Mechanisms that generate Gerono's lemniscate

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Abstract. It is presented Gerono's lemniscate with bibliographic references on the geometrical aspects of this curve vastly studied in literature. A mechanism of Artobolevski that draws this curve is studied. The desired lemniscate was obtained. We synthesize a new, original mechanism that generates this curve, starting from two references showing a geometric construction and animation. The mechanism obtained is of the type R-RPP-RPR-PRP, so that the relations of the positions are written by the contouring method and the desired lemniscate is obtained. There are changes in the mechanism resulting in similar but deformed curves.

1. Introduction

Gerono's lemniscate (the eight-shaped curve) is a curve constructed by the French mathematician Camille-Christophe Gerono, who lived between 1799 and 1891, a curve that is part of the group of polysomal curves, i.e. the median curves of two conical curves, having a common axis, a parallel and a perpendicular to this axis, the name being given by Cayley in 1868 [1]. By customization the curves are obtained: double-heart curve, slider-crank curves by Béard, regular bifolium, quartic by Bernoulli, Gerono's lemniscate. In [2], Gerono's lemniscate is detailed, is given the construction with animation and equations are given in different forms, the curve being a quart, and in [3] there are many such curves obtained by the transformation of Newton. This curve is further mathematically detailed in [4] and [5]. It is also shown in [6] how to construct this curve (figure 1): From a point P on a circle it goes horizontally until it cuts the parallel with the ordinate tangent to the circle at the point M. The intersection of the OM straight with the order passing through P gives the Q point that generates Gerono's lemniscate.

Figure 1. The geometric construction [6]
In [7] there is a demonstration (on Youtube) how to get this curve mathematically. Other calculations on the geometry of this curve are given in [8]. In [9] there are also mathematical calculations for this curve, also called Huygens lemniscate. By rotating this curve, [10] gives an interesting spatial surface. In the [11] is given a mechanism that generates this curve. Below, the mechanism in [11] is analyzed and an original mechanism generating this curve is constructed, starting from figure 1 and the animation in [2].

In [12, 13, 14, 15, 16] there are presented many original mechanisms that generate mathematical curves with aesthetic shapes. The structure and kinematics of some aesthetic curves were studied in papers [17, 18, 19].

This paper presents a new mechanism, original, that generates this curve (lemniscate). The mechanism can be applied to machines in the textile industry, to mechanical toys and to the industrial drawing of this curve.

2. Artobolevskii’s mechanism

The Artobolevskii’s mechanism is shown in Figure 2.

![Figure 2. Artobolevskii’s mechanism [11]](image)

For geometrical reasons, the author has set the following conditions:

\[ OE = EC = FG = AB = OB = \frac{a}{2}; \quad EF = OG \]  \hspace{1cm} (1)

and relationships:

\[ x = OA \sin \alpha = 2OB \cos \alpha \sin \alpha \]  \hspace{1cm} (2)

\[ y = 2OE \cos \alpha \]  \hspace{1cm} (3)

In relations (1) the values of the sides are expressed according to a length "a". For \( a = 10 \) mm, using the relations (2) and (3), the Gerono’s lemniscate of Figure 3 was obtained, and for \( a = 30 \) mm the one in Figure 4, the dimensions increasing to increase "a".
For the $\alpha = 0 ... 180$ degrees, the half of the curve is obtained and this is shown in Figure 5.

3. Synthesizing a new mechanism

It has gone from Figure 1 and from the animation in [2], constructing the kinematic scheme of the mechanism in Figure 6. The circle with the radius AB was drawn and the tangent to this parallel with the ordered, considered straight line. The right angle DBC was constructed with the variable length sides, taking point B somewhere in the original circle. The point of intersection of the BC side of the right angle with the tangent to the circle is C. This point is variable on the BC, therefore C was introduced in slider 4, but it must also move on the tangent, therefore it is a double slider of type PP (P comes from prismatic). It joins A with C, its length is variable, which requires another slide 5 in C, connected by a rotation joint to the element 4. The position of D is at the intersection of the BD side of the right angle with the variable radius AC so that the point D slides to the right BD, so slide 7 into C, but C also slips to the AC radius, that is, the sliding 6, connected to 7 by a rotation joint. Point D will shoot Gerono’s lemniscate.

The mechanism is quite complicated, as can be seen from the structural analysis of Figure 7. It is found that the mechanism consists of the leading element AB, dyad BC type RPP (R comes from rotation), dyad AC type RPR and dyad 6-7 of D, type PRP.
4. Relations and results

Based on Figure 6 the following equations are written:

\[ x_B = AB \cos \varphi; \quad y_B = AB \sin \varphi \]  
\[ x_c = \text{const} \]  
\[ BC = x_c - x_B \]  
\[ y_C = y_B \]  
\[ AC^2 = x_c^2 + y_C^2 \]  
\[ \cos \alpha = \frac{x_c}{AC} \]  
\[ x_D = x_B + BD \cos 270 = AD \cos \alpha = x_B \]  
\[ y_D = y_B + BD \sin 270 = AD \sin \alpha \]

From (4) equation results the position of B, and from (5), (6), (7) equations the position of C is obtained. With (8) equation it calculated the length of the AC, and with (9) equation it calculated the angle \( \alpha \). From (10) equation it obtained AD and \( x_D \), and from (11) equation, it obtained \( y_D \).

Figure 8 shows the mechanism obtained with the written program based on the above relations for a position, and Figure 9 shows the mechanism in successive positions.

The curve is shown in Figure 10. Only half of the curve was obtained, the one for the (+) sign in front of the radical by which it is calculated \( \alpha \) from equation (9). The other half of the curve results for the case where the adopted sign is (-), Figure 11.

The entire curve is obtained for the signs \( \pm \) in front of the radical (Figure 12). In the case of the real mechanism, it generates half a curve when it is initially positioned above the axis and the other half when it is positioned under the x-axis.

Figure 13 shows the variations of the coordinates of D for the sign (+), and Figure 14 for the sign (-). The curves are continuous, i.e. there are no breaks in the operation of the mechanism, the curve of \( x_D \) is the same in both charts, and the curve of \( y_D \) is inverted to the (-) sign.
Figure 8. The mechanism in a position

Figure 9. The mechanism in successive positions

Figure 10. Curve for the sign (+)

Figure 11. Curve for the sign (−)

Figure 12. The full curve

Figure 13. Curves for D at the sign (+)

Figure 14. Curves for D at the sign (−)

It was also attempted to change the position of the tangent by C, considering a line parallel to the order that is no longer tangent to the circle with the radius AB. Thus, for $x_C = 10$ mm, was obtained the curve of Figure 15, for $x_C = 25$ mm was obtained the curve of Figure 16, and for $x_C = 50$ mm was obtained the curve of Figure 17, i.e. the same kind of curves, but deformed in height. The curves from Figures 15, 16 and 17 result for the same values of $x_C$ but negative, i.e. C is positioned to the left of the y-axis.

Figure 15. Curve for $x_C=10$

Figure 16. Curve for $x_C=25$

Figure 17. Curve for $x_C=50$. 
5. Conclusions

It has departed from geometric considerations on Gerono’s curve, with many works showing equations and other geometric features in the literature. A mechanism of Artobolevskii was also studied, generating the curve of Gerono. Next, a geometric solution and an animation, given in the literature, made the synthesis of an original generating mechanism of this lemniscate. The structure of this new R-RPP-RPR-PRP mechanism was studied, the position relationships were plotted by the contouring method and the desired curve was obtained. Equations provide two solutions, each generating a half of the curve. From the diagrams obtained it is found that the movement of the mechanism is continuous. The mechanism has been modified by positioning a line parallel to the y-axis so that it is no longer tangent to the initial circle, obtaining similar but deformed curves. This original mechanism, obtained through structural and geometric synthesis, is added to similar mechanisms in the literature and can be used to build machinery from the textile industry, mechanical toys, when cutting parts.

6. References

[1] Ferreol R 2008 Mathcurve. Polyzomal curve. [http://www.mathcurve.com/courbes2d/polyzomale/polyzomale.shtml]
[2] Ferreol R and Mandonnet J 2017 Lemniscate of Gerono - Mathcurve. [http://www.mathcurve.com/courbes2d.gb/gerono/gerono.shtml]
[3] Ferreol R 2017 Hyperbolism and antihyperbolism of a curve newton transformation. Mathcurve. [http://www.mathcurve.com/courbes2d.gb/hyperbolisme/hyperbolisme.shtml]
[4] https://en.wikipedia.org/wiki/Lemniscate_of_Gerono
[5] http://mathworld.wolfram.com/EightCurve.html
[6] http://xahlee.info/SpecialPlaneCurves_dir/LemniscateOfGerono_dir/lemniscateOfGerono.html
[7] [https://www.youtube.com/watch?v=E7oR_JBgUzA]
[8] Weisstein E W 1999 Lemniscate of Gerono [http://archive.lib.msu.edu/crcmath/math/math/e/e039.htm]
[9] https://www.revolyv.com/main/index.php?s=Lemniscate%20of%20Gerono
[10] https://www.geogebra.org/m/X2U9Jv3K
[11] Artobolevskii I I 1959 Teoria mehanizmov dlia vosproizvedenia ploskih crivih. Izd. Academii Nauk, SSSR, Moskva.
[12] Luca L and Popescu I 2015 Paths and Laws of Motion of a Mechanism with Two Successive Conductive Elements and a Triad Applied Mechanics and Materials 772 Trans Tech Publications, Switzerland pp 344-349
[13] Luca L, Popescu I and Ghimisi S 2012 Studies regarding generation of aesthetics surfaces with mechanisms Proceedings of the 3-rd International Conference on Design and Product Development Montreux Elvetia pp 249-254
[14] Luca L, Ghimisi S and Popescu I 2012 Studies regarding the movement on the cochleoid BookSeries: Advanced Materials Research 463-464 pp 147-150
[15] Luca L and Popescu I 2012 Generation of aesthetic surfaces through trammel mechanism Fiability&DurabilityFiabilitatesiDurabilitate supplement pp 55-61
[16] Luca L and Popescu I 2013 Curves and aesthetic surfaces generated by the R-R-RTR mechanism Fiability&DurabilityFiabilitatesiDurabilitate supplement pp 28-34
[17] Popescu I, Luca L and Cherciu M 2011 Trajectories and motion laws of some mechanisms (Traiectorii si legi de miscare ale unor mecanisme) Sitech Publishing House Craiova
[18] Popescu I, Luca L and Mitsi S 2011 Geometry, structure and kinematics of some mechanisms (Geometria, structura si cinematica unor mecanisme) Sitech Publishing House Craiova
[19] Popescu I, Luca L and Cherciu M 2013 Structure and kinematics of mechanisms. Applications (Structura si cinematica mecanismelor. Aplicatii) Sitech Publishing House Craiova