**Moyal-Weyl deformations of DGA and DGCA**

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**Abstract** We consider a natural variant of the Moyal-Weyl product and show that it yields a functorial deformation of differential graded algebras and coalgebras.

**Introduction**

In the last decades deformation theories appeared in many branches of mathematics and physics. The original idea of deformation seems quite hard to trace back, but some of the original literature surely are [5], [1], [2], [11] and finally the impressive proof of deformation quantization [9].

In quantization procedures it is common to denote the formal deformation parameter by \( \hbar \) in honour of Max Planck. Of course in physics \( \hbar \approx 6.58211928 \times 10^{-16} \text{eV} \cdot \text{s} \) is not really a variable, neither a constant that we are able to determine exactly in experiments, therefore we expect quantizations to be rigid with respect to fluctuations of \( \hbar \). Notice that the dimension of \( \hbar \) is \( \text{[mass]} \cdot \text{[length]}^2 \;/\; \text{[time]} \) and inverse to the dimension of the ordinary Poisson bracket on the phase space \( T(R^d) \) we will soon describe in formula (2).

In the deformation quantization of a Poisson manifold \((M, \Pi)\) the definition of a \( \star \) product is central: A \( \star \) product is a \( C[\hbar] \)-bilinear associative operation \( \star : C^\infty(M)[[\hbar]] \times C^\infty(M)[[\hbar]] \rightarrow C^\infty(M)[[\hbar]] \) of the shape \( f \star g = \sum_{n=0}^\infty \hbar^n B_n(f, g) \) with bi-differential operators \( B_n \) and with the properties \( 1 \star f = f = f \star 1, B_0(f, g) = fg \) and \( B_1(f, g) = i\{f, g\} \) where \( \{\cdot, \cdot\} \) is the Poisson bracket. Two star products on \((M, \Pi)\) are equivalent if there exists a formal series \( S = \text{id} + \sum_{l=1}^\infty \hbar^l S_l \) of \( C[[\hbar]] \)-linear operators \( S_l : C^\infty(M)[[\hbar]] \rightarrow C^\infty(M)[[\hbar]] \) with \( S_l(1) = 0 \) for \( l \geq 1 \) and

\[
 f \star g = S^{-1}(Sf \star Sg) \forall f, g \in C^\infty(M)[[\hbar]] \tag{1}
\]

For the ordinary Poisson bracket on the phase space \( T(R^d) \) given by

\[
 \{f, g\} = \sum_{k=1}^d \left( \frac{\partial f}{\partial q^k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q^k} \right) \tag{2}
\]

the standard \( \star \) product, named Weyl-Moyal product, is the binary operation defined by

\[
 f \star g := \mu \circ \exp \left( -i\hbar \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial q^k} \right) f \otimes g
\]

where \( \mu \) is the multiplication [7]. The general Moyal-Weyl product formula for \((\mathcal{A}, \mu)\) an associative algebra and \( D^i, D_i : \mathcal{A} \rightarrow \mathcal{A} \) commuting derivations is

\[
 \mu \circ \exp \left( -i\hbar \sum_{i=1}^n D^i \otimes D_i \right) \tag{3}
\]
where we use the Koszul sign conventions
\[(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)\]
\[(f \otimes g) \circ (f' \otimes g') = (-1)^{|g'||f|} f \circ f' \otimes g \circ g'\]
and this formula defines a deformed associative product [8], for a detailed proof of the associativity of [8] we refer to [13]. This product is an important cornerstone in the Fedosov deformation quantization of symplectic manifolds, see [10] for a construction of Fedosov $\star$ products on Kähler manifolds of constant sectional curvature.

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1 Functorial deformation of differential graded algebras

This paper is inspired by deformation quantization, but considers a different input data. The deformation of the exterior algebra is one of many natural examples that appear in the mathematical literature and physics where one can apply the following proposition [4]. The deformation [4] is in some sense just the simplest incarnation of the Moyal-Weyl product [3] where we choose $n = 1$ and $D_i = D_i = d$ where $d$ squares to zero:

Proposition 1.0.1. Let $(A, \wedge)$ be a graded algebra over $\mathbb{C}$ and $d$ with $d^2 = 0$ be an odd degree super derivation. The following formula defines a deformed associative product

\[a_1 \wedge^d a_2 = a_1 \wedge a_2 + i\hbar(-1)^{|a_1|} da_1 \wedge da_2\]  

Proof. The proof only uses that $d$ squares to zero and is an odd degree super derivation of the graded product $\wedge$, for instance we assume the graded Leibniz compatibility

\[d(a_1 \wedge a_2) = da_1 \wedge a_2 + (-1)^{|a_1|} a_1 \wedge da_2\]

The detailed proof of the associativity goes as follows

\[a \wedge^d (b \wedge^d c) = a \wedge^d \left[b \wedge c + i\hbar(-1)^{|b|} db \wedge dc\right] = a \wedge b \wedge c + i\hbar \left[(-1)^{|a|} da \wedge d(b \wedge c) + (-1)^{|b|} a \wedge db \wedge dc\right] = a \wedge b \wedge c + \frac{(-1)^{|a|} da \wedge db \wedge c + (-1)^{|a|+|b|} da \wedge b \wedge dc + (-1)^{|b|} a \wedge db \wedge dc}{1/i\hbar} = a \wedge b \wedge c + i\hbar \left[(-1)^{|a|} da \wedge db \wedge c + (-1)^{|b|} da \wedge b \wedge dc\right] = \left[a \wedge b + i\hbar(-1)^{|a|} da \wedge db\right] \wedge^d c = (a \wedge^d b) \wedge^d c\]
Remarks

- With the Koszul sign convention we could also rewrite the deformation $i\hbar (-1)^{|a_1|} da_1 \land da_2$ only depends on $a_1$ and $a_2$ modulo closed terms. We can also write for example $a_1 \land^d a_2 = a_1 \land a_2 + i\hbar (-1)^{|a_1|} d(a_1 \land da_2)$ or the more symmetric version $a_1 \land^d a_2 = a_1 \land a_2 + i\hbar d(-d a_1 \land a_2 + (-1)^{|a_1|} a_1 \land da_2)$. The meaning of this rewriting is that the added deformation term is exact.

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- Clearly the product $\land^d$ is not graded in the sense of the $\land$ grading, but it is graded product if we as usual give the formal parameter $\hbar$ degree $-2$. Notice that this quantization is a first order deformation, hence it has no convergence problems or issues with physical dimensions of the deformation parameter $\hbar$.

- As for the wedge product the deformation $\land^d$ still inherits the property that a $\land^d$ product of two bosons $\in \Omega^{2\bullet}(M[[\hbar]])$ as well as the $\land^d$ product of two fermions $\in \Omega^{2\bullet+1}(M[[\hbar]])$ are “bosonic” sums and the $\land^d$ product of a boson with a fermion is a “fermionic” sum, here we have copied the boson-fermion language from Witten’s article on super-symmetry and Morse theory [13].

In the following we will assume that the deformation parameter $\hbar$ and differential $d$ are real, i.e. $\overline{\hbar} = \hbar$ and $\overline{d a} = d \overline{a}$. The “Pauli exclusion principle” $a_1 \land a_2 = (-1)^{|a_1||a_2|} a_2 \land a_1$ does no longer hold for $\land^d$ and we just have the “weak Pauli exclusion principle” $a_1 \land^d a_2 = (-1)^{|a_1||a_2|} a_2 \land^d a_1$. Two fermions in this model can a priori non trivially be in the same state $f$ and create the exact boson $f \land^d f = (\hbar/\mathrm{id}) (f \land d f)$. Besides this difference also for the deformed product the formulas $b \land^d f = f \land^d b, f_1 \land^d f_2 = -f_2 \land^d f_1$ and $b_1 \land^d b_2 = b_2 \land^d b_1$ hold, i.e. the bosons $(\Omega^{2\bullet}(M)[[\hbar]], \land^d)$ form a *-algebra over $\mathbb{C}$.

For the bosons there is another proof of the associativity of $\land^d$. With help of the invertible maps $S := \mathrm{id} + \sqrt{\hbar/\mathrm{id}}$ and $S^{-1} = \mathrm{id} - \sqrt{\hbar/\mathrm{id}}$ it is easy to verify that we have an associative product by the equivalence formula

$$S^{-1} (Sa_1 \land Sa_2) = a_1 \land a_2 + \sqrt{\hbar/\mathrm{id}} (1 - (-1)^{|a_1|}) a_1 \land da_2 + i\hbar (-1)^{|a_1|} da_1 \land da_2$$

- As usual introducing the anti-commuting coordinates $V^i = dx^i$ we can interpret the term $da \land db = V^i \partial_i a \land \partial_j b$ as a super-Poisson bracket, we thank D. Roytenberg for pointing out this interpretation.
Proposition 1.0.2. By \( i.e. \) degree respecting, compatible hence some of the previous observations can be summarized in a functorial statement:

Consider the category with objects differential graded algebras \((A, \wedge, d)[[\hbar]]\) over \(\mathbb{C}\) and with arrows structure compatible maps \(\varphi : (A, \wedge, d_A)[[\hbar]] \to (B, \wedge, d_B)[[\hbar]]\) \(i.e.\) degree respecting, compatible \(\mathbb{C}\)-linear morphisms.

**Proposition 1.0.2.** By \( D(A, \wedge, d_A)[[\hbar]] = (A, \wedge_{d_A}, d_A)[[\hbar]]\) and \( D(\varphi) = \varphi \) we have defined a faithful functor \( D : DGA[[\hbar]] \to DGA[[\hbar]]\). We can also consider \(\mathfrak{M}\) as a faithful functor \( D : DGA \to DGA[[\hbar]]\).

1.1 Example: Moyal-Weyl deformation of the category of complexes

We discuss an application of \(\mathfrak{M}\). From the deformation quantization point of view the most interesting DGAs are the space of polyvector fields \(T_{poly}(M) = \oplus_{k=1}^{\infty} \Gamma^k(\wedge^k TM)\) and the vector space \(D_{poly}(M) = \oplus_{k=1}^{\infty} \{\text{Polydifferential operators} : C^\infty(M) \otimes_{k+1} \to C^\infty(M)\}\) of polydifferential operators on a manifold \(M\): We denote by \([\cdot, \cdot]_{S-N}\) the Schouten-Nijenhuis bracket on \(T_{poly}(M)\) and a Poisson structure \(\Pi\) induces a flat derivation \(d_{\Pi} := [\Pi, \cdot]_{S-N} : T_{poly}(M) \to T_{poly}(M)\) of the \(\wedge\) product. The famous Hochschild differential \(d_{H} : D_{poly}^*(M) \to D_{poly}^{*+1}(M)\) is a graded flat derivation of the associative cup product \(\cup\). In other words the input data of a Poisson tensor endows \(T_{poly}(M)\) with a DGA structure and \(D_{poly}(M)\) carries a natural DGA structure. In both cases the direct application of \(\mathfrak{M}\) looks somehow strange, for example for \((T_{poly}(M), d_{\Pi})\) the in general not invertible Poisson structure \(\Pi\) appears “quadratic” in the first order of \(\hbar\). However, nevertheless let us mention the following example how to cure this strange “quadratic” deformation into the direction that \(\Pi\) can appear in some sense “linear”:

Consider the category with objects chain complexes \((C^*, d^*)\) over \(\mathbb{C}\), \(i.e.\) graded vector spaces \(C^*\) equipped with linear, square zero maps \(d : C^* \to C^{*+1}\) and the arrows are linear maps \(\varphi : C^*_k \to C^*_{k+1}\) where we do not assume that this maps intertwine the boundary maps \(d_k, d^*_k\). It is well-known that there is a differential of degree 1 acting on the space of graded linear maps \(\varphi : C^*_k \to C^*_{k+1}\) by the graded homotopy formula \(d_\varphi := d_k \circ \varphi - (-1)^{|\varphi|} \varphi \circ d_k\). This natural differential defines a graded derivation of the composition of linear maps, \(i.e.\) whenever it makes sense we have a graded, associative “product” defined by composition \(\varphi \circ \alpha\) for \(\alpha : C^*_j \to C^*_{j+1}\) and \(\varphi : C^*_k \to C^*_{k+1}\) and also the graded Leibniz rule \(d(\varphi \circ \alpha) = d_\varphi \circ \alpha + (-1)^{|\varphi|} \varphi \circ d_\alpha\) holds. Hence analog to \(\mathfrak{M}\) we can deform this composition by the formula

\[
\varphi \circ d^\alpha = \varphi \circ \alpha + i\hbar((-1)^{|\varphi|} d_l \circ \varphi - \varphi \circ d_k) \circ (d_k \circ \alpha - (-1)^{|\alpha|} \alpha \circ d_j)
\]

\[
= \varphi \circ \alpha + i\hbar((-1)^{|\varphi|} d_l \circ \varphi \circ d_k \circ \alpha - (-1)^{|\alpha|+|\varphi|} d_l \circ \varphi \circ \alpha \circ d_j + (-1)^{|\alpha|} \varphi \circ d_k \circ \alpha \circ d_j)
\]

This deformed composition still respects the identity morphisms and differentials in the sense \(\varphi \circ d^\alpha \text{id} = \varphi = \text{id} \circ d^\varphi\) and \(d_l \circ d^\varphi = d_l \circ \varphi\) and \(\varphi \circ d_k = \varphi \circ d_k\)


2 Deformation of graded coalgebras with codifferential

Proposition 2.0.1. Let $d$ be an odd degree flat coderivation of a graded coproduct $\Delta$ over $\mathbb{C}$. With the Koszul sign convention we have a deformed coproduct

$$\Delta^d := \Delta + i\hbar (d \otimes d)\Delta$$

Proof. This is just dually to [1.0.2] Again the proof of the coassociativity $(\text{id} \otimes \Delta^d) \Delta^d = (\Delta^d \otimes \text{id}) \Delta^d$ only uses that $d$ is assumed to be a flat coderivation, i.e. we have $\Delta d = (\text{id} \otimes d) \Delta + (d \otimes \text{id}) \Delta$ and $d^2 = 0$ quite analog to the previous proof, we have

$$(\text{id} \otimes \Delta^d) \Delta^d = \left( \text{id} + i\hbar \left[ d \otimes d \otimes 1 + d \otimes 1 \otimes d + 1 \otimes d \otimes d \right] \right) (\text{id} \otimes \Delta) \Delta$$

where the signs of $i\hbar$ are hidden in the Koszul sign conventions.

Consider the category DGCA with objects $(C, \Delta_C, d_C)$ where $C$ is a graded coalgebra with coproduct $\Delta_C : C \to C \otimes C$ and flat coderivation $d_C : C \to C$, and arrows in DC are by definition graded structure compatible $\mathbb{C}$-linear morphisms, i.e. for any $\varphi : (C, \Delta_C, d_C)[[\hbar]] \to (D, \Delta_D, d_D)[[\hbar]]$ we assume $\varphi d_C = d_D \varphi$ and $(\varphi \otimes \varphi) \Delta_C = \Delta_D \varphi$.

Proposition 2.0.2. The change of objects $D(C, \Delta_C, d_C) = (C, \Delta^d_C, d_C)[[\hbar]]$ with help of [2.0.1] defines a faithful deformation functor DC $\mapsto$ DC[[\hbar]].

3 “Defect” for differential graded Lie algebras

Here we consider the analog of and [2.1.1] for DGLAs, i.e. we assume a graded $\mathbb{C}$-bilinear bracket $[\cdot, \cdot]$ that satisfies the graded Jacobi identity

$$(-1)^{|a_1||a_3|} [a_1, [a_2, a_3]] + (-1)^{|a_2||a_1|} [a_2, [a_3, a_1]] + (-1)^{|a_3||a_2|} [a_3, [a_1, a_2]] = 0$$

and a graded $\mathbb{C}$-linear map of degree 1 that satisfies the graded Leibniz rule

$$d[a_1, a_2] = [da_1, a_2] + (-1)^{|a_1|} [a_1, da_2]$$

Proposition 3.0.3. The operation $[\cdot, \cdot]_d$ defined by

$$[a_1, a_2]_d := [a_1, a_2] + i\hbar (-1)^{|a_1|} [da_1, da_2]$$

has an exact defect of graded Jacobi identity.

Proof. By calculation

$$\frac{2i\hbar}{3} \left( (-1)^{|a_1||a_3|+|a_1|+|a_2|} [a_1, [a_2, da_3]] - (-1)^{|a_1||a_3|} [da_1, [a_2, a_3]] \\
+ (-1)^{|a_2||a_1|+|a_2|+|a_3|} [a_2, [a_3, da_1]] - (-1)^{|a_2||a_1|} [da_2, [a_3, a_1]] \\
+ (-1)^{|a_3||a_2|+|a_1|+|a_2|} [a_3, [a_1, da_2]] - (-1)^{|a_3||a_2|} [da_3, [a_1, a_2]] \right)$$

is an explicit primitive of the Jacobiator, hence the cohomology class of the defect vanishes.
Remark

- The question arises if it is possible to construct for $n \geq 4$ Taylor components $Q_n : S^n(g[1]) \to g[1]$ to upgrade this defect deformation to a functor from DGLA to the category of $L_\infty$ algebras. The previous question has already been considered in the literature, we thank D. Roytenberg for pointing this out: The version of higher derived brackets of Getzler [6] is a convenient positive answer and the higher derived brackets of Voronov [12] seem related, let us mention the article [4].

References

[1] Bayen F., Flato M., Frønsdal C., Lichnerowicz A., and Sternheimer D., *Deformation Theory and Quantization*, Springer-Verlag, Volume 94, Group Theoretical Methods in Physics, 280-289 (1979).

[2] Berezin F. A., *General Concept of Quantization*, Commun. Math. Phys. 40, 153-174 (1975).

[3] ———, *General Concept of Quantization*, Commun. Math. Phys. 40, 153-174 (1975).

[4] Cattaneo A.-S. and Schätz F., *Equivalence of higher derived brackets*, J. Pure Appl. Algebra 212, Issue 11, 2450-2460 (2008).

[5] Dirac P.A.M., *The principles of quantum mechanics first and second edition*, Clarendon press. Oxford, (1930).

[6] Getzler E., *Higher derived brackets*, arXiv: 1010.5859v2 (2010).

[7] Groenewold H.J., *On the Principles of elementary quantum mechanics*, Physica. 12, 405-460 (1946).

[8] Gutt S., *An explicit $\ast$-product on the cotangent bundle of a Lie group*, Lett. Math. Phys. 7, no. 3, 249-258, (1983).

[9] Kontsevich M., *Deformation quantization of Poisson manifolds*, arxiv (1999).

[10] Löffler J., *Fedosov differentials and Catalan numbers*, IOP J. Phys. A: Math. Theor. 43 235404, (2010).

[11] Moyal J.E. and Bartlett M.S., *Quantum mechanics as a statistical theory*, Mathematical Proceedings of the Cambridge Philosophical Society 45, 9-124, (1949).

[12] Voronov Th., *Higher derived brackets and homotopy algebras*, J. Pure and Appl. Algebra 202 issues 1 – 3, 113-153 (2005).

[13] Waldmann S., *Poisson-Geometrie und Deformationsquantisierung*, Springer-Verlag, (2007).

[14] Witten E., *Supersymmetry and Morse theory*, J. Diff. Geom 17, 353-386 (1982).

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