OPTIMAL CONTROL PROBLEMS FOR A NEUTRAL INTEGRO-DIFFERENTIAL SYSTEM WITH INFINITE DELAY

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Abstract. This work devotes to the study on problems of optimal control and time optimal control for a neutral integro-differential evolution system with infinite delay. The main technique is the theory of resolvent operators for linear neutral integro-differential evolution systems constructed recently in literature. We first establish the existence and uniqueness of mild solutions and discuss the compactness of the solution operator for the considered control system. Then, we investigate the existence of optimal controls for the both cases of bounded and unbounded admissible control sets under some assumptions. Meanwhile, the existence of time optimal control to a target set is also considered and obtained by limit arguments. An example is given at last to illustrate the applications of the obtained results.

1. Introduction. In this paper, we consider the following neutral integrodifferential evolution systems with infinite delay:

\[ \begin{aligned}
\frac{d}{dt} \left[ x(t) + \int_{-\infty}^{t} N(t-s)x(s)ds \right] &= Ax(t) + \int_{-\infty}^{t} \gamma(t-s)x(s)ds \\
&+ F(t, x_t) + B(t)u(t), \quad 0 \leq t \leq T, \\
x_0 &= \varphi \in \mathcal{B}, \quad t \in (-\infty, 0],
\end{aligned} \]

where the state function \( x(\cdot) \in X \) and the control function \( u(\cdot) \in U \) with \( X \) and \( U \) being two Hilbert spaces. The history functionals \( x_t(\cdot) : (-\infty, 0] \to X \) given, in the usual way, by \( x_t(\theta) = x(t + \theta) \) for \( \theta \leq 0 \), belong to some abstract phase space \( \mathcal{B} \) defined axiomatically. The (unbounded) linear operator \( A \) generates an analytic semigroup on \( X \), \( \gamma(t) \) is a family of closed linear operators on \( X \) with domain \( D(\gamma(t)) \supseteq D(A) \) and \( N(t) \) in the neutral term is a family of bounded linear operators on \( X \) for every \( t \geq 0 \). Additionally, \( \{B(t) : t \geq 0\} \) is a family of bounded linear operators from \( U \) to \( X \) and \( F : [0, T] \times \mathcal{B} \to X \) is a nonlinear Lipschitz continuous function to be described below.

As well known, the concept of optimal control is one of the fundamental concepts in mathematical control theory for infinite differential systems [24, 25]. Roughly
speaking, optimal control generally describes that the minimization for a cost function of the states and control inputs of the system over a set of admissible control functions. Meanwhile, the time optimal control problem is to find a control which transfers the trajectory of a control system from a given initial state to a specified final state in minimum time. Actually, the optimal and time optimal control have been commonly used in control theory. Hence, the problems of optimal and time optimal control for infinite-dimensional evolution systems gain much more attention in the past decades. In [13], Harrat et al. established some sufficient conditions for solvability and optimal controls of an impulsive nonlinear Hilfer fractional delay evolution inclusion in Banach spaces. Mokkedem and Fu [28] discussed the standard optimal control and time optimal control problems for a class of semilinear evolution systems with infinite delay by employing the theory of fundamental solution as in papers [18, 19, 30]. While Tucsnak et al. studied in [36], by weakening the regularity assumptions on the initial data with $z_0 \in X$, the numerical approximation of the solutions of a class of abstract parabolic time optimal control problems with unbounded control operator. For more related works on optimal and time optimal control problems for evolution equations, we refer to [1, 2, 3, 20, 22, 23, 26, 29, 33, 38, 39, 40] and the references cited therein.

On the other hand, System (1) is an abstract form of neutral partial functional integro-differential equations (NPFIDEs) with infinite delay. Indeed, NPFIDEs can be used to describe a lot of natural phenomena arising from many fields such as electronics, fluid dynamics, chemical kinetics and biological models. An effective way of studying NPFIDEs is to transfer them into integro-differential evolution equations with or without delay in abstract spaces. In [9, 10, 11], Grimmer et al. proved the existence of solutions of the following integro-differential evolution equation in Banach space $X$:

$$\begin{align*}
\begin{cases}
v'(t) = Av(t) + \int_0^t \gamma(t-s)v(s)ds + f(t), & \text{for } t \geq 0, \\
v(0) = v_0 \in X,
\end{cases}
\end{align*}$$

(2)

where $f : \mathbb{R}^+ \to X$ is a continuous function. The author(s) showed the representation, existence and uniqueness, of solutions of (2) via the resolvent operator associated to the following linear homogeneous equation

$$\begin{align*}
\begin{cases}
v'(t) = Av(t) + \int_0^t \gamma(t-s)v(s)ds, & \text{for } t \geq 0, \\
v(0) = v_0 \in X.
\end{cases}
\end{align*}$$

That is, the resolvent operator, replacing the role of $C_0$-semigroup for evolution equations, plays an important role in solving Eq. (2) in weak and strict senses. By means of the theory of resolvent operators, in these years much work on various topics, such as existence results, asymptotic properties, controllability and optimal control on various semilinear integro-differential systems have been investigated by many authors, see [4, 5, 15, 34, 39, 40], for example. In particular, Diallo et al. [4] obtained by using the theory of resolvent operator and set-valued mapping the existence and stability for the optimal control problem of the following semi-linear
integro-differential equation with compact control set

\[
\begin{aligned}
&y'(t) = Ay(t) + \int_0^t C(t - s)y(s)ds + f(t, y(t)) + Bu(t), \quad t \in [0, b], \\
y(0) = y_0.
\end{aligned}
\]

In [39] Yan and Lu also discussed the optimal control problem of semi-linear fractional stochastic integro-differential equation with infinite delay. The main tool there was the theory of analytic $\alpha-$resolvent operators.

Very recently, Dos Santos et al. [7, 14] established the theory of resolvent operators for neutral linear integro-differential equations (to be presented briefly in Section 2). It is clear that this theory of resolvent operators is a routine extension of that introduced by Grimmer et al. [9, 10, 11] and has a more comprehensive description for various dynamic behaviors of neutral integro-differential equations. For example, Dos Santos applied it in [6] to study the existence of solutions for abstract neutral integro-differential equations with state-dependent delay. Likewise, by using the theory of new resolvent operators and approximating technique, Jeet and Sukavanam [17] obtained the approximate controllability for nonlocal and impulsive neutral integro-differential equations. Up to now, there are many papers on neutral integro-differential equations by employing this new theory, see [32, 37].

In this article we intend to derive the existence of optimal and time optimal controls of the infinite-delayed neutral integro-differential system (1). Our purpose here is to develop the results for semi-linear differential equations established in [28] to neutral integro-differential equations. Certainly, neutral integro-differential equations are much more complicate than semi-linear differential systems to be dealt with. The essential technique in all our discussion, other than in [39], is the theory of resolvent operators for linear neutral integro-differential systems established in [7, 14]. As the considered system involves a nonlinear function, we will first prove the compactness of the solution operator $W$ mapping from each control $u(\cdot)$ to its corresponding mild solution. Then we obtain the existence of optimal and time optimal controls for the control system (1) by using the compactness of $W$ and standard convergence arguments. Due to the compactness of $W$, as in [28], we do not require the convexity assumptions on the integral cost function. Note that the convexity assumptions on integral cost functions were generally adopted in literature, see [2, 13, 20, 22, 30, 38, 40] for example. It is also worth stressing here that, to the best of our knowledge, the study of the optimal control problem for abstract neutral integro-differential with infinite delay of the form (1) is an untreated topic in the literature, this is actually an additional motivation of our work. Clearly our work can be regarded as extension and development of that in [4, 28, 30, 38] and other related papers mentioned above.

The organization of this work is as follows. Some basic notations and preliminary facts about the resolvent operators and phase space for $B$ for infinite-delayed equations are presented in Section 2. In Section 3, using the Banach contraction principle, we first prove the existence and uniqueness of mild solutions for System (1) represented via the resolvent operator. Then, we show the compactness of the solution operator $W$. We discuss in Section 4 the existence of optimal controls of certain fixed time integral cost function subject to the neutral integro-differential control system (1) in two cases of bounded and unbounded admissible control sets, respectively. In Section 5, we study the existence of optimal control which transfer
the mild solutions of the neutral integro-differential control system (1) from the initial data to a target set in the shortest time, namely, the time optimal control to a target set. Following the existence results, we give a theorem about the convergence of time optimal controls to a point target set. Finally, an example is provided to show the applications of the obtained results in Section 6.

2. Preliminaries. Let $X$ and $K$ be two separable Hilbert spaces and $\mathcal{L}(X; K)$ stands for the space of all bounded linear operators from $X$ into $K$, we abbreviate it to $\mathcal{L}(X)$ whenever $K = X$. For the closed linear operator $(A, D(A))$ on $X$ in Eq. (1), denote by $Y$ the Banach space $(D(A), \| \cdot \|)$ with the graph norm $\| \cdot \|_Y$ given by $\|x\|_Y = \|Ax\| + \|x\|$, for $x \in D(A)$.

We next state briefly the theory of resolvent operators for linear neutral integro-differential equations which was introduced in [7, 14].

**Definition 2.1.** (see [7]) A one parameter family of bounded linear operators $(R(t))_{t \geq 0}$ on $X$ is called a resolvent operator for

$$
\begin{aligned}
\frac{d}{dt} \left[ x(t) + \int_0^t N(t-s)x(s)ds \right] &= Ax(t) + \int_0^t \gamma(t-s)x(s)ds, \quad t > 0, \\
x(0) &= x_0 \in X,
\end{aligned}
$$

if the following conditions are verified.

(i) The function $R(\cdot) : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous, exponentially bounded and $R(0) = I$.

(ii) For $x \in Y$, $R(t)x \in C^1([0, T], X) \cap C([0, T], Y)$ and for $t \geq 0$ such that

$$
\begin{aligned}
\frac{d}{dt} \left[ R(t)x + \int_0^t N(t-s)R(s)xds \right] &= AR(t)x + \int_0^t \gamma(t-s)R(s)xds, \\
\frac{d}{dt} \left[ R(t)x + \int_0^t R(t-s)N(s)xds \right] &= R(t)Ax + \int_0^t R(t-s)\gamma(s)xds.
\end{aligned}
$$

In the sequel we always impose the following hypotheses on the operators appearing in the system (1) or (3).

(V$_1$) The operator $A : D(A) \subseteq X \to X$ generates an analytic semigroup $\{S(t)\}_{t \geq 0}$ on $X$ (for the theory of operator semigroups one can refer to the books [8] and [31]), and there are constants $M_0 > 0$ and $\theta \in (\pi/2, \pi)$ such that $\rho(A) \supseteq \Lambda_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg(\lambda)| < \theta \}$ and $\|R(\omega, A)\| \leq M_0|\lambda|^{-1}$ for each $\lambda \in \Lambda_\theta$.

(V$_2$) The function $N : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous and $\tilde{N}(\lambda)x$ is absolutely convergent for any $x \in X$ and $Re(\lambda) > 0$. There exist a constant $\alpha > 0$ and an analytic extension of $\tilde{N}(\lambda)$ (still denoted by $\tilde{N}(\lambda)$) to $\Lambda_\theta$ such that $\|\tilde{N}(\lambda)\| \leq N_0|\lambda|^{-\alpha}$ for each $\lambda \in \Lambda_\theta$ and $\|\tilde{N}(\lambda)\| \leq N_1|\lambda|^{-1}\|x\|_1$ for each $\lambda \in \Lambda_\theta$ and $x \in D(A)$. Here $N_0$ and $N_1$ are constants, the notation $\hat{f}$ represents the Laplace transform of function $f(t)$.

(V$_3$) For each $t \geq 0$, $\gamma(t) : D(\gamma(t)) \subseteq X \to X$ is linear and closed and $D(A) \subseteq D(\gamma(t))$ and the function $\gamma(\cdot)x$ is strongly measurable on $(0, \infty)$ for any $x \in D(A)$. There exists $b(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^+) \subset C^0(\mathbb{R}^+, \mathbb{R}^+)$ such that $\tilde{b}(\lambda)$ can be obtained for $Re\lambda > 0$ and $\|\gamma(t)x\| \leq b(t)\|x\|_1$ for each $t > 0$ and $x \in D(A)$. In addition, the function $\hat{\gamma} : \Lambda_{\pi/2} \to \mathcal{L}(D(A), X)$ has an analytical extension (still denoted by $\hat{\gamma}$) to $\Lambda_\theta$ such that $\|\hat{\gamma}(\lambda)x\| \leq \|\tilde{\gamma}(\lambda)\|\|x\|_1$ for each $x \in D(A)$, and $\|\hat{\gamma}(\lambda)\| \to 0$ as $|\lambda| \to \infty$. 
There is a subspace $D \subseteq D(A)$ which is dense in $D(A)$ and there exist constants $C_i > 0$, $i = 1, 2$, such that $A(D) \subseteq D(A)$, $\hat{\gamma}(\lambda)(D) \subseteq D(A)$, $\tilde{\gamma}(\lambda)(D) \subseteq D(A)$, $\|A\tilde{\gamma}(\lambda)x\| \leq C_1\|x\|$ and $\|\tilde{\gamma}(\lambda)x\|_1 \leq C_2|\lambda|^{-\alpha}\|x\|_1$ for each $x \in D$ and all $\lambda \in \Lambda_{\theta}$.

Then, it follows from [7, 14] that, under these conditions, there is a family of resolvent operator $(R(t))_{t \geq 0}$ for linear neutral integro-differential system (3) (the linear part of System (1)) defined by

$$R(0) = I$$

and

$$R(t)x = \frac{1}{2\pi i} \int_{\Gamma_{r,\vartheta}} e^{\lambda t}G(\lambda)x d\lambda, \quad t > 0,$$

where $G(\lambda) = (\lambda I + \lambda \tilde{\gamma}(\lambda) - A - \hat{\gamma}(\lambda))^{-1} \in \mathcal{L}(X)$ satisfying $\|G(\lambda)\| \leq N|\lambda|^{-1}$ for $\lambda \in \Lambda_{r,\vartheta}$ and some $N > 0$. Here $r > 0$, $\vartheta \in (\frac{\pi}{2}, \theta)$ are fixed numbers, and the contour $\Gamma_{r,\vartheta}$ can selected to be included in the region $\Lambda_{r,\vartheta} = \{\lambda \in \mathbb{C} \setminus \{0\}: |x| > r, \arg(\lambda) < \vartheta\}$, consisting of $\Gamma^1_{r,\vartheta}, \Gamma^2_{r,\vartheta}$ and $\Gamma^3_{r,\vartheta}$, where

$$\Gamma^1_{r,\vartheta} = \{te^{i\vartheta}: t \geq r\}, \quad \Gamma^2_{r,\vartheta} = \{re^{i\xi}: -\vartheta \leq \xi \leq \vartheta\}, \quad \Gamma^3_{r,\vartheta} = \{te^{-i\vartheta}: t \geq r\},$$

oriented so that $\text{Im}(\lambda)$ is increasing along $\Gamma^1_{r,\vartheta}$ and $\Gamma^2_{r,\vartheta}$.

The following theorem summarizes several important properties of the resolvent operator $R(t)$.

**Theorem 2.2.** For the resolvent operator $R(t)$ defined in (3), one has

1. The resolvent operator $R(t)$ is analytic and there exists $M \geq 1$ such that

$$\|R(t)\| \leq M, \quad 0 \leq t \leq T.$$

2. If $R(\lambda_0; A)$ is a compact operator for some $\lambda_0 \in \rho(A)$, then $R(t)$ is compact for all $t > 0$.

3. The resolvent operators $R(t)$ is continuous in $t > 0$ in the uniform operator topology of $\mathcal{L}(X)$.

4. If $R(t)$ is a compact operator for each $t > 0$, then $\|R(t)R(\nu) - R(\nu)\|_{\mathcal{L}(X)} \to 0$ as $t \to 0^+$ for each $\nu > 0$.

**Proof.** Assertions (i) and (ii) were established in [7], while Assertions (iii) and (iv) come from Lemma 2.6 and Lemma 4.1 in [17], respectively. □

Now we turn to introduce the axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato [12], adopting the terminologies used in Hino et al. [16]. That is, $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into $X$ endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$, which satisfies the following axioms:

1. (A) If $x: (-\infty, \sigma + a] \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a]$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a]$ the followings hold:

2. (i) $x_t$ is in $\mathcal{B}$;

3. (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$;

4. (iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma)\sup\{\|x(s)\|: \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$.

Here $H \geq 0$ is a constant, $K, M: [0, +\infty) \to [0, +\infty)$, $K(\cdot)$ is continuous and $M(\cdot)$ is locally bounded, and $H, K(\cdot), M(\cdot)$ are independent of $x(t)$.

5. (A1) For the function $x(\cdot)$ in (A), $x_t$ is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a]$.

6. (B) The space $\mathcal{B}$ is complete.
For example, let the phase space $\mathcal{B} = C_r \times L^p(\mathcal{G} : X)$, $r \geq 0$, $1 \leq p < \infty$, which consists of all classes of functions $\varphi \in (-\infty, 0] \to X$ such that $\varphi$ is continuous on $[-r, 0]$ and $g(\varphi(\cdot)) \in L^p$ is Lebesgue integrable on $(-\infty, -r)$, where $g : (-\infty, -r) \to \mathbb{R}$ is a positive Lebesgue integrable function. The seminorm in $\mathcal{B}$ is defined by

$$
\|\varphi\|_{\mathcal{B}} = \sup \{\|\varphi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p d\theta\right)^{\frac{1}{p}}.
$$

Then it satisfies all the axioms $(A)$, $(A_1)$ and $(B)$, see [16].

3. Existence and uniqueness of mild solutions. In this section, we prove the existence and uniqueness of mild solutions for System (1) and discuss the compactness of the solution operator. In what follows, the control function $u \in L^p(0, T; U)$ with $p > 1$, and $\varphi \in \mathcal{B}$ will be a given initial function. We introduce the functions $f_i : [0, T] \to X$, $i = 1, 2$, by

$$
f_1(t) = -\int_{-\infty}^{0} N(t - s)\varphi(s)ds \quad \text{and} \quad f_2(t) = \int_{-\infty}^{0} \gamma(t - s)\varphi(s)ds,
$$

and we always assume that $f_1$ is differentiable on $[0, T]$ and $f_1^{'}, f_2 \in L^1(0, T; X)$. Then the mild solutions of System (1) are defined as

**Definition 3.1.** A function $x(\cdot) : (-\infty, T] \to X$ is said to be a mild solution of System (1) on $[0, T]$, if $x_0 = \varphi$, and $x(\cdot)$ satisfies the integral equation

$$
x(t) = R(t)\varphi(0) + \int_{0}^{t} R(t - s) [F(s, x_s) + f_1'(s) + f_2(s) + B(s)u(s)] ds, \quad t \in [0, T].
$$

To carry on our discussion, we make the following assumptions on Eq. (1).

$(H_1)$ The operator $B(\cdot) \in L^\infty(0, T; \mathcal{L}(U, X))$ and $\|B(\cdot)\|_{\infty}$ stands for its usual norm on $L^\infty(0, T; \mathcal{L}(U, X))$.

$(H_2)$ The nonlinear function $F(\cdot, \cdot) : [0, T] \times \mathcal{B} \to X$ satisfies that

$(i)$ It is measurable in $t$ for each $\phi \in \mathcal{B}$.

$(ii)$ $F(\cdot, \cdot)$ is Lipschitz continuous with respect to the second variable, i. e., there exist a positive constant $L > 0$ such that

$$
\|F(t, \phi) - F(t, \psi)\| \leq L\|\phi - \psi\|_{\mathcal{B}}, \quad \text{for} \quad t \in [0, T], \quad \phi, \psi \in \mathcal{B},
$$

and there holds

$$
\|F(t, \phi)\| \leq L(\|\phi\|_{\mathcal{B}} + 1), \quad \text{for} \quad (t, \phi) \in [0, T] \times \mathcal{B}.
$$

Now we establish the result on existence and uniqueness of mild solutions of the control solution (1) as follows.

**Theorem 3.2.** Let $\varphi \in \mathcal{B}$ and $u \in L^p(0, T; U)$ be given. If the above conditions $(H_1)$ and $(H_2)$ are satisfied, then System (1) admits a unique mild solution $x(\cdot) : (-\infty, T] \to X$ which is continuous on $[0, T]$.

**Proof.** We start by proving the local existence of solutions of Eq. (1) by employing the Banach contraction principle.

First, for the functions $K(\cdot)$ and $M(\cdot)$ in Axiom $(A)(iii)$ we put, for any $b \in (0, T]$,

$$
K_b := \max_{t \in [0, b]} K(t) \quad \text{and} \quad M_b := \sup_{t \in [0, b]} M(t).
$$

Take some $T_1 \in (0, T]$ such that

$$
k_0 := MT_1LK_{T_1} < 1. \quad (4)
$$
Denote
\[
\rho_1 := \frac{1}{1-k_0} \left[ (MH + MT_1LMT_1) \|\varphi\|_\mathcal{B} + MT_1L + M \|f'_1 + f_2\|_{L^1(0,T;X)} \\
+ M\|B\|_\infty T_1^{\frac{p-1}{p}} \|u\|_{L^p(0,T;U)} \right],
\]
and defined the set \( E(T_1, \rho_1) \) by
\[
E(T_1, \rho_1) := \left\{ x(\cdot) \in C([0,T_1]; X) \mid x(0) = \varphi(0) \text{ and } \|x\|_{C([0,T_1];X)} \leq \rho_1 \right\}.
\]
It is clear that \( E(T_1, \rho_1) \) is a closed, bounded and convex subset of \( C([0,T_1]; X) \).

We then define an operator \( Q_1 \) on the set \( E(T_1, \rho_1) \) as
\[
(Q_1x)(t) = R(t)\varphi(0) + \int_0^t R(t-s)F(s,x_s) + f'_1(s) + f_2(s) ds \\
+ \int_0^t R(t-s)B(s)u(s) ds, \quad t \in [0,T_1].
\]

Obviously, \((Q_1x)(\cdot) \in C([0,T_1]; X)\) for any \( x(\cdot) \in E(T_1, \rho_1) \). From Theorem 2.2 (i), (H1), (H2), Axiom (A) and Hölder inequality we obtain that, for \( t \in [0,T_1] \),
\[
\|(Q_1x)(t)\| \leq \|R(t)\| \|\varphi(0)\| + \int_0^t \|R(t-s)\| \|F(s,x_s)\| ds \\
+ \int_0^t \|R(t-s)\| \|f'_1(s) + f_2(s)\| ds + \int_0^t \|R(t-s)\| \|B(s)u(s)\| ds \\
\leq M\|\varphi(0)\| + \int_0^t ML(KT_1\|x\|_{C([0,T_1];X)} + MT_1\|\varphi\|_\mathcal{B} + 1) ds \\
+ M\|f'_1 + f_2\|_{L^1(0,T;X)} + M\|B\|_\infty T_1^{\frac{p-1}{p}} \left( \int_0^t \|u(s)\|_{L^p(U)}^2 ds \right)^{\frac{1}{p}} \\
\leq (MH + MT_1LMT_1)\|\varphi\|_\mathcal{B} + MT_1L + M\|f'_1 + f_2\|_{L^1(0,T;X)} \\
+ M\|B\|_\infty T_1^{\frac{p-1}{p}} \|u\|_{L^p(0,T;U)} + MT_1LK_{T_1}\|x\|_{C([0,T_1];X)} \\
\leq \rho_1, \text{ (by (5))},
\]
that is,
\[
\|(Q_1x)\|_{C([0,T_1];X)} \leq \rho_1,
\]
which implies that \( Q_1 \) maps \( E(T_1, \rho_1) \) into itself since \( (Q_1x)(0) = \varphi(0) \). Moreover, from (6) we readily have that, for any \( x^1, x^2 \in E(T_1, \rho_1) \) and \( t \in [0,T_1] \),
\[
\|(Q_1x^1)(t) - (Q_1x^2)(t)\| \leq \int_0^t \|R(t-s)\| \left[ F(s,x^1_s) - F(s,x^2_s) \right] ds \\
\leq \int_0^t ML\|x^1_s - x^2_s\|_\mathcal{B} ds \\
\leq k_0\|x^1 - x^2\|_{C([0,T_1];X)},
\]
which leads to
\[
\|Q_1x^1 - Q_1x^2\|_{C([0,T_1];X)} \leq k_0\|x^1 - x^2\|_{C([0,T_1];X)}.
\]
That is, \( Q_1 \) is a contractive mapping on \( E(T_1, \rho_1) \) due to (4). Therefore, from the well-known Banach contraction principle, there exists a unique fixed point \( x(\cdot) \) for the operator \( Q_1 \) on \( E(T_1, \rho_1) \). We then extend \( x(\cdot) \) to \((-\infty, T_1]\) by putting \( x|_{(-\infty, 0]} = \varphi \) so that we get a solution \( x(\cdot) = x(\cdot; \varphi, u) \) of System (1) defined on \((-\infty, T_1]\) which is continuous on \([0, T_1]\) and satisfies \( x_0 = \varphi \).

Next we extend the mild solution to the interval \([T_1, 2T_1]\). To do this, let

\[
\rho_2 := \frac{1}{1 - k_0} \left[ \rho_1 + MT_1 L (MT_1 (K_{T_1} \rho_1 + M_{T_1} \| \varphi \|_{\mathcal{B}}) + 1) + M \| B \|_{\infty T_1} \| u \|_{L^p(0, T; U)} \right],
\]

where \( k_0 \) is from (4). We consider the set \( E(2T_1, \rho_2) \) given by

\[
E(2T_1, \rho_2) := \left\{ z(\cdot) \in C([T_1, 2T_1]; X) \mid z(T_1) = x(T_1; \varphi, u) \right. \text{ and } \| z \|_{C([T_1, 2T_1]; X)} \leq \rho_2 \},
\]
on which we define the operator \( Q_2 \) as

\[
(Q_2 z)(t) = R(t) \varphi(0) + \int_0^t R(t - s) F(s, x_s) ds
+ \int_0^t R(t - s) [f'_1(s) + f_2(s) + B(s)u(s)] ds
+ \int_0^t R(t - s) F(s, z_s) ds, \quad t \in [T_1, 2T_1].
\]

In the above expression we set \( z(s) = x(s) \) for \( t \in [0, T_1] \). Clearly, \((Q_2 z)(\cdot) \in C([T_1, 2T_1]; X) \) and similar to the previous arguments, we find by (8) that

\[
\| (Q_2 z)(t) \| \leq (MH + MT_1 LM_{T_1}) \| \varphi \|_{\mathcal{B}} + MT_1 L + M \| B \|_{\infty T_1} \| u \|_{L^p(0, T; U)}
+ MT_1 L + M \| B \|_{\infty T_1} \| u \|_{L^p(0, T; U)}
+ ML \sup_{s \in [T_1, 2T_1]} \| z(s) \| + MT_1 \| x_{T_1} \|_{\mathcal{B}} + 1 \right) ds
+ M \| B \|_{\infty T_1} \| u \|_{L^p(0, T; U)}
\leq \rho_1 + MT_1 L (MT_1 (K_{T_1} \rho_1 + M_{T_1} \| \varphi \|_{\mathcal{B}}) + 1)
+ MT_1 L + M \| B \|_{\infty T_1} \| u \|_{L^p(0, T; U)} + MT_1 L K_{T_1} \| z \|_{C([T_1, 2T_1]; X)}
\leq \rho_2, \quad \text{by (7)},
\]
which shows that \( Q_2 \) maps \( E(2T_1, \rho_2) \) into itself. And, on the other hand, by virtue of \((H_1)\) and \((H_2)\) it yields easily that, for any \( z^1, z^2 \in E(2T_1, \rho_2) \),

\[
\| (Q_2 z^1)(t) - (Q_2 z^2)(t) \| \leq \left\| \int_{T_1}^t R(t - s) \left[ F(s, z^1_s) - F(s, z^2_s) \right] ds \right\|
\leq \int_{T_1}^t ML \| z^1_s - z^2_s \|_{\mathcal{B}} ds
\leq k_0 \| z^1 - z^2 \|_{C([T_1, 2T_1]; X)},
\]
or
\[ \|Q_2 z^1 - Q_2 z^2\|_{C([T_1, 2T_1]; X)} \leq k_0 \|z^1 - z^2\|_{C([T_1, 2T_1]; X)}, \]
which from (4) implies that \(Q_2\) is also a contractive mapping on \(E(2T_1, \rho_2)\). Hence, by Banach contraction principle again, there exists a unique fixed point \(z\) for the operator \(Q_2\) in \(E(2T_1, \rho_2)\). Thus we obtain the mild solution \(z(\cdot; \varphi, u) : (-\infty, 2T_1) \rightarrow X\) of System (1) by setting \(z|_{-\infty, T_1} = x(\cdot; \varphi, u)\). Therefore, relabeling \(z(\cdot)\) as \(x(\cdot)\) we then get a solution \(x(\cdot; \varphi, u)\) of System (1) defined on \((-\infty, 2T_1)\) being continuous on \([0, 2T_1]\) and satisfying \(x_0 = \varphi\).

Repeating the above procedure on \([2T_1, 3T_1]\), \([3T_1, 4T_1]\), \ldots, in finite steps we can prove the existence of a solution for System (1) on \((-\infty, T]\) which is continuous on \([0, T]\) and verifies \(x_0 = \varphi\).

Finally, the uniqueness of the mild solutions can be prove readily by Gronwall inequality in a standard way. \(\square\)

In what follows, we denote the mild solution \(x(\cdot)\) of System (1) by \(x(\cdot; u)\) to show its dependence on the control function \(u\). Using this notation, from Theorem 3.2 we are able to define well the solution operator \(W : L^p(0, T; U) \rightarrow C([0, T]; X)\) as
\[ u(\cdot) \mapsto (W u)(\cdot) = x(\cdot; u). \] (9)

The next theorem reveals that if the resolvent operator \((R(t))_{t \geq 0}\) is compact, then the operator \(W\) is compact as well.

**Theorem 3.3.** Assume that \((H_1) - (H_2)\) hold and, additionally, the resolvent operator \((R(t))_{t \geq 0}\) is compact. Then the solution operator \(W\) defined by (9) is compact from \(L^p(0, T; U)\) to \(C([0, T]; X)\).

**Proof.** To prove this claim, we need to show that for any bounded sequence \(\{u_n\}_{n \geq 1} \subset L^p(0, T; U)\), there exists a subsequence \(\{u_{n_k}\}_{k \geq 1} \subset L^p(0, T; U)\) such that \(\{W u_{n_k}\}\) converges in \(C([0, T]; X)\). We do this by utilizing the well-known Ascoli-Arzela theorem.

First, by virtue of Theorem 3.2, it is easily seen that \(\{x(\cdot; u_n), n \geq 1\}\) is uniformly bounded in \(C([0, T]; X)\) since the sequence \(\{u_n\}_{n \geq 1}\) bounded in \(L^p(0, T; U)\). That is,
\[ \|x(\cdot; u_n)\|_{C([0, T]; X)} \leq \bar{p}, \] (10)
for some \(\bar{p} > 0\).

We then prove the equicontinuity of \(\{x(\cdot; u_n), n \geq 1\}\) in \(C([0, T]; X)\). To do so, let \(0 < t_1 < t_2 \leq T\) and \(\epsilon > 0\) be small enough such that \(0 < \epsilon < t_1\). Then
\[
\|x(t_2; u_n) - x(t_1; u_n)\| \\
\leq \|R(t_2) - R(t_1)\| \|\varphi(0)\| + \int_0^{t_1-\epsilon} \|R(t_2 - s) - R(t_1 - s)\| \|F(s, x_s(\cdot; u_n))\| ds \\
+ \int_0^{t_1-\epsilon} \|R(t_2 - s) - R(t_1 - s)\| \|f_1'(s) + f_2(s)\| ds \\
+ \|B\|_\infty \left( \int_0^{t_1-\epsilon} \|R(t_2 - s) - R(t_1 - s)\|^{\frac{p}{p-\alpha}} ds \right)^{\frac{p-\alpha}{p}} \|u_n\|_{L^p(0,T;U)} \\
+ \int_{t_1-\epsilon}^{t_1} \|R(t_2 - s) - R(t_1 - s)\| \|F(s, x_s(\cdot; u_n))\| ds \\
+ \int_{t_1-\epsilon}^{t_1} \|R(t_2 - s) - R(t_1 - s)\| \|f_1'(s) + f_2(s)\| ds
\]
\[ + \|B\|_{\infty} \left( \int_{t_1 - \epsilon}^{t_1} \|R(t_2 - s) - R(t_1 - s)\|^{\frac{p - 1}{p}} \, ds \right) \|u_n\|_{L^P(0, T; U)} \]

\[ + \int_{t_1}^{t_2} \|R(t_2 - s)\| \|F(s, x_s; u_n)\| \, ds \]

\[ + \int_{t_1}^{t_2} \|R(t_2 - s)\| f_1(s) + f_2(s) \, ds \]

\[ + \|B\|_{\infty} \left( \int_{t_1}^{t_2} \|R(t_2 - s)\|^p \, ds \right) \|u_n\|_{L^P(0, T; U)} \]

\[ := \|R(t_2) - R(t_1)\| \|\varphi(0)\| + \sum_{i=1}^{g} I_i. \]

In view of Theorem 2.2 (i), (H1) - (H2), (10) and Hölder inequality again, we give the following estimations:

\[ I_1 + I_4 + I_7 \leq L \left( KtT \|\varphi\|_{\infty} + 1 \right) \]

\[ \cdot \left( \int_{0}^{t_1 - \epsilon} \|R(t_2 - s) - R(t_1 - s)\| \, ds + 2M\epsilon + M(t_2 - t_1) \right), \]

\[ I_2 + I_5 + I_8 \leq \int_{0}^{t_1 - \epsilon} \|R(t_2 - s) - R(t_1 - s)\| \|f_1'(s) + f_2(s)\| \, ds \]

\[ + 2M \int_{t_1 - \epsilon}^{t_2} \|f_1'(s) + f_2(s)\| \, ds, \]

\[ I_3 + I_6 + I_9 \leq \|B\|_{\infty} \|u_n\|_{L^P(0, T; U)} \left[ \left( \int_{0}^{t_1 - \epsilon} \|R(t_2 - s) - R(t_1 - s)\|^p \, ds \right)^{\frac{p - 1}{p}} \right]

\[ + 2M(\epsilon)^{\frac{p - 1}{p}} + M(t_2 - t_1)^{\frac{p - 1}{p}} \right]. \]

So

\[ \|x(t_2; u_n) - x(t_1; u_n)\| \to 0, \text{ as } t_2 \to t_1 \text{ and } \epsilon \to 0^+, \]

independently of \( u_n \), since, due to Theorem 2.2 (iii), \( R(t) \) is uniformly continuous for \( t \in (0, T) \). Therefore, the set \( \{x(\cdot; u_n), n \geq 1\} \) is equi-continuous on \( (0, T) \). In addition, the set \( \{x(\cdot; u_n), n \geq 1\} \) is trivially equi-continuous at \( t = 0 \).

To check the relative compactness of \( \{x(t; u_n), n \geq 1\} \) in the space \( X \) for each \( t \in [0, T] \), it suffices to show it on \( (0, T) \) since \( \{x(t; u_n), n \geq 1\} = \{\varphi(0)\} \). Let \( t \in (0, T) \) be fixed. Then, for any \( u_n \in L^p(0, T; U) \), we get that

\[ x(t; u_n) = R(t)\varphi(0) + \int_{0}^{t} R(t - s) [F(s, x_s; u_n)] + f_1'(s) + f_2(s) + B(s)u_n(s) \, ds. \]

Let \( \epsilon > 0 \) be sufficiently small such that \( t - \epsilon > 0 \) and define

\[ x^\epsilon(t; u_n) = R(t)\varphi(0) + R(\epsilon) \int_{0}^{t - \epsilon} R(t - s) [F(s, x_s; u_n)] + f_1'(s) + f_2(s) + B(s)u_n(s) \, ds. \]

Observe that

\[ \left\| \int_{0}^{t - \epsilon} R(t - s) [F(s, x_s; u_n)] + f_1'(s) + f_2(s) + B(s)u_n(s) \, ds \right\| \]
from the fact that for each $t \in (0, T)$, the set

$$R(\epsilon) \int_0^{t-\epsilon} R(t-s) [F(s, x_s(\cdot; u_n)) + f_1(s) + f_2(s) + B(s)u_n(s)] ds,$$ 

is relatively compact in $X$ and hence the set $\{x^\epsilon(t; u_n), \ n \geq 1\}$ does so. By Theorem 2.2 (iv) we deduce easily that

$$\|x(t; u_n) - x^\epsilon(t; u_n)\|$$ 

$$\leq \left\| \int_0^{t-\epsilon} (R(t-s) - R(\epsilon)R(t-s))[F(s, x_s(\cdot; u_n)) + f_1(s) + f_2(s) + B(s)u_n(s)] ds \right\|$$ 

$$+ \left\| \int_{t-\epsilon}^t R(t-s) [F(s, x_s(\cdot; u_n)) + f_1(s) + f_2(s) + B(s)u_n(s)] ds \right\|$$ 

$$\leq \int_0^{t-\epsilon} \|R(t-s) - R(\epsilon)R(t-s)\| [L (K_T\varphi + M_T\|\varphi\|_{J^p} + 1) + \|f_1(s) + f_2(s)\|] ds$$ 

$$+ \|B\|_\infty \left( \int_0^{t-\epsilon} \|R(t-s) - R(\epsilon)R(t-s)\| \frac{ds}{t-\epsilon} \right)^{\frac{p-1}{p}} \|u_n\|_{L^p(0,T;U)}$$ 

$$+ M \left[ L (K_T\varphi + M_T|\varphi|_{J^p} + 1) + \int_{t-\epsilon}^t \|f_1(s) + f_2(s)\| ds \right]$$ 

$$\rightarrow 0 \quad (\epsilon \rightarrow 0),$$

it follows immediately that the set $\{x(t; u_n), \ n \geq 1\}$ is relatively compact in $X$ as well for every $t \in (0, T)$.

Accordingly, by Ascoli-Arzela theorem, we conclude that the solution operator $W$ is compact from $L^p(0, T; U)$ to $C([0, T]; X)$. This completes the proof. \hfill \square

Now, we define the so-called Nemitsky operator $\mathcal{F} : L^p(0, T; U) \rightarrow L^p(0, T; X)$ corresponding to the nonlinear function $F(\cdot, \cdot)$ by the formula

$$(\mathcal{F}u)(\cdot) = F(\cdot, (W u)(\cdot)).$$

Then using the similar arguments as the proof of Lemma 3.2 in [28], we can establish the following lemma.

**Lemma 3.4.** Under the assumptions of Theorem 3.3, the operator $\mathcal{F}$ is compact from $L^p(0, T; U)$ to $L^p(0, T; X)$.

4. **Optimal control.** The purpose of this section is to discuss the optimal control problem for System (1). Let the admissible set $U_{ad}$ be a closed convex subset of $L^p(0, T; U)$. As above, we denote the mild solutions of Eq. (1) by $x(\cdot; u)$ to express their dependence on $u \in U_{ad}$. Now, define the integral cost function as

$$\mathcal{J} = \mathcal{J}(u, x(\cdot; u)) := \phi_0(x(T; u)) + \int_0^T [\phi_1(t, x(t; u), x_\xi(t; u)) + \phi_2(t, u(t))] dt, \quad (11)$$
where the kernel functions $\phi_i$, $i = 0, 1, 2$, satisfy that

(H$_3$) (i) $\phi_0 : X \to \mathbb{R}$ is continuous.
(ii) $\phi_1 : [0, T] \times X \times \mathcal{B} \to \mathbb{R}$ is measurable in $t \in [0, T]$ for each $(x, \psi) \in X \times \mathcal{B}$ and continuous in $(x, \psi) \in X \times \mathcal{B}$ for a.e. $t \in [0, T]$.
(iii) $\phi_2 : [0, T] \times \mathcal{U} \to \mathbb{R}$ is integrable on $[0, T]$ for each $u \in U_{ad}$ and the functional $\Gamma : U_{ad} \to \mathbb{R}$ given by

$$
(\Gamma u) = \int_0^T \phi_2(t, u(t))dt
$$

is continuous and convex.

We consider here the problem of optimal control which is described as follows. 

(P$_1$) Find a control $u \in U_{ad}$ which minimizes the cost function $J$ subject to the constraint (1).

Such a control $u \in U_{ad}$ is called an optimal control and the pair $(u, x(\cdot; u))$ is called an optimal solution for $J$.

Subsequently, we study the problem (P$_1$) in two cases, one is the case that the admissible set $U_{ad}$ is bounded and the other case is that $U_{ad}$ is unbounded in $L^p(0, T; U)$. We first have the following existence result of (P$_1$) for the case of bounded admissible set $U_{ad}$.

**Theorem 4.1.** Suppose that (H$_1$) − (H$_3$) are all satisfied and $U_{ad}$ is bounded in $L^p(0, T; U)$. If the resolvent operator $(R(t))_{t \geq 0}$ is compact, then the problem (P$_1$) admits at least one solution, i.e., there exists at least one control $u \in U_{ad}$ which minimizes the cost function $J$ subject to (1).

**Proof.** We assume that $\inf \{J(u, x(\cdot; u)) \mid u \in U_{ad}\} = m < +\infty$. If not, there is nothing to prove.

From the definition of infimum there exists a minimizing sequence of feasible pair $\{(u_n, x(\cdot; u_n))\}_{n \geq 1} \subseteq A_{ad} := \{(u, x(\cdot; u)) \mid u \in U_{ad}\}$, such that $J(u_n, x(\cdot; u_n)) \to m$ as $n \to +\infty$. Since $\{u_n\}_{n \geq 1} \subseteq U_{ad}$ is bounded in $L^p(0, T; U)$, there exists a subsequence, relabeled still as $\{u_n\}_{n \geq 1}$, and $u_0 \in L^p(0, T; U)$ such that

$$
 u_n \to u_0 \text{ weakly in } L^p(0, T; U). \tag{12}
$$

Noting that $U_{ad}$ is closed and convex, by Marzur lemma we have $u_0 \in U_{ad}$ as well.

Rewrite $x(\cdot; u_n)$ as $x_n(\cdot) := (n \geq 1)$ (the mild solutions of System (1) corresponding to $u_n$), then $x_n(\cdot)$ satisfy the following integral equation

$$
x_n(t) = R(t)\varphi(0) + \int_0^t R(t-s)\left( [\mathcal{F} u_n](s) + f_1'(s) + f_2(s) + B(s)u_n(s) \right) ds, \quad t \in [0, T].
$$

Denote by $x^0$ the mild solution of Eq. (1) corresponding to the control $u_0$, i.e.,

$$
x^0(t) = R(t)\varphi(0) + \int_0^t R(t-s)\left( [\mathcal{F} u_0](s) + f_1'(s) + f_2(s) + B(s)u_0(s) \right) ds, \quad t \in [0, T].
$$

Then from (12) and the compactness of the operator $\mathcal{F}$ (Theorem 3.3) it follows that

$$
x_n(t) \to x^0(t) \text{ strongly in } X, \quad n \to +\infty, \tag{13}
$$

by taken a subsequence of $\{x_n(\cdot)\}_{n \geq 1}$ if necessary.

Next, we verify that $(u_0, x^0)$ is an optimal solution for $J$. From (13) and (H$_3$)(i) we infer that, at $t = T$,

$$
\lim_{n \to +\infty} \phi_0(x_n(T)) = \phi_0 (x^0(T)). \tag{14}
$$
Meanwhile, combining (A)(iii) and (H₃)(ii) yields readily that
\[ \lim_{n \to +\infty} \phi_1(t, x_n(t), (x_n)_l) = \phi_1(t, x^0(t), (x^0)_l), \quad t \in [0, T], \]
from which and Fatou’s lemma we further derive that
\[ \liminf_{n \to +\infty} \int_0^T \phi_1(t, x_n(t), (x_n)_l)dt \geq \int_0^T \liminf_{n \to +\infty} \phi_1(t, x_n(t), (x_n)_l)dt \]
\[ = \int_0^T \phi_1(t, x^0(t), (x^0)_l)dt. \]  
(15)

In addition, since the assumption (H₃)(iii) implies that the functional \( \Gamma \) is weak lower semi-continuous, namely,
\[ \liminf_{n \to +\infty} \Gamma u_n \geq \Gamma u_0 = \int_0^T \phi_2(t, u_0(t))dt. \]  
(16)

It hence follows from (14), (15) and (16) that
\[ m = \inf_{u \in U_{ad}} J(u, x(\cdot; u)) \geq \liminf_{n \to +\infty} \phi_0(x_n(T)) + \liminf_{n \to +\infty} \int_0^T \phi_1(t, x_n(t), (x_n)_l)dt \]
\[ + \liminf_{n \to +\infty} (\Gamma u_n) \]
\[ \geq \phi_0(x^0(T)) + \int_0^T \phi_1(t, x^0(t), (x^0)_l)dt + (\Gamma u_0) \]
\[ = J(u_0, x^0) > -\infty, \]
which means that \( J \) attains its infimum at \( (u_0, x^0) \in A_{ad}. \) The proof is proved. \( \square \)

**Remark 1.** We would like to emphasize that, since we have shown the compactness of the operator \( \mathscr{W} \), we do not require the functions \( \phi_0 \) and \( \phi_1 \) in the cost function \( J \) satisfy the convex condition which was used commonly in the literature such as [2, 13, 20, 22, 30, 38, 40].

Now we consider the case that \( U_{ad} \) is unbounded in \( L^p(0, T; U) \). To this end, besides the previous assumptions we also suppose that the followings hold true.

(H₄) (i) There exists a constant \( c_0 > 0 \) such that \( \phi_0(\cdot) \geq -c_0 \) on \( X \).

(ii) There exists a constant \( c_1 > 0 \) such that \( \phi_1(\cdot, \cdot, \cdot) \geq -c_1 \) on \( [0, T] \times X \times \mathscr{B} \).

(iii) There exists a monotonely increasing function \( \theta_0 \in C(\mathbb{R}^+; \mathbb{R}) \) such that
\[ \lim_{r \to \infty} \theta_0(r) = +\infty \] and
\[ (\Gamma u) = \int_0^T \phi_2(t, u(t))dt \geq \theta_0(||u||_{L^p(0, T; U)}) \], for \( u \in U_{ad}. \)

Under these conditions, we can prove the following result for the case that the admissible set \( U_{ad} \) is unbounded.

**Theorem 4.2.** Assume that the conditions (H₁) – (H₄) are all verified and \( U_{ad} \) is unbounded in \( L^p(0, T; U) \). If the resolvent operator \( (R(t))_{t \geq 0} \) is compact, then there exists at least one optimal control \( u \in U_{ad} \) that minimizes the cost function \( J \) subject to (1).

**Proof.** Let \( \{u_n\}_{n \geq 1} \subset U_{ad} \) be a minimizing sequence of \( J \) such that
\[ \inf_{u \in U_{ad}} J(u, x(\cdot; u)) = \lim_{n \to \infty} J(u_n, x(\cdot; u_n)) = m < +\infty. \]
Note that, by virtue of \((H_4)\),
\[
J(u, x(\cdot; u)) \geq \theta_0 \left( \|u\|_{L^p(0, T; U)} \right) - c_0 - c_1 T, \quad \text{for } u \in U_{ad},
\]
and \(\lim_{r \to \infty} \theta_0(r) = +\infty\), we see that the minimizing sequence \(\{u_n\}_{n \geq 1}\) is bounded in \(L^p(0, T; U)\). Hence, conducted the similar arguments as in the proof of Theorem 4.1, the assertion follows. \(\Box\)

5. **Time optimal control.** In this section, we study the time optimal control problem for System (1). Let the admissible set \(U_{ad}\) and the set \(W\) be bounded, closed and convex respectively in \(L^p(0, T; U)\) and \(X\). We set
\[
U_0 := \left\{ u \in U_{ad} \mid x(t; u) \in W \text{ for some } t \in [0, T] \right\},
\]
and suppose that \(U_0 \neq \emptyset\). For each \(u \in U_0\), we define the transition time to be the first time \(\tilde{t}(u)\) such that \(x(\tilde{t}; u) \in W\) and the set \(W\) is called a target set. Then the time optimal control problem considered in this part is described as
\[
(P_2) \quad \text{Find a control } \pi \in U_0 \text{ such that }
\]
\[
\tilde{t}(\pi) = \inf \left\{ \tilde{t}(u) : u \in U_0 \right\},
\]
subject to the constraint (1).

In \((P_2)\), such a \(\pi \in U_0\) is called a time optimal control and \(\tilde{t}(\pi)\) is called an optimal time.

Now we solve the problem \((P_2)\), namely, we prove the existence of a control which transfers the mild solutions of the constraint (1) from the initial data to a target set in the shortest time.

**Theorem 5.1.** Let \(\varphi \in \mathcal{B}\). Suppose that \((H_1)\) and \((H_2)\) hold and \(U_0 \neq \emptyset\). If the resolvent operator \((R(t))_{t \geq 0}\) is compact, then there exists a time optimal control \(\pi \in U_0\) for the problem \((P_2)\).

**Proof.** Let \(t_0 = \inf \left\{ \tilde{t}(u) : u \in U_0 \right\}\) and \(\{u_n, x_n\}_{n \geq 1} \subset U_0 \times C([0, T]; X)\) be a minimizing sequence, where \(x_n(\cdot) = x(\cdot; u_n) \quad (n = 1, 2, \cdots)\) are the mild solutions of Eq. (1) corresponding to the controls \(u_n\). Assume that \(t_n := \tilde{t}(u_n) \downarrow t_0\ (n \to +\infty)\). Then, \(x_n(t_n)\) satisfies the integral equation
\[
x_n(t_n) = R(t_n)\varphi(0) + \int_0^{t_n} R(t_n - s)[(\mathcal{F}u_n)(s) + f_1(s) + f_2(s) + B(s)u_n(s)]ds
\]
\[
= R(t_n)\varphi(0) + \int_0^{t_0} R(t_n - s)[(\mathcal{F}u_n)(s) + f_1(s) + f_2(s) + B(s)u_n(s)]ds
\]
\[
+ \int_{t_0}^{t_n} R(t_n - s)[(\mathcal{F}u_n)(s) + f_1(s) + f_2(s) + B(s)u_n(s)]ds
\]
\[
:= I_1 + I_2 + I_3.
\]
Clearly, since \(W\) and \(U_{ad}\) are bounded, closed and convex subsets in the Hilbert spaces \(X\) and \(L^p(0, T; U)\) respectively, \(\{x_n(t_n)\}_{n \geq 1}\) is a bounded sequence in \(W\), and moreover, there exist \(\hat{x} \in W\), \(\hat{u} \in U_{ad}\) and subsequences of \(\{x_n(t_n)\}_{n \geq 1}\) and \(\{u_n\}_{n \geq 1}\), still denoted by themselves, such that
\[
x_n(t_n) \to \hat{x} \quad \text{weakly in } X, \quad (18)
\]
\[
u_n \to \hat{u} \quad \text{weakly in } L^p(0, T; U). \quad (19)
\]
We subsequently show that \( \tilde{u} \) is the time optimal control for \((P_2)\) with the optimal time \( t_0 \). In fact, it is clear that \( I_1 = R(t_n)\varphi(0) \to R(t_0)\varphi(0) \) as \( n \to +\infty \), and meanwhile, from (19) and Lemma 3.4 we have

\[
I_2 \to \int_0^{t_0} R(t_0 - s)[(\mathcal{F}\tilde{u})(s) + f_1'(s) + f_2(s) + B(s)\tilde{u}(s)]ds \quad \text{as} \quad n \to +\infty.
\]

On the other hand, we find that

\[
\|I_3\| \leq M \left[ L(K_T\overline{p} + M_T\|\varphi\|_B + 1) (t_n - t_0) + \int_{t_0}^{t_n} \|f_1'(s) + f_2(s)\|ds \\
+ \|B\|_\infty \|u_n\|_{L^p(0,T;U)} (t_n - t_0)^\frac{p-1}{p} \right] \\
\to 0 \quad \text{as} \quad n \to +\infty.
\]

Hence, taking the limit for \( n \to +\infty \) on both sides of (17), it then yields

\[
x_n(t_n) \to R(t_0)\varphi(0) + \int_0^{t_0} R(t_0 - s)[(\mathcal{F}\tilde{u})(s) + f_1'(s) + f_2(s) + B(s)\tilde{u}(s)]ds
\]

in \( X \). Combining this with (18) we derive straightly that

\[\hat{x} = R(t_0)\varphi(0) + \int_0^{t_0} R(t_0 - s)[(\mathcal{F}\tilde{u})(s) + f_1'(s) + f_2(s) + B(s)\tilde{u}(s)]ds \in W,\]

which manifests that \( \tilde{u} \in U_0 \). It is clear by definition of optimal time that \( t_0 = \hat{t}(\tilde{u}) \leq \hat{t}(u) \) for all \( u \in U_0 \). Consequently, \( \tilde{u} \) is the time optimal control for \((P_2)\), which is our desired result.

\( \square \)

Next, as in [30], we also consider the case in which the target set \( W \) is singleton. Put \( W = \{w_0\} \) such that \( \varphi(0) \neq w_0 \). Since \( X \) is reflexive, we can choose a decreasing sequence of non-empty, bounded, closed and convex sets \( \{W_n\}_{n \geq 1} \) in \( X \) such that

\[
w_0 = \bigcup_{n=1}^{+\infty} W_n \quad \text{and} \quad \text{dist}(w_0, W_n) = \sup_{x \in W_n} |x - w_0| \to 0 \quad \text{as} \quad n \to +\infty. \tag{20}
\]

Assume that, for each \( n \geq 1 \),

\[
U^n_0 := \{u \in U_ad \mid x(t; u) \in W_n \text{, for some } t \in [0,T]\} \neq \emptyset. \tag{21}
\]

Then the time optimal control problem \((P_2)\) with the target set \( \{w_0\} \) can be solved as follows.

**Theorem 5.2.** Let \( \{W_n\}_{n \geq 1} \) be a decreasing sequence of non-empty, bounded, closed and convex sets in \( X \) satisfying (20) and (21). Let \( \{u_n\}_{n \geq 1} \) be a sequence such that each \( u_n \) is the time optimal control with the optimal time \( t_n \) to the target set \( W_n \). Then there exists a time optimal control \( u_0 \) with the optimal time \( t_0 = \sup \{t_n\} \) to the point target set \( \{w_0\} \) which is given by the weak limit of some subsequence of \( \{u_n\}_{n \geq 1} \) in \( L^p(0,t_0;U) \).

**Proof.** Let \( t_0 = \sup \{t_n\} \). Since (20) is satisfied and \( U_ad \) is bounded, closed and convex in \( L^p(0,T;U) \), there exist \( u_0 \in U_ad \) and subsequences, still denoted \( \{u_n\}, \{t_n\} \) and \( \{x_n(t_n, u_n)\} \) such that

\[
\begin{align*}
& u_n \rightharpoonup u_0 \text{ weakly in } L^p(0,T;U), \\
& t_n = t(u_n) \uparrow t_0 \text{ in } [0,T],
\end{align*}
\]

and

\[
\lim_{n \to +\infty} \left[ \int_0^{t_n} R(t_0 - s)[(\mathcal{F}\tilde{u})(s) + f_1'(s) + f_2(s) + B(s)\tilde{u}(s)]ds \right] = \int_0^{t_0} R(t_0 - s)[(\mathcal{F}\tilde{u})(s) + f_1'(s) + f_2(s) + B(s)\tilde{u}(s)]ds
\]
Thus, using similar arguments as in the proof of Theorem 5.1, we can easily prove that
\[ u_0 \in U_0 := \{ u \in U_{ad} \mid x(t_0; u) = w_0 \}. \]
Moreover, we can verify that \( u_0 \) is the time optimal control and \( t_0 \) is the optimal time to the target set \( \{ w_0 \} \). Indeed, if it is not true, then there exists a control \( v \in U_{ad} \) such that \( x(t^1; v) = w_0 \) with \( t^1 < t_0 \). We may choose an integer \( n_0 \) such that \( t^1 < t_{n_0} \leq t_0 \). From the definition of \( U_0^{n_0} \) there must have \( v \in U_0^{n_0} \). On the other hand, \( u_{n_0} \) is the time optimal control with the optimal time \( t_{n_0} \) to the target set \( W_{n_0} \). Hence \( t_{n_0} \leq t(u) \) for all \( u \in U_0^{n_0} \). Particularly, \( t_{n_0} \leq t(v) \leq t^1 \), which is a contradiction and hence the desired result follows.

6. An example. In this section, we apply the obtained results to investigate the optimal control problems for the following neutral partial integro-differential control system with infinite delay.

\[
\begin{aligned}
&\frac{d}{dt} \left[ z(t, x) + \int_{-\infty}^{t} a(t - s)z(s, x)ds \right] = \frac{\partial^2}{\partial x^2}z(t, x) \\
&\quad + \int_{-\infty}^{t} b(t - s)\frac{\partial^2}{\partial x^2}z(s, x)ds + \int_{-\infty}^{t} c(t, s)f(s, z(s, x))ds \\
&\quad + \int_{0}^{\pi} r(t, x, y)u(t, y)dy, \quad 0 \leq t \leq T, \quad 0 \leq x \leq \pi, \\
&z(t, 0) = z(t, \pi) = 0, \quad 0 \leq t \leq T, \\
&z(\theta, x) = \varphi(\theta, x), \quad \theta \leq 0, \quad 0 \leq x \leq \pi,
\end{aligned}
\]  

where \( a(\cdot), b(\cdot), c(\cdot, \cdot), f(\cdot, \cdot), r(\cdot, \cdot, \cdot), u(\cdot, \cdot) \) and \( \varphi(\cdot, \cdot) \) are functions to be described below. System (22) arises in the study of heat flow in materials of the so-called fading memory type ([27]). Here, \( z(t, x) \) represents the temperature of the point \( x \) at time \( t \). Various topics on this system have been studied in literature in the past decades. For instance, in [21] and [35] the authors have discussed respectively the existence and maximal regularity of solutions of this kind of equations. However, little is known on its optimal control problems. Here we can use the above results to obtain the optimal control and time optimal control for System (22) under some proper conditions.

To represent this problem as the form of Eq. (1), we take \( X = U = L^2([0, \pi]) \) and we define \( Z(t)(\cdot) := z(t, \cdot) \) and \( \varphi(t)(\cdot) := \varphi(t, \cdot) \). Let \( A : D(A) \rightarrow X \) be the operator given by

\[ A\xi = \xi'' , \]

with the domain

\[ D(A) = H^2([0, \pi]) \cap H_0^1([0, \pi]) = \left\{ \xi(\cdot) \in X : \xi' , \xi'' \in X, \xi(0) = \xi(\pi) = 0 \right\} . \]

Then \( A \) generates a strongly continuous semigroup \((S(t))_{t \geq 0}\) which is analytic, compact and self-adjoint. Furthermore, \( A \) has a discrete spectrum, the eigenvalues are \(-n^2, \ n \in \mathbb{N}^+\), with the corresponding normalized eigenvectors \( e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx), \) \( n = 1, 2, \cdots \). Then the following properties hold:
(i) If \( \xi \in D(A) \), then
\[
A\xi = \sum_{n=1}^{\infty} -n^2 \langle \xi, e_n \rangle e_n.
\]

(ii) For every \( \xi \in X \),
\[
S(t)\xi = \sum_{n=1}^{\infty} e^{-n^2 t} \langle \xi, e_n \rangle e_n.
\]

Here we take the phase space \( \mathcal{B} = C_0 \times L^2(g : X) \) which was described in Section 2 with \( r = 0 \) and \( p = 2 \). Thus the norm of \( \mathcal{B} \) is given by, for \( \varphi \in \mathcal{B} \),
\[
\|\varphi\|_{\mathcal{B}} = \|\varphi(0)\| + \left( \int_{-\infty}^{0} g(\theta) \|\varphi(\theta)\|^2 d\theta \right)^{\frac{1}{2}}.
\]

It is known that, by choosing a proper function \( g \), the phase space \( C_0 \times L^2(g : X) \) satisfies the Axioms \((A)\), \((A_1)\) and \((B)\).

We assume that the following conditions hold for the system \((22)\).

\((a_1)\) The functions \( a \in C^1([0; \infty)) \), \( b \in C([0; \infty)) \), \( \varphi \) and \( A\varphi \) belong to \( \mathcal{B} \). Moreover, suppose that
\[
l_1 := \sup_{0 \leq t \leq T} \int_{-\infty}^{0} \frac{1}{g(s)} |a(t-s)|^2 ds < \infty,
\]
\[
l_2 := \sup_{0 \leq t \leq T} \int_{-\infty}^{0} \frac{1}{g(s)} \left| \frac{\partial}{\partial t} a(t-s) \right|^2 ds < \infty
\]
and
\[
l_3 := \sup_{0 \leq t \leq T} \int_{-\infty}^{0} \frac{1}{g(s)} |b(t-s)|^2 ds < \infty,
\]
where \( g(\cdot) \) is from the definition of the phase space \( \mathcal{B} \).

\((a_2)\) There is a \( \theta \in (\pi/2, \pi) \) such that \( \|\tilde{a}(\lambda)\| \leq N_0|\lambda| \) for each \( \lambda \in \Lambda_0 = \{ \lambda \in \mathbb{C} \setminus \{0\} \mid \arg(\lambda) < \theta \} \). In addition, \( \tilde{b}(\lambda) \to 0 \) as \( |\lambda| \to \infty \), \( \lambda \in \Lambda_0 \).

\((a_3)\) The functions \( c : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous and satisfy that
\[
|c(t, t+\theta)| < h(\theta),
\]
with
\[
k := \int_{-\infty}^{0} \frac{1}{g(\theta)} |h(\theta)|^2 d\theta < \infty.
\]

\((a_4)\) The function \( f(\cdot, \cdot) : [0, T] \times \mathbb{R} \to \mathbb{R} \) is measurable in \( t \) for each \( z \in \mathbb{R} \). Moreover, there exists a constant \( l > 0 \) such that
\[
|f(t, x) - f(t, y)| \leq l|x - y|, \quad \text{for all} \quad (t, x, y) \in [0, T] \times \mathbb{R}^2,
\]
\[
|f(t, x)| \leq l(|x| + 1), \quad \text{for all} \quad (t, x) \in [0, T] \times \mathbb{R}.
\]

\((a_5)\) The function \( \varphi(t, x) \) belongs to the phase space \( \mathcal{B} \). More specifically, \( \varphi(0, \cdot) \in L^2([0, \pi]) \) and \( \varphi(t, \cdot) \in L^2 \left( g : L^2([0, \pi]) \right), \theta < 0 \).

\((a_6)\) \( r(\cdot, \cdot, \cdot) : [0, T] \times [0, \pi] \times [0, \pi] \to \mathbb{R} \) is a continuous function.

Now define the operator \( N(\cdot) : [0, T] \to \mathcal{L}(X) \), and the functions \( \gamma(t) : D(A) \subseteq X \to X \), \( f_i(\cdot) : [0, T] \to X \), \( i = 1, 2 \), and \( F(\cdot, \cdot) : [0, T] \times \mathcal{B} \to X \), respectively, as
\[
N(t) = a(t),
\]
\[
\gamma(t) = b(t)A,
\]
\[
f_i(t) = c(t, t+\theta)g(\theta)^{\frac{1}{2}}.
\]
\[ f_1(t)(x) = -\int_{-\infty}^{0} a(t - s)\varphi(\theta, x) d\theta, \]
\[ f_2(t)(x) = \int_{-\infty}^{0} b(t - s)A\varphi(\theta, x) d\theta, \]
\[ F(t, \varphi)(x) = F(t, \varphi(\cdot, x)) = \int_{-\infty}^{0} c(t, t + \theta)f(t + \theta, \varphi(\theta)(x)) d\theta. \]

Then under the assumptions \((a_1)\) and \((a_3)\) the functions \(F(\cdot, \cdot)\) and \(f_i(\cdot), i = 1, 2,\) are well defined. On the other hand, the assumptions \((a_2)\) imply that the conditions \((V_1) - (V_4)\) are verified with \(\hat{N}(\lambda) = \hat{a}(\lambda)I, \hat{\gamma}(\lambda) = \hat{b}(\lambda)A,\) and hence the corresponing linear neutral system of \((22)\) generates a resolvent operator \((R(t))_{t \geq 0},\) which is given by, for \(z \in X,\)
\[ R(t)z = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{r, \theta}} e^{\lambda t}(\lambda I + \lambda \hat{N}(\lambda) - A - \hat{\gamma}(\lambda))^{-1}z d\lambda, & t > 0, \\ 0, & t = 0, \end{cases} \]

where \(\Gamma_{r, \theta}\) is as described in Section 2. Moreover, in view of Theorem 2.2 \((ii)\), it is easily see that \(R(t)\) is compact for all \(t > 0\) since the semigroup \((S(t))_{t \geq 0}\) is so.

In addition, let \(U_{ad}\) be a admissible set which is closed and convex in \(L^2(0, T; U)\) and we define the operator \(B(\cdot)\) as
\[ B(t)u(t)(x) = \int_{0}^{\pi} r(t, x, y)u(t, y)dy, \quad u \in U_{ad}. \]

Then under these notations System \((22)\) is rewritten into the form
\[
\begin{aligned}
\frac{d}{dt} \left[ x(t) + \int_{0}^{t} a(t - s)x(s)ds - f_1(t) \right] &= Ax(t) + \int_{0}^{t} b(t - s)Ax(s)ds \\
&\quad + f_2(t) + F(t, x_t) + B(t)u(t), \quad 0 \leq t \leq T,
\end{aligned}
\]

\[ x_0 = \varphi \in \mathcal{B}, \quad t \in (-\infty, 0]. \]

From the condition \((a_1)\) it follows that \(f_1 \in C^1([0, T], X), f_2 \in C([0, T], X)\) and
\[ f_1'(t)(x) = -\int_{-\infty}^{0} \frac{\partial}{\partial t} a(t - s)\varphi(\theta, x) dx, \quad (t, x) \in [0, T] \times [0, \pi]. \]

Subsequently let us certify that for this system the conditions in Theorem 3.2 are all fulfilled. Firstly, the assumptions \((a_5)\) - \((a_5)\) ensure that the function \(F\) satisfies the hypotheses \((H_2)\) with \(L = l\sqrt{k}\.\) Actually, from the assumptions \((a_5)\) - \((a_5)\) we compute that, for any \(t \in [0, T]\) and \(\varphi_1, \varphi_2 \in \mathcal{B},\)
\[ |F(t, \varphi_1)(x) - F(t, \varphi_2)(x)|^2 \]
\[ \leq \left( \int_{-\infty}^{0} |c(t, t + \theta)| |f(t + \theta, \varphi_1(\theta, x)) - f(t + \theta, \varphi_2(\theta, x))| d\theta \right)^2 \]
\[ \leq \left( \int_{-\infty}^{0} |h(\theta)| l |\varphi_1(\theta, x) - \varphi_2(\theta, x)| d\theta \right)^2 \]
\[ \leq l^2 \left( \int_{-\infty}^{0} \frac{1}{g(\theta)} |h(\theta)|^2 d\theta \right) \left( \int_{-\infty}^{0} g(\theta) |\varphi_1(\theta, x) - \varphi_2(\theta, x)|^2 d\theta \right) \]
\[ = l^2 k \int_{-\infty}^{0} g(\theta) |\varphi_1(\theta, x) - \varphi_2(\theta, x)|^2 d\theta. \]
Hence
\[ \|F(t, \varphi_1) - F(t, \varphi_2)\|_X^2 = \int_0^\pi |F(t, \varphi_1)(x) - F(t, \varphi_2)(x)|^2 \, dx \]
\[ \leq \ell^2 k \int_{-\infty}^0 g(\theta) \int_0^\pi |\varphi_1(\theta, x) - \varphi_2(\theta, x)|^2 \, dx \, d\theta \]
\[ \leq \ell^2 k \|\varphi_1 - \varphi_2\|_{L^2}^2. \]

Moreover, the assumption \((a_0)\) guarantees that \((H_1)\) holds. Consequently, from Theorem 3.2, for any \(\varphi \in \mathcal{B}\), System (22) admits a unique mild solution, denoted by \(z(t, x; u)\) on \((-\infty, T]\) with \(z|_{[0, T]} \in C([0, T]; X)\).

Now, we can discuss the optimal control and the time optimal control problems subject to the constraint system (22) by applying the results established in Section 4 and Section 5.

We first consider the optimal control problem for \(U_{ad}\) is bounded. Choose a bounded, closed and convex subset \(V\) of \(U\) and define the admissible set as
\[ U_{ad} := \{ u \in L^2(0, T; U) \mid u \text{ is strongly measurable and } u(t) \in V \text{ a.e. } t \in [0, T] \}. \]
Clearly \(U_{ad}\) is also bounded, closed and convex in \(L^2(0, T; U)\). We consider the cost function
\[ \mathcal{J}_1 = \int_0^T \left( c\sqrt{\|z(t, x; u)\|_X + \|u(t, x)\|_{L^2}^2} \right) \, dt, \]
where \(c > 0\) is a given constant. Then, for this situation,
\[ \phi_0(z(T, x; u)) := 0, \]
\[ \phi_1(t, z(t, x; u), z_l(\cdot, x; u)) := c\sqrt{\|z(t, x; u)\|_X}, \]
\[ \phi_2(t, u(t, x)) := \|u(t, x)\|_{L^2}^2. \]

It is easily seen that all the functions \(\phi_i\) \((i = 0, 1, 2)\) satisfy the corresponding assumptions in \((H_3)\) (note that the function \(\phi_1\) is not convex). So, by Theorem 4.1, there exists an optimal control \(u_0 \in U_{ad}\) which minimizes the cost function \(\mathcal{J}_1\) subject to System (22).

Now we discuss the optimal control problem for the case that \(U_{ad}\) is unbounded. Let now \(U_{ad} = L^2(0, T; U)\) and define the cost function \(\mathcal{J}_2\) by
\[ \mathcal{J}_2 = \int_0^T \left( \sqrt{\|z(t, x; u)\|_X + \|z_l(\cdot, x; u)\|_{L^2} + \|u(t, x)\|_{L^2}^2} \right) \, dt - \|z(T, x; u)\|_X^2. \]

To rewrite the cost function \(\mathcal{J}_2\) in the form of (11), we put
\[ \phi_0(z(T, x; u)) := -\|z(T, x; u)\|_X^2, \]
\[ \phi_1(t, z(t, x; u), z_l(\cdot, x; u)) := \sqrt{\|z(t, x; u)\|_X + \|z_l(\cdot, x; u)\|_{L^2}}, \]
\[ \phi_2(t, u(t, x)) := \|u(t, x)\|_{L^2}^2. \]

Then it is easy to verify that the functions \(\phi_i\) \((i = 0, 1, 2)\) satisfy well the assumptions \((H_5)\) and \((H_4)\). Hence, by Theorem 4.2, there exists at least one optimal control \(u_0\) in \(L^2(0, T; U)\) that minimizes \(\mathcal{J}_2\) subject to the constraint (22).

Finally we look at the problem of time optimal control for the system (22). Let the admissible set \(U_{ad}\) and the target set \(W\) be two bounded, closed and convex subsets in \(L^2(0, T; U)\) and \(X\) respectively. If
\[ U_0 := \{ u \in U_{ad} \mid z(t, x; u) \in W \text{ for some } t \in [0, T] \} \neq \emptyset. \]
Then from Theorem 5.1 there exists a control \( \bar{u} \in U_0 \) such that
\[
\bar{t}(\bar{u}) \leq \bar{t}(u), \quad \text{for all } u \in U_0,
\]
subject to the constraint (22), i.e., there exists a time optimal control to the target set \( W \) subject to System (22). Particularly, if \( W = \{w_0\} \subset X \) is a singleton so that (21) is verified, then, by Theorem 5.2, there also exists a time optimal control to the point target set \( \{w_0\} \) subject to System (22).

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