ON THE ASYMPTOTIC REPRESENTATION FOR TRANSVERSE MAGNETIC MULTIPLE SCATTERING OF RADIATION BY AN INFINITE GRATING OF DIELECTRIC CIRCULAR CYLINDERS AT OBLIQUE INCIDENCE

ÖMER KAVAKLIOĞLU and BARUCH SCHNEIDER
Division of Electrophysics Research and Department of Mathematics, Faculty of Computer Sciences, Izmir University of Economics, Balçova, IZMIR 35330 TURKEY
(omer_kavaklioglu@yahoo.com; omer.kavaklioglu@ieu.edu.tr; baruch.schneider@ieu.edu.tr)

Abstract

In this article, we present the derivation of the ‘asymptotic forms’ of the equations corresponding to the ‘scattering coefficients of the exterior electric and magnetic fields of an infinite grating of insulating dielectric circular cylinders for vertically polarized and obliquely incident plane electromagnetic waves’. Exploiting the generalized forms of the “Twersky’s elementary function representations for Schlömilch series”, we have deducted an ‘Ansatz’ describing the behavior of the scattering coefficients of the electric and magnetic fields for obliquely incident waves when the grating spacing is much smaller than the wavelength of the incident electromagnetic radiation. Introducing the statement of this ‘Ansatz’ into the equations of the ‘scattering coefficients of the infinite grating at oblique incidence’, and expanding the scattering coefficients in the form of an ‘asymptotic series’ as a function of the ratio of the radius of the cylinders to the grating spacing, we have acquired two ‘new’ infinite sets of algebraic equations associated with the ‘scattering coefficients of the exterior electric and magnetic fields of the grating for vertically polarized and obliquely incident plane waves’.

Keywords: multiple scattering; infinite grating; Schlömilch series; Neumann iteration
PACS numbers: 41.20.Jb, 42.25.Fx, 42.79.Dj, 77.90.+k, 84.40.-x

1. Introduction

Lord Rayleigh (1881) first treated the classical electromagnetic problem of the incidence of plane electric waves on an insulating dielectric cylinder as long ago as 1881. He published the diffraction of a plane wave at normal incidence by a homogeneous dielectric cylinder (Lord Rayleigh 1881). His solution was generalized for obliquely incident plane waves when the magnetic vector of the incident wave is transverse to the axis of the cylinder by Wait (1955).
Twersky (1952a) first obtained the formal analytical solution for the scattering of a plane electromagnetic wave by an arbitrary configuration of parallel cylinders in terms of cylindrical wave functions, considering all possible contributions to the excitation of a particular cylinder by the radiation scattered by the remaining cylinders in the grating. In his solution, he expressed the scattered wave as an infinite sum of orders of scattering, and later extended his solution to consider the case where all the axes of cylinders lie in the same plane (Twersky 1952b). Twersky (1952c) then introduced the “multiple scattering theories” to the finite grating of cylinders, and employed “Green’s function methods” to represent the “multiple scattering amplitude of one cylinder within the grating” in terms of “the functional equation” and the “single scattering amplitude of an isolated cylinder” (Twersky 1956). Twersky (1962) obtained a set of algebraic equations for the multiple scattering coefficients in terms of the elementary function representations of Schlömilch series (Twersky 1961), and in terms of the known coefficients of an isolated cylinder.

Bogdanov et al. (1985a) constructed an algorithm for the problem of diffraction of a plane electromagnetic wave, incident arbitrarily on a periodic array of infinitely long dielectric rods of circular cross section, and presented the relations between the main diffraction characteristics of the array and its parameters. Bogdanov et al. (1985b, 1987, 1991) treated various configurations of the same problem.

More recent investigations in the area of scattering by the arrays of cylinders have been conducted by Nicorovici et al. (1994) who developed the spatial and spectral domain forms of the Green’s function for the diffraction of a plane wave at arbitrary incidence in the x-y plane on a grating oriented along the x axis. Nicorovici and McPhedran (1994) considered the spatial and spectral domain forms of the Green’s function appropriate in the electromagnetic diffraction of a plane wave incident at an arbitrary angle in the x-y plane on a singly periodic structure oriented along the x-axis, and established the expressions from which grating lattice sums can effectively be evaluated. In addition, Chin et al. (1994) investigated the techniques for representing in absolutely convergent forms of the lattice sums in doubly
periodic electromagnetic diffraction problems. Petit (1980) presented a more
generalized case of arbitrary incidence and discussed quasi-periodicity.

Problems dealing with two dimensional arrays have been treated in detail by
McPhedran et al. (2000) who investigated the lattice sums arising in quasi-periodic
Green’s functions, McPhedran and Nicorovici (2002) who investigated sums
arising in doubly quasiperiodic Green’s functions, McPhedran et al. (2004) who
studied two-dimensional lattice sums, McPhedran et al. (2005a) who considered
sums over the square lattice and provide formulas, McPhedran et al. (2005b) who
took into consideration the use of Poisson summation formula to obtain effective
formulas for sums arising in scattering problems for the case of an infinite number
of cylinders ordered periodically along a line in the form of an infinite array.

These theoretical ideas mentioned above have found substantial applications in
the studies of Botten et al. (2000) who developed a formulation for wave
propagation and scattering through stacked gratings comprising metallic and
dielectric cylinders. Furthermore, Botten et al. (2004) developed a semi-analytic
approach for analyzing photonic crystals by employing the Bloch mode scattering
matrix methods and White et al. (2004) applied this method to two-dimensional
photonic waveguide structures that consist of lattices of either parallel finite
dielectric cylinders in an air background or parallel finite air cylinders submerged in
a dielectric medium.

Cai and Williams (1999a, b) investigated the multiple scattering of anti-plane
shear waves in fiber-reinforced composite materials, and Cai (2006) treated the
‘layered multiple scattering method’ for anti-plane shear wave scattering from
multiple gratings consisting of parallel cylinders.

Previous investigations mentioned above do not include the most general case of
oblique incidence although the grating is illuminated by an incident electromagnetic
wave at an arbitrary angle to the x-axis. As far as can be ascertained by the writers,
Sivov (1961) first treated the diffraction by an infinite periodic array of perfectly
conducting cylindrical columns for the most generalized case of obliquely incident
plane polarized electromagnetic waves in order to determine the reflection and
transmission coefficients of the infinite grating of perfectly conducting cylinders in free space. The period of the grating was assumed to be small in comparison with the wavelength. Lee (1990) studied the scattering of an obliquely incident electromagnetic wave by an arbitrary configuration of parallel, non-overlapping infinite cylinders and presented the solution for the scattering of an obliquely incident plane wave by a collection of closely-spaced, radially-stratified parallel cylinders that can have an arbitrary number of stratified layers (Lee 1992). Kavaklioglu (2000, 2001, 2002) and Kavaklioglu and Schneider (2007) extended the results of Twersky (1956, 1962) for the multiple scattering of an obliquely incident plane electromagnetic wave by an infinite grating of dielectric circular cylinders. In a more recent investigation by Kavaklioglu (2007), the ‘direct Neumann iteration technique’ is employed in order to acquire the exact solutions for the scattering coefficients of an infinite grating in the form of an infinite series and an analogue of Twersky’s solution is acquired for obliquely incident plane electromagnetic waves.

The most generalized oblique incidence solution presented in this investigation, the direction of the incident plane wave makes an arbitrary oblique angle of arrival \( \theta_i \) with the positive z-axis as indicated in figure 1.

2. Problem formulation

2.1. Multiple scattering representations for an infinite grating of dielectric circular cylinders for obliquely incident E-polarized plane electromagnetic waves

A vertically polarized plane electromagnetic wave, which is obliquely incident upon the infinite array of insulating dielectric circular cylinders having infinite length with radius “a”, dielectric constant “\( \varepsilon_r \)”, and relative permeability “\( \mu_r \)”, can be expanded (Wait 1955; kavaklioglu 2000) in the cylindrical coordinate system \((R_s, \phi_s, z)\) of the \( s^{th} \) cylinder in terms of the cylindrical waves referred to the axis of \( s^{th} \) cylinder as

\[
E_{\psi}^{inc}(R_s, \phi_s, z) = \mathbf{\hat{v}}_1 E_{0\psi} e^{i2\pi x_s \sin \psi_s} \left\{ \sum_{n=-\infty}^{\infty} e^{-i\nu \psi_s} J_n(k_s R_s) e^{i\nu(k_s + \pi/2)} \right\} e^{-i\kappa z} \quad (1)
\]
The cylinders of the grating are placed perpendicularly to the x-y plane, and separated by a distance of “d ”, as indicated in figure 1. In the above description of the incident field, $\hat{\psi}_i$ denotes the vertical polarization vector associated with a unit vector having a component parallel to all the cylinders, $\phi_i$ is the angle of incidence in x-y plane measured from $x-$axis in such a way that $\psi_i = \pi + \phi_i$, implying that the wave is obliquely incident in the first quadrant of the coordinate system, and “ $J_n(x)$ ” stands for “Bessel function of order n.” In addition, we have the following definitions

$$k_r = k_0 \sin \theta_i$$  \hspace{1cm} (2a) \\
$$k_z = k_0 \cos \theta_i$$  \hspace{1cm} (2b)

“$e^{-i\omega t}$” time dependence is suppressed throughout the paper, where “$\omega$ ” stands for the angular frequency of the incident wave in radians per second and “$t$” represents time in seconds.

2.2. Expressions for the z-components of the exterior fields

The centers of the cylinders in the infinite grating are located at positions $r_0$, $r_1$, $r_2$,.., etc. The exact solution for the z-components of the electric field in the exterior of the grating belonging to this configuration can be expressed in terms of the incident electric field in the coordinate system of the $s^{th}$ cylinder located at $r_s$, plus a summation of cylindrical waves outgoing from each of the individual $m^{th}$ cylinder located at $r_m$, as $|r - r_m| \to \infty$, i. e.,

$$E_z^{(ext)}(r_s, \phi_s, z) = E_z^{inc}(r_s, \phi_s, z) + \sum_{m=-\infty}^{\infty} E_z^{(m)}(r_m, \phi_m, z)$$  \hspace{1cm} (3)

The external electric and magnetic field intensities associated with vertically polarized obliquely incident plane electromagnetic waves are then given in (Kavakloğu 2000) as

$$E_z^{(ex)}(r_s, \phi_s, z) = \left\{ e^{ik_0z \sin \theta_s} \sum_{n=-\infty}^{\infty} \left[ E_n + \sum_{m=-\infty}^{\infty} A_{nm} (k, d) \right] J_n(k_r r_s) \right\}$$
In the representation of the electric and magnetic fields above, \( \{ A_n, A_n^m \}_{n=-\infty}^{\infty} \) denotes the set of all multiple scattering coefficients of the infinite grating associated with “vertically polarized obliquely incident plane electromagnetic waves”, \( \forall n \in Z \), where “Z” represents the set of all integers. In expressions (4a, b), we have

\[
E^n = \sin \phi_i E_{0z} e^{-in\psi_i}
\]

(5a)

\[
I_n(2\pi\Delta) = \sum_{p=1}^{\infty} H_n^{(1)}(2\pi p\Delta) \left[ e^{2\pi p\Delta\sin\psi_i} (-1)^n + e^{-2\pi p\Delta\sin\psi_i} \right]
\]

(5b)

where \( \Delta \equiv \frac{k_d}{2\pi} \), and “\( H_n^{(1)}(x) \)” denotes the \( n^{th} \) order Hankel function of first kind, \( \forall n \in Z \), where “Z” represents the set of all integers. The series in expression (5b) is the generalized form of the ‘Schlömilch series for obliquely incident waves \( l_{n-m}(k, d) \)’ (Twersky 1961, Kavaklíoğlu 2002) and convergent provided that \( k_d(1 \pm \sin \psi_i) / 2\pi \) does not equal integers.

3. Derivation of the Asymptotic Equations for the Scattering Coefficients of the Infinite Grating at Oblique Incidence

This section is devoted to the derivation of the asymptotic equations for the scattering coefficients of the infinite grating of dielectric cylinders at oblique incidence. In order to demonstrate the procedure of obtaining the asymptotic equations, we have introduced the exact system of equations for the scattering coefficients \( \{ A_n; A_n^m \}_{n=-\infty}^{+\infty} \) of the infinite grating of dielectric circular cylinder associated with an obliquely incident vertically polarized plane wave by the
application of the boundary conditions on the surface of each cylinder within the grating in Kavaklıoğlu (2000) as

\[ b_n^\mu \left\{ A_n + c_n \left[ E_n^i + \sum_{m=-\infty}^{+\infty} A_m^i \mid_{n-m}(k, d) \right] \right\} = - \left[ A_n^H + a_n^\mu \sum_{m=-\infty}^{+\infty} A_m^H \mid_{n-m}(k, d) \right] \]

(6a)

\[ \forall n \in \mathbb{Z}, \text{ and} \]

\[ b_n^\varepsilon \left[ A_n^H + c_n \sum_{m=-\infty}^{+\infty} A_m^H \mid_{n-m}(k, d) \right] = A_n + a_n^\varepsilon \left[ E_n^i + \sum_{m=-\infty}^{+\infty} A_m^i \mid_{n-m}(k, d) \right] \]

(6b)

\[ \forall n \in \mathbb{Z}. \] The coefficients arising in this infinite set of linear algebraic equations are defined as

\[ c_n := \frac{J_n(k, a)}{H_n^{(1)}(k, a)} \]

(7)

\[ \forall n \in \mathbb{Z}, \text{ and two sets of constants } a_n^\varepsilon \text{ and } b_n^\varepsilon, \text{ in which } \zeta_r \in \{ \varepsilon, \mu \} \text{ stands for the relative permittivity and permeability of the dielectric cylinders respectively, are given as} \]

\[ a_n^\varepsilon = \left[ \frac{J_n(k, a) \dot{J}_n(k, a) - \zeta_r \begin{pmatrix} k_r \\ k_1 \end{pmatrix} \dot{J}_n(k, a)}{J_n(k, a) \dot{H}_n^{(1)}(k, a) - \zeta_r \begin{pmatrix} k_r \\ k_1 \end{pmatrix} H_n^{(1)}(k, a)} \right] \]

(8)

for \( \zeta \in \{ \varepsilon, \mu \} \), and \( \forall n \in \mathbb{Z} \); where \( k_1 \) is defined as \( k_1 = k_0 \sqrt{\varepsilon_r \mu_r - \cos^2 \theta} \), and

\[ b_n^\varepsilon = \sqrt{\frac{\varepsilon_0 \mu_0}{\zeta_0}} \left[ \frac{J_n(k, a) H_n^{(1)}(k, a)}{J_n(k, a) \dot{H}_n^{(1)}(k, a) - \zeta_r \begin{pmatrix} k_r \\ k_1 \end{pmatrix} H_n^{(1)}(k, a) \dot{J}_n(k, a)} \right] \left( \frac{\text{im} F(k, a)}{k_r} \right) \]

(9)
for $\zeta \in \{\varepsilon, \mu\}$, and $\forall n \in Z$, where $F$ in the expression above is a constant and given as

$$F = \frac{(\mu_i \varepsilon_r - 1) \cos \theta_i}{\mu_i \varepsilon_r - \cos^2 \theta_i}$$  \hspace{1cm} (10)

$\forall n \in Z$. In these equations $\varepsilon_r$ and $\mu_r$ denotes the dielectric constant and the relative permeability constant of the insulating dielectric cylinders; $\varepsilon_0$ and $\mu_0$ stands for the permittivity and permeability of the free space respectively, $A_n$ and $A_n^H$ correspond to the scattering coefficients for the electric field intensity and magnetic field intensity associated with obliquely incident plane E-polarized electromagnetic waves, respectively. The $J_n^i$, and $H_n^{(1)}$ in expressions (7-8) are defined as

$$J_n^i(\zeta) = \frac{d}{d\zeta} J_n(\zeta)$$  \hspace{1cm} (11a)

$$H_n^{(1)}(\zeta) = \frac{d}{d\zeta} H_n^{(1)}(\zeta)$$  \hspace{1cm} (11b)

which imply the first derivatives of the Bessel and Hankel functions of first kind and of order $n$ with respect to their arguments.

3.1. Derivation of the Approximate Equations for the Scattering Coefficients of the Infinite Grating at Oblique Incidence

The exact equations in (6a-b) can be solved for $A_n$ and $A_n^H$ when the distance between the cylinders of the infinite grating are smaller than the wavelength of the incident wave, i. e., for $k_d << 1$ the exact equations take the following form

$$\begin{pmatrix} A_{zn}^i \\ A_{zn}^H \end{pmatrix} \approx S_n^{-1} \begin{pmatrix} E_{zn}^i + \sum_{m=-\infty}^{\infty} A_m^i H_{z-n-m}(k, d) \\ \sum_{m=-\infty}^{\infty} A_m^H H_{z-n-m}(k, d) \end{pmatrix}$$  \hspace{1cm} (12)

where $S_n$ is a $(2 \times 2)$ matrix defined as
\[ S_n := \left( \frac{s_n^{\mu \mu} s_n^{\pi \pi}}{s_n^{\mu \pi} s_n^{\pi \mu}} \right) \frac{(k_d a)^i a}{D} \]  \hspace{1cm} (13)

and \( \mathcal{H}_n(k, d) \) stands for the approximations to the “exact form of the Schlömilch series \( I_n(k, d) \)” in the limiting case when for \( k_d << 1 \). Introducing (13) into (12), “the approximate set of equations for the scattering coefficients of the infinite grating at oblique incidence” can explicitly be written as

\[
\begin{pmatrix}
A_{zn} \\
A_{zn}^n
\end{pmatrix} \approx \frac{(k_d a)^i a}{D} \begin{pmatrix}
s_n^{\mu \mu} & s_n^{\pi \pi} \\
s_n^{\mu \pi} & s_n^{\pi \mu}
\end{pmatrix} \begin{pmatrix}
E_{zn} + \sum_{m=-\infty}^{\infty} A_m H_{z n - m}(k_r, d) \\
\sum_{m=-\infty}^{\infty} A_m^n H_{z n - m}(k_r, d)
\end{pmatrix}
\]  \hspace{1cm} (14)

In the above, we have

\[ D = \left[ 1 + \varepsilon_r \left( \frac{k_r}{k_1} \right)^2 \right] \left[ 1 + \mu_r \left( \frac{k_r}{k_1} \right)^2 \right] - F^2 \]  \hspace{1cm} (15)

The \( n \)-dependent constants appearing in (13-14) are defined as

\[ s_n^{\mu \mu} := \left[ \frac{in \pi}{(2^n n!)^2} \right] s_{\mu \mu} \]  \hspace{1cm} (16a)

\[ s_n^{\pi \pi} := \left[ \frac{in \pi}{(2^n n!)^2} \right] s_{\pi \pi} \]  \hspace{1cm} (16b)

\[ s_n^{\mu \pi} := \left[ \frac{in \pi}{(2^n n!)^2} \right] s_{\mu \pi} \]  \hspace{1cm} (16c)

\[ s_n^{\pi \mu} := \left[ \frac{in \pi}{(2^n n!)^2} \right] s_{\pi \mu} \]  \hspace{1cm} (16d)

\( \forall n \in \mathbb{N} \), where \( \mathbb{N} \) denotes the set of all natural numbers. The various constants appearing in the definitions (16) are expressed as

\[ s_{\mu \mu} = \left[ 1 - \varepsilon_r \left( \frac{k_r}{k_1} \right)^2 \right] \left[ 1 + \mu_r \left( \frac{k_r}{k_1} \right)^2 \right] + F^2 \]  \hspace{1cm} (17a)

\[ s_{\pi \pi} = \left[ 1 - \mu_r \left( \frac{k_r}{k_1} \right)^2 \right] \left[ 1 + \varepsilon_r \left( \frac{k_r}{k_1} \right)^2 \right] + F^2 \]  \hspace{1cm} (17b)

and
\[ s_{\pm n} = \pm 2i \xi_0 F \quad (18a) \]
\[ s_{2\eta} = \mp 2i \eta_0 F \quad (18b) \]
The elements of the matrix of coefficients in (14) can be calculated using the expressions (17-18), for instance \( \frac{s_{\mu\nu}}{D} \) and \( \frac{s_{\mu\epsilon}}{D} \) terms can be written as
\[
\frac{s_{\mu\nu}}{D} \equiv \left[ \begin{array}{c}
1 - \epsilon_r \left( \frac{\sin^2 \theta_i}{\mu, \epsilon_r - \cos^2 \theta_i} \right) \\
1 + \epsilon_r \left( \frac{\sin^2 \theta_i}{\mu, \epsilon_r - \cos^2 \theta_i} \right)
\end{array} \right] 
+ \frac{1}{\mu} \left[ \begin{array}{c}
\left( \mu, \epsilon_r - 1 \right) \cos \theta_i \\
\left( \mu, \epsilon_r - 1 \right) \cos \theta_i
\end{array} \right]^2
\]
\[
\frac{s_{\mu\epsilon}}{D} \equiv \left[ \begin{array}{c}
1 - \mu_r \left( \frac{\sin^2 \theta_i}{\mu, \epsilon_r - \cos^2 \theta_i} \right) \\
1 + \mu_r \left( \frac{\sin^2 \theta_i}{\mu, \epsilon_r - \cos^2 \theta_i} \right)
\end{array} \right] 
+ \frac{1}{\mu} \left[ \begin{array}{c}
\left( \mu, \epsilon_r - 1 \right) \cos \theta_i \\
\left( \mu, \epsilon_r - 1 \right) \cos \theta_i
\end{array} \right]^2
\]

In terms of the definitions of (16), “the approximate set of equations for the scattering coefficients of the infinite grating at oblique incidence” given in (14) takes the following form
\[
\begin{align*}
\begin{pmatrix} A_{\pm n}^\mu \\ A_{\pm n}^\nu
\end{pmatrix} & \approx \frac{1}{D} \begin{pmatrix} s_{\mu\nu} & s_{\mu\epsilon} \end{pmatrix} \begin{pmatrix} E_{\pm n}^i + \sum_{m=-\infty}^{\infty} A_m H_{\pm n-m}^i(k, d) \\ \sum_{m=-\infty}^{\infty} A_m H_{\pm n-m}^\nu(k, d)
\end{pmatrix} \left[ \frac{2 \pi \xi}{(2\pi n)!^2} \right] (k, d)^2n 
\end{align*}
\]
\[ \forall n \in N. \] These equations can be separated into two different sets, in which the first one contains only the odd coefficients and the second one contains only the even coefficients, as
\[
\begin{align*}
A_{\pm (2\eta-1)}^\mu & \approx \frac{(k, a)^{4n-2}}{D} \left[ s_{\mu\nu}^{\pm (2\eta-1)}(E_{\pm (2\eta-1)}^i) + \sum_{m=-\infty}^{\infty} H_{\pm (2\eta-1)-m} A_m^\mu \right] + s_{\mu\epsilon}^{\pm (2\eta-1)} \left[ \sum_{m=-\infty}^{\infty} H_{\pm (2\eta-1)-m} A_m^\nu \right] 
\end{align*}
\]
\[
\begin{align*}
A_{\pm (2\eta)}^\nu & \approx \frac{(k, a)^{4n-2}}{D} \left[ s_{\mu\nu}^{\pm (2\eta-1)}(E_{\pm (2\eta-1)}^i) + \sum_{m=-\infty}^{\infty} H_{\pm (2\eta-1)-m} A_m^\mu \right] + s_{\mu\epsilon}^{\pm (2\eta-1)} \left[ \sum_{m=-\infty}^{\infty} H_{\pm (2\eta-1)-m} A_m^\nu \right]
\end{align*}
\]

10
\( \forall n \in \mathbb{N} \) for the odd coefficients. Similarly, for the even coefficients, we have acquired the following two sets of infinite number of approximate equations for the undetermined scattering coefficients of the infinite grating as

\[
A_{\pm 2n} \approx \frac{(k a)^{4n}}{D} \left[ s_{\pm 2n}^\mu (E_{\pm 2n}^\mu + \sum_{m=-\infty}^{\infty} H_{\pm 2n-m} A_m^\mu) + s_{\pm 2n}^s (\sum_{m=-\infty}^{\infty} H_{\pm 2n-m} A_m^s) \right] \quad (22a)
\]

\[
A_{\pm 2n}^H \approx \frac{(k a)^{4n}}{D} \left[ s_{\pm 2n}^\eta (E_{\pm 2n}^\eta + \sum_{m=-\infty}^{\infty} H_{\pm 2n-m} A_m^\eta) + s_{\pm 2n}^{\mu s} (\sum_{m=-\infty}^{\infty} H_{\pm 2n-m} A_m^{\mu s}) \right] \quad (22b)
\]

\( \forall n \in \mathbb{N} \).

3. 2. Derivation of the approximate expressions of the ‘Schlömilch series’

The elementary function representations of the ‘Schlömilch series \( H_n = \int_n^\infty \)’ in the limit of \( k d << 1 \)

given in (5b) have been originally derived by Twersky (1956) for the ‘normal incidence’, and modified by Kavaklıoğlu (2002) for the ‘oblique incidence’. We will employ these elementary function representations for the evaluation of the asymptotic forms of the ‘Schlömilch series \( H_n = \int_n^\infty \)’ in the limit of \( k d << 1 \).

Twersky’s forms (Twersky 1956) are still valid for the case of obliquely incident waves (Kavaklıoğlu 2002) with a slight modification in their arguments.

We have obtained \( H_n \) for the special case of \( n = 0 \) as

\[
H_0 = -1 + \frac{1}{\pi \Delta} \sum_{\mu = -\mu}^{\mu} \frac{1}{\cos \phi_\mu} + \frac{2}{\pi} \ln \frac{\Delta \gamma}{2} + i \left( \sum_{\mu = 1}^{\mu} \right) \frac{1}{\mu} + \frac{1}{\pi} \sum_{\mu = \mu}^{\mu} \left[ \frac{1}{\Delta \sinh \eta_\mu^+} - \frac{1}{\mu} \right] \quad (23)
\]

and for the general case, we have derived \( H_n \), \( \forall n \in \mathbb{N} \) as

\[
H_{2n} = \frac{1}{\pi \Delta} \sum_{\mu = -\mu}^{\mu} \frac{\cos 2n \phi_\mu}{\cos \phi_\mu} + i \left( \sum_{\mu = 1}^{\mu} \right) \frac{1}{n} \left[ \frac{1}{n} + \sum_{m=1}^{n} \frac{(-1)^m 2^m (n+m-1)! B_{2m}(\Delta \sin \psi_i)}{(2m)! (n-m)!} \right] \\
+ \frac{1}{\pi \Delta} \left[ \sum_{\mu = 0}^{\mu} \sum_{m=1}^{\mu} \sin 2n \phi_\mu \frac{1}{\cos \phi_\mu} + (-1)^n \left( \sum_{\mu = \mu}^{\mu} \frac{e^{-2n \eta_\mu^+}}{\sinh \eta_\mu^+} + \sum_{\mu = \mu}^{\mu} \frac{e^{-2n \eta_\mu^-}}{\sinh \eta_\mu^-} \right) \right] \quad (24a)
\]

\[
H_{2n+1} = \frac{1}{\pi \Delta} \sum_{\mu = -\mu}^{\mu} \frac{\sin (2n+1) \phi_\mu}{\cos \phi_\mu} + \frac{2}{\pi} \sum_{m=0}^{n} \frac{(-1)^m 2^m (n+m)! B_{2m+1}(\Delta \sin \psi_i)}{(2m+1)! (n-m)!} \quad (24b)
\]
\[
+ \frac{1}{\pi \Delta} \left[ \sum_{\mu=0}^{\mu_+} \frac{\sinh(2n+1)\phi_{\mu}}{\sinh \phi_{\mu}} + (-1)^n \left( \sum_{\mu=\mu_+}^{\infty} e^{-(2n+1)\eta_{\mu}^+} - \sum_{\mu=\mu_+}^{\infty} e^{-(2n+1)\eta_{\mu}^-} \right) \right]
\]

where \(\mu_\pm\) is the upper and lower bounds for the propagating modes, and \(\eta_{\mu}^\pm\)'s are determined from the grating equation as \(\cosh \eta_{\mu}^\pm = \pm \sin \psi_i + \frac{\mu}{\Delta}\).

3.2.1. Approximations for \(H_0\), \(H_{2n}\) and \(H_{2n+1}\) in the limit of \(\Delta \ll 1\)

The real part of \(H_n\), \(\forall n \in \mathbb{Z}_+\) in (23, 224a, b), which is recognized as ‘Bessel series \(J_n\)’, can explicitly be written as

\[
J_{2n} = \left[ 2 \frac{2}{k_c d} \sum_{\mu=\mu_-}^{\mu_+} \cos 2n \phi_{\mu} - \delta_{n0} \right]; \quad \forall n \in \mathbb{Z}_+,
\]

\[
J_{2n+1} = \left[ 2 \frac{2}{ik_c d} \sum_{\mu=\mu_-}^{\mu_+} \frac{\sin(2n+1)\phi_{\mu}}{\cos \phi_{\mu}} \right]; \quad \forall n \in \mathbb{Z}_+,
\]

where \(\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}\).

3.2.2. Approximations for \(N_0\), \(N_{2n}\) and \(N_{2n+1}\) in the limit of \(\Delta \ll 1\)

The imaginary part of the ‘Schlömilch series \(H_n\)’, which is known as the ‘Neumann series \(N_n\)’ in (3), can be put into the following form for this limiting case as

\[
N_0 \approx -\frac{2}{\pi} \ln \frac{\gamma \Delta}{2} + \frac{1}{\pi \Delta} \left( \sum_{\mu=1}^{\mu} + \sum_{\mu=1}^{\mu} \right) \frac{1}{\mu} - \frac{1}{\pi \Delta} \sum_{\mu=\mu_+}^{\infty} \left( \frac{1}{\mu} \ln \left( \frac{\mu}{\Delta} \right) \right) \left( \frac{\mu}{\Delta} + \sin \psi_i - \frac{1}{2} \frac{\Delta}{\mu} \right)
\]

\[
- \frac{1}{\pi \Delta} \sum_{\mu=\mu_+}^{\infty} \left( \frac{\mu}{\Delta} \right) \left( \frac{\mu}{\Delta} + \sin \psi_i - \frac{1}{2} \frac{\Delta}{\mu} \right) \left( \frac{1}{\mu} - \frac{\Delta}{\mu} \right)
\]

\[
\approx \frac{2}{\pi} \ln \left( \frac{\gamma \Delta}{2} \right) + \frac{1}{\pi \Delta} \left( \sum_{\mu=1}^{\mu} + \sum_{\mu=1}^{\mu} \right) \frac{1}{\mu} - \frac{1}{\pi \Delta} \sum_{\mu=\mu_+}^{\infty} \left( \frac{1}{\mu} \ln \left( \frac{\mu}{\Delta} \right) \right) \left( \frac{1}{\mu} + \sin \psi_i - \frac{1}{2} \frac{\Delta}{\mu} \right)
\]

In addition, we can obtain the simplified expressions for \(N_{2n}\) and \(N_{2n+1}\) in (24a, b) as
$$\mathbb{N}_{2n} \equiv \frac{1}{n\pi} + \frac{1}{n\pi} \sum_{m=1}^{n} \frac{(-1)^{m} 2^{2m} (n + m - 1)! B_{2m}(\Delta \sin \psi_i)}{(2m)! (n - m)! \Delta^{2m}}$$

$$- \frac{1}{n\pi} \left( \sum_{\mu = -\mu_{-}}^{\mu_{+}} \sum_{\mu = 0}^{\mu_{+}} \sum_{m=1}^{n} \frac{(-1)^{m} 2^{2m-1} (n + m - 1)!}{(2m-1)! (n - m)! \Delta^{2m}} \right) (\mu + \Delta \sin \psi_i)^{2m-1}$$

$$- \frac{(-1)^{n}}{\pi \Delta} \sum_{\mu = \mu_{+} + 1}^{\infty} \left( \frac{\mu}{\Delta} + \sin \psi_i - \frac{1}{2} \frac{\Delta}{\mu} + O \left( \frac{\Delta}{\mu} \right) \right)^{2n} \left( \frac{\mu}{\Delta} \right)^{2} + O \left( \frac{\Delta}{\mu} \right)$$

$$\forall n \in \mathbb{N}, \text{ and}$$

$$\mathbb{N}_{2n+1} \equiv \frac{2}{i\pi} \sum_{m=0}^{n} \frac{(-1)^{m} 2^{2m} (n + m)! B_{2m+1}(\Delta \sin \psi_i)}{(2m+1)! (n - m)! \Delta^{2m+1}}$$

$$- \frac{1}{i\pi} \left( \sum_{\mu = -\mu_{-}}^{\mu_{+}} \sum_{\mu = 0}^{\mu_{+}} \sum_{m=0}^{n} \frac{(-1)^{m} 2^{2m} (n + m)!}{(2m)! (n - m)! \Delta^{2m+1}} \right) (\mu + \Delta \sin \psi_i)^{2m}$$

$$- \frac{(-1)^{n}}{i\pi \Delta} \sum_{\mu = \mu_{+} + 1}^{\infty} \left( \frac{\mu}{\Delta} + \sin \psi_i - \frac{1}{2} \frac{\Delta}{\mu} + O \left( \frac{\Delta}{\mu} \right) \right)^{2n} \left( \frac{\mu}{\Delta} \right)^{2} + O \left( \frac{\Delta}{\mu} \right)$$

$$- \sum_{\mu = \mu_{+} + 1}^{\infty} \left( \frac{\mu}{\Delta} - \sin \psi_i - \frac{1}{2} \frac{\Delta}{\mu} + O \left( \frac{\Delta}{\mu} \right) \right)^{2n} \left( \frac{\mu}{\Delta} \right)^{2} + O \left( \frac{\Delta}{\mu} \right)$$

$$\forall n \in \mathbb{Z}_{+}.$$
3. 3. Special case with $\mu_+ = \mu_- = 0$

If there is only one propagating mode, which corresponds to the physical problem when the scattering of wavelengths larger than the ‘grating spacing’, i.e., $k d (1 \pm \sin \psi_i) < 1$, then the ‘Bessel series’, for $\phi_0 = \pi + \psi_i$, reduces to

$$J_{2n} = \frac{2 \cos 2n \phi_0}{k d \cos \phi_0} - \delta_{n0}$$ \hspace{1cm} (28a)

$$J_{2n+1} = -\frac{2i \sin (2n+1) \phi_0}{k d \cos \phi_0}$$ \hspace{1cm} (28b)

$\forall n \in \mathbb{Z}_+$.

3. 3. 1. Approximations for $N_0$, $N_{2n}$ and $N_{2n+1}$ with the special case of $\mu_+ = \mu_- = 0$ in the limit of $\Delta << 1$

Inserting $\mu_+ = \mu_- = 0$ into (26, 27a, and b), the expression for $N_0$ in (26) reduces to

$$N_0 \approx -\frac{2}{\pi} \ln \frac{\gamma \Delta}{2} - \frac{1}{\pi \Delta} \sum_{\mu=1}^{\infty} \left\{ \left(1 + 2 \sin^2 \psi_i\right) - \frac{1}{2} \left(\frac{\Delta}{\mu}\right)^2 \right\} \left(\frac{\mu}{\Delta} \left[1 - \left(1 + \sin^2 \psi_i\right) \left(\frac{\Delta}{\mu}\right)^2 + \frac{1}{4} \left(\frac{\Delta}{\mu}\right)^4\right]\right)$$ \hspace{1cm} (29)

The approximation for the ‘Neumann series $N_0$’, for $\phi_0 = \pi + \psi_i$, up to terms of the order $(k d)^2$ can be obtained from (9) as

$$N_0 \approx -\frac{2}{\pi} \ln \frac{\gamma \Delta}{2} - \frac{(1 + 2 \sin^2 \psi_i) \Delta^2}{\pi} \sum_{\mu=1}^{\infty} \mu^{-1}$$ \hspace{1cm} (30)

We have $\sum_{\mu=1}^{\infty} \mu^{-1} \approx 1.202$ in (30). In the same range, the ‘Neumann series $N_n$’ reduces to

$$N_{2n} \approx \frac{1}{n \pi} + \frac{1}{\pi} \sum_{m=1}^{n} \frac{(-1)^n 2^{2m-1} (n + m - 1)!}{(2m-1)! (n-m)! (\Delta \sin \psi_i)^{2m-1}} \left[ B_{2m} (\Delta \sin \psi_i) m + (\Delta \sin \psi_i)^{2m-1}\right] + F_{2n} \hspace{1cm} (31a)$$
\forall n \in \mathbb{N}, and
\[ N^{2n+1} = \frac{1}{i\pi} \sum_{m=0}^{n} \frac{(-1)^m 2^m (n + m)!}{(2m)!} \left( \frac{B_{2m+1}(\Delta \sin \psi_i)}{m + \frac{1}{2}} + (\Delta \sin \psi_i)^{2m} \right) + F_{2n+1}^{n} \quad (31b) \]

\forall n \in \mathbb{Z}_+, where \( F \)'s are given by
\[ F_{2n} \approx \frac{(-1)^{n+1}}{\pi \Delta} \sum_{\mu=1}^{\infty} \frac{2^{2n-1}}{\mu} \left( \Delta \right)^{2n+1} \]
\[ F_{2n+1} \approx i \frac{(-1)^{n+1}}{\pi \Delta} \sin \psi_i \sum_{\mu=1}^{\infty} \frac{2^{2n}}{\mu} \left( \Delta \right)^{2n+3} \quad (32b) \]

3. 3. 2. Approximations for the ‘Schlömilch series’, \( H_n \rightarrow \frac{1}{n} + i \mathbb{N} \), with the special case of \( \mu_+ = \mu_- = 0 \) in the limit of \( \Delta \ll 1 \)

If \( k, d \) is small, that is to say if \( \frac{k, d}{2\pi} (1 \pm \sin \psi_i) < 1 \), then there is only one discrete propagating mode. In this range, using the expansions for the ‘Bessel and Neumann Series’ obtained in the previous sections, we can write the expansions for the ‘Schlömilch Series’, for \( \phi_0 = \pi + \psi_i \), as

\[ H_0 \approx \frac{2}{k, d \cos \phi_0} - \frac{2i}{\pi} \ln \frac{k, d}{4\pi} - 1 - \frac{(k, d)^2}{2\pi^3} \left( \frac{1}{2} + \sin^2 \phi_0 \right) 1.202i \quad (33a) \]

\[ H_1 \approx -2i \sin \phi_0 + \frac{2 \sin \phi_0}{k, d \cos \phi_0} + \frac{(k, d)^2}{4\pi^3} \sin \phi_0 1.202 \quad (33b) \]

\[ H_2 \approx \frac{4\pi}{3i(k, d)^2} + \frac{2 \cos 2\phi_0}{k, d \cos \phi_0} + i \frac{(1 - 2 \sin^2 \phi_0)}{\pi} \left( \frac{k, d)^2}{(2\pi)^3} \right) 1.202i \quad (33c) \]

\[ H_3 \approx -\frac{16\pi \sin \phi_0}{3(k, d)^2} - i \frac{2 \sin 3\phi_0}{k, d \cos \phi_0} + \frac{2 \sin \phi_0}{\pi} (1 - \frac{4}{3} \sin^2 \phi_0) \]
\[ -\frac{\sin \phi_0}{2} \frac{(k, d)^4}{(2\pi)^5} \sum_{\mu=1}^{\infty} \mu^5 \quad (33d) \]

\[ H_4 \approx \frac{2 \pi^3}{15i(k, d)^4} - \frac{16\pi}{(k, d)^2} \left( \frac{1}{6} - \sin^2 \phi_0 \right) + \frac{2 \cos 4\phi_0}{k, d \cos \phi_0} \]
\[ + \frac{i}{2\pi} (1 - 8 \sin^2 \phi_0 + 8 \sin^4 \phi_0) - i \frac{(k, d)^4}{4(2\pi)^5} \sum_{\mu=1}^{\infty} \mu^5 \quad (33e) \]
The leading terms of $H_n$ for large $\forall n \in N$ is given as

$$H_{2n} \approx 2^{4n-1} \left[ \frac{(-1)^n \pi^{2n-1} B_{2n}(0)}{(k,d)^{2n}} \right] \frac{i}{n}$$

(34a)

$$H_{2n+1} \approx 2^{4n+1} \left[ \frac{(-1)^n \pi^{2n-1} B_{2n}(0)}{(k,d)^{2n}} \right] \sin \phi_0$$

(34b)

where $B_n(0)$ corresponds to ‘Bernoulli Polynomial’. From (33, and 34), we can determine the leading terms of the ‘Schlömilch Series’ as

$$H_0 \approx \frac{h_0}{k,d} \quad \text{where} \quad h_0 \equiv 2 \sec \phi_0$$

(35a)

$$H_1 \approx \frac{h_1}{k,d} \quad \text{where} \quad h_1 \equiv -2i \tan \phi_0$$

(35b)

$$H_2 \approx \frac{h_2}{(k,d)^2} \quad \text{where} \quad h_2 \equiv \frac{4 \pi}{3i}$$

(35c)

$$H_3 \approx \frac{h_3}{(k,d)^2} \quad \text{where} \quad h_3 \equiv -\frac{16 \pi \sin \phi_0}{3}$$

(35d)

$$H_4 \approx \frac{h_4}{(k,d)^3} \quad \text{where} \quad h_4 \equiv \frac{2^5 \pi^3}{15i}$$

(35e)

$$H_5 \approx \frac{h_5}{(k,d)^3} \quad \text{where} \quad h_5 \equiv -\frac{2^8 \pi^3 \sin \phi_0}{15}$$

(35f)

The leading terms of $H_n$ for large $n$ are given by

$$H_{2n} \approx \frac{h_{2n}}{(k,d)^{2n}}$$

(36a)

$$H_{2n+1} \approx \frac{h_{2n+1}}{(k,d)^{2n}}$$

(36b)

where $h_{2n}$’s and $h_{2n+1}$’s for large $n$ are given as

$$h_{2n} \rightarrow \frac{i}{n} (-1)^n \pi^{2n-1} B_{2n}(0)$$

(37a)

and

$$h_{2n+1} \rightarrow (-1)^n \pi^{2n-1} B_{2n}(0) \sin \phi_0 \equiv -4i h_{2n} \sin \phi_0$$

(37b)
respectively. In the above expressions, $B^\prime$s are the ‘Bernoulli numbers’, and the relationship between ‘Bernoulli polynomial’ and ‘Bernoulli numbers’ is given as

$$B_{2\xi}(0) \equiv (-1)^{\xi-1} B^\xi$$  \hspace{1cm} (38)

4. Asymptotic expansions for the scattering coefficients of the infinite grating at oblique incidence in the limiting case of “$(a/d)\ll 1$”

In order to find a solution for the set of equations given in (21 and 22), we have introduced an ‘Ansatz’ for the scattering coefficients of the electric and magnetic fields of the infinite grating assuming $(k,a) \ll 1$, and $(k_{a,d}) \equiv \xi < \frac{1}{2}$ as

$$A_{\pm(2n-1)} \cong A_{\pm(2n-1),0}(k,a)^{2n}$$  \hspace{1cm} (39a)

$$A^H_{\pm(2n-1)} \cong A^H_{\pm(2n-1),0}(k,a)^{2n}$$  \hspace{1cm} (39b)

$\forall n \in N$ for the odd coefficients, and

$$A_{\pm2n} \cong A_{\pm2n,0}(k,a)^{2n+2}$$  \hspace{1cm} (39c)

$$A^H_{\pm2n} \cong A^H_{\pm2n,0}(k,a)^{2n+2}$$  \hspace{1cm} (39d)

$\forall n \in Z_+$ for the even coefficients. We have defined the overall effect of the multiple scattering terms when the wavelength is larger than the grating spacing, i.e., $(k,d) \ll 1$, and $(k_{a,d}) \equiv \xi < \frac{1}{2}$ as

$$G_{\pm,n} \equiv \sum_{m=-\infty}^{\infty} \hat{H}_{\pm n-m} A_m$$  \hspace{1cm} (40a)

for the electric field coefficients, and

$$G^H_{\pm,n} \equiv \sum_{m=-\infty}^{\infty} \hat{H}_{\pm n-m} A^H_m$$  \hspace{1cm} (40b)

for the magnetic field coefficients. Introducing the Schlömilch Series into (14), we can write the overall effect of the multiple scattering terms when the wavelength is larger than the grating spacing, i.e., $(k,d) \ll 1$, and $(k_{a,d}) \equiv \xi < \frac{1}{2}$ as

$$G_{\pm(2n-1),0} = \sum_{m=1}^{\infty} \left( \frac{a}{d} \right)^{2m} h_{\pm2(m+n-1)} A_{\pm(2m-1),0}$$  \hspace{1cm} (41a)
\[ G_{z(2n-1),0}^H = \sum_{m=1}^{\infty} \left( \frac{a}{d} \right)^{2m} h_{z(2m+n-1)} A_{r(2m-1),0}^H \]  
(41b)  
\[ \forall n \in \mathbb{N} \] for the odd coefficients;  
\[ G_{z2n,0} = \sum_{m=1}^{\infty} \left( \frac{a}{d} \right)^{2m} \left\{ h_{z(2m+n-1)} A_{r(2m-1),0}^H + h_{z(2m+2n-1)} A_{r(2m-1),0}^H \right\} \]  
(41c)  
\[ G_{z2n,0}^H = \sum_{m=1}^{\infty} \left( \frac{a}{d} \right)^{2m} \left\{ h_{z(2m+n-1)} A_{r(2m-1),0}^H + h_{z(2m+2n-1)} A_{r(2m-1),0}^H \right\} \]  
(41d)  
\[ \forall n \in \mathbb{N} \] for the even coefficients; and the special case for \( n = 0 \) is given by  
\[ G_{0,0} = \sum_{m=-1}^{\infty} h_{m} A_{-m,0} \]  
(42a)  
\[ G_{0,0}^H = \sum_{m=-1}^{\infty} h_{m} A_{-m,0}^H \]  
(42b)  
Defining the wavelength independent parts of the scattering matrices from (13) as  
\[ S := \left( k, a \right)^{2n} \]  
(43a)  
\[ S_{z,0} := \frac{1}{D} \begin{pmatrix} \delta_{z,n}^\mu & \delta_{z,n} \varepsilon \\ \delta_{z,n}^\varepsilon & \delta_{z,n}^{\mu \varepsilon} \end{pmatrix} \]  
(43b)  
\[ S_{z(2n-1),0} \equiv \frac{1}{D} \begin{pmatrix} \delta_{z(2n-1),0}^\mu & \delta_{z(2n-1),0} \varepsilon \\ \delta_{z(2n-1),0}^\varepsilon & \delta_{z(2n-1),0}^{\mu \varepsilon} \end{pmatrix} \]  
(43c)  
\[ \forall n \in \mathbb{N} \] corresponding to the odd, and  
\[ S_{z2n,0} \equiv \frac{1}{D} \begin{pmatrix} \delta_{z2n,0}^\mu & \delta_{z2n,0} \varepsilon \\ \delta_{z2n,0}^\varepsilon & \delta_{z2n,0}^{\mu \varepsilon} \end{pmatrix} \]  
(43d)  
\[ \forall n \in \mathbb{N} \] corresponding to the even part. In terms of these definitions of (43), and Upon introducing (41) into (21 and 22) and employing the ‘Kronocker delta \( \delta_{nm} \) symbol’, we have obtained the following set of equations for the approximations of the scattering coefficients as  
\[ \begin{pmatrix} A_{z(2n-1),0} \\ A_{z(2n-1),0}^H \end{pmatrix} = S_{z(2n-1),0} \begin{pmatrix} \delta_{z,n} E_{z(2n-1),0}^i + \left( a \over d \right)^{2(n-1)} G_{z(2n-1),0} \\ \left( a \over d \right)^{2(n-1)} G_{z(2n-1),0}^H \end{pmatrix} \]  
(44a)  
\[ \forall n \in \mathbb{N} \] corresponding to the odd scattering coefficients, and
\[
\begin{bmatrix}
A_{\pm 2n,0} \\
A^H_{\pm 2n,0}
\end{bmatrix} = S_{\pm \pm 2n,0} \begin{bmatrix}
\delta_{n1}E'_{\pm 2n,0} + \left( \frac{a}{d} \right)^{2(n-1)} G_{\pm 2n,0} \\
\left( \frac{a}{d} \right)^{2(n-1)} G^H_{\pm 2n,0}
\end{bmatrix}
\]  
(44b)

\( \forall n \in \mathbb{N} \) corresponding to the even scattering coefficients. Splitting the matrices in (44) into two parts, we have

\[
\begin{bmatrix}
A_{\pm (2n-1),0} \\
A^H_{\pm (2n-1),0}
\end{bmatrix} = \frac{\delta_{n1}}{D} \begin{bmatrix}
s^{\pm}_{2n-1,0} \\
s^\eta_{\pm 2n-1,0}
\end{bmatrix} E^{i}_{\pm (2n-1),0} + \left( \frac{a}{d} \right)^{2(n-1)} S_{\pm \pm (2n-1),0} \begin{bmatrix}
G_{\pm (2n-1),0} \\
G^H_{\pm (2n-1),0}
\end{bmatrix}
\]  
(45a)

\( \forall n \in \mathbb{N} \) for the odd scattering coefficients, and

\[
\begin{bmatrix}
A_{\pm 2n,0} \\
A^H_{\pm 2n,0}
\end{bmatrix} = \frac{\delta_{n1}}{D} \begin{bmatrix}
s^{\pm}_{2n,0} \\
s^\eta_{\pm 2n,0}
\end{bmatrix} E^{i}_{\pm 2n,0} + \left( \frac{a}{d} \right)^{2(n-1)} S_{\pm \pm 2n,0} \begin{bmatrix}
G_{\pm 2n,0} \\
G^H_{\pm 2n,0}
\end{bmatrix}
\]  
(45b)

\( \forall n \in \mathbb{N} \) for even scattering coefficients, respectively. From (41), we have established the following terms as

\[
\left( \frac{a}{d} \right)^{2(n-1)} G_{\pm (2n-1),0} = \sum_{m=1}^{\infty} \left( \frac{a}{d} \right)^{2(m+n-1)} h_{\pm 2(m+n-1)} A_{\mp (2m-1),0}
\]  
(46a)

for the multiple interactions corresponding to the scattering coefficients of the electric field,

\[
\left( \frac{a}{d} \right)^{2(n-1)} G^H_{\pm (2n-1),0} = \sum_{m=1}^{\infty} \left( \frac{a}{d} \right)^{2(m+n-1)} h_{\pm 2(m+n-1)} A^H_{\mp (2m-1),0}
\]  
(46b)

for the multiple interactions corresponding to the scattering coefficients of the magnetic field, \( \forall n \in \mathbb{N} \) for the odd scattering coefficients; and

\[
\left( \frac{a}{d} \right)^{2(n-1)} G_{\pm 2n,0} = \sum_{m=1}^{\infty} \left( \frac{a}{d} \right)^{2(m+n-1)} \left\{ h_{\pm 2(m+n-1)} A_{\mp (2m-2),0} + h_{\pm 2(m+n-1)} A^H_{\mp (2m-1),0} \right\}
\]  
(47a)

for the multiple interactions corresponding to the scattering coefficients of the electric field,

\[
\left( \frac{a}{d} \right)^{2(n-1)} G^H_{\pm 2n,0} = \sum_{m=1}^{\infty} \left( \frac{a}{d} \right)^{2(m+n-1)} \left\{ h_{\pm 2(m+n-1)} A^H_{\mp (2m-2),0} + h_{\pm 2(m+n-1)} A^H_{\mp (2m-1),0} \right\}
\]  
(47b)

for the multiple interactions corresponding to the scattering coefficients of the magnetic field, \( \forall n \in \mathbb{N} \) for the even scattering coefficients. Inserting (46 and 47)
into (45), we have finally obtained an infinite set of equations for the electric and magnetic scattering coefficients as

\[
\begin{bmatrix}
A_{x(2n-1),0} \\
A_{\eta(2n-1),0}
\end{bmatrix} = \frac{\delta_{n1}}{D} \begin{bmatrix}
S_{2n-1,0} \\
S_{2n-1,0}
\end{bmatrix} E_{z(2n-1),0}
+ \sum_{m=1}^{\infty} \frac{a}{d} \left( \frac{a}{d} \right)^{2(m+n-1)} h_{z(2m+n-1)} S_{z(2n-1),0} \begin{bmatrix}
A_{x(2m-1),0} \\
A_{\eta(2m-1),0}
\end{bmatrix}
\]

\((48a)\)

\(\forall n \in N\) for the odd scattering coefficients, and

\[
\begin{bmatrix}
A_{z2n,0} \\
A_{\eta2n,0}
\end{bmatrix} = \frac{\delta_{n1}}{D} \begin{bmatrix}
S_{2n,0} \\
S_{2n,0}
\end{bmatrix} E_{z2n,0}
+ \sum_{m=1}^{\infty} \frac{a}{d} \left( \frac{a}{d} \right)^{2(m+n-1)} S_{z2n,0} \begin{bmatrix}
A_{x(2m-1),0} \\
A_{\eta(2m-1),0}
\end{bmatrix}
\]

\((48b)\)

\(\forall n \in N\) for the even scattering coefficients. In equations (48a, b), we have noticed that the scattering coefficients of the electric and magnetic fields appeared as coupled to each others. We have finally attempt to express the equations of (48) more compactly by defining a new vector such as \(\vec{\alpha}_p\) where

\[
\vec{\alpha}_p \equiv \begin{bmatrix}
A_{p,0} \\
A_{\eta p,0}
\end{bmatrix}
\forall p \in Z
\]

\((49)\)

Inserting this definition of (49) into (48a, b), we have obtained

\[
\vec{\alpha}_{z(2n-1),0} = \frac{\delta_{n1}}{D} \begin{bmatrix}
S_{2n-1,0} \\
S_{2n-1,0}
\end{bmatrix} E_{z(2n-1),0}
+ \left( \frac{a}{d} \right)^{2(n-1)} S_{z(2n-1),0} \sum_{m=1}^{\infty} \left( \frac{a}{d} \right)^{2m} h_{z(2m+n-1)} \vec{\alpha}_{z(2m-1),0}
\]

\((50a)\)

\(\forall n \in N\) for the odd scattering coefficients, and

\[
\vec{\alpha}_{z2n,0} = \frac{\delta_{n1}}{D} \begin{bmatrix}
S_{2n,0} \\
S_{2n,0}
\end{bmatrix} E_{z2n,0}
+ \left( \frac{a}{d} \right)^{2(n-1)} S_{z2n,0} \sum_{m=1}^{\infty} \left( \frac{a}{d} \right)^{2m} \left\{ h_{z(2m+n-1)} \vec{\alpha}_{z(2m-1),0} + h_{z(2m+2n-1)} \vec{\alpha}_{z(2n-1),0} \right\}
\]

\((50b)\)

\(\forall n \in N\) for the even scattering coefficients.
**Conclusion**

In this investigation, we have presented a rigorous derivation of the asymptotic equations associated with ‘the scattering coefficients of an infinite grating of dielectric circular cylinders for obliquely incident vertically polarized plane electromagnetic waves’. We have predicted the asymptotic behavior of the scattering coefficients by exploiting the “Twersky's elementary function representations for Schlömilch series” when the wavelength of the scattered wave is much larger than the distance between the constituent cylinders of the grating, and then used these predicted forms for the determination of an ‘Ansatz’ which describes the behavior of the scattering coefficients as a function of $(a/d)$. Finally, we have acquired the asymptotic forms of the equations associated with the ‘electric and magnetic scattering coefficients of the infinite grating at oblique incidence’. Our results are the generalizations of those acquired by (Twersky 1962) for the non-oblique incidence case.

**Acknowledgments**

We are indebted to deceased Professor Emeritus Victor Twersky of the Department of Mathematics of the University of Illinois at Chicago for the kind concern he spared to our work. The first author takes this opportunity to express his sincere thanks to Professor Dr Roger Henry Lang for suggesting the problem and many fruitful discussions.

**References**

Bogdanov, F. G., Kevanishvili, G. Sh., Sikmashvili, Z. I. & Tsagareyshvili, O. P. 1985a Diffraction of a plane electromagnetic wave by an array of circular dielectric rods. Radiotehnika i Elektronika 30, No. 10, 1879-1884. (Engl. Translation in 1986 Soviet J. of Communications Technology and Electronics 31, No. 2, 6-11).

Bogdanov, F. G., Kevanishvili, G. Sh., Sikmashvili, Z. I., Tsagareishvili, O. P. & Chikhladze, M. N. 1985b Diffraction of a plane E-polarized wave from a lattice of coaxial cylinders. Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika 28, No. 2, 229-235. (Engl. Translation in 1985 Radiophysics and Quantum Electronics 28, No. 2, 157-162).
Bogdanov, F. G., Kevanishvili, G. Sh. & Chikhladze, M. N. 1987 Diffraction of a plane electromagnetic wave from a lattice of coaxial dielectric cylinders. *Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika* 30, No. 5, 637-642. (Engl. Translation in 1987 *Radiophysics and Quantum Electronics* 30, No. 5, 485-491).

Bogdanov, F. G., Kevanishvili, G. Sh., Meriakri, V. V. & Nikitin, I. P. 1991 Investigation of the resonance characteristics of arrays of dielectric cylinders. *Radiotekhnika i Elektronika* 36, No. 1, 48-51. (Engl. Translation in 1991 *Soviet J. of Communications Technology and Electronics* 36, No. 9, 78-81).

Botten, L. C., Nicorovici, N. -A. P., Asatryan, A. A., McPhedran, R. C., Martijn de Sterke, C., and Robinson, P. A. 2000 Formulation for electromagnetic scattering and propagation through grating stacks of metallic and dielectric cylinders for photonic crystal calculations. Part I. Method, *J. Opt. Soc. Am. A* 17, 2165-2176.

Botten, L. C., White, T. P., Asatryan, A. A., Langtry, T. N., Martijn de Sterke, C., and McPhedran, R. C. 2004 Bloch mode scattering matrix methods for modeling extended photonic crystal structures. I. Theory. *Physical Review E* 70(5), 056606(13).

White, T. P., Botten, L. C., Martijn de Sterke, C., McPhedran, R. C., Asatryan, A. A., and Langtry, T. N. 2004 Bloch mode scattering matrix methods for modeling extended photonic crystal structures. II. Applications. *Physical Review E* 70(5), 056607(10).

Cai, L.-W., and Williams, Jr., J. H. 1999a Large-scale multiple scattering problems. *Ultrasonics* 37(7), 453-462.

Cai, L.-W., and Williams, Jr., J. H. 1999b Full-scale simulations of elastic wave scattering in fiber reinforced composites. *Ultrasonics* 37(7), 463-482.

Cai, L.-W. 2006 Evaluation of layered multiple-scattering method for antiplane shear wave scattering from gratings. *J. Acoust. Soc. Am.* 120, 49-61.

Chin, S.K., Nicorovici, N. A. and McPhedran, R.C. 1994 Green’s function and lattice sums for electromagnetic scattering by a square array of cylinders. *Phys. Rev. E* 49, No. 5, 4590-4602.

Kavakhoğlu, Ö. 2000 Scattering of a plane wave by an infinite grating of circular dielectric cylinders at oblique incidence: E-polarization. *Int. J. Electron.* 87, 315-336.

Kavakhoğlu, Ö. 2001 On diffraction of waves by the infinite grating of circular dielectric cylinders at oblique incidence: Floquet representation. *J. Mod. Opt.* 48, 125-142.

Kavakhoğlu, Ö. 2002 On Schlömilch series representation for the transverse electric multiple scattering by an infinite grating of insulating dielectric circular cylinders at oblique incidence. *J. Phys. A: Math. Gen.* 35, 2229-2248.

Kavakhoğlu, Ö. 2007 On multiple scattering of radiation by an infinite grating of dielectric circular cylinders at oblique incidence (in review)
Kavaklioglu, Ö. and Schneider, B. 2007 On Floquet-Twersky representation for the diffraction of obliquely incident plane-H polarized electromagnetic waves by an infinite grating of insulating dielectric circular cylinders (in review).

Lee, S. C. 1990 Dependent scattering of an obliquely incident plane wave by a collection of parallel cylinders. *J. Appl. Phys.* **68**, 4952-4957.

Lee, S. C. 1992 Scattering by closely-spaced radially-stratified parallel cylinders. *J. Quant. Spectrosc. Radiat. Transfer* **48**, 119-130.

McPhedran, R. C., Nicorovici, N. A., Botten, L. C., and Grubits, K. A. 2000 Lattice sums for gratings and arrays. *J. Math. Phys.* **41**, 7808-7816.

McPhedran, R. C., and Nicorovici, N. A. 2002 Static Bloch sums for the square array. *J. Math. Phys.* **43**, 2802-2813.

McPhedran, R. C., Smith, G. H., Nicorovici, N. A., and Botten, L. C. 2004 Distributive and analytic properties of lattice sums. *J. Math. Phys.* **45**, 2560-2578.

McPhedran, R. C., Nicorovici, N. A., and Botten, L. C. 2005a Schlömilch series and grating sums. *J. Phys. A: Math. Gen.* **38**, 8353-8366.

McPhedran, R. C., Nicorovici, N. A., and Botten, L. C. 2005b Neumann series and lattice sums. *J. Math. Phys.* **46**, 083509.

Nicorovici, N. A., McPhedran, R. C. & Petit, R. 1994 Efficient calculation of the Green’s function for electromagnetic scattering by gratings. *Physical Review E* **49**, No. 5, 4563-4577.

Nicorovici, N. A., and McPhedran, R.C. 1994 Lattice sums for off-axis electromagnetic scattering by gratings. *Physical Review E* **50**, No. 4, 3143-3160.

Petit, R. 1980 in *Electromagnetic Theory of Gratings*, edited by R Petit, *Topics in Current Physics* **22** Berlin: Springer-Verlag.

Rayleigh, Lord, 1881 On the electromagnetic theory of light. *Philos. Mag. S.5* **12**, No. 73, 81, 101.

Rayleigh, Lord. 1918 The dispersal of light by a dielectric cylinder. *Philos. Mag. S. 6* **36**, No. 215, 365 376.

Sivov, A. N. 1961 Electrodynamic theory of a dense plane grating of parallel conductors. *Radiotekhnika i elektronika* **6**, No. 4, 483. (Engl. Translation in 1961 *Radio Eng. and Electronic Phys.* **6**, No. 4, 429-440).

Twersky, V. 1952a Multiple scattering of radiation by an arbitrary configuration of parallel cylinders. *J. Acoust. Soc. Am.* **24**, 42-46.

Twersky, V. 1952b Multiple scattering of radiation by an arbitrary planar configuration of parallel cylinders and by two parallel cylinders. *J. Appl. Phys.*, **23**, 407-414.

Twersky, V. 1952c On a multiple scattering theory of the finite grating and Wood anomalies. *J. Appl. Phys.* **23**, 1099-1118.
Twersky, V. 1956 On the scattering of waves by an infinite grating. *IRE Trans. on Antennas Propagat.* AP-4, 330-345.

Twersky, V. 1961 Elementary function representations of Schlömilch series. *Arch. for Rational Mech. and Anal.* 8, 323-332.

Twersky, V. 1962 On scattering of waves by the infinite grating of circular cylinders. *IRE Trans. on Antennas Propagat.* AP-10, 737-765.

Wait, J. R. 1955 Scattering of a plane wave from a circular dielectric cylinder at oblique incidence. *Canad. J. Phys.* 33, 189-195.