Application of information-percolation method to reconstruction problems on graphs

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Abstract

In this paper we propose a method of proving impossibility results based on applying strong data-processing inequalities to estimate mutual information between sets of variables forming certain Markov random fields. The end result is that mutual information between two “far away” (as measured by the graph distance) variables is bounded by the probability of existence of open path in a bond-percolation problem on the same graph. Furthermore, stronger bounds can be obtained by establishing mutual comparison results with an erasure model on the same graph, with erasure probabilities given by the contraction coefficients.

As application, we show that our method gives sharp threshold for partially recovering a rank-one perturbation of a random Gaussian matrix (spiked Wigner model), recovers (and generalizes) the best known upper bound on noise-level for group synchronization due to Abbe and Boix, and establishes new impossibility result for a $k$-community detection (stochastic block model).

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1 Introduction

As a generalization of ideas of Evans-Schulman [ES99], a method for upper-bounding the mutual information between sets of variables via a probability of existence of a percolation path was proposed by the authors in [PW17, Theorem 5]. This allows one to reuse results on critical threshold
for percolation to show vanishing of mutual information. Original bound was stated for Bayesian networks (known as directed graphical models) but in this paper we show that similar results can be obtained for certain Markov random fields too, especially those appearing in questions such as community detection and group synchronization.

Our original motivation was to obtain a simple proof of a result of Abbe and Boix [AB18], improving the earlier work of Abbe, Massoulie, Montanari, Sly and Srivastava [AMM+17] (the new result, for the case of a 2D square grid, is stated below as Corollary 5).

The paper is organized as follows. First, we present the idea in its simplest form in Section 2. This is, however, already sufficient to recover the result of Abbe and Boix [AB18]. Second, we extend the method in two different directions in Sections 3 and 4. These extensions are then applied in Section 5 to group synchronization, spiked Wigner model and stochastic block model with $k$-blocks. For the latter our results strengthen (in some regime) the best known impossibility results on correlated (partial) recovery for $k = 3$.

## 2 Information–percolation bound (basic version)

Consider a simple undirected graph $G = (V, E)$ with finite or countably-infinite $V$. Let $\{X_v : v \in V\}$ be i.i.d. Bern$(1/2)$ and let $\{Z_v : v \in E\}$ be i.i.d. Bern$(\delta)$. For each $e = (u, v) \in E$, let $Y_e = X_u + X_v + Z_e$. For any $S$, let $X_S = \{X_v : v \in S\}$.

We recall some basic notions from information theory. The mutual information $I(X; Y)$ between random variables $X$ and $Y$ with joint law $P_{XY}$ is $I(X; Y) = D(P_{XY} || P_X \otimes P_Y)$, where $D(P||Q) = \int dP \log \frac{dP}{dQ}$ if $P \ll Q$ and $\infty$ otherwise. Two properties of mutual information are particularly useful here: (a) Chain rule: $I(X; Y, Z) = I(X; Y) + I(X; Z|Y)$, where $I(X; Z|Y) = 0$. (b) Data processing inequality (DPI): whenever $W \to X \to Y$ forms a Markov chain, we have $I(W; Y) \leq I(W; X)$. Furthermore, a quantitative version of the DPI is the strong data processing inequality (SDPI), $I(W; Y) \leq \eta(P_Y|X)I(W; X)$ where $\eta(P_Y|X) \in [0, 1]$ is called the KL contraction coefficient of the channel. For example, if $P_{Y|X}$ is the binary symmetric channel (BSC) with flip probability $\delta$, denoted by $\text{BSC}(\delta)$, that is, $Y = X \oplus Z$, where $Z \sim \text{Bern}(\delta)$ is independent of $X$, we have $\eta(\text{BSC}(\delta)) = (1 - 2\delta)^2$. For more on SDPI, we refer the reader to the survey [PW17] and the references therein.

Let $\text{ER}(G, p)$ denote the random graph on the vertex set $V$ where each edge $e \in E$ is kept independently with probability $p$.

**Theorem 1.** For any subset $S \subset V$ and any vertex $v \in V$,

$$I(X_v; X_S, Y_E) \leq \text{perc}_G(v, S) \log 2,$$

where $\text{perc}_G(v, S) = P[v \text{ is connected to } S \text{ in } \text{ER}(G, \eta)]$, with

$$\eta \triangleq (1 - 2\delta)^2.$$

**Remark 1.** Notice that right-hand side of (1) can be seen as $I(X_v; X_S, \tilde{Y}_E)$ where $\tilde{Y}_e, e = (u, v)$ is a random variable equal to $X_u + X_v$ with probability $\eta$ and * (erasure) otherwise. This is not accidental – it can be shown via [PW17, Prop. 15, 16] that observations over the erasure channel $\text{BEC}(\eta)$ lead to strictly larger mutual informations: $I(X_{S_1}; Y_E|X_{S_2}) \leq I(X_{S_1}; \tilde{Y}_E|X_{S_2})$, regardless of the joint distribution $P_{X_{S_1}}$. This generalization is pursued in Section 4.
Proof. By the monotone convergence property of mutual information (and probability), it suffices to consider finite graph $G$.

Let $\tilde{X}_V = (1 + x_v \text{ mod } 2)_{v \in V}$. The symmetry of the problem shows that

$$(X_V, Y_E) \overset{d}{=} (\tilde{X}_V, Y_E).$$

In particular, we have

$$I(X_v; Y_E) = 0$$

for any $v$.

Fix $V$ and $v \in V$. We induct on the number of edges $|E|$. For the base case of $E = \emptyset$, by the independence of $\{X_v\}$, we have

$$I(X_v; X_S) = 1 \{v \in S\} \log 2 = \text{perc}_G(v, S) \log 2.$$

Next suppose (1) holds for all $G' = (V, E')$ with $|E'| < |E|$ and all $S$, i.e.

$$I(X_z; X_S, Y_{E'}) \leq \text{perc}_{G'}(z, S) \log 2. \tag{3}$$

We now show (1) holds for $E$. Fix $S$. Suppose there is no edge in $E$ incident to any vertex in $S$. Then both sides of (1) are zero by (2). Otherwise, there exists an edge $e = (u, z) \in E$ incident to some vertex $z \in S$. Set $E' = E \setminus e$ and $G' = (V, E')$.

Next we apply the strong data processing inequality (SDPI) for BSC (see [PW17] for a survey on SDPIs): since $Y_e = X_u + X_z + Z_e$, where $Z_e \sim \text{Bern}(\delta)$ is independent of everything else. Thus, conditioned on $(X_S, Y_{E'})$, we have the Markov chain: $X_v \rightarrow X_u \rightarrow Y_e$. Therefore

$$I(X_v; Y_e | X_S, Y_{E'}) \leq \eta I(X_v; X_u | X_S, Y_{E'}).$$

Adding $I(X_v; X_S, Y_{E'})$ to both sides gives

$$I(X_v; X_S, Y_E) \leq \eta I(X_v; X_S, X_u, Y_{E'}) + \bar{\eta} I(X_v; X_S, Y_{E'})$$

Applying the induction hypothesis (3) to the RHS of the above display, we have:

$$I(X_v; X_S, Y_E) \leq \eta \cdot \text{perc}_{G'}(v, S \cup \{u\}) + \bar{\eta} \cdot \text{perc}_{G'}(v, S) = \text{perc}(G)(v, S)$$

\[ \square \]

2.1 Simple example of tightness of the bound

Consider $G$ a complete infinite $d$-ary tree, with $X_\rho$ – root and $X_{S_k}$ – the set of all nodes at depth $k$. Then, by broadcasting on trees [EKPS00], it is easy to see that

$$\lim_{k \to \infty} I(X_\rho; Y_E, X_{S_k}) = \begin{cases} 0, & (1 - 2\delta)^2 d \leq 1 \\ > 0, & (1 - 2\delta)^2 d > 1 \end{cases} \tag{4}$$

The bound in Theorem 1 is tight in this case in the sense that the right-hand side of (1) converges to zero if and only if $(1 - 2\delta)^2 d \leq 1$. 

3
3 General version: information percolation

Consider a bipartite graph $G = (V, W, E)$ with parts $V, W$ and edges $E$, with finite or countably-infinite $V, W, E$. For any subset $W' \subset W$ we will denote $G[W']$ the induced subgraph on vertices $V \cup W'$.

Let $\{X_v : v \in V\}$ be a collection of independent discrete random variables. Let $\{Y_w : w \in W\}$ be a collection of random variables conditionally independent given $X_V$ and distributed each as

$$Y_w \sim P_{Y_w|X_{N(w)}} \quad \forall w \in W,$$

where $N(w) \subset V$ denote the neighborhood of $w$ in the bipartite graph. Let $\eta_w \triangleq \eta_{KL}(P_{Y_w|X_{N(w)}})$ be the SDPI constant corresponding to this channel [PW17].

Let $\text{ER}(G)$ denote the random subgraph $G[W']$ where each vertex $w \in W$ is included in $W'$ independently with probability $\eta_w$. For a pair of sets $S_1, S_2 \subset V$ we define the average number of vertices in $S_1$ that are connected to $S_2$:

$$\text{perc}_G(S_1, S_2) \triangleq \sum_{v \in S_1} \mathbb{P}[v \text{ is connected to } S_2 \text{ in } \text{ER}(G)].$$

We note the following easy to verify identity: if $w$ is such that $N(w) \cap S_2 \neq \emptyset$ then

$$\text{perc}_G(S_1, S_2) = \eta_w \text{perc}_{G[W\setminus{w}])(S_1, S_2 \cup N(w))} + (1 - \eta_w)\text{perc}_{G[W\setminus{w}])(S_1, S_2).}$$

(5)

To get back to the setting of the previous section, where graph was simple, we let bipartite graph be the incidence graph between vertices and edges (in this case degree of every $w \in W$ is 2).

**Theorem 2.** For any subsets $S_1, S_2$ of $V$, we have

$$I(X_{S_1}; X_{S_2}|Y_W) \leq \text{perc}_G(S_1, S_2) \cdot \sup_{v \in V} H(X_v).$$

(6)

**Remark 2.** Note that $I(X_{S_1}; Y_W) = 0$ does not hold even in the setting of the previous section, unless $S_1$ is a singleton (see (2)). Indeed, one may consider the graph $a - b - c$ in the context of Theorem 1. For $S_1 = \{a, c\}$, $I(X_{a,c}; Y_{ab, bc}) \geq I(X_a + X_c; Y_{ab} + Y_{bc}) \geq 1 - h(2\delta(1 - \delta)).$ Thus $I(X_{S_1}; X_{S_2}, Y_W) \neq I(X_{S_1}; X_{S_2}|Y_W)$ and the former does not satisfy the inequality in Theorem 2.

**Proof.** Again, because of the identity

$$I(X_{S_1}; X_{S_2}|Y_W) = I(X_{S_1}; X_{S_2}, Y_W) - I(X_{S_1}; Y_W)$$

and continuity of mutual information and percolation probability we may consider finite $S_1, S_2, W$ only.

We will prove (6) by induction on $|W|$. Assume that

$$H(X_v) \leq H_1 \quad \forall v \in V.$$

First, suppose that $W = \emptyset$. We have then:

$$I(X_{S_1}; X_{S_2}) = \sum_{i \in S_1 \cap S_2} H(X_i) \leq |S_1 \cap S_2|H_1 = \text{perc}_{G[W']}(S_1, S_2)H_1.$$

Next, suppose that we have shown (6) for all $G[W']$ with $|W'| < |W|$. Consider two cases.
Case 1. There does not exist \( w \in W \) such that \( N(w) \cap S_2 \neq \emptyset \). Then, we have

\[
I(X_{S_1}; X_{S_2}|Y_W) \leq I(X_{S_1}; Y_W; X_{S_2}) \leq I(X_{S_1}; X_{S_0}; X_{S_2}) \leq |S_1 \cap S_2| H_1,
\]

where \( S_0 = \bigcup_{w \in W} N(w) \) and the last equality is due to \( S_0 \cap S_2 = \emptyset \). Similarly, we have

\[
\text{perc}_G(S_1, S_2) = |S_1 \cap S_2|
\]

and (6) is established.

Case 2. There exists \( w \in W \) such that \( N(w) \cap S_2 \neq \emptyset \). Let \( W' = W \setminus w \) and consider the chain

\[
I(X_{S_1}; X_{S_2}, Y_{W'}, Y_w) = I(X_{S_1}; X_{S_2}, Y_{W'}) + I(X_{S_1}; Y_w|X_{S_2}, Y_{W'}) \leq I(X_{S_1}; X_{S_2}, Y_{W'}) + \eta_w I(X_{S_1}; X_{N(w)}|X_{S_2}, Y_{W'})
\]

(7)

\[
= (1 - \eta_w) I(X_{S_1}; X_{S_2}, Y_{W'}) + \eta_w I(X_{S_1}; X_{N(w)}|X_{S_2}, Y_{W'})
\]

(8)

\[
= (1 - \eta_w) I(X_{S_1}; X_{S_2}|Y_{W'}) + \eta_w I(X_{S_1}; X_{N(w)}|X_{S_2}|Y_{W'}) + I(X_{S_1}; Y_{W'})
\]

(9)

\[
= (1 - \eta_w) I(X_{S_1}; X_{S_2}|Y_{W'}) + \eta_w I(X_{S_1}; X_{N(w)}|X_{S_2}|Y_{W'}) - I(X_{S_1}; Y_W)
\]

(10)

where the inequality is an application of the SDPI, which is justified since given \( X_{S_2}, Y_{W'} \) we still have the Markov chain: \( X_{S_1} \to X_{N(w)} \to Y_w \).

Subtracting \( I(X_{S_1}; Y_{W'}) \) from both sides of (10) we get

\[
I(X_{S_1}; X_{S_2}|Y_W) \leq (1 - \eta_w) I(X_{S_1}; X_{S_2}|Y_{W'}) + \eta_w I(X_{S_1}; X_{N(w)}|X_{S_2}|Y_{W'}) + I(X_{S_1}; Y_{W'}) - I(X_{S_1}; Y_W)
\]

(11)

\[
\leq (1 - \eta_w) I(X_{S_1}; X_{S_2}|Y_{W'}) + \eta_w I(X_{S_1}; X_{N(w)}|X_{S_2}|Y_{W'})
\]

(12)

since \( I(X_{S_1}; Y_{W'}) \leq I(X_{S_1}; Y_W) \) by monotonicity of mutual information. From induction hypothesis and (5) we conclude the proof of (6). \( \square \)

4 General version: channel comparison

In the setting of Section 2 we have condition (2) which implies

\[
I(X_v; X_S, Y_E) = I(X_v; X_S|Y_E) = I(X_v; Y_E|X_S).
\]

Consequently, Theorem 1 (giving a bound on the first quantity) and Theorem 2 (giving a bound on the second one) are equivalent in the case of (2). However, Theorem 2 holds in wider generality. Can we also bound the third quantity? It turns out the answer is yes, and in fact this generalization allows to remove the most restrictive condition of Theorem 2 – the independence of \( X \)’s. (However, the two Theorems bound different quantities.) To focus ideas, we recommend revisiting Remark 1.

We proceed to describing the setting. Consider a bipartite graph \( G = (V, W, E) \) with parts \( V, W \) and edges \( E \), with finite or countably-infinite \( V, W, E \). For any subset \( W' \subset W \) we will denote \( G[W'] \) the induced subgraph on vertices \( V \cup W' \).

Let \( \{X_v : v \in V \} \) be a collection of discrete random variables (not necessarily independent). Let \( \{Y_w : w \in W \} \) and \( \{\bar{Y}_w : w \in W \} \) be two collection of random variables each conditionally independent given \( X_V \) and distributed as

\[
Y_w \sim P_{Y_w|X_{N(w)}} \quad \forall w \in W,
\]

(13)

\[
\bar{Y}_w \sim Q_{Y_w|X_{N(w)}} \quad \forall w \in W
\]

(14)

where \( N(w) \subset V \) denote the neighborhood of \( w \) in the bipartite graph.
We also remind the definition of less-noisy relation: stochastic matrix $Q_{\tilde{Y}|X}$ is less-noisy than $P_{Y|X}$ if for every distribution $P_{U,X}$ we have
\[ I(U;Y) \leq I(U;\tilde{Y}) \]
where mutual informations are computed under the joint distribution
\[ P_{U,X,Y}(u,x,y) = P_{U,X}(u,x)Q_{\tilde{Y}|X}(y|x)P_{Y|X}(y|x). \]
See [vD97, Theorem 2], [PW17, Prop. 14] and [MP16, Theorem 2, Prop. 8] for various characterizations of a less-noisy relation.

**Theorem 3.** If for every $w$ channel $Q_{\tilde{Y}_w|X_{N(w)}}$ is less-noisy than $P_{Y_w|X_{N(w)}}$ then for any subsets $S_1, S_2 \subset V$ we have
\[ I(X_{S_1};Y_{E|X_{S_2}}) \leq I(X_{S_1};\tilde{Y}_E|X_{S_2}). \quad (15) \]

**Remark 3.** Connection between Theorems 3 and 2 arises from [PW17, Proposition 15]: the channel $P_{Y|X}$ has $\eta_{KL}(P_{Y|X}) \leq 1 - \delta$ if and only if $P_{Y|X}$ is more-noisy than a channel $Q_{\tilde{Y}|X}$ which sets $\tilde{Y} = X$ with probability $1 - \delta$ and otherwise set $\tilde{Y} = ?$ (erasure).

**Remark 4.** One cannot replace the less-noisy condition with “more-capable”, a weaker notion (see [KM75]). Indeed, it is known that erasure channel with probability of erasure $1 - h(\delta)$ is more-capable than BSC(\(\delta\)). But then consider the example in Section 2.1. If the more-capable variation of Theorem 3 were true, we’d be able to reduce probability of an open bond from $(1 - 2\delta)^2$ to $1 - h(\delta)$ and thus contradict (4).

**Proof.** Conditioning on $X_{S_2}$ we get a Markov chain $X_{S_1} \to X_E \to Y_{E|X_{S_2}}$. By [PW17, Prop. 14] less-noisy relation tensorizes. That is, the channel $X_E \to Y_E$ is less-noisy than $X_E \to Y_E$.

Consequently, we get (15).

5 Applications to statistical reconstruction

5.1 Group synchronization over $\mathbb{Z}/2\mathbb{Z}$

**Corollary 4.** In the setting of Section 2, consider the problem of reconstructing $T_{uv} = X_v + X_u \mod 2$ for two (possibly non-adjacent) vertices $u, v \in V$ given $Y_E$ (observations of all edges). We have for any estimator $\hat{T}_{uv} = \hat{T}_{uv}(Y_E)$:
\[ \mathbb{P}[\hat{T}_{uv} \neq T_{uv}] \geq \frac{1}{2} - \sqrt{\frac{1}{2\log e} I(X_u;X_v,Y_E)} - \sqrt{\frac{\log 2}{2\log e} \text{perc}_G(v,u)} \]
\[ \quad \geq \frac{1}{2} - \sqrt{\frac{\log 2}{2\log e} \text{perc}_G(v,u)} \quad (16) \]
Consequently,
\[ \frac{1}{|V|^2} \sum_{u,v \in V} \mathbb{P}[\hat{T}_{uv} \neq T_{uv}] \geq \frac{1}{2} - o(1) \quad (17) \]
provided
\[ \sum_{u,v \in V} I(X_u;X_v,Y_E) = o(|V|^2) \quad \text{or} \quad \sum_{u,v \in V} \text{perc}_G(v,u) = o(|V|^2). \]
Remark 5. It is clear, from Theorem 2, that the result above extends to arbitrary channels \( P_{Y_e|X_u,X_v} \) for \( e = (u, v) \), arbitrary function \( T = T(X_u, X_v) \) and arbitrary (discrete) \( X_v \). The only general requirement we need to impose is validity of (2). The only change is that the first term \( \frac{1}{2} \) in the right-hand side of (16) should be replaced with \( 1 - \max_s \mathbb{P}[T(X_u, X_v) = s] \) and \( \log 2 \) in the denominator inside the square root with \( \max_v H(X_v) \). We put this corollary first, as it originally motivated writing of this note.

Proof. It suffices to show (16) as the rest follows from Jensen’s inequality. Next abbreviate \( T_{uv} \) as \( T \). Note that

\[
I(T; Y_E) \leq I(X_u, X_v; Y_E) = I(X_u; Y_E|X_v) + I(X_v; Y_E)
\]

where (a) is the data processing inequality for mutual information; (b) follows from (2); (c) follows from the assumption that \( X_u \perp \perp X_v \); (d) follows from Theorem 1.

On the other hand, for any estimator \( \hat{T} = \hat{T}(Y_E) \), let \( p = \mathbb{P}[\hat{T} = T] \) and \( q = \mathbb{Q}[\hat{T} = T] \), where \( \mathbb{Q} \) denote the probability measure where \( Y_E \) and \( T \) are independent. Thus \( q \leq P_{\max}(T) \triangleq \max_t \mathbb{P}[T = t] \). By the data processing inequality and the Pinsker inequality, we have

\[
I(T; Y_E) \geq d(p||q) \geq 2 \log e(p - q)^2.
\]

Thus,

\[
\mathbb{P}[\hat{T} = T] \leq P_{\max}(T) + \sqrt{\frac{\text{perc}_G(v, u) \log 2}{2 \log e}}.
\]

Using Kesten’s result on 2D-square grid percolation [Kes80], we get:

Corollary 5. Let \( G \) be an infinite 2D-grid and suppose the goal is to estimate \( T_n = X_{0,0} + X_{n,n} \mod 2 \) for large \( n \) given observations of all (infinitely many) edges \( Y_e \). If

\[
(1 - 2\delta)^2 \leq \frac{1}{2}
\]

then for any estimator \( \hat{T}_n = \hat{T}_n(Y_E) \) we have \( \mathbb{P}[\hat{T}_n \neq T_n] \to \frac{1}{2} \).

5.2 Spiked Wigner model

Consider the following statistical model for PCA:

\[
Y = \sqrt{\frac{\lambda}{n}} X X^\top + W \tag{18}
\]

where \( X = (X_1, \ldots, X_n) \in \{\pm 1\}^n \) consists of independent Rademacher entries, and \( W \) is a Wigner matrix which is symmetric consisting of independent standard normal off-diagonal entries. This ensemble is known as the spiked Wigner model (rank-one perturbation of the Wigner ensemble).
Observing the matrix $Y$, the goal is to achieve correlated recovery, i.e., to reconstruct $X$ (up to a global sign flip) better than chance, that is, find $\hat{X} = \hat{X}(Y) \in \{\pm 1\}^n$, such that
\[
\liminf_{n \to \infty} \frac{1}{n} \mathbb{E}[\|X, \hat{X}\|] > 0. \tag{19}
\]

It is known that for fixed $\lambda$, if $\lambda > 1$, spectral method (taking the signs of the first eigenvector of $Y$) achieves correlated recovery [BAP05]. Conversely, if $\lambda < 1$, correlated recovery is information-theoretically impossible.

As the next result shows, applying Theorem 1 together with classical results on Erdős-Rényi graphs immediately yields the optimal threshold previously obtained in [DAM15, Theorem 4.3]. Here, $o(1)$ is any vanishing factor so this result is the best possible.

**Corollary 6.** Correlated recovery in the sense of (19) is impossible if
\[
\lambda \leq 1 + o(1). \tag{20}
\]

**Proof.** Note that (19) is equivalent to
\[
\limsup_{n \to \infty} \frac{1}{n^2} \mathbb{E}\left[\left\|XX^\top - \hat{X}\hat{X}^\top\right\|^2_F\right] < 2. \tag{21}
\]

It is clear that the diagonal entries of $X$ are independent of $X$ and hence the problem reduces to the setting in Section 2 with $G$ being the complete graph on $n$ vertices and $Y_{ij} = \sqrt{\frac{2}{n}}X_iX_j + W_{ij}$ for $i < j$. Applying Theorem 1 together with Corollary 4, we conclude that: for any $i < j$,
\[
\inf_{\tilde{T}_{ij}(\cdot)} \mathbb{P}\left[ X_iX_j \neq \tilde{T}_{ij}(Y) \right] \geq \frac{1}{2} - O(\mathbb{P}[i \text{ and } j \text{ are connected in } G(n, \eta)]).
\]

where $\eta = \eta(N(-\sqrt{n}^{-1/2}), N(\sqrt{n}^{-1})) = \frac{\lambda}{n}(1 + o(1))$ in view of (49). Summing over $i \neq j$, we conclude that for any $\hat{X} = \hat{X}(Y) \in \{\pm 1\}^n$,
\[
\mathbb{E}\left[\left\|XX^\top - \hat{X}\hat{X}^\top\right\|^2_F\right] = 4 \sum_{i \neq j} \mathbb{P}\left[ X_iX_j \neq \hat{X}_i\hat{X}_j \right]
\geq 2n^2 - 2 \sum_{i \in [n]} \mathbb{E}[\text{size of the connected component in } G(n, \eta) \text{ containing } i]
\geq 2n^2 - n \mathbb{E}[C_{\max}],
\]

where $C_{\max}$ denotes the size of the largest connected component in $G(n, \eta)$. Existing results in the random matrix theory show that $\mathbb{E}[C_{\max}] = o(n)$ whenever $\eta = \frac{\lambda}{n}(1 + o(1))$, which implies the impossibility of (21). Specifically, let $\eta = \frac{n}{n^s}(n + s)$, where $s = o(n)$ by assumption. By monotonicity, it suffices to consider the case of $s = \omega(n^{2/3})$. By a result of Luczak [Luc90, Lemma 3] (see also [JLR00, Theorem 5.12]), we have $C_{\max} \leq c_0s$ with probability at least $1 - c_1n^{1/3}s^{-1/2}$ for some universal constants $c_0, c_1$. Since $C_{\max} \leq n$, this shows $\mathbb{E}[C_{\max}] = o(n)$, completing the proof. \qed

**Remark 6** (Channel universality). Consider a more general observation model than (18): Let $P(\cdot|\theta)$ be a family of conditional distributions parametrized by $\theta \in \mathbb{R}$, with conditional density $p_{\theta}(\cdot)$ with respect to some reference measure $\mu$. Given $M = \sqrt{\frac{2}{n}}XX^\top$, we observe the matrix $Y = (Y_{ij})$, where
each $Y_{ij}$ is obtained by passing $M_{ij}$ through the same channel independently, with the conditional distribution given by $P_{Y_{ij}|M_{ij}} = P(\cdot|M_{ij})$. The spiked Wigner model corresponds to the Gaussian channel $P(\cdot|\theta) = N(\theta,1)$.

Under appropriate regularity conditions on the channel, the sharp threshold (20) is replaced by the following:

$$\lambda \leq \frac{1}{J_0} + o(1)$$

where $J_0 \triangleq \int \frac{\partial^2}{\partial \theta^2} f\left( \frac{1}{p_0} d\mu \right) d\mu$ is the Fisher information. This follows from the relationship between the contraction coefficient and the Fisher information. To see why this is true intuitively, note that $M_{ij} \in \{\pm \epsilon\}$, with $\epsilon \triangleq \sqrt{\frac{2}{n}}$. Using the characterization (44) of the contraction coefficient for binary-input channels, we have $\eta = \sup_{\beta \in [0,1]} LC_\beta(p_\epsilon\|p_\delta)$, where $LC_\beta$ is an $f$-divergence\(^1\) with $f(x) = f_\beta(x) = \beta \frac{(x-1)^2}{\beta x + \beta}$. By the local expansion of $f$-divergence, we have $D_f(P_\delta\|P_\delta) = \frac{\delta^2(1 + o(1))}{1 - \beta + o(1)}$ as $\delta \to 0$. Note that $f''_\beta(1) = 2\beta \bar{\beta}$, maximized at $\beta = \frac{1}{2}$. It follows that $\eta = \frac{\lambda J_0 + o(1)}{n}$. Thus the same percolation bound used in Corollary 6 shows that (22) implies the impossibility of correlated construction. In the positive direction, it was suggested in [LKZ15, Section II-C] showed that spectral method applied to the score matrix succeeds provided that $\lambda > \frac{1}{J_0}$ (see also [KZ16] about the provable phase transition of the mutual information $I(M;Y)$ at this point).

5.3 Community detection: two communities

Consider a complete graph $K_n$ and $X_i \overset{i.i.d.}\sim Bern(1/2)$. Unlike the group-synchronization case, we have the following observation channel: for each edge $e = (u,v)$ we have

$$Y_e = \begin{cases} 
\text{Bern}(p), & X_u = X_v \\
\text{Bern}(q), & X_u \neq X_v 
\end{cases}$$

(23)

In other words, $Y$ is the adjacency matrix of a random graph (known as the stochastic block model), in which any pair of vertices are connected with probability $p$ if they are from the same community (with the same labels) or with probability $q$ otherwise.

Given the matrix $Y = (Y_{ij})$, the goal is to achieve correlated recovery, that is, estimating the labels up to a global flip better than random guess. In other words, construct $\hat{X} = \hat{X}(Y) \in \{0,1\}^n$, such that

$$\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}[\min\{d(\hat{X},X), n - d(\hat{X},X)\}] < \frac{1}{2},$$

(24)

where $d$ denotes the Hamming distance. Equivalently, the goal is to estimate $1_\{X_i = X_j\}$ for any pair $i,j$ on the basis of $Y$ with probability of error asymptotically (as $n \to \infty$) not tending to $1/2$. The exact region when this is impossible is known [MNS15, MNS13]: for $p = \frac{a}{n}$ and $q = \frac{b}{n}$ with fixed $a,b$, correlated recovery is possible if and only if

$$a + b < 2 \quad \text{or} \quad \frac{(a - b)^2}{2(a + b)} < 1.$$  

Applying the information-percolation method (namely Theorem 2) we get the following slightly suboptimal result (see Fig. 1).

---

\(^1\)Recall an $f$-divergence is defined as $D_f(P\|Q) = \mathbb{E}_P[f(\frac{dP}{dQ})]$ for convex $f$ with $f(1) = 0$ [Csi69].
Proposition 7. For the binary stochastic block model with edge probabilities \( p \) and \( q \), for any \( i \neq j \in [n] \), we have the following non-asymptotic bound:

\[
I(X_i; X_j, Y_E) \leq \Pr[i \text{ and } j \text{ are connected in } G(n, \eta)]
\]

where \( \eta = p + q - 2pq + 2\sqrt{p(1-p)q(1-q)} \). Furthermore, if \( p = \frac{a}{n} \) and \( q = \frac{b}{n} \), then correlated recovery (i.e., (24)) is impossible if

\[
(\sqrt{a} - \sqrt{b})^2 < 1 + o(1).
\]

**Proof.** The mutual information bound (25) follows from Theorem 1 and the exact expression for the contraction coefficients in (45), which satisfies

\[
\eta_{KL}(\text{Bern}(a/n), \text{Bern}(b/n)) = \frac{(\sqrt{a} - \sqrt{b})^2 + o(1)}{n},
\]

where the \( o(1) \) terms is uniform in \( (a, b) \) in view (48). The remaining proof is the same as Corollary 6 using the behavior of the giant component of the Erdős-Rényi graph. \( \square \)

5.4 Community detection: \( k \) communities

In the setting of previous section, suppose now that \( X_v \overset{i.i.d.}{\sim} \text{Unif}[k] \), with the same observation channel (23). This is the stochastic block model with \( k \) equal-sized communities, and the notion of correlated recovery is extended as follows: for any \( x, \hat{x} \in [k]^n \), define the following error metric:

\[
d(x, \hat{x}) \triangleq \min_{\pi \in S_k} \frac{1}{n} \sum_{i \in [n]} 1\{x_i \neq \pi(\hat{x}_i)\}
\]

that is, the number of classification errors up to a global permutation of labels. We say correlated recovery is possible if there exists a (sequence of) estimator \( \hat{X} \in [k]^n \) that outperforms random guessing, i.e.,

\[
\limsup_{n \to \infty} \mathbb{E}[d(X, \hat{X})] < \frac{k - 1}{k},
\]
For $k \geq 3$, the sharp threshold is not known. In terms of the impossibility result, the best known sufficient condition is [BMNN16, Theorem 1]

$$\frac{(a - b)^2}{a + (k - 1)b} < \frac{2k \log(k - 1)}{k - 1}$$

(30)

Now, it turns out that applying Theorem 1 would only yield a $k$-independent bound (26). To get an improved estimate, instead, we use the comparison theorem with the erasure model in Theorem 3 and then show the impossibility of reconstruction on the corresponding erasure model. The threshold is given by (31) in the next proposition and the numerical comparison with the bound of (30) is shown in Fig. 2. For $k = 3$, (31) improves over (30) in some regime but not for $k = 4$. For large $k$, (31) is suboptimal by a logarithmic factor.

**Proposition 8.** Correlated recovery in the sense of (29) is impossible if

$$(\sqrt{a} - \sqrt{b})^2 \leq \frac{k}{2}$$

(31)

**Proof.** We start by setting up the mutual comparison with the corresponding model per Theorem 3. Let $\eta = \frac{(\sqrt{a} - \sqrt{b})^2 + o(1)}{n}$ be given in (27). Define the corresponding erasure model on the same graph: for each $(u, v) \in \binom{[n]}{2}$, let $\tilde{Y}_{uv} = 1_{\{X_u = X_v\}}$ with probability $\eta$ and $\tilde{Y}_{uv} = ?$ with probability $1 - \eta$ independently. Equivalently, the reconstruction problem under the erasure model can be phrased as follows. Let $G = ([n], E)$ denote an Erdős-Rényi graph $G(n, \eta)$ independent of $X$. Then for each $(u, v) \in E$, we observe a deterministic function $\tilde{Y}_{uv} = 1_{\{X_u = X_v\}}$. By Theorem 3 and Remark 3, we have the following comparison result: for any $S \subset [n]$

$$I(X_S; Y) \leq I(X_S; \tilde{Y}).$$

(32)

By symmetry, $I(X_S; \tilde{Y})$ only depends on $|S|$. Next we assume $S = [m]$ and show that

$$I(X_S; \tilde{Y}) = o(1), \quad n \to \infty,$$

under the condition that $(\sqrt{a} - \sqrt{b})^2 \leq \frac{k}{2}$.
By the chain rule, we have
\[
I(X_S; \tilde{Y}) = I(X_1; \tilde{Y}) + I(X_2; \tilde{Y}|X_1) + \ldots + I(X_m; \tilde{Y}|X_1, \ldots, X_{m-1})
\]
\[
= \sum_{u=2}^{m} I(X_u; X_1, \ldots, X_{u-1}, \tilde{Y}),
\]
where we used the fact that \(X_i\)'s are independent and \(I(X_1; \tilde{Y}) = 0\).

Next using the local tree structure of \(G\), we show that for each \(u\), \(I(X_u; X_1, \ldots, X_{u-1}, \tilde{Y}) = o(1)\). Condition on the realization of \(G\). Fix \(t\) to be specified later. Let \(G_u^t\) denote the \(t\)-hop neighborhood of \(u\). Let \(R\) to be the boundary of \(G_u^t\), i.e., the set of vertices that are at distance \(t\) to \(u\). For any \(v\) whose distance to \(u\) exceeds \(t\), \(R\) forms a cut separating \(u\) and \(v\) in the sense that any path from \(u\) to \(v\) passes through \(S\). Then for any set of vertices \(U\) outside the \(t\)-hop neighborhood of \(r\), we have
\[
I(X_u; X_U, \tilde{Y}_E) \leq I(X_u; X_R, \tilde{Y}_E) = I(X_u; X_R, \tilde{Y}_{\leq t}),
\]
where \(\tilde{Y}_{\leq t} \triangleq \tilde{Y}_{E(G_u^t)}\). Indeed, the first inequality follows from the fact that \(X_u \rightarrow X_R \rightarrow X_S\) forms a Markov chain conditioned on \(\tilde{Y}_E\), and the second inequality follows from the independence of \(X_u\) and \(Y_{E(G_u^t)}\) conditioned on the \((X_R, \tilde{Y}_{\leq t})\).

By [PW16, Proposition 12], since \(X_u\) only takes \(k\) values, we can bound the mutual information by the total variation as follows:
\[
I(X_u; X_R, \tilde{Y}_{\leq t}) \leq \log(k-1)T(X_u; X_R, \tilde{Y}_{\leq t}) + h(T(X_u; X_R, \tilde{Y}_{\leq t}))
\]
where \(h(x) \triangleq x \log \frac{1}{x} + (1-x) \log \frac{1}{1-x}\), and
\[
T(X_u; X_R, \tilde{Y}_{\leq t}) \triangleq \mathbb{E}[d_{TV}(P_{X_u, \tilde{Y}_{\leq t}|X_u}, P_{X_R, \tilde{Y}_{\leq t}})] \leq \max_{x, x' \in [k]} d_{TV}(P_{X_R, \tilde{Y}_{\leq t}|X_u=x}, P_{X_R, \tilde{Y}_{\leq t}|X_u=x'})
\]
where the last inequality follows from the convexity of the total variation.

Now choose \(t = t_n\) such that \(t = \omega(1)\) and \(t = o(\log n)\). We show that
\[
\tau \triangleq \max_{x, x' \in [k]} d_{TV}(P_{X_R, \tilde{Y}_{\leq t}|X_u=x}, P_{X_R, \tilde{Y}_{\leq t}|X_u=x'}) = o(1).
\]
To this end, let \(T_u^t\) denote a depth-\(t\) Galton-Watson tree rooted at \(u\) with offspring distribution \(\text{Poi}(d)\), with \(d \triangleq n\eta\) is at most a constant by assumption. By the locally tree-like property of the Erdős-Rényi graph (see, e.g., [MNS15, Proposition 4.2] with \(p = q\)), there exists a coupling between \(T_u^t\) and \(G_u^t\) such that \(\mathbb{P}[G_u^t = T_u^t] = 1 - o(1)\). In the sequel we condition on the event of \(G_u^t = T_u^t\).

In particular, by standard results in branching process [AN72], the expected number of \(i\)th progeny is \(d^i\) and hence the expected size of the \(t\)-neighborhood of \(u\) is \(\frac{(d^t+1-1)}{d-1}\). By the Markov inequality, the size of the \(t\)-neighborhood of \(u\) is at most \(M \triangleq (Cd)^t = n^{o(1)}\) with probability \(1 - o(1)\). In other words, the majority of \(v\) are outside the \(t\)-neighborhood of \(u\). Next we conditioned on the event \(G_u^t = T_u^t\) and abbreviate \(T_u^t\) as \(T\). For each \(x \neq x'\), we construct a coupling \([X_u^+, X_u^- : v \in V(T)]\) and \([Y_v : e \in E(T)]\) so that \((X_{V(T)}^+, Y_{E(T)})\) and \((X_{V(T)}^-, Y_{E(T)})\) are distributed as the law of \((X_{V(T)}, Y_{E(T)})\) conditioned on the root \(X_u = x\) and \(X_u = x'\), respectively. The coupling is defined inductively as follows: First set \(X_u^+ = x\) and \(X_u^- = x'\). Next we generate each layer of observations recursively as follows: Given all the \(X_v^+\)'s and \(Y_v^+\)'s up to depth \(k\), draw \(Y_e = \text{Bern}(1/k)\) independently for all edges between the \(k\)th and the \((k+1)\)th layer. For each edge \(e = (i, j)\) that \(i\) is on \(k\)th layer and \(j\) is on \((k+1)\)th layer, if \(X_i^+ = X_j^-\), we couple all observations on the subtree rooted at \(i\) together, that is, set \(X_j^+ = X_j^- = X_i^+\) if \(Y_i = 1\) and \(X_j^+ = X_j^- = R\) if \(Y_i = 1\) where \(R\) is drawn uniformly at random from \([k] \setminus \{X_i^+\}\); if \(X_i^+ \neq X_i^-\), we proceed as follows:
if $Y_e = 1$, set $X_j^+ = X_i^+$ and $X_j^- = X_i^-$. 

- if $Y_e = 0$, with probability $\frac{k^2}{k-1}$, set $X_j^+ = X_j^- = R$ with $R$ drawn uniformly at random from $\{k\} \setminus \{X_i^+, X_i^-\}$, and with probability $\frac{1}{k-1}$ set $X_j^+ = X_i^-$, and $X_j^- = X_i^+$.

Note that for each $i$ and each of its child $j$, we have

$$
P[X_j^+ \neq X_j^- | X_i^+ \neq X_i^-] = P(Y_e = 1) + P(Y_e = 0) \frac{1}{k-1} = \frac{2}{k}.
$$

Thus, the number of uncoupled pairs $(X_i^+, X_i^-)$ evolves as a GW tree with offspring distribution $\text{Poi} (\frac{2d}{k})$, which dies out if $\frac{2d}{k} \leq 1$ (see, e.g., [AN72, Theorem 1]), in which case we have $d_{TV}(P_{XV(T), Y_{E(T)} | X_u = x, P_{XV(T), Y_{E(T)} | X_u = x'}) \leq \max \{X_R^+ \neq X_R^- \} \rightarrow 0$, as $t \rightarrow \infty$. This completes the proof of (37).

Combining (35)–(37), we have

$$I(X_S; X_1, \ldots, X_u-1, Y) \leq \log(k-1) + h(\tau) + (1 - \max \{E \cap E'\}) \log k
$$

for $S = \{m\}$ and hence any $S \in \binom{[n]}{m}$.

Finally, using (38) for appropriately chosen $m$, we show the impossibility of the correlated recovery (29). First of all, note that for any fixed $x, \hat{x} \in [k]^n$ and any $m \in [n]$ we have

$$d(x, \hat{x}) \geq \sum_{i \in S} \min_{\pi \in S_k} \sum_{i \in S} \min_{\pi \in S_k} \sum_{i \in S} 1 \{x_i = \pi(\hat{x}_i)\}
$$

where $S \sim \text{Unif} \binom{[n]}{m}$ and recall that for any $S$, we have $d(x_S, \hat{x}_S) = \frac{1}{|S|} \min_{\pi \in S_k} \sum_{i \in S} 1 \{x_i = \pi(\hat{x}_i)\}$ per (28). The inequality (39) simply follows from

$$d(x, \hat{x}) = \min_{\pi \in S_k} \sum_{i \in S} \min_{\pi \in S_k} \sum_{i \in S} 1 \{x_i = \pi(\hat{x}_i)\}
$$

Fix a constant $m$ independent of $n$. For any estimator $\hat{X} = \hat{X}(Y) \in [k]^n$, applying (39) yields

$$\mathbb{E}[d(X_S, \hat{X}_S)] \leq \mathbb{E}[d(X, \hat{X})].
$$

where $S$ is a random uniform $m$-set independent of $X, \hat{X}$.

By the data processing inequality, we have for any fixed $S$,

$$I(X_S; \hat{X}_S) \leq I(X_S; Y) \overset{(32)}{=} I(X_S; \hat{X}) \overset{(38)}{=} o(1).
$$
By Pinsker’s inequality, we have \( d_{TV}(P_{X_S \tilde{X}_S}, P_{X_S} \otimes P_{\tilde{X}_S}) \leq \sqrt{2I(X_S; \tilde{X}_S)} = o(1) \). Note that the loss function \( d \) defined in (28) is bounded by one. Thus

\[
E[d(X_S, \hat{X}_S)] \geq E[d(X_S, Z_S)] - d_{TV}(P_{X_S \hat{X}_S}, P_{X_S} \otimes P_{\hat{X}_S}) = E[d(X_S, Z_S)] + o(1),
\]

where \( Z_S \) has the same distribution as \( \hat{X}_S \) and is independent of \( X_S \). By Lemma 9 at the end of this subsection, we have

\[
E[d(X_S, Z_S)] \geq \left( \frac{k-1}{k} - m^{-1/3} \right)(1 - k!e^{-2m^{2/3}}).
\]

Combining (40), (41) and (42), sending \( n \to \infty \) followed by \( m \to \infty \), we arrive at

\[
\liminf_{n \to \infty} E[d(X, \hat{X})] \geq \frac{k-1}{k}.
\]

This completes the proof of the proposition.

**Lemma 9.** Let \( X \) be uniformly distributed on \([k]^m\) and \( Z \) is independent of \( X \) with an arbitrary distribution on \([k]^m\). For the loss function in (28), we have

\[
d(X, Z) \geq \frac{k-1}{k} - m^{-1/3}
\]

with probability at least \( 1 - (k!e^{-2m^{2/3}}) \).

**Proof.** For each fixed \( \pi \), the Hamming distance \( d_H(X, \pi(Z)) \sim \text{Binom}(m, \frac{k-1}{k}) \). From Hoeffding’s inequality we have

\[
P[d_H(X, \pi(Z)) < \frac{k-1}{k} - \delta] \leq e^{-2m\delta^2},
\]

and from the union bound

\[
P[\min_{\pi} d_H(X, \pi(Z)) < \frac{k-1}{k} - \delta] \leq k!e^{-2m\delta^2}.
\]

Setting \( \delta = m^{-1/3} \) completes the proof.

## A Contraction coefficients of some binary-input channels

Consider a binary input channel \( P_{Y|X} \), where \( P_{Y|X=0} = P \) and \( P_{Y|X=1} = Q \). Denote the contraction coefficient is denoted by \( \eta_{KL}(P_{Y|X}) \triangleq \eta_{KL}(P, Q) \). The following representation is given in [PW17, Proof of Theorem 21] in terms of the Le Cam divergence:

\[
\eta_{KL}(P, Q) = \sup_{\beta \in [0,1]} \beta \tilde{\beta} \int \frac{(P - Q)^2}{\beta P + \tilde{\beta} Q} \leq \text{LC}_{\beta}(P\|Q)
\]

Note that for any fixed \( k, m \) and any string \( x, z \in [k]^m \), we can always outperform random matching, i.e., \( d(x, z) < \frac{k-1}{k} \). The point of (43) is that this improvement is negligible for large \( m \).
where we denote $\bar{\beta} = 1 - \beta$. For example, for a binary-input binary-output channel, direction calculation gives

$$\eta_{KL}(\text{Bern}(p), \text{Bern}(q)) = p + q - 2pq - 2\sqrt{pq}$$

$$\leq (\sqrt{p} - \sqrt{q})^2 + 2\sqrt{pq}(p + q)$$

(45) (46)

It is shown in [PW17, Theorem 21] that squared Hellinger distance determines the contraction coefficient of binary-input channel up to a factor of two:

$$\frac{H^2(P, Q)}{2} \leq \eta(\{P, Q\}) \leq H^2(P, Q).$$

(47)

Thus, we have

$$\eta_{KL}(\text{Bern}(a/n), \text{Bern}(b/n)) \leq \frac{(\sqrt{a} - \sqrt{b})^2 + o(1)}{n}, \quad n \to \infty$$

$$\eta_{KL}(N(\delta, 1), N(\delta, 1)) \leq \delta^2(1 + o(1)), \quad \delta \to 0.$$  

(48) (49)

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