The existence and regularity of time-periodic solutions to the three-dimensional Navier–Stokes equations in the whole space

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Abstract
The existence, uniqueness and regularity of time-periodic solutions to the Navier–Stokes equations in the three-dimensional whole space are investigated. We consider the Navier–Stokes equations with a non-zero drift term corresponding to the physical model of a fluid flow around a body that moves with a non-zero constant velocity. The existence of a strong time-periodic solution is shown for small time-periodic data. It is further shown that this solution is unique in a large class of weak solutions that can be considered physically reasonable. Finally, we establish regularity properties for any strong solution regardless of its size.

Keywords: Navier–Stokes, time-periodic, strong solutions, uniqueness, regularity
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1. Introduction

We investigate the time-periodic Navier–Stokes equations with a non-zero drift term in the three-dimensional whole space. More specifically, we consider the system

\[
\begin{aligned}
\partial_t u - \Delta u - \lambda \partial_1 u + \nabla p + u \cdot \nabla u &= f & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\
\text{div } u &= 0 & \text{in } \mathbb{R}^3 \times \mathbb{R},
\end{aligned}
\] (1.1)
for an Eulerian velocity field \( u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \) and pressure term \( p : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \) as well as data \( f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \) that are all \( T \)-time-periodic, that is,
\[
\forall (x, t) \in \mathbb{R}^3 \times \mathbb{R} : \quad u(x, t) = u(x, t + T) \quad \text{and} \quad p(x, t) = p(x, t + T)
\] (1.2)
and
\[
\forall (x, t) \in \mathbb{R}^3 \times \mathbb{R} : \quad f(x, t) = f(x, t + T).
\] (1.3)
The time period \( T > 0 \) is a fixed constant. Physically, the system (1.1) originates from the model of a flow of an incompressible, viscous, Newtonian fluid past an object that moves with velocity \( \lambda e_1 \in \mathbb{R}^3 \). We shall consider the case \( \lambda \neq 0 \) corresponding to the case of an object moving with non-zero velocity.

The study of the time-periodic Navier–Stokes equations was suggested by Serrin in [25]. Serrin proposed applying a well-known method from dynamical systems theory to identify a time-periodic solution as a periodic orbit. The method is based on the postulate that for time-periodic data \( f \) and any initial value, the solution \( u(x, t) \) to the corresponding initial-value problem tends to a periodic orbit as \( t \to \infty \). Another approach was initiated by Prodi [23], Yudovich [33] and Prouse [24]. These authors proposed obtaining a time-periodic solution by considering the Poincaré map that takes an initial value into the state obtained by evaluation at time \( T \) of the solution to the corresponding initial-value problem, where \( T \) is the period of the prescribed data \( f \). A time-periodic solution is then identified via a fixed point of this Poincaré map. A further technique based on a representation formula derived from the Stokes semigroup was introduced by Kozono and Nakao in [14]. All the methods described above have in common that they utilize the theory for the initial-value problem. Over the years, a number of investigations based on these methods, or similar ideas involving the initial-value problem in some way, have been carried out: [4, 8–11, 13, 17–22, 26, 28–32, 34].

With the exception of Galdi and Silvestre [10,11] and the very recent papers by Galdi [8,9], none of the authors and papers mentioned above treat the question of existence and regularity of strong solutions in the case \( \lambda \neq 0 \) of a flow past an obstacle moving with non-zero velocity. In [10,11], the time-periodic flow past an object, moving with a prescribed and with a non-prescribed rigid motion, respectively, is investigated. The problem in [10,11] is therefore more general than the one under investigation here. In fact, the existence of a weak solution to (1.1)–(1.3) can be derived from the results in [10]. Under restrictions on the size of the data, it is further shown in [10] that the solution is strong in a local sense. The paper [8] deals with the two-dimensional problem corresponding to (1.1)–(1.3). The existence of a solution that is strong both locally and in the sense of global summability is established. In [9], the results of [8] are established for the two-dimensional exterior domain.

As the main result in this paper, the existence of a strong solution to the time-periodic Navier–Stokes equations in the three-dimensional whole space in the case \( \lambda \neq 0 \) is shown for time-periodic data sufficiently restricted in size—that is, a strong solution to (1.1)–(1.3). It is further shown that this solution is unique in a large class of weak solutions, that it obeys a balance of energy, and that it is as regular as is allowed for by the data. We shall employ a method that does not utilize the corresponding initial-value problem. Instead, we will reformulate (1.1)–(1.3) as an equivalent system on the group \( G := \mathbb{R}^3 \times \mathbb{R}/T\mathbb{Z} \). This approach allows us to derive a suitable representation of the solution in terms of a Fourier multiplier based on the Fourier transform \( \mathcal{F}_G \) associated with the group \( G \). We are thereby able to avoid the typical setting of function spaces associated with the initial-value problem and instead develop one that is better suited for the time-periodic case. Consequently, we obtain solutions that are strong both locally and in the sense of global summability.

The choice of function spaces is a delicate matter in the study of time-periodic Navier–Stokes equations in unbounded domains. Since time-independent solutions are trivially also
time-periodic, an investigation should preferably be carried out in a setting of function spaces containing also the steady states. In unbounded domains, the function spaces typically used to study the initial-value problem do not contain the full set of generic steady states. If one therefore applies a method based on the initial-value problem, the choice of function spaces in which to conduct the study poses an additional challenge. It was not until the work of Yamazaki [32] and Galdi and Sohr [4] that this challenge was properly overcome by introduction of appropriate function spaces rather than structural restrictions on the data. Both of these papers are focused on the case $\lambda = 0$ in a three-dimensional exterior domain. In the papers [8,9] on the two-dimensional problem, the challenge is also overcome with appropriate function spaces. A novelty of the current paper is the employing of function spaces that not only incorporate the steady-state setting, but are in fact optimal in a certain sense. In [16] the linearization of (1.1)–(1.3) was investigated and Banach spaces that establish maximal regularity in an $L^q$-setting were identified. We will show in this paper that the nonlinear term in (1.1) can be treated as a 'right-hand side' in the setting of these Banach spaces. Consequently, an ‘optimal’ setting offers many advantages. For example, it allows us to establish additional regularity in a solution by boot-strapping arguments. The construction of the Banach spaces in [16] is based on a decomposition of the problem into a steady-state problem and a time-periodic problem involving only vector fields with vanishing time average. We shall employ in this paper both the decomposition and the maximal regularity results established in [16].

2. A statement of the main results

We start by defining the function spaces needed to state the main theorems. We denote points in $\mathbb{R}^3 \times \mathbb{R}$ by $(x, t)$, and refer to $x$ as the spatial variable and $t$ as the time variable. We introduce the spaces of real functions

\[ C_{\infty}^0(\mathbb{R}^3 \times \mathbb{R}) := \{ U \in C^\infty(\mathbb{R}^3 \times \mathbb{R}) \mid \forall t \in \mathbb{R} : U(\cdot, t + T) = U(\cdot, t) \}, \]

\[ C_0,per(\mathbb{R}^3 \times [0, T]) := \{ u \in C_0(\mathbb{R}^3 \times [0, T]) \mid \exists U \in C_{\infty}^0(\mathbb{R}^3 \times \mathbb{R}) : U = U|_{\mathbb{R}^3 \times [0, T]} \}, \]

\[ C_{0,\sigma,per}(\mathbb{R}^3 \times [0, T]) := \{ u \in C_0^{\infty}(\mathbb{R}^3 \times [0, T]) \mid \text{div}_t u = 0 \}. \]

We introduce Lebesgue and Sobolev spaces as completions of the spaces above in different norms. Lebesgue spaces are defined for $q \in [1, \infty)$ by

\[ L^q_{per}(\mathbb{R}^3 \times (0, T)) := C_{0,per}^{\infty}(\mathbb{R}^3 \times [0, T])^{1/q}, \quad \| u \|_q := \| u \|_{L^q(\mathbb{R}^3 \times (0, T))}. \]

Clearly, $L^q_{per}(\mathbb{R}^3 \times (0, T))$ coincides with the classical Lebesgue space $L^q(\mathbb{R}^3 \times (0, T))$, and we will therefore omit the subscript per for Lebesgue spaces in the following. Sobolev spaces of $T$-time-periodic functions are defined for $k \in \mathbb{N}_0$ by

\[ W^{k,q}_{per}(\mathbb{R}^3 \times (0, T)) := C_{0,per}^{\infty}(\mathbb{R}^3 \times [0, T])^{1/q}, \]

\[ \| u \|_{k,q} := \left( \sum_{(\alpha, \beta) \in \mathbb{N}_0^3 \times \mathbb{N}_0} \| \partial_\alpha^\alpha \partial_\beta^\beta u \|_q^k \right)^{1/q}. \] (2.1)

In contrast to the Lebesgue spaces, the Sobolev spaces $W^{k,q}_{per}(\mathbb{R}^3 \times (0, T))$ do not coincide with the classical Sobolev spaces $W^{k,q}(\mathbb{R}^3 \times (0, T))$. For $q \in [1, \infty)$ we define the Lebesgue space of solenoidal vector fields

\[ L^q(\mathbb{R}^3 \times (0, T)) := C_{0,\sigma,per}^{\infty}(\mathbb{R}^3 \times [0, T])^{1/q}. \]
and the anisotropic Sobolev space of $T$-time-periodic, solenoidal, vector fields
\[
W^{2,1,q}_{\sigma,\text{per}}(\mathbb{R}^3 \times (0, T)) := C^{\infty}_{0,\sigma,\text{per}}(\mathbb{R}^3 \times [0, T])^{\perp,1,q},
\]
\[
\|u\|_{2,1,q} := \left( \sum_{(\alpha, \beta) \in \mathbb{N}_0^3} \|\partial_\alpha u\|_q^q + \|\partial_\beta u\|_q^q \right)^{1/q}.
\] (2.2)

In order to incorporate the decomposition described in the introduction on the level of function spaces, we define on functions $u : \mathbb{R}^3 \times (0, T) \to \mathbb{R}$ the operators
\[
\mathcal{P} u(x, t) := \frac{1}{T} \int_0^T u(x, s) \, ds
\]
and
\[
\mathcal{P}_\perp u(x, t) := u(x, t) - \mathcal{P} u(x, t)
\]
whenever these expressions are well-defined. Note that $\mathcal{P}$ and $\mathcal{P}_\perp$ decompose a time-periodic vector field $u$ into a time-independent part $\mathcal{P} u$ and a time-periodic part $\mathcal{P}_\perp u$ with vanishing time average over the period. Also note that $\mathcal{P}$ and $\mathcal{P}_\perp$ are complementary projections, that is, $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{P}_\perp = \text{Id} - \mathcal{P}$. As one may easily verify,
\[
\mathcal{P}, \mathcal{P}_\perp : C^\infty_{0,\text{per}}(\mathbb{R}^3 \times [0, T]) \to C^\infty_{0,\text{per}}(\mathbb{R}^3 \times [0, T]),
\]
and both projections extend by continuity to bounded operators on $L^q_\perp(\mathbb{R}^3 \times (0, T))$ and $W^{2,1,q}_{\sigma,\text{per}}(\mathbb{R}^3 \times (0, T))$. We can thus define
\[
L^q_{\sigma,\perp}(\mathbb{R}^3 \times (0, T)) := \mathcal{P}_\perp L^q(\mathbb{R}^3 \times (0, T)),
\]
\[
W^{2,1,q}_{\sigma,\text{per},\perp}(\mathbb{R}^3 \times (0, T)) := \mathcal{P}_\perp W^{2,1,q}_{\sigma,\text{per}}(\mathbb{R}^3 \times (0, T)).
\]

For convenience, we introduce for intersections of such spaces the notation
\[
L^q_{\sigma,\perp}(\mathbb{R}^3 \times (0, T)) := L^q_{\sigma,\perp}(\mathbb{R}^3 \times (0, T)) \cap L^q_{\sigma,\perp}(\mathbb{R}^3 \times (0, T)),
\]
\[
W^{2,1,q}_{\sigma,\text{per},\perp}(\mathbb{R}^3 \times (0, T)) := W^{2,1,q}_{\sigma,\text{per},\perp}(\mathbb{R}^3 \times (0, T)) \cap W^{2,1,q}_{\sigma,\text{per},\perp}(\mathbb{R}^3 \times (0, T)).
\]

For $q \in (1, 2)$ we let
\[
X^q_{\sigma,\text{Oseen}}(\mathbb{R}^3) := \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^3)^3 \mid \div v = 0, \|v\|_{q,\text{Oseen}} < \infty \right\},
\]
\[
\|v\|_{q,\text{Oseen}} := |\lambda|^\frac{q}{2} \|v\|_{\mathcal{H}_0^q} + |\lambda|^\frac{q}{2} \|
abla v\|_{\mathcal{H}_0^q} + |\lambda|\|\partial_1 v\|_q + \|
abla^2 v\|_q,
\] (2.4)

which is a Banach space intrinsically linked with the three-dimensional Oseen operator. For $q \in (1, 2)$ and $r \in (1, \infty)$ we put
\[
X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3) := \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^3)^3 \mid \div v = 0, \|v\|_{q,r,\text{Oseen}} < \infty \right\},
\]
\[
\|v\|_{q,r,\text{Oseen}} := \|v\|_{q,\text{Oseen}} + \|\nabla^2 v\|_r.
\] (2.5)

For $q \in (1, 3)$ and $r \in (1, \infty)$ we further define
\[
X^{q,r}_{\text{per}}(\mathbb{R}^3 \times (0, T)) := \left\{ p \in L^1_{\text{loc}}(\mathbb{R}^3 \times (0, T)) \mid \|p\|_{X^{q,r}_{\text{per}} \cap \mathcal{H}_0^q} < \infty \right\},
\]
\[
\|p\|_{X^{q,r}_{\text{per}}} := \left( \int_0^T \|p(\cdot, t)\|_{q,r}^q + \|
abla_x p(\cdot, t)\|_{q,r}^q \, dt \right)^{1/q} + \|
abla_x p\|_r.
\] (2.6)

Finally, we let
\[
D^{1,2}_{0,\sigma}(\mathbb{R}^3) := C^\infty_{0,\sigma}(\mathbb{R}^3)^{\perp,1,2} := \{ u \in L^6(\mathbb{R}^3)^3 \mid \nabla u \in L^2(\mathbb{R}^3)^{3 \times 3}, \div u = 0 \}
\]
denote the classical homogeneous Sobolev space of solenoidal vector fields (the latter equality above is due to a standard Sobolev embedding theorem) and
\[
W^{a,b,q}_{\text{loc}}(\mathbb{R}^3) := \{ u \in L^q_{\text{loc}}(\mathbb{R}^3) \mid \partial_\alpha u, \partial_\beta u \in L^q_{\text{loc}}(\mathbb{R}^3) \} \quad \text{for } |\alpha| \leq a, |\beta| \leq b
\]
for $q \in [1, \infty)$ and $a, b \in \mathbb{N}_0$.  

Throughout the paper, we shall frequently consider the restriction of $T$-time-periodic functions defined on $\mathbb{R}^3 \times \mathbb{R}$ to the domain $\mathbb{R}^3 \times (0, T)$. More specifically, without additional notation we implicitly treat $T$-time-periodic functions $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ as functions $f : \mathbb{R}^3 \times (0, T) \to \mathbb{R}$. If $f$ is independent of $t$, we may treat it as a function $f : \mathbb{R}^3 \to \mathbb{R}$.

We are now in a position to state the main results of the paper. The first theorem establishes the existence of a strong solution for sufficiently small data. It is further shown that this solution is unique in large class of weak solutions that can be considered physically reasonable. We first define this class.

**Definition 2.1.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R})^3$ satisfy (1.3). We say that $\mathcal{U} \in L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R})^3$ satisfying (1.2) is a physically reasonable weak time-periodic solution to (1.1) if

1. $\mathcal{U} \in L^2((0, T); D^{1,0,0}_{T,\text{per}}(\mathbb{R}^3))$,
2. $\mathcal{P}_\perp \mathcal{U} \in L^\infty((0, T); L^2(\mathbb{R}^3)^3)$,
3. $\mathcal{U}$ is a generalized $T$-time-periodic solution to (1.1) in the sense that for all test functions $\Phi \in C_{0,\text{per}}^\infty(\mathbb{R}^3 \times (0, T))$ it holds that

$\int_0^T \int_{\mathbb{R}^3} -\mathcal{U} : \partial_t \Phi + \nabla \cdot \mathcal{U} \cdot \nabla \Phi - \lambda \partial_t \mathcal{U} : \Phi + (\mathcal{U} : \nabla \mathcal{U}) : \Phi \, dx \, dt = \int_0^T \int_{\mathbb{R}^3} f \cdot \Phi \, dx \, dt,$

(2.7)

4. $\mathcal{U}$ satisfies the energy inequality

$\int_0^T \int_{\mathbb{R}^3} |\nabla \mathcal{U}|^2 \, dx \, dt \leq \int_0^T \int_{\mathbb{R}^3} f \cdot \mathcal{U} \, dx \, dt.$

(2.8)

**Remark 2.2.** The characterization of a solution satisfying (1)–(4) in definition 2.1 as a physically reasonable solution is justified by the physical properties that can be derived from property (2) and (4). More precisely, if we consider the fluid flow corresponding to the Eulerian velocity field $\mathcal{U}$ as the sum of a steady state $\mathcal{P} \mathcal{U}$ and a non-steady part $\mathcal{P}_\perp \mathcal{U}$, property (2) implies that the kinetic energy of the non-steady part of the flow is bounded. Property (4) states that the energy dissipated due to the viscosity of the fluid is less than the input of energy from the external forces.

We now state the first main theorem of the paper, which establishes the existence of a strong solution unique in the class of physically reasonable weak solutions. We shall further show that this solution satisfies an energy equality. The theorem reads:

**Theorem 2.3.** Let $q \in (1, \frac{3}{2})$, $r \in (4, \infty)$ and $\lambda \neq 0$. There is a constant $\varepsilon_0 > 0$ such that for any $f \in L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R})^3$ satisfying (1.3) and

$\|f\|_{L^q(\mathbb{R}^3 \times (0, T))} + \|f\|_{L^r(\mathbb{R}^3 \times (0, T))} \leq \varepsilon_0$

(2.9)

there is a solution $(u, p) \in W^{2,1,1}_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R})^3 \times W^{1,0,0}_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R})$ to (1.1)–(1.2) with $u = v + w$ and

$(v, w, p) \in X^{q,1,1}_{O,\text{per}}(\mathbb{R}^3) \times W^{0,1,1}_{\text{per},\perp}(\mathbb{R}^3 \times (0, T)) \times X^{q,1,1}_{\text{per}}(\mathbb{R}^3 \times (0, T)).$

(2.10)

1. We can consider the restriction $\mathcal{U} \in L^1_{\text{loc}}(\mathbb{R}^3 \times (0, T))$ as a vector-valued mapping $t \to \mathcal{U}(\cdot, t)$. Moreover, it is easy to see that $\mathcal{P} \mathcal{U}$ and thus also $\mathcal{P}_\perp \mathcal{U}$ are well-defined as elements in $L^1_{\text{loc}}(\mathbb{R}^3 \times (0, T))$. Consequently, we may also consider $\mathcal{P}_\perp \mathcal{U}$ as a vector-valued mapping $t \to \mathcal{P}_\perp \mathcal{U}(\cdot, t)$.

2. The integral on the right-hand side of (2.8) is not necessarily well-defined for $f \in L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R})^3$ and $\mathcal{U}$ satisfying (1)–(2). Included in the definition of a physically reasonable weak time-periodic solution is therefore an implicit condition that these vector fields possess enough integrability for this integral to be well-defined.

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Moreover, \( u \) belongs to and is unique in the class of physically reasonable weak solutions characterized by definition 2.1, and it satisfies the energy equality

\[
\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt = \int_0^T \int_{\mathbb{R}^3} f \cdot u \, dx \, dt.
\] (2.11)

The second main theorem of the paper concerns regularity properties of strong solutions. More specifically, it is shown that any additional regularity of the data translates into a similar degree of additional regularity for the solution.

**Theorem 2.4.** Let \( q \in \{1, \frac{4}{3}\}, r \in (8, \infty), \lambda \neq 0 \) and \( m \in \mathbb{N}_0 \). If \( f \in L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3) \) satisfies (1.3) and

\[
f \in W^{m,q}_{\text{per}}(\mathbb{R}^3 \times (0, T))^3 \cap W^{m,r}_{\text{per}}(\mathbb{R}^3 \times (0, T))^3,
\]

then a solution \((u, p) \in L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R})^3 \times L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}) \) to (1.1)–(1.2) in the class (2.10) (with \( u = v + w \)) satisfies

\[
\forall (\alpha, \beta, \kappa) \in \mathbb{N}_0^3 \times \mathbb{N}_0^3 \times \mathbb{N}_0, \ |\alpha| \leq m, \ |\beta| + |\kappa| \leq m:\]

\[
\begin{align*}
(\partial^\alpha v_0, \partial^\beta \partial_0^\alpha w, \partial^\beta \partial_0^\alpha p) & \in W^{2,1, r}_{\text{per}}(\mathbb{R}^3) \times W^{1,0, r}_{\text{per}}(\mathbb{R}^3) \times W^{1,0, r}_{\text{per}}(\mathbb{R}^3) \quad \text{with} \\
(\partial^\alpha v, \partial^\beta \partial_0^\alpha w, \partial^\beta \partial_0^\alpha p) & \in X^{q, r}_{\text{Oseen}}(\mathbb{R}^3) \times W^{2,1, q, r}_{\text{per}}(\mathbb{R}^3 \times (0, T)) \times X^{q, r}_{\text{pres}}(\mathbb{R}^3 \times (0, T)).
\end{align*}
\]

We state, as a corollary to theorem 2.4, that a strong solution is smooth if we have smooth data.

**Corollary 2.5.** Let \( q \in \{1, \frac{4}{3}\}, r \in (8, \infty) \) and \( \lambda \neq 0 \). If \( f \in C^\infty_{\text{per}}(\mathbb{R}^3 \times \mathbb{R})^3 \), then a solution \((u, p) \in L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R})^3 \times L^1_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}) \) to (1.1)–(1.2) in the class (2.10) (with \( u = v + w \)) satisfies \( u \in C^\infty_{\text{per}}(\mathbb{R}^3 \times \mathbb{R})^3 \) and \( p \in C^\infty_{\text{per}}(\mathbb{R}^3 \times \mathbb{R}) \).

### 3. Notation

Points in \( \mathbb{R}^3 \times \mathbb{R} \) are denoted by \((x, t)\) with \( x \in \mathbb{R}^3 \) and \( t \in \mathbb{R} \). We refer to \( x \) as the spatial variable and to \( t \) as the time variable.

For a sufficiently regular function \( u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \), we put \( \partial_t u := \partial_{x^0} u \). For any multi-index \( \alpha \in \mathbb{N}_0^3 \), we let \( \partial^\alpha u := \sum_{j=1}^3 \partial_{x^j}^\alpha u \) and put \( |\alpha| := \sum_{j=1}^3 \alpha_j \). Moreover, for \( x \in \mathbb{R}^3 \) we let \( x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \). Differential operators act only in the spatial variable unless otherwise indicated. For example, we denote by \( \Delta u \) the Laplacian of \( u \) with respect to the spatial variables, that is, \( \Delta u := \sum_{j=1}^3 \partial_{x^j}^2 u \). For a vector field \( u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \), we let \( \text{div } u := \sum_{j=1}^3 \partial_{x^j} u \) be the divergence of \( u \). For \( u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \) and \( v : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \) we let \((u \cdot \nabla v) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \) denote the vector field \((u \cdot \nabla v)_j := \sum_{j=1}^3 \partial_{x^j} u v_j \).

For two vectors \( a, b \in \mathbb{R}^3 \), we let \( a \otimes b \in \mathbb{R}^{3 \times 3} \) denote the tensor with \((a \otimes b)_{ij} := a_i b_j \). We denote by \( I \) the identity tensor \( I \in \mathbb{R}^{3 \times 3} \).

We use the symbol \( \hookrightarrow \) to denote an embedding \( X \hookrightarrow Y \) of one vector space \( X \) into another vector space \( Y \). In the case of topological vector spaces, embeddings are always required to be continuous.

Constants represented by capital letters in the proofs and theorems are global, while constants represented by lower case letters are local to the proof in which they appear.
4. Reformulation in a group setting

We let $G$ denote the group

$$G := \mathbb{R}^3 \times \mathbb{R}/T\mathbb{Z}$$

with addition as the group operation. Clearly, there is a natural correspondence between $T$-time-periodic functions defined on $\mathbb{R}^3 \times \mathbb{R}$ and functions defined on $G$. We shall take advantage of this correspondence and reformulate (1.1)–(1.3) and the main theorems in a setting of vector fields defined on $G$. For this purpose, we introduce a differentiable structure on $G$ and define appropriate Lebesgue and Sobolev spaces.

4.1. Differentiable structure, distributions and the Fourier transform

The topology and differentiable structure on $G$ are inherited from $\mathbb{R}^3 \times \mathbb{R}$. More precisely, we equip $G$ with the quotient topology induced by the canonical quotient mapping

$$\pi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}/T\mathbb{Z}, \quad \pi(x, t) := (x, [t]).$$

Equipped with the quotient topology, $G$ becomes a locally compact abelian group. We shall use the restriction

$$\Pi : \mathbb{R}^3 \times [0, T) \to G, \quad \Pi : \pi |_{\mathbb{R}^3 \times [0, T)}$$

to identify $G$ with the domain $\mathbb{R}^3 \times [0, T)$. $\Pi$ is clearly a (continuous) bijection.

Via $\Pi$, one can identify the Haar measure $\,dg$ on $G$ as the product of the Lebesgue measure on $\mathbb{R}^3$ and the Lebesgue measure on $[0, T)$. The Haar measure is unique up to a normalization factor, which we choose such that

$$\forall u \in C_0(G) : \quad \int_G u(g) \, dg = \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} u \circ \Pi(x, t) \, dx \, dt,$$

where $C_0(G)$ denotes the space of continuous functions of compact support. For the sake of convenience, we will omit the $\,dg$ of $G$-defined functions, that is, instead of $\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} u \circ \Pi(x, t) \, dx \, dt$ we simply write $\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} u(x, t) \, dx \, dt$.

Next, we define by

$$C^\infty(G) := \{ u : G \to \mathbb{R} | u \circ \pi \in C^\infty(\mathbb{R}^3 \times \mathbb{R}) \}$$

the space of smooth functions on $G$. For $u \in C^\infty(G)$ we define derivatives

$$\forall (\alpha, \beta, \gamma) \in \mathbb{N}_0^3 \times \mathbb{N}_0 : \quad \partial^\alpha \partial^\beta_t \partial^\gamma_x u := \left[ \partial^\alpha \partial^\beta_t \partial^\gamma_x (u \circ \pi) \right] \circ \Pi^{-1}.$$

It is easy to verify for $u \in C^\infty(G)$ that also $\partial^\beta_t \partial^\gamma_x u \in C^\infty(G)$.

With a differentiable structure defined on $G$ via (4.1), we can introduce the space of tempered distributions on $G$. For this purpose, we first recall the Schwartz–Bruhat space of generalized Schwartz functions; see for example [1]. More precisely, we define for $u \in C^\infty(G)$ the semi-norms

$$\rho_{\alpha, \beta, \gamma}(u) := \sup_{(x, t) \in G} |x^\gamma \partial^\beta_t \partial^\alpha_x u(x, t)| \quad \text{for } (\alpha, \beta, \gamma) \in \mathbb{N}_0^3 \times \mathbb{N}_0 \times \mathbb{N}_0^3,$$

and put

$$\mathcal{S}(G) := \{ u \in C^\infty(G) | \forall (\alpha, \beta, \gamma) \in \mathbb{N}_0^3 \times \mathbb{N}_0 \times \mathbb{N}_0^3 : \rho_{\alpha, \beta, \gamma}(u) < \infty \}.$$
Clearly, \( \mathcal{S}(G) \) is a vector space and \( \rho_{a,\beta,\gamma} \) a semi-norm on \( \mathcal{S}(G) \). We endow \( \mathcal{S}(G) \) with the semi-norm topology induced by the family \( \{ \rho_{a,\beta,\gamma} \mid (a, \beta, \gamma) \in \mathbb{N}_0^3 \times \mathbb{N}_0 \times \mathbb{N}_0 \} \). The topological dual space \( \mathcal{S}'(G) \) of \( \mathcal{S}(G) \) is then well-defined. We equip \( \mathcal{S}'(G) \) with the weak* topology and refer to it as the space of tempered distributions on \( G \). Observe that both \( \mathcal{S}(G) \) and \( \mathcal{S}'(G) \) remain closed under multiplication by smooth functions that have at most polynomial growth with respect to the spatial variables.

For a tempered distribution \( u \in \mathcal{S}'(G) \), distributional derivatives \( \partial^\beta \partial^\gamma_x u \in \mathcal{S}'(G) \) are defined by duality in the usual manner:

\[
\forall \psi \in \mathcal{S}(G) : \langle \partial^\beta \partial^\gamma_x u, \psi \rangle := \langle u, (-1)^{(a,\beta,\gamma)} \partial^\beta \partial^\gamma_x \psi \rangle.
\]

It is easy to verify that \( \partial^\beta \partial^\gamma_x u \) is well-defined as an element of \( \mathcal{S}'(G) \). For tempered distributions on \( G \), we keep the convention that differential operators like \( \Delta \) and \( \text{div} \) act only in the spatial variable \( x \) unless otherwise indicated.

We shall also introduce tempered distributions on \( G \)'s dual group \( \widehat{G} \). We associate each \((\xi, k) \in \mathbb{R}^3 \times \mathbb{Z}\) with the character \( \chi : G \to \mathbb{C}, \chi(x, t) := e^{ix \cdot \xi + ik \cdot t} \) on \( G \). It is standard to verify that all characters are of this form, and we can thus identify \( \widehat{G} = \mathbb{R}^3 \times \mathbb{Z} \). By default, \( \widehat{G} \) is equipped with the compact–open topology, which in this case coincides with the product of the Euclidean topology on \( \mathbb{R}^3 \) and the discrete topology on \( \mathbb{Z} \). The Haar measure on \( G \) is the semi-norm topology induced by the family of semi-norms \( \| (\alpha, \beta, \gamma) \| = \sup_{(\xi, k) \in \widehat{G}} |k^\beta \xi^\alpha \partial^\gamma_x \psi| \). For tempered distributions on \( G \), we keep the convention that differential operators like \( \Delta \) and \( \text{div} \) act only in the spatial variable \( x \) unless otherwise indicated.

A differentiable structure on \( \widehat{G} \) is obtained by introduction of the space

\[
C^\infty(\widehat{G}) := \{ w \in C(\widehat{G}) \mid \forall k \in \mathbb{Z} : w(\cdot, k) \in C^\infty(\mathbb{R}^3) \}.
\]

To define the generalized Schwartz–Bruhat space on the dual group \( \widehat{G} \), we further introduce for \( w \in C^\infty(\widehat{G}) \) the semi-norms

\[
\hat{\rho}_{a,\beta,\gamma}(w) := \sup_{(\xi, k) \in \widehat{G}} |(\xi, k)^{a,\beta,\gamma} w(\xi, k)| \quad \text{for} \quad (\alpha, \beta, \gamma) \in \mathbb{N}_0^3 \times \mathbb{N}_0 \times \mathbb{N}_0^3.
\]

We then put

\[
\mathcal{S}(\widehat{G}) := \{ w \in C^\infty(\widehat{G}) \mid \forall (\alpha, \beta, \gamma) \in \mathbb{N}_0^3 \times \mathbb{N}_0 \times \mathbb{N}_0^3 : \hat{\rho}_{a,\beta,\gamma}(w) < \infty \}.
\]

We endow the vector space \( \mathcal{S}(\widehat{G}) \) with the semi-norm topology induced by the family of semi-norms \( \{ \hat{\rho}_{a,\beta,\gamma} \mid (\alpha, \beta, \gamma) \in \mathbb{N}_0^3 \times \mathbb{N}_0 \times \mathbb{N}_0^3 \} \). The topological dual space of \( \mathcal{S}(\widehat{G}) \) is denoted by \( \mathcal{S}'(\widehat{G}) \). We equip \( \mathcal{S}'(\widehat{G}) \) with the weak* topology and refer to it as the space of tempered distributions on \( G \).

So far, all function spaces have been defined as real vector spaces of real functions. Clearly, we can define them analogously as complex vector spaces of complex functions. When a function space is used in context with the Fourier transform, which we shall introduce below, we consider it as a complex vector space.

The Fourier transform \( \mathcal{F}_G \) on \( G \) is given by

\[
\mathcal{F}_G : L^1(G) \to C(\widehat{G}), \quad \mathcal{F}_G(u)(\xi, k) := \hat{u}(\xi, k) := \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} u(x, t) e^{-ix \cdot \xi - it \cdot \frac{k}{T}} \, dx \, dt.
\]

If no confusion can arise, we simply write \( \mathcal{F} \) instead of \( \mathcal{F}_G \). The inverse Fourier transform is formally defined by

\[
\mathcal{F}^{-1} : L^1(\widehat{G}) \to C(G), \quad \mathcal{F}^{-1}(w)(x, t) := w^\vee(x, t) := \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^3} w(\xi, k) e^{ix \cdot \xi + it \cdot \frac{k}{T}} \, d\xi.
\]

It is standard to verify that \( \mathcal{F} : \mathcal{S}(G) \to \mathcal{S}(\widehat{G}) \) is a homeomorphism with \( \mathcal{F}^{-1} \) as the actual inverse, provided the Lebesgue measure \( d\xi \) is normalized appropriately. By duality, \( \mathcal{F} \) extends to a mapping \( \mathcal{F}'(G) \to \mathcal{F}'(\widehat{G}) \). More precisely, we define

\[
\mathcal{F} : \mathcal{S}'(G) \to \mathcal{S}'(\widehat{G}), \quad \forall \psi \in \mathcal{S}'(\widehat{G}) : \langle \mathcal{F}(u), \psi \rangle := \langle u, \mathcal{F}(\psi) \rangle.
\]
Similarly, we define
\[ \mathcal{F}^{-1} : \mathcal{S}'(G) \to \mathcal{S}'(\hat{G}), \quad \forall \psi \in \mathcal{S}'(G) : \langle \mathcal{F}^{-1}(u), \psi \rangle := \langle u, \mathcal{F}^{-1}(\psi) \rangle. \]
Clearly \( \mathcal{F} : \mathcal{S}'(G) \to \mathcal{S}'(\hat{G}) \) is a homeomorphism with \( \mathcal{F}^{-1} \) as the actual inverse.

The Fourier transform in the setting above provides us with a calculus connecting the differential operators on \( G \) and the polynomials on \( \hat{G} \). As one easily verifies, for \( u \in \mathcal{S}'(G) \) and \( \alpha \in \mathbb{N}^3_0, l \in \mathbb{N}_0 \) we have
\[ \mathcal{F}(\partial^l_t \partial^\alpha_x u) = i^{|\alpha|} \left( \frac{2\pi}{T} \right)^l k^l \xi^\alpha \mathcal{F}(u) \]
as an identity in \( \mathcal{S}'(\hat{G}) \).

### 4.2. Function spaces

Having introduced smooth functions on \( G \) in the form of the space \( C^\infty(G) \), we define function spaces of \( G \)-defined functions and vector fields corresponding to the Lebesgue and Sobolev spaces of \( T \)-time-periodic functions and vector fields introduced in section 2.

We start by putting
\[ C^\infty_0(G) := \{ u \in C^\infty(G) \mid \text{supp } u \text{ is compact} \}. \]
We let \( L^q(G) \) denote the usual Lebesgue space with respect to the Haar measure \( dg \), and let \( \| \cdot \|_q \) denote the norm. It is standard to verify that \( L^q(G) \subset \mathcal{S}'(G) \). Classical Sobolev spaces are then defined as
\[ W^{k,q}(G) := \{ u \in L^q(G) \mid \| u \|_{k,q} < \infty \}, \]
where \( \| \cdot \|_{k,q} \) is defined exactly as in (2.1), and the condition \( \| u \|_{k,q} < \infty \) expresses that the distributional derivatives of \( u \) appearing in the norm \( \| \cdot \|_{k,q} \) all belong to \( L^q(G) \). We note that \( W^{k,q}(G) = C^\infty_0(G) \| \cdot \|_{k,q} \) and \( L^q(G) = C^\infty_0(G) \| \cdot \|_q \), which can be shown by standard arguments.

Next, we let
\[ C^\infty_{0,\sigma}(G) := \{ u \in C^\infty_0(G) \mid \text{div } u = 0 \}, \]
and define the Banach spaces
\[ L^q_\sigma(G) := C^\infty_{0,\sigma}(G) \| \cdot \|_{2,1,q}, \quad W^{2,1,q}_\sigma(G) := C^\infty_{0,\sigma}(G) \| \cdot \|_{2,1,q} \]
of solenoidal vector fields, where the norm \( \| \cdot \|_{2,1,q} \) is defined as in (2.2). It can be shown that
\[ L^q_\sigma(G) = \{ u \in L^q(G) \mid \text{div } u = 0 \}. \tag{4.2} \]
This identity is well-known for when the underlying domain is \( \mathbb{R}^3 \); a proof can be found in [6, chapter III.4]. Simple modifications to this proof (see [15, lemma 3.2.1]) suffice to establish the identity in the case where \( \mathbb{R}^3 \) is replaced with \( G \). For convenience, we put
\[ L^{q,r}_\sigma(G) := L^q_\sigma(G) \cap L^r_\sigma(G), \quad \| \cdot \|_{L^{q,r}_\sigma(G)} := \| \cdot \|_q + \| \cdot \|_r, \]
\[ W^{2,1,q,r}_\sigma(G) := W^{2,1,q}_\sigma(G) \cap W^{2,1,r}_\sigma(G), \quad \| \cdot \|_{2,1,q,r} := \| \cdot \|_{2,1,q} + \| \cdot \|_{2,1,r}, \]
which are obviously Banach spaces in the associated norms.

Recalling (2.3), we define analogously the projection \( \mathcal{P} \) on \( G \)-defined functions:
\[ \mathcal{P} : C^\infty_0(G) \to C^\infty_0(G), \quad \mathcal{P}u(x, t) := \frac{1}{T} \int_0^T u(x, s) \, ds \]
and put \( \mathcal{P}_\perp := \text{Id} - \mathcal{P} \). We note the following properties:
Lemma 4.1. Let $q \in (1, \infty)$. The projection $\mathcal{P} : C_{0,\sigma}^\infty(G) \to C_{0,\sigma}^\infty(G)$ extends uniquely, by continuity, to a bounded projection $\mathcal{P} : L^q(G) \to L^q(G)$ and to a bounded projection $\mathcal{P} : W^{2,1,q}_\sigma(G) \to W^{2,1,q}_\sigma(G)$. The same is true for $\mathcal{P}_\perp$.

Proof. Boundedness of $\mathcal{P}$ in the norms of $L^q(G)$ and $W^{2,1,q}_\sigma(G)$ can easily be verified by employing Hölder’s and Minkowski’s integral inequalities; see also [16, lemma 4.5].

Clearly, both $\mathcal{P}$ and $\mathcal{P}_\perp$ extend uniquely to projections:

$$
\mathcal{P} : L^q_{\text{loc}}(G) \to L^q_{\text{loc}}(G), \quad \mathcal{P}_\perp : L^q_{\text{loc}}(G) \to L^q_{\text{loc}}(G).
$$

We observe:

Lemma 4.2. Let $f, g \in L^1_{\text{loc}}(G)$. Then

$$
\frac{1}{T} \int_0^T \mathcal{P} f(x, t) \cdot \mathcal{P}_\perp g(x, t) \, dt = 0 \quad \text{for a.e. } x \in \mathbb{R}^3.
$$

Proof. This is a simple consequence of the fact that $\mathcal{P} f$ is independent of $t$.

Lemma 4.3. The projections $\mathcal{P}$ and $\mathcal{P}_\perp$ extend uniquely, by continuity, to continuous operators $\mathcal{P} : \mathcal{F}(G) \to \mathcal{F}(G)$ and $\mathcal{P}_\perp : \mathcal{F}(G) \to \mathcal{F}(G)$ with

$$
\mathcal{P} f = \mathcal{F}_G^{-1} \left[ \kappa_0 \right] f,
$$

$$
\mathcal{P}_\perp f = \mathcal{F}_G^{-1} \left[ (1 - \kappa_0) \right] f,
$$

where

$$
\kappa_0 : \hat{G} \to \mathbb{C}, \quad \kappa_0(\xi, k) := \begin{cases} 
1 & \text{if } k = 0, \\
0 & \text{if } k \neq 0.
\end{cases}
$$

Proof. We simply observe for $f \in \mathcal{F}(G)$ that

$$
\mathcal{F}_G [\mathcal{P} f](\xi, k) = \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \frac{1}{T} \int_0^T f(x, s) \, ds \, e^{-i\xi \cdot \pi + ik \cdot t} \, dx \, dt
$$

$$
= \kappa_0(\xi, k) \int_{\mathbb{R}^3} \frac{1}{T} \int_0^T f(x, s) \, ds \, e^{-i\xi \cdot x} \, dx = \kappa_0(\xi, k) \hat{f}(\xi, 0) = \kappa_0(\xi, k) \hat{f}(\xi, k).
$$

The formula extends to $f \in \mathcal{F}'(G)$ by duality.

Having introduced the projections $\mathcal{P}$ and $\mathcal{P}_\perp$, we can now define

$$
L^q_{\sigma,\perp}(G) := \mathcal{P}_\perp L^q_{\sigma}(G),
$$

$$
L^q_{\sigma,\perp'}(G) := \mathcal{P}_\perp L^q_{\sigma'}(G) = L^q_{\sigma,\perp}(G) \cap L^q_{\sigma,\perp}(G),
$$

$$
W^{2,1,q}_{\sigma,\perp}(G) := \mathcal{P}_\perp W^{2,1,q}_{\sigma}(G),
$$

$$
W^{2,1,q}_{\sigma,\perp'}(G) := \mathcal{P}_\perp W^{2,1,q}_{\sigma'}(G) = W^{2,1,q}_{\sigma,\perp}(G) \cap W^{2,1,q}_{\sigma,\perp}(G).
$$

Since $\mathcal{P} u$ is independent of $t$, it is easy to verify that $\mathcal{P} L^q_{\sigma}(G) = L^q_{\sigma}(\mathbb{R}^3)$. It follows that $\mathcal{P}$ induces the decomposition

$$
L^q_{\sigma}(G) = L^q_{\sigma}(\mathbb{R}^3) \oplus L^q_{\sigma,\perp}(G).
$$

Next, we introduce the Helmholtz projection on the Lebesgue space $L^q(G)^3$ via a classical Fourier-multiplier expression:
Lemma 4.4. The Helmholtz projection

\[ \mathcal{P}_H : L^2(G)^3 \to L^2(G)^3, \quad \mathcal{P}_H f := \mathcal{F}^{-1}_G \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \hat{f} \]  

(4.7)

extends for any \( q \in [1, \infty) \) uniquely to a continuous projection \( \mathcal{P}_H : L^q(G)^3 \to L^q(G)^3 \). Moreover, \( \mathcal{P}_H L^q(G)^3 = L^q(G)^3 \).

Proof. The Fourier multiplier on the right-hand side in (4.7) is identical to the multiplier of the classical Helmholtz projection in the Euclidean \( \mathbb{R}^3 \)-setting. Boundedness of \( \mathcal{P}_H \) on \( L^q(G)^3 \) can thus be derived from boundedness of the classical Helmholtz projection on \( L^q(\mathbb{R}^3)^3 \). One readily verifies that \( \mathcal{P}_H \) is a projection, and that \( \text{div} \, \mathcal{P}_H f = 0 \). By (4.2), \( \mathcal{P}_H L^q(G)^3 \subset L^q(\mathbb{R}^3)^3 \) follows. On the other hand, since \( \text{div} \, f = 0 \) implies \( \xi_j \hat{f}_j = 0 \), we have \( \mathcal{P}_H f = f \) for all \( f \in L^q(G)^3 \). We conclude that \( \mathcal{P}_H L^q(G)^3 = L^q(G)^3 \). \( \square \)

Since \( \mathcal{P}_H : L^q(G)^3 \to L^q(G)^3 \) is a continuous projection, it decomposes \( L^q(G)^3 \) into a direct sum

\[ L^q(G)^3 = L^q_{\sigma}(G) \oplus \mathcal{Q}_q^0(G) \]

of closed subspaces with

\[ \mathcal{Q}_q^0(G) := (\text{Id} - \mathcal{P}_H) L^q(G)^3. \]

We further define

\[ \mathcal{Q}_q^0,r(G) := \mathcal{Q}_q^0(G) \cap \mathcal{Q}_r(G), \quad \|\cdot\|_{\mathcal{Q}_q^0,r(G)} := \|\cdot\|_q + \|\cdot\|_r, \]

which is clearly a Banach space with respect to the associated norm.

We introduce the convention that a \( G \)-defined function \( u : G \to \mathbb{R} \) can be considered an element of a function space of \( \mathbb{R}^3 \)-defined functions, say \( X(\mathbb{R}^3) \), if and only if \( u \) is independent of \( t \), and the restriction \( u|\mathbb{R}^3 \times \{0\} \) belongs to \( X(\mathbb{R}^3) \). In this context, we shall need, in addition to the spaces \( X_{\sigma, \text{Oseen}}(\mathbb{R}^3) \) defined in (2.5), also the homogeneous Sobolev spaces \( D^{m,q}(\mathbb{R}^3) \) and their associated semi-norms:

\[ D^{m,q}(\mathbb{R}^3) := \{ u \in L^1_{\text{loc}}(\mathbb{R}^3) \mid \forall \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| = m : \partial^\alpha u \in L^q(\mathbb{R}^3) \}, \]

\[ |u|_{m,q} := \left( \sum_{|\alpha| = m} \int_{\mathbb{R}^3} |\partial^\alpha u(x)|^q \, dx \right)^{1/q}. \]

Moreover, we will deploy the space of solenoidal vector fields

\[ L^q_{\sigma}(\mathbb{R}^3) := C^\infty_{0,\sigma}(\mathbb{R}^3)^3 \]

and

\[ L^{q,r}_{\sigma}(\mathbb{R}^3) := L^q_{\sigma}(\mathbb{R}^3) \cap L^r_{\sigma}(\mathbb{R}^3), \quad \|\cdot\|_{L^{q,r}_{\sigma}(\mathbb{R}^3)} := \|\cdot\|_q + \|\cdot\|_r, \]

which is obviously a Banach space in the given norm.

Finally, we define for \( G \)-defined functions the norm \( \|\cdot\|_{X_{\text{pres}}^{q,r}} \) exactly as in (2.6) and let

\[ X_{\text{pres}}^{q,r}(G) := \{ p \in L^1_{\text{loc}}(G) \mid \|p\|_{X_{\text{pres}}^{q,r}} < \infty \}. \]
4.3. Reformulation

Since the differentiable structure on $G$ is inherited from $\mathbb{R}^3 \times \mathbb{R}$, we can formulate (1.1)–(1.3) as a system of partial differential equations on $G$:

$$
\begin{cases}
\partial_t u - \Delta u - \lambda \partial_t u + \nabla p + u \cdot \nabla u = f & \text{in } G, \\
\text{div } u = 0 & \text{in } G,
\end{cases}
$$

(4.8)

with unknowns $u : G \to \mathbb{R}^3$ and $p : G \to \mathbb{R}$, and data $f : G \to \mathbb{R}^3$. Observe that in this formulation the periodicity conditions are no longer needed. Indeed, all functions defined on $G$ are necessarily $T$-time-periodic.

On the basis of the new formulation above, we obtain the following new formulations of theorems 2.3 and 2.4 in a setting of $G$-defined vector fields. For convenience, in the new formulation we split theorem 2.3 into three parts: the statement of existence, the balance of energy, and the statement of uniqueness.

**Theorem 4.5.** Let $q \in (1, \frac{4}{3}]$, $r \in (4, \infty)$ and $\lambda \neq 0$. There is a constant $\varepsilon_1 > 0$ such that for any $f \in L^q(G)^3 \cap L^r(G)^3$ with

$$\|f\|_{L^q(G)} + \|f\|_{L^r(G)} \leq \varepsilon_1$$

there is a solution $(u, p)$ to (4.8) with $u = v + w$ and

$$(v, w, p) \in X_{q, \text{Oseen}}^\perp(\mathbb{R}^3) \times W^{2,1,q,r}(G) \times X_{\text{pres}}^{q,r}(G).$$

(4.9)

**Definition 4.6.** Let $f \in L^q_\text{loc}(G)^3$. We say that $\mathcal{U} \in L^1_\text{loc}(G)^3$ is a physically reasonable weak solution to (4.8) if, considered as a mapping $t \mapsto \mathcal{U}(\cdot, t)$, it satisfies $\mathcal{U} \in L^2(0, T; D^1_{0}\text{Oseen}(\mathbb{R}^3))$, $\mathcal{P}_\perp \mathcal{U} \in L^\infty(0, T; L^2(\mathbb{R}^3)^3)$, $\mathcal{U}$ satisfies (2.7) for all $\Phi \in C^\infty_{0,0}(G)$, and $\mathcal{U}$ satisfies the energy inequality (2.8).

**Theorem 4.7.** Let $q \in (1, \frac{4}{3}]$, $r \in (4, \infty)$, $\lambda \neq 0$ and $f \in L^q(G)^3 \cap L^r(G)^3$. A solution $(u, p)$ to (4.8) in the class (4.10) (with $u = v + w$) satisfies the energy equation (2.11).

**Theorem 4.8.** Let $q \in (1, \frac{4}{3}]$, $r \in (4, \infty)$, $\lambda \neq 0$ and $f \in L^q(G)^3 \cap L^r(G)^3$. There is a constant $\varepsilon_2 > 0$ such that if $\|f\|_{L^q(G)} + \|f\|_{L^r(G)} \leq \varepsilon_2$, then a solution $(u, p)$ to (4.8) in the class (4.10) (with $u = v + w$) is unique in the class of physically reasonable weak solutions characterized by definition 4.6.

**Theorem 4.9.** Let $q \in (1, \frac{4}{3}]$, $r \in (8, \infty)$, $\lambda \neq 0$ and $m \in \mathbb{N}$. If $f \in W^{m,q}(G)^3 \cap W^{m,r}(G)^3$, then a solution $(u, p)$ to (4.8) in the class (4.10) (with $u = v + w$) satisfies

$$\forall (\alpha, \beta, \kappa) \in \mathbb{N}_0^3 \times \mathbb{N}_0^3 \times \mathbb{N}_0, \ |\alpha| \leq m, \ |\beta| + |\kappa| \leq m :$$

$$(\partial^\alpha_t v, \partial^\beta_x \partial^\kappa p, \partial^\beta_x \partial^\kappa p) \in X_{q, \text{Oseen}}^\perp(\mathbb{R}^3) \times W^{2,1,q,r}_{\text{pres}}(G) \times X_{\text{pres}}^{q,r}(G).$$

The main challenge will now be to prove the theorems above, which will be done in the next section. The advantage obtained at this point, by the reformulation of these theorems in the setting on the group $G$, is the ability by means of the Fourier transform $\mathcal{F}_G$ to employ multiplier theory.

5. Proof of the main theorems

We will now prove theorems 2.3 and 2.4. The proofs reduce to simple verifications once we have established theorems 4.5–4.9. First, however, we recall the maximal regularity results for the linearization of (4.8) from [16]. On the basis of the linear theory, the existence of a strong
Lemma 5.1. Let \( q \in (1, \infty) \) and \( r \in (3, \infty) \). Then
\[
\forall u \in D^{1,r}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3) : \quad \|u\|_\infty \leq C_1\|u\|_{1,r} + \|u\|_q \quad (5.1)
\]
with \( C_1 = C_1(q, r) \).

Proof. See [6, remark II.7.2]. \( \square \)

Lemma 5.2. Let \( q \in \left(1, \frac{4}{3}\right) \) and \( r \in (4, \infty) \). Then every \( v \in X_{\text{Oseen}}^{q,r}(\mathbb{R}^3) \) satisfies
\[
\|\nabla v\|_\infty \leq C_2\|v\|_{X_{\text{Oseen}}^{q,r}(\mathbb{R}^3)}, \quad (5.2)
\]
\[
\|\nabla v\|_r \leq C_3\|v\|_{X_{\text{Oseen}}^{q,r}(\mathbb{R}^3)}, \quad (5.3)
\]
\[
\|v\|_\infty \leq C_4\|v\|_{X_{\text{Oseen}}^{q,r}(\mathbb{R}^3)}, \quad (5.4)
\]
\[
\|\nabla v\|_2 \leq C_5\|v\|_{X_{\text{Oseen}}^{q,r}(\mathbb{R}^3)}, \quad (5.5)
\]
Moreover, every \( w \in W_{\text{Oseen}}^{3,q,r}(G) \) satisfies
\[
\|w\|_\infty \leq C_6\|w\|_{2,1,q,r}. \quad (5.6)
\]

Proof. Recall (5.1) and observe that
\[
\|\nabla v\|_\infty \leq C_1\left(\|v\|_{2,r} + \|\nabla v\|_{\frac{4}{3-q}}\right) \leq C_1\|v\|_{X_{\text{Oseen}}^{q,r}(\mathbb{R}^3)},
\]
which implies (5.2). It follows that \( \nabla v \in L^{\frac{4q}{4-q}}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \) and consequently, since \( \frac{4q}{4-q} < r < \infty \), by interpolation that
\[
\|\nabla v\|_r \leq c_1\left(\|\nabla v\|_\infty + \|\nabla v\|_{\frac{4}{3-q}}\right) \leq c_1\|v\|_{X_{\text{Oseen}}^{q,r}(\mathbb{R}^3)}.
\]
This shows (5.3). With (5.3) at our disposal, we again employ (5.1) and find that
\[
\|v\|_\infty \leq C_1\left(\|\nabla v\|_r + \|v\|_{\frac{4}{3-q}}\right) \leq c_2\|v\|_{X_{\text{Oseen}}^{q,r}(\mathbb{R}^3)}.
\]
Thus (5.4) follows. To show (5.5), observe, since \( q < \frac{4}{3} \) and thus \( \frac{4q}{4-q} \leq 2 \), that
\[
\|\nabla v\|_2 \leq c_3\left(\|\nabla v\|_r + \|\nabla v\|_\infty\right) \leq c_4\|v\|_{X_{\text{Oseen}}^{q,r}(\mathbb{R}^3)}.
\]
Finally, the Sobolev embedding \( W^{1,r}(G) \hookrightarrow L^\infty(G) \) for \( r > 4 \), which follows from the classical Sobolev embedding \( W^{1,r}(\mathbb{R}^3 \times (0, T)) \hookrightarrow L^\infty(\mathbb{R}^3 \times (0, T)) \) since \( \Pi \), by lifting, induces an embedding \( W^{1,r}(G) \hookrightarrow W^{1,r}(\mathbb{R}^3 \times (0, T)) \), implies (5.6). \( \square \)

Lemma 5.3. If \( u \in \mathcal{S}'(G) \) with \( \mathcal{P}u = 0 \) satisfies
\[
\partial_t u - \Delta u - \lambda \partial_1 u = 0 \quad \text{in} \ G, \quad (5.7)
\]
then \( u = 0 \).

Proof. Applying \( \mathcal{F}_G \) on both sides in (5.7), we deduce that \((i \frac{2\pi}{\lambda} k + |\xi|^2 - \lambda i \xi_1) \hat{u} = 0 \). Since the polynomial \( |\xi|^2 + i \frac{2\pi}{\lambda} k - \lambda \xi_1 \) vanishes only at \( (\xi, k) = (0, 0) \), we conclude that \( \text{supp} \hat{u} \subset \{(0, 0)\} \). However, since \( \mathcal{P}u = 0 \) we have \( \kappa_0 \hat{u} = 0 \), whence \( (\xi, 0) \notin \text{supp} \hat{u} \) for all \( \xi \in \mathbb{R}^3 \). Consequently, \( \text{supp} \hat{u} = \emptyset \). It follows that \( \hat{u} = 0 \) and thus \( u = 0 \). \( \square \)
Lemma 5.4. Let \( v \in L^q(\mathbb{R}^3) \) for some \( q \in [1, \infty) \). If
\[
-\Delta v - \lambda \partial_1 v = 0 \quad \text{in } \mathbb{R}^3,
\]
then \( v = 0 \).

Proof. Applying the Fourier transform \( \mathcal{F}_{\mathbb{R}^3} \) in (5.8), we see that \( (|\xi|^2 - \lambda i \xi_1) \hat{v} = 0 \). It follows that \( \text{supp} \hat{v} \subset \{0\} \), whence \( v \) is a polynomial. Since \( v \in L^q(\mathbb{R}^3) \), we must have \( v = 0 \). \( \square \)

Lemma 5.5. Let \( q \in (1, \infty) \). Then
\[
A_{TP} : W^{2,1,q}_{\sigma,1}(G) \to L^q_{\sigma,1}(G), \quad A_{TP}w := \partial_1 w - \Delta w - \lambda \partial_1 w
\]
is a homeomorphism. Moreover \( \|A_{TP}^{-1}\| \leq C_7 P(\lambda, T) \), where \( C_7 = C_7(q) \) and \( P(\lambda, T) \) is a polynomial in \( \lambda \) and \( T \).

Proof. See [16, theorem 4.8]. \( \square \)

Lemma 5.6. For \( q \in (1, 2) \),
\[
A_{Oseen} : X^{q,r}_{\sigma,0seen}(\mathbb{R}^3) \to L^{q,r}_{\sigma}(\mathbb{R}^3), \quad A_{Oseen}v := -\Delta v - \lambda \partial_1 v
\]
is a homeomorphism. Moreover \( \|A_{Oseen}^{-1}\| \leq C_8 \) with \( C_8 \) independent of \( \lambda \).

Proof. See [6, theorem VII.4.1]. \( \square \)

Lemma 5.7. If \( q \in (1, 2) \), \( r \in (4, \infty) \) and \( \lambda \neq 0 \), then
\[
A_{TP} : X^{q,r}_{\sigma,0seen}(\mathbb{R}^3) \times W^{2,1,q,r}_{\sigma,1}(G) \to L^{q,r}_{\sigma}(G),
A_{TP}(v, w) := \partial_1 w - \Delta (v + w) - \lambda \partial_1 (v + w)
\]
is a homeomorphism. Moreover
\[
\|A_{TP}^{-1}\| \leq C_9 P(\lambda, T), \tag{5.9}
\]
where \( P(\lambda, T) \) is a polynomial in \( \lambda \) and \( T \), and \( C_9 = C_9(q) \).

Proof. Recalling \( L^{q,r}_{\sigma}(G) = L^{q,r}_{\sigma}(\mathbb{R}^3) \oplus L^{q,r}_{\sigma,1}(G) \) from (4.6), lemma 5.5 and lemma 5.6 concludes the proof. \( \square \)

Lemma 5.8. Let \( q \in (1, 3) \) and \( r \in (1, \infty) \). Then
\[
\text{grad} : X^{q,r}_{\sigma, pres}(G) \to \mathcal{G}^{q,r}(G), \quad \text{grad } p := \nabla p
\]
is a homeomorphism.

Proof. See [16, lemma 5.4]. \( \square \)

Proof of theorem 4.5. We can use the Helmholtz projection to eliminate the pressure term \( \nabla p \) in (4.8). More precisely, we shall first study
\[
\begin{aligned}
\partial_1 u = & \Delta u - \lambda \partial_1 u + \mathcal{P}_H [u \cdot \nabla u] = \mathcal{P}_H f \quad \text{in } G, \\
\text{div } u = & 0 \quad \text{in } G.
\end{aligned}
\tag{5.10}
\]
After solving (5.10), a pressure term \( p \) can be constructed such that \( (u, p) \) solves (4.8).
We first show that any pair of vector fields \((v, w) \in X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3) \times W^{2,1,q,r}_{\sigma,\perp}(G)\) satisfies \((v + w) \cdot \nabla (v + w) \in L^q(G)^3 \cap L^r(G)^3\). Recalling (5.3) and (5.4), we find that
\[
\| v \cdot \nabla v \|_r \leq \| v \|_{\infty} \| \nabla v \|_r \leq c_1 \| v \|_{X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3)}^2.
\]  
(5.11)
Moreover, employing Hölder’s inequality and recalling (5.5) we deduce
\[
\| v \cdot \nabla v \|_q \leq \| v \|_{\infty} \| \nabla v \|_q \leq c_2 \| v \|_{X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3)}^2.
\]  
(5.12)
We also observe that
\[
\| v \cdot \nabla w \|_r \leq \| v \|_{\infty} \| \nabla w \|_r \leq c_3 \| v \|_{X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3)} \| w \|_{2,1,q,r}.
\]  
(5.13)
and
\[
\| v \cdot \nabla w \|_q \leq \| v \|_{\infty} \| \nabla w \|_q \leq c_4 \| v \|_{X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3)} \| w \|_{2,1,q,r}.
\]  
(5.14)
Similarly, recalling (5.2) we can estimate
\[
\| w \cdot \nabla v \|_r \leq \| w \|_r \| \nabla v \|_r \leq c_5 \| w \|_{2,1,q,r} \| v \|_{X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3)}.
\]  
(5.15)
and
\[
\| w \cdot \nabla v \|_q \leq \| w \|_q \| \nabla v \|_q \leq c_6 \| w \|_{2,1,q,r} \| v \|_{X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3)}.
\]  
(5.16)
By (5.6) it follows that also
\[
\| w \cdot \nabla w \|_r \leq \| w \|_{\infty} \| \nabla w \|_r \leq c_7 \| w \|_{2,1,q,r}^2.
\]  
(5.17)
and
\[
\| w \cdot \nabla w \|_q \leq \| w \|_q \| \nabla w \|_q \leq c_8 \| w \|_{2,1,q,r} \| v \|_{X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3)}^2.
\]  
(5.18)
Combining (5.11)–(5.18), we conclude that \((v + w) \cdot \nabla (v + w) \in L^q(G)^3 \cap L^r(G)^3\) and
\[
\| P_H [(v + w) \cdot \nabla (v + w)] \|_{L^q(G)} \leq c_9 \| (v, w) \|_{X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3) \times W^{2,1,q,r}_{\sigma,\perp}(G)}.
\]  
(5.19)
Recalling lemma 5.7, we can now define the map
\[
\mathcal{J} : X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3) \times W^{2,1,q,r}_{\sigma,\perp}(G) \to X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3) \times W^{2,1,q,r}_{\sigma,\perp}(G),
\]
\[
\mathcal{J}(v, w) := \Lambda_{\sigma,\perp}^{-1} (P_H f - P_H [(v + w) \cdot \nabla (v + w)]).
\]
By construction, a fixed point \((v, w) \in \mathcal{J}\) yields a solution \(u := v+w\) to (5.10). As \(X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3)\) and \(W^{2,1,q,r}_{\sigma,\perp}(G)\) are Banach spaces, we shall employ Banach’s fixed point theorem to show the existence of such a fixed point. To this end, we recall (5.9) and estimate
\[
\| \mathcal{J}(v, w) \| \leq c_{10} P(\lambda, T) \left( \| P_H f \|_{L^q(G)} + \| P_H [(v + w) \cdot \nabla (v + w)] \|_{L^q(G)} \right)
\]  
\[
\leq c_{10} P(\lambda, T) \left( \| v \|_{\infty} + \| w \|_{\infty} \right)^2 \| X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3) \times W^{2,1,q,r}_{\sigma,\perp}(G) \|.
\]  
(5.20)
Consequently, \(\mathcal{J}\) is a self-mapping on the ball \(\overline{B}_\rho \subset X^{q,r}_{\sigma,\text{Oseen}}(\mathbb{R}^3) \times W^{2,1,q,r}_{\sigma,\perp}(G)\) provided \(\rho\) and \(\epsilon_1\) satisfy
\[
c_{10} P(\lambda, T) \left( \epsilon_1 + \rho^2 \right) \leq \rho.
\]
The above inequality is satisfied if we choose, for example,
\[
\rho := \frac{1}{4c_{10} P(\lambda, T)}, \quad \epsilon_1 := \frac{1}{16c_{10}^2 P(\lambda, T)^2}.
\]  
(5.21)
With this choice of parameters, we further have, for \((v_1, w_1), (v_2, w_2) \in \overline{B}_\rho,
\[
\| \mathcal{J}(v_1, w_1) - \mathcal{J}(v_2, w_2) \| \leq c_{10} P(\lambda, T) \|(v_1, w_1) - (v_2, w_2)\|^2
\]
\[
\leq c_{10} P(\lambda, T) 2\rho \|(v_1, w_1) - (v_2, w_2)\|
\]
\[
\leq \frac{1}{2} \|(v_1, w_1) - (v_2, w_2)\|.
\]
(2923)
Thus, \( J \) becomes a contractive self-mapping. By Banach’s fixed point theorem, \( J \) then has a unique fixed point in \( B_p \).

Finally, we construct the pressure. By (5.19), \( u \cdot \nabla u \in L^4(G)^3 \cap L'(G)^3 \). Recalling lemma 5.8, the function

\[
p := \text{grad}^{-1}(\{\text{Id} - P_P\}(f - u \cdot \nabla u))
\]

belongs to \( X^0_{q, \tau}(G) \). Clearly, \((u, p)\) is a solution to (4.8).

\[\Box\]

**Lemma 5.9.** Let \( \lambda \neq 0 \) and \( U \in L^{1}_{\text{loc}}(G)^3 \) be a generalized solution to (4.8); that is, it satisfies (2.7) for all \( \Phi \in C^\infty_{0, \sigma}(G) \), with \( U \in L^2((0, T); D^{0, \sigma}_{0, \sigma}(\mathbb{R}^3)) \) and \( P_u U \in L^\infty((0, T); L^2(\mathbb{R}^3)^3) \). If for some \( \tilde{q} \in (1, \frac{5}{2}] \)

\[
f \in L^4(G)^3 \cap L^6(G)^3,
\]

then

\[
P_u U \in W^{2, 1, q}_{\sigma, \text{loc}}(G).
\]

If for some \( \tilde{q} \in (1, \frac{5}{3}] \)

\[
f \in L^5(G)^3 \cap L^4(G)^3,
\]

then

\[
P_u U \in X^0_{q, \text{Oseen}}(\mathbb{R}^3).
\]

**Proof.** We first assume (5.22) for some \( q \in (1, \frac{5}{2}] \). Put \( U := P_u U \) and \( W := P_u U \). By assumption, \( W \in L^2(G)^3 \) and \( \nabla W \in L^2(G)^3 \), whence

\[
W \cdot \nabla W \in L^1(G)^3.
\]

Employing first Hölder’s inequality and then a Sobolev-type inequality (see for example [6, lemma II.2.2] invoked with \( n = 3, q = 2 \) and \( r = \frac{10}{3} \)), we estimate

\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} |W \cdot \nabla W|^\frac{2}{3} \, dx \, dt \leq \frac{1}{T} \int_0^T \left\| \nabla W(\cdot, t) \right\|^\frac{2}{3} \left\| W(\cdot, t) \right\|^\frac{1}{3} \, dt
\]

\[
\leq c_1 \frac{1}{T} \int_0^T \left\| \nabla W(\cdot, t) \right\|^\frac{2}{3} \left( \left\| \nabla W(\cdot, t) \right\|^2 \left\| W(\cdot, t) \right\|^2 \right) \, dt
\]

\[
\leq c_2 (\text{ess sup}_{t \in (0, T)} \left\| W(\cdot, t) \right\|_2)^\frac{1}{3} \cdot \frac{1}{T} \int_0^T \left\| \nabla W(\cdot, t) \right\|_2^2 \, dx < \infty,
\]

whence

\[
W \cdot \nabla W \in L^\frac{4}{3}(G)^3.
\]

We further deduce, by employing first Minkowski’s integral inequality, then Hölder’s inequality, and finally the Sobolev embedding \( D^{1, 2}_{0, \sigma}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \), that

\[
\left( \int_{\mathbb{R}^3} \left| \frac{1}{T} \int_0^T |W \cdot \nabla W(\cdot, t)|^2 \, dx \right|^\frac{2}{3} \right)^\frac{3}{2} \leq \frac{1}{T} \int_0^T \left( \int_{\mathbb{R}^3} |W \cdot \nabla W|^2 \, dx \right)^\frac{1}{2} \, dt
\]

\[
\leq \frac{1}{T} \int_0^T \left( \int_{\mathbb{R}^3} |W|^6 \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^3} |\nabla W|^2 \, dx \right)^\frac{1}{2} \, dt
\]

\[
\leq \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} |\nabla W|^2 \, dx \, dt < \infty.
\]
Consequently, we have
\[ \mathcal{P} \left[ \nabla \cdot \nabla W \right] \in L^2(G)^3. \]  
(5.29)
Recalling lemma 4.2, it is easy to verify from the weak formulation (2.7) that
\[ \int_{\mathbb{R}^3} -\nabla \cdot \partial_t \Phi + \nabla \cdot \nabla \Phi = \nabla \cdot \nabla \Phi - \lambda \partial_t \nabla \cdot \Phi + \left( \nabla \cdot \nabla W + \mathcal{P} [ W \cdot \nabla W ] \right) \cdot \Phi \, \mathrm{d}x = \int_{\mathbb{R}^3} \mathcal{P} f \cdot \Phi \, \mathrm{d}x \]
for all \( \Phi \in C^0_{0, \sigma}(\mathbb{R}^3) \). This means that \( \nabla \in D^{1,2}_{0, \sigma}(\mathbb{R}^3) \) is a generalized solution to the steady-state problem
\[
\begin{cases}
-\Delta \nabla - \lambda \partial_t \nabla + \mathcal{P}_H [ \nabla \cdot \nabla \nabla ] = \mathcal{P}_H \mathcal{P} f - \mathcal{P}_H \mathcal{P} [ W \cdot \nabla W ] & \text{in } \mathbb{R}^3, \\
\text{div } \nabla = 0 & \text{in } \mathbb{R}^3.
\end{cases}
\]  
(5.30)
From (5.26), (5.29), and assumption (5.22), we deduce the summability
\[ \mathcal{P} f - \mathcal{P} [ W \cdot \nabla W ] \in L^q(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3 \]
for the right-hand side in (5.30). Known results for the steady-state Navier–Stokes problem (5.30) then imply
\[ \nabla \in X^q_{\sigma, \text{Oseen}}(\mathbb{R}^3)^3 \cap X^1_{\sigma, \text{Oseen}}(\mathbb{R}^3). \]  
(5.31)
More specifically, we can employ [7, lemma X.6.1] which, although formulated for a three-dimensional exterior domain, also holds for solutions to the whole-space problem (5.30). By the additional regularity for \( \nabla \) implied by (5.31), it follows that \( \nabla \nabla \in L^3(\mathbb{R}^3)^{3 \times 3} \). Since by assumption \( \nabla W \in L^2(G)^3 \), we thus have \( \nabla \nabla \in L^3(G)^{3 \times 3} \). In addition, we can deduce as in (5.27) that \( \nabla \cdot \nabla \nabla \in L^3(G)^3 \). Consequently, by interpolation
\[ \nabla \cdot \nabla \nabla \in L^3(G)^3. \]  
(5.32)
From (5.31) we further obtain \( \nabla \in L^\infty(\mathbb{R}^3)^3 \), which combined with \( \nabla W \in L^2(G)^3 \) yields
\[ \nabla \cdot \nabla \nabla \in L^2(G)^3. \]  
(5.33)
We have now derived enough summability properties for the terms appearing in (4.8) to finalize the proof. Recalling again lemma 4.2, it is easy to verify from the weak formulation (2.7) that
\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} -\nabla \cdot \partial_t \Phi + \nabla \cdot \nabla \Phi = \nabla \cdot \nabla \Phi - \lambda \partial_t \nabla \cdot \Phi + \left( \mathcal{P}_H \nabla \cdot \nabla W + \mathcal{P}_H [ W \cdot \nabla W + \mathcal{P}_H W \cdot \nabla W ] \right) \cdot \Phi \, \mathrm{d}x \, \mathrm{d}t = \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \mathcal{P}_H f \cdot \Phi \, \mathrm{d}x \, \mathrm{d}t
\]
for all \( \Phi \in C^0_{0, \sigma}(G) \). The summability of \( \nabla W \) and \( \nabla W \), together with the summability properties obtained for \( \nabla \cdot \nabla W \), \( \nabla \cdot \nabla \nabla \), and \( \nabla \cdot \nabla \nabla \) above, enables us to extend (5.34) to all \( \Phi \in \mathcal{H}(G) \) with \( \partial_t \Phi = 0 \). Thus the system
\[
\begin{cases}
\partial_t \nabla - \Delta \nabla - \lambda \partial_t \nabla = \mathcal{P}_H \mathcal{P}_+ f - \mathcal{P}_H \left[ \mathcal{P}_+ \left[ W \cdot \nabla W + \mathcal{P}_H [ W \cdot \nabla W + \mathcal{P}_H W \cdot \nabla W ] \right] \right] & \text{in } G, \\
\text{div } \nabla \nabla = 0 & \text{in } G
\end{cases}
\]
is satisfied as an identity in \( \mathcal{H}(G) \). From (5.26), (5.28), (5.32), (5.33), and the assumptions on \( f \), we conclude that
\[ \mathcal{P}_H \mathcal{P}_+ f - \mathcal{P}_H \left[ \mathcal{P}_+ \left[ W \cdot \nabla W + \mathcal{P}_H [ W \cdot \nabla W + \mathcal{P}_H W \cdot \nabla W ] \right] \right] \in L^q(G)^3. \]
Consequently, lemma 5.5 combined with lemma 5.3 implies (5.23).

\(^3\) Lemma X.6.1 is new in the latest edition of the monograph [7].
Finally, assume \((5.24)\) for some \(q \in (1, \frac{3}{2}]\). In view of \((5.26)\) and \((5.29)\), we deduce

\[
P f = \mathcal{P} [W \cdot \nabla W] \in L^4(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3.
\]

Recalling that \(V\) solves \((5.30)\), utilizing once more [7, lemma X.6.1] we conclude \((5.25)\). □

**Proof of theorem 4.7.** The proof relies on the summability properties of the solution \(u = v + w\) being sufficient for multiplying \((4.8)\) with \(u\) itself and subsequently integrating over space and time. Due to the different summability properties of \(v\) and \(w\), it is more convenient to carry out this process for \(v\) and \(w\) separately. Applying first \(\mathcal{P}_\perp\) and then \(\mathcal{P}_H\) to both sides in \((4.8)\), we obtain

\[
\partial_t w - \Delta w - \lambda \partial_t w = \mathcal{P}_H \mathcal{P}_\perp f - \mathcal{P}_H \left[ \mathcal{P}_\perp [w \cdot \nabla w] + w \cdot \nabla v + v \cdot \nabla w \right].
\]

which we multiply with \(w\) and integrate over \(G\). We can easily verify that the product of \(w\) with each term in \((5.35)\) is integrable over \(G\). For example, we observe that

\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} |\partial_t w \cdot w| \, dx \, dt \leq ||\partial_t w||_4 ||w||_4 \leq ||w||_{2,1,q,r}^2.
\]

Similarly, one can verify for all the other terms in \((5.35)\) that the product with \(w\) can be integrated over \(G\). We thus conclude that

\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \partial_t w \cdot w - \Delta w \cdot w - \lambda \partial_t w \cdot w \, dx \, dt = \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \mathcal{P}_\perp f \cdot w - \mathcal{P}_\perp [w \cdot \nabla w] \cdot w - (w \cdot \nabla v) \cdot w - (v \cdot \nabla w) \cdot w \, dx \, dt,
\]

where the Helmholtz projection \(\mathcal{P}_H\) can be omitted since \(w\) is solenoidal. Since \(w = \mathcal{P}_\perp w\) we can, recalling \((4.3)\), also omit the projection \(\mathcal{P}_\perp\) in the first two terms on the right-hand side. Moreover, the summability properties of \(w\) are sufficient for integrating by parts in each term above. Consequently, we obtain

\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \nabla v \cdot \nabla w \, dx \, dt = \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} f \cdot w - (w \cdot \nabla v) \cdot w \, dx \, dt.
\]

We now repeat the procedure with \(v\) in the role of \(w\). Applying first \(\mathcal{P}\) and then \(\mathcal{P}_H\) to both sides in \((4.8)\), we obtain

\[
- \Delta v - \lambda \partial_t v = \mathcal{P}_H f - \mathcal{P}_H \left[ \mathcal{P} [w \cdot \nabla w] + v \cdot \nabla v \right].
\]

which we multiply with \(v\) and integrate over \(\mathbb{R}^3\). Again it should be verified that the product of the terms in \((5.38)\) with \(v\) is integrable over \(\mathbb{R}^3\). This, however, is standard to show. For example, in view of \((5.4)\) and the fact that \(\frac{2q}{2-q} \leq \frac{q}{q-1}\) it follows that

\[
\int_{\mathbb{R}^3} \Delta v \cdot v \, dx \leq ||\Delta v||_q ||v||_q \leq ||v||_{2,1,q,r(\mathbb{R}^3)}^2.
\]

Similarly, one can verify for all the other terms in \((5.38)\) that the product with \(v\) can be integrated over \(\mathbb{R}^3\). We thus conclude that

\[
\int_{\mathbb{R}^3} -\Delta v \cdot v - \lambda \partial_t v \cdot v \, dx = \int_{\mathbb{R}^3} f \cdot v - (w \cdot \nabla w) \cdot v - (v \cdot \nabla v) \cdot v \, dx.
\]

One may also verify that the summability properties of \(v\) are sufficient for integrating by parts in \((5.39)\). We thereby obtain

\[
\int_{\mathbb{R}^3} \nabla v \cdot v \, dx = \int_{\mathbb{R}^3} f \cdot v + (w \cdot \nabla v) \cdot w \, dx.
\]
Adding together (5.37) and (5.40) we deduce
\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} |\nabla w|^2 + |\nabla v|^2 \, dx \, dt = \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} f \cdot (v + w) \, dx \, dt.
\]
Since
\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \nabla v : \nabla w \, dx \, dt = 0,
\]
we finally conclude (2.11). \hfill \Box

**Proof of theorem 4.8.** Choosing \( \varepsilon_2 \leq \varepsilon_1 \), we obtain by theorem 4.5 a solution \((u, p)\) in the class (4.10) (with \( u = v + w \)). From the proof of theorem 4.5, in particular (5.20)–(5.21), we recall that \( u \in \mathcal{T} \rho \subset X^{q,r}_{\alpha,\text{Oseen}} (\mathbb{R}^3) \times W^{2,1}_{q,r} (G) \) with \( \rho := \varepsilon_2 \varepsilon_1 \), which means that
\[
\| (u, w) \|_{X^{q,r}_{\alpha,\text{Oseen}} (\mathbb{R}^3) \times W^{2,1}_{q,r} (G)} \leq \varepsilon_2 \varepsilon_1.
\]
Now recall definition 4.6 and consider a physically reasonable weak solution \( U \) corresponding to the same data \( f \). Put \( V := \mathcal{P} U \) and \( W := \mathcal{P}_2 U \). We shall verify that the regularity of \( V \) and \( W \) ensured by lemma 5.9 enables us to use \( u = v + w \) as a test function in the weak formulation for \( U = V + W \). Observe for example that (5.25) implies \( V \in L^{2q/(q-2)} (\mathbb{R}^3)^3 \), from which it follows, since the H"older conjugate \( (2q/(q-2))' = 2q/(q-2) \) belongs to the interval \((q, r)\), that
\[
V \cdot \partial_t w \in L^1 ( G ).
\]
Moreover, since by assumption \( W \in L^2 ( G)^3 \), we also have
\[
W \cdot \partial_t w \in L^1 ( G ).
\]
In a similar manner, one may verify that
\[
\nabla V : \nabla v, \nabla V : \nabla w, \nabla W : \nabla v, \nabla W : \nabla w \in L^1 ( G^3 ).
\]
From (5.25) and the initial regularity of \( V \), we obtain \( \partial_t V \in L^q (\mathbb{R}^3)^3 \cap L^2 (\mathbb{R}^3)^3 \). Thus, since \( v \in L^{2q/(q-2)} (\mathbb{R}^3)^3 \) and the H"older conjugate \( (2q/(q-2))' = 2q/(q-2) \) belongs to the interval \((q, 2)\), we deduce
\[
\partial_t V \cdot v \in L^1 (\mathbb{R}^3)^3.
\]
In view of (5.23), the same argument yields
\[
\partial_t W \cdot v \in L^1 (\mathbb{G})^3.
\]
It is easy to see that
\[
\partial_t V \cdot w, \, \partial_t W \cdot v \in L^1 (G)^3.
\]
By lemma 5.2, we have \( v \in L^{2q/(q-2)} (\mathbb{R}^3) \cap L^\infty (\mathbb{R}^3)^3 \). Moreover, recalling the embedding \( D^{1,2}_{0,\sigma}(\mathbb{R}^3) \hookrightarrow L^6 (\mathbb{R}^3) \), we find that \( U \in L^{2q/(q-2)} (\mathbb{R}^3)^3 \cap L^6 (\mathbb{R}^3)^3 \). We thus see that \( u \), \( v \), \( W \) \in \( L^6 (\mathbb{R}^3)^3 \), from which one can deduce that
\[
( V \cdot \nabla V ) \cdot v, \, ( V \cdot \nabla V ) \cdot w, \, ( V \cdot \nabla W ) \cdot v, \, ( V \cdot \nabla W ) \cdot w \in L^1 ( G )^3.
\]
Lemma 5.2 also yields \( w \in L^\infty ( G )^3 \), whence
\[
( W \cdot \nabla V ) \cdot v, \, ( W \cdot \nabla V ) \cdot w, \, ( W \cdot \nabla W ) \cdot v, \, ( W \cdot \nabla W ) \cdot w \in L^1 ( G )^3.
\]
Finally, recalling that \( (2q/(q-2))' = 2q/(q-2) \in (q, 2) \), the summability of \( f \) implies
\[
f \cdot v, \, f \cdot w \in L^1 ( G ).
\]
From the summability properties (5.42)–(5.50), we conclude, by a standard approximation argument, that \( u = v + w \) can be used as a test function in the weak formulation for \( \mathcal{U} = \mathcal{V} + \mathcal{W} \) and thus obtain

\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} -\mathcal{W} \cdot \partial_t w + \nabla \mathcal{U} : \nabla u - \lambda \partial_t \mathcal{U} \cdot u + (\mathcal{U} \cdot \nabla \mathcal{U}) \cdot u \, dx \, dt = \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} f \cdot u \, dx \, dt.
\]

We now consider the equation

\[
\partial_t u - \Delta u - \lambda \partial_1 u + \nabla p + u \cdot \nabla u = f \quad \text{in } G
\]

satisfied by the strong solution. We shall multiply (5.52) with \( \mathcal{U} \) and integrate over \( G \). With the aid of lemmas 5.9 and 5.2, one can verify as above that the resulting integral is well-defined.

We thus obtain

\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \partial_t w \cdot \mathcal{W} - \Delta u \cdot \mathcal{U} - \lambda \partial_1 u \cdot \mathcal{U} + \nabla p \cdot \mathcal{U} + (u \cdot \nabla u) \cdot \mathcal{U} \, dx \, dt
\]

\[
= \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} f \cdot \mathcal{U} \, dx \, dt.
\]

Recalling (5.44)–(5.49), we see that the following integration by parts is valid:

\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \partial_t w \cdot \mathcal{W} + \nabla u : \nabla \mathcal{U} + \lambda u \cdot \partial_1 \mathcal{U} - (u \cdot \nabla \mathcal{U}) \cdot u \, dx \, dt = \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} f \cdot \mathcal{U} \, dx \, dt.
\]

Adding together (5.51) and (5.53), we deduce

\[
2 \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \nabla U : \nabla u \, dx \, dt = \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} f \cdot \mathcal{U} \, dx \, dt + \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} f \cdot u \, dx \, dt
\]

\[
+ \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} ((u - \mathcal{U}) \cdot \nabla U) \cdot u \, dx \, dt.
\]

Since

\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} |\nabla U - \nabla u|^2 \, dx \, dt
\]

\[
= \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} |\nabla \mathcal{U}|^2 + |\nabla u|^2 \, dx \, dt - 2 \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \nabla U : \nabla v \, dx \, dt,
\]

we can utilize (5.54) in combination with the energy equality (2.11) satisfied by \( u \) due to theorem 4.7 and the energy inequality (2.8) satisfied by \( \mathcal{U} \) to deduce

\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} |\nabla U - \nabla u|^2 \, dx \, dt \leq \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} (\mathcal{U} - u) \cdot \nabla \mathcal{U} \cdot u \, dx \, dt.
\]

Recalling (5.5), we see that \( \nabla u \in L^2(G)^{3\times3} \). We already observed that \( u, \mathcal{V} \in L^4(G)^3 \). Thus, an integration by parts yields

\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} (\nabla \cdot \nabla u) \cdot u \, dx \, dt = 0.
\]

Since \( \mathcal{V} \in L^2(G)^3 \) and \( u \in L^\infty(G) \), it further follows that

\[
\frac{1}{T} \int_0^T \int_{\mathbb{R}^3} (\mathcal{W} \cdot \nabla u) \cdot u \, dx \, dt = 0.
\]
Consequently, we can rewrite (5.55) as
\[ \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} (\mathcal{U} \cdot \nabla \mathcal{U}) \cdot \mathcal{U} \, dx \, dt = 0. \] (5.58)

Accordingly, we can rewrite (5.55) as
\[ \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} \frac{|\nabla \mathcal{U} - \nabla u|^2}{dx} \, dt \leq \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} (\mathcal{U} - u) \cdot \nabla (\mathcal{U} - u) \cdot \mathcal{U} \, dx \, dt. \] (5.59)

Recalling the embedding \( D_{1,2}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \), we estimate
\[ \left| \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} (\mathcal{U} - u) \cdot \nabla (\mathcal{U} - u) \cdot \mathcal{U} \, dx \, dt \right| \leq \frac{1}{T} \int_0^T \| (\mathcal{U}(t) - u(t)) \|_6 \| \nabla (\mathcal{U}(t) - u(t)) \|_{L^6(\mathbb{R}^3)} \| u(t) \|_3 \, dt \]
\[ \leq \text{ess sup}_{t \in (0,T)} \| u(t) \|_3 \frac{1}{T} \int_0^T \| \nabla (\mathcal{U}(t) - u(t)) \|_{L^6(\mathbb{R}^3)}^2 \, dt. \] (5.60)

Now we finally need the assumption \( q \leq \frac{6}{5} \), which implies \( \frac{2q}{q-1} \leq 3 \). Consequently, from the fact that \( \| v \|_{\frac{2q}{q-1}} \leq \| v \|_{X^{\sigma,q}_e(\mathbb{R}^3)} \) and, by lemma 5.2, \( \| v \|_{L^\infty(\mathbb{R}^3)} \leq C(q) \| v \|_{X^{\sigma,q}_e(\mathbb{R}^3)} \), we obtain
\[ \| v \|_{L^\infty(\mathbb{R}^3)} \leq C(q) \| u \|_{X^{\sigma,q}_e(\mathbb{R}^3)}. \] (5.61)

Since \( w \in W^{2,1,q}_{\sigma_\perp}(G) \hookrightarrow W^{1,3}(G) \hookrightarrow W^{1,3}(0, T; L^3(\mathbb{R}^3)) \), standard Sobolev embedding yields \( w \in L^\infty((0, T); L^3(\mathbb{R}^3)) \) with
\[ \| w \|_{L^\infty((0, T); L^3(\mathbb{R}^3))} \leq C_2 \| w \|_{W^{2,1,q}_{\sigma_\perp}(G)}. \] (5.62)

Combining (5.61) and (5.62), we obtain
\[ \| u \|_{L^\infty((0, T); L^3(\mathbb{R}^3))} \leq C_3 \| (v, w) \|_{X^{\sigma,q}_e(\mathbb{R}^3) \times W^{2,1,q}_{\sigma_\perp}(G)}. \]
This estimate together with (5.41), (5.59) and (5.60) finally yields
\[ \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} |\nabla \mathcal{U} - \nabla u|^2 \, dx \, dt \leq C_3 \varepsilon_2 \frac{1}{T} \int_0^T \int_{\mathbb{R}^3} |\nabla \mathcal{U} - \nabla u|^2 \, dx \, dt. \]
We conclude that \( \mathcal{U} = u \) if \( \varepsilon_2 < C_3^{-1} \). \( \square \)

**Remark 5.10.** The proof of theorem 4.8 follows an idea introduced by Galdi in [5]. The same method was also used in [26] to show a uniqueness result for the time-periodic Navier–Stokes problem in the case \( \lambda = 0 \).

**Proof of theorem 4.9.** By assumption, \((u, p)\) is a solution to (4.8) in the class (4.10) with \( u = v + w \). Applying first \( \mathcal{P}_\perp \) and then \( \mathcal{P}_H \) on both sides in (4.8) we obtain
\[
\begin{cases}
\partial_t w - \Delta w - \lambda \partial_t w = \mathcal{P}_H \mathcal{P}_\perp f - \mathcal{P}_H \mathcal{P}_\perp \left[ \left( w \cdot \nabla w \right) + w \cdot \nabla v + v \cdot \nabla w \right] \quad &\text{in } G, \\
\text{div } w = 0 \quad &\text{in } G.
\end{cases}
\] (5.63)

We shall ‘take half a derivative in time’ on both sides of (5.63). We therefore introduce the pseudo-differential operator
\[
\partial_t^\frac{1}{2} : \mathcal{S}(G) \to \mathcal{S}(G), \quad \partial_t^\frac{1}{2} \psi := \mathcal{F}_G^{-1} \left[ \left( \frac{2\pi}{T} k \right)^\frac{1}{2} \psi \right].
\]
which, by duality, extends to an operator \( \partial_t^\frac{1}{2} : \mathcal{S}'(G) \to \mathcal{S}(G) \). Note that \( w \cdot \nabla w = \text{div } w \otimes w \) for a solenoidal vector field \( w \). We thus find that

\[
\partial_t^\frac{1}{2} \left[ \mathcal{P}_L [w \cdot \nabla w] \right]_j = \mathcal{F}_G^{-1} \left[ \left( 1 - \kappa_0(\xi, k) \right) \left( \frac{1}{2\pi^2} k^2 \right)^\frac{1}{2} \left( \frac{\pi}{T} \right)^{\frac{1}{2}} \right] \hat{w}_j \hat{w}_i \mathcal{F}_G \left[ (\partial_t - \Delta) [w_i w_j] \right]
\]

with

\[
M_j : \hat{G} \to \mathbb{C}, \quad M_j(\xi, k) := \left( 1 - \kappa_0(\xi, k) \right) \left( \frac{1}{2\pi^2} k^2 \right)^\frac{1}{2} \left( \frac{\pi}{T} \right)^{\frac{1}{2}} \frac{\xi_j}{|\xi|^2 + \frac{\pi}{T} k}.
\]

Observe that the only zero of the polynomial denominator of \( M_j \) is \( (\xi, k) = (0, 0) \). When \( k = 0 \), however, the numerator vanishes due to the term \( 1 - \kappa_0(\xi, k) \). Consequently, we see that \( M_j \in C^\infty(\hat{G}) \) and that \( M_j \) is bounded. Using the same idea as was introduced in [16], based on the transference principle of Fourier multipliers, we will show that \( M_j \) is an \( L^p(G) \)-multiplier for all \( p \in (1, \infty) \). For this purpose, we let \( \chi \) be a ‘cutoff’ function with

\[
\chi \in C^\infty_0(\mathbb{R}; \mathbb{R}), \quad \chi(\eta) = 1 \quad \text{for } |\eta| \leq \frac{1}{2}, \quad \chi(\eta) = 0 \quad \text{for } |\eta| \geq 1,
\]

and define

\[
m_1 : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}, \quad m_1(\xi, \eta) := \frac{\left( 1 - \chi(\frac{\xi}{\eta} \eta) \right) |\xi_j|}{|\xi|^2 + |\eta|}.
\]

Observe that the numerator in the definition of \( m_1 \) vanishes in a neighbourhood of the only zero \( (\xi, \eta) = (0, 0) \) of the denominator. Away from \( (0, 0) \), \( m_1 \) is a rational function with non-vanishing denominator. Consequently, \( m_1 \) is smooth and bounded. Moreover, as one readily verifies, \( m_1 \) satisfies

\[
\sup_{\xi \in [0, 1]^4} \sup_{(\xi, \eta) \in \mathbb{R}^3 \times \mathbb{R}} |\xi^4 \xi_2^3 \xi_3^2 \xi_4 \hat{\partial}_1^2 \hat{\partial}_3 \hat{\partial}_4 m_1(\xi, \eta)| < \infty.
\]

This means that \( m_1 \) satisfies the condition of Marcinkiewicz’s multiplier theorem; see for example [12, corollary 5.2.5] or [27, chapter IV, section 6]. Consequently, \( m_1 \) is an \( L^p(\mathbb{R}^3 \times \mathbb{R}) \)-multiplier. Next, we introduce \( \Phi : \hat{G} \to \mathbb{R}^3 \times \mathbb{R}, \Phi(\xi, k) := (\xi, \frac{\pi}{T} k) \). Clearly, \( \Phi \) is a continuous homomorphism between the topological groups \( G \) and \( \mathbb{R}^3 \times \mathbb{R} \), the latter being considered a topological group in the canonical way. Observe that \( M_j = m_1 \circ \Phi \). Since \( m_1 \) is an \( L^p(\mathbb{R}^3 \times \mathbb{R}) \)-multiplier, it follows from the transference principle of Fourier multipliers on groups\(^4\) (see [3, theorem B.2.1] or [15, theorem 3.4.5]) that \( M_j \) is an \( L^p(G) \)-multiplier. Recalling (5.64), we thus obtain

\[
\forall p \in (1, \infty) : \quad \partial_t^\frac{1}{2} \left[ \mathcal{P}_L [w \cdot \nabla w] \right]_j \leq c_1 \| (\partial_t - \Delta) [w \otimes w] \|_p.
\]

Due to \( w \in W^{2,1,q,p}(G) \) and the fact that, by (5.6), \( w \in L^{\infty}(G) \), we have \( \partial_t w_j w_i \in L^q(G) \cap L^p(G) \) and \( \Delta w_j w_i \in L^q(G) \cap L^p(G) \). Moreover, since \( \frac{q}{2} > q \) we observe that

\[
\nabla w_j \cdot \nabla w_i \in L^q(G) \cap L^p(G).
\]

\(^4\) Originally, de Leeuw [2] established the principle of transference between the torus group and \( \mathbb{R} \). Edwards and Gaudry [3, theorem B.2.1] generalized the theorem of de Leeuw to arbitrary locally compact abelian groups. We employ this general version with groups \( \mathbb{R}^3 \times \mathbb{R} \) and \( G := \mathbb{R}^3 \times \mathbb{R}/\mathbb{T} \mathbb{Z} \). A proof of the theorem for this particular choice of groups can be found in [15, theorem 3.4.5].
Computing
\[(\partial_t - \Delta)[w_j w_l] = \partial_j w_j w_l + w_j \partial_l w_l - (\Delta w_j w_l + w_j \Delta w_l + 2 \nabla w_j \cdot \nabla w_l),\]
we conclude by (5.65) that
\[\dot{\partial}_t^{\frac{1}{2}} \left[ P_\perp (w \cdot \nabla w) \right] \in L^q(G) \cap L^r(G). \tag{5.67}\]
We now recall (5.2), (5.4), and (5.6) to deduce
\[\partial_1 w_j v_l, \Delta w_j v_l, \Delta v_j w_l, \nabla w_j \cdot \nabla v_l \in L^q(G) \cap L^r(G).\]
By the same argument as above, we obtain
\[\|\dot{\partial}_t^{\frac{1}{2}} [w \cdot \nabla v + v \cdot \nabla w]\|_q \in L^q(G) \cap L^r(G) \subset L^q(G) \cap L^r(G).\]
We now apply \(\dot{\partial}_t^{\frac{1}{2}}\) to both sides in (5.63). Clearly, all differential operators commute with \(\dot{\partial}_t^{\frac{1}{2}}\). Recalling definition (4.7) of the Helmholtz projection in terms of a Fourier multiplier, we also see that \(\dot{\partial}_t^{\frac{1}{2}}\) commutes with \(P_H\). Similarly, \(\dot{\partial}_t^{\frac{1}{2}}\) commutes with \(P_\perp\). Consequently, upon applying \(\dot{\partial}_t^{\frac{1}{2}}\) to both sides in (5.63), we obtain
\[\|\dot{\partial}_t^{\frac{1}{2}}[w \cdot \nabla v + v \cdot \nabla w]\|_q \in L^q(G) \cap L^r(G),\]
Combining now lemmas 5.5 and 5.3, we conclude
\[\partial_1^{\frac{1}{2}} w \in W^{2,1,q}_s \cap W^{2,1,r}_s (G). \tag{5.68}\]
Since
\[\|\partial_1^{\frac{1}{2}} w\|_p \leq c_2 \|\partial_1^{\frac{1}{2}} w\|_p,\]
in view of (5.68), we thus have
\[\|\partial_1^{\frac{1}{2}} w\|_p \leq c_2 \|\partial_1^{\frac{1}{2}} w\|_p. \tag{5.69}\]
Moreover, with the fact that \(w \in W^{2,1,q}_s \cap W^{2,1,r}_s (G)\), it follows that \(\nabla w \in W^{1,\frac{1}{2}}_s (G)\). Since \(\frac{1}{2} > 4\), classical Sobolev embedding yields \(W^{1,\frac{1}{2}}_s (G) \hookrightarrow L^\infty(G)\). Thus
\[\nabla w \in L^\infty(G). \tag{5.70}\]
With this information, we return to (5.66) and conclude that in fact
\[\nabla w_h \cdot \nabla w_m \in L^q(G) \cap L^r(G).\]
We therefore obtain improved regularity in (5.67), namely
\[\|\dot{\partial}_t^{\frac{1}{2}}[P_\perp (w \cdot \nabla w)]\|_q \in L^q(G) \cap L^r(G). \tag{5.71}\]
We shall now take a full derivative in time on both sides in (5.63). Concerning the terms that will then appear on the right-hand side, we observe, recalling (5.2), (5.4), (5.6), (5.70), and (5.71), that
\[
\partial_t w \cdot \nabla w, \quad w \cdot \nabla \partial_t w, \quad \partial_t w \cdot \nabla v, \quad v \cdot \nabla \partial_t w \in L^q(G) \cap L^r(G).
\]

Consequently, we have
\[
\partial_t \partial_j w - \Delta \partial_j w - \lambda \partial_t \partial_j w \in L^q(G) \cap L^r(G).
\]

Combining again lemmas 5.5 and 5.3, we conclude the improved regularity
\[
\partial_t w \in W^{2,1,q}_{\alpha} (G) \cap W^{2,1,r}_{\alpha} (G).
\]

of the time derivative of \(w\). We can establish the same improved regularity of spatial derivatives of \(w\). For this purpose we simply observe that
\[
\partial_j w \cdot \nabla w, \quad w \cdot \nabla \partial_j w, \quad \partial_j w \cdot \nabla v, \quad \partial_j v \cdot \nabla w, \quad v \cdot \nabla \partial_j w \in L^q(G) \cap L^r(G),
\]

which implies, on applying \(\partial_j\) on both sides in (5.70), that
\[
\partial_t \partial_j w - \Delta \partial_j w - \lambda \partial_t \partial_j w \in L^q(G) \cap L^r(G).
\]

Employing yet again lemmas 5.5 and 5.3, we obtain
\[
\nabla w \in W^{2,1,q}_{\alpha} (G) \cap W^{2,1,r}_{\alpha} (G).
\]

We now turn our attention to \(v\). Applying \(\mathcal{P}\) to both sides in (4.8), we deduce
\[
\begin{align*}
-\Delta v - \lambda \partial_t v &= \mathcal{P}_H f - \mathcal{P}_H \left[ \mathcal{P} [w \cdot \nabla w] + v \cdot \nabla v \right] \quad \text{in } \mathbb{R}^3, \\
\text{div} v &= 0 \quad \text{in } \mathbb{R}^3.
\end{align*}
\]

Recalling (5.2), (5.4), (5.6) and (5.70), one readily verifies
\[
\mathcal{P}[\partial_j w \cdot \nabla w], \quad \mathcal{P}[w \cdot \nabla \partial_j w], \quad \partial_j v \cdot \nabla v, \quad v \cdot \nabla \partial_j w \in L^q(\mathbb{R}^3) \cap L^r(\mathbb{R}^3).
\]

Thus, applying \(\partial_j\) on both sides in (5.74) we obtain
\[
-\Delta [\partial_j v] = \partial_t [\partial_j v] \in L^q(\mathbb{R}^3) \cap L^r(\mathbb{R}^3).
\]

By lemmas 5.6 and 5.4, we conclude that
\[
\nabla v \in X^{q,r}_{\alpha,\text{Oseen}}(\mathbb{R}^3). \tag{5.75}
\]

Summarizing (5.72), (5.73), and (5.75), we have established regularity for the first-order derivatives of \(w\) and \(v\) similar to what we had originally for \(w\) and \(v\). More precisely, we have
\[
\forall |\alpha| \leq 1, \quad |\beta| + |\kappa| \leq 1 : \quad (\partial^\alpha_x v, \partial^\beta_t \partial^\kappa_u w) \in X^{q,r}_{\alpha,\text{Oseen}}(\mathbb{R}^3) \times W^{2,1,q,r}_{\alpha} (G).
\]

Iterating the argument above with \((\partial^\alpha_x v, \partial^\beta_t \partial^\kappa_u w)\) in the role of \((v, w)\), we obtain the same regularity for all higher order derivatives as well, that is,
\[
\forall |\alpha| \leq m, \quad |\beta| + |\kappa| \leq m : \quad (\partial^\alpha_x v, \partial^\beta_t \partial^\kappa_u w) \in X^{q,r}_{\alpha,\text{Oseen}}(\mathbb{R}^3) \times W^{2,1,q,r}_{\alpha} (G). \tag{5.76}
\]

Concerning the pressure term \(p\), we clearly have
\[
\nabla p = (\text{Id} - \mathcal{P}_H) [f - u \cdot \nabla u]. \tag{5.77}
\]

From (5.76) one easily deduces
\[
\forall |\beta| + |\kappa| \leq m : \quad \partial^\beta_t \partial^\kappa_u [u \cdot \nabla u] \in L^q(G)^3 \cap L^r(G)^3.
\]

Taking derivatives in (5.77) and recalling lemma 5.8, we thus obtain
\[
\forall |\beta| + |\kappa| \leq m : \quad \partial^\beta_t \partial^\kappa_u p \in X^{q,r}_{\text{pres}}(G),
\]

which concludes the theorem. \(\square\)
We have now shown the main results for the reformulated version (4.8) of the system (1.1)–(1.2) in the setting of the group $G$. It remains to verify that the results carry over to the original time-periodic setting in $\mathbb{R}^3 \times \mathbb{R}$. For this purpose, we make two basic observations.

**Lemma 5.11.** Let $k \in \mathbb{N}_0$ and $q \in (1, \infty)$. The quotient mapping $\pi : \mathbb{R}^3 \times \mathbb{R} \to G$ induces, by lifting, an embedding $W^{k,q}(G) \hookrightarrow W^{k,q}_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R})$ with

$$\forall (\alpha, \beta) \leq k : \quad (\partial_\beta^\alpha \delta u) \circ \pi = \delta_\beta^\alpha (u \circ \pi).$$

(5.78)

Similarily, lifting by $\pi$ induces the embedding $W^{2,1,q}(G) \hookrightarrow W^{2,1,q}_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R})$, with the relevant derivatives satisfying (5.78), and for $r \geq q$ also $X_r^{q,r}(G) \hookrightarrow W^{1,r}_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R})$.

**Proof.** Consider $u \in W^{k,q}(G)$ and let $\psi \in C^\infty_0(\mathbb{R}^3 \times \mathbb{R})$. Let $\{\psi_k\}_{k \in \mathbb{Z}} \subset C^\infty_0(\mathbb{R})$ be a partition of unity subordinate to the open cover $\{(\frac{1}{2} T, \frac{1}{2} T + 1) \mid k \in \mathbb{Z}\}$ of $\mathbb{R}$. The $T$-periodicity of $u \circ \pi$ then implies

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}} u \circ \pi \cdot \partial_\beta^\alpha \delta u \, dt \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}} u \circ \Pi \cdot \partial_\beta^\alpha \delta u \left[ \sum_{k \in \mathbb{Z}} \psi_k \psi \right] \, dt \, dx$$

$$= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^3} \int_0^T u \circ \Pi \cdot \partial_\beta^\alpha \delta u \left[ \psi_k \psi \right] \left( x, t + \frac{k}{2} T \right) \, dt \, dx$$

$$= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} u \cdot \partial_\beta^\alpha \delta u \left[ \psi_k \psi \right] \left( x, t + \frac{k}{2} T \right) \circ \Pi^{-1} \, dg$$

$$= (-1)^{(|\alpha|, |\beta|)} \int_{\mathbb{R}^3} \partial_\beta^\alpha \delta u \circ \Pi \cdot \psi \, dt \, dx,$$

from which we deduce (5.78). The other statements follow analogously. \hfill $\Box$

**Lemma 5.12.** The mapping $\Pi = \pi |_{\mathbb{R}^3 \times (0, T)}$ induces, by lifting, a homeomorphism between $W^{k,q}_{\text{per}}(\mathbb{R}^3 \times (0, T))$, $W^{2,1,q,r}_{\sigma,\per}(G)$ and $W^{2,1,q,r}_{\sigma,\per,\per}(\mathbb{R}^3 \times (0, T))$, as well as between $X_r^{q,r}_{\text{per}}(G)$ and $X_r^{q,r}_{\text{per}}(\mathbb{R}^3 \times (0, T))$.

**Proof.** The spaces $C^\infty_0(G)$ and $C^\infty_{0,\text{per}}(\mathbb{R}^3 \times [0, T])$ are dense in the Sobolev spaces $W^{k,q}(G)$ and $W^{k,q}_{\text{per}}(\mathbb{R}^3 \times (0, T))$, respectively. By construction of the differentiable structure on $G$, lifting by $\Pi$ is a homeomorphism between $(C^\infty_0(G), ||\cdot||_{k,q})$ and the space $(C^\infty_{0,\text{per}}(\mathbb{R}^3 \times [0, T]), ||\cdot||_{k,q,\text{per}})$. It follows that this mapping extends to a homeomorphism between $W^{k,q}(G)$ and $W^{k,q}_{\text{per}}(\mathbb{R}^3 \times (0, T))$. The other statements follow analogously. \hfill $\Box$

**Proof of theorem 2.3.** It is easy to verify that lifting by $\Pi$ is a homeomorphism between $L^q(G)$ and $L^q(\mathbb{R}^3 \times (0, T))$ with $\|f \circ \Pi^{-1}\|_{q,G} = T^{-\frac{q}{2}} \|f\|_{q,\mathbb{R}^3 \times (0,T)}$. We thus choose $\epsilon_0 \leq (T^{-\frac{q}{2}} + T^{-\frac{1}{2}})^{-1} \epsilon_1$, where $\epsilon_1$ is the constant from theorem 4.5. Consider now a vector field $f \in L^q(\mathbb{R}^3 \times (0, T)) \cap L^r(\mathbb{R}^3 \times (0, T))^3$ satisfying (1.3) and (2.9). Then $\tilde{f} := f \circ \Pi^{-1}$ satisfies (4.9), whence there exists, by theorem 4.5, a solution $(\tilde{u}, \tilde{p})$ to (4.8) in the class (4.10) (with $\tilde{u} = \tilde{v} + \tilde{w}$). Letting $u := \tilde{u} \circ \pi$ and $p := \tilde{p} \circ \pi$, we deduce from lemmas 5.11 and 5.12 that $(u, p)$ is a solution to (1.1)–(1.2) in the class (2.10). By lemma 5.11 we further see that a vector field $\tilde{u}$ is a weak solution in the sense of definition 2.1 corresponding to $f$ if and
only if $\tilde{U} := U \circ \Pi^{-1}$ is a weak solution in the sense of definition 4.6 corresponding to $\tilde{f}$.

Consequently, uniqueness of $u$ in the class of physically reasonable weak solutions follows from theorem 4.8. Finally, we obtain directly from theorem 4.7 that $u$ satisfies the energy equality (2.11).

Proof of theorem 2.4. It follows directly from theorem 4.9 in combination with lemmas 5.11 and 5.12.

Proof of corollary 2.5. The corollary follows from theorem 2.4 by a standard localization argument combined with classical Sobolev embedding.

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