Unveiling bosonic string effects in Wilson loops via boundary action
-SU(2) Yang-Mills theory-

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Abstract

The density profile of the QCD flux-tube is investigated within the framework of the Lüscher-Weisz (LW) string action with two boundary terms. We present results of the fits of string model to the data of SU(2) gauge theory for two lattices at couplings $\beta = 2.5$, and $\beta = 2.635$ calculated by Bali et. al. [1]. The transverse action profile and potential between static quarks are calculated using Wilson’s loop overlap formalism at zero temperature. We find the predictions of the LW string matching the lattice data for the width of the energy-density and $Q\bar{Q}$ potential up to a small color-source separation of $R = 0.3$ fm.

Keywords: Boundary terms, Bosonic strings, Effective string theory, Lattice gauge theory, Flux tubes, string formation, Confinement

1. INTRODUCTION

A fundamental characteristic of quantum chromodynamics (QCD) and strong interactions is the confinement of quarks. There is no solid analytical construction for the phenomena of confinement starting from fundamental principles, despite the enormous research efforts to give mechanisms of quark confinement based on gluonic degrees of freedom of the QCD.

In the Monte-Carlo computations of QCD path integrals, the confinement property can be directly investigated. The linear rising feature [2, 3, 4] of confinement potential was disclosed by computer simulations of the infinitely heavy quark-antiquark $Q\bar{Q}$ bound state. It is evident from this observation that the linear increase in potential at great distances between two static $Q\bar{Q}$ is caused by the gluonic field, which is thought to have condensed into a flux tube with strings [6, 7, 8, 9, 10, 11, 1, 12, 13, 14].

Numerous highly correlated systems exhibit string creation, which may be explained by an effective string action following the roughening transition [15, 16, 17, 18]. The effective string action is a low energy effective field theory [19] that is constructed by integrating out the degrees of freedom of the Yang-Mills (YM) vacuum in the presence of two static quarks. This string description provides a tool to identify a collection of infrared (IR) observables that may be compared to the results from the numerical lattice data.

In addition to the static potential, the QCD vacuum’s energy profile [20] when confined color sources are present is a key source for investigating the confinement’s physics from first principles. Numerous lattice simulations [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31] of numerous gauge groups have demonstrated the expected logarithmic widening [32] of the confining strings and the linear behavior [33] close to the confinement point and at a large distances.

Nevertheless, the analysis of string’s fine structure in the lattice numerical data for the broadening profile revealed substantial deviations [34, 35] from the free-string Nambu-Goto (NG) model in the intermediate distance region. These intermediate distance deviations appear when employing Wilson loops at zero temperatures [1].

The Lorentz-invariant boundary corrections [36] to the static $Q\bar{Q}$ potential have shown viable in both the Wilson and the Polyakov-loop correlators case [37, 38, 39]. The boundary corrections to the static quark potential provide an explanation to the deviations lately discovered among predictions of the effective string and numerical outcomes [40].

The Lorentz-invariant boundary terms in the action should also correct the energy field’s widening profile. However, neither theoretical calculations nor comparisons with numerical lattice data have been done to determine the contributions of the boundary action to the width profile. These adjustments are intended to account for several characteristics of the fine structure of the QCD flux-tube profile in the IR region. These corrections are hoped to account to many features of the fine structure of the profile of QCD flux-tube in IR region. For example, It might explain the well-known variations at low temperature and relatively short distances.

In this paper, we aim to compare the width of the energy distribution in static meson with the analytic solution of the Lüscher-Weisz (LW) effective string width the boundary terms. An open string with a cylinder-shaped Dirichlet boundary condition is used for the numerical comparison. We consider the perturbative expansion of two boundary terms at the order of fourth and six derivative and evaluate the modification to the mean-square width around the free NG string. This is compat-
2. LÜSCHER-WEISZ STRING ACTION

The conjecture Yang-Mills (YM) vacuum admits the formation of a very thin string-like object [41] has originated in the context of the linear rise property of the confining potential between color sources. Translational and rotational symmetries are spontaneously broken when the formation of a string-like condensate in the Yang-Mills vacuum. By virtue of the Goldstone theorem [42], the symmetry breaking produces $(d - 2)$ massless transverse self-interacting modes.

A string action describing the massless Goldstone modes can be created using the derivative expansion of collective string co-ordinates meeting Poincare and parity invariance. One particular form of this action is taken in the physical string coordinates meeting Poincare and parity invariance. The world-sheet area can be created using the derivative expansion of collective fluctuations to transverse directions of the world-sheet $C$.

In Wilson loops [49, 40, 48] operators of the boundary action survives over both the spatial and temporal extends, the boundary term $S^b$ in Lüscher-Weisz action accounts for the symmetry breaking by the cylindrical boundary conditions owing to the Polyakov lines. The modification to the potential received when considering Dirichlet boundary condition are given by [48]

$$V^B = V^{b_2} + V^{b_1},$$

$$V^{b_2} = b_2(d - 2) \frac{\pi^2 L_T}{60 \eta T} E_4(\tau),$$

$$V^{b_1} = - b_1(d - 2) \frac{\pi^5 T}{126 \xi_1} E_6(\tau).$$

In Wilson loops [49, 40, 48] operators of the boundary action survives over both the spatial and temporal extends, the boundary action is then given by

$$S^b = b_2 \int_{\partial \Sigma_\tau} d_6 \partial_0 (\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X) + b_2 \int_{\partial \Sigma_\tau} d_6 \partial_1 (\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X),$$

this surface term lives on the boundary $\partial \Sigma$ which in the action Eq. (7) is conveniently chosen as a rectangle-shaped Wilson’s loop circumscribing the spatial-temporal area of $R \times L$. The curves $\partial \Sigma_\tau$ and $\partial \Sigma_\tau$ stands for temporal and spatial parts of the loop, respectively.

The direct calculation of the expectation value of the $\langle S_b \rangle$ entails contributions from both the spatial and temporal parts to the static quark potential [48]. As a consequence of the symmetry of the propagator [50], these two corrections give similar formulas [48]; however, with the role between the source separation, $R$, and temporal extent, $L$, exchanged. Moreover, the contribution from the temporal path is formally equivalent [48] to that of two Polyakov loop with the identification of the tem-
poral height of the Wilson’s loop $L$ instead of the cylindrical
time extend $L_T \rightarrow R$.

$$V^H = V^{b_2} + V^{b_1},$$

$$V^{b_2} = b_2(d - 2) \frac{\pi^3 L_T}{64 R^4} E_4(\tau) + \frac{b_2(d - 2)}{2} \frac{\pi^3 R}{60 L_T^4} E_4 \left( \frac{1}{\tau} \right),$$

$$V^{b_1} = - \frac{b_2(d - 2)}{2} \frac{\pi^3 L_T}{128 R^6} E_6(\tau) + \frac{b_2(d - 2)}{2} \frac{\pi^5 R}{126 L_T^6} E_6 \left( \frac{1}{\tau} \right).$$

(8)

The expression for the potential due to the next boundary term $V^{b_3}$ correction is derived in \[51\].

The expectation value \[52, 53\] of the mean-square width of the free bosonic string theory in two dimensions

$$W^2_{b_0}(\zeta_1, \tau) = \frac{d - 2}{2 \pi \sigma_0} \log \left( \frac{R}{R_0(\zeta_1)} \right) + \frac{d - 2}{2 \pi \sigma_0} \log \left( \frac{\sigma_2(\pi \zeta_1 / R; \tau)}{\sigma_1(0; \tau)} \right),$$

(9)

where $\theta$ are Jacobi elliptic functions and $\tau = L/T$ is the modular parameter of the cylinder, with $q_1 = e^{2\pi \tau}$, and $R_0^2$ is the UV cutoff which has been generalized to be dependent on distances from the sources.

The NLO term of the mean-square width from the low energy parameter expansion of NG action Eq.(7?) has been worked out in detail in Ref.\[54, 53\]. The width due to the self-interaction is modified with the term \[54, 53\]

$$W^2_{nlo} = \frac{(d - 2)\pi}{12\sigma^2 R^2} \left( \tau \frac{d}{d\tau} - \frac{d - 2}{12} E_2(\tau) \right) [E_2(2\tau) - E_2(\tau)]$$

$$- \frac{d - 2}{8\pi} E_2(\tau) + \frac{\pi}{12\sigma R^2} [E_2(\tau) - 4E_2(2\tau)]$$

$$\times \left( W^2_{b_0} - \frac{d - 2}{4\pi \sigma} \right).$$

(10)

The generalization of the expectation value of the width due to Wilson’s loop $L$ is accordingly evaluated as the expectation value of

$$W^2_{b_2} = \langle X^2 S_{b_2} \rangle_{\delta\Sigma} + \langle X^2 S_{b_2} \rangle_{\delta\Sigma},$$

(11)

for the boundary action given by Eq.(7). The spatial part of this expectation value is an integral of the diffused energy of the string along the spatial path the infinitesimal-time interval of either the creation/annihilation of the static quark-antiquark pair. As the mean-square width can be deduced to have the form

$$W^2_{b_2} = \frac{\pi b_2(d - 2)}{8e^2} \left[ \frac{1}{R^4} \left( 1 - \frac{1}{24} E_2 \left( \frac{iL}{R} \right) \right) + \frac{1}{L_T^4} \left( 1 - \frac{1}{24} E_2 \left( \frac{iR}{L_T} \right) \right) \right].$$

(12)

where $L$ is the temporal extend of Wilson loop.

On the other hand, the corresponding boundary corrections of the width of Wilson loop at coupling $b_2$ can be deduced following up the same line of reasoning leading to Eq.(12). The next-to-leading boundary correction for Wilson loop of rectangular area $L \times R$ is, accordingly, given by

$$W^2_{b_3} = \frac{\pi^3(d - 2)b_4}{64 e^2} \left[ \frac{1}{R^6} \left( E_2 \left( \frac{iL}{2R} \right) - \frac{5}{4} \right) \frac{11}{36} E_2 \left( \frac{iL}{4R} \right) \right.$$

$$\left. - \frac{5}{9} E_2 \left( \frac{iL}{2R} \right) - \frac{5}{14} \right] + \left[ \frac{1}{R^6} \left( E_2 \left( \frac{iR}{2L} \right) - \frac{5}{4} \right) \frac{11}{36} E_2 \left( \frac{iR}{4L} \right) \right. \right]$$

$$\left. - \frac{5}{9} E_2 \left( \frac{iR}{2L} \right) - \frac{5}{14} \right].$$

(13)

The leading non-vanishing boundary corrections to the flux-tube width Eq.(12) indicate an inverse decrease with the fourth power of the length scale $R$. This suggests effects that are more noticeable near the intermediate and short string length scale.

3. NUMERICAL RESULTS AND DISCUSSION

3.1. Lattice Gauge Theory Data

In the following we consider comparing lattice data calculated by Bali et. al. \[1\] with the predictions of string model with boundary action. The lattice gauge theory data represent sets of careful analysis of the QQ potential and energy distribution at different couplings.

The numerical calculations performed by Bali et al. \[1\] represent lattices with hypercubic geometry and periodic boundary conditions in all four directions with volumes $L^2 \times L_T$ and $\beta$. The simulation parameters are summarized in Table 1 with lattice spacing and string tension in lattice units at each $\beta$.

The gauge fields are updated through the standard Wilson action \[1\] via Fabricius-Haan heatbath sweeps \[55, 56\]. The updates are mixed with microcanonical overrelaxation step.

| $L^2 \times L_T$ | $\beta = 2.50$ | $\beta = 2.635$ | $\beta = 2.74$ |
|------------------|--------------|----------------|--------------|
| $L^2 \times L_T$ | $32^4$       | $48^4 \times 64$ | $32^4$       |
| $\sigma$         | $0.0350(12)$ | $0.01458(8)$    | $0.00830(6)$ |
| $a/\text{fm}$    | $0.0826(14)$ | $0.0541(2)$     | $0.0408(2)$  |

TABLE 1: The string tension and the corresponding lattice spacing at each coupling \[1\]. The physical scales have been computed \[1\] from the value $\sqrt{\sigma}a = 440$ MeV.

To increase signal to noise ratio, temporal link integration \[57, 58\] has been implemented \[1\] analytically. This amounts to the substitution of the temporal links with the average

$$V_4(n) = \frac{I_2(\beta f_p(n))}{f_p(n) I_1(\beta f_p(n))} f_p(n),$$

$$f_p(n) = \sqrt{\text{det}(F_p(n))}.$$ 

(14)

$I_n$ denote the modified Bessel functions.

To enhance the overlap with the ground state, Ape smearing has been applied \[1, 59\] on the spatial links $n_{iter} = 150$ iteration sweeps. That is each spatial link $U_i(n)$, occurring in the transporter, is substituted by a “fat” link

$$U_i(n) \rightarrow N \left( aU_i(n) + \sum_{j \neq i} U_i(n + j) U_j^d(n + j) \right)$$

(15)

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The smearing of the spatial links $U(0 \rightarrow R)$ used for the construction of the $Q\bar{Q}$ creation operator (Eq. 16) does not alter the eigenvalues of the transfer matrix. This allows for an optimal enhancement of the ground state overlap by a suitable superposition of paths, aiming at $C_0(R) \approx 1$.

$$\langle W(R, T) \rangle = \sum_n C_n(R)e^{-V_n(R)T} \rightarrow C_0(R)e^{-V(R)T},$$

which is valid in the limit of large $(T \rightarrow \infty)$.

The bounds of the operators used to create the quarks is one potential reason that might account for the discrepancies in the effective string description at the short and intermediate string lengths. In order to test the LW string prediction for the boundary terms. We first compare a set of Wilson loop correlators with the pure NG string. We performed this analysis for three different values of the bare coupling $\beta$.

The $Q\bar{Q}$ potential data are fitted to theoretical formulas of the NG string potential at the leading and next-to-leading orders Eqs. (3) and Eq. (5), respectively. We set the renormalization $\mu(T)$ and boundary parameters $b_2$ and $b_3$ as free fitting parameters while keeping the string tension $\sigma_0 a^2$ fixed to the standard values enlisted in Table 1.

Since the leading boundary terms in the Lüscher-Weisz action arise at the order of four derivative terms, it is anticipated that the boundary terms will have an impact on the physics of small color separation. Thus we consider LW string with the interaction terms at NLO in the low energy expansion switched on.

The following alternative combinations of NLO Nambu-Goto static potential with boundary terms are defined for the discussion of the numerical data of the static meson potential:

$$V_{nlo}^{\mu} = V_{nlo} + V^{b_2},$$

where sub-scripted $V_{nlo}$ denotes the NLO NG static potential, the super-scripted $V^{b_2}$ is, however, given by (8).

The $Q\bar{Q}$ potential data are fitted to the static potential with the possibly interesting combination of the boundary terms $V_{nlo}$ and $V^{b_2}$ given by Eq. (19). The inspection of each boundary term allows for exploring the viability and limitations of each boundary parameter.

The fits are obtained by optimizing the sets of the parameter space $\mu, b_2$ in the above-mentioned models such that the least square residuals

$$\chi^2(R_0, b_2) = \frac{1}{\ell(R)} \sum \left( \frac{V(R_i) - V_{model}(R_i; R_0, b_2)}{\ell(R_i)} \right)^2,$$

The corresponding returned values of $\chi^2$ and fit parameters are enlisted in Table 2 at three values of the coupling constants.

The fit of the numerical data to boundary corrected string potential Eq. (19) produces a reduction in the residuals by excluding short-distance points. The fit of the numerical data using the next-to-leading form of the potential Eq. (5) produces good $\chi^2_{4.0.f}$ for fit interval $R \in [5a, 8a]$ at $\beta = 2.5$. The string behavior according to this ansatz takes place on source separation scales commencing from $R = 0.4$ fm.

Despite of the reductions in the minima of $\chi^2_{4.0.f}$, the string models with boundary terms have no significantly different behavior with respect to the string tension parameter. This is consistent with the modular transforms where the inverse of the
FIGURE 2: The quark-antiquark Q\bar{Q} potential data at three different couplings \( \beta \). The lines correspond to the LW string with boundary terms \( V_{\text{nlo}}^{b_2} \) and \( V_{\text{nlo}}^{b_2,b_4} \) given by Eqs. (19) and Eq. (21), respectively.

The fit to the static potential \( V_{\text{nlo}}^{b_2} \), model of Eq. (19) returns values for the parameter \( b_2 \) which appear to depend on the fit interval and the corresponding coupling \( \beta \). This is a reflection of the large value of \( \chi^2_{\text{d.o.f.}} \) returned on such fit intervals. Lattices at coupling \( \beta = 2.63 \) appears to return good \( \chi^2_{\text{d.o.f.}} \) for the potential for color separation of \( R = 8a \) which corresponds to distance \( R = 0.432 \) fm. At lattice coupling \( \beta = 2.74 \) of slightly smaller lattice spacing, the string behavior sets in over distances of minimal length \( R = 0.4 \) fm.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\( V_{\text{nlo}}^{b_2} \) & Fit Interval & \( V_{\text{nlo}}^{b_2} \) & Fit Parameters & \( \chi^2 \) \\
\hline
\hline
\( \beta = 2.5 \) & [1,8] & -2.919(3) & -0.0591(3) & 106.7 \\
 & [2,8] & -3.011(5) & -0.198(5) & 30.2 \\
 & [4,8] & -0.321(2) & -1.0(1) & 1.8 \\
 & [5,8] & -0.331(6) & -1.6(3) & 0.1 \\
\hline
\end{tabular}
\caption{The \( \chi^2 \) values and the corresponding fit parameters \( b_2 \) and \( \mu \) returned from fits to the next to leading order (NLO) static potential with boundary terms \( V_{\text{nlo}}^{b_2} \) given by Eq. (19).}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( V_{\text{nlo}}^{b_2,b_4} \) & Fit Interval & \( V_{\text{nlo}}^{b_2,b_4} \) & Fit Parameters & \( \chi^2 \) \\
\hline
\hline
\( \beta = 2.5 \) & [1,8] & -3.03(5) & -0.2675(9) & -0.007(3) & 25.9 \\
 & [2,8] & -31.75(1) & -0.987(6) & -0.085(6) & 3.5 \\
 & [3,8] & -31.38(3) & -1.8(3) & -0.268(7) & 0.93 \\
\hline
\end{tabular}
\caption{The \( \chi^2 \) values and the corresponding fit parameters \( b_2 \), \( b_4 \) and \( \mu \) returned from fits to the next-to-leading order (NLO) static potential with boundary terms \( V_{\text{nlo}}^{b_2,b_4} \) given by Eq. (21).}
\end{table}

The \( Q\bar{Q} \) potential data are not accurately described by the effective description that is based only on the Nambu-Goto model. Even if the string ansatz beyond the free Gaussian approximation is utilized, the inclusion of the NLO terms does not provide the anticipated acceptable optimization at moderate distances. The fit ansatz with boundary term Eq. (19), however, reveals a flux tube behavior consistent with that LW string for separation distances \( R > 0.4 \) fm.

To further explore the viability of the boundary terms we examine the prospective effects of including \( b_4 \) term. The following define the LW string ansatz at NLO combined with both \( V_{\text{nlo}}^{b_2} \) and \( V_{\text{nlo}}^{b_4} \) boundary corrections.

\[
V_{\text{nlo}}^{b_2,b_4} = V_{\text{nlo}}^{b_2} + V_{\text{nlo}}^{b_4},
\]  

The fits parameters \( R_0, b_2, b_4 \) are obtained by minimizing

\[
\chi^2(R_0,b_2,b_4) = \sum_i \left( \frac{V(R_i) - V_{\text{model}}(R_i; R_0,b_2,b_4)}{\epsilon_i(R_i)} \right)^2,
\]  

The returned fit parameters are collected in Table 3 for the lattice at coupling \( \beta = 2.53 \).
In general, one can observe a further reduction of the $\chi^2$ d.o.f. when employing $V^{b_i b_j}_{\ell 0}$ compared to fits parameters in Table 2 for $V^{b_i b_j}_{\ell 0}$ at all coupling values $\beta$.

The comparison asserts the improved behavior of the LW string at NLO ansatz with the inclusion of additional boundary terms. The fits return good $\chi^2$ for the fit range $R \in [3a, 8a]$ at $\beta = 2.5$. For finer lattices at $\beta = 2.63$ and $\beta = 2.74$ the optimal fits are reproduced on the intervals $R \in [5a, 12a]$ and $R \in [5a, 14a]$, respectively. This extends the string behavior approximately on source separation $R \geq 0.3$ fm.

The panel in Fig. 2 collects three plots at each $\beta$. Each plot represent the lattice data of $QO$ of the potential together with the lines indicating the string model solutions $V^{b_0}_{\ell 0}$ of Eq. 19 and $V^{b_i b_j}_{\ell 0}$ of Eq. 19. The lines reflects the chosen fit interval where a good $\chi^2$ is received.

One clearly observe the improvement in the fit with respect to the LW string with one boundary term compared to that employing two boundary terms. As discussed above the plots in three figures at each coupling indicate at least three lattice spacing of improved match where the string behavior can be revealed with the second boundary term.

3.3. Energy-density profile

The Euclidean action-density on the lattice is characterized utilizing a plaquette operator defined by

$$\Box_{\mu \nu} (\vec{p}) = 1 - \frac{1}{3} R e \text{Tr} \left[ U_{\mu} (\vec{p}) U_{\nu} (\vec{p} + \vec{v}) U_{\mu}^\dagger (\vec{p} + \vec{v}) U_{\nu}^\dagger (\vec{p}) \right],$$

which corresponds to the minimal loop structure on the lattice with the indices $\mu$ and $\nu$ corresponding to Lorentz indices. The plaquette operator can be expanded in a power series [60] of the symmetric tensor $F^c_{\mu \nu}$ such that

$$\Box_{\mu \nu} (\vec{p}) = 1 - \frac{1}{3} R e \text{Tr} \left[ i g a^2 \sum_c F^c_{\mu \nu} (\vec{p}) T^c \right],$$

where $g$ is the coupling of Yang-Mills theory, the index $c$ is color indices and $T^c$ are the generators of Lie algebra of SU(2)$_c$,

$$S(\vec{p}) = \frac{1}{2} \left( E^2 (\vec{p}) - B^2 (\vec{p}) \right).$$

The chromoelectric and chromomagnetic components fields are related to the plaquette components at position $\vec{p}$ as

$$E^2 (\vec{p}) = \sum_i E_i^2 (\vec{p}) \rightarrow \sum_i \Box_{0i} (\vec{p}),$$

$$B^2 (\vec{p}) = \sum_i B_i^2 (\vec{p}) \rightarrow \sum_i \Box_{ji} (\vec{p}),$$

where the index $i$ of the magnetic field is the complement of the $j$ $k$ plaquette component.

A dimensionless scalar field characterizing the Euclidean action density distribution in the Polyakov vacuum, i.e., in the presence of color sources can be defined as

$$C(\vec{p}; \vec{r}_1, \vec{r}_2) = \frac{\langle P_{2Q}(\vec{r}_1, \vec{r}_2) \rangle \langle S(\vec{p}) \rangle - \langle P_{2Q}(\vec{r}_1, \vec{r}_2) S(\vec{p}) \rangle}{\langle P_{2Q}(\vec{r}_1, \vec{r}_2) \rangle \langle S(\vec{p}) \rangle},$$

with the vector $\vec{p}$ referring to the spatial position of the energy probe with respect to some origin, and the bracket (...) stands for averaging over gauge configurations and lattice symmetries. Other dimensionful definitions of the correlator (28) yield equivalent representation of the width (see, for example Ref. [14]).

The above equation is dimensionless. However, the field densities in physical units would read

$$\sum_c f_c (n) = \frac{1}{2\pi^2} \exp \left( -\frac{n^2}{\sigma^2} \right),$$

with the normalization

$$\sum_n f_c (n) \approx \int d^2 n f_c (n) = \int d^2 n f_c (n) = \frac{1}{2\sigma^2}.$$
FIGURE 3: (a) The data points correspond to the width of the flux-tube at the middle plane \( \frac{R}{2} \) at \( \beta = 2.5 \) for the depicted fit ranges. The lines are the fits of square-width of the effective NG string at NLO Eq. (32) compared to LW string with boundary terms Eq. (33) respectively. In (b) and (c) the comparison is held on variant interval at lattice coupling \( \beta = 2.63 \).

| \( \beta \) | Fit Range | \( \chi^2_{d.o.f.} \) |
|----------|-----------|-----------------|
| \( 2.63 \) | [12,20]   | 2.12896         |

TABLE 7: Same as Table6, however, the fit range is over the source separation \( R \in [12,20] \).

| \( \beta \) | Fit Range | \( \chi^2_{d.o.f.} \) |
|----------|-----------|-----------------|
| \( 2.63 \) | [4,20]    | 2.02            |

TABLE 8: Parameters from the fits to LW model with two boundary terms Eq. (33) at lattice spacing \( a = 0.054 \) fm.

| \( \beta \) | Fit Range | \( \chi^2_{d.o.f.} \) |
|----------|-----------|-----------------|
| \( 2.63 \) | [6,20]    | 0.62            |

TABLE 9: Same as Table 8 except that the fit range commences at \( R = 6a \).

which are given in accordance with Eqs.(9) and (10), respectively. We consider the mean-square width of LW string with two boundary corrections

\[
W^2 = W^2_{NG_{\beta=2.63}} + W^2_{NG_{\beta=2.63}} + W^2_b + W^2_{\beta},
\]

are minimized. In Eq.(34), \( e^2(R/2) \) is the square of the error in the measured mean-square width \( W^2(R/2) \) from the lattice simulation [1].

Here, the analysis of the fit behavior of the mean-square width data is discussed keeping the string tension fixed to the standard value [61] of \( \sqrt{s} = 440 \) MeV. Our interest is to divulge the parametrization behavior of the string with respect to numerical data. We consider the lattice data of the mean-square width over the source separation \( R \) which are provided by Eqs.(9), (10), (12) and (13).

The resultant minimum \( \chi^2 \) and the corresponding fit parameters \( R_0, b_2 \) and \( b_4 \) from the fits to the lattice data Eq.(22) are collected in Tables. 4 to 9. Each table corresponds to fits to the width of the flux tube at either one of the two lattices at couplings \( \beta = 2.5 \) and \( \beta = 2.63 \).

The characteristics of the energy profile of the QCD string should be understood in the context of the complementary IR observable, namely the the ground state potential \( QQ \). Despite the comparatively higher uncertainty in the action density Eq. (28) than in the \( QQ \) potential. However, the observation of the string over intervals commencing from same source sepa-
ration demonstrates the relevance of the boundary action to the confining flux tube. The $\chi^2_{\text{d.o.f.}}$ values in Table 4 indicate poor fits when using NLO approximation Eq. (32) of the pure NG string. One expects that takes significant source separations $R \geq 0.8$ fm before the LO approximation matches the numerical data optimally. The next table encapsulates the parameters retrieved by fitting the width data to the LW string with two boundary terms Eq. (32). Comparing these with the fits along the pure (NG) model, the results in Table 5 show a significant reduction of residuals with $\chi^2_{\text{d.o.f.}} = 1.9$. The fit parameters of the boundary action assume the values $b_2 = -1.9(5)$ fm and $b_4 = -0.2(6)$ fm. Upon the conversion of the returned values of $b_2$ and $b_4$ from lattice units into physical units we reproduce the values of the boundary parameters within the uncertainties. This is evident from opposing these with those values obtained from the $Q\bar{Q}$ potential in Table 2 and Table 3. It is remarkable that the fits to the energy width reproduces the parameters of the boundary action obtained from the $Q\bar{Q}$ potential fits within the uncertainties of measurements. Inspection of the fit analysis over the finer lattice spacing $a = 0.054$ of coupling $\beta = 2.63$ in Table 7 indicates large $\chi^2_{\text{d.o.f.}}$ values. The pure NG string model Eq. (32) does not reproduce optimal fit to the width up to separation distances $R \geq 14a$, despite of the improvement in the fits by excluding smaller distance. At this length scale, which in physical units corresponds to $R \geq 0.432$ fm, the NG string picture Eq. (9) is expected to hold. For small color source separation $R \in [4a, 18a]$, the rendering in Fig 3(a) of the fitted width of the pure NG string exhibits significant deviations from the lattice data. The model good fits at large distances $R \geq 0.8$ fm in Fig 3(b). The plots suggest the incompetence of the pure NG string action as a physical description integrating the subtle features of the QCD flux tubes at short distances. However, on the same fit intervals, the improvements with respect to the LW string models [Eqs. (33)] are clearly evident. The plots in Figs. 3(a) and (b) show the need to consider boundary terms to describe flux tube profile on source separation distances $R < 0.8$ fm. The plot in Fig 3(c) covers a subset of the intermediate source separation region $R \in [6a, 20a]$. This is typically the distance where we obtained the best fits to LW string with boundary terms.

4. CONCLUSION

In this paper, the quark-antiquark potential and energy profile of a static meson are compared to the theoretical predictions based on the Lüscher-Weisz (LW) string with two boundary terms. The link-integrated Wilson loop correlators with smeared spatial connections are used [1] to determine the static ($Q\bar{Q}$) quark-antiquark potential. At intermediate and short distances, we detect signatures of the two boundary terms of the Lüscher-Weisz (LW) string [48, 51] in the Monte-Carlo data of the static $Q\bar{Q}$ potential. The boundary corrected string model extends the region of validity of the string for color source separation $R \geq 0.3$ fm. The fits with regard to the pure Nambu-Goto string at NLO order poorly parameterize the width growth of the energy density. It takes, thereof, considerable source separations $R \geq 0.8$ fm before the NG string matches the numerical data optimally. The boundary-corrected width [50] with one boundary term $b_2$ for the (LW) action reduces the residuals of the fits to the lattice data of the action density, with good fits obtained for string length $R \geq 0.5$ fm. Inclusion of the second Lorentz-invariant boundary term $b_4$ discloses a good match with the LGT data for color sources separation $R \geq 0.3$ fm.

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