Dijkgraaf-Vafa conjecture and $\beta$-deformed matrix models

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Abstract

We study the $\beta$-deformed matrix models using the method of refined topological string theory. The refined holomorphic anomaly equation and boundary conditions near the singular divisors of the underlying geometry fix the refined amplitudes recursively. We provide exact test of the quantum integrality conjecture in the Nekrasov-Shatashvili limit. We check the higher genus exact formulae with perturbative matrix model calculations.
1 Introduction and Summary

Recently there have been some interests in refined topological string theory, which originate from the Ω deformation in supersymmetric gauge theory [26]. Certain matrix models for the \( \mathcal{N} = 2 \) gauge theory and refined topological string theory are derived and studied in e.g. [10, 29, 30]. It is expected that the Dijkgraaf-Vafa conjecture [9], which relates matrix models with topological strings on certain local Calabi-Yau manifolds, can be generalized to the refined case. Here the refinement will correspond to the \( \beta \)-deformation of the matrix models. The topological expansion of ordinary matrix model free energy has been constructed from the spectral curves [2, 12], and can be generalized to the \( \beta \)-deformed case [7, 8]. However, the topological recursion method is still too difficult for many practical calculations in the \( \beta \)-deformed case, see e.g. [5, 22]. Furthermore, the holomorphic anomaly equation can be derived from the topological recursion in the undeformed case [11]. For the \( \beta \)-deformed case such a derivation is not available at the moment in the literature. In this paper, we shall study the simple example of \( \beta \)-deformed cubic matrix model. Continuing in the direction of previous works [16, 18, 1], we provide some higher genus formulae from the refined topological string method of holomorphic anomaly equation and the gap boundary conditions near singular divisors [4, 20, 17].

2 Perturbative calculations

The partition function \( Z \) and free energy \( F \) of a Hermitian matrix model with polynomial potential \( W(\Phi) \) is defined by

\[
Z = e^F = \frac{1}{\Vol(U(N))} \int D\Phi e^{-\frac{1}{g_s} W(\Phi)} = \frac{1}{N!(2\pi)^N} \int (\prod_i d\lambda_i) \Delta^2(\lambda) e^{-\sum_i \frac{W(\lambda_i)}{g_s}},
\]

where \( g_s \) is a perturbative expansion parameter, \( \lambda_i (i = 1, 2, \ldots, N) \) are the eigenvalues of the Hermitian matrix \( \Phi \), and \( \Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j) \) is the well known Vandermonde determinant.
determinant. The $\beta$-deformation replaces the matrix integrand by its $\beta$ power, so the partition function becomes

$$Z(\beta) = e^{F(\beta)} = \frac{1}{N!(2\pi)^N} \int \left( \prod_i d\lambda_i \right) \Delta^{2\beta}(\lambda) e^{-\frac{g_s}{\beta} \sum_i W(\lambda_i)}. \quad (2.2)$$

In this paper we mostly consider a cubic polynomial potential

$$W(\Phi) = \frac{1}{2} \Phi^2 + \frac{1}{3} \Phi^3. \quad (2.3)$$

We can compute the free energy perturbatively for small $g_s$, by expanding the exponential and reducing the computation to the expectation values of multi-trace operators in Gaussian matrix model. For the correspondence with Dijkgraaf-Vafa conjecture and topological string theory, we consider two-cut solution of the matrix model in the large $N$ limit. The eigenvalues of the matrix $\Phi$ distribute continuously around the two extrema of the potential (2.3). Here one of the extrema is actually a local maximum, so we need to use an analytic continuation to anti-Hermitian matrix, so that the expansion around the local maximum is the usual Gaussian matrix model.

The perturbative calculations in the case without $\beta$-deformation, i.e. $\beta = 1$, were studied in detail in [19]. The idea is similar in the $\beta$-deformed case, and the partition function of the two-cut solution can be written as a two-matrix model

$$Z = \frac{1}{N_1!(2\pi)^{N_1} N_2!(2\pi)^{N_2}} \int \left( \prod_{i=1}^{N_1} d\mu_i \right) \Delta^{2\beta}(\mu) \left( \prod_{i=1}^{N_2} d\nu_i \right) \Delta^{2\beta}(\nu) e^{-W_1(\Phi_1) - W_2(\Phi_2) - W(\Phi_1, \Phi_2)}, \quad (2.4)$$

where $\mu_i$ and $\nu_j$ are the eigenvalues of the two Hermitian matrices $\Phi_1, \Phi_2$, and the potentials are

$$W_1(\Phi_1) = \text{tr} \left[ \frac{1}{2} \Phi_1^2 + \frac{1}{3} \left( \frac{g_s}{\beta} \right)^2 \Phi_1^3 \right],$$

$$W_2(\Phi_2) = -\text{tr} \left[ \frac{1}{2} \Phi_2^2 - \frac{1}{3} \left( \frac{g_s}{\beta} \right)^2 \Phi_2^3 \right],$$

$$W(\Phi_1, \Phi_2) = \frac{\beta N_2}{6g_s} - \beta N_1 N_2 \log(\frac{g_s}{\beta})$$

$$+ 2\beta \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{g_s}{\beta} \right)^{\frac{2}{3}} \sum_{p=0}^{k} (-1)^p \binom{k}{p} \text{tr}(\Phi_1^p) \text{tr}(\Phi_2^{k-p}). \quad (2.5)$$

Here the interaction term comes from the exponentiation of the interaction of the eigenvalues $\mu_i$ and $\nu_j$ in the Vandermonde determinant. We have rescaled the matrices $\Phi_1$ and $\Phi_2$ by a factor $\left( \frac{g_s}{\beta} \right)^{\frac{2}{3}}$ to normalize the Gaussian potential, and neglected the unimportant overall factor in the partition function from the scaling.

The correlators of multi-trace operators can be computed recursively by the loop equation of the $\beta$-deformed matrix model [24] [25]. Denote the correlators

$$C_{k_1, k_2, \ldots, k_m}(N, \beta) = \frac{\int D_\beta \Phi \text{tr}(\Phi^{k_1}) \text{tr}(\Phi^{k_2}) \cdots \text{tr}(\Phi^{k_m}) \exp(-\frac{\text{tr}(\Phi^2)}{2})}{\int D_\beta \Phi \exp(-\frac{\text{tr}(\Phi^2)}{2})}, \quad (2.6)$$

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where $\Phi$ is a $N \times N$ Hermitian matrix, and the $\beta$-deformed measure is $D_\beta \Phi \sim (\prod_i d\lambda_i) \Delta^{2\beta}(\lambda)$.

Here the indices $k_i$ are non-negative integers and the correlator is understood to be zero by convention if one of the indices is negative. For the cases of odd $\sum_{i=1}^m k_i$, the correlators vanish due to the reflection symmetry of the Gaussian potential. Also obvious is the situation of one of the indices $k_i = 0$, where we can simply pull out a $N$ factor since $\text{tr}(1) = N$. The loop equation, which is the Ward identity of matrix model, provides the following recursion relation for the correlators

$$C_{k_1,k_2,\ldots,k_m} = \beta \sum_{i=0}^{k_m-2} C_{k_1,\ldots,k_{m-1},i,k_{m-2}-i} + \sum_{i=1}^{m-1} k_i C_{k_1,\ldots,k_i+k_{m-2},\ldots,k_{m-1}}$$

$$+ (1 - \beta)(k_m - 1)C_{k_1,\ldots,k_{m-1},k_{m-2}}. \quad (2.7)$$

Here the index $k_m$ for the recursion should be strictly positive $k_m > 0$. For the undeformed case $\beta = 1$, the above recursion can be easily understood from the Feynman rule of $m$ vertices, where a line of the last vertex can be contracted with another line from itself or a line from another vertex. From the recursion relation (2.7) and the initial conditions $C_0 = N$, one can then compute the correlators recursively [25].

We would like to obtain the genus expansion of the free energy. It turns out that there is a nice shift of the t’Hooft parameters so that the odd terms in the expansion vanish [1]. This is similar to the shift of mass parameters $m_i \rightarrow m_i + \frac{\epsilon_1 + \epsilon_2}{2}$ in Seiberg-Witten theory studied in [21, 15].

There are two contributions to the free energy, known as the non-perturbative part and the perturbative part. The non-perturbative contributions come from the measure of the matrix integral and the $U(N)$ volume factor [28], which is obtained by evaluating the partition function (2.4), without the interaction term $W(\Phi_1, \Phi_2)$ and setting $g_s = 0$. The case of $\beta$ deformation is computed in [5]. Up to an unimportant constant from the analytic continuation to Gaussian potential $\Phi_2 \rightarrow i\Phi_2$, the result is

$$F_{n.p.} = \sum_{i=1}^{2} \left[ \frac{\beta^2 t_i^2}{2} (\log(t_i) - \frac{3}{2}) g_s^{-2} + \frac{\beta - 1}{2} (\log(\beta t_i) - 1) t_i g_s^{-1} \right. $$

$$+ \left. \frac{1 - 3\beta + \beta^2}{12\beta} \log(\beta t_i) + \frac{1 - \beta}{24\beta t_i} g_s + \frac{1 - 5\beta^2 + \beta^4}{720\beta^3 t_i^2} g_s^2 + O(g_s^3) \right],$$

where the t’Hooft parameters are defined as

$$t_i = g_s N_i, \quad i = 1, 2. \quad (2.9)$$

We see naively there are odd $g_s$ power terms in (2.8). In refined topological string theory, the free energy comes from the effective action of integrating out charged BPS particles in graviphoton background, and as such only even power terms appear. The situation is remedied by a shift

$$t_i \rightarrow t_i - \frac{\beta - 1}{2\beta} g_s, \quad i = 1, 2. \quad (2.10)$$
The shift cancels the odd power terms in the non-perturbative contribution \(^{(2.8)}\). We use the calligraphic symbol to denote the free energy after the shift

\[
\mathcal{F}_{n.p.} = \sum_{i=1}^{2} \left[ (\frac{\log t_i}{2} - \frac{3}{4} \frac{\beta t_i^2}{g_s^2} - \frac{\beta^2 + 1}{24 \beta} \log(t_i) - \frac{7 \beta^4 + 10 \beta^2 + 7 g_s^2}{5760 \beta^3} \right] + O(g_s^4). \tag{2.11}
\]

Next we compute the perturbative contributions. We can neglect the unimportant terms in the first line in \(W(\Phi_1, \Phi_2)\) in \(^{(2.5)}\), and keep the contributions that are perturbative for small \(g_s\). We expand the remaining exponents in \(^{(2.4)}\), and use the recursion \(^{(2.7)}\). We find the first few terms

\[
F_{pert} = (N_1 - N_2)[2 \beta^2 (2N_1^2 - 13N_1N_2 + 2N_2^2) - 9\beta(\beta - 1)(N_1 + N_2) + (5\beta^2 - 9\beta + 5)\frac{g_s}{6\beta} + [2\beta^3 (8N_1^4 - 91N_1^3N_2 + 177N_1^2N_2^2 - 91N_1N_2^3 + 8N_2^4) - \beta^2(\beta - 1)(59N_1^3 - 81N_1^2N_2 - 81N_1N_2^2 + 59N_2^3) + \beta(73\beta^2 - 132\beta + 73)(N_1^3 + N_2^3) - 2\beta(92\beta^2 - 153\beta + 92)N_1N_2 - (\beta - 1)(30\beta^2 - 43\beta + 30)(N_1 + N_2)] \frac{g_s^2}{6\beta^2} + O(g_s^3). \tag{2.12}
\]

Again we find that the shift \(^{(2.10)}\) eliminates the odd \(g_s\) power terms in the perturbative free energy. Putting the non-perturbative and perturbative contributions (after the shift) together, we can extract the higher genus refined amplitudes \(\mathcal{F}^{(n,g)}\) by the expansion

\[
\mathcal{F}_{n.p.} + \mathcal{F}_{pert} = \sum_{n,g=0}^{\infty} g_s^{2(n+1)-2-2n} (\beta - 1)^{2n} \mathcal{F}^{(n,g)}(t_1, t_2). \tag{2.13}
\]

We list the first few refined topological amplitudes. Here the unrefined cases of \(F^{(0,g)}\) have been computed in \(^{(19)}\), and for completeness we also display them here.

\[
\mathcal{F}^{(0,0)} = \sum_{i=1}^{2} \left( \frac{\log t_i}{2} - \frac{3}{4} \frac{\beta t_i^2}{g_s^2} + \frac{2}{3} \frac{\beta t_i^2}{g_s^2} - \frac{5 t_i^2 t_2 + 5 t_1 t_2^2 - \frac{2}{3} t_1^3}{2} + \frac{1}{3} (8t_1^3 - 91t_1^2t_2 + 177t_i^2 t_2^2 - 91t_1t_2^3 + 8t_2^4) \right) + O(t^4), \tag{2.14}
\]

\[
\mathcal{F}^{(1,0)} = - \sum_{i=1}^{2} \frac{\log t_i}{24} + \frac{19}{12} (t_1 - t_2) + \frac{1}{6} (97t_1^2 - 265t_1t_2 + 97t_2^2) + \frac{1}{18} (4004t_1^3 - 19401t_1^2t_2 + 19401t_1t_2^2 - 4004t_2^3) + O(t^4), \tag{2.15}
\]

\[
\mathcal{F}^{(0,1)} = - \sum_{i=1}^{2} \frac{\log t_i}{12} + \frac{1}{6} (t_1 - t_2) + \frac{1}{3} (7t_1^2 - 31t_1t_2 + 7t_2^2) + \frac{1}{9} (332t_1^3 - 2769t_1^2t_2 + 2769t_1t_2^2 - 332t_2^3) + O(t^4). \tag{2.16}
\]

\[
\mathcal{F}^{(2,0)} = - \sum_{i=1}^{2} \frac{7}{5760 t_i^2} + \frac{131}{48} + \frac{22709}{144} (t_1 - t_2) + \frac{1}{96} (581203t_1^2 - 1449550t_1t_2 + 581203t_2^2) + \frac{23420099(t_1^3 + t_2^3) - 100452495t_1t_2(t_1 - t_2)}{120} + O(t^4) \tag{2.17}
\]
The leading terms in $F(n,g)$ from the non-perturbative contributions are basically the singular terms appearing in integrating out a massless BPS particles in the general graviphoton background near a conifold point in refined topological string theory. We note that there is a $(-1)^n$ factor difference comparing with the convention in [17], probably due to some analytic continuation between these two kinds of calculations.

3 Review of Dijkgraaf-Vafa geometry

Dijkgraaf and Vafa conjectured [9] that the Hermitian matrix model with a degree $n + 1$ polynomial potential $W(\Phi)$ is dual to the topological string theory on a local Calabi-Yau three-fold geometry, defined by the following curve on the $\mathbb{C}^4$ coordinates $(u, v, y, x)$

$$uv = y^2 + W'(x)^2 + f(x),$$

(3.1)

where $f(x)$ is a degree $n - 1$ polynomial whose coefficients parametrize the complex structure moduli of the Calabi-Yau geometry. The A-periods of the Calabi-Yau geometry are identified with the filling fractions, i.e. the t'Hooft parameters in large $N$ limit, around the extrema of the potential in the $n$-cut solution of the matrix model.

We specialize to the case of cubic potential $W(x)$ in (2.3) and introduce some notations. The complex deformation $f(x)$ split the double roots $(a_1, a_2)$ in the equation $W'(x)^2 + f(x) = 0$, whose roots are denoted as $(a_1^-, a_1^+, a_2^-, a_2^+)$ $\equiv (x_1, x_2, x_3, x_4)$. We define the complex parameters

$$z_1 = \frac{(x_2 - x_1)^2}{4}, \quad z_2 = \frac{(x_4 - x_3)^2}{4},$$

(3.2)

which parametrize the complex structure moduli space of the Calabi-Yau geometry. The moduli space and its singular divisors are studied in detail in [16]. Here the divisor $z_1z_2 = 0$ behaves like the conifold divisor. There are two other singular divisors

$$I(z_1, z_2) \equiv \frac{1}{4}[(x_3 + x_4) - (x_1 + x_2)]^2 = 1 - 2(z_1 + z_2),$$

$$J(z_1, z_2) \equiv (x_1 - x_3)(x_2 - x_3)(x_1 - x_4)(x_2 - x_4)$$

$$= 1 - 6(z_1 + z_2) + 9z_1^2 + 14z_1z_2 + 9z_2^2,$$

(3.3)

where the singular divisor $J = 0$ also behave like the conifold divisor, and the higher genus topological string amplitudes expended around the divisor will have gap like behavior [13].
[18]. On the other hand, the singular divisor $I = 0$ is actually an essential singularity where the perturbative solutions to the Picard-Fuchs equation have zero radius of convergence.

The planar free energy $F^{(0)}$ of the matrix model are determined by the well known equation $\frac{\partial F^{(0)}}{\partial t_i} = \Pi_i$, where the periods are

$$t_i = \frac{1}{2\pi i} \int_{a_i^+}^{a_i^-} \lambda dx, \quad \Pi_i = \frac{1}{2\pi i} \int_{a_i^+}^{\Lambda} \lambda dx,$$  \hspace{1cm} (3.4)

as contour integrals over cycles of $x$-plane, with the one-form differential $\lambda dx = \sqrt{W(x)^2 + f(x)dx}$. The expansion for the A-periods to the first few orders are

$$t_1 = \frac{z_1}{4} - \frac{z_1}{8} (2z_1 + 3z_2) - \frac{z_1}{32} (4z_1^2 + 13z_1z_2 + 9z_2^2) + O(z^4),$$
$$t_2 = -\frac{z_2}{4} + \frac{z_2}{8} (2z_2 + 3z_1) + \frac{z_2}{32} (4z_2^2 + 13z_1z_2 + 9z_1^2) + O(z^4). \hspace{1cm} (3.5)$$

To compute the period integrals, one finds differential operators $L(z_1, z_2)$ such that $L\lambda$ is a total derivative of $x$ variable. One also needs to be careful about possible residua, because of which an operator $L$ may not annihilate the periods even though $L\lambda$ is a total derivative [16]. More precisely, up to the second order differentials, there are three linearly independent operators whose actions on $\lambda$ are total derivatives, which are the followings

$$L_1 = [(3 - 2z_1 - 6z_2)\partial_{z_1} - 2z_1(1 - 2z_1 - 6z_2)\partial_{z_1}^2 + (z_1 \leftrightarrow z_2)]$$
$$\hspace{1cm} + 2(1 - 5z_1 - 5z_2 + 6z_1^2 + 4z_1z_2 + 6z_2^2)\partial_{z_1}\partial_{z_2},$$
$$L_2 = -6 + 36(z_1 + z_2) - 6(9z_1^2 + 14z_1z_2 + 9z_2^2) + 2(z_1 + z_2)[-1 + 6z_1 + 6z_2 - 11z_1^2 - 10z_1z_2$$
$$\hspace{1cm} - 11z_2^2 + 6(z_1 - z_2)^2(z_1 + z_2)\partial_{z_1}\partial_{z_2} + [(7z_1 - 3z_2 - 39z_1^2 - 18z_1z_2 + 9z_2^2 + 46z_1^3)$$
$$\hspace{1cm} + 62z_1^2z_2 + 26z_1z_2^2 - 6z_2^3)\partial_{z_1} + 2z_1(1 - 6z_1 - 6z_2 + 13z_1^2 + 14z_1z_2 + 5z_2^2 - 10z_1^3$$
$$\hspace{1cm} - 6z_1^2z_2 + 10z_1z_2^2 + 6z_2^3)\partial_{z_1}^2 + (z_1 \leftrightarrow z_2)],$$
$$L_3 = [(5z_1 + 3z_2)\partial_{z_1} + 2z_1(2 - 5z_1 - 3z_2)\partial_{z_1}^2 - (z_1 \leftrightarrow z_2)]$$
$$\hspace{1cm} -2(z_1 - z_2)(1 - 3z_1 - 3z_2)\partial_{z_1}\partial_{z_2} \hspace{1cm} (3.6)$$

All three operators annihilate the A-periods $t_i$. After adding proper constants from regularizing the integrals, the B-periods $\Pi_i$ can be annihilated by the operators $L_1$ and $L_2$, but not by $L_3$.

The three point couplings are rational functions of $z_1, z_2$, and have been computed in [16]

$$C_{z_1, z_1, z_1} = \frac{1 - (6z_1 + 5z_2) + 3(3z_1^2 + 3z_1z_2 + 2z_2^2)}{16I(z_1, z_2)},$$
$$C_{z_1, z_1, z_2} = \frac{1 - 3z_1 - 5z_2}{16I(z_1, z_2)}, \hspace{1cm} (3.7)$$

and the others are related by symmetry.
4 Genus one formulae

For the genus one case, we have the following exact formulae from refined topological string

\[ \mathcal{F}^{(0,1)} = -\frac{1}{2} \log(\det(\frac{\partial t_i}{\partial z_j})) - \frac{1}{12} \log(z_1z_2) - \frac{1}{4} \log I(z_1, z_2) + \frac{1}{3} \log J(z_1, z_2), \]

\[ \mathcal{F}^{(1,0)} = -\frac{1}{24} \log(z_1z_2) - \frac{1}{12} \log J(z_1, z_2), \]

(4.1)

where \( I, J \) are the singular divisors in (3.3). These formulae can be checked perturbatively with the expansions (2.15), (2.16). For the case of \( \mathcal{F}^{(0,1)} \) amplitude, the exact formula can be also derived from the loop equation in matrix model [6]. However, the case of \( \mathcal{F}^{(1,0)} \) amplitude, which comes from the \( \beta \)-deformation, seems much more challenging to obtain exactly from the matrix model methods.

In [1], the perturbative expansion for \( \mathcal{F}^{(1,0)} \) amplitude is used to test the idea of quantum integrability in the Nekrasov-Shatashvili (NS) limit [27, 23]. Here we should follow the method in [14], and check the formula exactly. First we define the deformed periods

\[ \tilde{t}_i = \oint_{A_i} \tilde{\lambda} dx, \quad \tilde{\Pi}_i = \oint_{B_i} \tilde{\lambda} dx, \]

(4.2)

where the deformed one-form differential \( \tilde{\lambda} dx \) is determined by quantizing the curve

\[ p^2 \Psi(x) = [W'(x)^2 + f(x)]\Psi(x), \]

(4.3)

where \( \Psi(x) \equiv \exp(\frac{1}{\epsilon} \int^{x} \tilde{\lambda} dx) \) is a wave function. The canonical quantization relation is imposed so that the operator \( p = \epsilon \partial_x \). We can use the WKB expansion of the wave function

\[ \tilde{\lambda}(x) = \sum_{n=0}^{\infty} \lambda_n(x)e^n, \]

(4.4)

and solve perturbatively for the first few terms as follows

\[ \lambda(x) \equiv \lambda_0(x) = \sqrt{W'(x)^2 + f(x)}, \]

\[ \lambda_1(x) = -2\frac{\lambda'(x)}{\lambda(x)}, \quad \lambda_2(x) = \frac{2\lambda''(x)\lambda(x) - 3\lambda'(x)^2}{8\lambda(x)^3}, \]

(4.5)

The odd terms are total derivatives and vanish when integrating over a contour. We are interested in the even term contributions. To compute the contour integrals, we should relate the integrand \( \lambda_{2n}(x) \) to the action of some differential operators on \( \lambda(x) \) plus some total derivatives of \( x \). Such a differential operator is found in [1], whose derivatives are with respect to the roots \( x_i(i=1, 2, 3, 4) \) of the quartic equation \( W'(x)^2 + f(x) = 0 \). Here we should write the operator as derivatives of complex structure parameters \( z_1, z_2 \), which is more convenient for calculations. Furthermore, we find that because of the extra Picard-Fuchs operator with non-vanishing residue, the operator is not uniquely determined but is ambiguous by addition of this operator.
We consider the differential operator without the constant and the $\partial_{z_1}\partial_{z_2}$ terms, which can be always eliminated using the Picard-Fuchs operators in \([3,6]\). Up to a total derivative denoted as $h_2'(x)$, we find $\lambda_2 = D_2 \lambda + h_2'(x)$, where the operator

$$D_2 = 1/[6J(z_1, z_2)^3(1-5z_1-5z_2+6z_1^2+4z_1z_2+6z_2^2)]\{(1+12(z_1-z_2)-2(121z_1^2+80z_1z_2
-25z_1^2)+4(331z_1^3+543z_1^2z_2+229z_1z_2^2-15z_2^3)-(3303z_1^4+8084z_1^3z_2+7978z_1^2z_2^2
+2772z_1z_2^3+135z_2^4)+16(243z_1^5+756z_1^4z_2+1123z_1^3z_2^2+909z_1^2z_2^3+270z_1z_2^4+27z_2^5)
-12(144z_1^6+525z_1^5z_2+967z_1^4z_2^2+1314z_1^3z_2^3+894z_1^2z_2^4+225z_1z_2^5+27z_2^6)\} \partial_{z_1}
+[-1+2(13z_1+7z_2)-2(100z_1^2+141z_1z_2+39z_2^2)+4(169z_1^3+388z_1^2z_2+285z_1z_2^2
+54z_2^3)-3(353z_1^4+1098z_1^3z_2+1356z_1^2z_2^2+678z_1z_2^3+99z_2^4)+6(105z_1^5+403z_1^4z_2
+694z_1^3z_2^2+594z_1^2z_2^3+225z_1z_2^4+27z_2^5)]\partial_{z_2}\} \{z_1 \leftrightarrow z_2\}. \quad (4.6)$$

There is still an ambiguity due to linear combination of the Picard-Fuchs operators $L_2$ and $L_3$. We will consider the NS limit and use the perturbative expansion of $F^{(1,0)}$ amplitude \([2,15]\), in order to fix the ambiguity. The operator $D_2$ turns out to be the correct operator which computes the deformed period from the leading term. So we find

$$\tilde{t}_i = t_i + (D_2 t_i) \epsilon^2 + \mathcal{O}(\epsilon^4),$$
$$\tilde{\Pi}_i = \Pi_i + (D_2 \Pi_i) \epsilon^2 + \mathcal{O}(\epsilon^4). \quad (4.7)$$

We denote the deformed prepotential in the NS limit as

$$\tilde{F}^{(0)}(\tilde{t}_i) = \sum_{n=0}^{\infty} F^{(n,0)}(t_i)(\frac{\epsilon}{4})^n, \quad (4.8)$$

where we use a normalization factor of $4^n$ for later convenience. As in \([14]\) we expand the equation for $\tilde{F}^{(0)}$ in terms of deformed periods, to derive differential equations for higher genus terms in the NS limit

$$0 = \partial_{t_i} \tilde{F}^{(0)}(\tilde{t}_i) - \tilde{\Pi}_i$$
$$= \partial_{t_i} F^{(0,0)} - \Pi_i + \epsilon^2 \partial_{t_i} F^{(1,0)} \frac{4}{4} + (D_2 t_i)(\partial_{t_i} \partial_{t_j} F^{(0,0)}) - D_2 \Pi_i \} + \mathcal{O}(\epsilon^4). \quad (4.9)$$

After some algebra, the $\epsilon^2$ order equation can be shown to be equivalent to

$$\frac{1}{4} \partial_{z_i} F^{(1,0)} = C_{z_1z_2z_3} p_{z_1z_2}, \quad (4.10)$$

where $C_{z_1z_2z_3}$ are the three-point Yukawa couplings, and $p_{z_1z_3}$ are the coefficients of $\partial_{z_1} \partial_{z_3}$ in the differential operator $D_2$. We check the above equation \((4.10)\) is satisfied exactly, using the three point functions \((3.7)\) and the differential operator $D_2$ in \((4.6)\).

\section{Higher genus}

The refined holomorphic anomaly equation \([20,17]\) for higher genus $n + g \geq 2$ is a simple generalization of the original BCOV holomorphic anomaly equation

$$\delta_{\bar{z}_i} F^{(n,g)} = \frac{1}{2} C_{\bar{z}_i}^{z_1z_2} (D_{z_1} D_{z_2} F^{(n,g-1)}) + \sum_{r_1,r_2} D_{z_1} F^{(r_1,r_2)} D_{z_2} F^{(n-r_1,g-r_2)} \quad (5.1)$$
where the principal value of the sum over $r_1, r_2$ does not include $(r_1, r_2) = 0$ and $(r_1, r_2) = (n, g)$, and the amplitudes e.g. $F^{(r_1,r_2)}$ on the right hand side are understood to be zero when the indices $r_1 < 0$ or $r_2 < 0$. For the local model that we consider here, there is no contribution to the covariant derivative from the Kahler potential in the form $\partial_z K$. So the covariant derivatives in the second term in the r.h.s. of (5.1) are the same as ordinary derivatives. For the first term, the covariant derivative is $D_{z_l} D_{z_l} F = \partial_{z_l} \partial_{z_k} F - \Gamma_{\bar{z}_l}^{z_l} \partial_{\bar{z}_l} F$, where the Christoffel connection is defined by the Kahler metric as $\Gamma_{\bar{z}_l}^{z_l} = G_{\bar{z}_l \bar{z}_l} \partial_{\bar{z}_l} G_{\bar{z}_l \bar{z}_l}$.

The refined topological string amplitudes at genus $n + g \geq 2$ can be written as polynomials of the propagators $S_{z_i z_j}$ of degrees $2n + 3g - 3$, with rational functions of complex structure moduli $z_{1,2}$ as coefficients [3, 31]. Here for the local geometry we can set the Kahler potential to be constant and only need the double-index $S_{z_i z_j}$ propagators. The propagators are defined in [4] by their anti-holomorphic derivative relation with the three point functions $\partial_{z_k} S_{z_i z_j} = C_{z_k}^{z_i z_j}$. Using the well known special geometry relation, we can integrate

$$S_{z_i z_j} C_{z_k z_l z_i} = -\Gamma_{\bar{z}_l}^{z_l} + f_{z_k z_l}^i,$$  \hspace{1cm} (5.2)

where the $f_{z_k z_l}^i$ are holomorphic ambiguities from the integration, rational functions of $z_{1,2}$, and have been determined in [16] as follows

$$f_{z_1 z_2}^1 = \left[ 6 - (49 z_2 + 48 z_2) + (163 z_2 + 219 z_2 + 126 z_2^2) \right] / (20 z_1 I J),$$
$$f_{z_1 z_2}^2 = f_{z_2 z_1}^1 = \left[ 29 - (79 z_1 + 157 z_2) + (10 z_1^2 + 260 z_1 z_2 + 210 z_2^2) \right] / (20 I J),$$

$$f_{z_2 z_2} = \left[ 7 - (55 z_1 + 68 z_2) + (142 z_1 + 315 z_1 z_2 + 192 z_2^2) \right] / (20 z_2 I J),$$

where the divisors $I, J$ are available in [3, 34]. The other $f_{z_1 z_2}^j$'s follows from the exchange symmetry $(z_1 \leftrightarrow z_2)$. For example, we have $f_{z_2 z_2}^2 = f_{z_1 z_1}^1 (z_1 \leftrightarrow z_2)$.

The holomorphic derivatives of the propagators form a closed algebra

$$\partial_{z_k} S_{z_l z_j} = C_{z_k z_l z_m} S_{z_l z_m} - f_{z_k z_l}^i S_{z_l z_m} - f_{z_k z_m}^j S_{z_l z_m} + h_{z_k}^{z_l z_j},$$  \hspace{1cm} (5.4)

where $h_{z_k}^{z_l z_j}$ are also rational functions of $z_{1,2}$. We also fix them and find it is sufficient to use $J(z_1, z_2)^3$ as the denominator.

Assuming the algebraic independence of the anti-holomorphic derivatives of the propagators, we can write the refined holomorphic anomaly equation (5.1) in terms of partial derivatives with respect to propagators as

$$\partial_{S_{z_i z_j} F^{(n,g)}} = \frac{1}{2} \left( D_{z_l} \partial_{z_l} F^{(n,g-1)} + \sum_{r_1, r_2} \partial_{z_{r_1}} F^{(r_1, r_2)} \partial_{z_{r_2}} F^{(n-r_1, g-r_2)} \right).$$  \hspace{1cm} (5.5)

We note that because of the symmetry of propagators $S_{z_i z_j} = S_{z_j z_i}$, we can use only the propagators $S_{z_i z_j}$ with $i \leq j$ in the polynomial ansatz for $F^{(n,g)}$. As a result, the partial
derivative $\partial_{z_i z_j} F(z, w)$ in the equation (5.5) should be multiplied by a factor of $\frac{1}{2}$ for the case of $i \neq j$ due to double counting.

The derivatives of the genus one amplitudes can be written as

$$
\partial_{z_i} F^{(0,1)} = \frac{1}{2} S^{z_j z_k} C_{z_i z_j z_k},
\partial_{z_i} F^{(1,0)} = \frac{1}{24} \partial_{z_i} \log(z_1 z_2 J^2).
$$

We see that the derivative of the amplitude $F^{(0,1)}$ is a linear function of the propagators with rational function coefficients, while the derivative of the amplitude $F^{(1,0)}$ is simply a rational function of $z_1, z_2$. Utilizing the equations (5.2, 5.4), the right hand side of (5.5) can be computed as a polynomial of the propagators with rational functions of complex structure moduli $z_1, z_2$ as coefficients. So we can integrate this equation and fix the coefficients of the propagators in the refined amplitude $F^{(n,g)}$. For example, in the simplest case of the amplitude $F^{(2,0)}$, the holomorphic anomaly equation (5.5) is

$$
\partial_{S^{z_i z_j}} F^{(2,0)} = \frac{1}{2} (\partial_{z_i} F^{(1,0)}) (\partial_{z_j} F^{(1,0)}),
$$

where r.h.s. is simply a rational function, found in (5.6). We can easily integrate the equation

$$
F^{(2,0)} = \frac{1}{2} S^{z_i z_j} (\partial_{z_i} F^{(1,0)}) (\partial_{z_j} F^{(1,0)}) + f^{(2,0)}(z_1, z_2),
$$

where the integration constant $f^{(2,0)}$, known as the holomorphic ambiguity, is the remaining piece that is not fixed by the holomorphic anomaly equation.

We can further fix the holomorphic ambiguity by the gap boundary conditions near the singular divisors of the Dijkgraaf-Vafa geometry. Here the ansatz for holomorphic ambiguity at genus $(n, g)$ is

$$
f^{(n,g)}(z_1, z_2) = \frac{h^{(n,g)}(z_1, z_2)}{(z_1 z_2 J^2)^{2n+2g-2}},
$$

where $h^{(n,g)}(z_1, z_2)$ is a symmetric polynomial of $z_1$ and $z_2$. The regularity condition of the refined amplitude around $z_1, z_2 \sim \infty$ implies that the degree of $h^{(n,g)}(z_1, z_2)$ is no higher than $12(n + g - 1)$. However the situation turns out to be a little better. In [18], it is found empirically that the degree of the polynomial $h^{(0,n+g)}(z_1, z_2)$ in the unrefined case is no higher than $9(n + g - 1)$. The refined amplitude $F^{(n,g)}$ at genus $(n, g)$ has similar boundary behavior as the unrefined amplitude $F^{(0,n+g)}$, so we should expect that $h^{(n,g)}(z_1, z_2)$ has degree no higher than $9(n + g - 1)$. It turns out that this is indeed the case, i.e. we can fix the holomorphic ambiguity with the gap boundary conditions, using the (5.9) with $h^{(n,g)}(z_1, z_2)$ a symmetric polynomial of degree $9(n + g - 1)$. Although there is no rigorous proof, this empirical fact significantly simplifies the calculations, so we can use it at low genus with precaution as long as it works.
To utilize the gap condition near the conifold divisor $J$, we choose a point on the divisor from its intersection with the line $z_1 = z_2$. Near the point $(z_1, z_2) = \left( \frac{1}{8}, \frac{1}{8} \right)$, the good local coordinate is

$$z_{c,1} = z_1 - z_2, \quad z_{c,2} = 1 - 4(z_1 + z_2).$$

(5.10)

There are 3 power series solutions to the Picard-Fuchs equation near this point

$$w_1 = z_{c,1} \sqrt{1 + z_{c,2}},$$

$$w_2 = z_{c,2}^2 + 4z_{c,1}z_{c,2} + (4z_{c,1}^2 + 3z_{c,1}z_{c,2} + \frac{3}{32}z_{c,2}^4) + O(z_{c,2}^5),$$

$$w_3 = (4z_{c,1}^3 + 3z_{c,1}z_{c,2}^2) + (18z_{c,1}^2z_{c,2} + \frac{3}{2}z_{c,1}z_{c,2}^3) + O(z_{c,2}^5).$$

(5.11)

One chooses the first two solutions as the flat coordinates $t_{c,i} = w_i$ near this point [18].

The singular terms of the refined topological string amplitudes expanded near the divisors $z_1 z_2 = 0$ and $J = 0$ are

$$F^{(n,g)} = c^{n,g} \left( \frac{1}{t_1^{2n+2g-2}} + \frac{1}{t_2^{2n+2g-2}} \right) + O(t_1^0, t_2^0),$$

$$F^{(n,g)} = \frac{(-2)^{11(n+g-1)} c^{n,g}}{t_{c,2}^{2n+2g-2}} + O(t_{c,1}^0, t_{c,2}^0).$$

(5.12)

Here the factor of $(-2)^{11(n+g-1)}$ is due to a normalization of the flat coordinate $t_{c,2}$, and the constants $c^{n,g}$ appear e.g. in [2,17,2,18,2,19] for genus two. These gap conditions (5.12) are sufficient to fix the holomorphic ambiguities, since any non-zero ansatz in (5.9) with $h^{(n,g)}(z_1, z_2)$ a polynomial of degree less than $12(n + g - 1)$ can not cancel all the poles in the denominators, so would affect the singular terms.

We fix the ambiguities and find the exact refined formulae for the genus 2 amplitudes. For example, the holomorphic ambiguity in the formula (5.8) for $F^{(2,0)}$ is

$$f^{(2,0)} = \frac{1}{720z_1^2 z_2^2} \left[ -((11z_1^2 + 7z_1 z_2 + 11z_2^2) + (231z_1^3 + 389z_1^2 z_2 + \cdots) - (2079z_1^4 + 5547z_1^3 z_2 + \cdots) + 4604z_1^2 z_2^2 + \cdots) + 15(693z_1^5 + 2487z_1^4 z_2 + 2324z_1^3 z_2^2 + \cdots) - 9(3465z_1^6 + 15405z_1^5 z_2 + \cdots) + 19435z_1^4 z_2^2 + 8382z_1^3 z_2^3 + \cdots) + 3(1871z_1^7 + 98037z_1^6 z_2 + 172791z_1^5 z_2^2 + 21757z_1^4 z_2^3 + \cdots) + 3(1871z_1^8 + 111699z_1^7 z_2 + 266724z_1^6 z_2^2 + 50973z_1^5 z_2^3 + 272722z_1^4 z_2^4 + \cdots) + 27(1 - z_2)^2(891z_1^7 + 7695z_1^6 z_2 + 32625z_1^5 z_2^2 + 673333z_1^4 z_2^3 + \cdots) \right],$$

(5.13)

where the $\cdots$ denote the terms implied by the symmetry $z_1 \leftrightarrow z_2$. The formulas for $F^{(0,2)}$ and $F^{(1,1)}$ are listed in Appendix [A].

We expand the exact formula in terms of the flat coordinates $t_{1,2}$, using the formula for propagators (5.2) and inverting the expansions (5.5). Here the Christoffel connections can be computed in the holomorphic limit by $\Gamma_{ijk}^{z_{ij}} = (\partial_{t_1} z_k)(\partial_{z_1} \partial_{z_1} t_i)$. We checked the expansions near the point $(z_1, z_2) = (0, 0)$ agree with the higher order terms from perturbative matrix
model calculations in \cite{2.17,2.18,2.19}. Our exact formulas provide a much more efficient way to compute the higher order terms than the perturbative method in matrix model.

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A The formulas for $\mathcal{F}^{(0,2)}$ and $\mathcal{F}^{(1,1)}$

In the Appendix we write down the formulas for the other genus two amplitudes. The unrefined formula $\mathcal{F}^{(0,2)}$ has been obtained in the previous paper \cite{16}. Here we rewrite it in the polynomial formalism. To make the expression compact, we do not write the coefficients of the propagator polynomials completely as the explicit rational functions of $z_{1,2}$. Instead, we use various ingredients such as the three-point Yukawa couplings $C_{ijk}$ in \cite{3.7}, the derivatives of the amplitude $\mathcal{F}^{(1,0)}$ in \cite{5.6} and the rational functions $f^i_{jk}$, $h^i_{jk}$ appeared in \cite{5.3,5.4}.

The amplitudes $\mathcal{F}^{(1,1)}$ and $\mathcal{F}^{(0,2)}$ are respectively quadratic and cubic polynomials of the propagators, from the integration of the holomorphic anomaly equation. The expressions are

$$\mathcal{F}^{(1,1)} = \frac{1}{2} S^{ij} S^{kl} C_{ijk} \partial_i \mathcal{F}^{(1,0)} + \frac{1}{2} S^{ij} (\partial_i \partial_j \mathcal{F}^{(1,0)} - f^{jk}_i \partial_k \mathcal{F}^{(1,0)}) + \frac{1}{15} S^{ij} (\partial_i \mathcal{F}^{(1,0)}) \partial_j \log(z_{1,2}J^2) + f^{(1,1)}(z_{1,2}), \quad (A.1)$$

$$\mathcal{F}^{(0,2)} = S^{ij} S^{kl} S^{mn} \left[ \frac{1}{12} C_{ikm} C_{jln} + \frac{1}{8} C_{ijk} C_{lmn} \right] + \frac{1}{30} S^{ij} S^{kl} C_{ijk} (\partial_i \mathcal{F}^{(1,0)}) + \frac{1}{16} S^{ij} S^{kl} (\partial_i \partial_j C_{kl}) + \frac{1}{4} S^{ij} S^{kl} f^{ij}_{mk} C_{mln} + S^{ij} \left[ \frac{1}{4} C_{ikm} h^k_{ij} + \frac{1}{30} \partial_i \partial_j \log(z_{1,2}J^2) - \frac{1}{30} f^k_{ij} \partial_k \log(z_{1,2}J^2) \right] + \frac{S^{ij}}{450} \partial_i \log(z_{1,2}J^2) \partial_j \log(z_{1,2}J^2) + f^{(0,2)}(z_{1,2}). \quad (A.2)$$

The holomorphic ambiguities are fixed by the gap boundary conditions to be the followings

$$f^{(1,1)} = \frac{1}{9000 z_{1,2}^2 J^4} \left[ (8 z_{1,2}^2 + 28 z_{1,2}^2 + 8 z_{1,2}^2) - 7(24 z_{1,2}^2 + 137 z_{1,2}^2 + \cdots) + 2(756 z_{1,2}^4 + 5937 z_{1,2}^2 + 7450 z_{1,2}^2 + \cdots) - 45(168 z_{1,2}^5 + 1669 z_{1,2}^5 + 2403 z_{1,2}^5 + \cdots) + 24(945 z_{1,2}^6 + 11295 z_{1,2}^6 + 18585 z_{1,2}^6 + 15998 z_{1,2}^6 + \cdots) - 27(1512 z_{1,2}^7 + 21027 z_{1,2}^7 + 39767 z_{1,2}^7 + 23710 z_{1,2}^7 + \cdots) + 162(252 z_{1,2}^8 + 3981 z_{1,2}^8 + 8646 z_{1,2}^8 + 3379 z_{1,2}^8 - 3844 z_{1,2}^8) - 81(z_{1,2}^2)^2 (216 z_{1,2}^2 + 4239 z_{1,2}^2 + 17667 z_{1,2}^2 + 34198 z_{1,2}^2 + \cdots) \right], \quad (A.3)$$

$$f^{(0,2)} = \frac{1}{9000 z_{1,2}^2 J^2} \left[ -1253 + 10503 (z_{1,2} + z_{1,2}) - 27(1081 z_{1,2}^2 + 950 z_{1,2} + 1081 z_{1,2}^2) \right].$$
\[ +26865(z_1 + z_2)(z_1 - z_2)^2, \]  

(A.4)

where \( \cdots \)'s denote terms implied by the exchange symmetry \( z_1 \leftrightarrow z_2 \).

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