Densification Strategies for Anytime Motion Planning over Large Dense Roadmaps

Shushman Choudhury, Oren Salzman, Sanjiban Choudhury and Siddhartha S. Srinivasa

Abstract—We consider the problem of computing shortest paths in a dense motion-planning roadmap \( G \). We assume that \( n \), the number of vertices of \( G \), is very large. Thus, using any path-planning algorithm that directly searches \( G \), running in \( O(n^2) \) time, becomes unacceptably expensive. We are therefore interested in anytime search algorithms that obtain successively shorter feasible paths and converge to the shortest path in \( G \). Our key insight is to leverage existing path-planning algorithms and provide them with a sequence of subgraphs of \( G \). To do so, we study the space of all \( (r,\text{-disk}) \) subgraphs of \( G \). We then formulate and present two densification strategies for traversing this space which exhibit complementary properties with respect to problem difficulty. This inspires a third, hybrid strategy which has favourable properties regardless of problem difficulty. This general approach is then demonstrated and analyzed using the specific case where a low-dispersion deterministic sequence is used to generate the samples used for \( G \). Finally we empirically evaluate the performance of our strategies for random scenarios in \( \mathbb{R}^2 \) and \( \mathbb{R}^4 \) and on manipulation planning problems for a 7 DOF robot arm, and validate our analysis.

I. INTRODUCTION

Let \( G \) be a dense motion-planning roadmap with \( n \) vertices embedded in some configuration space (C-space). We consider the problem of finding a shortest path between two vertices of \( G \). Specifically, we are interested in settings, prevalent in motion planning, where testing if an edge of the graph is collision free or not is computationally expensive. We call such graphs Explicit graphs with Expensive Edge-Evaluation or \( E^4 \)-graphs. Moreover, we are interested in the case where \( n \) is very large. This makes any path-finding algorithm that directly searches \( G \), subsequently performing \( O(n^2) \) edge evaluations, impractical. We wish to obtain an approximation of the shortest path quickly and refine it as time permits. We refer to this problem as anytime planning on large \( E^4 \)-graphs.

Our problem is motivated by previous work (Sec. II) on sampling-based motion-planning algorithms [3], [20] that construct a fixed roadmap as part of a preprocessing stage [17], [1], [25], [26]. These methods are used to efficiently approximate the structure of the C-space using a graph, or a roadmap data structure. When the size of the roadmap is large, even finding a solution, let alone an optimal one, becomes a non-trivial problem requiring specifically-tailored search algorithms [26].

Our key insight for solving the anytime planning problem in large \( E^4 \)-graphs is to provide existing path-planning algorithms with a sequence of subgraphs. Specifically, we consider a sequence of increasingly dense subgraphs of \( G \) using some densification strategy. At each iteration, we run a shortest-path algorithm on the current subgraph to obtain an increasingly tighter approximation of the true shortest path. We present a number of such strategies, and we are interested in the following question:

How does the densification strategy affect the time at which the first solution is found, and the quality of the solutions obtained?

We focus on \( r \)-disk subgraphs of \( G \), i.e. graphs defined by a specific set of vertices where every two vertices are connected if their distance is at most \( r \). This induces a space of subgraphs (Fig. 1) defined by the number of vertices and the connection radius (which, in turn, defines the number of edges). We observe two natural ways to traverse this space. The first is to define each subgraph over the entire set of vertices and incrementally add batches of edges by increasing \( r \) (vertical line at \( |V| = n \) in Fig. 1). Alternatively, we can incrementally

---

*Work by Sh. C., O. S. and S. S. was (partially) funded by the National Science Foundation IIS (#1409003), Toyota Motor Engineering & Manufacturing (TEMA), and the Office of Naval Research.

The Robotics Institute, Carnegie Mellon University {shumshmac, osalzman, sanjibac, ss5} @andrew.cmu.edu

---

![Fig. 1: (a) Our meta algorithm leverages existing path-planning algorithms and provides them with a sequence of subgraphs. (b) To do so we consider densification strategies for traversing the space of \( r \)-disk subgraphs of the roadmap \( G \). The \( x \)-axis and the \( y \)-axis represent the number of vertices and the number of edges (induced by \( r \)) of the subgraph, respectively. A particular subgraph is defined by a point in this space. Edge batching searches over all samples and adds edges according to an increasing radius of connectivity. Vertex batching searches over complete graphs induced by progressively larger subsets of vertices. Hybrid batching uses the minimal connection radius \( f(|V|) \) required to ensure connectivity until it reaches \( |V| = n \) and then proceeds like edge batching. The harder a problem, i.e. the lower the clearance between obstacles, the more vertices are needed by vertex and hybrid batching (to get their first feasible solution).]
We introduce a family of hybrid batching algorithms that perform both vertex and edge batching, and, at each iteration, consider the complete graph \( r = r_{max} \) defined over the current set of vertices (parabolic arc \( |E| = O(|V|^2) \) in Fig. 1). We call these variants edge batching and vertex batching, respectively. Roughly speaking, vertex batching and edge batching are suitable for easy and hard problems, respectively, as visualized and explained in Fig. 2 and Fig. 3. This analysis motivates our hybrid batching strategy, which is more suitable for all ranges of problem difficulty.

Our main contribution is the formulation and analysis of various densification strategies to traverse the space of subgraphs \( G \) (Sec. IV). We demonstrate this general formulation and analysis for the specific case where \( G \) is constructed using a low-dispersion deterministic sequence (Sec. V). Specifically, we describe the structure of the space of subgraphs and analytically demonstrate the tradeoff between effort and bounded suboptimality for our densification strategies. Furthermore, we explain how this tradeoff varies with problem difficulty, which is measured in terms of the clearance of the shortest path in \( G \).

We discuss various implementation decisions and parameters that allow us to efficiently use our strategies in dense \( E \) graphs (Sec. VI). We then empirically validate our analysis on a number of random scenarios in \( \mathbb{R}^2 \) and \( \mathbb{R}^4 \) and on manipulation planning problems for a 7 DOF robot arm (Sec. VII). Finally, we discuss directions of future work (Sec. VIII).

II. RELATED WORK

A. Sampling-based motion planning

Sampling-based approaches have been used to solve motion-planning problems that were previously considered intractable [3], [20]. These planners build a graph, or a roadmap, in the C-space, where vertices are configurations and edges are local paths connecting configurations. A path is then found by traversing this roadmap while checking if the vertices and edges are collision free.

Initial sampling-based motion-planning algorithms such as PRM [17] and RRT [21] were concerned with finding a feasible solution. However, in recent years, there has been growing interest in finding high-quality solutions. Karaman and Frazzoli [15] introduced variants of the PRM and RRT algorithms, called PRM* and RRT*, respectively and proved that asymptotically, the solution obtained by these algorithms converges to the optimal solution. However, the running times of these algorithms is often significantly slower than their non-optimal counterparts. Thus, subsequent algorithms have been suggested to increase the rate of convergence to high-quality solutions. They use different approaches such as lazy computation [1], [14], informed sampling [9], pruning vertices [10], relaxing optimality [24], exploiting local information [4] and lifelong planning together with heuristics [5]. Indeed, in this work we employ several of these techniques in order to speed up the convergence rate of our algorithms.

B. Finite-time properties of sampling-based algorithms

Most analysis on sampling-based motion-planning algorithms has concentrated on asymptotic properties, i.e. properties such as connectivity and optimality when the number of samples tends to infinity [16], [15].

However, we are interested in bounding the quality of a solution obtained using a fixed roadmap. When the samples are generated from a deterministic sequence, Janson et. al. [13, Thm2] give a closed-form solution bounding the quality of the solution of a PRM whose roadmap is an \( r \)-disk graph. The bound is a function of \( r \), the number of vertices \( n \) and the dispersion \( s \) of the set of points used. (See Sec. III for an exact definition of dispersion and for the exact bound given by Janson et. al.).
Dobson et al. [8] provide similar bounds when randomly sampled i.i.d points are used. Specifically, they consider a PRM whose roadmap is an $r$-disk graph for a specific radius $r = c \cdot (\log n/n)^{1/d}$ where $n$ is the number of points, $d$ is the dimension and $c$ is some constant. They then give a bound on the probability that the quality of the solution will be larger than a given threshold.

Both bounds and analyses assume that the optimal path has some clearance $\delta$. Namely, that each point along the path is at least distance $\delta$ from any obstacle.

C. Efficient path-planning algorithms

Path-planning algorithms, which compute the shortest path between two vertices of a given graph, have been studied for many decades. We are interested in algorithms that attempt to reduce the amount of edge expansions performed in a search. This is typically done using heuristics such as for $A^*$ [12], for Iterative Deepening $A^*$ [19] and for Lazy Weighted $A^*$ [5]. Some of these algorithms, such as Lifelong Planning $A^*$ [18] allow recomputing the shortest path in an efficient manner when the graph undergoes changes. *Anytime* variants of $A^*$ such as Anytime Repairing $A^*$ [22] and Anytime Nonparametric $A^*$ [29] efficiently run a succession of $A^*$ searches, each with an inflated heuristic. This potentially obtain a fast approximation and refines its quality as time permits. Having said that, there is no formal guarantee that these approaches will decrease search time and they may still search all edges of a given graph [30]. For a unifying formalism of such algorithms relevant to $E^1$ graphs and additional references, see [7].

III. NOTATION, PROBLEM FORMULATION AND MATHEMATICAL BACKGROUND

We provide standard notation and define our problem concretely. We then provide necessary mathematical background about the *dispersion* of a set of points.

A. Notation and problem formulation

Let $\mathcal{X}$ denote a $d$-dimensional C-space, $\mathcal{X}_{\text{free}}$, the collision-free portion of $\mathcal{X}$, $\mathcal{X}_{\text{obs}} = \mathcal{X} \setminus \mathcal{X}_{\text{free}}$ its complement and let $\rho: \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be some distance metric. For simplicity, we assume that $\mathcal{X} = [0, 1]^d$ and that $\rho$ is the Euclidean norm. Let $S = \{s_1, \ldots, s_n\}$ be some sequence where $s_t \in \mathcal{X}$ and denote by $S(\ell)$ the first $\ell$ elements of $S$. We define the $r$-disk graph $\mathcal{G}(\ell, r) = (V_\ell, E_\ell, r)$ where $V_\ell = S(\ell)$, $E_\ell, r = \{(u, v) | u, v \in V_\ell \text{ and } \rho(u, v) \leq r\}$ and each edge $(u, v)$ has a weight $w(u, v) = \rho(u, v)$. See [15], [27] for various properties of such graphs in the context of motion planning. Finally, set $\mathcal{G} = \mathcal{G}(n, \sqrt{d})$, namely, the complete $1$ graph defined over $S$.

For ease of analysis we assume that the roadmap is complete, but our densification strategies and analysis can be extended to *dense* roadmaps that are not complete. Furthermore, our definition assumes that $\mathcal{G}$ is embedded in the C-space.

Thus, we will use the terms *vertices* and *configurations* as well as *edges* and *paths* in C-space interchangeably.

A query $\mathcal{Q}$ is a scenario with start and target configurations. Let the start and target configurations be $s_1$ and $s_2$, respectively. The obstacles induce a mapping $\mathcal{M}: \mathcal{X} \to \{\mathcal{X}_{\text{free}}, \mathcal{X}_{\text{obs}}\}$ called a collision detector which checks if a configuration or edge is collision-free or not. Typically, edges are checked by densely sampling along the edge, hence the term *expensive edge evaluation*. A feasible path is denoted by $\gamma: [0, 1] \to \mathcal{X}_{\text{free}}$ where $\gamma[0] = s_1$ and $\gamma[1] = s_2$. Slightly abusing this notation, set $\gamma(\mathcal{G}(\ell, r))$ to be the shortest collision-free path from $s_1$ to $s_2$ that can be computed in $\mathcal{G}(\ell, r)$, its clearance as $\delta(\mathcal{G}(\ell, r))$ and denote by $\gamma^* = \gamma(\mathcal{G})$ and $\delta^* = \delta(\mathcal{G})$ the shortest path and its clearance that can be computed in $\mathcal{G}$, respectively.

Our problem calls for finding a sequence of increasingly shorter feasible paths $\gamma_0, \gamma_1 \ldots$ in $\mathcal{G}$, converging to $\gamma^*$. Finally, because $n = |S|$ is very large, any path-finding algorithm that directly searches $\mathcal{G}$, thereby performing $O(n^2)$ calls to $\mathcal{M}$ may be too time-consuming to be practical.

B. Dispersion

The *dispersion* $D_n(S)$ of a sequence $S$ is defined as $D_n(S) = \sup_{x \in \mathcal{X}} \min_{s \in S} \rho(x, s)$. Intuitively, it can be thought of as the radius of the largest empty ball (by some metric) that can be drawn around any point in the space $\mathcal{X}$ without intersecting any point of $S$. A lower dispersion implies a better coverage of the space by the points in $S$. When $\mathcal{X}$ is the $d$-dimensional Euclidean space and $\rho$ is the Euclidean distance, deterministic sequences with dispersion of order $O(n^{-1/d})$ exist. A simple example is a set of points lying on grid or a lattice.

Other low-dispersion deterministic sequences exist which also have low discrepancy, i.e. they appear to be random for many purposes. One such example is the Halton sequence [11]. We will use them extensively for our analysis because they have been studied in the context of deterministic motion planning[20], [13], [2]. Halton sequences are constructed by taking $d$ prime numbers, called generators, one for each dimension. Each generator $g$ induces a sequence, called a Van der Corput sequence. The $k$'th element of the Halton sequence is then constructed by taking the $k$'th element of each of the $d$ Van der Corput sequences.

For Halton sequences, tight bounds on dispersion exist. Specifically, $D_n(S) \leq p_d \cdot n^{-1/d}$ where $p_d \approx d \ln d$ is the $d$'th prime number. In the rest of the paper, when the sequence $S$ is clear from the context, we will omit it and use $D_n$ (and not $D_n(S)$) to denote the dispersion of the first $n$ points of $S$.

Janson et al. bound the length of shortest path computed over an $r$-disk roadmap constructed using a low-dispersion deterministic sequence [13, Thm2]. Specifically, given start and target vertices, consider all paths $\Sigma$ connecting them which have $\delta$-clearance for some $\delta$. Set $\delta_{\text{max}}$ to be the maximal clearance over all such $\delta$. If $\delta_{\text{max}} > 0$, then for all $0 < \delta \leq \delta_{\text{max}}$ set $c^*(\delta)$ to be the cost of the shortest path in $\Sigma$ with $\delta$-clearance. Let $c(\ell, r)$ be the length of the path returned by a
shortest-path algorithm on $G(\ell, r)$ with $S(\ell)$ having dispersion $D_\ell$. For $2D_\ell < r < \delta$, we have that
\[ c(\ell, r) \leq \left( 1 + \frac{2D_\ell}{r - 2D_\ell} \right) \cdot e^a(\delta). \] (1)

Interestingly, for $n$ random i.i.d. points, the lower bound on the dispersion is $O((\log n/n)^{1/4})$ [23] which is strictly larger than for deterministic samples.

IV. APPROACH

We now discuss our general approach of searching over the space of all $(r$-disk) subgraphs of $G$. We start by characterizing the boundaries and different regions of this space. Subsequently, we introduce two densification strategies—edge batching and vertex batching. As we will see, these two are complementary in nature which motivates our third strategy, which we call hybrid batching.

A. The space of subgraphs

To perform an anytime search over $G$, we suggest iteratively considering a sequence of graphs $G_0(n_0, r_0) \subseteq G_1(n_1, r_1) \subseteq \ldots \subseteq G_m(n_m, r_m) = G$. At the $i^\text{th}$ iteration, we perform a shortest-path query over $G_i$.

We use an incremental path-planning algorithm that allows us to efficiently recompute shortest paths. However, any alternative shortest-path algorithm may be used. We emphasize that here we focus on approaches for the meta-algorithm of choosing which subgraphs to search. Further details on the implementation of these approaches are provided in Sec. VI.

Fig. 4 depicts the set of possible graphs $G(\ell, r)$ for all choices of $0 < \ell \leq n$ and $0 < r \leq \sqrt{d}$. Specifically, the graph depicts $|E_{\ell,r}|$ as a function of $|V_e|$. We discuss in detail Fig. 4 to motivate our approach for solving the problem of anytime planning in large $E^d$-graphs and the specific sequence of graphs we use. First, consider the curves that define the boundary of all possible graph $G$: The vertical line $|V| = n$, corresponds to subgraphs defined over the entire set of vertices where batches of edges are added as $r$ increases. The parabolic arc $|E| = |V| \cdot (|V| - 1)/2$, corresponds to complete subgraphs defined over increasingly larger sets of vertices.

Recall that we wish to approximate the path $\gamma^*$ which has some minimal clearance $\delta^*$. Given a specific graph, to ensure that a path that approximates $\gamma^*$ is found, two conditions should be met: (i) The graph includes some minimal number $n_{\min}$ of vertices. The exact value of $n_{\min}$ will be a function of the dispersion $D_n$ of the sequence $S$ and the clearance $\delta^*$. (ii) A minimal connection radius $r_0$ is used to ensure that the graph is connected. Its value will depend on the sequence $S$ (and not on $\delta^*$).

Requirement (i) induces a vertical line at $|V| = n_{\min}$. Any point to the left of this line corresponds to a graph with too few vertices to prove any guarantee that a solution will be found. We call this the vertex-starvation region. Requirement (ii) induces a curve $f(|V|)$ such that any point below this curve corresponds to a graph which may be disconnected. We call this the edge-starvation region. The exact form of the curve depends on the sequence $S$ that is used. The specific value of $n_{\min}$ and the form of $f(|V|)$ when Halton sequences are used is provided in Sec. V. For a comment on random i.i.d sequences, see Sec. VIII.

Any point outside the starvation regions represents a graph $G(\ell, r)$ such that the length of $\gamma(G(\ell, r))$ may be bounded. For a discussion on specific bounds, see Sec. II-B. For a visualization of the different regions, see Fig. 4.

B. Edge and vertex batching

Our goal is to search increasingly dense subgraphs of $G$. This corresponds to a sequence of points on the space of subgraphs (Fig. 4) that ends at the upper right corner of the space. Two natural strategies emerge from this. We defer the discussion on the choice of parameters used for each strategy to Sec. VI.

1) Edge batching: All subgraphs include the complete set of vertices $S$ and the edges are incrementally added via an increasing connection radius. Specifically, $\forall i, n_i = n$ and $r_i = \eta_i r_{i-1}$ where $\eta_i > 1$ and $r_0$ is some small initial radius. Here, we choose $r_0 = O(f(n))$ Using Fig. 4, this induces a sequence of points along the vertical line at $|V| = n$ starting from $|E| = O(n^2 r_0^d)$ and ending at $|E| = O(n^2)$.

2) Vertex batching: In this variant, all subgraphs are complete graphs defined over increasing subsets of $S$. Specifically $\forall i, r_i = \sqrt{d}$, $n_i = \eta_i n_{i-1}$ where $\eta_i > 1$ and $n_0$ is some small number of vertices. Using Fig. 4, this induces a sequence of points along the parabolic arc $|E| = |V| \cdot (|V| - 1)/2$ starting from $|V| = n_0$ and ending at $|E| = O(n^2)$. The vertices are chosen in the same order with which they are generated by $S$. So $G_0$ has the first $n_0$ samples of $S$, and so on.

Intuitively, the relative performance of these densification strategies depends on problem hardness. We use the clearance of the shortest path, $\delta^*$, to represent the hardness of the
problem. This, in turn, defines $n_{\text{min}}$ which bounds the vertex-starvation region. Specifically we say that a problem is easy (resp. hard) when $\delta^* \approx \sqrt{d}$ (resp. $\delta^* \approx \Omega(D_n(S))$). For easy problems, with larger gaps between obstacles vertex batching can find a solution quickly with fewer samples and long edges, thereby restricting the work done for future searches. In contrast, assuming that $n > n_{\text{min}}$, edge batching will find a solution on the first iteration but the time to do so may be far greater than for vertex batching because the number of samples is so large. For hard problems vertex batching requires multiple iterations until the number of samples it uses is larger than $n_{\text{min}}$ and it is out of the vertex-starvation region. Each of these searches would exhaust the fully connected subgraph before terminating. This cumulative effort is expected to exceed that required by edge batching for the same problem, which is expected to find a feasible path on the first search. A visual depiction of this intuition is given in Fig. 5.

C. Hybrid batching

Vertex and edge batching exhibit complementary properties for problems with varying difficulty. Yet, when a query $Q$ is given, the hardness of the problem is not known a-priori. In this section we propose a hybrid approach that exhibits favourable properties, regardless of the hardness of the problem.

This hybrid batching strategy commences by searching over a graph $G(n_0, r_0)$ where $n_0$ is the same as for vertex batching and $r_0 = O(f(n_0))$. As long as $n_i < n$, the next batch has $n_{i+1} = \min\{n, \eta_{i} n_{i}\}$ and $r_i = O(f(n_i))$. When $n_i = n$ and $r_i = O(f(n))$, all subsequent batches are similar to edge batching, i.e., $r_{i+1} = \eta_{i} r_i$ (and $n_{i+1} = n$).

This can be visualized on the space of subgraphs as sampling along the curve $f(|V|)$ from $|V| = n_0$ until $f(|V|)$ intersects $|V| = n$ and then sampling along the vertical line $|V| = n$. See Fig. 1 and Fig. 5 for a mental picture. As we will see in our experiments, hybrid batching typically performs comparably (in terms of path quality) to vertex batching on easy problems and to edge batching on hard problems.

V. Analysis for Halton Sequences

In this section we consider the space of subgraphs and the densification strategies that we introduced in Sec. IV for the specific case that $S$ is a Halton sequence. We start by describing the boundaries of the starvation regions. We then continue by simulating the bound on the quality of the solution obtained as a function of the work done for each of our heuristics.

A. Starvation-region bounds

To bound the vertex starvation region we wish to find $n_{\text{min}}$ after which bounded sub-optimality can be guaranteed to find the first solution. Note that $\delta^*$ is the clearance of the shortest path $\gamma^*$ in $G$ connecting $s_1$ and $s_2$, that $p_d$ denotes the $d^{th}$ prime and $D_n \leq p_d/n^{1/d}$ for Halton sequences. For Eq. (1) to hold we require that $2D_{n_{\text{min}}} < \delta^*$. Thus,

$$2D_{n_{\text{min}}} < \delta^* \Rightarrow 2 \frac{p_d}{n_{\text{min}}^{1/d}} \delta^* \Rightarrow n_{\text{min}} > \left( \frac{2p_d}{\delta^*} \right)^d.$$ 

Indeed, one can see that as the problem becomes harder (namely, $\delta^*$ decreases), $n_{\text{min}}$ and the entire vertex-starvation region grows.

We now show that for Halton sequences, the edge-starvation region has a linear boundary. Namely that $f(|V|) = O(|V|)$. Using Eq. (1) we have that the minimal radius $r_{\text{min}}(|V|)$ required for a graph with $|V|$ vertices is

$$r_{\text{min}}(|V|) > 2D_{|V|} \Rightarrow r_{\text{min}}(|V|) > 2p_d/|V|^{1/d}.$$ 

For any graph $G(\ell, r)$, the number of edges is $|E_{\ell, r}| = O(\ell^2 \cdot r^d)$. In our case we have that,

$$f(|V|) = O(|V|^2 \cdot r_{\text{min}}^d(|V|)) = O(|V|).$$
B. Effort-to-quality ratio

We now compare our densification strategies in terms of their worst-case anytime performance. Specifically, we plot the cumulative amount of work as subgraphs are searched, measured by the maximum number of edges that may be evaluated, as a function of the bound on the quality of the solution that may be obtained using Eq. (1). We fix a specific setting (namely $d$ and $n$) and simulate the work done and the suboptimality using the necessary formulae. This is done for an easy and a hard problem. See Fig. 6.

Indeed, this simulation coincides with our discussion on properties of both batching strategies with respect to the problem difficulty. Vertex batching outperforms edge batching on easy problems and vice versa. Hybrid batching lies somewhere in between the two approaches with the specifics depending on problem difficulty.

VI. IMPLEMENTATION

A. Search Parameters

We choose the parameters for each densification strategy such that the number of batches is $O(\log_2 n)$.

1) Edge Batching: We set $\eta_e = 2^{1/d}$. Recall that for $r$-disk graphs, the average degree of vertices is $n \cdot r^d$, therefore this value (and hence the number of edges) is doubled after each iteration. We set $r_0 = 3 \cdot n^{-1/d}$.

2) Vertex Batching: We set the initial number of vertices $n_0$ to be 100 and set $\eta_v = 2$. Namely, after each batch we double the number of vertices.

3) Hybrid Batching: The parameters are derived from those used for vertex and edge batching. We begin with $n_0 = 100$, and after each batch we increase the vertices by a factor of $\eta_v = 2$. For these searches, i.e. in the region where $n_i < n$, we use $r_i = 3 \cdot n^{-1/d}$. This ensures the same radius at $n$ as for edge batching. Subsequently, we increase the radius as $r_i = \eta_e r_{i-1}$, where $\eta_e = 2^{1/d}$.

B. Optimizations

Our analysis and intuition is agnostic to any specific algorithms or implementations. However, for these densification strategies to be useful in practice, we employ certain optimizations.

1) Search Technique: Each subgraph is searched using Lazy A* [5] with incremental rewiring as in LPA* [18]. For details, see the search algorithm used for a single batch of BIT* [10]. This lazy variant of A* has been shown to outperform other path-planning techniques for motion-planning search problems with expensive edge evaluations [7].

2) Caching Collision Checks: Each time the collision-detector $\mathcal{M}$ is called for an edge, we store the ID of the edge along with the result using a hashing data structure. Subsequent calls for that specific edge are simply lookups in the hashing data structure which incur negligible running time. Thus, $\mathcal{M}$ is called for each edge at most once.

3) Sample Pruning and Rejection: For anytime algorithms, once an initial solution is obtained, subsequent searches should be focused on the subset of states that could potentially improve the solution. When the space $\mathcal{X}$ is Euclidean, this, so-called “informed subset”, can be described by a prolate hyperspheroid [9]. For our densification strategies, we prune away all existing vertices (for all batching), and reject the newer vertices (for vertex and hybrid batching), that fall outside the informed subset. In practice, successive prunings due to intermediate solutions significantly reduces the average-case complexity of future searches [10]. We analyse this for edge batching under the assumption of free space.

Lemma 1 Running edge batching in an obstacle-free $d$-dimensional Euclidean space over a roadmap constructed using a deterministic low-dispersion sequence with $r_0 > 2D_n$ and $r_{i+1} = 2^{1/d} r_i$, while using sample pruning and rejection makes the worst-case complexity of the total search, measured in edge evaluations, $O(n^{1+1/d})$.

For a proof see Appendix A.

VII. EXPERIMENTS

Our implementations of the various strategies are based on the publicly available OMPL [6] implementation of BIT* [10]. Other than the specific parameters and optimizations mentioned earlier, we use the default parameters of BIT*. The notable ones are using Euclidean distance for heuristics, using an approximately sorted queue, and limiting graph pruning to changes in path length greater than 1%.

A. Random scenarios

The different batching strategies are compared to each other on problems in $\mathbb{R}^d$ for $d = 2, 4$. The domain is the unit hyper-cube $[0,1]^d$ while the obstacles are randomly generated axis-aligned $d$-dimensional hyper-rectangles. All problems have a start configuration of $[0.25, 0.25, \ldots]$ and a goal configuration of $[0.75, 0.75, \ldots]$. We used the first $n = 10^4$ and $n = 10^5$ points of the Halton sequence for the $2$ and $4$ problems, respectively.

Two parameters of the obstacles are varied to approximate the notion of problem hardness described earlier - the number of obstacles and the fraction of $\mathcal{X}$ which is in $\mathcal{X}_{\text{obs}}$, which we denote by $\zeta_{\text{obs}}$. Specifically, in $\mathbb{R}^2$, we have easy problems with 100 obstacles and $\zeta_{\text{obs}} = 0.33$, and hard problems with 1000 obstacles and $\zeta_{\text{obs}} = 0.75$. In $\mathbb{R}^4$ we maintain the same values for $\zeta_{\text{obs}}$, but use 500 and 3000 obstacles for easy and hard problems, respectively. For each problem setting ($\mathbb{R}^2, \mathbb{R}^4$; easy/hard) we generate 30 different random scenarios and evaluate each strategy with the same set of samples on each of them. Each random scenario has a different set of solutions, so we show a representative result for each problem setting in Fig. 7.

The results align well with our intuition about the relative performance of the densification strategies on easy and hard problems. Notice that the naive strategy of searching $\mathcal{G}$ directly, requires considerably more time to report the optimum
solution than any other strategy. We mention the numbers in the accompanying caption of Fig. 7 but avoid plotting them so as not to stretch the figures. It is also encouraging to observe that hybrid batching does perform the best on average across all problems and difficulty levels.

B. Manipulation problems

We also run simulated experiments on HERB [28], a mobile manipulator designed and built by the Personal Robotics Lab at Carnegie Mellon University. The planning problems are for the 7-DOF right arm, on the problem scenarios shown in Fig. 8. We use a roadmap of $10^5$ vertices defined using a Halton sequence $\mathcal{S}$ which was generated using the first 7 prime numbers.

In addition to the batching strategies, we also evaluate the performance of BIT* [10], which has been shown to achieve anytime performance superior to contemporary anytime algorithms. We use the default parameters of BIT* as available from the OMPL implementation, only modifying it to use the same set of samples $\mathcal{S}$. The hardness of the problems in terms of clearance is difficult to visualize in terms of the C-space of the arm, but the goal regions are considerably constrained. As our results show (Fig. 8), all densification strategies solve the difficult planning problem in reasonable time, and generally outperform the BIT* strategy on the same set of samples.

VIII. CONCLUSION AND FUTURE WORK

We now discuss several interesting questions that arise from our work which we wish to pursue in the future. In this work we demonstrate our analysis for the case where the set of samples is generated from a low-dispersion deterministic sequence. A natural extension is to provide a similar analysis for a sequence of random i.i.d. samples. Some changes such as the structure of the edge-starvation region are straightforward to analyze. Here, $f(\|V\|) = O(\log |V|)$ [15] instead of $O(|V|)$. When out of the starvation regions we would like to bound the quality obtained similar to the bounds provided by Eq. (1). A starting point would be to leverage recent results by Dobson et al. [8] for Random Geometric Graphs under expectation. However, their analysis is stated only for a specific radius $r$ and should be extended in order to be used in our framework.

A second line of research we wish to pursue is alternative possibilities to traverse the subgraph space of $\mathcal{G}$. As depicted in Fig. 1, our densification strategies are essentially ways to traverse this space. We discuss three techniques that traverse relevant boundaries of the space. But there are innumerable trajectories that a strategy can follow to reach the optimum. It would be interesting to compare our current batching methods, both theoretically and practically, to those that go through the interior of the space.

APPENDIX

We provide a proof of Lemma 1, which we repeat for completeness.

**Lemma** Running edge batching in an obstacle-free $d$-dimensional Euclidean space over a roadmap constructed using a deterministic low-dispersion sequence with $r_0 > 2D_n$ and $r_{i+1} = 2^{1/d}r_i$, while using sample pruning and rejection
makes the worst-case complexity of the total search, measured in edge evaluations, $O(n^{1+1/d})$.

Proof: Let $c_{\text{best}}^i$ denote the cost of the solution obtained after $i$ iterations by our edge batching algorithm, and $c_{\text{min}} = \rho(s_1, s_2) \leq \sqrt{d}$ denote the cost of the optimal solution. Using Eq. (1),

$$c_{\text{best}}^i \leq (1 + \varepsilon_i) c_{\text{min}},$$  

where $\varepsilon_i = \frac{2D_n}{r_i - 2D_n}$. Using the parameters for edge batching,

$$\varepsilon_{i+1} = \frac{2D_n}{r_{i+1} - 2D_n} = \frac{2D_n}{2\pi r_i - 2D_n} \leq \frac{\varepsilon_i}{2\pi}.$$  

Let $i_{\text{max}}$ be the maximum number of iterations and recall that we have $i_{\text{max}} = O(\log_2 n)$.

Note that the fact that vertices and edges are pruned away, does not change the bound provided in Eq. (2). To compute the actual number of edges considered at the $i$th iteration, we bound the volume of the the prolate hyperspheriod $X_{\text{chbest}}^i$ in $\mathbb{R}^d$ (see [9]) by,

$$\mu \left( X_{\text{chbest}}^i \right) = \left( c_{\text{best}}^i - c_{\text{min}}^2 \right) \frac{\varepsilon_{i+1}^{d-1}}{2^d} \frac{\varepsilon_{i+1}^d}{\Gamma_d},$$  

where $\varepsilon_d$ is the volume of an $\mathbb{R}^d$ unit-ball. Using Eq. (2) in Eq. (4),

$$\mu \left( X_{\text{chbest}}^i \right) \leq \varepsilon_{i+1}^{d-1} \left( 1 + \varepsilon_i \right) \left( 2 + \varepsilon_0 \right) \frac{\varepsilon_{i+1}^d}{\Gamma_d} \leq \frac{\varepsilon_i^{d-1}}{\Gamma_d} \left( 1 + \varepsilon_i \right) \left( 2 + \varepsilon_0 \right) \frac{\varepsilon_{i+1}^d}{\Gamma_d} \leq 2^{\frac{i-(d-1)}{2d}} \mu \left( X_{\text{chbest}}^0 \right).$$  

Furthermore, we choose $r_0$ such that $\mu \left( X_{\text{chbest}}^0 \right) \leq \mu (X)$. Now, the number of vertices in $X_{\text{chbest}}^i$ can be bounded by,

$$n_{i+1} = \frac{\mu \left( X_{\text{chbest}}^i \right)}{\mu (X)} n \leq 2^{\frac{i-(d-1)}{2d}} n.$$  

Recall that we measure the amount of work done by the search at iteration $i$ using $|E_i|$, the number of edges considered. Thus,

$$|E_i| = O(n^2 r_i^d) = O \left( n^2 2^{\frac{i-(d-1)}{2d}} r_0 2^{\frac{d}{2}} \right) = O \left( n^{2\frac{d}{d+1}} \right).$$  

Finally, the total work done by the search over all iterations is

$$O \left( \sum_{i=0}^{\log_2 n} n 2^{\frac{d}{d+1}} \right) = O \left( n \sum_{i=0}^{\log_2 n} 2^{i/d} \right) = O \left( n^{1+\frac{1}{d}} \right).$$  

REFERENCES

[1] R. Bohlin and L. E. Kavraki. Path planning using lazy PRM. In IEEE International Conference on Robotics and Automation, pages 521–528, 2000.

[2] M. S. Branicky, S. M. LaValle, K. Olson, and L. Yang. Quasi-randomized path planning. In IEEE International Conference on Robotics and Automation, pages 1481–1487, 2001.

[3] H. Choset, K. M. Lynch, S. Hutchinson, G. Kantor, W. Burgard, L. E. Kavraki, and S. Thrun. Principles of Robot Motion: Theory, Algorithms, and Implementation. MIT Press, June 2005.

[4] S. Choudhury, J. D. Gammell, T. D. Barfoot, S. S. Srinivasa, and S. Scherer. Regionally accelerated batch informed trees (rabit*): A framework to integrate local information into optimal path planning. In 2016 IEEE International Conference on Robotics and Automation (ICRA), pages 4207–4214, IEEE, 2016.

[5] B. J. Cohen, M. Phillips, and M. Likhachev. Planning single-arm manipulations with n-arm robots. In Robotics: Science and Systems, 2014.

[6] I. A. Sucan, M. Moll, and L. E. Kavraki. The Open Motion Planning Library. IEEE Robotics & Automation Magazine, 2012.

[7] C. M. Dellin and S. S. Srinivasa. A unifying formalism for shortest path problems with expensive edge evaluations via lazy best-first search over paths with edge selectors. In International Conference on Automated Planning and Scheduling, pages 459–467, 2016.

[8] A. Dobson, G. V. Moustakides, and K. E. Bekris. Geometric probability results for bounding path quality in sampling-based roadmaps after finite computation. In IEEE International Conference on Robotics and Automation, 2015.

[9] J. D. Gammell, S. S. Srinivasa, and T. D. Barfoot. Informed RRT*: Optimal sampling-based path planning focused via direct sampling of an admissible ellipsoidal heuristic. In IEEE/RSJ International Conference on Intelligent Robots and Systems, pages 2997–3004, 2014.

[10] J. D. Gammell, S. S. Srinivasa, and T. D. Barfoot. Batch informed trees (BIT*): Sampling-based optimal planning via the heuristically guided search of implicit random geometric graphs. In IEEE International Conference on Robotics and Automation, pages 3067–3074, 2015.

[11] J. H. Halton. On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals. Numer. Math., 2(1):84–90, 1960.

[12] P. E. Hart, N. J. Nilsson, and B. Raphael. A formal basis for the heuristic determination of minimum cost paths. IEEE Transactions on Systems, Science, and Cybernetics, SSC-4(2):100–107, 1968.

[13] L. Janson, B. Ichter, and M. Pavone. Deterministic sampling-based motion planning: Optimality, complexity, and performance. CoRR, abs/1505.00023, 2015.

[14] L. Janson, E. Schmerling, A. Clark, and M. Pavone. Fast marching tree: A fast marching sampling-based method for optimal motion planning in many dimensions. I. J. Robotics Res., pages 883–921, 2015.

[15] S. Karaman and E. Frazzoli. Sampling-based algorithms for optimal motion planning. I. J. Robotics Res., 30(7):846–894, 2011.

[16] L. E. Kavraki, M. N. Kolountrakis, and J. Latombe. Analysis of probabilistic roadmaps for path planning. IEEE Trans. Robotics and Automation, 14(1):166–171, 1998.

[17] L. E. Kavraki, P. Svestka, J.-C. Latombe, and M. H. Overmars. Probabilistic roadmaps for path planning in high dimensional configuration spaces. IEEE Trans. Robotics, 12(4):566–580, 1996.

[18] S. Koenig, M. Likhachev, and D. Furcy. Lifelong planning A*: Artificial Intelligence, 155(1):93–146, 2004.

[19] R. E. Korf. Iterative-deepening-A*: An optimal admissible tree search. In Joint Conference on Artificial Intelligence, pages 1034–1036, 1985.

[20] S. M. LaValle. Planning algorithms. Cambridge University Press, 2006.

[21] S. M. LaValle and J. J. Kuffner. Randomized kinodynamic planning. In IEEE International Conference on Robotics and Automation, pages 459–467, 2001.

[22] S. Scherer. Regionally accelerated batch informed trees (rabit*): A randomized path planning. In IEEE International Conference on Robotics and Automation, pages 3067–3074, 2015.

[23] M. Likhachev, G. J. Gordon, and S. Thrun. ARA*: Anytime A* with provable bounds on sub-optimality. In Advances in Neural Information Processing Systems, pages 767–774, 2003.

[24] M. Niedercorn. Random Number Generation and quasi-Monte Carlo Methods. Society for Industrial and Applied Mathematics, 1992.

[25] O. Salzman and D. Halperin. Asymptotically near-optimal RRT for fast, high-quality motion planning. IEEE Trans. Robotics, 32(3):473–483, 2016.
[25] O. Salzman, K. Solovey, and D. Halperin. Motion planning for multi-link robots by implicit configuration-space tiling. *IEEE Robotics and Automation Letters*. to appear.

[26] K. Solovey, O. Salzman, and D. Halperin. Finding a needle in an exponential haystack: Discrete RRT for exploration of implicit roadmaps in multi-robot motion planning. *I. J. Robotics Res.*, 35(5):501–513, 2016.

[27] K. Solovey, O. Salzman, and D. Halperin. New perspective on sampling-based motion planning via random geometric graphs. In *Robotics: Science and Systems*, 2016.

[28] S. S. Srinivasa, D. Ferguson, C. J. Helfrich, D. Berenson, A. Collet, R. Diankov, G. Gallagher, G. Hollinger, J. J. Kuffner, and M. V. Weghe. HERB: a home exploring robotic butler. *Autonomous Robots*, 28(1):5–20, 2010.

[29] J. van den Berg, R. Shah, A. Huang, and K. Y. Goldberg. Anytime nonparametric A. In *Association for the Advancement of Artificial Intelligence*, pages 105–111, 2011.

[30] C. M. Wilt and W. Ruml. When does weighted A* fail? In *Symposium on Combinatorial Search*, pages 137–144, 2012.