Irrelevance of memory in the minority game

Andrea Cavagna*

Theoretical Physics, University of Oxford, 1 Keble Road, Oxford, OX1 3NP, UK

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Abstract: By means of extensive numerical simulations we show that all the distinctive features of the minority game introduced by Challet and Zhang (1997), are completely independent from the memory of the agents. The only crucial requirement is that all the individuals must posses the same information, irrespective of the fact that this information is true or false.

Originally inspired by the El Farol problem stated by Arthur in [1], it has been introduced in [2] a model system for the adaptive evolution of a population of interacting agents, the so called minority game. This is a toy model where inductive, rather than deductive, thinking, in a population of bounded rationality, gives rise to cooperative phenomena.

The setup of the minority game is the following: \( N \) agents have to choose at each time step whether to go in room 0 or 1. Those agents who have chosen the less crowded room (minority room) win, the other loose, so that the system is intrinsically frustrated.

A crucial feature of the model is the way by which agents choose. In order to decide in what room to go, agents use strategies. A strategy is a choosing device, that is an object that processes the outcomes of the winning room in the last \( m \) time steps (each outcome being 0 or 1) and accordingly to this information prescribes in what room to go the next step. The so-called memory \( m \) defines \( 2^m \) potential past histories (for instance, with \( m=2 \) there are four possible pasts, 11, 10, 01 and 00). A strategy is thus formally a vector \( \nu \), with \( \mu=1, \ldots, 2^m \), whose elements can be 0 or 1. The space \( \Gamma \) of the strategies is a hypercube of dimension \( D=2^m \) and the total number of strategies is \( 2^D \).

At the beginning of the game each agent draws randomly a number \( s \) of strategies from the space \( \Gamma \) and keeps them forever, as a genetic heritage. The problem is now to fix which one, among these \( s \) strategies, the agent is going to use\(^1\) The rule is the following. During the game the agent gives point to all his/her strategies according to their potential success: at each time step a strategy gets a point only if it has forecasted the correct winning room, regardless of having been actually used or not. At a given time the agent chooses among his/her \( s \) strategies the most successful one up to that moment (i.e. the one with the highest number of points) and uses it in order to choose the room. The adaptive nature of the game relies in the time evolution of the best strategy of each single agent.

In this way the game has a well defined deterministic time evolution, which only depends on the initial distribution of strategies and on the random initial string of \( m \) bits necessary to start the game.

Among all the possible observables, a special role is played by the variance \( \sigma \) of the attendance \( A \) in a given room \( \Gamma \). We can consider, for instance, room 0 and define \( A(t) \) as the number of agents in this room at time \( t \). We have,

\[
\sigma^2 = \lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} dt' \left( A(t') - \frac{N}{2} \right)^2,
\]

where \( N/2 \) is the average attendance in the room and \( t_0 \) is a transient time after which the process is stationary\(^2\). In all the simulations presented in this Letter it has been taken \( t = t_0 = 10,000 \) for a maximum value of \( N = 101 \) and it has been verified that the averages were saturated over these times.

The importance of \( \sigma \) (called volatility in financial context) is simple to understand: the larger is \( \sigma \), the larger is the global waste of resources by the community of agents. Indeed, only with an attendance \( A \) as near as possible to its average value there is the maximum distribution of points to the whole population. Moreover, from a financial point of view, it is clear that a low volatility \( \sigma \) is of great importance in order to minimize the risk.

If all the agents were choosing randomly, the variance would simply be \( \sigma^2 = N/4 \). An important issue is therefore: in what conditions is the variance \( \sigma \) smaller than \( \sigma_r \)? In other words, is it possible for a population of selfish individuals to collectively behave in a better-than-random way? What has been found first in [3] is that the volatility \( \sigma \) as a function of \( m \) has a remarkable behaviour, since actually there is a regime where \( \sigma \) is smaller than the random value \( \sigma_r \). In this phase the collective behaviour is such that less resources are globally wasted by the population of agents. A deep understanding of this feature is therefore important.

From the very definition of the model and from the behaviour of \( \sigma(m) \) described above, it seems clear that the memory \( m \) is a crucial quantity for the two following

\(^1\)E-mail: a.cavagna1@physics.ox.ac.uk
\(^2\)We will consider only the non-trivial case \( s > 1 \).
First, from a geometrical point of view, \( m \) defines the dimension of the space of strategies \( \Gamma \) and therefore it is related to the probability that strategies drawn randomly by different agents could give similar predictions: the larger is \( m \), the bigger is \( \Gamma \) and the lower is the probability that different players have some strategies in common. Since the non-random nature of the game relies in the presence of correlated choices, that is, exactly in the possibility that different agents use the same strategies, it follows that for very large \( m \) the game proceeds in a random way [3–6].

Secondly, \( m \) is supposed to be a real memory. Actually, the whole game is constructed around the role of \( m \) as a memory: at time \( t \) agents use strategies which process the last \( m \) events in the past. As a consequence of this, a new minority room will come out and at time \( t + 1 \) there will be a new \( m \)-bits past which will differ from the old one for the outcome at time \( t \). Thus, agents, or better, strategies, choose by remembering the last \( m \) steps of time history, so that \( m \) is a natural time scale of the system. Due to this, an explanation of the behaviour of \( \sigma(m) \) has been proposed in [3], where the decay rate of the time correlations in the system is compared and related to \( m \), thus supporting the key interpretation of \( m \) as a real memory. This memory role of \( m \) complicates greatly the nature of the problem, since it induces an explicit dynamical feedback in the evolution of the system, such that the process is not local in time.

The purpose of this Letter is to show that the memory of the agents is irrelevant. We shall prove that there is no need of an explicit time feedback, to obtain all the distinctive features of the model.

In order to prove this statement we consider the same model introduced in [2] and described above, but with the following important difference: at each time step, the past history is just invented, that is, a random sequence of \( m \) bits is drawn, to play the role of a fake time history. This is the information that all the agents process with their best strategies to choose the room. As we are going to show, this oblivious version of the model gives exactly the same results as the original one, thus proving that the role of \( m \) is purely geometrical.

In Fig. 1, the variance \( \sigma \) as a function of \( m \) is plotted both for the case with and without memory. The two models give the same results, not only qualitatively, but also quantitatively (see also the data of [3, 3]). In particular, the minimum of \( \sigma \) as a function of \( m \) is found even without memory and cannot therefore be related to it.

The dependence of the whole function \( \sigma(m) \) on the individual number of strategies \( s \) is another important point. It has been shown for the first time in [4] that the minimum of this curve is shallower the larger is the value of \( s \). In Fig. 2 we show that this same phenomenon occurs for the model without memory.

From a technical point of view, note that, once eliminated the role of \( m \) as a memory, the only quantity involved in the actual implementation of the model is \( D \), the dimension of the space of strategies \( \Gamma \). Therefore, instead of drawing a random sequence of \( m \) bits, it is much easier to draw a random component \( \mu \in [1, D] \) to mimic the past history: each agent uses component \( \mu \) of his/her
best strategy to choose the room. The main consequence of this is that there is no need for being $D = 2^m$, since we can choose any integer value of $D$. In [3] it has been introduced a method by which it is possible to consider non-integer values of $m$ in the model with memory. This is useful, since it permits to study the shape of $\sigma(m)$ around its minimum, with a better resolution in $m$. In the present context, it is trivial to consider non-integer values of $m$, since we simply have $m = \log_2 D$. In this way results identical to [5] are obtained.

Once fixed $s$, let $m_c$ be the value of $m$ where the minimum of $\sigma(m)$ occurs. In [3] it has been pointed out that for $m < m_c$ the variance $\sigma$ grows as $N$, where $N$ is the number of agents, while for $m > m_c$ it grows as $N^{1/2}$. In Fig.3, $\sigma$ as a function of $N$ is plotted for the model without memory. The same behaviour as in the model with memory is found.

An interesting question is whether $\sigma$ is a function of a single scaling variable $z$ constructed with $m$, $N$ and $s$. It has been shown in [3] that by considering as a scaling variable $z = 2^m/N = D/N$ all the data for $\sigma$ at various $m$ and $N$ collapse on the same curve. In this case the relevant parameter is thus the dimension $D$ of $\Gamma$, over the number $N$ of playing strategies. On the other hand, it has been proposed in [4] a different scaling variable, that is $z' = 2 \cdot 2^m/sN = 2D/sN$. In this way, the relevant parameter would be the density on $\Gamma$ of the total number of strategies $sN$. In Fig.4 we plot $\sigma^2/N$ as a function of $z'$, at different values of $D$, $N$ and $s$, for the model without memory. We see that the correct scaling parameter is $z$ and not $z'$, since the data with different values of $s$ collapse on different curves. The same result is obtained if we perform the simulation with the memory (see [5]).

The two models give once again the same results. Note from Fig.4 that the scaling is not perfect at very low values of $z'$, that is for very small $D$. This is just a trace of the integer nature of the model.

From what shown above it is reasonable to conclude that, in order to obtain all the crucial features of the minority game, the presence of an individual memory of the agents is irrelevant. The parameter $m$ still plays a major role, but only for being related to the dimension $D = 2^m$ of the strategies space $\Gamma$. A consequence of this fact is that any attempt to explain the properties of this model, relying on the role of $m$ as a memory, can hardly be correct. On the other hand, as already said, the geometrical role of $m$ remains. Indeed, some recent attempts to give an analytic description of the model (see [4,6]) are only grounded on geometrical considerations about the distribution of strategies in the space $\Gamma$ and go therefore, in our opinion, in the correct direction.

The most important result of the present Letter is the existence of a regime where the whole population of agents still behaves in a better-than-random way, even if the information they process is completely random, that is wrong, if compared to the real time history. The crucial thing is that everyone must possess the same information. Indeed, if we invent a different past history for each different agent, no coordination emerges at all and the results are the same as if the agents were behaving randomly (this can be easily verified numerically). In other words, if each individual is processing a different information, the features of the system are completely identical to the random case, irrespective of the values of $m$ and $s$.

The conclusion is the following: the crucial property is not at all the agents’ memory of the real time history, but...
rather the fact that they all share the same information, whatever false or true this is. As a consequence, there is no room in this model for any kind of forecasting of the future based on the “understanding” of the past.

We hope this result to be useful for a future deeper understanding of this kind of adaptive systems. Indeed, before trying to explain the rich structure of a quite complicated model, it is important in our opinion to clear up what are the truly necessary ingredients of such a model and what, on the contrary, is just an irrelevant complication, which can be dropped. In the case of the so-called memory (or brain size, or intelligence), $m$, there also has been a problem of terminology: given the original formulation of the model, it seemed that the very nature of a variable encoding the memory or the intelligence of the agents, could warrant by itself a relevance to it relevance which, as we have seen, was not deserved. Notwithstanding this, we consider the present model still to be very interesting and far from being trivial.

Finally, let us note that the passage from a model with memory to a model without memory, is equivalent to substitute a deterministic, but very complicated system, with a stochastic, but much simpler one, which nevertheless gives the same results as the original case and which is therefore indistinguishable from it for all the practical purposes. The use of a stochastic/disordered model to mimic a deterministic/ordered one, is similar in the spirit to what happens in the context of glassy systems, where some disordered models of spin glasses are often used in order to have a better understanding of structural glasses, which contain in principle no quenched disorder.

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