Numerical Radius Inequalities for Hilbert Space Operators

Mohammad W. Alomari

Received: 4 March 2019 / Accepted: 6 September 2021 / Published online: 21 September 2021
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2021

Abstract
In this work, an improvement of Hölder–McCarty inequality is established. Based on that, several refinements of the generalized mixed Schwarz inequality are obtained. Consequently, some new numerical radius inequalities are proved. New inequalities for numerical radius of $n \times n$ matrix of Hilbert space operators are proved as well. Some refinements of some earlier results were proved in literature are also given. Some of the presented results are refined and it shown to be better than earlier results were proved in literature.

Keywords Numerical radius · Operator norm · Mixed Schwarz inequality · Hölder–McCarty inequality

Mathematics Subject Classification Primary 47A12 · 47A30; Secondary 15A60 · 47A63

1 Introduction

Let $\mathcal{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathcal{H}; \langle \cdot, \cdot \rangle)$ with the identity operator $1_{\mathcal{H}}$ in $\mathcal{B}(\mathcal{H})$. A bounded linear operator $A$ defined on $\mathcal{H}$ is selfadjoint if and only if $\langle Ax, x \rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$. The spectrum of an operator $A$ is the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - A$ does not have a bounded linear operator inverse, and is denoted by $\text{sp}(A)$. Consider the real vector space $\mathcal{B}(\mathcal{H})_{sa}$ of self-adjoint operators on $\mathcal{H}$ and its positive cone $\mathcal{B}(\mathcal{H})^+$ of positive operators on $\mathcal{H}$. Also, $\mathcal{B}(\mathcal{H})_{sa}^I$ denotes the convex set of bounded self-adjoint operators on the Hilbert space $\mathcal{H}$ with spectra in a real interval $I$. A partial order is naturally equipped on $\mathcal{B}(\mathcal{H})_{sa}$ by defining $A \leq B$ if and only...
if $B - A \in \mathcal{B}(\mathcal{H})^+$. We write $A > 0$ to mean that $A$ is a strictly positive operator, or equivalently, $A \geq 0$ and $A$ is invertible. When $\mathcal{H} = \mathbb{C}^n$, we identify $\mathcal{B}(\mathcal{H})$ with the algebra $\mathfrak{M}_{n \times n}$ of $n$-by-$n$ complex matrices. Then, $\mathfrak{M}_{n \times n}^+$ is just the cone of $n$-by-$n$ positive semidefinite matrices.

For a bounded linear operator $T$ on a Hilbert space $\mathcal{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathcal{H}$ under the quadratic form $x \to \langle Tx, x \rangle$ associated with the operator. More precisely,

$$W(T) = \{ \langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1 \}$$

Also, the numerical radius is defined to be

$$w(T) = \sup \{ |\lambda| : \lambda \in W(T) \} = \sup_{\|x\|=1} |\langle Tx, x \rangle| .$$

The spectral radius of an operator $T$ is defined to be

$$r(T) = \sup \{ |\lambda| : \lambda \in \sigma(T) \}$$

We recall that, the usual operator norm of an operator $T$ is defined to be

$$\|T\| = \sup \{ \|Tx\| : x \in \mathcal{H}, \|x\| = 1 \} .$$

and

$$\ell(T) := \inf \{ \|Tx\| : x \in \mathcal{H}, \|x\| = 1 \} = \inf \{ |\langle Tx, y \rangle| : x, y \in \mathcal{H}, \|x\| = \|y\| = 1 \} .$$

It’s well known that the numerical radius is not submultiplicative, but it is satisfies $w(TS) \leq 4w(T)w(S)$ for all $T, S \in \mathcal{B}(\mathcal{H})$. In particular if $T, S$ are commute, then $w(TS) \leq 2w(T)w(S)$. Moreover, if $T, S$ are normal then $w(\cdot)$ is submultiplicative $w(TS) \leq w(T)w(S)$. Denote $|T| = (T^*T)^{1/2}$ the absolute value of the operator $T$. Then we have $w(|T|) = \|T\|$. It’s convenient to mention that, the numerical radius norm is weakly unitarily invariant; i.e., $w(U^*TU) = w(T)$ for all unitary $U$. Also, let us not miss the chance to mention the important property that $w(T) = w(T^*)$ and $w(TT^*) = w(TT^*)$ for every $T \in \mathcal{B}(\mathcal{H})$.

A popular problem is the following: does the numerical radius of the product of operators commute, i.e., $w(TS) = w(ST)$ for any operators $T, S \in \mathcal{B}(\mathcal{H})$?

This problem has been given serious attention by many authors and in several resources (see [14], for example). Fortunately, it has been shown recently that, for one of such operators must be a multiple of a unitary operator, and we need only to check $w(TS) = w(ST)$ for all rank one operators $S \in \mathcal{B}(\mathcal{H})$ to arrive at the conclusion. This fact was proved by Chien et al. [7]. For other related problems involving numerical ranges and radius see [7,8] as well as the elegant work of Li [28] and the references therein. For more classical and recent properties of numerical range and radius, see [7,8,28] and the comprehensive books [5,15,16].
On the other hand, it is well known that $w(\cdot)$ defines an operator norm on $\mathcal{B}(\mathcal{H})$ which is equivalent to operator norm $\| \cdot \|$. Moreover, we have
\[
\frac{1}{2} \| T \| \leq w(T) \leq \| T \| \tag{1.1}
\]
for any $T \in \mathcal{B}(\mathcal{H})$. The inequality is sharp.

In 2003, Kittaneh [20] refined the right-hand side of (1.1), where he proved that
\[
w(T) \leq \frac{1}{2} \left( \| T \| + \| T^2 \|^{1/2} \right) \tag{1.2}
\]
for any $T \in \mathcal{B}(\mathcal{H})$.

After that in 2005, the same author in [18] proved that
\[
\frac{1}{4} \| A^* A + A A^* \| \leq w^2(T) \leq \frac{1}{2} \| A^* A + A A^* \|. \tag{1.3}
\]
The inequality is sharp. This inequality was also reformulated and generalized in [13] but in terms of Cartesian decomposition.

In 2007, Yamazaki [31] improved both (1.1) and (1.2) by proving that
\[
w(T) \leq \frac{1}{2} \left( \| T \| + w(\tilde{T}) \right) \leq \frac{1}{2} \left( \| T \| + \| T^2 \|^{1/2} \right) \tag{1.4}
\]
where $\tilde{T} = |T|^{1/2} U |T|^{1/2}$ with unitary $U$.

In 2008, Dragomir [12] used Buzano inequality to improve (1.1), as follows:
\[
w^2(T) \leq \frac{1}{2} \left( \| T \| + w(T^2) \right) \tag{1.5}
\]
This result was also recently generalized by Sattari et al. [30].

This work, is divided into three sections, after this introduction, Sect. 2 is devoted to recall some facts about superquadratic functions and the mixed Schwarz inequality. In Sect. 3, we refine the Jesnen and Hölder–McCarty inequalities for positive operators which in turn allow us to refine the generalized mixed Schwarz inequality with of its some consequences. In Sect. 4, new inequalities for numerical radius of $n \times n$ matrix of Hilbert space operators are proved. Some refinements of some earlier results were proved in literature are also given.

2 Lemmas

2.1 Superquadratic Functions

A function $f : J \to \mathbb{R}$ is called convex iff
\[
f(t \alpha + (1-t) \beta) \leq tf(\alpha) + (1-t)f(\beta),
\]
for all points $\alpha, \beta \in J$ and all $t \in [0, 1]$. If $-f$ is convex then we say that $f$ is concave. Moreover, if $f$ is both convex and concave, then $f$ is said to be affine.

Geometrically, for two point $(x, f(x))$ and $(y, f(y))$ on the graph of $f$ are on or below the chord joining the endpoints for all $x, y \in I, x < y$. In symbols, we write

$$f(t) \leq \frac{f(y) - f(x)}{y - x}(t - x) + f(x)$$

for any $x \leq t \leq y$ and $x, y \in J$.

Equivalently, given a function $f : J \rightarrow \mathbb{R}$, we say that $f$ admits a support line at $x \in J$ if there exists a $\lambda \in \mathbb{R}$ such that

$$f(t) \geq f(x) + \lambda(t - x)$$

for all $t \in J$.

The set of all such $\lambda$ is called the subdifferential of $f$ at $x$, and it’s denoted by $\partial f$. Indeed, the subdifferential gives us the slopes of the supporting lines for the graph of $f$. So that if $f$ is convex then $\partial f(x) \neq \emptyset$ at all interior points of its domain.

From this point of view Abramovich et al. [1] extend the above idea for what they called superquadratic functions. Namely, a function $f : [0, \infty) \rightarrow \mathbb{R}$ is called superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$f(t) \geq f(x) + C_x(t - x) + f(|t - x|)$$

for all $t \geq 0$. We say that $f$ is subquadratic if $-f$ is superquadratic. Thus, for a superquadratic function we require that $f$ lie above its tangent line plus a translation of $f$ itself.

Prima facie, superquadratic function looks to be stronger than convex function itself but if $f$ takes negative values then it may be considered as a weaker function. Therefore, if $f$ is superquadratic and non-negative. Then $f$ is convex and increasing [1].

Moreover, the following result holds for superquadratic function.

**Lemma 1** [1] Let $f$ be superquadratic function. Then

1. $f(0) \leq 0$
2. If $f$ is differentiable and $f(0) = f'(0) = 0$, then $C_x = f'(x)$ for all $x \geq 0$.
3. If $f(x) \geq 0$ for all $x \geq 0$, then $f$ is convex and $f(0) = f'(0) = 0$.

The next result gives a sufficient condition when convexity (concavity) implies super(sub)quadraticity.

**Lemma 2** [1] If $f'$ is convex (concave) and $f(0) = f'(0) = 0$, then is super(sub)quadratic. The converse of is not true.

**Remark 1** Subquadraticity does always not imply concavity; i.e., there exists a subquadratic function which is convex. For example, $f(x) = x^p, x \geq 0$ and $1 \leq p \leq 2$ is subquadratic and convex. For more about subquadratic see [24].
Among others, Abramovich et al. [1] proved that the inequality
\[
f \left( \int \varphi \, d\mu \right) \leq \int \left( f(\varphi(s)) - f\left( \left\| \varphi(s) - \int \varphi \, d\mu \right\| \right) \right) \, d\mu(s) \tag{2.1}
\]
holds for all probability measures \( \mu \) and all nonnegative, \( \mu \)-integrable functions \( \varphi \) if and only if \( f \) is superquadratic. This inequality plays a main role overall our presented results below.

### 2.2 The Mixed Schwarz Inequality

The mixed Schwarz inequality was introduced in [27], as follows:

**Lemma 3** Let \( A \in \mathcal{B}(\mathcal{H})^+ \), then
\[
|\langle Ax, y \rangle|^2 \leq \left| |A|^{2\alpha} x, x \right| \left| A^* \right|^{2(1-\alpha)} y, y \right|, \quad 0 \leq \alpha \leq 1. \tag{2.2}
\]
for any vectors \( x, y \in \mathcal{H} \).

In order to generalize (2.2), Kittaneh [23] used the key lemma

**Lemma 4** Let \( A, B \in \mathcal{B}(\mathcal{H})^+ \). Then \[
\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}
\]
is positive in \( \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \) if and only if \( |\langle Cx, y \rangle|^2 \leq \langle Ax, x \rangle \langle By, y \rangle \) for every vectors \( x, y \in \mathcal{H} \),
to prove that

**Lemma 5** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) such that \( |A|B = B^*|A| \). If \( f \) and \( g \) are nonnegative continuous functions on \( [0, \infty) \) satisfying \( f(t)g(t) = t \ (t \geq 0) \), then
\[
|\langle ABx, y \rangle| \leq r(B) \|f(|A|)x\| \|g\left(|A^*|\right)y\| \tag{2.3}
\]
for any vectors \( x, y \in \mathcal{H} \).

Clearly, by setting \( B = 1_{\mathcal{H}} \) and choosing \( f(t) = t^\alpha \), \( g(t) = t^{1-\alpha} \) we refer to (2.2).

The following interesting estimates of spectral radius also obtained by Kittaneh [19].

**Lemma 6** If \( A, B \in \mathcal{B}(\mathcal{H}) \). Then
\[
\begin{align*}
\text{r}(AB) & \leq \frac{1}{4} \left( \|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4 \min \{\|A\| \|BAB\|, \|B\| \|ABA\|\}} \right) \tag{2.4}
\end{align*}
\]

In some of our results we need the following two fundamental norm estimates, which are:
\[
\|A + B\| \leq \frac{1}{2} \left( \|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \|A^{1/2}B^{1/2}\|} \right). \tag{2.5}
\]
and
\[ \|A^{1/2}B^{1/2}\| \leq \|AB\|^{1/2}. \] (2.6)

Both estimates are valid for all positive operators \( A, B \in \mathcal{B}(\mathcal{H}) \).

### 3 Refining Hölder–McCarty Inequality and Mixed Schwarz Inequality

In this part we give some new refinements of the ‘mixed’ Schwarz inequality and its generalization based on a new refinement of Hölder–McCarty inequality. The next lemma plays a main role in our main results.

**Lemma 7** Let \( A \in \mathcal{B}(\mathcal{H})^+ \), then
\[ (Ax, x)^p \leq \{A^p x, x\} - \|A - (Ax, x) 1_{\mathcal{H}}\|^p x, x \leq \{A^p x, x\} \] (3.1)
for all \( p \geq 2 \), and
\[ (Ax, x)^p \geq \{A^p x, x\} - \|A - (Ax, x) 1_{\mathcal{H}}\|^p x, x \] (3.2)
for all \( 0 < p < 2 \) and every \( x \in \mathcal{H} \).

**Proof** Since \( A \) is positive then there \( B \in \mathcal{B}(\mathcal{H}) \) such that \( A = B^*B \). Also, since \( B^*B \) is always positive and selfadjoint, thus by the spectral representation theorem \( A \) can be represented as \( A = \int_0^\infty t dE(t) \). Employing the inequality (2.1) for the superquadratic function \( f(t) = t^p, t \in [0, \infty) \) \( p \geq 2 \), then we have
\[
(Ax, x)^p = \left( \int_0^\infty t dE(t) x, x \right)^p \\
\leq \int_0^\infty t^p dE(t) x, x - \int_0^\infty t - \int_0^\infty s dE(s) x, x \right)^p dE(t) x, x \\
= \{A^p x, x\} - \|A - (Ax, x) 1_{\mathcal{H}}\|^p x, x .
\]
The inequality (3.2) follows in similar manner by applying the reverse of (3.1) for the subquadratic function \( f(t) = t^p, 0 < p \leq 2 \).

The inequalities (3.1) and (3.2) were proved in [4] in different context and only for positive selfadjoint operators. Also, we should note that, a stronger version for positive selfadjoint operators was proved earlier in [26] (see also [25]) where different approach were used. Our presented proof above is more general and completely different.

**Remark 2** Let \( A \in \mathcal{B}(\mathcal{H})^+ \), then the McCatry inequality reads that
\[ (Ax, x)^p \geq \{A^p x, x\}, 0 < p \leq 1 . \] (3.3)
Using (3.2), we have the following refinement
\[
\langle Ax, x \rangle^p \geq \langle A^p x, x \rangle \geq \langle A^p x, x \rangle - \langle |A - \langle Ax, x \rangle 1_H|^{p} x, x \rangle, \quad 0 < p \leq 1
\]
for every \( x \in \mathcal{H} \).

The following refinement of Cauchy–Schwarz inequality holds.

**Lemma 8** Let \( A \in \mathcal{B}(\mathcal{H})^+ \), then
\[
|\langle Ax, y \rangle|^2 \leq \left[ |\langle A^p x, x \rangle - \langle |A - \langle Ax, x \rangle 1_H|^{p} x, x \rangle \right] \times \left[ |\langle A^p y, y \rangle - \langle |A - \langle Ay, y \rangle 1_H|^{p} y, y \rangle \right]
\]
\[
\leq \langle A^p x, x \rangle \langle A^p y, y \rangle
\]
(3.4)
for all \( p \geq 2 \) and every \( x, y \in \mathcal{H} \).

**Proof** By Cauchy–Schwarz inequality we have
\[
|\langle Ax, y \rangle|^2 \leq \langle Ax, x \rangle \langle Ay, y \rangle
\]
for every \( x, y \in \mathcal{H} \), and this implies that
\[
|\langle Ax, y \rangle|^2 \leq (Ax, x)^p \langle Ay, y \rangle^p, \quad p \geq 2.
\]
Employing (3.1) we get the desired result. \(\Box\)

**Corollary 1** If \( T \in \mathcal{B}(\mathcal{H}) \), then
\[
|\langle Tx, y \rangle|^2 \leq \left[ \langle |T^p x, x \rangle - \langle |T| - \langle |T|x, x \rangle 1_H|^{p} x, x \rangle \right] \times \left[ \langle |T^*|^p y, y \rangle - \langle |T^*| - \langle |T^*|y, y \rangle 1_H|^{p} y, y \rangle \right]
\]
\[
\leq \langle |T^p x, x \rangle \langle |T^*|y, y \rangle
\]
(3.5)
for all \( p \geq 2 \). In particular, we have
\[
|\langle Tx, x \rangle| \leq \left[ \langle |T^p x, x \rangle - \langle |T| - \langle |T|x, x \rangle 1_H|^{p} x, x \rangle \right]^{1/p}
\]
(3.6)

**Proof** Recall that, if \( T \in \mathcal{B}(\mathcal{H}) \), then
\[
\begin{bmatrix}
|T|

T^*

\end{bmatrix}
\]
is positive in \( \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \), (see [23]). Therefore, by (2.1) we have
\[
|\langle Tx, y \rangle|^2 \leq \langle |T|x, x \rangle \langle |T^*|y, y \rangle
\]
and this gives by (3.1) that;

$$\langle Tx, y \rangle^2 \leq \langle |T| x, x \rangle^p \langle |T^*| y, y \rangle^p$$

$$\leq \left[ \langle |T|^p x, x \rangle - \langle |T| - \langle |T| x, x \rangle 1_{\mathcal{H}} |^p x, x \rangle \right]$$

$$\times \left[ \langle |T^*|^p y, y \rangle - \langle |T^*| - \langle |T^*| y, y \rangle 1_{\mathcal{H}} |^p y, y \rangle \right]$$

$$\leq \langle |T|^p x, x \rangle \langle |T^*|^p y, y \rangle$$

as desired. □

A generalization of the above result in Kittaneh like inequality (2.3) is considered in the following result.

**Corollary 2** Let T, S ∈ B(ℋ) such that |T|S = S^∗ T|. If f and g are nonnegative continuous functions on [0, ∞) satisfying f(t)g(t) = t (t ≥ 0), then

$$\langle TSx, y \rangle | \leq r(S) 2p^2 \sqrt{f^2(|T|)x, x - f^2(|T|) - g^2(|T^*|)y, y}$$

$$\times 2p^2 \sqrt{g^2(|T^*|)y, y - g^2(|T^*|) - g^2(|T^*|)y, y} 1_{\mathcal{H}} |^p x, x \rangle$$

$$\leq r(S) 2p^2 \sqrt{f^2(|T|)x, x} 2p^2 \sqrt{g^2(|T^*|)y, y}$$

for all p ≥ 2 and any vectors x, y ∈ ℋ.

**Proof** Using (2.3), and by employing (3.4) we have

$$\langle TSx, y \rangle | \leq r(S) \|f(|T|)x\| \|g(|T^*|)y\|$$

$$= r(S) \left[ f^2(|T|)x, x \right]^{1/2} \left[ g^2(|T^*|) y, y \right]^{1/2}$$

$$\leq r(S) 2p^2 \sqrt{f^2(|T|)x, x - f^2(|T|) - g^2(|T^*|)y, y} 1_{\mathcal{H}} |^p x, x \rangle$$

$$\times 2p^2 \sqrt{g^2(|T^*|)y, y - g^2(|T^*|) - g^2(|T^*|)y, y} 1_{\mathcal{H}} |^p y, y \rangle$$

which proves the result. □

**Corollary 3** Let T, S ∈ B(ℋ) such that |T|S = S^∗ T|. Then

$$\langle TSx, y \rangle | \leq r(S) 2p^2 \sqrt{|T|^{2p\alpha} x, x - |T|^{2\alpha} x, x}$$

$$\times 2p^2 \sqrt{|T^*|^{2p(1-\alpha)} x, x - |T^*|^{2(1-\alpha)} x, x} 1_{\mathcal{H}} |^p x, x \rangle$$

$$\leq r(S) 2p^2 \sqrt{|T|^{2p\alpha} x, x} \sqrt{|T^*|^{2p(1-\alpha)} x, x}$$

(3.8)
for all $p \geq 2$ and any vectors $x, y \in \mathcal{H}$. In particular, we have

$$
|\langle TSx, y \rangle|
\leq r(S) \sqrt[p]{\frac{4}{p} \left( \frac{1}{p} \left| |T|^2 x, x \right| - \frac{1}{2} \left( \|T\|^2 x, x \right) \right)^2}
\times \sqrt[p]{\frac{4}{p} \left( \frac{1}{p} \left| |T^*|^2 x, x \right| - \frac{1}{2} \left( \|T^*\|^2 x, x \right) \right)^2},
$$

(3.9)

**Proof** Setting $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ in (3.7) we get the inequality (3.8). Choosing $p = 2$ and $\alpha = \frac{1}{2}$ in (3.8), we get the second inequality (3.9).

\[\Box\]

4 Numerical Radius Inequalities

This section is divided into two parts; the first part concerning numerical inequalities for general Hilbert space operators. The second part deals with numerical radius inequalities for $n \times n$ matrix operators.

4.1 Numerical Radius Inequalities

In this section, some numerical radius inequalities based on results of Sect. 2 are obtained. Before that, we need to recall that in some recent works, some authors used the concept of infimum norm (or $\ell$-norm) which is defined as:

$$
\ell(T) := \inf \{ \|Tx\| : x \in \mathcal{H}, \|x\| = 1 \}
= \inf \{ \langle Tx, y \rangle : x, y \in \mathcal{H}, \|x\| = \|y\| = 1 \}.
$$

The next result gives a numerical radius bound of product of two operators based on the refinement of Kittaneh inequality (3.7).

**Theorem 1** Let $T, S \in \mathcal{B}({\mathcal{H}})$ such that $|T|S = S^*|T|$. If $f$ and $g$ are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ ($t \geq 0$), then

$$
w(TS) \leq \frac{1}{2} \left( \|S\| + \left\| S^2 \right\|^{1/2} \right) \cdot \left[ \left\| f^p (|T|) \right\|^2 - \ell \left( \left\| f^2 (|T|) - \left\| f (|T|) \right\|^2 \right\| \right) \right]^{\frac{1}{2p}}
\times \left[ \left\| g^p (|T^*|) \right\|^2 - \ell \left( \left\| g^2 (|T^*|) - \left\| g (|T^*|) \right\|^2 \right\| \right) \right]^{\frac{1}{2p}}
$$

(4.1)

for all $p \geq 2$. 
Proof From the first inequality in (3.7), we have

\[
|\langle TSx, y \rangle|^{2p} \leq r^{2p}(S) \left[ \left( f^{2p}(|T|)x, x \right) - \left( f^{2}(|T|) - f^{2}(|T|)x, x \right) 1_{\mathcal{H}} \right]^{p} \times \left[ \left( g^{2p}(|T^*|)y, y \right) - \left( g^{2}(|T^*|) - g^{2}(|T^*|)y, y \right) 1_{\mathcal{H}} \right]^{p}
\]

Let \( y = x \) and taking the supremum over \( x \in \mathcal{H} \), we observe that

\[
\sup_{\|x\|=1} |\langle TSx, x \rangle|^{2p} \leq r^{2p}(S) \sup_{\|x\|=1} \left[ \left( f^{2p}(|T|)x, x \right) - \left( f^{2}(|T|) - f^{2}(|T|)x, x \right) 1_{\mathcal{H}} \right]^{p} \times \left[ \left( g^{2p}(|T^*|)x, x \right) - \left( g^{2}(|T^*|) - g^{2}(|T^*|)x, x \right) 1_{\mathcal{H}} \right]^{p}
\]

\[
\leq r^{2p}(S) \left\{ \sup_{\|x\|=1} \left( f^{2p}(|T|)x, x \right) \right. \\
- \inf_{\|x\|=1} \left( f^{2}(|T|) - \sup_{\|x\|=1} \left( f^{2}(|T|)x, x \right) 1_{\mathcal{H}} \right) \left. \right\}^{p} \times \left\{ \sup_{\|x\|=1} \left( g^{2p}(|T^*|)x, x \right) \right. \\
- \inf_{\|x\|=1} \left( g^{2}(|T^*|) - \sup_{\|x\|=1} \left( g^{2}(|T^*|)x, x \right) 1_{\mathcal{H}} \right) \left. \right\}^{p}
\]

\[
\leq r^{2p}(S) \cdot \left[ \| f^{p}(|T|) \|^2 - \ell \left( \left[ f^{2}(|T|) - \| f \| \| T \| \|^2 \right]^{p} \right) \right]
\times \left[ \| g^{p}(|T^*|) \|^2 - \ell \left( \left[ g^{2}(|T^*|) - \| g \| \| T^* \| \|^2 \right]^{p} \right) \right].
\]

Now, from Lemma 6 with \( A = S, B = 1_{\mathcal{H}} \), we have

\[
\rho(S) \leq \frac{1}{4} \left( 2\|S\| + \sqrt{4 \min \left\{ \| S^2 \|^2, \| S \|^2 \right\}} \right) = \frac{1}{2} \left( \| S \| + \| S^2 \|^{1/2} \right).
\]

Substituting in the above inequality we obtain the result in (4.1).
**Corollary 4** Let $T, S \in \mathcal{B}(\mathcal{H})$ such that $|T| S = S^* |T|$. Then

$$w(TS) \leq r(S) \cdot \left[ \left( \| |T| \|^p \|^{2} - \ell^{2} \left( \left| \left( |T| \right)^{2\alpha} - \left( |T| \right)^{\alpha} \right| \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \right.$$

$$\times \left[ \left( \| (T^*)^{p(1-\alpha)} \|^2 - \ell^{2} \left( \left| (T^*)^{2(1-\alpha)} - \left( |T^{*}| \right)^{1-\alpha} \right| \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \right.$$  

$$\leq r(S) \cdot \left( \| |T| \|^{p} \|^{\frac{1}{p}} \| (T^*)^{p(1-\alpha)} \|^\frac{1}{p} \right)$$  

(4.2)

for all $p \geq 2$. In particular, we have

$$w(TS) \leq r(S) \cdot \left[ \| T \|^{2} - \ell^{2} \left( \left| \left( |T| \right)^{1/2} \right| \right)^{\frac{1}{2}} \right.$$  

$$\times \left[ \| T \|^{2} - \ell^{2} \left( \left| \left( T^* \right)^{1/2} \right| \right)^{\frac{1}{2}} \right.$$  

$$\leq r(S) \cdot \| T \|.$$  

(4.3)

**Proof** Setting $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ in (4.1), we get the inequality (4.2). Choosing $p = 2$ and $\alpha = \frac{1}{2}$ in (4.2) and use the fact that $\| |T| \| = \| |T^*| \| = \|T\|$, we get the second inequality (4.3). \qed

Another generalization of the above inequalities under Kittaneh’s assumptions is embedded as follows:

**Corollary 5** Under the assumptions of Theorem 1, we have

$$w(TS) \leq \frac{1}{4} \left( \| S \| + \| S^2 \|^{1/2} \right) \| f^{2p} (|T|) \| + g^{2p} (|T^*|) \|$$

$$\leq \frac{1}{8} \left( \| S \| + \| S^2 \|^{1/2} \right) \cdot \left\{ \left( \| f^P (|A|) \| \right)^2 + \| g^P (|A^*|) \| \right\}$$

$$+ \sqrt{\left( \| f^{2p} (|A|) \| - \| g^{2p} (|A^*|) \| \right)^2 + 4 \| f^P (|A|) \| g^P (|A^*|) \|}$$

for all $p \geq 2$. 
Proof In the second inequality in (3.7), let $x = y$ then we have

$$
|\langle TSx, x \rangle| \leq r(S) \sqrt{\left| f^{2p} (|T|)x, x \right|} \sqrt{\left| g^{2p} (|T^*|)x, x \right|}
$$

$$
\leq \frac{1}{2} r(S) \left( \left| f^{2p} (|T|)x, x \right| + \left| g^{2p} (|T^*|)x, x \right| \right) \quad \text{(by AM-GM inequality)}
$$

$$
\leq \frac{1}{2} r(S) \left\| f^{2p} (|T|) + g^{2p} (|T^*|) \right\| 
$$

$$
\leq \frac{1}{4} r(S) \left( \left\| f^{2p} (|A|) \right\| + \left\| g^{2p} (|A^*|) \right\| \right)
$$

$$
+ \frac{1}{4} \sqrt{\left( \left\| f^{2p} (|A|) \right\| - \left\| g^{2p} (|A^*|) \right\| \right)^2 + 4 \left\| f^{\frac{p}{2}} (|A|) g^{\frac{p}{2}} (|A^*|) \right\|^2}
$$

Now, using (2.5) and (2.6) in the last inequality and use the inequality

$$
r(S) \leq \frac{1}{2} \left( \|S\| + \|S^2\|^{1/2} \right).
$$

Substituting all together in the last inequality and taking the supremum for all $x \in \mathcal{H}$, we get the desired result. \qed

4.2 Numerical Radius Inequalities for $n \times n$ Matrix Operators

On the other hand, several refinements inequalities for numerical radius of $n \times n$ operator matrices have been recently obtained by many other authors see for example [3,9–11,20–22,29]. Among others, three important facts are obtained by different authors are summarized together in the following result.

Let $A = [A_{ij}] \in \mathcal{B} \left( \bigoplus_{i=1}^{n} \mathcal{H}_i \right)$. Then

$$
\omega \left( \left[ t_{ij}^{(1)} \right] \right), \quad \text{Hou & Du in [18]}
$$

$$
\omega \left( \left[ t_{ij}^{(2)} \right] \right), \quad \text{BaniDomi & Kittaneh in [6] ;} \quad (4.4)
$$

$$
\omega \left( \left[ t_{ij}^{(3)} \right] \right), \quad \text{AbuOmar & Kittaneh in [2]}
$$

where

$$
t_{ij}^{(1)} = \omega \left( \left\| T_{ij} \right\| \right), \quad t_{ij}^{(2)} = \left\{ \begin{array}{ll}
\frac{1}{2} \left( \|T_{ii}\| + \|T_{ii}^{2}\|^{1/2} \right), & i = j \\
\|T_{ij}\|, & i \neq j
\end{array} \right.
$$
and

\[ t^{(3)}_{ij} = \begin{cases} \omega(T_{ii}), & i = j \\ \|T_{ij}\|, & i \neq j \end{cases} \]

Our next result gives a new bound for Numerical radius of \( n \times n \) matrix Hilbert Operators.

**Theorem 2** Let \( A = \begin{bmatrix} A_{ij} \end{bmatrix} \in \mathcal{B} \left( \bigoplus_{i=1}^{n} \mathcal{H}_i \right) \) and \( f, g \) be as in Lemma 5. Then

\[ w(A) \leq w \left( \begin{bmatrix} a_{ij} \end{bmatrix} \right) \quad (4.5) \]

where

\[ a_{ij} = \begin{cases} \frac{1}{4}B_{ii}, & i = j \\ \|A_{ij}\|, & i \neq j \end{cases} \]

such that

\[ B_{ii} = \sqrt{\| f^2 (|A_{ii}|) \| + \| g^2 (|A_{ii}^*|) \|} \]

\[ + \sqrt{\| f^2 (|A_{ii}|) \| - \| g^2 (|A_{ii}^*|) \|}^2 + 4 \sqrt{ f (|A_{ii}|) \ g (|A_{ii}^*|) } \]

**Proof** Let \( x = (x_1 \ x_2 \ldots \ x_n)^T \in \bigoplus_{i=1}^{n} \mathcal{H}_i \), with \( \|x\| = 1 \). Then we have

\[ |\langle Ax, x \rangle| \]

\[ = \left| \sum_{i,j=1}^{n} \langle A_{ij}x_j, x_i \rangle \right| \]

\[ \leq \sum_{i,j=1}^{n} |\langle A_{ij}x_j, x_i \rangle| \]

\[ = \sum_{i=1}^{n} |\langle A_{ii}x_i, x_i \rangle| + \sum_{j \neq i}^{n} |\langle A_{ij}x_j, x_i \rangle| \]

\[ \leq \sum_{i=1}^{n} \left( f^2(|A_{ii}|) \|x_i\|^2 \right)^{1/2} \left( g^2(|A_{ii}^*|) \|x_i\|^2 \right)^{1/2} + \sum_{j \neq i}^{n} |\langle A_{ij}x_j, x_i \rangle| \quad \text{(by (2.3))} \]

\[ \leq \frac{1}{2} \sum_{i=1}^{n} \left( f^2(|A_{ii}|) + g^2(|A_{ii}^*|) \right) \|x_i\|^2 + \sum_{j \neq i}^{n} \|A_{ij}\| \|x_i\| \|x_j\| \quad (4.6) \]
(by AM-GM inequality)
\[
\leq \frac{1}{4} \sum_{i=1}^{n} \left( \left\| f^2(|A_{ii}|) \right\| + \left\| g^2(|A_{ii}^*|) \right\| \right) \tag{by (2.5)}
+ \sqrt{\left( \left\| f^2(|A_{ii}|) \right\| - \left\| g^2(|A_{ii}^*|) \right\| \right)^2 + 4 \left\| f(|A_{ii}|) g \left(|A_{ii}^*| \right)^2 \right\| \| x_i \|^2}
+ \sum_{j \neq i} \| A_{ij} \| \| x_i \| \| x_j \|
= \{ [a_{ij}] y, y \}
\]
where \( y = (\| x_1 \| \| x_2 \| \cdots \| x_n \|)^T \). Taking the supremum for all \( x \in \mathcal{H} \), we get the desired result. \( \square \)

**Corollary 6** If \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \) in \( \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \), then
\[
w(A) \leq w(\{ \hat{a}_{ij} \}) \tag{4.7}
\]
where
\[
\hat{a}_{ij} = \begin{cases} \frac{1}{4} \hat{B}_{ii}, & i = j \\ \| A_{ij} \|, & i \neq j \end{cases}
\]
such that
\[
\hat{B}_{ii} = \left\| A_{ii} \right\|^{2\alpha} + \left\| A_{ii}^* \right\|^{2(1-\alpha)}
+ \sqrt{\left( \left\| A_{ii} \right\|^{2\alpha} - \left\| A_{ii}^* \right\|^{2(1-\alpha)} \right)^2 + 4 \left\| A_{ii} \right\| \left\| A_{ii}^* \right\|^{(1-\alpha)}}
(:= \hat{B}_{ii}(\alpha))
\]

**Proof** Setting \( f(t) = t^\alpha \) and \( g(t) = t^{1-\alpha} \) in (4.5), then we get
\[
w(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) \leq w(\begin{bmatrix} \frac{1}{4} \hat{B}_{11} & \| A_{12} \| \\ \| A_{21} \| & \frac{1}{4} \hat{B}_{22} \end{bmatrix})
= \frac{1}{2} \begin{bmatrix} \frac{1}{2} \hat{B}_{11} & \| A_{12} \| + \| A_{21} \| \\ \| A_{21} \| + \| A_{12} \| & \frac{1}{2} \hat{B}_{22} \end{bmatrix}
= \frac{1}{4} \left( \hat{B}_{11} + \hat{B}_{22} + \sqrt{(\hat{B}_{11} - \hat{B}_{22})^2 + (\| A_{12} \| + \| A_{21} \|)^2} \right)
\]
which gives the required result. \( \square \)
**Remark 3** Setting \( \alpha = \frac{1}{2} \) in (4.7) and employing the facts (2.5) and (2.6), so that we get (1.2).

**Theorem 3** Let \( A = [A_{ij}] \in \mathcal{B} \left( \bigoplus_{i=1}^{n} \mathcal{H}_i \right) \) and \( f, g \) be as in Lemma 3. Then

\[
 w(A) \leq w \left( [h_{ij}] \right) \tag{4.8}
\]

where

\[
h_{ij} = \begin{cases} 
\frac{1}{4} (D_{ii} - d_{ii}) , & i = j \\
\|A_{ij}\| , & i \neq j
\end{cases}
\]

such that

\[
D_{ii} = \frac{1}{2} \left( \| f^4 (|A_{ii}|) \| + \| g^4 (|A_{ii}^*|) \| 
+ \sqrt{\left( \| f^4 (|A_{ii}|) \| - \| g^4 (|A_{ii}^*|) \| \right)^2 + 4 \| f^2 (|A_{ii}|) g^2 (|A_{ii}^*|) \|^{1/2}} \right)
\]

and

\[
d_{ii} = \left\| f^2 (|A_{ii}|) - \| f^2 (|A_{ii}|) \| \right\|^2 + \left\| g^2 (|A_{ii}^*|) - \| g^2 (|A_{ii}^*|) \| \right\|^2
\]

**Proof** From (4.6) we have

\[
|\langle Ax, x \rangle| \leq \sum_{i=1}^{n} \left( f^2 (|A_{ii}|) x_i, x_i \right)^{1/2} \left( g^2 (|A_{ii}^*|) x_i, x_i \right)^{1/2} + \sum_{j \neq i}^{n} \left| \langle A_{ij} x_j, x_i \rangle \right|
\]

\[
\leq \sum_{i=1}^{n} \left\{ \sqrt{f^4 (|A_{ii}|) x, x} - \left( f^2 (|A_{ii}|) - \langle f^2 (|A_{ii}|) x, x \rangle 1_{\mathcal{H}} \right)^2 x, x \right\}
\]

\[
\times \sqrt{g^4 (|A_{ii}^*|) y, y} - \left( g^2 (|A_{ii}^*|) - \langle g^2 (|A_{ii}^*|) y, y \rangle 1_{\mathcal{H}} \right)^2 y, y \cdot \|x_i\|^2 \right\}
\]

\[+ \sum_{j \neq i}^{n} \|A_{ij}\| \|x_i\| \|x_j\| \quad \text{(by (3.9) with } S = 1_{\mathcal{H}})\]

\[
\leq \frac{1}{4} \sum_{i=1}^{n} \left\{ \| f^4 (|A_{ii}|) + g^4 (|A_{ii}^*|) \| \right\}
\]

\[\leq \left\{ \| f^2 (|A_{ii}|) - \| f^2 (|A_{ii}|) \| \right\}^2 + \left\| g^2 (|A_{ii}^*|) - \| g^2 (|A_{ii}^*|) \| \right\|^2 \|x_i\|^2
\]
\[ + \sum_{j \neq i}^{n} \|A_{ij}\| \|x_i\| \|x_j\| \] (by GM − AM inequality)

\[ \leq \sum_{i=1}^{n} \left\{ \frac{1}{8} \left( \|f^4(|A_{ii}|)| + \|g^4(|A_{ii}^*|)| \right) \right. \\
+ \sqrt{\left( \|f^4(|A_{ii}|)| - \|g^4(|A_{ii}^*|)| \right)^2 + 4 \|f^2(|A_{ii}|)g^2(|A_{ii}^*|)|^2} \right. \\
- \frac{1}{4} \left\{ \|f^2(|A_{ii}|)| - \|g^2(|A_{ii}^*|)| \right\} \cdot \|x_i\|^2 \\
\left. \sum_{j \neq i}^{n} \|A_{ij}\| \|x_i\| \|x_j\| \right\}

= \frac{1}{8} \sum_{i=1}^{n} \left( \|f^4(|A_{ii}|)| + \|g^4(|A_{ii}^*|)| \right) \\
+ \sqrt{\left( \|f^4(|A_{ii}|)| - \|g^4(|A_{ii}^*|)| \right)^2 + 4 \|f^2(|A_{ii}|)g^2(|A_{ii}^*|)|^2} \cdot \|x_i\|^2 \\
- \sum_{i=1}^{n} \frac{1}{4} \left\{ \|f^2(|A_{ii}|)| - \|g^2(|A_{ii}^*|)| \right\} \cdot \|x_i\|^2 \\
\left. \sum_{j \neq i}^{n} \|A_{ij}\| \|x_i\| \|x_j\| \right\}

= \langle [h_{ij}] y, y \rangle

where \( y = (\|x_1\| \|x_2\| \cdots \|x_n\|)^T \). Taking the supremum for all \( x \in \mathcal{H} \), we get the desired result. \( \Box \)

**Corollary 7** If \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \) in \( \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2) \), then

\[
\omega \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right) \leq \frac{1}{4} \left\{ (\tilde{D}_{11} - \tilde{d}_{11}) + (\tilde{D}_{22} - \tilde{d}_{22}) \\
+ \sqrt{((\tilde{D}_{11} - \tilde{d}_{11}) - (\tilde{D}_{22} - \tilde{d}_{22}))^2 + (\|A_{12}\| + \|A_{21}\|)^2} \right\}
\]

where

\[
\tilde{h}_{ij} = \begin{cases} \frac{1}{4} (\tilde{D}_{ii} - \tilde{d}_{ii}) , & i = j \\ \|A_{ij}\| , & i \neq j \end{cases}
\]
such that
\[\tilde{D}_{ii} = \frac{1}{2} \left( \|A_{ii}\|^{4\alpha} + \|A_{ii}^*\|^{4(1-\alpha)} \right) + \sqrt{\left( \|A_{ii}\|^{4\alpha} - \|A_{ii}^*\|^{4(1-\alpha)} \right)^2 + 4 \|A_{ii}\|^2 \|A_{ii}^*\|^{2(1-\alpha)}^2} \]

and
\[\tilde{d}_{ii} = \left\| A_{ii}^{2\alpha} - \|A_{ii}^{2\alpha}\| \right\|^2 + \left| A_{ii}^{2(1-\alpha)} \right| - \left( A_{ii}^{2(1-\alpha)} \right)^2 \]

**Proof** Setting \( f(t) = t^\alpha \) and \( g(t) = t^{1-\alpha} \) in (4.8), then we get
\[
w \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right) \leq \left( \begin{bmatrix} \frac{1}{2} (\tilde{D}_{11} - \tilde{d}_{11}) & 1 \|A_{12}\| \\ \|A_{21}\| & \frac{1}{2} (\tilde{D}_{22} - \tilde{d}_{22}) \end{bmatrix} \right) \]
\[
= \frac{1}{2} r \left( \left( \frac{1}{2} (\tilde{D}_{11} - \tilde{d}_{11}) \|A_{12}\| + \|A_{21}\| \right) + \frac{1}{2} (\tilde{D}_{22} - \tilde{d}_{22}) \right) \]
\[
= \frac{1}{4} \left\{ (\tilde{D}_{11} - \tilde{d}_{11}) + (\tilde{D}_{22} - \tilde{d}_{22}) \right. + \sqrt{((\tilde{D}_{11} - \tilde{d}_{11}) - (\tilde{D}_{22} - \tilde{d}_{22}))^2 + (\|A_{12}\| + \|A_{21}\|)^2} \}
\]

The following results refines the first and the second inequalities in (4.4)

**Corollary 8** If \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \) in \( \mathcal{B} (\mathcal{H}_1 \oplus \mathcal{H}_2) \), then
\[
w \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right) \leq \frac{1}{4} \left( \tilde{R}_{11} + \tilde{R}_{22} + \sqrt{(\tilde{R}_{11} - \tilde{R}_{22})^2 + (\|A_{12}\| + \|A_{21}\|)^2} \right),
\]

where
\[\tilde{h}_{ij} = \begin{cases} R_{ii}, & i = j \\ \|A_{ij}\|, & i \neq j \end{cases} \]
such that \( R_{ii} = \frac{1}{2} \|A_{ii}^2\| - \frac{1}{4} \|A_{ii}\| - \|A_{ii}\|^2 + \|A_{ii}^*\| - \|A_{ii}\|^2 \]

**Proof** Setting \( \alpha = \frac{1}{2} \) in Corollary 7.

Clearly, the obtained bounds in Corollary 8 are better than the first and the second bounds in (4.4).
References

1. Abramovich, S., Jameson, G., Sinnamon, G.: Refining Jensen’s inequality. Bull. Math. Soc. Sci. Math. Roumanie 47, 3–14 (2004)
2. Abu-Omar, A., Kittaneh, F.: Numerical radius for \( n \times n \) operator matrices. Linear Algebra Appl. 468, 18–26 (2015)
3. Abu-Omar, A., Kittaneh, F.: Estimates for the numerical radius and the spectral radius of the Frobenius companion matrix and bounds for the zeros of polynomials. Ann. Funct. Anal. 5, 56–62 (2014)
4. Alomari, M.W.: Operator Popviciu’s inequality for superquadratic and convex functions of selfadjoint operators in Hilbert spaces. Adv. Pure Appl. Math. (accepted)
5. Bhatia, R.: Matrix Analysis. Springer, New York (1997)
6. Bani-Domi, W., Kittaneh, F.: Numerical radius inequalities for operator matrices. Linear Multilinear Algebra 57, 421–427 (2009)
7. Chien, M.-T., Gau, H.-L., Li, C.-K., Tsai, M.-C., Wang, K.-Z.: Product of operators and numerical range. Linear Multilinear Algebra 64(1), 58–67 (2016)
8. Chien, M.-T., Ko, C.-L., Nakazato, H.: On the numerical ranges of matrix products. Appl. Math. Lett. 23, 732–737 (2010)
9. Dragomir, S.S.: Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces, Springer Briefs in Mathematics (2013)
10. Dragomir, S.S.: Some inequalities for the Euclidean operator radius of two operators in Hilbert spaces. Linear Algebra Appl. 419, 256–264 (2006)
11. Dragomir, S.S.: Power inequalities for the numerical radius of a product of two operators in Hilbert spaces. Sarajevo J. Math. 5 (18)(2), 269–278 (2009)
12. Dragomir, S.S.: Some inequalities for the norm and the numerical radius of linear operator in Hilbert spaces. Tamkang J. Math. 39(1), 1–7 (2008)
13. El-Haddad, M., Kittaneh, F.: Numerical radius inequalities for Hilbert space operators. II. Studia Math. 182(2), 133–140 (2007)
14. Gustafson, K.E., Rao, D.K.: Numerical Range. Springer, New York (1997)
15. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge (1985)
16. Horn, R.A., Johnson, C.R.: Topics in Matrix Analysis. Cambridge University Press, Cambridge (1991)
17. Hou, J.C., Du, H.K.: Norm inequalities of positive operator matrices. Integral Equ. Oper. Theory 22, 281–294 (1995)
18. Kittaneh, F.: Numerical radius inequalities for Hilbert space operators. Studia Math. 168(1), 73–80 (2005)
19. Kittaneh, F.: Spectral radius inequalities for Hilbert space operators. Proc. Am. Math. Soc. 134(2), 385–390 (2005)
20. Kittaneh, F.: A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix. Studia Math. 158, 11–17 (2003)
21. Kittaneh, F.: Bounds for the zeros of polynomials from matrix inequalities. Arch. Math. 81, 601–608 (2003). ((Basel))
22. Kittaneh, F.: Norm inequalities for certain operator sums. J. Funct. Anal. 143(2), 337–348 (1997)
23. Kittaneh, F.: Notes on some inequalities for Hilbert Space operators. Publ. Res. Inst. Math. Sci 24(2), 283–293 (1988)
24. Krič, M., Lovrinčević, N., Pečarić, J., Perić, J.: Superadditivity and monotonicity of the Jensen-type functionals: New Methods for improving the Jensen-type Inequalities in Real and in Operator Cases. Element, Zagreb (2016)
25. Kian, M.: Operator Jensen inequality for superquadratic functions. Linear Algebra Appl. 456, 82–87 (2014)
26. Kian, M., Dragomir, S.S.: Inequalities involving superquadratic functions and operators. Mediterr. J. Math. 11(4), 1205–1214 (2014)
27. Kato, T.: Notes on some inequalities for linear operators. Math. Ann. 125, 208–212 (1952)
28. Li, C.-K., Tsai, M.-C., Wang, K.-Z., Wong, N.-C.: The spectrum of the product of operators, and the product of their numerical ranges. Linear Algebra Appl. 469, 487–499 (2015)
29. Omidvar, M.E., Moslehian, M.S., Niknam, A.: Some numerical radius inequalities for Hilbert space operators. Involve 2(4), 471–478 (2009)
30. Sattari, M., Moslehian, M.S., Yamazaki, T.: Some genaralized numerical radius inequalities for Hilbert space operators. Linear Algebra Appl. 470, 1–12 (2014)
31. Yamazaki, T.: On upper and lower bounds of the numerical radius and an equality condition. Studia Math. 178, 83–89 (2007)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.