frac{3(n-1)}{2n(n+2)} \cdot \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})} = \frac{3||s||^4}{n(n+2)}. \tag{66}
$$

To prove (52), it remains to use (64), (66) and Cauchy-Schwartz inequality ($\mathbb{E}[XY]^2 \leq \mathbb{E}[X^2] \cdot \mathbb{E}[Y^2]$):

$$
\mathbb{E}[(s, e)^2||e||^2_2] \leq \sqrt{\mathbb{E}[(s, e)^4]} \cdot \mathbb{E}[||e||^4_2] \leq \sqrt{3} ||s||^2 \min\{2q - 1, 32 \ln n - 8\} n^{2q-2}.
$$

Appendix H. Technical Results

Lemma 10  Let $a_0, \ldots, a_{N-1}, b, R_1, \ldots, R_{N-1}$ be non-negative numbers such that

$$
R_l \leq \sqrt{2} \cdot \sqrt{\left( \sum_{k=0}^{l-1} a_k + \frac{1}{2} \sum_{k=1}^{l-1} \alpha_{k+1} R_k \right)^2} \quad l = 1, \ldots, N, \tag{67}
$$

where $\alpha_{k+1} = \frac{k+2}{96n^2 \rho_n L_2}$ for all $k \in \mathbb{N}$. Then for $l = 1, \ldots, N$

$$
\sum_{k=0}^{l-1} a_k + \frac{1}{2} \sum_{k=1}^{l-1} \alpha_{k+1} R_k \leq \left( \sum_{k=0}^{l-1} a_k + \sqrt{2} b \cdot \frac{l^2}{96n^2 \rho_n L_2} \right)^2. \tag{68}
$$
**Proof.** For \( l = 1 \) it is trivial inequality. Assume that (68) holds for some \( l < N \) and prove it for \( l + 1 \). From the induction assumption and (67) we obtain

\[
R_l \leq \sqrt{2} \left( \sqrt{\sum_{k=0}^{l-1} a_k} + \sqrt{2} b \cdot \frac{l^2}{96n^2 \rho_L^2} \right),
\]

whence

\[
\sum_{k=0}^{l} a_k + b \sum_{k=1}^{l} \alpha_k R_k = \sum_{k=0}^{l-1} a_k + b \sum_{k=1}^{l-1} a_{k+1} R_k + a_l + b \alpha_{l+1} R_l
\]

\[
\leq \left( \sqrt{\sum_{k=0}^{l-1} a_k} + \sqrt{2} b \cdot \frac{l^2}{96n^2 \rho_L^2} \right)^2 + a_l
\]

\[+ \sqrt{2} b \alpha_{l+1} \left( \sqrt{\sum_{k=0}^{l-1} a_k} + \sqrt{2} b \cdot \frac{l^2}{96n^2 \rho_L^2} \right)
\]

\[= \sum_{k=0}^{l} a_k + 2 \sqrt{\sum_{k=0}^{l-1} a_k} \cdot \sqrt{2} b \left( \frac{l^2}{96n^2 \rho_L^2} \right) + b^2 \left( \frac{l^4}{(96n^2 \rho_L^2)^2} + \alpha_{l+1} \cdot \frac{l^2}{96n^2 \rho_L^2} \right)
\]

\[\leq \sum_{k=0}^{l} a_k + 2 \sqrt{\sum_{k=0}^{l-1} a_k} \cdot \sqrt{2} b \left( \frac{(l+1)^2}{96n^2 \rho_L^2} \right) + 2b^2 \left( \frac{(l+1)^4}{(96n^2 \rho_L^2)^2} \right)
\]

\[= \left( \sqrt{\sum_{k=0}^{l} a_k} + \sqrt{2} b \cdot \frac{(l+1)^2}{96n^2 \rho_L^2} \right)^2,
\]

where \( \circ \) follows from the induction assumption and (69), \( \circ \) is because \( \sum_{k=0}^{l-1} a_k \leq \sum_{k=0}^{l} a_k \) and

\[
\frac{l^2}{96n^2 \rho_L^2} + \frac{(l+1)^2}{96n^2 \rho_L^2} \leq \frac{(l+1)^2}{96n^2 \rho_L^2},
\]

\[
\frac{2l^2}{192n^2 \rho_L^2} + \frac{(l+1)^2}{96n^2 \rho_L^2} \leq \frac{l^4 + (l+2)^2}{(96n^2 \rho_L^2)^2} \leq \frac{(l+1)^4}{(96n^2 \rho_L^2)^2}.
\]

**Lemma 11** Let \( a_0, \ldots, a_{N-1}, b, R_1, \ldots, R_{N-1} \) be non-negative numbers such that

\[
R_l \leq \sqrt{2} \cdot \sqrt{\sum_{k=0}^{l-1} a_k + b \alpha \sum_{k=1}^{l-1} R_k} \quad l = 1, \ldots, N.
\]

Then for \( l = 1, \ldots, N \)

\[
\sum_{k=0}^{l-1} a_k + b \alpha \sum_{k=1}^{l-1} R_k \leq \left( \sqrt{\sum_{k=0}^{l-1} a_k + \sqrt{2} b \alpha l} \right)^2.
\]
**Proof.** For \( l = 1 \) it is trivial inequality. Assume that (71) holds for some \( l < N \) and prove it for \( l + 1 \). From the induction assumption and (70) we obtain

\[
R_l \leq \sqrt{2} \left( \sqrt{\sum_{k=0}^{l-1} a_k + \sqrt{2}b\alpha l} \right),
\]  

(72)

whence

\[
\sum_{k=0}^{l} a_k + b\alpha \sum_{k=1}^{l} R_k = \sum_{k=0}^{l-1} a_k + b\alpha \sum_{k=1}^{l-1} R_k + a_l + b\alpha R_l
\]

\[
\begin{aligned}
&\overset{\circ}{=} \left( \sqrt{\sum_{k=0}^{l-1} a_k + \sqrt{2}b\alpha l} \right)^2 + a_l + \sqrt{2}b\alpha \left( \sqrt{\sum_{k=0}^{l-1} a_k + \sqrt{2}b\alpha l} \right) \\
&= \sum_{k=0}^{l} a_k + 2\sqrt{\sum_{k=0}^{l-1} a_k \cdot \sqrt{2}b\alpha l + 2b^2\alpha^2 l^2} + \sqrt{2}b\alpha \left( \sqrt{\sum_{k=0}^{l-1} a_k + \sqrt{2}b\alpha l} \right) \\
&= \sum_{k=0}^{l} a_k + 2\sqrt{\sum_{k=0}^{l-1} a_k \cdot \sqrt{2}b\alpha \left( l + \frac{1}{2} \right) + 2b^2\alpha^2 \left( l^2 + l \right)} \\
&\overset{\circ}{=} \sum_{k=0}^{l} a_k + 2\sqrt{\sum_{k=0}^{l} a_k \cdot \sqrt{2}b\alpha (l + 1) + 2b^2\alpha^2 (l + 1)^2} \\
&= \left( \sqrt{\sum_{k=0}^{l} a_k + \sqrt{2}b\alpha (l + 1)} \right)^2,
\end{aligned}
\]

where \( \overset{\circ}{=} \) follows from the induction assumption and (72), \( \overset{\circ}{=} \) is because \( \sum_{k=0}^{l-1} a_k \leq \sum_{k=0}^{l} a_k \).