A Generalized Positive Energy Theorem for Spaces with Asymptotic SUSY Compactification

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Abstract

In this short note, we prove a generalized positive energy theorem for spaces with asymptotic SUSY compactification involving non-symmetric data. This work is motivated by the work of Dai [D1, D2], Hertog-Horowitz-Maeda [HHM], and Zhang [Z].

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1 Introduction and Statement of the Result

In 1960, Arnowitt-Deser-Misner made a detailed study of isolated gravitational systems from the Hamiltonian point of view [ADM]. They discovered a conserved quantity given precisely by an integral and they concluded that this conserved quantity

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is the total energy of this isolated system. Mathematically rigorous proof of the conjecture that the total energy for asymptotically flat spaces is nonnegative was firstly given by Schoen and Yau \cite{SY1, SY2, SY3}. Shortly thereafter, Witten raised a simple proof using spinors from ‘spacetime’ view \cite{Wi, PT}. Later, various results have been established: Bartnik \cite{B} defined the ADM mass for higher dimensional spin manifolds and generalized this theorem to that case; Zhang \cite{Z} globally defined the concept of angular momentum and proved a positive mass theorem involving this nonsymmetric data which gave an answer to the 120\textsuperscript{th} problem of Yau in his problem section \cite{Y}.

In string theory \cite{CHSW}, our universe is modelled by a ten dimensional manifold which asymptotically approaches the product of a flat Minkowski space $M^3$ with a compact Calabi-Yau 3-fold $X$. This is the so-called Calabi-Yau compactification which motivates the spaces we discuss here. Hertog-Horowitz-Maeda constructed classical configuration which has regions of negative energy density as seen from four dimensional perspective \cite{HHM}. This guides us to revisit the concept of the ADM mass (or the total energy) in string theory. A positive mass theorem for such spaces was established by Dai \cite{D1} and its Lorentzian version was discussed in \cite{D2}.

In this short note, we formulate and prove a generalized positive energy theorem for spaces with asymptotic SUSY compactification which involves non-symmetric initial data.

We consider the complete Riemannian manifold $(M^n, g_{ab}, p_{ab})$ with non-symmetric data $p_{ab}$. Suppose $M = M_0 \cup M_\infty$ with $M_0$ compact and $M_\infty \simeq (\mathbb{R}^k - B_R(0)) \times X$ for some $R > 0$ and $X$ a compact simply connected Calabi-Yau manifold. We will call $(M^n, g_{ab}, p_{ab})$ a space with asymptotic SUSY compactification if the metric on the end $M_\infty$ satisfies the following asymptotic conditions

$$g = \bar{g} + h, \quad \bar{g} = g_{\mathbb{R}^k} + g_X,$$  \hspace{1cm} (1.1)

$$h = O(r^{-\tau}), \quad \nabla h = O(r^{-\tau-1}), \quad \nabla \nabla h = O(r^{-\tau-2}),$$ \hspace{1cm} (1.2)

and

$$p = O(r^{-\tau-1}), \quad \nabla p = O(r^{-\tau-2})$$ \hspace{1cm} (1.3)
where \( p_{ab} \) is an arbitrary two-tensor satisfying \( p_{\beta\alpha} = p_{\beta i} = p_{i\beta} = 0 \), \( \bar{\nabla} \) is the Levi-Civita connection with respect to \( \bar{g} \), \( \tau > 0 \) is the asymptotic order, \( r \) is the Euclidean distance to a base point, and the index \( \alpha, \beta \) run over the compact factor while the index \( i \) runs over Euclidean part.

For such a space \((M^n, g_{ab}, p_{ab})\), the total energy is defined as
\[
E = \lim_{R \to \infty} \frac{1}{4\omega_k \text{vol}(X)} \int_{S_{R \times X}} (\partial_i g_{ij} - \partial_j g_{aa}) \ast dx_j d\text{vol}(X), \tag{1.4}
\]
and the total momentum is defined as
\[
P_k = \lim_{R \to \infty} \frac{1}{4\omega_k \text{vol}(X)} \int_{S_{R \times X}} 2(p_{kj} - \delta_{kj} p_{ii}) \ast dx_j d\text{vol}(X). \tag{1.5}
\]
Here the \( \ast \) operator is the one on the Euclidean factor, the index \( i, j, k \) run over the Euclidean factor while the index \( a, b \) run over the full index of the manifold.

We say that \((M^n, g_{ab}, p_{ab})\) satisfies the dominant energy condition if
\[
\mu \geq \max \left\{ \sqrt{\sum_a (\omega_a)^2}, \sqrt{\sum_a (\omega_a + \chi_a)^2} \right\} + \sqrt{\sum_{1 \leq a \leq n-3} \kappa_a^2}. \tag{1.6}
\]
Here, local energy density is defined as
\[
\mu = \frac{1}{2} (R + (\sum_a p_{aa})^2 - \sum_{a,b} p_{ab}^2) \tag{1.7}
\]
where \( R \) is the scalar curvature, and local momentum densities are defined as
\[
\omega_a = \sum_b (\nabla_b p_{ab} - \nabla_a p_{ab}), \tag{1.8}
\]
\[
\chi_a = 2 \sum_b \nabla_b \tilde{p}_{ba}, \tag{1.9}
\]
\[
\kappa_a^2 = \sum_{b,c,d : c > d > b > a} (\tilde{p}_{ab} \tilde{p}_{cd} + \tilde{p}_{ac} \tilde{p}_{db} + \tilde{p}_{ad} \tilde{p}_{bc})^2, \tag{1.10}
\]
where \( \tilde{p}_{ab} = p_{ab} - p_{ba} \).

Our main result is

**Main Theorem.** Let \((M^n, g_{ab}, p_{ab})\) be a complete spin manifold as above and the
asymptotic order $\tau > \frac{k-2}{2}$ and $k \geq 3$. If $(M^n, g_{ab}, p_{ab})$ satisfies the dominant energy condition \[(1.6)\], then one has
\[
E \geq |P|.
\] (1.11)
Moreover, if $E = 0$ and $k = n$, then the following equation holds on $M$
\[
\sum_{c<d}(R_{abcd} + p_{ac}p_{bd} - p_{ad}p_{bc})e^ce^d - \sqrt{-1}\sum_c(\nabla_a p_{bc} - \nabla_b p_{ac})e^c
\]
\[
= -\sqrt{-1}(\sum_{c,d:a\neq c\neq d\neq b \neq a} \nabla_a p_{cd}e^b e^d - \sum_{c,d:a\neq c\neq d\neq b \neq a} \nabla_b p_{cd}e^a e^d)
\]
\[
- (\sum_{f,c,d:a\neq f\neq c\neq d \neq b \neq a} p_{cd}p_{af}e^b e^f e^c e^d - \sum_{f,c,d:a\neq f\neq c\neq d \neq b \neq a} p_{cd}p_{bf}e^a e^f e^c e^d)
\]
(1.12)
as an endomorphism of the spinor bundle $S$, where $R_{abcd}$ is the Riemann curvature tensor of the manifold $(M^n, g_{ab}, p_{ab})$.

Remarks:
1. This theorem extends without change to the case of $X$ with any other special holonomy except $Sp(m) \cdot SP(1)$.
2. In particular, if the data $p_{ab}$ is symmetric, then this theorem reduces to the result in \cite{D2}.
3. This theorem corresponds to the result in \cite{Z} in the asymptotically flat case.

2 The Bochner-Lichnerowicz-Weitzenbock Formula

Our argument is inspired by Witten \cite{Wi} \cite{PT}. We will adapt the spinor method \cite{Z} \cite{D1} \cite{D2} to our situation. The crucial point is that we use the Dirac-Witten operator $\tilde{D}$ which is defined in \cite{Z}. Our positive energy theorem is a consequence of a nice generalized Bochner-Lichnerowicz-Weitzenbock formula.

Fix a point $p \in M$ and an orthonormal basis $\{e_a\}$ of $T_pM$ such that $(\nabla_a e_b)_p = 0$, where $\nabla$ is the Levi-Civita connection of $M$. Let $\{e^a\}$ be the dual frame. Let $S$ be the spinor bundle of $M$ with Hermitian metric $\langle \cdot, \cdot \rangle$. The connection $\nabla$ of $M$
induces a connection on $S$. Define the modified connections $\tilde{\nabla}$ and $\nabla$ on $S$ as

$$\tilde{\nabla}_a = \nabla_a + \frac{\sqrt{-1}}{2} \sum_b p_{ab} e^b,$$  \hspace{1cm} (2.1)

$$\nabla_a = \nabla_a + \frac{\sqrt{-1}}{2} \sum_b p_{ab} e^b - \frac{\sqrt{-1}}{2} \sum_{b,c; a \neq b \neq c \neq a} p_{bc} e^a e^b e^c.$$ \hspace{1cm} (2.2)

Then the Dirac operator $D$ and the Dirac-Witten operator $\tilde{D}$ are defined as

$$D = \sum_a e^a \nabla_a,$$ \hspace{1cm} (2.3)

$$\tilde{D} = \sum_a e^a \tilde{\nabla}_a$$ \hspace{1cm} (2.4)

respectively. Moreover, we have the following formulae:

$$d(<\phi, \psi> int(e^a) dvol) = (<\tilde{\nabla}_a \phi, \psi> + <\phi, (\tilde{\nabla}_a - \sqrt{-1} \sum_b p_{ab} e^b) \psi>) dvol)$$ \hspace{1cm} (2.5)

$$= (<\nabla_a \phi, \psi> + <\phi, (\nabla_a - \sqrt{-1} \sum_b p_{ab} e^b) \psi>) dvol),$$ \hspace{1cm} (2.6)

$$d(<e^a \phi, \psi> int(e^a) dvol) = (<\tilde{D} \phi, \psi> - <\phi, (\tilde{D} + \sqrt{-1} \sum_a p_{aa}) \psi>) dvol).$$ \hspace{1cm} (2.7)

We denote the adjoint operators by

$$\tilde{\nabla}_a^* = -\tilde{\nabla}_a + \sqrt{-1} \sum_b p_{ab} e^b,$$ \hspace{1cm} (2.8)

$$\nabla_a^* = -\nabla_a + \sqrt{-1} \sum_b p_{ab} e^b,$$ \hspace{1cm} (2.9)

$$\tilde{D}^* = \tilde{D} + \sqrt{-1} \sum_a p_{aa}.$$ \hspace{1cm} (2.10)

Now we recall two nice formulae in [Z].
Proposition 2.1 One has
\[
\tilde{D}^*\tilde{D} = \nabla^\dagger \nabla + \frac{1}{2} (\mu + \sqrt{-1} \sum_b \omega_b e^b) + \frac{1}{2} \mathcal{F},
\]
(2.11)
\[
\tilde{D}\tilde{D}^* = \nabla \nabla^\dagger + \frac{1}{2} (\mu - \sqrt{-1} \sum_b (\omega_b + \chi_b) e^b) - \frac{1}{2} \mathcal{F},
\]
(2.12)
where
\[
\mathcal{F} = \sum_{a \neq b \neq c \neq d \neq a} p_{abpcde^ae^be^c}.
\]
We are going to derive the integral form of the generalized Bochner-Lichnerowicz-Weitzenbock formula.

Lemma 2.1 One has
\[
\int_{\partial M} \langle \phi, \nabla_a \phi + e^a \tilde{D} \phi \rangle_{\text{int}} (e^a) d\text{vol}(g) = \int_M |\nabla \phi|^2 + \frac{1}{2} \langle \phi, (\mu + \sqrt{-1} \sum_a \omega_a e^a) \phi \rangle + \int_M \frac{1}{2} \langle \phi, \mathcal{F} \phi \rangle - |\tilde{D} \phi|^2.
\]
(2.13)
Proof. By (2.11),
\[
\text{RHS} = \int_M |\nabla \phi|^2 + \langle \phi, \tilde{D}^* \tilde{D} \phi \rangle - |\tilde{D} \phi|^2 - \langle \phi, \nabla \nabla \phi \rangle
\]
\[
= \int_{\partial M} \langle \phi, \nabla_a \phi \rangle_{\text{int}} (e^a) d\text{vol}(g) - \int_{\partial M} \langle e^a \phi, \tilde{D} \phi \rangle_{\text{int}} (e^a) d\text{vol}(g) = \text{LHS}.
\]

3 Manifolds with Parallel Spinors

Recall that the spin manifold \( M = M_0 \cup M_\infty \) with \( M_0 \) compact and \( M_\infty \simeq (\mathbb{R}^k - B_R(0)) \times X \) for some \( R > 0 \). Since \( k \geq 3 \) and \( X \) is simply connected, the end \( M_\infty \) is also simply connected and therefore has a unique spin structure which comes from the product of the restriction of the spin structure on \( \mathbb{R}^k \) and the spin structure on \( X \). One has the following result in [Wa].

Proposition 3.1 Let \((M, g)\) be a complete, simply connected, irreducible Riemannian spin manifold and \( N \) be the dimension of parallel spinors. Then \( N > 0 \) if and only if the holonomy group of \( M \) is one of SU\((m)\), Sp\((m)\), Spin\( (7)\), G\(_2\).
Remark. In physics language, manifolds with parallel spinors are said to be super-symmetric (SUSY).

We denote by \( \{e^0_a\} \) the orthonormal basis of \( \mathcal{O}g \) which consists of \( \{\partial/\partial x_i\} \) followed by an orthonormal basis \( \{f_\alpha\} \) of \( g_X \). Orthonormalizing the orthonormal frame \( \{e^0_a\} \) with respect to \( \mathcal{O}g \) yields an orthonormal frame \( \{e_a\} \) with respect to \( g \). Moreover,

\[
e_a = e^0_a - \frac{1}{2}h_{ab}e^0_b + O(r^{-2\tau}).
\]

(3.1)

This provides a gauge transformation \( A \) of the tangent bundles on the end \( M_\infty \):

\[
A : SO(\mathcal{O}g) \to SO(g)
\]

\[
e^0_a \mapsto e_a.
\]

Hence it induces a map from the spinor bundles.

Now we pick a unit norm parallel spinor \( \psi_0 \) of \((\mathbb{R}^k, g_{\mathbb{R}^k})\) and a unit parallel spinor \( \psi_1 \) of \((X, g_X)\). Then \( \phi_0 = A(\psi_0 \otimes \psi_1) \) defines a spinor of \( M_\infty \). We extend \( \phi_0 \) smoothly inside and note that

\[
\nabla \phi_0 = O(r^{-\tau-1})
\]

(3.2)

which is a consequence of an asymptotic formula in \([D1]\).

4 Fibred Boundary Calculus and the Dirac-Witten Equation

In this section, we will use the fibred boundary calculus of Melrose-Mazzeo \([MM]\) to solve the Dirac-Witten equation. The argument is following Dai’s \([D1]\).

Let \( \overline{M} \) be a smooth compact manifold with boundary and suppose that \( x \) is a boundary defining function such that \( x \) vanishes on \( \partial \overline{M} \) and \( dx \neq 0 \) there. Assume further that the boundary \( \partial \overline{M} \) comes with a fibration structure \( F \to \partial \overline{M} \xrightarrow{\pi} B \) with fiber \( F \). Then the metric \( g \) is called a fibred boundary metric if in a neighborhood of the boundary \( \partial \overline{M} \), the metric \( g \) takes the form

\[
g = \frac{dx^2}{x^4} + \frac{\pi^*(g_B)}{x^2} + g_F
\]

(4.1)
where $g_B$ is a metric on the base $B$ and $g_F$ is a family of fiberwise metrics.

In the setting of spaces with asymptotic SUSY compactification, the change of variable $x = \frac{1}{r}$ gives a trivial fibration $S^{k-1} \times X$.

Sometimes we use the notation $M$ and $\overline{M}$ interchangeably. For a manifold with boundary, we introduce two Lie algebras of vector fields:

- **b-vector fields**

  $$\mathcal{V}_b(M) := \{V \mid V \text{ tangent to the boundary } \partial M\}$$  \hspace{1cm} (4.2)

  and

- **fibred boundary vector fields**

  $$\mathcal{V}_{fb}(M) := \{V \in \mathcal{V}_b(M) \mid V \text{ tangent to the fiber } F \text{ at } \partial M, \ V x = O(x^2)\}.$$  \hspace{1cm} (4.3)

The Sobolev space $L^{p,2}(M, S)$ is defined as

$$L^{p,2}(M, S) := \{\phi \in L^2(M, S) \mid \nabla V_1 \cdots \nabla V_j \phi \in L^2(M, S), \ \forall j \leq p, \ V_i \in \mathcal{V}_b(M)\}. \hspace{1cm} (4.4)$$

Let $\gamma \in \mathbb{R}$ and we define the space of conormal sections of order $\gamma$ by

$$\mathcal{A}^{\gamma}(M, S) := \{\phi \in C^\infty(M, S) \mid |\nabla V_1 \cdots \nabla V_j \phi| \leq C x^\gamma, \ \forall j, \ V_i \in \mathcal{V}_b(M)\}, \hspace{1cm} (4.5)$$

and the subspace of polyhomogeneous sections by

$$\mathcal{A}^{*}_{phg}(M, S) := \{\phi \in \mathcal{A}^{*}(M, S) \mid \phi \sim \sum_{\gamma_j \to \infty} \sum_{k=0}^{N_j} \psi_{jk} x^{\gamma_j} (\log x)^k, \ \psi_{jk} \in C^\infty(\partial M, S)\}. \hspace{1cm} (4.6)$$

These expansions are meant in the usual asymptotic sense as $x \to 0$ and hold along with all derivatives. The superscript * may be replaced by an index set $I$ containing all pairs $(\gamma_j, N_j)$ which appear in this expansion.

Denote by $\Pi_0 : L^2(M, S) \to Ker D_F$ the natural orthogonal projector and let $\Pi_\perp := Id - \Pi_0$.

The following proposition is a summary of the results in [HHMa] (See also [DH], Theorem 3.1).
**Proposition 4.1** Suppose that $a$ is not an indicial root of $\Pi_0 x^{-1} D \Pi_0$. Then

$$D : x^a L^{1,2}(M, S) \to x^{a+1} \Pi_0 L^2(M, S) \oplus x^a \Pi_\perp L^2(M, S)$$

is Fredholm. If $D \phi = 0$ for $\phi \in x^a L^2(M, S)$, then $\phi$ is polyhomogeneous with exponents in its expansion determined by the indicial roots of $\Pi_0 x^{-1} D \Pi_0$ and truncated at $a$. If $D \xi = \psi$ for $\psi \in \mathcal{A}^a(M, S)$ and $\xi \in x^{c-1} \Pi_0 L^{1,2}(M, S) \oplus x^c \Pi_\perp L^{1,2}(M, S)$ and $c < a$, then $\xi \in \Pi_0 \mathcal{A}_{phg}^a(M, S) + \mathcal{A}^a(M, S)$.

**Remarks.**

1. Strictly speaking, only the metric $\tilde{g}$ is a fibred boundary metric. However, it is easy to see that the results generalize to the metric $g$ (see [11]). The metric perturbation produces only a lower order term.

2. In our situation, note that $\tilde{D} = D + \sqrt{-1} \sum_{a,b} p_{ab} e^a e^b = D + O(r^{-\tau-1})$. It follows from the decay condition of the initial data $p_{ab}$ that the Dirac-Witten operator $\tilde{D}$ is also a Fredholm operator from $x^a L^{1,2}(M, S)$ to $x^{a+1} \Pi_0 L^2(M, S) \oplus x^a \Pi_\perp L^2(M, S)$.

3. The precise forms of these results for the Dirac-Witten operators $\tilde{D}$ and $\tilde{D}^*$ are somewhat different, but one still has the regularity property.

4. For the precise definition of the indicial root, we refer to [MM] [HHMa]. For our purpose, we only note that the set of indicial roots is discrete.

To prove that the Dirac-Witten operator $\tilde{D}$ is an isomorphism under certain conditions, we need the following lemma inspired by [PT] and [Z].

**Lemma 4.1** Suppose $(M^n, g_{ab}, p_{ab})$ is a complete spin manifold as above and the spinor $\phi$ satisfying either $\nabla \phi = 0$ or $\nabla^* \phi = 0$. If $\lim_{r \to \infty} \phi = 0$, then $\phi = 0$.

**Proof.** By the assumptions, we have $|d|\phi|^2| < 2| < Re\nabla \phi, \phi > \leq C|p||\phi|^2$ where $C$ is some constant. This implies $|d \log |\phi|| \leq C r^{-\tau-1}$ outside a compact set. Fix a point $(r_0, y_0)$ and integrate along a path from $(r_0, y_0)$ with respect to $r$. Then one has

$$|\phi(r, y_0)| \geq |\phi(r_0, y_0)| e^{Cr_0^{-\tau-1} - r^{-\tau}}.$$

Taking $r \to \infty$ or taking $(r, y_0)$ to be the zero of $\phi$, we get $\phi(r_0, y_0) = 0$. Hence $\phi = 0$ when $r$ is large enough. It follows from the unique continuation property that $\phi = 0$ since $\phi$ satisfies the Dirac-Witten equation. We complete the proof of this lemma.
Lemma 4.2 If the dominant energy condition (1.6) holds and \( a > \frac{k-2}{2} \) is not an indicial root, then

\[
\tilde{D} : x^a L^{1,2}(M, S) \to x^{a+1} \Pi_0 L^2(M, S) \oplus x^a \Pi_1 L^2(M, S)
\]

is an isomorphism.

Proof. The argument here is similar to Dai’s (see [D1], Section 3). We first see that \( \tilde{D} \) is injective. If \( \phi \in \text{Ker} \tilde{D} \subset x^a L^{1,2}(M, S) \), then by elliptic regularity, \( \phi \in \mathcal{A}_{p_{phg}}(M, S) \).

By the Weitzenbock formula (2.13),

\[
\int_{\Omega} \{ |\nabla \phi|^2 + \frac{1}{2} < \phi, (\mu + \sqrt{-1} \sum_a \omega_a e^a) \phi > + \frac{1}{2} < \phi, \mathcal{F} \phi > \} \, d\text{vol} = \int_{\partial \Omega} < \phi, \nabla_a \phi + e^a \tilde{D} \phi > \text{int} \, (e^a) \, d\text{vol}.
\]

By taking \( \Omega \) so that \( \partial \Omega = S_r \times X \) and \( r \to \infty \) we see that the right hand side of the above equality tends to zero since \( \phi \in \mathcal{A}_{p_{phg}}(M, S) \) and \( a > \frac{k-2}{2} \). It follows from the dominant energy condition (1.6) that \( \nabla \phi = 0 \) and hence \( \phi = 0 \) by Lemma 4.1.

The same argument as above applies to the adjoint operator \( \tilde{D}^* \). By the Fredholm property, the surjectivity of \( \tilde{D} \) follows from the injectivity of \( \tilde{D}^* \) which is a consequence of the Weitzenbock formula (2.12) as well as Lemma 4.1. This proves the lemma.

Now we are ready to solve the Dirac-Witten equation.

Lemma 4.3 There exists a smooth spinor \( \phi \) such that \( \tilde{D} \phi = 0 \) and \( \phi = \phi_0 + O(r^{-\tau}) \).

Proof. We construct the wanted spinor by setting \( \phi = \phi_0 + \xi \) and solve \( \tilde{D} \xi = -\tilde{D} \phi_0 = O(r^{-\tau-1}) \). By Lemma 4.2 adjusting \( \tau \) slightly if necessary so that it is not one of the indicial root, we have a solution \( \xi = O(r^{-\tau}) \).

5 Proof of the Main Theorem

Lemma 5.1 If a spinor \( \phi \) is asymptotic to \( \phi_0 : \phi = \phi_0 + O(r^{-\tau}) \), then one has

\[
\lim_{R \to \infty} \int_{S_R \times X} < \phi, \nabla_a \phi + e^a \tilde{D} \phi > \text{int}(e_a) \, d\text{vol}(g) = \omega_k \text{vol}(X) < \phi_0, E \phi_0 + \sqrt{-1} P_i dx^i \phi_0 >.
\]

(5.1)
Proof.

\[
\begin{align*}
\int_{S_R \times X} \langle \phi, \nabla_a \phi + e^a \tilde{D} \phi \rangle > & \text{int}(e_a) d\text{vol}(g) \\
= \int_{S_R \times X} \langle \phi, \nabla_a \phi + \sum_{b} p_{ab} e^b \rangle + \sum_{b < c; a \neq b \neq c} p_{bc} e^a e^b e^c \phi \rangle > \text{int}(e_a) d\text{vol}(g) \\
+ \int_{S_R \times X} \langle \phi, e^a \sum_{b} e^b (\nabla_b + \frac{\sqrt{-1}}{2} \sum_{c} p_{bc} e^c) \phi \rangle > \text{int}(e_a) d\text{vol}(g), \\
= \int_{S_R \times X} \langle \phi, \nabla_a \phi + e^a D \phi \rangle > \text{int}(e_a) d\text{vol}(X) \\
+ \int_{S_R \times X} \langle \phi, \frac{\sqrt{-1}}{2} (\sum_{b} p_{ab} e^b - \sum_{b < c; a \neq b \neq c} p_{bc} e^a e^b e^c) \phi \rangle > \text{int}(e_a) d\text{vol}(g) \quad (5.2)
\end{align*}
\]

The first term in (5.2) is computed in [D1] which tends to \( \omega_k \text{vol}(X) < \phi_0, E \phi_0 > \) as \( r \to \infty \). The second term is

\[
\begin{align*}
\int_{S_R \times X} \langle \phi, \frac{\sqrt{-1}}{2} (\sum_{b} p_{ab} e^b + \sum_{a = b; c \neq b} + \sum_{b \neq c} + \sum_{b = c}) p_{bc} e^a e^b e^c) \phi \rangle > \text{int}(e_a) d\text{vol}(g) \\
= \int_{S_R \times X} \langle \phi, \frac{\sqrt{-1}}{2} (\sum_{b} p_{ab} e^b + \sum_{b \neq a} p_{ab} e^a e^b + \sum_{b \neq c} p_{bc} e^b e^a + \sum_{b} p_{bc} e^a e^b e^c) \phi \rangle > \text{int}(e_a) d\text{vol}(g) \\
= \int_{S_R \times X} \langle \phi, \frac{\sqrt{-1}}{2} (\sum_{b} p_{ab} e^b - \sum_{b \neq a} p_{ab} e^b + \sum_{b \neq a} p_{ba} e^b - \sum_{c} p_{cc} e^a) \phi \rangle > \text{int}(e_a) d\text{vol}(g) \\
= \int_{S_R \times X} \langle \phi, \frac{\sqrt{-1}}{2} (\sum_{b} p_{ba} e^b - \delta_{ba} p_{cc} e^b) \phi \rangle > \text{int}(e_a) d\text{vol}(g)
\end{align*}
\]

which goes to \( \omega_k \text{vol}(X) < \phi_0, \sqrt{-1} P_i dx^i \phi_0 > \) as \( r \to \infty \).

**Proof of the main theorem:** Now we are ready to prove our main result. Note that \( \sqrt{-1} P_i dx^i \) has eigenvalues \( \pm |P| \). We take \( \phi_0 \) as the unit eigenspinor of eigenvalue \( -|P| \). It follows from the Weitzenbock formula (2.13) that

\[
E \geq |P|.
\]

The proof of the second part is the same as in [Z].
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