This item is the archived preprint of:

The geometry of representations of 3-dimensional Sklyanin algebras

Reference:
De Laet Kevin, Le Bruyn Lieven.- The geometry of representations of 3-dimensional Sklyanin algebras
Algebras and representation theory - ISSN 1386-923X - 18(2015), p. 761-776
DOI: http://dx.doi.org/doi:10.1007/s10468-014-9515-6
THE GEOMETRY OF REPRESENTATIONS OF
3-DIMENSIONAL SKLYANIN ALGEBRAS

KEVIN DE LAET AND LIEVEN LE BRUYN

Abstract. The representation scheme \( \text{rep}_A \) of the 3-dimensional Sklyanin algebra \( A \) associated to a plane elliptic curve and \( n \)-torsion point contains singularities over the augmentation ideal \( m \). We investigate the semi-stable representations of the noncommutative blow-up algebra \( B = A \oplus mt \oplus m^2 t^2 \oplus \ldots \) to obtain a partial resolution of the central singularity
\[
\text{proj } Z(B) \longrightarrow \text{spec } Z(A)
\]
such that the remaining singularities in the exceptional fiber determine an elliptic curve and are all of type \( \mathbb{C} \times \mathbb{C}^2 / \mathbb{Z}_n \).

1. Introduction

Three dimensional Sklyanin algebras appear in the classification by M. Artin and W. Schelter \[2\] of graded algebras of global dimension 3. In the early 90ties this class of algebras was studied extensively by means of noncommutative projective algebraic geometry, see for example \[3\], \[4\], \[5\], \[9\] and \[15\]. Renewed interest in this class of algebras arose recently as they are superpotential algebras and as such relevant in supersymmetric quantum field theories, see \[6\] and \[16\].

Consider a smooth elliptic curve \( E \) in Hesse normal form \( V((a^3 + b^3 + c^3)XYZ - abc(X^3 + Y^3 + Z^3)) \hookrightarrow \mathbb{P}^2 \) and the point \( p = [a : b : c] \) on \( E \). The 3-dimensional Sklyanin algebra \( A \) corresponding to the pair \((E, p)\) is the noncommutative algebra with defining equations
\[
\begin{align*}
axy + byx + cz^2 &= 0, \\
ayz + bzy + cx^2 &= 0, \\
axz + bxz + cy^2 &= 0.
\end{align*}
\]

The connection comes from the fact that the multi-linearization of these equations defines a closed subscheme in \( \mathbb{P}^2 \times \mathbb{P}^2 \) which is the graph of translation by \( p \) on the elliptic curve \( E \), see \[4\]. Alternatively, one obtains the defining equations of \( A \) from the superpotential \( W = axyz + byxz + 2c(x^3 + y^3 + z^3) \), see \[10\], Ex. 2.3.

The algebra \( A \) has a central element of degree 3, found by a computer calculation in \[2\]
\[
c_3 = c(a^3 - c^3)x^3 + a(b^3 - c^3)xyz + b(c^3 - a^3)yxz + c(c^3 - b^3)y^2,
\]
with the property that \( A/(c_3) \) is the twisted coordinate ring of the elliptic curve \( E \) with respect to the automorphism given by translation by \( p \), see \[4\]. We will prove an intrinsic description of this central element, answering a MathOverflow question from 2013 (see \[8\]).
Theorem 1. The central element $c_3$ of the 3-dimensional Sklyanin algebra $A$ corresponding to the pair $(E, p)$ can be written as

$$a(b^3 - c^3)(xyz + yxz + zyx) + b(c^3 - a^3)(yxz + xzy + zyx) + c(a^3 - b^3)(x^3 + y^3 + z^3)$$

and is the superpotential of the 3-dimensional Sklyanin algebra $A'$ corresponding to the pair $(E, [-2]p)$.

Next, we turn to the geometry of finite dimensional representations of $A$ in the special case when $A$ is a finite module over its center. This setting is important in physics in order to understand the Calabi-Yau geometry of deformed $N=4$ SYM theories. We refer the interested reader to the introduction of [16] for more details.

It is well known that $A$ is a finite module over its center $Z(A)$ and a maximal order in a central simple algebra of dimension $n^2$ if and only if the point $p$ is of finite order $n$, see [5], Thm II. We will further assume that $(n, 3) = 1$ in which case J. Tate and P. Smith proved in [15], Thm. 4.7 that the center $Z(A)$ is generated by $c_3$ and the reduced norms of $x, y$ and $z$ (which are three degree $n$ elements, say $x', y', z'$) satisfying one relation of the form

$$c_3^n = \text{cubic}(x', y', z').$$

It is also known that $\text{proj} Z(A) \simeq \mathbb{P}^2$ with coordinates $[x' : y' : z']$ in which the $\text{cubic}(x', y', z')$ defines the isogenous elliptic curve $E' = E/\langle p \rangle$, see [9], Section 2. We will use these facts to give explicit matrices for the simple $n$-dimensional representations of $A$ and show that $A$ is an Azumaya algebra away from the isolated central singularity.

However, the scheme $\text{rep}_n A$ of all (trace preserving) $n$-dimensional representations of $A$ contains singularities in the nullcone. We then try to resolve these representation singularities by considering the noncommutative analogue of a blow-up algebra

$$B = A \oplus mt \oplus m^2t^2 \oplus \ldots \subset A[t, t^{-1}]$$

where $m = (x, y, z)$ is the augmentation ideal of $A$. We will prove

Theorem 2. The scheme $\text{rep}^{ss}_n B$ of all semi-stable $n$-dimensional representations of the blow-up algebra $B$ is a smooth variety.

This allows us to compute all the (graded) local quivers in the closed orbits of $\text{rep}^{ss}_n B$ as in [10] and [7]. This information then leads to the main result of this paper which gives a partial resolution of the central isolated singularity.

Theorem 3. The exceptional fiber $\mathbb{P}^2$ of the canonical map

$$\text{proj } Z(B) \longrightarrow \text{spec } Z(A)$$

contains $E' = E/\langle p \rangle$ as the singular locus of $\text{proj } Z(B)$. Moreover, all these singularities are of type $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$ with $\mathbb{C}^2/\mathbb{Z}_n$ an Abelian quotient surface singularity.

2. Central elements and superpotentials

The finite Heisenberg group of order 27

$$\langle u, v, w \mid [u, v] = w, [u, w] = [v, w] = 1, u^3 = v^3 = w^3 = 1 \rangle$$
has a 3-dimensional irreducible representation $V = \mathbb{C}x + \mathbb{C}y + \mathbb{C}z$ given by the action

$$
u \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \nu \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{bmatrix}, \quad w \mapsto \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{bmatrix},$$

with $\rho^3 = 1$ a primitive 3rd root of unity. One verifies that $V \otimes V$ decomposes as three copies of $V^*$, that is,

$$V \otimes V \simeq \wedge^2(V) \oplus S^2(V) \simeq V^* \oplus (V^* \oplus V^*),$$

where the three copies can be taken to be the subspaces

$$V_1 = \mathbb{C}(yz - zy) + \mathbb{C}(zx - xz) + \mathbb{C}(xy - yx),$$

$$V_2 = \mathbb{C}(yz + zy) + \mathbb{C}(zx + xz) + \mathbb{C}(xy + yx),$$

$$V_3 = \mathbb{C}x^2 + \mathbb{C}y^2 + \mathbb{C}z^2.$$

Taking the quotient of $\mathbb{C}(x, y, z)$ modulo the ideal generated by $V_1 = \wedge^2 V$ gives the commutative polynomial ring $\mathbb{C}[x, y, z]$. Hence we can find analogues of the polynomial ring in three variables by dividing $\mathbb{C}(x, y, z)$ modulo the ideal generated by another copy of $V^*$ in $V \otimes V$ and the resulting algebra will inherit an action by $H_3$. Such a copy of $V^*$ exists for all $[A : B : C] \in \mathbb{P}^2$ and is spanned by the three vectors

$$A(yz - zy) + B(yz + zy) + Cx^2,$$

$$A(zx - xz) + B(zx + xz) + Cy^2,$$

$$A(xy - yx) + B(xy + yx) + Cz^2,$$

and by taking $a = A + B, b = B - A$ and $c = C$ we obtain the defining relations of the 3-dimensional Sklyanin algebra. In particular there is an $H_3$-action on $A$ and the canonical central element $c_3$ of degree 3 must be a 1-dimensional representation of $H_3$. It is obvious that $c_3$ is fixed by the action of $v$ and a minor calculation shows that $c_3$ is also fixed by $u$. Therefore, the central element $c_3$ given above, or rather $3c_3$, can also be represented as

$$a(b^3 - c^3)(xyz + yzx + zxy) + b(c^3 - a^3)(yzx + xzy + zyx) + c(a^3 - b^3)(x^3 + y^3 + z^3).$$

Now, let us reconsider the superpotential $W = axyz + byxz + \frac{c}{3}(x^3 + y^3 + z^3)$ for a $[a : b : c] \in \mathbb{P}^2$. This superpotential gives us three quadratic relations by taking cyclic derivatives with respect to the variables

$$\partial_x W = ayz + bzy + cx^2,$$

$$\partial_y W = azx + bzx + cy^2,$$

$$\partial_z W = axy + byx + cz^2,$$

giving us the defining relations of the 3-dimensional Sklyanin algebra. We obtain the same equations by considering a more symmetric form of $W$, or rather of $3W$

$$a(xyz + yzx + zxy) + b(yxz + xzy + zyx) + c(x^3 + y^3 + z^3).$$

We see that the form of the central degree 3 element and of the superpotential are similar but with different coefficients. This means that the central element is the superpotential defining another 3-dimensional Sklyanin algebra and theorem 1 clarifies this connection.
\textbf{Proof of Theorem 1 :} The 3-dimensional Sklyanin algebra associated to the superpotential $3c_3$ is determined by the point $[a_1 : b_1 : c_1] = [a(b^3 - c^3) : b(c^3 - a^3) : c(a^3 - b^3)] \in \mathbb{P}^2$ (instead of $[a : b : c]$ for the original). Therefore, the associated elliptic curve has defining Hesse equation
\[ V(\alpha(x^3 + y^3 + z^3) - \beta xyz) \subseteq \mathbb{P}^2 \]
where
\[ \begin{cases} \alpha = a_1b_1c_1 = a(b^3 - c^3)b(c^3 - a^3)c(a^3 - b^3), \\ \beta = a_1^3 + b_1^3 + c_1^3 = (a(b^3 - c^3))^3 + (b(c^3 - a^3))^3 + (c(a^3 - b^3))^3, \end{cases} \]
but by a Maple computation one verifies that, up to a scalar, this is the original curve
\[ E = V(abc(x^3 + y^3 + z^3) - (a^3 + b^3 + c^3)xyz). \]
The tangent line to $E$ in the point $p = [a : b : c]$ has equation
\[ (2a^3bc - b^3c - bc^3)(x - a) + (2ab^3c - a^4c - ac^4)(y - b) + (2abc^3 - a^4b - ab^4)(z - c) = 0 \]
and so the third point of intersection is
\[ [-2p] = [a(b^3 - c^3) : b(c^3 - a^3) : c(a^3 - b^3)], \]
which are the parameters of the algebra associated to the the superpotential $3c_3$. □

3. Resolving representation singularities

Let $R$ be a graded $\mathbb{C}$-algebra, generated by finitely many elements $x_1, \ldots, x_m$ where $\text{deg}(x_i) = d_i \geq 0$, which is a finite module over its center $Z(R)$. Following [IL, 2.3] we say that $R$ is a Cayley-Hamilton algebra of degree $n$ if there is a $Z(R)$-linear gradation preserving trace map $tr : R \rightarrow Z(R)$ such that for all $a, b \in R$ we have
\[ \begin{align*} \bullet & \quad tr(ab) = tr(ba) \\ \bullet & \quad tr(1) = n \\ \bullet & \quad \chi_{n,a}(t) = 0 \end{align*} \]
where $\chi_{n,a}(t)$ is the $n$-th Cayley-Hamilton identity expressed in the traces of powers of $a$. Maximal orders in a central simple algebra of dimension $n^2$ are examples of Cayley-Hamilton algebras of degree $n$.

In particular, a 3-dimensional Sklyanin algebra $A$ associated to a couple $(E, p)$ where $p$ is a torsion point of order $n$, and the corresponding blow-up algebra $B = A \oplus mt \oplus m^2t^2 \oplus \ldots$ are finitely generated graded Cayley-Hamilton algebras of degree $n$ equipped with the (gradation preserving) reduced trace map.

If $R$ is an finitely generated graded Cayley-Hamilton algebra of degree $n$ we define $\text{rep}_n R$ to be the affine scheme of all $n$-dimensional trace preserving representations, that is of all algebra morphisms
\[ R \xrightarrow{\phi} M_n(\mathbb{C}) \quad \text{such that} \quad \forall a \in R : \phi(tr(a)) = Tr(\phi(a)), \]
where $Tr$ is the usual trace map on $M_n(\mathbb{C})$. Isomorphism of representations defines a $GL_n$-action of $\text{rep}_n R$ and a result of Artin’s [IL, 12.6], asserts that the closed orbits under this action, that is the points of the GIT-quotient scheme $\text{rep}_n R//GL_n$, are precisely the isomorphism classes of $n$-dimensional trace preserving semi-simple representations of $R$. The reconstruction result of Procesi [IL, Thm. 2.6] asserts that in this setting
\[ \text{spec } Z(R) \simeq \text{rep}_n R//GL_n. \]
The gradation on $R$ defines an additional $\mathbb{C}^*$-action on $\text{rep}_n^s R$ commuting with the $\text{GL}_n$-action. With $\text{rep}_n^s R$ we denote the Zariski open subset of all semi-stable trace preserving representations $\phi : R \longrightarrow M_n(\mathbb{C})$, that is, such that there is an homogeneous central element $c$ of positive degree such that $\phi(c) \neq 0$. We have the following graded version of Procesi’s reconstruction result, see amongst others [7, Section 8]

$$\text{proj } Z(R) \simeq \text{rep}_n^s R//\text{GL}_n \times \mathbb{C}^*.$$ As a $\text{GL}_n \times \mathbb{C}^*$-orbit is closed in $\text{rep}_n^s R$ if and only if the $\text{GL}_n$-orbit is closed we see that points of $\text{proj } Z(R)$ classify one-parameter families of isoclasses of trace-preserving $n$-dimensional semi-simple representations of $R$. In case of a simple representation such a one-parameter family determines a graded algebra morphism

$$R \longrightarrow M_n(\mathbb{C}[t,t^{-1}])((0,\ldots,0,1,\ldots,1,e-1,\ldots,e-1))^{m_0 \ldots m_2 \ldots m_{e-1}}$$

where $e$ is the degree of $t$, the $m_i$ are natural numbers with $\sum_{i=0}^{e-1} m_i = n$ and where we follow [13] in defining the shifted graded matrix algebra $M_n(S)(a_1,\ldots,a_n)$ by taking its homogeneous part of degree $i$ to be

$$\begin{bmatrix}
S_i & S_{i-a_1+a_2} & \cdots & S_{i-a_1+a_n} \\
S_{i-a_2+a_1} & S_i & \cdots & S_{i-a_2+a_n} \\
\vdots & \vdots & \ddots & \vdots \\
S_{i-a_n+a_1} & S_{i-a_n+a_2} & \cdots & S_i
\end{bmatrix}$$

The $\text{GL}_n \times \mathbb{C}^*$-stabilizer subgroup of any of the simples $\phi$ in this family is then isomorphic to $\mathbb{C}^* \times \mu_e$ where the cyclic group $\mu_e$ has generator $(g_\zeta, \zeta) \in \text{GL}_n \times \mathbb{C}^*$ where $\zeta$ is a primitive $e$-th root of unity and

$$g_\zeta = \text{diag}(1,\ldots,1,\zeta,\ldots,\zeta^{e-1},\zeta^{e-1},\ldots,\zeta^{e-1}),$$

see [7, lemma 4]. If, in addition, $\phi$ is a smooth point of $\text{rep}_n^s R$ then the normal space

$$N(\phi) = T_\phi \text{rep}_n^s R//T_\phi \text{GL}_n . \phi$$

to the $\text{GL}_n$-orbit decomposes as a $\mu_e$-representation into a direct sum of 1-dimensional simples

$$N(\phi) = \mathbb{C}_{i_1} \oplus \cdots \oplus \mathbb{C}_{i_d}$$

where the action of the generator on $\mathbb{C}_{i_d}$ is by multiplication with $\zeta^{i_d}$. Alternatively, $\phi$ determines a (necessarily smooth) point $[\phi] \in \text{spec } Z(R)$ and because $N(\phi)$ is equal to $\text{Ext}_R^1(S_{\phi}, S_{\phi})$ and because $R$ is Azumaya in $[\phi]$ it coincides with $\text{Ext}_R^1(Z(R))([S_{\phi}, S_{\phi}])$ (where $S_{[\phi]}$ is the simple 1-dimensional representation of $Z(R)$ determined by $[\phi]$) which is identical to the tangent space $T_{[\phi]} \text{spec } Z(R)$. The action of the stabilizer subgroup $\mu_e$ on $\text{Ext}_R^1(S_{\phi}, S_{\phi})$ carries over to that on $T_{[\phi]} \text{spec } Z(R)$.

The one-parameter family of simple representations also determines a point $\bar{\phi} \in \text{proj } Z(R)$ and an application of the Luna slice theorem [12] asserts that for all $t \in \mathbb{C}$ there is a neighborhood of $(\bar{\phi},t) \in \text{proj } Z(R) \times \mathbb{C}$ which is étale isomorphic to a neighborhood of 0 in $N(\phi)/\mu_e$, see [7, Thm. 5].
3.1. From $\text{Proj}(A)$ to $\text{rep}_n^A$. In noncommutative projective algebraic geometry, see for example \cite{[4],[5] and [3]}, one studies the Grothendieck category $\text{Proj}(A)$ which is the quotient category of all graded left $A$-modules modulo the subcategory of torsion modules. In the case of 3-dimensional Sklyanin algebras the linear modules, that is those with Hilbert series $(1-t)^{-1}$ (point modules) or $(1-t)^{-2}$ (line modules) were classified in \cite{[5]}, Section 6. Identify $P^2$ with $P_{nc}^2 = P(A_1^*)$, then

\[ \begin{align*}
E \cong P_{nc}^2 & \xrightarrow{\mathcal{N}} P_{c}^2 = P(Z(A)^*) \\
\mathcal{N} & \hookrightarrow \mathcal{N}/(p) \xrightarrow{\mathcal{N}/(p)} E' = E/\langle p \rangle
\end{align*} \]

Points $\pi \in P^2 - E'$ determine fat points $F_\pi$ with graded endomorphism ring isomorphic to $M_n(\mathbb{C}[t,t^{-1}])$ with $\deg(t) = 1$, and hence determine a one-parameter family of simple n-dimensional representations in $\text{rep}_{n}^{\mathbb{C}}A$ with $\mathbb{GL}_n \times \mathbb{C}^*$-stabilizer subgroup $\mathbb{C}^* \times 1$. There is an effective method to construct $F_\pi$, see \cite{[5]}, Section 3. Write $\pi$ as the intersection of two lines $\mathcal{V}(z) \cap \mathcal{V}(z')$ and let $\mathcal{V}(z') \cap E' = \{ q_1, q_2, q_3 \}$ be the intersection with the elliptic curve $E'$. Then by lifting the $q_i$ through the isogeny to $n$ points $p_{ij} \in E$ we see that we can lift the line $\mathcal{V}(z')$ to $n^2$ lines in $P_{nc}^2 = P(A_1^*)$, that is, there are $n^2$ one-dimensional subspaces $Cl \subset A_1$ with the property that $\mathbb{C}N(l) = \mathbb{C}z'$. The fat point corresponding to $\pi$ is then the shifted quotient of a line module determined by $l$

\[ F_\pi \simeq \frac{A}{A.l + A.z}[p]. \]

On the other hand, if $q$ is a point on $E'$, then lifting $q$ through the isogeny results in an orbit of $n$ points of $E$, $\{ r, r + p, r + [2]p, \ldots, r + [n-1]p \}$. If $P$ is the point module corresponding to $r \in E$, then the fat point module corresponding to $q$ is

\[ F_q = P \oplus P[1] \oplus P[2] \oplus \ldots \oplus P[n-1] \]

and the corresponding graded endomorphism ring is isomorphic to $M_n(\mathbb{C}[t,t^{-1}]) \langle 0,1,2,\ldots,n-1 \rangle$ where $\deg(t) = n$ and hence corresponds to a one-parameter family of simple $n$-dimensional representations in $\text{rep}_{n}^{\mathbb{C}}A$ with $\mathbb{GL}_n \times \mathbb{C}^*$-stabilizer subgroup generated by $\mathbb{C}^* \times 1$ and a cyclic group of order $n$

\[ \mu_n = \langle \left[ \begin{array}{c} 1 \\
\zeta \\
\vdots \\
\zeta^{n-1} \end{array} \right] \rangle \]

with $\zeta$ a primitive $n$-th root of unity. In fact, we can give a concrete matrix-representation of these simple modules. Assume that $r - [i]p = [a_i : b_i : c_i] \in P_{nc}^2$,
then the fat point module $F_q$ corresponds to the quiver-representation

and the map $A \rightarrow M_n(\mathbb{C}[t,t^{-1}])/(0,1,2,\ldots,n-1)$ sends the generators $x,y$ and $z$ to the degree one matrices

$$
\begin{bmatrix}
0 & 0 & \ldots & a_{n-1}t \\
0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & a_n \\
0 & \ldots & a_{n-2} & 0
\end{bmatrix}
\quad
\begin{bmatrix}
b_0 & 0 & \ldots & b_{n-1}t \\
b_0 & 0 & \ldots & 0 \\
b_0 & \ldots & 0 & b_n \\
b_0 & \ldots & b_{n-2} & 0
\end{bmatrix}
\quad
\begin{bmatrix}
c_0 & 0 & \ldots & c_{n-1}t \\
c_0 & 0 & \ldots & 0 \\
c_0 & \ldots & 0 & c_n \\
c_0 & \ldots & c_{n-2} & 0
\end{bmatrix}
$$

**Lemma 1.** The three matrices define a simple $n$-dimensional representation of $A$ for each choice of $t \in \mathbb{C}^*$. 

**Proof.** The point modules of $A$ are given by the elliptic curve $E$ and the automorphism is determined by summation with the point $p = [a:b:c]$. Choose $r \in E$ and let $r_i = r - [i]p = [a_i : b_i : c_i] \in \mathbb{P}_n^3$. Then by definition of point modules and the associated automorphism, we have

$$
\begin{align*}
&aa_{i+1}b_i + bb_{i+1}a_i + cc_{i+1}c_i = 0, \\
&ab_{i+1}c_i + bc_{i+1}b_i + ca_{i+1}a_i = 0, \\
&ac_{i+1}a_i + ba_{i+1}c_i + cb_{i+1}b_i = 0.
\end{align*}
$$

Therefore, a quick calculation shows that for each $t \in \mathbb{C}^*$, these 3 matrices define a $n$-dimensional representation of $A$. This representation is not an extension of the trivial representation for the center should then be mapped to 0, which is not the case. Using [16], Thm. 3.7, we conclude that this representation is indeed simple. \hfill \square

**Theorem 4.** Let $A$ be a 3-dimensional Sklyanin algebra corresponding to a couple $(E,p)$ where $p$ is a torsion point of order $n$ and assume that $(n,3) = 1$. Consider the GIT-quotient

$$
\text{rep}_n A \rightarrow \text{spec} Z(A) = \text{rep}_n A / / \text{GL}_n.
$$

Then we have

1. $\text{rep}_n^s A$ is a smooth variety of dimension $n^2 + 2$.
2. $A$ is an Azumaya algebra away from the isolated singularity $\tau \in \text{spec} Z(A)$,
3. the nullcone $\pi^{-1}(\tau)$ contains singularities.

**Proof.** We know that $\mathbb{P}^2 = \text{proj} Z(A) = \text{rep}_n^s A / / \text{GL}_n \times \mathbb{C}^*$ classifies one-parameter families of semi-stable $n$-dimensional semi-simple representations of $A$. To every point $\rho \in \mathbb{P}^2$ we have associated a one-parameter family of simples, so all semi-stable $A$-representations are in fact simple as the semi-simplification $M^{ss}$ of a semi-stable
representation still belongs to $\text{rep}^s_n A$. But then, all non-trivial semi-simple $A$-representations are simple and therefore the GIT-quotient

$$\text{rep}^s_n A \longrightarrow \text{spec} Z(A) - \{\tau\} = \text{rep}^s_n A//\text{GL}_n$$

is a principal $\text{PGL}_n$-fibration in the étale topology. This proves (1).

The second assertion follows as principal $\text{PGL}_n$-fibrations in the étale topology correspond to Azumaya algebras. For (3), if $\text{rep}_n A$ would be smooth, the algebra $A$ would be Cayley-smooth as in [10]. There it is shown that the only type of central singularity that can arise for Cayley-smooth algebras with a 3-dimensional center is the conifold singularity. This is not the case as for the conifold singularity there need to be at least 2 simple representations lying above $\tau$, but there is only one. □

In general, if $R$ is Cayley-smooth in $m \in \text{spec} Z(R)$ and if $M = S_1^{e_1} \oplus \ldots \oplus S_k^{e_k}$ is an isotypical decomposition of the corresponding semi-simple representation $M$, then we know that the tangent space $T_M(\text{rep}_n R)$ is the vectorspace of all (trace preserving) algebra maps $\psi$

$$R \xrightarrow{\psi} M_n(C[c]) \longrightarrow M_n(C)$$

such that the composition with the canonical epimorphism to $M_n(C)$ is the representation $\phi_M$ determined by $M$. Likewise, the normal space $N_M$ to the $\text{GL}_n$-orbit coincides with the vectorspace of all trace preserving extensions $\text{Ext}^1_R(M, M)$. From [10 §4.2] we recall that this vectorspace, together with the natural action of $\text{Stab}(M) = (\text{GL}_{e_1} \times \ldots \times \text{GL}_{e_k})/C^*(1_{e_1}, \ldots, 1_{e_k})$, is given by the representation space $\text{rep}(Q^*, \alpha_M)$ of a (marked) quiver setting $(Q^*, \alpha_M)$ where $Q^*$ is a directed graph on $k$ vertices, corresponding to the distinct simple components $S_i$ of $M$ where some of the loops may be marked, the dimension vector $\alpha_M = (e_1, \ldots, e_k)$ encodes the multiplicities of the simple components in $M$ and the representation space is the usual quiver-representation space modulo the requirement that matrices corresponding to marked loops are required to have trace zero. This allows us to compute a defect against $R$ being Cayley-smooth in $m$. With notations as before, this defect is

$$\text{defect}_m(R) = \dim C \text{Ext}^1_R(M, M) + (n^2 - \sum_{i=1}^k e_i^2) - \dim \text{rep}_n R.$$  

For example, if $R_m$ is an Azumaya algebra over $Z(R)_m$, then $R$ is Cayley-smooth in $m$ if and only if $m$ is a smooth point of $\text{spec} Z(R)$.

In the previous section we have seen that there are two different types of simple $n$-dimensional representations of $A$ corresponding to whether or not the maximal ideal $m$ lies over a point of $E' \subset P^2_\mathbb{C}$ or not. Still, their marked quiver-settings are the same (as $A$ is Azumaya in this point and $m$ is a smooth point of the center). In order to distinguish between the two types we have to bring in the extra $\mathbb{C}^*$-action coming from the gradation and turn these (marked) quiver-settings into weighted quiver-settings as in [7].

If the $\text{GL}_n \times \mathbb{C}^*$-orbit of the simple representation $M$ corresponding to $m$ determines a point not lying on the elliptic curve $E'$, its stabiliser subgroup is reduced to $\mathbb{C}^* 1_n \times 1$, whereas if it determines a point in $E'$ the stabiliser subgroup is generated
by $\mathbb{C}^*1_n \times 1$ together with a cyclic group of order $n$

$$\mu_n = \langle (\begin{smallmatrix} 1 \\ \zeta \\ \ddots \\ \zeta^{n-1} \end{smallmatrix}), \zeta \rangle$$

where $\zeta$ is a primitive $n$-th root of unity. This can be easily verified using the quiver-representation description of the matrices given before.

As a consequence, this finite group acts on the normal-space to the orbit and hence all three loops correspond to a one-dimensional eigenspace for the $\mu_n$-action with eigenvalue $\zeta^i$ for some $i$. To encode this extra information we will weight the corresponding loop by $i$. Let us work through the special case of quaternionic Sklyanin algebras:

**Example 1** (Quaternionic Sklyanin algebras). *It is easy to see that 3-dimensional Sklyanin algebras $A_\lambda$ determined by a point $\tau$ of order 2 have defining equations

$$\begin{cases}
xy + yx = \lambda z^2 \\
yz + zy = \lambda x^2 \\
zx + xz = \lambda y^2
\end{cases}$$

with $-27\lambda^3 \neq (2 - \lambda^3)^2, \lambda \neq 0$. An alternative description of $A_\lambda$ is as a Clifford algebra over $\mathbb{C}[u, v, w]$ (with $u = x^2, v = y^2$ and $w = z^2$) associated to the rank 3 bilinear form determined by the symmetric matrix

$$\begin{bmatrix}
2u & \lambda w & \lambda v \\
\lambda w & 2v & \lambda u \\
\lambda v & \lambda u & 2w
\end{bmatrix}$$

The center $Z(A_\lambda)$ is generated by $u, v, w$ and the determinant of the matrix which gives the equation of the elliptic curve $E'$ in $\mathbb{P}^2$

$$\lambda^2(u^3 + v^3 + w^3) - (4 + \lambda^3)uvw = 0$$

A 2-dimensional simple representation $M_1$ of $A_\lambda$ corresponding to the point $[0 : 0 : 1] \in \mathbb{P}^2 - E'$ is given by the matrices

$$x \mapsto \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}, \quad y \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

A simple representation $M_2$ corresponding to the point $[1 : -1 : 0] \in E'$ is given by the matrices

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad y \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad z \mapsto \begin{bmatrix} 0 & 0 \\ -\lambda & 0 \end{bmatrix}$$

Note that $M_2$ corresponds to the orbit $\{[1 : -1 : 0], [1 : 1 : -\lambda]\}$ of points on the elliptic curve $E \subset \mathbb{P}^2$ given by the equation

$$\lambda(x^3 + y^3 + z^3) - (\lambda^3 - 2)xyz = 0$$

The tangent space in $M_2$ to $\text{rep}_2(A_\lambda)$ is determined by trace-preserving maps

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{bmatrix}, \quad y \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{bmatrix}, \quad z \mapsto \begin{bmatrix} 0 & 0 \\ -\lambda & 0 \end{bmatrix} + \epsilon \begin{bmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{bmatrix}$$
satisfying the three quadratic defining relations of $A_\lambda$. The $\epsilon$-terms of these equalities give the independent linear relations
\[
\begin{align*}
-\lambda^2c_2 &= a_2 - a_3 + b_2 + b_3, \\
\lambda(a_2 + a_3 + b_2) &= c_2 - c_3, \\
c_2 + c_3 - \lambda a_2 &= \lambda(b_2 - b_3).
\end{align*}
\]
which implies that this tangentspace is indeed (as required) 6-dimensional. The additional $\mu_2$-stabilizer for the $\text{PGL}_2 \times \mathbb{C}^*$-action is generated by
\[
\left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),
\]
which acts on a trace zero matrix by sending it to
\[
\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -a & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -a & b \\ c & a \end{bmatrix}.
\]
Observe that the three linear equations above are fixed under this action so correspond to eigenspaces of weight $0$. Hence we can encode the tangentspace together with the action of the stabiliser subgroup by the weighted quiver setting
\[
\begin{array}{c}
1 \to 1 \to 1
\end{array}
\]
where unadorned loops correspond to weight zero. To compute the tangent space to the $\text{GL}_n$-orbit we have to determine the subspace of the tangent space given by the $\epsilon$-terms of
\[
(1_2 + \epsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix})(\phi_{M_2}(x), \phi_{M_2}(y), \phi_{M_2}(z))(1_2 - \epsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix})
\]
which gives us the three dimensional subspace consisting of matrix triples
\[
\begin{bmatrix} b - c & a - d \\ d - a & c - b \end{bmatrix}, \begin{bmatrix} b + c & d - a \\ d - a & -b - c \end{bmatrix}, \begin{bmatrix} -\lambda b & 0 \\ \lambda(a - d) & \lambda b \end{bmatrix}
\]
and under the action of $\mu_2$ this space is spanned by one eigenvector of weight zero $a - d$ and two of weight one $b$ and $c$, whence the tangent space to the orbit can be represented by the weighted quiver-setting
\[
\begin{array}{c}
1 \to 1
\end{array}
\]
and hence the weighted quiver-setting corresponding to the normal space is represented by
\[
N \leftrightarrow \begin{array}{c}
1 \to 1
\end{array}
\]
Having a precise description of the center $Z(A)$ we can shortcut such tangentspace computations, even for general order $n$ Sklyanin algebras:
Lemma 2. If $S$ is a simple $A$-representation with $\mathfrak{gl}_n \times \mathbb{C}^*$-orbit determining a fat point $F_S$ with $q \in E'$, then the normal space $N(S)$ to the $\mathfrak{gl}_n$-orbit decomposes as representation over the $\mathfrak{gl}_n \times \mathbb{C}^*$-stabilizer subgroup $\mu_n$ as $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}_3$, or in the terminology of [7], the associated local weighted quiver is

$$\begin{align*}
\begin{tikzpicture}[baseline=(current point), scale=1]
\node (v1) at (0,0) {1};
\node (v2) at (1,0) {3};
\node (v3) at (2,0) {2};
\node (v4) at (3,0) {3};
\draw (v1) to (v2);
\draw (v2) to (v3);
\draw (v3) to (v4);
\end{tikzpicture}
\end{align*}$$

Proof. From [15, Thm. 4.7] we know that the center $Z(A)$ can be represented as

$$Z(A) = \frac{\mathbb{C}[x', y', z', c_3]}{(c_3^n - \text{cubic}(x', y', z'))}$$

where $x', y', z'$ are of degree $n$ (the reduced norms of $x, y, z$) and $c_3$ is the canonical central element of degree 3. The simple $A$-representation $S$ determines a point $s \in \text{spec}Z(A)$ such that $c_3(s) = 0$. Again, as $A$ is Azumaya over $s$ we have that $N(S) = \text{Ext}^1_S(S, S)$ coincides with the tangent space $T_s \text{spec}Z(A)$. Gradation defines a $\mu_n$-action on $Z(A)$ leaving $x', y', z'$ invariant and sending $c_3$ to $c_3^3c_3$. The stabilizer subgroup of this action in $s$ is clearly $\mu_n$ and computing the tangent space gives the required decomposition. \hfill $\square$

3.2. $A$ is Cayley-smooth. Because $A$ is a finitely generated module over $Z(A)$, it defines a coherent sheaf of algebras $A$ over $\text{proj}Z(A) = \mathbb{P}^2$. In this subsection we will show that $A$ is a sheaf of Cayley-smooth algebras of degree $n$.

As $\langle n, 3 \rangle = 1$ it follows that the graded localisation $Q_{x'}^n(A)$ at the multiplicative set of central elements $\{1, x', x'^2, \ldots\}$ contains central elements $t$ of degree one and hence is isomorphic as a graded algebra to

$$Q_{x'}^n(A) = (Q_{x'}^n(A))_0[t, t^{-1}].$$

For $u \in Z(A)$, let $\mathcal{X}(u) = \{u \neq 0\} \subset \mathbb{P}^2$. By definition $\Gamma(\mathcal{X}(x'), A) = (Q_{x'}^n(A))_0$ and by the above isomorphism it follows that $\Gamma(\mathcal{X}(x'), A)$ is a Cayley-Hamilton domain of degree $n$ and is Auslander regular of dimension two and consequently a maximal order. Repeating this argument for the other standard opens $\mathcal{X}(y')$ and $\mathcal{X}(z')$ we deduce

Proposition 1. $A$ is a coherent sheaf of Cayley-Hamilton maximal orders of degree $n$ which are Auslander regular domains of dimension 2 over $\text{proj}Z(A) = \mathbb{P}^2$.\hfill $\square$

Thus, $A$ is a maximal order over $\mathbb{P}^2$ in a division algebra $\Sigma$ over $\mathbb{C}(\mathbb{P}^2)$ of degree $n$. By the Artin-Mumford exact sequence (see for example [10], Thm. 3.11) describing the Brauer group of $\mathbb{C}(\mathbb{P}^2)$ we know that $\Sigma$ is determined by the ramification locus of $A$ together with a cyclic $\mathbb{Z}_n$-cover over it.

Again using the above local description of $A$ as a graded algebra over $Z(A)$ we see that the fat point module corresponding to a point $p \notin E'$ determines a simple $n$-dimensional representation of $A$ and therefore $A$ is Azumaya in $p$. However, if $p \in E'$, then the corresponding fat point is of the form $P \oplus P[1] \oplus \cdots \oplus P[n-1]$ and this corresponds to a semi-simple $n$-dimensional representation which is the direct sum of $n$ distinct one-dimensional $A$-representations, one component for each point of $E$ lying over $p$. Hence, we see that the ramification divisor of $A$ coincides with $E'$ and, naturally, the division algebra $\Sigma$ is the one corresponding to the cyclic $\mathbb{Z}_n$-cover $E \longrightarrow E' = E/\langle \tau \rangle$.

Because $A$ is a maximal order with smooth ramification locus, we deduce from [10] §5.4
Proposition 2. \( A \) is a sheaf of Cayley-smooth algebras over \( \mathbb{P}_c^2 \) and hence \( \text{rep}_n(A) \) is a smooth variety of dimension \( n^2 + 1 \) with GIT-quotient
\[
\text{rep}_n(A) \xrightarrow{\pi} \mathbb{P}_c^2 = \text{rep}_n(A)//\text{GL}_n
\]
and is a principal \( \text{PGL}_n \)-fibration over \( \mathbb{P}_c^2 - E' \).

3.3. The non-commutative blow-up. Consider the augmentation ideal \( \mathfrak{m} = (x, y, z) \) of the 3-dimensional Sklyanin algebra \( A \) corresponding to a couple \( (E, p) \) with \( p \) a torsion point of order \( n \). Define the non-commutative blow-up algebra to be the graded algebra
\[
B = A \oplus \mathfrak{m}t \oplus \mathfrak{m}^2 t^2 \oplus \ldots \subset A[t]
\]
with degree zero part \( A \) and where the commuting variable \( t \) is given degree 1. Note that \( B \) is a graded subalgebra of \( A[t] \) and therefore is again a Cayley-Hamilton algebra of degree \( n \). Moreover, \( B \) is a finite module over its center \( Z(B) \) which is a graded subalgebra of \( Z(A)[t] \). Observe that \( B \) is generated by the degree zero elements \( x, y, z \) and by the degree one elements \( X = xt, Y = yt \) and \( Z = zt \). Apart from the Sklyanin relations among \( x, y, z \) and among \( X, Y, Z \) these generators also satisfy commutation relations such as \( Xr = xX, Yr = yX, Zr = zX \) and so on.

With \( \text{rep}^*_n B \) we will denote again the Zariski open subset of \( \text{rep}_n B \) consisting of all trace-preserving \( n \)-dimensional semi-stable representations, that is, those on which some central homogeneous element of \( Z(B) \) of strictly positive degree does not vanish. Theorem 2 asserts that \( \text{rep}^*_n B \) is a smooth variety of dimension \( n^2 + 3 \).

Proof of Theorem 2: As before, we have a \( \text{GL}_n \times \mathbb{C}^* \)-action on \( \text{rep}^*_n B \) with corresponding GIT-quotient
\[
\text{proj}Z(B) \simeq \text{rep}^*_n B//\text{GL}_n \times \mathbb{C}^*
\]
Composing the GIT-quotient map with the canonical morphism (taking the degree zero part) \( \text{proj}Z(B) \longrightarrow \text{spec}Z(A) \) we have a projection
\[
\gamma : \text{rep}^*_n B \longrightarrow \text{spec}Z(A).
\]
Let \( \mathfrak{p} \) be a maximal ideal of \( Z(A) \) corresponding to a smooth point, then the graded localization of \( B \) at the degree zero multiplicative subset \( Z(A) - \mathfrak{p} \) gives
\[
B_{\mathfrak{p}} \simeq A_{\mathfrak{p}}[t, t^{-1}]
\]
whence \( B_{\mathfrak{p}} \) is an Azumaya algebra over \( Z(A)[t, t^{-1}] \) and therefore over \( \text{spec}Z(A) - \{\tau\} \) the projection \( \gamma \) is a principal \( \text{PGL}_n \times \mathbb{C}^* \)-fibration and in particular the dimension of \( \text{rep}^*_n B \) is equal to \( n^2 + 3 \).

This further shows that possible singularities of \( \text{rep}^*_n B \) must lie in \( \gamma^{-1}(\tau) \) and as the singular locus is Zariski closed we only have to prove smoothness in points of closed \( \text{GL}_n \)-orbits in \( \gamma^{-1}(\tau) \). Such a point \( \phi \) must be of the form
\[
x \mapsto 0, \quad y \mapsto 0, \quad z \mapsto 0, \quad X \mapsto K, \quad Y \mapsto L, \quad Z \mapsto M.
\]
By semi-stability, \( (K, L, M) \) defines a simple \( n \)-dimensional representation of \( A \) and its \( \text{GL}_n \times \mathbb{C}^* \)-orbit defines the point \( \{\text{det}(K): \text{det}(L): \text{det}(M)\} \in \mathbb{P}_c^2 \). hence we may assume for instance that \( K \) is invertible.

The tangent space \( T_{\phi} \text{rep}^*_n B \) is the linear space of all trace-preserving algebra maps \( B \longrightarrow M_n(\mathbb{C}[e]) \) of the form
\[
x \mapsto 0 + eU, y \mapsto 0 + eV, z \mapsto 0 + eW, X \mapsto K + eR, Y \mapsto L + eS, Z \mapsto M + eT
\]
and we have to use the relations in $B$ to show that the dimension of this space is at most $n^2 + 3$. As $(K,L,M)$ is a simple $n$-dimensional representation of the Sklyanin algebra, we know already that $(R,S,T)$ depend on at most $n^2 + 2$ parameters. Further, from the commutation relations in $B$ we deduce the following equalities (using the assumption that $K$ is invertible)

- $xX = Xx \Rightarrow UK = KU$,
- $xY = Yx \Rightarrow UL = KV \Rightarrow K^{-1}UL = V$,
- $xZ = Zx \Rightarrow UM = KW \Rightarrow K^{-1}UM = W$,
- $Yx = yX \Rightarrow LU = VK \Rightarrow LK^{-1}U = V$,
- $Zx = zX \Rightarrow MU = WK \Rightarrow MK^{-1}U = W$.

These equalities imply that $K^{-1}U$ commutes with $K, L$ and $M$ and as $(K,L,M)$ is a simple representation and hence generate $M_n(\mathbb{C})$ it follows that $K^{-1}U = \lambda_1 n$ for some $\lambda \in \mathbb{C}$. But then it follows that

$$U = \lambda K, \quad V = \lambda L, \quad W = \lambda M$$

and so the triple $(U,V,W)$ depends on at most one extra parameter, showing that $T_{\phi} \text{rep}_n^a B$ has dimension at most $n^2 + 3$, finishing the proof. 

**Remark 1.** The statement of the previous theorem holds in a more general setting, that is, $\text{rep}_n^a B$ is smooth whenever $B = A + A^* t \oplus (A^*)^t t^2 \oplus \ldots$ with $A$ a positively graded algebra that is Azumaya away from the maximal ideal $A^*$ and $Z(A)$ smooth away from the origin.

Unfortunately this does not imply that $\text{proj} Z(B) = \text{rep}_n^a B//\text{GL}_n \times \mathbb{C}^*$ is smooth as there are closed $\text{GL}_n \times \mathbb{C}^*$ orbits with stabilizer subgroups strictly larger than $\mathbb{C}^* \times 1$. This happens precisely in semi-stable representations $\phi$ determined by

$$x \mapsto 0, \quad y \mapsto 0, \quad z \mapsto 0, \quad X \mapsto K, \quad Y \mapsto L, \quad Z \mapsto M$$

with $[\text{det}(K):\text{det}(L):\text{det}(M)] \in E'$. In which case the matrices $(K,L,M)$ can be brought into the form

$$\begin{bmatrix}
0 & 0 & \cdots & a_{n-1} t \\
0 & a_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{n-2}
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 & \cdots & b_{n-1} t \\
0 & b_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b_{n-2}
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 & \cdots & c_{n-1} t \\
0 & c_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & c_{n-2}
\end{bmatrix}$$

and the stabilizer subgroup is generated by $\mathbb{C}^* \times 1$ together with the cyclic group of order $n$

$$\mu_n = \langle \left[ \begin{array}{c} 1 \\
\zeta \\
\zeta^2 \\
\vdots \\
\zeta^{n-1} \end{array} \right], \zeta \rangle.$$ 

**Lemma 3.** If $\phi$ is a representation as above, then the normal space $N(\phi)$ to the $\text{GL}_n$-orbit decomposes as a representation over the $\text{GL}_n \times \mathbb{C}^*$-stabilizer subgroup $\mu_n$ as $\mathbb{C}_0 \oplus \mathbb{C}_0 \oplus \mathbb{C}_3 \oplus \mathbb{C}_{-1}$, that is, the associated local weighted quiver is

```
1 [3]
-1
```
Proof: The extra tangential coordinate \( \lambda \) determines the tangent-vectors of the three degree zero generators
\[
x \mapsto 0 + \epsilon \lambda K, \quad y \mapsto 0 + \epsilon \lambda L, \quad z \mapsto 0 + \epsilon \lambda M
\]
and so the generator of \( \mu_n \) acts as follows

\[
\begin{pmatrix}
1 \\
\zeta \\
\vdots \\
\zeta^{n-1}
\end{pmatrix}
(\epsilon \lambda (K,L,M))
\begin{pmatrix}
1 \\
\zeta \\
\vdots \\
\zeta^{n-1}
\end{pmatrix} = \epsilon \zeta^{n-1} \lambda (K,L,M)
\]

and hence accounts for the extra component \( C_{-1} \).

We have now all information to prove Theorem 3 which asserts that the canonical map
\[
\text{proj} Z(B) \longrightarrow \text{spec} Z(A)
\]
is a partial resolution of singularities, with singular locus \( E' = E/\langle p \rangle \) in the exceptional fiber, all singularities of type \( \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n \). In other words, the isolated singularity of \( \text{spec} Z(A) \) ‘sees’ the elliptic curve \( E' \) and the isogeny \( E \longrightarrow E' \) defining the 3-dimensional Sklyanin algebra \( A \).

**Proof of Theorem 3:** The GIT-quotient map
\[
\text{rep}_n^* B \longrightarrow \text{proj} Z(B)
\]
is a principal \( \text{PGL}_n \times \mathbb{C}^* \)-bundle away from the elliptic curve \( E' \) in the exceptional fiber whence \( \text{proj} Z(B) - E' \) is smooth. The application to the Luna slice theorem of [7, Thm. 5] asserts that for any point \( \bar{\phi} \in E' \longrightarrow \text{proj} Z(B) \) and all \( t \in \mathbb{C} \) there is a neighborhood of \( (\bar{\phi}, t) \in \text{proj} Z(B) \times \mathbb{C} \) which is étale isomorphic to a neighborhood of \( 0 \) in \( N(\phi)/\mu_n \). From the previous lemma we deduce that
\[
N(\phi)/\mu_n \simeq \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n
\]
where \( \mathbb{C}[\mathbb{C}^2/\mathbb{Z}_n] \simeq \mathbb{C}[u, v, w]/(w^3 - uv^3) \), finishing the proof.

As \( B \) is a finite module over its center, it defines a coherent sheaf of algebras over \( \text{proj} Z(B) \). From Theorem 3 we obtain

**Corollary 1.** The sheaf of Cayley-Hamilton algebras \( B \) on \( \text{proj} Z(B) \) is Azumaya away from the elliptic curve \( E' \) in the exceptional fiber \( \pi^{-1}(m) = \mathbb{P}^2 \) and hence is Cayley-smooth on this open set. However, \( B \) is not Cayley-smooth.

**Proof.** For a point \( p \) in the exceptional fiber \( \pi^{-1}(m) - E' \) we already know that \( \text{proj} Z(B) \) is smooth and that \( B \) is Azumaya, which implies that \( \text{rep}_n^* B \) is smooth in the corresponding orbit. However, for a point \( p \in E' \) we know that \( \text{proj} Z(B) \) has a non-isolated singularity in \( p \). Therefore, \( \text{rep}_n^* B \) can not be smooth in the corresponding orbit, as the only central singularity possible for a Cayley-smooth order over a center of dimension 3 is the conifold singularity, which is isolated. \( \square \)
THE GEOMETRY OF REPRESENTATIONS OF 3-DIMENSIONAL SKLYANIN ALGEBRAS

References

[1] Michael Artin, On Azumaya algebras and finite dimensional representations of rings, J. Alg. 11 (1969) 532-563
[2] Michael Artin and William Schelter, Graded algebras of global dimension 3, Adv. in Math. 66, 171-216 (1987)
[3] Michael Artin, Geometry of quantum planes, in Azumaya algebras, actions and modules (Bloomington, 1990), Contemp. Math. 124, 1-15, AMS, Providence (1992)
[4] Michael Artin, John Tate and Michel Van den Bergh, Some algebras associated to automorphisms of elliptic curves, in The Grothendieck Festschrift I, Progress in Math. 86, 33-85, Birkhäuser-Boston (1990)
[5] Michael Artin, John Tate and Michel Van den Bergh, Modules over regular algebras of dimension 3, Invent. Math. 106 335-388 (1991)
[6] David Berenstein, Reverse geometric engineering of singularities, hep-th/0201093 (2002)
[7] Raf Bocklandt and Stijn Symens, The local structure of graded representations, Communications in Algebra 34, 12, 4401-4426, Taylor & Francis (2006)
[8] Euklid345, Central element in Sklyanin algebras? MathOverflow question, april 25 (2013)
[9] Lieven Le Bruyn, Sklyanin algebras and their symbols, K-theory 8 3-17, 1994.
[10] Lieven Le Bruyn, Noncommutative geometry and Cayley-smooth orders, Pure and Applied Mathematics 290, Chapman & Hall/CRC (2008)
[11] Lieven Le Bruyn, Representation stacks, D-branes and noncommutative geometry, Communications in Algebra 40(10):3636-3651 (2010)
[12] Domingo Luna, Slices étales, Sur les groupes algébriques. Bull. Soc. Math. France, Mémoire 33 (1973) 81-105
[13] Constantin Nastasescu and Fred Van Oystaeyen, Graded and filtered rings and modules, Springer Lect. Notes Math. 758 (1980)
[14] Claudio Procesi, A formal inverse to the Cayley-Hamilton theorem, J. Algebra 107 63-74 (1987)
[15] S. Paul Smith and John Tate, The center of the 3-dimensional and 4-dimensional Sklyanin algebras, K-theory, 8,19-63 (1994)
[16] Chelsea Walton, Representation theory of three-dimensional Sklyanin algebras, Nuclear Physics B 860, 1, 167-185, Elsevier (2012)

Mathematics and statistics, Hasselt University, Agoralaan - Building D, B-3590 Diepenbeek (Belgium), kevin.delaet@uhasselt.be

Department of Mathematics, University of Antwerp, Middelheimlaan 1, B-2020 Antwerp (Belgium), lieven.lebruyn@uantwerpen.be