Tight contact structures on some bounded Seifert manifolds with minimal convex boundary

Fan Ding, Youlin Li and Qiang Zhang

Abstract: We classify positive tight contact structures, up to isotopy fixing the boundary, on the manifolds \( N = M(D^2; r_1, r_2) \) with minimal convex boundary of slope \( s \) and Giroux torsion 0 along \( \partial N \), where \( r_1, r_2 \in (0, 1) \cap \mathbb{Q} \), in the following cases:

1. \( s \in (-\infty, 0) \cup [2, +\infty) \);
2. \( s \in [0, 1) \) and \( r_1, r_2 \in [1/2, 1) \);
3. \( s \in [1, 2) \) and \( r_1, r_2 \in (0, 1/2) \);
4. \( s = \infty \) and \( r_1 = r_2 = 1/2 \).

We also classify positive tight contact structures, up to isotopy fixing the boundary, on \( M(D^2; 1/2, 1/2) \) with minimal convex boundary of arbitrary slope and Giroux torsion greater than 0 along the boundary.

Keywords: contact structure; bounded Seifert manifolds

2010 Mathematical Subject Classification: 57M50; 53D10.

1 Introduction

If \( M \) is an oriented 3-manifold, a contact structure on \( M \) is a completely non-integrable 2-plane distribution \( \xi \) given as the kernel of a global 1-form \( \alpha \) such that \( \alpha \wedge d\alpha \neq 0 \) at every point of \( M \). Throughout this paper, we assume the contact structures are positive, i.e., given by one form \( \alpha \) satisfying \( \alpha \wedge d\alpha > 0 \), and oriented.

Classification of tight contact structures on oriented 3-manifolds is a fundamental problem in contact topology. See [1], [7], [8], [9] and [10]. The classification of tight contact structures on small Seifert manifolds has been the object of intensive study in the last few years. See [14], [3], [4] and [2]. In [9], Honda classified tight contact structures on the solid torus \( S^1 \times D^2 \) and the thickened torus \( T^2 \times I \). In [13], Tanya classified tight contact structures on \( \Sigma_2 \times I \) where the boundary condition is specified by a single, nontrivial separating dividing curve on each boundary component. In this article, we classify tight contact structures on some bounded Seifert manifolds.

Let \( N \) be a small bounded Seifert manifold \( M(D^2; r_1, r_2) \), where \( r_i \in (0, 1) \cap \mathbb{Q}, i = 1, 2 \). We concentrate on tight contact structures on \( N \) with minimal convex boundary, i.e., the number of dividing curves on \( \partial N \) is 2. Suppose \( s \in \mathbb{Q} \). Denote the greatest integer not greater than \( s \) by \( \lfloor s \rfloor \). Let \( s - \lfloor s \rfloor = \frac{b}{a} \), where \( a > b \geq 0 \) are integers and \( g.c.d.(a, b) = 1 \). If \( \frac{1}{1-s} \) is not an integer, then write \( \frac{1}{1-s} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{a_4 - \cdots}}} \), where \( a_j \)'s are integers and
Theorem 1.1 The number of tight contact structures on N with minimal convex boundary of slope s and Giroux torsion 0 along ∂N, up to isotopy fixing the boundary, is the number of tight contact structures, up to isotopy, on the small Seifert manifold M(−1−[s]; r_1, r_2, r_3) in the following cases:

1. s ∈ (−∞, 0);
2. s ∈ [0, 1) and r_1, r_2 ∈ [1, 1);
3. s ∈ [1, 2) and r_1, r_2 ∈ (1, 2);
4. s ∈ [2, +∞).

The idea of the proof of Theorem 1.1 is roughly as follows. Similar to the arguments in [13], [3], [4], we get an upper bound for the number of tight contact structures, up to isotopy fixing the boundary, on N with given conditions. This upper bound is the same as the number of tight contact structures, up to isotopy, on M(−1−[s]; r_1, r_2, r_3). For any tight contact structure η on M(−1−[s]; r_1, r_2, r_3), we can decompose M(−1−[s]; r_1, r_2, r_3) into N and a solid torus V_3, and isotope η so that ∂N is minimal convex with dividing curves of slope s. When measured in the coordinates of ∂V_3, this slope is −1. Thus, by the uniqueness of tight contact structures, up to isotopy fixing the boundary, on a solid torus with minimal convex boundary of slope −1, we conclude that the number of tight contact structures, up to isotopy fixing the boundary, on N with given conditions.

Using the fact that a double cover of M(D^2; 1, 1) is the thickened torus T^2 × I and the classification of tight contact structures on T^2 × I, we have the following classification.

Theorem 1.2 (1) We can divide the set of tight contact structures on M(D^2; 1, 1) with minimal convex boundary of slope ∞ and Giroux torsion 0 along the boundary, up to isotopy fixing the boundary, into two subsets. The tight contact structures in one subset are in 1-1 correspondence with Z. The other subset contains two elements.

(2) For any integer t > 0 and any number s ∈ Q ∪ {∞}, there are exactly 2 tight contact structures on M(D^2; 1, 1) with minimal convex boundary of slope s and Giroux torsion t along the boundary, up to isotopy fixing the boundary.

In Section 2, we give some preliminaries. In Section 3, we prove Theorem 1.1 in cases 1 and 2, and in Section 4, we prove Theorem 1.1 in cases 3 and 4. In Section 5, we prove Theorem 1.2. The reader is assumed to be familiar with convex surfaces theory (cf. [6], [5]) and bypasses (cf. [9]).
2 Preliminaries

For $r_1, r_2, r_3 \in \mathbb{Q} \setminus \mathbb{Z}$, the Seifert manifolds $M(D^2; r_1, r_2)$ and $M(r_1, r_2, r_3)$ are described as follows. Let $\Sigma$ be an oriented pair of pants, and identify each connected component of

$$-(\partial \Sigma \times S^1) = T_1 \cup T_2 \cup T_3$$

with $\mathbb{R}^2/\mathbb{Z}^2$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ gives the direction of $-\partial(\Sigma \times \{1\})$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ gives the direction of the $S^1$ factor. For $i = 1, 2, 3$, let $V_i = D^2 \times S^1$, and identify $\partial V_i$ with $\mathbb{R}^2/\mathbb{Z}^2$ so that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ gives the direction of the meridian $\partial(D^2 \times \{1\})$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ gives the direction of the $S^1$ factor. Then $M(D^2; r_1, r_2)$ (respectively, $M(r_1, r_2, r_3)$) is obtained from $\Sigma \times S^1$ by gluing $V_i$ to $T_i$, $i = 1, 2$ (respectively, $i = 1, 2, 3$), using the map $\varphi_i : \partial V_i \to T_i$ defined by the matrix

$$\varphi_i = \begin{pmatrix} p_i & u_i \\ -q_i & -v_i \end{pmatrix},$$

where $\frac{q_i}{p_i} = r_i$, $u_i q_i - p_i v_i = 1$, and $0 < u_i < p_i$.

Note that if $r_3 = n + \frac{1}{a_1-a_2-\ldots-a_{m-1}+a_m}$, where $m \geq 2$, $n$ and $a_j$'s are integers, $a_j \geq 2$ for $1 \leq j < m$ and $a_m \geq 1$, then

$$\varphi_3 = \begin{pmatrix} p_3 & u_3 \\ -q_3 & -v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ -1 & 0 \end{pmatrix} \ldots \begin{pmatrix} a_{m-1} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a_m & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -a & 1 \\ a-b & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} a_1 & 1 \\ -1 & 0 \end{pmatrix} \ldots \begin{pmatrix} a_{m-1} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a_m+1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix} = \begin{pmatrix} -a & 1 \\ a-b & 1 \end{pmatrix},$$

where $a > b \geq 0$ are integers so that $g.c.d.(a, b) = 1$ and $n = 1 - \frac{1}{a_1-a_2-\ldots-a_{m-1}+a_m}$. Thus we have

**Proposition 2.1** In the notations above, $\varphi_3 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -a \\ na + a - b \end{pmatrix}$. \[\square\]

Let $n_1, n_2$ be integers. The Seifert manifolds $M(D^2; r_1, r_2)$ and $M(D^2; r_1+n_1, r_2+n_2)$ are orientation-preserving diffeomorphic. This can be seen as follows. Let $f : \Sigma \times S^1 \to \Sigma \times S^1$ be an orientation-preserving diffeomorphism such that $f$ sends each $T_i$ to itself and on each $T_i$, $f$ is given by the matrix $f_i = \begin{pmatrix} 1 & 0 \\ -n_i & 1 \end{pmatrix}$, where $n_3 = -n_1-n_2$. ($f$ can be constructed by using a smooth function $g : \Sigma \to SO(2)$ such that for $x \in \Sigma$, $z \in S^1$, $f(x, z) = (x, g(x)z)$.) $f$ can be extended to an orientation-preserving diffeomorphism, still denoted by $f$, from $M(D^2; r_1, r_2)$ to $M(D^2; r_1+n_1, r_2+n_2)$. Since $f_3 = \begin{pmatrix} 1 & 0 \\ n_1+n_2 & 1 \end{pmatrix}$, we have
Proposition 2.2 Under $f$, a simple closed curve of slope $s$ in $T_3$ of $M(D^2; r_1, r_2)$ changes to a simple closed curve of slope $s + n_1 + n_2$ in $T_3$ of $M(D^2; r_1 + n_1, r_2 + n_2)$.

Similarly, the Seifert manifolds $M(r_1, r_2, r_3)$ and $M(r_1 + n_1, r_2 + n_2, r_3 - n_1 - n_2)$ are orientation-preserving diffeomorphic. They are also denoted by $M(e_0; r_1 - [r_1], r_2 - [r_2], r_3 - [r_3])$, where $e_0 = [r_1] + [r_2] + [r_3]$.

On $T^2 \times [0, 1] \cong \mathbb{R}^2 / \mathbb{Z}^2 \times [0, 1]$ with coordinates $((x, y), t)$, consider $\xi_n = \ker(\sin(\pi nt)dx + \cos(\pi nt)dy)$, with the boundary adjusted so it becomes convex with two dividing curves on each component, where $n \in \mathbb{Z}^+$. Let $(M, \xi)$ be a contact 3-manifold and $T \subset M$ an embedded torus. The Giroux torsion along $T$ is the supremum, over $n \in \mathbb{Z}^+$, for which there exists a contact embedding $\phi : (T^2 \times [0, 1], \xi_n) \hookrightarrow (M, \xi)$, where $\phi(T^2 \times \{t\})$ is isotopic to $T$. (We set the Giroux torsion to be 0 if there is no such embedding). One can consult [11] for this definition.

The main invariant in the classification of tight contact structures on Seifert manifolds is the maximal twisting number. One can consult [2] for the definition.

For the rest of the paper, $r_1, r_2 \in (0, 1) \cap \mathbb{Q}$, $s \in \mathbb{Q}$, $a, b, a_j (j = 1, \ldots, m)$ and $r_3$ are defined as in the Introduction. For $i = 1, 2$, suppose $-\frac{1}{r_i} = a_i^j - \frac{1}{a_i^{j+1}} - \frac{1}{a_i^{j+2}}$, where $a_i^j$'s are integers and $a_i^j \leq -2$ for $j \geq 0$. When we consider the number of tight contact structures up to isotopy or up to isotopy fixing the boundary, we usually omit the phrase “up to isotopy” or “up to isotopy fixing the boundary”.

3 Proof of Theorem 1.1 in cases 1 and 2

For $i = 1, 2$, let $\varphi_i = \begin{pmatrix} p_i & u_i \\ -q_i & -v_i \end{pmatrix}$, where $\frac{u_i}{p_i} = r_i$, $u_i q_i - p_i v_i = 1$ and $0 < u_i < p_i$.

Let $\xi$ be a tight contact structure on $N = M(D^2; r_1, r_2)$ with minimal convex boundary of slope $s(T_3)$ and Giroux torsion 0 along $\partial N$. We first isotope $\xi$ to make each $V_i$ ($i = 1, 2$) a standard neighborhood of a Legendrian circle isotopic to the $i$th singular fiber with twisting number $t_i < 0$, i.e., $\partial V_i$ is convex with two dividing curves each of which has slope $\frac{1}{t_i}$ when measured in the coordinates of $\partial V_i$. Then, when measured in the coordinates of $T_i$, the slope $s_i = -\frac{u_i}{p_i} + \frac{1}{p_i(t_i p_i + u_i)} < -\frac{u_i}{p_i}$.

The proof of the following lemma is similar to the proof of [14] Lemma 2.2.

Lemma 3.1 On $M(D^2; r_1, r_2)$, if $s(T_3) \leq \max\{r_1, r_2\}$, or if $0 < s(T_3) < 1$ and $r_i \geq \frac{1}{2}$ ($i = 1, 2$), then any tight contact structure with minimal convex boundary of slope $s(T_3)$ admits a vertical Legendrian circle $L$ with twisting number 0.

Now suppose $s(T_3) < 0$. Using the vertical Legendrian circle $L$, we can thicken $V_i$ ($i = 1, 2$) to $V_i'$ such that $V_i''$'s are pairwise disjoint, and $T_i' = \partial V_i'$ is a minimal convex torus.
with vertical dividing curves when measured in coordinates of \( T_i \). Also, we can thicken \( T_3 \) to \( L_3 = T_3 \times [0, 1] \) such that \( T_3 \times \{0\} = T_3 \) and \( T_3 \times \{1\} = T_3' \) is a minimal convex torus with vertical dividing curves when measured in the coordinates of \( T_3 \). Choose \( t_i \ll -1 \) so that \(-\infty < \frac{1}{a_5+1} < s_i \) for \( i = 1, 2 \). By \([9]\) Proposition 4.16, for \( i = 1, 2 \), there exists a minimal convex torus \( T_i'' \) in the interior of \( V_i'' \setminus V_i \) isotopic to \( T_i \) that has dividing curves of slope \( \frac{1}{a_5+1} \).

Let \( V''_i \) be the solid torus bounded by \( T''_i \), and \( \Sigma'' \times S^1 = N \setminus (V''_1 \cup V''_2) \).

First we consider \( V''_1 \) and \( V''_2 \). Since \( \varphi_i^{-1} \left( \begin{array}{c} a_0^i + 1 \\ 1 \end{array} \right) = \left( \begin{array}{c} -(a_0^i + 1)v_i - u_i \\ (a_0^i + 1)q_i + p_i \end{array} \right) \) (here \( u_i, v_i \) correspond respectively to \(-u_i, -v_i\) in the proof of \([14]\) Theorem 1.6), the dividing curves of \( T''_i \) (\( i = 1, 2 \)) have slope \( \frac{-(a_0^i + 1)v_i - u_i}{(a_0^i + 1)q_i + p_i} \) when measured in the coordinates of \( \partial V_i \). By a similar argument as in the proof of \([14]\) Theorem 1.6, there are exactly \( \prod_{j=1}^{\ell_i} |a_j^i + 1| \) tight contact structures on \( V''_1 \) that satisfy the given boundary condition.

Then we consider \( N \setminus (V''_1 \cup V''_2) = \Sigma'' \times S^1 \). Let \( L_i \) (\( i = 1, 2 \)) be the thickened torus which is bounded by \( T''_i \) and \( T''_i \), then \( L_i \) has boundary slopes \( \infty \) and \( \frac{1}{a_5+1} \). By \([9]\) Theorem 2.2, there are exactly \( |a_0^i| \) minimally twisting tight contact structures on \( L_i \) that satisfy the given boundary condition. The two boundary slopes of the thickened torus \( L_3 \) are \( \infty \) and \( s(T_3) \) respectively.

**Case 1.** \( s \in (\infty, 0) \).

We divide it into two subcases.

**Case 1(a).** \( s \in (\infty, -1) \).

Let \( s(T_3) = s \). We decompose \( L_3 \) into \( m \) continued fraction blocks (some blocks may be invariant neighborhoods of convex tori). The first continued fraction block has two boundary slopes \( \infty \) and \( [s] + 1 - \frac{1}{a_1-1} \), the second continued fraction block has two boundary slopes \( [s] + 1 - \frac{1}{a_1-1} \) and \( [s] + 1 - \frac{1}{a_2-1} \), ..., the \( m \)th continued fraction block has two boundary slopes \( [s] + 1 - \frac{1}{a_1-1} \) and \( [s] + 1 - \frac{1}{a_2-1} \) and \( s(T_3) = s = [s] + 1 - \frac{1}{a_{m-1}-1} \). By shuffling, there are at most \( a_1(a_2 - 1) \ldots (a_{m-1} - 1)a_m \) minimally twisting tight contact structures on \( L_3 \). By a similar argument as in the first paragraph of \([14]\) page 241, the upper bound of the number of tight contact structures on \( N \) with minimal convex boundary of slope \( s \) and Giroux torsion 0 along \( \partial N \) is \( |a_0^i| \prod_{j=1}^{\ell_i} (a_j^i+1)a_1(a_2 - 1) \ldots (a_{m-1} - 1)a_m \).

Consider the closed Seifert manifold \( M(r_1, r_2, -1 - [s] + r_3) \). Since \(-1 - [s] > 0 \), by \([14]\) Theorem 1.6, it admits \( |a_0^i| \prod_{j=1}^{\ell_i} (a_j^i+1)a_1(a_2 - 1) \ldots (a_{m-1} - 1)a_m \) tight contact structures. By \([14]\) Theorem 1.3, for any tight contact structure \( \eta \) on \( M(r_1, r_2, -1 - [s] + r_3) \), there is a vertical Legendrian circle with twisting number 0. We isotope \( \eta \) so that there is a vertical Legendrian circle \( L \) with twist number 0 in the interior of \( \Sigma \times S^1 \), and \( V_3 \) is a standard neighborhood of a Legendrian circle isotopic to the 3rd singular fiber with twisting number \( t \) \( \leq 0 \), i.e., \( \partial V_3 \) is convex with two dividing curves each of which has slope \( \frac{1}{t} \) when measured in the coordinates of \( \partial V_3 \). Let \( \varphi_3 = \begin{pmatrix} p_3 & u_3 \\ -q_3 & v_3 \end{pmatrix} \), where \( \frac{q_3}{p_3} = -1 - [s] + r_3 \), \( u_3q_3 - p_3v_3 = 1 \) and
0 \leq u_3 < p_3$. Then, when measured in the coordinates of $T_3$, the slope $s_3 = -\frac{q_3}{p_3} + \frac{1}{p_3(t_3 p_3 + u_3)}$.

Using $L$, we can thicken $V_3$ to $V'_3$, such that $T'_3 = \partial V'_3$ is a minimal convex torus with vertical dividing curves when measured in the coordinates of $T_3$. Since $1 - \frac{b}{a} = \frac{1}{a_1 - a_2 - \ldots - a_{m-1} - \frac{1}{a_m}} > \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \ldots - \frac{1}{a_{m-1} - \frac{1}{a_m}}}}} = r_3$, we have $-\infty < s = [s] + \frac{b}{a} < [s] + 1 - r_3$. Thus $-\infty < s < s_3$ for sufficiently small $t$. By [9, Proposition 4.16], there exists a minimal convex torus $T''_3$ in the interior of $V'_3 \setminus V_3$ isotopic to $T_3$ that has dividing curves of slope $s$. Thus we can isotope $\eta$ so that $T_3$ is minimal convex with dividing curves of slope $s$ when measured in the coordinates of $T_3$. Note that $M(r_1, r_2, -[s] + r_3)$ has a decomposition $N \cup \varphi_3 V_3$. By Proposition 2.1, $\varphi_3^{-1} \left( \begin{array}{c} -a \\ -[s]a - b \end{array} \right) = \left( \begin{array}{c} -1 \\ 1 \end{array} \right)$. Thus the slope of the dividing curves on $\partial V_3$ is $-1$ when measured in the coordinates of $\partial V_3$. There is exactly one tight contact structure on $V_3$ with minimal convex boundary of slope $-1$. Note that $\eta$, when restricted to $N$, has Giroux torsion $0$ along $\partial N$. Hence the number of tight contact structures on $N$ with given conditions is at least $|a_0^3a_0^2 \prod_{i=1}^2 \prod_{j=1}^{l_i} (a_j^i + 1)|a_1(a_2 - 1) \ldots (a_{m-1} - 1)a_m$.

Therefore, there are exactly $|a_0^3a_0^2 \prod_{i=1}^2 \prod_{j=1}^{l_i} (a_j^i + 1)|a_1(a_2 - 1) \ldots (a_{m-1} - 1)a_m$ tight contact structures on $N$ with minimal convex boundary of slope $s$ and Giroux torsion $0$ along $\partial N$.

**Case 1(b).** $s \in (-1, 0)$.

Let $s(T_3) = s$. Note that the outermost continued fraction block of $L_3$ has two boundary slopes $\infty$ and $-\frac{1}{a_1 - 1}$, and hence contains $a_1 - 1$ basic slices. By a similar argument as in the proof of [3, Theorem 2.4], there are at most $|a_0^3a_0^2a_1 - (a_0^3 + 1)(a_0^2 + 1)(a_1 - 1)(a_2 - 1) \ldots (a_{m-1} - 1)a_m \prod_{i=1}^2 \prod_{j=1}^{l_i} |a_j^i + 1|$ tight contact structures on $N$ with minimal convex boundary of slope $s$ and Giroux torsion $0$ along $\partial N$.

Consider the small Seifert manifold $M(r_1, r_2, r_3)$. By [3, Theorem 1.1], it admits exactly $|a_0^3a_0^2a_1 - (a_0^3 + 1)(a_0^2 + 1)(a_1 - 1)(a_2 - 1) \ldots (a_{m-1} - 1)a_m \prod_{i=1}^2 \prod_{j=1}^{l_i} |a_j^i + 1|$ tight contact structures. Let $\varphi_3 = \left( \begin{array}{cc} p_3 & u_3 \\ -q_3 & -v_3 \end{array} \right)$, where $\frac{p_3}{q_3} = r_3$, $u_3q_3 - p_3v_3 = 1$ and $0 < u_3 < p_3$. Note that $M(r_1, r_2, r_3)$ has a decomposition $N \cup \varphi_3 V_3$. By a similar argument as in Case 1(a), for any tight contact structure $\eta$ on $M(r_1, r_2, r_3)$, we can isotope $\eta$ so that $T_3$ is minimal convex with dividing curves of slope $s$ when measured in the coordinates of $T_3$. By Proposition 2.1, $\varphi_3^{-1} \left( \begin{array}{c} -a \\ a - b \end{array} \right) = \left( \begin{array}{c} -1 \\ 1 \end{array} \right)$. Thus the slope of the dividing curves on $\partial V_3$ is $-1$ when measured in the coordinates of $\partial V_3$. Similar to Case 1(a), we conclude that the number of tight contact structures on $N$ with given conditions is at least $|a_0^3a_0^2a_1 - (a_0^3 + 1)(a_0^2 + 1)(a_1 - 1)(a_2 - 1) \ldots (a_{m-1} - 1)a_m \prod_{i=1}^2 \prod_{j=1}^{l_i} |a_j^i + 1|$.

Therefore, there are exactly $|a_0^3a_0^2a_1 - (a_0^3 + 1)(a_0^2 + 1)(a_1 - 1)(a_2 - 1) \ldots (a_{m-1} - 1)a_m \prod_{i=1}^2 \prod_{j=1}^{l_i} |a_j^i + 1|$ tight contact structures on $N$ with the given boundary condition and Giroux torsion $0$ along $\partial N$.

**Case 2.** $s \in [0, 1)$ and $r_1, r_2 \in \left( \frac{4}{3}, 1 \right)$.  

By Lemma 3.1, any tight contact structure on $M(D^2; r_1, r_2)$ with minimal convex boundary of slope $s$ contains a Legendrian vertical circle with twisting number 0.

By Proposition 2.2, a tight contact structure on $M(D^2; r_1, r_2)$ with minimal convex boundary of slope $s$ corresponds to a tight contact structure on $M(D^2; -1 + r_1, r_2)$ with minimal convex boundary of slope $s - 1$. We consider tight contact structures on $M(D^2; -1 + r_1, r_2)$ with minimal convex boundary of slope $s - 1$. Without loss of generality, assume that $r_1 \geq r_2$.

Suppose $\xi_0$ is a tight contact structure on $N_0 = M(D^2; -1 + r_1, r_2)$ with minimal convex boundary of slope $s - 1$ and Giroux torsion 0 along $\partial N_0$. Using a vertical Legendrian circle with twisting number 0, we can thicken standard neighborhoods of two Legendrian singular fibers to $U_1$ and $U_2$ such that the slopes of the dividing curves on $\partial U_1$ and $\partial U_2$ are $\infty$ when measured in the coordinates of $T_1$ and $T_2$, respectively. We can thicken $T_3$ to a thickened torus $L_3$ so that the slope of the other boundary component of $L_3$ is $\infty$.

Consider the closed Seifert manifold $M(-1 + r_1, r_2, r_3)$. Let $\varphi_3 = \begin{pmatrix} p_3 & u_3 \\ -q_3 & -v_3 \end{pmatrix}$, where $\frac{q_3}{p_3} = r_3, u_3q_3 - p_3v_3 = 1$ and $0 < u_3 < p_3$. $M(r_1, r_2, r_3)$ has a decomposition $N \cup_{\varphi_3} V_3$. By Proposition 2.1, $\varphi_3^{-1} \begin{pmatrix} -a \\ a - b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Thus we can find layers $N_j$ of $M(-1 + r_1, r_2, r_3)$ in [4] page 1432] in $(N_0, \xi_0)$. Note that $L_3$ corresponds to $U_3 \setminus V_3'$ in [4] page 1432]. We can obtain similar results as in [4] Propositions 6.1 and 6.3] for $(N_0, \xi_0)$. Since $\frac{1}{r_3}$ is not an integer, $L_3$ contains at least two layers and we can obtain a similar result as in [4] Proposition 6.4] for $(N_0, \xi_0)$ (we only encounter Case 2 in the proof of [4] Proposition 6.4]). Therefore we obtain an upper bound of the number of tight contact structures on $N_0$ with given conditions, which is the same as the number of tight contact structures on $M(-1 + r_1, r_2, r_3)$.

By [4] Proposition 5.1, for any tight contact structure $\eta$ on $M(-1 + r_1, r_2, r_3)$, there is a Legendrian vertical circle with twisting number 0. Then similar to Case 1(a), we can isotopy $\eta$ so that $T_3 = \partial V_3$ is minimal convex with dividing curves of slope $-1$ when measured in the coordinates of $\partial V_3$. Then we conclude that the number of tight contact structures on $M(-1 + r_1, r_2, r_3)$ is less than or equal to the number of tight contact structures on $N_0$ with minimal convex boundary of slope $s - 1$ and Giroux torsion 0 along $\partial N_0$.

Therefore, the number of tight contact structures on $N$ with given conditions is exactly the number of tight contact structures on $M(-1; r_1, r_2, r_3)$.

4 Proof of Theorem 1.1 in cases 3 and 4

By Proposition 2.2, a tight contact structure on $M(D^2; r_1, r_2)$ with minimal convex boundary of slope $s$ corresponds to a tight contact structure on $M(D^2; r_1 - 1, r_2 - 1)$ with minimal convex boundary of slope $s - 2$.

Now consider the manifold $M = M(D^2; r_1 - 1, r_2 - 1)$ and let $s(T_3) = s - 2$. For $i = 1, 2$,
\[ r_i - 1 = -1 - \frac{1}{a_{i-1} - a_{i-1}}. \] Suppose \( r_i - 1 = -\frac{q_i}{p_i}, \) where \( p_i, q_i \) are integers, \( 0 < q_i < p_i \) and \( \gcd(p_i, q_i) = 1. \) Let \( \varphi_i = \left( \begin{array}{cc} p_i & u_i \\ q_i & v_i \end{array} \right), \) where \( p_i > u_i > 0 \) and \( p_i v_i - q_i u_i = 1. \)

Let \( \xi \) be a tight contact structure on \( M \) with minimal convex boundary of slope \( s(T_3) \) and Girou\x32 torsion 0 along \( \partial M. \) The proof of the following lemma is similar to the proof of [14, Theorem 1.4].

**Lemma 4.1** On \( M = M(D^2; -\frac{q_1}{p_1}, -\frac{q_2}{p_2}), \) if \( s(T_3) \geq -1, \) then any tight contact structure with minimal convex boundary of slope \( s(T_3) \) and Girou\x32 torsion 0 along \( \partial M \) does not admit Legendrian vertical circles with twisting number 0. \( \Box \)

The proof of the following lemma is similar to the proof of the corresponding result contained in the proof of [14, Theorem 1.7].

**Lemma 4.2** On \( M = M(D^2; -\frac{q_1}{p_1}, -\frac{q_2}{p_2}), \) if \( s(T_3) \geq 0, \) or if \( -1 \leq s(T_3) < 0 \) and \( \frac{q_i}{p_i} > \frac{1}{2} \) \((i = 1, 2), \) then the maximal twisting number of Legendrian vertical circles in \( (M, \xi) \) is \( -1. \)

Now assume that \( s(T_3) \geq 0, \) or \(-1 \leq s(T_3) < 0 \) and \( \frac{q_i}{p_i} > \frac{1}{2} \) \((i = 1, 2). \) By Lemma 4.2 after an isotopy of \( \xi, \) we can find a Legendrian vertical circle \( L \) in the interior of \( \Sigma \times S^1 \) with twisting number \(-1. \) Then we make each \( V_i \) \((i = 1, 2) \) a standard neighborhood of a Legendrian circle which is isotopic to the \( i \)th singular fiber with twisting number \( t_i < -2, \) i.e., \( \partial V_i \) is convex with two dividing curves each of which has slope \( \frac{1}{t_i} \) when measured in the coordinates of \( \partial V_i. \) Let \( s_i \) be the slope of the dividing curves of \( T_i = \varphi_i(\partial V_i) \) measured in the coordinates of \( T_i. \) Then we have \( s_i = \frac{q_i t_i + v_i}{p_i t_i + u_i} = \frac{q_i}{p_i} + \frac{1}{p_i(p_i t_i + u_i)}. \) The fact that \( t_i < -2 \) implies that \( 0 < s_i < \frac{q_i}{p_i}. \) In particular, if \( \frac{q_i}{p_i} > \frac{1}{2} \) \((i = 1, 2), \) then \( \frac{1}{2} < s_i < \frac{q_i}{p_i}. \)

We can assume that \( T_1 = \varphi_1(\partial V_1) \) \((i = 1, 2) \) and \( T_3 \) have Legendrian rulings of slope \( \infty \) when measured in the coordinates of \( T_1 \) and \( T_3, \) respectively. Using \( L, \) we can thicken \( V_i \) to \( V_i' \) \((i = 1, 2) \) and \( T_3 \) to \( L_3 \) to get a decomposition, \( M = (\Sigma' \times S^1) \cup (V_1' \cup V_2' \cup L_3), \) such that \( T_i' = \partial V_i' \) \((i = 1, 2) \) has two dividing curves of slope \( s_i = 0 \) when measured in the coordinates of \( T_i \), and the thickened torus \( L_3 \) has two boundary slopes \( s(T_3) \) and \( s(T_3) \) (cf. the proof of [14, Theorem 1.7]).

By [14, Lemma 5.1], there are exactly \( 2 + \lfloor s(T_3) \rfloor \) tight contact structures on \( \Sigma' \times S^1 \) satisfying the boundary condition and admitting no Legendrian vertical circles with twisting number 0.

The slope of the dividing curves on \( \partial V_i' \) is \( -\frac{q_i}{v_i} \) when measured in the coordinates of \( \partial V_i. \) So by a similar argument as in the proof of [14, Theorem 1.7], there are exactly \( \prod_{j=0}^{t_i} |a_j^1 + 1| \) tight contact structures on \( V_i' \) satisfying such boundary condition.

We consider Case 4 first.

**Case 4.** \( s \in [2, +\infty). \)
We decompose $L_3$ into $m$ continued fraction blocks (some blocks may be invariant neighborhoods of convex tori). The first one has boundary slopes $[s] - 2$ and $[s] - 1 - \frac{1}{a_1 - 1}$, the second one has boundary slopes $[s] - 1 - \frac{1}{a_1 - a_2}$ and $[s] - 1 - \frac{1}{a_1 - a_2 - 1}$, ..., the last one has boundary slopes $[s] - 1 - \frac{1}{a_1 - a_2 - \cdots - a_{m-1}}$ and $[s] - 1 - \frac{1}{a_1 - a_2 - \cdots - a_{m-1} - 1}$ = $s(T_3)$. By shuffling, there are at most $(a_1 - 1)(a_2 - 1) \ldots (a_{m-1} - 1)a_m$ minimally twisting tight contact structures on $L_3$ (except $s = [s]$). So there are at most $[s] \prod_{i=1}^{l} \prod_{j=0}^{l_i} |a_j^i + 1|(a_1 - 1)(a_2 - 1) \ldots (a_{m-1} - 1)a_m$ tight contact structures on $M$ with minimal convex boundary of slope $s - 2$ and Giroux torsion $0$ along $\partial M$.

Consider the small closed Seifert manifold $M(r_1 - 1, r_2 - 1, -[s] + 1 + r_3) = M(-\frac{a_1}{p_1}, -\frac{a_2}{p_2}, -[s] + 1 + r_3)$. Since $[s] \geq 2$, applying [14] Theorem 1.7, there are exactly $[s] \prod_{i=1}^{l} \prod_{j=0}^{l_i} |a_j^i + 1|(a_1 - 1)(a_2 - 1) \ldots (a_{m-1} - 1)a_m$ tight contact structures on $M(-\frac{a_1}{p_1}, -\frac{a_2}{p_2}, -[s] + 1 + r_3)$. According to the proof of [14] Theorem 1.7, for any tight contact structure $\eta$ on $M(-\frac{a_1}{p_1}, -\frac{a_2}{p_2}, -[s] + 1 + r_3)$, the maximal twisting number of a Legendrian vertical circle is $-1$. After an isotopy of $\eta$, we can find a vertical Legendrian circle $L'$ with twist number $-1$ in the interior of $\Sigma \times S^1$ and make $V_3$ a standard neighborhood of a Legendrian circle isotopic to the 3rd singular fiber with twisting number $t < 0$, i.e., $\partial V_3$ is convex with two dividing curves each of which has slope $\frac{1}{t}$ when measured in the coordinates of $\partial V_3$. Let $\varphi_3 = \left(\begin{array}{cc} p_3 & u_3 \\ q_3 & v_3 \end{array}\right)$, where $\frac{a_3}{p_3} = |s| - 1 - r_3$, $p_3q_3 - u_3q_3 = 1$ and $0 < u_3 < p_3$. Then, when measured in the coordinates of $T_3$, the slope $s_3 = \frac{a_3}{p_3} + \frac{1}{p_3(t_{p_3} + p_3)}$.

Using $L'$, we can thicken $V_3$ to $V_3'$ such that $T_3' = \partial V_3'$ has two dividing curves of slope $[s] - 2$. Since $|s| - 2 \leq s - 2 < |s| - 1 - r_3$, $|s| - 2 \leq s - 2 < s_3$ for sufficiently small $t$. By [9] Proposition 4.16, there is a convex torus $T_3''$ in the interior of $V_3' \setminus V_3$ which is parallel to $T_3$ and has two dividing curves of slope $s(T_3) = s - 2$. Thus we can isotopy $\eta$ so that $T_3$ is minimal convex with dividing curves of slope $s(T_3)$ when measured in the coordinates of $T_3$. $M(r_1 - 1, r_2 - 1, -[s] + 1 + r_3)$ has a decomposition $M \cup \varphi_3 V_3$.

By Proposition 2.1, $\varphi_3^{-1} \left(\begin{array}{cc} -a \\ -((s) - 2)a - b \end{array}\right) = \left(\begin{array}{cc} -1 \\ 1 \end{array}\right)$. Thus the slope of the dividing curves on $\partial V_3$ is $-1$ when measured in the coordinates of $\partial V_3$. Similar to the argument in Case 1(a), the number of tight contact structures on $M$ with given conditions is at least $[s] \prod_{i=1}^{l} \prod_{j=0}^{l_i} |a_j^i + 1|(a_1 - 1)(a_2 - 1) \ldots (a_{m-1} - 1)a_m$.

Therefore, there are exactly $[s] \prod_{i=1}^{l} \prod_{j=0}^{l_i} |a_j^i + 1|(a_1 - 1)(a_2 - 1) \ldots (a_{m-1} - 1)a_m$ tight contact structures on $M$ with minimal convex boundary of slope $s - 2$ and Giroux torsion $0$ along $\partial M$.

**Case 3.** $s \in (1, 2)$ and $r_1, r_2 \in (0, \frac{1}{2})$.

Since $r_i \in (0, \frac{1}{2})$ ($i = 1, 2$), $\frac{a_3}{p_3} = 1 - r_i > \frac{1}{2}$ ($i = 1, 2$). Similar to Case 4, there are at most $\prod_{i=1}^{l} \prod_{j=0}^{l_i} |a_j^i + 1|(a_1 - 1)(a_2 - 1) \ldots (a_{m-1} - 1)a_m$ tight contact structures on $M$ with given conditions.

Consider the small closed Seifert manifold $M(r_1 - 1, r_2 - 1, r_3) = M(-\frac{a_1}{p_1}, -\frac{a_2}{p_2}, r_3)$. We claim that this small closed Seifert manifold is an $L$-space (see [12] for the definition). Note
that since \( r_1 + r_2 + r_3 - 2 \neq 0 \), \( M(r_1 - 1, r_2 - 1, r_3) \) is a rational homology sphere. By [12, Theorem 1.1], it suffices to show that \( -M(\frac{-q_1}{p_1}, \frac{-q_2}{p_2}, q_3) = M(-1; \frac{q_1}{p_1}, \frac{q_2}{p_2}, 1 - r_3) \) carries no positive, transverse contact structures. Suppose otherwise, by [2, Theorem 4.5], there are integers \( h_1, h_2, h_3 \) and \( k > 0 \), such that (1) \( \frac{h_1}{k} < \frac{-q_1}{p_1}, \frac{h_2}{k} < \frac{-q_2}{p_2}, \frac{h_3}{k} < r_3 - 1 \), and (2) \( \frac{h_1 + h_2 + h_3}{k} = -1 - \frac{1}{k} \). Let \( n \) be a positive integer such that \( 1 - r_3 \geq \frac{1}{n} \). Combining (1) and (2), we have \( -1 - \frac{1}{k} < \frac{-q_1}{p_1} - \frac{-q_2}{p_2} + r_3 - 1 < -1 - \frac{1}{n} \). So \( 1 \leq k \leq n - 1 \). If \( k \) is even, then \( h_1 \leq \frac{k}{2} - 1, h_2 \leq \frac{k}{2} - 1 \) and \( h_3 \leq -1 \). Thus we have \( -1 - \frac{1}{k} = \frac{h_1 + h_2 + h_3}{k} \leq \frac{-k - 3}{k} = -1 - \frac{3}{k} \). This is absurd. If \( k \) is odd, then \( h_1 \leq \lfloor \frac{-k}{2} \rfloor = \frac{-k - 1}{2}, h_2 \leq \lfloor \frac{-k}{2} \rfloor = \frac{-k - 1}{2} \) and \( h_3 \leq -1 \). Thus we have \( -1 - \frac{1}{k} = \frac{h_1 + h_2 + h_3}{k} \leq \frac{2(-\frac{k - 1}{2}) - 1}{k} = -1 - \frac{2}{k} \). This is also absurd.

By [2, Theorem 1.3], there are exactly \( \prod_{i=1}^{2} \prod_{j=0}^{h_i} |a_j + 1|(a_1 - 1)(a_2 - 1) \ldots (a_{m-1} - 1)a_m \) tight contact structures on \( M(r_1 - 1, r_2 - 1, r_3) \). Moreover, in each of these tight contact structures, there is a Legendrian vertical circle with twisting number \(-1\). By a similar argument as in Case 4, there are at least \( \prod_{i=1}^{2} \prod_{j=0}^{h_i} |a_j + 1|(a_1 - 1)(a_2 - 1) \ldots (a_{m-1} - 1)a_m \) tight contact structures on \( M \) with given conditions. So, in this case, there are exactly \( \prod_{i=1}^{2} \prod_{j=0}^{h_i} |a_j + 1|(a_1 - 1)(a_2 - 1) \ldots (a_{m-1} - 1)a_m \) tight contact structures on \( M \) with minimal convex boundary of slope \( s - 2 \) and Giroux torsion 0 along \( \partial M \).

## 5 Proof of Theorem [1.2]

Consider the thickened torus \( S^1 \times S^1 \times I \), where \( I = [0, 1] \). Denote by \((x, y, z)\) the coordinates of \( S^1 \times S^1 \times I \), where \( x \in \mathbb{R}/2\pi\mathbb{Z}, y \in \mathbb{R}/2\pi\mathbb{Z} \) and \( z \in [0, 1] \). For convenience, let \( x \in [-\pi, \pi] \), \( y \in [-\pi, \pi] \), and \( -\pi \) is identified with \( \pi \). Let \( N \) be the quotient space of \( S^1 \times S^1 \times I \) by identifying \((x, y, z)\) with \((x + \pi, -y, 1 - z)\). Let \( p : S^1 \times S^1 \times I \to N \) be the covering projection. The covering transformation \((x, y, z) \mapsto (x + \pi, -y, 1 - z)\) is denoted by \( \tau \). \( N \) can be identified with \( M(D^2; -\frac{1}{2}, \frac{1}{2}) \) (the images under \( p \) of \( S^1 \times \{0\} \times \{\frac{1}{2}\} \) and \( S^1 \times \{\pi\} \times \{\frac{1}{2}\} \) correspond to the singular fibers). On the boundary \( T_3 = -\partial N = S^1 \times S^1 \times \{0\}, S^1 \times \{pt\} \times \{0\} \) gives the fiber direction (i.e., corresponds to \((0, 1)\) in the notation of Section 2) and \( \{pt\} \times S^1 \times \{0\} \) corresponds to \((-1, 0)\) in the notation of Section 2. By Proposition [2.2] a simple closed curve of slope \( s \) in \( T_3 \) of \( N = M(D^2; -\frac{1}{2}, \frac{1}{2}) \) corresponds to a simple closed curve of slope \( s + 1 \) in \( T_3 \) of \( M(D^2; \frac{1}{2}, \frac{1}{2}) \).

We regard \( N \) as the quotient space of a thickened cylinder \([0, \pi] \times S^1 \times I\) by identifying \((0, y, z)\) with \((\pi, -y, 1 - z)\). See Figure 1. The coordinates of the four points \( P, Q, R \) and \( S \) at the left end are \((0, -\frac{\pi}{2}, 0), (0, -\frac{\pi}{2}, 1), (0, \frac{\pi}{2}, 1) \) and \((0, \frac{\pi}{2}, 0)\) respectively. The coordinates of the two points \( X \) and \( Y \) at the left end are \((0, 0, \frac{1}{2})\) and \((0, \pi, \frac{1}{2})\) respectively.
Let $\xi$ be a tight contact structure on $N$ with minimal convex boundary of slope $s(T_3) = \infty$. This means that the dividing curves of $T_3$ consist of two simple closed curves parallel to $S^1 \times \{pt\} \times \{0\}$. Note that the image of $\{pt\} \times S^1 \times I$ under $p$ is an essential annulus in $N$ and the metric closure of its complement in $N$ is a solid torus. Assume that $T_3$ is a convex torus in standard form with dividing curves $S^1 \times \{0\} \times \{0\}$ and $S^1 \times \{\pi\} \times \{0\}$, and $\{pt\} \times S^1 \times \{0\}$, $pt \in S^1$, are the Legendrian rulings. See Figure 1. The upper bold line and the upper dashed line form a dividing curve, and the lower bold line and the lower dashed line form the other dividing curve. The plus sign “+” in Figure 1 denotes the region $p(S^1 \times [0, \pi] \times \{0\})$ in $T_3$ bounded by the two dividing curves.

Let $A$ denote the annulus which is the image of $\{0\} \times S^1 \times I$ under $p$. After perturbation, $A$ is convex with Legendrian boundary. Also assume that $\# \Gamma_A$, the number of connected components of the dividing set $\Gamma_A$ of $A$, is minimal among all convex annuli in its isotopy class relative to the boundary. $\Gamma_A$ contains exactly two properly embedded arcs. Without loss of generality, we assume that the endpoints of these two dividing arcs are $P$, $Q$, $R$ and $S$.

**Case 5.1.** Both of the two dividing arcs in $\Gamma_A$ connect the two different components of $\partial A$.

The two dividing arcs in $\Gamma_A$ must connect the points $P$, $Q$ and $R$, $S$ respectively. As shown in Figures 2 and 3, when we cut $N$ along the convex annulus $A$ and round the edges, we obtain a solid torus with two dividing curves on the boundary. Moreover, each of these two dividing curves intersects a meridian of this solid torus exactly once. There exists a unique tight contact structure on this solid torus by [9 Proposition 4.3]. This implies that in this case, for every choice of $\Gamma_A$, there exists at most one tight contact structure.
Similar to the proof of [9 Proposition 4.9], we define the holonomy $k(A)$ by passing to the cover $\{0\} \times \mathbb{R} \times I \subset S^1 \times \mathbb{R} \times I$ and letting $k(A)$ be the integer such that there is a dividing curve which connects from $(0, \frac{\pi}{2}, 0)$ to $(0, 2k(A)\pi + \frac{\pi}{2}, 1)$. For example, the holonomy in Figure 2 is 0, and the holonomy in Figure 3 is $-1$. 
Let $\alpha_0 = \cos y dx + \sin y dz$ on $S^1 \times S^1 \times I$. Then $\xi_0 = \ker \alpha_0$ is the $I$-invariant neighborhood of a convex $S^1 \times S^1$ with dividing curves $S^1 \times \{0\}$ and $S^1 \times \{\pi\}$. If we take $\xi_0$ and isotope $S^1 \times S^1 \times \{0\}$ via $(x, y) \mapsto (x, y - k\pi)$ and isotope $S^1 \times S^1 \times \{1\}$ via $(x, y) \mapsto (x, y + k\pi)$, while fixing $S^1 \times S^1 \times \{\frac{1}{2}\}$, namely, we take a self-diffeomorphism of $S^1 \times S^1 \times I$ by sending $(x, y, z)$ to $(x, y \pm 2k\pi(z - \frac{1}{2}), z)$, then we obtain a tight contact structure $\xi_k$ on $S^1 \times S^1 \times I$ with holonomy $k$ (in the sense of [9, Proposition 4.9]), and the corresponding contact form $\alpha_k = \cos(y \pm 2k\pi(z - \frac{1}{2})) dx + \sin(y \pm 2k\pi(z - \frac{1}{2})) dz$.

Since $\tau^*(\alpha_k) = \alpha_k$, each nonrotative tight contact structure $\xi_k$ on $S^1 \times S^1 \times I$ is $\tau$-invariant. So $\xi_k$ induces a tight contact structure on $N$ with holonomy $k(A) = k$. By [9, Proposition 4.9], the nonrotative tight contact structures $\xi_k$, $k \in \mathbb{Z}$, on $S^1 \times S^1 \times I$ are non-isotopic, so they induce non-isotopic tight contact structures on $N$. All these tight contact structures on $N$ have Giroux torsion 0 along $\partial N$ since each $\xi_k$ has Giroux torsion 0 along the boundary. These tight contact structures on $N$ form the subset in Theorem 1.2(1) whose elements are in 1-1 correspondence with $\mathbb{Z}$. Note also that for all these tight contact structures, a convex torus parallel to $\partial N$ must have slope $\infty$ since each $\xi_k$ is nonrotative.

**Case 5.2.** The two endpoints of each dividing arc in $\Gamma_A$ belong to the same component of $\partial A$.

If $\Gamma_A$ contains an odd number of closed dividing curves, see Figure 4, then, when we cut $N$ along $A$ and perform edge-rounding, we find two dividing curves which bound disks. This contradicts Giroux’s criterion (see [9, Theorem 3.5]). So $\Gamma_A$ must contain an even number of closed dividing curves.

Suppose $\Gamma_A$ contains $2t$ closed dividing curves, where $t \geq 0$. As shown in Figure 5, when we cut $N$ along the convex annulus $A$ and round the edges, we obtain a solid torus $S^1 \times D^2$ with $4t + 2$ vertical dividing curves.
Next cut $S^1 \times D^2$ along a meridional disk $D$ after modifying the boundary to be standard with horizontal rulings. Since $\sharp \Gamma_A$ is minimal, by a similar argument as in the proof of [9, Lemma 5.2], the dividing set of the convex meridional disk $D$ has a unique configuration as follows. Let $\gamma_0$ and $\gamma_1$ be the two dividing curves on $\partial (S^1 \times D^2)$ which intersect $\partial N$. Then all $\gamma \in \Gamma_D$ must separate $D \cap \gamma_1$ from $D \cap \gamma_0$ (hence the dividing curves of $D$ are parallel segments, with only two boundary-parallel components, each containing one $D \cap \gamma_i$ in the interior); otherwise there would exist a bypass which allows for a reduction in the number of dividing curves on $A$.

Therefore, the tight contact structure $\xi$ on $N$ depends only on $\Gamma_A$, which in turn is determined by the sign of the boundary-parallel components of $A$ along $\partial N$, together with $t + 2 = \sharp \Gamma_A$. So in this case, for each $t \geq 0$, there exist at most two tight contact structures on $N$.

For each $t \in \{0\} \cup \mathbb{Z}^+$, let $\eta_t$ be the contact structure on $S^1 \times S^1 \times I$ given by 1-form $\beta_t = \sin((2t + 1)\pi z)dx + \cos((2t + 1)\pi z)dy$, with the boundary adjusted so it becomes convex with two dividing curves on each component. Let $\eta'_t$ denote the contact structure given by $-\beta_t$. By [9, Lemma 5.3], any two of these tight contact structures on $S^1 \times S^1 \times I$ are distinct. For each $t \in \{0\} \cup \mathbb{Z}^+$, since $\tau^*(\beta_t) = \beta_t$, both $\eta_t$ and $\eta'_t$ are $\tau$-invariant, and hence induce contact structures $\zeta_t$ and $\zeta'_t$ on $N$ respectively. Since these two induced contact structures on $N$ lift to distinct tight contact structures on $S^1 \times S^1 \times I$, they are tight and distinct. Moreover, both $\zeta_t$ and $\zeta'_t$ have minimal convex boundary of slope $\infty$ and Giroux torsion $t$ along $\partial N$ by the explicit formula of $\beta_t$ and [11, Proposition 3.4].

Similar to [9, Lemma 5.2] and [11, Proposition 3.2], if $\Gamma_A$ contains $2t$ closed curves, then $\xi$ is $\zeta_t$ or $\zeta'_t$. $\zeta_0$ and $\zeta'_0$ form the subset in Theorem 1.2(1) which contains two elements.
This completes the proof of Theorem 1.2(1). For \( t \geq 1 \), there are exactly two tight contact structures, namely, \( \zeta \) and \( \zeta' \), on \( N \) with minimal convex boundary of slope \( \infty \) and Giroux torsion \( t \) along \( \partial N \). This proves Theorem 1.2(2) when \( s = \infty \).

Now let \( \xi \) be a tight contact structure on \( N \) with minimal convex boundary of slope \( s(T_3) = s \in \mathbb{Q} \) and Giroux torsion \( t \geq 1 \) along \( \partial N \). There is a minimal convex torus \( T' \) in the interior of \( N \) which is parallel to \( T_3 \) and has slope \( s \), such that the restriction of \( \xi \) on the thickened torus \( U' \) bounded by \( T' \) and \( T_3 \) has Giroux torsion \( t \). According to [9, Lemma 5.2], \((U', \xi|_{U'})\) is universally tight.

There is a minimal convex torus \( T \) in the interior of \( U' \) which is parallel to \( T_3 \) and has slope \( \infty \). We assume that the restriction of \( \xi \) on the thickened torus \( U \) bounded by \( T \) and \( T_3 \) is minimally twisting. Note that \( U \subset U' \).

The contact submanifold \((N \setminus U, \xi|_{N \setminus U})\) belongs to Case 5.2. If the contact submanifold \((N \setminus U, \xi|_{N \setminus U})\) belongs to Case 5.1, then each convex torus in \( N \setminus U \) which is parallel to \( T \) has slope \( \infty \), contradicting the fact that \( T' \) has slope \( s \). Note that for the contact structure \( \zeta_0 \) on \( N \), the slope of a convex torus parallel to \( T_3 \) is greater than or equal to \( 0 \). Thus if \( s \geq 0 \), then the Giroux torsion of \((N \setminus U, \xi|_{N \setminus U})\) is \( t - 1 \), and if \( s < 0 \), then the Giroux torsion of \((N \setminus U, \xi|_{N \setminus U})\) is \( t \). So there are at most two possibilities of \((N \setminus U, \xi|_{N \setminus U})\).

Since \( U \subset U' \) and \((U', \xi|_{U'})\) is universally tight, \((U, \xi|_{U})\) is universally tight. By [9, Proposition 5.1], there are at most two possibilities of \((U, \xi|_{U})\). Moreover, these two possibilities are distinguished by their relative Euler classes. If \((U, \xi|_{U})\) is given, then at most one possibility of \((N \setminus U, \xi|_{N \setminus U})\) can make \((N, \xi)\) tight by [9, Lemma 5.2]. Hence there are at most two tight contact structures on \( N \) with the given conditions.

For a given \( t' \in \mathbb{Z}^+ \cup \{0\} \), let \( 0 < w < \frac{1}{2t' + 1} \) satisfy that \( -s = \cot ((2t' + 1)\pi w) \). Let \( \eta_{t'} \) (we use the same notation as in Case 5.2) be the tight contact structure on \( S^1 \times S^1 \times [-w, 1 + w] \) given by 1-form \( \beta_{t'} = \sin((2t' + 1)\pi z)dx + \cos((2t' + 1)\pi z)dy \), with the boundary adjusted so it becomes convex with two dividing curves on each component. \( \eta_{t'} \) is given by the 1-form \( -\beta_{t'} \). Think of \( N \) as the quotient space of \( S^1 \times S^1 \times [-w, 1 + w] \) by identifying \((x, y, z)\) with \((x + \pi, -y, 1 - z)\). The transformation \((x, y, z) \mapsto (x + \pi, -y, 1 - z)\) on \( S^1 \times S^1 \times [-w, 1 + w] \) is still denoted by \( \tau \). Since \( \beta_s \) is \( \tau \)-invariant, \( \eta_{t'} \) and \( \eta_{t'}' \) induce tight contact structures \( \zeta_{t'} \) and \( \zeta_{t'}' \) on \( N \) with minimal convex boundary of slope \( s(T_3) = s \). \( \zeta_{t'} \) and \( \zeta_{t'}' \) are distinct since \( \eta_{t'} \) and \( \eta_{t'}' \) are distinct.

Note that the restriction of the contact structure \( \eta_{t'} \) on \( S^1 \times S^1 \times [-w, 0] \) is minimally twisting. We conclude that if \( s < 0 \), then the Giroux torsion along \( \partial N \) of \( \zeta_{t'} \) and \( \zeta_{t'}' \) is \( t' \) and if \( s \geq 0 \), then the Giroux torsion along \( \partial N \) of \( \zeta_{t'} \) and \( \zeta_{t'}' \) is \( t' + 1 \).

Therefore for each \( t \in \mathbb{Z}^+ \), there are exactly two tight contact structures on \( N \) with minimal convex boundary of slope \( s \) and Giroux torsion \( t \) along \( \partial N \). This finishes the proof of Theorem 1.2(2).

**Acknowledgements.** The first author is partially supported by grant no. 10631060 of the National Natural Science Foundation of China. The second author is partially supported by grant no. 1100171 of the National Natural Science Foundation of China.
References

[1] Y. Eliashberg, Contact 3-manifolds twenty years since J. Martinet’s work, Ann. Inst. Fourier 42 (1992), 165–192.

[2] P. Ghiggini, On tight contact structures with negative maximal twisting number on small Seifert manifolds, Algebr. Geom. Topol. 8 (2008), no. 1, 381–396.

[3] P. Ghiggini, P. Lisca, A. Stipsicz, Classification of tight contact structures on small Seifert 3-manifolds with $e_0 \geq 0$, Proc. Amer. Math. Soc. 134 (2006), no. 3, 909–916 (electronic).

[4] P. Ghiggini, P. Lisca, A. Stipsicz, Tight contact structures on some small Seifert fibered 3-manifolds, Amer. J. Math. 129 (2007), no. 5, 1403–1447.

[5] H. Geiges, An introduction to contact topology, Cambridge Studies in Advanced Mathematics, vol. 109, Cambridge University Press, Cambridge, 2008.

[6] E. Giroux, Convexité en topologie de contact, Comment. Math. Helv. 66 (1991), 637–677.

[7] E. Giroux, Structures de contact en dimension trois et bifurcations des feuilletages de surfaces, Invent. Math. 141 (2000), 615–689.

[8] E. Giroux, Structures de contact sur les variétés fibrées en cercles audessus d’une surface, Comment. Math. Helv. 76 (2001), 218–262.

[9] K. Honda, On the classification of tight contact structures I, Geom. Topol. 4 (2000), 309–368 (electronic).

[10] K. Honda, On the classification of tight contact structures II, J. Differential Geom. 55 (2000), no. 1, 83–143.

[11] K. Honda, W. Kazez, G. Matic, Convex decomposition theory, Int. Math. Res. Not. 2002, no. 2, 55–88.

[12] P. Lisca, A. Stipsicz, Ozsváth-Szabó invariants and tight contact 3-manifolds, III, J. Symplectic Geom. 5 (2007), no. 4, 357–384.

[13] C. Tanya, A class of tight contact structures on $\Sigma_2 \times I$, Algebr. Geom. Topol. 4 (2004), 961–1011 (electronic).

[14] H. Wu, Legendrian vertical circles in small Seifert spaces, Commun. Contemp. Math. 8 (2006), no. 2, 219–246.

School of Mathematical Sciences, Peking University, Beijing 100871, China

E-Mail address: dingfan@math.pku.edu.cn

Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, China
E-mail address: liyoulin@sjtu.edu.cn

School of Science, Xi’an Jiaotong University, Xi’an 710049, China
E-mail address: zhangq.math@mail.xjtu.edu.cn