Spherically symmetric solutions of the 6th order SU(N) Skyrme models.

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Abstract

Following the construction described in [1], we use the rational map ansatz to construct analytically some topologically non-trivial solutions of the generalised SU(3) Skyrme model defined by adding a sixth order term to the usual Lagrangian. These solutions are radially symmetric and some of them can be interpreted as bound states of Skyrmions. The same ansatz is used to construct low-energy configuration of the SU(N) Skyrme model.

1 Introduction

The Skyrme model [2] is widely accepted as an effective theory to describe the low-energy properties of nucleons. It was indeed shown [3, 4, 5] that in the large $N_c$ limit, the Skyrme model is the low-energy limit of QCD. The classical static solutions of the model describe the bound states of nucleons and every configuration is characterised by a topological charge which following Skyrme’s idea, is interpreted as the baryon charge.

The Skyrme model can be used to predict the properties of the nucleons within 10 to 20% [4, 5]. To improve these phenomenological predictions various extensions of the models have been proposed mostly by adding higher order terms [6, 7, 8, 9] or extra fields [10] to the Lagrangian.

The study of the classical solutions of the Skyrme model has been done mostly using numerical methods, but recently Houghton et al. [11] showed that the classical solutions of

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the $SU(2)$ model can be well approximated by using an ansatz that involves the harmonic maps from $S^2$ to $S^3$. The harmonic map describes the angular distribution of the solution while a profile function describes its radial distribution. This construction was later generalised [12] for the $SU(N)$ model using harmonic maps from $S^2$ to $CP^{N-1}$. Moreover, it was shown that using a further generalisation of this ansatz one can construct exact spherically symmetric solutions of the $SU(N)$ Skyrme model.

The same method was also used in [13] to construct solutions of another $SU(N)$ 4th order Skyrme model. In this paper, we use the same generalised ansatz to construct solutions of the sixth order $SU(3)$ Skyrme model and low-energy configurations of the $SU(N)$ models defined in [14].

2 The sixth order Skyrme model

The Skyrme model is described by an $SU(N)$ valued field $U(\vec{x}, t)$ which, to ensure finiteness of the energy, is required to satisfy the boundary condition $U \to I$ as $|\vec{x}| \to \infty$, where $I$ is the unit matrix. This boundary condition implies that the three dimensional Euclidean space on which the model is defined can be compactified into $S^3$ and as a result, the Skyrme field $U$ corresponds to mappings from $S^3$ into $SU(N)$. As $\pi_3(SU(N)) = \mathbb{Z}$ each configuration is characterised by its winding number, or topological charge, which can be obtained explicitly by evaluating the expression

$$B = \frac{1}{24\pi^2} \int_{R^3} d\vec{x}^3 \varepsilon^{ijk} Tr(R_i R_j R_k),$$

(1)

where $R_{\mu} = (\partial_{\mu} U) U^{-1}$ is the right chiral current. Skyrme’s ideas was to interpret the winding number associated with these topologically non-trivial mappings as the baryon charge.

The generalised sixth order Skyrme model is defined by the Lagrangian

$$E = -\frac{1}{12\pi^2} \int d\vec{x}^3 \left( \frac{1}{2} Tr R_i^2 + \frac{1 - \lambda}{16} Tr[R_i, R_j]^2 + \frac{1}{96} \lambda Tr[R_i, R_j][R_j, R_k][R_k, R_i] \right),$$

(2)

where this parametrisation of the model is chosen such that $\lambda \in [0, 1]$ is a mixing parameter between the Skyrme term and the sixth order term: when $\lambda = 0$ the model reduces to the usual pure Skyrme model while for $\lambda = 1$ the Skyrme term vanishes and the model reduces to what we refer to in what follows as the pure Sk6 model.
The Euler-Lagrange equations derived from (2) for the static solutions are given by
\[ \partial_i \left( R_i - \frac{1}{4} (1 - \lambda) \left[ R_j, [R_j, R_i] \right] - \frac{1}{16} \lambda [R_j, [R_j, R_k][R_k, R_i]] \right) = 0. \] (3)
and the following inequality holds for every configuration
\[ \tilde{E} \geq \sqrt{1 - \lambda B} . \] (4)

The multi-Skyrmion solutions of the SU(2) Skyrme model have been studied in [14] where it was shown that they have the same symmetry as the pure Skyrme model. It was also shown that the harmonic map ansatz gives a good approximation to the solutions.

In the next section we describe the harmonic map ansatz. In the third section we prove that due to a constraint coming from the sixth order term, the multi-projector harmonic map ansatz provides exact solutions only for the SU(3) generalised model. We then show that one can nevertheless use the ansatz to construct low-energy configurations of the SU(N) models. In the fourth section we look at these configurations for the SU(4) model, while in the last section we look at some special ansatz for the SU(N) model.

3 Harmonic map ansatz

The rational map ansatz, introduced by Houghton et al. [11] is a generalisation of the hedgehog ansatz found by Skyrme [2], to approximate multi-Skyrmion solution of the SU(2) model. The ansatz was later generalised by Ioannidou et al. [1] to approximate solutions of the SU(N) Skyrme model using harmonic maps from S^2 into CP^{N-1}. This generalised ansatz is given by
\[ U(r, \theta, \varphi) = e^{2if(r)(P(\theta, \varphi) - I/N)} = e^{-2if(r)/N \left( I + (e^{2if(r)} - 1)P(\theta, \varphi) \right)} \] (5)
where \( r, \theta \) and \( \varphi \) are the usual polar coordinates. The profile function \( f(r) \) must satisfy the boundary conditions \( f(0) = \pi \) and \( \lim_{r \to \infty} f(r) = 0 \) and \( P(\theta, \varphi) \) is a projector in \( C^N \) which must be a harmonic map from \( S^2 \) into \( CP^{N-1} \) or equivalently a classical solution of the 2 dimensional \( CP^{N-1} \sigma \) model. These solutions are well known [15, 16] and to construct them it is convenient to introduce the complex coordinate \( \xi = \tan(\theta/2)e^{i\varphi} \) which corresponds to the stereographic projection of the unit sphere onto the complex plane.
In these coordinates, $P$ must satisfy the equation
\[ P \frac{\partial P}{\partial \xi} = 0. \] (6)
and the solutions of that equation are given by any projector of the form
\[ P(f) = \frac{h \otimes h^\dagger}{|h|^2} \] (7)
where $h \in C^N$ is holomorphic
\[ \frac{\partial h}{\partial \xi} = 0. \] (8)

The topological charge for the ansatz (5), with the prescribed boundary conditions for $f(r)$, is given by the winding number of the $S^2 \to CP^{N-1}$. This winding number which is itself given by the degree of the harmonic function $h$ [15, 16] which must then be a rational function of $\xi$.

To approximate a solution, one plugs the ansatz (5) into the energy (2) and notices that if $P$ satisfies (3), the integration over the polar angles and the radius decouple. One then has to minimise the integral over the polar angles of an expression which depends only on $P$. Taking for $P$ the most general harmonic map of the desired degree, one then has to find the parameters of the general map which minimise that integral. Having done this, the profile function $f$ is obtained by solving the Euler Lagrange equation derived from the effective energy.

A special case of this construction is the so-called hedgehog ansatz for the $SU(2)$ model corresponding to one Skyrmion. In this case, we have $h = (1, \xi)^t$ and after inserting (7) into (2) the energy reduces to
\[ E = \frac{1}{3\pi} \int dr \left( f_r^2 r^2 + 2 \sin^2 f (1 + (1 - \lambda)f_r^2) + (1 - \lambda)\frac{\sin^4 f}{r^2} + \lambda \frac{\sin^4 f}{r^2} f_r^2 \right) \] (9)
and the equation for $f$ is given by
\[ f_{rr} \left( 1 + 2 (1 - \lambda) \frac{\sin^2 f}{r^2} + \lambda \frac{\sin^4 f}{r^4} \right) + \frac{2}{r} f_r \left( 1 - \lambda \frac{\sin^4 f}{r^4} \right) + \frac{\sin 2g}{r^2} \left( (1 - \lambda)f_r^2 - 1 + \frac{\sin^2 f}{r^2} (\lambda f_r^2 - 1 + \lambda) \right) = 0. \] (10)
This actually corresponds to an exact solution of the model and it is radially symmetric. In Figure 1 we present the $\lambda$ dependence of the energy and in Figure 2 we show the profile function $f$ and the energy density for the pure Skyrme model, $\lambda = 0$, and the pure Sk6 model, $\lambda = 1$. 
Figure 1: Total energy of the 1 Skyrmion solution.

Figure 2: Function profile $f$ and energy density for the 1 Skyrmion solution of the pure Skyrme model, $\lambda = 0$, and the pure Sk6 model, $\lambda = 1$. 
4 Spherically symmetric solutions for the $SU(N)$ model

In this section we will follow the construction described in [1], to attempt to construct solutions of the extended $SU(N)$ Skyrme model using a generalisation of the harmonic map ansatz (3).

To build the new ansatz we need to introduce an operator $P_+$ which acts on any complex vector $u \in C^N$ and is defined as

$$P_+u = \partial_\xi u - u \frac{u^\dagger \partial_\xi u}{|u|^2}. \quad (11)$$

Taking a holomorphic vector $h(\xi)$ we then define $P_+^0 h = h$ and by induction $V_k = P_+^k h = P_+(P_+^{k-1} h)$. These $N$ vectors are mutually orthogonal [16] and so the corresponding projectors

$$P_k = P(P_+^k h) = \frac{P_+^k h(P_+^k h)^\dagger}{|P_+^k h|^2} \quad k = 0, \ldots, N-1, \quad (12)$$

satisfy the orthogonality relations

$$P_k P_j = \delta_{ij} P_k \quad \sum_{k=0}^{N-1} P_k = 1 \quad (13)$$

as well as other properties discussed in detail in [1].

The generalised harmonic map ansatz is then defined as

$$U = \exp\{ig_0(P_0 - \frac{I}{N}) + ig_1(P_1 - \frac{I}{N}) - \ldots + ig_{N-2}(P_{N-2} - \frac{I}{N})\}$$

$$= e^{-ig_0/N}(I + A_0 P_0) e^{-ig_1/N}(I + A_1 P_1) \ldots e^{-ig_{N-2}/N}(I + A_{N-2} P_{N-2}) \quad (14)$$

where $g_k(r)$ are $N - 1$ profile functions and $A_k = e^{ig_k} - 1$. Moreover for the ansatz to be well defined, the profile functions $g_k(r)$ must be a a multiple of $2\pi$ at the origin and at infinity.

To proceed with our construction, it is convenient to rewrite the Euler-Lagrange equations of the model (3) using the usual spherical coordinates

$$\partial_r \left\{ r^2 R_r + \frac{1 - \lambda}{4} \left( A_{\theta r \theta} + \frac{1}{\sin^2 \theta} A_{\varphi r \varphi} \right) + \frac{1}{16} \lambda \left[ \frac{1}{\sin^2 \theta} (B_{\theta \theta \varphi r \varphi} + B_{\varphi \varphi \theta r \theta}) \right] \right\} +$$

$$\frac{1}{\sin \theta} \partial_\theta \left\{ \sin \theta \left[ R_\theta + \frac{1 - \lambda}{4} \left( A_{r \theta r} + \frac{1}{r^2 \sin^2 \theta} A_{\varphi \theta \varphi} \right) \right] + \frac{\lambda}{16 r^2 \sin^2 \theta} (B_{rr \theta \varphi \varphi} + B_{\varphi \varphi r \theta \theta}) \right\} +$$

$$\frac{1}{\sin^2 \theta} \partial_\varphi \left\{ R_\varphi + \frac{1 - \lambda}{4} \left( A_{r \varphi r} + \frac{1}{r^2} A_{\theta \varphi \theta} \right) + \frac{\lambda}{16 r^2} (B_{rr \varphi \varphi} + B_{\varphi \varphi r \theta}) \right\} = 0 \quad (15)$$
where \( A_{ij} \equiv [R_j, [R_i, R_j]] \) and \( B_{jkl} \equiv [R_j, [R_i, R_k] [R_l, R_k]] \). It is fairly easy to show that
\[
R_r = i \sum_{j=0}^{N-2} \hat{g}_j \left( P_j - \frac{I}{N} \right),
\]
(16)
where \( \hat{g}_j \) is the derivative of \( g_j(r) \) with respect to \( r \). Using the complex coordinates \( \xi \) and \( \bar{\xi} \) introduced before we have
\[
R_\xi = \sum_{i=1}^{N-1} \left[ e^{i(\theta_i - \theta_{i-1})} - 1 \right] \frac{V_i V_{i-1}^\dagger}{|V_{i-1}|^2}
\]
(17)
and the derivatives with respect to \( \theta \) and \( \varphi \) are given by
\[
\partial_\theta = \frac{1 + |\xi|^2}{2 \sqrt{|\xi|^2}} \left( \xi \partial_\xi + \bar{\xi} \partial_{\bar{\xi}} \right), \quad \partial_\varphi = i \left( \xi \partial_\xi - \bar{\xi} \partial_{\bar{\xi}} \right).
\]
(18)
Substituting the above into equations (15) we get
\[
\partial_r \left[ r^2 R_r + (1 - \lambda) \frac{(1 + |\xi|^2)^2}{8} (A_{\xi r} \xi + A_{\xi r} \bar{\xi}) \right] + \frac{(1 + |\xi|^2)^2}{2} \left( (R_\xi)_{\xi} + (R_\xi)_{\bar{\xi}} \right) + (1 - \lambda) \frac{(1 + |\xi|^2)^3}{8r^2} (A_{\xi \bar{\xi} \xi} \xi - \bar{\xi} A_{\xi \bar{\xi} \bar{\xi} \bar{\xi}}) + (1 - \lambda) \frac{(1 + |\xi|^2)^4}{16r^2} \left( [A_{\xi \bar{\xi} \xi}]_{\xi} - [A_{\xi \bar{\xi} \bar{\xi} \bar{\xi}}]_{\xi} \right) + \lambda \frac{(1 + |\xi|^2)^4}{4} \left( B_{\xi \bar{\xi} \xi r} \xi - B_{\xi \bar{\xi} \xi r} \bar{\xi} \right) + \frac{(1 + |\xi|^2)^2}{4r^2} \left( \partial_\xi \left[ (1 + |\xi|^2)^2 B_{r r \xi \bar{\xi}} \right] - \partial_{\bar{\xi}} \left[ (1 + |\xi|^2)^2 B_{r r \xi \bar{\xi}} \right] \right) + \frac{(1 + |\xi|^2)^2}{2|\xi|^2 r^2} \left( \partial_\xi \left[ \frac{(1 + |\xi|^2)^2}{4|\xi|^2} (-\xi \xi B_{\xi \bar{\xi} \xi r} \bar{\xi}) \right] + \partial_{\bar{\xi}} \left[ \frac{(1 + |\xi|^2)^2}{4|\xi|^2} (-\bar{\xi} \xi B_{\xi \bar{\xi} \xi r} \xi \bar{\xi}) \right] \right) + \frac{(1 + |\xi|^2)^2}{8r^2} \left( \partial_\xi \left[ (1 + |\xi|^2)^2 (B_{\xi \bar{\xi} \xi r} \xi + B_{\xi \bar{\xi} \xi r} \bar{\xi} - B_{\xi \bar{\xi} \xi r} \xi) + B_{\xi \bar{\xi} \xi r} \xi \right] \right) = 0.
\]
(19)
In (1) it is shown that if one takes the special holomorphic vector
\[
V_0 = h = (h_0, h_1, \ldots, h_{N-1})^t
\]
(20)
where
\[
h_k = \xi^k \sqrt{C_k^{N-1}}
\]
(21)
and where \( C_k^{N-1} \) denotes the binomial coefficients, then the terms in (19) coming from the usual Skyrme model, i.e. all terms except the ones proportional to \( \lambda/16 \), are all proportional to \( P_i - P_{i-1} \) and \( P_i - \frac{I}{N} \). Using (13) one can get rid of the projector \( P_{N-1} \) and (19) will then be the sum of the \( N - 1 \) terms \( P_i - \frac{I}{N} \) for \( i = 0 \ldots N - 2 \), with coefficients that depend only on \( r \). This implies that the equations for the Skyrme model reduce to
\[ N - 1 \] ordinary differential equations for the profile functions \( g_i \) and their solutions, if they exist, will provide us with exact solutions of the \( SU(N) \) Skyrme model.

In what follows we will show that the angular dependence of the terms proportional to \( \lambda \) in (19), i.e. the terms coming from the sixth order term, is also coming exclusively from the projectors \( P_i - \frac{\xi}{N} \) or \( P_i - P_{i-1} \) but that we have to impose an extra constraint on the profile functions \( g_i \).

We start by noting that

\[
\begin{align*}
[R_{\xi}, R_{\bar{\xi}}] &= -\sum_{i=1}^{N-1} a_i^2 \frac{|V_i|^2}{|V_{i-1}|^2} \left( \frac{V_i V_i^\dagger}{|V_i|^2} - \frac{V_{i-1} V_{i-1}^\dagger}{|V_{i-1}|^2} \right) \quad (22) \\
[R_r, R_{\xi}] &= i \sum_{i=1}^{N-1} (\dot{g}_i a_i - \dot{g}_{i-1} a_i) \frac{V_i V_{i-1}^\dagger}{|V_{i-1}|^2} = \sum_{i=1}^{N-1} K_i \frac{V_i V_{i-1}^\dagger}{|V_{i-1}|^2} \quad (23) \\
[R_r, R_{\bar{\xi}}] &= i \sum_{i=1}^{N-1} (\dot{g}_i a_i - \dot{g}_{i-1} a_i) \frac{V_{i-1} V_i^\dagger}{|V_{i-1}|^2} = \sum_{i=1}^{N-1} K_i \frac{V_{i-1} V_i^\dagger}{|V_{i-1}|^2} \quad (24)
\end{align*}
\]

where \( a_i = e^{i(g_i-g_{i-1})} - 1 \). It is then straightforward to check that

\[
B_{\xi\bar{\xi}\bar{\xi}r} \equiv B_{\xi\bar{\xi}\bar{\xi}r} = \sum_{i=1}^{N-1} \left( b_i \frac{|V_{i-1}|^2}{|V_i|^2} \frac{|V_i|^2}{|V_{i-1}|^2}^2 + c_i \frac{|V_i|^4}{|V_{i-1}|^4} + d_i \frac{|V_i|^2}{|V_{i-1}|^2} \right) (P_i - P_{i-1}) \quad (25)
\]

where \( b_i, c_i \) and \( d_i \) are functions of \( g_k \) only. However, as shown in [1], if \( V_0 \) is given by (20) and (21) then \( \frac{|V_i|^2}{|V_{i-1}|^2} \propto (1 + |\xi|^2)^{-2} \) and thus

\[
\frac{(1 + |\xi|^2)^4}{4} \left( B_{\xi\bar{\xi}\bar{\xi}r} - B_{\xi\bar{\xi}\bar{\xi}r} \right) \propto (P_i - P_{i-1}) \quad (26)
\]

Furthermore, we have

\[
B_{r\bar{r}\xi\bar{\xi}} = i \sum_{i=1}^{N-1} \left( e_i \frac{|V_i|^2}{|V_{i-1}|^2} + s_i \frac{|V_{i-1}|^2}{|V_i|^2} \right) \frac{V_i V_{i-1}^\dagger}{|V_{i-1}|^2} \quad (27)
\]

with \( e_i = e(g_i) \) and \( s_i = s(g_i) \). But in equation (19) this term appears as

\[
\partial_{\xi} \left[ (1 + |\xi|^2)^2 B_{r\bar{r}\xi\bar{\xi}} \right] = 2\xi (1 + |\xi|^2) B_{r\bar{r}\xi\bar{\xi}} + (1 + |\xi|^2)^2 \partial_{\bar{\xi}} (B_{r\bar{r}\xi\bar{\xi}}). \quad (28)
\]

Since \( \partial_{\bar{\xi}} \frac{|V_i|^2}{|V_{i-1}|^2} \propto -2\xi (1 + |\xi|^2)^{-3} \) the only parts of (28) that are non zero are the ones that involve the derivatives of \( \frac{V_i V_{i-1}^\dagger}{|V_{i-1}|^2} \) with respect to \( \bar{\xi} \). Since it can be shown that the latter are proportional to \( \sum_{i=1}^{N-1} C_i (1 + |\xi|^2)^{-2} (P_i - P_{i-1}) \) with \( C_i = C(g_i) \), then one sees that the term that involves \( B_{r\bar{r}\xi\bar{\xi}} \) in (19) is proportional to \( (P_i - P_{i-1}) \).

Using similar arguments, it is easy to check that the terms involving \( B_{r\bar{r}\xi\bar{\xi}}, B_{\xi\bar{\xi}\bar{\xi}r}, B_{\xi\bar{\xi}\bar{\xi}r}, B_{\xi\bar{\xi}\bar{\xi}r}, B_{\xi\bar{\xi}\bar{\xi}r} \) and \( B_{\xi\bar{\xi}\bar{\xi}r} \) factorise in the same way.
There are a few terms in (13) which we still have to consider. They involve the expressions

\[ B_{\xi \xi_r \xi_r} = \sum_{i=3}^{N-1} \left( a_i K_{i-1} K_{i-2} - a_{i-2} K_i K_{i-1} \right) \frac{V_i V_i^\dagger}{|V_{i-3}|^2} \]

(29)

\[ B_{\xi \xi_r \xi_r} = \sum_{i=3}^{N-1} \left( a_i K_{i-1} K_{i-2} - a_{i-2} K_i K_{i-1} \right) \frac{V_{i-3} V_i^\dagger}{|V_{i-3}|^2} \]

(30)

where \( K_i = i (\dot{g}_i a_i - \dot{g}_{i-1} a_i) \). It is clear that these terms will always give a \( \xi, \bar{\xi} \) dependence besides the projectors \( P_i \) and hence, if we want (19) to reduce to \( N - 1 \) equations that involve only the profile functions \( g_i \), then we have to make sure that (29) and (30) vanish ie we must impose the conditions

\[ a_i K_{i-1} K_{i-2} - a_{i-2} K_i K_{i-1} = 0 \quad \Rightarrow \quad \dot{g}_i = \dot{g}_{i-2}. \]

(31)

This last constraint which is a result of the addition of the sixth order term, implies that we can only consider two profile functions \( g_0 \) and \( g_1 \) and that we should thus have only two equations. Unfortunately we have \( N - 1 \) equations which are not compatible with each other. From this we see that the ansatz (3) will provide exact solutions of the generalised Skyrme model for the \( SU(2) \) and the \( SU(3) \) model only. For larger values of \( N \), the ansatz will nevertheless give some low-energy radially symmetric configurations. The \( SU(2) \) case is nothing but the usual hedgehog ansatz and we will focus on the solutions of the \( SU(3) \) model in the next section.

In order to derive the equations for the profile functions, it is convenient to write the energy density of the model in terms of \( (\xi, \bar{\xi}) \):

\[
E = -\frac{i}{12\pi^2} \int r^2 dr d\xi d\bar{\xi} T r \left( \frac{1}{1 + |\xi|^2} R_r^2 + \frac{1}{r^2} |R_\xi|^2 + \frac{1 - \lambda}{4r^2} [R_r, R_\xi][R_r, R_\xi] 
- (1 - \lambda) \frac{(1 + |\xi|^2)^2}{16r^4} [R_\xi, R_\xi]^2 + \lambda \frac{(1 + |\xi|^2)^2}{64r^4} \left[ [R_r, R_\xi], [R_r, R_\xi] \right] [R_\xi, R_\xi] \right).
\]

(32)

Defining

\[ F_i = g_i - g_{i+1} \quad \text{for} \quad i = 0, \ldots, N - 3, \]

\[ F_{N-2} = g_{N-2} \]

(33)

as well as \( W_i = \frac{|V_i|^2}{|V_{i-1}|} (1 - \cos(F)) \) and \( W_{N-1} = \frac{|V_{N-1}|^2}{|V_{N-2}|} (1 - \cos(g)) \) the terms in the above expression can be rewritten as

\[ T r R_r^2 = \frac{1}{N} \left( \sum_{i=0}^{N-2} \dot{g}_i \right)^2 - \sum_{i=0}^{N-2} \dot{g}_i^2, \]

(34)

\[ T r |R_\xi|^2 = -2 \sum_{i=1}^{N-1} W_i, \]

(35)
and from this we see that all the terms in (32) are proportional to $(1 + |\xi|^2)$.

In [1] it was shown that

$$E = \frac{1}{6\pi} \int r^2 dr \left\{ -\frac{1}{N} \left( \sum_{i=0}^{N-2} g_i \right)^2 + \frac{2}{r^2} \sum_{k=1}^{N-1} Z_k + \sum_{k=1}^{N-1} (\hat{g}_k - \hat{g}_{k-1})^2 Z_k \right\},$$

where $Z_k = k(N - k)(1 - \cos(F_{k-1}))$.

In [1] the fields $F_i$ defined by (33) were used, and very special solutions were obtained by taking $F_0 = F_1 = \ldots = F_{N-2}$. It was observed that when $F_i(0) = 2\pi$ and $F_i(\infty) = 0$ this solution of the $SU(N)$ pure Skyrme model has a topological charge $B = \frac{N}{6}(N^2 - 1)$ and has an energy equal exactly to $\frac{N}{6}(N^2 - 1)$ times the energy of the single Skyrmion solutions. It is easy to show that, if one uses the same ansatz for the sixth order Skyrme model, the profile $f = F_0/2$ satisfies the hedgehog profile equation (40) and the energy of the configuration is given by $E(\lambda) = 4E_0(\lambda)$ where $E_0(\lambda)$ is the energy of the hedgehog solution for the generalised model. These configurations are not exact solutions, except for the $SU(3)$ model.

To consider the most general ansatz, one can derive from (40) the following equations for the profile functions $F_i$, $l = 0, \ldots (N - 2)$.

$$-\frac{2(l + 1)}{N} \sum_{i=0}^{N-2} (i + 1)\tilde{F}_i + 2 \sum_{k=0}^{l} \sum_{i=k}^{N-2} \tilde{F}_i + \frac{(1 - \lambda)}{r^2} \tilde{F}_l(l + 1)(N - l - 1)(1 - \cos F_l) + \tilde{F}_l(l + 1)(N - l - 1)(1 - \cos F_l) + \ldots$$
\[
\frac{2}{r} \left( -\frac{2(l+1)}{N} \sum_{i=0}^{N-2} (i+1) \hat{F}_i + 2 \sum_{k=0}^{l} \left( \sum_{i=k}^{N-2} \hat{F}_i \right) \right) + \frac{(1-\lambda)}{2r^2} \hat{F}_i^2 (l+1)(N-l-1) \sin F_i + \frac{2}{r^2} (l+1)(N-l-1) \sin F_i - \frac{(1-\lambda)}{r^4} (l+1)(N-l-1)^2 (1-\cos F_i) \sin F_i + \frac{(1-\lambda)}{2r^4} (l+1)(N-l-1) \sin F_i \left[ l(N-l)(1-\cos F_{l-1}) + (l+2)(N-l-2)(1-\cos F_{l+1}) \right] \\
\quad + \frac{\lambda}{8r^4} \left\{ 2 \hat{F}_i (l+1)^2 (N-l-1)^2 (1-\cos F_i)^2 - \hat{F}_{l-1} l(l+1)(N-l)(N-l-1) \\
(1-\cos F_{l-1})(1-\cos F_l) - \hat{F}_{l+1} (l+1)(l+2)(N-l-1)(N-l-2) \\
(1-\cos F_l)(1-\cos F_{l+1}) \right\} + \frac{\lambda}{4r^5} \left\{ 2 \hat{F}_i (l+1)^2 (N-l-1)^2 (1-\cos F_i)^2 - \hat{F}_{l-1} \\
l(l+1)(N-l)(N-l-1)(1-\cos F_{l-1})(1-\cos F_l) - \hat{F}_{l+1} (l+1)(l+2)(N-l-1) \\
(N-l-2)(1-\cos F_l)(1-\cos F_{l+1}) \right\} + \frac{\lambda}{8r^4} \left\{ 2 \hat{F}_i^2 (l+1)^2 (N-l-1)^2 (1-\cos F_i) \\
\sin F_i - \hat{F}_{l-1}^2 l(l+1)(N-l)(N-l-1) \sin F_{l-1} (1-\cos F_i) - \hat{F}_{l+1}^2 (l+1)(l+2) \\
(N-l-1)(N-l-2)(1-\cos F_l) \sin F_{l+1} \right\} = 0. \quad (41)
\]

When \( N = 3 \), the solution of the 2 equations lead to exact solutions of the model, while for larger values of \( N \), the ansatz (44) corresponds to low-energy configurations.

We would like to point out at this stage that as proved in [1], the topological charge \( B \) for the configuration (44) is given by

\[
B = \sum_{i=0}^{N-2} \mathcal{D}_k (F_i - \sin F_i)_{r=\infty} \quad (42)
\]

where

\[
\mathcal{D}_k = -i \frac{1}{4\pi^2} \int \frac{|P_k^+ h|^2}{|P_k h|^2} d\xi d\bar{\xi} \quad (43)
\]

takes integer values given by the degree in \( \xi \) of the wedge product \( \mathcal{L} \) of \( h \) and its derivatives

\[
\mathcal{D}_k = \frac{1}{2\pi} \text{deg}(h^{(k)}) \quad h^{(k)} = h \wedge \partial h \wedge ... \wedge \partial^k h \quad k = 0, ..N-1. \quad (44)
\]

Each configuration is thus characterised by the boundary conditions for the profile function \( F_i \) and we can without loss of generality impose the condition \( \lim_{r \to \infty} F_i(r) = 0 \). For the configuration to be well-defined at the origin we must also impose a condition of the type

\[
F_i(0) = n_i 2\pi \quad (45)
\]

where the \( n_i \in N \).
5 Radially symmetric $SU(3)$ Solutions

To describe the solution of the $SU(3)$ model, we use the profile $F = F_0$ and $g = F_1$ and the energy (40) simplifies to

$$E = \frac{1}{6\pi} \int r^2 dr \left\{ \frac{2}{3} (\dot{g}^2 + \dot{F}^2 + \dot{g} \dot{F}) + \frac{1}{r^2} \left( (1 - \cos F)(1 - \lambda) \dot{F}^2 + 4 \right) + (1 - \cos g)((1 - \lambda)\dot{g}^2 + 4) \right. + (1 - \lambda) \frac{2}{r^4} \left( (1 - \cos F)^2 - (1 - \cos F)(1 - \cos g) \right) + \left. (1 - \cos g)^2 \right\}.$$ \hfill (46)

The equations for the profile function $F$ and $g$ are then given by

$$g_{rr} + \frac{1}{2} F_{rr} + \frac{F_r}{r} + 2 \frac{g_r}{r} + \frac{3}{2r^2} \left( (1 - \lambda)(1 - \cos F)g_{rr} + \frac{1}{2} \sin g((1 - \lambda)g_r^2 - 4) \right) + \frac{1}{2} \sin g((1 - \lambda)g_r^2 - 4) + (1 - \lambda) \frac{3}{2r^4} \left( (1 - \cos F) - 2(1 - \cos g) \right) \sin(g) + \frac{3\lambda}{8r^4} (1 - \cos g) \left( 2(\sin gg_r^2 + (1 - \cos g)(g_{rr} - 2\frac{g_r}{r})) \right)$$

$$\sin FF_r^2 - (1 - \cos F)(F_{rr} - 2\frac{F_r}{r}) = 0$$ \hfill (47)

$$F_{rr} + \frac{1}{2} g_{rr} + 2 \frac{F_r}{r} + \frac{g_r}{r} + \frac{3}{4r^2} \left( \sin F((1 - \lambda)F_r^2 - 4) + 2(1 - \lambda)(1 - \cos F)F_{rr} \right) - (1 - \lambda) \frac{3}{2r^4} \left( 2(1 - \cos F) - (1 - \cos g) \right) \sin F + \frac{3\lambda}{8r^4} (1 - \cos F) \left( 12(\sin F F_r^2 + (1 - \cos F)(F_{rr} - 2\frac{F_r}{r})) - \sin g g_r^2 - (1 - \cos g)(g_{rr} - 2\frac{g_r}{r}) \right) = 0.$$ \hfill (48)

The topological charge of the solution now reads

$$B = \frac{1}{\pi} \left( (F - \sin F) \big|_{r=\infty}^{r=0} + (g - \sin(g)) \big|_{r=\infty}^{r=0} \right)$$ \hfill (49)

and if we take the boundary conditions

$$F(0) = n_F 2\pi$$
$$g(0) = n_g 2\pi$$ \hfill (50)

where $n_F$ and $n_g$ are integers, we have $B = 2(n_F + n_g)$. When $n_F$ and $n_g$ are of opposite signs, we can interpret the solutions as a mixture of Skyrmions and anti Skyrmions.
In Table 1, we give the energy of the hedgehog solution ($B = 1$) for the $SU(2)$ model. This solution is an embedded solution of any $SU(N)$ model and it is the solution with the lowest energy. We thus use it as the reference energy for all the other solutions.

In Table 2 we present the properties of the different solutions for the $SU(3)$ models. The first two columns specify the boundary condition of the solution, and the third columns gives the topological charge of that solution. In column 4 and 5 we give the energy of the solutions for the pure Skyrme model and the pure Sk6 model while column 6 and 7 give the corresponding relative energy per Skyrmion, that is the energy divided by the energy of the single Skyrmion and the total number of Skyrmions. For the solutions corresponding to the superposition of Skyrmions and anti-Skyrmion, we define the total number of Skyrmions as the total number of Skyrmions and anti-Skyrmions. Notice that the cases $n_g = 0, n_F = 1$ and $n_g = 1, n_F = 0$ correspond to the same solution modulo an internal rotation.

In Figure 3, we present the energy of the 3 different types of solution as a function of $\lambda$.

![Figure 3: Energy of the $SU(3)$ solution for the boundary conditions (A) $n_F = 0, n_g = 1$, (B) $n_F = 1, n_g = 0$, (C) $n_F = 1, n_g = -1$, (D) $n_F = 1, n_g = 1$.](image)

6 Low Energy $SU(4)$ Configurations

As was shown in the last two sections, the ansatz (14) provides an exact solution of the sixth order model only for the $SU(3)$ model, or when $\lambda = 0$, that is for the usual
| $SU(2)$ | Energy |
|---|---|
| $n_g$ | $B$ | $E(0)$ | $E(1)$ |
| 1 | 1 | 1.2315 | 0.9395 |

Table 1: Topological charge and Energy of the hedgehog $SU(2)$ solution.

| $SU(3)$ | Total Energy | Relative Energy |
|---|---|---|
| $n_F$ | $n_g$ | $B$ | $E(0)$ | $E(1)$ | $E_B(0)/(|B|E_1(0))$ | $E_B(1)/(|B|E_1(1))$ |
| 1 | 1 | 4 | 4.928 | 3.758 | 1 | 1 |
| 1 | 0 | 2 | 2.377 | 1.819 | 0.965 | 0.968 |
| 0 | 1 | 2 | 2.377 | 1.819 | 0.965 | 0.968 |
| 1 | -1 | 2-2 | 3.862 | 3.191 | 0.784 | 0.849 |

Table 2: Topological charge and Energy of some $SU(3)$ solutions.

Skyrme model. For the $SU(N)$ model with $N \geq 4$, the ansatz still produces low-energy configurations. In particular, when $\lambda$ is small, we can expect the ansatz to be very close to an exact solution. In this section we look at some configurations of the $SU(4)$ model. For this model, we have three profile functions $F_0$, $F_1$ and $F_2$ and the energy for the general ansatz (14) is explicitly given by

$$E = \frac{1}{6\pi} \int r^2 dr \left\{ \frac{1}{4} \left( 3\dot{F}_0^2 + 4\dot{F}_1^2 + 3\dot{F}_2^2 + 4\ddot{F}_0 F_1 + 4\ddot{F}_1 F_2 + 2\ddot{F}_0 \ddot{F}_2 \right) + \frac{2}{r^2} [3(1-\cos F_0) + 4(1-\cos F_1) + 3(1-\cos F_2)] + (1-\lambda) \left\{ \frac{1}{2r^2} \left[ 3\dot{F}_0^2 (1-\cos F_0) + 4\dot{F}_1^2 (1-\cos F_1) + 3\dot{F}_2^2 (1-\cos F_2) \right] + \frac{1}{2r^2} \left[ 9(1-\cos F_0)^2 + 16(1-\cos F_1)^2 + 9(1-\cos F_2)^2 \right] - 12(1-\cos F_0)(1-\cos F_1) - 12(1-\cos F_1)(1-\cos F_2) \right\} \right\}$$

$$+ \frac{\lambda}{8r^4} \left\{ 9\dot{F}_0^2 (1-\cos F_0)^2 + 16\dot{F}_1^2 (1-\cos F_1)^2 + 9\dot{F}_2^2 (1-\cos F_2)^2 - 12F_0 F_1 (1-\cos F_0)(1-\cos F_1) - 12F_1 F_2 (1-\cos F_1)(1-\cos F_2) \right\} \right\}$$

From which we can derive the following equations

$$\left( \frac{3\lambda(1-\cos F_0)^2}{2r^4} + \frac{2(1-\lambda)(1-\cos F_0)}{r^2} + 1 \right) \dddot{F}_0 + \left( \frac{2}{3} - \frac{\lambda(1-\cos F_0)(1-\cos F_1)}{r^4} \right) \dddot{F}_1$$

$$+ \frac{1}{3} \dddot{F}_2 - \frac{4 \sin F_0}{r^2} + \frac{6 \dot{F}_0 + 4 \dot{F}_1 + 2 \dot{F}_2}{3r} + \frac{(1-\lambda)\dot{F}_0^2 \sin F_0}{r^2}$$
\begin{align*}
+(1 - \lambda) \frac{\sin F_0}{r^4} (4(1 - \cos F_1) - 6(1 - \cos F_0)) + \lambda \frac{(1 - \cos F_0)}{r^4} \left( \frac{3}{2} \dot{F}_0^2 \sin F_0 - \dot{F}_1^2 \sin F_1 \right) \\
- \lambda \frac{(1 - \cos F_0)}{r^5} \left( 3\dot{F}_0(1 - \cos F_0) - 2 \dot{F}_1(1 - \cos F_1) \right) = 0, 
\end{align*}

\begin{align*}
\left( \frac{1}{2} - \frac{3\lambda(1 - \cos F_0)(1 - \cos F_1)}{4r^4} \right) \ddot{F}_0 + \left( 1 + \frac{2\lambda(1 - \cos F_1)^2}{r^4} + \frac{2(1 - \lambda)(1 - \cos F_1)}{r^2} \right) \ddot{F}_1 \\
+ \left( \frac{1}{2} - \frac{3\lambda(1 - \cos F_1)(1 - \cos F_2)}{4r^4} \right) \ddot{F}_2 + \left( \frac{(1 - \lambda)\dot{F}_1^2 \sin F_1}{r^2} + \dot{F}_0 + 2 \dot{F}_1 + \dot{F}_2 \right) - 4 \frac{\sin F_1}{r^2} \\
+(1 - \lambda) \frac{\sin F_1}{r^4} \left( 3(1 - \cos F_0) + 3(1 - \cos F_2) - 8(1 - \cos F_1) \right) \\
- \frac{\lambda}{r^7}(1 - \cos F_1) \left( 4 \dot{F}_1(1 - \cos F_1) - \frac{3}{2} \dot{F}_0(1 - \cos F_0) - \frac{3}{2} \dot{F}_2(1 - \cos F_2) \right) \\
+ \frac{\lambda}{r^7}(1 - \cos F_1) \left( 2 \dot{F}_1^2 \sin F_1 - \frac{3}{4} \dot{F}_0^2 \sin F_0 - \frac{3}{4} \dot{F}_2^2 \sin F_2 \right) = 0 
\end{align*}

and

\begin{align*}
\left( \frac{2}{3} - \frac{\lambda(1 - \cos F_1)(1 - \cos F_2)}{r^4} \right) \ddot{F}_1 + \left( \frac{3\lambda(1 - \cos F_2)^2}{2r^4} + \frac{2(1 - \lambda)(1 - \cos F_2)}{r^2} + 1 \right) \ddot{F}_2 \\
+ \frac{1}{3} \dddot{F}_0 + \frac{2\ddot{F}_0 + 4\dddot{F}_1 + 6 \dddot{F}_2}{3r} - 4 \frac{\sin F_2}{r^2} + \left( \frac{(1 - \lambda)\ddot{F}_1^2 \sin F_2}{r^2} + (1 - \lambda) \frac{\sin F_2}{r^4} \right) (4(1 - \cos F_1) \\
- 6(1 - \cos F_2)) - \frac{\lambda(1 - \cos F_2)}{r^5} \left( 3 \ddot{F}_2(1 - \cos F_2) - 2 \ddot{F}_1(1 - \cos F_1) \right) \\
+ \lambda \frac{(1 - \cos F_2)}{r^4} \left( \frac{3}{2} \ddot{F}_2^2 \sin F_2 - \ddot{F}_1^2 \sin F_1 \right) = 0. 
\end{align*}

Describing the boundary condition for the profile functions as before, \( F_i(0) = n_i 2\pi \), the topological charge is given by

\begin{equation}
B = 3n_0 + 4n_1 + 3n_2. 
\end{equation}

In Table 3 we present the energy values of various types of configurations when \( \lambda = 0 \) and \( \lambda = 1 \). We notice that when \( \lambda = 0 \), the solutions are symmetric under the exchange \( f_0 \leftrightarrow f_2 \), but that the sixth order term breaks the symmetry. This results in a difference of energy between the configuration with \( n_0 = 0, n_1 = 0, n_2 = 1 \) and \( n_0 = 1, n_1 = 0, n_2 = 0 \) as well as between the configurations with \( n_0 = 1, n_1 = 1, n_2 = 0 \) and \( n_0 = 0, n_1 = 1, n_2 = 1 \). In Figure 4, we present the curve for the energy of the configurations as a function of \( \lambda \).
| SU(4) | Total Energy | Relative Energy |
|-------|--------------|-----------------|
|       | E(0)         | E(1)            | E_B(0)/(|B|E₁(0)) | E_B(1)/(|B|E₁(1)) |
| n₀  | n₁  | n₂  | B   |       |                 |                 |                 |
| 0   | 0   | 1   | 3   | 3.51739 | 2.66653 | 0.95210 | 0.94598 |
| 1   | 0   | 0   | 3   | 3.51739 | 2.72915 | 0.95210 | 0.96819 |
| 0   | 1   | 0   | 4   | 4.78807 | 6.33322 | 0.97204 | 1.68507 |
| 1   | 0   | 1   | 6   | 7.22464 | 6.04604 | 0.97780 | 1.07244 |
| 1   | 1   | 0   | 7   | 8.45219 | 6.62998 | 0.98052 | 1.00802 |
| 0   | 1   | 1   | 7   | 8.45219 | 7.28058 | 0.98052 | 1.10694 |
| 1   | 1   | 1   | 10  | 12.311  | 9.39605 | 1         | 1         |

Table 3: Topological charge and Energy of some SU(4) configurations.

Figure 4: Energy density of the SU(4) multi-projector ansatz (a) n₀ = 0, n₁ = 0, n₂ = 1; (b) n₀ = 1, n₁ = 0, n₂ = 0; (c) n₀ = 0, n₁ = 1, n₂ = 0; (d) n₀ = 1, n₁ = 0, n₂ = 1; (e) n₀ = 1, n₁ = 1, n₂ = 0; (f) n₀ = 0, n₁ = 1, n₂ = 1.
After inserting the ansatz (5) in the full equation for the $SU(N)$ model, we found that we had only two independent profile functions $g_0$ and $g_1$ and that the ansatz would only provide solutions for the $SU(3)$ model. One can nevertheless use the $SU(N)$ ansatz to compute low energy configurations. For example if we consider the reduced ansatz defined by (5) together with the constraint $g_i = g_{i+2}$ and define the profiles $F = g_0 - g_1$ and $g = g_{N-2}$ we can minimise the energy (30) and solve the equations for $F$ and $g$ for various boundary conditions. We found that to get configurations corresponding to a bound state, i.e. a configuration with an energy per Skyrmion smaller than the energy of the hedgehog solution, we must take $n_F = 0$ and $n_g = 1$. The energies that we found are given in Table 3.

| Model | Total Energy | Relative Energy |
|-------|--------------|-----------------|
|       | $B$ | $E(0)$ | $E(1)$ | $E_B(0)/(|B|E_1(0))$ | $E_B(1)/(|B|E_1(1))$ |
| $SU(3)$ | 2 | 2.377 | 1.819 | 0.965 | 0.968 |
| $SU(4)$ | 3 | 3.624 | 2.759 | 0.981 | 0.979 |
| $SU(5)$ | 4 | 4.811 | 3.632 | 0.977 | 0.966 |
| $SU(6)$ | 5 | 6.015 | 4.518 | 0.977 | 0.962 |

Table 4: Topological charge and energy for the reduced ansatz with $n_F = 0$ and $n_g = 1$.

In Figures 5 and 6 we present the profile and the energy density for different values of $N$ and for $\lambda = 0.5$. It shows that the energy density has the shape of a hollow sphere of radius $r = 0.7\sqrt{N}$. The profile $g$ has the same shape for all values of $N$ but is shifted to the right as $N$ increases. The profile $F$ on the other hand is also shifted as the shell radius increases, but its amplitude decreases like $1/N^2$; note that in Figure 6, the profile for $N = 100$ and $N = 200$ have been multiplied by 100 to make them visible. For other values of $\lambda$ the graphics look very much the same except that the shell radius and width are slightly different, but the conclusions remain the same.

Figure 6.b suggests to simplify the ansatz further for large $N$ by taking $F(r) = 0$. This implies that $g_i = g \forall i$ and the multi-projector ansatz (5) becomes

$$U = \exp \left( -ig(P_{N-1} - I/N) \right)$$

(56)
Figure 5: Energy density of the multi-projector solution with $n_F = 0$, $n_g = 1$, $\lambda = 0.5$. (A) $N=10$, (B) $N=20$, (C) $N=50$, (D) $N=100$, (E) $N=200$.

Figure 6: Profile (a) $g$ and (b) $F$ of the multi-projector solution with $n_F = 0$, $n_g = 1$, $\lambda = 0.5$. (A) $F$ for $N=10$, (B) $F$ for $N=20$, (C) $F$ for $N=50$, (D) $100 \times F$ for $N=100$, (E) $100 \times F$ for $N=200$. 
where \( P_{N-1} \) can also be written as
\[
P_{N-1} = \frac{\tilde{h}\tilde{h}^\dagger}{|h|^2} \tag{57}
\]
where \( \tilde{h} \) is equal, up to a unitary rotation, to the complex conjugate of the holomorphic vector \( V_0 \) defined in (20-21): \( \tilde{h} = AV_0 \) for some \( A \in SU(N) \) with \( \partial_\xi A = \partial_{\bar{\xi}} A = 0 \). This is shown by using the fact that \( P_{N-1} \) is an anti-holomorphic projector [16] and that solving (39) recursively we have
\[
|V_k|^2 = \frac{k!(N-1)!}{(N-1-k)!} |1 + |\xi|^2|^{N-1-2k} \tag{58}
\]
and so \( |V_{N-1}|^2 = (N-1)! |1 + |\xi|^2|^{1-N} \). Knowing that up to an overall coefficient \( |V_{N-1}|^2 \) is a polynomial in \( \bar{\xi} \) of degree \( N-1 \), we can conclude that up to a unitary iso-rotation, \( V_{N-1} \) is equal to the complex conjugate of \( V_0 \).

The topological charge of the anti-holomorphic projector \( P_{N-1} \) is equal to \( 1 - N \) and as the profile function is \(-g\), the baryon number for this configuration is \( N - 1 \). The ansatz (56) is not a solution, but its energy
\[
E = \frac{1}{6\pi} \int r^2 dr \left\{ \frac{N-1}{N} \tilde{g}^2 + \frac{1}{2r^2} + (N-1)(1 - \cos g)((1 - \lambda)\tilde{g}^2 + 4)
\right.
\]
\[
+ \frac{1}{2r^4}(N-1)^2(1 - \cos g)^2 \left( (1 - \lambda) + \frac{\lambda}{4r^2}\tilde{g}^2 \right) \right\} \tag{59}
\]
can easily be computed by solving the equation
\[
2g_{rr} + 4\frac{g_r}{r} + \frac{N}{r^2} \left( (1 - \lambda)(1 - \cos g)g_{rr} + \frac{1}{2} \sin g((1 - \lambda)g_r^2 - 4) \right)
\]
\[
+ \frac{\lambda}{4r^4}N(N-1)(1 - \cos g)(\sin gg_r^2 + (1 - \cos g)(g_{rr} - 2\frac{g_r}{r})) = 0. \tag{60}
\]

In Figure 7, we present the relative energy, \( E(\lambda)/(E_{B=1}(\lambda)(N-1)) \), of this configuration as a function of \( N \) for different values of \( \lambda \). We see that this configuration corresponds to a bound state of Skyrmions and that the energy per Skyrmion decreases with \( N \). The energy of this configuration corresponds to an upper bound for the energy of the \( B = N-1 \) radially symmetric solution of the \( SU(N) \) model and these configurations correspond to bound states of Skyrmions for all values of \( N \) and all values of \( \lambda \). As every \( SU(p) \) solution can be trivially embedded in an \( SU(q) \) solution when \( p \leq q \) we can claim that for every \( B < N \) the \( SU(N) \) model has a radially symmetric solution of charge \( B \) corresponding to a bound state. With the exception of the hedgehog solutions, these solutions are expected to be unstable when the radial symmetry is broken as their energies are larger than the known \( SU(2) \) solutions [14].
Figure 7: Energy $E/(E_{B=1}(N-1))$ of the $SU(N)$, configuration (56) for (a) $\lambda = 0$, (b) $\lambda = 0.25$, (c) $\lambda = 0.5$, (d) $\lambda = 0.75$, (e) $\lambda = 1$

8 Conclusions

In this paper we have shown how to construct some radially symmetric solutions of the $SU(3)$ sixth order Skyrme model. The construction is similar to the one used for the pure Skyrme model in \[1\] except that, because of an extra constraint, the construction only works for the $SU(3)$ model. The same ansatz can nevertheless be used to compute low-energy configurations of the $SU(N)$ model. In particular we showed that for every $N$ there is a radially symmetric solution of charge $B < N$ which corresponds to a bound state of Skyrmion.

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