Renormalization of the static-light axial current

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We discuss the determination of the heavy-light axial current renormalization in the static approximation, using a new method based on the Schrödinger Functional (SF). Previous perturbative results for the renormalization constant are confirmed.

1. Introduction and strategy

The static limit is of interest in the computation of $f_B$ and the control of the systematic errors involved. In order to obtain a truly non-perturbative answer for $f_B$ in that limit, one first needs to solve the renormalization problem for the static axial current. This is of practical importance as underlined in S. Hashimoto’s and Y. Kuramashi’s reviews [1,2].

The ALPHA-collaboration developed a strategy for non-perturbative renormalization and applied it to the coupling and the quark masses of QCD [3]. The same strategy applies to the static axial current. The final goal in this case is to compute the renormalization constant relating the bare current in the lattice regularization to the renormalization group invariant (RGI) current

$$(A_{\text{RGI}})^0 \propto \lim_{\mu \to \infty} \left[ g(\mu) \right]^{-d_0/b_0} (A_{\text{R}}^{\text{stat}})^0,$$

which – in contrast to $(A_{\text{R}}^{\text{stat}})^0$ renormalized at scale $\mu$ – is scale and scheme independent.

Following [3], we propose to reach this goal through the steps:

1. Define the renormalized static axial current in the SF scheme, i.e. through correlation functions in a finite space-time volume of extent $L$ with SF boundary conditions. The renormalization scale is $\mu = 1/L$.

2. Compute the renormalization constant at low energy, $\mu_0 = 1/L_0$, as a function of the bare coupling.

3. Compute the step scaling function connecting the currents renormalized at different $\mu$ and use it to evolve the current from $\mu_0$ to the perturbative regime of $\mu = O(100 \text{GeV})$.

4. Use perturbation theory to evolve further to infinite energy to obtain $(A_{\text{RGI}})^0$.

Apart from 4., all steps have to be done non-perturbatively (by MC). In the important part 2. the continuum limit can and should be taken. Once one arrives at 4. all dependence on the intermediate SF-scheme is gone and the current depends only on the bare coupling as well as the details of the discretization.

Up to now, we have used perturbation theory to study 1. and to obtain the two-loop anomalous dimension in the SF scheme, which is necessary to render higher order perturbative effects negligible in 4. We summarize these investigations in the following sections.

2. Static approximation and the SF

The SF is defined on a space-time cylinder $L^3 \times L$. All details as well as notation pertaining to the relativistic fields are taken over from [4].

In the static limit, $m_h \to \infty$, the relativistic Dirac Lagrangian is replaced by

$$L_h(x) = \bar{\psi}_h(x)D_0\psi_h(x),$$

with a static quark field, $\psi_h(x)$, satisfying

$$\frac{1}{2}(1 + \gamma_0)\psi_h = \psi_h, \quad \frac{1}{2}(1 - \gamma_0)\psi_h = 0,$$

and the time component $D_0$ of the covariant derivative. In the time direction, the following boundary conditions are imposed on the heavy quark field:

$$\psi_h(x) |_{x_0=0} = \rho_h(x), \quad \bar{\psi}_h(x) |_{x_0=L} = \bar{\rho}_h(x).$$
In space, both the static and the relativistic quark fields are periodic up to a phase \( \theta \):

\[
\psi_h(x + \hat{L}) = e^{i \theta} \psi_h(x).
\]

For the heavy quark field, boundary fields

\[
\frac{\delta}{\delta \rho_h(x)} \zeta_h(x) = \frac{\delta}{\delta \bar{\rho}_h(x)},
\]

are introduced as for the light quarks following [6]. In particular, the derivatives are taken at \( \rho_h = \bar{\rho}_h = 0 \).

The static-light axial current,

\[
A_{0}^{\text{stat}}(x) = \bar{\psi}(x) \gamma_0 \gamma_5 \psi_h(x),
\]

involves a relativistic anti quark field \( \bar{\psi}_j \).

The renormalization of the SF with static quarks needs to be discussed. Here we make the usual assumption that the SF is finite after adding [6]

- local counterterms of dimension \( d \leq 4 \) to the bulk action
- and surface terms composed again out of local fields, now with \( d \leq 3 \), integrated over the surfaces \( x_0 = 0 \) and \( x_0 = L \).

With these assumptions, we find that apart from the renormalization of the relativistic SF [6], we need a multiplicative renormalization of the static boundary quark fields \( \zeta_h, \zeta'_h \) and a mass counterterm,

\[
\delta m \bar{\psi}_h(x) \psi_h(x).
\]

Of course, the static current is renormalized multiplicatively,

\[
(A_R^{\text{stat}})_0 = Z_A^{\text{stat}} A_0^{\text{stat}},
\]

not depending on the boundary conditions.

Above, we have used continuum notation; the lattice action for the static quark fields, i.e. the discretization of (1), is chosen as in refs. [2, 3].

3. Correlation functions and their renormalization properties

Starting from the correlation functions,

\[
\begin{align*}
A^{\text{stat}}_A(x_0) & = -a_0^6 \sum_{y,z} \frac{1}{2} \langle A_0^{\text{stat}}(x) \bar{\zeta}_h(y) \gamma_5 \zeta_i(z) \rangle, \\
A_1^{\text{stat}} & = -a_0^{12} L^6 \sum_{u,v,y,z} \frac{1}{2} \langle \bar{\zeta}_h(u) \gamma_5 \zeta'_i(v) \rangle \times \bar{\zeta}_h(y) \gamma_5 \zeta_i(z),
\end{align*}
\]

we define

\[
X(u,a/L) = \frac{\langle A_1^{\text{stat}}(L/2) \rangle}{\sqrt{\langle A_0^{\text{stat}} \rangle}} \mid_{\hat{g}^2(\mu = 1/L) = u}.
\]

In this ratio, both the wave function renormalization constants and the static quark mass counterterm cancel, such that \( X \) is renormalized by current renormalization only: \( X_R = Z_A^{\text{stat}} X \).

As a check of the assumptions made in section 2 we computed, in perturbation theory, the matching of the ratio \( X_R \) to the corresponding quantity \( Y_R(u,a/L, z) \) defined for two relativistic quark flavours, where the \( \overline{\text{MS}} \)-mass of one quark flavour is \( m_{\overline{\text{MS}}} = z/L \), and the other relativistic quark mass is set to zero.

Defining also the finite parts of the current renormalizations in the \( \overline{\text{MS}} \) scheme, the matching condition

\[
Y_R = X_R + O(\frac{a}{L}) + O(\frac{1}{z}),
\]

holds. Here, \( O(\frac{1}{z}) \) stands for \( O((\log z)^n/z) \), with \( n \leq 1 \) at 1-loop.

We have checked eq. (2) explicitly at 1-loop order, controlling both the \( O(\frac{a}{L}) \) and the \( O(\frac{1}{z}) \) terms by extrapolations. This shows that the renormalization works as expected and at the same time confirms the result of [6] for \( Z_A^{\text{stat}} \) (without the need of an infrared regulator).

4. Renormalization in the SF scheme

In the SF scheme, the finite parts of the renormalization constants are defined by the renormalization condition

\[
X_R = X^{(0)} \rightarrow Z_A^{\text{stat}} = \frac{X^{(0)}}{X}.
\]
Choosing zero background field as in [3], the constant $Z_{A,\text{stat}}^{\text{static}}$ still depends on the scale $1/L$ and on the parameter $\theta$, which remains free. To determine the scale dependence of the renormalization constant, the step scaling function $\Sigma_{A,\text{stat}}$, defined by

$$Z_{A,\text{stat}}(2L) = \Sigma_{A,\text{stat}}(u, a/L) Z_{A,\text{stat}}(L),$$

$$u = \bar{g}^2 \left( \mu = 1/L \right),$$

is introduced. The continuum limit, $\sigma(u)$, satisfies

$$\Sigma(u, a/L) = \sigma(u) + O(a/L),$$

$$\sigma(u) = 1 + d_0 \log(2) u + \ldots,$$

where $d_0$ is the 1-loop coefficient of the static axial current’s anomalous dimension.

As an estimate for the discretization errors,

$$\delta(u, a/L) = \frac{\Sigma(u, a/L) - \sigma(u)}{\sigma(u)}$$

$$= 0 + \delta^{(1)}(a/L) u + \ldots$$

is computed in perturbation theory (figure [1]). At one-loop level, the discretization errors are as small as a few per cent, when $O(a)$ improvement is employed. We therefore expect that $\Sigma$, computed by MC-simulations, can be extrapolated to its continuum limit $\sigma$.

The relation to the $\overline{\text{MS}}$ scheme is given by

$$(A_{\text{stat},\overline{\text{MS}}})_0 = (A_{\text{stat},\overline{\text{SF}}})_0 (1 + c_1(\theta) g^2 + \ldots),$$

$$\mu = 1/L,$$

with $c_1(\theta)$ independent of the regularization. As an example we found

$$c_1(0.5) = -0.0352(2).$$

From (3) and the two-loop anomalous dimension in the $\overline{\text{MS}}$ scheme [9], we computed the two-loop anomalous dimension in the SF scheme,

$$d_{\text{SF}}^2(\theta = 0.5) = \frac{1}{(4\pi)^2} \{ 0.066(4) - 0.0455(3) N_f \},$$

with $N_f$ relativistic quark flavours.

5. Concluding remarks

We have defined a renormalization scheme, which, through steps 1.-4., should allow to solve the renormalization problem for the static axial current. Cutoff effects in the associated step scaling function and the relation to other schemes have been computed to one-loop but the signals in Monte Carlo computations remain to be investigated.

We finally remark that it may be advantageous to also compute the decay constant $f_{\text{stat}}^B$ using the ratio $X$, but now for large values of the time-extent of the SF [10].

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