ON THE AFFINE SURFACE AREA

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Abstract. It is shown that at least two expressions that extend the definition of the affine surface area to all convex bodies coincide.

1. Introduction

In the monograph [2] the affine surface area of a convex body $C$ in $\mathbb{R}^3$ with sufficiently smooth boundary is introduced by $\int_{\partial C} k(x)^{1/4} d\mu(x)$ where $k(x)$ is the Gauss-Kronecker curvature and $\mu$ is the surface measure on $\partial C$. It is then shown that this expression equals

$$\lim_{\delta \to 0} \sqrt{\pi} \frac{\text{vol}_3(C) - \text{vol}_3(C_\delta)}{\delta};$$

$C_\delta$ denotes the floating body of $C$: Every supporting hyperplane of $C_\delta$ cuts off a set of volume $\delta$ from $C$. It was shown by Leichtweiß [4] that these expressions generalize in the case of higher dimensions to

$$\int_{\partial C} k(x)^{1/(n+1)} d\mu(x),$$

$$\lim_{\delta \to 0} c_n \frac{\text{vol}_n(C) - \text{vol}_n(C_\delta)}{\delta^{2/(n+1)}}$$

where $c_n = 2(\text{vol}_{n-1}(B_2^{n-1}(0,1))/(n+1)^{2/(n+1)}$, provided that $C$ has a $C^2$-boundary and $k(x)$ is always positive. Leichtweiß also showed that these expressions are equal. The expressions (1) and (2) do not exist for all convex bodies. Therefore, Leichtweiß suggested the following [5] as the definition for the affine surface area:

$$\lim_{\varepsilon \to 0} \lim_{\delta \to 0} nc_n \delta^{-2/(n+1)}(\text{vol}_n(C + B_2^n(0, \varepsilon)) - V((C + B_2^n(0, \varepsilon))),$$

$$\ldots, (C + B_2^n(0, \varepsilon)), (C + B_2^n(0, \varepsilon))_\delta))$$

where $V(\ldots)$ denotes the mixed volume.
At the same time Lutwak [8] gave the following as the definition for the affine surface area:

\[ (4) \quad \inf_{L \in S^n_+} \left\{ \left( \int_{\partial B^n_2} \frac{1}{\rho_L(x)} \, dS(x) \right) \left( n \, \text{vol}_n(L) \right)^{1/n} \right\}^{n/(n+1)} \]

where \( L \) is a star body and \( \rho_L \) its radius.

Leichtweiß [6, 7] proved that (3) is smaller than or equal to (4). It is conjectured that both expressions are equal. In [11] the convex floating body \( C_\delta \) was studied, i.e., the intersection of all halfspaces \( H^+ \) with \( \text{vol}_n(C \cap H^-) = \delta \). Clearly \( C_\delta \) exists for all \( C \) and \( \delta \) and is equal to the floating body whenever it exists. It was shown that

\[ (5) \quad \int_{\partial C} \kappa(x)^{1/(n+1)} \, d\mu(x) = \lim_{\delta \to 0} \frac{\text{vol}_n(C) - \text{vol}_n(C_\delta)}{\delta^{2/(n+1)}} \]

where \( \kappa(x) \) denotes the generalized Gauss-Kronecker curvature [10, p. 25]. A convex function \( \Phi \) on an open subset of \( \mathbb{R}^n \) is said to be twice differentiable in a generalized sense at \( x_0 \) if there is a linear map \( d^2\Phi(x_0) \) from \( \mathbb{R}^n \) into itself so that we have for all \( x \) in a neighborhood \( U(x_0) \) and all subdifferentials \( d\Phi(x_0) \)

\[ ||d\Phi(x) - d\Phi(x_0) - d^2\Phi(x_0)(x - x_0)||_2 \leq C(||x - x_0||_2)||x - x_0||_2, \]

where \( C \) is a function with \( \lim_{t \to 0} C(t) = 0 \). As curvature radius we take the product of the principal axes of the ellipsoid or ellipsoidal cylinder generated by \( d^2\Phi(x_0) \). It follows that (3) equals

\[ (3') \quad \lim_{\varepsilon \to 0} \frac{\text{vol}_n(C + B^n_2(0, \varepsilon)) - \text{vol}_n((C + B^n_2(0, \varepsilon))_{[\delta]})}{\varepsilon^{2/(n+1)}}. \]

We show that the expressions (3) and (5) are equal. Then we show that (5) and, thus, (3) are valuations, a question raised by Leichtweiß [6].

2. Preliminaries

The \( n \)-dimensional volume \( \text{vol}_n(A) \) of a subset \( A \) of \( \mathbb{R}^n \) is the Lebesgue measure, and the \((n - 1)\)-dimensional volume \( \text{vol}_{n-1}(A) \) is the \((n - 1)\)-dimensional Hausdorff measure of \( A \). The surface measure on the boundary of a convex set is the restriction of the \((n - 1)\)-dimensional Hausdorff measure to the boundary. We also note that the Hausdorff measure is Borel regular [3]. \( B^n_2(x, r) \) denotes the Euclidean ball with radius \( r \) and center \( x \) in \( \mathbb{R}^n \).

A convex surface is almost everywhere twice differentiable in a generalized sense [1]. As a consequence the indicatrix of Dupin exists almost everywhere, and thus we can define a generalized Gauss-Kronecker curvature \( \kappa(x) \) that exists almost everywhere [10].

For every \( x \) in the boundary \( \partial C \) of a convex body \( C \) that has a unique normal we define \( \Delta(C, x, \delta) \) or \( \Delta(x, \delta) \) to be the width of a slice of volume \( \delta \) whose defining hyperplane is orthogonal to the normal at \( x \). We have [11]

\[ \kappa(x) = \lim_{\delta \to 0} c^n \frac{\Delta(x, \delta)}{\delta^{2/(n+1)}} \]

where \( c_n \) is as in (2).
For a convex body $C$ in $\mathbb{R}^n$ the nearest point projection $q$ from $\mathbb{R}^n$ onto $C$ is defined by $\|q(x) - x\|_2 = \inf_{y \in C} \|y - x\|_2$. Let $\tilde{C}$ be a convex body containing $C$, and let $p$ be the restriction of $q$ to $\partial \tilde{C}$. Then we have for all Borel subsets $A$ of $\partial \tilde{C}$ that

\begin{equation}
\text{vol}_{n-1}(p(A)) \leq \text{vol}_{n-1}(A).
\end{equation}

A cap of $C$ at $x$ with height $h$ is denoted by cap$(C, x, h)$.

### 3. The equality of (3) and (5)

**Proposition 1.** The expressions (3) and (5) are equal.

Proposition 1 follows from the next lemma. One has to use that $C_\delta$ and $C_{[\delta]}$ coincide whenever $C_{[\delta]}$ exists.

**Lemma 2.** Let $C$ be a convex body in $\mathbb{R}^n$. Then we have

\begin{equation}
\lim_{\delta \to 0} \frac{\text{vol}_n(C) - \text{vol}_n(C_\delta)}{\delta^{2/(n+1)}} = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \delta^{-2/(n+1)} \{\text{vol}_n(C + B^n_2(0, \varepsilon)) - \text{vol}_n((C + B^n_2(0, \varepsilon))_\delta)\}.
\end{equation}

**Proof of Lemma 2.** We show first that the right-hand expression of (8) is smaller than the left-hand expression. We have a.e.

\begin{equation}
\Delta(C + B^n_2(0, \varepsilon), x, \delta) \leq \Delta(C, p(x), \delta)
\end{equation}

where $p$ is the restriction of the nearest point projection from $\partial(C + B^n_2(0, \varepsilon))$ to $\partial C$. Equation (9) follows from

\begin{equation}
cap(C, p(x), h) + \varepsilon N(x) \subseteq \cap(C + B^n_2(0, \varepsilon), x, h).
\end{equation}

If a convex body $C$ in $\mathbb{R}^n$ contains a Euclidean ball of radius $r$ then

\begin{equation}
\text{vol}_{n-1}(\partial C) \leq \text{vol}_{n-1}(\partial(C + B^n_2(0, \varepsilon))) \leq (1 + \varepsilon/r)^{n-1} \text{vol}_{n-1}(\partial C)
\end{equation}

because $C \subseteq C + B^n_2(0, \varepsilon) \subseteq (1 + \varepsilon/r)C$.

Let $A_j$ be measurable subsets of $\partial(C + B^n_2(0, \varepsilon))$ and $a_j \geq 0$ so that

\begin{equation}
\sum_{j=1}^N a_j \chi_{A_j}(x) \leq \lim_{\delta \to 0} c_n \delta^{-2/(n+1)} \Delta(C + B^n_2(0, \varepsilon), x, \delta)
\end{equation}

holds almost everywhere and

\begin{align*}
(1 - \eta) \int_{\partial(C + B^n_2(0, \varepsilon))} \lim_{\delta \to 0} c_n \delta^{-2/(n+1)} \Delta(C + B^n_2(0, \varepsilon), x, \delta) \, d\mu_{C + B^n_2(0, \varepsilon)} \\
\leq \int_{\partial(C + B^n_2(0, \varepsilon))} \sum_{j=1}^N a_j \chi_{A_j} \, d\mu_{C + B^n_2(0, \varepsilon)}.
\end{align*}
Then we get
\[
(1 - \eta) \int_{\partial(C + B_2^n(0, \varepsilon))} \lim_{\delta \to 0} c_\eta \delta^{-2/(n+1)} \Delta(C + B_2^n(0, \varepsilon), x, \delta) \, d\mu_{C + B_2^n(0, \varepsilon)}
\]
\[
\leq \sum_{j=1}^N a_j \mu_{C + B_2^n(0, \varepsilon)}(A_j)
\]
\[
= \sum_{j=1}^N a_j \mu_C(p(A_j)) + \sum_{j=1}^N a_j (\mu_{C + B_2^n(0, \varepsilon)}(A_j) - \mu_C(p(A_j))).
\]
By (9) and
\[
\Delta(B_2^n(0, \varepsilon), (\varepsilon, 0, \ldots , 0), \delta) \leq \Delta(C + B_2^n(0, \varepsilon), x, \delta)
\]
we get that the last expression is smaller than
\[
\int_{\partial C} \lim_{\delta \to 0} c_\eta \delta^{-2/(n+1)} \Delta(C, y, \delta) \, d\mu_C(y)
\]
\[
+ \varepsilon^{-(n-1)/(n+1)}(\operatorname{vol}_{n-1}(\partial(C + B_2^n(0, \varepsilon))) - \operatorname{vol}_{n-1}(\partial C)).
\]
Because of (10), the second summand can be estimated by
\[
\varepsilon^{-(n-1)/(n+1)}((1 + \varepsilon/\delta)^{n-1} - 1) \operatorname{vol}_{n-1}(\partial C).
\]
Therefore, we get altogether
\[
\limsup_{\varepsilon \to 0} \int_{\partial(C + B_2^n(0, \varepsilon))} \lim_{\delta \to 0} c_\eta \delta^{-2/(n+1)} \Delta(C + B_2^n(0, \varepsilon), x, \delta) \, d\mu_{C + B_2^n(0, \varepsilon)}
\]
\[
\leq \int_{\partial C} \lim_{\delta \to 0} c_\eta \delta^{-2/(n+1)} \Delta(C, y, \delta) \, d\mu_C.
\]
In view of (6) we may plug in \(\kappa(x)\).
In order to show that the right-hand side of (8) is larger than the left-hand side we require a lemma.

\textbf{Lemma 3.} Let \(x \in \partial(C + B_2^n(0, \varepsilon))\), and suppose that the indicatrix of Dupin at \(p(x) \in \partial C\) is an ellipsoid with radius \(R = (R_1, \ldots , R_{n-1})\). Then we have
\[
\kappa(\partial(C + B_2^n(0, \varepsilon)), x) = \prod_{i=1}^{n-1} (R_i(p(x)) + \varepsilon)^{-1}.
\]
The set \(\{y \in \partial C | \kappa(y) > 0\}\) is measurable since \(\kappa(y)^{1/(n+1)} \in L^1(\partial C)\) [11]. Since the Hausdorff measure is Borel regular, there is a subset \(A\) of \(\{y \in \partial C | \kappa(y) > 0\}\) that is a Borel set having the same measure. By Lemma 3 we obtain
\[
\int_{\partial(C + B_2^n(0, \varepsilon))} \kappa(x)^{1/(n+1)} \, d\mu_{C + B_2^n(0, \varepsilon)}
\]
\[
\geq \int_{p^{-1}(A)} \prod_{i=1}^{n-1} (R_i(p(x)) + \varepsilon)^{-1/(n+1)} \, d\mu_{C + B_2^n(0, \varepsilon)}.
\]
As above we get that the last expression is larger than or equal to

$$\int_A \prod_{i=1}^{n-1} (R_i(y) + \varepsilon)^{-1/(n+1)} d\mu_C.$$ 

Applying Fatou’s lemma we get

$$\liminf_{\varepsilon \to 0} \int_{\partial(C + B_2^+(0, \varepsilon))} \kappa(x)^{1/(n+1)} d\mu_{C + B_2^+(0, \varepsilon)} \geq \int_{\partial C} \kappa(y)^{1/(n+1)} d\mu_C. \quad \square$$

4. The affine surface area is a valuation

A map $T$ from the family of convex bodies into $\mathbb{R}$ is called a valuation if

$$T(K \cup L) + T(K \cap L) = T(K) + T(L)$$

whenever $K \cup L$ is convex.

**Proposition 4.** The affine surface area is a valuation.

**Lemma 5.** Let $K$ and $L$ be convex bodies in $\mathbb{R}^n$, and suppose that $K \cup L$ is a convex body. Then we have for all $x \in \partial K \cap \partial L$ where all the curvatures $\kappa_{K \cup L}$, $\kappa_{K \cap L}$, $\kappa_K$, and $\kappa_L$ exist that

$$\kappa_{K \cup L}(x) = \min\{\kappa_K(x), \kappa_L(x)\},$$

$$\kappa_{K \cap L}(x) = \max\{\kappa_K(x), \kappa_L(x)\}.$$ 

Please note that the set where one of the curvatures does not exist is a null set [10].

For the proof of Lemma 5 we only have to observe that the indicatrix of Dupin of $K \cup L$ at $x$ is the union of those of $K$ and $L$ at $x$. Moreover, the indicatrix of $K \cap L$ at $x$ is the intersection of those of $K$ and $L$. Then one uses that the intersection or union of two ellipsoids is again an ellipsoid if and only if one ellipsoid is contained in the other.

**Proof of Proposition 4.** The affine surface area of a convex body $M$ equals

$$\int_{\partial M} \kappa_M(x)^{1/(n+1)} d\mu_M.$$ 

We apply this formula to the bodies $K \cup L$, $K \cap L$, $K$, and $L$, and decompose the surfaces

\begin{align*}
\partial(K \cup L) &= \{\partial K \cap \partial L\} \cup \{\partial K \cap L^c\} \cap \{\partial L \cap K^c\}, \\
\partial(K \cap L) &= \{\partial K \cap \partial L\} \cup \{\partial K \cap L^c\} \cup \{\partial L \cap K\}, \\
\partial K &= \{\partial K \cap \partial L\} \cup \{\partial K \cap L^c\} \cup \{\partial K \cap L\}, \\
\partial L &= \{\partial K \cap \partial L\} \cup \{\partial L \cap K^c\} \cup \{\partial L \cap K\},
\end{align*}

where $K^c$ is the complement of $K$ and $K^\circ$ is the interior of $K$.

Since all sets (except possibly $\partial K \cap \partial L$) are open subsets of $\partial K$, $\partial L$, $\partial(K \cap L)$, and $\partial(K \cup L)$ and since the curvature is a local invariant, the
integrals over those sets cancel out. It remains to show

$$\int_{\partial K \cap \partial L} \kappa_{K \cup L}(x) \, d\mu_{K \cup L} + \int_{\partial K \cap \partial L} \kappa_{K \cap L}(x) \, d\mu_{K \cap L}$$

$$= \int_{\partial K \cap \partial L} \kappa_K(x) \, d\mu + \int_{\partial K \cap \partial L} \kappa_L(x) \, d\mu.$$

This follows from Lemma 5. □

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