Phase Transition Behavior in Knowledge Compilation

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Abstract. The study of phase transition behaviour in SAT has led to deeper understanding and algorithmic improvements of modern SAT solvers. Motivated by these prior studies of phase transitions in SAT, we seek to study the behaviour of size and compile-time behaviour for random \(k\)-CNF formulas in the context of knowledge compilation. We perform a rigorous empirical study and analysis of the size and runtime behavior for different knowledge compilation forms (and their corresponding compilation algorithms): d-DNNFs, SDDs and OBDDs across multiple tools and compilation algorithms. We employ instances generated from the random \(k\)-CNF model with varying generation parameters to empirically reason about the expected and median behavior of size and compilation-time for these languages. Our work is similar in spirit to the early work in CSP community on phase transition behavior in SAT/CSP. In a similar spirit, we identify the interesting behavior with respect to different parameters: clause density and solution density, a novel control parameter that we identify for the study of phase transition behavior in the context of knowledge compilation. Furthermore, we summarize our empirical study in terms of two concrete conjectures; a rigorous study of these conjectures will possibly require new theoretical tools.

1 Introduction

Phase transition is concerned with a sudden change in the behavior of a property of interest of an object pertaining to variations of a parameter of interest. In the context of combinatorial problems, the phase transition behavior was first demonstrated in random graphs in the seminal work of Erdos and Renyi \cite{erdos1960evolution}. With the advent of SAT as a modeling language, the initial studies observed phase transition in the satisfiability of random \(k\)–CNF formulas and the seminal work of Mitchell, Selman, and Levesque \cite{mitchell1992empirical} demonstrated empirical hardness around the phase transition region for modern SAT solvers. Theoretical investigations into determining the location of the phase transition region have led to several exciting results that yield insights into the algorithmic behavior of modern SAT heuristics \cite{de2002phase}. In a significant theoretical breakthrough, the existence of the phase transition behavior for random \(k\)–CNF for large \(k\) was theoretically proved; the question for small \(k\) (\(>2\)) is still open \cite{chung2005phase}.
The success of SAT solvers has led the development of tools and techniques for problems and sub-fields broadly classified under the umbrella of Beyond NP. This has led to an interest in the behavior of solvers through the lens of phase transition [16][17][31]. Motivated by the success of these studies in uncovering surprising insights into the solution space structure and regions of hardness for the modern SAT solvers, we turn our focus to another sub-field in Beyond NP that has found several practical applications: Knowledge Compilation.

Knowledge compilation broadly refers to the approaches that seek to compile propositional formulae into tractable representations; tractability is defined with respect to queries that can be performed in polynomial time over the compiled form. As the runtime of queries such as counting, sampling, equivalence is often polytime in the size of a representation, efficient compilation gains importance. Several compilations forms have been proposed showcasing the tradeoff between tractability and succinctness: OBDDs (Ordered Binary Decision Diagrams) [9], SDDs (Sentential Decision Diagrams) [13] and d-DNNFs (deterministic-Decomposable Negation Normal Forms) [11] and others. We refer the reader to [14] for a detailed survey on the size and tractability for several target languages. While every tractable language known so far has exponential size complexity in the worst case when the input is presented in CNF form, the size of compiled forms can often be only polynomially larger than the input CNF formula, which has highlighted the need for more detailed study of the size and runtime complexity of compilation procedures.

In this work, we undertake a rigorous empirical study and analysis of phase transition behavior in knowledge compilation. Our experimental setup employs instances generated from the random \( k \)-CNF model with varying generation parameters (number of variables and clauses as well as length of clauses). Our study is multidimensional, comparing and contrasting phase transition behaviors spanning:

- knowledge compilations: d-DNNFs, SDDs and OBDDs;
- properties of interest: size and compilation times;
- compilation algorithms and tools:
  - C2D [12], D4 [23], DSharp [25] for d-DNNF,
  - MiniC2D [29] and TheSDDPackage [33] for SDD,
  - CUDD with different variable ordering heuristics [32][19] for BDD.

A primary contribution of the seminal work of Mitchell et al [26] was the establishment of clause density as a popular parameter of interest in the study of SAT solving. In a similar spirit, one of the key contributions of this work is the proposal of solution density as a new control parameter, along with clause density, for studying knowledge compilations. Solution density is defined as the ratio of the logarithm of the number of satisfying assignments to the number of variables. We show that while clause density is linked with solution density in expectation, the size and compilation-time varies significantly with varying solution density for a fixed clause density. We discover that for low clause densities, varying solution density has minimal effect on the size of compilations. In
contrast, for high clause densities, solution density dictates the size as there is a minimal variation with clause density for a given fixed solution density.

Based on our experiments, we make two conjectures for compiled structures in a language $L$ that is a subset of DNNF:

1. Over a population of $k$-CNF formulas, $F_k(n, \lceil r_1n \rceil)$ of $k$-clauses over $n$ variables with a clause density $r$, for all integers $k \geq 2$, there exists a clause density $r_t$ that witnesses a phase transition on the size of the compiled structures in a language $L$; that is, there always exists a clause density $r_t$ such that:
   (a) for each pair $(r_1, r_2)$, such that $r_1 < r_2 < r_t$, the expected size of the compiled structure in $L$ for $F_k(n, r_1n)$ is strictly smaller than that for clause density $r_2$;
   (b) for each pair $(r_1, r_2)$, such that $r_1 < r_2 < r_t$, the expected size of the compiled structure in $L$ for $F_k(n, r_1n)$ is strictly larger than that for clause density $r_2$.

2. Over a population of $k$-CNF formulas, $G_k(n, \lceil 2^{\alpha n} \rceil)$ of $k$-clauses over $n$ variables and having $\lceil 2^{\alpha n} \rceil$ solutions, for all integers $k \geq 2$, there exists a solution density $\alpha_k$ that witnesses a phase transition on the size of the compiled structures in a language $L$; that is, there always exists a solution density $\alpha_t$ such that:
   (a) for each pair $(\alpha_1, \alpha_2)$ such that $0 \leq \alpha_1 < \alpha_2 < \alpha_k$, the expected size of the compiled structure in $L$ for $G_k(n, 2^{\alpha_1n})$ is strictly smaller than that for solution density $\alpha_2$;
   (b) for each pair $(\alpha_1, \alpha_2)$ such that $\alpha_k < \alpha_1 < \alpha_2 \leq 1$, the expected size of the compiled structure in $L$ for $G_k(n, 2^{\alpha_1n})$ is strictly larger than that for solution density $\alpha_2$.

The phase-transition behavior for the satisfiability of CNF constraint has been instrumental in driving several breakthroughs in the design of new solvers and better understanding the problem structure. We hope that our experimental study of phase transitions for knowledge compilations would lead to similar developments in the knowledge compilation community. Our results in this work are similar in spirit to the seminal work by the CSP community in empirical identification of phase transition phenomenon in the early 1990s and their summarization in the form of conjectures. It is worth emphasizing that theoretical proofs of these conjectures were presented nearly 20 years since the first empirical studies, and the efforts to establishing these conjectures contributed to the development of several theoretical tools of widespread applicability. We hope our empirical results will inspire similar efforts in the context of knowledge compilation.

The organization of the rest of the paper is as follows: Section 2 describes the notations and preliminaries, along with a survey of prior work. Section 3 describes the design of our experiments while Sections 4 and 5 provide detailed observations with respect to clause density and solution density respectively. Due to space restrictions, this article is limited to experiments using the following tools: D4 for d-DNNF, TheSDDPackage for SDD, and CUDD (with SIFT
variable reordering) for BDD. We, however, observed similar behaviors across all the other tools and summarise them in Section 6. Furthermore, throughout the article, we present only representative plots and an extended collection of corresponding plots is deferred to Appendix.

2 Notations and Preliminaries

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of propositional variables. A literal is a propositional variable ($x_i$) or its negation ($\neg x_i$). For a formula $F$ defined over $X$, a satisfying assignment or witness of $F$ is an assignment of truth values to the variables in $X$ such that $F$ evaluates to true. The total number of witnesses for a formula $F$ is denoted by $\#F$.

Let a $k$-clause be a disjunction of $k$ literals drawn over $X$ without repetition. Given $k \in \mathbb{N}$, $n \in \mathbb{N}$ and $r \in \mathbb{R}_{>0}$, let the random variable $F_k(n, \lceil rn \rceil)$ denote a Boolean formula consisting of the conjunction of $\lceil rn \rceil$ $k$-CNF clauses selected uniformly and independently from $\binom{n}{k} 2^k$ possible $k$-clauses over $n$ variables. For a given $k$, $k$-CNF denotes the set of all possible boolean formulas $F_k(n, \lceil rn \rceil)$. For any given formula $F$, we denote the clause density ($r$) as the ratio of number of clauses to variables and the solution density ($\alpha$) as the ratio of logarithm of the number of satisfying assignments to the number of variables, i.e. $\alpha = \log(\#F)/n$. We denote the SAT phase transition clause density for random $k$-CNF by $r_p^k$.

Given an $\alpha \in [0, 1]$, we define another random variable $G_k(n, \lceil 2^{\alpha n} \rceil)$ that denotes a randomly chosen boolean formula from $k$-CNF with $n$ variables and $\lceil 2^{\alpha n} \rceil$ satisfying assignments.

2.1 Target Compilation Languages

We briefly describe some of the prominent target compilation languages that we study—d-DNNFs, OBDDs and SDDs [14].

Definition 1. [14] Let $V$ be the set of propositional variables. A formula in NNF is a rooted, directed acyclic graph (DAG) where each leaf node is labelled with true, false, $x$ or $\neg x$, $x \in V$; and each internal node is labelled with $\lor$ or $\land$ and can have arbitrarily many children.

Deterministic Decomposable Negation Normal Form (d-DNNFs), a subset of NNF, satisfies determinism (operands of $\lor$ are mutually inconsistent) and decomposition (the operands of $\land$ are expressed on a mutually disjoint set of variables).

Ordered Binary Decision Diagrams (OBDD) is a subset of d-DNNFs where the root node is a decision node and the order of decision variables is same for all paths from the root to a leaf; OBDDs are canonicalised by the variable ordering. A decision node is either a constant ($T/F$), or of the form $(X \land \alpha) \lor (\neg X \land \beta)$, on decision nodes $\alpha$ and $\beta$, and decision variables $X$. 
Sentential Decision Diagrams (SDDs) are a subset of d-DNNFs that hold the properties of structured decomposability and strongly deterministic decompositions. The idea of structured decomposability is captured by the notion of vtrees; vtrees are binary trees whose leaves correspond to variables of the formula and internal nodes mark the decomposition into variables given by their left and right child. Vtrees seek to generalise variable ordering and precisely specify a decomposition scheme to be followed by the corresponding SDD. Strongly deterministic decompositions seek to generalise Shannon’s expansion in which decision is made on a single variable. For a formula $F$ where $\forall U, V \subseteq \text{vars}(F)$ such that $\text{vars}(F) = U \cup V$ and $U \cap V = \emptyset$, representing $F = (p_1(U) \land s_1(V)) \lor \ldots \lor (p_n(U) \land s_n(V))$ where $p_i$’s and $q_i$’s are boolean functions and $p_i(U) \land p_j(U) = \bot$ for $i \neq j$ captures the idea of strongly deterministic decompositions. Here, the decomposition of $\text{vars}(F)$ into $U$ and $V$ is the decomposition scheme.

We denote a target language by $\mathcal{L} \in \{\text{d-DNNF, OBDD, SDD}\}$. For a formula $F$ compiled to $\mathcal{L}$, let $N_{\mathcal{L}}(F)$ denote the size (in the number of nodes) of the target representation and $T_{\mathcal{L}}(F)$ denote the compilation time. The expected size for $k$-CNF with clause density $r$ and target language $\mathcal{L}$ can, therefore, be represented as $E(N_{\mathcal{L}}(F_k(n,rn)))$ and the expected time as $E(T_{\mathcal{L}}(F_k(n,rn)))$.

2.2 Related Work

Cheeseman et al. [10] studied the phase transition behavior for several CSP problems, noting an easy-hard-easy behavior with respect to an order parameter for these problems. For Hamiltonian circuits, graph coloring and k-SAT, the average connectivity of the graph (which translates to clause density for k-SAT) was found to be a good order parameter demonstrating phase transition behavior. Subsequently, exploring phase transition behaviors with various random generative models has drawn considerable attention. [3,21,22,23,34,27].

Aguirre and Vardi [4] identified an “easy-hard-less hard” behaviour in OBDD compilations with clause density and discovered a phase transition from polynomial running time to exponential running time for SAT. Huang and Darwiche [24] proposed a new top-down algorithm for OBDD compilation based on the DPLL algorithm typically used for SAT solving and observed that the expected size of OBDD peaks around clause density equal to 2. Later, Gao, Yin, and Xu [20] conducted an experimental study for phase transitions in random $k$-CNF formulas taking d-DNNFs, OBDDs and DFAs (Deterministic Finite Automata) as the target compilation languages. They observe a phase transition behavior with respect to size of compilations and draw a conjecture stating that all subsets of DNNF show a “small-large-small” behavior with a unique peak. Our study is more comprehensive and we discuss in Section 4 several behaviors that were not explored by Gao et al.

Birnbaum and Lozinskii [6] presented a procedure called as Counting Davis-Putnam (CDP) for $\#k$SAT whose trace lies in FBDD, a stricter subset of d-DNNF language. They found that the median number of recursive calls of CDP reaches its peak when for clause density is 1.2 for random 3-CNF formulas. A different DPLL extension for solving $\#k$SAT, called Decomposing Davis-Putnam
(DDP) leveraging component decomposition, was developed by Bayardo and Pehoushek [5]. The trace of DPP-model counter lies in d-DNNF language and the authors observed phase transition around the clause density of 1.5.

3 Design of experiments

We aim to dive deeper into the size and compile-time behavior for d-DNNF, SDD and OBDD compilations owing to their wide range of applications. We measure the size of compilations by the number of nodes (edges also display a very similar behavior). To study the phase transition behavior, we identify a new parameter, solution density, the ratio of log of the number of satisfying assignments to the number of variables. While it is trivial to generate a random instance $F_k(n,rn)$ given $r$, a direct way to generate $G_k(n,2^{\alpha n})$ given $\alpha$ is unknown. Therefore, to analyze the effect of varying $\alpha$ in compilations, we study the variation in size and compile-time against $\alpha$ for individual instances generated by varying $r$. Our experimental observations with varying clause density are given in Section 4 while those with solution density follow in Section 5.

Note that, in case of SAT which is a decision problem, satisfiability phase transition is often characterized by existence of a unique cross-over point, at the intersection of the curves representing probability of true (SAT) decision against clause density for different number of variables. However, in our case, we are dealing with a functional problem and therefore, we characterize phase transition with respect to the gradient of the function, i.e., when the underlying function achieves a local maxima. More concretely, we will be concerned with the size and runtime of the underlying knowledge compilation form and its corresponding compiler.

We carried out our experiments on a high performance compute cluster whose each node is an Intel E5-2690 v3 CPU with 24 cores and 96GB of RAM. We utilize a single core per benchmark instance. Note that knowledge compilation is significantly harder and memory intensive than satisfiability, which restricts the scale of our experiments. Therefore, we have conducted experiments up to the largest number of variables for which we could gather all the needed statistics. We utilized more than 40,000 computational hours for our experimentation.

3.1 The variables of our study

We conducted our experiments on a large number of random $k$-CNF formulas with a varying number of variables and clauses. For studying variations with clause density, we aggregated the results over 1000 instances of the random variable $F_k(n,rn)$ for each $r$ given a fixed $n$ and $k$. Here, $r$ is incremented with step size of at least 0.1 in a range containing 0 to $r^p_k$. For studying variations with solution density, we aggregated the results over at least 5 instances of the random variable $F_k(n,rn)$ for each $r$ given a fixed $n$ and $k$. Here, $r$ is incremented with step size of $1/n$ in a range from 0 to $r^p_k$. As clause density and solution density are linked in expectation, we vary $r$ finely for study with solution density to
facilitate uniform distribution of instances with $\alpha$. Specifically, a representative subset of the following variations are a part of our study:

- number of variables ($n$): 20 − 70 in general, upto an exponential number ($\lfloor 1.3^{37} \rfloor$) for small $r$
- length of clauses ($k$): 2 − 7
- target languages: d-DNNFs, OBDDs, SDDs
- knowledge compilers: D4 [25], CUDD [32,19], TheSDDPackage [33]

4 Phase transition with clause density

In this section, we present our observations for variations in size and compile times with d-DNNFs, SDDs, and OBDDs as target languages. We also dive deeper into the complexity of compilations. As noted earlier, for exposition, we will first present plots for a representative subset of the experimental results, and a comprehensive collection of plots appears in the Appendix.

4.1 Observing the phase transitions

Size of compilations Figure [1] shows the variations in the mean and median of the number of nodes for d-DNNF, SDD, and OBDD with respect to clause density. Figure [2] shows variations in mean number of nodes for d-DNNF on a log scale versus number of variables. The small-large-small pattern in the size of compilations is reminiscent of the empirical hardness of SAT near the phase transition, which is marked by an exponential increase in SAT solvers' runtimes near a specific (phase transition) clause density.

Runtime of compilation From an empirical usage perspective, the runtime of compilation assumes paramount importance. It is worth emphasizing that the size of the compiled form does not solely determine the runtime. In particular, the runtime of compilation broadly depends upon: (1) the search space itself, (2) the time spent in heuristics. In state-of-the-art knowledge compilers, newer heuristics increasingly attempt to prune the search space leading to reduced memory and runtime for subsequent exploration. However, they incur an increased overhead of applying heuristics. Therefore, to provide a holistic view of the hardness of compilations, Figure [1] also shows the variation in average runtimes against clause density for d-DNNF, SDD, and OBDD. We observe that the location of phase transition appears slightly shifted (within ±0.3) compared to that for the size of compilations in the case of d-DNNF and SDD. However, in the case of OBDD, we have a starkly different behavior due to the different compilation procedures, even though both SDDs and OBDDs are compiled by repeatedly conjoining the clauses one by one using the polytime APPLY operation.

A plausible explanation is that while SDD compilation using TheSDDPackage involves dynamic vtree search with clause reordering, OBDD compilation using
CUDD supports only dynamic variable ordering. This can lead to an easy-hard-less hard behavior for SDD compilations as clause reordering facilitates early pruning of intermediate SDDs [33]. On the other hand, OBDD compilation involves large intermediate OBDDs near phase transition clause density irrespective of the total number of clauses as clauses are selected randomly for the
APPLY operation. This can lead to a sharp step transition in runtimes as the addition of more clauses after phase transition prunes the OBDD, translating to reduced cost of successive APPLY operations.

We refer the reader to Appendix for variation with respect to clause length. We sum up our observations via the following conjecture

**Conjecture 1.** For every integer \( k \geq 2 \), given the number of variables \( n \) and a target language \( \mathcal{L} \) that is a subset of DNNF, there exists a positive real number \( r_k \) such that

For each pair \((r_1, r_2)\), if \( r_1 < r_2 < r_k \) then,

\[
E(\mathcal{N}_\mathcal{L}(F_k(n, r_1n))) < E(\mathcal{N}_\mathcal{L}(F_k(n, r_2n)))
\]

For each pair \((r_1, r_2)\), if \( r_k < r_1 < r_2 \) then,

\[
E(\mathcal{N}_\mathcal{L}(F_k(n, r_1n))) > E(\mathcal{N}_\mathcal{L}(F_k(n, r_2n)))
\]
4.2 Diving deep into the asymptotic complexity

The size and runtime of compilations are exponential in the number of variables in the worst case. We are, however, interested in more precise relationship as size and runtime play crucial roles in many applications. While it is known that the exponent increases towards phase transition clause density [20], it is not clear if the exponent is linear or sublinear in the number of variables and whether this behavior changes with changing clause density.

Since, the size of OBDDs (and d-DNNFs and SDDs being more succinct [7]) is bounded by \( O(2^n) \), we plot \( \log(E(N_{\text{d-DNNF}}(F_k(n, rn))))/n \) for different \( n \) against varying \( r \) in Figure 2 and 3 for further investigation. From Figure 2 we observe phase transition with respect to clause density even though the location of the phase transition point seems to shift slightly with different \( n \). Next, we turn our attention to Figure 3 wherein we observe that \( \log(E(N_{\text{d-DNNF}}(F_k(n, rn))))/n \)
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Fig. 6: Variation in size with solution density for individual instances with $r = 3.0$ for d-DNNF

decreases while showing signs of possible convergence with increasing $n$ for a given $r$ for all compilations in our study. In order to understand it better, we scale our experiments to a larger $n$ for small $r$ in Figure 3b. However, even in this case, the answer remains unclear if $\log(\mathbb{E}(N_{d-DNNF}(F_k(n, r_n))))/n$ shall converge to a constant $> 0$. Therefore, we can only say that for a constant, $c \geq 0$,

$$\lim_{n \to \infty} \frac{\log(\mathbb{E}(N_{d-DNNF}(F_k(n, r_n))))}{n} = c.$$ 

While the question if $c = 0$ remains open for size, the case for runtime of compilation is a different story: Figure 5 shows that $\mathbb{E}(T_L(F_k(n, r_n)))$ increases with $n$. Knowing that the runtime in worst case is $\text{poly}(2^n)$ and extrapolating the observation, we conjecture that $\lim_{n \to \infty} \frac{\mathbb{E}(T_L(F_k(n, r_n)))}{n} = c$ where $c > 0$ and depends on $r$. In other words, $\mathbb{E}(T_L(F_k(n, r_n))) = \theta(2^{cn})$ for state-of-the-art compilers.

Polynomial to Exponential size phase transition. Gao et al. [20] had shown the existence of a polynomial to exponential phase transition for size of compilations around $r = 0.3$ by showing increase in slope for $r > 0.3$ and near constant slope for $r < 0.3$ on a log-log graph. Relationships of the form $y = ax^k$ appear as straight lines in a log-log graph. Our observations in Figure 3 and 4a show that while the instances with $r = 0.2$ are indeed very easy compared to $r = 0.4$, the behavior is still exponential or quasi-polynomial for $r = 0.2$ which becomes dominant for large enough number of variables. In Figure 4a the behavior appears to change from a straight line to a line with increasing slope around $n = 7482$. On extrapolating the behavior for even smaller $r$, we can conjecture that $\forall r$, $\mathbb{E}(N_{d-DNNF}(F_k(n, r_n)))$ is at least quasi-polynomial in $n$. 
5 Phase transitions with solution density

Since knowledge compilations are a compact way of representing solutions, one can expect that they show variations in sizes with respect to the solution density as well. The solution density, however, is not independent of clause density for random $k$-CNF as given a clause density, expected solution density is fixed, and vice versa. Notwithstanding, we observe (Figure 6) that solution density also appears to be a fundamental parameter given a fixed clause density, instances with different solution density have marked changes in their size of d-DNNF compilations. We, now, look at the size and compile-time behavior with respect to solution density.

5.1 Observing the phase transition

Figure 7 shows the small-large-small variation in size of compilations with respect to solution density. We can observe that there exists a region of critical solution density for each target language around which the size of instances are very large. The location of phase transition appears to depend upon the target compilation but it is difficult to comment upon the precise location and an extended study is required to generate independent instances for a given solution density.

We have seen that solution density is a major parameter affecting the size and runtime of compiled instances. In this context, we seek to understand the impact of other parameters on the phase transition location with respect to solution density. We focus on two such parameters: clause length and the number of variables.

Impact of clause length Figure 8 shows that as we increase the clause length, the location of phase transition point with respect to solution density moves closer to 1. It is worth remarking that in the context of the satisfiability, the location of phase transition, albeit with respect to clause density, is known to depend on the clause length, so a similar behavior in the context of knowledge compilation is indeed not surprising.

Size with number of variables From Figure 9 we observe that the distribution of Nodes in d-DNNF compilation becomes sharper around the phase transition solution density with increasing number of variables.

Runtime of compilation From Figure 7 we observe that the distribution of runtimes follows a similar easy-hard-easy pattern for d-DNNFs as well as SDDs but not OBDDs. The runtime behavior is similar to that of distribution of the number of nodes, as discussed in section 4.

We sum up our observations in the following conjecture:

Conjecture 2. For every integer $k \geq 2$, given the number of variables $n$ and a target language $\mathcal{L}$ that is a subset of DNNF, there exists a positive real number
Fig. 7: Number of Nodes and compile-times for individual instances of 3-CNF against solution density

For each pair \((\alpha_1, \alpha_2)\), if \(0 \leq \alpha_1 < \alpha_2 < \alpha_k\) then,

\[
E(N_{\mathcal{C}}(G_k(n, 2^{\alpha_1} n))) < E(N_{\mathcal{C}}(G_k(n, 2^{\alpha_2} n)))
\]
For each pair \((\alpha_1, \alpha_2)\), if \(\alpha_k < \alpha_1 < \alpha_2 \leq 1\) then,

\[
E(N_L(G_k(n, 2^{\alpha_1 n})) > E(N_L(G_k(n, 2^{\alpha_2 n})))
\]

### 5.2 Combined effect of clause and solution density

We have seen that the density of solutions and clauses play a pivotal role, affecting the size and runtime. Also, given a clause density, the expected solution density is fixed and vice versa. This makes one wonder if there is a more complex relationship at play that affects the phase transition behavior of knowledge compilations. To investigate, we plot a heatmap (Figure 10) on \(\alpha \times r\) grid where the colours indicate the size of compilations. We employ random 3-CNF instances with 70 variables used for comparisons with clause density, as described earlier in section 3. For each cell in the grid, we take an average of instances with the corresponding clause density that lie within the interval of solution density.
Fig. 10: log(Nodes) in d-DNNF compilation in solution vs clause density grid for 3-CNF

marked by the cell. Since the number of such instances can differ across the cells, we mark the average for a cell only if the number of instances is greater than 5 to minimize the variance to a feasible extent. From Figure 10, we observe that for low clause densities, varying solution density has minimal effect on the size of compilations. In contrast, for high clause densities, solution density has a dominant effect on the size as there is a minimal variation with clause density. On the other hand, near phase transition, both the parameters play a significant role, and the precise relationship is murkier.

6 Effect of different tools

Heuristics and compilation algorithms play a crucial role in perceived hardness as elaborated in Section 4.1. We experimented with bottom up (TheSDDPackage) as well as top down (MiniC2D) compilation strategies for SDDs, predefined total variable ordering against dynamic ordering for OBDDs (CUDD) and different decomposition techniques for d-DNNFs (C2D, Dsharp and D4). Notably, target language for MiniC2D is decision-SDD and CUDD with predefined total variable order is OBDD \(_{\text{SU}}\) [30,14], which are less succinct than SDD and OBDD respectively. We state our representative observations for 3-CNF here. We observed that both MiniC2D and The SDD Package show maximum number of nodes around clause density 1.8 and solution density around 0.62. However, runtime for MiniC2D peaks around clause density 1.8 while TheSDDPackage peaks around clause density 2.0. In case of BDDs, we observed that disabling dynamic variable reordering shifts the peak (number of nodes) clause density from 2.0 to 1.5 and peak (number of nodes) solution density from 0.62 to 0.75. The observations for runtimes of OBDDs are much more involved due to reasons discussed in Section 4.1. For d-DNNFs, we observe that the peak (number of nodes) clause density stays around 1.8 and peak solution density stays around

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(a) \( r \in [0.1, 1.4] \)

(b) \( r \in [3.1, 5.0] \)
0.62 irrespective of the hypergraph partitioning algorithm. Similar observations were recorded for runtime as well in case of d-DNNF. Summing up, we observe that while the precise behaviour of phase transition (for example, its location) can depend upon the heuristics employed in the process, the general behaviour persists irrespectively.

7 Conclusion

Our study provides evidence of phase transition behavior with respect to clause as well as solution density. While, both these parameters are linearly linked in expectation, it is interesting that varying the number of solutions on a fixed clause density leads to significant variation in the expected size of compilations. In terms of the complexity of compilations, we found the expected size is at least quasi-polynomial and expected runtime is exponential in the number of variables with varying clause density for state-of-the-art knowledge compilers. We believe that this paper opens up new directions for theoretical studies in an attempt to explain our empirical observations and conjectures.

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APPENDIX

In this Appendix, we provide some detailed results from our experiments. To aid readability, some of the MainFigures from the main paper are also included here.

A Phase transition with clause density

In this section, we show the variations in size and compile times for d-DNNFs, SDDs and OBDDs with clause density from our experiments.

A.1 Observing the phase transitions

**Size with number of variables** We look at the effect of variation in the number of variables on phase transition in Figures 11, 12, 13. Overall, the transition appears sharper with increase in number of variables. Figure 14 compares log(nodes)/variables for different number of variables on same graph.

![Fig. 11: Nodes in d-DNNF vs clause density with different number of variables](image1)

(a) 60 vars  (b) 40 vars  (c) 20 vars

![Fig. 12: Nodes in SDD vs clause density with different number of variables](image2)

(a) 40 vars  (b) 30 vars  (c) 20 vars
Fig. 13: Nodes in OBDD vs clause density with different number of variables

Fig. 14: log(nodes)/variables vs clause density for different target languages

Runtime of compilation Figures 16, 17, 18 shows the variation in average runtimes against clause density for d-DNNF, SDD and OBDD for different number of variables.

Fig. 15: Average runtime for 3-CNF
Fig. 16: Compile-time for d-DNNF vs clause density with different number of variables

Fig. 17: Compile-time for SDD vs clause density with different number of variables

Fig. 18: Compile-time for OBDD vs clause density with different number of variables
A.2 Diving deep into nature of complexity

We plot \( \log(\#\text{Nodes}/\text{variables}) \) for different compilations against varying number of variables for small clause densities in Figures 19, 20 and 27.

We compare \( \log(\text{nodes in compilations}) \) against \( \log(\text{number of variables}) \) for different compilations in Figures 21, 24, 28.

![Graph showing log(nodes in d-DNNF)/variables vs variables for small clause densities](image1)

(a) clause densities 0.6 and 0.4

![Graph showing log(nodes in SDD)/variables vs variables for small clause density 0.2](image2)

(b) clause density 0.2

Fig. 19: \( \log(\text{nodes in d-DNNF})/\text{variables} \) vs variables for small \( r \)

We plot \( \log(\text{runtime for compilations}/\text{variables}) \) against varying number of variables for smaller clause densities in Figures 22, 25 and 29.

We compare \( \log(\text{runtime for compilations}) \) against \( \log(\text{number of variables}) \) for different compilations in Figures 23, 26, 30.

![Graph showing log(runtimes vs variables for small clause densities)](image3)

Fig. 20: \( \log(\text{nodes in SDD})/\text{variables} \) vs variables for \( r = 0.2 \)
Phase Transition Behavior in Knowledge Compilation

Fig. 21: log-log graph for nodes in d-DNNF compilation with clause density 0.2

(a) clause densities 0.6 and 0.4 
(b) clause density 0.2

A.3 Impact of clause length on phase transition

Figures 31, 32 and 33 show the variation in size with clause density for different clause lengths. Figures 34, 35 and 36 show the variation in runtime with clause density for different clause lengths.
Fig. 23: log-log graph for runtime of d-DNNF compilation with clause density 0.2

Fig. 24: log-log graph for nodes in SDD compilation with clause density 0.2
Fig. 25: $\log(\text{runtime for SDD})/\text{variables}$ vs variables for $r = 0.2$

Fig. 26: log-log graph for runtime of SDD compilation with clause density 0.2
Fig. 27: log(nodes in OBDD)/variables vs variables for small $r = 0.2$

Fig. 28: log-log graph for nodes in OBDD compilation with clause density 0.2
Fig. 29: log(runtime for OBDD)/variables vs variables for small $r = 0.2$

Fig. 30: log-log graph for runtime of OBDD compilation with clause density 0.2
Fig. 31: Nodes in d-DNNF vs clause density for different clause lengths ($k$)

Fig. 32: Nodes in SDD vs clause density for different clause lengths ($k$)

Fig. 33: Nodes in BDD vs clause density for different clause lengths ($k$)

Fig. 34: Compile-time for d-DNNF vs clause density for different clause lengths ($k$)
Fig. 35: Compile-time for SDD vs clause density for different clause lengths($k$)

(a) 2-CNF and 200 vars    (b) 4-CNF and 30 vars    (c) 7-CNF and 20 vars

Fig. 36: Compile-time for BDD vs clause density for different clause lengths($k$)

(a) 2-CNF and 200 vars    (b) 4-CNF and 30 vars    (c) 7-CNF and 20 vars
B  Phase transtion with solution density

In this section, we show the variations in size and compile-times obtained empirically for d-DNNFs, SDDs and OBDDs with solution density.

B.1  Observing the phase transition

Size with number of variables  Figures 37, 38 and 39 show the variation of nodes in d-DNNF, SDD and OBDD compilations with solution density for different number of variables.

Fig. 37: Nodes in d-DNNF vs solution density($\alpha$) for different number of variables

Runtime of compilation  Figures 41, 42 and 43 show the distribution in run-times of compilation for d-DNNFs, SDDs and OBDDs with different number of variables.
Fig. 39: Nodes in OBDD vs solution density($\alpha$) for different number of variables

(a) 40 vars  
(b) 30 vars  
(c) 20 vars

Fig. 40: Compile-time for individual instances of 3-CNF against solution density

(a) d-DNNF, 70 vars  
(b) SDD, 50 vars  
(c) OBDD, 50 vars

Fig. 41: Compile-time for d-DNNF vs solution density($\alpha$) for different number of variables

(a) 60 vars  
(b) 40 vars  
(c) 20 vars

Fig. 42: Compile-time for SDD vs solution density($\alpha$) for different number of variables

(a) 40 vars  
(b) 30 vars  
(c) 20 vars
Fig. 43: Compile-time for OBDD vs solution density ($\alpha$) for different number of variables
B.2 Impact of clause length on phase transition

Figures 44, 45 and 46 show the variation in the size with solution density for different clause lengths and number of variables. Figures 47, 48 and 49 show the variation in the runtime with solution density for different clause lengths and number of variables.

![Figure 44](image1)
(a) 2-CNF and 600 vars  
(b) 4-CNF and 40 vars  
(c) 7-CNF and 25 vars

Fig. 44: Nodes in d-DNNF vs solution density for different clause lengths($k$)

![Figure 45](image2)
(a) 2-CNF and 200 vars  
(b) 4-CNF and 30 vars  
(c) 7-CNF and 20 vars

Fig. 45: Nodes in SDD vs solution density for different clause lengths($k$)
Fig. 46: Nodes in OBDD vs solution density for different clause lengths ($k$)

(a) 2-CNF and 200 vars  (b) 4-CNF and 40 vars  (c) 7-CNF and 20 vars

Fig. 47: Runtime of d-DNNF compilation vs solution density for different clause lengths ($k$)

(a) 2-CNF and 600 vars  (b) 4-CNF and 40 vars  (c) 7-CNF and 25 vars

Fig. 48: Runtime of SDD compilation vs solution density for different clause lengths ($k$)

(a) 2-CNF and 200 vars  (b) 4-CNF and 30 vars  (c) 7-CNF and 20 vars

Fig. 49: Runtime of OBDD compilation vs solution density for different clause lengths ($k$)

(a) 2-CNF and 200 vars  (b) 4-CNF and 30 vars  (c) 7-CNF and 20 vars
B.3 Combined effect of clause and solution density

In Figures 50 and 51 we plot a heatmap on $\alpha \times r$ grid where the colours indicate the size and runtimes of compilations, respectively. For each cell in the grid, we take the average of instances with the corresponding clause density that lie within a small interval of solution density marked by the cell. Since the number of such instances can differ across the cells, we mark the average for a cell only if the number of instances is greater than 5 (to minimize the variance to a feasible extent).

![Fig. 50: log(#Nodes) in d-DNNF compilation in solution vs clause density grid for 3-CNF](image)

(a) $r \in [0.1, 1.4]$  
(b) $r \in [3.1, 5.0]$

![Fig. 51: log(runtime) for d-DNNF compilation in solution vs clause density grid for 3-CNF](image)

(a) $r \in [0.1, 1.4]$  
(b) $r \in [3.1, 5.0]$