SOME PROPERTIES OF THE REPRESENTATION CATEGORY OF TWISTED DRINFELD DOUBLES OF FINITE GROUPS

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Abstract. A criterion for a simple object of the representation category $\text{Rep}(D^\omega(G))$ of the twisted Drinfeld double $D^\omega(G)$ to be a generator is given, where $G$ is a finite group and $\omega$ is a 3-cocycle on $G$. A description of the adjoint category of $\text{Rep}(D^\omega(G))$ is also given.

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1. Introduction

Modular tensor categories arise in several diverse areas such as quantum group theory, vertex operator algebras, and rational conformal field theory. Let $G$ be a finite group, let $D(G)$ denote the Drinfeld double of $G$, a quasi-triangular semisimple Hopf algebra, and let $\text{Rep}(D(G))$ denote the category of finite-dimensional complex representations of $D(G)$. The category $\text{Rep}(D(G))$ is a modular tensor category [1], and it is perhaps the most accessible constructions of a modular tensor category. As such, it is desirable to have a thorough understanding of this category. In this paper, we make a contribution towards this goal. The category $\text{Rep}(D(G))$ is equivalent to the $G$-equivariantization of $\text{Vec}_G$, and it is also equivalent to the center $\mathcal{Z}(\text{Vec}_G)$ of the tensor category $\text{Vec}_G$ of finite-dimensional $G$-graded complex vector spaces.

In the papers [3,4], R. Dijkgraaf, V. Pasquier, and P. Roche introduce a quasi-triangular semisimple quasi-Hopf algebra $D^\omega(G)$, often called the twisted Drinfeld double of $G$, where $\omega$ is a 3-cocycle on $G$. When $\omega = 1$ this quasi-Hopf algebra coincides with the Drinfeld double $D(G)$ considered above. The category $\text{Rep}(D^\omega(G))$ of finite-dimensional complex representations of $D^\omega(G)$ is a modular tensor category. Analogous to the $\omega = 1$ case, the category $\text{Rep}(D^\omega(G))$ is equivalent to the $G$-equivariantization of $\text{Vec}_G^\omega$, and it is also equivalent to the center $\mathcal{Z}(\text{Vec}_G^\omega)$ of the tensor category $\text{Vec}_G^\omega$ of finite-dimensional $G$-graded complex vector spaces with associativity constraint defined using $\omega$. Every braided group-theoretical fusion
category is equivalent to a full fusion subcategory of some \( \text{Rep}(D^\omega(G)) \), and all such subcategories were parametrized in the paper [12].

This paper contains two main results, stated below. The first gives a criterion for a simple object of \( \text{Rep}(D^\omega(G)) \) to be a generator, and the second gives a description of the adjoint category of \( \text{Rep}(D^\omega(G)) \).

**Theorem.** Let \( G \) be a finite group, let \( \omega \) be a normalized 3-cocycle on \( G \), and let \((a, \chi)\) be a simple object of \( \text{Rep}(D^\omega(G)) \). Then \((a, \chi)\) is a generator of \( \text{Rep}(D^\omega(G)) \) if and only if the following two conditions hold.

(a) The normal closure of \( a \) in \( G \) is equal to \( G \).

(b) For all \( b \in Z(G) \) and \( \chi' \in \text{Irr}_{\beta b}(G) \), if \( \chi(b)\chi'(a) = \deg \chi \deg \chi' \) (equivalently, \((a, \chi)\) and \((b, \chi')\) centralize each other), then \( b = e \) and \( \chi' = 1 \).

**Theorem.** Let \( G \) be a finite group, and let \( \omega \) be a normalized 3-cocycle on \( G \). Then

\[
\text{Rep}(D^\omega(G))_{\text{pt}} = S(Z_\omega(G), [G, G], B)
\]

and

\[
\text{Rep}(D^\omega(G))_{\text{ad}} = S([G, G], Z_\omega(G), (B^{\text{op}})^{-1})
\]

where \( B : Z_\omega(G) \times [G, G] \to \mathbb{C}^\times \) is the \( G \)-invariant \( \omega \)-bicharacter defined in Lemma 4.4.

**Organization:**

In Section 2, we recall basic facts about the modular tensor category \( \text{Rep}(D^\omega(G)) \). In Section 3, we prove the first theorem above, and in Section 4, we prove the second theorem.

**Convention and notation:**

Throughout this paper we work over the field \( \mathbb{C} \) of complex numbers. The multiplicative group of nonzero complex numbers is denoted \( \mathbb{C}^\times \). Let \( G \) be a finite group. The identity element of \( G \) is denoted \( e \), and the center of \( G \) is denoted \( Z(G) \). For any character \( \chi \) of \( G \), the degree of \( \chi \) is denoted \( \deg \chi \), the complex conjugate of \( \chi \) is denoted \( \overline{\chi} \), and the kernel of \( \chi \) is denoted \( \text{Ker} \chi \). Let \( \mu \) be a 2-cocycle on \( G \) with coefficients in \( \mathbb{C}^\times \). The set of irreducible \( \mu \)-characters of \( G \) is denoted \( \text{Irr}_\mu(G) \). When \( \mu = 1 \), we write \( \text{Irr}(G) \) instead of \( \text{Irr}_1(G) \). Finally, the coboundary operator on the space of cochains of \( G \) with coefficients in \( \mathbb{C}^\times \) is denoted \( d \).
2. Drinfeld doubles of finite groups

Let $G$ be a finite group. As stated earlier, the category $\text{Rep}(D(G))$ of finite-dimensional representations of the Drinfeld double $D(G)$ is a modular tensor category [1]. The simple objects of $\text{Rep}(D(G))$ are in bijection with the set of pairs $(a, \chi)$, where $a$ is a representative of a conjugacy class of $G$, and $\chi$ is an irreducible character of the centralizer $C_G(a)$ of $a$ in $G$. The $S$-matrix and the $T$-matrix of $\text{Rep}(D(G))$ are square matrices indexed by the simple objects of $\text{Rep}(D(G))$, and are given by the following formulas [1,2].

\[
S_{(a, \chi), (b, \chi')} = \frac{1}{|C_G(a)||C_G(b)|} \sum_{g \in G(a,b)} \chi(gbg^{-1})\overline{\chi'}(g^{-1}ag),
\]

\[
T_{(a, \chi), (b, \chi')} = \delta_{a,b} \delta_{\chi, \chi'} \deg \chi,
\]

where $G(a,b)$ denotes the set $\{g \in G \mid agbg^{-1} = gb^g a\}$.

Let $\omega : G \times G \times G \to \mathbb{C}^\times$ be a normalized 3-cocycle. Then
\[
\omega(b, c, d) \omega(a, bc, d) \omega(a, b, c) = \omega(ab, c, d) \omega(a, b, cd)
\]

for all $a, b, c, d \in G$, and $\omega(a, b, c) = 1$ if $a, b$, or $c$ is the identity element. Replacing $\omega$ by a cohomologous 3-cocycle, if necessary, we may assume that the values of $\omega$ are roots of unity.

For each $a \in G$, define a function $\beta_a : G \times G \to \mathbb{C}^\times$ by
\[
\beta_a(x, y) = \frac{\omega(a, x, y)\omega(x, y, y^{-1}x^{-1}axy)}{\omega(x, x^{-1}ax, y)}. \tag{1}
\]

The 3-cocycle condition on $\omega$ ensures that the relation
\[
\beta_{a^{-1}ax}(y, z)\beta_a(x, yz) = \beta_a(xy, z)\beta_a(x, y)
\]
holds for all $a, x, y, z \in G$. Therefore, for any $a \in G$, the restriction of $\beta_a$ to the centralizer $C_G(a)$ of $a$ in $G$ is a normalized 2-cocycle, that is,
\[
\beta_a(y, z)\beta_a(x, yz) = \beta_a(xy, z)\beta_a(x, y)
\]
for all $x, y, z \in C_G(a)$, and $\beta_a(x, y) = 1$ if $x$ or $y$ is the identity element.

For each $a \in G$, define a function $\gamma_a : G \times G \to \mathbb{C}^\times$ by
\[
\gamma_a(x, y) = \frac{\omega(x, y, a)\omega(a, a^{-1}xa, a^{-1}ya)}{\omega(x, a, a^{-1}ya)}.
\]

Direct calculations using the 3-cocycle condition of $\omega$ show that
\[
\frac{\beta_a(x, y)\beta_b(x, y)}{\beta_{ab}(x, y)} = \frac{\gamma_{xy}(a, b)}{\gamma_x(a, b)\gamma_y(x^{-1}ax, x^{-1}bx)}.
\]
for all $a, b, x, y \in G$. For all $a \in G$, the functions $\beta_a$ and $\gamma_a$ are equal when restricted to $C_G(a)$. Therefore, we have

$$\frac{\beta_a(x, y)\beta_b(x, y)}{\beta_{ab}(x, y)} = \frac{\beta_{xy}(a, b)}{\beta_x(a, b)\beta_y(a, b)}$$

for all $a, b \in Z(G)$, and $x, y \in G$.

As stated earlier, the category $\text{Rep}(D^\omega(G))$ of finite-dimensional representations of the twisted Drinfeld double $D^\omega(G)$ is a modular tensor category. The simple objects of $\text{Rep}(D^\omega(G))$ are in bijection with the set of pairs $(a, \chi)$, where $a$ is a representative of a conjugacy class of $G$, and $\chi$ is an irreducible $\beta_a$-character of the centralizer $C_G(a)$ of $a$ in $G$. The $S$-matrix and the $T$-matrix of $\text{Rep}(D^\omega(G))$ are square matrices indexed by the simple objects of $\text{Rep}(D(G))$, and are given by the following formulas [2].

$$S_{(a,\chi),(b,\chi')} = \sum_{g \in C\ell G(a) \cap C_G(g)} \frac{\beta_a(x, g')\beta_a(xg', x^{-1})\beta_b(y, g)\beta_b(yy^{-1})}{\beta_a(x, x^{-1})\beta_b(y, y^{-1})} \chi(xg'x^{-1})\chi'(yy^{-1})$$

$$T_{(a,\chi),(b,\chi')} = \delta_{a,b} \delta_{\chi,\chi'} \frac{\chi(a)}{\deg \chi}$$

where $g = x^{-1}ax$, $g' = y^{-1}by$, and $C\ell G(a)$ denotes the conjugacy class of $a$ in $G$.

3. Tensor generators

In this section, we give a criterion for a simple object $(a, \chi)$ of $\text{Rep}(D^\omega(G))$ to be a tensor generator, that is, the full fusion subcategory given by the intersection of all fusion subcategories of $\text{Rep}(D^\omega(G))$ that contain $(a, \chi)$ is $\text{Rep}(D^\omega(G))$.

Let $\mathcal{C}$ be a modular tensor category with braiding $c$. Two objects $X, Y \in \mathcal{C}$ centralize each other if

$$c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}.$$  

Let $\mathcal{D}$ be a full (not necessarily tensor) subcategory of $\mathcal{C}$. In the paper [10], M. Müger defined the centralizer of $\mathcal{D}$ in $\mathcal{C}$ as the full subcategory of $\mathcal{C}$, denoted $\mathcal{D}'$, consisting of all objects in $\mathcal{C}$ that centralize every object in $\mathcal{D}$. That is,

$$\text{Obj}(\mathcal{D}') = \{X \in \text{Obj}(\mathcal{C}) \mid c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y} \text{ for all } Y \in \text{Obj}(\mathcal{D})\}.$$  

It was shown in [10] that $\mathcal{D}'$ is a fusion subcategory, and that if $\mathcal{D}$ is a fusion subcategory, then $\mathcal{D}' = \mathcal{D}$; we refer to this result as the double centralizer theorem.

We recall the following result from [11].
Proposition 3.1. Let $G$ be a finite group, and let $(a, \chi)$ and $(b, \chi')$ be simple objects of $\text{Rep}(D(G))$. Then $(a, \chi)$ and $(b, \chi')$ centralize each other if and only if the following two conditions hold.

(a) The conjugacy classes of $a$ and $b$ commute elementwise.

(b) For all $g \in G$, $\chi(gbg^{-1})\chi'(g^{-1}ag) = \deg \chi \deg \chi'$.

Below, we record a special case of the result above.

Proposition 3.2. Let $G$ be a finite group, and let $(a, \chi)$ and $(b, \chi')$ be simple objects of $\text{Rep}(D(G))$, where $b$ lies in the center of $G$, so that $\chi' \in \text{Irr}(G)$. Then $(a, \chi)$ and $(b, \chi')$ centralize each other if and only if the following holds.

(i) $\chi(b)\chi'(a) = \deg \chi \deg \chi'$.

If $b = e$ or $\chi = 1$, then the condition above is equivalent to the condition

(i') $a \in \ker \chi'$.

Proof. If $b = e$ or $\chi = 1$, then the equality in condition (i) is equivalent to the equality $\chi'(a) = \deg \chi'$, which is equivalent to condition (i').

Suppose that $(a, \chi)$ and $(b, \chi')$ centralize each other. Putting $g = e$ in condition (b) of Proposition 3.1, we get $\chi(b)\chi'(a) = \deg \chi \deg \chi'$, which is condition (i).

Conversely, suppose that condition (i) holds. Since $b$ lies in the center of $G$, we know that $C_G(b) = G$ and $\chi'$ is a character of $G$. Condition (a) of Proposition 3.1 clearly holds. For all $g \in G$, we have $\chi(gbg^{-1})\chi'(g^{-1}ag) = \chi(b)\chi'(a)$, since $b$ is in the center of $G$ and $\chi'$ is a class function on $G$. By supposition, $\chi(b)\chi'(a) = \deg \chi \deg \chi'$, and so condition (b) of Proposition 3.1 holds. Hence $(a, \chi)$ and $(b, \chi')$ centralize each other. □

Theorem 3.3. Let $G$ be a finite group, and let $(a, \chi)$ be a simple object of $\text{Rep}(D(G))$. Then $(a, \chi)$ is a generator of $\text{Rep}(D(G))$ if and only if the following two conditions hold.

(a) The normal closure of $a$ in $G$ is equal to $G$.

(b) For all $b \in Z(G)$ and $\chi' \in \text{Irr}(G)$, if $\chi(b)\chi'(a) = \deg \chi \deg \chi'$ (equivalently, $(a, \chi)$ and $(b, \chi')$ centralize each other), then $b = e$ and $\chi' = 1$.

Proof. By the double centralizer theorem, the simple object $(a, \chi)$ is a generator of $\text{Rep}(D(G))$ if and only if the only simple object that centralizes $(a, \chi)$ is the trivial simple object $(e, 1)$.

Suppose that conditions (a) and (b) in the statement of the theorem hold, and let $(b, \chi')$ be a simple object of $\text{Rep}(D(G))$ that centralizes $(a, \chi)$. By Proposition 3.1, the conjugacy classes of $a$ and $b$ commute elementwise; combining this fact with
condition (a), we deduce that \( b \) lies in the center of \( G \), and so \( \chi' \) lies in \( \text{Irr}(G) \). By Proposition 3.2, we must have \( \chi(b)\chi'(a) = \deg \chi \deg \chi' \). Applying condition (b), we get \( b = e \) and \( \chi' = 1 \), and it follows that \( (a, \chi) \) is a generator of \( \text{Rep}(D(G)) \).

Conversely, suppose that \( (a, \chi) \) is a generator of \( \text{Rep}(D(G)) \). Then the only simple object that centralizes \( (a, \chi) \) is the trivial simple object \( (e, 1) \). Let \( b \in Z(G) \), let \( \chi' \in \text{Irr}(G) \), and suppose that \( \chi(b)\chi'(a) = \deg \chi \deg \chi' \). Then the simple objects \( (a, \chi) \) and \( (b, \chi') \) centralize each other, by Proposition 3.2. Since the only simple object that centralizes \( (a, \chi) \) is the trivial simple object \( (e, 1) \), it follows that \( b = e \) and \( \chi' = 1 \), showing that condition (b) holds.

To see that condition (a) holds, let \( H \) denote the normal closure of \( a \) in \( G \), and suppose that \( H \neq G \). By Proposition 3.2, for all \( \chi' \in \text{Irr}(G) \), the simple objects \( (a, \chi) \) and \( (e, \chi') \) centralize each other if and only if \( a \in \text{Ker} \chi' \), equivalently, \( H \leq \text{Ker} \chi' \). Since \( H \) is proper in \( G \), the action of \( G \) on the coset space \( G/H \) is not trivial, and so the corresponding representation contains a nontrivial irreducible constituent; let \( \chi' \) denote the character of this constituent. Since \( H \) is normal in \( G \), it acts trivially on \( G/H \), and so \( H \leq \text{Ker} \chi' \). It follows that \( (a, \chi) \) and \( (e, \chi') \) centralize each other, a contradiction. Hence \( H = G \), showing that condition (a) holds.

**Corollary 3.4.** Let \( G \) be a finite group with trivial center, and let \( (a, \chi) \) be a simple object of \( \text{Rep}(D(G)) \). Then \( (a, \chi) \) is a generator of \( \text{Rep}(D(G)) \) if and only if the normal closure of \( a \) in \( G \) is equal to \( G \).

**Proof.** Suppose that the normal closure of \( a \) in \( G \) is equal to \( G \). To see that condition (b) of Theorem 3.3 holds, let \( \chi' \in \text{Irr}(G) \), and suppose that the simple objects \( (a, \chi) \) and \( (e, \chi') \) centralize each other. By Proposition 3.2, the element \( a \) belongs to \( \text{Ker} \chi' \). Since \( \text{Ker} \chi' \) is a normal subgroup of \( G \), the supposition forces \( \text{Ker} \chi' = G \), equivalently, \( \chi' = 1 \), and so condition (b) of Theorem 3.3 holds. Hence \( (a, \chi) \) is a generator of \( \text{Rep}(D(G)) \).

The converse clearly holds, by Theorem 3.3.

Next, we address the twisted case. Of course, the untwisted case above is a special case of the twisted case below, but we find it instructive to treat the untwisted case separately, as in [11] and [12]. We recall the following result from [11].

**Proposition 3.5.** Let \( G \) be a finite group, let \( \omega \) be a normalized 3-cocycle on \( G \), and let \( (a, \chi) \) and \( (b, \chi') \) be simple objects of \( \text{Rep}(D^\omega(G)) \). Then \( (a, \chi) \) and \( (b, \chi') \) centralize each other if and only if the following two conditions hold.

(a) The conjugacy classes of \( a \) and \( b \) commute elementwise.
Lemma 3.6. Let $G$ be a finite group, let $ω$ be a normalized 3-cocycle on $G$, and let $a, b, x \in G$. If $ab = ba$, then

$$\frac{\beta_a(x, y^{-1})}{\beta_b(x, b)\beta_a(xb, x^{-1})} = \frac{\beta_b(x^{-1}, x)}{\beta_b(x^{-1}, a)\beta_b(x^{-1}a, x)}.$$  

Lemma 3.7. Let $G$ be a finite group, and let $µ$ be a normalized 2-cocycle on $G$.

(a) For all $a, x, y \in G$,

$$\frac{µ(y, x^{-1}a)µ(xy^{-1}ax, y^{-1})µ(xy^{-1}, yx^{-1}axy^{-1})}{µ(y, y^{-1})µ(a, xy^{-1})} = \frac{µ(x, x^{-1}a)µ(x^{-1}a, a)}{µ(x, a, x)} = \frac{µ(x, x^{-1}a)µ(x^{-1}a, a)}{µ(a, x)}.$$  

(b) For all $a, x \in G$,

$$\frac{µ(x, x^{-1})}{µ(x^{-1}, a)µ(x^{-1}a, x)} = \frac{µ(x^{-1}a, a)}{µ(x^{-1}a, x)} = \frac{µ(x^{-1}a, a)}{µ(a, x)}.$$  

Proof. That $µ$ is a normalized 2-cocyle means that

$$µ(y, z)µ(x, yz) = µ(xy, z)µ(x, y)$$

for all $x, y, z \in G$, and $µ(x, y) = 1$ if $x$ or $y$ is the identity element. Applying the 2-cocycle condition of $µ$ to the triple $(xy^{-1}, yx^{-1}ax, y^{-1})$ gives

$$µ(xy^{-1}ax, y^{-1})µ(xy^{-1}, yx^{-1}axy^{-1}) = µ(ax, y^{-1})µ(xy^{-1}, yx^{-1}ax).$$

Making this substitution in the expression

$$\frac{µ(y, x^{-1}a)µ(xy^{-1}ax, y^{-1})µ(xy^{-1}, yx^{-1}axy^{-1})}{µ(y, y^{-1})µ(a, xy^{-1})}$$

yields

$$\frac{µ(y, x^{-1}a)µ(ax, y^{-1})µ(xy^{-1}, yx^{-1}ax)}{µ(y, y^{-1})µ(a, xy^{-1})}.$$

Applying the 2-cocycle condition of $µ$ to the triple $(a, x, y^{-1})$ gives

$$\frac{µ(ax, y^{-1})}{µ(a, xy^{-1})} = \frac{µ(ax, y^{-1})}{µ(a, xy^{-1})}. $$

Making this substitution in the expression above yields

$$\frac{µ(y, x^{-1}a)µ(xy^{-1}, yx^{-1}ax)µ(x, y^{-1})}{µ(y, y^{-1})µ(a, x)}.$$  

Applying the 2-cocycle condition of $µ$ to the triple $(y, y^{-1}, yx^{-1}ax)$ gives

$$\frac{µ(y, x^{-1}a)µ(xy^{-1}, yx^{-1}ax)}{µ(y, y^{-1})µ(a, x)}.$$  

Making this substitution in the expression above yields

$$\frac{µ(xy^{-1}, yx^{-1}ax)µ(x, y^{-1})}{µ(a, x)µ(y^{-1}, yx^{-1}ax)}.$$
Applying the 2-cocycle condition of $\mu$ to the triple $(x, y^{-1}, yx^{-1}ax)$ gives
\[
\frac{\mu(xy^{-1}, yx^{-1}ax)\mu(x, y^{-1})}{\mu(y^{-1}, yx^{-1}ax)} = \mu(x, x^{-1}ax).
\]
Making this substitution in the expression above yields
\[
\frac{\mu(x, x^{-1}ax)}{\mu(a, x)},
\]
establishing (a).

Applying the 2-cocycle condition of $\mu$ to the triple $(x, x^{-1}, a)$ gives
\[
\frac{\mu(x, x^{-1})}{\mu(x^{-1}, a)\mu(x^{-1}a, x)}
\]
yields
\[
\frac{\mu(x, x^{-1}a)}{\mu(x^{-1}a, x)}.
\]
Applying the 2-cocycle condition of $\mu$ to the triple $(x, x^{-1}a, x)$ gives
\[
\frac{\mu(x, x^{-1}a)}{\mu(a, x)},
\]
and so the expression above is equal to
\[
\frac{\mu(x, x^{-1}ax)}{\mu(a, x)},
\]
establishing (b).

Proposition 3.8. Let $G$ be a finite group, let $\omega$ be a normalized 3-cocycle on $G$, and let $(a, \chi)$ and $(b, \chi')$ be simple objects of $\text{Rep}(D^\omega(G))$, where $b$ lies in the center of $G$, so that $\chi' \in \text{Irr}_\beta b(G)$. Then $(a, \chi)$ and $(b, \chi')$ centralize each other if and only if the following holds.

(i) $\chi(b)\chi'(a) = \deg \chi \deg \chi'$.

If $b = e$, then the condition above is equivalent to the condition

(i') $a \in \text{Ker} \chi'$.

Proof. If $b = e$, then $\chi'$ is an ordinary character, and the equality in condition (i) is equivalent to the equality $\chi'(a) = \deg \chi$, which is equivalent to condition (i').

Suppose that $(a, \chi)$ and $(b, \chi')$ centralize each other. Putting $x = y = e$ in condition (b) of Proposition 3.5, we get $\chi(b)\chi'(a) = \deg \chi \deg \chi'$, which is condition (i).

Conversely, suppose that condition (i) holds. Since $b$ lies in the center of $G$, we know that $C_G(b) = G$, $\beta_b$ is a 2-cocycle on $G$, and $\chi'$ is a $\beta_b$-character of $G$. Condition (a) of Proposition 3.5 clearly holds. It remains to show that condition
(b) of Proposition 3.5 holds. Let \( x, y \in G \). Since \( b \) lies in the center of \( G \), the
left-hand side of condition (b) of Proposition 3.5 reduces to
\[
\frac{\beta_b(x, b) \beta_b(x b, x^{-1}) \beta_b(y, x^{-1} a x) \beta_b(y x^{-1} a x, y^{-1})}{\beta_b(x, x^{-1}) \beta_b(y, y^{-1})} \chi(b) \chi'(y x^{-1} a x y^{-1}).
\] (3)
Let \( \rho : G \to \text{GL}(V) \) be a projective \( \beta_b \)-representation of \( G \) whose character is \( \chi' \),
and let \( z \in G \). Then
\[
\rho(a) \rho(z) = \beta_b(a, z) \rho(a z) = \beta_b(a, z) \rho(z)(z^{-1} a z),
\]
and so
\[
\rho(z^{-1} a z) = \frac{\beta_b(z, z^{-1} a z)}{\beta_b(a, z)} [\rho(z)]^{-1} \rho(a) \rho(z).
\]
Taking the trace of both sides, we get
\[
\chi'(z^{-1} a z) = \frac{\beta_b(z, z^{-1} a z)}{\beta_b(a, z)} \chi'(a).
\]
Putting \( z = xy^{-1} \) in the equation above, we get
\[
\chi'(y x^{-1} a x y^{-1}) = \frac{\beta_b(xy^{-1}, y x^{-1} a x y^{-1})}{\beta_b(a, x y^{-1})} \chi'(a).
\]
Substituting the expression above in (3), we get
\[
\frac{\beta_b(x, b) \beta_b(x b, x^{-1}) \beta_b(y, x^{-1} a x) \beta_b(y x^{-1} a x, y^{-1}) \beta_b(xy^{-1}, y x^{-1} a x y^{-1})}{\beta_b(x, x^{-1}) \beta_b(y, y^{-1}) \beta_b(a, xy^{-1})} \chi(b) \chi'(a).
\]
Using Lemma 3.6, we see that the expression above is equal to
\[
\frac{\beta_b(x^{-1}, a) \beta_b(x^{-1} a, x) \beta_b(y, x^{-1} a x) \beta_b(y x^{-1} a x, y^{-1}) \beta_b(xy^{-1}, y x^{-1} a x y^{-1})}{\beta_b(x^{-1}, x) \beta_b(y, y^{-1}) \beta_b(a, xy^{-1})} \chi(b) \chi'(a).
\]
Applying Lemma 3.7 with \( \mu = \beta_b \), and noting that \( \beta_b(x^{-1}, x) = \beta_b(x, x^{-1}) \), we see
that the expression above reduces to \( \chi(b) \chi'(a) \). By supposition, \( \chi(b) \chi'(a) = \deg \chi \deg \chi' \),
and so condition (b) of Proposition 3.5 holds. Hence \( (a, \chi) \) and \( (b, \chi') \)
centralize each other. \( \square \)

**Theorem 3.9.** Let \( G \) be a finite group, let \( \omega \) be a normalized 3-cocycle on \( G \), and let
\( (a, \chi) \) be a simple object of \( \text{Rep}(D^\omega(G)) \). Then \( (a, \chi) \) is a generator of \( \text{Rep}(D^\omega(G)) \)
if and only if the following two conditions hold.

(a) The normal closure of \( a \) in \( G \) is equal to \( G \).

(b) For all \( b \in Z(G) \) and \( \chi' \in \text{Irr}_{\beta_b}(G) \), if \( \chi(b) \chi'(a) = \deg \chi \deg \chi' \) (equivalently, \( (a, \chi) \) and \( (b, \chi') \) centralize each other), then \( b = e \) and \( \chi' = 1 \).
Proof. The proof to be given is almost identical to the one given for the untwisted case. By the double centralizer theorem, the simple object \((a, \chi)\) is a generator of \(\text{Rep}(D^\omega(G))\) if and only if the only simple object that centralizes \((a, \chi)\) is the trivial simple object \((e, 1)\).

Suppose that conditions (a) and (b) in the statement of the theorem hold, and let \((b, \chi')\) be a simple object of \(\text{Rep}(D^\omega(G))\) that centralizes \((a, \chi)\). By Proposition 3.5, the conjugacy classes of \(a\) and \(b\) commute elementwise; combining this fact with condition (a), we deduce that \(b\) lies in the center of \(G\), and so \(\chi'\) lies in \(\text{Irr}_{\beta_b}(G)\). By Proposition 3.8, we must have \(\chi(b)\chi'(a) = \deg \chi \deg \chi'\). Applying condition (b), we get \(b = e\) and \(\chi' = 1\), and it follows that \((a, \chi)\) is a generator of \(\text{Rep}(D^\omega(G))\).

Conversely, suppose that \((a, \chi)\) is a generator of \(\text{Rep}(D^\omega(G))\). Then the only simple object that centralizes \((a, \chi)\) is the trivial simple object \((e, 1)\). Let \(b \in \text{Z}(G)\), let \(\chi' \in \text{Irr}_{\beta_b}(G)\), and suppose that \(\chi(b)\chi'(a) = \deg \chi \deg \chi'\). Then the simple objects \((a, \chi)\) and \((b, \chi')\) centralize each other, by Proposition 3.8. Since the only simple object that centralizes \((a, \chi)\) is the trivial simple object \((e, 1)\), it follows that \(b = e\) and \(\chi' = 1\), showing that condition (b) holds.

To see that condition (a) holds, let \(H\) denote the normal closure of \(a\) in \(G\), and suppose that \(H \neq G\). By Proposition 3.8, for all \(\chi' \in \text{Irr}(G)\), the simple objects \((a, \chi)\) and \((e, \chi')\) centralize each other if and only if \(a \in \text{Ker} \chi'\), equivalently, \(H \leq \text{Ker} \chi'\). Since \(H\) is proper in \(G\), the action of \(G\) on the coset space \(G/H\) is not trivial, and so the corresponding representation contains a nontrivial irreducible constituent; let \(\chi'\) denote the character of this constituent. Since \(H\) is normal in \(G\), it acts trivially on \(G/H\), and so \(H \leq \text{Ker} \chi'\). It follows that \((a, \chi)\) and \((e, \chi')\) centralize each other, a contradiction. Hence \(H = G\), showing that condition (a) holds.

Corollary 3.10. Let \(G\) be a finite group with trivial center, let \(\omega\) be a normalized 3-cocycle on \(G\), and let \((a, \chi)\) be a simple object of \(\text{Rep}(D^\omega(G))\). Then \((a, \chi)\) is a generator of \(\text{Rep}(D^\omega(G))\) if and only if the normal closure of \(a\) in \(G\) is equal to \(G\).

Example 3.11. Let \(G\) be a finite group, and let \(\omega\) be a normalized 3-cocycle on \(G\). If \(G\) has trivial center, and \(a\) is an element of \(G\) whose normal closure is \(G\), then, by Corollary 3.10, for every irreducible \(\beta_a\)-character of \(C_G(a)\), the simple object \((a, \chi)\) is a generator of \(\text{Rep}(D^\omega(G))\). We give three related examples below.

(a) Take \(G = S_n\), the symmetric group on \(n\) letters, with \(n \geq 3\). Then \(S_n\) has trivial center, and the normal closure of the transposition \(\sigma = (12)\) is \(S_n\). Therefore, for every irreducible \(\beta_\sigma\)-character of \(C_{S_n}(\sigma)\), the simple object \((\sigma, \chi)\) is a generator of \(\text{Rep}(D^\omega(S_n))\).
(b) Let $n \geq 3$ be an odd integer, and take $G = \text{Dih}_n$, the dihedral group of order $2n$ generated by the elements $a$ and $b$ subject to the relations $a^n = e$, $b^2 = e$, and $ba = a^{-1}b$. Then $\text{Dih}_n$ has trivial center, and the normal closure of the element $b$ is $\text{Dih}_n$. Therefore, for every irreducible $\beta_b$-character of $C_{\text{Dih}_n}(b) = \{e, b\}$, the simple object $(b, \chi)$ is a generator of $\text{Rep}(D^\omega(\text{Dih}_n))$.

Note that, since the Schur multiplier of a cyclic group is trivial, the 2-cocycle $\beta_b$ is cohomologically trivial, and so $\chi$ may be identified with an ordinary character.

(c) Suppose that $G$ is nonabelian and simple. Then $G$ has trivial center, and the normal closure of every nontrivial element $a$ is $G$. Therefore, for every nontrivial element $a$, and for every irreducible $\beta_a$-character of $C_G(a)$, the simple object $(a, \chi)$ is a generator of $\text{Rep}(D^\omega(G))$.

Example 3.12. Let $p$ be an odd prime, and consider the special linear group $\text{SL}(2, p)$ consisting of all $2 \times 2$ matrices of determinant 1 whose entries belong to the field of $p$ elements. This group has order $p^3 - p^2$. The matrices

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generate the group $\text{SL}(2, p)$. The center of $\text{SL}(2, p)$ is a subgroup of order 2, consisting of the matrices $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The character table of $\text{SL}(2, p)$ was first obtained by F. G. Frobenius. Later, I. Schur [13] and independently H. Jordan [9] obtained the characters of the special linear groups over arbitrary finite fields [8]. We use the exposition given in [5]. The group $\text{SL}(2, p)$ has exactly $p + 4$ distinct irreducible characters. For the purpose of this example, we will only need a portion of the character table of $\text{SL}(2, p)$. Set $\epsilon = (-1)^{(p-1)/2}$. The table below gives the values of the irreducible characters evaluated at the identity matrix $I$ and at the matrix $Y$, omitting identical columns.

| $I$ | $1$ | $p$ | $p + 1$ | $p - 1$ | $\frac{p + 1}{2}$ | $\frac{p - 1}{2}$ | $\frac{1 + \sqrt{p}}{2}$ | $\frac{1 - \sqrt{p}}{2}$ | $\frac{-1 + \sqrt{p}}{2}$ | $\frac{-1 - \sqrt{p}}{2}$ |
|-----|-----|-----|--------|--------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $Y$ | $1$ | $0$ | $1$ | $-1$ | $1 + \sqrt{p}$ | $1 - \sqrt{p}$ | $-1 + \sqrt{p}$ | $-1 - \sqrt{p}$ |

It is easily verified that $X = Y^{-1}(XY^{-1}X^{-1})Y^{-1}$, and since the matrices $X$ and $Y$ generate $\text{SL}(2, p)$, it follows that the normal closure of $Y$ is $\text{SL}(2, p)$.

The conjugacy class of $Y$ contains $(p^2 - 1)/2$ elements, and so the centralizer of $Y$ in $\text{SL}(2, p)$ has order $2p$. The matrix $Y$ has order $p$, so the matrix $-Y$ has order $2p$, and it follows that the centralizer of $Y$ in $\text{SL}(2, p)$ is a cyclic group with generator $-Y$. 
Let \( \zeta \) be a primitive \( 2p \)-th root of unity. For each \( 1 \leq i \leq 2p \), let \( \chi_i : (-Y) \to \mathbb{C}^\times \) denote the group homomorphism that sends \(-Y\) to \( \zeta^i \). Then the \( \chi_i \) constitute all of the irreducible characters of the centralizer of \( Y \) in \( SL(2,p) \). We have

\[
\chi_i(-I) = \chi_i((-Y)^p)) = \zeta^{pi} = (-1)^i.
\]

Suppose that \( i \) is odd. We will show that the simple object \((Y, \chi_i)\) is a generator of \( \text{Rep}(D(SL(2,p))) \). We have shown above that condition (a) of Theorem 3.3. To see that condition (b) of Theorem 3.3 holds, let \( \chi' \) be an irreducible character of \( SL(2,p) \). Suppose that the simple objects \((Y, \chi_i)\) and \((I, \chi')\) centralize each other. Then \( Y \in \text{Ker} \chi' \), by Proposition 3.2. Since the normal closure of \( Y \) is \( SL(2,p) \), it follows that \( \chi' = 1 \). We have

\[
\chi_i(-I)\chi'(Y) = -\chi'(Y)
\]

and

\[
\deg \chi_i, \deg \chi' = \deg \chi'.
\]

Inspecting the partial character table given above, we see that \(-\chi'(Y) \neq \deg \chi'\), and so the simple objects \((Y, \chi_i)\) and \((-I, \chi')\) do not centralize each other, by Proposition 3.2. It follows that condition (b) of Theorem 3.3 also holds, proving that \((Y, \chi_i)\) is a generator of \( \text{Rep}(D(SL(2,p))) \).

Note that if \( i \) is even, then the simple objects \((Y, \chi_i)\) and \((-I, 1)\) centralize each other, and so \((Y, \chi_i)\) is not a generator of \( \text{Rep}(D(SL(2,p))) \), by Theorem 3.3.

4. Adjoint category

In this section, we describe the adjoint category of \( \text{Rep}(D^\omega(G)) \). The case where \( \omega = 1 \) was addressed in the paper [12]. For a fusion category \( \mathcal{C} \), adjoint category of \( \mathcal{C} \), denoted \( \mathcal{C}_{\text{ad}} \), is the full fusion subcategory of \( \mathcal{C} \) generated by all subobjects of \( X \otimes X^* \), where \( X \) runs through simple objects of \( \mathcal{C} \). For example, for a finite group \( G \), we have \( \text{Rep}(G)_{\text{ad}} \cong \text{Rep}(G/Z(G)) \).

**Lemma 4.1.** Let \( G \) be a finite group, and let \( \omega \) be a normalized 3-cocycle on \( G \). The set

\[
Z_\omega(G) = \{ a \in Z(G) \mid \beta_a \text{ is cohomologically trivial} \}
\]

is a subgroup of \( Z(G) \).

**Proof.** Since \( \beta_e = 1 \), the identity element \( e \) lies in \( Z_\omega(G) \). Let \( a, b \in Z_\omega(G) \). Define a function \( \tau_{a,b} : G \to \mathbb{C}^\times \) by \( \tau_{a,b}(x) = \beta_x(a, b) \). It follows from (2) that

\[
\beta_{ab} = \beta_a \cdot \beta_b \cdot d \tau_{a,b},
\]
showing that $\beta_{ab}$ and $\beta_a \beta_b$ are cohomologous, where $d$ denotes the coboundary operator. Since $\beta_a$ and $\beta_b$ are cohomologically trivial, the same is true for $\beta_{ab}$, and so $ab \in Z_\omega(G)$. □

The following definition is taken from [12].

**Definition 4.2.** Let $G$ be a finite group, let $\omega$ be a normalized 3-cocycle on $G$, let $K$ and $H$ be normal subgroups of $G$ that commute elementwise, and let $B : K \times H \to \mathbb{C}^\times$ be a function. We say that $B$ is an $\omega$-bicharacter on $K \times H$ if

(a) $B(x, uv) = \beta_x^{-1}(u,v)B(x,u)B(x,v)$, and

(b) $B(xy, u) = \beta_{ux}(x,y)B(x,u)B(y,u)$

for all $x, y \in K$ and $u, v \in H$. We say that $B$ is $G$-invariant if

$$B(g^{-1}xg, u) = \frac{\beta_x(g, u)\beta_x(gu, g^{-1})}{\beta_x(g, g^{-1})}B(x, gu^{-1})$$

for all $g \in G, x \in K$, and $u \in H$.

We refer the reader to [12] for an explanation of the $G$-invariance property and the apparent lack of symmetry. It was shown in [12] that the fusion subcategories of $\text{Rep}(D^\omega(G))$ are parametrized by triples $(K,H,B)$, where $K$ and $H$ are normal subgroups of $G$ that commute elementwise, and $B$ is an $\omega$-bicharacter on $K \times H$.

Given such a triple $(K,H,B)$, denote by $S(K,H,B)$ the full abelian subcategory generated by simple objects $(a,\chi)$ such that $a \in K$ and $\chi(h) = B(a,h)\deg \chi$ for all $h \in H$. It was shown in [12] that $S(K,H,B)$ is, in fact, a fusion subcategory of $\text{Rep}(D^\omega(G))$, and

$$S(K,H,B)' = S(H,K,(B^{op})^{-1}),$$

where $(B^{op})^{-1} : H \times K \to \mathbb{C}^\times$ is defined by $(B^{op})^{-1}(h,k) = B(k,h)^{-1}$.

**Lemma 4.3.** Let $G$ be a finite group, and let $\omega$ be a normalized 3-cocycle on $G$.

(a) For all $a, g, x, y \in G$,

$$\beta_a(gxg^{-1}, gyy^{-1}) = \frac{\beta_a(g, g^{-1})}{\beta_a(g, xy)\beta_a(gx, g^{-1})} = \frac{\beta_x(g, u)\beta_x(gu, g^{-1})}{\beta_x(g, g^{-1})} \cdot \frac{\beta_a(g, g^{-1})}{\beta_x(g, x)\beta_x(gx, g^{-1})} \cdot \frac{\beta_a(g, g^{-1})}{\beta_a(g, y)\beta_a(gy, g^{-1})}.$$

(b) For all $a, g, x, y \in G$, if $a$ lies in $Z_\omega(G)$, then

$$\beta_a(gxg^{-1}, gyy^{-1}) = \beta_a(x, y).$$
Proof. Part (a) was proved in [12]. To see (b), let $g, x, y \in G$, and let $a \in Z_\omega(G)$. Applying Lemma 3.7 with $\mu = \beta_a$ to the equality in (a), we get

$$\frac{\beta_a(g^{-1}, gxy)}{\beta_a(gxy, g^{-1})} = \frac{\beta_a(g^{-1}, gy)}{\beta_a(gx, g^{-1})} \cdot \frac{\beta_a(g^{-1}, g)}{\beta_a(gy, g^{-1})}.$$

Since $\beta_a$ is cohomologically trivial, it is symmetric, and so the equation above reduces to

$$\beta_a(g^{-1}, gyy^{-1}) = \beta_a(x, y),$$

proving (b).

Lemma 4.4. Let $G$ be a finite group, let $\omega$ be a normalized 3-cocycle on $G$, let $K$ be a subgroup of $Z_\omega(G)$, and let $H$ be a subgroup of the commutator subgroup $[G, G]$ of $G$. For each $a \in K$, choose a function $\sigma_a : G \to \mathbb{C}^\times$ such that $d\sigma_a = \beta_a$.

The function $B : K \times H \to \mathbb{C}^\times$ defined by $B(a, x) = \sigma_a(x)$ does not depend on the choice of the $\sigma_a$, and it is a $G$-invariant $\omega$-bicharacter on $K \times H$.

Proof. Since the restriction of any homomorphism $G \to \mathbb{C}^\times$ to the subgroup $H$ is trivial, we deduce that for any two functions $f_1 : G \to \mathbb{C}^\times$ and $f_2 : G \to \mathbb{C}^\times$, if $df_1 = df_2$, then $f_1$ and $f_2$ are equal when restricted to $H$. It follows that the function $B$ does not depend on the choice of the $\sigma_a$.

The condition $d\sigma_a = \beta_a$ is equivalent to the first condition in the definition of $\omega$-character. To see that the second condition in the definition of $\omega$-character holds, let $a, b \in K$. Define a function $\tau_{a,b} : G \to \mathbb{C}^\times$ by $\tau_{a,b}(x) = \beta_a(a, b)$. As seen in the proof of Lemma 4.1,

$$\beta_{ab} = \beta_a \cdot \beta_b \cdot d\tau_{a,b} = d(\sigma_a \cdot \sigma_b \cdot \tau_{a,b}).$$

Since we also have $\beta_{ab} = d\sigma_{ab}$, we deduce that the functions $\sigma_{ab}$ and $\sigma_a \cdot \sigma_b \cdot \tau_{a,b}$ are equal when restricted to $H$, that is, for all $x \in H$,

$$\sigma_{ab}(x) = \sigma_a(x)\sigma_b(x)\tau_{a,b}(x),$$

equivalently,

$$B(ab, x) = \beta_a(a, b)B(a, x)B(b, x),$$

which is the second condition in the definition of $\omega$-character.

To see that $B$ is $G$-invariant, let $g \in G$, let $a \in K$, and let $x \in H$. Applying the definition of $B$ and Lemma 3.7 with $\mu = \beta_a$ to the expression

$$\frac{\beta_a(g, x)\beta_a(gx, g^{-1})}{\beta_a(g, g^{-1})} B(a, gxg^{-1}),$$
we get

\[ \frac{\beta_a(g^{-1}, gx)}{\beta_a(gx, g^{-1})} \sigma_a(gxg^{-1}). \]

Since \( \beta_a \) is cohomologically trivial, it is symmetric, and so the expression above reduces to \( \sigma_a(gxg^{-1}) \). By Lemma 4.3, \((\beta_a)^g = \beta_a\), and so \( d(\sigma_a)^g = (\beta_a)^g = \beta_a = d\sigma_a \), where the superscript denotes the conjugation action. Therefore, the functions \((\sigma_a)^g\) and \( \sigma_a \) are equal when restricted to \( H \), and so \( \sigma_a(gxg^{-1}) = \sigma_a(x) = B(a, b) \), proving that \( B \) is \( G \)-invariant.

A fusion category \( \mathcal{C} \) is said to be **pseudounitary** if its categorical dimension coincides with its Frobenius-Perron dimension [6]. In this case, \( \mathcal{C} \) admits a canonical spherical structure with respect to which categorical dimensions of objects coincide with their Frobenius-Perron dimensions [6]. The category \( \text{Rep}(D^\omega(G)) \) is pseudounitary. In the paper [7], S. Gelaki and D. Nikshych showed that, for a pseudounitary modular category \( \mathcal{C} \), the adjoint subcategory \( \mathcal{C}_{ad} \) and the full maximal pointed subcategory \( \mathcal{C}_{pt} \) are centralizers of each other, that is,

\[ \mathcal{C}_{ad} = (\mathcal{C}_{pt})'. \]

**Theorem 4.5.** Let \( G \) be a finite group, and let \( \omega \) be a normalized 3-cocycle on \( G \). Then

\[ \text{Rep}(D^\omega(G))_{pt} = S(Z_\omega(G), [G, G], B) \]

and

\[ \text{Rep}(D^\omega(G))_{ad} = S([G, G], Z_\omega(G), (B^{\text{op}})^{-1}) \]

where \( B : Z_\omega(G) \times [G, G] \to \mathbb{C}^\times \) is the \( G \)-invariant \( \omega \)-bicharacter defined in Lemma 4.4.

**Proof.** The dimension of a simple object \((a, \chi)\) of \( \text{Rep}(D^\omega(G)) \) is \( \frac{|G|}{|CG(a)|} \deg \chi \), which is equal to 1 if and only if \( a \) lies in the center of \( G \) and \( \deg \chi = 1 \). The latter condition implies that \( \beta_a \) is cohomologically trivial. It follows that \( \text{Rep}(D^\omega(G))_{pt} = S(Z_\omega(G), [G, G], B) \). Applying (4) and (5) to the previous equation, we get

\[ \text{Rep}(D^\omega(G))_{ad} = S([G, G], Z_\omega(G), (B^{\text{op}})^{-1}). \]

**References**

[1] B. Bakalov and A. Kirillov Jr., Lectures on Tensor Categories and Modular Functors, University Lecture Series, 21, American Mathematical Society, Providence, RI, 2001.

[2] A. Coste, T. Cannon and P. Ruelle, Finite group modular data, Nuclear Phys. B, 581(3) (2000), 679-717.
[3] R. Dijkgraaf, V. Pasquier and P. Roche, *Quasi-quantum groups related to orbifold models*, Modern quantum field theory (Bombay, 1990), World Sci. Publ., River Edge, NJ, (1991), 375-383.

[4] R. Dijkgraaf, V. Pasquier and P. Roche, *Quasi-Hopf algebras, group cohomology, and orbifold models*, Integrable systems and quantum groups (Pavia, 1990), World Sci. Publ., River Edge, NJ, (1992), 75-98.

[5] L. Dornhoff, Group Representation Theory, Part A, M. Dekker (1971).

[6] P. Etingof, D. Nikshych and V. Ostrik, *On fusion categories*, Ann. of Math. (2), 162 (2005), 581-642.

[7] S. Gelaki and D. Nikshych, *Nilpotent fusion categories*, Adv. Math., 217 (2008), 1053-1071.

[8] J. E. Humphreys, *Representation of SL(2, p)*, Amer. Math. Monthly, 82 (1975), no. 1, 21-39.

[9] H. Jordan, *Group characters of various types of linear groups*, Amer. J. Math., 29 (1907), 387-405.

[10] M. Müger, *On the structure of modular categories*, Proc. London Math. Soc., 87(2) (2003), 291-308.

[11] D. Naidu and D. Nikshych, *Lagrangian subcategories and braided tensor equivalences of twisted quantum doubles of finite groups*, Comm. Math. Phys., 279 (2008), 845-872.

[12] D. Naidu, D. Nikshych and S. Witherspoon, *Fusion subcategories of representation categories of twisted quantum doubles of finite groups*, Int. Math. Res. Not. IMRN, 22 (2009), 4183-4219.

[13] I. Schur, *Untersuchungen über die Darstellungen der endlichen Gruppen durch gebrochene lineare Substitutionen*, J. Reine Angew. Math., 132 (1907) 85-137.

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