Notes on Multi-Trace Operators and Holographic Renormalization Group\(^1\).

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Abstract

It is shown that the Holographic Renormalization Group can be formulated universally within Quantum Field Theory as (the quantization of) the Hamiltonian flow on the cotangent bundle to the space of gauge-invariant single-trace operators supplied with the canonical symplectic structure. The classical Hamiltonian dynamics is recovered in the large \( N \) limit.

1. There are many places in modern Quantum Field Theory in which one finds the relation as follows:

\[
\left\langle \exp \left\{ i \int d^D x g^n(x) O_n [\Phi(x)] \right\} \right\rangle \propto \Psi^{(D+1)}_N [g(x)],
\]

where in the LHS the average is taken with a weight \( \exp\{i S_0[\Phi]\} \) in a \( D \)-dimensional boundary Quantum Field Theory (bQFT). The content of fields in bQFT is denoted for simplicity by \( \Phi \), \( O_n[\Phi] \) is a basis of local gauge invariant operators in the theory. At the same time on the RHS of (1) \( \Psi^{(D+1)}_N \) is a wave function in a \((D+1)\)-dimensional Bulk Quantum Field Theory (BQFT) corresponding to a quantum number \( N \).

Known examples of such a relation are:

- The relation between three-dimensional Chern-Simons theory and two-dimensional WZNW model \([1]\);
- AdS/CFT correspondence and its relatives \([2, 3, 4]\);
- As well some what similar is the relation between two-dimensional conformal topological models and old matrix models (see e.g. \([5]\));

In these examples it appears that supplying the fields in bQFT with \((N \times N)\) matrix indexes gives a natural parameter in the \((D + 1)\)-dimensional BQFT. In fact, the phase space of the BQFT theory is given by the functionals on the space of the gauge-invariant operators \( O_n \) and corresponding sources \( g^n \) of the theory. This space has a natural symplectic structure. For

\(^1\)Talk given at "Particle, Strings and Fields" Vancouver, July 2000; "30 years of SUSY" Minneapolis, October, 2000; "Integrable Models, Strings and Quantum Gravity", Cennai, January 2002. Based on the unpublished work done in collaboration with A.Gerasimov.
instance in the $D = 0$ case there is the duality between coupling constants $\{g^n\}$ and the traces of the powers of the fundamental field $O_n = \{\frac{1}{n!} Tr \Phi^n\}$. In the limit $N \to \infty$ these are dual sets of variables connected by Fourier transform (Legandre transform for arbitrary $D$) defined by the integral kernel:

$$Z \left( \{g\} ; \{O(\Phi)\} \right) = \sum_{n=0}^{\infty} e^{\sum_{n=0}^{\infty} g^n O_n}$$

(2)

Note that if we do not take the large $N$ limit there is a finite number of independent $O_n$ and infinite set of $g^n$.

Hence, the configuration space $\{g^n\}$ is a linear vector space and the symplectic structure

$$\omega = \delta g^n \wedge \delta O_n$$

(3)

is non-degenerate only in the limit $N \to \infty$. Note that as well $g^n$ and $\Phi$ are good coordinates on the phase space, but in the $g^n$ and $O_n$ coordinates the symplectic structure takes the simplest form (3).

More specifically, there are three regimes. First regime is when $N$ is finite, then we can always take finite number of $g^n$ and the symplectic structure (3) is non-degenerate. However, it seems that this corresponds to the situation when the configuration space is not a linear space. Hence, the aforementioned kernel of the Fourier transform is not defined in the simple form (2).

Second regime is when $N = \infty$ and the configuration space becomes a linear vector space with the well defined Fourier kernel of the type (2). Moreover, in this case we obtain the classical approximation for BQFT [2]. Third regime is when we expand around $N = \infty$, i.e. $N \to \infty$ but finite. Then we obtain the quantization of the second regime.

What is most important for us is that the formula (4) reveals the deep relation of the Euclidian time evolution in BQFT and the conformal properties of the bQFT [2]. In other words there is a relation between equations of motion in BQFT and Renormalization Group (RG) flow in bQFT [7, 8, 9].

In particular it is tempting to connect the effective action at the momentum scale $p^2 = M^2(u)$ in bQFT and the wave function at the constant "time" slice $u = const$ in BQFT. The function $M(u)$ is given by the solution of the classical equations of motions of BQFT. When this function is invertible (as in the AdS/CFT case) it is natural to identify the scale of the boundary theory with the "time" coordinate in BQFT. In the AdS/CFT correspondence the connection is [2]:

$$e^{\phi(u)} \propto \frac{u}{R}$$

(4)

(here $R$ is the "radius" of the AdS space — UV regulator for D-dimensional gauge theory. This connection may be obtained from the requirement of conformal invariance of the term $\sqrt{det(h)} Tr[\Phi, \Phi]^2$ in the action of $D = 4$ $N = 4$ SYM.) However in general the transformation is more involved. The simplest non-trivial case is the SUGRA solution corresponding to several groups of D-branes [2, 10].

This conjectured connection between quantum theories in different dimension gives rise to the following natural questions:
• What is the correct configuration space \( \{g^n\} \), i.e. generating function of what kind of operators in bQFT is appropriate for the relation (1)?

• How RG equations in bQFT, which are normally first order differential equations, become second order differential classical equations in BQFT.

• How RG flow becomes reversible? In fact, one have to relate somehow seeming irreversibility of RG flows with the Holography.

• What is the meaning of the quantum number \( N \) from the point of view of bQFT? Or more generally, what is the meaning of an arbitrary wave-function in BQFT in terms of bQFT?

The purpose of this note is to give answers to these questions via consideration of a concrete example of bQFT.

2. We begin with some simple remarks on the RG flow in field theory. For definiteness from now on we consider the matrix field theory

\[
S_0 = \int d^D x N \, Tr \left( \frac{1}{2} m^2 |\Phi|^2 + g^{(4)} |\Phi|^4 \right),
\]

(5)

where \( \Phi \) is a field taking values in the adjoint representation of \( SU(N) \).

Consider RG flow in this theory in the Wilson’s picture \([11]\). The basic ingredient of the Wilson’s approach is the effective action defined at some scale. To obtain the latter, one separates the field \( \Phi \) into slow classical modes \( \Phi_0 \) dependent on the momenta \( p^2 < u^2 \) and quantum fluctuations \( \varphi \) dependent on the momenta \( p^2 > u^2 \): \( \Phi = \Phi_0 + \varphi \). Then one takes the functional integral over \( \varphi \). Along this way one obtains the effective action which is a linear combination of all possible operators in the theory \([12]\):

\[
S(g^n, O_n) = \int d^D x d^D y N \, K(x, y \mid u) \, Tr \left[ \Phi(x) \, \Phi(y) \right] + \int d^D x g^n O_n[\Phi]
\]

(6)

where \( K(x, y \mid u) \) is a regularized propagator in the theory \([5]\). Note that the index \( n \) in this formula can include \( D \)-dimensional tensor indexes as well.

Thus, the field theory at the normalization point \( p^2 \sim u^2 \) is characterized by the coupling constants dependent on the momenta \( p^2 > u^2 \) and the ”classical” fields dependent on the harmonics with the momenta \( p^2 < u^2 \). One can change the normalization point by integrating out some of the classical field modes \( \Phi_0 \). Note that by keeping the whole infinite set of \( g^n \)'s, one keeps information about all higher frequency modes \( \varphi \), i.e. makes the RG flow reversible. Actually, as we show below it is enough to keep only some infinite subset of all possible \( g^n \)'s.

Having in mind that \( g^n \) and \( O_n \) are coordinates on a phase space it is natural to establish the relation of the type \([11]\). In fact, the exponent of the action in the theory with the given bare sources and given background fields averaged over the fluctuations with the momenta \( u_0^2 > p^2 > u^2 \) may be interpreted as the transition amplitude in a BQFT with ”time-direction” along \( u \):
\[ K \left( \{ g^n(u_0) \}, \{ O_n(\Phi_0, u) \} \right) \equiv \exp \left\{ \int_0^{u_0} du \mathcal{H}(u) \right\} \{ O_n(\Phi_0) \} = \int \prod_{u^2 < p^2 < u_0^2} d\varphi_p \exp \left\{ i S_0[\Phi_0 + \varphi] + i \int g^n(x) O_n[\Phi_0 + \varphi] \right\}, \tag{7} \]

(in the AdS/CFT case \( u_0 \propto R/\alpha' \)) where \( \mathcal{H}(u) \) is a Hamiltonian in a \((D+1)\)-dimensional theory with dynamical variables \( g^n \) and \( O_n \). As we see, the wave function \( \Psi^{(D+1)} \) can be expanded in the basis of wave functions:

\[ \langle g^n|O_n \rangle = \exp \left\{ i \int d^D x \, g^n \right\}, \tag{8} \]

which appear from \( K \) when \( u \to u_0 \).

At this point it is easy to see the meaning of the quantum number \( \mathcal{N} \) in (1). In fact, taking \( u \to 0 \), we observe that \( \Psi^{(D+1)} \) is characterized by quantum numbers \( \mathcal{N} = \{ \langle O_n \rangle \} \), i.e. by the generalized momenta in the BQFT or the VEV’s of \( O_n \)’s in the bQFT. It is an interesting problem to find within bQFT an explicit form of the operator which can change \( \mathcal{N} \) (inside the Hilbert space of BQFT).

Most transparently RG flow in the Wilson approach may be represented in the form of the Callan-Simanzik-Polchinski equations [12]. Now we are going to show that such equations can be represented as Hamiltonian equations in a theory with the phase space \( \{ g^n \}, \{ O_n(\Phi_0) \} \).

For the theory in question the Polchinski equation is as follows [12]:

\[ \int d^D x \left( u \partial_a g^n \right) \left\{ O_n[\Phi_0 + \varphi] \right\} + \frac{1}{N} \int d^D x d^D y \left( u \partial_a K^{-1}(x, y | u) \right) \left\{ e^{-i S_1} \frac{\partial^2}{\partial \Phi^{ij}(x) \partial \Phi^{ji}(y)} e^{i S_1} \right\} = 0 \tag{9} \]

where \( S_1 = \int g^n O_n(\Phi_0 + \varphi) \) and the quantum average is taken over \( \varphi \) with the weight \( \exp\{i S_0\} \).

Using decomposition of the gauge invariant operators, we obtain (similarly to [13]):

\[ u \partial_a g^n O_n(\Phi_0) - \beta^n(g) O_n(\Phi_0) - \frac{1}{2N} \gamma^{nm}(g) O_n(\Phi_0) O_m(\Phi_0) + \ldots = 0, \tag{10} \]

Here \( \beta^n(g) \) and \( \gamma^{nm}(g) \) are some non-zero model dependent functions. It is straightforward to see that \( \beta \)’s in (10) are \( \beta \)-functions for the corresponding coupling constants.

There are two options now. If we use the full linear basis of the operators in (6) then, taking into account operator product expansion:

\[ O_n(x) O_m(y)|_{x \to y} = \sum_k C^k_{nm} O_k(x), \tag{11} \]

we have a linear differential equation for the sources:
\[ u \partial_u g^n - \beta^n(g) - \gamma^{km}(g) C^n_{km} = 0. \]  \hspace{1cm} (12)

This equation defines the RG flow on the sources in the theory.

However, if we use the basis in the space of the single trace operators in (6), we obtain the differential equations in the Hamiltonian form:

\[ u \partial_u g^n = \beta^n(g) + \frac{1}{2N} \gamma^{nm}(g) \text{Tr} O_m(\Phi_0) \]
\[ u \partial_u \text{Tr} O_n(\Phi_0) = -\partial_n \beta^m(g) \text{Tr} O_m(\Phi_0) - \frac{1}{2N} \partial_n \gamma^{mk}(g) \text{Tr} O_m(\Phi_0) \text{Tr} O_k(\Phi_0) \]  \hspace{1cm} (13)

where the second equation appears from (9) when we vary it with respect to \( g^n \). Note that we are explicitly showing powers of traces in these equations.

Corresponding Hamiltonian function has the following form:

\[ \mathcal{H}(g^n, \pi_m) = \beta^n(g) \pi_n + \frac{1}{2N} \gamma^{nm}(g) \pi_n \pi_m \]  \hspace{1cm} (14)

and generates the RG-flow with respect to the canonical symplectic structure (3) (\( \pi_n \propto \text{Tr} O_n(\Phi_0) \propto \frac{\delta}{\delta g^n} \)). Thus, the proper choice of the configuration space \( g^n \) is given by the sources for only single trace operators. Furthermore, as we noticed in the introduction, the presence of the well defined symplectic structure (3) and well defined Legendre transform implies the linearity (proper choice) of the configuration space.

Now the Hamiltonian (14) defines the equation on the wave function:

\[ \left[ u \partial_u - \beta^n(g) \frac{\delta}{\delta g^n} - \frac{1}{2N} \gamma^{nm}(g) \frac{\delta}{\delta g_n} \frac{\delta}{\delta g_m} \right] \Psi(g, u) = 0 \]  \hspace{1cm} (15)

This equation is the right substitution of the RG-flow equations for the nonzero \( \gamma \). In the special case of vanishing \( \gamma \) coefficients, we have:

\[ \left[ u \partial_u - \beta^n(g) \frac{\delta}{\delta g^n} \right] \Psi(g, u) = 0, \quad \Psi(g, u) = Z(g, u) \]  \hspace{1cm} (16)

Then the explicit solution of this equation is given in terms of the RG-flow equations:

\[ Z(g^n, u) = Z[g^n_*(g_n, u)] \]
\[ u \partial_u g^n_* = \beta^n(g_*) \]
\[ g^n_*(u \to \infty) = g^n \]  \hspace{1cm} (17)

Thus, in this special case we have the direct connection with the standard RG flow.

It is interesting that combining scale transformation properties of the theory with \( N \to \infty \) limit we obtain the classical equations of motion for the Hamiltonian defined above instead of the first order differential equations. Moreover these
Hamiltonian equations are of higher order which is a manifestation of the appearance of multi-trace operators.

In the AdS/CFT correspondence the BQFT contains graviton as a dynamical variable and thus formally has zero Hamiltonian\(^2\). This gives the Hamiltonian constraint on the wave function instead of the evolution equation. This Hamiltonian constraint includes the derivatives over the metric on a slice and thus contains the information on the scaling properties of the bQFT. However the Hamiltonian is non-linear over the momentum and thus gives rise to the higher order equation on the metric. To reconcile this with the equations discussed above one could use the energy-momentum pseudo-tensor which may be defined in some fixed coordinate system. This new Hamiltonian gives rise to the "time" evolution and was implicitly used in the calculations of \(^3\) (see the next section).

3. Let us show here that AdS/CFT correspondence fits well in to our picture.

Note first, that superconfomal theory of the SYM constrains the corresponding BQFT uniquely (at least in the case of 8 SUSY). This can not be done so easily for other examples of bQFT (note, at least, that the Hamiltonian \[^{14}\] is defined perturbatively in \(g^n\) through \(\beta\)'s and \(\gamma\)'s).

AdS/CFT correspondence predicts the following RG flow equation for the dilaton field in the linear approximation:

\[
\left[ z^3 \partial_z \frac{1}{z^3} \partial_z + \Delta^{(D)} \right] g = 0, \tag{18}
\]

where \(\Delta^{(D)}\) is Laplace operator in \(D\)-dimensions and we denote the dilaton field by \(g\) to show the relation with our previous considerations. Corresponding action functional in the first order formalism is given by:

\[
S \propto \int \frac{dz}{z} \left[ \pi z \partial_z g - \frac{1}{2} \pi^2 z^4 - \frac{1}{2z^2} \partial_\mu g \partial^\mu g \right] \tag{19}
\]

Note that the corresponding evolution equations generate quadratic in momenta terms (\(z \propto 1/u\), i.e. the UV limit \(u \to \infty\) corresponds to \(z \to 0\)).

Equations of motion in the Hamiltonian form are:

\[
z \partial_z g = \pi z^4 \tag{20}
\]

\[
\partial_z \pi = \frac{1}{z^3} \Delta^{(D)} g \tag{21}
\]

In the special case of \(\Delta^{(D)} g = 0\) we have:

\[
z \partial_z g = \pi z^4 \tag{22}
\]

\[
\partial_z \pi = 0 \tag{23}
\]

\(^2\)Note that conformal invariance, which mixes \(x\)'s and \(u\) should restore general covariance in \(D+1\)-dimensions, which is explicitly broken in the Hamiltonian approach.
with the solution:

\[ g = g_0 + \pi_0 z^4 \]  
\[ \pi = \pi_0 \]  

(24)

(25)

\((g_0, \pi_0 \) are integration constants). More generally [10]:

\[ g^{(\Delta)} \propto z^{D-\Delta} g_0^{(\Delta)} + z^\Delta \frac{1}{2\Delta - D} \pi_0^{(\Delta)} \]  

Here we may note the mixing of the dual variables \( g^n \) and \( O_n (\pi^n_0 \propto O^n) \).

As we noticed in the previous section the action (19) solves the Jacoby equation (see e.g. [8]):

\[ \int \sqrt{G} \left( \pi_{\mu \nu} \pi_{\mu \nu} - \frac{1}{3} \pi_{\mu}^{\mu} \pi_{\nu}^{\nu} \right) + \frac{1}{2} \sqrt{G} \pi^2 + \frac{1}{2} \sqrt{G} \partial_\mu g \partial^\mu g + \sqrt{G} \mathcal{R} \right] e^{-S_{\text{min}}} = 0, \]

\[ \Psi_{cl}(g) = e^{-S_{\text{min}}(g)} \]  

(27)

if minimized on the solutions of (18). Here \( \mathcal{R} \) is four-dimensional curvature and \( \pi^{\mu \nu} \) is the one conjugate to the component \( G_{\mu \nu} \) of the metric:

\[ ds^2 = \left( d \ln(z) \right)^2 + G_{\mu \nu}(z, x) dx^\mu dx^\nu \]  

(28)

and we have set the AdS radius \( R = 1 \) for simplicity. In our case \( G_{\mu \nu} = 1/z^2 \delta_{\mu \nu} \) and variations with respect to the metric are translated into variations with respect to \( z \propto \frac{1}{u} \). Thus, along this way we obtain “time” evolution with respect to \( u \) from the Hamiltonian constraint (27) in the theory with general covariance. Note that the equation (27) is of the second order in \( \partial_u \), while (15) is of the first order. The possible resolution of this discrepancy is related to the restoration of general covariance in the BQFT, which is explicitly broken in the Hamiltonian formalism. Apriory, however, for arbitrary bQFT the general covariance (in the corresponding BQFT) should not be restored. For example, it could that the general covariance is just explicitly broken by some VEV’s of tensor fields in BQFT.

4. It is worth mentioning that the described picture is an attempt to combine the approach of loop equations [14] and RG approach. Which was successfully done in the old matrix models [5]. Apart from that a related to our considerations is the appearance of the group theory interpretation of the RG flow [15]. Note as well the recent interest in the multi-trace operators [16] and their relevance for the AdS/CFT correspondence.

I would like to acknowledge discussions with I.Kogan, P.M.Ho, T.Wiseman, H.Verlinde, A.Losev, A.Rosly, A.Gorsky, A.Morozov, A.Marshakov and A.Young. Especially I would like to thank A.Gerasimov for intensive collaboration. This work was partially supported by the grant RFBR 01-01-00548 and by the grant for support of young scientists 02-01-06360 and by INTAS-00-390.
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