CUMULANTS, FREE CUMULANTS AND HALF-SHUFFLES

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Abstract. Free cumulants were introduced as the proper analog of classical cumulants in the theory of free probability. There is a mix of similarities and differences, when one considers the two families of cumulants. Whereas the combinatorics of classical cumulants is well expressed in terms of set partitions, the one of free cumulants is described, and often introduced in terms of non-crossing set partitions. The formal series approach to classical and free cumulants also largely differ.

It is the purpose of the present article to put forward a different approach to these phenomena. Namely, we show that cumulants, whether classical or free, can be understood in terms of the algebra and combinatorics underlying commutative as well as noncommutative (half-)shuffles and (half-)unshuffles. As a corollary, cumulants and free cumulants can be characterized through linear fixed point equations, from which various new properties thereof can be automatically deduced. As a first step in that direction, we study the exponential solutions of these linear fixed point equations, which display well the commutative, respectively noncommutative, character of classical, respectively free, cumulants.

Keywords: Cumulants, free cumulants, double bar construction, half-shuffles, codendr imorphic coalgebra.

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1. Introduction

D. Voiculescu introduced in the 1980s the theory of free probability [28]. In this theory the classical concept of probabilistic independence is replaced by the algebraic notion of freeness, i.e., the absence of relations. Briefly, the definition of a noncommutative probability space consist of a pair $(A, \phi)$, where $A$ is a complex algebra with unit $1_A$. The map $\phi$ is a $\mathbb{C}$-valued linear form on $A$, such that $\phi(1_A) = 1$. The elements of $A$ play the role of random variables, while the map $\phi$ should be considered as the expectation map, similar to classical probability theory. Let $I$ be a set of indices, and $B_i$, for $i \in I$, be subalgebras of $A$, containing the unit. The family of
algebras $B_i$, $i \in I$, will be called free if $\phi(a_1 \cdots a_n) = 0$ everytime $\phi(a_j) = 0 \, \forall j = 1, \ldots, n$ and $a_j \in B_{i_j}$ for some indices $i_1 \neq i_2 \neq \cdots \neq i_n$.

R. Speicher introduced the notion of free cumulants as the proper analog of classical cumulants in the theory of free probability. See e.g. [22], the standard reference on the subject. There is a mix of similarities and differences between the two families of cumulants. Indeed, whereas the combinatorics of classical cumulants is naturally expressed in terms of set partitions, the one of free cumulants is described and often introduced in terms of non-crossing set partitions. The formal series approach to cumulants and free cumulants also largely differ.

It is the purpose of the present article to develop a different approach to the algebraic and combinatorial structures underlying free and classical cumulants. Namely, we show that cumulants, both classical and free, can be understood algebraically in terms of commutative and noncommutative (un-)shuffles (noncommutative shuffles and unshuffles are often referred to in the literature as dendrimorphic products and coproducts). As a corollary, cumulants and free cumulants happen to solve linear fixed point equations, from which various new algebraic and combinatorial properties thereof can be deduced.

To begin with, we focus in the present article on the moment/cumulant relationship from an algebraic point of view – other aspects of the combinatorial theory will be addressed in forthcoming papers. We show that the aforementioned linear fixed point equations can be solved in terms of proper exponentials using the pre-Lie Magnus expansion [11, 12]. On this level, the basic difference between classical and free cumulants can be described analogously to the case of scalar- versus matrix-valued linear initial value problems. Indeed, classical cumulants correspond to an exponential solution of a linear fixed point equation in a commutative setting, whereas free cumulants correspond to solutions in a noncommutative context.

In the following $k$ denotes a ground field of characteristic zero. This is basically the interesting case, and a convenient hypothesis to avoid cumbersome distinctions. However, we point out that this assumption is not strictly necessary for all the results in the article. Indeed, many equations we will consider are defined and can be solved over the integers. We also assume any $k$-algebra $A$ to be associative and unital, if not stated otherwise. The unit in $A$ is denoted $1$. Identity morphisms are written $I$.

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2. Shuffle algebras

Recall first various classical results and definitions related to shuffle algebras. In the classical literature, shuffles refer to the (commutative) combinatorial shuffles arising from products of (functional) iterated integrals that also appear in the theory of free Lie algebras [25]. They refer, however, as well to topological shuffles, the latter being noncommutative (they are commutative only up to homotopy). These notions can be traced back at least to the 1950’s – the period in which both families of shuffle products were axiomatized in the works of Eilenberg–MacLane and Schützenberger [15, 26].
We follow the usual practice and reserve the name “shuffle algebra” for the commutative case. The names “nc-shuffle algebra” (noncommutative shuffle algebra) and “dendrimorphic algebra”\(^1\) are used in the noncommutative case.

Recall the definition of a nc-shuffle, or dendrimorphic algebra. It is a \(k\)-vector space \(D\) together with two bilinear compositions \(<\) and \(>\) (the left and right half-shuffle products) subject to three axioms

\[
\begin{align*}
(a < b) < c &= a < (b \mathcal{W} c) \\
(a > b) < c &= a > (b < c) \\
a > (b > c) &= (a \mathcal{W} b) > c,
\end{align*}
\]

where the bilinear product

\[a \mathcal{W} b := a < b + a > b.\]

We call \(\mathcal{W}\) shuffle product on \(D\), in both the commutative and noncommutative cases, this should not result in any confusion.

A shuffle algebra, sometimes also called Zinbiel algebra (in reference to the Bloh–Cuvier \([2, 8, 9]\) dual notion of Leibniz algebra), but we will stick to the classical terminology, is a noncommutative shuffle algebra, where the left and right half-shuffles are identified:

\[x > y = y < x,\]

so that in particular the shuffle product \(\mathcal{W}\) is then commutative: \(x \mathcal{W} y = x < y + x > y = y \mathcal{W} x.\)

The axioms (1-3) imply that any nc-shuffle algebra is an associative algebra for the shuffle product (4). This observation actually underlies the classical and celebrated abstract proof of the associativity of the topological shuffle products by Eilenberg–MacLane \([15]\). Let us mention that one could actually show that the axioms of noncommutative shuffle algebras encode exactly products of topological simplices. This is due to the equivalence between the computation of these products and computations in symmetric group algebras, i.e., in the Malvenuto–Reutenauer Hopf algebra \([20, 23]\), together with the property of the latter to be free as a noncommutative shuffle algebra \([17]\).

Let us introduce some useful notations. Let \(L_{a >} (b) = a > b = R_{b <} (a)\). From the axioms, we get

\[L_{a >} L_{b >} = L_{a \mathcal{W} b >}, \quad R_{< a} R_{< b} = R_{< b \mathcal{W} a}.\]

Recall that a left pre-Lie algebra \([5, 21]\) is a vector space \(V\) equipped with a bilinear product \(\triangleright\), such that for arbitrary \(a, b, c \in V\)

\[a \triangleright (b \triangleright c) - (a \triangleright b) \triangleright c = b \triangleright (a \triangleright c) - (b \triangleright a) \triangleright c.\]

It implies that the bracket \([a, b] := a \triangleright b - b \triangleright a\) satisfies the Jacobi identity. For several reasons, largely due to the general theory of integration encoded by Rota–Baxter algebras (see \([14]\)), pre-Lie algebras play a key role, e.g., in the understanding of dendrimorphic equations of interest to physics. The next lemma follows directly from the axioms (1-3) of shuffle products.

**Lemma 1.** Let \(D\) be a nc-shuffle algebra. The product \(\triangleright\colon D \otimes D \to D\)

\[a \triangleright b := a > b - b < a\]

is left pre-Lie. We write its left action \(L_{a >} (b) = a > b = L_{a >} - R_{< a}\).

\(^1\)Note that we prefer the word “dendrimorphic” over the commonly used terminology, “dendriform”, which fails to meet the standard criteria of name-giving in mathematics by mixing greek and latin roots.
Note that \([a,b] = a \triangleright b - b \triangleright a = a \shuffle b - b \shuffle a\) for all \(a,b \in D\). The pre-Lie product is trivial (null) on shuffle algebras, since we then have \(a \triangleright b = b \lessdot a\).

Dendrimorphic and shuffle algebras are not naturally unital. This is because it is impossible to “split” the unit equation, \(1 \shuffle a = a \shuffle 1 = a\), into two equations involving the half-shuffle products \(\triangleright\) and \(\lessdot\). This issue is circumvented by using the “Schützenberger trick”, that is, for \(D\) a dendrimorphic algebra, \(\overline{D} := D \oplus k.1\) denotes the dendriformic algebra augmented by a unit 1, such that

\[
a \preccurlyeq 1 := a =: 1 \triangleright a \quad \quad 1 \lessdot a := 0 =: a \triangleright 1,
\]

implying \(a \shuffle 1 = 1 \shuffle a = a\). By convention, \(1 \shuffle 1 = 1\), but \(1 \preccurlyeq 1\) and \(1 \lessdot 1\) cannot be defined consistently in the context of the axioms of shuffle and dendrimorphic algebras.

The following set of so-called dendrimorphic words in \(\overline{D}\) are defined recursively for fixed elements \(x_1, \ldots, x_n \in D, n \in \mathbb{N}\)

\[
\begin{align*}
w^{(0)}_<(x_1, \ldots, x_n) & := 1 =: w^{(0)}_{\preccurlyeq}(x_1, \ldots, x_n) \\
w^{(n)}_<(x_1, \ldots, x_n) & := x_1 \lessdot (w^{(n-1)}_{\preccurlyeq}(x_2, \ldots, x_n)) \\
w^{(n)}_{\preccurlyeq}(x_1, \ldots, x_n) & := (w^{(n-1)}_{\preccurlyeq}(x_1, \ldots, x_{n-1})) \triangleright x_n.
\end{align*}
\]

In case that \(x_1 = \cdots = x_n = x\) we simply write \(w^{(n)}_{\preccurlyeq}(x) = w^{(n)}_{\lessdot}(x, \ldots, x)\) and \(w^{(n)}_{\preccurlyeq}(x) = w^{(n)}_{\triangleright}(x, \ldots, x)\).

In the unital algebra \(\overline{D}\) both the exponential and logarithm maps are defined in terms of the associative product \(\bullet\)

\[
\exp_{\shuffle}(x) := 1 + \sum_{n>0} \frac{x \shuffle w^{(n)}}{n!} \quad \text{resp.} \quad \log_{\shuffle}(1 + x) := -\sum_{n>0} (-1)^n \frac{x \shuffle w^{(n)}}{n}.
\]

Notice that we do not consider convergence issues: in practice we will apply such formal power series computations either in a purely algebraical setting (formal convergence arguments would then apply), either when dealing with graded algebras (then the series will reduce to a finite number of nonzero terms when restricted to a given graded component).

It is also convenient to introduce the so-called “time-ordered” exponential

\[
\exp_{\triangleright}(x) := 1 + \sum_{n>0} w^{(n)}_{\triangleright}(x).
\]

It corresponds to the usual time-ordered exponential in physics, when the shuffle product is defined with respect to products of, say, matrix- or operator-valued iterated integrals. See for instance [3] Sect. 1], where the links between products of iterated integrals and the “shifted”, noncommutative shuffle product in the Malvenuto–Reutenauer Hopf algebra are detailed. In [14] a detailed study of time-ordered exponentials from an abstract algebraic point of view is presented.

Similarly, we also define \(\exp_{\preccurlyeq}(x) := \sum_{n>0} w^{(n)}_{\preccurlyeq}(x)\). Notice that \(X = \exp_{\triangleright}(x)\) and \(Z = \exp_{\preccurlyeq}(x)\) are respectively the formal solutions of the two linear recursions

\[X = 1 + x \prec X\quad \text{resp.} \quad Z = 1 + x \preccurlyeq Z.\]

Both the time-ordered exponential as well as the proper exponential map \([9]\) will be key ingredients in our approach to cumulants. This point of view paves the way to new formal results on the structure and combinatorics of cumulants.
Let us show, for example, how the classical group-theoretic properties of the flow map for, say, matrix-valued linear differential equations, translate almost immediately into the computation of a multiplicative inverse of the time-ordered exponential:

**Lemma 2.** Let $A$ be a dendrimorphic algebra, and $\overline{A}$ its augmentation by a unit $1$. For $x \in A$ we have
\[ \exp^\prec(-x) \overline{A} \exp^\prec(x) = 1. \]

**Proof.** Indeed, we see that
\[ \exp^\prec(-x) \overline{A} \exp^\prec(x) - 1 = \sum_{n,m \geq 1} (-1)^n \{ (x^n) \prec (x^{-m}) + (x^{-n}) \succ (x^m) \} \]
\[ = \sum_{n,m \geq 0} (-1)^n (x^n) \prec (x^{-m}) + \sum_{n \geq 0, m > 0} (-1)^n (x^n) \succ (x^m). \]

Now, since $(-1)^n (x^n) \prec (x^{-m}) = (-1)^n ((x^n-1) \succ x) \prec (x^{-m}) = (-1)^n (x^n-1) \succ (x^{-m+1})$, the proof follows.
\[ \square \]

Another useful result follows from the computation of the composition inverse of the time-ordered exponential.

**Lemma 3.** Let $A$ be a dendrimorphic algebra, and $\overline{A}$ its augmentation by a unit $1$. For $x \in A$ and $X := 1 + Y := \exp^\prec(x)$, then
\[ x = Y \prec \left( \sum_{n \geq 0} (-1)^n Y^m \right). \]

**Proof.** We follow [18]. From $X = 1 + \sum_{n=0} \overline{x^n}$, we get $X - 1 = Y = x \prec X$. On the other hand, the (formal) inverse of $X$ for the shuffle product is given by $X^{-1} = \overline{1 + Y} = \sum_{k \geq 0} (-1)^k Y^k$. We finally obtain
\[ x = x \prec 1 = x \prec (X \overline{A} X^{-1}) = (x \prec X) \prec X^{-1} = Y \prec \left( \sum_{n \geq 0} (-1)^n Y^m \right). \]
\[ \square \]

There is an abundance of literature on noncommutative shuffles and on dendrimorphic identities. The interested reader is referred to, e.g., [6, 11, 13] for further insights and examples.

### 3. Unshuffling the Double Bar Construction

The notion dual to the one of shuffle product, i.e., the unshuffle coproduct – commutative and noncommutative, – has been considered only recently from an abstract axiomatic point of view. It plays a key role in the seminal works of L. Foissy, and especially in his proof of the Duchamp–Hivert–Thibon “free Lie algebra” conjecture. We refer to his work for further details [17].

**Definition 1.** A counital codendrimorphic coalgebra (or counital unshuffle coalgebra) is a coaugmented coalgebra $\overline{C} = C \oplus k$ with coproduct
\[ \Delta(c) := \overline{\Delta}(c) + c \otimes 1 + 1 \otimes c, \tag{10} \]
such that, on $C$, $\overline{\Delta} = \Delta_\prec + \Delta_\succ$ with
\[ (\Delta_\prec \otimes I) \circ \Delta_\prec = (I \otimes \Delta) \circ \Delta_\prec \tag{11} \]
moments in free probability, as viewed from the point of view of shuffle products. The word \( w \) is written as words \( a \) and \( \Delta \) product

\[
\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)
\]

\[
\Delta^+(a \cdot b) = \Delta^+(a) \cdot \Delta(b),
\]

where

\[
\Delta^+(a) := \Delta(a) + a \otimes 1
\]

\[
\Delta^+(a) := \Delta(a) + 1 \otimes a.
\]

We shall omit the definition of a codendrimorphic coalgebra. The latter is obtained by removing the unit, that is, \( \Delta \) is acting on \( C \), and has a splitting into two half-coproducts, \( \Delta_\prec \) and \( \Delta_\succ \), which obey relations (11), (12) and (13).

**Definition 2.** A codendrimorphic bialgebra is a unital and counital bialgebra \( B = B \oplus k \) with product \( \cdot \) and coproduct \( \Delta \). At the same time \( B \) is a counital codendrimorphic coalgebra with \( \Delta = \Delta_\prec + \Delta_\succ \). The following compatibility relations hold

\[
\Delta_\prec(a \cdot b) = \Delta_\prec(a) \cdot \Delta_\prec(b)
\]

\[
\Delta_\prec(a \cdot b) = \Delta_\prec(a) \cdot \Delta_\prec(b),
\]

We introduce now the algebraic structures encoding the relation between cumulants and moments in free probability, as viewed from the point of view of shuffle products.

Let \( A \) be an associative \( k \)-algebra. Define \( T(A) := \oplus_{n \geq 0} A^{\otimes n} \) to be the nonunital tensor algebra over \( A \). The full tensor algebra is denoted \( \overline{T}(A) := \oplus_{n \geq 0} A^{\otimes n} \). Elements in \( T(A) \) are written as words \( a_1 \cdots a_n \in T(A) \) (to avoid ambiguities we denote \( a_1 \cdots a \in A^{\otimes n} \) by \( a^{\otimes n} \), and the product of the \( a_i \)s in \( A \) is written \( a_1 \cdot a_2 \). The algebra \( T(A) \), equipped with the concatenation product of words (for \( w = a_1 \cdots a_n \) and \( w' = b_1 \cdots b_m \), \( w \cdot w' := a_1 \cdots a_n b_1 \cdots b_m \), is naturally graded. The word \( w = a_1 \cdots a_n \) is an element of degree \( n \), and we write \( w \in T_n(A) \).

We also set \( T(T(A)) := \oplus_{n \geq 0} T(A)^{\otimes n} \), and use the bar-notation to denote elements \( w_1 | \cdots | w_n \in T(T(A)) \), \( w_i \in T(A) \), \( i = 1, \ldots, n \). The algebra \( T(T(A)) \) is equipped with the concatenation product. For \( a = w_1 | \cdots | w_n \) and \( b = w'_1 | \cdots | w'_m \) we denote their concatenation product in \( T(T(A)) \) by \( a[ b \), that is, \( a[ b := w_1 | \cdots | w_n | w'_1 | \cdots | w'_m \). This algebra is multigraded, \( T(T(A))_{n_1, \ldots, n_k} := T_{n_1}(A) \otimes \cdots \otimes T_{n_k}(A) \), as well as graded. The degree \( n \) part is \( T(T(A))_n := \bigoplus_{n_1 + \cdots + n_k = n} T(T(A))_{n_1, \ldots, n_k} \). Similar observations hold for the unital case, that is, \( \overline{T}(T(A)) = \bigoplus_{n \geq 0} T(A)^{\otimes n} \), and we will identify without further comments a bar symbol such as \( w_1 | 1 | w_2 \) with \( w_1 | w_2 \) (formally, using the canonical map from \( \overline{T}(T(A)) \) to \( T(T(A)) \)).

When \( A \) is commutative (or graded commutative in the sense of algebraic topology), then \( T(T(A)) \) is classically involved in the definition of the double bar construction on \( A \). This is a differentially graded algebra structure appearing in homological algebra as well as in the study of \( K(\Pi, n) \) spaces – the latter can be seen as the very motivation underlying the Eilenberg–MacLane study of shuffle products in \([15]\). See, e.g., \([24]\) for a modern account. The terminology “double bar” refers to the fact, that one may represent tensors \( a_1 \otimes \cdots \otimes a_n \) using bars, \( a_1 \cdots | a_n \), instead of using the word notation. The representation of elements in \( T(T(A)) \) would then involve double bars. We point out that the combinatorial operations we are going to define and study on \( T(T(A)) \) are different from the classical structures existing on the double bar construction, even for a commutative algebra \( A \).

Given two (canonically ordered) subsets \( S \subseteq T \) of the set of integers \( \mathbb{N} \), we call connected component of \( S \) relative to \( T \) a maximal sequence \( s_1, \ldots, s_n \) in \( S \) such that there are no \( 1 \leq i < n \)
and \( t \in T \), such that \( s_i < t < s_{i+1} \). In particular, a connected component of \( S \) in \( N \) is simply a maximal sequence of successive elements \( s, s + 1, \ldots, s + n \) in \( S \).

Consider a word \( a_1 \cdots a_n \in T(A) \). For \( S := \{ s_1, \ldots, s_p \} \subseteq [n] \), we set \( a_S := a_{s_1} \cdots a_{s_p} \) (resp. \( a_\emptyset := 1 \)). Denoting \( J_1, \ldots, J_k \) the connected components of \([n] - S\), we also set \( a_{J_1[n]} := a_{J_1} \cdots a_{J_k} \). More generally, for \( S \subseteq T \subseteq [n] \), set \( a_{J_S[n]} := a_{J_1} \cdots a_{J_k} \), where the \( a_{J_j} \) are now the connected components of \( S \) in \( T \).

**Definition 3.** The map \( \Delta : T(A) \to \overline{T}(A) \otimes \overline{T}(T(A)) \) is defined by

\[
\Delta(a_1 \cdots a_n) := \sum_{S \subseteq [n]} a_S \otimes a_{J_1[n]} \cdots a_{J_k[n]}.
\]

The coproduct is then extended multiplicatively to all of \( \overline{T}(T(A)) \)

\[
\Delta(w_1) \cdots \Delta[w_m] := \Delta(w_1) \cdots \Delta[w_m],
\]

with \( \Delta(1) := 1 \otimes 1 \).

**Theorem 4.** The graded algebra \( \overline{T}(T(A)) \) equipped with the coproduct \( (18) \) is a connected graded noncommutative and noncocommutative Hopf algebra.

**Proof.** By construction, \( \overline{T}(T(A)) \) is a graded algebra, and the map \( (18) \) respects the gradation and is both multiplicative and counital. It remains to show that \( \Delta \) is coassociative. Note that the multiplicativity of \( \Delta \) implies that it is enough to check the property on elements of \( T(A) \).

We get:

\[
(\Delta \otimes I) \circ \Delta(a_1 \cdots a_n) = (\Delta \otimes I)(\sum_{T \subseteq [n]} a_T \otimes a_{f_T[n]}) = \sum_{S \subseteq T \subseteq [n]} a_S \otimes a_{J_S[n]} \otimes a_{f_T[n]} = (I \otimes \Delta) \circ \Delta(a_1 \cdots a_n).
\]

The crucial observation is that coproduct \( (18) \) can be split into two parts as follows. On \( T(A) \) define the **left half-coproduct** by

\[
\Delta_<(a_1 \cdots a_n) := \sum_{1 \in S \subseteq [n]} a_S \otimes a_{J_S[n]},
\]

and

\[
\Delta_>(a_1 \cdots a_n) := \Delta_+(a_1 \cdots a_n) - a_1 \cdots a_n \otimes 1.
\]

The **right half-coproduct** is defined by

\[
\Delta_+(a_1 \cdots a_n) := \sum_{1 \notin S \subseteq [n]} a_S \otimes a_{J_S[n]},
\]

and

\[
\Delta_>(a_1 \cdots a_n) := \Delta_+(a_1 \cdots a_n) - 1 \otimes a_1 \cdots a_n.
\]

Which yields \( \Delta = \Delta_+ + \Delta_\geq \), and

\[
\Delta(w) = \Delta_<(w) + \Delta_>(w) + w \otimes 1 + 1 \otimes w.
\]
This is extended to $T(T(A))$ by defining
\[ \Delta^+(w_1 | \cdots | w_m) := \Delta^+(w_1) \Delta(w_2) \cdots \Delta(w_m), \]
\[ \Delta^\circ(w_1 | \cdots | w_m) := \Delta^+(w_1) \Delta(w_2) \cdots \Delta(w_m). \]

**Theorem 5.** The bialgebra $T(T(A))$ equipped with $\Delta_\prec$ and $\Delta_\prec$ is a codendriform bialgebra.

**Proof.** From $\Delta(a|b) = \Delta(a)\Delta(b)$, we get
\[ \Delta^+(a|b) = \Delta^+(a)\Delta(b), \quad \Delta^\circ(a|b) = \Delta^+(a)\Delta(b). \]

We know that the coproduct $\Delta$ is coassociative. For an element $w_1 | \cdots | w_k \in T(T(A))$, let us write $w_i = a_{s_i^1} \cdots a_{s_i^{n_i}}$ with $S_i = \{s_i^1, \ldots, s_i^{n_i}\}$. We get:
\[ (\Delta \otimes I) \circ \Delta(w_1 | \cdots | w_k) = (I \otimes \Delta) \circ \Delta(w_1 | \cdots | w_k) \]
\[ = \sum_{X_i \subseteq T_i \subseteq S_i} (a_{X_1} | \cdots | a_{X_k}) \otimes (a_{j_{T_1}} | \cdots | a_{j_{T_k}}) \otimes (a_{j_{S_1}} | \cdots | a_{j_{S_k}}). \]

Applying $\Delta_\prec$ instead of $\Delta$ to $w_1 | \cdots | w_k$ amounts to limiting the range of variation of the $T_i$ in
\[ \Delta(w_1 | \cdots | w_k) = \sum_{T_i \subseteq S_i} (a_{T_1} | \cdots | a_{T_k}) \otimes (a_{j_{S_1}} | \cdots | a_{j_{S_k}}) \]
by requiring $1 \in T_1$ and $\bigsqcup T_i \neq \bigsqcup S_i$. Similarly for higher order compositions of half-coproduts.

Eventually, we get
- $(\Delta_\prec \otimes I) \circ \Delta_\prec(w_1 | \cdots | w_k)$ and $(I \otimes \Delta) \circ \Delta_\prec(w_1 | \cdots | w_k)$ are equal, and both are obtained by restricting the domain of the summation operator in (23) to the $X_i, T_i, S_i$ such that $1 \in X_1, \bigsqcup X_i \neq \bigsqcup T_i \neq \bigsqcup S_i$.
- $(\Delta_\prec \otimes I) \circ \Delta_\prec(w_1 | \cdots | w_k)$ and $(I \otimes \Delta_\prec) \circ \Delta_\prec(w_1 | \cdots | w_k)$ are equal, and both are obtained by restricting the domain of the summation operator in (23) to the $X_i, T_i, S_i$ such that $1 \in T_1, 1 \notin X_i, \bigsqcup X_i \neq \bigsqcup T_i \neq \bigsqcup S_i$.
- $(\Delta \otimes I) \circ \Delta_\prec(w_1 | \cdots | w_k)$ and $(I \otimes \Delta_\prec) \circ \Delta_\prec(w_1 | \cdots | w_k)$ are equal and both obtained by restricting the domain of the summation operator in (23) to the $X_i, T_i, S_i$ such that $1 \in S_1, 1 \notin T_1, \bigsqcup X_i \neq \bigsqcup T_i \neq \bigsqcup S_i$.

\[ \square \]

4. Convolution and characters

Recall that the ultimate purpose of free probability theory is the study of linear forms on $T(A)$. However, this is equivalent to the study of linear forms that are multiplicative maps $\Phi$ on $T(T(A))$. This observation motivates the present section; the link with cumulant-moments relations in free probability will be made precise in the next section.

The following proposition is the natural generalization to codendriform bialgebras of the fact that the convolution product equips the space of linear endomorphisms of a classical Hopf algebra with an associative algebra structure \cite{4}. We refer to \cite{18} for an application of these ideas to the study of the structure of shuffle bialgebras.

Indeed, recall that the set of linear maps, $\text{Lin}(T(T(A)), k)$, is a $k$-algebra with respect to the convolution product defined in terms of the coproduct \cite{18}, i.e., for $f, g \in \text{Lin}(T(T(A)), k)$
\[ f \ast g := m_k \circ (f \otimes g) \circ \Delta, \]
where \( m_k \) stands for the product map in \( k \). We define accordingly the left and right half-convolution products:

\[
\begin{align*}
\ell \lhd g & := m_k \circ (f \otimes g) \circ \Delta_\prec, \\
\ell \rhd g & := m_k \circ (f \otimes g) \circ \Delta_\succ.
\end{align*}
\]

**Proposition 6.** The space \((\Lambda := \text{Lin}(T(T(A)), k), \prec, \succ)\) is a dendrimorphic algebra.

We recall its proof: for arbitrary \( f, g, h \in \Lambda \),

\[
(f \prec g) \prec h = m_k \circ ((f \prec g) \otimes h) \circ \Delta_\prec = m_k^{[3]} \circ (f \otimes g \otimes h) \circ (\Delta_\prec \otimes I) \circ \Delta_\prec,
\]

where \( m_k^{[3]} \) stands for the product map from \( k^{(3)} \) to \( k \). Similarly

\[
f \prec (g \shuffle h) = m_k \circ (f \otimes (g \shuffle h)) \circ \Delta_\prec = m_k^{[3]} \circ (f \otimes g \otimes h) \circ (I \otimes \Delta) \circ \Delta_\prec,
\]

so that the identity \((f \prec g) \prec h = f \prec (g \shuffle h)\) follows from \((\Delta_\prec \otimes I) \otimes \Delta_\prec = (I \otimes \Delta) \circ \Delta_\prec\), and similarly for the other identities characterizing noncommutative shuffle algebras.

As usual, we equip the dendrimorphic algebra \( \Lambda \) with a unit, that is, in \( \overline{\Lambda} := \Lambda \oplus k \cong \text{Lin}(\overline{T}(T(A)), k) \), where in the last isomorphism the unit \( 1 \in \overline{\Lambda} \) is identified with the augmentation map \( e \in \text{Lin}(\overline{T}(T(A)), k) \) — the null map on \( T(T(A)) \) and the identity map on \( T(A)^{(0)} \cong k \). That is, for an arbitrary \( f \) in \( \overline{\Lambda} \),

\[
f \prec e = f = e \succ f, \quad e \prec f = 0 = f \succ e.
\]

Let now \( \phi \) be a linear form on \( T(A) \). It extends uniquely to a multiplicative linear form \( \Phi \) on \( T(T(A)) \) by setting

\[
\Phi(w_1 \cdots | w_n) := \phi(w_1) \cdots \phi(w_n),
\]

(or to a unital and multiplicative linear form on \( \overline{T}(T(A)) \)). Conversely any such multiplicative map \( \Phi \) gives rise to a linear form on \( T(A) \) by restriction of its domain.

This motivates the following definition, which generalizes to codendrimorphic bialgebras the classical link between characters and infinitesimal characters in the theory of classical Hopf algebras. For the latter, we refer to [10], where the equivalence between the two families of characters is studied in detail.

**Definition 4.** A linear form \( \Phi \in \overline{\Lambda} \) is called a character if it is unital, \( \Phi(1) = 1 \), and multiplicative, i.e., for all \( a, b \in \overline{T}(T(A)) \)

\[
\Phi(a|b) = \Phi(a)\Phi(b).
\]

A linear form \( \kappa \in \overline{\Lambda} \) is called infinitesimal character, if \( \kappa(1) = 0 \), and if for all \( a, b \in T(T(A)) \)

\[
\kappa(a|b) = 0.
\]

We write \( Ch(\kappa) \) for the obvious extension of a linear form on \( T(A) \) (e.g. the restriction to \( T(A) \) of an infinitesimal character) to a character, defined by \( Ch(\kappa)(1) := 1 \), \( Ch(\kappa)(w_1 \cdots | w_k) := \kappa(w_1) \cdots \kappa(w_k) \). Conversely, for an arbitrary \( F \in \overline{\Lambda} \), let us write \( Res(F) \) for the infinitesimal character, which is defined as the restriction of \( F \) to \( T(A) \), and the null map on other tensor powers of \( T(A) \) in \( \overline{T}(T(A)) \).
Theorem 7. There exists another natural bijection $B$ between $G(A)$, the set of characters, and $g(A)$, the set of infinitesimal characters on $T(T(A))$. More precisely, for $\Phi \in G(A)$, $\exists \kappa \in g(A)$ such that

$$\Phi = e + \kappa \prec \Phi = \exp^\prec(\kappa),$$

and conversely, for $\kappa \in g(A)$

$$\Phi := \exp^\prec(\kappa)$$
is a character.

Let us use in the following the shortcut “Hopf-type” notation $\Delta_\prec(w) =: w^{1, \prec} \otimes w^{2, \prec}$ (which is abusive, but its proper use should not result in wrong equations).

Proof. We know from Lemma 3 that the implicit equation $\Phi = e + \kappa \prec \Phi = \exp^\prec(\kappa)$ has a unique solution $\kappa$ in $\mathbb{C} \mathcal{A}$. Let us consider the infinitesimal character $\mu := \text{Res}(\kappa)$, and let us show that $\mu$ also solves $\Phi = e + \mu \prec \Phi$; the first part of the Theorem will follow.

Indeed, for an arbitrary $a \in T(T(A)), a = w_1 \cdots w_n$, notice first that by definition of the product $\prec$, and due to the vanishing of $\mu$ on any $T(A)^{\otimes k}$, for $k \neq 1$, we have:

$$(\mu \prec \Phi)(a) = \mu(w_1 \prec)\Phi(w_2 \prec | w_2 | w_1 \cdots | w_n) = \kappa(w_1 \prec)\Phi(w_2 \prec | w_2 | w_1 \cdots | w_n).$$

We immediately obtain, since

$$\Phi(w_1) = (e + \kappa \prec \Phi)(w_1) = \kappa(w_1 \prec)\Phi(w_2 \prec) = \mu(w_1 \prec)\Phi(w_2 \prec)$$

that, for any $i > 1$

$$\Phi(w_1| \cdots | w_n) = \Phi(w_1)\Phi(w_2| \cdots | w_n) = \mu(w_1 \prec)\Phi(w_2 \prec | w_2 | w_1 \cdots | w_n) = (e + \mu \prec \Phi)(w_1| \cdots | w_n),$$

from which the property follows.

Conversely:

$$\exp^\prec(\kappa)(w_1| \cdots | w_n) = (e + \kappa \prec \exp^\prec(\kappa))(w_1| \cdots | w_n) = \kappa(w_1 \prec)\exp^\prec(\kappa)(w_2 \prec | w_2 | w_1 | \cdots | w_n).$$

Assuming by induction that the property $\exp^\prec(\kappa)(w_1| \cdots | w_k) = \exp^\prec(\kappa)(w_1| \cdots \exp^\prec(\kappa)(w_k)$ holds for elements $w_1| \cdots | w_k \in T(T(A))$ of total degree less than the degree of $w_1| \cdots | w_n$, yields

$$\exp^\prec(\kappa)(w_1| \cdots | w_n) = \kappa(w_1 \prec)\exp^\prec(\kappa)(w_2 \prec | w_2 | w_1 | \cdots | w_n) = \exp^\prec(\kappa)(w_1)\exp^\prec(\kappa)(w_2 | \cdots | w_n).$$

$\square$

5. Free Cumulants as infinitesimal characters

Recall now the definition of free cumulants [1][27], which motivated our previous developments. A pair $(A, \phi)$, where $A$ is an associative $k$-algebra with unit and $\phi$ a linear form on $A$, is by definition a noncommutative probability space. The linear form is extended to $T(A)$, for all words $a_1 \cdots a_n \in A^{\otimes n} \quad \phi(a_1 a_2 a_3 \cdots a_n) := \phi(a_1 \cdot_A a_2 \cdot_A a_3 \cdots \cdot_A a_n).$

Viewing $a \in A$ as a noncommutative random variable, the moments of $A$ are defined by

$$m_n := \phi(a^n) = \phi(a_{\otimes n}),$$

whereas the free cumulants $k_n$ are obtained from the identity

$$C(zM(z)) = M(z),$$

(24)
Theorem 8. Let \( \Phi : A \rightarrow k \) be a unital map, and \( \Phi \) its extension to \( \overline{T}(T(A)) \) as above. Let the map \( \kappa : \overline{T}(T(A)) \rightarrow k \) be the infinitesimal character solving the linear fixed point equation (24) \( \Phi = e + \kappa \prec \Phi \). For \( a \in A \) we set \( k_n := \kappa(a^\otimes n) \), \( n \geq 1 \) and \( m_n := \Phi(a^\otimes n) = \phi(a^n) \), \( n \geq 0 \). Then

\[
m_n = \sum_{s=1}^{n} \sum_{\sum_{i_1 + \cdots + i_s = n-s} k_s m_{i_1} \cdots m_{i_s}},
\]

where the \( i_j \) run over the integers (i.e. the value \( i_j = 0 \) is allowed).

Our main claim is that the fixed point equation (24) is a consequence of the fixed point equation \( \Phi = e + \kappa \prec \Phi \) introduced in the previous section. Moreover, the same approach, properly abelianized, holds for classical cumulants, legitimizing in a new way the claim that free cumulants are a noncommutative version of classical cumulants.

To fix the ideas and illustrate concretely the half-shuffle approach, let us start with low-dimensional computations. Let \( \phi \) be the linear form on \( T(A) \) associated to a noncommutative probability space \( A \), and extended to \( \overline{T}(T(A)) \) multiplicatively, \( \Phi : \overline{T}(T(A)) \rightarrow k \)

\[
\Phi(1) := 1, \; \Phi(w_1 \cdots |w_m) := \phi(w_1) \cdots \phi(w_m).
\]

Let \( \kappa : \overline{T}(T(A)) \rightarrow k \) be the infinitesimal character solving the linear fixed point equation (25)

\[
\Phi = e + \kappa \prec \Phi.
\]

We calculate a few simple examples. Let \( a \in A \subset T(A) \). Then \( \Delta^+_k(a) = a \otimes 1 \), and hence, with \( \Phi(1) = 1 \)

\[
\Phi(a) = \kappa(a) = k_1.
\]

Next we look at the two letters word \( aa \in T_2(A) \). The left-coproduct reads \( \Delta^+_k(aa) = aa \otimes 1 + a \otimes a \), such that

\[
\Phi(aa) = \kappa(aa) + \kappa(a)\kappa(a) := k_2 + k_1 k_1.
\]

For \( aaa \in T_3(A) \) the left-coproduct reads

\[
\Delta^+_k(aaa) = aaa \otimes 1 + a \otimes aa + 2aa \otimes a,
\]

such that

\[
\Phi(aaa) = \kappa(aaa) + 3\kappa(aa)\kappa(a) + \kappa(a)\kappa(a)\kappa(a) = k_3 + 3k_2 k_1 + k_1 k_1 k_1.
\]

Let \( aaaa \in T_4(A) \). The left-coproduct reads

\[
\Delta^+_k(aaaa) = aaaa \otimes 1 + a \otimes aaaa + 2aa \otimes aa + aa \otimes a|a + 3aaa \otimes a.
\]

This then gives

\[
\Phi(aaaa) = \kappa(aaaa) + 4\kappa(aaa)\kappa(a) + 2\kappa(aa)\kappa(aa) + 6\kappa(aa)\kappa(a)\kappa(a) + \kappa(a)\kappa(a)\kappa(a)\kappa(a).
\]

We used that \( \Phi(a|a) = \Phi(a)\Phi(a) \). These equations coincide with the moments-cumulants relation for non-crossing partitions up to order four. More generally, we have

Theorem 8. Let \( \phi : A \rightarrow k \) be a unital map, and \( \Phi \) its extension to \( \overline{T}(T(A)) \) as above. Let the map \( \kappa : \overline{T}(T(A)) \rightarrow k \) be the infinitesimal character solving \( \Phi = e + \kappa \prec \Phi \). For \( a \in A \) we set \( k_n := \kappa(a^\otimes n) \), \( n \geq 1 \) and \( m_n := \Phi(a^\otimes n) = \phi(a^n) \), \( n \geq 0 \). Then

\[
m_n = \sum_{s=1}^{n} \sum_{\sum_{i_1 + \cdots + i_s = n-s} k_s m_{i_1} \cdots m_{i_s}}.
\]

In particular, the \( k_n \) identify with the free cumulants of \( a \in (A, \phi) \).
Proof. Indeed, notice first that subsets \( \{ 1 = s_1, \ldots, s_i \} = S \subseteq [n] \) are in bijection with sequences of (possibly null) integers of length \( i, s_2 - s_1 - 1, \ldots, s_i - s_{i-1} - 1, n - s_i \), and of total sum \( n - i \). The nonzero terms of the sequence compute the lengths of the connected components of \( [n] \setminus S \) in \( [n] \). We get

\[
\Delta_{\prec}(a^\otimes n) = \sum_{i=1}^{n} a_i^\otimes j_1 \otimes \cdots \otimes a_i^\otimes j_i,
\]

with the convention that tensor powers \( a^\otimes 0 \) have to be erased.

Applying this to \( \Phi(a^\otimes n) = (e + \kappa \prec \Phi)(a^\otimes n) \), we get the expected identity

\[
m_n = \sum_{s=1}^{n} \sum_{i_1 + \cdots + i_s = n - s} k_n m_{i_1} \cdots m_{i_s}.
\]

\( \square \)

6. Classical Cumulants from half-unshuffles.

As the notion of codendrimorphic coalgebra is dual to the one of dendrimorphic algebra, one can dualize the notion of shuffle algebra. It is a codendrimorphic coalgebra, in which \( \Delta_{\prec} = \tau \circ \Delta_{\succ} \). Here \( \tau \) denotes the twist map, \( \tau(x \otimes y) := y \otimes x \). We will call a codendrimorphic bialgebra satisfying this property a dual shuffle bialgebra or unshuffle bialgebra.²

In the following, \( A \) is an arbitrary unital associative algebra equipped with a linear form \( \phi : A \to k \) extended, as in the previous section, to a unital linear form on \( T(A) \) by \( \phi(a_1 a_2 \cdots a_n) := \phi(a_1 \cdot A a_2 \cdot A \cdots \cdot A a_n) \). However, since our interest is oriented toward the moment/cumulant relation, the reader should have in mind for \( A \) a commutative algebra of scalar random variables admitting moments of all orders, and for \( \phi = E \) the expectation operator. In the later case, the moments of \( a \in A \) are given by \( m_n := E(a^n) \), and the series of cumulants \( c_n \) is determined through

\[
\sum_{n \geq 1} c_n \frac{z^n}{n!} := \log\left( \sum_{n \geq 0} m_n \frac{z^n}{n!} \right).
\]

We will use, however, the equivalent definition of cumulants by means of the equations

\[
(26) \quad m_n = c_n + \sum_{m=1}^{n-1} \binom{n-1}{m-1} c_m m_{n-m}.
\]

The unshuffle bialgebra structure on \( \overline{T}(A) \) is defined by dualizing the one of shuffle algebra on \( T(A) \) (or simply by using the concatenation product on \( T(A) \), and by splitting the unshuffle coproduct into two pieces). We refer to [25] for details on the shuffle bialgebra and its dual.

Concretely, the cocommutative coproduct is given by \( \Delta : \overline{T}(A) \to T(A) \otimes T(A) \)

\[
\Delta_{\prec}(a_1 \cdots a_n) := \sum_{J \subseteq [n]} a_J \otimes a_{[n] \setminus J}.
\]

This coproduct splits into left and right half-coproducts

\[
(27) \quad \Delta_{\prec}(a_1 \cdots a_n) := \sum_{1 \in J \subseteq [n]} a_J \otimes a_{[n] \setminus J}
\]

²We feel that this terminology is more natural than the one of cozinbiel Hopf algebra used, e.g., in Fischer’s thesis, to which we refer for details on the subject [16].
and

\[ \Delta^\omega (a_1 \cdots a_n) := \tau \circ \Delta^\omega \Delta^\omega (a_1 \cdots a_n) = \sum_{1 \in J \subset [n]} a_J \otimes a_{[n]-J}. \]

Together with the concatenation product, these maps define a structure of unshuffle bialgebra on \( T(A) \).

Notice that the relation \( \Delta^\omega = \tau \circ \Delta^\omega \) implies that, for arbitrary \( f, g \in \text{Lin}(T(A), k) \), we have:

\[ f \prec g = g \succ f, \]

with the usual conventions \( f \prec g := m_k \circ (f \otimes g) \circ \Delta \) respectively \( f \succ g := m_k \circ (f \otimes g) \circ \Delta \), so that \( \text{Lin}(T(A), k) \) is a shuffle algebra for the left and right half-convolution products \( \prec, \succ \).

Let now \( \phi : T(A) \to k \) be a unital map in \( \text{Lin}(T(A), k) \), and consider the linear fixed point equation

\[ \phi = e + c \prec \phi. \]  

Here, the map \( e : T(A) \to k \) is the identity map on \( A^{\otimes 0} \), and the null map on the other tensor powers of \( A \). Let us calculate a few examples. Let \( a \in T(A) \) be a single letter different from the empty word. Then \( \Delta^\omega (a) = a \otimes 1 \), and hence with \( \phi(1) = 1 \)

\[ \phi(a) = c(a) := c_1. \]

Next we look at the word \( aa \in T_2(A) \). The left-coproduct \( \Delta^\omega (aa) = aa \otimes 1 + a \otimes a + 2a \otimes a \), such that

\[ \phi(aa) = c(aa) + c(a)c(a) := c_2 + c_1^2. \]

For \( aaa \in T_3(A) \) the left-coproduct reads

\[ \Delta^\omega (aaa) = aaa \otimes 1 + a \otimes aaa + 2aa \otimes a, \]

such that

\[ \phi(aaa) = c(aaa) + 3c(aa)c(a) + c(a)c(a)c(a) = c_3 + 3c_2c_1 + c_1c_1c_1. \]

Let \( aaaa \in T_4(A) \). The left-coproduct reads

\[ \Delta^\omega (aaaa) = aaaa \otimes 1 + a \otimes aaaa + 2aa \otimes aaa + 3aaa \otimes a. \]

This then gives

\[ \phi(aaaa) = c(aaaa) + 4c(aaa)c(a) + 3c(aa)c(aa) + 2c(aa)c(a)c(a) + c(a)c(a)c(a)c(a) \]

These identities coincide with the moments-cumulants relations up to order four.

In general, we have

\[ \Delta^\omega \left( a^{\otimes n} \right) = \sum_{S \subseteq [n]} a^{\otimes |S|} \otimes a^{\otimes n-|S|} = \sum_{i=0}^{n} \binom{n}{i} a^{\otimes i} \otimes a^{\otimes n-i} \]

and

\[ \Delta^\omega \left( a^{\otimes n} \right) = \sum_{1 \in S \subseteq [n]} a^{\otimes |S|} \otimes a^{\otimes n-|S|} = \sum_{i=0}^{n-1} \binom{n-1}{i} a^{\otimes i+1} \otimes a^{\otimes n-i-1}, \]

from which, with \( c_n := c(a^{\otimes n}) \) and \( m_n := \phi(a^{\otimes n}) = \phi(a^n) \), one finds that \([29]\) gives the moment-cumulant relation

\[ m_n = \phi(a^n) = \sum_{j=0}^{n-1} \binom{n-1}{j} c_{j+1} \phi(a^{n-j-1}) = \sum_{j=0}^{n-1} \binom{n-1}{j} c_{j+1} m_{n-j-1}, \]
7. Exponentials and cumulants

Recall that the series of moments and cumulants are related by the logarithm and exponential maps. We explain why this nice relationship breaks down in the noncommutative framework of free probabilities.

Let $A$ be an arbitrary dendrimorphic algebra, and $\overline{A}$ its augmentation by a unit $e$. Let us return to Theorem 7, and recall from [11, 12], that the solution of the linear fixed point equation (31)

$$\Phi = e + \kappa \prec \Phi$$

is also given in terms of the proper exponential [9]. Indeed, it can be shown that

$$\Phi = \exp(\Omega'(\kappa)).$$

The map $\Omega'(\kappa)$ is called pre-Lie Magnus expansion. Let us mention that it can also be understood from the point of view of enveloping algebras of pre-Lie algebras [7]. It obeys the following recursive equation

$$\Omega'(\kappa) = \frac{L_{\Omega'}}{\exp(L_{\Omega'}) - 1}(\kappa) = \sum_{m \geq 0} \frac{B_m}{m!} L_{\Omega'}^m(\kappa),$$

where the $B_m$'s are the Bernoulli numbers. We recall that $L_{\Omega'}(\kappa) = \kappa - \frac{1}{2} \kappa \triangleright \kappa + \sum_{m \geq 2} \frac{B_m}{m!} L_{\Omega'}^m(\kappa)$.

$\Omega'$ reduces in that case to the identity map, i.e., $\Omega'(c) = c$. Hence, in a shuffle algebra the exponential solution of (31) reduces to

$$\phi = \exp(\mu(c)).$$

This phenomenon is strictly analogous to what happens with ordinary scalar and matrix first order linear differential equations. Indeed, the first ones are solved by the exponential map, whereas the latter are solved by means of the Magnus formula, see e.g. [12] for details.

Let us now consider the case $D = Lin(T(A), k)$ and show how this last formula expands combinatorially, which allows to recover the usual exponential computation of the generating series of moments from the one of cumulants. Notice first that, due to the set theoretical definition of the unshuffle coproduct in $T(A)$, for an arbitrary $a \in A$, we have

$$c_{\triangleright}(a^\otimes n) = m_k(c \otimes c) \Delta_{\triangleright}(a^\otimes n) = \sum_{p+q=n} \binom{n}{p} c(a^\otimes p)c(a^\otimes q) = \sum_{p+q=n} \binom{n}{p} c_{p}c_{q},$$

and more generally

$$c_{\triangleright}(a^\otimes n) = \sum_{i_1 + \cdots + i_k = n} \binom{n}{i_1, i_2, \ldots, i_k} c_{i_1} \cdots c_{i_k}.$$

We get

$$m_n = \phi(a^\otimes n) = \sum_{k \geq 1} \frac{1}{k!} \sum_{i_1 + \cdots + i_k = n} \binom{n}{i_1, i_2, \ldots, i_k} c_{i_1} \cdots c_{i_k},$$
which is the degree $n$ component of the cumulant/moment relation (26).

In conclusion, for a given noncommutative probability space $(A, \phi)$, the character $\Phi \in G(A) \subset \text{Lin}(\mathcal{T}(\mathcal{T}(A)), k)$, which is defined as a multiplicative extension of the moment linear form $\phi$, can be written as the solution $\Phi = \exp_{\mathcal{T}}(\Omega'(\kappa))$ of the linear fixed point equation (25). The infinitesimal character $\kappa \in g(A)$ defines free cumulants. The classical analog of this situation is defined over $\text{Lin}(\mathcal{T}(A), k)$. The moment map is then given in terms of the cumulants map via the commutative exponential, $\Phi = \exp_{\mathcal{T}}(c)$. From this perspective, the difference between free and classical cumulants-moments is once again displayed in the noncommutative and commutative $k$-algebras $\text{Lin}(\mathcal{T}(\mathcal{T}(A)), k)$ and $\text{Lin}(\mathcal{T}(A), k)$, respectively.

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