SOME RESULTS ON DEGENERATE HARMONIC NUMBERS AND DEGENERATE FUBINI POLYNOMIALS

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ABSTRACT. In recent years, some degenerate versions of quite a few special numbers and polynomials are introduced and investigated by means of various methods. The aim of this paper is to study some results on degenerate harmonic numbers, degenerate hyperharmonic numbers, degenerate Fubini polynomials and degenerate $r$-Fubini polynomials from a general identity which is valid for any two formal power series and involves the degenerate $r$-Stirling numbers of the second kind.

1. INTRODUCTION

It was Carlitz who initiated the study of degenerate versions of some special numbers and polynomials, namely the degenerate Bernoulli and degenerate Euler numbers (see [4]). This pioneering work regained interests of some mathematicians, various versions of many special numbers and polynomials were investigated and some interesting arithmetical and combinatorial results were found along with some of their applications (see [9-11,13-22] and the references therein).

The aim of this paper is to study some results on degenerate harmonic numbers, degenerate hyperharmonic numbers, degenerate Fubini polynomials and degenerate $r$-Fubini polynomials from a general identity which is valid for any two formal power series and involves the degenerate $r$-Stirling numbers of the second kind.

The outline of this paper is as follows. In Section 1, we recall the degenerate exponentials, their compositional inverses called the degenerate logarithms, the degenerate Stirling numbers of the first kind and the degenerate Stirling numbers of the second kind. We mention the degenerate $r$-Stirling numbers of the first kind, the degenerate Stirling $r$-Stirling numbers of the second kind and the degenerate unsigned $r$-Stirling numbers of the first kind as further generalizations of aforementioned numbers. We recall the reader of the degenerate Bell polynomials, the degenerate $r$-Bell polynomials, the degenerate harmonic numbers, the degenerate hyperharmonic numbers and the degenerate Fubini polynomials. Section 2 is the main results of this paper. In Theorem 1, we derive an identity of operators involving the degenerate $r$-Stirling numbers of the second kind. A general identity valid for any two formal power series is derived in Theorem 2 by using Theorem 1, where again the degenerate $r$-Stirling numbers of the second kind appear. Applying the result in Theorem 2 with $f(x) = x^m$ and $g(x) = \frac{1}{1-x}$, we get an identity on the degenerate $r$-Fubini polynomials which are slight generalization of the degenerate Fubini polynomials (see (35)). In Theorem 4, we deduce a differential equation satisfied by the degenerate Fubini polynomials with the help of a recurrence relation for the degenerate Stirling numbers of the second kind. In Theorem 5, the degenerate hyperharmonic numbers are expressed in terms of the degenerate unsigned $r$-Stirling numbers of the first kind. Theorem 6 shows that the $k$-th derivative of a certain formal power series is equal to an expression involving the degenerate harmonic number $H_{k,\lambda}$ and the degenerate logarithm whose value at 0 is simply $k!H_{k,\lambda}$. In Corollary 7, we derive a relation between the degenerate hyperharmonic numbers and the degenerate harmonic numbers from Theorem 6 and some other previous

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As the inversion formula of (6), the degenerate Stirling numbers of the second kind are defined by
\[ (x)_{n,\lambda} = \sum_{k=0}^{n} S_2(n,k)(x)_k, \quad (n \geq 0), \quad (n \geq 1). \]

Recently, the degenerate Stirling numbers of the first kind are defined by Kim-Kim as
\[ (x)_n = \sum_{k=0}^{n} S_1(n,k)(x)_k, \quad (n \geq 0), \quad (n \geq 1). \]

As the inversion formula of (6), the degenerate Stirling numbers of the second kind are defined by
\[ (x)_{n,\lambda} = \sum_{k=0}^{n} S_1(n,k)(x)_{k,\lambda}, \quad (n \geq 0), \quad (n \geq 9). \]

From (6) and (7), we note that
\[ \frac{1}{k!} \left( \log_\lambda (1 + t) \right)^k = \sum_{n=k}^{\infty} S_1(n,k) \frac{t^n}{n!}, \quad \text{and} \]
\[ \frac{1}{k!} \left( e_\lambda (t) - 1 \right)^k = \sum_{n=k}^{\infty} S_2(n,k) \frac{t^n}{n!}, \quad \text{(see [9, 15]).} \]
For $r \geq 0$, the degenerate $r$-Stirling numbers of the first kind are defined by

$$
(x+r)_n = \sum_{k=0}^{n} S_{1,\lambda}^{(r)}(n+r,k+r)(x)_{k,\lambda}, \quad (n \geq 0), \quad \text{(see [18, 20, 21])}.
$$

In view of (5), the degenerate $r$-Stirling numbers of the second kind are defined by

$$
(x+r)_{n,\lambda} = \sum_{k=0}^{n} \left\{ \frac{n+r}{k+r} \right\}_{r,\lambda} (x)_{k}, \quad \text{(see [18, 21])}.
$$

The degenerate unsigned $r$-Stirling numbers of the first kind are given by

$$
\langle x \rangle_n = x(x+1) \cdots (x+n-1), \quad (n \geq 1), \quad \langle x \rangle_0 = 1,
$$

$$
\langle x \rangle_{n,\lambda} = x(x+\lambda) \cdots (x+(n-1)\lambda), \quad (n \geq 1), \quad \langle x \rangle_{0,\lambda} = 1.
$$

From (9), (10) and (11), we note that

$$
\frac{1}{k!} (1+t)^k \left( \log_\lambda (1+t) \right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}^{(r)}(n+r,k+r) \frac{t^n}{n!},
$$

$$
\frac{1}{k!} \left( e_\lambda(t) - 1 \right)^k e_\lambda^r(t) = \sum_{n=k}^{\infty} \left\{ \frac{n+r}{k+r} \right\}_{r,\lambda} \frac{t^n}{n!},
$$

$$
\frac{1}{k!} \left( - \log_\lambda (1-t) \right)^k (1-t)^r = \sum_{n=k}^{\infty} \left\{ \frac{n+r}{k+r} \right\}_{r,\lambda} \frac{t^n}{n!},
$$

(see [18, 20]).

The degenerate Bell polynomials are defined by

$$
e_k^{(x_1 \cdots x_k)} = \sum_{n=k}^{\infty} \phi_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{(see [15, 16])}.
$$

By (8) and (13), we get

$$
\phi_{n,\lambda}(x) = \sum_{k=0}^{n} S_{2,\lambda}(n,k) x^k, \quad (n \geq 0), \quad \text{(see [15, 16])}.
$$

More generally, for $r \geq 0$, the degenerate $r$-Bell polynomials are given by

$$
e_k^{(x_1 \cdots x_k)} = \sum_{n=k}^{\infty} \phi_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}.
$$

When $x = 1$, $\phi_{n,\lambda}^{(r)} = \phi_{n,\lambda}^{(r)}(1)$ are called the degenerate $r$-Bell numbers.

From (12) and (15), we note that

$$
\phi_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \left\{ \frac{n+r}{k+r} \right\}_{r,\lambda} x^k, \quad (n \geq 0), \quad \text{(see [18, 21])}.
$$

The harmonic numbers are defined by

$$
H_0 = 1, \quad H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \quad (n \in \mathbb{N}).
$$

From (17), we have

$$
- \frac{\log(1-x)}{1-x} = \sum_{n=1}^{\infty} H_n x^n, \quad \text{(see [5, 13, 14, 23])}.
$$
Recently, the degenerate harmonic numbers are introduced as

\begin{equation}
H_{0,\lambda} = 0, \quad H_{n,\lambda} = \sum_{k=1}^{n} \frac{\lambda}{k} (-1)^{k-1}, \quad (n \in \mathbb{N}), \quad \text{(see [13]).}
\end{equation}

Note that \( \lim_{\lambda \to 0} H_{n,\lambda} = H_n, \quad (n \geq 1). \)

From (19), we have

\begin{equation}
-\frac{1}{1-t} \log \lambda (1-t) = \sum_{n=1}^{\infty} H_{n,\lambda} t^n, \quad \text{(see [12, 13]).}
\end{equation}

For \( n \geq 0, \quad r \geq 1, \) the degenerate hyperharmonic numbers are given by

\begin{equation}
H_{n,\lambda}^{(1)} = H_{n,\lambda}, \quad H_{0,\lambda}^{(r)} = 0, \quad (r \geq 1), \quad H_{n,\lambda}^{(r)} = \sum_{k=1}^{n} H_{k,\lambda}^{(r-1)}, \quad (r \geq 2, n \geq 1), \quad \text{(see [14]).}
\end{equation}

Thus, by (21), we get

\begin{equation}
-\frac{\log \lambda (1-t)}{(1-t)^r} = \sum_{n=1}^{\infty} H_{n,\lambda}^{(r)} t^n, \quad \text{(see [14]).}
\end{equation}

It is well known that the Fubini polynomials are defined by

\begin{equation}
\frac{1}{1-x(e^t-1)} = \sum_{n=0}^{\infty} F_n(x) \frac{t^n}{n!}, \quad \text{(see [5, 6, 8, 17, 19]).}
\end{equation}

Note that \( F_n(x) = \sum_{k=0}^{n} S_2(n,k) k! x^k, \quad (n \geq 0). \)

The degenerate Fubini polynomials are defined by (see [17])

\begin{equation}
\frac{1}{1-x(e^t-1)} = \sum_{n=0}^{\infty} F_{n,\lambda}(x) \frac{t^n}{n!}.
\end{equation}

Note that \( \lim_{\lambda \to 0} F_{n,\lambda}(x) = F_n(x) \) and \( F_{n,\lambda}(x) = \sum_{k=0}^{n} S_2(\lambda, n,k) k! x^k, \quad (n \geq 0). \)

2. SOME RESULTS ON DEGENERATE HARMONIC NUMBERS AND DEGENERATE FUBINI POLYNOMIALS

In this section, we assume that \( r \) is a fixed nonnegative integer.

For \( f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[x], \) we define

\begin{equation}
f_{\lambda}(x) = \sum_{n=0}^{\infty} a_n(x)_{n,\lambda} \in \mathbb{C}[x],
\end{equation}

where \( \lambda \) is any fixed real number. First, we observe that

\begin{equation}
\left( x \frac{d}{dx}\right)_{m,\lambda} x^r f(x) = \left( x \frac{d}{dx}\right)_{m,\lambda} \sum_{k=0}^{m} a_k x^{k+r} = \sum_{k=0}^{m} a_k (k+r)_{m,\lambda} x^{k+r}
\end{equation}

\begin{align}
&= \sum_{k=0}^{m} a_k \sum_{l=0}^{m} \binom{m+r}{l+r}_{r,\lambda} x^{k+r} \\
&= \sum_{l=0}^{m} \binom{m+r}{l+r}_{r,\lambda} x^{l+r} f^{(l)}(x),
\end{align}

where \( f^{(l)}(x) = \left( \frac{d}{dx}\right)^l f(x). \)
From (26), we note that
\[
\left(x \frac{d}{dx}\right)_{m-r,\lambda} x^r \left(\frac{d}{dx}\right)^l f(x) = \sum_{l=0}^{m-r} \binom{m}{l+r} \left(\frac{d}{dx}\right)^{l+r} f(x)
\]
(27)
\[
= \sum_{l=r}^{m} \binom{m}{l} x^l \left(\frac{d}{dx}\right)^l f(x).
\]
Therefore, by (26) and (27), we obtain the following theorem.

**Theorem 1.** For \(m, r \in \mathbb{Z}\) with \(m \geq r \geq 0\), we have
\[
\left(x \frac{d}{dx}\right)_{m-r,\lambda} x^r \left(\frac{d}{dx}\right)^l f(x) = \sum_{l=0}^{m-r} \binom{m}{l+r} x^r \left(\frac{d}{dx}\right)^l f(x).
\]
In particular, we also have
\[
\left(x \frac{d}{dx}\right)_{m-r,\lambda} x^r \left(\frac{d}{dx}\right)^l = \sum_{l=0}^{m-r} \binom{m}{l} x^l \left(\frac{d}{dx}\right)^l.
\]
Assume that
\[
f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[x], \quad g(x) = \sum_{n=0}^{\infty} b_n x^n \in \mathbb{C}[x].
\]
By (28), we easily get \(a_k = \gamma^{(k)}(0)\), \(b_k = \mu^{(k)}(0)\).

From Theorem 1 and (28), we have
\[
x^r \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n+r}{k+r} x^k g^{(k)}(x) = \sum_{n=0}^{\infty} a_n \left(x \frac{d}{dx}\right)_{n,\lambda} x^r g(x)
\]
(29)
\[
= \sum_{n=0}^{\infty} a_n \left(x \frac{d}{dx}\right)_{n,\lambda} \sum_{l=0}^{\infty} b_l x^{l+r} = \sum_{n=0}^{\infty} b_l \sum_{n=0}^{\infty} a_n \left(x \frac{d}{dx}\right)_{n,\lambda} x^{l+r}
\]
\[
= \sum_{n=0}^{\infty} b_l \sum_{n=0}^{\infty} a_n (l+r)_{n,\lambda} x^{l+r} = x^r \sum_{l=0}^{\infty} b_l f_x(l+r)x^l.
\]
In particular, by Theorem 1 and (28), we get
\[
\sum_{k=r}^{m} \binom{m}{k} x^k \left(\frac{d}{dx}\right)^k g^{(k)}(x) = \left(x \frac{d}{dx}\right)_{m-r,\lambda} x^r \left(\frac{d}{dx}\right)^l \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n
\]
(30)
\[
= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \left(x \frac{d}{dx}\right)_{m-r,\lambda} x^r \left(\frac{d}{dx}\right)^l = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} (n)_{r} \left(x \frac{d}{dx}\right)_{m-r,\lambda} x^n
\]
\[
= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} n! \frac{n!}{r!} (n)_{r} \left(x \frac{d}{dx}\right)_{m-r,\lambda} x^n.
\]
By (30), we get
\[
\sum_{m=r}^{\infty} \frac{f^{(m)}(0)}{m!} \left(\sum_{k=r}^{m} \binom{m}{k} x^k g^{(k)}(x)\right) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} n! \frac{n!}{r!} \left(\sum_{m=r}^{\infty} \frac{f^{(m)}(0)}{m!} (n)_{m-r,\lambda}\right) x^n.
\]
Therefore, by (29) and (31), we obtain the following theorem.
Let us take $g(x) = \sum_{n=0}^{\infty} a_n x^n$ and $f(x) = \sum_{n=0}^{\infty} b_n x^n$. Then we have
\[
\sum_{n=0}^{\infty} \frac{f(n)(0)}{n!} \sum_{k=0}^{n+r} \binom{n+r}{k+r} x^k g^{(k)}(x) = \sum_{n=0}^{\infty} \frac{g(n)(0)}{n!} f_\lambda(n+r)x^n.
\]
In particular, we also have
\[
\sum_{m=r}^{\infty} \frac{f^{(m)}(0)}{m!} \left( \sum_{k=r}^{m} \binom{m}{k} x^k \right) g^{(k)}(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \binom{n}{r} r! \left( \sum_{m=r}^{\infty} \frac{f^{(m)}(0)}{m!} (n)_{m-r,\lambda} \right) x^n.
\]

From Theorem 2, we have
\[
\phi_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{m} \binom{m}{k+r} x^k, \quad (n \geq 0).
\]
Let us take $g(x) = e^x$ and $f(x) = x^m$, $(m \in \mathbb{N})$. Note that $f_\lambda(x) = (x)_{m,\lambda}$.

From Theorem 2, we have
\[
e^x \sum_{k=0}^{m} \binom{m+r}{k} x^k = \sum_{n=0}^{\infty} n! (n+r)_{m,\lambda} x^n.
\]
Thus, by (16) and (33), we get
\[
\phi_{m,\lambda}^{(r)}(x) = \sum_{k=0}^{m} \binom{m+r}{k+r} x^k = e^{-x} \sum_{n=0}^{\infty} \frac{1}{n!} (n+r)_{m,\lambda} x^n.
\]
In view of (15), we consider the degenerate $r$-Fubini polynomials which are given by
\[
\sum_{n=0}^{\infty} F^{(r)}_{n,\lambda}(x) \frac{t^n}{n!} = e_\lambda^x(t) \frac{1}{1-x(e_\lambda^x(t) - 1)}.
\]
By (12) and (35), we get
\[
\sum_{n=0}^{\infty} F^{(r)}_{n,\lambda}(x) \frac{t^n}{n!} = e_\lambda^x(t) \frac{1}{1-x(e_\lambda^x(t) - 1)} = \sum_{n=0}^{\infty} x^k \frac{k!}{k!} e_\lambda^x(t) (e_\lambda^x(t) - 1)^k = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n+r} \binom{n+r}{k+r} x^k \right) \frac{t^n}{n!}
\]
Comparing the coefficients on both sides of (36), we have
\[
F^{(r)}_{n,\lambda}(x) = \sum_{k=0}^{n} \binom{n+r}{k+r} k! x^k, \quad (n \geq 0).
\]
Let us take $g(x) = \frac{1}{1-x}$ in Theorem 2. Then we have
\[
\sum_{n=0}^{\infty} \frac{f(n)(0)}{n!} \sum_{k=0}^{n+r} \binom{n+r}{k+r} x^k \frac{k!}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} f_\lambda(n+r)x^n.
\]
Let us take $f(x) = x^m$ in (38). Then we have
\[
\frac{1}{1-x} \sum_{k=0}^{m} \binom{m+r}{k+r} \frac{k!}{1-x} \left( \frac{x}{1-x} \right)^k = \sum_{n=0}^{\infty} (n+r)_{m,\lambda} x^n.
\]
By (37) and (39), we get
\[
\frac{1}{1-x} F^{(r)}_{m,\lambda} \left( \frac{x}{1-x} \right) = \sum_{n=0}^{\infty} (n+r)_{m,\lambda} x^n.
\]
Therefore, by (40), we obtain the following theorem.
Theorem 3. For \( m \geq 0 \), we have
\[
\frac{1}{1-x} F_{m,\lambda}^{(r)} \left( \frac{x}{1-x} \right) = \sum_{n=0}^{\infty} (n+r)_{m,\lambda} x^n.
\]
In particular, we also have
\[
F_{m,\lambda}^{(r)} = F_{m,\lambda}^{(r)} (1) = \sum_{n=0}^{\infty} (n+r)_{m,\lambda} \left( \frac{1}{2} \right)^{n+1},
\]
where \( F_{m,\lambda}^{(r)} \) are called the \( r \)-Fubini numbers.

By (7) and (8), we easily get
\[
S_{2,\lambda}(n+1,k) = S_{2,\lambda}(n,k-1) + (k-n\lambda)S_{2,\lambda}(n,k),
\]
where \( n, k \in \mathbb{N} \) with \( n \geq k \).

From (24), we note that
\[
x \frac{d}{dx} F_{n,\lambda}(x) + x \frac{d}{dx} (xF_{n,\lambda}(x)) - n\lambda F_{n,\lambda}(x)
\]
\[
= \sum_{k=1}^{n} kS_{2,\lambda}(n,k)k!x^k + x \sum_{k=0}^{n} S_{2,\lambda}(n,k)k!x^k
\]
\[
+ x^2 \sum_{k=1}^{n} S_{2,\lambda}(n,k)k!x^{k-1} - n\lambda \sum_{k=0}^{n} S_{2,\lambda}(n,k)k!x^k
\]
\[
= \sum_{k=1}^{n} kS_{2,\lambda}(n,k)k!x^k + \sum_{k=1}^{n+1} S_{2,\lambda}(n,k-1)(k-1)!x^k
\]
\[
+ \sum_{k=2}^{n+1} S_{2,\lambda}(n,k-1)(k-1)!x^k - n\lambda \sum_{k=0}^{n} S_{2,\lambda}(n,k)k!x^k
\]
\[
= \sum_{k=1}^{n+1} \left\{ (k-n\lambda)S_{2,\lambda}(n,k) + S_{2,\lambda}(n,k-1) \right\} k!x^k
\]
\[
= \sum_{k=1}^{n+1} S_{2,\lambda}(n+1,k)k!x^k = F_{n+1,\lambda}(x).
\]
Therefore, by (42), we obtain the following theorem.

Theorem 4. For \( n \geq 0 \), we have
\[
x \frac{d}{dx} F_{n,\lambda}(x) + x \frac{d}{dx} (xF_{n,\lambda}(x)) - n\lambda F_{n,\lambda}(x) = F_{n+1,\lambda}(x).
\]
From (3), we note that
\[
\frac{d}{dx} \left( \log_{\lambda} (1-x) \right)^2
\]
\[
= -\log_{\lambda} (1-x) \left( \lambda \log_{\lambda} (1-x) + 1 \right) = -\frac{\lambda}{1-x} \left( \log_{\lambda} (1-x) \right)^2 - \frac{\log_{\lambda} (1-x)}{1-x}
\]
\[
= -2\lambda \sum_{n=2}^{\infty} \frac{n+1}{1,\lambda} \frac{x^n}{n!} + \sum_{n=1}^{\infty} H_{n,\lambda} x^n
\]
\[
= \sum_{n=1}^{\infty} \left( -2\lambda \left[ \frac{n+1}{1,\lambda} + n!H_{n,\lambda} \right] \right) \frac{x^n}{n!}.
\]
On the other hand, by (12), we get
\[
\frac{1}{2!} \frac{d}{dx} \left( \log \lambda (1 - x) \right)^2 = \frac{d}{dx} \sum_{n=2}^{\infty} S_{1,\lambda}(n,2) \frac{(-1)^n x^n}{n!}
\]
\[
= \sum_{n=2}^{\infty} S_{1,\lambda}(n,2) \frac{(-1)^n x^{n-1}}{(n-1)!} = \sum_{n=1}^{\infty} S_{1,\lambda}(n+1,2) (-1)^n x^n
\]
\[
= \sum_{n=1}^{\infty} \left[ \frac{n+1}{2} \right] x^n
\]
where \( \left[ \frac{n}{k} \right] \lambda = (-1)^{n-k} S_{1,\lambda}(n,k) \) are called the unsigned degenerate Stirling numbers of the first kind.

By (12) and (22), we get
\[
-\log \lambda (1 - t) = \sum_{n=1}^{\infty} \binom{r}{n+1} \frac{t^n}{n!}
\]
\[
\sum_{n=1}^{\infty} \binom{n+r}{n+1} \frac{t^n}{n!}
\]
From (43), (44) and (45), we have
\[
n! H_{n,\lambda}^{(r)} = \binom{n+r}{r+1},
\]
\[
n! H_{n,\lambda} = 2 \lambda \left[ \frac{n+1}{3} \right] \lambda + \left[ \frac{n+1}{2} \right] \lambda, \quad (n \geq 1).
\]
Therefore, by (46), we obtain the following theorem.

**Theorem 5.** For \( n \in \mathbb{N} \), we have
\[
n! H_{n,\lambda}^{(r)} = \binom{n+r}{r+1},
\]
\[
n! H_{n,\lambda} = 2 \lambda \left[ \frac{n+1}{3} \right] \lambda + \left[ \frac{n+1}{2} \right] \lambda,
\]
where \( \left[ \frac{n+1}{k} \right] \lambda = (-1)^{n-k} S_{1,\lambda}(n+1,2) \) are the unsigned degenerate Stirling numbers of the first kind.

**Remark 6.** Note that Theorem 5 shows the following which holds between the degenerate 1-Stirling numbers of the first kind and the unsigned degenerate Stirling numbers of the first kind.
\[
\left[ \frac{n+1}{2} \right] \lambda = 2 \lambda \left[ \frac{n+1}{3} \right] \lambda + \left[ \frac{n+1}{2} \right] \lambda.
\]
Let \( g(t) = -\frac{1}{1-t} \log \lambda (1 - t) \). Then, by (19), we get
\[
g'(t) = \frac{1}{(1-t)^2} \log \lambda (1 - t) + \frac{1}{(1-t)^2} \left( \lambda \log \lambda (1 - t) + 1 \right)
\]
\[
= -\frac{(1-\lambda)}{(1-t)^2} \log \lambda (1 - t) + \frac{1}{(1-t)^2}
\]
\[
= \frac{1}{(1-t)^2} \left( -1 - \lambda \right) \log \lambda (1 - t) + H_{1,\lambda}.
\]
Continuing this process, we have

\[ g''(t) = \frac{d^2}{dt^2} g'(t) = -\frac{2(1 - \lambda)}{(1 - t)^3} \log_{\lambda}(1 - t) + \frac{(1 - \lambda)(\lambda \log_{\lambda}(1 - t) + 1)}{(1 - t)^3} + \frac{2!}{(1 - t)^3} \]

\[ = -\frac{(1 - \lambda)(2 - \lambda)}{(1 - t)^3} \log_{\lambda}(1 - t) + \frac{2!}{(1 - t)^3} \left( 1 + \frac{1 - \lambda}{2} \right) \]

\[ = \frac{2!}{(1 - t)^3} \left( - \frac{2 - \lambda}{2} \log_{\lambda}(1 - t) + \sum_{k=1}^{\infty} \frac{1}{\lambda} \left( \frac{\lambda}{k} \right)(-1)^{k-1} \right) \]

\[ = \frac{2!}{(1 - t)^3} \left( - \frac{2 - \lambda}{2} \log_{\lambda}(1 - t) + H_{2,\lambda} \right), \]

and

\[ g^{(3)}(t) = \frac{d^3}{dt^3} g''(t) = -\frac{3(1 - \lambda)(2 - \lambda)}{(1 - t)^4} \log_{\lambda}(1 - t) \]

\[ + \frac{(1 - \lambda)(2 - \lambda)(\lambda \log_{\lambda}(1 - t) + 1)}{(1 - t)^4} + \frac{3!}{(1 - t)^4} \left( \frac{1 - \lambda}{2} + 1 \right) \]

\[ = \frac{3!}{(1 - t)^4} \left( - \frac{3 - \lambda}{3} \log_{\lambda}(1 - t) + \sum_{k=1}^{\infty} \frac{1}{\lambda} \left( \frac{\lambda}{k} \right)(-1)^{k-1} \right) \]

\[ = \frac{3!}{(1 - t)^4} \left( - \frac{3 - \lambda}{3} \log_{\lambda}(1 - t) + H_{3,\lambda} \right). \]

Continuing this process, we have

\[ g^{(k)}(t) = \frac{k!}{(1 - t)^{k+1}} \left( - \frac{k - \lambda}{k} \log_{\lambda}(1 - t) + H_{k,\lambda} \right), \]

and

\[ g^{(k)}(0) = k! H_{k,\lambda}, \quad (k \in \mathbb{N}). \]

Therefore, by (48), we obtain the following theorem.

**Theorem 7.** Let \( g(t) = -\frac{1}{1 - t} \log_{\lambda}(1 - t) \). For \( k \in \mathbb{N} \), we have

\[ g^{(k)}(t) = \left( \frac{d}{dt} \right)^k g(t) = \frac{k!}{(1 - t)^{k+1}} \left( - \frac{k - \lambda}{k} \log_{\lambda}(1 - t) + H_{k,\lambda} \right), \]

and

\[ g^{(k)}(0) = k! H_{k,\lambda}. \]

From (20), (22) and Theorem 6, we note that

\[ \sum_{n=k}^{\infty} n(n - 1) \cdots (n - k + 1) H_{n,\lambda} t^{n-k} = \frac{d^k}{dt^k} \left( - \frac{\log_{\lambda}(1 - t)}{1 - t} \right) \]

\[ = \frac{k!}{(1 - t)^{k+1}} \left( H_{k,\lambda} - \left( \frac{k - \lambda}{k} \log_{\lambda}(1 - t) \right) \right) = \frac{k!}{(1 - t)^{k+1}} \left( H_{k,\lambda} - \left( \frac{k - \lambda}{k} \right) \log_{\lambda}(1 - t) \right) \]

\[ = k! \left\{ \sum_{n=0}^{\infty} \binom{n+k}{n} H_{k,\lambda} t^n + \left( \frac{k - \lambda}{k} \right) \sum_{n=1}^{\infty} H_{n,\lambda}^{(k+1)} t^n \right\}. \]

Thus, by (49), we get

\[ \sum_{n=0}^{\infty} \binom{n+k}{k} H_{n+k,\lambda} t^n \]

\[ = \sum_{n=0}^{\infty} k! \binom{n+k}{k} H_{k,\lambda} t^n + k! \left( \frac{k - \lambda}{k} \right) \sum_{n=1}^{\infty} H_{n,\lambda}^{(k+1)} t^n. \]
By \((50)\), we get
\[
\sum_{n=1}^{\infty} \binom{n+k}{k} k! H_{n+k,\lambda} x^n = \frac{1}{n!} \sum_{k=0}^{n} \binom{n+k}{k} H_{k,\lambda} + k! \left( \binom{k-\lambda}{k} H_{n,\lambda}^{(k+1)} \right) x^n.
\]

From \((51)\), we obtain the following theorem.

**Corollary 8.** For \(n, k \in \mathbb{N}\), we have
\[
\left( \binom{k-\lambda}{k} H_{n,\lambda}^{(k+1)} \right) x^n = \left( \binom{n+k}{k} \right) \left( H_{n+k,\lambda} - H_{k,\lambda} \right).
\]

Let us take \(g(x) = -\frac{1}{\lambda} \log(1-x)\) in Theorem 2. Then we have
\[
\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \binom{n+r}{k} k! \left( \frac{1-x}{1-\lambda x} \right)^k \left( \binom{k-\lambda}{k} \log(1-x) + H_{k,\lambda} \right)
\]
\[
= \sum_{n=0}^{\infty} H_{n,\lambda} f(x) (n+r)x^n.
\]

For \(f(x) = x^m\), by \((52)\), we get
\[
\sum_{n=0}^{\infty} H_{n,\lambda} (n+r)m! \lambda x^n
\]
\[
= \frac{1}{1-x} \sum_{k=0}^{m} \binom{m+r}{k} k! \left( \frac{x}{1-x} \right)^k \log(1-x) \sum_{k=0}^{m} \binom{m+r}{k} k! \left( \frac{x}{1-x} \right)^k \binom{k-\lambda}{k}
\]
\[
= \frac{1}{1-x} \sum_{k=0}^{m} \binom{m+r}{k} k! \left( \frac{x}{1-x} \right)^k \left( H_{k,\lambda} - \binom{k-\lambda}{k} \log(1-x) \right).
\]

Therefore, by \((53)\), we obtain the following theorem.

**Theorem 9.** For \(m \geq 0\), we have
\[
\sum_{n=0}^{\infty} H_{n,\lambda} (n+r)m! \lambda x^n
\]
\[
= \frac{1}{1-x} \sum_{k=0}^{m} \binom{m+r}{k} k! \left( \frac{x}{1-x} \right)^k \left( H_{k,\lambda} - \binom{k-\lambda}{k} \log(1-x) \right).
\]

### 3. Conclusion

A general identity valid for any two formal power series was derived, where the degenerate \(r\)-Stirling numbers of the second kind appear. Applying that result to appropriate formal power series, we obtained an identity on the degenerate \(r\)-Fubini polynomials. We deduced a differential equation satisfied by the degenerate Fubini polynomials with the help of a recurrence relation for the degenerate Stirling numbers of the second kind. The degenerate hyperharmonic numbers were expressed in terms of the degenerate unsigned \(r\)-Stirling numbers of the first kind. We derived a relation between the degenerate hyperharmonic numbers and the degenerate harmonic numbers. By applying the general identity with suitable formal power series, we obtained an identity expressing a power series with coefficients given by the product of the degenerate harmonic numbers and the degenerate falling factorials in terms of the degenerate \(r\)-Stirlings of the second, the degenerate harmonic numbers and the degenerate logarithms.
It is one of our future research projects to continue to study various degenerate versions of some degenerate special numbers and polynomials and to find their applications in physics, science and engineering as well as in mathematics.

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