THE IWASAWA ALGEBRA $\Omega_G$ AND ITS DUAL ARTIN COALGEBRA

ZHENG FANG AND FENG WEI

Abstract. For any compact $p$-adic Lie group $G$, the Iwasawa algebra $\Omega_G$ over finite field $\mathbb{F}_p$ is a complete noetherian semilocal algebra. It is shown that $\Omega_G$ is the dual algebra of an artinian coalgebra $C$. We induce a duality between the derived category $\mathcal{D}^{b}_{fg}(\Omega_G \mathcal{M})$ of bounded complexes of left $\Omega_G$-modules with finitely generated cohomology modules and the derived category $\mathcal{D}^{b}_{qf}(C \mathcal{M})$ of bounded complexes of left $C$-comodules with quasi-finite cohomology comodules.

L’algèbre d’Iwasawa $\Omega_G$ et Son Coalgèbre d’Artin Duale.

Résumé
Pour tout groupe de Lie $G$ $p$-adique compact, l’algèbre d’Iwasawa $\Omega_G$ sur un corps fini $\mathbb{F}_p$ est une algèbre noethérienne, semilocale et complète. Il est montré que $\Omega_G$ est l’algèbre duale d’une coalgèbre artinienne $C$. On induit une dualité entre la catégorie dérivée $\mathcal{D}^{b}_{fg}(\Omega_G \mathcal{M})$ des complexes bornés de $\Omega_G$-modules à gauche avec des modules cohomologie de type fini et la catégorie dérivée $\mathcal{D}^{b}_{qf}(C \mathcal{M})$ des complexes bornés de $C$-comodules à gauche dont leur comodules de cohomologie sont quasi-finis.

Version française abrégée
Dans [25], Yekutieli introduit le concept de complexe dualisant non commutatif. Avec les complexes dualisants, on peut trouver plusieurs propriétés assez remarquables sur les algèbres noethériennes semilocale et complète (voir [21], [22] and [23]). Pour tout groupe de Lie $G$ $p$-adique compact, l’algèbre d’Iwasawa $\Omega_G$ sur le corps fini $\mathbb{F}_p$ est semilocale noethérienne complète (voir [2], [11]). Dans [21], on a utilisé le complexe dualisant pour étudier les propriétés homologiques de l’algèbre d’Iwasawa $\Omega_G$. Il a été montré que la formule d’Auslander-Buchsbaum, le théorème de Bass et le théorème des « non trous » maintiennent pour l’algèbre d’Iwasawa $\Omega_G$. Une version duale correspondant de ces résultats a été obtenue par la théorie de dualité de Morita. De plus, il a été prouvé que l’algèbre d’Iwasawa $\Omega_G$ est auto-duale au sens de Morita. On renvoie le lecteur à [21] pour ces résultats et pour plus de détails.

D’autre part, il n’est pas difficile de montrer que le radical de Jacobson $J(\Omega_G)$ de l’algèbre d’Iwasawa $\Omega_G$ est cofini, c’est-à-dire, l’algèbre quotient $\Omega_G / J(\Omega_G)$ de $\Omega_G$ avec l’égard de ses Jacobson radical $J(\Omega_G)$ est de dimension finie. Heyneman et Radford ([10]) ont démontré qu’une algèbre noethérienne complète dont le radical de Jacobson est cofini est l’algèbre duale d’une coalgèbre artinienne. Cela nous motive à considérer quelques propriétés de dualité entre la catégorie de $\Omega_G$-modules et la
catégorie de $C$-modules. Dans cet article, on induit une dualité entre la catégorie dérivée $\mathcal{D}^\text{b}(\Omega_G, \mathcal{M})$ des complexes bornés de $\Omega_G$-modules à gauche à cohomologie de type fini, et la catégorie dérivée $\mathcal{D}^\text{b}(C, \mathcal{M})$ des complexes bornés de $C$-comodules à gauche dont leurs comodules de cohomologie sont quasi-finis.

0. Introduction

In the recent years, there has been an increasing interest in the noncommutative Iwasawa algebras of compact $p$-adic Lie groups. The main motivation to study the Iwasawa algebras $\Lambda_G$ and $\Omega_G$ of a compact $p$-adic Lie group $G$ is due to their connections with number theory and arithmetic algebraic geometry (see [6], [7], [8], [12], [18], [19], [20]). Throughout, let $p$ be a fixed prime integer, $\mathbb{Z}_p$ be the ring of $p$-adic integers and $\mathbb{F}_p$ be the field of $p$ elements. Given a compact $p$-adic Lie group, the Iwasawa algebra of $G$ over $\mathbb{Z}_p$ is the completed group algebra

$$\Lambda_G := \lim_{\leftarrow} \mathbb{Z}_p[G/N],$$

where the inverse limit is taken over the open normal subgroups $N$ of $G$. Closely related to $\Lambda_G$ is the Iwasawa algebra of $G$ over $\mathbb{F}_p$, which is defined as

$$\Omega_G := \lim_{\leftarrow} \mathbb{F}_p[G/N].$$

It is well-known that $\Lambda_G$ and $\Omega_G$ are complete noetherian semilocal algebras and are in general noncommutative. These algebras associated with certain topological setting were defined and studied by Lazard in his seminal paper [11] at first. We would like to point out that the Iwasawa algebras $\Lambda_G$ and $\Omega_G$, despite its great interest in number theory and arithmetic, seems to have been neglected as two concrete examples for the application of noncommutative methods. Some papers are specially contributed to the ring-theoretic (or module-theoretic) and homological properties of Iwasawa algebras (see [2], [3], [4], [7], [16], [17], [19], [21]). We refer to [2] and [11] for the basic properties of $\Lambda_G$ and $\Omega_G$.

Let $G$ be a compact $p$-adic analytic group and $N$ be a closed normal subgroup of $G$. Passman in [13] showed that if $R = S * G$ is a crossed product of a ring $S$ with a finite group $G$, then $J(S) * G$ is contained in $J(R)$. Thus the augmentation ideal $w_{N,G} = \ker(\Omega_G \to \mathbb{F}_p[G/N]) = J(\Omega_N) * G$ is contained in $J(\Omega_G)$. On the other hand, $\Omega_G/w_{N,G} \cong \mathbb{F}_p[G/N]$ is finite dimensional. Hence $\Omega_G/J(\Omega_G)$ is also finite dimensional, being a quotient of $\Omega_G/w_{N,G}$. In this situation, we say that $\Omega_G$ has cofinite Jacobson radical. Heyneman and Radford [10] observed that any noetherian complete algebra $A$ with cofinite Jacobson radical is the dual algebra of an artinian coalgebra $C$. Therefore we conclude that for any compact $p$-adic analytic group $G$, the Iwasawa algebra $\Omega_G$ is the dual algebra of an artinian coalgebra $C$ (see Proposition 1.1 and Corollary 1.2).

The main purpose of this article is to describe certain duality properties between the category of $\Omega_G$-modules and the category of $C$-comodules.

1. The Main Results

Let us first recall some basic definitions and existing facts. A subspace $W$ of a vector space $V$ over a field $K$ is called cofinite in $V$, or simply cofinite, when the quotient space $V/W$ is finite dimensional.

Let $A$ be an algebra over a field $K$ and $M$ be an $A$-module. $M$ is said to be almost noetherian if every cofinite submodule of $M$ is finitely generated. $A$ is called left
almost noetherian if it is almost noetherian as a left $A$-module, i.e., every cofinite left ideal of $A$ is finitely generated. $A$ is an almost noetherian algebra if $A$ is both left and right almost noetherian. We refer the reader to the reference [10] for basic properties of almost noetherian modules and those of almost algebras.

Some topological preliminaries are necessary for our later work. Suppose that $V$ is a vector space over a field $K$. We will regard the dual space $V^*$ of all linear functionals on $V$ as a topological vector space with the weak*-topology; that is, $V^*$ is given the least fine topology such that the vectors of $V$ induce continuous functionals (where we regard $V$ as embedded in $V^{**}$ and give the discrete topology to the scalars $k$). The closed (respectively, open) linear subspaces of $V^*$ are then the annihilators $W^\perp$ of arbitrary (respectively, finite-dimensional) subspaces $W$ of $V$.

Let $A$ be an algebra over a field $K$. If $I$ is a subspace of $A$, then
\[ I^\perp = \{ f \in A^* | f(I) = 0 \}. \]
Recall [15, Section 6] that
\[ A^0 = \{ I^\perp | I \text{ runs over the cofinite ideals of } A \} \]
inherits a natural coalgebra structure from the transpose of the algebra structure map $A \otimes A \to A$. Now let $C$ be a coalgebra. If $M$ is a finite dimensional $C^*$-module and $\pi : C^* \to \text{End } M$ is the corresponding algebra map, then $I_M = \text{Ker } \pi$ is a cofinite ideal of $C^*$. By [10, Lemma 1.2 (e)] it follows that $(\pi^*) = I^*_M$, the annihilator of $I_M$ in $(C^*)^*$, so that the image $(\pi^*)$ is contained in $(C^*)^0$.

A module $M$ is said to be rational if the image $(\pi^*)$ is contained in $C$ (of course $C$ is a subspace of $A^0$). Since we have seen in [10, Lemma 1.2 (b)] that a cofinite subspace $I$ of $A$ is closed when $I^\perp$, the annihilator of $I$ in $A^*$, is actually contained in $C$, it follows that $M$ is a rational $A$-module when $I_M = \text{Ker } \pi$ is closed in $A$.

Let $C_0$ be the sum of the simple subcoalgebras of a coalgebra $C$. We inductively define an increasing coalgebra filtration by
\[ C_n = C_0 \wedge C_{n-1} = \{ x \in C | \Delta x \in C_0 \otimes C + C \otimes C_{n-1} \}. \]
$C$ is said to be finite type when $C_1 = C_0 \wedge C_0$ is finite dimensional.

**Proposition 1.1.** [10, Proposition 4.3.1] Let $A$ be a left almost noetherian algebra with cofinite Jacobson radical. Then $C = A^0$ is a coalgebra of finite type, and the algebra map $A \to C^*$ can be identified with the map $A \to \hat{A}$ (completion with respect to the $J$-adic topology). In particular $C^* = \hat{A}$ is again an almost noetherian algebra with cofinite Jacobson radical.

This proposition means that any complete noetherian algebra $A$ with cofinite Jacobson radical is the dual algebra of an artinian coalgebra $C$. Moreover, the artinian coalgebra $C$ is of finite type. We will now apply the above results to the Iwasawa algebra $\Omega_G$, which is our work context.

**Corollary 1.2.** For any compact $p$-adic analytic group $G$, the Iwasawa algebra $\Omega_G$ is the dual algebra of an artinian coalgebra $C$.

Let $C$ be the dual artinian coalgebra of the Iwasawa algebra $\Omega_G$. $\Omega_G M$ denotes the category of left $\Omega_G$-modules and $\text{Rat}(\Omega_G)$ denotes the subcategory of $\text{Mod}(\Omega_G)$ consisting of all rational left $\Omega_G$-modules. It is well known that the abelian category
Proposition 1.3. The following statements are equivalent:

(a) $\Omega_{G} C$ is injective in $\Omega_{G} \mathcal{M}$;
(b) $\Omega_{G} C$ is an injective cogenerator of $\Omega_{G} \mathcal{M}$;
(c) $\Omega_{G} C$ is artinian;
(d) $\Omega_{G}$ is right noetherian;
(e) The injective hull of a rational left $\Omega_{G}$-module is rational.

Throughout this section $C$ is always a left and right artinian coalgebra. Let $A$ and $B$ be two algebras with its opposite algebras $A^e$ and $B^o$. Let $A E B$ be an $(A, B)$-bimodule. We say that $E$ induces a Morita duality between $A$ and $B$ if

1. $A E$ and $E B$ are injective cogenerators in the categories of left $A$-modules and right $B$-modules, respectively;
2. the canonical ring homomorphisms $A \to \text{End} E B$ and $B^e \to \text{End} A E$ are isomorphisms.

In this case we say that $A$ is left Morita and $B$ is right Morita, and that $A$ is Morita dual to $B$ (or $A$ and $B$ are in Morita duality). If $A = B$, then $A$ is Morita self-dual, or has a Morita self-duality. We refer to [1] and [24] for some basic properties of a Morita duality.

For the dual coalgebra $C$ of the Iwasawa algebra $\Omega_{G}$, let us consider the $\Omega_{G}$-bimodule $\Omega_{G} C_{\Omega_{G}}$. We easily observe that $\text{End}(\Omega_{G} C) \cong \Omega_{G} C$ and $\text{End}(C_{\Omega_{G}}) \cong \Omega_{G}$. Moreover, if $C$ is both left and right artinian, then $\Omega_{G} C$ and $C_{\Omega_{G}}$ are both injective cogenerators by Proposition 1.3. It follows from [1, Theorem 24.1] that the $\Omega_{G}$-bimodule $\Omega_{G} C_{\Omega_{G}}$ defines a Morita duality. Let $\Omega_{G} \mathcal{R}$ (resp. $\mathcal{R}_{\Omega_{G}}$) be the subcategory of $\Omega_{G} \mathcal{M}$ (resp. $\mathcal{M}_{\Omega_{G}}$) consisting of $\Omega_{G} C_{\Omega_{G}}$-reflexive modules. Thus we obtain a duality

$$
\begin{array}{c}
\Omega_{G} \mathcal{R} \\
\text{Hom}_{\Omega_{G}}(-, C) \\
\Omega_{G} \mathcal{R} \\
\text{Hom}_{\Omega_{G}}(-, C)
\end{array}
\xrightarrow{\cong}
\begin{array}{c}
\mathcal{R}_{\Omega_{G}} \\
\text{Hom}_{\Omega_{G}}(-, C) \\
\mathcal{R}_{\Omega_{G}} \\
\text{Hom}_{\Omega_{G}}(-, C)
\end{array}
\tag{\phantom{\text{Proposition 1.3}}}
$$

Let $\Omega_{G} \mathcal{M}_{fg}$ be the category of all finitely generated left $\Omega_{G}$-modules and $C. \mathcal{M}_{af}$ be the category of all quasi-finite left $C$-comodules. Since the Iwasawa algebra $\Omega_{G}$ is noetherian and semiperfect, we get $\text{gl. dim } C = \text{gl. dim } \Omega_{G}$ by [3, Proposition 3.4]. Applying this fact and the duality (\phantom{\text{Proposition 1.3}}) yields

Proposition 1.4. Let $C$ be the dual artinian coalgebra of $\Omega_{G}$. Then there exists a duality between abelian categories

$$
\begin{array}{c}
\Omega_{G} \mathcal{M}_{fg} \\
\text{Hom}_{\Omega_{G}}(-, C) \\
\Omega_{G} \mathcal{M}_{fg} \\
\text{Hom}_{\Omega_{G}}(-, C)
\end{array}
\xrightarrow{\cong}
\begin{array}{c}
C. \mathcal{M}_{af} \\
\text{Hom}_{\Omega_{G}}(-, C) \\
C. \mathcal{M}_{af} \\
\text{Hom}_{\Omega_{G}}(-, C)
\end{array}
$$

For any left $\Omega_{G}$-module $\Omega_{G} M$, there is a right $\Omega_{G}$-module morphism:

$$
\eta_{M} : \text{Hom}_{\Omega_{G}}(M, C) \to M^* \\
f \mapsto \varepsilon \circ f.
$$
Lemma 1.5. The above transformation $\eta$ is a natural isomorphism.

If $\Omega_G M$ is a finitely generated free module, then $\eta_M : \text{Hom}_{\Omega_G}(M, C) \rightarrow \text{Rat}(M)^*$ is obviously an isomorphism. For a general finitely generated module $M$, $M$ is finitely presented since $\Omega_G$ is noetherian

$$\bigoplus_{\text{finite}} \bigoplus_{\text{finite}} \rightarrow M \rightarrow 0.$$ 

The statement follows from the following commutative diagram:

$$
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}_{\Omega_G}(M, C) \\
\downarrow \eta_M & & \downarrow \eta \\
0 & \rightarrow & \text{Rat}(M)^* \\
\end{array}
$$

Let $A$ be a noetherian algebra with Jacobson radical $J$ such that $A_0 = A/J$ is finite dimensional. Let $A M$ be an $A$-module. An element $m \in M$ is called a torsion element if $J^n m = 0$ for all $n > 0$. Let us set $\Gamma(M) = \{ m \in M \mid m \text{ is a torsion element} \}$. Then $\Gamma(M)$ is a submodule of $M$. In fact, we obtain an additive functor

$$\Gamma : A \mathcal{M} \rightarrow A \mathcal{M},$$

by sending an $A$-module to its maximal torsion submodule. Clearly, $\Gamma$ is a left exact functor. The functor $\Gamma$ has another representation $\Gamma(M) = \lim Hom_A(A/J^n, M)$. We use $\Gamma^o$ to denote the torsion functor on the category of right $A$-modules.

Now let $C$ be the dual coalgebra of Iwasawa algebra $\Omega_G$, and let

$$\text{Rat} : \Omega_G \mathcal{M} \rightarrow \Omega_G \mathcal{M}$$

be the rational functor. Since $C$ is artinian, every finite dimensional $\Omega_G$-module is rational by [10, Proposition 3.1.1 and Remarks 3.1.2]. Therefore, for a left $\Omega_G$-module, it follows that $\text{Rat}(M)$ is the sum of all the finite dimensional submodules of $M$. It should be remarked that the Jacobson radical $J = C^+_0$ and $C_0$ is finite dimensional. Thus $\Gamma(M)$ is also the sum of all the finite dimensional submodules of $M$. Hence $\Gamma(M) \cong \text{Rat}(M)$. This gives that the functor $\Gamma$ is naturally isomorphic to the rational functor $\text{Rat}$. In what follow, we identify the right derived functor $R\Gamma$ with $R\text{Rat}$.

Let $C$ be the dual coalgebra of the Iwasawa algebra $\Omega_G$. Then $\text{Soc}(C)$ is finite dimensional. This implies that there are only finitely many non-isomorphic simple right (or left) $C$-comodules. If $C M$ is quasi-finite, then $\text{Soc}(M)$ is finite dimensional. Thus $C M$ is finitely cogenerated. This means that $C \mathcal{M}$ is a thick subcategory of $\mathcal{M}$. So $\mathcal{D}_G^+(C \mathcal{M})$, the derived category of bounded below complexes of left $C$-comodule with quasi-finite cohomology comodules, is a full triangulated subcategory of $\mathcal{D}_G^+(C \mathcal{M})$. Also, since $\Omega_G$ is noetherian, $\mathcal{D}_G^{-}(\Omega_G \mathcal{M})$, the derived
The category of bounded above complexes of left $\Omega_G$-modules with finitely generated cohomology modules, is a full triangulated subcategory of $\mathcal{D}^-(\Omega_G\mathcal{M})$. The duality in Proposition 1.4 gives rise to a duality of derived categories.

**Theorem 1.6.** Let $C$ be the artinian coalgebra of the Iwasawa algebra $\Omega_G$. We have dualities of triangulated categories:

$$
\begin{align*}
\mathcal{D}^-_{fg}(\Omega_G\mathcal{M}) & \xrightarrow{R\Gamma \circ (\cdot)^*} \mathcal{D}^+_{qf}(C\mathcal{M}), & \mathcal{D}^b_{fg}(\Omega_G\mathcal{M}) & \xrightarrow{R\Gamma \circ (\cdot)^*} \mathcal{D}^b_{qf}(C\mathcal{M}).
\end{align*}
$$

By the fact that $C$ is artinian, we know that $\mathcal{D}^+_{qf}(C\mathcal{M})$ is equivalent to the derived category $\mathcal{D}^+(C\mathcal{M}_{qf})$ of complexes of quasi-finite comodules. Since $\Omega_G$ is noetherian, $\mathcal{D}^-_{fg}(\Omega_G\mathcal{M})$ is equivalent to $\mathcal{D}^-((\Omega_G\mathcal{M})_{fg})$. In view of Proposition 1.4 we obtain the following duality

$$
\mathcal{D}^-((\Omega_G\mathcal{M})_{fg}) \xrightarrow{\text{Hom}_{\Omega_G}(\cdot, C)^*} \mathcal{D}^+(C\mathcal{M}_{qf}).
$$

Applying Lemma 1.5 we easily verify that the composition

$$
\mathcal{D}^-_{fg}(\Omega_G\mathcal{M}) \xrightarrow{\cong} \mathcal{D}^-((\Omega_G\mathcal{M})_{fg}) \xrightarrow{\text{Hom}_{\Omega_G}(\cdot, C)^*} \mathcal{D}^+(C\mathcal{M}_{qf}) \xrightarrow{\cong} \mathcal{D}^+_{qf}(C\mathcal{M})
$$

is isomorphic to the functor $R\Gamma \circ (\cdot)^*$. Furthermore, one can observe that $R\Gamma \circ (\cdot)^*$ sends bounded complexes to bounded complexes.

**Corollary 1.7.** Let $C$ be the dual artinian coalgebra of the Iwasawa algebra $\Omega_G$. Then we have a duality of triangulated categories:

$$
\begin{align*}
\mathcal{D}^-_{fd}(\Omega_G\mathcal{M}) & \xrightarrow{R\Gamma \circ (\cdot)^*} \mathcal{D}^+_{fd}(C\mathcal{M}).
\end{align*}
$$

It is enough to show that $R\Gamma(\cdot)^* \in \mathcal{D}^b_{fd}(C\mathcal{M})$ for any finite dimensional left $\Omega_G$-module $M$. Since $\Omega_G$ is noetherian and complete with respect to the $J$-adic filtration, the Jacobson radical $J$ of $\Omega_G$ satisfies Artin-Rees condition. Applying [9, Theorem 3.2] yields that the injective envelop of a $J$-torsion module is still $J$-torsion. Now $M^*$ is a $J$-torsion module. Thus we have an injective resolution of $M^*$ with each component being $J$-torsion. Therefore $R\Gamma(M^*)$ is quasi-isomorphic to $M^*$, that is, $R\Gamma(M^*) \in \mathcal{D}^b_{fd}(C\mathcal{M})$.

### 2. Conclusion

It is well-known that there is an informal idea in the theory of derived categories of coalgebras: ‘the derived category associated to a coalgebra keeps the homological relevant information, so if two coalgebras have equivalent derived categories, they should have isomorphic cohomology’. In view of the beautiful structure and properties of the Iwasawa algebra $\Omega_G$, $C$ as the dual artinian coalgebra of $\Omega_G$ should have elegant properties which build stable basis for the further study of derived categories of bounded complexes of $C$-comodules. Let $\mathcal{D}^b_{qf}(C\mathcal{M})$ (respectively, $\mathcal{D}^b_{qf}(\mathcal{M}^C)$) be the derived category of bounded complexes of left (respectively, right) $C$-comodules with quasi-finite cohomology modules. We wonder whether it is possible for us
to establish a duality of the derived categories of left $C$-comodules and of right $C$-comodules by the results obtained in Section 1. More precisely speaking, we have the following question:

**Question 2.1.** Let $C$ be the dual artinian coalgebra of the Iwasawa algebra $\Omega_G$. Are there dualities of triangulated categories

$$D^b_C(M) \xrightarrow{F} D^b_C(M^C) \xleftarrow{G}$$

Here $F = R\Gamma \circ (\cdot)^*$ and $G = R\Gamma^c \circ (\cdot)^*$.

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**References**

[1] A. F. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Second Edition, Graduate Texts in Mathematics, 13 Springer-Verlag, New York, 1992.

[2] K. Ardakov and K. A. Brown, *Ring-theoretic properties of Iwasawa algebras: a survey*, Documenta Math., **Extra Volume Cotaes** (2006), 7-33.

[3] K. Ardakov, F. Wei and J. J. Zhang, *Reflexive ideals in Iwasawa algebras*, Adv. Math., 218 (2008), 865-901.

[4] K. Ardakov, F. Wei and J. J. Zhang, *Nonexistence of reflexive ideals in Iwasawa algebras of Chevalley type*, J. Algebra, 320 (2008), 259-275.

[5] T. Brzezinski and R. Wisbauer, *Corings and Comodules*, London Math. Soc. Lecture Note Series 309, Cambridge University Press, 2003.

[6] J. Coates, *Iwasawa algebras and arithmetic*, Séminaire Bourbaki. Vol. 2001/2002. Astérisque, 290 (2003), Exp. No. 896, , 37–52.

[7] J. Coates, P. Schneider and R. Sujatha, *Modules over Iwasawa algebras*, J. Inst. Math. Jussieu 2 (2003), 73–108.

[8] J. Coates, T. Fukaya, K. Kato, R. Sujatha and O. Venjakob, *The GL$_2$ main conjecture for elliptic curves without complex multiplication*, Publ. Math. IHES 101 (2005), 163-208.

[9] J. Cuadra, C. Năstăsescu and F. Van Oystaeyen, *Graded almost noetherian rings and applications to coalgebras*, J. Algebra, 256 (2002), 97-110.

[10] R. G. Heyneman and D. E. Radford, *Reflexivity and coalgebras of finite type*, J. Algebra, 28 (1974), 215-246.

[11] M. Lazard, *Groupes analytiques p-adiques*, Publ. Math. IHES., 26 (1965), 389-603.

[12] Y. Ochi and O. Venjakob, *On the structure of Selmer groups over p-adic Lie extensions*, J. Algebraic Geom. 11 (2002), 547–580.

[13] D. S. Passman, *Infinite Crossed Products*, Pure and Applied Mathematics, Vol. 135, Academic Press, San Diego, 1989.

[14] B. Stenström, *Rings of Quotients*, Springer-Verlag, New York, 1975.

[15] M. Sweedler, *Hopf Algebras*, Mathematics Lecture Note Series, Benjamin, New York, 1969.

[16] O. Venjakob, *Iwasawa theory of p-adic Lie extensions*, PhD. thesis, University of Heidelberg, 2000.

[17] O. Venjakob, *On the structure theory of the Iwasawa algebra of a p-adic Lie group*, J. Eur. Math. Soc.(JEMS), 4 (2002), 271-311.
[18] O. Venjakob, A noncommutative Weierstrass preparation theorem and applications to Iwasawa theory, J. Reine. Angew. Math., 559 (2003), 153-191.

[19] O. Venjakob, On the Iwasawa theory of p-adic Lie extensions, Compositio Math., 138 (2003), 1-54.

[20] O. Venjakob, Characteristic elements in noncommutative Iwasawa theory, J. Reine. Angew. Math., 583 (2005), 193-236.

[21] F. Wei, Homological properties of noncommutative Iwasawa algebras, C. R. Math. Acad. Sci. Paris, 349 (2011), 15-20.

[22] Q. -S. Wu and J. J. Zhang, Dualizing complexes over noncommutative local algebras, J. Algebra, 239 (2001), 513-548.

[23] Q. -S. Wu and J. J. Zhang, Homological identities for noncommutative rings, J. Algebra, 242 (2001), 516-535.

[24] W. -M. Xue, Rings with Morita Duality, Lecture Notes in Mathematics, 1523, Springer-Verlag, Berlin, 1992.

[25] A. Yekutieli, Dualizing complexes over noncommutative graded algebras, J. Algebra 153 (1992), 41-84.

Fang: School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, 100081, P. R. China
E-mail address: 514139626@qq.com

Wei: School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, 100081, P. R. China
E-mail address: daoshuo@hotmail.com
E-mail address: daoshuoweii@gmail.com