STABILITY FOR EVOLUTION EQUATIONS GOVERNED BY A NON-AUTONOMOUS FORM

OMAR EL-MENNAOUI AND HAFIDA LAASRI

Abstract. This paper deals with the approximation of non-autonomous evolution equations of the form
\[ \dot{u}(t) + A(t)u(t) = f(t), \quad t \in [0, T], \quad u(0) = u_0, \]
where \( A(t), t \in [0, T] \) arise from a non-autonomous sesquilinear forms \( a(t; \cdot, \cdot) \) on a Hilbert space \( H \) with constant domain \( V \subset H \). Assuming the existence of a sequence \( a_n : [0, T] \times V \times V \to \mathbb{C}, n \in \mathbb{N} \) of non-autonomous forms such that the associated Cauchy problem has \( L^2 \)-maximal regularity in \( H \) and \( a_n(t, u, v) \) converges to \( a(t, u, v) \) as \( n \to \infty \), then among others we show under additional assumptions that the limit problem has \( L^2 \)-maximal regularity. Further we show that the convergence is uniformly on the initial data \( u_0 \) and the inhomogeneity \( f \).

Introduction

Throughout this paper \( H, V \) are two separable Hilbert spaces over \( \mathbb{K} = \mathbb{C} \). We denote by \( (\cdot, \cdot)_V \) the scalar product and \( \| \cdot \|_V \) the norm on \( V \) and by \( (\cdot, \cdot), \| \cdot \| \) the corresponding quantities in \( H \). Moreover, we assume that \( V \) is densely and continuously embedded into \( H \). Let \( V' \) denote the dual of \( V \) and \( (\cdot, \cdot)_V \) the duality between \( V' \) and \( V \). As usual, by identifying \( H \) with \( H' \), we have \( V \hookrightarrow H \cong H' \hookrightarrow V' \) with continuous and dense embedding. Let \( T > 0 \). Let \( a : [0, T] \times V \times V \to \mathbb{C} \) be a closed non-autonomous sesquilinear form, i.e., \( a(\cdot, \cdot, \cdot) \) is measurable for all \( u, v \in V \), and \( a(t; \cdot, \cdot) \) is a sesquilinear form with
\[ \| a(t; u, v) \| \leq M \| u \|_V \| v \|_V, \quad \text{and} \quad \Re a(t; u, u) + \beta \| u \|^2_H \geq \alpha \| u \|^2_V \]
for all \( t \in [0, T], u, v \in V \) and for some constants \( \beta \in \mathbb{R}, \alpha, M > 0 \). By Lax-Milgram Theorem, for each \( t \in [0, T] \) there exists an isomorphism \( A(t) : V \to V' \) such that
\[ (A(t)u, v) = a(t; u, v), \quad u, v \in V. \]
We call \( A(t) \) the operator associated with \( a(t; \cdot, \cdot) \) on \( V' \). Seen as an unbounded operator on \( V' \) with domain \( D(A(t)) = V \), the operator \(-A(t)\) generates a holomorphic \( C_0 \)-semigroup \( T \) on \( V' \). Further, we denote by \( A(t) \) the part of \( A(t) \) on \( H \); i.e.,
\[ D(A(t)) := \{ u \in V : A(t)u \in H \}, \quad A(t)u := A(t)u, \quad \text{for} \ u \in D(A(t)). \]
It is a known fact that \(-A(t)\) generates a holomorphic \( C_0 \)-semigroup \( T \) on \( H \) and \( T = T_H \) is the restriction of the semigroup generated by \(-A\) to \( H \). Then \( A(t) \) is the operator induced by \( a(t; \cdot, \cdot) \) on \( H \). See, e.g., [2,15,22 Chap. 2] and [8].

Consider the non-autonomous Cauchy problem
\[ \dot{u}(t) + A(t)u(t) = f(t), \quad \text{a.e on} \ [0, T], \quad u(0) = u_0. \]
Then the following \( L^2 \)-maximal regularity in \( V' \) result has been proved by J. L. Lions on 1961:

Theorem 0.1. (Lions 1961) The non-autonomous Cauchy problem \( \dot{u}(t) + A(t)u(t) = f(t), \) a.e. on \([0, T], u(0) = u_0\), \( f \in L^2(0, T; V') \) has \( L^2 \)-maximal regularity in \( V' \), i.e., for given \( f \in L^2(0, T; V') \) and \( u_0 \in H \), \( \dot{u}(t) + A(t)u(t) = f(t), \) a.e. on \([0, T], u(0) = u_0\), \( f \in L^2(0, T; V') \) has a unique solution \( u \) in \( MR_2(V, V') := L^2(0, T; V') \cap H^1(0, T; V') \). Moreover, there exists a constant \( c_H > 0 \) depending only on \( \alpha, \beta, M \) and \( c_H \) such that
\[ \| u \|_{MR_2(V, V')} \leq c_H \| f \|_{L^2(0, T; V')} + \| u_0 \|_H, \]
where \( c_H \) is the continuous embedding constant of \( V \) into \( H \).

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Lions proved this result in [17] (see also [23, Chapter 3]) using a representation theorem of linear functionals due to him self and usually known in the literature as Lions’s representation Theorem and using Galerkin’s method in [15, XVIII Chapter 3, p. 620]. We refer [24, Section 5.5] and [21] for other proofs. The theorem of Lions requires only the measurability of \( t \rightarrow a(t; u, v) \) for all \( u, v \in V \). However, in applications to boundary problems maximal regularity in \( V' \) is not sufficient because it is only the part \( A(t) \) of \( A(t) \) in \( H \) that realizes the boundary conditions in question. Precisely one is more interested on \( L^2\)-maximal regularity in \( H \), i.e., the solution \( u \) of (2) belongs to \( H^1(0, T; H) \) if \( f \in L^2(0, T; H) \) and \( u_0 \in V \). The problem of \( L^2\)-maximal regularity in \( H \) was initiated by Lions in [17, p. 68] for \( u_0 = 0 \) and \( \alpha \) is symmetric. In general, we have to impose more regularity on the form \( \alpha \) then measurability of the form is not sufficient [10, 8]. However, under additional regularity assumptions on the form \( \alpha \), the initial value \( u_0 \) and the inhomogeneity \( f \), some positive results were already done by Lions in [17, p. 68, p. 94, 17], Theorem 1.1, p. 129] and [17, Theorem 5.1, p. 138] and by Bardos [7]. More recently, this problem has been studied with some progress and different approaches [4, 5, 14, 11, 13, 12, 14]. Results on multiplicative perturbation are established in [4, 11, 6]. See also the recent review paper [3] for more details and references.

Exploiting the ideas and proofs of a recent result of Arendt and Monniaux [5] we study in this paper stability and the uniform approximation of the non-autonomous Cauchy problems (2). More precisely, assume that there exist a sequence of closed non-autonomous sesquilinear forms \( \omega_n : [0, T] \times V \times V \rightarrow \mathbb{C} \) of non-autonomous forms such that Cauchy problem

\[
\begin{align*}
\dot{u}_n(t) + A_n(t)u_n(t) &= f(t), \quad u(0) = u_0 \\
\end{align*}
\]

(4)

associated with \( \omega_n \) has \( L^2\)-maximal regularity in \( H \) and \( \omega_n(t, u, v) \) converges to \( a(t, u, v) \) as \( n \to \infty \). Then our aim is to study weather \( L^2\)-maximal regularity is iterated by the limit problem (2) and weather the sequence \( (u_n)_{n \in \mathbb{N}} \) of solutions of (3) converges uniformly on \( u_0 \) and \( f \) to the the solution of (2). Let \( 0 < \gamma < 1 \). Let \( \omega_n : [0, T] \rightarrow [0, +\infty), n \in \mathbb{N}, \) be a sequence of non-decreasing continuous function and let \( (d_n)_{n \in \mathbb{N}} \) be a zero real sequence such that

\[
\begin{align*}
&|\omega_n(t, u, v) - a(t, u, v)| \leq d_n\|u\|_V\|v\|_V, \quad t \in [0, T], \quad u, v \in V, \\
&|\omega_n(t, u, v) - \omega_n(s, u, v)| \leq \omega_n(t - s)\|u\|_V\|v\|_V, \quad t \in [0, T], \quad u, v \in V,
\end{align*}
\]

for all \( t, s \in [0, T], n \in \mathbb{N} \) and for all \( u, v \in V \), where \( V_{\gamma} := [H, V]_{\gamma} \) is the complex interpolation space. Then we show in Section 2 that the limit problem (2) has also \( L^2\)-maximal regularity in \( H \) and the sequence \( (u_n)_{n \in \mathbb{N}} \) of solutions of (3) converges weakly in \( MR_2(V, H) \) to the solution of (2). This convergence holds for the strongly topology of \( MR_2(V, H) \) and uniformly on \( u_0 \) and \( f \) provided the sequence \( (d_n)_{n \in \mathbb{N}} \) is decreasing, \( \lim_{n \to \infty} d_n n^{\gamma/2} = 0 \) and \( \lim_{n \to \infty} \int_0^1 \frac{\omega_n(t)}{t^{1+\gamma/2}} dt = 0 \), see Section 3. Moreover we show that similar results holds on the space \( C(0, T; V) \) if \( (u_n)_{n \in \mathbb{N}} \subset C(0, T; V) \). In the last section we provide an explicit approximation of \( a \) that satisfies the above required hypothesis. The reader interested in examples of application is referred to above cited papers and the references therein.

1. Preliminary results: uniform approximation on \( V' \)

In this section \( a : [0, T] \times V \times V \rightarrow C \) is a closed non-autonomous sesquilinear form. Moreover, we assume that there exist a sequence of closed non-autonomous sesquilinear forms \( a_n : [0, T] \times V \times V \rightarrow C \) satisfying (1) with the same constants \( \beta, \alpha \) and \( M > 0 \) and a zero real sequence \( (d_n)_{n \in \mathbb{N}} \) such that the following assumption holds:

\[
(H_0) \quad |a(t; u, v) - a_n(t; u, v)| \leq d_n\|u\|_V\|v\|_V, \quad t \in [0, T], \quad u, v \in V.
\]
For each $t \in [0,T]$ and $n \in \mathbb{N}$, let $A_n(t) \in \mathcal{L}(V,V')$ be the operator associated with $a_n(t; \cdot, \cdot)$ on $V'$ and consider the approximation Cauchy problems

$$\dot{u}_n(t) + A_n(t)u_n(t) = f(t), \quad a.e \text{ on } [0,T], \quad u_n(0) = u_0, \quad (n \in \mathbb{N}).$$

Note that the maximal regularity space $MR_2(V,V')$ is continuously embedded into $C([0,T];H)$ [20, p. 106]. Moreover, the result of Lions implies that $H$ coincides with the trace space, that is

$$H = Tr_2(V,V') := \{u(0) \mid u \in MR_2(V,V')\}.$$

The following theorem is the main result of this section.

**Theorem 1.1.** Let $u, u_n \in MR_2(V,V')$ be the solutions of (4) and (5), respectively. Then the following inequalities

$$\|u_n - u\|_{MR_2(V,V')} \leq c_n \left( \|f\|_{L^2(0,T;V')} + \|u_0\|_H \right), \quad n \in \mathbb{N}, \quad \text{and}$$

$$\|u_n - u\|_{C([0,T],H)} \leq c_n \left( \|f\|_{L^2(0,T;V')} + \|u_0\|_H \right), \quad n \in \mathbb{N}$$

hold for some positive constant $c > 0$ depending only on $M, \alpha, c_H$ and $T$. The sequence $(u_n)_{n \in \mathbb{N}}$ thus converges in $MR_2(V,V') \cap C([0,T],H)$ to $u$ uniformly on the data $f, u_0$.

**Proof.** For simplicity, we will in the sequel denote all positive constants depending on $M, \alpha, c_H$ and $T$ by $c > 0$. In view of the above Remark, it suffices to prove the first inequality. To that purpose, consider the unbounded linear operators $A, A_n$ and $B$ with domains $D(A) = D(A_n) = L^2(0,T;V)$ and $D(B) = \{u \in H^1(0,T;V'), u(0) = 0\}$ defined by

$$(A\varphi)(t) = A_n(t)\varphi(t), \quad (A\varphi)(t) = A_n(t)\varphi(t) \quad \text{and} \quad (B\varphi)(t) = -\varphi(t)$$

for almost every $t \in [0,T]$. Thus the Cauchy problem (4), respectively (5), has $L^2$-maximal regularity in $V'$ if and only if the unbounded operator $A + B$, respectively $A_n + B$, with domain

$$D(A + B) = D(A_n + B) := \{u \in MR_2(V,V') \mid u(0) = 0\}$$

is invertible. Consider first the case where $u_0 = 0$. Then we have $u = (A + B)^{-1}f$ and $u_n = (A_n + B)^{-1}f$. From Theorem 1.1 and (H0) we have

$$\|u_n - u\|_{MR_2(V,V')} = \|(A + B)^{-1}f - (A_n + B)^{-1}f\|_{MR_2(V,V')}$$

$$= \|(A + B)^{-1}(A_n - A)(A_n + B)^{-1}f\|_{MR_2(V,V')}$$

$$\leq c_n \|f\|_{L^2(0,T;V')}.$$

Let now $0 \neq u_0 \in H$. Choose $\vartheta \in MR_2(V,V')$ such that $\vartheta(0) = u_0$ and $\|\vartheta\|_{MR_2(V,V')} \leq 2\|u_0\|_H$. Set $g_n := -\dot{\vartheta}(\cdot) - A_n\vartheta(\cdot) + f(\cdot)$ and $g := -\dot{\vartheta}(\cdot) - A\vartheta(\cdot) + f(\cdot) \in L^2(0,T;V')$. Then there exist $v_n, u \in MR_2(V,V')$ such that

$$\dot{v}_n(t) + A_n(t)v_n(t) = g_n(t) \quad a.e \text{ on } [0,T], \quad v_n(0) = 0,$$

and

$$\dot{\vartheta}(t) + A(t)v(t) = g(t) \quad a.e \text{ on } [0,T], \quad v(0) = 0.$$
Corollary 1.2. Assume that the approximation problems \(5\) has \(L^2\)-maximal regularity in \(H\). Let \(u \in V\) and \(f \in L^2(0,T;H)\) and let \((u_n)_{n \in \mathbb{N}}\) be the sequence of solutions of \(5\). If \((u_n)_{n \in \mathbb{N}}\) converges weakly in \(MR^2(V;H)\), then the limit problem \(5\) has also \(L^2\)-maximal regularity in \(H\) and \(u\) is equal to the weak limits of \((u_n)_{n \in \mathbb{N}}\).

2. \(L^2\)-MAXIMAL REGULARITY IN \(H\): A WEAK APPROXIMATION

Let \(a, a_n : [0,T] \times V \times V \rightarrow \mathbb{C}\) be a closed non-autonomous forms satisfying \((\mathbf{1})\) with the same constants \(\beta, \alpha\) and \(M > 0\). In this section we assume that there exist \(0 \leq \gamma < 1\), a sequence of non-decreasing continuous function \(\omega_n : [0,T] \rightarrow [0,\infty), n \in \mathbb{N}\), and zero real sequence \((a_n)_{n \in \mathbb{N}}\) such that the following assumptions hold.

\[
\begin{align*}
\text{(H1)} & \quad |a_n(t, u, v) - a(t, u, v)| \leq d_n ||v||_{V^{'}} ||v||_V, \quad t \in [0,T], \quad u, v \in V, \\
\text{(H2)} & \quad |a_n(t, u, v) - a_n(s, u, v)| \leq \omega_n(|t-s|)||v||_{V^{'}} ||v||_V, \quad t, u, v \in V, \\
\text{(H3)} & \quad \sup_{t \in [0,T], n \in \mathbb{N}} \frac{\omega_n(t)}{t^{\gamma/2}} < \infty, \quad \text{and} \quad \sup_{n \in \mathbb{N}} \int_{0}^{T} \frac{\omega_n(t)}{t^{1+\gamma/2}} dt < \infty, \\
\text{(H4)} & \quad \text{The approximation problem} \ 5 \ 4 \ \text{has} \ \mathbb{L}^2\text{-maximal regularity in} \ H \ \text{for every} \ n \in \mathbb{N}.
\end{align*}
\]

where \(V_{\gamma} := [H, V]_{\gamma}\) is the complex interpolation space. Note that

\[V \hookrightarrow V_{\gamma} \hookrightarrow H \hookrightarrow V_{\gamma}' \hookrightarrow V'
\]

with continuous embeddings. Remark that condition \((\text{H2})\) implies that \(A_n(t) - A_n(s) \in \mathbb{L}(V, V_{\gamma}')\) and \(a_n(t, u, v) - a_n(s, u, v) \leq \omega_n(|t-s|)||u||_V||v||_{V^{'}}\), \(t, s \in [0,T], n \in \mathbb{N}\).

The following proposition is of great interest for this paper.

Proposition 2.1. \([5\text{ Section 2}]\) Let \(b\) be any sesquilinear form that satisfies \((\mathbf{1})\) with the same constants \(M, \alpha, \gamma\) and \(c_H\) and \(\gamma \in [0,1]\). Let \(B\) and \(b\) be the associated operators on \(V'\) and \(H\), respectively. Then there exists a constant \(c > 0\) which depends only on \(M, \alpha, \gamma\) and \(c_H\) such that

\[
\begin{align*}
(1) & \quad \| (\lambda - B)^{-1} \|_{\mathbb{L}(V_{\gamma}'; H)} \leq \frac{c}{\lambda}, \\
(2) & \quad \| (\lambda - B)^{-1} \|_{\mathbb{L}(V)} \leq \frac{c}{1 + |\lambda|^2}, \\
(3) & \quad \| (\lambda - B)^{-1} \|_{\mathbb{L}(H; V)} \leq \frac{c}{1 + |\lambda|^2}, \\
(4) & \quad \| (\lambda - B)^{-1} \|_{\mathbb{L}(V'; H)} \leq \frac{c}{1 + |\lambda|^2}, \\
(5) & \quad \| (\lambda - B)^{-1} \|_{\mathbb{L}(V'; V')} \leq \frac{c}{1 + |\lambda|^2}, \\
(6) & \quad \| e^{-sB} \|_{\mathbb{L}(V_{\gamma}'; H)} \leq \frac{c}{s^{1/2}}, \\
(7) & \quad \| e^{-sB} \|_{\mathbb{L}(V_{\gamma}'; V)} \leq \frac{c}{s^{1/2}}, \\
(8) & \quad \| e^{-sB} \|_{\mathbb{L}(V'; V')} \leq \frac{c}{s^{1/2}}, \\
(9) & \quad \| B e^{-sB} \|_{\mathbb{L}(H)} \leq \frac{c}{s}, \\
(10) & \quad \| B e^{-sB} \|_{\mathbb{L}(V)} \leq \frac{c}{s}
\end{align*}
\]

for each \(t \in [0,T], s \geq 0\) and \(\lambda \notin \Sigma_\theta := \{ re^{i\varphi} : r > 0, |\varphi| < \theta\} \).

Remark 2.2. All estimates in Proposition \((2.1)\) holds for \(A_n(t)\) and \(A(t)\) with constant independent of \(n\) and \(t \in [0,T]\), since \(a, a\) satisfies \((\mathbf{1})\) with the same constants \(M, \beta, \alpha,\) also \(\gamma\) and \(c_H\) does not depend on \(n\) and \(t \in [0,T]\).

Notation 2.3. To keep notations simple as possible we will in the sequel denote all positive constants depending on \(M, \alpha, \gamma, c_H\) and \(T\) that appear in proofs and theorems uniformly as \(c > 0\).

For each \(f \in L^2(0,T;H)\) and \(u_0 \in V\), the solutions \(u_n, n \in \mathbb{N}\), of \((5)\) satisfies the following key formula

\[
\begin{align*}
u_n(t) = e^{-tA_n(t)}u_0 + \int_{0}^{t} e^{-(t-s)A_n(t)}f(s)ds + \int_{0}^{t} e^{-(t-s)A_n(t)}(A_n(t) - A_n(s))u_n(s)ds
\end{align*}
\]
for all \( t \in [0, T] \). This formula is due to Acquistapace and Terreni [1] and was proved in a more general setting in [5] Proposition 3.5. In the sequel we will use the following notations:

\[
\begin{align*}
\tag{8} u_{n,1}(t) &:= e^{-tA_n(t)}u_0, \\
u_{n,2}(t) &:= \int_0^t e^{-(t-s)A_n(t)}f(s)ds.
\end{align*}
\]

With this notation we can state the main result of this section which, in particular, shows that the limit problem [2] also has \( L^2 \)-maximal regularity in \( H \).

**Theorem 2.4.** Assume that the assumptions \((H_1)-(H_4)\) holds. Then the problem [2] also has \( L^2 \)-maximal regularity in \( H \). Moreover, if \( f \in L^2(0, T; H) \) and \( u_0 \in V \) and \( (u_n)_{n\in\mathbb{N}} \subset MR_2(V,H) \) is the sequence of the unique solutions of [3], then \( (u_n)_{n\in\mathbb{N}} \) converges weakly in \( MR_2(V,H) \) and \( u := w-\lim_{n \to \infty} u_n \) satisfies [2].

For the proof we need first some preliminary lemmas. Using the same argument as in the proof of [5] Theorem 4.1], the next two lemmas follow thanks to \((H_1)-(H_4)\) and Remark 2.2.

**Lemma 2.5.** Assume that the assumptions \((H_1)-(H_4)\) holds. Let \( Q_n^\mu : L^2(0, T; H) \to L^2(0, T; H) \) denotes the linear operator defined for all \( g \in L^2(0, T; H) \) and \( \mu \geq 0 \) by

\[
\begin{align*}
\tag{9} (Q_n^\mu)(t) &:= \int_0^t (A_n(t) + \mu)e^{-(t-s)(A_n(t)+\mu)}(A_n(t) - A_n(s))(A_n(s)+\mu)^{-1}g(s)ds \\
t-a.e.
\end{align*}
\]

Then \( \lim_{\mu \to \infty} \|Q_n^\mu\|_{L^2(0,T;H)} = 0 \) uniformly on \( n \) and thus \( I - Q_n^\mu \) is invertible on \( L^2(0,T;H) \) for \( \mu \) large enough and for all \( n \).

**Lemma 2.6.** Assume that the assumptions \((H_1)-(H_3)\) holds. The following tow estimates

\[
\begin{align*}
\|A_n u_{n,1}\|_{L^2(0,T,H)} &\leq c\|u_0\|_{V}, \\
\|A_n u_{n,2}\|_{L^2(0,T,H)} &\leq c\|f\|_{L^2(0,T;H)}
\end{align*}
\]

hold.

Now we can give the proof of Theorem 2.4.

**Proof.** (of Theorem 2.4) According to Lemma 2.5 and replacing \( A_n(t) \) with \( A_n(t) + \mu \), we may assume without loss of generality that \( Q_n = Q_n^\mu \) satisfies \( \|Q_n\|_{L(L^2(0,T;H))} < 1 \), and then \( I - Q_n \) is invertible by the Neumann series. We deduce from (7) that

\[
\dot{u}_n = A_n u_n = (I - Q_n)^{-1}(A_n u_{n,1} + A_n u_{n,2}).
\]

This equality and Lemma 2.6 yield the estimate

\[
\|\dot{u}_n\|_{L^2(0,T,H)} \leq c\left[\|u_0\|_{V} + \|f\|_{L^2(0,T;H)}\right].
\]

Since for all \( t \in [0, T] \) one has \( u_n(t) = u_n(0) + \int_0^t \dot{u}_n(s)ds \), we conclude that

\[
\|u_n\|_{H^1(0,T,H)} \leq c\left[\|u_0\|_{V} + \|f\|_{L^2(0,T;H)}\right].
\]

Then there exists a subsequence of \((u_n)\), still denoted by \((u_n)\) that converges weakly to some \( v \in H^1(0,T;H) \).

On the other hand, the Cauchy problem [2] has a unique solution \( u \in MR_2(V,V') \), and \((u_n)\) converges strongly to \( u \) on \( MR_2(V,V') \) by Theorem 1.1. We conclude by uniqueness of limits that \( u = v \in H^1(0,T;H) \). This completes the proof.

3. \( L^2 \)-maximal regularity in \( H \): uniform approximation

Assume that \( a \) and \( u_n \) are as in Section 2. Let \((f, u_0) \in L^2(0,T;H) \times V \) and let \( u, u_n \in MR_2(V,H) \) be the solutions of [2] and [4] respectively. In the previous section we have seen that \((u_n)_{n\in\mathbb{N}} \) converges weakly to \( u \) with respect to the norm of \( MR_2(V,H) \). The aim of this section is to prove that this convergence
holds for the strong topology of $MR_2(V, H)$ and uniformly on the initial data $u_0$ and $f$. To this end, we impose the following additional conditions:

$$(H_5) \lim_{n \to \infty} d_n n^{\gamma/2} = 0 \text{ and the sequence } (d_n)_{n \in \mathbb{N}} \text{ is decreasing.}$$

$$(H_6) \lim_{n \to \infty} \int_0^{1/n} \omega_n(r) r^{1+\gamma/2} \, dr = 0.$$  

Recall that $-A_n(t)$ generates a holomorphic $C_0$-semigroup (of angle $\theta := \frac{\pi}{2} - \arctan(A))$ $e^{-s A_n(t)}$ on $H$ which is the restriction to $H$ of $e^{-s A_n(t)}$, and we have

$$e^{-A_n(t)} = \frac{1}{2\pi} \int_{\Gamma} e^{\mu (\mu + A_n(t))^{-1}} \, d\mu$$  

where $\Gamma := \{re^{\pm \varphi} : r > 0\}$ for some fixed $\varphi \in (\theta, \frac{\pi}{2})$.

**Theorem 3.1.** Assume that the assumptions $(H_1)$-$(H_6)$ holds. Then there exists a positive constant $c > 0$ depending only on $M, \alpha, \gamma$ and $c_H$ such that

$$\|\hat{u} - \hat{u}_n\|_{L^2(0,T; H)} \leq c \left[ (1 + n^{\gamma/2})d_n + \int_0^{1/n} \frac{\omega_n(r)}{r^{1+\gamma/2}} \, dr \right] \left[ \|f\|_{L^2(0,T; H)} + \|u_0\|_V \right].$$  

Thus $(u_n)_{n \in \mathbb{N}}$ converges to $u$ for the strong topology of $MR_2(V, H)$ and uniformly on the initial data $u_0$ and $f$.

**Proof.** We only have to prove (13) the uniform convergence with respect to $u_0, f$ in $MR_2(V, H)$ becomes obvious. Indeed, we known from Theorem 1.1 that $u_n \rightarrow u$ in $L^2(0,T; V)$ uniformly on the initial data $u_0$ and the homogeneity $f$.

We will use the representation formula (7) and (8). We proceed by several steps. Let $m, k \in \mathbb{N}$ and set $n := m + k$ and $d_{n,m} := d_n + d_m$.

(a) First, we estimate $A_n u_{n,1} - A_m u_{m,1}$ in $L^2(0,T; H)$. Let $t \neq 0$. Using $(H_1)$ we obtain the estimates (11) and (12) in Proposition 2.1 that

$$\begin{align*}
\|A_n(t)u_{n,1}(t) - A_m(t)u_{m,1}(t)\|_H &= \|A_n(t)e^{-tA_n(t)}u_0 - A_m(t)e^{-tA_m(t)}u_0\|_H \\
&\leq \|e^{-tA_n(t)}[A_n(t)u_0 - A_m(t)u_0]\|_H + \|[e^{-tA_n(t)} - e^{-tA_m(t)}]A_n(t)u_0\|_H \\
&= \|e^{-tA_n(t)}[A_n(t)u_0 - A_m(t)u_0]\|_H + \int_0^t \|e^{-(t-s)A_n(t)}(A_n(t) - A_m(t))e^{-sA_n(t)}u_0\|_H \\
&\leq c d_{n,m} \left( \frac{1}{\Gamma/2} + \int_0^t \frac{1}{s^{1/2}} \, ds \right) \|u_0\|_V.
\end{align*}$$

Similarly, combining the estimates (11) and (12) in Proposition 2.1 and the estimate (15) in Proposition 2.1 we obtain

$$\begin{align*}
\|A_n(t)u_{n,2}(t) - A_m(t)u_{m,2}(t)\|_H &\leq \int_0^t \|[A_n(t)e^{-(t-s)A_n(t)} - A_m(t)e^{-(t-s)A_m(t)}]f(s)\|_H \, ds \\
&\leq c \int_0^t d_{n,m} \int_\Gamma \|e^{-(t-s)A_n(t)} - A_n(t)\|_H \, d\lambda ds \\
&= cd_{n,m} \int_0^t \|f(s)\|_H \int_0^\infty e^{-(t-s)r\cos(\nu)} \, dr ds \\
&= cd_{n,m} \int_0^t \|f(s)\|_H \int_0^{\pi} \frac{e^{-r\cos(\theta)}}{r^{1/2}} \, d\theta ds \\
&\leq cd_{n,m} \int_0^t \|f(s)\|_H (t-s)^{-\frac{\gamma}{2}} \, ds.
\end{align*}$$

This completes the proof.
Again by estimates (6) and (9) in Proposition 2.1, we obtain for the second term respectively,

\[ \|A_n u_{n,1} - A_m u_{m,1}\|_{L^2(0,T;H)} \leq c d_{n,m} \|u_0\|_V \]

and

\[ \|A_n u_{n,2} - A_m u_{m,2}\|_{L^2(0,T;H)} \leq c d_{n,m} \|f\|_{L^2(0,T;H)}. \]

(b) Next, we prove the following estimate

\[ \|Q_n - Q_m\|_{L^2(0,T;H)} \leq c \left[ d_{n,m} + m^{\gamma/2} d_m + n^{\gamma/2} d_n + \int_0^1 \frac{\omega_n(r)}{r^{1+\gamma/2}} dr + \int_0^1 \frac{\omega_m(r)}{r^{1+\gamma/2}} dr \right] \]

where \( Q_n : L^2(0,T;H) \rightarrow L^2(0,T;H) \) is defined \([13]\). To this end, for \( g \in L^2(0,T;H) \) and \( t \in [0,T] \) we write

\[ \|(Q_n g)(t) - (Q_m g)(t)\|_H \]

\[ \leq \int_0^t \|A_n(t) e^{-(t-s)A_n(t)} (A_n(t) - A_m(t)) (A_n^{-1}(s) - A_m^{-1}(s)) g(s)\|_H ds \]

\[ + \int_0^t \|A_n(t) e^{-(t-s)A_n(t)} (A_n(t) - A_m(t)) - A_n(t) + A_m(t)\|_H ds \]

\[ + \int_0^t \|\left( A_n(t) e^{-(t-s)A_n(t)} - A_n(t) e^{-(t-s)A(t)} \right) (A_m(t) - A_m(t)) A_n^{-1}(s) g(s)\|_H ds \]

\[ = I_{n,m,1}(t) + I_{n,m,2}(t) + I_{n,m,3}(t) \]

Replacing \( A_m(s) \) by \( A_m(s) + \mu \) and according to Proposition 2.1 we may assume \( \|A_n^{-1}(s)\|_{\mathcal{L}(V'_V,V)} \leq c \) and \( \|A_m^{-1}(s)\|_{\mathcal{L}(H,V)} \leq c \). Next, by the estimates \([13]\) and \([13]\) in Proposition 2.1 together with \((H_1) - (H_2)\), we obtain

\[ I_{n,m,1}(t) = \int_0^t \|A_n(t) e^{-(t-s)A_n(t)} - A_n(t) e^{-(t-s)A_n(t)} (A_n(t) - A_m(t)) (A_n^{-1}(s) - A_m^{-1}(s)) g(s)\|_H ds \]

\[ \leq c \int_0^t \frac{\omega_n(t-s)}{(t-s)^{1+\gamma/2}} \|A_n^{-1}(s) - A_m^{-1}(s)\|_V ds \]

\[ = c \int_0^t \frac{\omega_n(t-s)}{(t-s)^{1+\gamma/2}} \|A_n^{-1}(s) (A_n(s) - A_m(s)) A_n^{-1}(s) g(s)\|_V ds \]

\[ \leq c d_{n,m} \int_0^t \frac{\omega_n(t-s)}{(t-s)^{1+\gamma/2}} \|g(s)\|_H ds \]

\[ = c d_{n,m} \|h_n\|_{L^1(\mathbb{R})} \leq c. \]

where \( h_n(t) := \omega_n(t) t^{-1+\gamma/2} \) for \( t \in [0,T] \) and \( h_n(t) := 0 \) for \( t \in (-\infty,0]\). The assumption \((H_3)\) implies that \( h_n \in L^1(\mathbb{R}) \) and that \( \|h_n\|_{L^1(\mathbb{R})} \) is bounded. Therefore, we obtain

\[ \int_0^T T_{n,m,1}(s) ds \leq c d_{n,m} \int_0^T \|g(s)\|_H^2 ds. \]

Again by estimates \([13]\) and \([13]\) in Proposition 2.1 we obtain for the second term \( I_{n,m,2} \)

\[ I_{n,m,2}(t) := \int_0^t \|A_n(t) e^{-(t-s)A_n(t)} (A_n(t) - A_m(t)) - A_n(t) + A_m(t)\|_{\mathcal{L}(V'_V,H)} \|g(s)\|_H ds \]

\[ \leq c \int_0^t \|A_n(t) - A_m(t) - A_n(t) + A_m(t)\|_{\mathcal{L}(V'_V,H)} \|g(s)\|_H ds \]

\[ \leq c \int_0^t \kappa_{n,m}(t-s) \|g(s)\|_H ds \]

where

\[ \kappa_{n,m}(t) := \begin{cases} \frac{\omega_n(t) + \omega_m(t)}{4d_{n,m}t^{-(1+\gamma/2)}} & \text{if } 0 \leq t < \frac{1}{n}, \\ \frac{\gamma}{n} & \text{if } \frac{1}{n} \leq t \leq T, \\ 0 & \text{if } t \in (-\infty,0]\cap[T, +\infty]. \end{cases} \]
Thanks to \((H_3)\) and \((H_5)\), \(t \mapsto \kappa_{n,m}(t)\) belongs to \(L^1(\mathbb{R})\), and by a simple calculation we obtain

\[(18) \quad \|\kappa_{n,m}\|_{L^1(\mathbb{R})} \leq c \left[ m^{\gamma/2}d_m + n^{\gamma/2}d_n + \int_0^{1/n} \rho^{1/\gamma/2} \, d\rho + \int_0^{1/m} \rho^{1/\gamma/2} \, d\rho \right].\]

Therefore,

\[(19) \quad \int_0^T I_{n,m,3}(s) \, ds \leq c \left[ m^{\gamma/2}d_m + n^{\gamma/2}d_n + \int_0^{1/n} \rho^{1/\gamma/2} \, d\rho + \int_0^{1/m} \rho^{1/\gamma/2} \, d\rho \right]^2 \int_0^T \|g(s)\|^2_H \, ds.

For the last term \(I_{n,m,3}(t)\), we set \(g_m(t, \cdot) := (A_m(t) - A_m(\cdot))A_m^{-1}(\cdot)g(\cdot)\). Again by assumptions \((H_3)\) and \((H_4)\) from Proposition 2.1 and we obtain

\[I_{n,m,3}(t) := \int_0^t \| (A_m(t)e^{-(t-s)}A_m(t) - A_m(t)e^{-(t-s)}A_m(t)) \tilde{g}_m(t, s) \|_H \, ds \]

\[\leq \frac{1}{2\pi} \int_0^t \int_\Gamma |\lambda| e^{-(t-s)} \Re \lambda \| (\lambda - A_m(t))^{-1} (A_m(t) - A_m(t)) (\lambda - A_m(t))^{-1} \tilde{g}_m(t, s) \|_H \, d\lambda \, ds \]

\[\leq \frac{1}{2\pi} \int_0^t \int_\Gamma |\lambda| e^{-(t-s)} \Re \lambda \| (\lambda - A_m(t))^{-1} \|_L(\nu', \nu) \| (A_m(t) - A_m(t)) (\lambda - A_m(t))^{-1} \tilde{g}_m(t, s) \|_{\nu}\nu' \, d\lambda \, ds \]

\[\leq \frac{1}{2\pi} \int_0^t \int_\Gamma |\lambda| e^{-(t-s)} \Re \lambda \| (\lambda - A_m(t))^{-1} \|_L(\nu', \nu) \| g_m(t, s) \|_{\nu'} \, d\lambda \, ds \]

\[\leq c \int_0^t \int_\Gamma |\lambda| e^{-(t-s)} \Re \lambda \frac{d_{n,m}}{(1 + |\lambda|)^{1/2}} \| (\lambda - A_m(t))^{-1} \|_L(\nu', \nu) \| g_m(t, s) \|_{\nu'} \, d\lambda \, ds \]

\[\leq c d_{n,m} \int_0^t \int_\Gamma |\lambda| e^{-(t-s)} \Re \lambda \| g_m(t, s) \|_{\nu'} \, d\lambda \, ds \]

\[\leq c d_{n,m} \int_0^t \int_\Gamma |\lambda| e^{-(t-s)} \cos(\nu) \| g_m(t, s) \|_{\nu'} \, d\lambda \, ds \]

Next, since \(\|g_m(t, s)\|_{\nu'} \leq \omega_m(|t-s|)\|A_m^{-1}(t)\|_{L(\nu, \nu')}\|g(s)\|_H\),

it follows

\[I_{n,m,3}(t) \leq c d_{n,m} \int_0^t \int_\Gamma |\lambda| e^{-(t-s)} \cos(\nu) \| g_m(t, s) \|_{\nu'} \, d\lambda \, ds \]

\[= c d_{n,m} \int_0^t \int_\Gamma |\lambda| e^{-(t-s)} \cos(\nu) \| g_m(t, s) \|_{\nu'} \, d\lambda \, ds \]

Using the same argument as that used above for \((17)\) one obtain

\[[20] \quad \int_0^T I_{n,m,3}(s) \, ds \leq c d_{n,m} \int_0^T \|g(s)\|^2_H \, ds.

This together with \((16)\) and \((17)\) give the desired estimate \((18)\).

(c) Using Lemma 2.6 we conclude from \((a) \rightarrow (b))\) that

\[\|A_n u_n - A_m u_m\|_{L^2(0,T;H)}\]

\[\leq \| (I - Q_n)^{-1} (A_n - A_m) \|_{L^2(0,T;H)} + \| (I - Q_n)^{-1} (A_n - A_m) \|_{L^2(0,T;H)}\]

\[+ \| (I - Q_n)^{-1} (Q_m - Q_n)(I - Q_m)^{-1}(A_m + A_m) \|_{L^2(0,T;H)}\]

\[\leq c \left[ d_{n,m} + n^{\gamma/2}d_n + m^{\gamma/2}d_m + \int_0^{1/n} \rho^{1/\gamma/2} \, d\rho + \int_0^{1/m} \rho^{1/\gamma/2} \, d\rho \right] \left[ \|u_0\|_H + \|f\|_{L^2(0,T;H)} \right].\]

Finally, since \((u_n)_{n \in \mathbb{N}}\) satisfies \((5)\), we conclude that

\[(21) \quad \| \hat{u}_{k+m} - \hat{u}_m \|_{L^2(0,T;H)} \leq c_{k,m} \left[ \|u_0\|_H + \|f\|_{L^2(0,T;H)} \right]\]

with
\[ c_{n,m} := c\left[ d_{k+m} + d_m + (k + m)^{\gamma/2} d_{k+m} + m^{\gamma/2} d_m + \int_0^{\pi/\gamma} \frac{\omega_{k+m}(r)}{r^{1+\gamma/2}} \, dr + \int_0^{\pi/\gamma} \frac{\omega_m(r)}{r^{1+\gamma/2}} \, dr \right]. \]

Thus \( (\hat{u}_n) \) is a Cauchy sequence. The limits coincide with \( u \) according to Corollary \( \text{(3)} \). This completes the proof of \( \text{(13)} \) by taking \( k \rightarrow \infty \) in \( \text{(21)} \).

\[ \square \]

4. Uniform Approximation on \( C(0, T; V) \)

Let \( a, a_n : [0, T] \times V \times V \rightarrow \mathbb{C} \) are as in Section \( \text{[5]} \). Additionally, we assume that \( (u_n)_n \subset C([0, T], V) \). Then we show in this section that \( (u_n)_n \) converges in \( C([0, T], V) \) uniformly on \( (f, u_0) \).

**Proposition 4.1.** With the notations of Section \( \text{[3]} \) the following estimate holds
\[ \|u_{m+k} - u_m\|_{C(0, T; V)} \leq c_{k,m} \left[ \|u_0\|_V + \|f\|_{L^2(0, T; H)} \right] \]
for all \( k, m \in \mathbb{N} \).

In view of Theorem \( \text{(11)} \), the following is then true and follows immediately from Proposition \( \text{(11)} \).

**Corollary 4.2.** Let \( (f, u_0) \in L^2(0, T; H) \times V \) and let \( u \in MR_2(V, H) \) be the solution of \( \text{(3)} \). Then \( u \in C(0, T; V) \) and
\[ \|u - u_n\|_{C(0, T; V)} \leq c \left[ d_n + n^{\gamma/2} d_m \right] \left[ \|u_0\|_V + \|f\|_{L^2(0, T; H)} \right] \]
holds.

**Proof.** We will proceed similarly as in the proof of Theorem \( \text{[3, 1]} \). Let \( m, k \in \mathbb{N} \) and set \( n := m + k \) and \( d_{n,m} := d_n + d_m \).

**Step a:** By using \( \text{(2)} \) and \( \text{(5)} \) in Proposition \( \text{[2, 1]} \) for \( \lambda - A_n(t)^{-1} \) and \( \lambda - A_m(t)^{-1} \), respectively, and \( (H_1) \) we obtain for every \( t \in [0, T] \) that
\[ \|u_{1,n}(t) - u_{1,m}(t)\| \leq \frac{1}{2\pi} \int_{\Gamma} e^{-t Re \lambda} \|\lambda - A_n(t)^{-1}(A_n(t) - A_m(t))(\lambda - A_m(t)^{-1}u_0\|_V d\lambda \]
\[ \leq c d_{n,m} \int_{\Gamma} \frac{e^{-t Re \lambda}}{(1 + |\lambda|)^{\frac{\gamma}{2}}} d\lambda \|u_0\|_V \]
\[ \leq c d_{n,m} \int_{0}^{\infty} \frac{1}{(1 + r)^{\frac{\gamma}{2}}} \, dr \|u_0\|_V. \]

**Step b:** Again the estimates \( \text{(4)} \) and \( \text{(5)} \) in Proposition \( \text{[2, 1]} \) and formula \( (H_1) \) imply that
\[ \|\|\lambda - A_n(t)^{-1}(A_n(t) - A_m(t))(\lambda - A_m(t)^{-1}f(s)\|_V \leq \frac{c d_{n,m}}{(1 + |\lambda|)^{\frac{\gamma}{2}}} \|f(s)\|_H. \]
Therefore, we obtain by using Fubini’s theorem that for all \( \lambda \in \Gamma \setminus \{0\} \)
\[ \|u_{2,n}(t) - u_{2,m}(t)\|_V = \|\int_{0}^{t} \frac{1}{2\pi} \int_{\Gamma} e^{-(t-s)\lambda} (\lambda - A_n(t)^{-1}(A_n(t) - A_m(t))(\lambda - A_m(t)^{-1}f(s) - f(s) d\lambda ds) \]
\[ \leq c d_{n,m} \int_{0}^{\infty} \frac{1}{(1 + r)^{\frac{\gamma}{2}}} \left( \int_{0}^{t} e^{-(t-s)\lambda} (\lambda - A_n(t)^{-1}(A_n(t) - A_m(t))(\lambda - A_m(t)^{-1}) \right) d\lambda ds \]
\[ \leq c d_{n,m} \|f\|_{L^2(0,T;H)} \int_{0}^{\infty} \frac{1}{(1 + r)^{\frac{\gamma}{2}}} \left( \int_{0}^{t} e^{-(t-s)\lambda} (\lambda - A_n(t)^{-1}(A_n(t) - A_m(t))(\lambda - A_m(t)^{-1}) \right) \frac{1}{2} ds \]
\[ \leq c d_{n,m} \sqrt{2 \cos(r)} \int_{0}^{\infty} \sqrt{\pi} \frac{1}{(1 + r)^{\frac{\gamma}{2}}} \, dr \|f\|_{L^2(0,T;H)} \leq c d_{n,m} \|f\|_{L^2(0,T;H)}. \]

**Step c:** For each \( h \in C(0, T; V) \) we set
\[ (P_n h)(t) := \int_{0}^{t} e^{-(t-s)A_n(t)} (A_n(t) - A_m(t)) h(s) \, ds. \]

From \( \text{[5, Lemma 4.5]} \) we have \( (P_n h)_n \subset C(0, T; V) \). Thanks to Proposition \( \text{[5, 2]} \) and assumptions \( (H_2)-(H_3) \) one can prove in a similar way as in Step 3 of the proof of \( \text{[5, Theorem 4.4]} \) (see also Step 3 of the
proof of Lemma 2.5 that \(\|P_n\|_{L(L(C(0,T))} \leq 1/2\) and thus \(I - P_n\) is invertible on \(L(C(0,T;V))\). Therefore, we obtain by using the representation formula 11

\[
\begin{align*}
\mathbf{P} - \mathbf{P} \quad \text{on} \\
\mathbf{m}
\end{align*}
\]

(24)

\[
\begin{align*}
\mathbf{u}_n - \mathbf{v}_m = (I - P_n)^{-1}(u_{n,1} - u_{m,1}) + (I - P_m)^{-1}(u_{n,2} - u_{m,2}) \\
+ (I - P_n)^{-1}(P_n - P_m)(I - P_m)^{-1}(u_{m,1} + u_{m,2})
\end{align*}
\]

The term on the right hand side of (24) is treated in Step a)-b). We need only to estimate the difference \(P_n - P_m\) on \(L(C(0,T;V))\). For each \(h \in C(0,T;V)\) and \(t \in [0,T]\) we have

\[
(P_n h - P_m h)(t)
\]

\[
\begin{align*}
\int_0^t e^{-(t-s)A_n(t)}[A_n(t) - A_n(s) - \mathcal{A}_m(t) + \mathcal{A}_m(s)]h(s)ds \\
+ \int_0^t e^{-(t-s)A_n(t)}[A_n(t) - A_n(s)]h(s)ds.
\end{align*}
\]

\[
\begin{align*}
\int_0^t e^{-(t-s)A_n(t)}[A_n(t) - A_n(s) - \mathcal{A}_m(t) + \mathcal{A}_m(s)]h(s)ds \\
+ \int_0^t \frac{1}{2\pi} \int_{\Gamma} e^{-(t-s)}(\lambda - A_n)^{-1}(A_n(t) - A_n(t))(\lambda - A_m)^{-1}(A_m(t) - A_m(s))h(s)ds.
\end{align*}
\]

From (H1)-(H2) and the estimate 5 in Proposition 2.1 we have

\[
\|(\lambda - A_n)^{-1}(A_n(t) - A_m(t))(\lambda - A_m)^{-1}(A_m(t) - A_m(s))h(s)\|_V \\
\leq c_{d,m}\frac{\omega_m(t-s)}{(1 + |\lambda|)^{1 - \gamma}}\|h(s)\|_V.
\]

Thus using (H3), it follows

\[
\begin{align*}
\int_0^t e^{-(t-s)}(\lambda - A_n)^{-1}(A_n(t) - A_m(t))(\lambda - A_m)^{-1}(A_m(t) - A_m(s))h(s)ds
\end{align*}
\]

\[
\int_0^t \frac{1}{2\pi} \int_{\Gamma} \frac{e^{-(t-s)}r \cos(\rho)(\omega_m(t-s))}{\rho^{1-\gamma}}\|h(s)\|_V drds
\]

\[
\int_0^t \frac{1}{2\pi} \int_{\Gamma} \frac{e^{-r \cos(\rho)}}{\rho^{1-\gamma}}(t-s)^{1-\gamma}\|h(s)\|_V d\rho ds
\]

\[
\int_0^t \int_{\Gamma} \frac{e^{-r \cos(\rho)} \omega_m(t-s)}{(t-s)^{1-\gamma}}\|h(s)\|_V d\rho ds
\]

\[
\int_0^t \int_{\Gamma} e^{-r \cos(\rho)} \omega_m(t-s)\|h(s)\|_V d\rho ds
\]

\[
\int_0^t \frac{1}{2\pi} \int_{\Gamma} \frac{e^{-(t-s)}r \cos(\rho)(\omega_m(t-s))}{\rho^{1-\gamma}}\|h(s)\|_V drds
\]

Next, writing

\[
e^{-(t-s)A_n(t)}[A_n(t) - A_n(s) - \mathcal{A}_m(t) + \mathcal{A}_m(s)]h(s)
\]

\[
= A_n^{-1/2}(t)A_n^{1/2}(t)e^{-(t-s)A_n(t)}[A_n(t) - A_n(s) - \mathcal{A}_m(t) + \mathcal{A}_m(s)]
\]

then from 7 and 5 in Proposition 2.1 and the fact that \(e^{-A_n(t)}\) is an analytic \(C_t\)-semigroup on \(V\) we obtain

\[
\int_0^t \|e^{-(t-s)A_n(t)}[A_n(t) - A_n(s) - \mathcal{A}_m(t) + \mathcal{A}_m(s)]h(s)\|_V ds
\]

\[
\leq c \int_0^t \kappa_{n,m}(s)\|h(t-s)\|_V ds
\]

where \(\kappa_{n,m}\) is defined in the proof of Theorem 3.1. Therefore, using 18 we conclude

\[
\|u_{m+k}(t) - u_m(t)\|_V \leq c_{n,m} \left[\|u_0\|_V + \|f\|_{L^2(0,T;H)}\right]
\]

This complete the proof of the proposition. \(\square\)
5. Example: an affine approximation

The aim of this section is to provide an explicit approximation of a that satisfies the required hypothesis \((H_1) - (H_6)\). Recall that \(V, H\) denote two separable complex Hilbert spaces and \(a : [0, T] \times V \times V \to \mathbb{C}\) is a non-autonomous closed form satisfying (1). Assume moreover that there exists \(0 \leq \gamma < 1\) and a non-decreasing continuous function \(\omega : [0, T] \to [0, +\infty)\) with

\[
\sup_{t \in [0, T]} \frac{\omega(t)}{t^{\gamma/2}} < \infty,
\]

\[
\int_0^T \frac{\omega(t)}{t^{1+\gamma/2}} dt < \infty
\]

and

\[
|a(t, u, v) - a(s, u, v)| \leq \omega(|t - s|) \|u\|_V \|v\|_V, \quad t, s \in [0, T], u, v \in V.
\]

Remark 5.1. We note that, the main example of a continuity modulus \(\omega\) introduced above is the function \(\omega(t) = t^\eta\) where \(\eta > \gamma/2\). This main example moreover satisfies

\[
\lim_{t \to 0} \frac{\omega(t)}{t^{\gamma/2}} = 0.
\]

For general case, thanks to (27), one can always find a null sequence \((t_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+\) with \(\lim_{n \to +\infty} \omega(t_n) t_n^{-\gamma/2} = 0\). This is true because \(\liminf_{t \to 0} \omega(t) t^{-\gamma/2} = 0\), since otherwise we would have \(\int_0^T \frac{\omega(s)}{s^{1+\gamma/2}} ds = \infty\) which contradict (27).

Now let \(\Lambda = (0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n+1} = T)\) be a uniform subdivision of \([0, T]\), i.e.,

\[
|\Lambda| := \sup |\lambda_{k+1} - \lambda_k| = |\lambda_{k+1} - \lambda_k| \quad \text{for each } k = 0, 1, \ldots, n,
\]

and consider a family of sesquilinear forms \(a_k : V \times V \to \mathbb{C}\) given by

\[
a_k(u, v) := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} a(r; u, v) dr, \quad u, v \in V
\]

for each \(k = 0, 1, \ldots, n\). Remark that \(a_k\) satisfies (14) for all \(k = 0, 1, \ldots, n\). Then \(a_\Lambda : [0, T] \times V \times V \to \mathbb{C}\) defined for \(t \in [\lambda_k, \lambda_{k+1}]\) by

\[
a_\Lambda(t; u, v) := \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} a_k(u, v) + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} a_{k+1}(u, v), \quad u, v \in V,
\]

is a non-autonomous closed sesquilinear forms satisfying (14) with the same constants \(\alpha, \beta\) and \(M\). The associated time dependent operator is denoted by

\[
A_\Lambda(.) : [0, T] \to \mathcal{L}(V, V')
\]

and is given for \(t \in [\lambda_k, \lambda_{k+1}]\) by

\[
A_\Lambda(t) := \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} A_k + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} A_{k+1}
\]

where

\[
A_k u := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} A_m(r) u dr, \quad u \in V, \quad k = 0, 1, \ldots, n.
\]

In what follows we extend \(\omega\) to \([0, 2T]\) by setting \(\omega(t) = \omega(T)\) for \(T \leq t \leq 2T\).

Proposition 5.2. For all \(u, v \in V, \ t, s \in [0, T]\) we have

\[
|a_\Lambda(t, u, v) - a_\Lambda(s, u, v)| \leq \omega_\Lambda(|t - s|) \|u\|_V \|v\|_V,
\]

where \(\omega_\Lambda : [0, T] \to [0, +\infty]\) is defined by

\[
\omega_\Lambda(t) := \begin{cases} \frac{4\omega(4|\Lambda|)}{T} & \text{for } 0 \leq t \leq 2|\Lambda|, \\ 2\omega(2t) & \text{for } 2|\Lambda| < t \leq T. \end{cases}
\]
Moreover, \( A_\lambda(t) - A_\lambda(s) \in \mathcal{L}(V, V'_t) \),

\( \| A_\lambda(t) - A_\lambda(s) \|_{\mathcal{L}(V, V'_t)} \leq \omega_\lambda(|t - s|) \)

and

\( \| A_\lambda(t) - A(t) \|_{\mathcal{L}(V, V'_t)} \leq 2\omega(2|\Lambda|) \)

for each \( t, s \in [0, T] \).

**Proof.** Let \( u, v \in V \) and \( t, s \in [0, T] \). For the proof of (36) we distinguish three cases

**Case 1:** If \( \lambda_k \leq s < t \leq \lambda_{k+1} \) for some fixed \( k \in \{0, 1, \cdots, n\} \). Then we obtain, using (28) and the fact that \( \omega \) is non-decreasing, that

\[
|a_\lambda(t, u, v) - a_\lambda(s, u, v)| = \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} |a_k(u, v) - a_{k+1}(u, v)| + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} |a_{k+1}(u, v) - a_k(u, v)|
\]

\[
\leq \frac{(t-s)}{|\lambda|} \int_0^{\lambda} |a(r + \lambda_k, u, v) - a(r + \lambda_{k+1}, u, v)| \, dr
\]

\[
\leq \frac{(t-s)}{|\lambda|} \int_0^{\lambda} \omega(\lambda_{k+1} - \lambda_k) \|u\|_V \|v\|_{V'_t} \, dr = \frac{(t-s)}{|\lambda|} \omega(|\lambda|) \|u\|_V \|v\|_{V'_t}
\]

**Case 2:** If \( \lambda_k \leq s < \lambda_{k+1} \leq t \leq \lambda_{k+2} \), then we deduce from Step 1 that

\[
|a_\lambda(t, u, v) - a_\lambda(s, u, v)| \leq |a_\lambda(t, u, v) - a_\lambda(\lambda_k, u, v)| + |a_\lambda(\lambda_{k+1}, u, v) - a_\lambda(s, u, v)|
\]

\[
\leq \frac{t - \lambda_{k+1}}{|\lambda|} \omega(|\lambda|) \|u\|_V \|v\|_{V'_t} \leq \frac{t - \lambda_{k+1} - s}{|\lambda|} \omega(|\lambda|) \|u\|_V \|v\|_{V'_t}
\]

**Case 3:** If now \( \lambda_k \leq s < \lambda_{k+1} < \cdots < \lambda_l \leq t \leq \lambda_{l+1} \). Then \( \lambda_l - \lambda_{k+1} \leq t - s \leq \lambda_{l+1} - \lambda_k \) and thus

\( |t - s + \lambda_{k+1} - \lambda_{l+1}| \leq |\lambda| \)

It follows that

\[
a_\lambda(t, u, v) - a_\lambda(s, u, v)
\]

\[
= \frac{\lambda_{l+1} - t}{|\lambda|} |a_l(u, v) - a_{l+1}(u, v)| + \frac{t - \lambda_l}{|\lambda|} |a_{l+1}(u, v) - a_{l}(u, v)|
\]

\[
+ \frac{\lambda_{l+1} - \lambda_{k+1} + s - t}{|\lambda|} |a_{k+1}(u, v) - a_{k}(u, v)|
\]

Because of (36) and since \( \lambda_l - \lambda_l = \lambda_{l+1} - \lambda_{l+1} \), we deduce that

\[
|a_\lambda(t, u, v) - a_\lambda(s, u, v)| \leq \frac{\lambda_{l+1} - t}{|\lambda|} \omega(|\lambda_l - \lambda_k|) + \frac{t - \lambda_l}{|\lambda|} \omega(|\lambda_{l+1} - \lambda_{k+1}|)
\]

\[
+ \frac{|t - s + \lambda_{l+1} - \lambda_{k+1}|}{|\lambda|} \omega(|\lambda_{l+1} - \lambda_l|)
\]

\[
\leq \omega(|\lambda_l - \lambda_k|) + \omega(|\lambda_{l+1} - \lambda_l|)
\]

\[
\leq 2\omega(2(t-s)).
\]
This completes the proof of (33). Next, (34) follows from (33). For the second statement, let \( t \in [0, T] \) and let \( k \in \{0, 1, \cdots, n\} \) be such that \( t \in [\lambda_k, \lambda_{k+1}] \). Then
\[
A_\Lambda(t) - m(t) = \frac{\lambda_{k+1} - t}{\lambda_{k+1} - \lambda_k} [A_k - A_m(t)] + \frac{t - \lambda_k}{\lambda_{k+1} - \lambda_k} [A_{k+1} - A_m(t)]
\]

Then using (28) and the fact that (38) \( \dot{u} \) has the square root property if and only if \( \Lambda \) has it. This is essentially based on the abstract result due to Arendt and Monniaux [5, Proposition 2.5]. They proved that for two sesquilinear forms \( a_1, a_2 : V \times V \rightarrow \mathbb{C} \) associated with the operator \( B \) on \( H \) has the Kato square root property if
\[
D(B^{1/2}) = V.
\]
We prove in Proposition 5.3 below that \( a_\Lambda(t, \cdot, \cdot) \) has the square root property for all \( t \in [0, T] \) if \( a_\Lambda(0; \cdot, \cdot) \) has it. This is essentially based on the abstract result due to Arendt and Monniaux [5, Proposition 2.5]. They proved that for two sesquilinear forms \( a_1, a_2 : V \times V \rightarrow \mathbb{C} \) which satisfies (1), the form \( a_1 \) has the square root property if and only if \( a_2 \) has it provided that
\[
|a_1(u, v) - a_2(u, v)| \leq c\|u\|_V\|v\|_{V^*}, \ u, v \in V
\]
for some constant \( c > 0 \).

**Proposition 5.3.** Assume \( a(0, \cdot, \cdot) \) has the square root property. Then \( a_\Lambda(t, \cdot, \cdot) \) has the square root properties for all \( t \in [0, T] \), too.

**Proof.** Let \( t \in [0, T] \) and let \( k \in \{0, 1, \cdots, n\} \) be such that \( t \in [\lambda_k, \lambda_{k+1}] \). Then assumption (28) implies that
\[
|a_\Lambda(t, u, v) - a(0, u, v)| \leq \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} |a(r; u, v) - a(0, u, v)| dr
\]

\[
+ \frac{1}{\lambda_{k+2} - \lambda_{k+1}} \int_{\lambda_{k+1}}^{\lambda_{k+2}} |a(r; u, v) - a(0, u, v)| dr
\]

\[
\leq \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} \omega(r)\|u\|_V\|v\|_{V^*} dr + \frac{1}{\lambda_{k+2} - \lambda_{k+1}} \int_{\lambda_{k+1}}^{\lambda_{k+2}} \omega(r)\|u\|_V\|v\|_{V^*} dr
\]

\[
\leq 2 \sup_{r \in [0, T]} \omega(r)\|u\|_V\|v\|_{V^*}.
\]

Now the claim follows from [5, Proposition 2.5]. \( \square \)

Let \( A_\Lambda \) be given by (37) and consider the Cauchy problem
\[
(38) \quad \dot{u}_\Lambda(t) + A_\Lambda(t)u_\Lambda(t) = f(t) \quad \text{a.e. on} \ [0, T], \ u_\Lambda(0) = u_0.
\]

Next, we use the above results to prove that \( a_\Lambda \) satisfies all assumption \( (H_1)-(H_6) \) by taking \( d_n = \omega(2|\Lambda|) = \omega(2T/n) \) and \( \omega_n(\cdot) = \omega(\cdot) \).

**Proposition 5.4.** Assume that \( a \) satisfies (22)-(23) and that \( a(0, \cdot, \cdot) \) has the square properties. Then \( a_\Lambda \) satisfies assumptions \( (H_1)-(H_5) \), and satisfies also \( (H_6) \) if moreover (26) holds. Furthermore, the solution \( u_\Lambda \) of (38) belongs to \( C([0, T]; V) \) for each given \( u_0 \in V \) and \( f \in L^2(0, T; H) \).
Proof. According to Proposition 5.2, \( a_\Lambda \) satisfies (H1) and (H2). By the definition of \( \omega_\Lambda \) it follows
\[
\int_0^T \frac{\omega_\Lambda(t)}{t^{1+\gamma/2}} dt = \int_0^{2|\Lambda|} \frac{\omega(4|\Lambda|)}{|\Lambda|} t^{-\gamma/2} dt + \int_0^T \frac{\omega(t)}{t^{1+\gamma/2}} dt \\
\leq c \frac{\omega(4|\Lambda|)}{(4|\Lambda|)^{\gamma/2}} + c \int_0^{2T} \frac{\omega(t)}{t^{1+\gamma/2}} dt \\
\leq c \sup_{t \in [0,T]} \frac{\omega(t)}{t^{1/2}} + c \int_0^{2T} \frac{\omega(t)}{t^{1+\gamma/2}} dt < \infty
\]
which is finite by (26) and (27). Next, it is easy to prove that
\[
\sup_{t \in [0,T]} \frac{\omega_\Lambda(t)}{t^{1/2}} \leq c \sup_{t \in [0,T]} \frac{\omega(t)}{t^{1/2}} < \infty.
\]
holds. On the other hand, the function \( t \mapsto a_\Lambda(\cdot, u, v) \) is piecewise \( C^1 \) for all \( u, v \in V \) and \( a_\Lambda(t, \cdot, \cdot) \), \( t \in [0, T] \), has the Kato square property by Lemma 5.3. Then the Cauchy problem (35) has \( 2\)-maximal regularity in \( H \) and \( u_n \in C(0, T; V) \) for each \( u_0 \in V \) and \( f \in L^2(0, T; H) \) [8, Theorem 4.2]. Therefore, (H3) and (H4) are also satisfied by \( a \). Assume now that (28) holds. Then we obtain that \( \omega(2|\Lambda|)|\Lambda|^{-\gamma/2} \) and
\[
\int_0^{2|\Lambda|} \frac{\omega_\Lambda(t)}{t^{1+\gamma/2}} dt = \frac{\omega(4|\Lambda|)}{|\Lambda|} \int_0^{2|\Lambda|} \frac{dt}{t^{1+\gamma/2}} = 2 \frac{\omega(4|\Lambda|)}{(2|\Lambda|)^{1+\gamma/2}}
\]
converge to 0 as \( |\Lambda| \to 0 \). Thus the fact that \( \omega \) is non-decreasing complete the proof.

The next provides in particular an alternative proof of some results in [5].

\begin{corollary}
Assume that \( \lambda \) satisfies (26, 27) and that \( \lambda(0, \cdot, \cdot) \) has the square properties. Then (28) has \( L^2 \)-maximal regularity in \( H \) and for each \( u_0 \in V \) and \( f \in L^2(0, T; H) \) the solution \( (u_\Lambda)_\lambda \) converges weakly in \( MR_2(V, H) \) as \( |\Lambda| \to 0 \), and \( u := w - \lim_{|\Lambda| \to 0} u_\Lambda \) satisfies (3). If moreover, (29) holds then \( u_\Lambda \to u \) strongly in \( MR_2(V, H) \cap C(0, T; V) \) and uniformly on \( (u_0, f) \) as \( |\Lambda| \to 0 \). Further, the following estimates
\[
\|u_\Lambda - u\|_{MR_2(V, H)} \leq c \left( 1 + \frac{1}{|\Lambda|^{\gamma/2}} \right) d_n + \int_0^{2|\Lambda|} \frac{\omega(t)}{t^{1+\gamma/2}} dt \left( \|f\|_{L^2(0, T; H)} + \|u_0\|_V \right).
\]
and
\[
\|u_\Lambda - u\|_{C(0, T; V)} \leq c \left( 1 + \frac{1}{|\Lambda|^{\gamma/2}} \right) d_n + \int_0^{2|\Lambda|} \frac{\omega(t)}{t^{1+\gamma/2}} dt \left( \|f\|_{L^2(0, T; H)} + \|u_0\|_V \right).
\]
holds.
\end{corollary}

Proof. The proof follows from Theorem 3.1, Proposition 5.4 and Corollary 4.2.

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\begin{thebibliography}{99}
\item[1] P. Acquistapace, B. Terreni. A unified approach to abstract linear nonautonomous parabolic equations, Rend. Sem. Mat. Univ. Padova 78 (1987), 47-107.
\item[2] W. Arendt. Heat kernels. 9th Internet Seminar (ISEM) 2005/2006. Available at. https://www.uni-ulm.de/mawi/iaa/members/professors/arendt.html
\item[3] W. Arendt, D. Sier, S. Fackler. J. L. Lions’ Problem on Maximal Regularity. Arch. Math. (Basel) 109, no. 1, 59-72, 2017.
\item[4] W. Arendt, D. Sier, H. Laasri and E. M. Ouhabaz. Maximal regularity for evolution equations governed by non-autonomous forms, Adv. Differential Equations 19 (2014), no. 11-12, 1043-1066.
\item[5] W. Arendt, S. Monniaux. Maximal regularity for non-autonomous Robin boundary conditions. Math. Nachr. 2016. DOI: 10.1002/mana.201400319
\item[6] B. Augner, B. Jacob and H. Laasri On the right multiplicative perturbation of non-autonomous \( L^p \)-maximal regularity. J. Operator Theory 74 (2015), 391-415.
\item[7] C. Bardos. A regularity theorem for parabolic equations. J. Functional Analysis, 7 (1971), 311-322.
\item[8] H. Brézis. Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, Berlin 2011.
\item[9] R. Dautray and J.L. Lions. Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques. Vol. 8, Masson, Paris, 1988.
\end{thebibliography}
D. Dier. Non-Autonomous Maximal Regularity for Forms of Bounded Variation. J. Math. Anal. Appl. 425 (2015), no. 1, 33–54.

D. Dier. Non-autonomous evolutionary problems governed by forms: maximal regularity and invariance. PhD-Thesis, Ulm, 2014.

D. Dier, R. Zacher. Non-autonomous maximal regularity in Hilbert spaces. J. Evol. Equ. (2016), doi:10.1007/s00028-016-0343-5.

O. El-Mennaoui, H. Laasri. On evolution equations governed by non-autonomous forms. Archiv der Mathematik (2016), 1-15, DOI 10.1007/s00013-016-0903-5

S. Fackler J.-L. Lions’ problem concerning maximal regularity of equations governed by non-autonomous forms. Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (2017)

T. Kato. Perturbation theory for linear operators. Springer-Verlag, Berlin 1992.

B. Haak, E. M. Ouhabaz. Maximal regularity for non-autonomous evolution equations. Math. Ann. 363 (2015), no. 3-4, 1117–1145.

J.L. Lions. Equations Différentielles Opérationnelles et Problèmes aux Limites. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1961.

E. M. Ouhabaz. Maximal regularity for non-autonomous evolution equations governed by forms having less regularity. Arch. Math. 105 (2015), 79-91.

E. M. Ouhabaz and C. Spina. Maximal regularity for non-autonomous Schrödinger type equations. J. Differential Equation 248 (2010), 1668-1683.

R. E. Showalter. Monotone Operators in Banach Space and Nonlinear Partial Differential Equations. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997.

A. Sani, H. Laasri, Evolution Equations governed by Lipschitz Continuous Non-autonomous Forms. Czechoslovak Mathematical Journal. 65 (140) (2015), 475-491.

H. Tanabe. Equations of Evolution. Pitman 1979.

S. Thomaschewski. Form Methods for Autonomous and Non-Autonomous Cauchy Problems, PhD Thesis, Universität Ulm 2003.