A Review of Volatility and Option Pricing

by

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Abstract

The literature on volatility modelling and option pricing is a large and diverse area due to its importance and applications. This paper provides a review of the most significant volatility models and option pricing methods, beginning with constant volatility models up to stochastic volatility. We also survey less commonly known models e.g. hybrid models. We explain various volatility types (e.g. realised and implied volatility) and discuss the empirical properties.

Key words: Option pricing, volatility models, risk neutral valuation, empirical volatility.

1. Introduction and Outline

This paper provides a review of the most significant volatility models and their related option pricing models, where we survey the development from constant up to stochastic volatility. We define volatility, the volatility types and study the empirical characteristics e.g. leverage effect. We discuss the key attributes of each volatility modelling method, explaining how they capture theoretical and empirical characteristics of implied and realised volatility e.g. time scale variance. We also discuss less commonly known models.

The study of volatility has become a significant area of research within financial mathematics. Firstly, volatility helps us understand price dynamics since it is one of the key variables in a stochastic differential equation governing an asset price. Secondly, volatility is the only variable in the Black-Scholes option pricing equation that is unobservable, hence the ability to model volatility is crucial to option pricing.

Thirdly, volatility is a crucial factor in a wide range of research areas. For example, contagion effects involve the “transmission” of volatility from one country to another.
Volatility can explain extreme events as Blake [Bla90] explains that the October 1987 crash could have resulted from volatility changes.

Finally, volatility has a wide range of industrial applications from pricing exotic derivatives to asset pricing models [Reb04]. Shiller [Shi89] argues the market’s volatility dynamics can be applied to macroeconomic variables, particularly as the stock market is a well-known leading indicator of the economy. Shiller [Sch81] also claims volatility can be used as a measure of market efficiency.

Option pricing in itself has become an important research area. Research interest in options pricing began with the Black-Scholes option pricing paper [BS73]; since then the derivatives market has grown into a multi-trillion dollar industry [Sto99]. Options have become important to industry, particularly as they can be used to hedge out risk. In fact in many situations it is more attractive to speculators and hedgers to trade an option rather than an underlying due to the limited loss. Additionally, option trading can normally be executed on a far higher level of leverage compared to trading stocks, therefore offering potentially higher returns for the same initial deposit.

The outline of the paper is as follows. Firstly, we review basic financial mathematics theory, which is essential for the study of volatility modelling and option pricing. Next, we introduce the differing types of volatility and discuss their empirical behaviour e.g. leverage effect. We then discuss the key models of volatility and their associated option pricing methods. We finally end with a conclusion.

2. Review of Financial Mathematics Theory

This section provides a brief exposition of standard financial mathematics theory. The recommended references on this area are Björk [Bjö04], Øksendal [Øks03], Neftci [Nef96] and Delbaen and Schachermayer [DS06].

2.1. Stochastic Calculus

2.1.1. Stochastic Differential Equations

From the time of Bachelier’s work [Bac00], the most popular method of modelling asset prices has been using stochastic differential equations. Let \( X(t) \) denote a stochastic process (such as a stock price) and we define it as a diffusion if it approximately follows the stochastic difference equation:

\[
X(t + \Delta t) - X(t) = \mu(t, X(t))\Delta t + \sigma(t, X(t))\Delta W(t),
\]

where:

- \( \Delta W(t) = W(t + \Delta t) - W(t) \);
• $\mu(t, X(t))$ denotes the drift of $X(t)$;

• $\sigma(t, X(t))$ denotes the volatility (also known as the diffusion term).

The increment $\Delta W(t)$ is from $W$, which is called a Wiener process.

**Definition 1.** A stochastic process $W(t)$ is called a Wiener process if it satisfies the following conditions:

1. $W(0) = 0$;
2. $W(t)$ has independent increments. In other words $W(u) - W(t)$ and $W(s) - W(r)$ are independent for $r < s \leq t < u$;
3. $W(t)$ has continuous trajectories;
4. $W(t) - W(s) \sim \mathcal{N}(0, \sqrt{t - s})$ for $s < t$.

The stochastic process is called a diffusion because the equation models diffusions in physics. For continuous time modelling we let $\Delta t \to 0$ and so the stochastic difference equation becomes:

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW,$$

(2)

for $X(0) = a$, (3)

where $a$ is a constant. Alternatively, in integral form the equation becomes:

$$X(t) = a + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dW.$$  

(4)

It is worth noting in equation (4) that the first integral is a standard Riemann integral whereas the second integral is a stochastic integral, that is, an integral with respect to a Wiener process.

**2.1.2. Stochastic Integrals and Ito’s Lemma**

To integrate a stochastic process $X(t)$ we must know the conditions that guarantee the existence of the stochastic integral. We can guarantee the integral’s existence if the process $X(t)$ belongs to the class $\mathcal{L}^2$. To define the class $\mathcal{L}^2$ we must introduce the idea of filtration.

**Definition 2.** Let $\{\mathcal{F}_t\}$ denote the set of information that is available to the observer. If

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}_T, \forall s, t \text{ with } s < t < T,$$

(5)

then the set $\{\mathcal{F}_t\}, t \in [0, T]$ is called a filtration.
For a given stochastic process \( X(t) \), increasingly more information is revealed to an observer as time progresses. Hence at time \( t=a \) (where \( a \) is a constant) some information is revealed and this information is known with certainty at any future time \( t > a \). To keep track of the information flow that is revealed at time \( t \) we introduce the filtration \( \mathcal{F}_t^X \).

**Definition 3.** The notation \( \mathcal{F}_t^X \) denotes the information generated by process \( X(t) \) on the interval \([0,t]\). If based upon the observations of the trajectory of \( X(s) \) over the interval \( s \in [0,t] \) it is possible to determine if event \( A \) has occurred, then we say \( A \) is \( \mathcal{F}_t^X \)-measurable and write \( A \in \mathcal{F}_t^X \). If the stochastic process \( Z \) can be determined based upon the trajectory of \( X(s) \) over the interval \( s \in [0,t] \) and \( Z(t) \in \mathcal{F}_t^X, \forall t \geq 0 \), then we say \( Z \) is adapted to the filtration \( \{ \mathcal{F}_t^X \}_{t \geq 0} \).

We now define the class \( \mathcal{L}^2 \).

**Definition 4.** A stochastic process \( X(s) \) belongs to the class \( \mathcal{L}^2[a,b] \) if the following conditions are satisfied

- \( X(s) \) is adapted to the \( \mathcal{F}_t^X \) filtration;
- \( \int_a^b E[X^2(s)]ds < \infty \).

If we wish to obtain stochastic differential equations we may require the stochastic version of the classical chain rule of differentiation. This is known as Ito’s lemma.

**Theorem 1.** (Ito’s Lemma) Assume that \( X(t) \) is a stochastic process with the stochastic differential given by

\[
dX(t) = \mu(t)dt + \sigma(t)dW, \tag{6}
\]

where \( \mu \) and \( \sigma \) are adapted processes. Let \( Z(t) \) be a new process defined by \( Z(t) = f(X(t)) \) and \( f \) is a twice differentiable function, then \( Z \) has the stochastic differential:

\[
df(X(t),t) = \left( \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial X} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial X^2} \right) dt + \sigma \frac{\partial f}{\partial X} dW. \tag{7}
\]

### 2.1.3. Geometric Brownian Motion

To model asset prices it is tempting to apply Bachelier’s model. On studying the empirical behaviour of stock prices over time, Bachelier in his doctoral thesis \([Bac00]\) first suggested modelling asset prices \( X(t) \) with the following equation:

\[
dX(t) = \mu X dt + \sigma dW, \tag{8}
\]

for \( X(0) = a \), \( \tag{9} \)

where \( a, \mu \) and \( \sigma \) are constants. However, Bachelier’s equation was not satisfactory; theoretically stock prices are always non-negative yet Bachelier’s equation allowed negative stock prices. We would expect percentage returns to be independent of stock price
X(t) yet Bachelier’s equation is not. Samuelson in 1965 [Sam65] introduced the Geometric Brownian motion (GBM) of stock prices, which is the standard model for stock prices to date:

\[
dX/X = \mu dt + \sigma dW, \tag{10}
\]

\[
X(t) = X(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W}. \tag{11}
\]

In equation (10) \(X(t)\) is nonnegative with probability one and if \(X(t_1) = 0\) then

\[
X(t) = 0, \forall t \geq t_1. \tag{12}
\]

This property reflects financial “bankruptcy” since once \(X(t)\) equals 0 it will remain permanently 0 thereafter.

If we have a portfolio of \(n\) assets \(X(t) = (X_1(t), X_2(t), \ldots, X_n(t))\), governed by \(n\) independent Wiener processes \(W = (W_1(t), W_2(t), \ldots, W_n(t))\), we have the model [Eth02]:

\[
dX_i/X_i = \mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j, \; i=1,\ldots,n \text{ and } j=1,\ldots,n, \tag{13}
\]

\[
X_i(t) = X_i(0)\exp\left(\left(\mu_i - \frac{1}{2} \sum_{j=1}^{n} \sigma_{ij}^2\right) t + \sum_{j=1}^{n} \sigma_{ij} W_j\right), \tag{14}
\]

where

- \(\sigma_{ij}\) is the volatility matrix

\[
\begin{pmatrix}
\sigma_{11} & \cdots & \sigma_{1n} \\
\vdots & \ddots & \vdots \\
\sigma_{n1} & \cdots & \sigma_{nn}
\end{pmatrix}; \tag{15}
\]

- the volatility of each stock \(X_i(t)\) is \(\sigma_i(t)\) where

\[
\sigma_i = \sqrt{\sum_{j=1}^{n} \sigma_{ij}^2}; \tag{16}
\]

- instantaneous correlation between \(W_i\) and \(W_k\) is given by

\[
corr(dW_i, dW_k) = \rho_{ik}(t)dt = \frac{1}{\sigma_i(t)\sigma_k(t)} \sum_{j=1}^{n} \sigma_{ij}(t)\sigma_{kj}(t). \tag{17-18}
\]
2.2. Black-Scholes Option Pricing

The Black-Scholes analysis of European options \cite{BS73} yields a closed form solution to option pricing, only requiring observable variables (except for volatility). The key insight into their analysis was to construct a dynamically replicating portfolio of a European option to value it. Options have become one of the most important and frequently traded derivatives in finance. We give a definition of a derivative security as follows \cite{Bin04}.

**Definition 5.** A derivative security (also known as a contingent claim) is a financial contract whose value at expiration time $T$ is precisely determined by the price of an underlying asset at time $T$.

As the name implies, an option gives the right (but not the obligation) to buy or sell an asset at a pre-determined price and time. A call option gives the right to buy the asset whereas a put gives the right to sell the asset at a predetermined price. One can purchase European options, which are options that can only be exercised at expiration $T$, or American options, which can be exercised any time during the life of the option.

The Black-Scholes portfolio replication argument for European call options begins by considering a market consisting of two assets - a riskless bond $B(t)$ and a stock $X(t)$ with equations respectively:

\[
\begin{align*}
\frac{dB}{B(t)} &= rB(t)dt, \quad (19) \\
\frac{dX}{X(t)} &= \mu dt + \sigma dW, \quad (20)
\end{align*}
\]

where $\sigma$ and $\mu$ are constants and $r$ is the risk free rate of return. Black and Scholes managed to determine the closed form value of a European call option $C$ on the assumption of no arbitrage (to be defined in section 2.4):

\[
C(X(t), t, T, r, \sigma, K) = X(t)\Psi(d_1) - Ke^{-r(T-t)}\Psi(d_2),
\]

where

\[
d_1 = \frac{\ln(X(t)/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad (22)
\]

\[
d_2 = \frac{\ln(X(t)/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad (23)
\]

\[
d_1 - \sigma\sqrt{T-t}. \quad (24)
\]

In $C(X(t), t, T, r, \sigma, K)$ $t$ is the time at which $C$ is being priced, $T$ is the expiration date, $\Psi(\cdot)$ is the standard normal cumulative distribution function and $K$ is the strike price. For a more detailed discussion of the Black-Scholes option pricing equation the reader is referred to \cite{Duf01}.
2.3. Risk Neutral Valuation

2.3.1. Martingales

In financial mathematics, martingale processes are important to understanding key concepts, such as the idea of completeness. We therefore define martingales now.

**Definition 6.** An $\mathcal{F}_t$–adapted process $\mathcal{M}_t$ is an $\mathcal{F}_t$–martingale if

- $E[|\mathcal{M}_t|] < \infty, \forall t \geq 0$;
- $E[\mathcal{M}_s | \mathcal{F}_t] = \mathcal{M}_t$, for $s \geq t$.

If we replace the equality with $\leq$ (or $\geq$) then $\mathcal{M}_t$ is a supermartingale (or submartingale respectively).

Whereas a martingale can be considered a fair game, submartingales and supermartingales can be considered as favourable and unfavorable games, respectively, from the gambler’s perspective. A trivial example of a martingale is a constant $k$; let $X(t) = k$ then:

$$E[X(t + s)|\mathcal{F}_t] = X(t) = k.$$  \hfill (25)

However in financial mathematics we are interested in stochastic processes since we wish to model asset prices. We therefore define martingales for stochastic processes.

**Proposition 1.** A stochastic process $X$ is a martingale if and only if the stochastic differential is of the form

$$dX(t) = f(t)dW,$$  \hfill (26)

where $f$ is any process satisfying the condition:

$$\int_0^t E[f^2(s)]ds \leq \infty.$$  \hfill (27)

Therefore for stochastic processes to be martingales they must be “driftless”.

2.3.2. Girsanov’s Theorem: Change of Measure

In addition to the Black-Scholes equation (equation (21)), there exists another method for valuing European options, known as risk neutral valuation. Samuelson showed that the price of a call option is equal to its discounted expected payoff:

$$C(X(t), K, t, T) = e^{-\tilde{r}(T-t)}E^\mathbb{P}[X(T) - K]^+, \hfill (28)$$

where $\mathbb{P}$ is the probability measure and $\tilde{r}$ is the discount factor for risk. The difficulty with this approach is that the discount factor $\tilde{r}$ and the probability measure $\mathbb{P}$ vary.
according to the risk preference of an investor, so were unknown and typically arbitrarily chosen. For instance a risk neutral (or risk indifferent) investor does not require an incentive or disincentive to take on a risky investment, therefore if the discounted expected value remains constant he will pay the same price for an investment regardless of the risk. Furthermore for a risk neutral investor we have \( \bar{r} \) equals the risk free rate of return (which is observable), however \( \mathbb{P} \) was unknown at the time Samuelson proposed his option formula.

Cox et al. in \([CRR79]\) realised that under a Black-Scholes model, option pricing is independent of risk preferences. The investor’s risk aversion increases with the stock’s drift \( \mu \); yet \( \mu \) does not appear in the Black-Scholes option pricing equation (equation (21)). By using equation (28), it was shown that all option prices would give the same option price regardless of the risk preference chosen provided \( \bar{r} \) and \( \mathbb{P} \) are chosen consistently. Therefore equation (28) and the Black-Scholes formula would always give the same answer.

To value options as a risk neutral investor, we discount at the risk free rate and take expectations under the risk neutral measure. To take expectations under the risk neutral measure we need to change \( \mathbb{P} \) in the stochastic differential equation, which requires Girsanov’s Theorem. Girsanov’s Theorem tells us how a stochastic differential equation (SDE) changes as probability measure \( \mathbb{P} \) changes. Essentially, Girsanov’s Theorem tells us a change in \( \mathbb{P} \) corresponds to a change in drift \( \mu \) and the rest of the SDE remains unchanged. To explain how we change probability measures, let us define probability spaces.

**Definition 7.** The triple \( \{\Omega, \mathcal{F}, \mathbb{P}\} \) is called a probability space where \( \Omega \) denotes the sample space, the set of all possible events, \( \mathcal{F} \) denotes a collection of subsets of \( \Omega \) or events and \( \mathbb{P} \) is the probability measure on \( \mathcal{F} \) or events. The quadruple \( \{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}\} \) is called a filtered probability space.

Assume we have the probability space \( \{\Omega, \mathcal{F}, \mathbb{P}\} \) then a change of measure from \( \mathbb{P} \) to \( \mathbb{Q} \) means we have probability space \( \{\Omega, \mathcal{F}, \mathbb{Q}\} \). We now state Girsanov’s Theorem for change of probability measures for SDEs.

**Definition 8.** (Girsanov’s Theorem) Suppose we have a family of information sets \( \mathcal{F}_t \) over a period \( [0,T] \) where \( T < \infty \). Define over \( [0,T] \) the random process (known as the Doleans exponential)

\[
\xi_t = \exp \left\{ - \int_0^t \lambda(u) dW^\mathbb{P}(u) - \frac{1}{2} \int_0^t \lambda^2(u) du \right\},
\]

(29)

where \( W^\mathbb{P}(t) \) is the Wiener process under probability measure \( \mathbb{P} \) and \( \lambda(t) \) is an \( \mathcal{F}_t \)-measurable process that satisfies the Novikov condition

\[
E^\mathbb{P} \left[ \exp \left\{ \frac{1}{2} \int_0^t \lambda^2(u) du \right\} \right] < \infty, \ t \in [0,T].
\]
Then \( W^Q \) is a Wiener process with respect to \( \mathcal{F}_t \) under probability measure \( Q \), where \( W^Q \) is defined by

\[
W^Q(t) = W^P(t) + \int_0^t \lambda(u)du, \ t \in [0, T].
\]

The probability measure \( Q \) is defined by

\[
Q[A] = \int_A \xi_t dP, A \in \mathcal{F}_t,
\]

with \( A \) being an event in \( \mathcal{F}_t \).

Heuristically, Girsanov’s Theorem tells us that a change of measure is a subtraction or addition of an \( \mathcal{F}_t \)-adapted drift in the Wiener process [BR96]:

\[
dW^Q = dW^P + \lambda(t)dt.
\]

Our discounted risk neutral process can be expressed as

\[
dX/X = (\mu - r)dt + \sigma dW^P,
\]

\[
= (\mu - r)dt + \sigma (dW^Q - \lambda(t)dt).
\]

If we choose

\[
\lambda(t) = (\mu - r)/\sigma;
\]

we obtain equation

\[
dX/X = \sigma dW^Q,
\]

which is a martingale (see section 2.3.1) since \( \lambda(t) \) cancels the drift. This choice of \( \lambda(t) \) is also known as the market price of risk and the Sharpe ratio (discovered by Sharpe [Sha66]). Note that only one choice of \( \lambda(t) \) in this SDE gives a martingale (or alternatively eliminates the drift) hence \( Q \) is known as the unique equivalent martingale measure. The equivalent martingale measure is also known as the risk neutral measure and pricing under this measure is known as risk neutral valuation.

2.3.3. Multidimensional or Multifactor Girsanov’s Theorem

To change measures for multidimensional SDEs we require the multidimensional Girsanov’s Theorem, which is very similar to the one dimensional version.

**Definition 9. (Multidimensional Girsanov’s Theorem)** Suppose we have a family of information sets \( \mathcal{F}_t \). Let \( \theta(t) = (\lambda_1(t), \lambda_2(t), ..., \lambda_n(t)) \) be an \( n \)-dimensional process that is \( \mathcal{F}_t \)-measurable and satisfies the Novikov condition:

\[
E^P \left[ \exp \left\{ \frac{1}{2} \int_0^t \sum_{i=1}^n \lambda_i^2(u)du \right\} \right] < \infty.
\]
We define the random process $\xi_t$:

$$\xi_t = \exp \left\{ \sum_{i=1}^{n} \left( - \int_0^t \lambda_i(u) dW^P_i(u) - \frac{1}{2} \int_0^t \lambda_i^2(u) du \right) \right\},$$

(38)

where $dW^P_i$ for $i=1, \ldots, n$ is an $n$-dimensional Wiener process under probability measure $\mathbb{P}$. Then under the measure $Q$, $W^Q_i$ is a multidimensional Wiener process defined by:

$$W^Q_i = W^P_i + \int_0^t \lambda_i(u) du, \text{ for } i = 1, 2, \ldots n.$$  

(39)

The probability measure $Q$ is defined by

$$Q[A] = \int_A \xi_t dP, A \in \mathcal{F}_t.$$  

(40)

We wish to obtain an equivalent martingale measure $Q$ related to the vector valued Wiener process $W = (W_1, W_2, \ldots, W_n)$. Therefore each stock in the multiple stock model must be a martingale with respect to $Q$. Since changing measure means changing drift we have a vector $\theta(t) = (\lambda_1(t), \ldots, \lambda_n(t))$ to give an $n$-dimensional $Q$-Wiener process

$$W^Q_i = W^P_i + \int_0^t \lambda_i(s) ds.$$  

(41)

In the one dimensional case we obtained a martingale by setting $\lambda = \frac{\mu - r}{\sigma}$, in the multiple stock model we apply a similar argument but in every dimension:

$$\lambda_i(t) = \frac{(\mu_i - r)}{\left( \sum_{j=1}^{n} \sigma_{ij}(t) \right)}, \text{ for } i = 1, \ldots, m.$$  

(42)

When the volatility matrix is invertible then unique $\lambda_i$ solutions exist, admitting a unique risk neutral process. Therefore the discounted risk neutral process for the multiple stock model is:

$$dX_i/X_i = \sum_{j=1}^{n} \sigma_{ij} dW^Q_i.$$  

(43)

2.4. Fundamental Theorems of Finance

Before stating the fundamental theorems of finance, we must define arbitrage and a self-financing portfolio. A portfolio $V(t)$ is self-financing if the change in value of the portfolio is a result of the assets’ changing values and no external withdrawals or additions to the portfolio are made. For instance a portfolio $V(t)$ consisting of $k_1$ shares of $X(t)$ and $k_2$ bonds of $B(t)$ is a self-financing portfolio if:

$$dV(t) = k_1 dX(t) + k_2 dB(t).$$  

(44)

We now define arbitrage.
Definition 10. An arbitrage possibility in a financial market is a self-financed portfolio $V(t)$ such that:

- $V(0) \leq 0$;
- $V(T) \geq 0$ almost surely and
- $E[V(T)] \geq 0$.

In words arbitrage is an event where it is possible to make a profit without the possibility of incurring a loss. We say we have a “fair price” when an asset is free from arbitrage. We are now in a position to state the two key theorems of financial mathematics.

Theorem 2. (First Fundamental Theorem of Finance) The market is arbitrage free if and only if there exists an equivalent martingale measure.

Theorem 3. (Second Fundamental Theorem of Finance) Assume that the market is arbitrage free. The market is then complete if and only if there exists a unique equivalent martingale measure.

In a no arbitrage market that is incomplete we have a variety of no arbitrage option prices since we have a variety of martingale measures. Each martingale measure will produce a different price, in general.

Completeness and arbitrage are separate properties and we can illustrate this following the heuristic argument of Björk [Bjo04]. Let $\varpi_2$ denote the number of independent sources of randomness and let $\varpi_1$ denote the number of tradable assets. Now the market is arbitrage free if $\varpi_1 \leq \varpi_2$ and complete if $\varpi_1 \geq \varpi_2$. Hence the market is complete and arbitrage free if $\varpi_1 = \varpi_2$.

For no arbitrage we must have $\varpi_1 \leq \varpi_2$ since each new tradable asset gives an opportunity to create an arbitrage portfolio. For example, let $\varpi_2=1$ and $\varpi_1=2$ and each asset follows GBM that are identical (including the same Wiener process since $\varpi_2=1$) but have different drifts:

$$
\frac{dX_i}{X_i} = \mu_i dt + \sigma dW.
$$

Then one can produce a riskless profit by shorting the lower drift asset $X_1$ and using the proceeds to purchase the higher drift asset $X_2$. For completeness we must have $\varpi_1 \geq \varpi_2$ to enable us to trade or replicate every possible claim.

The no arbitrage definition enables us to state the put-call parity relationship for European options, which does not require any assumptions other than the market is arbitrage free.
Proposition 2. (Put-Call Parity)
Assume the market is arbitrage free. If a European call $C(t,X)$ and a European put $P(t,X)$ with the same underlying asset $X$, strike $K$ and expiration $T$ exist then we have the put-call parity relation:

$$P(t, X) = Ke^{-r(T-t)} + C(t, X) - X(t) + D,$$

where $D$ is the cash dividend received from the underlying stock during the life of the option.

2.5. Feynman-Kac Stochastic Representation Formula

The Feynman-Kac theorem provides a link between the partial differential equation of a diffusion process and its expectation. This is useful because we can solve important analytical problems either by taking expectations under a risk neutral measure or by solving the partial differential equation.

Theorem 4. (Feynman-Kac Theorem)

Let $X(t)$ satisfy the equation

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW,$$

with initial value at initial time $s$

$$X(s) = x,$$

and let $f(s,x) \in \mathcal{L}^2$ be a function which satisfies

$$\int_0^T E \left[ \left( \sigma(t, X(t)) \frac{\partial f}{\partial x}(t, X(t)) \right)^2 \right] dt < \infty$$

and boundary condition

$$f(T, x) = h(x).$$

If the function $f(s,x)$ is a solution to the boundary value problem

$$\frac{\partial f}{\partial s} + \frac{1}{2} \sigma^2(s, x) \frac{\partial^2 f}{\partial x^2} + \mu(s, x) \frac{\partial f}{\partial x} - rf(s, x) = 0,$$

then $f$ has the representation:

$$f(s, x) = e^{-r(T-s)} E[h(X(T))|X(s) = x].$$

The Feynman-Kac formula can be extended to the case of $W = (W_1, W_2)$ where $W_1, W_2$ are independent Wiener processes and we have two stochastic differentials in $X_1$ and $X_2$. Following Shreve [Shr04], let the stochastic processes follow

$$dX_1(t) = \mu_1(t, X_1(t), X_2(t))dt + \sigma_{11}(t, X_1(t), X_2(t))dW_1 + \sigma_{12}(t, X_1(t), X_2(t))dW_2,$$

$$dX_2(t) = \mu_2(t, X_1(t), X_2(t))dt + \sigma_{21}(t, X_1(t), X_2(t))dW_1 + \sigma_{22}(t, X_1(t), X_2(t))dW_2,$$

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with initial values at initial time \( s \)

\[
X_1(s) = x_1, \tag{55}
\]
\[
X_2(s) = x_2, \tag{56}
\]

with conditions

\[
\int_0^T E[(\sigma_{11}(t, X_1(t), X_2(t)) \frac{\partial f}{\partial x_1}(t, X_1(t), X_2(t)))^2] \, dt < \infty,
\]
\[
\int_0^T E[(\sigma_{21}(t, X_1(t), X_2(t)) \frac{\partial f}{\partial x_2}(t, X_1(t), X_2(t)))^2] \, dt < \infty,
\]

with boundary condition

\[
f(T, x_1, x_2) = h(x_1, x_2) \tag{57}
\]

and satisfy:

\[
f(s, x_1, x_2) = e^{-r(T-s)} E[h(X_1(T), X_2(T))|X_1(s) = x_1, X_2(s) = x_2]. \tag{58}
\]

We then have the associated partial differential equation:

\[
0 = \frac{\partial f}{\partial s} + \mu_1(s, x_1, x_2) \frac{\partial f}{\partial x_1} + \mu_2(s, x_1, x_2) \frac{\partial f}{\partial x_2} + \frac{1}{2} \left( \sigma^2_{11}(s, x_1, x_2) + \sigma^2_{12}(s, x_1, x_2) \right) \frac{\partial^2 f}{\partial x_1^2}
\]
\[
+ \left( \sigma_{11}(s, x_1, x_2) \sigma_{21}(s, x_1, x_2) + \sigma_{12}(s, x_1, x_2) \sigma_{22}(s, x_1, x_2) \right) \frac{\partial^2 f}{\partial x_1 \partial x_2}
\]
\[
+ \frac{1}{2} \left( \sigma^2_{21}(s, x_1, x_2) + \sigma^2_{22}(s, x_1, x_2) \right) \frac{\partial^2 f}{\partial x_2^2} - rf(s, x_1, x_2).
\]

3. An Introduction to Volatility

3.1. Different Types of Volatility

Wilmott in [Wil01] distinguishes between four different types of volatility, although in practise little distinction is made between them. The four types of volatility are:

- volatility \( \sigma \) in equation [2]. It is a measure of randomness in asset return and since it exists at each moment in time it has no “timescale” associated with it. For example volatility can be 20\%, 5\%, 60\%.
• historic volatility (also known as realised volatility): this is a measure of volatility using past empirical price data and will be explained further in section 3.2.

• implied volatility: this is the volatility associated with empirical option prices and will be explained further in section 3.2.

• forward volatility: this is the volatility obtained from some forward instrument.

We note that all four types of volatility in theory should not differ since they all refer to the same variable $\sigma$, however in practise they may be different. For example some researchers believe actual volatility and implied volatility are 2 separate variables and treat them differently (see Schonbucher’s model in section 4.4 for instance).

3.2. Historic and Implied Volatility

In theory volatility should not depend on the method of measurement. However, in practice this is not the case. Volatility $\sigma$ can be empirically measured by two methods: historic volatility and implied volatility. Historic volatility is calculated from empirical (and therefore discrete) stock price data $X(t_0), ..., X(t_n)$ where $\Delta t = t_i - t_{i+1}$ denotes the chosen sampling interval. To estimate historic volatility $\hat{\sigma}$ we calculate the standard deviation of an asset’s continuously compounded return per unit time:

$$\hat{\sigma} = \frac{\sqrt{V_X}}{\sqrt{\Delta t}},$$

(59)

where

• $V_X$ is the sample variance

$$V_X = \frac{1}{n-1} \sum_{i=1}^{n} (\chi_i - \bar{\chi}).$$

(60)

Note that sample variance contains $n-1$ in the denominator, whereas variance of a theoretical distribution contains $n$;

• $\chi_i = \ln \left( \frac{X(t_i)}{X(t_{i-1})} \right)$;

(61)

• $\bar{\chi}$ is the sample mean

$$\bar{\chi} = \frac{1}{n} \sum_{i=1}^{n} \chi_i.$$

(62)
In contrast to historic volatility we obtain implied volatility values from empirical options data. Using Black-Scholes option pricing, call options $C$ are a function of $C(X,t,T,r,\sigma,K)$, with all the independent variables observable except $\sigma$. Since the quoted option price $C^{\text{obs}}$ is observable, using the Black-Scholes formula we can therefore calculate or imply the volatility that is consistent with the quoted options prices and observed variables. We can therefore define implied volatility $I$ by:

$$C_{BS}(X, t, T, r, I, K) = C^{\text{obs}},$$

where $C_{BS}$ is the option price calculated by the Black-Scholes equation (equation (21)). Implied volatility surfaces are graphs plotting $I$ for each call option’s strike $K$ and expiration $T$. Theoretically options whose underlying is governed by GBM should have a flat implied volatility surface, since volatility is a constant; however in practise the implied volatility surface is not flat and $I$ varies with $K$ and $T$.

Implied volatility plotted against strike prices from empirical data tends to vary in a “u-shaped” relationship, known as the volatility smile, with the lowest value normally at $X=K$ (called “at the money” options). The opposite graph shape to a volatility smile is known as a volatility frown due to its shape. The smile curve has become a prominent feature since the 1987 October crash (see for instance [Bat00] and [CP98]).

Various explanations have been proposed to account for volatility smiles. Firstly, it has been suggested that options are priced containing information about future short term volatility that is not already contained within past price information (see for instance [Jor95]). Secondly, implied volatility is influenced more by market sentiment rather than by pure fundamentals, for instance the VIX (a weighted average of implied volatilities from S&P 500 index options) is used as a gauge of market sentiment [SW01].

Thirdly, the transaction costs involved in trading options is significantly higher and more complicated compared to their associated underlyings, creating volatility smiles (see for instance [PRS99]). Finally, Jarrow and Turnbull [JT99] discuss non-simultaneous price observation; since option and stock prices are from two different financial markets, there will always be observation time differences. Such time differences can cause substantial differences in the estimated implied volatility.

3.3. Empirical Characteristics of Volatility

Volatility has been observed to exhibit consistently some empirical characteristics, which we will now discuss. Firstly, one of the most well known empirical characteristics of volatility is the leverage effect, as first proposed by Black [Bla76]. Black observed that volatility is negatively correlated with stock price and accounted for this through
the concept of leverage. Leverage (also known as the gearing or debt to equity ratio) is defined by:

\[
\text{Leverage} = \frac{l}{MKT},
\]

where \(l\) is the company’s total debt and MKT is the market capitalisation (number of shares \(\times\) share price). As the share price drops the company becomes riskier, since a greater percentage of the company is debt financed, hence increasing volatility.

Black argued leverage could not entirely explain volatility since companies with little or no debt still exhibit high volatility. Other explanations of volatility’s negative corelation with stock price include portfolio rebalancing, where investors are forced to liquidate their assets if they fall below a threshold price. Alternatively, threshold prices may act as triggers for sales when interpreted in terms of prospect theory, as proposed by Kahneman and Tversky \([KT79]\). Also there exists ownership concentration, where an investor owns a substantial percentage of a company’s stock and sells all his holdings at once (see for instance \([CCW08]\)). In both cases a large volume of selling pushes prices down further, increasing volatility.

Secondly, volatility (and return distributions) show dependency on the time scale \(\Delta t\) chosen to measure it, as defined in equation \((59)\). The return distribution becomes increasingly more Gaussian as \(\Delta t\) increases \([Con01]\), known as “Gaussian aggregation”, yet under GBM volatility is theoretically scale invariant. Such empirical observations have motivated researchers to seek models that exhibit scale variation and is one of the benefits of mean reverting stochastic volatility models.

Thirdly, Mandelbrot \([Man63]\) and Fama \([Fam65]\) were the first to observe volatility clusters (positively autocorrelates) with time; large (small) price changes tend to follow large (small) price changes. Such observations motivated GARCH and stochastic volatility models (see section 4.3) and for this reason volatility clustering is sometimes known as the “GARCH effect” \([Con04]\). The autocorrelation is significant over time scales \(\Delta t\) of days and weeks but insignificant over longer time scales. This is because the autocorrelation strength decays following a power law with \(\Delta t\) increasing.

The autocorrelation’s slow power law of decay has been cited as evidence of the existence of long memory in volatility \([Con01]\) and empirical evidence can be found in \([LGS99]\). A stationary stochastic process \(X(t)\) has long memory if its covariance \(v(\tau) = \text{cov}(X(t), X(t + \tau))\) follows the power law of decay \([Tan06]\)

\[
v(\tau) = \tau^d, \text{ for } -1 < d < 0,
\]

(64)
and
\[ \sum_{\tau=0}^{\infty} v(\tau) = \infty. \] (65)

Fourthly, volatility tends to be correlated with significant financial or economic variables. For instance Schwert [Sch89b] empirically shows that excessive volatility increases during financial crises; during the Great Depression (1929-39) stock volatility increased by a factor two or three times its typical values.

Finally, in commodities and other asset classes seasonal effects and calendar time play a significant role in return and volatility distributions. For instance, French [Fre80] noticed stock returns measured over the weekend on average are more negative than any other day of the week. In the past decade the increasing availability of intraday data has enabled researchers to show times of the day impact volatility, therefore complicating intraday volatility estimation [GSW01].

Other volatility empirical characteristics include mean reversion, where volatility tends to revert around some long term value [SS91]. This has motivated mean reverting volatility models (see section 4.3 for more detail). Volatility tends to be correlated with high trading volume [SX03] and company specific news (e.g. earnings announcements) [Ros06]. Finally Shiller claims market volatility can be explained in terms of psychological factors such as fashion, herding effects and overreaction to new information [Shi89]. Shiller in [Shi89] also proposes a stock price model taking into account psychological factors.

4. A Review of Volatility and Option Pricing Models

As described in section 2.1.3 Bachelier [Bac00] proposed a model for stock prices, with constant volatility (see equation (8)). Bachelier reasoned that investing was theoretically a “fair game” in the sense that statistically one could neither profit nor lose from it. Hence Bachelier included the Wiener process to incorporate the random nature of stock prices. Osborne [Osb59] conducted empirical work supporting Bachelier’s model. Samuelson [Sam65] continued the constant volatility model under the Geometric Brownian Motion stock price model (see equation (10)) on the basis of economic justifications.

Over time, empirical data and theoretical arguments found constant volatility to be inconsistent with the observed market behaviour (such as the leverage effect, as discussed in section 3.3). A plot of the empirical daily volatility of the S&P 500 index clearly shows volatility is far from constant. This consequently led to the develop-
ment of dynamic volatility modelling. Volatility modelling may be classified into four categories:

1. Constant volatility $\sigma$;
2. Time dependent volatility $\sigma(t)$;
3. Local volatility: volatility dependent on the stock price $\sigma(X(t))$;
4. Stochastic volatility: volatility driven by an additional random process $\sigma(\omega)$.

These models will now be discussed in some detail in the subsequent sections.

It is worth mentioning that some models do not use Wiener processes to capture price movements. For example, Madan and Seneta [MS90] propose a Variance-Gamma model where price movements are purely governed by a discontinuous jump process. Carr et al. discuss various jump processes in [CGMY02] such as the pure jump process of Cox and Ross [CR76]. The advantages of pure jump models are: firstly, they realistically reflect the fact that trading occurs discontinuously, secondly price jumps are consistent with information releases according to Fama’s “Efficient Market Hypothesis” [Fam65] and finally, stock price processes jump even on an intraday time scale [CT04].

4.1. Time Dependent Volatility Model

As mentioned in section 3.3, it was empirically observed that implied volatility varied with an option’s expiration date. Consequently, a straightforward extension proposed to the constant volatility model was time dependent volatility modelling [W+98]:

$$dX/X = \mu dt + \sigma(t) dW.$$  \hspace{1cm} (66)

Merton [Mer73] was the first to propose a formula for pricing options under time dependent volatility. The option price associated with $X$ is still calculated by the standard Black-Scholes formula (equation (21)) except we set $\sigma = \sigma_c$ where:

$$\sigma_c = \sqrt{\frac{1}{T-t} \int_t^T \sigma^2(\tau) d\tau},$$  \hspace{1cm} (67)

i.e. $d_1$ and $d_2$ in the Black-Scholes equation become:

$$d_1 = \frac{\log \left( \frac{X}{K} \right) + \mu(T-t) + \frac{1}{2} \int_t^T \sigma^2(\tau) d\tau}{\sqrt{\int_t^T \sigma^2(\tau) d\tau}},$$ \hspace{1cm} (68)

$$d_2 = \frac{\log \left( \frac{X}{K} \right) + \mu(T-t) - \frac{1}{2} \int_t^T \sigma^2(\tau) d\tau}{\sqrt{\int_t^T \sigma^2(\tau) d\tau}}.$$  \hspace{1cm} (69)
The equation (67) converts $\sigma(t)$ to its constant volatility equivalent $\sigma_c$ over time period $t$ to $T$. The distribution of $X(t)$ is given by:

$$\log\left(\frac{X_T}{X_t}\right) \sim N\left((\mu - \frac{1}{2} \sigma_c^2)(T - t), \sigma_c^2(T - t)\right).$$  \hspace{1cm} (70)

Note that the constant volatility $\sigma_c$ changes in value as $t$ and $T$ change. This property enables time dependent volatility to account for empirically observed implied volatilities increasing with time (for a given strike).

4.2. Local Volatility Models

4.2.1. Definition and Characteristics

In explaining the empirical characteristics of volatility, a time dependent volatility model was found to be insufficient. For instance, time dependent volatility did not explain the volatility smile, nor the leverage effect (see section 3.3), since volatility cannot vary with price. Therefore volatility as a function of price (and optionally time) was proposed, that is local volatility is $\sigma = f(X, t)$:

$$\frac{dX}{X} = \mu dt + \sigma(X, t)dW.$$ \hspace{1cm} (71)

The term “local” arises from knowing volatility with certainty “locally” or when $X$ is known - for a stochastic volatility model we never know the volatility with certainty.

The advantages of local volatility models are that firstly, no additional (or untradable) source of randomness is introduced into the model. Hence the models are complete, unlike stochastic volatility models. It is theoretically possible to perfectly hedge contingent claims. Secondly, local volatility models can be also calibrated to perfectly fit empirically observed implied volatility surfaces, enabling consistent pricing of the derivatives (an example is given in [Dup97]). Thirdly, the local volatility model is able to account for a greater degree of empirical observations and theoretical arguments on volatility than time dependent volatility (for instance the leverage effect). We will now look at some common local volatility models.

4.2.2. Constant Elasticity of Variance Model (CEV)

CEV was proposed by Cox and Ross [CR76]:

$$\frac{dX}{X} = \mu dt + \sigma(X)dW,$$ \hspace{1cm} (72)

$$\sigma(X) = aX^{n-1},$$ \hspace{1cm} (73)

where $0 \leq n \leq 1, a > 0$ are constants. For $n = 0$ we obtain Bachelier’s model (equation (8)) while $n = 1$ gives the Geometric Brownian Motion model (equation (10)). Therefore $n$ can be seen as a parameter for intuitively choosing between the two
extreme models. Additionally, \( n \) captures the level of leverage effect since volatility increases as stock price decreases, as can be seen from equation (73).

The CEV model is analytically tractable, which is in contrast to the local volatility model in section 4.2.4 where numerical solutions for derivatives become analytically intractable [W+98]. Additionally, by appropriate choices of \( a \) and \( n \) we can fit CEV to volatility smiles (see for instance Beckers [Bec80]).

The CEV model has been developed over the years giving various CEV modified models. For instance the square root CEV model [CR76], which is similar to the CIR interest rate model [CIJR85], Schroder [Sch89a] re-expresses the CEV model in terms of a chi-squared distribution, enabling derivation of closed form solutions. Hsu et al. [HLL08] determine the CEV model’s probability density function while Lo et al. [LYH00] derive the option pricing formula for CEV with time dependent parameters.

4.2.3. Mixture Distribution Models

Mixture distributions have been applied to numerous statistical applications for many years outside financial mathematics (Everitt [EH81] gives a survey of such applications). It had been known for some time that one could capture many empirical characteristics of stock prices such as volatility smiles through mixture distributions (see for instance Bingham and Kiesel [BK02] and Melick and Thomas [MT97]). However Brigo and Mercurio [BM00] were the first to prove the theoretical relation between volatility and a mixture of lognormal distributions.

Let us assume the risk neutral probability density function \( p(t, X) \) at time \( t \) of a stock price \( X(t) \) is a weighted sum of \( N \) lognormal probability densities \( \tilde{p}_i(t, X) \):

\[
p(t, X) = \sum_{i=1}^{N} \tilde{w}_i(t) \tilde{p}_i(t, X),
\]

(74)

where \( \sum_{i=1}^{N} \tilde{w}_i(t) = 1, \tilde{w}_i(t) \geq 0, \forall t. \)

(75)

The term \( \tilde{w}_i(t) \) is a weighting for each component \( i \) at a point in time \( t \), \( \tilde{p}_i(t, X) \) denotes the probability density of component \( i \) at a specific point in time for a specific stock price \( X(t) \). We assume each \( \tilde{p}_i(t, X) \) has the same mean \( \mu \) but different variances \( \sigma_i^2(t) \).

Brigo and Mercurio then proved that the stock price follows the process:

\[
dX/X = \mu dt + \sigma(X, t)dW,
\]

(76)

where \( \sigma^2(X, t) = \sum_{i=1}^{N} w_i(t)\sigma_i^2(t). \)

(77)

The mixture distribution model has been developed further by Brigo et al. (see for instance [BM02], [BMS03], [BMRS04]). Alexander has also developed the mix-
ture distribution model; Alexander has applied it to the areas of stress testing portfolios [AS08], combining it with GARCH processes [AL06] and bivariate option pricing [AS04]. GARCH processes will be covered in more detail in section 4.4.

The mixture distribution models have been used to model more complicated volatility models due to their ability to capture a variety of distributions. For example, Leisen [Lei05] shows how a mixture distribution model approximates Merton’s jump diffusion model (to be covered in section 4.4) amongst others, Lewis [Lew02] shows how mixture models can approximate stochastic volatility models.

4.2.4. Implied Local Volatility

Dupire [Dup94], Derman and Kani [DK94] proved for local volatility that a unique risk-neutral process existed that was consistent with option data. The price of a European call option by risk neutral valuation is given by:

\[ C = e^{-rT}E^Q[X(T) - K]^+, \]

\[ = e^{-rT} \int_K^\infty (X(T) - K)p(X(T))dX, \]

where \( p(X(T)) \) is the risk neutral probability density function for \( X(T) \). Breeden and Litzenberger [BL78] then showed from equation (79) that the risk neutral cumulative distribution function \( F(\cdot) \) at \( K \) is:

\[ \frac{\partial C}{\partial K} = -e^{-rT}F(X(T) \geq K). \]

Furthermore, the risk neutral probability density function \( p(X(T)=K) \) is

\[ \frac{\partial^2 C}{\partial K^2}e^{rT} = p(X(T) = K). \]

Hence we can recover the risk neutral density \( p(X(T)) \) from option data. This probability can be interpreted as the current view of the future outcome of the stock price.

Dupire [Dup94] then showed by applying \( p(X(T)) \) from equation (81) (obtained from Breeden and Litzenberger’s work) to the Fokker-Planck equation, one could obtain Dupire’s equation:

\[ \frac{\partial C}{\partial T} = \sigma^2(X, T) \frac{X^2}{2} \frac{\partial^2 C}{\partial X^2} - (r - D)X \frac{\partial C}{\partial X} - DC, \]

where \( D \) is the dividend. Rearranging equation (82) gives:

\[ \sigma(X, T) = \sqrt{\frac{\frac{\partial C}{\partial T} + (r - D)X \frac{\partial C}{\partial X} + DC}{\frac{X^2}{2} \frac{\partial^2 C}{\partial X^2}}} \].
Therefore the local volatility $\sigma(X, T)$ can be fully extracted from option data.

It can be seen from equation (83) that calculating $\sigma$ requires partial differentials with respect to $T$ and $K$. We therefore require a continuous set of options data for all $K$ and $T$. This is highly unrealistic and quoted option prices tend to suffer from significant illiquidity effects, affecting option bid-ask spreads [Nor03]. Furthermore, Pinder [Pin03] shows that option bid-ask spreads are related to volatility, expiry and trading volume. Since option data is discrete we require some interpolation method to convert it to continuous data, for example Monteiro et al. [MTV08] apply a cubic spline method. However, Wilmott [Wil00] states that local volatility computation is highly sensitive to interpolation methods.

Dupire’s model implicitly assumes the options data contains all the information on the underlying’s volatility if we calibrate to options data alone. However there is evidence to show historic and implied volatility differ significantly [CP98]. Furthermore, calibrating the local volatility surface to the options data tends to be unstable with time, since the surface significantly changes from one week to another [DFW98].

Numerical computation of local volatility has been implemented by Andersen and Brotherton-Ratcliffe using a finite difference method [ABR98]. Derman and Kani (DKC96, DKZ96) and Rubinstein Rub94 determine local volatilities by fitting a unique binomial tree to the observed option prices. Tree fitting also has the computational advantage of not being affected by different interpolation methods.

4.3. Stochastic Volatility Models

4.3.1. Definition and Characteristics

Although local volatility models were an improvement on time dependent volatility, they possessed certain undesirable properties. For example, volatility is perfectly correlated (positively or negatively) with stock price yet empirical observations suggest no perfect correlation exists. Stock prices empirically exhibit volatility clustering but under local volatility this does not necessarily occur. Consequently after local volatility development, models were proposed that allowed volatility to be governed by its own stochastic process. We now define stochastic volatility.

**Definition 11.** Assume $X$ follows the stochastic differential equation

$$dX/X = \mu dt + \sigma(\omega)dW_1.$$  \hspace{1cm} (84)

Volatility is stochastic if $\sigma(\omega)$ is governed by a stochastic process that is driven by another (but possibly correlated) random process, typically another Wiener process $dW_2$. The probability space $(\Omega, \mathcal{F}, P)$ is $\Omega = C([0, \infty) : \mathbb{R}^2)$, with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ representing information on two Wiener processes $(W_1, W_2)$.

The process governing $\sigma(\omega)$ must always be positive for all values since volatility can
only be positive. The Wiener processes have instantaneous correlation \( \rho \in [-1, 1] \) defined by:

\[
\text{corr}(dW_1(t), dW_2(t)) = \rho dt.
\]

Empirically \( \rho \) tends to be negative in equity markets due to the leverage effect (see section 3.3) but close to 0 in the currency markets. Although \( \rho \) can be a function of time we assume it is a constant throughout this thesis.

The key difference between local and stochastic volatility is that local volatility is not driven by a random process of its own; there exists only one source of randomness \((dW_1)\). In stochastic volatility models, volatility has its own source of randomness \((dW_2)\) making volatility intrinsically stochastic. We can therefore never definitely determine the volatility’s value, unlike in local volatility.

The key advantages of stochastic volatility models are that they capture a richer set of empirical characteristics compared to other volatility models [MR05]. Firstly, stochastic volatility models generate return distributions similar to what is empirically observed. For example, the return distribution has a fatter left tail and peakedness compared to normal distributions, with tail asymmetry controlled by \( \rho \) [Dur07]. Secondly, Renault and Touzi [RT96] proved volatility that is stochastic and \( \rho=0 \) always produces implied volatilities that smile (note that volatility smiles do not necessarily imply volatility is stochastic).

Thirdly, historic volatility shows significantly higher variability than would be expected from local or time dependent volatility, which could be better explained by a stochastic process. A particular case in point is the dramatic change in volatility during the 1987 October crash (Schwert [Sch90] gives an empirical study on this). Finally, stochastic volatility accounts for the volatility’s empirical dependence on the time scale measured (as discussed in section 3.3), which should not occur under local or time dependent volatility.

The disadvantages of stochastic volatility are firstly that these models introduce a non-tradable source of randomness, hence the market is no longer complete and we can no longer uniquely price options or perfectly hedge. Therefore the practical applications of stochastic volatility are limited. Secondly stochastic volatility models tend to be analytically less tractable. In fact, it is common for stochastic volatility models to have no closed form solutions for option prices. Consequently option prices can only be calculated by simulation (for example Scott’s model in section 4.3.3).

4.3.2. Stochastic Volatility and the Driving Process

Stochastic volatility models fundamentally differ according to their driving mechanisms for their volatility process. Different driving mechanisms may be favoured due
to their tractability, theoretical or empirical appeal and we can categorise stochastic volatility models according to them. Many stochastic volatility models favour a mean reverting driving process. A mean reverting stochastic volatility process is of the form [FPS00a]:

\[ \sigma = f(Y), \]  
\[ dY = \alpha(m - Y)dt + \beta dW_2, \]

where:

- \( \beta \geq 0 \) and \( \beta \) is a constant;
- \( m \) is the long run mean value of \( \sigma \);
- \( \alpha \) is the rate of mean reversion.

Mean reversion is the tendency for a process to revert around its long run mean value. We can economically account for the existence of mean reversion through the cob-web theorem, which claims prices mean revert due to lags in supply and demand [LH92]. The inclusion of mean reversion (\( \alpha \)) within volatility is important, in particular, it controls the degree of volatility clustering (burstiness) if all other parameters are unchanged. Volatility clustering is an important empirical characteristic of many economic or financial time series [Eng82], which neither local nor time dependent volatility models necessarily capture. Additionally, a high \( \frac{1}{\alpha} \) can be thought of as the time required to decorrelate or “forget” its previous value.

The equation (86) is an Ornstein-Uhlenbeck process in \( Y \) with known solution:

\[ Y(t) = m + (Y(0) - m)e^{-\alpha t} + \beta \int_0^t e^{-\alpha t}dW_2, \]

where \( Y(t) \) has the distribution

\[ Y(t) \sim \mathcal{N}\left(m + (Y(0) - m)e^{-\alpha t}, \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})\right). \]

Note that alternative processes to equation (86) could have been proposed to define volatility as a mean reverting stochastic volatility model, for example the Feller or Cox-Ingersoll-Ross (CIR) process [FPS00b]:

\[ dY = \alpha(m - Y)dt + \beta \sqrt{Y}dW_2. \]

However with \( \sigma = f(Y) \) in equation (86) we can represent a broad range of mean reverting stochastic volatility models in terms of a function of \( Y \).
4.3.3. Significant Stochastic Volatility Models

There is no generally accepted canonical stochastic volatility model and a large number of them exist, therefore we review here the most significant ones.

1. Johnson and Shanno Model

Johnson and Shanno in 1987 [JS87] proposed one of the first stochastic volatility models:

\[
\begin{align*}
  dX &= \mu_1 X dt + \sigma^n dW_1, \quad n \geq 0, \\
  d\sigma &= \mu_2 \sigma dt + \sigma^k \beta dW_2, \quad k \geq 0,
\end{align*}
\]

where \( \text{corr}(dW_1(t), dW_2(t)) = \rho dt \). A Monte Carlo method is proposed to determine the price of options under the stochastic volatility process by risk neutral valuation. Johnson’s and Shanno’s computational results show that their option prices are consistent with what is empirically observed (see section 3.3), that is they exhibit a volatility smile and an increase in value with expiry.

2. Scott Model

Scott in 1987 [Sco87] considered the case where one assumes a geometric process for stock prices and an Ornstein-Uhlenbeck process for the volatility:

\[
\begin{align*}
  dX/X &= \mu dt + \sigma dW_1, \\
  d\sigma &= \alpha(m - \sigma) dt + \beta dW_2.
\end{align*}
\]

Scott proposed a stochastic mean reverting process based on empirical stock price returns and describes a parameter estimation method based on moments matching. Scott assumes \( \rho=0 \) to facilitate computation of option prices. The option prices for Scott’s process are calculated by Monte Carlo simulation and he observes a marginal improvement in option pricing accuracy compared to standard Black-Scholes option pricing.

3. Hull-White Model

Hull and White modelled volatility as follows [HW87]:

\[
\begin{align*}
  dX/X &= \mu_1 dt + \sigma dW_1, \\
  d\sigma^2/\sigma^2 &= \mu_2 dt + \beta dW_2.
\end{align*}
\]

The Hull-White model is an important contribution since it provides a closed form solution to European option prices when \( \rho=0 \) and for a given risk neutral measure \( \gamma \). Let us define

\[
\tilde{\sigma}^2 = \frac{1}{T-t} \int_t^T \sigma^2(s) ds,
\]
where $\tilde{\sigma}^2$ is a random variable. The option price is computed using the standard Black-Scholes formula, under a risk neutral measure $\gamma$, with volatility $\sqrt{\sigma^2}$. In other words:

$$C(t, X, K, T, \sigma(\omega)) = E^\gamma[C_{BS}(t, X, K, T, \sqrt{\sigma^2})].$$  

(97)

The option pricing equation (97) provides results consistent with empirically observed currency options, where empirically $\rho \simeq 0$. Furthermore equation (97) is still valid for any stochastic volatility process provided $\rho = 0$ [FPS00a]. For correlated volatility, option prices are obtained using Monte Carlo simulation.

4. Stein and Stein Model

Stein and Stein in 1991 [SS91] proposed an Ornstein-Uhlenbeck process for volatility based on tractability and empirical considerations:

$$dX/X = \mu dt + \sigma dW_1, \quad (98)$$

$$d\sigma = -\alpha(\sigma - m)dt + \beta dW_2, \quad (99)$$

where $\rho = 0$. Note that although equation (99) implies volatility can be negative, Stein and Stein state that only $\sigma^2$ is ever applied in calculation [SS91]. A closed form solution to option prices is derived for particular choices of risk neutral measures, which is in contrast to Johnson and Shanno [JS87] and Scott [Sco87], who only provide numerical methods to option pricing.

5. Heston model

Heston’s model [Hes93] created in 1993 stands out from other stochastic volatility models because there exists an analytical solution for European options that takes in account correlation between $dW_1$ and $dW_2$ (although this requires assumptions on the risk neutral measure):

$$dX/X = \mu dt + \sigma dW_1, \quad (100)$$

$$d\sigma^2 = \alpha(m - \sigma^2)dt + \beta \sigma dW_2. \quad (101)$$

The $d\sigma^2$ modelling process originated from CIR interest rate model [CIJR85]. To price an option under risk neutral valuation, we must specify a risk neutral measure (due to market incompleteness) and this is chosen on economic justifications [Hes93]. Heston then finds an analytical solution for options using Fourier transforms; the reader is referred to Heston [Hes93] and Musiela and Rutkowski [MR05] for a derivation.

Due to the existence of an analytic solution to option pricing the Heston model has been subsequently developed by various researchers. For example, Scott in-
includes stochastic interest rates [Sco97], Pan includes stochastic dividends [Pan02], and Bates adds jumps to the stochastic process [Bat96].

4.4. Other Volatility Models

Apart from Bates [Bat96], the models discussed so far all have continuous sample paths yet empirical stock prices appear to exhibit “jumps”. Additionally, the frequency and magnitude of the jumps are too large to be explained by lognormal distributions, which would be obtained under a GBM process. Merton therefore proposed the jump diffusion model [Mer76], which added a random jump component to the GBM process:

\[
dX/X = (\mu - \theta_k)dt + \sigma dW + dP,
\]

where

- \( \theta \) is the average number of jumps/year;
- \( k \) is the average jump size measured as a percentage of the asset price \( X \);
- \( P \) is a Poisson process and independent of \( dW \).

A Poisson process counts the number of events that occur in a given period. We define a Poisson process \( P(\nu) \) as [LM66]:

\[
p(X = k) = \frac{e^{-\nu t} (\nu t)^k}{k!}, \quad k = 0, 1, 2, \ldots
\]

where

- \( \nu \) is called the rate parameter, where \( \nu \) is the expected number of events per unit time;
- \( X \) is a random variable denoting the number of events;
- \( p(X=k) \) is the probability that the random number of events from 0 to time \( t \) equals \( k \).

In the case of the jump diffusion model the events are the jumps themselves. To obtain a closed form solution for option prices under the jump diffusion model, Merton makes a key argument as follows. We know that a replicating portfolio constructed in the derivation of the original Black-Scholes equation ([BS73]) will eliminate the risk arising from the GBM and so must earn the risk free rate of return. Next Merton assumes the jump process represents nonsystematic risk and risk that is not priced into the market. Therefore the same replicating portfolio applied in the Black-Scholes equation must
earn the risk free rate of return. Merton then proved that the option price is the sum of an infinite series of options, valued by the standard Black-Scholes option pricing equation without the jumps.

\[
C(X(t), K, t, T, r, \sigma) = \sum_{n=0}^{\infty} e^{-\theta' \tau} (\theta' \tau)^n \frac{C_{BS}(X(t), K, t, T, r_n, \sigma_n)}{n!},
\]

where

\[
\tau = T - t,
\]

\[
\theta' = \theta(1 + k),
\]

\[
r_n = r - \theta k + \frac{n \ln(1 + k)}{\tau},
\]

\[
\sigma_n^2 = \sigma^2 + \frac{n \varsigma^2}{\tau}.
\]

The \(n^{th}\) option in the series is valued by the standard Black-Scholes equation, \(C_{BS}\), assuming \(n\) jumps have occurred before expiry. Since the series converges exponentially it can be implemented computationally \[CT04\]. The term \(\varsigma^2\) arises from the fact that a Poisson process can be approximated by a normal distribution. Therefore we can assume the logarithm of the jumps are normally distributed with variance \(\varsigma^2\). The jumps fatten the return distribution’s tail, therefore the model is more consistent with empirically observed distributions compared to GBM. Merton accounts for jumps as the arrival of new information that have more than a marginal effect on price movements.

Another class of volatility models is to use a stochastic process to model the evolution of the implied volatility directly. In all the models discussed so far, we have assumed implied volatility and the underlying’s volatility are the same. Implied volatility methods model an option’s (or any derivative’s) implied volatility \(\sigma^*\) separately from the underlying’s volatility \(\sigma\). One significant implied volatility model is Schonbucher’s stochastic implied volatility model \[Sch99\]. Schonbucher models the implied volatility \(\sigma^*\) of a vanilla option as a SDE and the underlying’s SDE separately:

\[
d\sigma^* = \mu_2 dt + \sigma_2 dW_1 + \beta dW_2.
\]

The underlying’s SDE is:

\[
dX/X = \mu_1 dt + \sigma_1 dW_1.
\]

Another class of volatility models is the lattice approach, with its origins from binomial trees by Cox et al. \[CRR79\]. Britten-Jones and Neuberger \[BJN00\] model stochastic volatility by fitting a lattice model that is consistent with observed option prices. The model is parameterised by up and down price movements and their associated probabilities for each branch, using empirical option data as its input. This is
an improvement on Derman’s and Kani’s model, who fit a tree to local volatility only. Britten-Jones and Neuberger also show how a variety of stochastic volatility models (such as regime switching volatility) can be fitted to be consistent with observed option price data.

Discrete time volatility models are another class of models that exist, such as GARCH(p,q) [Bol86], the generalised autoregressive conditional heteroscedasticity model:

$$\sigma_t^2 = a_0 + \sum_{i=1}^{q} a_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} b_i \sigma_{t-i}^2, \text{ for } \varepsilon_t \sim \mathcal{N}(0, \sigma_t^2),$$

where $a_i$ and $b_i$ are weighting constants. We obtain the ARCH(q) [Eng82] model by setting $p=0$. Due to the success of GARCH in econometrics it has been substantially extended by various researchers, for example Nelson [Nel91], Sentana [Sen95] and Zakoian [Zak94]. Whereas with continuous time modelling one is able to derive closed form solutions, which can reduce computation and provide new insights, this is generally not possible with discrete time models.

Finally, one class of volatility models are “hybrid” models; combining various volatility models into one. For instance Alexander [Ale04] extends Brigo’s and Mercurio’s mixture model [BM00] by combining it with the binomial tree of Cox et al. [CRR79] to incorporate stochastic volatility. The SABR model [HKLW02] is a stochastic extension of the CEV model [CR76], where SABR is an abbreviation for stochastic alpha, beta and rho in its equations. The SABR model captures the dynamics of a forward price $F$ of some asset (e.g. stock) under stochastic volatility; its risk neutral dynamics under measure $\mathbb{Q}$ is:

$$dF(t) = \sigma(t)F^\beta(t)dW^\mathbb{Q}_1,$$

$$d\sigma(t) = \alpha \sigma(t)dW^\mathbb{Q}_2,$$

where $\text{corr}(dW_1,dW_2) = \rho \in [-1,1], 0 \leq \beta \leq 1, \alpha \geq 0.$

A common shortfall in all the volatility models reviewed so far has been that all these models are short term volatility models. The models implicitly ignore any long term or broader economic factors influencing the volatility model, which is empirically unrealistic and theoretically inconsistent. Furthermore, although some models may specify a closed form solution for option pricing, they provide no method or recommendation for calibration, which is important to modelling and option pricing.

5. Conclusions

This paper has surveyed the key volatility models and developments, highlighting the innovation associated with each new class of volatility models. In conclusion it can
be seen from our review of volatility models that the development of has progressed in a logical order to address key shortcomings of previous models.

Time dependent models addressed option prices varying with expiration dates, local volatility also addressed volatility smiles and the leverage effect, whereas stochastic volatility could incorporate all the effects captured by local volatility and a range of other empirical effects e.g. greater variability in observed volatility. However the trade-off associated with improved volatility modelling has been at the expense of analytical tractability.
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