Brainstorming about the geometric structures in the Wolfram model of fundamental physics

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Abstract

In this essay, we explore the geometric structures involved in the Wolfram model of fundamental physics. Furthermore, we propose some directions of research aiming to get the bosons and fermions out of this framework.

1 Introduction

The mathematical problems that have been solved or techniques that have arisen out of physics in the past have been the lifeblood of mathematics.

Michael Atiyah

On April 14, 2020, the physicist and CEO of Wolfram Research, S. Wolfram, announced on the official YouTube channel [Wolc] of his company his ambitious project [Wolb] of deriving fundamental physics from a renewed version of the computational framework that he developed his book *A new kind of science* [Wol02]. Wolfram’s approach has been previously criticized
by S. Aaronson [Aar02] by showing some weak points concerning relativity and quantum mechanics. Nevertheless, this time, Wolfram’s team provided two papers arguing that, in their current model, both relativity [Gor20b] and quantum mechanics [Gor20a], emerge in a rather natural way.

Wolfram’s approach to physics can be described as an inversion of the first line in D. Deutsch’s and R. Jozsa’s paper [DJ92] on quantum algorithms:

The operation of any computing machine is necessarily a physical process.

Indeed, in the Wolfram model, all the physical processes emerge from the operations of an “immaterial” computer. This computer is immaterial in the same sense that the strings of string theory are: their status is more fundamental than that of the elementary particles that constitute matter.

In order to know of how the Wolfram model fits into the framework of mainstream physics, we will use a summary of this science due to E. Witten, one of the most qualifiers experts for such a difficult task. Following E. Witten [Wit87 page 280], the main geometric structures in which fundamental physics is built are:

(i) A pseudo-Riemannian manifold $M$, modeling spacetime.

(ii) A principal bundle $X$ over $M$ with a non-abelian structure group $G$ (gauge group), responsible for the interactions among matter particles [Has13 page 1101].

(iii) An associated bundle, specially a Clifford or a spinor bundle, responsible of the matter in the universe.

The way in which the Wolfram model fit into the point (i) was already discussed in J. Gorard’s paper [Gor20b]. Concerning the points (ii), J. Gorard wrote [Gor20a page 45]:

[...] local gauge invariance in the multiway evolution [...] follows as an inevitable consequence of causal invariance of the underlying replacement rules.

1A clever philosopher may find a way to justify that both the strings and the computations in the Wolfram’s model are material in some sense, but we will avoid this discussion in the present essay.
The aim of this essay is to explore the role of principal and associated bundles in the Wolfram model. In section 1, we give an overview of the Wolfram model. In section 2, we explain how spacetime emerges from the rewriting rules. In section 3, we show how principal bundles are related to the bosons and how they may emerge in the Wolfram model. In section 4, we explain the link between associated bundles and fermions and propose a possible way to get these particles in the Wolfram model.

All the results in this essay are due to other researchers, the only role of the author is to emphasize their connection. The author is only interested in the mathematical structures and remains neutral concerning the physical consequences. The suggestions of directions of research in the Wolfram model are not precise conjectures, but just brainstorming.

2 Emergence of complexity

In 1944, E. Schrödinger [Sch67] pointed out the apparent paradox that living systems increase their organization whereas it is expected for any closed system to converge to a state of thermodynamic equilibrium. In 1981, S. Wolfram began a systematic study of how complex patterns emerge in nature despite the second law of thermodynamics. In 1983, he published a paper [Wol83] about the thermodynamic properties of extremely simple computations, e.g., the so-called Rule 30 cellular automaton, shown in figure 1, which is manifested in nature [Coo09] as the patterns in the cone snail species *Conus textile* shown in figure 2.

![Figure 1: Rule 30 cellular automaton.](image)
In a speculative twist, from the observed fact that simple computations may produce extremely complex patterns, S. Wolfram \cite{Wol02} conjectured that this is the only way in which all the complexity of the universe emerges. In particular, the differential geometric structures which are used to describe fundamental physics should be - according to Wolfram - asymptotic approximations of the long term behavior of simple computations happening at the smallest scales in the universe, far beyond the Plank length.

In the Wolfram model, the arena where all physical processes take place is a combinatorial structure known as spacial hypergraph (hypergraph for short), which is just a visual way to represent the set of all nonempty ordered subsets of a given set. There is a rewriting rule, known as the hypergraph replacement rule (rule for short\footnote{When we loosely speak about the properties of a “rule”, or a “which hypergraph replacement rule”, when we want to make a distinction with other uses of the word “rule”, we assume some initial conditions together with the rule.}), which determine how the hypergraph should be transformed at each step in the computation. Sometimes this rule is non-deterministic and there are several way of doing the replacement, which are recorded in a graph known as the causal graph.

The expressiveness of the Wolfram model have been criticized by Scott Aaronson \cite{Bec20}:

"Its this sort of infinitely flexible philosophy where, regardless of what anyone said was true about physics, they could then assert, ‘Oh, yeah, you could graft something like that onto our model’."
Nevertheless, the Wolfram model is not a specific theory of how Nature behaves in detail but a framework for such a theory: each statement “this rule describes the universe” is a theory in the Wolfram model. When asked the question “How could your model be proved wrong?”, S. Wolfram [Wola] answered:

Any particular rule could be proved wrong by disagreeing with observations, for example predicting particles that do not exist. But the overall framework of our models is something more general, and not as directly amenable to experimental falsification. Asking how to falsify our framework is similar to asking how one would prove that calculus could not be a model for physics. An obvious answer would be another model successfully providing a fundamental theory of physics, and being proved incompatible.

The following hierarchy, in increasing order according to the level of structure, seems to be a natural way to proceed in the mathematical development of the Wolfram Physics Project:

1. Defining and proving **combinatorial** properties of the rules, including graph-theoretical properties.
2. Defining and proving **algebraic topological** properties of the rules, including the cohomology and the homotopy of the hypergraph.
3. Defining and proving **differential geometric** properties of the rules, e.g., when the spacial hypergraph converges to a manifold.
4. Defining and proving **physical** properties of the rules, e.g., the emergence of particles, in particular, of both bosons and fermions.

For example, in order to study the electrons (fundamental physics) using the Wolfram model, we need to find a rule from which the spinors emerge (differential geometry), but in order to do that, we need a rule in which the second Stiefel-Whitney class of spacetime vanishes (algebraic topology) and in order to define the Stiefel-Whitney classes we need to study the combinatorial properties of the hypergraph (discrete mathematics).

The study of the properties of the rules is an extremely difficult subject. Nevertheless there have been some progress motivated by the prizes of $
25,000 and $30,000 offered by S. Wolfram for proofs of some conjectures concerning the rules.

M. Gromov [Gro92a] suggested several ways to rigorously study the asymptotic geometry emerging from a change in scale, also known as looking the “space at infinite”. I. Polterovich, A. Shnirelman [PS98], obtained the space at infinite of the hyperbolic plane. These techniques may be useful in order to formalize and prove of the observed asymptotic behaviors of some rules.

3 Emergence of spacetime

In some ancient civilizations, like the Greece of Plato [Pla19], space and time were considered as separated entities. In other Ancient civilizations, like the Incas (South-America) space and time were regarded as a single concept [Qui].

In the mainstream approach to physics the tradition is to consider space and time as a continuum, named spacetime. This unification was due to H. Minkowski (1908) and it was exploited by A. Einstein (1915) in his theory of gravity. Indeed, for A. Einstein, what it seems to be the gravitational force is just the effect of the deformation of spacetime caused by a massive object. In retrospective, it is also possible to reformulate Newtonian gravity as a theory of curved spacetime, where the curvature is, roughly speaking, in the time direction. So, the curvature of spacetime in general relativity is not the consequence of relativity, in the sense of finite maximum speed, but of gravity.

The Lagrangian of general relativity is [Wit87]

\[ S_{GR} = \frac{1}{16\pi G} \int_M R, \]

where \( R \) is Ricci scalar of \( M \) and \( G \) is the gravitational constant.

One of the standpoints in S. Wolfram’s approach to physics is to separate space and time as it was before H. Minkowski. Indeed, according to S. Wolfram [Wolb]:

If I wanted to pick a possible wrong term in the history of physics, which is probably about 100 years ago, it would be when people started saying that space and time are the same kinds of things.

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3 https://www.wolframscience.com/prizes/tm23/
4 https://www.rule30prize.org/
In the Wolfram model, the universe is like a computer and time is one step in a (discrete) computation. This “computer” is processing information, but unlike our daily life experience, this information is not an abstraction realized by a physical device. On the contrary, the evolution of this rather immaterial information, which precedes everything except time, is what generates space. Furthermore, space generates the elementary particles and reality as we know it emerges. In particular, according to them, Minkowski spacetimes emerges as the result of the “immaterial” computation that they postulated as fundamental.

According to S. Wolfram [Wolc], in the same way that the fluid appearance of water emerges from the interaction of discrete components ($H_2O$ molecules), the illusion of living in a continuum spacetimes is caused by the fact that the combinatorial structures conforming the universe asymptotically behave as manifolds. So, there is the possibility that the dimension of the physical space is not actually $3$, but just an approximation of $3$, e.g., $2.99999$ or $3.00001$. Indeed, Wolfram asked for a generalization of the concept of manifold such that the dimension can be fractal and it may change from one point to another. Also, he asked for a generalized notion of a Lie group associated to such a fractal manifold which becomes a traditional Lie group when the dimension is a positive integer. Furthermore, S. Wolfram [Wolb] and J. Gorard [Gor20b] conjectured that general relativity, in his model, may be reformulated using the fractal dimension of the tangent space in place of using curvature as usual. This idea came from Laurent Nottale’s scale relativity [Not11].

The mathematical formalization of the “flat” spacetime of special relativity is as a 4-dimensional real vector space endowed with a nondegenerate, symmetric bilinear form with signature $(− + + +)$. In the same vein, the “curved” spacetime of general relativity is a differentiable manifold $M$ of dimension $4$, endowed with a covariant, second-degree, symmetric tensor, known as the metric tensor and denoted $g$, which is assumed to be nondegenerate with signature $(− + + +)$.

The metric tensor $g$, which determines causality, time, distance, volume, curvature, angle, and separation of the future and the past, is a symmetric bilinear form on each tangent space of $M$ that smoothly change from point to point. Thus, even at this early stage in Witten’s summary of physics [Wit87, page 280], the presence of fiber bundles over spacetime, in this case, tangent bundles, is fundamental. The role of fiber bundles will be less and less evident in the case of bosons and fermions as we will show in the next
In general relativity, the way in which matter/energy determines the curvature of spacetime is given by Einstein field equations,

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},
\]

where \( R_{\mu\nu} \) is the Ricci curvature tensor, \( T_{\mu\nu} \) is the stress energy tensor, \( \Lambda \) is the cosmological constant and \( c \) is the speed of light in the vacuum. This equation is a generalization of the law of gravity due to I. Newton.

We have seen that the arena where physics takes place is a pseudo-Riemannian manifold (spacetime) and that its curvature is determined by the mass/energy according to Einstein field equation. Now, we will provide the law of motion of particles, known as the geodesic equation,

\[
d^2 x^\mu ds^2 + \Gamma^\mu_{\alpha\beta} dx^\alpha ds dx^\beta = 0,
\]

which is a generalization of Newton’s laws of motion.

Wolfram’s team was able to prove that their model, under some natural assumptions, satisfies Einstein field equations. Hence, the movement of a “particle”, which according to their definition is a structure of the hypergraph that is locally stable during the computation (evolution of the universe), will satisfy the geodesic equation in the asymptotic limit. The idea behind their proof is formally analogous to Chapman-Enskog theory [Gor20b, page 38].

Let’s finish the present section with an example. The hypergraph replacement rule

\[
\{\{1, 2, 2\}, \{3, 2, 4\}\} \longrightarrow \{\{1, 5, 1\}, \{5, 4, 4\}, \{3, 2, 5\}\},
\]

shown in figure 3, applied to the initial condition (spacial hypergraph)

\[
\{\{1, 1, 1\}, \{1, 1, 1\}\},
\]

after 198 steps generates the spacetime (causal graph) and the space (spacial hypergraph) shown in figures 4 and 5 respectively. The first steps in the evolution are shown in picture 6 (the first spacial hypergraph is the initial condition). The open problem is to determine whether the spacetime and the space generated by the rule converge, in some sense still to be defined, to some generalized version of pseudo-Riemannian manifolds (maybe admitting fractal dimension, smoothly changing from one point to another).
4 Emergence of bosons

Our first step toward gauge theory is to distinguish between global and local symmetries. By global symmetry physicists mean symmetries of the laws of nature, which are the same over any point of spacetime, e.g., translation and rotational invariance, whereas a local symmetry varies from point to point. Global symmetries are connected to conservation laws via Noether theorem, but this is not the case of local symmetries, e.g., the rotational invariance of the laws of motions is connected to the conservation of the angular momentum, but the $U(1)$-gauge invariance of the Lagrangian of quantum electrodynamics does not produce a new conservation law [Gro92b, page 957].

According to David Gross [Gro92b, page 971], global symmetries seems to be accidental features of low energy physics and they are likely to be broken. So, they are not fundamental, unlike the local symmetries, which can always be restored at high temperature or pressure.

Roughly speaking, the standard model is a $U(1) \times SU(2) \times SU(3)$ gauge theory. This description is rather inaccurate, because the gauge group should be the quotient $(U(1) \times SU(2) \times SU(3)) \Gamma$, where $\Gamma$ is one of the following additive groups

\[ \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \{0\}. \]
The determination of which of the groups above is the candidate for $\Gamma$ has not been determined yet neither by theory nor by experiments \cite{Ton17, page 1].

According to E. Witten \cite{Wit87}, over the spacetime $M$ there is a principal bundle $X$, with a structure group $G$ such that the Lie algebra of $G$ contains the Lie algebra of $U(1) \times SU(2) \times SU(3)$ (Witten made no claim concerning the global structure of $G$). The Lagrangian given by Yang-Mills theory is \cite{Wit87}

$$S_{YM} = -\frac{1}{4e^2} \int_M |F|^2, \quad |F|^2 = g^{ij} g^{jj'} \langle F_{ij} | F_{ij'} \rangle$$
where $F$ is the curvature of the two form of a connection in the bundle, $\langle \cdot | \cdot \rangle$ is the Cartan-Killing form on the Lie algebra of $G$ and $e$ is the Yang-Mills coupling constant.

A principal bundle \[\text{[Ble81, page 26]}\] is a 4-tuple $(X, G, \pi, M)$, where $X$ (total space) and $M$ (base space) are differentiable manifolds, $G$ (structure group) is a Lie group and $\pi : X \to M$ (projection) is a surjective differentiable map, satisfying the following following properties:

(A) The structure group acts freely and differentiably on the total space to the right.

(B) Any fiber is diffeomorphic to the structure group, but there is no natural group structure on the fiber.

(C) The total space admits a local trivialization.

It is important to remark that there is an essential distinction between the gauge gravitation theory and the different versions of Yang-Mills theory in particle physics. On the one hand, the gravitation theory is developed on the total space of the principal bundle of the tangent frames of spacetime. On the other hand, the typical gauge transformations in Yang-Mills theory are vertical automorphisms of the total space of a principal bundle, leaving its base space fixed.

Following J. Gorard’s observation \[\text{[Gor20a, page 45]}\],

\[\text{[... for each vertex in a spatial hypergraph, there are many possible orientations in which a hypergraph replacement rule could be applied to that vertex [...], and we may interpret each such orientation as corresponding to a particular choice coordinate basis (i.e. some local section of a fiber bundle), which will subsequently place constraints on the set of possible orientations for other purely spacelike-separated rule applications. Thus, we can interpret the hypergraph itself as corresponding to some base space, with each vertex corresponding to a fiber [...]}\]

we will associate $M$ with a spacial hypergraph and the fiber over $x \in M$ with the set of orientations in which the hypergraph replacement rule could...
be applied to $x$, labeled with $x$ in order to recover this information from $X$ using $\pi$. The only ingredient that is missing is $G$, which should be associated to a finite set of permutations. One of the open problems in the Wolfram model is to formalize the convergence of this discrete system to the actual principal bundle $(X, G, \pi, M)$.

Notice that we are acting in the opposite way to C. N. Yang’s approach [Gro92b, page 958],

[local gauge] symmetry dictates the form of the interaction.

Indeed, the interaction (hypergraph replacement rule) is given to us from the beginning and we want to know the corresponding local gauge invariance, whereas, for C. N. Yang, the local gauge invariance is postulated from the beginning in order to reduce the number of “hypergraph replacement rules” (using the terminology of S. Wolfram) which can be considered in a particular situation.

5 Emergence of fermions

Using a geometric language, the main idea of Dirac’s theory of fermions is that they are the sections in the associated bundle (spinor bundle) of a spin group principal bundle and the space of spinors as fiber. Let’s begin with the rather technical definition of associated bundle.

Let $G$ be a Lie group. Given a left group action of $G$ on a differentiable manifold $F$ and a principal $G$-bundle $X \rightarrow M$, the associated bundle to $X \rightarrow M$ with fiber $F$ is the fiber bundle given by

$$\frac{X \times F}{\sim} \rightarrow M,$$

where the quotient $\frac{X \times F}{\sim}$ is determined by the equivalence relation defined on $X \times F$ by

$$\forall g \in G. \quad (x, f) \sim (xg, g^{-1}f).$$

The spinor bundle mentioned at the beginning of this section is a vector bundle whose fiber at each point is the Clifford module generated by Dirac $\gamma$-matrices [Wit87, page 274]. The possibility of defining this bundle can be characterized by the second Stiefel-Whitney class of spacetime, which is a topological invariant.
We recall that the Stiefel-Whitney classes of a vector bundle $\pi : X \to M$ are the cohomology classes $w_k(\xi) \in H^k(M; \mathbb{Z}/2\mathbb{Z})$ verifying the following axioms:

(i) $w_0(\pi) = 1_M$.

(ii) $w_k(\pi) = 0$ if $\pi$ is an $n$-dimensional vector bundle and $k > n$.

(iii) $w_k(\pi) = f^*(w_k(\eta))$ if there is a bundle map $\pi \to \eta$ with base map $f$ (naturality).

(iv) $w_k(\pi \oplus \eta) = \sum_{j=0}^k w_{k-j}(\pi) w_j(\eta)$ (Whitney product).

(v) $w_1(\gamma_1^1) \neq 0$, where $\gamma_1^1$ is the canonical line bundle over the projective space $\mathbb{P}^1$ (nontriviality).

In order to study whether or not a given hypergraph replacement rule generates fermions in the limit, it is natural to follow the steps:

1. Define a cohomology with coefficients $\mathbb{Z}/2\mathbb{Z}$ on the hypergraph.

2. Use the axioms above in order to define the Stiefel-Whitney classes on the hypergraph.

3. Verify whether or not the second Stiefel-Whitney class vanishes.

If it is proved that the Stiefel-Whitney class of the hypergraph vanishes, then the emergence of fermions should be expected from this rule. The cohomology may give a clue about the spinor bundle obtained in the limit.

If on the contrary, the Stiefel-Whitney class of the hypergraph does not vanish, maybe the rule under study does not correspond to the one that is supposed to describe physical reality or maybe the cohomology defined by as was not the right one and we need to change for another one. To change the cohomology means to give another interpretation of the relationship between the rule and physical reality.
6 Eric Weinstein’s Geometric Unity

Some days before S. Wolfram’s announcement [Wolc], the mathematician E. Weinstein [Weib] also announced a new approach to fundamental physics known as Geometric Unity. One recurrent question asked by people aware of both projects is whether or not there is a connection between them.

Roughly speaking, Geometric Unity is the project of developing a framework in which Einstein’s General Relativity, Yang-Mills Theory and Dirac’s Theory of Fermions can be derived from a single and yet unknown geometric principle. As E. Weinstein [Weib] pointed out, each of the fundamental equations of physics is optimal in itself, but maybe not optimal concerning the compatibility with other equations. Let’s assume that such a project is achieved and call the resulting theory also Geometric Unity for a lack of a better word. Then, according to the point of view that we developed in this essay, Geometric Unity is just a set of asymptotic restrictions that the rule corresponding to our universe should satisfy.

Using a metaphor from computer science (imperative vs declarative programming): the Wolfram model is an attempt of generating our universe via imperative programming whereas Geometric Unity is an attempt for reaching the same goal but using declarative programming. Another way to put it: E. Weinstein is looking for the constrains that the source code of our universe should satisfy whereas S. Wolfram is looking for the physical properties that general source codes satisfy in order to select the one which generates our universe.

Beside Geometric Unity, it would be interesting to compare the Wolfram model with String Theory.

7 Conclusions

The Wolfram model gives a motivation for generalizing several of concepts from differential geometry depending on the notion of integer dimension, in order to admit fractal dimension, e.g., Lie groups, pseudo-Riemannian structure, principal and associated bundles. Furthermore, it provides an intuitive framework for the systematic study of the convergence of discrete structures from computer science to continuous structures appearing fundamental physics, e.g., the convergence of the local symmetries of the hypergraph replacement rule (finite set of permutations) to the gauge invariance
(Lie group) in Yang-Mills theory.

According to the opinion of the author, independently of the role that the Wolfram Physics Project may play in the history of physics, there is not doubt that it is important in order to make connections between the mathematical methods used in fundamental physics and in computer science, specially in abstract rewriting systems, computational complexity, information theory and distributive computing. This project may be the beginning of a cross fertilization between both fields.

Summarizing the main idea of this essay in one sentence: Witten’s summary of physics \cite{Wit87, page 280} seems to be an appropriated framework in order to derive all fundamental physics using the Wolfram model. The main mathematical obstacle is to define and prove the convergence of the discrete structures in the Wolfram model to the geometric structures in mainstream physics.

To close this essay in a rather poetic way, let’s allow us to dream for one second. If the Wolfram model becomes the predominant approach to physics, maybe someday this science will be redefined as the study of the asymptotic properties of string rewriting systems.

8 Acknowledgments

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\url{https://www.wolframscience.com/nks/p27--how-do-simple-programs-behave/}

Figure 2 is licensed under the Creative Commons Attribution 2.0 Generic license \url{https://creativecommons.org/licenses/by/2.0/deed.en} The author is James St. John. The only changes made in this picture were a resize and a rotation in order to fit the text: \url{https://commons.wikimedia.org/wiki/File:Conus_textile_(textile_cone_snail)_2_(24425421465).jpg}

Figures 3, 4, 5 and 6 are copyrighted images used with permission from Wolfram Research, Inc.\cite{https://www.wolframphysics.org/universes/wm2644/}

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