Equations of Motion and Galilei Invariance in D-Particle Dynamics

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Abstract

As a continuation of our previous work on the multi-body forces of D-particles in supergravity and Matrix theory, we investigate the problem of motion. We show that the scattering of D-particles including recoil derived in Matrix theory is precisely reproduced by supergravity with the discrete light-cone prescription up to the second order in 11 dimensional Newton constant. An intimate connection of recoil and Galilei invariance in supergravity is pointed out and elucidated.

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1. Introduction

Matrix theory [1] as a candidate for the non-perturbative formulation of M-theory has passed many nontrivial checks. In particular, the correspondence with 11 dimensional supergravity may be regarded as a cardinal test. If Matrix theory would be proven to be well defined non-perturbatively and to agree with supergravity in the low-energy limit, that would be convincing evidence for establishing Matrix-theory conjecture.

In our previous work [2] which is hereafter referred to as I, we have presented a detailed comparison between Matrix theory and supergravity with respect to the multi-body interactions of D-particles. We have shown that the effective actions on both sides precisely agree up to 3-body interaction terms. The effective actions are determined under a particular approximation which is common on both sides. Namely, we have completely neglected the recoil of D-particles and compared the actions with the criterion that they should give the same eikonal phase shift. On Matrix-theory side, this allows us to perform the simple loop calculation with a fixed background which corresponds to straight-line trajectories of D-particles. On supergravity side, on the other hand, the eikonal approximation amounts to the approximation that the energy-momentum tensor of D-particles is assumed to be that of free D-particles. In so doing, it was crucial that the recoil effect can be separated from the rest in the particular gauge we have adopted.

However, it is important to remember that the eikonal approximation is not a systematic expansion scheme in any rigorous sense. For this reason, we must take into account the recoil effect to make the computations on both sides completely self-consistent and must check whether the agreement persists after that. The main purpose of the present note is to present such a check as promised in I. It is also an important exercise to see explicitly how the general relativistic equation of motion, i.e. geodesic equation which is nothing but the integrability condition for the gravitational field equation, fits into the loop expansion of Matrix-theory calculations where we cannot see, at least directly, the gravitational field degrees of freedom which emerge only as a loop effect. Furthermore, obtaining the effective action using the eikonal approximation implicitly assumes that there exists the effective action which is local with respect to time. Therefore it is desirable to confirm directly that the recoil effect calculated from the equations of motion obtained by
the variation of the effective action coincides with the contribution of the tadpole diagram of Matrix theory. These checks would provide firm evidence that Matrix theory indeed describes the dynamics of D-particles beyond their restricted properties associated with BPS or supersymmetric constraint.

In section 2, we first explain the apparent violation of Galilei invariance in the eikonal approximation in supergravity in the simplest case of two clusters of coincident D-particles and its resolution after taking into account the recoil effect. We then extend the arguments to the multi D-particle system treated in I. The effective action including the recoil correction is formulated in section 3 and the precise agreement between supergravity and Matrix theory is demonstrated. Although the treatment of recoil in Matrix theory is more or less self-evident from the general formalism of effective actions, we feel that our discussion emphasizing its connection with general relativistic equations of motion will be useful as a basis for future works toward further systematic comparisons of higher order effects in supergravity and Matrix theory.

2. Galilei Invariance and Recoil in Supergravity

2.1 System of two clusters of coincident D-particles

Let us start from considering the simplest case of two clusters of coincident D-particles, assuming that one of them, source, with $p_+ = N_1/R$ is much ‘heavier’ ($N_1 \gg N_2$) than the other, probe, with $p_- = N_2/R$. The source can then be treated to be at rest in the transverse space, and the effective action for the probe with (transverse) velocity $v$ in the lowest nontrivial approximation is given by

$$S = \int ds \left[ \frac{N_2}{2R} v^2 + \frac{15N_1N_2}{16M^9R^3} \frac{v^4}{r^7} \right].$$

(2.1)

Here the time $s$ is identified with the light-cone time in 11 dimension as $s = x^+/2 = (x^{11} + t)/2$. The second term is the gravitational energy of the probe corresponding to the one-graviton exchange contribution

$$\int d^{11} x \frac{1}{2} \zeta_{\mu\nu}(x) \sigma_2^{\mu\nu}(x),$$

(2.2)

\[\]
where \( \tau^{\mu\nu}_2 \) and \( \zeta_{1\mu\nu} \) are the lowest order energy-momentum tensor of the probe and the gravitational field produced by the fixed source, respectively,

\[
\tau^{\mu\nu}_2(x) = \frac{N_2}{2\pi R^2} \delta^9(x - x(s)) s_2^\mu s_2^\nu, \tag{2.3}
\]

\[
\zeta_{1\mu\nu}(x) = \frac{15}{(2\pi)^4} \kappa_{11}^2 \frac{N_1}{2\pi R^2} \frac{s_1^\mu s_1^\nu}{r^i}, \tag{2.4}
\]

with \( r \) being the relative transverse distance between the source and probe. In our convention, the 11D Newton constant is \( \kappa_{11}^2 = 16\pi^5/M^9 \) (\( M^3 = g_s\ell_s^3 \)) and the compactification radius along the \( x^- = x^{11} - t \) direction is \( R (= g_s\ell_s) \). These expressions are obtained by averaging over the \( x^- \) direction from 0 to \( 2\pi R \). The velocity vectors \( s^\mu \equiv dx^\mu/ds \) are

\[
(s_{1+}, s_{1-}, s_{11}) = (0, 1, 0), \quad (s_{2+}^i, s_{2-}^i, s_{22}^i) = (2, -\frac{1}{2}v^2, v^i), \tag{2.5}
\]

for the source and probe, respectively.

Now the variation of the above effective action gives the equation of motion for the probe

\[
\frac{N_2}{R} \frac{d^2v^i}{ds^2} + \frac{15N_1N_2}{4M^9R^3} \frac{d}{ds} \left( \frac{v^i v^2}{r^7} \right) + \frac{105N_1N_2}{16M^9R^3} \frac{v^4 r^i}{r^9} = 0. \tag{2.6}
\]

On the other hand, the equation of motion in General Relativity is the geodesic equation, which should be regarded as the integrability condition for the field equation. It is not entirely obvious whether the latter coincides with the former variational equation of motion, because we have derived the effective action in supergravity using the eikonal approximation.

Since the above effective action or energy-momentum tensor corresponds to the following probe action

\[
S_2 = \frac{N_2}{2R} \int ds \ g_{\mu\nu}(x(s)) \frac{dx^\mu(s)}{ds} \frac{dx^\nu(s)}{ds}, \tag{2.7}
\]

with \( g_{\mu\nu} = \eta_{\mu\nu} + \zeta_{1\mu\nu} \), we should adopt the variational equation of this action as the geodesic equation. However, we find that the spatial component of the geodesic equation is then

\[
\frac{N_2}{R} \frac{d^2x^i(s)}{ds^2} - \frac{N_2}{2R} \partial_i \zeta_{1-\cdot}(x(s))(s_2^-)^2 = 0, \tag{2.8}
\]

where use is made of the fact that only nonzero component of the gravitational field of the source D-particle system is \( \zeta_{1-\cdot} \). Comparing with the variational equation of motion (2.6)
derived from the effective action, the geodesic equation of motion contains only the first and the last terms. The second term of (2.6) which comes from the velocity dependence of the interaction is missing in (2.8).

Let us next consider the equation of motion in the different Galilean frame in which the probe is at rest in transverse space. Then the velocity vectors must be replaced by

\[(s_{1+}, s_{1-}, s_{1i}) = (-\frac{1}{4}v^2, 1, -v^i), \quad (s_{2+}, s_{2-}, s_{2i}) = (2, 0, 0). \tag{2.9}\]

The geodesic equation is now

\[\frac{N_2}{R} \left( \frac{d^2 x^i(s)}{ds^2} + 2 \frac{d \zeta_{1i}(x(s))}{ds} \right) - \frac{2N_2}{R} \partial_i \zeta_{1++}(x(s)) = 0, \tag{2.10}\]

which coincides with (2.6) after substituting the velocity vectors (2.9). Note that in this case we have used the fact that only nonzero component of the velocity vector of the probe is \(s_2^+ = 2\).

Thus we found that the geodesic equation is not Galilei invariant in the above naive argument based on the eikonal approximation. The resolution of this apparent contradiction is as follows. Since the D-particle is assumed to be a Kaluza-Klein mode of the graviton, its trajectory must satisfy the massless condition,

\[g_{\mu\nu}(x(s)) \frac{dx^\mu(s)}{ds} \frac{dx^\nu(s)}{ds} = 0, \tag{2.11}\]

and the momentum constraint,

\[p_+ = \frac{N}{R}. \tag{2.12}\]

In reference [3], the massless condition is imposed by taking the zero mass limit \(m \to 0\) starting from the usual action

\[S_{bbpt} = -m \int ds \sqrt{-g_{\mu\nu}(x(s))} \frac{dx^\mu(s)}{ds} \frac{dx^\nu(s)}{ds}, \]

and the momentum constraint is taken into account by moving to the Routhian

\[S_{bbpt} \to S_{bbpt} - \int ds p_- \frac{dx^-}{ds}. \]

In quantum theory, the transition to the Routhian is explained as the Fourier wave function corresponding to the change from the coordinate to momentum representation.
To deal with the above difficulty, however, it is more appropriate to start from the action with a Lagrange multiplier corresponding to the massless condition (2.11) as adopted in I,

\[ S_D = \frac{N}{2R} \int ds \lambda(s) g_{\mu\nu}(x(s)) \frac{dx^\mu(s)}{ds} \frac{dx^\nu(s)}{ds} - \int ds p_- \frac{dx^-}{ds}, \]  

(2.13)

where \( s \) is an arbitrary parametrization along the D-particle trajectory. The reparametrization invariance allows us to adopt the gauge \( s = x^+/2 \). Note that, by defining the Routhian, the independent variables in the longitudinal direction are \( x^- \) and \( p_- \), and the equation of motion for \( x^- \) must be reinterpreted as the momentum constraint

\[ p_- = \frac{N}{R} \lambda(s) g_{\mu-} \frac{dx^\mu}{ds} = \frac{N}{R} g_{\mu-} \frac{dx^\mu}{d\tau} = \frac{N}{R} = \text{constant}, \]

(2.14)

where we have set

\[ d\tau = \frac{ds}{\lambda(s)}. \]

(2.15)

The parameter \( \tau \) is invariant under reparametrization of the D-particle trajectory. If the action and the energy-momentum tensor of D-particles are expressed in terms of \( \tau \), we can eliminate the Lagrange multiplier \( \lambda(s) \).

Let us confirm the consistency of the above momentum constraint with the equations of motion and the field equation. In the classical approximation, we have to require that the variation of the Routhian with respect to \( x^- \) keeping \( p_- \) fixed must vanish without surface term. We then have, denoting the first term (integrand) of (2.13) by \( L \),

\[ \frac{\partial L}{\partial (dx^-/ds)} \bigg|_{x^-} \frac{\partial (dx^-/ds)}{\partial x^-} \bigg|_{p_-} + \frac{\partial L}{\partial x^-} \bigg|_{x^-} - p_- \frac{\partial (dx^-/ds)}{\partial x^-} \bigg|_{p_-} = 0, \]

(2.16)

which, combined with (2.14), leads to

\[ \frac{\partial L}{\partial x^-} \bigg|_{x^-} = 0. \]

(2.17)

On the other hand, the energy-momentum tensor derived from the above action is

\[ T^{\mu\lambda}(x) = \frac{N}{R} \int ds \lambda(s) \frac{dx^\mu(s)}{ds} \frac{dx^\lambda(s)}{ds} \frac{1}{\sqrt{-g(x(s))}} \delta^{11}(x - x(s)). \]

(2.18)

\[ ^\text{§} \]

We note that, to the order we are interested in, the determinant of the space-time metric can always be treated as \(-1\) because of the massless condition \( s^\mu s_\mu = 0 \).
The conservation, $D_\mu T^{\mu\nu} = 0$, of the energy-momentum tensor which is nothing but the integrability condition of the field equation requires that the trajectories of D-particles must satisfy the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0. \quad (2.19)$$

The geodesic equation is equal to the variational equation derived by assuming the action consisting of only the first term in (2.13). The equation (2.17) is then equivalent to $dp_-/ds = 0$, as it should be to be compatible with the momentum constraint. By examining the compatibility of this condition and the geodesic equation (2.19), we find that the constancy of $p_-$ is satisfied for arbitrary velocities of D-particle when

$$\partial_- g_{\mu\nu} = 0, \quad (2.20)$$

which is again as it should be, since there is no exchange of $p_-$ momentum, and is consistent with the averaging prescription along the compactification circle in defining the energy-momentum tensor as in (2.3). Physically, this prescription is justified since $R$ is small for small $g_s$. Note that the actual expansion parameter in our calculation is $\kappa_1^2/R^2 \propto g_s^3/g_s^2 = g_s$.

Let us now consider the momentum constraint (2.14). In our gauge $s = x^+/2$, it reduces to

$$1 = \lambda(s)(1 + \zeta_{1\mu-} \frac{dx^\mu}{ds}). \quad (2.21)$$

This indicates that, once the geodesic equation of motion is taken into account, the choice $\lambda(s) = 1$ is allowed only when the relative velocity of D-particles are zero. In particular, for the probe system in the above discussion, the relation (2.21) becomes, to the first nontrivial order,

$$\lambda(s) = 1 - \zeta_{1\mu-} \frac{dx^\mu}{ds} = 1 + \frac{15N_1}{4M^9 R^2} \frac{v^2}{r^7}. \quad (2.22)$$

It is easy to check that this also ensures the consistency of the gauge condition $x^+ = 2s$ with the + component of the geodesic equation (2.19). We also note that the geodesic equation is compatible with the massless condition (2.11). This ensures that the $x^-$ expressed in terms of the transverse components solving the massless constraint satisfies the geodesic equation automatically.
The spatial component of the geodesic equation of the probe with the light-cone time
\( s = x^+ / 2 \), is, in the present approximation,
\[
\frac{N_2}{R} \frac{d}{ds} \left( \left( 1 + \frac{15N_1}{4M^9R^2 r^7} \right) \frac{dx^i(s)}{ds} \right) - \frac{N_2}{2R} \partial_\xi (x(s))(s^-)^2 = 0,
\]
(2.23)
coinciding with the variational equation of motion corresponding to the effective action
(2.1). Note that the shift of \( \lambda(s) \) for the probe does not affect its Routhian, since \( dx^- / ds \) is
obtained by solving the massless constraint which does not contain the Lagrange multiplier
directly. For the heavy source, the correction to \( \lambda(s) = 1 \) is proportional to \( N_2(\ll N_1) \) and
can be neglected compared with (2.22). Thus the shift of \( \lambda(s) \) does not change the form
of the effective action. However, Galilei invariance of the geodesic equation of motion on
the side of supergravity and the agreement with the equation of motion derived from the
effective action are recovered only by properly taking into account the Lagrange multiplier
corresponding to the massless condition. Galilei invariance is of course expected in the
light-front formulation of supergravity adopting \( x^+ \) as the time.

This is a part of the recoil effect which has been neglected in previous calculations
of D-particle interactions, including our work I. The eikonal approximation
amounts to treating the D-particle trajectories as if they were completely independent of
each other. Then the only natural choice of the energy-momentum tensor would be that of
the free D-particle corresponding to the assumption \( \lambda(s) = 1 \ (\tau = s = x^+/2) \) as has been
adopted in I following [3]. Since the Lagrange multiplier \( \lambda(s) \) essentially represents the
arbitrariness of parametrization for the trajectories of D-particles, its shift means that
the invariant parameter \( \tau \) of each D-particle must be deviated from the external time
parameter \( x^+ (= 2s) \) depending on the relative motions of D-particles. The total recoil
effect \( \delta_{rc}x^i(s) \) obtained from the equation of motion (2.6) for the probe system can be
expressed as the correction for the velocity vector
\[
\frac{d\delta_{rc}x^i}{ds} = - \frac{15N_1}{4M^9R^2 r^7} v^i - \int_{-\infty}^{s} ds' \frac{105N_1}{16M^9R^2 r^9} v^{\alpha} r^i,
\]
(2.24)
to the first order in \( \kappa^2_{11} \). The first term in the right hand side can be interpreted as due
to the change of parametrization
\[
ds \to (1 + \frac{15N_1}{4M^9R^2 r^7}) ds,
\]
which is required to keep the gauge \( x^+ = 2s \) after the recoil effect is taken into account.

It is quite remarkable that the simple-looking effective action and the associated equation of motion for D-particles automatically account for these subtle properties of D-particle motions in supergravity, in spite of the fact that the effective action itself was derived using the eikonal approximation.

\[ (2.25) \]

### 2.2 Multi-cluster system

It is straightforward to extend our analysis to multi-cluster systems. The effective action is

\[
S_{\text{eff}} = \int ds \left[ \sum_a \frac{N_a}{R} \left( \frac{dx^i_a}{ds} \right)^2 + \frac{1}{2} \sum_{a,b} \frac{15N_aN_b}{16R^3M^9} \frac{v_{ab}^4}{r_{ab}^5} + L_3 + \cdots \right],
\]

where \( L_3 \) is the second order term as derived in \[3\]

\[
L_3 = L_V + L_Y,
\]

\[
L_V = -\sum_{a,b,c} \frac{(15)^2 N_aN_bN_c}{64R^5M^{18}} v_{ab}^2v_{ca}^2 (v_{ca} \cdot v_{ab}) \frac{1}{r_{ab}^5} \frac{1}{r_{ca}^5},
\]

\[
L_Y = -\sum_{a,b,c} \frac{(15)^3 N_aN_bN_c}{24(2\pi)^4R^5M^{18}} \left[ -(s_b \cdot s_c)(s_c \cdot s_a)(s_b \cdot \tilde{\partial}_c)(s_a \cdot \tilde{\partial}_c)
\right.
\]

\[
+ \frac{1}{2}(s_c \cdot s_a)^2(s_b \cdot \tilde{\partial}_c)^2 + \frac{1}{2}(s_b \cdot s_c)^2(s_a \cdot \tilde{\partial}_c)^2
\]

\[
- \frac{1}{2}(s_b \cdot s_a)(s_a \cdot s_c)(s_b \cdot \tilde{\partial}_c)(s_c \cdot \tilde{\partial}_b)
\]

\[
+ \frac{1}{4}(s_b \cdot s_c)^2(s_a \cdot \tilde{\partial}_b)(s_a \cdot \tilde{\partial}_c) \Delta(a,b,c),
\]

where

\[
\Delta(a,b,c) \equiv \int d^3y \frac{1}{|x_a - y|^7|x_b - y|^7|x_c - y|^7}
\]

\[
= \frac{64(2\pi)^3}{(15)^3} \int_0^\infty d^3\sigma (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)^{3/2} \exp(-\sigma_1|x_a - x_b|^2 - \sigma_2|x_b - x_c|^2 - \sigma_3|x_c - x_a|^2),
\]

\[
(2.29)
\]

\text{Here and in what follows we assume the convention that when the sums over particles hit the case} \( a = b \) \text{leading vanishing relative distance} \( r_{ab} = 0 \), \text{the contribution identically vanishes because of the vanishing relative velocity} \( v_{ab} \equiv 0 \). \text{We note that, in Matrix-theory calculations, there does not occur any infrared singularity associated with the massless (diagonal) modes with this prescription.
with \( d^3\sigma = d\sigma_1 d\sigma_2 d\sigma_3 \) and the notation \( \tilde{\partial} \) is defined by \( \tilde{\partial}_\mu = (\partial_+ , 0 , -\partial_t) \). As we will see below, to compare with the available Matrix-theory calculations, it is sufficient to take into account the recoil correction to the first nontrivial order, namely to the first order in 11D Newton constant \( \kappa_{11}^2 \).

The D-particle part of the action is now

\[
S_D = \sum_{a} \frac{N_a}{2R} \int ds \, \lambda_a(s) \langle x_a(s) \rangle \frac{d x_\mu}{d s} \langle x_a(s) \rangle \frac{d x_\nu}{d s} - \sum_{a} \int d s p_{a} \frac{d x^-}{d s}. \tag{2.30}
\]

The eikonal approximation corresponds to the choice \( \lambda_a(s) = 1 \) with straight-line trajectories \( dx^i_a/ds = v^i_a = \text{constant} \) in the gauge \( s = \tau = x^+/2 \). The momentum constraint is now

\[
1 = \lambda_a(s) g_{\mu\nu}(x_a(s)) \frac{d x_\mu}{d s}. \tag{2.31}
\]

In the present approximation, the solution to this equation is given by

\[
\lambda_a(s) = 1 + \sum_{b} \frac{15 N_b}{4 M^9 R^2} \frac{v_{ab}^2}{r_{ab}^7}, \tag{2.32}
\]

where \( v_{ab} \) and \( r_{ab} \) are relative transverse velocity and distance, respectively. As in the previous case of two clusters, (2.32) ensures the consistency of the gauge condition with the geodesic equation of motion. The spatial component of the geodesic equation of motion to the first order in \( \kappa_{11}^2 \) is then

\[
N_a \frac{d}{d s} \left( v^i_a + \sum_{b} \frac{15 N_b}{4 M^9 R^2} \frac{v_{ab}^2}{r_{ab}^7} v^i_b \right) - \frac{N_a}{2R} \partial_k \zeta_{\mu\nu}(x_a(s)) s^\mu_a s^\nu_a = 0, \tag{2.33}
\]

where the gravitational field is

\[
\zeta_{\mu\nu}(x_a) = \sum_{b} \frac{15 N_b}{2 M^9 R^2} \frac{s_{b\mu} s_{b\nu}}{r_{ab}^7}. \tag{2.34}
\]

It is easy to check that the geodesic equation (2.33) coincides with the variational equation derived from the effective action (2.25) in the present approximation. We note that, if the deviation of the Lagrange multiplier \( \lambda_a(x) \) from the eikonal approximation \( \lambda_a(s) = 1 \) were neglected in supergravity, the second term in the parenthesis would have been

\[
- \sum_{b} \frac{15 N_b}{4 M^9 R^2} \frac{v_{ab}^2}{r_{ab}^7} v^i_b,
\]
which violates Galilei invariance.

Furthermore, the shift of $\lambda_a(s)$ does not affect the form of the effective action itself (2.25), since the change of the two-body term induced by the shift of the Lagrange multiplier is canceled between the two contributions coming from the pure gravity part and the D-particle part $\int ds p_\mu dx^\mu / ds$ in the present approximation. This cancellation explains why the equation of motion obtained by the variation of the effective action in the eikonal approximation agrees with the geodesic equation. The total shift induced by recoil is thus determined only by the shift $\delta_{rc}x^i_a$ of the trajectories of D-particles as

$$\frac{d\delta_{rc}x^i_a}{ds} = -\sum_b \left( \frac{15 N_b}{4 M^3 R^2} v^{2}_{ab} r^i_{ab} + \int_{-\infty}^{s} ds' \frac{105 N_b}{16 M^3 R^2} v^{4}_{ab} r^i_{ab} \right).$$  \hspace{1cm} (2.35)

3. Effective Action Including Recoil Corrections

Our next task is to study the change of the value (i.e. scattering phase shift) of the effective action caused by recoil. Before going into the calculation, we have to pay attention to one subtlety associated with surface terms. Namely, in computing the scattering phase shift, we have to take into account the wave function $\exp \left( -i \sum_a p^i_a x^i_a \bigg|_{t=\pm \infty} \right)$ of the initial and final states. In classical approximation (or the lowest WKB approximation), the surface terms with $p^i_a = N_a v^i_a / R$ are necessary in order to ensure that the first variation around the classical trajectories vanishes even at the asymptotic past and future. When we include the shift induced by recoil, we have to assume that these surface terms are simultaneously shifted corresponding to the change of the initial or final states of D-particles. Since the surface terms are arranged such that the surface contributions under the variation vanish, we can then safely neglect the surface contributions in evaluating the effect of recoil using partial integration. We note that this remark applies to both supergravity and Matrix theory calculations.

\hspace{1cm} \footnote{This is easily seen by examining the equations (2.47) and (2.48) of I. The coefficients $-1/4$ and $1/2$, respectively, in these expressions are responsible for the cancellation. Note that as already mentioned in the previous subsection, the form of the Routhian does not directly receive correction from $\lambda$.}

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3.1 Recoil correction of the effective action

Let us now proceed to calculate the effective action including recoil. We denote the quantities before recoil correction by putting subscript 0. Thus the trajectory is now $x^i_a(s) = x^i_{a,0}(s) + \delta x^i_a(s)$. We are interested in the effective action to the second order in Newton constant $\kappa^2_{11}$, or more appropriately, to the second order in $g_s$ comparing with the lowest order (which is of order $g_s^{-1}$) term, since the actual expansion parameter is $\kappa^2_{11}/R^2 \propto g_s$ because of the compactification along the $x^-$ direction. It is then sufficient to consider the equation of motion including only the first order interaction term. We denote the terms of the effective action by putting the superscript indicating the order with respect to the string coupling $g_s$. For notational simplicity, we set $\ell_s = 1 (M^{-3} = g_s, R = g_s)$.

\[
S^{(-1)} = \int ds \sum_a \frac{N_a}{2g_s} v^2_{a,0}, \quad (3.36)
\]

\[
S^{(0)} = \frac{1}{2} \int ds \sum_{a,b} \frac{15N_a N_b v^4_{ab,0}}{16 r^7_{ab,0}}, \quad (3.37)
\]

By substituting the trajectory with the recoil corrections, the effective action $S^{(-1)} + S^{(0)}$ becomes

\[
S^{(-1)} + S^{(0)} \rightarrow \int ds \sum_a \frac{N_a}{2g_s} (v_{a,0} + \delta v_a)^2 + \frac{1}{2} \int ds \sum_{a,b} \frac{15N_a N_b (v_{ab,0} + \delta v_{ab})^4}{(r_{ab,0} + \delta r_{ab})^7}
\]

\[
= S_0^{(-1)} + S_0^{(0)} + S^{(1)} + O(g_s^2), \quad (3.38)
\]

where

\[
S_0^{(-1)} = \int ds \sum_a \frac{N_a}{2g_s} v^2_{a,0}, \quad (3.39)
\]

\[
S_0^{(0)} = \frac{1}{2} \int ds \sum_{a,b} \frac{15N_a N_b v^4_{ab,0}}{16 r^7_{ab,0}}, \quad (3.40)
\]

\[
S^{(1)} = - \int ds \sum_a \frac{N_a}{2g_s} (\delta v_a)^2. \quad (3.41)
\]

In deriving this result, we have used the fact that $x^i_a(s) = x^i_{a,0}(s) + \delta x^i_a(s)$ is the solution of the variational equation of the effective action. The surface term in performing partial integration can be discarded as discussed above. The last term represents the recoil correction to the action and hence to the phase shift. If further partial integrations
are allowed, we can formally rewrite this expression in terms of the recoil acceleration
\( \delta \alpha_a = d \delta v_a / ds \) as

\[
S'^{(1)} = \frac{1}{2} \sum_a N_a \int ds_1 ds_2 \delta \alpha_a(s_1) D_x(s_1, s_2) \delta \alpha_a(s_2),
\]

(3.42)

where

\[
\frac{d^2}{dt_1^2} D_x(t_1, t_2) = \frac{1}{g_s} \delta(t_1 - t_2).
\]

(3.43)

This formal expression is more convenient for comparison with Matrix theory. It should be noted that the recoil correction is of the same order as the 3-body interaction term
\( S^{(1)} = \int ds L_3 \).

### 3.2 Effective action formalism and recoil in Matrix theory

We next discuss the treatment of recoil in the path integral formalism of Matrix theory. Let us first remind ourselves some general facts about the method of effective action. To avoid unnecessary complexity in the presentation, we use simplified notations for the products of correlation functions and their functional derivatives, suppressing the integrals over the time variable. There should not arise any confusion for such text-book matters.

The effective action \( \Gamma[\phi] \) is defined by

\[
e^{iW[J]} = \mathcal{N}^{-1} \int [d\varphi] \ e^{iS[\varphi] - iJ\varphi} = e^{i(\Gamma[\phi] - J\phi)},
\]

(3.44)

\[
\phi \equiv \langle \varphi \rangle [J] = \frac{\int [d\varphi] \ \varphi \ e^{iS[\varphi] - iJ\varphi}}{\int [d\varphi] \ e^{iS[\varphi] - iJ\varphi}}.
\]

(3.45)

Note that here we discriminate the field \( \varphi \), which is path-integrated, from its expectation value \( \phi \). Of course, the field \( \varphi \) represents all of the different fields of Matrix theory. In terms of Feynman diagrams, the effective action \( \Gamma[\phi] \) is the generating functional of 1PI diagrams with their external lines are given by the function \( \phi \). The familiar properties of the effective action are

\[
\frac{\delta \Gamma[\phi]}{\delta \phi} = J,
\]

(3.46)

\[
i \frac{\delta}{\delta J} \frac{\delta \Gamma[\phi]}{\delta \phi} = i \frac{\delta \phi \ \delta^2 \Gamma[\phi]}{\delta J \delta \phi^2} = \langle \varphi \varphi \rangle e \frac{\delta^2 \Gamma[\phi]}{\delta \phi^2} = i.
\]

(3.47)

The last equation of course means that the second derivative of the effective action is the inverse of the propagator (=connected two-point function).
Assume that the classical action has a solution \( \phi_0 \),
\[
\frac{\delta S[\phi]}{\delta \phi} \bigg|_{\phi=\phi_0} = 0.
\]
(3.48)

What we obtain in our eikonal approximation is the phase shift defined by
\[
\Delta_{\phi_0} \equiv \Gamma[\phi_0],
\]
(3.49)
since the calculation amounts to computing the 1PI diagrams by taking the external lines to be the straight line trajectories after setting \( J = 0 \). Namely, the eikonal effective action is just the special value of the exact effective action which is evaluated by setting the would-be expectation value \( \langle \varphi \rangle \) to be the classical background. Of course, the classical background does not satisfy the variational equation of the exact effective action,
\[
\frac{\delta \Gamma[\phi]}{\delta \phi} \bigg|_{\phi=\phi_0} \neq 0.
\]
(3.50)

In the calculation of scattering phase shift, the external source term is zero in the bulk and is only nonzero at infinite future and past, leading to the surface term as explained in the beginning of the present section. With this understanding, we have the identity,
\[
e^{i\Gamma[\phi]} = N^{-1} \int [d\tilde{\varphi}] \, e^{iS[\phi_0 + \tilde{\varphi}]},
\]
(3.51)
as a consequence of the definition of the effective action. Here the exact vacuum expectation value,
\[
\phi = \phi_0 + \langle \tilde{\varphi} \rangle,
\]
(3.52)
satisfies the quantum equation of motion
\[
\frac{\delta \Gamma[\phi]}{\delta \phi} = 0.
\]
(3.53)

This shows that obtaining the exact effective action for the correct trajectory which satisfies the equation of motion, we have to take into account the 1P reducible part. The difference between the effective action in the eikonal approximation and the exact effective action represents the recoil effect,
\[
\Gamma_{\text{recoil}}[\phi_0] \equiv \Gamma[\phi] - \Delta_{\phi_0}.
\]
(3.54)
This can be expanded as
\[
\left. \frac{\delta \Gamma[\phi]}{\delta \phi} \right|_{\phi = \phi_0} \langle \tilde{\phi} \rangle + \frac{1}{2} \langle \tilde{\phi} \rangle \left. \frac{\delta^2 \Gamma[\phi]}{\delta \phi^2} \right|_{\phi = \phi_0} \langle \tilde{\phi} \rangle + \cdots.
\]

The one-point function \( \langle \tilde{\phi} \rangle \) is of first order with respect to the loop expansion parameter. Thus we have the loop expansion as,
\[
\Gamma[\phi] = \mathcal{S}[\phi] + \Gamma^{(1)}[\phi] + \Gamma^{(2)}[\phi] + \cdots, \tag{3.55}
\]
\[
\phi = \phi_0 + \phi^{(1)} + \phi^{(2)} \cdots, \tag{3.56}
\]
where the superscript \((i)\) denote the order with respect to the loop expansion. This leads to
\[
\Gamma_{\text{recoil}}[\phi_0] = \left. \frac{\delta \Gamma^{(1)}[\phi]}{\delta \phi} \right|_{\phi = \phi_0} \phi^{(1)} + \frac{1}{2} \langle \tilde{\phi} \rangle \left. \frac{\delta^2 \mathcal{S}[\phi]}{\delta \phi^2} \right|_{\phi = \phi_0} \phi^{(1)} + \cdots. \tag{3.57}
\]
Thus the lowest order recoil effect starts from the two-loop order. On the other hand, by the definition of the effective action,
\[
\frac{\delta^2 \mathcal{S}[\phi]}{\delta \phi^2} = -N \frac{d^2}{g_s ds^2}, \tag{3.58}
\]
\[
0 = \left. \frac{\delta \Gamma[\phi]}{\delta \phi} \right|_{\phi = \phi_0} = \langle \tilde{\phi} \rangle \left. \frac{\delta \Gamma^{(1)}[\phi]}{\delta \phi} \right|_{\phi = \phi_0} + \frac{1}{2} \langle \tilde{\phi} \rangle \left. \frac{\delta^2 \mathcal{S}[\phi]}{\delta \phi^2} \right|_{\phi = \phi_0} \phi^{(1)} + \cdots. \tag{3.59}
\]
Then, the final form of the lowest order recoil effect is expressed as
\[
\Gamma^{(2)}_{\text{recoil}}[\phi_0] = \frac{i}{2} \langle \tilde{\phi} \rangle \left. \frac{\delta \Gamma^{(1)}[\phi]}{\delta \phi} \right|_{\phi = \phi_0} \phi^{(1)} + \frac{1}{2} \langle \tilde{\phi} \rangle \left. \frac{\delta \Gamma^{(1)}[\phi]}{\delta \phi} \right|_{\phi = \phi_0} \frac{g_s}{2N} \left. \frac{\delta^2 \mathcal{S}[\phi]}{\delta \phi^2} \right|_{\phi = \phi_0} \left( \frac{d}{ds} \right)^{-2} \left. \frac{\delta \Gamma^{(1)}[\phi]}{\delta \phi} \right|_{\phi = \phi_0}. \tag{3.60}
\]
This coincides with the supergravity result by identifying the recoil acceleration of the D-particle as
\[
\delta \alpha = \frac{g_s}{N} \left. \frac{\delta \Gamma^{(1)}[\phi]}{\delta \phi} \right|_{\phi = \phi_0} = \frac{d^2}{ds^2} \phi^{(1)}. \tag{3.61}
\]
Note again that the recoil correction \( \Gamma^{(2)}_{\text{recoil}}[\phi_0] \) is of the same order as the 1PI part \( \Gamma^{(2)}[\phi_0] \).

The above argument shows that in order to check the agreement of recoil between supergravity and Matrix theory up to two-loop order, it is sufficient to compute the one-loop tadpole diagrams for the matrix fields. The agreement requires that the tadpoles vanish for the gauge field and the off-diagonal components of the Higgs fields, and that the diagonal components of the Higgs tadpole must coincide with the recoil shifts \( \delta \phi_0 \). Since
the effective action for the diagonal Higgs fields is derived using the eikonal approximation, it is not completely tautological to check that the tadpole of the diagonal Higgs field is given by the variational equation of motion derived from our effective action. Conversely, the explicit check of these properties provides a strong support for our method starting from the eikonal approximation.

3.3 Matrix-theory calculation

Let us now confirm that all the above requirements are indeed satisfied in Matrix theory. The one-point functions of ghost, anti-ghost and fermionic fields trivially vanish and that of the gauge field also vanishes by explicit calculation. Taking the background configuration of the Higgs field as

\[ \delta_{ij}(v_i \tau + \tilde{x}_i) \equiv \delta_{ij}(\tau) \]

the propagators of all the fluctuating fields can be expressed in terms of proper-time scalar propagator,

\[ \Delta_{ij}(\sigma, \tau_1, \tau_2) \equiv \exp \left[ -\sigma(-\partial^2_{\tau_1} + r_{ij}(\tau_1)^2) \right] \delta(\tau_1 - \tau_2), \]  

(3.62)

where \( r_{ij} = \tilde{r}_i - \tilde{r}_j \) and we used the Euclidean metric. To avoid confusion, we note that in this subsection the indices \((i, j, \ldots)\) are matrix indices. The blocks with equal velocities and initial positions on the other hand are denoted by using the indices \((a, b, \ldots)\). The only nonvanishing one-point function is that of diagonal components of the fluctuating Higgs field \( Y_{ii} \). We found

\[ \langle \tilde{Y}_{ii}(\tau) \rangle = g_s \int_{-\infty}^{\infty} d\tau' \Delta_0(\tau - \tau') \sum_j \int_0^\infty d\sigma \left[ -32 \tilde{V}_{ij}(\sigma)^4 \tilde{r}_{ij}(\tau') \Delta_{ij}(\sigma, \tau', \tau') 
-32 \tilde{C}_{ij}(\sigma) \tilde{V}_{ij}(\sigma)^2 \tilde{r}_{ij}(\sigma) \partial_{\tau'} \Delta_{ij}(\sigma, \tau', \tau') \right] + O(g_s^2), \]  

(3.63)

where

\[ \tilde{v}_{ij} \equiv \tilde{v}_i - \tilde{v}_j, \quad \tilde{V}_{ij}(\sigma) \equiv \frac{\tilde{v}_{ij}}{\tilde{v}_{ij}} \sinh \frac{\sigma \tilde{v}_{ij}}{2}, \quad \tilde{C}_{ij}(\sigma) \equiv \cosh \frac{\sigma \tilde{v}_{ij}}{2}, \]  

(3.64)

and \( \Delta_0(\tau - \tau') \) satisfies

\[ -\partial^2_{\tau_1} \Delta_0(\tau_1 - \tau_2) = \delta(\tau_1 - \tau_2). \]  

(3.65)

The recoil acceleration

\[ \langle \partial^2_{\tau} \tilde{Y}_{ii}(\tau) \rangle = g_s \sum_j \int_0^\infty d\sigma \left[ 32 \tilde{V}_{ij}(\sigma)^4 \tilde{r}_{ij}(\tau) \Delta_{ij}(\sigma, \tau, \tau) \right] \]  

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\[ +32 \bar{C}_{ij}(\sigma) \bar{V}_{ij}(\sigma)^2 \bar{V}_{ij}(\sigma) \partial_\tau \Delta_{ij}(\sigma, \tau, \tau) \] + O(g_s^2), \tag{3.66} \]

is local with respect to time and its leading contribution with respect to the relative velocities is

\[ \langle \partial_t^2 \tilde{Y}_{ii}(\tau) \rangle_{\text{leading}} = g_s \sum_j \left[ \frac{105}{16} \bar{v}_{ij}^4 \tilde{\sigma}_{ij}(\tau) + \frac{15}{4} \bar{v}_{ij}^2 \tilde{\sigma}_{ij} \frac{d}{d\tau} \frac{1}{r_{ij}(\tau)^2} \right] + O(g_s^2), \tag{3.67} \]

which is in Minkowski metric \((\tau \to i s)\),

\[ \langle \partial_s^2 \tilde{Y}_{ii}(s) \rangle_{\text{leading}} = -g_s \sum_j \left[ \frac{105}{16} v_{ij}^4 \tilde{\sigma}_{ij}(s) + \frac{15}{4} v_{ij}^2 \tilde{\sigma}_{ij} \frac{d}{ds} \frac{1}{r_{ij}(s)^2} \right] + O(g_s^2). \tag{3.68} \]

After rewriting the sum over the diagonal indices \((i, j, \ldots)\) to that \((a, b, \ldots)\) over the diagonal blocks corresponding to the clusters of D-particles, this is precisely the form we have derived from the effective Lagrangian.

4. Concluding Remarks

We have completed the comparison of 11 dimensional supergravity in its classical approximation and Matrix theory up to two-loop order by properly taking into account the recoiled motion of D-particles. There are many directions to extend our works in I and the present paper. We enumerate some of them.

1. Investigate the case with nontrivial gravitational background fields. On Matrix-theory side, it is not at all clear how to treat nontrivial background fields. At least some parts of the gravitational degrees of freedom are contained in the off-diagonal part of the matrix fields. Deeper understanding of the dynamics of off-diagonal part might be useful to answer the problem pointed out in [8].

2. Extend the computation beyond classical approximation. It is not obvious how to take into account the higher-dimension effects \([9]\) on supergravity side. From the viewpoint of type IIA superstring, this amounts to considering the higher effects in \(\alpha'\) and genus-expansions. On Matrix theory side, such higher order effects are contained in the subleading contributions with respect to the velocity \(\alpha' v/r^2\) and also to the \(1/N\) expansion. To clarify the situation of higher order effects is important
to deal with the question posed in [10]. We have to discriminate two different origins of higher dimension effects related to the string extension and string loop effects. It should also be kept in mind that in going to higher orders corresponding to quantum loops of supergravity, the subtlety of the zero modes [11] might become really relevant.

3. Extend the comparison between supergravity and Matrix theory to other higher-dimensional branes. For such examples, see ref. [12]. It is also important to include the spin effect [13] and make supersymmetric completion [14] of the effective multi-body interactions.

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