SPECTRALITY OF POLYTOPES AND EQUIDECOMPOSABILITY
BY TRANSLATIONS

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Abstract. Let \( A \) be a polytope in \( \mathbb{R}^d \) (not necessarily convex or connected). We say that \( A \) is spectral if the space \( L^2(A) \) has an orthogonal basis consisting of exponential functions. A result due to Kolountzakis and Papadimitrakis (2002) asserts that if \( A \) is a spectral polytope, then the total area of the \((d - 1)\)-dimensional faces of \( A \) on which the outward normal is pointing at a given direction, must coincide with the total area of those \((d - 1)\)-dimensional faces on which the outward normal is pointing at the opposite direction. In this paper, we prove an extension of this result to faces of all dimensions between 1 and \( d - 1 \). As a consequence we obtain that any spectral polytope \( A \) can be dissected into a finite number of smaller polytopes, which can be rearranged using translations to form a cube.

1. Introduction

1.1. Let \( A \subset \mathbb{R}^d \) be a bounded, measurable set of positive Lebesgue measure. It is said to be spectral if there exists a countable set \( \Lambda \subset \mathbb{R}^d \) such that the system of exponential functions

\[
E(\Lambda) = \left\{ e_\lambda \right\}_{\lambda \in \Lambda}, \quad e_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle},
\]

is orthogonal and complete in \( L^2(A) \), that is, the system is an orthogonal basis for the space. Such a set \( \Lambda \) is called a spectrum for \( A \). The classical example of a spectral set is the unit cube \( A = [-\frac{1}{2}, \frac{1}{2}]^d \), for which the set \( \Lambda = \mathbb{Z}^d \) serves as a spectrum.

Interest in spectral sets has been inspired for many years by an observation due to Fuglede [Fug74], that the notion of spectrality is closely related to another, geometrical notion – the tiling by translations. We say that \( A \) tiles the space by translations if there exists a countable set \( \Lambda \subset \mathbb{R}^d \) such that the collection of sets \( \{A + \lambda\}, \lambda \in \Lambda \), consisting of translated copies of \( A \), constitutes a partition of \( \mathbb{R}^d \) up to measure zero.

Fuglede originally conjectured that a set \( A \subset \mathbb{R}^d \) is spectral if and only if it can tile the space by translations. While it is still an open problem whether this conjecture holds e.g. for convex domains\(^1\) (see [Kol00, IKT01, IKT03, GL17, GL18]), nowadays we know that the conjecture is not true in general, even if \( A \) is assumed to be a finite union of cubes [Tao04]. Nevertheless, with time it became apparent that spectral sets behave in many ways like sets which can tile by translations. In particular, many results

\(^{1}\)Note added in proof: After the first version of this paper was submitted, the Fuglede conjecture for convex domains was settled in the affirmative, see [LM19].
about spectral sets have analogous results for sets which can tile, and vice versa. For example, Fuglede proved in [Fug74] that a set $A$ tiles the space with respect to a lattice translation set $\Lambda$ if and only if the dual lattice $\Lambda^*$ is a spectrum for $A$.

1.2. In this paper we establish a connection between spectrality, and a geometrical notion which is closely related to tiling – the equidecomposability by translations. In this context, we will assume the set $A$ to be a polytope, although not necessarily a convex or a connected one.

Recall that a polytope in $\mathbb{R}^d$ is a set which can be represented as the union of a finite number of simplices with disjoint interiors, where a simplex is the convex hull of $d+1$ points in $\mathbb{R}^d$ which do not all lie in some hyperplane.

If $A$ and $B$ are two polytopes in $\mathbb{R}^d$, then they are said to be equidecomposable (or dissection equivalent, or scissors congruent) if the polytope $A$ can be partitioned, up to measure zero, into a finite number of smaller polytopes which can be rearranged using rigid motions to form, again up to measure zero, a partition of the polytope $B$. If the pieces of the partition can be rearranged using translations only, then we say that $A$ and $B$ are equidecomposable by translations.

It has long been known that if a polytope $A \subset \mathbb{R}^d$ can tile the space by translations, then $A$ must be equidecomposable by translations to a cube of the same volume. This result was first proved by M"urner in [M"ur75], and was later rediscovered in [LM95a]. In this paper, we establish that the analogous result for spectral sets is true:

**Theorem 1.1.** Let $A$ be a polytope in $\mathbb{R}^d$ (not necessarily convex or connected). If $A$ is spectral, then $A$ is equidecomposable by translations to a cube of the same volume.

This result can be understood informally as saying that a spectral polytope $A \subset \mathbb{R}^d$ can “nearly” tile the space by translations. This conclusion is best possible in a sense, since there are examples of spectral polytopes which cannot tile (as shown in [Tao04]).

One can easily verify that equidecomposability by translations constitutes an equivalence relation on the set of all polytopes in $\mathbb{R}^d$. Theorem 1.1 yields the conclusion that all the spectral polytopes of a given volume lie in the same equivalence class.

We will obtain Theorem 1.1 as a consequence of another result, which will also be proved in this paper, and which will be described next.

1.3. In [KP02], Kolountzakis and Papadimitrakis proved the following result: Let $A$ be a polytope in $\mathbb{R}^d$ (again, $A$ may be non-convex or even disconnected). If $A$ is spectral, then the total area of the $(d-1)$-dimensional faces of $A$ on which the outward normal is pointing at a given direction, must coincide with the total area of those $(d-1)$-dimensional faces on which the outward normal is pointing at the opposite direction.

In this paper, we will prove an extension of this result to faces of all dimensions between 1 and $d-1$. The statement of our result involves certain functions which are called the Hadwiger functionals, and whose definition will now be given. For more details we refer the reader to [Bol78, Sections 2.10, 3.19] where a friendly introduction to Hadwiger functionals in dimensions two and three can be found.

Let $r$ be an integer, $1 \leq r \leq d-1$, and suppose that

$$V_r \subset V_{r+1} \subset \cdots \subset V_{d-1} \subset V_d = \mathbb{R}^d$$

(1.2)
is a sequence of linear subspaces such that $V_j$ has dimension $j$. Each subspace $V_j$ ($r \leq j \leq d - 1$) in the sequence divides the next one $V_{j+1}$ into two half-spaces; let us call one of them the positive half-space, and the other one the negative half-space. Such a sequence of nested linear subspaces, endowed with a choice of positive and negative half-spaces, will be called an $r$-flag, and will be denoted by $\Phi$.

Now let $A$ be a polytope in $\mathbb{R}^d$, and suppose that $A$ has a sequence of faces

$$F_r \subset F_{r+1} \subset \cdots \subset F_{d-1} \subset F_d = A,$$

where $F_j$ is a $j$-dimensional face of $A$ which is parallel to $V_j$ ($r \leq j \leq d - 1$). To each face $F_j$ we associate a coefficient $\varepsilon_j$, defined in the following way: $\varepsilon_j = +1$ if the face $F_{j+1}$ adjoins its surface $F_j$ from the same side where the positive half-space of $V_{j+1}$ adjoins $V_j$; while $\varepsilon_j = -1$ if $F_{j+1}$ adjoins $F_j$ from the opposite side. We then define

$$H_\Phi(A) = \sum \varepsilon_r \varepsilon_{r+1} \cdots \varepsilon_{d-1} \text{Vol}_r(F_r),$$

where the sum goes through all sequences of faces of $A$ as above, and where $\text{Vol}_r(F_r)$ denotes the $r$-dimensional volume of $F_r$. If no sequence of faces of $A$ as above exists, then we define the value of $H_\Phi(A)$ to be zero. We call $H_\Phi$ the Hadwiger functional associated to the $r$-flag $\Phi$.

For example, if $\Phi$ is a $(d-1)$-flag, then the value of $H_\Phi(A)$ is equal to the difference between the total area of the $(d-1)$-dimensional faces of $A$ on which the outward normal is perpendicular to the hyperplane $V_{d-1}$ and is pointing at the direction of the negative half-space determined by $V_{d-1}$, and the total area of those $(d-1)$-dimensional faces on which the outward normal is pointing at the opposite direction. Hence the result from [KP02] can be equivalently stated by saying that if $A$ is spectral, then we must have $H_\Phi(A) = 0$ for every $(d-1)$-flag $\Phi$.

We will prove that much more is actually true. Our main result is the following:

**Theorem 1.2.** Let $A$ be a polytope in $\mathbb{R}^d$ (not necessarily convex or connected). If $A$ is spectral, then $H_\Phi(A) = 0$ for every $r$-flag $\Phi$ ($1 \leq r \leq d - 1$).

This theorem thus extends the result in [KP02] to $r$-dimensional faces of $A$, for every $r$ between 1 and $d - 1$.

1.4. In the special case when the polytope $A$ is convex, the result in [KP02] says that if $A$ is spectral, then each one of the $(d-1)$-dimensional faces of $A$ has a parallel face of the same area. By a classical theorem of Minkowski, this condition is equivalent to $A$ being centrally symmetric. Hence any spectral convex polytope must be centrally symmetric. This result was obtained for the first time in [Kol00], using a different method.

Moreover, in [GL17] Section 4 it was proved that if a convex, centrally symmetric polytope $A$ is spectral, then all the $(d-1)$-dimensional faces of $A$ must also be centrally symmetric. This conclusion can also be stated in terms of the Hadwiger functionals; indeed, it is equivalent to the statement that $H_\Phi(A) = 0$ for every $(d-2)$-flag $\Phi$.

In fact, in [Mur77] Section 3.3 it is shown that for a convex polytope $A \subset \mathbb{R}^d$, the condition that $H_\Phi(A) = 0$ for every $r$-flag $\Phi$ ($1 \leq r \leq d - 1$), is equivalent to $A$ being centrally symmetric and having centrally symmetric $(d-1)$-dimensional faces. Thus one can view Theorem 1.2 as an extension to non-convex polytopes of the result which
states that if a convex polytope $A$ is spectral, then $A$ must be centrally symmetric and have centrally symmetric $(d - 1)$-dimensional faces.

Our proof of Theorem 1.2 is inspired by both [KP02] and [GL17, Section 4]. The proof involves an application of a Stokes-type theorem, which provides an expansion of the Fourier transform $\hat{1}_A$ of the indicator function $1_A$ of a polytope $A \subset \mathbb{R}^d$ in terms of the Fourier transforms of $r$-dimensional volume measures on $r$-dimensional faces of $A$. By identifying the main terms versus error terms in this expansion, we obtain an approximate expression for the function $\hat{1}_A$ which is valid in certain directions. The analysis gets more involved for smaller values of the face dimension $r$, since then there exist more different types of error terms, and for each type a different estimate is required in order to show that the term is small.

1.5. We will now clarify the relationship between our two results stated above, namely, Theorems 1.1 and 1.2. In fact, we will see that the first result is a consequence of the second one.

We start by recalling that the theory of equidecomposability of polytopes originated from Hilbert’s third problem – one of the famous 23 problems posed by Hilbert at the International Congress of Mathematicians in 1900. It is obvious that if two polytopes $A$ and $B$ are equidecomposable, then they must have the same volume. Hilbert’s third problem was concerned with the converse assertion: if $A$ and $B$ are two polytopes of the same volume, are they necessarily equidecomposable by rigid motions? It has been known earlier that in two dimensions, any two polygons of equal area are equidecomposable. However, in the same year 1900 it was shown by Dehn that in three dimensions, such a result is no longer true (a comprehensive exposition can be found in [Bol78]).

Dehn’s solution to Hilbert’s third problem involved an important notion in the theory of equidecomposability – the notion of additive invariants. Let $G$ be a group of rigid motions of $\mathbb{R}^d$. A function $\varphi$, defined on the set of all polytopes in $\mathbb{R}^d$, is said to be an additive $G$-invariant if (i) it is additive, namely, if $A$ and $B$ are two polytopes with disjoint interiors then $\varphi(A \cup B) = \varphi(A) + \varphi(B)$; and (ii) it is invariant under motions from the group $G$, that is, $\varphi(A) = \varphi(g(A))$ whenever $A$ is a polytope and $g \in G$.

It is obvious that for two polytopes $A$ and $B$ to be equidecomposable using motions from $G$, it is necessary that $\varphi(A) = \varphi(B)$ for any additive $G$-invariant $\varphi$. A general problem is to construct a “complete system” of additive $G$-invariants, that is, invariants which together provide a condition which is both necessary and sufficient for two polytopes of the same volume to be equidecomposable using motions from the group $G$.

In his solution to Hilbert’s third problem, Dehn constructed an additive invariant with respect to the group of all rigid motions of $\mathbb{R}^3$, which allowed him to show that a regular tetrahedron and a cube of the same volume are not equidecomposable [Deh01]. Dehn invariants for polytopes in $\mathbb{R}^d$ have also been studied [Had54], and shown to form a complete system in dimensions $d = 3, 4$ [Syd65, Jes72]. It remains an open problem as to whether these invariants are complete also in dimensions $d \geq 5$.

Equidecomposability with respect to the group of translations was first studied by Hadwiger. He introduced the Hadwiger functionals $H_k$ defined above, and proved that they form a system of additive invariants with respect to translations [Had52, Had57]. Moreover, it was shown that the Hadwiger invariants form a complete system, so that together they provide a necessary and sufficient condition for two polytopes of the same
volume to be equidecomposable by translations. This was proved by Hadwiger and Glur in dimension two [HG51], by Hadwiger in dimension three [Had68], and by Jessen and Thorup [JT78], and independently Sah [Sah79], in every dimension.

This clarifies why Theorem 1.1 is a consequence of Theorem 1.2. Indeed, Theorem 1.2 asserts that if a polytope \( A \subset \mathbb{R}^d \) is spectral, then we must have \( H_\Phi(A) = 0 \) for every \( r \)-flag \( \Phi \) (\( 1 \leq r \leq d - 1 \)). Let \( B \) be a cube of the same volume as \( A \), then it is easy to check that also \( H_\Phi(B) = 0 \) for every flag \( \Phi \). We thus obtain that \( H_\Phi(A) = H_\Phi(B) \) for all flags \( \Phi \). By the completeness of the Hadwiger invariants we can therefore conclude that \( A \) and \( B \) must be equidecomposable by translations, and so Theorem 1.1 follows.

We remark that the proof given in [M"ur75] (or in [LM95a]) of the fact that a polytope \( A \subset \mathbb{R}^d \) which can tile by translations must be equidecomposable by translations to a cube, relies on the same consideration. First it is proved that the tiling assumption implies that \( H_\Phi(A) = 0 \) for all flags \( \Phi \), and then the completeness of the Hadwiger invariants is used to conclude that \( A \) is equidecomposable by translations to a cube.

The rest of the paper is devoted to the proof of Theorem 1.2.

2. Preliminaries

2.1. Notation. We will use \( \langle \cdot, \cdot \rangle \) and \( |\cdot| \) to denote respectively the standard scalar product and norm in \( \mathbb{R}^d \). We denote by \( \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_d \) the standard basis vectors in \( \mathbb{R}^d \), and by \( x_1, x_2, \ldots, x_d \) the coordinates of a vector \( x \in \mathbb{R}^d \).

If \( A \subset \mathbb{R}^d \) and \( \tau \) is a vector in \( \mathbb{R}^d \), then we let \( A + \tau = \{ a + \tau : a \in A \} \) denote the translate of \( A \) by the vector \( \tau \). If \( A, B \) are two subsets of \( \mathbb{R}^d \), then \( A + B \) and \( A - B \) denote respectively their set of sums and set of differences.

For each \( \xi \in \mathbb{R}^d \) we denote by \( e_\xi \) the exponential function \( e_\xi(x) := e^{2\pi i \langle \xi, x \rangle}, x \in \mathbb{R}^d \).

By the Fourier transform of a function \( f \in L^1(\mathbb{R}^d) \) we mean the function

\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e_{\xi}(x) \, dx,
\]

and similarly, the Fourier transform of a finite, complex measure \( \mu \) on \( \mathbb{R}^d \) is the function

\[
\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e_{\xi}(x) \, d\mu(x).
\]

2.2. Spectra. If \( A \) is a bounded, measurable set in \( \mathbb{R}^d \) of positive measure, then by a spectrum for \( A \) we mean a countable set \( \Lambda \subset \mathbb{R}^d \) such that the system of exponential functions \( E(\Lambda) \) defined by (1.1) is orthogonal and complete in the space \( L^2(A) \).

For any two points \( \lambda, \lambda' \in \mathbb{R}^d \) we have \( \langle e_\lambda, e_{\lambda'} \rangle_{L^2(A)} = \hat{1}_A(\lambda' - \lambda) \), where \( \hat{1}_A \) is the Fourier transform of the indicator function \( 1_A \) of the set \( A \). The orthogonality of the system \( E(\Lambda) \) in \( L^2(A) \) is therefore equivalent to the condition

\[
(\Lambda - \Lambda) \setminus \{0\} \subset \{ \xi \in \mathbb{R}^d : \hat{1}_A(\xi) = 0 \}.
\] (2.1)

A set \( \Lambda \subset \mathbb{R}^d \) is said to be uniformly discrete if there is \( \delta > 0 \) such that \( |\lambda' - \lambda| \geq \delta \) for any two distinct points \( \lambda, \lambda' \) in \( \Lambda \). The condition (2.1) implies that every spectrum \( \Lambda \) of \( A \) is a uniformly discrete set.
2.3. Polytopes and equidecomposability. A simplex in $\mathbb{R}^d$ is the convex hull of $d+1$ points which do not all lie in some hyperplane. A polytope in $\mathbb{R}^d$ is a set which can be represented as the union of a finite number of simplices with disjoint interiors. Remark that a polytope is not necessarily a convex, nor even a connected, set.

The property of $\Lambda$ being a spectrum for $A$ is invariant under translations of both $A$ and $\Lambda$. If $M$ is a $d \times d$ invertible matrix, then $\Lambda$ is a spectrum for $A$ if and only if the set $(M^{-1})^\top(\Lambda)$ is a spectrum for $M(A)$.

Let $A$ and $B$ be two polytopes in $\mathbb{R}^d$. We say that $A$ and $B$ are equidecomposable if there exist finite decompositions of $A$ and $B$ of the form

$$A = \bigcup_{j=1}^N A_j, \quad B = \bigcup_{j=1}^N B_j$$

where $A_1, \ldots, A_N$ are polytopes with pairwise disjoint interiors, $B_1, \ldots, B_N$ are also polytopes with pairwise disjoint interiors, and for each $j$ the polytope $B_j$ is the image of $A_j$ under some rigid motion. If for each $j$ there is a vector $\tau_j \in \mathbb{R}^d$ such that $B_j = A_j + \tau_j$ (that is, $B_j$ is the image of $A_j$ under translation), then we say that the polytopes $A$ and $B$ are equidecomposable by translations.

2.4. Flags. If $r$ is an integer, $0 \leq r \leq d - 1$, then an $r$-flag $\Phi$ in $\mathbb{R}^d$ is defined to be a sequence of linear subspaces

$$V_r \subset V_{r+1} \subset \cdots \subset V_{d-1} \subset V_d = \mathbb{R}^d$$

such that $V_j$ has dimension $j$. Each subspace $V_j$ ($r \leq j \leq d - 1$) in the sequence divides the next one $V_{j+1}$ into two half-spaces; we assume that $\Phi$ is endowed with a choice of one of these half-spaces being called positive, and the other being called negative.

It will be convenient to define also a $d$-flag in $\mathbb{R}^d$ to be the sequence which consists of just one subspace $V_d = \mathbb{R}^d$.

Let $A$ be a polytope in $\mathbb{R}^d$, and suppose that we have a sequence

$$F_r \subset F_{r+1} \subset \cdots \subset F_{d-1} \subset F_d = A,$$

where $F_j$ is a $j$-dimensional face of $A$ ($r \leq j \leq d - 1$). Such a sequence will be called an $r$-sequence of faces of the polytope $A$, and will be denoted by $\mathcal{F}_r$.

Let $\Phi$ be an $r$-flag determined by a sequence of linear subspaces $V_r \subset V_{r+1} \subset \cdots \subset V_d$, and let $\mathcal{F}_r$ be an $r$-sequence of faces $F_r \subset F_{r+1} \subset \cdots \subset F_d$ of $A$. We say that the face $F_j$ is parallel to the subspace $V_j$ if the affine hull of $F_j$ is a translate of $V_j$. We say that the $r$-sequence $\mathcal{F}_r$ is parallel to the $r$-flag $\Phi$ if $F_j$ is parallel to $V_j$ for each $r \leq j \leq d - 1$.

Each $r$-flag $\Phi$ ($1 \leq r \leq d - 1$) determines a function $H_\Phi$ defined on the set of all polytopes in $\mathbb{R}^d$, which is given by (1.3). The function $H_\Phi$ is additive, and it is invariant with respect to translations. It will be called the Hadwiger functional associated to the $r$-flag $\Phi$.

Notice that if two $r$-flags $\Phi$ and $\Psi$ correspond to the same sequence of linear subspaces $V_r \subset V_{r+1} \subset \cdots \subset V_d$, then either $H_\Phi = H_\Psi$ or $H_\Phi = -H_\Psi$ (depending on the choice of
positive and negative half-spaces). Hence each sequence of linear subspaces essentially corresponds to one Hadwiger functional.

If $\Phi$ is a $d$-flag, then its associated Hadwiger functional $H_\Phi$ is defined by $H_\Phi(A) = \text{Vol}_d(A)$ for any polytope $A \subset \mathbb{R}^d$.

(We do not consider Hadwiger functionals associated to 0-flags, as these functionals vanish identically and thus they do not provide any information.)

2.5. Flag measures. Let $\Phi$ be an $r$-flag in $\mathbb{R}^d$ ($0 \leq r \leq d$), determined by a sequence of linear subspaces (2.2). To each polytope $A \subset \mathbb{R}^d$ we associate a signed measure $\mu_{A,\Phi}$ on $\mathbb{R}^d$ given by

$$
\mu_{A,\Phi} = \sum_{F_r} \varepsilon_r \varepsilon_{r+1} \cdots \varepsilon_{d-1} \text{Vol}_r|_{F_r},
$$

(2.3)

where $F_r$ goes through all $r$-sequences of faces $F_r \subset F_{r+1} \subset \cdots \subset F_d$ of the polytope $A$ that are parallel to $\Phi$, the $\varepsilon_j$ are the $\pm 1$ coefficients associated to the $r$-sequence $F_r$ with respect to $\Phi$ in the same way as in (1.4), and $\text{Vol}_r|_{F_r}$ denotes the $r$-dimensional volume measure restricted to the face $F_r$.

If $r = 0$, then by an $r$-dimensional face of $A$ we mean a vertex of $A$, and by the measure $\text{Vol}_r_{|_{F_r}}$, we mean the Dirac measure at the vertex $F_r$. Hence the flag measure $\mu_{A,\Phi}$ associated to a 0-flag $\Phi$ is a discrete measure supported on vertices of $A$.

If $\Phi$ is a $d$-flag, then $\mu_{A,\Phi} = \text{Vol}_d|_{A}$ (the Lebesgue measure restricted to $A$).

It follows from (1.4) and (2.3) that the measure $\mu_{A,\Phi}$ satisfies

$$
\int d\mu_{A,\Phi} = H_\Phi(A)
$$

(2.4)

for any $r$-flag $\Phi$ ($1 \leq r \leq d$).

(If $\mu_{A,\Phi}$ is the flag measure associated to a 0-flag $\Phi$, then $\int d\mu_{A,\Phi} = 0$.)

3. Stokes-type theorem for Fourier transforms of flag measures

The main result obtained in this section (Theorem 3.1) provides an expansion of the Fourier transform of a $k$-dimensional flag measure, in terms of Fourier transforms of $(k-1)$-dimensional flag measures. It is basically an application of Stokes theorem, which allows us to replace integration over $k$-dimensional faces of a polytope, by integration over the relative boundaries of these faces (see also [Bar02, p. 341], for instance).

In [LL18, Section 4] we proved a similar result but in a more refined context, where the equidecomposability of polytopes was studied with respect to a proper subgroup of all the translations. For the completeness of our exposition, we reproduce here the arguments in a self-contained version that is suitable for our present context.

3.1. Let $A$ be a polytope in $\mathbb{R}^d$, and let $\Phi_k$ be a $k$-flag ($1 \leq k \leq d$) determined by a sequence of linear subspaces $V_k \subset V_{k+1} \subset \cdots \subset V_d$. The Fourier transform of the measure $\mu_{A,\Phi_k}$ is given by

$$
\hat{\mu}_{A,\Phi_k}(\xi) = \int e^{\xi \cdot \mu_{A,\Phi_k}} = \sum_{F_k} \varepsilon_k \varepsilon_{k+1} \cdots \varepsilon_{d-1} \int_{F_k} e^{\xi \cdot \mu_{A,\Phi_k}},
$$

(3.1)
where $\mathcal{F}_k$ goes through all $k$-sequences of faces $F_k \subset F_{k+1} \subset \cdots \subset F_d$ of the polytope $A$ that are parallel to $\Phi_k$, the $\varepsilon_j$'s are the $\pm 1$ coefficients associated to the $k$-sequence $\mathcal{F}_k$ with respect to $\Phi_k$, and the integral on the right hand side is taken with respect to the $k$-dimensional volume measure on the face $F_k$.

Let $\partial F_k$ denote the relative boundary of the face $F_k$, and for each $x \in \partial F_k$ let $n(x)$ be a vector in the linear subspace $V_k$ which is outward unit normal to $F_k$ at the point $x$. Then for every $v \in V_k$ we have

$$-2\pi i \langle \bar{\xi}, v \rangle \int_{F_k} n(v) \cdot d\xi = \int_{\partial F_k} \langle n(v), v \rangle \cdot d\xi,$$

(3.2)

which follows by applying the divergence theorem to the function $f(x) = \bar{\xi}(x) \cdot v$ over the face $F_k$. The relative boundary $\partial F_k$ consists of a finite number of $(k-1)$-dimensional faces $F_{k-1}$ of $F_k$. Hence, using (3.1) and (3.2), we get

$$-2\pi i \langle \bar{\xi}, v \rangle \hat{\mu}_{A, \Phi_k}(\xi) = \sum_{\mathcal{F}_k} \varepsilon_k \varepsilon_{k+1} \cdots \varepsilon_{d-1} \int_{\partial F_k} \langle n(v), v \rangle \cdot d\xi,$$

(3.3)

$$= \sum_{\mathcal{F}_{k-1}} \varepsilon_k \varepsilon_{k+1} \cdots \varepsilon_{d-1} \int_{F_{k-1}} \langle n(v), v \rangle \cdot d\xi,$$

(3.4)

where $F_{k-1}$ goes through the $(k-1)$-dimensional subfaces of the $k$-dimensional face $F_k$ from the sequence $\mathcal{F}_k$, and $n$ is the outward unit normal to $F_k$ on $F_{k-1}$.

Let $\mathcal{E}$ be the collection of all the $(k-1)$-sequences of faces $F_{k-1} \subset F_k \subset \cdots \subset F_d$ of the polytopes $A$, such that $F_j$ is parallel to $V_j$ ($k \leq j \leq d-1$). We define an equivalence relation on $\mathcal{E}$ by saying that two elements $\mathcal{F}_{k-1}$ and $\mathcal{F}'_{k-1}$ from $\mathcal{E}$ are equivalent if the $(k-1)$-dimensional face $F_{k-1}$ from the sequence $\mathcal{F}_{k-1}$ is parallel to the $(k-1)$-dimensional face $F'_{k-1}$ from $\mathcal{F}'_{k-1}$. Then $\mathcal{E}$ can be partitioned into a finite number of equivalence classes $\mathcal{E}^1, \mathcal{E}^2, \ldots, \mathcal{E}^N$ induced by this equivalence relation.

To each equivalence class $\mathcal{E}^l$ ($1 \leq l \leq N$) we associate a $(k-1)$-flag $\Phi^l_{k-1}$, defined in the following way. The flag $\Phi^l_{k-1}$ is determined by a sequence of linear subspaces

$$V^l_{k-1} \subset V_k \subset V_{k+1} \subset \cdots \subset V_d = \mathbb{R}^d,$$

where $V_k, V_{k+1}, \ldots, V_d$ are the linear subspaces that determine the $k$-flag $\Phi_k$, while $V^l_{k-1}$ is a new linear subspace of dimension $k-1$. The subspace $V^l_{k-1}$ is chosen such that it is parallel to all the $(k-1)$-dimensional faces $F_{k-1}$ belonging to sequences $\mathcal{F}_{k-1}$ from the equivalence class $\mathcal{E}^l$. It is obvious from the definition of the equivalence relation on $\mathcal{E}$ that the subspace $V^l_{k-1}$ exists and that it is unique. We endow the $(k-1)$-flag $\Phi^l_{k-1}$ with a choice of positive and negative half-spaces, by saying that the positive and negative half-spaces of $V^l_{j+1}$ determined by the subspace $V^l_j$ coincide with those from the $k$-flag $\Phi_k$ for all $k \leq j \leq d-1$; while the positive and negative half-spaces of $V^l_k$ that are determined by the new subspace $V^l_{k-1}$ are selected in an arbitrary way.

For each $1 \leq l \leq N$, let $\sigma^l$ denote the (unique) unit vector in the linear subspace $V_k$ which is normal to $V^l_{k-1}$ and is pointing towards the negative half-space of $V_{k+1}$ determined by $V^l_{k-1}$. We then observe that if $\mathcal{F}_{k-1}$ is a sequence of faces $F_{k-1} \subset F_k \subset \cdots \subset F_d$ belonging to the equivalence class $\mathcal{E}^l$, and if $n$ is the outward unit normal to $F_k$ on $F_{k-1}$, then we have $n = \varepsilon_{k-1} \sigma^l$, where $\varepsilon_{k-1} = +1$ if $F_{k-1}$ adjoins $F_{k-1}$ from the positive side of $V_k$ which is determined by $V^l_{k-1}$, and $\varepsilon_{k-1} = -1$ if $F_{k-1}$ adjoins $F_{k-1}$ from the negative
where. It follows that the sum in (3.4) is equal to
\[
\sum_{l=1}^{N} \langle \sigma^l, v \rangle \sum_{F_{k-1}} \varepsilon_{k-1} \varepsilon_{k} \varepsilon_{k+1} \cdots \varepsilon_{d-1} \int_{F_{k-1}} \bar{\varepsilon}_{\xi},
\]
where \(F_{k-1}\) goes through all \((k-1)\)-sequences of faces \(F_{k-1} \subset F_k \subset \cdots \subset F_d\) of the polytope \(A\) that are parallel to \(\Phi_{k-1}\), and the \(\varepsilon\)'s are the \pm 1 coefficients associated to the \((k-1)\)-sequence \(F_{k-1}\) with respect to \(\Phi_{k-1}\). But now the inner sum in (3.5) is just the integral of the function \(\bar{\varepsilon}_{\xi}\) with respect to the measure \(\mu_{A,\Phi_{k-1}}\). Hence combining (3.3), (3.4), (3.5) we finally arrive at the following result:

**Theorem 3.1.** Let \(A\) be a polytope in \(\mathbb{R}^d\), and let \(\Phi_k\) be a \(k\)-flag \((1 \leq k \leq d)\) determined by a sequence of linear subspaces \(V_k \subset V_{k+1} \subset \cdots \subset V_d\). Then for every \(\xi \in \mathbb{R}^d\) and every \(v \in V_k\) we have
\[
-2\pi i \langle \xi, v \rangle \hat{\mu}_{A,\Phi_k}(\xi) = \sum_{l=1}^{N} \langle \sigma^l, v \rangle \hat{\mu}_{A,\Phi_{k-1}}(\xi),
\]
where the flags \(\Phi_{k-1}\) and vectors \(\sigma^l\) are as above.

**Remark 3.2.** It may happen that the polytope \(A\) does not have any \(k\)-sequences of faces \(F_k\) that are parallel to the \(k\)-flag \(\Phi_k\). In this case, \(\mu_{A,\Phi_k}\) is the zero measure, and the right hand side of (3.6) is understood to be an empty sum.

4. **Asymptotics of Fourier transform**

In this section we use the flag measures \(\mu_{A,\Phi}\) to analyze the asymptotic behavior of the Fourier transform \(\hat{1}_A\) of the indicator function of a polytope \(A \subset \mathbb{R}^d\). The main result of this section (Theorem 4.1) provides approximate expressions for \(\hat{1}_A\) which are valid in certain unbounded domains, in terms of the Fourier transforms \(\hat{\mu}_{A,\Phi}\) of the flag measures.

4.1. Let \(\Phi_r\) be an \(r\)-flag \((0 \leq r \leq d-1)\). We will say that \(\Phi_r\) is in **standard position** if it is determined by the sequence of linear subspaces \(V_r, V_{r+1}, \ldots, V_d\) given by
\[
V_j = \{ x \in \mathbb{R}^d : x_{j+1} = x_{j+2} = \cdots = x_d = 0 \}, \quad r \leq j \leq d-1,
\]
and the positive and negative half-spaces of \(V_{r+1}\) that are determined by \(V_j\) are chosen such that \(V_{j+1} \cap \{ x : x_{j+1} < 0 \}\) is the positive half-space, while \(V_{j+1} \cap \{ x : x_{j+1} > 0 \}\) is the negative half-space, for all \(r \leq j \leq d-1\).

Given an integer \(0 \leq r \leq d-1\), and three positive real numbers \(\alpha, \delta, \text{ and } L\) such that \(0 < 2\delta < \alpha < 1\), we denote by \(K(r, \alpha, L, \delta)\) the set of all vectors \(\xi \in \mathbb{R}^d\) satisfying the following three conditions:
\[
|\xi_j| \leq \alpha |\xi_{j+1}| \quad (1 \leq j \leq r), \quad (4.2)
\]
\[
L \leq |\xi_{r+1}|, \quad (4.3)
\]
\[
|\xi_j| \leq 2\delta |\xi_{j+1}| \quad (r+1 \leq j \leq d-1). \quad (4.4)
\]

In this section, our goal is to prove:
**Theorem 4.1.** Let $A$ be a polytope in $\mathbb{R}^d$, and let $\Phi_r$ be an $r$-flag in standard position ($0 \leq r \leq d - 1$). Then there exists $\alpha > 0$, such that for any $\eta > 0$ one can find $\delta$ and $L$ such that
\[
\left| \widehat{\mu}(\xi) - \hat{\mu}_{A,\Phi_r}(\xi) \right| < \eta, \quad \xi \in K(r, \alpha, L, \delta). \tag{4.5}
\]

This result allows us to approximate $\widehat{\mu}$ in the domain $K(r, \alpha, L, \delta)$ in terms of the Fourier transform of the flag measure $\mu_{A,\Phi_r}$. This shows that the behavior of the Fourier transform $\widehat{\mu}$ in the domain $K(r, \alpha, L, \delta)$ is essentially governed only by the contribution of those $r$-dimensional faces $F_r$ of $A$ that belong to some $r$-sequence $F_r, F_{r+1}, \ldots, F_d$ of faces which is parallel to the $r$-flag $\Phi_r$.

Notice that the estimate (4.5) yields different information for different values of $r$. Namely, for smaller $r$ we obtain a more accurate approximation for the Fourier transform $\widehat{\mu}$, but the domain in which this approximation is valid is also smaller.

The requirement in Theorem 4.1 that the $r$-flag $\Phi_r$ be in standard position, is done merely in order to simplify the notation in the statement. Indeed, a similar result for an arbitrary $r$-flag (that is, an $r$-flag which is not necessarily in standard position) can be deduced easily, by using the fact that any $r$-flag in $\mathbb{R}^d$ can be mapped by an invertible linear transformation onto an $r$-flag in standard position.

The rest of the section is devoted to the proof of Theorem 4.1. We divide the proof into a series of lemmas.

### 4.2.

**Lemma 4.2.** Let $A$ be a polytope in $\mathbb{R}^d$, let $0 \leq r \leq d - 1$, and let $\Psi_k$ be a $k$-flag $(1 \leq k \leq d)$ determined by a sequence of linear subspaces $W_k \subset W_{k+1} \subset \cdots \subset W_d$. Let $m$ be the smallest element of the set $\{0, 1, 2, \ldots, d\}$ such that
\[
W_k \subset \{ x \in \mathbb{R}^d : x_{m+1} = x_{m+2} = \cdots = x_d = 0 \}, \tag{4.6}
\]
and suppose that
\[
m \geq r + 1. \tag{4.7}
\]

Then there exist $\alpha > 0$, a constant $C$, and $(k-1)$-flags $\Psi^1_{k-1}, \Psi^2_{k-1}, \ldots, \Psi^N_{k-1}$ such that for any $\delta$ and $L$ we have
\[
\left| -2\pi i \xi_m \hat{\mu}_{A,\Psi_k}(\xi) \right| \leq C \sum_{l=1}^N \left| \hat{\mu}_{A,\Psi^l_{k-1}}(\xi) \right|, \quad \xi \in K(r, \alpha, L, \delta). \tag{4.8}
\]

**Proof.** Since $W_k$ is a linear subspace of dimension $k$, we must have $m \geq k$. Then it follows from the definition of $m$ that we can find a vector $v \in W_k$ such that $v_m \neq 0$. By multiplying $v$ on an appropriate scalar we may assume that $v_m > 1$.

Let $\xi \in K(r, \alpha, L, \delta)$. It follows from (4.5) that $v_{m+1} = v_{m+2} = \cdots = v_d = 0$, hence
\[
|\langle \xi, v \rangle| = \left| \sum_{j=1}^m \xi_j v_j \right| \geq |\xi_m v_m| - \left| \sum_{j=1}^{m-1} \xi_j v_j \right|. \tag{4.9}
\]
The conditions (4.2), (4.4), (4.7) ensure that if we choose $\alpha > 0$ small enough (in a way that depends on the vector $v$ but does not depend on $\xi$), then the right hand side of (4.9) will be not less than $|\xi_m|$. We thus obtain that

$$|\langle \xi, v \rangle| \geq |\xi_m|, \quad \xi \in K(r, \alpha, L, \delta). \quad (4.10)$$

We now apply Theorem 3.1 to the $k$-flag $\Psi_k$ and to the vector $v$. The theorem gives

$$-2\pi i \langle \xi, v \rangle \hat{\mu}_{A,\Psi_k}(\xi) = \sum_{l=1}^{N} \langle \sigma^l, v \rangle \hat{\mu}_{A,\Psi_{k-l}}(\xi). \quad (4.11)$$

Combining this with (4.10) and the estimate $|\langle \sigma^l, v \rangle| \leq |v|$, implies that (4.8) holds. \qed

4.3.

**Lemma 4.3.** Let $A$ be a polytope in $\mathbb{R}^d$, and let $\Psi_r$ be an $r$-flag $(1 \leq r \leq d - 1)$ determined by a sequence of linear subspaces $W_r \subset W_{r+1} \subset \cdots \subset W_d$. Assume that $W_r$ does not coincide with the subspace

$$V_r = \{ x \in \mathbb{R}^d : x_{r+1} = x_{r+2} = \cdots = x_d = 0 \}. \quad (4.12)$$

Then there exists $\alpha > 0$, such that for any $\eta > 0$ one can find $L$ such that

$$|\hat{\mu}_{A,\Psi}(\xi)| < \eta, \quad \xi \in K(r, \alpha, L, \delta). \quad (4.13)$$

**Proof.** We wish to apply Lemma 4.2 with $k = r$. Indeed, the assumption that $W_r$ does not coincide with the subspace $V_r$ in (4.12) implies that condition (4.7) is satisfied, hence we may use Lemma 4.2. The lemma yields that the estimate (4.8) is true, provided that $\alpha > 0$ is sufficiently small and the constant $C$ is sufficiently large.

If $\xi \in K(r, \alpha, L, \delta)$, then (4.3), (4.4) imply that $|\xi_m| \geq |\xi_{r+1}| \geq L$. So from (4.8) we get

$$2\pi L |\hat{\mu}_{A,\Psi}(\xi)| \leq C \sum_{l=1}^{N} |\hat{\mu}_{A,\Psi_{r-l}}(\xi)|, \quad \xi \in K(r, \alpha, L, \delta). \quad (4.14)$$

Notice that the right hand side of the inequality in (4.14) is bounded as a function of $\xi$. Hence given $\eta > 0$, if we choose $L$ sufficiently large then (4.13) holds. \qed

4.4.

**Lemma 4.4.** Let $A$ be a polytope in $\mathbb{R}^d$, let $0 \leq r \leq d - 1$, and let $\Psi_k$ be a $k$-flag $(r+1 \leq k \leq d)$. Then there exist $\alpha > 0$ and a constant $C$, such that for any $\delta$ and $L$ we have

$$|\hat{\mu}_{A,\Psi_k}(\xi) \prod_{j=r+1}^{k} (-2\pi i \xi_j)| \leq C, \quad \xi \in K(r, \alpha, L, \delta). \quad (4.15)$$

**Proof.** Again we wish to apply Lemma 4.2. Since we have $m \geq k \geq r+1$, the condition (4.7) is satisfied, and the lemma yields that the estimate (4.8) is true, provided that $\alpha > 0$ is sufficiently small and the constant $C$ is sufficiently large.

If $\xi \in K(r, \alpha, L, \delta)$, then (4.4) implies that $|\xi_m| \geq |\xi_k|$. Hence (4.8) implies that

$$|(-2\pi i \xi_k)\hat{\mu}_{A,\Psi_k}(\xi)| \leq C \sum_{l=1}^{N} |\hat{\mu}_{A,\Psi_{k-l}}(\xi)|, \quad \xi \in K(r, \alpha, L, \delta). \quad (4.16)$$
We notice that the right hand side of the inequality in (4.16) is bounded as a function of \( \xi \). This confirms that (4.15) is true in the special case when \( k = r + 1 \).

It remains to prove (4.15) also in the case when \( r + 2 \leq k \leq d \). This will be done by induction on \( k \). We multiply each side of (4.16) by the absolute values of the terms \(-2\pi i \xi_j \ (r + 1 \leq j \leq k - 1)\), and obtain

\[
\left| \hat{\mu}_{A,\Psi_k}(\xi) \prod_{j=r+1}^{k} (-2\pi i \xi_j) \right| \leq C \sum_{l=1}^{N} \left| \hat{\mu}_{A,\Psi_{k-1}}(\xi) \prod_{j=r+1}^{k-1} (-2\pi i \xi_j) \right|. \tag{4.17}
\]

By the inductive hypothesis, each one of the terms in the sum on the right hand side of (4.17) is bounded in the domain \( K(r, \alpha, L, \delta) \), provided that \( \alpha > 0 \) is sufficiently small. Hence also the left hand side is bounded, and again we arrive at (4.15). \( \square \)

4.5.

**Lemma 4.5.** Let \( A \) be a polytope in \( \mathbb{R}^d \), let \( 0 \leq r \leq d - 1 \), and let \( \Psi_k \) be a \( k \)-flag \( (r + 1 \leq k \leq d) \) determined by a sequence of linear subspaces \( W_k \subset W_{k+1} \subset \cdots \subset W_d \). Assume that \( W_k \) does not coincide with the subspace

\[
V_k = \{ x \in \mathbb{R}^d : x_{k+1} = x_{k+2} = \cdots = x_d = 0 \}. \tag{4.18}
\]

Then there exists \( \alpha > 0 \), such that for any \( \eta > 0 \) one can find \( \delta > 0 \) such that

\[
\left| \hat{\mu}_{A,\Psi_k}(\xi) \prod_{j=r+1}^{k} (-2\pi i \xi_j) \right| < \eta, \quad \xi \in K(r, \alpha, L, \delta). \tag{4.19}
\]

**Proof.** Once more we wish to apply Lemma 4.2. The assumption that \( W_k \) does not coincide with the subspace (4.18) implies that the number \( m \) from the lemma satisfies the condition \( m \geq k + 1 \). In particular, (4.7) holds and we may apply the lemma, which yields that the estimate (4.3) is true, provided that \( \alpha > 0 \) is sufficiently small and the constant \( C \) is sufficiently large.

Let \( \xi \in K(r, \alpha, L, \delta) \). Then the conditions \( k \geq r + 1 \) and \( m \geq k + 1 \) imply, using (4.4), that \( |\xi_m| \geq (2\delta)^{-1}|\xi_k| \). So it follows from (4.8) that

\[
\left| (-2\pi i \xi_k) \hat{\mu}_{A,\Psi_k}(\xi) \right| \leq 2C\delta \sum_{l=1}^{N} \left| \hat{\mu}_{A,\Psi_{k-1}}(\xi) \right|. \tag{4.20}
\]

The sum on the right hand side is bounded as a function of \( \xi \). Hence given \( \eta > 0 \), if we choose \( \delta > 0 \) small enough then we can make the right hand side of (4.20) smaller than \( \eta \) in the domain \( K(r, \alpha, L, \delta) \). This yields (4.19) in the case when \( k = r + 1 \).

In the case when \( r + 2 \leq k \leq d \), we multiply each side of (4.20) by the absolute values of the terms \(-2\pi i \xi_j \ (r + 1 \leq j \leq k - 1)\), and obtain

\[
\left| \hat{\mu}_{A,\Psi_k}(\xi) \prod_{j=r+1}^{k} (-2\pi i \xi_j) \right| \leq 2C\delta \sum_{l=1}^{N} \left| \hat{\mu}_{A,\Psi_{k-1}}(\xi) \prod_{j=r+1}^{k-1} (-2\pi i \xi_j) \right|. \tag{4.21}
\]

The sum on the right hand side of (4.21) is bounded as a function of \( \xi \), according to Lemma 4.4. Hence again, given \( \eta > 0 \) we can choose \( \delta > 0 \) such that (4.19) holds. \( \square \)
Lemma 4.6. Let $A$ be a polytope in $\mathbb{R}^d$, and let $\Phi_r$ be an $r$-flag, and $\Phi_k$ be a $k$-flag $(0 \leq r < k \leq d)$, both in standard position. Then there exists $\alpha > 0$, such that for any $\eta > 0$ one can find $\delta$ and $L$ such that

$$\left| \left( \hat{\mu}_{A,\Phi_k}(\xi) \prod_{j=r+1}^{k} (-2\pi i \xi_j) \right) - \hat{\mu}_{A,\Phi_r}(\xi) \right| < \eta, \quad \xi \in K(r, \alpha, L, \delta). \quad (4.22)$$

Proof. Let $V_r, V_{r+1}, \ldots, V_d$ be the linear subspaces given by (4.1). We apply Theorem 3.1 to the $k$-flag $\Phi_k$ and to the vector $v = \vec{e}_k$ which belongs to $V_k$. Then from (3.6) we get

$$-2\pi i \xi_k \hat{\mu}_{A,\Phi_k}(\xi) = \hat{\mu}_{A,\Phi_{k-1}}(\xi) + \sum_{l=1}^{N} \langle \sigma_l, \vec{e}_k \rangle \hat{\mu}_{A,\Psi_{k-1}^l}(\xi), \quad (4.23)$$

where $\Phi_{k-1}$ is a $(k - 1)$-flag in standard position, and each $\Psi_{k-1}^l$ is a $(k - 1)$-flag determined by a sequence $W_{k-1}^l, V_k, \ldots, V_d$, such that $W_{k-1}^l$ is a $(k - 1)$-dimensional linear subspace of $V_k$ which is different from $V_{k-1}$. Notice that the first term on the right hand side of (4.23) corresponds to one of the $(k - 1)$-flags in (3.6) being in standard position, possibly after re-choosing the positive and negative half-spaces of $V_k$. We can assume that this is the case, since if neither of the $(k - 1)$-flags corresponds to this term, then $\mu_{A,\Phi_{k-1}}$ must be the zero measure and again (4.22) is true.

If $r = 0$ and $k = 1$, then there is a unique $(k - 1)$-dimensional linear subspace of $V_k$, namely, the subspace $V_{k-1} = \{0\}$. Hence in this case there are no $(k - 1)$-dimensional linear subspaces which are different from $V_{k-1}$, so the sum on the right hand side of (4.23) is empty. Thus we obtain that $-2\pi i \xi_k \hat{\mu}_{A,\Phi_k}(\xi) = \hat{\mu}_{A,\Phi_r}(\xi)$ for every $\xi \in \mathbb{R}^d$, which in particular implies (4.22).

If $k = r + 1$ and $r \geq 1$, then we apply Lemma 4.3 to each one of the $(k - 1)$-flags $\Psi_{k-1}^l$. We may apply the lemma since the subspace $W_{k-1}^l$ does not coincide with $V_{k-1}$. We obtain from the lemma that if $\alpha > 0$ is small enough (not depending on $\eta$) and if $L$ is large enough, then

$$|\hat{\mu}_{A,\Psi_{k-1}^l}(\xi)| < N^{-1} \eta, \quad \xi \in K(r, \alpha, L, \delta), \quad (4.24)$$

for all $1 \leq l \leq N$. Then (4.23), (4.24) and the estimate $|\langle \sigma_l, \vec{e}_k \rangle| \leq 1$ imply (4.22).

Finally, it remains to prove the lemma in the case when $r + 2 \leq k \leq d$. We do this by induction on $k$. We multiply both sides of (4.23) by the terms $-2\pi i \xi_j \ (r+1 \leq j \leq k-1)$, and obtain

$$\hat{\mu}_{A,\Phi_k}(\xi) \prod_{j=r+1}^{k} (-2\pi i \xi_j) = \hat{\mu}_{A,\Phi_{k-1}}(\xi) \prod_{j=r+1}^{k-1} (-2\pi i \xi_j) \quad (4.25)$$

$$+ \sum_{l=1}^{N} \langle \sigma_l, \vec{e}_k \rangle \hat{\mu}_{A,\Psi_{k-1}^l}(\xi) \prod_{j=r+1}^{k-1} (-2\pi i \xi_j), \quad (4.26)$$

By the inductive hypothesis, the right hand side of (4.25) satisfies

$$\left| \left( \hat{\mu}_{A,\Phi_{k-1}}(\xi) \prod_{j=r+1}^{k-1} (-2\pi i \xi_j) \right) - \hat{\mu}_{A,\Phi_r}(\xi) \right| < \eta/2, \quad \xi \in K(r, \alpha, L, \delta), \quad (4.27)$$
provided that $\alpha > 0$ is small enough (not depending on $\eta$), $\delta$ is small enough and $L$ is large enough. Next, we estimate the sum in (4.26) by applying Lemma 4.5 to each one of the $(k - 1)$-flags $\Psi_{k-1}^l$. We may apply the lemma since $W_{k-1}^l$ does not coincide with $V_{k-1}$. We obtain from the lemma that if $\delta > 0$ is small enough, then

$$\left| \hat{\mu}_{A, \Psi_{k-1}^l}(\xi) \prod_{j=r+1}^{k-1} (-2\pi i \xi_j) \right| < (2N)^{-1} \eta, \quad \xi \in K(r, \alpha, L, \delta),$$

(4.28)

for all $1 \leq l \leq L$. Then using (4.25), (4.26), (4.27), (4.28) and the estimate $|\langle \sigma^l, \tilde{e}_k \rangle| \leq 1$, we obtain that (4.22) holds. \hfill \square

4.7.

Proof of Theorem 4.1. We apply Lemma 4.6 with $k = d$. If $\Phi_d$ is a $d$-flag, then the measure $\mu_{A, \Phi_d}$ is equal to $\text{Vol}_d|_A$ (that is, the Lebesgue measure restricted to $A$). In particular we have $\hat{\mu}_{A, \Phi_d} = \hat{1}_A$, so the condition (4.5) is a special case of (4.22) obtained when $k = d$. Hence Theorem 4.1 is just a special case of Lemma 4.6. \hfill \square

Remark 4.7. The above proof of Theorem 4.1 yields a quantitative estimate on how small should $\delta$ be, and how large should $L$ be, in order that (4.5) becomes valid. Indeed, it can be inferred from the proof that there is a constant $c = c(A, \Phi_r) > 0$ such that (4.5) is true if $\delta = c\eta$ and $L = (c\eta)^{-1}$.

5. Auxiliary lemmas

In this section we prove two auxiliary lemmas needed for the proof of Theorem 1.2.

5.1.

Lemma 5.1. Let $A$ be a polytope in $\mathbb{R}^d$, and let $\Phi_r$ be an $r$-flag in standard position ($1 \leq r \leq d - 1$). Then the function $\hat{\mu}_{A, \Phi_r}$ has the form

$$\hat{\mu}_{A, \Phi_r}(\xi) = \sum_{k=1}^N \varphi_k(\xi_1, \xi_2, \ldots, \xi_r) \exp \left( -2\pi i \sum_{j=r+1}^d \tau_{k,j} \xi_j \right), \quad \xi \in \mathbb{R}^d,$$

(5.1)

where $\tau_{k,j}$ are real numbers, and $\varphi_k$ are continuous functions on $\mathbb{R}^r$ vanishing at infinity.

Proof. Let $V_r, V_{r+1}, \ldots, V_d$ be the linear subspaces given by (4.1), and suppose that $F_r$ is an $r$-dimensional face of $A$ that is parallel to the subspace $V_r$. Then there are real numbers $\tau_{r+1}, \tau_{r+2}, \ldots, \tau_d$ such that

$$F_r \subset \{ x \in \mathbb{R}^d : x_{r+1} = \tau_{r+1}, \ x_{r+2} = \tau_{r+2}, \ldots, x_d = \tau_d \}.$$

The Fourier transform of the measure $\sigma := \text{Vol}_r|_{F_r}$ (the $r$-dimensional volume measure restricted to $F_r$) is therefore given by

$$\hat{\sigma}(\xi) = \varphi(\xi_1, \xi_2, \ldots, \xi_r) \exp \left( -2\pi i \sum_{j=r+1}^d \tau_j \xi_j \right), \quad \xi \in \mathbb{R}^d,$$

(5.2)

where the function $\varphi$ is the Fourier transform of the indicator function of the polytope in $\mathbb{R}^r$ obtained by projecting the face $F_r$ on the $(x_1, x_2, \ldots, x_r)$ coordinates. In particular, $\varphi$ is a continuous function on $\mathbb{R}^r$ vanishing at infinity.
Now the measure $\mu_{A, \Phi_r}$ is a linear combination (with $\pm 1$ coefficients) of measures of the form $\text{Vol}_r|_F$, where $F_r$ belongs to a sequence of faces $F_r \subset F_{r+1} \subset \cdots \subset F_d$ such that $F_j$ is a $j$-dimensional face of $A$ which is parallel to $V_j$ ($r \leq j \leq d-1$). Hence the Fourier transform $\hat{\mu}_{A, \Phi_r}$ of the measure $\mu_{A, \Phi_r}$ is a linear combination of functions of the form (5.2). This implies that $\hat{\mu}_{A, \Phi_r}$ has the form (5.1) as claimed. □

5.2.

**Lemma 5.2.** Let $p(t)$ be a trigonometric polynomial given by

$$p(t) = \sum_{k=1}^{N} c_k e^{2\pi i \tau_k t} \quad (t \in \mathbb{R})$$

where $\tau_k$ are real numbers, and $c_k$ are complex numbers. For any $\eta > 0$ there exists a relatively dense set $T \subset \mathbb{R}$, such that $|p(t' - t) - p(0)| < \eta$ for any two elements $t, t' \in T$.

We give two proofs, one relies on the theory of almost periodic functions (in the same spirit as in [KP02]), while the other on a result from dynamical systems.

**First proof of Lemma 5.2.** The trigonometric polynomial $p$ is a linear combination of periodic functions, and so it is an almost periodic function, see for instance [Kat04, Section VI.5]. According to the definition of an almost periodic function, this implies that given $\eta > 0$ there exists a relatively dense set $T \subset \mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}} |p(x + t) - p(x)| < \eta/2, \quad t \in T.$$

Then for any two elements $t, t' \in T$ we have

$$|p(t' - t) - p(0)| \leq \sup_{x \in \mathbb{R}} |p(x + t') - p(x + t)| \leq \sup_{x \in \mathbb{R}} |p(x + t') - p(x)| + \sup_{x \in \mathbb{R}} |p(x + t) - p(x)| < \eta. \quad \square$$

**Second proof of Lemma 5.2.** For $\delta > 0$, let $T(\delta) = T(\delta; \tau_1, \ldots, \tau_N)$ denote the set of integers $t$ for which the condition $\text{dist}(\tau_k t, \mathbb{Z}) < \delta$ holds for all $1 \leq k \leq N$. Then $T(\delta)$ is a relatively dense set, see for instance [Fur81, Theorem 1.21]. For any two elements $t, t' \in T(\delta)$ we have

$$|e^{2\pi i \tau_k (t' - t)} - 1| \leq 2\pi \text{dist}(\tau_k (t' - t), \mathbb{Z}) < 4\pi \delta \quad (1 \leq k \leq N),$$

and therefore

$$|p(t' - t) - p(0)| \leq \sum_{k=1}^{N} |c_k| \cdot |e^{2\pi i \tau_k (t' - t)} - 1| \leq 4\pi \delta \sum_{k=1}^{N} |c_k|.$$

Hence if $\delta = \delta(p, \eta)$ is chosen sufficiently small, this implies that $|p(t' - t) - p(0)| < \eta. \quad \square$

6. **Proof of Theorem 1.2**

We now give the proof of Theorem 1.2 using the results obtained above. The proof strategy extends the one that was introduced in [KP02] and further developed in [GL17, Section 4].
6.1. Let $A$ be a spectral polytope in $\mathbb{R}^d$, and let $\Phi_r$ be an $r$-flag ($1 \leq r \leq d - 1$). We must show that $H_{\Phi_r}(A) = 0$. By applying an invertible linear transformation, we may assume that $\Phi_r$ is in standard position.

Suppose to the contrary that $H_{\Phi_r}(A) \neq 0$. Choose a number $\eta$ such that
\begin{equation}
0 < 3\eta < |H_{\Phi_r}(A)|. \tag{6.1}
\end{equation}
According to Theorem 4.1 we can find $\alpha, \delta$ and $L$ such that (4.5) holds. Let $v = v(r, \delta)$ be the vector in $\mathbb{R}^d$ given by
\begin{equation}
v := \sum_{j=r+1}^{d} \delta^{d-j} \vec{e}_j, \tag{6.2}
\end{equation}
and define
\begin{equation}
p(t) := \hat{\mu}_{A,\Phi_r}(tv), \quad t \in \mathbb{R}. \tag{6.3}
\end{equation}
By Lemma 5.1 the function $\hat{\mu}_{A,\Phi_r}$ is of the form (5.1), and so we have
\begin{equation}
p(t) = \sum_{k=1}^{N} \varphi_k(0, 0, \ldots, 0) \exp \left( -2\pi it \sum_{j=r+1}^{d} \tau_{k,j} \delta^{d-j} \right). \tag{6.4}
\end{equation}
Hence $p(t)$ is a trigonometric polynomial of the form (5.3). By Lemma 5.2 there is a relatively dense set $T \subset \mathbb{R}$ such that
\begin{equation}
|p(t' - t) - p(0)| < \eta, \quad t, t' \in T. \tag{6.5}
\end{equation}
Since the function $\hat{\mu}_{A,\Phi_r}$ is uniformly continuous on $\mathbb{R}^d$ (being the Fourier transform of a finite measure), there is $\varepsilon > 0$ such that
\begin{equation}
|\hat{\mu}_{A,\Phi_r}(\xi') - \hat{\mu}_{A,\Phi_r}(\xi)| < \eta \quad \text{whenever} \quad \xi, \xi' \in \mathbb{R}^d, \quad |\xi' - \xi| < 2\varepsilon. \tag{6.6}
\end{equation}
Define
\begin{equation}
E := \{tv + w : t \in T, w \in \mathbb{R}^d, |w| < \varepsilon \}. \tag{6.7}
\end{equation}
Then the set $E$ consists of the union of open balls of radius $\varepsilon$ centered at the points of the form $tv$ ($t \in T$). These points constitute a relatively dense subset of the line spanned by the vector $v$.

6.2. We now claim that
\begin{equation}
|\hat{\mu}_{A,\Phi_r}(\xi)| > \eta, \quad \xi \in E - E. \tag{6.8}
\end{equation}
Indeed, let $\xi$ be a point in $E - E$. Then we may write $\xi = (t' - t)v + w$, where $t, t' \in T$ and $|w| < 2\varepsilon$. Hence using (6.3), (6.5), (6.6) it follows that
\begin{equation}
|\hat{\mu}_{A,\Phi_r}(\xi)| > |\hat{\mu}_{A,\Phi_r}((t' - t)v)| - \eta = |p(t' - t)| - \eta > |p(0)| - 2\eta. \tag{6.9}
\end{equation}
Note that
\begin{equation}
p(0) = \hat{\mu}_{A,\Phi_r}(0) = \int d\mu_{A,\Phi_r} = H_{\Phi_r}(A). \tag{6.10}
\end{equation}
Hence (6.1), (6.9) and (6.10) imply that (6.8) holds as claimed.
6.3. For each $h > 0$, we let $S(h)$ denote the cylinder of radius $h$ along the line spanned by the vector $v$, that is, 

$$S(h) := \{tv + w : t \in \mathbb{R}, w \in \mathbb{R}^d, |w| < h\}.$$ 

Notice that 

$$E - E \subset S(2\varepsilon).$$ 

(6.11) 

It is straightforward to check, using (6.2), that there is $R > 0$ such that 

$$S(2\varepsilon) \setminus B_R \subset K(r, \alpha, L, \delta),$$ 

(6.12) 

where $B_R$ denotes the open ball of radius $R$ centered at the origin.

6.4. Let $\Lambda$ be a spectrum for $A$. We claim that for any $\tau \in \mathbb{R}^d$, if $\lambda, \lambda'$ are two points in $\Lambda \cap (E + \tau)$, then $|\lambda' - \lambda| < R$. Indeed, if not, then it follows from (6.11), (6.12) that 

$$\lambda' - \lambda \in (E - E) \setminus B_R \subset K(r, \alpha, L, \delta).$$ 

On the other hand, by (2.1) we have $\hat{1}_A(\lambda' - \lambda) = 0$, hence (4.5) implies that we must have $|\hat{\mu}_{A,\Phi}(\lambda' - \lambda)| < \eta$. However this is not possible, due to (6.8).

Since $\Lambda$ is a uniformly discrete set, it follows that $\Lambda \cap (E + \tau)$ is a finite set, for every $\tau \in \mathbb{R}^d$. Since $\Lambda$ is a relatively dense set, there is $M > 0$ such that every ball of radius $M$ intersects $\Lambda$. The cylinder $S(M)$ can be covered by a finite number of translates of $E$, hence $\Lambda \cap S(M)$ is also a finite set. It follows that $S(M)$ must contain a ball of radius $M$ free from points of $\Lambda$, a contradiction. Theorem 1.2 is thus proved. 

\[\square\]

7. Remark

The assumption in Theorem 1.2 (and in Theorem 1.1) that the polytope $A$ is spectral, was used only in order to know that there is a relatively dense set of frequencies $\Lambda \subset \mathbb{R}^d$ such that the exponential system $E(\Lambda)$ is orthogonal in the space $L^2(A)$. Hence the result remains valid under this weaker assumption. In other words, we have actually proved the following more general version of the result:

**Theorem 7.1.** Let $A$ be a polytope in $\mathbb{R}^d$ (not necessarily convex or connected). Assume that there is a relatively dense set $\Lambda \subset \mathbb{R}^d$ such that the exponential system $E(\Lambda)$ is orthogonal in the space $L^2(A)$. Then $H_\Phi(A) = 0$ for every $r$-flag $\Phi$ ($1 \leq r \leq d - 1$). As a consequence, $A$ is equidecomposable by translations to a cube of the same volume.

In the special case when the polytope $A$ is convex, the conclusion implies that $A$ must be centrally symmetric and have centrally symmetric facets. This recovers a result stated in [GL18, Theorem 5.5].

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