Truth Discovery via Proxy Voting

RESHEF MEIR, Technion- Israel Institute of Technology, Israel
OFRA AMIR, Technion- Israel Institute of Technology, Israel
GAL COHENSIUS, Technion- Israel Institute of Technology, Israel
OMER BEN-PORAT, Technion- Israel Institute of Technology, Israel
LIRONG XIA, RPI, USA

Truth discovery is a general name for a broad range of statistical methods aimed to extract the correct answers to questions, based on multiple answers coming from noisy sources. For example, workers in a crowdsourcing platform. In this paper, we design simple truth discovery methods inspired by proxy voting, that give higher weight to workers whose answers are close to those of other workers.

We prove that under standard statistical assumptions, proxy-based truth discovery (P-TD) allows us to estimate the true competence of each worker, whether workers face questions whose answers are real-valued, categorical, or rankings. We then demonstrate through extensive empirical study on synthetic and real data that P-TD is substantially better than unweighted aggregation, and competes well with other truth discovery methods, in all of the above domains.

“All happy families are alike; each unhappy family is unhappy in its own way.”
— Leo Tolstoy, Anna Karenina

Authors’ addresses: Reshef Meir, Technion- Israel Institute of Technology, Bloomfield Building, Technion City, Haifa, 3200003, Israel, reshefm@ie.technion.ac.il; Ofra Amir, Technion- Israel Institute of Technology, Bloomfield Building, Technion City, Haifa, 3200003, Israel, oamir@technion.ac.il; Gal Cohensius, Technion- Israel Institute of Technology, Bloomfield Building, Technion City, Haifa, 3200003, Israel, @ie.technion.ac.il; Omer Ben-Porat, Technion- Israel Institute of Technology, Bloomfield Building, Technion City, Haifa, 3200003, Israel, reshefm@ie.technion.ac.il; Lirong Xia, RPI, ? Troy, NY, ? USA, xial@cs.rpi.edu.

arXiv:1905.00629v1 [cs.AI] 2 May 2019
1 INTRODUCTION

Consider a standard crowdsourcing task such as image labeling [12, 28] or corpus annotation [32]. Such tasks are often used to construct large databases, that can later be used to train and test machine learning algorithms. Crowdsourcing workers are usually not experts, thus answers obtained this way often contain many mistakes [23, 37, 39]. A simple approach to improve accuracy is to ask the same question to a number of workers and to aggregate their answers by some aggregation rule.

Truth discovery is a general name for a broad range of methods that aim to extract some underlying ground truth from noisy answers. While the mathematics of truth discovery dates back to the early days of statistics, at least to the Condorcet Jury Theorem [10], the rise of crowdsourcing platforms suggests an exciting modern application for truth discovery.

A simple approach to improve accuracy in crowdsourcing applications is to ask the same question a number of workers and to aggregate their answers by some aggregation rule. The use of aggregation rules suggests a natural connection between truth discovery and social choice, which deals with aggregation of voters’ opinions and preferences. Indeed, voting rules have come useful in the design of truth discovery and crowdsourcing techniques [6, 11, 27]. It is our intention to further explore and exploit these connections in the current paper.

In political and organizational elections, a common practice is to allow voting-by-proxy, where some voters let others vote on their behalf [5, 15]. Thus the aggregation is performed over a subset of “active” voters, who are weighted by the number of “inactive” voters with similar opinions. In a recent paper, Cohensius et al. 2017 showed that under some assumptions on the distribution of preferences, proxy voting reduces the variance of the outcome, and thus requires fewer active voters to reach the socially-optimal alternative. Cohensius et al. suggested that an intuitive explanation for the effectiveness of proxy voting lies in the fact that the more ‘representative’ voters tend to also be similar to one another, but did not provide formal justifications of that claim. Further, in their model the designer seeks to approximate the subjective “preferences of the society,” whereas truth-discovery is concerned with questions for which there is an objective ground truth.

In this paper, we consider algorithms for truth discovery that are inspired by proxy voting, with crowdsourcing as our main motivation. Our goal is to develop a simple approach for tackling the following challenge in a broad range of domains: We are given a set of workers, each answering multiple questions, and want to: (a) identify the competent workers; and (b) aggregate workers’ answers such that the outcome will be as close as possible to the true answers. These challenges are tightly related: a good estimate of workers’ competence allows us to use better aggregation methods (e.g. by giving higher weight to good workers); and aggregated answers can be used as an approximation of the ground truth to assess workers’ competence. Indeed, several current approaches in truth discovery tackle these goals jointly (see Related Work below).

Our approach decouples the above goals. For (a), we apply proxy voting to estimate each worker’s competence, where each worker increases the estimated competence of similar workers. For the truth discovery problem (b), we then use a straightforward aggregation function (e.g. Majority or Average), giving a higher weight to more competent workers (i.e. who are closer to others).

We depart from previous work on proxy voting mentioned above by dropping the assumption that each worker relegates her vote to the (single) nearest proxy. While this requirement makes sense in a political setting so as to keep the voting process fair (1 vote per person), it is somewhat arbitrary when our only goal is a good estimation of workers’ competence and the ground truth.

Crowdsourcing is also used for a variety of tasks in which there is no “ground truth,” such as studying vocabulary size [24], rating the quality of an image [31], etc. In this paper, we focus only on questions for which there is a well defined true answer, as in the examples above.
For analysis purposes, we use *distances* rather than proximity. We assume that each worker has some underlying *fault level* $f_i$ that is the expected distance between her answers and the ground truth. The question of optimal aggregation when competence/fault levels are *known* is well studied in the literature, and hence our main challenge is to estimate fault levels. To capture the positive influence of similar workers, we define the *proxy distance* of each worker, as her average distance from all other workers. While the similarity between workers has been considered in the literature (see Related Work), we are unaware of any systematic study of its uses. Our main theoretical result can be written as follows:

**Theorem (Anna Karenina principle, informal).** The expected proxy distance of each worker is linear in her fault level $f_i$.

Essentially, the theorem says that as in Tolstoy’s novel, “good workers are all alike,” (thereby boosting one another proxy distances), whereas “each bad worker is bad in her own way” and thus not particularly close to other workers. The exact linear function depends on the statistical model, and in particular on whether the data is categorical or continuous.

The Anna Karenina principle suggests a natural proxy-based truth-discovery (P-TD) algorithm, that first estimates fault levels based on proxy distances, and then uses standard techniques from the truth discovery literature to assign workers’ weights and aggregate their answers. We emphasize that a good estimate of $f_i$ may be of interest regardless of the aggregation procedure (this is goal (a) above). For example, the operators of the crowdsourcing platform may use it to decide on payments for workers or for terminating the contract with low-quality workers.

1.1 Contribution and paper structure

Our main theoretical contribution is a formal proof of Theorem 1 in the following domains: (i) when answers are continuous with independent normal noise, and where the answers of each worker $i$ have variance $f_i$; (ii) when answers are categorical and each worker $i$ fails each question independently with probability $f_i$; (iii) when answers are rankings of alternatives sampled from the Condorcet noise model with parameter $f_i$. In all three domains, the parameters of the linear function depend on the distribution from which fault levels are sampled. We show conditions under which our estimates of the true fault levels and of the ground truth are statistically consistent. In the continuous domain, we further show that our proxy-based method generalizes another common approach for fault estimation.

We devote one section to each domain (continuous, categorical, rankings). In each section, the theoretical results are followed by an extensive empirical evaluation of truth discovery methods on synthetic and real data. We compare P-TD to standard (unweighted) aggregation and to other approaches suggested in the truth discovery literature. We show that P-TD is substantially better than straight-forward (unweighted) aggregation, and often beats other approaches of weighted aggregation. We also show how to extend P-TD to an iterative algorithm that competes well with more advanced approaches.

Due to space constraints and to allow continuous reading, most proofs are deferred to appendices. Appendices also contain additional figures that show the findings brought in the paper apply broadly.

2 PRELIMINARIES

We denote by $\mathbb{1}[Z]$ the indicator variable of the Boolean condition $Z$. $\Delta(Z)$ is the set of probability distributions over set $Z$.

A *domain* is a tuple $(X, d)$ where $X$ is a set of possible world states; and $d : X \times X \to \mathbb{R}_+$ is a distance measure.
We assume there is a fixed set of \( n \) workers, denoted by \( N \). An instance \( I = \langle S, z \rangle \) in domain \( \langle X, d \rangle \) is a set of reports \( S = (s_i)_{i \in N} \) where \( s_i \in X \) for all \( i \in N \); and a ground truth \( z \in X \). To make things concrete, we will consider three domains in particular:

- Continuous domain. Here we have \( m \) questions with real-valued answers. Thus \( X = \mathbb{R}_+^m \); and we define \( d_E \) to be the squared normalized Euclidean distance.
- Categorical domain. Here we have \( m \) questions with categorical answers in some finite set \( A \). Thus \( X = A^m \); and we define \( d_H \) as the normalized Hamming distance.
- Rankings domain. Here \( X \) is all orders over a set of element \( C \), with the Kendall-tau distance \( d_{KT} \).

A noise model in domain \( \langle X, d \rangle \) is a function \( h \) from \( X \times \mathbb{R}_+ \) to \( \Delta(X) \). We consider noise models that are informative, in the sense that: (I) \( h(z, 0) \) returns \( z \) w.p. 1 (i.e., without noise); and (II) \( E_{s \sim h(z, f)}[(d(z, s))^t] \) is strictly increasing in \( f \), for any moment \( t \) (i.e., higher \( f \) means more noise).

A population in domain \( \langle X, d \rangle \) is a set of \( n \) workers, each with a fault level \( f_i \). A Proto-population is a distribution \( \mathcal{F} \) over fault levels. We denote by \( \mu(\mathcal{F}) \), \( \delta(\mathcal{F}) \) the mean and variance of distribution \( \mathcal{F} \), respectively. We omit \( \mathcal{F} \) when clear from the context. Note that the higher \( \mu \) is, we should expect more erroneous answers.

Generated instances. Fix a particular domain \( \langle X, d \rangle \). Given a population \( f = (f_i)_{i \in N} \), a noise model \( h \), and a ground truth \( z \in X \), we can generate instance \( I = \langle S, z \rangle \) by sampling each answer \( s_i \) independently from \( h(z, f_i) \). We can similarly generate instances from a proto-population \( \mathcal{F} \) by first sampling each \( f_i \) from \( \mathcal{F} \), and then sampling an instance from the resulted population.

Aggregation. An aggregation function in a domain \( \langle X, d \rangle \) is a function \( r : X^n \rightarrow X \). In this work we consider simple aggregation functions: \( r^M \) is the Mean function in the continuous domain; and \( r^P \) is the Plurality function in the categorical domain. The functions are applied to each question independently. Formally: \( (r^M(S))_j = \frac{1}{n} \sum_{i \in N} s_{ij} \) and \( (r^P(S))_j = \arg \max_{x \in A} \left| \{s_{ij} = x\} \right| \)

In the ranking domain we apply several popular voting rules,\(^2\) including Plurality, Borda, Veto, Copeland, and Kemeny. All the aggregation functions we consider have natural weighted analogs, denoted as \( r(S, w) \) for a weight vector \( w \).

Aggregation errors. Given an instance \( I = \langle S, z \rangle \) and aggregation function or algorithm \( r \) in domain \( \langle X, d \rangle \), the error of \( r \) on \( I \) is defined as \( d(z, r(S)) \). The goal of truth discovery is to find aggregation functions that tend to have low error.

For the convenience of the reader, a list of notation and acronyms is available at the end of the appendix.

### 2.1 A general Proxy-based Truth-Discovery scheme

Given an instance \( I \) and a basic aggregation function \( r \), we apply the following work flow, whose specifics depend on the domain.

1. Collect the answers \( S = (s_i)_{i \in N} \) from all workers.
2. Compute the pairwise distance \( d(s_i, s_j) \) for each pair of workers.
3. For each worker \( i \):
   - Compute proxy distance \( \pi_i \) by averaging over all pairwise distances:
     \[
     \pi_i := \frac{1}{n-1} \sum_{j \neq i} d(s_i, s_j).
     \]
   - Estimate the fault level \( \hat{f}_i \) from \( \pi_i \).

\(^2\)The more accurate term for functions that aggregate several rankings into a single ranking is social welfare functions [4].
By Theorem 1, the (expected) proxy distance of each agent is a linear transformation of her fault level. Thus, we implement step 3b by reversing the linear transformation to get an estimated fault level $\hat{f}_i$, and then use known results to obtain the (estimated) optimal weight in step 3c. The details are given in the respective sections, where we refer to the algorithms that return $\hat{f}$ and $\hat{z}$ as \textit{Proxy-based Estimation of Fault Levels (P-EFL) and Proxy-based Truth Discovery (P-TD)}, respectively.

3 CONTINUOUS ANSWERS DOMAIN
As specified in the preliminaries, each report is a vector $s \in \mathbb{R}^m$. The distance measure we use is the normalized squared Euclidean distance:

$$d_E(s, s') = \frac{1}{m} \sum_{j \leq m} (s_j - s'_j)^2.$$ 

The Independent Normal Noise model. For our theoretical analysis, we assume an independent normal noise (INN). Formally, $s_i := z + \epsilon_i$, where $\epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{im}) \sim N(0, \Sigma_i)$ is a $m$-dimensional noise. In the simplest case the noise of each worker is i.i.d. across dimensions, whereas workers are also independent but with possibly different variance. We further assume that questions are equally difficult\(^4\) and not correlated given the fault level, i.e. $\Sigma_i = \mathbb{I} \cdot f_i$. Note that $E[d_E(s_i, \epsilon_i)|f_i] = Var[\epsilon_i|f_i] = f_i$.

3.1 Estimating Fault Levels
Given an instance $I = \langle S, z \rangle$, our first goal is to get a good estimate of the true fault level.

\textit{Fault estimation by distance from the empirical mean.} In situations where there is a simple aggregation method $r$, a simple approach that is a step in many truth discovery algorithms, is estimating the quality of each worker according to their distance from the aggregated outcome [26]. We name this approach \textit{Estimation Fault Levels by Distance from the Outcome (D-EFL)}.

In the continuous domain, we use D-EFL (Alg. 1) where $r$ is the mean function and $d$ is the square Euclidean distance, however, we leave the notation general as the algorithm can be used in other domains with appropriate distance and aggregation functions.

We later analyze the properties of D-EFL in Section 3.2. But before, we will describe our own approach that relies on the proxy distance.

\textit{Fault estimation by proxy distance.} Applying Eq. (1) to the continuous domain, we get that the proxy distance of each worker is $\pi_i(I) = \frac{1}{n-1} \sum_{i' \neq i} d_E(s_i, s_{i'}) = \frac{1}{(n-1)m} \sum_{i' \neq i} \sum_{j \leq m} (s_{ij} - s'_{ij})^2$.

Note that once $f_i$ is fixed, the proxy distance $\pi_i$ is a random variable that depends on two separate randomizations. First, the sampling of the other workers’ fault levels from the proto-population $F$; and second, the realization of a particular instance $I = \langle z, S \rangle$, where $s_i \sim h^{INN}(z, f_i)$.

\textbf{Theorem 1 (Anna Karenina principle for the INN model).} \textit{Suppose that instance $I = \langle S, z \rangle$ is sampled from proto-population $F$ via the INN model. For every worker $i$, $E[\pi_i(I)|f_i] = f_i + \mu$.}

\(^3\)The squared Euclidean distance in not a true metric (violates triangle inequality), but this is not required for our needs. The squared Euclidean distance is often used as a dissimilarity measure in various clustering applications [7, 8, 25].

\(^4\)In the INN model the equal-difficulty assumption is a normative decision, since we can always scale the data. Essentially, it means that we measure errors in standard deviations, giving the same importance to all questions.
\textbf{ALGORITHM 1:} 
Estimate-Fault-Levels-by-Distance-from-outcome (D-EFL) 

\textbf{Input:} Dataset S 
\textbf{Output:} Fault levels estimation \((f_i^0)_{i \in \mathbb{N}}\) 

Estimate the ground truth as \(y^0 \leftarrow r(S)\); 
\textbf{for each worker} \(i \in \mathbb{N}\) \textbf{do} 
\hspace{0.5cm} Set \(f_i^0 \leftarrow d(s_i, y^0)\); 
\textbf{end}

\textbf{ALGORITHM 2:} Proxy-based-Estimate-Fault-Levels (P-EFL for continuous answers) 

\textbf{Input:} Dataset S; parameter \(u\). 
\textbf{Output:} Fault levels estimation \((\hat{f}_i)_{i \in \mathbb{N}}\) 

Compute \(d_{ij} \leftarrow d_E(s_i, s_j)\) for every pair of workers; 
\textbf{for each worker} \(i \in \mathbb{N}\) \textbf{set} \(PS_i \leftarrow \frac{1}{n-1} \sum_{i' \neq i} d_{ij}\); 
\textbf{Set} \(\hat{\mu} \leftarrow \text{EstimateMu}(S)\); 
\textbf{for each worker} \(i \in \mathbb{N}\) \textbf{set} \(\hat{f}_i \leftarrow PS_i - \hat{\mu}\);

---

**Proof.** Denote \(d_{ij'} := d_E(s_i, s_{ij'})\), which is a random variable.

\[
E[\pi_i(I)|f_i] = E_{f_{i'} \sim \mathcal{F}^n}|E_{s_i \sim N(z,f_i), \ldots, s_n \sim N(z,f_n)} \left[ \frac{1}{n-1} \sum_{i' \neq i} d_{ij'}|f_i, f_i'| \right] [f_i] 
\]

\[
= E_{f_{i'}} \left[ \frac{1}{n-1} \sum_{i' \neq i} E_{s_i, s_{ij'}}[d_{ij'}|f_i, f_i'] \right] = \frac{1}{n-1} \sum_{i' \neq i} E_{f_{ij'} \sim \mathcal{F}}[E_{s_i, s_{ij'}}(\frac{1}{m} \sum_{j=1}^m (s_{ij} - s_{ij'})^2|f_i, f_i')] 
\]

We use the fact that the difference of two independent normal variables is also a normal variable whose expectation is the difference of expectations, and whose variance is the sum of the two variances. Denote by \(x = \epsilon_{ij} - \epsilon_{ij'}\) then \(E[x|f_i, f_i'] = E[s_{ij}|f_i] - E[s_{ij'}|f_i'] = z_i - z_i' = 0\); since \(f_i\) is the variance of \(s_{ij}\) for all \(j\),

\[
\text{VAR}[x|f_i, f_i'] = \text{VAR}[s_{ij}|f_i] + \text{VAR}[s_{ij'}|f_i'] = f_i + f_i'. 
\]  

(2)

We continue by bounding the inner expression for \(i, i'\).

\[
E_{f_{ij'} \sim \mathcal{F}}[E_{s_i, s_{ij'}}(\frac{1}{m} \sum_{j=1}^m (s_{ij} - s_{ij'})^2|f_i, f_i')] = E_{f_{ij'} \sim \mathcal{F}}[E_{s_i, s_{ij'}}((s_{ij} - s_{ij'})^2|f_i, f_i')] 
\]

\[
= E_{f_{ij'} \sim \mathcal{F}}[E[x^2|f_i, f_i']] = E_{f_{ij'} \sim \mathcal{F}}[\text{VAR}[x|f_i, f_i'] + (E[x|f_i, f_i'])^2] 
\]

\[
= E_{f_{ij'} \sim \mathcal{F}}[\text{VAR}[x|f_i, f_i']] = E_{f_{ij'} \sim \mathcal{F}}[f_i + f_i'] = f_i + E_{f_{ij'} \sim \mathcal{F}}[f_i'] = f_i + \mu. 
\]

Finally,

\[
E_{s_{ij}, R}[\pi_i(I)|f_i] = \frac{1}{n-1} \sum_{i' \neq i} (f_i + \mu) = f_i + \mu. 
\]

as required. \(\square\)

By Theorem 1, given \(\pi_i(I)\) and an estimate of \(\mu\), we can extract an estimate of \(f_i\), which suggests the P-EFL algorithm (Alg. 2).

Estimating parameters. What should be the value for \(\hat{\mu}\)? If we know \(\mu(\mathcal{F})\), we can of course use it. Otherwise, we can estimate it from the data. We first argue that lower values of \(\hat{\mu}\) would result in a more conservative estimation of \(f_i\): Consider two workers with \(f_1 > f_2\) and some \(\pi_1, \pi_2\). Denote by \(\hat{f}_1^{\hat{u}}\) the estimate we get when using some parameter \(\hat{\mu}\). Then it is easy to see that the ratio between \(\hat{f}_1^{\hat{u}}\) and \(\hat{f}_2^{\hat{u}}\) gets closer to 1 as we pick smaller \(\hat{\mu}\).

We define Algorithm 3 for estimating \(\mu\) as \(\hat{\mu} := u \cdot \sum_{i \in \mathbb{N}} d(s_i, y^0)\) (where \(y^0 = r(S)\)). If we use \(u = \frac{1}{n}\), then \(\hat{\mu}\) is the average of \(f_i^0\). By the argument above, lower values of \(u\) would result in more conservative estimation, therefore a default conservative value we could use is \(u = 0\), in which case the estimated fault level \(\hat{f}_i\) in D-EFL would equal \(\pi_i\).
We refer to this simple algorithm as the 

When fault levels are known, the best aggregation method is well understood.

We show in the next subsections that the value \( \frac{1}{n-1} \) (which is less conservative than both) is also of interest.

### 3.2 Equivalence and Consistency of P-EFL and D-EFL

**Theorem 2 (Equivalence of P-EFL and D-EFL).** Denote by \( \hat{f} \), \( \hat{f}^0 \) the output of algorithms \( \frac{1}{n-1} \)-P-EFL and D-EFL, respectively. For any instance \( I \), and any worker \( i \), \( \hat{f}_i = \frac{n}{n-1} \hat{f}_i^0 \).

Note that \( \frac{n}{n-1} \) does not depend on the particular instance or on the identity of the worker. Moreover, since in the continuous domain only relative fault matters (multiplying all \( f_i \) by a constant is just a change of scale), we get D-EFL as a special case of the proxy-based algorithm. Note that this equivalence does not depend on any statistical assumptions.

Theorem 1 does not guarantee that estimated fault levels are good estimates. We want to verify that at least for large instances from the INN model, they converge to the correct value. More precisely, an algorithm is consistent under a statistical model, if for any ground truth parameter and any \( \tau > 0 \), the probability for the outcome of the algorithm to be \( \tau \) away from the ground truth according to some measure goes to 0 as the data size grows.

**Theorem 3 (Consistency of D-EFL (continuous)).** When \( F \) has bounded support and \( z \) is bounded, D-EFL is consistent as \( n \to \infty \), \( m = \omega(\log(n)) \) and \( n = \omega(\log(m)) \). That is, \( |\hat{f}_i^0 - f_i| \to 0 \) for all \( i \in N \), as \( n \to \infty \), \( m = \omega(\log(n)) \) and \( n = \omega(\log(m)) \).

### 3.3 Aggregation

When fault levels are known, the best aggregation method is well understood.

**Proposition 4 ([1]).** Under the Independent Normal Noise model, \( x^* = \sum_{i \leq n} \frac{1}{n} s_i \) is minimizing \( E(d_E(z, x)|s_1, \ldots, s_n] \).

That is, the optimal way to aggregate the data is by taking a weighted mean of the answers of each question, where the weight of each worker is inversely proportional to her variance (i.e. to her fault level).

Prop. 4 suggests the following algorithmic skeleton (Alg. 4) for aggregating continuous answers.

A common approach to truth discovery is to combine Algorithm 4 with Algorithm 1 (D-EFL). We refer to this simple algorithm as the Distance-based Truth Discovery (D-TD) algorithm.

We define the Proxy Truth Discovery (P-TD) algorithm for the continuous domain, by similarly combining Algorithm 4 with \( u \)-P-EFL (Algorithm 2). When using \( u = \frac{1}{n-1} \), the P-TD and D-TD algorithms coincide by Thm. 2.

Note that both algorithms are well defined for any instance, whether the assumptions of the INN model hold or not. Moreover, in the INN model, Theorems 1 and 3 guarantee that with enough workers and questions, \( \hat{f}_i \) is a good estimate of the real fault level \( f_i \).

This of course does not yet guarantee that either algorithm returns accurate answers. For this, we need the following two results. The first says that good approximation of \( f \) entails a good approximation of \( z \); and the second says that in the limit, D-TD (and thus P-TD) return the correct answers. Recall that \( x^* = \frac{1}{n} \sum_{i \in N} w^*_i s_i \) is the best possible estimation of \( z \) by Prop. 4.

**Theorem 5.** For any instance, such that \( \forall i \in N, \hat{f}_i \in (1 \pm \delta) f_i \) for some \( \delta \log 0.25 \); it holds that \( d(\hat{z}, z) \leq d(x^*, z) + O(\delta \cdot \max_{i,j}(s_{ij})^2) \).
**Algorithm 3: Estimate-Mu**

**Input:** Dataset $S$; parameter $u$.

**Output:** Estimation $\hat{\mu}$

Set $\hat{f}^0 \leftarrow$ D-EFL ($S$);
Set $\hat{\mu} \leftarrow u \cdot \sum_{i \in N} \hat{f}_i^0$.

---

**Theorem 6 (Consistency of D-TD (continuous)).** When $F$ has bounded support and $z$ is bounded, D-TD is consistent as $n \to \infty$, $m = \omega(\log(n))$ and $n = \omega(\log(m))$. That is, for any $\tau > 0$, $\Pr[d(\hat{z}, z) > \tau] \to 0$ for all $i \in N$, as $n \to \infty$, $m = \omega(\log(n))$ and $n = \omega(\log(m))$.

### 3.4 Empirical Results

We compared the performance of $\frac{1}{n-1}$-P-TD (which coincides with D-TD) to the baseline method UA on synthetic and real data. In addition, we created an “Oracle Aggregation” (OA) baseline, which runs the aggregation skeleton with the true fault level $f_i$ when available, or the empirical fault $d(s_i, z)$ otherwise.

We also tried other values for the parameter $u$, which had similar results.

We generated instances from the INN model, where $F = N(1, 1)$ (additional distributions in Appendix B). Each instance was generated by first sampling a population $f$ from $\mathcal{F}$, and then generating the instance from $f$. The Buildings dataset was collected via Amazon Mechanical Turk. The Triangles dataset is from [18] (see Appendix B.1 for further details). We used each such dataset as a distribution over instances, where for each instance we sampled $m$ questions and $n$ workers uniformly at random with replacement. We then normalized each question so that its answers have mean 0 and variance 1. For every combination of $n$ and $m$ we sampled 500 instances.

We can see in Fig. 2 that the P-TD/D-TD method is substantially better than unweighted mean almost everywhere.
ALGORITHM 4: Aggregation skeleton (continuous)

**Input:** Dataset $S$; parameter $u$.

**Output:** Answers $\hat{z}$

1. $f \leftarrow \text{EstimateFaultLevels}(S, u)$;
2. $\forall i \in N$, set $w_i \leftarrow \frac{1}{f_i}$;
3. Set $\hat{z} \leftarrow \mu^M(S, w)$;

---

Fig. 2. A comparison of weighted to unweighted aggregation on a synthetic distribution (left) and two real datasets. In each cell, the number on the color scale indicates the ratio of average errors. Areas where the ratio $< 1$ are blue with ▽, and indicate an advantage to P-TD (which is equivalent to D-TD); areas where the ratio is $> 1$ are red with ▲ and indicate advantage to UA. Gray means a tie, and * means that both error rates are negligible, or a missing data point.

### 4 CATEGORICAL ANSWERS DOMAIN

In this setting we have $m$ categorical (multiple-choice) questions. The ground truth is a vector $z \in A^m$, where $|A| = k$. The distance measure we use is the normalized Hamming distance (note that for binary labels it coincides with the squared Euclidean distance):

$$d_H(s, s') := \frac{1}{m} |\{j \leq m : s_j \neq s'_j\}| = \frac{1}{m} \sum_{j \leq m} [s_j \neq s'_j].$$

We follow the same scheme of Section 3: given an instance $I = \langle S, z \rangle$, we first show how to estimate workers’ fault levels under a simple noise model, then transform them to weights and aggregate multiple answers.

**Independent Errors Model.** For our theoretical analysis, we assume an independent error (IER) model. Formally, for every question $j$, and every worker $i \in N$, $s_{ij}, z_j$ with probability $f_i$; and for any $x \in A \setminus \{z_j\}$, $s_{ij} = x$ w.p. $f_{k-1}$. That is, all wrong answers occur with equal probability. Note that $E[d_H(s_i, z)|f_i] = f_i$.

We denote by $\theta := \frac{1}{k-1}$ the probability that two workers who are wrong select the same answer.

#### 4.1 Estimating Fault Levels

**Fault estimation by distance from the Plurality outcome.** As in the continuous case, it is a common practice to use a simple aggregation method (in this case Plurality) to estimate fault levels. We similarly denote the estimated fault level by $\hat{f}_i^0 := d(s_i, r^P(S))$, and refer to it as the D-EFL algorithm for the categorical domain.

**Fault estimation by proxy distance.** Applying the definition of the proxy distance (Eq. (1)) to the categorical domain, we get:

$$\pi_I(I) = \frac{1}{n-1} \sum_{i \neq i'} d_H(s_i, s_{i'}) = \frac{1}{(n-1)m} \sum_{i \neq i'} \sum_{j \leq m} [s_{ij} \neq s'_{ij}].$$
Algorithm 5: Proxy-based-Estimate-Fault-Levels (P-EFL, categorical)

**Input:** Dataset $S$; parameter $u$

**Output:** Fault levels estimation $\hat{f}$

Set $d_{ij'} \leftarrow d(s_i, s_{i'})$ for every pair $i, i'$;

Set $\hat{\mu} \leftarrow \text{EstimateMu}(S, u)$;

for each $i \in N$ do
    Set $\pi_i \leftarrow \frac{1}{n-1} \sum_{i' \neq i} d_{ii'}$;
    Set $\hat{f}_i \leftarrow \frac{\pi_i - \hat{\mu}}{1-(1+\theta)\hat{\mu}}$;
end

Fig. 3. The figures present the same information as in Fig. 1, for generated populations of $n = 40$ workers and $m = 50$ yes/no questions. Each figure shows one instance from Normal distributions with different $\mu$ and $\vartheta$. The bottom right figure shows the correlation between $\hat{f}$ and $f$ (blue line); and between $\hat{f}_0$ and $f$ (dashed orange line), as the number of workers $n$ grows (every datapoint is an average over 1000 instances).

**Theorem 7 (Anna Karenina principle for the IER model).** Suppose that instance $I = \langle S, z \rangle$ is sampled from proto-population $F$ via the IER model. For every worker $i$,

$$E[\pi_i(I)|f_i] = \mu + (1 - (1 + \theta)\mu)f_i.$$  \hfill (4)

The proof is somewhat more nuanced than in the continuous case. We first show that for every pair of workers,

$$Pr[s_{ij} = s_{i'j}|f_i, f_{i'}] = 1 - f_i - f_{i'} + (1 + \theta)f_if_{i'}.$$  

Then, for every population,

$$E[\pi_i|\pi] = f_i + (1 - (1 + \theta)f_i) \frac{1}{n-1} \sum_{i' \neq i} f_{i'},$$  \hfill (5)

and then we take expectation again over populations to prove the claim.

We get that there is a positive relation between $f_i$ and $\pi_i$ exactly when $\mu(F) < \frac{1}{1+\theta} = 1 - \frac{1}{k}$, i.e. when the average fault levels are below those of completely random answers.

To estimate $f_i$ from the data, the P-EFL algorithm (Alg. 5) reverses the linear relation.

**Setting parameter values.** As in the continuous domain, setting $u = \frac{1}{n}$ means that $\hat{\mu}$ is the average of $\hat{f}_i^0$. Also as in the continuous domain, we can use $u = 0$ as a conservative default value, in which case $\hat{f}_i = \pi_i(I)$.

In contrast to the continuous case, it is obvious that the estimates we get from the P-EFL and the D-EFL algorithms are not at all equivalent. To see why, note that a small change in the report of a single worker may completely change the Plurality outcome (and thus the fault estimates of all workers in the D-EFL algorithm), but only has a gradual effect on P-EFL.

We do not know whether D-EFL is consistent, but we can show that P-EFL is.

**Theorem 8 (Consistency of P-EFL).** Suppose the support of $F$ is a closed subset of $(0, 1]$ and $\mu(F) < \frac{k-1}{k}$. Then $\frac{1}{n}$-P-EFL is consistent as $n \to \infty$ and $m = \omega(\log n)$. That is, for any $\delta > 0$, $Pr[|\hat{f}_i - f_i| > \delta] \to 0$ for all $i \in N$, as $n \to \infty$ and $m = \omega(\log n)$. 10
Evaluation. How good is this estimation of P-EFL for a given population? Rephrasing Theorem 7, we can write
\[ E[\pi_i | f_i] = f_i + (1 - (1 + \theta)f_i)\mu. \]
That is, the proxy distance is a sum of two components. The first is the actual fault level \( f_i \) (the “signal”). The second one decreases the signal proportionally to \( \mu \). Thus the lower \( \mu \) is, the better estimation we get on average. We can see this effect in Fig. 3: the top left figure presents an instance with lower \( \mu \) than the figure to its right, so the dependency of \( \pi_i \) on \( f_i \) is stronger and we get a better estimation. The top right figure has the same \( \mu \) as the middle one, but higher \( \theta \), thus fault levels are more spread out and easier to estimate. The two figures on the bottom left demonstrate that a good fit is not necessarily due the the IER model, as the estimated faults for the real dataset on the middle are much more accurate.

The bottom right figure shows that in P-EFL is somewhat more accurate that D-EFL on average.

4.2 Aggregation

The vector \( x^* \) that minimizes the expected distance to the ground truth \( z \) is also the MLE under equal prior. This is since we simply try to find the most likely answer for each question. When fault levels are known, this was studied extensively in a binary setting [16, 29, 35]. Specifically, Grofman et al. [16] identified the optimal rule for binary aggregation. Ben-Yashar and Paroush [2] extended these results to questions with multiple answers.\(^5\)

For a worker with fault level \( f_i \), we denote \( w_i^* := \log \left( \frac{(1-f_i)(k-1)}{f_i} \right) \). We refer to \( w^* \) as the Grofman weights of population \( f \).

**Proposition 9 ([2, 16]).** Suppose that \((z, S)\) is a random instance from the IER model. Let \( x_i^* := \arg\max_{x_j \in A} \sum_{i \in N} w_i^* [s_{ij} = x_j] \). Then \( x^* \) is the maximum likelihood estimator of \( z \) (and thus also minimizes \( d(x, z) \) in expectation).

That is, \( x^* \) is the result of a weighted plurality rule, where the optimal weight of \( i \) is her Grofman weight \( w_i^* \) (note that it depends only on \( f_i \)). Note that workers whose fault level is above random error (\( f_i > 1 - \frac{1}{k} \)) get a negative weight. Of course, since we have no access to the true fault level, we cannot use Grofman weights directly.

Prop. 9 suggests a simple aggregation skeleton, which is the same as Alg. 4, except it uses \( r^p \) instead of \( r^m \), and sets weights to \( w_i := \log \left( \frac{(1-f_i)(k-1)}{f_i} \right) \).\(^6\) D-TD and \( u-P-TD \) are the combinations of this categorical skeleton with D-EFL and with \( u-P-EFL \), respectively.

As in the continuous case, the algorithm is well-defined for any categorical dataset, but in the special case of the IER noise model we get that the workers’ weights are a reasonable estimate of the Grofman weights due to Theorems 7 and 8. Lastly for this section, we show that P-TD is consistent.

**Theorem 10 (Consistency of P-TD).** Suppose the support of \( \mathcal{F} \) is a closed subset of \((0, 1)\) and \( \mu_\mathcal{F} < \frac{k-1}{k} \). Then \( \frac{1}{n}-P-TD \) is consistent as \( n \to \infty \) and \( m = \omega(\log n) \). That is, for any \( \tau > 0 \), \( \Pr[d(\hat{z}, z) > \tau] \to 0 \) as \( n \to \infty \) and \( m = \omega(\log n) \).

4.3 Empirical results

We compared the performance of P-TD to the competing methods UA (which returns \( r^p(S) \)) and D-TD on synthetic and real data. In all simulations we used the default parameter \( u = 0 \) (i.e.,

\(^5\)Ben-Yashar and Paroush [2] also considered other extensions including unequal priors, distinct utilities for the decision maker, and general confusion matrices instead of equal-probability errors. In all these cases the optimal decision rule is not necessarily a weighted plurality rule, and generally requires comparing all pairs of answers.

\(^6\)Also appears as Alg. 8 in the appendix for completeness.
Fig. 4. Performance of UA, D-TD, and P-TD. Results are shown for 3 synthetic distributions and one real dataset. IP-TD is discussed later in Sec. 4.4.

Fig. 5. Each heatmap in the top row compares P-TD to UA (as in Fig. 2), and each heatmap in the middle row compares P-TD to D-TD, varying \( n \) and \( m \) (1000 samples for each). The bottom row is discussed in Sec. 4.4.

0-P-TD). Averages are over 1000 samples for each \( n \) and \( m \). The oracle benchmark OA returns \( r^*(S, w^*) \). Note that OA and UA are not affected by the number of questions.

For synthetic data we generated instances from the IER model: one distribution with Yes/No questions \((k = 2)\), where \( F = N(0.45, 0.1) \); and another with multiple-choice questions \((k = 4)\), where \( F = N(0.7, 0.1) \) (additional distributions in Appendix D). In addition, we used three datasets from [34] (Flags, GoldenGate, Dogs) and one that we collected (DotsBinary). Their description is in Appendix D.1.

Fig. 4 shows that both P-TD and D-TD have a strong advantage over unweighted aggregation. Fig. 5 directly compares P-TD to UA (top row) and to D-TD (mid row). We can see that P-TD dominates except in some regions.

### 4.4 Iterative methods

There are more advanced methods for truth discovery that are based on the following reasoning: a good estimate of workers’ fault levels leads to aggregated answers that are close to the ground truth (by appropriately weighing the workers); and a good approximation of the ground truth can get us a good estimate of workers’ competence (by measuring their distance from the approximate answers). Thus we can iteratively improve both estimates (see e.g. in [26], Section 2.2.2). The Iterative D-TD
algorithm (ID-TD), see Alg. 6) captures this reasoning. Note that the D-TD algorithm is a special case of ID-TD with a single iteration.

Iterative P-EFL. We can adopt a similar iterative approach for estimating fault levels using proxy voting. Intuitively, in each iteration we compute the proxy distance of a worker according to her expectation, the nominator holds:

\[ \hat{w} \]

Thus ideally, we would like to make this second part smaller to strengthen the signal. We argue that weighted proxy distance obtains just that. We do not provide a formal proof, but rather approximate calculations that should provide some intuition. Exact calculations are complicated due to correlations among terms.

We assume that fault levels are already determined, and take expectation only over realization of workers’ answers.

**Lemma 11.** In step t of the iterative algorithm,

\[ E[\pi_i | f_i] = f_i + (1 - 2f_i)\mu. \tag{6} \]

Ideally, we would like to make this second part smaller to strengthen the signal. We argue that weighted proxy distance obtains just that. We do not provide a formal proof, but rather approximate calculations that should provide some intuition. Exact calculations are complicated due to correlations among terms.

We assume that fault levels are already determined, and take expectation only over realization of workers’ answers.

**Algorithm 6:** Iterative Distance-from-outcome Truth Discovery (ID-TD)

**Input:** number of iterations \( T \); dataset \( S \)

**Output:** Fault levels \( \hat{f} = (\hat{f}_i)_{i \in N} \), answers \( \hat{z} \)

Initialize \( w^0 = \frac{1}{n} \);

for \( t = 0, 1, 2, \ldots, T - 1 \) do

\[ y^t \leftarrow r^P(S, w^t); \]

\[ \forall i \in N, \text{ set } \hat{f}_i^t \leftarrow d_H(s_i, y^t); \]

\[ \forall i \in N, \text{ set } w_i^t \leftarrow \log \left( \frac{1 - f_i^t}{f_i^t} \right); \]

end

Set \( \hat{f} \leftarrow \hat{f}^T; \)

Set \( \hat{z} \leftarrow r^P(S, w^T); \)

**Algorithm 7:** Iterative-Proxy-based-Estimate-Fault-Levels (IP-EFL)

**Input:** number of iterations \( T \); dataset \( S \)

**Output:** Fault levels \( \hat{f} = (\hat{f}_i)_{i \in N} \)

Initialize \( w^0 = \frac{1}{n} \);

Compute \( d_{i'\ell} \leftarrow d(s_j, s_{i'}) \) for every pair of workers;

for \( t = 0, 1, 2, \ldots, T - 1 \) do

for every worker \( i \in N \) do

Set \( \hat{f}_i^t = \pi_i^t \leftarrow \sum_{i' \neq i} w_{i'} d_{i'i'} \);

Set \( w_i^{t+1} \leftarrow \log \left( \frac{1 - f_i^t}{f_i^t} \right); \)

end

end

Set \( \hat{f} \leftarrow \hat{f}^T; \)
Similarly, in expectation, the denominator holds $\mathbb{E}[\sum_{i' \neq i} w_{i'}^* f_{i'}] = \frac{1-\mu}{n-1}$.

If we neglect both the correlation among nominator and denominator, and the fact that $w_i^*$ is only an approximation of $w^*$, we get that:

$$\mathbb{E}\left[\frac{\sum_{i' \neq i} w_i^* f_{i'}}{\sum_{i' \neq i} w_{i'}^*}\right] \approx \mathbb{E}\left[\frac{\sum_{i' \neq i} w_i^* f_{i'}}{\sum_{i' \neq i} w_{i'}^*}ight] \approx \mathbb{E}\left[\frac{\sum_{i' \neq i} w_i^* f_{i'}}{\sum_{i' \neq i} w_{i'}^*}\right] \approx \mathbb{E}\left[\frac{\sum_{i' \neq i} w_i^* f_{i'}}{\sum_{i' \neq i} w_{i'}^*}\right] \approx \frac{1}{2} - \frac{\theta}{2} = \mu - \frac{\theta}{2}.$$ 

We conclude from Lemma 11 and the above discussion that after enough iterations, $\mathbb{E}[\pi_i^t | f_i] = \mathbb{E}[\pi_i^t | f_i] \approx f_i + (1 - 2f_i) \left(\mu - \frac{\theta}{2}\right)$.

Since $\mu < \frac{1}{2}$, this noise term is not larger than the noise in the unweighted P-EFL algorithm ($\mu$ in Eq. (6)). We therefore expect that if $w^t$ is already a reasonable estimation of $w^*$ then accuracy will grow with further iterations.

**Empirical results for iterative algorithms.** The Iterative Proxy-based Truth Discovery (IP-TD) algorithm, combines our aggregation skeleton with IP-EFL. We can see how adding iterations affects the performance of P-TD on synthetic data in Fig. 4. We further compare the performance of IP-TD to ID-TD on more distributions and datasets in the third row of Fig. 5 (and in Appendix D). For both algorithms we used $T = 8$ iterations. A higher number of iterations had little effect on the results. Note that as we use $u = 0$, our IP-TD algorithm never explicitly estimates $\mu$ or $\theta$, yet it manages to take advantage of the variance among workers. We do see instances however where the initial estimation is off, and any additional iteration makes it worse.

5 **RANKING DOMAIN**

Consider a set of alternatives $C$, where $L = L(C)$ is the set of all rankings (permutations) over $C$. Each pairwise relation over $C$ corresponds to a binary vector $x \in \{-1, 1\}^m$ where $m = (|C|!)/2!$ (i.e. each dimension is a pair of candidates). In particular, every ranking $L \in L$ has a corresponding vector $x_L \in \{-1, 1\}^m$. A vector $x \in \{-1, 1\}^m$ is called transitive if it corresponds to some ranking $L_x$ s.t. $x_L = x$. The ground truth is a transitive vector $z$ (equivalently, a ranking $L_z \in L$). A natural metric over rankings is the Kendall-tau distance (a.k.a swap distance): $d_{KT}(L, L') := d_H(x_L, x_{L'})$.

**Independent Condorcet Noise Model.** According to Independent Condorcet noise (ICN) model, an agent with fault level $f_i \in [0, 1]$ observes a vector $s_i \in \{-1, 1\}^m$ where for every pair of candidates $j = (a, b)$, we have $s_{ij} \neq z_j$ with probability $f_i$. In particular, $s_i$ may not be transitive.

**Mallows Models.** The down side of the Condorcet noise model is that it may result in nontransitive answers. Mallows model is similar, except it is guaranteed to produce transitive answers (rankings).

Formally, given ground truth $L_z \in L$ and parameter $\phi_i > 0$, the probability of observing order $L_i \in L$ is proportional to $\phi_i^{d(L_i, L_z)}$. Thus for $\phi = 1$ we get a uniform distribution, whereas for low $\phi_i$ we get orders concentrated around $L_z$.

In fact, if we throw away all non-transitive samples, the probability of getting rank $L_i$ under the Condorcet noise model with parameter $f_i$ (conditional on the outcome being transitive) is exactly the same as the probability of getting $L_i$ under Mallows Model with parameter $\phi_i = \frac{1-f_i}{f_i}$.

5.1 **Estimating Fault Levels**

By definition, the ICN model is a special case of the IER model, where $k = 2$, $m = (|C|!)/2!$, and the ground truth is a transitive vector. We thus get the following result as an immediate corollary of Theorem 7, and can therefore use P-EFL directly.

Classically, the Condorcet model assumes all voters have the same parameter [43].
**Theorem 12 (Anna Karenina principle for the Condorcet model).** Suppose that instance \( I = (S, z) \) is sampled from population \( (f_i)_i \in \mathcal{N} \) via the ICN model, where all \( f_i \) are sampled independently from proto-population \( \mathcal{F} \) with expected value \( \mu \). For every worker \( i, E[\pi_i(I)|f_i] = \mu + (1 - 2\mu)f_i \).

5.2 Aggregation

Note that while our results on fault estimation from the binary domain directly apply (at least to the ICN model), aggregation is more tricky: an issue-by-issue aggregation may result in a non-transitive (thus invalid) solution. The voting rules we consider are guaranteed to output a valid ranking.

The problem of retrieving \( L_z \) given \( n \) votes \( L_1, \ldots, L_n \) is a classical problem, and in fact any social welfare function \( r: \mathcal{L}^n \rightarrow \mathcal{L} \) offers a possible solution [4].

There is a line of work that deals with finding the MLE under various assumptions on the noise model [2, 13]. In general, these estimators may take a complicated form that depends on all parameters of the distribution. Yet some cases are simpler.

**The Kemeny rule and optimal aggregation.** It is well known that for both Condorcet noise model and Mallows model, when all voters have the same fault level, the maximum likelihood estimator of \( L_z \) is obtained by applying the Kemeny-Young voting rule on \( S \) [43].

The Kemeny-Young rule \( r^{KY} \) (henceforth, Kemeny) computes the binary vector \( y^0 \) that corresponds to the majority applied separately on every pair of candidates (that is, \( y^0 := \text{sign} \sum_{i \leq n} s_i \)); then \( r^{KY}(S) := \arg\min_{L \in \mathcal{L}} d_H(x_L, y^0) \).

In particular it can be applied when \( S \) is composed of transitive vectors.

A natural question is whether there is a weighted version of KY that is an MLE and/or minimizes the expected distance to \( L^* \) when fault levels \( f_i \) are known. We did not find any explicit reference to this question or to the case of distinct fault levels in general. There are some other extensions: [13] deal with a different variation of the noise model where it is less likely to swap pairs that are further apart. [42] extend the Kemeny rule to deal with more general noise models and partial orders.

As it turns out, using weighted KY with (binary) Grofman weights \( w^*_i = \log \frac{1 - f_i}{f_i} \) provides us with (at least) an approximately optimal outcome.

**Proposition 13.** Suppose that the ground truth \( L_z \) is sampled from a uniform prior on \( L \). Suppose that instance \( I = (S, z) \) is sampled from population \( (f_i)_i \in \mathcal{N} \) via the ICN model. Let \( L^* := r^{KY}(S, w^*(f)) \).

Let \( L' \in \mathcal{L} \) be any random variable that may depend on \( S \). Then \( E[d_{KT}(L^*, L_z)] \leq 2E[d_{KT}(L', L_z)] \), where expectation is over all instances.

**Proof.** Consider the ground truth binary vector \( z \). Let \( y := \text{sign} \sum_{i \leq n} w_i s_i \). Let \( x' \in \{-1, 1\}^m \) be an arbitrary random variable that may depend on the input profile. For every \( j \leq m \) (i.e., every pair of elements), we know from Prop. 9 that \( y_j \) is the MLE for \( z_j \), and thus \( \Pr[x'_j = z_j] \leq \Pr[y_j = z_j] \).

Now, recall that by definition of the KY rule, \( L^* \) is the closest ranking to \( L_y \). Also denote \( x^* = x_{L^*} \).

\[
E[d_{KT}(L^*, L_z)] = E[d_H(x^*, z)] \leq E[d_H(x^*, y) + d_H(y, z)]
\]

\[
\leq E[2d_H(y, z)] \quad \text{(since} \ x^* \text{is the closest transitive vector to} \ y)\]

\[
= \frac{2}{m} \sum_{j \leq m} \Pr[y_j \neq z_j] \leq \frac{2}{m} \sum_{j \leq m} \Pr[x'_j \neq z_j] = 2E[d_H(x', z)].
\]

In particular, this holds for any transitive vector \( x' = x_{L'} \) corresponding to ranking \( L' \), thus

\[
E[d_{KT}(L^*, L_z)] \leq 2E[d_H(x_{L'}, z)] = 2E[d_{KT}(L', L_z)],
\]

as required. \( \square \)
Prop. 13 provides some justification to use Kemeny voting rule and Grofman weights for aggregating rankings when the $f_i$’s are known. We can now apply a similar reasoning as in the binary case to estimate $f_i$. Given a set of rankings $S$ and any voting rule $r$ (not necessarily Kemeny!), the P-TD $r$ algorithm is a combination of Alg. 8 with $k = 2$ and $r$ instead of $r^P$, and Alg. 5 with the Kendall-tau distance and $u = 0$. Since the meaning of negative weights is not clearly defined, we replace every negative weight with 0 (the full description appears as Alg. 9 in the appendix for completeness).

5.3 Empirical results

We compared the performance of P-TD $r$ using 8 different voting rules: Borda, Copeland, Kemeny-Young (with weighted and unweighted majority graph), Plurality, Veto, Random dictator, and Best dictator.\footnote{For formal definition of the voting rules, see \cite{4}, or Appendix E.1.}

We generated instances from Mallows model, where $F = N(0.85, 0.15)$ (additional distributions in Appendix E.3). For every combination of $n$ and $m$ we sampled 1000 instances, each from a different population. We can see the results in Fig. 6 and 7 (left). The advantage of P-TD is particularly visible in the four latter voting rules, which are simpler. This is since the intermediate estimation $y^0 = r(S)$ is poor and misguides the D-TD algorithm.

We also used a dataset collected by Mao et al. \cite{27}, where groups of 20-30 subjects were asked to order 4 images according to the number of dots in them (DOTS dataset); or according to the number of steps required to solve the 8-puzzle board (PUZZLES dataset). We compared P-TD to D-TD on all of these groups, with each of the eight voting rules. Fig. 7 (right) shows a clear advantage to P-TD.

6 DISCUSSION

Proxy voting can be used as a general scheme to estimate workers’ competence (or fault level) in a broad range of truth discovery and crowdsourcing scenarios, as long as there is a natural notion of distance between answers. For three such domains (real-valued answers, categorical answers, and rankings) we showed formally a linear version of the “Anna Karenina principle”, i.e. that the average distance from a worker to other workers is linear in her fault level, under common noise models.

It is interesting that under several rather different noise models, we get that the proxy distance itself ($\pi_i(I)$) is a reasonable estimate of the fault level $f_i$. This suggests that using the proxy distance may be a good fallback option when estimating workers’ fault levels in domains when we do not have an explicit noise model and/or basic aggregation function to work with. Results in the continuous and ranking domains show substantial improvement, whereas in the Categorical domain this is more nuanced, especially when there are few questions. This may be due to workers that are worse than random (note that consistency is not guaranteed in this case).
6.1 Related Work

Similarity of workers (or “peer consistency”) has been previously considered in the context of crowdsourcing. It has been most commonly used in “Games with a Purpose” which incorporated “output agreement” mechanism [36], where two players are given a task and progress in the game if their answers match. Inspired by this approach, Huang et al. [21] used such a mechanism to incentivize workers by telling workers that their answers will be compared to those of a random worker. They showed that this incentive scheme can enhance workers’ performance. However, they did not use peer consistency for aggregation nor do they provide any theoretical analysis.

The iterative algorithm ID-TD that we mentioned belongs to a large class of iterative methods, that couple competence estimation and truth discovery (goals (a) and (b) from our introduction) [19, 22, 33, 40, 41]. These approaches typically assume an underlying model of worker competence and estimate the parameters of this model based on workers’ responses to assess their reliability. They then utilize this information to aggregate labels (goal (b)) or to determine which workers to assign to new tasks. Many more algorithms for truth discovery have been suggested (see two recent surveys in [26, 38]), and full comparison with all of them is outside the scope of this work.

In contrast to these iterative techniques, our competence estimation methods (including our iterative IP-EFL algorithm) can be completely independent of any aggregation rule—consider our comment above on using \( \hat{\mu} = 0 \). They can thus be applied in scenarios where answers are complicated and no obvious aggregation method is available (e.g., complex labeling of text, images, or video). In fact, we apply it in the ranking domain where there are many aggregation rules but no consensus on what is the right one to use.

Another common approach to assess workers’ competence (i.e. goal (a)) in the crowdsourcing literature, is incorporating “gold questions” for which the ground truth answer is known; these questions are then used to filter low-quality workers [14, 44]. Several works proposed approaches to make the use of gold questions more effective by adaptively choosing whether to ask a worker additional gold questions [3, 30].

We argue that proxy-based truth discovery is not intended to replace the above arsenal of tools for crowdsourcing, but to complement it. Indeed, it is very easy to weight workers according to their
proxy distance,⁹ and combine this with any aggregation rule and gold questions, active selection, and/or various incentive schemes tools that are being used in crowdsourcing platforms.

This is important, as the tools in use may be subject to various computational and incentive constraints, legacy code, interpretability requirements, and so on. We thus believe it is important to be able to boost the performance of a broad range of aggregation rules in diverse scenarios. Indeed, the only thing we need to apply the Anna Karenina principle is a distance measure. Curiously, the empirical performance we observed e.g. in the ranking domain, was for the simplest voting rules, that had no theoretical guarantees.

6.2 Conclusion and future work

Recall that our initial inspiration was proxy voting in social choice settings. Having now a deeper understanding of the reasons underlying the success of proxy voting in truth discovery tasks, we would like to apply these insights back to social choice and political settings.

Apart from that, we believe that proxy voting could be very easily integrated into almost any truth discovery or crowdsourcing system, and in many cases provide a significant boost in performance at a very low cost. A unifying “Anna Karenina” theorem that is not domain specific (perhaps uses only abstract properties of the distance measure and the statistical model) would be of great help in this direction. Another important direction is to consider the economic incentives in crowdsourcing system, and see how proxy voting affects equilibrium behavior of participants.

On the practical side, we are currently engaged in several collaborations that apply proxy voting to real crowdsourcing problem with complex input, and hope it would help to bring substantial benefit to the world.

REFERENCES

[1] AC Aitkin. On least squares and linear combination of observations. Proceedings of the RSE, 55:42–48, 1935.
[2] Ruth Ben-Yashar and Jacob Paroush. Optimal decision rules for fixed-size committees in polychotomous choice situations. Social Choice and Welfare, 18(4):737–746, 2001.
[3] Jonathan Bragg, Daniel S Weld, et al. Optimal testing for crowd workers. In AAMAS’16, pages 966–974. IFAAMAS, 2016.
[4] Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D Procaccia. Handbook of computational social choice. Cambridge University Press, 2016.
[5] Markus Brill. Interactive democracy. In AAMAS’18, pages 1183–1187, 2018.
[6] Ioannis Caragiannis, Ariel D Procaccia, and Nisarg Shah. When do noisy votes reveal the truth? In EC’13, pages 143–160. ACM, 2013.
[7] Randy L Carter, Robin Morris, and Roger K Blashfield. On the partitioning of squared euclidean distance and its applications in cluster analysis. Psychometrika, 54(1):9–23, 1989.
[8] Sung-Hyuk Cha. Comprehensive survey on distance/similarity measures between probability density functions. International journal of mathematical models and methods in applied sciences, 1(2):1, 2007.
[9] Gal Cohensius, Shie Mannor, Reshef Meir, Eli Meirom, and Ariel Orda. Proxy voting for better outcomes. In AAMAS’16, pages 858–866. IFAAMAS, 2017.
[10] Marie J Condorcet et al. Essai sur l’application de l’analyse à la probabilité des décisions rendues à la pluralité des voix, volume 252. American Mathematical Soc., 1785.
[11] Vincent Conitzer and Tuomas Sandholm. Common voting rules as maximum likelihood estimators. In Proceedings of the Twenty-First Conference on Uncertainty in Artificial Intelligence, pages 145–152. AUAI Press, 2005.
[12] Jia Deng, Olga Russakovsky, Jonathan Krause, Michael S Bernstein, Alex Berg, and Li Fei-Fei. Scalable multi-label annotation. In Proceedings of the SIGCHI Conference on Human Factors in Computing Systems, pages 3099–3102. ACM, 2014.
[13] Mohamed Drissi-Bakhkhat and Michel Truchon. Maximum likelihood approach to vote aggregation with variable probabilities. Social Choice and Welfare, 23(2):161–185, 2004.
[14] Matthew R Gormley, Adam Gerber, Mary Harper, and Mark Dredze. Non-expert correction of automatically generated relation annotations. In NAACL HLT Workshop, pages 204–207. ACL, 2010.
[15] James Green-Armytage. Direct voting and proxy voting. Constitutional Political Economy, 26(2):190–220, 2015.

⁹Admittedly, calling P-TD an “algorithm” is stretching the term!
[16] Bernard Grofman, Guillermo Owen, and Scott L. Feld. Thirteen theorems in search of the truth. *Theory and Decision*, 15(3):261–278, 1983.

[17] Ronald K. Hambleton, Hariharan Swaminathan, and H. Jane Rogers. *Fundamentals of item response theory*, volume 2. Sage, 1991.

[18] Yuval Hart, Moira R. Dillon, Andrew Marantan, Anna L. Cardenas, Elizabeth Spelke, and L. Mahadevan. The statistical shape of geometric reasoning. *Scientific reports*, 8(1):12906, 2018.

[19] Chien-Ju Ho, Shahin Jabbari, and Jennifer Wortman Vaughan. Adaptive task assignment for crowdsourced classification. In *ICML*, pages 534–542, 2013.

[20] John J. Horton. The dot-guessing game: A fruit fly for human computation research. *SSRN*, 2010.

[21] Shih-Wen Huang and Wai-Tat Fu. Enhancing reliability using peer consistency evaluation in human computation. In *CSCW*, pages 639–648. ACM, 2013.

[22] David R. Karger, Sewoong Oh, and Devavrat Shah. Iterative learning for reliable crowdsourcing systems. In *NIPS*, pages 1953–1961, 2011.

[23] Gabriella Kazai, Jaap Kamps, Marijn Koolen, and Natasa Milic-Frayling. Crowdsourcing for book search evaluation: impact of hit design on comparative system ranking. In *SIGIR* ’11, pages 205–214. ACM, 2011.

[24] Emmanuel Kruele, Michael Stevens, Pawel Mandera, and Marc Brysbaert. Word knowledge in the crowd: Measuring vocabulary size and word prevalence in a massive online experiment. *Quarterly Journal of Experimental Psychology*, 68(8):1665–1692, 2015. PMID: 25715025.

[25] Esve Kosman and KJ Leonard. Similarity coefficients for molecular markers in studies of genetic relationships between individuals for haploid, diploid, and polyploid species. *Molecular ecology*, 14(2):415–424, 2005.

[26] Yaliang Li, Jing Gao, Chuishi Meng, Qi Li, Lu Su, Bo Zhao, Wei Fan, and Jiawei Han. A survey on truth discovery. *ACM SIGKDD Explorations Newsletter*, 17(2):1–16, 2016.

[27] Andrew Mao, Ariel D. Procaccia, and Yiling Chen. Better human computation through principled voting. In *AAAI*. Citeseer, 2013.

[28] Elliot McLaughlin. Image overload: Help us sort it all out, nasa requests. CNN.com. Retrieved at 18/9/2014.

[29] Shmuel Nitzan and Jacob Paroush. Collective decision making: an economic outlook. *Molecular ecology*, 14(2):415–424, 2005.

[30] Maximilien Servajean, Alexis Joly, Dennis Shasha, Julien Champ, and Esther Pacitti. Crowdsourcing thousands of specialized labels: a bayesian active training approach. *IEEE Transactions on Multimedia*, 19(6):1376–1391, 2017.

[31] Flavio Ribeiro, Diniel Florencio, and Vitor Nascimento. Crowdsourcing subjective image quality evaluation. In *Image Processing (ICIP), 2011 18th IEEE International Conference on*, pages 3097–3100. IEEE, 2011.

[32] Marta Sabou, Kalina Bontcheva, Leon Derczynski, and Arno Scharl. Corpus annotation through crowdsourcing: Towards best practice guidelines. In *LREC*, pages 859–866, 2014.

[33] Lloyd Shapley and Bernard Grofman. Optimizing group judgmental accuracy in the presence of interdependencies. *Public Choice*, 43(3):329–343, 1984.

[34] Luis Von Ahn and Laura Dabbish. Designing games with a purpose. *Communications of the ACM*, 51(8):58–67, 2008.

[35] Jeroen Vuuren, Arjen P. de Vries, and Carsten Eickhoff. How much spam can you take? an analysis of crowdsourcing worker’s reliability. In *AAMAS’18*, pages 1486–1494. IFAAMAS, 2018.

[36] Flavio Ribeiro, Diniel Florencio, and Vitor Nascimento. Crowdsourcing subjective image quality evaluation. In *Image Processing (ICIP), 2011 18th IEEE International Conference on*, pages 3097–3100. IEEE, 2011.

[37] Nihar Bhadresh Shah and Denny Zhou. Double or nothing: Multiplicative incentive mechanisms for crowdsourcing. In *NIPS*, pages 1–9, 2015.

[38] Lloyd Shapley and Bernard Grofman. Optimizing group judgmental accuracy in the presence of interdependencies. *Public Choice*, 43(3):329–343, 1984.

[39] Luis Von Ahn and Laura Dabbish. Designing games with a purpose. *Communications of the ACM*, 51(8):58–67, 2008.

[40] Jeroen Vuuren, Arjen P. de Vries, and Carsten Eickhoff. How much spam can you take? an analysis of crowdsourcing worker’s reliability. In *AAMAS’18*, pages 1486–1494. IFAAMAS, 2018.

[41] Jacob Whitehill, Ting-fan Wu, Jacob Bergsma, Javier R Movellan, and Paul L. Ruvolo. Whose vote should count more: Optimal integration of labels from labelers of unknown expertise. In *NIPS workshop on CIR’ 11*, pages 21–26, 2011.

[42] Dalia Atta Waguih and Laure Berti-Equille. Truth discovery algorithms: An experimental evaluation. *arXiv preprint arXiv:1409.6428*, 2014.

[43] Paul Wais, Shivaram Lingamneni, Duncan Cook, Jason Fennell, Benjamin Goldenberg, Daniel Lubarov, David Marin, and Hari Simons. Towards building a high-quality workforce with mechanical turk. *NIPS workshop*, pages 1–5, 2010.

[44] Peter Welinder and Pietro Perona. Online crowdsourcing: rating annotators and obtaining cost-effective labels. In *Computer Vision and Pattern Recognition Workshops (CVPRW), 2010 IEEE Computer Society Conference on*, pages 25–32. IEEE, 2010.

[45] Jacob Whitehill, Ting-fan Wu, Jacob Bergsma, Javier R Movellan, and Paul L. Ruvolo. Whose vote should count more? Optimal integration of labels from labelers of unknown expertise. In *NIPS*, pages 2035–2043, 2009.

[46] Lirong Xia and Vincent Conitzer. A maximum likelihood approach towards aggregating partial orders. In *AJA*, pages 1953–1961, 2011.

[47] Paul Wais, Shivaram Lingamneni, Duncan Cook, Jason Fennell, Benjamin Goldenberg, Daniel Lubarov, David Marin, and Hari Simons. Towards building a high-quality workforce with mechanical turk. *NIPS workshop*, pages 1–5, 2010.

[48] Peter Welinder and Pietro Perona. Online crowdsourcing: rating annotators and obtaining cost-effective labels. In *Computer Vision and Pattern Recognition Workshops (CVPRW), 2010 IEEE Computer Society Conference on*, pages 25–32. IEEE, 2010.

[49] Jacob Whitehill, Ting-fan Wu, Jacob Bergsma, Javier R Movellan, and Paul L. Ruvolo. Whose vote should count more? Optimal integration of labels from labelers of unknown expertise. In *NIPS*, pages 2035–2043, 2009.

[50] Lirong Xia and Vincent Conitzer. A maximum likelihood approach towards aggregating partial orders. In *AJA*, pages 1953–1961, 2011.
A PROOFS FOR CONTINUOUS DOMAIN

Theorem 2. Consider the result of $\frac{1}{n-1}$-P-EFL. For any instance $I$, and any worker $i$, \( \hat{f}_i = \frac{n}{n-1} \hat{f}_i^0 \).

Proof. First note that

\[
\hat{f}_i^0 = d_E(s, \frac{1}{n} \sum_{I' \in N} s_{I'}) = \frac{1}{m} \sum_{j \leq m} (s_{ij} - \frac{1}{n} \sum_{I' \in N} s_{I'j})^2
= \frac{1}{m} \sum_{j \leq m} \left( s_{ij}^2 - \frac{2}{n} \sum_{I' \in N} s_{I'j} + \frac{1}{n} \sum_{I' \in N} s_{I'j}^2 \right)
\]

Now, the estimated fault level based on proxy score is

\[
\frac{n-1}{n} \hat{f}_i = \frac{n-1}{n} (\pi_i(I) - \hat{\mu}) = \frac{1}{n} \sum_{I' \in N} d_E(s_i, s_{I'}) - \frac{1}{n} \sum_{I' \in N} f_{I'}^0
= \frac{1}{n} \sum_{I' \in N} \left( \frac{1}{m} \sum_{j \leq m} (s_{ij} - s_{I'j})^2 - \frac{1}{n} \sum_{I' \in N} \left( s_{I'j}^2 - s_{I'j}^2 \frac{2}{n} \sum_{I'' \in N} s_{I''j} + \frac{1}{n} \sum_{I'' \in N} s_{I''j}^2 \right) \right)
= \frac{1}{m} \sum_{j \leq m} \left( \frac{1}{n} \sum_{I' \in N} \left( s_{I'j}^2 - 2s_{I'j} s_{I'j} - (s_{I'j}^2 - s_{I'j} - \frac{2}{n} \sum_{I'' \in N} s_{I''j} + \frac{1}{n} \sum_{I'' \in N} s_{I''j}^2) \right) \right)
= \frac{1}{m} \sum_{j \leq m} \left( s_{I'j}^2 - 2s_{I'j} s_{I'j} + \frac{1}{n} \sum_{I'' \in N} s_{I''j} + \frac{1}{n} \sum_{I'' \in N} s_{I''j}^2 \right)
= \frac{1}{m} \sum_{j \leq m} \left( s_{I'j}^2 - 2s_{I'j} s_{I'j} + 2\left( \frac{1}{n} \sum_{I'' \in N} s_{I''j} \right)^2 - \left( \frac{1}{n} \sum_{I'' \in N} s_{I''j} \right)^2 \right)
= \frac{1}{m} \sum_{j \leq m} \left( \frac{1}{n} \sum_{I' \in N} s_{I'j} + \frac{1}{n} \sum_{I' \in N} s_{I'j}^2 \right) = \hat{f}_i^0,
\]

as required. \( \square \)

Theorem 3. When $F$ has bounded support and $z$ is bounded, D-EFL is consistent as $n \to \infty$, $m = \omega(\log(n))$ and $n = \omega(\log m)$. That is, $|\hat{f}_i - f_i| \to 0$ for all $i \in N$, as $n \to \infty$, $m = \omega(\log(n))$ and $n = \omega(\log m)$.

Proof. We prove in steps: For each $j \leq m$, $y_j^0 = \frac{1}{n} \sum_{I' \in N} s_{I'j}$ is close to $z_j$ with high probability. This can be seen as the average of $n$ i.i.d. random variables such that the mean of each random variable is $z_j$ and the variance of each random variable is $\mu(F)$. Therefore, by Hoeffding’s inequality for sub-Gaussian random variables, we have that for any $T_1 > 0$:

\[
\Pr(|y_j^0 - z_j| > T_1) < \exp\{-C_1 T_1^2 n\},
\]

for constant $C_1 > 1$. 20
For each \( j \leq m \), given \(|y_j^0 - z_j| \leq T_1\), then we have that \( \sum_{j=1}^{m} \frac{(s_{ij} - y_j^0)^2}{m} \) is close to \( f_i \) with high probability.

\[
\sum_{j=1}^{m} \frac{(s_{ij} - y_j^0)^2}{m} = \sum_{j=1}^{m} \frac{s_{ij}^2}{m} + \sum_{j=1}^{m} \frac{y_j^0^2}{m} - 2 \sum_{j=1}^{m} \frac{y_j^0 s_{ij}}{m} \\
\leq \sum_{j=1}^{m} \frac{s_{ij}^2}{m} + \sum_{j=1}^{m} \frac{(z_j + T_1)^2}{m} - 2 \sum_{j=1}^{m} \frac{z_j s_{ij}}{m} + 2T_1 \sum_{j=1}^{m} \frac{s_{ij}}{m} \\
= \sum_{j=1}^{m} \frac{(s_{ij} - z_j)^2}{m} + T_1^2 + 2T_1 \sum_{j=1}^{m} \frac{(s_{ij} + z_j)}{m} \\

\]

Because for each \( j \leq m \), \( s_{ij} - z_j \) is a sample of the Gaussian distribution with variance \( f_i \), for any \( T_2 > 0 \), we have:

\[
\Pr(|\sum_{j=1}^{m} (s_{ij} - z_j)/m| > T_2) < \exp\{-C_2 T_2^2 m\},
\]

for constant \( C_2 > 0 \), where \( C_2 \) is related to the minimum variance in \( F \).

Because for each \( j \leq m \), \( E[(s_{ij} - z_j)^2] = f_i \) and \( \text{Var}[(s_{ij} - z_j)^2] = 2f_i^2 \), for any \( T_3 > 0 \), we have:

\[
\Pr(|\sum_{j=1}^{m} (s_{ij} - z_j)^2/m - f_i| > T_3) < \exp\{-C_3 T_3^2 m\},
\]

for constant \( C_3 > 0 \).

By union bound, with probability at least \( 1 - m \exp\{-C_1 T_1^2 n\} + n \exp\{-C_2 T_2^2 m\} + n \exp\{-C_3 T_3^2 m\} \), we have:

- For every \( j \leq m \), \( |y_j^0 - z_j| \leq T_1 \).
- For every \( i \leq n \), \( |\sum_{j=1}^{m} (s_{ij} - z_j)/m| \leq T_2 \).
- For every \( i \leq n \), \( |\sum_{j=1}^{m} (s_{ij} - z_j)^2/m - f_i| \leq T_3 \)

This means that

\[
\hat{f}_i = \sum_{j=1}^{m} \frac{(s_{ij} - y_j^0)^2}{m} \\
= \sum_{j=1}^{m} \frac{(s_{ij} - z_j)^2}{m} + T_1 \sum_{j=1}^{m} \frac{(s_{ij} + z_j)}{m} \\
\leq f_i + T_3 + T_1^2 + 2T_1 \sum_{j=1}^{m} \frac{(s_{ij} - z_j)/m}{m} + 4T_1 \sum_{j=1}^{m} \frac{z_j}{m} \\
\leq f_i + T_3 + T_1^2 + 2T_1 T_2 + 4T_1 (\sum_{j=1}^{m} \frac{z_j}{m})
\]

Similarly, we can prove that with probability at least \( 1 - 2(m \exp\{-C_1 T_1^2 n\} + n \exp\{-C_2 T_2^2 m\} + n \exp\{-C_3 T_3^2 m\}) \), for all \( i \leq n \), we have:

\[
|\hat{f}_i - f_i| \leq T_3 + T_1^2 + 2T_1 T_2 + 4T_1 (\sum_{j=1}^{m} \frac{z_j}{m}),
\]
Consistency follows after setting letting $T_1 = \omega(1)\sqrt{\log \frac{m}{n}}$, $T_2 = T_3 = \omega(1)\sqrt{\log \frac{n}{m}}$. □

**Lemma 14.** Suppose that $\hat{f}_i \in (1 \pm \delta)f_i$ for some $\delta \in [0, 1]$. Then $w_i$ is in the range $[w_i^*(1-\delta), w_i^*(1+2\delta)]$.

**Proof.** Note first that by Lemma 14, each $w_i$ is in the range $[w_i^*(1-\delta), w_i^*(1+2\delta)]$. W.l.o.g., $\sum_{i \in N} w_i^* = 1$. Thus

$$(1-\delta) \leq \sum_{i' \in \mathcal{I}} \hat{w}_{i'} \leq (1+2\delta),$$

and

$$\frac{1}{\sum_{i' \in \mathcal{I}} \hat{w}_{i'}} \in (1-4\delta, 1+4\delta)$$

as well.

Denote $\bar{s} = \max_{ij} s_{ij}$. For each $j \leq m$,

$$(\hat{z}_j - z_j)^2 = \left(\frac{1}{\sum_{i' \in \mathcal{I}} \hat{w}_{i'}} \sum_{i \in N} \hat{w}_i s_{ij} - z_j\right)^2$$

$$= \left(\frac{1}{\sum_{i' \in \mathcal{I}} \hat{w}_{i'}} \sum_{i \in N} \hat{w}_i^*(1 + \tau_i) s_{ij} - z_j\right)^2$$

(for some $\tau_i \in [-2\delta, 2\delta]$)

$$= \left(\sum_{i \in N} w_i^*(1 + \tau_i') s_{ij} - z_j\right)^2$$

(for some $\tau_i' \in [-8\delta, 8\delta]$)

$$= \sum_{i \in N} \sum_{i' \in N} w_i^*(1 + \tau_i') s_{ij} w_i^*(1 + \tau_i') s_{ij} + z_j^2 - 2z_j \sum_{i \in N} w_i^*(1 + \tau_i') s_{ij}$$

$$= \sum_{i \in N} \sum_{i' \in N} w_i^* s_{ij} w_i^* s_{ij} (1 + \tau_i' + \tau_i' + \tau_i' \tau_i') + z_j^2 - 2z_j \sum_{i \in N} w_i^*(1 + \tau_i') s_{ij}$$

$$= \sum_{i \in N} \sum_{i' \in N} w_i^* s_{ij} w_i^* s_{ij} (1 + \tau_i'') + z_j^2 - 2z_j \sum_{i \in N} w_i^* s_{ij}$$

For $|\tau_i''| < 25\delta$. Similarly,

$$(x^* - z_j)^2 = \sum_{i \in N} \sum_{i' \in N} w_i^* s_{ij} w_i^* s_{ij} + z_j^2 - 2z_j \sum_{i \in N} w_i^* s_{ij}.$$

22
Next,
\[
d(\tilde{z}, z) - d(x^*, z) = \frac{1}{m} \sum_{j=1}^{m} \left((\tilde{z}_j - z_j)^2 - (x^*_j - z_j)^2\right)
\]
\[
= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{i \in N} \sum_{i' \in N} w^*_is_{ij}w^*_is_{i'j}\right) \tau_{ii'} + 2z_j \sum_{i \in N} w^*_is_{ij}\tau_{i'} 
\]
\[
\leq 25\delta \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{i \in N} \sum_{i' \in N} w^*_is_{ij}w^*_is_{i'j}\right) + 16\delta \sum_{i \in N} w^*_i(\tilde{s})^2 
\]
\[
\leq 25\delta \frac{1}{m} \sum_{j=1}^{m} \sum_{i \in N} \sum_{i' \in N} w^*_i w^*_i(\tilde{s})^2 + 16\delta \sum_{i \in N} w^*_i(\tilde{s})^2 
\]
\[
\leq 50\delta(\tilde{s})^2,
\]
as required. \(\square\)

**Theorem 6.** When \(F\) has bounded support and \(z\) is bounded, \(D\)-TD is consistent as \(n \to \infty, m = \omega(\log(n))\) and \(n = \omega(\log m)\). That is, for any \(\tau > 0\), \(\Pr[d(\tilde{z}, z) > \tau] \to 0\) for all \(i \in N\), as \(n \to \infty, m = \omega(\log(n))\) and \(n = \omega(\log m)\).

**Proof.** By Theorem 3, we know that as \(m \to \infty\), \(\hat{f}_i\) is close to \(f_i\) (with small multiplicative error). More specifically, we showed that
\[
|\hat{f}_i - f_i| \leq T_3 + T_1^2 + 2T_1T_2 + 4T_1(\frac{m}{n})z_j/m).
\]
Equivalently,
\[
1 - G(T_1, T_2, T_3) \leq \frac{\hat{f}_i}{f_i} \leq 1 + G(T_1, T_2, T_3),
\]
where \(G(T_1, T_2, T_3) = (T_3 + T_1^2 + 2T_1T_2 + 4T_1z_{\text{max}})/f_{\text{min}}\).

Therefore, for any \(j \leq m\) and any \(G(T_1, T_2, T_3) < \frac{2}{3}\), we have
\[
\frac{\sum_{i=1}^{n} s_{ij}/\hat{f}_i}{\sum_{i=1}^{n} 1/\hat{f}_i} \leq \frac{\sum_{i=1}^{n} s_{ij}/f_i}{\sum_{i=1}^{n} 1/f_i} \times \frac{1 + G(T_1, T_2, T_3)}{1 - G(T_1, T_2, T_3)} \leq \frac{\sum_{i=1}^{n} s_{ij}/f_i}{\sum_{i=1}^{n} 1/f_i} (1 + 3G(T_1, T_2, T_3))
\]
and
\[
\frac{\sum_{i=1}^{n} s_{ij}/\hat{f}_i}{\sum_{i=1}^{n} 1/\hat{f}_i} \geq \frac{\sum_{i=1}^{n} s_{ij}/f_i}{\sum_{i=1}^{n} 1/f_i} \times \frac{1 - G(T_1, T_2, T_3)}{1 + G(T_1, T_2, T_3)} \leq \frac{\sum_{i=1}^{n} s_{ij}/f_i}{\sum_{i=1}^{n} 1/f_i} (1 - 3G(T_1, T_2, T_3))
\]
It follows that
\[
|\frac{\sum_{i=1}^{n} s_{ij}/\hat{f}_i}{\sum_{i=1}^{n} 1/\hat{f}_i} - \frac{\sum_{i=1}^{n} s_{ij}/f_i}{\sum_{i=1}^{n} 1/f_i}| \leq 3G(T_1, T_2, T_3) \frac{\sum_{i=1}^{n} s_{ij}/f_i}{\sum_{i=1}^{n} 1/f_i}
\]
Therefore, for any \(n\), if \(\frac{\sum_{i=1}^{n} s_{ij}/f_i}{\sum_{i=1}^{n} 1/f_i}\) is no more than \(\tau\) away from the ground truth with probability at least \(1 - \delta\), then \(\frac{\sum_{i=1}^{n} s_{ij}/\hat{f}_i}{\sum_{i=1}^{n} 1/\hat{f}_i}\) is no more than \((\tau + 3G(T_1, T_2, T_3)(z_{\text{max}} + \tau)) - 3G(T_1, T_2, T_3)\) away from the ground truth with probability at least \(1 - (\delta + m \exp\{-C_1T_1^2n\} + n \exp\{-C_2T_2^2m\} + n \exp\{-C_3T_3^2m\})\).

The consistency of \(\frac{\sum_{i=1}^{n} s_{ij}/\hat{f}_i}{\sum_{i=1}^{n} 1/\hat{f}_i}\) follows after the fact that \(\frac{\sum_{i=1}^{n} s_{ij}/f_i}{\sum_{i=1}^{n} 1/f_i}\) is consistent and by letting
\[
T_1 = \omega(1)\sqrt{\frac{\log m}{n}}, T_2 = T_3 = \omega(1)\sqrt{\frac{\log n}{m}}.
\]
\(\square\)
Fig. 8. In this figure we can see the performance P-TD (which equals D-TD) on several distributions from the INN model.

**B **MORE EMPIRICAL RESULTS FOR THE CONTINUOUS DOMAIN

**B.1 Datasets**

We used datasets from two sources. The Buildings dataset we collected via Amazon Turk, see Appendix D.1 for details.

The Triangles dataset\(^{10}\) is taken from a study of people’s geometric reasoning [18]. Participants in the study were shown the base of a triangle (two vertices and their angles) and were asked to position the third vertex. That is, their answer to each question is an x-coordinate and y-coordinate of a vertex. In our analysis we treat each of the coordinates as a separate question.

In Fig. 8 we can see that P-TD works substantially better than unweighted mean on nearly all datasets we tried.

**C PROOFS FOR THE CATEGORICAL DOMAIN**

**C.1 Fault Estimation**

**Theorem 7.** Suppose that instance \(I = \langle S, z \rangle\) is sampled from proto-population \(\mathcal{F}^r\) via the IER model. For every worker \(i\),

\[
E[\pi_i(I)|f_i] = \mu + (1 - (1 + \theta)\mu)f_i.
\]

**Proof.** Note first that for any two agents \(i, i'\) with fixed fault levels \(f_i, f_{i'}\), and any question \(j\):

\[
Pr[s_{ij} = s_{ij'}|f_i, f_{i'}] = Pr[s_{ij} = z_j, s_{ij'} = z_j|f_i, f_{i'}] + Pr[s_{ij} \neq z_j, s_{ij'} \neq z_j, s_{ij} = s_{ij'}|f_i, f_{i'}]
\]

\[
= Pr[s_{ij} = z_j|f_i] \cdot Pr[s_{ij'} = z_j|f_{i'}] + Pr[s_{ij} \neq z_j|f_i] \cdot Pr[s_{ij'} = z_j|f_{i'}] \cdot Pr[s_{ij} = s_{ij'}|s_{ij} \neq z_j, s_{ij'} \neq z_j]
\]

\[
= (1 - f_i)(1 - f_{i'}) + f_i f_{i'} \theta = 1 - f_i - f_{i'} + (1 + \theta)f_if_{i'}
\]

\(^{10}\)https://github.com/StatShapeGeometricReasoning/StatisticalShapeGeometricReasoning
Denote by \( d_{iv} \) the random variable \( d(s_i, s_{iv}) \). Thus,
\[
E[d_{iv'][f_i, f_{iv}]] = E\left[\frac{1}{m} \sum_{j \leq m} \mathbb{1}\{s_{ij} \neq s_{ij'}\} | f_i, f_{iv}\right] = \frac{1}{m} \sum_{j \leq m} \mathbb{P}(s_{ij} \neq s_{ij'}) | f_i, f_{iv}\]
\[
= \frac{1}{m} \sum_{j \leq m} \left(1 - \mathbb{P}(s_{ij} = s_{ij'}) | f_i, f_{iv}\right) = \frac{1}{m} \sum_{j \leq m} (f_i + f_{iv} - (1 + \theta)f_i f_{iv}) = f_i + f_{iv} - (1 + \theta)f_i f_{iv}.
\]

For a fixed population (i.e. taking expectation only over realized answers):
\[
E[\pi_i | f_i] = E\left[\frac{1}{n - 1} \sum_{i' \neq i} d_{iv}[f] \right] = \frac{1}{n - 1} E[d_{iv}[f_i, f_{iv}]] = \frac{1}{n - 1} \sum_{i' \neq i} (f_i + f_{iv} - (1 + \theta)f_i f_{iv})
\]
\[
= f_i + \frac{1}{n - 1} \sum_{i' \neq i} f_{iv} - (1 + \theta)f_i \frac{1}{n - 1} \sum_{i' \neq i} f_{iv} = f_i + (1 - (1 + \theta)f_i) \frac{1}{n - 1} \sum_{i' \neq i} f_{iv} \tag{8}
\]

Finally, taking expectation over sampled populations,
\[
E[\pi_i | f_i] = E_{f_{i-1} \sim \mathcal{F}^{n-1}} [E[\pi_i | f_i]] = E_{f_{i-1} \sim \mathcal{F}^{n-1}} \left[f_i + (1 - (1 + \theta)f_i) \frac{1}{n - 1} \sum_{i' \neq i} f_{iv} | f_i\right]
\]
\[
= f_i + (1 - (1 + \theta)f_i) E\left[\frac{1}{n - 1} \sum_{i' \neq i} f_{iv} | f_i\right] = f_i + (1 - (1 + \theta)f_i) \mu = \mu + (1 - (1 + \theta)\mu)f_i,
\]
as required. \(\square\)

**Theorem 8.** Suppose the support of \(\mathcal{F}\) is a closed subset of \((0, 1)\) and \(\mu_{\mathcal{F}} < \frac{k - 1}{k}\). Then \(\frac{1}{n}\)-P-EFL is consistent. That is, for any \(\delta > 0\), \(\mathbb{P}(|\hat{f}_i - f_i| > \delta) \to 0\) for all \(i \in \mathbb{N}\), as \(n \to \infty\) and \(m = \omega(\log n)\).

**Proof.** The proof proceeds by bounding the probability for the following four events as \(n \to \infty\) and \(m = \omega(n)\). We will show that by choosing the parameters in the following three events appropriately, the output of P-EFL converge to the ground truth fault levels \(\hat{f}_i\).

**Event A:** \(r^0(s) = z\). That is, the unweighted plurality rule reveals the ground truth.

**Event B:** Given \(T_1 > 0\), \(\hat{\mu}\) in Algorithm 5 is no more than \(T_1\) away from \(\mu_{\mathcal{F}}\).

**Event C:** Given \(T_2 > 0\), for all \(i \leq n\) and \(i' \leq n\) with \(i \neq i'\), \(d_{iv}\) is no more than \(T_2\) away from \(f_i + f_{iv} - \frac{k}{k - 1} f_i f_{iv}\).

We first show that conditioned on events A, B, C, and D holding simultaneously, the output of P-EFL is close to \(\hat{f}_i\). Because event C holds, for every \(i \leq n\), \(\pi_i\) is no more than \(T_2\) away from \(f_i + \frac{\sum_{i' \neq i} f_{iv}}{n - 1} - \frac{k}{k - 1} f_i \frac{\sum_{i' \neq i} f_{iv}}{n - 1}\), which is no more than \((1 + \frac{k}{k - 1})T_2 + \frac{1}{n - 1}\) away from \(f_i + \mu_{\mathcal{F}} - \frac{k}{k - 1} f_i \mu_{\mathcal{F}}\) because event B holds. Therefore, as \(T_1 \to 0\) and \(T_2 \to 0\), we have that for every \(i \leq n\), \(\hat{f}_i \to f_i\).

We now show that as \(n \to \infty\) and \(m = \omega(n)\), we can choose \(T_1(n)\) and \(T_2(n)\) as functions of \(n\) such that (1) \(T_1(n)\) and \(T_2(n)\) converge to 0, and (2) the probability for events A, B, and C hold simultaneously goes to 1.

First, as \(n \to \infty\), the probability of event A goes to 1. For any \(j \leq m\), the expected score difference between the ground truth and any given different alternative is \(\int_0^1 (1 - f) - \frac{1}{n - 1} \mathcal{F}(f) df = 1 - \frac{k}{k - 1} \mu_{\mathcal{F}} > 0\), where the last inequality follows after the assumption that \(\mu_{\mathcal{F}} < \frac{k - 1}{k}\). Due to the union bound, as \(n \to \infty\), the probability for the plurality rule to reveal the ground truth of all \(m\) questions goes to 1.

Second, given any ground truth \(z\) and any \(f, \sum_{i=1}^n d_H(s_i, z)/n\) can be seen as the average of \(mn\) independent Bernoulli trials: for each \(i \leq n\), there are \(m\) i.i.d. Bernoulli trials whose probability of
We note that the upper bounds go to $\infty$ as $n \to \infty$. By the union bound, as $n \to \infty$, with probability 1 event A holds and (9) and (10) do not hold, which imply that event B hold with probability 1.

Third, given any $i \neq i'$, $d_{ii'}$ can be seen as the average of $m$ i.i.d. random variables, each of which takes 1 with probability $1 - (1 - f_i)(1 - f_{i'}) - f_i f_{i'}/(k - 1)$ and takes 0 otherwise. Therefore, by Hoeffding’s inequality, we have:

$$\Pr(|\bar{d}_{ii'} - (f_i + f_{i'} - \frac{k}{k-1}f_i f_{i'})| > T_2) \leq 2e^{-2T_2^2m}$$

By the union bound, the probability for (11) to hold for any pairs of $i \neq i'$ is no more than $2n^2 e^{-2T_2^2m}$. Therefore, as $n \to \infty$ and $m = \omega(\log n)$, the probability for event C to hold goes to 0.

Finally, it follows that for any $\delta > 0$, we can choose sufficiently small $T_1$ and $T_2$ so that the estimate of the fault level of any agent by P-EFL is no more than $\delta$ away from the true level $f_i$ when events A, B, and C hold simultaneously, which happens with probability that goes to 1 as $n \to \infty$ and $m = \omega(\log n)$. This proves the theorem.

\[\square\]

\section{C.2 Aggregation}

\textbf{Algorithm 8: Aggregation Skeleton (Categorical)}

\textbf{Output:} Estimated true answers $\hat{z}$

$(\hat{z}_i)_{i \in N} \leftarrow \text{EstimateFaultLevels}(S)$;

\forall i \in N, set $w_i \leftarrow \log \left(\frac{(1-f_i)(k-1)}{f_i}\right)$;

Set $\hat{z} \leftarrow \tau^P(S, w)$;

\textbf{Theorem 15 (Consistency of Plurality with Grofman Weights).} Plurality with Grofman weights is consistent for IER with observed fault levels, if and only if $\mathcal{F}(\frac{k-1}{k}) \neq 1$.

\textbf{Proof.} W.l.o.g. let $m = 1$ and $z = 0$. Let $W^*$ (respectively, $W$) denote the random variable for weighted vote to alternative 0 (respectively, alternative 1) under IER with observed fault levels. In other words, for any $0 < f < 1$, conditioned on the fault level being $f$ (which happens with probability $\mathcal{F}(f)$), we have $W^* = \log \left(\frac{1-f}{f}\right)$ with probability $1 - f$, and $W^* = 0$ otherwise. For any $0 < f < 1$, conditioned on the fault level being $f$, we have $W = \log \left(\frac{1-f}{f}\right)$ with probability $\frac{f}{k-1}$, and $W = 0$ otherwise. Therefore, we have:

$$\mathbb{E}(W^* - W) = \int (1 - \frac{k}{k-1}f) \log \left(\frac{1-f}{f}\right) \frac{1-f}{f} d\mathcal{F}(f)$$

We note that $1 - \frac{k}{k-1}f$ and $\log \left(\frac{1-f}{f}\right)$ always have the same sign (positive, negative, or 0), and both of them are 0 if and only if $f = \frac{k-1}{k}$. Therefore, when $\mathcal{F}(\frac{k-1}{k}) < 1$, we have (12) > 0. Let $\hat{W}^* = \sum_{i=1}^{n} W^*_i / n$ and let $\hat{W} = \sum_{i=1}^{n} W^*_i / n$. By the Law of Large Numbers, we have $\lim_{n \to \infty} \Pr(\hat{W}^* - \hat{W} > 0) = 1$. This means that as $n$ increases, the total weighted votes of 0 is strictly larger than the
total weighted votes of 1 with probability close to 1. It follows from the union bound that as \( n \to \infty \), the probability for 0 to be the winner goes to 1, which proves the consistency of the algorithm.

When \( \mathcal{F}(\frac{k-1}{k}) = 1 \), with probability 1 the votes are uniformly distributed with the same weights regardless of the ground truth. Therefore the algorithm is not consistent. \( \square \)

**Theorem 10.** Suppose the support of \( \mathcal{F} \) is a closed subset of \((0, 1)\) and \( \mu_{\mathcal{F}} < \frac{k-1}{k} \). Then \( \frac{1}{n} \)-P-TD is consistent. That is, for any \( \tau > 0 \), \( \Pr[d(\hat{z}, z) > \tau] \to 0 \) as \( n \to \infty \) and \( m = \omega(\log n) \).

**Proof.** Because the support of \( \mathcal{F} \) is a closed subset of \((0, 1)\), the estimated Grofman weight is bounded. By Theorem 8, for any \( \tau > 0 \) and \( \delta > 0 \), there exists \( n^* \) such that for all \( n > n^* \) and \( m = \omega(\log n) \),

\[
\Pr(|\hat{w}_i - \omega_i^*| < \tau \text{ for all } i \leq n) > 1 - \delta,
\]

where \( \hat{w}_i \) and \( \omega_i^* \) are the estimated Grofman weight for agent \( i \) obtained by using the output of P-EFL, the true Grofman weight, respectively. Let \( \hat{w} = (\hat{w}_1, \ldots, \hat{w}_n) \) and \( \omega^* = (\omega_1^*, \ldots, \omega_n^*) \). For any alternative \( l \in A \), let \( \text{Score}(l, \hat{w}) \) denote the weighted score of alternative \( l \) with weights \( \hat{w} \).

This means that with probability at least \( 1 - \delta \), for each \( l \in A \), we have

\[
\frac{|\text{Score}(l, \hat{w}) - \text{Score}(l, \omega^*)|}{n} < \tau
\]

The consistency of PTD follows after the fact that \( r^P(\cdot, \omega^*) \) is consistent and the gap between the average score of the winner and the average score of any other alternative is bounded below by the constant in (12) (which is strictly positive because \( \mu_{\mathcal{F}} < \frac{k-1}{k} \)) with probability that goes to 1 as \( n \to \infty \). \( \square \)

**D MORE EMPIRICAL RESULTS FOR THE CATEGORIAL DOMAIN**

**D.1 Datasets**

In the datasets we collected, participants were given short instructions, then they had to answer \( m = 25 \) questions. We recruited participants through Amazon Mechanical Turk. We restricted participation to workers that had at least 50 assignments approved. We planted in each survey a simple question that can be easily answered by anyone who understand the instructions of the experiment (known as Gold Standards questions). Participants who answered correctly the gold standard question received a payment of $0.3. Participants did not receive bonuses for accuracy. The study protocol was approved by the Institutional Review Board at the Technion.

- **BinaryDots** Subjects were asked which picture had more dots (binary questions).
- **Buildings** Subjects were asked to mark the height of the building in the picture on a slide bar (continuous answers).

In addition, we used four datasets from [34], in those experiments workers had to identify an object in pictures. Workers got paid by the Double or Nothing payment scheme described in their paper.

- **GoldenGate** subjects were asked whether the golden gate bridge in the picture (binary questions). 35 workers, 21 questions.
- **Dogs** subjects were asked to identify the breed of the dog in the picture (10 possible answers). 31 workers, 85 questions. We omitted 4 workers due to missing data in their reports.
- **HeadsOfCountries** subjects were asked to identify heads of countries (4 possible answers). 32 workers, 20 questions. We omitted 3 workers due to missing data in their report. On this datasets almost all workers got perfect results, and thus we did not use it in our simulations.
Fig. 9. Performance of several methods for varying values of $\mu$ and fixed $\vartheta$. Notice that the scales of the vertical axis are different.

Fig. 10. Performance of several methods for varying values of $\vartheta$ and fixed $\mu$.

**Flags** subjects were asked to identify country to which shown flag belongs (4 possible answers). 35 workers, 126 questions.

In addition, we used two datasets from [27], in those experiments workers had to rank four alternatives. The two datasets consists 6400 rankings from 1300 unique workers. Workers got paid 0.1$ per ranking.

**Pictures of dots** subjects were asked to rank four pictures of dots, from fewest dots to most dots. This task has been suggested as a benchmark task for human computation in [20].

**Sliding puzzles** subjects were asked to rank four 8-puzzles by the least number of moves the puzzles are from the solution, from closest to furthest.

### D.2 Empirical comparison of methods

We compare all truth discovery methods mentioned in the paper on a variety of synthetic distributions: UA, P-TD, and D-TD, as well as IP-TD and ID-TD with 8 iterations each. We also include OA as a benchmark.

Unless explicitly stated otherwise, we use $u = 0$ in all versions of P-TD.\(^\text{11}\) Our bar plots include all of the above methods.

**Binary data.** The first distributions conform to the IER model. First, we considered additional Normal distributions, varying the mean $\mu$ and the variance $\vartheta$.

\(^\text{11}\)Using other values typically hurt performance on low $n$ and $m$, and otherwise has negligible effect.
As expected, the error of P-TD improves as $\mu$ decreases (Fig. 9), and as $\vartheta$ increases (Fig. 10). However the latter effect is much weaker.

We also considered other distributions of fault levels (still under the IER model), specifically Uniform distribution, Triangular distribution, and bi-modal distribution with 20% of “good” workers ($f_i = 0.2$), and 80% of “bad” workers ($f_i = 0.52$).

In general the results in the categorical domain are not as clear cut as in the continuous and ranking domains. Possibly because of the workers with negative (rather than just weak) influence.

**Item Response Theory.** Next, we considered distributions that exceed the IER model, allowing questions of different difficulty level. More specifically, for every worker $i$ we generated a parameter $a_i$ from either a normal or bi-modal distribution as above, and for every question $j$, we sampled its difficulty parameter $b_j$ from a Normal distribution. We then generated answers according to a (simplified) Item Response Theory model [17]: The probability that $i$ answered correctly question $j$ was set to $\frac{1}{k} + \frac{k-1}{k} \frac{1}{1+\exp(a_i-b_j)}$. Wrong answers were selected uniformly at random.

Somewhat surprisingly, the performance of P-TD is not very different that on instances that do conform to the IER model (Fig. 11). This suggests that independence among workers is not necessarily a key requirement. We do see however that performance is weak on bi-modal distributions (2 left figures) when there are few questions.

**Multiple choice questions.** We repeated the above simulations, with $k = 4$ answers per question. We also generated populations with $k = 8$ (with only 250 samples) and $k = 12$ answers (125 samples). It is hard to say anything conclusive about the regions where P-TD performs poorly, but there are very few areas where it does substantially worse than the other methods, and many areas where it is much better (Fig. 12).
Fig. 12. Further heatmaps of the P-TD performance, with categorial data from the IER noise model and multiple choice questions. CATXX specifies the number of categories (e.g. in the left most simulation there are $k = 12$ answers per question).

D.3 Iterative P-TD

Recall our (loose) theoretical prediction of the weighted proxy score from Section 4.4:

$$E[\pi_t^f|f] \equiv f_i + (1 - 2f_i)(\mu - \frac{\theta}{2} - \mu).$$

In Fig. 13 we compare this theoretical prediction to simulation results. There is a reasonable fit when $\theta$ is low or moderate (two left columns), but for higher $\theta$ the approximation breaks, we get a negative noise term, and the theoretical estimation does not make sense.

As the $\theta(F)$ becomes larger (fault levels are more dispersed), the noise in the weighted variant of the proxy score (Eq. (7)) becomes smaller, and the fault estimation becomes more accurate. Recall that higher dispersion of fault levels is also what allows Algorithm 8 to improve aggregation results (regardless of which estimation method is used).

E RANKING DOMAIN

E.1 Definitions of voting rules

All voting rules get as input a voting profile $S = (L_1, \ldots, L_n)$ and non-negative normalized weights $w$. We denote by $L_i(c)$ the rank of candidate $c$ by voter $i$ (lower is better).

For a pair of candidates $c, c'$, we denote: $c \prec_i c'$ if $L_i(c) < L_i(c')$; $v(c, c') := \sum_{i: L_i(c) < L_i(c')} w_i$ is the fraction of voters preferring $c$ to $c'$; and $y(c, c') := \lceil w(c, c') > 0.5 \rceil$ means that $c$ beats $c'$ in a pairwise match.

Note that $v$ and $y$ are vectors of length $|C|^2$. Recall that $x_L$ is the binary vector corresponding to rank $L$.

Every voting rule $r$ generates a score $q^r(c, S, w)$ for each candidate $c$, then ranks candidates by increasing score (breaking ties at random). Therefore we need to specify how each voting rule determines the score.

- **Borda** $q^r(c, S, w) = \sum_{i \in N} w_i (L_i(c) - 1) = \sum_{c' \neq c} w(c', c)$.
- **Copeland** $q^r(c, S, w) = \sum_{c' \neq c} y(c', c)$.
- **Kemeny** $q^r(c, S, w) = L^*(c)$, where $L^* = \arg\min_{L \in L} d_H(x_L, y)$.
- **Weighted Kemeny** $q^r(c, S, w) = L^*(c)$, where $L^* = \arg\min_{L \in L} d_{\ell_1}(x_L, v)$.
- **Plurality** $q^r(c, S, w) = n - \sum_{i \in N: L_i(c) = 1} w_i$.
- **Veto** $q^r(c, S, w) = \sum_{i \in N: L_i(c) = m} w_i$.
- **Best Dictator** $q^r(c, S, w) = L_{i^*}(c)$, where $i^* = \arg\max_{i \in N} w_i$. 

30
Fig. 13. The top row is as in Fig. 3. All distributions have \( \mu = 0.4 \) but right figures have higher variance of \( \mathcal{F} \). The bottom row shows the proxy score \( \pi^t_i \) after \( t = 8 \) iterations. The dashed line is \( f_i + (1 - 2f_i)(\mu - \frac{\varphi}{\xi - \mu}) \).

**Random Dictator** Similar to Best Dictator, except that \( i^* \) is selected at random, proportionally to its weight \( w_i \).

### E.2 Full description of algorithms

**ALGORITHM 9: Proxy-based Truth Discovery** (for rankings)

**Input:** Voting rule \( r \); Rankings \( S = (L_1, \ldots, L_n) \)

**Output:** Estimated true answers \( \hat{z} \)

Compute \( d_{ii'} \leftarrow d_{K_T}(L_i, L_{i'}) \) for every pair of workers;

for each \( i \in N \) do

- Set \( \pi_i \leftarrow \frac{1}{n-1} \sum_{i' \neq i} d_{ii'} \);
- Set \( \hat{f}_i \leftarrow \pi_i \);
- Set \( w_i \leftarrow \max \{0, \log \left( \frac{1 - \hat{f}_i}{\hat{f}_i} \right) \} \);

end

Set \( \hat{z} \leftarrow r(S, w) \);

### E.3 More empirical results in the Ranking Domain

We generated more instances from Mallows model, with different distributions of the parameter \( \phi_i \). Recall that \( \phi_i \) is not exactly the fault level, but it is related to \( \frac{1 - f_i}{f_i} \) as explained in the main text.

Fig. 14 shows heatmaps comparing P-TD to D-TD on a dataset with “good” and “bad” workers. As can be seen in Fig. 15, P-TD and IP-TD outperform other methods in most settings on datasets generated from different distributions.
Fig. 14. A comparison of P-TD and D-TD with different voting rules.

Fig. 15. The mean error of different methods when varying voting rules (columns) and the distributions from which datasets were generated (rows).
F  LIST OF PARAMETERS, ALGORITHMS, SYMBOLS AND ACRONYMS:

$P - EFL = \text{Proxy-based Estimation of Fault Levels (Algorithms 2 and 5)}$

$P - TD = \text{proxy-based truth discovery}$

$IP - EFL, IP - TD = \text{Iterative P-EFL and P-TD (Alg. 7)}$

$D - EFL = \text{Estimate-Fault-Levels-by-Distance-from-outcome (Algorithm 1)}$

$D - TD = \text{Distance-based truth discovery}$

$ID - TD = \text{Iterative D-TD (Alg. 6)}$

$UA = \text{Unweighted Aggregation}$

$OA = \text{Oracle Aggregation}$

$INN = \text{Independent normal noise, see Section 3}$

$IER = \text{Independent Error model, see Section 4}$

$f_i = \text{Fault level of worker } i$

$f_i^0 = \text{Estimate of fault when using parameter } \hat{\mu} = 0$

$r(S, w) = \text{Aggregate answer using voting rule } r, \text{ over opinion set } S \text{ with weights } w$

$\mathcal{F} = \text{The proto-population, a distribution over fault levels.}$

$\mu(\mathcal{F}) = \text{Mean fault of the population}$

$\sigma(\mathcal{F}) = \text{Fault’s variance over the population}$

$\pi_i = \text{Proxy distance}$

$z = \text{Ground truth}$

$\hat{z} = \text{Estimate of the ground truth}$

$y^0 = \text{Estimated of the ground truth by } r(S) \text{ (unweighted)}$

$u = \text{Input parameter in algorithm 3, } \hat{\mu} = u \cdot \sum_{i \in N} f_i^0$

$n = \text{number of workers}$

$m = \text{number of questions}$

$A = \text{set of answers in categorical questions}$

$k = \text{size of } A$

$w^* = \text{optimal weights}$

$\hat{w} = \text{estimated weights}$

$C = \text{set of candidates or alternatives to rank}$

$L_i = \text{Ranking of worker } i$. 