ON THE SPECTRUM OF DENSE RANDOM GEOMETRIC GRAPHS

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Abstract. In this paper we study the spectrum of the random geometric graph $G(n, r)$, in a regime where the graph is dense and highly connected. In the Erdős-Rényi $G(n, p)$ random graph it is well known that upon connectivity the spectrum of the normalized graph Laplacian is concentrated around 1. We show that such concentration does not occur in the $G(n, r)$ case, even when the graph is dense and almost a complete graph. In particular, we show that the limiting spectral gap is strictly smaller than 1. In the special case where the vertices are distributed uniformly in the unit cube and $r = 1$, we show that for every $0 \leq k \leq d$ there are at least $\binom{d}{k}$ eigenvalues near $1 - 2^{-k}$, and the limiting spectral gap is exactly $1/2$. We also show that the corresponding eigenfunctions in this case are tightly related to the geometric configuration of the points.

Keywords: Random geometric graphs, spectral measure, homological connectivity.

1. Introduction

Let $G$ be an undirected graph on the vertex set $[n] := \{1, 2, \ldots, n\}$, and let $A$ be its adjacency matrix. The degree of the vertex $i$ is then $d_i = \sum_{j \neq i} A_{i,j}$, and the graph Laplacian is defined as $L := D - A$, where $D$ is the diagonal matrix with $d_1, \ldots, d_n$ on the diagonal. The symmetrically normalized graph Laplacian is defined as

$$\mathcal{L} := D^{-\frac{1}{2}} L D^{-\frac{1}{2}} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}.$$  

We are interested in the eigenvalues of $\mathcal{L}$ denoted $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$.

It is well known, cf. [3, 25], that $\gamma_i \in [0, 2]$ for all $i$. In addition, $\gamma_1 = 0$, and the graph is connected if, and only if, $\gamma_2 > 0$. The value of $\gamma_2$ is typically referred to as the spectral gap of the graph.

Graph Laplacians and their spectra contain important information about the connectivity structure of graphs and the behavior of random walks on them, see for example [1, 3, 8, 17, 31]. Graph spectra and harmonics also play key roles in various applications such as network analysis and machine learning [6, 27, 32].

In this paper we study the spectrum of a random geometric graph. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a probability density function on $\mathbb{R}^d$, and let $X_n = \{X_1, \ldots, X_n\} \overset{i.i.d.}{\sim} f$. Let $\| \cdot \|$ be a norm on $\mathbb{R}^d$.

The random geometric graph $G(n, r)$ is defined as the undirected graph with vertex set $[n]$, where $i$ is connected to $j$, abbreviated $i \sim j$, if, and only if, $\|X_i - X_j\| \leq r$. That is, the entries of the adjacency matrix are of the form $A_{i,j} = h_r(X_i, X_j)$, where

$$h_r(x, y) := 1_{\|x-y\| \leq r}.$$  

Suppose that $f$ is uniform on compact $S \subset \mathbb{R}^d$. In that case, it can be shown [25] that there exists a constant $C_S$ such that, if $r = r(n) \geq C_S \left( \log n / n \right)^{1/d}$, then, as $n \to \infty$, with high probability, $G(n, r)$ is connected. Since the graph is connected we know that $\gamma_2 > 0$. However, even if $r$ is much larger than $(\log n / n)^{1/d}$, as long as $r = r(n) \to 0$, using Cheeger’s inequality it can be shown [29] that a.s. $\gamma_2 \to 0$. This, in particular, implies that such graphs are not expanders, cf. [6, 17]. From at least one aspect, this behavior is somewhat counter-intuitive, as $r$ can be chosen large enough so that the graph is $k$-connected with $k = k(n) \to \infty$ [26].

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Date: April 13, 2020.

(1) KA was supported in part by a Zeff Fellowship, a Viterbi Fellowship and the Israel Science Foundation, Grants 2539/17 and 771/17.

(2) RJA was supported in part by the Israel Science Foundation, Grant 2539/17.

(3) OB was supported in part by the Israel Science Foundation, Grant 1965/19.

(4) RR was supported in part by the Israel Science Foundation, Grant 771/17 and the US-Israel Binational Science Foundation, Grant 2018330.
This behavior is very different to that occurring in some other models of random graphs. In particular, for the the Erdős-Rényi random graph $G(n, p)$, it was shown that above the connectivity threshold ($p = \log n / n$) the entire spectrum of the graph (except for the $\gamma_1$) is concentrated around $1$, and, in particular that $\gamma_2 \to 1$.

In this paper we want to study a regime where the spectral gap of $G(n, r)$ is bounded away from zero. Thus, we have to take $r$ to be uniformly bounded away from zero, and, in particular, will take $r$ to be constant, independent of $n$. We take $S$ to be the cube $[-1, 1]^d$, equipped with the $L^\infty$ norm,

$$||x|| := ||x||_{\infty} = \max_{1 \leq k \leq d} |x_k|, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$  

Note that $[-1, 1]^d$ is in fact the unit ball in the $L^\infty$ norm, and so we will denote it by $B^d$.

The motivation behind these seemingly arbitrary choices ($L^\infty$ and $B^d$) is twofold. The first is that of mathematical tractability, since these specific choices allow us to compute concrete estimates for the limiting spectrum of $G(n, r)$ which, for geometric reasons, would be much harder to compute with, for example, an Euclidean ball equipped with the $L^2$ metric. The second reason for this choice arises from one of the key motivations for this work as a whole, which we now describe.

**Homological connectivity in random Vietoris-Rips complexes.** A *simplicial complex* is a generalization of a graph, where in addition to vertices and edges, it is possible to include triangles, tetrahedra, and higher dimensional simplexes (finite subsets of vertices). Given a graph $G$, its corresponding flag (or clique) complex is constructed by adding a $k$-dimensional simplex (subsets of vertices of size $k + 1$) for every $(k + 1)$-clique in the graph. When $G = G(n, r)$ this complex is known as the random Vietoris-Rips (VR) complex. In [19], the homology groups (algebraic-topological structures describing cycles in various dimensions) of random VR-complexes were studied.

One of the main open questions in this area is about the homological connectivity of these complexes. In particular, one is interested in ‘phase transitions’ (as $n$ and $r_n$ change) leading to geometric complexes for which one of these homology groups suddenly becomes trivial (in a suitable sense). This is a higher-dimensional analogue of the traditional graph-connectivity property. This phenomenon was studied recently [14] for a different type of a geometric complex, known as the random Čech complex. The proof there, however, heavily relies on Morse theory, which is not applicable to the VR case. Therefore, a different approach is required.

In [20] Kahle studied homological connectivity in random flag complexes generated by the Erdős-Rényi graph $G(n, p)$. Kahle’s proof uses the so-called ‘Garland’s method’ [14], that translates questions about the homology of a simplicial complex into questions about the graph Laplacian of its links. Combining Garland’s method with concentration results for the spectrum of $G(n, p)$ [16], leads to the proof of a phase-transition for homological connectivity. Garland’s method was also used in the study of homological connectivity of other models of random simplicial complexes such as the high-dimensional Erdős-Rényi model (the Linial-Mehulam model), cf. [15] [16].

In the random VR complex, using scaling invariance, the relevant links can be shown to form random geometric graphs in the intersection of a finite number of unit balls. Consequently, we believe that the analysis we provide here for $S = B^d$ could be used to prove homological connectivity for the VR complex (in the $L^\infty$ norm). This remains for future work.

### 2. Main results

Throughout the paper, $d \geq 1$ will denote dimension. Let $X = \{X_1, X_2, \ldots, X_n\}$ be i.i.d. uniformly distributed random variables in $[-1, 1]^d$, and let $G(n, r)$ be the random geometric graph generated by $X$, using the $L^\infty$ norm [3], as described above. We will focus on $G_{n,r} := G(n, r)$, for a fixed $r > 0$ independent of $n$. Define

$$h_r(x, y) = 1_{||x - y|| \leq r},$$

let $A_{n,r}$ be the adjacency matrix of $G_{n,r}$, i.e. $(A_{n,r})_{ij} = h_r(X_i, X_j)$, and let $L_{n,r}$ be the corresponding symmetrically normalized graph Laplacian [1]. Finally, let

$$W_{n,r} = I_n - L_{n,r} = D_{n,r}^{-\frac{1}{2}} A_{n,r} D_{n,r}^{-\frac{1}{2}}.$$


where $I_n$ denotes the $n \times n$ identity matrix, and $D_{n,r}$ is the diagonal matrix of vertex degrees (i.e. $D_{n,r} = \text{diag}(d_1, \ldots, d_n)$ where $d_i = \sum_j (A_{n,r})_{i,j}$).

Let $\lambda^{(n)}_1 \geq \lambda^{(n)}_2 \geq \cdots \geq \lambda^{(n)}_n$ be the (ordered) eigenvalues of $W_{n,r}$, and define the empirical eigenvalue measure

$$\mu_n(I) = \sum_{i=1}^n \delta_{\lambda^{(n)}_i}(I),$$

where $\delta_x$ denotes the Dirac delta measure on $\mathbb{R}$. Observe that, for all $r > 0$, $\lambda^{(n)}_1 = 1$, with corresponding eigenvector $(\sqrt{d_1}, \ldots, \sqrt{d_n})$.

**Remark 2.1.** When $2 \leq r < \infty$ $G(n, r)$ is fully connected, and so all the entries of $A_{n,r}$ are equal to 1. In this case, $\lambda^{(n)}_1 = 0$ for all $i > 1$, so that $\mu_n = (n-1)\delta_0 + \delta_1$, and there is nothing interesting to study. Consequently, we will always assume that $0 < r < 2$.

With basic notation out of the way, we can now summarise our main results, which provide detailed information about the structure of the spectrum of $W_{n,r}$ (and consequently $\mathcal{L}_{n,r}$), as well as some its harmonics, for large $n$.

When $r = 1$, we show that in the limit (as $n \to \infty$) the spectrum of $W_{n,r}$ contains the values $1/2^k$ with multiplicity $\binom{n}{k}$ for all $0 \leq k \leq d$. In addition, we show that the remaining $(n - 2^d)$ eigenvalues are concentrated in the interval $(-0.3, 0.3)$.

When $r \in (1, 2)$, we show that the entire spectrum (except for $\lambda^{(n)}_1$) is contained in $(-1/2, 1/2)$.

Finally, when $r \in (0, 1)$, we show that the limit of $\lambda^{(n)}_2$ is larger than 1/2.

One consequence of these results is that the spectral gap of $\mathcal{L}_{n,r}$ either converges to 1/2 ($r = 1$), is strictly larger than 1/2 ($r \in (1, 2)$), or strictly smaller than 1/2 ($r \in (0, 1)$).

Here are the formal statements. The first result provides estimates for the case $r = 1$.

**Theorem 2.2.** For $r = 1$, the following holds almost surely.

1. For any $\delta > 0$, and $0 \leq k \leq d$, define the open interval $I_{k,\delta} = (2^{-k} - \delta, 2^{-k} + \delta)$. Then,

$$\lim_{n \to \infty} \mu_n(I_{k,\delta}) \geq \binom{d}{k}.$$

2. Let $I \subset \mathbb{R}\setminus[-0.3,0.3]$, then

$$\lim_{n \to \infty} \mu_n(I) = \delta_1(I) + d\delta_{1/2}(I).$$

These results imply that, for large enough (random) $n$, the normalized Laplacian $\mathcal{L}_{n,r}$ has at least $\binom{d}{k}$ eigenvalues around 1 $- 2^{-k}$. Similarly, in the interval $[0, 0.7]$ the only eigenvalues of $\mathcal{L}_{n,r}$ are 0 and 1/2, and there are no eigenvalues in the interval $(1.3, 2]$.

The next two results provide estimates for the cases where $r \neq 1$.

**Theorem 2.3.** Let $1 < r < 2$, then almost surely there exists $N > 0$ such that all $n \geq N$,

$$|\lambda^{(n)}_k| < \frac{1}{2}, \quad \text{for } k = 2, \ldots, n.$$

**Theorem 2.4.** Let $0 < r < 1$. Then, almost surely, there exists $N > 0$ such that for all $n \geq N$,

$$\lambda^{(n)}_2 > \frac{1}{2}.$$
As alluded to in the Introduction, this behavior is very different to that in the case in the Erdős-Rényi random graph $G(n, p)$. For, $G(n, p)$ we know \cite{16} that when the expected vertex-degree is a little above $\log n$, the spectrum of $\mathcal{L}_{n,r}$ is concentrated around 1. In particular, the spectral gap converges to 1 (in probability). In the setting of $G(n, r)$, the graph is considerably denser, as the expected degree is proportional to $n$, yet the spectral gap is much lower, and there is an entire sequence of eigenvalues between 0 and 1.

The remainder of the paper is dedicated to proving Theorems 2.2–2.4 and their corollary.

3. Spectral convergence

The proofs of Theorems 2.2–2.4 rely heavily on a suitable definition convergence for the eigenvalues of a matrix. For this we exploit results from \cite{30} on the convergence of self-adjoint operators. In this section we provide the essential background.

Let $(V, \nu)$ be a probability space, and denote by

$$\mathcal{H} := L^2(V, \nu) = \left\{ f : V \to \mathbb{R} : \int_V |f(x)|^2 d\nu(x) < \infty \right\},$$

the Hilbert space with the inner product

$$\langle f, g \rangle = \int_V f(x)g(x) d\nu(x),$$

and associated norm $\|f\|_2 := \sqrt{\langle f, f \rangle}$. Let $K : V \times V \to \mathbb{R}$ be a kernel function in $L^2(V \times V, \nu \times \nu)$, and let $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ be the Hilbert-Schmidt kernel operator for the kernel $K$, defined by

$$\mathcal{K}f(x) = \int_V K(x,y)f(y)d\nu(y), \text{ for } f \in \mathcal{H}.$$ 

Since $K \in L^2(V \times V, \nu \times \nu)$, the operator $\mathcal{K}$ is compact, and hence its spectrum is given by a sequence of eigenvalues converging to zero. Furthermore, if $K(x,y) = K(y,x)$, then the operator $\mathcal{K}$ is self-adjoint, and hence all of its eigenvalues are real. Throughout the paper we will use $\text{spec}(\cdot)$ to refer to the set of eigenvalues of a matrix or an operator, where eigenvalues are repeated according to their multiplicity.

The cut norm of $K$ is defined by

$$\|K\|_{\square} = \sup_{S,T} \left| \int_{S \times T} K(x,y)d\nu(x)d\nu(y) \right|,$$

where $S,T$ run through all pairs of measurable sets in $V$. Note that

$$\|K\|_{\square} \leq \|K\|_1 := \int_{V \times V} |K(x,y)|d\nu(x)d\nu(y).$$

The following result is an extension of Lemma 1.11 in \cite{30}, and will be used in the proof of Theorem 2.2.

**Lemma 3.1** (Lemma 1.11 in \cite{30} – extended). Let $\{\mathcal{K}_n\}$ be a sequence of self-adjoint Hilbert-Schmidt kernel operators in $\mathcal{H}$ with corresponding kernels $\{K_n\}$, such that sup$_n \|K_n\|_\infty \leq C$, for some $C > 0$. Suppose that $K_n \to K$ in the cut norm. Let $\mathcal{K}$ be the Hilbert-Schmidt kernel operator for the kernel $K$. Then, for every $\lambda > 0$ such that $\pm \lambda \notin \text{spec}(\mathcal{K})$,

$$\lim_{n \to \infty} |\text{spec}(\mathcal{K}_n) \cap (\lambda, \infty)| = |\text{spec}(\mathcal{K}) \cap (\lambda, \infty)|,$$

$$\lim_{n \to \infty} |\text{spec}(\mathcal{K}_n) \cap (-\infty, -\lambda)| = |\text{spec}(\mathcal{K}) \cap (-\infty, -\lambda)|,$$

where $| \cdot |$ denotes cardinality.

The proof for Lemma 3.1 is a modification of the proof in \cite{30}. In order to keep the paper self contained, we provide a proof in Appendix B.
4. Outline for the proofs of Theorems 2.2, 2.3, 2.4

Before starting the proofs in detail, we will outline the main steps required for proving Theorems 2.2, 2.3 and 2.4. Lemma 3.1 plays a key role in our proofs, where in our settings $V = B^d := [-1, 1]^d$ and $\nu_d$ is the uniform probability measure on $B^d$. The proof will then consist of three main steps.

**Step I**: We construct a special sequence of kernels $K_{n,r} : B^d \times B^d \to \mathbb{R}$ and show that the corresponding Hilbert-Schmidt operators satisfy $\text{spec}(K_{n,r}) = \text{spec}(W_{n,r})$. This is achieved by using Lemma 6.1 ($d = 1$), Lemma 7.1 ($d = 2$), and Lemma 8.1 ($d \geq 3$).

**Step II**: Recall the definition of $h_r$ from (4), and define $K^d_r(x,y) : B^d \times B^d \to \mathbb{R}$ by

$$K^d_r(x,y) := \frac{h_r(x,y)}{\sqrt{H_r(x)H_r(y)}},$$

where $0 < r < 2$ and

$$H_r(x) := \int_{B^d} h_r(x,u)d\nu_d(u).$$

We show that $\int \int |K^d_r(x,y)|^2 d\nu_d(x)d\nu_d(y) < \infty$, from which it follows that the corresponding integral operator $K^d_r$ is a Hilbert-Schmidt kernel operator. Furthermore, we show that $K^d_r$ has at least $\left(\frac{d}{2}\right)$ eigenvalues at $1/2^k$, for all $0 \leq k \leq d$, and that the remaining eigenvalues lie in $(-0.3, 0.3)$. This is the content of Lemma 5.7. We also show that all the eigenvalues of $K^d_r$ (except for 1) lie in $(-0.5, 0.5)$, for all $1 < r < 2$. This is the content of Lemma 5.11. On the other hand we show that the second largest eigenvalue of $K^d_r$ is larger than 0.5 for $0 < r < 1$. See Lemma 5.12.

**Step III**: We show that $K_{n,r} \to K^d_r$ in the cut-norm, almost surely. This is carried out in Lemma 6.5 ($d = 1$), Lemma 7.5 ($d = 2$), and Lemma 8.5 ($d \geq 3$).

Combining these three steps and Lemma 3.1 gives us Theorems 2.2, 2.3 and 2.4.

The rest of the paper is organized as follows. In Section 5 we calculate the eigenvalues of the limiting operator $K^d_r$, for $0 < r < 2$. In Section 6 we construct the kernels $K_{n,r}$ for $d = 1$, and show their convergence. In Section 7 we construct the kernels for $d = 2$, and show their convergence. In Section 8 we provide the details needed to generalize the two-dimensional case to arbitrary $d \geq 3$. Finally, the proofs for Theorems 2.2, 2.3 and 2.4 and Corollary 2.5 are given in Section 9.

To conclude this section, we present explicit formulae for $H_r(x)$ that will be useful for us later. For $0 < r \leq 1$, we have that

$$H_r(x) = \begin{cases} \frac{1+r-|x|}{r} & 1 - r \leq |x| \leq 1, \\ 1 - r & |x| \leq 1 - r. \end{cases}$$

For $1 \leq r < 2$, we have

$$H_r(x) = \begin{cases} \frac{1+r-|x|}{r} & r - 1 \leq |x| \leq 1, \\ 1 & |x| \leq r - 1. \end{cases}$$

In particular, when $r = 1$, we have $H_r(x) = 1 - |x|/2$.

5. The Spectrum of the Limiting Operators

Recall the definition of the integral operator $K^d_r$ with kernel $K^d_r$ given in (9). In this section we estimate the eigenvalues of this operator for arbitrary $d \geq 1$. First, we show that the operator is indeed self-adjoint and Hilbert-Schmidt.

**Lemma 5.1.** For every $d \geq 1$ the kernel $K^d_r$ satisfies $K^d_r(x,y) = K^d_r(y,x)$ for all $x, y \in B^d$ and

$$\int \int_{B^d \times B^d} |K^d_r(x,y)|^2 d\nu_d(x)d\nu_d(y) < \infty.$$

That is, $K^d_r$ is a self-adjoint compact Hilbert-Schmidt operator.

**Proof.** Note that, for $x = (x_1, \ldots, x_d) \in B^d$ and $y = (y_1, \ldots, y_d) \in B^d$, we have

$$h_r(x,y) = \prod_{i=1}^d h_r(x_i, y_i), \quad \text{and} \quad H_r(x) = \prod_{i=1}^d H_r(x_i).$$
Consequently,

\[ K^n_r(x, y) = \prod_{i=1}^{d} K^1_i(x_i, y_i). \]  

(13)

Thus, it suffices to prove the result for \( K^1_r \). From the definition and the fact that \( h_r(x, y) = h_r(y, x) \) for every \( x, y \in B^1 \), it follows that \( K^1_r \) is real and \( K^1_r(x, y) = K^1_r(y, x) \). Hence \( K^1_r \) is symmetric.

Next, from (11) and (12), for all \( 0 < r < 2 \), we have \( H_r(x) \geq \frac{r}{r^2} > 0 \), and hence \( K^1_r(x, y) \leq 2/r \).

The last equality then gives

\[ \int_{B^1 \times B^1} |K^1_r(x, y)|^2 d\nu_1(x) d\nu_1(y) \leq \frac{4}{r^2} < \infty, \]

as required. \( \blacksquare \)

5.1. General statements. In this section we present a few lemmas that are true for all \( r \in (0, 2) \).

Denote by \((\lambda_i)_{i\geq1}\) the eigenvalues of \( W_{n, r} \) in decreasing order. Since we know that the spectrum of \( \mathcal{L}_{n, r} \) is in \([0, 2]\), we have that the spectrum of \( W_{n, r} \) is in \([-1, 1]\). In the following sections we provide the proofs for our estimates of the spectrum for different values of \( r \). The first eigenvalue, however, is the same for all \( r \in (0, 2) \).

**Lemma 5.2.** Let \( K^1_r \) be as defined above. Then \( \lambda_1 = 1 \) and \( \sqrt{H_r(x)} \) is the corresponding eigenfunction.

**Proof.** Let \( f(x) = \sqrt{H_r(x)} \). Then

\[ K^1_r f(x) = \int_{B^1} \frac{h_r(x, y)}{\sqrt{H_r(x) H_r(y)}} \sqrt{H_r(y)} d\nu_1(y) = \frac{H_r(x)}{\sqrt{H_r(x)}} = f(x). \]

\( \blacksquare \)

The behavior of the remaining eigenvalues will be studied in the following sections, depending on the value of \( r \).

The next lemma shows that the space of eigenfunctions of \( K^1_r \) is spanned by a collection of even and odd functions.

**Lemma 5.3.** Let \( \lambda \) be an eigenvalue of \( K^1_r \) with corresponding eigenfunction \( f \). Then \( f^*(x) \) := \( f(-x) \) is also an eigenfunction with the same eigenvalue \( \lambda \). Consequently, both of the functions \( f(x) \) and \( f(x) - f^*(x) \) are also eigenfunctions corresponding to the eigenvalue \( \lambda \), provided that they do not vanish.

**Proof.** Let \( \lambda \) be an eigenvalue of \( K^1_r \) with corresponding eigenfunction \( f \). Recalling that \( B^1 = [-1, 1] \), we have

\[ \lambda f(x) = \int_{-1}^{1} \frac{h_r(x, y)}{\sqrt{H_r(x) H_r(y)}} f(y) d\nu_1(y). \]

(14)

Note that

\[ H_r(-x) = \int_{-1}^{1} h_r(-x, y) d\nu_1(y) = \int_{-1}^{1} h_r(-x, -y) d\nu_1(y) = \int_{-1}^{1} h_r(x, y) d\nu_1(y) = H_r(x), \]

where we used the change of variables \( y \to (-y) \) and the fact that \( h_r(-x, y) = h_r(x, y) \). Therefore \( H_r \) is even. Setting \( f^*(x) = f(-x) \), we then have

\[ \lambda f^*(x) = \lambda f(-x) \]

\[ = \int_{-1}^{1} \frac{h_r(-x, y)}{\sqrt{H_r(-x) H_r(y)}} f(y) d\nu_1(y) \]

\[ = \int_{-1}^{1} \frac{h_r(-x, -y)}{\sqrt{H_r(-x) H_r(-y)}} f(-y) d\nu_1(y) \]

\[ = \int_{-1}^{1} \frac{h_r(x, y)}{\sqrt{H_r(x) H_r(y)}} f^*(y) d\nu_1(y), \]

where, as before, we used the change of variables \( y \to (-y) \) and the fact that \( h_r(-x, -y) = h_r(x, y) \), as well as the fact that \( H_r \) is even. \( \blacksquare \)
To prove some of our statements below, we will need to use an auxiliary kernel

$$K'_r(x, y) = \frac{h_r(x, y)}{H_r(x)}, \quad (15)$$

and $K'_r : L^2([-1, 1], \nu_1) \to L^2([-1, 1], \nu_1)$ the Hilbert-Schmidt kernel operator associated with $K'_r$.

The next lemma shows a spectral equivalence between $K'_r$ and $K^d_1$.

**Lemma 5.4.** Let $0 < r < 2$. A function $f : B^1 \to \mathbb{R}$ is an eigenfunction of $K'_r$ with eigenvalue $\lambda$ if and only if $\sqrt{H_r(x)}f(x)$ is an eigenfunction of $K^d_1$ with the same eigenvalue $\lambda$.

**Proof.** A function $f$ is an eigenfunction of $K'_r$ with eigenvalue $\lambda$ if, and only if, for every $x \in [-1, 1]$

$$\lambda H_r(x)f(x) = \int_{-1}^{1} h_r(x, y) f(y) d\nu_1(y) = \int_{-1}^{1} \frac{h_r(x, y)}{\sqrt{H_r(y)}} \sqrt{H_r(y)} f(y) d\nu_1(y).$$

In turn, the above is true if, and only if, for every $x \in [0, 1]$ (recall that by (11) and (12) $H_r(x) > 0$ for every $x \in [0, 1]$)

$$\lambda \sqrt{H_r(x)}f(x) = \int_{-1}^{1} h_r(x, y) \frac{\sqrt{H_r(y)}}{H_r(x)} f(y) d\nu_1(y) = \int_{-1}^{1} K^d_1(x, y) \frac{\sqrt{H_r(y)}}{H_r(y)} f(y) d\nu_1(y),$$

which holds if, and only if, $\sqrt{H_r(x)}f(x)$ is an eigenfunction of $K^d_1$ with eigenvalue $\lambda$. \hfill \blacksquare

In the next lemma we show that the eigenfunctions of $K^d_1$ are continuous.

**Lemma 5.5.** Let $f$ be an eigenfunction of $K^d_1$ corresponding to a non-zero eigenvalue. Then $f$ is continuous on $[-1, 1]$.

**Proof.** It is enough to show that the result holds for $d = 1$, due to the product form of the eigenfunctions of $K^d_1$.

Let $f$ be an eigenfunction of $K^d_1$ with corresponding, non-zero, eigenvalue $\lambda$. Without loss of generality we assume that $\|f\| = 1$. Therefore, we have

$$f(x) = \frac{1}{\lambda} \int_{B^1} \frac{h_r(x, y)}{\sqrt{H_r(x)H_r(y)}} f(y) d\nu_1(y),$$

implying that

$$|f(x) - f(x')| \leq \frac{1}{|\lambda|} \int_{B^1} \left| \frac{h_r(x, y)}{\sqrt{H_r(x)H_r(y)}} - \frac{h_r(x', y)}{\sqrt{H_r(x')H_r(y)}} \right| |f(y)| d\nu_1(y)$$

$$\leq \frac{2\sqrt{2}}{|\lambda|r^{1/2}} \int |\sqrt{H_r(x')}h_r(x, y) - \sqrt{H_r(x)}h_r(x, y')| |f(y)| d\nu_1(y). \quad (16)$$

The last equation follows from the fact that $H_r(x) \geq r/2$. By the triangle inequality we have

$$\left| \sqrt{H_r(x')}h_r(x, y) - \sqrt{H_r(x)}h_r(x', y) \right| \leq \left| \sqrt{H_r(x')} - \sqrt{H_r(x)} \right| |h_r(x, y) + \sqrt{H_r(x)}h_r(x, y) - h_r(x', y)|.$$

Note that $H_r(x)$ is continuous in $[-1, 1]$. Therefore, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \sqrt{H_r(x')} - \sqrt{H_r(x)} \right| \leq \epsilon, \quad \text{if } |x - x'| < \delta.$$

Note that $h_r(x, y) \leq 1$ and $H_r(x) \leq 1$, for $0 < r < 2$. Therefore, if $|x - x'| < \delta$, then

$$\left| \sqrt{H_r(x')}h_r(x, y) - \sqrt{H_r(x)}h_r(x', y) \right| \leq \epsilon + |h_r(x, y) - h_r(x', y)|. \quad (17)$$

Observe that, if $|x - x'| < \delta$, then

$$|h_r(x, y) - h_r(x', y)| \leq \begin{cases} 1 & \text{if } y \in (x \land x' + r, x \lor x' + r) \cup (x \land x' - r, x \lor x' - r), \\ 0 & \text{otherwise}, \end{cases}$$
where \( x \land x' = \min\{x, x'\} \) and \( x \lor x' = \max\{x, x'\} \). Thus \( \nu_1(\{y : |h_r(x, y) - h_r(x', y)| = 1\}) \leq \delta \). Therefore, for \( |x - x'| < \delta \), by (17) and the Cauchy-Schwarz inequality, we have
\[
\int \left| \sqrt{H_r(x')}h_r(x, y) - \sqrt{H_r(x')}h_r(x', y) \right| |f(y)| d\nu_1(y) \\
\leq \epsilon \int |f(y)| d\nu_1(y) + \int |h_r(x, y) - h_r(x', y)||f(y)| d\nu_1(y) \\
\leq \epsilon \|f\| + \|f\| \sqrt{\int |h_r(x, y) - h_r(x', y)|^2 d\nu_1(y)} \leq \epsilon + \sqrt{\delta}.
\]
The last inequality follows from the fact that \( \nu_1(\{y : |h_r(x, y) - h_r(x', y)| = 1\}) \leq \delta \) and \( \|f\| = 1 \). Thus (16) implies the result, as \( \epsilon, \delta > 0 \) can be chosen arbitrarily small.

5.2. The spectrum of \( K^d_1 \). The goal of this subsection is analyze the spectrum in the case \( r = 1 \).

 Lemma 5.6. Denote by \( (\lambda_i)_{i \geq 1} \) the eigenvalues of \( K^d_1 \), in decreasing order. Then \( \lambda_2 = 1/2 \) with matching eigenfunction \( x \sqrt{H_1(x)} \), and for all \( i \geq 3 \) we have \( \lambda_i \in (-0.3, 0.3) \).

 Proof. Denote by \( (\varphi_i)_{i \geq 1} \) the orthonormal eigenfunctions corresponding to the eigenvalues \( (\lambda_i)_{i \geq 1} \). By the spectral theorem for self-adjoint, compact operators
\[
K^d_1(x, y) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(x) \varphi_i(y),
\]
which implies that
\[
\iint_{B^1 \times B^1} |K^d_1(x, y)|^2 d\nu_1(x) d\nu_1(y) = \sum_{i=1}^{\infty} \lambda_i^2.
\]
On the one hand, for \( r = 1 \), we have \( H_1(x) = 1 - |x|/2 \), and therefore
\[
\iint_{B^1 \times B^1} |K^d_1(x, y)|^2 d\nu_1(x) d\nu_1(y) = \int_{-1}^{1} \int_{-1}^{1} \frac{1_{|x-y| \leq 1}}{(2 - |x|)(2 - |y|)} dy dx \\
= \int_{-1}^{1} \int_{-1}^{1} \frac{1}{(2 - |x|)(2 - |y|)} dy dx + \int_{1}^{1} \int_{x-1}^{1} \frac{1}{(2 - |x|)(2 - |y|)} dy dx.
\]
Noting that
\[
\int_{-1}^{1} \frac{1}{(2 - |y|)} dy = 2 \log 2 - \log(1 - x), \quad \text{and} \quad \int_{1}^{1} \frac{1}{(2 - |y|)} dy = 2 \log 2 - \log(1 + x),
\]
we conclude
\[
\iint_{B^1 \times B^1} |K^d_1(x, y)|^2 d\nu_1(x) d\nu_1(y) = 4(\log 2)^2 - 2 \int_{0}^{1} \frac{\log(1 + x)}{2 - x} dx \approx 1.33299.
\]
Next, it is easy to see that \( \sqrt{H_1(x)} \) and \( x \sqrt{H_1(x)} \) are eigenfunctions of \( K^d_1 \), with eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = 1/2 \), respectively. Therefore,
\[
\sum_{i=3}^{\infty} \lambda_i^2 \approx 0.0829 < 0.09,
\]
and so \( |\lambda_i| < 0.3 \) for all \( i \geq 3 \), as required.

 Lemma 5.7. For every \( 0 \leq k \leq d \) the operator \( K^d_k \) has eigenvalue \( 1/2^k \) with multiplicity at least \( \binom{d}{k} \). Moreover, the rest of the eigenvalues lie in \((-0.3, 0.3)\).

 Proof of Lemma 5.7. Let \( (\lambda_i)_{i \geq 1} \) be the eigenvalues of \( K^d_1 \) listed with multiplicities in decreasing order, and \( (\varphi_i)_{i \geq 1} \) the corresponding orthonormal eigenfunctions. Since \( K^d_1 \) is a compact and self-adjoint operator, the spectral theorem implies that \( (\varphi_i)_{i \geq 1} \) form an orthonormal basis of \( L^2(B^d, \nu_d) \). Recall that \( L^2(B^d, \nu_d) \) is the space of functions on \( B^1 \times \cdots \times B^1 \) \((d\text{-times})\) with respect to the product measure \( \nu_1 \times \cdots \times \nu_1 \) \((d\text{-times})\). Therefore \( (\varphi_{i_1, \ldots, i_d})_{i_1, \ldots, i_d \in \mathbb{N}} \) is an orthonormal basis for \( L^2(B^d, \nu_d) \), where \( \varphi_{i_1, \ldots, i_d}(x) := \varphi_{i_1}(x_1) \cdots \varphi_{i_d}(x_d) \) for all \( x \in B^d \).
Using [13], for \( i_1, \ldots, i_d \in \mathbb{N} \) and \( x \in B^d \) we have
\[
K^d_{i_1} \varphi_{i_1, \ldots, i_d}(x) = \int K^d_{i_1}(x, y) \varphi_{i_1, \ldots, i_d}(y) d\nu_d(y) = \lambda_{i_1} \cdots \lambda_{i_d} \varphi_{i_1, \ldots, i_d}(x).
\]
Hence \((\lambda_{i_1} \cdots \lambda_{i_d})_{i_1, \ldots, i_d \in \mathbb{N}}\) forms the complete list of eigenvalues of \( K^d \) including multiplicities. In particular, by Lemma 5.6 if there exists \( 1 \leq k \leq d \) such that \( i_k \geq 3 \) then
\[
|\lambda_{i_1} \cdots \lambda_{i_d}| < 0.3.
\]

Lemma 5.6 also implies that \( \lambda_1 = 1 \) and \( \lambda_2 = 1/2 \). Thus, by considering all the eigenvalues corresponding to \( i_1, \ldots, i_d \in \{1, 2\} \), we get \( 1/2^k \) as an eigenvalue of \( K^d_1 \) with multiplicity at least \( \binom{d}{k} \), for \( k = 0, \ldots, d \). This completes the proof. \( \blacksquare \)

5.3. The spectrum of \( K^d_r \) for \( 1 < r < 2 \). In this subsection we estimate the eigenvalues of \( K^d_r \) for \( 1 < r < 2 \). Recall that \( K^d_r \) is the Hilbert-Schmidt kernel operator with the kernel \( K^d_r \), as defined in [9]. We start by analyzing the spectrum in the case \( d = 1 \).

Lemma 5.8. Let \( 1 < r < 2 \), and denote by \((\lambda_i)_{i \geq 1}\) the eigenvalues of \( K^1_r \) in decreasing order. Then
\[
|\lambda_i| < \frac{1}{r}, \quad i \geq 2.
\]

We will show that the statement of Lemma 5.8 holds for even and odd eigenfunctions separately. Using Lemma 5.3, this will suffice to cover all the eigenfunctions.

Lemma 5.9. Let \( 1 < r < 2 \), and \( \lambda \) be an eigenvalue \( K^1_r \) with an odd eigenfunction. Then \( |\lambda| < \frac{1}{2} \).

Proof. Let \( S_{\text{odd}} = \{ f \in S : f \text{ is odd and } \|f\|_2 = 1 \} \), where \( S \) denotes the space of eigenfunctions of \( K^1_r \). Since \( \lambda \) is an eigenvalue with an odd eigenfunction,
\[
|\lambda| \leq \sup\{|\langle K^1_r f, f \rangle| : f \in S_{\text{odd}}\}.
\]

For every \( f \in S_{\text{odd}} \)
\[
\langle K^1_r f, f \rangle = \int_0^1 \int_0^{(r-x)^\wedge 1} f(x) f(y) d\nu_1(x) d\nu_1(y)
\]
\[
= \int_0^\frac{(r-x)^\wedge 1}{(x-r)^\wedge 1} \frac{f(y)}{\sqrt{H_r(x)H_r(y)}} d\nu_1(y)
\]
\[
= 2 \int_0^\frac{(r-x)^\wedge 1}{(x-r)^\wedge 1} \frac{f(y)}{\sqrt{H_r(x)H_r(y)}} d\nu_1(y),
\]
where for the last equality we used a change of variables and the fact that \( H_r \) is even and \( f \) is odd. Since for \( f \in S_{\text{odd}} \), we also have
\[
\int_0^{(r-x)^\wedge 1} \frac{f(y)}{\sqrt{H_r(y)}} d\nu_1(y) = 0,
\]
we conclude that
\[
\langle K^1_r f, f \rangle = 2 \int_0^\frac{(r-x)^\wedge 1}{(x-r)^\wedge 1} \frac{f(x) f(y)}{\sqrt{H_r(x)H_r(y)}} d\nu_1(x) d\nu_1(y).
\]

Next, denote \( S_{\text{odd}}^+ = \{ f \in S_{\text{odd}} : f(x) \geq 0 \text{ for } x \in [0, 1] \} \). We claim that
\[
\sup_{f \in S_{\text{odd}}^+} |\langle K^1_r f, f \rangle| = \sup_{f \in S_{\text{odd}}^+} \langle K^1_r f, f \rangle.
\]
Indeed, since $S^+_{\text{odd}} \subset S_{\text{odd}}$ the inequality $\geq$ holds trivially. As for the other direction, given $f \in S_{\text{odd}}$ define $\hat{f} \in S^+_{\text{odd}}$ by $\hat{f}(x) = |f(x)|$ for $x \in [0, 1]$ and $\hat{f}(x) = -|f(x)|$ for $x \in [-1, 0]$. Note that

$$|\langle K_1 f, f \rangle| = \left| 2 \int_0^1 \int_{(r-x)\wedge 1}^1 \frac{f(x)f(y)}{\sqrt{H_r(x)H_r(y)}} \, dv_1(x) \, dv_1(y) \right|$$

$$\leq 2 \int_0^1 \int_{(r-x)\wedge 1}^1 \sqrt{H_r(x)H_r(y)} \, dv_1(x) \, dv_1(y)$$

$$= \langle K_r \hat{f}, \hat{f} \rangle,$$

where we used the fact that $1 < r < 2$, and hence that $r - x \geq 0$ for all $x \in [0, 1]$.

Finally, note that, for $1 < r < 2$, we know that $[(r - x) \wedge 1, 1] \subset [1 - x, 1]$ and also that, by (12), $H_r(x) \geq H_1(x)$. Hence, for $f \in S^+_{\text{odd}}$

$$\langle K_1 f, f \rangle = 2 \int_0^1 \int_{(r-x)\wedge 1}^1 \frac{f(x)f(y)}{\sqrt{H_r(x)H_r(y)}} \, dv_1(x) \, dv_1(y)$$

$$\leq 2 \int_0^1 \int_{1-x}^1 \frac{f(x)f(y)}{\sqrt{H_1(x)H_1(y)}} \, dv_1(x) \, dv_1(y)$$

(20)

$$= \langle K_1 f, f \rangle \leq \frac{1}{2},$$

where in the inequality we used Lemma 5.6.

**Lemma 5.10.** Let $1 < r < 2$, and $\lambda$ be an eigenvalue $K_1$ with even eigenfunction $f$. If $f$ is orthogonal to the eigenfunction $\sqrt{H_r}$ (see Lemma 5.5), then $|\lambda| < \frac{1}{2}$. In particular, the multiplicity of the eigenvalue 1 is one.

**Proof.** Let $f$ be an even eigenfunction with eigenvalue $\lambda$, which is orthogonal to $\sqrt{H_r}$ and define $g = f/\sqrt{H_r}$. By Claim 5.4, the function $g$ is an eigenfunction with eigenvalue $\lambda$ of $K_1$, and therefore, for $0 \leq x \leq 1$,

$$\lambda H_r(x)g(x) = \int_{(x-r)\wedge (-1)}^1 g(y) \, dv_1(y) = \int_0^1 g(y) \, dv_1(y) + \int_0^1 g(y) \, dv_1(y).$$

(21)

From the assumption that $f$ is orthogonal to $\sqrt{H_r}$ also know that

$$\int_{-1}^1 g(y)H_r(y) \, dv_1(y) = 0. $$

(22)

Together with the assumption that $f$ is even and the fact that $H_r$ is even, it follows that

$$\int_0^1 g(y)H_r(y) \, dv_1(y) = 0. $$

(23)

Define $\|f\|_{\infty} := \sup \{|f(x)| : 0 \leq x \leq 1\}$. Lemma 5.5 implies that $f$ is continuous, and hence $g$ is a continuous, non-trivial, eigenfunction. Therefore, we can find $x_0 \in [0, 1]$ such that $|g(x_0)| = \|g\|_{\infty} > 0$. Without loss of generality we assume that $g(x_0) > 0$. The rest of the argument depends on the location of $x_0$.

**Case 1** ($r - 1 \leq x_0 \leq 1$). For $x \in [r - 1, 1]$, $r - 1 \leq r - x \leq 1$. Using (21) and (23) we have that, for all $\alpha \in \mathbb{R}$,

$$2\lambda H_r(x)g(x) = \int_0^{r-x} g(y) \, dy + \int_0^1 g(y) \, dy$$

$$= \int_0^{r-x} g(y) \, dy + \int_0^1 g(y) \, dy + \alpha \int_0^1 g(y)H_r(y) \, dy$$

$$= \int_0^{r-1} g(y)(2 + \alpha H_r(y)) \, dy + \int_{r-1}^{r-x} g(y)(2 + \alpha H_r(y)) \, dy + \int_{r-x}^1 g(y)(1 + \alpha H_r(y)) \, dy.$$
Choosing \( \alpha = -2 \) and using (12), we obtain that \( 2 + \alpha H_r(y) = 0 \) for \( 0 \leq y \leq r - 1 \). Consequently, applying (12) again, for \( x \in [r - 1, 1] \),
\[
2\lambda H_r(x)g(x) = \int_{r-x}^{r-1} g(y)(2 - 2H_r(y))dy + \int_{r-x}^{1} g(y)(1 - 2H_r(y))dy \\
= \int_{r-x}^{r-1} g(y)(2 - (r + 1 - y))dy + \int_{r-x}^{1} g(y)(1 - (r + 1 - y))dy \\
= \int_{r-x}^{r-1} g(y)(y - (r - 1))dy + \int_{r-x}^{1} g(y)(y - (r + y))dy \\
\leq \|g\|\int_{r-x}^{r-1} (y - (r - 1))dy + \|g\|\int_{r-x}^{1} (y - (r - 1))dy \\
= \|g\| \int_0^{x_0} (2 - r)\|g\|_\infty,
\]
(24)

Taking \( x = x_0 \) in (24) gives
\[
|\lambda| \|(r + 1 - x_0)\|\|g\|_\infty = |\lambda| \|(r + 1 - x_0)\|g(x_0)\| \leq \frac{r}{2}(2 - r)\|g\|_\infty,
\]
and, since \( r + 1 - x_0 \geq r \), that
\[
|\lambda| \leq \frac{2 - r}{2} < \frac{1}{2},
\]
where in the last inequality we used the assumption that \( 1 < r < 2 \).

**Case 2** \( 0 \leq x_0 \leq r - 1 \). Let \( x \in [0, r - 1] \). Then \( r - x \geq 1 \). Using (21) and (23), for every \( \alpha \in \mathbb{R} \) we have
\[
\lambda H_r(x)g(x) = \int_0^{1} g(y)dy + \alpha \int_0^{r-1} g(y)H_r(y)dy \\
= \int_0^{r-1} g(y)(1 + \alpha H_r(y))dy + \int_{r-1}^{1} g(y)(1 + \alpha H_r(y))dy.
\]

From (12) we have that \( H_r(x) = 1 \) for \( x \leq r - 1 \). Thus, taking \( \alpha = -1 \), we have
\[
\lambda g(x) = \int_{r-1}^{1} g(y)(1 - H_r(y))dy \\
= \frac{1}{2} \int_{r-1}^{1} g(y)(y - (r - 1))dy \\
\leq \|g\|\int_{r-x}^{1} (y - (r - 1))dy \\
= \|g\| \int_0^{x_0} (2 - r)^2.
\]

In particular, for \( x = x_0 \),
\[
|\lambda| \cdot \|g\|_\infty = |\lambda|g(x_0) \leq \|g\| \int_0^{x_0} (2 - r)^2 \leq \frac{1}{4}\|g\|_\infty,
\]
and hence \( |\lambda| \leq \frac{1}{4} \).

Finally, we can prove Lemma 5.8.

**Proof of Lemma 5.8** Note that Lemma 5.3 implies that the eigenfunctions are generated by only odd and even functions. Thus, combining Lemmas 5.2, 5.9 and 5.10 and the fact that the eigenfunction form an orthonormal basis, the result follows. \( \square \)

Now that we have estimates for the spectrum in the one-dimensional case, we can treat the case of arbitrary dimension.

**Lemma 5.11.** Let \( 1 < r < 2 \). Then \( \lambda = 1 \) is an eigenvalue of \( K_r^d \) with multiplicity 1, and all other eigenvalues \( \lambda \) satisfy \( |\lambda| < 1/2 \).
Proof. Fix $1 < r < 2$ and denote by $(\lambda_i)_{i \geq 1}$ the eigenvalues of $K^1_r$ in decreasing order. Repeating the argument in the proof of Lemma 5.7, we see that the eigenvalues of $K^d_r$ for $0 < r < 1$. Hence the result follows from Lemma 5.8.

5.4. The spectrum of $K^d_r$ for $0 < r < 1$. In this subsection we estimate the eigenvalues of $K^d_r$ for $0 < r < 1$. In particular, we prove the following result.

Lemma 5.12. Fix $0 < r < 1$, and let $K^d_r$ be as defined above. Then the second largest eigenvalue of $K^d_r$ is strictly greater than $1/2$.

Proof. Due to the product form of the eigenvalues of $K^d_r$, it suffices to show that the second largest eigenvalue of $K^d_r$ is strictly larger than $1/2$ for every $0 < r < 1$.

Fix $0 < r < 1$ and denote by $\lambda_2$ the second largest eigenvalue of $K^d_r$. Since $K^d_r$ is a self-adjoint operator, and $1$ is an eigenvalue with eigenfunction $\sqrt{H_r}$,

$$\lambda_2 = \sup_{f \in S'} \frac{\langle K^d_r f, f \rangle}{\langle f, f \rangle},$$

where $S' = \{ f \in L^2[-1, 1] : f \text{ is orthogonal to } \sqrt{H_r} \}$. Hence, in order to prove the statement, it suffices to find a function $f \in S'$ satisfying

$$\frac{\langle K^d_r f, f \rangle}{\langle f, f \rangle} > \frac{1}{2}.$$  

Let $f : [-1, 1] \to \mathbb{R}$ be given by $f(x) = x \sqrt{H_r(x)}$ for $x \in [-1, 1]$. Since $\sqrt{H_r}$ is even and bounded, it follows that $f \in S'$, as $f$ is odd. On the one hand, (11) implies that, for $0 < r < 1$,

$$\langle f, f \rangle = \int_{-1}^{1} |f(x)|^2 \nu_1(x)$$

$$= \int_{0}^{1} x^2 H_r(x) dx$$

$$= r \int_{0}^{1-r} x^2 dx + \frac{1}{2} \int_{1-r}^{1} x^2 (1 + r - x) dx.$$  

On the other hand, using the definition of the kernel $K^d_r$,

$$\langle K^d_r f, f \rangle = \int \int K^d_r(x, y) f(x) f(y) d\nu_1(x) d\nu_1(y)$$

$$= \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} x y h_r(x, y) dxdy$$

$$= \frac{1}{2} \int_{0}^{1} \int_{-1}^{1} x y h_r(x, y) dxdy.$$  

Recalling that $h_r(x, y) = 1_{|x-y| \leq r}$ gives

$$\int_{0}^{1} \int_{-1}^{1} x y h_r(x, y) dxdy = \int_{0}^{1-r} \int_{-r}^{r} xy dy dx + \frac{1}{2} \int_{1-r}^{1} \int_{-r}^{r} x y dx dy$$

$$= 2r \int_{0}^{1-r} x^2 dx + \frac{1}{2} \int_{1-r}^{1} x (1 - x^2 + 2r - r^2) dx$$

$$= \langle f, f \rangle + r \int_{0}^{1-r} x^2 dx + \frac{(1-r)}{2} \int_{1-r}^{1} (1 + r - x) dx$$

$$\geq \langle f, f \rangle + \frac{r(1-r)}{3} + \frac{(1-r)^2}{2},$$  

where the last equality follows from (25), and the last inequality follows from the fact that $1 + r - x \geq r$. Combining (27) and (28), we conclude that for $0 < r < 1$

$$\langle K^d_r f, f \rangle > \frac{1}{2} \langle f, f \rangle,$$

as required.
The following lemma will be used in the proof of Corollary 2.5.

**Lemma 5.13.** Fix $0 < r < 2$, and let $K^d_r$ be as defined above. Suppose $\lambda_2$ is the second largest eigenvalue of $K^d_r$. Then $0 < \lambda_2 < 1$.

**Proof.** We first show that $\lambda_2 < 1$. Due to the product form of the eigenvalues of $K^d_r$, it is enough to show that the result holds for $d = 1$.

Let $\lambda$ be an eigenvalue of $K^1_r$ with corresponding eigenfunction $f$, where $f$ is orthogonal to $\sqrt{H_r}$. It suffices to show that $|\lambda| < 1$, as eigenfunctions of the self-adjoint operator $K^1_r$ are orthogonal and $\sqrt{H_r}$ is the eigenfunction for $\lambda = 1$, by Lemma 5.2. Without loss of generality we assume that $f(x) = g(x) \sqrt{H_r(x)}$. Since $\lambda$ is an eigenvalue of $K^1_r$ with eigenfunction $f$, we have

$$\lambda f(x) = \int_{-1}^{1} K^1_{r}(x,y) f(y) d\nu_1(y).$$

Consequently, as $H_r(x) > 0$, we have that

$$\lambda g(x) = \frac{1}{H_r(x)} \int_{-1}^{1} h_r(x,y) g(y) d\nu_1(y), \quad \text{for } x \in [-1,1]. \quad (29)$$

Since $f$ is orthogonal to $\sqrt{H_r}$, we also have that

$$\langle f, \sqrt{H_r} \rangle = \int_{-1}^{1} g(x) H_r(x) d\nu_1(x) = 0. \quad (30)$$

Observe that (30) implies that $g$ is a non-constant function in $[-1,1]$. Also note that $g$ is continuous, as $f$ and $\sqrt{H_r}$ are continuous by Lemma 5.5 and (11). Therefore, there exists $x_0 \in [-1,1]$ such that $g$ is not constant in the interval $[-1 \vee (x_0 - r), (x_0 + r) \wedge 1]$ and $|g(x_0)| = \|g\|_\infty$, where $\|g\|_\infty := \sup\{|g(x)| : -1 \leq x \leq 1\}$. Thus, for $x = x_0$, from (29) we have that

$$|\lambda| |g(x_0)| \leq \frac{1}{H_r(x_0)} \int_{-1}^{1} h_r(x_0,y) |g(y)| d\nu_1(y) < \frac{\|g\|_\infty}{H_r(x_0)} \int_{-1}^{1} h_r(x_0,y) d\nu_1(y).$$

The strict inequality in the last equation follows from the fact that $g$ is not constant in the interval $[-1 \vee (x_0 - r), (x_0 + r) \wedge 1]$. Therefore

$$|\lambda| < \frac{\|g\|_\infty}{|g(x_0)|} = 1.$$  

Next, we show that $\lambda_2 > 0$. As before, it is enough to show the result holds for $d = 1$. Since $K^1_r$ is a self-adjoint operator and $\sqrt{H_r}$ is the eigenfunction for the largest eigenvalue 1, we have

$$\lambda_2 = \sup_{f \in S} \frac{\langle K^1_r f, f \rangle}{\langle f, f \rangle},$$

where $S' = \{ f \in L^2[-1,1] : \langle f, \sqrt{H_r} \rangle = 0 \}$. Taking $f(x) = x \sqrt{H_r(x)}$, for $x \in [-1,1]$, then $\langle f, \sqrt{H_r} \rangle = 0$, and

$$\lambda_2 \geq \frac{\langle K^1_r f, f \rangle}{\langle f, f \rangle} > 0.$$  

Hence the result. \qed

6. **Construction of kernels for $d = 1$**

In this section and those to follow we construct, for each $n$, a kernel whose spectrum is the same as the spectrum of the symmetrically normalized adjacency operator of the random geometric graph $G(n,r)$, and show that they converge in the cut norm to the limiting integral operators $K^d_r$ of the previous section. This section is devoted to the proof in the case $d = 1$ and the following sections are dedicated to the cases $d = 2$ and general $d$. The main reason for this partition is pedagogical, as we wish to present the proofs in an incremental level of difficulty, allowing each step to rely on the preceding ones.
Throughout this section fix $0 < r < 2$. We start by defining a partition of $[-1, 1]$ into subintervals. For $n \geq 1$, define $(L^n_i)_{i=1}^{n}$ by

\[
L^n_1 = \left[ -1 + \frac{2(i-1)}{n}, -1 + \frac{2i}{n} \right], \quad 1 \leq i \leq n-1, \\
L^n_n = \left[ 1 - \frac{2}{n}, 1 \right],
\]

so that the $L^n_i$ are disjoint intervals, with $\nu_i(L^n_i) = 1/n$ for all $i$. For brevity, throughout this section, we write $L_i$ for $L^n_i$.

Let $X_1, \ldots, X_n$ be a sequence of i.i.d. uniformly distributed random variables in $B^1 = [-1, 1]$, and let $X^{(1)}, \ldots, X^{(n)}$ be their order statistics, i.e., $X^{(1)} \leq \cdots \leq X^{(n)}$. For $n \geq 1$, define the random functions $h_{n,r} : B^1 \times B^1 \to \mathbb{R}$, as

\[
h_{n,r}(x,y) = \sum_{i,j=1}^{n} h_r(X^{(i)}, X^{(j)}) \mathbf{1}_{L_i}(x) \mathbf{1}_{L_j}(y).
\]

Next, define a sequence of random kernels $K_{n,r} : B^1 \times B^1 \to \mathbb{R}$ by

\[
K_{n,r}(x,y) = \frac{h_{n,r}(x,y)}{H_{n,r}(x)H_{n,r}(y)},
\]

where

\[
H_{n,r}(x) := \int_{B^1} h_{n,r}(x,u)d\nu_1(u) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{L_i}(x) \sum_{p=1}^{n} h_r(X^{(i)}, X^{(p)}).
\]

Hence,

\[
K_{n,r}(x,y) = \sum_{i,j=1}^{n} \frac{nh_r(X^{(i)}, X^{(j)}) \mathbf{1}_{L_i}(x) \mathbf{1}_{L_j}(y)}{\sum_{p,q=1}^{n} h_r(X^{(i)}, X^{(p)})h_r(X^{(j)}, X^{(q)})}.
\]

Let $K_{n,r} : L^2(B^1, \nu_1) \to L^2(B^1, \nu_1)$ be the Hilbert-Schmidt kernel operator for the kernel $K_{n,r}$, i.e.

\[
K_{n,r}f(x) = \int_{B^1} K_{n,r}(x,y)f(y)d\nu_1(y).
\]

Note that $K_{n,r}$ is a random operator, as $K_{n,r}$ is a random function. As mentioned before, the goal of this section is to prove: (a) the operator $K_{n,r}$ has the same spectrum as the operator $W_{n,r}$, and (b) almost surely, $K_{n,r} \to K^1_r$ in the cut-norm as $n \to \infty$.

6.1. The spectrum of $K_{n,r}$. We start by showing that $K_{n,r}$ and $W_{n,r}$ have the same spectrum.

**Lemma 6.1.** Let $d = 1$ and $0 < r < 2$. For $n \geq 1$, let $K_{n,r}$ be as defined in (34) and $W_{n,r}$ as defined in (5). Then

\[
\text{spec}(K_{n,r}) = \text{spec}(W_{n,r}).
\]

**Proof.** Let $\tilde{A}_{n,r}$ and $\tilde{W}_{n,r}$ be the adjacency and symmetrically normalized adjacency matrices for the vertex set $X^{(1)}, \ldots, X^{(n)}$. Since we only changed the order of the vertices, $\text{spec}(\tilde{W}_{n,r}) = \text{spec}(W_{n,r})$. Abbreviate $a_{i,j} = (\tilde{A}_{n,r})_{i,j} = h_r(X^{(i)}, X^{(j)})$, and $d_i = \sum_{j=1}^{n} a_{i,j}$. Then the $(i,j)$-th entry of $\tilde{W}_{n,r}$ is given by

\[
w_{i,j} := \frac{a_{i,j}}{d_i d_j}.
\]

Using this notation with (33), we can also write

\[
K_{n,r}(x,y) = n \sum_{i,j=1}^{n} w_{i,j} \mathbf{1}_{L_i}(x) \mathbf{1}_{L_j}(y).
\]

Let $f$ be an eigenfunction of $K_{n,r}$ with eigenvalue $\lambda$. Then, for every $x \in B^1$,

\[
\lambda f(x) = n \sum_{i=1}^{n} \left( \sum_{j=1}^{n} w_{i,j} (f, \mathbf{1}_{L_j}) \right) \mathbf{1}_{L_i}(x).
\]
In other words, \( f \) must be piecewise constant on the intervals \( L_i \), and we can write
\[
f(x) = \sum_{i=1}^{n} c_i 1_{L_i}(x),
\]
for some values \( c_i \). Hence
\[
\mathcal{K}_{n,r} f(x) = \sum_{i,j=1}^{n} w_{i,j} c_j 1_{L_i}(x).
\]
Therefore, \( f(x) \) is an eigenfunction of \( \mathcal{K}_{n,r} \) with eigenvalue \( \lambda \) if, and only if,
\[
\sum_{i=1}^{n} \left( \sum_{j=1}^{n} w_{i,j} c_j \right) 1_{L_i}(x) = \lambda \sum_{i=1}^{n} c_i 1_{L_i}(x).
\]
The last equation holds if, and only if, the vector \( c = (c_1, \ldots, c_n) \) is an eigenvector of \( \widetilde{W}_{n,r} \), with eigenvalue \( \lambda \). That is, there is a one-to-one correspondence between eigenfunctions of \( \mathcal{K}_{n,r} \) and the eigenvectors of \( \widetilde{W}_{n,r} \), with matching eigenvalues. This concludes the proof. ■

6.2. Concentration of order statistics. The following lemma shows that the order statistics of the uniformly distributed random variables in \([-1, 1]\) are concentrated around their means, which we will use to show that, almost surely, \( \mathcal{K}_{n,r} \to \mathcal{K}_1 \) in the cut-norm.

Lemma 6.2. Let \( X^{(k)} \) be the order statistics of \( n \) i.i.d. uniformly distributed points in \( B^1 \). Then, almost surely, there exists \( N > 0 \) such that, for all \( n \geq N \), we have
\[
\sup_{k=1, \ldots, n} \left| X^{(k)} - \mathbb{E}[X^{(k)}] \right| \leq \frac{1}{n^{1/3}},
\]
where
\[
\mathbb{E}[X^{(k)}] = -1 + \frac{2k}{n + 1} \in L_i. \tag{35}
\]
The proof of Lemma 6.2 can be found in [12, Lemma 2]. For completeness, we provide a simpler proof for this result in Appendix A.

6.3. The convergence of \( \mathcal{K}_{n,r} \). In this subsection we show that \( \mathcal{K}_{n,r} \) converges to \( \mathcal{K}_1 \) in the cut-norm almost surely as \( n \to \infty \). We start by defining, for \( 0 < r < 2 \) and \( \varepsilon > 0 \), the sets
\[
\mathcal{G}^1_{\varepsilon} = \{(i,j) : |x - y| < r - \varepsilon, \text{ for all } (x,y) \in L_i \times L_j\},
\]
\[
\mathcal{G}^2_{\varepsilon} = \{(i,j) : |x - y| > r + \varepsilon, \text{ for all } (x,y) \in L_i \times L_j\}. \tag{36}
\]

Figure 1. The box \([-1, 1]^2\) divided into the cells \( L_i \times L_j \). The gray cells (middle block) correspond to the set \( \mathcal{G}^1_{\varepsilon} \) while the red cells (corners) correspond to \( \mathcal{G}^2_{\varepsilon} \).
Lemma 6.3. For every $\varepsilon > 0$, almost surely, there exists (a random) $N_\varepsilon \in \mathbb{N}$ such that, for all $n \geq N_\varepsilon$, the following two statements are true:

1. If $(i, j) \in G_1$ then $h_\varepsilon(X^{(i)}, X^{(j)}) = 1$.
2. If $(i, j) \in G_2$ then $h_\varepsilon(X^{(i)}, X^{(j)}) = 0$.

Proof. Fix $\varepsilon > 0$. By Lemma 6.2, there exists (almost surely) $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\sup_{1 \leq i \leq n} \left| X^{(i)} - \mathbb{E}[X^{(i)}] \right| \leq n^{-1/3}.$$ 

Furthermore, by increasing the value of $N$, we can almost surely find $N_\varepsilon \in \mathbb{N}$ such that $n^{-1/3} < \varepsilon/2$ for all $n \geq N_\varepsilon$. For all $1 \leq i, j \leq n$, we then have

$$|X^{(i)} - X^{(j)}| \leq |X^{(i)} - \mathbb{E}[X^{(i)}]| + |\mathbb{E}[X^{(i)}] - \mathbb{E}[X^{(j)}]| + |\mathbb{E}[X^{(j)}] - X^{(j)}| \leq |\mathbb{E}[X^{(i)}] - X^{(j)}| + \varepsilon.$$ 

Finally, note that, by (35), $\mathbb{E}[X^{(i)}] \in L_1$ for all $1 \leq i \leq N$, which implies $|\mathbb{E}[X^{(i)}] - \mathbb{E}[X^{(j)}]| < r - \varepsilon$ for all $(i, j) \in G_1^+$, and hence

$$|X^{(i)} - X^{(j)}| < r - \varepsilon + \varepsilon = r,$$

as required.

Similarly, we can show that $h_\varepsilon(X^{(i)}, X^{(j)}) = 0$, i.e., $|X^{(i)} - X^{(j)}| > r$ for all $(i, j) \in G_2$, completing the proof.

Lemma 6.4. For $0 < r < 2$, let $H_r$ and $H_{n,r}$ be as defined above in (10) and (32). Then, almost surely,

$$\lim_{n \to \infty} \sup_{x \in B_1} |H_{n,r}(x) - H_r(x)| = 0.$$

Proof. Using (10) and (32), if $x \in L_i$, then

$$|H_{n,r}(x) - H_r(x)| = \left| \frac{1}{n} \sum_{j=1}^{n} h_\varepsilon(X^{(i)}, X^{(j)}) - \int_{B_1} h_\varepsilon(x, u) d\nu_1(u) \right| \leq \sum_{j=1}^{n} \left| \frac{1}{n} h_\varepsilon(X^{(i)}, X^{(j)}) - \int_{L_j} h_\varepsilon(x, u) d\nu_1(u) \right|.$$

Fix $\varepsilon > 0$. By Lemma 6.3, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$, if $(i, j) \in G_1^+$, then $h_\varepsilon(X^{(i)}, X^{(j)}) = 1$ and if $(i, j) \in G_2^+$, then $h_\varepsilon(X^{(i)}, X^{(j)}) = 0$. In addition, since $x \in L_i$ if $(i, j) \in G_1^+$, then for all $u \in L_j$ we have $h_\varepsilon(x, u) = 1$ and for all $(i, j) \in G_2$, we have $h_\varepsilon(x, u) = 0$. Consequently, if $(i, j) \in G_1^+ \cup G_2^+$, then

$$\left| \frac{1}{n} h_\varepsilon(X^{(i)}, X^{(j)}) - \int_{L_j} h_\varepsilon(x, u) d\nu_1(u) \right| = 0,$$

where we used the fact that $\nu_1(L_i) = n^{-1}$ for all $1 \leq i \leq n$.

Let $B^{(i)} = \{ j : (i, j) \notin G_1^+ \cup G_2^+ \}$. By the previous argument

$$|H_{n,r}(x) - H_r(x)| \leq \sum_{j \in B^{(i)}} \left| \frac{1}{n} h_\varepsilon(X^{(i)}, X^{(j)}) - \int_{L_j} h_\varepsilon(x, u) d\nu_1(u) \right| \leq \frac{|B^{(i)}|}{n}.$$ 

Note that for a fixed $i$, if $j \in B^{(i)}$ then there exists $y_0 \in L_j$ such that $r - \varepsilon \leq |x - y_0| \leq r + \varepsilon$, and hence, for all $y \in L_j$,

$$r - \varepsilon - \frac{2}{n} < |x - y_0| - |y - y_0| \leq |x - y| \leq |x - y_0| + |y - y_0| \leq r + \varepsilon + \frac{2}{n}.$$ 

In particular, for every $n \geq N_\varepsilon$, and every $x \in B_1^+$ such that $x \in L_i$, if $j \in B^{(i)}$, then $r - 2\varepsilon \leq |x - y| \leq r + 2\varepsilon$, for all $y \in L_j$. Let

$$\Omega_{2\varepsilon} = \{(x, y) \in [-1, 1]^2 : r - 2\varepsilon \leq |x - y| \leq r + 2\varepsilon\}.$$
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Since the length of each set $L_j$ is $2/n$, we obtain the bound
\[ |B^{(i)}| \leq \frac{|\Omega_{2\varepsilon,r}|}{2/n} \leq \frac{8(2-r)\varepsilon}{2/n} = 4(2-r)\varepsilon n. \]

Therefore, we have
\[ \sup_{x \in B_1} |H_{n,r}(x) - H_r(x)| \leq 4(2-r)\varepsilon. \]
Since $\varepsilon$ was arbitrary, we conclude that, almost surely, the limit is zero. ■

Finally, we are ready to prove the main result of this section.

**Lemma 6.5.** For every $0 < r < 2$, almost surely, $K_{n,r} \rightarrow K_r^1$ in the cut-norm.

**Proof.** Since the cut-norm is bounded by the $L^1$ norm, it is enough to show that, almost surely, $K_{n,r} \rightarrow K_r^1$ in $L^1$, i.e. that
\[ \int_{B^1 \times B^1} |K_{n,r}(x,y) - K_r^1(x,y)| d\nu_1(x) d\nu_1(y) \rightarrow 0, \]
as $n \rightarrow \infty$.

For $x \in B^1$, let $1 \leq i(x) \leq n$ to be the unique index such that $x \in L_i(x)$. Fix $\varepsilon > 0$, recall the notation in (36) and define $B^\varepsilon = \{(i,j) \in [n]^2 : (i,j) \notin G^1_1 \cup G^2_2 \}$. Then, almost surely, there exists $N_\varepsilon \in \mathbb{N}$ such that, for all $n \geq N_\varepsilon$,
\[ |K_{n,r}(x,y) - K_r^1(x,y)| \leq \begin{cases} \frac{6\varepsilon}{3r} & (i(x),i(y)) \in G^1_1 \\ 0 & (i(x),i(y)) \in G^2_2 \\ \frac{6}{r} & (i(x),i(y)) \in B^\varepsilon \end{cases}. \]

In fact, if $(i(x),i(y)) \in G^1_1$, then $h_r(x,y) = 1$, and from Lemma 6.3 $h_r(X^{(i)}, X^{(j)}) = 1$. Therefore,
\[ |K_{n,r}(x,y) - K_r^1(x,y)| = \frac{1}{\sqrt{H_{n,r}(x)H_{n,r}(y)}} - \frac{1}{\sqrt{H_r(x)H_r(y)}} = \frac{N_r(x,y)}{D_r^{(1)}(x,y)D_r^{(2)}(x,y)}, \]
where
\[ N_r(x,y) := |H_r(x)H_r(y) - H_{n,r}(x)H_{n,r}(y)| \]
\[ D_r^{(1)}(x,y) := \sqrt{H_r(x)H_r(y) + H_{n,r}(x)H_{n,r}(y)} \]
\[ D_r^{(2)}(x,y) := \sqrt{H_{n,r}(x)H_{n,r}(y)H_r(x)H_r(y)}. \]
and
\[ H_{n,r}(x) = \frac{1}{n} \sum_{i=1}^{n} h_r(X^{(i)}(x), X^{(i)}) \quad \text{and} \quad H_{n,r}(y) = \frac{1}{n} \sum_{q=1}^{n} h_r(X^{(q)}(y), X^{(q)}). \]

Recall from (11) and (12) that \( \frac{r}{4} \leq H_r(x) \leq 1 \) for \( x \in B^1 \). Hence, by Lemma 6.4, we have that, almost surely, for all sufficiently large \( n \),
\[ \frac{r}{4} \leq H_r(x) - \frac{r}{4} \leq H_{n,r}(x), \tag{38} \]
and
\[ \sup_{x \in B^1} |H_{n,r}(x) - H_r(x)| \leq \varepsilon. \]
Observe that \( H_{n,r}(x) \leq 1 \). Hence we conclude that
\[ D_r^{(1)}(x,y) \geq \frac{3r}{4} \quad \text{and} \quad D_r^{(2)}(x,y) \geq \frac{r^2}{8}, \]
and
\[ N_r(x, y) \leq |H_r(x)||H_r(y) - H_{n,r}(y)| + |H_{n,r}(y)||H_r(x) - H_{n,r}(x)| \leq 2\varepsilon. \]

Combining all of the above we conclude that almost surely, for all sufficiently large \( n \)
\[ |K_{n,r}(x, y) - K_r^1(x, y)| \leq \frac{2\varepsilon}{\frac{3r}{4}, \frac{r^2}{8}} \leq \frac{64\varepsilon}{3r^3}. \]

Turning to the second case in (37), note that if \( (i(x), i(y)) \in G_2^1 \), then \( h_r(x, y) = 0 \), and from Lemma 6.3 also \( h_r(X^{(i)}, X^{(j)}) = 0 \). Therefore, \( |K_{n,r}(x, y) - K_r^1(x, y)| = 0 \).

Finally, if \( (i(x), i(y)) \not\in B^2 \), then from (38), the fact that \( |h_r(x, y)| \leq 1 \) for all \( x, y \in B^1 \) and the definitions of \( K_{n,r} \) and \( K_r^1 \), it follows that
\[ |K_{n,r}(x, y) - K_r^1(x, y)| \leq |K_{n,r}(x, y)| + |K_r^1(x, y)| \leq \frac{4}{r} + \frac{2}{r} = \frac{6}{r}. \]

Combining (37) with the fact that \( (\nu_1 \times \nu_1)(L_i \times L_j) = n^{-2} \), we conclude that
\begin{equation} \int_{B^1 \times B^1} |K_{n,r}(x, y) - K_r^1(x, y)|d\nu_1(x)d\nu_1(y) \leq \frac{64\varepsilon}{3r^3} \frac{|G_2^1|}{n^2} + \frac{6|B^2|}{n^2} \leq \frac{64\varepsilon}{3r^3} + \frac{6|B^2|}{rn^2}, \tag{39} \end{equation}
where in the last inequality we used the fact that \( |G_2^1| \leq n^2 \).

Next, we bound the size of \( B^2 \). Note that, for every \( x, y \in B^1 \), if \( (i(x), i(y)) \in B^2 \), then \( r - 2\varepsilon < |x - y| < r + 2\varepsilon \), and so
\[ (\nu_1 \times \nu_1)(\{(x, y) : (i(x), i(y)) \in B^2\}) \leq \nu_1 \times \nu_1(\Omega_{2r}) = 2(2 - r)^2. \]

Since the sets \( (\nu_1 \times \nu_1)(L_i \times L_j) \) are disjoint, cover \( B^1 \times B^1 \) and \( (\nu_1 \times \nu_1)(L_i \times L_j) = n^{-2} \) for all \( 1 \leq i, j \leq n \), it follows that
\[ |B^2| \leq \frac{(\nu_1 \times \nu_1)(\{(x, y) : (i(x), i(y)) \in B^2\})}{n^2} \leq 2(2 - r)n^2. \]

Combining the last bound together with (39), we conclude that for all sufficiently large \( n \)
\begin{equation} \int_{B^1 \times B^1} |K_{n,r}(x, y) - K_r^1(x, y)|d\nu_1(x)d\nu_1(y) \leq \frac{64\varepsilon}{3r^3} + \frac{12(2 - r)}{r} \varepsilon. \end{equation}

Since \( \varepsilon > 0 \) was arbitrary, the result follows.

7. CONSTRUCTION OF THE KERNEL FOR \( K_{n,r} \) \( d = 2 \)

Our next goal is to generalize the results from the previous section to arbitrary dimension \( d \geq 2 \).

That is, to construct a sequence of kernels \( K_{n,r} : B^d \times B^d \to \mathbb{R} \) that possess the same spectra as the symmetrically normalized adjacency operators and converge in the cut norm to \( K_r^d \).

Recall that the kernel \( K_{n,r} \), defined in (37), for \( d = 1 \), uses an ordering of the points based on their (single) coordinate value. The main challenge now is how to choose a similar ordering on the points, when \( d \geq 2 \). Throughout this section, instead of considering the kernels \( K_{n,r} \) for arbitrary choice of \( n \), we only examine the case where \( n = m^d \) for some \( m \in \mathbb{N} \). This will help us devising the required ordering on the points. Later on, in Section 9, we will show how to extend the results from \( n = m^d \) to any \( n \). Finally, since the case \( d = 2 \) is considerably simpler than the general case,
we start by providing all the details for \( d = 2 \) in this section. The general construction, which is done similarly, is outlined in Section 5.

7.1. Kernel definition and spectrum. As mentioned above, the challenging part here is to define a useful ordering on the \( d \)-dimensional points \( X_1, \ldots, X_n \). Assume that \( n = m^2 \) for some \( m \in \mathbb{N} \), and let \( X_i = (X_{i,1}, X_{i,2}) \), \( i = 1, \ldots, n \), be i.i.d. uniformly distributed random variables in \( B^2 = [-1, 1]^2 \). We order the points \( X_1, \ldots, X_n \) and rename them in the following way:

**Step I:** We order \( X_1, \ldots, X_n \) according to the order statistics of the first coordinates \( X_{1,1}, \ldots, X_{n,1} \), and denote the resulting points by \( X^{(1)}, \ldots, X^{(n)} \). In other words, if \( X^{(i)} = (X^{(i)}_1, X^{(i)}_2) \) for \( i = 1, \ldots, n \), then, for \( i < j \),
\[
X^{(i)}_1 \leq X^{(j)}_1.
\]

**Step II:** We take the first \( m \) variables \( X^{(1)}, \ldots, X^{(m)} \), and re-order them according to the order statistics of the second coordinate \( X^{(1)}_2, \ldots, X^{(m)}_2 \). The resulting ordering is denoted by \( X^{(1)}, \ldots, X^{(m)} \), so that, if \( X^{(i)} = (X^{(i)}_1, X^{(i)}_2) \) for \( i = 1, \ldots, m \), then, for \( 1 \leq i < j \leq m \),
\[
X^{(i)}_2 \leq X^{(j)}_2.
\]

**Step III:** We order each of the \( m \)-tuples in a similar fashion. For \( p = 2, \ldots, m \), take \( X^{(p-1)m+1}, \ldots, X^{(pm)} \), and sort them according to the order statistics of the second coordinate \( X^{(p-1)m+1}_2, \ldots, X^{(pm)}_2 \). The resulting ordered random variables are denoted by \( X^{(p,1)}, \ldots, X^{(p,m)} \), so that \( X^{(p,i)} = (X^{(p,i)}_1, X^{(p,i)}_2) \), \( i = 1, \ldots, m \), and, for every \( 1 \leq i < j \leq m \),
\[
X^{(p,i)}_2 \leq X^{(p,j)}_2.
\]

The result is a collection of indexed variables \( (X^{(i,j)})_{i,j=1}^m \) with the property such that \( X^{1,k}_1 \leq X^{j,l}_1 \) for all \( 1 \leq i \leq j \leq m \) and \( 1 \leq k \leq l \leq m \), and \( X^{k,j}_1 \leq X^{k,j}_2 \) for all \( q \leq i, j \leq m \) and all \( 1 \leq k \leq m \). This new ordering of the points will play a crucial role for the construction of \( K_n \).

Recall that in order to define the kernel \( K_{n,r} \) for \( d = 1 \) we divided \( B^1 \) into the intervals \( L_i \), (cf. (31)) and that the bulk of the convergence proof relied on the fact that \( X^{(i)} \in L_i \) for all \( i \) (almost surely for large enough \( n \)). For \( d = 2 \), we use a similar construction, where we divide \( B^2 \) into boxes \( L^n_{p,q} \), and show that almost surely, for \( n \) large enough, we have \( X^{(p,q)} \in L^n_{p,q} \) for all \( p, q \). More concretely, recall the definition of \( L^n_2 \) in (31), and, for every \( 1 \leq p, q \leq m \), define \( L^n_{p,q} = L^n_{p,q} \) by
\[
L^n_{p,q} = L^n_p \times L^n_q \subset B^2.
\]

Note that \( B^2 = \bigsqcup_{p,q=1}^m L^n_{p,q} \), and \( \nu_2(L^n_{p,q}) = m^{-2} = n^{-1} \).

The kernels defined in this subsection, are similar to the ones from Section 6 where instead of \( X^{(i)} \) and \( L^n_i \) we use \( X^{(p,q)} \) and \( L^n_{p,q} \).

Recall that \( n = m^2 \), and define a sequence of random functions \( h_{n,r} : B^2 \times B^2 \to \mathbb{R} \) by
\[
h_{n,r}(x, y) := m \sum_{p, q, p' \neq q'} \nu_r(X^{(p,q)}, X^{(p',q')}) 1_{L^n_{p,q}}(x) 1_{L^n_{p',q'}}(y).
\]

Furthermore, for \( x \in B^2 \), denote
\[
H_{n,r}(x, y) = \int_{B^2} h_{n,r}(x, y) d\nu_2(y) \quad \text{and} \quad h_{n,r}(x) = \frac{1}{n} \sum_{p, q=1}^m 1_{L^n_{p,q}}(x) \sum_{p', q'=1}^m \nu_r(X^{(p,q)}, X^{(p',q')}).
\]

where we used the fact that \( \nu_2(L^n_{p',q'}) = n^{-1} \).

Next, define a sequence of random kernels \( K_{n,r} : B^2 \times B^2 \to \mathbb{R} \) by
\[
K_{n,r}(x, y) := \frac{h_{n,r}(x, y)}{H_{n,r}(x) H_{n,r}(y)},
\]
or, equivalently,
\[
K_{n,r}(x, y) = \sum_{p, q, p', q'=1}^m \frac{nh_r(X^{(p,q)}, X^{(p',q')}) 1_{L^n_{p,q}}(x) 1_{L^n_{p',q'}}(y)}{\sum_{a_1, a_2, a_3, a_4=1}^m h_r(X^{(p,q)}, X^{(a_1, a_2)}) h_r(X^{(p',q')}, X^{(a_3, a_4)})}.
\]
Finally, let \( K_{n,r} : L^2(B^2, \nu_2) \to L^2(B^2, \nu_2) \) be the Hilbert-Schmidt kernel operator corresponding to the kernel \( K_{n,r} \), i.e.

\[
K_{n,r}f(x) = \int_{B^2} K_{n,r}(x,y)f(y)d\nu_2(y).
\]  

(43)

The following is the 2-dimensional analogue of Lemma 6.1

**Lemma 7.1.** Suppose that \( d = 2 \) and \( 0 < r < 2 \), and let \( K_{n,r} \) be as defined in (43). Then \( \text{spec}(K_{n,r}) = \text{spec}(W_{n,r}) \), where \( W_{n,r} \) is as defined in (5).

**Proof.** The proof is very similar to that of Lemma 6.1 and so we only highlight the differences.

Let \( \tilde{A}_n \) and \( \tilde{W}_n \) be the matrices describing the graph generated by the sorted points \( X^{(p,q)} \), where we use the lexicographic ordering on the pairs \( (p,q) \) as described above. Using a slight abuse of notation, we use quadruplets \( (p,q,p',q') \), as entry indices for the matrices \( \tilde{A}_n, \tilde{W}_n \). Since we only changed the order of the original vertices \( X_1, \ldots, X_n \), we have \( \text{spec}(W_n) = \text{spec}(W_n) \).

For \( 1 \leq p, q, p', q' \leq m \), denote

\[
a_{(p,q),(p',q')} = (\tilde{A}_n)_{(p,q),(p',q')} = h_t(X^{(p,q)}, X^{(p',q')}),
\]

and

\[
d_{(p,q)} = \sum_{p',q'=1}^m a_{(p,q),(p',q')}.
\]

Then the \( (p,q,p',q') \)-th entry of \( \tilde{W}_n \) can be written as

\[
w_{(p,q),(p',q')} = \frac{a_{(p,q),(p',q')}}{\sqrt{d_{(p,q)d_{(p',q')}}}}.
\]

Using this notation with (42), we can also write

\[
K_{n,r}(x,y) = n \sum_{p,q,p',q'=1}^m w_{(p,q),(p',q')} 1_{L_{p,q}}(x)1_{L_{p',q'}}(y).
\]

The rest of the proof is identical to that of Lemma 6.1 \( \square \)

7.2. **Concentration statements.** Similarly to the case \( d = 1 \), we want to show that \( X^{(p,q)} \in L^n_{p,q} \).

**Lemma 7.2.** Let \( (X^{(p,q)})_{p,q=1}^m \) be the ordering defined above. Then, almost surely, there exists \( N \in \mathbb{N} \) such that, for all \( n \geq N \),

\[
\sup_{1 \leq p,q \leq m} \left\| X^{(p,q)} - \mathbb{E}[X^{(p,q)}] \right\|_\infty \leq n^{-1/6},
\]

where \( \mathbb{E}[X^{(p,q)}] \in L^n_{p,q} \).

**Proof.** The bound in this lemma can be obtained using [28] equation (1.1). See [23] Section 4 for the proof, which in fact gives a better bound. For the sake completeness we provide an alternative proof of this bound using order statistics arguments in Appendix A. The remainder of the proof is dedicated to show that, indeed, \( \mathbb{E}[X^{(p,q)}] \in L^n_{p,q} \).

Denote by \( X_1 = (X_1^{(1)},\ldots,X_1^{(m)}) \) the vector of first coordinates of all points, and suppose that \( X_1 \) is given. In this case, by Step II and Step III, for all \( p = 1, \ldots, m \), the values of \( X_1^{(p-1)}, X_1^{(p)}, \ldots, X_1^{(m)} \) are the same as those in the sequence \( X_2^{((p-1)m+1)}, \ldots, X_2^{(pm)} \), under a random permutation (since they are ordered according to the values of the second coordinates \( X_2^{((p-1)m+1)}, \ldots, X_2^{(pm)} \), which are i.i.d. and independent of the first coordinate). Therefore,

\[
\mathbb{E}[X_1^{(p,q)} \mid X_1] = \frac{1}{m} \sum_{s=1}^m X_1^{((p-1)m+s)}.
\]

(44)
Next, from Step I we have that $X_1^{(1)}, \ldots, X_1^{(n)}$ are the order statistics of $n = m^2$ i.i.d. uniformly distributed random variables in $[-1, 1]$. Thus, using (45),

$$
\mathbb{E}[X_1^{(p,q)}] = \frac{1}{m} \sum_{s=1}^{m} \frac{2(p-1)m + s}{n+1} - 1
$$

$$=-1 + \frac{2pm - m + 1}{n+1},
$$

and it follows that $\mathbb{E}[X_1^{(p,q)}] \in L_{p,m}$.

Next, fix $p$, and notice that given $X_1$ we have that $X_2^{(p,1)} \leq \cdots \leq X_2^{(p,m)}$ are the order statistics of $m$ i.i.d. random variables, uniformly distributed in $[-1, 1]$. Therefore, using (45) again, gives

$$
\mathbb{E}[X_2^{(p,q)} | X_1] = -1 + \frac{2q}{m+1},
$$

which implies that $\mathbb{E}[X_2^{(p,q)}] = -1 + \frac{2q}{m+1}$, $L_{q,m}$. To conclude, we showed that for all $1 \leq p, q \leq m$,

$$(\mathbb{E}[X_1^{(p,q)}], \mathbb{E}[X_2^{(p,q)}]) \in L_p^m \times L_q^m = L_{p,q}^n,$$

as required.

### 7.3. The convergence of $K_{n,r}$

In this section we show that $K_{n,r}$ converges to $K_2^r$ in the cut-norm, almost surely, as $n \to \infty$. The proofs leading to this statement follow steps similar to those in Section 6.3 and so we only highlight the main differences.

Fix $0 < r < 2$ and $\epsilon > 0$. Similarly to (26), we start by defining the sets

$$
G_1^r := \{(p,q), (p',q') \} : |x - y| < r - \epsilon, \text{ for all } (x, y) \in L_{p,q} \times L_{p',q'}^r;
$$

$$
G_2^r := \{(p,q), (p',q') \} : |x - y| > r + \epsilon, \text{ for all } (x, y) \in L_{p,q} \times L_{p',q'}^r.
$$

We start by proving the analogue of Lemma 6.3.

**Lemma 7.3.** Almost surely, there exists (random) $N_\epsilon > 0$ such that, for all $n \geq N_\epsilon$, the following two statements are true:

1. If $((p,q), (p',q')) \in G_1^r$, then $h_r(X^{(p,q)}, X^{(p',q')}) = 1$.
2. If $((p,q), (p',q')) \in G_2^r$, then $h_r(X^{(p,q)}, X^{(p',q')}) = 0$.

**Proof.** For $1 \leq p, q, p', q' \leq m$,

$$
\|X^{(p,q)} - X^{(p',q')}\|_\infty
$$

$$\leq \|X^{(p,q)} - \mathbb{E}[X^{(p,q)}]\|_\infty + \|\mathbb{E}[X^{(p,q)}] - \mathbb{E}[X^{(p',q')}]\|_\infty + \|\mathbb{E}[X^{(p',q')} - X^{(p',q')}]\|_\infty.
$$

Lemma 7.2 implies that $\mathbb{E}[X^{(p,q)}] \in L_{p,q}$ and $\mathbb{E}[X^{(p',q')}] \in L_{p',q'}$. Thus, if $((p,q), (p',q')) \in G_1^r$, then

$$
\|\mathbb{E}[X^{(p,q)}] - \mathbb{E}[X^{(p',q')}]\|_\infty < r - \epsilon.
$$

Lemma 7.2 also implies that a.s. there exists $N_1$ such that, for $n \geq N_1$, and for all $1 \leq p, q \leq m$,

$$
\|X^{(p,q)} - \mathbb{E}[X^{(p,q)}]\|_\infty \leq n^{-1/6}.
$$

Choosing $N_1$ such that $N_1^{-1/6} < \epsilon/2$ and combining the last two estimates, we have that, for $n \geq \max\{N_0, N_1\}$,

$$
\sup_{((p,q), (p',q')) \in G_1^r} \|X^{(p,q)} - X^{(p',q')}\|_\infty < r,
$$

implying that $h_r(X^{(p,q)}, X^{(p',q')}) = 1$ for all $((p,q), (p',q')) \in G_1^r$.

A similar computation shows that $h_r(X^{(p,q)}, X^{(p',q')}) = 0$ for all $((p,q), (p',q')) \in G_2^r$, thus completing the proof with $N_\epsilon = \max\{N_0, N_1\}$.

Next, we prove a result analogous to Lemma 6.4.

**Lemma 7.4.** Let $H_r$ and $H_{n,r}$ be as defined in (10) and (11) respectively. Then, almost surely

$$
\lim_{n \to \infty} \sup_{x \in B_2} |H_{n,r}(x) - H_r(x)| = 0.
$$

**Proof.** The proof here is identical to that of Lemma 6.4.

\[\square\]
Finally, we prove the main result of this section.

Lemma 7.5. Let $K_{n,r}$ be as defined above. Then $K_{n,r} \to K^2_r$, with respect to the cut-norm, almost surely, as $n \to \infty$.

Proof. The proof is similar to that of Lemma 6.5 and again we highlight only the necessary changes. Fix $0 < r < 2$ and $\varepsilon > 0$, and define

$$B^c = \{(p, q), (p', q') \in [m]^4 : ((p, q), (p', q')) \notin G^1_r \cup G^2_r\}.$$ 

For $x \in B^2$, define $p(x), q(x)$ to be the unique integers in $[m]$ such that $x \in L^m_{p(x), q(x)}$. A similar argument to the one in the one-dimensional case shows that

$$|\mathcal{K}_{n,r}(x, y) - \mathcal{K}^2_r(x, y)| \leq \begin{cases} \frac{512\varepsilon}{3r^6} & ((p(x), q(x)), (p(y), q(y))) \in G^1_r, \\ 0 & ((p(x), q(x)), (p(y), q(y))) \in G^2_r, \\ \frac{12}{r^2} & ((p(x), q(x)), (p(y), q(y))) \in B^c \end{cases},$$

and therefore

$$\int_{B^2 \times B^2} |\mathcal{K}_{n,r}(x, y) - \mathcal{K}^2_r(x, y)| dv_2(x) dv_2(y) \leq \frac{512\varepsilon}{3r^6} \cdot \frac{|G^1_r|}{n^2} + \frac{12}{r^2} \cdot \frac{|B^c|}{n^2} \leq \frac{512\varepsilon}{3r^6} + \frac{12}{r^2} \cdot \frac{|B^c|}{n^2}.$$  

Thus, it remains to bound the size of $B^c$. Note that if $(x, y) \in B^2 \times B^2$ is a pair of points such that $((p(x), q(x)), (p(y), q(y))) \in B^c$, then $r - 2\varepsilon \leq \|x - y\|_\infty \leq r + 2\varepsilon$, and therefore, either

$$r - 2\varepsilon < |x_1 - y_1| < r + 2\varepsilon$$

or $r - 2\varepsilon < |x_2 - y_2| < r + 2\varepsilon$. Hence,

$$\nu_2 \times \nu_2 \left(\{(x, y) \in B^2 \times B^2 : ((p(x), q(x)), (p(y), q(y))) \in B^c\}\right)$$

$$\leq \nu_2 \times \nu_2 \left(\{(x, y) \in B^2 \times B^2 : r - 2\varepsilon < |x_1 - y_1| < r + 2\varepsilon\}\right) + \nu_2 \times \nu_2 \left(\{(x, y) \in B^2 \times B^2 : r - 2\varepsilon < |x_2 - y_2| < r + 2\varepsilon\}\right)$$

$$\leq \left(2(2 - r)\varepsilon + 2(2 + r)\varepsilon\right) = 4(2 - r)\varepsilon.$$

Since $(L^m_{p,q} \times L^m_{p',q'})_{p,q,p',q'=1}^m$ are disjoint, cover $B^2 \times B^2$ and each one satisfies $\nu_2 \times \nu_2 (L^m_{p,q} \times L^m_{p',q'}) = n^{-2}$, it follows that

$$|B^c| \leq \frac{\nu_2 \times \nu_2 (B^c)}{n^2} = 4(2 - r)\varepsilon n^2.$$ 

Substituting the last bound into (48) shows that, for all $n \geq N_{\varepsilon}$,

$$\int_{B^2 \times B^2} |\mathcal{K}_{n,r}(x, y) - \mathcal{K}^2_r(x, y)| dv_2(x) dv_2(y) \leq \frac{512\varepsilon}{3r^6} + \frac{48(2 - r)\varepsilon}{r^2},$$

and, since $\varepsilon > 0$ was arbitrary, the result follows.

8. Outline of the construction of $K_{n,r}$ for general $d$.

In this section we show how to construct the kernel $K_{n,r}$ for $d \geq 3$ and for $n = m^d$ for some $m \in \mathbb{N}$. Later, we will show how to prove the results for arbitrary values of $n \in \mathbb{N}$. The construction as well as the proofs are similar to the case $d = 2$, but a bit more technically involved. Therefore, in this section we only wish to provide an outline for the general case, without repeating all the details and proofs.

Let $X_1, \ldots, X_n$ be $n$ i.i.d. points, uniform in $B^d$, and denote $X_1 = (X_{1,1}, \ldots, X_{1,d})$. As in the $d = 2$ case, the tricky part here is to provide a useful ordering on the vertices. This is done in a sequence of $d$ steps as follows.

Step 1: Order $X_1, \ldots, X_n$ according to the first coordinate, and denote the result by $X^{(1)} \ldots X^{(n)}$. Thus, if $X^{(i)} = (X^{(i)}_1, \ldots, X^{(i)}_d)$ then for all $1 \leq i < j \leq n$,

$$X^{(i)}_1 \leq X^{(j)}_1.$$

Step 2: Take the variables $X^{(1)}, \ldots, X^{(m^d-1)}$ and order them using the second coordinates $X^{(1)}_2, \ldots, X^{(m^d-1)}_2$. Similarly, for all $i_1 = 1, \ldots, m$, take the $i_1$-th collection of the $m^{d-1}$ variables
Let $X((i_1-1)m^{d-1}+1), \ldots, X((i_m m^{d-1})$, and order them according to the values in the second coordinate. Denote the result $X(i_1 \ldots i_2)$, where $1 \leq i_1 \leq m$, and $1 \leq i_2 \leq m^{d-1}$.

In the end of this sorting process, from the first step we have that for all $1 \leq i_1 < i_1', \leq m$, and for all $1 \leq i_2, i_2' \leq m^{d-1}$,
\[ X^{(i_1,i_2)}_1 \leq X^{(i_1',i_2)}_1. \]

In addition, if we fix $1 \leq i_1 \leq m$, then, from the second step, for all $1 \leq i_2 < i_2' \leq m^{d-1}$,
\[ X^{(i_1,i_2)}_2 \leq X^{(i_1,i_2')}_2. \]

**Step 3:** Take the variables $X^{(1,1)}, \ldots, X^{(1,m^{d-2}}$, and order them according to the third coordinates $X^{(1,1)}, \ldots, X^{(1,m^{d-2}}$. Similarly, for all $1 \leq i_1, i_2 \leq m$, take the collection of the $m^{d-2}$ variables $X^{(i_1,i_2-1)m^{d-2}+1)}, \ldots, X^{(i_1,i_2 m^{d-2}}$ and order them according to the third coordinate. Denote the result $X(i_1,i_2,i_3)$, for $1 \leq i_1, i_2 \leq m$, and $1 \leq i_3 \leq m^{d-2}$.

In the end of this sorting process, from the first step we have that for all $1 \leq i_1 < i_1', \leq m$, for all $1 \leq i_2, i_2' \leq m$, and for all $1 \leq i_3, i_3' \leq m^{d-2}$, we have
\[ X^{(i_1,i_2,i_3)}_1 \leq X^{(i_1',i_2',i_3)}_1. \]

Next, fixing $1 \leq i_1 \leq m$, then from the second step for all $1 \leq i_2 < i_2' \leq m$, and for all $1 \leq i_3, i_3' \leq m^{d-2}$, we have
\[ X^{(i_1,i_2,i_3)}_2 \leq X^{(i_1,i_2',i_3)}_2. \]

Finally, fixing $1 \leq i_1, i_2 \leq m$, then from the third step for all $1 \leq i_3 < i_3' \leq m^{d-2}$ we have
\[ X^{(i_1,i_2,i_3)}_3 \leq X^{(i_1,i_2,i_3')}_3. \]

**Step k:** We keep performing these sorting procedure in a similar way. For the $k$-th step, for every choice of $1 \leq i_1, \ldots, i_k-1 \leq m$, we take collections of the $m^{d-k+1}$ variables from the previous step, i.e. $X^{(i_1, \ldots, i_{k-2},(i_{k-1}-1)m^{d-k+1}+1)}, \ldots, X^{(i_1, \ldots, i_{k-2},i_k-1m^{d-k+1}}$, and order them according to the $k$-th coordinate. The result is denoted $X(i_1, \ldots, i_k)$. This will be done for all $k \leq d$.

Concluding this procedure, we take the $d$-dimensional variables $X_1, \ldots, X_n$ and order them in a sequence of $d$ steps, coordinate by coordinate, until we reach the sorted sequences $X^{(i_1, \ldots, i_d)}$, where the indices are $1 \leq i_1, \ldots, i_d \leq m$. For brevity we will use $i = (i_1, \ldots, i_d)$, and $X(i) = X^{(i_1, \ldots, i_d)}$. We also define $1 = (1, \ldots, 1)$ and $m = (m, \ldots, m)$, and we use ‘$\leq$’ to denote lexicographic order.

Similarly to the $d = 2$ case, our next step is to define a useful partition of $B^d$. Suppose that $i = (i_1, \ldots, i_d)$ is such that $1 \leq i \leq m$. Using the definition of $L_{i,n}$ [7], we define
\[ L_{i,n} = L_{i_1,m} \times L_{i_2,m} \times \cdots \times L_{i_d,m} \subset B^d. \]

In this case we have that $B^d = \bigcup_{j=1}^m L_{j,n}$, and $n = (L_{i,n}) = 1/n$. As before, we denote $L_i = L_{i,n}$.

Next, we define the kernels, for $x, y \in B^d$, as
\[ K_{n,r}(x,y) = \frac{h_{n,r}(x,y)}{\sqrt{H_{n,r}(x)H_{n,r}(y)}}, \]
where
\[ h_{n,r}(x,y) = \sum_{i,j=1}^m h_r(X^{(i)}, X^{(j)})1_{L_i}(x)1_{L_j}(y), \]
and
\[ H_{n,r}(x) = \int_{B^d} h_{n,r}(x,y)\,d\nu_d(y) = \frac{1}{n} \sum_{i} 1_{L_i}(x) \sum_{p} h_r(X^{(i)}, X^{(p)}). \]

To prove that $K_{n,r} \rightarrow K^d$ we will have to prove lemmas corresponding to those in Sections [6] and [7]. We will present the lemmas and discuss the needed adjustments for the proofs.

**Lemma 8.1.** Let $K_{n,r}$ be the Hilbert-Schmidt kernel operator on $L^2(B^d, \nu_d)$ corresponding to $K_{n,r}$ defined above. Then spec$(K_{n,r}) = \text{spec}(W_{n,r})$.

**Proof.** The proof here is identical to that of Lemma [7].
Lemma 8.2. Let $X^{(i)}$ be as defined above. Then, almost surely, there exists $N > 0$ such that, for all $n \geq N$, we have,
\[ \sup_{1 \leq i \leq m} \left| X^{(i)} - \mathbb{E}[X^{(i)}] \right| \leq \frac{1}{n^{1/3d}}, \]
where \[ \mathbb{E}[X^{(i)}] \in L_i. \]

Proof. This bound can be proved using [28, Theorem 1.1]. But for completeness we give a proof in Appendix A. We will explain the steps needed to bound \( \mathbb{E}[X^{(i)}] \).

With, as before, \( i = (i_1, \ldots, i_d) \), for every \( 1 \leq k \leq d \) we need to show that
\[ \mathbb{E}[X^{(i)}] \in L_{i_k, m}. \]

Denote by \( X_k \) the collection of all variables \( \{X_{i,j}\}_{1 \leq i \leq n, 1 \leq j \leq k} \). Notice that our sorting algorithm is such that given \( X_k \) we can apply steps 1 through \( k \) above and thus the values of \( X_j^{(i_1, \ldots, i_k)} \) for all \( 1 \leq i_1, \ldots, i_{k-1} \leq m, 1 \leq i_k \leq m^{d-k+1} \), and \( 1 \leq j \leq d \) are known.

Next, fix \( 1 \leq k \leq d \), and \( 1 \leq i_1, \ldots, i_k \leq m \). Recall that given \( X_k \), the set of \( m^{d-k} \) variables \( \{X_k^{(i_1, \ldots, i_{k-1}, i_k m^{d-k})}, \ldots, X_k^{(i_1, \ldots, i_{k-1}, i_k m^{d-k})}\} \) by a sequence of random permutations (given in steps \( k + 1, \ldots, d \)) where all the permutations are determined by independent sequences of i.i.d. variables. Therefore, each individual variable \( X_k^{(i_1, \ldots, i_{k-1}, i_k m^{d-k})} \) can take the value of any of the variables \( X_k^{(i_1, \ldots, i_{k-1}, (i_k - 1)m^{d-k} + 1)} \) for \( j = 1, \ldots, m^{d-k} \), with equal probability. Thus,
\[ \mathbb{E}[X_k^{(i)}] = \mathbb{E}\left[ \mathbb{E}[X_k^{(i)} \mid X_k]\right] = \frac{1}{m^{d-k}} \sum_{j=1}^{m^{d-k}} \mathbb{E}[X_k^{(i_1, \ldots, i_{k-1}, (i_k - 1)m^{d-k} + j)}]. \] (52)

Next, recall that, as described in step \( k \), \( X_k^{(i_1, \ldots, i_{k-1}, 1)}, \ldots, X_k^{(i_1, \ldots, i_{k-1}, m^{d-k} + 1)} \) are the order statistics of \( m^{d-k+1} \) i.i.d. variables, uniform in \([-1, 1]\). Therefore,
\[ \mathbb{E}\left[ X_k^{(i_1, \ldots, i_{k-1}, j)}\right] = -1 + \frac{2j}{m^{d-k+1} + 1}. \]

Putting this into (52), we have
\[ \mathbb{E}[X_k^{(i)}] = -1 + \frac{2i_k m^{d-k} - m^{d-k} + 1}{m^{d-k+1} + 1}. \]

All that remains to verify is that the last value is indeed in \( L_{i_k, m} \), and this easy step completes the proof.

For the next step, take \( G_1^n \) and \( G_2^n \) as in (46).

Lemma 8.3. Almost surely, there exists (random) \( N_\varepsilon > 0 \) such that, for all \( n \geq N_\varepsilon \), the following two statements are true:

1. If \( (i, j) \in G_1^n \), then \( h_r(X^{(i)}, X^{(j)}) = 1 \).
2. If \( (i, j) \in G_2^n \), then \( h_r(X^{(i)}, X^{(j)}) = 0 \).

Proof. Using Lemma 8.2, the proof is identical to that of Lemma 7.3.

Lemma 8.4. Let \( H_r \) and \( H_{n,r} \) be as defined above in (10) and (51). Then, almost surely,
\[ \lim_{n \to \infty} \sup_{x \in B^d} |H_{n,r}(x) - H_r(x)| = 0. \]

Proof. The proof here is identical to that of Lemma 6.3.

Lemma 8.5. Let \( 0 < r < 2 \), and \( K_{n,r} \) be as defined above. Then \( K_{n,r} \to K_r^\alpha \) with respect to the cut-norm, almost surely, as \( n \to \infty \).
Proof. The proof is similar to that of Lemma 7.5 and we will only highlight the required updates. We use similar notation as in the proof of Lemma 7.5. Therefore, (17) is replaced by

$$|K_{n,r}(x,y) - K_r^2(x,y)| \leq \begin{cases} \frac{2^d(d+1)\varepsilon}{3r^{3d}} & ((\hat{i}(x), \hat{i}(y)) \in \mathcal{G}_r^2, (\hat{i}(x), \hat{i}(y)) \in \mathcal{G}_r^2, \ (\hat{i}(x), \hat{i}(y)) \in B^2 \end{cases}$$

and so

$$\int_{B^2 \times B^2} |K_{n,r}(x,y) - K_r^2(x,y)| dv_2(x) dv_2(y) \leq \frac{2^d(d+1)\varepsilon}{3r^{3d}} + \frac{|\mathcal{G}_r^2|}{n^2} + \frac{3 \cdot 2d}{r^d} \cdot \frac{|B^2|}{n^2}$$

In addition, we have, for \( \varepsilon > 0 \),

$$|B^2| \leq 2d(2 - r)\varepsilon n^2. $$

Substituting the last bound into (54) shows that, for all \( n \geq N_\varepsilon \),

$$\int_{B^2 \times B^2} |K_{n,r}(x,y) - K_r^2(x,y)| dv_2(x) dv_2(y) \leq \frac{2^d(d+1)\varepsilon}{3r^{3d}} + \frac{6d(2 - r)2^d}{r^d} \varepsilon.$$

Observe that in the special case \( d = 2 \) we obtain the bounds derived in the proof of Lemma 7.5. Since \( \varepsilon > 0 \) is arbitrary, we are done.

9. Proofs of Theorems 2.2 and 2.3

In this section we finally complete the proofs of Theorems 2.2 and 2.3 using the eigenvalue interlacing theorem, see Theorem 4.3.28 in [18].

Theorem 9.1 (Eigenvalue Interlacing Theorem). Suppose \( A \) is a real symmetric \( n \times n \) matrix. Let \( B \) be a \( m \times m \) principal submatrix (obtained by deleting both the \( i \)-th row and the \( i \)-th column for some values of \( i \)). Suppose \( A \) has eigenvalues \( \alpha_1 \geq \cdots \geq \alpha_n \) and \( B \) has eigenvalues \( \beta_1 \geq \cdots \geq \beta_m \). Then, for every \( 1 \leq k \leq m \)

$$\alpha_{k+n-m} \leq \beta_k \leq \alpha_k.$$

Proof of Theorem 2.2. The case of \( d = 1 \) follows from the discussion in Sections 5 and 6. In fact, from Lemma 6.1, we have that \( \text{spec}(W_{n, r}) = \text{spec}(K_{n, r}) \) from Lemma 6.5 we have that \( K_{n, r} \rightarrow K_r^1 \) almost surely in cut norm, and, from Lemma 5.7 we have that, with the exception of the eigenvalues \( 1/2 \) and \( 1 \), all eigenvalues of \( K_r^1 \) lie in \((-0.3, 0.3)\). Finally, applying Lemma 5.1 proves the result.

For \( d \geq 2 \), using Lemmas 8.1, 8.5, 5.7, and Lemma 3.1 implies the result for all \( n = m^d \). We are left to prove that the statement holds for any sequence of \( n \).

Suppose that \( n > 0 \) is not in the form \( n = m^d \). Then there exists \( m > 0 \) such that \( (m - 1)^d < n < m^d \). Let \( \lambda_{1,n} \geq \cdots \geq \lambda_{n,n} \) be the eigenvalues of \( W_{n,1} \). Then Theorem 9.1 implies that

$$\{ j : |\lambda_{j,(m-1)^d}| > \lambda \} \subseteq \{ j : |\lambda_{j,n}| > \lambda \} \subseteq \{ j : |\lambda_{j,m^d}| > \lambda \}.$$

Taking \( m \to \infty \), and using the convergence of the eigenvalues for \( n = m^d \), concludes the proof.

Proof of Theorem 2.3. Combine Lemmas 5.11, 8.1, 8.5 and 3.1 as in the proof of Theorem 2.2.

Proof of Theorem 2.4. Combine Lemmas 5.12, 8.1, 8.5 and 3.1 as in the proof of Theorem 2.2.

Proof of Corollary 2.3. Recall that \( \gamma_2^{(n)} = 1 - \lambda_2^{(n)} \), where \( \lambda_2^{(n)} \) is the second largest eigenvalue of \( W_{n,r} \). The proof is in three parts, one for each of the claims of the Corollary.

Proof of first claim. Let \( r = 1 \). Theorem 2.2 (2) implies that, for every \( \epsilon > 0 \), almost surely there exists \( N > 0 \) such that, for all \( n \geq N \),

$$\lambda_2^{(n)} \in \left( \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon \right).$$

Hence the result, as \( \epsilon \) is arbitrary and \( \gamma_2^{(n)} = 1 - \lambda_2^{(n)} \).
Proof of second claim. Let \( r \in (1, 2) \). Let \( \lambda_2 \) be the second largest eigenvalue of \( \mathcal{K}_r^d \). Lemmas 5.11 and 5.13 imply that there exists \( \epsilon > 0 \) such that
\[
\epsilon < \lambda_2 < \frac{1}{2} - \epsilon.
\]
Therefore, by Lemmas 8.1 5.5 and 3.1, almost surely there exists \( N > 0 \) such that, for all \( n \geq N \),
\[
\frac{\epsilon}{2} \leq \lambda_2^{(n)} \leq \frac{1}{2} - \frac{\epsilon}{2}.
\]
(55)
This implies that, for \( n \geq N \),
\[
\frac{1}{2} + \frac{\epsilon}{2} \leq \gamma_2^{(n)} \leq 1 - \frac{\epsilon}{2}.
\]
Hence the result.

Proof of third claim. Let \( r \in (0, 1) \). Lemmas 5.12 and 5.13 imply that there exists \( \epsilon > 0 \) such that
\[
\frac{1}{2} + \epsilon < \lambda_2 < 1 - \epsilon.
\]
By Lemmas 8.1 5.5 and 3.1 almost surely there exists \( N_3 > 0 \) such that, for all \( n > N \),
\[
\frac{1}{2} + \frac{\epsilon}{2} \leq \lambda_2^{(n)} \leq 1 - \frac{\epsilon}{2},
\]
(56)
which implies that
\[
\frac{\epsilon}{2} \leq \gamma_2^{(n)} \leq \frac{1}{2} - \frac{\epsilon}{2}.
\]
Hence the result. \( \square \)

10. Conclusion

We have shown that, almost surely, the second largest eigenvalue of \( W_{n,r} \) is larger (smaller) than \( 1/2 \) if \( 0 < r < 1 \) (respectively, \( 1 < r < 2 \)) for all large \( n \). We also proved that, if \( r = 1 \), then \( W_{n,r} \) has at least \( \binom{d}{2} \) many eigenvalues around \( 1/2^k \). In Section 5, in order to study the eigenvalues of \( W_{n,r} \), we studied the eigenvalues of the limiting operator \( \mathcal{K}_r^d \). We proved that \( \mathcal{K}_r^d \) is a self-adjoint and compact operator with the largest eigenvalue \( 1 \), and the second largest eigenvalue is larger (smaller) than \( 1/2 \) for \( 0 < r < 1 \) (respectively, \( 1 < r < 2 \)). We conjecture that the second largest eigenvalue of \( \mathcal{K}_r^d \) is both continuous and monotonically decreasing in \( 0 < r < 2 \).

In the above discussion two vertices in the graph are connected if they lie in a cube of side-length \( r \). We note that our results can be extended to the case where the cube is replaced by general box. More precisely, let \( r_1, \ldots, r_d \in (0, 2) \). Define, for \( x, y \in [-1, 1]^d \),
\[
h_{r_1,\ldots,r_d}(x,y) = \prod_{i=1}^d 1(|x_i-y_i| \leq r_i).
\]
Let \( G_n \) be a random graph with \( n \) points \( \{X_1, \ldots, X_n\} \), where \( X_1, \ldots, X_n \) are i.i.d. uniformly distributed random variables in \([-1, 1]^d\), such that two vertices \( X_i, X_j \) are connected if, and only if, \( h_{r_1,\ldots,r_d}(X_i,X_j) = 1 \). Let \( A_n = (a_{ij}) = (h_{r_1,\ldots,r_d}(X_i,X_j)) \) be the adjacency matrix of \( G_n \). Define
\[
W_n = D_n^{-\frac{1}{2}} A_n D_n^{-\frac{1}{2}},
\]
where \( D_n = \text{diag}(d_1, \ldots, d_n) \) with \( d_i = \sum_{k=1}^{n} a_{ik} \). Then it can be shown that the second largest eigenvalue of \( W_n \) is almost surely smaller (larger) than \( 1/2 \) when \( r_1, \ldots, r_d \in (1, 2) \) (respectively, if \( r_i \in (0, 1) \) for some \( 1 \leq i \leq d \)) for all large \( n \). In order to prove this claim one needs to study the eigenvalues of the integral kernel operator \( \mathcal{K}_{r_1,\ldots,r_d} \) in \( L^2([-1, 1]^d, \nu_d) \) with kernel
\[
\mathcal{K}_{r_1,\ldots,r_d}(x,y) = \prod_{i=1}^d \frac{h_{r_i}(x_i,y_i)}{\sqrt{H_{r_i}(x_i)H_{r_i}(y_i)}}, \quad x, y \in [-1, 1]^d.
\]
Let \((\lambda_{i,k})_{k \in \mathbb{N}}, \) for \(i = 1, \ldots, d,\) be the eigenvalues of \(K_{r_i},\) where \(K_{r_i}\) is an integral kernel operator in \(L^2([-1,1],\mu)\) with respect to the kernel
\[
K_{r_i}(x,y) = \frac{h_{r_i}(x,y)}{\sqrt{H_{r_i}(x)H_{r_i}(y)}}, \quad x, y \in [-1,1].
\]

Then, following the proof of Lemma 5.7 it can be shown that \((\lambda_{1,k}, \ldots, \lambda_{d,k})_{k \in \mathbb{N}}\) are the eigenvalues of \(K_{r_1}, \ldots, r_d.\) As a consequence, from Lemma 5.12 it follows that the second largest eigenvalue of \(K_{r_1}, \ldots, r_d\) is larger than 1/2 if \(r_i \in (0,1)\) for some \(i \in \{1, \ldots, d\},\) and from Lemma 5.8 it follows that all the eigenvalues (except 1) of \(K_{r_1}, \ldots, r_d\) are strictly smaller than 1/2 when \(r_1, \ldots, r_d \in (1,2).\)

Finally, we considered here only the \(L^\infty\) norm, which made the details of the calculations easier. We conjecture that qualitatively similar results should be true if we replace the \(L^\infty\)-norm by other norms, including \(L^2.\)

**Appendix A. Proofs of Lemmas 6.2, 7.2, 8.2**

Let \(U_1, \ldots, U_n\) be i.i.d. uniformly distributed random variables in \([0,1],\) and let \(U^{(1)}, \ldots, U^{(n)}\) be their order statistics, i.e., \(U^{(1)} \leq \cdots \leq U^{(n)}\). It is well known that the \(k\)-th order statistics is a beta random variable, or more precisely,
\[
U^{(k)} \sim \text{Beta}(k,n+1-k),
\]
which implies that
\[
E[U^{(k)}] = \frac{k}{n+1}.
\]

In our situation, we have \(X_1, \ldots, X_n\) i.i.d. uniformly distributed in \([-1,1].\) Since we can write \(X_i = 2U_i - 1,\) with \(U_i\) as above, then the order statistics \(X^{(1)} \leq \cdots \leq X^{(n)}\) can also be written as \(X^{(k)} = 2U^{(k)} - 1.\) Thus, we have
\[
E[X^{(k)}] = -1 + \frac{2k}{n+1}.
\]

To prove the lemmas we use the sub-Gaussian property of the beta distribution. A random variable \(X\) with finite mean \(\mu = E[X]\) is said to be sub-Gaussian if there is a \(\sigma > 0\) such that, for all \(\lambda \in \mathbb{R},\)
\[
E[e^{\lambda(X-\mu)}] \leq e^{\frac{\lambda^2\sigma^2}{2}}.
\]

The constant \(\sigma^2\) is called a proxy variance, and we say that \(X\) is \(\sigma^2\) sub-Gaussian.

Let \(X\) be a \(\sigma^2\) sub-Gaussian random variable. Then Markov’s inequality, together with (58), implies that, for any \(\lambda, t > 0,\)
\[
P[X - \mu > t] = P[e^{\lambda(X-\mu)} > e^{\lambda t}] \leq e^{-\lambda t + \frac{\lambda^2\sigma^2}{2}}.
\]

Optimizing the upper bound over \(\lambda\) yields,
\[
P[X - \mu > t] \leq e^{-\frac{\lambda^2}{2\sigma^2}}.
\]

Similarly, it can be shown that if \(X\) is \(\sigma^2\) sub-Gaussian, then, for all \(t > 0\)
\[
P[X - \mu < -t] \leq e^{-\frac{\lambda^2}{2\sigma^2}}.
\]

Therefore, we conclude that if \(X\) is \(\sigma^2\) sub-Gaussian, then, for all \(t > 0,\)
\[
P[|X - \mu| > t] \leq 2e^{-\frac{\lambda^2}{2\sigma^2}}.
\]

To prove Lemma 6.2 we will use the following result.

**Theorem 1 (Theorem 1 in [24]).** The Beta(\(\alpha, \beta\)) distribution is \((4(\alpha + \beta + 1))^{-1}\) sub-Gaussian.

We can now prove Lemma 6.2.
Proof of Lemma 6.2. For any $\delta > 0$, we have
\[
\mathbb{P} \left[ |X^{(k)} - \mathbb{E}[X^{(k)}]| > \delta \right] = \mathbb{P} \left[ |U^{(k)} - \mathbb{E}[U^{(k)}]| > \frac{\delta}{2} \right].
\]
Recall that $U^{(k)} \sim \text{Beta}(k, n - k + 1)$. Theorem 1 implies that for all $k = 1, \ldots, n$, $U^{(k)}$ is sub-Gaussian with $\sigma^2 = (4(n + 2))^{-1}$. Therefore, applying (59), we have
\[
\mathbb{P} \left[ |X^{(k)} - \mathbb{E}[X^{(k)}]| > \frac{1}{n^{1/3}} \right] \leq 2e^{-\frac{\delta^2}{2}}.
\]
Using the union bound, we have
\[
\mathbb{P} \left[ \bigcup_{k=1}^{n} \left\{ |X^{(k)} - \mathbb{E}[X^{(k)}]| > \frac{1}{n^{1/3}} \right\} \right] \leq 2ne^{-\frac{\delta^2}{2}}.
\]
Thus,
\[
\sum_{n=1}^{\infty} \mathbb{P} \left[ \bigcup_{k=1}^{n} \left\{ |X^{(k)} - \mathbb{E}[X^{(k)}]| > \frac{1}{n^{1/3}} \right\} \right] < \infty,
\]
and the result follows from the Borel-Cantelli lemma.

Proof of Lemma 7.2. Let $X^{(i)} = (X^{(i)}_1, X^{(i)}_2)$. Since we will bound each of the coordinates separately.

We start with $X^{(i)}_1$. Recall that $X^{(i)}_1, \ldots, X^{(i)}_n$ are the order statistics of $X_1, \ldots, X_n$. Using Lemma 6.2, almost surely there exists $N_1 > 0$ such that, for $n \geq N_1$,
\[
\sup_{k} |X^{(k)}_1 - \mathbb{E}[X^{(k)}_1]| \leq \frac{1}{n^{1/3}}.
\]
Denote by $E$ the almost-sure event described above and fix $\omega \in E$. Next, fix $p$, and recall that $X^{(p,1)}, \ldots, X^{(p,m)}_i$ is a permutation of $X_1^{(p-1)m+1}, \ldots, X_1^{(p)m}$, which implies that, for all $q = 1, \ldots, m$, we have that $X^{(p,q)}_1 = X^{((p-1)m+r)}_1$ for some $1 \leq r \leq m$. Thus,
\[
|X^{(p,q)} - \mathbb{E}[X^{(p,q)}]| \leq |X_1^{((p-1)m+r)} - \mathbb{E}[X^{((p-1)m+r)}]| + |\mathbb{E}[X^{((p-1)m+r)}] - \mathbb{E}[X^{(p,q)}]|.
\]
Next, recall that $\mathbb{E}[X^{((p-1)m+r)}] = -1 + \frac{2((p-1)m+r)}{n+1}$ (cf. (58)), and $\mathbb{E}[X^{(p,q)}] = -1 + \frac{2m-m+1}{n+1}$ (cf. (55)). Therefore, for $n \geq N_1$, we have
\[
|X^{(p,q)} - \mathbb{E}[X^{(p,q)}]| \leq \frac{1}{n^{1/3}} + \frac{1}{m} \leq \frac{1}{n^{1/6}}.
\]
Since this is true for all $p, q$, we have, for all $n \geq N_1$,
\[
\sup_{i} |X^{(i)}_1 - \mathbb{E}[X^{(i)}_1]| \leq \frac{1}{n^{1/6}}.
\]
We proceed with bounding $X^{(i)}_2$. Suppose that $X^{(1)}_1$ is given. Then, for every $p$, we have that $X^{(p,q)}_2$ is the $q$-th order statistic of $m$ i.i.d. uniform random variables in $[-1, 1]$. By Lemma 6.2, for every $p$, almost surely there exists $N_2(p) > 0$ such that, for all $n \geq N_2(p)$, we have
\[
\sup_{k=1, \ldots, m} |X^{(p,k)}_2 - \mathbb{E}[X^{(p,k)}_2]| \leq \frac{1}{n^{1/3}} = \frac{1}{n^{1/6}}.
\]
Taking $N_2 = \max_{1 \leq p \leq m} N_2(p)$, then for $n \geq N_2$ we have
\[
\sup_{i} |X^{(i)}_2 - \mathbb{E}[X^{(i)}_2]| \leq \frac{1}{n^{1/6}} = \frac{1}{n^{1/6}}.
\]
To conclude, we showed that almost surely there exists $N = \max(N_1, N_2)$ such that, for all $n \geq N$, both (60) and (61) hold. This concludes the proof.
Proof of Lemma 3.2  Fix $1 \leq k \leq d$, and recall that, for every $i_1, \ldots, i_{k-1}$, in Step $k$ of our construction we had that $X_k^{(i_1, \ldots, i_{k-1})} \leq \cdots \leq X_k^{(i_1, \ldots, i_{k-1}, m^{d-k+1})}$ are the order statistics of $m^{d-k+1}$ i.i.d. uniformly distributed random variables in $[−1, 1]$. Therefore, by Lemma 6.2 almost surely there exists $N_k(i_1, \ldots, i_{k-1})$ such that, for all $n \geq N_k(i_1, \ldots, i_{k-1})$, we have

$$\sup_{j=1, \ldots, m^{d-k+1}} |X_k^{(i_1, \ldots, i_{k-1}, j)} - E[X_k^{(i_1, \ldots, i_{k-1}, j)}]| \leq \frac{1}{m^{(d-k+1)/2}}.$$  

Next, fix $i = (i_1, \ldots, i_d)$, and let $E$ be the almost sure event above. Fix $\omega \in E$ and suppose that $n \geq N_k(i_1, \ldots, i_{k-1})$. Recall that the variable $X_k^{(i)}$ is equal to one of the variables $X_k^{(i_1, \ldots, i_{k-1}, (i_k-1)m^{d-k+1})}, \ldots, X_k^{(i_1, \ldots, i_{k-1}, i_km^{d-k})}$. Let $r = r(\omega)$ be such that $X_k^{(i_1, \ldots, i_d)} = X_k^{(i_1, \ldots, i_{k-1}, (i_k-1)m^{d-k} + r)}$. Then,

$$|X_k^{(i)} - E[X_k^{(i)}]| \leq |X_k^{(i_1, \ldots, i_{k-1}, (i_k-1)m^{d-k} + r)} - E[X_k^{(i_1, \ldots, i_{k-1}, (i_k-1)m^{d-k} + r)}]|$$

$$+ |E[X_k^{(i_1, \ldots, i_{k-1}, (i_k-1)m^{d-k} + r)}] - E[X_k^{(i_1, \ldots, i_d)}]|$$

Since both $E[X_k^{(i_1, \ldots, i_{k-1}, (i_k-1)m^{d-k} + r)}]$ and $E[X_k^{(i_1, \ldots, i_d)}]$ lie in $L_{i_k,m}$, we have

$$|E[X_k^{(i_1, \ldots, i_{k-1}, (i_k-1)m^{d-k} + r)}] - E[X_k^{(i_1, \ldots, i_d)}]| \leq \frac{1}{m}.$$  

In addition, since we assume $n \geq N_k(i_1, \ldots, i_{k-1})$, we have

$$|X_k^{(i_1, \ldots, i_{k-1}, (i_k-1)m^{d-k} + r)} - E[X_k^{(i_1, \ldots, i_{k-1}, (i_k-1)m^{d-k} + r)}]| \leq \frac{1}{m^{(d-k+1)/2}},$$

and therefore,

$$|X_k^{(i)} - E[X_k^{(i)}]| \leq \frac{1}{m^{(d-k+1)/2}} + \frac{1}{m} \leq \frac{1}{n^{1/3d}}.$$  

Taking $N_k = \max_{i_1, \ldots, i_{k-1}} N_k(i_1, \ldots, i_{k-1})$ and $n \geq N_k$, we have

$$\sup_i |X_k^{(i)} - E[X_k^{(i)}]| \leq \frac{1}{n^{1/3d}}. \quad (62)$$  

Finally, let $N = \max \{N_k : 1 \leq k \leq d\}$. Then (62) holds for all $n \geq N$, and we are done.

### Appendix B. Integral kernel operators

We now provide a proof for Lemma 3.1 which extends Lemma 1.11 in [30]. The proof will make use of two lemmas.

Recall that $H$ denotes the Hilbert space $L^2(V, \nu)$. A sequence $\{f_n\}_{n=1}^\infty$ is called weakly convergent if $\{(f_n, g)\}_{n=1}^\infty$ converges for every $g \in H$.

**Lemma B.1** (Lemma 1.10 in [30]). Let $K$ be the cut norm limit of $\{K_n\}_{n=1}^\infty$. Let $\{f_n\}_{n=1}^\infty$ be a weakly convergent sequence in $H$ with limit $f$ such that $\|f_n\|_2 = 1$ for every $n$ and $K_n f_n = \lambda_n f_n$, where $\lim_{n \to \infty} \lambda_n = \lambda \neq 0$. Then $\{f_n\}_{n=1}^\infty$ converges in $L_2$ to $f$ and $K f = \lambda f$.

**Proof of Lemma B.1**. Let $\{\lambda_{n,j}\}_{j=1}^\infty$ be the eigenvalues of $K_n$, listed with multiplicities. If $K_n$ is a finite rank operator then we put an infinite number of zeroes at the end. We assume that $\{|\lambda_{n,j}|\}_{j=1}^\infty$ is a decreasing sequence. Since $K_n$ is symmetric, using the spectral decomposition theorem for $K_n$, the kernel function $K_n$ can be expressed as

$$K_n(x, y) = \sum_{j=1}^\infty \lambda_{n,j} \varphi_{n,j}(x) \varphi_{n,j}(y),$$

where $\{\varphi_{n,j}\}_{i, j \neq 0}$ is an orthonormal system in $H$. For $\lambda_{n,j} = 0$ we take $\varphi_{n,j}$ to be an arbitrarily chosen function of unit length.

Note that

$$\int \int |K_n(x, y)|^2 d\nu(x) d\nu(y) = \sum_{j=1}^\infty \lambda_{n,j}^2,$$
and since we assume that \( \| K_n \| \leq C \) we also have that
\[
\sum_{j=1}^{\infty} \lambda_{n,j}^2 \leq C.
\]
For every \( j \), \( \{\lambda_{n,j}\}_{n=1}^{\infty} \) is bounded. In addition, every bounded sequence in a Hilbert space contains a weakly convergent subsequence. Therefore, for every fixed \( j \) we can find a subsequence \( \{n_i\}_{i=1}^{\infty} \) such that \( \{\varphi_{n_i,j}\}_{i=1}^{\infty} \) is weakly convergent in \( \mathcal{H} \) and \( \{\lambda_{n_i,j}\}_{i=1}^{\infty} \) is convergent. Let \( \varphi_j \) be the weak limit of \( \{\varphi_{n_i,j}\}_{i=1}^{\infty} \) and \( \lambda_j \) be the limit of \( \{\lambda_{n_i,j}\}_{i=1}^{\infty} \). Then it can be shown that (a) \( \sum_{j=1}^{\infty} \lambda_j^2 \leq C \), and (b) \( \{\lambda_j\}_{j=1}^{\infty} \) is a decreasing sequence. These two facts imply that \( |\lambda_j| \leq \frac{\sqrt{C}}{\sqrt{j}} \) for every \( j \). Indeed, for \( j_0 \in \mathbb{N} \),
\[
\sum_{j=1}^{j_0} \lambda_j^2 \leq C \quad \Rightarrow \quad |\lambda_{j_0}| \leq \sqrt{\frac{C}{j_0}}.
\]
If \( \lambda_j \neq 0 \), then Lemma B.1 implies that
\[
\lim_{t \to \infty} \int_{1}^{t} |\varphi_{n_{i,j}}(x) - \varphi_j(x)|^2 d\nu(x) \to 0.
\]
Using the triangle and Cauchy-Schwarz inequalities gives that, for \( \lambda_{j_1}, \lambda_{j_2} \neq 0 \), we have
\[
\lim_{t \to \infty} \int_{1}^{t} |\varphi_{n_{i,j_1}}(x) - \varphi_{n_{i,j_2}}(x) - (\varphi_{j_1}, \varphi_{j_2})| = 0.
\]
This implies that, if \( \lambda_{j_1} \) and \( \lambda_{j_2} \) are non-zero, then \( \langle \varphi_{j_1}, \varphi_{j_2} \rangle = 0 \). Therefore \( \{\varphi_j\}_{j: \lambda_j \neq 0} \) is an orthogonal system of functions. Define
\[
K'(x, y) = \sum_{j: \lambda_j \neq 0} \lambda_j \varphi_j(x) \varphi_j(y).
\]
We show that \( K(x, y) = K'(x, y) \). Let \( t \in \mathbb{N} \). Then
\[
\| K_n - K' \| \leq \left\| \sum_{j=1}^{t} (\lambda_{n,j} \varphi_{n,j}(x) \varphi_{n,j}(y) - \lambda_j \varphi_j(x) \varphi_j(y)) \right\| + \left\| \sum_{j=t+1}^{\infty} \lambda_{n,j} \varphi_{n,j}(x) \varphi_{n,j}(y) \right\| + \left\| \sum_{j=t+1}^{\infty} \lambda_j \varphi_j(x) \varphi_j(y) \right\|.
\]
To bound the first term in (65), fix \( 1 \leq j \leq t \) and note that, since \( \lambda_{n,j} \to \lambda_j \), there exists a \( i_0 > 0 \) such that, for all \( i > i_0 \) we have
\[
|\lambda_{n,j} - \lambda_j| \leq \frac{1}{(t+1)^2}.
\]
Using (64), (66) and triangle inequality, we have for all \( i > i_0 \),
\[
\left\| \sum_{j=1}^{t} (\lambda_{n,j} \varphi_{n,j}(x) \varphi_{n,j}(y) - \lambda_j \varphi_j(x) \varphi_j(y)) \right\| \leq \frac{C}{\sqrt{t+1}}.
\]
for some positive constant \( C' \).
To bound the other two terms in (65) we use the notion of spectral radius. Let \( M \) be a self-adjoint kernel operator, and \( \{\lambda_j(M)\}_{j=1}^{\infty} \) be its eigenvalues. Then the spectral radius of \( M \) is defined as
\[
\text{rad}(M) = \sup_j |\lambda_j(M)|.
\]
Lemma 1.5 in [30] states that \( \| M \| \leq \text{rad}(M) \). Note that since the absolute eigenvalues are decreasing, and using (63), for both the second and the third terms in (65) we have that the spectral radius is bounded by \( \sqrt{C}/\sqrt{t+1} \).
To conclude, we can show that there exists \( C'' > 0 \) such that for a large enough \( i \) we have
\[
\| K_n - K' \| \leq \frac{C''}{\sqrt{t+1}}.
\]
Letting \( t \to \infty \) we get that \( K_{n_i} \) converges to \( K' \) in the cut norm. Since we also know that \( K_{n_i} \to K \), we conclude that \( K' \equiv K \).

Take \( \lambda > 0 \) such that \( \pm \lambda \not\in \text{spec}(K) \), and let \( t \) be an integer greater than \( C \lambda^{-2} \). Note that, for \( j > t \),

\[
\lambda_{n_i,j} \leq \frac{\sqrt{C}}{\sqrt{j}} \leq \frac{\sqrt{C}}{\sqrt{t}} \leq \lambda.
\]

Let \( m_t = \min(|\lambda - |\lambda_j|| : 1 \leq j \leq t) > 0 \). Note that \( m_t > 0 \), as \( \lambda \) and \(-\lambda\) are not eigenvalues of \( K \). In addition, there exists \( i_0 > 0 \) such that, for all \( i > i_0 \) and \( 1 \leq j \leq t \),

\[
|\lambda_{n_i,j} - \lambda_j| \leq \frac{m_t}{2}.
\]

Therefore, we conclude that \( \lambda_j > \lambda \Leftrightarrow \lambda_{n_i,j} > \lambda \) and \( \lambda_j < -\lambda \Leftrightarrow \lambda_{n_i,j} < -\lambda \), for \( 1 \leq j \leq t \) as \( i \to \infty \). Hence (\( F \)) is true for \( \{K_{n_i}\} \). That is,

\[
\lim_{i \to \infty} |\{\text{spec}(K_{n_i}) \cap (\lambda, \infty)\}| = |\{\text{spec}(K) \cap (\lambda, \infty)\}|,
\]

\[
\lim_{i \to \infty} |\{\text{spec}(K_{n_i}) \cap (-\infty, -\lambda)\}| = |\{\text{spec}(K) \cap (-\infty, -\lambda)\}|.
\]

To conclude, we have to show convergence when \( n \to \infty \) (as opposed to \( n_i \to \infty \)). Suppose that we can choose an infinite subsequence such that (\( F \)) does not hold. This leads to an immediate contradiction, since from such a subsequence we cannot choose a subsequence which satisfies the result, and we are done.

\[ \square \]

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