BLOW UP AND LIFESPAN ESTIMATES FOR SYSTEMS OF SEMI-LINEAR WAVE EQUATIONS WITH DAMPING AND POTENTIAL

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ABSTRACT. In this paper, we consider the semi-linear wave systems with power-nonlinearities and a large class of space-dependent damping and potential. We obtain the same blow-up regions and the lifespan estimates for three types wave systems, compared with the systems without damping and potential.

1. INTRODUCTION

In this paper, we study the finite time blow-up phenomenon of three kinds of semi-linear wave systems with space dependent damping and potential. More precisely, we consider the following small data Cauchy problem with power-nonlinearities

\begin{equation}
\begin{aligned}
&u_{tt} - \Delta u + D_1(x)u_t + V_1(x)u = N_1(v, v_t), \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
v_{tt} - \Delta v + D_2(x)v_t + V_2(x)v = N_2(u, u_t), \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
u(0, x) = \varepsilon u_0(x), \quad u_t(0, x) = \varepsilon u_1(x), \quad x \in \mathbb{R}^n, \\
v(0, x) = \varepsilon v_0(x), \quad v_t(0, x) = \varepsilon v_1(x), \quad x \in \mathbb{R}^n.
\end{aligned}
\end{equation}

Here \((u_0, u_1), (v_0, v_1) \in C^\infty_c(\mathbb{R}^n)\), and the small parameter \(\varepsilon > 0\) measures the size of the data. As usual, to show blow up, we assume both \((u_0, u_1)\) and \((v_0, v_1)\) are nontrivial, nonnegative and supported in \(B_R := \{x \in \mathbb{R}^n : r \leq R\}\) for some \(R > 0\), where \(|x| = r\). For the coefficient of dampings \(D_i(x)\) and potentials \(V_i(x)\), we assume \(V_i(x), D_i(x) \in C(\mathbb{R}^n) \cap C^\delta(B_\delta)\) for some \(\delta > 0\). In addition, we assume

\begin{align}
0 \leq D_i(x) &= D_i(|x|) \leq \alpha_1 (1 + |x|)^{-\beta}, \quad 0 \leq \alpha_1 \in \mathbb{R}, 1 < \beta \in \mathbb{R}, i = 1, 2, \\
0 \leq V_i(x) &= V_i(|x|) \leq \alpha_2 (1 + |x|)^{-\omega}, \quad 0 \leq \alpha_2 \in \mathbb{R}, 2 < \omega \in \mathbb{R}, i = 1, 2.
\end{align}

In order to motivate the study of (1.1), let us recall some semi-linear models which are strongly related to this weakly coupled system. We begin with the following system

\begin{equation}
\begin{aligned}
&u_{tt} - \Delta u = G_1(v, v_t), \quad t > 0, x \in \mathbb{R}^n, \\
v_{tt} - \Delta v = G_2(u, u_t), \quad t > 0, x \in \mathbb{R}^n, \\
u(0, x) = \varepsilon u_0(x), \quad u_t(0, x) = \varepsilon u_1(x) \\
v(0, x) = \varepsilon v_0(x), \quad v_t(0, x) = \varepsilon v_1(x)
\end{aligned}
\end{equation}

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When \( G_1(v, v_t) = |v|^p, G_2(u, u_t) = |u|^q \), Santo-Georgiev-Mitidieri [6] proved there exist a critical curve in \((p, q)\)-plane,

\[
\Gamma_{SS}(n, p, q) = \max \{\frac{p + 2 + q^{-1}}{pq - 1}, \frac{q + 2 + p^{-1}}{pq - 1}\} - \frac{n - 1}{2}.
\]

They showed that when \( \Gamma_{SS}(n, p, q) < 0 \), \( n \geq 2 \), the system (1.4) has a global in time solution for sufficiently small \( \varepsilon \), while a solution for some positive initial data blows up in finite time if \( \Gamma_{SS}(n, p, q) > 0 \). Palmieri-Takamura [1] obtained the upper bound of lifespan estimates:

\[
(1.5) \quad T_{\varepsilon} \lesssim \begin{cases} 
\varepsilon^{-\Gamma_{SS}^{-1}(n, p, q)}, & \Gamma_{SS}(n, p, q) > 0, \\
\exp(\varepsilon^{-\min\{p(pq-1), q(pq-1)\}}), & \Gamma_{SS}(n, p, q) = 0, p \neq q, \\
\exp(\varepsilon^{-\min\{p(pq-1)\}}), & \Gamma_{SS}(n, p, q) = 0, p = q = \rho_{SS}(n).
\end{cases}
\]

See also [5] [9] [10] for relevant works.

When \( G_1(v, v_t) = |v|^p, G_2(u, u_t) = |u|^q \), it has been investigated by Deng [12], Ikeda-Sobojima-Wakasa [10], Xu [23]. Where, Deng [12] has obtained the blow up region of system (1.4) in

\[
1 < pq < \infty \begin{cases} 
1 < p < \infty, & n = 1, \\
1 < p \leq 2, & n = 2, \\
p = 1, & n = 3.
\end{cases}
\]

\[
1 < pq \leq \frac{(n + 1)p}{(n - 1)p - 2} \begin{cases} 
2 < p \leq 3, & n = 2, \\
1 < p \leq 2, & n = 3, \\
1 < p \leq \frac{n + 1}{n - 1}, & n \geq 4.
\end{cases}
\]

Ikeda-Sobajima-Wakasa [10] obtained the finite time blow up results when

\[
\Gamma_{GG}(p, q, n) := \max \{\frac{p + 1}{pq - 1}, \frac{q + 1}{pq - 1}\} - \frac{n - 1}{2} \geq 0, \quad p, q > 1, \quad n \geq 2,
\]

and obtain the upper bound estimates:

\[
(1.6) \quad T_{\varepsilon} \lesssim \begin{cases} 
\varepsilon^{-\Gamma_{GG}^{-1}(n, p, q)}, & \Gamma_{GG}(n, p, q) > 0, \\
\exp(\varepsilon^{-\min\{p(pq-1)\}}), & \Gamma_{GG}(n, p, q) = 0, \ p \neq q, \\
\exp(\varepsilon^{-\min\{p(pq-1)\}}), & \Gamma_{GG}(n, p, q) = 0, \ p = q.
\end{cases}
\]

When \( G_1(v, v_t) = |v|^q, G_2(u, u_t) = |u|^p \), Hidano-Yokoyama [8] showed the upper bound of lifespan estimates:

\[
(1.7) \quad T_{\varepsilon} \lesssim \varepsilon^{-(p + 2)(\frac{p(pq-1)}{pq-1})(pq-1)},
\]

when \( n \geq 2, \ 1 < q, \ 1 < p < \frac{2n}{n - 1} \) and \( \frac{(n - 1)p}{2} - 1 < pq - 1 < p + 2 \). Ikeda-Sobojima-Wakasa [10] introduced two kind of exponent for the system (1.4),

\[
F_{SG, 1}(n, p, q) = \frac{1}{p + 1 + q}(pq - 1)^{-1} - \frac{n - 1}{2}, \\
F_{SG, 2}(n, p, q) = \frac{1}{q + 2}(pq - 1)^{-1} - \frac{n - 1}{2}.
\]

They showed the solution for some positive data blows up in finite time if

\[
\Gamma_{SG}(n, p, q) = \max \{F_{SG, 1}(n, p, q), F_{SG, 2}(n, p, q)\} \geq 0,
\]
and obtained the upper bound estimates:

\begin{equation}
T \varepsilon \lesssim \begin{cases}
\varepsilon^{-\frac{1}{p-1}(n,p,q)}, & \Gamma_{SG}(n,p,q) > 0, \\
\exp(-q(p-1)), & F_{SG,1}(n,p,q) = 0 > F_{SG,2}(n,p,q), \\
\exp(-q(p-1)), & F_{SG,2}(n,p,q) = 0 > F_{SG,1}(n,p,q), \\
\exp(-q(p-1)), & F_{SG,2}(n,p,q) = 0 = F_{SG,1}(n,p,q).
\end{cases}
\end{equation}

Concerning the influence of dampings and potentials to the blow up region of semilinear wave equations, Lai-Tu [17] considered the small data Cauchy problems

\begin{equation}
\begin{aligned}
&u_{tt} - \Delta u + \frac{n}{(1+|x|^2)} u_t = N(u, u_t), (t, x) \in (0, T) \times \mathbb{R}^n, \\
u(0, x) = \varepsilon f(x), & u_t(0, x) = \varepsilon g(x)
\end{aligned}
\end{equation}

where $\mu > 0$, $\beta > 2$ are constants. They showed that such damping terms does not effect the upper bound of lifespan estimates:

\[ N(u, u_t) = |u|^p, \quad T \varepsilon \lesssim \begin{cases}
\varepsilon^{-\frac{2(p-1)}{p-1}}, & 1 < p \leq \frac{n}{n-1}, \\
\varepsilon^{-\frac{2(p-1)}{2n-p}}, & \frac{n}{n-1} < p < p_S(n).
\end{cases} \]

\[ N(u, u_t) = |u_t|^p, \quad T \varepsilon \lesssim \begin{cases}
\varepsilon^{-\frac{(p-1)-\frac{n}{p-1}}{p-1}}, & 1 < p \leq p_G(n), \\
\exp(-p-1), & p = p_G(n).
\end{cases} \]

Here, $p_S(n)$ is the Strauss exponent (see [13]) and $p_S$ is the positive root of the quadratic equation

\[ 2 + (n+1)p - (n-1)p^2 = 0. \]

And $p_G(n) = 1 + 2/(n-1)$ is Glassy exponent (see [21]). Recently, Lai-Liu-Tu-Wang [18] considered the following semi-linear wave equation

\begin{equation}
\begin{aligned}
&u_{tt} - \Delta u + D(x) u_t + V(x) u = |u|^p, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
u(0, x) = \varepsilon f(x), & u_t(0, x) = \varepsilon g(x)
\end{aligned}
\end{equation}

where $n \geq 2$. They showed the problem exists a large class of damping and potential functions of critical/long range in the blow up part. In particular, they showed that for the coefficient of dampings and potentials satisfying (1.2), (1.3), the solution of (1.10) will blow up under the Strauss exponent $p_S(n)$. Based on this work, we can imagine such damping and potential will not effect the blow up region of system (1.1).

In this work, we obtain the blow up results and the upper bound lifespan estimates to three types of system (1.1) under the exponent $\Gamma_{SS}$, $\Gamma_{GG}$, $\Gamma_{SG}$ with a large class of damping and potential functions (1.2) (1.3).

Before proceeding, we give a definition of the weak solution.

**Definition of weak solutions:** Suppose that $u, v \in C([0, T), H^1(\mathbb{R}^n)) \cap C^1([0, T), L^2(\mathbb{R}^n))$ and $u \in L^p_{\text{loc}}((0, T) \times \mathbb{R}^n)$, $v \in L^p_{\text{loc}}((0, T) \times \mathbb{R}^n)$ for Theorem 1.1, $u_t \in L^p_{\text{loc}}((0, T) \times \mathbb{R}^n)$, $v_t \in L^p_{\text{loc}}((0, T) \times \mathbb{R}^n)$ for Theorem 1.2,
Remark 1.1. When damping and potential functions satisfying (1.2), (1.3), we conjecture the critical curve of the solution to system (1.1) is the same as system (1.4)(with corresponding nonlinear term). That is to say, such damping and potential functions does not effect the critical curve of system (1.1).
2. PRELIMINARY

In this section, we collect some Lemmas we will use later. For the strategy of the proof, we use test function methods, see, e.g., [19], [3], [10].

**Lemma 2.1 (Lemma 2.2 in [18]).** Support $0 \leq V \in C(\mathbb{R}^n)$, $V = V = O((r)^{-\omega})$ for some $\omega > 2$, and $V \in C^\delta(B_k)$ for some $\delta > 0$. In addition, assume $V$ is nontrivial when $n = 2$. Then there exist a $C^2$ solution of

$$\Delta \phi_0 = V \phi_0, \quad x \in \mathbb{R}^n, n \geq 2,$$

satisfying

$$\phi_0 \simeq \begin{cases} \ln(r + 2), & n = 2, \\ 1, & n \geq 3. \end{cases}$$

Moreover, when $n = 2$ and $V = 0$, it is clear that $\phi_0 = 1$ is a solution (2.1).

**Lemma 2.2 (Lemma 2.4 in [18]).** Let $V = V(r)$, $D = D(r) \in C(\mathbb{R}^n) \cap C^\delta(B_k)$ for some $\delta > 0$. Suppose $V \geq 0$, $D \geq -\lambda - \frac{\lambda_1}{\lambda}$, and also that for some $d_\infty \in \mathbb{R}$, $R > 1$ and $r \geq R$, we have

$$V(r) \in L^1([R, \infty)), \quad D(r) = \frac{d_\infty}{r} + D_\infty(r), \quad D_\infty \in L^1([R, \infty)).$$

Then there exists a $C^2$ solution of

$$\Delta \phi_\lambda = (\lambda^2 + \lambda D + V)\phi_\lambda, \quad \lambda > 0, \quad x \in \mathbb{R}^n,$$

satisfying

$$\phi_\lambda \simeq \langle r \rangle^{-\frac{n-1-d_\infty}{2}} e^{\lambda r}.$$

**Lemma 2.3.** Concerning (1.2) (1.3), we have $d_\infty = 0$ in Lemma 2.2. Let $a > 0$. Define

$$b_a(t, x) = \int_0^1 e^{-\lambda t} \phi_\lambda(x)\lambda^{-1-a} d\lambda, \quad x \in \mathbb{R}^n$$

$b_a(t, x)$ lie on the facts that they satisfy

$$\partial_t^2 b_a - \Delta b_a - D_i(x)\partial_i b_a + V_i(x)b_a = 0, \quad i = 1, 2.$$  

and enjoy the asymptotic behavior for $n \geq 2$ and $r \leq t + R$

$$b_a(t, x) \gtrsim (t + R)^{-a}, \quad a > 0,$$

$$b_a(t, x) \lesssim \begin{cases} (t + R)^{-a}, & 0 < a < \frac{n-1}{2}, \\ (t + R)^{-\frac{n-1}{2}} (t + R + 1 - |x|)^{\frac{n-1}{2} - a}, & a > \frac{n-1}{2}. \end{cases}$$

**Proof.** See the proof of (6.7) (6.8) in [18].

**Lemma 2.4 (Lemma 3.1 in [17]).** If $\beta > 0$, then for any $\alpha \in \mathbb{R}$ and a fixed constant $R$, we have

$$\int_0^{t+R} (1 + r)^\alpha e^{-\beta(t-r)} dr \lesssim (t + R)^\alpha.$$  

Finally, we need the following ODE Lemma to show the finite time blow up in some critical cases.
Lemma 2.5 (Lemma 3.10 in [10]). Let $2 < t_0 < T$, $0 \leq \phi \in C^1([t_0, T])$. Assume that

\[
\begin{align*}
\delta &\leq K_1 t \phi'(t), \quad t \in (t_0, T) \\
\phi(t)^{p_1} &\leq K_2 t (\ln t)^{p_2-1} \phi'(t), \quad t \in (t_0, T)
\end{align*}
\]

with $\delta, K_1, K_2 > 0$ and $p_1, p_2 > 1$. If $p_2 < p_1 + 1$, then there exists positive constant $\delta_0$ and $K_3$ (independent of $\delta$) such that

\[
T \leq \exp(K_3 \delta^{-\frac{p_1-1}{p_1-p_2+1}})
\]

for all $\delta \in (0, \delta_0)$.

Set up: We take $\lambda = 1$ in Lemma 2.2 and $\phi(x) = \phi_1(x)$, $\Phi(t, x) = e^{-t} \phi(x)$, it is easy to see that $\Phi$ is the solution of the Linear wave equation

\[
\partial_t^2 \Phi - \Delta \Phi - D_i(x) \partial_i \Phi + V_i \Phi = 0, \quad i = 1, 2.
\]

3. Proof of Theorems 1.1-1.3

In this section, we give the proof of Theorems 1.1-1.3.

3.1. Proof of Theorem 1.1.

3.1.1. Blow up region $\Gamma_{SS}(n, p, q) > 0$. Let $\eta(t) \in C^\infty([0, \infty))$ satisfies

\[
\eta(t) = \begin{cases} 
1, & 0 < t \leq \frac{1}{2}, \\
\text{decreasing}, & \frac{1}{2} < t < 1, \\
0, & t \geq 1,
\end{cases}
\]

and

\[
\eta_M(t) = \eta\left(\frac{t}{M}\right), \quad M \in (1, T_\varepsilon).
\]

In addition, we take $T \in [M, T_\varepsilon)$.

(I) Test function 1: $\Psi(t, x) = \eta_M^{2p'}(t) \phi_0(x)$

Firstly, we choose the test function $\Psi(t, x) = \eta_M^{2p'}(t) \phi_0(x)$ to replace $\Psi(t, x)$ in (1.11) and applying Lemma 2.1, we get that

\[
\begin{align*}
\varepsilon \int_{\mathbb{R}^n} u_1(x) \phi_0 dx + \varepsilon \int_{\mathbb{R}^n} u_0(x) D_1(x) \phi_0 dx + &\int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2p'} \phi_0 dx dt \\
= &\int_0^M \int_{\mathbb{R}^n} u_0 \left(\partial_t^2 \eta_M^{2p'} - D_1(x) \partial_t \eta_M^{2p'}\right) dx dt - \int_0^M \int_{\mathbb{R}^n} u_0^{2p'} (\Delta \phi_0 - V_1(x) \phi_0) dx dt \\
= &\int_0^M \int_{\mathbb{R}^n} u_0 \left(\partial_t \eta_M^{2p'} - D_1(x) \partial_t \eta_M^{2p'}\right) dx dt \\
= &I_1 + I_2
\end{align*}
\]
Noting that when \( p \leq q \) we have \( 2p'q - 2q \geq 2p' \). We apply Hölder inequality and (2.2) to the following estimates

\[
|I_1| \lesssim M^{-2} \int_0^M \int_{\mathbb{R}^n} u(t, x) \theta_M^{2p'-2} dx dt
\]
\[
\lesssim M^{-2} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q(2p'-2)} dx dt \right)^{\frac{1}{q'}} \left( \int_{\mathbb{R}^n} dx dt \right)^{\frac{1}{q'}} \quad (p \leq q)
\]
\[
|I_2| \lesssim M^{-1} \int_0^M \int_{\mathbb{R}^n} u(t, x) D_1(x) \theta_M^{2p'-1} dx dt
\]
\[
\lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q(2p'-1)} dx dt \right)^{\frac{1}{q'}} \left( \int_{\mathbb{R}^n} (D_1(x))^q dx dt \right)^{\frac{1}{q'}}
\]
\[
\lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2p'}} dx dt \right)^{\frac{1}{q'}} \left( \int_{\mathbb{R}^n} (1 + r)^{-\beta q' r^n-1} dr dt \right)^{\frac{1}{q'}}
\]
\[
\lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2p'}} dx dt \right)^{\frac{1}{q'}} \left\{ \begin{array}{ll}
(M \ln(M))^{\frac{1}{q'}} & , n - \beta q' = 0,
M^{(n+1-\beta q')\frac{1}{q'}} & , n - \beta q' > 0,
M^{\frac{1}{q'}} & , n - \beta q' < 0.
\end{array} \right.
\]
\[
\lesssim M^{\frac{nq-n-q-1}{q'}} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2p'}} dx dt \right)^{\frac{1}{q'}}.
\]

By combining (3.2) and (3.3), we get that

\[
(4.4) \quad \varepsilon C(u_0, u_1) + \int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2p'} dx dt \lesssim M^{\frac{nq-n-q-1}{q'}} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2p'}} dx dt \right)^{\frac{1}{q'}}
\]

then we get that

\[
(4.5) \quad \int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2p'} dx dt \lesssim M^{\frac{nq-n-q-1}{q'}} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2p'}} dx dt \right)^{\frac{1}{q'}}.
\]

By a similar way, we choose the test function with \( \Psi(t, x) = \eta_M^{2p'}(t)\phi_0(x) \) to replace \( \Psi(t, x) \) in (1.12), we can get

\[
(4.6) \quad \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2p'}} dx dt \lesssim M^{\frac{nq-n-q-1}{q'}} \left( \int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2p'} dx dt \right)^{\frac{1}{q'}}.
\]

Applying (3.5) and (3.6) yields

\[
(4.7) \quad \int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2p'} dx dt \lesssim M^{\frac{np-2p-n-pq-1}{pq-1}}.
\]
We apply Hölder inequality to the following estimates

\[ (3.8) \quad \int_0^M \int_{\mathbb{R}^n} |u|^q \eta_M^{2p'} \, dx \, dt \lesssim M^{\frac{n-q}{p'-1}}. \]

\(\text{(II) Test function} 2: \Psi(t, x) = \eta_M^{2p'}(t) e^{-t} \phi(x) = \eta_M^{2p'}(t) \Phi(t, x)\)

Secondly, we choose another test function \(\Psi = \eta_M^{2p'} \Phi(t, x)\) to replace \(\Psi(t, x)\) in (1.11) and applying Lemma 2.2, we get that

\[
\varepsilon C(u_0, u_1) + \int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2p'} \Phi \, dx \, dt \\
= -\int_0^M \int_{\mathbb{R}^n} u_t \partial_t (\eta_M^{2p'} \Phi) \, dx \, dt + \int_0^M \int_{\mathbb{R}^n} \nabla u (\eta_M^{2p'} \Phi) \, dx \, dt \\
- \int_0^M \int_{\mathbb{R}^n} D_1(x) u \partial_t (\eta_M^{2p'} \Phi) \, dx \, dt + \int_0^M \int_{\mathbb{R}^n} u V_1(x) \eta_M^{2p'} \Phi \, dx \, dt \\
= \int_0^M \int_{\mathbb{R}^n} u \left( (\partial_t^2 \eta_M^{2p'}) \Phi + 2(\partial_t \eta_M^{2p'}) (\partial_t \Phi) - D_1(x) (\partial_t \eta_M^{2p'}) \Phi \right) \, dx \, dt \\
+ \int_0^M \int_{\mathbb{R}^n} u \eta_M^{2p'} (\partial_t^2 \Phi - \Delta \Phi - D_1(x) \partial_t \Phi + V_1(x) \Phi) \, dx \, dt \\
= \int_0^M \int_{\mathbb{R}^n} u \left( (\partial_t^2 \eta_M^{2p'}) \Phi + 2(\partial_t \eta_M^{2p'}) (\partial_t \Phi) - D_1(x) (\partial_t \eta_M^{2p'}) \Phi \right) \, dx \, dt
\]

We apply Hölder inequality to the following estimates

\[ |I_3| \lesssim M^{-2} \int_0^M \int_{\mathbb{R}^n} |\theta_M^{2p'} \Phi| \, dx \, dt \]
\[
\lesssim M^{-2} \left( \int_0^M \int_{\mathbb{R}^n} |u|^q |\theta_M^{2p'}|^q \, dx \, dt \right)^{\frac{1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} |\Phi| \, dx \, dt \right)^{\frac{1}{q'}} (p < q) \]
\[
\lesssim M^{-2} \left( \int_0^M \int_{\mathbb{R}^n} |u|^q |\theta_M^{2p'}| \, dx \, dt \right)^{\frac{1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} |\Phi| \, dx \, dt \right)^{\frac{1}{q'}} (\text{Lemma 2.2}) \]
\[
\lesssim M^{-2} \left( \int_0^M \int_{\mathbb{R}^n} |u|^q |\theta_M^{2p'}| \, dx \, dt \right)^{\frac{1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} (r + 1)^{n-1-\frac{2q'}{q}} e^{-(t-r)q'} \, dr \, dt \right)^{\frac{2}{q'}} (\text{Lemma 2.4}) \]
\[
\lesssim M^{-2+(n-\frac{(n-1)q'}{q})} \left( \int_0^M \int_{\mathbb{R}^n} |u|^q |\theta_M^{2p'}| \, dx \, dt \right)^{\frac{1}{q}}, \]

\[ |I_4| \lesssim M^{-1+(n-\frac{(n-1)q'}{q})} \left( \int_0^M \int_{\mathbb{R}^n} |u|^q |\theta_M^{2p'}| \, dx \, dt \right)^{\frac{1}{q}}, \]
where we set
That is to say

\(|I_5| \lesssim M^{-1} \int_0^M \int_{\mathbb{R}^n} |u\theta_M^{2p'-1}\Phi D_1(x)| dx dt\)

\(\lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2p'} dx dt \right)^{\frac{1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} D_1'(x) \Phi' dx dt \right)^{\frac{1}{p'}}\)

\(\lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2p'} dx dt \right)^{\frac{1}{q}} \left( \int_0^M (1 + t)^{n-1 - \frac{2}{q} + \beta} \Phi' dx dt \right)^{\frac{1}{p'}} \) (Lemma 2.4)

\(\lesssim M^{-2 + \left(n - \frac{(n-1)p'}{2}\right)} \left( \int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2p'} dx dt \right)^{\frac{1}{q}}.

This in turn implies that

\(|C(u_0, u_1)\varepsilon|^q \lesssim M^{\frac{n-1}{2}} \varepsilon^{\frac{1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2p'} dx dt \right)^{\frac{1}{q}}\).

In an analogy way, we choose test function \(\Psi = \eta_M^{2p'} \Phi(t, x)\) to replace \(\Phi(t, x)\) in (1.12), we have that

\(|C(v_0, v_1)\varepsilon|^p \lesssim M^{\frac{n-1}{q}} \varepsilon^{\frac{1}{p}} \left( \int_0^M \int_{\mathbb{R}^n} |v|^p \theta_M^{2p'} dx dt \right)^{\frac{1}{p}}\).

(3.9) and (3.8) yield

\(M \lesssim \varepsilon^{-\frac{1}{(p-q-1+2)(q-1)-1 - \frac{n-1}{2}}}\).

(3.7) and (3.10) yield

\(M \lesssim \varepsilon^{-\frac{1}{(q+p-1+2)(q-1)-1 - \frac{n-1}{2}}}\).

Note that, by replacing \(p\) with \(q\) in the test functions, we could have the corresponding estimates under the assumption \(p \geq q\). Hence for any \(M \in (1, T_\varepsilon)\), we get that

\(M \lesssim \min \left\{ \varepsilon^{-\frac{1}{(p-q-1+2)(q-1)-1 - \frac{n-1}{2}}} \varepsilon^{-\frac{1}{(q+p-1+2)(q-1)-1 - \frac{n-1}{2}}} \right\}\).

That is to say

\(T_\varepsilon \lesssim \min \left\{ \varepsilon^{-\frac{1}{(p-q-1+2)(q-1)-1 - \frac{n-1}{2}}} \varepsilon^{-\frac{1}{(q+p-1+2)(q-1)-1 - \frac{n-1}{2}}} \right\}\).

So we come to the estimate

\(T_\varepsilon \lesssim \varepsilon^{-\Gamma_{SS}^\frac{1}{2}(n, p, q)}, \Gamma_{SS}(n, p, q) > 0\).

We obtain the first lifespan estimate in Theorem 1.1.

3.1.2. Blow up region \(\Gamma_{SS}(n, p, q) = 0\). Let

\(\theta(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ \eta(t), & t \geq \frac{1}{2}, \end{cases}\)

where \(\theta_M(t) = \theta(\frac{t}{M})\) and \(M \in (1, T_\varepsilon)\). For nonnegative function \(\omega \in L^1_{loc}([0, T); \mathcal{L}^1(\mathbb{R}^n))\), we set

\(Y[\omega(t, x)](M) = \int_1^M \left( \int_0^T \int_{\mathbb{R}^n} \omega(t, x) \theta_M^{2p'}(t) dx dt \right) \sigma^{-1} ds\).

Then we can get that

\(Y'[\omega(t, x)](M) = \frac{d}{dM} Y[\omega(t, x)](M) = M^{-1} \left( \int_0^T \int_{\mathbb{R}^n} \omega(t, x) \theta_M^{2p'}(t) dx dt \right)\).
By direct computation, we get that

\[ Y[\omega(x)] = \int_0^1 \left( \int_0^T \omega(x) \theta^2_p(t) dt \right) ds^{-1} ds dt = \int_0^T \int_{\mathbb{R}^n} \omega(x) \int_t^T \theta^2_p(s) s^{-1} ds dt \]

\[ \lesssim \int_0^T \int_{\mathbb{R}^n} \omega(x) \int_t^T \theta^2_p(s) s^{-1} ds dt + \int_0^T \int_{\mathbb{R}^n} \omega(x) \int_t^T \theta^2_p(s) s^{-1} ds dt \]

\[ \lesssim \ln 2 \int_0^T \int_{\mathbb{R}^n} \omega(x) \int_t^T \frac{\theta^2_p(s)}{M} \int_1^s s^{-1} ds dt \]

Then we can obtain that

\[ Y[\omega(t, x)](M) \lesssim \ln 2 \int_0^T \int_{\mathbb{R}^n} \omega(t, x) \eta^2_{M}(t) dt. \]

**(III) Test function 3:** \( \Psi(t, x) = \eta^2_{M}(t) b_a(t, x) \)

Let \( a = \frac{n-1}{2} - \frac{1}{q} > 0 \), by the lower bound of \( b_a \) in Lemma 2.3, we have that

\[ \int_0^M \int_{\mathbb{R}^n} |v|^p \theta^2_q b_a dt \geq M^{-a} \int_0^M \int_{\mathbb{R}^n} |v|^p \theta^2_q b_a dt. \]

By applying (3.6) and (3.9) (by replacing \( p \) with \( q \) in \( \theta^2_q b_a \)), we know that

\[ \int_0^M \int_{\mathbb{R}^n} |v|^p \theta^2_q b_a dt \geq \varepsilon^q M^{n+p+1-n-1/2 pq - a} = \Gamma_{SS}(p, q, n) = 0. \]

Noting that when \( p > q \), \( \Gamma_{SS} = (p + 2 + 1/q)/(pq - 1) \), then we have

\[ n + p + 1 - \frac{n-1}{2} pq = \Gamma_{SS}(p, q, n) = 0. \]

Hence we get that

\[ (3.14) \quad \int_0^M \int_{\mathbb{R}^n} |v|^p \theta^2_q b_a dt \gtrsim \varepsilon^q. \]

Next we choose test function \( \Psi = \eta^2_{M} b_a \) to replace \( \Psi(t, x) \) in (1.11) and applying (2.6), we get that

\[ \varepsilon C(u_0, u_1) + \int_0^M \int_{\mathbb{R}^n} |v|^p \eta^2_{M} b_a dt \]

\[ = \int_0^T \int_{\mathbb{R}^n} \left( (\partial_t \eta^2_{M}) b_a + 2(\partial_t \eta^2_{M})(\partial_t b_a) - D_1(x)(\partial_t \eta^2_{M}) b_a \right) dt \]

\[ = I_6 + I_7 + I_8 \]
We apply Hölder inequality to the following estimates

\[ |I_6| \lesssim M^{-2} \int_0^M \int_{\mathbb{R}^n} |u(t,x)\theta_M^{2q-2}(t)b_a| \, dx \, dt \]

\[ \lesssim M^{-2} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2q}} \, dx \, dt \right)^{\frac{1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} b_a^{q'} \, dx \, dt \right)^{\frac{1}{q'}} \]

\[ \lesssim M^{-2} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2q}} \, dx \, dt \right)^{\frac{1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} (t + R)^{\frac{1}{q} - \frac{n-1}{2}} q' \, dx \, dt \right)^{\frac{1}{q'}} \quad \text{(Lemma 2.3)} \]

\[ \lesssim M^{\frac{n + 2n - 2n + 1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2q}} \, dx \, dt \right)^{\frac{1}{q}} , \]

\[ |I_7| \lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2q}} \, dx \, dt \right)^{\frac{1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} b_a^{q'} \, dx \, dt \right)^{\frac{1}{q'}} \]

\[ \lesssim M^{\frac{n + 2n - 2n + 1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2q}} \, dx \, dt \right)^{\frac{1}{q}} , \]

\[ |I_8| \lesssim M^{-1} \int_0^M \int_{\mathbb{R}^n} |u(t,x)\theta_M^{2q-1}(t)b_aD_1(x)| \, dx \, dt \]

\[ \lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2q}} (t) \, dx \, dt \right)^{\frac{1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} D_1^{q'} (x)b_a^{q'} \, dx \, dt \right)^{\frac{1}{q'}} \]

\[ \lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2q}} (t) \, dx \, dt \right)^{\frac{1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} (t + R)^{\frac{1}{q} - \frac{n-1}{2}} q' \, dx \, dt \right)^{\frac{1}{q'}} \]

\[ \lesssim M^{\frac{n + 2n - 2n + 1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2q}} (t) \, dx \, dt \right)^{\frac{1}{q}} , \]

Hence, we get that

\[ (3.15) \quad \int_0^M \int_{\mathbb{R}^n} |v|^p n_M^{2q-1}(t) b_a \, dx \, dt \lesssim M^{\frac{n + 2n - 2n + 1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2q}} \, dx \, dt \right)^{\frac{1}{q}} , \]

\[ (3.16) \quad \int_0^M \int_{\mathbb{R}^n} |u|^{q\theta_M^{2q}} (t) b_a \, dx \, dt \lesssim M^{\frac{n + 2n - 2n + 1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} |v|^p \theta_M^{2q} \, dx \, dt \right)^{\frac{1}{p}} . \]

Plugging (3.6) into (3.15) and by the lower bound of \( b_a \) in Lemma 2.3, we get that

\[ (3.17) \quad \left( \int_0^M \int_{\mathbb{R}^n} |v|^p n_M^{2q-1} \, dx \, dt \right)^{\frac{1}{p}} \lesssim (\ln M)^{\mu(q-1)} \int_0^M \int_{\mathbb{R}^n} |v|^p \, dx \, dt . \]

Let

\[ Y[b_a|v|^p](M) = \int_1^M \left( \int_0^\tau \int_{\mathbb{R}^n} |v|^p b_a \theta_M^{2q} (t) \, dx \, dt \right) \sigma^{-1} \, d\sigma \]

By combining (3.12), (3.13), (3.14) and (3.17), we have that

\[ \varepsilon^{pq} \lesssim MY'[b_a|v|^p](M) \]

\[ [Y[b_a|v|^p](M)]^{pq} \lesssim (\ln M)^{\mu(q-1)}MY'[b_a|v|^p](M) \]
Exploiting Lemma 2.5 with \( p_2 = p(q - 1) + 1, p_1 = pq, \delta = \varepsilon^pq \), we have that
\[
M \lesssim \exp(\varepsilon^{-q(pq-1)}), \quad p > q.
\]

Hence for any \( M \in (1, T_\varepsilon) \), that is to say
\[
T_\varepsilon \lesssim \exp(\varepsilon^{-q(pq-1)}), \quad p > q.
\]

By the symmetrical characteristic of \( p \) and \( q \), we could get that
\[
T_\varepsilon \lesssim \exp(\varepsilon^{-p(pq-1)}), \quad p < q.
\]

Then we get that
\[
T_\varepsilon \lesssim \exp \left( \varepsilon^{-\min\{p(pq-1), q(pq-1)\}} \right), \quad \Gamma_{SS}(n, p, q) = 0, \quad p \neq q.
\]

Next, we give the lifespan estimate when \( \Gamma_{SS}(n, p, q) = 0, \quad p = q = p_S(n) \). In the following, we take \( D(x) = D_1(x) + D_2(x) \). By combining (3.9), (3.10), we have
\[
\varepsilon C(u_0, u_1) + \varepsilon C(v_0, v_1) + \int_0^M \int_{\mathbb{R}^n} |u + v|^p \eta_M^{2p'} \Phi \, dx dt \\
\lesssim \varepsilon C(u_0, u_1) + \varepsilon C(v_0, v_1) + \int_0^M \int_{\mathbb{R}^n} (|u|^p + |v|^p) \eta_M^{2p'} \Phi \, dx dt \\
\lesssim \int_0^M \int_{\mathbb{R}^n} (u + v) \left( (\partial_t^2 \eta_M^{2p'}) \Phi + 2(\partial_t \eta_M^{2p'}) (\partial_t \Phi) - D(x) (\partial_t \eta_M^{2p'}) \Phi \right) \, dx dt \\
\lesssim M^{\frac{(n-1)}{2} \frac{1}{p}} \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p \theta_M^{2p'} b_adxdt \right)^{\frac{1}{p}} \\
\lesssim M^{\frac{(n-1)}{2} \frac{1}{p}} M^{\frac{n-1}{2p}} \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p \theta_M^{2p'} b_a dx dt \right)^{\frac{1}{p}}
\]

Noting that when \( \Gamma_{SS}(n, p, q) = 0, \quad p = q = p_S(n) \), we have that
\[
n - \frac{(n-1)p}{2} = \frac{n - 1}{2} - \frac{1}{p}.
\]

Which leads to
\[
(3.18) \quad \varepsilon^p \lesssim \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p \theta_M^{2p'} b_a dx dt \right).
\]

Similarly, we can get that
\[
\varepsilon + \int_0^M \int_{\mathbb{R}^n} |v + u|^p \eta_M^{2p'} (t) b_a dx dt \\
\lesssim \int_0^M \int_{\mathbb{R}^n} (u + v) \left( (\partial_t^2 \eta_M^{2p'}) b_a + 2(\partial_t \eta_M^{2p'}) (\partial_t b_a) - D(x) (\partial_t \eta_M^{2p'}) b_a \right) dx dt \\
\lesssim I_9 + I_{10} + I_{11}
\]
We apply Hölder inequality to the following estimates

\[ |J_9| \lesssim M^{-2} \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p \theta_M^{2p'} \| b_0 \|_{L^1} \ dx \ dx dt \right)^{\frac{1}{p}} \left( \int_0^M \int_{\mathbb{R}^n} b_0 \ dx \ dx dt \right)^{\frac{2}{p'}} \]

\[ \lesssim M^{-2} \left( \int_0^M \int_{\mathbb{R}^n} (t + R)^{-\left(\frac{n-1}{2p'} - \frac{1}{p}\right)} |u|^r \ dx \ dx dt \right)^{\frac{p-1}{r}} \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p \theta_M^{2p'} \| b_0 \|_{L^1} \ dx \ dx dt \right)^{\frac{1}{p'}} \]

\[ \lesssim \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p \theta_M^{2p'} \| b_0 \|_{L^1} \ dx \ dx dt \right)^{\frac{1}{p'}}. \]

\[ |J_{10}| \lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p \theta_M^{2p'} \| b_0 \|_{L^1} \ dx \ dx dt \right)^{\frac{1}{p}} \left( \int_0^M \int_{\mathbb{R}^n} b_0^{-\left(\frac{1}{p'} - \frac{1}{p}\right)} \ dx \ dx dt \right)^{\frac{p-1}{p'}} \]

\[ \lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} (t + R)^{-\left(\frac{n-1}{2p'} - \frac{1}{p}\right)} |u|^r \ dx \ dx dt \right)^{\frac{p-1}{r}} \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p \theta_M^{2p'} \| b_0 \|_{L^1} \ dx \ dx dt \right)^{\frac{1}{p'}} \]

\[ \lesssim (\ln M)^{\frac{p-1}{p'}} \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p \theta_M^{2p'} \| b_0 \|_{L^1} \ dx \ dx dt \right)^{\frac{1}{p'}}. \]

\[ |J_{11}| \lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p \theta_M^{2p'} \| b_0 \|_{L^1} \ dx \ dx dt \right)^{\frac{1}{p}} \left( \int_0^M \int_{\mathbb{R}^n} D^p \ (x) b_0 \ dx \ dx dt \right)^{\frac{p-1}{p'}} \]

\[ \lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} (t + R)^{-\left(\frac{n-1}{2p'} - \frac{1}{p}\right)} (1 + r)^{-\beta_0 r} \ dx \ dx dt \right)^{\frac{p-1}{r}} \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p \theta_M^{2p'} \| b_0 \|_{L^1} \ dx \ dx dt \right)^{\frac{1}{p'}} \]

\[ \lesssim \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p \theta_M^{2p'} \| b_0 \|_{L^1} \ dx \ dx dt \right)^{\frac{1}{p'}}. \]

Therefore we conclude from the estimates \( I_9 - I_{11} \) that

\[ \left( \int_0^M \int_{\mathbb{R}^n} |v + u|^p \theta_M^{2p'} \ (t) b_0 \ dx \ dx dt \right)^{\frac{1}{p}} \lesssim (\ln M)^{p-1} \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p \theta_M^{2p'} \| b_0 \|_{L^1} \ dx \ dx dt \right). \]

Let

\[ Y[b_a | u + v|^p](M) = \int_1^M \left( \int_0^M \int_{\mathbb{R}^n} |u + v|^p b_0 \theta_M^{2p'} \ (t) \ dx \ dx dt \right) \sigma^{-1} \ dx \ dx dt. \]

By combining (3.12), (3.13), (3.18) and (3.19), we get that

\[ \varepsilon^p \lesssim MY'[b_a | u + v|^p](M) \]

\[ [Y[b_a | u + v|^p](M)]^p \lesssim (\ln M)^{p-1} MY'[b_a | u + v|^p](M) \]

By exploiting Lemma 2.5 with \( p_2 = p_1 = p, \delta = \varepsilon^p, \) we have that \( M \lesssim \exp(\varepsilon^{-p(p-1)}), \) \( p = q = p_S(n). \)

Hence for any \( M \in (1, T_\varepsilon), \) that is to say

\[ T_\varepsilon \lesssim \exp(\varepsilon^{-p(p-1)}), \] \( p = q = p_S(n). \)

We finish the proof of Theorem 1.1.
3.2. Proof of Theorem 1.2.

3.2.1. Blow up region $\Gamma_{GG}(n, p, q) > 0$.

(I) Test function 1: $\Psi(t, x) = \eta_M^{2p'}(t)\phi_0(x)$

Firstly, we choose test function $\Psi = \eta_M^{2p'}(t)\phi_0$ to replace $\Psi(t, x)$ in (1.11) we get

\[
\varepsilon \int_{\mathbb{R}^n} u_1(x)\phi_0 dx + \varepsilon \int_{\mathbb{R}^n} u_0(x)D_1(x)\phi_0 dx + \int_0^M \int_{\mathbb{R}^n} |v_t|^p \eta_M^{2p'}(t)\phi_0 dx dt
\]

\[
= - \int_0^M \int_{\mathbb{R}^n} u_t\phi_0(\partial_t \eta_M^{2p'}(t)) dx dt + \int_0^M \int_{\mathbb{R}^n} u_t(D_1(x))\eta_M^{2p'}(t)\phi_0 dx dt
\]

(3.20) \quad - \int_0^M \int_{\mathbb{R}^n} u\eta_M^{2p'}(\Delta \phi_0 - V_1(x)\phi_0) dx dt

\[= I_{12} + I_{13}\]

We apply Hölder inequality to the following estimates

\[
|I_{12}| \lesssim M^{-1} \int_0^M \int_{\mathbb{R}^n} |u_t\eta_M^{2p'-1}| dx dt
\]

\[
\lesssim M^{\frac{n+1}{4}} \left( \int_0^M \int_{\mathbb{R}^n} |u_t\eta_M^{2p'} dx dt \right)^{\frac{4}{n}},
\]

\[
|I_{13}| \lesssim \int_0^M \int_{\mathbb{R}^n} |u_t\eta_M^{2p'} D_1(x)| dx dt
\]

\[
\lesssim \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^q \eta_M^{2p'} dx dt \right)^{\frac{1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} (D_1(x))^{p'} dx dt \right)^{\frac{1}{p}}
\]

\[
\lesssim \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^q \eta_M^{2p'} dx dt \right)^{\frac{1}{q}} \left\{ \begin{array}{ll}
(M \ln(M))^{\frac{1}{p'}}, & n - \beta q = 0, \\
M^{(n+1-\beta q)/p'}, & n - \beta q > 0, \\
M^{\frac{1}{p'}}, & n - \beta q < 0.
\end{array} \right.
\]

So we get that

\[
\int_0^M \int_{\mathbb{R}^n} |v_t|^p \eta_M^{2p'} dx dt \lesssim M^{\frac{n+1-\beta q}{q}} \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^q \eta_M^{2p'} dx dt \right)^{\frac{1}{q}}
\]

Similarly, we have that

\[
\int_0^M \int_{\mathbb{R}^n} |u_t|^q \eta_M^{2p'} dx dt \lesssim M^{\frac{n+1-\beta q}{p'}} \left( \int_0^M \int_{\mathbb{R}^n} |v_t|^p \eta_M^{2p'} dx dt \right)^{\frac{1}{p'}}
\]

Applying (3.21) and (3.22) yields

\[
\int_0^M \int_{\mathbb{R}^n} |v_t|^p \eta_M^{2p'} dx dt \lesssim M^{\frac{n+1-\beta q}{qp'-1}},
\]

\[
\int_0^M \int_{\mathbb{R}^n} |u_t|^q \eta_M^{2p'} dx dt \lesssim M^{\frac{n+1-\beta q}{q'-1}}.
\]

(II) Test function 2: $\Psi = -\partial_t(\eta_M^{2p'}(t)\Phi(t, x))$
Secondly, we choose another test function $\Psi = \partial_t (\eta_{M}^{2p'} \Phi)$ to replace $\Psi$ in (1.11). It is worth observing that

$$\Psi(t, x) = -\partial_t (\eta_{M}^{2p'} \Phi) = \eta_{M}^{2p'} \Phi - 2p' \eta_{M}^{2p'-1} (\partial_t \eta_{M}) \Phi \geq \eta_{M}^{2p'} \Phi > 0.$$

Direct calculation shows

$$\varepsilon C(u_0, u_1) + \int_0^M \int_{\mathbb{R}^n} |v_i|^p (\eta_{M}^{2p'} \Phi - 2p' \eta_{M}^{2p'-1} (\partial_t \eta_{M}) \Phi) dx \, dt$$

$$= \int_0^M \int_{\mathbb{R}^n} u_i \left( (\partial_t^2 \eta_{M}^{2p'}) \Phi + 2 (\partial_t \eta_{M}^{2p'}) (\partial_t \Phi) - D_1(x)(\partial_t \eta_{M}^{2p'}) \Phi \right) dx \, dt$$

(3.25)

$$= I_{14} + I_{15} + I_{16}$$

By the same procedure in (3.2)-(3.3), we get that

(3.26) \quad $$\varepsilon C(u_0, u_1) \lesssim M^{-1 + \left(\frac{n - 2p}{p - 2p'}\right)} \left( \int_0^M \int_{\mathbb{R}^n} |v_i|^q \eta_{M}^{2p'} dx \, dt \right)^\frac{1}{q},$$

(3.27) \quad $$\varepsilon C(v_0, v_1) \lesssim M^{-1 + \left(\frac{n - 2p}{p - 2p'}\right)} \left( \int_0^M \int_{\mathbb{R}^n} |v_i|^q \eta_{M}^{2p'} dx \, dt \right)^\frac{1}{q}.$$}

On the one hand, by (3.21), (3.22), (3.26) and (3.27), we can obtain the first lifespan estimation in Theorem 1.2.

On the other hand, by exploiting the estimates of $\Phi$, we have that

$$|I_{14}| \lesssim \int_0^M \int_{\mathbb{R}^n} |u_i (\partial_t^2 \eta_{M}^{2p'}) \Phi| dx \, dt$$

$$\lesssim M^{-2} \left( \int_0^M \int_{\mathbb{R}^n} \theta_{M}^{2p'} |u_i|^q \Phi dx \, dt \right)^\frac{1}{q} \left( \int_0^M \int_{\mathbb{R}^n} \Phi dx \, dt \right)^\frac{1}{p}$$

$$\lesssim M^{-2} \left( \int_0^M \int_{\mathbb{R}^n} \theta_{M}^{2p'} |u_i|^q \Phi dx \, dt \right)^\frac{1}{q} \left( \int_0^M (t + 1)^{\frac{n-1}{2}} dx \, dt \right)^\frac{1}{p}$$

$$\lesssim M^{\frac{nq-n-3p-1}{4p}} \left( \int_0^M \int_{\mathbb{R}^n} \theta_{M}^{2p'} |u_i|^q \Phi dx \, dt \right)^\frac{1}{q},$$

$$|I_{15}| \lesssim \int_0^M \int_{\mathbb{R}^n} |u_i (\partial_t \eta_{M}^{2p'}) \Phi| dx \, dt$$

$$\lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} \theta_{M}^{2p'} |u_i|^q \Phi dx \, dt \right)^\frac{1}{q} \left( \int_0^M \int_{\mathbb{R}^n} \Phi dx \, dt \right)^\frac{1}{p}$$

$$\lesssim M^{\frac{nq-n-4q-1}{4q}} \left( \int_0^M \int_{\mathbb{R}^n} \theta_{M}^{2p'} |u_i|^q \Phi dx \, dt \right)^\frac{1}{q},$$

$$|I_{16}| \lesssim |I_{15}| \lesssim M^{\frac{nq-n-3p-1}{4p}} \left( \int_0^M \int_{\mathbb{R}^n} \theta_{M}^{2p'} |u_i|^q \Phi dx \, dt \right)^\frac{1}{q}.$$
Then we get that
\begin{equation}
(3.28)\varepsilon C(u_0, u_1) + \int_0^M \int_{\mathbb{R}^n} |v_t|^p \eta_M^{2p'} \Phi \, dx \, dt \lesssim M^{nq \eta - n - \frac{1}{2p} - \frac{1}{2p'}} \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^q \vartheta_M^{2p'} \Phi \, dx \, dt \right)^{\frac{1}{q'}}.
\end{equation}
\begin{equation}
(3.29)\varepsilon C(v_0, v_1) + \int_0^M \int_{\mathbb{R}^n} |u_t|^q \eta_M^{2p'} \Phi \, dx \, dt \lesssim M^{nq \eta - n - \frac{1}{2p} - \frac{1}{2p'}} \left( \int_0^M \int_{\mathbb{R}^n} |v_t|^p \vartheta_M^{2p'} \Phi \, dx \, dt \right)^{\frac{1}{p'}}.
\end{equation}
Combining above two inequalities, we have that
\begin{equation}
(3.30)\varepsilon C(u_0, u_1) + \int_0^M \int_{\mathbb{R}^n} |v_t|^p \eta_M^{2p'} \Phi \, dx \, dt \lesssim M^{nq \eta - n - \frac{1}{2p} - \frac{1}{2p'}} \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^q \vartheta_M^{2p'} \Phi \, dx \, dt \right)^{\frac{1}{q'}}.
\end{equation}
\begin{equation}
(3.31)\varepsilon C(v_0, v_1) + \int_0^M \int_{\mathbb{R}^n} |u_t|^q \eta_M^{2p'} \Phi \, dx \, dt \lesssim M^{nq \eta - n - \frac{1}{2p} - \frac{1}{2p'}} \left( \int_0^M \int_{\mathbb{R}^n} |v_t|^p \vartheta_M^{2p'} \Phi \, dx \, dt \right)^{\frac{1}{p'}}.
\end{equation}
We set $A = \int_0^M \int_{\mathbb{R}^n} |v_t|^p \eta_M^{2p'} \Phi \, dx \, dt$, $B = \int_0^M \int_{\mathbb{R}^n} |u_t|^q \eta_M^{2p'} \Phi \, dx \, dt$, then by above two inequalities, we apply the Young inequality
\[\varepsilon + A \lesssim M^{nq \eta - n - \frac{1}{2p} - \frac{1}{2p'}} A^{\frac{1}{p'}} \leq \frac{A}{pq} + CM^{nq \eta - n - \frac{1}{2p} - \frac{1}{2p'}} \left( \frac{1}{pq} \right)^{\frac{1}{p'}}.\]
Then
\[\varepsilon + (1 - \frac{1}{pq})A \lesssim M^{nq \eta - n - \frac{1}{2p} - \frac{1}{2p'}} \left( \frac{1}{pq} \right)^{\frac{1}{p'}}.\]
which lead to
\[M \leq T_\varepsilon \lesssim \varepsilon^{-\frac{1}{nq \eta - n - \frac{1}{2p} - \frac{1}{2p'}}}, \quad p < q.\]
By the symmetrical characteristic of $p$ and $q$, we get that
\[T_\varepsilon \lesssim \begin{cases} \varepsilon^{-\frac{1}{nq \eta - n - \frac{1}{2p} - \frac{1}{2p'}}}, & p < q, \\ \varepsilon^{-\frac{1}{nq \eta - n - \frac{1}{2p} - \frac{1}{2p'}}}, & q < p. \end{cases}\]
So we come to the estimate
\[T_\varepsilon \lesssim \varepsilon^{-\Gamma_{GG}(n,p,q)^{-1}}, \quad \Gamma_{GG}(n,p,q) > 0.\]
Which is first part of lifespan estimate in Theorem 1.2.

3.2.2. **Blow up region** $\Gamma_{GG}(n,p,q) = 0$. For the case $\Gamma_{GG}(n,p,q) = 0$, $p \neq q$. By (3.30) and (3.12), (3.13), we have
\begin{equation}
(3.32)[C\varepsilon + Y[\Phi[v_t]](M)]^p \lesssim MY'([\Phi[v_t]](M)) .
\end{equation}
Let $Z(M) = C\varepsilon + Y[\Phi[v_t]](M)$, then $Z(1) = C\varepsilon$ and $Z'(M) = Y'(M)$, by above inequality, we have
\[MZ'(M) \gtrsim Z^{pq}, Z(1) = C\varepsilon.\]
This is a typical ODE inequality which will blow up in finite time. By direct computation, we have that
\[M \leq T_\varepsilon \lesssim \exp(\varepsilon^{-(pq-1)}).\]
This is the second lifespan estimate in Theorem 1.2. For the case \( \Gamma_{GC}(n, p, q) = 0 \), \( p = q = p_G(n) \), we set

\[
Y[\Phi | v_t + u_t|^p](M) = \int_1^M \left( \int_0^T \int_{\mathbb{R}^n} |v_t + u_t|^p \Phi \theta^{2p'}(t) dx dt \right) \sigma^{-1} d\sigma.
\]

From (3.28) and (3.29), we get that

\[
[C\varepsilon + Y[\Phi | v_t + u_t|^p] \lesssim MY'[\Phi | v_t + u_t|^p](M).
\]

Which lead to

\[
M \leq T_\varepsilon \lesssim \exp(\varepsilon^{-(p-1)}),
\]

by similar procedure above. We have completed all the proofs of Theorem 1.2.

3.3. Proof of Theorem 1.3.

3.3.1. Blow up region \( \Gamma_{SG}(n, p, q) > 0 \).

(I) Test function 1: \( \Psi(t, x) = \theta^{2p'}_M(t)\phi_0(x), \Psi(t, x) = \eta^{2p'}_M(t)\phi_0(x) \)

Firstly, we choose test function \( \Psi = \theta^{2p'}_M(t)\phi_0(x), \eta^{2p'}_M(t)\phi_0(x) \). By substituting \( \Psi(t, x) \) in (1.11) and (1.12) respectively, and applying Lemma 2.1, we get that

\[
\int_0^M \int_{\mathbb{R}^n} |v|^q \theta^{2p'}_M \phi_0 dx dt = \int_0^M \int_{\mathbb{R}^n} u_t \phi_0 \left( -\partial_t \theta^{2p'}_M + D_1(x)\theta^{2p'}_M \right) dx dt = I_17 + I_18
\]

\[
\varepsilon \int_{\mathbb{R}^n} v_1(x)\phi_0 dx + \varepsilon \int_{\mathbb{R}^n} v_0(x)D_2(x)\phi_0 dx + \int_0^M \int_{\mathbb{R}^n} |u_t|^p \eta^{2p'}_M \phi_0 dx dt = I_19 + I_20
\]

By Hölder’s inequality, we have

\[
\int_0^M \int_{\mathbb{R}^n} |v|^q \theta^{2p'}_M dx dt \lesssim M^{\frac{nq-n-1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta^{2p'}_M dx dt \right)^\frac{1}{p},
\]

and

\[
\int_0^M \int_{\mathbb{R}^n} |u_t|^p \eta^{2p'}_M dx dt \lesssim M^{\frac{nq-n-1}{q}} \left( \int_0^M \int_{\mathbb{R}^n} |v|^q \theta^{2p'}_M dx dt \right)^\frac{1}{p}.
\]

Then by above two inequalities, we can get that

\[
\int_0^M \int_{\mathbb{R}^n} |v|^q \theta^{2p'}_M dx dt \lesssim M^{\frac{nq-n-2q-1}{pq-1}},
\]

\[
\int_0^M \int_{\mathbb{R}^n} |u|^p \eta^{2p'}_M dx dt \lesssim M^{\frac{nq-n-2q-1}{pq-1}}.
\]

(II) Test function 2: \( \Psi(t, x) = -\partial_t (\eta^{2p'}_M(x)\Phi(t, x)), \eta^{2p'}_M \Phi(t, x) \)
Then we apply Hölder inequality to the above terms, we can get that
\[ t, x \Psi(\cdot) \]

Utilizing (3.40) and \( n, p, q \) in (1.3), we obtain the first lifespan estimation in Theorem 1.3.

Thirdly, we choose test function \( \Psi = -\partial_t(\eta_M^{2p'-1} \Phi) \). By substituting \( \Psi(t,x) \) in (1.11) and \( \Psi = \eta_M^{2p'} \Phi \) in (1.12), we get that
\[
\varepsilon C(u_0, u_1) + \int_0^M \int_{\mathbb{R}^n} |v|^q \partial_t \left( -\eta_M^{2p'} \Phi \right) dxdt
\]
\[
= \int_0^M \int_{\mathbb{R}^n} u_t \left( (\partial^2_t \eta_M^{2p'}) \Phi + 2(\partial_t \eta_M^{2p'}) (\partial_t \Phi) - D_1(x)(\partial_t \eta_M^{2p'}) \Phi \right) dxdt
\]
\[
= I_{21} + I_{22} + I_{23}
\]
\[
\varepsilon C(v_0, v_1) + \int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2p'} \Phi dxdt
\]
\[
= \int_0^M \int_{\mathbb{R}^n} v \left( (\partial^2_t \eta_M^{2p'}) \Phi + 2(\partial_t \eta_M^{2p'}) (\partial_t \Phi) - D_2(x)(\partial_t \eta_M^{2p'}) \Phi \right) dxdt
\]
\[
= I_{24} + I_{25} + I_{26}
\]
Then we apply Hölder inequality to the above terms, we can get that
\[
[\varepsilon C(u_0, u_1)]^p \lesssim M^{\frac{2}{n-2p}} \int_0^M \int_{\mathbb{R}^n} |u_t|^p \eta_M^{2p'} dxdt,
\]
\[
[\varepsilon C(v_0, v_1)]^q \lesssim M^{\frac{nq-4-2n}{2}} \int_0^M \int_{\mathbb{R}^n} |v|^q \eta_M^{2p'} dxdt.
\]
Hence we have by combining (3.38) and (3.39) that
\[
M \lesssim \varepsilon^{-F_{\Sigma,1}(n,p,q)}.
\]
Utilizing (3.37) and (3.40), we obtain the estimate
\[
M \lesssim \varepsilon^{-F_{\Sigma,2}(n,p,q)}.
\]
Since for any \( M \in (1, T_\varepsilon) \) we get that
\[
T_\varepsilon \lesssim \min \left\{ \varepsilon^{-F_{\Sigma,1}(n,p,q)}, \varepsilon^{-F_{\Sigma,2}(n,p,q)} \right\}.
\]
So we come to the estimate
\[
T_\varepsilon \lesssim \varepsilon^{-F_{\Sigma,2}(n,p,q)}.
\]
We obtain the first lifespan estimation in Theorem 1.3.

3.3.2. Blow up region \( \Gamma_{SG}(n,p,q) = 0 \).

(III) Test function 3: \( \Psi(t,x) = -\partial_t(k_M^{2p'} (t) b_a(t,x)) \)

Thirdly, we choose test function \( \Psi = -\partial_t(\eta_M^{2p'} b_a) \), where \( a = \frac{n+1}{2} - \frac{1}{p} \). Noting that
\[
\Psi(t,x) = -\partial_t(\eta_M^{2p'} b_a) = -\partial_t \eta_M^{2p'} b_a - \eta_M^{2p'} \partial_t b_a \geq -\partial_t \eta_M^{2p'} b_a > 0.
\]
By substituting \( \Psi(t,x) \) in (1.11), we have
\[
\varepsilon C(u_0, u_1) + M^{-1} \int_0^M \int_{\mathbb{R}^n} |v|^q \eta_M^{2p'} b_a dxdt
\]
\[
\lesssim \int_0^M \int_{\mathbb{R}^n} u_t \left( (\partial^2_t \eta_M^{2p'}) b_a + 2(\partial_t \eta_M^{2p'}) (\partial_t b_a) - D_1(x)(\partial_t \eta_M^{2p'}) b_a \right) dxdt
\]
\[
= I_{27} + I_{28} + I_{29}
\]
We apply Hölder inequality and Lemma 2.3 to get that

$$|I_{27}| \lesssim M^{-2} \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_f M^p dxdt \right)^\frac{1}{p} \left( \int_0^M \int_{\mathbb{R}^n} \left( t + R \right)^{-\frac{n-1}{2} \rho' - n^{-1}} dr dt \right)^\frac{1}{p'}$$

$$\lesssim M^{-2} \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_f M^p dxdt \right)^\frac{1}{p} \left( \int_0^M \int_{\mathbb{R}^n} \left( t + R \right) \left( t + R + 1 - r \right)^{-1} r^{-\frac{n-1}{2}} dr dt \right)^\frac{1}{p'}$$

$$\lesssim M \frac{n-2n}{2n-3n} (\ln M)^{\frac{p-1}{p}} \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_f M^p dxdt \right)^\frac{1}{p},$$

$$|I_{28}| \lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_f M^p dxdt \right)^\frac{1}{p} \left( \int_0^M \int_{\mathbb{R}^n} \left( t + R \right)^{-\frac{n-1}{2} \rho' - n^{-1}} \ln \left( t + R \right) \left( t + R + 1 - r \right)^{-\frac{1}{2}} dr dt \right)^\frac{1}{p'}$$

$$\lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} \left( t + R \right)^{-\frac{n-1}{2} \rho' - n^{-1}} \ln \left( t + R \right) \left( t + R + 1 - r \right)^{-\frac{1}{2}} dr dt \right)^\frac{1}{p'} \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_f M^p dxdt \right)^\frac{1}{p}$$

$$\lesssim M \frac{n-2n}{2n-3n} (\ln M)^{\frac{p-1}{p}} \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_f M^p dxdt \right)^\frac{1}{p},$$

$$|I_{29}| \lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_f M^p dxdt \right)^\frac{1}{p} \left( \int_0^M \int_{\mathbb{R}^n} \left( t + R \right)^{-\frac{n-1}{2} \rho' - n^{-1}} \left( t + R + 1 - r \right)^{-\frac{1}{2}} dr dt \right)^\frac{1}{p'}$$

$$\lesssim M^{-1} \left( \int_0^M \int_{\mathbb{R}^n} \left( t + R + 1 \right)^{-\frac{n-1}{2} \rho' - n^{-1}} \left( t + R + 1 - r \right)^{-\frac{1}{2}} \left( t + R + 1 \right)^{-\beta' \rho'} dr dt \right)^\frac{1}{p'} \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_f M^p dxdt \right)^\frac{1}{p}$$

$$\lesssim M \frac{n-2n}{2n-3n} \beta' \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_f M^p dxdt \right)^\frac{1}{p}$$

$$\lesssim M \frac{n-2n}{2n-3n} (\ln M)^{\frac{p-1}{p}} \left( \int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_f M^p dxdt \right)^\frac{1}{p}.$$
Exploiting Lemma 2.5 with $p_2 = q(p - 1) + 1$, $p_1 = pq$, $\delta = \varepsilon^{pq}$, which lead to

$$T_{\varepsilon} \leq \exp(\varepsilon^{-p(p-1)}).$$

(IV) Test function 4: $\Psi(t, x) = \eta_M^{2p'}(t)b_a(t, x)$

We choose test function $\Psi = \eta_M^{2p'}b_a$, where $a = \frac{n-1}{2} - \frac{1}{q}$. By substituting $\Psi(t, x)$ in (1.12), we get that

$$\varepsilon C(v_0, v_1) + \int_0^M \int_{\mathbb{R}^n} |u_i|^{p} \eta^{2p'}M b_a dx dt$$

(3.44)$$= \int_0^M \int_{\mathbb{R}^n} v((\partial_t \eta^{2p'}M) b_a + 2(\partial_t \eta^{2p'}M)(\partial_t b_a) - D_2(x)(\partial_t \eta^{2p'}M) b_a) dx dt$$

$$= I_{30} + I_{31} + I_{32}$$

For the case $F_{SG,2}(n, p, q) = 0 > F_{SG,1}(n, p, q)$. Observing that the condition $F_{SG,2}(n, p, q) = 0$ yields

$$\left(n - \frac{n-1}{2}p - \left(\frac{n-1}{2} - \frac{1}{q}\right)\right) + \left(n - \frac{n-1}{2}q - \left(\frac{n-1}{2} - \frac{1}{p}\right)\right)p = -F_{SG,2}(n, p, q) = 0.$$

Therefor we conclude from the estimates $I_{30}-I_{32}$ that

$$\left(\int_0^M \int_{\mathbb{R}^n} |u_i|^{p} \eta^{2p'}M b_a dx dt\right)^{pq} \lesssim (\ln M)^{p(q-1)} \left(\int_0^M \int_{\mathbb{R}^n} |u_i|^{p} \theta^{2p'}M b_a dx dt\right).$$

By combining (3.35), (3.40) and Lemma 2.3, we acquire

$$\varepsilon^{pq} \lesssim \int_0^M \int_{\mathbb{R}^n} |u_i|^{p} \theta^{2p'}M b_a dx dt.$$

From (3.12), (3.13), (3.45) and (3.46), we can get that

$$\varepsilon^{pq} \lesssim MY'[\eta_M|u_i|^p](M),$$

$$[MY[\eta_M|u_i|^p](M)]^{pq} \lesssim (\ln M)^{p(q-1)}MY'[\eta_M|u_i|^p](M).$$

Exploiting Lemma 2.5 with $p_2 = q(p - 1) + 1$, $p_1 = pq$, $\delta = \varepsilon^{pq}$, which lead to

$$T_{\varepsilon} \leq \exp(\varepsilon^{-p(p-1)}).$$

For the case $F_{SG,2}(n, p, q) = 0 = F_{SG,1}(n, p, q)$, that is

$$n - \frac{n-1}{2}p = \frac{n-1}{2} - \frac{1}{q}, \quad n - \frac{n-1}{2}q = \frac{n+1}{2} - \frac{1}{p}.$$

By combining (3.39) and the lower bound of $b_a$ in Lemma 2.3, we have that

$$\varepsilon^p \leq \int_0^M \int_{\mathbb{R}^n} |u_i|^{p} b_a \theta^{2p'}M dx dt.$$

From (3.12), (3.13), (3.45) and (3.47), we can get that

$$\varepsilon^p \lesssim MY'[\eta_M|u_i|^p](M),$$

$$[MY[\eta_M|u_i|^p](M)]^{pq} \lesssim (\ln M)^{p(q-1)}MY'[\eta_M|u_i|^p](M).$$

Exploiting Lemma 2.5 with $p_2 = q(p - 1) + 1$, $p_1 = pq$, $\delta = \varepsilon^p$, which lead to

$$T_{\varepsilon} \lesssim \exp(\varepsilon^{-(pq-1)}).$$

We complete the proof of Theorem 1.3.
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