We study the binary mass distribution for the recently predicted thermally fissile neutron-rich uranium and thorium nuclei using statistical model. The level density parameters needed for the study are evaluated from the excitation energies of temperature dependent relativistic mean field formalism. The excitation energy and the level density parameter for a given temperature are employed in the convolution integral method to obtain the probability of the particular fragmentation. As representative cases, we present the results for the binary yield of $^{250}$U and $^{254}$Th. The relative yields are presented for three different temperatures $T = 1, 2$ and $3$ MeV.

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I. INTRODUCTION

Fission phenomenon is one of the most interesting subject in the field of nuclear physics. To study the fission properties, a large number of models have been proposed. The fissioning of a nucleus is successfully explained by the liquid drop model and the semi-empirical mass formula is the best and simplest oldest tool to get a rough estimation of the energy released in a fission process. The pioneering work of Vautherin and Brink [1], who has applied the Skyrme interaction in a self-consistent method for the calculation of ground state properties of finite nuclei opened a new dimension in the quantitative estimation of nuclear properties. Subsequently, the Hartree-Fock and time dependent Hartree-Fock formalisms [2] are also implemented to study the properties of fission. Most recently, the microscopic relativistic mean field approximation, which is another successful theory in nuclear physics is also used for the study of nuclear fission [3].

From last few decades, the availability of neutron rich nuclei in various laboratories across the globe opened up new research in the field of nuclear physics, because of their exotic decay properties. The effort for the synthesis of superheavy nuclei in the laboratories like, Dubna (Russia), GSI (Germany), RIKEN (Japan) and BNLL (USA) is also quite remarkable. Due to all these, the periodic table is extended till date up to atomic number $Z = 118$ [4]. The decay modes of these superheavy nuclei are very different than the usual modes. Mostly, we understand that, a neutron rich nucleus has a large number of neutron than the light or medium mass region of the periodic table. The study of these neutron-rich superheavy nuclei is very interesting, because of their ground state structures and various mode of decays, including multi-fragment fission (more than two) [3]. Another interesting feature of some neutron rich uranium and thorium nuclei is that similar to $^{238}$U, $^{235}$U and $^{239}$Pu, the nuclei $^{246–264}$U and $^{244–262}$Th are also thermally fissile, which are extremely important for the energy production in fission process. If the neutron rich uranium and thorium nuclei are the viable sources, then these nuclei will be more effective to achieve the critical condition in a controlled fission reaction.

Now the question arises, how we can get a reasonable estimation of the mass yield in the spallation reaction of these neutron rich thermally fissile nuclei. As mentioned earlier in this section, there are many formalisms available in the literature to study these cases. Here, we adopt the statistical model developed by Fong [5]. The calculation is further extended by Rajasekaran and Devananthan [6] to study the binary mass distributions using the single particle energies of the Nilsson model. The obtained results are well in agreement with the experimental data. In the present study, we would like to replace the single particle energies with the excitation energies of a successful microscopic approach, the relativistic mean field (RMF) formalism. For last few decades, the relativistic mean field (RMF) formalism [7–11] with various parameter sets have successfully reproduced the bulk properties, such as binding energies, root mean square radii, quadrupole deformation etc. not only for nuclei near the $\beta$–stability line but also for nuclei away from it. Further, the RMF formalism is successfully applied to the study of clusterization of known cluster emitting heavy nucleus [12–14] and the fission of hyper-hyper deformed $^{56}$Ni [15]. Rutz et al. [16] reproduced the double, triple humped fission barrier of $^{240}$Pu, $^{232}$Th and the asymmetric ground states of $^{220}$Ra using RMF formalism. Moreover, the symmetric and asymmetric fission modes are also successfully reproduced. Patra et. al. [3] studied the neck configuration in the fission decay of neutron rich U and Th isotopes. The main goal of this present paper is to understand the binary fragmentation yield of such neutron rich thermally fissile superheavy nuclei. $^{250}$U and $^{254}$Th are taken for further calculations as the representative cases.

The paper is organized as follows: In Section II, the statistical model and relativistic mean field theory are presented briefly. In subsection A of this section, the level density parameter and its relation with the relative mass yield are outlined. In subsection B of II, the equation of motion of the nucleon and meson fields obtained from the relativistic mean field Lagrangian and the temperature dependent of the equa-
tions are adopted through the occupation number of protons and neutrons. The results are discussed in Section III and compared with the finite range droplet model (FRDM) predictions. The summary and concluding remarks are given in Section IV.

II. FORMALISM

The possible binary fragments of the considered nucleus is obtained by equating the charge to mass ratio of the parent nucleus to the fission fragments as [17]:

\[
\frac{Z_P}{A_P} \approx \frac{Z_i}{A_i},
\]

with \(A_P, Z_P\), and \(A_i, Z_i\) (\(i = 1\) and \(2\)) correspond to mass and charge numbers of the parent nucleus and the fission fragments [6]. The constraints, \(A_1 + A_2 = A, Z_1 + Z_2 = Z\) and \(A_1 \geq A_2\) are imposed to satisfy the conservation of charge and mass number in a nuclear fission process and to avoid the repetition of fission fragments. Another constraint i.e., the binary charge numbers from \(Z_2 \geq 26\) to \(Z_1 \leq 66\) is also taken into consideration from the experimental yield [18] to generate the combinations, assuming that the fission fragments lie within these charge range.

A. Statistical theory

The statistical theory [5, 19] assumes that the probability of the particular fragmentation is directly proportional to the folded level density \(\rho_{12}\) of that fragments with the total excitation energy \(E^*\), i.e., \(P(A_j, Z_j) \propto \rho_{12}(E^*)\). Where,

\[
\rho_{12}(E^*) = \int_0^{E^*} \rho_1(E_1^*)\rho_2(E^* - E_1^*) \, dE_1^*,
\]

and \(\rho_i\) is the level density of two fragments (\(i = 1, 2\)). The nuclear level density [20, 21] is expressed as a function of fragment excitation energy \(E_i^*\) and the single particle level density parameter \(a_i\), which is given as:

\[
\rho_i(E_i^*) = \frac{1}{12} \left(\frac{\pi^2}{a_i} \right)^{1/4} E_i^{(-5/4)} \exp\left(2\sqrt{\frac{a_i E_i^*}{T}}\right). \tag{3}
\]

In Refs. [17, 22], we calculate the excitation energies of the fragments using the ground state single particle energies of finite range droplet model (FRDM) [23] at a given temperature \(T\) keeping the total number of proton and neutron fixed. In the present study, we apply the self consistent temperature dependent relativistic mean field theory to calculate the \(E_i^*\) of the fragments. The excitation energy is calculated as,

\[
E_i^*(T) = E_i(T) - E_i(T = 0). \tag{4}
\]

The level density parameter \(a_i\) is given as,

\[
a_i = \frac{E_i^*}{T^2}. \tag{5}
\]

The relative yield is calculated as the ratio of the probability of a given binary fragmentation to the sum of the probabilities of all the possible binary fragmentations and it is given by,

\[
Y(A_j, Z_j) = \frac{P(A_j, Z_j)}{\sum_j P(A_j, Z_j)}, \tag{6}
\]

where \(A_j\) and \(Z_j\) are referred to the binary fragmentations involving two fragments with mass and charge numbers \(A_1, A_2\) and \(Z_1, Z_2\) obtained from Eq. (1). The competing basic decay modes such as neutron/proton emission, \(\alpha\) decay and ternary fragmentation are not considered. In addition to these approximations, we have also not included the dynamics of the fission reaction, which are really important to get a quantitative comparison with the experimental measurements. The presented results are the prompt disintegration of a parent nucleus into two fragments (democratic breakup). The resulting excitation energy would be liberated as prompt particle emission or delayed emission, but such secondary emissions are also ignored.

B. RMF Formalism

The RMF theory assume that the nucleons interact with each other via meson fields. The nucleon - meson interaction is given by the Lagrangian density [7–9, 11, 24, 25],

\[
\mathcal{L} = \overline{\psi}_i \left( i \gamma^\mu \partial_\mu - M \right) \psi_i + \frac{1}{2} g^{\mu\nu} \sigma \partial_\mu \sigma - \frac{1}{2} m_\sigma^2 \sigma^2 - \frac{1}{3} \frac{g_2}{2} \sigma^3 - \frac{1}{4} \frac{g_3}{3} \sigma^4
\]

\[
- \frac{1}{4} \Omega^{\mu\nu} \Omega_{\mu\nu} + \frac{1}{2} m_\rho^2 V^{\mu} V_\mu - g_\omega \overline{\psi}_i \gamma^\mu \psi_i V_\mu
\]

\[
- \frac{1}{4} \tilde{B}^{\mu\nu} \tilde{B}_{\mu\nu} + \frac{1}{2} m_\sigma^2 \tilde{R}^{\mu} \tilde{R}_\mu - g_{\omega \rho} \overline{\psi}_i \gamma^\mu \tilde{\sigma} \psi_i \tilde{R}^{\mu}
\]

\[
+ \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - e \overline{\psi}_i \gamma^\mu \left(1 - \gamma_5\right) \psi_i A_\mu. \tag{7}
\]

Where, \(\psi_i\) is the single particle Dirac spinor. The arrows over the letters in the above equation represent the isovector quantities. The nucleon, the \(\sigma, \omega, \rho\) meson masses are denoted by \(M, m_\sigma, m_\omega\) and \(m_\rho\) respectively. The meson and the photon fields are termed as \(\sigma, V_\mu, \tilde{B}_{\mu\nu}\) and \(A_\mu\) for \(\sigma, \omega, \rho\) mesons and photon respectively. The \(g_\sigma, g_\omega, g_\rho\) and \(\gamma_5\) are the coupling constants for the \(\sigma, \omega, \rho\) mesons and photon fields with nucleons respectively. The strength of the constants \(g_2\) and \(g_3\) are responsible for the nonlinear couplings of \(\sigma, \omega, \rho\) mesons. The field tensors of the isovector mesons and the photon are given by,

\[
\Omega^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu, \tag{8}
\]

\[
\tilde{B}^{\mu\nu} = \partial^\mu \tilde{R}^\nu - \partial^\nu \tilde{R}^\mu - g_{\omega \rho}(\tilde{R}^{\mu} \times \tilde{R}^{\nu}), \tag{9}
\]

\[
F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{10}
\]

The classical variational principle gives the Euler-Lagrange equation and we get the Dirac-equation with potential terms.
for the nucleons and Klein-Gordon equations with source terms for the mesons. We assume the no-sea approximation, so we neglect the antiparticle states. We are dealing with the static nucleus, so the time reversal symmetry and the conservation of parity simplifies the calculations. After simplifications, the Dirac equation for the nucleon is given by,
\[
\{-i\alpha \cdot \nabla + V(r) + \beta [M + S(r)]\} \psi_i = \epsilon_i \psi_i, \tag{11}
\]
where \(V(r)\) represents the vector potential and \(S(r)\) is the scalar potential,
\[
V(r) = g_\omega \omega_0 + g_\rho \tau_3 \rho_0(r) + e \frac{(1 - \tau_3)}{2} A_0(r),
\]
\[
S(r) = g_\sigma \sigma(r), \tag{12}
\]
which contributes to the effective mass,
\[
M^*(r) = M + S(r). \tag{13}
\]

The Klein-Gordon equations for the mesons and the electromagnetic fields with the nucleon densities as sources are,
\[
\{-\Delta + m_{\sigma}^2\} \sigma(r) = -g_\sigma \rho_0(r) - g_2 \sigma^2(r) - g_3 \sigma^3(r), \tag{14}
\]
\[
\{-\Delta + m_{\omega}^2\} \omega_0(r) = g_\omega \rho_0(r), \tag{15}
\]
\[
\{-\Delta + m_{\rho}^2\} \rho_0(r) = g_\rho \rho_3(r), \tag{16}
\]
\[
-\Delta A_0(r) = e \rho_c(r). \tag{17}
\]

The corresponding densities such as scalar, baryon (vector), isovector and proton (charge) are given as
\[
\rho_\sigma(r) = \sum_i n_i \psi_i^\dagger(r) \psi_i(r), \tag{18}
\]
\[
\rho_\omega(r) = \sum_i n_i \psi_i^\dagger(r) \gamma_0 \psi_i(r), \tag{19}
\]
\[
\rho_3(r) = \sum_i n_i \psi_i^\dagger(r) \tau_3 \psi_i(r), \tag{20}
\]
\[
\rho_c(r) = \sum_i n_i \psi_i^\dagger(r) \left(1 - \frac{\tau_3}{2}\right) \psi_i(r). \tag{21}
\]

To solve the Dirac and Klein-Gordon equations, we expand the Boson fields and the Dirac spinor in an axially deformed harmonic oscillator basis with \(\beta_0\) as the initial deformation parameter. The nucleon equation along with different meson equations form a set of coupled equations, which can be solved by iterative method. The center of mass correction is calculated with the non-relativistic approximation. The quadrupole deformation parameter \(\beta_2\) is calculated from the resulting quadrupole moments of the proton and neutron. The total energy is given by \([10, 26, 27]\),
\[
E(T) = \sum_i \epsilon_i n_i + E_\sigma + E_{\sigma NL} + E_\omega + E_\rho
\]
\[
+ E_C + E_{\text{pair}} + E_{c.m.} - AM, \tag{22}
\]

with
\[
E_\sigma = -\frac{1}{2} g_\sigma \int d^3 r \rho_\sigma(r) \sigma(r), \tag{23}
\]
\[
E_{\sigma NL} = -\frac{1}{2} \int d^3 r \left\{ \frac{2}{3} g_2 \sigma_3^2(r) + \frac{1}{2} g_3 \sigma_3^4(r) \right\}, \tag{24}
\]
\[
E_\omega = -\frac{1}{2} g_\omega \int d^3 r \rho_\omega(r) \omega_0^2(r), \tag{25}
\]
\[
E_\rho = -\frac{1}{2} g_\rho \int d^3 r \rho_\rho(r) \rho_3^2(r), \tag{26}
\]
\[
E_C = -\frac{e^2}{8\pi} \int d^3 r \rho_c(r) A_0^2(r), \tag{27}
\]
\[
E_{\text{pair}} = -\Delta \sum_{i>0} u_i v_i = -\Delta \frac{2}{G}, \tag{28}
\]
\[
E_{c.m.} = -\frac{3}{4} \times 41 A^{-1/3}. \tag{29}
\]

Here, \(\epsilon_i\) is the single particle energy, \(n_i\) is the occupation probability and \(E_{\text{pair}}\) is the pairing energy obtained from the simple BCS formalism.

C. Pairing and temperature dependent RMF formalism

The pairing correlation plays a distinct role in open-shell nuclei. The effect of pairing correlation is markedly seen with increase in mass number \(A\). Moreover it helps in understanding the deformation of medium and heavy nuclei. It has a lean effect on both bulk and single particles properties of lighter mass nuclei because of the availability of limited pairs near the Fermi surface. We take the case of \(T=1\) channel of pairing correlation i.e, pairing between proton-proton and neutron-neutron. In this case, a nucleon of quantum states \(|jm_z\rangle\) pairs with another nucleons having same \(I_z\) value with quantum states \(|j-m_z\rangle\), since it is the time reversal partner of the other. In both nuclear and atomic domain the ideology of BCS pairing is the same. The even-odd mass staggering of isotopes was the first evidence of its kind for the pairing energy. Considering the mean-field formalism, the violation of the particle number is seen only due to the pairing correlation. We find terms like \(\bar{\psi}^\dagger \psi\) (density) in the RMF Lagrangian density but we put an embargo on terms of the form \(\bar{\psi}^\dagger \psi\) or \(\bar{\psi} \psi\) since it violates the particle number conservation. We apply externally the BCS constant pairing gap approximation for our calculation to take the pairing correlation into account. The pairing interaction energy in terms of occupation probabilities \(u_i^2\) and \(v_i^2 = 1 - u_i^2\) is written as \([28, 29]\):
\[
E_{\text{pair}} = -G \left[ \sum_{i>0} u_i v_i \right]^2, \tag{30}
\]
with \(G\) is the pairing force constant. The variational approach with respect to the occupation number \(u_i^2\) gives the BCS equation \([29]\):
\[
2\epsilon_i u_i v_i - \Delta (u_i^2 - v_i^2) = 0, \tag{31}
\]
with the pairing gap $\Delta = G \sum_{i>0} u_i v_i$. The pairing gap ($\Delta$) of proton and neutron is taken from the empirical formula [10, 30]:

$$\Delta = 12 \times A^{-1/2}. \quad (32)$$

The temperature introduced in the partial occupancies in the BCS approximation is given by,

$$n_i = v_i^2 = \frac{1}{2} \left[ 1 - \frac{\epsilon_i - \lambda}{\epsilon_i} [1 - 2f(\tilde{\epsilon}_i, T)] \right], \quad (33)$$

with

$$f(\tilde{\epsilon}_i, T) = \frac{1}{(1 + \exp[\tilde{\epsilon}_i/T])} \quad \text{and}$$

$$\tilde{\epsilon}_i = \sqrt{(\epsilon_i - \lambda)^2 + \Delta^2}. \quad (34)$$

The function $f(\tilde{\epsilon}_i, T)$ represents the Fermi Dirac distribution for quasi particle energy $\tilde{\epsilon}_i$. The chemical potential $\lambda_p(\lambda_n)$ for protons (neutrons) is obtained from the constraints of particle number equations

$$\sum_i n_i^Z = Z,$$
$$\sum_i n_i^N = N. \quad (35)$$

The sum is taken over all proton and neutron states. The entropy is obtained by,

$$S = -\sum_i [n_i \ln(n_i) + (1 - n_i) \ln(1 - n_i)]. \quad (36)$$

The total energy and the gap parameter are obtained by minimizing the free energy,

$$F = E - TS. \quad (37)$$

In constant pairing gap calculations, for a particular value of pairing gap $\Delta$ and force constant $G$, the pairing energy $E_{\text{pair}}$ diverges, if it is extended to an infinite configuration space. In fact, in all realistic calculations with finite range forces, $\Delta$ is not constant, but decreases with large angular momenta states above the Fermi surface. Therefore, a pairing window in all the equations are extended up-to the level $|\epsilon_i - \lambda| \leq 2(41A^{-1/3})$ as a function of the single particle energy. The factor 2 has been determined so as to reproduce the pairing correlation energy for neutrons in $^{118}\text{Sn}$ using Gogny force [10, 28, 31].

III. RESULTS AND DISCUSSIONS

In our very recent work [32], we have calculated the ternary mass distributions for $^{254}\text{Cf}$, $^{242}\text{Pu}$ and $^{236}\text{U}$ with the fixed third fragments $A_3 = ^{48}\text{Ca}$, $^{26}\text{O}$ and $^{16}\text{O}$ respectively for the three different temperatures $T = 1, 2$ and $3$ MeV within the TRMF formalism. The structure effects of binary fragments are also reported in Ref. [33]. In this article, we study the mass distribution of $^{250}\text{U}$ and $^{254}\text{Th}$ as a representative cases from the range of neutron-rich thermally fissile nuclei $^{246-264}\text{U}$ and $^{244-262}\text{Th}$. Because of the neutron-rich nature of these nuclei, a large number of neutrons emit during the fission process. These nucleons help to achieve the critical condition much sooner than the normal fissile nuclei.

To assure the predictability of the statistical model, we also study the binary fragmentation of naturally occurring $^{238}\text{U}$ and $^{232}\text{Th}$ nuclei. The possible binary fragments are obtained using the Eq. (1). To calculate the total binding energy at a given temperature, we use the axially symmetric harmonic oscillator basis expansion $N_F$ and $N_B$ for the Fermion and Boson wave-functions to solve the Dirac Eq. (11) and the Klein Gordon Eqs. (14 - 17) iteratively. It is reported [34] that the effect of basis space on the calculated binding energy, quadrupole deformation parameter ($\beta_2$) and the rms radii of nucleus are almost equal for the basis set $N_F = N_B = 12$ to 20 in the mass region $A \sim 200$. Thus, we use the basis space $N_F = 12$ and $N_B = 20$ to study the binary fragments up to mass number $A \sim 182$. The binding energy is obtained by minimizing the free energy, which gives the most probable quadrupole deformation parameter $\beta_2$ and the proton (neutron) pairing gaps $\Delta_p(\Delta_n)$ for the given temperature. At finite temperature, the continuum corrections due to the excitation of nucleons to be considered. The level density in the continuum depends on the basis space $N_F$ and $N_B$ [35]. It is shown that the continuum corrections need not be included in the calculations of level densities up-to the temperature $T \sim 3$ MeV [36, 37].

A. Level density parameter and level density within TRMF and FRDM formalisms

In TRMF, the excitation energies $E^*$ and the level density parameters $\alpha_i$ of the fragments are obtained self consistently from Eqns. (4) to (5). The FRDM calculations are also done for comparison. In this case, level density of the fragments are evaluated from the ground state single particle energies of the finite range droplet model (FRDM) of Möller et. al. [38] which are retrieved from the Reference Input Parameter Library (RIPL-3) [39]. The total energy at a given temperature is calculated as $E(T) = \sum n_i \epsilon_i; \epsilon_i$ are the ground state single particle energies and $n_i$ are the Fermi-Dirac distribution function. The $T$ dependent energies are obtained by varying the occupation numbers at a fixed particle number for a given temperature and given fragment. The level density parameter $\alpha$ is a crucial quantity in the statistical theory for the estimation of yields. These values of $\alpha$ for the binary fragments of $^{236}\text{U}$, $^{250}\text{U}$, $^{232}\text{Th}$ and $^{254}\text{Th}$ obtained from TRMF and FRDM are depicted in Fig. 1. The empirical estimation $\alpha = A/K$ are also given for comparison, with $K$, the inverse level density parameter. In general, the $K$ value varies from 8 to 13 with the increasing temperature. However, the level density parameter is considered to be constant up-to $T \approx 4$ MeV. Hence, we take the practical value of $K = 10$ as mentioned.
in Ref. [40]. The $a$ values of TRMF are close to the empirical level density parameter. The FRDM level density parameters are appreciably lower than the referenced $a$. Further, in both models at $T = 1$ MeV, there are more fluctuations in the level density parameter due to the shell effects of the fragments. At $T = 2$ and 3 MeV, the variations are small. This may be due to the fact that the shell become degenerate at the higher temperatures. All fragments becomes spherical at temperature $T \approx 3$ MeV as shown in Ref. [33]. The level density parameter $a$ is evaluated in two different ways using excitation energy and the entropy of the system as:

$$a_E = \frac{E^*}{T^2}, \quad (38)$$

$$a_S = \frac{S}{2T}.$$  

For instance, the inverse level density parameters $K_E$ and $K_S$ of $^{236}$U, $^{250}$U, $^{232}$Th and $^{254}$Th within TRMF formalism are depicted in Fig. 2. Both $K_S$ and $K_E$ have maximum fluctuation upto 30 MeV at $T = 1$ MeV. These values reduce to $10 - 13$ MeV at temperature $T = 2$ MeV or above. It is to be noted that at $T = 3$ MeV, the inverse level density parameter substantially lower around the mass number $A \sim 130$ in all

FIG. 1: (Color online) The level density parameter $a$ for the binary fragmentation of $^{236}$U, $^{250}$U, $^{232}$Th and $^{254}$Th at temperature $T = 1$, 2 and 3 MeV within the TRMF (solid lines) and FRDM (dash lines) formalisms.

FIG. 2: (Color online) The inverse level density parameters $K_E$ (solid lines) and $K_S$ (dash lines) are obtained for $^{236}$U, $^{250}$U, $^{232}$Th and $^{254}$Th at temperatures $T = 1, 2$ and 3 MeV.
FIG. 3: (Color online) The level density of the binary fragmentations of $^{236}$U, $^{250}$U, $^{232}$Th and $^{254}$Th at temperature $T = 1, 2$ and $3$ MeV within the TRMF (solid lines) and FRDM (dash lines) formalisms. Higher $\rho$ values. A further inspection reveals that the level density of the closed shell nucleus around $A \sim 130$ has higher value than the neighboring nuclei for both $^{236, 250}$U, but it has lower yield due to the smaller level density of the corresponding partners. At $T = 3$ MeV, the level density of the fragments around mass number $A \sim 72$ and $130$ have larger values compared to other fragments of $^{236}$U. On the other hand, the level density in the vicinity of neutron number $N = 82$ and proton number $Z = 50$ for the fragments of the neutron-rich $^{250}$U nucleus is quite high, because of the close shell of the fragments. This is evident from the small kink in the level density of $^{130}$Cd ($N = 82$), $^{132}$In ($N \sim 82$) and $^{135}$Sn ($Z = 50$). Again, for $^{232}$Th, the level densities are found to be maximum around mass number $A \sim 81$ and $100$ for $T = 2$ MeV. In case of $^{254}$Th, the level density parameters are calculated using Eq. (3). From the fragment level densities $\rho_i$, the folding density $\rho_{12}$ is calculated using the convolution integral as in Eq. (2) and the relative yields are calculated using Eq. (6). The total yields are normalized to the scale 2.

The mass yield of normal nuclei $^{236}$U and $^{232}$Th are briefly explained first, followed by the detailed description of the neutron rich nuclei. The results of most favorable fragments yield of $^{236, 250}$U and $^{232, 254}$Th are listed in Table I at three different temperatures $T = 1, 2$ and $3$ MeV.

The symmetric binary fragmentation $^{118}$Pd $+^{118}$Pd for $^{236}$U is the most favorable combination. In TRMF, the fragments with close shell ($N = 100$ and $Z = 28$) combinations are more probable at the temperature $T = 2$ MeV. The blend region of neutron and proton close shell ($N \approx 82$ and $Z \approx 50$) has the considerable yield values at $T = 3$ MeV. The fragmentations $^{151}$Pr $+^{85}$As, $^{142}$Cs $+^{94}$Rb and $^{144}$Ba $+^{92}$Kr are the favorable combinations at temperature $T = 1$ MeV in FRDM formalism. For higher temperatures $T = 2$ and $3$ MeV, the closed shell or near closed shell fragments ($N = 82, 50$ and $\ldots$)

cases. This may be due to the neutron closed shell ($N = 82$) in the fission fragments of $^{236}$U and $^{232}$Th and the neutron rich nuclei $^{250}$U and $^{254}$Th. The level density for the fission fragments of $^{236}$U, $^{250}$U, $^{232}$Th and $^{254}$Th are plotted as a function of mass number in Fig. 3 within the TRMF and FRDM formalisms at three different temperatures $T = 1, 2$ and $3$ MeV.

The level density $\rho$ has maximum fluctuations at $T = 1$ MeV for all considered nuclei in TRMF model similar to the level density parameter $a$. The $\rho$ values are substantially lower at mass number $A \sim 130$ for all nuclei. In Fig. 3, one can notice that the level density has small kinks in the mass region $A \sim 71 - 81$ of $^{236}$U and $A \sim 77 - 91$ of $^{250}$U, comparing with the neighboring nuclei at temperature $T = 2$ MeV. Consequently, the corresponding partner fragments have also
Z = 28) have larger yields. From Fig. 5 in TRMF formalism, the combinations $^{118}$Pd + $^{114}$Ru and $^{140}$Xe + $^{92}$Kr are the possible fragments at $T = 1$ MeV for the nucleus $^{232}$Th. At $T = 2$ MeV, we find maximum yields for the fragments with the close shell or near close shell combinations ($N = 82, 50$). For higher temperature $T = 3$ MeV, near the neutron close shell ($N \sim 82$), $^{132}$Sb + $^{100}$Y is the most favorable fragmentation pair compared with all other yields. Similar fragmentations are found in the FRDM formalism at $T = 2$ and 3 MeV. In addition, the probability of the evaluation of $^{120}$Sn + $^{103}$Zr is also quite substantial in the fission process. For $T = 1$ MeV, the yield is more or less similar with the TRMF model.

From Fig. 4, for $^{250}$U the fragment combinations $^{140, 141}$Te + $^{110, 109}$Zr have the maximum yields at $T = 1$ MeV in TRMF. This is also consistent with the evolution of the sub-close shell proton $Z = 40$ in $Zr$ isotopes [41]. Contrary to this almost symmetric binary yield, the mass distribution of this nucleus in FRDM formalism have the asymmetric evolution of the fragment combinations like $^{150, 159}$Pr + $^{90, 91}$As, $^{163, 162}$Nd + $^{87, 88}$Ge and $^{150}$Cs + $^{100}$Rb. Interestingly, at $T = 2$ and 3 MeV, the more favorable fragment combinations have one of the closed shell nuclei. At $T = 2$ MeV, $^{109}$Pr + $^{91}$As, $^{162}$Nd + $^{88}$Ge and $^{173}$Gd + $^{77}$Ni are the more probable fragmentations (see Fig. 4(c)). It is reported by Satpathy et al [42] and experimentally verified by Patel et al [43] that $N = 100$ is a neutron close shell for the deformed region, where $Z = 62$ acts like a magic number. In FRDM, $^{125}$Ag
The mass distributions or the relative yields calculated using TRMF and FRDM approaches mainly arise due to the differences in the level densities associated with these approaches. The mean values and the fluctuations in the level density parameter and the corresponding level density are even qualitatively different in both the approaches considered. This is possibly stemming from the fact that the single-particle energies in the FRDM based model are temperature independent. The temperature dependence of the excitation energy as required to calculate the level density parameter comes only from the modification of the single-particle occupancy due to the Fermi distribution. In the TRMF approach, the excitation energy for each fragment at a given temperature is calculated self-consistently. Therefore, the deformation and the single-particle energies changes with temperature.

For the neutron-rich nuclei, the fragments having neutron/proton close shell $N = 50, 82$ and $100$ have maximum possibility of emission at $T = 2$ and $3$ MeV (for both nuclei $^{250}\text{U}$ and $^{254}\text{Th}$). This is a general trend, we could expect for all neutron-rich nuclei. It is worthy to mention some of the recent reports and predictions of multi-fragment fission for neutron-rich uranium and thorium nuclei. When such a neutron-rich nucleus breaks into nearly two fragments, the products exceed the drip-line leaving few nucleons (or light nuclei) free. As a result, these free particles along with the scission neutrons enhance the chain reaction in a thermonuclear device. These additional particles (nucleons or light nuclei) responsible to reach the critical condition much faster than the usual fission for normal thermally fission nuclei. Thus, the neutron-rich thermally fissionable nuclei, which are in the case of $^{246-264}\text{U}$ and $^{244-262}\text{Th}$ will be very useful for energy production.

IV. SUMMARY AND CONCLUSIONS

The fission mass distributions of $\beta$-stable nuclei $^{236}\text{U}$ and $^{232}\text{Th}$ and the neutron-rich thermally fissionable nuclei $^{250}\text{U}$ and $^{254}\text{Th}$ are studied within the statistical theory. The possible combinations are obtained by equating the charge to mass ratio of the parents to that of the fragments. The excitation energies of fragments are evaluated from the temperature dependent self-consistent binding energies at the given $T$ and
the ground state binding energies which are calculated from the relativistic mean field model. The level densities and the yields combinations are manipulated from the convolution integral approach. The fission mass distributions of the aforementioned nuclei are also evaluated from the FRDM formalism for comparison. The level density parameter $a$ and inverse level density parameter $K$ are also studied to see the difference in results with these two methods. Besides fission fragments, the level densities are also discussed in the present paper. For $^{236}\text{U}$ and $^{232}\text{Th}$, the symmetric and nearly symmetric fragmentations are more favorable at temperature $T = 1, 2$ and $3$ MeV. Interestingly, in most of the cases we find one of the favorable fragment is a close shell or near close shell configuration ($N = 82, 50$ and $Z = 28$) at temperature $T = 2$ and $3$ MeV. This result ascertains with our earlier predictions. Further, Zr isotopes has larger yield values for $^{250}\text{U}$ and $^{254}\text{Th}$ with their accompanied possible fragments at $T = 1$ MeV. The Ba and Cs isotopes with their partners are also more possible for $^{254}\text{Th}$. This could be due to the deformed close shell in the region $Z = 52 - 66$ of the periodic table [46]. The Ni isotopes and the neutron close shell ($N \sim 100$) nuclei are some of the prominent yields for both $^{250}\text{U}$ and $^{254}\text{Th}$ at temperature $T = 2$ MeV. At $T = 3$ MeV, the neutron close shell ($N = 82$) is one of the largest yield fragments. The symmetric fragmentation $^{252}\text{Rh} + ^{252}\text{Rh}$ is possible for $^{254}\text{Th}$ due to the $N = 82$ close shell occurs in binary fragmentation. For $^{250}\text{U}$, the larger yield values are confined to the junction of neutron and proton closed shell nuclei.

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