Conformal anomaly in non-hermitian quantum mechanics

Pulak Ranjan Giri

Theory Division, Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Calcutta 700064, India

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A model of an electron and a Dirac monopole interacting through an axially symmetric non-hermitian but $\mathcal{PT}$-symmetric potential is discussed in detail. The intriguing localization of the wave-packet as a result of the anomalous breaking of the scale symmetry is shown to provide a scale for the system. The symmetry algebra for the system, which is the conformal algebra $SO(2,1)$, is discussed and is shown to belong to the enveloping algebra of the combined algebra, composed of the Virasoro algebra, $\{L_n, n \in \mathbb{N}\}$ and an abelian algebra, $\{P_n, n \in \mathbb{N}\}$.

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I. INTRODUCTION

The massless scalar field theory $\mathcal{L}$ in $(d+1)$-dimensions with interaction $\mathcal{L}_{int} = -g_0^2(d+1)/(d-1)$ is known to have conformal symmetry $SO(2,1)$, generated by dilation $D$, the Hamiltonian itself $H$ and generator for conformal transformation $K$. This model defined by the Lagrangian $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - g_0^2(d+1)/(d-1)$ plays a crucial role in the context of conformal symmetry in quantum mechanics, when the scalar field is considered in $(0+1)$-dimension.

Since then a huge number of works $[2, 3, 4, 5]$ have been reported in the quantum mechanical settings, studying conformal symmetry $\mathcal{L}$ and related issue like anomaly $\mathcal{L}$. The basic ingredient in almost all the cases is the interaction potential of the form $V_I = C/r^{\alpha t}$, $D : r \rightarrow \alpha r$, of the potential compared to the kinetic energy term. The same scale transformation for the potential $D : V_I \rightarrow \alpha^{-2} V_I$ as the kinetic term $D : p^2/2m \rightarrow \alpha^{-2} p^2/2m$ makes the Lagrangian scales as $D : L \rightarrow \alpha^{-2} L$, which is sufficient to keep the action, $S = \int dt L$, invariant, $D : S \rightarrow S$. The invariance of the system under scale transformation, $D$, has a consequence on the observables like bound state eigenvalue and phase-shift of the scattering states. Scale symmetry implies that, the ground state of the system is not bounded from below, i.e., $E_{g,s} = -\infty$. Then the system is not stable and therefore will collapse into the singularity. It is however possible to make these systems stable against collapse by suitable quantization. The quantization procedure provide a scale for the system and shows up as a lower bound to the bound state eigenvalue.

It can be noted that the scale transformation in spherical co-ordinates, $D : t \rightarrow \alpha^2 t$, $D : r \rightarrow \alpha r$, $D : \theta \rightarrow \theta$, $D : \phi \rightarrow \phi$, does not effect the angular coordinates $\theta$ and $\phi$. One can therefore generalize potential, still remaining scale covariant, like $V_{\theta, \phi} = C(\theta, \phi)/r^{\alpha s}$, where now instead of being constant coefficient, $C(\theta, \phi)$ is both function of $\theta$ and $\phi$. Note the scale transformation $D : V_{\theta, \phi} \rightarrow \alpha^{-2} V_{\theta, \phi}$, which is same as the previous scale covariant potential $V_I$. One can also generalize the kinetic term to include magnetic vector potential as long as it remains scale covariant. One can easily find a magnetic vector potential $A$, such that the generalized kinetic term $(p + cA)^2/2m$ transform the same way as $p^2/2m$. In our present article, we discuss such a system, where an electrically charged particle is moving in the background field of a magnetic monopole. We also include an interaction potential, which is axially symmetric $V_{\mathcal{PT}} = \frac{c_1}{r(r+z)} + \frac{c_2}{r(r-z)}$, where $c_1$ and $c_2$ are two complex valued constant parameters such that $c_1 = c_2^*$. This system is obtained from the generalized MIC-Kepler system $[6, 10]$, which is the system of two dyons with the axially symmetric potential $V_{MIC} = \frac{c_1}{r(r+z)} + \frac{c_2}{r(r-z)} - \alpha_s/r + s^2/r^2$. We set the Coulomb term and the extra inverse square term zero, i.e., $\alpha_s = s^2 = 0$ and and generalize the two constants $c_1$ and $c_2$ to complex numbers in $V_{MIC}$. Note that the complex potential, $V_{\mathcal{PT}}$, although makes the system non-hermitian, it still remains $\mathcal{PT}$-symmetric.

This article is organized in the following way: We discuss the model in the next section and offer a physically realizal solution for the problem. The scale symmetry of the classical version of the problem is discussed in Sec. III, and it is shown that scale symmetry goes anomalous breaking in our quantization process. The algebraic property of the model is discussed in Sec. IV, where it is shown that $SO(2,1)$ algebra is a subalgebra of an enveloping algebra. Finally we conclude in Sec. V.

II. ELECTRON AND DIRAC MONOPOLE SYSTEM IN $\mathcal{PT}$ -SYMMETRIC POTENTIAL

The formal Hamiltonian for an electron moving in the background field of a Dirac monopole and interacting with the potential $V_{\mathcal{PT}}$ is written in the form ($\hbar = e = c = 2 \times \text{reduced mass} = 1$)

$$H = \left( -i \nabla - sA \right)^2 + \frac{c_1}{r(r+z)} + \frac{c_2}{r(r-z)},$$

where according to Dirac quantization condition $s = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots$. The vector potential, $A$, due to the
magnetic monopole field, \( \mathbf{B} = r/r^{-3} \), has been taken as \( \mathbf{A} = (x^2 - r z)^{-1}(y, -x, 0) \). Note that we drop the inverse square term \( s^2/(\sqrt{2}r)^2 \) from our model Hamiltonian, which was put in by hand in order to restore the \( SO(4) \) and \( SO(1,3) \) symmetry for the bound state and scattering state respectively. See Ref. 11 for detail discussion on it. The Hamiltonian (11), defined on the Hilbert space \( L^2(\mathbb{R}^3, r^2drd\Omega) \in \mathcal{H} \) can be separated in radial and angular part as

\[
H \equiv H(r) + r^{-2}\Sigma(\theta, \phi). \tag{2}
\]

Note that the radial and angular Hamiltonians \( H(r) = -[\partial^2_1 - \frac{1}{2} s^2 - \Sigma(\theta, \phi)] \) act over the Hilbert spaces \( L^2(\mathbb{R}^3, r^2dr) \) and \( L^2(S^2, \Omega) \) respectively, where \( L^2(\mathbb{R}^3, r^2drd\Omega) \equiv L^2(\mathbb{R}, r^2dr) \otimes L^2(S^2, d\Omega) \). We now consider a similarity transformation (unitary) \( U(r) : L^2(\mathbb{R}^3, r^2drd\Omega) \rightarrow L^2(\mathbb{R}^3, r^2dr) \otimes L^2(S^2, d\Omega) \), so that the radial Hamiltonian is obtained in a convenient form

\[
H_{UL} \equiv U(r)^{-1}HU(r) = -\partial^2_1 + (\alpha - s^2)/r^2, \tag{3}
\]

where \( \Sigma(\theta, \phi) \) has been replaced by its corresponding eigenvalue \( \alpha \), obtained from \( \Sigma(\theta, \phi)Y(\theta, \phi) = \alpha Y(\theta, \phi) \). The explicit form of the angular Hamiltonian can be found in 9, 10, 12, but for our present purpose it is not required. Note that the Hamiltonian (4) is a well known operator appeared in diverse fields in theoretical physics. It is an example of a class of operators where both the method of self-adjoint extensions (SAE) and re-normalization technique are successfully applied in order to get physically realizable solutions. The specific technique used depends on the value of the effective coupling constant \( (\alpha - s^2) \) of the inverse square interaction. In our case \( (\alpha - s^2) \) is always positive. Since the re-normalization technique is useful for coupling \( -1/4 \) 3, we rule out the the re-normalization technique from our consideration because effective coupling is \( (\alpha - s^2) > 0 \).

For potential of the form \( V_1 = Cr^{-2} \) in 1-dimensional, one can show that there is a window in the coupling constant, \( -1/4 \leq C \leq 3/4 \), where the problem under consideration is not self-adjoint for a very simple domain and needs a self-adjoint extensions (SAE). Our model Hamiltonian \( H_U \) therefore deserves SAE for \( -1/4 \leq (\alpha - s^2) \leq 3/4 \). The usual prescription is to define the Hamiltonian \( H_U \) over a very restricted domain

\[
D(H_U) = \{ \psi(r) \in L^2(\mathbb{R}^3, dr), \psi(0) = \psi'(0) = 0 \}, \tag{4}
\]

so that the Hamiltonian \( H_U \) easily becomes symmetric, \( (\chi_1, H_U \chi_2) = (H_U \chi_1, \chi_2) \) for \( \chi_1, \chi_2 \in D_U \). Then one needs to go for a consistent method to get a SAE for the Hamiltonian \( H_U \). We use the von Neumann’s method of SAE for our purpose. It helps us to construct a self-adjoint domain

\[
D^*(H_U) = \{ D(H_U) + \psi^\omega, |\psi^\omega \in D(H_U^2) \}, \tag{5}
\]

where the explicit form of the function \( \psi^\omega \) is the linear combination \( \psi^\omega = \psi^+ + \exp(i\omega)\psi^- \) of the two deficiency space solutions \( H_U^1\psi^\pm = \pm i\psi^\pm \) (\( H_U^1 \) is the adjoint of \( H_U \)). The Hamiltonian \( H_U \) is now self-adjoint over the domain \( D^*(H_U) \).

The bound state energy and bound state eigenfunction, for \( 0 < \zeta^2 < 1/4 = \alpha - s^2 \leq 3/4 \), are respectively given by

\[
E(L^{-2}, \omega) = -L^{-2}F(\omega), \quad \psi(r) \equiv K_\zeta \left( \sqrt{E(L^{-2}, \omega)r} \right), \tag{6}
\]

where \( K_\zeta \) is the modified bessel function. \( L \) is the length scale which comes from self-adjoint extensions and \( F(\omega) \) is a periodic function whose explicit form can be found by matching the limiting value of the eigenfunction (6) with the domain \( D^*(H_U) \) at \( r \rightarrow 0 \),

\[
F(\omega) = \sqrt{\frac{\cos \frac{1}{2}(2\omega + \zeta \pi)}{\cos \frac{1}{2}(2\omega - \zeta \pi)}}, \tag{7}
\]

Note that the periodic function \( F(\omega) = F(\omega + \pi) \) also depends on the coupling constant \( \zeta \), besides the SAE parameter \( \omega \). The bound state does not exist for two extremes for the periodic function, when \( |F(\omega = (1 - \zeta^2/2)\pi)| = 0 \) (this is the condition for threshold) or \( |F(\omega = (1 + \zeta^2/2)\pi)| = \infty \) (this is the condition when the bound state collapses into singularity).

### III. ANOMALOUS SYMMETRY BREAKING

We now discuss the scaling symmetry breaking in our model. We start with the corresponding classical Hamiltonian

\[
H_{CI} = \mathcal{D}_{CI}^2 + V_A , \tag{8}
\]

where now \( \mathcal{D}_{CI} = (p - sA) \). The lagrangian obtained from the Hamiltonian (3) is found to be \( L_{CI} = 1/2r^2 - A.v - V_A \). It can be noted that in terms of dimensions the relation \( [H_{CI}] = [L_{CI}] = [t^{-1}] = [r^{-2}] \) evidently makes the action \( S = \int L_{CI}dt \) dimensionless. Consider the scale transformation \( T : r \rightarrow s r, t \rightarrow \tilde{t}t \). The action \( S \) is invariant under this transformation, \( T : S \rightarrow S \), which in turn implies the existence of a conserved charge according to the Noether theorem, known as Dilation 13, 14

\[
D_{CI} = \sum \frac{\partial L_{CI}}{\partial \dot{x}_i} \Delta x - T^{00} \Delta t = H_{CI}t - (1/4) [r, p_r]_+ \tag{9}
\]

(in symmetrized form), where \( T^{00} = \sum \frac{\partial L_{CI}}{\partial \dot{x}_i} \dot{x}_i - L_{CI} \). In classical physics the transformation related to dilation \( D_{CI} \), can be shown 13, 14 to be responsible for generating infinitesimal scale transformation.

In order to see whether the scale symmetry, we just discussed, goes through unbroken even after quantization of the classical system \( H_{CI} \), we have to know the possible consequence of the scale symmetry which we could be able to identify in quantum system. It can be easily
shown that in order the scale symmetry to be unbroken even after quantization, the system does not have any lower bound of the energy, which implies that there is no bound state for the system. The proof goes as follows: Consider the eigenvalue equation $H\psi(r) = E\psi(r)$. The function $\psi(r)$ is also an eigen-state with eigenvalue $E/g^2$. This shows that the eigen-state $\psi(r)$ can be continuously pushed towards the center to collapse to the singularity in the limit $g \to 0$; $\lim_{g\to 0} \psi(r), \lim_{g\to 0} E/g^2 = \infty$.

In our case we showed in the previous section that there are two extremes: one is threshold at $(1 - \zeta/2)\pi$ and other is at $(1 + \zeta/2)\pi$ where the bound state collapses, indicating that scaling symmetry still survives in case of two inequivalent quantizations.

IV. THE ENVELOPING ALGEBRA AND ITS CORRESPONDING PROPERTY

The model we are discussing in this article has a radial eigen-value equation which possesses $SO(2,1)$ symmetry generated by the Hamiltonian $H_U$, the dilatation $D = iD$ and the conformal generator $K$. The explicit forms of two of the $SO(2,1)$ generators, $H_U$ and $D$, are known in our case so far. The explicit form of the generator $K$ of the algebra ($h = 1$) is found to be $K = Ht^2 - (1/2) [r, p_r]_+ + (1/4)r^2$. It has been shown in the literature that the Hamiltonian of form $H_U$ are part of the enveloping algebra of an algebra $A$, made up with the two sub-algebras. One is the Virasoro algebra (with generators $L_m, m \in \mathbb{Z}$)

$$[L_m, L_n] = (m - n)L_{m+n}, \quad (11)$$

and other is an abelian algebra (with generators $P_m, m \in \mathbb{Z}$)

$$[P_m, P_n] = 0, \quad (12)$$

with commutators between the elements of the two different sub-algebras is

$$[L_m, P_n] = nP_{m-n}. \quad (13)$$

Consider the representations $L_m = -r^{m+1}\partial_r$ and $P_n = 1/r^n$. One can write down the above discussed generators of the $SO(2,1)$ algebra, $H_U, D$ and $K$ in terms of $L_m$ and $P_n$ for some values of $m$ and $n$,

$$H_U = (-L_{-1} + \vartheta P_1)(L_{-1} + \vartheta P_1), \quad H_U = H_U t^2 + 1/4(L_0 + L_{-1} P_{-1}), \quad K = H_U t^2 + 1/2(L_0 + L_{-1} P_{-1})t + (1/2)P_{-2}. \quad (16)$$

where $\vartheta = \zeta + 1/2$. It can be noted that the $SO(2,1)$ generators are products of the elements of the algebra $A$ given in Eqs. (14), (12) and (13). So it does not belong to the algebra $A$, however they belong to its enveloping algebra $\tilde{A}$. One can think $SO(2,1)$ as a sub-algebra of the enveloping algebra $\tilde{A}$. The commutation relation of the Hamiltonian with the generators of the algebra $A$ can be written as

$$[L_n, H_U] = -(n + 1) [L_{n-1}, L_{-1}]_+ + 2\vartheta(\vartheta - 1)P_{2-n}, \quad [P_n, H_U] = n [L_{-1}, P_{1+n}]_+ \quad (17)$$

The commutation relation of the remaining $SO(2,1)$ generators with the elements of the algebra, $A$, can be similarly evaluated as

$$[L_n, K] = [L_n, H_U] t^2 + n/2L_n + (n + 1)/2L_{n-1}P_{-1} - 1/2L_{n-1}P_{-2-n} - mP_{-2-n}, \quad (18)$$

$$[P_n, K] = [P_n, H_U] t^2 - n/2P_n - 1/2P_{n+1}P_{-1}, \quad (19)$$

$$[L_n, -iD] = [L_n, H_U] t + n/42L_n + (n + 1)/4L_{n-1}P_{-1} - 1/4L_{n-1}P_{-1-n}, \quad (20)$$

$$[P_n, -iD] = [P_n, H_U] t - n/4P_n - 1/4P_{n+1}P_{-1}. \quad (21)$$

Note that all the above three commutators are written in terms of the nonlinear sums of the of elements of $A$.

V. CONCLUSION

In conclusion, we discussed the dynamics of an electron in the field of a Dirac monopole and interacting with an axially symmetric $PT$-symmetric potential $V_{PT}$. We find bound state solutions due to the anomalous breaking of the scaling symmetry of the system by self-adjoint extensions. We show that the $so(2,1)$ algebra of the system belong to the enveloping algebra, $\tilde{A}$, of an algebra, $A$, which is a combination of the Virasoro algebra, $\{L_n, n \in \mathbb{N}\}$ and an abelian algebra, $\{P_n, n \in \mathbb{N}\}$.

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