EXPECTED SIGNATURE OF GAUSSIAN PROCESSES WITH STRICTLY REGULAR KERNELS

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Abstract. We compute the expected signature of a class of Gaussian processes which is a subclass of the Gaussian processes with regular kernels, in the sense of [AMN01].

1. Introduction

The expected signature of a stochastic process was first studied by T. Fawcett in [Faw03] and N. Victoire in [LV04], who independently calculated the expected signature of Brownian motion. This is used in [LV04] to calculate the cubature measure which approximates the Wiener measure. Since then, interests in the subject have grown. In [LH11], Ni and Lyons expressed the expected signature of Brownian motion in a disc up to the first exit time in terms of the solution of a PDE. In [Wer12], the first three gradings of the expected signature of the Chordal SLE measure were explicitly calculated. In this article we calculate the expected signature of Gaussian processes with strictly regular kernel (defined in next section). These are Gaussian processes with regular kernels (see [AMN01]) which do not have Brownian components.

The original motivation for undertaking this study was to find the expected signature of fractional Brownian motions for $H > \frac{1}{2}$. After posting this result, we were informed that the calculation of expected signature for fractional Brownian motions for Hurst parameter $H > \frac{1}{3}$ has already appeared in [BC07]. Therefore, this article is a generalisation of the calculation in [BC07].

2. Main result

We will first recall some notations.

Let $T > 0$ be fixed throughout this note.

Let $\triangle' := \{(t,s) \in \mathbb{R}^2 : 0 \leq s < t \leq T\}$.

Let $K(\cdot, r) : \triangle' \to \mathbb{R}$ be a function such that:

(K1) For each $r$, $K(\cdot, r)$ is absolutely continuous on $(r, T]$.

(K2) $K(r^+, r) = 0 \forall r \in [0, T]$.

(K3) Let $|K|(r, T, r)$ denote the total variation of $K(\cdot, r)$ on $(r, T]$. Then

$$\int_0^\infty |K|(r, T, r)^2 \, dr < \infty.$$ 

If we remove the condition (K2) and weaken the absolute continuity condition in (K1) to bounded total variation, then we recover the notion of regular kernel in [AMN01]. These extra conditions mean that our Gaussian processes have to be strictly smoother than Brownian motion.

Let $\partial_1 K$ denote the derivative of $K$ with respect to its first coordinate.
Let \( W_t \) be a Gaussian process of the form

\[
W_t := \int_0^t K(t, r) \, dB_r, \quad t \in [0, T],
\]

where \( dB_r \) denotes the integration in the Itô’s sense.

We shall call a Gaussian process \( W \) of the form (2.1), where \( K \) satisfies (K1), (K2) and (K3), a Gaussian process with a strictly regular kernel \( K \).

In section 3, we shall recall the properties of Gaussian rough paths.

In section 4, we shall prove the main result Theorem 2.

In section 5, we show that our formula coincide with the formula for the fractional Brownian motion case in [BC07].

In section 6, we show that the expected signature is right continuous at \( H = \frac{1}{2} \).

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3. Signature of Gaussian process

3.1. Gaussian process. Recall that \( \triangle':=\{(t,s)\in \mathbb{R}^2: 0\leq s < t \leq T\} \).

Let \( K(\cdot,\cdot):\triangle'\rightarrow \mathbb{R} \) be a function satisfying the conditions (K1), (K2) and (K3) in Section 1.1, and \( W: [0, T] \rightarrow \mathbb{R} \) be a Gaussian process satisfying (2.1).

Then \( W_t \) is a Gaussian process with the covariance function

\[
R(t,s):=\int_0^{t\wedge s} K(t,r)K(s,r)\,dr
\]

The integral in (3.1) exists by condition (K3).

By (K1) and (K2), we have

\[
\partial_1K(\cdot, r) \in L^1([r, T])
\]

and

\[
K(t, u) = \int_r^t \partial_1 K(u, r) \, du.
\]

Thus (3.1) becomes

\[
R(t, s) = \int_0^{t\wedge s} \int_r^t \int_r^s \partial_1 K(u, r) \partial_1 K(v, r) \, du \, dv \, dr.
\]

By Tonelli’s theorem, we have the following equality:

\[
R(t, s) = \int_0^t \int_0^s \left[ \int_0^{u\wedge v} [\partial_1 K(u, r)][\partial_1 K(v, r)] \, dr \right] \, du \, dv.
\]

We shall denote the function \( \int_0^{u\wedge v} [\partial_1 K(u, r)][\partial_1 K(v, r)] \, dr \) by \( f(u, v) \).

Note in particular that \( f(\cdot, \cdot) \in L^1([0, T] \times [0, T]) \) by Tonelli’s theorem and (K3).

To summarise, we have

\[
R(t, s) = \int_0^t \int_0^s f(u, v) \, du \, dv
\]

where \( f \in L^1([0, T] \times [0, T]) \).

This means that for \( \sigma \leq \tau \) and \( s \leq t \) in \( [0, T] \), we have

\[
\mathbb{E}[(W_t - W_s) (W_\tau - W_\sigma)] = \int_\sigma^\tau \int_s^t f(u, v) \, du \, dv.
\]

This expression will be key to our computation.

By the definition of \( f \), we also have \( f(u, v) = f(v, u) \).

We summarise our calculations in the following lemma:

**Lemma 3.** Let \( W: [0, T] \rightarrow \mathbb{R} \) be a Gaussian process with a strictly regular kernel \( K \). Then there exists an integrable function \( f: [0, T]^2 \rightarrow \mathbb{R} \) such that \( f(u, v) = f(v, u) \), and

\[
\mathbb{E}[(W_t - W_s) (W_\tau - W_\sigma)] = \int_\sigma^\tau \int_s^t f(u, v) \, du \, dv.
\]

3.2. Geometric rough paths. Let \( T^n(\mathbb{R}^d) \) and \( T(\mathbb{R}^d) \) denote the graded algebras on \( \mathbb{R}^d \) defined by

\[
T^n(\mathbb{R}^d):=\oplus_{k=0}^n (\mathbb{R}^d)^{\otimes k}
\]

and

\[
T(\mathbb{R}^d):=\oplus_{k=0}^\infty (\mathbb{R}^d)^{\otimes k}
\]

where \( (\mathbb{R}^d)^{\otimes 0}: = \mathbb{R} \).
We shall define three projection maps as follow:

(1) $\pi_n$ will denote the projection map from $T(\mathbb{R}^d)$ to $(\mathbb{R}^d)^{\otimes n}$.
(2) If $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, then $\pi_I$ denote the projection map onto the basis $e_{i_1} \otimes \cdots \otimes e_{i_n}$.
(3) $\pi^{(n)}$ will denote the projection of an element of $T(\mathbb{R}^d)$ onto $T^n(\mathbb{R}^d)$.

We equip $(\mathbb{R}^d)^{\otimes k}$ with a metric by identifying $(\mathbb{R}^d)^{\otimes k}$ with $\mathbb{R}^{k^n}$, and we equip $T^n(\mathbb{R}^d)$ with the metric

$$|w|_{T^n(\mathbb{R}^d)} := \max_{1 \leq k \leq n} |\pi_k(w)|$$

Let $p \geq 1$ and let $\mathcal{V}^p(\mathbb{R}^d)$ denote the set of all continuous functions $f : [0, T] \to \mathbb{R}^d$ with finite $p$-variation, i.e.

$$(3.3) \quad \|f\|_p^p := \sup_{P} \sum_k |f(t_{k+1}) - f(t_k)|^p < \infty$$

where the supremum is taken over all finite partitions $P := (t_0, t_1, \ldots, t_{n-1}, t_n)$, with $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$.

We will now define the lift of functions in $\mathcal{V}^1$.

**Definition 4.** Let $\gamma \in \mathcal{V}^1(\mathbb{R}^d)$ and let $\Delta_n(s, t) := \{(t_1, \ldots, t_n) : s < t_1 < \cdots < t_n < t\}$. The lift of $\gamma$ is a function $S(\gamma) : \{(s, t) : 0 \leq s \leq t\} \to T(\mathbb{R}^d)$ defined by

$$(3.4) \quad S(\gamma)_{s,t} = 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(s,t)} d\gamma_{t_1} \otimes \cdots \otimes d\gamma_{t_n}$$

where the sum + is the direct sum operation in $T(\mathbb{R}^d)$ and the integrals are taken in the Lebesgue-Stieltjes sense.

The signature of a path $\gamma \in \mathcal{V}^1(\mathbb{R}^d)$ is defined to be $S(\gamma)_{0,1}$.

Note in particular that $\pi_n(S(\gamma)_{s,t}) = \int_{\Delta_n(s,t)} d\gamma_{t_1} \otimes \cdots \otimes d\gamma_{t_n}$ and will be called the $n$-th grading of the lift of $\gamma$. We will denote $\pi^{(n)}(S(\gamma)_{s,t})$ by $S_n(\gamma)_{s,t}$.

The signature of a path satisfies the Chen’s identity:

$$S(\gamma)_{s,u} \otimes S(\gamma)_{u,t} = S(\gamma)_{s,t} \quad \forall 0 \leq s \leq u \leq t \leq T$$

For paths which do not have finite variation, such as sample paths of Brownian motion, the signatures have to be defined using geometric rough paths. They are constructed using the following metric. Let $\triangle := \{(s, t) : 0 \leq s \leq t \leq T\}$.

**Definition 5.** Let $n \in \mathbb{N}$ and $p \geq 1$. Let $\mathcal{V}^p(T^n(\mathbb{R}^d))$ denote the set of all continuous functions $w$ from $\triangle$ to $T^n(\mathbb{R}^d)$ such that

1. $\pi_0(w_{s,t}) \equiv 1$.
2. $w$ satisfies

$$\max_{1 \leq k \leq n} \sup_P \left( \sum_{l} |\pi_k(w_{t_{l-1}, t_l})|^{\frac{p}{k}} \right)^{\frac{k}{p}} < \infty$$

where $\sup_P$ runs over all partitions $t_0 = 0 < t_1 < t_2 < \ldots < t_n = T$. 
Let \( w^1, w^2 \) be elements of \( \mathcal{V}^p \left( T^n \left( \mathbb{R}^d \right) \right) \). We define, for each \( p \geq 1 \), a distance function between \( w^1 \) and \( w^2 \) by

\[
\rho_{p-\text{var}} \left( w^1, w^2 \right) = \max_{1 \leq k \leq n} \sup_D \left( \sum_l \left| \pi_k \left( w^1_{t_{l-1}, t_l} \right) - \pi_k \left( w^2_{t_{l-1}, t_l} \right) \right| \right)^{\frac{1}{p}}
\]

where \( \sup_D \) runs over all partitions \( t_0 = 0 < t_1 < t_2 < \ldots < t_n = T \).

For \( p \geq 1 \), let \( \lfloor p \rfloor \) be the integer part of \( p \). The metric \( \rho_{p-\text{var}} \left( \cdot, \cdot \right) \) defined on \( \mathcal{V}^p \left( T^{\lfloor p \rfloor} \left( \mathbb{R}^d \right) \right) \) is known as the \( p \)-variation metric and will be denoted by \( d_p \left( \cdot, \cdot \right) \).

Let \( \mathcal{C}^{p-\text{var}} \left( [0, T], T^n \left( \mathbb{R}^d \right) \right) \) denote the set of all continuous function \( w \) from \( \Delta \) to \( T^n \left( \mathbb{R}^d \right) \) such that \( d_p \left( w, 0 \right) < \infty \). Let \( L \left( T^n \left( \mathbb{R}^d \right) \right) \) denote the set \( \left\{ (s, t) \to \pi^{(n)} \left( S \left( w \right)_{s, t} \right) : w \in \mathcal{V}^1 \left( \mathbb{R}^d \right) \right\} \).

Let \( G\Omega_p \left( \mathbb{R}^d \right) \) be the completion of the set \( L \left( T^{\lfloor p \rfloor} \left( \mathbb{R}^d \right) \right) \) under the \( p \)-variation metric in \( \mathcal{V}^p \left( T^{\lfloor p \rfloor} \left( \mathbb{R}^d \right) \right) \). \( G\Omega_p \left( \mathbb{R}^d \right) \) is called the space of \( p \)-geometric rough paths.

This means that a continuous function \( w : [0, T] \to T^{\lfloor p \rfloor} \left( \mathbb{R}^d \right) \) lies in \( G\Omega_p \left( \mathbb{R}^d \right) \) if we can approximate \( w \) in the \( p \)-variation metric by signatures of paths which have finite variations. One candidate of such approximation is the piecewise linear approximation, defined as below:

**Definition 6.** Let \( w : [0, T] \to \mathbb{R}^d \) be a continuous function.

Let \( D = \{ 0 = t_0 < t_1 < \ldots < t_n = T \} \) be a partition of \( [0, T] \). We define the piecewise linear interpolation of \( w \) with respect to the partition \( D \) by

\[
w_D \left( t \right) := w \left( t_i \right) + \frac{w \left( t_{i+1} \right) - w \left( t_i \right)}{t_{i+1} - t_i} \left( t - t_i \right) \quad t \in \left[ t_i, t_{i+1} \right].
\]

One final fact we need about geometric rough paths is that:

**Theorem 7.** \( \left[ \text{[Lyn98]} \right] \) Let \( p \geq 1 \). Let \( w \in \mathcal{V}^p \left( T^{\lfloor p \rfloor} \left( \mathbb{R}^d \right) \right) \) and that \( w \) is multiplicative, in the sense that

\[
w_s \otimes w_{u, t} = w_{s, t}
\]

for all \( 0 \leq s \leq u \leq t \). Then for all \( N \geq \lfloor p \rfloor \), there exists a unique multiplicative functional \( S_N \left( w \right) \in \mathcal{V}^p \left( T^n \left( \mathbb{R}^d \right) \right) \) such that

\[
\pi^{(\lfloor p \rfloor)} \left( S_N \left( w \right) \right) = w
\]

3.3. **Signature of Gaussian processes.** Recall that \( \Delta \) denotes \( \{ (s, t) : 0 \leq s \leq t \leq T \} \). Let \( f : \Delta \times \Delta \to \mathbb{R} \) be a function. We shall follow \( \left[ \text{[FV10]} \right] \) and use the notation

\[
f \left( \begin{array}{c}
\scriptstyle s, t \\
\scriptstyle u, v
\end{array} \right)
\]

to denote, for \( s \leq t \) and \( u \leq v \),

\[
f \left( \begin{array}{c}
\scriptstyle s, t \\
\scriptstyle u, v
\end{array} \right) := f \left( s, u \right) + f \left( t, v \right) - f \left( s, v \right) - f \left( t, u \right)
\]

and a function \( f : [0, T]^2 \to \mathbb{R} \) is said to have finite \( p \)-variation if \( \left| f \right|_{p-\text{var}: [0, T]^2} < \infty \), where

\[
\left| f \right|_{p-\text{var}: [s, t] \times [u, v]} = \sup_{\{ (t_i) \in D \left( [s, t] \right) \}} \left( \sum_{i,j} \left| f \left( \begin{array}{c}
\scriptstyle t_i, t_{i+1} \\
\scriptstyle t'_j, t'_{j+1}
\end{array} \right) \right| \right)^{\frac{1}{p}}
\]
and $D([a, b])$ be the set of all partitions of the interval $[a, b]$.

For a function $\omega : \triangle \times \triangle \rightarrow [0, \infty)$, we shall denote, for $[s, t] \times [u, v] \in [0, T]^2$,

$$\omega \left( [s, t] \times [u, v] \right) := \omega (s, t, u, v)$$

**Definition 8.** A 2D control is a continuous function $\omega : \triangle \times \triangle \rightarrow [0, \infty)$ such that for all rectangles $R_1, R_2, R$ in $[0, T] \times [0, T]$, such that if $R_1 \cup R_2 \subset R$, $R_1 \cap R_2 = \emptyset$, then

$$\omega (R_1) + \omega (R_2) \leq \omega (R)$$

and that for all rectangles $R$ with zero Lebesgue measure in $\mathbb{R}^2$,

$$\omega (R) = 0$$

**Definition 9.** Let $\omega : \triangle \times \triangle \rightarrow [0, \infty)$ be a 2D control and let $f : \triangle \times \triangle \rightarrow \mathbb{R}$ be a continuous function. We say that $\omega$ controls the $p$-variation of $f$ if for all $[s, t] \times [u, v] \in [0, T]^2$,

$$\left| f \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) \right|^p \leq \omega \left( [s, t] \times [u, v] \right)$$

**Lemma 10.** ([FY10], Lemma 5.56) Let $f : \triangle \times \triangle \rightarrow \mathbb{R}$ be a continuous function. Then $f$ has finite $p$-variation if and only if there exists a 2D control $\omega$ such that $\omega$ controls the $p$-variation of $f$.

We will now recall the existence and some properties of the signatures of Gaussian processes with strictly regular kernels.

Let $X$ be a $d$-dimensional Gaussian process, then its covariance function is defined as $R_X (s, t) := E[X_s \otimes X_t]$.

The following theorem gives the existence of the signatures of some Gaussian processes and the approximation result that the limit of $E \left[ S \left( X^D \right)_{0,T} \right]$ as $\|D\| \rightarrow 0$ is $E \left[ S \left( X \right)_{0,T} \right]$.

**Theorem 11.** ([FR]) Let $X = (X^1, \ldots, X^d) : [0, T] \rightarrow \mathbb{R}^d$ be a centered, continuous Gaussian process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $X^i, X^j$ being independent if $i \neq j$. Assume that the covariance function of $X$, denoted by $R_X$, has finite $p$-variation for some $\rho \in [1, 2)$ and that there exists a finite constant $K$ such that $\|R_X\|_{p{\text{-var, } [0, T]^2}} \leq K$ (see (3.7)). Then there exists a process $X$ with sample paths almost surely in $\mathcal{V}^p \left( T^{[p]} (\mathbb{R}^d) \right)$ for all $p \in (2\rho, 4)$, such that for any $\gamma > \rho$, $\frac{1}{\gamma} + \frac{1}{\rho} > 1$ and any $q > 2\gamma$ and $N \in \mathbb{N}$ there exists a constant $C = C(q, \rho, \gamma, K, N, T)$ such that for all $r \geq 1$,

$$\left| \rho^{(N)}_q \left( S_N \left( X^{(k)} \right), S_N \left( X^{(k)} \right) \right) \right|_{L^r} \leq C r^\frac{N}{2} \sup_{0 \leq t \leq T} \left| X_t^{(k)} - X_t \right|_{L^2}^{1 - \frac{q}{2}}$$

for any piecewise linear interpolation $X^{(k)}$ of $X$, where the width of the partition is smaller than $\frac{1}{T}$.

**Definition 12.** Let $X$ be a process satisfying the conditions in Theorem [11] we shall denote the process $X$ in Theorem [11] by $S (X)$ and the expected signature of $X$ on $[0, T]$ is defined to be $E \left[ S \left( X \right)_{0,T} \right]$. 

Proposition 13. Let \( X \) be a \( d \)-dimensional Gaussian process with a strictly regular kernel. Then the covariance function of \( X \) has finite 1-variation.

Moreover, let \( D^m \) be the dyadic partition \( D^m = \left( 0, \frac{T}{2^m}, \frac{2T}{2^m}, \ldots, \frac{(2^m-1)T}{2^m}, T \right) \). Then for all \( N \in \mathbb{N} \),

\[
\left| \mathbb{E} \left[ S_N \left( X^{D^m} \right) \right] - \mathbb{E} \left[ S_N (X) \right] \right| \to 0
\]
as \( m \to \infty \), where \(| \cdot |\) is the norm induced by identifying \((\mathbb{R}^d)^{\otimes n}\) with \( \mathbb{R}^d \).

**Proof.** By Lemma 3 and its notation, the 1-variation of the covariance function of \( X \), \( R_X \), is controlled by

\[
\omega ([s,t] \times [u,v]) := \int_s^t \int_u^v |f(x,y)| \, dx \, dy
\]

Thus by Lemma 10 \( R_X \) has finite 1-variation. Moreover, the total 1-variation of \( R_X \) on \([0,T]^2\) is bounded above by \( \int_0^t \int_0^T |f(x,y)| \, dx \, dy \). Therefore, Theorem 11 applies.

Observe that we have, in the pathwise sense, the following inequality for all \( N \in \mathbb{N}, q > 2 \):

\[
\left| S_N \left( X^{D^m} \right)_{0,T} - S_N (X)_{0,T} \right| \leq \left| \rho^{(N)}_{q-\text{var}} \left( S_N \left( X^{D^m} \right), S_N (X) \right) \right|
\]

Thus by Theorem 11 the only thing to prove is

\[
\sup_{0 \leq t \leq T} \left| X^{D^m}_t - X_t \right|_{L^2} \to 0
\]

Let \( t^m_k := \frac{k}{2^m} T \). Then for \( t^m_k \leq t \leq t^m_{k+1} \), we have

\[
\left| X^{D^m}_t - X_t \right| \leq \left| X_{t^m_{k+1}} - X_t \right| \left( \frac{t - t^m_k}{t^m_{k+1} - t^m_k} \right) + \left| X_{t^m_k} - X_t \right| \left( \frac{t^m_{k+1} - t}{t^m_{k+1} - t^m_k} \right)
\]

Thus by Lemma 3 and its notation,

\[
\left| X^{D^m}_t - X_t \right|_{L^2} \leq \left| X_{t^m_{k+1}} - X_t \right|_{L^2} \left( \frac{t - t^m_k}{t^m_{k+1} - t^m_k} \right) + \left| X_{t^m_k} - X_t \right|_{L^2} \left( \frac{t^m_{k+1} - t}{t^m_{k+1} - t^m_k} \right)
\]

\[
\leq \left( \frac{t - t^m_k}{t^m_{k+1} - t^m_k} \right) \left( \int_{[t, t^m_{k+1})} |f(u,v)| \, du \, dv \right)^{\frac{1}{2}} + \left( \frac{t^m_{k+1} - t}{t^m_{k+1} - t^m_k} \right) \left( \int_{[t^m_k, t)} |f(u,v)| \, du \, dv \right)^{\frac{1}{2}}
\]

\[
\leq 2 \left( \int_{[t^m_k, t^m_{k+1})} |f(u,v)| \, du \, dv \right)^{\frac{1}{2}}
\]

Note

\[
\left( \sup_{0 \leq t \leq 1} \left| X^{D^m}_t - X_t \right|_{L^2} \right)^2 \leq 4 \max_{1 \leq k \leq 2^m} \int_{[t^m_{k}, t^m_{k+1})} |f(u,v)| \, du \, dv
\]

\[
\leq 4 \int_{\cup_{k=1}^{2^m} [t^m_{k}, t^m_{k+1})} |f(u,v)| \, du \, dv
\]
The set \( U_k \) converge to a set with Lebesgue measure zero as \( m \to \infty \), and \( f \) is integrable on \( \mathbb{R}^2 \). Thus

\[
\left( \sup_{0 \leq t \leq 1} \left| X_t^{D_m} - X_t \right|_{L^2} \right)^2 \to 0
\]
as \( m \to \infty \). \( \square \)

4. Computation

In this section we shall calculate the expected signatures of Gaussian processes with regular kernels.

A key tool in our calculation is the following Wick's formula (also known as Isserlis' theorem):

**Theorem 14.** \([?]\) Let \((W_1, \ldots, W_n)\) be a Gaussian vector. Then

\[
\mathbb{E}(\prod_{i=1}^n W_i) = \begin{cases} 
0 & \text{if } n \text{ is odd}. \\
\sum_{\pi \in \Pi_n} \prod_{(i,j) \in \pi} \mathbb{E}(W_i W_j) & \text{if } n = 2k \text{ is even}.
\end{cases}
\]

We will now carry out some preliminary calculations. Recall that \( f \) is the function defined in Lemma 3.

Let \( t_{i,k}^0 \) denote the dyadic partition point \( \frac{i}{2^k} T \).

Define \( c_{i,j} \), with \( 1 \leq i \leq 2^m, 1 \leq j \leq 2^m \), by

\[
c_{i,j} = \int_{\mathbb{R}^2} f(u,v) 1_{[t_{i-1}^m,t_i^m] \times [t_{j-1}^m,t_j^m]} \, du \, dv
\]

Recall that given a continuous path \( W : [0,T] \to \mathbb{R}^d \), we define the dyadic approximation of \( W \) by the function

\[
W(m)_t = \begin{cases} 
W_0 & t = 0 \\
\sum_i \left( W_{i,n}^r + \frac{2^m}{2} \left( W_{i,n}^r - W_{i-1,n}^r \right) \right) e_i, & t \in (t_{i-1}^m, t_i^m]
\end{cases}
\]

Let \( W \) be a \( d \)-dimensional Gaussian process with a strictly regular kernel, and let \( W(m) \) be the random process obtained from the dyadic linear interpolation of its sample paths.

Let \((k_1, \ldots, k_{2N})\) be a finite sequence of natural numbers, satisfying \( 1 \leq k_1 \leq k_2 \leq \ldots \leq k_{2N} \leq 2^m \). Then the symbol \(|\#k = 1|\) will denote the number of elements in the set \( \{j : k_j = 1\} \), and in general, let \(|\#k = i|\) be the number of elements in the set \( \{j : k_j = i\} \).

The following lemma summarises our preliminary calculation:

**Lemma 15.** If \( k = 2n \) for some \( n \) and \( i_1, \ldots, i_{2n} \in E_{2n} \), then the projection onto the basis \( e_{i_1} \otimes \cdots \otimes e_{i_k} \) of \( \mathbb{E}(S(W(m))_{0,T}) \) is

\[
\sum_{\pi \in \Pi_{n_1,\ldots,n_{2n}}} \sum_{1 \leq k_1 \leq \ldots \leq k_{2n} \leq 2^m} \frac{1}{|\#k = 1|! \ldots |\#k = 2^m|!} \cdot \Pi_{(j,l) \in \pi_{c_{k_1,k_j}}}
\]

otherwise the projection onto \( e_{i_1} \otimes \cdots \otimes e_{i_k} \) of \( \mathbb{E}(S(W(m))_{0,T}) \) is zero.

**Remark 16.** In fact we can prove by induction that

\[
\frac{1}{|\#k = 1|! \ldots |\#k = 2^m|!} = \int_{[t_{k_1-1}^0,t_{k_1}^0] \times \cdots \times [t_{k_{2N}-1}^0,t_{k_{2N}}^0]} \cap \Delta_{2^N} 1 \, du_1 \ldots du_{2N}
\]
but we shall not need it here.

**Proof.** Define \( X_k : [t^m_{k-1}, t^m_k] \to \mathbb{R}^d \) by

\[
X_k(t) = \sum_i \left( W^i_{t^m_k} + \frac{2m}{T} \left( W^i_{t^m_k} - W^i_{t^m_{k-1}} \right) \right) e_i
\]

Then

\[
S(X_k) = 1 + \sum_{n=1}^{\infty} \int_{t^m_{k-1} < s_1 < \ldots < s_n < t^m_k} dX_{s_1} \otimes \ldots \otimes dX_{s_n}
\]

\[
= 1 + \sum_{n=1}^{\infty} \left[ \sum_{i=1}^{2m} \left( W^i_{t^m_k} - W^i_{t^m_{k-1}} \right) \right] \otimes^n d s_1 \ldots d s_n
\]

\[
= 1 + \sum_{n=1}^{\infty} \left[ \sum_{i=1}^{2m} \left( W^i_{t^m_k} - W^i_{t^m_{k-1}} \right) \right] \otimes^n d s_1 \ldots d s_n
\]

By Chen’s identity,

\[
(4.1) \quad S(W(m)_t) = \mathbb{E} \left[ e^{\sum_i (W^i_t - W^i_0)} e_1 \otimes \ldots \otimes e^{\sum_i (W^i_{2m} - W^i_{2m-1})} e_i \right]
\]

We have

\[
e^{\sum_i (W^i_t - W^i_{t-1})} e_i = \sum_{j=0}^{\infty} \frac{1}{j!} \left( \sum_i \left( W^i_t - W^i_{t-1} \right) e_i \right)^j
\]

The coefficient of \( e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_n} \) in the expansion of

\[
\otimes_{k=1}^{2m} \sum_i (W^i_{t_k} - W^i_{t_{k-1}}) e_i = \otimes_{k=1}^{2m} \sum_{j=0}^{\infty} \frac{1}{j!} \left( \sum_i \left( W^i_{t_k} - W^i_{t_{k-1}} \right) e_i \right)^j
\]

is

\[
\sum_{1 \leq k_1 \leq k_2 \leq \ldots \leq k_n \leq 2m} \Pi_{j=1}^{n} \left( W^i_{t_{kj}} - W^i_{t_{kj-1}} \right)
\]

If \( n \) were odd, then as the expected value of the product of an odd number of Gaussian random variables is zero, we have the coefficient of \( e_{i_1} \otimes \ldots \otimes e_{i_n} \) in \( \mathbb{E} \left( S(W(m))_{0,t} \right) \) being zero.

If \((i_1, \ldots, i_{2n}) \notin E_{2n}\), then since the process \((W_t : t \geq 0)\) has independent components and that the expected value of the product of an odd number of Gaussian random variables is zero, thus

\[
\mathbb{E} \left( \Pi_{j=1}^{n} \left( W^i_{t_{kj}} - W^i_{t_{kj-1}} \right) \right) = 0,
\]

which in turn implies that

\[
\pi_{i_1, \ldots, i_{2n}} \mathbb{E} \left[ S(W(m)_{0,1}) \right] = 0
\]

when \((i_1, \ldots, i_{2n}) \notin E_{2n}\).
We now calculate the projection to $e_{i_1} \otimes \cdots \otimes e_{i_{2n}}$ of $E \left( S \left( W \left( m \right) \right)_{0,t} \right)$, which by Wick's formula equals

$$
\sum_{\pi \in \Pi_{i_1, \ldots, 2n}} \left[ E \left( S \left( W \left( m \right) \right)_{0,t} \right) \right] \Pi_{(i,j) \in \pi}
$$

(4.2)

$$
\sum_{\pi \in \Pi_{i_1, \ldots, 2n}} \frac{1}{\left| \Pi_{(i,j) \in \pi} \right|} \left( W_{m}^{i,j} - W_{m}^{i,j-1} \right) \left( W_{m}^{i,j} - W_{m}^{i,j-1} \right)
$$

where $\sum_{\pi \in \Pi_{i_1, \ldots, 2n}}$ is the sum over all possible pairings $(l, j)$ from the set $\Pi_{i_1, \ldots, 2n}$.

The following lemma is crucial to our calculation of the sum in Lemma 15.

**Lemma 17.** For $i \geq 1$ and any pairing $\pi$ of $\{1, \ldots, 2n\}$,

$$
\sum_{1 \leq k_i \leq \cdots \leq k_i = k_i+1 \leq k_i+2 \cdots \leq k_{2n} \leq 2^m} \Pi_{(i,j) \in \pi} k_i, k_j \to 0 \text{ as } m \to \infty
$$

Proof. If $(i, i+1) \in \pi$, then

(4.3)

$$
\sum_{1 \leq k_i \leq \cdots \leq k_i = k_i+1 \leq k_i+2 \cdots \leq k_{2n} \leq 2^m} \Pi_{(i,j) \in \pi} k_i, k_j 
\leq \sum_{k=1}^{2^m} |c_{k,k}| \times \Pi_{(i,j) \in \pi \setminus (i, i+1)} \sum_{k_i=1}^{2^{k-1}} |c_{k_i, k_j}|
$$

Note that for $(j, l) \neq (i, i+1)$,

(4.4)

$$
= \sum_{k=1}^{2^m} \sum_{k_i=1}^{2^{k-1}} \int_{\mathbb{R}^2} |f(u, v)| 1_{\left[ m, t^m_{i, k_i-1} \right]} \times 1_{\left[ m, t^m_{j, k_j-1} \right]} dv
$$

and that

$$
\sum_{k=1}^{2^m} |c_{k,k}| \leq \sum_{k=1}^{2^m} \int_{\mathbb{R}^2} |f(u, v)| 1_{\left[ m, t^m_{i, k_i-1} \right]} \times 1_{\left[ m, t^m_{j, k_j-1} \right]} dv
$$

If $f(\cdot , \cdot) \in L^1 \left( [0, T]^2 \right)$ and the set $\bigcup_{k=1}^{2^m} \left[ t^m_{k} , t^m_{k-1} \right] \times \left[ t^m_{k} , t^m_{k-1} \right]$ converges to a null set in $\mathbb{R}^2$ as $m \to \infty$, we have

$$
\sum_{k=1}^{2^m} |c_{k,k}| \to 0
$$

as $m \to \infty$.

Thus by (4.3) we have

$$
\sum_{1 \leq k_i \leq \cdots \leq k_i = k_i+1 \leq k_i+2 \cdots \leq k_{2n} \leq 2^m} \Pi_{(i,j) \in \pi} k_i, k_j
\leq \int_{[0,T] \times [0,T]} |f(u, v)| dv
\to 0 \text{ as } m \to \infty.
$$

Now if $(i, i+1) \notin \pi$, then let $\pi \left( i \right), \pi \left( i+1 \right)$ denote the unique integers satisfying $(i, \pi \left( i \right)), (i+1, \pi \left( i+1 \right)) \in \pi$. 

Assume now that \((i, i + 1) \notin \pi\), then we have

\[
\sum_{1 \leq k_1 \leq \ldots \leq k_n = k_{i+1} \leq k_{i+2} \ldots \leq k_{2n} \leq 2^2m} \Pi_{(l,j) \in \pi} |c_{lk}| \leq \\
\Pi_{(l,j) \in \pi} \sum_{1 \leq k_1 \leq \ldots \leq k_n = k_{i+1} \leq k_{i+2} \ldots \leq k_{2n} \leq 2^2m} \prod_{k=1}^{2^2m} |c_{lk}| = 1 \sum_{k_1 = 1}^{2^2m} \prod_{k=2}^{2^2m} |c_{lk}| \cdot \sum_{k_1 = 1}^{2^2m} \prod_{k=2}^{2^2m} |c_{lk}|.
\]

Let \(F(v) := \int_0^T |f(u, v)| \, du\). Then as \(f(\cdot, \cdot) \in L^1[0, T]^2\), we have \(F(\cdot) \in L^1[0, T]\). Thus

\[
\sum_{k=1}^{2^m} \int_{\mathbb{R}^2} |f(u, v)| 1_{[\sigma_k^{m}, \sigma_{k+1}^{m}] \times [\tau_k^{m}, \tau_{k-1}^{m}]} \, dudv = \int_0^T F(v) 1_{[\sigma_k^{m}, \sigma_{k+1}^{m}]} (v) \, dv.
\]

Note that since \(F \geq 0\), and that \(F(\cdot) \in L^1\), the integral

\[
\int_0^T \int_0^T F(u) F(v) \, dudv = \int_0^T F(u) \, du \int_0^T F(v) \, dv
\]

exist and thus \((u, v) \rightarrow F(u) F(v)\) is integrable on \([0, T]^2\).

Hence

\[
\sum_{k=1}^{2^m} \int_{\mathbb{R}^2} F(v) F(u) 1_{[\sigma_k^{m}, \sigma_{k+1}^{m}] \times [\tau_k^{m}, \tau_{k-1}^{m}]} (u, v) \, dudv.
\]

As \((u, v) \rightarrow F(u) F(v)\) is integrable on \(L^1[0, T]^2\), and the set \(\bigcup_{k=1}^{2^m} [\sigma_k^{m}, \sigma_{k+1}^{m}] \times [\tau_k^{m}, \tau_{k-1}^{m}]\) converges to a null set in \(\mathbb{R}^2\), so that

\[
\sum_{k=1}^{2^m} \int_{\mathbb{R}^2} F(v) F(u) 1_{[\sigma_k^{m}, \sigma_{k+1}^{m}] \times [\tau_k^{m}, \tau_{k-1}^{m}]} (u, v) \, dudv \to 0,
\]
as \(m \to \infty\).

Together with (4.4) and (4.5) we have

\[
\sum_{1 \leq k_1 \leq \ldots \leq k_n = k_{i+1} \leq k_{i+2} \ldots \leq k_{2n} \leq 2^2m} \Pi_{(l,j) \in \pi} |c_{lk}| \leq \\
\left( \int_{[0,T] \times [0,T]} |f(u, v)| \, dudv \right)^{n-2} \to 0,
\]
as \( m \to \infty \).

This completes the proof of the lemma. \( \square \)

We now prove our main result Theorem 2.

**Proof.** (of Theorem 2) By Proposition 13, the expected signature is given by the limit as \( m \to \infty \) of the sum in Lemma 15.

However, by Lemma 17 for any \( i \geq 1 \), as \( m \to \infty \),

\[
\sum_{1 \leq k_1 \leq \ldots \leq k_i = k_{i+1} \leq k_{i+2} \ldots \leq k_{2n} \leq 2^m} \frac{1}{\#k = 1! \ldots \#k = 2^m} \cdot \Pi_{(i,j) \in \pi} |c_{k_i k_j}| \to 0
\]

Thus

\[
\lim_{m \to \infty} \sum_{1 \leq k_1 \leq \ldots \leq k_i = k_{i+1} \leq k_{i+2} \ldots \leq k_{2n} \leq 2^m} \frac{1}{\#k = 1! \ldots \#k = 2^m} \cdot \Pi_{(i,j) \in \pi} c_{k_i k_j} = \lim_{m \to \infty} \sum_{1 < k_1 < \ldots < k_n \leq 2^m} \Pi_{(j,l) \in \pi} c_{k_i k_j}
\]

where the last equality uses the fact that for \( k_1 < k_2 < \ldots < k_{2n} \), we have \(|\#k = 1! \ldots \#k = 2^m| = 1\).

Note that

\[
\Pi_{(j,l) \in \pi} \int_{\mathbb{R}^2} f(u,v) 1_{\left[t_{k_1}^m, t_{k_1}^{-m}\right] \times \left[t_{k_j}^m, t_{k_j}^{-m}\right]} du dv
\]

\[
= \int_{\mathbb{R}^2} \Pi_{(j,l) \in \pi} f(u, u_t) 1_{\left[t_{k_1}^m, t_{k_1}^{-m}\right] \times \left[t_{k_2}^m, t_{k_2}^{-m}\right]} (u_t, \ldots, u_{2n}) du_t \ldots du_{2n}.
\]

Thus

\[
\lim_{m \to \infty} \sum_{1 < k_1 < \ldots < k_n \leq 2^m} \Pi_{(j,l) \in \pi} c_{k_i k_j}
\]

\[
= \lim_{m \to \infty} \sum_{1 < k_1 < \ldots < k_n \leq 2^m} \int_{\left[t_{k_1}^m, t_{k_1}^{-m}\right] \times \left[t_{k_2}^m, t_{k_2}^{-m}\right]} \Pi_{(j,l) \in \pi} f(u_j, u_t) du_j \ldots du_{2n}.
\]

Let \( \triangle_{2n}(0,T) \) denote the simplex \( \{(u_1, \ldots, u_{2n}) \in \mathbb{R}^{2n} : 0 \leq u_1 < \ldots < u_{2n} \leq T\} \).

Then,

\[
\int_{\left[t_{k_1}^m, t_{k_1}^{-m}\right] \times \left[t_{k_2}^m, t_{k_2}^{-m}\right]} \cap \triangle_{2n}(0,T) \Pi_{(j,l) \in \pi} f(u_j, u_t) du_j \ldots du_{2n}
\]

\[
\leq \int_{\left[t_{k_1}^m, t_{k_1}^{-m}\right] \times \left[t_{k_2}^m, t_{k_2}^{-m}\right]} \cap \triangle_{2n}(0,T) \Pi_{(j,l) \in \pi} f(u_j, u_t) du_j \ldots du_{2n}
\]

\[
= \Pi_{(j,l) \in \pi} |c_{k_i k_j}|.
\]

However, by Lemma 17

\[
\sum_{1 \leq k_1 \leq \ldots \leq k_i = k_{i+1} \leq k_{i+2} \ldots \leq k_{2n} \leq 2^m} \Pi_{(j,l) \in \pi} |c_{k_i k_j}| \to 0 \text{ as } m \to \infty.
\]

Thus for any \( 1 \leq i \), as \( m \to \infty \),

\[
\sum_{1 \leq k_1 \leq \ldots \leq k_i = k_{i+1} \leq k_{i+2} \ldots \leq k_{2n} \leq 2^m} \int_{\left[t_{k_1}^m, t_{k_1}^{-m}\right] \times \left[t_{k_2}^m, t_{k_2}^{-m}\right]} \cap \triangle_{2n}(0,T) \Pi_{(j,l) \in \pi} f(u_j, u_t) du_j \ldots du_{2n}
\]

converges to zero.
Hence,

\[
\lim_{m \to \infty} \sum_{1 \leq k_1, \ldots, k_{2n} \leq 2^m} \int_{t_{k_1}^m}^{t_{k_2}^m} \cdots \int_{t_{k_{2n-1}}^m}^{t_{k_{2n}}^m} \Pi_{(j,l) \in \pi} f(u_j, u_l) du_1 \cdots du_{2n}
= \lim_{m \to \infty} \sum_{1 \leq k_1, \ldots, k_{2n} \leq 2^m} \int_{t_{k_1}^m}^{t_{k_2}^m} \cdots \int_{t_{k_{2n-1}}^m}^{t_{k_{2n}}^m} \cap \Delta_{2n}(0,T) \Pi_{(j,l) \in \pi} f(u_j, u_l) du_1 \cdots du_{2n}
= \int_{\Delta_{2n}} \Pi_{(i,j) \in \pi} f(u_j, u_l) du_1 \cdots du_{2n}.
\]

Finally by Lemma 13 and Lemma 13

\[
\pi_{i_1, \ldots, i_{2n}} \left( E \left( S \left( W \right) \right) \right)_{0,T}
= \lim_{m \to \infty} \pi_{i_1, \ldots, i_{2n}} \left( E \left( S \left( W \left( m \right) \right) \right) \right)_{0,T}
= \sum_{\pi \in \Pi_{i_1, \ldots, i_{2n}}} \int_{\Delta_{2n}} \Pi_{(i,j) \in \pi} f(u_j, u_l) du_1 \cdots du_{2n}.
\]

\[\square\]

5. The fractional Brownian motion with Hurst parameter \( H > \frac{1}{2} \)

We will now show that the formula we give coincide with the following formula for the expected signature of fractional Brownian motion with Hurst parameter \( H > \frac{1}{2} \) calculated in [BC07].

**Proposition 18.** (See [BC07], Theorem 31) Let \( H > \frac{1}{2} \). If \( k = 2n \) for some \( n \in \mathbb{N} \), and \( (i_1, \ldots, i_{2n}) \in E_{2n} \), then the projection to the basis \( e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_k} \) of the expected signature of fractional Brownian Motion with Hurst parameter \( H \) up to time \( T \) is

\[
\sum_{\pi \in \Pi_{i_1, \ldots, i_{2n}}} (H(2H-1)T^H)^n \int_{\Delta_{2n}(1)} \Pi_{(i,j) \in \pi} (u_j - u_l)^{2H-2} du_1 \cdots du_{2n}
\]

where \( \Delta_{2n}(1) \) denotes the simplex \( \{(u_1, \ldots, u_{2n}) \in \mathbb{R}^{2n} : 0 \leq u_1 < \ldots < u_{2n} \leq 1\} \).

The projection to the basis \( e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_k} \) is zero otherwise.

**Proof.** (of Proposition 18) For \( H > \frac{1}{2} \), let \( K_H(t,s) \) be defined, for \( t > s \), as

\[
K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t |u-s|^{H-\frac{1}{2}} u^{H-\frac{1}{2}} du
\]

where \( c_H = \left[ \frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right]^\frac{1}{2} \) and \( \beta \) denotes the beta function.

Then by [NVV99], the fractional Brownian motion \( B^H_t \) can be represented as

\[
B^H_t = \int_0^t K_H(t,s) dB_s
\]

where \( dB_s \) denotes integration in the sense of Itô.

Note that \( K(s^+, s) = 0 \) for all \( s \), \( K(\cdot, s) \) is differentiable and has a positive, integrable derivative, and thus \( K_H \) satisfies (K1) and (K2).
Now note
\[
K_H(t, s) \leq \frac{c_H}{H - \frac{1}{2}} s^\frac{1}{2} t^\frac{1}{2} (t - s)^{H - \frac{1}{2}} \leq \frac{c_H}{H - \frac{1}{2}} s^\frac{1}{2} t^2 H^{-1}
\]
and thus
\[
K_H(t, s)^2 \leq \left[ \frac{c_H}{H - \frac{1}{2}} \right]^2 s^{1 - 2H} t^{H - 2}
\]
and the right hand side is integrable in \(s\).
As \(K(\cdot, s)\) is increasing and \(K(s^+, s) = 0\), \(|K|((s, t], s) = K(t, s) - K(s^+, s) = K(t, s)\) and (K3) is satisfied.

Therefore, Theorem 2 applies and Proposition 18 follows from a change of variable. \(\square\)

6. Right continuity at \(H = \frac{1}{2}\)

We shall prove that the formula for the expected signature in Proposition 18 reconciles with the expected signature of Brownian motion when we take limit as \(H \to \frac{1}{2}\). By the self-similarity property of fractional Brownian motions, it is sufficient to establish this continuity in the case \(T = 1\). First we recall the expected signature of Brownian motion up to time 1:

**Proposition 19.** ([Faw03, LV04]) The \(n^{th}\) level term of the expected signature of Brownian motion up to time 1 is
\[
\frac{1}{2^n n!} \left( \sum_{i=1}^{d} e_i \otimes e_i \right)^n
\]

Equivalently, the projection to any basis of the form \(e_{i_1} \otimes e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_{i_n}\) is equal to \(\frac{1}{2^n n!}\).

Note that a term \(e_{i_1} \otimes e_{i_2} \otimes e_{i_2n}\) would appear in the expansion of \(\left( \sum_{i=1}^{d} e_i \otimes e_i \right)^n\) if and only if the pairing \(\pi_n := \{(1, 2), (3, 4), \ldots, (2n-1, 2n)\}\) is in \(\Pi_{i_1, \ldots, i_{2n}}\). Thus to prove the continuity of the expected signature as \(H \to \frac{1}{2}\), it suffices to prove

1. Let \(\pi_n\) denote the pairing \(\{(1, 2), (3, 4), \ldots, (2n-1, 2n)\}\). Then the integral
\[
I_n := (H(2H - 1))^n \int_{\Delta_{2n}} \Pi_{(l,j) \in \pi_n} (u_j - u_l)^{2H - 2} du_1 \ldots du_{2n}
\]
converges to \(\frac{1}{2^n n!}\) as \(H \to \frac{1}{2}\).
2. The integral
\[
(H(2H - 1))^n \int_{\Delta_{2n}} \Pi_{(l,j) \in \pi} (u_j - u_l)^{2H - 2} du_1 \ldots du_{2n},
\]
converges to 0 as \(H \to \frac{1}{2}\) for any \(\pi \in E_{2n} \setminus \pi_n\).
We will calculate first the integral $I_n$, which we may write explicitly as

$$(H (2H - 1))^n \int_{\Delta_{2n}} \Pi_{j=1}^{n} (u_{2j} - u_{2j-1})^{2H-2} du_1 \ldots du_{2n}$$

By Fubini’s theorem, and using the notation $u_{2n+1} = 1$, we can “integrate $u_{2n}$ first” to obtain:

$$(H (2H - 1))^n \int_{\Delta_{2n}} \Pi_{j=1}^{n} (u_{2j} - u_{2j-1})^{2H-2} du_1 \ldots du_{2n} = (H (2H - 1))^n \int_{0}^{u_{2n-1}} \ldots \int_{0}^{u_{3}} \Pi_{j=1}^{n} (u_{2j} - u_{2j-1})^{2H-2} du_1 \ldots du_{2n-1}

= H^n \int_{0}^{1} \Pi_{j=1}^{n} (u_{2j+1} - u_{2j-1})^{2H-1} du_1 du_3 \ldots du_{2n-1}.$$ 

Taking limit as $H \to \frac{1}{2}$ and using the bounded convergence theorem, we have

$$\lim_{H \to \frac{1}{2}} H^n \int_{0}^{1} \Pi_{j=1}^{n} (u_{2j+1} - u_{2j-1})^{2H-1} du_1 du_3 du_5 \ldots du_{2n-1} = \left( \frac{2}{3} \right)^n \frac{1}{2^{n-1}}.$$ 

Now we consider other pairings $\pi$ in $\Pi_{i=1}^{2n}$, that is when there exists a $k,i$ such that $k - i > 1$ but $(i,k) \in \pi$. We have

$$\nonumber (6.1) \quad (H (2H - 1))^n \int_{\Delta_{2n}} \Pi_{(i,j) \in \pi} (u_j - u_i)^{2H-2} du_1 \ldots du_{2n} \leq (H (2H - 1))^n \int_{\Delta_{2n-2}} \Pi_{(i,j) \in \pi \setminus (i,k)} [(u_j - u_i)^{2H-2} du_i du_j] \times (H (2H - 1)) \int (u_k - u_i)^{2H-2} 1_{[u_{i-1}, u_{i+1}] \times [u_{k-1}, u_{k+1}]} (u_i, u_k) du_i du_k.$$

Note that as $k - 1 \geq i + 1$,

$$
H (2H - 1) \int_{\mathbb{R}^2} (u_k - u_i)^{2H-2} 1_{[u_{i-1}, u_{i+1}] \times [u_{k-1}, u_{k+1}]} (u_i, u_k) du_i du_k \leq H (2H - 1) \int_{\mathbb{R}^2} (u_k - u_i)^{2H-2} 1_{[0, u_{k-1}] \times [u_{k-1}, 1]} (u_i, u_k) du_i du_k
\leq \frac{1}{2} \left[ 1 - u_{k-1}^{2H-2} - (1 - u_{k-1})^{2H} \right]
\leq \frac{1}{2} \left( 1 - \left( \frac{1}{2} \right)^{2H} \right),
$$

where the final inequality holds because $0 \leq 1 - x^{2H} - (1 - x)^{2H} \leq 1 - (\frac{1}{2})^{2H-1}$ for $H > \frac{1}{2}$.

Thus by (6.1),

$$
(H (2H - 1))^n \int_{\Delta_{2n}} \Pi_{(i,j) \in \pi} (u_j - u_i)^{2H-2} du_1 \ldots du_{2n} \leq \frac{1}{2} \left( 1 - \left( \frac{1}{2} \right)^{2H-1} \right) (H (2H - 1))^{n-1} \int_{\Delta_{2n-2}} \Pi_{(i,j) \in \pi \setminus (i,k)} [(u_j - u_i)^{2H-2} du_i du_j].
$$
Note first that
\[
(H(2H - 1))^{n-1} \int_{\Delta_{2n-2}} \Pi_{(i,j) \in \pi \setminus (i,k)} \left[ (u_j - u_i)^{2H-2} du_i du_j \right]
\]
\[
\leq (H(2H - 1))^n \int_{[0,1]^{2n}} \Pi_{(i,j) \in \pi} |u_j - u_i|^{2H-2} du_1 \ldots du_{2n}
\]
\[
= (2H(2H - 1))^n \left( \int_0^1 \int_0^1 |y - x|^{2H-2} dy dx \right)^n
\]
\[
= (2H(2H - 1))^n \left( \int_0^1 \int_0^y |y - x|^{2H-2} dy dx \right)^n
\]
\[
= 1,
\]
therefore,
\[
(H(2H - 1))^n \int_{\Delta_{2n}} \Pi_{(i,j) \in \pi} (u_j - u_i)^{2H-2} du_1 \ldots du_{2n}
\]
\[
\leq \frac{1}{2} \left( 1 - \left( \frac{1}{2} \right)^{2H-1} \right)
\]
\[
\rightarrow 0
\]
as $H \to \frac{1}{2}$.

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