Multigraded Betti numbers of simplicial forests

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Abstract

We prove that multigraded Betti numbers of a simplicial forest are always either 0 or 1. Moreover a nonzero multidegree appears exactly in one homological degree in the resolution. Our work generalizes work of Bouchat [2] on edge ideals of graph trees.

1 Introduction

The Betti numbers of edge ideals of graph forests were studied by several authors ([2], [7], [8], [9], [10]). Kimura [10] combinatorially characterized the graded Betti numbers for a graph forest. In [2] Bouchat proved that multigraded Betti numbers of graph trees are always 0 or 1 by using the mapping cone construction. Ehrenborg and Hetyei [3] showed that the independence complex of graph forests are simple-homotopy equivalent to a single vertex or to a sphere. By the well-known formula of Hochster this implies that multigraded Betti numbers of graph forests appear in at most one homological degree. We shall generalize these results about multigraded Betti numbers to simplicial forests.

Note that multigraded Betti numbers of edge ideals are not necessarily 0 or 1 in general. Also a multidegree can appear at more than one homological degree, see the example below.

Example 1.1. For $I = (ab, ae, be, cd, ce, de)$ one can check with Macaulay2 [6] that $b_{2,abcde}(I) = 2$ and $b_{3,abcde}(I) = 1$.

2 Background material

2.1 Resolutions

Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $k$. A minimal free resolution of a monomial ideal $I$ is an exact sequence of free $S$-modules

$$0 \rightarrow F_r \xrightarrow{d_r} \cdots \xrightarrow{d_1} F_1 \xrightarrow{d_0} F_0 \xrightarrow{d_0} I \rightarrow 0$$

such that $d_{i+1}(F_{i+1}) \subseteq (x_1, \ldots, x_n)F_i$ for all $i \geq 0$. The rank of $F_i$ is called the $i^{th}$ total Betti number of $I$ and is denoted by $b_i^S(I)$. Moreover if the differential maps preserve the
(standard) degrees, then the resolution is called a **minimal graded free resolution**. In this case the resolution is of the form

\[
0 \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{b^S_{i,j}(I)} \xrightarrow{d_r} \cdots \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{b^S_{i,j}(I)} \xrightarrow{d_1} \bigoplus_{j \in \mathbb{Z}} S(-j)^{b^S_{i,j}(I)} \xrightarrow{d_0} I \to 0
\]

where the integers \(b^S_{i,j}(I)\) are the **graded Betti numbers** of \(I\).

One usually also considers \(\mathbb{N}^n\)-grading (multigrading) on \(S\) where \(\mathbb{N} = \{0, 1, 2, \ldots\}\). Note that with this grading the degree of a monomial \(m = x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}\) is equal to \(m = (a_1, \ldots, a_n)\). If the differential maps of a minimal free resolution preserve the multidegrees then it takes the following form:

\[
0 \to \bigoplus_{m \in \mathbb{N}^n} S(-m)^{b^S_{i,m}(I)} \xrightarrow{d_r} \cdots \to \bigoplus_{m \in \mathbb{N}^n} S(-m)^{b^S_{i,m}(I)} \xrightarrow{d_1} \bigoplus_{m \in \mathbb{N}^n} S(-m)^{b^S_{i,m}(I)} \xrightarrow{d_0} I \to 0
\]

which is called a **minimal multigraded free resolution**. The associated ranks \(b^S_{i,m}(I)\) are called **multigraded Betti numbers** of \(I\).

Clearly the Betti numbers are related with the following equations.

\[
b^S_i(I) = \sum_{j \in \mathbb{Z}} b^S_{i,j}(I)
\]

\[
b^S_{i,j}(I) = \sum_{\deg(m) = j} b^S_{i,m}(I)
\]

where \(\deg(m)\) stands for the standard degree of \(m\), i.e., \(\deg(x_1^{a_1} \cdots x_n^{a_n}) = a_1 + \cdots + a_n\). For simplicity, we shall use a monomial \(m\) and its \(\mathbb{N}^n\)-degree \(m\) interchangeably.

### 2.2 Simplicial complexes and homology

An abstract simplicial complex \(\Gamma\) on a set of vertices \(V(\Gamma) = \{x_1, \ldots, x_n\}\) is a collection of subsets of \(V(\Gamma)\) such that \(\{x_i\} \in \Gamma\) for all \(i\) and, \(F \in \Gamma\) implies that all subsets of \(F\) are also in \(\Gamma\). The elements of \(\Gamma\) are called **faces** and the maximal faces under inclusion are called **facets**.

Since the simplicial complex \(\Gamma\) is determined by its facets \(F_1, \ldots, F_q\) we say that \(F_1, \ldots, F_q\) **generate** \(\Gamma\) and, write \(\Gamma = \langle F_1, \ldots, F_q \rangle\) or Facets(\(\Gamma\)) = \(\{F_1, \ldots, F_q\}\). A subcollection of \(\Gamma\) is a subcomplex generated by a subset of the facets of \(\Gamma\). The simplicial complex obtained by **removing the facet** \(F_i\) from \(\Gamma\) is the simplicial complex \(\Gamma \setminus \langle F_i \rangle = \langle F_1, \ldots, \hat{F}_i, \ldots, F_q \rangle\). If \(A\) is a subset of \(V(\Gamma)\), the **induced subcollection on** \(A\) is defined as \(\Gamma_A = \langle F \in \text{Facets}(\Gamma) \mid F \subseteq A \rangle\)

Two facets \(F\) and \(G\) of \(\Gamma\) are **connected** if there exists a chain of facets of \(\Gamma\), \(F_0 = F, F_1, \ldots, F_m = G\) such that every two consecutive facets has nonempty intersection. The simplicial complex \(\Gamma\) is called **connected** if any two of its facets are connected.
Let $F$ be a facet of $\Gamma$. The **connected component** of $F$ in $\Gamma$ is denoted by $\text{conn}_\Gamma(F)$. If $\text{conn}_\Gamma(F) \setminus \langle F \rangle = (F_1, ..., F_p)$, then the **the reduced connected component** of $F$ in $\Gamma$ denoted by $\overline{\text{conn}_\Gamma(F)}$ will be the simplicial complex

$$
\overline{\text{conn}_\Gamma(F)} = (F_i \setminus F \mid (F_j \setminus F) \not\subseteq (F_i \setminus F), j \neq i, 1 \leq i, j \leq p).
$$

In other words, the facets of $\overline{\text{conn}_\Gamma(F)}$ are the minimal nonempty sets among all sets $G \setminus F$, where $G$ is a facet of $\text{conn}_\Gamma(F)$.

A facet $F$ of $\Gamma$ is a **leaf** if either $F$ is the only facet of $\Gamma$, or there exists a facet $G \in \Gamma$ such that $F \cap F' \subseteq G$ for every facet $F' \neq F$. By definition, every leaf $F$ of $\Gamma$ contains a vertex $v$ such that $v \notin F'$ for every facet $F' \neq F$ of $\Gamma$. Such a vertex is called a **free vertex**. A connected simplicial complex $\Gamma$ is a **tree** if every nonempty subcollection of $\Gamma$ has a leaf. We say $\Gamma$ is a **forest** if every connected component of $\Gamma$ is a tree.

The **facet ideal** $\mathcal{F}(\Gamma)$ of $\Gamma$ is the monomial ideal in $\mathbb{k}[x_1, ..., x_n]$ which is generated by $\{\prod_{x \in F} x \mid F$ is a facet of $\Gamma\}$. Using the following correspondence

$$
m = x_{i_1} ... x_{i_s}, a \text{ squarefree monomial} \iff A = \{x_{i_1}, ..., x_{i_s}\} \subseteq \mathcal{V}(\Gamma)
$$

we shall use the squarefree monomials and nonempty subsets of $\mathcal{V}(\Gamma)$ interchangeably.

A **simplex** is a simplicial complex with only one nonempty facet. For each integer $i$, the $\mathbb{k}$-vector space $\overline{H_i}(\Gamma, \mathbb{k})$ is the $i^{th}$ **reduced homology** of $\Gamma$ over $\mathbb{k}$.

### 2.3 The Taylor complex

Let $I$ be an ideal in $S = \mathbb{k}[x_1, ..., x_n]$ which is minimally generated by the monomials $m_1, ..., m_s$. In [1], Taylor constructed an explicit multigraded free resolution of $I$ which is usually nonminimal. This construction was generalized then to simplicial resolutions in [1]. Taylor’s resolution is an example of a simplicial resolution where the underlying simplicial complex is a full simplex over the vertex set labeled with $\{m_1, ..., m_s\}$, called the **Taylor simplex** of $I$. The Betti numbers of $I$ can be determined by the dimensions of reduced homologies of certain subcomplexes of the Taylor simplex. Before stating this precisely we need one more definition.

Let $\Theta$ be the Taylor simplex whose vertices are labeled with monomials $m_1, ..., m_s$. If $\tau = \{m_{i_1}, ..., m_{i_r}\}$ is a face of $\Theta$, then by lcm($\tau$) we mean lcm($m_{i_1}, ..., m_{i_r}$). For any monomial $m$ in $S$ the simplicial subcomplex $\Theta_{<m}$ is defined as

$$
\Theta_{<m} = \{\tau \in \Theta \mid \text{lcm(}\tau\text{)} \text{ strictly divides } m\}.
$$

**Example 2.1.** For $I = (x_1x_2, x_1x_3, x_1x_4, x_3x_4)$ the Taylor simplex $\Theta$ and a subcomplex $\Theta_{<x_1x_2x_3x_4}$ are illustrated in Figures [1] and [2] respectively.

**Theorem 2.2** ([1]). Let $I$ be a monomial ideal of $S$ which is minimally generated by the monomials $m_1, ..., m_s$. Denote by $\Theta$ the Taylor simplex of $I$. For $i \geq 0$, the multigraded Betti numbers of $I$ are given by

$$
b^S_{i,m}(I) = \begin{cases} 
\dim \overline{H}_{i-1}(\Theta_{<m}; \mathbb{k}), & \text{if } m \text{ divides } \text{lcm}(m_1, ..., m_s) \\
0, & \text{otherwise.}
\end{cases}
$$

(2)
Remark 2.3. By Theorem 2.2, we are allowed to not specify a polynomial ring $S$ when we deal with Betti numbers. We can think of a facet ideal $\mathcal{F}(\Gamma)$ lying in a polynomial ring over $k$ that contains at least as many variables as the vertices of $\Gamma$. Therefore we drop $S$ and write $b_{i,m}(\mathcal{F}(\Gamma))$ and $b_{i,j}(\mathcal{F}(\Gamma))$ for the Betti numbers.

Remark 2.4. If $I = (m_1, ..., m_s)$ and $q = \deg \text{lcm}(m_1, ..., m_s)$ then for any $r > q$ we have $b_{i,r}(I) = 0$ for all $i$. That is, $q$ is the largest possible grade at which the Betti number can be nonzero. Therefore we call the numbers $b_{i,q}(I)$, $i \in \mathbb{Z}$ as the top grade Betti numbers. Clearly, for a facet ideal $\mathcal{F}(\Gamma)$, the top grade is the number of vertices of $\Gamma$.

Remark 2.5. If $I$ is generated by a single monomial $m$ then its multigraded resolution is

$$0 \rightarrow S(-m) \rightarrow I \rightarrow 0.$$ 

3 Betti numbers of simplicial forests

Lemma 3.1. If $m$ is a squarefree monomial of degree $j$, then $b_{i,m}(\mathcal{F}(\Gamma)) = b_{i,j}(\mathcal{F}(\Gamma_m))$.

Proof. Let $\Theta$ and $\Lambda$ be Taylor simplices of $\mathcal{F}(\Gamma)$ and $\mathcal{F}(\Gamma_m)$ respectively. Then clearly we have $\Theta_{<m} = \Lambda_{<m}$. So by Theorem 2.2 $b_{i,m}(\mathcal{F}(\Gamma)) = b_{i,m}(\mathcal{F}(\Gamma_m))$. But by Equation (1) we get $b_{i,m}(\mathcal{F}(\Gamma_m)) = b_{i,j}(\mathcal{F}(\Gamma_m))$ since $m$ is the only possible squarefree monomial of degree $j$ that can divide the lcm of the generators of $\mathcal{F}(\Gamma_m)$. \qed

Lemma 3.2. If $I_1, I_2, ..., I_N$ are squarefree monomial ideals whose minimal generators contain no common variable then

$$b_{i,j}(I_1 + I_2 + ... + I_N) = \sum_{\substack{u_1 + ... + u_N = i \\ v_1 + ... + v_N = j}} b_{u_1,v_1}(I_1) ... b_{u_N,v_N}(I_N). \quad (3)$$

Moreover, if each $I_i$ has minimal generators whose least common multiple is of degree $q_i$, then

$$b_{i,q_1 + ... + q_N}(I_1 + I_2 + ... + I_N) = \sum_{u_1 + ... + u_N = i} b_{u_1,q_1}(I_1) ... b_{u_N,q_N}(I_N). \quad (4)$$
Proof. The case $N = 2$ of Equation (3) is Corollary 2.2 of [9] and, the general case follows from an easy induction on $N$. To see (4), note that we have

$$b_{i,q_1+\ldots+q_N}(I_1 + I_2 + \ldots + I_N) = \sum_{\substack{u_1+\ldots+u_N = i \\ v_1+\ldots+v_N = q_1+\ldots+q_N}} b_{u_1,v_1}(I_1)\ldots b_{u_N,v_N}(I_N).$$

by Equation (3). Suppose that $v_1 + \ldots + v_N = q_1 + \ldots + q_N$. If $v_\ell \neq q_\ell$ for some $\ell$ then there exists a $j$ such that $v_j > q_j$ whence $b_{u_j,v_j}(I_j) = 0$ since $b_{u_j,q_j}$ is a top grade Betti number. In this case the term $b_{u_1,v_1}(I_1)\ldots b_{u_N,v_N}(I_N)$ vanishes. So we can rewrite the sum above as

$$\sum_{\substack{u_1+\ldots+u_N = i \\ v_1=q_1,\ldots,v_N=q_N}} b_{u_1,v_1}(I_1)\ldots b_{u_N,v_N}(I_N)$$

and this completes the proof. □

We will make use of the following results on simplicial trees.

Lemma 3.3 ([7, 5]). Let $F$ be a facet of a forest $\Gamma$. Then $\text{conn}_\Gamma(F)$ is a forest.

If $\Gamma$ is a simplicial tree, one can order its facets as $F_0, F_1, \ldots, F_q$ so that each facet $F_i$ is a leaf of the simplicial tree $\Gamma_i = (F_0, \ldots, F_i)$ for $0 \leq i \leq q$. In [4], based on such an order, a refinement of the recursive formula for graded Betti numbers of simplicial forests (Theorem 5.8, [7]) of Hà and Van Tuyl was given.

Theorem 3.4 (Proposition 4.9, [4]). Let $\Gamma$ be a simplicial tree whose facets $F_0, F_1, \ldots, F_q$ are ordered such that each facet $F_i$ is a leaf of the simplicial tree $\Gamma_i = (F_0, \ldots, F_i)$ for $0 \leq i \leq q$. Then for all $i \geq 1$ and $j \geq 0$

$$b_{i,j}(\mathcal{F}(\Gamma)) = b_{i,j}(\mathcal{F}(\langle F_0 \rangle)) + \sum_{u=1}^{q} b_{i-1,j-|F_u|}(\mathcal{F}(\text{conn}_u(F_u)))$$

where we adopt the convention that $b_{-1,j}(I)$ is 1 if $j = 0$ and is 0 otherwise for any ideal $I$.

We now prove the main result of this paper.

Theorem 3.5. Let $\Gamma$ be a simplicial forest. Then multigraded Betti numbers of $\mathcal{F}(\Gamma)$ are either zero or one. Moreover, if for some monomial $m$ we have $b_{i,m}(\mathcal{F}(\Gamma)) \neq 0$, then $b_{j,m}(\mathcal{F}(\Gamma)) = 0$ for all $j \neq i$.

Proof. We prove the given statements by induction on the number of vertices of $\Gamma$. The case when $\Gamma$ has only one vertex is clear by Remark 2.3. Suppose that the given statements hold for any simplicial forest whose number of vertices is $s$ or less. Now let $\Gamma$ be a simplicial forest on $s + 1$ vertices and take a monomial $m$ which divides the product of vertices of $\Gamma$. Then the induced subcollection $\Gamma_m$ is also a forest by definition. Note that we have $b_{i,m}(\mathcal{F}(\Gamma)) = b_{i,j}(\mathcal{F}(\Gamma_m))$ by Lemma 3.1 where $j = \text{deg}(m)$. If $j$ is greater than the
number of vertices of $\Gamma_m$, then $b_{i,m}(\mathcal{F}(\Gamma)) = b_{i,j}(\mathcal{F}(\Gamma_m)) = 0$ by Remark 3.4. So we assume $|\mathcal{V}(\Gamma_m)| = j = \deg(m)$.

If $\Gamma_m$ is not a tree, then its connected components $\Upsilon_1, \ldots, \Upsilon_t$ satisfy the induction hypothesis. For ideals $\mathcal{F}(\Upsilon_1), \ldots, \mathcal{F}(\Upsilon_t)$ we can apply Lemma 3.2 to get

$$b_{i,j}(\mathcal{F}(\Gamma_m)) = \sum_{\gamma_1 + \ldots + \gamma_t = i} b_{\gamma_1, \ldots, \gamma_t}(\mathcal{F}(\Upsilon_1)) \ldots b_{\gamma_t, \ldots, \gamma_t}(\mathcal{F}(\Upsilon_t))$$

where $l_v$ is the number of vertices of $\Upsilon_v$ for each $1 \leq v \leq t$. Since for each $l_v$ there exists at most one $\gamma_v$ such that $b_{\gamma_v, l_v}(\mathcal{F}(\Upsilon_v)) \neq 0$, we see that there must be at most one $i$ such that $b_{i,j}(\mathcal{F}(\Gamma_m)) \neq 0$. And, in such a case

$$b_{i,j}(\mathcal{F}(\Gamma_m)) = \prod_{v=1}^{t} b_{\gamma_v, l_v}(\mathcal{F}(\Upsilon_v)) = \prod_{v=1}^{t} 1 = 1$$

as desired. Therefore we assume that $\Gamma_m$ is a tree and $j = |\mathcal{V}(\Gamma_m)|$.

Suppose that the facets $F_0, F_1, \ldots, F_q$ of $\Gamma_m$ are ordered as in Theorem 3.4 Then we have $j = |\cup_{r=0}^{q} F_r|$ as $\Gamma_m$ is a simplicial complex on $j$ vertices. Now we have

$$b_{i,j}(\mathcal{F}(\Gamma_m)) = b_{i,j}(\mathcal{F}(\{F_0\})) + \sum_{u=1}^{q} b_{i-1,j-|F_u|}(\mathcal{F}(\text{com}(\Gamma_m)_{u}(F_u)))$$

by Theorem 3.4. If $F_0$ is the only facet of $\Gamma_m$ then we are done by Remark 2.5. So assume that $q \geq 1$ and note that the set of facets of $\text{com}(\Gamma_m)_{u}(F_u)$ is a subset of $\{F_0 \setminus F_u, \ldots, F_{u-1} \setminus F_u\}$ for every $1 \leq u \leq q$.

Since $F_q$ has a free vertex in $\Gamma_m, |\mathcal{V}(\Gamma_m)_{u}| < j$ for $u < q$. In particular, $|F_0| < j$ and $|\mathcal{V}(\text{com}(\Gamma_m)_{u}(F_u))| < j - |F_u|$ when $u < q$. Hence by Remark 2.4, Equation (5) turns into

$$b_{i,m}(\mathcal{F}(\Gamma)) = b_{i-1,j-|F_q|}(\mathcal{F}(\text{com}(\Gamma_m)_{q}(F_q))).$$

Observe that by definition of $\text{com}(\Gamma_m)_{q}(F_q)$ some of $F_0 \setminus F_q, \ldots, F_{q-1} \setminus F_q$ might have already been omitted when forming the facet set of $\text{com}(\Gamma_m)_{q}(F_q)$. So, $j - |F_q|$ is greater than or equal to the number of vertices of $\text{com}(\Gamma_m)_{q}(F_q)$. If it is greater, then

$$b_{i,m}(\mathcal{F}(\Gamma)) = b_{i-1,j-|F_q|}(\mathcal{F}(\text{com}(\Gamma_m)_{q}(F_q))) = 0$$

and nothing is left to prove. Otherwise, $\text{com}(\Gamma_m)_{q}(F_q)$ is a simplicial forest on $j - |F_q|$ vertices by Lemma 3.3. Since $j \leq s + 1$, $\text{com}(\Gamma_m)_{q}(F_q)$ satisfies the induction hypothesis. The proof follows by observing that $b_{i-1,j-|F_q|}(\mathcal{F}(\text{com}(\Gamma_m)_{q}(F_q)))$ is also a multigraded Betti number.

$\square$
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