Accessible maps in a group of classical or quantum channels

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We study the problem of accessibility in a set of classical and quantum channels admitting a group structure. Group properties of the set of channels, and the structure of the closure of the analyzed group $G$ plays a pivotal role in this regard. The set of all convex combinations of the group elements contains a subset of channels that are accessible by a dynamical semigroup. We demonstrate that accessible channels are determined by probability vectors of weights of a convex combination of the group elements, which depend neither on the dimension of the space on which the channels act, nor on the specific representation of the group. Investigating geometric properties of the set $\mathcal{A}$ of accessible maps we show that this set is non-convex, but it enjoys the star-shape property with respect to the uniform mixture of all elements of the group. We demonstrate that the set $\mathcal{A}$ covers a positive volume in the polytope of all convex combinations of the elements of the group.

Dedicated to the memory of Prof. Andrzej Kossakowski (1938 – 2021)

I. INTRODUCTION

The Schrödinger equation describes the time evolution of an isolated quantum system. However, for a system open to an environment, one needs a broader framework. Gorini, Kossakowski and Sudarshan [1], and independently Lindblad [2], derived such an equation of motion governing the evolution of open quantum systems – see [3] for historical remarks. The most general form of such a GKLS generator, acting on $N$ dimensional systems in Hilbert space $\mathcal{H}_N$ and implying Markovian dynamics, is given by

$$L(\cdot) = -i[H,\cdot] + \Lambda(\cdot) - \frac{1}{2}\{\Lambda^*(\mathbb{1}),\cdot\},$$

(1)

where $H$ is the effective Hamiltonian, $\Lambda$ is a completely positive map, and $\ast$ denotes the dual in the Heisenberg picture. The first term on the right-hand side is responsible for the unitary part of the evolution, whereas the remaining terms describe the dissipation. Any operation $L$ may be written in the GKLS form (1) if and only if it is (a) Hermiticity preserving, $L(X^\dagger) = L(X)^\dagger$, (b) trace suppressing, $L^*(\mathbb{1}) = 0$, and (c) conditionally completely positive, i.e. $L \otimes (|\psi_+\rangle\langle\psi_+|)$ is positive semidefinite in the subspace orthogonal to the maximally entangled state, $|\psi_+\rangle = \frac{1}{\sqrt{N}} \sum_i |ii\rangle$ [4]. All terms in the above equation can generally be time-dependent. While a time-dependent Lindblad operation, $L_t$, governs a broader class of evolutions, the simple structure of a semigroup is predicated by a time-independent case, $L_t = L$.

On the other hand, one may adopt yet another approach, namely quantum channels, applicable in a more general situation. Indeed, this method works whenever the physical system and its interacting the environment are initially disjoint [5], and even correlated in some cases [6–10]. By a quantum channel, we mean a completely positive and trace-preserving map sending the convex set of $N$-dimensional quantum states into itself. Assuming that the time evolution of the system and environment is governed by the time-dependent unitary operator $U_t$, one has $\mathcal{E}_t(\rho) = \text{Tr}_E [U_t(\rho \otimes \sigma) U_t^\dagger]$. The Stinespring dilation [11] then guarantees that $\mathcal{E}_t$ is a quantum channel. Accordingly, quantum channels may cover a broader set of evolutions rather than Markovian ones. Note that since $U_t$ satisfies the Schrödinger equation, $\mathcal{E}_t$ is a one parameter quantum channel continuous on $t$.

The question of Markovianity is then brought up. It was introduced into quantum theory in 2008 [4]; however, the corresponding classical problem has a long history [12–15]. The question, if a given classical map described by a stochastic matrix is embeddable, so a continuous Markov process can generate it remains open, and it was recently generalized for the quantum case [16].

Markovianity asks for a given quantum channel $\mathcal{E}$ whether there exists a Lindblad generator of the form (1) whose resulting quantum map is equal to $\mathcal{E}$ at some time $t$. For a general time-dependent Lindblad generator, the answer
to this question is positive if and only if $\mathcal{E}$ is an infinitesimal divisible channel \cite{17, 18}. By definition, an infinitesimal divisible channel is one that can be written as a concatenation of quantum maps arbitrarily close to identity \cite{17}.

Here, we are more interested in the case with a time-independent Lindblad generator. Thus the central question for an assumed quantum channel $\mathcal{E}$ is to find whether there exists a dynamical semigroup starting from identity and equal to $\mathcal{E}$ at some time $t = T$, i.e. if $\mathcal{E} = \exp\{TL\}$ where $L$ is a time-independent Lindblad generator in the form (1). Equivalently, one may ask if there exists a logarithm for a quantum channel $\mathcal{E}$ satisfying properties (a)-(c) mentioned above \cite{4}. However, the non-uniqueness of logarithm for an assumed matrix leaves this a highly difficult question. Indeed, it has been proved to be an NP-hard problem from the computational perspective as well \cite{19}.

As a consequence, while for qubit channels more facts are revealed \cite{20–23}, less is known about Markovianity of quantum channels in higher dimensions \cite{4, 24–26}. For instance, in the case of Pauli channels, $\mathcal{E}(\rho) = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i^\dagger$ where $\sigma_0 = \mathbb{1}_2$ and $\sigma_i$ for $i \in \{1, 2, 3\}$ is one of Pauli matrices, the subset of Markovian channels can be divided into two classes \cite{20}. The first class contains three measure-zero subsets of the channels with two negative degenerated eigenvalues $\lambda_1$ satisfying $\lambda_2^2 \leq \lambda_1$, where $\lambda_1$ is the other nontrivial positive eigenvalue of the channel. While the second class is a subset of channels with positive eigenvalues such that eigenvalues satisfy $\lambda_i \lambda_j \leq \lambda_k$ for all combinations of different $i,j,k$. The latter set occupies $3/32$ of the whole set of Pauli channels \cite{25}. Interestingly, in the case of Pauli channels by taking a convex combination of three Lindblad generators of the form $L_i(\rho) = \sigma_i \rho \sigma_i^\dagger - \rho$ for $i = 1, 2, 3$ as the generator of the semigroup, one can exhaust the entire volume of Markovian maps \cite{21}. Motivated by this fact, a slightly different and simplified version of the Markovianity problem called accessibility has been introduced \cite{25}.

Given a set of quantum channels $\mathcal{S} = \{\mathcal{E}_i\}$ with a not necessarily finite number of elements, in the accessibility problem one asks which quantum channels can be generated by a Lindblad generator of the form $L(\rho) = \sum q_i \mathcal{E}_i(\rho) - \rho$ for some probabilities $q_i$.

The accessibility problem was studied for mixed unitary channels given by a convex combination of Weyl unitary operators \cite{27} – a unitary generalization of Pauli matrices to higher dimensions – which form the set $\mathcal{S}$ of extremal channels \cite{25}. Thus accessible channels form a subset $\mathcal{A}$ of the polytope of all convex combinations of the elements of the generating set $\mathcal{S}$. The set $\mathcal{A}$ can be obtained by proper Lindblad generators related to the elements of the set $\mathcal{S}$ of Weyl channels itself. Although the set $\mathcal{A}$ of accessible channels is a proper subset of Markovian ones by definition, like the Pauli channels, the accessible Weyl channels recover the full measure of Markovian maps. This volume is reduced by an increase in the dimension of the system \cite{25}. In the general case, the set $\mathcal{A}$ of accessible maps occupies a positive volume of the set of channels.

Another relevant fact about Markovian and accessible Pauli channels concerns the rank of these channels. It is known that Pauli channels of Choi rank 3 are neither accessible nor Markovian \cite{20, 21}. In higher dimensions, to our best knowledge, the rank problem has been solved only for a mixture of qutrit Weyl channels accessible by a Lindblad semigroup confirming the existence of accessible channels only of rank 1, 3, 9 \cite{25}. The accessibility rank has been an open problem for Weyl channels in other dimensions as well as other quantum channels.

The aim of this work is to study the problem of accessibility for a large class of quantum channels admitting a semigroup structure. Such a set contains Weyl channels as a special subset. We show that the group properties play an essential role in the accessibility problem. Applying such an approach, we can solve the accessibility rank for Weyl channels of any dimension and show which channels with admissible accessible ranks are Markovian. Furthermore, we obtain analytic results for the relative volume of the set $\mathcal{A}$ of accessible channels in some cases.

The structure of the paper is as follows. In Section II the necessary notions of Lindblad dynamics and accessible channels are introduced. Subsequent Section III provides a detailed description of the set $\mathcal{A}$ of accessible maps generated by a given group $G$ of quantum channels. Some key results of this work are presented in Section IV, in which we show that the set $\mathcal{A}$ is non-convex, but it has the star-shape property with respect to the uniform mixture of all elements of the group $G$. The following section discusses some further results on quantum channels, while the case of stochastic matrices and classical maps is described in Sec. VI. The final Section VII concludes the work and presents a list of open problems. Derivation of the relative volume of the set $\mathcal{A}$ for cyclic and non-cyclic groups of order $g = 4$ is provided in Appendix A.
II. GENERAL PICTURE

Consider a quantum channel $E$ acting on $N$ dimensional states and described by a set $\{K_a\}$ of Kraus operators. The superoperator is then represented by a matrix of order $N^2$, which reads $\Phi_E = \sum_a K_a \otimes \overline{K}_a$, where overline denotes complex conjugation.

Let $B_\alpha$ with $\alpha \in \{0, \ldots, N^2-1\}$ form a basis of $N^2$ matrices of order $N$ such that $\text{Tr}(B_\alpha B_\beta^\dagger) = \delta_{\alpha\beta}$ for any $\alpha, \beta$. Hence any operator $X$ of dimension $N$ can be expanded as $X = \sum x_\alpha B_\alpha$. If a quantum channel $E$ acts on $X$ as the input, the output reads $E(X) = \sum (\Phi_E \alpha, \beta) x_\alpha B_\alpha$. Here the $N^2$ dimensional matrix $(\Phi_E)_{\alpha, \beta} = \text{Tr}(B_\alpha^\dagger E(B_\beta))$ introduces the channel effects to the vector $x$ of same dimension formed by coefficients $x_\alpha$ of the input. Selecting $B_0 = |i\rangle\langle i|$ as the basis, the superoperator takes the standard form $\sum K_a \otimes \overline{K}_a$. Using the Hermitian basis of generalized Gell-Mann matrices, with $B_0 = \frac{1}{\sqrt{N}} I_N$, the affine parameterization of quantum channels in terms of the generalized Bloch vector is obtained,

$$\Phi_E = \begin{pmatrix} 1 & 0 \\ t & M \end{pmatrix}. \tag{2}$$

Here $M$ denotes a real matrix of dimension $N^2 - 1$ called the distortion matrix, while $t$ is a real translation vector of the same dimension presenting how the channel $E$ shifts the identity. The same approach gives the superoperator assigned to a Lindblad generator $L$ given in (1), as $(\mathcal{L}_\Phi)_{\alpha, \beta} = \text{Tr}(B_\alpha^\dagger E(B_\beta))$. Hereafter, quantum channels and Lindblad generators are denoted by their corresponding superoperators, $\Phi$ and $L$, and we will drop the subscripts $E$ and $L$ for the sake of brevity.

If $\Phi$ denotes a quantum channel, then $\mathcal{L}_\Phi = \Phi - 1$ is a Hermiticity preserving and trace suppressing map. Moreover, complete positivity of $\Phi$ imposes conditional complete positivity on $\mathcal{L}_\Phi$ resulting in a valid Lindblad generator assigned to any quantum channel [18]. It is possible to show this fact through Eq. (1) by choosing $H = I_N$ and $\Lambda = E$. Being trace preserving, $E$ admits $\Lambda^*(1) = E^*(1) = 1$ which implies $L_{\mathcal{E}}(\delta) = E(\delta) - \frac{1}{2} (I_{N^2})$. However, the inverse is not valid, i.e. not all Lindblad generators can be obtained by subtracting the identity from a quantum channel. The operation $\mathcal{L}_\Phi = \Phi - 1$ is a generator of a dynamical semigroup, $e^{t\mathcal{L}_\Phi}$, which provides a completely positive and trace-preserving map at each moment, $t \geq 0$, starting from $e^{0\mathcal{L}_\Phi} = 1$ and tending to a point in the set of quantum channels at $t \to \infty$. The quantum channel $\Phi_t$, based on which $\mathcal{L}_\Phi$ is defined, does not necessarily belong to the trajectory of $e^{t\mathcal{L}_\Phi}$.

**Proposition 1.** Let $\Phi$ denote a quantum channel and $\mathcal{L}_\Phi = \Phi - 1$ be the corresponding Lindblad generator. The quantum channel $e^{t\mathcal{L}_\Phi}$ at $t \to \infty$ is a projective map that preserves the invariant states of $\Phi$.

**Proof.** According to the Perron–Frobenius theorem, the spectrum of a superoperator $\Phi$ corresponding to a completely positive and trace-preserving map is confined to the unit disk, and it contains a leading eigenvalue equal to unity. Hence the eigenvalues of the Lindblad generator $\mathcal{L}_\Phi = \Phi - 1$ are either zero or have negative real parts. Therefore, the exponential $e^{t\mathcal{L}_\Phi}$ approaches the operator projecting on the vector subspace on which $\Phi$ acts as identity. This subspace is spanned by the eigenvectors $x$ satisfying $\Phi x = x$. As an example, note that a unitary channel, preserves the diagonal entries and therefore has (at least) $N$ such eigenvectors; in such cases, $\lim_{t \to \infty} e^{t\mathcal{L}_\Phi}$ will therefore be the decoherence channel.

**Corollary 2.** For almost every channel $\Phi$, the limit point $\lim_{t \to \infty} e^{t\mathcal{L}_\Phi}$ is the completely depolarizing channel, i.e. the channel that sends all states to the maximally mixed one.

**Definition 3.** A channel written as $\Omega_t = \exp(t\mathcal{L})$, where $\mathcal{L} = \Phi - 1$ is a time-independent Lindblad generator determined by a quantum channel $\Phi$ from a referred set $S$ of channels, is called accessible. The set of all accessible channels is denoted by $\mathcal{A}$.

There is no limitation on choosing the set $S$ of channels; for example, it can be the set of all channels, all unital channels, or a subset of channels determined based on what one can apply in a lab. Here we demand that the set $S$ be the semigroup formed by the convex hull of a group of channels. Hence the set $S$ here consists of mixed unitary channels; therefore, all elements are unital.

Let $G$ be a group consisting of $g = |G|$ quantum channels acting on states in $\mathcal{H}_N$, which means that all $\Phi \in G$ are unitary quantum channels, i.e. $\Phi = U \otimes \overline{U}$. As a convention, let us take $\Phi_0 = 1$ as the identity (neutral) element of the group. We define the corresponding set of Lindblad generators as $\mathcal{F}_G(G) \coloneqq \{\mathcal{L}_\Phi = \Phi - 1\}$ where $\Phi \in G - \{1\}$. Note that we have excluded by hand the useless generator corresponding to the identity from the set.
Now, we can investigate the accessibility problem, i.e. ask which quantum channels obtained by a convex combination of the elements of $F_G(G)$ belong to the set $A$ of accessible channels. As the following theorem shows, the set $A$ is a subset of the convex hull of the group elements $C(G) := \{ \Phi = \sum \mu \Phi_\mu \}$ with $\sum \mu p_\mu = 1$, which is a semigroup itself.

**Theorem 4.** Consider an identity map, $\Phi_0 = 1$, and a set of maps $\Phi_\mu$ which form a group $G$. Then the dynamical semigroup $e^{tL}$ generated by $L = \sum_{\mu} q_\mu L_{\Phi_\mu}$, where $L_{\Phi_\mu} = \Phi_\mu - 1$, is a convex combination of the group members, 

$$e^{tL} = \sum_{\mu=0}^{g-1} w_\mu(t) \Phi_\mu,$$

where the weights $w_\mu$ depend on time.

*Proof.* Consider a power series,

$$e^{tL} = e^{t(\sum_{\mu} q_\mu \Phi_\mu - 1)} = e^{-t} \sum_{m=0}^{\infty} \frac{\left( \sum_{\mu} t q_\mu \Phi_\mu \right)^m}{m!}.$$

Closure of the group guarantees that the last expression on the right-hand side can be expanded based on group members with non-negative coefficients. Additionally, the left-hand side of this equation describes a trace-preserving quantum map at each moment in time. The above implies that the coefficients form a probability vector which completes the proof.

The explicit time-dependent form of the probabilities $w_\mu(t)$ entering Eq. (3) relies on the group structure and not the specific representation of the group $G$. Therefore, any faithful representation with any dimension, which may not even present a quantum channel, can be adapted to compute the weights $w_\mu(t)$. For example, one may think about the regular representation of a group. We remind the reader that the regular representation is written based on the standard form of the Cayley table of the group. In this special representation we assign a permutation matrix of dimension $g$, denoted by $R_\alpha$, to each element of the group $G$ with the same cardinality such that for any $\alpha$ and $\beta$ we have $\text{Tr}(R_\alpha R_\beta^\dagger) = g \delta_{\alpha, \beta}$, see Example 8 of [28]. Hence, $w_\mu(t)$ can be obtained through the following lemma, which is a consequence of the orthogonality of the permutations $R_\alpha$.

**Lemma 5.** Let $G_\Phi = \{ \Phi_0 = 1, \cdots, \Phi_{g-1} \}$ be a representation of the group $G$ in terms of quantum channels. The group $G$ also admits a regular representation based on orthogonal permutations of dimension $g$ as $G = \{ R_0 = 1, \cdots, R_{g-1} \}$. An accessible quantum channel is defined by Eq. (3), in which

$$w_\mu(t) = \frac{1}{g} \text{Tr}\left( R_\mu^\dagger \exp \left( t \sum_{\nu=1}^{g-1} q_\nu R_\nu - 1 \right) \right) = \frac{e^{-t}}{g} \text{Tr}\left( R_\mu^\dagger \exp \left( t \sum_{\nu=1}^{g-1} q_\nu R_\nu \right) \right).$$

Additionally, as the group, $G_\Phi$, is assumed to be finite with unitary elements $\Phi_\mu$, a convex polytope with exactly $g$ extreme points in the set of all quantum channels can be constructed from them. Theorem 4 tells us the set of accessible maps also belongs to this polytope. The question concerning the position of the trajectory $e^{tL}$ in the polytope of the group is highly related to the group structure, its subgroups, and the non-zero interaction times (non-zero $q_\mu$). However, the trajectory is in the interior of the polytope once all $q_\mu$ are non-zero. Moreover, it tends to the centre of the polytope as $t \to \infty$.

**Proposition 6.** Let $G_\Phi = \{ \Phi_\mu \}^{g-1}_{\mu=0}$ be a group of $g$ unitary quantum channels. The trajectory $e^{tL}$ generated by a generic generator $L = \sum q_\mu \Phi_\mu - 1$ ends in the center of the polytope formed by the maps $\Phi_\mu$, i.e. the uniform mixture of all group members $\Phi_* = \frac{1}{g} \sum_{\mu=0}^{g-1} \Phi_\mu$.

*Proof.* It has been shown in Proposition 1 that the trajectory $e^{tL}$ ends in the projector $1_{V_0}$ that projects on $V_0$, the invariant subspace of $\Phi = \sum q_\mu \Phi_\mu$, i.e.

$$V_0 \equiv \{ x \mid \Phi_\mu x = x, \ \forall \Phi_\mu \in G_\Phi \}.$$

Now write $\mathbb{H}$ as a direct sum of irreducible representations as

$$\mathbb{H} = \bigoplus_k \mathbb{V}_k.$$
On the other hand, note that for all \( \mu \) we get \( \Phi^\mu \Phi = \Phi^\mu \Phi = \Phi^\mu \) due to group rearrangement theorem. Applying Schur’s lemma, we get

\[
\Phi^\mu = \bigoplus_k c_k \mathbb{1} = \mathbb{1}.
\]

The equality \( \Phi^\mu \Phi = \Phi^\mu \) implies \( c_k = 0 \) for all \( k \neq 0 \) and for \( k = 0 \), it is possible to directly verify \( c_0 = 1 \). This shows \( \Phi^\mu \) is the projector to the intersection of the invariant subspace of all group members and completes the proof.

In the case with some vanishing weights, \( q_\mu \), the situation will be completely different. However, one gets the salient result from Theorem 4 that the trajectory remains in the subset characterized by the smallest subgroup containing all non-vanishing quantum channels appearing in the Lindblad generator. Such a curve ends in the centre of the polytope formed by the smallest subgroup as well. As an example, let us remind readers the group properties imply that every element \( \Phi^\mu \in G^\Phi \) has order equal to the smallest integer and the positive number \( h^\mu \) such that \( \Phi^{h^\mu} = \mathbb{1} \).

The trajectory generated by \( \mathcal{L}_{\Phi^\mu} = \Phi^\mu - \mathbb{1} \) is actually a cyclic subgroup of \( G^\Phi \) is equal to the smallest integer and the positive number \( h^\mu \) has order equal to the smallest integer and the positive number \( h^\mu \) such that \( \Phi^{h^\mu} = \mathbb{1} \). The equality \( \Phi^{h^\mu} = \mathbb{1} \) implies \( c_k = 0 \) for all \( k \neq 0 \) and for \( k = 0 \), it is possible to directly verify \( c_0 = 1 \). This shows \( \Phi^{h^\mu} \) is the projector to the intersection of the invariant subspace of all group members and completes the proof.
where \( h_\mu \), and \( h_\mu \) are the order of \( \Phi_\mu \), and \( \Phi_\mu^i \), respectively. In the last equation \( l = (ij \mod h_\mu) \) and

\[
\Phi(t_\mu) \equiv \prod_{i=1}^{h_\mu-1} \sum_{j=0}^{h_\mu-1} p_j(t_\mu). \tag{14}
\]

Hence, we can find the time-dependent probabilities in the convex combination. However, this is only one of the possibilities since, in a general case, the polytope formed by the set elements \( G_\Phi \) can differ from the simplex.

Consider now another quantum channel \( \Phi'' \) from the set \( G_\Phi \) such that there is no cyclic subgroup of \( G_\Phi \) including both \( \Phi_\mu \) and \( \Phi''_\mu \). To emphasize this fact, we will denote \( \Phi''_\mu \) by \( \Theta_\mu \). If we want to add \( L_\Theta \) to the set discussed in Eq. (13) two different cases are possible. First, if we assume \([\Phi_\mu, \Theta_\mu] = [L_{\Phi_\mu}, L_{\Theta_\mu}] = 0\), then

\[
\exp(t_\alpha L_{\Theta_\alpha} + \sum t_\mu L_{\Phi_\mu}) = \exp(t_\alpha L_{\Theta_\alpha}) \exp(\sum t_\mu L_{\Phi_\mu}) = \sum p_k(t_\alpha) P(t_\mu) \Theta_\alpha^k \Phi_\mu^l, \tag{15}
\]

where \( h_\alpha \) and \( h_\mu \) are the order of \( \Theta_\alpha \) and \( \Phi_\mu \), and \( p_k(t_\alpha) \) and \( P(t_\mu) \) are introduced in Eqs. (10) and (14), respectively. Note that commutativity of \( \Theta_\alpha \) and \( \Phi_\mu \) imposes compatibility on any power of them, i.e. all elements of the cyclic subgroups \( H_{\Theta_\alpha} \) and \( H_{\Phi_\mu} \) commute. Moreover, as we assumed that there is no cyclic subgroup including both \( \Theta_\alpha \) and \( \Phi_\mu \), thus \( H_{\Theta_\alpha} \) and \( H_{\Phi_\mu} \) have no element in common but the identity. This implies that the product of \( H_{\Theta_\alpha} \) and \( H_{\Phi_\mu} \) is a subgroup of \( G_\Phi \) of order \( h_\alpha h_\mu \), i.e. \( \forall k,l: \Theta_\alpha^k \Phi_\mu^l \) belongs to a subgroup of \( G_\Phi \) formed by multiplication of its cyclic and commutative subgroups. Applying the above arguments, we can generalize these results as follows.

**Corollary 7.** Let quantum channels \( \Theta \) and \( \Phi \) respectively denote the generators of the cyclic groups \( H_\Theta \) of order \( h_\Theta \) and \( H_\Phi \) of order \( h_\Phi \). Assume that \([\Theta, \Phi] = 0\) and that \( H_\Theta \) and \( H_\Phi \) have no element in common but the identity. Consider any convex combination of the elements of \( \{\Theta^i - 1\}_{i=1}^{h_\Theta-1} \cup \{\Phi^j - 1\}_{j=1}^{h_\Phi-1} \).

Then the dynamical semigroups generated by such Lindblad generators belong to the polytope constructed by the convex hull of \( h_\Theta h_\Phi \) extreme points related to different elements of \( H_\Theta \times H_\Phi \). The trajectory, \( \exp(tL) \), starts from \( \Theta^0 = \Phi^0 = 1 \) when all interaction times \( t_{\Theta^i} \) and \( t_{\Phi^j} \) are zero, it goes inside the polytope if there exist \( i \) and \( j \) for which \( t_{\Theta^i} \neq 0 \) and \( t_{\Phi^j} \neq 0 \), and it tends to the projector at the centre of polytope at \( t \to \infty \).

**Corollary 8.** Consider an abelian group \( G_\Phi \) represented by the maps \( \Phi_\mu \) and the Lindblad generators of the form \( L = \sum \Phi_\mu \Phi_\mu^i - 1 \). Then the dynamical semigroup \( e^{tL} \), generated by \( L \), can be decomposed convexly based on the elements of a subgroup of \( G_\Phi \) obtained by the product of different cyclic subgroups, each formed by powers of all the maps \( \Phi_\mu \) appearing in \( L \).

This Corollary can be proved through the fundamental theorem of abelian groups and the above discussion.

The condition on commutativity in Corollary 7 may be relaxed in a general scenario, i.e. \([\Theta, \Phi] \neq 0\). In that case, \( H_\Theta \times H_\Phi \) is not necessarily a subgroup. Therefore, we need to add some other quantum channels from \( G_\Phi \) to the set \( H_\Theta \times H_\Phi \) so it satisfies closure. The smallest subgroup \( H \) of \( G_\Phi \) that includes all members of the set \( H_\Theta \times H_\Phi \) expands convexly the dynamical semigroups generated by Lindblad generators of our interest. This is a direct consequence of the closure property, as the group structure plays a crucial role for Theorem 4.

### III. EXEMPLARY GROUPS OF CHANNELS AND CORRESPONDING SETS OF ACCESSIBLE MAPS

In this section we will apply the aforementioned theorems and relations for certain constructive examples. For some groups associated with quantum channels acting on \( N \)-dimensional systems and represented by matrices of size \( N^2 \), we demonstrate how to find the quantum channels, their polytopes, and the subset of accessible maps inside the polytope.

Before proceeding with the examples let us mention that if \( G_\Phi = \{\Phi_\mu = U_\mu \otimes U_\mu^*\}_{\mu=0}^{g-1} \) is a group of \( g \) unitary channels, then the set of the corresponding Kraus operators denoted by \( \tilde{G}_U = \{U_\mu\}_{\mu=0}^{g-1} \) is a group up to a phase, which is also called a **projective representation** of group \( G \). Explicitly, in the set \( \tilde{G}_U \): (i) there are no two elements which are equal up to a phase, i.e. if \( U \in G_U \), then \( e^{i\phi}U \notin G_U \), (ii) the set is closed up to a phase, i.e. if \( U_1, U_2 \in G_U \), then \( e^{i\phi}U_1U_2 \in \tilde{G}_U \) for some \( \phi \), (iii) it contains an inverse up to a phase for each element, i.e. for any \( U_1 \in \tilde{G}_U \) there exists an element \( U_2 \in \tilde{G}_U \) and a phase \( \phi \) such that \( U_1U_2 = e^{i\phi}1 \). Clearly, the last two conditions imply that (iii) the set possesses a neutral element up to a phase. Moreover, the set \( G_\Phi \) is abelian if and only if \( \tilde{G}_U \) is abelian up to a phase, i.e. \( \forall U_1, U_2 \in \tilde{G}_U \) one has \( U_1U_2 = e^{i\phi}U_2U_1 \) for some phase \( \phi \).
Example 1 (The group of order \( g = 2 \)). The first non-trivial example is a group of order 2, \( G_\Phi = Z_2 = \{ \Phi_0^Z = \mathbb{1}, \Phi_1^Z \} \) which is a cyclic group. The superscript \( Z_2 \) is to emphasis that the group is a cyclic group of order 2. Since \( \Phi_1^Z \) is of group order \( h_1 = 2 \), its spectrum contains only \( \pm 1 \). In order to have two distinguished elements in the group, there has to be \(-1\) in the spectrum. Indeed, for \( m \) from the set \{1, \ldots, \lfloor N/2 \rfloor \}, \( 2m(N - m) \) out of \( N^2 \) eigenvalues of \( \Phi_1^Z \) may be equal to \(-1\). To see that, let us denote by \( \tilde{G}_U^Z = \{ U_0^Z = \mathbb{1}_N, U_1^Z \} \) the group up to a phase of Kraus operators. To have two distinguished elements in \( \tilde{G}_U^Z \) at least one of the eigenvalues of \( U_1^Z \) should possess a phase difference equal to \( \pi \) from other eigenvalues which are equal. However, there might be \( m \) eigenvalues with such a phase difference in the spectrum in the general case where \( m \) to the aforementioned set. Note that \( 2m(N - m) \) negative eigenvalues will appear in the spectrum of the quantum channel \( \Phi_1^Z = U_1^Z \otimes \mathbb{1}_1 \).

The polytope related to the group, \( G_\Phi = Z_2 \), is the simplex of dimension one, i.e. a line segment. The set of accessible maps, in this case, can be written as (11) with probabilities given by Eq. (10)

\[
\Omega_t = \frac{1 + e^{-2t}}{2} \mathbb{1} + \frac{1 - e^{-2t}}{2} \Phi_1^Z.
\] (16)

Hence the trajectory starts from the identity and goes to the projector at the centre of the line at \( t \to \infty \), see Fig.1(a). Thus, any map of the form \( \Phi = w \mathbb{1} + (1 - w) \Phi_1^Z \) with \( w \geq 1/2 \) is accessible. Note that the results do not depend on the dimension of the quantum channels, and such a simplex exists in all dimensions. As an explicit example, however, one may get the qubit channel \( \Phi_1^Z = \sigma_i \otimes \sigma_j \) with one of the Pauli matrices.

Example 2 (The group of order \( g = 3 \)). A group of order 3 is still a cyclic and so an abelian group, \( G_\Phi = Z_3 = \{ \Phi_0^Z = \mathbb{1}, \Phi_1^Z, \Phi_2^Z \} \), where the superscript \( Z_3 \) denotes the cyclic group of order 3. Since all the members of this group are unitary channels and so the extreme points of the set of quantum channels, these channels do not lie on the same line. Thus, they are linearly independent and the polytope of the group is a triangle for \( N \)-dimensional quantum channels. This triangle is regular with respect to the Hilbert-Schmidt distance between two unitary maps \( \Phi_i \) and \( \Phi_j \),

\[
D(\Phi_i, \Phi_j) = \| \Phi_i - \Phi_j \|_2^2 = 2N^2 - \text{Tr}(\Phi_i \Phi_j^\dagger) - \text{Tr}(\Phi_i^\dagger \Phi_j),
\] (17)

In the case of the group of order 3 we get

\[
\forall i \neq j : \quad D(\Phi_i^Z_1, \Phi_j^Z_1) = 2N^2 - \text{Tr}(\Phi_i^Z_1) - \text{Tr}(\Phi_j^Z_1),
\] (18)

showing all elements of the group are equidistant. Therefore, the triangle is regular for \( N \)-dimensional quantum channels, and we deal with a 2-simplex. On the other hand, the order of both non-trivial elements are three, \( h_1 = h_2 = 3 \). Thus, through Eq. (13) and Eq. (14), accessible maps are given by \( \Omega_t = w_0(t) \mathbb{1} + w_1(t) \Phi_1^Z + w_2(t) \Phi_2^Z \), where \( t = t_1 + t_2 \) and

\[
w_0(t) = \frac{1}{3} \left( 1 + 2e^{\frac{i}{3} \pi (t_1 + t_2)} \cos \left( \frac{\sqrt{3}}{2} (t_1 - t_2) \right) \right),
\]

\[
w_1(t) = \frac{1}{3} \left( 1 - e^{\frac{i}{3} \pi (t_1 + t_2)} \cos \left( \frac{\sqrt{3}}{2} (t_1 - t_2) \right) + \sqrt{3} e^{\frac{i}{3} \pi (t_1 + t_2)} \sin \left( \frac{\sqrt{3}}{2} (t_1 - t_2) \right) \right),
\]

\[
w_2(t) = \frac{1}{3} \left( 1 - e^{\frac{i}{3} \pi (t_1 + t_2)} \cos \left( \frac{\sqrt{3}}{2} (t_1 - t_2) \right) - \sqrt{3} e^{\frac{i}{3} \pi (t_1 + t_2)} \sin \left( \frac{\sqrt{3}}{2} (t_1 - t_2) \right) \right).
\] (19)

These equations are already introduced in [25] in the study of accessible Weyl channels of dimension three. The simplex representing this group and the subset \( \mathcal{A} \) of accessible maps are shown in Fig. 1b. Since the group has no non-trivial subgroup, there is no accessible map on the edges, as shown in this figure. Observe that this result is independent of the dimension of the quantum channels. Again, as an explicit example we may think of a group of three single-qubit unitary channels based on the group up to a phase, \( G_U^Z = \{ \mathbb{1}_2, \text{diag}[e^{\frac{i}{3} \pi}, e^{\frac{2i}{3} \pi}], \text{diag}[e^{\frac{i}{3} \pi}, e^{\frac{2i}{3} \pi}] \} \). Existence of a faithful representation of qubit channels for this group guarantees the existence of a faithful representation in higher dimensions. Note that the same non-convex subset of the equilateral triangle, plotted in Fig. 1b and bounded by two logarithmic spirals [25], was earlier identified in search of classical semigroups [16, 29] in the space of bistochastic matrices of order three.

Concerning the spectrum, the fact that non-trivial elements are of order 3 implies that eigenvalues of these maps are \( E_\Phi = \{ 1, e^{\pm i \frac{2}{3} \pi} \} \). Perron-Frobenius theorem in the space of quantum channels suggests all of these elements are
FIG. 1: The set $\mathcal{A}$ of accessible maps, indicated by blue, in (a) 1-simplex of a cyclic group of order 2, (b) 2-Simplex of a cyclic group of order 3, and the set of channels in the regular polygons of linearly dependent channels forming a cyclic group of order (c) 4, (d) 5, (e) 6, and (f) 7. The symbol $\mathbb{I}_{N^2}$ denotes the identity map and the subscript $N^2$ is added to emphasize that it can be considered in any dimension $N$. The map $\Phi_*$ indicates the center of each polygon. All sets $\mathcal{A}$ are star-shaped.

Example 3 (The groups of order $g = 4$). A group of order 4 is isomorphic either to a cyclic group $G_\Phi = \mathbb{Z}_4 = \{\Phi_0^1, \Phi_0^2, \Phi_0^3, \Phi_0^4\}$ or to a non-cyclic one, $G_\Psi = \{\Psi_0 = \mathbb{I}, \Psi_1, \Psi_2, \Psi_3\}$ with $\Psi_i^2 = \mathbb{I}$ and $\Psi_i \Psi_j = \Psi_k$ for different $i, j, k \in \{1, 2, 3\}$. Both of these groups are still abelian so one can find a set of common basis in which all the elements are diagonal. Let us first consider the cyclic group, $G_\Phi$.

1. The cyclic group of order $g = 4$

The polytopes of this group can be two-dimensional or three-dimensional related to the linear dependency of the elements. The spectra of the group can determine this up to a phase of Kraus operators.

Lemma 9. Consider the cyclic group $G_\Phi = \mathbb{Z}_4$ of order 4 of quantum channels acting on $N$-dimensional systems. Let $\Phi_i^{Z_4} = U_1^{Z_4} \otimes U_1^{Z_4}$ denote its generator, i.e. $\Phi_i^{Z_4} = (\Phi_i^{Z_4})^i$ for $i \in \{0, \ldots, 3\}$. So $U_1^{Z_4}$ is the generator of a cyclic group up to a phase of order 4, $G_\mathbb{C}^{Z_4}$, whose spectrum up to a general phase is from the set $E_{U_1} = \{\pm 1, \pm i\}$. Moreover, both real and imaginary elements should be present in the spectrum of $U_1^{Z_4}$ to have four distinguished elements in the group $G_\Phi$. Then the elements of $G_\Phi = \mathbb{Z}_4$ are linearly dependent if and only if all the real elements appearing in the spectrum of $U_1^{Z_4}$ have the same sign and all the imaginary eigenvalues have also the same sign (can be different from the sign of real ones).
These numbers allow us to express the Hilbert-Schmidt distance (17) between the channels, in Lemma 9. In a similar way let

\[ \text{Corollary 10.} \]

when we need to emphasise this property. An immediate result of the above lemma is the following corollary.

We will change the superscript \( Z_4 \) as \( Z_{AD} \) and \( Z_{UI} \) for the linearly dependent and linearly independent case, respectively, when we need to emphasise this property. An immediate result of the above lemma is the following corollary.

**Corollary 10.** For the cyclic group of order 4 formed by qubit channels, acting on systems of dimension \( N = 2 \), elements are linearly dependent.

However, in higher dimensions both two and three dimensional polytopes are possible. In order to study the regularity of the polytope, let \( r_+ \) and \( r_- \) denote the number of eigenvalues +1 and −1, respectively, in the spectrum of \( U_{1}^{Z_4} \), defined in Lemma 9. In a similar way let \( i_+ \) and \( i_- \) denote the number of eigenvalues +i and −i, so that \( r_+ + r_- + i_+ + i_- = N \). These numbers allow us to express the Hilbert-Schmidt distance (17) between the channels,

\[
D(\Phi_0^{Z_4}, \Phi_{1}^{Z_4}) = D(\Phi_0^{Z_4}, \Phi_{2}^{Z_4}) = D(\Phi_1^{Z_4}, \Phi_{2}^{Z_4}) = D(\Phi_0^{Z_4}, \Phi_{3}^{Z_4})
\]

\[
= 2N^2 - Tr\Phi_0^{Z_4} - Tr\Phi_{1}^{Z_4} - Tr\Phi_2^{Z_4} = 8r_+ r_- + 8i_+ i_- + 4(r_+ + r_-) (i_+ + i_-),
\]

\[
D(\Phi_0^{Z_4}, \Phi_{2}^{Z_4}) = D(\Phi_1^{Z_4}, \Phi_{3}^{Z_4}) = 2N^2 - 2Tr\Phi_2^{Z_4} = 8(r_+ + r_-) (i_+ + i_-).
\]

The above relations imply if the polytope is of dimension 2, it forms a square like the one in Fig. 1(c), while the three-dimensional case can be a (non-)regular tetrahedron – see Fig. 3. In particular, for a cyclic group of order 4 of linearly-independent qutrit channels one out of four numbers \( r_\pm \) and \( i_\pm \) is zero due to Lemma 9, while the remaining three numbers are equal to 1. This implies that both distances in Eq. (20) are equal. Hence, the polytope is a regular tetrahedron for a cyclic group of order 4 formed by linearly independent quantum channels acting on \( N = 3 \) dimensional systems. In higher dimensions, however, both regular and non-regular tetrahedrons, as well as the square, are possible, see Fig. 3. To emphasize the regularity of the polytope, we will denote cyclic groups of four linearly-independent channels with a regular and non-regular tetrahedrons by \( G_{\Phi}^{Z_4} = \{ \Phi_0^{Z_4} = \mathbf{1}, \Phi_1^{Z_4}, \Phi_2^{Z_4}, \Phi_3^{Z_4} \} \) and \( G_{\Gamma}^{Z_4} = \{ \Gamma_0^{Z_4} = \mathbf{1}, \Gamma_1^{Z_4}, \Gamma_2^{Z_4}, \Gamma_3^{Z_4} \} \), respectively.

The shape of the polytope formed by the elements of the group \( G \) in the space of quantum channels and the dimension of the channels do not play any role by describing the boundary of the set \( \mathcal{A} \) of accessible maps. Hence the group \( G_{\Phi}^{Z_4} \) denotes one of three different types of the group: \( G_{\Phi}^{Z_4} \), \( G_{\Phi}^{Z_4} \), and \( G_{\Gamma}^{Z_4} \). Due to Corollary 8, the order of the cyclic

![FIG. 2: The regular polygons in the complex plane represent support of spectra of the quantum channels forming a cyclic group of order (a) 3, (b) 4, and (c) 5. The blue region in each polygon determines the support of spectra of the accessible maps in the corresponding set. Note that the shape and boundaries of the blue regions above are exactly the same as the quantum channels in Fig. 1(b)-d.](image-url)
subgroups (the order of elements) plays a pivotal role. Assuming $\Phi_Z^4$ as the generator of $G_Z^4$, we get $h_1 = h_3 = 4$ while $h_2 = 2$. This implies that

$$e^{i t_1 L_{Z_1}} = \frac{e^{-t_1}}{2} \left[ (\cosh t_1 + \cos t_1) I + (\sinh t_1 + \sin t_1) \Phi^Z_1 + (\cosh t_1 - \cos t_1) \Phi^Z_2 + (\sinh t_1 - \sin t_1) \Phi^Z_3 \right], \quad (21)$$

while $e^{i t_2 L_{Z_2}}$ is the same as Eq. (16) with substitution of $t_2$ for $t$, and noting that $\Phi^Z_2 = \Phi^Z_1$. The trajectory $e^{i t_3 L_{Z_3}}$ is the same as $e^{i t_1 L_{Z_1}}$ by replacing $t_1$ by $t_3$ and $\Phi^Z_1$ by $\Phi^Z_3$. Hence for an accessible map of a cyclic group of order 4, we get $\Omega = \sum w_i(t) \Phi^Z_i$ with $t = t_1 + t_2 + t_3$ independently of the shape of its polytope and dimension of the channels. In this way we arrive at equations

$$w_0(t) = \frac{1}{4} \left( 1 + e^{-(t_1 + t_3)} + 2e^{-2t_2} \cos(t_1 - t_3) \right),$$
$$w_1(t) = \frac{1}{4} \left( 1 - e^{-(t_1 + t_3)} + 2e^{-(t_1 + 2t_2 + t_3)} \sin(t_1 - t_3) \right),$$
$$w_2(t) = \frac{1}{4} \left( 1 + e^{-(t_1 + t_3)} - 2e^{-2t_2} \cos(t_1 - t_3) \right),$$
$$w_3(t) = \frac{1}{4} \left( 1 - e^{-(t_1 + t_3)} - 2e^{-(t_1 + 2t_2 + t_3)} \sin(t_1 - t_3) \right), \quad (22)$$

which allow us to represent any element of the set $A$ of accessible maps as a convex combination of the group members. In the case of the group $G_Z^2$ the set $A$ of accessible maps forms a non-convex subset in the square – see Fig. 1(c). The spectra of superoperators obtained by a convex combination of the elements of this group are also embedded in a square in the complex plane whose vertices are $(\pm 1, \pm i)$. The spectra corresponding to accessible maps are restricted to a shape equivalent to the shape of accessible maps in the set of quantum channels, see Fig. 2(b). As a simple example for such a group, one may think of the set of four qubit channels constructed by the group up to a phase $G_{U}^{Z^4} = \{1_2, \text{diag}[e^{\pm i}, e^{\pm i}], \text{diag}[i, -i], \text{diag}[e^{\pm i}, e^{\pm i}]\}$. Having a qubit realization, this group can also find representations in higher dimensions.

Fig. 3 shows the subset of accessible maps in a regular and a non-regular tetrahedron. As mentioned in Corollary 10, it is impossible to find such polytopes satisfying a cyclic group structure in the space of qubit channels. Let us first consider the regular tetrahedron, Fig. 3 (a) and (b). The simplest example of a cyclic group of order four with linearly-independent elements is a set of qutrit channels based on the group up to a phase, $G_{U}^{Z^4} = \{1_3, \text{diag}[1, -1, i], \text{diag}[1, 1, -1], \text{diag}[1, -1, -i]\}$. Due to Lemma 9 existence of both 1 and $-1$ in the spectrum of these unitary operators implies that they are linearly independent. One can find other examples of qutrit channels as well as channels in higher dimensions for this group. However, the shape of accessible maps inside the tetrahedron is independent of a particular example or even dimension of the channels.

We shall now analyze the case of a non-regular tetrahedron, presented in Fig. 3 (c) and (d). This special non-regular tetrahedron is the polytope of a cyclic group of four linearly-independent members for which the distance of the channels are given by $2N^2 - \text{Tr} \Gamma_1^{Z^4} - \text{Tr} \Gamma_3^{Z^4} = \frac{3}{4} \left( 2N^2 - 2 \text{Tr} \Gamma_2^{Z^4} \right)$, see Eq. (20). As it is not possible to represent such a group with qubit and qutrit channels, we are going to analyze the set of channels acting on 4-dimensional systems. A particular example is given by a group, $G_{U}^{Z^4} = \{1_4, \text{diag}[1, 1, i, -i], \text{diag}[1, 1, -1, -1], \text{diag}[1, 1, -i, i]\}$. Since both $i$ and $-i$ appear simultaneously in the spectrum of the generator $V_4$, the elements of this group are linearly independent, although the elements of the group up to a phase are not linearly independent.

Recall that a cyclic group of order 4 has one non-trivial subgroup of order 2. In all corresponding polytopes of this group, there exist accessible maps belonging to the line connecting the identity map to the channel corresponding to the generator of this subgroup. In tetrahedrons shown in Fig. 3 this edge of the polytope contains accessible maps, while the diagonal of the square in Fig. 1(c) also contains accessible channels. Moreover, let us emphasize that independently of the shape of the polytope representing a cyclic group of order 4, the spectra of superoperators corresponding to accessible channels form the same set in the complex plane as accessible maps in the square of the quantum channel itself, see Fig. 2 (b).
FIG. 3: The subset of accessible quantum channels specified inside the 3-simplex: (a) and (b) related to a cyclic group of order 4 from two different perspectives, and (c) and (d) inside the tetrahedron of a cyclic group of order 4 whose elements are not equidistant. The length of the dashed black edges of the tetrahedron is 1, and for the blue dashed ones, the length is equal to $\sqrt{3}/2$. Explicit examples of quantum channels forming such groups are provided in Section III 1. The volume of the subset of accessible maps in both regular and non-regular shapes is $3(1 - e^{-4\pi})/32$ of its volume of the corresponding tetrahedron.

2. The non-cyclic group of order 4

As mentioned above the non-cyclic group $G_\Psi = \{\Psi_0 = 1, \Psi_1, \Psi_2, \Psi_3\}$ of order four is abelian. Due to the fundamental theorem of abelian groups, it can be obtained by the product of two cyclic groups of order 2 whose elements are compatible with each other. Since all non-trivial group members are of order 2, their spectrum consists of $\pm 1$. There exist distinguished elements of the group if there are subspaces in which the eigenvalues of all channels are not of the same sign. As for these non-trivial elements, relation $\Psi_i\Psi_j = \Psi_k$ holds; the negative eigenvalue should appear in the spectrum of two maps simultaneously in the same subspace, so the third channel possesses a positive eigenvalue. Taking into account all possibilities, one can prove that to have $\sum c_i\Psi_i = 0$ the only possibility is $c_i = 0$ for any $i$. This means the elements of $G_\Psi$ are linearly independent. Thus the polytope for this group, regardless of the dimension of the channels, is a tetrahedron, see Fig. 4. Moreover, one needs to study the distance of the channels to find whether this tetrahedron is regular or not. In this case, the Hilbert-Schmidt distance (17) for different maps are given by

$$D(\Psi_0, \Psi_i) = 2(N^2 - \text{Tr}\Psi_i), \quad \text{for } i \in \{1, 2, 3\},$$

$$D(\Psi_i, \Psi_j) = 2(N^2 - \text{Tr}\Psi_k), \quad \text{for } i \neq j \neq k \text{ and } i, j, k \in \{1, 2, 3\}. \quad (23)$$
As mentioned in Example 1, the number of negative eigenvalues in a unitary channel of order \( h = 2 \) is \( 2m(N - m) \) for an \( m \in \{1, \ldots, [N/2]\} \). In particular, for qubit and qutrit channels, the numbers of negative eigenvalues can be 2 and 4, respectively. This means unitary qubit channels of order 2 are traceless and for any unitary qutrit channels of this order, the trace of all such channels is equal to 5 - 4 = 1. Thus for qubit and qutrit channels, the tetrahedron spanned by the elements of the group is regular – see Fig. 4 (a) and (b).

The group of Pauli channels is the well-known example of a non-cyclic group of order 4 whose accessibility has been studied in [20, 21]. As an example of qutrit channels forming such a group, consider three different quantum channels corresponding to three different quantum channels. To get a non-regular tetrahedron consider a group up to a phase, \( G_U = \{U_0, U_1, U_2, U_3\} \) where \( U_0 = I \), \( U_1 = \text{diag}[1, -1, -1, 1] \), \( U_2 = \text{diag}[1, 1, 1, -1] \), and \( U_3 = \text{diag}[-1, 1, 1, 1] \). This tetrahedron is similar to the non-regular tetrahedron of Example 3, in which the length of two non-connected edges are \( 2/\sqrt{3} \) times the length of the other edges.

Note that the property of accessibility of a channel is independent of the dimension and the form of the polytope, as it depends on the group structure. In the case of a non-cyclic group of order four all non-trivial elements are of order 2, so individual trajectories, with only a single non-zero interaction time, are of the form (16). For an accessible map, \( \Psi_t = \sum w_i(t) \Psi_i \), with \( t = t_1 + t_2 + t_3 \) we get

\[
\begin{align*}
    w_0(t) &= \frac{1}{4} \left( 1 + e^{-2(t_2 + t_3)} + e^{-2(t_1 + t_3)} + e^{-2(t_1 + t_2)} \right), \\
    w_1(t) &= \frac{1}{4} \left( 1 + e^{-2(t_2 + t_3)} - e^{-2(t_1 + t_3)} - e^{-2(t_1 + t_2)} \right), \\
    w_2(t) &= \frac{1}{4} \left( 1 - e^{-2(t_2 + t_3)} + e^{-2(t_1 + t_3)} - e^{-2(t_1 + t_2)} \right), \\
    w_3(t) &= \frac{1}{4} \left( 1 - e^{-2(t_2 + t_3)} - e^{-2(t_1 + t_3)} + e^{-2(t_1 + t_2)} \right).
\end{align*}
\]

These results have already been reported for the Pauli channels [20, 21, 25]. However, here we see that they can be gained for any set of quantum channels admitting the same group structure. The subset of accessible maps in the regular and non-regular tetrahedrons are presented in Fig. 4. Note that a non-cyclic group of order 4 has three non-trivial subgroups, each of which is of order 2. Consequently, we see on the edges connecting the identity map to other three elements, accessible maps can be found.

IV. GEOMETRIC PROPERTIES OF THE SET OF ACCESSIBLE MAPS

In this section, we will study the geometric properties of the set \( \mathcal{A} \) of accessible maps, which are obtained by a mixture of unitary maps forming a discrete group. Proposition 1 implies that any Lindblad operator determined by a quantum channel, \( \mathcal{L}_\Phi = \Phi - I \), generates a trajectory in the set of quantum channels which converges to the projector onto the invariant subspace of the channel \( \Phi \). For any quantum channel \( \Phi \), the line connecting the identity with \( \Phi \) is a tangent line to the trajectory generated by \( \mathcal{L}_\Phi \) at the beginning of the evolution. To see this, note that for the dynamical semigroup \( \exp(t\mathcal{L}_\Phi) \), one gets

\[
\frac{d}{dt} \left. e^{t\mathcal{L}_\Phi} \right|_{t=0} = \mathcal{L}_\Phi e^{t\mathcal{L}_\Phi} \bigg|_{t=0} = \Phi - I,
\]

which proves the claim that \( (1 - p)I + p\Phi \) defines the tangent line for the evolution at \( t = 0 \). Analyzing a finite group of quantum channels and the semigroup formed by convex combinations of elements of this group, a distinguished trajectory joins the identity with the centre \( \Phi_* \) of the polytope, which follows the direction determined by the tangent line at the beginning of evolution. Stated in other words, we have the following result.

Proposition 11. Let \( G_\Phi = \{\Phi_\mu\} \) define a finite group of \( g \) quantum channels. The line connecting identity to the centre of the polytope, \( \Phi_* = \frac{1}{g} \sum_{\mu=0}^{g-1} \Phi_\mu, \) always belongs to the set \( \mathcal{A} \) of accessible channels in this group.

Proof. To show this we will start with the Lindbladian \( \mathcal{L}_* = \frac{1}{g^2} \sum_{\mu=0}^{g-1} \mathcal{L}_\Phi - I \). Note that in this definition we ignored the trivial generator \( \mathcal{L}_{\Phi_0} = 0 \) to follow the convention adopted here. So \( \mathcal{L}_* \) is not equal to \( \Phi_* - I \), but \( \mathcal{L}_* = \frac{g}{g-1} (\Phi_* - I) \).
FIG. 4: The set $\mathcal{A}$ of accessible maps specified in the space of quantum channels forming a non-cyclic group of order 4 for (a) a 3-simplex, regular tetrahedron which includes the case of Pauli channels \([20, 21, 25]\), and (b) a non-regular tetrahedron. Black dashed edges are of length 1, while blue dashed edges have the length $\sqrt{3}/2$. Explicit examples of these groups are discussed in Section III 2. Accessible maps in each of these polytopes occupy $3/32$ of the volume of their corresponding tetrahedrons.

However, they generate the same dynamics up to a rescaling of time.

\[
\varphi(t) := e^{t\mathcal{L}_\epsilon} = e^{\frac{it\pi}{\gamma-1}} e^{\frac{it\pi}{\gamma-1} \Phi_*} = e^{\frac{it\pi}{\gamma-1}} \left( 1 + \sum_{m=1}^{\infty} \left( \frac{4\pi}{\gamma-1} \right)^m \frac{m!}{\Phi^m} \right) = e^{\frac{it\pi}{\gamma-1}} \left( 1 + (1 - e^{\frac{it\pi}{\gamma-1}}) \Phi_* \right),
\]

where we called the resultant channel $\varphi(t)$ for future reference. Furthermore, in the first equation of the second line, we used the fact that $\Phi_*$ is a projector; see Proposition 6. Eq. (26) shows the line connecting the identity to the centre of polytope always belongs to the set of accessible maps and completes the proof. \hfill $\square$

The above proposition leads us to the following important result.

**Theorem 12.** The set $\mathcal{A}$ of accessible maps occupies a positive measure in the polytope formed by the convex hull of the corresponding group elements.

**Proof.** To prove, we will show that for any non-ending point in the interval connecting identity to the centre of the polytope, one can always find a ball that belongs to the set of accessible maps. According to above proposition, any such non-ending point can be defined as $\exp[(T + \delta T)\mathcal{L}_\epsilon]$ for time $t = T$ such that there exists $\delta T << T$. Now let $\forall \mu \in 1, \ldots, g - 1$ we have $|\epsilon_\mu| << 1$ and $\sum_{\mu=1}^{g-1} \epsilon_\mu = 0$. So we can define a family of valid $\epsilon$-dependent Lindbladians as $\mathcal{L}_\epsilon = \sum_{\mu=1}^{g-1} \left( \frac{1}{g-1} + \epsilon\mu \right) \Phi_\mu - \mathbb{1}$. Hence, for any $\epsilon$ satisfying above conditions $\exp(t\mathcal{L}_\epsilon)$ belongs to the set of accessible maps at each moment of time. Thus, for $t = T + \delta T$ we get

\[
e^{(T+\delta T)\mathcal{L}_\epsilon} = e^{(T+\delta T)(\mathcal{L}_\epsilon + \Sigma \epsilon\mu \Phi_\mu)} = e^{(T+\delta T)\mathcal{L}_\epsilon} e^{(T+\delta T)\Sigma \epsilon\mu \Phi_\mu}
\approx \varphi(T + \delta T) \left( 1 + (T + \delta T) \sum_{\mu=1}^{g-1} \epsilon\mu \Phi_\mu \right)
\approx \varphi(T) + e^{-\frac{\pi\epsilon}{\gamma-1}} \sum_{\mu=0}^{g-1} \epsilon\mu \Phi_\mu,
\]

(27)
where \( \epsilon'_0 = -\delta T \) and \( \epsilon'_\mu = T \epsilon_\mu + \delta T / (g-1) \) for \( \mu \in \{1, \cdots, g-1\} \). In writing these equations, we considered only the first order of \( \delta T \) and \( \epsilon' \) and also ignored their products. Moreover, one may notice that \( \Phi_* \) and consequently \( \mathcal{L}_* \) commute with any channel in the polytope of the group as stated in Proposition 6. The left-hand side of Eq. (27) belongs to the set of accessible maps of the group \( G_\Phi \), so does a neighbourhood of any \( \varphi(T) \) for a finite \( T \) in any direction. This completes the proof.

The above result was intuitive because the trajectory generated by Lindbladian corresponding to any quantum channel starts in the tangent plane of the line connecting the identity element to the assumed channel. So the set of accessible maps inside a polytope is defined by trajectories that go in all possible directions at the beginning. However, we will generalize the above approach concerning the point \( \varphi(T) \) in the sequel to show for an abelian group at a finite time for any point on the trajectory generated by a Lindbladian corresponding to a channel from the interior of the polytope such a ball exists. Before proceeding with such an extension, let us conclude the discussion on the volume by explicit calculation of the volume for the examples mentioned in the previous section.

**Proposition 13.** The ratio of the volume of the set \( A \) of accessible maps to the volume of the corresponding polytope for the examples discussed in Section III reads:

a) For any choice of a cyclic group of order \( g \) with exactly three linearly independent elements corresponding to a regular \( g \)-polygon in the plane, see Fig. 1, the ratio is given by

\[
\frac{V_{\text{acc}}}{V_{G^2\delta T}} = \frac{1 - e^{-2\pi \tan(\pi/g)}}{2g \sin^2(\pi/g)} = 1 - \frac{\pi^2}{g} + \mathcal{O}\left(\frac{1}{g^2}\right).
\]

(28)

b) For a cyclic group of order \( g = 4 \) with all linearly independent elements, for both regular and non-regular tetrahedrons, see Fig. 3, the ratio is given by

\[
\frac{V_{\text{acc}}}{V_{G^2\delta T}} = \frac{3}{32} \left(1 - e^{-4\pi}\right) \approx 0.0937497.
\]

(29)

c) For the non-cyclic group of order \( g = 4 \) the ratio is given by

\[
\frac{V_{\text{acc}}}{V_{G^2\delta T}} = \frac{3}{32} = 0.09375,
\]

independently of the regularity of tetrahedron, see Fig. 4.

The proof of this proposition is provided in Appendix A. Note that Eqs. (28) and (30) were already presented in [25] for the ratio of the spectra of accessible to the entire set of Weyl maps and for accessible Pauli channels, respectively. However, we should emphasize that these results are valid for any representation of the aforementioned groups independently of the dimension of the maps. In Remark 19 we show that for the specific example of Weyl channels, the volume of the set of accessible and Markovian Weyl channels are the same. Accordingly, in this case, the set \( A \) of the accessible maps forms a good approximation of the set of Markovian channels. Now we will apply the same approach of Theorem 12 to get the boundaries of the set \( A \) assigned to an abelian group.

**Theorem 14.** The boundary \( \partial A \) of the set of accessible channels in the polytope of a finite abelian group of unitary maps is formed by Lindblad operators \( \mathcal{L}_\Phi = \Phi - \mathbb{1} \), where \( \Phi \) belongs to the boundaries of the polytope.

**Proof.** Let \( \mathcal{L} = \sum q_\mu \Phi_\mu - \mathbb{1} \) denote the Lindblad generator such that the channels \( \Phi_\mu \) along with identity form a group of order \( g \) and \( q_\mu \neq 0 \) for all \( \mu \). Then it is always possible to find a probability \( q'_\mu = q_\mu + \epsilon_\mu \) for small enough \( \epsilon_\mu \) such that \( \sum \epsilon_\mu = 0 \). Similar to the proof of the previous theorem, we can define a family of valid \( \epsilon \)-dependent Lindblad generators \( \mathcal{L}_{\epsilon} = \sum_{\mu=1}^{g-1} (q_\mu + \epsilon_\mu) \Phi_\mu - \mathbb{1} \). So for any finite time \( t = T \) there is \( \delta T \) such that \( |\delta T| << 1 \) and \( \exp[(T + \delta T)\mathcal{L}_{\epsilon}] \) belongs to the set of accessible maps. On the other hand, we have

\[
eq e^{(T+\delta T)\mathcal{L}_{\epsilon}} = e^{(T+\delta T)\mathcal{L}_{\epsilon}} e^{(T+\delta T)\mathcal{L}_{\epsilon}} \sum_{\epsilon_\mu} \epsilon_\mu \Phi_\mu
\]

\[
\approx e^{T\mathcal{L}} \left( (1 - \delta T) \mathbb{1} + (T + \delta T) \sum_{\mu=1}^{g-1} q_\mu \Phi_\mu \right) \left( \mathbb{1} + T \sum_{\mu=1}^{g-1} \epsilon_\mu \Phi_\mu \right)
\]

\[
\approx e^{T\mathcal{L}} + e^{T\mathcal{L}} \left( -\delta T \mathbb{1} + \sum_{\mu=1}^{g-1} (T \epsilon_\mu + q_\mu \delta T) \Phi_\mu \right)
\]

\[
\approx e^{T\mathcal{L}} + \sum_{\alpha=0}^{g-1} \sum_{\mu=0}^{g-1} w_{\alpha}(T) \epsilon_\mu \Phi_\alpha \Phi_\mu = e^{T\mathcal{L}} + \sum_{i=0}^{g-1} f_i(\vec{w}(T), \epsilon') \Phi_1.
\]

(31)
where $\epsilon'_0 = -\delta T$, $\epsilon'_\mu = T\epsilon_\mu + q_\mu \delta T$ for $\mu \in \{1, \ldots, g - 1\}$, and $w_\alpha(T)$ is the time-dependent probability by which the trajectory of $\exp(T \mathcal{L})$ is defined, see Theorem 4. Note that due to rearrangement theorem for any fixed $\alpha$ the term $\Phi_\alpha \Phi_\mu$ produces all group elements with different order according to the group multiplication table. Thus we can define the $g$-dimensional vector $\tilde{f}(\tilde{w}(T), \tilde{\epsilon}')$ by its element, $f_j(\tilde{w}(T), \tilde{\epsilon}')$ which can be gained by

$$\tilde{f}(\tilde{w}(T), \tilde{\epsilon}') = (\sum_{\alpha=0}^{g-1} w_\alpha(T) R_\alpha) \tilde{\epsilon}'$$

where $R_\alpha$ is the regular representation of element $\alpha$ of the group $G = \{ \Phi_0 = 1, \ldots, \Phi_{g-1} \}$ of dimension $g \times g$ defined based on the group multiplication table see Lemma 5. Therefore, one can claim the vector $\tilde{f}(\tilde{w}(T), \tilde{\epsilon}')$ can define a ball around the point $\exp(T \mathcal{L})$ if the $g$-dimensional matrix $\sum_{\alpha=0}^{g-1} w_\alpha(T) R_\alpha$ is invertible. To show that we use Lemma 5 that leads to

$$\sum_{\alpha=0}^{g-1} w_\alpha(T) R_\alpha = \exp \left[ T \left( \sum_{\mu=1}^{g-1} q_\mu R_\mu - I \right) \right],$$

This is an invertible matrix as long as $T$ is finite, which is the case here, completing the proof.

We should notice that the inverse of the theorem is not necessarily valid, i.e. not any point from the boundaries of a polytope formed by a finite abelian group of quantum channels can give a trajectory at the boundaries of the set of accessible maps. For a counterexample, see all the red dashed lines in Fig. 1c-f. These are trajectories generated by single Lindbladians corresponding to the channels at different vertices of the polygons. Our conjecture is that the inverse is true when the elements are linearly-independent.

To proceed with investigating more geometrical properties of accessible maps, in what follows, we show that this set is star-shaped with respect to the centre of its corresponding polytope. First, note that due to Proposition 11, the line segment $[\mathbb{1}, \Phi_*]$ is accessible by the generator $\mathcal{L}_*$ that commutes with all channels in the polytope. Applying this fact, we get that for any accessible $\Omega_t$

$$p\Omega_t + (1 - p)\Phi_* = \Omega_t(p \mathbb{1} + (1 - p)\Phi_*) = e^{t\mathcal{L}} e^{t' \mathcal{L}_*} = e^{t' \mathcal{L}_*},$$

and therefore, the line segment $[\Omega_t, \Phi_*]$ is accessible, proving the following result.

**Theorem 15.** The set $\mathcal{A}$ of accessible maps in a polytope formed by a discrete finite group of quantum channels is star-shaped with respect to the centre of the polytope.

**Corollary 16.** If a set $\mathcal{A}$ of accessible maps is planar, then accessible maps are star-shaped with respect to the whole interval $[\mathbb{1}, \Phi_*]$.

**Proof.** This follows from the fact, that for any accessible $\Omega_T = e^{T \mathcal{L}}$, the set

$$\{pe^{t\mathcal{L}} + (1 - p)\Phi_*\}_{p \in [0,1], t \in [0,T]}$$

is accessible. Next using the fact, that this is a subset of a plane, we have for all $q \in [0,1]$, it contains intervals of a form

$$\{pe^{t\mathcal{L}} + (1 - p)(q \mathbb{1} + (1 - q)\Phi_*)\}_{p \in [0,1]},$$

Which shows the star shapeness with respect to the interval $[\mathbb{1}, \Phi_*]$.

---

**V. ACCESSIBILITY RANK, PAULI AND WEYL CHANNELS**

In this Section, we explicitly study some further examples of quantum channels forming a group and discuss the rank of accessible channels.
Example 4. Let $\lambda_j = e^{i\theta_j}$ with $j = 1, \ldots, N^2$ denote eigenvalues of a superoperator $\Phi_U = U \otimes \overline{U}$ corresponding to a unitary $U$. If $\forall j: \theta_j = 2\pi r_j$, where $r_j = m_j/n_j$ is an irreducible fraction of a rational number, then $\Phi$ can generate a discrete and finite cyclic group of order equal to the lowest common multiple of $n_j$, denoted by $n$, i.e., $G_n = \langle \{ \Phi_i \} \rangle_{i=0}^{n-1}$.

As an example of such a group, one may consider a group of rotations around a fixed axis with discrete angles. Let us focus on qubit channels. In this case a rotation around $n$ of magnitude $\theta$ is gained by applying $U = e^{i 2\pi n \hat{\sigma}_z}$.

Let us take $\theta$ to be one of the following angles $\theta_j = 2\pi j/n$ with $j = 0, 1, \ldots, l - 1$. It is easy to check that $\{ U_j = \exp(i \theta j \hat{\sigma}_z) \}$ is an abelian group up to a phase of order $l$. Thus $G_n = \langle \Phi = U_j \otimes \overline{U}_j \rangle_{j=0}^{l-1}$ is an abelian (more precisely a cyclic) group of the same order. Moreover, for every divisor $d$ of $l$ ($l = dd'$), $G_n$ has at most one cyclic subgroup $H_{d'}$ of order $d$ generated by different powers of $\Phi_{d'} = U_{d'} \otimes \overline{U}_{d'}$. Therefore, one can always find an accessible map of order $h_{d'} = d$ which has a convex expansion in terms of the subgroup members $H_{d'}$. The space of such a group is an $l$-polygon. For the $l = 2, 3, 7$, the set of channels and their corresponding accessible maps are given in Fig. 1.

Example 5 (Group of commutative quantum maps forming an orthogonal basis). An abelian group of $N^2$ unitary quantum maps forming an orthogonal basis, $\text{Tr}(U_j U_j^\dagger) = N \delta_{j\mu}$, is one of the simplest non-trivial examples to investigate. Let $G_U = \{ U_{ij} \}_{i,j=0}^{N^2-1}$ denote an abelian group up to a phase of such $N^2$ unitary matrices. Weyl unitary matrices [27] provide an example of such a set. Since we are dealing with an abelian group here, Corollary 8 clarifies the problem of accessibility. Since unitary matrices form an orthogonal basis in the Hilbert-Schmidt space of matrices and thus are linearly independent, the analyzed polytope forms a simplex in dimension $N^2 - 1$. This implies that the convex combinations we get through Corollary 7 or Corollary 8 for the accessible maps are unique. Therefore, we can call the number of vertices appearing in such a unique convex combination the rank of the channel (and it equals to the Choi rank as well since the vertices are orthogonal maps). Accordingly, an immediate result of Corollary 8 is that not all quantum maps with any assumed rank can be accessible.

Proposition 17. The set $\mathcal{A}_N^Q$ of accessible quantum maps of the form $\Phi = \sum p_{\mu} \Phi_{\mu}$, in which $\Phi_{\mu}$'s are unitary channels belong to an abelian group and satisfy $\text{Tr}(\Phi_{\mu} \Phi_{\mu}^\dagger) = N^2 \delta_{\mu\nu}$, is formed by the maps of (Choi) rank equal to the order of possible cyclic subgroups of $G_{\Phi}$ or their multiplications.

Through the fundamental theorem of abelian groups this the theorem can be rephrased as follows. The set of accessible channels satisfying the above conditions is formed by the maps of Choi rank equal to the order of possible subgroups of $G_{\Phi}$. Note that through Corollary 8 it is possible not only to indicate the rank of accessible maps but also to find which $\Phi_{\mu}$ participated in the combination when an assumed $t_{\nu}$ is not zero. In summary, we should expand convexly an assumed quantum channel $\Phi$ in terms of extreme points $\Phi_{\mu}$. Such an expansion is unique here. If the extreme points participating in this expansion do not form a subgroup of $G_{\Phi}$, then the map $\Phi$ is not accessible.

As an example let us consider Weyl channels more explicitly. Weyl channels are a unitary generalization of Pauli maps. These channels are based on $N^2$ unitary operators of the form $U_{kl} = X^k Z^l$ for $k, l \in \{ 0, \ldots, N - 1 \}$ where $X|j⟩ = |j \oplus 1⟩$ and $Z = \text{diag}[1, ω_N, \ldots, ω_N^{N-1}]$ with $ω_N = \exp(i 2\pi /N)$. The accessibility in this set is already studied in [25]. In the case $N = 2$ of Pauli channels $G_{\Phi}$ is of order 4. There are three different cyclic subgroups of order 2, $H_i = \{ 1, \sigma_i \otimes \overline{\sigma}_i \}$ for $i \in \{ 1, 2, 3 \}$ multiplication of any two of them exhausts the group $G_{\Phi}$. Moreover, the identity element itself is always the only trivial cyclic subgroup of order 1. Therefore, there are: (i) The identity map as the only accessible map of rank 1. (ii) Accessible maps of rank two in the form of $\Phi = pI + (1 - p)\sigma_i \otimes \overline{\sigma}_i$ for $i \in \{ 1, 2, 3 \}$ which are gained when the only non-vanishing interaction time is $t_i$. (iii) The full-rank accessible maps, which are available when there are at least two different $t_i \neq 0$. Fig. 4(a) presents the accessible maps among all Pauli channels.

In the case of qutrit Weyl maps, $G_{\Phi}$ is of order 9. There are four different cyclic subgroups of order 3, $H_0 = \{ 1, \Phi_{kl}, \Phi_{-k\oplus N, -l\oplus N} \}$. Again, the multiplication of any two subgroups makes the group $G_{\Phi}$. In addition, there is not any other nontrivial subgroup for $G_{\Phi}$. Therefore, the accessible maps belong to one of the following sets: (i) The identity map as the only rank one accessible map. (ii) Accessible maps of rank 3 which can be found on the face of the simplex specified by $\{ 1, \Phi_{kl}, \Phi_{-k\oplus N, -l\oplus N} \}$ when $t_{kl} \neq 0$ or $t_{-k\oplus N, -l\oplus N} \neq 0$ and all other interaction times are zero. Fig. 1(b) can be considered as a cross section presenting such a subgroup. (iii) The full-rank accessible maps which are available when there are at least two different $t_{kl} \neq 0$ and $t_{kl'} \neq 0$ where $(k', l') = (-k \oplus N, -l \oplus N)$.

The above results concerning the Weyl channels are mentioned in [25]. However, the group properties of Weyl channels help us to generalize these results to a general case of $N$-dimension. Any group of Weyl unitary maps acting on $N$-dimensional systems, formed by $N + 1$ cyclic subgroups of order $N$ with no element in common but the identity. This implies that the direct product of any two of them forms the group entirely. So we get

Corollary 18. In the case of Weyl quantum maps acting on $N$-level systems, there always exist accessible maps of rank 1, $N$ and $N^2$. When $N$ is prime, these are the only choices, however, for a composite $N$ other ranks are possible.
too. The ranks correspond to the cardinality of the subgroups.

Before proceeding with the following example concerning local channels, let us mention the following simple and yet significant remark about accessible Weyl channels.

**Remark 19.** The accessible Weyl channels can recover the full measure of Markovian Weyl channels but not the entire set.

The supporting evidence for this statement comes from the fact that the necessary and sufficient condition for accessibility of Weyl channels introduced in [25] is the same as the necessary and sufficient condition stated in [4] for Markovianity of channels with non-negative eigenvalues. Note that in the case of channels with negative eigenvalue, Markovianity can only be defined for the maps whose negative eigenvalues are even-fold degenerated [20]. Such a set is by definition of measure zero.

**Example 6 (Tensor product of quantum channels).** Let us now generalize our results to channels acting locally in higher dimension. Note that the set $\tilde{T}$ transition matrix, $\{G_U \otimes V_V\}$ is an abelian group up to a phase if $\tilde{G}_U = \{U_\mu\}$ and $\tilde{G}_V = \{V_\alpha\}$ are. Moreover, it contains a complete set of orthogonal matrices, i.e. $\text{Tr}\left((U_\mu \otimes V_\alpha)(U_\nu \otimes V_\beta)\right) = \text{Tr}(U_\mu^\dagger U_\nu^\dagger)\text{Tr}(V_\alpha V_\beta) = MN\delta_{\mu\nu}\delta_{\alpha\beta}$, provided that $\tilde{G}_U$ and $\tilde{G}_V$ are two sets of orthogonal basis in the space of matrices of order $M$ and $N$, respectively. Let $G_T$ denote the group formed by $\tilde{G}_U \otimes \tilde{G}_V$. We will also assume that $\tilde{G}_U$ and $\tilde{G}_V$ are the sets of orthogonal basis and are abelian up to a phase. Therefore, the subgroups of $G_T$ are of order $h_u h_v$, where $h_u$ and $h_v$ are the cardinalities of the subgroups of $G_\Phi = \{\Phi_\alpha = U_\alpha \otimes U_\alpha\}$ and $G_\Psi = \{\Psi_\mu = V_\mu \otimes V_\mu\}$, respectively, and each of which is a group of orthogonal quantum maps.

In a very particular case, when we restrict ourselves to $G_\Phi = G_\Psi$ being the group related to Pauli channels, we get $(1, 2, 4)^{\otimes 2} = \{1, 2, 4, 8, 16\}$ as the rank of an accessible map of the form $\mathcal{E}_A \otimes \mathcal{E}_B$ acting on a two-qubit system. It is easy to generalize this result to the system of $k$ qubits, each of them is subjected to an evolution described by a Pauli map.

**Corollary 20.** Let $G^\otimes k$ define the group obtained by a tensor product of Pauli channels. Hence each element is a map acting on a $k$-qubit system. Then channels accessible by Pauli semigroups can only be of rank equal to $2^m$ with $m = 0, \ldots, 2k$.

**VI. CLASSICAL STOCHASTIC MAPS**

Any stochastic matrix $T$ describes a Markov chain – a transition in space of $N$-point probability distributions, $p^t = Tp$. By construction, each column on $T$ forms an $N$-point probability vector itself. Return now back to the space of density matrices. The process of decoherence transforms a quantum state $\rho$ into a classical probability vector embedded in its diagonal entries, $\rho_d = \Delta(\rho) = \sum \rho_{i|i}$, where $\Delta$ denotes the decoherence channel. Similarly, a transition matrix $T$ can be obtained from a quantum channel suffering the super-decoherence [30],

$$T_{ij}(\mathcal{E}) = \langle i|\mathcal{E}(|j\rangle\langle j|)|i\rangle.$$

(37)

This effect can be considered as a decoherence applied to the Choi matrix $C = d(\mathcal{E} \otimes \mathbb{I}) P_+$, where $P_+ = |\psi_+\rangle\langle \psi_+|$ with $|\psi_+\rangle = \sum_i |i, i\rangle/\sqrt{N}$ denotes the maximally entangled state. Hence the classical transition matrix $T$ arises from reshaping of the $N^2$ diagonal entries of $C$ into a square matrix of size $N$. Super decoherence of a quantum channel $\mathcal{E}$ can also be described by sandwiching it between repeated action of the decoherence channel,

$$T(\mathcal{E}) = \Delta \circ \mathcal{E} \circ \Delta.$$

(38)

If $\{K_a\}$ denotes the set of Kraus operators of the channel $\mathcal{E}$, then $T(\mathcal{E})$ is given by Hadamard product, $T(\mathcal{E}) = \sum_a K_a \otimes \overline{K_a}$. Thus super-decoherence transforms a unitary quantum channel, $\Phi_U = U \otimes U$, into a unistochastic transition matrix, $T_U = U \otimes U$. Moreover, one can apply the special ordering of Gell-Mann basis where $B_i$’s for $i = 1, \ldots, N - 1$ are the diagonal elements, and for $i = N, \ldots, N^2 - 1$ denote other elements. In that case, we can divide the distortion matrix $M$ of size $N^2 - 1$ defined in (2) into four blocks and the translation vector into two parts,

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ k & D & Q \\ k' & Q' & D' \end{pmatrix}.$$  

(39)
where $D$ and $D'$ are square matrices of order $N - 1$ and $N^2 - N$, respectively, and $\mathbf{k}$, $\mathbf{k}'$ are vectors with the same respective length. Off-diagonal blocks $Q$ and $Q'$ are rectangular so that the dimension of $\Phi$ is $N^2$. Thus, Eq. (38) implies that the process of super-decoherence in the above basis results in a projection into a $N$-dimensional space [31],

$$ T(\mathcal{E}) = \begin{pmatrix} 1 & 0 \\ \mathbf{k} & D \end{pmatrix}. $$

(40)

It is worth mentioning that super-decoherence sends Lindblad generators to the set of Kolmogorov operators, i.e. the generators of classical Markovian evolutions. This fact can be observed through the GKLST form (1). The Kolmogorov generators $K$, also called ‘transition rate matrices’, satisfy conditions: (a) $K_{ij} \geq 0$ for all $i \neq j$, and (b) $\sum_i K_{ij} = 0$ for all $j$.

**Lemma 21.** Let $\Phi$ (39) be a quantum channel and $T(\mathcal{E})$ (40) denote its corresponding classical transition gained by super-decoherence, then $T^n(\mathcal{E}) = T(\mathcal{E}^n)$ for all $n \in \{1, \ldots, r\}$ if and only if (i) $Q(D')^n 2^{K'} = 0$, and (ii) $Q(D')^n 2^{Q'} = 0$ for all $n \in \{2, \ldots, r\}$.

Proof. The proof of this lemma follows from Eq. (38) and the fact that in the Bloch representation (39) for the decoherence map $\Delta$ we have $D_\Delta = 1_{N-1}$, while other parameters vanish.

If the above condition for a quantum channel holds, then the cyclic group generated by different powers of this channel has its classical counterpart of the same order, such that each element finds its respective element in the classical set through super-decoherence.

A trivial example of quantum channels satisfying Lemma 21 are those for which $Q = Q' = 0$ and $K' = 0$. This implies

$$ \Delta \circ \mathcal{E} \circ \Delta = \mathcal{E} \circ \Delta. $$

(41)

The set of quantum channels satisfying above equation are known as Maximally Incoherent Operation (MIO) in the context of resource theory of quantum coherence [32]. This is the largest set of operations which are not able to generate coherence in an incoherent state. So taking classical probabilities embedded in diagonal entries of an incoherent state $\rho_d$, the set MIO can be seen as classical operations. Note that the super-decoherence map from quantum channels to stochastic matrices is a surjective and non-injective function. However, we can always find a maximally incoherent operation which is sent to any assumed classical transition through super-decoherence. Moreover, it is worth mentioning if $\mathcal{E}$ satisfies Eq. (41), then the corresponding Lindblad generator also satisfies the same condition. Such a generator can be thus called an incoherent generator. Interestingly, for any Kolmogorov generator $K$ one can always find an incoherent Lindblad generator $L$ [16] whose trajectory is an incoherent quantum channel at each moment of time.

In the classical case, $G_T = \{T_i\}$ defines a group if $T_i$ are permutation matrices, i.e. the extreme points of Birkhoff polytope. So the group $G_T$ is of order $N!$ and an arbitrary convex combination is the most general bistochastic matrix. We assign to each element of $G_T$ the Kolmogorov generator $K_{T_i} = T_i - 1_N$. As any classical transition matrix is equivalent to an MIO, and its Kolmogorov operator is equivalent to an incoherent Lindblad generator, all the results of Section II are also valid for classical transitions. However, we arrive at the same conclusion noting that the aforementioned results are obtained by adopting the group properties and not the fact that $\Phi$ is a quantum channel. Hence this approach can be applied to analyze the accessibility of classical bistochastic matrices by classical dynamical semigroups generated by a Kolmogorov generator.

Based on the results of these work, one can easily show that a classical dynamical semigroup $e^{tK}$ is a convex combination of the group members $T_i$. If we take the cyclic permutations as the assumed group, since cyclic permutations are orthogonal and thus linearly independent, the classical bistochastic matrices gained by their convex combination enjoy the following property. The set $A_{NC}$ of accessible circulant bistochastic matrices is formed by the matrices of rank equal to the order of possible cyclic subgroups of $G_T$ or their multiplications.

The most general finite group in the classical domain is the set of all permutations forming the Birkhoff polytope $B_N$. The first nontrivial case, which forms a non-abelian group, is the permutation group $S_3$. The 6 permutation matrices and their convex combinations make the Birkhoff polytope $B_3$. This is a 4-dimensional set comprising the convex combination of two equilateral triangles in two orthogonal $2D$ planes. To demonstrate an exemplary application of our approach, we discuss the set of accessible bistochastic matrices of order three, initially analyzed in [33].
accessible set is found by exponentiating different Lindblad generators formed by subtracting the identity from a map picked from the Birkhoff polytope $B_3$. The set of accessible classical channels remains in $B_3$. We demonstrate this set in Fig. 5 by producing a million such accessible channels and projecting them on the orthogonal equilateral triangles.

Note that the triangle with even permutations on its corners forms the convex hull of the cyclic group of order 3. Therefore we expect the projection of the accessible maps inside the $B_3$ to at least cover the interior of the red curve – see Fig. 5 (a). The data suggest that there are no accessible maps whose projections fall outside this curve. The case for the odd subspace is different as there are no accessible channels that lie on the odd subspace other than the centre of the polytope. This is true because the spectrum of a matrix $B = \alpha P_{23} + \beta P_{13} + \gamma P_{12}$ comprises of the values $\pm|\alpha + \beta e^{2\pi i/3} + \gamma e^{-2\pi i/3}|$ and unity. Since a positive determinant is necessary for accessibility, we find that any combination except for $\alpha = \beta = \gamma = 1/3$ is not accessible.

Another way to visualize the accessible matrices is to draw cross-sections with 2D planes parallel to the even or odd subspace with different displacement vectors in the orthogonal subspace. Fig. 6 includes some of these cross-sections.

The volume of the Birkhoff polytope for $N = 3$ is known to be $9/8$ [34]. Using a Monte Carlo sampling method based on a sample consisting of $2.2 \times 10^7$ points that were uniformly distributed in the Birkhoff Polytope $B_3$, we estimated the relative volume of the set $A_3$ of accessible maps as

$$\frac{\text{vol}(A_3)}{\text{vol}(B_3)} \approx (4.398 \pm 0.004) \times 10^{-2}$$

(42)
FIG. 6: Several cross sections of the set of accessible points (blue) with different 2D planes. a, b, c) cross-sections parallel to the even subspace; d, e, f) cross-sections parallel to the odd subspace. The grey triangles correspond to the cross-sections of the polytope $B_3$. The red dot inside the small triangles, shows the position of the planes in the orthogonal subspace.

In order to decide if a particular bistochastic map was accessible or not, we computed the matrix logarithm and checked if it was possible to write it as a positive combination of the five generators in the form $(P - \mathbb{1})$.

VII. CONCLUDING REMARKS

The problem of characterization of Markovian quantum channels has attracted considerable attention in recent years [4, 24, 26]. The simplest case of single-qubit maps is already well-understood [20–23], while the general problem, in quantum and classical setup, remains open.

Following [25] we analyzed in this work the set $A$ of accessible quantum channels obtained by a Lindblad generator of the form, $L(\rho) = \sum q_i E_i(\rho) - \rho$, for some selected maps $E_i$ and probabilities $q_i$. This set forms a subset of the set of Markovian channels of a positive volume. Instead of analyzing the structure of $A$ for a fixed dimension $N$ of the system and a concrete choice of the maps $E_i$, we studied the case in which the maps belong to a particular group $G$.

The key result in this work consists in identifying properties of the set $A$ of accessible maps which do not depend on the dimensionality $N$ of the system but are determined by the properties of the group $G$. We demonstrated that this set has a positive volume and the star-shape property with respect to the uniform mixture of all the maps $E_i$ forming the group. We computed the volume of the set of accessible maps and analyzed its structure for several groups – see Fig. 1 showing set $A$ for linearly dependent channels forming cyclic groups of order $2 - 7$ and Fig. 3(b) obtained for cyclic groups of order 4 of linearly independent channels.

It is worth mentioning that some of the results presented here are obtained only based on a group’s closure
property. Thus, they also hold once the set $S$ of extremal channels is a semigroup. In this way, we can also generalize the results to the nonunital channels. For example consider the set $S = \{ \Phi_0, \Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5 \}$ of qubit channels where $\Phi_0$ is the identity map, $\Phi_i$ for $i \in \{1, 2, 3\}$ are Pauli channels, the nonunital maps $\Phi_4$ and $\Phi_5$ are completely contractive channels into the states $|0\rangle \langle 0|$ and $|1\rangle \langle 1|$, respectively. This set is a semigroup formed by six extreme channels. Hence, their convex hull has six vertices. However, the elements are not all linearly independent. They satisfy $\sum_{i=0}^{3} \Phi_i = 2(\Phi_3 + \Phi_4)$. In this case, the set of accessible maps is still confined in the convex hull of the semigroup. There are, however, some results that need other group properties. For example, note that the trajectory $\exp\{t(\sum q_i \Phi_i - 1)\}$ does not necessarily end in the centre of the polytope. A counterexample happens when all probabilities $q_i$ are equal to zero but $q_3$. In this case, the trajectory will end in the vertex assigned to $\Phi_3$.

Furthermore, we showed that when the vertices of the convex hull of the channels are linearly independent, the set of accessible channels contains quantum channels with (convexity) ranks satisfying certain constraints. The figures mentioned above illustrate the following three facts:

(i) In the simplest case of $g = 2$, accessible maps are of ranks one and two.
(ii) In the case of $g = 3$, the group of order 3, has only two trivial subgroups; identity and the group itself. Hence there exists an accessible map of rank 1, the identity map, and of rank 3. There is no accessible map of rank two, which correspond to the edges of the triangle.
(iii) For $g = 4$, the group has (at least) a subgroup of order two. Thus, when we have a tetrahedron, the set consists of maps of (convexity) rank 1 (the identity map), rank 2 on the edges assigned to the maps generating the subgroup and maps of full rank $r = 4$. Note that there are no maps of rank three corresponding to the faces of the tetrahedrons shown in Fig. 3 and Fig. 4.

We have also established more general results concerning ranks of accessible maps acting on larger systems.

(iv) For quantum Weyl maps acting on $N$-level systems, the set $\mathcal{A}$ contains maps of rank 1, $N$ and $N^2$. If $N$ is prime, this list is complete.
(v) For a group of Pauli channels acting on a $k$-qubit system the set $\mathcal{A}$ contains maps of rank $2^m$ with $m = 0, \ldots, 2k$.

Our approach can also be applied to the classical case. We also discussed when classical and quantum accessible maps could be compared directly. The present study leaves several questions open. Let us list here some of them.

1. Characterize the set of points such that the set $\mathcal{A}$ has the star-shape property with respect to them.
2. Find a criterion allowing one to decide whether a given channel is accessible with respect to a given set $S$ of quantum maps $\Phi_i$.
3. What is the relative measure of the set $\mathcal{A}$ of accessible channels with respect to Markovian channels?

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Appendix A: The Group Matrix/Table; Proof of Proposition 13

In this Appendix, we try to calculate the accessibility volume fraction for some abelian groups thereby providing proof for the results presented in Proposition 13.

The volume measure is induced from the Euclidean metric on the dynamical maps (see (2)). For unitary channels, or a mixture of unitary channels, the vector $t$ is zero, and we need only consider the affine matrix $M$. In fact, the affine matrices themselves form a representation of our group of interest. Furthermore, for abelian groups, we may work in a basis in which all the affine matrices are diagonal, thereby making all the convex mixtures diagonal as well. Finally, since we are only dealing with diagonal matrices, it sounds reasonable to write down the diagonal entries of each affine matrix in the group in columns of a table. The table will therefore form a matrix with $N^2 - 1$ rows (remember that the affine matrices are of order $N^2 - 1$ where $N$ is the dimension of the Hilbert space.) and $|G| = g$ columns. For example, a possible group table for a qubit representation of the non-cyclic group with four members $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is the following $3 \times 4$ matrix

\[
\begin{pmatrix}
\left\langle \Phi_0 \right\rangle & \left\langle \Phi_1 \right\rangle & \left\langle \Phi_2 \right\rangle & \left\langle \Phi_3 \right\rangle \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
Therefore, the volume is hence

In general the $Z_2 \times Z_2$ table looks like below

\[
\begin{pmatrix}
+1 & +1 & -1 & -1 \\
+1 & -1 & -1 & +1 \\
+1 & -1 & +1 & -1
\end{pmatrix},
\]

(A1)

where $1_a$ denotes an $a$ dimensional column vector with all its components being equal to unity and the height of the table is given by $a + b + c + d = N^2 - 1$. Note that changing the values of $a$, $b$, $c$ (and therefore $d$) changes the shape and size of the tetrahedron; however, as we will see, the accessibility volume fraction will remain invariant as long as the $a, b, c, d$ parameters all remain positive. Part (c) of Proposition 13 can now be restated as regardless of the values $(a, b, c, d)$ for the group $Z_2 \times Z_2$, the accessible channels cover a fraction $3/32$ of the corresponding tetrahedron’s volume.

Proof. Let $\mathbf{I}$ be the origin of the coordinate system and define the vectors

\[
\mathbf{e}_1 \equiv (-2) \begin{pmatrix} 1_a \\ 1_b \\ 1_c \\ 1_d \end{pmatrix}; \quad \mathbf{e}_2 \equiv (-2) \begin{pmatrix} 0_a \\ 1_b \\ 1_c \\ 1_d \end{pmatrix}; \quad \mathbf{e}_3 \equiv (-2) \begin{pmatrix} 1_a \\ 0_b \\ 1_c \\ 1_d \end{pmatrix}.
\]

(A3)

Then the tetrahedron consists of the diagonal matrices $\mathbf{I} + \text{diag} \{x^1 \mathbf{e}_1 + x^2 \mathbf{e}_2 + x^3 \mathbf{e}_3\}$ with the constraints

\[
x^1, x^2, x^3 \geq 0; \quad x^1 + x^2 + x^3 \leq 1.
\]

(A4)

The line-element is $ds^2 = h_{ij} dx^i dx^j$ with the metric

\[
h_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = 4 \begin{pmatrix} a + b & b & a \\ b & b + c & c \\ a & c & a + c \end{pmatrix}.
\]

(A5)

Therefore the volume of the tetrahedron is given by

\[
V_{\text{tot.}} = \int dx^1 dx^2 dx^3 \sqrt{|h|} = \frac{8}{3} \sqrt{abc}.
\]

(A6)

It remains to calculate the volume of the accessible channels. These are found by element-wise exponentiation of the diagonal vectors $\exp(x^1 \mathbf{e}_1 + x^2 \mathbf{e}_2 + x^3 \mathbf{e}_3)$ This time the only constraint is $x^i \geq 0$ and the metric is

\[
h = 4 \begin{pmatrix} a' + b' & b' & a' \\ b' & b' + c' & c' \\ a' & c' & a' + c' \end{pmatrix},
\]

(A7)

with

\[
a' = ae^{-4(x_1 + x_3)}, \quad b' = be^{-4(x_1 + x_2)}, \quad c' = ce^{-4(x_2 + x_3)},
\]

(A8)

hence

\[
|h| = 4^2 (abc)e^{-8(x_1^2 + x_2^2 + x_3^2)}.
\]

(A9)

Therefore, the volume is

\[
V_{\text{acc.}} = \int dx^1 dx^2 dx^3 \sqrt{|h|} = \frac{1}{4} \sqrt{abc}.
\]

(A10)

Finally

\[
\frac{V_{\text{acc.}}}{V_{\text{tot.}}} = \frac{3}{32}.
\]

(A11)
It is possible to repeat the same procedure for the $\mathbb{Z}_2^n$ groups to prove the following.

**Proposition 22.** The accessibility volume ratios for $\mathbb{Z}_2^n$ are given by

$$\frac{(2^n)!}{2^n2^n}$$

For 3D cyclic groups of order 4, the result is slightly different. Indeed, part (b) of Proposition 13 is now rephrased as for all values of $(a,b,c)$ with $b > 0$, the accessible channels cover a fraction $\frac{3}{32}(1 - e^{-4\pi})$ of the tetrahedron’s volume.

**Proof.** Let us start again with the group table.

$$
\begin{pmatrix}
1_a & i1_a & -1_a & -i1_a \\
1_a & -i1_a & -1_a & +i1_a \\
1_b & -1_b & +1_b & -1_b \\
1_c & +1_c & +1_c & +1_c
\end{pmatrix}.
$$

(A12)

Since we are working with a real vector space, it is convenient to merge the diagonal imaginary matrix

$$
\begin{pmatrix}
i \\
-i
\end{pmatrix}
$$

(A13)

with the real matrix

$$J \equiv \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(A14)

The group table then becomes

$$
\begin{pmatrix}
1_{2a} & J_a & -1_{2a} & -J_a \\
1_b & -1_b & +1_b & -1_b \\
1_c & +1_c & +1_c & +1_c
\end{pmatrix}
$$

(A15)

with $2a + b + c = N^2 - 1$.

Defining $\mathbf{e}_i$ as we did in the previous proof, we get the metric

$$h_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = 4 \begin{pmatrix} a + b & a & b \\ a & 2a & a \\ b & a & a + b \end{pmatrix},$$

(A16)

which leads to

$$V_{tot} = \frac{8}{3}a\sqrt{b}.$$  

(A17)

Now for the accessible channels, if we use the $x^i$ coordinates, we get

$$\exp(x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3) = \left( e^{-x^1-2x^2-x^3}[\cos(x^1-x^3) + J\sin(x^1-x^3)]_{2a} e^{-2(x^1+x^3)}1_b \right)_{2a}.$$  

However, these coordinates are not injective: the periodic functions in the first part will re-address the same point many times, leading to an overestimating of the volume by evaluating the integral as before. So let us use $(y, z, \phi)$ defined as

$$x^1 \equiv \frac{z + \phi}{2}; \quad x^2 = y; \quad x^3 = \frac{z - \phi}{2}. $$

(A18)

The constraints $x^i \geq 0$ become

$$y \geq 0; \quad z \geq 0; \quad |\phi| \leq \min(\pi, z).$$

(A19)
Note that the angle $\phi$ is manually constrained to stay in the interval $[-\pi, \pi)$. The metric is
\[ ds^2 = 4be^{-4z}dz^2 + 2ae^{-2z-4y}(d\phi^2 + dz^2 + 4dy^2 + 4dydz). \]
Therefore
\[ \sqrt{|h|} = 8a\sqrt{b}e^{-4z-4y}. \]
Finally, the accessible volume is
\[
V_{\text{acc.}} = 8a\sqrt{b} \int_0^\infty e^{-4y}dy \int_0^\infty dz \ e^{-4z} \left[ 2 \min(\pi, z) \right]
= 4a\sqrt{b} \left( \int_0^\pi ze^{-4z}dz + \pi \int_\pi^\infty e^{-4z}dz \right)
= \frac{1}{4}a\sqrt{b}(1 - e^{-4\pi}).
\]
\[ \square \]
A unitary channel for $N$-dimensional quantum systems has an affine map $(R, 0)$ where $R$ is an $N^2 - 1$ dimensional rotation matrix. Dealing with Abelian subgroups allows us to diagonalize the rotation matrices simultaneously and only worry about the spectral configuration of the group members, see also section 5 of [28].

**Proposition 23.** For the 2 dimensional cyclic group $\{\mathbb{I}, R, \cdots, R^{p-1}\}$, where all the members lie on a 2D plane, a fraction
\[
\frac{1 - e^{-2\pi \tan(\pi/2)}}{2g \sin^2(\pi/2)} = 1 - \frac{\pi^2}{g} + O(1/g^2)
\]
of the 2D regular polygon is covered by exponentiating the Lindblad generators.

**Proof.** Forgetting about the scale of the actual polygon, we may identify it with the complex polygon with vertices $\{1, e^{2\pi i/2}, \cdots, e^{-2\pi i/p}\}$. Then the area of the polygon is
\[ A = g \sin(\pi/2) \cos(\pi/2). \]
By exponentiating the Lindblad generators, we get the complex set
\[
\{e^{-x+i\theta} \mid x \geq 0, \ |\theta| < \frac{x}{\tan(\pi/n)} \} = \{re^{i\phi} \mid 0 \leq r \leq 1, \ |\phi| < \frac{-\log r}{\tan(\pi/n)} \}.
\]
A standard calculation of the area of this set leads to the claimed result.

**Observation:** Any complex number $z$ with $|z| < r^* = e^{-\pi \tan(\pi/2)}$ is infinitesimally divisible. It may be interesting to calculate $r^*$ for other groups as well. \[ \square \]

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