Research Article

Commutators of Multilinear Calderón–Zygmund Operator on Weighted Herz-Morrey Spaces with Variable Exponents

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In this paper, we acquire the boundedness of commutators generated by multilinear Calderón-Zygmund operator and BMO functions on products of weighted Herz-Morrey spaces with variable exponents.

1. Introduction

The space of all Schwartz functions on \( \mathbb{R}^n \) was denoted by \( \mathcal{S}(\mathbb{R}^n) \), and the space of all tempered distributions on \( \mathbb{R}^n \) was denoted by \( \mathcal{S}'(\mathbb{R}^n) \). The space of compactly supported bounded functions denoted by \( L^\infty_c(\mathbb{R}^n) \), and the support set of function \( f \) was denoted by \( \text{supp} (f) \). On the \( m \)-fold of the Schwartz function space \( \mathcal{S}(\mathbb{R}^n) \), we also set \( T \) as an \( m \)-linear operator originally defined and \( m \geq 2 \), and its value belongs to \( \mathcal{S}'(\mathbb{R}^n) \):

\[
T : \mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \times \cdots \times \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{1}
\]

We say that \( T \) is an \( m \)-linear Calderón-Zygmund operator, if for some \( p_1, \cdots, p_m \in [1, \infty) \), it extends to a bounded multilinear operator from \( L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m} \) to \( L^1 \) with \( 1/p_1 + 1/p_2 + \cdots + 1/p_m = 1/p \), and for \( f_1, f_2, \cdots, f_m \in L^\infty_c(\mathbb{R}^n) \), \( x \notin \bigcap_{j=1}^{m} \text{supp} (f_j) \)

\[
T(f_1, f_2, \cdots, f_m)(x) = \int_{\mathbb{R}^n} K(x, y_1, y_2, \cdots, y_m) \prod_{i=1}^{m} f_i(y_i) dy_1 dy_2 \cdots dy_m, \tag{2}
\]

where kernel \( K \) is a function in \( (\mathbb{R}^n)^{m+1} \) away from the diagonal \( x = y_1 = y_2 = \cdots = y_m \) and there exist positive constants \( \epsilon, A \) satisfies the following:

\[
|K(x, y_1, \cdots, y_m)| \leq A \left( \sum_{i=1}^{m} |x - y_i| \right)^{\epsilon},
\]

\[
|K(x, y_1, \cdots, y_m) - K(x', y_1, \cdots, y_m)| \leq \frac{A|x - x'|^\epsilon}{\left( \sum_{i=1}^{m} |x - y_i| \right)^{\epsilon}},
\]

whenever \( |x - x'| \leq 1/2 \max \{ |x - y_1|, |x - y_2|, \cdots, |x - y_m| \} \), and for all \( 1 \leq i \leq m \),

\[
|K(x, y_1, \cdots, y_i, \cdots, y_m) - K(x, y_1, \cdots, y_i', \cdots, y_m)| \leq \frac{A|y_i - y_i'|^\epsilon}{\left( \sum_{i=1}^{m} |x - y_i| \right)^{\epsilon}},
\]

where \( |y_i - y_i'| \leq 1/2 \max \{ |x - y_1|, |x - y_2|, \cdots, |x - y_m| \} \).

If \( b \in L^1_{\text{loc}}(\mathbb{R}^n) \), set

\[
\|b\|_* = \sup_{B} \frac{1}{|B|} \int_{B} |b(x) - b_B| dx, \tag{5}
\]

where \( b_B = (1/|B|) \int_{B} b(y) dy \) and the supremum is taken over all \( B \subset \mathbb{R}^n \), and what follows \( |B| \) is the Lebesgue measure of measurable set \( B \subset \mathbb{R}^n \). A function \( b \) is called bounded mean
oscillation if \( \|b\|_{L^p} < \infty \). Denote by \( \text{BMO}(\mathbb{R}^n) \) the set of all bounded mean oscillation functions on \( \mathbb{R}^n \).

Although our method suits any multilinear operator, only the bilinear Calderón-Zygmund operator will be considered here for the sake of simplicity. Specifically, we will discuss the commutator of a bilinear Calderón-Zygmund operator \( T \), BMO functions \( b_1 \) and \( b_2 \), and suitable functions \( f_1 \) and \( f_2 \),

\[
[b_1, b_2, T](f_1, f_2)(x) = b_1(x)b_2(x)T(f_1, f_2)(x) - b_1(x)T(f_1, b_2f_2)(x) - b_2(x)T(b_1f_1, f_2)(x) + T(b_1f_1, b_2f_2).
\]

Many analyses of linear commutators have been extended to other fields, such as weighted space, homogeneous space, multiparameter, and multilinear settings. Huang and Xu [1] obtained boundedness of multilinear singular integrals and their commutators from products of variable exponent Lebesgue spaces to variable exponent Herz-Morrey spaces with variable exponents. Huet al. [2] proved the boundedness of commutators generated by fractional integrals and BMO on generalized Herz spaces with general Muckenhoupt weights. Tang et al. [3] obtained the boundedness of a commutator generated by the multilinear Calderón-Zygmund operator and BMO functions in Herz-Morrey spaces with variable exponents. Chen et al. [4] studied multiple weighted norm inequalities for maximal vector-valued multilinear singular operator and maximal commutators. Wang et al. [5] proved the boundedness for a class of multisublinear singular integral operators on the product central Morrey spaces with variable exponents.

Motivated by the mentioned works, we will consider the boundedness of commutators generated by multilinear Calderón-Zygmund operator and BMO functions on products of weighted Herz-Morrey spaces with variable exponents.

2. Notations and Main Result

In this section, we recall some notations and definitions; then, we describe our results. Assume \( p(\cdot) \) be a measurable function on \( \mathbb{R}^n \) and take values in \( [1, \infty) \), the Lebesgue space with variable exponent \( L^{p(\cdot)}(\mathbb{R}^n) \) is acquired by

\[
L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable} : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx < \infty \text{ for some } \lambda > 0 \right\}.
\]

The norm is defined by

\[
\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\}.
\]

On a Banach function space, the Lebesgue space \( L^{p(\cdot)}(\mathbb{R}^n) \) is equipped with the norm \( \|f\|_{L^{p(\cdot)}} \). The space \( L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n) \) is defined by

\[
L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n) = \left\{ f : f_{|K} \in L^{p(\cdot)}(\mathbb{R}^n) \text{ for all compact subsets } K \subset \mathbb{R}^n \right\},
\]

where and what follows, \( \chi_A \) denotes the characteristic function of a measurable set \( A \subset \mathbb{R}^n \).

Let \( p(\cdot) : \mathbb{R}^n \to (0, \infty) \), we denote

\[
p_- = \inf_{x \in \mathbb{R}^n} p(x), \quad p_+ = \sup_{x \in \mathbb{R}^n} p(x).
\]

The set \( \mathcal{P}(\mathbb{R}^n) \) consists of all \( p(\cdot) \) satisfying \( p_+ > 1 \) and \( p_+ < \infty \). \( \mathcal{P}_0(\mathbb{R}^n) \) consists of all \( p(\cdot) \) satisfying \( p_- > 0 \) and \( p_+ < \infty \). \( L^{p(\cdot)} \) can be equally defined as above for \( p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \). \( q'(\cdot) \) is the conjugate exponent of \( p(\cdot) \), defined pointwise by \( 1/p(\cdot) + 1/q'(\cdot) = 1 \).

Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and \( w \) be a weight which is a nonnegative measurable function on \( \mathbb{R}^n \). Then, the weighted variable exponent Lebesgue space \( L^{p(\cdot)}(w) \) is the set of all complex-valued measurable function \( f \) such that \( fw \in L^{p(\cdot)} \). The space \( L^{p(\cdot)}(w) \) is a Banach space equipped with the norm

\[
\|f\|_{L^{p(\cdot)}(w)} := \|fw\|_{L^{p(\cdot)}}.
\]

Let \( f \in L^{1(\cdot)}(\mathbb{R}^n) \). Then, the standard Hardy-Littlewood maximal function of \( f \) is defined by

\[
Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy, \forall x \in \mathbb{R}^n,
\]

where the supremum is taken over all balls containing \( x \) in \( \mathbb{R}^n \). Generally speaking, on weighted variable Lebesgue spaces, the Hardy-Littlewood maximal operator is not bounded. But if it meets certain conditions, it will be established. Namely, let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and meet the following global log-Hölder continuous and \( w \in A_{p(\cdot)} \) such that \( M \) is bounded on \( L^{p(\cdot)}(w) \), see [6].

**Definition 1.** Assume \( \alpha(\cdot) \) be a real-valued measurable function on \( \mathbb{R}^n \).

(i) We say that \( \alpha(\cdot) \) satisfies the local log-Hölder continuity condition if there exists a constant \( C_1 \) such that

\[
|\alpha(x) - \alpha(y)| \leq \frac{C_1}{\log (e + (1/|x-y|))}, \quad x, y \in \mathbb{R}^n, |x-y| < \frac{1}{2}
\]

(13)
Lemma 4

\[ |a(x) - a(0)| \leq \frac{C_2}{\log (e + (1/|x|))}, \forall x \in \mathbb{R}^n \]  

(14)

Denote by \( \mathcal{S}^\text{log}_{\text{loc}}(\mathbb{R}^n) \) the set of all log-Hölder continuous functions at the origin.

(iii) We say that \( a(\cdot) \) satisfies the log-Hölder continuous at the infinity if there exists \( a_{\infty} \in \mathbb{R} \) and a constant \( C_3 \) such that

\[ |a(x) - a_{\infty}| \leq \frac{C_3}{\log (e + |x|)}, \forall x \in \mathbb{R}^n \]  

(15)

Denote by \( \mathcal{S}^\text{log}(\mathbb{R}^n) \) the set of all log-Hölder continuous functions at infinity.

(iv) We say that \( a(\cdot) \) satisfies the global log-Hölder continuous if \( a(\cdot) \) is both log-Hölder continuous and locally log-Hölder continuous at infinity. We denote by \( \mathcal{S}^\text{glob}(\mathbb{R}^n) \) the set of all global log-Hölder continuous functions.

**Definition 2.** Given \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and a positive measurable function \( w \), we say that \( w \in A_p(\cdot) \) if there exists a positive constant \( C \) for all balls \( B \) in \( \mathbb{R}^n \) such that

\[ \frac{1}{|B|} \| wX_B \|_{L^p} \| w^{-1}X_B \|_{L^{p'}(w)} \leq C. \]  

(16)

**Remark 3.** In [7], Cruz-Uribe et al. obtained that if \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and \( w \in A_p(\cdot) \), then \( w^{-1} \in A_{p'}(\cdot) \).

The Muckenhoupt \( A_p \) class with constant exponent \( p \in (1, \infty) \) was firstly proposed by Muckenhoupt in [8]. The variable Muckenhoupt \( A_{p(\cdot)} \) was considered in [7, 9–12].

**Lemma 4** (see [7, Theorem 1.5]). If \( p(\cdot) \in \mathcal{P}^\text{log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n) \) and \( w \in A_{p(\cdot)} \), then there is a positive constant \( C \) such that for each \( f \in L^p(\cdot)(w) \),

\[ \| (Mf)w \|_{L^p(\cdot)(w)} \leq C \| fw \|_{L^p(\cdot)(w)}. \]  

(17)

Next, we define the weighted Herz-Morrey space with variable exponents, and we use the following concepts. Let \( k \in \mathbb{Z} \), we define

\[ B_k = \{ x \in \mathbb{R}^n : |x| \leq 2^k \}, \]

\[ D_k = B_k \setminus B_{k-1}, \]

\[ X_k = X_{D_k}, \]

\[ \check{X}_m = X_m, m \geq 1, \]

\[ \check{X}_0 = X_{D_0}. \]

**Definition 5.** Let \( q(\cdot), p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n), \lambda \in [0, \infty) \). Let \( a(\cdot) \) be a bounded real-valued measurable function on \( \mathbb{R}^n \). The non-homogeneous weighted Herz-Morrey space \( MK_{q(\cdot),\lambda}^{a(\cdot),p(\cdot)}(w) \) and homogeneous weighted Herz-Morrey space \( MK_{q(\cdot),\lambda}^{a(\cdot),p(\cdot)}(w) \) are defined, respectively, by

\[ MK_{q(\cdot),\lambda}^{a(\cdot),p(\cdot)}(w) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n, w) : \| f \|_{MK_{q(\cdot),\lambda}^{a(\cdot),p(\cdot)}(w) < \infty} \right\}, \]

\[ MK_{q(\cdot),\lambda}^{a(\cdot),p(\cdot)}(w) = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{ 0 \}, w) : \| f \|_{MK_{q(\cdot),\lambda}^{a(\cdot),p(\cdot)}(w) < \infty} \right\}, \]  

(19)

and where

\[ \| f \|_{MK_{q(\cdot),\lambda}^{a(\cdot),p(\cdot)}(w)} = \sup_{k \in \mathbb{Z}} 2^{-k \lambda} \left\| \left( 2^{a(\cdot)k} f X_k \right)_{k \in \mathbb{Z}} \right\|_{L^p(\cdot)(w)}, \]

\[ \| f \|_{MK_{q(\cdot),\lambda}^{a(\cdot),p(\cdot)}(w)} = \sup_{k \in \mathbb{N}_0} 2^{-k \lambda} \left\| \left( 2^{a(\cdot)k} f X_k \right)_{k = 0}^L \right\|_{L^p(\cdot)(w)}. \]

(20)

Let \( B \) and \( C \) be two real numbers. If there exists a constant \( K > 0 \) such that \( B \leq KC \), we denote \( B \leq C \). If \( B \leq C \) and \( C \leq B \), we denote \( B \approx C \).

**Proposition 6** (see [13, Proposition 1]). Let \( p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n), \lambda \in [0, \infty) \), and \( a(\cdot) \in L^\infty(\mathbb{R}^n) \).

(i) If \( a(\cdot) \in \mathcal{P}^\text{log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n), \) then for all \( f \in L^p(\cdot)(\mathbb{R}^n \setminus \{ 0 \}, w), \)

\[ \| f \|_{MK_{q(\cdot),\lambda}^{a(\cdot),p(\cdot)}(w)} = \max \left\{ \sup_{k \in \mathbb{Z} \cup \{ 0 \}} 2^{-k \lambda} \left\| \left( 2^{a(\cdot)k} f X_k \right)_{k \in \mathbb{Z} \cup \{ 0 \}} \right\|_{L^p(\cdot)(w)}, \right\} \]

\[ + \sup_{k \in \mathbb{Z} \cup \{ 0 \}} 2^{-k \lambda} \left\| \left( 2^{a(\cdot)k} f X_k \right)_{k \in \mathbb{Z} \cup \{ 0 \}} \right\|_{L^p(\cdot)(w)} \].

(21)

where and hereafter, \( q_0 := q(0) \).

(ii) If \( a(\cdot) \in \mathcal{P}^\text{log}_{\text{loc}}(\mathbb{R}^n) \), then
Lemma 7 has been proved by Noi and Izuki in [14, 15].

**Lemma 7.** If \( p(\cdot) \in \mathcal{P}^{0}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n) \) and \( w \in A_{p(\cdot)} \), then there exist constants \( \delta_1, \delta_2 \in (0, 1) \) and \( C > 0 \) such that for all balls \( B \subset \mathbb{R}^n \) and all measurable subsets \( S \subset B \),

\[
\|X_S\|_{L^{p}(\omega)} \leq C\frac{|S|}{|B|}^{\delta_1},
\]
\[
\|X_S\|_{L^{p}(\omega)} \leq C\frac{|S|}{|B|}^{\delta_2}.
\]

The following is the main result.

**Theorem 8.** Let \( T \) be a bilinear Calderón-Zygmund operator and let \( b_1 \) and \( b_2 \) be BMO functions. Given \( p_1(\cdot), p_2(\cdot) \in \mathcal{P}^{0}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n) \) satisfying \( 1/p(x) = 1/p_1(x) + 1/p_2(x) \) for \( x \in \mathbb{R}^n \). Let \( w_1, w_2 \) be weights, \( w = w_1 w_2 \), \( w_1 \in A_{p_1(\cdot)} \), \( w_2 \in A_{p_2(\cdot)} \), \( i = 1, 2 \). Assume that \( a(\cdot) \in L^{1+}(\mathbb{R}^n) \) and weights \( w_i \), \( i = 1, 2 \). Assume that \( \lambda \in \mathbb{N}, \lambda > 0, 0 \leq \lambda_1 < \lambda_2 < \infty, \delta_1, \delta_2 \in (0, 1) \) are the constants in Lemma 7 for exponents \( p_1(\cdot) \) and weights \( w_i \), \( i = 1, 2 \). Let \( \lambda - n \delta_1 < \lambda_1, \lambda_1 = 0 \) and \( p_1(\cdot) \). Assume that \( \lambda_1, \lambda_2 \in (0, 1) \) are the constants in Lemma 7 for exponents \( p_1(\cdot) \) and weights \( w_i \), \( i = 1, 2 \). Let \( \lambda - n \delta_2 < \lambda_2, \lambda_2 = n \delta_2, i = 1, 2 \), then

\[
\|b_1, b_2, T(f_1, f_2)\|_{M_{L^{p}(\omega)}^{(\delta_1)}} \leq \|f_1\|_{M_{L^{p}(\omega)}^{(\delta_1)}} \|f_2\|_{M_{L^{p}(\omega)}^{(\delta_1)}}.
\]

**3. Proof of Theorem 8**

Before we prove Theorem 8, we need to introduce some lemmas.

**Lemma 9** (see [1, Theorem 2.3]). Let \( p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}^{0}(\mathbb{R}^n) \) such that \( 1/p(x) = 1/p_1(x) + 1/p_2(x) \) for \( x \in \mathbb{R}^n \). Then, there exists a constant \( C_{p_1(\cdot)} \) independent of functions \( f \) and \( g \) such that

\[
\|fg\|_{L^{p}(\omega)} \leq C_{p_1(\cdot)} \|f\|_{L^{p}(\omega)} \|g\|_{L^{p}(\omega)}.
\]

holds for every \( f \in L^{p}(\omega) \) and \( g \in L^{p}(\omega) \).

If \( p \in \mathcal{P}(\mathbb{R}^n) \), \( w \in A_{p(\cdot)} \) with \( w = w_1 w_2 \), then by the Hölder inequality, we have

\[
\|fg\|_{L^{p}(\omega)} \leq C_{p(\cdot)} \|f\|_{L^{p}(\omega)} \|g\|_{L^{p}(\omega)}.
\]

**Lemma 10** (see [16, Corollary 3.11]). Let \( b \in \text{BMO}(\mathbb{R}^n) \), \( p(\cdot) \in \mathcal{P}^{0}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n) \), \( w \in A_{p(\cdot)} \), \( t \in [1, \infty) \), and \( k, i \in \mathbb{N} \) such that \( k > i \), then one has

\[
\|b - b_{k}\|_{X_{k,t}(\omega)} \leq C(k - i)\|b\|_{L^{p}(\omega)}.
\]
Proof of Theorem 8. Assume $f_1$ and $f_2$ are bounded functions with compact support and write

$$f_i = \sum_{l=-\infty}^{\infty} f_i, i = 1, 2.$$  \hspace{1cm} (35)$$

By Proposition 6, we have

$$\||b_1, b_2, T(f_1, f_2)\|_{\mathcal{M}p^{q}(\omega)} \leq \sup_{L \in \mathbb{N}} \left\{ \sup_{L \in \mathbb{N}} \left| E \right|, H = \sup_{L \in \mathbb{N}} \left| F + G \right|, F = 2^{-L} L \left\| \left( 2^{k+q}\chi \right) \right\|_{p_0, \mathbb{N}, w}, G = 2^{-L} L \left\| \left( 2^{k+q}\chi \right) \right\|_{p_0, \mathbb{N}, w} \right\} \right.$$  \hspace{1cm} (36)$$

$$E \leq C \sum_{i=1}^{9} E_i, G \leq C \sum_{i=1}^{9} G_i,$$  \hspace{1cm} (38)$$

where

$$E_i = \sup_{k \in \mathbb{N}} \left\{ \left( 2^{k+q}\chi \right) \right\|_{p_0, \mathbb{N}, w}, E = \sup_{L \in \mathbb{N}} \left\{ \left( 2^{k+q}\chi \right) \right\|_{p_0, \mathbb{N}, w} \right\} \right.$$  \hspace{1cm} (39)$$

We shall use the following estimates. If $l \leq k - 1$, then pass Hölder’s inequality, we have

$$\int_{\mathbb{R}^n} |b_i(x) - b_j(x)| f_i dy_i \leq \int_{\mathbb{R}^n} |b_i(x) - (b_j)_{\mathbb{R}}| f_i dy_i + \int_{\mathbb{R}^n} (b_i(x) - (b_j)_{\mathbb{R}}) f_i dy_i \leq \int_{\mathbb{R}^n} |b_i(x) - (b_j)_{\mathbb{R}}| f_i dy_i + \int_{\mathbb{R}^n} f_i X_i \left\| X_i \right\|_{L^{p_0, \mathbb{N}, w}^{q_0} (w^{-1})} \left\| b_i - (b_j)_{\mathbb{R}} \right\|_{L^{q_0, \mathbb{N}, w}^{p_0, \mathbb{N}, w}} \right.$$  \hspace{1cm} (40)$$

By Lemmas 7 and 10, Hölder’s inequality, and Definition 2, we acquire that
\[\left\|2^{kn} \int_{\mathbb{R}^n} |b_i(x) - b_j(y)| f_d \, dy \, X_k \right\| \leq 2^{kn} \left\| \left( b_i(x) - b_j(y) \right) \cdot X_k \right\|_{L^{p_k}(w_k)} + 2^{kn} \left\| \left( b_i(x) - b_j(y) \right) X_i \right\|_{L^{p_k}(w_k)} \|X_k\|_{L^{q_k}(w_k)} \]

If \( l = k \), then

\[2^{kn} \left\| \left( b_i(x) - b_j(y) \right) \cdot X_k \right\|_{L^{p_k}(w_k)} + 2^{kn} \left\| \left( b_i(x) - b_j(y) \right) X_i \right\|_{L^{p_k}(w_k)} \|X_k\|_{L^{q_k}(w_k)} \]

If \( l \geq k + 1 \), then pass Hölder’s inequality, we have

\[\int_{\mathbb{R}^n} |b_i(x) - b_j(y)| f_d \, dy \, f_{d' k'} \]

By Lemmas 7 and 10, Hölder’s inequality, and Definition 2, we acquire that

\[\left\|2^{kn} \int_{\mathbb{R}^n} |b_i(x) - b_j(y)| f_d \, dy \, X_k \right\| \leq 2^{kn} \left\| \left( b_i(x) - b_j(y) \right) \cdot X_k \right\|_{L^{p_k}(w_k)} + 2^{kn} \left\| \left( b_i(x) - b_j(y) \right) X_i \right\|_{L^{p_k}(w_k)} \|X_k\|_{L^{q_k}(w_k)} \]

By the interchange of \( f_1 \) and \( f_2 \), we see that the estimates of \( E_2, E_3, \) and \( E_6 \) are similar to \( E_1, E_7, \) and \( E_8 \), respectively. Thus, we only estimate \( E_1, E_2, E_3, E_4, E_6, \) and \( E_9 \).

To estimate \( E_1 \), due to \( l, j \leq k - 2 \), we infer that for \( i = 1, 2 \),

\[|x - y| \geq |x| - |y| > 2^{k-1} - 2^{min \{l,j\}} \geq 2^{k-2}, x \in D_k, y_1 \in D_l, y_2 \in D_j. \]

Therefore, for \( x \in D_k \), we have

\[|K(x, y_1, y_2)| \leq C(|x - y_1| + |x - y_2|)^{2n} \leq C 2^{-2kn}. \]
Thus, according to Hölder’s inequality, we have

\[
\left\| \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left[ b_1, b_2, T \right] \left( f_{1j}, f_{2j} \right) X_k \right\|_{L^p(\mu)} \\
\leq \left\| \sum_{k=-\infty}^{\infty} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1j}(y_1)| dy_1 \\
\times \sum_{j=-\infty}^{\infty} 2^{-kn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_1)| |f_{2j}(y_1)| dy_2 X_k \right\|_{L^p(\mu)}
\] (48)

\[
\leq \left\| \sum_{k=-\infty}^{\infty} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1j}(y_1)| dy_1 X_k \right\|_{L^p(\mu)} \\
\times \left\| \sum_{j=-\infty}^{\infty} 2^{-kn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_1)| |f_{2j}(y_1)| dy_2 X_k \right\|_{L^p(\mu)}
\]

\[
\leq \left\| \sum_{k=-\infty}^{\infty} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1j}(y_1)| dy_1 \right\|_{L^p(\mu)} \\
\times \left\| \sum_{j=-\infty}^{\infty} 2^{-kn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_1)| |f_{2j}(y_1)| dy_2 \right\|_{L^p(\mu)}
\]

(49)

where

\[ E_{1j} = \sup_{k \in \mathbb{L}, j \in \mathbb{Z}} 2^{-kn} \left\| \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \sum_{j=-\infty}^{\infty} 2^{k \alpha_2}(0,0) |b_1(x) - b_1(y_1)| |f_{1j}(y_1)| dy_1 \right\|_{L^p(\mu)}
\]

(50)

Since \( n\delta_{\omega} - \alpha_1(0) > 0 \), by (41) and Lemma 11, we acquire that

\[
E_{11} \leq \sup_{k \in \mathbb{L}, j \in \mathbb{Z}} 2^{-kn} \left\{ \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \left\| b_1 \right\|_{L^p(\mathbb{R}^n)} \sum_{j=-\infty}^{\infty} 2^{k \alpha_2}(0,0) \left\| f_1 \right\|_{L^p(\mathbb{R}^n)} \right\} \left\| \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \right\|_{L^p(\mathbb{R}^n)}
\]

\[
= \left\| b_1 \right\|_{L^p(\mathbb{R}^n)} \sup_{k \in \mathbb{L}, j \in \mathbb{Z}} 2^{-kn} \left\{ \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \left\| f_1 \right\|_{L^p(\mathbb{R}^n)} \right\} \left\| \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \right\|_{L^p(\mathbb{R}^n)}
\]

\[ \leq \left\| b_1 \right\|_{L^p(\mathbb{R}^n)} \sup_{k \in \mathbb{L}, j \in \mathbb{Z}} 2^{-kn} \left( \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \left\| f_1 \right\|_{L^p(\mathbb{R}^n)} \right) \left\| \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \right\|_{L^p(\mathbb{R}^n)}
\]

(51)

Thus, we obtain that

\[
E_1 \leq \left\| b_1 \right\|_{L^p(\mathbb{R}^n)} \left\| f_1 \right\|_{L^p(\mathbb{R}^n)}
\]

To estimate \( E_2 \), due to \( l \leq k - 2, k - 1 \leq j \leq k + 1 \) for \( i = 1, 2 \), then we have

\[
|x - y_1| \geq |x| - |y_1| \geq 2^{k-2}, x \in \mathcal{D}_k, y_1 \in \mathcal{D}_l.
\]

(53)

So, according to Hölder’s inequality, we have

\[
\left\| \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left[ b_1, b_2, T \right] \left( f_{1j}, f_{2j} \right) X_k \right\|_{L^p(\mu)} \\
\leq \left\| \sum_{k=-\infty}^{\infty} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1j}(y_1)| dy_1 \right\|_{L^p(\mu)} \\
\times \left\| \sum_{j=-\infty}^{\infty} 2^{-kn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_1)| |f_{2j}(y_1)| dy_2 X_k \right\|_{L^p(\mu)}
\]

\[
\leq \left\| \sum_{k=-\infty}^{\infty} 2^{-kn} \int_{\mathbb{R}^n} |b_1(x) - b_1(y_1)| |f_{1j}(y_1)| dy_1 \right\|_{L^p(\mu)} \\
\times \left\| \sum_{j=-\infty}^{\infty} 2^{-kn} \int_{\mathbb{R}^n} |b_2(x) - b_2(y_1)| |f_{2j}(y_1)| dy_2 \right\|_{L^p(\mu)}
\]

(54)

Since \( 1/q(0) = 1/q_1(0) + 1/q_2(0) \), \( \lambda = \lambda_1 + \lambda_2 \), by Hölder’s inequality, we acquire that

\[
E_2 \leq \sup_{k \in \mathbb{L}, j \in \mathbb{Z}} 2^{-kn} \left\{ \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \sum_{j=-\infty}^{\infty} 2^{k \alpha_2}(0,0) |b_1(x) - b_1(y_1)| |f_{1j}(y_1)| dy_1 \right\} \left\| \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \right\|_{L^p(\mathbb{R}^n)}
\]

\[
\times \left\{ \sum_{k=-\infty}^{\infty} 2^{k \alpha_2}(0,0) \right\} \left\| \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \left\| f_1 \right\|_{L^p(\mathbb{R}^n)} \right\} \left\| \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \right\|_{L^p(\mathbb{R}^n)}
\]

\[
\leq \sup_{k \in \mathbb{L}, j \in \mathbb{Z}} 2^{-kn} \left\{ \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \right\} \left\| \sum_{k=-\infty}^{\infty} 2^{k \alpha_2}(0,0) \right\}
\]

\[
\times \left\{ \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \right\} \left\| \sum_{k=-\infty}^{\infty} 2^{k \alpha_2}(0,0) \right\}
\]

\[
\times \left\{ \sum_{k=-\infty}^{\infty} 2^{k \alpha_1}(0,0) \right\} \left\| \sum_{k=-\infty}^{\infty} 2^{k \alpha_2}(0,0) \right\}
\]

(55)

It is obviously that

\[
E_{2,1} = E_{1,1} \leq \left\| b_1 \right\|_{L^p(\mathbb{R}^n)} \left\| f_1 \right\|_{L^p(\mathbb{R}^n)}
\]

(56)

Now we estimate \( E_{2,2} \). Taking (41), (42), and (44) together, we have
where we used \(2^{-n_2} < 1\) and \(2^{-(j-k)n} < 2^{n_2},\ j \in \{k-1, k, k+1\}\) for (41) and (44), respectively. Therefore, we acquire that

\[
E_3 \leq \|b_1\|_s \|b_2\|_s \|f_1\|_{MK^{n_1(n_2+1)}_{\frac{1}{p_1}}(u_1)} \|f_2\|_{MK^{n_1(n_2+1)}_{\frac{1}{p_2}}(w_2)}.
\]

To estimate \(E_3\), since \(l \leq k-2, j \geq k+2\), then, we have

\[
|x - y_1| \geq |x| - |y_1| \geq 2^{j-2}, \quad |x - y_2| \geq |y_2| - |x| > 2^{j-2}, \quad x \in D_k, y_1 \in D_{l_1}, y_2 \in D_j.
\]

Therefore, \(\forall x \in D_k, l \leq k-2, j \geq k+2\), we get

\[
\left| b_1, b_2, \mathcal{I} \right| f_{i_1}, f_{i_2} \right|(x)
\]

\[
\leq 2^{-k} \sum_{n=1}^{\infty} \left| b_1(x) - b_1(y_1) \right| \left| B(x, y_1, y_2) \right| \left| f_{i_1}(y_1) \right| \left| f_{i_2}(y_2) \right| dy_1 dy_2
\]

\[
\leq 2^{-k} 2^{j-2} \int_{\mathbb{R}^n} \left| b_1(x) - b_1(y_1) \right| \left| f_{i_1}(y_1) \right| \left| f_{i_2}(y_2) \right| dy_1 dy_2.
\]

So, according to Hölder’s inequality, we acquire that

\[
\left\| \sum_{k=0}^{\infty} \sum_{j=k+2}^{\infty} \left| b_1, b_2, \mathcal{I} \right| f_{i_1}, f_{i_2} \right| X_k \right\|_{L^p\left(u_1\right)}
\]

\[
\leq \left\| \sum_{k=0}^{\infty} \sum_{j=k+2}^{\infty} \left| b_1(x) - b_1(y_1) \right| \left| f_{i_1}(y_1) \right| \left| f_{i_2}(y_2) \right| dy_1 \right\|_{L^p\left(u_1\right)}
\]

\[
\leq \left\| \sum_{k=0}^{\infty} \sum_{j=k+2}^{\infty} \left| b_1(x) - b_1(y_1) \right| \left| f_{i_1}(y_1) \right| \left| f_{i_2}(y_2) \right| dy_1 \right\|_{L^p\left(u_1\right)}
\]

Since \(1/q(0) = 1/q_1(0) + 1/q_2(0)\), according to Hölder’s inequality, we acquire that
We consider \( I_2 \). Since \( n\delta_{21} + \alpha_2(0) - \lambda_2 > 0 \), we obtain that

\[
I_2 \leq \|b\|_{-} \sup_{L \in \partial \mathcal{C}^{(a)}} 2^{-L_k} \left( \sum_{k=0}^{\infty} 2^{k(n\delta_{21} + \alpha_2(0))} \|f_{k}X_k\|_{L^2((\omega_2))} \right) \leq \|b\|_{-} \sup_{L \in \partial \mathcal{C}^{(a)}} 2^{-L_k} \left( \sum_{k=0}^{\infty} 2^{k(n\delta_{21} + \alpha_2(0))} \|f_{k}X_k\|_{L^2((\omega_2))} \right) \leq \|b\|_{-} \sup_{L \in \partial \mathcal{C}^{(a)}} 2^{-L_k} \left( \sum_{k=0}^{\infty} 2^{k(n\delta_{21} + \alpha_2(0))} \|f_{k}X_k\|_{L^2((\omega_2))} \right)
\]

(66)

We consider \( I_3 \). Since \( n\delta_{21} + \alpha_2(0) - \lambda_2 > 0 \) and \( n\delta_{21} + \alpha_2(0) - \lambda_2 > 0 \), we obtain

\[
I_3 \leq \|b\|_{-} \sum_{L \in \partial \mathcal{C}^{(a)}} 2^{-L_k} \left( \sum_{k=0}^{\infty} 2^{k(n\delta_{21} + \alpha_2(0))} \|f_{k}X_k\|_{L^2((\omega_2))} \right) \leq \|b\|_{-} \sum_{L \in \partial \mathcal{C}^{(a)}} 2^{-L_k} \left( \sum_{k=0}^{\infty} 2^{k(n\delta_{21} + \alpha_2(0))} \|f_{k}X_k\|_{L^2((\omega_2))} \right) \leq \|b\|_{-} \sum_{L \in \partial \mathcal{C}^{(a)}} 2^{-L_k} \left( \sum_{k=0}^{\infty} 2^{k(n\delta_{21} + \alpha_2(0))} \|f_{k}X_k\|_{L^2((\omega_2))} \right)
\]

(67)

Thus, we have

\[
E_3 \leq \|b\|_{-} \|b\|_{+} \|f_{1}\|_{\mathcal{M}^{\delta_{21}}_{\mathcal{R}_1(\omega_1)}} \|f_{2}\|_{\mathcal{M}^{\delta_{21}}_{\mathcal{R}_2(\omega_2)}} .
\]

(68)

To estimate \( E_3 \), according to Lemma 13 and Hölder’s inequality, we have

\[
E_3 \leq \|b\|_{-} \|b\|_{+} \|f_{1}\|_{\mathcal{M}^{\delta_{21}}_{\mathcal{R}_1(\omega_1)}} \|f_{2}\|_{\mathcal{M}^{\delta_{21}}_{\mathcal{R}_2(\omega_2)}} .
\]

Since \( 1/q(0) = 1/q_1(0) + 1/q_2(0), \lambda = \lambda_1 + \lambda_2 \), according to Hölder’s inequality, we have

(72)
By the interchange of \( f_1 \) and \( f_2 \), we acquire that the estimate of \( E_{6,1} \) is analogical to the estimate of \( E_{6,2} \) and \( E_{6,3} = E_{6,3} \).

To estimate \( E_{6,2} \), since \( l_j \geq k + 2 \), then we get

\[
\|x - y\| > 2^{k-2}, x \in D_k, y_1 \in D_{k-1}, y_2 \in D_{k-2}.
\]

Therefore, \( \forall x \in D_k, l_j \geq k + 2 \), we have

\[
\|b_1, b_2, T(f_1, f_2)(x)\| \leq \int_{R^n} \prod_{i=1}^n \|b_1(x) - b_1(y_i)\|K(x, y_1, y_2)\|f_1(y_1)\|f_2(y_2)\|dy_1dy_2 \leq 2^{-2^{k}}\int_{R^n} \prod_{i=1}^n \|b_1(x) - b_1(y_i)\|K(x, y_1, y_2)\|f_1(y_1)\|f_2(y_2)\|dy_1dy_2.
\]

So, according to Hölder’s inequality, we obtain that

\[
\|b_1, b_2, T(f_1, f_2)\| \leq \int_{R^n} \prod_{i=1}^n \|b_1(x) - b_1(y_i)\|K(x, y_1, y_2)\|f_1(y_1)\|f_2(y_2)\|dy_1dy_2
\]

\[
\leq 2^{-2^{k}}\int_{R^n} \prod_{i=1}^n \|b_1(x) - b_1(y_i)\|f_1(y_1)\|f_2(y_2)\|dy_1dy_2
\]

\[
\leq \int_{R^n} \prod_{i=1}^n \|b_1(x) - b_1(y_i)\|f_1(y_1)\|f_2(y_2)\|dy_1dy_2.
\]

Since \( 1/q(0) = 1/q_1(0) + 1/q_2(0) \), \( \lambda = \lambda_1 + \lambda_2 \), according to Hölder’s inequality, we have

\[
E_{6,2} \leq \sup_{I \in G_6} 2^{-\lambda I} \left( \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} 2^{-\lambda I} \left( \sum_{n=0}^{\infty} \left[ \sum_{i=1}^{\infty} 2^{-\lambda I} \int_{R^n} \|b_1(x) - b_1(y_i)\|f_1(y_1)\|f_2(y_2)\|dy_1dy_2 \right] \right) \right).
\]

In order to continue, we need further preparation. If \( l < 0 \), since Proposition 6, we obtain that

\[
\|f_{il}X_i\|_{L^{\lambda I}(w_i)} \leq 2^{-2^{-\lambda I}(0)} \left( \sum_{l=-\infty}^{0} \left[ \sum_{n=0}^{\infty} \left[ \sum_{i=1}^{\infty} \|f_{il}X_i\|_{L^{\lambda I}(w_i)} \right] \right] \right)^{1/q(0)}
\]

\[
\leq 2^{-2^{-\lambda I}(0)} \left( \sum_{l=-\infty}^{0} \left[ \sum_{n=0}^{\infty} \left[ \sum_{i=1}^{\infty} \|f_{il}X_i\|_{L^{\lambda I}(w_i)} \right] \right] \right)^{1/q(0)}
\]

\[
\leq 2^{(\lambda_1 - \alpha_0)l} \left( 2^{-2^{-\lambda I}(0)} \left( \sum_{l=-\infty}^{0} \left[ \sum_{n=0}^{\infty} \left[ \sum_{i=1}^{\infty} \|f_{il}X_i\|_{L^{\lambda I}(w_i)} \right] \right] \right) \right)^{1/q(0)}
\]

Conclusively, we estimate \( G \), according to the interchange of \( f_1 \) and \( f_2 \), we see that the estimates of \( G_2, G_3, \) and \( G_6 \) are similar to \( G_6, G_6, \) and \( G_6, \) respectively. Thus, it was only necessary to estimate \( G_1, G_2, G_3, G_5, G_6, \) and \( G_6. \)
To estimate $G_1$, due to $l_j \leq k - 2$, $1/q_\alpha = 1/q_{1\alpha} + 1/q_{2\alpha}$, $\lambda = \lambda_1 + \lambda_2$, by (48) and Hölder's inequality, we obtain that
\[
G_1 \leq 2^{-L_k} \left( \sum_{k = 0}^{L_k} \left\{ \sum_{j = 0}^{L_k} \left\| b_j \right\| \frac{\left\| f_j \right\|_{L^{q_{1\alpha}}(\omega_1)}}{2^{l_j} \left( k \right) \alpha_j \left( \omega_1 \right)} \right\} \right)^{1/q_{1\alpha}} \times \left\| b_j \right\| \left( \sum_{k = 0}^{L_k} \frac{1}{2^{l_j} \left( k \right) \alpha_j \left( \omega_1 \right)} \right)^{1/q_{2\alpha}} \times \left\| b_j \right\| \left( \sum_{k = 0}^{L_k} \frac{1}{2^{l_j} \left( k \right) \alpha_j \left( \omega_1 \right)} \right)^{1/q_{2\alpha}} = G_{1,1} \times G_{1,2}.
\]

where
\[
G_{1,1} = 2^{-L_k} \left( \sum_{k = 0}^{L_k} \frac{1}{2^{l_j} \left( k \right) \alpha_j \left( \omega_1 \right)} \right)^{1/q_{1\alpha}}.
\]

We consider $G_{1,2}$. By (41), we obtain that
\[
G_{1,2} \leq \left\{ \sum_{k = 0}^{L_k} \left\| f_j \right\|_{L^{q_{1\alpha}}(\omega_1)} \left( \sum_{j = 0}^{L_k} \frac{1}{2^{l_j} \left( k \right) \alpha_j \left( \omega_1 \right)} \right)^{1/q_{1\alpha}} \right\} \times \left\{ \sum_{k = 0}^{L_k} \frac{1}{2^{l_j} \left( k \right) \alpha_j \left( \omega_1 \right)} \right\}^{1/q_{2\alpha}}
\]
\[
\times \left\{ \sum_{k = 0}^{L_k} \frac{1}{2^{l_j} \left( k \right) \alpha_j \left( \omega_1 \right)} \right\} \times \left\{ \sum_{k = 0}^{L_k} \frac{1}{2^{l_j} \left( k \right) \alpha_j \left( \omega_1 \right)} \right\} = I_4 + I_5.
\]

If $q_{1\alpha} \geq 1$, since $n \delta_2 - \alpha_{1\alpha} > 0$ and $n \delta_2 - \alpha(0) > 0$, then by the Minkowski inequality and (79), we have
\[
I_4 = \left\{ \sum_{k = 0}^{L_k} \left\| f_j \right\|_{L^{q_{1\alpha}}(\omega_1)} \left( \sum_{j = 0}^{L_k} \frac{1}{2^{l_j} \left( k \right) \alpha_j \left( \omega_1 \right)} \right)^{1/q_{1\alpha}} \right\} \times \left\{ \sum_{k = 0}^{L_k} \frac{1}{2^{l_j} \left( k \right) \alpha_j \left( \omega_1 \right)} \right\}^{1/q_{2\alpha}}
\]
\[
\times \left\{ \sum_{k = 0}^{L_k} \frac{1}{2^{l_j} \left( k \right) \alpha_j \left( \omega_1 \right)} \right\} \times \left\{ \sum_{k = 0}^{L_k} \frac{1}{2^{l_j} \left( k \right) \alpha_j \left( \omega_1 \right)} \right\} = G_{2,1} \times G_{2,2}.
\]

It is obviously that
\[
G_{2,1} = G_{1,1} \leq \left\{ \sum_{k = 0}^{L_k} \left\| f_j \right\| \left( \sum_{j = 0}^{L_k} \frac{1}{2^{l_j} \left( k \right) \alpha_j \left( \omega_1 \right)} \right) \right\}.
\]
Now, we estimate $G_{2,2}$. Combining (41), (42), and (44), we have

$$
G_{2,2} \leq \|b_2\|_2 \sum_{k=0}^{L_2} \left( \sum_{n=0}^{2^{(k-j)\delta_{21}}} \sum_{p \in \mathbb{T}} \left( \sum_{j=0}^{L_2} \left( \sum_{l=0}^{2^{(k-j)\delta_{21}}} \|f_{2jX_l}\|_{L^{p_2}((\omega_j))} \right)^2 \right)^{1/2} \right)^{1/2}.
$$

(89)

where we used $2^{-\delta_{21}} < 1$ and $2^{(j-k)\delta_{21}} + 1 < 2^{(j-k)n}$ for (41) and (44), respectively.

Thus, we acquire that

$$
G_2 \leq \|b_1\|_2 \|b_2\|_2 \|f_1\|_{MK^{(1/4,4)}_{1/2}((\omega_j))} \|f_2\|_{MK^{(1/4,4)}_{1/2}((\omega_j))},
$$

(90)

To estimate $G_3$, since $l \leq k - 2, j \geq k + 2, 1/q_{\infty} = 1/q_{\infty} + 1/q_{\infty} = \lambda_1 + \lambda_2$, using (61) and Hölder’s inequality, we obtain that

$$
G_3 \leq \|b_2\|_2 \left( \sum_{k=0}^{L_2} \sum_{l=0}^{2^{(k-j)\delta_{21}}} \left( \sum_{n=0}^{2^{(k-j)\delta_{21}}} \sum_{p \in \mathbb{T}} \left( \sum_{j=0}^{L_2} \left( \sum_{l=0}^{2^{(k-j)\delta_{21}}} \|f_{lX_j}\|_{L^{p_2}((\omega_j))} \right)^2 \right)^{1/2} \right)^{1/2} \right)^{1/2}.
$$

(91)

It is obviously that

$$
G_{3,1} = G_{l,1} \leq \|b_1\|_2 \|f_1\|_{MK^{(1/4,4)}_{1/2}((\omega_j))},
$$

(92)

Since $n\delta_{21} + \alpha_{2c0} > 0$, by (44) and Lemma 11, we obtain that

$$
G_{3,2} \leq \|b_2\|_2 \sum_{k=0}^{L_2} \left( \sum_{l=0}^{2^{(k-j)\delta_{21}}} \left( \sum_{n=0}^{2^{(k-j)\delta_{21}}} \left( \sum_{p \in \mathbb{T}} \left( \sum_{j=0}^{L_2} \left( \sum_{l=0}^{2^{(k-j)\delta_{21}}} \|f_{lX_j}\|_{L^{p_2}((\omega_j))} \right)^2 \right)^{1/2} \right)^{1/2} \right)^{1/2} \right)^{1/2}.
$$

(93)

We consider $I_6$. By Lemma 11, we obtain that

$$
I_6 \leq \|b_2\|_2 \left( \sum_{k=0}^{L_2} \sum_{l=0}^{2^{(k-j)\delta_{21}}} \left( \sum_{n=0}^{2^{(k-j)\delta_{21}}} \|f_{lX_j}\|_{L^{p_2}((\omega_j))} \right)^2 \right)^{1/2} \left( \sum_{k=0}^{L_2} \sum_{l=0}^{2^{(k-j)\delta_{21}}} \|f_{lX_j}\|_{L^{p_2}((\omega_j))} \right)^{1/2}.
$$

(94)

where we wrote $2^{(j-k)\delta_{21} + \alpha_{2c0}} = 2^{(j-k)\delta_{21}}$ for $\ell_2 = n\delta_{21} + \alpha_{2c0} > 0$.

We consider $I_7$. Since $n\delta_{21} + \alpha_{2c0} - \lambda_2 > 0$, we have

$$
I_7 \leq \|b_2\|_2 \left( \sum_{k=0}^{L_2} \sum_{l=0}^{2^{(k-j)\delta_{21}}} \left( \sum_{n=0}^{2^{(k-j)\delta_{21}}} \|f_{lX_j}\|_{L^{p_2}((\omega_j))} \right)^2 \right)^{1/2} \left( \sum_{k=0}^{L_2} \sum_{l=0}^{2^{(k-j)\delta_{21}}} \|f_{lX_j}\|_{L^{p_2}((\omega_j))} \right)^{1/2}.
$$

(95)

Thus, we get

$$
G_3 \leq \|b_1\|_2 \|b_2\|_2 \|f_1\|_{MK^{(1/4,4)}_{1/2}((\omega_j))} \|f_2\|_{MK^{(1/4,4)}_{1/2}((\omega_j))}.
$$

(96)

To estimate $G_5$, according to Lemma 13 and Hölder’s inequality, we have

$$
G_5 \leq \|b_2\|_2 \left( \sum_{k=0}^{L_2} \sum_{l=0}^{2^{(k-j)\delta_{21}}} \left( \sum_{n=0}^{2^{(k-j)\delta_{21}}} \left( \sum_{p \in \mathbb{T}} \left( \sum_{j=0}^{L_2} \left( \sum_{l=0}^{2^{(k-j)\delta_{21}}} \|f_{lX_j}\|_{L^{p_2}((\omega_j))} \right)^2 \right)^{1/2} \right)^{1/2} \right)^{1/2} \right)^{1/2}.
$$

(97)
To estimate $G_{i}$, due to $k - 1 \leq i \leq k + 1$ and $j \geq k + 2$, $1/q_{6,0} = 1/q_{1,0} + 1/q_{2,0}$, using (72) and Hölder’s inequality, we obtain that

$$
G_{i} \lesssim 2^{-iL_{1}} \left( \sum_{k=0}^{\infty} 2^{kn_{1}} \left( \sum_{j=n_{2}}^{\infty} 2^{jn_{2}} \int_{R^{n}} |f_{i}(x)| |f_{j}(y)| dy \right)^{q} \right)^{1/q} \lesssim \lambda_{i,2},
$$

(98)

Since the symmetry of $f_{1}$ and $f_{2}$, we can know that the estimate $G_{6,1}$ is analogical to the estimated $G_{1,2}$ and $G_{6,2} = G_{3,2}$.

To estimate $G_{9}$, due to $lj \geq k + 2$, $1/q_{6,0} = 1/q_{1,0} + 1/q_{2,0}$, $\lambda = \lambda_{1} + \lambda_{2}$, using (76) and Hölder’s inequality, we obtain that

$$
G_{i} \lesssim 2^{-iL_{1}} \left( \sum_{k=0}^{\infty} 2^{kn_{1}} \left( \sum_{j=n_{2}}^{\infty} 2^{jn_{2}} \int_{R^{n}} |f_{i}(x)| |f_{j}(y)| dy \right)^{q} \right)^{1/q} \lesssim \lambda_{i,2},
$$

(99)

clearly, the estimate $G_{9,i}$ is analogical to the estimated $G_{3,2}$ for $i = 1, 2$.

Combining all the estimates of $G_{i}$ together, $i = 1, 2, \ldots, 9$, we obtain that

$$
G \lesssim \|f_{1}\|_{M_{K_{2}}^{p_{1}^{(1)},q_{1}^{(1)}}(\omega_{1})} \|f_{2}\|_{M_{K_{2}}^{p_{2}^{(1)},q_{2}^{(1)}}(\omega_{2})},
$$

(100)

Combining the above estimates for $E$, $F$, and $G$, the proof of Theorem 8 is completed.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no competing interests.

**Authors’ Contributions**

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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