CATEGORIES OF TWO-COLORED PAIR PARTITIONS
PART II: CATEGORIES INDEXED BY SEMIGROUPS

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Abstract. Within the framework of unitary easy quantum groups, we study an analogue of Brauer’s Schur-Weyl approach to the representation theory of the orthogonal group. We consider concrete combinatorial categories whose morphisms are formed by partitions of finite sets into disjoint subsets of cardinality two; the points of these sets are colored black or white. These categories correspond to “half-liberated easy” interpolations between the unitary group and Wang’s quantum counterpart. We complete the classification of all such categories demonstrating that the subcategories of a certain natural halfway point are equivalent to additive subsemigroups of the natural numbers; the categories above this halfway point have been classified in a preceding article. We achieve this using combinatorial means exclusively. Our work reveals that the half-liberation procedure is quite different from what was previously known from the orthogonal case.

INTRODUCTION

Given partitions of an upper and a lower finite sequence of two-colored points into disjoint sets, one can create new partitions of this kind by vertical and horizontal concatenation as well as exchanging the roles of upper and lower row. In this article we proceed with classifying the sets of partitions invariant under these operations. They resemble the structures introduced by Brauer [Bra37] in order to study the representation theory of the orthogonal group; see Section 2.3. The classification program of such sets of partitions was begun in [TW17a] and since continued in [MW18] and [Gro18]. See also [Fre17b]. Such categories are of importance in Banica and Speicher’s path ([BS09], [Web16] and [Web17a]) towards compact quantum groups in Woronowicz’s sence ([Wor87a], [Wor87b], [Wor91] and [Wor98]). However, we use combinatorial means exclusively. The quantum-algebraic implications of the combinatorial result are discussed in Section 9.

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We deal with subcategories of a specified category $\mathcal{P}_{2nb}^\circ$ of pair partitions which in addition conform with a certain rule on their coloration. In [MW18], we showed that a subcategory $\mathcal{S}_0$ of $\mathcal{P}_{2nb}^\circ$ exists such that for every category $\mathcal{C} \subseteq \mathcal{P}_{2nb}^\circ$ holds $\mathcal{S}_0 \subseteq \mathcal{C}$ or $\mathcal{C} \subseteq \mathcal{S}_0$. All the categories satisfying the latter condition we determined in [MW18]. In the present article we address the subcategories of $\mathcal{S}_0$ and classify them. For a short introduction to two-colored partitions, see Section 2.

1. Main Results

We define and characterize a class of categories of two-colored partitions equivalent to the additive subsemigroups of $\mathbb{N}_0$.

Main Theorem 1. (a) For each subsemigroup $D$ of $(\mathbb{N}_0, +)$, in short: $D \in \mathcal{D}$, a category of two-colored partitions is given by the set $\mathcal{I}_D$ of all two-colored pair partitions with the following properties satisfied when the partition is rotated to one line:

1. Each block contains one point each of every color.
2. Between the two legs of any block lie as many black points as white ones.
3. Two points of opposite color may not belong to crossing blocks if the following condition is met: The difference in the numbers of black and white points between them amounts to an element of $D$.

(b) The categories $(\mathcal{I}_D)_{D \in \mathcal{D}}$ are pairwise distinct.

(c) For all $D, D' \in \mathcal{D}$ holds

\[ D \subseteq D' \implies \mathcal{I}_D \supseteq \mathcal{I}_{D'} \]

In particular, $\mathcal{I}_{\mathbb{N}_0} \subseteq \mathcal{I}_D \subseteq \mathcal{I}_{\mathbb{N}}$ holds for every $D \in \mathcal{D}$.

(d) For every finite subset $w$ of $\mathbb{N}$ (with $0 \notin \mathbb{N}$), define:

\[ \text{Br}_\bullet(w) := \text{symmetry axis } A_{\text{vert}} \]

\[ \max(w) \]

\[ j \quad i \quad 0 \]

\[ \text{if } i \notin w: \text{ block crosses } A_{\text{hor}} \]

\[ \text{if } j \in w: \text{ block crosses } A_{\text{vert}} \]

For every $D \in \mathcal{D}$, the category $\mathcal{I}_D$ is generated

- by the partition $\text{Br}_\bullet(\mathbb{N}\setminus D)$ if $\mathbb{N}\setminus D$ is finite,
- by the partitions $\{\text{Br}_\bullet(\{1, \ldots, v\}\setminus D) \mid v \in \mathbb{N}\}$ if $\mathbb{N}\setminus D$ is infinite
- and, in both cases, in addition to that, by $\mathcal{D}$ if $0 \notin D$. 

At the same time, we show that these categories are in fact all categories of two-colored partitions that exist below the unique maximal element of the family.

**Main Theorem 2.** For every category $C$ with

$$\langle \emptyset \rangle \subseteq C \subseteq \mathcal{I}_\emptyset,$$

there exists $D \in \mathcal{D}$ such that $C = \mathcal{I}_D$. In particular, $\langle \emptyset \rangle = \mathcal{I}_{\mathbb{N}_0}$.

These two theorems warrant a corollary when combined with the results of [MW18].

For every $w \in \mathbb{N}_0$, denote by $S_w$ the set given by all two-colored pair partitions with the following properties satisfied once the partition is rotated to one line:

1. Each block contains one point each of every color.
2. Between the two legs of any block the difference in the numbers of black and white points is a multiple of $w$.

In [MW18] the following facts were shown: The sets $(S_w)_{w \in \mathbb{N}_0}$ are pairwise distinct categories of two-colored partitions with

$$S_0 = \bigcap_{w' \in \mathbb{N}} S_{w'} \subseteq S_w \subseteq S_1 = \langle \emptyset, \emptyset \rangle$$

for every $w \in \mathbb{N}_0$ and with

$$w\mathbb{Z} \subseteq w'\mathbb{Z} \implies S_w \subseteq S_{w'}$$

for all $w, w' \in \mathbb{N}_0$. For every $w \in \mathbb{N}$, the category $S_w$ is generated by

```
\begin{array}{c}
\bullet \\
\vdots \\
\bullet \\
\end{array}
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$w$ times

The category $S_0$ is cumulatively generated by the partitions

$$\emptyset \emptyset$$

and $\text{Br}_\bullet(\{ v \})$ for all $v \in \mathbb{N}$.

We combine the results from [MW18] with the above theorems, yielding a full classification.

**Corollary.** For every category $C$ with

$$\langle \emptyset \rangle \subseteq C \subseteq \langle \emptyset, \emptyset \rangle$$

there exist either $D \in \mathcal{D}$ such that $C = \mathcal{I}_D$ or $w \in \mathbb{N}_0$ such that $C = S_w$. The categories gathered in the two families $(\mathcal{I}_D)_{D \in \mathcal{D}}$ and $(S_w)_{w \in \mathbb{N}}$ are all pairwise distinct and for every $D \in \mathcal{D}$ and all $w \in \mathbb{N}_0$ holds

$$\langle \emptyset \rangle = \mathcal{I}_{\mathbb{N}_0} \subseteq \mathcal{I}_D \subseteq \mathcal{I}_\emptyset = S_0 \subseteq S_w \subseteq S_1 = \langle \emptyset, \emptyset \rangle.$$
2. Reminder on Two-Colored Partitions and their Categories

For a more detailed introduction to two-colored partitions and their categories, confer \[TW17a\], and, more specifically for this article, for a treatment of two-colored partitions with neutral blocks, including more examples and illustrations, see \[MW18\].

2.1. Two-Colored Partitions. By a (two-colored) partition we mean a combinatorial object specified by the following data: two finite sets, the upper and lower row, a total order on each of them (from left, less, to right, greater), an exhaustive decomposition into mutually disjoint subsets, the blocks, of the disjoint union of the upper and lower row (the points) and, lastly, a two-valued (black, ●, or white, ○) map on the points, assigning to every point its color.

If a block contains both upper and lower points, we call it a through block and a non-through block otherwise. We say that ○ and ● are inverse to each other.

Partitions are represented graphically by two parallel lines of black and white dots connected by a collection of strings. The set of all partitions is denoted by $P_{○●}$. A partition each of whose blocks has two elements is called a pair partition, an element of $P_{○●}^2$. We restrict ourselves to pair partitions in this article.

2.2. Operations. From given partitions $p$ and $p'$, a new partition, the tensor product $p \otimes p'$, is created by appending each row of $p'$ at the right end of the respective row of $p$.

By exchanging the roles of the upper and the lower row of $p \in P_{○●}$, we obtain the involution $p^*$ of $p$.

A pairing $(p,p')$ of partitions $p,p' \in P_{○●}$ is composable if the lower row of $p'$ and the upper row of $p$ agree in size and coloration. Under these conditions vertical concatenation is possible and yields the composition $pp'$ of $(p,p')$: The lower row of $p$ also becomes the lower row of $pp'$, whereas the upper row is carried over from $p'$; Existing non-through blocks of $p$ on the lower row and, likewise, of $p'$ on the upper row are retained; The other blocks of $pp'$ are induced by the partition $s$ which is the least upper bound of, on the one hand, $p'$ restricted to its lower and, on the other hand, $p$ restricted to its upper row; Namely, for every block $B$ of $s$, the points from the upper row of $p'$ and the lower row of $p$ whose former blocks had a non-empty intersection with $B$ form a block in $pp'$.

The color inversion of $p \in P_{○●}$ replaces ○ with ● and vice versa for all points.

Reversing the total orders of both rows of $p \in P_{○●}$ produces the reflection $\hat{p}$ of $p$. The color inversion of $\hat{p}$ is called the verticolor reflection $\tilde{p}$ of $p$.

Four basic kinds of rotation can be defined: To obtain $p'$, we remove the leftmost point $\alpha$ on the upper row of $p \in P_{○●}$ and add a point $\beta$ of the opposite color of $\alpha$ to the lower row left to its leftmost point. The point $\beta$ replaces $\alpha$ as far as the blocks are concerned. Transferring the rightmost point of the upper row to the right end
of the lower row gives the rotation $p^\lambda$. Analogously, $p^\xi$ and $p^\delta$ result from moving points up from the lower row, instead.

Defining $p^\circ := (p^\lambda)^\circ$ and $p^\bullet := (p^\lambda)^\bullet$ yields clockwise and counter-clockwise cyclic rotations.

Given $p \in \mathcal{P}^{\circ\bullet}$ and a set $S$ of points in $p$, the erasing $E(p, S)$ of $S$ from $p$ is obtained by removing $S$ and combining all blocks of $p$ which contained an element of $S$ into one new block.

2.3. Categories. A category (of partitions) is a subset of $\mathcal{P}^{\circ\bullet}$ which is closed under tensor products, compositions and involutions and contains the partitions $\emptyset$, $\begin{array}{c} \circ \cr \bullet \end{array}$ and $\begin{array}{c} \bullet \cr \circ \end{array}$. While categories are then also invariant under verticolor reflection as well as basic and cyclic rotations, this need not be the case for reflection and color inversion. For any set $\mathcal{G} \subseteq \mathcal{P}^{\circ\bullet}$, we denote the smallest category containing $\mathcal{G}$ by $\mathcal{G}$ and call it the category generated by $\mathcal{G}$. Categories of (uncolored) partitions were first introduced in [BS09]. Note that the composition of uncolored pair partitions with the same amount of upper and lower points yields the multiplication in the Brauer algebra [Bra37] or the Temperley-Lieb algebra [TL71] up to a scalar factor.

2.4. Orientation. On the points of $p \in \mathcal{P}^{\circ\bullet}$ with lower row $L$ and upper row $U$, a cyclic order, the orientation, is defined by the condition that it concur with the total order $\leq_L$ on $L$, but with the exact reverse of the total order $\leq_U$ on $U$, that the minimum of $\leq_U$ be succeeded by the minimum of $\leq_L$ and that the maximum of $\leq_U$ be preceded by the maximum of $\leq_L$. Intervals with respect to this cyclic order are denoted by, e.g., $[\alpha, \beta]_p$, $]\alpha, \beta[_p$, etc. for points $\alpha$ and $\beta$ in $p$. See [MW18, Sect. 3.1].

2.5. Sectors. Given a proper subset $S$ of the points of $p$ that can be written as an interval with respect to the cyclic order, we call the set containing exactly the first and last point of $S$ the boundary $\partial S$ of $S$. In contrast, the set $\text{int}(S) := S \setminus \partial S$ is referred to as the interior of $S$. If $\partial S$ is a block of $p$, the set $S$ is called a sector in $p$. The sectors $S'$ in $p$ with $S' \subseteq \text{int}(S)$ are the subsectors of $S$. See [MW18, Sect. 3.4].

2.6. Color Sum. Based on the native coloration, we define the normalized color of any given point of a partition to congrue with the native color in the case of a lower point, but to be the inverse color of any upper point.

The color sum $\sigma_p$ of $p \in \mathcal{P}^{\circ\bullet}$ is the signed measure with density 1 and $-1$ given to the normalized colors $\circ$ and $\bullet$ respectively. The null sets of $\sigma_p$ we call neutral.

Every category of partitions is closed under the erasing of neutral intervals. A neutral interval of length 2 is called a turn. See [MW18, Sect. 3.3].

2.7. Connectedness. Two blocks $B$ and $B'$ in $p \in \mathcal{P}^{\circ\bullet}$ are said to cross if there are four pairwise distinct points $\alpha, \beta \in B$ and $\gamma, \delta \in B'$ occurring in the order $(\alpha, \gamma, \beta, \delta)$ with respect to the orientation. If no two blocks cross in $p$, then we say that $p$ is non-crossing, in short: $p \in \mathcal{NC}^{\circ\bullet}$ (see [TW17a] for all subcategories of $\mathcal{NC}^{\circ\bullet}$).
We call the blocks $B$ and $B'$ connected if $B = B'$, if $B$ and $B'$ cross or if there are pairwise different blocks $B_1, \ldots, B_m$ in $p$ such that $B$ crosses $B_1$, such that $B_i$ crosses $B_{i+1}$ for every $i \in \mathbb{N}$ with $i < m$, and such that $B_m$ crosses $B'$.

The classes of this equivalence relation are the connected components of $p$. And we say that $p$ is connected if it has only a single connected component. Erasing the complement of any connected component $S$ of $p$ yields the factor partition of $S$. See [MW18, Sect. 3.2].

2.8. Pair Partitions with Neutral Blocks. We denote by $P_{2,\text{nb}}^\bullet$ the set of all pair partitions all of whose blocks are neutral sets. Furthermore, denote by $S_0$ the set of all $p \in P_{2,\text{nb}}^\bullet$ such that $\sigma_p(S) = 0$ for all sectors $S$ in $p$. See [MW18, Sect. 4 and Main Thm. 1] for more on $S_0$.

3. Definition of $I_D$ and Set Relationships

[Main Theorem 1 (c)]

To define the sets $I_D$, which are the subject matter of this article, we introduce the notion of color distance.

Definition 3.1. Let $p \in P_{2,\text{nb}}^\bullet$ be arbitrary and let $\alpha$ and $\beta$ be points in $p$. We call

$$\delta_p(\alpha, \beta) : = \begin{cases} 
\sigma_p([\alpha, \beta],p), & \text{if } \alpha \text{ and } \beta \text{ have different normalized colors,} \\
\sigma_p([\alpha, \beta]_p), & \text{if } \alpha \text{ and } \beta \text{ have the same normalized color,}
\end{cases}$$

the signed color distance from $\alpha$ to $\beta$ in $p$ and

$$d_p(\alpha, \beta) := |\delta_p(\alpha, \beta)|.$$

the (absolute) color distance from $\alpha$ to $\beta$ in $p$.

While only the absolute color distance is required to define the sets $I_D$, it is the signed color distance which enables the proofs. The following lemma shows that, given $p \in P_{2,\text{nb}}^\bullet$, the name “distance” is appropriately chosen for $\delta_p$ and $d_p$. Note that the set of all points of $p$, being the disjoint union of neutral blocks, is neutral as well.

Lemma 3.2. Let $\alpha$, $\beta$ and $\gamma$ be points in $p \in P_{2,\text{nb}}^\bullet$.

(a) It holds $\delta_p(\alpha, \alpha) = 0$.
(b) It holds $\delta_p(\alpha, \beta) = -\delta_p(\beta, \alpha)$.
(c) It holds $\delta_p(\alpha, \gamma) = \delta_p(\alpha, \beta) + \delta_p(\beta, \gamma)$.
(d) The map $d_p$ is a pseudo-metric on the set of points of $p$.

Proof. (a) The definition of $\delta_p$ yields $\delta_p(\alpha, \alpha) = \sigma_p([\alpha, \alpha],p) = 0$.
(b) We can rewrite the definition of $\delta_p$ as

$$\delta_p(\alpha, \beta) = \sigma_p([\alpha, \beta],p) + \frac{1}{2} \left( \sigma_p([\alpha]) - \sigma_p([\beta]) \right).$$

Using $\sigma_p([\alpha, \beta],p) = -\sigma_p([\beta, \alpha],p)$ now proves the claim.
(c) We compute, employing the formula for $\delta_p$ from the proof of Claim (b),
\[
\delta_p(\alpha, \beta) + \delta_p(\beta, \gamma) = \sigma_p([\alpha, \beta]) + \sigma_p([\beta, \gamma]) + \frac{1}{2}(\sigma_p([\alpha]) - \sigma_p([\beta])) + \frac{1}{2}(\sigma_p([\beta]) - \sigma_p([\gamma])).
\]
Thus, from $\sigma_p([\alpha, \beta]) + \sigma_p([\beta, \gamma]) = \sigma_p([\alpha, \gamma])$ follows the claim.

(d) Claim (d) is implied by the previous three. \[\square\]

**Remark 3.3.** Without the assumption $p \in P_{\mathbb{2}, \ab}$, Lemma 3.2 (a)–(c) remains true for arbitrary $p \in P_{\mathbb{2}}$ if we replace equality by congruence modulo $\Sigma(p) := \sigma_p(P_p)$, where $P_p$ denotes the set of all points of $p$.

For special partitions, color distance respects the block structure as the following lemma shows.

**Lemma 3.4.** Let $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$ be blocks in $p \in P_{\mathbb{2}, \ab}$. If $p \in S_0$, then
\[
\delta_p(\alpha, \alpha') = \delta_p(\alpha, \beta') = \delta_p(\beta, \alpha') = \delta_p(\beta, \beta') = 0,
\]
because $p \in S_0$ means $\delta_p(\alpha, \beta) = \delta_p(\alpha', \beta') = 0$, see Section 2.8. \[\square\]

Hence, for $p \in S_0$, we can actually regard $\delta_p$ and $d_p$ as defining color distances not only for points but also for blocks.

**Definition 3.5.** Let $B$ and $B'$ be two blocks in $p \in S_0$. We call
\[
\delta_p(B, B') := \delta_p(\alpha, \alpha') \quad \text{and} \quad d_p(B, B') := |\delta_p(B, B')|,
\]
where $\alpha \in B$, $\alpha' \in B'$, the signed respectively (absolute) color distance from $B$ to $B'$.

The properties of the signed and absolute color distance of points from Lemma 3.2 carry over to the signed and absolute color distance of blocks.

Now, we are in a position to define the sets $\mathcal{I}_D$ from the [Main Theorems].

**Definition 3.6.** For every subsemigroup $D$ of $(\mathbb{N}_0, +)$, denote by $\mathcal{I}_D$ the set of all partitions $p \in S_0$ such that, for all blocks $B$ and $B'$ in $p$, whenever $d_p(B, B') \in D$, the blocks $B$ and $B'$ do not cross each other in $p$.

For example, the two crossing blocks $B$ and $B'$ in the partition on the left hand side have color distance 2. All color distances occurring between blocks in the partition are 0, 1 or 2. There are also crossings between blocks of distance 1, but no blocks with color distance 0 cross each other, making the partition an element of $\mathcal{I}_{\mathbb{N}_0\setminus\{1, 2\}}$.

In comparison, all three blocks in the partition $\mathcal{I}_N$ have color distance 0, which is why this partition is not an element of $\mathcal{I}_{\mathbb{N}_0\setminus\{1, 2\}}$.\[\square\]
Part (c) of Main Theorem 1 follows immediately from Definition 3.6.

**Proposition 3.7.**  
(a) It holds \( \mathcal{I}_{\mathbb{N}_0} = \mathcal{NC} \cap \mathcal{P}_{2,\text{nb}} = (\emptyset) \).  
(b) It holds \( \mathcal{I}_\emptyset = \mathcal{S}_0 \).  
(c) For all subsemigroups \( D, D' \) of \( (\mathbb{N}_0, +) \) holds  
\[ D \subseteq D' \implies \mathcal{I}_D \supseteq \mathcal{I}_{D'} \].

**Proof.**  
(a) If blocks \( B \) and \( B' \) in \( p \in \mathcal{S}_0 \) may only cross if \( d(B, B') \notin \mathbb{N}_0 \), then \( p \) must be non-crossing. Conversely, recognize that \( \mathcal{NC} \cap \mathcal{P}_{2,\text{nb}} \subseteq \mathcal{S}_0 \) because \( \text{int}(S) \) must be a subpartition for every sector \( S \) in a non-crossing \( p \in \mathcal{P}_{2,\text{nb}} \), implying \( \sigma_p(S) = 0 \). It was shown in [TW17a, Proposition 3.3 a)] that \( \mathcal{NC} \cap \mathcal{P}_{2,\text{nb}} = (\emptyset) \).

(b) Likewise, the condition that blocks \( B \) and \( B' \) in \( p \in \mathcal{S}_0 \) are forbidden from crossing unless \( d(B, B') \in \mathbb{N}_0 \) is no restriction at all. So, \( \mathcal{I}_\emptyset = \mathcal{S}_0 \).

(c) If blocks \( B \) and \( B' \) may not cross in \( p \in \mathcal{S}_0 \) if \( d(B, B') \in D' \) and if \( D \subseteq D' \), then, especially, they cannot cross if \( d(B, B') \in D \).

\[ \square \]

4. Category Property of \( \mathcal{I}_D \)  

**[Main Theorem 1 (a)]**

In the following, it is convenient to gather together all the color distances which occur between crossing blocks in a given partition.

**Definition 4.1.** For all \( p \in \mathcal{S}_0 \) define  
\[ A(p) := \{ d_p(B, B') \mid B, B' \text{ crossing blocks in } p \} . \]

This notation can be used to express membership in one of the sets \( \mathcal{I}_D \) from Definition 3.6 more compactly:

**Remark 4.2.** For all subsemigroups \( D \) of \( (\mathbb{N}_0, +) \) holds  
\[ \mathcal{I}_D = \{ p \in \mathcal{S}_0 \mid A(p) \subseteq \mathbb{N}_0 \setminus D \} . \]

The next lemma shows how the map \( A \) behaves under category operations.

**Lemma 4.3.** Let \( p, p' \in \mathcal{S}_0 \) be arbitrary.  
(a) It holds \( A(p^*) = A(p) \).  
(b) It holds \( A(p \otimes p') = A(p) \cup A(p') \).  
(c) If \( (p, p') \) is composable, then \( A(pp') \subseteq A(p) \cup A(p') \).

**Proof.**  
(a) Exchanging the roles of the upper and the lower row of \( p \) does not affect color distances: Both the sign of the color sum measure and the cyclic order effectively reverse and the two effects cancel each other. Hence, \( A(p^*) = A(p) \).

(b) On the one hand, no crossings exist between the two subpartitions of \( p \otimes p' \) corresponding to \( p \) and to \( p' \) respectively. So all crossings in \( p \otimes p' \) stem from crossings either in \( p \) or in \( p' \). On the other hand, the color distances between crossings from \( p \) and \( p' \) are unaltered when passing to \( p \otimes p' \) because the subpartitions of \( p \otimes p' \) resulting from \( p \) and \( p' \) are neutral as a whole due to \( p, p' \in \mathcal{S}_0 \).
(c) Let \( A \) and \( B \) be two blocks crossing each other in \( pp' \). By passing to \( (pp')^* = (p')^*p^* \) if necessary, we can assume that both \( A \) and \( B \) intersect the lower row, i.e. each have a point in common with the set of all lower points. We treat the case of both \( A \) and \( B \) being through blocks. The other cases are similar.

In that situation, there exist \( m, n \in \mathbb{N} \) and sequences \( A_1, \ldots, A_m, B_1, \ldots, B_n \) of blocks in \( p \) and \( A_1', \ldots, A_m', B_1', \ldots, B_n' \) of blocks in \( p' \) such that the following conditions are met: Block \( A \) intersects both \( A_1 \) and \( A_m' \); Block \( B \) intersects both \( B_1 \) and \( B_n' \); If we identify the lower row of \( p' \) with the upper row of \( p \), then, for all \( i, j \in \mathbb{N} \) with \( i < m \) and \( j < n \), in each of the following four pairs of blocks the two blocks intersect each other: \( (A, A_i'), \ (A_i', A_{i+1}) \), \( (B_j, B_j') \) and \( (B_j', B_{j+1}) \).

![Diagram](image)

The fact that all the sectors of all the blocks \( A_1, A_1', \ldots, A_m, A_m', B_1, B_1', \ldots, B_n, B_n' \) are neutral can be used to prove by induction with the help of Lemma 3.2 (c) first

\[ 0 = \delta_p(A_{i_1}, A_{i_2}) = \delta_{p'}(A'_{i_1}, A'_{i_2}) = \delta_p(B_{j_1}, B_{j_2}) = \delta_{p'}(B'_{j_1}, B'_{j_2}) \]

for all \( i_1, i_2, j_1, j_2 \in \mathbb{N} \) with \( i_1, i_2 \leq m \) and \( j_1, j_2 \leq n \), and thus

\[ \delta_{pp'}(A, B) = \delta_p(A_i, B_j) = \delta_{p'}(A'_i, B'_j) \]

for all \( i, j \in \mathbb{N} \) with \( i \leq m \) and \( j \leq n \). Because \( A \) and \( B \) cross in \( pp' \), there must exist \( i, j \in \mathbb{N} \) such that \( A_i \) and \( B_j \) cross in \( p \) or \( A'_i \) and \( B'_j \) cross in \( p' \). That means \( d_{pp'}(A, B) \in A(p) \cup A(p') \), which concludes the proof.

We employ for all sets \( X \) and \( Y \), all maps \( f : X \to Y \) and all subsets \( S \subseteq X \) the notation \( f(S) := \{ f(x) \mid x \in S \} \). Moreover, for all systems \( X \) of sets use \( \cup X := \bigcup_{Y \in X} Y \).

**Remark 4.4.** For all sets \( S \subseteq S_0 \) holds \( \cup A((S)) = \bigcup A(S) \).

**Proof.** The set \( C = \{ p \in S_0 \mid A(p) \subseteq \bigcup A(S) \} \) satisfies \( S \subseteq C \). Because \( C \) is a category by Lemma 4.3, we conclude \( (S) \subseteq C \). That implies \( \bigcup A((S)) \subseteq \bigcup A(C) \subseteq \bigcup A(S) \).

Lemma 4.3 is the key to proving Part (a) of Main Theorem [1].

**Proposition 4.5.** For every subsemigroup \( D \) of \( (\mathbb{N}_0, +) \), the set \( D \) is a category of partitions.
Proof. For \( p, p' \in \mathcal{I}_D \) holds \( A(p) \cup A(p') \subseteq \mathbb{N}_0 \setminus D \) by Remark 4.2. Lemma 4.3 hence proves \( A(p^*) = A(p) \subseteq \mathbb{N}_0 \setminus D, A(p \otimes p') = A(p) \cup A(p') \subseteq \mathbb{N}_0 \setminus D \) and, if \( (p, p') \) is composable, \( A(pp') \subseteq A(p) \cup A(p') \subseteq \mathbb{N}_0 \setminus D \).

\( \square \)

Note that Lemma 4.4 gives no clue as to which sets \( \cup A(C) \) actually occur for categories \( C \in S_0 \). The fact that only subsemigroups of \( (\mathbb{N}_0, +) \) are possible requires an entirely different argument, to be given in the subsequent sections.

5. Reminder on Brackets

We recall the definitions and results from [MW18, Sect. 6] about bracket partitions required in the subsequent sections of this article. With the help of brackets we will be able to give explicit generators of the categories \( \mathcal{I}_D \) and classify all subcategories of \( S_0 \).

5.1. Brackets. All categories \( C \) with \( C \in \mathcal{P}^{\bullet}_{2,\text{nb}} \) will in fact be described solely in terms of the classes of the following equivalence relation.

**Definition 5.1.** [MW18, Def. 6.2] Given \( p, p' \in \mathcal{P}^{\bullet}_{2,\text{nb}} \) and sectors \( S \) in \( p \) and \( S' \) in \( p' \), we number the points in \( \text{int}(S) \) and \( \text{int}(S') \) with respect to the cyclic order. We say that \( (p, S) \) and \( (p', S') \) are equivalent if the following four conditions are met:

1. The sectors \( S \) and \( S' \) are of equal size.
2. The same normalized colors occur in the same order in \( S \) and \( S' \).
3. For all \( i \), the \( i \)-th point of \( S \) belongs to a block crossing \( \partial S \) in \( p \) if and only if the \( i \)-th point of \( S' \) belongs to a block crossing \( \partial S' \) in \( p' \).
4. For all \( i, j \), the \( i \)-th and \( j \)-th points of \( S \) form a block in \( p \) if and only if the \( i \)-th and \( j \)-th points of \( S' \) form a block in \( p' \).

In other words: \( p \) restricted to \( S \) coincides with \( p' \) restricted to \( S' \) (up to rotation).

Particular representatives of the equivalence classes are bracket partitions.

**Definition 5.2.** [MW18, Def. 6.1] We call \( p \in \mathcal{P}^{\bullet}_{2,\text{nb}} \) a bracket if \( p \) is projective, i.e. \( p = p^* \) and \( p^2 = p \), and if the lower row of \( p \) is a sector in \( p \).

**Definition 5.3.** [MW18, Def. 6.3] Let \( S \) be a sector in \( p \in \mathcal{P}^{\bullet}_{2,\text{nb}} \). We refer to the (uniquely determined) bracket \( q \) with lower row \( M \) which satisfies that \( (p, S) \) and \( (q, M) \) are equivalent as the bracket \( B(p, S) \) associated with \( (p, S) \).

Categories are closed under passing to associated brackets.

**Lemma 5.4.** [MW18, Lem. 6.4] For all sectors \( S \) in \( p \in \mathcal{P}^{\bullet}_{2,\text{nb}} \) holds \( B(p, S) \in \langle p \rangle \).
5.2. **Residual Brackets.** It can be seen that every category is generated by its set of brackets. But that result can be significantly refined to *residual brackets*. Recall the definition of the verticolor reflection \( \tilde{p} \) of a partition \( p \in \mathcal{P}_\circ\overline{\bullet} \) from Section 2.2 and of a turn from Section 2.6.

**Definition 5.5.** We call a partition \( p \in \mathcal{P}_\circ\overline{\bullet} \) *verticolor-reflexive* if \( \tilde{p} = p \).

Especially, verticolor-reflexive partitions have evenly many points in both their rows, which is why the following definition makes sense.

**Definition 5.6.** [MW18, Def. 6.10] We refer to a bracket \( p \) with lower row \( S \) as *dualizable* if \( p \in S_0 \), if \( p \) is verticolor-reflexive, if \( \text{int}(S) \) is non-empty and if the two middle points of \( \text{int}(S) \) form a turn and belong to through blocks.

Rotating a dualizable bracket cyclically by a quarter times the number of its points produces again a bracket (and both directions of rotation give identical partitions).

**Definition 5.7.** [MW18, Def. 6.11] For a dualizable bracket \( p \) with \( n \) points, we call the bracket \( p^\dagger = p\overline{\circ}^\dagger = p\overline{\circ}^\dagger \) the *dual bracket* of \( p \).

With the bracket \( p \in \mathcal{P}_{2,\text{nb}} \), being dualizable, so is its dual \( p^\dagger \) and it holds \((p^\dagger)^\dagger = p\) and \((p) = (p^\dagger)\).

**Definition 5.8.** [MW18, Def. 6.12]

(a) Let \( p \) be a bracket with lower row \( S \).

(1) We call \( p \) *residual of the first kind* if \( p \) is connected and if \( \text{int}(S) \) contains no turns of \( p \).

(2) We call \( p \) *residual of the second kind* if \( p \) is connected and dualizable and if \( \text{int}(S) \) contains exactly one turn of \( p \).

(3) We call \( p \) *residual* if \( p \) is residual of the first or the second kind.

(b) The set of all residual brackets is denoted by \( B_{\text{res}} \).

Let the lower row \( S \) of \( p \in B_{\text{res}} \) start with a point of color \( c \). If \( p \) is residual of the first kind, there exists \( w \in \mathbb{N} \) such that \( S \) is given by

\[
\begin{align*}
\overbrace{c \ldots c}^w & \quad \text{or} \quad \overbrace{\overline{c} \ldots \overline{c}}^w \\
\end{align*}
\]
whereas, if \( p \) is residual of the second kind, we find \( v \in \mathbb{N} \) such that \( S \) has the coloration
\[
\begin{array}{c}
\underbrace{c \ldots c}_v \underbrace{\overline{c} \ldots \overline{c}}_v \text{ or } \underbrace{c \overline{c} \ldots c}_v \underbrace{\overline{c} \ldots c}_v.
\end{array}
\]

With the set \( B_{\text{res}} \) of residual brackets we have found a “universal generator set”:

**Proposition 5.9.** [MW18, Prop. 6.13] For every category \( C \subseteq \mathcal{P}^{\circ\bullet}_{2,\text{nb}} \) holds
\[
C = (C \cap B_{\text{res}}).
\]

### 5.3. Bracket Arithmetics.

To further reduce the set \( B_{\text{res}} \), to filter out residual brackets generating the same categories (see Section 6), we need to know how to generate new residual brackets from old ones via category operations.

**Definition 5.10.** [MW18, Def. 6.14]

(a) If \( p \in \mathcal{P}^{\circ\bullet} \) is a bracket, the projective partition which is obtained from \( p \) by erasing in every row the left- and the rightmost point, is called the **argument** of \( p \).

(b) Conversely, for each projective \( a \in \mathcal{P}^{\circ\bullet}_{2,\text{nb}} \), and every color \( c \in \{\circ, \bullet\} \), denote by \( \text{Br}(c \mid a \mid \overline{c}) \) the bracket whose leftmost lower point is of color \( c \) and which has the argument \( a \).

We can define two ways of altering the starting color of a bracket while, in some sense, preserving its argument. Write \( \text{Id}(\circ) := \circ \) and \( \text{Id}(\bullet) := \bullet \).

**Definition 5.11.** [MW18, Def. 6.17]

For every \( c \in \{\circ, \bullet\} \) and projective \( a \in \mathcal{P}^{\circ\bullet}_{2,\text{nb}} \), we call
\[
\text{WIn} (\text{Br}(c \mid a \mid \overline{c})) := \text{Br}(\overline{c} \mid \text{Br}(c \mid a \mid \overline{c}) \mid c)
\]
the **weak inversion** and
\[
\text{SIn} (\text{Br}(c \mid a \mid \overline{c})) := \text{Br}(\overline{c} \mid \text{Id}(c) \otimes a \otimes \text{Id}(\overline{c}) \mid c)
\]
the **strong inversion** of \( \text{Br}(c \mid a \mid \overline{c}) \).

These two transformations can indeed be performed using category operations.
Lemma 5.12. [MW18, Lem. 6.18 d–f)] Let $p, p'$ be two brackets starting with the same color.

(a) It holds $(\langle \bigcirc \bullet \bigcirc \rangle) = (\langle \bigcirc \bullet \bigcirc \rangle) = (\langle \bigcirc \bullet \bigcirc \rangle) = (\langle \bigcirc \bullet \bigcirc \rangle)$.

(b) Weak inversion is a reversible category operation: $(p) = (\text{WIn}(p))$.

(c) Strong inversion is reversible as well, but it is only available in certain categories: $(p, \langle \bigcirc \bullet \bigcirc \rangle) = (\text{SIn}(p))$.

For further operations on the set of all brackets see [MW18, Lem. 6.18].

6. Minimal Brackets and Bracket Patterns

From Proposition 5.9 we know that categories $\mathcal{C} \subseteq \mathcal{P}_{2,\text{nb}}$ are generated by their sets $\mathcal{C} \cap \mathcal{B}_{\text{res}}$ of residual brackets. In this section we improve on this result by showing that subcategories $\mathcal{C}$ of $\mathcal{S}_0$ are determined by their sets of minimal brackets (Proposition 6.11), proper subsets of $\mathcal{C} \cap \mathcal{B}_{\text{res}}$. In addition, we prove that the set of minimal brackets contained in a given category is closed under three generic operations. In fact, in Proposition 6.16 (a) and (b) we will see that considering which minimal brackets beget which by means of these transformations is sufficient to prove the remaining Parts (b) and (d) of Main Theorem and Main Theorem 2.

6.1. Minimal Brackets. With the aim of refining Proposition 5.9 we advance from considering residual brackets to studying minimal brackets.

Definition 6.1. We call a bracket $p$ with lower row $S$ minimal if $p$ is connected and dualizable and if $S$ contains exactly one turn and starts with a $\bullet$-colored point. The set of all minimal brackets is denoted by $\mathcal{B}_{\text{min}}$.

It is immediate from the definition that minimal brackets are residual of the second kind. In fact, residual brackets of the first kind are of no concern to us since we want to classify subcategories of $\mathcal{S}_0$:

Lemma 6.2. Let $p \in \mathcal{P}_{2,\text{nb}}$ be a residual bracket. Then, $p \in \mathcal{S}_0$ if and only if $p$ is residual of the second kind.

Proof. If $p$ is residual of the first kind, then the property that $\text{int}(S)$ contains no turns necessitates $\sigma_p(S) \neq 0$, implying $p \notin \mathcal{S}_0$. On the other hand, any residual bracket of the second kind is in $\mathcal{S}_0$ by Definition 6.6.

Given a bracket $p \in \mathcal{P}_{2,\text{nb}}$ with lower row $S$, note an important difference in the definitions of $p$ being residual of the second kind and of $p$ being minimal: In both cases we ask a set to contain exactly one turn, but in the former this set is $\text{int}(S)$, while in the latter it is $S$. The following lemma explains the difference between the two classes of brackets in detail. Recall that we denote the color inversion of $q \in \mathcal{P}_{2,\text{nb}}$ by $\overline{q}$, see Section 2.2.

Lemma 6.3. The residual brackets $\mathcal{B}_{\text{res}} \cap \mathcal{S}_0$ of $\mathcal{S}_0$ are precisely the partitions $\langle \bigcirc \bullet \bigcirc \rangle, p, \overline{p}, \text{WIn}(p), \text{WIn}(\overline{p}), \text{SIn}(p), \text{SIn}(\overline{p})$. 

for minimal brackets \( p \in B_{\min} \).

**Proof.** Let \( q \in B_{\text{res}} \cap S_0 \) have lower row \( S \). Lemma 6.2 shows that \( q \) is residual of the second kind. Hence \( \text{int}(S) \) contains exactly one turn. That means that at most three turns can exist in all of \( S \), two of which must intersect \( \partial S \). If \( S \) has just one turn, then either \( q \) or \( \overline{q} \) is minimal already. So, suppose that \( S \) has three turns. If \( S \) has four points then, \( q = \overline{q} \) or \( q = \overline{\overline{q}} \). Hence, let \( S \) have at least six points. As \( q \) is verticolor-reflexive, either \( \text{int}(S) \) is a subsector of \( S \) or \( \partial(\text{int}(S)) \) intersects through blocks exclusively. In the first case we can write \( q = WIn(p) \) for some bracket \( p \), in the second \( q = SIn(p) \). The partition \( p \) inherits being residual of the second kind from \( q \). If the lower row of \( p \) starts with \( \bullet \)-colored point, \( p \) is minimal; otherwise \( \overline{p} \) is. \( \Box \)

In the light of the preceding result and Lemma 5.12 (a) we make the following distinction.

**Definition 6.4.** Let \( C \subseteq S_0 \) be a category of partitions.

(a) We say that \( C \) is in the **monoid case** if \( \overline{\overline{C}} \notin C \).

(b) We say that \( C \) is in the **non-monoid case** if \( \overline{\overline{C}} \in C \).

Then, we can draw the ensuing conclusion from the above characterization of the set \( B_{\text{res}} \cap S_0 \), refining Proposition 5.9.

**Proposition 6.5.** Let \( C \subseteq S_0 \) be a category.

(a) If \( C \) is in the monoid case, then \( C = \langle C \cap (B_{\min} \cup B_{\overline{\min}}) \rangle \).

(b) If \( C \) is in the non-monoid case, then \( C = \langle C \cap (B_{\min} \cup B_{\overline{\min}}), \overline{\overline{C}} \rangle \).

**Proof.** By Proposition 5.9 holds \( C = \langle C \cap B_{\text{res}} \rangle \). Now, Lemma 5.12 together with Lemma 6.3 yields the result. \( \Box \)

One goal of the following subsection is to show that we can omit \( B_{\overline{\min}} \) in the above result, see Proposition 6.11.

### 6.2. Bracket Patterns and Their Induced Partitions

To speak about individual minimal brackets, we introduce the language of *bracket patterns*. In the following, note our convention \( 0 \notin \mathbb{N} \) and \( 0 \in \mathbb{N}_0 \).

**Definition 6.6.** (a) A **bracket pattern** is a non-empty finite subset \( w \) of \( \mathbb{N} \).

(b) If \( w \) is a bracket pattern, we call \( \|w\| := \max(w) \) the **frame** of \( w \).

(c) For every bracket pattern \( w \) and every color \( c \in \{\circ, \bullet\} \), we define the **\( c \)-bracket** \( Br_c(w) \) of \( w \) as the unique residual bracket of the second kind with \( 2(\|w\| + 1) \) points in its lower row \( S \), exactly one turn in \( S \) and with the property that, if we label the points of the left half of \( S \) right to left from 0 to \( \|w\| \), the block at position \( k \) crosses the horizontal symmetry axis if \( k \notin w \) and crosses
the vertical symmetry axis if \( j \in w \).

\[
\begin{array}{c}
\text{symmetry axis } A_{\text{vert}} \\
\begin{array}{c}
\cdots \cdots \cdots \cdots \cdots \cdots \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{symmetry axis } A_{\text{hor}} \\
\begin{array}{c}
\cdots \cdots \cdots \cdots \cdots \cdots \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\|w\| \\
\begin{array}{c}
j \\
i \\
0 \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{if } j \notin w: \text{ block crosses } A_{\text{hor}} \\
\text{if } j \in w: \text{ block crosses } A_{\text{vert}}
\end{array}
\]

In this notation we can characterize the minimal brackets as follows.

**Lemma 6.7.** It holds \( \mathcal{B}_{\text{min}} = \{ \text{Br}_c(w) \mid w \text{ bracket pattern} \} \).

We define three operations on bracket patterns.

**Definition 6.8.** Let \( w \) and \( w' \) be bracket patterns.

(a) Let the *superposition of \( w \) and \( w' \)* be given simply by union of sets,

\[
w \cup w' := \{ i \mid i \in w \text{ or } i \in w' \}.
\]

(b) For all \( j \in w \), denote by

\[
\cap_j w := \{ i \mid i \in w, i \leq j \}
\]

the *\( j \)-projection of \( w \).*

(c) Lastly, define the *dual of \( w \)* by

\[
w^\dag := \{ \|w\| - i \mid i \in \mathbb{N}_0, i < \|w\|, i \notin w \}.
\]

To relate these operations on bracket patterns to category operations on the associated minimal brackets we need the following technical result.

**Lemma 6.9.** Let \( w \) be an arbitrary bracket pattern.

(a) It holds \( \|w\| = \|w^\dag\| \).

(b) It holds \( (w^\dag)^\dag = w \).

(c) It holds \( (w \cup (w^\dag))^\dag = w \cap (w^\dag) \).

**Proof.** (a) Since \( 0 \notin w \), the definition of \( w^\dag \) implies \( \|w\| = \|w^\dag\| \).

(b) Let \( i \in \mathbb{N} \) with \( i < \|w\| \) be arbitrary. Part [a] shows that it suffices to prove that \( i \in (w^\dag)^\dag \) if and only if \( i \in w \). Because \( 0 < i < \|w\| \), by definition of \( (w^\dag)^\dag \) the statement \( i \in (w^\dag)^\dag \) is equivalent to there existing \( j \in \mathbb{N} \) with \( j < \|w\| \) such that \( j \notin w^\dag \) and \( i = \|w\| - j \). In other words, \( i \in (w^\dag)^\dag \) if and only if \( \|w\| - i \notin w^\dag \). And,
by definition of \( w^\dagger \), an index \( j \in \mathbb{N} \) with \( j < \| w \| \) satisfies \( j \notin w^\dagger \) if and only if for all \( k \in \mathbb{N} \) with \( k < \| w \| \) holds \( k \in w \) whenever \( j = \| w \| - k \). Applying this to \( j := \| w \| - i \) shows that \( i \in (w^\dagger)^\dagger \) if and only if \( i \in w \).

(c) Once more, Part \([a]\) allows us to confine ourselves to proving \( i \in (w \cup (w^\dagger))^\dagger \) if and only if \( i \in w \cap (w^\dagger) \) for all \( i \in \mathbb{N} \) with \( i < \| w \| \). As seen in the proof of Part \([b]\) for such \( i \) the statement \( i \in (w \cup (w^\dagger))^\dagger \) is equivalent to \( \| w \| - i \notin w \cup (w^\dagger) \). This in turn is the same as saying both \( \| w \| - i \notin w \) and \( \| w \| - i \notin w^\dagger \). Using Part \([b]\) we can reformulate the last statement equivalently as \( \| w \| - i \notin (w^\dagger)^\dagger \) and \( \| w \| - i \notin \w^\dagger \). By the definitions of \( (w^\dagger)^\dagger \) and \( w^\dagger \), this is then true if and only if \( i \in w^\dagger \) and \( i \in w \), i.e. \( i \in w \cap (w^\dagger) \). □

Note that, as opposed to Claims \([a]\) and \([b]\), the colors on the left and the right hand side of the identities in Claim \([c]\) of the following lemma do not agree. Claim \([d]\) remedies that.

**Lemma 6.10.** Let \( w \) and \( w' \) be bracket patterns and \( c \in \{\circ, \bullet\} \).

- **(a)** It holds \( \text{Br}_c(w \cup w') \in \{\text{Br}_c(w), \text{Br}_c(w')\} \).
- **(b)** For all \( j \in w \) holds \( \text{Br}_c(\cap_j w) \in \{\text{Br}_c(w)\} \).
- **(c)** It holds \( \text{Br}_c(w^\dagger) = \text{Br}_c(w)^\dagger \). Hence, \( \{\text{Br}_c(w^\dagger)\} = \{\text{Br}_c(w)\} \).
- **(d)** It holds \( \{\text{Br}_c(w)\} = \{\text{Br}_c(w)\} \).

**Proof.**

(a) We can assume \( \| w' \| \leq \| w \| \). Then, the pairing

\[
(\text{Br}_c(w), \text{Id}(c) \otimes (\| w \| - \| w' \|) \otimes \text{Br}_c(w') \otimes \text{Id}(\overline{c}) \otimes (\| w \| - \| w' \|))
\]

is composable and the composition equals \( \text{Br}_c(w \cup w') \).

(b) The set of subsectors of the lower row \( S \) of \( \text{Br}_c(w) \) is totally ordered by \( \leq \). Let \( S_j \) denote the \( j \)-th smallest subsector of \( S \). Then the identity \( B(\text{Br}_c(w), S_j) = \text{Br}_c(\cap_j w) \) in conjunction with Lemma 5.4 proves the claim.

(c) This is clear from the definitions.

(d) We prove the claim by induction over the frame \( \| w \| \) of \( w \). For \( \| w \| = 1 \), the assertion is true because \( \text{Br}_c(\{1\}) = \{\circ, \bullet\} \) and \( \text{Br}_c(\{1\}) = \{\circ, \bullet\} \) are duals of each other. So, let \( \| w \| \geq 2 \) and suppose that the claim holds for all bracket patterns whose frame is at most \( \| w \| - 1 \). Let \( c \in \{\circ, \bullet\} \) be arbitrary. We show \( \text{Br}_c(w) \in C \simeq \{\text{Br}_c(w)\} \).

If \( w = w^\dagger \), then Part \([c]\) shows \( \text{Br}_c(w) = \text{Br}_c(w)^\dagger \in C \). Hence, we can assume \( w \setminus (w^\dagger) \neq \emptyset \) or \( w \setminus w^\dagger \neq \emptyset \). We first treat the most involved case of both \( w \setminus (w^\dagger) \neq \emptyset \) and \( w^\dagger \setminus w \neq \emptyset \). Define in this situation the three bracket patterns

\[
w_1 := \cap_{\| w \| \setminus (w^\dagger)} w \quad \text{and} \quad w_2 := \cap_{\| w \| \setminus w} (w^\dagger)
\]

and

\[
w_3 := (w_1 \cup (w^\dagger))^\dagger \cup w_2.
\]

In two steps we prove first \( \text{Br}_c(w_3^\dagger) \in C \) and then \( w = w_3^\dagger \).

**Step 1:** Part \([b]\) implies \( \text{Br}_c(w_1) \in C \). From \( \| w \| \in w \cap (w^\dagger) \) we infer \( \| w_1 \| = \| (w \setminus (w^\dagger)) \| < \| w \| \). Hence, by the induction hypothesis it follows \( \text{Br}_c(w_1) \in C \). By
Proposition 6.11. Let $\mathcal{C} \subseteq \mathcal{S}_0$ be a category.

(a) If $\mathcal{C}$ is in the monoid case, then $\mathcal{C} = (\mathcal{C} \cap \mathcal{B}_{\min})$.
(b) If $\mathcal{C}$ is in the non-monoid case, then $\mathcal{C} = (\mathcal{C} \cap \mathcal{B}_{\min}, \circ)$.

PROOF. Combine Proposition 6.5, Lemma 6.7 and Lemma 6.10 (d). □

Lastly, we note how the set of color distances occurring between the blocks of a minimal bracket can be expressed using the language of bracket patterns.

Definition 6.12. For every bracket pattern $w$, define the completion of $w$ by

$$A(w) := \{ j - i \mid j \in w, i \in \mathbb{N}_0, i \notin w, i < j \}.$$ 

No harm will come from overloading the symbol $A$ from Definition 4.1 as the following lemma shows.

Lemma 6.13. For all bracket patterns $w$ holds

$$A(\text{Br}(w)) = A(w).$$

PROOF. Follows immediately from Definitions 4.1, 6.6 and 6.12 □

6.3. Bracket Patterns of a Category. While Lemma 6.10 was crucial for proving Proposition 6.11 by showing that minimal brackets generate the same categories as their color inversions and vice versa, it also motivates the following notion.

Definition 6.14. Let $\mathfrak{B}$ be a set of bracket patterns.
(1) We call \( W \) a bracket pattern category if it is closed under superposition, dualisation and projections.

(2) By \( \langle W \rangle \) we denote the bracket pattern category generated by the set \( W \).

**Definition 6.15.** For every category \( C \subseteq S_0 \), we call the set

\[ \mathcal{B}_C := \{ w \mid w \text{ bracket pattern, } \text{Br}_*(w) \in C \} \]

the bracket patterns of \( C \).

In the light of Lemma 6.7, the above definition implies \( C \cap \mathcal{B}_{\min} = \{ \text{Br}_*(w) \mid w \in \mathcal{B}_C \} \) for all categories \( C \subseteq S_0 \). Thus, we draw the following conclusion.

**Proposition 6.16.** Let \( C \subseteq S_0 \) be a category.

(a) The bracket patterns \( \mathcal{B}_C \) of \( C \) form a bracket pattern category.

(b) Let \( \mathcal{G}_C \) be a set of bracket patterns satisfying \( \mathcal{B}_C = \langle \mathcal{G}_C \rangle \).

(1) If \( C \) is in the monoid case, then \( C = \{ \text{Br}_*(w) \mid w \in \mathcal{G}_C \} \).

(2) If \( C \) is in the non-monoid case, then \( C = \{ \text{Br}_*(w), \text{Br}_*(w)^0 \mid w \in \mathcal{G}_C \} \).

**Proof.**

(a) This is the combined result of all parts of Lemma 6.10.

(b) First, recognize that for all sets \( \mathcal{G} \) of bracket patterns holds

\[ \{ \text{Br}_*(w) \mid w \in \langle \mathcal{G} \rangle \} \subseteq \{ \text{Br}_*(w) \mid w \in \mathcal{G} \}, \]

again thanks to Lemma 6.10. Using this for the first inclusion below, we find

\[ \{ \text{Br}_*(w) \mid w \in \mathcal{B}_C \} = \{ \text{Br}_*(w) \mid w \in \langle \mathcal{G}_C \rangle \} \]

\[ \subseteq \{ \text{Br}_*(w) \mid w \in \mathcal{G}_C \} \subseteq \{ \text{Br}_*(w) \mid w \in \mathcal{B}_C \} \subseteq C. \]

Applying Proposition 6.11 we infer, if \( C \) is in the monoid case,

\[ C = \{ \text{Br}_*(w) \mid w \in \mathcal{B}_C \} \subseteq \{ \text{Br}_*(w) \mid w \in \mathcal{G}_C \} \subseteq C. \]

and, if \( C \) is in the non-monoid case,

\[ C = \{ \text{Br}_*(w), \text{Br}_*(w)^0 \mid w \in \mathcal{B}_C \} \subseteq \{ \text{Br}_*(w), \text{Br}_*(w)^0 \mid w \in \mathcal{G}_C \} \subseteq C, \]

which is what we needed to show. \( \square \)

The preceding result demonstrates that there exists an injection from the set of subcategories of \( S_0 \) on the one hand to two copies of the set of all bracket pattern categories on the other hand (one for the monoid case and one for the non-monoid case). In addition, Proposition 6.16 reveals that we can find generators of a given category \( C \subseteq S_0 \) by determining a generator of the corresponding bracket pattern category \( \mathcal{B}_C \). In Section 8 we will prove the injection to be surjective as well by combining Propositions 4.5 and 6.16 with the results of the following section. That will return the full classification of subcategories of \( S_0 \).
7. Classification of Bracket Pattern Categories

In Proposition 6.16 we saw that the problem of classifying subcategories of $S_0$ and finding their generators can be devolved to the analogous tasks for another, simpler class of combinatorial objects, bracket pattern categories (see Subsection 7.1). These two reduced problems are the object of this section. Consequently, it mentions no partitions at all and can be read entirely independently from the rest of the article.

7.1. Bracket Patterns and their Categories. For the convenience of the reader we repeat the relevant definitions (Definition 6.6, 6.8 and 6.14). Mind $0 \notin \mathbb{N}$, whereas $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

**Definition 7.1.** (a) A bracket pattern is a non-empty finite subset of $\mathbb{N}$ and $\|w\|$ denotes its largest element, its frame.

(b) We define three operations on bracket patterns $w, w'$.

1. Let the superposition of $w$ and $w'$ be given simply by union of sets,

   $$ w \cup w' := \{ i \mid i \in w \text{ or } i \in w' \}.
   $$

2. For all $j \in w$, denote by

   $$ \cap_j w := \{ i \mid i \in w, i \leq j \}
   $$

   the $j$-projection of $w$.

3. Lastly, define the dual of $w$ by

   $$ w^\dagger := \{ \|w\| - i \mid i \in \mathbb{N}_0, i < \|w\|, i \notin w \}.
   $$

(c) A set $\mathcal{W}$ of bracket patterns is called a bracket pattern category if it is closed under superposition, dualisation and projections in the above sense.

(d) For every set $\mathcal{W}$ of bracket patterns, denote by $\langle \mathcal{W} \rangle$ the bracket pattern category generated by $\mathcal{W}$. If $\mathcal{W} = \{ w \}$ for a bracket pattern $w$, we slightly abuse notation by writing $\langle w \rangle$ instead of $\langle \{ w \} \rangle$.

Note that $\emptyset$ is a bracket pattern category but not a bracket pattern. We repeat Lemma 6.9 stating elementary facts about these operations.

**Lemma 7.2.** Let $w$ be an arbitrary bracket pattern.

(a) It holds $\|w\| = \|w^\dagger\|$.

(b) It holds $(w^\dagger)^\dagger = w$.

(c) It holds $(w \cup (w^\dagger))^\dagger = w \cap (w^\dagger)$.

7.2. Completion of Bracket Patterns. Reiterating Definition 6.12 the following mapping will be key to classifying bracket pattern categories.

**Definition 7.3.** For every bracket pattern $w$, define the completion of $w$ by

$$ A(w) := \{ j - i \mid j \in w, i \in \mathbb{N}_0, i \notin w, i < j \}.$$
The next lemma shows especially that $A$ is extensive. Eventually it will be proven that $A$ is idempotent. However, note that, in general, $w \subseteq w'$ does not imply $A(w) \subseteq A(w')$ for bracket patterns $w$ and $w'$. For example, $\{4\} \subseteq \{1,2,4\}$, but $A(\{4\}) = \{1,2,3,4\} \notin \{1,2,4\} = A(\{1,2,4\})$.

**Lemma 7.4.** For every bracket pattern $w$ holds
\[ w \subseteq A(w), \quad 1 = \min(A(w)) \quad \text{and} \quad \|A(w)\| = \|w\|. \]
In particular, $A(w) \subseteq A(A(w))$.

**Proof.** The facts that $w \subseteq A(w)$, that $1 \leq \min(A(w))$ and that $\max(A(w)) = \|w\|$ are clear from the definition of $A(w)$. If $w = \{i \mid i \in \mathbb{N}, i \leq \|w\|\}$, then $1 \in A(w)$.

So suppose there exists $i \in \mathbb{N}$ such that $i \leq \|w\|$ and $i \notin w$. We can choose $i$ to be maximal such. Then, $i < \|w\|$ and $i + 1 \in w$. It follows $1 = (i + 1) - i \in A(w)$ by definition of $A(w)$. \hfill \square

We now show, amongst other things, that bracket pattern categories are invariant under the completion operation. This takes a few auxiliary results, some of which will also be used later on for other purposes.

**Lemma 7.5.** Let $w$ and $w'$ be bracket patterns and let $j \in w$ be arbitrary.
\[(a) \text{ It holds } A(w \cup w') \subseteq A(w) \cup A(w'). \]
\[(b) \text{ It holds } A(\cap_j w) \subseteq A(w). \]
\[(c) \text{ It holds } A(w^+) = A(w). \]

**Proof.** Claims \[(a) \] and \[(b) \] are immediate from the definition. To prove Part \[(c) \] it suffices to show $A(w^+) \subseteq A(w)$ since $(w^+)^+ = w$. So, let $j \in w^+$ and $i \in \mathbb{N}_0$ with $i \notin w^+$ and $i < j$, i.e. $j - i \in A(w^+)$, be arbitrary. By definition of $w^+$ we find $k \in \mathbb{N}_0$ with $k < \|w\|$, $k \notin w$ and $j = \|w\| - k$. Then, $i < j$ implies $k < \|w\| - i$. Moreover, since $\|w^+\| = \|w\|$ and $i \notin w^+$, we can infer $\|w\| - i \in (w^+)^+ = w$. Hence, $j - i = (\|w\| - k) - i = (\|w\| - i) - k$ yields $j - i \in A(w)$ as $\|w\| - i \in w$, $k \notin w$ and $k < \|w\| - i$. \hfill \square

The next two propositions are central tools in the proof of the classification of bracket pattern categories. The first one holds for all sets of bracket patterns, the latter only for bracket pattern categories. Recall $\cup X = \cup_{Y \in X} Y$ for all systems $X$ of sets.

**Proposition 7.6.** For all sets $\mathbb{W}$ of bracket patterns holds
\[ \bigcup A(\langle \mathbb{W} \rangle) = \bigcup A(\mathbb{W}). \]

Especially, $\bigcup A(A(\|w\|)) = A(w)$ for every bracket pattern $w$.

**Proof.** Lemma 7.5 shows $\bigcup A(\langle \mathbb{W} \rangle) \subseteq \bigcup A(\mathbb{W})$. The other inclusion holds since $\mathbb{W} \subseteq \langle \mathbb{W} \rangle$. \hfill \square

**Proposition 7.7.** If $\mathbb{W}$ is a bracket pattern category, then
\[ \bigcup A(\mathbb{W}) = \{\|w\| \mid w \in \mathbb{W}\}. \]
The inclusion \( \{ \| w \| \mid w \in \mathcal{W} \} \subseteq \bigcup A(\mathcal{W}) \) follows from Lemma 7.4. Conversely, let \( w \in \mathcal{W}, j \in w \) and \( i \in \mathbb{N}_0 \) with \( i < \| w \|, i \not\in w \) and \( i < j \), i.e., \( j - i \in A(w) \), be arbitrary. From \( j \in w \) follows \( (\cap_j w)^\dagger \in \mathcal{W} \). Now, \( i \not\in w \) and \( i < j \) imply \( j - i \in (\cap_j w)^\dagger \). Hence,
\[
j - i = \| \cap_j - i ((\cap_j w)^\dagger) \|
\]
and \( \cap_j - i ((\cap_j w)^\dagger) \in \mathcal{W} \) prove the claim. 

7.3. Generators of Bracket Pattern Categories. We now see, using these two results, that bracket pattern categories are indeed closed under the completion operation.

**Lemma 7.8.** For all bracket patterns \( w \) holds
\[
A(w) = \bigcup \{ \|w\| \in \|w\| \}.
\]

**Proof.** The set
\[
\tilde{w} := \bigcup \{ \|w\| \}
\]
is a bracket pattern contained in the category \( \|w\| \) because \( \|w\| \) is finite and closed under superposition. We show \( \tilde{w} = A(w) \).

According to Propositions 7.6 and 7.7,
\[
\{ \| w' \| \mid w' \in \|w\| \} = \bigcup A(\|w\|) = A(w).
\]

As \( \|w'\| \in w' \not\subseteq \tilde{w} \) for every \( w' \in \|w\| \), we conclude \( \tilde{w} \supseteq A(w) \).

Conversely, \( \tilde{w} \in \|w\| \) implies \( A(\tilde{w}) \in A(\|w\|) \) and thus \( A(\tilde{w}) \subseteq \bigcup A(\|w\|) = A(w) \).

By Lemma 7.4 holds \( \tilde{w} \subseteq A(\tilde{w}) \). Thus we have shown \( \tilde{w} \subseteq A(w) \), which completes the proof. 

In the following most important step of the classification we show that the completion allows us to give a full characterization of all singly generated bracket pattern categories.

**Lemma 7.9.** For every bracket pattern \( w \) holds
\[
\|w\| = \{ w' \mid w' \text{ bracket pattern}, A(w') \subseteq A(w) \}.
\]

**Proof.** It is clear that \( \|w\| \subseteq \{ w' \mid w' \text{ bracket pattern}, A(w') \subseteq A(w) \} \) because of Proposition 7.6. The reverse inclusion we prove in two steps. Let \( w' \) be a bracket pattern with \( A(w') \subseteq A(w) \).

**Case 1:** First, assume \( \|w'\| = \|w\| \). Then, we define
\[
\tilde{w} := \left( A(w)^\dagger \cup \bigcup_{i \in A(w) \setminus w'} (\cap_{\|w\| - i} A(w))^\dagger \right)^\dagger
\]
and show

(i) For all \( i \in A(w) \setminus w' \) holds \( \|w\| - i \in A(w) \) and \( \|\cap_{\|w\| - i} A(w)\| = \|w\| - i \).
(ii) \( \tilde{w} \in \|w\| \).
(iii) \( \|\tilde{w}\| = \|w\| \).
(iv) \( w' = \tilde{w} \).
Together (ii) and (iv) then prove \( w' = \tilde{w} \in \langle w \rangle \).

(i) Let \( i \in A(w) \setminus w' \) be arbitrary. Then, \( \| w \| - i \in A(w') \) since we have assumed \( \| w' \| = \| w' \| \in w' \) and \( i \notin w' \). Now, the other assumption \( A(w') \subseteq A(w) \) implies the first part \( \| w \| - i \in A(w) \) of Claim (i). The remaining assertion \( \cap_{|w|-i} A(w) \| = \| w \| - i \) is then clear by definition of the projection \( \cap_{|w|-i} A(w) \).

(ii) By Lemma 7.8 we know \( A(w) \in \langle w \rangle \). Using Claim (i) we can then conclude \( \cap_{|w|-i} A(w) \in \langle w \rangle \) for all \( i \in A(w) \setminus w' \), and thus \( \tilde{w} \in \langle w \rangle \) because \( \langle w \rangle \) is a bracket pattern category.

(iii) Lemma 7.2 (a) shows \( \| (\cap_{|w|-i} A(w))' \| = \| \cap_{|w|-i} A(w) \| \) for all \( i \in A(w) \setminus w' \), which implies

\[
\bigg\| \bigcup_{i \in A(w) \setminus w'} (\cap_{|w|-i} A(w))' \bigg\| = \max_{i \in A(w) \setminus w'} (\| w \| - i) < \| w \|
\]

by Claim (i) and Lemma 7.4. In contrast, Lemma 7.2 (a) guarantees \( \| A(w)' \| = \| A(w) \| = \| w \| \). So, both together imply \( \| \tilde{w}' \| = \| w \| \) and thus, using Lemma 7.2 (a) one last time, \( \| \tilde{w} \| = \| w \| \), as claimed.

(iv) Let \( j \in \mathbb{N} \) with \( j < \| w \| \) be arbitrary. We show \( j \in \tilde{w} \) if and only if \( j \in w' \). For all bracket patterns \( w_0 \) and all \( k \in \mathbb{N} \) with \( k < \| w_0 \| \) holds by definition of the dual: \( k \notin w_0 \) if and only if \( \| w_0 \| - k \notin w_0 \). Hence, for all \( i \in A(w) \setminus w' \) such that \( i < j \) follows by Claim (i)

\[
\| w \| - j \notin (\cap_{|w|-i} A(w))' \iff (\| w \| - i) - (\| w \| - j) \in \cap_{|w|-i} A(w) \iff j - i \in A(w).
\]

At the same time, \( \| w \| - j \notin (\cap_{|w|-i} A(w))' \). Combining the two, we conclude for all \( i \in A(w) \setminus w' \)

\[
\| w \| - j \notin (\cap_{|w|-i} A(w))' \iff i \neq j \text{ and } (i < j \implies j - i \in A(w)).
\]

Thus, using Claim (iii) we infer

\[
j \in \tilde{w} \iff \| w \| - j \notin A(w)' \text{ and for all } i \in A(w) \setminus w' : \| w \| - j \notin (\cap_{|w|-i} A(w))' \iff j \in A(w) \text{ and for all } i \in A(w) \setminus w' : i \neq j \text{ and } (i < j \implies j - i \in A(w)).
\]

We prove now that this last statement is equivalent to \( j \in w' \).

On the one hand, if \( j \in A(w) \) and \( i \neq j \) for all \( i \in A(w) \setminus w' \), then \( j \in w' \) since \( w' \subseteq A(w') \subseteq A(w) \) by Lemma 7.4 and assumption on \( w' \). Hence, \( j \in \tilde{w} \) implies \( j \in w' \).

Conversely, if \( j \in w' \), then it holds both \( j \in A(w) \) and \( j - i \in A(w') \subseteq A(w) \) for all \( i \in A(w) \setminus w' \) with \( i < j \) by definition of \( A(w') \) and, again, our assumption on \( w' \). So, \( j \in w' \) also implies \( j \in \tilde{w} \).

That concludes the proof of Claim (iv) and thus shows \( w' \in \langle w \rangle \) in case \( \| w' \| = \| w \| \).

**Case 2:** Now, suppose \( \| w' \| \neq \| w \| \). Lemma 7.4 and the assumption \( A(w') \subseteq A(w) \) guarantee \( \| w' \| = \| A(w') \| \leq \| A(w) \| \). Hence and because \( \| w' \| \in A(w') \subseteq A(w) \), we obtain a well-defined bracket pattern of \( \langle A(w) \rangle \) by putting

\[
\tilde{w} := \cap_{|w'|-i} A(w).
\]
We now apply the result of Case 1 to \((w', A(\tilde{w}))\) here in the roles of \((w', w)\) there. This is possible for the following two reasons: Firstly, \(\|w'\| = \|\tilde{w}\| = \|A(\tilde{w})\|\) by Lemma 7.4. Secondly, \(A(w') \subseteq A(A(\tilde{w}))\) holds as well: From \(A(w') \subseteq \tilde{w}\) because \(A(w') \subseteq A(w)\), \(\tilde{w} \subseteq A(w)\) and \(\|A(w')\| = \|w'\|\) follows \(A(w') \subseteq \tilde{w} \subseteq A(\tilde{w}) \subseteq A(A(\tilde{w}))\) where we have employed Lemma 7.4 twice. So, indeed, Case 1 yields \(w' \in \langle A(\tilde{w}) \rangle\). We conclude since \(\tilde{w} \in \langle A(w) \rangle\).

We can generalize Lemma 7.9 to characterize all bracket pattern categories. In addition, for finite bracket pattern categories which are finite we can identify a canonical generator.

**Proposition 7.10.** Let \(\mathcal{W}\) be an arbitrary bracket pattern category.

(a) It holds
\[
\mathcal{W} = \{w \mid w \text{ bracket pattern}, A(w) \subseteq \bigcup A(\mathcal{W})\}.
\]

(b) If \(\mathcal{W}\) is finite, then \(\bigcup \mathcal{W} \in \mathcal{W}\) and
\[
\mathcal{W} = \langle \bigcup \mathcal{W} \rangle.
\]

Especially, finite bracket pattern categories are singly generated.

**Proof.** (a) The inclusion \(\mathcal{W} \subseteq \{w \mid w \text{ bracket pattern}, A(w) \subseteq \bigcup A(\mathcal{W})\}\) is clear by definition. Conversely, let \(w\) be a bracket pattern with \(A(w) \subseteq \bigcup A(\mathcal{W})\). By Proposition 7.7, \(A(w) \subseteq \{\|w'\| \mid w' \in \mathcal{W}\}\). Because \(A(w)\) is finite, there exists a finite set \(\mathcal{W}_w \subseteq \mathcal{W}\) of bracket patterns such that \(A(w) \subseteq \{\|w'\| \mid w' \in \mathcal{W}_w\}\). As \(\mathcal{W}_w\) is finite and \(\mathcal{W}\) is closed under superpositions, \(\bigcup \mathcal{W}_w \in \mathcal{W}\). Because \(\{\|w'\| \mid w' \in \mathcal{W}_w\}\) \(\subseteq \bigcup \mathcal{W}_w\) by definition, \(A(w) \subseteq \bigcup \mathcal{W}_w\). By Lemma 7.4 then, \(A(w) \subseteq \bigcup \mathcal{W}_w \subseteq A(\bigcup \mathcal{W}_w)\). Lemma 7.9 hence shows \(w \in \langle \bigcup \mathcal{W}_w \rangle\). Now, \(\bigcup \mathcal{W}_w \in \mathcal{W}\) shows \(\langle \bigcup \mathcal{W}_w \rangle \subseteq \mathcal{W}\) and thus \(w \in \mathcal{W}\), which is what we needed to prove.

(b) Since \(\mathcal{W}\) is closed under finite superpositions, \(\bigcup \mathcal{W} \in \mathcal{W}\), which also implies \(\langle \bigcup \mathcal{W} \rangle \subseteq \mathcal{W}\). For the converse inclusion we use the characterization of \(\langle \mathcal{W} \rangle\) from Lemma 7.9. Hence, by Part (a) all we have to prove is \(\bigcup A(\mathcal{W}) \subseteq A(\bigcup \mathcal{W})\). But by Lemma 7.4 we know \(\{\|w\| \mid w \in \mathcal{W}\}\) \(\subseteq \bigcup \mathcal{W} \subseteq A(\bigcup \mathcal{W})\) and Proposition 7.7 completes the proof. \(\square\)

**Remark 7.11.** The proof of Proposition 7.10 (a) actually reveals \(\bigcup A(\mathcal{W}) = A(\bigcup \mathcal{W})\) for all finite bracket pattern categories \(\mathcal{W}\) since \(A(\bigcup \mathcal{W}) \in A(\mathcal{W})\).

As announced, we prove that the completion operation is idempotent, a fact we will need later.

**Lemma 7.12.** For all bracket patterns \(w\) holds \(A(A(w)) = A(w)\). In particular, \(\langle A(w) \rangle = \langle w \rangle\).
Proof. Combining $A(w) \in \langle w \rangle$, by Lemma 7.8 and, thus, $A(A(w)) \subseteq \bigcup A(\langle w \rangle) = A(w)$ by Proposition 7.6 shows $A(A(w)) \subseteq A(w)$. Lemma 7.4 guarantees $A(w) \subseteq A(A(w))$, from which the claim now follows. Finally, $\langle A(w) \rangle = \langle w \rangle$ holds by Lemma 7.9.

7.4. Link to Submonoids of $(\mathbb{N}_0, +)$. The definition of the completion gives it essential properties which help to reveal the nature of bracket pattern categories in the following result. Recall that a monoid is a semigroup with neutral element.

Lemma 7.13. (a) If $w$ is a bracket pattern, $\mathbb{N}_0 \setminus A(w)$ is a submonoid of $(\mathbb{N}_0, +)$.

(b) For every submonoid $M$ of $(\mathbb{N}_0, +)$ such that $\mathbb{N}_0 \setminus M$ is finite holds $A(\mathbb{N}_0 \setminus M) = \mathbb{N}_0 \setminus M$.

Especially, for a bracket pattern $w$ holds $w = A(w)$ if and only if $w = \mathbb{N}_0 \setminus M$ for some submonoid $M$ of $(\mathbb{N}_0, +)$.

Proof. (a) First, note that $0 \notin A(w)$ and hence $0 \in \mathbb{N}_0 \setminus A(w)$. Now, let $x, y \in \mathbb{N}_0 \setminus A(w)$ be arbitrary. If $\|w\| < x + y$, then $x + y \in \mathbb{N}_0 \setminus A(w)$ holds by Lemma 7.4. So, suppose $1 \leq x + y \leq \|w\|$ and $1 \leq y$. Then, $x < x + y$. Hence, if $x + y \in A(w)$ held, it would follow $y = (x + y) - x \in A(A(w))$ by definition of $A(A(w))$, contradicting $y \notin A(w)$, see Lemma 7.12. In conclusion, $x + y \in \mathbb{N}_0 \setminus A(w)$.

(b) By Lemma 7.4, we only need to show $A(\mathbb{N}_0 \setminus M) \subseteq \mathbb{N}_0 \setminus M$: Suppose $j \in \mathbb{N}_0 \setminus M$, $i \in \mathbb{N}_0$, $i \notin \mathbb{N}_0 \setminus M$ and $i < j$. If $j - i \in M$ were true, then, $M$ being a semigroup, the assumption $i \in M$ would imply $j = (j - i) + i \in M$, contradicting the other premise $j \notin M$. Hence, $j - i \notin \mathbb{N}_0 \setminus M$, which is what we needed to show.

The preceding result easily generalizes to arbitrary bracket pattern categories.

Proposition 7.14. For every bracket pattern category $\mathcal{W}$, the set $\mathbb{N}_0 \setminus \bigcup A(\mathcal{W})$ is a submonoid of $(\mathbb{N}_0, +)$.

Proof. As $\mathbb{N}_0 \setminus \bigcup A(\mathcal{W}) = \bigcap_{w \in \mathcal{W}} (\mathbb{N}_0 \setminus A(w))$ and as intersection respects the submonoid structure, Lemma 7.13 (a) proves the claim.

We will now show that the following sets comprise all possible bracket pattern categories and identify their generators.

Definition 7.15. For every submonoid $M$ of $(\mathbb{N}_0, +)$, we call

$$\mathcal{W}_M := \{ w \mid w \text{ bracket pattern}, A(w) \in \mathbb{N}_0 \setminus M \}$$

the bracket pattern category of $M$.

For the following lemma, note that we allow set differences $X \setminus Y$ even if $Y \notin X$.

Lemma 7.16. Let $\mathcal{M}$ denote the set of submonoids of $(\mathbb{N}_0, +)$.

(a) For every $M \in \mathcal{M}$, the system $\mathcal{W}_M$ is a bracket pattern category.
(b) It holds $\mathcal{W}_{\mathbb{N}_0} = \emptyset$. If $M \neq \mathbb{N}_0$, then

$$\mathcal{W}_M = \langle \mathbb{N}_0 \setminus M \rangle$$

if $\mathbb{N}_0 \setminus M$ is finite,

and

$$\mathcal{W}_M = \langle \{0, \ldots, v\} \setminus M \mid v \in \mathbb{N} \rangle$$

if $\mathbb{N}_0 \setminus M$ is infinite.

(c) For every $M \in \mathcal{M}$ holds

$$M = \mathbb{N}_0 \setminus \bigcup A(\mathcal{W}_M) = \mathbb{N}_0 \setminus \bigcup \mathcal{W}_M.$$

**Proof.** (a) Lemma 7.5 shows that $\mathcal{W}_M$ is a bracket pattern category for every submonoid $M \in \mathcal{M}$.

(b) The defining condition $A(w) \subseteq \emptyset$ of $\mathcal{W}_{\mathbb{N}_0}$ cannot be satisfied by any bracket pattern $w$. Hence, $\mathcal{W}_{\mathbb{N}_0} = \emptyset$. Now, let $M \in \mathcal{M} \setminus \{\mathbb{N}_0\}$ be arbitrary.

**Case 1:** First, let $\mathbb{N}_0 \setminus M$ be finite. In that case, the definition of $\mathcal{W}_M$ and Lemma 7.13 (b) imply

$$\mathcal{W}_M = \{w \mid w \text{ bracket pattern}, A(w) \subseteq A(\mathbb{N}_0 \setminus M)\}.$$

Lemma 7.9 hence allows us to conclude $\mathcal{W}_M = \langle \mathbb{N}_0 \setminus M \rangle$.

**Case 2:** If, on the other hand, $\mathbb{N}_0 \setminus M$ is infinite, we argue as follows: Let $v \in \mathbb{N}$ be arbitrary. The sets $M_v := \{0\} \cup \{i \mid i \in \mathbb{N}, i > v\}$ and $M \cup M_v$ are submonoids of $(\mathbb{N}_0, +)$. In addition, $\mathbb{N}_0 \setminus (M \cup M_v) = \{0, \ldots, v\} \setminus M$ is finite. We conclude

$$\mathcal{W}_{M \cup M_v} = \langle \{0, \ldots, v\} \setminus M \rangle$$

by Case 1. Moreover, by definition holds

$$\mathcal{W}_{M \cup M_v} = \mathcal{W}_M \cap \mathcal{W}_{M_v}.$$

This and the fact that

$$\mathcal{W}_{M_v} = \{w \mid w \text{ bracket pattern}, \|w\| \leq v\},$$

which holds by definition of $\mathcal{W}_{M_v}$ and Lemma 7.4, imply

$$\mathcal{W}_M = \mathcal{W}_M \cap \bigcup_{v \in \mathbb{N}} \mathcal{W}_{M_v} = \bigcup_{v \in \mathbb{N}} (\mathcal{W}_M \cap \mathcal{W}_{M_v}) = \bigcup_{v \in \mathbb{N}} \mathcal{W}_{M \cup M_v} = \bigcup_{v \in \mathbb{N}} \langle \{0, \ldots, v\} \setminus M \rangle \subseteq \langle \{0, \ldots, v\} \setminus M \mid v \in \mathbb{N} \rangle.$$

Conversely, by Lemma 7.13 (b) and definition of $\mathcal{W}_M$, the inclusion

$$A(\{0, \ldots, v\} \setminus M) = A(\mathbb{N}_0 \setminus (M \cup M_v)) = \mathbb{N}_0 \setminus (M \cup M_v) = \{0, \ldots, v\} \setminus M \subseteq \mathbb{N}_0 \setminus M$$

proves $\{0, \ldots, v\} \setminus M \in \mathcal{W}_M$ for every $v \in \mathbb{N}$. We have thus proven

$$\mathcal{W}_M = \langle \{0, \ldots, v\} \setminus M \mid v \in \mathbb{N} \rangle$$

as claimed.
From $\mathcal{M}_n = \emptyset$ follows $N_0 = N_0 \setminus A(\emptyset) = N_0 \setminus A(2^n)$. Now, let $M \in \mathcal{M} \setminus \{N_0\}$ be arbitrary. Once more, we distinguish two cases.

**Case 1:** Suppose $N_0 \setminus M$ is finite. Then, it follows

$$N_0 \setminus M = A(N_0 \setminus M) = \bigcup A(\{N_0 \setminus M\}^\bot) = \bigcup A(\mathcal{M}_M).$$

by Lemma 7.13 (b), Proposition 7.6 and Claim (b).

**Case 2:** Now, let $N_0 \setminus M$ be infinite. Because $A(\{0, \ldots, v\} \setminus M) = \{0, \ldots, v\} \setminus M$ as seen in the proof of Claim (b), we infer

$$N_0 \setminus M = \bigcup \{\{0, \ldots, v\} \setminus M \mid v \in \mathbb{N}\} = \bigcup A(\{0, \ldots, v\} \setminus M) \setminus \{0, \ldots, v\} \mid v \in \mathbb{N}\} = \bigcup A(\{0, \ldots, v\} \setminus M) \setminus \{0, \ldots, v\} \mid v \in \mathbb{N}\}$$

by Proposition 7.6 and Claim (b).

**Proposition 7.17.** There is a one-to-one correspondence between submonoids of $(\mathbb{N}_0, +)$ and bracket pattern categories, given by the map $M \mapsto \mathcal{M}_M$.

**Proof.** By Lemma 7.16 (a) the map $M \mapsto \mathcal{M}_M$ is well-defined. By Lemma 7.16 (c) it is injective. Finally, in combination, Propositions 7.10 (a) and 7.14 prove that every bracket pattern category $\mathcal{M}$ is of the form $\mathcal{M}_M(\mathcal{M})$ for the submonoid $M(\mathcal{M}) := N_0 \setminus A(\mathcal{M})$.

**Remark 7.18.** Submonoids $M$ of $(\mathbb{N}_0, +)$ with finite $N_0 \setminus M$, also known as numerical semigroups, are interesting as such (see [DGR13] for a survey on open problems in this area). Given a finitely generated bracket pattern category $\mathcal{M}$, the set $N_0 \setminus A(\mathcal{M})$ is the largest numerical semigroup disjoint from $\mathcal{M}$. (In fact, one can show $\mathcal{M}(\mathcal{M}_M) = \mathcal{M}_M(\mathcal{M})$.) Its gap set is $\mathcal{M}(\mathcal{M}_M)$, its genus $|A(\mathcal{M}_M)|$ and its Frobenius number $\|\mathcal{M}_M\|$.

8. Generators of $\mathcal{I}_D$ and Classification

[Main Theorems 1 (b), (d), 2]

As seen in Proposition 6.16, the mapping $\mathcal{C} \mapsto \mathcal{B}_C$ (see Definition 6.15) injects the set of all subcategories $\mathcal{C}$ of $\mathcal{S}_0$ into two copies of the set of all bracket pattern categories (one for the monoid- and one for the non-monoid case). Moreover, by Proposition 6.16 (b) for every category $\mathcal{C} \in \mathcal{S}_0$ each generator of the corresponding bracket pattern category $\mathcal{B}_C$ induces a generator of $\mathcal{C}$. We combine now with these results from Section 6 our knowledge about bracket pattern categories and their generators won in Section 7. By showing that the injection $\mathcal{I}_D \mapsto \mathcal{B}_C$ is also surjective we thus obtain a full classification of all subcategories of $\mathcal{S}_0$. At the same time, identifying which bracket pattern categories correspond to which subcategories of $\mathcal{S}_0$ yields generators of the latter.
Theorem 8.3. Let $B$ shown also be expressed as Brence that $B_r$ all of partitions $C$ ∈ $D$ category (Proposition 6.16 (a)). Moreover, $D \cup \{0\}$ is a submonoid of $(N_0, +)$, implying that also the set $\mathcal{W}_{D,0}(0)$ (Definition 7.15) is a bracket pattern category (see Lemma 7.16 (a)).

Lemma 8.1. Let $D$ be a subsemigroup of $(N_0, +)$.
(a) The category $\mathcal{I}_D$ is in the monoid case if and only if $0 \in D$.
(b) It holds $\mathcal{B}_D = \mathcal{W}_{D,0}(0)$.

Proof. (a) We prove the contraposition of the claim. It holds $A(\mathcal{S}_C) = \{0\}$, see the example after Definition 3.6. By Definition 6.4, the category $\mathcal{I}_D$ is in the non-monoid case if and only if $\mathcal{S}_C \in \mathcal{I}_D$. In turn, by Remark 4.2 of $\mathcal{I}_D$ that is true if and only if $\{0\} = A(\mathcal{S}_C) \not\subset N_0 \setminus D$. But the latter statement is equivalent to $0 \not\in D$.
(b) For every bracket pattern $w$, Remark 4.2 and Lemma 6.13 show the equivalence that $Br(w) \in \mathcal{I}_D$ holds if and only if $A(w) \not\subset N_0 \setminus D$. The latter statement can also be expressed as $A(w) \not\subset N_0 \cup \{0\}$ since $0 \not\in A(w)$. In other words, we have shown $\mathcal{B}_D = \mathcal{W}_{D,0}(0)$.

Theorem 8.2. Let $D$ denote the set of subsemigroups of $(N_0, +)$.
(a) Let $D \in \mathcal{D}$ be arbitrary.
(i) If $0 \in D$ and $|\mathcal{N} \setminus D| < \infty$, then $\mathcal{I}_D = \{Br(\mathcal{N} \setminus D)\}$.
(ii) If $0 \not\in D$ and $|\mathcal{N} \setminus D| < \infty$, then $\mathcal{I}_D = \{Br(\mathcal{N} \setminus D) \cup \mathcal{S}_C\}$.
(iii) If $0 \in D$ and $|\mathcal{N} \setminus D| = \infty$, then $\mathcal{I}_D = \{Br(\{1, \ldots, v\} \setminus D) \mid v \in \mathcal{N}\}$.
(iv) If $0 \not\in D$ and $|\mathcal{N} \setminus D| = \infty$, then $\mathcal{I}_D = \{Br(\{1, \ldots, v\} \setminus D) \cup \mathcal{S}_C \mid v \in \mathcal{N}\}$.
(b) The categories $(\mathcal{I}_D)_{D \in \mathcal{D}}$ are pairwise distinct.

Proof. (a) Minding $\mathcal{N} \setminus D = N_0 \setminus (D \cup \{0\})$, Propositions 6.16 (b) and 7.16 (b) and Lemma 8.1 prove Claim (a).
(b) Suppose $D_1, D_2 \in \mathcal{D}$ and $\mathcal{I}_{D_1} = \mathcal{I}_{D_2}$. Lemma 8.1 (b) showed $\mathcal{B}_{\mathcal{I}_{D_1}} = \mathcal{W}_{D_1,0}(0)$ for all $i \in \{1, 2\}$. So, from $\mathcal{I}_{D_1} = \mathcal{I}_{D_2}$ follows $\mathcal{W}_{D_1,0(0)} = \mathcal{W}_{D_2,0(0)}$. We conclude $D_1 \cup \{0\} = D_2 \cup \{0\}$ by Proposition 7.17. Once more, as seen in Lemma 8.1 (a) for all $i \in \{1, 2\}$, the category $\mathcal{I}_{D_i}$ is in the monoid case if and only if $0 \in D_i$. In other words, $0 \in D_1$ if and only if $0 \in D_2$. Hence, $\mathcal{I}_{D_1} = \mathcal{I}_{D_2}$.

Theorem 8.3. Let $D$ denote the set of subsemigroups of $(N_0, +)$. For every category of partitions $C \in \mathcal{S}_0$ exists $D \in \mathcal{D}$ such that $C = \mathcal{I}_D$.

Proof. Let $C \in \mathcal{S}_0$ be an arbitrary category. The bracket patterns $\mathcal{B}_C$ of $C$ in the sense of Definition 6.15 are a bracket pattern category by Proposition 6.16 (a) Employing the classification result of Proposition 7.17, we conclude that there exists a submonoid $M$ of $(N_0, +)$, i.e. $M \in \mathcal{D}$ with $0 \in M$, such that $\mathcal{B}_C = \mathcal{W}_M$, in the notation of Definition 7.15. Put $E = \varnothing$ if $C$ is in the monoid case and define $E = \{0\}$ otherwise. Then, $M \setminus E \in \mathcal{D}$. We show $C = \mathcal{I}_{M \setminus E}$. 
By definition of $E$ and by Lemma 8.1 (a) the category $\mathcal{I}_{M\setminus E}$ is in the monoid case if and only if $C$ is. Moreover, Lemma 8.1 (b) shows $\mathcal{B}_{\mathcal{I}_{M\setminus E}} = \mathcal{W}_M = \mathcal{B}_C$. Hence, Proposition 6.16 (b) proves $\mathcal{C} = \mathcal{I}_{M\setminus E}$. That concludes the proof. 

Subsemigroups $D$ of $(\mathbb{N}_0, +)$ with $|\mathbb{N}\setminus D| = \infty$ are precisely the sets $\varnothing, \{0\}$ as well as $n\mathbb{N}_0$ and $n\mathbb{N}$ for $n \in \mathbb{N}$ with $n \geq 2$ (see [RG09, Chpt. 1]). In particular, we can confirm the existence of non-finitely-generated categories.

**Corollary 8.4.** Let $D$ be a subsemigroup of $(\mathbb{N}_0, +)$ with $|\mathbb{N}\setminus D| = \infty$. Then, for all sets $\mathcal{G} \in \mathcal{P}^{\circ \ast}$ with $\langle \mathcal{G} \rangle = \mathcal{I}_D$ holds $|\mathcal{G}| = \infty$.

**Proof.** As seen in Lemma 8.1 (b), $\mathcal{B}_{\mathcal{I}_D} = \mathcal{W}_{D \cup \{0\}}$. That implies $\bigcup A(\{\text{Br}_\ast(w) \mid w \in \mathcal{B}_{\mathcal{I}_D}\}) = \bigcup A(\{\text{Br}_\ast(w) \mid w \in \mathcal{W}_{D \cup \{0\}}\})$. With the help of Lemma 6.13 we conclude $\bigcup A(\{\text{Br}_\ast(w) \mid w \in \mathcal{B}_{\mathcal{I}_D}\}) = \bigcup A(\{w \mid w \in \mathcal{W}_{D \cup \{0\}}\}) = \bigcup A(\mathcal{W}_{D \cup \{0\}})$. Now, Lemma 7.16 assures us that $D \cup \{0\} = \mathbb{N}_0 \cup A(\mathcal{W}_{D \cup \{0\}})$, from which it follows that $\bigcup A(\{\text{Br}_\ast(w) \mid w \in \mathcal{B}_{\mathcal{I}_D}\}) = \mathbb{N}_0 \setminus (D \cup \{0\}) = \mathbb{N}\setminus D$. We have thus proven $\mathbb{N}\setminus D \subseteq \bigcup A(\mathcal{I}_D)$ by Proposition 6.16 (b). So, especially $\bigcup A(\mathcal{I}_D)$ is infinite. Let $\mathcal{G} \in \mathcal{P}^{\circ \ast}$ satisfy $\langle \mathcal{G} \rangle = \mathcal{I}_D$. By Remark 4.4 we infer $\mathbb{N}\setminus D \subseteq \bigcup A(\mathcal{I}_D) = \bigcup A(\langle \mathcal{G} \rangle) = \bigcup A(\mathcal{G})$, proving that $\bigcup A(\mathcal{G})$ is infinite. That is only possible if $\mathcal{G}$ is infinite. 

9. Concluding Remarks

9.1. **Half-Liberations of $U_n$.** Banica and Speicher showed that categories of one-colored partitions correspond to certain quantum subgroups of Wang’s free orthogonal quantum group $Q_n$ (see [BS09, Web16] and [Web17a]). Categories of two-colored partitions are in bijection with so-called unitary easy quantum groups (cf. [TW17a]), certain compact quantum subgroups of Wang’s free unitary quantum group $U_n^+$. Since $O_n^+$ is a quantum version, a kind of “liberation”, of the orthogonal group $O_n$, a natural question is to find all (orthogonal) easy quantum groups between $O_n$ and $O_n^+$, i.e. all “half-liberations” of $O_n$. Only one (orthogonal) easy quantum group exists here, namely the half-liberated orthogonal quantum group $O_n^+$ (see [BS09]). Here the commutation relations $ab = ba$ holding in $O_n$ are not yet dropped entirely, as is the case eventually in $O_n^+$, but are relaxed to the half-commutation relations $acb = bca$. Equivalently, the category of (one-colored) partitions corresponding to $O_n^+$ is generated by

Research into generalizing this half-liberation procedure to the unitary case (see [BDD11], [BD13], [Bho+14], [BB16] and [BB17]), where the generators are no longer self-adjoint and thus adjoints, i.e. colors, come in, effectively went in the direction of coloring the points of this partition in different ways or, eventually, those of the
similar-looking partition

essentially, these partitions represent relations $ac_1 \ldots c_kb = bc_1 \ldots c_ka$ provided certain conditions on the factors $c_1 \ldots c_k$ are satisfied.

Note that finding all unitary easy quantum groups between $U_n$ and $U_n^\times$, i.e. all unitary half-liberations, amounts to classifying all categories $C$ with $\emptyset \leq C \leq \mathbb{E}$, as is the contents of our classification result in Section 1.

9.2. **Comparison with the Previous Research on Half-Liberations of $U_n$.**

Our results obtained in [MW18] and the present article reproduce in combinatorial terms and extend the previous quantum algebraic research on half-liberations of $U_n$:

1. In [BDD11, Def. 5.5] and [Bho+14, Definition 2.8], Bhowmick, D’Andrea, Das and Dabrowski introduced the first half-liberation of $U_n$, whose algebra they denoted by $A_1^u(n)$. Later, Banica and Bichon wrote $U_n^\times$ for the corresponding quantum group [BB17, Def. 3.2 (3)]. Its associated category is generated by the partition

It holds

$$\emptyset = \emptyset = \emptyset = \emptyset = \emptyset = \emptyset$$

(see [MW18, Lemma 6.18 d]), parts of which can be seen as follows:

Hence, the quantum group $U_n^\times$ pertains to the category $I_{n_0 \setminus \{0\}}$ in our notation (see Definition 3.6). Alternatively, one could write $I_{\{0\}}^\epsilon$ for this category.
(2) A second half-liberation of $U_n$ was given by Bichon and Dubois-Violette. In [BD13, Ex. 4.10] they define an algebra $A_n^*(n)$ whose intertwiner spaces are generated by the partition

In [BB16] and [BB17] Banica and Bichon wrote $U_n^{**}$ for the corresponding quantum group. And $U_n^{**}$ appears again as a special case of an entire family of quantum groups introduced by Banica and Bichon in [BB16, Def. 7.1]. For every $k \in \mathbb{N}$ (with $0 \notin \mathbb{N}$), their $k$-half-liberated unitary quantum group $U_{n,k}^*$ (also $U_{n,k}$) has its intertwiner spaces generated by:

![Diagram]

It holds $U_n = U_{n,1}^*$ and $U_n^{**} = U_{n,2}^*$. In [MW18, Section 9], we showed that $U_{n,k}^*$ corresponds to our category $S_k$ (see after Main Theorem 2 of the present article) for every $k \in \mathbb{N}$. However, the quantum group associated with the category $S_0$ is not considered in [BB16] nor [BB17].

(3) In contrast, another half-liberation of $U_n$ which is studied by Banica and Bichon is $U_n^*$ ([BB16, Definition 8.3]), also denoted $U_{n,\infty}^*$ ([BB17, Definition 4.1 (3)]). They obtain it as some quantum algebraic limit case of their series $(U_{n,k}^*)_k \in \mathbb{N}$ and show that its associated category is generated by the two partitions

![Diagram]

Cyclically rotating both these partition counterclockwise once, reveals them to generate the same category as $\begin{array}{c}
\begin{array}{cc}
& * \\
& *
\end{array}
\end{array}$ and $\begin{array}{c}
\begin{array}{cc}
& * \\
& *
\end{array}
\end{array}$. Since $\begin{array}{c}
\begin{array}{cc}
& * \\
& *
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{cc}
& * \\
& *
\end{array}
\end{array}$ as seen above, it is our category $\mathcal{I}_{\mathbb{N}\setminus\{0,1\}}$ (also denoted $\mathcal{I}_{\{0,1,\infty\}}$) that is associated with the quantum group $U_n^*$. Especially, $U_n^*$ does not correspond to $S_0 = \mathcal{I}_\emptyset$, which, arguably, is a limit case of $(S_w)_{w \in \mathbb{N}}$ in a combinatorial sense.

Summarizing, the half-liberations of $U_n$ previously known correspond precisely to our categories $(S_w)_{w \in \mathbb{N}}, \mathcal{I}_{\mathbb{N}\setminus\{0\}}$ and $\mathcal{I}_{\{0,1,\}}$. Equivalently, the categories $S_0 = \mathcal{I}_\emptyset$ and $\mathcal{I}_D$ for $D \in \mathcal{D}$ with $D \nsubseteq \mathbb{N}\setminus\{0\}$, $\mathbb{N}\setminus\{0,1\}$ from [MW18, Def. 4.1] and Definition 3.6 were heretofore unknown. Especially we see that the quantum groups $U_n^*$ and $U_n^{**}$ (in the notation of [BB16]) represent merely special cases of our family $(\mathcal{I}_D)_{D \in \mathcal{D}}$. Moreover, from a combinatorial viewpoint, the half-liberations $(S_k)_{k \in \mathbb{N}} \sim (U_{n,k}^*)_{k \in \mathbb{N}}$.
give rise as a limit case to $\mathcal{I}_\varnothing = \mathcal{S}_0$ rather than $\mathcal{I}_{\mathbb{N}_0\setminus\{0,1\}} \sim U_{n,\infty}^* = U_n^*$. This is not the only way our results provide a different viewpoint on the previous research into the topic of half-liberating $U_n$.

9.3. New Perspective on the Half-Liberation Procedure. Our article [MW18] and the present follow-up to it shed a different light on the idea of half-liberating the classical unitary group:

When considering possible half-liberations of $U_n$, a natural starting point is to simply use the half-commutation relations $abc = cba$ in the unitary case as well. This leads to the category $\mathcal{S}_2 = \langle \varnothing, 0 \rangle$. For $k \geq 3$, the quantum groups $U_{n,k}^* \sim \mathcal{S}_k$ expand on these relations in the following way: For all $w \in \mathbb{N}$, the partition $\cdots \overset{w-1}{\cdots}$ can be seen to generate the category $\mathcal{S}_w$ ([MW18, Sect. 8.1]). This partition corresponds to the relations $ab_1 \ldots b_{w-1}c = cb_1 \ldots b_{w-1}a$. Here, we see the half-commutation relations generalized to what might be called a “paradigm of transpositions”: Only two factors switch places. The quantum groups $U_{n,k}^* \sim \mathcal{I}_{\mathbb{N}_0/\{0\}}(\mathbb{N}_0)$ and $U_{n,k}^* \sim \mathcal{I}_{\mathbb{N}_0\setminus\{0\}}$ (in the notation of [BB17]) are still following this principle exactly. The category $\mathcal{I}_{\mathbb{N}_0\setminus\{0\}}$ is generated by $\varnothing$. And, likewise, the partition $\cdots \overset{v}{\cdots}$ can be shown to generate $\mathcal{I}_{\mathbb{N}_0\setminus\{0,1\}}$. And one can continue this line of thinking to describe the entire family $(\mathcal{I}_D)_{D \in \mathcal{D}, 0 \in D}$, where $\mathcal{D}$ denotes the set of all subsemigroups of $(\mathbb{N}_0,+)$. For example, one can prove that the partition $\cdots \overset{v+1}{\cdots}$ generates $\mathcal{I}_{\mathbb{N}_0\setminus\{0,\ldots,v\}}$ for a given arbitrary $v \in \mathbb{N}_0$. That means all half-liberations known previously to the present article, including the ones from [MW18], are subsumed by this transpositional paradigm.

But the process of half-liberating $U_n$ on the way to $U_n^*$ is not exhausted by this scheme of transposing single pairs of factors. This is first evidenced in this article by the family $(\mathcal{I}_D)_{D \in \mathcal{D}, 0 \in D}$, the categories in the “monoid case” (see Definiton 6.4).
The category $\mathcal{I}_{\mathbb{N}_0\setminus\{1,\ldots,v\}}$, for example, is generated by the partition

\[
\begin{array}{c}
\bullet \quad \cdots \quad \bullet \\
\cdots \\
\bullet
\end{array}
\]

\[v \text{ times} \quad \quad v \text{ times}\]

which cannot be written in the form of a transposition. The relations of the associated easy quantum group do not take the shape $ac_1^* \ldots c_v^* d_1 \ldots d_v b = bc_1^* \ldots c_v^* d_1 \ldots d_v a$ as in the case of $\mathcal{I}_{\mathbb{N}_0\setminus\{0,\ldots,v\}}$; see also Section 9.4.

In light of this fact, one is tempted to further generalize the paradigm of transpositions to one of permutations: The generator of $\mathcal{I}_{\mathbb{N}_0\setminus\{1,\ldots,v\}}$ can at least be expressed in the form of the – non-transpositional – permutation

\[
\begin{array}{c}
\bullet \quad \cdots \\
\cdots \\
\bullet
\end{array}
\]

\[v \text{ times} \quad \quad v \text{ times}\]

meaning that relations of the sort $ab^* c_1^* \ldots c_v^* d_1 \ldots d_v = c_1^* \ldots c_v^* d_1 \ldots d_v ab^*$ hold in the corresponding easy quantum group. Still, some kind of partial commutativity, permutability characterizes them compared to $U_n^+$. While this modification works for $\mathcal{I}_{\mathbb{N}_0\setminus\{1,\ldots,v\}}$; for general $D \in \mathcal{D}$ with $0 \in D$, we cannot write the generator of $\mathcal{I}_D$ as a permutation: Take $\mathcal{I}_{\mathbb{N}_0\setminus\{1,2,5\}}$ as a counterexample:

\[
\begin{array}{c}
\bullet \quad \cdots \\
\cdots \\
\bullet
\end{array}
\]

\[v \text{ times} \quad \quad v \text{ times}\]

It is impossible to write this generator in a form where all strings connect the two rows. There is no permutation $\pi$ such that one could express the relations in the quantum group as $a_1 \ldots a_m = a_{\pi(1)} \ldots a_{\pi(m)}$.

In conclusion, except for special cases, half-liberation does not mean half-commutation in the unitary case, not even in the sense of non-transpositional permutation.
It seems that none of the shapes capture the true spirit of half-liberation in the unitary case. Rather, it is the bracket structure more precisely, that one should consider. So, heuristically, the generator is the right one to import from the orthogonal case and an appropriate way to view half-liberation as presented in this article.

9.4. \textbf{C*-Algebraic Relations.} It is straightforward to identify the $C^*$-algebras of the quantum groups associated with the categories $(S_w)_{w \in \mathbb{N}_0}$ and $(I_D)_{D \in \mathcal{D}}$, see [TW17b] and [Web17b; Web17c]. For the convenience of the reader, we list the $C^*$-algebraic relations corresponding to the generators of the categories $(S_w)_{w \in \mathbb{N}_0}$ and $(I_D)_{D \in \mathcal{D}}$:

1. The partition induces the following relations: For all $a, b, c \in \{u_{i,j}\}_{i,j=1}^n$

\[ ab^*c = cb^*a. \]

2. For $w \in \mathbb{N}$, the following relations are imposed on the algebra of the quantum group with category $S_w$ by the generator or rather its rotation (see
Section 9.3: For all \(a, b_1, \ldots, b_{w-1}, c \in \{u_{i,j}\}_{i,j=1}^n\)

\[
ab_1 \cdots b_{w-1} c = cb_1 \cdots b_{w-1} a.
\]

(3) For the quantum group whose intertwiner spaces are generated by the category \(S_0 = \mathcal{I}_\emptyset\) with the generators \(\{\text{Br}_v(\{v\}) \mid v \in \mathbb{N}\}\) and \(\bigotimes\) (or rather their combination \(\bigotimes\bigotimes\)) we can express the fundamental relations as follows: For all \(a, b, c, d_1, \ldots, d_v \in \{u_{i,j}\}_{i,j=1}^n\)

\[
ab^* c_1^* \cdots c_v^* d_1 \cdots d_v = c_1^* \cdots c_v^* d_1 \cdots d_v ab^*.
\]

in addition to the relations induced by \(\bigotimes\bigotimes\)

(4) For general subsemigroups \(D\) of \((\mathbb{N}_0, +)\) the relations of the quantum group with associated category \(\mathcal{I}_D\) cannot be expressed as compactly.

We adopt the convention \(-0 \neq 0\) and

\[-v < -(v - 1) < \ldots < -1 < -0 < 0 < 1 < \ldots < v - 1 < v\]

and define \(I_v := \{-v, \ldots, -1, 0, 1, \ldots, v\}\) for all \(v \in \mathbb{N}\). Moreover, for all \(m \in \mathbb{N}\) abbreviate \(\mathbb{N}_0(m) = \{0, \ldots, m\}\) and \(\mathbb{N}(m) = \{1, \ldots, m\}\). Then, the relations induced by \(\text{Br}_v(\mathbb{N}_0(m) \setminus D)\) for a submonoid \(D\) of \((\mathbb{N}_0, +)\), i.e. \(0 \in D\), and \(v \in \mathbb{N}\) with \(v \notin D\) are the following: For all \(\alpha, \beta : I_v \to \mathbb{N}(n)\)

\[
\sum_{\gamma : I_v \to \mathbb{N}(n)} \left( \prod_{j \in \mathbb{N}_0(v)} \delta_{\gamma_j, \gamma_j'} \delta_{\beta_j, \beta_j'} \right) \left( \prod_{i \in \mathbb{N}_0(v)} \delta_{\gamma_i, \beta_i} \delta_{\gamma_i, \beta_i'} \right) u_{\gamma_v, \alpha - v}^* \cdots u_{\gamma_0, \alpha - 0}^* u_{\gamma_0, \alpha_0} \cdots u_{\gamma_v, \alpha_v}^* = \sum_{\gamma' : I_v \to \mathbb{N}(n)} \left( \prod_{j \in \mathbb{N}_0(v)} \delta_{\gamma'_j, \gamma'_j} \delta_{\alpha_j - 0} \delta_{\alpha_j} \right) \left( \prod_{i \in \mathbb{N}_0(v)} \delta_{\gamma'_i, \alpha_i} \delta_{\gamma'_i, \alpha_i'} \right) u_{\beta_v, \gamma'_v}^* \cdots u_{\beta_0, \gamma'_0}^* u_{\beta_0, \gamma_0} \cdots u_{\beta_v, \gamma_v}.
\]

For example, the partition \(\text{Br}_v(\{1, 2, 5\})\)
induces the following relations: For all $a_1, a_2, a_3, b_1, b_2, b_3 \in \{u_{i,j}\}_{i,j=1}^n$ and all $i, j, i', j' : \mathbb{N}(3) \to \mathbb{N}(n)$
\[
\delta_{i_1,j_1}' \delta_{i_2,j_2}' \delta_{i_3,j_3}' \sum_{k_1,k_2,k_3=1}^n u_{k_1,i_1}^* a_1^* a_2^* u_{k_2,i_2}^* u_{k_3,i_3}^* a_3^* b_3 u_{k_3,j_3} a_3 b_3 u_{k_3,j_3} u_{k_2,j_2} b_2 b_1 u_{k_1,j_1} = \delta_{i_1,j_1} \delta_{i_2,j_2} \delta_{i_3,j_3} \sum_{k_1', k_2', k_3'=1}^n u_{i_1,k_1}^* a_1^* a_2^* u_{i_2,k_2}^* u_{i_3,k_3}^* a_3^* b_3 u_{j_3,k_3} a_3 b_3 u_{j_3,k_3} u_{j_2,k_2} b_2 b_1 u_{j_1,k_1}.
\]

Put $u := (u_{i,j})_{i,j=1}^n$ and $\overline{u} := (u_{i,j}^*)_{i,j=1}^n$. Note that, although we hence know that $\mathcal{S}_w$ corresponds to the quantum group with $C^*$-algebra
\[C^*(\{u_{i,j}\}_{i,j=1}^n | u, \overline{u} \text{ unitary, } \forall a, b_1, \ldots, b_{n-1} : (u_{i,j})_{i,j=1}^n : \text{relations (9.1) hold}),\]
while $\mathcal{I}_D$ corresponds to
\[C^*(\{u_{i,j}\}_{i,j=1}^n | u, \overline{u} \text{ unitary, } \forall v \in \mathbb{D} : \forall \alpha, \beta : I_v \rightarrow \mathbb{N}(n) : \text{relations (9.2) hold}),\]
we know basically nothing about these quantum groups. In particular, it would be enlightening to construct these quantum groups from known ones, if possible, in the sense of \cite{TW17b}.

9.5. Further Questions.

(1) With all easy quantum groups $G$ classified in the regions $U_n \subseteq G \subseteq U_n^+$ (present article and \cite{MW18}) and $O_n \subseteq G \subseteq O_n^+$ (\cite{BS09}) as well as $O_n \subseteq G \subseteq U_n$ (\cite{TW17a}) and $O_n^* \subseteq G \subseteq U_n^+$ (\cite{TW17a}), a natural question is to find all unitary easy quantum groups $G$ with $O_n \subseteq G \subseteq U_n^+$. This is an ongoing project of the authors.

(2) In general, the complete classification of categories of two-colored partitions is still an open question (see also \cite{TW17a}; see \cite{RW16} for the complete classification in the orthogonal case). The question of all such categories with $\overline{\mathcal{C}} \otimes \mathcal{C}$ has recently been settled by Gromada in \cite{Gro18}.

(3) In the orthogonal case, $O_n^*$ is not only the only easy quantum group $G$ such that $O_n \subseteq G \subseteq O_n^*$, but also the only compact quantum group in the region $O_n \subseteq G \subseteq O_n^*$ (\cite{Ban+13}). One wonders whether there exist further compact quantum groups $G$ with $U_n \subseteq G \subseteq U_n^+$ besides the unitary easy quantum groups.

(4) Our results might yield clues to advancing with the classification of bi-easy ("busy") geometries begun in \cite{Ban18b}. Also, the study of affine homogeneous spaces of the free complex sphere (see \cite{Ban18a}) might benefit from the results in the present article.

(5) In \cite{Fre14, Fre17a} and \cite{Fre17b}, Freslon investigates a different notion of colored partitions. And some of the results of \cite{MW18} and this article, especially \cite{MW18} Lemma, 6.7], generalize to his setting. One could try to apply the same methods to classify all the categories of this kind.
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