AVERAGE ELLIPTIC BILLIARD INVARIANTS WITH SPATIAL INTEGRALS

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Abstract. We compare invariants of N-periodic trajectories in the elliptic billiard, classic and new, to their aperiodic counterparts via a spatial integral evaluated over the boundary of the elliptic billiard. The integrand is weighed by a universal measure equal to the density of rays hitting a given boundary point. We find that aperiodic averages are smooth and monotonic on caustic eccentricity, and perfectly match N-periodic average invariants at the discrete caustic parameters which admit a given N-periodic family.

1. Introduction

The two classic invariants of Poncelet N-periodics in the elliptic billiard are perimeter $L$ and quantity known as Joachimsthal’s constant $J$; see Figure 1. The former implies a billiard trajectory is an extremum of the perimeter function while the latter is equivalent to stating that all trajectory segments are tangent to a confocal caustic [14].

Experiments have unearthed a few additional “dependent” invariants including (i) the sum of cosines, (ii) the product of outer polygon cosines, (iii) certain ratios of areas, etc. [12]. These have been subsequently proved [1, 3, 4]. More recently, the list of conjectured invariants has grown to many dozen [13].

With a small perturbation of the caustic, an N-periodic trajectory becomes aperiodic (space-filling); see Figure 2. A key question we explore is: given a discrete invariant computed for an N-periodic, what is its analogue in the space-filling case? In the latter case, the sum or product of a given quantity can diverge. Fortunately, in both cases we can compare their finite averages.

Main Result. Our contribution is to accurately and efficiently estimate aperiodic averages using a spatial integral evaluated over the caustic’s boundary. We weigh the integrand by a the elliptic billiard universal measure [2, Section 51], [11, 10, 8] which yields the aperiodic density of rays hitting a particular point on the billiard boundary.

Referring to Table 1, we will examine one classic (perimeter) and two “new” invariants (sum of cosines and product of exterior cosines). We will compare their averages (average chord length, average cosine, and geometric mean of exterior cosines) within the continuum of aperiodic trajectories.

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1A billiard N-periodic is fully specified by $L, J$, so any “new” invariants are ultimately dependent on them.
In Section 2 we review preliminary concepts and definitions. In Sections 3 to 5 we calculate, via spatial integrals, the (i) average sidelength (i.e., average perimeter), (ii) average cosine, and (iii) geometric mean of outer cosines. We then compare them with the values predicted by either closed form or numeric computation of the original quantities in the N-periodic case, showing that they lie at consistent locations within the continuum of confocal caustics. Unanswered questions and/or future work appear in Section 6.
2. Preliminaries

Let \((\mathcal{E}, \mathcal{E}_c)\) denote the outer and inner ellipses in the confocal pair given by:

\[
\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \mathcal{E}_c : \frac{x^2}{a_c^2} + \frac{y^2}{b_c^2} = 1
\]

Let \(c^2 = a^2 - b^2 = a_c^2 - b_c^2\). Let \(A = \text{diag}[1/a^2, 1/b^2]\). Joachimsthal’s constant at a point \(\delta\) on \(\mathcal{E}\) is given by \(\langle A.P, v \rangle\), where \(v\) is the unit incoming vector \([14]\). \(J\) is also given by:

\[
J = \frac{\sqrt{\lambda}}{ab}
\]

where \(\lambda = a^2 - a_c^2 = b^2 - b_c^2\).

Referring to Fig. 3, let \(P_c\) be a point on the caustic, and \(P_1, P_2\) be the intersections of the tangent through \(P_c\) with the outer ellipse. These are given by:

\[
\begin{align*}
P_1 &= (x_1, y_1) = \frac{1}{\psi} \left[ a_c^2 b_c^4 (a^4 x_c - \zeta b y_c), b_c^2 b (b a_c^4 y_c + \zeta a x_c) \right] \\
P_2 &= (x_2, y_2) = \frac{1}{\psi} \left[ a_c^2 a (b a_c^4 x_c + \zeta b y_c), b_c^2 b (b a_c^4 y_c - \zeta a x_c) \right]
\end{align*}
\]

\[
\zeta = \sqrt{a^2 - a_c^2} \sqrt{b_c^2 x_c^2 + a_c^4 y_c^2} \\
\psi = a_c^2 b_c^4 x_c^2 + b^2 a_c^4 y_c^2
\]

Given a confocal caustic (say parametrized by its minor axis \(b_c\)) let \(P_i\) be the vertices of an associated aperiodic trajectory, \(i = 1, \ldots, \infty\). The asymptotic average \(\overline{g}\) of some vertex-evaluated quantity \(g(P_i)\) is given by:
One can evaluate \( g(s) \), for example at either intersection \( P_1 \) or \( P_2 \) in Figure 3.

The billiard map is an involution of the pair \((P, P_c)\) of a point on \( E \) and \( E_c \) respectively to new points. There is change of variables \( s \to x \) which linearizes the billiard map, \( x \to x + \tau \) [5, Chapter 13], [9, 6, 15].

Let \( \rho \) be the density of an invariant measure normalized such that \( \oint \rho(x) \, ds = 1 \).

This can be regarded as the density of rays associated with \( x \). The following universal measure has been derived for the elliptic billiard, independent of \( \tau \):

\[
\rho = \frac{dx}{\kappa_c^{2/3} ds} = \frac{(a_c b_c)^{2/3}}{\sqrt{a^2 y^2 + b^2 x^2 / a^2_c}} = \frac{(a^2 - \lambda)^{1/2} (b^2 - \lambda)^{1/2}}{\sqrt{a^2 - \lambda - (a^2 - b^2) \cos^2 u}}
\]

Below we will be also expressing certain average quantities in terms of the following Jacobi elliptic functions of the first and third kind, respectively [7, Introduction]:

\[
K(m) = \int_0^{\pi/2} \frac{da}{\sqrt{1 - m^2 \sin^2 \alpha}}
\]

\[
\Pi(n, m) = \int_0^{\pi/2} \frac{da}{(1 - n^2 \sin^2 \alpha) \sqrt{1 - m^2 \sin^2 \alpha}}
\]

3. Average Sidelength

We construct a spatial integral to compute \( L \), the average sidelength in an aperiodic trajectory and compare it with \( L/N \) for an N-periodic.

The distance between two consecutive points \( P_1 \) and \( P_2 \) of a billiard orbit parametrized by the point \( P_c = [x_c, y_c] \) in the confocal caustic is given by:
Therefore, $P_c = [\sqrt{a^2 - \lambda \cos u}, \sqrt{b^2 - \lambda \sin u}]$ leads to

$$l_{12} = 2ab\sqrt{a^2b^2c^2 + a^4b^2y_e^2 - \frac{a^4b^4}{a^2b^2c^2} + \frac{a^4b^4}{a^2b^2c^2}}$$

Therefore, $P_c = [\sqrt{a^2 - \lambda \cos u}, \sqrt{b^2 - \lambda \sin u}]$ leads to

$$l_{12}(u) = 2ab\sqrt{\lambda c^2 \cos^2 u - (a^2 - \lambda)}$$

$$l(u) = l_{12}\kappa_c^{2/3} ds = \frac{c_1\sqrt{a^2 - \lambda} - c^2 \cos^2 u}{b^2(a^2 - \lambda) - \lambda c^2 \cos^2 u}$$

$$c_1 = 2ab\sqrt{\lambda \sqrt{a^2 - \lambda} \sqrt{b^2 - \lambda}}$$

(6)

$$\mathcal{L} = \frac{1}{\frac{1}{\kappa_c^{2/3} ds}} \int_0^{2\pi} l(u) du$$

In terms of the elliptic integrals $K$ and $\Pi$ we have that:

$$\int \kappa_c^{2/3} ds = \int_0^{2\pi} \frac{(a^2 - \lambda)^{1/2}(b^2 - \lambda)^{1/2}}{\sqrt{s_3\sqrt{1 - s_3 \cos^2 u}}} du$$

$$= \frac{4(a^2 - \lambda)^{1/2}(b^2 - \lambda)^{1/2}}{\sqrt{s_3}} K(\sqrt{s_3})$$

$$\int_0^{2\pi} l(u) du = \frac{c_1}{b^2} \left( \frac{2s_5 - 2s_3}{s_5} \right) \Pi(s_5, \sqrt{s_3}) + 2s_3K\left(\sqrt{s_3}\right)$$

Therefore,

$$\mathcal{L} = \frac{2a}{\sqrt{\lambda} K(\sqrt{s_3})} \left( (-b^2 + \lambda) \Pi\left( \frac{\lambda s_3}{b^2}, \sqrt{s_3} \right) + K(\sqrt{s_3})b^2 \right)$$

Numerical results are shown in Figure 4 for three different billiard aspect ratios. Notice points on each curve report the average perimeters $L/N$ obtained with $N$-periodics at the required caustic parameters $\lambda$.

4. Average Cosine

We evaluate the average cosine $\bar{C}$ for aperiodics with a spatial integral.

Using the Joachimstall invariant we obtain:

$$\cos \theta_1 = \frac{J^2a^4b^4}{2(a^4y_1^2 + b^4x_1^2)} - 1 = \frac{\lambda a^2b^2}{2(a^4y_1^2 + b^4x_1^2)} - 1$$

(7)

$$= \frac{\lambda}{2a_1d_2} - 1, \quad d_1 = |P_1 - F_1|, \quad d_2 = |P_3 - F_2|$$

Let $\cos \theta(u) = (\cos \theta_1(u) + \cos \theta_2(u))/2$. Let $a_c = \sqrt{a^2 - \lambda}$, $b_c = \sqrt{b^2 - \lambda}$ and $(x_c, y_c) = (a_c \cos u, b_c \sin u)$.

Using (1) and (2) it follows that:
The value of the average sidelengths vs $1-\lambda$, $b=1$, and three values of $a$. The dots show agreement of the value with $L/N$ for various non-intersecting $N$-periodics. When $1-\lambda$ is zero, the average perimeter tends to $2a$.

Substituting $\cos \theta$ above for $g$ in (5) and obtain the spatial integral for the average cosine. Therefore it follows that:

$$
\int dx = \int_0^{2\pi} \kappa_2^2 ds = \int_0^{2\pi} \frac{(a^2 - \lambda)^\frac{1}{2}(b^2 - \lambda)^\frac{1}{2}}{\sqrt{s_3 \sqrt{1 - s_3 \cos^2 u}}} du
$$

In terms of the elliptic integrals $K$ and $\Pi$ it follows that:

$$
C = \frac{r_3}{r_4} \left( s_2 - s_1 \right) \Pi \left( s_2, \sqrt{s_3} \right) + s_1 K \left( \sqrt{s_3} \right)
$$

In [12, 1, 3] the following expression was presented for the invariant sum of cosines in $N$-periodics:
Figure 5. The average cosine vs $1 - \lambda$, $b = 1$ for three values of $a$, with $b = 1$. The dots show agreement of the value with $JL/N - 1$ for various non-intersecting N-periodics. When $1 - \lambda$ is one (resp. zero), the average cosine tends to 1 (resp. −1).

Therefore the average cosine for N-periodics is simply $JL/N - 1$. Figure 5 shows results obtained with spatial integration, and that they agree with the values predicted for N-periodics at the appropriate locations.

**Sum of curvatures to two-thirds.** In [13] we show conservation of $\sum \kappa^2/3 = \text{constant}$ is a corollary to the sum of cosines, where $\kappa_i$ denotes the curvature of the outer ellipse at the $i$th vertex. One can express $\kappa^2/3$ as a linear function of $\cos \theta$:

$$\kappa^2/3 = (ab)^{-3/4}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^{-1} = (ab)^{3/4}d_1d_2 = \frac{4(ab)^{3/4}}{|\nabla f|^2} = (ab)^{-3/4}\left(1 + \cos \theta\right)^{-1}$$

Therefore, the sum of $\kappa^2/3$ is also invariant and its average value will be given by:

$$\overline{\kappa^2/3} = \frac{1}{\int_{\kappa_c}^{2/3} \kappa_c^2/3 \, ds} \int_{\kappa_c}^{2/3} \kappa_c^2/3 \, ds.$$ 

**5. Geometric Mean of Outer Cosines**

Referring to Figure 1, let $\theta'_i$ denote the $i$th internal angle of the outer polygon whose sides are tangent to the elliptic billiard at the vertices of an N-periodics. The product of $\theta'_i$ is invariant over N-periodics, for all N [1, 3]. The geometric mean $\overline{\cos \theta}$ of $\theta'_i$ is given asymptotically by:

$$\overline{\cos \theta} = \lim_{k \to \infty} \left( \prod_{i=1}^{k} \cos \theta'_i \right)^{1/k}$$

To work with spatial integrals we must first convert the above to a sum:
Figure 6. Average cosines (red) and geometric mean of outer cosines (green) vs. $b_c$, the minor semiaxes of the caustic. Here $a = 5$, $b = 1$. Blue dashed vertical lines mark the $b_c$ for non-intersecting orbits. Dashed green: past the $N = 4$ caustic, the latter is reflected about the $x$ axis showing proximity to the average cosine.

$$\log \overline{C'} = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \log |\cos \theta'_{i}|$$

As before, replace the above time average by the following spatial integral:

$$\log \overline{C'} = \frac{1}{\oint \kappa_c(s)^{2/3} ds} \oint \log |\cos \theta'(s)| \kappa_c^{2/3} ds$$

A quick look on a picture shows that in order to compute $\cos \theta'$ it suffices to make the scalar product of the normalized gradients at the points $P_1, P_2$.

$$\cos \theta' = \frac{x_1 x_2/a^2 + y_1 y_2/b^2}{(x_1^2/a^2 + y_1^2/b^2)^{1/2}(x_2^2/a^2 + y_2^2/b^2)^{1/2}}$$

$$\cos \theta' = -c_a \sqrt{a^2 \cos^2 u + (a^2 - \lambda)^2}$$

$$\sqrt{c^2 c_a^2 \cos^2 u - (2 b^2 \lambda + c_a^2)^2 (a^2 - \lambda)}$$

$$c_a = a^2 b^2 - \lambda (a^2 + b^2), \quad \text{sign}(\cos \theta') = -\text{sign}(c_a)$$

Numerical results for both average cosines and geometric mean of outer cosines are shown in Figure 6 for $a = 5$ (smaller $a$ make the two spatial averages become to close to each other). For values of $b_c$ where the trajectory is periodic, results obtained with spatial averages perfectly match numerically-estimated discrete averages computed numerically with $N$-periodics.

6. Questions

The following questions are still unanswered:

- Why is the geometric mean of outer aperiodic cosines so close to the average aperiodic cosines?
• Is there a universal measure expressed in terms of the outer ellipse?
• Can we use this framework to estimate aperiodic averages for cases where the caustic is a hyperbola?
• A third invariant introduced in [12] was the ratio of outer-to-orbit areas. These do not seem amenable to a discrete sum of individual quantities. Would there be counterpart be for aperiodic areal averages?

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