Perturbative renormalization factors of three- and four-quark
operators for domain-wall QCD

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Abstract

Renormalization factors for three- and four-quark operators, which appear
in the low energy effective Lagrangian of the proton decay and the weak in-
teractions, are perturbatively calculated in domain-wall QCD. We find that
the operators are multiplicatively renormalizable up to one-loop level without
mixing with any other operators that have different chiral structures. As an
application, we evaluate a renormalization factor for $B_K$ at the parameters
where previous simulations have been performed, and find one-loop correc-
tions to $B_K$ are 1-5% in these cases.

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I. INTRODUCTION

Calculation of hadron matrix elements of phenomenological interest represents an inevitable application of lattice QCD. In the past decade much efforts have been devoted for the calculation of three- and four-quark hadron matrix elements relevant to the proton decay amplitude and the weak interaction ones using the Wilson and the Kogut-Susskind(KS) quark actions. However, the satisfactorily precise measurement of the matrix elements has not been achieved so far because of the inherent defects in these quark actions: the explicit chiral symmetry breaking in the Wilson quark action causes the non-trivial operator mixing between different chiralities and for the KS quark it is hard to treat the heavy-light cases due to the flavor symmetry breaking.

The domain-wall quark formulation in lattice QCD, which is based on the introduction of many heavy regulator fields, was proposed by Shamir [1,2] anticipating superior features over other quark formulations: no need of the fine tuning to realize the chiral limit and no restriction for the number of flavors. Recent simulation results seem to support the former feature non-perturbatively [3,4]. It is also perturbatively shown that the massless mode at the tree level still remains stable against the quantum correction [6]. These advantageous features fascinate us to the application of the domain-wall quark for calculation of the three- and four-quark hadron matrix elements.

In order to convert the matrix elements obtained by lattice simulations to those defined in some continuum renormalization scheme(e.g., MS), we must know the renormalization factors connecting the lattice operators to the continuum counterparts defined in some renormalization scheme. In this article we make a perturbative calculation of the renormalization factors for the three- and four-quark operators consisting of physical quark fields in the domain-wall QCD(DWQCD). This work is an extension of the previous paper [7], in which we developed a perturbative renormalization procedure for DWQCD demonstrating the calculation of the renormalization factors for quark wave function, mass and bilinear operators. We focus on whether or not the renormalization of the three- and four-quark operators in DWQCD is free from the notorious operator mixing problem. In the Wilson case it is well known that the mixing problem is not adequately manipulated by the perturbation theory, leading to an “incorrect” value for the $B_K$ matrix element.

This paper is organized as follows. In Sec. II we briefly introduce the DWQCD action and the Feynman rules relevant for the present calculation to make this paper self-contained. In Sec. III our calculational procedure of the renormalization factors for the four-quark operators is described in detail. We also evaluate the renormalization factors for three-quark operators in Sec. IV. In Secs. II and IV numerical results for one-loop coefficients of the renormalization factors are given with and without the mean field improvement. In Sec. V, using our results, we analyze a renormalization factor for $B_K$. Our conclusions are summarized in Sec. VI.

The physical quantities are expressed in lattice units and the lattice spacing $a$ is suppressed unless necessary. We take SU($N$) gauge group with the gauge coupling $g$ and the second Casimir $C_F = \frac{N^2 - 1}{2N}$, while $N = 3$ is specified in the numerical calculations.
II. ACTION AND FEYNMAN RULES

We take the Shamir’s domain-wall fermion action \([1]\),

\[
S_{DW} = \sum_{n} \sum_{s=1}^{N_s} \left[ \frac{1}{2} \sum_{\mu} \left( \bar{\psi}(n)s(-r + \gamma_\mu)U_\mu(n)\psi(n + \mu)s + \bar{\psi}(n)s(-r - \gamma_\mu)U_\mu^\dagger(n - \mu)\psi(n - \mu)s \right) \\
+ \frac{1}{2} \left( \bar{\psi}(n)s(1 + \gamma_5)\psi(n)s_{s+1} + \bar{\psi}(n)s(1 - \gamma_5)\psi(n)s_{s-1} \right) + (M - 1 + 4r)\bar{\psi}(n)s\psi(n)s \right] \\
+ m \sum_{n} \left( \bar{\psi}(n)sP_R\psi(n)s_1 + \bar{\psi}(n)s_1P_L\psi(n)s \right),
\]

(1)

where \(n\) is a four dimensional space-time coordinate and \(s\) is an extra fifth dimensional or “flavor” index, the Dirac “mass” \(M\) is a parameter of the theory which we set \(0 < M < 2\) to realize the massless fermion at tree level, \(m\) is a physical quark mass, and the Wilson parameter is set to \(r = -1\). It is important to notice that we have boundaries for the flavor space; \(1 \leq s \leq N_s\). In our one-loop calculation we will take \(N_s \to \infty\) limit to avoid complications arising from the finite \(N_s\). \(P_{R/L}\) is a projection matrix \(P_{R/L} = (1 \pm \gamma_5)/2\). For the gauge part we employ a standard four dimensional Wilson plaquette action and assume no gauge interaction along the fifth dimension.

In the DWQCD the zero mode of domain-wall fermion is extracted by the “physical” quark field defined by the boundary fermions

\[
q(n) = P_R\psi(n)s_1 + P_L\psi(n)s_{N_s}, \\
\bar{q}(n) = \bar{\psi}(n)s_{N_s}P_R + \bar{\psi}(n)s_1P_L.
\]

(2)

We will consider the QCD operators constructed from this quark fields, since this field has been actually used in the previous simulations. Moreover our renormalization procedure is based on the Green functions consisting of only the “physical” quark fields, in which we have found that the renormalization becomes simple \([1]\).

Weak coupling perturbation theory is developed by expanding the action in terms of gauge coupling. The gluon propagator can be written as

\[
G^{AB}_{\mu\nu}(k) = \delta_{\mu\nu}\delta_{AB} \frac{1}{4 \sin^2(k/2) + \lambda^2}
\]

(3)

in the Feynman gauge with the infrared cut-off \(\lambda^2\), where \(\sin^2(k/2) = \sum_\mu \sin^2(k_\mu/2)\). Quark-gluon vertices are also identical to those in the \(N_s\) flavor Wilson fermion. We need only one gluon vertex for our present calculation:

\[
V^A_{\mu}(k, p)_{st} = -igT^A \{ \gamma_\mu \cos(-k_\mu/2 + p_\mu/2) - ir \sin(-k_\mu/2 + p_\mu/2) \} \delta_{st},
\]

(4)

where \(k\) and \(p\) represent incoming momentum into the vertex (see Fig. 1 of Ref. \([1]\)). \(T^A\) \((A = 1, \ldots, N^2 - 1)\) is a generator of color \(SU(N)\).

The fermion propagator originally takes \(N_s \times N_s\) matrix form in \(s\)-flavor space. In the present one-loop calculation, however, we do not need the whole matrix elements because we consider Green functions consisting of the physical quark fields. The relevant fermion propagators are restricted to following three types:
\[ \langle q(-p)\overline{q}(p) \rangle = \frac{-i\gamma_\mu \sin p_\mu + (1 - We^{-\alpha})m}{(1 - e^aW) + m^2(1 - We^{-\alpha})} = S_q(p), \]
\[ \langle q(-p)\overline{\psi}(p, s) \rangle = \frac{1}{F} \left( i\gamma_\mu \sin p_\mu - m \left( 1 - We^{-\alpha} \right) \right) \left( e^{-\alpha(N_s-s)}P_R + e^{-\alpha(s-1)}P_L \right) \]
\[ + \frac{1}{F} \left[ m \left( i\gamma_\mu \sin p_\mu - m \left( 1 - We^{-\alpha} \right) \right) - F \right] e^{-\alpha} \left( e^{-\alpha(s-1)}P_R + e^{-\alpha(N_s-s)}P_L \right), \]
\[ \langle \psi(-p, s)\overline{q}(p) \rangle = \frac{1}{F} \left( e^{-\alpha(N_s-s)}P_L + e^{-\alpha(s-1)}P_R \right) \left( i\gamma_\mu \sin p_\mu - m \left( 1 - We^{-\alpha} \right) \right) \]
\[ + \frac{1}{F} \left( e^{-\alpha(s-1)}P_L + e^{-\alpha(N_s-s)}P_R \right) e^{-\alpha} \left[ m \left( i\gamma_\mu \sin p_\mu - m \left( 1 - We^{-\alpha} \right) \right) - F \right] \]

with
\[ W = 1 - M - r \sum_\mu (1 - \cos p_\mu), \]
\[ \cosh(\alpha) = \frac{1 + W^2 + \sum_\mu \sin^2 p_\mu}{2|W|}, \]
\[ F = 1 - e^aW - m^2 \left( 1 - We^{-\alpha} \right), \]

where the argument \( p \) in the factors \( \alpha \) and \( W \) is suppressed.

In the perturbative calculation of Green functions the external quark momenta and masses are assumed to be much smaller than the lattice cut-off, so that we can expand the external quark propagators in terms of them. We have the following expressions as leading term of the expansion:
\[ \langle q\overline{q}\rangle(p) = \frac{1 - w_0^2}{i\phi + (1 - w_0^2)m}, \]
\[ \langle q\overline{\psi}_s(p) \rangle = \langle q\overline{q}\rangle(p) \left( w_0^{s-1}P_L + w_0^{N_s-s}P_R \right), \]
\[ \langle \psi_s\overline{q}(p) \rangle = \left( w_0^{s-1}P_R + w_0^{N_s-s}P_L \right) \langle q\overline{q}\rangle(p), \]

where \( w_0 = 1 - M \).

### III. Renormalization Factors for Four-Quark Operators

We consider the following four-quark operators:
\[ O_\pm = \frac{1}{2} \left[ (\bar{q}_1\gamma_\mu^L q_2)(\bar{q}_3\gamma_\mu^L q_4) \pm (\bar{q}_1\gamma_\mu^L q_2)(\bar{q}_3\gamma_\mu^L q_2) \right], \]
\[ O_1 = -C_F (\bar{q}_1\gamma_\mu^L q_2)(\bar{q}_3\gamma_\mu^R q_4) + (\bar{q}_1 T^A \gamma_\mu^L q_2)(\bar{q}_3 T^A \gamma_\mu^R q_4), \]
\[ O_2 = \frac{1}{2N} (\bar{q}_1\gamma_\mu^L q_2)(\bar{q}_3\gamma_\mu^R q_4) + (\bar{q}_1 T^A \gamma_\mu^L q_2)(\bar{q}_3 T^A \gamma_\mu^R q_4), \]

where \( \gamma_\mu^L,R = \gamma_\mu P_{L,R} \). Summation over repeated indices such as \( \mu \) and \( A \) is assumed. We note that \( q_i \) \( (i = 1, 2, 3, 4) \) are boundary quark fields in DWQCD. For the convenience of calculation we rewrite the above operators as
\[ \mathcal{O}_\pm = \frac{1}{2} \left[ 1 \otimes 1 \pm 1 \overline{\otimes} 1 \right]^{abcd} \left[ (\bar{q}_1 \gamma_\mu L q_2^b)(\bar{q}_3 \gamma_\mu R q_4^d) \right] , \quad (17) \]
\[ \mathcal{O}_1 = \frac{1}{2} \left[ -N1 \otimes 1 + 1 \overline{\otimes} 1 \right]^{abcd} \left[ (\bar{q}_1 \gamma_\mu L q_2^b)(\bar{q}_3 \gamma_\mu L q_4^d) \right] , \quad (18) \]
\[ \mathcal{O}_2 = \frac{1}{2} \left[ 1 \overline{\otimes} 1 \right]^{abcd} \left[ (\bar{q}_1 \gamma_\mu L q_2^b)(\bar{q}_3 \gamma_\mu R q_4^d) \right] , \quad (19) \]

where \( a, b, c, d \) are color indices, and \( \otimes, \overline{\otimes} \) represent the tensor structures in the color space:

\[ \left[ 1 \otimes 1 \right]^{abcd} \equiv \delta_{ab} \delta_{cd} , \quad (20) \]
\[ \left[ 1 \overline{\otimes} 1 \right]^{abcd} \equiv \delta_{ad} \delta_{cb} . \quad (21) \]

To derive these formula, we have used the Fierz transformation for \( \mathcal{O}_\pm \) and the formula

\[ \sum_A T^A \overline{T}^A = \frac{1}{2} \left[ -\frac{1}{N} 1 \otimes 1 + 1 \overline{\otimes} 1 \right] . \quad (22) \]

for \( \mathcal{O}_{1,2} \).

We calculate the following Green function:

\[ \langle \mathcal{O}_\Gamma \rangle_{\alpha\beta\gamma\delta}^{ijkl} \equiv \langle \mathcal{O}_\Gamma(q_1)_\alpha(q_2)_\beta(q_3)_\gamma(q_4)_\delta \rangle , \quad (23) \]

where \( \Gamma = \pm, 1, 2 \). Spinor indices are labeled by \( \alpha, \beta, \gamma, \delta \) and color ones by \( i, j, k, l \). Truncating the external quark propagators from \( \langle \mathcal{O}_\Gamma \rangle \), where we multiply \( \langle \mathcal{O}_\Gamma \rangle \) by \( ip_i + (1 - w_0^2) m \), we obtain the vertex functions, which is written in the following form up to the one-loop level

\[ (1 - w_0^2)^4 \left( \Lambda_\Gamma \right)_{\alpha\beta\gamma\delta}^{ijkl} = (1 - w_0^2)^4 \left( \Lambda_\Gamma^{(0)} + \Lambda_\Gamma^{(1)} \right)_{\alpha\beta\gamma\delta}^{ijkl} , \quad (24) \]

where the superscript \( (i) \) refers to the \( i \)-th loop level and the trivial factor \( (1 - w_0^2)^4 \) is factored out for the convenience. We suppress the external momenta \( p_i \) since the renormalization factor does not depend on them.

The tree level vertex functions \( \Lambda_\Gamma^{(0)} \) are given by

\[ \Gamma = \pm, \quad \frac{1}{2} \left[ \gamma_\mu L \otimes \gamma_\mu R \right]_{\alpha\beta\gamma\delta} \left[ 1 \otimes 1 \pm 1 \overline{\otimes} 1 \right]^{ijkl} , \quad (25) \]
\[ \Gamma = 1, \quad \frac{1}{2} \left[ \gamma_\mu L \otimes \gamma_\mu R \right]_{\alpha\beta\gamma\delta} \left[ -N1 \otimes 1 + 1 \overline{\otimes} 1 \right]^{ijkl} , \quad (26) \]
\[ \Gamma = 2, \quad \frac{1}{2} \left[ \gamma_\mu L \otimes \gamma_\mu R \right]_{\alpha\beta\gamma\delta} \left[ 1 \overline{\otimes} 1 \right]^{ijkl} , \quad (27) \]

where \( \otimes \) acts on the Dirac spinor space representing \( [\gamma_X \otimes \gamma_Y]_{\alpha\beta\gamma\delta} \equiv (\gamma_X)_{\alpha\beta} (\gamma_Y)_{\gamma\delta} \).

The one-loop vertex corrections are illustrated by six diagrams in Fig. 1, the sum of which yields the one-loop level vertex function

\[ \Lambda_\Gamma^{(1)} = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \left( I_\Gamma^0 + \ldots + I_\Gamma^\prime \right) . \quad (28) \]
In order to obtain the expressions for the integrands $I_1^a, \ldots, I_1^c$ we should note that the internal quark propagators appearing in the diagrams are multiplied by the damping factor which comes from eqs. (12) and (13). The following formula are useful.

$$\langle q\bar{\psi}_s \rangle \left( w_{0}^{s-1} P_L + w_{0}^{N_s} P_R \right) = \left( w_{0}^{s-1} P_R + w_{0}^{N_s} P_L \right) \langle \psi_s \bar{\psi} \rangle = \frac{i \gamma_\mu \sin p_\mu}{F \cdot F_0} \equiv \mathcal{C},$$

$$\langle q\bar{\psi}_s \rangle \left( w_{0}^{s-1} P_R + w_{0}^{N_s} P_L \right) = \left( w_{0}^{s-1} P_L + w_{0}^{N_s} P_R \right) \langle \psi_s \bar{\psi} \rangle = -\frac{1}{F_0} \equiv \tilde{G},$$

where $\tilde{F} = e^{-\alpha} - W$ and $\tilde{F}_0 = e^{\alpha} - w_0$. Here we set $m = p_\mu = 0$ for the internal propagator.

The contribution from Fig. 1a takes the form

$$I_1^a = \frac{1}{2} J_{a}^{AB} \left\{ \nabla_\mu (k) \mathcal{G}(k) + \bar{\nabla}_\mu (k) \tilde{\mathcal{G}}(k) \right\} \Gamma \left( \mathcal{G}(k) \nabla_\nu (k) + \tilde{\mathcal{G}}(k) \bar{\nabla}_\nu (k) \right) \otimes \Gamma_Y G^{\mu \nu}_Y (k),$$

where $\Gamma_X = \gamma_\mu L$, $\Gamma_Y = \gamma_\mu^R$ or $\gamma_\mu^L$, and the interaction vertices are

$$\nabla_\mu = -ig \gamma_\mu \cos (k_\mu / 2), \quad \bar{\nabla}_\mu = -ig \sin (k_\mu / 2).$$

The color factors are represented by $J_{a}^{AB}$, which are listed in Table II. In a similar way the contributions from Fig. 1b and Fig. 1c are given by

$$I_1^b = \frac{1}{2} J_{b}^{AB} \left\{ \nabla_\mu (k) \mathcal{G}(k) + \bar{\nabla}_\mu (k) \tilde{\mathcal{G}}(k) \right\} \Gamma \left( \mathcal{G}(k) \nabla_\nu (k) + \tilde{\mathcal{G}}(k) \bar{\nabla}_\nu (k) \right) \otimes \Gamma_Y G^{\mu \nu}_Y (k),$$

$$I_1^c = \frac{1}{2} J_{c}^{AB} \left\{ \nabla_\mu (k) \mathcal{G}(k) + \bar{\nabla}_\mu (k) \tilde{\mathcal{G}}(k) \right\} \Gamma \left( \mathcal{G}(k) \nabla_\nu (k) + \tilde{\mathcal{G}}(k) \bar{\nabla}_\nu (k) \right) \otimes \Gamma_Y G^{\mu \nu}_Y (k).$$

After a little algebra the expressions of $I_1^{a,b,c}$ are reduced to

$$I_1^a = \frac{1}{2} g^2 J_{a}^{AA} K \left[ T + A_{VA} \right] \left[ \Gamma_X \otimes \Gamma_Y \right],$$

$$I_1^b = -\frac{1}{2} g^2 J_{b}^{AA} \left[ T T_X \otimes \Gamma_Y + \cos^2 (k_\mu / 2) \sin^2 k_\alpha (\gamma_\mu \gamma_\alpha \Gamma_X) \otimes (\gamma_\mu^L \gamma_\alpha^R) \right],$$

$$I_1^c = \frac{1}{2} g^2 J_{c}^{AA} K \left[ T T_X \otimes \Gamma_Y + \cos^2 (k_\mu / 2) \sin^2 k_\alpha (\gamma_\mu \gamma_\alpha \Gamma_X) \otimes (\Gamma_Y^L \gamma_\alpha^R) \right],$$

where

$$K = \frac{1}{F^2 F_0^2 (4 \sin^2 (k/2) + \lambda^2)},$$

$$T = r \sin^2 (k/2) F^2 + r \sin^2 k F,$$

$$A_{VA} = \sum_\mu \cos^2 (k_\mu / 2) \sin^2 k_\mu.$$

In order to rewrite the second term of $I_1^{b,c}$ we apply the Fierz transformation:

$$\left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) \otimes \left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) = -\gamma_\nu^L \otimes \gamma_\nu^L = \gamma_\nu^L \otimes \gamma_\nu^L,$$

$$\left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) \otimes \left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) = -\gamma_\nu^L \otimes \gamma_\nu^L (1 - 2 \delta_{\alpha \nu} / (1 - 2 \delta_{\mu \nu})),

$$\left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) \otimes \left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) = -\gamma_\nu^L \otimes \gamma_\nu^L (1 - 2 \delta_{\alpha \nu} / (1 - 2 \delta_{\mu \nu})) + 2 \gamma_\nu^L \otimes \gamma_\nu^L \delta_{\alpha \nu},$$

$$\left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) \otimes \left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) = \left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) \otimes \left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) = 2 \gamma_\nu^L \otimes \gamma_\nu^L \delta_{\alpha \nu},$$

$$\left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) \otimes \left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) = \left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) \otimes \left( \gamma_\mu \gamma_\alpha \gamma_\nu \right) = 2 \gamma_\nu^L \otimes \gamma_\nu^L \delta_{\alpha \nu}.$$
where $[\gamma_X \otimes \gamma_Y]_{\alpha\beta;\gamma\delta} \equiv (\gamma_X)_{\alpha\delta} (\gamma_Y)_{\gamma\beta}$, and summation over $\nu$ is taken. The Fierz transformation is again used for the second equality. We omit the tensor term in eq. (44) since it vanishes in the integral.

Choosing $\Gamma_{X,Y} = \gamma^L_\nu$ in eqs. (53), (56) and (57) we first consider the case of $O_\pm$. After simplifying the expressions of the color factors we obtain

$$I^a_\pm = \frac{1}{2} g^2 K (T + A_{VA}) \left[ \gamma^L_\nu \otimes \gamma^L_\nu \right] \left[ (C_F + \frac{1}{2}) \tilde{1} \tilde{1} + \frac{1}{2N} \tilde{1} \tilde{1} \right],$$

(46)

$$I^b_\pm = -\frac{1}{2} g^2 K (T + A_{SP}) \left[ \gamma^L_\nu \otimes \gamma^L_\nu \right] \left[ (-\frac{1}{2N} \pm \frac{1}{2}) \tilde{1} \tilde{1} + (\frac{1}{2} \pm \frac{1}{2N}) \tilde{1} \tilde{1} \right],$$

(47)

$$I^c_\pm = \frac{1}{2} g^2 K (T + A_{VA}) \left[ \gamma^L_\nu \otimes \gamma^L_\nu \right] \left[ -\frac{1}{2N} \tilde{1} \tilde{1} + (\frac{1}{2} \pm C_F) \tilde{1} \tilde{1} \right].$$

(48)

The total contribution becomes

$$I^a_+ + I^b_+ + I^c_+ = \frac{1}{2} g^2 K \left[ \gamma^L_\nu \otimes \gamma^L_\nu \right]_{\alpha\beta;\gamma\delta} \left[ 1 \tilde{1} + 1 \tilde{1} \right]^{ij,kl} \times \{ TC_F + A_{VA} (C_F + \frac{1}{2} - \frac{1}{2N}) - A_{SP} (\frac{1}{2} - \frac{1}{2N}) \},$$

(49)

and

$$I^a_- + I^b_- + I^c_- = \frac{1}{2} g^2 K \left[ \gamma^L_\nu \otimes \gamma^L_\nu \right]_{\alpha\beta;\gamma\delta} \left[ 1 \tilde{1} - 1 \tilde{1} \right]^{ij,kl} \times \{ TC_F + A_{VA} (C_F - \frac{1}{2} - \frac{1}{2N}) + A_{SP} (\frac{1}{2} + \frac{1}{2N}) \},$$

(50)

where

$$A_{SP} = \cos^2 (k/2) \cdot \sin^2 k.$$  

(51)

We should note that the other three contributions $I^{q', b', c'}_\pm$ from Fig. 1a', 1b' and 1c' are equal to $I^{a, b, c}_\pm$ respectively, therefore the factors 1/2 in eqs. (43) and (40) disappear in the total contributions of all.

Comparing the one-loop results to the tree level ones we obtain

$$\Lambda_+ = \left[ 1 + g^2 \frac{N-1}{N} \left\{ <T>(N+1) + <A_{VA}>(N+2) - <A_{SP}> \right\} \right] \Lambda_+^{(0)},$$

(52)

$$\Lambda_- = \left[ 1 + g^2 \frac{N+1}{N} \left\{ <T>(N-1) + <A_{VA}>(N-2) + <A_{SP}> \right\} \right] \Lambda_-^{(0)}$$

(53)

with

$$<X> = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} K(k) X(k)$$

(54)

for $X = T, A_{VA}, A_{SP}$. We remark that $C_F < T + A_{VA} >$ and $C_F < T + A_{SP} >$ correspond to the one-loop vertex corrections to the (axial) vector current and the (pseudo) scalar density which are expressed as $(T_{VA} - 1)$ and $(T_{SP} - 1)$ in Ref. [1]. The expressions of eqs. (52) and (53) show an important property of $O_\pm$ in the DWQCD formalism: the one-loop vertex
corrections are multiplicative. This is contrary to the Wilson case, in which the mixing
operators with different chiralities appears at the one-loop level [8].

We next turn to the case of \( \mathcal{O}_{1,2} \). For \( \mathcal{O}_1 \) the vertex corrections of eqs.(55), (56) and (57) with \( \Gamma_X = \gamma^L_\mu \) and \( \Gamma_Y = \gamma^R_\mu \) are written as

\[
I^a_1 = \frac{1}{2} g^2 K (T + A_{VA}) \left[ \gamma^L_\mu \otimes \gamma^R_\nu \right] \left[ (-N C_F + \frac{1}{2}) 1 \tilde{\otimes} 1 - \frac{1}{2N} 1 \tilde{\otimes} 1 \right], \tag{55}
\]
\[
I^b_1 = -\frac{1}{2} g^2 K (T + A_{VA}) \left[ \gamma^L_\mu \otimes \gamma^R_\nu \right] \left[ \frac{1}{2} + \frac{1}{2} 1 \tilde{\otimes} 1 + \left( -\frac{N}{2} - \frac{1}{2N} \right) 1 \tilde{\otimes} 1 \right], \tag{56}
\]
\[
I^c_1 = \frac{1}{2} g^2 K (T + A_{SP}) \left[ \gamma^L_\mu \otimes \gamma^R_\nu \right] \left[ \frac{1}{2} 1 \tilde{\otimes} 1 + \left( -\frac{N}{2} + C_F \right) 1 \tilde{\otimes} 1 \right]. \tag{57}
\]

The total contribution including those from Fig. 1a', 1b' and 1c' is given by

\[
2 (I^a_1 + I^b_1 + I^c_1) = 2 \frac{1}{2} g^2 K \left[ \gamma^L_\mu \otimes \gamma^R_\nu \right]_{\alpha \beta \gamma \delta} \left[ -N 1 \tilde{\otimes} 1 + 1 \tilde{\otimes} 1 \right]^{ijkl} \times \left[ T C_F + A_{VA} \frac{N}{2} - A_{SP} \frac{1}{2N} \right]. \tag{58}
\]

Using the tree level result in eq.(52) the vertex function up to the one-loop level is expressed as

\[
\Lambda_1 = \left[ 1 + g^2 \frac{1}{N} \left\{ < T > (N^2 - 1) + < A_{VA} > N^2 - < A_{SP} > \right\} \right] \Lambda_1^{(0)}. \tag{59}
\]

This result shows that the operator \( \mathcal{O}_1 \) is multiplicatively renormalizable in DWQCD, which is in contrast with the Wilson case [8].

In a similar way we write the vertex corrections for \( \mathcal{O}_2 \).

\[
I^a_2 = \frac{1}{2} g^2 K (T + A_{VA}) \left[ \gamma^L_\mu \otimes \gamma^R_\nu \right] \left[ \frac{1}{2} 1 \tilde{\otimes} 1 - \frac{1}{2N} 1 \tilde{\otimes} 1 \right], \tag{60}
\]
\[
I^b_2 = -\frac{1}{2} g^2 K (T + A_{VA}) \left[ \gamma^L_\mu \otimes \gamma^R_\nu \right] \left[ \frac{1}{2} 1 \tilde{\otimes} 1 - \frac{1}{2N} 1 \tilde{\otimes} 1 \right], \tag{61}
\]
\[
I^c_2 = \frac{1}{2} g^2 K (T + A_{SP}) \left[ \gamma^L_\mu \otimes \gamma^R_\nu \right] \left[ C_F 1 \tilde{\otimes} 1 \right]. \tag{62}
\]

The total contribution including those from Fig. 1a', 1b' and 1c' becomes

\[
2 (I^a_2 + I^b_2 + I^c_2) = 2 \frac{1}{2} g^2 K \left[ \gamma^L_\mu \otimes \gamma^R_\nu \right]_{\alpha \beta \gamma \delta} \left[ 1 \tilde{\otimes} 1 \right]^{ijkl} \times C_F \left[ T + A_{SP} \right], \tag{63}
\]

which leads to

\[
\Lambda_2 = \left[ 1 + g^2 \frac{N^2 - 1}{N} \left\{ < T > + < A_{SP} > \right\} \right] \Lambda_2^{(0)}. \tag{64}
\]

We again find that the vertex correction is multiplicative up to the one-loop level as opposed to the Wilson case [8].

The contribution from the fermion self-energy has already been evaluated [8,9] and the total lattice renormalization factor is now obtained:
\[ Z_{\text{lat}}^\text{lat} = (1 - w_0^2)^2 Z^2 \Gamma^2 V, \]  
\[ \text{where} \]
\[ Z_2 = 1 + \frac{g^2}{16\pi^2} C_F \left[ \log(\lambda a)^2 + \Sigma_1 \right], \]  
\[ V_\Gamma = 1 + \frac{g^2}{16\pi^2} \left[ -\delta_\Gamma \log(\lambda a)^2 + v_\Gamma \right], \]
\[ v_\Gamma = \frac{16\pi^2(N^2 - 1)}{N} \left[ \langle T \rangle + \langle \langle A_{VA} \rangle \rangle \right] + \delta_1 \log \pi^2, \]  
\[ v_2 = \frac{16\pi^2(N^2 - 1)}{N} \left[ \langle T \rangle + \langle \langle A_{SP} \rangle \rangle \right] + \delta_2 \log \pi^2 \]

with
\[ \delta_\Gamma = \begin{cases} 
\frac{(N - 1)(N - 2)}{N} & \Gamma = + \\
\frac{(N + 1)(N + 2)}{N} & \Gamma = - \\
\frac{(N + 2)(N - 2)}{N} & \Gamma = 1 \\
\frac{4(N + 1)(N - 1)}{N} & \Gamma = 2 
\end{cases} \]

The infrared singularity of \( \langle A_X \rangle \) is subtracted as
\[ \langle \langle A_X \rangle \rangle = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left[ K(k)A_X(k) - c_X \frac{1}{(k^2)^2} \theta(\pi^2 - k^2) \right] \]

with \( c_{SP} = 4 \) and \( c_{VA} = 1 \).

Numerical values of \( v_\Gamma \) are evaluated by two independent methods. In one method the momentum integration is performed by a mode sum for a periodic box of a size \( L^4 \) after transforming the momentum variable through \( k_\mu = q_\mu - \sin q_\mu \). We employ the size \( L = 64 \) for integrals. In the other method the momentum integration is carried out by the Monte Carlo integration routine VEGAS, using 20 samples of 1000000 points each. We find that both results agree very well. Numerical values of \( v_\Gamma \) are presented in Table [II] as a function of \( M \).

We have to also calculate the corresponding continuum wave-function renormalization factor and vertex corrections in the \( \overline{\text{MS}} \) scheme employing the same gauge and the same infrared regulator as the lattice case. For the present calculation it seems preferable to choose
Dimensional Reduction (DRED) as the ultraviolet regularization, in which the loop momenta of the Feynman integrals are defined in $D < 4$ dimensions while keeping the Dirac matrices in four dimensions. In the DRED scheme we can use the same calculational techniques for the vertex corrections as the lattice case thanks to applicability of the Fierz transformation for the Dirac matrices. For the wave-function renormalization factor a simple calculation gives

$$Z_2^{\text{MS}} = 1 + \frac{g^2}{16\pi^2} C_F \left[ \log(\lambda/\mu)^2 - 1/2 \right],$$  \hspace{1cm} (72)

where $\mu$ is a renormalization scale. This result leads to $\Sigma_1^{\text{MS}} = -1/2$. For the vertex corrections we obtain

$$V_1^{\text{MS}} = 1 + \frac{g^2}{16\pi^2} \delta \Gamma \left[ -\log(\lambda/\mu)^2 + 1 \right],$$  \hspace{1cm} (73)

giving $\Sigma_1^{\text{MS}} = \delta \Gamma$. Here we should remark that the one-loop vertex corrections yield the evanescent operators which vanish in $D = 4$ for the DRED scheme. It is meaningless to give results without mentioning the definition of evanescent operators, because the constant terms at the one-loop level depend on the definition of the evanescent operators. Our choice is as follows:

$$E_{\pm}^{\text{DRED}} = \delta_{\mu\nu} \gamma_\mu (1 - \gamma_5) \otimes \gamma_\nu (1 - \gamma_5) - \frac{D}{4} \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu (1 - \gamma_5),$$  \hspace{1cm} (74)

$$E_{1,2}^{\text{DRED}} = \delta_{\mu\nu} \gamma_\mu (1 - \gamma_5) \otimes \gamma_\nu (1 + \gamma_5) - \frac{D}{4} \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu (1 + \gamma_5),$$  \hspace{1cm} (75)

where $\delta_{\mu\nu}$ is the $D$-dimensional metric tensor which emerges inevitably in the evaluation of the Feynman integrals.

Combining these results with the previous lattice ones we obtain

$$O_1^{\text{MS}}(\mu) = \frac{1}{(1 - w_0^2)^2 Z_w^2} Z_1(\mu a) O_1^{\text{lat}}(1/a),$$  \hspace{1cm} (76)

where

$$Z_1(\mu a) = \frac{(Z_2^{\text{MS}})^2 V_1^{\text{MS}}}{(Z_2)^2 V_1^\Gamma}$$

$$= 1 + \frac{g^2}{16\pi^2} \left[ (\delta \Gamma - 2C_F) \log(\mu a)^2 + z_1 \right],$$  \hspace{1cm} (77)

$$z_1 = v_1^{\text{MS}} - v_1 + 2C_F \{ \Sigma_1^{\text{MS}} - \Sigma_1 \}.$$  \hspace{1cm} (78)

Numerical values of $z_1$ are given in Table III and the results for the mean-field improved one, $z_1^{MF}$, are also given in Table IV.

Although the results for the DRED scheme are presented here, it is an easy task to obtain those for the Naive Dimensional Regularization (NDR) scheme. In Appendix B we summarize the finite parts of the wave-function renormalization factor and vertex corrections in the NDR scheme.
IV. RENORMALIZATION FACTORS FOR THREE-QUARK OPERATORS

The three-quark operators relevant to the proton decay amplitude are given by

\[ (O_{PD})_\delta = \varepsilon^{abc} \left( \left( \bar{q}_1^C \right)_a \Gamma_X (q_2)_b \right) \left( \Gamma_Y (q_3)_c \right)_\delta, \]  

(79)

where \( q^C = -q^T C^{-1} \) with \( C = \gamma_0 \gamma_2 \) is a charge conjugated field of \( q \) and \( \Gamma_X \otimes \Gamma_Y = P_R \otimes P_R, P_R \otimes P_L, P_L \otimes P_R, P_L \otimes P_L \). The summation over repeated color indices \( a, b, c \) is assumed. We should note that the domain-wall fermion action \( \Pi \) is transformed identically where \( \bar{\Pi} \) is written as follows:

\[ igT^A \to -ig(T^A)^T, \]

(80)

where the superscript \( T \) means the transposed matrix.

In order to evaluate the vertex corrections we consider the following Green function:

\[ \langle (O_{PD})_{\delta}^{ijk} \rangle_{i\beta j\gamma} \equiv \langle (O_{PD})_{\delta} \left( \bar{q}_1^C \right)_i (\bar{q}_2)_j (\bar{q}_3)_k \rangle, \]

(81)

where \( \alpha, \beta, \gamma \) and \( i, j, k \) are spinor and color indices respectively. Truncating the external quark propagators of \( \langle O_{PD} \rangle \) we obtain the vertex function

\[ (1 - w_0^2)^3 (\Lambda_{PD})^{ijk}_{\alpha\beta\gamma} = (1 - w_0^2)^3 \left( \Lambda_{\Gamma}^{(0)} + \Lambda_{\Gamma}^{(1)} \right)^{ijk}_{\alpha\beta\gamma}, \]

(82)

where the trivial factor \( (1 - w_0^2)^3 \) is factored out for the convenience. We suppress the external momenta \( p_i \) since the renormalization factor does not depend on them.

At the tree level the vertex function takes the form

\[ \Lambda_{\Gamma}^{(0)} = \varepsilon^{ijk} [\Gamma_X \otimes \Gamma_Y]_{\alpha\beta\gamma}, \]

(83)

where \( [\Gamma_X \otimes \Gamma_Y]_{\alpha\beta\gamma} \equiv (\Gamma_X)_{\alpha\beta} (\Gamma_Y)_{\gamma} \).

The one-loop vertex corrections are shown in Figs. 2a, 2b and 2c, the sum of which gives the one-loop level vertex function

\[ \Lambda_{\Gamma}^{(1)} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left( I_{PD}^a + I_{PD}^b + I_{PD}^c \right). \]

(84)

Using the notations in eqs. (29), (30) and (32), the integrands \( I_{PD}^{a,b,c} \) are written as follows:

\[ I_{PD}^a = \varepsilon^{abc} (-T^A)^{ia} T^{B}_{bj} \times \left\{ \left( \nabla_\mu (k) \bar{G}(k) + \bar{V}_\mu (k) \bar{G}(k) \right) \Gamma_X \left\{ \bar{G}(k) \nabla_\nu (k) + \bar{G}(k) \bar{V}_\nu (k) \right\} \otimes \Gamma_Y G^{AB}_{\mu\nu} (k), \right. \]

(85)

\[ I_{PD}^b = \varepsilon^{ibc} T^{A}_{bj} T^{B}_{ck} \times \Gamma_X \left\{ \bar{G}(k) \nabla_\nu (k) + \bar{G}(k) \bar{V}_\nu (k) \right\} \otimes \Gamma_Y \left\{ \bar{G}(-k) \nabla_\nu (-k) + \bar{G}(-k) \bar{V}_\nu (-k) \right\} G^{AB}_{\mu\nu} (k), \]

(86)

\[ I_{PD}^c = \varepsilon^{a} T^{A}_{ia} T^{B}_{ck} \times \left\{ \nabla_\mu (k) \bar{G}(k) + \bar{V}_\mu (k) \bar{G}(k) \right\} \Gamma_X \otimes \Gamma_Y \left\{ \bar{G}(k) \nabla_\nu (k) + \bar{G}(k) \bar{V}_\nu (k) \right\} G^{AB}_{\mu\nu} (k). \]

(87)
A little algebra yields

\[ I_{PA}^a = g^2 N + 1 \varepsilon^{ijk} K [T + A_{SP}] [\Gamma_X \otimes \Gamma_Y], \] (88)

\[ I_{PA}^b = g^2 N + 1 \varepsilon^{ijk} K [T(\Gamma_X \otimes \Gamma_Y) + \cos^2(k_\mu/2) \sin^2 k_\alpha (\Gamma_X \gamma_\alpha \gamma_\mu) \otimes (\Gamma_Y \gamma_\alpha \gamma_\mu)], \] (89)

\[ I_{PA}^c = g^2 N + 1 \varepsilon^{ijk} K [T(\Gamma_X \otimes \Gamma_Y) + \cos^2(k_\mu/2) \sin^2 k_\alpha (\gamma_\mu \gamma_\alpha \Gamma_X) \otimes (\Gamma_Y \gamma_\alpha \gamma_\mu)], \] (90)

where \( K, T \) and \( A_{SP} \) are given in eqs.(38), (39) and (51). It is noted that a sum of \( I_{PA}^b \) and \( I_{PA}^c \) becomes

\[ g^2 N + 1 \varepsilon^{ijk} K [2T(\Gamma_X \otimes \Gamma_Y) + \cos^2(k_\mu/2) \sin^2 k_\alpha (\gamma_\mu \gamma_\alpha \Gamma_X + \Gamma_X \gamma_\alpha \gamma_\mu) \otimes (\Gamma_Y \gamma_\alpha \gamma_\mu)], \] (91)

for \( \Gamma_X = P_R \) or \( P_L \), therefore no Fierz transformation is necessary to simplify the spinor structure of the total contribution. Finally we obtain

\[ I_{PA}^a + I_{PA}^b + I_{PA}^c = g^2 N + 1 \varepsilon^{ijk} K [\Gamma_X \otimes \Gamma_Y] [3T + A_{SP} + 2A_{VA}]. \] (92)

Compared with the tree level result of eq.(83) we find that the vertex correction is multiplicative up to the one-loop level:

\[ \Lambda_{PD} = \left[ 1 + g^2 N + 1 \varepsilon^{ijk} K [3 < T > + < A_{SP} > + 2 < A_{VA} >] \right] \Lambda_{PD}^{(0)}, \] (93)

where \( < X > (X = T, A_{VA}, A_{SP}) \) are defined in eq.(54). We remark that in the Wilson case \( O_{PD} \) mixes with other operators which have different chiral structures under renormalization [10].

Taking account of the contribution of the wave function the lattice renormalization factor for \( O_{PD} \) is expressed as

\[ Z_{PD}^{lat} = (1 - w_0^2)^{3/2} Z_0^{3/2} \Delta_{PD}, \] (94)

where

\[ V_{PD} = 1 + \frac{g^2}{16\pi^2} \left[ -\delta_{PD} \log(\lambda a)^2 + v_{PD} \right], \] (95)

\[ v_{PD} = \frac{16\pi^2(N + 1)}{2N} [3 < T > + < A_{SP} > + 2 < A_{VA} >] + \delta_{PD} \log \pi^2 \] (96)

with \( \delta_{PD} = \frac{6(N + 1)}{2N} \). Numerical values for \( v_{PD} \), evaluated as before, are given in Table IV as a function of \( \lambda \).

The corresponding continuum renormalization factors in the \( \overline{\text{MS}} \) scheme are calculated employing the DRED scheme as the regularization in the Feynman gauge with the fictitious gluon mass \( \lambda \). The vertex correction for \( O_{PD} \) is
\[ V_{PD}^{\text{MS}} = 1 + \frac{g^2}{16\pi^2} \delta_{PD} \left[ - \log(\lambda/\mu)^2 + 1 \right], \] (97)
giving \( v_{PD}^{\text{MS}} = \delta_{PD} \). We remark that in this case the evanescent operator does not appear at the one-loop level.

Combining this result with the previous lattice one we finally obtain the relation between the operators \( \mathcal{O}_{PD}^{\text{MS}} \) and \( \mathcal{O}_{PD}^{\text{lat}} \):
\[
\mathcal{O}_{PD}^{\text{MS}}(\mu) = \frac{1}{(1 - w_0^2)^{3/2} Z_{PD}^{3/2} Z_{PD}(\mu)} Z_{PD}(\mu) \mathcal{O}_{PD}^{\text{lat}}(1/a),
\] (98)
with
\[
Z_{PD}(\mu) = \left( \frac{Z_2^{\text{MS}} v_{PD}^{\text{MS}}}{Z_2^{\text{lat}}} \right) \left( \frac{1}{(1 - w_0^2)^{3/2} Z_{PD} Z_{PD}(\mu)} \right) \]
\[
= 1 + \frac{g^2}{16\pi^2} \left[ (\delta_{PD} - 3C_F/2) \log(\mu a)^2 + z_{PD} \right],
\] (99)
\[
z_{PD} = v_{PD}^{\text{MS}} - v_{PD} + \frac{3}{2} C_F \{ \Sigma_1^{\text{MS}} - \Sigma_1 \}. \] (100)

We present numerical values for \( z_{PD} \) in Table [III] and those for the mean-field improved one, \( z_{PD}^{\text{MF}} \), in Table [IV].

V. RENORMALIZATION FACTOR FOR \( B_K \)

As an application of results in the previous sections, we estimate a renormalization factor for the kaon \( B \) parameter \( B_K \), defined by
\[
B_K = \frac{\langle K^0 | \mathcal{O}_+ | K^0 \rangle}{\frac{2}{3} \langle K^0 | A_4 | 0 \rangle \langle 0 | A_4 | K^0 \rangle}
\] (101)
with \( q_1 = q_3 = s \) and \( q_2 = q_4 = d \) in \( \mathcal{O}_+ \).

Denoting the renormalization factor between the continuum \( B_K \) at scale \( \mu \) and the lattice one at scale \( 1/a \) as \( Z_{B_K}(\mu) \), we obtain
\[
Z_{B_K}(\mu a) = \frac{(1 - w_0^2)^{-2} Z_{w_0}^{-2} Z_{+}(\mu a)}{(1 - w_0)^{-2} Z_{w}^{-2} Z_{A}(\mu a)^2} = \frac{Z_{+}(\mu a)}{Z_{A}(\mu a)^2},
\] (102)
where
\[
Z_{+}(\mu a) = 1 + \frac{g^2}{16\pi^2} \left[ -4 \log(\mu a) + z_{+} \right]
\] (103)
from eq. (77) in this paper, and
\[
Z_{A}(\mu a) = 1 + \frac{g^2 C_F}{16\pi^2} z_{A}
\] (104)
from Ref. [4], so that
\[ Z_{B_K}(\mu a) = 1 + \frac{g^2}{16\pi^2} [-4 \log(\mu a) + z_+ - 2C_F z_A]. \] (105)

Note that \( z_A \) in Ref. [7] is evaluated in the NDR scheme while the DRED scheme is used for \( z_+ \) in this paper. From the result in Appendix B we have

\[ z_A(\text{DRED}) = z_A(\text{NDR}) + 1/2, \quad z_+(\text{DRED}) = z_+(\text{NDR}) + 3. \] (106)

In Ref. [3] \( B_K \) has been evaluated at \( \beta = 5.85, 6.0 \) with \( M = 1.7 \) and \( \beta = 6.3 \) with \( M = 1.5 \), using domain-wall QCD with the quenched approximation. Here we explicitly calculate \( Z_{B_K}(\mu a) \) for these parameters. From Table III and the previous result [7], \( z_+ = -41.854(-42.399), z_A = -17.039(-16.827) \) and \( z_+ - 2C_F z_A = 3.583(2.473) \) for \( M = 1.7(1.5) \) in the DRED scheme, and \( z_+ = -44.854(-45.399), z_A = -17.539(-17.327) \) and \( z_+ - 2C_F z_A = 1.917(0.8063) \) for \( M = 1.7(1.5) \) in the NDR scheme. Taking \( \mu = 1/a \) and \( g^2 = g^2_{\text{MS}}(1/a) \), estimated by the formula

\[ \frac{1}{g^2_{\text{MS}}} (1/a) = P\beta - 0.13486 \] (107)

for the quenched QCD with \( P \) being the average value of the plaquette, we have \( Z_{B_K} = 1.053 (1.029), 1.049 (1.026) \) and \( 1.030 (1.010) \) at \( \beta = 5.85, 6.0 \) and \( 6.3 \), respectively, in the DRED (NDR) scheme. Sizes of one-loop corrections for \( B_K \) are not so large, \( 1-5\% \), at these \( \beta \) values even without mean-field improvement, since the large contribution, which comes from a \( (1-w_0)Z_w \) factor, cancels out in the ratio of \( O_1^+ \) and \( A_1^2 \).

If we employ the mean-field improvement by replacing \( M \to \tilde{M} = M + 4(u - 1) \) with \( u = P^{1/4} \), we obtain \( Z_{B_K} = 1.018 (0.994), 1.017 (0.994) \) and \( 1.009 (0.988) \) at \( \beta = 5.85, 6.0 \) and \( 6.3 \), respectively, in the DRED (NDR) scheme. See Appendix A for some remarks.

Note that there is no mean-field improvement factor for \( B_K \) in actual simulations since, as mentioned before, it is defined by the ratio. Therefore the difference between values of \( Z_{B_K} \) with and without mean-field improvement comes from higher order ambiguity in perturbation theory.

Necessary informations for the analysis in this section are given in Table V, together with values of \( Z_{B_K} \).

### VI. CONCLUSION

In this paper we have calculated the one-loop contributions for the renormalization factors of the three- and four-quark operators in DWQCD. We have demonstrated that the three- and four-quark operators in DWQCD can be renormalized without any operator mixing between different chiralities as opposed to the Wilson case. This desirable property in DWQCD would practically surpass the cost of the introduction of an unphysical fifth dimension. The numerical values for the finite parts \( z_X \) with \( X = \pm, 1, 2, \text{PD} \) settle in reasonable magnitude with the mean-field improvement, while unimproved values are rather large in general.

In this work we do not treat the operators which yield the so-called “penguin” diagram. It seems feasible to carry out the calculation of their renormalization factors, which we leave to future investigation.
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APPENDIX A: MEAN-FIELD IMPROVEMENT

The mean-field improvement \([11]\) in our paper uses

\[ u = 1 - \frac{g^2 C_F}{2} T \]

with \( T = 0.15493 \), which is the value for the link in Feynman gauge. It may be better to use \( u \) from \( K_c \) or plaquette in DWQCD. In that case

\[ u = 1 - \frac{g^2 C_F}{2} (T + \delta T), \]

where \( \delta T = 0.00793 \) for \( K_c \), or \( \delta T = -0.02993 \) for plaquette. Accordingly we have to modify renormalization factors as follows:

\[
\begin{align*}
\Sigma_{w}^{\text{MF}}(T + \delta T) &= \Sigma_{w}^{\text{MF}}(T) + \frac{2w_0}{1 - w_0^3}16\pi^2 \times 2\delta T, \\
v_{+}^{\text{MF}}(T + \delta T) &= \frac{3}{2} \times \frac{3}{2} - \frac{2N - 3}{N + 2}, \\
v_{-}^{\text{MF}}(T + \delta T) &= \frac{3}{2} - \frac{2N + 3}{N - 2}, \\
v_{ij}^{\text{MF}, \text{bilinear}}(T + \delta T) &= \frac{3}{2} \times \frac{3}{2} - \frac{2(N^2 - 5)}{N^2 - 4}, \\
v_{e}^{\text{MF}, \text{3}}(T + \delta T) &= \frac{3}{2} \times \frac{3}{2} - \frac{2(N^2 - 5)}{N^2 - 4}, \\
v_{i}^{\text{MF}, \text{4}}(T + \delta T) &= \frac{3}{2} \times \frac{3}{2} - \frac{2(N^2 - 5)}{N^2 - 4}, \\
v_{PD}^{\text{MF}}(T + \delta T) &= \frac{3}{2} \times \frac{3}{2} - \frac{2(N^2 - 5)}{N^2 - 4}, \\
v_{+}^{\text{MF}}(T + \delta T) &= \frac{3}{2} \times \frac{3}{2} - \frac{2(N^2 - 5)}{N^2 - 4}, \\
v_{-}^{\text{MF}}(T + \delta T) &= \frac{3}{2} \times \frac{3}{2} - \frac{2(N^2 - 5)}{N^2 - 4}, \\
v_{ij}^{\text{MF}, \text{bilinear}}(T + \delta T) &= \frac{3}{2} \times \frac{3}{2} - \frac{2(N^2 - 5)}{N^2 - 4}, \\
v_{e}^{\text{MF}, \text{3}}(T + \delta T) &= \frac{3}{2} \times \frac{3}{2} - \frac{2(N^2 - 5)}{N^2 - 4}, \\
v_{i}^{\text{MF}, \text{4}}(T + \delta T) &= \frac{3}{2} \times \frac{3}{2} - \frac{2(N^2 - 5)}{N^2 - 4}, \\
v_{PD}^{\text{MF}}(T + \delta T) &= \frac{3}{2} \times \frac{3}{2} - \frac{2(N^2 - 5)}{N^2 - 4},
\end{align*}
\]

APPENDIX B: NAIVE DIMENSIONAL REGULARIZATION (NDR)

In this appendix we compile the finite part of the renormalization constant in the \( \overline{\text{MS}} \) subtraction scheme with the Naive Dimensional Regularization:

\[
\Sigma_{1}^{\overline{\text{MS}}} = 1/2,
\]

\[
v_{+}^{\overline{\text{MS}}} = \delta_+ \times \left\{ \frac{3}{2} - \frac{2N + 3}{N - 2} \right\},
\]

\[
v_{-}^{\overline{\text{MS}}} = \delta_- \times \left\{ \frac{3}{2} - \frac{2N - 3}{N + 2} \right\},
\]

\[
v_{ij}^{\overline{\text{MS}}} = \left( \delta_1 \times \left\{ 3/2 - \frac{2(N^2 - 5)}{N^2 - 4} \right\}, \delta_2 \times 3/4 \right),
\]

\[
v_{PD}^{\overline{\text{MS}}} = \delta_{PD} \times 2/3,
\]

\begin{align*}
\Sigma_{1}^{\overline{\text{MS}}} &= 1/2, \\
v_{+}^{\overline{\text{MS}}} &= \delta_+ \times \left\{ \frac{3}{2} - \frac{2N + 3}{N - 2} \right\}, \\
v_{-}^{\overline{\text{MS}}} &= \delta_- \times \left\{ \frac{3}{2} - \frac{2N - 3}{N + 2} \right\}, \\
v_{ij}^{\overline{\text{MS}}} &= \left( \delta_1 \times \left\{ 3/2 - \frac{2(N^2 - 5)}{N^2 - 4} \right\}, \delta_2 \times 3/4 \right), \\
v_{PD}^{\overline{\text{MS}}} &= \delta_{PD} \times 2/3,
\end{align*}
where \( v_{ij} \) with \( i, j = 1, 2 \) is a matrix, which represents the mixing of the finite part for \( O_{1,2} \). The one-loop vertex corrections for \( O_\Gamma \) (\( \Gamma = \pm, 1, 2 \)) require to specify their evanescent operators, which originates from the property that the Fierz transformation can not be defined in the NDR scheme. We employ

\[
E_{\pm}^{\text{NDR}} = \gamma_\rho \gamma_\sigma \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu (1 - \gamma_5) \gamma_\delta \gamma_\rho - (2 - D)^2 \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu (1 - \gamma_5),
\]

\[
E_{1,2}^{\text{NDR}} = \gamma_\rho \gamma_\sigma \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu (1 + \gamma_5) \gamma_\delta \gamma_\rho - D^2 \gamma_\mu (1 - \gamma_5) \otimes \gamma_\mu (1 + \gamma_5),
\]

where \( D \) is the reduced space-time dimension. On the other hand, the evanescent operator does not appear in the one-loop vertex correction of \( O_{PD} \).

For later convenience, values of the finite part of quark bilinear operators are also given here. For NDR scheme

\[
z^{\overline{\text{MS}}}_{V,A} = 0, \quad z^{\overline{\text{MS}}}_{S,P} = 5/2, \quad z^{\overline{\text{MS}}}_{T} = 1/2,
\]

while for DRED scheme

\[
z^{\overline{\text{MS}}}_{V,A} = 1/2, \quad z^{\overline{\text{MS}}}_{S,P} = 7/2, \quad z^{\overline{\text{MS}}}_{T} = -1/2,
\]

where the evanescent operators are

\[
E_{\mu \nu}^{\text{DRED}} = \delta_{\mu \nu} \gamma_\nu - \frac{D}{4} \gamma_\mu,
\]

\[
E_{\mu \nu \gamma_5}^{\text{DRED}} = \delta_{\mu \nu} \gamma_\nu \gamma_5 - \frac{D}{4} \gamma_\mu \gamma_5.
\]
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TABLES

TABLE I. Color factors for $I_{\Gamma}^{a,b,c}$ ($\Gamma = \pm, 1, 2$).

| $\Gamma$ | $J_{a}^{AB}$ | $J_{b}^{AB}$ | $J_{c}^{AB}$ |
|---------|-------------|-------------|-------------|
| $\pm$   | $T^{A}T^{B}\otimes1 \pm T^{A}\otimes T^{B}$ | $T^{A}\otimes T^{B} \pm T^{A}\otimes T^{B}$ | $T^{A}\otimes T^{B} \pm T^{A}T^{B}\otimes1$ |
| 1       | $-N T^{A}T^{B}\otimes1 + T^{A}\otimes T^{B}$ | $-N T^{A}\otimes T^{B} + T^{A}\otimes T^{B}$ | $-N T^{A}\otimes T^{B} + T^{A}T^{B}\otimes1$ |
| 2       | $T^{A}\otimes T^{B}$ | $T^{A}\otimes T^{B}$ | $T^{A}T^{B}\otimes1$ |
TABLE II. Numerical values for $V_T$ ($\Gamma = \pm, 1, 2, PD$) as a function of $M$.

| $M$  | $V_+$     | $V_-$     | $V_1$      | $V_2$      | $V_{PD}$  |
|------|-----------|-----------|------------|------------|-----------|
| 0.05 | 13.9096(8)| 10.847(8) | 13.3992(19)| 8.805(12)  | 8.646(5)  |
| 0.10 | 13.5696   | 11.537    | 13.2309    | 10.182     | 8.992     |
| 0.15 | 13.2941   | 12.098    | 13.0948    | 11.301     | 9.273     |
| 0.20 | 13.0548   | 12.587    | 12.9768    | 12.275     | 9.518     |
| 0.25 | 12.8391   | 13.029    | 12.8708    | 13.155     | 9.740     |
| 0.30 | 12.6404   | 13.438    | 12.7734    | 13.970     | 9.946     |
| 0.35 | 12.4542   | 13.822    | 12.6822    | 14.734     | 10.139    |
| 0.40 | 12.2775   | 14.188    | 12.5960    | 15.462     | 10.323    |
| 0.45 | 12.1083   | 14.539    | 12.5135    | 16.160     | 10.499    |
| 0.50 | 11.9449   | 14.880    | 12.4341    | 16.837     | 10.671    |
| 0.55 | 11.7861   | 15.212    | 12.3571    | 17.496     | 10.838    |
| 0.60 | 11.6307   | 15.538    | 12.2819    | 18.143     | 11.002    |
| 0.65 | 11.4779   | 15.859    | 12.2081    | 18.780     | 11.164    |
| 0.70 | 11.3269   | 16.178    | 12.1354    | 19.412     | 11.325    |
| 0.75 | 11.1770   | 16.495    | 12.0634    | 20.041     | 11.485    |
| 0.80 | 11.0275   | 16.813    | 11.9917    | 20.670     | 11.645    |
| 0.85 | 10.8779   | 17.131    | 11.9201    | 21.300     | 11.806    |
| 0.90 | 10.7276   | 17.452    | 11.8484    | 21.935     | 11.968    |
| 0.95 | 10.5760   | 17.777    | 11.7762    | 22.578     | 12.133    |
| 1.00 | 10.4225   | 18.107    | 11.7033    | 23.230     | 12.300    |
| 1.05 | 10.2659   | 18.437    | 11.6278    | 23.885     | 12.466    |
| 1.10 | 10.1076   | 18.790    | 11.5547    | 24.579     | 12.646    |
| 1.15 | 9.9443    | 19.139    | 11.4768    | 25.269     | 12.822    |
| 1.20 | 9.7768    | 19.505    | 11.3981    | 25.990     | 13.007    |
| 1.25 | 9.6037    | 19.880    | 11.3165    | 26.732     | 13.198    |
| 1.30 | 9.4244    | 20.272    | 11.2323    | 27.504     | 13.396    |
| 1.35 | 9.2375    | 20.680    | 11.1446    | 28.309     | 13.603    |
| 1.40 | 9.0419    | 21.109    | 11.0530    | 29.153     | 13.820    |
| 1.45 | 8.8361    | 21.560    | 10.9567    | 30.042     | 14.049    |
| 1.50 | 8.6183    | 22.037    | 10.8547    | 30.983     | 14.291    |
| 1.55 | 8.3863    | 22.547    | 10.7464    | 31.987     | 14.550    |
| 1.60 | 8.1375    | 23.093    | 10.6301    | 33.063     | 14.827    |
| 1.65 | 7.8685    | 23.683    | 10.5043    | 34.227     | 15.127    |
| 1.70 | 7.5747    | 24.328    | 10.3668    | 35.496     | 15.454    |
| 1.75 | 7.2502    | 25.038    | 10.2149    | 36.897     | 15.814    |
| 1.80 | 6.8864    | 25.832    | 10.0440    | 38.463     | 16.216    |
| 1.85 | 6.4706    | 26.737    | 9.8482     | 40.247     | 16.675    |
| 1.90 | 5.9812    | 27.794    | 9.6168     | 42.337     | 17.210    |
| 1.95 | 5.3749    | 29.093    | 9.3281     | 44.907     | 17.867    |
TABLE III. Numerical values for \( z_\Gamma \) (\( \Gamma = \pm, 1, 2, \text{PD} \)) as a function of \( M \).

| \( M \) | \( z_+ \)          | \( z_- \)          | \( z_1 \)    | \( z_2 \)    | \( z_{\text{PD}} \) |
|-------|----------------|----------------|-------------|-------------|----------------|
| 0.05  | -49.908(10)    | -40.845(11)    | -48.397(8)  | -34.803(15) | -32.144(7)    |
| 0.10  | -49.332        | -41.300        | -47.994     | -35.945     | -32.314       |
| 0.15  | -48.847        | -41.651        | -47.647     | -36.583     | -32.437       |
| 0.20  | -48.416        | -41.949        | -47.338     | -37.637     | -32.539       |
| 0.25  | -48.025        | -42.215        | -47.057     | -38.341     | -32.629       |
| 0.30  | -47.663        | -42.461        | -46.796     | -38.993     | -32.713       |
| 0.35  | -47.325        | -42.693        | -46.553     | -39.605     | -32.792       |
| 0.40  | -47.007        | -42.918        | -46.326     | -40.191     | -32.870       |
| 0.45  | -46.706        | -43.137        | -46.111     | -40.758     | -32.948       |
| 0.50  | -46.419        | -43.354        | -45.908     | -41.311     | -33.027       |
| 0.55  | -46.145        | -43.571        | -45.716     | -41.855     | -33.108       |
| 0.60  | -45.883        | -43.790        | -45.534     | -42.395     | -33.191       |
| 0.65  | -45.631        | -44.012        | -45.361     | -42.933     | -33.279       |
| 0.70  | -45.388        | -44.239        | -45.196     | -43.473     | -33.370       |
| 0.75  | -45.153        | -44.472        | -45.040     | -44.017     | -33.467       |
| 0.80  | -44.927        | -44.712        | -44.891     | -44.569     | -33.570       |
| 0.85  | -44.708        | -44.961        | -44.750     | -45.130     | -33.679       |
| 0.90  | -44.496        | -45.220        | -44.617     | -45.704     | -33.795       |
| 0.95  | -44.290        | -45.491        | -44.490     | -46.292     | -33.918       |
| 1.00  | -44.091        | -45.776        | -44.372     | -46.899     | -34.051       |
| 1.05  | -43.898        | -46.070        | -44.260     | -47.517     | -34.190       |
| 1.10  | -43.710        | -46.392        | -44.157     | -48.181     | -34.347       |
| 1.15  | -43.528        | -46.723        | -44.061     | -48.853     | -34.510       |
| 1.20  | -43.352        | -47.079        | -43.973     | -49.565     | -34.688       |
| 1.25  | -43.180        | -47.457        | -43.893     | -50.308     | -34.880       |
| 1.30  | -43.014        | -47.862        | -43.822     | -51.094     | -35.088       |
| 1.35  | -42.853        | -48.296        | -43.760     | -51.924     | -35.315       |
| 1.40  | -42.697        | -48.763        | -43.708     | -52.808     | -35.561       |
| 1.45  | -42.545        | -49.269        | -43.666     | -53.751     | -35.831       |
| 1.50  | -42.399        | -49.818        | -43.635     | -54.763     | -36.127       |
| 1.55  | -42.257        | -50.417        | -43.617     | -55.857     | -36.453       |
| 1.60  | -42.119        | -51.074        | -43.611     | -57.044     | -36.813       |
| 1.65  | -41.985        | -51.800        | -43.621     | -58.343     | -37.214       |
| 1.70  | -41.854        | -52.607        | -43.646     | -59.776     | -37.663       |
| 1.75  | -41.725        | -53.513        | -43.690     | -61.372     | -38.170       |
| 1.80  | -41.595        | -54.541        | -43.753     | -63.172     | -38.748       |
| 1.85  | -41.460        | -55.727        | -43.838     | -65.237     | -39.417       |
| 1.90  | -41.311        | -57.124        | -43.946     | -67.666     | -40.207       |
| 1.95  | -41.121        | -58.840        | -44.074     | -70.652     | -41.177       |
TABLE IV. Numerical value for $z_{T}^{MF}$ ($\Gamma = \pm, 1, 2, PD$) as a function of $M$.

| $M$ | $z_{+}^{MF}$ | $z_{-}^{MF}$ | $z_{1}^{MF}$ | $z_{2}^{MF}$ | $z_{PD}^{MF}$ |
|-----|---------------|---------------|--------------|--------------|---------------|
| 0.05| -17.287(10)   | -8.224(11)    | -15.776(8)   | -2.182(15)   | -7.679(7)     |
| 0.10| -16.712       | -8.679        | -15.373      | -3.324       | -7.848        |
| 0.15| -16.226       | -9.030        | -15.027      | -4.232       | -7.972        |
| 0.20| -15.796       | -9.328        | -14.718      | -5.016       | -8.074        |
| 0.25| -15.404       | -9.594        | -14.436      | -5.720       | -8.164        |
| 0.30| -15.042       | -9.840        | -14.175      | -6.372       | -8.247        |
| 0.35| -14.705       | -10.073       | -13.933      | -6.98        | -8.326        |
| 0.40| -14.386       | -10.297       | -13.705      | -7.57        | -8.404        |
| 0.45| -14.085       | -10.516       | -13.490      | -8.13        | -8.482        |
| 0.50| -13.799       | -10.734       | -13.288      | -8.69        | -8.561        |
| 0.55| -13.525       | -10.951       | -13.096      | -9.23        | -8.642        |
| 0.60| -13.262       | -11.169       | -12.913      | -9.77        | -8.726        |
| 0.65| -13.010       | -11.391       | -12.740      | -10.3        | -8.813        |
| 0.70| -12.767       | -11.618       | -12.575      | -10.8        | -8.905        |
| 0.75| -12.532       | -11.851       | -12.419      | -11.3        | -9.002        |
| 0.80| -12.306       | -12.091       | -12.270      | -11.9        | -9.104        |
| 0.85| -12.087       | -12.340       | -12.129      | -12.5        | -9.213        |
| 0.90| -11.875       | -12.600       | -11.996      | -13.0        | -9.329        |
| 0.95| -11.670       | -12.871       | -11.870      | -13.6        | -9.453        |
| 1.00| -11.470       | -13.155       | -11.751      | -14.2        | -9.585        |
| 1.05| -11.278       | -13.449       | -11.639      | -14.8        | -9.725        |
| 1.10| -11.089       | -13.772       | -11.536      | -15.5        | -9.882        |
| 1.15| -10.908       | -14.103       | -11.440      | -16.2        | -10.044       |
| 1.20| -10.731       | -14.459       | -11.352      | -16.9        | -10.223       |
| 1.25| -10.559       | -14.836       | -11.272      | -17.6        | -10.414       |
| 1.30| -10.393       | -15.241       | -11.201      | -18.4        | -10.623       |
| 1.35| -10.232       | -15.675       | -11.139      | -19.3        | -10.849       |
| 1.40| -10.076       | -16.143       | -11.087      | -20.1        | -11.096       |
| 1.45| -9.925        | -16.648       | -11.045      | -21.13       | -11.365       |
| 1.50| -9.778        | -17.197       | -11.014      | -22.14       | -11.661       |
| 1.55| -9.636        | -17.796       | -10.996      | -23.23       | -11.987       |
| 1.60| -9.498        | -18.453       | -10.991      | -24.42       | -12.347       |
| 1.65| -9.364        | -19.179       | -11.000      | -25.72       | -12.749       |
| 1.70| -9.233        | -19.986       | -11.025      | -27.15       | -13.198       |
| 1.75| -9.104        | -20.892       | -11.069      | -28.75       | -13.705       |
| 1.80| -8.975        | -21.920       | -11.132      | -30.55       | -14.283       |
| 1.85| -8.840        | -23.106       | -11.217      | -32.61       | -14.952       |
| 1.90| -8.690        | -24.503       | -11.326      | -35.04       | -15.742       |
| 1.95| -8.500        | -26.219       | -11.453      | -38.03       | -16.711       |
| $\beta$ | 5.85 | 6.0 | 6.3 |
|---|---|---|---|
| $M$ | 1.70 | 1.70 | 1.50 |
| $P$ | 0.57506 | 0.59374 | 0.62246 |
| $g_{\text{MS}}^2(1/a)$ | 2.3484 | 2.1792 | 1.9278 |
| $\nu$ | 0.87082 | 0.87781 | 0.88823 |
| $\tilde{M}$ | 1.20 | 1.20 | 1.05 |
| $z_+$ | DRED | NDR | DRED | NDR | DRED | NDR |
| $z_A$ | -41.854 | -44.854 | -41.854 | -44.854 | -42.399 | -45.399 |
| $z_+ - 2C_Fz_A$ | 3.583 | 1.917 | 3.583 | 1.917 | 2.473 | 0.806 |
| $Z_{B_K}(\mu a = 1)$ | 1.053 | 1.029 | 1.049 | 1.026 | 1.030 | 1.010 |
| $z_{MF}^+$ | DRED | NDR | DRED | NDR | DRED | NDR |
| $z_{MF}^-$ | -17.033 | -20.033 | -17.033 | -20.033 | -17.580 | -20.580 |
| $z_{MF}^A$ | -6.853 | -7.353 | -6.853 | -7.353 | -6.864 | -7.364 |
| $z_{MF}^+ - 2C_Fz_{MF}^A$ | 1.242 | -0.425 | 1.242 | -0.425 | 0.724 | -0.943 |
| $Z_{MF}^+$ | DRED | NDR | DRED | NDR | DRED | NDR |
| $Z_{MF}^+$ | 1.018 | 0.994 | 1.017 | 0.994 | 1.009 | 0.988 |
FIG. 1. One-loop vertex corrections for the four-quark operator. $\alpha, \beta, \gamma, \delta$ and $i, j, k, l$ label Dirac and color indices respectively.
FIG. 2. One-loop vertex corrections for the three-quark operator. $\alpha, \beta, \gamma$ and $i, j, k$ label Dirac and color indices respectively.