Turán number for odd-ballooning of bipartite graphs

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Abstract

Given a graph $H$ and an odd integer $t$ ($t \geq 3$), the odd-ballooning of $H$, denoted by $H(t)$, is the graph obtained from replacing each edge of $H$ by an odd cycle of length at least $t$ where the new vertices of the cycles are all distinct. In this paper, we determine the range of Turán numbers for odd-ballooning of bipartite graphs when $t \geq 5$. As applications, we may deduce the Turán numbers for odd-ballooning of stars, paths and even cycles.

Keywords: Extremal graphs, Turán number, Odd-ballooning, Bipartite graph

1. Introduction

In this paper, we consider simple graphs without loops and multiedges. The order of a graph $H = (V(H), E(H))$ is the number of its vertices denoted by $\nu(H)$, and the size of a graph $H$ is the number of its edges denoted by $e(H)$. For a vertex $v \in V(H)$, the neighborhood of $v$ in $H$ is denoted by $N_H(v) = \{u \in V(H) : uv \in E(H)\}$. Let $N_H[v] = \{v\} \cup N_H(v)$. The degree of the vertex $v$ is written as $d_H(v)$ or simply $d(v)$. $\Delta(H)$ is the maximum degree of $H$ and $\delta(H)$ is the minimum degree of $H$. Usually, a path of order $n$ is denoted by $P_n$, a cycle of order $n$ is denoted by $C_n$. A star of order $n + 1$ is denoted by $S_n$ ($n \geq 2$), and the vertex of degree larger than one is called the center vertex. The maximum number of edges in a matching of $H$ is called the matching number of $H$ and denoted by $\alpha'(H)$. For $U \subseteq V(H)$, let $H[U]$ be the subgraph of $H$ induced by $U$, $H - U$ be the graph obtained by deleting all the vertices in $U$ and their incident edges.

Given two graphs $G$ and $H$, the union of graphs $G$ and $H$ is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The union of $k$ copies of $P_2$ is denoted by $kP_2$. The join of $G$ and $H$, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by adding all edges between $V(G)$ and $V(H)$. The graph $K_p(i_1, i_2, \ldots, i_p)$ denotes the complete $p$-partite graph with parts of order $i_1, i_2, \ldots, i_p$. Denoted by $T_p(n)$, the $p$-partite Turán graph is the complete $p$-partite graph on $n$ vertices with the order of each partite set as equal as possible.

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Given a family of graphs \( \mathcal{L} \), a graph \( H \) is \( \mathcal{L} \)-free if it does not contain any graph \( L \in \mathcal{L} \) as a subgraph. The Turán number, denoted by \( \text{ex}(n, \mathcal{L}) \), is the maximum number of edges in a graph of order \( n \) that is \( \mathcal{L} \)-free. The set of \( \mathcal{L} \)-free graphs of order \( n \) with \( \text{ex}(n, \mathcal{L}) \) edges is denoted by \( \text{EX}(n, \mathcal{L}) \) and call a graph in \( \text{EX}(n, \mathcal{L}) \) an extremal graph for \( \mathcal{L} \). In 1966, Erdős and Simonovits [5] proved a classic theorem showing that the Turán number is closely related to the chromatic number. The chromatic number of \( H \) is denoted by \( \chi(H) \). For a family of graphs \( \mathcal{L} \), the subchromatic number of \( \mathcal{L} \) is defined by \( p(\mathcal{L}) = \min \{ \chi(L) : L \in \mathcal{L} \} - 1 \).

**Theorem 1.1** (Erdős and Simonovits [5]). Given a family of graphs \( \mathcal{L} \), \( p = p(\mathcal{L}) \), if \( p > 0 \), then

\[
\text{ex}(n, \mathcal{L}) = \left( 1 - \frac{1}{p} \right) \binom{n}{2} + o(n^2).
\]

Erdős and Stone [4] proved the following theorem, which shows that if the size of a graph satisfies some conditions, it contains a Turán graph as a subgraph.

**Theorem 1.2** (Erdős and Stone [4]). For all integers \( p \geq 2 \) and \( N \geq 1 \), and every \( \epsilon > 0 \), there exists an integer \( n_1 \) such that every graph with \( n > n_1 \) vertices and at least \( e(T_{p-1}(n)) + \epsilon n^2 \) edges contains \( T_p(pN) \) as a subgraph.

In 2003, Chen, Gould and Pfender [2] determined the Turán numbers for \( F_{k,r} \), a graph consists of \( k \) complete graphs of order \( r \) which intersect in exactly one common vertex. In 2016, Hou, Qiu and Liu [7] determined the Turán numbers for intersecting odd cycles with the same length. Later, Hou, Qiu and Liu [8] considered the Turán numbers for \( H_{s,t} \), a graph consists of \( s \) triangles and \( t \) odd cycles with length at least 5 which intersect in exactly one common vertex. For an odd integer \( t \geq 3 \), the odd-ballooning of a graph \( H \), denoted by \( H(t) \), is the graph obtained from \( H \) by replacing each edge of \( H \) with an odd cycle of length at least \( t \) where the new vertices of the odd cycles are all different. It is easy to see that \( H_{s,t} \) can be seen as an odd-ballooning of a star \( S_{s+t} \). In 2020, Zhu, Kang and Shan [13] determined the Turán numbers for odd-ballooning of paths and cycles. Recently, Zhu and Chen determined the Turán numbers for odd-ballooning of general bipartite graphs when \( t \geq 5 \) by using progressive induction.

A covering of a graph \( H \) is a set of vertices which meets all edges of \( H \). The minimum number of vertices in a covering of \( H \) is denoted by \( \beta(H) \). An independent covering of a bipartite graph \( H \) is an independent set which meets all edges. The minimum number of vertices in an independent covering of a bipartite graph \( H \) is denoted by \( \gamma(H) \). For any connected bipartite graph \( H \), let \( A \) and \( B \) be its two color classes with \( |A| \leq |B| \). Moreover, if \( H \) is disconnected, we always partition \( H \) into \( A \cup B \) such that (1) \( |A| \) is as small as possible;
(2) \(\min\{d_H(x) : x \in A\}\) is as small as possible subject to (1). In this paper, we study the Turán numbers for odd-ballooning of bipartite graph \(H\).

**Lemma 1.1** (Yuan [12]). Let \(H\) be a bipartite graph, \(V(H) = A \cup B\), then we have \(\gamma(H) = |A|\) and each independent covering of \(H\) contains either all the vertices of \(A\) or all the vertices of \(B\).

Given a family of graphs \(\mathcal{L}\), the following three parameters \(q(\mathcal{L})\), \(S(\mathcal{L})\) and \(B(\mathcal{L})\) are proposed in [12]. The independent covering number \(\gamma(\mathcal{L})\) of \(\mathcal{L}\) is defined as

\[ q(\mathcal{L}) = \min\{\gamma(L) : L \in \mathcal{L} \text{ is bipartite}\}. \]

The independent covering family \(S(\mathcal{L})\) of \(\mathcal{L}\) is the family of independent coverings of bipartite graphs \(L \in \mathcal{L}\) of order \(q(\mathcal{L})\). The subgraph covering family \(B(\mathcal{L})\) of \(\mathcal{L}\) is the set of subgraphs induced by a covering of \(L \in \mathcal{L}\) with order less than \(q(\mathcal{L})\). If \(\beta(L) \geq q(\mathcal{L})\) for each \(L \in \mathcal{L}\), then we set \(B(\mathcal{L}) = \{K_{q(L)}\}\).

**Definition 1.1** (Ni, Kang and Shan [10]). Given a family of graphs \(\mathcal{L}\), define \(p = p(\mathcal{L})\). For any integer \(p' : 2 \leq p' \leq p\), let \(\mathcal{M}_{p'}(\mathcal{L})\) be the family of minimal graphs \(M\) for which there exist an \(L \in \mathcal{L}\) and a \(t = t(L)\) such that there is a copy of \(L\) in \(M \setminus K_{p'-1}(t, t, \ldots, t)\) where \(M' = M \cup I_t\). We call this the \(p'\)-decomposition family of \(\mathcal{L}\).

For a bipartite graph \(H\), we have \(\chi(H(t)) = 3\). Therefore, in this paper, we mainly use 2-decomposition family of \(H(t)\).

Given a graph \(H\), by the definition of \(\mathcal{M}_2(H(t))\), for any \(M \in \mathcal{M}_2(H(t))\) there exist two independent sets \(Y_1, Y_2\) such that there is a copy of \(H(t)\) as a subgraph in \((M \cup Y_1) \setminus Y_2\). Let \(H_M\) be a copy of \(H\) and \(H_M\) satisfy that \(H_M(t) \subseteq (M \cup Y_1) \setminus Y_2\) is a copy of \(H(t)\). \(f\) is a bijection: \(V(H_M) \to V(H)\) such that \(uv \in E(H_M)\) if and only if \(f(u)f(v) \in E(H)\). We may directly obtain the following lemma.

**Proposition 1.1.** Suppose \(H\) is a graph and \(t \geq 3\) is an odd integer. For any \(M \in \mathcal{M}_2(H(t))\), \(M\) satisfies the following properties:

(i) \(e(M) = e(H_M) = e(H)\);
(ii) each odd cycle of \(H_M(t)\) expanded from an edge of \(H_M\) contains exactly one edge in \(M\);
(iii) \(V(M) \subseteq V(H_M(t))\);
(iv) the vertex of \(M\) which is in at least two odd cycles expanded from edges of \(H_M\) is the vertex of \(H_M\).

In the following part of this paper, we always write \(\bar{q}(H) = q(\mathcal{M}_2(H(t)))\), \(\bar{S}(H) = S(\mathcal{M}_2(H(t)))\), \(\bar{B}(H) = B(\mathcal{M}_2(H(t)))\), \(k(H) = \min\{d_M(x) : x \in S, S \in \bar{S}(H)\}\), where \(M \in \mathcal{M}_2(H(t))\) has the independent covering set \(S\).
Set $F(n, q) = I_{q-1} \vee T_2(n-q+1)$ and $f(n, q) = e(F(n, q))$. For a family of graphs $\mathcal{L}$, denote by $\mathcal{F}(n, q, k, \mathcal{L})$, the set of graphs which are obtained by taking an $F(n, q)$, putting a copy of $K_k, k$ in one class of $T_2(n-q+1)$ and putting a copy of a member of $\text{EX}(q-1, \mathcal{L})$ in $I_{q-1}$. Our main results are as follows.

**Theorem 1.3.** Let $H$ be a bipartite graph, $t \geq 5$ be an odd integer and $n$ be a sufficiently large integer. Then

$$f(n, \tilde{q}(H)) + \text{ex}(\tilde{q}(H)-1, \tilde{B}(H)) \leq \text{ex}(n, H(t)) \leq f(n, \tilde{q}(H)) + \text{ex}(\tilde{q}(H)-1, \tilde{B}(H)) + (k(H)-1)^2.$$  

Moreover, if $\text{ex}(n, H(t)) = f(n, \tilde{q}(H)) + \text{ex}(\tilde{q}(H)-1, \tilde{B}(H)) + (k(H)-1)^2$ holds, then the graphs in $\mathcal{F}(n, \tilde{q}(H), k(H)-1, \tilde{B}(H))$ are the only extremal graphs for $H(t)$.

2. Characterizations of $\mathcal{M}_2(H(t))$ and $\tilde{B}(H)$

Given a graph $H$, the **vertex division** on some non-pendent vertex $v$ of $H$ is defined as follows: $v$ is replaced by an independent set $\{v', v_1, v_2, \ldots, v_m\}$ ($1 \leq m \leq d_H(v)-1$) in which $v_i$ ($1 \leq i \leq m$) is adjacent to exactly one distinct vertex in $N_H(v)$ and $v'$ is adjacent to the remaining neighbors of $v$ in $H$. In particular, if $m = d_H(v) - 1$, it is called vertex split in [9, 10]. Denote by $\mathcal{D}(H)$, the family of graphs which can be obtained by applying vertex division on some vertex set $U \subseteq V(H)$. An isolated edge is an edge whose endpoints has degree 1. Lemma 2.1 shows that the 2-decomposition family of odd-ballooning for any graph $H$ is actually the family of graphs obtained from dividing some vertices of $H$.

**Lemma 2.1.** Let $H$ be any graph and $t \geq 5$ be an odd integer, then $\mathcal{M}_2(H(t)) \subseteq \mathcal{D}(H)$ holds.

**Proof.** To prove Lemma 2.1 we may show that any graph in $\mathcal{M}_2(H(t))$ can be obtained by using vertex division on some vertices of $V(H)$. For any graph $M \in \mathcal{M}_2(H(t))$, by Proposition 1.1(i), we have $e(M) = e(H)$. $H_M$ is a copy of $H$ and there exist two independent sets $Y_1$ and $Y_2$ such that $H_M(t) \subseteq (M \cup Y_1) \cup Y_2$ holds. Furthermore, $d_M(v) \geq 1$ holds for any vertex $v \in V(M)$. If there is an isolated vertex $v$ in $M$, then we may add a vertex $v'$ in $Y_1$ to replace $v$, and we have $H_M(t) \subseteq ((V(M) \backslash \{v\}) \cup (Y_1 \cup \{v'\})) \cup Y_2$ which contradicts the minimality of $M$.

For any vertex $v$ of $M$, first, we suppose $v \in V(M) \cap V(H_M)$. If $d_M(v) > d_{H_M}(v)$, then $d_M(v) - d_{H_M}(v)$ edges are not in $H_M(t)$ and it contradicts the minimality of $M$. Hence we have $d_M(v) \leq d_{H_M}(v)$. When $d_M(v) < d_{H_M}(v)$, then $Y_2$ contains $x$ neighbors $(d_{H_M}(v) - d_M(v) \leq x \leq d_{H_M}(v))$ of $v$ in $H_M$. Each edge between $Y_2$ and $v$ can be expanded into an odd cycle by using one edge in $M$. By the minimality of $M$, in $M$ there is a star $S_{d_H(v)}$ and $d_{H_M}(v) - d_M(v)$ distinct edges can be used to obtain an $S_{d_{H_M}(v)}(t)$. Since the new vertices...
Therefore, $e_M(v) - V(S_{dH_M}(v))$ are all different, these $d_{H_M}(v) - d_M(v)$ edges are independent. Therefore, to obtain $M$, we may divide the vertex $f(v)$ of $H$ into a vertex with degree $d_M(v)$ and an independent set of order $d_H(v) - d_M(v)$. When $d_M(v) = d_{H_M}(v)$, the adjacency relation of $v$ in $M$ is the same as $f(v)$ in $H$.

Now we suppose $v \in V(M) \cap (V(H_M(t)) \setminus V(H_M))$. The fact that the new vertices of odd cycles are all different implies the edges incident to $v$ are in the same odd cycle. By Proposition [11] we have $d_M(v) = 1$. Suppose $v$ is in an odd cycle expanded from the edge $uw$ of $H_M$. Then the edge $uw$ is not in $M$, otherwise this odd cycle has two edges in $M$, a contradiction. Suppose $uw$ is between $M$ and $Y_2$, $u \in V(M)$, $w \in Y_2$. Since $N_{H_M}(w) \subseteq V(M) \cup Y_1$, each odd cycle of $S_{d_H}(t)$ contains exactly one edge in $M$, there are $d_{H_M}(w)$ independent edges in $M$. Thus, to obtain $M$, we may use vertex split on $f(w)$ of $H$. If $uw$ is between $Y_1$ and $Y_2$, $u \in Y_1$, $w \in Y_2$, then $N_{H_M}(u) \subseteq Y_2$. Noting that $S_{d_H}(t)$ spanned by $N_{H_M}[u]$ in $H_M(t)$ contains exactly $d_{H_M}(u)$ independent edges in $M$. We may deduce that the edge which contains vertex $v$ is an isolated edge in $M$. Hence to obtain $M$ we may use vertex split on both $f(u)$ and $f(w)$ of $H$.

Therefore, we have $M \in \mathcal{D}(H)$. As $M$ is arbitrary, we have $\mathcal{M}_2(H(t)) \subseteq \mathcal{D}(H)$. \qed

Lemma 2.2. Let $H$ be a bipartite graph, $t \geq 5$ be an odd integer. Then $e(H)P_2 \in \mathcal{M}_2(H(t))$ holds.

Proof. Suppose $Y_1$ and $Y_2$ are two independent sets and large enough. Let $H'$ be a copy of $H$, $V(H') = A' \cup B'$. $A'$ corresponds to $A$, $B'$ corresponds to $B$. Let $A' \subseteq V(e(H)P_2)$ and they are independent in $e(H)P_2$, $B' \subseteq Y_2$, then we have $H' \subseteq e(H)P_2 \vee Y_2$. In the graph $(e(H)P_2 \cup Y_1) \vee Y_2$, the edge of $H'$ can be expanded into an odd cycle by using an edge in $e(H)P_2$ and some vertices of $Y_1$ and $Y_2$, then we have $H'(t) \subseteq (e(H)P_2 \cup Y_1) \vee Y_2$. Therefore, $e(H)P_2$ contains a subgraph in $\mathcal{M}_2(H(t))$. Moreover, Proposition [11] (i) implies that $e(H)P_2 \in \mathcal{M}_2(H(t))$ holds. \qed

Let $H$ be a bipartite graph, $t \geq 5$ be an odd integer. Denoted by $\mathcal{N}(H(t)) \subseteq \mathcal{M}_2(H(t))$ is the family of graphs $M$ with $\gamma(M) = \bar{q}(H)$.

Lemma 2.3. Let $H$ be a bipartite graph, $M \in \mathcal{M}_2(H(t))$ be a graph with an independent covering $S \in \bar{\mathcal{S}}(H)$. If $S$ contains a vertex with degree one in $M$, then $\mathcal{N}(H(t))$ contains a graph with an isolated edge.

Proof. Let $H_M$ be a copy of $H$ and there exist two independent sets $Y_1$ and $Y_2$ such that $H_M(t) \subseteq (M \cup Y_1) \vee Y_2$ holds. If there is an isolated edge in $M$, the conclusion holds. Now suppose there is no isolated edge in $M$.

Let $v' \in S$ and $N_M(v') = \{u\}$ hold. If $d_M(u) = 1$, then the edge $uv'$ is an isolated edge in $M$, a contradiction. Thus we have $d_M(u) \geq 2$, then by Proposition [11] (iv), $u$ is a vertex
of $H_M$. Let $M'$ be the graph obtained from dividing vertex $u$ into \{$u'$, $u''$\}, $u'$ is adjacent to $u'$ with degree one, $u''$ is adjacent to the remaining neighbors of the original vertex $u$ in $M$. In the graph $(M' \cup Y_1) \cup Y_2$, there is a vertex $w \in Y_2$ adjacent to $u''$, the edge $u''w$ can be expanded into an odd cycle by using the edge $v'u'$ in $M'$. As there is an $H_M(t)$ in $(M \cup Y_1) \cup Y_2$, and the vertices in $M'$ except $v'$, $u'$, $u''$ have the same neighbors of the vertices in $M$, $(M' \cup Y_1) \cup Y_2$ contains a copy of $H(t)$ as a subgraph. It is easy to see that $e(M) = e(M')$ and there is an isolated edge in $M'$, thus we have $M' \in \mathcal{M}_2(H(t))$. As $S$ is the independent covering of $M$, $v'$ is a vertex in $S$, we have $u \notin S$. When we divide the vertex $u$ of $M$, $S$ is also an independent covering of $M'$. From the definition of $\tilde{S}(H)$, we have $|S| = \tilde{q}(H)$ and then $\gamma(M') \leq \tilde{q}(H)$. Because $\tilde{q}(H)$ is the minimum size of the independent covering of graphs in $\mathcal{M}_2(H(t))$ and $M' \in \mathcal{M}_2(H(t))$, we may deduce $\gamma(M') = \tilde{q}(H)$ and then $M' \in \mathcal{N}(H(t))$. \hfill \Box

**Lemma 2.4.** Let $H$ be a bipartite graph with $V(H) = A \cup B$, $M \in \mathcal{M}_2(H(t))$ be a graph with an independent covering $S \in \tilde{\mathcal{S}}(H)$ and $\min \{d_M(x) : x \in S\} = k(H)$. If each vertex of $S$ has degree at least 2 in $M$, then $|A| = \tilde{q}(H)$ and $\min \{d_H(x) : x \in A\} = k(H)$.

**Proof.** Let $H_M$ be a copy of $H$ and there exist two independent sets $Y_1$ and $Y_2$ such that $H_M(t) \subseteq (M \cup Y_1) \cup Y_2$ holds. By Proposition \[1.1] (iv), we have $S \subseteq V(H_M)$. Suppose $e$ is an edge of $H_M$. If $e \in E(M)$, then $e$ is covered by $S$. If $e$ is between $Y_1$ and $Y_2$, there is an isolated edge in $M$, a contradiction. If $e$ is between $M$ and $Y_2$, then let $xy = e$, $x \in V(M)$, $y \in Y_2$. Since there is no isolated edge in $M$, there is an edge $xy'' \in E(M)$ to be used to expand $e$ into an odd cycle, where $N_M(y'') = \{x\}$, then we have $x \in S$. Hence the edge $e$ is covered by $S$. Therefore, $S$ is an independent covering of $H_M$ and then $|A| = \tilde{q}(H)$ holds.

Suppose $w$ is a vertex in $S$. If $d_M(w) < d_{H_M}(w)$, $N_{H_M}(w)$ contains a vertex $w'$ in $Y_2$. The edge $w'w$ can be expanded into an odd cycle by using an edge $e$ in $M$ and $e$ is in $H_M(t) - V(H_M)$. The fact that the new vertices of the odd cycles are all distinct implies $e$ is an isolated edge in $M$, a contradiction. If $d_M(w) > d_{H_M}(w)$, then $M$ contains $d_M(w) - d_{H_M}(w)$ edges not in $H_M(t)$, a contradiction to the minimality of $M$. So $d_M(w) = d_{H_M}(w)$ holds. As $w$ is arbitrary, we have $\min \{d_H(x) : x \in A\} = \min \{d_{H_M}(x) : x \in f^{-1}(A)\} = \min \{d_M(x) : x \in S\} = k(H)$. \hfill \Box

**Example 1** For the star $S_a$, by Lemma \[2.1], we have $\mathcal{M}_2(S_a(t)) \subseteq \mathcal{D}(S_a)$. On the other hand, for any $M \in \mathcal{D}(S_a)$, we have $M = S_x \cup (a - x)P_2$ (1 \leq x \leq a). $Y_1$ and $Y_2$ are two independent sets and large enough. In the graph $(M \cup Y_1) \cup Y_2$, by using the vertices of $Y_1$ and $Y_2$, different edges of $M$ can be used to expand different odd cycles of $S_a(t)$. Since $e(M) = e(S_a)$, we have $S_a(t) \subseteq (M \cup Y_1) \cup Y_2$ and $M \in \mathcal{M}_2(S_a(t))$. As $M$ is arbitrary, we have $\mathcal{D}(S_a) \subseteq \mathcal{M}_2(S_a(t))$. Hence we have $\mathcal{M}_2(S_a(t)) = \mathcal{D}(S_a)$. Then we may imply $\tilde{q}(S_a) = 1$, $\tilde{S}(S_a)$ is the center vertex of the star and $k(S_a) = a$. For any graph $M \in \mathcal{M}_2(S_a(t))$, $\beta(M) \geq \tilde{q}(S_a)$, hence we have $\tilde{B}(S_a) = \{K_1\}$.


Example 2 For the path $P_{m+1}$, by Lemma 2.1, $\mathcal{M}_2(P_{m+1}(t)) \subseteq \mathcal{D}(P_{m+1})$. On the other hand, for any $M \in \mathcal{D}(P_{m+1})$, $M$ is a union of some paths and $e(M) = m$. Set $Y_1$ and $Y_2$ be two independent sets and large enough. In the graph $(M \cup Y_1) \cup Y_2$, by using the vertices of $Y_1$ and $Y_2$, different edges of $M$ can be used to expanded different odd cycles of $P_{m+1}(t)$. Since $e(M) = m$, we have $P_{m+1}(t) \subseteq (M \cup Y_1) \cup Y_2$ and $M \in \mathcal{M}_2(P_{m+1}(t))$. Hence we have $\mathcal{D}(P_{m+1}) \subseteq \mathcal{M}_2(P_{m+1}(t))$. Therefore $\mathcal{D}(P_{m+1}) = \mathcal{M}_2(P_{m+1}(t))$ holds. When $m$ is even, $\tilde{q}(P_{m+1}) = \frac{m}{2}$, $\tilde{S}(P_{m+1})$ consists of the independent coverings of the graphs in $\mathcal{M}_2(P_{m+1}(t))$ such that each component of them is a path with even edges, and $k(P_{m+1}) = 2$. For any graph $M \in \mathcal{M}_2(P_{m+1}(t))$, $\beta(M) \geq \tilde{q}(P_{m+1})$, hence we have $\tilde{B}(P_{m+1}) = \{K_{\frac{m}{2}}\}$. When $m$ is odd, we have $\tilde{q}(P_{m+1}) = \frac{m+1}{2}$. Then $\tilde{S}(P_{m+1})$ consists of the independent coverings of the graphs in $\mathcal{M}_2(P_{m+1}(t))$ such that each component of them is a path with even edges except one component is a path with odd edges and $k(P_{m+1}) = 1$. For any graph $M \in \mathcal{M}_2(P_{m+1}(t))$, $\beta(M) \geq \tilde{q}(P_{m+1})$, hence we have $\tilde{B}(P_{m+1}) = \{K_{\frac{m+1}{2}}\}$.

As stars, paths and even cycles satisfy the conditions of Theorem 1.3, Theorem 1.3 implies the Turán numbers for odd-ballooning of stars, paths and even cycles.

Corollary 2.1 (Hou, Qiu and Liu 8). If $n$ is sufficiently large and $t \geq 5$ is an odd integer, then

$$\text{ex}(n, S_a(t)) = e(T_2(n)) + (a - 1)^2$$

holds and the only extremal graph for $S_a(t)$ is the graph obtained from $T_2(n)$ by putting a copy of $K_{a-1, a-1}$ in one class of $T_2(n)$.

Proof. Let $F$ be the graph obtained by putting a copy of $K_{a-1, a-1}$ in one class of $T_2(n)$. From Example 1, we have $\mathcal{M}_2(S_a(t)) = \{S_x \cup (a - x)P_2 \mid 1 \leq x \leq a\}$. If $S_a(t) \subseteq F$, then we have $S_a(t) \subseteq (K_{a-1, a-1} \cup I_m) \cup I_m$ where $m = \lceil \frac{a}{2} \rceil - 2a$. So $K_{a-1, a-1}$ contains a subgraph as a copy of a member of $\mathcal{M}_2(S_a(t))$. For any graph $M \in \mathcal{M}_2(S_a(t))$, $M = S_x \cup (a - x)P_2$ ($1 \leq x \leq a$), we have $x + a'(M - V(S_x)) = a$. However, $x + a'(K_{a-1, a-1} - V(S_x)) = a - 1 < a$.

So $F$ is $S_a(t)$-free and

$$\text{ex}(n, S_a(t)) \geq e(T_2(n)) + (a - 1)^2.$$ 

On the other hand, from Example 1, we have $\tilde{q}(S_a) = 1$, $k(S_a) = a$ and $\tilde{B}(S_a) = \{K_1\}$. By applying Theorem 1.3 we have

$$\text{ex}(n, S_a(t)) \leq f(n, 1) + (a - 1)^2 = e(T_2(n)) + (a - 1)^2.$$

Therefore, $\text{ex}(n, S_a(t)) = e(T_2(n)) + (a - 1)^2$ holds. Noting that $\mathcal{F}(n, 1, a - 1, K_1) = \{F\}$ holds, hence $F$ is the only extremal graph for $S_a(t)$.

Corollary 2.2 (Zhu, Kang and Shan 13). Let $n$ be a sufficiently large integer, $t$ be an odd integer at least 5. We have the following:
(i) If $m$ is even, let $d = \frac{m}{2}$, then
\[ \text{ex}(n, P_{m+1}(t)) = e(T_2(n - d + 1) \lor K_{d-1}) + 1 \]
holds and the only extremal graph for $P_{m+1}(t)$ is the graph obtained from $T_2(n - d + 1) \lor K_{d-1}$ by putting an edge in one class of $T_2(n - d + 1)$.

(ii) If $m$ is odd, let $d = \frac{m+1}{2}$, then
\[ \text{ex}(n, P_{m+1}(t)) = e(T_2(n - d + 1) \lor K_{d-1}) \]
holds and $T_2(n - d + 1) \lor K_{d-1}$ is the unique extremal graph for $P_{m+1}(t)$.

Proof. (i) When $m$ is even, let $F$ be obtained from $T_2(n - d + 1) \lor K_{d-1}$ by putting an edge in one class of $T_2(n - d + 1)$. In $F$, each vertex of $K_{d-1}$ is contained in at most two odd cycles of $P_{m+1}(t)$ and the edge in one class of $T_2(n - d + 1)$ can be contained in only one odd cycle of $P_{m+1}(t)$. Therefore, in $F$, the number of odd cycles in the odd-ballooning of $P_{m+1}$ is at most $2(d - 1) + 1 = m - 1 < m$. So $F$ is $P_{m+1}(t)$-free and
\[ \text{ex}(n, P_{m+1}(t)) \geq e(T_2(n - d + 1) \lor K_{d-1}) + 1 = e(F). \]
On the other hand, from Example 2, we have $\tilde{q}(P_{m+1}) = \frac{m}{2}$, $k(P_{m+1}) = 2$, $\tilde{B}(P_{m+1}) = \{K_{\frac{m}{2}}\}$. By applying Theorem 1.3 we have
\[ \text{ex}(n, P_{m+1}(t)) \leq f'(n, d) + \text{ex}(d - 1, K_d) + 1 = e(T_2(n - d + 1) \lor K_{d-1}) + 1. \]
Therefore $\text{ex}(n, P_{m+1}(t)) = e(T_2(n - d + 1) \lor K_{d-1}) + 1$ holds. Noting that $\mathcal{F}(n, d, 1, K_{d-1}) = \{F\}$, hence $F$ is the only extremal graph for $P_{m+1}(t)$.

(ii) When $m$ is odd, from Example 2, we have $\tilde{q}(P_{m+1}) = \frac{m+1}{2}$, $k(P_{m+1}) = 1$, $\tilde{B}(P_{m+1}) = \{K_{\frac{m+1}{2}}\}$. By applying Theorem 1.3 the lower bound and the upper bound of the inequality in Theorem 1.3 are the same, we have
\[ \text{ex}(n, P_{m+1}(t)) = f(n, d) + \text{ex}(d - 1, K_d) = e(T_2(n - d + 1) \lor K_{d-1}) \]
and $T_2(n - d + 1) \lor K_{d-1}$ is the unique extremal graph for $P_{m+1}(t)$.

Using the similar arguments as the proof of Corollary 2.2 (i), we have the following corollary.

**Corollary 2.3** (Zhu, Kang and Shan [13]). Given an even integer $m \geq 4$, an odd integer $t \geq 5$ and a sufficiently large integer $n$, we have $\text{ex}(n, C_m(t)) = e(T_2(n - d + 1) \lor K_{d-1}) + 1$ where $d = \frac{m}{2}$. The only extremal graph for $C_m(t)$ is the graph obtained from $T_2(n - d + 1) \lor K_{d-1}$ by putting an edge in one class of $T_2(n - d + 1)$.
Let $T$ be a tree and $V(T) = A \cup B$. An odd-balooning $T(t)(t \geq 3)$ of $T$ is good if all edges which are expanded into triangles are the edges who have one endpoint with degree one and the non-leaf vertices are in $A$. Recently, Zhu [14] gave the exact value of $ex(n, T(t))(t \geq 3)$ when $T(t)$ is a good odd-balooning of $T$. As $\mathcal{M}_2(T(t)) \subseteq \mathcal{D}(T)$, when $T(t)$ is a good odd-balooning, we have $\overline{q}(T) = |A|$. Hence if $A$ has a vertex with degree one in $T$, we have the following corollary.

**Corollary 2.4** (Zhu and Chen [14]). Let $T$ be a tree with $V(T) = A \cup B$ and $a = |A|$. Suppose $T(t)$ is a good odd-balooning of $T$ where $t \geq 5$ is an odd cycle. If $A$ contains a vertex $u$ with $d_T(u) = 1$, then

$$ex(n, T(t)) = f(n, a) + ex(a - 1, \overline{B}(T)).$$

Moreover the extremal graphs for $T(t)$ are in $\mathcal{F}(n, a, 0, \overline{B}(T))$.

3. Proof of Theorem [1.3]

This section is devoted to the proof of Theorem [1.3]. First, we prove the lower bound of Theorem [1.3].

**Lemma 3.1.** Let $H$ be a bipartite graph, $n$ be a sufficiently large integer, $t \geq 5$ be an odd integer, then we have

$$ex(n, H(t)) \geq f(n, \overline{q}(H)) + ex(\overline{q}(H) - 1, \overline{B}(H)).$$

**Proof.** If there is an $H(t) \subseteq F \in \mathcal{F}(n, \overline{q}(H), 0, \overline{B}(H))$ i.e.,

$$H(t) \subseteq (Q \lor I_m) \lor I_m \subseteq ((Q \lor I_m) \cup I_m) \lor I_m,$$

where $Q \in EX(\overline{q}(H) - 1, \overline{B}(H))$ and $m = \lceil \frac{n - \overline{q}(H) + 1}{2} \rceil$. By the definition of $\mathcal{M}_2(H(t))$, $(G \lor I_m) \lor I_m$ contains a copy of $H(t)$, then $G$ contains a subgraph as a member of $\mathcal{M}_2(H(t))$. Thus $Q \lor I_m$ contains a subgraph as a member, say $M$ of $\mathcal{M}_2(H(t))$. Since $I_m$ is an independent set, $Q$ contains a subgraph induced by a covering of $M$. However, when $\beta(M) < \overline{q}(H)$ holds, since the graphs in $EX(\overline{q}(H) - 1, \overline{B}(H))$ are $\overline{B}(H)$-free, $Q$ contains no subgraph induced by a covering of $M$; when $\beta(M) \geq \overline{q}(H)$, since the order of $Q \in EX(\overline{q}(H) - 1, \overline{B}(H))$ is $\overline{q}(H) - 1$, $Q$ contains no subgraph induced by a covering of $M$. This contradiction shows any graph in $\mathcal{F}(n, \overline{q}(H), 0, \overline{B}(H))$ is $H(t)$-free.

Note that for any $F$ in $\mathcal{F}(n, \overline{q}(H), 0, \overline{B}(H))$, we have $e(F) = f(n, \overline{q}(H)) + ex(\overline{q}(H) - 1, \overline{B}(H))$. Therefore, $ex(n, H(t)) \geq f(n, \overline{q}(H)) + ex(\overline{q}(H) - 1, \overline{B}(H))$ holds. \qed

To prove Theorem 3.1 we need the following lemmas. $c(H)$ is the number of components of $H$. 9
Lemma 3.2 (Hou, Qiu and Liu [2]). Let $H$ be a graph with no isolated vertex. If $\Delta(H) \leq 2$, then
\[ \alpha'(H) \geq \frac{\nu(H) - e(H)}{2}. \]

Lemma 3.3 (Hou, Qiu and Liu [2]). Let $H$ be a graph with no isolated vertex. If for all $x \in V(H)$, $d(x) + \alpha'(H - N[x]) \leq k$, then $e(H) \leq k^2$. Moreover, the equality holds if and only if $H = K_k, k$.

Define $\varphi(\alpha', \Delta)=\max\{e(H) : \alpha'(H) \leq \alpha', \Delta(H) \leq \Delta\}$. Chvátal and Hanson [3] proved the following theorem which is useful to estimate the number of edges of a graph with restricted degrees and matching number.

Lemma 3.4 (Chvátal and Hanson [3]). For any graph $H$ with maximum degree $\Delta \geq 1$ and matching number $\alpha' \geq 1$, we have
\[ e(H) \leq \varphi(\alpha', \Delta) = \alpha'\Delta + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lceil \frac{\alpha'}{\Delta/2} \right\rceil \leq \alpha'\Delta + \alpha'. \]

Theorem 3.1. Let $n, b$ be sufficiently large positive integers, $H$ be a bipartite graph. Let $G$ be a graph of order $n$ with a partition of vertices into three parts $V(G) = V_0 \cup V_1 \cup V_2$ satisfying the following conditions:

(i) there exist $V'_1 \subseteq V_1$, $V'_2 \subseteq V_2$ such that $G[V'_1 \cup V'_2] = T_2(2b)$;
(ii) $|V_0| = \tilde{q}(H) - 1$ and each vertex of $V_0$ is adjacent to each vertex of $T_2(2b)$;
(iii) each vertex of $V''_i = V_i \setminus V'_i$ is adjacent to at least $c_0|V'_{3-i}|$ ($\frac{1}{2} < c_0 \leq 1$) vertices of $V'_{3-i}$, and is not adjacent to any vertex of $V'_i$ ($i = 1, 2$).

If $G$ is $H(t)$-free, then
\[ e(G) \leq f(n, \tilde{q}(H)) + \text{ex}(\tilde{q}(H) - 1, \overline{B}(H)) + (k(H) - 1)^2 \]
and the equality holds if and only if $G \in \mathcal{F}(n, \tilde{q}(H), k(H) - 1, \overline{B}(H))$.

Proof. Let $M$ be a graph in $\mathcal{M}_2(H(t))$ with an independent set $S \subseteq \tilde{S}(H)$ and $\min\{d_M(x) : x \in S\} = k(H)$. $\overline{B}(H)$ is the subgraph covering family of $\mathcal{M}_2(H(t))$.

If $G[V_0]$ contains a graph in $\overline{B}(H)$, there exists an $M \subseteq G[V_0 \cup V'_1]$ such that $M \in \mathcal{M}_2(H(t))$ holds. Then we have $H(t) \subseteq G[V_0 \cup V'_1 \cup V'_2] \subseteq G$, which is a contradiction. Thus, $e(G[V_0]) \leq \text{ex}(\tilde{q}(H) - 1, \overline{B}(H))$.

If $S$ contains a vertex with degree one in $M$, by Lemma 2.3, we know that $\mathcal{M}_2(H(t))$ contains a graph $M'$ with an isolated edge, say $uv$, and $\gamma(M') = \tilde{q}(H) = \gamma(M' - uv) = \tilde{q}(H) - 1$. And in this case, we have $k(H) = 1$. If there is an edge in $G[V''_1]$, noting that $|V_0| = \tilde{q}(H) - 1$ and each vertex of $V_0$ is adjacent to each vertex of $V'_1$, then $G[V_0 \cup V_1]$ contains a copy of $M'$. The number of vertices in $V'_2$ adjacent to $V''_1$ is sufficiently large. Then we have a copy.
of $H(t) \subseteq G[V_0 \cup V_1 \cup V_2'] \subseteq G$ which is a contradiction. Therefore there is no edge in $G[V_1]$. Similarly, there is no edge in $G[V_2]$. Then \( e(G) \leq f(n, \tilde{q}(H)) + \text{ex}(\tilde{q}(H) - 1, \tilde{B}(H)) \) holds. Together with Lemma 3.1 \( e(G) = f(n, \tilde{q}(H)) + \text{ex}(\tilde{q}(H) - 1, \tilde{B}(H)) \) holds and $G$ is in $\mathcal{F}(n, \tilde{q}(H), 0, \tilde{B}(H))$.

Now suppose that each vertex of $S$ has degree at least 2 in $M$, then by Lemma 2.4, we have \( |A| = \gamma(H) = \tilde{q}(H) \) and there is a vertex in $A$ with degree $k(H)$. Noting that \( |V_0| = \tilde{q}(H) - 1 \) holds and each edge between $V_0$ and $V_1'$ or $V_2'$ can be expanded into an odd cycle by using vertices of $V_1'$ and $V_2'$. If there is a copy of $S_k(H)(t) \subseteq G[V_1 \cup V_2]$ and $V_1', V_2'$ contain the neighbors of the center vertex of $S_k(H)$, there is a copy of $H(t)$ in $G$. Therefore, we may suppose $G[V_1 \cup V_2]$ contains no such $S_k(H)(t)$.

Let $V_1'' \subseteq V_1'$, $V_2'' \subseteq V_2'$ be the vertex sets, which are not isolated vertices in $G[V_1'']$ and $G[V_2'']$ respectively. In the following part of the proof, denote by $G_1 = G[V_1''], G_2 = G[V_2'']$, $G' = G_1 \cup G_2, m = e(G')$. For a vertex $x \in V(G_i)$, denote by $E_{3-i}(x) = \{e \in E(G_{3-i}) \mid V(e) \cap N_G(x) \neq \emptyset \}$.

If \( e(G) < f(n, \tilde{q}(H)) + \text{ex}(\tilde{q}(H) - 1, \tilde{B}(H)) + (k(H) - 1)^2 \) holds, the conclusion follows. Now suppose
\[
e(G) \geq f(n, \tilde{q}(H)) + \text{ex}(\tilde{q}(H) - 1, \tilde{B}(H)) + (k(H) - 1)^2
= e(T_2(n - \tilde{q}(H) + 1)) + (\tilde{q}(H) - 1)(n - \tilde{q}(H) + 1) + \text{ex}(\tilde{q}(H) - 1, \tilde{B}(H)) + (k(H) - 1)^2.\]

(3.1)

We have the following claims.

Claim 1. For every vertex $x \in V(G_i)$, we have $d_{G_i}(x) + \alpha'(G_i - N_G[x]) + \alpha'(G[E_{3-i}(x)]) \leq k(H) - 1$ (i = 1, 2).

Suppose to the contrary that there exists some $x \in V(G_i)$ such that $d_{G_i}(x) + \alpha'(G_i - N_G[x]) + \alpha'(G[E_{3-i}(x)]) \geq k(H)$ holds. Without loss of generality, we may suppose $x \in V(G_1)$. Let $x_1, x_2, \ldots, x_s \ (0 \leq s \leq k(H))$ be $s$ neighbors of $x$ in $V_1''$; $y_{s+1}, z_{s+1}, \ldots, y_u, z_u \ (s \leq u \leq k(H))$ be a matching in $G_1 - N_{G_1}[x]$; $y_{u+1}, z_{u+1}, \ldots, y_{k(H)}, z_{k(H)}$ be a matching in $G[E_2(x)]$ where $x$ is adjacent to $y_{u+1}, \ldots, y_{k(H)}$. Since the number of vertices in $V_1' \cup V_2'$ is sufficiently large and each vertex of $V_1''$ is adjacent to at least $c_0 |V_{3-i}'|$ vertices of $V_{3-i}'$ (i = 1, 2), we may find $k$ odd cycles intersecting in vertex $x$. When $1 \leq j \leq s$, let $C_{t_j} = xP_{t_j-2}x_1x$. $P_{t_j-2}$ is a path between $V_1'$ and $V_2'$ and the endpoints of the path are in $V_2'$. When $s + 1 \leq j \leq u$, let $C_{t_j} = xP_{t_j-4}z_jy_jw_jx$, $P_{t_j-4}$ is a path between $V_1'$ and $V_2'$ and the endpoints of the path are in $V_2'$. When $u + 1 \leq j \leq k(H)$, let $C_{t_j} = xP_{t_j-3}z_jy_jx, P_{t_j-3}$ is a path between $V_1'$ and $V_2'$ and one of the endpoints of the path is in $V_1'$, the other endpoint of the path is in $V_2'$. For any $j \in [1, k(H)], t_j \geq t$ is odd, the vertices of the paths in $C_{t_j}$ are different, $w_j$ is not in any paths in these cycles. Then there is a copy of $S_k(H)(t) \subseteq G[V_1 \cup V_2]$ and $V_1'$ or $V_2'$ contains the neighbors of the center vertex of $S_k(H)$. This implies that there is a copy of $H(t) \subseteq G$, a contradiction.
Claim 2. \( \alpha'(G_1) + \alpha'(G_2) \leq k(H) - 1. \)

Suppose to the contrary that \( \alpha'(G_1) + \alpha'(G_2) \geq k(H) \). Let \( \{x_1y_1, x_2y_2, \ldots, x_sy_s\} \) (0 \( \leq s \leq k(H) \)) be a matching in \( G_1 \) and \( \{x_{s+1}y_{s+1}, \ldots, x_{k(H)}y_{k(H)}\} \) be a matching in \( G_2 \). Since the number of vertices of \( V'_1 \cup V'_2 \) is sufficiently large and each vertex of \( V''_i \) (\( i = 1, 2 \)) is adjacent to at least \( c_0|V'_3-i| \) vertices of \( V'_3-i \), we may find a vertex \( v_0 \in V'_1 \) such that \( v_0 \in \cap_{i=s+1}^{k(H)} N_{V'_1}(x_i) \) holds and \( k(H) \) odd cycles intersect in exactly one vertex \( v_0 \). When \( 1 \leq j \leq s \), let \( C_{t_j} = v_0P_{t_j-4}x_jy_jw_jv_0, P_{t_j-4} \) is a path between \( V'_1 \) and \( V'_2 \) and the endpoints of the path are in \( V'_2 \), \( w_j \) is the vertex in \( V'_2 \). When \( s+1 \leq j \leq k(H) \), let \( C_{t_j} = v_0x_jy_jP_{t_j-3}v_0, P_{t_j-3} \) is a path between \( V'_1 \) and \( V'_2 \) and one of the endpoints of the path is in \( V'_1 \), the other endpoint of the path is in \( V'_2 \). For any \( j \in [1, k(H)] \), \( t_j \geq t \) is odd, the vertices of the paths in \( C_{t_j} \) are different, \( w_j \) is not in any paths in these cycles. Then there is a copy of \( S_{k(H)}(t) \subseteq G[V'_1 \cup V'_2] \) and \( V'_1 \) or \( V'_2 \) contains the neighbors of the center vertex of \( S_{k(H)} \). This implies that there is a copy of \( H(t) \subseteq G \), which is a contradiction. The result follows.

Claim 3. \( \max \{\Delta(G_1), \Delta(G_2)\} = k(H) - 1. \)

By Claim 1 we have \( \Delta(G_i) \leq k(H) - 1 \) (\( i = 1, 2 \)). If \( \max \{\Delta(G_1), \Delta(G_2)\} \leq k(H) - 2 \), then
\[
m = e(G_1) + e(G_2) \\
\leq \varphi(\alpha'(G_1), k(H) - 2) + \varphi(\alpha'(G_2), k(H) - 2) \\
\leq \varphi(\alpha'(G_1) + \alpha'(G_2), k(H) - 2) \\
\leq \varphi(k(H) - 1, k(H) - 2).
\]

By the construction of \( G \), we deduce \( e(G) \leq f(n, \tilde{q}(H)) + ex(\tilde{q}(H)-1, \tilde{B}(H)) + m \). Combining with (3.1), we have \( m \geq (k(H) - 1)^2 \). If \( k(H) = 2 \), then we have \( m \leq \varphi(1, 0) = 0 \), a contradiction. If \( k(H) \) is odd, then we have \( m \leq \varphi(k(H) - 1, k(H) - 2) < (k - 1)^2 \), a contradiction. If \( k(H) \) is even and \( k(H) \neq 4 \), we have \( m \leq \varphi(k(H) - 1, k(H) - 2) < (k(H) - 1)^2 \), a contradiction.

If \( k(H) = 4 \) holds, then we have \( m \leq \varphi(3, 2) = (k(H) - 1)^2 = 9 \). By (3.1), we have \( m = (k(H) - 1)^2 = 9 \). Then \( G' \) is a graph with \( e(G') = 9, \Delta(G') = 2 \) and \( \alpha'(G') = 3 \). Since \( \Delta(G') = 2 \) and \( G' \) has no isolated vertex, \( \nu(G') \geq e(G') = 9 \), the equality holds if and only if \( G' \) is 2-regular, and \( c(G') \leq \alpha'(G') = 3 \). On the other hand by Lemma 3.2, we obtain
\[
3 = \alpha'(G') \geq \frac{\nu(G') - c(G')} {2} \geq \frac{\nu(G') - 3} {2}.
\]

Hence \( \nu(G') \leq 9 \) and then \( \nu(G') = 9 \) holds. Therefore \( G' \) consists of three vertex-disjoint triangles. As \( m = (k(H) - 1)^2 \), then we have \( e(G) = f(n, \tilde{q}(H)) + ex(\tilde{q}(H) - 1, \tilde{B}(H)) + (k(H) - 1)^2 \). Therefore, each vertex of \( V''_1 \) is adjacent to each vertex of \( V''_2 \). Then for any vertex \( x \in V(G') \), we have \( d_{C_i}(x) + \alpha'(G_i - N_{C_i}[x]) + \alpha'(G[E_3-i(x)]) = 4 = k(H) \) (\( i = 1, 2 \)), a contradiction to Claim 1. Therefore, we have \( \max \{\Delta(G_1), \Delta(G_2)\} = k(H) - 1. \)
Claim 4. $e(G_1) \cdot e(G_2) = 0$.

First we have

$$m = e(G_1) + e(G_2)$$

$$\leq \varphi(\alpha'(G_1), k(H) - 1) + \varphi(\alpha'(G_2), k(H) - 1)$$

$$\leq \varphi(\alpha'(G_1) + \alpha'(G_2), k(H) - 1)$$

$$\leq \varphi(k(H) - 1, k(H) - 1)$$

$$\leq k(H)(k(H) - 1).$$

From Claim 3 we may suppose $\Delta(G_1) = k(H) - 1$, and $x$ is in $V_1''$ with $d_{G_1}(x) = k(H) - 1$. If $e(G_2) \geq 1$, then $\alpha'(G_2) \geq 1$. By Claim 2, $\alpha'(G_1) \leq k(H) - 1 - \alpha'(G_2) \leq k(H) - 2$. By Claim 1, we obtain $\alpha'(G[E_2(x)]) = 0$ which implies $V_2'' \cap N_G(x) = \emptyset$. Hence, for every $v \in V_2''$, $v$ is not adjacent to $x$. Let $n' = n - \tilde{q}(H) + 1$. So

$$e(V_1, V_2) \leq |V_1||V_2| - |V_2''| \leq e(T_2(n')) - |V_2''|.$$

Thus we have

$$e(T_2(n')) + (k(H) - 1)^2 \leq e(G[V_1 \cup V_2]) \leq e(T_2(n')) - |V_2''| + m.$$

Therefore, $|V_2''| \leq m - (k(H) - 1)^2$, and

$$m \leq \varphi(\alpha'(G_1), \Delta(G_1)) + \varphi(\alpha'(G_2), \Delta(G_2))$$

$$\leq \alpha'(G_1)(\Delta(G_1) + 1) + \alpha'(G_2)(\Delta(G_2) + 1)$$

$$\leq k(H)\alpha'(G_1) + (k(H) - 1 - \alpha'(G_1))|V_2''|$$

$$= \alpha'(G_1)(k(H) - |V_2''|) + (k(H) - 1)|V_2''|$$

$$\leq (k(H) - 2)(k(H) - |V_2''|) + (k(H) - 1)|V_2''|$$

$$= (k(H) - 1)^2 + |V_2''| - 1$$

$$\leq (k(H) - 1)^2 + m - (k(H) - 1)^2 - 1$$

$$= m - 1.$$

This contradiction shows $e(G_2) = 0$. Therefore, we have proved $e(G_1) \cdot e(G_2) = 0$.

By Claim 1 and Claim 4, for any vertex $x \in V_i$ we know that $d_{G_i}(x) + \alpha'(G_i - N_{G_i}[x]) + \alpha'(G[E_{3-i}(x)]) \leq k(H) - 1$ and $e(G_1) \cdot e(G_2) = 0$ hold. Then we have $d_{G'}(x) + \alpha'(G' - N_{G'}[x]) \leq k(H) - 1$. Applying Lemma 3.3, we deduce $e(G') \leq (k(H) - 1)^2$. The equality holds if and only if $G' = K_{k(H)-1, k(H)-1}$. Therefore, we have $e(G) = f(n, \tilde{q}(H)) + \exp(\tilde{q}(H) - 1, B(H)) + (k(H) - 1)^2$ and $G \in F(n, \tilde{q}(H), k(H) - 1, B(H))$. The proof is complete.

We mainly use the so-called progressive induction to prove the upper bound of Theorem 1.3 and this technique is borrowed from [12].
Theorem 3.2 (Simonovites [11]). Let \( \mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i \) be a set of given elements, such that \( \mathcal{U}_i \) are disjoint finite subsets of \( \mathcal{U} \). Let \( P \) be a condition or property defined on \( \mathcal{U} \) which means the elements of \( \mathcal{U} \) may satisfy or not satisfy \( P \). Let \( \phi(x) \) be a function defined on \( \mathcal{U} \) such that \( \phi(x) \) is a non-negative integer and

(i) if \( x \) satisfies \( P \), then \( \phi(x) = 0 \);
(ii) there is an \( n_0 \) such that if \( n > n_0 \) and \( x \in \mathcal{U}_n \) then either \( x \) satisfies \( P \) or there exist an \( n' \) and an \( x' \) such that

\[
\frac{n}{2} < n' < n, \quad x' \in \mathcal{U}_{n'} \quad \text{and} \quad \phi(x) < \phi(x').
\]

Then there exists an \( n_0 \) such that if \( n > n_0 \), every \( x \in \mathcal{U}_n \) satisfies \( P \).

Proof of Theorem 1.3. Let \( G_n \) be an extremal graph for \( H(t) \) of order \( n \), \( F_n \in \mathcal{F}(n, \bar{q}(H), k(H) - 1, \bar{B}(H)) \), then \( e(F_n) = f(n, \bar{q}(H)) + \text{ex}(\bar{q}(H) - 1, \bar{B}(H)) + (k(H) - 1)^2 \). Let \( \mathcal{U}_n \) be the set of extremal graphs for \( H(t) \) of order \( n \), \( P \) be the property defined on \( \mathcal{U} \) satisfying that \( e(G_n) \leq e(F_n) \) and the equality holds if and only if \( G_n \in \mathcal{F}(n, \bar{q}(H), k(H) - 1, \bar{B}(H)) \). Define \( \phi(G_n) = \max\{e(G_n) - e(F_n), 0\} \). If \( G_n \) satisfies \( P \), then \( \phi(G_n) = 0 \), which implies the condition (i) in Theorem 3.2 is satisfied.

In the following part we may prove either \( G_n \) satisfies \( P \) or there exist an \( n' \) and a \( G_{n'} \) such that

\[
\frac{n}{2} < n' < n, \quad G_{n'} \in \mathcal{U}_{n'} \quad \text{and} \quad \phi(G_n) < \phi(G_{n'}).\]

By Theorem 1.2 and the fact \( e(G_n) \geq f(n, \bar{q}(H)) + \text{ex}(\bar{q}(H) - 1, \bar{B}(H)) \geq \frac{n^2}{4} \), there is an \( n_1 \) such that if \( n > n_1 \), \( G_n \) contains \( T_2(2n_2) \) (\( n_2 \) is sufficiently large) as a subgraph. By Lemma 2.2 we have \( e(H)P_2 \in \mathcal{M}_2(H(t)) \). In \( G_n \) each class of \( T_2(2n_2) \) contains no copy of \( e(H)P_2 \). Otherwise it follows from the definition of the 2-decomposition family that \( G_n \) contains a copy of \( H(t) \), a contradiction. Hence, there is an induced subgraph \( T_2(2n_3) \) (\( n_3 \) is also sufficiently large) of \( G_n \) by deleting \( 2e(H) \) vertices of each class of \( T_2(2n_2) \).

Let \( c \) be a sufficiently small constant and \( T_0 = T_2(2n_3), X = V(G_n) \setminus V(T_0) \). We pick vertices \( x_t \in X \) and graphs \( T_t \) recursively: \( x_t \) is the vertex which has at least \( c'n_3 \) neighbors in each class of \( T_{t-1} \), and \( T_t = T_2(2c^n_3) \) is the subgraph of \( T_{t-1} \) induced by the neighbors of \( x_t \). \( B_1^t \) and \( B_2^t \) are the vertex sets of two classes of \( T_t \). The progress stops after at most \( \bar{q}(H) - 1 \) steps. If \( t \geq \bar{q}(H), G_n[\{x_1, x_2, \cdots, x_{\bar{q}(H)}\} \cup B_1^{\bar{q}(H)}] \) contains a copy of \( M \), then \( G_n \) contains a copy of \( H(t) \). Therefore, we may suppose the progress ends at \( x_s \) and \( T_s \) where \( s \leq \bar{q}(H) - 1 \). Denote by \( Y = \{x_1, \cdots, x_s\} \).

Next we divide \( V(G_n) \setminus (V(T_s) \cup Y) \). If \( x \in V(G_n) \setminus (V(T_s) \cup Y) \) is adjacent to less than \( c^{s+1}n_3 \) vertices of \( B_1^s \) and is adjacent to at least \( (1 - \sqrt{c})c^n_3 \) vertices of \( B_2^{s-1} \), then we put \( x \) in \( C_i \) (\( i = 1, 2 \)). If \( x \in V(G_n) \setminus (V(T_s) \cup Y) \) is adjacent to less than \( c^{s+1}n_3 \) vertices
of \( B^*_i \) and is adjacent to less than \((1 - \sqrt{c})c^s n_3\) vertices of \( B^*_{3-i} \), for some \( i \in \{1, 2\} \), then we put \( x \) in \( D \). Then \( V(G) \setminus (V(T_s) \cup Y) = C_1 \cup C_2 \cup D \) holds.

The number of independent edges in \( G_n[\mathcal{B}^*_i \cup C_i] \) is less than \( e(H) \). Otherwise, if \( e(H)P_2 \subseteq G_n[\mathcal{B}^*_i \cup C_i] \), \( G_n \) contains a copy of \( H(t) \). Consider the edges joining \( \mathcal{B}^*_i \) and \( C_i \) and select a maximal set of independent edges, say \( y_1 z_1, \ldots, y_m z_m \) with \( y_j \in \mathcal{B}^*_i \), \( z_j \in C_i \) and \( 1 \leq j \leq m \), \( 1 \leq m < \ell \), where \( \ell = e(H) \). The number of vertices of \( \mathcal{B}^*_i \) joining to at least one of \( z_1, z_2, \ldots, z_m \) is less than \( c^s + 1 \ell n_3 \) and the remaining vertices of \( \mathcal{B}^*_i \) are not adjacent to any vertex of \( C_i \). Therefore there are at least \((1 - c\ell)c^s n_3\) vertices of \( \mathcal{B}^*_i \) which are not adjacent to any vertices of \( C_i \). We may move these \( c^s + 1 \ell n_3 \) vertices of \( \mathcal{B}^*_i \) to \( C_i \) to obtain \( \mathcal{B}^*_i \) and \( C_i' \) such that \( \mathcal{B}^*_i \subseteq \mathcal{B}^*_i \), \( C_i \subseteq C_i' \) and there are no edges between \( \mathcal{B}^*_i \) and \( C_i' \).

In conclusion, the vertices of \( G_n \) can be partitioned into \( V(T'_s), C'_1, C'_2, D \) and \( Y \), where \( T'_s = T_2(2n_4) \) with classes \( B'_i \) and \( B'_2 \), \( n_4 = c^s n_3 - c^s + 1 \ell n_3 \).

(i) \(|Y| = s \) and each \( v \in Y \) is adjacent to each vertex of \( T_2(2n_4) \).

(ii) Each vertex of \( C'_i \) is adjacent to at least \((1 - \sqrt{c} - c\ell)c^s n_3\) vertices of \( B^*_{2-i} \) and is not adjacent to any vertex of \( B'_i \) (\( i = 1, 2 \)).

(iii) Each vertex of \( D \) is adjacent to less than \( c^s + 1 \ell n_3 \) vertices of \( B'_i \) and is adjacent to less than \((1 - \sqrt{c})c^s n_3\) vertices of \( B^*_{3-i} \) for some \( i \in \{1, 2\} \).

Denote by \( \hat{G} = G_n - V(T'_s) \). Since \( \hat{G} \) does not contain a copy of \( H(t) \), we have \( e(\hat{G}) \leq e(G_n - 2n_4) \). There is a \( T'_s \) contained in \( F_n \). Denote by \( \hat{F} = F_n - V(T'_s) \). Then

\[
e(G) - e(F) = e(T'_s) + e \left( V(\hat{G}), V(T'_s) \right) + e(\hat{G}) - e(T'_s) + e \left( V(\hat{F}), V(T'_s) \right) + e(\hat{F})
\]

\[
\leq e(\hat{G}) - e(\hat{F}) + e \left( V(\hat{G}), V(T'_s) \right) - e \left( V(\hat{F}), V(T'_s) \right)
\]

Then we have \( \phi(G_n) \leq \phi(G_n - 2n_4) + e \left( V(\hat{G}), V(T'_s) \right) - e \left( V(\hat{F}), V(T'_s) \right) \).

On the other hand

\[
e \left( V(\hat{G}), V(T'_s) \right) - e \left( V(\hat{F}), V(T'_s) \right)
\]

\[
\leq 2sn_4 + (n - 2n_4 - |D|)n_4 + |D| [c^s + 1n_3 + (1 - \sqrt{c})c^s n_3] - [\ell q(H) - 1]2n_4 + n_4 (n - \ell q(H) + 1 - 2n_4)
\]

\[
= [2s + n - 2n_4 - |D| - 2(\ell q(H) - 1) - (n - \ell q(H) + 1)]n_4
\]

\[
+ |D| c^s + 1 + |D| (1 - \sqrt{c}) c^s n_3
\]

\[
= (s - \ell q(H) + 1)n_4 + (c(\ell + 1) - \sqrt{c}) c^s n_3 |D|
\]

\[
\leq 0.
\]

If \( e \left( V(\hat{G}), V(T'_s) \right) - e \left( V(\hat{F}), V(T'_s) \right) < 0 \), then \( \phi(G_n) < \phi(G_n - 2n_4) \) holds. Since \( n - 2n_4 > \frac{n}{2} \), the condition (ii) in Theorem 3.2 is satisfied.
If \( e\left(V(\tilde{G}), V(T'_3)\right) - e\left(V(\tilde{F}), V(T'_3)\right) = 0 \), then \( s = \tilde{q}(H)-1, |D| = 0 \), each vertex of \( C'_i \) is adjacent to at least \( (1 - \sqrt{c - c_0 n_4})c s n_3 \) (not less than \( c_0 n_4, c_0 \in (\frac{1}{2}, 1] \) is a constant) vertices of \( B_{3-i}' \) (\( i = 1, 2 \)). By Theorem 3.1 we have \( e(G_n) \leq f(n, \tilde{q}(H)) + \mathrm{ex}(\tilde{q}(H) - 1, \tilde{B}(H)) + (k(H) - 1)^2 \), and the equality holds if and only if \( G_n \in F(n, \tilde{q}(H), k(H) - 1, \tilde{B}(H)) \).

Since \( n \) is sufficiently large, there exists an \( n_0 \) such that \( n > n_0 \). Hence \( \mathrm{ex}(n, H(t)) \leq f(n, \tilde{q}(H)) + \mathrm{ex}(\tilde{q}(H) - 1, \tilde{B}(H)) + (k(H) - 1)^2 \), and the equality holds if and only if the extremal graphs for \( H(t) \) of order \( n \) are in \( F(n, \tilde{q}(H), k(H) - 1, \tilde{B}(H)) \). The proof is complete.

**Remark**

This paper determines the range of Turán numbers for odd-ballooning of general bipartite graphs obtained from replacing each edge by an odd cycle of order \( t \) where \( t \geq 5 \) is an odd integer. Given an integer \( p \), the edge blow-up of a graph \( H \), denoted by \( H^{p+1} \), is the graph obtained from replacing each edge in \( H \) by a clique of order \( p+1 \), and the new vertices of the cliques are all distinct. Yuan in [12] determined the range of Turán numbers for edge blow-up of all bipartite graphs when \( p \geq 3 \). The Turán numbers for \( H(3) \) has been determined when \( H \) is a star, a path or an even cycle. While the Turán numbers for \( H(3) \) when \( H \) is a general bipartite graph are unclear.

**Declaration**

The authors have declared that no competing interest exists.

**References**

[1] N. Balachandean, N. Khare, Graphs with restricted valency and matching number, Discrete Math. 309 (2009) 4176-4180.

[2] G. Chen, R.J. Gould, F. Pfender, B. Wei, Extremal graphs for intersecting cliques, J. Combin. Theory Ser. B 89 (2003) 159-171.

[3] V. Chvátal, D. Hanson, Degrees and matchings, J. Combin. Theory Ser. B 20 (1976) 128-138.

[4] P. Erdős, A. H. Stone, On the structure of linear graphs, Bull. Am. Math. Soc. 52 (1946) 1089-1091.

[5] P. Erdős, M. Simonovits, A limit theorem in graph theory, Studia Sci. Math. Hung. 1 (1966) 51-57.

[6] T. Gallai, Neuer Beweis eines Tutte’schen Satzes, Magy. Tud. Akad. Mat. Kut. Intéz. Közl. 8 (1963) 135-139.
[7] X. Hou, Y. Qiu, B. Liu, Extremal graph for intersecting odd cycles, Electron. J. Combin. 23 (2016) 2. 29.

[8] X. Hou, Y. Qiu, B. Liu, Turán number and decomposition number of intersecting odd cycles, Discrete Math. 341 (2018) 126-137.

[9] H. Liu, Extremal graphs for blow-ups of cycles and trees, Electron. J. Combin. 20 (1) (2013) 65.

[10] Z. Ni, L. Kang, E. Shan, H. Zhu, Extremal graphs for blow-ups of keyrings, Graphs Combin. 36 (2020) 1827-1853.

[11] M. Simonovites, A method for solving extremal problems in graph theory, stability problems, in: Theory of Graphs, Proc. Colloq., Tihany, 1966, Academic Press, New York, 1968, pp. 279-319.

[12] L. Yuan, Extremal graphs for edge blow-up of graphs, J. Combin. Theory Ser. B 152 (2022) 397-398.

[13] H. Zhu, L. Kang, E. Shan, Extremal graphs for odd-ballooning of paths and cycles, Graphs Combin. 36 (2020) 755-756.

[14] X. Zhu, Y. Chen, Turán number for odd-ballooning of trees, J. Graph Theory, (2023). https://doi.org/10.1002/jgt.22959.