Markov’s Transformation of Series and the WZ Method

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Abstract

In a well forgotten memoir of 1890, Andrei Markov devised a convergence acceleration technique based on a series transformation which is very similar to what is now known as the Wilf-Zeilberger (WZ) method. We review Markov’s work, put it in the context of modern computer-aided WZ machinery, and speculate about possible reasons of the memoir being shelved for so long.

Keywords: Wilf-Zeilberger method; A.A. Markov, Sr.; series transformation; convergence acceleration; hypergeometric series; basic hypergeometric series; double series; discrete Green formula; Apéry constant.

1 Introduction

By this publication we aim to resurface the memoir [16] by the Russian mathematician Andrei Andreevich Markov (1856–1922), who is best known as the inventor of Markov’s chains in probability theory. However, by the time Markov began his studies in probability, he was a distinguished analyst and a member of the (Russian) Emperor’s Academy of Sciences.

Why would the old paper be worth attention of today’s mathematical community? All of the sudden, it appears very relevant in the context of a powerful technique of series transformation known as the Wilf-Zeilberger
(WZ) method, and just as relevant in the context of recent sport about faster and faster evaluation of the constant

\[ \zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots, \]

called the Apéry constant after R. Apéry proved its irrationality in 1978 [1]. (Not too far away is an actively pursued challenge — irrationality of further odd zeta values, cf. [33] and references therein.)

To appreciate the following results, try (if you never did) to obtain 7 correct decimals of \( \zeta(3) \) with a non-programmable calculator!

The formula

\[ \zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)^3 n^3} \tag{1} \]

is often attributed to Apéry, but it wasn’t him who first discovered it. The review [25] points out the result [12] reported in 1953, and here is formula (14) from Markov’s memoir

\[ \sum_{n=0}^{\infty} \frac{1}{(a+n)^3} = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n n!^6}{(2n+1)!} \frac{5(n+1)^2 + 6(a-1)(n+1) + 2(a-1)^2}{[a(a+1)\ldots(a+n)]^4}, \tag{2} \]

which is a generalization of (1). The series (1), (2) converge at the geometric rate with ratio 1/4. A series convergent at the geometric rate with ratio 1/27,

\[ \zeta(3) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2 n^3} \frac{56n^2 - 32n + 5}{(2n-1)^2 n^3} \frac{(n!)^3}{(3n)!}. \tag{3} \]

is "automatically" derived in [2], together with formula (1), using the WZ method. Interestingly, Markov has an equivalent of (3) on page 9 of his memoir.

Note for reference that [2] contains an even faster convergent representation for \( \zeta(3) \) with ratio \( 2^2/4^4 = 1/64 \). A series of a non-hypergeometric type, convergent at the geometric rate with ratio \( e^{-2\pi} \approx 1/535 \) is essentially due to Ramanujan [6, p.30, (59)]. And the largest number of decimals in \( \zeta(3) \), currently 520,000, to our knowledge, was obtained by means of the nice formula derived in [3]

\[ \zeta(3) = \sum_{n=0}^{\infty} \frac{(-1)^n n!^10 (205n^2 + 250n + 77)}{64 (2n+1)^{15}}. \]
The ratio of convergence here is $2^{-10}$.

These highly nontrivial results have been obtained by the same method, which is deceitfully simple in an abstract form. It can be viewed either as a generalization of the elementary telescoping trick:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots = 1,$$

or as a finite-difference analog of Green’s formula for circulation of a vortex-free vector field.

One may believe in existence of interesting applications of the discrete Green formula to series transformations, but it isn’t easy to bring forth a convincing example. Markov has demonstrated prolificity of that approach in about a dozen of striking identities. The subtlety that makes it work is a proper choice of certain auxiliary factors unknown in advance. (One may think of them as integrating factors).

In Sect. 2 we outline the memoir’s scope and review Markov’s method (or rather its visible side). Sect. 3 goes into details of one of Markov’s examples. We’ll show close parallels between its treatment in the memoir and by the modern computerized WZ.

That said, one should not get an impression that Markov knew the entire WZ theory hundred years earlier. The most apparent omission in the memoir, as well as in the later textbook [18], is scope of the method; the related how to (construct such examples) and what else questions remain unanswered. The creators of the modern technique put a great effort into clarification, generalization, and algorithmization (see [22, 30, 31] and perhaps the most consonant to this context [32]). Also, Markov was concerned only about convergence acceleration, while the WZ pretends to certify, in a well-defined sense, nearly all ”concrete mathematics”.

In the memoir, we don’t see a slightest hint to anything resembling Gosper’s algorithm (for integrating, if possible, linear difference equations with polynomial coefficients) — a crucial subroutine of Zeilberger’s algorithm, which, in turn, is an inborn ingredient of the WZ method.

It would be unfair to criticize Markov for not inventing all these things. Unfortunate — and hard to explain — is the fact that no one of Markov’s contemporaries picked up his technique. We speculate about possible reasons in Sect. 4.

One of us came across the textbook [18] in 1995 while studying convergence acceleration methods for purposes of an applied project [24]. It is how
the memoir \textsuperscript{16} was revealed; it is cited in \textsuperscript{24}. Unfortunately, we were not aware of the WZ method up until April 2002 and it took more than a year for us to set on writing a detailed presentation of Markov’s work after the first published announcement \textsuperscript{14}. \textsuperscript{1}

2 A review of Markov’s memoir

We begin with a translation of Section 1 of the memoir \textsuperscript{16}.

"Recall at first the proposition, which is easily derived by considering a double sum:

If two functions $U_{x,z}$ and $V_{x,z}$ of independent variables $x$ and $z$ are bound by the condition

$$U_{x,z} - U_{x+1,z} = V_{x,z} - V_{x,z+1}, \quad (4)$$

then

$$U_{0,0} + U_{0,1} + \ldots + U_{0,j-1} - U_{i,0} - U_{i,1} - \ldots - U_{i,j-1}$$

$$= V_{0,0} + V_{1,0} + \ldots + V_{i-1,0} - V_{0,j} - V_{1,j} - \ldots - V_{i-1,j}, \quad (5)$$

$i$ and $j$ being arbitrary positive integers.

In all the cases occurring in this memoir, the series with terms

$$U_{0,0}, U_{0,1}, \ldots, U_{0,j}, \ldots$$

$$V_{0,0}, V_{1,0}, \ldots, U_{i,0}, \ldots$$

are convergent and the sums

$$U_{i,0} + U_{i,1} + \ldots + U_{i,j-1}, \ldots$$

and

$$V_{0,j}, V_{1,j} + \ldots + U_{i-1,j}, \ldots$$

tend to zero as $i$ and $j$ increase indefinitely.

That stated, the formula (5) will give

$$U_{0,0} + U_{0,1} + \ldots + U_{0,j} + \ldots = V_{0,0} + V_{1,0} + \ldots + V_{i,0} + \ldots.” \quad (6)$$

\textsuperscript{1}In February 2003 Alexandru Lupas independently suggested at an Internet discussion board ([http://groups.google.com/groups?q=WZ-Theory](http://groups.google.com/groups?q=WZ-Theory)) that traces of the WZ could be found in \textsuperscript{7} \textsuperscript{18} (source: \textsuperscript{17}).
Markov works with hypergeometric and basic hypergeometric series in his memoir, although he avoids calling them so. To make formulae more concise and comprehensible, let us recall appropriate definitions and notation, cf. [1, 6].

The **rising factorial** is defined as

\[(a)_n = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \geq 0.\]

In particular, \((1)_n = n!\). Denote for brevity

\[(a_1, \ldots, a_r)_n = \prod_{j=1}^r (a_j)_n.\]

A **hypergeometric (HG) term** is an expression of the form

\[\frac{(a_1, \ldots, a_r)_n}{(b_1, \ldots, b_s)_n} z^n,\]

and a hypergeometric series is a series of the form

\[rF_s \left( \frac{a_1, \ldots, a_r}{b_1, \ldots, b_s}; z \right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r)_n}{(b_1, \ldots, b_s, 1)_n} z^n.\]

If \(z = 1\), it is common to omit the argument \(z\).

Basic hypergeometric (BHG) terms and series contain an additional parameter \(q\), called the **base**. The **\(q\)-rising factorial** is the product

\[(a; q)_n = (1 - a)(1 - qa) \cdots (1 - q^{n-1}a),\]

The expression \((q; q)_n\) is called the **\(q\)-factorial** of \(n\). The product of several \(q\)-rising factorials is abbreviated as

\[(a_1, \ldots, a_r; q)_n = \prod_{j=1}^r (a_j; q)_n.\]

A **basic hypergeometric term** is an expression of the form

\[\frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} z^n,\]
and a basic hypergeometric series is a series of the form
\[ r \phi_s \left( a_1, \ldots, a_r ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s, 1; q)_n} z^n \left( (-1)^n q^{n(n-1)/2} \right)^{1+s-r}. \]

The base \( q \) is usually omitted in the notation, unless BHG series with different bases are discussed in the same context. Also, as for HG series, the argument \( z \) is often omitted in the special case \( z = 1 \).

The ordinary hypergeometry is a limiting case of the basic one:
\[ \lim_{q \to 1} \frac{(q^a; q)_n}{(q^b; q)_n} = \frac{(a)_n}{(b)_n}. \]  

(7)

In the basic case, Markov assumes \(|q| > 1\), while the modern convention strongly prefers \(|q| < 1\). For this reason we re-denote Markov’s \( q \) to \( q \) and adopt the base \( q = q^{-1} \). Thus in the sequel \(|q| < 1\) and \(|q| > 1\).

**Structure of the functions \( U_{x,z} \) and \( V_{x,z} \) in Markov’s examples**

All the examples, in their most general form, deal with convergence-accelerating transformations of hypergeometric or basic hypergeometric series. Every time we have a HG or BHG term \( F_{x,z} \) such that the series \( \sum_z F_{0,z} \) is to be summed. Dependence of \( F_{x,z} \) on \( x \) is characterized by the multiplicative pattern
\[ \frac{(a)_z}{(b)_{x+z}} ; \]

in the basic case an additional factor \( q^{f(x,z)} \) is present, where \( f \) is a quadratic polynomial in \( x \). In all cases, \( F_{x,z} \) and the sums over \( z \) have limit 0 as \( x \to \infty \).

The function \( U_{x,z} \) has one of the following three forms:

\[ U_{x,z} = F_{x,z} A_x \quad \text{in } \S\,2,3 \]  

(8)

or

\[ U_{x,z} = F_{x,z} (A_x + B_x z) \quad \text{in } \S\,4,8 \]  

(9)

or

\[ U_{x,z} = F_{x,z} (A_x + \tilde{B}_x z + \tilde{C}_x z^2) \quad \text{in } \S\,9. \]  

(10)

In the first case \( F_{x,z} \) is a BHG term, and in the latter two cases \( F_{x,z} \) is a HG term. In addition, \( B_0 = 0 \) and \( \tilde{B}_0 = \tilde{C}_0 = 0 \). Trying to present Markov’s
patterns in a unified form, we use symbols with tildes where our notation is not identical to that in [16].

The function \( V_{x,z} \) is sought in the form
\[
V_{x,z} = F_{x,z} M_{x,z},
\]
(11)
where in the case (8)
\[
M_{x,z} = B_x + C_x q^z; \tag{8}
\]
in the case (9)
\[
M_{x,z} = C_x + D_x z + F_x z^2; \tag{9}
\]
and finally in the case (10)
\[
M_{x,z} = F_x + \tilde{G}_x z + \tilde{H}_x z^2. \tag{10}
\]

List of the series dealt with in the Memoir

§ 2: \( 2\phi_1 \left( \frac{a, 1}{b}; t \right) = 1 + \frac{1-a}{1-b} t + \frac{(1-a)(1-aq)}{(1-b)(1-bq)} t^2 + \cdots. \)

§ 3: \( 3\phi_2 \left( \frac{a, b, 1}{c, d}; \frac{cd}{abq} \right) \) and the limiting (Schellbach’s) case \( 3F_2 \left( \frac{a, b, 1}{c, d}; \right). \)

§ 4: \( 4F_3 \left( \frac{a, a+h, a-h, 1}{b, b+h, b-h}; \right) = 1 + \frac{a}{b} \cdot \frac{a^2-h^2}{b^2-h^2} \)
\[
+ \frac{a}{b} \cdot \frac{a+1}{b+1} \cdot \frac{a^2-h^2}{b^2-h^2} \cdot \frac{(a+1)^2-h^2}{(b+1)^2-h^2} + \cdots.
\]

§ 5: Special case of § 4: \( a = 1, b = 2; \) then further specialization \( h = 0 \)
yields the series defining \( \zeta(3). \) Formula (3) is found in this §.

§ 6: Special case of § 4: \( h = 0, b = a+1, \) yielding the Hurwitz zeta series \( \zeta(3, a). \)

§ 7: Special case of § 4: Kummer’s sum
\[
4F_3 \left( \frac{9/2, 9/2, 9/2, 1}{5, 5, 5}; \right) = \sum_{n=0}^{\infty} \left( \frac{(2n+1)!!}{(2n)!!} \right)^3.
\]

§ 8: The series \( 3F_2 \left( \frac{a, b, 1}{c, d}; -1 \right) \) with \( c-a = d-b, \) and particular cases.

§ 9: A well-poised [4, 9] series \( 4F_3 \left( \frac{a, a, a, 1}{b, b, b}; -1 \right). \) Among considered particular cases there are Stirling’s series \( \sum_{k}^{\infty} (-1)^n n^{-k}, \) \( k = 2, 3. \)
Three more formulae, (2) being the simplest, are contained in the last §10. Details of the transformations, in particular, the form of the functions $U$, $V$, are not provided.

Method for obtaining the transformations

Once the parametric form of the functions $U_{x,z}$ and $V_{x,z}$ is set, it remains to choose the undetermined constants in order to satisfy Eq. (4). The main question is why exactly that many parameters are needed in the particular situation. We suppose that Markov simply used a trial and error approach, starting with minimal number of parameters and extending the family of parameters until a solution was found. But there may exist a clever reasoning of which we are not aware.²

3 Example: Transformation of a $3\phi_2$ series

Consider the series

$$3\phi_2\left(a, b, 1 \atop c, d ; q, t\right) = \sum_{z=0}^{\infty} \frac{(a; q)_z(b; q)_z}{(c; q)_z(d; q)_z} t^z, \quad t = \frac{cd}{abq}, \quad |t| < 1. \quad (12)$$

from §3 of Markov’s memoir. We made a random choice between this example and the one in §2, but we deliberately chose an example that falls under the case (8), where the auxiliary factor is $A_x$ is $z$-independent. Our intention is to compare Markov’s procedure with the WZ one.

Markov writes the series in the form (as agreed, we rename his $q$ to $q = q^{-1}$)

$$1 + \frac{(r - 1)(r' - 1)}{(s - 1)(s' - 1)} q + \frac{(r - 1)(rq - 1)(r' - 1)(r'q - 1)}{(s - 1)(sq - 1)(s' - 1)(s'q - 1)} q^2 + \ldots . \quad (13)$$

In view of the relation

$$(u - 1)(uq - 1) \ldots (uq^{n-1} - 1) = (u^{-1}; q)_n u^n q^{n(n-1)/2},$$

the series (12) and (13) are equivalent, if their parameters are related as follows

$$a = \frac{1}{r}, \quad b = \frac{1}{r'}, \quad c = \frac{1}{s}, \quad d = \frac{1}{s'}, \quad t = \frac{cd}{abq} = \frac{rr'q}{ss'}. \quad (14)$$

²Written before we had a chance to study [19].
First, let us derive Markov’s result (see (22) below) in our notation by hand. Set $A_0 = 1$ in (8). Markov takes an extension $F_{x,z}$ of the term $F_{0,z} = U_{0,z}$ in the form

$$F_{x,z} = \frac{(a; q)_x(b; q)_z t^x}{(c; q)_{x+z}(d; q)_{x+z}} (cdq^{2z})^x q^{x(x-1)}.$$  

(14)

Such a pattern doesn’t appear obvious when using the modern form with $|q| < 1$, but it is naturally suggested by the original form (13): replace the two products of $z$ factors in the denominator by the products of $(x+z)$ factors. In particular,

$$F_{x,0} = q^{x(x-1)} \frac{(cd)^x}{(c, d; q)_x}.$$  

With (14) and (8), the condition (4) becomes

$$A_x \frac{(a, b; q)_z t^x}{(c, d; q)_{x+z}} (cdq^{2z})^x q^{x(x-1)} - A_{x+1} \frac{(a, b; q)_z t^x}{(c, d; q)_{x+z+1}} (cdq^{2z})^{x+1} q^{x(x+1)}$$

$$= \left[ (B_x + C_x q^x) \frac{(a, b; q)_z t^x}{(c, d; q)_{x+z}} - (B_x + C_x q^{x+1}) \frac{(a, b; q)_{x+1} t^{x+1}}{(c, d; q)_{x+z+1}} q^{2x} \right]$$

$$\times (cdq^{2z+x-1})^x.$$  

Taking out the common factor $(a, b; q)_z/(c, d; q)_{x+z+1} t^x (cdq^{2z+x-1})^x$, we obtain an equation of degree 3 in $q^x$. To satisfy condition (14), all the coefficients of that equation must vanish, that is

$$A_x = B_x (1 - tq^{2x}),$$  

(15)

$$-A_x (c + d) q^x = C_x - B_x (c + d) q^x + B_x (a + b) q^{2x} t - C_x q^{2x+1} t,$$  

(16)

$$(A_x - A_{x+1}) cd q^{2x} = B_x (cd - abt) q^{2x} + C_x \left( (a + b) q^{2x+1} t - (c + d) q^x \right),$$  

(17)

$$0 = C_x (cd q^{2x} - ab q^{2x+1} t).$$  

(18)

Eq. (18) holds automatically. Equations (15), (16) imply

$$\frac{B_x}{A_x} = (1 - tq^{2x})^{-1},$$  

$$\frac{C_x}{A_x} = \frac{tq^{2x} [(c + d) q^x - (a + b)]}{(1 - tq^{2x})(1 - tq^{2x+1})}.$$  

(19)
and therefore, by (8)

\[
M_{x,0}/A_x = \frac{B_x + q^0C_x}{A_x} = \frac{1 - tq^{2x}(a + b + q) + tq^{3x}(c + d)}{(1 - tq^{2x})(1 - tq^{2x+1})}. \tag{20}
\]

Substitution of (19) to (17) yields the recurrence for \(A_x\)

\[
A_{x+1}/A_x = \frac{(1 - \frac{c}{a}q^x)(1 - \frac{c}{b}q^x)(1 - \frac{d}{a}q^x)(1 - \frac{d}{b}q^x)}{q(1 - tq^{2x})(1 - tq^{2x+1})}. \tag{21}
\]

Since \(A_0 = 1\), we obtain

\[
A_x = \frac{(\frac{c}{a}, \frac{c}{b}, \frac{d}{a}, \frac{d}{b}; q)_x}{q^x (t; q)_{2x}}.
\]

Finally, we find the general term of the transformed series in the r.h.s. of (6)

\[
V_{x,0} = \frac{M_{x,0}/A_x}{A_x}F_{x,0} = \frac{(\frac{c}{a}, \frac{c}{b}, \frac{d}{a}, \frac{d}{b}; q)_x}{(c, d; q)_x} (cd)^x q^{x(x-2)} \frac{1 - tq^{2x}(a + b + q) + tq^{3x}(c + d)}{(t; q)_{2x+2}}. \tag{22}
\]

For \(x = 0\) or 1, the terms are consistent with those in formula (7) in [16], where further terms are not written out, while they are not easy to guess.

Following Markov, we proceed to consider the limiting case \(q \to 1\). Re- denote \(a, b, c, d, t\) respectively to \(q^a, q^b, q^c, q^d, q^t\). The relation \(tq = (cd)/(ab)\) is replaced by the following:

\[t + 1 = c + d - a - b.\]

Then, by (7),

\[
\lim_{q \to 1} \frac{(q^{c-a}, q^{c-b}, q^{d-a}, q^{d-b}; q)_x}{(q^c, q^d; q)_x (t; q)_{2x}} = \frac{(c - a, c - b, d - a, d - b)_x}{(c, d)_x (t)_{2x}}.
\]

The limit of the remaining factor in (22) is found by applying L'Hospital Rule two times:

\[
\lim_{q \to 1} \frac{1 - q^{2x+t}(a + b + q) + q^{3x+t}(c + d)}{(1 - q^{2x+t})(1 - q^{2x+t+1})} = \frac{p(a, b, c, d, x)}{(2x + t)(2x + t + 1)}.
\]
where
\[ p(a, b, c, d, x) = (2x + t + a)(2x + b + t) - (x + c - 1)(x + d - 1) \]
\[ = (c + d - a - 1 + 2x)(c + d - b - 1 + 2x) - (c - 1 + x)(d - 1 + x). \]
The result is Schellbach's formula
\[
\_3F_2 \left( \begin{array}{c} a, b, 1 \\ c, d \end{array} \right) = \sum_{x=0}^{\infty} \frac{(c-a, c-b, d-a, d-b)_x}{(c, d)_x} \frac{p(a, b, c, d, x)}{(c + d - a - b - 1)_{2x+2}}.
\]
The left-hand side converges as \( \sum n^{-t-1} \) (assuming that \( t = c + d - a - b - 1 > 0 \)). The right-hand side converges geometrically; namely, the term with subscript \( x \) has the asymptotics \( 4^{-x} \cdot K x^{-a-b+1/2} \) with
\[
K = (3/2^{t+1})\sqrt{\pi} \Gamma(t) \Gamma(c) \Gamma(d) / (\Gamma(c-a) \Gamma(c-b) \Gamma(d-a) \Gamma(d-b)).
\]
We turn now to the Wilf-Zeilberger approach, more specifically, to its computer-aided version. Speaking pragmatically, all one needs is to type in the expression \( (14) \) in Maple, feed it to the \texttt{qEKHAD} program, and analyze the results. The substitution \( t = (cd) / (abq) \) must be made in advance in \( (14) \).

The program produces a recurrence operator \( \Omega(X, x) \) and a certificate \( R(x, z) \). We believe that the reader can't avoid looking into \cite{22} anyway, but below we give a self-contained account of the procedure in this case.

The recurrence operator outputted by \texttt{qEKHAD} has the structure
\[
\Omega(X, x) = P(x) + Q(x)X.
\]
Here \( X \) is the operator of forward shift in \( x \), that is
\[
(\Omega(X, x)F)_{x,z} = P(x)F_{x,z} + Q(x)F_{x+1,z}. \tag{23}
\]
The certificate \( R(x, z) \) is a rational function of \( q^x, q^z \) such that the function
\[
G(x, z) = R(x, z)F_{x,z}
\]
satisfies the equation
\[
(\Omega(X, x)F)_{x,z} = G(x, z) - G(x, z - 1).\] For comparison purposes, it is more convenient to deal with forward \( z \)-difference in the right-hand side, so we denote
\[
\tilde{G}(x, z) = G(x, z - 1) = \tilde{R}(x, z)F_{x,z}.
\]
where
\[ \tilde{R}(x, z) = R(x, z - 1) \frac{F_{x, z-1}}{F_{x, z}}. \]

Now
\[ (\Omega(X, x)F)_{x, z} = \tilde{G}(x, z + 1) - \tilde{G}(x, z). \]  
(24)

We will actually need only values \( \tilde{R}(x, 0) \). Taking the output of qEKHAD and transforming it this way, we find (with \( t = (cd)/(abq) \), as before)
\[ \tilde{R}(x, 0) = \frac{1 - tq^{2x}(a + b + q) + tq^{3x}(c + d)}{1 - tq^{2x+1}}. \]  
(25)

The values of \( P(x) \) and \( Q(x) \) in (23) produced by qEKHAD are
\[ P(x) = 1 - tq^{2x}, \quad Q(x) = \frac{(1 - \frac{c}{a}q^x)(1 - \frac{b}{a}q^x)(1 - \frac{d}{a}q^x)(1 - \frac{d}{b}q^x)}{q(1 - tq^{2x+1})}. \]  
(26)

Equations (25), (26) have much in common with (20), (21), though they are not identical. Of course, the similarity is not occasional. It is explored below in detail.

If we fix \( F_{x, z} \) and try to satisfy Eq. (4) using substitutions of the form (8), (11), the following equation comes up:
\[ A_{x+1}F_{x+1, z} - A_xF_{x, z} = M_{x, z+1}F_{x, z+1} - M_{x, z}F_{x, z}. \]  
(27)

On the other hand, Eq. (24) in expanded notation reads
\[ Q(x)F_{x+1, z} + P(x)F_{x, z} = \tilde{R}(x, z + 1)F_{x, z+1} - \tilde{R}(x, z)F_{x, z}. \]  
(28)

Suppose that the certificate \( \tilde{R}(x, z) \) and the operators \( \Omega(x, z) \) are known. Let us find \( A_x \) and \( M_{x, z} \). Introduce as yet undetermined coefficients \( \Phi(x) \) such that multiplication by \( \Phi(x) \) turns Eq. (28) into (27). Thus,
\[ M_{x, z} = \Phi(x)\tilde{R}(x, z) \]
and
\[ A_x = \Phi(x)P(x), \quad A_{x+1} = \Phi(x)Q(x). \]

Therefore,
\[ \frac{M_{x, z}}{A_x} = \frac{\tilde{R}(x, z)}{P(x)}, \quad \frac{A_{x+1}}{A_x} = \frac{Q(x)}{P(x)}. \]

The right-hand sides in these equations follow from (25), (26). The obtained equations for \( A_x \) and \( M_{x, z} \) are identical to (20) and (21), from which we (following Markov) have found the terms (22) of the transformed series.
4 How did Markov miss his audience?

This section is mostly speculative. A thorough study of Markov’s works, letters, and other documents, which may reveal circumstances of the appearance of the memoir in question and of its abandonment, is yet to be undertaken.

Having been deeply involved in studies on continued fractions throughout the 1880s, Markov corresponded with T. J. Stieltjes [21] and closely watched his publications. In 1887 Stieltjes [26] published a table of the values of the Riemann Zeta function $\zeta(k)$ with 32 decimals for integral values of $k$ from 2 to 70. Markov might have felt challenged by that achievement and by Stieltjes’ convergence acceleration technique. Apparently, it was this challenge and rivalry that prompted Markov to develop his new acceleration method. In a brief note [15], he gives two formulae, one of them equivalent to (3), and obtains 20 decimals of $\zeta(3)$ taking 13 terms in his series. Afterwards he jealously beat Stieltjes’ record, taking 22 terms and obtaining the result with 33 decimals in [16]:

$$\zeta(3) = 1,202056903159594285399738161511450.$$  

The second formula published in [15] is a $27^{-k}$-fast convergent representation

$$\zeta(2) = \frac{5}{3} + \sum_{k=1}^{\infty} \frac{(-1)^k(2k-1)!!^3}{(6k-1)!!} \left( \frac{1}{4k^2} + \frac{5}{(6k+1)(6k+3)} \right).$$

Claiming that Markov missed his audience, as it eventually turned out, we don’t mean that the series transformation he proposed remained unnoticed. Markov himself tried to popularize it. In the textbook [18] there is a chapter devoted to this transformation with a number of examples, although examples with basic hypergeometric series are not included. References to Markov’s work are found in the well-known textbooks [5, III.24], [13]. The latter contains a section (Ch. VIII, § 33) on Markov’s transformation, at the beginning of which we find, among all, a reference to Stirling’s work [27], the starting point of Gosper’s seminal paper [8a], which laid out the foundation of automated identity proving. Both Stirling’s and Markov’s methods are treated in detail in another text [7], which seems to be left out nowadays, perhaps undeservedly.

The evidence that T. J. Bromwich [5] was aware of Markov’s work is especially interesting, since it was England where the research in hypergeometric and combinatorial identities enjoyed its most fruitful period in the first two
decades of the 20th century. Did Rogers and MacMahon see Markov’s memoir? Ramanujan might have appreciated formula (1) had he noticed it in \[5\], but Hardy [11, II.14] doubts Ramanujan having seen that book.

It is perhaps even more surprising that the memoir of 1890 had been completely forgotten in Russia. Markov’s name and works were well known and highly regarded in the Soviet Union.\(^3\) The biography [10] contains an appendix, where Markov’s works in various directions are reviewed by experts in the respected areas. In the Analysis section (as well as anywhere else), the convergence acceleration topic is not even mentioned! We managed to find only one reference to [16] in mathematical literature of the Soviet period: a rarity textbook [23]. It contains a section on Markov’s transformation and the exposition there, as the author indicates, closely follows that in [13]. ”Markov’s theorem”, see below, is also found in a widely circulated treatise [28]; however, no exact reference and no applications are given.

In our opinion, the latter theorem is partly to blame for draining the key issue of the 1890 memoir. The theorem is also contained in the cited texts [13] and [23], and it goes back to Markov’s lecture notes [18]. The formulation below is taken from [13].

**Theorem.** Let a convergent series \(\sum_{k=0}^{\infty} z^{(k)}\) be given with each of its terms itself expressed as a convergent series:

\[
z^{(k)} = a_0^{(k)} + a_1^{(k)} + \ldots + a_n^{(k)} + \ldots \quad (k = 0, 1, 2, \ldots).
\]

(29)

Let the individual columns \(\sum_{k=0}^{\infty} a_n^{(k)}\) of the array [29] so formed represent convergent series with sum \(s^{(n)}\), \(n = 0, 1, 2, \ldots\), so that the remainders

\[
r_m^{(k)} = \sum_{n=m}^{\infty} a_n^{(k)} \quad (m \geq 0)
\]

of the series in the horizontal rows also constitute a convergent series

\[
\sum_{k=0}^{\infty} r_m^{(k)} = R_m \quad (m \text{ fixed}).
\]

\(^3\)To avoid a confusion, we are talking about A. A. Markov, Sr. His son, Andrei Andreevich Markov, Jr. (1903–1979), was also a prominent mathematician, a member of the USSR Academy of Sciences, and one of the founders of Computer Science in the Soviet Union.
In order that the sums by vertical columns should form a convergent series \( \sum s^{(n)} \), it is necessary and sufficient that \( \lim R_m = R \) should exist; and in order that the relation
\[
\sum_{n=0}^{\infty} s^{(n)} = \sum_{k=0}^{\infty} z^{(k)} \tag{30}
\]
should hold as well, it is necessary and sufficient that this limit should be 0.

Compare the introductory section of the memoir and this theorem. The latter is as simple in essence as the former but how much harder it is to grasp! It is positioned, in the first instance, as a convergence theorem and the equation (30) is just yet another switch-the-order formula. The theorem per se expresses a nice and possibly useful analytical criterion, but it completely overshadows the original point.

This may partly explain an underestimation of Markov’s work, but another component is the strikingly different level (compared to the nearly trivial general idea) of concrete formulae, and a lack of Markov’s elaboration on the forms of the series and the auxiliary factors. In neither of the cited books did their authors offer their own examples! And, since Markov didn’t make any precise statements regarding the applicability range of his transformation, we got to observe the tendency to phase out vague and complicated applications and emphasize the simple and well-rounded theorem. But does calculus exist for the sake of convergence theorems?

Ch. Hermite, the then-editor of *Comptes Rendus*, replied to [15]: "Par quelle voie vous êtes parvenu à une telle transformation, je ne puis même de loin l’entrevoir, et il me faut vous laisser votre secret." It might have sound as a compliment, but shouldn’t it be heard by Markov as a warning? A nice hint would have helped Markov’s readers (and himself?): an advise to investigate specifically series of hypergeometric and basic hypergeometric type. Restriction to these two classes yielded the development of effective algorithms whose traces are implicit in [16] for determining auxiliary factors (certificates, in the WZ version) and ensured a huge success of the modern WZ method. Availability of a software ([EKHAD, qEKHAD — see [22]]) makes it tremendously helpful for everybody who deals with hypergeometric functions, partitions, and the like.

\[\text{4}^{\text{“I can’t even remotely guess the way you arrived at such a transformation, and it remains to leave your secret with you.”}}\]
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