NONNOETHERIAN SINGULARITIES AND THEIR NONCOMMUTATIVE BLOWUPS

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ABSTRACT. We establish a new fundamental class of varieties in nonnoetherian algebraic geometry related to the central geometry of dimer algebras. Specifically, given an affine algebraic variety $X$ and a finite collection of non-intersecting positive dimensional algebraic sets $Y_i \subset X$, we construct a nonnoetherian coordinate ring whose variety coincides with $X$ except that each $Y_i$ is identified as a distinct positive dimensional closed point. We then show that the noncommutative blowup of such a singularity is a noncommutative desingularization, in a suitable geometric sense.

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1. INTRODUCTION

The primary objectives of this article are (i) to extend the framework of depictions, introduced in [B4], to a much larger class of varieties with nonnoetherian coordinate rings; and (ii) to show that noncommutative blowups of these varieties are noncommutative desingularizations, in a suitable sense. This framework was originally developed to provide the geometric tools needed to understand the representation theory of a class of quiver algebras called non-cancellative dimer algebras (e.g., [B2, B3, B5]). Dimer algebras arose in string theory [HK, FHMSVW], and have found wide application to many areas of mathematics (e.g., [BKM, Br, FHKV, He, IN, IU, MR]). Depictions have enabled various notions in noncommutative algebraic geometry—such as noncommutative crepant resolutions [V], homological homogeneity [BH], and Azumaya loci—to be generalized to tiled matrix algebras that are not finitely generated.

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modules over their centers \[B2, B3\]; we will consider some of these generalizations here. The underlying ideas of nonnoetherian algebraic geometry also suggest possible directions towards a new theory of quantum gravity \[B1, B6\].

Throughout, let \(k\) be an algebraically closed field, and let \(R\) be a subalgebra of an affine coordinate ring \(S\) over \(k\). It is generally believed that nonnoetherian algebras do not admit concrete geometric descriptions. For example, consider the subalgebras of the polynomial rings \(S_1 = k[x, y]\) and \(S_2 = k[x, y, z]\),

\[
R_1 = k[x] + x(x - 1)(x - 2)S_1, \\
R_2 = k[x^2 - y - z^2] + (x^2 - y, z - 5)(x - z, y)S_2.
\]

We may ask, informally, what their maximal spectra \(\text{Max } R\) ‘look like’, but such a question initially appears hopeless, at least in terms of geometries we can visualize.

We could instead consider the simpler subalgebras

\[
R_1' = k + x(x - 1)(x - 2)S_1, \\
R_2' = k + (x^2 - y, z - 5)(x - z, y)S_2.
\]

Both are of the form \(R = k + I\), where \(I\) is an ideal of \(S\). A geometric description of such subalgebras was introduced in \[B4\]: the maximal spectrum \(\text{Max } R\) of \(R\) coincides with the algebraic variety \(\text{Max } S\), except that the zero locus \(\mathcal{Z}(I) \subset \text{Max } S\) is identified as a single ‘smeared-out’ point.

In particular, we may view the variety \(\text{Max } R_1'\) as \(\mathbb{A}^2_k\), with the union of the three lines

\[
(1) \quad \mathcal{Z}(x) = \{x = 0\}, \quad \mathcal{Z}(x - 1) = \{x = 1\}, \quad \mathcal{Z}(x - 2) = \{x = 2\},
\]

identified as a single 1-dimensional point. Similarly, we may view the variety \(\text{Max } R_2'\) as \(\mathbb{A}^3_k\), with the union of the two curves

\[
(2) \quad \mathcal{Z}(x^2 - y, z - 5) \quad \text{and} \quad \mathcal{Z}(x - z, y)
\]

identified as a single 1-dimensional point.

These geometric pictures are made precise using depictions and geometric dimension. A depiction of a nonnoetherian domain \(R\) is a finitely generated \(k\)-algebra \(S\) that is as close to \(R\) as possible, in a suitable geometric sense (Definition 2.1). In particular, if \(R\) is depicted by \(S\), then \(R\) and \(S\) have equal Krull dimension, and their maximal spectra are birationally equivalent \[B4\, Theorem 2.5\]. Furthermore, the locus where \(R\) and \(S\) locally coincide,

\[
U_{S/R} := \{n \in \text{Max } S \mid R_n \cap R = S_n\},
\]

is open dense in \(\text{Max } S\) \[B4\, Proposition 2.4\].

Algebras of the form \(R = k + I\), with \(\dim S/I \geq 1\), comprise an elementary class of examples in nonnoetherian algebraic geometry. Two ideals \(I_1, I_2 \subset S\) are said to be coprime if \(I_1 + I_2 = S\); equivalently, their zero loci in \(\text{Max } S\) do not intersect,

\[
\mathcal{Z}(I_1) \cap \mathcal{Z}(I_2) = \emptyset.
\]
In this article, we consider the question: given a collection of pairwise coprime ideals $I_1,\ldots,I_n \subset S$, is there a nonnoetherian ring $R$ for which $\text{Max} \ R$ coincides with $\text{Max} \ S$, except that each $\mathbf{Z}(I_i)$ is identified as a distinct closed point of $\text{Max} \ R$? We will show that this question has a positive answer, with $R$ given by the intersection
\[ R = \cap_i (k + I_i). \]

Our first main theorem is the following.

**Theorem A.** (Propositions 3.3, 3.4, and Theorem 3.14.) Let $X$ be an affine algebraic variety over $k$ with coordinate ring $S$. Consider a collection of pairwise non-intersecting algebraic sets $Y_1,\ldots,Y_n$ of $X$, where each ideal $I(Y_i)$ is proper, nonzero, and non-maximal. Then the maximal spectrum of the ring
\[ R := \cap_i (k + I(Y_i)) \]
coincides with $X$ except that each $Y_i$ is identified as a distinct closed point. In particular, the locus $U_{S/R} \subset X$ is given by the intersection of the complements $Y_i^c$,
\[ U_{S/R} = \cap_i Y_i^c. \]
Furthermore, we have:

(i) $R$ is nonnoetherian if and only if there is some $i$ for which $\dim Y_i \geq 1$.
(ii) $R$ is depicted by $S$ if and only if for each $i$, $\dim Y_i \geq 1$.

Theorem A answers our initial question in a surprisingly simple way: observe that the subalgebras $R_1$ and $R_2$ are of the form
\[ R_1 = k[x] + x(x-1)(x-2)S_1 = (k + xS_1) \cap (k + (x-1)S_1) \cap (k + (x-2)S_1), \]
and
\[ R_2 = k[x^2-y-z^2] + (x^2-y,z-5)(x-z,y)S_2 = (k + (x^2-y,z-5)S_2) \cap (k + (x-z,y)S_2). \]
The variety $\text{Max} R_1$ therefore looks exactly like $\mathbb{A}_k^2$, except that each of the three lines in $\mathbb{P}$ is identified as a distinct 1-dimensional point. Similarly, $\text{Max} R_2$ looks exactly like $\mathbb{A}_k^3$, except that each of the curves in $\mathbb{P}$ is identified as a distinct 1-dimensional point.

To note, it is peculiar that by adjoining to $R_2'$ the polynomial $x^2-y-z^2$,
\[ R_2 = R_2'[x^2-y-z^2], \]
the single 1-dimensional point of Max $R_2'$ separates into two distinct 1-dimensional points, while all other points of Max $R_2'$ are left unchanged.

Theorem A also implies the following generalization of the fact that, given any maximal ideal $\mathfrak{n}$ of $S$, $S$ decomposes as the sum $S = k + \mathfrak{n}$. 


Corollary B. Let $I$ be a proper non-maximal nonzero radical ideal of $S$, and set $R = k + I$. The following are equivalent:

(i) $\dim S/I \geq 1$.
(ii) $R$ is nonnoetherian.
(iii) $R$ is depicted by $S$.

In particular, $R = k + I$ is noetherian if and only if $\dim S/I = 0$, that is,

$$I = n_1 \cap \cdots \cap n_\ell$$

for some maximal ideals $n_1, \ldots, n_\ell \in \text{Max } S$. The implication (ii) $\Rightarrow$ (i) was also shown by Stafford in [St, Lemma 1.4] using different methods.

In Section 4, we define a sheaf of depictions on an affine scheme $X$ to be a sheaf of algebras that is a depiction on each principal open set of $X$. We show that the sheafification of a depiction $S$ of $R$ is a sheaf of depictions on $\text{Spec } R$.

In Section 5, we consider nonnoetherian coordinate rings in the setting of noncommutative algebraic geometry. Let $S$ be a finite type normal integral domain, let $Y_1, \ldots, Y_n$ be positive dimensional proper subvarieties of $\text{Max } S$ that intersect the smooth locus, and denote by $I_i := I(Y_i)$ their radical ideals in $S$. By Theorem A, $R := \cap_i (k + I_i)$ is a nonnoetherian coordinate ring with $n$ positive dimensional closed points,

$$m_i := I_i \cap R \in \text{Spec } R.$$ Following [L, Section R], we call the endomorphism ring

$$A := \text{End}_R(R \oplus \bigoplus_i m_i)$$

the ‘noncommutative blowup’ of $\text{Max } R$ at the points $m_1, \ldots, m_n$. We would like to know whether $A$ is a desingularization of its center $R$.

A resolution of a singularity $X$ is a proper birational morphism of schemes $Y \to X$ such that $Y$ is smooth. If we omit the requirement of properness, then we may say that $Y \to X$ is a desingularization of $X$. We note the following:

(a) Birationality implies that $X$ and $Y$ have isomorphic function fields,

$$\text{Frac } k[X] \cong \text{Frac } k[Y].$$

(b) Let $\text{Spec } S$ be an affine open subset of $Y$. Then $\text{Spec } S$ is smooth over $\text{Spec } k$ at a closed point $n \in \text{Spec } S$ if and only if the global dimension of $S_n$, the projective dimension of the residue field $S_n/n \cong k$, and the Krull dimension of $S_n$ all coincide $[AB1, AB2, S]$,

$$\text{gldim } S_n = \text{pd}_{S_n}(S_n/n) = \text{dim } S_n.$$

Since we are assuming $k$ algebraically closed, $\text{Spec } S$ is smooth at $n$ if and only if $S_n$ is regular $[H, III.10.0.3]$.  

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Following Brown and Hajarnavis’s notion of a homologically homogeneous ring \([BH]\), and Van den Bergh’s notion of a noncommutative crepant resolution \([V]\), we say that a noncommutative algebra \(A\), module-finite over its noetherian center \(R\), is a noncommutative desingularization of \(R\) if the following two conditions hold:

\((a')\) Frac \(R\) and \(A \otimes_R \text{Frac} \ R\) are Morita equivalent.

\((b')\) For each closed point \(m \in \text{Spec} \ R\), the central localization \(A_m := A \otimes_R R_m\) satisfies

\[
gldim A_m = \text{pd}_{A_m}(A_m/m) = \dim R_m.
\]

However, the singularities we are considering here are nonnoetherian, and their noncommutative blowups are not module-finite over their centers (just as the case for non-cancellative dimer algebras). Condition \((b')\) must therefore be modified to allow for this generality. Such a modification is possible for tiled matrix algebras using the notions of ‘cycle algebra’ and ‘cyclic localization’, introduced in \([B3, B4]\) (Definition 2.2). In cases of interest, if the center \(R\) is noetherian, then the cycle algebra and center coincide, and cyclic localization is the same as central localization \([B4, \text{Theorem 4.1}]\). We thus replace \((b')\) with the following condition:

\((b'')\) Let \(S\) be the cycle algebra of \(A\). For each closed point \(m \in \text{Spec} \ R\) and each minimal prime \(q \in \text{Spec} \ S\) over \(m\), the cyclic localization \(A_q\) satisfies

\[
gldim A_q = \text{pd}_{A_q}(A_q/q) = \dim S_q.
\]

Our second main theorem is the following.

**Theorem C.** (Theorem 5.17.) Let \(A\) be the endomorphism ring in \([4]\), and let \(S\) be its cycle algebra. If each \(Y_i\) is irreducible, or \(n = 1\), then \(A\) is a noncommutative desingularization of its center \(R\):

- Frac \(R\) and \(A \otimes_R \text{Frac} \ R\) are Morita equivalent, and
- for each \(i \in [1, n]\) and minimal prime \(q \in \text{Spec} \ S\) over \(m_i\), we have

\[
gldim A_q = \text{pd}_{A_q}(A_q/q) = \dim S_q.
\]

Furthermore, the Azumaya locus of \(A\) and the noetherian locus \(U_{S/R}\) of \(R\) coincide.

### 2. Preliminary definitions

Given an integral domain \(k\)-algebra \(S\), denote by \(\text{Max} \ S\), \(\text{Spec} \ S\), \(\text{Frac} \ S\), and \(\dim S\) the maximal spectrum (or variety), prime spectrum (or affine scheme), fraction field, and Krull dimension of \(S\) respectively. For a subset \(I \subseteq S\), set \(Z(I) := \{n \in \text{Max} \ S \mid n \supseteq I\}\).

Given a (not-necessarily-commutative) \(k\)-algebra \(A\) and an \(A\)-module \(V\), denote by \(\text{gldim} \ A\) and \(\text{pd}_A(V)\) the left global dimension of \(A\) and projective dimension of \(V\), respectively. By module we mean left module, unless stated otherwise.

The following definitions have been instrumental in studying dimer algebras (e.g., \([B2, B3, B5]\)).
**Definition 2.1.** [B4, Definition 3.1] Let $S$ be an integral domain and a finitely generated $k$-algebra, and let $R$ be a subalgebra of $S$.

- We say $S$ is a depiction of $R$ if the morphism $\iota_{S/R} : \text{Spec } S \to \text{Spec } R$, $q \mapsto q \cap R$, is surjective, and

$$U_{S/R} := \{ n \in \text{Max } S \mid R_{n \cap R} = S_n \} = \{ n \in \text{Max } S \mid R_{n \cap R} \text{ is noetherian} \} \neq \emptyset.$$  

- The geometric height of $p \in \text{Spec } R$ is the minimum

$$\text{ght}(p) := \min \left\{ \text{ht}_S(q) \mid q \in \iota_{S/R}^{-1}(p), \ S \text{ a depiction of } R \right\}.$$  

The geometric dimension of $p$ is

$$\text{gdim } p := \dim R - \text{ght}(p).$$

For brevity, we will often write $\iota$ for $\iota_{S/R}$. To note, if $R$ is depicted by $S$, then $R$ is noetherian if and only if $R = S$ [B4, Theorem 3.12].

Now let $B$ be an integral domain and $k$-algebra, and let $A = [A^{ij}] \subset M_n(B)$ be a tiled matrix ring, that is, each diagonal $A^i := A ii$ is a unital subalgebra of $B$.

The following definitions, with the exception of residue module, were introduced in [B3]; the notion of residue module we are considering here is new.

**Definition 2.2.** [B3, Definition 3.1] Set

$$R := k[\cap_i A^i] \quad \text{and} \quad S := k[\cup_i A^i].$$

We call $S$ the cycle algebra of $A$, and in cases of interest, $R$ is the center of $A$ [B4, Theorem 4.1]. The cyclic localization of $A$ at a prime $q \in \text{Spec } S$ is the algebra

$$A_q := \left\langle \begin{bmatrix} A_{q \cap A^1}^1 & A_{q \cap A^2}^{12} & \cdots & A_{q \cap A^n}^{1n} \\ A_{q \cap A^2}^{21} & A_{q \cap A^2}^{22} & \cdots & A_{q \cap A^n}^{2n} \\ \vdots & \ddots & \ddots & \vdots \\ A_{q \cap A^n}^{n1} & A_{q \cap A^n}^{n2} & \cdots & A_{q \cap A^n}^{nn} \end{bmatrix} \right\rangle \subset M_n(\text{Frac } B).$$

The residue module $A_q / q$ of $A$ at $q$ is the quotient of $A_q$ by the ideal

$$A_q \left[ \begin{bmatrix} q \cap A^1 & 0 & \cdots & 0 \\ 0 & q \cap A^2 & 0 & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q \cap A^n \end{bmatrix} \right] A_q.$$

\(^2\)Recall that if $S$ is an integral domain and a finitely generated $k$-algebra, then for each $q \in \text{Spec } S$, we have $\dim S/q = \dim S - \text{ht}(q)$. 


Remark 2.3. If $R = S$, that is, $A^i = A^j$ for each $i, j$, then cyclic localization coincides with the usual notion of central localization:

$$A_q \cong A \otimes_R R_q \quad \text{and} \quad A_q/q \cong A \otimes_R R_q/q.$$ 

Definition 2.4. [B3, Definition 3.2] We say $A$ is cycle regular at $m \in \text{Max } R$ if for each minimal prime $q \in \text{Spec } S$ over $m$, we have

$$\text{gldim}(A_q) = \text{pd}_{A_q}(A_q/q) = \dim S_q.$$ 

If, in addition, Frac $R$ and $A \otimes_R \text{Frac } R$ are Morita equivalent, then we say $A$ is a noncommutative desingularization of $R$.

3. Nonnoetherian coordinate rings with multiple positive dimensional points

Let $S$ be an integral domain and a finitely generated $k$-algebra. Let $I_1, \ldots, I_n$ be a collection of proper non-maximal nonzero radical ideals of $S$ such that, for each $i \neq j$, $Z(I_i) \cap Z(I_j) = \emptyset$; equivalently, $I_i$ and $I_j$ are coprime: $I_i + I_j = S$. Unless stated otherwise, we denote by $R$ the algebra

$$R := \cap_i (k + I_i).$$

Remark 3.1. If some $I_j$ were a maximal ideal of $S$, then $k + I_j = S$, whence $R = \cap_{i \neq j} (k + I_i)$. The assumption that each $I_i$ is proper and nonzero implies that $\dim S \geq 1$.

Lemma 3.2. Suppose $n \geq 2$. For each $i \in [1, n]$, there are elements $a, b \in R$ satisfying

$$a \in I_i \setminus (\cup_{j \neq i} I_j), \quad b \in (\cap_{j \neq i} I_j) \setminus I_i,$$

and which sum to unity, $a + b = 1$.

Proof. Fix $i \in [1, n]$. By assumption, we have

$$Z(1) = \emptyset = \cup_{j \neq i} (Z(I_i) \cap Z(I_j)) = Z(I_i) \cap (\cup_{j \neq i} Z(I_j)) = Z(I_i + \cap_{j \neq i} I_j).$$

Whence

$$1 \in I_i + \cap_{j \neq i} I_j.$$ 

Thus there is some $a \in I_i$ and $b \in \cap_{j \neq i} I_j$ such that $a + b = 1$. In particular,

$$a = 1 - b \in I_i \cap (\cap_{j \neq i} (k + I_j)) \subset R.$$ 

It also follows that for each $j \neq i$,

$$a = 1 - b \in I_i \setminus I_j \quad \text{and} \quad b = 1 - a \in I_j \setminus I_i.$$

\[\square\]

3In [B3], we defined $A$ to be cycle regular at $m \in \text{Max } R$ if, for each minimal prime $q \in \text{Spec } S$ over $m$ and each simple $A_q$-module $V$, we have $\text{gldim}(A_q) = \text{pd}_{A_q}(V) = \dim S_q$. In this article, we replace the set of simple $A_q$-modules $V$ with the residue module $A_q/q$, which, in our case, is a direct sum of all such simples (see Propositions 5.14 and 5.15).
Proposition 3.3. Each ideal $I_i \cap R$ is a distinct closed point of Spec $R$.

Proof. Fix $i$. For each $a \in R \subseteq (k + I_i)$, there is some $\alpha_i \in k$ and $b_i \in I_i$ such that $a = \alpha_i + b_i$. In particular, there is an algebra epimorphism

$$R \to k, \quad a \mapsto \alpha_i,$$

with kernel $I_i \cap R$; whence an algebra isomorphism $R/(I_i \cap R) \cong k$. Furthermore, there exists some $a \in (I_i \cap R) \setminus (\cup_{j \neq i} I_j)$, by Lemma 3.2. Thus, for each $j \neq i$,

$$I_j \cap R \neq I_i \cap R.$$

Therefore each $I_i \cap R$ is a distinct maximal ideal of $R$. □

Proposition 3.4. The locus $U_{S/R} := \{ n \in \text{Max} S \mid R_{n \cap R} = S_n \}$ is given by

$$U_{S/R} = (\cup_i Z(I_i))^c.$$

Proof. (i) We first claim that $U_{S/R} \subseteq (\cup_i Z(I_i))^c$. Indeed, let $n \in \cup_i Z(I_i)$. Then $n$ contains some $I_i$. By assumption, $I_i$ is a non-maximal radical ideal of $S$. Thus there is another maximal ideal $n' \neq n$ of $S$ which contains $I_i$. Whence

$$I_i \cap R \subseteq n \cap R \neq R \quad \text{and} \quad I_i \cap R \subseteq n' \cap R \neq R.$$

But $I_i \cap R$ is a maximal ideal of $R$ by Proposition 3.3. Therefore

$$n \cap R = I_i \cap R = n' \cap R.$$

Now fix $c \in n \setminus n'$. Assume to the contrary that $c \in R_{n \cap R}$. Then there is some $a \in R$ and $b \in R \setminus (n \cap R)$ such that $c = \frac{a}{b}$. Whence

$$a = bc \in n \cap R = n' \cap R.$$

In particular, $bc \in n'$ with $b, c \in S$. Therefore

$$b \in n',$$

since $c \not\in n'$ and $n'$ is a prime ideal of $S$. But $b \in R$ and

$$b \not\in n \cap R = n' \cap R.$$

Whence $b \not\in n'$, a contradiction to (5). Thus $c \in S_n \setminus R_{n \cap R}$. Therefore $n \in U_{S/R}^c$.  

(ii) We now claim that $U_{S/R} \supseteq (\cup_i Z(I_i))^c$. Let $n \in (\cup_i Z(I_i))^c$. For each $i$, $n \not\supseteq I_i$. In particular, for each $i$ there is some $c_i \in I_i \setminus n$. Furthermore, since $n$ is prime, we have

$$c := c_1 \cdots c_n \in (\cap_i I_i) \setminus n.$$

Now let $\frac{a}{b} \in S_n$, with $a \in S$ and $b \in S \setminus n$. Then by (6),

$$ac \in R \quad \text{and} \quad bc \in R \setminus (n \cap R).$$

Thus

$$\frac{a}{b} = \frac{ac}{bc} \in R_{n \cap R}.$$  

This claim was proven in the special case $n = 1$ in [BA, Proposition 2.8].
Whence
\[ S_n \subseteq R_{n \cap R} \subseteq S_n. \]
Therefore \( R_{n \cap R} = S_n. \)

**Lemma 3.5.** If \( J \) is a proper ideal of \( R \) and \( \mathcal{Z}(J) \cap U_{S/R} = \emptyset \), then \( J \) is contained in some \( I_i \).

**Proof.** Suppose the hypotheses hold, and let \( n \in \mathcal{Z}(J) \). Then \( n \in U_{S/R}^c \). Whence \( n \in \cup_j \mathcal{Z}(I_j) \) by Proposition 3.4. Thus \( n \) contains some \( I_i \). Consequently,
\[ I_i \cap R \subseteq n \cap R \neq R. \]
Whence \( I_i \cap R = n \cap R \) since \( I_i \cap R \) is maximal, hence prime, ideal of \( k + I_i \). Conversely, \( R_{I_i \cap R} = (k + I_i)_{I_i} \).

It follows that
\[ (k + I_i)_{I_i} \subseteq R_{I_i \cap R}. \]

Conversely,
\[ R_{I_i \cap R} = (\cap_j (k + I_j))_{I_i \cap R} \subseteq \cap_j (k + I_j)_{I_i \cap (k + I_j)} \subseteq (k + I_i)_{I_i \cap (k + I_i)} = (k + I_i)_{I_i}. \]
Therefore (7) holds.

For the following, note that if \( n_1, \ldots, n_\ell \) are maximal ideals of \( S \), then
\[ I = n_1 \cap \cdots \cap n_\ell = \sqrt{n_1 \cdots n_\ell} \]
is a radical ideal of \( S \) satisfying \( \text{dim } S/I = 0 \).

**Lemma 3.7.** Suppose \( I \) is a radical ideal of \( S \) satisfying \( \text{dim } S/I = 0 \). Then the ring \( R = k + I \) is noetherian.
Proof. Suppose \( R \) is nonnoetherian. We claim that
\[
\dim S/I = \dim \mathcal{Z}(I) \overset{(i)}{=} \dim U^c_{S/R} \overset{(ii)}{=} 1.
\]
Indeed, (i) holds since by Proposition 3.4
\[
(8) \quad \mathcal{Z}(I) = U^c_{S/R}.
\]
To show (ii), recall [B4, Theorem 3.13.2] if \( R \) is a nonnoetherian subalgebra of a finitely generated \( k \)-algebra \( S \), and there is some \( m \in \iota(U^c_{S/R}) \) satisfying \( \sqrt{mS} = m \), then
\[
\dim U^c_{S/R} \geq 1.
\]
In our case, \( R = k + I \) is nonnoetherian and \( \sqrt{IS} = I \). Moreover, \( I \) is in \( \iota(U^c_{S/R}) \): for \( n \in \mathcal{Z}(I) \), we have
\[
I \overset{(A)}{=} n \cap R = \iota(n) \in \iota(\mathcal{Z}(I)) \overset{(a)}{=} \iota(U^c_{S/R}),
\]
where (A) holds since \( I \) is maximal in \( R \), and (a) holds by (8). Therefore (ii) holds.
\[\square\]

Proposition 3.8. Suppose each \( I_i \) is a radical ideal of \( S \).

(1) If \( \dim S/I_i = 0 \) for each \( i \), then \( R \) is noetherian.

(2) If \( \dim S/I_i = 0 \), then the localization \( R_{I_i} \cap R \) is noetherian.

Proof. (1) Suppose \( \dim S/I_i = 0 \) for each \( i \). Set
\[
R^m := \cap_{i=1}^m (k + I_i).
\]
We proceed by induction on \( m \).

By Lemma 3.7, \( R^1 \) is noetherian. So suppose \( R^m \) is noetherian; we claim that \( R^{m+1} \) is noetherian.

Indeed, recall that a ring \( T \) is noetherian if there is a finite set of elements \( a_1, \ldots, a_m \in T \) such that \( (a_1, \ldots, a_m)T = T \), and each localization \( T_{a_i} := T[a_i^{-1}] \) is noetherian (e.g., [H, Proposition III.3.2]).

By Lemma 3.2, \( R^{m+1} \) contains elements
\[
(9) \quad a \in I_{m+1} \setminus (\cup_{i=1}^m I_i) \quad \text{and} \quad b \in (\cap_{i=1}^m I_i) \setminus I_{m+1}
\]
satisfying \( a + b = 1 \). In particular,
\[
(a, b)R^{m+1} = R^{m+1}.
\]
Furthermore, (9) implies
\[
(10) \quad R^{m+1}_a = R^m_a \quad \text{and} \quad R^{m+1}_b = (k + I_{m+1})_b.
\]
\[\text{In the published version of [B4, Theorem 3.13.2], } S \text{ is assumed to be a depiction of } R, \text{ but this is not used in the proof of the theorem.}\]
But $R^m$ is noetherian by assumption, and $(k + I_{m+1})$ is noetherian by Lemma 3.7. Thus the localizations (10) are noetherian. Therefore $R^{m+1}$ is noetherian, proving our claim.

(2) Now suppose $\dim S/I_i = 0$. Then the ring $k + I_i$ is noetherian by Lemma 3.7. Thus the localization $(k + I_i)_{I_i}$ is noetherian. But $R_{I_i \cap R} = (k + I_i)_{I_i}$ by Lemma 3.6. Therefore $R_{I_i \cap R}$ is noetherian.

\begin{flushright}
\textbf{□}
\end{flushright}

\textbf{Proposition 3.9.} Suppose $I$ is a nonzero radical ideal of $S$ satisfying $\dim S/I \geq 1$. Then the ring $R = k + I$ is nonnoetherian and $I$ contains a strict infinite ascending chain of ideals of $R$.\footnote{This proposition is erroneously claimed as a corollary to [B4, Theorem 3.13, published version]. [B4, Theorem 3.13] assumes that $S$ is a depiction of $R$, but if $R$ is noetherian, then $S$ will not be a depiction of $R$. Indeed, in this case the only depiction of $R$ will be itself [B4, Theorem 3.12], and $R \neq S$ if $I$ is a non-maximal ideal of $S$.}

\textbf{Proof.} Since $\dim S/I \geq 1$, $I$ is a non-maximal ideal of $S$. Thus there is a maximal ideal $n$ of $S$ for which $n \supset I$. Since $I$ is a maximal ideal of $R$ and $I \subset n$, we have

\begin{equation}
(11) \quad n \cap R = I.
\end{equation}

Furthermore, since $I$ is a radical of $S$, there are primes $p_1, \ldots, p_n$ of $S$ such that $I = p_1 \cap \cdots \cap p_n$, by the Lasker-Noether theorem. Fix $h \in n \setminus (p_1 \cup \cdots \cup p_n)$. Then for $f \in S$, we have

\begin{equation}
(12) \quad fh \in I \quad \Rightarrow \quad f \in I.
\end{equation}

Indeed, if $fh \in I$, then $fh \in p_i$ for each $i$. Whence $f \in p_i$ for each $i$, and therefore $f \in I$.

By assumption, $I \neq 0$. Fix $g \in I \setminus 0$, and consider the chain of ideals of $R$,

$$0 \subset gR \subseteq (g, gh)R \subseteq (g, gh, gh^2)R \subseteq \cdots \subseteq I.$$  

We claim that each inclusion is proper. Indeed, assume to the contrary that there is some $\ell \geq 0$ and $r_0, \ldots, r_{\ell} \in R$ such that

$$gh^{\ell+1} = \sum_{j=0}^{\ell} r_j gh^j.$$  

Then since $S$ is an integral domain,

$$h^{\ell+1} = \sum_{j=0}^{\ell} r_j h^j.$$  

Whence

\begin{equation}
(13) \quad h^{\ell+1} - \sum_{j=1}^{\ell} r_j h^j = r_0 \in R.
\end{equation}
But $h \in \mathfrak{n}$. Therefore $r_0 \in \mathfrak{n} \cap R = I$ by (11). Furthermore, since $R = k + I$, for each $j \in [0, \ell]$ there is some $\beta_j \in k$ and $t_j \in I$ such that $r_j = \beta_j + t_j$. Since $r_0$ and each $t_j h^j$ are in $I$, (13) yields

$$t := h^{\ell+1} - \sum_{j=1}^{\ell} \beta_j h^j = r_0 + \sum_{j=1}^{\ell} t_j h^j \in I \subset \mathfrak{n}. \quad (14)$$

The left-hand side implies that $t$ is a polynomial in $k[h]$. Therefore, since $k$ is algebraically closed, $t$ splits

$$t = h^{\ell+1} - \sum_{j=1}^{\ell} \beta_j h^j = h^n(h - \alpha_1) \cdots (h - \alpha_{\ell-m}),$$

where $m \geq 1$ and $\alpha_1, \ldots, \alpha_{\ell-m} \in k \setminus 0$. Set $f := (h - \alpha_1) \cdots (h - \alpha_\ell)$. By (14) we have $hf = t \in I$. Thus, by (12), $f \in I$. Consequently, $f \in \mathfrak{n}$. But this is not possible by Hilbert’s Nullstellensatz, since $\alpha_1, \ldots, \alpha_{\ell-m}$ are nonzero scalars, $h$ is in $\mathfrak{n}$, and $\mathfrak{n}$ is a maximal ideal of $S$. \[\square\]

**Proposition 3.10.** Suppose $\dim S/I_i \geq 1$ for some $i$. Then $R = \cap_i (k + I_i)$ is nonnoetherian.

**Proof.** Suppose $\dim S/I_i \geq 1$. By Proposition 3.9, $I_i$ contains a strict infinite ascending chain of ideals of $k + I_i$,

$$J_1 \subset J_2 \subset J_3 \subset \cdots \subset I_i.$$

(i) We claim that each $J_\ell$ is an $R$-module. Let $r \in R$. Then $r \in k + I_i$. Whence $J_\ell r \subseteq J_\ell$ since $J_\ell$ is an ideal of $k + I_i$, proving our claim.

(ii) Now let $a \in \cap J_i$. Then each $aJ_\ell$ is in $\cap_j I_j \subset R$. Thus each $aJ_\ell$ is an ideal of $R$ by Claim (i).

Consider the chain of ideals of $R$,

$$aJ_1 \subseteq aJ_2 \subseteq aJ_3 \subseteq \cdots \quad (15)$$

Assume to the contrary that for some $\ell$,

$$aJ_\ell = aJ_{\ell+1}.$$ 

Then for each $b \in J_{\ell+1} \setminus J_\ell$, there is some $c \in J_\ell$ such that

$$ab = ac.$$

But $S$ is an integral domain. Whence

$$b = c \in J_\ell,$$

a contradiction to our choice of $b$. Thus the chain (15) is strict. Therefore $R$ is nonnoetherian. \[\square\]

We recall the following elementary facts.
Lemma 3.11. Let $R$ be an integral domain, and let $p, m \in \text{Spec } R$ be ideals satisfying $p \subseteq m$. Then

1. $pR_m \cap R = p$.
2. $pR_m \in \text{Spec } R_m$.

Again let $R = \cap_i (k + I_i)$.

Lemma 3.12. If $p \in \text{Spec } R$ and $p \subseteq I_i$ for some $i$, then $pS \cap R = p$.

Proof. Suppose the hypotheses hold. Let $ab \in pS \cap R$, with $a \in p$ and $b \in S$. We claim that $ab \in p$. Indeed, by Lemma 3.2 there is some $c \in \cap_j I_j \cap R \setminus I_i$. Then $ac \in \cap_j I_j$ since $a \in p \subseteq I_i$. Thus for any $s \in S$, $acs \in \cap_j I_j \subseteq R$. In particular, $acb^2 \in R$.

Thus, since $a \in p$, $(ab)^2 \cdot c = a \cdot (acb^2) \in p$.

But $c \in R \setminus p$ and $(ab)^2 \in R$. Thus $(ab)^2 \in p$ since $p$ is prime in $R$. Therefore $ab \in p$, again since $p$ is prime in $R$. □

Proposition 3.13. The morphism $\iota: \text{Spec } S \to \text{Spec } R$, $q \mapsto q \cap R$, is surjective.

We prove Lemma 3.11 for completeness.

(1) It suffices to show that $pR_m \cap R \subseteq p$. Let $\frac{a}{b} \in pR_m \cap R$, with $a \in p$ and $b \in R \setminus m$. Then $b \cdot \frac{a}{b} = a \in p$.

Thus, since $b, \frac{a}{b} \in R$ and $p$ is prime in $R$, we have $b \in p$ or $\frac{a}{b} \in p$. But $b \not\in p$ since $b \not\in m$ and $p \subseteq m$.

Therefore $\frac{a}{b} \in p$.

(2) Let $\frac{a_1}{b_1}, \frac{a_2}{b_2} \in R_m$, with $a_1, a_2 \in R$ and $b_1, b_2 \in R \setminus m$. Suppose

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \in pR_m.$$ 

We claim that $\frac{a_1}{b_1}$ or $\frac{a_2}{b_2}$ is in $pR_m$. Indeed, there is some $c \in p$ and $d \in R \setminus m$ such that

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{c}{d}. $$

Whence

$$a_1a_2d = b_1b_2c \in p.$$ 

Now $d \not\in p$ since $d \not\in m$ and $p \subseteq m$. Thus $a_1a_2c \in p$ since $p$ is prime in $R$. In particular, $a_1 \in p$ or $a_2 \in p$; say $a_1 \in p$. Then $\frac{a_1}{b_1} \in pR_m$, proving our claim.
Proof. Let \( p \in \text{Spec} \, R \). We claim that there is some \( q \in \text{Spec} \, S \) such that \( q \cap R = p \).

(i) First suppose \( \mathcal{Z}(p) \cap U_{S/R} = \emptyset \). Then there is some \( i \) for which \( p \subseteq I_i \), by Lemma 3.3. Set

\[
\bar{t} := p(k + I_i)_{I_i} \cap (k + I_i).
\]

Recall that \( I_i \cap R \in \text{Spec} \, R \) by Proposition 3.3.

(i.a) We have \( p = \bar{t} \cap R \) since

\[
p = p(\mathcal{Z}(p) \cap U_{S/R} = \emptyset). \quad \text{by Claim (i.a)}
\]

where (i) holds by Lemma 3.6.

(i.b) We claim that \( \bar{t} \in \text{Spec}(k + I_i) \) and \( \bar{t} \subseteq I_i \).

By Lemma 3.11.2,

\[
pR_{I_i \cap R} \in \text{Spec} \, R_{I_i \cap R}.
\]

Thus by Lemma 3.6

\[
p(k + I_i)_{I_i} \in \text{Spec}(k + I_i)_{I_i}.
\]

Therefore \( \bar{t} \in \text{Spec}(k + I_i) \), since the intersection of a prime ideal with a subalgebra is a prime ideal of the subalgebra.

Furthermore,

\[
\bar{t} = p(k + I_i)_{I_i} \cap (k + I_i) \subseteq I_i(k + I_i)_{I_i} \cap (k + I_i) \subseteq I_i,
\]

where (i) holds by Lemma 3.11.1 since \( I_i \in \text{Spec}(k + I_i) \).

(i.c) We claim that

\[
p = \sqrt{\bar{t}S} \cap R.
\]

Indeed,

\[
p = \bar{t} \cap R \subseteq \sqrt{\bar{t}S} \cap R \subseteq \sqrt{\bar{t}S} \cap R = \sqrt{\bar{t}S} \cap (k + I_i) \cap R = \sqrt{\bar{t}S} \cap R \cap (k + I_i) \cap R = \sqrt{\bar{t}S} \cap R = p,
\]

where (i) and (iv) hold by Claim (i.a); (ii) holds since if \( s^n \notin tS \) and \( s \in R \), then \( s \in \sqrt{tS} \cap R \); and (iii) holds by Claim (i.b) together with Lemma 3.12 (with \( k + I_i \) in place of \( R \)).

(i.d) Since \( S \) is noetherian, the Lasker-Noether theorem implies that there are ideals \( q_1, \ldots, q_m \in \text{Spec} \, S \), minimal over \( \sqrt{\bar{t}S} \), such that

\[
\sqrt{\bar{t}S} = q_1 \cap \cdots \cap q_m.
\]

Thus

\[
p = \sqrt{\bar{t}S} \cap R = (q_1 \cap \cdots \cap q_m) \cap R = (q_1 \cap R) \cap \cdots \cap (q_m \cap R),
\]

where (i) holds by Claim (i.c). Furthermore, each \( q_j \cap R \) is a prime ideal of \( R \) since \( q_j \in \text{Spec} \, S \) and \( R \subset S \) (e.g., [11, Lemma 2.1]).

Assume to the contrary that for each \( j \in [1, m] \),

\[
q_j \cap R \neq p.
\]
Then for each $j$ there is some
\[ a_j \in (q_j \cap R) \setminus p. \]
Whence
\[ a_1 \cdots a_m \in \cap_j (q_j \cap R) = p, \]
where (1) holds by (16). But $p$ is prime in $R$, a contradiction. Thus there is some $j$
for which
\[ q_j \cap R = p. \]
Our desired ideal is therefore $q := q_j \in \text{Spec } S$.

(ii) Now suppose $\mathcal{Z}(p) \cap U_{S/R} \neq \emptyset$; say $n \in \mathcal{Z}(p) \cap U_{S/R}$. Set
\[ q := pS_n \cap S. \]
We claim that
\[ q \cap R = p \quad \text{and} \quad q \in \text{Spec } S. \]

First observe that
\[ p \overset{(i)}{=} p_{n \cap R} \cap R \overset{(ii)}{=} pS_n \cap R = pS_n \cap S \cap R = q \cap R, \]
where (i) holds by Lemma 3.11.1, and (ii) holds since $n \in U_{S/R}$. Furthermore, since $p \in \text{Spec } R$, we have $p_{n \cap R} \in \text{Spec}(R_{n \cap R})$ by Lemma 3.11.2. Whence $pS_n \in \text{Spec } S_n$ since $n \in U_{S/R}$. Therefore $q = pS_n \cap S \in \text{Spec } S$. \[ \square \]

**Theorem 3.14.** Let $I_1, \ldots, I_n$ be a set of proper non-maximal nonzero radical ideals of $S$ which are pairwise coprime, and set $R := \cap_i (k + I_i)$. Then

1. $R$ is nonnoetherian if and only if there is some $i$ for which $\dim S/I_i \geq 1$.
2. $R$ is depicted by $S$ if and only if for each $i$, $\dim S/I_i \geq 1$.

**Proof.** (1): The implications $\Rightarrow$ and $\Leftarrow$ are respectively Propositions 3.8.1 and 3.10.

(2): The morphism $\iota : \text{Spec } S \to \text{Spec } R$ is surjective by Proposition 3.13. Furthermore, $U_{S/R}$ is nonempty since $U_{S/R} = (\cup_i \mathcal{Z}(I_i))^c$ is an open dense subset of $\text{Max } S$, by Proposition 3.4. It thus suffices to show that
\[ U_{S/R} = \cup_i \mathcal{Z}(I_i) \subseteq \{ n \in \text{Max } S \mid R_{n \cap R} \text{ is nonnoetherian} \}, \]
where the inclusion holds if and only if $\dim S/I_i \geq 1$ for each $i$.

Suppose $n \in \cup_i \mathcal{Z}(I_i)$. Then $n$ contains some $I_j$. Whence $n \cap R = I_j \cap R$ by Proposition 3.3. Thus by Lemma 3.6
\[ R_{n \cap R} = R_{I_j \cap R} = (k + I_j)I_j. \]

- First suppose $\dim S/I_j = 0$. Then $R_{n \cap R} = R_{I_j \cap R}$ is noetherian by Proposition 3.8.2. Therefore the inclusion in (17) does not hold.
- Now suppose $\dim S/I_j \geq 1$. Then $I_j$ contains a strict infinite ascending chain of ideals of $k + I_j$, by Proposition 3.9. Therefore the localization $R_{n \cap R} = (k + I_j)I_j$ is nonnoetherian. In particular, if $\dim S/I_i \geq 1$ for each $i$, then the inclusion in (17) holds. \[ \square \]
Corollary 3.15. If \( \dim S/I_i \geq 1 \) for each \( i \), then each of the closed points \( I_i \cap R \) of \( \text{Spec} \, R \) has positive geometric dimension.

Proof. By Theorem 3.14, \( S \) is a depiction of \( R \). Therefore for each \( i \),
\[
gdim(I_i \cap R) \geq \dim S/I_i \geq 1.
\]
\[\square\]

4. Sheaves of depictions

Let \((X, \mathcal{O})\) be an affine scheme, and set \( R := \mathcal{O}(X) \). We introduce the following definition.

Definition 4.1. A sheaf of depictions \( \tilde{S} \) on \((X, \mathcal{O})\) is a sheaf of algebras such that on each principal open set \( D(a) \subset X \), \( a \in R \), the algebra \( \tilde{S}(D(a)) \) is a depiction of \( \mathcal{O}(D(a)) \).

A sheaf \( \mathcal{M} \) on \( X \) is said to be a sheaf of modules if, on each open set \( U \subset X \), \( \mathcal{M}(U) \) is an \( \mathcal{O}(U) \)-module, and for each inclusion of open sets \( U \subset V \), the restriction \( \mathcal{M}(V) \to \mathcal{M}(U) \) is an \( \mathcal{O}(V) \)-module homomorphism. The sheafification of an \( R \)-module \( M \) is the sheaf of modules \( \tilde{M} \) defined on each principal open set \( D(a) \) by
\[
\tilde{M}(D(a)) := M \otimes_{\mathcal{O}(X)} \mathcal{O}(D(a)) = M \otimes_R R[a^{-1}],
\]
and on a general open set \( U \) by the inverse limit
\[
\tilde{M}(U) := \lim_\leftarrow D(a) \subset U \tilde{M}(D(a)).
\]

In this section we show that the sheafification of a depiction is a sheaf of depictions.

Let \( S \) be an integral domain and \( k \)-algebra. For an element \( a \in S \) and ideal \( I \subset S \), set \( S_a := S[a^{-1}] \) and \( I_a := IS[a^{-1}] \).

Lemma 4.2. Fix \( a \in S \).

(1) If \( q \in \text{Spec} \, S \) and \( a \notin q \), then \( q_a \in \text{Spec} \, S_a \).

(2) If \( n \in \text{Max} \, S \) and \( a \notin n \), then \( n_a \in \text{Max} \, S_a \).

Proof. (1) Suppose \( q \in \text{Spec} \, S \) and \( a \notin q \). Since \( S_a \) is a flat \( S \)-module, the short exact sequence \( 0 \to q \to S \to S/q \to 0 \) induces the short exact sequence
\[
0 \to q \otimes_S S_a \to S \otimes_S S_a \cong S_a \to S/q \otimes_S S_a \to 0.
\]
Whence
\[
S/q \otimes_S S_a \cong S_a/q_a.
\]

But \( S/q \) is an integral domain since \( q \) is prime. Furthermore, \( S/q \otimes_S S_a \) is not the zero ring since \( a^n \notin q \) for all \( n \geq 0 \). Thus \( S/q \otimes_S S_a \) is also an integral domain. Therefore \( q_a \) is a prime of \( S_a \), by (18).
(2) Suppose \( n \in \text{Max} \, S \) and \( a \not\in n \). By Claim (1), we have
\[
S/n \otimes_S S_a \cong S_a/n_a \neq 0.
\]
Furthermore, \( S/n \otimes S_a \) is a field since \( n \) is a maximal ideal of \( S \). Consequently, \( n_a \) is a maximal ideal of \( S_a \).

Let \( R \) be a subalgebra of \( S \).

**Lemma 4.3.** Fix \( a \in R \). If
\[
\iota_{S/R} : \text{Spec} \, S \to \text{Spec} \, R
\]
is surjective, then so is
\[
\iota_{S_a/R_a} : \text{Spec} \, S_a \to \text{Spec} \, R_a.
\]

**Proof.** Suppose \( \iota_{S/R} \) is surjective. Let \( \tilde{p} \in \text{Spec} \, R_a \), and set \( p := \tilde{p} \cap R \). Then \( p \) is in \( \text{Spec} \, R \). Thus there is a prime \( q \in \text{Spec} \, S \) such that \( q \cap R = p \), by the surjectivity of \( \iota_{S/R} \). Furthermore, the ideal \( q_a \) is in \( \text{Spec} \, S_a \), by Lemma 4.2.1.

We want to show that \( q_a \cap R_a = \tilde{p} \), from which the lemma follows.

(i) We first claim that \( q_a \cap R_a \supseteq \tilde{p} \).
Let \( g \in \tilde{p} \). Then for \( \ell \geq 0 \) sufficiently large, \( a^\ell g \) is in \( R \). Whence \( a^\ell g \in \tilde{p} \cap R = p \).
Thus \( a^\ell g \in q \). Therefore \( g = a^{-\ell} a^\ell g \in q_a \).

(ii) We now claim that \( q_a \cap R_a \subseteq \tilde{p} \).
Let \( g \in q_a \cap R_a \). Then again for \( \ell \geq 0 \) sufficiently large, \( a^\ell g \) is in \( q \) and \( R \). Thus,
\[
a^\ell g \in q \cap R = p = \tilde{p} \cap R.
\]
Consequently, \( g = a^{-\ell} a^\ell g \in \tilde{p} \).

**Proposition 4.4.** Fix \( a \in R \). If \( S \) is a depiction of \( R \), then \( S_a \) is a depiction of \( R_a \).

**Proof.** Suppose \( S \) is a depiction of \( R \).

(i) The morphism \( \iota_{S_a/R_a} : \text{Spec} \, S_a \to \text{Spec} \, R_a \) is surjective by Lemma 4.3.

(ii) Let \( n \in \text{Max} \, S_a \), and suppose \( (R_a)_{n \cap R_a} \) is noetherian. We claim that
\[
(R_a)_{n \cap R_a} = (S_a)_n.
\]
Since \( n \) is a proper ideal of \( S_a \), we have \( n \not\supseteq a \). Therefore
\[
(R_a)_{n \cap R_a} \overset{(i)}{=} R_{n \cap R} \overset{(ii)}{=} S_{n \cap S} \overset{(iii)}{=} (S_a)_n,
\]
where (i) and (iii) hold since \( a \in R \setminus n \); and (ii) holds since \( R_{n \cap R} = (R_a)_{n \cap R_a} \) is noetherian and \( S \) is a depiction of \( R \).

(iii) Finally, we claim that the locus \( U_{S_a/R_a} \) is nonempty.
Let \( D_S(a) := \{ n \in \text{Max} \, S \mid n \not\supseteq a \} \) denote the complement of the vanishing locus of \( a \) in \( \text{Max} \, S \). Then
\[
U_{S_a/R_a} = U_{S/R} \cap D_S(a) \not\emptyset
\]
since \( U_{S/R} \) and \( D_S(a) \) are open dense sets of \( \text{Max} \, S \).
Corollary 4.5. Suppose \( S \) is a depiction of \( R \). Then the sheafification \( \tilde{S} \) of the \( R \)-module \( S \) on \( \text{Spec} \, R \) is a sheaf of depictions on \( \text{Spec} \, R \).

5. Noncommutative blowups of nonnoetherian singularities

Let \( S \) be a normal integral domain and a finitely generated \( k \)-algebra. Let \( Y_1, \ldots, Y_n \) be positive dimensional proper subvarieties of \( \text{Max} \, S \) that intersect the smooth locus. For each \( i \in [1, n] \), denote by \( I_i := I(Y_i) \) the corresponding radical ideal of \( S \). Consider the nonnoetherian coordinate ring \( R := \cap_i (k + I_i) \) and its set of positive dimensional closed points (Proposition 3.3), \( m_i := I_i \cap R \in \text{Spec} \, R \).

Following \([L, \text{Section} \, R]\), we call the endomorphism ring

\[
A := \text{End}_R(RR \oplus \bigoplus_i m_i)
\]

the `noncommutative blowup' of \( \text{Max} \, R \) at the points \( m_1, \ldots, m_n \). These points are precisely the nonnoetherian points of \( R \) (that is, the points \( m \in \text{Max} \, R \) for which \( R_m \) is nonnoetherian), by Theorem 3.14 and Proposition 3.4. Our main theorem in this section is that if either (i) each \( Y_i \) is irreducible, or (ii) \( n = 1 \), then \( A \) is a noncommutative desingularization of its center \( R \). Furthermore, \( S \) is the cycle algebra of \( A \), and thus \( A \) provides a means to retrieve \( S \) from the knowledge of \( R \) alone. In particular, \( R \) is depicted by the cycle algebra of \( A \).

In the following lemma we do not assume \( S \) is normal.

Lemma 5.1. Let \( I \) be a nonzero ideal of a noetherian integral domain \( S \), and suppose \( I \) is also an ideal of an overring \( T \subset \text{Frac} \, S \) of \( S \). Then \( T \) is contained in the integral closure \( \overline{S} \) of \( S \).

Proof. Let \( s \in I \setminus \{0\} \) and \( t \in T \). By assumption, \( t^\ell s \in I \) for each \( \ell \geq 0 \). Consider the ascending chain of ideals of \( S \)

\[
sS \subseteq (s, ts)S \subseteq (s, ts, t^2 s)S \subseteq (s, ts, t^2 s, t^3 s) \subseteq \cdots .
\]

Since \( S \) is noetherian, there is some \( m \geq 1 \) and \( \sigma_0, \ldots, \sigma_{m-1} \in S \) such that

\[
t^m s = \sum_{j=0}^{m-1} \sigma_j t^j s.
\]

Thus, since \( S \) is an integral domain and \( s \neq 0 \), we have

\[
t^m - \sum_{j=0}^{m-1} \sigma_j t^j = 0.
\]

Consequently, \( t \) is in the integral closure \( \overline{S} \) of \( S \). \( \square \)
Again let $S$ be a normal finitely generated domain. For brevity, set
\[ R^i := S \cap (\cap_{j \neq i} (k + I_j)) . \]
We include $S$ in the intersection for the case $n = 1$.

**Lemma 5.2.** For each $i \in [1, n]$, we have
\[ \text{Hom}_R(m_i, m_i) = \text{Hom}_R(\mathfrak{m}_i, R) = R^i. \]

**Proof.** (i) We first claim that $\text{Hom}_R(\mathfrak{m}_i, R) \subseteq S$.
Indeed, $\text{Hom}_S(I_i, I_i)$ is the largest overring of $S$ for which $I_i$ is an ideal. Thus, since $S$ is normal, Lemma 5.1 implies
\[ \text{(19)} \quad \text{Hom}_S(I_i, I_i) \subseteq S. \]
Let $x \in \text{Hom}_R(m_i, R)$ and $w \in I_1I_2 \cdots I_n$. Then $x^\ell w$ is in $\text{Hom}_S(I_i, I_i)$ for each $\ell \geq 1$, since $wI_i \subseteq \mathfrak{m}_i$. Whence $x^\ell w$ is in $S$ by (19). But since $S$ is a normal noetherian domain, the same argument given in the proof of Lemma 5.1 with $x$ and $w$ in place of $t$ and $s$, shows that $x$ itself is in $S$.

(ii) We now claim that $\text{Hom}_R(\mathfrak{m}_i, R) \subseteq R$.
Consider $x \in \text{Hom}_R(m_i, R)$ and $y \in m_i$. Then for each $j \in [1, n]$, $xy$ is in $k + I_j$. Furthermore, since $y$ is in $R$, there is a $c \in k$ and $z \in I_j$ such that $y = c + z \in k + I_j$. In particular, $xz$ is in $I_j$, since $x$ is in $S$ by Claim (i). Thus $x$ itself is in $k + I_j$, since $cx + xz = xy$ is in $k + I_j$. But $j$ was arbitrary, and therefore $x$ is in $R$.

(iii) Finally, we claim that $R^i \subseteq \text{Hom}_R(\mathfrak{m}_i, m_i)$.
Since $\mathfrak{m}_i \subseteq R \subseteq k + I_j$ for each $j$, and $R^i \subseteq S$, we have $R^i \mathfrak{m}_i \subseteq R$. Furthermore, $R^i \subseteq S$ implies $R^i \mathfrak{m}_i \subseteq I_i$. Therefore $R^i \mathfrak{m}_i \subseteq I_i \cap R = \mathfrak{m}_i$.

(iv) We have
\[ R^i \subseteq \text{Hom}_R(\mathfrak{m}_i, m_i) \subseteq \text{Hom}_R(\mathfrak{m}_i, R) \subseteq R \subseteq R^i, \]
where (i) holds by Claim (iii), and (ii) holds by Claim (ii).

**Lemma 5.3.** Let $p, q \in \text{Spec } R$ be coprime ideals. Then
\[ \text{Hom}_R(p, q) = q. \]

**Proof.** Since $p, q$ are ideals of $R$, $\text{Hom}_R(p, q)$ is isomorphic as an $R$-module to the maximum $R$-module $C \subseteq \text{Frac } R$ satisfying $Cp \subseteq q$. In particular, $C \supseteq q$.
To show the reverse inclusion, let $c \in C$. Since $p, q$ are coprime, there is an $a \in p$ and $b \in q$ such that $a + b = 1$. Whence
\[ c(1 - b) = ca \in Cp \subseteq q. \]
But $q$ is prime and $1 - b \notin q$. Thus, $c \in q$. Therefore $C = q$. \[ \square \]
Proposition 5.4. There is an algebra isomorphism

\begin{equation}
A = \text{End}_R(RR \oplus \bigoplus_i m_i) \cong \begin{bmatrix}
R & m_1 & m_2 & \cdots & m_n \\
R^1 & R^1 & m_2 & \cdots & m_n \\
R^2 & m_1 & R^2 & \cdots & m_n \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
R^n & m_1 & m_2 & \cdots & R^n
\end{bmatrix}.
\end{equation}

Proof. Each $m_i$ is a prime ideal of $R$, by Proposition 3.3. Furthermore, for each $i \neq j$, there is some $a \in I_i \cap R = m_i$ and $b \in I_j \cap R = m_j$ such that $a + b = 1$, by Lemma 3.2. Thus the set of ideals $m_1, \ldots, m_n$ are pairwise coprime. The isomorphism (20) therefore holds by Lemmas 5.2 and 5.3. \□

Remark 5.5. The endomorphism ring of the right $R$-module $RR \oplus \bigoplus_i m_i$ is the transpose of the matrix ring given in (20), and it is not known whether it is cycle regular. (As a right (resp. left) $R$-module, $RR \oplus \bigoplus_i m_i$ may be viewed as an $n+1$ column (resp. row) vector.)

Remark 5.6. In the case $n = 1$, we have $m = I$ (omitting the subscript $i$), and the tiled matrix ring (20) simplifies to

\[ A = \text{End}_R(RR \oplus I) \cong \begin{bmatrix} R & I \\ S & S \end{bmatrix}. \]

Proposition 5.7. The cycle algebra of $A$ is $S$.

Proof. By Proposition 5.4, the cycle algebra of $A$ is $\tilde{S} := k[R + R^1 + \cdots + R^n]$. By Remark 5.6, it suffices to suppose that $n \geq 2$.

We first claim that for any subset $K \subseteq \{1, \ldots, n\}$ with $|K| \geq 2$, we have

\begin{equation}
\sum_{i \in K} \bigcap_{j \in K \setminus \{i\}} I_j = S.
\end{equation}

We proceed by induction on $|K|$. Let $K \ni 1, 2$.

First suppose $|K| = 2$. Then (21) reduces to $I_1 + I_2 = S$, and this holds since $I_1$ and $I_2$ are coprime ideals of $S$.
Now suppose (21) holds for $|K| \leq N$, and let $|K| = N + 1$. Set $K_1 := K \setminus \{1\}$ and $K_2 := K \setminus \{2\}$. Then

$$S^{(i)} = I_1 + I_2 = I_1 \cap S + I_2 \cap S$$

$$\subseteq I_1 \cap \left( \sum_{i \in K_1} \bigcap_{j \in K_1 \setminus \{i\}} I_j \right) + I_2 \cap \left( \sum_{i \in K_2} \bigcap_{j \in K_2 \setminus \{i\}} I_j \right)$$

$$\subseteq \sum_{i \in K} \bigcap_{j \in K \setminus \{i\}} I_j \subseteq S,$$

where (i) holds since $I_1$ and $I_2$ are coprime, and (ii) holds by induction. This proves our claim.

Thus,

$$S = \sum_{i = 1}^{n} \bigcap_{j \neq i} I_j \subseteq \sum_{i = 1}^{n} R^i \subseteq \tilde{S} \subseteq S.$$

Therefore $\tilde{S} = S$. □

Fix $1 \leq i \leq n$, and let $q \in \text{Spec} \, S$ be a minimal prime over $m_i$. Since $m_i$ is a maximal ideal of $R$, we have

$$m_i = I_i \cap R = q \cap R. \quad (22)$$

**Lemma 5.8.** Suppose $q \in \text{Spec} \, S$ is a minimal prime over $m_i$. Then $I_i \subseteq q$. Consequently, if $I_i$ is prime in $S$, then $I_i = q$.

**Proof.** We first claim that $I_i \subseteq q$. Let $a \in S \setminus q$; we want to show that $a \not\in I_i$.

Assume to the contrary that $a \in I_i$. By Lemma 3.2, there is some $b \in R$ that is in $\bigcap_{j \neq i} I_j \setminus I_i$. Whence, $ab \in \bigcap_{j \neq i} I_j \subset R$. Furthermore, since $b \in R \setminus I_i$, we have $b \not\in q \cap R$ by (22). In particular, $b \not\in q$. Since $a$ and $b$ are not in $q$ and $q$ is prime, their product $ab$ is not in $q$. Thus,

$$a^{-1} = b(ab)^{-1} \in R_{q \setminus R} \overset{(i)}{=} (k + I_i)_{I_i},$$

where (i) holds by Lemma 3.6. Whence $a^{-1} \in (k + I_i)_{I_i}$. But $a \in I_i$, and thus $a$ is not invertible in $(k + I_i)_{I_i}$, a contradiction. Therefore $I_i \subseteq q$. □

**Lemma 5.9.** For each minimal prime $q \in \text{Spec} \, S$ over $m_i$ and $j \neq i$, the following hold:

$$R_{q \cap R}^j = R_{q \cap R}^i = m_j(k + I_i)_{I_i} = (k + I_i)_{I_i} \quad \text{and} \quad m_j S_q = S_q.$$

Furthermore, if either $I_i$ is prime in $S$ or $n = 1$, then

$$R_{q \cap R}^i = S_q \quad \text{and} \quad m_i S_q = q S_q.$$
Proof. (i) By Lemma 3.6, we have $R_{q|R} = (k + I_i)_{I_i}$, and for $j \neq i$, $R_{q|R}^j = (k + I_i)_{I_i}$.

(ii) Let $j \neq i$. We claim that $m_j(k + I_i)_{I_i} = (k + I_i)_{I_i}$. Fix $b \in m_j \setminus I_i$. Then $b \in k + I_i$ since $b \in R$. Whence, $b^{-1} \in (k + I_i)_{I_i}$. Therefore

$$1 = b b^{-1} \in m_j(k + I_i)_{I_i}.$$

(iii) Let $j \neq i$. We claim that $m_j S_q = S_q$. By Lemma 3.2, there is some $b \in (I_j \cap R) \setminus I_i = m_j \setminus m_i$. Whence, $b \not\in q$ by (22). Therefore

$$1 = b b^{-1} \in m_j S_q.$$  

(iv) Suppose $I_i$ is prime in $S$. We claim that $R_{q|R}^i = S_q$. Clearly, $R_{q|R}^i \subseteq S_q$.

To show the reverse inclusion, suppose $\frac{a}{b} \in S_q$ with $a \in S$ and $b \in S \setminus q$. By Lemma 3.2, there is some $c \in (\cap_{j \neq i} I_j) \setminus I_i$. Thus, $ac$ and $bc$ are in $\cap_{j \neq i} I_j \subset R_i$. Furthermore, $c \not\in q$ since $q = I_i$, by Lemma 5.8. Whence $bc \not\in q$ since $q$ is prime. Therefore

$$\frac{a}{b} = \frac{ac}{bc} \in R_{q|R}^i,$$

proving our claim.

(v) Again suppose $I_i$ is prime in $S$. We claim that $m_i S_q = q S_q$. Clearly, $m_i S_q \subseteq q S_q$.

To show the reverse inclusion, let $a \in q = I_i$. Fix $b \in \cap_{j \neq i} I_j \setminus I_i$. Then

$$ab \in \cap_{j \neq i} J_j \subset R.$$

Whence, $ab \in I_i \cap R = m_i$. Furthermore, $b \in S \setminus q$ since $q = I_i$. Therefore

$$a = ab b^{-1} \in m_i S_q.$$  

(vi) Finally, suppose $n = 1$, in which case $m = I$ (we omit the subscript $i$). We claim that $IS_q = q S_q$. The inclusion $IS_q \subseteq q S_q$ follows from Lemma 5.8.

To show the reverse inclusion, let $a \in q$. Consider the set of minimal primes over $I$,

$$q_1 := q, q_2, \ldots, q_m \in \text{Spec } S.$$

In particular, $I = \cap q_j$ since $I$ is radical.

For each $2 \leq j \leq m$, fix $b_j \in q_j \setminus q$. Then $b_2 \cdots b_m \in S \setminus q$ since $q$ is prime. Therefore

$$a = (ab_2 \cdots b_m)(b_2 \cdots b_m)^{-1} \in (\cap q_j) S_q = IS_q.$$

Set

$$\tilde{R} := (k + I_i)_{I_i} + q S_q.$$  

If $I_i$ is prime in $S$, then by Lemma 5.8 this reduces to

$$\tilde{R} = (k + q)_{\tilde{q}} + q S_q.$$

Lemma 5.10. $\tilde{R}$ is a subalgebra of $S_q$. 


Proof. Since $k + I_i \subset S$, it suffices to show that if $a$ is invertible in $(k + I_i)_i$, then $a$ is also invertible in $S_q$. So suppose $a \in (k + I_i) \setminus I_i$. Then $a = c + \alpha$, where $c \in k^\times$ and $\alpha \in I_i$. But $I_i \subseteq q$ by Lemma 5.8. Whence $a \in S \setminus q$. □

Index the rows and columns of $A_q$ by $0, 1, \ldots, n$. Denote by $e_{ij} \in M_{n+1}(\text{Frac} S)$ the matrix with a 1 in the $ij$-th slot and zeros elsewhere, and set $e_i := e_{ii}$.

Proposition 5.11. Suppose each $I_i \subset S$ is prime, or $n = 1$. Fix $p \in \text{Spec } R$, and let $q \in \text{Spec } S$ be a minimal prime over $p$.

(1) If $R_p$ is noetherian, then the cyclic localization $A_q$ at $q$ is the full matrix ring

$$A_q = M_{n+1}(R_p) \cong A \otimes_R R_p.$$ 

(2) If $R_p$ is nonnoetherian, then $p = m_i$ for some $i$, and

$$A_q = \begin{bmatrix} \tilde{R} & \cdots & \tilde{R} & qS_q & \tilde{R} & \cdots & \tilde{R} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \tilde{R} & \cdots & \tilde{R} & qS_q & \tilde{R} & \cdots & \tilde{R} \\ S_q & \cdots & S_q & S_q & S_q & \cdots & S_q \\ \tilde{R} & \cdots & \tilde{R} & qS_q & \tilde{R} & \cdots & \tilde{R} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \tilde{R} & \cdots & \tilde{R} & qS_q & \tilde{R} & \cdots & \tilde{R} \end{bmatrix} \subset M_{n+1}(\text{Frac } S)$$

where the $i$th row and column are respectively

$$e_iA_q = \begin{bmatrix} S_q & \cdots & S_q \end{bmatrix} = S_q^{\oplus n+1},$$

$$A_q e_i = \begin{bmatrix} qS_q & \cdots & qS_q & S_q & qS_q & \cdots & qS_q \end{bmatrix}^t,$$

and all other entries are $\tilde{R}$.

Proof. (1) Suppose $R_p$ is noetherian. Then $R_p = S_q$ since $S$ is a depiction of $R$.

(1.i) We first claim that the diagonal entries of $A_q$ are all $S_q$. Fix $i \in [1, n]$. We have

$$S_q = R_{q \cap R} \subseteq R_{q \cap R}^i \subseteq S_q,$$

where (i) holds since $S$ is a depiction of $R$; (ii) holds since $R \subset R'$; and (iii) holds since $R' \subseteq S$. Therefore $R_{q \cap R}^i = S_q$.

(1.ii) We now claim that the off-diagonal entries of $A_q$ are also all $S_q$.

Fix $i \in [1, n]$, and assume to the contrary that $m_i \subseteq q$. Then

$$m_i = m_i \cap R \subseteq q \cap R = p.$$ 

Whence, $m_i = p$ since $m_i$ is maximal. But $R_{m_i}$ is nonnoetherian by Theorem 3.14 and Proposition 3.4, contrary to our choice of $p$. Thus $m_i \not\subseteq q$. Hence, there is some
a \in m_i \setminus q$. Consequently, $1 = aa^{-1} \in m_i S_q$. Therefore $m_i S_q = S_q$. Together with (1.i), this implies that the off-diagonal entries of $A_q$ in columns $1, \ldots, n$ are $S_q$.

Finally, the off-diagonal entries in column $0$ are also $S_q$: since $1 \in R_i$ and $R_i \subseteq S$, we have $R_i S_q = S_q$.

(2) Follows from Lemmas 5.9 and 5.10. □

Fix $i \in [1, n]$ and a minimal prime $q \in \text{Spec } S$ over $m_i$.

**Lemma 5.12.** Let $j \in [0, n]$, let $P$ be a projective $A_q$-module, and let

$$\delta : A_q e_j \to P$$

be an $A_q$-module homomorphism. Suppose $e_{ij} \in A_q$. If $\delta(e_{ij}) = 0$, then $\delta = 0$.

**Proof.** Set $\Lambda := A_q$, and suppose $\delta(e_{ij}) = 0$. Let $\ell \geq 1$ be minimal such that $P$ is a direct summand of $\Lambda^{\oplus \ell}$. Let $a_1, \ldots, a_\ell \in \Lambda$ be such that

$$\delta(e_j) = (a_1, \ldots, a_\ell) \in \Lambda^{\oplus \ell}.$$

Each $a_k$ is in $e_j \Lambda$ since

$$(a_1, \ldots, a_\ell) = \delta(e_j) = \delta(e_j^2) = e_j \delta(e_j) \in e_j \Lambda^{\oplus \ell}.$$}

Furthermore, each product $e_{ij} a_k$ is zero since

$$(e_{ij} a_1, \ldots, e_{ij} a_\ell) = e_{ij} (a_1, \ldots, a_\ell) = e_{ij} \delta(e_j) = \delta(e_{ij}) = 0.$$

But $e_{ij} \alpha \neq 0$ for all nonzero $\alpha$ in $e_j \Lambda$. Therefore each $a_k$ is zero. □

**Proposition 5.13.** The left global dimension of $A_q$ is bounded above by the Krull dimension of $S_q$,

$$\text{gldim } A_q \leq \dim S_q.$$

**Proof.** Set $\Lambda := A_q$ and $d := \dim S_q - 1$. Let $V$ be a $\Lambda$-module. We claim that

(23) $$\text{pd}_\Lambda(V) \leq d + 1.$$

It suffices to show that there is a projective resolution $P_\bullet$ of $V$,

$$\cdots \longrightarrow P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} V \longrightarrow 0,$$

for which the $(d + 1)$th syzygy $\ker \delta_d$ is a projective $\Lambda$-module [R Proposition 8.6.iv].

Since $\{e_0, \ldots, e_n\}$ is a complete set of orthogonal idempotents of $\Lambda$, we may assume that for each $\ell \geq 0$ and $j \in [0, n]$, there is some $m_{\ell j} \geq 0$ such that

$$P_\ell = \bigoplus_{j : m_{\ell j} \geq 1} (\Lambda e_j)^{\oplus m_{\ell j}} = \bigoplus_{j : m_{\ell j} \geq 1} \bigoplus_{t \in [1, m_{\ell j}]} \Lambda e_j \varepsilon_t,$$

where $e_j \varepsilon_t$ generates the $t$-th $\Lambda e_j$ summand of $(\Lambda e_j)^{\oplus m_{\ell j}}$ over $\Lambda$.

Now $e_i \Lambda$ is a left $S_q$-module since $e_i \Lambda e_i = e_i S_q$. Furthermore, $e_i \Lambda$ is a projective, hence flat, right $\Lambda$-module. Thus, setting $\otimes := \otimes_\Lambda$, the sequence of $S_q$-modules

$$\cdots \longrightarrow e_i \Lambda \otimes P_2 \xrightarrow{1 \otimes \delta_2} e_i \Lambda \otimes P_1 \xrightarrow{1 \otimes \delta_1} e_i \Lambda \otimes P_0 \xrightarrow{1 \otimes \delta_0} e_i \Lambda \otimes V \longrightarrow 0$$
is exact. Moreover, each term $e_i \Lambda \otimes P_e$ is a free $S_q$-module since
\begin{equation}
(24) \quad e_i \Lambda \otimes P_e = \bigoplus_{j \in S_q} (e_i \Lambda \otimes \Lambda e_j)^{\oplus m_{ij}} \cong \bigoplus_{j \in S_q} (e_i \Lambda e_j)^{\oplus m_{ij}} = \bigoplus_{j \in S_q} (e_i j S_q)^{\oplus m_{ij}}.
\end{equation}
It follows that $e_i \Lambda \otimes P_e$ is a free resolution of the $S_q$-module $e_i \Lambda \otimes V \cong e_i V$. Thus, since $S_q$ is a regular local ring of dimension $d + 1$, the $(d + 1)$th syzygy $\ker(1 \otimes \delta_d)$ of $e_i \Lambda \otimes P_e$ is a free $S_q$-module. Therefore, since $\ker(1 \otimes \delta_d)$ is a submodule of $\bigoplus_{j \in S_q} e_i j S_q^{\oplus m_{dj}}$, for each $j \in [0, n]$ there is some $r_j \in [0, m_{dj}]$ such that
\begin{equation}
(25) \quad \ker(1 \otimes \delta_d) \cong \bigoplus_{j : r_j \geq 1} (e_i j S_q)^{\oplus r_j}.
\end{equation}
Again since $e_i \Lambda$ is a flat right $\Lambda$-module, the sequence
\begin{equation*}
0 \rightarrow e_i \Lambda \otimes \ker \delta_d \rightarrow e_i \Lambda \otimes P_d \xrightarrow{1 \otimes \delta_d} e_i \Lambda \otimes P_{d-1}
\end{equation*}
is exact. Whence
\begin{equation}
(26) \quad e_i \Lambda \otimes \ker \delta_d = \ker(1 \otimes \delta_d).
\end{equation}
But (25) and (26) together imply
\begin{equation}
(27) \quad e_i \ker \delta_d = \bigoplus_{j : r_j \geq 1} (e_i j S_q)^{\oplus r_j} = e_i \bigoplus_{j : r_j \geq 1} (\Lambda e_j)^{\oplus r_j} = e_i \bigoplus_{j : r_j \geq 1} \Lambda e_j \varepsilon_t.
\end{equation}
In particular, for each $t \in [1, r_j]$ we have $\delta_d(e_i j \varepsilon_t) = 0$. Thus by Lemma 5.12
\begin{equation*}
\delta_d(\Lambda e_j \varepsilon_t) = 0.
\end{equation*}
Therefore
\begin{equation}
(28) \quad \ker \delta_d \supseteq \bigoplus_{j : r_j \geq 1} \Lambda e_j \varepsilon_t.
\end{equation}

To show the reverse inclusion, fix $j \in [0, n]$ satisfying $m_{dj} \geq 1$, and let $t \in [1, m_{dj}]$. Suppose $e_k j \varepsilon_t \in \ker \delta_d$. Then, since $1 \in \Lambda^k$ for each $k \in [0, n]$, we have
\begin{equation*}
\delta_d(e_i j \varepsilon_t) = \delta_d(e_ik e_k j \varepsilon_t) = e_ik \delta_d(e_k j \varepsilon_t) = 0.
\end{equation*}
Whence $e_i j \varepsilon_t \in e_i \ker \delta_d$. Thus $t \in [1, r_j]$, by (27). Therefore
\begin{equation}
(29) \quad \ker \delta_d \subseteq \bigoplus_{j : r_j \geq 1} \Lambda e_j \varepsilon_t.
\end{equation}
(28) and (29) together imply that $\ker \delta_d = \bigoplus_{j : r_j \geq 1} (\Lambda e_j)^{\oplus r_j}$. Consequently, (23) holds. \hfill \Box

**Proposition 5.14.** The cyclic localization $A_q$ has precisely two simple modules up to isomorphism,
\begin{equation*}
V := A_q e_i / A_q (1 - e_i) A_q e_i \cong S_q / q,
\end{equation*}
\begin{equation}
(30) \quad W := A_q e_0 / A_q e_i A_q e_0 \cong \bigoplus_{j \neq i} ke_{j0} \cong (R/q)^{\oplus n}.
\end{equation}
Their projective dimensions are respectively
\[ \text{pd}_{A_q}(V) = \dim S_q \quad \text{and} \quad \text{pd}_{A_q}(W) = 1. \]

**Proof.** Set \( \Lambda := A_q \).

(i) We claim that the simple \( \Lambda \)-modules are precisely the modules \( V \) and \( W \) in (30). For each \( j \in [0, n] \setminus \{i\} \), there is a (left) \( \Lambda \)-module isomorphism
\[ \Lambda e_0 \xrightarrow{-e_0} \Lambda e_j. \]
Furthermore, \( W \) is simple since for each \( j,k \in [0, n] \setminus \{i\} \), the matrix entry \( A^{jk} \) contains \( 1 \in R \); whence
\[ e_{jk}e_{k0} = e_{j0} \quad \text{and} \quad e_{kj}e_{j0} = e_{k0}. \]

(ii) We claim that \( \text{pd}_\Lambda(V) = \dim S_q \). Indeed, we have
\[ \dim S_q = \text{pd}_{S_q}(S_q/q) \leq \text{pd}_\Lambda(V) \leq \dim S_q, \]
where (i) holds since \( S_q \) is a regular local ring, and (iii) holds by Proposition 5.13.

(ii) holds since if \( P_\bullet \) is a projective resolution of \( V \) over \( \Lambda \), then \( e_i \Lambda \otimes_\Lambda P_\bullet \) is a free resolution of \( V \cong S_q/q \) over \( e_i \Lambda e_i \cong S_q \), as shown in (24).

(iii) We claim that \( \text{pd}_\Lambda(W) = 1 \). Consider the complex
\[ 0 \rightarrow \Lambda e_i \xrightarrow{-e_0} \Lambda e_0 \rightarrow W \rightarrow 0. \]

The module homomorphism \( \Lambda e_i \xrightarrow{-e_0} \Lambda e_0 \) maps onto the kernel of \( \Lambda e_0 \rightarrow W \), namely \( \Lambda e_0 \Lambda e_0 \), since \( \Lambda^{ii} = S_q = \Lambda^{i0} \). Thus the complex (31) is exact. \( \square \)

**Proposition 5.15.** The residue module at \( q \) decomposes as
\[ A_q/q = V \oplus W^{\oplus n}, \]
and has projective dimension
\[ \text{pd}_{A_q}(A_q/q) = \dim S_q. \]

**Proof.** Set \( \Lambda := A_q \).

The direct sum decomposition (32) follows from Proposition 5.14, where we view \( V \) and \( W \) as columns of the \((n+1) \times (n+1)\) tiled matrix ring \( \Lambda/(q \cap \Lambda) \).

The projective dimension of \( \Lambda/(q \cap \Lambda) \) equals the Krull dimension of \( S_q \) since
\[ \dim S_q = \text{pd}_\Lambda(V) \leq \text{pd}(\Lambda/(q \cap \Lambda)) \leq \text{gl.dim} \Lambda \leq \dim S_q, \]
Indeed, (i) holds by Proposition 5.14, (ii) holds by (32), since the projective dimension of a module \( M \) is greater than or equal to the projective dimension of any direct summand of \( M \); and (iii) holds by Proposition 5.13. \( \square \)
Let $A$ be a $k$-algebra with prime center $Z$, and let $m \in \text{Max } Z$. Then $A_m = A \otimes_Z Z_m$ is said to be Azumaya over its center $Z_m$ if $A_m$ is a free $Z_m$-module of finite rank, and the algebra homomorphism

$$A_m \otimes_{Z_m} A_m^{op} \rightarrow \text{End}_{Z_m}(A_m)$$

$$a \otimes b \mapsto (x \mapsto axb)$$

is an isomorphism [McR, 13.7.6], [BG, III.1.3]. The Azumaya locus of $A$ is the set of points $m \in \text{Max } Z$ for which $A_m$ is an Azumaya algebra.

**Remark 5.16.** It is well-known that if $A_m$ is free of finite rank over $Z_m$, then $A_m$ is Azumaya if and only if $A_m/mA_m$ is a central simple algebra over $k$, if and only if $A_m/mA_m \sim M_n(k)$ for some $n \geq 1$ (assuming $k$ is algebraically closed).

**Theorem 5.17.** Let $S$ be a finite type normal integral domain. Let $I_1, \ldots, I_n$ be a set of proper non-maximal nonzero radical ideals of $S$ such that either $n = 1$, or the closed sets $Z(I_i) \subset \text{Max } S$ are irreducible and pairwise non-intersecting.

Set $R := \cap_{i=1}^n (k + I_i)$, and consider its nonnoetherian points $m_i := I_i \cap R \in \text{Max } R$. Then

1. $S$ can be retrieved from $R$ as the cycle algebra of the endomorphism ring

   $$A = \text{End}_R (R \oplus m_1 \oplus \cdots \oplus m_n).$$

   Furthermore, the center of $A$ is $R$.

2. The Azumaya locus of $A$ and the noetherian locus $U_{S/R}$ of $R$ coincide.

3. If each $Z(I_i)$ intersects the smooth locus of Max $S$, then $A$ is a noncommutative desingularization of its center $R$:
   (a) Frac $R$ and $A \otimes_R \text{Frac } R$ are Morita equivalent, and
   (b) for each $i \in [1, n]$ and minimal prime $q \in \text{Spec } S$ over $m_i$, we have

   $$\text{gldim } A_q = \text{pd } A_q(A_q/q) = \dim S_q.$$ 

**Proof.** (1) The algebra $A$ has center $R$ by Proposition 5.4 and cycle algebra $S$ by Proposition 5.7.

(2) The noetherian locus $U_{S/R}$ of $R$ is contained in the Azumaya locus of $A$ by Proposition 5.13.1 and Remark 5.16. Conversely, if $n \in \text{Max } S \setminus U_{S/R}$, then $A \otimes_R R_n \cap R$ is not a free $R_n \cap R$-module by Proposition 5.4. Whence $n \cap R \in \text{Max } R$ is not in the Azumaya locus of $A$.

(3.a) We claim that Frac $R$ and $A \otimes_R \text{Frac } R$ are Morita equivalent. By Theorem 3.14, $S$ is a depiction of $R$; in particular, $U_{S/R} \neq \emptyset$. Thus Frac $R = \text{Frac } S$. Therefore

$$A \otimes_R \text{Frac } R = A \otimes_R \text{Frac } S \overset{(1)}{=} M_{n+1}(\text{Frac } S) = M_{n+1}(\text{Frac } R),$$

where (1) holds by Proposition 5.4. The claim follows.

(3.b) We have $\text{gldim } A_q \leq \dim S_q$ by Proposition 5.13 and $\text{gldim } A_q \geq \dim S_q$ by Proposition 5.14. Furthermore, $\text{pd } A_q(A_q/q) = \dim S_q$ by Proposition 5.15. □
Remark 5.18. The advantage of the noncommutative blowup $A$ over the depiction $S$ is given by Theorem 5.17: the noetherian locus $U_{S/R}$ of $R$ is intrinsic to $A$ since it is encoded in the representation theory of $A$, whereas the noetherian locus is invisible to $S$. Furthermore, $A$ ‘sees’ both $R$ and $S$: they appear as the center and cycle algebra of $A$ respectively.

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References

[AB1] M. Auslander, D. A. Buchsbaum, Homological dimension in noetherian rings, Proc. Nat. Acad. Sci. U.S.A., 42(1):36-38, 1956.

[AB2] ______, Homological dimension in local rings, Trans. Amer. Math. Soc., 85:390-405, 1957.

[BKM] K. Baur, A. King, R. Marsh, Dimer models and cluster categories of Grassmannians, Proc. London Math. Soc., 113(2):213-260, 2016.

[B1] C. Beil, The Bell states in noncommutative algebraic geometry, Int. J. of Quantum Inf., 12(5), 2014.

[B2] ______, Morita equivalences and Azumaya loci from Higgsing dimer algebras, J. Algebra, 453:429-455, 2016.

[B3] ______, Nonnoetherian homotopy dimer algebras and noncommutative crepant resolutions, Glasgow Math. J., 60(2):447-479, 2018.

[B4] ______, Nonnoetherian geometry, J. Algebra Appl., 15(09), 2016.

[B5] ______, On the central geometry of nonnoetherian dimer algebras, J. Pure Appl. Algebra, 225(8), 2021.

[B6] ______, Nonnoetherian Lorentzian manifolds, [arXiv:2103.3743]

[Br] N. Broomhead, Dimer models and Calabi-Yau algebras, Memoirs AMS, 1011, 2012.

[BG] K. A. Brown, K. R. Goodearl, Lectures on Algebraic Quantum Groups, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser Verlag, Basel, 2002.

[BH] K. Brown, C. Hajarnavis, Homologically homogeneous rings, Trans. Amer. Math. Soc., 281:197-208, 1984.

[FHKV] B. Feng, Y.-H. He, K. Kennaway, C. Vafa, Dimer models from mirror symmetry and quivering amoebae, Adv. Theor. Math. Phys., 12(3):489-545, 2008.

[FHMSVW] S. Franco, A. Hanany, D. Martelli, J. Sparks, D. Vegh, B. Wecht, Gauge theories from toric geometry and brane tilings, J. High Energy Phys., 1:128, 2006.

[HK] A. Hanany, K. D. Kennaway, Dimer models and toric diagrams, arXiv:0503149.

[H] R. Hartshorne, Algebraic geometry, Springer, 1977.

[He] Y.-H. He, Calabi-Yau Varieties: from Quiver Representations to Dessins d’Enfants, [arXiv:1611.09398]

[IN] O. Iyama, Y. Nakajima, On steady non-commutative crepant resolutions, J. Noncommut. Geom., 12(2):457-471, 2018.

[IU] A. Ishii, K. Ueda, Dimer models and the special McKay correspondence, Geometry and Topology, 19:3405-3466, 2015.

[L] G. Leuschke, Non-commutative crepant resolutions: scenes from categorical geometry, Progress in commutative algebra 1, 293-361, de Gruyter, Berlin, 2012.
[McR] J. C. McConnell and J. C. Robson, *Noncommutative noetherian rings*, Amer. Math. Soc. volume 30, Wiley-Interscience, 1987.

[MR] S. Mozgovoy, M. Reineke, On the noncommutative Donaldson-Thomas invariants arising from dimer models, *Adv. Math.*, 223:1521-1544, 2010.

[R] J. Rotman, *An introduction to homological algebra*, Springer, 2009.

[S] J.-P. Serre, Sur la dimension homologique des anneaux et des modules noeth’eriens. *Proceedings of the international symposium on algebraic number theory*, 175-189, Tokyo and Nikko, 1955. Science Council of Japan, Tokyo, 1956.

[St] J. T. Stafford, On the Ideals of a Noetherian Ring, *Trans. Amer. Math. Soc.*, 289(1):381-392, 1985.

[V] M. Van den Bergh, Non-commutative crepant resolutions, *The legacy of Niels Henrik Abel*, 749-770, Springer, Berlin, 2002.

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