Stability of a nonlinear elastic plate under uniaxial loading with respect to finite perturbations

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Abstract. The stability of a plate made of nonlinear elastic material with respect to finite perturbations is considered. The main process of plate deformation is reduced to the solution of the nonlinear boundary value problem with respect to finite perturbations. Solutions for perturbations of displacements are chosen in the form of eigenfunction series, that are solutions of the corresponding linearized problems and satisfy the geometric boundary conditions. After applying the principle of possible displacements, the question on the stability of the ground state of the nonlinear problem is reduced to the study of the stability of zero solution of an infinite system of ordinary differential equations with constant coefficients, the number of terms in which is specified by the elastic potential. For the obtained system of equations under certain restrictions on initial perturbations, a Lyapunov function is constructed. The dimension of the strange attractor of the dynamical system is found which allows to limit the number of terms in the Bubnov-Galerkin series.

1. Introduction
In this paper we consider the stability of a rectangular plate made of nonlinear elastic material under uniaxial load with respect to finite perturbations [1-3]. A fairly wide computational experiment was performed. The permissible limits of the region with respect to the finite, initial perturbations for the given parameters of loading and structures are established. Finite sequences of bifurcation points are constructed, confirming, in contrast to stability at small perturbations, the existence of a hierarchy of stable equilibrium states. The dimensions of the strange attractor for nonlinear systems are found. The estimation of the classical linearized stability theory is made. New phenomena and characteristic effects are established.

2. Materials and methods
Let us consider a rectangular plate \(0 \leq x_1 \leq L; \ 0 \leq x_2 \leq L_2; \ -h \leq x_3 \leq +h\) of nonlinear elastic material.

Let us suppose that the plate is loaded along the \(x_3\)-axis, then

\[ S_{11}^0 = S_{22}^0 = 0. \]  \(1\)

The ground state is supposed to be uniform, that is
\[ u^n_n = \delta_n (\lambda_i - 1) x_i. \] (2)

Let the Lagrangian coordinate system in the body before deformation is the Cartesian, and there are no perturbations of mass and surface forces.

Relations for the case (2) will be written in the form [1]

\[ 2\varepsilon^i_0 = \delta^i_j (\lambda^2_j - 1), \quad A^0_j = \frac{1}{2} (\lambda^2_j \lambda^2_n - 3), \quad A^0_n = \frac{1}{4} (\lambda^2_n - 1) (\lambda^2_n - 1), \quad A^0_i = \frac{1}{8} (\lambda^2_n - 1) (\lambda^2_n - 1) (\lambda^2_n - 1). \] (3)

For the Murnaghan potential \( W = \frac{1}{2} \lambda A^2_1 + \mu A^2_2 + \frac{a}{3} A^3_1 + b A_1 A_2 + \frac{c}{3} A_3 \), from the ratios for the components of the stress tensor [1], taking into account (3), we obtain the expressions for the components of the stress tensor in the ground state:

\[ S^0_{11} = \lambda A^0_1 + a A^0_1 + b A^0_1 + (\lambda^2_1 - 1) (\mu + b A^0_1) + 3 (\lambda^2_1 - 1) \left( a A^0_2 + \frac{c}{3} \right) = 0; \]
\[ S^0_{22} = \lambda A^0_2 + a A^0_2 + b A^0_2 + (\lambda^2_2 - 1) (\mu + b A^0_2) + 3 (\lambda^2_2 - 1) \left( a A^0_2 + \frac{c}{3} \right) = 0; \] (4)
\[ S^0_{33} = \lambda A^0_3 + a A^0_3 + b A^0_3 + (\lambda^2_3 - 1) (\mu + b A^0_3) + 3 (\lambda^2_3 - 1) \left( a A^0_2 + \frac{c}{3} \right). \]

Subtracting from the first equation (4) the second equation, we obtain that \( \lambda_1 = \lambda_2 \). Substituting the relation (3) in the first and second equation (4), and assuming in the second equation \( \lambda_1 = \lambda_2 \), we get the equations connecting \( \lambda_1, \lambda_2 \) and \( \lambda_3 \):

\[ 3a (\lambda^2_1 - 1)^4 + 3a (\lambda^2_2 - 1) (\lambda^2_3 - 1)^3 + \left[ \frac{3}{4} a (\lambda^2_3 - 1)^2 + c + \frac{3}{2} b + a \right] (\lambda^2_3 - 1)^2 \\
+ \left[ \frac{b}{2} + a \right] (\lambda^2_3 - 1) + \lambda + \mu \left( \lambda^2 - 1 \right) + \frac{a + b}{4} (\lambda^2 - 1)^2 = 0; \]
\[ 3a (\lambda^2_2 - 1)^4 + 3a (\lambda^2_3 - 1) (\lambda^2_3 - 1)^3 + \left[ \frac{3}{4} a (\lambda^2_3 - 1)^2 + c + \frac{3}{2} b + a \right] (\lambda^2_3 - 1)^2 \\
+ \left[ \frac{b}{2} + a \right] (\lambda^2_3 - 1) + \lambda + \mu \left( \lambda^2 - 1 \right) + \frac{a + b}{4} (\lambda^2 - 1)^2 = 0. \] (5)

From the equations (5), \( \lambda_1 \) and \( \lambda_2 \) can be obtained through \( \lambda_3 \), which allows to change the value of the parameter only at uniaxial loading \( \lambda_3 \). The values \( \lambda_1, \lambda_2 \) will be obtained as a result of solution for the equations (5) by numerical methods.

For two-constant potential \( W = \frac{1}{2} \lambda A^2_1 + \mu A^2_2 \), the stress tensor components are obtained from (4), when \( a = b = c = 0 \).

\[ S^0_{11} = \lambda A^0_1 + \mu (\lambda^2 - 1) = 0; \quad S^0_{22} = \lambda A^0_2 + \mu (\lambda^2 - 1) = 0; \quad S^0_{33} = \lambda A^0_3 + \mu (\lambda^2 - 1). \] (6)

Substituting (3) in to (6), and taking into account that \( \lambda_1 = \lambda_2 \), from the first and second equations (5) we obtain the equations connecting \( \lambda_1 \) and \( \lambda_2 \) with \( \lambda_3 \):
\[(\lambda + \mu)(\lambda_i^2 - 1) + \frac{\lambda}{2}(\lambda_i^2 - 1) = 0; \quad (\lambda + \mu)(\lambda_i^2 - 1) + \frac{\lambda}{2}(\lambda_i^2 - 1) = 0,\]  
(7)

where

\[\lambda_i^2 = 1 - \frac{\lambda}{2(\lambda + \mu)}(\lambda_i^2 - 1); \quad \lambda_i^2 = 1 - \frac{\lambda}{2(\lambda + \mu)}(\lambda_i^2 - 1).\]  
(8)

In the case of the Mooney potential \(W = 2\lambda A_1 + 2\mu(2A_1 + A_1^2 - A_2)\) we obtain the stress tensor components in the ground state:

\[S_{11}^0 = 2(\lambda + \mu) + 4\mu A_1^0 + \lambda_1^{-2}p_0 - 2\mu(\lambda_1^2 - 1) = 0;\]
\[S_{22}^0 = 2(\lambda + \mu) + 4\mu A_1^0 + \lambda_2^{-2}p_0 - 2\mu(\lambda_2^2 - 1) = 0;\]  
(9)
\[S_{33}^0 = 2(\lambda + \mu) + 4\mu A_1^0 + \lambda_3^{-2}p_0 - 2\mu(\lambda_3^2 - 1).

For an incompressible material (2), the relation between the elongations follows:

\[\lambda_1\lambda_2\lambda_3 = 1.\]  
(10)

Taking into account the symmetry of the expressions (9) with respect to elongations \(\lambda_1\) and \(\lambda_2\), using the conditions (10), we find

\[\lambda_1 = \lambda_2 = \frac{1}{2}\lambda_3^{0.5}.\]  
(11)

Subtracting the first equation from the second equation in the system (9), we obtain:

\[(\lambda_2^{-2} - \lambda_1^{-2})p_0 = 2\mu(\lambda_2^2 - \lambda_1^2).\]

Using the relations (11), we derive:

\[p_0 = -2\mu\lambda_3^{-2}.\]  
(12)

In the case of the Treloar potential \(W = 2EA_1\), for a homogeneous subcritical state, the stress tensor components are as follows:

\[S_{11}^0 = 2E + \lambda_1^{-2}p_0 = 0; \quad S_{22}^0 = 2E + \lambda_2^{-2}p_0 = 0; \quad S_{33}^0 = 2E + \lambda_3^{-2}p_0.\]  
(13)

For extensions \(\lambda_1, \lambda_2\) will also be performed the relation (11) from where, adding the first and second equations (13), we obtain:

\[p_0 = -2E\lambda_3^{-2}.\]  
(14)

We introduce the dimensionless quantities, referring to all quantities having the dimension of stresses to \(2(\lambda + \mu)\), and the quantities having the dimension of length to \(L\)

\[\frac{\lambda}{2(\lambda + \mu)} = \nu; \quad \frac{\mu}{2(\lambda + \mu)} = 0.5 - \nu; \quad \frac{\lambda + 2\mu}{2(\lambda + \mu)} = 1 - \nu;\]
\[\frac{a}{2(\lambda + \mu)} = a_1; \quad \frac{b}{2(\lambda + \mu)} = b_1; \quad \frac{c}{2(\lambda + \mu)} = c_1;\]
\[
E = \frac{\lambda (1 + \nu)(1 - 2\nu)}{2\nu(\lambda + \mu)} = E_i; \quad (15)
\]
\[
S_{ij}^{0} = \frac{S_{ij}^{0}}{2(\lambda + \mu)}, \quad S_{ij}^{0} = \frac{S_{ij}^{0}}{2(\lambda + \mu)} = S_{ij}^{0};
\]
\[
x_1 = \frac{x_1}{L} \quad (0 \leq x_1 \leq 1); \quad x_2 = \frac{x_2}{L} \quad \left(0 \leq x_2 \leq \frac{L}{L}\right); \quad x_3 = \frac{x_3}{L} \quad \left(0 \leq x_3 \leq \frac{b}{L}\right).
\]
Then, a non-zero component (4) of the stress tensor for the Murnaghan potential is:
\[
S_{ij}^{0} = \nu A_i^{0} + a_i A_i^{0} + b_i A_i^{0} + \left(\lambda_i^2 - 1\right)\left(0.5 - \nu + b_i A_i^{0}\right) + 3\left(\lambda_i^2 - 1\right)\left(a_i A_i^{0} + \frac{c_i}{3}\right).
\]
Equations (5) will be rewritten as
\[
3a_i\left(\lambda_i^2 - 1\right)^4 + 3a_i\left(\lambda_i^2 - 1\right)\left(\lambda_i^2 - 1\right)^3 + \left[\frac{3}{4}a_i\left(\lambda_i^2 - 1\right)^2 + c_i + \frac{3}{2}b_i + a_i\right]\left(\lambda_i^2 - 1\right)^2
\]
\[
+ \left[\frac{b_i + a_i}{2}\right]\left(\lambda_i^2 - 1\right) + 0.5\left(\lambda_i^2 - 1\right)^2 + \frac{a_i + b_i}{4}\left(\lambda_i^2 - 1\right)^2 = 0;
\]
\[
3a_i\left(\lambda_i^2 - 1\right)^4 + 3a_i\left(\lambda_i^2 - 1\right)\left(\lambda_i^2 - 1\right)^3 + \left[\frac{3}{4}a_i\left(\lambda_i^2 - 1\right)^2 + c_i + \frac{3}{2}b_i + a_i\right]\left(\lambda_i^2 - 1\right)^2
\]
\[
+ \left[\frac{b_i + a_i}{2}\right]\left(\lambda_i^2 - 1\right) + 0.5\left(\lambda_i^2 - 1\right)^2 + \frac{a_i + b_i}{4}\left(\lambda_i^2 - 1\right)^2 = 0. \quad (17)
\]
Non-zero component (16) of the stress tensor for a two-constant potential in dimensionless form will take the form:
\[
S_{33}^{0} = \nu A_i^{0} + (0.5 - \nu)\left(\lambda_i^2 - 1\right).
\]
The relations (8) will be rewritten in the form:
\[
\lambda_i^2 = \lambda_i^2 = 1 - \nu\left(\lambda_i^2 - 1\right). \quad (19)
\]
For the Mooney potential, the non-zero component (9) of the stress tensor will take the form
\[
S_{33}^{0} = 1 + 4(0.5 - \nu)\lambda_i^{0} + \lambda_i^{0} p_0 - 2(0.5 - \nu)\left(\lambda_i^{0} - 1\right). \quad (20)
\]
Accordingly, for the Treloar potential we have:
\[
S_{33}^{0} = 2E_i + \lambda_i^{0} p_0. \quad (21)
\]
In the relation (20), the value of \(p_0\) is:
\[
p_0 = -2(0.5 - \nu)\lambda_i^{0}. \quad (22)
\]
Accordingly, in the relation (21) \(p_0\) has the form
\[
p_0 = -2E_i\lambda_i^{0}. \quad (23)
We will look for the solution of the nonlinear boundary value problem in perturbations in the known form \([1]\). Taking into account that in the problem under consideration the coordinate system in the body before deformation is Cartesian, the solution is rewritten in the form:

\[
\phi_l(x, t) = \sum_{m} f_m(t) \phi_{lm}(x); \quad (l = 1, 2, 3, 4; \quad m = 1, 2, \ldots, \infty).
\]

As a complete system of basis functions \(\phi_{lm}(x)\) we choose the known forms of deflections of the stability problem with respect to small perturbations, which have the form \([4]\):

\[
\phi_{lm} = \frac{\partial \Psi}{\partial x_2} - \frac{\partial^2 \chi}{\partial x_1 \partial x_2}; \quad \phi_{2lm} = \frac{\partial \Psi}{\partial x_1} - \frac{\partial^2 \chi}{\partial x_1 \partial x_3}; \quad \phi_{3lm} = \frac{\lambda a_{i1} + S_{i1}^0 \lambda_{i3}}{a_{i2} + S_{i1}^0} \frac{\partial^2 \chi}{\partial x_3};
\]

\[
\phi_{4lm} = \lambda_{i1}^3 \left[ (2 \lambda_{i1}^3 \mu_{i2} - \lambda_{i1}^5 a_{i1} - \lambda_{i1}^5 \mu_{i3} + \lambda_{i2}^5 a_{i1} + S_{i1}^0 \lambda_{i1}^2) \Delta + \left( \lambda_{i1}^5 \mu_{i3} + S_{i3}^0 \lambda_{i1}^2 \right) \right] \frac{\partial \chi}{\partial x_3},
\]

where

\[
\Psi = \left[ A_1 \exp(-\gamma_{0,1}^x x_3) + A_2 \exp(-\gamma_{0,1}^y x_3) \right] \cos(m \pi x_1) \cos(n \pi x_2);
\]

\[
\chi = \left[ A_3 \exp(-\gamma_{0,2}^x x_3) + A_4 \exp(-\gamma_{0,2}^y x_3) + A_5 \exp(\gamma_{0,2}^x x_3) \right] \sin(m \pi x_1) \sin(n \pi \frac{L}{L_0});
\]

\[
\gamma_{0}^2 = (m \pi)^2 + \left( n \pi \frac{L}{L_0} \right)^2.
\]

Let us note that in the relations (25) the function \(\phi_{4lm}\) is added only when considering the incompressible material. The first three functions (25) are taken for the compressible material.

The values of \(a_d\) and \(\mu_d\) for a nonlinear elastic body are defined in \([4]\). We write them down for the types of the potentials considered above.

For the Murnaghan potential:

\[
a_{ik} = \nu + 2a_i A_i^0 + 2b_i \left( \lambda_i^2 - 1 \right) + \delta_{ik} \left[ 2 \left( 0.5 - \nu + b_i A_i^0 \right) + c_i \left( \lambda_i^2 - 1 \right) \right],
\]

\[
\mu_{ik} = (0.5 - \nu) + b_i A_i^0 + c_i \left( \lambda_i^2 + \lambda_i^2 - 2 \right).
\]

For two-constant potential:

\[
a_{ik} = \nu + \delta_{ik} \left( 0.5 - \nu \right), \quad \mu_{ik} = 0.5 - \nu.
\]

For Mooney potential:

\[
a_{ik} = 4 \left( 0.5 - \nu \right) - 4 \delta_{ik} \left( 0.5 - \nu \right) - 2 \delta_{ik} p_0 \lambda_i^4, \quad \mu_{ik} = -2 \left( 0.5 - \nu \right) - p_0 \left( \lambda_i \lambda_i - \lambda_i - \lambda_i \right).
\]

For Treloar potential:

\[
a_{ik} = -2 \delta_{ik} p_0 \lambda_i^4, \quad \mu_{ik} = -p_0 \left( \lambda_i \lambda_i - \lambda_i - \lambda_i \right).
\]

In the functions (26), the following notations are introduced:
\[
\eta_{2}^{2} = \frac{\mu_{2} + S_{11}^{0} \lambda_{1}^{-2}}{\mu_{2} + S_{11}^{0} \lambda_{1}^{-2}}, \quad \eta_{2}^{2} = c \pm \left[ e^{2} - \frac{a_{11} + S_{11}^{0} \lambda_{1}^{-2}}{a_{11} + S_{11}^{0} \lambda_{1}^{-2}} \right]^{\frac{1}{2}},
\]
Where
\[
2c = \frac{a_{13} + S_{13}^{0} \lambda_{1}^{-2}}{\mu_{13} + S_{13}^{0} \lambda_{1}^{-2}} \frac{a_{11} + S_{11}^{0} \lambda_{1}^{-2}}{a_{11} + S_{11}^{0} \lambda_{1}^{-2}} - \frac{(a_{13} + \mu_{13})^{2}}{(a_{11} + \mu_{11})^{2}}.
\]

(31)

Since, in general, the values of \( \eta, \eta_2, \eta_3 \) are complex, the functions \( \Psi, \chi \) can be written as

\[
\Psi = \psi_1 + i\psi_2, \quad \chi = \chi_1 + i\chi_2,
\]

(32)

where

\[
\Psi_1 = \left[ A_{1} \exp \left( \frac{Re \eta_{1} \gamma_{x_{3}}}{M_{1}} \right) + A_{2} \exp \left( -\frac{Re \eta_{2} \gamma_{x_{3}}}{M_{1}} \right) \right] \cos \left( \frac{Im \eta_{1} \gamma_{x_{3}}}{M_{1}} \right) \cos \left( m \pi x_{1} \right) \cos \left( n \pi x_{2} \right);
\]

\[
\Psi_2 = \left[ A_{1} \exp \left( \frac{Re \eta_{1} \gamma_{x_{3}}}{M_{1}} \right) + A_{2} \exp \left( -\frac{Re \eta_{2} \gamma_{x_{3}}}{M_{1}} \right) \right] \sin \left( \frac{Im \eta_{1} \gamma_{x_{3}}}{M_{1}} \right) \cos \left( m \pi x_{1} \right) \cos \left( n \pi x_{2} \right);
\]

\[
\Psi_3 = \left[ A_{1} \exp \left( \frac{Re \eta_{1} \gamma_{x_{3}}}{M_{1}} \right) + A_{2} \exp \left( -\frac{Re \eta_{2} \gamma_{x_{3}}}{M_{1}} \right) \right] \cos \left( \frac{Im \eta_{1} \gamma_{x_{3}}}{M_{1}} \right) \sin \left( m \pi x_{1} \right) \sin \left( n \pi x_{2} \right);
\]

\[
\chi_1 = \left[ A_{1} \exp \left( \frac{Re \eta_{1} \gamma_{x_{3}}}{M_{2}} \right) + A_{2} \exp \left( -\frac{Re \eta_{2} \gamma_{x_{3}}}{M_{2}} \right) \right] \cos \left( \frac{Im \eta_{2} \gamma_{x_{3}}}{M_{2}} \right) \sin \left( m \pi x_{1} \right) \sin \left( n \pi x_{2} \right);
\]

\[
\chi_2 = \left[ A_{1} \exp \left( \frac{Re \eta_{1} \gamma_{x_{3}}}{M_{2}} \right) + A_{2} \exp \left( -\frac{Re \eta_{2} \gamma_{x_{3}}}{M_{2}} \right) \right] \cos \left( \frac{Im \eta_{2} \gamma_{x_{3}}}{M_{2}} \right) \sin \left( m \pi x_{1} \right) \sin \left( n \pi x_{2} \right);
\]

\[
M_1 = \left( Re \eta_1 \right)^2 + \left( Im \eta_1 \right)^2; \quad M_2 = \left( Re \eta_2 \right)^2 + \left( Im \eta_2 \right)^2;
\]

(33)

In the relations (25) similarly to (32) we have:

\[
\varphi_{1m} = \varphi_{1m}^1 + i\varphi_{1m}^2; \quad \varphi_{2m} = \varphi_{1m}^2 + i\varphi_{2m}^2; \quad \varphi_{3m} = \varphi_{1m}^3 + i\varphi_{3m}^3.
\]

(34)

Taking into account the relations (34), the coefficients of the system of equations will be complex [2]. Taking the real and imaginary parts in the equation, we obtain two systems of equations:

\[
A_{1m} f_k + C_{1m} f_k + D_{1m} f_k f_d + L_{1m} f_k f_d f_r + \ldots = 0;
\]

\[
A_{2m} f_k + C_{2m} f_k + D_{2m} f_k f_d + L_{2m} f_k f_d f_r + \ldots = 0; \quad p, k, d, r, \ldots = 1, 2, 3, \ldots
\]

(35)

Here index 1 is assigned to the real part of the coefficients in the form (11), and index 2 is assigned to the imaginary part.

Repeating the reasoning (35) we obtain that the functions
\[ \Pi_1 = \frac{1}{2} C_{p_1} f_p f_k + \frac{1}{3} D_{p_1} f_p f_k f_d + \cdots; \]
\[ \Pi_2 = \frac{1}{2} C_{p_2} f_p f_k + \frac{1}{3} D_{p_2} f_p f_k f_d + \cdots; \quad p, k, d, \ldots = 1, 2, \ldots, \infty \]  

(36)

must be positive [1-3] in the perturbation region determined from the system of equations:

\[ C_{p_1} f_k + D_{p_1} f_k f_d + L_{p_1} f_k f_d f_r + \cdots = 0; \]
\[ C_{p_2} f_k + D_{p_2} f_k f_d + L_{p_2} f_k f_d f_r + \cdots = 0. \]  

(37)

A zero solution of the system of equations (35) corresponding to the main process of the deterministic problem [1] will be stable for the given values of the load parameter and the physical and mechanical characteristics of the medium, if the functions (36) \( \Pi_1 \) and \( \Pi_2 \) are positively defined in the perturbation region.

To eliminate the uncertainty in the choice of functions \( \varphi_{mp} \), arising from the presence of six arbitrary constants in functions (36), we require that the deflection forms with respect to small perturbations satisfy not only the geometric boundary conditions on the unloaded surface, but also the exact boundary conditions on the loaded surface, which will be performed in the integral meaning.

For stability problems with respect to small perturbations at \( x_i = \pm \frac{h}{L} \) the following boundary conditions take place [4]:

\[ \left( a_{11} - a_{12} \right) \frac{\partial^2 \Psi}{\partial x_1 \partial x_2} + \left[ \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \frac{a_{11} \lambda_1^2}{a_{13} + \mu_3} \left( \Delta + \frac{S_{33}^0 \lambda_1^2}{a_{13} + \mu_3} \right) \right] \frac{\partial}{\partial x_3} \chi = 0; \]
\[ - \frac{\partial^2}{\partial x_1^2} \Psi - 2 \frac{\partial^3 \chi}{\partial x_1 \partial x_2 \partial x_3} = 0; \]
\[ \frac{\partial^2 \Psi}{\partial x_2 \partial x_3} - \left[ \frac{a_{11}}{a_{13} + \mu_3} \Delta + \frac{S_{33}^0 \lambda_1^2 - a_{13}}{a_{13} + \mu_3} \frac{\partial^2}{\partial x_3^2} \right] \frac{\partial}{\partial x_1} \chi = 0. \]  

(38)

Obviously, the boundary conditions (38) are satisfied for both the real and imaginary parts of the functions \( \Psi \) and \( \chi \). Substituting (26) in to (38), we obtain six homogeneous linear equations with respect to six arbitrary constants. Namely, we can express five arbitrary constants through the sixth, so:

\[ A_1 = k_1 A_6; \quad A_2 = k_2 A_6; \quad A_3 = k_3 A_6; \quad A_4 = k_4 A_6; \quad A_5 = k_5 A_6; \quad A_6 = \delta, \]  

(39)

where \( \delta \) is an undefined constant.

In this way the relations (33) take the form:

\[ \Psi_1 = \delta \left[ k_1 \exp \left( \frac{\text{Re} \eta_1 m \pi x_i}{M_1} \gamma_3 \right) + k_2 \exp \left( -\frac{\text{Re} \eta_1 m \pi x_i}{M_1} \gamma_3 \right) \right] \cos \left( \frac{\text{Im} \eta_1 m \pi x_i}{M_1} \gamma_3 \right) \cos (m \pi x_i) \cos (n \pi x_i); \]
\[ \Psi_2 = \delta \left[ k_1 \exp \left( \frac{\text{Re} \eta_1 m \pi x_i}{M_1} \gamma_3 \right) + k_2 \exp \left( -\frac{\text{Re} \eta_1 m \pi x_i}{M_1} \gamma_3 \right) \right] \sin \left( \frac{\text{Im} \eta_1 m \pi x_i}{M_1} \gamma_3 \right) \cos (m \pi x_i) \cos (n \pi x_i); \]
\[
\chi_1 = \delta \left[ k_4 \exp \left( \frac{\text{Re} \eta_2}{M_2} \gamma x_1 \right) + k_4 \exp \left( -\frac{\text{Re} \eta_2}{M_2} \gamma x_1 \right) \cos \left( \frac{\text{Im} \eta_2}{M_2} \gamma x_1 \right) \right] \\
+ \left[ k_4 \exp \left( \frac{\text{Re} \eta_3}{M_3} \gamma x_1 \right) + \exp \left( -\frac{\text{Re} \eta_3}{M_3} \gamma x_1 \right) \cos \left( \frac{\text{Im} \eta_3}{M_3} \gamma x_1 \right) \right] \cos (m \pi x_1) \cos (n \pi x_1); (40)
\]

\[
\chi_2 = \delta \left[ k_4 \exp \left( \frac{\text{Re} \eta_2}{M_2} \gamma x_1 \right) - k_4 \exp \left( -\frac{\text{Re} \eta_2}{M_2} \gamma x_1 \right) \cos \left( \frac{\text{Im} \eta_2}{M_2} \gamma x_1 \right) \right] \\
+ \left[ k_4 \exp \left( \frac{\text{Re} \eta_3}{M_3} \gamma x_1 \right) + \exp \left( -\frac{\text{Re} \eta_3}{M_3} \gamma x_1 \right) \sin \left( \frac{\text{Im} \eta_3}{M_3} \gamma x_1 \right) \right] \sin (m \pi x_1) \sin (n \pi x_2);
\]

\[
M_1 = (\text{Re} \eta_1)^2 + (\text{Im} \eta_1)^2; \quad M_2 = (\text{Re} \eta_2)^2 + (\text{Im} \eta_2)^2; \quad M_3 = (\text{Re} \eta_3)^2 + (\text{Im} \eta_3)^2.
\]

Taking into account (40), the equations (35) will be rewritten as:

\[
\delta A_{p1} L^p f + \delta C_{p1} f_k + \delta^2 D_{p1}^d f_k f_d + \delta^3 L_{p1}^{d2} f_k f_d f_k f_d + \ldots = 0;
\]

\[
\delta A_{p2} L^p f + \delta C_{p2} f_k + \delta^2 D_{p2}^d f_k f_d + \delta^3 L_{p2}^{d2} f_k f_d f_k f_d + \ldots = 0; \quad p, k, d, r, \ldots = 1, 2, 3, \ldots \quad (41)
\]

Since the values of \( f_k \) are arbitrary, we denote \( \delta f_k = f_k \). Then, the system of equations (41) will coincide in form with the system of equations (35) and, therefore, the stability condition will be the positivity of functions (36) for the perturbation region found from (37). It is obvious that the determination of the boundary of the region of permissible perturbations from the system of equations (37), under given loads (elongations) and the parameters of the medium, as well as the evaluation of positive certainty of the functions (36) can be performed as a result of a computational experiment. In this case, for the compressible material in the numerical simulations, we will take polystyrene with the parameters [6]:

\[
\nu = 0.338; \quad \lambda = 0.138 \cdot 10^5; \quad \mu = 0.0604 \cdot 10^6 \text{ kg cm}^{-2};
\]

\[
a = -1.1 \cdot 10^5; \quad b = -0.8 \cdot 10^5; \quad c = -1.0 \cdot 10^4 \text{ kg cm}^{-2}; \quad \rho = 0.12 \cdot 10^{-2} \text{ kg cm}^{-3}.
\]

For incompressible material we will take a rubber 2959 with parameters [6]

\[
E = 0.528 \cdot 10^3 \text{ kg cm}^{-2}; \quad \rho = 0.12 \cdot 10^{-2} \text{ kg cm}^{-3}.
\]

**3. Results and discussion**

The results of numerical simulations are presented in figure 1, figure 2. In figure 1a, figure 2a the dependences of the permissible initial perturbation \( |f| \) on the elongation parameter \( \lambda_3 \) at \( \frac{h}{L} = 10^{-3} \); \( \pi \frac{L_4}{L} = 1 \), and figure 1b, figure 2b at \( \frac{h}{L} = 10^{-3} \); \( \pi \frac{L_4}{L} = 0.5 \).
Figure 1. The dependence of the permissible initial perturbation $|f|$ on the elongation parameter $\lambda$, at $\frac{h}{L} = 10^{-3}$; $a - \pi \frac{L_a}{L} = 1$, $b - \pi \frac{L_a}{L} = 0.5$. 1, 1' correspond to the Murnaghan potential, 2, 2' correspond two-constant potential.

In figure 1 and 2, here and below, curves 1 and 1' correspond to the Murnaghan potential, curves 2 and 2' correspond to the two-constant potential, curves 3 and 3' correspond to the Mooney potential, curves 4 and 4' correspond to the Treloar potential. In the calculations was taken to be $n = 1$, that is, the number of half waves along the $x_3$-axis, was considered equal to one. Note that in the numerical simulations from the system of equations (37) the permissible values of the initial perturbations were determined, which were substituted in the expressions for the function $\Pi$ and the positivity condition of this function was checked. This procedure was repeated at each step according to the load parameter (elongation $\lambda_i$). Solid curves 1 - 4, 1' - 4' on the plane $|f| - \lambda_i$ represent the upper boundary of the stability region, that is, an infinite sequence of stable equilibrium states.

Thus, the stability regions are enclosed between the horizontal axis and the corresponding curves with numbers 1, 2, 3, 4, (1', 2', 3', 4'). Results of the numerical simulations experiment in the case of nonlinear elastic stochastic heterogeneous material with a given dispersion of inhomogeneity $\langle f^2 \rangle << 0.1$ are shown in figure 1, 2 by the dashed curves 1' - 4'.
Figure 2. The dependence of the permissible initial perturbation $|f|$ on the elongation parameter $\lambda_3$ at $h/L = 10^{-3}$; a - $\pi L_1/L = 1$, b - $\pi L_4/L = 0.5$. 3, 3' - the Mooney potential, 4, 4' - the Treloar potential.

As it is shown in figure 1, 2 the dependence between the average perturbation $\langle |f| \rangle$ and an average elongation $\langle \lambda_3 \rangle$ in qualitative terms, the same to homogeneous material. The effect of inhomogeneity on the critical parameters $\langle |f| \rangle - \langle \lambda_3 \rangle$, when the variance of the inhomogeneity $\langle f^{*2} \rangle << 0.1$, is negligible.

The areas in the right part of figures 1, 2 correspond to the flexural form of the buckling, the areas in the left part of the figures correspond to the shape of the buckling with the formation of the neck. The value of $M'_i$ ($i = 1, 2, 3, 4$) in figure 1, 2 correspond to the linearized stability theory.

It is seen that with the approach to the point of the corresponding bifurcation point $M'_i$ for problems of relatively small perturbations, the region of stability with respect to finite perturbations narrows and near the bifurcation point shrinks to a point. From figures 1, 2 it is possible to find the maximum permissible perturbation for specific values of the load parameter, and vice versa, if the maximum perturbations are known, then it is possible to find the interval of change of the load parameter in which these perturbation values will not be exceeded and for this set of load parameters and perturbations the undisturbed state will be stable.
Thus, in contrast to the three-dimensional linearized stability theory, which defines a single point of bifurcation, in the stability theory at finite perturbations a region is found, in which there is a hierarchy of stable states.

From the analysis of the curves in figures 1, 2 it follows that the reduction of the linear dimensions ratios, in particular the decrease in the width of the plate, narrows the stability region both with respect to elongations and with respect to initial perturbations.

![Figure 3](image)

**Figure 3.** The dependence of the dimension of the strange attractor $\gamma_m$ on the elongation $\lambda_a$ at $\frac{h}{L} = 10^{-3}$; $a - \pi \frac{L_1}{L} = 1$, $b - \pi \frac{L_1}{L} = 0.5$.

Note, that from the equations (37) we obtain a set of roots $\{f_i\}$ of this system of equations, the minimum of which determines the permissible value of the boundary of the stability region with respect to perturbations at a fixed load (elongation) and the initial geometric parameters of the plate. The remaining roots of this system of equations are used in the calculation of the dimension of the strange attractor [1, 7-9] of this dynamical system, which allows to determine the dimension of the phase space of the embedding. The dependences of the dimension of the strange attractor $\gamma_m$ on the elongation $\lambda_a$ (mean elongation $\left\langle \lambda_a \right\rangle$) are shown in figures 3, in figure 3a at $\frac{h}{L} = 10^{-3}$; $\pi \frac{L_1}{L} = 1$, and in figure 3b at $\frac{h}{L} = 10^{-3}$; $\pi \frac{L_1}{L} = 0.5$. Solid curves in figure 3 correspond to a homogeneous material, and dashed curves to stochastically inhomogeneous material [10], when $\left\langle f^2 \right\rangle \ll 0.1$. It can be seen from the figures that with the increase of the load parameter the dimension of the strange attractor decreases from the value of approximately eight to three. This is obviously explained by the fact that the bifurcation values used to find the dimension also capture the region of attraction of the strange attractor. By increasing the load parameter, the roots of the system of equations (37) are closer to each other and therefore the dimension of the strange attractor is significantly smaller. The constructed graphic dependences in figure 3 allow us to give recommendations for solving stability problems with respect to finite perturbations about how many members of the series (24) should be taken when working in a fixed range of elongations. If the range of elongation in which the material operates is known in advance, the range of stress and strain changes will be known. The calculation of the correlation dimension of the strange attractor allows to limit the embedding phase space for the set of initial values of the perturbation amplitudes and their velocities, which allows to reasonably limit the number of terms in the Bubnov-Galerkin series and, consequently, to solve the question on the convergence of this series.
4. Conclusions
From the analysis of the given figures it should be noted the following:

- reduction the ratio of size of structures reduces the stability region as relative to the initial perturbation, and with respect to elongation; the replacement of the compressible material in an incompressible leads to the same results; for a compressible material, the Murnaghan potential gives the upper boundary of region of stability in comparison with two-constant;
- for an incompressible material, the Treloar potential gives a lower bound on the stability region in comparison with the Mooney potential;
- for all the considered cases, a decrease in the dimension of the strange attractor with an increase in the value of the load parameter is observed.

We also emphasize that the dimension of the strange attractor, found for this problem, allows us to recommend the number of terms in a Bubnov-Galerkin series when the structure in different ranges of initial stresses and strains.

The use of the stability criterion with respect to finite perturbations always allows for a specific value of the load parameter to obtain a limited sequence of permissible values of the initial perturbations in which the main deformation process will be stable and, at the same time, there is a hierarchy of stable equilibrium states.

Setting the maximum permissible value of the initial perturbations leads to finding the area of variation of the load parameter in which the main deformation process will be stable.

Thus, the use of the stability criterion, consisting in the application of the second Lyapunov method with respect to finite perturbations, allows for a specific value of the load parameter to obtain a limited sequence of permissible values of initial perturbations, in which the main deformation process will be stable and, at the same time, there is a hierarchy of stable equilibrium states, which determines the functioning of complexes and systems in normal mode.

Note, that in the framework of three-dimensional linearized stability theory similar problems were considered in [4].

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