Einstein, Wigner, and Feynman:
From $E = mc^2$ to Feynman’s decoherence
via Wigner’s little groups

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Abstract

The 20th-century physics starts with Einstein and ends with Feynman. Einstein introduced the Lorentz-covariant world with $E = mc^2$. Feynman observed that fast-moving hadrons consist of partons which act incoherently with external signals. If quarks and partons are the same entities observed in different Lorentz frames, the question then is why partons are incoherent while quarks are coherent. This is the most puzzling question Feynman left for us to solve. In this report, we discuss Wigner’s role in settling this question. Einstein’s $E = mc^2$, which takes the form $E = \sqrt{m^2 + p^2}$, unifies the energy-momentum relations for massive and massless particles, but it does not take into account internal space-time structure of relativistic particles. It is pointed out Wigner’s 1939 paper on the inhomogeneous Lorentz group defines particle spin and gauge degrees of freedom in the Lorentz-covariant world. Within the Wigner framework, it is shown possible to construct the internal space-time structure for hadrons in the quark model. It is then shown that the quark model and the parton model are two different manifestations of the same covariant entity. It is shown therefore that the lack of coherence in Feynman’s parton picture is an effect of the Lorentz covariance.

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1 Introduction

Let us start with Einstein. If the momentum of a particle is much smaller than its mass, the energy-momentum relation is \( E = p^2/2m + mc^2 \). If the momentum is much larger than the mass, the relation is \( E = cp \). These two different relations can be combined into one covariant formula. This aspect of Einstein’s \( E = mc^2 \) is well known.

In the quantum world, particles have internal space-time variables. Massive particles have spins while massless particles have their helicities and gauge variables. Our first question is whether this aspect of space-time variables can be unified into one covariant concept. The answer to this question is Yes. Wigner’s little group does the job, as is illustrated in Table I.

In addition, particles can have space-time extensions. For instance, in the quark model, hadrons are bound states of quarks. However, the hadrons appear as collections of partons when they move with speed close to the velocity of light. Quarks and partons seem to have quite distinct properties. The most serious difference is that the partons interact incoherently with external signals while the quark are coherent particles. The purpose of this report is to address this issue, after reviewing what Wigner did and what Feynman did to understand the Lorentz-covariant world.

By “further contents of \( E = mc^2 \)”, we mean that the internal space-time structures of massive and massless particles can be unified into one covariant package, as \( E = \sqrt{m^2 + p^2} \) does for the energy-momentum relation. The mathematical framework of this program was developed by Eugene Wigner in 1939 [1]. He constructed the maximal subgroups of the Lorentz group whose transformations will leave the four-momentum of a given particle invariant. These groups are known as Wigner’s little groups.

Thanks to high-energy accelerators, we can do experiments with massive particles, such as protons and heavy ions, which move with relativistic speed. After Gell-Mann invented the quark model where all hadrons are quantum bound-states of quarks, Feynman came up with an idea that a hadron appears like a collection of partons when it moves with a velocity close to that of light. Then the question is whether the quark model and the parton model are two different manifestations of the same covariant entity.

In order to have a theory of extended particles, we need bound-state wave functions of the quarks inside the hadron. These wave functions have to be covariant. This is the most fundamental problem. Neither the present form of quantum mechanics nor the quantum field theory addresses this issue. Let
us start with a well-localized wave function in one Lorentz frame. Then how would this look to an observer in a different Lorentz frame? Here, Feynman was right in guessing that the first covariant wave function has to be that of harmonic oscillators, as in the case of most of new theories. He and his coauthors started constructing such functions, and showed that the hadronic mass spectra are consistent with the degeneracies of the three-dimensional oscillators.

However, Feynman et al. did not succeed in constructing a covariant formalism. Indeed, this is possible if we construct Wigner’s little group for massive particles. The wave functions in this representation is covariant, and we use these wave functions to show that the quark model and the parton model are two different manifestations of one covariant model. The scope of this report is summarized in Table 1.

Table 1: Further contents of Einstein’s $E = mc^2$.

| Massive, Slow | COVARIANCE | Massless, Fast |
|---------------|-------------|----------------|
| Energy-Momentum | $E = p^2/2m$ | Einstein’s | $E = [p^2 + m^2]^{1/2}$ |
| Internal space-time symmetry | $S_3$ | Wigner’s Little Group | $S_3$ |
| Relativistic Extended Particles | Quark Model | Covariant Model | Partons |
2 Formulation of the Problem

It was Eugene Wigner who observed that the space-time symmetry of relativistic particles is dictated by the Poincaré group, the group of inhomogeneous Lorentz transformations, namely Lorentz transformations preceded or followed by space-time translations [1]. In particular, Wigner studied the maximal subgroups of the Lorentz group whose transformations leave the four-momentum of a given free particle. These subgroups are called the little groups. Since the little group leaves the four-momentum invariant, it governs the internal space-time symmetries of relativistic particles. Wigner shows in his paper that the internal space-time symmetries of massive and massless particles are dictated by the little groups which are locally isomorphic to the three-dimensional rotation group and the two-dimensional Euclidean groups respectively.

The group of Lorentz transformations consists of three boosts and three rotations. The rotations therefore constitute a subgroup of the Lorentz group. If a massive particle is at rest, its four-momentum is invariant under rotations. Thus the little group for a massive particle at rest is the three-dimensional rotation group. Then what is affected by the rotation? The answer to this question is very simple. The particle in general has its spin. The spin orientation is going to be affected by the rotation!

If we use the four-vector coordinate \( (x, y, z, t) \), the Lorentz group is generated by three rotation generators \( J_i \) and three boost generators \( K_i \). They satisfy the commutation relations

\[
[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k.
\] (1)

This means that the three rotation generators form a closed set of commutation relations. Indeed, they are the generators of the \( O(3) \)-like little group for a massive particle at rest. If the particle is at rest, its momentum is invariant under rotations. However, its spin direction becomes rotated. Therefore, the \( O(3) \)-like little group defined the spin degree of freedom.

It is not possible to bring a massless particle to its rest frame, but we can consider a massive particle moving along the \( z \) direction without loss of generality. In his 1939 article, Wigner observed that the little group for this massless particle is generated by

\[
J_3, \quad N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1,
\] (2)
They satisfy the commutation relations

\[ [N_1, N_2] = 0, \quad [J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1. \]  \tag{3}

In order to understand the mathematical basis of the above commutation relations, let us consider transformations on a two-dimensional plane with the \( xy \) coordinate system. We can then make rotations around the origin and translations along the \( x \) and \( y \) directions. If we write these generators as \( L, P_x \) and \( P_y \) respectively, they satisfy the commutation relations

\[ [P_x, P_y] = 0, \quad [L, P_x] = iP_y, \quad [L, P_y] = -iP_x. \]  \tag{4}

This is a closed set of commutation relations for the generators of the \( E(2) \) group. If we replace \( N_1 \) and \( N_2 \) of Eq.\((3)\) by \( P_x \) and \( P_y \), and \( J_3 \) by \( L \), the commutations relations for the generators of the \( E(2) \)-like little group becomes those for the \( E(2) \)-like little group. This is precisely why we say that the little group for massless particles are like \( E(2) \).

It is not difficult to associate the rotation generator \( J_3 \) with the helicity degree of freedom of the massless particle. Then what physical variable is associated with the \( N_1 \) and \( N_2 \) generators? Indeed, Wigner was the one who discovered the existence of these generators, but did not give any physical interpretation to these translation-like generators. For this reason, for many years, only those representations with the zero-eigenvalues of the \( N \) operators were thought to be physically meaningful representations \[3\]. It was not until 1971 when Janner and Janssen reported that the transformations generated by these operators are gauge transformations \[4, 5\]. The role of this translation-like transformation has also been studied for spin-1/2 particles, and it was concluded that the polarization of neutrinos is due to gauge invariance \[6, 7\].

3 Contraction of \( O(3) \)-like to \( E(2) \)-like Little Groups

The \( O(3) \)-like little group remains \( O(3) \)-like when the particle is Lorentz-boosted. Then, what happens when the particle speed becomes the speed of light? The energy-momentum relation \( E = \sqrt{m^2 + p^2} \) become \( E = p \). Is there then a limiting case of the \( O(3) \)-like little group? Since those little groups are like the three-dimensional rotation group and the two-dimensional
Euclidean group respectively, we are first interested in whether $E(2)$ can be obtained from $O(3)$. This will then give a clue to obtaining the $E(2)$-like little group as a limiting case of $O(3)$-like little group. With this point in mind, let us look into this geometrical problem.

In 1953, Inonu and Wigner formulated this problem as the contraction of $O(3)$ to $E(2)$ \[8\]. Let us see what they did. We always associate the three-dimensional rotation group with a spherical surface. Let us consider a circular area of radius 1 kilometer centered on the north pole of the earth. Since the radius of the earth is more than 6,450 times longer, the circular region appears flat. Thus, within this region, we use the $E(2)$ symmetry group for this region. The validity of this approximation depends on the ratio of the two radii.

How about then the little groups which are isomorphic to $O(3)$ and $E(2)$? It is reasonable to expect that the $E(2)$-like little group be obtained as a limiting case for of the $O(3)$-like little group for massless particles. In 1981, it was observed by Ferrara and Savoy that this limiting process is the Lorentz boost \[9\]. In 1983, using the same limiting process as that of Ferrara and Savoy, Han et al showed that transverse rotation generators become the generators of gauge transformations in the limit of infinite momentum and/or zero mass \[10\].

Let us see how this happens when the system is Lorentz-boosted along the $z$ direction. The $J_3$ generator is not affected by the boost whose transformation matrix takes the form

$$B = \exp (-i\eta K_3).$$  \hspace{1cm} (5)$$

On the other hand, the $J_1$ and $J_2$ matrices become

$$N_1 = e^{-\eta B^{-1}}J_2B, \quad N_2 = -e^{-\eta B^{-1}}J_1B,$$  \hspace{1cm} (6)$$

and they become $N_1$ and $N_2$ given in Eq. (2). The generators $N_1$ and $N_2$ are the contracted $J_2$ and $J_1$ respectively in the infinite-momentum/zero-mass limit. In 1987, Kim and Wigner studied this problem in more detail and showed that the little group for massless particles is the cylindrical group which is isomorphic to the $E(2)$ group \[11\].

This completes the second row in Table 1 where Wigner’s little group unifies the internal space-time symmetries of massive and massless particles. The transverse components of the rotation generators become generators of gauge transformations in the infinite-momentum/zero-mass limit.
4 Covariant Harmonic Oscillators

We are now interested in constructing the third row in Table I. As we promised in Sec. 1 we will be dealing with hadrons which are bound states of quarks with space-time extensions. For this purpose, we need a set of covariant wave functions consistent with the existing laws of quantum mechanics, including of course the uncertainty principle and probability interpretation. The first wave function which comes to our mind is the harmonic oscillator wave function. If we are interested in Lorentz-transforming them, the most straight-forward method is to construct representations of the Poincaré group using harmonic oscillators wave functions [12, 13, 14, 15].

In this report, we start with the Lorentz-invariant differential equation of Feynman, Kislinger, and Ravndal [15]. It is a linear partial differential equation which has many different solutions depending on boundary conditions. Unlike in the case of Feynman et al., we use normalizable wave functions which constitute a representation of the $O(3)$-like little group [2].

Let us consider a bound state of two particles. For convenience, we shall call the bound state the hadron, and call its constituents quarks. Then there is a Bohr-like radius measuring the space-like separation between the quarks. There is also a time-like separation between the quarks, and this variable becomes mixed with the longitudinal spatial separation as the hadron moves with a relativistic speed. There are no quantum excitations along the time-like direction. On the other hand, there is the time-energy uncertainty relation which allows quantum transitions. It is possible to accommodate these aspects within the framework of the present form of quantum mechanics. The uncertainty relation between the time and energy variables is the c-number relation [16], which does not allow excitations along the time-like coordinate. We shall see that the covariant harmonic oscillator formalism accommodates this narrow window in the present form of quantum mechanics.

For a hadron consisting of two quarks, we can consider their space-time positions $x_a$ and $x_b$, and use the variables

$$X = (x_a + x_b)/2, \quad x = (x_a - x_b)/2\sqrt{2}. \quad (7)$$

The four-vector $X$ specifies where the hadron is located in space and time, while the variable $x$ measures the space-time separation between the quarks. In the convention of Feynman et al. [15], the internal motion of the quarks bound by a harmonic oscillator potential of unit strength can be described
by the Lorentz-invariant equation
\[
\frac{1}{2} \left\{ x_\mu^2 - \frac{\partial^2}{\partial x_\mu^2} \right\} \psi(x) = \lambda \psi(x).
\] (8)

It is now possible to construct a representation of the Poincaré group from the solutions of the above differential equation [2].

The coordinate \( X \) is associated with the overall hadronic four-momentum, and the space-time separation variable \( x \) dictates the internal space-time symmetry or the \( O(3) \)-like little group. Thus, we should construct the representation of the little group from the solutions of the differential equation in Eq. (8). If the hadron is at rest, we can separate the \( t \) variable from the equation. For this variable we can assign the ground-state wave function to accommodate the \( c \)-number time-energy uncertainty relation [16]. For the three space-like variables, we can solve the oscillator equation in the spherical coordinate system with usual orbital and radial excitations. This will indeed constitute a representation of the \( O(3) \)-like little group for each value of the mass. The solution should take the form
\[
\psi(x, y, z, t) = \psi(x, y, z) \left( \frac{1}{\pi} \right)^{1/4} \exp \left( -t^2/2 \right),
\] (9)

where \( \psi(x, y, z) \) is the wave function for the three-dimensional oscillator with appropriate angular momentum quantum numbers. Indeed, the above wave function constitutes a representation of Wigner’s \( O(3) \)-like little group for a massive particle [2].

Since the three-dimensional oscillator differential equation is separable in both spherical and Cartesian coordinate systems, \( \psi(x, y, z) \) consists of Hermite polynomials of \( x, y, \) and \( z \). If the Lorentz boost is made along the \( z \) direction, the \( x \) and \( y \) coordinates are not affected, and can be temporarily dropped from the wave function. The wave function of interest can be written as
\[
\psi^n(z, t) = \left( \frac{1}{\pi} \right)^{1/4} \exp \left( -t^2/2 \right) \psi_n(z),
\] (10)

with
\[
\psi^n(z) = \left( \frac{1}{\pi n! 2^n} \right)^{1/2} H_n(z) \exp(-z^2/2),
\] (11)

where \( \psi^n(z) \) is for the \( n \)-th excited oscillator state. The full wave function \( \psi^n(z, t) \) is
\[
\psi^n_0(z, t) = \left( \frac{1}{\pi n! 2^n} \right)^{1/2} H_n(z) \exp \left\{ -\frac{1}{2} \left( z^2 + t^2 \right) \right\}.
\] (12)
The subscript 0 means that the wave function is for the hadron at rest. The above expression is not Lorentz-invariant, and its localization undergoes a Lorentz squeeze as the hadron moves along the $z$ direction [2].

It is convenient to use the light-cone variables to describe Lorentz boosts. The light-cone coordinate variables are

$$u = (z + t)/\sqrt{2}, \quad v = (z - t)/\sqrt{2}. \quad (13)$$

In terms of these variables, the Lorentz boost along the $z$ direction,

$$\left( \begin{array}{c} z' \\ t' \end{array} \right) = \left( \begin{array}{cc} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{array} \right) \left( \begin{array}{c} z \\ t \end{array} \right), \quad (14)$$

takes the simple form

$$u' = e^\eta u, \quad v' = e^{-\eta} v, \quad (15)$$

where $\eta$ is the boost parameter and is $\tanh^{-1}(v/c)$. Indeed, the $u$ variable becomes expanded while the $v$ variable becomes contracted. This is the squeeze mechanism illustrated discussed extensively in the literature [17, 18].

The wave function of Eq.(12) can be written as

$$\psi_n(z,t) = \psi_0(z,t) = \left( \frac{1}{\pi n!^2} \right)^{1/2} H_n\left( (u + v)/\sqrt{2} \right) \exp \left\{ -\frac{1}{2} (u^2 + v^2) \right\}. \quad (16)$$

If the system is boosted, the wave function becomes

$$\psi_n(z,t) = \left( \frac{1}{\pi n!^2} \right)^{1/2} H_n\left( (e^{-\eta} u + e^{\eta} v)/\sqrt{2} \right) \times \exp \left\{ -\frac{1}{2} \left( e^{-2\eta} u^2 + e^{2\eta} v^2 \right) \right\}. \quad (17)$$

In both Eqs. (16) and (17), the localization property of the wave function in the $uv$ plane is determined by the Gaussian factor, and it is sufficient to study the ground state only for the essential feature of the boundary condition. The wave functions in Eq.(16) and Eq.(17) then respectively become

$$\psi_0(z,t) = \left( \frac{1}{\pi} \right)^{1/2} \exp \left\{ -\frac{1}{2} (u^2 + v^2) \right\}. \quad (18)$$

If the system is boosted, the wave function becomes

$$\psi_n(z,t) = \left( \frac{1}{\pi} \right)^{1/2} \exp \left\{ -\frac{1}{2} (e^{-2\eta} u^2 + e^{2\eta} v^2) \right\}. \quad (19)$$
We note here that the transition from Eq. (18) to Eq. (19) is a squeeze transformation. The wave function of Eq. (18) is distributed within a circular region in the $uv$ plane, and thus in the $zt$ plane. On the other hand, the wave function of Eq. (19) is distributed in an elliptic region. This is how the wave function is Lorentz-boosted.

5 Feynman’s Parton Picture

It is safe to believe that hadrons are quantum bound states of quarks having localized probability distribution. As in all bound-state cases, this localization condition is responsible for the existence of discrete mass spectra. The most convincing evidence for this bound-state picture is the hadronic mass spectra which are observed in high-energy laboratories [2, 15]. However, this picture of bound states is applicable only to observers in the Lorentz frame in which the hadron is at rest. How would the hadrons appear to observers in other Lorentz frames?

In 1969, Feynman observed that a fast-moving hadron can be regarded as a collection of many “partons” whose properties do not appear to be identical to those of quarks [19]. For example, the number of quarks inside a static proton is three, while the number of partons in a rapidly moving proton appears to be infinite. The question then is how the proton looking like a bound state of quarks to one observer can appear different to an observer in a different Lorentz frame? Feynman made the following systematic observations.

a). The picture is valid only for hadrons moving with velocity close to that of light.

b). The interaction time between the quarks becomes dilated, and partons behave as free independent particles.

c). The momentum distribution of partons becomes widespread as the hadron moves very fast.

d). The number of partons seems to be infinite or much larger than that of quarks.

Because the hadron is believed to be a bound state of two or three quarks, each of the above phenomena appears as a paradox, particularly b) and c)
together. We would like to resolve this paradox using the covariant harmonic oscillator formalism.

For this purpose, we need a momentum-energy wave function. If the quarks have the four-momenta $p_a$ and $p_b$, we can construct two independent four-momentum variables:

$$P = p_a + p_b, \quad q = \sqrt{2}(p_a - p_b).$$

(20)

The four-momentum $P$ is the total four-momentum and is thus the hadronic four-momentum. $q$ measures the four-momentum separation between the quarks.

We expect to get the momentum-energy wave function by taking the Fourier transformation of Eq.(19):

$$\phi_\eta(q_z, q_0) = \left(\frac{1}{2\pi}\right)^2 \int \psi_\eta(z,t) \exp\{-i(q_z z - q_0 t)\} dx dt.$$  

(21)

Let us now define the momentum-energy variables in the light-cone coordinate system as

$$q_u = (q_0 - q_z)/\sqrt{2}, \quad q_v = (q_0 + q_z)/\sqrt{2}.$$  

(22)

In terms of these variables, the Fourier transformation of Eq.(21) can be written as

$$\phi_\eta(q_z, q_0) = \left(\frac{1}{2\pi}\right)^2 \int \psi_\eta(z,t) \exp\{-i(q_u u + q_v v)\} dudv.$$  

(23)

The resulting momentum-energy wave function is

$$\phi_\eta(q_z, q_0) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2} \left(e^{-2\eta q_u^2} + e^{-2\eta q_v^2}\right)\right\}.$$  

(24)

Since we are using the harmonic oscillator, the mathematical form of the above momentum-energy wave function is identical to that of the space-time wave function. The Lorentz squeeze properties of these wave functions are also the same, as are indicated in Fig. 1. These squeeze transformations perfectly consistent with the algorithms of the Poincaré group [20].

When the hadron is at rest with $\eta = 0$, both wave functions behave like those for the static bound state of quarks. As $\eta$ increases, the wave functions become continuously squeezed until they become concentrated along their
respective positive light-cone axes. Let us look at the z-axis projection of the space-time wave function. Indeed, the width of the quark distribution increases as the hadronic speed approaches that of the speed of light. The position of each quark appears widespread to the observer in the laboratory frame, and the quarks appear like free particles.

Furthermore, interaction time of the quarks among themselves become dilated. Because the wave function becomes wide-spread, the distance between one end of the harmonic oscillator well and the other end increases as is indicated in Fig. 1. This effect, first noted by Feynman [19], is universally observed in high-energy hadronic experiments. The period is oscillation is increases like $e^{\eta}$. On the other hand, the interaction time with the external signal, since it is moving in the direction opposite to the direction of the hadron, it travels along the negative light-cone axis. If the hadron contracts along the negative light-cone axis, the interaction time decreases by $e^{-\eta}$. The ratio of the interaction time to the oscillator period becomes $e^{-2\eta}$. The energy of each proton coming out of the Fermilab accelerator is $900 GeV$. This leads the ratio to $10^{-6}$. This is indeed a small number. The external signal is not able to sense the interaction of the quarks among themselves inside the hadron. This is the reason why the partons appear to be incoherent to external signals. Indeed, Feynman’s decoherence is an effect of the Lorentz covariance.

**Concluding Remarks**

Due to Einstein, this world, at least the physics world, became Lorentz-covariant. The lack of coherence in Feynman’s parton picture is the most puzzling question in covariance. It is a pleasure to report that Wigner’s formulation of the internal space-time symmetries of relativistic particles provide a resolution to this problem.

In this report, we discussed Wigner’s 1939 paper on the representations of the Poincaré group. Wigner wrote many other papers. They were also discussed at this conference. We are grateful to Professors Joszef Janszky and Peter Adam for organizing this historical conference. The author would like to thank Jiri Kvita for pointing out an typographical error in the original version.
Figure 1: Lorentz-squeezed space-time and momentum-energy wave functions. As the hadron’s speed approaches that of light, both wave functions become concentrated along their respective positive light-cone axes. These light-cone concentrations lead to Feynman’s parton picture.
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