Differential Galois Theory of Algebraic Lie-Vessiot Systems

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Abstract

In this paper we develop a differential Galois theory for algebraic Lie-Vessiot systems in algebraic homogeneous spaces. Lie-Vessiot systems are non autonomous vector fields that are linear combinations with time-dependent coefficients of fundamental vector fields of an algebraic Lie group action. Those systems are the building blocks for differential equations that admit superposition of solutions. Lie-Vessiot systems in algebraic homogeneous spaces include the case of linear differential equations. Therefore, the differential Galois theory for Lie-Vessiot systems is an extension of the classical Picard-Vessiot theory. In particular, algebraic Lie-Vessiot systems are solvable in terms of Kolchin’s strongly normal extensions. Therefore, strongly normal extensions are geometrically interpreted as the fields of functions on principal homogeneous spaces over the Galois group. Finally we consider the problem of integrability and solvability of automorphic differential equations. Our main tool is a classical method of reduction, somewhere cited as Lie reduction. We develop and algebraic version of this method, that we call Lie-Kolchin reduction. Obstructions to the application are related to Galois cohomology.

1 Introduction

A Lie-Vessiot system, as defined in [3], is a system of non-autonomous differential equations,

\[ \dot{x}_i = F_i(t, x_1, \ldots, x_n), \tag{1.1} \]

such that there exist \( r \) functions \( f(t) \) of the parameter \( t \) verifying:

\[ F_i(t, x_1, \ldots, x_n) = \sum_{j=1}^{r} f_j(t)(A_j x_i), \]

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where $A_1, \ldots, A_s$ are autonomous vector fields which infinitesimally span a pretransitive Lie group action. Such systems were introduced by S. Lie at the end of 19th century (see, for instance [24]). The differential equation (1.1), interpreted as a non-autonomous vector field, in a manifold $M$, is a linear combination of the infinitesimal generators of the action of $G$ in $M$:

$$\vec{X} = \frac{\partial}{\partial t} + \sum f_j(t)A_j.$$ 

In [3], it is proven that a differential equation admits a superposition law if and only if it is a Lie-Vessiot system related to a pretransitive Lie group action (this is the global version of a classical result exposed in [24]). The orbits by a pretransitive group action are homogeneous $G$-spaces, so that we can decompose a Lie-Vessiot system in a family of systems on homogeneous spaces. Therefore, Lie-Vessiot systems on homogeneous spaces are the building blocks of differential equations admitting superposition laws.

Here, we study Lie-Vessiot systems on algebraic homogeneous spaces $M$ with coefficients $f_j$ in a differential field $K$ whose field of constants $C$ is the field of definition of the phase space $M$. In this frame, a Lie-Vessiot system is seen as a derivation of the scheme $M_\mathbb{C}$, compatible with the canonical derivation of $\mathbb{C}$.

**Notation and Conventions**

We denote differential and ordinary fields and rings by calligraphic letters $\mathbb{C}, \mathbb{K}, \ldots$. The canonical derivation of a differential ring $\mathbb{K}$ is denoted by $\partial_{\mathbb{K}}$ or just $\partial$ whenever it does not lead to confusion. Algebraic varieties are denoted by capital letters $M, G, \ldots$. The structure sheaf of $M$ is denoted by $\mathcal{O}_M$. If $M$ is a $\mathbb{C}$-algebraic variety and $\mathbb{C} \subset \mathbb{K}$, the space of $\mathbb{K}$-points of an algebraic variety $M$ is denoted by $M(\mathbb{K})$. We write $M_{\mathbb{K}}$ for the $\mathbb{K}$-algebraic variety obtained after base change $M \times_{\mathbb{C}} \text{Spec}(\mathbb{K})$. If $p$ is a point of $M$ we denote by $\kappa(p)$ its quotient field and $p^\flat$ the valuation morphism $p^\flat : \mathcal{O}_{M,p} \to \kappa(p)$.

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2 Algebraic Groups and Homogeneous Spaces

2.1 Algebraic Groups

Let us consider a field \( \mathbb{C} \) and its algebraic closure \( \bar{\mathbb{C}} \). By an algebraic variety over \( \mathbb{C} \) we mean a reduced and separated scheme of finite type over \( \mathbb{C} \). Along this text an algebraic group means an algebraic variety endowed with an algebraic group law and inversion morphism. In particular, algebraic groups over fields of characteristic zero are smooth varieties ([28] pp. 101–102).

The functor of points of an algebraic group takes values on the category of groups. If \( G \) is a \( \mathbb{C} \)-algebraic group, and \( \mathcal{K} \) is a \( \mathbb{C} \)-algebra, then the set \( G(\mathcal{K}) \) of \( \mathcal{K} \)-points of \( G \) is naturally endowed with a structure of group.

An algebraic group is an affine group if it is an affine algebraic variety. The main example of an affine algebraic group is the General Linear Group, \( \text{GL}(n, \mathbb{C}) = \text{Spec}(\mathbb{C}[x_{ij}, \Delta]), \quad \Delta = 1 \mid x_{ij} \mid \).

We call algebraic linear groups to the Zariski closed subgroups of \( \text{GL}(n, \mathbb{C}) \). It is well known that any affine algebraic group is isomorphic to an algebraic linear group.

2.2 Lie Algebra of an Algebraic Group

Let us consider \( \mathfrak{X}(G) \) the space of regular vector fields in \( G \), id est, derivations of the sheaf \( \mathcal{O}_G \) vanishing on \( \mathbb{C} \). The Lie bracket of regular vector fields is a regular vector field, so \( \mathfrak{X}(G) \) is a Lie algebra.

**Definition 2.1** Let \( A \) be a regular vector field in \( G \), and \( \psi: G \to G \) an automorphism of algebraic variety. Then, we define \( \psi(A) \) the transformed vector field \( \psi(A) = (\psi^\sharp)^{-1} \circ A \circ \psi^\sharp \).

\[
\begin{array}{c}
\mathcal{O}_G \xrightarrow{\psi(A)} \mathcal{O}_G \\
\downarrow \psi^\sharp \quad \quad \quad \downarrow (\psi^\sharp)^{-1} \\
\mathcal{O}_G \xrightarrow{A} \mathcal{O}_G
\end{array}
\]

Any \( \mathbb{C} \)-point \( \sigma \) of \( G \) induces right and left translations, \( R_\sigma \) and \( L_\sigma \), which are automorphisms of the algebraic variety \( G \). A \( \bar{\mathbb{C}} \)-point \( \bar{\sigma} \) of \( G \), induces translations in \( G_{\bar{\mathbb{C}}} \).

**Definition 2.2** The Lie algebra \( \mathfrak{R}(G) \) of \( G \) is the space of all regular vector fields \( A \in \mathfrak{X}(G) \) such that for all \( \mathbb{C} \)-point \( \sigma \in G(\mathbb{C}) \), \( R_\sigma(A \otimes 1) = A \otimes 1 \). In the same way, we define the Lie algebra \( \mathfrak{L}(G) \) of left invariant vector fields.

The Lie bracket of two right invariant vector field is a right invariant vector field. The same is true for left invariant vector fields, so \( \mathfrak{R}(G) \) and \( \mathfrak{L}(G) \) are
Lie sub-algebras of $\mathfrak{X}(G)$. For a point $x \in G$ its tangent space $T_x G$ is defined as the space of $C$-derivations from the ring of germs of regular functions, $\mathcal{O}_{G,\sigma}$ with values in its quotient field $\kappa(\sigma)$. It is a $\kappa(\sigma)$-vector space of the same dimension than $G$. Any regular vector field $\vec{X}$ in $\mathfrak{X}(G)$, can be seen as a map $\sigma \mapsto \vec{X}_\sigma \in T_\sigma G$. Let us consider $e$ the identity element of $G$. If $\mathcal{C}$ is algebraically closed, for any vector field $\vec{v}$ in $\mathfrak{X}(G)$ there are unique invariant vector fields $\vec{R} \in \mathcal{R}(G)$ and $\vec{L} \in \mathcal{L}(G)$ such that $\vec{R}_e = \vec{L}_e = \vec{v}$ (see [28] pp. 98–99).

2.3 Algebraic Homogeneous spaces

**Definition 2.3** Let $G$ be a $C$-algebraic group. A $G$-space $M$ is an algebraic variety over $C$ endowed with an algebraic action of $G$, $G \times_C M \rightarrow M$, $(\sigma, x) \mapsto \sigma \cdot x$.

Let $M$ be a $G$-space. Then for each extension $\mathcal{C} \subset K$, the group $G(K)$ acts on the set $M(K)$. Therefore it is a $G(K)$-set in the set theoretic sense. Given a point $x \in M$ its isotropy subgroup is an algebraic subgroup of $G$ that we denote by $H_x$. It is defined by equation $H_x \cdot x = x$. Note that it is not necessary for $x$ to be a rational point.

The intersection of the isotropy subgroups of all closed points of $M$ is a normal algebraic subgroup $H_M \triangleleft G$. The action of $G$ is called faithful if $H_M$ is the identity element $\{e\}$, and it is called free if for any rational point $x$, $H_x = \{e\}$. It is called transitive if for each pair of rational points $x, y \in M$ there is a $\sigma \in G$ such that $\sigma \cdot x = y$; id est there is only one orbit.

**Definition 2.4** Let us consider the induced morphism, $(a \times Id) : G \times_C M \rightarrow M \times_C M$, $(\sigma, x) \mapsto (\sigma x, x)$ then,

1. $M$ is an homogeneous $G$-space if $(a \times Id)$ is surjective.
2. $M$ is a principal homogeneous $G$-space if $(a \times Id)$ is an isomorphism.

If $\mathcal{C}$ is algebraically closed, an homogeneous $G$-space is simply a transitive $G$-space and a principal homogeneous $G$-space is a free and transitive $G$-space. In such case, any principal homogeneous $G$-space over is isomorphic to $G$.

2.4 Existence of quotients: Chevalley’s theorem

Let $V$ be a $\mathcal{C}$-vector space, and $GL(V)$ the group of linear transformations of $V$. It is an $\mathcal{C}$-algebraic group, and it acts algebraically on any tensor space over $V$. Given a tensor $T$ we call stabilizer subgroup of $T$ to the group of linear transformations $\sigma \in GL(V)$ for whom there exist a scalar $\lambda \in C$ such that $\sigma(t) = \lambda T$. In other words, the stabilizer subgroup of $T$ is the isotropy subgroup of the line $\langle T \rangle$ spanned by $T$ in the projectivization of the tensor space.
Theorem 2.5 (Chevalley, see [14] p. 80) Let $V$ be a $C$-vector space of finite dimension, and let $H \subset GL(V)$ be an algebraic subgroup. There exist a tensor,

$$T \in \bigoplus_i \left(V^\otimes n_i \otimes_C (V^\otimes m_i)^*\right)$$

such that $H$ is the stabilizer of $T$,

$$H = \{\sigma \in GL(V) | \langle \sigma(T) \rangle = \langle T \rangle\}$$

From this result we obtain that for a linear algebraic group $G$ and an algebraic subgroup $H$, the quotient space $G/H$ is isomorphic to the orbit $O_{\langle T \rangle}$ in the projective space $\mathbb{P} \left( \bigoplus_i \left(V^\otimes n_i \otimes_C (V^\otimes m_i)^*\right) \right)$. It is a quasiprojective algebraic variety.

There is a lack in the literature of an existence theorem for arbitrary quotients of an non-linear algebraic group over an arbitrary field. However, there is a result, due to M. Rosenlicht [33], saying that for any action of an algebraic group $G$ on an algebraic variety $V$, there exist a $G$-invariant open subset $U \subset V$ such that the geometrical quotient $U/G$ in the sense of Mumford exists. In the case of a subgroup $G'$ acting on $G$, this open subset must be right-invariant, and then it coincides with $G$.

2.5 Galois Cohomology

In this section, we assume that $C$ is a perfect field; note that this holds if $C$ is of characteristic zero, which is the case we are interested in. In such case, any algebraic extension can be embedded into a Galois extension. Therefore, the algebraic closure $\bar{C}$ is the inductive limit of all Galois extensions of $C$. The group of $C$-automorphisms of $\bar{C}$ is then identified with the projective limit of all Galois groups, of algebraic extensions of $C$. With the initial topology of the family of projections onto finite Galois groups, this is a compact totally disconnected group, that we denote $Gal(\bar{C}/C)$.

Let $G$ be a $C$-algebraic group. The group of automorphisms acts on $G(\bar{C})$ by composition. Let us consider $G_k$ the set of continuous maps from $Gal(\bar{C}/C)^k$ onto $G(\bar{C})$. In such case $G_0 = G(\bar{C})$. We consider the sequence:

$$0 \rightarrow G^0 \xrightarrow{\delta_0} G^1 \xrightarrow{\delta_1} G^2,$$  \hspace{1cm} (2.1)

where the codifferential of $x \in G^0$ is $(\delta_0 x)(\sigma) = x^{-1}.\sigma(x)$, and the codifferential of $\varphi \in G^1$ is $(\delta_1 \varphi)(\sigma, \tau) = \varphi(\sigma \cdot \tau)^{-1} \cdot \varphi(\sigma) \cdot \sigma(\varphi(\tau))$. An element in the image of $\delta_0$ is called a coboundary, the set of coboundaries is denoted by $B^1(G, C)$. An element $\varphi \in G^1$ is called a 1-cocycle if $\delta_1 \varphi$ vanish. The set of 1-cocycles is denoted $Z^1(G, C)$. Two 1-cocycles are called cohomologous if there is $x \in G^0$ such that $\varphi(\sigma) = x^{-1} \cdot \psi(\sigma) \cdot \sigma(x)$. This is an equivalence relation in $Z^1(G, C)$. The quotient set $Z^1(G, C)/\sim$ is a pointed set, with distinguished point the class of coboundaries. Note that when $G$ is an abelian group the sequence (2.1) is a differential complex and this quotient is the first cohomology group.
Definition 2.6 The zero Galois cohomology set of $G$ with coefficients in $\mathcal{C}$, $H^0(G, \mathcal{C})$, is the kernel of $\delta_0$. It is a pointed set with distinguished point the identity. The first Galois cohomology set of $G$ with coefficients in $\mathcal{C}$, $H^1(G, \mathcal{C})$, is the pointed set $Z^1(G, \mathcal{C})/\sim$.

From the definition of $\delta_0$ it is clear that $x \in H^0(G, \mathcal{C})$ if and only if it is invariant under the action of Gal($\overline{\mathcal{C}}/\mathcal{C}$). The fixed field of $\overline{\mathcal{C}}$ in precisely $\mathcal{C}$, therefore the zero Galois cohomology set coincides with the set of $\mathcal{C}$-points $G(\mathcal{C})$.

Let $G'$ be an algebraic subgroup of $G$. In such case $H^0(G/G', \mathcal{C})$ is a pointed set, with distinguished point the class of the identity. An element $x \in H^0(G/G', \mathcal{C})$ is a $\mathcal{C}$-point of the homogeneous space $G/G'$. This $x$ is the class of a unique $\overline{\mathcal{C}}$-point $\overline{x}$ of $G$. The coboundary $\delta_0\overline{x}$ is a cocycle in $G'$, and its cohomology class $[\overline{x}] \in H^1(G', \mathcal{C})$ does not depend on the election of $x$. We have a morphism of pointed sets $H^0(G/G', \mathcal{C}) \rightarrow H^1(G, \mathcal{C})$ called the connecting morphism. We obtain an exact sequence of pointed sets:

$$0 \rightarrow H^0(G', \mathcal{C}) \rightarrow H^0(G, \mathcal{C}) \rightarrow H^0(G/G', \mathcal{C}) \rightarrow H^1(G', \mathcal{C}) \rightarrow H^1(G, \mathcal{C})$$

and when $G'$ is a normal subgroup of $G$, the sequence

$$H^1(G', \mathcal{C}) \rightarrow H^1(G, \mathcal{C}) \rightarrow H^1(G/G', \mathcal{C})$$

is also exact (see [19], p. 277–288).

Using the previous exact sequence it is relatively easy to compute the first Galois cohomology set of several algebraic groups. We say that the first cohomology set of $G$ with coefficients in $\mathcal{C}$ vanish if it consists of an only point. In particular the following results are well known:

- The first cohomology set of the additive group $H^1((\mathcal{C}, +), \mathcal{C})$ vanish.
- The first cohomology set of the multiplicative group $H^1(\mathcal{C}^*, \cdot, \mathcal{C})$ vanish.
- $H^1(GL(n, \mathcal{C}), \mathcal{C})$ vanish.
- $H^1(SL(n, \mathcal{C}), \mathcal{C})$ vanish.
- If $G$ is linear connected solvable group then $H^1(G, \mathcal{C})$ vanish.
- If $\mathcal{C}$ is algebraically closed then for any algebraic group $H^1(G, \mathcal{C})$ vanish.
- If $S$ is a Riemann surface and $\mathcal{M}(S)$ is its field of meromorphic function then for any linear connected $\mathcal{M}(S)$-algebraic group $G$, $H^1(G, \mathcal{M}(S))$ vanish (this is a particular case of fields of dimension lower or equal than one, treated in [35]).
- If $S$ is an open Riemann surface then for any connected $\mathcal{M}(S)$-algebraic group $H^1(G, \mathcal{M}(S))$ vanish (Grauert theorem, see [36]).
The first Galois cohomology set classifies the principal homogeneous spaces over $G$. This classification was first obtained by Châtelet for some particular cases, here we follow Kolchin [19] (see p. 281–283). The main fact is that if the first Galois cohomology set vanish then all principal homogeneous spaces have rational points.

**Theorem 2.7** Let $G$ be a $C$-algebraic group and $M$ a principal homogeneous $G$-space. Then $M$ defines a class $[M]$ in $H^1(G,C)$. This cohomology class classifies $M$ up to $C$-isomorphisms. $M$ is isomorphic to $G$ if and only if $[M]$ is the distinguished point of $H^1(G,C)$. Reciprocally any cohomology class of $H^1(G,C)$ is the class of certain homogeneous $G$-space.

### 2.6 Fundamental Fields

Consider a right invariant vector field $A \in R(G)$. Then, $\vec{A} \otimes 1$ is a regular vector field in $G \times_C M$. This vector field is projectable by the action of $G$ in $M$,

$$a : G \times_C M \to M, \quad \vec{A} \otimes 1 \mapsto \vec{A}^M.$$

**Definition 2.8** We call algebra of fundamental field $R(G, M)$ to the Lie algebra of regular vector fields in $M$ spanned by the projections $\vec{A}^M$ of vector fields $\vec{A} \otimes 1$, being $\vec{A}$ right invariant vector field in $G$.

There is a canonical surjective Lie algebra morphism,

$$R(G) \to R(G, M), \quad \vec{A} \to \vec{A}^M,$$

the kernel of this morphism is the Lie algebra of the kernel of the action $H_M$, $R(H_M) \subset R(G)$. In particular, the Lie algebra of fundamental fields $R(G,G)$ in $G$ coincides with $R(G)$.

### 3 Differential Algebraic Geometry

We can state that the differential algebraic geometry is with respect to the differential algebra the same than the classical algebraic geometry is with respect to the commutative algebra. In this sense, the differential algebraic geometry is the study of geometric objects associated with differential rings. Here we present the theory of schemes with derivations, which has been developed by Buium [7], and the theory of differential schemes, which is due to Keigher [16, 17], Carra’ Ferro (see [9]), and Kovacic [21].

#### 3.1 Differential Algebra

We present here some preliminaries in differential algebra. The main references for this subject are [32], [15], [19].

A differential ring is a commutative ring $\mathcal{A}$ and a derivation $\partial_{\mathcal{A}}$. By a derivation we mean an application verifying the Leibnitz rule, $\partial_{\mathcal{A}}(ab) = a \cdot \partial_{\mathcal{A}}(b) + \partial_{\mathcal{A}}(a) \cdot b$. 
\[ \partial_A(b) + b \cdot \partial_A(a) \]. An element \( a \in A \) is called a constant if it has vanishing derivative \( \partial a = 0 \). Whenever it does not lead to confusion, we will write \( \partial \) instead of \( \partial_A \). The subset \( C_A \) of constants elements is a subring of \( A \). When \( A \) is a field we call it a differential field. In such a case, the constant ring \( C_A \) is a subfield of \( A \). An ideal \( I \subset A \) is a differential ideal if \( \partial(I) \subset I \).

Note that if \( I \) is a differential ideal, then the quotient \( A/I \) is also a differential ring. For a subset \( S \subset A \) we denote \( [S] \) for the smallest differential ideal containing \( S \), and \( \{S\} \) for the smallest radical differential ideal containing \( S \). For an ideal \( I \subset A \) we denote \( \mathfrak{I} \) for the smallest differential ideal containing \( I \), namely: \( \mathfrak{I} = \sum_i \partial^i(I) \). Localization by arbitrary multiplicative systems is also suitable in differential rings. A ring morphism is called differential if it is compatible with the derivation. In the category of differential rings, tensor product is also well defined.

Consider \( K \) a differential field. A differential ring \( A \) endowed with a morphism \( K \hookrightarrow A \) is called a differential \( K \)-algebra. If \( A \) is a differential field then we say that it is a differential extension of \( K \).

### 3.2 Keigher Rings

If \( I \subset A \) is an ideal, we denote its radical ideal by \( \sqrt{I} \), the intersection of all prime ideals containing \( I \). In algebraic geometry, there is a one-to-one correspondence between the set of radical ideals of \( A \) and the set of Zariski closed subsets of \( \text{Spec}(A) \), the prime spectrum of \( A \). In order to perform an analogous systematical study of the set of differential ideals – id est differential algebraic geometry – we should require radicals of differential ideals to be also differential ideals. This property does not hold in the general case. We have to introduce a suitable class of differential rings. This class was introduced by Keigher (see [16]; we call them Keigher rings.

**Definition 3.1** A Keigher ring is a differential ring verifying that for each differential ideal \( \mathfrak{I} \), its radical \( \sqrt{\mathfrak{I}} \) is also a differential ideal.

**Definition 3.2** For any ideal \( I \subset A \) we define its differential core as \( I^\sharp = \{ a \in I : \forall n (\partial^n a \in I) \} \).

Keigher rings can be defined in several equivalent ways. The following theorem of characterization includes different possible definitions (see [21], proposition 2.2.).

**Theorem 3.3** Let \( A \) be a differential ring. The following are equivalent:

(a) If \( p \subset A \) is a prime ideal, then \( p^\sharp \) is a prime differential ideal.

(b) If \( I \subset A \) is a differential ideal, and \( S \) is a multiplicative system disjoint from \( I \), then there is a prime maximal differential ideal containing \( I \) disjoint with \( S \).

(c) If \( I \subset A \) is a differential ideal, then so is \( \sqrt{I} \).
(d) If $S$ is any subset, then $\{S\} = \sqrt{|S|}$.

(e) $A$ is a Keigher ring.

By a Ritt algebra we mean a differential ring including the field $\mathbb{Q}$ of rational numbers. When studying differential equations in characteristic zero, differential rings considered are mainly Ritt algebras. A main property of Ritt algebras is that the radical of a differential ideal is a differential ideal (see for instance [15]), therefore Ritt algebras are Keigher rings.

**Proposition 3.4** If $A$ is a Keigher ring then for any differential ideal $I$, $A/I$ is Keigher and for any multiplicative system $S$, $S^{-1}A$ is Keigher.

**Proof.** Assume $A$ is Keigher. First, let us prove that $A/I$ is Keigher. Consider the projection $\pi: A \to A/I$. Let $a$ be a differential ideal of $A/I$. Then $\sqrt{a} = \pi(\sqrt{\pi(a)})$ is a differential ideal.

Second, consider a localization morphism $l: A \to S^{-1}A$. Let $a \subset S^{-1}A$ be a differential ideal. Let us denote by $b$ the preimage $l^{-1}(a)$; it is a differential ideal and $l(b) \cdot S^{-1}A = a$. $A$ is Keigher, and then $\partial a \in \sqrt{b}$. Therefore $(\partial a)^m \in b$, so that $(\frac{\partial}{a})^m \in \sqrt{a}$.

3.3 New Constants

From now on let $K$ be a differential field, and let $C$ be its field of constants. We assume that $C$ is algebraically closed. A classical lemma of differential algebra (see [19] p. 87 Corollary 1) says that if $A$ is a differential $K$-algebra, then the ring of constant $C_A$ is linearly disjoint over $C$ with $K$. Let us set this classical lemma in a more geometric frame.

**Lemma 3.5** Let $A$ be an integral finitely generated differential $K$-algebra. Then there is an affine subset $U \subset \text{Spec}(A)$ such that the ring of constants $C_{A_U}$ is a finitely generated algebra over $C$.

**Proof.** Consider $Q(A)$ the field of fractions of $A$. The extension $K \subset Q(A)$ is of finite transcendency degree. Then, $K \subset K \cdot C_{Q(A)} \subset Q(A)$ are extensions of finite transcendency degree, and there are $\lambda_1, \ldots, \lambda_s$ in $C_{Q(A)}$ such that $K(\lambda_1, \ldots, \lambda_s) = K \cdot C_{Q(A)}$. Constants $\lambda_1, \ldots, \lambda_s$ are fractions $\frac{A}{g_i}$. Consider the affine open subset obtained by removing from $\text{Spec}(A)$ the zeroes of the denominators,

$$U = \text{Spec}A \setminus \bigcup_{i=1}^{s} (g_i)_0.$$
Then, \( \lambda_i \in \mathcal{A}_U \) and \( \mathcal{K}[\mathcal{C}_\mathcal{A}_U] = \mathcal{K}[\lambda_1, \ldots, \lambda_s] \). We will prove that \( \mathcal{C}_\mathcal{A}_U = \mathcal{C}[\lambda_1, \ldots, \lambda_s] \). Let \( \lambda \in \mathcal{C}_\mathcal{A}_U \). It is certain polynomial in the variables \( \lambda_i \) with coefficients in \( \mathcal{K} \):

\[
\lambda = \sum_{I \in \Lambda} a_I \lambda^I, \quad a_I \in \mathcal{K};
\]

where \( \Lambda \) is a suitable finite set of multi-indices. We can take this set in such way that the \( \{ \lambda^I \}_{I \in \Lambda} \) are linearly independent over \( \mathcal{K} \), and then so they are over \( \mathcal{C} \). \( \{ \lambda, \lambda^I \}_{I \in \Lambda} \) is a subset of \( \mathcal{C} \)-linearly dependents elements of \( \mathcal{C}_\mathcal{A}_U \). By [19] (p. 87 corollary 1) then they are \( \mathcal{C} \)-linearly dependent. Hence, \( \lambda \) is \( \mathcal{C} \)-linear combination of \( \{ \lambda^I \}_{I \in \Lambda}, \lambda \in \mathcal{C}[\lambda_1, \ldots, \lambda_s] \) and finally \( \mathcal{C}_\mathcal{A}_U = \mathcal{C}[\lambda_1, \ldots, \lambda_s] \). □

### 3.4 Differential Spectra

**Definition 3.6** Let \( \mathcal{A} \) be a differential ring. \( \text{DiffSpec}(\mathcal{A}) \) is the set of all prime differential ideals \( p \subset \mathcal{A} \).

Let \( S \subset \mathcal{A} \) any subset. We define the differential locus of zeroes of \( S \), \( \{ S \}_0 \subset \text{DiffSpec}(\mathcal{A}) \) as the subset of prime differential ideals containing \( S \). This family of subsets define a topology (having these subsets as closed subsets), that we call the Kolchin topology or differential Zariski topology. Note that \( \{ S \}_0 = (S)_0 \cap \text{DiffSpec}(\mathcal{A}) \). From that if follows:

**Proposition 3.7** \( \text{DiffSpec}(\mathcal{A}) \) with Kolchin topology is a topological subspace of \( \text{Spec}(\mathcal{A}) \) with Zariski topology.

From now on, let us consider the following notation: \( X = \text{Spec}(\mathcal{A}) \), and \( X' = \text{DiffSpec}(\mathcal{A}) \).

Let us recall that a topological space is said reducible if it is the non-trivial union of two closed subsets. It is said irreducible if it is not reducible. A point of an irreducible topological space is said generic if it is included in each open subset. The following properties of the differential spectrum are proven in [17] (see Proposition 2.1):

**Proposition 3.8** \( X' \) verifies:

1. \( X' \) is quasicompact.
2. \( X' \) is \( T_0 \) separated.
3. Every closed irreducible subspace of \( X' \) admits a unique generic point. The map \( X' \to 2^{X'} \), that maps each point \( x \) to its Kolchin closure \( \overline{\{x\}} \) is a bijection between points of \( X' \) and irreducible closed subspaces of \( X' \).

Here we review some of the topological properties of the differential spectrum of Keigher rings.

**Lemma 3.9** Assume that \( \mathcal{A} \) is a Keigher ring. Then each minimal prime ideal is a differential ideal.
Proof. Then, let $p$ be a minimal prime. By Theorem 3.3 (a), $p_\sharp$ is a prime differential ideal and $p_\sharp \subseteq p$. □

**Proposition 3.10** Assume that $\mathcal{A}$ is Keigher. Then, $X$ is an irreducible topological space if and only if $X'$ is an irreducible topological space.

Proof. Just note that the irreducible components of $X'$ are the Kolchin closure of minimal prime ideals of $\mathcal{A}$. □

**Proposition 3.11** Assume $\mathcal{A}$ is Keigher. If $X'$ is connected, then $X$ is connected.

Proof. Assume that $X = Y \sqcup Z$, then we have an isomorphism of rings

$$(p_1, p_2): \mathcal{A} \mapsto \mathcal{O}_X(Y) \times \mathcal{O}_X(Z), \quad a \mapsto (a|_X, a|_Y),$$

the kernel of each restriction $p_i$ is intersection of minimal prime ideals, so by Lemma 3.9 they are differential ideals. Hence, the rings $\mathcal{O}_X(Y)$ and $\mathcal{O}_X(Z)$ are also differential rings. Then,

$$X' = Y' \sqcup Z',$$

being $Y' = \text{DiffSpec}(\mathcal{O}_X(Y))$, $Z' = \text{DiffSpec}(\mathcal{O}_X(Z))$. We have proven that if $X$ disconnects, then $X'$ disconnects. □

### 3.5 Structure Sheaf

We define the structure sheaf $\mathcal{O}_{X'}$ as in [21]. Let us consider the projection,

$$\pi: \bigsqcup_{x \in X'} A_x \rightarrow X'.$$

being $\bigsqcup_{x \in X'} A_x$ the disjoint union of all the localized rings $A_x$. We say that a section $s$ of $\pi$ defined in an open subset $U \subset X'$ is a regular function if it verifies the following: for all $x \in U$ there exist an open neighborhood $x \in U_x$ and $a, b \in \mathcal{A}$ with $b(x) \neq 0$ ($b \notin x$), such that for all $y \in U_x$ with $b(y) \neq 0$, $s(y) = \frac{a}{b} \in A_y$. Thus, a regular function is a section which is locally representable as a quotient. We write $\mathcal{O}_{X'}$ for the sheaf of regular functions in $X'$. By the above construction we can state:

**Proposition 3.12** The stalk $\mathcal{O}_{X', x}$ is a ring isomorphic to $A_x$.

**Theorem 3.13** Let us consider the natural inclusion $j: X' \hookrightarrow X$. The sheaf of regular functions $\mathcal{O}_{X'}$ is the restriction $\mathcal{O}_X|_{X'}$ of the sheaf of regular function in $X$. 

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Proof. First, let us define a natural morphism of presheaves of rings on $X'$ between the inverse image presheaf $j^{-1}\mathcal{O}_X$ and $\mathcal{O}_{X'}$. Let us consider an open subset $U \subset X'$ and a section $s$ of the presheaf $j^{-1}\mathcal{O}_X$ defined in $U$. By definition of inverse image, there is an open subset $W$ of $X$ such that $W \cap X' \cap U$ and for what $s$ is written as a fraction $\frac{a}{b} \in \mathcal{A}_W$. This fraction is a section of $\mathcal{O}_{X'}(U)$, and it defines the presheaf morphism

$$j^{-1}\mathcal{O}_X \rightarrow \mathcal{O}_{X'}.$$ 

This presheaf morphism induces a morphism between associated sheaves $\mathcal{O}_X|_{X'}$ and $\mathcal{O}_{X'}$. It is clear that this natural morphism induce the identity between fibers $(j^{-1}\mathcal{O}_X)_x = A_x \rightarrow \mathcal{O}_{X',x} = A_x$, and then it is an isomorphism. □

3.6 Global Sections

One of the main facts of the differential algebraic geometry is that the ring of global regular sections of $X'$ does not coincide with the differential ring $A$. Of course there is a canonical morphism from $A$ to $\mathcal{O}_{X'}(X')$. However there are non-vanishing elements giving rise to the zero section and non invertible elements giving rise to invertible sections. An element $a$ of $A$ is called a differential zero if its annihilator ideal is not contained in any proper differential ideal. The set of differential zeroes is denoted by $\mathfrak{Z}$. An element is called a differential unit if it is not contained in any proper differential ideal. The set of differential units is denoted by $\mathfrak{U}$. Then, there is a canonical injective morphism, $\mathfrak{U}^{-1}A/\mathfrak{Z} \rightarrow \mathcal{O}_{X'}(X')$. But in general this morphism is not surjective, i.e., there are regular functions that are not representable as fractions of $A$. Therefore, the differential spectrum of $\mathcal{O}_{X'}(X')$ is not always isomorphic to $X'$. This problem is extensively discussed in [2].

3.7 Differential Schemes

The study of differential schemes started within the work of Keigher [16, 17] and was continued by Carra' Ferro [9], Buium [7] and Kovacic [21]. Definitions are slightly different in each author approach, here we follow Kovacic.

Let us remind that a locally ringed space is a topological space $X$ endowed with an structure sheaf of rings $\mathcal{O}_X$ such that for all $x \in X$ the stalk $\mathcal{O}_{X,x}$ is a local ring. Thus, a locally differential ringed space is a locally ringed space whose structure sheaf $\mathcal{O}_X$ is a sheaf of differential rings. A morphism of locally differential ringed spaces $f: X \rightarrow Y$ consist of a continous map together with a sheaves morphism $f^2: \mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$. For the differential ring $A$ it is clear that its differential spectrum $X'$ endowed with the structure sheaf $\mathcal{O}_{X'}$ is a locally differential ringed space.

Definition 3.14 An affine differential scheme is a locally differentially ringed space $X$ which is isomorphic to $\text{DiffSpec}(A)$ for some differential ring $A$. 

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Definition 3.15 A differential scheme is a locally differentially ringed space $X$ in which every point has a neighborhood that is an affine differential scheme.

Remark 3.16 Schemes are differential schemes, endowed with the trivial derivation. The category of differential schemes is an extension of the category of schemes, in the same way that the category of differential rings is an extension of the category of rings.

By a morphism of differential schemes $f : X \rightarrow Y$ we mean a morphism of locally ringed spaces, such that $f^* : O_Y \rightarrow f_* O_X$ is a morphism of sheaves of differential rings.

Let $K$ be a differential field. A $K$-differential scheme is a differential scheme $X$ provided with a morphism $X \rightarrow \text{DiffSpec}(K)$, it means that $O_X$ is a sheaf of differential $K$-algebras.

A morphism of differential schemes $f : X \rightarrow Y$ between two differential $K$-schemes is a morphism of differential $K$-schemes if the sheaf morphism $f^* : O_Y \rightarrow f_* O_X$ is a morphism of sheaves of differential $K$-algebras.

### 3.8 Product of Differential Schemes

There is not a direct product in the category of differential schemes relative to a given basic differential scheme. This problem is discussed in [21]. However, in the case of differential schemes over a differential field $K$ we can construct the direct product by patching tensor products, as it is usually done in algebraic geometry. Therefore,

$$\text{DiffSpec}(A) \times_K \text{DiffSpec}(B) = \text{DiffSpec}(A \otimes_K B).$$

Moreover, if $X$ and $Y$ are reduced differential $K$-schemes then $X \times_K Y$ is also reduced (see [22] Proposition 25.2).

### 3.9 Split of Differential Schemes

Definition 3.17 Let $X$ be a differential scheme. Define the presheaf of rings $C_X$ on $X$ by the formula,

$$C_X(U) = C_{O_X(U)},$$

for any open subset $U \subseteq X$.

From this definition it follows that $C_X$ is a sheaf of rings and its fiber $C_{X,x}$ is isomorphic to the ring of constants $C_{O_X,x}$. In particular, if $X$ is a $K$-differential scheme $C_X$ is a sheaf of $C_K$-algebras.

Definition 3.18 We call space of constants of $X$, $\text{Const}(X)$ to the locally ringed space $(X, C_X)$.

Definition 3.19 We say that $X$ is an almost-constant differential scheme if its space of constants $\text{Const}(X)$ is a scheme.
Let $X$ be an almost-constant scheme. Then, each open subset $U \subset X$ is also almost-constant. If $Y$ is a reduced closed subscheme of $X$ then $Y$ is almost-constant. In this way if $Y$ is a locally closed reduced subscheme of $X$, then $Y$ is almost-constant.

Let $K$ be a differential field, and $C$ its field of constants.

**Definition 3.20** A differential $K$-scheme $X$ splits if there is a $C$-scheme $Y$ and an isomorphism of $K$-differential schemes,

$$\phi: X \xrightarrow{\sim} Y \times_C \text{DiffSpec}(K).$$

The isomorphism $\phi$ is called a splitting isomorphism for $X$.

**Proposition 3.21** If $X$ is reduced and splits, then it is almost-constant and

$$X \xrightarrow{\sim} \text{Const}(X) \times_C \text{DiffSpec}(K).$$

**Proof.** [22] proposition 28.2. □

### 3.10 Strongly Normal Extensions

Strongly normal extensions are introduced by Kolchin [18]. They are differential field extensions whose group of automorphisms admits a structure of algebraic group. This notion has been recently characterized in terms of differential schemes by Kovacic [23]. This characterization is more convenient for our presentation of differential Galois theory, so that we will use it as a new definition.

**Definition 3.22** $K \rightarrow \mathcal{L}$ is a strongly normal extension if and only if the differential scheme $\text{DiffSpec}(\mathcal{L} \otimes_K \mathcal{L})$ splits. In such case denote $\text{Gal}(\mathcal{L}/K)$ to the scheme $\text{Const}(\text{DiffSpec}(\mathcal{L} \otimes_K \mathcal{L}))$.

Note that prime differential ideals of $\mathcal{L} \otimes_K \mathcal{L}$ whose quotient field is $\mathcal{L}$, correspond to $K$-automorphisms of $\mathcal{L}$. If $\sigma$ is a $K$-automorphism of $\mathcal{L}$, the kernel of the differential $K$-algebra morphism,

$$\mathcal{L} \otimes_K \mathcal{L} \rightarrow \mathcal{L}, \quad a \otimes b \mapsto a\sigma(b),$$

is a prime differential ideal $p_\sigma$. Then, the set of rational points of $\text{DiffSpec}(\mathcal{L} \otimes_K \mathcal{L})$ is naturally endowed with a group structure. This group structure descent to a structure of $C$-algebraic group precisely when $\text{DiffSpec}(\mathcal{L} \otimes_K \mathcal{L})$ splits. In such case the space of constant $\text{Gal}(\mathcal{L}/K)$ is endowed with an structure of algebraic group. This problem is inexhaustively treated in [23].

This approach gives us a parallelism with Galois extensions in classical theory of fields. Note that a field extension $k \rightarrow K$ is a Galois extension if and only if $\text{Spec}(K \otimes_k K) = G \times_k \text{Spec}(K)$ (see [34]). We also obtain the scheme structure of the Galois group: it is the scheme of constants of $\text{DiffSpec}(\mathcal{L} \otimes_K \mathcal{L})$. 

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3.11 Galois Correspondence for Strongly Normal Extensions

Let us consider as above $K \subset \mathcal{L}$ a strongly normal extension of differential fields. To each subgroup $H \subset \text{Gal}(\mathcal{L}/K)$ we assign the intermediate extension $K \subset \mathcal{L}^H \subset \mathcal{L}$ of $H$-invariants. Reciprocally to each intermediate extension $K \subset \mathcal{F} \subset \mathcal{L}$ we assign the subgroup $\text{Gal}(\mathcal{L}/\mathcal{F}) \subset \text{Gal}(\mathcal{L}/K)$ of automorphisms of $\mathcal{L}$ that are differential $\mathcal{F}$-algebra automorphism. The Galois correspondence between closed subgroups and intermediate extensions is first shown by Kolchin (see [18] and [19]).

Theorem 3.23 The maps

$$H \mapsto \mathcal{L}^H \subset \mathcal{L}$$

from group subschemes of $\text{Gal}(\mathcal{L}/K)$ to intermediate differential extensions and

$$\mathcal{F} \mapsto \text{Gal}(\mathcal{L}/\mathcal{F}) \subset \text{Gal}(\mathcal{L}/K)$$

from intermediate differential extensions subgroup schemes, are bijective and inverse each other. The extension $K \subset \mathcal{F}$ is strongly normal if and only if $\text{Gal}(\mathcal{L}/\mathcal{F})$ is a normal subgroup of $\text{Gal}(\mathcal{L}/K)$. In such case $\text{Gal}(\mathcal{F}/K)$ is isomorphic to the quotient $\text{Gal}(\mathcal{L}/K)/\text{Gal}(\mathcal{L}/\mathcal{F})$.

3.12 Lie Extensions

The algebraic differential approach to Lie-Vessiot systems, in terms of differential fields, was initiated by K. Nishioka [31]. He relates the differential extensions generated by solutions of a Lie-Vessiot system with algebraic dependence on initial conditions; a concept introduced by H. Umemura [40] in relation with the analysis of Painlevé differential equations. He also introduces the notion of Lie extension, a differential field extension that carry the infinitesimal structure of a Lie-Vessiot system. Here we review some of his results, in order to relate them with the Galois theory of automorphic systems. Consider a differential field $\mathcal{K}$ of characteristic zero with algebraically closed constant field $\mathcal{C}$. Any considered differential field $\mathcal{K}$ is a subfield of certain fixed universal extension of $\mathcal{K}$.

Definition 3.24 We say that a differential extension $\mathcal{K} \subset \mathcal{R}$ depends rationally on arbitrary constants if there exist a differential field extension $\mathcal{K} \subset \mathcal{M}$ such that $\mathcal{R}$ and $\mathcal{M}$ are free over $\mathcal{K}$ and $\mathcal{R} \cdot \mathcal{M} = \mathcal{M} \cdot C_{\mathcal{R}, \mathcal{M}}$.

For a differential extension $\mathcal{K} \subset \mathcal{L}$ denote $\text{Der}_\mathcal{K}(\mathcal{L})$ the space of derivations of $\mathcal{L}$ that vanish over $\mathcal{K}$. This space is a $\mathcal{K}$-Lie algebra.

Definition 3.25 We say that a differential extension $\mathcal{K} \subset \mathcal{L}$ is a Lie extension if $\mathcal{L} = C_{\mathcal{L}}$, there exists a $\mathcal{C}$-Lie sub algebra $\mathfrak{g} \subset \text{Der}_\mathcal{K}(\mathcal{L})$ such that $[\partial, \mathfrak{g}] \subset \mathcal{K}\mathfrak{g}$, and $\mathcal{L}\mathfrak{g} = \text{Der}_\mathcal{K}(\mathcal{L})$.

Theorem 3.26 ([31]) Suppose that $\mathcal{K}$ is algebraically closed. Then every intermediate differential field of a strongly normal extension of $\mathcal{K}$ is a Lie extension.
3.13 Schemes with Derivation

In this section we present some facts of the theory of schemes with derivations. This is mainly the point of view of [7]. However we consider only regular derivations whereas A. Buium considers the more general case of meromorphic derivations. Our purpose is to relate schemes with derivations to differential schemes. Note that the regularity of the derivation is essential to Theorem 3.28 below; hence it does not hold under Buium’s definition.

Let $X$ be a scheme. A derivation $\partial_X$ of the structure sheaf $\mathcal{O}_X$ is a law that assigns to each open subset $U \subset X$ a derivation $\partial_X(U)$ of the ring $\mathcal{O}_X(U)$. This law is assumed to be compatible with restriction morphisms.

**Definition 3.27** A scheme with derivation is a pair $(X, \partial_X)$ consisting of a scheme $X$ and a derivation $\partial_X$ of the structure sheaf $\mathcal{O}_X$.

Thus, a scheme with derivation is a scheme such that its structure sheaf is a sheaf of differential rings. A morphism of schemes with derivation is a scheme morphism such that induces a morphism of sheaves of differential rings.

Let $K$ be a differential field. A $K$-scheme with derivation is a scheme with derivation $(X, \partial_X)$ together with a morphism $(X, \partial_X) \rightarrow (\text{Spec}(\mathcal{O}_X(U)), \partial)$.

Let $(X, \partial_X)$, $(Y, \partial_Y)$ be two $K$-schemes with derivation. Then the direct product $X \times_K Y$ admits the derivation $\partial_X \otimes 1 + 1 \otimes \partial_Y$. Then, $(X \times_K Y, \partial_X \otimes 1 + 1 \otimes \partial_Y)$ is the direct product of $(X, \partial_X)$ and $(Y, \partial_Y)$ in the category of schemes with derivation.

3.14 Differential Schemes and Schemes with Derivation

**Theorem 3.28** Given a scheme with derivation $(X, \partial)$ there exist a unique topological subspace $X' \subset X$ verifying

1. $X'$ endowed with the structure sheaf $\mathcal{O}_X|_{X'}$ and the derivation $\partial|_{X'}$ is a differential scheme. This differential scheme will be denoted $\text{Diff}(X, \partial)$.

2. For each open affine subset $U \subset X$, $U \cap X' \simeq \text{DiffSpec}(\mathcal{O}_X(U), \partial)$.

Furthermore, each morphism of schemes with derivation $(X, \partial_X) \rightarrow (Y, \partial_Y)$ induces a morphism of differential schemes $\text{Diff}(X, \partial_X) \rightarrow \text{Diff}(Y, \partial_Y)$. The assignation $(X, \partial) \mapsto \text{Diff}(X, \partial)$ is functorial.

**Proof.** If $X$ is an affine scheme then the theorem holds, and $X' = \text{DiffSpec}(\mathcal{O}_X(X))$.

Let us consider the non-affine case. Let $(X, \partial_X)$ be an scheme with derivation, and let $\{U_i\}_{i \in \Lambda}$ be a covering of $X$ by affine subsets. The ring of sections
$\mathcal{O}_X(U_i)$ is a differential ring for all $i \in \Lambda$, and its spectrum $\text{Spec}(\mathcal{O}(U_i))$ is canonically isomorphic to $U_i$.

For each $i \in \Lambda$ we take $U'_i$ the differential spectrum $\text{DiffSpec}(\mathcal{O}_X(U_i))$, which is a topological subspace of $U_i$. Then $U'_i \subset U_i \subset X$. Let us define $X' = \bigcup_{i \in \Lambda} U'_i$. Thus, $X'$ is a locally differential ringed space with the sheaf $\mathcal{O}_X|_{X'}$.

Let us prove that $X'$ is a differential scheme.

First, let us prove that $U_i \cap X' = U'_i$. By construction we have, $U'_i \subset U_i \cap X'$. Let us consider $x \in U_i \cap X'$. It means that for some $j \in \Lambda$, $x \in U_i \cap U_j$, and $x \in U'_j \subset U_j$. Let us consider an affine neighborhood $U_x$ of $x$ contained in such intersection. Because the inclusion $U_x \to U_j$, we have that $x \in U'_x = \text{DiffSpec}(\mathcal{O}_X(U_x))$. Then we have inclusions and restriction as follows:

$$
\begin{array}{cccc}
U_x & U_i & \mathcal{O}_X(U_x) & U'_x \\
& U_j & \mathcal{O}_X(U_j) & U'_j \\
\end{array}
$$

We conclude that $x \in U'_i$.

Secondly, let us prove that for any affine subset $U$, the intersection $U \cap X'$ is an affine differential scheme $\text{DiffSpec}(\mathcal{O}_X(U))$. Let $U$ be an affine subset, and let us denote $U'$ the differential spectrum $\text{DiffSpec}(\mathcal{O}_X(U))$ that we consider as a subset of $U$. Let us consider $x \in U'$. Then, for certain $i \in \Lambda$, $x \in U \cap U_i$. Let $U_x$ be an affine neighborhood of $x$ such that $U_x \subset U \cap U_i$. Denote by $U'_x$ the differential spectrum of $\mathcal{O}_X(U_x)$. We have that $U'_x \subset U'_i$, and then $x \in U \cap X'$. Reciprocally let us consider $x \in U \cap X'$. Then for certain $i \in \Lambda$ we have $x \in U'_i$. By the same argument, we have that $x \in U$ is a prime differential ideal of $\mathcal{O}_X(U)$.

The derivation $\partial$ induces derivations on the structure sheaf of $U \cap X$ for each affine open subset $U \subset X$. Then, it induce a derivation $\partial : \mathcal{O}_{X'} \to \mathcal{O}_{X'}$ and $\text{Diff}(X, \partial) = (X', \mathcal{O}_{X'}, \partial|_{X'})$ is a differential scheme.

Finally, let us consider $f : (X, \partial_X) \to (Y, \partial_Y)$ a morphism of schemes with derivation. If we assume that they are both affine schemes, then the theorem holds. In the general case, we cover $Y$ by affine subsets $\{U_i\}_{i \in \Lambda}$, and each fiber $f^{-1}(U_i)$ by affine subsets $\{V_{ij}\}_{i \in \Lambda, j \in \Pi}$. Then $f$ is induced by the family of differential ring morphisms

$$f^j_{ij} : \mathcal{O}_Y(U_j) \to \mathcal{O}_X(V_{ij}).$$

These morphisms induce morphisms,

$$f^j_{ij} : V_{ij} \to U'_i,$$

of locally differential ringed spaces which coincide on the intersections, and then they induce a unique morphism,

$$f' : X' \to Y'.$$

\[\square\]
**Definition 3.29** Let \((X, \partial)\) be an scheme with derivation. We will say that \(x \in X\) is a differential point if \(x \in \text{Diff}(X, \partial)\).

**Corollary 3.30** Let us consider \((X, \partial)\) an scheme with derivation, and \(x\) a point of \(X\). Then, the following are equivalent:

(a) \(x \in X\) is a differential point.

(b) For each affine neighborhood \(U\), \(x\) correspond to a differential ideal of \(\mathcal{O}_X(U)\).

(c) The maximal ideal \(m_x\) of the local ring \(\mathcal{O}_{X,x}\) is a differential ideal.

(d) The derivation \(\partial\) induces a structure of differential field in quotient field \(\kappa(x)\).

(e) The derivation \(\partial\) restricts to the Zariski closure of \(x\).

### 3.15 Split of Schemes with Derivation

Let \(Z\) be a scheme provided with the zero derivation. Then we will write \(Z\) instead of the pair \((Z, 0)\). Consider a differential field \(\mathcal{K}\) and let \(\mathcal{C}\) be its field of constants.

**Definition 3.31** We say that a \(\mathcal{K}\)-scheme with derivation \((X, \partial)\) splits, if there is a \(\mathcal{C}\)-scheme \(Y\), and an isomorphism

\[
\phi: (X, \partial) \xrightarrow{\sim} Y \times_{\mathcal{C}} (\text{Spec}(\mathcal{K}), \partial),
\]

\(\phi\) is called a splitting isomorphism for \((X, \partial)\).

**Definition 3.32** The space of constants \(\text{Const}(X, \partial)\) is locally ringed space defined as follows: it is the topological subspace of differential points of \(X\), endowed with restriction of the sheaf of constant regular functions.

**Proposition 3.33** Suppose \((X, \partial)\) is Keigher, then

\[
\text{Const}(X, \partial) = \text{Const}(\text{Diff}(X, \partial)).
\]

**Proof.** As topological subspaces of \(X\) they coincide by construction. Let \(X' = \text{Diff}(X, \partial)\). If \(X\) is Keigher then \(\mathcal{O}_{X'}(U) = \lim_{U \subseteq V} \mathcal{O}_X(V)\) (see [9]). And because of that we have,

\[
C \left( \lim_{U \subseteq V} \mathcal{O}_X(V) \right) = \lim_{U \subseteq V} C_{\mathcal{O}_X(V)},
\]

and we finish. \(\Box\)

**Definition 3.34** \((X, \partial)\) is almost-constant if \(\text{Const}(X, \partial)\) is a scheme.
Proposition 3.35 If \((X, \partial)\) splits, then Diff\((X, \partial)\) splits. If \((X, \partial)\) is reduced and split, then it is almost-constant and
\[
(X, \partial) \sim \text{Const}(X, \partial) \times_C (\text{Spec}(K), \partial).
\]

**Proof.** Let us consider the splitting isomorphism \((X, \partial) \to Y \times_C (\text{Spec}(K), \partial)\). It is clear that Diff\((Y \times_C (\text{Spec}(K), \partial)) = Y \times_C \text{DiffSpec}(K)\). Then the above splitting isomorphism induces the splitting isomorphism of the differential scheme Diff\((X, \partial)\). If \(X\) is reduced, then Diff\((X, \partial)\) is also reduced, and then we apply Proposition 3.21. \(\square\)

4 Galois theory of Algebraic Lie-Vessiot Systems

In this chapter we discuss the Galois theory of Lie-Vessiot systems on algebraic homogeneous spaces. The field of functions of the independent variable is here a differential field \(K\) of characteristic zero and with a field of constants \(C\) that we assume to be algebraically closed. We modelize algebraic Lie-Vessiot systems with coefficients in \(K\) as certain \(K\)-schemes with derivation. We study the general solution of algebraic Lie-Vessiot systems. It means that we study the differential extensions of \(K\) that allow us to split the Lie-Vessiot system, and the associated automorphic system. We find that they are strongly normal extensions in the sense of Kolchin [13], and then we can apply Kovacic’s approach to Kolchin’s differential Galois theory. In fact, the Galois theory presented here should be seen as a generalization of the classical Picard-Vessiot theory, obtained by replacing the general linear group by an arbitrary algebraic group. However, the particular case of Picard-Vessiot theory contains all obstructions to solvability, because the non-linear part of an algebraic group over \(C\) is an abelian variety: abelian groups do not give obstruction to integration by quadratures.

4.1 Differential Algebraic Dynamical Systems

Here we establish a parallelism between dynamical systems and differential algebraic terminology. **From now on let us consider a differential field \(K\), and \(C\) its field of constants. We assume that \(C\) is algebraically closed and of characteristic zero.** We modelize non-autonomous dynamical systems as schemes with derivation. The phase space is an algebraic variety \(M\) over the constant field \(C\), and the extended phase space is \(M_K = M \times_C \text{Spec}(K)\). Therefore, non-autonomous dynamical system on \(M\) with coefficients in \(K\) is a derivation on \(M_K\).

**Definition 4.1** A differential algebraic dynamical system is a \(K\)-scheme with derivation \((M, \partial_M)\) such that \(M\) is an algebraic variety over \(K\). We say that \((M, \partial_M)\) is non-autonomous if \(K\) is a non-constant differential field.
There is a huge class of dynamical systems that can be seen as differential algebraic dynamical systems, as polynomial or meromorphic vector fields. It includes Lie-Vessiot systems in algebraic homogeneous spaces, hence it also includes systems of linear differential equations. Furthermore, a differential algebraic study of a dynamical system is suitable in the most general case, but results depend on the choice of an adequate differential field $\mathcal{K}$.

For a differential algebraic dynamical system $(M, \partial M)$ we have the associated differential scheme $\text{Diff}(M, \partial M)$. As a topological space this differential scheme is the set of all irreducible algebraic invariant subsets of the dynamical system. By algebraic, we mean that they are objects defined by algebraic equations with coefficients in $\mathcal{K}$.

Let us recall that for a $\mathcal{K}$-algebra $L$ we denote by $M(L)$ the set of $L$-points of $M$. This sets consist of all the morphisms of $\mathcal{K}$-schemes from $\text{Spec}(L)$ to $M$, or equivalently, of all the rational points of the extended scheme

$$M_L = M \times_{\mathcal{K}} \text{Spec}L.$$ 

**Definition 4.2** Let $(M, \partial M)$ be a $\mathcal{K}$-scheme with derivation. We call rational solution of $(M, \partial M)$ any rational differential point $x \in \text{Diff}(M, \partial M)$. Let us consider a differential extension $\mathcal{K} \subset L$. A solution with coefficients in $L$ is an $L$-point $x \in M(L)$ such that the morphism

$$x: (\text{Spec}(L), \partial) \to (M, \partial M),$$

is a morphism of schemes with derivation. In such a case the image $x(0) = \bar{x}$ of the ideal $(0) \subset L$ by $x$ is a differential point $\bar{x} \in \text{Diff}(M, \partial M)$ and its quotient field $\kappa(\bar{x})$ is an intermediate extension,

$$\mathcal{K} \subset \kappa(\bar{x}) \subset L,$$

we say that $\kappa(\bar{x})$ is the differential field generated by $x \in M(L)$.

As in classical algebraic geometry, there is a one-to-one correspondence between solutions with coefficients in $L$ of $(M, \partial M)$ and rational solutions of the differential algebraic dynamical system after a base change, $(M, \partial M) \times_{\mathcal{K}} (\text{Spec}(L), \partial)$.

**Definition 4.3** Let us consider two differential algebraic dynamical systems over $\mathcal{K}$, $(M, \partial)$ and $(N, \partial)$. We say that $(M, \partial)$ reduces to $(N, \partial)$ if there is an algebraic variety $Z$ over $\mathcal{C}$ and,

$$(M, \partial) = (N, \partial) \times_{\mathcal{C}} Z.$$  

The notion of reduction is a generalization of the notion of split. In particular, to split means reduction to $(\text{Spec}(\mathcal{K}), \partial)$.

Given a differential algebraic dynamical system; what does it mean to integrate the dynamical system? As algebraists, we shall use this term for writing
down the general solution of the dynamical system by terms of known operations, mainly algebraic operations and quadratures. However, in the general context of dynamical systems there is not a general definition for integrability.

We are tempted to say that integrability is equivalent to split. Notwithstanding, there are several situations in which the general solution can be given, but there is not a situation of split. For example, algebraically completely integrable Hamiltonian systems [1]. In such cases the flux is tangent to a global lagrangian bundle, and the generic fibers of this bundle are affine subsets of abelian varieties. It allows us to write down the global solution by terms of Riemann theta functions and Jacobi’s inversion problem. However, this general solution can not be expressed in terms of the splitting of a scheme with derivation.

Split is the differential algebraic equivalent to Lie’s canonical form of a vector field. The scheme with derivation $Z \times_C (\text{Spec}(K), \partial)$ should be seen as an extended phase space, and $\partial$ as the derivative with respect to the time parameter. The splitting morphism,

$$(M, \partial) \rightarrow Z \times_C (\text{Spec}(K), \partial),$$

can be seen as Lie’s canonical form, usually referred to, in dynamical system argot, as the flux box reduction. Then $Z$ is simultaneously the algebraic variety of initial conditions, and the space of global solutions of the dynamical system. Our conclusion is that the split differential algebraic dynamical systems are characterized by following the property: its space of solutions is parameterized by a scheme over the constants.

In the context of algebraic Lie-Vessiot systems we will see that algebraic solvability of the problem, is equivalent to the notion of split (Theorem 4.19). And then, this notion plays a fundamental role in our theory. We will see that generically, a Lie-Vessiot equation does not split. If we want to solve it, then we need to admit some new functions by means of a differential extension of $K \subset L$. Thus, the dynamical system splits after a base change to $L$. The Galois theory will provide us with the techniques for obtaining such extensions and studying their algebraic properties (Proposition 4.24).

4.2 Algebraic Lie-Vessiot Systems

From now on we will consider a fixed characteristic zero differential field $K$ whose field of constants $C$ is algebraically closed. Let $G$ be a $C$-algebraic group, and $M$ a faithful homogeneous $G$-space.

**Definition 4.4** A non-autonomous algebraic vector field $\vec{X}$ in $M$ with coefficients in $K$ is an element of the vector space $\mathfrak{X}(M) \otimes_C K$.

A non-autonomous algebraic vector field $\vec{X}$ in $M$ is written in the form,

$$\vec{X} = \sum_{i=1}^{s} f_i \vec{X}_i,$$
for certain elements $f_i \in \mathcal{K}$ and $X_i \in \mathfrak{X}(M)$. We define the derivation $\partial_X$ associated to $X$ as the following derivation of the extended scheme $M_K$:

$$\partial_X : \mathcal{K} \otimes \mathcal{C} \mathcal{O}_M \to \mathcal{K} \otimes \mathcal{C} \mathcal{O}_M, \quad a \otimes f \mapsto \partial a \otimes f + \sum_{i=1}^s (af_i \otimes X_i f).$$

**Definition 4.5** A non-autonomous algebraic vector field $X$ in $M$ with coefficients in $\mathcal{K}$ is called a Lie-Vessiot vector field if belongs to $\mathcal{R}(G,M) \otimes \mathcal{K}$. The differential algebraic dynamical system $(M_K, \partial_X)$ is called a Lie-Vessiot system in $M$ with coefficients in $\mathcal{K}$.

The group $G$ is, in particular, a faithful homogeneous $G$-space. Let us recall that the Lie algebra of fundamental fields on the group $G$ coincides with the Lie algebra of right invariant vector field $\mathcal{R}(G)$. Then, a Lie-Vessiot vector field in $G$ with coefficients in $\mathcal{K}$ is an element of $\mathcal{R}(G) \otimes \mathcal{C} \mathcal{K}$.

**Definition 4.6** We call automorphic vector fields to the Lie-Vessiot vector fields in $G$. An automorphic vector field $\tilde{A}$ in $G$ with coefficients in $\mathcal{K}$ is an element of $\mathcal{R}(G) \otimes \mathcal{C} \mathcal{K}$.

The canonical isomorphism between $\mathcal{R}(G)$ and $\mathcal{R}(G,M)$ allows us to translate Lie-Vessiot vector fields in $M$ to automorphic vector fields in $G$.

**Definition 4.7** We call automorphic system associated to $(M, \partial_X)$ to the Lie-Vessiot system $(G_K, \partial_X)$, where $\tilde{A}$ is the automorphic vector field whose corresponding Lie-Vessiot vector field in $M$ is $X$.

From now on let $X$ be a Lie-Vessiot vector field in $M$, with coefficients in $\mathcal{K}$, and let $\tilde{A}$ be the associated automorphic vector field in $G$.

### 4.3 Logarithmic Derivative

A $\mathcal{K}$-point of the algebraic group $G$ has coefficients in a differential field, so that it can be differentiated. The derivative of a $\mathcal{K}$-point of $G$ gives a tangent vector at a $\mathcal{K}$-point of $G_K$. If we translate this tangent vector to a right invariant vector field, we obtain the logarithmic derivative. In order to do so we identify systematically the Lie algebra $\mathcal{R}(G)$ with the tangent space $T_e G = \text{Der}_\mathcal{C}(\mathcal{O}_{G,e}, \mathcal{C})$. It is also important to remark that the tangent space is compatible with extensions of the base field in the following way:

$$\mathcal{R}(G) \otimes \mathcal{C} \mathcal{K} \sim \sim\sim T_e (G_K) = \text{Der}_\mathcal{K}(\mathcal{O}_{G_K,e}, \mathcal{K}).$$

In classical algebraic geometry it is assumed that derivations of $T_e (G_K)$ vanish on $\mathcal{K}$. However, automorphic systems are by definition compatible with the derivation $\partial$ of $\mathcal{K}$. Thus, the restriction of an automorphic vector field $\partial \tilde{A}$
to \( e \in G_K \) is not a tangent vector of \( T_e(G_K) \): it is shifted by \( \partial \). We have identifications of \( K \)-vector spaces:

\[
\mathcal{R}(G) \otimes_K K \xrightarrow{\sim} \mathcal{R}(G) \otimes_K K + \partial \xrightarrow{\sim} T_e(G_K)
\]


Let us consider \( \sigma \in G(K) \) and the canonical morphism \( \sigma^\#: \mathcal{O}_{G_K, \sigma} \to K, f \mapsto f(\sigma) \).

Let us remember that there is a canonical form of extension of the derivation \( \partial \) in \( K \) to a derivation in \( G_K \). We consider the direct product \( G \times_K (\text{Spec}(K), \partial) \) in the category of schemes with derivation. By abuse of notation we denote by \( \partial \) this canonical derivation in \( G_K \). By construction we have that \((G_K, \partial)\) splits – the identity is the splitting morphism – and \( \text{Const}(G_K, \partial) = G \). Let us consider the following non-commutative diagram,

\[
\begin{array}{ccc}
\mathcal{O}_{G_K, \sigma} & \xrightarrow{\sigma^\#} & K \\
\partial \downarrow & & \downarrow \partial \\
\mathcal{O}_{G_K, \sigma} & \xrightarrow{\sigma^\#} & K
\end{array}
\]

**Lemma 4.8** The commutator \( \sigma' = [\partial, \sigma^\#] \) of the diagram (4.1) is a derivation vanishing on \( K \), and then \( \sigma' \) belong to the tangent space \( T_\sigma(G_K) \) (id est, the space of derivations \( \text{Der}_K(\mathcal{O}_{G_K, \sigma}, K) \)).

**Proof.** \([\partial, \sigma^\#]\) is the difference between two derivations, and then it is a derivation. Let us consider \( f \in K \subset \mathcal{O}_{G_K, \sigma} \), then \( \sigma'(f) = \partial f - \partial f = 0 \). \( \square \)

If \( \sigma \) is a geometric point of \( G_K \), then \( R_{\sigma^{-1}} \) is an automorphism of \( G_K \) sending \( \sigma \) to \( e \). It induces an isomorphism between the ring of germs \( \mathcal{O}_{G_K, \sigma} \) and \( \mathcal{O}_{G_K, e} \), and then an isomorphisms between the corresponding spaces of derivations:

\[
T_\sigma(G_K) \xrightarrow{R_{\sigma^{-1}}} T_e(G_K) \simeq \mathcal{R}(G) \otimes_K K
\]

**Definition 4.9** Let \( \sigma \) be a geometric point of \( G_K \); we call logarithmic derivative of \( \sigma \), \( l\partial(\sigma) \), to the automorphic vector field \( R_{\sigma^{-1}}([\partial, \sigma^\#]) \). The logarithmic derivative is then a map:

\[
l\partial: G(K) \to \mathcal{R}(G) \otimes_K K.
\]

**Proposition 4.10** Properties of logarithmic derivative:
(1) Logarithmic derivative is functorial in \( K \); for each differential extension \( K \subset L \) we have a commutative diagram:

\[
\begin{array}{ccc}
G(K) & \longrightarrow & R(G) \otimes_C K \\
\downarrow & & \downarrow \\
G(L) & \longrightarrow & R(G) \otimes_C L
\end{array}
\]

(2) Let us consider \( \sigma \) and \( \tau \) in \( G(K) \):

\[
l\partial(\sigma\tau) = l\partial(\sigma) + \text{Adj}_\sigma(l\partial(\tau))
\]

(3) Let us consider \( \sigma \in G(K) \):

\[
l\partial(\sigma^{-1}) = -\text{Adj}_\sigma(l\partial(\sigma)).
\]

**Proof.** (1) comes directly from the differential field extension, (2) comes from the right invariance, and (3) is corollary to (2). □

### 4.4 Automorphic Equation

**Theorem 4.11** Let us consider \( K \subset L \) a differential extension. Then \( \sigma \in G(L) \) is a solution of the differential algebraic dynamical system \((G_K, \partial)\) if and only if \( l\partial(\sigma) = \bar{A} \).

**Proof.** Let us consider \( \sigma \in G(L) \), and let \( \bar{B} \) be its logarithmic derivative. The space \( R(G) \otimes_C L \) is canonically identified with the Lie algebra of right invariant vector fields on the base extended \( L \)-algebraic group \( G_L \):

\[
R(G) \otimes_C L = R(G_L).
\]

By this identification, the automorphic vector field \( \bar{B} \) is seen as a derivation \( \bar{B} \) of the structure sheaf \( O_{G_L} \). The germ \( \bar{B}(\sigma) \) at \( \sigma \) of \( \bar{B} \) is a derivation of the ring \( O_{G_L,\sigma} \). The composition with \( \sigma^2 \) give us the tangent vector \( \bar{B}_\sigma \in T_\sigma(G_L) \):

\[
\begin{array}{ccc}
O_{G_L,\sigma} & \xrightarrow{\bar{B}(\sigma)} & O_{G_L,\sigma} \\
\downarrow & & \downarrow \sigma^2 \\
K & \xrightarrow{\bar{B}_\sigma} & \mathcal{K}
\end{array}
\]

The value of \( \bar{B} \) at the identity point is, by definition, \( l\partial(\sigma) \). Since \( \bar{B} \) is a right invariant vector field we have \( l\partial(\sigma) = R_{\sigma^{-1}}(B) = \sigma^2 \circ \bar{B}(\sigma) \circ R_{\sigma^{-1}}^2 \) hence
\( \vec{B}_\sigma \) is equal to the commutator \([\partial, \sigma^\sharp]\) of Definition 4.9. Then, \( \vec{B}(\sigma) \) is the defect of the diagram (4.1); therefore the following diagram commutes:

\[
\begin{array}{c}
\mathcal{O}_{G_\mathcal{K},\sigma} \xrightarrow{\sigma^\sharp} \mathcal{K} \\
\downarrow \partial + \vec{B}(\sigma) \quad \downarrow \partial \\
\mathcal{O}_{G_\mathcal{K},\sigma} \xrightarrow{\sigma^\sharp} \mathcal{K}
\end{array}
\]

Furthermore, \( \vec{B} \) is determined by the commutator \( \vec{B}_\sigma = [\partial, \sigma^\sharp] \) and then it is unique right invariant vector field in \( G_\mathcal{L} \) that forces the diagram to commute.

Let us note that the commutation of the above diagram holds if and only if the kernel \( m_\sigma \) of \( \sigma^\sharp \) is a differential ideal. Then \( \vec{B} \) is the unique right invariant vector field in \( G_\mathcal{L} \) such that the maximal ideal \( m_\sigma \) is a differential ideal. Let us note also that, this derivation \( \partial + \vec{B}_\sigma \) is the germ in \( \sigma \) of the automorphic derivation

\[
\partial_{\vec{B}} = \partial + \vec{B},
\]

we conclude that \( \vec{B} \), the logarithmic derivative of \( \sigma \), is the unique element of \( \mathcal{R}(G) \otimes_{\mathcal{L}} \mathcal{L} \) such that \( \sigma \) is a differential point of \( (G_\mathcal{L}, \partial_{\vec{B}}) \). □

Because of that we can substitute the automorphic system \( \vec{A} \), for the so-called automorphic equation:

\[
\lambda \partial(x) = \vec{A} \quad (4.2)
\]

### 4.5 Solving Lie-Vessiot Systems

**Definition 4.12** Let us consider \( \sigma \in G(\mathcal{K}) \). We call gauge transformation induced by \( \sigma \) to the left translation \( L_{\sigma} : G_\mathcal{K} \to G_\mathcal{K} \).

**Lemma 4.13** \((G_\mathcal{K}, \partial_\mathcal{K})\) splits if and only if the automorphic equation \( \circ \) has at least one solution in \( G(\mathcal{K}) \).

**Proof.** Assume \((G_\mathcal{K}, \partial_\mathcal{K})\) splits. Let us consider the splitting isomorphism

\[
\psi : (G_\mathcal{K}, \partial_\mathcal{K}) \to Z \times_\mathcal{L} (\text{Spec}(\mathcal{K}), \partial)
\]

Let \( x \) be a \( \mathcal{L} \)-rational point of \( Z \). Let us denote by \( x_\mathcal{K} \) the corresponding \( \mathcal{K} \)-point of \( G_\mathcal{K} \) obtained after the extension of the base field. Thus, \( \psi^{-1}(x_\mathcal{K}) \) is a solution of \( \circ \). Reciprocally, let us assume that there exists a solution \( \sigma \) of \( \circ \) in \( G(\mathcal{K}) \). Let us consider the gauge transformation:

\[
L_{\sigma^{-1}} : G_\mathcal{K} \to G_\mathcal{K}.
\]

It applies \( \sigma \) onto the identity element \( e \in G_\mathcal{K} \). But the logarithmic derivative \( \lambda \partial(e) \) vanishes, so that \( L_{\sigma^{-1}} \) transforms \( \partial_\mathcal{K} \) into the canonical derivation \( \partial \). We conclude that \( L_{\sigma^{-1}} \) is an splitting isomorphism. □
Lemma 4.14 Assume that \((G_K, \partial_X)\) splits. In such case we can choose the splitting isomorphism between the gauge transformations of \(G_K\). This gauge transformation induces the split of any associated Lie-Vessiot system \((M_K, \partial_X)\).

Proof. We use the same argument as above. If it splits,

\[
s: (G_K, \partial_X) \to G \times_C (\text{Spec}(K), \partial) = (G, \partial),
\]

then the preimage of the identity element \(s^{-1}(e) = \sigma\) is a solution of the automorphic system. So that the gauge transformation \(L_{\sigma^{-1}}: \sigma \mapsto e\) maps solutions of \((G_K, \partial_X)\) to solutions of \((G, \partial)\) and it is an splitting isomorphism. For any associated Lie-Vessiot system \((M_K, \partial_X)\), and any point \(x_0 \in M(C)\) we have that \(L_{\sigma}(x_0)\) is a solution of \((M_K, \partial_X)\). So that \(L_{\sigma}\) sends solutions of the canonical derivation \(\partial\) to solutions of \(\partial_X\). Thus, its inverse \(L_{\sigma^{-1}}\) is an splitting isomorphism for \((M_K, \partial_X)\). \(\square\)

Lemma 4.15 Let \(Z\) be a \(C\)-algebraic variety and \((Z_K, \vec{D})\) a non-autonomous differential algebraic dynamical system over \(K\). If \((Z_K, \vec{D})\) splits then \((Z_K, \vec{D})\) is almost-constant and \(\text{Const}(Z_K, \vec{D}) \simeq Z\).

Proof. Assume that \((Z_K, \vec{D})\) splits. It implies that there exist an \(C\)-scheme \(Y\), such that \(Z_K = Y \times_C \text{Spec}(K)\). We have that \(Z_K \simeq Y_K\), and then \(Z \simeq Y\). \(\square\)

Lemma 4.16 Let \(Z\) be a reduced \(C\)-scheme. There is a one-to-one correspondence between closed subschemes of \(Z\) and closed subschemes with derivation of \((Z_K, \partial) = Z \times_C (\text{Spec}(K), \partial)\).

Proof. First, let us consider the affine case. Assume \(Z = \text{Spec} R\) for a \(C\)-algebra \(R\). The ring of constants \(C_R \otimes_C K\) is \(R\) itself. It follows that \(\text{Const}(Z_K, \partial) = Z\). It is clear that \(R \otimes_C K\) is an almost-constant ring: each radical differential ideal is generated by constants. Because of that there is an one-to-one correspondence between radical ideals of \(R\) and radical differential ideals of \(K\).

In the non-affine case, let us consider \(Y\) a closed sub-\(C\)-scheme of \(Z\). The canonical immersion \((Y_K, \partial) \subset (Z_K, \partial)\) identifies \(Y\) with a closed sub-\(K\)-scheme with derivation of \((Z_K, \partial)\). Reciprocally, let \((Y, \partial|_Y)\) be a closed sub-\(K\)-scheme with derivation of \((Z_K, \partial)\). Let us consider \(\{U_i\}_{i \in A}\) an affine covering of \(Z\). The collection \(\{V_i\}_{i \in A}\) with \(V_i = U_i \times_C K\) is then an affine covering of \(Z_K\). Each intersection \(Y_i = Y|_{V_i}\) is an affine closed sub-\(K\)-scheme of \(V_i\). We are in the affine case: by the above argument there are closed sub-\(C\)-schemes \(Y_i \subset U_i\) such that \((Y_i, \partial|_{Y_i}) = Y_i \times_C (\text{Spec}(K), \partial)\). This family defines a covering of a closed sub-\(C\)-scheme \(Y = \bigcup_{i \in A} Y_i\) of \(Z\). \(\square\)

Lemma 4.17 Let \(Z\) be a \(C\)-algebraic variety and \((Z_K, \vec{D})\) a non autonomous algebraic dynamical system over \(K\). Let \(Y \subset Z\) a locally closed subvariety, and assume that \(\vec{D}\) is tangent to \(Y\), so that \((Y_K, \vec{D}|_Y)\) is a sub-\(K\)-scheme with derivation. If \((Z_K, \vec{D})\) splits then \((Y_K, \vec{D}|_Y)\) splits.
Proof. By substituting $Z$ for certain open subset we can assume that $Y$ is closed. Let us consider the splitting isomorphism,

$$
\psi: (Z_K, \mathcal{D}) \rightarrow Z \times_C (\text{Spec}(K), \partial).
$$

The image $\psi(Y_K, \mathcal{D}|_Y)$ is a locally closed subscheme with derivation of $Z \times_C (\text{Spec}(K), \partial)$. By Lemma 4.10 it splits. □

Lemma 4.18 Assume that the action of $G$ on $M$ is faithful. Then $(G_K, \partial_{\mathcal{A}})$ splits if and only if $(M_K, \partial_{\mathcal{X}})$ splits.

Proof. Lemma 4.14 says that if $(G_K, \partial_{\mathcal{A}})$ splits, then $(M_K, \partial_{\mathcal{X}})$ splits. Reciprocally, let us assume that $(M_K, \partial_{\mathcal{X}})$ splits. For each positive number $r$ we consider the natural lifting to the cartesian power $(M_K^r, \partial_{\mathcal{X}}^r)$. The splitting of $(M_K, \partial_{\mathcal{X}})$ induces the splitting of those cartesian powers differential algebraic dynamical system $(M_K^r, \partial_{\mathcal{X}}^r)$. For $r$ big enough there is a point $x \in M^r$ such that its orbit $O_\epsilon$ is a principal homogeneous space isomorphic to $G$. Then $(O_{x,K}, \partial_{\mathcal{X}})$ is a locally closed sub-$K$-scheme with derivation of $(M_K^r, \partial_{\mathcal{X}}^r)$. By Lemma 4.17 it splits. We also know that $(O_{x,K}, \partial_{\mathcal{X}})$ is isomorphic to $(G_K, \partial_{\mathcal{A}})$. Finally, $(G_K, \partial_{\mathcal{A}})$ splits. □

Theorem 4.19 Assume that the action of $G$ on $M$ is faithful. Then the following are equivalent.

1. The automorphic equation (4.2) has a solution in $G(K)$
2. $(G_K, \partial_{\mathcal{A}})$ splits.
3. There is a gauge transformation of $G_K$ sending $\mathcal{A}$ to 0.
4. $(M_K, \partial_{\mathcal{X}})$ splits.
5. $(G_K, \partial_{\mathcal{A}})$ splits, is almost-constant, and $\text{Const}(G_K, \partial_{\mathcal{A}}) \cong G$.
6. $(M_K, \partial_{\mathcal{X}})$ splits, is almost-constant, and $\text{Const}(M_K, \partial_{\mathcal{X}}) \cong M$.

Proof. Equivalence between (1) and (2) comes from Lemma 4.13. Equivalence between (2) and (3) comes from Lemma 4.14. (2) and (4) are equivalent by Lemma 4.15. By Lemma 4.14 they all imply (5) and (6). □

4.6 Splitting Field of an Automorphic System

Note that a differential extension $K \subset L$, induces a canonical inclusion,

$$
\mathcal{R}(G, M) \otimes_C K \subset \mathcal{R}(G, M) \otimes_C L;
$$

so that a Lie-Vessiot vector field with coefficients in $K$ is a particular case of a Lie-Vessiot vector field with coefficients in $L$. So that if $(M_K, \partial_{\mathcal{X}})$ is a Lie-Vessiot system, then $(M_K, \partial_{\mathcal{X}})$ makes sense.
Definition 4.20  We say that a differential extension $K \subset L$ is a splitting extension for $(M_K, \partial_{\vec{X}})$ if $(M_L, \partial_{\vec{X}})$ splits.

From theorem 4.19 we know that $K \subset L$ is a splitting extension of $(M_K, \partial_{\vec{X}})$ if and only it is a splitting extension of $(G_K, \partial_{\vec{A}})$. Then we will center our attention in the automorphic vector field $\vec{A}$.

4.7 Action of $G(C)$ on $G_K$

For each $\sigma \in G(C)$, $R_\sigma$ is an automorphism of $G_K$. The composition law is an action of $G$ on $G_K$ by the right side,

$$G_K \times_C G \to G_K.$$  

The vector field $\vec{A}$ is right invariant, so that we expect the differential points of $(G_K, \partial_{\vec{A}})$ to be invariant under right translations. In fact, the above morphism is a morphism of schemes with derivation,

$$(G_K, \partial_{\vec{A}}) \times_C G \to (G_K, \partial_{\vec{A}}).$$

We apply the functor Diff, and then we obtain an action of the $C$-algebraic group $G$ on the differential scheme $\text{Diff}(G_K, \partial_{\vec{A}})$,

$$\text{Diff}(G_K, \partial_{\vec{A}}) \times_C G \to \text{Diff}(G_K, \partial_{\vec{A}}).$$

Assume that $(G_K, \partial_{\vec{A}})$ split. In such case, when we apply the functor Const to the previous morphism, we obtain a morphism of schemes,

$$\text{Const}(G_K, \partial_{\vec{A}}) \times_C G \to \text{Const}(G_K, \partial_{\vec{A}}).$$

Because of the split we already knew that $\text{Const}(G_K, \partial_{\vec{A}})$ is a $C$-scheme isomorphic to $G$. Furthermore, the above morphism says that the action of $G$ by the right side on this $G$-scheme is canonical. We have proven the following:

Lemma 4.21  Assume that $(G_K, \partial_{\vec{A}})$ splits. Then $\text{Const}(G_K, \partial_{\vec{A}})$ is a principal $G$-homogeneous space by the right side.

4.8 Existence and Uniqueness of the Splitting Field

Lemma 4.22  There is a differential point $p \in \text{Diff}(G_K, \partial_{\vec{A}})$ which is closed in the Kolchin topology.

Proof. Let us consider the generic point $p_0 \in G_K$. In particular it is a differential point $p_0 \in \text{Diff}(G_K, \partial_{\vec{A}})$. If $p_0$ is Kolchin closed, then we finish and the result holds. If not, then the Kolchin closure of $p_0$ contains a differential point point $p_1$ such that $p_0$ specializes on it $p_0 \to p_1$. We continue this process with $p_1$. As $G_K$ is an algebraic variety, and then a noetherian scheme, this process finish in a finite number of steps and lead us to a Kolchin closed point. □
Lemma 4.23 Let \( \mathfrak{x} \in \text{Diff}(G_K, \partial_{\vec{A}}) \) be a closed differential point. Then its field of quotients \( \kappa(\mathfrak{x}) \) is a differential extension of \( K \) with the same field of constants; \( C_{\kappa(\mathfrak{x})} = C \).

Proof. Reasoning by \textit{reductio ad absurdum} let us assume that there exists \( c \in C_{\kappa(\mathfrak{x})} \) not in \( C \). Let us consider an affine open neighborhood \( U \) of \( \mathfrak{x} \) and denote by \( A \) its ring of regular functions. We identify \( \mathfrak{x} \) with a maximal differential ideal \( \mathfrak{x} \subset A \). Denote by \( B \) the quotient ring \( A/\mathfrak{x} \). \( B \) is a differential subring of the differential field \( \kappa(\mathfrak{x}) \). By Lemma 3.5 there exist \( b \in B \) such that the ring constants \( C_Bb \) of the localized ring \( B_{b} \) is a finitely generated \( C \)-algebra. By reducing our original neighborhood \( U \) – removing the zeros of \( b \) – we can assume that \( b \) is invertible and then the localized ring \( B_{b} \) is just \( B \). \( C_B \) is a non-trivial finitely generated \( C \)-algebra over \( C \), because it contains an element \( c \) not in \( C \). So that there is a non-invertible element \( c_2 \in C_B \). The principal ideal \( (c_2) \) is a non trivial differential ideal in \( B \). Let us consider a regular function \( a_2 \) such that \( a_2(\mathfrak{x}) = c_2 \). Then \( b\partial_{\vec{A}}a_2 \in \mathfrak{x} \) and \( (a, \mathfrak{x}) \) is a non-trivial differential ideal of \( A \) strictly containing \( \mathfrak{x} \). We arrive to contradiction with the maximality of \( \mathfrak{x} \). \( \square \)

Proposition 4.24 Let \( \mathfrak{x} \in \text{Diff}(G_K, \partial_{\vec{A}}) \) be a closed point. Then \( K \subset \kappa(\mathfrak{x}) \) is a splitting extension of \( (G_K, \vec{A}) \).

Proof. Let \( \mathfrak{x} \) be a closed point. Then the canonical morphism \( \mathfrak{r}^t \) of taking values in \( \mathfrak{x} \), \( \mathfrak{r}^t: O_{G_K, \mathfrak{x}} \rightarrow \kappa(\mathfrak{x}) \) is a morphism of differential rings. Let \( U \) be an affine neighborhood of the image of \( \pi(\mathfrak{x}) \) by the canonical projection \( \pi: G_K \rightarrow G \). By composition we construct a morphism \( \text{Spec}(\kappa(\mathfrak{x})) \rightarrow \mathfrak{r}^t : O_{G_{K, \mathfrak{x}}} \rightarrow \kappa(\mathfrak{x}) \).

The morphism \( \mathfrak{r}^t \) is the dual of a morphisms \( \sigma \) from \( \text{Spec}(\kappa(\sigma)) \) to \( U \). In other words, \( \sigma \) is a point of \( G(\kappa(\mathfrak{x})) \). We consider \( \sigma \) as a rational differential point of \( (G_{\kappa(\mathfrak{x})}, \partial_{\vec{A}}) \), and then it is a solution of the automorphic equation. By Lemma 4.13 \((G_{\kappa(\mathfrak{x})}, \partial_{\vec{A}}) \) splits. \( \square \)

Definition 4.25 We say that \( \sigma \), as defined in the above proof, is the fundamental solution of \( \vec{A} \) associated with the closed differential point \( \mathfrak{x} \).

Let us consider the action of \( G \) on \( G_K \) by right translations. The derivation \( \partial_{\vec{A}} \) is invariant by right translations, and then it is a morphism of schemes with derivation:

\[(G_K, \partial_{\vec{A}}) \times_C G \rightarrow (G_K, \partial_{\vec{A}})\]

We apply the functor \text{Diff}, thus we obtain a morphism of differential schemes which is an algebraic action of \( G \) on the set of differential points:

\[\text{Diff}(G_K, \partial_{\vec{A}}) \times_C G \rightarrow \text{Diff}(G_K, \partial_{\vec{A}})\]

29
Proposition 4.26  The action of $G(\mathbb{C})$ on the set of closed points of $\text{Diff}(G_K, \partial \vec{A})$ is transitive.

Proof. Let us consider a Kolchin closed point $x \in \text{Diff}(G_K, \partial \vec{A})$. Let $\mathcal{L}$ be the rational field of $x$. It is an splitting field for $(G_K, \partial \vec{A})$. We have that $(G_L, \partial \vec{A})$ splits, hence $\text{Diff}(G_L, \partial \vec{A})$ is an almost-constant differential scheme. Thus $\text{Diff}(G_L, \partial \vec{A})$ is homeomorphic to the principal homogeneous $G$-space $\text{Const}(G_L, \partial \vec{A})$.

The differential extension $K \subset L$ induces a commutative diagram of schemes, 

\[
\begin{array}{ccc}
(G_L, \partial \vec{A}) \times _G \mathbb{C} & \rightarrow & (G_L, \partial \vec{A}) \\
\downarrow & & \downarrow \pi_1 \\
(G_K, \partial \vec{A}) \times _G \mathbb{C} & \rightarrow & (G_K, \partial \vec{A})
\end{array}
\]

and thus, a commutative diagram of differential schemes, 

\[
\begin{array}{ccc}
\text{Diff}(G_L, \partial \vec{A}) \times _G \mathbb{C} & \rightarrow & \text{Diff}(G_L, \partial \vec{A}) \\
\downarrow & & \downarrow \pi_2 \\
\text{Diff}(G_K, \partial \vec{A}) \times _G \mathbb{C} & \rightarrow & \text{Diff}(G_K, \partial \vec{A})
\end{array}
\]

Let $s$ be a Kolchin closed point of $\text{Diff}(G_K, \partial \vec{A})$. The projection $\pi_2$ of the above diagram is exhaustive. Consider any $p \in \pi_2^{-1}(s)$, and let us consider a Kolchin closed point $x$ in the closure $\{p\}$. Thus, $\pi_2(x)$ is in the closure $\{s\}$. As $s$ is a Kolchin closed point we know that $\pi_2(x) = s$. Hence, there is a Kolchin closed point $x \in \text{Diff}(G_K, \partial \vec{A})$ such that $\pi_2(x) = s$.

Consider two Kolchin closed points $s, \eta \in \text{Diff}(G_K, \partial \vec{A})$. Because of the above argument there are two Kolchin closed points $x, y \in \text{Diff}(G_L, \partial \vec{A})$ such that $\pi_2(x) = s$ and $\pi_2(y) = \eta$. The set of Kolchin closed points of $\text{Diff}(G_L, \partial \vec{A})$ is a $G(\mathbb{C})$-homogeneous space in the set theoretical sense. Then there is $\sigma \in G(\mathbb{C})$ such that $x \cdot \sigma = y$, and by the commutativity of the diagram we have $s \cdot \sigma = \eta$.

\[\square\]

Corollary 4.27  Let $\eta$ and $\eta$ be two closed points of $\text{Diff}(G_K, \partial \vec{A})$. Then there exists an invertible $\mathcal{K}$-isomorphism of differential fields $\kappa(\eta) \simeq \kappa(\eta)$.

Proof. There is a closed point $\sigma \in G$, such that $\eta \cdot \sigma = \eta$. Then

\[R_\sigma : (G_K, \partial \vec{A}) \rightarrow (G_K, \partial \vec{A})\]

is an automorphism that maps $\eta$ to $\eta$. Then it induces an invertible $\mathcal{K}$-isomorphism

\[R_\sigma^*: \kappa(\eta) \rightarrow \kappa(\eta)\]

\[\square\]
**Definition 4.28** For each closed point \( \xi \in \text{Diff}(G_K, \partial_{\vec{A}}) \) we say that the differential extension \( K \subset \kappa(\xi) \) is a Galois extension associated to the non-autonomous differential algebraic dynamical system \((G_K, \partial_{\vec{A}})\).

**Notation.** As we have proven, all Galois extensions associated to \((G_K, \partial_{\vec{A}})\) are isomorphic. From now on let us choose a closed point \( \xi \) and denote by \( K \subset L \) its corresponding Galois extension.

**Proposition 4.29** A Galois extension is a minimal splitting extension for \((G_K, \partial_{\vec{A}})\) in the following sense: If \( K \subset S \) is any splitting extension for \((G_K, \partial_{\vec{A}})\) then there is a \( K \)-isomorphism of differential fields \( L \cong S \).

**Proof.** If \( K \subset S \) is a splitting extension, then \((G_S, \partial_{\vec{A}})\) splits. Hence, for each Kolchin closed differential point \( x \in \text{Diff}(G_S, \partial_{\vec{A}}) \) the rational field of \( x \) is \( S \). Let us consider the natural projection \( \pi : (G_S, \partial_{\vec{A}}) \to (G_K, \partial_{\vec{A}}) \). We can choose a Kolchin closed point \( x \in \text{Diff}(G_K, \partial_{\vec{A}}) \) such that \( \pi(x) = \xi \). We have a morphism of \( K \)-differential algebras between the corresponding rational fields \( \pi^*: L \to S \).

**Example 4.30 (Picard-Vessiot extensions)** Let us consider system of \( n \) linear differential equations

\[
\partial x = Ax, \quad A \in \text{gl}(n, K),
\]

and let us denote \( a_{ij} \) for the matrix elements of \( A \). The algebraic construction of the Picard-Vessiot extension is done as follows (cf. [19] and [41]):

Let us consider the algebra \( K[u_{ij}, \Delta] \), being \( \Delta = |u_{ij}|^{-1} \) the inverse of the determinant. Note that it is the algebra of regular functions on the affine group \( GL(n, K) \). If is an affine group, and then it is isomorphic to the spectrum

\[
GL(n, K) = \text{Spec}(K[u_{ij}, \Delta]).
\]

We define the following derivation,

\[
\partial_{\vec{A}} u_{ij} = \sum_{k=1}^{n} a_{ik} u_{jk},
\]

that gives to \( K[u_{ij}, \Delta] \) the structure of differential \( K \)-algebra, and to \((GL(n, K), \partial_{\vec{A}})\) the structure of automorphic system. The set of Kolchin closed differential points of \( \text{Diff}(GL(n, K), \partial_{\vec{A}}) \) is the set of maximal differential ideals of \( \mathcal{R} \). A Picard-Vessiot algebra is a quotient algebra \( K \subset K[u_{ij}, \Delta]/\mathfrak{m} \), and a Picard-Vessiot extension is a rational differential field \( K \subset \kappa(\mathfrak{m}) \). It is self-evident that the Picard-Vessiot extension is the particular case of Galois extension when the considered group is the general linear group.

**Lemma 4.31** Let \( K \subset S \) be a splitting extension. The canonical projection

\[
\pi : \text{Diff}(G_S, \partial_{\vec{A}}) \to \text{Diff}(G_K, \partial_{\vec{A}})
\]

is a closed map.
Proof. It is enough to prove that the projection $\eta = \pi(y)$ of a closed point $y \in \text{Diff}(G_S, \partial \vec{A})$ is a closed point. Let us take a closed point $\tilde{y} \in \{\eta\}$. Then $\pi^{-1}(\tilde{y})$ is closed and there is a closed point $z \in \pi^{-1}(\tilde{y})$. $\text{Diff}(G_S, \partial \vec{A})$ is a principal homogeneous $G$-space, there is a $\sigma \in G(C)$ such that $z \cdot \sigma = y$, and then $\tilde{y} \cdot \sigma = \eta$. $G(C)$ acts transitively in the space of closed points, and $\tilde{y}$ is closed, so that we have proven that $\eta$ is closed. In fact $\eta$ and $\tilde{y}$ are the same differential point. \[ \square \]

Proposition 4.32 Let us consider any intermediate differential extension, $K \subset \mathcal{F} \subset S$, with $K \subset S$ an splitting extension. The projection,

$$\pi: \text{Diff}(G_\mathcal{F}, \partial \vec{A}) \to \text{Diff}(G_K, \partial \vec{A}),$$

is a closed map.

Proof. Let us consider the following diagram of projections:

$$
\begin{array}{ccc}
\text{Diff}(G_S, \partial \vec{A}) & \xrightarrow{\pi_1} & \text{Diff}(G_K, \partial \vec{A}) \\
\downarrow{\pi_2} & & \downarrow{\pi} \\
\text{Diff}(G_\mathcal{F}, \partial \vec{A}) & &
\end{array}
$$

By Lemma 4.31 $\pi_1$ and $\pi_2$ are closed and surjective. Then $\pi$ is closed. \[ \square \]

Lemma 4.33 Let $K \subset \mathcal{F} \subset \mathcal{L}$ be an intermediate differential extension of the Galois extension of $(G_K, \partial \vec{A})$, and $\sigma$ the fundamental solution associated to $y$. Let us consider the sequence of base changes,

$$
\begin{array}{ccc}
\text{Diff}(G_\mathcal{L}, \partial \vec{A}) & \xrightarrow{\pi_1} & \text{Diff}(G_\mathcal{F}, \partial \vec{A}) & \xrightarrow{\pi_2} & \text{Diff}(G_K, \partial \vec{A}) \\
\sigma & & \eta & & \gamma ,
\end{array}
$$

then $\eta$ is closed in Kolchin topology, $\kappa(\eta)$ is the Galois extension $\mathcal{L}$ and $\sigma$ is the fundamental solution associated with $\eta$.

Proof. By Proposition 4.31 $\pi_1$ is a closed map, so that $\eta$ is a closed point. The chain of projections induces a chain of differential extensions $\kappa(y) \subseteq \kappa(\eta) \subseteq \kappa(\sigma)$ but $\kappa(y) = \kappa(\sigma)$, and then we have the equality. \[ \square \]

4.9 Galois Group

Here we give a purely geometrical definition for the Galois group associated to a Kolchin closed differential point. We prove strong normality of the Galois extensions, and identify our geometrically-defined Galois group with the group of automorphisms of the Galois extension. Let us consider the action of $G$ on $\text{Diff}(G_K, \partial \vec{A})$ shown in Subsection 4.7

$$
\text{Diff}(G_K, \partial \vec{A}) \times_G G \to \text{Diff}(G_K, \partial \vec{A}).
$$
Definition 4.34 Let \( x \in \text{Diff}(G_K, \partial_{\vec{A}}) \) be a Kolchin closed differential point. We call Galois group of the system \((G_K, \partial_{\vec{A}})\) in \( x \) to the isotropy subgroup of \( x \) in \( G \) by the above action, and denote it by \( \text{Gal}_x(G_K, \partial_{\vec{A}}) \).

Proposition 4.35 \( \text{Gal}_x(G_K, \partial_{\vec{A}}) \) is an algebraic subgroup of \( G \).

Proof. Denote by \( H_x \) the Galois group in \( x \). Let us consider the projection \( \pi_1 : G_K \to G \) induced by the extension \( C \subset K \). Denote by \( x \) the point \( \pi_1(x) \), and let \( U \) be an affine neighborhood of \( x \). Then \( U = G \setminus Y \) with \( Y \) closed in \( G \).

\( U_K \) is an affine neighborhood of \( x \) in \( G_K \). We have that the ring of regular functions in \( U_K \) is the tensor product \( \mathcal{O}_G(U) \otimes C \).

We identify \( x \) with a maximal prime differential ideal \( x \subset \mathcal{O}_G(U) \otimes C \).

Let us consider a \( C \)-point \( \sigma \) of \( G \).

Then, for each \( f \in \mathcal{O}_G(U) \otimes C \) we have that the right translate \( R^\sigma(f) \) is in \( \mathcal{O}_{G \cdot \sigma^{-1}} \otimes C \).

The morphism
\[
\pi_2 : G \to G, \quad \sigma \mapsto R^\sigma(x),
\]
is algebraic, and let \( W \) be the complementary in \( G \) of \( \pi_2^{-1}(Y) \),
\[
W = G \setminus \pi_2^{-1}(Y),
\]
then for each \( f \in \mathcal{O}_G(U) \otimes C \) we have that the right translate \( R^\sigma(f) \) is in \( \mathcal{O}_{G \cdot \sigma^{-1}} \otimes C \).

We will prove that the equations of \( H_x \) in \( W \) are algebraic. Let us consider \( W_1 \) an affine open subset in \( W \). Let \( \{\xi_1, \ldots, \xi_r\} \) be a system of generators of \( \mathcal{O}_G(W) \) as \( C \)-algebra.

The composition is algebraic,
\[
\pi_3 : U \times_C W_1 \to G, \quad (y, \sigma) \mapsto y \cdot \sigma,
\]
and it induces a morphism,
\[
\pi_3^1 : \mathcal{O}_{G,x} \to (\mathcal{O}_G(U) \otimes C \mathcal{O}(W_1))_{\pi_3^{-1}(x)},
\]
and then for each \( f \in \mathcal{O}_{G,x} \), \( \pi_3^1(f) = F(\xi) \), is a rational function in the \( \xi_i \) with coefficients in \( \mathcal{O}_{G,x} \).

We identify \( x \) with a prime ideal of \( \mathcal{O}_G(U) \otimes C \).

We consider a system of generators,
\[
x = (\eta_1, \ldots, \eta_r), \quad \eta_i \in \mathcal{O}_G(U) \otimes C \mathcal{K}.
\]

Property (b) says that by the natural inclusion,
\[
j : \mathcal{O}_G(U) \otimes C \mathcal{K} \to (\mathcal{O}_G(U) \otimes C \mathcal{O}(W_1))_{\pi_3^{-1}(x)} \otimes C \mathcal{K},
\]

33
$j(\mathfrak{r})$ spans a non trivial ideal of $(\mathcal{O}_G(U) \otimes_{\mathcal{C}} \mathcal{O}(W_1))_{\pi_3^{-1}(x)} \otimes_{\mathcal{C}} \mathcal{K}$, and then we have a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{O}_G(U) \otimes_{\mathcal{C}} \mathcal{K} & \xrightarrow{\kappa(\mathfrak{r})} & (\mathcal{O}_G(U) \otimes_{\mathcal{C}} \mathcal{O}(W_1))_{\pi_3^{-1}(x)} \otimes_{\mathcal{C}} \mathcal{K} \\
\downarrow & & \downarrow_{\pi_4} \\
(\kappa(\mathfrak{r}) \otimes_{\mathcal{C}} \mathcal{O}(W_1))_{\pi_3^{-1}(x)} & \xrightarrow{\pi_4} & (\kappa(\mathfrak{r}) \otimes_{\mathcal{C}} \mathcal{O}(W_1))_{\pi_3^{-1}(x)}
\end{array}
\]

An element $\sigma \in W_1$ stabilizes $\mathfrak{r}$ if and only if $R^\sharp_\sigma(\eta_i) \in \mathfrak{r}$, and this is so if and only if $\pi_4(j(\eta_i)) = 0$ for $i = 1, \ldots, r$. Let us consider a basis $\{e_\lambda\}_{\lambda \in \Lambda}$ of $\kappa(\mathfrak{r})$ over $\mathcal{C}$. For each $i$, we have a finite sum:

$\pi_4(j(\eta_i)) = \sum_\alpha G_{i\alpha}(\xi)e_\alpha$, 

and then $G_{i\alpha}(\xi) \in \mathcal{O}(W_1)$ are the algebraic equations of $H_x$ in $W_1$.

**Remark 4.36** Let $\mathfrak{r}$ be a Kolchin closed differential point as above, and $H \subset G$ the Galois group of $(G_K, \partial_{\mathfrak{A}})$ in $\mathfrak{r}$. Then $H_K = H \times_{\mathcal{C}} \text{Spec}(K)$ is the stabilizer subgroup of $\{\mathfrak{r}\}$, the Zariski closure of $\mathfrak{r}$, by the action of composition by the right side:

$G_K \times_K G_K \rightarrow G_K$.

However, the morphisms $R_\sigma$ for $\sigma \in H_K$ are not in general morphisms of schemes with derivation. In the same sense, for any field extension $K \subset L$, $H_L \subset G_L$ is the stabilizer group of $\pi^{-1}(\mathfrak{r})$, the Zariski closure of the preimage of $\mathfrak{r}$, where $\pi$ is the natural projection from $G_L$ to $G_K$. This means that $H_L$ stabilizes the fiber, in the following sense: for each $L$-point $\sigma \in H_L$, $R_\sigma : G_L \rightarrow G_L$ induces,

$R_\sigma|_{\pi^{-1}(\mathfrak{r})} : \pi^{-1}(\mathfrak{r}) \rightarrow \pi^{-1}(\mathfrak{r})$.

**Proposition 4.37** Consider two Kolchin closed differential points $\mathfrak{r}, \eta$ in $\text{Diff}(G_K, \partial_{\mathfrak{A}})$. The groups $\text{Gal}_L(G_K, \partial_{\mathfrak{A}})$ and $\text{Gal}_\eta(G_K, \partial_{\mathfrak{A}})$ are isomorphic conjugated algebraic subgroups of $G$.

**Proof.** The group of $\mathcal{C}$-points of $G$ acts transitively in the set of closed differential points. Hence, there exists $\sigma \in G(\mathcal{C})$ with $\mathfrak{r} \cdot \sigma = \eta$, and then $H_\mathfrak{r} \cdot \sigma = \sigma \cdot H_\eta$.

**Theorem 4.38** The Galois extensions associated to $(G_K, \partial_{\mathfrak{A}})$ are strongly normal extensions.

**Proof.** Let us consider a Galois extension $K \subset L$. Thus, $L$ is the rational field of certain Kolchin closed differential point that we denote by $\mathfrak{r}$. Let us consider $\sigma \in G_L$ the fundamental solution associated to $\mathfrak{r}$. We have that $\sigma$ projects onto $\mathfrak{r}$ and the gauge transformation $L_{\sigma^{-1}}$ is a splitting morphism. We define
the morphism \( \psi \) of schemes with derivation through the following commutative diagram:

\[
\begin{array}{ccc}
(G_L, \partial) & \xrightarrow{\pi} & (G_K, \partial) \\
L \sigma \downarrow & & \downarrow \\
(G_L, \partial) = G \times_C (\text{Spec}(L), \partial)
\end{array}
\]

Denote by \( H \) the Galois group in \( \mathfrak{r} \). We have that \( (H_L, \partial) \subset (G_L, \partial) \) is a closed subscheme with derivation. The group \( H_L \) is the preimage of \( H \) by the projection from \( G_L \) to \( G \). By remark \[4.36\] \( H_L \) is the stabilizer of the \( \pi^{-1}(\mathfrak{r}) \) in \( G_L \). It means that for any point \( z \) of \( G_L \) whose projection is adherent to \( x \) and any \( L \)-point \( \tau \) of \( H_L \), the right translate \( z \cdot \tau \) is also adherent to \( x \). In particular we have that \( \psi(\tau) = x \), and then

\[
(H_L, \partial) \subset \psi^{-1}(x).
\]

Reciprocally, let us consider an \( L \)-point \( \tau \in \psi^{-1}(x) \). Therefore \( \pi(\sigma \cdot \tau) \) is adherent to \( x \). The following diagram is commutative:

\[
\begin{array}{ccc}
G_L \times_C G_L & \xrightarrow{\pi} & G_L \\
\downarrow & & \downarrow \\
G_K \times_K G_K & \rightarrow & G_K
\end{array}
\]

We deduce that, for any other preimage \( \bar{\sigma} \) of \( x \) by \( \pi \), the right translated \( \bar{\sigma} \cdot \tau \) also projects onto \( \{x\} \). Thus, \( \tau \) stabilizes \( \pi^{-1}(x) \), so that \( \tau \in (H_L, \partial) \). Finally we have the identity:

\[
\psi^{-1}(x) = (H_L, \partial) = H \times_C (\text{Spec}(L), \partial).
\]

On the other hand we apply the affine stalk formula (Proposition \[A.4\], that comes from the classical stalk formula, Theorem \[A.1\] in Appendix \[A\]) to \( x \). We obtain the isomorphism:

\[
\pi^{-1}(x) \simeq (\text{Spec}(L \otimes_K L), \partial).
\]

From the definition of \( \psi \) we know that \( L \sigma \) gives us an isomorphism between the fibers \( \pi^{-1}(x) \) and \( \psi^{-1}(x) \). This restricted morphism \( L \sigma \mid (H_L, \partial) \) is a splitting morphism

\[
\begin{array}{ccc}
(\text{Spec}(L \otimes_K L), \partial) & \xrightarrow{\pi} & \{x\} \\
L \sigma \mid (H_L, \partial) \downarrow & & \downarrow \psi \\
H \times_C (\text{Spec}(L), \partial)
\end{array}
\]

of the tensor product \( L \otimes_K L \). All differential point \( \tau \in (\text{Spec}(L \otimes_K L), \partial) \) must be be in the preimage of \( x \), because of the maximality of \( x \) as differential point.
if $G_{\mathcal{L}}$. If follows that $\text{Diff}(\text{Spec}(\mathcal{L} \otimes_{\mathcal{K}} \mathcal{L}, \partial) = \text{DiffSpec}(\mathcal{L} \otimes_{\mathcal{K}} \mathcal{L})$. And then, we obtain an isomorphism

$$\text{DiffSpec}(\mathcal{L} \otimes_{\mathcal{K}} \mathcal{L}) \rightarrow H \times_{\mathcal{C}} \text{DiffSpec}(\mathcal{L}),$$

it follows that $\mathcal{K} \subset \mathcal{L}$ is strongly normal. □

**Remark 4.39** Following [22], $\text{DiffSpec}(\mathcal{L} \otimes_{\mathcal{K}} \mathcal{L})$ is the set of admissible $\mathcal{K}$-isomorphism of $\mathcal{L}$, modulo generic specialization. In the case of a strongly normal extension $\mathcal{K} \subset \mathcal{L}$ the space of constants $\text{Const}(\text{DiffSpec}(\mathcal{L} \otimes_{\mathcal{K}} \mathcal{L}))$ is an algebraic group and its closed points correspond to differential $\mathcal{K}$-algebra automorphisms of $\mathcal{L}$. Let us consider the previous splitting morphism,

$$H \times_{\mathcal{C}} (\text{Spec}(\mathcal{L}, \partial) \rightarrow (\text{Spec}(\mathcal{L} \otimes_{\mathcal{K}} \mathcal{L}), \partial)$$

if we apply the constant functor $\text{Const}$, we obtain a isomorphism of $\mathcal{C}$-algebraic varieties,

$$H \rightarrow \text{Gal}(\mathcal{L}/\mathcal{K}),$$

where $H$ and $\text{Gal}(\mathcal{L}/\mathcal{K})$ are algebraic groups. To each $\tau \in H$, we have $\bar{\tau} \cdot \tau = \tau$, and the $R^2_{\tau} : \mathcal{L} \rightarrow \mathcal{L}$. We have $R^2_{\bar{\tau}} \circ R^2_{\tau} = R^2_{\tau \bar{\tau}}$ and it realizes $H$ as a group of differential $\mathcal{K}$-algebra automorphisms of $\mathcal{L}$.

**Theorem 4.40** The Galois group $\text{Gal}_x(G_{\mathcal{K}}, \partial_{\bar{\mathcal{K}}})$ is the group of differential $\mathcal{K}$-algebra automorphisms of the Galois extension $\mathcal{K} \subset \kappa(\bar{\mathcal{L}})$.

**Proof.** Denote, as above, by $H \subset G$ the Galois group and by $\mathcal{L}$ the Galois extension $\kappa(\bar{\mathcal{L}})$. We consider the isomorphism $s$ stated in remark 4.39. Let us prove that $s$ is an isomorphism of algebraic groups over $\mathcal{C}$, and that for $\tau \in H(\mathcal{C})$, $s(\tau)$ is the automorphism $R^2_{\tau}$ of $\mathcal{L}$, induced by the translation $R_{\tau}$.

We already know that $s$ is a scheme isomorphism. We have to prove that it is a group morphism. For $\tau \in H$, let us compute $s(\tau)$. First, let us denote by $\bar{\tau}$ the point of $H_{\mathcal{L}}$ obtained from $\tau$ after the base extension from $\mathcal{C}$ to $\mathcal{L}$. It is a differential point of $(H_{\mathcal{L}}, \partial)$. Then $L_{\tau}(\bar{\tau}) = R_{\tau}(\sigma) \in \pi^{-1}(\bar{\tau})$. We identify $R_{\tau}(\sigma)$ with a differential point of $\pi^{-1}(\bar{\tau})$. By the stalk formula we have that $\pi^{-1}(\bar{\tau}) = (\text{Spec}(\mathcal{O}_{G_{\mathcal{K}}, \bar{\mathcal{L}}}), \partial)$. We identify $R_{\tau}(\sigma)$ with a prime differential ideal of $\mathcal{O}_{G_{\mathcal{K}}, \bar{\mathcal{L}}}$. Because $\pi(R_{\tau}(\sigma)) = \bar{\tau}$, the morphism $R_{\tau}(\sigma) \psi$ factorizes,

$$\begin{array}{cccc}
\mathcal{O}_{G_{\mathcal{K}}, \bar{\mathcal{L}}} & \xrightarrow{\tau^{\#} \text{Id}} & \kappa(\bar{\mathcal{L}}) \otimes_{\mathcal{K}} \mathcal{L} & \xrightarrow{\psi} \mathcal{L} \\
\downarrow R_{\tau}(\sigma) \psi & & & \\
\kappa(\bar{\mathcal{L}}) \otimes_{\mathcal{K}} \mathcal{L} & \xrightarrow{\psi} \mathcal{L}
\end{array}$$

and then the kernel of $\psi$ is the prime differential ideal defining the automorphism $s(\tau)$,

$$\psi(a \otimes b) = s(\tau)(a) \cdot b$$
Let us consider the right translation $R_\tau$, 

$$
\begin{array}{ccc}
G_L & \xrightarrow{R_\tau} & G_L \\
\downarrow & & \downarrow \\
G_K & \xrightarrow{R_\tau} & G_K
\end{array}
\quad
\begin{array}{ccc}
\sigma & \xrightarrow{R_\tau} & L_\sigma(\bar{\tau}) \\
\downarrow & & \downarrow \\
\bar{\tau} & = & \bar{\tau}
\end{array}
$$

we have a commutative diagram between the local rings,

$$
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{Id} & \mathcal{L} \\
\sigma & \downarrow & \sigma \\
\mathcal{O}_{G_\mathcal{L},\pi^{-1}(x)} & \xrightarrow{R_\tau^L} & \mathcal{O}_{G_\mathcal{L},\pi^{-1}(x)} \\
\downarrow & & \downarrow \\
\mathcal{O}_{G_\mathcal{K},s} & \xrightarrow{R_\tau^L} & \mathcal{O}_{G_\mathcal{K},s}
\end{array}
$$

where $\mathcal{O}_{G_\mathcal{L},\pi^{-1}(x)} = \mathcal{O}_{G_\mathcal{K},s} \otimes_\mathcal{L} \mathcal{L}$, and the morphism $R_\tau^L$ on these rings is defined as follows:

$$\mathcal{O}_{G_\mathcal{K},s} \otimes_\mathcal{L} \mathcal{L} \to \mathcal{O}_{G_\mathcal{K},s} \otimes_\mathcal{L} \mathcal{L}, \quad a \otimes b \mapsto R_\tau^L(a) \cdot b.$$ 

It is then clear that morphism $\psi$ defined above sends,

$$\psi: (a \otimes b) \mapsto R_\tau^L(a) \cdot b$$

and then its kernel defines the automorphism $R_\tau^L$ and we finally have found $R_\tau^L = s(\tau)$. □

4.10 Galois Correspondence

There is a Galois correspondence for strongly normal extensions (theorem 3.23). It is naturally transported to the context of algebraic automorphic systems. Let $\mathcal{L}$ be a Galois extension, which is the rational field $\kappa(\varpi)$ of a Kolchin closed point $\varpi$ as above. Let $\mathcal{F}$ be an intermediate differential extension,

$$\kappa \subset \mathcal{F} \subset \mathcal{L}.$$ 

We make base extensions sequentially so that we obtain a sequence of schemes with derivations,

$$(G_\mathcal{L}, \partial_\mathcal{A}) \to (G_\mathcal{F}, \partial_\mathcal{A}) \to (G_\mathcal{K}, \partial_\mathcal{A}),$$

and the associated sequence of differential schemes,

$$\text{Diff}(G_\mathcal{L}, \partial_\mathcal{A}) \to \text{Diff}(G_\mathcal{F}, \partial_\mathcal{A}) \to \text{Diff}(G_\mathcal{K}, \partial_\mathcal{A}).$$

Let $\sigma \in G(\mathcal{L})$ be the fundamental solution induced by $\varpi$. We obtain a sequence of differential points:

$$\sigma \mapsto \eta \mapsto \varpi.$$
They are Kolchin closed and $\sigma$ is the fundamental solution associated to $x$ and $y$ (Lemma 4.33). The stabilizer subgroup of $y$ is a subgroup of the stabilizer subgroup of $x$. We have inclusions of algebraic groups,

$$\text{Gal}_y(G_F, \partial A) \subset \text{Gal}_x(G_K, \partial A) \subset G.$$ 

In particular we have that $K \subset F$ is a strongly normal extension if and only if

$$\text{Gal}_y(G_F, \partial A) \triangleleft \text{Gal}_x(G_K, \partial A).$$

**Proposition 4.41** Assume that $\text{Gal}_x(G_K, \partial A)$ is the whole group $G$, and $K \subset F$ is a strongly normal extension. Then the quotient group

$$\tilde{G} = G/\text{Gal}_y(G_F, \partial A)$$

exists. Let $\tilde{B}$ be the projection of $\tilde{A}$ in $R(\tilde{G}) \otimes_{\mathbb{C}} K$. Then, there is a unique closed differential point $\tilde{z} \in \text{Diff}(\tilde{G}_K, \partial \tilde{B})$, and,

$$\text{Gal}_{\tilde{z}}(\tilde{G}_K, \partial \tilde{B}) = \tilde{G}.$$ 

**Proof.** The quotient realizes itself as the group of automorphisms of the differential $K\text{-algebra } F$. The extension $K \subset F$ is strongly normal, and then this group is algebraic by Galois correspondence (Theorem 3.23). The induced morphism

$$\tau: \text{Diff}(G_K, \partial A) \to \text{Diff}(\tilde{G}_K, \partial \tilde{A})$$

restricts to the differential points, and it is surjective. The hypothesis $\text{Gal}_x(G_K, \tilde{A}) = G$ implies that $\text{Diff}(G_K, \partial A)$ consist in the only point $\{x\}$, and then $\text{Diff}(\tilde{G}_K, \partial \tilde{A}) = \{\tilde{z}\}$. Hence, $\tilde{z}$ is the generic point of $G_K$ and the Galois group is the total group. □

Reciprocally let us consider an algebraic subgroup $H \subset \text{Gal}_x(G_K, \partial A)$. Then $H$ is a subgroup of differential $K\text{-algebra automorphisms of } L$. Let $F = L^H$ be its field of invariants. We have again a sequence of non-autonomous algebraic dynamical systems

$$(G_L, \partial A) \to (G_F, \partial A) \to (G_K, \partial A).$$

Let again $\sigma$ be the fundamental solution induced by $x$, we have the sequence of closed differential points,

$$\sigma \mapsto y \mapsto x.$$ 

**Proposition 4.42** Let us consider an intermediate differential field,

$$K \subset F \subset L,$$

as above, and $H = \text{Aut}(L/F)$, then

$$\text{(a) } H \text{ is the Galois group } \text{Gal}_y(G_F, \partial A) \subset \text{Gal}_x(G_K, \partial A).$$
K ⊂ F is strongly normal if and only if $H \trianglelefteq \text{Gal}_x(G_K, \partial \vec{A})$. In such case $\text{Aut}(F/K) = \text{Gal}_x(G_K, \partial \vec{A})/H$.

**Proof.** By considering the identification of the Galois group with the group of automorphisms, the result is a direct translation of the Galois correspondence for strongly normal extensions (see [22] Theorem 20.5, Theorem 3.23 in this text). □

In particular, each algebraic group admits a unique normal subgroup of finite index, the connected component of the identity. Let $\text{Gal}^0_1(G_K, \partial \vec{A})$ be the connected component of the identity of $\text{Gal}_1(G_K, \partial \vec{A})$ and,

$$\text{Gal}^1_1(G_K, \partial \vec{A}) = \text{Gal}_1(G_K, \partial \vec{A})/\text{Gal}^0_1(G_K, \partial \vec{A}),$$

which is a finite group. In such case we have:

(a) The invariant field $L^{\text{Gal}^0_1(G_K, \partial \vec{A})}$ is the relative algebraic closure $K^o$ of $K$ in $L$.

(b) $K \subset K^o$ is an algebraic Galois extension of Galois group $\text{Gal}^1_1(G_K, \partial \vec{A})$.

(c) $\text{Gal}_0(G_K^o, \partial \vec{A}) = \text{Gal}^0_1(G_K, \partial \vec{A})$.

Thus, we can set out:

**Proposition 4.43** $K$ is relatively algebraically closed in $L$ if and only if its Galois group is connected.

### 4.11 Galois Correspondence and Group Morphisms

Here, we relate the Galois correspondence and the projection of automorphic vector fields through algebraic group morphisms. It is self evident that a group morphism $\pi: G \rightarrow \tilde{G}$ sends an automorphic system $\tilde{A}$ in $G$ with coefficients in $K$ to an automorphic system $\pi(\tilde{A})$ in $\tilde{G}$ with coefficients in $K$. Furthermore we know that $\pi(\tilde{A})$ is an automorphic system in the image of $\pi$ which is a subgroup of $\tilde{G}$. By restricting our analysis to this image, we can assume that $\pi$ is a surjective morphism.

**Theorem 4.44** Let $\pi: G \rightarrow \tilde{G}$ be a surjective morphism of algebraic groups, and $\tilde{B}$ the projected automorphic system $\pi(\tilde{A})$. Then:

1. $\eta = \pi(x)$ is a closed differential point of $\text{Diff}(G_K, \partial B)$.
2. $\kappa(\eta)$ is a strongly normal intermediate extension of $K \subset \kappa(\eta) \subset L$.
3. $\text{Gal}_0(\tilde{G}_K, \partial \tilde{B}) = \text{Gal}_1(G_K, \partial \tilde{A})/(\ker(\pi) \cap \text{Gal}_1(G_K, \partial \tilde{A}))$.
4. Let $\mathfrak{z}$ be a Kolchin closed point of $(G_{\kappa(\eta)}, \partial \tilde{A})$ in the fiber of $x$. Then $\text{Gal}_1(G_{\kappa(\eta)}, \partial \tilde{A}) = \ker(\pi) \cap \text{Gal}_1(G_K, \partial \tilde{A})$. 39
Proof. (1) Let \( s \) be a closed point of Diff(\( \bar{G}_K, \partial_B \)) adherent to \( y \). Then \( \pi^{-1}(x) \) is a closed subset of Diff(\( G_K, \partial_A \)) and it contains a closed point \( z \). \( G(\mathbb{C}) \) acts transitively in the set of closed points, and then there is \( \tau \in G(\mathbb{C}) \) such as \( x = z \cdot \tau \). Thus, \( s = \pi^{-1}(\tau) \), so that \( s = x \), and furthermore \( \pi(\tau) \in \text{Gal}_y(\bar{G}_K, \partial_B) \).

(2) \( \pi^\#: \kappa(y) \to L \) is a differential \( K \)-algebra morphism, and \( \kappa(y) \) is realized as an intermediate extension \( K \subset \kappa(y) \subset L \). It is a strongly normal if and only if the subgroup of \( \text{Gal}(L/K) \) fixing \( \kappa(y) \) is a normal subgroup. We identify \( \text{Gal}(L/K) \) with \( \text{Gal}_x(\bar{G}_K, \partial_A) \). Then \( \tau \) fixes \( \kappa(y) \) if and only if \( \pi(\tau) = e \). This subgroup fixing \( \kappa(y) \) is \( \ker(\pi) \cap \text{Gal}_x(\bar{G}_K, \partial_A) \). By hypothesis, \( \ker(\pi) \) is a normal subgroup of \( G \), and then its intersection with \( \text{Gal}_x(\bar{G}_K, \partial_A) \) is a normal subgroup.

Finally, be obtain (3) and (4) by Galois correspondence. □

4.12 Lie Extension Structure on Intermediate Fields

Differential field approach to Lie-Vessiot systems was initiated by K. Nishioka, in terms of the notions of rational dependence on arbitrary constants and Lie extensions (see definitions 3.24 and 3.25). Here we relate our results with these notions.

Theorem 4.45 Assume one of the following:

(a) \( K \) is algebraically closed.

(b) The Galois group of \( (G_K, \partial_A) \) is \( G \).

Let \( y \) be a particular solution of \( (M_K, \partial_{\bar{X}}) \) with coefficients in a differential field extension \( K \subset R \). Assume that \( R \) is generated by \( y \). Then:

(i) \( K \subset R \) depends rationally on arbitrary constants.

(ii) \( K \subset R \) is a Lie extension.

Proof. (i) \( R \) is an intermediate extension of the splitting field of the automorphic system which is a strongly normal extension. It is a stronger condition than the one of Definition 3.24, thus \( R \) depends rationally on arbitrary constants.

(ii) If \( K \) is algebraically closed, then the result comes directly from Theorem 3.24. For the case (b), some analysis on the infinitesimal structure of \( R \) must be done. If the Galois group is \( G \), then there are not non-trivial differential points in \( G_K \), nor in \( M_K \). Then \( R \) coincides with \( M(M_K) \), the field of meromorphic functions in \( M_K \). Fundamental vector fields of the action of \( G \) on \( M \) induce derivations of the corresponding fields of meromorphic functions so that we have a Lie algebra morphism,

\[
R(G) \to \text{Der}_K(R), \quad A_i \mapsto X_i,
\]

and the derivation in \( \partial \) in \( R \) is seen in \( M(R) \) as the Lie-Vessiot system

\[
\tilde{\partial} = \partial + \sum_{i=1}^{r} f_i X_i.
\]
From that, we have that,

\[ [\bar{\partial}, \mathcal{R}(G)] \subset \mathcal{R}(G) \otimes_{\mathcal{K}} \mathcal{K}, \]

and because the vector fields \( \vec{X}_i \) span the tangent vector space to \( M \), we have that the morphism,

\[ \mathcal{R}(G) \otimes_{\mathcal{C}} \mathcal{R} \to \text{Der}_{\mathcal{K}}(\mathcal{R}) \]

is surjective. According to Definition [3.25], we conclude that \( \mathcal{R} \) is a Lie extension.

\[ \square \]

5 Algebraic Reduction and Integration

Here we present the algebraic theory of reduction and integration of algebraic automorphic and Lie-Vessiot systems. Our main tool is an algebraic version of Lie’s reduction method, that we call Lie-Kolchin reduction. Once we have developed this tool we explore different applications.

5.1 Lie-Kolchin Reduction Method

In [3], when discussing the general topic of analytic Lie-Vessiot systems, we have shown the Lie’s method for reducing an automorphic equation to certain subgroups, once we know certain solution of a Lie-Vessiot associated system. This method is local, because it is assumed that we can choose a suitable curve in the group for the application of the algorithm. A germ of such a curve exists, but it is not true that a suitable global curve exists in the general case. In the algebraic realm we will find obstructions to the applicability of this method, highly related to the structure of principal homogeneous spaces over a non algebraically closed field, and then to Galois cohomology.

We will show that the application of the Lie’s method in the algebraic case leads us directly to Kolchin reduction theorem of a linear differential system to the Lie algebra of its Galois group. Because of this, we decided to use the nomenclature of Lie-Kolchin reduction method.

5.2 Lie-Kolchin Reduction

From now on, let us consider a differential field \( \mathcal{K} \) of characteristic zero. The field of constant is \( \mathcal{C} \), that we assume to be algebraically closed. Let \( G \) be an algebraic group over \( \mathcal{C} \), and let \( \vec{A} \) be an algebraic automorphic vector field in \( G \) with coefficients in \( \mathcal{K} \). We also fix a Kolchin closed point \( \vec{\lambda} \) of \( \text{Diff}(G_{\mathcal{K}}, \partial_{\vec{\lambda}}) \) and denote by \( \mathcal{L} \) its associated Galois extension.

**Lemma 5.1** Let \( G' \subset G \) be an algebraic subgroup, and let \( M \) be the quotient homogeneous space \( G/G' \). Then:

(a) \( M_{\mathcal{K}} = G_{\mathcal{K}}/G'_{\mathcal{K}} \)
Let us consider the natural projection morphism $\pi_K: G_K \to M_K$. For each rational point $x \in M_K$, $\pi_K^{-1}(x) \subset G_K$ is an homogeneous space of group $G_K'$.

**Proof.** (a) $\mathcal{C}$ is algebraically closed, and then the geometric quotient is universal; (a) is the fundamental property of geometric universal quotients (see [34]).

(b) The isotropy subgroup $H_x$ of $x$ is certain algebraic subgroup isomorphic and conjugated with $G_K'$. The action of $(H_x)_K$ on $G$ preserves the stalk $\pi_K^{-1}(x)$,

$$\psi: (H_x)_K \times_K \pi_K^{-1}(x) \to \pi_K^{-1}(x),$$

the induced morphism

$$(\psi \times \text{Id}): (H_x)_K \times_K \pi_K^{-1}(x) \to \pi_K^{-1}(x) \times_K \pi_K^{-1}(x)$$

is the restriction of the isomorphism

$$G_K \times_K G_K \to G_K \times_K G_K, \quad (\tau, \sigma) \mapsto (\tau \cdot \sigma, \sigma),$$

and then it is an isomorphism. \(\square\)

Let $M$ be an homogenous space over $G$, and $\vec{X}$ the Lie-Vessiot vector field induced in $M$ by the automorphic vector field $\vec{A}$. Let us fix a rational point $x_0$ of $M$ and denote by $H_{x_0}$ the isotropy subgroup at $x_0$.

**Lemma 5.2** Assume that $x_0 \in M$ is a constant solution of $(M_K, \partial \vec{X})$. Then:

$$\vec{A} \in \mathcal{R}(H_{x_0}) \otimes_K \mathcal{K}.$$

**Proof.** There is a solution $\tau$ of $\vec{A}$ with coefficients in $\mathcal{L}$ such that $x_0 = \tau \cdot x_0$. Therefore $\tau \in (H_{x_0})_\mathcal{L}$ and its logarithmic derivative is an automorphic vector field in $H_{x_0}$,

$$l\partial(\tau) \in \mathcal{R}(H_{x_0}) \otimes \mathcal{L}.$$

Taking into account that $l\partial(\tau) = \vec{A}$, we obtain $\vec{A} \in \mathcal{R}(H_{x_0}) \otimes_K \mathcal{K}$. \(\square\)

**Theorem 5.3 (Main Result)** Let us assume that $(M_K, \partial \vec{X})$ has a solution $x$ with coefficients in $\mathcal{K}$. If $H^1(H_{x_0}, \mathcal{K})$ is trivial, then there exists a gauge transformation $L_\tau$ of $G_K$ that sends the automorphic vector field $\vec{A}$ to:

$$\vec{B} = \text{Adj}_{\tau}(\vec{A}) + l\partial(\tau),$$

with $\vec{B} \in \mathcal{R}(H_{x_0}) \otimes_K \mathcal{K}$ an automorphic vector field in $H_{x_0}$.

**Proof.** Let us consider the canonical isomorphism $G/H_{x_0} \to M$ that sends the class $[\sigma]$ to $\sigma \cdot x_0$. Now, let us consider the base extended morphism,

$$\pi: G_K \to M_K, \quad \tau \mapsto \tau \cdot x_0.$$
We are under the hypothesis of Lemma 5.1 (b). Therefore the stalk $\pi^{-1}(x)$ is a principal homogeneous space of group $(H_K)_x$ which is a subgroup of $G_K$ conjugated to $(H_{x_0})_K$. Because of the vanishing of the Galois cohomology, there exist a rational point $\tau_1 \in \pi^{-1}(x)$, and then $\tau_1 \cdot x_0 = x$. Define $\tau = \tau_1^{-1}$. Let us consider the gauge transformation,

\[ L_{\tau}: (G_K, \partial_{\tilde{A}}) \to (G_K, \partial_{\tilde{B}}) \]

\[ L_{\tau}: (M_K, \partial_{\tilde{X}}) \to (M_K, \partial_{\tilde{Y}}), \]

where $\tilde{Y}$ is the Lie-Vessiot vector field in $M$ induced by $\tilde{B}$. We have that $\tau \cdot x = x_0$ is a constant solution of $(M_K, \partial_{\tilde{X}})$. By Lemma 5.2 $\tilde{B}$ is an automorphic field in $H_{x_0}$. □

**Proposition 5.4** Assume that there is a rational point $x_0 \in M$ such that $\text{Gal}_x(G_K, \partial_{\tilde{A}}) \subset H_{x_0}$, then there exists a rational solution $x \in M(K)$ of $\tilde{X}$.

**Proof.** Let us consider the fundamental solution $\sigma$ associated to $x$. We consider it as an $L$-point of $G$,

\[ \sigma: \text{Spec}(L) \to G_K. \]

It is determined by the canonical morphism of taking values in $\sigma$,

\[ \sigma^\sharp: \mathcal{O}_{G_K, \tilde{x}} \to L = \kappa(\tilde{x}). \]

Now, let us consider the projection $\pi: G \to M$, $\tau \mapsto \tau \cdot x_0$. It induces a morphism $\pi: G_K(L) \to M_K(L)$. Let us consider $x = \pi(\sigma)$. This point $x$ is an $L$ point of $M$ and then it is a morphism

\[ x: \text{Spec}(L) \to M_K. \]

Let $\tilde{x} \in M_K$ be the image of $x$; then $x$ is determined by the morphism $x^\sharp$ defined by the following composition:

\[ \mathcal{O}_{M_K, \tilde{x}} \xrightarrow{\pi^\sharp} \mathcal{O}_{G_K, \tilde{x}} \xrightarrow{x^\sharp} \mathcal{O}_{G_K, \tilde{x}} \xrightarrow{\sigma^\sharp} \mathcal{L} \]

We are going to prove that $x$ is a rational point of $M_K$. Let us consider $\tau \in \text{Gal}_x(G_K, \partial_{\tilde{A}})$. Therefore we have $R_\tau(x) = x$, and the following diagram is commutative:

\[ \mathcal{O}_{M_K, \tilde{x}} \xrightarrow{x^\sharp} \mathcal{O}_{G_K, \tilde{x}} \xrightarrow{\sigma^\sharp} \mathcal{L} \]

\[ \mathcal{O}_{M_K, \tilde{x}} \xrightarrow{x^\sharp} \mathcal{O}_{G_K, \tilde{x}} \xrightarrow{(\sigma \tau)^\sharp} \mathcal{L} \]

\[ \mathcal{L} \]

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For each $f \in \mathcal{O}_{X_{\mathcal{K}}, \check{\tau}}$, we have $x^\check{\tau}(f) = R^\check{\tau}_x(x^\tau(f))$. This equality holds for all $\tau \in H_{\mathcal{K}}$. Hence, $x^\check{\tau}(f)$ an element of $\mathcal{L}$ that is invariant for any differential $\mathcal{K}$-algebra automorphism of $\mathcal{L}$. In virtue of the Galois correspondence the fixed field of $\mathcal{L}$ by the action of $\text{Gal}_x(G_{\mathcal{K}}, \partial_{A})$ is $\mathcal{K}$. Thus, $x^\check{\tau}(f) \in \mathcal{K}$. □

Theorem 5.5 Let us consider an algebraic subgroup $G'$ of $G$ verifying:

1. $\text{Gal}_x(G_{\mathcal{K}}, \partial_{A}) \subset G'$,
2. $H^1(H, \mathcal{K})$ is trivial.

Then there exist a gauge isomorphism $L_\tau$ of $G$ with coefficients in $\mathcal{K}$ reducing the automorphic system $\check{A}$ to an automorphic system in $H$,

$$\check{B} = \text{Adj}_\tau(\check{A}) + l\partial(\tau),$$

belongs to $\mathcal{R}(G') \otimes_\mathcal{C} \mathcal{K}$.

PROOF. By Proposition 5.4 there exists a rational solution of the Lie-Vessiot system in $M$ associated to $\check{A}$. Theorem 5.3 says that such a reduction exists. □

Denote by $\text{Gal}^0_x(G_{\mathcal{K}}, \partial_{A})$ the connected component of the identity of the Galois group $\text{Gal}_x(G_{\mathcal{K}}, \partial_{A})$.

Corollary 5.6 Let $\mathcal{K}^\circ$ be the relatively algebraic closure of $\mathcal{K}$ in $\mathcal{L}$. Assume that $H^1(\text{Gal}_x^0(G_{\mathcal{K}}, \partial_{A}), \mathcal{K}^\circ)$ is trivial. Then there is a gauge transformation $L_\tau$, $\tau$ with coefficients in $\mathcal{K}^\circ$ such that

$$\check{B} = \text{Adj}_\tau(\check{A}) + l\partial(\tau)$$

belongs to $\mathcal{R}(\text{Gal}_x^0(G_{\mathcal{K}}, \partial_{A})) \otimes_\mathcal{C} \mathcal{K}^\circ$.

PROOF. We know that the Galois group of the automorphic system with coefficients in $\mathcal{K}^\circ$ is precisely $\text{Gal}_x^0(G_{\mathcal{K}}, \partial_{A})$ (see, for instance, remark (c) in [?], below Proposition 18). We apply then Theorem 5.5. □

Corollary 5.7 If $H^1(\text{Gal}_x(G_{\mathcal{K}}, \partial_{A}), \mathcal{K})$ is trivial then $\text{Gal}_x(G_{\mathcal{K}}, \partial_{A})$ is connected.

PROOF. If $H^1(\text{Gal}_x(G_{\mathcal{K}}, \partial_{A}), \mathcal{K})$ is trivial, then we can reduce the automorphic system to an automorphic system in $\mathcal{R}(\text{Gal}_x(G_{\mathcal{K}}, \partial_{A})) \otimes_\mathcal{C} \mathcal{K}$. Note that $\text{Gal}_x^0(G_{\mathcal{K}}, \partial_{A})$ and $\text{Gal}_x(G_{\mathcal{K}}, \partial_{A})$ have the same Lie algebra. Therefore the Galois group of the reduced equation is contained in $\text{Gal}_x^0(G_{\mathcal{K}}, \partial_{A})$. □

The following is an extension of the classical result of Kolchin on the reduction a system of linear differential equations to the Lie algebra of its Galois group [19].
Theorem 5.8 (Kolchin) Let us consider the relative algebraic closure $K^{\circ}$ of $K$ in $L$. There is a gauge transformation $L_{\tau}$, $\tau$ with coefficients in $K^{\circ}$, such that,

$$\bar{B} = \text{Adj}_{\tau}(\bar{A}) + l\partial(\tau)$$

belongs to $\mathcal{R}(\text{Gal}(G_{K}, \partial \bar{A})) \otimes_{C} K^{\circ}$.

Proof. Denote by $H$ the Galois group $\text{Gal}(G_{K}, \partial \bar{A})$. Let us consider $M = G/H$, and let us denote by $x_{0} \in M$ the origin which is the class of $H$ in $M$. Let $\bar{Y}$ be the Lie-Vessiot vector field in $M$ associated to $\bar{A}$. In virtue of Proposition 5.4, the canonical projection $G(L) \to M(L)$ sends the fundamental solution $\sigma$ to a solution $x$ of $(M, \partial \bar{Y})$ with coefficients in $K$. Let us consider the projection:

$$\pi: G_{K} \to M_{K}.$$ 

Lemma 5.1 says that the stalk $\pi^{-1}(x)$ is a principal homogeneous space modeled over the group $H_{K}$. Let us denote by $P \subset G_{K}$ such homogeneous space. Note that $P$ is $\{x\}$, the closure of $x$ in Zariski topology. We have the isomorphism,

$$\psi: P \times_{K} H_{K} \to P \times_{K} P, \quad (\tau, g) \to (\tau, \tau g),$$

Let $\tau$ be a closed point of $P$. Its rational field $\kappa(\tau)$ is an algebraic extension of $K$. We have that $x = \tau \cdot x_{0}$. Thus, we can apply Lie-Kolchin reduction method. $L_{\tau-1}$ is a gauge transformation with coefficients in $\kappa(\tau)$:

$$L_{\tau-1}: G_{K} \to G_{\kappa(\tau)},$$

that sends the automorphic vector field $\bar{A}$ to an automorphic vector field $\bar{B}$ in $H$ with coefficients in $\kappa(\tau)$.

In order to finish the proof we have to see that $\kappa(\tau)$ is a subfield of the relative algebraic closure $K^{\circ}$ of $K$ in $L$. It is enough to see that $K \subset \kappa(\tau)$ is an intermediate differential extension of $K \subset L$. Furthermore, if $\kappa(\tau)$ is an intermediate differential extension then it coincides with $K^{\circ}$ because of the Galois correspondence.

Let us consider then the following base extension and natural projection,

$$P_{\kappa(\tau)} = P \times_{K} \text{Spec}(\kappa(\tau)), \quad \pi_{1}: P_{\kappa(\tau)} \to P.$$ 

The product $P_{\kappa(\tau)}$ is a principal homogeneous space modeled over $H_{\kappa(\tau)}$. Moreover, $\tau$ induces a rational point of $P_{\kappa(\tau)}$. Hence, the Galois cohomology cohomology class of $P_{\kappa(\tau)}$ is trivial, so that it is isomorphic to $H_{\kappa(\tau)}$ as homogeneous space. $P_{\kappa(\tau)}$ has as many connected components as $H_{\kappa(\tau)}$. We write it as the disjoint union of its connected components.

$$P_{\kappa(\tau)} = \bigsqcup_{i \in \Lambda} P_{i}.$$ 

For each $i \in \Lambda$, the restriction $P_{i} \to P$ is an isomorphism of $K$-schemes, and $\pi_{1}$ is a trivial covering. But each $P_{i}$ is a $\kappa(\tau)$-scheme, and then each component
induces in $P$ a structure of $\kappa(\tau)$-scheme. Hence we have a realization of $\kappa(\tau)$ as intermediate extension

$$K \subset \kappa(\tau) \subset L.$$  
Thus, $\kappa(\tau) = K^\circ$. $\square$

### 5.3 Integrability by Quadratures

To integrate an automorphic system by quadratures means to write down a fundamental solution by terms of a formula. This formula should involve the solutions of certain simpler equations. We assume that we have a geometrical meccano to express these solutions. We refer to elements of such a meccano as *quadratures*. Those simpler equations are like the building blocks of our integrability theory. Depending of which simpler equations we consider as *integrable* we obtain different theories integrability. In theory of Lie-Vessiot systems the elements of our formulas are the *exponential maps of Lie groups* and *indefinite integrals*.

From a geometric point of view, it is reasonable to consider automorphic systems in *abelian groups as integrable*. Let us consider an abelian Lie group $G$. Then, the exponential map,

$$\exp: \mathcal{R}(G) \to G,$$

is a group morphism, and moreover, $\mathcal{R}(G)$ is the universal covering of $G$. An automorphic equation,

$$\frac{d \log}{dt}(x) = \sum_{i=1}^{n} f_i(t) \tilde{A}_i, \quad \tilde{A}_i \in \mathcal{R}(G)$$

is integrated by the formula,

$$\sigma(t) = \exp \left( \sum_{i=1}^{n} \left( \int_{t_0}^{t} f_i(\xi) d\xi \right) \tilde{A}_i \right).$$

This formula involves the integral of $t$ dependent functions, and the exponential map of the Lie group. Assuming that we are able of realize these operations a reasonable point of view is to consider all automorphic equations in *abelian groups integrable*. This assumption is done in [43], and followed in [6]. On the other hand, the algebraic case has a new kind of richness. An abelian Lie group splits in direct product of circles an lines, but an abelian algebraic group can carry a higher complexity, for example in the case of abelian varieties. In such case the exponential map is the solution of the Abel-Jacobi inversion problem. In [18] Kolchin develops a theory of integrability generalizing Liouville integrability, in which just quadratures in one dimensional abelian groups are allowed. It reduces the case to quadratures in the additive group, the multiplicative group and elliptic curves.
5.4 Quadratures in the Additive Group

Let us consider an automorphic equation in the additive group \( C \). The additive group is its own Lie algebra, and the logarithmic derivative is the usual derivative. Thus, the automorphic equations are written in the following form:

\[
\partial x = a, \quad a \in K.
\]

(5.1)

**Definition 5.9** An extension of differential fields \( K \subset L \) is an integral extension if \( L = K(b) \), with \( \partial b \in K \). We say that \( b \) is an integral element over \( K \).

It is obvious that the Galois extension of equation (5.1) is an integral extension of \( K \), with \( b = \int a \). The additive group (of a field of characteristic zero) has no algebraic subgroups. Therefore, if \( a \) is algebraic over \( K \), then \( a \in K \). Hence we have two different possibilities for integral extensions:

- \( b \in K \), \( \text{Gal}(L/K) = \{ e \} \),
- \( b \notin K \), \( \text{Gal}(L/K) = C \).

5.5 Quadratures in the Multiplicative Group

Let us consider now an automorphic equation in the multiplicative group. For the complex numbers \( C^* \) the exponential map is the usual exponential. In the general case of an algebraically closed field of characteristic zero, we can build the exponential map for \( C^* \). However, it does not take values in \( C^* \) but in a bigger group. We avoid such a construction, and then we consider the exponential just as an algebraic symbol. The logarithmic derivative in \( C^* \) coincides with the classical notion of logarithmic derivative,

\[
K^* \to K, \quad x \mapsto \frac{\partial x}{x}.
\]

The general automorphic equation in the multiplicative group is written as follows:

\[
\frac{\partial x}{x} = a, \quad a \in K.
\]

(5.2)

**Definition 5.10** An extension of differential fields \( K \subset L \) is an exponential extension if \( L = K(b) \), with \( \frac{\partial b}{b} \in K \). We say that \( b \) is an exponential element over \( K \).

\( C^* \) has cyclic finite subgroups. Then, we can obtain exponential extensions that are algebraic. There appears the following casuistic:

- \( \text{Gal}(L/K) \) is the multiplicative group \( C^* \) if \( b \) is transcendent over \( K \).
- \( \text{Gal}(L/K) \) is a cyclic group \( (\mathbb{Z}_n)^* \) if \( b^n \in K \) for certain \( n \). It means that there is \( c \in K \) that \( \frac{\partial c}{c} = a \). In such case, \( b^n = c \).
Reciprocally, any algebraic Galois extension of \( K \) with a cyclic Galois group is an exponential extension. Here, it is a an essential point that \( C \) is algebraically closed.

### 5.6 Quadratures in Abelian Varieties

Abelian varieties provide us examples of non linearizable automorphic systems. For the following discussion, let us assume that the constant field of \( K \) is the field of complex numbers \( C \). Let \( G \) be a complex abelian variety of complex dimension \( g \). Let us consider a basis of holomorphic differentials \( \omega_1, \ldots, \omega_g \), and \( A_1, \ldots, A_g, B_1, \ldots, B_g \) a basis of the homology of \( G \), we can assume that \( \int_{A_i} \omega_j = \delta_{ij} \). Define the Jacobi-Abel map,

\[
G \xrightarrow{\sim} \mathbb{C}^g / \Lambda, \quad p \mapsto \left( \int_{C_1} \omega_1, \ldots, \int_{C_g} \omega_g \right).
\]

The exponential map is given by the exponential universal covering of the torus and the inversion of the Jacobi-Abel map.

\[
\begin{array}{c}
\mathbb{C}^g \\
\downarrow \\
G \xrightarrow{\exp} \mathbb{C}^g / \Lambda
\end{array}
\]

A projective immersion of \( G \) in \( \mathbb{P}(\mathbb{C}, d) \), for \( d \) big enough, is given by terms of theta functions, \( z \mapsto (\theta_0(z) : \ldots : \theta_d(z)) \). Hence there are some homogeneous polynomial constrains \( \{ P(\theta_0, \ldots, \theta_d) = 0 \} \). The quotient \( \frac{\theta_j}{\theta_0} \) defines a meromorphic abelian function in \( G \) (see [28] Chapter 1, Section 3, p. 30). Let us consider affine coordinates in \( G \), \( x_i = \frac{\theta_i}{\theta_0} \). We can project the vector fields of \( \mathcal{R}(\mathbb{C}^g) \) to \( G \),

\[
\frac{\partial}{\partial z_i} \mapsto \sum_j F_{ij}(x_1, \ldots, x_d) \frac{\partial}{\partial x_j}, \quad F_{ij}(x_1, \ldots, x_d) = \frac{\partial \theta_j}{\partial x_i} \theta_0 - \frac{\partial \theta_i}{\partial x_j} \theta_j \theta_0^2
\]

being \( F_{ij} \) abelian functions, and then rational functions in the \( x_j \). The automorphic system in \( \mathbb{C}^g \)

\[
\sum_i a_i \frac{\partial}{\partial z_i}, \quad a_i \in \mathcal{K}
\]

is seen in \( A \) as a non linear system an \( A \),

\[
\dot{x}_j = \sum_i a_i F_{ij}(x_1, \ldots, x_d), \quad \{ P(1, x_1, \ldots, x_d) = 0 \}, \quad (5.3)
\]

If \( b_1, \ldots, b_d \) are integral elements over \( \mathcal{K} \) such that \( \partial b_i = a_i \), then the solution of the automorphic system \( (5.3) \) is:

\[
x_j = \frac{\theta_j(b)}{\theta_0(b)}, \quad (\theta_0(b) : \ldots : \theta_d(b)).
\]

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Definition 5.11 A strongly normal extension $K \subset L$ whose Galois group is an abelian variety is called an abelian extension.

For an automorphic system in an abelian variety $A$ we have that the Galois group is an algebraic subgroup of $A$. Then its identity component is an abelian variety. The Galois extension is then,

$$K \subset K^o \subset L,$$

being $K^o \subset L$ an abelian extension.

Example 5.12 Let us consider an algebraically completely integrable hamiltonian system in the sense of Adler, Van Moerbecke and Vanhaecke (see [1]) \{H, H_2, \ldots, H_n\} in $\mathbb{C}^{2n}$. Assume that \{H_i(x, y) = h_i\} are the equations of the affine part of an abelian variety $G$. The Hamilton equations,

$$\dot{x}_i = \frac{\partial H}{\partial y_i}, \quad \dot{y}_i = -\frac{\partial H}{\partial x_i}, \quad H_i(x, y) = h_i \quad (5.4)$$

are an automorphic system $\vec{H}$ in $G$ with constant coefficients $K = \mathbb{C}$. In the generic case, $G$ is a non-resonant torus, and then it is densely filled by a solution curve of the equations \{5.4\}. We conclude that $(G, \partial \vec{H})$ has not proper differential points: its differential spectrum consist only of the generic point. In such case, the Galois extension of the system is $\mathbb{C} \subset \mathcal{M}(G)$, the field of meromorphic functions in $G$.

Example 5.13 Automorphic systems in elliptic curves: Let us examine the case of an elliptic curve $E$ over $\mathbb{C}$. Assume that $E$ is given as a projective subvariety of $\mathbb{P}(2, \mathbb{C})$ in Weierstrass normal form.

$$t_0t_2^2 = 4t_1^3 - g_2t_1t_0 - g_3t_0^2$$

We take affine coordinates $x = \frac{t_1}{t_0}$ and $y = \frac{t_2}{t_0}$. The Lie algebra $\mathcal{R}(E)$ is then generated by the vector field,

$$\vec{v} = y \frac{\partial}{\partial x} + (12x^2 - g_2) \frac{\partial}{\partial y}$$

Every automorphic vector field in $E$ with coefficients in $K$ is written in the form $a\vec{v}$ with $a \in K$. A solution of the automorphic equation is a point of $E$ with values in the Galois extension $L$. Such solution have homogeneous coordinates $(1: \xi: \eta)$ such that $\eta = a^{-1}\partial \xi$, and $\xi$ is a solution of the single differential equation,

$$(\partial \xi)^2 = a^2(4\xi^2 - g_1\xi - g_2). \quad (5.5)$$

If we know a particular solution $b$ of \{5.5\} then we can write down the general solution $(1: \xi: \eta)$ of the automorphic equation by means of the addition law in $E$ (see [15] p. 804 eq. 9), depending of an arbitrary point $(1:x_0:y_0) \in E(\mathbb{C})$:

$$\text{Sol} \{5.5\} \times E(\mathbb{C}) \rightarrow E(\mathbb{C}), \quad (b, (1:x_0:y_0)) \mapsto (1: \xi: \eta)$$
\[ \xi(x_0, y_0) = -b - x_0 - \frac{1}{4} \left( \frac{\partial b - a y_0}{a(b - x_0)} \right)^2 \tag{5.6} \]

\[ \eta(x_0, y_0) = -\frac{\partial b + a y_0}{2a} + \frac{6}{2} (b + x_0) \frac{\partial b - a y_0}{a(b - x_0)} - \frac{1}{4} \left( \frac{\partial b - a y_0}{a(b - x_0)} \right)^3 \tag{5.7} \]

**Definition 5.14** Let \( K \subset L \) a differential field extension. We say that \( b \in L \) is a Weierstrassian element if there exist \( a \in K \), and \( g_1, g_2 \in \mathbb{C} \), with the polynomial \( 4x^3 - g_1 x - g_2 \) having simple roots and such that, \( (\partial b)^2 = a^2 (4b^2 - g_1 x - g_2) \). The differential extension \( K \subset K(b, \partial b) \) is called an elliptic extension.

The Galois extension of the automorphic equation (5.5) is an elliptic extension of \( K \). It can be transcendental or algebraic. If it is transcendental then its Galois group is the elliptic curve \( E \), if it is algebraic then its Galois group is a finite subgroup of \( E \).

**Remark 5.15** Let us examine the case of complex numbers: assume that the field of constants of \( K \) is \( \mathbb{C} \). The solution of Weierstrass equation is the elliptic function \( \wp \), and it gives rise to the universal covering of \( E \),

\[ \pi: \mathbb{C} \to E, \quad z \mapsto (1: \wp(z): \wp'(z)). \]

The automorphic vector field \( a\vec{v} \) in \( E \) is the projection of the automorphic vector field \( a \frac{\partial}{\partial z} \) in \( \mathbb{C} \). The solution of the equation in the additive group is given by an integral element \( \int a \). Then the a solution of the projected system in \( E \) \( \wp(x) = \wp(\int a) \). Then \( a = \wp(\int a) \) is the Weierstrass element of the Galois extension. Formulas (5.6) and (5.7) are the addition formulas for the Weierstrass \( \wp \) and \( \wp' \) functions.

**Example 5.16** We obtain the previous situation in the case of one degree of freedom, algebraic complete integrable hamiltonian systems. Let us consider the pendulum equation:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= \sin(x)
\end{align*}
\]

\[ \begin{array}{c}
\begin{cases}
y^2 - 2 \cos(x) = h 
\end{cases}
\end{array} \tag{5.8} \]

It is written as a simple ordinary differential equation depending of the energy parameter \( h \),

\[ \left( \frac{dx}{dt} \right)^2 = 2h + 2 \cos(x), \]

by setting \( z = e^{ix} \), we obtain the algebraic form of such equation, which is an automorphic equation in an elliptic curve for all values of \( h \) except for \( h = \pm 1 \); \[ \left( \frac{dz}{dt} \right)^2 = -z^3 - 2hz^2 - 1. \]
The Weierstrass normal form is attained by setting
\[ u = \frac{-z^4}{16h}, \]

\[ \left( \frac{du}{dt} \right)^2 = 4u^3 - \frac{h^2}{3}u - \left( \frac{h^3}{27} + \frac{1}{16} \right). \]

Hence, the general solution is written in terms of the \( \wp \) functions of invariants \( g_2 = \frac{h^2}{3} \) and \( g_3 = \frac{h^3}{27} + \frac{1}{16} \), for \( h \neq \pm 1 \):

\[ z(t) = -4\wp(t + t_0) - \frac{2}{3}h; \quad x(t) = \log \left( -4\wp(t - t_0) - \frac{4h + 3\pi i}{6} \right). \]

### 5.7 Liouville and Kolchin Integrability

**Definition 5.17** Let \( K \subset \mathcal{F} \) a differential field extension. Let us break it up into a tower of differential fields:

\[ K = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_d = \mathcal{L}. \]

We say that \( K \subset \mathcal{F} \) is . . .

(1) . . . a Liouvillian extension if the differential fields \( \mathcal{F}_i \) can be chosen in such way that \( \mathcal{F}_i \subset \mathcal{F}_{i+1} \) is an algebraic, exponential or integral extension.

(2) . . . a strict-Liouvillian extension if the differential fields \( \mathcal{F}_i \) can be chosen in such way that \( \mathcal{F}_i \subset \mathcal{F}_{i+1} \) is an exponential or integral extension.

(3) . . . a Kolchin extension the differential fields \( \mathcal{F}_i \) can be chosen in such way that \( \mathcal{L}_i \subset \mathcal{F}_{i+1} \) is algebraic, elliptic, exponential or integral extension.

Liouvillian and strict-Liouvillian extensions are Picard-Vessiot extensions. An elliptic curve can not be a subquotient of an affine group. Hence, if \( K \subset \mathcal{F} \) is a Kolchin extension and \( \text{Gal}(\mathcal{F}/K) \) is an affine group, then it is a Liouville extension. From this perspective, the following classical result is almost self evident:

**Theorem 5.18 (Drach-Kolchin)** Let \( K \) be a field of meromorphic functions of the complex plane \( \mathbb{C} \). Assume that the Weierstrass’s \( \wp \) function is not algebraic over \( K \). Then \( \wp \) is not the solution of any linear differential equation with coefficients in \( K \).

**Proof.** Let us assume that this equation exist, and let \( K \subset \mathcal{F} \) na associated its Galois extension. Its Galois group \( \text{Gal}(\mathcal{F}/K) \) is an affine group. We have an intermediate extension:

\[ K \subset K(\wp, \wp') \subset \mathcal{F}, \]

This intermediate extension \( K \subset K(\wp, \wp') \) is strongly normal and its Galois group is an elliptic curve. Thus, there is a normal subgroup \( H \triangleleft \text{Gal}(\mathcal{F}/K) \) and an exact sequence,

\[ 0 \rightarrow H \rightarrow \text{Gal}(\mathcal{F}/K) \rightarrow \mathcal{E} \rightarrow 0 \]

but the quotient group of an affine group is an affine group, and then \( \mathcal{E} \) is affine. \( \Box \)
From the Galois correspondence and some elemental properties of algebraic groups we also have immediately the characterization of Liouvillian and Kolchin extensions in terms of their Galois groups.

**Proposition 5.19** Let $K \subset L$ be a strongly normal extension.

1. $K \subset L$ is a Kolchin extension if and only if there is a sequence of normal subgroups in $\text{Gal}(L/K)$, 
   
   $$ H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_n = \text{Gal}(L/K), $$  

   such that $\dim_{\mathbb{C}} H_i/H_{i+1} \leq 1$.

2. $K \subset L$ is a strict-Liouville extension if and only if $\text{Gal}(L/K)$ is an affine solvable group.

3. $K \subset L$ is a Liouvillian extension if and only if the identity component $\text{Gal}^0(L/K)$ is a linear solvable group.

**Proof.** For (1) and (3) see [18]. Let us proof that linear solvable Galois group implies strict Liouville. Let us consider a resolution of the Galois group $H_0 \triangleleft \ldots \triangleleft H_n$ such that each quotient $H_{i+1}/H_i$ is a cyclic group, a multiplicative group or an additive group. This resolution exist by means of Lie-Kolchin theorem. This resolution split the extension $K \subset L$ in a tower of differential fields

$$ K_n \subset K_{n-1} \subset \ldots \subset K_0, $$

Each differential extension of the tower is an exponential, integral or algebraic extension with cyclic Galois group. But an algebraic extension with cyclic group is a radical extension. The field $\mathcal{C}$ is algebraically closed, hence such radical extension is generated by the radical $\sqrt[n]{a}$ of a non-constant element of $a$, and then it is the Picard-Vessiot extension of the equation,

$$ \partial x = \frac{\partial a}{na} x, $$

which is an exponential extension. □

### 5.8 Integration by Quadratures in Solvable Groups

Let us remind that along this chapter we are considering an automorphic vector field $\vec{A}$ with coefficients in $\mathcal{K}$ in an algebraic group $G$ defined over $\mathcal{C}$. We also consider a Kolchin closed differential point $\bar{\tau} \in \text{Diff}(G_\mathcal{K}, \partial_{\vec{A}})$ and the associated Galois extension $K \subset L$. We are going to explain the classical integration by quadratures in terms of Lie-Kolchin reduction method and Galois correspondence.

Let us consider a normal subgroup $H \triangleleft G$, and the quotient group $\bar{G} = G/H$. Let $\eta$ be the projection in $\bar{G}_\mathcal{K}$ of $\bar{\tau}$. In virtue of Theorem 4.44 we know that,

$$ K \subset \kappa(\eta) \subset L, $$

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is an intermediate strongly normal extension. Furthermore, the Galois group in \( \eta \) of the automorphic system with coefficients if \( \kappa(\eta) \) is the intersection of the Galois group \( \text{Gal}_x(G_K, \partial_X) \) with \( H \).

**Theorem 5.20** Assume that there is a resolution of \( G \),

\[
H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_n = G,
\]

such that \( \dim C H_i / H_{i+1} = 1 \), then \( K \subset L \) is a Kolchin extension.

**Proof.** Let us consider the quotients \( \bar{G}_i = H_n - i + 1 / H_n - i \). They are algebraic groups of dimension one. Each \( G_i \) is isomorphic to one of the following: the additive group, the multiplicative group, or an elliptic curve. Each one corresponds to an integral, exponential, or Weierstrassian quadrature. We prove the theorem by induction in the length of the resolution. Let us consider the projection \( \pi : G \to G / H_{n-1} \). Define \( \eta = \pi(\chi) \) and let \( K_1 \) be the relative algebraic closure of \( \kappa(\chi) \) in \( L \). Then \( K \subset \kappa(\eta) \) is an integral, exponential or elliptic extension and \( \kappa(\eta) \subset K_1 \) is an algebraic extension. Hence, \( K \subset K_1 \) is a Kolchin extension.

Let \( \mathfrak{z} \) be a closed differential point of \((G_K, \partial_X)\) in the fiber of \( x \). By Theorem 4.44 \( \text{Gal}_x(G_K, \partial_X) \subset H_{n-1} \), and then by Theorem 5.8 there is a gauge transformation \( L \tau \) with coefficients in \( K_1 \) reducing the automorphic field to an automorphic field in \( H_{n-1} \). Any Galois extension associated to this last equation is \( K_1 \)-isomorphic to \( L \). By the induction hypothesis the extension \( K_1 \subset L \) is a Kolchin extension, hence \( K \subset L \) is a Kolchin extension.

**Theorem 5.21** Assume that \( G \) is affine and solvable. Then \( K \subset L \) is a strict-Liouville extension.

**Proof.** The Galois group is a subgroup of \( G \), and then it is a solvable group. The result comes from Proposition 5.19 (2) together with Theorem 5.20.

**Proposition 5.22** If there is a connected affine solvable group \( H \subset G \) such that \( \text{Gal}_x(G_K, \partial_X) \subset H \), then \( K \subset L \) is a strict-Liouville extension.

**Proof.** \( H \) is connected affine solvable an then it has trivial Galois cohomology. We can reduce to the group \( H \) by means of theorem 5.5. Hence, we are in the hypothesis of theorem 5.21.

### 5.9 Linearization

There exist non-linear non-linearizable algebraic groups. An algebraic group that does not admit any linear representation is called quasi-abelian. In other words, a quasi-abelian variety is an algebraic group \( G \) such that \( \mathcal{O}_G(G) = \mathcal{C} \). Algebraic groups over an algebraic closed base field \( \mathcal{C} \), which are complete and connected, are called abelian varieties. Since they are complete varieties, they
do not admit non-constant global regular functions and then they are quasi-abelian.

The following results give us the structure of the algebraic groups by terms of linear and quasi-abelian algebraic groups. See, for instance [34].

**Theorem 5.23 (Rayleigh decomposition)** Let $G$ be an algebraic group. There is a unique subgroup $X \in G$ such that, $X$ is quasi-abelian and $G/X$ is an affine group.

**Theorem 5.24 (Chevalley-Barsotti-Sancho)** Let $G$ be a connected algebraic group over $\mathbb{C}$, with $\mathbb{C}$ an algebraically closed field of characteristic zero. Then there is a unique normal affine subgroup $N \subset G$ such that the quotient $G/N$ is an abelian variety.

### 5.10 Reduction by means Chevalley-Barsotti-Sancho Theorem

In virtue of Chevalley-Barsotti-Sancho theorem [5.23 in appendix B), there is a unique linear normal connected algebraic group $N \triangleleft G$ such that the quotient $G/N$ and is an abelian variety $V$. Let us consider the projection $\pi: G \to V$. Let $\tilde{B}$ be the projected automorphic system $\pi(\tilde{A})$ in $V$, and denote by $y$ the image of $x$ by $\pi$. We state the following:

**Theorem 5.25** Let $\mathcal{M}$ be the field of meromorphic functions in $V_K$. Assume that $Gal_\eta(V_K, \partial_{\tilde{B}}) = V$, and one of the following hypothesis:

1. $H^1(N, \mathcal{M})$ is trivial.
2. $K$ is relatively algebraically closed in $\mathcal{L}$.

Then, there is a gauge transformation of $G$ with coefficients in $\mathcal{M}$ reducing the automorphic system $\tilde{A}$ to $N$.

**Proof.** Let us consider $\tilde{A}$ as an automorphic vector field in $G$ with coefficients in $\mathcal{M}$. By Galois correspondence we have:

$$Gal(\mathcal{L}/\mathcal{M}) \simeq Gal_\eta(G_K, \partial_{\tilde{A}}) \cap N.$$ 

If hypothesis (1) holds, then the statement is a particular case of Theorem 5.5. Let us prove the result in the case of hypothesis (2). By Theorem 5.8 there exists a gauge transformations whose coefficients are algebraic over $\mathcal{M}$. By hypothesis $Gal_\eta(G_K, \partial_{\tilde{A}})$ is connected. This group $Gal_\eta(G_K, \partial_{\tilde{A}})$ realizes itself as a principal bundle over $V$ whose structural group is $Gal(\mathcal{L}/\mathcal{M})$. It implies that $Gal(\mathcal{L}/\mathcal{M})$ is also connected. So that $\mathcal{M}$ is relatively algebraically closed in $\mathcal{L}$. The coefficients of the considered gauge transformation are in $\mathcal{M}$, as we wanted to prove. □
5.11 Linearization by means of Adjoint Representation

We consider $GL(\mathcal{R}(G))$ the group of $\mathcal{C}$-linear automorphisms of the Lie algebra $\mathcal{R}$. It is an algebraic group over $\mathcal{C}$. The adjoint representation

$$\text{Adj}: G \to GL(\mathcal{R}(G))$$

is a morphism of algebraic groups. It gives us a linearization of the equations.

Let us consider the center $\mathfrak{Z}(G)$ and the exact sequence:

$$0 \to \mathfrak{Z}(G) \to G \to GL(\mathcal{R}(G)) \to 0$$

Denote by $\vec{B}$ the projection of the automorphic vector field $\vec{A}$ by the morphism $\text{Adj}$. It is a linear system and then its Galois extension $\mathcal{K} \subset \mathcal{P}$ is a Picard-Vessiot intermediate extension of $\mathcal{K} \subset \mathcal{L}$.

**Proposition 5.26** $\mathcal{P} \subset \mathcal{L}$ is a strongly normal extension and $\text{Gal}(\mathcal{L}/\mathcal{P})$ is an abelian group.

**Proof.** The extension $\mathcal{P} \subset \mathcal{L}$ is a Galois extension of $\vec{A}$ with coefficients in $\mathcal{P}$, so that it is strongly normal. Its Galois group is, by the Galois correspondence, the intersection of the Galois group of $\text{Gal}_\mathcal{K}(G_{\mathcal{K}}, \partial \vec{A})$ with the center $\mathfrak{Z}(G)$; it is an abelian group. □

5.12 Linearization by means of Global Regular Functions

The ring of global regular functions $\Gamma(\mathcal{O}_G, G)$ is a Hopf algebra, and then its spectrum is a linear algebraic group $L = \text{Spec}(\Gamma(\mathcal{O}_G, G))$. The kernel $C$ of the canonical morphism $\pi: G \to L$ is, by definition a quasi-abelian variety (see [34]). Let us consider the exact sequence:

$$0 \to C \to G \to L \to 0.$$

We proceed as we did in Proposition 5.26, and then we obtain the following result.

**Proposition 5.27** Let $\mathcal{K} \subset \mathcal{P}$ be the Picard-Vessiot extension of the automorphic system $\pi(\vec{A})$ in $L$. Then $\mathcal{P} \subset \mathcal{L}$ is a strongly normal extension, and the connected component of the identity of its Galois group is a quasi-abelian variety.

6 Integrability of Linear Equations

This section is devoted to the Liouville integrability of linear differential equations. Since the development of Picard-Vessiot system it is a rich field of research, let us cite some important specialized literature [20], [37], [38], [39], [12], [13]. Here, we adopt a slightly different point of view on linear differential
equations. We see them as automorphic systems. It gives us some insight into the geometric mechanisms that allows quadratures. In this way we are able to measure the solvability of the Galois groups, in terms of equations in flag varieties and grassmanians (Theorem 6.2). They are the natural geometrical generalization of Riccati equations.

From now on let $G$ be a linear connected algebraic group over $\mathbb{C}$. We consider $\vec{A}$ an automorphic vector field in $G$ with coefficients in $\mathcal{K}$.

6.1 Flag Variety

We call Borel subgroup of $G$ to any maximal connected solvable group of $G$. Borel subgroups are all conjugated and isomorphic subgroups. The quotient space $G/B$ is a complete variety (see [34] p. 163, th. 10.2).

**Definition 6.1** We call flag variety of $G$ to the homogeneous space quotient $G/B$, being $B$ a Borel subgroup of $G$.

The flag variety of $G$ is defined up to isomorphism of $G$-homogeneous spaces. Let us consider $\text{Flag}(G)$ a flag variety of $G$, and let $(\text{Flag}(G), \partial \vec{F})$ be the induced Lie-Vessiot system.

Let us see a natural generalization of the well-known theorem of J. Liouville that relates the integrability by Liouvillian functions of the second order linear homogeneous differential equation with the existence of an algebraic solution of an associated Riccati equation. This classical result is the particular case of $GL(2, \mathbb{C})$ in the following general Liouville’s theorem.

**Theorem 6.2** The Galois extension $\mathcal{K} \subset \mathcal{L}$ is Liouvillian if and only if the flag Lie-Vessiot system $(\text{Flag}(G), \partial \vec{F})$ has an algebraic solution with coefficients in $\mathcal{K}^0$, the algebraic relative closure of $\mathcal{K}$ in $\mathcal{L}$.

**Proof.** By the Galois correspondence we have that the Galois group of $(G_{\mathcal{K}^0}, \partial \vec{A})$ is the connected identity component of the Galois group of $(G_{\mathcal{K}}, \partial \vec{A})$. Assume that $(\text{Flag}(G), \partial \vec{F})$ has an algebraic solution $x \in \text{Flag}(G)(\mathcal{K}^0)$. We are under the hypothesis of Theorem 5.8. There is a gauge transformation of $G_{\mathcal{K}^0}$ that send $\vec{A}$ to an automorphic vector field $\vec{B}$ in the Borel subgroup $B$. Then the Galois group of $\vec{B}$ with coefficients in $\mathcal{K}^0$ is contained in a Borel subgroup. Then the connected component of $\text{Gal}_2(G_{\mathcal{K}^0}, \partial \vec{A})$ is solvable.

Reciprocally, let us assume that $\mathcal{K} \subset \mathcal{L}$ is a Liouvillian extension. In such case the identity connected component of the Galois group is contained in a Borel subgroup $B$. By Proposition 6.4 there is a solution with coefficients in $\mathcal{K}^0$ of $\vec{F}$. □

6.2 Automorphic Equations in the General Linear Group

6.3 Grassmanians

Let us consider $E$ as $n$-dimensional vector space. Along this text $m$-plane will mean $m$-dimensional linear subspace. For all $m \leq n$ the linear group $GL(E)$
acts transitively in the set of $m$-planes. For an $m$-plane $E_m$, the stabilizer subgroup is an algebraic group, and then the set of $m$-planes define an algebraic homogeneous space.

**Definition 6.3** We call grassmanian of $m$-planes of $E$, $\text{Gr}(E,m)$, to the homogeneous space whose closed points are the $m$-planes of $E$. Denote $\text{Gr}(C,n,m)$ the grassmanian of $m$-planes of $C^n$.

**Example 6.4** $	ext{Gr}(C,n,1)$ is the space of lines in $C^n$, and then if its the projective space of dimension $n-1$, $\mathbb{P}(n-1,C)$. The $\text{Gr}(C,n,n-1)$ is the space of hyperplanes and then it is the dual projective space $\mathbb{P}(n-1,C)^*$. 

In general, $m$-planes of $E$ are in one-to-one correspondence with $(n-m)$-planes of the dual space $E^*$, and then we have the projective duality

$$\text{Gr}(E,m) \cong \text{Gr}(E^*,n-m).$$

The action of $GL(E)$ on $\text{Gr}(E,m)$ is not faithful. Each scalar matrix of the center of $GL(C,n)$ fix all $m$-planes. Thus, the non faithful action of $GL(E)$ is reduced to a faithful action of the projective group $PGL(E)$.

All grassmanian are projective varieties. There is a canonical embedding of $\text{Gr}(E,m)$ into the projective space of dimension $(\binom{n}{m})-1$, called the plücker embedding:

$$\text{Gr}(E,m) \to \mathbb{P}(E^{\wedge n}), \quad \langle e_1, \ldots, e_m \rangle \mapsto \langle e_1 \wedge e_1 \wedge \ldots \wedge e_m \rangle.$$

For computation in the grassmanian spaces we will use plückerian coordinates. This system of coordinates is subordinated to a basis in $E$. Thus, let us consider a basis $\{e_1, \ldots, e_n\}$. Let $E_1 = \langle e_1, \ldots, e_m \rangle$ be the $m$ plane spanned by the first $m$ elements of the basis, and define $E_2 = \langle e_{m+1}, \ldots, e_n \rangle$ its complementary. Let us consider the projection $\pi: E \to E_2$ of kernel $E_1$. We define the open subset $U \subset \text{Gr}(E,m)$,

$$U = \{F: F \oplus E_2 = E\}.$$

For $F \in U$ the splitting of the space induces an isomorphism $i_F: E_1 \to F$. We have an isomorphism

$$U \cong \text{Hom}_C(E_1, E_2), \quad F \mapsto \pi \circ i_F.$$

We define the plückerian coordinates of $F$ as the matrix elements of $\pi \circ i_F$ in the above mentioned basis. By permuting the elements of the basis we construct a covering of $\text{Gr}(E,m)$ by $(\binom{n}{m})$ affine open subsets isomorphic to $C^{n(n-m)}$.

Let us compute plückerian coordinates in $\text{Gr}(C,m,n)$ related to the canonical basis. Let us consider $F \in \text{Gr}(C,m,n)$, and a basis of $F$, $\{\bar{x}_1, \ldots, \bar{x}_m\}$, $\bar{x}_i =$
\((x_{1i}, \ldots, x_{ni})\). The matrix,
\[
\begin{pmatrix}
x_{11} & \cdots & x_{1m} \\
x_{21} & \cdots & x_{2m} \\
\vdots & \ddots & \vdots \\
x_{n1} & \cdots & x_{nm}
\end{pmatrix}
\]
is of maximal rank. Thus, there is a non vanishing minor of rank \(m\). In particular, \(F\) is in the open subset \(U\) if and only if the minor corresponding to the first \(m\) rows does not vanish. In such case we define the numbers \(\lambda^{(m)}_{ij}\)
\[
\begin{pmatrix}
x_{11} & \cdots & x_{1m} \\
x_{21} & \cdots & x_{2m} \\
\vdots & \ddots & \vdots \\
x_{n1} & \cdots & x_{nm}
\end{pmatrix}^{-1} =
\begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
\lambda^{(m)}_{11} & \cdots & \lambda^{(m)}_{1m} \\
\vdots & \ddots & \vdots \\
\lambda^{(m)}_{n-m,1} & \cdots & \lambda^{(m)}_{n-m,m}
\end{pmatrix}
\]
that are the plückerian coordinates of \(E_m \in \text{Gr}(\mathbb{C}, m,n)\) in the open affine subset \(U\) related to the split of \(\mathbb{C}^n\) as \(E_1 \otimes E_2\).

### 6.4 Flag Variety of the General Linear Group

A flag of subspaces of \(\mathbb{C}^n\), is a sequence,
\[
E_1 \subset E_2 \subset \ldots \subset E_{n-1}, \quad \dim_{\mathbb{C}} E_i = i
\]
of linear subspaces of \(\mathbb{C}^n\). The space \(\text{Flag}(\mathbb{C}, n)\) of flags of \(\mathbb{C}^n\) is an homogeneous space of \(\text{GL}(\mathbb{C}, n)\), and it is faithful for the action of \(\text{PGL}(\mathbb{C}, n)\). There is a canonical morphism,
\[
\text{Flag}(\mathbb{C}, n) \to \prod_{m=1}^{n-1} \text{Gr}(\mathbb{C}, n,m), \quad E_1 \subset E_2 \subset E_{n-1} \mapsto (E_1, \ldots, E_{n-1}).
\]

By Lie-Kolchin theorem the isotropy subgroup of a flag is also a Borel subgroup. Then, we can state \(\text{Flag}(\mathbb{C}, n)\) is the flag variety of the general linear group. Let us introduce a system of coordinates in \(\text{Flag}(\mathbb{C}, n)\). Let us consider \(\{e_1, \ldots, e_n\}\) the canonical basis of \(\mathbb{C}^n\). Each \(\sigma \in \text{GL}(\mathbb{C}, n)\) defines a flag \(F(\sigma)\) as follows:
\[
(\sigma(e_1)) \subset (\sigma(e_1), \sigma(e_2)) \subset \ldots \subset (\sigma(e_1), \ldots, \sigma(e_{n-1})).
\]

There is a canonical flag corresponding to the identity element. Its isotropy group is precisely \(T(\mathbb{C}, n)\) the group of upper triangular matrices. Then two matrices \(A, B \in \text{GL}(\mathbb{C}, n)\) define the same flag if and only if \(A = BU\) for certain
$U \in T(\mathcal{C}, n)$. Then let us consider the affine subset of $GL(\mathcal{C}, n)$ of matrices with non-vanishing principal minors. For such a matrix there exist a unique $LU$ decomposition such that $U \in T(\mathcal{C}, n)$ and is a lower triangular matrix as follows,

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \lambda_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & 1 \end{pmatrix}$$

Hence the matrix elements $\lambda_i$ define a system of affine coordinates in $Flag(\mathcal{C}, n)$, in certain affine open subset. We construct an open covering of the flag space by permuting the vectors of the canonical base. The canonical morphism

$$Flag(\mathcal{C}, n) \to \prod_m \text{Gr}(\mathcal{C}, m, n)$$

is easily written in Plückerian coordinates:

$$\lambda^{(m)}_{ij} = \lambda_{i+m,j} - \sum_{k=1}^m \lambda_{i+m,k} \lambda_{kj}.$$  

6.5 Matrix Riccati Equations

Let us consider an homogeneous linear differential equation

$$\dot{x} = Ax, \quad A \in gl(\mathcal{K}, n).$$

It is seen as an automorphic system that induces Lie-Vessiot systems in each homogeneous space. Let us compute the induced Lie-Vessiot systems in the grassmanian spaces. First, the linear system induces a linear system in $(\mathbb{C}^n)^m$.

$$\dot{X} = AX, \quad (6.1)$$

where $X$ is a $n \times m$ matrix. We write $X = (Y^U)$, being $U$ a $m \times m$ matrix and $Y$ a $(n-m) \times m$ matrix. $\Lambda_m = YU^{-1}$ is the matrix of Plückerian coordinates of the space generated by the $m$ column vectors of the matrix $X$. Then, $\dot{\Lambda}_m = YU^{-1} - \Lambda_m U^* U^{-1}$. If we decompose the matrix $A$ in four submatrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

being $A_{11}$ of type $m \times m$, $A_{12}$ of type $m \times (n-m)$, $A_{21}$ of type $(n-m) \times m$, and $A_{22}$ if type $m \times m$. Then the matrix linear equation (6.1) splits as a system of matrix linear differential equations,

$$\dot{U} = A_{11} + A_{12}Y, \quad \dot{Y} = A_{21}U + A_{22}Y,$$

from which we obtain the differential equation for affine coordinates in the grassmanian,

$$\dot{\Lambda}_m = A_{21} + A_{22}\Lambda_m - \Lambda_m A_{11} - \Lambda_m A_{12} \Lambda_m \quad (6.2)$$
which is a quadratic system. We call such a system a **matrix Riccati equation**
associated to the linear system.

\[ \Lambda_m = \begin{pmatrix}
\lambda_{11}^{(m)} & \cdots & \lambda_{1,m}^{(m)} \\
\vdots & \ddots & \vdots \\
\lambda_{n-m,1}^{(m)} & \cdots & \lambda_{n-m,m}^{(m)} 
\end{pmatrix} \]

\[ \dot{\lambda}_{ij}^{(m)} = a_{m+i,j} + \sum_{k=1}^{n-m} a_{m+i,m+k} \lambda_{kj}^{(m)} - \sum_{k=1}^{m} \lambda_{ik}^{(m)} a_{kj} - \sum_{r=1}^{n-m} \lambda_{ik}^{(m)} a_{k,r+m} \lambda_{rj}^{(m)} \]

**Example 6.5** Let us compute the matrix Riccati equations associated to the general linear system of rank 2 and 3. First, let us consider a general linear system of rank 2,

\[ \dot{x}_1 = a_{11} x_1 + a_{12} x_2, \quad \dot{x}_2 = a_{21} x_1 + a_{22} x_2. \]

There is one only grassmanian \( \text{Gr}(C, 1, 2) \), which is precisely the projective line. The associated matrix Riccati equation is an ordinary Riccati equation

\[ \dot{x} = a_{21} + (a_{22} - a_{11}) x - a_{12} x^2. \]

In the case of a general system of rank 3,

\[ \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \]

there are two grassmanian spaces, \( \text{Gr}(C, 1, 3) \) and \( \text{Gr}(C, 2, 3) \), being the projective plane \( \mathbb{P}^2(C) \) and the projective dual plane \( \mathbb{P}^2(C)^\ast \) respectively. Then we obtain two quadratic systems,

\[ \mathbb{P}(2, C) \begin{cases} \dot{x} = a_{21} + (a_{22} - a_{11}) x + a_{23} y - a_{12} x^2 - a_{13} xy \\ \dot{y} = a_{31} + (a_{33} - a_{11}) x + a_{32} y - a_{12} x^2 - a_{13} xy \end{cases} \]

\[ \mathbb{P}(2, C)^\ast \begin{cases} \dot{\xi} = a_{31} + (a_{33} - a_{11}) \xi + a_{21} \eta - a_{23} \xi \eta - a_{13} \xi^2 \\ \dot{\eta} = a_{32} + (a_{33} - a_{22}) \eta + a_{12} \xi - a_{13} \xi \eta - a_{23} \eta^2 \end{cases} \]

called the associated projective Riccati equations.

### 6.6 Flag Equation

From the relation between plückerian coordinates and affine coordinates in the flag variety we can deduce the equations of the induced Lie-Vessiot system in \( \text{Flag}(C, n) \), from the matrix Riccati equations. We will obtain a Riccati quadratic equation for \( n = 2 \), and a cubic system for \( n \geq 3 \).

\[ \dot{\lambda}_{ij} = a_{ij} + \sum_{k=j+1}^{n} a_{ik} \lambda_{kj} - \sum_{k=1}^{j} \lambda_{ik} a_{kj} + \sum_{k=1}^{j} \sum_{r=k+1}^{j} \lambda_{ir} \lambda_{rk} a_{kj} \]
\[-\sum_{k=1}^{j} \sum_{r=j+1}^{n} \lambda_{ik} a_{kr} \lambda_{rj} + \sum_{k=1}^{j} \sum_{r=j+1}^{n} \lambda_{rk} a_{kr} \lambda_{rj},\]

Setting \(\lambda_{ii} = 1\) for all \(i\), we can simplify these equations.

\[
\dot{\lambda}_{ij} = \sum_{k=j}^{n} a_{ik} \lambda_{kj} - \sum_{k=1}^{j} \sum_{r=j}^{n} \lambda_{ik} a_{kr} \lambda_{rj} + \sum_{k=1}^{j} \sum_{r=k+1}^{n} \sum_{s=j}^{n} \lambda_{ir} \lambda_{ks} a_{kr} \lambda_{sj}\]

Such as cubic system can be seen as a hierarchy of projective Riccati equations.

The equation corresponding to the first column \(\lambda_{i1}, i = 2, \ldots, n\) is a projective Riccati equation in \(P(n-1,\mathbb{C})\). The equation corresponding to the second column is a projective Riccati equation in \(P(n-2,\mathbb{C}^{(\lambda_{i1})})\), and so on.

**Example 6.6** Let us compute the flag equation for the general differential linear system of rank 3. Denote \(x = \lambda_{21}, y = \lambda_{31}, z = \lambda_{32}\).

\[
\begin{align*}
\dot{x} &= a_{21} + (a_{22} - a_{11})x + a_{23}y - a_{12}x^2 - a_{13}xy \\
\dot{y} &= a_{31} + a_{32}x + (a_{33} - a_{11})y - a_{12}xy - a_{13}y^2 \\
\dot{z} &= a_{32} - a_{12}y + (a_{33} - a_{22} + a_{12}y - a_{13}y)z + (a_{13}y - a_{23})z^2.
\end{align*}
\]

### 6.7 Equations in the Special Orthogonal Group

Automorphic equations in special orthogonal group have been deeply studied since 19th century \([44], [10]\). In particular Darboux related these equation with Riccati equation. He stated that the integration of (6.5) is reduced to the integration of (6.11). Here we show that the Flag equation of an automorphic equation in \(SO(3,\mathbb{C})\) is precisely the Riccati equation, and then the solutions of (6.5) are Liouvillian if and only if there are algebraic solutions for (6.11).

The Lie algebra so(3,\(\mathbb{C}\)) is the algebra of skew-symmetric matrices of \(gl(3,\mathbb{C}\)). Then an automorphic system in \(SO(3,\mathbb{C})\) is written in the following form.

\[
\begin{pmatrix}
\dot{x}_0 \\
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
a & b & 0 \\
-a & c & 0 \\
-b & -c & 0
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_1 \\
x_2
\end{pmatrix},
\]

where the void spaces represent the vanishing elements in the matrix.

### 6.8 On the Structure of the Special Orthogonal Group

The special orthogonal group is the group of linear transformations preserving the quadratic form \(x_0^2 + x_1^2 + x_2^2\). Let us consider the non degenerated quadric in the projective space \(S_2 \subset \mathbb{P}(3,\mathbb{C})\), defined by homogeneous equation \(\{t_0^2 + t_1^2 + t_2^2 = 0\}\). In affine coordinates \(x_i = \frac{t_i}{t_3}\), its affine part is a sphere of radius 1. Thus \(SO(3,\mathbb{C})\) is a subgroup of algebraic automorphisms of the quadric; \(SO(3) \subset Aut(S_2)\).
Each non-degenerate quadric in the projective space over an algebraically closed field is a hyperbolic ruled surface. It has two systems of generatrices, being each system parameterized by a projective line. Denote $P_1$, $P_2$ these projective lines, $p \in P_1$, and $q \in P_2$ are lines $S_2$, and they intersect in a unique point $s(p, q) \in p \cap q$. We have a decomposition of $S_2$ which is a particular case of Segre isomorphism,

$$P_1 \times c P_2 \sim S_2 \subset \mathbb{P}(3, \mathcal{C})$$

$$(u_0 : u_1, (v_0 : v_1)) \mapsto (t_0 : t_1 : t_2 : t_3) \begin{cases} 
t_0 = u_0v_1 + u_1v_0 \\
t_1 = u_1v_1 - u_0v_0 \\
t_2 = i(u_1v_1 + u_0v_0) \\
t_3 = u_0v_1 - u_1v_0 \end{cases}$$

Let us consider any algebraic automorphism of $S_2$, $\tau \colon S_2 \rightarrow S_2$. In particular, it must carry a system of generatrices to a system of generatrices. Let us denote $P_1$, $P_2$ to the two system of generatrices of $S_2$. Hence, $\tau$ is induces by a pair of projective transformations $(\tau_1, \tau_2)$, where

$$\tau_1 \colon P_1 \rightarrow P_1, \quad \tau_2 \colon P_2 \rightarrow P_2$$

or

$$\tau_1 \colon P_1 \rightarrow P_2, \quad \tau_2 \colon P_2 \rightarrow P_1.$$ 

We conclude that the group of automorphism of $S_2$ is isomorphic to the following algebraic group,

$$\text{Aut}(S_2) = PGL(1, \mathcal{C}) \times c PGL(1, \mathcal{C}) \times c \mathbb{Z}/2\mathbb{Z}.$$ 

Let us compute the image of the canonical monomorphism $SO(3, \mathcal{C}) \subset \text{Aut}(S_2)$. We take affine coordinates in the pair of projective lines, $x = \frac{u_0}{u_1}$, $y = \frac{v_0}{v_1}$. This is the system of symmetric coordinates of the sphere introduced by Darboux [10].

$$x_0 = \frac{1 - xy}{x - y}, \quad x_1 = \frac{1 + xy}{x - y}, \quad x_2 = \frac{x + y}{x - y} \quad (6.6)$$

$$x = \frac{x_0 + ix_1}{1 - x_2}, \quad y = \frac{x_2 - 1}{x_1 - ix_2} \quad (6.7)$$

Let us write a general element of $SO(3, \mathcal{C})$ in affine coordinates,

$$R_{\lambda, \mu, \nu} = \begin{pmatrix} 1 \\
\frac{\lambda + \lambda^{-1}}{2} \\
\frac{\lambda^{-1} - \lambda}{2i} \\
\frac{\lambda^{-1} + \lambda}{2i} \end{pmatrix} \begin{pmatrix} \mu + \mu^{-1} \\
\mu - \mu^{-1} \\
\nu - \nu^{-1} \\
\nu + \nu^{-1} \end{pmatrix} \begin{pmatrix} 1 \\
\frac{\mu + \mu^{-1}}{2} \\
\frac{\nu - \nu^{-1}}{2} \\
\frac{\nu + \nu^{-1}}{2} \end{pmatrix}$$

where, in the complex case $\lambda = e^{i\alpha}$, $\mu = e^{i\beta}$, $\nu = e^{i\gamma}$ are the exponentials of the Euler angles. Direct computation gives us,

$$R_{\lambda, \mu, \nu} \begin{pmatrix} x \\
y \end{pmatrix} \mapsto \begin{pmatrix} (\lambda \mu \nu + \lambda \nu \nu + \lambda \nu \nu - \lambda \mu - \lambda \nu - \lambda - 1)x + \lambda \mu \nu + \lambda \nu \nu - \lambda \mu + \lambda \nu - \lambda - 1 \\
(\lambda \mu \nu + \lambda \nu \nu - \lambda \mu - \lambda \nu - \lambda - 1)x + \lambda \mu \nu + \lambda \nu \nu - \lambda \mu + \lambda \nu + \lambda + 1 \\
(\lambda \mu \nu + \lambda \nu \nu + \lambda \nu \nu - \lambda \mu - \lambda \nu - \lambda - 1)y + \lambda \mu \nu + \lambda \nu \nu - \lambda \mu + \lambda \nu + \lambda + 1 \\
(\lambda \mu \nu + \lambda \nu \nu - \lambda \mu - \lambda \nu - \lambda - 1)y + \lambda \mu \nu + \lambda \nu \nu + \lambda \mu + \lambda \nu + \lambda + 1 \end{pmatrix} = r_{\lambda, \mu, \nu}(x)$$

$$r_{\lambda, \mu, \nu} \begin{pmatrix} x \\
y \end{pmatrix} = \begin{pmatrix} x \\
y \end{pmatrix}$$

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and then \( R_{\lambda,\mu,\nu} \) induces the same projective transformation \( r_{\lambda,\mu,\nu} \) for \( x \) and \( y \). Hence,

\[
SO(3) \subseteq PGL(1, \mathbb{C}) \subset Aut(S_2).
\]

In particular, we have the following formulae for rotations around euclidean axis:

\[
\begin{pmatrix}
1 & \frac{\lambda + \lambda^{-1}}{2} & \frac{\lambda^{-1} - \lambda}{2i} \\
\frac{\lambda - \lambda^{-1}}{2i} & 1 & \frac{\lambda^{-1} - \lambda}{2i} \\
\frac{\lambda + \lambda^{-1}}{2} & \frac{\lambda^{-1} - \lambda}{2i} & 1
\end{pmatrix} : x \mapsto \frac{(\lambda + 1)x + (\lambda - 1)}{(\lambda - 1)x + (\lambda + 1)} \quad (6.8)
\]

\[
\begin{pmatrix}
\frac{\lambda + \lambda^{-1}}{2} & \frac{\lambda^{-1} - \lambda}{2i} & 1 \\
\frac{\lambda - \lambda^{-1}}{2i} & 1 & \frac{\lambda^{-1} - \lambda}{2i} \\
\frac{\lambda + \lambda^{-1}}{2} & \frac{\lambda^{-1} - \lambda}{2i} & 1
\end{pmatrix} : x \mapsto \lambda x \quad (6.9)
\]

\[
\begin{pmatrix}
\frac{\lambda + \lambda^{-1}}{2} & \frac{\lambda^{-1} - \lambda}{2i} & 1 \\
\frac{\lambda - \lambda^{-1}}{2i} & 1 & \frac{\lambda^{-1} - \lambda}{2i} \\
\frac{\lambda + \lambda^{-1}}{2} & \frac{\lambda^{-1} - \lambda}{2i} & 1
\end{pmatrix} : x \mapsto \frac{(\lambda + \lambda^{-1} + 1/2)x - i(\lambda - \lambda^{-1})}{i(\lambda^{-1} - \lambda)x - (\lambda + \lambda^{-1} + 1/2)} \quad (6.10)
\]

An the following formulae for the induced Lie algebra morphism – the are computed by derivation of previous formulae with \( \lambda = 1 + i \varepsilon \). Here the Lie algebra \( pgl(1, \mathbb{C}) \) is identified with \( sl(2, \mathbb{C}) \):

\[
\begin{pmatrix}
-1 & 1 \\
0 & -1
\end{pmatrix} \mapsto \begin{pmatrix}
\frac{i}{2} & -i \\
-i & \frac{i}{2}
\end{pmatrix}
\]

Reciprocally, a projective transformation

\[
x \mapsto \frac{u_{11}x + u_{12}}{u_{21}x + u_{22}}; \quad y \mapsto \frac{u_{11}y + u_{12}}{u_{21}y + u_{22}}
\]

induces a linear transformation in the affine coordinates \( x_0, x_1, x_2 \) (see [10] p. 34). \( SO(\mathbb{C}, 3) \) is precisely the group of automorphisms of \( S_2 \) that are linear in those coordinates. We have proven the following proposition which is due to Darboux.

**Proposition 6.7** The special orthogonal group \( SO(3, \mathbb{C}) \) over an algebraically closed field is isomorphic to the projective general group \( PGL(1, \mathbb{C}) \). The isomorphism is given by formulae (6.8), (6.9), (6.10).
6.9 Flag Equation

The flag variety of $SO(3, \mathbb{C})$ is a projective line. Any of the Darboux symmetric coordinates,

$$x : S_2 \rightarrow P_1$$

gives us a realization of the action of $SO(3)$ on $P_1$. By substituting the equation (6.5) in the identities (6.6), (6.7) we deduce the Riccati differential equation satisfied by this symmetric coordinate, which is the flag equation of equation (6.5):

$$\dot{x} = \frac{-b - ic}{2} - iax + \frac{-b + ic}{2} x^2.$$

(6.11)

In [10], Darboux reduces the integration of the equation (6.5) to finding two different particular solutions of the Riccati equation (6.11). By application of our generalization of Liouville’s theorem we obtain an stronger result.

**Theorem 6.8 (Darboux)** The Galois extension of the equation (6.5) is a Liouvillean extension of $K$ if and only if the Riccati equation (6.11) has an algebraic solution.

**Proof.** It is a particular case of Theorem 6.2.$\square$

A Stalk formula for affine morphisms

A.1 Stalk Formula for Ring Morphisms

Let us consider a ring morphism $\varphi : \mathcal{R} \rightarrow \mathcal{R}'$, and $a \subset \mathcal{R}$ an ideal. We write $\varphi(a) \cdot \mathcal{R}'$ for the ideal of $\mathcal{R}'$ spanned by the image of $a$ by $\varphi$.

**Theorem A.1 (Stalk formula)** Let us consider $x \in \text{Spec}(\mathcal{R})$. The stalk $(\varphi^*)^{-1}(x) \subset \text{Spec}(\mathcal{R}')$ is homeomorphic to the spectrum of

$$\mathcal{R}'_{\varphi(x) \cdot \mathcal{R}'} / (\varphi(x) \cdot \mathcal{R}') \cdot \mathcal{R}' = (\mathcal{R}' / \varphi(x) \cdot \mathcal{R}')_{\varphi(x) \cdot \mathcal{R}'} = \mathcal{R}' \otimes_{\mathcal{R}} \kappa(x).$$

Let us note that we do two different processes in the computation of the stalk. First there is a process of localization: the spectrum of $\mathcal{R}'_{\varphi(x) \cdot \mathcal{R}'} = \mathcal{R}' \otimes_{\mathcal{R}} \mathcal{R}_x$ is identified with the set of prime ideals $y \subset \mathcal{R}'$ verifying $\varphi(y) \subseteq x$. Second there is a process of restriction, the spectrum of $\mathcal{R}' / \varphi(x) \cdot \mathcal{R}' = \mathcal{R}' \otimes_{\mathcal{R}} \mathcal{R} / x$ is identified with the set of prime ideals $y \subset \mathcal{R}'$ verifying $\varphi(y) \supseteq x$. These processes commute. When we take both together we obtain $\mathcal{R}' \otimes_{\mathcal{R}} \kappa(x)$. As expected, the canonical morphism $\mathcal{R}' \rightarrow \mathcal{R}' \otimes_{\mathcal{R}} \kappa(x)$, $a \mapsto a \otimes 1$ induces de immersion of the stalk into $\text{Spec}(\mathcal{R}')$. 

A.2 Stalk Formula for Change of Base Field

Definition A.2 Let $X$ be an $k$-scheme, and $k \hookrightarrow A$ a $k$-algebra. We write $X(A)$ for the set of $k$-scheme homomorphisms $\text{Spec}(A) \to X$. The functor

$$X : A \rightsquigarrow X(A) = \text{Hom}_k(\text{Spec}(A), X)$$

of the category of $k$-algebras in the category of sets, is called the functor of points of $X$. An element $x \in X(A)$ is called an $A$-point of $X$.

First, note that for each field extension $k \hookrightarrow K$ there is a map,

$$X(k) \to X, \quad x \mapsto x((0)), \quad (0) \subset K$$

following this map, $X(k)$ is identified with the set of points of $X$ whose rational field $\kappa(x)$ is $k$. We call these points rational points of $X$.

For any field extension $k \subset K$, the map $X(K) \to X$ is surjective onto the subset of points $x \in X$ for whom that there exist a commutative diagram,

$$
\begin{array}{ccc}
        & K & \\
\downarrow & & \downarrow \\
\kappa(x) & & \\
\end{array}
$$

and moreover, $X(K)$ is identified with the set of $K$-rational points of the $K$-scheme $X_K$:

$$
\begin{array}{ccc}
X \times \text{Spec}(K) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\text{Spec}(K) & \longrightarrow & X \\
\end{array}
$$

If $X$ is of finite type, then $X(\bar{k}) \hookrightarrow |X|_{\text{cl}} \subset |X|$ is surjective onto the subset of closed points of $X$.

Theorem A.3 There is a canonical one-to-one correspondence between the set $X(K)$ of $K$-points of $X$ and the set of rational points of the extended scheme $X_K$.

Proposition A.4 (Base change formula) Let $X$ be a $k$-scheme, $x \in X$, and $k \subset A$ a $k$-algebra. The stalk $\pi^{-1}(x)$ of $x$ by $\pi : X_A \to X$, is isomorphic to $\text{Spec}(\kappa(x) \otimes_k A)$.

Proof. First, assume that $X = \text{Spec}(B)$ is affine. Then, by stalk formula, we have

$$\pi^{-1}(x) = \text{Spec}(A \otimes_k B \otimes \kappa(x)) = \text{Spec}(A \otimes_k \kappa(x)),$$

the homeomorphism is induced by the ring morphism

$$A \otimes_k B \to A \otimes \kappa(x), \quad a \otimes f \mapsto a \otimes f(x).$$

If $X$ is not affine, then we cover it with affine subsets $U_i$. If $\pi(y) = x$, and $x \in U_i$, then $y \in U_i \times_k \text{Spec}(A)$ and the previous argument is sufficient. □
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