The Rayleigh-Benard instability in an enclosure having finite thickness walls

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Abstract. Natural convection in an enclosure having finite thickness heat-conducting walls at local heating at the bottom of the cavity has been numerically studied. Heat exchange with an environment due to convection and radiation has been considered on one of external sides of the decision region. The governing unsteady three-dimensional flow equations in the Boussinesq approximation for the gas cavity and heat conduction equation for the solid walls, written in dimensionless variables such as vector potential functions, the vorticity vector and the temperature, have been solved using finite difference method. Results have been obtained for a Prandl number of 0.7 and for a Grashof number ranging from $10^4$ to $10^6$.

Nomenclature

- $Bi$: Biot number, \((= hL_z/k)\)
- $Fo$: Fourier number, \((= \alpha t_0/L_z^3)\)
- $g$: gravitational acceleration (m/s²)
- $Gr$: Grashof number, \((= g\beta(T_{hs} - T_0)L_z^5/\nu^3)\)
- $h$: heat-transfer coefficient (W/m² K)
- $H$: vertical (along the z-axis) dimension of the solution region (m)
- $k$: thermal conductivity (W/m K)
- $k_{ij}$: thermal conductivity ratio, \((= k_i/k_j)\)
- $L_x$: horizontal (along the x-axis) dimension of the solution region (m)
- $L_y$: horizontal (along the y-axis) dimension of the solution region (m)
- $Sk$: Stark number, \((= \varepsilon\sigma L_z(T_{hs} - T_0)^3/k)\)
- $Nu_{avg}$: average Nusselt number
- $Pr$: Prandtl number, \((= \nu/\alpha)\)
- $Ra$: Rayleigh number, \((= g\beta(T_{hs} - T_0)L_z^5/\nu\alpha)\)
- $t$: time (s)
- $t_0$: time scale (s)
1. Introduction
The Rayleigh-Benard convection as one of kinds of a thermal instability of a fluid layer under the gravity is a flow pattern that attracts the greatest attention of researchers [1-10]. Recently heightened interest is linked to creation of the system for validation of new computing algorithms of mechanics of continua [5, 7, 10]. At the same time, effects of both finite heat-conducting walls of an enclosure and nonuniform heat exchange with an environment are not considered in [5, 6, 10]. As is well known [11] such effects can lead to essential changes of both velocity fields and temperature fields. Investigation of similar systems is important for microelectronics. Modern progress trends of the microelectronic technique, linked to reduction of overall dimensions, lead to increase in specific heat-flux density. The latter has the negative effect on operating efficiency of all system [12, 13].

The purpose of the present work is mathematical simulation of unsteady three-dimensional conjugate convective-conductive heat transfer in an enclosure with local heat source in conditions of convective-radiative heat exchange on one of the external sides of the decision region (Figure 1).
2. Mathematical formulation

We considered a boundary-value problem of an unsteady conjugate heat transfer in a region as shown in figure 1. The heat source located at the bottom of the gas cavity is kept at constant temperature during the whole process. The convective-radiative heat exchange with an environment is modeled on one of the external sides ($x = 0$). Other external sides are assumed to be adiabatic.

Thermophysical properties of solid material elements and gas are assumed to be temperature-independent, and the flow regime is laminar. The gas is supposed to be a Newtonian incompressible fluid satisfying the Boussinesq approximation. The gas motion and heat transfer in the internal volume are assumed three-dimensional, the heat exchange by an emission from the heat source and between the walls is assumed to be negligibly small as compared to the convective heat exchange, the gas is assumed to be absolutely transparent for thermal emission.

Figure 1. Schematic view of the problem: 1 – walls; 2 – gas; 3 – heat source

Heat transfer process in the considered area (Fig. 1) is governed by the system of unsteady three-dimensional convection equations in the Boussinesq approximation in the gas cavity [14–16]. The unsteady three-dimensional energy equation [17] with nonlinear boundary conditions is used for simulation of heat conduction in the solid walls.

The mathematical model is formulated in the dimensionless variables such as vector potential functions, vorticity vector and temperature [18–20].

The length of the gas cavity along $x$-axis is chosen as the scale distance. For the reduction to the dimensionless form of the equations system following correlations are used:

\[
X = x/L_x, \quad Y = y/L_y, \quad Z = z/L_z, \quad \tau = t/t_0, \quad U = u/V_0, \quad V = v/V_0, \quad W = w/V_0, \\
\Theta = (T - T_0)/(T_{bs} - T_0), \quad \Psi_x = \psi_x/\psi_0, \quad \Psi_y = \psi_y/\psi_0, \quad \Psi_z = \psi_z/\psi_0, \quad \Omega_x = \omega_x/\omega_0, \\
\Omega_y = \omega_y/\omega_0, \quad \Omega_z = \omega_z/\omega_0, \quad \omega_0 = V_0/L_x, \quad \psi_0 = V_0L_x, \quad V_0 = \sqrt{g_0\beta(T_{bs} - T_0)L_x}.
\]

Based on the above-mentioned assumptions, the non-dimensional form of the governing equations for the fluid can be written as follows:
\[
\frac{\partial \Omega_x}{\partial \tau} + U \frac{\partial \Omega_x}{\partial X} + V \frac{\partial \Omega_x}{\partial Y} + W \frac{\partial \Omega_x}{\partial Z} - \Omega_y \frac{\partial U}{\partial X} - \Omega_z \frac{\partial U}{\partial Z} = \frac{1}{\sqrt{Gr}} \left( \frac{\partial^2 \Omega_x}{\partial X^2} + \frac{\partial^2 \Omega_x}{\partial Y^2} + \frac{\partial^2 \Omega_x}{\partial Z^2} \right) + \frac{\partial \Theta}{\partial Y} \tag{1}
\]

\[
\frac{\partial \Omega_y}{\partial \tau} + U \frac{\partial \Omega_y}{\partial X} + V \frac{\partial \Omega_y}{\partial Y} + W \frac{\partial \Omega_y}{\partial Z} - \Omega_x \frac{\partial V}{\partial X} - \Omega_z \frac{\partial V}{\partial Z} = \frac{1}{\sqrt{Gr}} \left( \frac{\partial^2 \Omega_y}{\partial X^2} + \frac{\partial^2 \Omega_y}{\partial Y^2} + \frac{\partial^2 \Omega_y}{\partial Z^2} \right) - \frac{\partial \Theta}{\partial X} \tag{2}
\]

\[
\frac{\partial \Omega_z}{\partial \tau} + U \frac{\partial \Omega_z}{\partial X} + V \frac{\partial \Omega_z}{\partial Y} + W \frac{\partial \Omega_z}{\partial Z} - \Omega_x \frac{\partial W}{\partial X} - \Omega_y \frac{\partial W}{\partial Y} = \frac{1}{\sqrt{Gr}} \left( \frac{\partial^2 \Omega_z}{\partial X^2} + \frac{\partial^2 \Omega_z}{\partial Y^2} + \frac{\partial^2 \Omega_z}{\partial Z^2} \right) \tag{3}
\]

\[
\frac{\partial^2 \Psi_x}{\partial X^2} + \frac{\partial^2 \Psi_x}{\partial Y^2} + \frac{\partial^2 \Psi_x}{\partial Z^2} = -\Omega_x \tag{4}
\]

\[
\frac{\partial^2 \Psi_y}{\partial X^2} + \frac{\partial^2 \Psi_y}{\partial Y^2} + \frac{\partial^2 \Psi_y}{\partial Z^2} = -\Omega_y \tag{5}
\]

\[
\frac{\partial^2 \Psi_z}{\partial X^2} + \frac{\partial^2 \Psi_z}{\partial Y^2} + \frac{\partial^2 \Psi_z}{\partial Z^2} = -\Omega_z \tag{6}
\]

\[
\frac{\partial \Theta}{\partial \tau} + U \frac{\partial \Theta}{\partial X} + V \frac{\partial \Theta}{\partial Y} + W \frac{\partial \Theta}{\partial Z} = \frac{1}{Pr \sqrt{Gr}} \left( \frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2} + \frac{\partial^2 \Theta}{\partial Z^2} \right) \tag{7}
\]

Energy equation for the solid walls
\[
\frac{\partial \Theta_i}{\partial F \alpha_i} = \frac{\partial^2 \Theta_i}{\partial X^2} + \frac{\partial^2 \Theta_i}{\partial Y^2} + \frac{\partial^2 \Theta_i}{\partial Z^2} \tag{8}
\]

Equations (1)–(8) are subjected to the following initial and boundary conditions.

Initial conditions are

\[
\Psi_x(X, Y, Z, 0) = 0, \quad \Psi_y(X, Y, Z, 0) = 0, \quad \Psi_z(X, Y, Z, 0) = 0,
\]

\[
\Omega_x(X, Y, Z, 0) = 0, \quad \Omega_y(X, Y, Z, 0) = 0, \quad \Omega_z(X, Y, Z, 0) = 0,
\]

\[
\Theta(X, Y, Z, 0) = 0 \quad \text{except temperature for the heat source on which } \Theta = 1 \quad \text{during the whole process time.}
\]

Boundary conditions are:

- convective-radiative heat exchange with an environment is modeled at the wall \(X = 0\)

\[
\frac{\partial \Theta_i}{\partial X} = B_i \cdot \Theta_i - B_i \cdot \Theta^e + S_k \left[ \left( \Theta_i + \zeta \right)^4 - \left( \frac{T^e}{T_{in} - T_0} \right)^4 \right], \quad X = 0;
\]

- at the rest external walls for the equation (8) heat insulation conditions are set

\[
\frac{\partial \Theta_i}{\partial X^i} = 0, \quad X^1 \equiv X, \quad X^2 \equiv Y, \quad X^3 \equiv Z;
\]

- at the solid-fluid interfaces parallel to plane \(XZ\):
\[ \Psi_x = \frac{\partial \Psi}{\partial Y} = \Psi_z = 0, \]
\[ \begin{align*}
\Theta_1 &= \Theta_2, \\
\frac{\partial \Theta_1}{\partial Y} &= k_{z,1} \frac{\partial \Theta_2}{\partial Y}; \\
\frac{\partial \Theta_1}{\partial Z} &= k_{y,1} \frac{\partial \Theta_2}{\partial Z}; \\
\frac{\partial \Theta_1}{\partial X} &= k_{x,1} \frac{\partial \Theta_2}{\partial X}.
\end{align*} \]

- at the solid-fluid interfaces parallel to plane \( XY \):
\[ \Psi_x = \Psi_y = \frac{\partial \Psi}{\partial Z} = 0, \]
\[ \begin{align*}
\Theta_1 &= \Theta_2, \\
\frac{\partial \Theta_1}{\partial Y} &= k_{z,1} \frac{\partial \Theta_2}{\partial Y}; \\
\frac{\partial \Theta_1}{\partial Z} &= k_{y,1} \frac{\partial \Theta_2}{\partial Z}; \\
\frac{\partial \Theta_1}{\partial X} &= k_{x,1} \frac{\partial \Theta_2}{\partial X}.
\end{align*} \]

- at the solid-fluid interfaces parallel to plane \( YZ \):
\[ \frac{\partial \Psi}{\partial X} = \Psi_y = \Psi_z = 0, \]
\[ \begin{align*}
\Theta_1 &= \Theta_2, \\
\frac{\partial \Theta_1}{\partial Y} &= k_{z,1} \frac{\partial \Theta_2}{\partial Y}; \\
\frac{\partial \Theta_1}{\partial Z} &= k_{x,1} \frac{\partial \Theta_2}{\partial Z}; \\
\frac{\partial \Theta_1}{\partial X} &= k_{y,1} \frac{\partial \Theta_2}{\partial X}.
\end{align*} \]

Equations (1)–(8) with corresponding initial and boundary conditions have been solved by means of finite differences method [21–24]. Each time step was started from the temperature field computation both in the gas cavity and in solid walls [Eqs. (7) and (8)], then the Poisson equations for vector potential functions were solved [Eqs. (4)–(6)]. The boundary conditions for the vorticity vector components were determined thereafter and equations (1)–(3) were solved.

The locally one-dimensional scheme of Samarskii [21] was used to solve energy equations [Eqs. (7) and (8)] and equations for the vorticity vector components [Eqs. (1)–(3)] numerically. In this scheme, the solution of a three-dimensional scheme reduces to sequential solution of one-dimensional systems. In this case, the solution of one-dimensional system reduces to a sequential solution of systems of difference equations with tridiagonal matrices by the Thomas method. An implicit difference scheme was used. The evolutionary term represented a one-sided difference in time and had the first order of accuracy in time step. All the derivatives with respect to spatial coordinates were approximated with the second order of accuracy in the step along the coordinate.

2.1. Benchmark solutions

Accuracy of the program developed by the authors was checked by preparing the benchmark solutions both for non-conjugate and conjugate problems. In case of non-conjugate analysis, well-known benchmarks of 3D natural convection in a differently heated cubical enclosure with adiabatic side walls [25, 26] and with linear temperature law at side walls [27, 28]. These benchmark results are shown in Tables 1 and 2.

**Table 1. Variations of average Nusselt number of heat wall with Rayleigh number (the first benchmark)**

| \( Ra \) | \([25]\) | \([26]\) | \begin{tabular}{c@{}c@{}c}
Present & uniform grid \\
50\(\times\)50\(\times\)50 & 60\(\times\)60\(\times\)60 \\
\end{tabular} |
|---|---|---|---|
| \( 10^4 \) | 2.055 | 2.100 | 2.075 | 2.071 |
| \( 10^5 \) | 4.339 | 4.361 | 4.494 | 4.446 |
| \( 10^6 \) | 8.656 | 8.770 | 9.719 | 9.432 |
Table 2. Variations of average Nusselt number of heat wall with Rayleigh number (the second benchmark)

| $Ra$ | [27] uniform grid $95 \times 95 \times 95$ | [28] uniform grid $50 \times 50 \times 50$ | Present uniform grid $60 \times 60 \times 60$ |
|------|---------------------------------|---------------------------------|---------------------------------|
| $10^4$ | 1.520±0.015 | 1.497 | 1.517 | 1.513 |
| $10^5$ | 3.097±0.028 | 3.106 | 3.229 | 3.194 |
| $10^6$ | 6.383±0.070 | 6.681 | 7.254 | 7.027 |

For conjugate problem benchmark solution has been obtained by using results [29]. Table 3 shows the good comparison between the results.

Table 3. Variations of average Nusselt number with Grashof number and heat conductivity ratio

| $Gr$ | $\frac{\lambda_s}{\lambda_f}$ | [11] | [29] | Present |
|------|----------------|------|------|--------|
| 1 | 0.877 | 0.87 | 0.872 |
| $10^3$ | 5 | – | 1.02 | 1.023 |
| | 10 | – | 1.04 | 1.046 |
| | 1 | 2.082 | 2.08 | 2.116 |
| $10^5$ | 5 | – | 3.42 | 3.421 |
| | 10 | – | 3.72 | 3.781 |
| | 1 | 2.843 | 2.87 | 3.002 |
| $10^6$ | 5 | – | 5.89 | 6.306 |
| | 10 | – | 6.81 | 6.935 |

3. Results and discussion
Numerical analysis of the boundary value problem (1)–(8) has been carried out at following dimensionless complexes such as $Gr=10^3–10^6$, $Pr=0.7$, $k_{2,1}=3.7 \cdot 10^{-2}$, $3.7 \cdot 10^{-3}$, $3.7 \cdot 10^{-4}$ describing the basic modes of conjugate convective heat transfer in enclosures. Dimensionless defining temperatures were $\Theta_s = -1$, $\Theta_{bs} = 1$, $\Theta_b = 0$.

3.1. Effect of the Grashof number
Streamlines and temperature fields at different values of the Grashof number at $\tau = 60$, $X = 0.6$ are presented in figure 2. The direction of gas motion in the gas cavity is indicated by arrows on streamlines.

Two convective cells are formed in the gas cavity at $X = 0.6$ for $Gr=10^4$ (figure 2)a). The reason for the appearance of these cells is both the heat source located on the bottom of the gas cavity and diffusion of perturbations deep into the gas cavity from solid walls. The field of temperature is uniformly in the gas cavity that is linked to dominance of conductive heat transfer. Heating of solid walls from the heat source is observed. In the central part of the cavity ($Y = 0.58$) the thermal plume caused by influence of the heat source is shaped. The increase in Grashof number (figure 2b) leads to change of both a configuration of streamlines and intensity of gas circulation in the cavity. Increase in flow velocities and displacement of a core of the convective cells in a vertical direction are observed. At the same time the temperature field is changed. The essential increase in temperature is observed in top layers of the gas cavity.
Figure 2. Streamlines $\Psi$ and isotherms $\Theta$ at $X = 0.6, \tau = 60, k_{2,1} = 3.7 \cdot 10^{-2}$: $Gr = 10^4 - a, Gr = 10^5 - b$

Streamlines and temperature fields at $Z = 0.6$ are presented in figure 3. The Grashof number ranging from $10^4$ to $10^5$ is reflected on the increase in the circulation velocities of the convective cells. The temperature field is changed. The thermal plume is formed in the centre of the gas cavity at $Gr = 10^5$. There is the heating of the cavity from the thermal plume in a radial direction.

The temperature profiles at $X = 0.6, Z = 0.9, \tau = 60$ and at different values of the Grashof number are shown in figure 4.
Figure 3. Streamlines $\Psi$ and isotherms $\Theta$ at $Z = 0.6$, $\tau = 60$, $k_{2,1} = 3.7 \cdot 10^{-2}$: $Gr = 10^4 - a$, $Gr = 10^5 - b$

Figure 4 shows the effect of the buoyancy force on the temperature field in the gas cavity. The increase in the Grashof number $10^3 \leq Gr < 10^5$ leads to uniform increase in the temperature in the central part of the cavity where there is the thermal plume. The essential change of the temperature profile at $Gr = 10^5$ and $Gr = 5 \cdot 10^5$ is linked to the increase in the gas motion velocities. It should be noted that the increase in Grashof number leads to the diminution of the boundary layer thickness.
The analysis of the Grashof number influence on the generalized heat transfer coefficient (the average Nusselt number \( \text{Nu}_{\text{avg}} \)) on the heat source surface has been carried out (figure 5).

The presented graphic dependences of the average Nusselt number as a function of the Grashof number evidently show the typical increase in the heat transfer intensity on the heat source surface at \( \tau = 60 \).
the Grashof number ranging $10^4 \leq Gr < 10^6$. The increase in a role of the buoyancy force in comparison with the internal friction force leads to the heat transfer intensification on the heater surface. The reason for this fact is both the increase in the motion velocities and more essential cooling of the descending gas flows. The latter leads to the significant heat sink from the heat source surface.

3.2. Effect of the transient factor
The transient factor in the conjugate heat transfer problems plays an essential role [20, 24] as it reflects not only dynamics of the velocity and temperature fields in the gas cavity, caused by formation, evolution and dissipation of the vortex structures from an initial quiescent state, but also it characterizes the thermal sluggishness of the solid walls in conditions of the environment influence. At the same time the advantage of such statement is definition of the temperature field at the solid-fluid interface on the basis of conservation laws without additional empirical data, for example, for the heat transfer coefficient. In turn the approach based on use of empirical heat transfer coefficients does not allow considering the transient factor as these coefficients are time functions.

The dynamics of the streamlines formation and temperature fields formation at $Y = 0.6$ for $Gr = 5 \cdot 10^4$ are shown in figure 6.

There are two convective cells of small intensity in the gas cavity at $\tau = 12$ (figure 6,a). This fact is explained by the initial stage of the flow evolution. The temperature distribution is already non-uniformly. There is an intensive heating of the bottom of the cavity.

The increase in the time up to $\tau = 36$ (figure 6,b) leads to the increase in the velocities of the gas circulation. The intensity of a motion in the left vortex and its sizes are greater in comparison with another vortex. This fact can be explained as the low temperature front has reached the gas cavity in a zone of the center of the left wall. The latter leads to the displacement of the thermal plume to the right wall. The further increase in the dimensionless time (figure 6,c) is reflected on dissipation of the right convective cell and leads to full displacement of the thermal plume (figure 7) that is caused by cooling of the left wall.

The temperature profiles at $Y = Z = 0.6$ for $Gr = 5 \cdot 10^4$ depending on time points are presented in figure 7. The increase in the dimensionless time leads to the displacement of the thermal plume to the right wall. There is reduction in the temperature at $X = 0$ and, accordingly, cooling of the solid wall such as $0 \leq X \leq 0.06$ as the ambient temperature is below initial temperature of the decision region.

The graphic dependences of the generalized heat transfer coefficient on the heat source surface versus dimensionless time and the Grashof number are presented in figure 8. Decrease in the average Nusselt number $Nu_{\text{avg}}$ with the dimensionless time can be explained by diminution of the temperature gradient at a surface of the heat source.

3.3. Effect of the heat conductivity ratio
The thermal conductivity ratio characterizes the heat exchange conditions on the solid-fluid interface at analysis of the conjugate heat transfer. Accordingly, the variation of this parameter enables to define a range of change of the required characteristics.

The decrease in the thermal conductivity ratio (figures 9, 10) leads to both an intensification of the heat transfer in the solid walls and suppression of the convective motion in the gas cavity. The latter can be explained by intensive heat elimination into the solid walls. At the same time the average Nusselt number on the heat source surface decreases (figure 5).
Figure 6. The dynamics of streamlines $\Psi$ and temperature fields $\Theta$ at $Y=0.6$ for $Gr=5 \cdot 10^4$, $k_{2,1}=3.7 \cdot 10^{-3}$: $\tau=12 - a$, $\tau=36 - b$, $\tau=60 - c$
Figure 7. The temperature profiles at $Y = Z = 0.6$ for $Gr = 5 \cdot 10^4$, $k_{2,1} = 3.7 \cdot 10^{-3}$

Figure 8. Variation of the average Nusselt number with the dimensionless time at different values of the Grashof number
Figure 9. The temperature profiles at $Y = Z = 0.6$ for $Gr = 10^4$ depending on the heat conductivity ratio.

Figure 10. Streamlines $\Psi$ and isotherms $\Theta$ at $X = 0.6$, $\tau = 24$, $Gr = 5 \cdot 10^4$: $k_{2,1} = 3.7 \cdot 10^{-3} - a$, $k_{2,3} = 3.7 \cdot 10^{-4} - b$
4. Conclusion
Mathematical simulation of unsteady conjugate heat transfer in an enclosure having finite thickness heat-conducting walls in the presence of heat source under the condition of convective-radiative heat exchange with an environment has been carried out. Distributions of streamlines and temperature fields in wide enough range of defining parameters $10^4 \leq Gr < 10^6$, $Pr = 0.7$ have been obtained. The influence of the defining parameters such as the Grashof number, the transient factor and the heat conductivity ratio on formation of thermo-hydrodynamic modes has been analysed. The scopes of the nonlinear environmental effect, owing to conduction in the solid walls of the enclosure (Fig. 3, 6, 7) have been determined. It should be noted that the decrease in the heat conductivity ratio leads to reduction of the average Nusselt number on the heat source surface (Fig. 5).

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