Violation of Chandrasekhar mass-limit in noncommutative geometry: A strong possible explanation for the super-Chandrasekhar limiting mass white dwarfs

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One of the most celebrated discoveries of twentieth century is the existence of limiting mass of white dwarfs, which is one of the compact objects formed once nuclear burning stops inside the star. On approaching this limiting mass \( \sim 1.4M_\odot \), called the Chandrasekhar mass-limit, a white dwarf is believed to spark off with an explosion called type Ia supernova, which is considered to be a standard candle. However, observations of several over-luminous, peculiar type Ia supernovae indicate that the Chandrasekhar mass-limit to be significantly larger. By considering noncommutativity of components of position and momentum variables, hence uncertainty in their measurements, at the quantum scales, we show that the mass of white dwarfs could be significantly super-Chandrasekhar and thereby arrive at a new mass-limit \( \sim 2.6M_\odot \), explaining a possible origin of over-luminous peculiar type Ia supernovae. The idea of noncommutativity, apart from the Heisenberg’s uncertainty principle, is there for quite sometime, without any observational proof however. Our finding offers a plausible astrophysical evidence of noncommutativity, arguing for a possible second standard candle, which has many far reaching implications.

1. INTRODUCTION

Einstein’s theory of general relativity (GR) and quantum mechanics are considered to be among the greatest discoveries in the twentieth century. GR is undoubtedly the most panoramic theory to explain the theory of gravity. It can easily explain a large number of phenomena where Newtonian gravity falls short. It also helps to understand the stability of Chandrasekhar’s mass-limit for the white dwarf with finite radius. White dwarf is the end state of a star with mass \( \lesssim 8M_\odot \), where the inward gravitational force is balanced by the force due to outward electron degeneracy pressure arising from Fermi statistics. Moreover, if the white dwarf has a binary partner, it starts pulling matter out off the partner due to its high gravity, resulting in the increase in the mass of the white dwarf. When it gains sufficient amount of matter, beyond a certain mass, known as the Chandrasekhar mass-limit (currently accepted value \( \sim 1.4M_\odot \) [1] for a carbon-oxygen non-magnetized and non-rotating white dwarf), this pressure/force balance is no longer sustained and it sparks off to produce type Ia supernova (SNIa) [2]. The luminosity of SNIa is very important as it is used as one of the standard candles to measure the luminosity distance of various objects in cosmology.

However, recent observations have questioned the complete validity of GR near the compact objects. For example, in the past decade, a number of peculiar over-luminous SNIa, viz. SN 2003fg, SN 2006gz, SN 2007if, SN 2009dc [3, 4] etc. have been observed, which were inferred to be originating from white dwarfs of super-Chandrasekhar mass as high as \( 2.8M_\odot \). In this scenario, the Chandrasekhar mass-limit is well violated. Different theories and models have been proposed to explain this class of the white dwarfs. Our group started exploring the significant violation of the Chandrasekhar mass-limit based on the effect of magnetic fields [5, 6]. Subsequently, there are enormous interest in re-exploring the Chandrasekhar mass-limit by introducing various new physical effects in white dwarfs. Some such physics are (1) effects of strong magnetic field leading to significantly super-Chandrasekhar mass; quantum, through Landau orbital effects above the Schwinger limit \( 4.414 \times 10^{13} \) G, which affects the equation of state (EoS) [7], and classical: through the field pressure affecting the macroscopic structural properties [8–11]; (2) modified gravity effect, leading to significantly sub- and super-Chandrasekhar mass-limits [12–14]; (3) ungravity effect [15]; (4) consequence of total lepton number violation in magnetized white dwarfs [16]; (5) charged white dwarfs leading to super-Chandrasekhar mass [17]; (6) generalized Heisenberg uncertainty principle [18]; (7) effects of momentum-momentum noncommutativity in the white dwarf matter and hence the equation of state, leading to super-Chandrasekhar mass-limit [19]; and many more.

In the present work, we plan to analyze the possible noncommutativity effects. Many researchers earlier used the idea of noncommutativity to explain the physics of various systems [20–35]. However, unfortunately, there is no direct way to confirm the natural evidence of such noncommutativity and hence it still remains as a hypothesis. Nevertheless, our observable universe abides by position-position and momentum-momentum commutative rules, which implies that two position coordinates and two momentum coordinates can be measured simultaneously. However, at a very small length scale (and/or at a very high energy regime), the position and corresponding conjugate momentum follow the Heisenberg’s uncertainty principle. Nevertheless, there are proposals that at a very high energy regime, e.g. at Planck’s scale, position-
position noncommutativity arises \[24, 25, 29, 34, 35\]. On the other hand, the density and the corresponding energy scale of white dwarfs are significantly lower than those at the Planck’s scale and, hence, any implementation of the position-position noncommutativity in white matter is still in the level of strong hypothesis. Moreover, the Chandrasekhar mass-limit arises from the interplay of pressures due to fermionic statistics and gravitational attraction. One of the important outcomes of noncommutative (NC) geometry is that the statistics of particles gets modified due to the starproduct \[36, 37\]. Effectively a fermion behaves less of a fermion and, hence, the pressure is reduced allowing collapse continuing till smaller radius with more mass accumulated. Therefore, although the scale of NC geometry in quantum spacetime namely Planck length is too small, the coherent effect of large density of white dwarf can enhance the effective NC scale to a larger value. This will be argued with the realistic densities of white dwarfs taken into account.

One way of interpreting this noncommutativity is the existence of spacetime magnetic field, almost equivalent to Landau quantization. This states that in the presence of external magnetic field, position coordinates perpendicular to the direction of magnetic field become NC, and hence the corresponding generalized momentum coefficients also become NC. It is a single parameter, field strength, which controls the noncommutativity of position and momentum coordinates. Now, the hypothesis in NC geometry is that in place of external field, there is an effective inherent magnetic field in the spacetime itself at the microscopic level. If so, at which length scale such a field, equivalent to external field producing Landau orbitals, becomes significant is a big question mark. However, if noncommutativity is present, with the analogy of external field effect, a single parameter should control both the position and momentum noncommutativities apart from the Heisenberg’s uncertainty principle. Note that the position-position noncommutativity is more fundamental in order to describe the NC universe and the momentum-momentum noncommutativity may arise as a consequence of it. Indeed it is a matter of fact that the curvature in position space leads to the noncommutativity in the conjugate momentum variables. Interestingly, in the energy dispersion relation, only the momentum-momentum noncommutativity parameter appears explicitly; hence mathematically speaking, whether position coordinates commute or not, that does not matter. However, given the lack of information at this scale, it is not generic to assume only momentum-momentum NC relation.

Recently, momentum-momentum noncommutativity has been hypothesized, keeping however position-position commutativity intact \[19\], which has also been argued to be a dynamical noncommutativity. However, as discussed above, NC geometry primarily offers a position-space noncommutativity which may lead to a momentum-momentum noncommutativity as well, similar to the effect of external magnetic field in Landau quantization. Moreover, the said work \[19\] argues the mass-limit of white dwarf to be 4.68\(M_\odot\), which is counter intuitive to the observations. One may assume that at a high central density \(\sim 10^{10}\text{ g cm}^{-3}\), such possibility may arise, if the noncommutativity effect is triggered at such density/energy. Indeed, it was shown earlier that such a high mass-limit of white dwarfs is not ruled out in the presence of magnetic field \[38\]. Nevertheless, the idea those authors proposed was that the field pressure (acting outward) and field density as well as tension (may act inward) would help to gravitate the star and hence to hold extra mass \[38\]. It is a macroscopic effect, unlike what is proposed based on the momentum-momentum noncommutativity by the authors \[19\], which is purely microscopic. However, at a low density, e.g. near the surface of white dwarf or the low central density white dwarfs, which are often evident observationally \[39\], the same noncommutativity effect should be inactive and the known observations should be explained.

The present work fairly overcomes both the shortcomings along with the misleading previous results \[19\] and shows that a single NC parameter can control both the position- and momentum-space noncommutativities, similar to the magnetic field, which affects the underlying microphysics and thereby the stellar structure. Furthermore, we constrain the NC parameter appropriately to reveal the known low density results by the same formalism. We show that with proper constraints, the mass-limit is actually \(\sim 2.6M_\odot\), which has already been reported from observations. The plan of the paper is as follows. In Section 2, we discuss the formalism of the noncommutativity based on which we calculate further the energy spectrum or the dispersion relation in Section 3. Eventually we use this relation to obtain the degenerate equation of state for the electrons in Section 4. In this section, we also discuss about the noncommutativity parameter which defines the energy spectrum. Further in Section 5, we obtain the mass-radius relation along with new mass-limit of the white dwarf in the presence of noncommutativity before we conclude in Section 6.
2. FORMALISM OF NONCOMMUTATIVITY

The NC algebra on the two dimensional plane has a direct link with the Landau quantization in the presence of magnetic field. The Landau problem is perhaps the simplest example of a system exhibiting spatial or momentum noncommutativity. By explicit investigation, one can find that the projection of coordinates to the Landau levels results in a NC algebra between the position coordinates of a particle in a two-dimensional plane. For relativistic electrons of mass $m_e$ moving in a three-dimensional space where $x-$ and $y-$coordinates constitute a NC Moyal plane [40, 41], the NC Heisenberg algebra (NCHA), satisfied by the operators $(\hat{x}, \hat{p})$, goes as follows

$$[\hat{p}_j, \hat{p}_k] = i\theta \epsilon_{jk}, \quad [\hat{x}_j, \hat{p}_k] = i\hbar \delta_{jk}, \quad [\hat{x}_j, \hat{x}_k] = \frac{i\theta \epsilon_{jk}^2}{4\hbar^2},$$

$$[\hat{x}_j, \hat{z}] = [\hat{p}_j, \hat{p}_k] = 0, \quad [\hat{z}, \hat{p}_j] = i\hbar,$$

$$[\hat{x}_j, \hat{p}_z] = 0, \quad [\hat{p}_j, \hat{z}] = 0,$$  \hspace{1cm} (2.1)

for $j, k = 1, 2$ where subscripts 1 and 2 respectively imply $x-$ and $y-$components of respective variables. Here $\theta$ is the NC parameter, $\eta$ an arbitrary constant which takes care of the dimension of the position-position noncommutativity and $\hbar$ the reduced Planck’s constant. Since only the $x-y$ plane is noncommutative, the motion along $z$-direction is free and commutes with the $x$ and $y$ coordinates. This apparently violates the SO(3) symmetry. This can however be restored for a system with $\theta = 0$.

The standard approach in the literature to deal with such problems is to form an equivalent commutative description of the NC theory by employing some non-canonical transformation, the so-called Bopp shift, which relates the NC operators $\hat{x}_j, \hat{p}_j$ following equation (2.1) to ordinary commutative operators $x_j, p_j$, satisfying the usual Heisenberg algebra

$$[x_j, p_k] = i\hbar \delta_{jk}, \quad [x, y] = 0 = [p_x, p_y].$$  \hspace{1cm} (2.2)

In our subsequent discussion, we denote NC operators with the hat notation and commutative operators without hat, and to satisfy the above NC algebra, we use the following generalized Bopp-shift transformations which is given by

$$\hat{p}_j = p_j + \frac{\theta}{2\hbar} \epsilon_{jk} x_k, \quad \hat{x}_j = x_j + \frac{\eta}{2\hbar} \hat{p}_j,$$  \hspace{1cm} (2.3)

If the total Hamiltonian of the system is $\hat{H}$, the Dirac equation for an electron moving in the NC plane satisfying the NCHA reads

$$\hat{H} \psi = i\hbar \frac{\partial \psi}{\partial t} = E \psi,$$  \hspace{1cm} (2.4)

where $\psi$ is a two-component spinor of components $\phi$ and $\chi$ with the Dirac Hamiltonian is given by

$$\hat{H} = \alpha \cdot \hat{p} c + \beta m_e c^2,$$  \hspace{1cm} (2.5)

where $c$ is the speed of light and $\alpha$ and $\beta$ have their usual meaning. The above gives a pair of equations

$$(E - m_e c^2)\phi = \sigma \cdot \hat{p} c \chi,$$

$$(E + m_e c^2)\chi = \sigma \cdot \hat{p} c \phi,$$  \hspace{1cm} (2.6)

where $\sigma$ is the Pauli matrix in vector form. On combining them, we obtain

$$\begin{align*}
(E^2 - m_e c^2)^2 &= \sigma \cdot (\hat{p} \times \hat{p}) c^2 \\
&= \sigma \cdot \frac{\sigma}{4\hbar^2} (x^2 + y^2) + \frac{\theta^2}{4\hbar^2} (x^2 + y^2) \\
&+ \frac{\theta}{\hbar} (yp_x - xp_y) \bigg] c^2. \hspace{1cm} (2.7)
\end{align*}$$

Therefore, we obtain an equivalent commutative Hamiltonian in terms of the commutative variables (quantum mechanical operators) which describes the original system defined over the NC plane.

3. ENERGY SPECTRUM

To compute the spectrum of a charged particle in such a NC spacetime, first of all we need to construct the ladder operators which will diagonalize the following part of right hand side of equation (2.7), given by

$$\hat{H}' = \left[ (p_x^2 + p_y^2) + \frac{\theta^2}{4\hbar^2} (x^2 + y^2) + \frac{\theta}{\hbar} (yp_x - xp_y) \right] c^2.$$  \hspace{1cm} (3.1)

The ladder operators involving the commutative phase-space variables (operators) $x, y, p_x, p_y$, given by

$$a_j = \frac{1}{\sqrt{\theta}} \left( p_j - i \frac{\theta}{2\hbar} x_j \right),$$

$$a_j^\dagger = \frac{1}{\sqrt{\theta}} \left( p_j + i \frac{\theta}{2\hbar} x_j \right),$$

satisfy the commutation relations

$$[a_1, a_2^\dagger] = 1 = [a_2, a_1^\dagger].$$  \hspace{1cm} (3.2)

Further defining a pair of operators

$$\hat{b}_1 = \frac{a_1 + i a_2}{\sqrt{2}}, \quad \hat{b}_2 = \frac{a_1 - i a_2}{\sqrt{2}},$$  \hspace{1cm} (3.3)

which satisfy the commutation relations

$$[\hat{b}_1, \hat{b}_1^\dagger] = 1 = [\hat{b}_2, \hat{b}_2^\dagger],$$  \hspace{1cm} (3.4)

the Hamiltonian given by equation (3.1) can be recast in the diagonal form as

$$\hat{H}' = \theta (2\hbar b_1^\dagger b_1 + 1) c^2.$$  \hspace{1cm} (3.5)

Therefore, combining (2.7) and (3.5), the total energy of the system is given by

$$E^2(\nu) = p_x^2 c^4 + m_e^2 c^4 + 2\nu \theta c^2,$$  \hspace{1cm} (3.6)

where for spin-up ($s = \frac{1}{2}$), $\nu = n_1$ and for spin-down ($s = -\frac{1}{2}$), $\nu = n_1 + 1$, when $n_1$ is the eigenvalue of the number operator $b_1^\dagger b_1$. 
4. DEGENERATE EQUATION OF STATE

Although the EoS, for the above dispersion relation, has already been found earlier [6, 19], just for completeness, in this section, we briefly discuss about it. Using equation (3.6), the Fermi energy $E_F$ of electrons for the $\nu$th level is given by

$$E_F^\nu(v) = \varepsilon_F^\nu(v) c^2 + m_e^2 e^4 + 2 \nu \theta c^2,$$

where $p_{zF}$ is the Fermi momentum of the electrons. In dimensionless form, it can be recast as follows

$$\varepsilon_F^\nu(v) = x_F^\nu(v) + 1 + 2 \nu \theta_D,$$

where $\theta_D = \frac{\theta}{m_e^4}$, $\varepsilon_F = \frac{E_F}{m_e^2}$ and $x_F(v) = \frac{p_{zF}(v)}{m_e}$.  

Due to the quantization of the energy levels in the $x-y$ plane, the modified density of state becomes $(4 \pi \theta / h^3) dp_z$. Hence the electron number density and electron energy density at zero temperature are respectively given by

$$n_e = \sum_{\nu=0}^{\nu_{\text{max}}} \frac{4 \pi m_e^3 c^7 \theta_D}{h^3} g_\nu x_F(v),$$

$$u = \frac{4 \pi m_e^3 c^7 \theta_D}{h^3} \sum_{\nu=0}^{\nu_{\text{max}}} g_\nu \int_0^{x_F} E(\nu) dx(\nu),$$

where $g_\nu$ is the degeneracy such that $g_\nu = 1$ for $\nu = 0$ and $g_\nu = 2$ for $\nu \geq 1$, which is taken to be the nearest lowest integer of $(\varepsilon_F^\nu - 1)/2 \theta_D$ for every $\varepsilon_F$ and $\theta_D$. Therefore the pressure of the Fermi gas for the electrons is given by

$$P = n_e E_F - u = \sum_{\nu=0}^{\nu_{\text{max}}} \frac{2 \pi m_e^4 c^5 \theta_D}{h^3} g_\nu \left[ \varepsilon_F x_F(v) - (1 + 2 \nu \theta_D) \log \left( \frac{\varepsilon_F + x_F(v)}{1 + 2 \nu \theta_D} \right) \right],$$

and the mass density is given by

$$\rho = \mu_e m_p n_e,$$

where $\mu_e$ is the mean molecular weight per electron and $m_p$ is the mass of a proton.

Now we assume that all the electrons are filled in the lowest Landau level, which implies that $\nu = 0$. The validity and condition of this assumption are given below. For $\nu = 0$,

$$\rho = Q x_F(0),$$

and the EoS given by equation (4.5) reduces to

$$P = \frac{h^3}{8 \pi \mu_e^2 m_p^2 m_e^2 c^2 \theta_D} \left( \rho \sqrt{\rho^2 + Q^2} - Q^2 \log \frac{\rho + \sqrt{\rho^2 + Q^2}}{Q} \right),$$

where

$$Q = \frac{4 \pi \mu_e m_p^3 c^3}{h^3} \theta_D.$$  

Let us now look at the asymptotic behavior of this EoS. For $x_F(0) >> 1$, which corresponds to $\rho^2 >> Q^2$, EoS further reduces to the following simpler polytropic form

$$P = \frac{h^3}{8 \pi \mu_e^2 m_p^2 m_e^2 c^2 \theta_D} \rho^2 = K_{nc} \rho^2 = K_{nc} \rho^{1+1/n},$$

with the polytropic index $n = 1$.

However, for the present case, also $x_F^\nu = \varepsilon_F^\nu - 1 > 0$ which implies that

$$\varepsilon_F^\nu = 2 \nu \theta_D + 1,$$

where $0 \leq \nu_1 < 1$, particularly at the center and for ground level (equivalent to the lowest Landau level for the magnetic case) $\sqrt{\varepsilon_F^\nu} - 1 = x_F(0) = \sqrt{2 \nu_1 \theta_D}$, which implies from equation (4.7)

$$\rho = \frac{4 \pi \mu_e m_p m_e^3 c^3}{h^3} \theta_D^{3/2} \sqrt{2 \nu_1},$$

when $\nu_1$ can have any value below unity for all electrons to be in the ground level. Rewriting $\theta_D$ in terms of $\rho$ and substituting it in equations (4.8) and (4.9), we have the EoS of the degenerate matter of the white dwarf, which is given by

$$P = \left( \frac{Av_1}{\rho^2} \right)^{1/3} \left[ \rho \sqrt{\rho^2 + Q^2} - Q^2 \log \frac{\rho + \sqrt{\rho^2 + Q^2}}{Q} \right],$$

$$Q = \left( \frac{B}{\nu_1} \right)^{3/2},$$

with

$$A = \frac{2 h^3 c^3}{32 \pi \mu_e^2 m_p^4}, \quad B = \frac{4 \pi \mu_e m_p^3 c^3}{2 h^3}.$$  

At large density limit, the above EoS given by equation (4.13) reduces to

$$P = \left( \frac{h^3 c^3 (2 \nu_1)}{32 \pi \mu_e^2 m_p^4} \right)^{1/3} \rho^{1/3} = K_{ncm} \nu_1^{1/3} \rho^{4/3}.$$

This is the highly relativistic EoS for degenerate electron gas, which we further use to compute the limiting mass of the white dwarf in the presence of noncommutativity. Here $\nu_1$ defines how much a Landau level (here the ground level) fills in.

4.1. Fixing noncommutativity parameter

From equation (4.12) with central density $\rho_c = 2 \times 10^{10} V$ g cm$^{-3}$, where $V$ is a parameter allowing to
change the central density, we can set $\theta_D$ at the center of the star given by

$$\theta_D = \left( \frac{2 \times 10^{10} \hbar^3}{4\pi\mu_e m_p m_e^3 c^3 \sqrt{2 m_1 V}} \right)^{2/3} \approx \frac{456}{(V \mu_e)^{2/3} \nu_f^{1/3}}.$$  

(4.15)

Hence, for $\mu_e = 2$ (carbon-oxygen white dwarfs) and $V = 1$, $\theta_D \sim 287.3$ from equation (4.15) at the center. This clearly confirms from equation (4.12) that from the center to surface, average distance between the electrons increases as the density decreases and, hence, the position and momentum spaces, both tend to become commutative.

Let us now pay attention to the values of the noncommutativity parameters. As discussed above, at the center of a white dwarf with $\rho_e = 2 \times 10^{10}$ g cm$^{-3}$, we have $\theta_D \sim 287.3$, which implies $\theta = \theta_D m_e^2 c^2 \sim 2.1 \times 10^{-31}$ g$^2$ cm$^2$/s$^2$. However, the Planck energy is $\sim 2 \times 10^{16}$ ergs, which implies the Planck momentum to be $\sim 6.5 \times 10^5$ g cm/s. It clearly indicates that our regime of interest is very far from the Planck's scale, but it still does not exactly follow the commutative algebra. This small deviation from the commutative algebra is enough to show the violation of the Chandrasekhar mass-limit. It is also clear from equation (4.12) that in case of neutron stars whose $\rho_e$ is much higher as compared to the white dwarf, the effect of noncommutativity will be much more significant. In both the cases, as density decreases from the center to surface, it is expected that the physics would be dominant by the usual commutative algebra, which is also evident from equation (4.12). At the surface, density decreases practically to zero, which indicates no noncommutativity and the usual commutative algebra is restored. Moreover, the constant $\eta$, introduced for the position-position noncommutativity in equation (2.1), represents the typical length scale of the system at which the position-position noncommutativity is significant. The value of $\theta/\hbar^2$ is $\sim 4.8 \times 10^{22}$ cm$^{-2}$ for a white dwarf with $\rho_e = 2 \times 10^{10}$ g cm$^{-3}$ and is $\sim 2.2 \times 10^{25}$ cm$^{-2}$ for a neutron star with $\rho_e = 2 \times 10^{14}$ g cm$^{-3}$. From equation (2.1), the value of $\eta$ has to be chosen in such a way that the position noncommutativity is much smaller as compared to the Planck’s scale.

4.2. Scale of noncommutativity in white dwarfs

The scale of NC parameter emerges from inherent coarse grained/foamy structure of the spacetime whose fundamental length is obtained as the Planck length [42]. However, quantum physics of matter through coherent effects brings additional structures to qualitatively enhance this length. This kind of expectation of quantum gravity effects, reflecting in stellar objects under extreme conditions, have been studied earlier. For example, the uncertainty in measurement of particular distance $L$ was anticipated to be $(L L_P^2)$, where $L_P$ being Planck's length. Such an argument is motivated by the arguments of Salecker and Wigner [43].

The following arguments are given by Ng [44, 45]. Let us assume the distance between two points $A$ and $B$ to be $L$. We can consider a clock of mass $m$ described by a wavepacket with $\delta$ as its spread. It sends a light signal from $A$ to $B$ which gets reflected and returns back to $A$ at a time $\frac{2L}{c}$. The spread now results in a new spread $\delta + \frac{2\pi c}{m \delta}$. In addition, GR provides another uncertainty namely the clock must have size $\delta \geq \frac{\hbar}{m c}$. These two arguments together yield

$$\delta \geq (L L_P^2)^{\frac{1}{2}}.$$  

Although it is heuristic, it combines both quantum mechanical and gravity effects for massive compact objects with large densities where interparticle distances are comparable or even much less than the Compton wavelength.

We can look for measuring the distance between atoms in white dwarfs. For a white dwarf with density $\rho \sim 10^7$ g cm$^{-3}$, the inter-atomic distance turns out to be $L \sim (10^7/\rho_{\text{p}})^{-1/3}$. This works out to be $L \sim 10^{-10}$ cm, which is of the order of the Compton wavelength of electrons. Now, for higher densities such as $\rho \gtrsim 10^{10}$ g cm$^{-3}$, which is for typical super-Chandrasekhar white dwarfs, we obtain $L \sim 10^{-12}$ cm, which is two orders of magnitude less than the Compton wavelength. The nature of statistics changing due to NC geometry is expected to play a crucial role. If we use the scale provided by the previous argument, we find $\delta \sim 10^{-26}$ cm, which is expected to be increased further due to many particle effects in a white dwarf. For usual white dwarfs, the inter-electron distance will be more than the Compton wavelength, thereby conventional Fermi statistics will be sufficient for getting the equation of state. A word of caution may be in order here however. The scale of uncertainty in no way provides the details of NC geometry. Therefore, we assume simple generalization of Moyal plane.

5. LIMITING MASS

To obtain the interior solution of any star, one needs to solve simultaneously the mass and momentum balance equations (together known as Tolman-Oppenheimer-Volkoff equations or in short TOV equations in GR) with appropriate boundary conditions. In GR, for static, non-magnetized fluid, the energy-momentum tensor is given by

$$T_{\mu}^{\nu} = \text{diag}(-\rho c^2, P, P, P),$$

with $\rho$ being the density and $P$ being the pressure of the fluid at an arbitrary $r$. Using the conservation of the energy-momentum tensor $\nabla_{\mu} T_{\mu}^{\nu} = 0$, the TOV equations
Defining the dimensionless density $\Theta$ as $\rho = \rho_c \Theta^n$, the above equation can be written as

$$\frac{P}{\rho c^2} = \frac{K}{c^2} \rho^{1+1/n} = \sigma \Theta,$$  

with $\sigma = \rho_c^{1/n} K/c^2$. Now, let us introduce the dimensionless distance $\xi$ and dimensionless mass $\mu$ such that $r = a \xi$ and $\mu(\xi) = (1/4\pi \rho_c a^3) M(r)$, where

$$a = \left[\frac{(n+1)K \rho_c^{1-n}}{4\pi G}\right]^{1/2}.\tag{5.3}$$

Using these dimensionless variables, equations (5.1) can be recast as

$$\frac{d\mu}{d\xi} = \xi^2 \Theta^n,\tag{5.4}$$

$$-\xi^2 \frac{d\Theta}{d\xi} = \frac{(\mu + \sigma \Theta \xi d\mu/d\xi)(1 + \sigma \Theta)}{1 - 2\sigma(n+1)\mu/\xi} \tag{5.5}.$$ 

These set of equations are known as the Lane-Emden equation in GR. Note that, for $\sigma \to 0$, the above equations reduce to the Newtonian Lane-Emden equations. Now, the mass of the white dwarf is given by

$$M_* = \int_0^R 4\pi r^2 \rho dr = 4\pi a^3 \rho_c \int_0^{\xi_1} \xi^2 \Theta^n d\xi,$$

where $\xi_1$ is defined as $R_\ast = a \xi_1$. From equation (4.14), it is observed that for the present case, $n = 3$ and $K = K_{nem} \nu^{1/3}$. Hence the mass of the white dwarf is given by

$$M_* = 4\pi a^3 \rho_c \int_0^{\xi_1} \xi^2 \Theta^3 d\xi$$

$$= 4\pi \left[ \frac{K_{nem} \nu_1}{(\pi G)^3} \right]^{1/2} \int_0^{\xi_1} \xi^2 \Theta^3 d\xi$$

$$= 4\pi \left[ \frac{K_{nem} \nu_1}{(\pi G)^3} \right]^{1/2} \mu(\xi_1),$$

$$\simeq 4.5\sqrt{\nu_1} \ M_\odot,$$

which is the limiting mass of the white dwarf. For $\nu_1 = 1/3$, the limiting mass of the white dwarf turns out to be $M_{\ast} \sim 2.6 M_\odot$, which is also confirmed from Figure 1, indicating super-Chandra sekhar limiting mass. This is similar to the limiting mass proposed by Das and Mukhopadhyay in the presence of magnetic field [7].

Let us now discuss about the significance of the NC parameter $\nu_1$. From equation (4.6) and (4.12), we have

$$\nu_1 = \frac{n_e^2 h^6}{32\pi^2 g^3} \tag{5.6}.$$ 

Let us now recall the quantum Hall effect, which occurs at high magnetic field and at low temperature in a two dimensional system. In quantum Hall effect, Hall resistivity $\rho_{xy}$ is quantized and it is given by [49]

$$\rho_{xy} = \frac{h}{e^2} \frac{1}{j}, \tag{5.7}$$

where $f$ is the quantization number (also known as the filling factor) and $e$ the charge of electron. In other words,
FIG. 1: Upper panel: The mass-radius relation, Lower panel: The variation of central density with mass of the white dwarf. Black solid line represents the mass-radius relation found by Chandrasekhar, whereas other lines represent the same in presence of noncommutativity.

the variation of $\rho_{xy}$ with magnetic field $B$ shows multiple plateau like structure with the values of $\rho_{xy}$ at the plateaus strictly depending on $f$. The values of $B$ at the center of each of these plateaus are given by

$$B = \frac{h n_e}{f e}. \quad (5.8)$$

As discussed earlier, in case of NC geometry, $B$ is equivalent of $\theta$, which implies $B = k\theta$ with $k$ being a dimensionful constant. Hence the above expression can be written as

$$f = \frac{h n_e}{e k \theta}. \quad (5.9)$$

Now, from equations (5.6) and (5.9), we have

$$n_e \nu_1 = \frac{h^3 e^3 k^3}{32 \pi^2} f^3. \quad (5.10)$$

It is observed that for a fixed $n_e$, $\nu_1 \propto f^3$. This implies that $\nu_1$ mimics as the filling factor and hence it represents how much a level is filled in.

From the above discussion, it is important to recall that the NC parameter $\theta$ is equivalent to the magnetic field leading to the Landau quantization. Only difference is that in the conventional Landau quantization, magnetic field is plausibly imposed externally, while in the noncommutativity, it is inherent and spin dependent. The NC parameter is originated from local small-scale curvature owing the gravitational interaction between the electrons and is thus a function of inter-electron separation, hence of the local density. Therefore, $\theta$ decreases from higher density center to lower density surface, as is evident from equation (4.12), leading to the commutative picture as expected at low density. This scenario is similar to the presence of the magnetic field inside stars and of its strength. Therefore, white dwarfs with lower central density should follow Chandrasekhar’s EoS only and the corresponding mass-radius relation, unlike what is emerged in other exploration [19], which appears to be unphysical.
While the idea of noncommutativity of space as well as momentum coordinates is there for quite sometime, there is no direct observational evidence which argues for its indispensable presence. It has been believed for a long time that the effect of noncommutativity is prominent only in the early universe, particularly at the Planck scale, when the density of matter was extremely high. On the other hand, the evidences for at least a dozen of over-luminous peculiar SNeIa argue for highly super-Chandrasekhar progenitor white dwarfs with a limiting mass much larger than the Chandrasekhar-limit. Such a highly super-Chandrasekhar mass-limit is quite evident if the components of position and linear momentum in a plane are assumed to be noncommutative (but in a much weaker scale than those in the Planck regime). This in turn affects the white dwarf matter statistically and the underlying EoS. This modified EoS leads to a super-Chandrasekhar limiting mass. Earlier attempts to explain super-Chandrasekhar white dwarfs and new limiting mass based on magnetic fields, modifying Einstein’s gravity, etc., encounter their respective uncertainties which may further need to pay attention to repair based on additional physics. In the NC premise, only required hypothesis is that position and momentum coordinates are related differently at a smaller length scale. Thus, over-luminous peculiar SNeIa suggest possible observational evidences for noncommutativity.

In this analysis based on noncommutativity both in the position and momentum variables, we have arrived at a new mass-limit of the white dwarfs which is $\sim 2.6 M_\odot$. This is completely viable with the current observation data, unlike the earlier authors found the mass-limit to be very high [19], which is totally unrealistic. Moreover, we have shown that the NC parameter $\nu_1$ behaves like the filling factor for the Landau levels in presence of the magnetic field. To obtain the correct mass-limit of the white dwarfs, it is extremely important to figure out the exact value of $\nu_1$. We have obtained the value of $\nu_1 = 1/3$ by matching the mass-radius relation for the low density white dwarfs. However, generically electrons in white dwarfs throughout not necessarily be lying in the ground level, hence $\nu$ need to be always 0 unlike the present simplistic approach.

The present result also enlightens the progenitors of SNeIa which is still a big question. If high density offers NC geometry leading to a new super-Chandrasekhar mass-limit explaining peculiar over-luminous SNeIa, conventional SNeIa, particularly relatively low luminous SNeIa obeying the pure detonation limit or even combined detonation and deflagration processes [50, 51], may be double-degenerate scenario. Also at what length scale exactly how coarse grained/foamy structure emerges in the spacetime is quite uncertain, which is expected to result in a wide range of SNeIa luminosities.

In future, these super-Chandrasekhar white dwarfs may be detected by the space-based gravitational wave detectors like LISA, DECIGO, etc. [11], and then the significance of noncommutativity in white dwarfs will become prominent.

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