Invariance of Brownian motion associated with exponential functionals

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Abstract

It is well known that Brownian motion enjoys several distributional invariances such as the scaling property and the time reversal. In this paper, we prove another invariance of Brownian motion that is compatible with the time reversal. The invariance, which seems to be new to our best knowledge, is described in terms of an anticipative path transformation involving exponential functionals as anticipating factors. Some related results are also provided.

1 Introduction

Let $B = \{B_t\}_{t \geq 0}$ be a one-dimensional standard Brownian motion and, for every $\mu \in \mathbb{R}$, denote by $B^{(\mu)} = \{B^{(\mu)}_t := B_t + \mu t\}_{t \geq 0}$ the Brownian motion with drift $\mu$. We define $\{A^{(\mu)}_t\}_{t \geq 0}$ to be the quadratic variation of the geometric Brownian motion $\{e^{B^{(\mu)}_t}\}_{t \geq 0}$, namely,

$$A^{(\mu)}_t := \int_0^t e^{2B^{(\mu)}_s} \, ds.$$ 

These exponential functionals of Brownian motion have importance in a variety of fields in probability theory such as mathematical finance, diffusion processes in random media, and probabilistic studies of Laplacians on hyperbolic spaces; see the detailed surveys [8, 9] by Matsumoto and Yor.

Let $C([0, \infty); \mathbb{R})$ be the space of continuous functions $\phi : [0, \infty) \to \mathbb{R}$, on which we define

$$A_t(\phi) := \int_0^t e^{2\phi_s} \, ds, \quad t \geq 0,$$

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so that $A^\mu_t = A_t(B^\mu_t)$, $t \geq 0$. When $\mu = 0$, with slight abuse of notation, we simply write $A_t$ for $A^0_t$. For every fixed $t > 0$, we have introduced in [5] the family $\{T^t_z\}_{z \in \mathbb{R}}$ of anticipative path transformations defined by

$$T^t_z(\phi)(s) := \phi_s - \log \left\{ 1 + \frac{A_s(\phi)}{A_t(\phi)} \left( e^z - 1 \right) \right\}, \quad 0 \leq s \leq t, \quad (1.1)$$

mapping the space $C([0, t]; \mathbb{R})$ of real-valued continuous functions $\phi$ over $[0, t]$ into itself. If there is no risk of ambiguity, we suppress the superscript $t$ from the notation and suppose that each $T_z$ acts on $C([0, t]; \mathbb{R})$. Let $\beta = \{\beta(s)\}_{s \geq 0}$ be another one-dimensional standard Brownian motion that is independent of $B$. In [5], we have shown the following identity in law which exhibits an invariance of the law of Brownian motion in the presence of the independent element $\beta$: for every $x \in \mathbb{R}$, the process

$$T_{x+B_t-\text{Argsh}(e^{B_t} \sinh x + \beta(A_t))}(B)(s), \quad 0 \leq s \leq t, \quad (1.2)$$

is identical in law with $\{B_s\}_{0 \leq s \leq t}$. Here

$$\text{Argsh} x \equiv \log \left( x + \sqrt{1 + x^2} \right), \quad x \in \mathbb{R},$$

is the inverse function of the hyperbolic sine function. The above invariance extends Bougerol’s celebrated identity in law ([1]):

$$\beta(A_t) \overset{(d)}{=} \sinh B_t; \quad (1.3)$$

indeed, evaluating (1.2) at $s = t$ and taking $x = 0$ leads to $\text{Argsh} \beta(A_t) \overset{(d)}{=} B_t$, hence to (1.3).

This paper is a continuation of [5] and aims at providing an invariance similar to the above but without the presence of an independent element. We retain $t > 0$ fixed and define another anticipative path transformation $\mathcal{T} \equiv \mathcal{T}^t$ on $C([0, t]; \mathbb{R})$ by

$$\mathcal{T}(\phi)(s) := T_{2\phi_t}(\phi)(s), \quad 0 \leq s \leq t, \quad (1.4)$$

for $\phi \in C([0, t]; \mathbb{R})$. Then, as in the theorem below, the law of Brownian motion is invariant under $\mathcal{T}$, which, as far as we are concerned, has remained unseen until now.

**Theorem 1.1.** It holds that

$$\{\mathcal{T}(B)(s)\}_{0 \leq s \leq t} \overset{(d)}{=} \{B_s\}_{0 \leq s \leq t}.$$
Theorem 1.1. It holds that
\[ \left\{ \frac{1}{A_s} + e^{2B_t} \right\}_{0 < s \leq t} = \left\{ \frac{1}{A_s} \right\}_{0 < s \leq t}. \]

The expression of the left-hand side is due to Proposition 2.1(ii). Notice that, thanks to the time reversal of Brownian motion,
\[ \left( e^{2B_t} \right)_{A_t} = 1_{A_t}; \tag{1.5} \]
see Remark 2.2. From the above identity in law, we see in particular that
\[ \mathbb{E}\left[ e^{2B_t} - 1_{A_t} \right] = 0, \]
which is consistent with Theorem 1.1. As for the integrability of the random variables
\[ 1/As, s > 0, \]
we refer the reader to [2]. (In fact, it is true that \( \mathbb{E}[\exp\{\theta/(2A_s)\}] < \infty \) for all \( \theta < 1 \), whenever \( s > 0 \); see, e.g., [5, Lemma 4.2].) We also note that identity (1.5) is a particular case of Theorem 1.1 evaluated at \( s = t \).

As a corollary to Theorem 1.1, by noting that \( T(B(t)) = -B_t \) (see Proposition 2.1(i)), the Cameron–Martin formula immediately entails the following relation between Brownian motions with opposite drifts:
\[ \left\{ T\left(B(-\mu)\right)(s)\right\}_{0 \leq s \leq t} \overset{(d)}{=} \left\{ B^\mu_s \right\}_{0 \leq s \leq t}. \tag{1.6} \]
for every \( \mu \in \mathbb{R} \). Thanks to the fact that \( T \) is an involution, we may put the above identity in law into a more symmetric form as stated in the corollary below.

Corollary 1.1. For every \( \mu \in \mathbb{R} \), it holds that
\[ \left\{ (T(B(-\mu))(s), B^\mu_s) \right\}_{0 \leq s \leq t} \overset{(d)}{=} \left\{ (B^\mu_s, T(B(-\mu))(s)) \right\}_{0 \leq s \leq t}. \]

Let \( \mu > 0 \) and set \( A^{(-\mu)} := \lim_{t \to \infty} A_t^{(-\mu)} \). Dufresne’s identity in law \([4\text{, Proposition 4.4.4(b)]}\] asserts that
\[ A^{(-\mu)} \overset{(d)}{=} \frac{1}{2\gamma_\mu}, \tag{1.7} \]
where \( \gamma_\mu \) is a gamma random variable with parameter \( \mu \):
\[ \mathbb{P}(\gamma_\mu \in dx) = \frac{1}{\Gamma(\mu)} x^{\mu - 1} e^{-x} dx, \quad x > 0. \]
Here \( \Gamma(\cdot) \) is the gamma function. Recall from Donati-Martin–Matsumoto–Yor [3] the family \( \{T_\alpha\}_{\alpha \geq 0} \) of (non-anticipative) path transformations on \( C([0, \infty); \mathbb{R}) \) defined by
\[ T_\alpha(\phi)(s) := \phi_s - \log \{1 + \alpha A_s(\phi)\}, \quad s \geq 0, \quad \phi \in C([0, \infty); \mathbb{R}). \tag{1.8} \]
Another distributional relationship between \( B^{(\mu)} \) and \( B^{(-\mu)} \) is shown in Matsumoto–Yor [7], which, in terms of the above notation, is stated as
Proposition 1.1 ([7, Theorem 2.3]). Suppose that \( \mu > 0 \). Then it holds that

\[
\left\{ \left( B_s^{(-\mu)} - \log \left( 1 - \frac{A_s^{(-\mu)}}{A_\infty^{(-\mu)}} \right), B_s^{(-\mu)} \right) \right\}_{s \geq 0} \overset{(d)}{=} \left\{ \left( B_s^{(\mu)}, T_{2\gamma_\mu}(B_s^{(\mu)})(s) \right) \right\}_{s \geq 0},
\]

where, on the right-hand side, \( \gamma_\mu \) is independent of \( B \).

In Section 4, we see that relation (1.9) may be deduced from Corollary 1.1. Combining the above proposition with Corollary 1.1, we also obtain the following joint invariance of the law of Brownian motion with drift in the presence of an independent gamma random variable.

Proposition 1.2. Let \( \mu > 0 \) and suppose that \( \gamma_\mu \) is independent of \( B \). Fix \( t > 0 \).

(1) Denote \( T_{\log\left(e^{2B_t^{(\mu)}}/\left(1+2\gamma_\mu A_t^{(\mu)}\right)\right)}(B^{(\mu)}) \) by \( X^1 \):

\[
X^1_s := B_s^{(\mu)} - \log \left\{ 1 + \frac{A_s^{(\mu)}}{A_t^{(\mu)}} \left( \frac{e^{2B_t^{(\mu)}}}{1 + 2\gamma_\mu A_t^{(\mu)}} - 1 \right) \right\}, \quad 0 \leq s \leq t.
\]

Then it holds that

\[
\left\{ (X^1_s, B_s^{(\mu)}) \right\}_{0 \leq s \leq t} \overset{(d)}{=} \left\{ (B_s^{(\mu)}, X^1_s) \right\}_{0 \leq s \leq t}.
\]

(2) Denote \( T_{\log\left(e^{2B_t^{(-\mu)}+2\gamma_\mu A_t^{(-\mu)}}\right)}(B^{(-\mu)}) \) by \( X^2 \):

\[
X^2_s := B_s^{(-\mu)} - \log \left\{ 1 + \frac{A_s^{(-\mu)}}{A_t^{(-\mu)}} \left( e^{2B_t^{(-\mu)}} + 2\gamma_\mu A_t^{(-\mu)} - 1 \right) \right\}, \quad 0 \leq s \leq t.
\]

Then it holds that

\[
\left\{ (X^2_s, B_s^{(-\mu)}) \right\}_{0 \leq s \leq t} \overset{(d)}{=} \left\{ (B_s^{(-\mu)}, X^2_s) \right\}_{0 \leq s \leq t}.
\]

Identity (1.10) suggests that the process

\[
\log \left\{ 1 + \frac{A_s^{(\mu)}}{A_t^{(\mu)}} \left( \frac{e^{2B_t^{(\mu)}}}{1 + 2\gamma_\mu A_t^{(\mu)}} - 1 \right) \right\}, \quad 0 \leq s \leq t,
\]

and hence its derivative with respect to \( s \) as well, is symmetric, which is of independent interest; a similar remark also applies to (1.11). Here we say that a real-valued continuous process \( X = \{ X_s \}_{0 \leq s \leq t} \) over the interval \([0, t]\) is symmetric if \( -X \overset{(d)}{=} X \). Moreover, by letting \( t \to \infty \) in (1.11), we see in particular that the process

\[
B_s^{(-\mu)} - \log \left\{ 1 + A_s^{(-\mu)} \left( 2\gamma_\mu - \frac{1}{A_\infty^{(-\mu)}} \right) \right\}, \quad s \geq 0,
\]
is identical in law with $B^{(-\mu)}$, which recovers [5] Proposition 4.1. A further remark on Proposition 1.2 is forwarded to Remark 4.1 after the end of the proof of the proposition.

The rest of the paper is organized as follows. In Section 2 we summarize properties of the transformation $T$ defined by (1.4), which are referred to throughout the paper. In Section 3 we prove Theorem 1.1 and Corollary 1.1; we give two proofs of Theorem 1.1 in Subsections 3.1 and 3.2 while Corollary 1.1 is proven in Subsection 3.3. In Section 4 after giving a proof of Proposition 1.1 by means of Corollary 1.1, we prove Proposition 1.2.

In the final section, we provide some related results such as extensions of Theorem 1.1 deduced from Proposition 1.2.

For each $t > 0$, we denote by $C([0, t]; \mathbb{R}^2)$ the space of $\mathbb{R}^2$-valued continuous functions over $[0, t]$. We equip each of the two spaces $C([0, t]; \mathbb{R})$ and $C([0, t]; \mathbb{R}^2)$ with topology of uniform convergence; real-valued functionals on these spaces are said to be measurable if they are Borel-measurable with respect to those topologies. In the sequel, unless otherwise specified, $t > 0$ is fixed and the path transformations $T$ and $T_z$, $z \in \mathbb{R}$, refer to those on $C([0, t]; \mathbb{R})$ defined respectively by (1.4) and (1.1).

2 Properties of the transformation $T$

In this section, based on those of $T_z$, $z \in \mathbb{R}$, investigated in [5], we explore properties of the transformation $T$.

Following the notation used in [3] by Donati-Martin, Matsumoto and Yor, we define the path transformation $Z$ by

$$Z_t(\phi) := e^{-\phi_t} A_t(\phi), \quad t \geq 0,$$

(2.1)

for $\phi \in C([0, \infty); \mathbb{R})$; as in the case of the exponential additive functional $\{A_t\}_{t \geq 0}$ of $B$, we will simply write $Z_t$ for $Z_t(B)$ for each $t \geq 0$. Notice that the two transformations $A$ and $Z$ on $C([0, \infty); \mathbb{R})$ are related via

$$\frac{d}{dt} \frac{1}{A_t(\phi)} = -\left\{\frac{1}{Z_t(\phi)}\right\}^2, \quad t > 0,$$

(2.2)

or, equivalently, for an arbitrarily fixed $t > 0$,

$$\frac{1}{A_s(\phi)} = \int_s^t \frac{du}{\{Z_u(\phi)\}^2} + \frac{e^{-\phi_u}}{Z_t(\phi)}, \quad 0 < s \leq t,$$

(2.3)

for any $\phi \in C([0, \infty); \mathbb{R})$.

Again, we fix $t > 0$ from now on, and let the path transformations $T$ and $T_z$, $z \in \mathbb{R}$, be those on $C([0, t]; \mathbb{R})$. Accordingly, the two transformations $A$ and $Z$ are restricted on $C([0, t]; \mathbb{R})$. We also consider the time reversal, which we denote by $R$, defined by

$$R(\phi)(s) := \phi_{t-s} - \phi_t, \quad 0 \leq s \leq t, \quad \phi \in C([0, t]; \mathbb{R}).$$

(2.4)
It is well known that the law of \( \{B_s\}_{0 \leq s \leq t} \) is invariant under \( R \):

\[
\{R(B)(s)\}_{0 \leq s \leq t} \overset{(d)}{=} \{B_s\}_{0 \leq s \leq t}.
\]

(Notice that the usual time reversal of Brownian motion refers to \( \{-R(B)(s)\}_{0 \leq s \leq t} \) in our notation.) The following properties of \( T_z, \ z \in \mathbb{R} \), are investigated in [5].

**Lemma 2.1 ([5 Proposition 2.1])**. We have the following (i)–(v).

(i) For every \( z \in \mathbb{R} \) and \( \phi \in C([0, t]; \mathbb{R}) \), \( T_z(\phi)(t) = \phi_t - z \).

(ii) For every \( z \in \mathbb{R} \) and \( \phi \in C([0, t]; \mathbb{R}) \),

\[
\frac{1}{A_s(T_z(\phi))} = \frac{1}{A_s(\phi)} + \frac{e^z - 1}{A_t(\phi)}, \quad 0 < s \leq t;
\]

in particular, \( A_t(T_z(\phi)) = e^{-z}A_t(\phi) \).

(iii) \( Z \circ T_z = Z \) for any \( z \in \mathbb{R} \).

(iv) (Semigroup property) \( T_z \circ T_{z'} = T_{z+z'} \) for any \( z, z' \in \mathbb{R} \); in particular,

\[ T_z \circ T_{-z} = T_0 = \text{Id} \quad \text{for any } z \in \mathbb{R}, \]

where \( \text{Id} \) is the identity map on \( C([0, t]; \mathbb{R}) \) as in Section 7.

(v) For every \( z \in \mathbb{R} \), \( T_z \circ R \circ T_z = R \), and hence

\[ R \circ T_z = T_{-z} \circ R. \]

The above properties may be verified by a direct computation; see the proof of [5 Proposition 2.1] for details. Since, among others, property (iii) is frequently used in the paper, we provide its proof for the reader’s convenience, together with a proof of (iv) that slightly differs from the one given in [5]. Notice that, for every \( z \in \mathbb{R} \) and \( \phi \in C([0, t]; \mathbb{R}) \), we may rewrite the displayed identity in property (ii) as

\[
\frac{1}{A_s(T_z(\phi))} = \int_s^t \frac{du}{\{Z_u(\phi)\}^2} + \frac{e^{-\phi_t+z}}{Z_t(\phi)}, \quad 0 < s \leq t;
\]

because of relation (2.2) and the definition (2.1) of \( Z \).

**Proofs of (iii) and (iv) of Lemma 2.1**. (iii) In view of relation (2.2), taking the derivative with respect to \( s \) on both sides of (2.6) leads to

\[
\{Z_s(T_z(\phi))\}^{-2} = \{Z_s(\phi)\}^{-2}, \quad 0 < s \leq t,
\]

which entails the claim by the positivity of \( Z \).
(iv) It suffices to prove that, for each \( \phi \in C([0, t]; \mathbb{R}) \),

\[
A_s((T_z \circ T_{z'})(\phi)) = A_s(T_{z+z'}(\phi)), \quad 0 \leq s \leq t;
\]

indeed, once this identity is proven, then taking the derivative with respect to \( s \) on both sides verifies the claim. To this end, for every \( 0 < s \leq t \), successive use of property (iii) in (2.3) yields

\[
\frac{1}{A_s((T_z \circ T_{z'})(\phi))} = \int_s^t \frac{du}{\{Z_u(\phi)\}^2} + \frac{e^{-(T_z \circ T_{z'})(\phi)(t)}}{Z_t(\phi)}.
\]

Using property (i) twice, we see that

\[
-(T_z \circ T_{z'})(\phi)(t) = -T_{z'}(\phi)(t) + z = -\phi_t + z' + z,
\]

which proves (2.7) by relation (2.6). 

Using the above lemma, we prove

**Proposition 2.1.** The transformation \( \mathcal{T} \) has the following properties.

(i) For every \( \phi \in C([0, t]; \mathbb{R}) \), \( \mathcal{T}(\phi)(t) = -\phi_t \).

(ii) For every \( \phi \in C([0, t]; \mathbb{R}) \),

\[
\frac{1}{A_s(\mathcal{T}(\phi))} = \frac{1}{A_s(\phi)} + \frac{e^{2\phi_t} - 1}{A_t(\phi)}, \quad 0 < s \leq t;
\]

in particular, \( A_t(\mathcal{T}(\phi)) = e^{-2\phi_t}A_t(\phi) \).

(iii) \( Z \circ \mathcal{T} = Z \).

(iv) For every \( z \in \mathbb{R} \), \( \mathcal{T} \circ T_z \circ \mathcal{T} = \text{Id} \); in particular, by taking \( z = 0 \),

\[ \mathcal{T} \circ \mathcal{T} = \text{Id}. \]

Moreover,

\[ (\mathcal{T} \circ T_z)(\phi) = T_{2\phi_t - z}(\phi) \]

for any \( \phi \in C([0, t]; \mathbb{R}) \).

(v) Restricted on the space of \( \phi \)'s vanishing at the origin, \( \mathcal{T} \) is commutative with \( R \); namely, for any \( \phi \in C([0, t]; \mathbb{R}) \) with \( \phi_0 = 0 \),

\[ (R \circ \mathcal{T})(\phi) = (\mathcal{T} \circ R)(\phi). \]
Proof. In view of the definition (1.4) of $\mathcal{T}$, properties (i), (ii) and (iii) follow by taking $z = 2\phi_t$ in (i), (ii) and (iii) of Lemma 2.1, respectively.

(iv) As in the proof of Lemma 2.1(iv), in order to prove the first half of the assertion, it suffices to show that, for each $\phi \in C([0, t]; \mathbb{R})$,

$$A_s((T \circ T_z \circ T \circ T_z)(\phi)) = A_s(\phi), \quad 0 \leq s \leq t. \tag{2.9}$$

To this end, repeated use of property (i) and Lemma 2.1(i) yields

$$(T \circ T_z \circ T \circ T_z)(\phi)(t) = -(T \circ T_z)(\phi)(t) + z$$

$$= -T_z(\phi)(t) + z$$

$$= \phi_t,$$

where we used property (i) for the first and third lines and Lemma 2.1(i) for the second and fourth. Then, for every $0 < s \leq t$, we have, by property (iii) and Lemma 2.1(iii), together with relation (2.3),

$$\frac{1}{A_s((T \circ T_z \circ T \circ T_z)(\phi))} = \int_s^t \frac{du}{\{Z_u(\phi)\}^2} + \frac{e^{-\langle T \circ T_z \circ T \circ T_z \rangle(\phi)(t)}}{Z_t(\phi)}$$

$$= \int_s^t \frac{du}{\{Z_u(\phi)\}^2} + \frac{e^{-\phi_t}}{Z_t(\phi)},$$

which agrees with $1/A_s(\phi)$ by (2.3) again, proving (2.9). To show the latter half, by Lemma 2.1(iv), note that

$$(T \circ T_z)(\phi) = (T \circ T_z)^{-1}(\phi)$$

$$= (T_z \circ T)(\phi),$$

which is equal to $T_{-z+2\phi_t}(\phi)$ by the definition (1.4) of $T$ and the semigroup property in Lemma 2.1(iv).

(v) By taking $z = 2\phi_t$ in Lemma 2.1(v), we have, for every $\phi \in C([0, t]; \mathbb{R})$,

$$(R \circ T)(\phi) = T_{-2\phi_t}(R(\phi)).$$

Since $R(\phi)(t) = \phi_0 - \phi_t$ by the definition (2.1) of $R$, we may rewrite the right-hand side of the above identity as $T_{2R(\phi)(t) - 2\phi_t}(R(\phi))$, and hence as $T(R(\phi))$ when $\phi_0 = 0$. \qed

We give a remark on property (v) in the above proposition.

Remark 2.1. (1) Property (v) is compatible with Theorem 1.1 and the time reversal (2.5) of Brownian motion: both $(R \circ T)(B)$ and $(T \circ R)(B)$ are identical in law with \{$B_s\}_{0 \leq s \leq t}$ by Theorem 1.1 and (2.5).

(2) Without the restriction $\phi_0 = 0$ in property (v), we have the relation

$$(R \circ T \circ R \circ R)(\phi) = (T \circ R)(\phi).$$
for any \( \phi \in C([0, t]; \mathbb{R}) \). To see that, since \((R \circ R)(\phi)(s) = \phi_s - \phi_0, 0 \leq s \leq t\), we have \((R \circ R)(\phi)(0) = 0\) and \(R \circ R \circ R = R\). Thereby we apply property (v) to \((R \circ R)(\phi)\) to conclude that

\[
(R \circ \mathcal{T})(R \circ R)(\phi) = \mathcal{T}(R \circ R)(R \circ R)(\phi) = \mathcal{T}(R(\phi)).
\]

The composition of the transformations \( \mathcal{T} \) of different durations \( t \) is also worth mentioning.

**Proposition 2.2.** For every \( u \geq 0 \), it holds that, for any \( \phi \in C([0, t+u]; \mathbb{R}) \),

\[
\mathcal{T}^t(T^{t+u}(\phi))(s) = T_{\phi_t + T^{t+u}(\phi)(t)}(\phi)(s), \quad 0 \leq s \leq t.
\]

Since, in the case \( u = 0 \), \( \phi_t + T^{t+u}(\phi)(t) = 0 \) by Proposition 2.1(i), the above proposition gives an extension of the property \( \mathcal{T} \circ \mathcal{T} = \text{Id} \) in Proposition 2.1(iv).

**Proof of Proposition 2.2.** For every \( 0 \leq s \leq t \), we have, by the definition (1.4) of \( \mathcal{T}^t \),

\[
\mathcal{T}^t(T^{t+u}(\phi))(s) = T_{2T^{t+u}(\phi)(t)}(T^{t+u}(\phi))(s). \tag{2.10}
\]

It is also readily seen from the definition (1.4) of \( \{ T^t \}_{t \in \mathbb{R}} \) that

\[
\mathcal{T}^{t+u}(\phi)(v) = T^t_{\phi_t - T^{t+u}(\phi)(t)}(\phi)(v), \quad 0 \leq v \leq t.
\]

Therefore, by the semigroup property in Lemma 2.1(iv), the right-hand side of (2.10) is equal to

\[
T^t_{2T^{t+u}(\phi)(t) + \phi_t - T^{t+u}(\phi)(t)}(\phi)(s),
\]

which proves the claim. \( \Box \)

We end this section with a remark concerning identity (1.5).

**Remark 2.2.** By (2.5), it holds that

\[
(e^{B_t}, A_t) \overset{(d)}{=} (e^{-B_t}, e^{-2B_t} A_t).
\]

Indeed, the left-hand side is identical in law with

\[
(e^{R(B)(t)}, A_t(R(B))),
\]

which is equal to the right-hand side by the definition (2.4) of \( R \). The above identity in law may also be deduced from Theorem 1.1 in such a way that

\[
(e^{B_t}, A_t) \overset{(d)}{=} (e^{T(B)(t)}, A_t(T(B))) = (e^{-B_t}, e^{-2B_t} A_t),
\]

thanks to properties (i) and (ii) in Proposition 2.1 for the second line.
3 Proofs of Theorem 1.1 and Corollary 1.1

In this section, we prove Theorem 1.1 and Corollary 1.1. We give two proofs of Theo-
rem 1.1; although the second proof is more direct, the first one, which utilizes Lemma 3.1
below, is of interest in its own right.

3.1 First proof of Theorem 1.1

For every $x \in \mathbb{R}$, we denote by $b_x^s \{b_x^s \} \in C(0, t; \mathbb{R})$ a Brownian bridge of duration $t$ starting
from 0 and ending at $x$. Notice that

$$
\lim_{s \to t} b_x^s = x.
$$

We begin with the following lemma.

Lemma 3.1. For every $x, z \in \mathbb{R}$, we have, for any nonnegative measurable functional $F$ on
$C([0, t]; \mathbb{R})$,

$$
\exp \left( \frac{-z^2}{2t} \right) E \left[ \exp \left( - \frac{\cosh x}{Z_t(b^s_x)} \right) F(T_{z-x}(b^s_x)) \right] = \exp \left( \frac{-x^2}{2t} \right) E \left[ \exp \left( - \frac{\cosh z}{Z_t(b^s_x)} \right) F(b^s_x) \right].
$$

Observe that, in the left-hand side, by Lemma 2.1(i), $T_{z-x}(b^s_x)(t) = z - (z - x) = x$,
which is consistent with the right-hand side, meaning that $b_x^t = x$.

Fix $z \in \mathbb{R}$ arbitrarily. Recall from [5, Theorem 1.2] that, for any nonnegative measurable functional $F$ on $C([0, t]; \mathbb{R})$,

$$
E[F(T_{z}(B)) = E \left[ \exp \left( \frac{\cosh B_t - \cosh(z + B_t)}{Z_t} \right) F(B) \right].
$$

We replace $F$ by a functional of the form

$$
\exp \left( \frac{-\cosh \phi_t}{Z_t(\phi)} \right) F(\phi), \quad \phi \in C([0, t]; \mathbb{R}),
$$

with $F$ a nonnegative measurable functional again; then, by properties (i) and (iii) in
Lemma 2.1, the above identity turns into

$$
E \left[ \exp \left( \frac{-\cosh(B_t - z)}{Z_t} \right) F(T_{z}(B)) \right] = E \left[ \exp \left( \frac{-\cosh(z + B_t)}{Z_t} \right) F(B) \right].
$$

Proof of Lemma 3.1. In the last equation, we substitute into $F$ a functional of the form

$$
f(\phi)F(\phi), \quad \phi \in C([0, t]; \mathbb{R}),
$$
where \( f : \mathbb{R} \rightarrow [0, \infty) \) is a measurable function, and we suppose, to begin with, that \( F : C([0, t]; \mathbb{R}) \rightarrow [0, \infty) \) is bounded and continuous. Then we have, by noting Lemma 2.1(i) as to the left-hand side,

\[
\int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi t}} \exp \left( -\frac{x^2}{2t} \right) f(x) \mathbb{E} \left[ \exp \left\{ -\frac{\cosh(x-z)}{Z_t(b^x)} \right\} F(T_z(b^x)) \right]
\]

= \[
\int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi t}} \exp \left( -\frac{x^2}{2t} \right) f(x) \mathbb{E} \left[ \exp \left\{ -\frac{\cosh(z+x)}{Z_t(b^x)} \right\} F(b^z) \right].
\]

By translation, the left-hand side is rewritten as

\[
\int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi t}} \exp \left\{ -\frac{(x+z)^2}{2t} \right\} f(x) \mathbb{E} \left[ \exp \left\{ -\frac{\cosh(x)}{Z_t(b^{x+z})} \right\} F(T_z(b^{x+z})) \right]
\]

Therefore, because of the arbitrariness of \( f \), we have, for a.e. \( x \in \mathbb{R} \),

\[
\exp \left\{ -\frac{(x+z)^2}{2t} \right\} \mathbb{E} \left[ \exp \left\{ -\frac{\cosh(x)}{Z_t(b^{x+z})} \right\} F(T_z(b^{x+z})) \right] = \exp \left( -\frac{x^2}{2t} \right) \mathbb{E} \left[ \exp \left\{ -\frac{\cosh(z+x)}{Z_t(b^x)} \right\} F(b^x) \right].
\]

In view of (3.1) and thanks to the boundedness and continuity of \( F \), the bounded convergence theorem entails that both sides are continuous in \( x \). Hence the last equality holds for any \( x \), and for any \( z \) as well since \( z \) was arbitrarily fixed. Consequently, replacing \( z \) by \( z - x \) proves identity (3.2) when \( F \) is bounded and continuous. Then, density and monotone class arguments extend \( F \) to any nonnegative measurable function as claimed.

We are in a position to prove Theorem 1.1.

**First proof of Theorem 1.1**. For each \( x \in \mathbb{R} \), in (3.2), we substitute \(-x\) into \( z \) and replace \( F \) by a functional of the form

\[
\exp \left\{ \frac{\cosh x}{Z_t(\phi)} \right\} F(\phi), \quad \phi \in C([0, t]; \mathbb{R}),
\]

with \( F \) an arbitrary nonnegative measurable functional. Then, thanks to Lemma 2.1(iii) as to the left-hand side, identity (3.2) becomes

\[
\mathbb{E} \left[ F(T_{-2x}(b^{-x})) \right] = \mathbb{E} [F(b^x)].
\]

Integration both sides with respect to \( P(B_t \in dx) \) over \( \mathbb{R} \) and using the symmetry \(-B \equiv B\) on the left-hand side leads to the conclusion. \( \square \)
3.2 Second proof of Theorem 1.1

We denote by \( \{Z_s\}_{s \geq 0} \) the natural filtration of the process \( \{Z_s\}_{s \geq 0} \). The proof of Theorem 1.1 given below hinges upon the observation that the conditional law of \( B_t \) given \( Z_t \) is symmetric as in the next lemma; see also Remark 3.1 at the end of this subsection.

**Lemma 3.2.** It holds that

\[
(B_t, \{Z_s\}_{0 \leq s \leq t}) \overset{(d)}{=} (-B_t, \{Z_s\}_{0 \leq s \leq t}).
\]

**Proof.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a bounded measurable function and \( F : C([0,t];\mathbb{R}) \to \mathbb{R} \) a bounded measurable functional. We have, conditionally on \( Z_t \),

\[
\mathbb{E}[f(B_t)F(Z)] = \mathbb{E}[\mathbb{E}[f(B_t) | Z_t] F(Z)].
\]

We know from [6, Proposition 1.7] that the conditional expectation in the right-hand side is equal a.s. to

\[
\mathbb{E}[f(z_u)] \big|_{u=\frac{1}{Z_t}},
\]

where, for each \( u > 0 \), \( z_u \) refers to a real-valued random variable whose law is given by

\[
\frac{1}{2K_0(u)}e^{-u \cosh x} dx, \quad x \in \mathbb{R}.
\]

Here \( K_0 \) is the modified Bessel function of the third kind (or the Macdonald function) of order 0. Since the above law is symmetric, we have \( \mathbb{E}[f(z_u)] = \mathbb{E}[f(-z_u)] \) for every \( u > 0 \), and hence from (3.2) and (3.3),

\[
\mathbb{E}[f(B_t)F(Z)] = \mathbb{E}[f(-B_t)F(Z)].
\]

As \( f \) and \( F \) are arbitrary, we have the claim. \( \Box \)

By using the above conditional symmetry of \( B_t \), the second proof of Theorem 1.1 proceeds as follows.

**Second proof of Theorem 1.1.** If we have proven

\[
\{A_s(\mathcal{T}(B))\}_{0 \leq s \leq t} \overset{(d)}{=} \{A_s\}_{0 \leq s \leq t},
\]

then, taking the derivative with respect to \( s \) on each side of the above identity leads to the conclusion. To this end, notice that, for each \( 0 < s \leq t \),

\[
\frac{1}{A_s(\mathcal{T}(B))} = \int_s^t \frac{du}{Z_u^2} + \frac{e^{-B_t}}{Z_t} + \frac{e^{2B_t} - 1}{A_t}
= \int_s^t \frac{du}{Z_u^2} + \frac{e^{B_t}}{Z_t},
\]
where we have used Proposition 2.1(ii) as well as relation (2.3) for the first line and the definition of \( Z_t \) (see (2.1)) for the second. By Lemma 3.2 the last expression entails that

\[
\left\{ \frac{1}{A_s(T(B))} \right\}_{0 < s \leq t} \overset{(d)}{=} \left\{ \int_s^t \frac{du}{Z_u^2} + e^{-B_t} \right\}_{0 < s \leq t}
= \left\{ \frac{1}{A_s} \right\}_{0 < s \leq t},
\]

which proves (3.6). Here we used relation (2.3) again for the last equality.

The above proof confirms that Theorem 1.1′ is indeed equivalent to Theorem 1.1 as indicated just above Theorem 1.1′.

Remark 3.1. We may associate Lemma 3.2 with the fact [6, Theorem 1.6(ii)] that, for any \( \mu \in \mathbb{R} \),

\[
\{ Z_s(B^{(\mu)}) \}_{s \geq 0} \overset{(d)}{=} \{ Z_s(B^{(-\mu)}) \}_{s \geq 0},
\]

in such a way that, by the Cameron–Martin formula,

\[
\mathbb{E}[e^{\mu B_t} F(Z)] = \mathbb{E}[e^{-\mu B_t} F(Z)]
\]

for every bounded measurable functional \( F \) on \( C([0, t]; \mathbb{R}) \). Then the injectivity of the Mellin transform entails the lemma. Identity (3.7) may be explained by the identity in law between the second coordinates in (1.9), and by the fact that \( Z \circ T_\alpha = Z \) for every \( \alpha \geq 0 \) ([3, Proposition 2.1(iii)]).

3.3 Proof of Corollary 1.1

First we show identity (1.6). Let \( F \) be a nonnegative measurable functional on \( C([0, t]; \mathbb{R}) \). For every \( \mu \in \mathbb{R} \), it holds that, by Theorem 1.1

\[
\mathbb{E}[F(T(B))e^{\mu T(B)(t)}] = \mathbb{E}[F(B)e^{\mu B_t}] .
\]

Multiplying both sides by \( e^{-\mu^2 t/2} \) and noting \( T(B)(t) = -B_t \) by Proposition 2.1(i), we have, by the Cameron–Martin formula,

\[
\mathbb{E}[F(T(B^{(-\mu)}))] = \mathbb{E}[F(B^{(\mu)})],
\]

which verifies (1.6).

Proof of Corollary 1.1. It follows from (1.6) that

\[
\{ (T(B^{(-\mu)})(s), (T \circ T)(B^{(-\mu)})(s)) \}_{0 \leq s \leq t} \overset{(d)}{=} \{ (B^{(\mu)}_s, T(B^{(\mu)})(s)) \}_{0 \leq s \leq t} .
\]

Since \( T \circ T = \text{Id} \) as stated in Proposition 2.1(iv), we have the claim.
Note that Corollary 1.1 may also be obtained by integrating both sides of (3.3) with respect to the probability measure
\[ \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(x - \mu t)^2}{2t} \right\} dx \]
over \( \mathbb{R} \), for any drift \( \mu \in \mathbb{R} \).

4 Proof of Proposition 1.2

This section is devoted to the proof of Proposition 1.2 in order to make the paper self-contained as much as possible, we also give a proof of Proposition 1.1 which will be done by using Corollary 1.1. Except for the proof of Proposition 1.1 we suppose that \( t > 0 \) is fixed.

We begin this section with the

Proof of Proposition 1.1 via Corollary 1.1. Let \( \mu > 0 \) and fix \( u > 0 \) arbitrarily. Set the process
\[ X_v := B_v^{(\mu)} - B_u^{(\mu)}, \]
which has the same law as \( B^{(\mu)} \) and is independent of \( \{B_s\}_{0 \leq s \leq u} \). We let \( t > 0 \) be such that \( u < t \). Then, by the definition (1.4) of \( T \), Corollary 1.1 entails that the two-dimensional process
\[ \left( B_s^{(\mu)}, B_s^{(\mu)} - \log \left\{ 1 + A_s^{(\mu)} \frac{e^{2B_s^{(\mu)} e^{2Z_t-u}} - 1}{A_s^{(\mu)} + e^{2B_s^{(\mu)} A_{t-u}(X)}} \right\} \right), \quad 0 \leq s \leq u, \]
is identical in law with \( \{(T(B^{(-\mu)})(s), B_s^{(-\mu)})\}_{0 \leq s \leq u} \). Rewrite
\[ \frac{e^{2B_s^{(\mu)} e^{2Z_t-u}} - 1}{A_s^{(\mu)} + e^{2B_s^{(\mu)} A_{t-u}(X)}} = \frac{e^{2B_s^{(\mu)} e^{-2Z_t-u}} - e^{-2Z_t-u}}{e^{-2Z_t-u} A_s^{(\mu)} + e^{2B_s^{(\mu)} e^{-2Z_t-u} A_{t-u}(X)}}, \quad (4.1) \]
and observe that \( e^{-2Z_t-u} \rightarrow 0 \) a.s. as \( t \rightarrow \infty \) because \( \mu > 0 \). Moreover, by the time reversal of Brownian motion,
\[ e^{-2Z_{t-u}} A_{t-u}(X) \stackrel{(d)}{=} A_{t-u}^{(-\mu)}, \]
which converges in law to \( 1/(2\gamma_\mu) \) as \( t \rightarrow \infty \) by Dufresne’s identity (1.7). Hence, owing to the independence of \( \{B_s\}_{0 \leq s \leq u} \) and \( X \), the pair of the process \( \{B_s^{(\mu)}\}_{0 \leq s \leq u} \) and the random variable (4.1) jointly converges in law to that of \( \{B_s^{(\mu)}\}_{0 \leq s \leq u} \) and \( 2\gamma_\mu \), with \( \gamma_\mu \) being independent of \( B \). Therefore we obtain the identity in law
\[ \{(B_s^{(\mu)}, B_s^{(-\mu)} - \log \{1 + 2\gamma_\mu A_s^{(\mu)}\})\}_{0 \leq s \leq u} \overset{(d)}{=} \left\{ \left( B_s^{(-\mu)} - \log \left( 1 - \frac{A_s^{(-\mu)}}{A_{\infty}^{(-\mu)}} \right), B_s^{(-\mu)} \right) \right\}_{0 \leq s \leq u}, \]
where the expression of the first coordinate in the right-hand side is due to the definition (1.4) of $\mathcal{T}$ and the fact that $e^{2R(t)} \to 0$ a.s. as $t \to \infty$. Since $u > 0$ is arbitrary, the last identity in law proves the proposition by the definition (1.8) of $\{T_\alpha\}_{\alpha \geq 0}$.

We proceed to the proof of Proposition 1.2. For every $\alpha \geq 0$, note that the expression $T(\mathcal{T}(\phi)) (s), \quad 0 \leq s \leq t,$

makes sense for any $\phi \in C([0,t]; \mathbb{R})$, and so does the expression $T_\alpha(\mathcal{T}(\phi)) (s), \quad 0 \leq s \leq t,$

because $T_\alpha$ is a non-anticipative transformation. Notice that, for any $\phi \in C([0,t]; \mathbb{R})$, $T_\alpha(\phi)$ is represented as

$\begin{aligned}
T_\alpha(\phi)(s) &= T_{\log\{1+\alpha_2(t)\}}(\phi)(s), \quad 0 \leq s \leq t, \\
&= T_{2\phi t - \log\{1+\alpha_2(t)\}}(\phi)(s),
\end{aligned}$

by the definition (1.1) of $\{T_z\}_{z \in \mathbb{R}}$; indeed,

$\begin{aligned}
T_{\log\{1+\alpha_2(t)\}}(\phi)(s) &= \phi_s - \log \left\{ 1 + \frac{\alpha_2(t)}{\alpha_2(t)} (1 + \alpha_2(\phi) - 1) \right\} \\
&= T_{\alpha}(\phi)(s).
\end{aligned}$

**Lemma 4.1.** (1) Expression (4.2) admits the representation

$\mathcal{T}(T_\alpha(\phi))(s), \quad 0 \leq s \leq t,$

(2) Expression (4.3) admits the representation

$T_\alpha(\mathcal{T}(\phi))(s), \quad 0 \leq s \leq t.$

**Proof.** Fix $0 \leq s \leq t$ arbitrarily.

(1) By (1.4) and relation (2.8) in Proposition 2.1(iv),

$\begin{aligned}
\mathcal{T}(T_\alpha(\phi))(s) &= \mathcal{T}(T_{\log\{e^{2\phi t}/(1+\alpha_2(t))\}}(\phi))(s) \\
&= T_{2\phi t - \log\{1+\alpha_2(t)\}}(\phi)(s),
\end{aligned}$

which proves the claim.

(2) By (1.4),

$\begin{aligned}
T_\alpha(\mathcal{T}(\phi))(s) &= T_{\log\{1+\alpha_2(t)\}}(\mathcal{T}(\phi))(s),
\end{aligned}$

which, by Proposition 2.1(i), is rewritten as

$\begin{aligned}
T_{\log\{1+\alpha e^{-2\phi t} A_2(t)\}}(\mathcal{T}(\phi))(s) &= T_{\log\{1+\alpha e^{-2\phi t} A_2(t)\} + 2\phi}(\phi)(s),
\end{aligned}$

proving (2). Here the last equality is due to the semigroup property in Lemma 2.1(iv) and the definition (1.4) of $\mathcal{T}$. \qed
Using the above lemma, we prove Proposition 1.2. First observe that, for each fixed \( t > 0 \), by the Markov property of Brownian motion and Dufresne’s identity (1.7), we have the identity in law
\[
\left( A_{\infty}^{(-\mu)}, \{ B_s^{(-\mu)} \}_{0 \leq s \leq t} \right) \overset{(d)}{=} \left( A_t^{(-\mu)} + e^{2B_t^{(-\mu)}}/(2\gamma_\mu), \{ B_s^{(-\mu)} \}_{0 \leq s \leq t} \right),
\]
where, in the right-hand side, \( \gamma_\mu \) is independent of \( B \). This observation entails that, in view of the definition (1.1) of \( \{ T_z \}_{z \in \mathbb{R}} \), Proposition 1.1 may be restated as the identity in law between the two two-dimensional processes
\[
\left( -A_t^{(-\mu)} + e^{2B_t^{(-\mu)}}/(2\gamma_\mu), \left\{ B_{s}^{(-\mu)} \right\}_{0 \leq s \leq t} \right),
\]
\[
\left( 0 \leq s \leq t, \right.
\]
and
\[
\left( B_s^{(\mu)}, T_s (B_{s}^{(\mu)}) \right),
\]
\[
\left( 0 \leq s \leq t, \right.
\]
because of the equality
\[
\frac{A_s^{(-\mu)}}{A_t^{(-\mu)} + e^{2B_t^{(-\mu)}}/(2\gamma_\mu)} = \frac{A_s^{(-\mu)}}{A_t^{(-\mu)}} \left( \frac{e^{2B_t^{(-\mu)}}}{e^{2B_t^{(-\mu)}} + 2\gamma_\mu A_t^{(-\mu)}} - 1 \right)
\]
for every \( 0 \leq s \leq t \).

**Proof of Proposition 1.2.** (1) Since the process
\[
\left( B_s^{(\mu)}, \mathcal{T} (T_2 B_{s}^{(\mu)}) (s) \right), \quad 0 \leq s \leq t,
\]
is nothing but the right-hand side of (1.10) by Lemma 4.1, it suffices to prove, in view of (4.6) and (4.7), that the process
\[
\left( \mathcal{T}_{2B_t^{(-\mu)} - \log \left( e^{2B_t^{(-\mu)}} + 2\gamma_\mu A_t^{(-\mu)} \right)} (B^{(-\mu)}), B_s^{(-\mu)} \right), \quad 0 \leq s \leq t,
\]
and
\[
\left( B_s^{(\mu)}, T_{2\gamma_\mu} (B_{s}^{(\mu)}) \right), \quad 0 \leq s \leq t,
\]
are identical in law with the left-hand side of (1.10). To this end, for each \( 0 \leq s \leq t \), we rewrite the first coordinate in (4.8) in such a way that
\[
\mathcal{T}_{- \log \left( 1 + 2\gamma_\mu A_t (B^{(-\mu)}) \right)} (\mathcal{T} (B^{(-\mu)}) (s) )
\]
by Proposition 2.1(ii) and by the fact that \( \mathcal{T} \) is an involution (Proposition 2.1(iv)). Then, by Corollary 1.1 we see that (4.8) is identical in law with
\[
\left( \mathcal{T}_{- \log \left( 1 + 2\gamma_\mu A_t^{(\mu)} \right)} (\mathcal{T} (B_s^{(\mu)}) (s) ), B_s^{(\mu)} \right), \quad 0 \leq s \leq t,
\]
which coincides with the left-hand side of (1.10) by the semigroup property of \( \{ \mathcal{T}_z \}_{z \in \mathbb{R}} \) in Lemma 2.1(iv) and the definition (1.4) of \( \mathcal{T} \).
(2) Since, for each \(0 \leq s \leq t\), we may express the first coordinate in (4.6) as
\[
(\mathcal{T} \circ \mathcal{T} \log\{e^{2\mu t} + 2\gamma_\mu A_t^{(-\mu)}\})(B^{(\mu)})(s)
\]
due to (2.8) in Proposition 2.1(iv), it suffices to prove, in view of (4.6) and (4.7), that the process
\[
(\mathcal{T}(B^{(\mu)})(s), T_{2\gamma_\mu}(B^{(\mu)})(s)), \quad 0 \leq s \leq t,
\]
is identical in law with the right-hand side of (1.11) owing to the fact that \(\mathcal{T}\) is an involution. Rewriting the last displayed process as
\[
(\mathcal{T}(B^{(\mu)})(s), T_{2\gamma_\mu}((\mathcal{T} \circ \mathcal{T})(B^{(\mu)}))(s)), \quad 0 \leq s \leq t,
\]
we see that it is identical in law with
\[
(B^{(-\mu)}_s, T_{2\gamma_\mu}(\mathcal{T}(B^{(-\mu)}))(s)), \quad 0 \leq s \leq t,
\]
by Corollary 1.1, which, thanks to Lemma 4.1(2), coincides with the right-hand side of (1.11) as claimed.

**Remark 4.1.** By extracting the gamma variable \(\gamma_\mu\) from the first coordinate in the left-hand side, identity (1.11) is equivalently rephrased as the joint identity in law
\[
(\gamma_\mu, \{(X^1_s, B^{(\mu)}_s)\}_{0 \leq s \leq t}) \overset{(d)}{=} \left(\frac{\gamma_\mu e^{2B^1_t}}{1 + 2\gamma_\mu A_t^{(\mu)}}, \{(B^{(\mu)}_s, X^1_s)\}_{0 \leq s \leq t}\right);
\]
similarly, identity (1.11) is equivalent to
\[
(\gamma_\mu, \{(X^2_s, B^{(-\mu)}_s)\}_{0 \leq s \leq t}) \overset{(d)}{=} \left(\frac{\gamma_\mu}{e^{2B^1_t} + 2\gamma_\mu A_t^{(-\mu)}}, \{(B^{(-\mu)}_s, X^2_s)\}_{0 \leq s \leq t}\right).
\]
In each of the above two identities, the identity in law between the first components may be explained in the following manner:
\[
\frac{\gamma_\mu e^{2B^1_t}}{1 + 2\gamma_\mu A_t^{(\mu)}} \overset{(d)}{=} \frac{\gamma_\mu}{e^{-2B^1_t} + 2\gamma_\mu e^{2B^1_t} A_t^{(\mu)}} \overset{(d)}{=} \frac{\gamma_\mu}{e^{2B^1_t} + 2\gamma_\mu A_t^{(-\mu)}} \overset{(d)}{=} \frac{1}{2A^{(-\mu)}_\infty},
\]
which is identical in law with \(\gamma_\mu\) by Dufresne’s identity (1.7). Here the second line is due to the time reversal of Brownian motion (see (2.5)) and the third line is nothing but the identity in law between the first components in (4.5).
Combining Corollary 1.1 and Proposition 2.2 enables us to obtain in part a generalization of Proposition 1.2. Let \( \tilde{B} = \{ \tilde{B}_s \}_{s \geq 0} \) be a one-dimensional standard Brownian motion that is independent of \( B \). For every drift \( \mu \in \mathbb{R} \), we denote

\[
\tilde{A}_s^{(\mu)} = A_s(\tilde{B}(\mu)), \quad s \geq 0,
\]

with \( \tilde{B}^{(\mu)} = \{ \tilde{B}_s^{(\mu)} = \tilde{B}_s + \mu s \}_{s \geq 0} \). Then we have the following distributional invariance of \( B^{(\mu)} \) in the presence of the independent element \( \tilde{B}^{(\mu)} \):

**Proposition 4.1.** For every \( \mu \in \mathbb{R} \), it holds that, for any \( u \geq 0 \), the process

\[
B_s^{(\mu)} - \log \left\{ 1 + \frac{A_s^{(\mu)}}{A_t^{(\mu)}} \left( e^{2B_t^{(\mu)} \tilde{A}_u^{(\mu)} + A_t^{(\mu)}} - 1 \right) \right\}, \quad 0 \leq s \leq t,
\]

is identical in law with \( \{ B_s^{(\mu)} \}_{0 \leq s \leq t} \).

**Proof.** Since there is nothing to prove in the case \( u = 0 \), we let \( u > 0 \). In view of Corollary 1.1 and Proposition 2.2, the process

\[
B_s^{(\mu)} - \log \left\{ 1 + \frac{A_s^{(\mu)}}{A_t^{(\mu)}} \left( e^{2B_t^{(\mu)} \tilde{A}_u^{(\mu)} + A_t^{(\mu)}} - 1 \right) \right\}, \quad 0 \leq s \leq t,
\]

has the same law as \( \{ B_s^{(\mu)} \}_{0 \leq s \leq t} \). Because Brownian motion has independent increments, we see that the pair of \( \{ B_s^{(\mu)} \}_{0 \leq s \leq t} \) and the random variable

\[
1 + \frac{A_t^{(\mu)}}{A_t^{(\mu)} + e^{2B_t^{(\mu)} \tilde{A}_u^{(\mu)} + A_t^{(\mu)}}} (e^{2B_t^{(\mu)} \tilde{A}_u^{(\mu)} + A_t^{(\mu)}} - 1)
\]

is identical in law with that of \( \{ B_s^{(\mu)} \}_{0 \leq s \leq t} \) and

\[
1 + \frac{A_t^{(\mu)}}{A_t^{(\mu)} + e^{2B_t^{(\mu)} \tilde{A}_u^{(\mu)} + A_t^{(\mu)}}} (e^{2B_t^{(\mu)} \tilde{A}_u^{(\mu)} + A_t^{(\mu)}} - 1).
\]

Since the last displayed random variable equals

\[
\frac{e^{2B_t^{(\mu)} \tilde{A}_u^{(\mu)} + A_t^{(\mu)}}}{A_t^{(\mu)} + e^{2B_t^{(\mu)} \tilde{A}_u^{(\mu)} + A_t^{(\mu)}}},
\]

we have the claim. \( \square \)

To see that the above proposition generalizes Proposition 1.2 partly, notice that, when \( \mu > 0 \),

\[
\frac{e^{2B_t^{(\mu)} \tilde{A}_u^{(\mu)} + A_t^{(\mu)}}}{A_t^{(\mu)} + e^{2B_t^{(\mu)} \tilde{A}_u^{(\mu)} + A_t^{(\mu)}}} = \frac{e^{2B_t^{(\mu)} (e^{-2B_u^{(\mu)} \tilde{A}_u^{(\mu)} + A_t^{(\mu)}} - 1)} A_t^{(\mu)}}{A_t^{(\mu)} + e^{2B_t^{(\mu)} \tilde{A}_u^{(\mu)} + A_t^{(\mu)}}}
\]

in (4.9) converges in law to
\[
\frac{e^{2B_t^{(\mu)}/(2\gamma\mu)}}{1/(2\gamma\mu) + A_t^{(\mu)}} = \frac{e^{2B_t^{(\mu)}}}{1 + 2\gamma\mu A_t^{(\mu)}}
\]
as \(u \to \infty\), owing to Dufresne’s identity (1.7), with \(\gamma\mu\) being independent of \(B^{(\mu)}\); recall the reasoning after equation (4.1) in the proof of Proposition 1.1. The case \(\mu < 0\) is similar but treated more readily.

5 Some related results and extensions

In this section, we provide some results related to those introduced above, including extensions of Theorem 1.1.

To begin with, for every point \(a \in \mathbb{R}\) and \(\phi \in C([0, \infty); \mathbb{R})\), we denote by \(\tau_a(\phi)\) the first hitting time of \(\phi\) to the level \(a\):
\[
\tau_a(\phi) := \inf\{s \geq 0; \phi_s = a\},
\]
with the convention \(\inf\emptyset = \infty\). Let \(\beta = \{\beta(s)\}_{s \geq 0}\) and \(\hat{B} = \{\hat{B}_s\}_{s \geq 0}\) be two one-dimensional standard Brownian motions that are independent of \(B\). Let \(x \in \mathbb{R}\) be fixed and denote
\[
\tau^x := \tau_{\cosh(x+B_t)}(\hat{B}(\cosh x/Z_t)) \tag{5.1}
\]
for simplicity. As recalled in Section 1, it is shown in [5] that the process
\[
\mathbb{T}_{x+B_t-\text{Argsh}(e^{B_t \sinh x + \beta(A_t)})}(B)(s), \quad 0 \leq s \leq t,
\]
is a Brownian motion; more precisely, what in fact we have proven is

**Proposition 5.1 ([5, Theorem 3.1])**. Under the above setting, we have the following for every \(x \in \mathbb{R}\):

(i) the pair of the process
\[
\mathbb{T}_{x+B_t-\text{Argsh}(e^{B_t \sinh x + \beta(A_t)})}(B)(s), \quad 0 \leq s \leq t,
\]
and the random variable \(A_t\) is identical in law with that of \(\{B_s\}_{0 \leq s \leq t}\) and \(\tau^x\);

(ii) the pair of the process
\[
\mathbb{T}_{\log(A_t/\tau^x)}(B)(s), \quad 0 \leq s \leq t,
\]
and the random variable \(\log(A_t/\tau^x)\) is identical in law with that of \(\{B_s\}_{0 \leq s \leq t}\) and
\[
\text{Argsh} \left( e^{B_t \sinh x + \beta(A_t)} \right) - x - B_t.
\]
By virtue of Theorem 1.1, similar distributional identities to the above hold as in the following proposition.

**Proposition 5.2.** Under the same setting as in Proposition 5.1, we have the following for every $x \in \mathbb{R}$:

(i) the pair of the process

$$T_{\operatorname{Argsh} (e^{B_t \sinh x + \beta(A_t)}) - x + B_t}(B)(s), \quad 0 \leq s \leq t,$$

and the random variable $A_t$ is identical in law with that of $\{B_s\}_{0 \leq s \leq t}$ and

$$\tau_{\cosh(x-B_t)}(\hat{B}(\cosh x/Z)) ;$$

(ii) the pair of the process

$$T_{\log(e^{2B_t \tau_x / A_t})}(B)(s), \quad 0 \leq s \leq t,$$

and the random variable $\log(A_t / \tau_x)$ is identical in law with that of $\{B_s\}_{0 \leq s \leq t}$ and

$$\operatorname{Argsh} (e^{-B_t \sinh x + e^{-B_t \beta(A_t)}}) - x + B_t.$$ 

**Proof.** (i) Because of Proposition 5.1(i) and the relation that, for each $0 \leq s \leq t$,

$$\mathcal{T}(T_{x+B_t - \operatorname{Argsh} (e^{B_t \sinh x + \beta(A_t)})}(B))(s) = T_{\operatorname{Argsh} (e^{B_t \sinh x + \beta(A_t)}) - x + B_t}(B)(s)$$

due to relation (2.8) in Proposition 2.1(iv), it suffices to show that

$$\left(\{T(s)\}_{0 \leq s \leq t}, \tau^x\right) \overset{(d)}{=} \left(\{B_s\}_{0 \leq s \leq t}, \tau_{\cosh(x-B_t)}(\hat{B}(\cosh x/Z))\right). \quad (5.2)$$

To this end, notice that, by the definition (5.1) of $\tau^x$ and properties (i) and (iii) in Proposition 2.1,

$$\tau^x = \tau_{\cosh(x-\mathcal{T}(B)(t))}(\hat{B}(\cosh x/Z(\mathcal{T}(B)))) .$$

As a consequence, by Theorem 1.1 and the independence of $B$ and $\hat{B}$, we have the claimed identity (5.2).

(ii) Similarly to (i), by virtue of Proposition 5.1(ii), it suffices to prove that

$$\left(\{\mathcal{T}(s)\}_{0 \leq s \leq t}, \operatorname{Argsh} (e^{B_t \sinh x + \beta(A_t)}) - x - B_t\right) \overset{(d)}{=} \left(\{B_s\}_{0 \leq s \leq t}, \operatorname{Argsh} (e^{-B_t \sinh x + e^{-B_t \beta(A_t)}} - x + B_t)\right) . \quad (5.3)$$

Note that, by properties (i) and (ii) in Proposition 2.1 the second component in the left-hand side may be written as

$$\operatorname{Argsh} (e^{-\mathcal{T}(B)(t) \sinh x + \beta(e^{-2\mathcal{T}(B)(t)} A_t(\mathcal{T}(B))))} - x + \mathcal{T}(B)(t),$$

which entails that the left-hand side of the claimed identity (5.3) is identical in law with

$$\left(\{B_s\}_{0 \leq s \leq t}, \operatorname{Argsh} (e^{-B_t \sinh x + \beta(e^{-2B_t} A_t)}) - x + B_t\right)$$

owing to Theorem 1.1. This verifies (5.3) by the scaling property of Brownian motion and the independence of $B$ and $\beta$. \qed
Remark 5.1. The identity in law between the second components in (5.2) may also be explained by means of the time reversal (2.5); indeed, by (2.5), the random variable \( \tau^x \) has the same law as

\[
\tau_{\cosh(x+R(B)(t))}(\hat{B}(\cosh x/Z_t(R(B)))),
\]

which is nothing but the second component in the right-hand side of (5.2) because \( R(B)(t) = -B_t \) and

\[
Z_t(R(B)) = e^{-R(B)(t)A_t(R(B))} = e^{B_t e^{-2B_t A_t}} = Z_t
\]

by the definition (2.1) of the transformation \( Z_t \).

Next we will see that Proposition 1.2 may be rephrased as

**Proposition 5.3.** For every \( x \geq 0 \), it holds that, for any nonnegative measurable functional \( F \) on \( C([0,t];\mathbb{R}^2) \),

\[
\mathbb{E}[F(T_{\log(e^{2B_t/(1+2\gamma A_t})}(B), B)] = \mathbb{E}\left[ \frac{e^{2B_t}}{e^{2B_t} - 2x A_t} \exp\left( x - \frac{x}{e^{2B_t} - 2x A_t} \right) F(B, T_{\log(e^{2B_t-2x A_t})}(B)); \frac{e^{2B_t}}{2A_t} > x \right] (5.4)
\]

and

\[
\mathbb{E}[F(T_{\log(e^{2B_t+2\gamma A_t})}(B), B)] = \mathbb{E}\left[ \frac{1}{1 - 2x A_t} \exp\left( x - \frac{xe^{2B_t}}{1 - 2x A_t} \right) F(B, T_{\log(e^{2B_t/(1-2\gamma A_t})}(B)); \frac{1}{2A_t} > x \right]. (5.5)
\]

The above proposition extends Theorem 1.1 in the sense that the theorem is recovered by taking \( x = 0 \):

\[
\{(T(B)(s), B_s)\}_{0 \leq s \leq t} \overset{(d)}{=} \{(B_s, T(B)(s))\}_{0 \leq s \leq t}, \quad (5.6)
\]

that is, the case \( \mu = 0 \) in Corollary 1.1.

**Proof of Proposition 5.3.** Since the two identities (5.4) and (5.5) are proven in the same way, we only give a proof for the latter.

For every nonnegative measurable functional \( F \) on \( C([0,t];\mathbb{R}^2) \), it follows from Proposition 1.2(2) that, by the Cameron–Martin formula,

\[
\mathbb{E}[F(T_{\log(e^{2B_t+2\gamma A_t})}(B), B)e^{-\mu B_t}] = \mathbb{E}[F(B, T_{\log(e^{2B_t+2\gamma A_t})}(B))e^{-\mu B_t}].
\]
Replacing $F$ by a nonnegative functional of the form $F(\phi^1, \phi^2)e^{\mu \phi^2}$, $(\phi^1, \phi^2) \in C([0, t]; \mathbb{R}^2)$, we have

$$
\mathbb{E}\left[F\left(T_{\log(e^{2B_t+2\gamma_t}A_t)}(B), B\right)\right] = \mathbb{E}\left[F\left(B, T_{\log(e^{2B_t+2\gamma_t}A_t)}(B)\right)\left(e^{2B_t} + 2\gamma_t A_t\right)^{-\mu}\right],
$$

where the expression of the right-hand side is due to property (i) in Lemma 2.1 and, in addition to the nonnegativity, we assume, for the time being, that $F$ is bounded and continuous. By the independence of $B$ and $\gamma$, and by Fubini’s theorem, the right-hand side of the above identity is rewritten as

$$
\frac{1}{\Gamma(\mu)} \mathbb{E}\left[\int_0^\infty dy y^{\mu-1} e^{-y} F\left(B, T_{\log(e^{2B_t+2yA_t})}(B)\right)\left(e^{2B_t} + 2y A_t\right)^{-\mu}\right],
$$

which, by changing the variables with $y/(e^{2B_t} + 2y A_t) = x$, $0 < x < 1/(2A_t)$, is further rewritten as

$$
\frac{1}{\Gamma(\mu)} \mathbb{E}\left[\int_0^{1/(2A_t)} dx x^{\mu-1} \exp\left(-\frac{xe^{2B_t}}{1 - 2x A_t}\right) F\left(B, T_{\log(e^{2B_t/(1-2x A_t)})}(B)\right)\right].
$$

Therefore, identity (5.7) is rephrased as

$$
\int_0^\infty dx x^{\mu-1} e^{-x} \mathbb{E}\left[F\left(T_{\log(e^{2B_t+2x A_t})}(B), B\right)\right] = \int_0^\infty dx x^{\mu-1} \mathbb{E}\left[\frac{1}{1 - 2x A_t} \exp\left(-\frac{xe^{2B_t}}{1 - 2x A_t}\right) F\left(B, T_{\log(e^{2B_t/(1-2x A_t)})}(B); \frac{1}{2A_t} > x\right)\right],
$$

where we used the independence of $B$ and $\gamma$ for the left-hand side, and Fubini’s theorem again for the right-hand side. Since the above identity holds for any $\mu > 0$, the injectivity of the Mellin transform entails that, for a.e. $x \geq 0$,

$$
e^{-x} \mathbb{E}\left[F\left(T_{\log(e^{2B_t+2x A_t})}(B), B\right)\right] = \mathbb{E}\left[\frac{1}{1 - 2x A_t} \exp\left(-\frac{xe^{2B_t}}{1 - 2x A_t}\right) F\left(B, T_{\log(e^{2B_t/(1-2x A_t)})}(B); \frac{1}{2A_t} > x\right)\right].\tag{5.8}
$$

It is clear that, in view of the definition (1.1) of $\{T_z\}_{z \in \mathbb{R}}$, the left-hand side is continuous in $x$ by the bounded convergence theorem, because of the fact that $F$ is assumed to be bounded and continuous. On the other hand, notice that, in the right-hand side, the integrand

$$
\frac{1}{1 - 2x A_t} \exp\left(-\frac{xe^{2B_t}}{1 - 2x A_t}\right) F\left(B, T_{\log(e^{2B_t/(1-2x A_t)})}(B)\right)
$$

for $0 \leq x < 1/(2A_t)$ and 0 otherwise, is continuous in $x$ a.s., thanks to the boundedness and continuity of $F$; moreover, it is bounded from above by the integrable random variable

$$M \max \{2e^{-2B_t} A_t, 1\}.$$
where $M := \sup\{F(\phi^1, \phi^2); (\phi^1, \phi^2) \in C([0, t]; \mathbb{R}^2)\}$. Therefore the right-hand side of (5.5) also gives rise to a continuous function in $x$ by the dominated convergence theorem, ensuring that identity (5.5) holds for all $x \geq 0$. Standard arguments of density and monotone class then extend $F$ to any nonnegative measurable functional and complete the proof of (5.5). \hfill \Box

We give two remarks on Proposition 5.3.

**Remark 5.2.** Proposition 5.3 suggests that we have the following two relations for any $\psi \in C([0, t]; \mathbb{R})$ such that $e^{2\psi t} - 2xA_t(\psi) > 0$ for the former and that $1 - 2xA_t(\psi) > 0$ for the latter:

\begin{align*}
T & \log \frac{e^{2\psi t}/(1+2xA_t(\phi))}{e^{2\psi t} - 2xA_t(\psi)} \bigg|_{\phi = T \log (e^{2\psi t} - 2xA_t(\psi))} = \psi; \quad (5.9) \\
T & \log \frac{e^{2\psi t} + 2xA_t(\phi)}{e^{2\psi t} - 2xA_t(\psi)} \bigg|_{\phi = T \log (e^{2\psi t}/(1-2xA_t(\psi)))} = \psi. \quad (5.10)
\end{align*}

Indeed, a direct computation verifies these two relations. As for (5.9), observe that, by properties (i) and (ii) in Lemma 2.1:

$$
\frac{e^{2\psi t}}{1 + 2xA_t(\phi)} \bigg|_{\phi = T \log (e^{2\psi t} - 2xA_t(\psi))} = \left\{ \frac{e^{\psi t}}{e^{2\psi t} - 2xA_t(\psi)} \right\}^2 \times \frac{1}{1 + 2xA_t(\psi)/\{ e^{2\psi t} - 2xA_t(\psi) \}} = \frac{1}{e^{2\psi t} - 2xA_t(\psi)}.
$$

Therefore the left-hand side of (5.9) is written as

$$
T - \log \frac{e^{2\psi t} - 2xA_t(\psi)}{1 + 2xA_t(\phi)} \bigg|_{\phi = T \log (e^{2\psi t} - 2xA_t(\psi))} = \psi,
$$

which, by Lemma 2.1(iv), equals $\psi$ as claimed in (5.9). Similarly, as for (5.10),

$$
\left\{ \frac{e^{2\psi t} + 2xA_t(\phi)}{e^{2\psi t}} \right\}^2 \times \frac{1 - 2xA_t(\psi)}{e^{2\psi t} - 2xA_t(\psi)} = \frac{1 - 2xA_t(\psi)}{e^{2\psi t}},
$$

and hence the left-hand side of (5.10) is written as

$$
T - \log \frac{e^{2\psi t}/(1-2xA_t(\psi))}{1 - 2xA_t(\psi)} \bigg|_{\phi = T \log (e^{2\psi t}/(1-2xA_t(\psi)))} = \psi,
$$

which equals $\psi$ and verifies (5.10).
Remark 5.3. The two relations (5.4) and (5.5) are equivalent and they are related via Theorem 1.1 (or, more precisely, identity (5.6)). For instance, to see that the former entails the latter, we replace $F$ by a functional of the form $F(T(\phi^1), T(\phi^2)), (\phi^1, \phi^2) \in C([0,t]; \mathbb{R}^2)$. Then, in view of Lemma 4.1(1), the left-hand side of (5.4) turns into

$$
\mathbb{E}[F(T(T(T_2x(B)))), T(B))] = \mathbb{E}[F(T_2x(B), T(B))] = \mathbb{E}[F(T_2x(T(B)), B)],
$$

which agrees with the left-hand side of (5.5) thanks to Lemma 4.1(2). Here we used the property $T \circ T = Id$ in Proposition 2.1(iv) for the first line and (5.6) for the second.

On the other hand, as for the right-hand side, observe the relation

$$
(e^R, A_t) = (e^{-T(B)(t)}, e^{-2T(B)(t)}A_t(T(B)))
$$

and the fact that

$$
T(T_{\log(e^{2R_t-2xA_t})}B)(s) = T_{2R_t-\log(e^{2R_t-2xA_t})B}(s) = T_{-\log(1-2xA_t(T(B)))}(B)(s)
$$

for each $0 \leq s \leq t$. Relation (5.12) is due to properties (i) and (ii) in Proposition 2.1, while in (5.13), we used (2.8) for the first line and (5.12) for the second. Then, thanks to these two observations (5.12) and (5.13), the above replacement of $F$ turns the right-hand side of (5.4) into

$$
\mathbb{E}\left[\frac{1}{1-2xA_t(T(B))} \exp\left\{x - \frac{xe^{2T(B)(t)}}{1-2xA_t(T(B))}\right\} F(T(B), T_{-\log(1-2xA_t(T(B)))}(B)); \frac{1}{2A_t(T(B))} > x\right]
$$

which is equal to

$$
\mathbb{E}\left[\frac{1}{1-2xA_t} \exp\left(x - \frac{xe^{2R_t}}{1-2xA_t}\right) F(B, T_{-\log(1-2xA_t)}(T(B))); \frac{1}{2A_t} > x\right]
$$

by (5.6). Since

$$
T_{-\log(1-2xA_t)}(T(B)) = T_{\log(e^{2R_t/(1-2xA_t)})}(B)
$$

by the definition (1.4) of $T$ and the semigroup property of $\{T_z\}_{z \in \mathbb{R}}$ (Lemma 2.1(iv)), the last expectation agrees with the right-hand side of (5.5).

Taking $F$ as a functional of the first coordinate, for every fixed $x \in \mathbb{R}$, we obtain from Proposition 5.3 Girsanov-type formulas for the two anticipative transforms $T_{\log(e^{2R_t/(1+2xA_t)})}(B)$ and $T_{\log(e^{2R_t+2xA_t})}(B)$, which is of interest from the viewpoint of Malliavin calculus as they would provide examples in which the associated Fredholm
determinants are explicitly calculated; we refer to [5, Section 6] and references cited therein in this respect.

We conclude this paper with a remark concerning the non-anticipative transform $T_{2x}(B)$, as treated in (5.11), of the Brownian motion $B$ up to time $t$.

Remark 5.4. In view of (5.11), it is also revealed in Remark 5.3 that, for each $x \geq 0$,

$$\mathbb{E}[F(T_{2x}(B), T(B))] = \mathbb{E}\left[\frac{1}{1 - 2x \sigma^2(t)} \exp\left(\frac{x - xe^{2Bt}}{1 - 2x \sigma^2(t)}\right) F(B, T_{\log\{e^{2Bt} / (1 - 2x \sigma^2(t))\}}(B)); \frac{1}{2A_t} > x\right].$$

Replacing $F$ by a functional of the form $F(\phi^1, T(\phi^2))$, $(\phi^1, \phi^2) \in C([0, t]; \mathbb{R}^2)$, we obtain the relation

$$\mathbb{E}[F(T_{2x}(B), B)] = \mathbb{E}\left[\frac{1}{1 - 2x \sigma^2(t)} \exp\left(\frac{x - xe^{2Bt}}{1 - 2x \sigma^2(t)}\right) F(B, T_{\log\{1 - 2x \sigma^2(t)\}}(B)); \frac{1}{2A_t} > x\right].$$

If, for every $\mu \in \mathbb{R}$, we substitute into $F$ a functional of the form $F(\phi^1, \phi^2)e^{\mu x^2}$, $(\phi^1, \phi^2) \in C([0, t]; \mathbb{R}^2)$, then we have, by the Cameron–Martin formula,

$$\mathbb{E}[F(T_{2x}(B^{(\mu)}), B^{(\mu)})] = \mathbb{E}\left[\frac{1}{\{1 - 2x \sigma(\mu)^2(t)\}^{\mu+1}} \exp\left\{\frac{x - xe^{2B^{(\mu)}t}}{1 - 2x \sigma(\mu)^2(t)}\right\} F(B^{(\mu)}, T_{\log\{1 - 2x \sigma(\mu)^2(t)\}}(B^{(\mu)})); \frac{1}{2A^{(\mu)}_t} > x\right]$$

for any $x \geq 0$, which extends [3, Theorem 1.5], in particular, to the case of negative drifts $\mu$. We also note that, by (4.3), the left-hand side may be expressed as

$$\mathbb{E}\left[F\left(T_{\log\{1 + 2x \sigma(\mu)^2(t)\}}(B^{(\mu)}), B^{(\mu)}\right)\right]$$

in terms of the transformations $T_z$, $z \in \mathbb{R}$.

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