Abstract

We consider one-dimensional quantum spin chain, which is called XX model (XX0 model or isotropic XY model) in a transverse magnetic field. We study the model on the infinite lattice at zero temperature. We are interested in the entropy of a subsystem [a block of L neighboring spins] it describes entanglement of the block with the rest of the ground state. For large blocks entropy scales logarithmically \cite{10,11}. We prove the logarithmic formula for the leading term and calculate the next term. We discovered that the dependence on the magnetic field interacting with spins is very simple: the magnetic field effectively reduce the size of the subsystem. We also calculate entropy of a subsystem of a small size. We also evaluated Rényi and Tsallis entropies of the subsystem. We represented the entropy in terms of a Toeplitz determinant and calculated the asymptotic analytically.
1 Introduction

In this paper we study entropy of a spin chain. The entropy is the main object in thermodynamics and statistical physics. It is also interesting for information theory \cite{1, 2, 3}. We study the how entropy of a subsystem scales with the size of the subsystem. We discover an interesting dependence on the magnetic field interacting with spins. The physical system we consider is XX model in a transverse magnetic field. Hamiltonian for this model can be written as

$$H_{XX}(h) = -\sum_{n=1}^{N} (\sigma^x_n\sigma^x_{n+1} + \sigma^y_n\sigma^y_{n+1} + h\sigma^z_n).$$

Here $\sigma^x_n$, $\sigma^y_n$, $\sigma^z_n$ are Pauli matrices, which describe spin operators on $n$-th lattice site, $h$ is the magnetic field and $N$ is the number of total lattice sites of spin chain (we take $N \to \infty$ in this paper). This model has been solved by E. Lieb, T. Schultz and D. Mattis in zero-magnetic field case \cite{4} and by E. Barouch and B.M. McCoy in the presence of a constant magnetic field \cite{5}. Some exact calculation of time-dependent properties also exists, for example, see Ref. \cite{6} by E. Barouch, B.M. McCoy and Refs. \cite{7} by M. Dresden and D.B. Abraham, E. Barouch, G. Gallavotti and A. Martin-Löf. We shall consider the model in thermodynamic limit $N \to \infty$. The ground state is ferromagnetic for $|h| > 2$ while it’s critical for $|h| < 2$. The ground state $|GS\rangle$ is unique. So the entropy of the whole infinite ground state is zero, but it can be positive for a subsystem [a part of the ground state]. We shall calculate the entropy of $L$ neighboring spins. We shall call the first $L$ neighboring spins as sub-system A and the rest as sub-system B. We shall consider von Neumann entropy ($S(\rho_A)$), Rényi \cite{2} and Tsallis \cite{8} entropies for sub-system A:

$$S_{\text{von Neumann}} = S(\rho_A) = -Tr(\rho_A \ln \rho_A),$$

$$S_{\text{Rényi}} = S_\alpha(\rho_A) = \frac{1}{1-\alpha} \ln Tr(\rho_A^\alpha), \quad \alpha \neq 1 \quad \text{and} \quad \alpha > 0. \quad (3)$$

$$S_{\text{Tsallis}} = \frac{\ln Tr(\rho_A^\alpha) - 1}{1-\alpha} \quad (4)$$

Von Neumann entropy is the standard one. Tsallis and Rényi entropy may also be important for both information theory and statistical physics, see \cite{8} and \cite{2}. When $\alpha \to 1$, Tsallis and Rényi entropy turns into von Neumann entropy.\footnote{Tsallis entropy and Rényi entropy are algebraically related: $S_{Tsallis} = \frac{1}{1-\alpha} \frac{(1-\alpha)S_\alpha - 1}{1-\alpha}$} Here $\rho_A$ is the
reduced density matrix of sub-system A:

$$\rho_A = Tr_B(\rho_{AB})$$

and the density matrix of the whole system is

$$\rho_{AB} = |GS\rangle\langle GS|$$

for zero temperature. Since calculations for von Neumann entropy and Rényi entropy are much similar, we give the detail calculation for von Neumann entropy only. The explicit result for Rényi entropy will be given without derivation.

G. Vidal, J.I. Latorre, E. Rico, and A. Kitaev emphasized the role of the entropy of the subsystem in information theory [it describes entanglement of the subsystem with the rest of the ground state]. They showed that for subsystems of the large size $L$ von Neumann entropy scales logarithmically $S(\rho_A) \sim (1/3) \ln L$, see [10, 11]. In this paper we prove this formula and calculate the next term of asymptotic decomposition. We also evaluated Rényi entropy of the subsystem. Before we give the full derivation in the following sections, we first summarize our results here. We discovered that one can introduce a universal scaling variable:

$$\mathcal{L} = 2L \sqrt{1 - \left(\frac{h}{2}\right)^2}$$

We consider the magnetic field to be less than critical value for $|h| < h_c$. When the magnetic field is larger than the critical value $h_c = 2$ then the ground state is ferromagnetic [all spins are parallel], it does not have any entropy. We calculated von Neumann entropy and Rényi entropy of block spins for the magnetic field smaller than critical value. Here we present the expression for entropies for large $\mathcal{L}$ and small $\mathcal{L}$

$$S_\alpha(\rho_A) \approx \begin{cases} 
\frac{1}{1-\alpha} \ln \left( \left(\frac{\mathcal{L}}{2\pi}\right)^\alpha + \left(1 - \frac{\mathcal{L}}{2\pi}\right)^\alpha \right) & (\alpha \neq 1) \\
\frac{\mathcal{L}}{2} \ln \frac{\mathcal{L}}{2} & (\alpha = 1)
\end{cases} \quad \text{if } 0 < \mathcal{L} < 1$$

$$\frac{1+\alpha^{-1}}{\alpha} \ln \mathcal{L} + \Upsilon_1^{(\alpha)} \quad \text{if } \mathcal{L} \gg 1$$

Here $\Upsilon_1^{(\alpha)}$ is a constant defined in Eq. [64]. When $\alpha = 1$, Rényi entropy $S_\alpha(\rho_A)$ becomes von Neumann entropy, the coefficient for log $\mathcal{L}$ in large $\mathcal{L}$ expression becomes $\frac{1}{3}$ and $\Upsilon_1^{(\alpha)}$ becomes

$$\Upsilon_1 = -\int_0^\infty dt \left\{ \frac{e^{-t}}{3t} + \frac{1}{t \sinh^2(t/2)} - \frac{\cosh(t/2)}{2 \sinh^3(t/2)} \right\}.$$
Following Ref. [4], let us introduce two Majorana operators

\[ c_{2l-1} = (\prod_{n=1}^{l-1} \sigma_n^z)\sigma_l^x \quad \text{and} \quad c_{2l} = (\prod_{n=1}^{l-1} \sigma_n^z)\sigma_l^y, \]  

(6)
on each site of the spin chain. Operators \( c_n \) are hermitian and obey the anti-commutation relations \( \{c_m, c_n\} = 2\delta_{mn} \). In terms of operators \( c_n \), Hamiltonian \( H_{XX} \) can be rewritten as

\[ H_{XX}(h) = i \sum_{n=1}^{N} (c_{2n}c_{2n+1} - c_{2n-1}c_{2n+2} + hc_{2n-1}c_{2n}). \]  

(7)

Here different boundary effects can be ignored because we are only interested in cases with \( N \to \infty \). This Hamiltonian can be subsequently diagonalized by linearly transforming the operators \( c_n \). It has been obtained [4, 5] (also see [10, 11]) that

\[ \langle GS|c_m|GS \rangle = 0, \quad \langle GS|c_m c_n|GS \rangle = \delta_{mn} + i(B_N)_{mn}. \]  

(8)

Here matrix \( B_N \) can be written in a block form as

\[ B_N = \begin{pmatrix} \Pi_0 & \Pi_{-1} & \ldots & \Pi_{1-N} \\ \Pi_{1} & \Pi_{0} & \ldots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \Pi_{N-1} & \ldots & \ldots & \Pi_{0} \end{pmatrix}, \quad \text{and} \quad \Pi_i = \frac{1}{2\pi} \int_0^{2\pi} d\theta \ e^{-i\theta} G(\theta), \]  

(9)

where both \( \Pi_i \) and \( G(\theta) \) (for \( N \to \infty \)) are \( 2 \times 2 \) matrix,

\[ G(\theta) = \begin{pmatrix} 0 & g(\theta) \\ -g(\theta) & 0 \end{pmatrix}, \quad g(\theta) = \begin{cases} 1, & -k_F < \theta < k_F, \\ -1, & k_F < \theta < (2\pi - k_F) \end{cases} \]  

(10)

and \( k_F = \arccos(|h|/2) \). Other correlations such as \( \langle GS|c_m \cdots c_n|GS \rangle \) are obtainable by Wick theorem. The Hilbert space of sub-system A can be spanned by \( \prod_{i=1}^{N} \{\sigma_i^+\}^p |0\rangle_F \), where \( \sigma_i^\pm \) is Pauli matrix, \( p_i \) takes value 0 or 1, and vector \( |0\rangle_F \) denotes the ferromagnetic state with all spins up. We are also able to construct a set of fermionic operators \( b_i \) and \( b_i^+ \) by defining

\[ d_m = \sum_{n=1}^{2L} v_{mn}c_n, \quad m = 1, \ldots, 2L; \quad b_l = (d_{2l} + id_{2l+1})/2, \quad l = 1, \ldots, L \]  

(11)

with \( v_{mn} \equiv (V)_{mn} \). Here matrix \( V \) is an orthogonal matrix. It’s easy to verify that \( d_m \) is hermitian operator and

\[ b_i^+ = (d_{2l} - id_{2l+1})/2, \quad \{b_i, b_j\} = 0, \quad \{b_i^+, b_j^+\} = 0, \quad \{b_i^+, b_j\} = \delta_{i,j}. \]  

(12)
In terms of fermionic operators $b_i$ and $b_i^+$, the Hilbert space can also be spanned by $\prod_{i=1}^{L} \{b_i^+\}^{p_i} |0\rangle_{vac}$. Here $p_i$ takes value 0 or 1, 2L fermionic operators $b_i$, $b_i^+$ and vacuum state $|0\rangle_{vac}$ can be constructed by requiring
\begin{equation}
    b_l |0\rangle_{vac} = 0, \quad l = 1, \cdots, L. \tag{13}
\end{equation}

We shall choose a specific orthogonal matrix $V$ later.

# 2 Density Matrix of Sub-system A

Let $\{\psi_I\}$ be a set of orthogonal basis for Hilbert space of any physical system. Then the most general form for density matrix of this physical system can be written as
\begin{equation}
    \rho = \sum_{I,J} c(I, J) |\psi_I\rangle \langle \psi_J|. \tag{14}
\end{equation}

Here $c(I, J)$ are complex coefficients. We can introduce a set of operators $P(I, J)$ by $P(I, J) \propto |\psi_I\rangle \langle \psi_J|$ and $\tilde{P}(I, J)$ satisfying
\begin{equation}
    \tilde{P}(I, J) P(J, K) = \delta_{I,K} |\psi_I\rangle \langle \psi_I|, \quad P(I, J) \tilde{P}(J, K) = \delta_{I,K} |\psi_I\rangle \langle \psi_I| \tag{15}.
\end{equation}

There is no summation over repeated index in these formula. We shall use an explicit summation symbol through the whole paper. Then we can write the density matrix as
\begin{equation}
    \rho = \sum_{I,J} \tilde{c}(I, J) P(I, J), \quad \tilde{c}(I, J) = Tr(\rho \tilde{P}(J, I)). \tag{16}
\end{equation}

Now let us consider quantum spin chain defined in Eq. 11. For the sub-system A, the complete set of operators $P(I, J)$ can be generated by $\prod_{i=1}^{L} O_i$. Here we take operator $O_i$ to be any one of the four operators $\{b_i^+, b_i^+ b_i, b_i, b_i^+\}$ (Remember that $b_i$ and $b_i^+$ are fermionic operators defined in Eq. 11). It’s easy to find that $\tilde{P}(J, I) = (\prod_{i=1}^{L} O_i)^\dagger$ if $P(I, J) = \prod_{i=1}^{L} O_i$. Here $\dagger$ means hermitian conjugation. Therefore, the reduced density matrix for sub-system A can be represented as
\begin{equation}
    \rho_A = \sum_{I,J} Tr_{AB} \left( \rho_{AB}(\prod_{i=1}^{L} O_i)^\dagger \right) \prod_{i=1}^{L} O_i. \tag{17}
\end{equation}

Here the summation is over all possible different terms $\prod_{i=1}^{L} O_i$. For the whole system to be in pure state $|GS\rangle$, the density matrix $\rho_{AB}$ is represented by $|GS\rangle \langle GS|$. Then
we have the expression for $\rho_A$ as following

$$\rho_A = \sum\langle GS|\prod_{i=1}^L O_i^\dagger|GS\rangle \prod_{i=1}^L O_i.$$  \hspace{1cm} (18)

This is the expression of density matrix with the coefficients related to multi-point correlation functions. These correlation functions are well studied in the physics literature \[9\]. Now let us choose matrix $V$ in Eq. 11 so that the set of fermionic basis $\{b_i^+\}$ and $\{b_i\}$ satisfy an equation

$$\langle GS|b_ib_j|GS\rangle = 0,$$  
$$\langle GS|b_i^+b_j|GS\rangle = \delta_{i,j}\langle GS|b_i^+b_i|GS\rangle.$$ \hspace{1cm} (19)

Then the reduced density matrix $\rho_A$ represented as sum of products in Eq. 18 can be represented as a product of sums

$$\rho_A = \prod_{i=1}^L (\langle GS|b_i^+b_i|GS\rangle b_i^+b_i + \langle GS|b_i^+b_i|GS\rangle b_i^+).$$ \hspace{1cm} (20)

Here we used the equations $\langle GS|b_i|GS\rangle = 0 = \langle GS|b_i^+|GS\rangle$ and Wick theorem. This fermionic basis was suggested by G. Vidal, J.I. Latorre, E. Rico and A. Kitaev in Ref. \[10, 11\]. A similar result for the density matrix of a subsystem in terms of free spinless fermion model was obtained by C.A. Cheong and C.L. Henley in Ref. \[12\].

### 3 Closed Form for The Entropy

Now let us find a matrix $V$ in Eq. 11 which will block-diagonalize correlation functions of Majorana operators $c_n$. From Eqs. 11 and 9 we have the following expression for correlation function of $d_n$ operators:

$$\langle GS|d_md_n|GS\rangle = \sum_{i=1}^{2L} \sum_{j=1}^{2L} v_{mi} \langle GS|c_ic_j|GS\rangle v_{jn},$$
$$\langle GS|c_mc_n|GS\rangle = \delta_{mn} + i(B_L)_{mn},$$
$$\langle GS|d_md_n|GS\rangle = \delta_{mn} + i(\tilde{B}_L)_{mn}.$$ \hspace{1cm} (21)

The last equation is the definition of a matrix $\tilde{B}_L$. Matrix $B_L$ is the sub-matrix of $B_N$ defined in Eq. 9 with $m, n = 1, 2, \ldots, L$. We also require $\tilde{B}_L$ to be the form \[10, 11\]

$$\tilde{B}_L = VB_LV^T = \oplus_{m=1}^L \nu_m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \Omega \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ \hspace{1cm} (22)
Here matrix $\Omega$ is a diagonal matrix with elements $\nu_m$ (all $\nu_m$ are real numbers). Therefore, choosing matrix $V$ satisfying Eq. 22 in Eq. 11 we obtain $2L$ operators $\{b_l\}$ and $\{b_l^+\}$ with following expectation values

$$
\langle GS | b_m | GS \rangle = 0, \langle GS | b_m b_n | GS \rangle = 0, \langle GS | b_m^+ b_n | GS \rangle = \delta_{mn} \frac{1 + \nu_m}{2}.
$$

Using the simple expression for reduced density matrix $\rho_A$ in Eq. 20 we obtain

$$
\rho_A = \prod_{i=1}^{L} \left( \frac{1 + \nu_i b_i^+ b_i + \frac{1 - \nu_i}{2} b_i b_i^+}{2} \right).
$$

This form immediately gives us all the eigenvalues $\lambda_{x_1 x_2 \cdots x_L}$ of reduced density matrix $\rho_A$,

$$
\lambda_{x_1 x_2 \cdots x_L} = \prod_{i=1}^{L} \frac{1 + (-1)^{x_i} \nu_i}{2}, \quad x_i = 0, 1 \quad \forall i.
$$

Note that in total we have $2^L$ eigenvalues. Hence, the entropy of $\rho_A$ from Eq. 2 becomes

$$
S(\rho_A) = \sum_{m=1}^{L} e(1, \nu_m)
$$

with

$$
e(x, \nu) = -\frac{x + \nu}{2} \ln\left(\frac{x + \nu}{2}\right) - \frac{x - \nu}{2} \ln\left(\frac{x - \nu}{2}\right).
$$

We shall use this result further to obtain analytical asymptotic. Function $e(1, \nu)$ in Eq. 26 is equal to the Shannon entropy function $H(\frac{1 + \nu}{2})$. However, in the following calculation (Eq. 32), we will need the more general function $e(x, \nu)$ instead of $H(\nu)$. Notice further that matrix $B_L$ can have a direct product form, i.e.

$$
B_L = G_L \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{with} \quad G_L = \begin{pmatrix} g_0 & g_{-1} & \cdots & g_{1-L} \\ g_1 & g_0 & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{L-1} & \cdots & \cdots & g_0 \end{pmatrix},
$$

where $g_l$ is defined as

$$
g_l = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-il\theta} g(\theta) \quad \text{and} \quad g(\theta) = \begin{cases} 1, & -k_F < \theta < k_F, \\ -1, & k_F < \theta < (2\pi - k_F) \end{cases}
$$

(29)
and $k_F = \arccos(|h|/2)$. From Eqs. \ref{eq:22} and \ref{eq:28} we conclude that all $\nu_m$ are just the eigenvalues of real symmetric matrix $G_L$.

However, to obtain all eigenvalues $\nu_m$ directly from matrix $G_L$ is a non-trivial task. Let us introduce

$$D_L(\lambda) = \det(\tilde{G}_L(\lambda) \equiv \lambda I_L - G_L).$$

(30)

Here $\tilde{G}_L$ is a Toeplitz matrix (see \[17\]) and $I_L$ is the identity matrix of dimension $L$. Obviously we also have

$$D_L(\lambda) = \prod_{m=1}^{L} (\lambda - \nu_m).$$

(31)

From the Cauchy residue theorem and analytical property of $e(x, \nu)$, then $S(\rho_A)$ can be rewritten as

$$S(\rho_A) = \lim_{\epsilon \to 0^+} \lim_{\delta \to 0^+} \frac{1}{2\pi i} \oint_{c(\epsilon, \delta)} e(1 + \epsilon, \lambda) d \ln D_L(\lambda).$$

(32)

Here the contour $c(\epsilon, \delta)$ in Fig. 1 encircles all zeros of $D_L(\lambda)$, but function $e(1 + \epsilon, \lambda)$ is analytic within the contour. Just like Toeplitz matrix $G_L$ is generated by function $g(\theta)$ in Eqs. \ref{eq:28} and \ref{eq:29} [see next section], Toeplitz matrix $\tilde{G}_L(\lambda)$ is generated by function $\tilde{g}(\theta)$ defined by

$$\tilde{g}(\theta) = \begin{cases} 
\lambda - 1, & -k_F < \theta < k_F, \\
\lambda + 1, & k_F < \theta < (2\pi - k_F).
\end{cases}$$

(33)

Notice that $\tilde{g}(\theta)$ is a piecewise constant function of $\theta$ on the unit circle, with jumps at $\theta = \pm k_F$. Hence, if one can obtain the determinant of this Toeplitz matrix analytically, one will be able to get a closed analytical result for $S(\rho_A)$ which is our new result.

Now, the calculation of $S(\rho_A)$ reduces to the calculation of the determinant of Toeplitz matrix $\tilde{G}_L(\lambda)$. Before we calculate the determinant of Toeplitz matrix $\tilde{G}_L(\lambda)$, we also want to point out two special cases which allow us to obtain an explicit form for these eigenvalues $\nu_m$ and hence the entropy $S(\rho_A)$. These are cases with small lattice size of subsystem $A$ and magnetic $h$ close to critical values $\pm 2$. Let us take $\tilde{k}_F = k_F$ for $k_F < \pi/2$ or $\tilde{k}_F = \pi - k_F$ for $k_F > \pi/2$. For cases $\tilde{k}_F L \ll 1$, Toeplitz matrix $G_L$ can be well approximated by a matrix with diagonal elements $(2\tilde{k}_F/\pi - 1)$ and all other matrix
Figure 1: The contour $c(\epsilon, \delta)$. Bold lines $(-\infty, -1-\epsilon)$ and $(1+\epsilon, \infty)$ are the cuts of integrand $e(1+\epsilon, \lambda)$. Zeros of $D_L(\lambda)$ (Eq. 31) are located on bold line $(-1, 1)$ and this line becomes the cut of $d\log D_L(\lambda)$ for $L \to \infty$ (Eq. 48). The arrow is the direction of the route of integral we take and $R$ is the radius of circles.

Elements equal to $2\tilde{k}_F/\pi$. Hence, we can obtain all eigenvalues for Toeplitz matrix $G_L$ as $\{2L\tilde{k}_F/\pi - 1, -1, -1, \cdots, -1\}$ and $S(\rho_A)$ becomes

$$S(\rho_A) \approx \frac{2L\tilde{k}_F}{\pi} \ln \frac{\pi}{2L\tilde{k}_F}, \quad 0 < \tilde{k}_F L \ll 1. \quad (34)$$

Expression above can also be re-expressed in terms of $h$ as

$$S(\rho_A) \approx \frac{2L(1-h^2/4)^{1/2}}{\pi} \ln \frac{\pi}{2L(1-h^2/4)^{1/2}}, \quad 0 < (1-h^2/4)^{1/2}L \ll 1. \quad (35)$$

## 4 Toeplitz Matrix and Fisher-Hartwig Conjecture

Toeplitz matrix $T_L[\phi]$ is said to be generated by function $\phi(\theta)$ if

$$T_L[\phi] = (\phi_{i-j}), \quad i, j = 1, \cdots, L-1 \quad (36)$$

where

$$\phi_l = \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) e^{-ilt} d\theta \quad (37)$$
is the $l$-th Fourier coefficient of generating function $\phi(\theta)$. The determinant of $T_L[\phi]$ is denoted by $D_L$. The asymptotic behavior of the determinant of Toeplitz matrix with singular generating function was initiated by T.T. Wu \cite{wu1964} in his study of spin correlation in two-dimensional Ising model and later incorporated into a more general conjecture, i.e., the famous Fisher-Hartwig conjecture \cite{fisher1968, hartwig1972, hartwig1973, hartwig1973a}.

**Fisher-Hartwig Conjecture:** Suppose the generating function of Toeplitz matrix $\phi(\theta)$ is singular in the following form

$$
\phi(\theta) = \psi(\theta) \prod_{r=1}^{R} t_{\beta_r, \theta_r}(\theta) u_{\alpha_r, \theta_r}(\theta)
$$

where

$$
t_{\beta_r, \theta_r}(\theta) = \exp[-i\beta_r(\pi - \theta + \theta_r)], \quad \theta_r < \theta < 2\pi + \theta_r
$$

$$
u_{\alpha_r, \theta_r}(\theta) = \left(2 - 2 \cos(\theta - \theta_r)\right)^{\alpha_r}, \quad \Re\alpha_r > -\frac{1}{2}
$$

and $\psi$: $\textbf{T} \to \textbf{C}$ is a smooth non-vanishing function with zero winding number. Then as $n \to \infty$, the determinant of $T_L[\phi]

$$
D_L = (\mathcal{F}[\psi])^L \left( \prod_{i=1}^{R} \Gamma^{\alpha_i^2 - \beta_i^2} \right) \mathcal{E}[\psi, \{\alpha_i\}, \{\beta_i\}, \{\theta_i\}], \quad L \to \infty.
$$

Here $\mathcal{F}[\psi] = \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \ln \psi(\theta) d\theta \right)$. Further assuming that there exists Weiner-Hopf factorization

$$
\psi(\theta) = \mathcal{F}[\psi] \psi_+(\exp(i\theta)) \psi_-(\exp(-i\theta)),
$$

then constant $\mathcal{E}[\psi, \{\alpha_i\}, \{\beta_i\}, \{\theta_i\}]$ in Eq. \ref{eq:fisher-hartwig} can be written as

$$
\mathcal{E}[\psi, \{\alpha_i\}, \{\beta_i\}, \{\theta_i\}] = \mathcal{E}[\psi] \prod_{i=1}^{R} G(1 + \alpha_i + \beta_i)G(1 + \alpha_i - \beta_i)/G(1 + 2\alpha_i)
$$

$$
\times \prod_{i=1}^{R} \left( \psi_+(\exp(i\theta_i)) \right)^{-\alpha_i - \beta_i} \left( \psi_-(\exp(-i\theta_i)) \right)^{-\alpha_i + \beta_i}
$$

$$
\times \prod_{1 \leq i \neq j \leq R} \left( 1 - \exp(i(\theta_i - \theta_j)) \right)^{-(\alpha_i + \beta_i)(\alpha_j - \beta_j)},
$$

$G$ is the Barnes $G$-function, $\mathcal{E}[\psi] = \exp(\sum_{k=1}^{\infty} k \pi_k s_k s_{-k})$, and $s_k$ is the $k$-th Fourier coefficient of $\ln \psi(\theta)$. The Barnes $G$-function is defined as

$$
G(1 + z) = (2\pi)^{z/2} e^{-(z+1)z/2 - \gamma z^2/2} \prod_{n=1}^{\infty} \left\{ (1 + z/n)^n e^{-z^2/(2n)} \right\},
$$
where \( \gamma_E \) is Euler constant and its numerical value is 0.5772156649 \( \cdots \). This conjecture has not been proven for general case. However, there are various special cases for which the conjecture was proven.

For our case, the generating function \( \tilde{g}(\theta) \) has two jumps at \( \theta = \pm k_F \) and it has the following canonical factorization

\[
\tilde{g}(\theta) = \psi(\theta) t_{\beta_1(\lambda), k_F} t_{\beta_2(\lambda), -k_F} \tag{45}
\]

with

\[
\psi(\theta) = (\lambda + 1) \left( \frac{\lambda + 1}{\lambda - 1} \right)^{-k_F/\pi}, \quad \beta(\lambda) = -\beta_1(\lambda) = \beta_2(\lambda) = \frac{1}{2\pi i} \ln \frac{\lambda + 1}{\lambda - 1}. \tag{46}
\]

The function \( t \) was defined in Eq. 39. We fix the branch of the logarithm in the following way

\[
-\pi \leq \arg \left( \frac{\lambda + 1}{\lambda - 1} \right) < \pi. \tag{47}
\]

For \( \lambda \notin [-1, 1] \), we know that \( |\Re(\beta_1(\lambda))| < \frac{1}{2} \) and \( |\Re(\beta_2(\lambda))| < \frac{1}{2} \) and Fisher-Hartwig conjecture was PROVEN by E.L. Basor for this case[15]. Therefore, we will call it the theorem instead of conjecture for our application. Hence following the theorem in Eq. 41, the determinant \( D_L(\lambda) \) of \( \lambda I_L - G_L \) can be asymptotically represented as

\[
D_L(\lambda) = \left( 2 - 2 \cos(2k_F) \right)^{-\beta^2(\lambda)} \left\{ G(1 + \beta(\lambda)) G(1 - \beta(\lambda)) \right\}^2 \left( (\lambda + 1)/(\lambda - 1) \right)^{-k_F/\pi} L^{-2\beta^2(\lambda)}. \tag{48}
\]

Here \( L \) is the length of sub-system A and \( G \) is the Barnes G-function and

\[
G(1 + \beta(\lambda)) G(1 - \beta(\lambda)) = e^{-\left( 1 + \gamma_E \right) \beta^2(\lambda)} \prod_{n=1}^{\infty} \left\{ 1 - \frac{\beta^2(\lambda)}{n^2} \right\} e^{\beta^2(\lambda)/n^2}. \tag{49}
\]

### 5 Asymptotic Form of The Entropy

Now, let us come back to the calculation of entropy \( S(\rho_A) \). For later convenience, let us define

\[
\Upsilon(\lambda) = \sum_{n=1}^{\infty} \frac{n^{-1} \beta^2(\lambda)}{n^2 - \beta^2(\lambda)}. \tag{50}
\]
Taking logarithmic derivative of $D_L(\lambda)$ (Eq. 48), we obtain
\[
\frac{d \ln D_L(\lambda)}{d \lambda} = \left( \frac{1 - k_F/\pi}{1 + \lambda} - \frac{k_F/\pi}{1 - \lambda} \right) L - \frac{4 \beta(\lambda)}{i \pi (1 + \lambda)(1 - \lambda)} \left( \ln L + \ln(2|\sin k_F|) + (1 + \gamma_E) + \Upsilon(\lambda) \right). \quad (51)
\]
Let us substitute the asymptotic form above for $d \ln D_L(\lambda)/d \lambda$ into Eq. 32 for entropy $S(\rho_A)$:
\[
S(\rho_A) = \lim_{\epsilon \to 0^+} \lim_{\delta \to 0^+} \frac{1}{2 \pi i} \oint_{c(\epsilon, \delta)} e(1 + \epsilon, \lambda) \left( \frac{1 - k_F/\pi}{1 + \lambda} - \frac{k_F/\pi}{1 - \lambda} \right) L + \beta(\lambda) \left( \ln L + \ln(2|\sin k_F|) + (1 + \gamma_E) + \Upsilon(\lambda) \right), \quad (52)
\]
where the contour is taken as shown in Fig. 1. The first integral in Eq. 52 can be carried out by using the residue theorem and the definition of function $e(x, \nu)$ in Eq. 27. We found that the linear term in $L$ for entropy $S(\rho_A)$ vanishes. The second integral can be calculated as follows: First, we notice that
\[
\oint_{c(\epsilon, \delta)} d\lambda (\cdots) = \left( \int_{AF} + \int_{FED} + \int_{DC} + \int_{CB\bar{A}} \right) d\lambda (\cdots) \quad (53)
\]
Second, we can show that the contribution of integral from the circular arcs $FED$ and $CB\bar{A}$ vanishes. Therefore, the entropy (Eq. 52) can be written as
\[
S(\rho_A) = \lim_{\epsilon \to 0^+} \lim_{\delta \to 0^+} \frac{2}{\pi^2} \left( \int_{1-i0^+}^{1+i0^+} + \int_{-1+i0^+}^{-1-i0^+} \right) d\lambda \frac{e(1 + \epsilon, \lambda) \beta(\lambda)}{(1 + \lambda)(1 - \lambda)} \left( \ln L + \ln(2|\sin k_F|) + (1 + \gamma_E) + \Upsilon(\lambda) \right). \quad (54)
\]
For further simplification, we shall use the fact that
\[
\beta(x + i0^+) = \frac{1}{2i \pi} \left( \frac{1 + x}{1 - x} + i(\pi - 0^+) \right) = -i W(x) + \frac{1}{2} - 0^+ \quad (55)
\]
for $x \in (-1, 1)$ and
\[
W(x) = \frac{1}{2 \pi \ln \frac{1 + x}{1 - x}}. \quad (56)
\]
We can now write the entropy $S(\rho_A)$ as
\[
S(\rho_A) = \frac{2}{\pi^2} \int_{-1}^{1} dx \frac{e(1, x)}{1 - x^2} \left( \frac{1}{2} + i W(x) \right)^3 \left( \frac{1}{n^2 - (\frac{1}{2} + i W(x))^2} + \frac{1}{n^2 - (\frac{1}{2} - i W(x))^2} \right) \quad (57)
\]
where $e(1, x)$ is defined in Eq. 27. This expression for $S(\rho_A)$ contains two integrals. The first integral can be done exactly as

$$\frac{2}{\pi^2} \int_{-1}^{1} dx \left( -\frac{1+x}{2} \ln \frac{1+x}{2} - \frac{1-x}{2} \ln \frac{1-x}{2} \right) \frac{1}{1-x^2} = \frac{1}{3}. \quad (58)$$

The second integral in Eq. 57 can be written as

$$\Upsilon_0 = \sum_{n=1}^{\infty} \frac{n^{-1}}{\pi^2} \int_{-1}^{1} dx \left( -\frac{1-x}{2} \ln \frac{1-x}{2} - \frac{1+x}{2} \ln \frac{1+x}{2} \right) \times \left( \frac{(\frac{1}{2} + iW(x))^3}{n^2 - (\frac{1}{2} + iW(x))^2} + \frac{(\frac{1}{2} - iW(x))^3}{n^2 - (\frac{1}{2} - iW(x))^2} \right), \quad (59)$$

which can be further simplified [18]. Finally we have that

$$S(\rho_A) = \frac{1}{3} \ln L + \frac{1}{6} \ln \left( 1 - \left( \frac{h}{2} \right)^2 \right) + \frac{\ln 2}{3} + \Upsilon_1, \quad L \to \infty \quad (60)$$

with

$$\Upsilon_1 = -\int_{0}^{\infty} dt \left( \frac{e^{-t}}{3t} + \frac{1}{t \sinh^2(t/2)} - \frac{\cosh(t/2)}{2 \sinh^3(t/2)} \right). \quad (61)$$

for XX model. The leading term of asymptotic of the entropy $\frac{1}{3} \ln L$ in Eq. 60 was first obtained based on numerical calculation and a simple conformal argument in Ref. [10, 11] in the context of entanglement. We also want to mention that a complete conformal derivation for this entropy was found in Ref. [19]. One can numerically evaluate $\Upsilon_1$ to very high accuracy to be $0.4950179 \cdots$. For zero magnetic field ($h = 0$) case, the constant term $\Upsilon_1 + \ln 2/3$ for $S(\rho_A)$ is close to but different from $(\pi/3) \ln 2$, which can be found by taking numerical accuracy to be more than five digits.

6 Summary

In this paper, we study the entropy of a block of L neighboring spins in XX model with the presence of the transverse magnetic field. We obtain Eq. 35 and Eq. 60 for the von Neumann entropy of a block of L neighboring spins in XX with small L and large L respectively. It’s interesting to note that there is a natural length scale $L_h = 1/ \left( 1 - \left( \frac{h}{2} \right)^2 \right)^{\frac{1}{2}}$ for $|h| < 2$ to incorporating the magnetic field effects. When $|h|$ increases from less than 2 into larger than 2, the system evolves from critical phase
into ferromagnetic phase. The ferromagnetic phase does not have any entropy. The scale \( L_h \) shows singular behavior at the critical value of the magnetic field \( h_c = 2 \). We discovered that one can introduce the universal scaling variable \( \mathcal{L} = 2L/L_h \):

\[
\mathcal{L} \equiv 2L \left( 1 - \left( \frac{h}{2} \right)^2 \right)^{\frac{1}{2}}
\]

for \( |h| < 2 \). Then we can express the von Neumann entropy of \( L \) neighboring spins in following simple form:

\[
S(\rho_A) = \begin{cases} 
\frac{L}{\pi} \ln \frac{\pi}{L} & \text{if } 0 < \mathcal{L} < 1 \\
\frac{1}{3} \ln \mathcal{L} + \Upsilon_1 & \text{if } \mathcal{L} \gg 1
\end{cases}
\tag{62}
\]

with

\[
\Upsilon_1 = - \int_0^\infty dt \left\{ \frac{e^{-t}}{3t} + \frac{1}{t \sinh^2(t/2)} - \frac{\cosh(t/2)}{2 \sinh^3(t/2)} \right\}.
\]

For small lattice and magnetic field close to \( \pm 2 \), we obtain the result directly. To obtain the result for large \( L \) asymptotically, we first expressed the entropy in terms of the determinant of a Toeplitz matrix. Then we used a special case of Fisher-Hartwig conjecture \[14\] and this special case was PROVEN in \[15\].

From similar calculation, we also obtain the Rényi entropy in Eq. 3 to be

\[
S_\alpha(\rho_A) = \begin{cases} 
\frac{1}{1-\alpha} \ln \left( \left( \frac{\pi}{L} \right)^{\alpha} + \left( 1 - \frac{\pi}{L} \right)^{\alpha} \right) & \text{if } 0 < \mathcal{L} < 1 \\
\frac{1+\alpha^{-1}}{6} \ln \mathcal{L} + \Upsilon_1^{(\alpha)} & \text{if } \mathcal{L} \gg 1
\end{cases}
\tag{63}
\]

Here

\[
\Upsilon_1^{(\alpha)} = - \frac{1}{\pi^2} \int_{-1}^1 dx \frac{s_\alpha(x)}{1-x^2} \left( \psi\left( \frac{1}{2} - iW(x) \right) + \psi\left( \frac{1}{2} + iW(x) \right) \right),
\tag{64}
\]

\[
s_\alpha(x) = \frac{1}{1-\alpha} \ln \left( \left( \frac{1+x}{2} \right)^{\alpha} + \left( \frac{1-x}{2} \right)^{\alpha} \right), \quad \alpha \neq 1,
\tag{65}
\]

\[
\psi(x) \equiv \frac{d}{dx} \ln \Gamma(x) = -\gamma_E + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+x} \right),
\tag{66}
\]

\[
W(x) = \frac{1}{2\pi} \ln \frac{1+x}{1-x}
\tag{67}
\]

with \( \gamma_E \) the Euler Constant and \( \Gamma(x) \) the well-known Gamma Function.
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Appendix: Simplification of Formula

In this appendix, we show more details for simplification of $\Upsilon_0$ (59) in detail. In order to simplify $\Upsilon_0$, we will use the Function $\psi(x)$, which is defined as

$$\psi(x) \equiv \frac{d}{dx} \ln \Gamma(x) = -\gamma_E + \sum_{n=0}^{\infty} \frac{1}{n+1} - \sum_{n=0}^{\infty} \frac{1}{n+x}$$

with $\gamma_E$ the Euler Constant and $\Gamma(x)$ the well-known Gamma Function, and the property

$$\psi(x + 1) = \psi(x) + \frac{1}{x}.$$ 

Introducing $z(\overline{z}) \equiv \frac{1}{2} + (-i)W(x)$ and using Eqs. (68) and (69), we obtain

$$\sum_{n=1}^{\infty} n^{-1} \left( \frac{\left(\frac{1}{2} + iW(x)\right)^3}{n^2 - \left(\frac{1}{2} + iW(x)\right)^2} + \frac{\left(\frac{1}{2} - iW(x)\right)^3}{n^2 - \left(\frac{1}{2} - iW(x)\right)^2} \right)$$

$$= \psi(1) - 1 - \frac{1}{2} \psi\left(\frac{1}{2} - iW(x)\right) - \frac{1}{2} \psi\left(\frac{1}{2} + iW(x)\right)$$

by using Eq. (69) and definition for $z$ and $\overline{z}$. Hence, we obtain

$$\Upsilon_0 = \frac{1}{\pi^2} \int_{-1}^{1} dx \left( - \frac{1}{1-x} \ln \frac{1+x}{2} - \frac{1}{1+x} \ln \frac{1-x}{2} \right)$$

$$\times \left[ \psi(1) - 1 - \frac{1}{2} \psi\left(\frac{1}{2} - iW(x)\right) - \frac{1}{2} \psi\left(\frac{1}{2} + iW(x)\right) \right]$$

$$= \Upsilon_1 - \frac{1 + \gamma_E}{3}$$

(71)
with $\Upsilon_1$ defined as

$$
\Upsilon_1 = -\frac{1}{2\pi^2} \int_{-1}^{1} dx \left( -\frac{1}{1-x} \ln \frac{1+x}{2} - \frac{1}{1+x} \ln \frac{1-x}{2} \right) \\
\times \left[ \psi \left( \frac{1}{2} - iW(x) \right) + \psi \left( \frac{1}{2} + iW(x) \right) \right] .
$$

(72)

We now perform a change of variable using $w = \frac{1}{2\pi} \ln \frac{1+x}{1-x}$:

$$
\Upsilon_1 = -\frac{2}{\pi} \int_{0}^{\infty} dw \left( \ln [2 \cosh(\pi w)] - \pi w \tanh(\pi w) \right) \\
\times \left[ \psi \left( \frac{1}{2} - iw \right) + \psi \left( \frac{1}{2} + iw \right) \right] .
$$

(73)

We note that

$$
\ln [2 \cosh(\pi w)] - \pi w \tanh(\pi w) = \left( 1 - \frac{d}{d\alpha} \right) \ln \left( 1 + e^{-2\pi w \alpha} \right) \bigg|_{\alpha=1} .
$$

(74)

Hence we can rewrite

$$
\Upsilon_1 = -\frac{2i}{\pi} \int_{0}^{\infty} dw \left( \ln [2 \cosh(\pi w)] - \pi w \tanh(\pi w) \right) \cdot \left( \frac{d}{dw} \right) \ln \frac{\Gamma \left( \frac{1}{2} - iw \right)}{\Gamma \left( \frac{1}{2} + iw \right)} .
$$

(75)

Using the following expression for the Logarithm of the Gamma Function:

$$
\ln \Gamma(z) = \int_{0}^{\infty} \left[ z - 1 - \frac{1 - e^{-(z-1)t}}{1 - e^{-t}} \right] \frac{e^{-t}}{t} dt
$$

(76)

which is particularly convenient because we need only the imaginary part of it:

$$
\ln \frac{\Gamma \left( \frac{1}{2} - iw \right)}{\Gamma \left( \frac{1}{2} + iw \right)} = -i \int_{0}^{\infty} \left[ 2we^{-t} - \frac{\sin(wt)}{\sinh(t/2)} \right] \frac{dt}{t} .
$$

(77)

After some elementary but tedious calculation, finally we obtain

$$
\Upsilon_1 = -\int_{0}^{\infty} dt \left\{ \frac{e^{-t}}{3t} + \frac{1}{t \sinh^2(t/2)} - \frac{\cosh(t/2)}{2 \sinh^3(t/2)} \right\} .
$$

(78)