THE FIRST CHEEGER CONSTANT OF A SIMPLEX

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Abstract. The coboundary expansion generalizes the classical graph expansion to the case of the general simplicial complexes, and allows the definition of the higher-dimensional Cheeger constants $h_k(X)$ for an arbitrary simplicial complex $X$, and any $k \geq 0$. In this paper we investigate the value of $h_1(\Delta[n])$ - the first Cheeger constant of a simplex with $n$ vertices. It is known, due to the pioneering work of Meshulam and Wallach, [MW09], that

$$\left\lceil \frac{n}{3} \right\rceil \geq h_1(\Delta[n]) \geq \frac{n}{3},$$

for all $n$, and that the equality $h_1(\Delta[n]) = \frac{n}{3}$ is achieved when $n$ is divisible by 3.

Here we expand on these results. First, we show that

$$h_1(\Delta[n]) = \frac{n}{3},$$

whenever $n$ is not a power of 2. So the sharp equality holds on a set whose density goes to 1. Second, we show that

$$h_1(\Delta[n]) = \frac{n}{3} + O(1/n),$$

when $n$ is a power of 2. In other words, as $n$ goes to infinity, the value $h_1(\Delta[n]) - \frac{n}{3}$ is either 0 or goes to 0 very rapidly.

Our methods include recasting the original question in purely graph-theoretic language, followed by a detailed investigation of a specific graph family, the so-called staircase graphs. These are defined by associating a graph to every partition, and appear to be especially suited to gain information about the first Cheeger constant of a simplex.

1. Introduction

The graph expanders are classical and well-studied mathematical objects with many applications, see, e.g., the surveys [HLW06, Lu12]. More recently, there have been different definitions of higher-dimensional expanders, see [Lu14]. This paper is concerned with the so-called coboundary expanders, which first made their appearance in the paper by Linial and Meshulam, [LiM06], and which were later independently defined by Gromov, see [Gr10]. Until now, the major objective of the research on coboundary expansion has been to find asymptotically good expanders, see, e.g., [DK12, LuM15, LMM16], with computing the precise values of Cheeger constants playing the secondary role. In this paper we deviate from this approach.

More specifically, the work we present here has a twofold purpose. Primarily, we are focused on taking the first step in the general program of precise computation, or, at the very least, finding sharp bounds for the higher Cheeger constants of standard simplicial complexes. Currently, we do not even know the precise value of the Cheeger constants for a simplex. In this paper, we attempt to change that at least for the first Cheeger constant. To do that, we reformulate the original questions for expansion in purely graph-theoretical terms. Furthermore, in order to get the actual estimates, we need to perform an in-depth analysis of certain graph families.

Key words and phrases. simplicial complexes, cohomology, coboundary expanders, Cheeger constant.
Our second, more general purpose is to describe and to emphasize the deep connection between the question of estimating the higher Cheeger constants and questions in extremal graph and hypergraph theory. We hope that this way the questions about coboundary between the question of estimating the higher Cheeger constants and questions in expander may gain popularity and thus further progress on their understanding can be achieved. In the conclusion of the paper we formulate several explicit purely combinatorial conjectures.

Let us start by summarizing what is known about the Cheeger constants of a simplex with \( n \geq 3 \) vertices. First, a word about our notations. Usually writing \( \Delta^n \) is reserved for the simplex of dimension \( n \), that is the one having \( n + 1 \) vertices. On the other hand, for an arbitrary set \( V \) one uses the notation \( \Delta^V \) to denote the simplex whose set of vertices is \( V \). Since we also have the set notation \( [n] := \{1, \ldots, n\} \), we find it consistent to use \( \Delta^n \) to denote the simplex with \( n \) vertices. The Cheeger constants \( h_k(\Delta^n) \) are then defined for all \( 0 \leq k \leq n - 2 \), and so we are facing the task of determining the numbers \( h_0(\Delta^n), \ldots, h_{n-2}(\Delta^n) \).

The 0-th Cheeger constant is just the classical case and it is very easy to calculate that \( h_0(\Delta^n) = [(n + 1)/2] \), for all \( n \). On the other extreme, trivially one can see that \( h_{n-2}(\Delta^n) = 1 \), for all \( n \). Furthermore, it is not difficult to show, see Proposition 6.5, that \( h_{n-3}(\Delta^n) = 2 \), for all \( n \). In general, we know, due to the work of Meshulam and Wallach, see [MW09], that

\[
[n/k] \geq h_{k-2}(\Delta^n) \geq n/k,
\]

for all \( 3 \leq k \leq n \). Meshulam and Wallach also showed that the lower bound is achieved when \( k \) divides \( n \). On the other hand, we see that the upper bound is sharp when \( k = n - 1 \).

In this paper we are primarily concerned with the first Cheeger constant \( h_1(\Delta^n) \). In this case \( k = 3 \), and (1.1) specializes to \( [n/3] \geq h_1(\Delta^n) \geq n/3 \), for all \( n \geq 3 \), with equality \( h_1(\Delta^n) = n/3 \) attained, whenever \( n \) is divisible by 3. Enhancing that information, we actually show that \( h_1(\Delta^n) = n/3 \), for all \( n \), with a definite exception of the cases \( n = 4 \) and \( n = 8 \), and a probable exception of the case when \( n \) is equal to other powers of 2. Furthermore, even when \( n \) is a power of 2 we show that not only is \( h_1(\Delta^n) \) contained in the interval between \( n/3 \) and \( [n/3] \), but it actually converges to \( n/3 \) very rapidly. More specifically, we show that \( h_1(\Delta^n) = n/3 + O(1/n) \).

We finish this introductory chapter by describing briefly the plan of the paper. In Section 2 we recall the definition of the coboundary expansion and the Cheeger constants. We then show how the calculation of the first Cheeger constant can equivalently we formulated as a graph-theoretic question. Section 3 is the core of the paper. Here a family of graphs, which we call the staircase graphs is introduced and studied, computing all the information which is relevant for the coboundary expansion. In Section 4 we apply the results of the previous section, both to make precise calculation of the first Cheeger constant in the case \( n \) is not a power of 2, as well as to derive sharp bounds in the case \( n \) is a power of 2. We introduce the concept of a Cheeger graph and find several of them realized as staircase graphs. Finally, in Section 5 we state several open questions in extreme graph and hypergraph theory, which are motivated by the coboundary expansion. Section 6 is the Appendix containing loose ends, including the proof that \( h_{n-3}(\Delta^n) = 2 \), and recasting the coboundary computation of Wallach and Meshulam in the graph-theoretical language.
2. Setting up the board

2.1. The terminology of coboundary expanders.

Let $X$ be a finite simplicial complex. In this paper, we shall consider the associated chain and cochain complexes with $\mathbb{Z}_2$-coefficients only, so we will suppress $\mathbb{Z}_2$ from the notations, and simply write $C_*(X)$ and $C^*(X)$. Let now $\sigma$ be an arbitrary chain of $X$, say $\sigma = \sigma_1 + \cdots + \sigma_d$, where $\sigma_i$ are generators indexed by the simplices of $X$, for $1 \leq i \leq d$, and $\sigma_i \neq \sigma_j$, whenever $i \neq j$. We set $||\sigma|| := d$ and call this the norm of $\sigma$. Dually, assume we have a cochain $c \in C^*(X)$, such that $c = c_1 + \cdots + c_d$, where $c_i$'s are generators indexed by the distinct simplices of $X$; each $c_i$ is the characteristic function of a $k$-simplex $\sigma_i$. Then, we set $||c|| := d$, which we also call the norm of $c$.

**Definition 2.1.** For an arbitrary $k$-chain $\sigma$ we consider $\min_\tau ||\sigma + \partial \tau||$, where the minimum is taken over all $(k+1)$-chains $\tau$. We call that number the systolic norm of $\sigma$, and denote it by $||\sigma||_{\text{sys}}$. Furthermore, a systolic form of $\sigma$ is any $\tilde{\sigma} = \sigma + \partial \tau$, such that $||\tilde{\sigma}|| = ||\sigma||_{\text{sys}}$. We let $\text{sys}(\sigma)$ denote the set of all systolic forms of $\sigma$. A chain is called a systole if $||\sigma|| = ||\sigma||_{\text{sys}}$.

Dually, assume $c$ is a $k$-cochain. We call $\min_d ||c + \partial^* d||$, where the minimum is taken over all $(k-1)$-cochains the cosystolic norm of $c$, and denote it by $||c||_{\text{csy}}$. Let $\text{csy}(c)$ denote the set of all cosystolic forms of $c$. A cosystolic form of $c$ is any $\tilde{c} = c + \partial^* d$, such that $||\tilde{c}|| = ||c||_{\text{csy}}$. A chain is called a cosystole if $||c|| = ||c||_{\text{csy}}$.

The cosystolic norm of a chain can be quite difficult to compute in general.

**Definition 2.2.** Assume we are given a simplicial complex $X$. For any $k$-cochain $c$ of $X$, which is not a coboundary, the **coboundary expansion** of $c$ is

$$||c||_{\text{exp}} := ||\partial c||/||c||_{\text{csy}}.$$  

The $k$-th **Cheeger constant** of $X$ is then

$$h_k(X) := \min_{c \in \text{csy}} ||c||_{\text{exp}}.$$  

Clearly, in (2.1) we might as well restrict ourselves to cosystoles, when taking the minimum. Finally, when $c$ is a cosystole such that $||c||_{\text{exp}} = h_k(X)$, then we shall call $c$ a **Cheeger cosystole**.

2.2. Simplicial complex of cut-minimal graphs.

Let us now introduce some graph terminology in order to give an alternative definition of the first Cheeger constant. We shall use the notation $G = (V, E)$, meaning that the graph $G$ has the set of vertices $V$ and the set of edges $E$. For any two, not necessarily disjoint, subsets $A, B \subseteq V$, we set $E(A, B) := \{|(v, w) \in E | v \in A, w \in B\}$, and $NE(A, B) := \{|(v, w) \notin E | v \in A, w \in B\}$, so $E(A, B) + NE(A, B) = |A| \cdot |B|$.

**Definition 2.3.** A graph $G = ([n], E)$ is called **cut-minimal** if for any proper subset $S \subseteq [n]$ we have

$$|E(S, [n] \setminus S)| \leq |NE(S, [n] \setminus S)|.$$  

In other words, at most half of the $|S|(n - |S|)$ potential edges connecting vertices from $S$ to vertices from $[n] \setminus S$ belong to $G$.

We call the cut $(S, [n] \setminus S)$ **perfect** if equality is achieved in (2.2).

In particular, the valencies of vertices of a cut-minimal graph with $n$ vertices can be at most $(n - 1)/2$. All the graphs which are shown on Figure 4.1 are cut-minimal. The way we think about the condition (2.2) is as follows. Imagine we are given a graph $G$ and we
are allowed to split the vertex set \([n]\) into two parts: \(S\) and \([n]\ \setminus S\). We take all the potential edges between these two parts, and think of them as a cut \(C\). We are now allowed change \(G\) by inverting the being the edge of \(G\) relationship within \(C\). In other words, we obtain a new graph by keeping all the edges in \(G\) which are outside of \(C\), removing all the edges of \(G\) which are in \(C\) and adding as edges all the non-edges of \(G\) which are in \(C\). The graph is then cut-minimal if no such operation can decrease the number of edges of \(G\); which explains our choice of terminology.

Note that removing some edges from a cut-minimal graph will certainly yield a cut-minimal graph again. Following the general ideology of combinatorial topology, see [Ko07], this observation leads to a definition of a natural combinatorial simplicial complex.

**Definition 2.4.** Let us fix \(n \geq 2\). The abstract simplicial complex \(CM(n)\) is defined as follows:

- the vertices are indexed by unordered pairs \(\{i, j\}\), \(i, j \in [n], i \neq j\);
- the set of vertices forms a simplex of \(CM(n)\) if and only if the corresponding graph is cut-minimal.

We see that \(CM(2)\) is empty, \(CM(3)\) has 3 vertices and no edges, and the complex \(CM(4)\) has 6 vertices and 3 disjoint edges. The complex \(CM(5)\) is more interesting. It has dimension 3 and its \(f\)-vector is \((10, 45, 100, 10)\). In particular, \(CM(5)\) has 10 vertices and a full 1-skeleton. It can be obtained from a full 2-skeleton by deleting 20 triangles and adding 10 tetrahedra. Its maximal simplices are these 10 tetrahedra, together with 60 triangles. It can be shown by direct inspection, using a combination of techniques from [Ko07], that \(CM(5)\) is homotopy equivalent to a wedge of 54 spheres of dimension 2.

In general, clearly \(CM(n)\) has \(\binom{n}{2}\) vertices, for all \(n \geq 3\). Furthermore, it is non-pure for all \(n \geq 5\). It would be interesting to understand more the simplicial structure or topology of these complexes. For example, the dimension of \(CM(n)\) is obtained by subtracting 1 from the maximal number of edges which a cut-minimal graph may have. This number has been computed precisely in the upcoming work [KR17].

### 2.3. The graph-theoretic definition of the first Cheeger constant.

Assume now we are given a simplicial complex \(X\). Its 1-skeleton \(G := X(1)\) is a graph, whose set of vertices is \(V := X(0)\) and whose set of edges is \(E := X(1)\). Here we follow very handy notations of Linial and Meshulam, [LiM06], by letting \(X(k)\) denote the set of all \(k\)-simplices of \(X\).

The edges of this graph \(G\) are in 1-to-1 correspondence with the generators of the group of 1-cochains \(C^1(X)\): associate to each edge \(e\) its characteristic cochain which evaluates to 1 on \(e\) and to 0 on all other edges. For simplicity we identify each edge with the associated characteristic cochain. Since we are working over \(\mathbb{Z}_2\), the arbitrary cochains can be identified with the sets of edges of \(X\), or, which is the same, with the subgraphs of \(G\).

In the same way, the vertices of \(G\) are in 1-to-1 correspondence with the generating 0-cochains of \(X\), and sets of vertices of \(G\) are in 1-to-1 correspondence with arbitrary 0-cochains. Taking the coboundary has graph-theoretic translation too. Given an arbitrary 0-cochain \(c\) corresponding to a set of vertices \(S\), its coboundary is the 1-cochain which corresponds to the edge set \(E(S, V \setminus S)\). The norm of the 0-cochain is \(|S|\), and the norm of the 1-cochain is \(|E(S, V \setminus S)|\).

**Proposition 2.5.** The correspondence above restricts to a 1-to-1 correspondence between the sets of cosystoles and cut-minimal graphs.
We have \( \sum_{(2.4)} \) \( T \) \( G \) from angles which contain an odd number of edges from \( G \) that the boxes are left-justified, the first row is of length \( \lambda \) \( c \) is taken over all cosystoles. (2.6) can alternatively be phrased as follows. Let \( T \) \( G \) denote the set of all “triangles” which contain an odd number of edges from \( G \), i.e.,

\[
T(G) := \{ (v, e) \mid v \in V, e = (w, u) \in E, v \notin e, \{(v, w), (v, u), (u, w)\} \cap E \text{ is odd} \}
\]

We have \( \sum_{e \in E} t(e) = |T(G)| \). This is because, by (2.3), if a triangle from \( T(G) \) has one edge from \( G \), then this edge gives a contribution \( 1 \) to the sum \( \sum_{e \in E} t(e) \), and if a triangle from \( T(G) \) has three edges from \( G \), then each of these edges gives a contribution \( 1/3 \) to that sum.

We therefore have the alternative formula

\[
(2.5) \quad |E| \cdot h(G) = |T(G)|.
\]

We are now ready to give a graph-theoretic description of the first Cheeger constant of a simplex.

**Proposition 2.7.** For any \( n \geq 3 \) we have

\[
h_1(\Delta^n) = \min_G h(G),
\]

where the minimum is taken over all cut-minimal graphs \( G \) with \( n \) vertices.

**Proof.** By definition, the constant \( h_1(\Delta^n) \) is equal to \( \min_c \|\partial^* c\|/\|c\| \), where the minimum is taken over all cosystoles \( c \). As mentioned above, being a cosystole precisely corresponds to cut-minimal graphs, and computing the value \( h(G) \) is exactly the same as computing \( \min_c \|\partial^* c\|/\|c\| \). □

It is rather straightforward to extend this description to the first Cheeger constant of an arbitrary simplicial complex.

**3. Staircase graphs**

**3.1. Terminology of partitions.** A partition \( \lambda \) is any ordered tuple of positive integers \( (\lambda_1, \ldots, \lambda_t) \), such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \). In such a case, we always set the default values \( \lambda_q := 0 \), for all \( q > t \). The Ferrers diagram of a partition \( \lambda = (\lambda_1, \ldots, \lambda_t) \) is the arrangement of square boxes in \( t \) rows, such that the boxes are left-justified, the first row is of length \( \lambda_1 \), the second row is of length \( \lambda_2 \), and so on. When referring to the individual boxes in the diagram, we shall count both rows and columns starting with 1, counting rows from top to bottom and counting columns from left to right.

To abbreviate our writing, we shall use the power notation for the diagram, i.e., using formal powers to denote multiple parts of the same cardinality, for example: \( (3^2, 2^3, 1) = \)
(3, 3, 2, 2, 1). For \( \lambda = (\lambda_1, \ldots, \lambda_t) \), we set \(|\lambda| := \sum_{k=1}^{t} \lambda_k \). We also set box(\( \lambda \)) := \( \lambda_1 + t \), say box(3\( ^{(2)} \), 2\( ^{(3)} \), 1) = 9.

Given a partition \( \lambda = (\lambda_1, \ldots, \lambda_t) \), a \textit{conjugate partition} \( \lambda^* = (\mu_1, \ldots, \mu_m) \) is defined as follows: we set \( m := \lambda_1 \), and for every \( 1 \leq k \leq m \), we set \( \mu_k \) to be equal to the maximal index \( i \) such that \( \lambda_i \geq k \). In particular of course \( \mu_1 = t \). In terms of the Ferrers diagram we just switch rows and columns of \( \lambda \). For an arbitrary partition \( \lambda \), we have \((\lambda^*)^* = \lambda \), \(|\lambda^*| = |\lambda| \), and box(\( \lambda \)) = box(\( \lambda^* \)). As an example, we have \((3\( ^{(2)} \), 2\( ^{(3)} \), 1)\) = (6, 5, 2).

For an arbitrary \( t \geq 1 \), we let \( \text{cor}(t) \) denote the partition \((t, t-1, \ldots, 2, 1)\). We clearly have box(\( \text{cor}(t) \)) = \( 2t \), \( |\text{cor}(t)| = (t+1)/2 \), and \( \text{cor}(t)^* = \text{cor}(t) \).

**Definition 3.1.** Let \( \lambda = (\lambda_1, \ldots, \lambda_t) \) be an arbitrary partition. The \textit{depth} of \( \lambda \), denoted depth(\( \lambda \)), is the maximal number \( d \) such that the Ferrers diagram of \( \text{cor}(d) \) is contained in the Ferrers diagram of \( \lambda \).

Alternatively, the depth of \( \lambda \) can be described as the unique value \( d \), such that

1. \( \lambda_1 \geq d, \lambda_2 \geq d - 1, \ldots, \lambda_k \geq d - k + 1, \ldots, \lambda_t \geq 1 \).
2. There exists \( 1 \leq k \leq d + 1 \), such that \( \lambda_k = d - k + 1 \).

A convenient way to think about depth(\( \lambda \)) is to notice that it is equal to the minimal number of rows and columns which will cover the entire Ferrers diagram of \( \lambda \), or, expressed algebraically, we have

\[
\text{depth}(\lambda) = \min_{0 \leq k \leq l} (k + \lambda_{k+1}),
\]

where we use the convention \( \lambda_{t+1} = 0 \).

Of course, we have depth(\( \text{cor}(t) \)) = \( t \).

3.2. **The definition of staircase graphs.**
The following family of graphs is central to our approach.

**Definition 3.2.** Assume we are given a partition \( \lambda \) and an integer \( n \), such that \( n \geq \text{box}(\lambda) \). The \textit{staircase graph} \( G_n(\lambda_1, \ldots, \lambda_t) = G_n(\lambda) \) is defined as follows:

- The set of vertices of \( G_n(\lambda) \) is a disjoint union \( V \cup W \cup U \), where \( V = \{v_1, \ldots, v_l\} \), with \( l = \lambda_1 \), \( W = \{w_1, \ldots, w_r\} \), and \( U = \{u_1, \ldots, u_s\} \), with \( r = n - l - t \);
- For each \( 1 \leq i \leq l \), \( 1 \leq j \leq t \), the vertices \( v_i \) and \( w_j \) are connected by an edge if \( \lambda_j \geq i \); all other pairs of vertices are not connected by an edge.

In particular, we see that \( G_n(\lambda) \) is always bipartite, \( V \) and \( W \) can be taken as two sides of the bipartition, and vertices of \( U \) are isolated. Note that \( l, t \neq 0 \), whereas \( r \) might be 0; this will happen if \( n = \text{box}(\lambda) \). Clearly, we also have \( |V(G_n(\lambda))| = n \), and \( |E(G_n(\lambda))| = |\lambda| \). Figure 3.1 shows a staircase graph, several further examples can be found on Figure 4.1. Finally, note that \( G_n(\lambda) \) is isomorphic to \( G_n(\lambda^*) \).

3.3. **Structure theory of staircase graphs.**
Assume we are given a partition \( \lambda = (\lambda_1, \ldots, \lambda_t) \), and \( 1 \leq k \leq t \), and \( 1 \leq m \leq \lambda_1 \).

**Definition 3.3.** For arbitrary index sets \( I \subseteq [t] \) and \( J \subseteq [\lambda_1] \), such that \(|I| = k \), and \(|J| = m \), let \( B_{\lambda}(I, J) \) denote the total number of boxes in the Ferrers diagram of \( \lambda \), which are either contained in one of the \( k \) rows indexed by \( I \), or contained in one of the \( m \) columns indexed by \( J \), but not in both.

If we consider a cut of \( G_n(\lambda) \) with the vertices corresponding to the row and column indices from \( I \cup J \), then \( B_{\lambda}(I, J) \) is precisely the number of edges across that cut.
Definition 3.4. Assume we are given a partition $\lambda$, and an integer $n \geq \text{box}(\lambda)$. Assume $\lambda^* = (\mu_1, \ldots, \mu_q)$. The partition $\lambda$ is called legal with respect to $n$ if the following three conditions are satisfied:

1. for all $1 \leq k \leq t$, we have $B_\lambda([k], \emptyset) = \sum_{j=1}^k A_j \leq k(n-k)/2$;
2. for all $1 \leq m \leq q$, we have $B_\lambda(\emptyset, [m]) = \sum_{j=1}^m \mu_j \leq m(n-m)/2$;
3. $|\lambda| \leq \text{depth}(\lambda)(n - \text{depth}(\lambda))/2$.

Let us say a few words on the intuition behind Definition 3.4. Conditions (1) and (2) make sure that the cut-minimality holds when we cut off the vertices corresponding to the first $k$ rows or the first $m$ columns. Condition (3) is rather concerned with the cuts where we choose $k$ first rows and $\lambda_{k+1}$ first columns. This condition could actually be strengthened to require that

$$|\lambda| - k\lambda_{k+1} \leq (k + \lambda_{k+1})(n - k - \lambda_{k+1})/2,$$

for all $k = 1, \ldots, t - 1$. We do not need this strengthening here and find it technically simpler to work with the condition in Definition 3.4.

It turns out that legality of a partition has the following strong implication.

Lemma 3.5. Assume $n$ is a natural number, and a partition $\lambda$ is legal with respect to $n$. Choose arbitrary index sets $I \subseteq [t]$ and $J \subseteq [q]$, such that $|I| + |J| \leq n/2$, and set $k := |I|, m := |J|$. We have

$$B_\lambda(I, J) \leq (k + m)(n - k - m)/2.$$  

Proof. Without loss of generality we can shift all the rows upwards and all the columns to the left. If after this they cover the entire Ferrers diagram of $\lambda$, then $B_\lambda(I, J) \leq |\lambda|$ and $k + m \geq \text{depth}(\lambda)$. We then get

$$B_\lambda(I, J) \leq \text{depth}(\lambda)(n - \text{depth}(\lambda))/2 \leq (k + m)(n - k - m)/2,$$

where the first inequality is given by condition (3) of Definition 3.4, and the second inequality follows from the fact that $\text{depth}(\lambda) \leq k + m \leq n/2$.

If, on the other hand, the Ferrers diagram is not covered completely, we have

$$B_\lambda(I, J) = (\lambda_1 + \cdots + \lambda_k) + (\mu_1 + \cdots + \mu_m) - 2km$$

$$\leq k(n-k)/2 + m(n-m)/2 - 2km$$

$$= (k+m)(n-k-m)/2 - km \leq (k+m)(n-k-m)/2,$$

where the first inequality follows from conditions (1) and (2) of Definition 3.4. □

Clearly, if a partition $\lambda$ is legal with respect to some $n$, and $n' \geq n$, then $\lambda$ is also legal with respect to $n'$. This observation motivates the following definition.
Lemma 3.8. If partition $t$, so, since $\text{cor}(t)$, we set $\text{deficiency}$ which we call the $\text{cut minimal}$. For example, one can compute that $N_{3,3,1} = 8$, and $N_{6,5,2} = 13$. Of course, we have $N_\lambda = N_{\lambda^*}$. Given $\lambda = (\lambda_1, \ldots, \lambda_k)$, we set
\[
N_\lambda(t) := \max_{1 \leq k \leq t} \left( k + \left\lceil \frac{2(\lambda_1 + \cdots + \lambda_k)}{t} \right\rceil \right),
\]
\[
N_\lambda := \text{depth}(\lambda) + \left\lceil \frac{2|\lambda|}{\text{depth}(\lambda)} \right\rceil.
\]
The following lemma gives us a precise formula for computing $N(\lambda)$.

Lemma 3.7. For an arbitrary partition $\lambda$, we have
\[
N(\lambda) = \max \{ N_\lambda(\lambda), N_\lambda(\lambda^*), N_\lambda \}.
\]

Proof. Simply rewrite the inequalities of Definition 3.4. As an example, for any $t \geq 1$, we get
\[
N_\lambda(t) = t + \left\lceil \frac{2(t + 1)}{t} \right\rceil = t + (t + 1) = 2t + 1.
\]
\[
N_\lambda = \text{max}_{1 \leq k \leq t} \left( k + \left\lceil \frac{2(t + \cdots + (t - k + 1))}{k} \right\rceil \right) = \text{max}(k + (2t - k + 1)) = 2t + 1,
\]
so, since $\text{cor}(t) = \text{cor}(t)^*$, we conclude that $N(\text{cor}(t)) = 2t + 1$.

Lemma 3.8. If partition $\lambda$ is legal with respect to some number $n$, then the graph $G_n(\lambda)$ is cut minimal.

Proof. Cutting the set $[n]$ into the subsets $S$ and $[n] \setminus S$, such that $|S| \leq n/2$, is the same as choosing subsets $I$ and $J$, with $k = |I|$ and $m = |J|$, such that $n - \text{box}(\lambda) \geq |S| - k - m \geq 0$. Then $B_\lambda(I, J)$ is the number of edges across the cut, and we have $(k + m)(n - k - m)/2 \leq |S|(n - |S|)/2$, so (3.1) implies the cut-minimality.

Definition 3.9. For any $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda^* = (\mu_1, \ldots, \mu_m)$, we set
\[
|\lambda^2| := \frac{1}{2} \left( \sum_{i=1}^{k} \lambda_i^2 + \sum_{j=1}^{m} \mu_j^2 \right),
\]
and furthermore, we set
\[
h(\lambda) := N(\lambda) - \frac{2|\lambda^2|}{|\lambda|}.
\]

We remark that our notation $|\lambda^2|$ is the special case of $|\lambda^p| := \left( \sum_{i=1}^{k} \lambda_i^p + \sum_{j=1}^{m} \mu_j^p \right)/2$, which for $p = 1$ also gives our notion $|\lambda|$. For future reference, for an arbitrary partition $\lambda$, we set
\[
\text{def}(\lambda) := h(\lambda) - N(\lambda)/3 = \frac{2}{3} \left( N(\lambda) - \frac{3|\lambda^2|}{|\lambda|} \right),
\]
which we call the deficiency of $\lambda$.

Taking $\lambda = \text{cor}(t)$ as a specific example, we can see
\[
|\text{cor}(t)^2| = t^2 + \cdots + t^2 = t(t + 1)(2t + 1)/6,
\]
which for $t = 3, 5, 7, 8$ gives the numbers $8, 13, 20, 25$, respectively.
which implies
\[ h(\text{cor}(t)) = 2t + 1 - 2 \cdot \frac{t(t+1)(2t+1)}{6} = 2t + 1 - \frac{2}{3}(2t+1) = \frac{2t+1}{3}, \]
and hence \( \text{def}(\text{cor}(t)) = 0. \)

**Lemma 3.10.** We have \( h(\lambda) = h(G_{N(\lambda)}(\lambda)) \).

**Proof.** In our notations, we have
\[ t(e_{i,j}) = (N(\lambda) - \lambda_i - \mu_j) + (\mu_1 - \mu_j) + (\lambda - \lambda_i) - N(\lambda) - \lambda_i - \mu_j, \]
where we recall Definition 2.6. Therefore
\[ (3.2) \quad h(G_{N(\lambda)}(\lambda)) = \frac{1}{|\lambda|} \sum_{i,j} t(e_{i,j}) = N(\lambda) - \frac{1}{|\lambda|} \sum_{i,j} (\lambda_i + \mu_j) = h(\lambda), \]
where the sum is taken over all \( i \) and \( j \), which correspond to boxes in the Ferrers diagram of \( \lambda \). Note, that the last equality follows from the fact that \( \sum_{i,j} \lambda_i = \sum_i \lambda_i^2 \) (this is because for each \( i \) the number of summands \( \lambda_i \) on the left hand side is equal to the number of boxes in the \( i \)th row of the Ferrers diagram of \( \lambda \), and this number is of course precisely \( \lambda_i \)), and analogously \( \sum_{i,j} \mu_j = \sum_j \mu_j^2 \). □

3.4. **The partition \( c\lambda \).**
The staircase partitions can be blown up using the following simple operation.

**Definition 3.11.** Given a partition \( \lambda = (\lambda_1, \ldots, \lambda_t) \), and a natural number \( c \), we set
\[ c\lambda := (c\lambda_1, \ldots, c\lambda_1, \ldots, c\lambda_t, \ldots, c\lambda_t) = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{ct}), \]
where \( \tilde{\lambda}_q = \frac{c\lambda_q}{c\lambda_q/c\lambda_q} \) for all \( 1 \leq q \leq ct \).

The next lemma relates the data associated to the partition \( c\lambda \) to the data associated to the partition \( \lambda \).

**Lemma 3.12.** For an arbitrary partition \( \lambda \) and an arbitrary natural number \( c \) we have the following equalities:
\[ (3.3) \quad \text{depth}(c\lambda) = c \cdot \text{depth}(\lambda), \]
\[ (3.4) \quad |c\lambda| = c^2 |\lambda|, \]
\[ (3.5) \quad |(c\lambda)^2| = c^3 \cdot |\lambda|^2 \]
\[ (3.6) \quad N_d(c\lambda) = c \cdot N_d(\lambda) \]
and inequalities
\[ (3.7) \quad N_d(c\lambda) \leq c \cdot N_d(\lambda), \]
\[ (3.8) \quad N(c\lambda) \leq c \cdot N(\lambda) \]
\[ (3.9) \quad h(c\lambda) \leq c \cdot h(\lambda) \]
Proof. We start by showing (3.3). For brevity, set \( d := \text{depth}(\lambda) \), and let us show that \( cd \) satisfies conditions (1) and (2) in the Definition 3.1 for \( c\lambda = (\lambda_1, \ldots, \lambda_{c}) \). First, for all \( 1 \leq \lambda_{q} \leq cd \), we see that
\[
\lambda_{q} = c\lambda_{q/c} \geq c(d - \lfloor q/c \rfloor + 1) \geq cd - (q + c - 1) + c = cd - q + 1,
\]
where we used the inequality \( c \cdot \lfloor q/c \rfloor \leq q + c - 1 \). This verifies condition (1). Second, assume \( \lambda_{k} = d - k + 1 \), for some \( 1 \leq k \leq d + 1 \). Then, \( 1 \leq c(k - c + 1) \leq dc + 1 \), and we have
\[
\lambda_{ck-c+1} = c\lambda_{k} = cd - c(k - c + 1) + 1,
\]
which verifies condition (2).

The equalities (3.4), (3.5), and (3.6), are direct computations which we leave to the reader.

Let us show the inequality (3.7). We have
\[
N_{r}(c\lambda) = \max_{1 \leq k \leq t} \left( k + \left\lfloor \frac{2(\lambda_{1} + \cdots + \lambda_{k})}{k} \right\rfloor \right).
\]
To start with, for \( k = cm \), for \( 1 \leq m \leq t \), we have
\[
ck + \left\lfloor \frac{2(\lambda_{1} + \cdots + \lambda_{cm})}{cm} \right\rfloor = cm + \left\lfloor \frac{2(\lambda_{1} + \cdots + \lambda_{m})}{m} \right\rfloor \leq cm + c \left\lfloor \frac{2(\lambda_{1} + \cdots + \lambda_{m})}{m} \right\rfloor \leq cN_{r}(\lambda),
\]
where the penultimate inequality follows from the fact that \( \lfloor cx \rfloor \leq \lfloor x \rfloor \), whenever \( c \) is an integer.

Next, consider the special case \( 1 \leq k \leq c - 1 \). Then, we have
\[
k + \left\lfloor \frac{2(\lambda_{1} + \cdots + \lambda_{k})}{k} \right\rfloor = k + \left\lfloor \frac{2kc\lambda_{1}}{k} \right\rfloor = k + 2c\lambda_{1} < c(1 + 2\lambda_{1}) \leq c \cdot N(\lambda).
\]
Assume now that \( k = cp + r \), where \( 1 \leq p \leq t - 1 \), \( 1 \leq r \leq c - 1 \). Set \( \tilde{r} := r/c \), so \( 0 < \tilde{r} < 1 \). We have
\[
\tilde{r} + \left\lfloor \frac{2s + \tilde{r}\lambda_{p+1}}{p + \tilde{r}} \right\rfloor \leq \tilde{r} \left( p + \left\lfloor \frac{2s}{p} \right\rfloor + \tilde{r} \left( p + 1 + \left\lfloor \frac{2s}{p + 1} \right\rfloor \right) \right).
\]
Set \( s := \lambda_{1} + \cdots + \lambda_{p} \), and note that \( \lambda_{p+1} < s/p \). We now claim that
\[
\tilde{r} \left( p + \left\lfloor \frac{2s}{p} \right\rfloor + \tilde{r} \left( p + 1 + \left\lfloor \frac{2s}{p + 1} \right\rfloor \right) \right) \leq (1 - \tilde{r}) \left( p + \left\lfloor \frac{2s}{p} \right\rfloor + \tilde{r} \left( p + 1 + \left\lfloor \frac{2s}{p + 1} \right\rfloor \right) \right).
\]
Clearly this would finish our proof of (3.7), since the right hand side of (3.10) is bound above by \( (1 - \tilde{r})N_{r}(\lambda) + \tilde{r}N_{r}(\lambda) = N_{r}(\lambda) \). On the other hand, the inequality (3.10) is equivalent to
\[
\left\lfloor \frac{2s + \tilde{r}\lambda_{p+1}}{p + \tilde{r}} \right\rfloor \leq (1 - \tilde{r}) \left( \frac{2s}{p} \right) + \tilde{r} \left( \frac{2s + \lambda_{p+1}}{p + 1} \right).
\]
A direct calculation, using the fact that \( \lambda_{p+1} < s/p \) shows that
\[
\frac{2s + \tilde{r}\lambda_{p+1}}{p + \tilde{r}} \leq (1 - \tilde{r}) \frac{2s}{p} + \tilde{r} \left( \frac{2s + \lambda_{p+1}}{p + 1} \right).
We now apply \([-\cdot]\) to both sides of (3.12), and use the fact that \([x + y] \leq [x] + [y]\) to verify (3.11).

Finally, the inequalities (3.8) and (3.9) are both obtained by a direct substitution. 

**Corollary 3.13.** Assume that we are given a partition \(\lambda\), such that \(N(\lambda) = N_d(\lambda)\), then we actually have the equalities \(N(c\lambda) = c \cdot N(\lambda)\), and \(h(c\lambda) = c \cdot h(\lambda)\).

**Proof.** We have
\[
c \cdot N(\lambda) \geq N(c\lambda) \geq N_d(c\lambda) = c \cdot N_d(\lambda) = c \cdot N(\lambda),
\]
where the first inequality is (3.8), the second inequality is the definition of \(N(-)\), the penultimate equality is (3.6), and the last equality is the assumption of the corollary. This shows that \(N(c\lambda) = c \cdot N(\lambda)\), and the equality \(h(c\lambda) = c \cdot h(\lambda)\) is an immediate consequence. 

4. **Applications of staircase graphs**

4.1. **Exact value of the first Cheeger constant for the simplex whose number of vertices is not a power of 2.**

Let us set \(h(n) := h_1(\Delta^{[n]}) = \min G h(G)\), for \(n \geq 3\), where the minimum is taken over all cut-minimal graphs \(G\) with \(n\) vertices.

**Theorem 4.1.** (Meshulam-Wallach bound, [MW09])

For any \(n\) we have
\[
\left\lceil \frac{n}{3} \right\rceil \geq h(n) \geq \frac{n}{3}.
\]

In particular, if \(3\) divides \(n\) then we have \(h(n) = \frac{n}{3}\).

We provide a short write-up of the proof of the lower bounds of Theorem 4.1 using our notations in subsection 6.3. Extending this result, our next theorem shows that the lower bound of Theorem 4.1 is true for the vast majority of the values of \(n\). We will give two proofs of the following theorem, one here using a direct computation, and one in the appendix as Corollary 6.4.

Theorem 4.2. Assume \(n = c(2t + 1)\), such that \(t \geq 1\), then \(h(n) = \frac{n}{3}\).

**Proof.** Consider the partition \(\lambda := c \cdot \text{cor}(t)\). Lemma 3.12 implies that \(|\lambda| = c^2 t(t + 1)/2\), \(\text{depth}(\lambda) = ct\), and \(N_d(\lambda) = c(2t + 1) = n\). Since \(N_d(\text{cor}(t)) = 2t + 1 = N(\text{cor}(t))\), Corollary 3.13 implies that \(N(\lambda) = n\) and \(h(\lambda) = n/3\).

Let us now consider the corresponding staircase graph \(G = G_n(\lambda)\). By Lemma 3.10, we have \(h(G) = h(\lambda) = n/3\). It follows immediately from Theorem 4.1 that \(h(G) = h(n) = n/3\). 

**Corollary 4.3.** If \(n \neq 2^k\), then we have \(h(n) = n/3\).

**Proof.** If \(n\) is not a power of 2, then it can be written as \(n = c(2t + 1)\), where \(t \geq 1\).

The following terminology seems natural, allowing us to talk about the graphs that are optimal with respect to the first Cheeger constant.

**Definition 4.4.** A graph \(G\) with \(n\) vertices is called a Cheeger graph if \(h(G) = h(n)\).

We have already found a family of Cheeger graphs.

**Corollary 4.5.** Assume \(c\) and \(t\) are arbitrary natural numbers. Set \(\lambda := c \cdot \text{cor}(t)\), and recall that \(N(\lambda) = c(2t + 1)\). Then the graph \(G_{c(2t+1)}(\lambda)\) is a Cheeger graph with \(c(2t + 1)\) vertices.
Proof. This follows directly from the proof of Theorem 4.2.

Note, how the special case \( t = 1 \) yields graphs considered in the previous work of Meshulam and Wallach. In general, it would be interesting to describe the set of all Cheeger graphs.

4.2. Bounds for the first Cheeger constant for the simplex whose number of vertices is a power of 2.

Since \( n \geq 3 \), the first relevant power of 2 is \( 2^2 = 4 \), in which case we have \( h(4) = 2 \). Furthermore, specific examples, and in case \( n = 8 \), exhaustive case analysis show that

\[
\begin{align*}
h(8) - \frac{8}{3} &= \frac{4}{21} \approx 0.19, \\
h(16) - \frac{16}{3} &\leq \frac{8}{93} \approx 0.086, \\
h(32) - \frac{32}{3} &\leq \frac{16}{381} \approx 0.042.
\end{align*}
\]

In general we have the following upper bound.

Theorem 4.6. Assume \( n = 2^d \), for some \( d \geq 3 \), then we have

\[
(4.1) \quad h(n) - n/3 \leq \frac{4n}{3(n^2 - 8)} = \frac{4}{3n} + \frac{32}{3n(n^2 - 8)} = \frac{1}{3}(4n^{-1} + 32n^{-3} + 256n^{-5} + \ldots).
\]

Proof. Set \( t := n/4 \geq 2 \). Set

\[ \lambda := ((2t - 1)^{(2)}, (2t - 3)^{(2)}, \ldots, 3^{(2)}, 1). \]

Note that the conjugate partition is given by

\[ \lambda^* = (2t - 1, (2t - 2)^{(2)}, \ldots, 2^{(2)}). \]
We have
\[|\lambda| = |\lambda^*| = 2 \sum_{k=1}^{t} (2k - 1) - 1 = 2t^2 - 1,\]
and
\[|\lambda^3| = 2 \sum_{k=1}^{2t-1} k^2 + (2t - 1)^2 - 1 = \frac{1}{3}t(8t^2 - 5).\]
A direct check shows that \(N_d(\lambda) = N_r(\lambda) = N_r(\lambda^*) = N(\lambda) = 4t = n.\) Hence the staircase graph \(G_{N(\lambda)}(\lambda) = G_n(\lambda)\) is cut-minimal.

Substituting the obtained values into the formula for \(h(\lambda),\) we obtain the following calculation:
\[h(\lambda) = 4t - \frac{2t(8t^2 - 5)}{3(2t^2 - 1)} - \frac{2t(4t^2 - 1)}{3(2t^2 - 1)},\]
We conclude that
\[\text{def}(\lambda) = h(\lambda) - n/3 = \frac{2t}{3(2t^2 - 1)} = \frac{4n}{3(n^2 - 8)} = \frac{4}{3n} + \frac{32}{3n(n^2 - 8)},\]
showing the inequality (4.1). \(\square\)

The next corollary is immediate.

**Corollary 4.7.** We have \(\lim_{n \to \infty} (h(n) - n/3) = 0.\)

5. **Conjectures and open problems**

We know that the following Conjecture 5.1 is true for \(\alpha = 1, 2,\) and 3.

**Conjecture 5.1.** We have \(h(n) > n/3,\) for all \(n = 2^\alpha, \alpha \geq 1.\)

In all of our examples, the constant \(h(G)\) for optimal graphs \(G\) never had a contribution coming from a triangle with all 3 edges in \(G,\) in other words, the second line of (2.3) was invoked. We conjecture that this holds in general.

**Conjecture 5.2.** All Cheeger graphs are triangle-free.

We actually believe that a stronger statement is true.

**Conjecture 5.3.** All Cheeger graphs are bipartite.

The next conjecture is very daring, and would clearly imply Conjectures 5.2 and 5.3.

**Conjecture 5.4.** All Cheeger graphs except for \(G_2(1) \sqcup G_2(1)\) can be represented as staircase graphs.

We finish this section with two open problems, which are probably rather hard, but which might help to stimulate further research.

**Open problem 5.5.** Classify all Cheeger graphs, for \(n \geq 9.\)

**Open problem 5.6.** Determine the topology of the simplicial complexes \(CM(n),\) for \(n \geq 6.\)

We mention, that recently, see [Me16], the asymptotics of these simplicial complexes, and, more generally, of the simplicial complexes of \(k\)-cosystoles, has been understood.
This humble section contains some facts and elementary proofs which we feel would be useful to fix in writing for future reference.

6. Appendix

6.1. Blowing up the graphs.

Let us now generalize the blowing of partitions, which we did in subsection 3.4, to blowing up arbitrary graphs.

Definition 6.1. Let $G = (V, E)$ be an arbitrary graph, and let $c$ be any natural number. We let $cG = (\bar{V}, \bar{E})$ to be the graph defined as follows:

- we set $\bar{V} := V \times [c]$;
- for any $v, w \in V$, and $i, j \in [c]$, we have $((v, i), (w, j)) \in \bar{E}$ if and only if $(v, w) \in E$.

In particular, we have $|cV| = c|V(G)|$ and $|E(cG)| = c^2|E(G)|$. To connect this to our partition notations, we note that for any partition $\lambda$ we have $cG_n(\lambda) = G_{cn}(\lambda)$.

Proposition 6.2. For an arbitrary graph $G$ and any natural number $c$, we have

$h(cG) = c \cdot h(G)$.

Proof. It follows directly from the definition in (2.4), that $T(cG) = c^3 \cdot T(G)$. Hence (2.5) implies that

$h(cG) = \frac{T(cG)}{E(cG)} = \frac{c^3 \cdot |T(G)|}{c^2 \cdot |E(G)|} = c \cdot h(G)$.

\[\square\]

The next theorem is the main result of this subsection.

Theorem 6.3. Assume $G = (V, E)$ is a cut-minimal graph, and $c$ is an arbitrary natural number. Then, the graph $cG$ is also cut-minimal.

Proof. Before proceeding with a formal argument, we would like to give an informal idea of how the proof goes. If $cG$ is not cut-minimal then it must have a “bad” cut. This cut cannot nicely go around the blown up vertices, as in these cuts the number of edges simply changes proportionally and the original graph was cut-minimal. So the bad cut must cut at least one of the blown up vertices. Now, shifting vertices between the two parts of the cut within the blown up vertices changes the number of edges which cross the cut linearly, while the total number of potential edges in the cut changes along a concave function. This means that one of these changes will yield a bad cut again, and so eventually we will get a bad cut which does not cut any of the blown up vertices, leading to a contradiction.

Let us now make this argument rigorous. For simplicity of notations, we set $cV := V(cG)$ and $cE := E(cG)$. Let us take an arbitrary proper subset $S \subset cV$. Assume first that $S = T \times [c]$, for some proper subset $T \subset V$. Then

$|E(S, cV \setminus S)| = c^2 \cdot |E(T, V \setminus T)| \leq c^2 \cdot |NE(T, V \setminus T)| = |NE(S, cV \setminus S)|$,

where the sets of edges and non-edges are always taken in the appropriate graphs. This verifies the condition (2.2) for the set $S$ and graph $cG$.

Assume that $cG$ is not cut-minimal. It follows from the previous paragraph that we can pick $S \subset cV$, such that the condition (2.2) is not satisfied, and there does not exist any proper subset $T \subset V$, such that $S = T \times [c]$. This means that we can pick $v \in V$, such that both sets $A := S \cap ((v) \times [c])$ and $B := (cV \setminus S) \cap ((v) \times [c])$ are non-empty. Clearly $(v) \times [c]$ is a disjoint union of $A$ and $B$. 

Set \( S^- := S \setminus A \) and \( S^+ := S \cup B \). We shall use a concavity argument to show that condition (2.2) is not satisfied for at least one of the sets \( S^- \) and \( S^+ \). For the short-hand notations we set \( s := |S|, a := |A|, b := |B|, n := |V|, s^+ := |S^+| = s + b, s^- := |S^-| = s - a, e := E(S, cV \setminus S), e^+ := E(S^+, cV \setminus S^+), \) and \( e^- := E(S^-, cV \setminus S^-) \). Note that \( a + b = c \).

Let \( w \) be any vertex from \([v] \times [c] \), and set \( \beta := |E(w, S)| \), and \( \gamma := |E(w, cV \setminus S)| \). Note that these numbers do not depend on the choice of \( w \). Also \( \beta + \gamma = c \cdot \text{val}(v) \), but we will not need that. Note that moving such a vertex \( w \) between \( A \) and \( B \) changes the number of the edges of the graph which cross the cut by \( \beta - \gamma \), so we have
\[
(6.1) \quad e^- = e + a(\beta - \gamma) \quad \text{and} \quad e^+ = e + b(\gamma - \beta),
\]
which yields
\[
(6.2) \quad ae^+ + be^- = ce.
\]

Assume both \( S^+ \) and \( S^- \) satisfy condition (2.2). This means that \( e^+ \leq s^+(cn - s^+)/2 \) and \( e^- \leq s^-(cn - s^-)/2 \). Combining these with (6.2) we get
\[
(6.3) \quad \frac{a}{a + b} s^+(cn - s^+) + \frac{b}{a + b} s^-(cn - s^-) \geq e.
\]
The function \( f(x) = x(cn - x) \) is concave, which means that
\[
\frac{a}{a + b} f(s + b) + \frac{b}{a + b} f(s - a) \leq f \left( \frac{a}{a + b} (s + b) + \frac{b}{a + b} (s - a) \right) = f(s).
\]
This translates to
\[
\frac{a}{a + b} s^+(cn - s^+) + \frac{b}{a + b} s^-(cn - s^-) \leq s(cn - s),
\]
which together with (6.3) contradicts the fact that condition (2.2) is not satisfied for \( S \).

Repeating this argument we can modify \( S \) until it has a form \( T \times [c] \), while the condition (2.2) is still not satisfied. This clearly contradicts the first paragraph of this proof, so we are done.

We can now derive a generalization of (3.9) as a simple corollary of Theorem 6.3.

**Corollary 6.4.** For any \( c \geq 1 \), and any \( n \geq 3 \), we have
\[
(6.4) \quad h(cn) \leq c \cdot h(n).
\]

**Proof.** Take any Cheeger graph \( G \) with \( n \) vertices. We have \( h(G) = h(n) \). The graph \( cG \) has \( cn \) vertices, and, by Theorem 6.3 it is cut-minimal. It follows that \( h(cG) \geq h(cn) \). On the other hand, by Proposition 6.2, we have \( h(cG) = c \cdot h(G) = c \cdot h(n) \), hence (6.4) follows. \( \square \)

Note, that (6.4) implies that if \( h(n) = n/3 \), then \( h(cn) = cn/3 \), for all natural numbers \( c \). This yields another, and simple proof of Theorem 4.2, since we can limit ourselves to the analysis of the staircase graphs associated to \( \text{cor}(t) \), which, in turn, is rather straightforward.

### 6.2. Computing the penultimate Cheeger constant of a simplex.

As promised, we now provide a simple argument for precise computation of \( h_{n-3}(\Delta^n) \). In this case, the upper Meshulam-Wallach bound is realized.

**Proposition 6.5.** For any \( n \geq 3 \), we have \( h_{n-3}(\Delta^n) = 2 \).

**Proof.** By definition, we have \( h_{n-3}(\Delta^n) = \min \| \partial f \| ||e||_{\text{key}} \), where the minimum is taken over all \( c \in C^{n-3}(\Delta^n) \), \( c \neq 0 \cdot f \). The group of cochains \( C^{n-3}(\Delta^n) \) is generated by characteristic cochains of simplices of codimension 2, i.e., simplices with \( n - 2 \) vertices. For all
Comparing with (2.4), we immediately see that \( c_{kl} \) evaluates to 1 on that simplex and it evaluate to 0 on all other \((n-3)\)-simplices. We have \( c_{kl} = c_{lk} \). Each cochain \( c \in C^{n-3}(\Delta[n]) \) has a unique presentation as a sum \( c_{k_1l_1} + \cdots + c_{k_nl_n} \), such that \( \{k_i, l_i\} \neq \{k_j, l_j\} \), for all \( i \neq j \).

For \( i = 1, \ldots, n \), let \( d_i \in C^{n-2}(\Delta[n]) \) denote the characteristic \((n-2)\)-cochain of the \((n-2)\)-simplex \([n] \setminus \{i\}\). We clearly have

\[
\partial^* c_{kl} = d_k + d_l.
\]

We have a bijection between the generators \( c_{kl} \) and edges of a complete graph on \( n \) vertices \( K_n \). If we extend this bijection to the one between \( d_i's \) and vertices of \( K_n \), then the coboundary equation (6.5) translates to taking the boundary of an edge in that graph. Note, that (6.5) means that for all \( c \in C^{n-3}(\Delta[n]) \), not just the characteristic ones, we know that \( \|\partial^* c\| \) must be even, since all the cancellations happen in pairs. The fact that \( H^{n-3}(\Delta[n]) = 0 \), implies that if \( c \) is not a coboundary, then it is not a cocycle, i.e., \( \partial^* c \neq 0 \), so \( \|\partial^* c\| \geq 2 \).

Finally, let \( f_{klm} \in C^{n-4}(\Delta[n]) \) denote the characteristic cochain of the simplex \([n] \setminus \{k, l, m\}\), for all \( 1 \leq k, l, m \leq n, k \neq l \neq m \). We clearly have

\[
\partial^* f_{klm} = c_{kl} + c_{km} + c_{lm}.
\]

Let us now pick an arbitrary non-zero cosystole \( c \in C^{n-3}(\Delta[n]) \), and write \( c = c_{k_1l_1} + \cdots + c_{k_nl_n} \), with \( \{k_i, l_i\} \neq \{k_j, l_j\} \), for all \( i \neq j \). Assume that not all the numbers in the set \( \{k_1, \ldots, k_i, l_1, \ldots, l_i\} \) are distinct. Then, without loss of generality, we can assume that \( k = k_1 = k_2 \). The equation (6.6) implies that \( c_{kl} + c_{kl} + c_{kl} \) is a coboundary. Adding this expression to \( c \) would decrease the norm, which contradicts the fact that we picked \( c \) to be a cosystole. Thus, we can assume that all the numbers in the set \( \{k_1, \ldots, k_i, l_1, \ldots, l_i\} \) are distinct. Since \( \partial^* c = d_{a_1} + \cdots + d_{a_n} + d_{b_1} + \cdots + d_{b_n} \), we conclude that \( \|\partial^* c\| = 2t \), and hence \( \|c\|_{\text{exp}} = 2t/t = 2 \).

On the other hand, non-zero \((n-3)\)-dimensional cosystoles clearly exist. For example, \( c_{12} \) is such a cosystole. Indeed, \( \partial^* c_{12} = d_1 + d_2 \neq 0 \), so \( c_{12} \) is not a coboundary, i.e., \( \|c_{12}\|_{\text{sy}} \geq 1 \). On the other hand 1 = \( \|c_{12}\| \geq \|c_{12}\|_{\text{sy}} \), hence \( \|c_{12}\|_{\text{sy}} = 1 \).

We conclude that in dimension \( n-3 \), all non-zero cosystoles are in fact Cheeger cosystoles, and that \( h_{n-3}(\Delta[n]) = 2 \).

6.3. The proof of the lower bound in Meshulam-Wallach theorem using our notations.

We restrict ourselves to proving the lower bound from Theorem 4.1, since the upper bound is improved by other results in this paper. The argument below follows closely the ideas of the original coboundary computation by Meshulam and Wallach, [MW09]. Still we find it instructive, and potentially useful, to phrase it in our elementary language.

Proof of the lower bound in Theorem 4.1. Assume we are given a cut-minimal graph \( G = (V, E) \), such that \( |V| = n \). Let \( M \) denote the set of all ordered pairs \((v, e)\), where \( v \in V \), and \( e = (w, u) \in E \), such that \( v \notin e \), and the number of edges of \( G \) among \((v, w)\), \((v, u)\), and \((w, u)\) is odd, in other words, set

\[
M := \{(v, e) \mid v \in V, e = (w, u) \in E, v \notin e, \{(v, w), (v, u), (u, w)\} \cap E \text{ is odd}\}.
\]

Comparing with (2.4), we immediately see that \(|M| = 3 \cdot |T(G)|\), hence (2.5) implies that

\[
|M| = 3 \cdot |E| \cdot h(G).
\]

On the other hand, for a fixed \( v \in V \), let \( M_v \) denote the set of edges \( e \), such that \((v, e) \in M\); clearly \(|M| = \sum_{v \in V(G)} |M_v|\). We now show that \(|M_v| \geq |E|\), for all \( v \in V \). Set \( A := \{w \in \)
\( V \setminus \{(v, w) \in E\} \) and \( B := \{w \in V \mid (v, w) \notin E\} \), see Figure 6.1. If \( A = \emptyset \), then \( M_v = E(B, B) = E \), so we might as well assume \( A \neq \emptyset \), and \( (A, V \setminus A) \) is a proper cut. We have

\[
M_v = E(A, A) \sqcup E(B, B) \sqcup NE(A, B),
\]

which is to be compared with

\[
E = E(A, A) \sqcup E(V \setminus A, V \setminus A) \sqcup E(A, V \setminus A) = E(A, A) \sqcup E(B, B) \sqcup E(A, V \setminus A).
\]

By definition of \( A \), we have \( NE(A, V \setminus A) \neq NE(A, B) \). Hence, the cut-minimality condition (2.2), when applied to the cut \((A, V \setminus A)\), yields \( E(A, V \setminus A) \leq NE(A, B) \). Together with (6.8) and (6.9), this proves \(|M_v| \geq |E|\), for all \( v \in V \). Since \(|M| = \sum_{v \in V(G)} |M_v|\), we get \(|M| \geq n \cdot |E|\). Together with (6.7) this yields \( h(G) \geq n/3 \), for all cut-minimal graphs \( G \), and hence \( h(n) \geq n/3 \). \( \square \)

\[ \tag{6.8} \]

Figure 6.1. Proof of the lower Meshulam-Wallach bound.

Note, that one can also see from our proof of the lower bound in Theorem 4.1, that the sharp bound \( h(n) = n/3 \) is achieved by a cut-minimal graph \( G \) if and only if for every non-isolated vertex \( v \) the corresponding cut \((A, V \setminus A)\) is perfect. This observation gives us a quick-and-dirty argument for the strict inequality \( h(8) > 8/3 \). Indeed, the size of \( A \) is a valency of \( v \), so if \( v \) is not isolated, it is equal to 1, 2, or 3, as \( G \) is cut-minimal. If \((A, V \setminus A)\) is a perfect cut, then \(|A|\) must be even, otherwise \(|A| \cdot (8 - |A|)\) would have been odd. This means that \(|A| = 2\), and all non-isolated vertices of \( G \) have valency 2. The graph \( G \) is a disjoint union of isolated vertices and cycles, and \( h(G) \geq n - 4 \) for such graphs. Here this means \( h(G) \geq 4 > 8/3 \).

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