DEGENERATION OF SHRINKING RICCI SOLITONS

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Abstract. Let \((Y,d)\) be a Gromov-Hausdorff limit of closed shrinking Ricci solitons with uniformly upper bounded diameter and lower bounded volume. We prove that off a closed subset of codimension at least 2, \(Y\) is a smooth manifold satisfying a shrinking Ricci soliton equation.

1. Introduction

Let \((Y,d)\) be a metric space obtained as a Gromov-Hausdorff limit of a sequence of complete \(n\)-dimensional Riemannian manifolds \((M_k,g_k)\). When the Ricci curvatures of \(M_k\) are uniformly bounded below, the work of Cheeger and Colding \([7, 8, 9]\) gives a beautiful description for the structure of \(Y\). In particular they proved in the noncollapsing case that \(Y\) is bi-Hölder equivalent to connected smooth Riemannian manifold outside of a singular set of codimension \(\geq 2\). When the Ricci curvatures of \(M_k\) are absolutely bounded, they also proved that the singular set of \(Y\) is closed.

As a generalization of notion, Bakry-Émery Ricci curvature plays an important role in the theory of smooth measure metric space, cf. \([27]\) and the references therein. In view of Cheeger and Colding’s work, one naturally asks what will happen to \(Y\) when it is obtained as a limit of Riemannian manifolds with bounded Bakry-Émery Ricci curvature. In this paper, we are going to prove a partial result to this question, for the special case when \(M_k\) are closed shrinking Ricci solitons. The argument here also applies to the noncompact Ricci solitons, either for shrinking, steady or expanding case.

The Ricci solitons are manifolds with constant Bakry-Émery Ricci curvature. Namely, a Ricci soliton is a Riemannian manifold \((M,g)\) satisfying

\[
\text{Ric} + \text{Hess}(f) = \epsilon g
\]

for some \(\epsilon \in \mathbb{R}\) and \(f \in C^\infty(M; \mathbb{R})\). The associated function \(f\) is called the potential function. We assume \(\epsilon = +\frac{1}{2}, -\frac{1}{2}\) or 0 after a normalization, in each case the Ricci soliton is called a shrinking, steady or expanding one respectively. Ricci solitons are also known as the self-similar solutions to

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the Ricci flow, where the metric varies via rescalings and the diffeomorphic transformations, cf. [12].

Ricci solitons are generalizations of Einstein manifolds. Suppose $M_k$ as in the later case, Cheeger and Colding proved in [7] that, under the noncollapsing hypothesis, $Y$ is Einstein off the singular set and the convergence is smooth off the singular set. We will prove the following Ricci soliton analogy of Cheeger and Colding’s result in the present paper.

**Theorem 1.1.** Let $(M_k, g_k)$ be a sequence of $n$-dimensional closed shrinking Ricci solitons satisfying (1) with $\epsilon = \frac{1}{2}$. Suppose that

\begin{enumerate}
\item $\text{Vol}_{g_k}(M_k) \geq v,$
\item $\text{diam}(M_k, g_k) \leq D$
\end{enumerate}

uniformly hold for some $0 < v, D < \infty$. Then passing a subsequence if necessary, the manifolds $(M_k, g_k)$ converge in the Gromov-Hausdorff sense to a length metric space $(Y, d_\infty)$.

The singular set $S$ is closed in $Y$ and has Hausdorff codimension at least 2; the regular set $\mathcal{R} = Y \setminus S$ is connected. On $\mathcal{R}$, the metric is induced from a smooth Riemannian metric which satisfies a shrinking Ricci soliton equation and the convergence takes place smoothly.

By definition, cf. [7], a point $y$ belongs to $S$ iff there is a tangent cone at $y$ which is not isometric to the $n$-Euclidean space.

The proof of the theorem proceeds as follows: We first show the theorem up to a conformal change of the shrinking Ricci solitons, by using Cheeger and Colding’s theorem on degeneration for Ricci curvature [7]; then we show that the degeneration property for the shrinking Ricci solitons does not depend on the conformal changes. The point is that the approximation maps associated to the Gromov-Hausdorff convergence do not depend on the conformal change of the metrics.

When the curvatures have bounded $L^{n/2}$-norms, the singularities are finite and of orbifold type; see [25], [11] or [30]. Applying Theorem 1.1 together with Theorem 2.6 of [2], we give a more direct proof of this result; see §3.6. We remark that the first orbifold type convergence theorem for Ricci solitons was given by Cao and Sesum in the Kähler case [6], under additional assumption of lower bounded Ricci curvature. Later, X. Zhang proved the analogy for real case [28]. The orbifold type compactness theorem is a generalization of Einstein manifolds, see [1], [4] and [23].

In the Kähler category, one may hope to know more about the structure of $Y$, as in the Ricci curvature case (cf. [10]). Unfortunately, our method in this paper can not applied to the Kähler manifolds.
However, by a different argument, we can improve the orbifold compactness theorem mentioned above as in the following theorem for Kähler Ricci solitons. The result is a shrinking Ricci soliton version of Tian’s compactness theorem for Kähler Einstein manifolds [24]. It can also be seen as a sharpening of Cao and Sesum’s orbifold compactness theorem for shrinking Kähler Ricci solitons [6] of dimension at least 3.

**Theorem 1.2.** Let \((M_k, g_k)\) be a sequence of Kähler Ricci solitons with positive first Chern class, of dimension \(n = \dim C M_k \geq 3\). Suppose (2) and (3) hold and that

\[
\int_{M_k} |Rm(g_k)|^n dv_{g_k} \leq C
\]

for some \(C\) independent of \(k\), then passing a subsequence, \((M_k, g_k)\) converge smoothly to another Kähler Ricci soliton of positive first Chern class.

**Remark 1.3.** Under the hypothesis of bounded \(L^{n/2}\)-norm of curvature tensor, the condition (2) together with (3) is equivalent to the boundedness of Perelman’s \(\nu\) functional; see [6] or [29] for a lower bound of \(\nu\) in terms of (2) and (3), and see [30] for a lower bound of volume and upper bound of diameter via \(\nu\) and (27).

At the end of the introduction, we give a remark on the geometry of Bakry-Émery Ricci curvature.

**Remark 1.4.** Bakry-Émery Ricci curvature shares many similar properties as the Ricci curvature, especially when the potential function admits a \(C^0\) bound. The key point is that, when Laplacian is replaced by a modified elliptic operator, the Laplace comparison theorem for distance function remains true up to a perturbation which is controlled by the \(C^0\) norm of the potential, cf. [15]. In addition, the Bochner formula for Bakry-Émery Ricci curvature remains valid [3]; see also [15, 27]. With these two things in hand, many results for Ricci curvature can be extended to Bakry-Émery Ricci curvature, such as the splitting theorem [15, 26].

The paper is organized as follows: In §2, we recall some preliminaries that will be used; in §3, we give a proof of Theorem 1.1. In §4, we consider the Kähler case and present a proof of Theorem 1.2.

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2. Preliminaries

2.1. Pseudolocality theorem for Ricci flow. We state another version of Perelman’s pseudolocality theorem which is proved in \[16\]. The difference from Perelman’s theorem \[20\] is that here we use the local almost Euclidean volume growth instead of almost Euclidean isoperimetric estimate.

**Theorem 2.1** (\[16\]). There exist universal constants \(\delta_0, \epsilon_0 > 0\) with the following property. Let \(g(t), t \in [0, (\epsilon_0 r_0)^2]\), be a solution to the Ricci flow

\[
\frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t))
\]

on a closed \(n\)-manifold \(M\) and \(x_0 \in M\) be a point. If the scalar curvature

\[R(x, t) \geq -r_0^{-2}\] whenever \(\text{dist}_{g(t)}(x_0, x) \leq r_0,\]

and the volume

\[
\text{Vol}_{g(t)}(B_{g(t)}(x, r)) \geq (1 - \delta_0)\text{Vol}(B(r)) \quad \text{for all } B_{g(t)}(x, r) \subset B_{g(t)}(x_0, r_0),
\]

where \(B(r)\) denotes a ball of radius \(r\) in the \(n\)-Euclidean space and \(\text{Vol}(B(r))\) denotes its Euclidean volume, then the Riemannian curvature tensor satisfies

\[|Rm|_{g(t)}(x, t) \leq t^{-1}, \quad \text{whenever } \text{dist}_{g(t)}(x_0, x) < \epsilon_0 r_0.\]

In particular, \(|Rm|_{g(t)}(x_0, t) \leq t^{-1}\) for all time \(t \in (0, (\epsilon_0 r_0)^2]\).

We use the pseudolocality theorem to prove the smooth convergence on the regular part of the limit space.

2.2. One technical lemma. Let \((M, g)\) be a shrinking Ricci soliton satisfying \[2\] and \[3\]. Then the potential function has a uniform \(C^1\) norm bound; see \[18\] in §3.1. Using the mean curvature and relative volume comparison theorems for Bakry-Émery Ricci curvature, cf. \[26\], one easily derives the following technical lemma.

**Lemma 2.2.** For all \(\epsilon > 0\), there exists \(c = c(n, v, D, \epsilon) > 0\) such that the following holds. Let \(E \subset M\) be a submanifold with smooth boundary and \(x_1, x_2 \in M\) be two points such that

\[
B(x_1, \epsilon) \cup B(x_2, \epsilon) \subset M \setminus E.
\]

If every minimal geodesic from \(x_1\) to point in \(B(x_2, \epsilon)\) intersects \(E\), then

\[
\text{Vol}_g(\partial E) \geq c.
\]

**Proof.** The proof follows directly from the argument in Page 525 of \[17\]; see also \[8\]. We omit the details here. \(\square\)

This lemma is applied to prove that on the regular part of the limit space, the intrinsic metric coincides with the extrinsic metric; see also \[7\].
2.3. Isoperimetric inequality. Let \((M,g)\) be an \(n\)-dimensional Riemannian manifold. The global isoperimetric constant is defined to be the minimal constant \(C_I(M,g)\) such that
\[
\min (\text{Vol}(\Omega)^{n-1}, \text{Vol}(M\setminus\Omega)^{n-1}) \leq C_I(M,g) \text{Vol}(\partial\Omega)^n
\]
for all domain \(\Omega \subset M\) with smooth boundary. It is well known that \(C_I(M,g)\) is equivalent to the Sobolev constant, which is defined to be the minimal constant \(C_S(M,g)\) such that
\[
\inf_{a \in \mathbb{R}} \left( \int_M (\psi - a)^{\frac{n}{n-1}} dv \right)^{\frac{n-1}{n}} \leq C_S(M,g) \int_M |\nabla \phi|
\]
for all smooth function \(\phi\). It is a direct calculation that for equivalent metrics \(g\) and \(\tilde{g}\) with relation
\[
C^{-1}g \leq \tilde{g} \leq Cg,
\]
the Sobolev constants satisfies
\[
C^{1-n} \cdot C_S(M,g) \leq C_S(M,\tilde{g}) \leq C^{n-1} \cdot C_S(M,g).
\]
The isoperimetric constants for \(g\) and \(\tilde{g}\) are also equivalent to each other.

If \((M,g)\) is a closed shrinking Ricci soliton which satisfies (2) and (3), then the metric \(\tilde{g} = e^{-\frac{2n}{n-2}f}g\) satisfies
\[
C_2^{-1}g \leq \tilde{g} \leq C_2g, \quad |\text{Ric}(\tilde{g})|_{\tilde{g}} \leq C_3,
\]
for some constants \(C_2, C_3\) depending on only \(n\) and \(D\), see Section 3. Thus the volume \(\text{Vol}(\tilde{g}) \geq C_2^{-n/2}v\) and the diameter \(\text{diam}(\tilde{g}) \leq C_2^{1/2}D\). So, by one of Croke’s classical theorem [14], \(C_I(M,\tilde{g})\) as well as \(C_S(M,\tilde{g})\) has a uniform upper bound. Passing to \(g\), we obtain an upper bound
\[
C_I(M,g) \leq C_0 = C_0(n,v,D).
\]

3. Degeneration of Ricci solitons

Let \((M,g)\) be a closed shrinking Ricci soliton with potential function \(f\):
\[
\text{Ric} + \text{Hess}(f) = \frac{1}{2}g.
\]
Tracing the soliton equation we get
\[
R + \nabla f = \frac{n}{2},
\]
then doing integration yields
\[
\frac{1}{\text{Vol}_g(M)} \int_M R dv = \frac{n}{2}.
\]
The following equation will also be frequently used, cf. [13] Eq. 4.13:
\[
R + |\nabla f|^2 = f + \text{const.},
\]
Assume that \( \text{const.} = 0 \) after a translation.

3.1. **Bound the potential.** We suppose in this section that the diameter satisfies

\[
\text{diam}(M,g) \leq D.
\]

By Ivey [19], a closed shrinking Ricci soliton has positive scalar curvature. (15) tells us that the infimums of \( f \) and \( R \) are attained at the same point. As a consequence, \( 0 < \inf f = \inf R \leq \frac{n}{2} \). Then using (15) once again one derives the upper bound of \( f \):

\[
\sup f \leq \frac{1}{4}(D + \frac{n}{2})^2.
\]

It follows immediately from (15) that for \( C_1 = \frac{2}{3}(D + \frac{n}{2})^2 \),

\[
\sup f + \sup |\nabla f|^2 + \sup R \leq C_1.
\]

3.2. **Conformal change of the soliton metric.** Define a conformal metric \( \tilde{g} = e^{-\frac{2}{n-2}f}g \). It follows an easy computation that

\[
C_2^{-1}g(x) \leq \tilde{g}(x) \leq g(x),
\]

where \( C_2 = e^{\frac{(D+\frac{n}{2})^2}{2(n-2)}} \) depending on \( n, D \). Let \( \tilde{\text{Ric}} \) denote the Ricci tensor of \( \tilde{g} \), then, together with using equations (13) and (15), we get that, cf. [5],

\[
\tilde{\text{Ric}} = \text{Ric} + \text{Hess}(f) + \frac{1}{n-2}df \otimes df + \frac{1}{n-2}(\Delta f - |\nabla f|^2)g
\]

\[
= \frac{1}{2}g + \frac{1}{n-2}df \otimes df + \frac{1}{n-2}(\frac{n}{2} - R(g) - |\nabla f|^2)g
\]

\[
= \frac{1}{n-2}(df \otimes df + (n-1-f)g).
\]

So we can bound the Ricci curvature after a conformal change:

**Lemma 3.1.** There exists \( C_3 \) depending only on \( n \) and \( D \) such that the Ricci curvature of \( \tilde{g} \) satisfies the bound

\[
|\tilde{\text{Ric}}|_{\tilde{g}} \leq C_3.
\]

**Proof.** By a straightforward computation,

\[
|\tilde{\text{Ric}}|_{\tilde{g}} \leq \frac{1}{n-2}(|df \otimes df|_{\tilde{g}} + |n-1-f| \cdot |g|_{\tilde{g}})
\]

\[
\leq \frac{C_2}{n-2}(|\nabla f|^2_{\tilde{g}} + n|n-1-f|).
\]

By (18), one can choose \( C_3 = \frac{C_2}{n-2}(n(n-1) + (n+1)C_1) \). \( \square \)
3.3. **Bound the volume ratio.** Suppose further that
\[(21) \quad \text{Vol}_g(M) \geq v.\]
Then \(\text{Vol}_\tilde{g}(M) \geq C_2^{-n/2} \text{Vol}_g(M) \geq C_2^{-n/2} v.\) By relative volume comparison theorem, together with using that \(\tilde{g}\) and \(g\) are uniformly equivalent, there exists a positive constant \(\kappa = \kappa(n, v, D)\) such that
\[(22) \quad \kappa r^n \leq \text{Vol}_g(B(r)) \leq \kappa^{-1} r^n\]
for all metric ball \(B(r)\) of radius \(r \leq D\) in \(M\).

3.4. **Convergence modulo conformal changes.** Let \((M_k, g_k)\) be a sequence of \(n\)-dimensional closed shrinking Ricci solitons with potential functions \(f_k\). Suppose that \((M_k, g_k)\) satisfies (2) and (3). Let \(\tilde{g}_k = e^{-\frac{2}{n-2} f_k} g_k\).

Applying Cheeger-Colding’s theorem, cf. [7, Thm. 7.2], gives the following

**Theorem 3.2.** Passing a subsequence if necessary, the Riemannian manifolds \((M_k, \tilde{g}_k)\) converge in the Gromov-Hausdorff sense to a compact \(n\)-dimensional length metric space \((Y, \tilde{d}_\infty)\).

The singular set \(S\) is a closed subset which has Hausdorff codimension at least 2. On \(Y \setminus S\), the metric is induced by a \(C^{1,\alpha}\) Riemannian metric for all \(\alpha < 1\). Furthermore, the convergence is in the \(C^{1,\alpha}\) sense on \(Y \setminus S\).

Based on this theorem, we can finally give a

3.5. **Proof of Theorem 1.1.** Let \((M_k, g_k)\) be a sequence of \(n\)-dimensional closed shrinking Ricci solitons satisfying (12) with potential functions \(f_k\). Suppose further that \((M_k, g_k)\) satisfies (2) and (3).

Let \(\tilde{g}_k = e^{-\frac{2}{n-2} f_k} g_k\) and \((Y, \tilde{d}_\infty)\) be the Gromov-Hausdorff limit of \((M_k, \tilde{g}_k)\). Let \(\mathcal{R}\) be the regular part of \(Y\) and \(S = M \setminus \mathcal{R}\) be the singular set. \(\mathcal{R}\) is a \(C^{1,\alpha}\) manifold on which the convergence is in the \(C^{1,\alpha}\) sense. Let \(\tilde{g}_\infty\) be the Riemannian metric defined on \(\mathcal{R}\).

Let \(\{K_k\}_{k=1}^\infty\) be any given exhaustion of \(\mathcal{R}\) by compact subsets, such that \(K_k \subset K_{k+1}\) for all \(k\) and \(\bigcup_{k=1}^\infty K_k = \mathcal{R}\). Then, by definition of \(C^{1,\alpha}\) convergence, there exists a sequence of smooth embeddings
\[\psi_k : K_k \to M_k\]
such that \(\psi_k^* \tilde{g}_k \xrightarrow{C^{1,\alpha}} \tilde{g}_\infty\) as \(k \to \infty\) on any compact sets of \(\mathcal{R}\).

**Claim 3.3.** Passing a subsequence if necessary, \(\psi_k^* g_k\) converges in the \(C^\alpha\) sense to a \(C^\alpha\) metric, say \(g_\infty\), on \(\mathcal{R}\).

**Proof.** It follows from that \(\|f_k\|_{C^1}\) is uniformly bounded on \(M_k\), and that the metrics
\[\psi_k^* g_k = \psi_k^* (e^{-\frac{2}{n-2} f_k} \tilde{g}_k) = e^{-\frac{2}{n-2} f_k} \circ \psi_k^* \tilde{g}_k\]
have a uniform \(C^1\) bound on any compact subset of \(\mathcal{R}\). \(\square\)
In view of [19], the metrics $g_\infty$ is uniformly equivalent to $\tilde{g}_\infty$ on $\mathcal{R}$:

$$C^{-1}_2 g_\infty \leq \tilde{g}_\infty \leq g_\infty.$$  

This concludes that, as intrinsic metric spaces, $(\mathcal{R}, g_\infty)$ and $(\mathcal{R}, \tilde{g}_\infty)$ have the same metric completions in the set point of view. In view of Theorem 7.2 in [7] and Theorem 3.9 in [8], the intrinsic metric defined on the same metric completions in the set point of view.

In view of (19), the metrics $d_\infty$, $\tilde{d}_\infty$, and $\gamma$ coincide with the extrinsic metric $\tilde{d}_\infty$, i.e., for all $y_1, y_2 \in \mathcal{R}$,

$$\tilde{d}_\infty(y_1, y_2) = \inf \{ L_{\tilde{g}_\infty}(\gamma) | \gamma \subset \mathcal{R} \text{ is a curve joining } y_1 \text{ and } y_2 \}.$$  

Thus the completion space of $(\mathcal{R}, g_\infty)$, as a set, equals $Y$.

Denote by $d_\infty$ the distance function on $Y$ as the metric completion of $(\mathcal{R}, g_\infty)$, then $\mathcal{R}$ is still an open set in $(Y, d_\infty)$ by the equivalence of $g_\infty$ and $\tilde{g}_\infty$. In addition, the distance function $d_\infty$ and $\tilde{d}_\infty$ satisfies

$$C^{-1/2}_2 d_\infty(y_1, y_2) \leq \tilde{d}_\infty(y_1, y_2) \leq d_\infty(y_1, y_2)$$

for all $y_1, y_2 \in Y$. So the set $\mathcal{S} = Y \setminus \mathcal{R}$ is closed and has Hausdorff dimension $\dim(\mathcal{S}) \leq n - 2$ with respect to $d_\infty$ as well. We next claim that

**Claim 3.4.** $(Y, d_\infty)$ is a path metric space.

**Proof.** By definition of metric completion, for any $y_1, y_2 \in Y$, there exist sequences of points $\{y_{1,i}, y_{2,i}\}_{i=1}^\infty$ and minimal geodesics $\gamma_i \subset \mathcal{R}$ connecting $y_{1,i}$ and $y_{2,i}$ such that

$$y_{1,i} \to y_1, \ y_{2,i} \to y_2, \ L_{g_\infty}(\gamma_i) \to d_\infty(y_1, y_2)$$

as $i \to \infty$. Obviously the metric space $(Y, d_\infty)$ is compact, so $\gamma_i$ can be chosen such that they converge to a minimal geodesic connecting $y_1$ and $y_2$ by Arzela-Ascoli theorem. This implies that $(Y, d_\infty)$ is a path metric space. \qed

From now on, we assume that $(M_k, \tilde{g}_k) \xrightarrow{dGH} (Y, \tilde{d}_\infty)$; assume further, as in Claim 3.3, that $\psi_k^* g_k \xrightarrow{C^m} g_\infty$ on $\mathcal{R}$. The later convergence is uniform on any compact subset. Denote by $d_k$ and $\tilde{d}_k$ the distance function induced by $g_k$ and $\tilde{g}_k$ respectively on $M_k$.

Let $\epsilon > 0$ be a fixed constant and $A \subset \mathcal{R}$ be an $\epsilon$-dense set of $(Y, \tilde{d}_\infty)$, i.e., the $\epsilon$-neighborhood of $A$ covers $Y$. Obviously $A$ is $C^{1/2}_1 \epsilon$-dense in $(Y, d_\infty)$. The goal for a while is to show that $\psi_k$ defines a $3\epsilon$-approximation from $(A, d_\infty)$ to $(M_k, \tilde{g}_k)$ whenever $k$ is large enough.

**Claim 3.5.** $\psi_k(A)$ is $3\epsilon$-dense in $(M_k, \tilde{g}_k)$ for $k$ large enough.

**Proof.** Let $K \subset \mathcal{R}$ be a suitable chosen compact submanifold with smooth boundary such that $A \subset K$ and that $\text{Vol}_{\tilde{g}_\infty}(\partial K)$ is as small as possible. This is can be done since the singular set is of codimension at least 2 in
(Y, \tilde{g}_\infty). We first show that \psi_k(K) is \epsilon-dense in (M_k, \tilde{g}_k). Let \{U_{k,j}\}^{N_k}_{j=1} be the components of \( M_k \setminus \psi_k(K) \). Then noting that for k large enough, K is contained in the interior of \( K_k \). Thus \( \partial(M_k \setminus \psi_k(K)) = \partial \psi_k(K) = \psi_k(\partial K) \) is a smooth submanifold of \( M_k \), so

\[ \partial U_{k,j_1} \cap \partial U_{k,j_2} = \emptyset \text{ whenever } j_1 \neq j_2. \]

Using the isoperimetric inequality \([8]\) to each domain \( U_{k,j} \), and noticing that \( \text{Vol}_{\tilde{g}_\infty}(\partial K) \) is as any small as possible, we obtain that

\[ \text{Vol}_{\tilde{g}_k}(U_{k,j}) \leq C_1(M_k, \tilde{g}_k) \cdot \text{Vol}_{\tilde{g}_k}(\partial U_{k,j}) \frac{n}{n-1} < \kappa \epsilon^n \]

whenever \( k \) is large enough. This, together with \([22]\), implies that \( M_k \setminus \psi_k(K) \) contains no balls of radius \( \epsilon \). Thus \( \psi_k(K) \) is \( \epsilon \)-dense in \( (M_k, \tilde{g}_k) \).

On the other hand, since \( A \) is \( \epsilon \)-dense in \( K \), we get that \( \psi_k(A) \) is \( 2\epsilon \)-dense in \( (\psi_k(K), \tilde{d}_k) \). Actually, for any \( y \in K \), there exists a curve \( \gamma \subset \mathcal{R} \) joining \( y \) to a point in \( A \) such that

\[ L_{\tilde{g}_\infty}(\gamma) \leq \tilde{d}_\infty(y, A) + \epsilon/2 < 3\epsilon/2. \]

Thus the distance

\[ \tilde{d}_k(\psi_k(y), \psi_k(A)) \leq L_{\tilde{g}_k}(\psi_k(\gamma)) \leq \epsilon/2 + L_{\tilde{g}_\infty}(\gamma) < 2\epsilon \]

for \( k \) large enough, since \( \psi_k^*\tilde{g}_k \) converges to \( \tilde{g}_\infty \) in \( C^{1,\alpha} \) sense on compact subsets of \( \mathcal{R} \). Then using the fact that \( K \) is compact one deduces the desired result.

Summing up the results obtained completes the proof of the claim. \( \square \)

Next we show that \( \psi_k \) is almost an isometry on \( A \).

**Claim 3.6.** For all \( a_1, a_2 \in A \) and \( k \) large enough,

\[ \tilde{d}_k(\psi_k(a_1), \psi_k(a_2)) \leq \tilde{d}_\infty(a_1, a_2) + 2\epsilon. \]  (25)

*Proof.* Indeed, for \( a_1, a_2 \in A \subset \mathcal{R} \), there exists a curve \( \gamma \subset \mathcal{R} \) joining \( a_1, a_2 \) such that

\[ L_{\tilde{g}_k}(\gamma) \leq \tilde{d}_\infty(a_1, a_2) + \epsilon. \]

On the other hand, the length of curves \( \psi_k(\gamma) \) with respect to \( \tilde{g}_k \) is bigger than \( \tilde{d}_k(\psi_k(a_1), \psi_k(a_2)) \). Thus, using that \( \psi_k^*\tilde{g}_k \) converges to \( \tilde{g}_\infty \) uniformly, we have for \( k \) large enough

\[ \tilde{d}_k(\psi_k(a_1), \psi_k(a_2)) \leq L_{\tilde{g}_k}(\psi_k(\gamma)) \leq L_{\tilde{g}_\infty}(\gamma) + \epsilon \leq \tilde{d}_\infty(a_1, a_2) + 2\epsilon. \]

This proves \( (25) \). \( \square \)

Next we prove the other part:

**Claim 3.7.** For all \( a_1, a_2 \in A \) and \( k \) large enough,

\[ \tilde{d}_k(\psi_k(a_1), \psi_k(a_2)) \geq \tilde{d}_\infty(a_1, a_2) - 2\epsilon. \]  (26)
Passing to the sequence, we have that

\[
\text{Claim 3.8.} \quad \text{such that } V \ vol_{\tilde{M}^k} \text{ both Gromov-Hausdorff convergence (GH)} \text{ contained on } \epsilon \text{ modification of the } \kappa \text{ for } \psi \text{ such that } V \ vol_{\tilde{M}^k} \text{ large enough. Then for given } a_1, a_2 \in A, \text{ if any minimal geodesic from } \psi_k(a_2) \text{ to points in } B_{\tilde{g}_k}(\psi_k(a_1), \epsilon/5) \text{ intersects } M_k \backslash \psi_k(K), \text{ then Lemma 2.2 yields that}
\]

\[
\kappa \cdot (\epsilon/5)^n \leq V \ vol_{\tilde{g}_k}(B_{\tilde{g}_k}(\psi_k(a_1), \epsilon/5)) \leq C(n, v, D, \epsilon) \cdot V \ vol_{\tilde{g}_k}(\psi_k(\partial K)),
\]

which can not happen if \( V \ vol_{\tilde{g}_k}(\partial K) \) is less than a quantity depending on \( n, v, D \) and \( \epsilon \). Thus for any \( a_1, a_2 \in A \) and \( k \) large enough, there exists one minimal geodesic \( \gamma_k \) joining \( \psi_k(a_2) \) to a point in \( B_{\tilde{g}_k}(\psi_k(a_1), \epsilon/5) \) which is contained in \( \psi_k(K) \). As \( k \to \infty \), \( \gamma_k \) converge to one minimal geodesic \( \gamma_\infty \) contained on \( K \) which connects \( a_2 \) and one point in \( B_{\tilde{g}_\infty}(a_1, \epsilon/5) \). Thus

\[
\tilde{d}_k(\psi_k(a_1), \psi_k(a_2)) \geq L_{\tilde{g}_k}(\gamma_k) - \epsilon/5 \geq L_{\tilde{g}_\infty}(\gamma_\infty) - \epsilon \geq \tilde{d}_\infty(a_2, \gamma_\infty(1)) - \epsilon \geq \tilde{d}_\infty(a_2, a_1) - 2\epsilon
\]

whenever \( k \) is large enough. This proves the claim.

We mention that the key point here is that, the submanifold \( K \) can be chosen such that its boundary has volume as small as possible. \( \square \)

Passing to the the metrics \( g_k, \psi_k(A) = 3C_1^{1/2} \epsilon \)-dense in \( M_k \). Applying the same argument as in Claim 3.6 and 3.7, by the \( C^\alpha \) convergence on \( \mathcal{R} \), one can show that

\[
|d_k(\psi_k(a_1), \psi_k(a_2)) - d_\infty(a_1, a_2)| \leq 2\epsilon
\]

for all \( a_1, a_2 \in A \) and \( k \) large enough. This means that \( \psi_k \) defines a \( 3C_1^{1/2} \epsilon \)-approximation from \((A, d_\infty)\) to \((M_k, d_k)\). As a consequence, noting that \( A \) is \( C_1^{1/2} \epsilon \)-dense in \((Y, d_\infty)\) and then letting \( \epsilon \to 0 \), we finally obtain

Claim 3.8. \( (M_k, g_k) \) converges to \((Y, d_\infty)\) in the Gromov-Hausdorff sense.

From the arguments above, the maps \( \psi_k \) define the approximation for both Gromov-Hausdorff convergence \((M_k, g_k) \xrightarrow{dGH} (Y, d_\infty)\) and \((M_k, \tilde{g}_k) \xrightarrow{dGH} (Y, \tilde{d}_\infty)\). Thus the convergent sequence of \((M_k, g_k)\) coincides with that of \((M_k, \tilde{g}_k)\). That’s, for \( x_k \in M_k \) and \( x_\infty \in Y \), \( x_k \to x_\infty \) under \((M_k, g_k) \xrightarrow{dGH} (Y, d_\infty)\) if \( x_k \to x_\infty \) under \((M_k, \tilde{g}_k) \xrightarrow{dGH} (Y, \tilde{d}_\infty)\).
Next we show that $y$ is a singular point of $(Y,d_\infty)$ iff $y$ is a singular point of $(Y,d_\infty)$. This means that $S$ is the singular set of $(Y,d_\infty)$. It suffices to prove the following claim.

**Claim 3.9.** The tangent cones of $(Y,d_\infty)$ coincide with that of $(Y,\tilde{d}_\infty)$ up to rescalings by constants.

**Proof of the Claim.** Let $y \in Y$ and $r_j \to 0$, we want to show that, modulo a rescaling, the limits

$$
\lim_{j \to \infty} (Y,r_j^{-1}d_\infty,y) = \lim_{j \to \infty} (Y,r_j^{-1}\tilde{d}_\infty,y)
$$

in the pointed Gromov-Hausdorff topology, if either the limit exists, that’s, the associated tangent cone at $y$ with respect to $d_\infty$ and $\tilde{d}_\infty$ are the same.

Passing a subsequence if necessary, the potentials $f_k$ converge to a Lipschitz function $f_\infty$ on $Y$. Then $\tilde{g}_\infty = e^{-\frac{3}{2}f_\infty}g_\infty$ on $\mathcal{R}$. By (18), we have $\text{Lip}(f_k),\text{Lip}(f_\infty) \leq C_0$ for some $C_0 = C_0(n,D)$. Let $\rho$ be any fixed positive constant. The distance functions $d_\infty$ and $\tilde{d}_\infty$ satisfy the relative comparison on $B_{\tilde{g}_\infty}(y,r_j\rho)$, $\beta_j^{-1} \leq \alpha(y) \cdot \frac{\tilde{d}_\infty(x_1,x_2)}{d_\infty(x_1,x_2)} \leq \beta_j$, for all $x_1,x_2 \in B_{\tilde{g}_\infty}(y,r_j\rho)$, where $\alpha(y) = e^{-\frac{1}{n-2}f_\infty(y)}$ is a fixed constant, while $\beta_j = e^{\frac{3}{n-2}C_0r_j\rho}$ tends to 1 as $j \to \infty$.

Let $(Y_y,d_y,o) = \lim_{j \to \infty} (Y,r_j^{-1}d_\infty,y)$. By the definition of convergence, for any $\epsilon > 0$, there exists an $\epsilon$-approximation $\psi_j : (B_{d_y}(o,\rho),d_y) \to (B_{\tilde{d}_\infty}(y,r_j\rho),r_j^{-1}d_\infty)$ for all $j$ large enough. Thus, whenever $j$ is large enough, the map $\psi_j$ defines a $10((\beta_j-1)\rho+\epsilon\beta_j)$-approximate from $(B_{d_y}(o,\rho),d_y)$ to $B_{\alpha(y)\tilde{d}_\infty}(y,(1+\beta_j^{1/3})r_j\rho)$. This shows that

$$(Y,\alpha(y)r_j^{-2}\tilde{g}_\infty,y) \xrightarrow{d_{GH}} (Y_y,d_y,o)$$

in the pointed Gromov-Hausdorff sense. It is equivalent to that

$$(Y,r_j^{-2}\tilde{g}_\infty,y) \xrightarrow{d_{GH}} (Y_y,\alpha(y)^{-1}d_y,o)$$

in the pointed Gromov-Hausdorff sense. This proves of the claim. $\square$

At last, we confirm the smoothness of $g_\infty$ on $\mathcal{R}$ and finishes the proof of Theorem [14]. We will use the pseudolocality theorem in the argument.

**Claim 3.10.** $g_\infty$ is smooth and satisfies a shrinking Ricci soliton equation on $\mathcal{R}$. 

Proof of the Claim. Given any small number \( r > 0 \), define

\[
K_r = \{ x \in \mathcal{R} | d_{\infty}(x, S) \geq r \}.
\]

Then \( K_r \subset K_k \) for all \( k \) large enough.

Noting that \( g_\infty \) is a \( C^\alpha \) Riemannian metric on \( \mathcal{R} \), we have for some small constant \( \rho = \rho(r) \leq \epsilon_0 r \)

\[
\text{Vol}(\partial \Omega)^n \geq (1 - \frac{1}{2}\delta_0) c_n \text{Vol}(\Omega)^{n-1}, \quad \forall \Omega \subset B_{g_\infty}(x_\infty, \rho), x_\infty \in K_r
\]

where \( c_n \) is the Euclidean isoperimetric constant and \( \epsilon_0, \delta_0 \) are constants in the Pseudolocality theorem \( \PageIndex{2.1} \). Passing to the sequence \( (M_k, g_k) \), since \( \psi_k^* g_k \) converge to \( g_\infty \) uniformly in \( C^\alpha \) on \( K_k \), we may assume that any domain \( \Omega_k \subset B_{g_\infty}(\psi_k(x_\infty), \frac{1}{2}\rho) \), where \( x_\infty \in K_r \), satisfies the following isoperimetric inequality

\[
\text{Vol}(\partial \Omega_k)^n \geq (1 - \delta_0) c_n \text{Vol}(\Omega_k)^{n-1},
\]

whenever \( k \) is large enough.

Let \( g_k(t) \) be the Ricci flow solution with initial metric \( g_k(0) = g_k \). It’s well known that \( g_k(t) = (1-t)\phi_k(t)^* g_k \) for a parameter family of diffeomorphisms \( \phi_k(t) \in \text{Diff}(M_k) \) which is generated by the gradient field of \( f_k \), cf. \( \PageIndex{12} \). Because \( f_k \) is \( C^1 \) bounded, \( \phi_k(t)(\psi_k(K_{2r})) \subset \psi_k(K_r) \) for all \( t \) in a small time interval \( [0, \eta] \) where \( \eta = \eta(r) \) does not depend on \( k \) whenever \( k \) is large enough. Applying the Pseudolocality theorem \( \PageIndex{2.1} \) to points in \( \psi_k(K_{2r}) \) at time \( t(r) = \min(\epsilon_0 r, \eta) \), we get a uniform curvature bound

\[
|Rm(g_k)|(x) \leq (1-t(r)) \cdot (t(r)^{-1} + (\epsilon_0 r)^{-2}), \quad \forall x \in \psi_k(K_{2r})
\]

Then Shi’s gradient estimate \( \PageIndex{22} \) to the Ricci flow \( g_k(t) \) on the ball \( B_{g_k}(x, r) \), where \( x \in \psi_k(K_{3r}) \), gives the higher derivation estimate for curvature

\[
|\nabla^l Rm(g_k)|(x) \leq C(n, l, r), \quad \forall x \in \psi_k(K_{3r}), l \geq 0
\]

for some constant \( C \) depending only on \( n, r \) and positive integer \( l \), but does not depend on specified \( k \) which is large enough. It follows from Cheeger-Gromov’s compactness theorem that, passing a subsequence once again, the metrics \( \psi_k^* g_k \), modulo changes of diffeomorphisms on subsets of \( M_k \), converge in the \( C^\infty \) sense to \( g_\infty \) on \( K_{3r} \). The arbitrariness of \( r > 0 \) implies that \( g_\infty \) is indeed smooth on the whole \( \mathcal{R} \).

In view of the soliton equation \( \PageIndex{12} \), the gradient estimate for curvature of \( g_k \) concludes the gradient estimate for potential functions as well:

\[
|\nabla^l f_k|(x) \leq \tilde{C}(n, l, r), \quad \forall x \in \psi_k(K_{3r}),
\]

where \( \tilde{C} \) are constants depending only on \( n, r \) and \( l \). On the other hand, one has the \( C^1 \) uniform bound of \( f_k \) over \( M_k \). Combining these we obtain that \( \psi_k^* f_k \) converge to a \( C^\alpha \) function \( f_\infty \) on \( Y \) which is smooth on \( \mathcal{R} \).
Furthermore, by the smooth convergence, the shrinking soliton equation

\[ \text{Ric}(g_\infty) + \text{Hess}(f_\infty) = \frac{1}{2} g_\infty \]

also holds on \( \mathcal{R} \). This completes the proof of the claim. \( \square \)

**Remark 3.11.** One may hope that the analogy for Kähler Ricci solitons remains true. That’s, the limit space of Kähler Ricci solitons with positive first Chern class, under (2) and (3), has singular set of Hausdorff codimension at least 4. To get this result, one needs more analysis on the tangent cone at the singular point, cf. [10].

### 3.6. Integral curvature bounds.

The following corollary has appeared in several papers, see [25], [30] and [11]. We reprove it here as a corollary of our main theorem.

**Corollary 3.12.** Let \((M_k, g_k)\) be a sequence of closed shrinking Ricci solitons which satisfies (2), (3) and

\[ \int_{M_k} |Rm(g_k)|^{n/2} dv_{g_k} \leq \Lambda \]

for some \(\Lambda\) independent of \(k\). Then \((M_k, g_k)\) converges along a subsequence to a compact orbifold shrinking Ricci soliton.

Here, an orbifold shrinking Ricci soliton is defined to be a smooth Riemannian orbifold which satisfying equation (12) for some smooth function in the orbifold sense.

**Proof.** Let \((M, g) = (M_k, g_k)\) for a fixed \(k\) and \(f = f_k\) be the associated potential function, then in local normal coordinate \((x^1, \cdots, x^n)\), the curvature tensor of \(\tilde{g}\) is given by, cf. [5],

\[ \tilde{R}_{ijkl} = e^{-\frac{2}{n-2}f} (R_{ijkl} + \frac{1}{(n-2)^2} g \circ ((n-2) \nabla^2 f + df \otimes df - \frac{1}{2} |\nabla f|^2 g)) \]

where \(\circ\) denotes the Kulkani-Nomizu product which is defined by

\[ (u \circ v)_{ijkl} = u_{ik}v_{jl} + u_{jl}v_{ik} - u_{il}v_{jk} - u_{jk}v_{il} \]
for any symmetric (2,0)-tensors $u$ and $v$. Substituting into the identity
\[
\nabla^2 f = \frac{1}{2} g - \Ric
\]
and integrating over $M$, we get
\[
\int_M |Rm(\tilde{g})|^{n/2}_{\tilde{g}} dv_{\tilde{g}}
\]
\[
= \int_M |R_{ijkl} + \frac{1}{(n-2)^2} g \circ ((n-2)\nabla^2 f + df \otimes df - \frac{1}{2} |\nabla f|^2 g)^{n/2} dv_{g_k}
\]
\[
\leq C(n) \cdot \int_M (|Rm|^{n/2} + |\nabla^2 f|^{n/4} + |\nabla f|^{n/2} + |\nabla f|^n) dv
\]
\[
\leq C(n) \cdot \int_M (|Rm|^{n/2} + |\nabla f|^n + 1) dv.
\]
By (18) and (22) we get a uniform bound $C = C(n,v,D,\Lambda)$ such that
\[
\int_M |Rm(\tilde{g})|^{n/2}_{\tilde{g}} dv_{\tilde{g}} \leq C.
\]

By Theorem 1.1 together with Theorem 2.6 of [2], we know that $(M_k, g_k)$ converge along a subsequence to a limit metric space, say $(Y, g_{\infty})$, with finite orbifold singularities. The metric $g_{\infty}$ is globally $C^0$ and smooth on $\mathcal{R}$, the regular part of $Y$. Furthermore, $g_{\infty}$ satisfies the Ricci soliton equation (12) on $\mathcal{R}$ for some function $f_{\infty}$ which is globally Lipschitz and locally smooth on $\mathcal{R}$. The remaining is to show that modulo a diffeomorphic transformation, the lifted metric of $g_{\infty}$ on the resolving domain admits a smooth extension across the singular point. The approach is standard for Einstein manifolds by now, see [1], [4] or [23] for instance; the treatment for Ricci soliton cases are similar, we refer to [6] and [28] for details.

So we finish the proof of the theorem. \[\square\]

**Remark 3.13.** In dimension four, the upper bound of $L^{n/2}$-norm of curvature tensor in the theorem can be replaced by the bound of the second Betti number; see [30].

4. **The Kähler case**

In this section we give a proof of Theorem 1.2. The proof adopts a different method, so we put it in a new section.

**Proof of Theorem 1.2.** By Cheeger-Gromov’s compactness theorem, it suffices to give a uniform bound for the sectional curvature of manifolds satisfying assumptions in the theorem. We adopt a blowing up argument to prove this. The idea comes from [21].

Suppose otherwise, there exists a sequence of shrinking Kähler Ricci solitons of positive first Chern classes $(M_k, g_k)$ satisfying (2), (3) and (27), and
a sequence of points $p_k \in M_k$ such that
\[ |Rm(g_k)|(p_k) = \sup |Rm(g_k)| \to \infty \]
as $k \to \infty$. Then the rescaled sequence of pointed manifolds $(M_k, Q_k g_k, p_k)$, where $Q_k = |Rm(g_k)|(p_k)$, converge smoothly along a subsequence to a complete Ricci flat Kähler manifold $(M_\infty, g_\infty, p_\infty)$ such that $|Rm(g_\infty)|(p_\infty) = 1$. Furthermore, $g_\infty$ has Euclidean volume growth and integral curvature bound:
\[ \text{Vol}(B_{g_\infty}(p_\infty, r)) \geq \kappa r^{2n}, \quad \forall r > 0, \]
\[ \int_{M_\infty} |Rm(g_\infty)|^n dv_{g_\infty} \leq C, \]
for some $\kappa > 0$. Recall that $\dim_C(M_\infty) = n \geq 3$, by Theorem 2 in [24], one knows that $(M_\infty, g_\infty)$ is a resolution of $\mathbb{C}^n/\Gamma$ for some finite subgroup of $SU(n)$ which acts freely on $\mathbb{C}^n \setminus \{0\}$. $\Gamma$ is nontrivial since otherwise $M_\infty$ is simply connected at infinity, then by Theorem 3.5 in [1], $M_\infty$ is flat, which contradicts with $|Rm(g_\infty)|(p_\infty) = 1$. In particular, there exists a compact subvariety in $M_\infty$, say $V$, which represents one integral homology class in $M_\infty$. Then adopting an argument step by step as in [21], one gets a contradiction. This finishes the proof of the theorem.

**Remark 4.1.** Y.G. Zhang told the author that the treatment can also be applied to the Kähler Einstein manifolds with non-positive Einstein constants whose Kähler class lies into the integral cohomological classes. For the negative Kähler Einstein case, this is proved in [24].

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