On the String Consensus Problem and the Manhattan Sequence Consensus Problem

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Abstract

In the Manhattan Sequence Consensus problem (MSC problem) we are given $k$ integer sequences, each of length $\ell$, and we are to find an integer sequence $x$ of length $\ell$ (called a consensus sequence), such that the maximum Manhattan distance of $x$ from each of the input sequences is minimized. For binary sequences Manhattan distance coincides with Hamming distance, hence in this case the string consensus problem (also called string center problem or closest string problem) is a special case of MSC. Our main result is a practically efficient $O(\ell)$-time algorithm solving MSC for $k \leq 5$ sequences. Practicality of our algorithms has been verified experimentally. It improves upon the quadratic algorithm by Amir et al. (SPIRE 2012) for string consensus problem for $k = 5$ binary strings. Similarly as in Amir’s algorithm we use a column-based framework. We replace the implied general integer linear programming by its easy special cases, due to combinatorial properties of the MSC for $k \leq 5$. We also show that for a general parameter $k$ any instance can be reduced in linear time to a kernel of size $k!$, so the problem is fixed-parameter tractable. Nevertheless, for $k \geq 4$ this is still too large for any naive solution to be feasible in practice.

1 Introduction

In the sequence consensus problems, given a set of sequences of length $\ell$ we are searching for a new sequence of length $\ell$ which minimizes the maximum distance to all the given sequences in some particular metric. Finding the consensus sequence is a tool for many clustering algorithms and as such has applications in unsupervised learning, classification, databases, spatial range searching, data mining etc [4]. It is also one of popular methods for detecting data commonalities of many strings (see [1]) and has a considerable number of applications in coding theory [6, 8], data compression [11] and bioinformatics [12, 15]. The consensus problem has previously been studied mainly in $\mathbb{R}^\ell$ space with the Euclidean distance and in $\Sigma^\ell$ (that is, the space of sequences over a finite alphabet $\Sigma$) with the Hamming distance. Other metrics were considered in [2]. We study the sequence consensus problem for Manhattan metric ($\ell_1$ norm) in correlation with the Hamming-metric variant of the problem.
The Euclidean variant of the sequence consensus problem is also known as the bounding sphere, enclosing sphere or enclosing ball problem. It was initially introduced in 2 dimensions (i.e., the smallest circle problem) by Sylvester in 1857 [21]. For an arbitrary number of dimensions, a number of approximation algorithms [4, 14, 20] and practical exact algorithms [7, 10] for this problem have been proposed.

The Hamming distance variant of the sequence consensus problem is known under the names of string consensus, center string or closest string problem. The problem is known to be NP-complete even for binary alphabet [8]. The algorithmic study of Hamming string consensus (HSC) problem started in 1999 with the first approximation algorithms [15]. Afterwards polynomial-time approximation schemes (PTAS) with different running times were presented in [3, 18, 19]. A number of exact algorithms have also been presented. Many of these consider a decision version of the problem, in which we are to check if there is a solution to HSC problem with distance at most \( d \) to the input sequences. Thus FPT algorithms with time complexities \( O(k\ell + kd^{d+1}) \) and \( O(k\ell + kd(16|\Sigma|)^d) \) were presented in [12] and [18], respectively.

An FPT algorithm parameterized only by \( k \) was given in [12]. It uses Lenstra’s algorithm [16] for a solution of an integer linear program of size exponential in \( k \) (which requires \( O(k!4.5k\ell) \) operations on integers of magnitude \( O(k!\ell) \), see [1]) and due to extremely large constants is not feasible for \( k \geq 4 \). This opened a line of research with efficient algorithms for small constant \( k \). A linear-time algorithm for \( k = 3 \) was presented in [12], a linear-time algorithm for \( k = 4 \) and binary alphabet was given in [5], and recently an \( O(\ell^2) \)-time algorithm for \( k = 5 \) and also binary alphabet was developed in [1].

For two sequences \( x = (x_1, \ldots, x_\ell) \) and \( y = (y_1, \ldots, y_\ell) \) the Manhattan distance (also known as rectilinear or taxicab distance) between \( x \) and \( y \) is defined as follows:

\[
\text{dist}(x, y) = \sum_{j=1}^{\ell} |x_j - y_j|.
\]

The Manhattan version of the consensus problem is formally defined as follows:

**Manhattan Sequence Consensus problem**

**Input:** A collection \( \mathcal{A} \) of \( k \) integer sequences \( a_i \), each of length \( \ell \);

**Output:** \( \text{OPT}(\mathcal{A}) = \min_{x} \max_{1 \leq i \leq k} \{ \text{dist}(x, a_i) \} \), the corresponding integer consensus sequence \( x \).

We assume that integers \( a_{i,j} \) satisfy \( |a_{i,j}| \leq M \), and all \( \ell, k, M \) fit in a machine word, so that arithmetics on integers of magnitude \( O(\ell M) \) take constant time.

For simplicity in this version of the paper we concentrate on computing \( \text{OPT}(\mathcal{A}) \) and omit the details of recovering the corresponding consensus sequence \( x \). Nevertheless, this step is included in the implementation provided.

**Example 1.** Let \( \mathcal{A} = ((120, 0, 80), (20, 40, 130), (0, 100, 0)) \). Then \( \text{OPT}(\mathcal{A}) = 150 \) and a consensus sequence is \( x = (30, 40, 60) \), see also Fig. 1.

Our results are the following:

- We show that **Manhattan Sequence Consensus problem** has a kernel with \( \ell \leq k! \) and is fixed-parameter linear with the parameter \( k \).
We present a practical linear-time algorithm for the Manhattan Sequence Consensus problem for \( k = 5 \) (which obviously can be used for any \( k \leq 5 \)).

Note that binary HSC problem is a special case of MSC problem. Hence, the latter problem is NP-complete. Moreover, the efficient linear-time algorithm presented here for MSC problem for \( k = 5 \) yields an equally efficient linear-time algorithm for the binary HSC problem and thus improves the result of [1].

**Organization of the paper.** Our approach is based on a reduction of the MSC problem to instances of integer linear programming (ILP). For general constant \( k \) we obtain a constant, though a very large, number of instances with a constant number of variables that we solve using Lenstra’s algorithm [16] which works in constant time (the constant coefficient of this algorithm is also very large). This idea is similar to the one used in the FPT algorithm for HSC problem [12], however for MSC it requires an additional combinatorial observation. For \( k \leq 5 \) we obtain a more efficient reduction of MSC to at most 20 instances of very special ILP which we solve efficiently without applying a general ILP solver.

In Section 2 we show the first steps of the reduction of MSC to ILP. In Section 3 we show a kernel for the problem of \( O(k!) \) size. In Section 4 we perform a combinatorial analysis of the case \( k = 5 \) which leaves 20 simple types of the sequence \( x \) to be considered. This analysis is used in Section 5 to obtain 20 special ILP instances with only 4 variables. They could be solved using Lenstra’s ILP solver. However, there exists an efficient algorithm tailored for this type of special instances. It can be found in Section A. Finally we analyze the performance of a C++ implementation of our algorithm in the Conclusions section.

### 2 From MSC Problem to ILP

Let us fix a collection \( \mathcal{A} = (a_1, \ldots, a_k) \) of the input sequences. The elements of \( a_i \) are denoted by \( a_{i,j} \) (for \( 1 \leq j \leq \ell \)). We also denote \( \text{dist}(x, \mathcal{A}) = \max \{ \text{dist}(x, a_i) : 1 \leq i \leq k \} \).

For \( j \in \{ 1, \ldots, n \} \) let \( \pi_j \) be a permutation of \( \{ 1, \ldots, k \} \) such that \( a_{\pi_j(1), j} \leq \cdots \leq a_{\pi_j(k), j} \), i.e. \( \pi_j \) is the ordering permutation of elements \( a_{1,j}, \ldots, a_{k,j} \). We also set \( s_{i,j} = a_{\pi_j(i), j} \), see Example 2. For some \( j \) there might be several possibilities for \( \pi_j \) (if \( a_{i,j} = a_{i',j} \) for some \( i \neq i' \)), we fix a single choice for each \( j \).

**Example 2.** Consider the following 3 sequences \( a_i \) and sequences \( s_i \) obtained by sorting columns:

\[
[a_{i,j}] = \begin{bmatrix}
120 & 0 & 80 \\
20 & 40 & 130 \\
0 & 100 & 0
\end{bmatrix}, \quad [s_{i,j}] = \begin{bmatrix}
0 & 0 & 0 \\
20 & 40 & 80 \\
120 & 100 & 130
\end{bmatrix}.
\]

The Manhattan consensus sequence is \( x = (30, 40, 60) \), see Fig. 1. In the figure, the circled numbers in \( j \)-th column are \( \pi_j(1), \pi_j(2), \ldots, \pi_j(k) \) (top-down).

**Definition 1.** A basic interval is an interval of the form \( [i, i + 1] \) (for \( i = 1, \ldots, k - 1 \)) or \( [i, i] \) (for \( i = 1, \ldots, k \)). The former is called proper, and the latter degenerate. An interval system is a sequence \( \mathcal{I} = (I_1, \ldots, I_\ell) \) of basic intervals \( I_j \).
Figure 1: Illustration of Example 2; \( \pi_1 = (3, 2, 1), \pi_2 = (1, 2, 3), \pi_3 = (3, 1, 2) \).

For a basic interval \( I_j \) we say that a value \( x_j \) is consistent with \( I_j \) if \( x_j \in \{ s_{i,j}, \ldots, s_{i+1,j} \} \) when \( I_j = [i, i+1] \) is proper, and if \( x_j = s_{i,j} \) when \( I_j = [i, i] \) is degenerate. A sequence \( \mathbf{x} \) is called consistent with an interval system \( \mathcal{I} = (I_j)_{j=1}^{\ell} \) if for each \( j \) the value \( x_j \) is consistent with \( I_j \).

For an interval system \( \mathcal{I} \) we define \( \text{OPT}(\mathcal{A}, \mathcal{I}) \) as the minimum \( \text{dist}(\mathbf{x}, \mathcal{A}) \) among all integer sequences \( \mathbf{x} \) consistent with \( \mathcal{I} \). Due to the following trivial observation, for every \( \mathcal{A} \) there exists an interval system \( \mathcal{I} \) such that \( \text{OPT}(\mathcal{A}) = \text{OPT}(\mathcal{A}, \mathcal{I}) \).

**Observation 1.** If \( \mathbf{x} \) is a Manhattan consensus sequence then for each \( j \), \( s_{1,j} \leq x_j \leq s_{k,j} \).

**Transformation of the input to an ILP.** Note that for all sequences \( \mathbf{x} \) consistent with a fixed \( \mathcal{I} \), the Manhattan distances \( \text{dist}(\mathbf{x}, \mathbf{a}_i) \) can be expressed as \( d_i + \sum_{j=1}^{\ell} e_{i,j}x_j \) with \( e_{i,j} = \pm 1 \). Thus, the problem of finding \( \text{OPT}(\mathcal{A}, \mathcal{I}) \) can be formulated as an ILP, which we denote \( \text{ILP}(\mathcal{I}) \). If \( I_j \) is a proper interval \( [i, i+1] \), we introduce a variable \( x_j \in \{ s_{i,j}, \ldots, s_{i+1,j} \} \). Otherwise we do not need a variable \( x_j \). The \( i \)-th constraint of \( \text{ILP}(\mathcal{I}) \) algebraically represents \( \text{dist}(\mathbf{x}, \mathbf{a}_i) \), see Example 3.

**Observation 2.** The optimal value of \( \text{ILP}(\mathcal{I}) \) is equal to \( \text{OPT}(\mathcal{A}, \mathcal{I}) \).

**Example 3.** Consider the following 5 sequences of length 7:

\[
[a_{i,j}] = \begin{bmatrix}
20 & 18 & 20 & 10 & 16 & 8 & 10 \\
11 & 6 & 7 & 17 & 14 & 14 & 17 \\
14 & 12 & 18 & 13 & 11 & 6 & 12 \\
19 & 8 & 16 & 18 & 12 & 19 & 19 \\
16 & 15 & 11 & 15 & 6 & 17 & 11
\end{bmatrix}
\]

and an interval system \( \mathcal{I} = ([2, 3], [2, 3], [2, 3], [3, 4], [3, 4], [3, 3], [3, 4]) \). An illustration of both can be found in Fig. 2.

We obtain the following ILP(\( \mathcal{I} \)), where \( x_1 \in [14, 16], x_2 \in [8, 12], x_3 \in [11, 16], x_4 \in [15, 17], x_5 \in [12, 14], x_7 \in [12, 17] \) and the sequence \( \mathbf{x} \) can be retrieved as \( \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, 14, x_7) \):

\[
\begin{array}{cccccccc}
20 - x_1 & + & 18 - x_2 & + & 20 - x_3 & + & x_4 - 10 & + & 16 - x_5 & + & 6 & + & x_7 - 10 & \leq & z \\
x_1 - 11 & + & x_2 - 6 & + & x_3 - 7 & + & 17 - x_4 & + & 14 - x_5 & + & 0 & + & 17 - x_7 & \leq & z \\
x_1 - 14 & + & 12 - x_2 & + & 18 - x_3 & + & x_4 - 13 & + & x_5 - 11 & + & 8 & + & x_7 - 12 & \leq & z \\
19 - x_1 & + & x_2 - 8 & + & 16 - x_3 & + & 18 - x_4 & + & x_5 - 12 & + & 5 & + & 19 - x_7 & \leq & z \\
16 - x_1 & + & 15 - x_2 & + & x_3 - 11 & + & x_4 - 15 & + & x_5 - 6 & + & 3 & + & x_7 - 11 & \leq & z
\end{array}
\]
Fact 1. Let $P$ from $\mathbb{R}^n$. Let $P$ be a program obtained from $P$ by replacing a variable $x_j$ with $-x_j$, i.e. setting $E_P(x_j) = -E_P(x_j)$ and $R_P(x_j) = -R_P(x_j)$. Then $\text{OPT}(P) = \text{OPT}(P')$.

Fact 2. Let $P$ be a program obtained from $P$ by removing the variable $x_j$ and replacing $x_j$ with $x_j + x_j'$, i.e. setting $R_P(x_j) = R_P(x_j) + R_P(x_j')$. Then $\text{OPT}(P') = \text{OPT}(P')$.

Proof. Let $(z, x_1, \ldots, x_n)$ be a feasible solution of $P$. Then setting $x_j := x_j + x_j'$ and removing the variable $x_j'$ we obtain a feasible solution of $P'$. Therefore $\text{OPT}(P') \leq \text{OPT}(P')$. For the proof of the other inequality, take a feasible solution $(z, x_1, \ldots, x_n)$ (with $x_j'$ missing) of $P'$. Note that $x_j \in R_P(x_j) = R_P(x_j) + R_P(x_j')$. Therefore one can split $x_j$ into $x_j + x_j'$ so that $x_j \in R_P(x_j)$ and $x_j' \in R_P(x_j')$. This way we obtain a feasible solution of $P$ and thus prove that $\text{OPT}(P) \leq \text{OPT}(P')$.

Figure 2: Illustration of Example 3: 5 sequences of length 7 together with an interval system. Notice that $I_6$ is a degenerate interval.

Note that $P = \text{ILP}(\mathcal{I})$ has the following special form, which we call $(\pm)\text{ILP}$:

$$\min z$$

$$d_i + \sum_j x_j e_{i,j} \leq z$$

$$x_j \in R_P(x_j)$$

where $e_{i,j} = \pm 1$ and $R_P(x_j) = \{\ell_j, \ldots, r_j\}$ for integers $\ell_j \leq r_j$. Whenever we refer to variables, it does not apply to $z$, which is of auxiliary character. Also, “$x_j \in R_P(x_j)$” are called variable ranges rather than constraints. We say that $(e_1, j, \ldots, e_k, j)$ is a coefficient vector of $x_j$ and denote it as $E_P(x_j)$. If the program $P$ is apparent from the context, we omit the subscript.

Simplification of ILP. The following two facts are used to reduce the number of variables of a $(\pm)\text{ILP}$. For $A, B \subseteq \mathbb{Z}$ we define $-A = \{-a: a \in A\}$ and $A + B = \{a + b: a \in A, b \in B\}$.

Fact 1. Let $P$ be a $(\pm)\text{ILP}$. Let $P'$ be a program obtained from $P$ by replacing a variable $x_j$ with $-x_j$, i.e. setting $E_P(x_j) = -E_P(x_j)$ and $R_P(x_j) = -R_P(x_j)$. Then $\text{OPT}(P) = \text{OPT}(P')$.

Fact 2. Let $P$ be a $(\pm)\text{ILP}$. Assume $E_P(x_j) = E_P(x_j')$ for $j \neq j'$. Let $P'$ be a program obtained from $P$ by removing the variable $x_j'$ and replacing $x_j$ with $x_j + x_j'$, i.e. setting $R_P(x_j) = R_P(x_j) + R_P(x_j')$. Then $\text{OPT}(P) = \text{OPT}(P')$. 

Proof. Let $(z, x_1, \ldots, x_n)$ be a feasible solution of $P$. Then setting $x_j := x_j + x_j'$ and removing the variable $x_j'$ we obtain a feasible solution of $P'$. Therefore $\text{OPT}(P') \leq \text{OPT}(P')$. For the proof of the other inequality, take a feasible solution $(z, x_1, \ldots, x_n)$ (with $x_j'$ missing) of $P'$. Note that $x_j \in R_P(x_j) = R_P(x_j) + R_P(x_j')$. Therefore one can split $x_j$ into $x_j + x_j'$ so that $x_j \in R_P(x_j)$ and $x_j' \in R_P(x_j')$. This way we obtain a feasible solution of $P$ and thus prove that $\text{OPT}(P) \leq \text{OPT}(P')$. 

\[\square\]
Lemma 1. For a \((\pm)\)ILP with \(k\) constraints one can compute in linear time an equivalent \((\pm)\)ILP with \(k\) constraints and up to \(2^{k-1}\) variables.

Proof. We apply Fact 1 to obtain \(e_{1,1} = e_{1,2} = \ldots = e_{1,\ell}\), this leaves at most \(2^{k-1}\) different coefficient vectors. Afterwards we apply Fact 2 as many times as possible so that there is exactly one variable with each coefficient vector.

Example 4. Consider the \((\pm)\)ILP \(P\) from Example 3. Observe that \(E_P(x_4) = E_P(x_7) = -E_P(x_2)\) and thus Facts 1 and 2 let us merge \(x_2\) and \(x_7\) into \(x_4\) with

\[
R_{P'}(x_4) = R_P(x_4) + R_P(x_7) - R_P(x_2) = [15, 17] + [12, 17] + [-12, -8] = [15, 26].
\]

Simplifying the constant terms we obtain the following \((\pm)\)ILP \(P'\):

\[
\begin{align*}
-x_1 - x_3 + x_4 - x_5 & + 60 \leq z \\
+x_1 + x_3 - x_4 - x_5 & + 24 \leq z \\
+x_1 - x_3 + x_4 + x_5 & - 12 \leq z \\
-x_1 - x_3 - x_4 + x_5 & + 57 \leq z \\
-x_1 + x_3 + x_4 + x_5 & - 9 \leq z
\end{align*}
\]

3 Kernel of MSC for Arbitrary \(k\)

In this section we give a kernel for the MSC problem parameterized with \(k\), which we then apply to develop a linear-time FPT algorithm. To obtain the kernel we need a combinatorial observation that if \(\pi_j = \pi_{j'}\) then the \(j\)-th and the \(j'\)-th column in \(A\) can be merged. This is stated formally in the following lemma.

Lemma 1. Let \(A = (a_1, \ldots, a_k)\) be a collection of sequences of length \(\ell\) and assume that \(\pi_j = \pi_{j'}\) for some \(1 \leq j < j' \leq \ell\). Let \(A' = (a'_1, \ldots, a'_{\ell})\) be a collection of sequences of length \(\ell - 1\) obtained from \(A\) by removing the \(j'\)-th column and setting \(a'_{i,j} = a_{i,j} + a_{i,j'}\). Then \(OPT(A) = OPT(A')\).

Proof. First, let us show that \(OPT(A') \leq OPT(A)\). Let \(x\) be a Manhattan consensus sequence for \(A\) and let \(x'\) be obtained from \(x\) by removing the \(j'\)-th entry and setting \(x'_j = x_j + x_{j'}\). We claim that \(dist(x', A') \leq dist(x, A)\). Note that it suffices to show that \(|x'_i - a'_{i,j}| \leq |x_j - a_{i,j}| + |x_{j'} - a_{i,j'}|\) for all \(i\). However, with \(x'_j = x_j + x_{j'}\) and \(a'_{i,j} = a_{i,j} + a_{i,j'}\), this is a direct consequence of the triangle inequality.

It remains to prove that \(OPT(A) \leq OPT(A')\). Let \(x'\) be a Manhattan consensus sequence for \(A'\). By Observation 1, \(x'_j\) is consistent with some proper basic interval \([i, i+1]\). Let \(d_{i,j} = x'_j - s'_{i,j}\) and \(D_{i,j} = s'_{i+1,j} - s'_{i,j}\). Also, let \(D_{i,j} = s_{i+1,j} - s_{i,j}\) and \(D_{i,j'} = s_{i+1,j'} - s_{i,j'}\). Note that, since \(\pi_j = \pi_{j'}\), \(D'_{i,j} = D_{i,j} + D_{i,j'}\). Thus, one can partition \(d_{i,j} = d_{i,j} + d_{i,j'}\) so that both \(d_{i,j}\) and \(d_{i,j'}\) are non-negative integers not exceeding \(D_{i,j}\) and \(D_{i,j'}\) respectively. We set \(x_j = s_{i,j} + d_{i,j}\) and \(x_{j'} = s_{i,j'} + d_{i,j'}\), and the remaining components of \(x\) correspond to components of \(x'\). Note that \(x'_j = x_j + x_{j'}\) and that both \(x_j\) and \(x_{j'}\) are consistent with \([i, i+1]\). Consequently for any sequence \(a_m\) it holds that \(dist(x, a_m) = dist(x', a_m)\) and therefore \(dist(x, A) = dist(x', A')\), which concludes the proof.

Finally, note that the procedures given above can be used to efficiently convert between the optimum solutions \(x\) and \(x'\).
By Lemma 1, to obtain the desired kernel we need to sort the elements in columns of $A$ and afterwards sort the resulting permutations $\pi_j$:

**Theorem 1.** In $O(\ell k \log k)$ time one can reduce any instance of MSC to an instance with $k$ sequences of length $\ell'$, with $\ell' \leq k!$.

**Remark 1.** For binary instances, if permutations $\pi_j$ are chosen appropriately, it holds that $\ell' \leq 2^k$.

**Theorem 2.** For any integer $k$, the MANHATTAN SEQUENCE CONSENSUS problem can be solved in $O(\ell k \log k + 2^{k!} k + O(k^2) \log M)$ time.

**Proof.** We solve the kernel from Theorem 1 by considering all possible interval systems $I$ composed of proper intervals. The sequences in the kernel have length at most $k!$, which gives $(k-1)^k! (\pm)\text{ILPs}$ of the form $\text{ILP}(I)$ to solve.

Each of the $(\pm)\text{ILPs}$ initially has $k$ constraints on $k!$ variables but, due to Corollary 1, the number of variables can be reduced to $2^{k-1}$. Lenstra’s algorithm with further improvements [13, 9, 17] solves ILP with $p$ variables in $O(p^{2.5p+o(p)} \log L)$ time, where $L$ is the bound on the scope of variables. In our case $L = O(\ell M)$, which gives the time complexity of:

$$O\left((k-1)^k! \cdot 2^{(k-1)(2.5 \cdot 2^{k-1} + o(2^{k-1}))} \log M\right) = O\left(2^{k! \log k + O(k^2)} \log M\right).$$

This concludes the proof of the theorem.

\[ \square \]

## 4 Combinatorial Characterization of Solutions for $k = 5$

In this section we characterize those Manhattan consensus sequences $x$ which additionally minimize $\sum_i \text{dist}(x, a_i)$ among all Manhattan consensus sequences. Such sequences are called here sum-MSC sequences. We show that one can determine a collection of 20 interval systems, so that any sum-MSC sequence is guaranteed to be consistent with one of them. We also prove some structural properties of these systems, which are then useful to efficiently solve the corresponding $(\pm)\text{ILPs}$.

We say that $x_j$ is in the center if $x_j = s_{3,j}$, i.e. $x_j$ is equal to the column median. Note that if $x_j \neq s_{3,j}$, then moving $x_j$ by one towards the center decreases by one $\text{dist}(x, a_i)$ for at least three sequences $a_i$.

**Definition 2.** We say that $a_i$ governs $x_j$ if $x_j$ is in the center or moving $x_j$ towards the center increases $\text{dist}(x, a_i)$. The set of indices $i$ such that $a_i$ governs $x_j$ is denoted as $G_j(x)$.

Observe that if $x_j$ is in the center, then $|G_j(x)| = 5$, and otherwise $|G_j(x)| \leq 2$; see Fig. 3. For $k = 5$ we have 4 proper basic intervals: $[1, 2]$, $[2, 3]$, $[3, 4]$ and $[4, 5]$. We call $[1, 2]$ and $[4, 5]$ border intervals, and the other two middle intervals. We define $G_j([1, 2]) = \{\pi_j(1)\}$, $G_j([2, 3]) = \{\pi_j(1), \pi_j(2)\}$, $G_j([3, 4]) = \{\pi_j(4), \pi_j(5)\}$ and $G_j([4, 5]) = \{\pi_j(5)\}$. Note that if we know $G_j(x)$ and $|G_j(x)| \leq 2$, then we are guaranteed that $x_j$ is consistent with the basic interval $I_j$ for which $G_j(I_j) = G_j(x)$.

Observe that if $x$ is a Manhattan consensus sequence, then $G_j(x) \neq \emptyset$ for any $j$. If we additionally assume that $x$ is a sum-MSC sequence, we obtain a stronger property.

**Lemma 2.** Let $x$ be a sum-MSC sequence. Then $G_j(x) \cap G_{j'}(x) \neq \emptyset$ for any $j, j'$. 

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Proof. For a proof by contradiction assume \( G_j(x) \) and \( G_{j'}(x) \) are disjoint. This implies that neither \( x_j \) nor \( x_{j'} \) is in the center and thus \( |G_j(x)|, |G_{j'}(x)| \leq 2 \). Let us move both \( x_j \) and \( x_{j'} \) by one towards the center. Then \( dist(x, a_i) \) remains unchanged for \( i \in G_j(x) \cup G_{j'}(x) \) (by disjointness), and decreases by two for the remaining sequences \( a_i \). There must be at least one such remaining sequence, which contradicts our choice of \( x \).

Additionally, if a sum-MSC sequence \( x \) has a position \( j \) with \( |G_j(x)| = 1 \), the structure of \( x \) needs to be even more regular.

**Definition 3.** A sequence \( x \) is called an \( i \)-border sequence if for each \( j \) it holds that \( x_j = a_{i,j} \) or \( G_j(x) = \{i\} \).

**Lemma 3.** Let \( x \) be a sum-MSC sequence. If \( G_j(x) = \{i\} \) for some \( j \), then \( x \) is an \( i \)-border sequence.

**Proof.** For a proof by contradiction assume \( G_j(x) \neq \{i\} \) and \( x_{j'} \neq a_{i,j'} \) for some \( j' \). Let us move \( x_j \) towards the center and \( x_{j'} \) towards \( a_{i,j'} \) both by one. Then for any \( i' \) it holds that \( dist(x, a_{i'}) \) does not increase. By Lemma 2 \( i \in G_{j'}(x) \), so \( x_{j'} \) is moved away from the center. Moreover, \( x_j \) is moved towards some \( a_{i',j} \) with \( i' \neq i \), since \( G_{j'}(x) \neq \{i\} \). Consequently, \( dist(x, a_{i'}) \) decreases by two, which contradicts our choice of \( x \).

**Definition 4.** A sequence \( x \) is called an \( i \)-middle sequence if for each \( j \) it holds that \( i \in G_j(x) \) and \( |G_j(x)| \geq 2 \).

**Definition 5.** For a 3-element set \( \Delta \subseteq \{1, \ldots, 5\} \) a sequence \( x \) is called a \( \Delta \)-triangle sequence if for each \( j \) it holds that \( |\Delta \cap G_j(x)| \geq 2 \).

**Lemma 4.** Let \( x \) be a sum-MSC sequence. Then \( x \) is a border sequence, a middle sequence or a triangle sequence.

**Proof.** Recall that \( G_j(x) \neq \emptyset \) for each \( j \). By Lemma 3, if \( G_j(x) = \{i\} \) for some \( j \), then \( x \) is an \( i \)-border sequence. This lets us assume \( |G_j(x)| \geq 2 \), i.e. \( |G_j(x)| \in \{2, 5\} \), for each \( j \). Let \( F \) be the family of 2-element sets among \( G_j(x) \). By Lemma 2 every two of them intersect, so we can apply the following easy set-theoretical claim.

**Claim 1.** Let \( G \) be a family of 2-element sets such that every two sets in \( G \) intersect. Then sets in \( G \) share a common element or \( G \) contains exactly three sets with three elements in total.

If all sets in \( F \) share an element \( i \), then \( x \) is clearly an \( i \)-middle sequence. Otherwise \( x \) is a \( \Delta \)-triangle sequence for \( \Delta = \bigcup F \).
Fact 3. There exist 20 interval systems $B_i, M_i$ (for $i \in \{1, \ldots, 5\}$) and $T_{\Delta}$ (for 3-element sets $\Delta \subseteq \{1, \ldots, 5\}$) such that:

(a) $B_i$ is consistent with all $i$-border sequences, $G_j(B_{i,j}) = \{i\}$ for proper $B_{i,j}$;

(b) $M_i$ is consistent with all $i$-middle sequences and if $M_{i,j}$ is proper then $|G_j(M_{i,j})| = 2$ and $i \in G_j(M_{i,j})$;

(c) $T_{\Delta}$ is consistent with all $\Delta$-triangle sequences and if $T_{\Delta,j}$ is proper then $|G_j(T_{\Delta,j})| = 2$ and $G_j(T_{\Delta,j}) \subseteq \Delta$.

Proof. (a) Let us fix a position $j$. For any $i$-border sequence $x$, we know that $x_j = a_{i,j}$ or $G_j(x) = \{i\}$. If either of the border intervals $I$ satisfies $G_j(I) = \{i\}$, we set $B_{i,j} := I$ (observe that $a_{i,j}$ is then consistent with $I$). Otherwise we choose $B_{i,j}$ so that it is degenerate and corresponds to $x_j = a_{i,j}$.

(b) Again fix $j$. For any $i$-middle sequence $x$, we know that $x_j$ is consistent with at least one of the two middle intervals (both if $x_j$ is in the center). If either of the middle intervals $I$ satisfies $i \in G_j(I)$, we choose $M_{i,j} := I$. (Note that this condition cannot hold for both middle intervals).

(c) We act as in (b), i.e. if either of the middle intervals $I$ satisfies $|G_j(I) \cap \Delta| = 2$, we choose $T_{\Delta,j} := I$ (because sets $G_j(I)$ are disjoint for both middle intervals, this condition cannot hold for both of them). Otherwise, we set $T_{\Delta,j} := [3, 3]$, since $x_j$ is guaranteed to be in the center for any $\Delta$-triangle sequence $x$. □

5 Practical Algorithm for $k \leq 5$

It suffices to consider $k = 5$. Using Fact 3 we reduce the number of interval systems from $(k - 1)^k = 4^5 > 1072$ to 20 compared to the algorithm of Section 3. Moreover, for each of them ILP($\mathcal{I}$) admits structural properties, which lets us compute $\text{OPT}(\mathcal{A}, \mathcal{I})$ much more efficiently than using a general ILP solver.

Definition 6. A $(\pm)$ILP is called easy if for each constraint the number of $+1$ coefficients is 0, 1 or $n$, where $n$ is the number of variables.

Lemma 5. For each $\mathcal{I}$ being one of the 20 interval systems $B_i, M_i$ and $T_{\Delta}$, ILP($\mathcal{I}$) can be reduced to an equivalent easy $(\pm)$ILP with up to 4 variables.

Proof. Recall that for degenerate intervals $I_j$, we do not introduce variables. On the other hand, if $I_j$ is proper, possibly negating the variable $x_j$ (Fact 1), we can make sure that the coefficient vector $E(x_j)$ has $+1$ entries corresponding to $i \in G_j(I_j)$ and $-1$ entries for the remaining $i$. Moreover, merging the variables (Fact 2), we end up with a single variable per possible value $G_j(I_j)$. Now we use structural properties stated in Fact 3 to claim that the $(\pm)$ILP we obtain this way, possibly after further variable negations, becomes easy.

Border sequences. By Fact 3(a), if $B_{i,j}$ is proper, then $G_j(B_{i,j}) = \{i\}$ and thus the $(\pm)$ILP has at most 1 variable and consequently is easy.

Middle sequences. By Fact 3(b), if $M_{i,j}$ is proper, then $G_j(M_{i,j}) = \{i, i'\}$ for some $i' \neq i$. Thus there are up to 4 variables, the constraint corresponding to $i$ has only $+1$ coefficients, and the remaining constraints have at most one $+1$. 


**Triangle sequences.** By Fact 3(c), if $T_{\Delta,j}$ is proper, then $G_j(T_{\Delta,j})$ is a 2-element subset of $\Delta$, and thus there are up to three variables. Any $(\pm)$ILP with up to two variables is easy, and if we obtain three variables, then the constraints corresponding to $i \in \Delta$ have exactly two $+1$ coefficients, while the constraints corresponding to $i \notin \Delta$ have just $-1$ coefficients. Now, negating each variable (Fact 1), we get one $+1$ coefficient in constraints corresponding to $i \in \Delta$ and all $+1$ coefficients for $i \notin \Delta$.

The algorithm of Lenstra [16] with further improvements [13, 9, 17], which runs in roughly $n^{2.5n+o(n)}$ time, could perform reasonably well for $n = 4$. However, there is a simple $O(n^2)$-time algorithm designed for easy $(\pm)$ILP. It can be found in Section A.

In conclusion, the algorithm for MSC problem first proceeds as described in Fact 3 to obtain the interval systems $B_i, M_i$ and $T_{\Delta}$. For each of them it computes $ILP(I)$, as described in Section 2, and converts it to an equivalent easy $(\pm)$ILP following Lemma 5. Finally, it uses the efficient algorithm to solve each of these 20 $(\pm)$ILPs. The final result is the minimum of the optima obtained.

### 6 Conclusions

We have presented an $O(\ell k \log k)$-time kernelization algorithm, which for any instance of the MSC problem computes an equivalent instance with $\ell' \leq k!$. Although for $k \leq 5$ this gives an instance with $\ell' \leq 120$, i.e. the kernel size is constant, solving it in a practically feasible time remains challenging. Therefore for $k \leq 5$ we have designed an efficient linear-time algorithm.

We have implemented the algorithm, including retrieving the optimum consensus sequence (omitted in the description above) \footnote{It is available at http://www.mimuw.edu.pl/~kociumaka/files/msc.cpp.}. For random input data with $\ell = 10^6$ and $k = 5$, the algorithm without kernelization achieved the running time of 1.48015s, which is roughly twice the time required to read the data (0.73443s, not included in the former). The algorithm pipelined with the kernelization achieved 0.33415s. The experiments were conducted on a MacBook Pro notebook (2.3 Ghz Intel Core i7, 8 GB RAM).

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A Solving easy (±)ILP

Recall the definition of an easy (±)ILP:

\[
\min z \\
d_i + \sum_j x_j e_{i,j} \leq z \\
x_j \in R(x_j)
\]

where \(e_{i,j} = \pm 1\), and for each constraint the number of +1 coefficients is 0, 1 or \(n\), where \(n\) is the number of variables. Moreover \(R(x_j) = \{\ell_j, \ldots, r_j\}\) for integers \(\ell_j \leq r_j\).

We define a canonical form of easy (±)ILP as follows:

\[
\min z \\
y_1 + y_2 + \ldots + y_n + K' \leq z \\
y_1 - y_2 - \ldots - y_n + K_1 \leq z \\
- y_1 + y_2 - \ldots - y_n + K_2 \leq z \\
- y_1 - y_2 - \ldots + y_n + K_n \leq z \\
y_i \in \{0, \ldots, D_i\}
\]

where \(D_i \in \mathbb{Z}_{\geq 0}\) and \(K', K_1, \ldots, K_n\) are of the same parity.

**Lemma 6.** Any easy (±)ILP with \(n\) variables and at least one constraint can be reduced to two easy (±)ILPs in the canonical form, with \(n + 1\) variables, so that the optimum value of the input (±)ILP is equal to the smaller optimum value of the two output (±)ILPs.

**Proof.** First, observe that the substitution \(x_j = y_j + \ell_j\) affects only the constant terms and the ranges of variables, and gives a (±)ILP where the range of \(y_j\) is \(\{0, \ldots, D_j\}\) for \(D_j = r_j - \ell_j\).

Recall that the number of +1 coefficients in a constraint of an easy (±)ILP is 0, 1 or \(n\). Note that if two constraints have equal coefficients for each variable, then the constraint with smaller constant term is weaker than the other, and thus it can be removed. Consequently, there are at most \(n + 2\) constraints: one (denoted \(C_+\)) only with coefficients +1, one (denoted \(C_-\)) only with coefficients −1, and for each variable \(y_j\) one constraint (denoted \(C_{y_j}\)) with coefficient +1 for \(y_j\) and −1 for the remaining variables. Also, unless there are no constraints at all, if some of these \(n + 2\) constraints are missing, one may add it, setting the constant term to a sufficiently small number, e.g. \(K - 2 \sum_i D_i\), where \(K\) is the constant term of any existing constraint \(C\). Then, with such a constant term, the introduced constraint follows from the existing one.

Finally, we introduce a dummy variable \(y_{n+1}\) with \(D_{n+1} = 0\) and coefficient +1 in \(C_-\) and \(C_+\), and coefficient −1 in the constraints \(C_{y_j}\). This way \(C_-\) can be interpreted as \(C_{n+1}\) and we obtain an (±)ILP in the canonical form, which does not satisfy the parity condition yet.

Now, we replace the (±)ILP with two copies, in one of them we increment (by one) each odd constant term, and in the other we increment (by one) each even constant term. Clearly, both these (±)ILPs are stronger than the original (±)ILP. Nevertheless we claim that the originally feasible solution remains feasible for one of them.
Consider a feasible solution \((z, y_1, \ldots, y_{n+1})\) and the parity of \(P = z + y_1 + \ldots + y_{n+1}\). Note that changing a +1 coefficient to a −1 coefficient does not change the parity, so only constraints with constant terms of parity \(P \mod 2\) may be tight. For the remaining ones, we can increment the constant term without violating the feasibility. This is exactly how one of the output (±)ILPs is produced. This completes the proof of the lemma.

In the following we concentrate on solving easy (±)ILPs in the canonical form. Moreover, we assume that \(K_1 \leq \ldots \leq K_n\) (constraints and variables can be renumbered accordingly). We can solve such ILP using function \texttt{Solve}. This function is quite inefficient, we improve it later on. An explanation of the pseudocode is contained in the proof of the following lemma.

\[
\begin{align*}
1 & \text{ if } n = 1 \text{ then return } \max(K', K_1) \text{ while } K' < K_n \text{ do } \\
2 & \quad \ell := \max\{i : K_i = K_1\}; \\
3 & \quad \text{for } i = 1 \text{ to } \ell \text{ do } \\
4 & \quad \quad \text{if } D_i = 0 \text{ then } \\
5 & \quad \quad \quad \text{return } \texttt{Solve}(K', K_1, \ldots, K_{i-1}, K_{i+1}, \ldots, K_n, D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_n); \\
6 & \quad K' := K' + 1; \\
7 & \quad \text{for } i = 1 \text{ to } \ell - 1 \text{ do } K_i := K_i - 1 \ K_\ell := K_\ell + 1; \\
8 & \quad D_\ell := D_\ell - 1; \\
9 & \quad \text{for } i = \ell + 1 \text{ to } n \text{ do } K_i := K_i - 1 \ \\
10 & \text{return } K'; \\
\end{align*}
\]

\textbf{Function} \texttt{Solve}(\(K', K_1, \ldots, K_n, D_1, \ldots, D_n\))

\begin{lemma}
\textbf{Lemma 7.} \texttt{Solve} correctly determines the optimum value of the easy (±)ILP in the canonical form, provided that \(K_1 \leq \ldots \leq K_n\).
\end{lemma}

\begin{proof}
If \(n = 1\) then the constraints are \(y_1 + K' \leq z\) and \(y_1 + K_1 \leq z\), therefore the optimum is clearly \(y_1 = 0\) and \(z = \min(K', K_1)\). Consequently line 1 is correct and in the following we may assume that \(n \geq 2\).

Due to the constraint \(\sum_i y_i + K' \leq z\) and the fact that \(y_i\) are non-negative, the value \(K'\) is clearly a lower bound on \(z\). Moreover, if \(K_n \leq K'\) then taking \(y_1 = \ldots = y_n = 0\) and \(z = K'\) is feasible, since in that case all the constraints reduce to \(K_i \leq K'\) which is true due to monotonicity of \(K_i\). Thus, we may exit the while-loop and return \(K'\) in line 12 when \(K_n \leq K'\).

Let us define \(\ell\) as in line 3. If \(D_i = 0\) for some \(1 \leq i \leq \ell\), then \(y_i\) must be equal to 0, and thus this variable can be removed. Then the constraint \(C_i\) becomes of the form \(-\sum_{j \neq i} y_j + K_i \leq z\). However, since \(n \geq 2\), there is another constraint \(C_{i'}\) of the form \(y_{i'} - \sum_{j \neq i'} y_j + K_{i'} \leq z\), which becomes \(2y_{i'} - \sum_{j \neq i'} y_j + K_{i'} \leq z\) after \(y_i\) is removed. Because of the inequality \(K_{i'} \geq K_i\) and due to non-negativity of \(y_{i'}\), \(C_{i'}\) gives a stronger lower bound on \(z\) than \(C_i\), i.e. the constraint \(C_i\) can be removed. Thus lines 3-6 are correct. Also, note that in the recursive call the monotonicity assumption still holds, also constant terms are still of the same parity.

Consequently, in lines 7-11 we may assume that \(D_i > 0\) for \(i \leq \ell\). The arithmetic operations in these lines correspond to replacing \(y_\ell\) with \(y_\ell + 1 \in \{1, \ldots, D_\ell\}\). This operation is valid provided that the following claim holds.

\begin{claim}
There exists an optimal solution \((z, y_1, \ldots, y_n)\) with \(y_\ell \geq 1\).
\end{claim}
Proof. Let us take an optimal solution \((z, y_1, \ldots, y_n)\). If \(y_\ell \geq 1\) we are done, so assume \(y_\ell = 0\). We consider several cases. First, assume that there exists some \(i\) with \(y_i \geq 1\). Replace \(y_i\) with \(y_i - 1\) and set \(y_\ell\) to 1. By \(D_\ell \geq 1\) this is feasible with respect to variable ranges. The only constraints that change are \(C_\ell\) and \(C_i\). Set \(S = \sum_j y_j\), which also does not change. Note that \(C_\ell\) changes from \(z \geq K_\ell - S\) to \(z \geq K_\ell - S + 2\) and \(C_i\) changes from \(z \geq K_i - S + 2y_i\) to \(z \geq K_i - S + 2y_i - 2\). Thus \(C_i\) only weakens and the new \(C_\ell\) is weaker than the original \(C_i\), since \(y_i \geq 1\) and \(K_i \geq K_\ell\). Consequently, unless \(y_1 = \ldots = y_n = 0\) is the only optimum, there exists an optimum with \(y_\ell = 1\).

Thus, assume \(y_1 = \ldots = y_n = 0\) is an optimum. Since \(K' < K_n\), we have \(z = K_n\) for the optimum. Now, we consider two cases. First, assume \(\ell < n\). We claim that after setting \(y_\ell := 1\), the solution is still feasible. The only constraints that become stronger are \(C_+\) and \(C_{\ell}\). For the former we need to show that \(K' + 1 \leq K_n\), which is satisfied since \(K' < K_n\). For the latter we need \(K_\ell + 1 \leq K_n\), which is also true since, by \(\ell < n\), we have \(K_\ell < K_n\).

Now, assume \(\ell = n\). By \(n \geq 2\) this means that \(\ell > 1\). Therefore we can increment both \(y_1\) and \(y_\ell\) to 1 and observe that only \(C_+\) became stronger, and we need to show that \(K' + 2 \leq K_n\). However, since \(K'\) and \(K_n\) are of the same parity, this is a consequence of \(K' < K_n\).

The claim proves that lines 7-11 are correct. Because \(K_\ell\) and \(K_{\ell+1}\) satisfied \(K_\ell < K_{\ell+1}\), that is, \(K_\ell + 2 \leq K_{\ell+1}\), before these lines were executed and each \(K_i\) changed by at most 1, they now satisfy \(K_\ell \leq K_{\ell+1}\), so the monotonicity is also preserved. Also, observe that after operations in lines 7-11 all constant terms have the same parity.

Finally, observe that each iteration decreases \(\sum_i (1 + D_i)\) and thus the procedure terminates.

Now we transform the \texttt{Solve} function into a more efficient version which we call \texttt{Solve2}. The latter simulates selected series of iterations of the while-loop of the former in single steps of its while-loop.

```plaintext
if \(n = 1\) then return \(\max(K', K_1)\) while \(K' < K_n\) do
  \(\ell := \max\{i : K_i = K_1\}\);
  for \(i = 1\) to \(\ell\) do
    if \(D_i = 0\) then
      return \texttt{Solve2}(K', K_1, \ldots, K_{i-1}, K_{i+1}, \ldots, K_n, D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_n);
    if \(\ell < n\) and \(K_n - K' < 2\ell\) then return \(\frac{1}{2}(K' + K_n)\) if \(\ell = n\) and \(K_n - K' < 2(n - 1)\)
      then return \(1 + \frac{1}{2}(K' + K_n)\) \(K' := K' + \ell;\)
  for \(i = 1\) to \(\ell\) do
    \(K_i := K_i + 2 - \ell;\)
    \(D_i := D_i - 1;\)
  for \(i = \ell + 1\) to \(n\) do \(K_i := K_i - \ell\)
return \(K'\);
```

Function \texttt{Solve2}(K', K_1, \ldots, K_n, D_1, \ldots, D_n)

Lemma 8. Functions \texttt{Solve2} and \texttt{Solve} are equivalent for all inputs corresponding to easy (±)ILPs in the canonical form with \(K_1 \leq \ldots \leq K_n\).
Proof. Lines 1-6 of Solve and Solve2 coincide, so we can restrict to inputs for which both functions reach line 7. Recall from the proof of Lemma 7 that this implies \( n \geq 2, K' < K_n \) and \( D_1, \ldots, D_\ell \geq 1 \).

We claim that lines 7-13 simulate the first \( \ell \) iterations of the while-loop of Solve (or all \( < \ell \) iterations if the loop execution is interrupted earlier). Note that the value of \( \ell \) decrements by 1 in each of these iterations excluding the last one. Also, in each of these iterations the condition in line 5 is false, since in the previous iteration the respective \( D_i \) variables did not change. Thus, after an iteration with \( \ell > 1 \), the execution of the following \( \ell - 1 \) iterations of the while-loop may only be interrupted due to the \( K' < K_n \) condition (line 2).

Note that if \( \ell < n \) then the difference \( K_n - K' \) decreases by exactly 2 per iteration, as \( K_n \) is decremented while \( K' \) is incremented. Thus, starting with \( \ell < n \) the loop fails to make \( \ell \) steps only if \( K_n - K' < 2\ell \) and it happens after \( \frac{1}{2}(K_n - K') \) iterations (due to equal parity of constant terms this is an integer). Then, the value \( K' \), incremented at each iteration, is \( \frac{1}{2}(K' + K_n) \) in terms of the original values.

On the other hand, if \( \ell = n \), then in the first iteration \( K_n - K' \) does not change, since \( K' \) and \( K_n \) are both incremented. The while-loop execution is interrupted before \( \ell \) iterations only if \( K_n - K' < 2(n - 1) \), and this happens after \( 1 + \frac{1}{2}(K_n - K') \) iterations, when \( K' \), incremented at each iteration, is \( 1 + \frac{1}{2}(K' + K_n) \) in terms of the original values.

Consequently, lines 7 and 8 are correct, and if Solve2 proceeds to line 9, we are guaranteed that Solve would execute at least \( \ell \) iterations, with value \( \ell \) decremented by 1 in each iteration. It is easy to see that lines 9-13 of Solve2 simulate execution of lines 7-11 of Solve in these \( \ell \) iterations, which corresponds to simultaneously replacing \( y_1, \ldots, y_\ell \) with \( y_1 + 1, \ldots, y_\ell + 1 \).

Finally we perform the final improvement of the algorithm for solving our special ILP, which yields an efficient function Solve3. Again what actually happens is that series of iterations of the while-loop from Solve2 are now performed at once.

1 if \( n = 1 \) then return \( \max(K', K_1) \) while \( K' < K_n \)
2 \quad \ell := \max\{i : K_i = K_1\};
3 \quad for \( i = 1 \) to \( \ell \) do
4 \quad \quad if \( D_i = 0 \) then
5 \quad \quad \quad return Solve3 \((K', K_1, \ldots, K_{i-1}, K_{i+1}, \ldots, K_n, D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_n)\);
6 \quad if \( \ell < n \) then
7 \quad \quad \Delta := \min(D_1, \ldots, D_\ell, \frac{1}{2}(K_{\ell+1} - K_\ell), \left\lfloor \frac{1}{2}(K_n - K') \right\rfloor);
8 \quad \quad if \( \Delta = 0 \) then return \( \frac{1}{2}(K' + K_n) \)
9 \quad else
10 \quad \quad \Delta := \min(D_1, \ldots, D_n, \left\lfloor \frac{1}{2(n-1)}(K_n - K') \right\rfloor);
11 \quad \quad if \( \Delta = 0 \) then return \( 1 + \frac{1}{2}(K' + K_n) \)
12 \quad \quad \quad K' := K' + \Delta \cdot \ell;
13 \quad \quad for \( i = 1 \) to \( \ell \) do
14 \quad \quad \quad K_i := K_i + \Delta(2 - \ell);
15 \quad \quad \quad D_i := D_i - \Delta;
16 \quad \quad for \( i = \ell + 1 \) to \( n \) do \( K_i := K_i - \Delta \cdot \ell \)
17 return \( K' \);

\textbf{Function} Solve3(\( K', K_1, \ldots, K_n, D_1, \ldots, D_n \))
Lemma 9. Functions $\text{Solve3}$ and $\text{Solve2}$ are equivalent for all inputs corresponding to easy $(\pm)$ILPs in the canonical form with $K_1 \leq \ldots \leq K_n$. Moreover, $\text{Solve3}$ runs in $\mathcal{O}(n^2)$ time (including recursive calls).

Proof. Lines 1-6 of $\text{Solve2}$ and $\text{Solve3}$ coincide, so we can restrict to inputs for which both functions reach line 7. This in particular means that $D_1, \ldots, D_\ell \geq 1$. Also, the definition of $\ell$ and the equal parity of constant terms mean that $K_{\ell+1} \geq K_\ell + 2$. Thus, if $\Delta = 0$ in line 9, then $K_n - K' < 2\ell$, and similarly in line 12, $\Delta = 0$ only if $K_n - K' < 2(n-1)$. These are exactly conditions that $\text{Solve2}$ checks in lines 7 and 8, and consequently $\text{Solve3}$ simulates the execution of $\text{Solve2}$ if one of these conditions is satisfied.

Consequently, in the following we may assume that $\Delta > 0$. Observe that during the execution of the while-loop in $\text{Solve2}$, the value $\ell$ may only increase. Indeed, the operations in lines 9-13 preserve the equality $K_1 = \ldots = K_\ell$ and decrease by 2 the difference $K_{\ell+1} - K_\ell$, which leads to an increase of $\ell$ after $\frac{1}{2}(K_{\ell+1} - K_\ell)$ steps. Also, in a single iteration, the difference $K_n - K'$ decreases by either $2\ell$ if $\ell < n$ or $2(n-1)$ if $\ell = n$. Consequently, unless the execution is interrupted, each value included in the minima in lines 8 and 11 decreases by exactly 1. Moreover, the execution is interrupted only if one of these values reaches 0, so we are guaranteed that $\Delta$ iterations will not be interrupted and can simulate them in bulk. This is what happens in lines 13-17 of $\text{Solve3}$, which correspond to lines 9-13 of $\text{Solve2}$. Thus, $\text{Solve2}$ and $\text{Solve3}$ are indeed equivalent. It remains to justify the quadratic running time of the latter.

Observe that each iteration of the while-loop of $\text{Solve3}$ can be executed in $\mathcal{O}(n)$ time. Thus, it suffices to show that, including the recursive calls, there are $\mathcal{O}(n)$ iterations in total. We show that $2n - \ell$ decreases not less frequently than every second iteration. First, note that the recursive call decrements both $n$ and $\ell$ by one, and thus it also decrements $2n - \ell$. Therefore it suffices to consider an iteration which neither calls $\text{Solve3}$ recursively nor terminates the whole procedure. Then lines 13-17 are executed. If $\Delta$ was chosen so that $\Delta = D_1$, the next iteration performs a recursive call, and thus $2n - \ell$ decreases after this iteration. On the other hand, if $\Delta = \frac{1}{2}(K_{\ell+1} - K_\ell)$, then after executing lines 13-17 it holds that $K_\ell = K_{\ell+1}$ and thus in the next iteration $\ell$ is increases, while $n$ remains unchanged, so $2n - \ell$ decreases. Finally, if the previous conditions do not hold, then $\Delta = \left\lfloor \frac{K_n - K'}{2\min(n, n-1)} \right\rfloor$ and in the next iteration line 7 is reached again, but then in line 8 or 11 $\Delta$ is set to 0, and the procedure is terminated in line 9 or 12. This shows that it could not happen that three consecutive iterations (possibly from different recursive calls) do not decrease $2n - \ell$. This completes the proof that $\text{Solve3}$ executes $\mathcal{O}(n)$ iterations in total.

Lemmas 7-9 together show that $\text{Solve3}$ correctly solves any $(\pm)$ILP in the canonical form in $\mathcal{O}(n^2)$ time. Moreover, note that the algorithm is simple and has a small constant hidden in the $\mathcal{O}$ notation. Also, with not much effort, the optimal solution $(y_1, \ldots, y_n)$ can be recovered. Combined with a reduction of Lemma 6, we obtain the following result.

Corollary 2. Any $n$-variate easy $(\pm)$ILP can be solved in $\mathcal{O}(n^2)$ time.

The natural open question is whether our approach extends to larger values of $k$, e.g. to $k = 6$. It seems that while combinatorial properties could still be used to dramatically reduce the number of interval systems required to be processed, the resulting $(\pm)$ILPs are not easy and have more variables. Consider, for example, a special case where for each column there are just 2 different values, each occurring 3 times. Then, there is a single interval sequence consistent with each $x$.
satisfying Observation 1, it also corresponds to taking a median in each column. Nevertheless, after simplifying the ILP there are still \( \frac{1}{2} \binom{6}{3} = 10 \) variables, and the \((\pm)ILP\) is not an easy \((\pm)ILP\).