ON OPERATIONS AND CHARACTERISTIC CLASSES

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Abstract. We define for any abelian category with tensor and exterior products the Grothendieck ring functor. Furthermore we use exterior products to define gamma operations and virtual Chern classes and virtual Segre classes for arbitrary elements in the Grothendieck ring. We apply this to define characteristic classes with values in algebraic K-theory and K-theory of connections.

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1. Introduction

The aim of this paper is to give a direct and elementary construction of virtual characteristic classes with values in the K-theory of an arbitrary abelian category with tensor and exterior products using operations. We apply this to construct characteristic classes with values in algebraic K-theory and K-theory of connections.

To construct characteristic classes of locally free sheaves with values in algebraic and topological K-theory one use the projective bundle formula. In the more general case of the K-theory of an abelian category with tensor and exterior product there is no such formula available. To define characteristic classes using exterior products is an alternative approach to the construction of characteristic classes not relying on the projective bundle formula.

The main results of the paper are the following theorems: Let $\text{Cat}^\otimes$ be the category of abelian categories with tensor and exterior products and morphisms. Let $\lambda$-Rings denote the category of finite dimensional augmented $\lambda$-rings and morphisms.

Theorem 1.1. There is a covariant functor

$$K_0^\lambda : \text{Cat}^\otimes \to \lambda\text{-Rings}$$
defined by
\[ K_0^λ(C) = \{ K_0(C), \{ λ^n \}_{n \geq 0} \}. \]
Here \( K_0(C) \) is the Grothendieck ring of \( C \) and \( λ^n \) is the \( n \)'th exterior product.

Let \( C \in Ob(Cat^⊗) \). We define for any \( l \geq 0 \) and \( x ∈ K_0(C) \) the virtual characteristic class \( c_l(x) ∈ K_0(C) \).

**Theorem 1.2.** Let \( x, y ∈ K_0(C) \). The following formulas hold:

1. \( c_l(x + y) = \sum_{i+j=l} c_i(x)c_j(y) \) (1.2.1)
2. \( c_l(x) = 0 \) if \( x \) is effective and \( l > e(x) \) (1.2.2)
3. \( f^*(c_l(x)) = c_l(f^*(x)) \) (1.2.3)

Here \( f : D \to C \) is a morphism in \( Cat^⊗ \).

2. Exterior products and characteristic classes

In this section we prove existence of virtual Chern and Segre classes \( c_i(x), s_i(x) \in K_0(C) \) for any integer \( i \geq 0 \) and any \( x ∈ K_0(C) \) where \( C \) is an arbitrary abelian category with tensor and exterior products.

Let \( C \) be a small abelian category. We say that \( C \) is an abelian category with tensor and exterior products (for short ACTEP) if the following holds. There is a tensor product

\[ \otimes : C \times C \to C \]

satisfying the following properties: The triple \( \{ C, ⊕, \otimes \} \) is an abelian tensor category. There are canonical isomorphisms

\[ x \otimes y \cong y \otimes x \]

and

\[ (x \otimes y) \otimes z \cong x \otimes z \oplus y \otimes z \]

for any objects \( x, y, z ∈ Ob(C) \). There is a unique object \( 1^⊗ ∈ Ob(C) \) and canonical isomorphisms

\[ 1^⊗ \otimes x \cong x \otimes 1^⊗ \cong x \]

for any object \( x ∈ Ob(C) \). Moreover the endofunctor \( x \otimes − \) is exact for any object \( x ∈ Ob(C) \).

There is a function

\[ rk : Ob(C) \to \mathbb{N} \]

such that \( rk(1^⊗) = 1 \) and \( rk \) is additive on exact sequences. There is for every \( k \geq 0 \) an endofunctor - exterior product

\[ \lambda^k : C \to C \]

with the following properties: \( \lambda^i(x) = 0 \) for \( i > rk(x) \), \( λ^0(x) = x \) for all \( x ∈ Ob(C) \). Moreover, for any exact sequence

\[ 0 \to x' \to x \to x'' \to 0 \]

in \( C \) and any \( p ≥ 2 \) there is a filtration

\[ 0 = F_{p+1} \subseteq F_p \subseteq \cdots \subseteq F_1 \subseteq F_0 = λ^p(x) \]
and canonical isomorphisms
\[ F_i/F_{i+1} \cong \lambda^i(x') \otimes \lambda^{p-i}(x'') \]
in \( C \). We say a functor
\[ f : D \to C \]
between ACTEP’s is a morphism of ACTEP’s if \( f \) is an additive tensor functor commuting with the exterior product. This means for any \( l \geq 0 \) there are canonical isomorphisms
\[ f(\lambda^l(x)) \cong \lambda^l(f(x)) \]
in \( \text{Ob}(C) \).

**Definition 2.1.** Let \( \text{Cat}^\otimes_{\Lambda} \) denote the category with ACTEP’s as objects and morphisms of ACTEP’s as morphisms.

**Theorem 2.2.** There is a covariant functor
\[ K_0 : \text{Cat}^\otimes_{\Lambda} \to \text{Rings} \]
defined by
\[ K_0(C) \to K_0(C) \]
where \( K_0(C) \) is the Grothendieck ring of the category \( C \).

**Proof.** Let for any \( C \in \text{Ob}(\text{Cat}^\otimes_{\Lambda}) \) \( K_0(C) \) be the Grothendieck ring of the category \( C \). Direct sum induce an addition operation and tensor product induce a multiplication with \([1^\otimes]\) as multiplicative unit. One checks that for any morphism \( f : D \to C \) there is an induced map of rings
\[ f^* : K_0(D) \to K_0(C) \]
Finally for two composable morphisms \( f, g \) it follows \((g \circ f)^* = g^* \circ f^*\) and the theorem follows.

Define for any element \( x \in \text{Ob}(C) \) and any integer \( l \geq 0 \) \( \lambda^l[x] = [\lambda^l(x)] \) where \( \lambda^l \) is the \( l \)’th exterior power. It follows that for any exact sequence
\[ 0 \to x' \to x \to x'' \to 0 \]
in \( C \) there is for all \( p \geq 2 \) an equality
\[ \lambda^p[x] = \sum_{i+j=p} \lambda^i[x']\lambda^j[x''] \]
in \( K_0(C) \). Let \( K_0(C)[[t]] \) be the ring of formal power series in \( t \) with coefficients in \( K_0(C) \). Let \( 1 + K_0(C)[[t]] \) be the multiplicative subgroup of \( K_0(C)[[t]] \) consisting of formal power series with constant term equal to one. Let \( \Phi(C) \) denote the abelian monoid on \( C \) with direct sum as addition operation. Define the following map
\[ \lambda_t : \Phi(C) \to 1 + K_0(C)[[t]] \]
by
\[ \lambda_t(x) = \sum_{l \geq 0} \lambda^l(x)t^l. \]
For any exact sequence
\[ 0 \to x' \to x \to x'' \to 0 \]
in \( C \) we get an equality of formal power series
\[ \lambda_t(x) = \lambda_t(x')\lambda_t(x''). \]
We get a well defined map of abelian groups
\[ \lambda_t : K_0(C) \to 1 + K_0(C)[[t]] \]
defined by
\[ \lambda_t(n[x] - m[y]) = \lambda_t(x)^n \lambda_t(y)^{-m}. \]
We can define for any element \( \omega \in K_0(C) \) a formal power series
\[ \lambda_t(\omega) = \sum_{l \geq 0} \lambda^l(\omega)t^l. \]

Let \( Op(K_0) \) be the set of natural transformations of the underlying set valued functor of \( K_0 \). It follows \( \lambda^n \in Op(K_0) \)

Lemma 2.3. The set \( Op(K_0) \) is an associative ring.

Proof. The proof is left to the reader as an exercise. \( \square \)

Let \( \lambda \)-Rings denote the category of finite dimensional augmented \( \lambda \)-rings.

Theorem 2.4. There is a covariant functor \( K_0^\lambda : \text{Cat} \to \lambda \)-Rings

defined by
\[ K_0^\lambda(C) = \{K_0(C), \{\lambda^n\}_{n \geq 0}\}. \]
Here \( K_0(C) \) is the Grothendieck ring of \( C \) and \( \lambda^n \) is the operation defined above.

Proof. One checks that for any category \( C \) \( \{K_0(C), \{\lambda^n\}_{n \geq 0}\} \) is a \( \lambda \)-ring. Any morphism \( f : D \to C \) induce a morphism \( f^* : K_0^\lambda(D) \to K_0^\lambda(C) \) of \( \lambda \)-rings because \( f \) commutes with exterior products. Moreover since any object in \( C \) has finite rank it follows that \( K_0^\lambda(C) \) is a finite dimensional \( \lambda \)-ring. Finally the rank function \( rk \) defines a map
\[ rk : K_0(C) \to \mathbb{Z} \]
which is a map of \( \lambda \)-rings where \( \mathbb{Z} \) has the canonical \( \lambda \)-structure, and the Theorem is proved. \( \square \)

Let \( u = t/1 - t \) and let \( \gamma_l(x) = \lambda_u(x) = \sum_{l \geq 0} \gamma^l(x)t^l \). It follows that for any exact sequence in \( C \)
\[ 0 \to x' \to x \to x'' \to 0 \]
there is an equality
\[ \gamma_l(x) = \gamma_l(x') \gamma_l(x''). \]
It follows that for any \( l \geq 0 \) we have \( \gamma^l \in Op(K_0) \).

Definition 2.5. Let \( \omega = \sum_i n_i[x_i] \in K_0(C) \). Define \( e : K_0(C) \to \mathbb{Z} \)
by
\[ e(\omega) = \sum_i n_irk(x_i). \]
Define \( d : K_0(C) \to K_0(C) \)
by
\[ d(\omega) = e(\omega)[1^\otimes] \]
where $1^\otimes$ is the unit object for $\otimes$. Define for all $l \geq 0$
\[ c_l(\omega) = (-1)^l \gamma^l(\omega - d(\omega)) \]
to be the $l$'th characteristic class of $\omega \in K_0(C)$. We say an element $\omega$ is effective if
$n_i \geq 1$ for all $i$.

**Theorem 2.6.** Let $x, y \in K_0(C)$. The following formulas hold:
\[ c_l(x + y) = \sum_{i+j=l} c_i(x)c_j(y) \] (2.6.1)
\[ c_l(x) = 0 \text{ if } x \text{ is effective and } l > e(x) \] (2.6.2)
\[ f^*(c_l(x)) = c_l(f^*(x)) \] (2.6.3)
Here $f : D \to C$ is a morphism in $\text{Cat}_\otimes^\omega$.

**Proof.**

We prove (2.6.1)
\[ c_l(x + y) = (-1)^l \gamma^l(x + y - d(x + y)) = (-1)^l \gamma^l(x - d(x) + y - d(y)) = \]
\[ (-1)^l \sum_{i+j=l} \gamma^i(x - d(x))\gamma^j(y - d(y)) = \sum_{i+j=l} (-1)^i \gamma^i(x - d(x))(-1)^j \gamma^j(y - d(y)) = \]
\[ \sum_{i+j=l} c_i(x)c_j(y), \]
and (2.6.1) is proved.

We next prove (2.6.2) Note: $u = t/1 - t$. We see that
\[ \gamma_t(1^\otimes) = \lambda_u(1^\otimes) = 1 + [1^\otimes]u = 1 + t/1 - t = 1/1 - t. \]
We get:
\[ \gamma_t(x - d(x)) =\gamma_t(x)\gamma_t(1^\otimes)^{-e(x)} = \gamma_t(x)(1 - t)^e \]
where $e = e(x)$ and $x = \sum_i n_i x_i = [\oplus_i x_i] = [\omega] \in K_0(C)$. We get:
\[ \gamma_t(x - d(x)) = \sum_{i \geq 0} \gamma^i(x - d(x)) = \gamma_t(x)(1 - t)^e = \]
\[ \lambda_u(\omega)(1 - t)^e = (1 + \lambda^1(\omega)u + \lambda^2(\omega)u^2 + \cdots + \lambda^e(\omega)u^e)(1 - t)^e = (1 - t)^e + \lambda^1(\omega)t(1 - t)^{e-1} + \cdots + \lambda^e(\omega)t^e. \]
It follows that
\[ \gamma_t(x - d(x)) = \sum_{i \geq 0} \gamma^i(x - d(x)) = p_0 + p_1 t + \cdots + p_e t^e \]
hence $\gamma^i(x - d(x)) = 0$ for $l \geq e = e(x)$ and hence
\[ c_l(x) = (-1)^l \gamma^l(x - d(x)) = 0 \]
for $l > e(x)$, and (2.6.2) is proved.

We next prove (2.6.3)
\[ f^*(c_l(x)) = f^*((-1)^l \gamma^l(x - d(x))) = (-1)^l \gamma^l(f^*(x) - f^*(e(x)[1^\otimes])) = \]
\[ (-1)^l \gamma^l(f^*(x) - e(f^*(x))[1^\otimes]) = c_l(f^*(x)), \]
and (2.6.3) is proved. □
Definition 2.7. Let \( x \in K_0(C) \) be an element. Let 
\[
    c_t(x) = \sum_{k \geq 0} c_k(x)t^k \in 1 + K_0(C)[[t]]
\]
be the Chern power series of the element \( x \).

It follows from Theorem 2.6 that for any functor \( f : D \to C \) in \( \text{Cat}^\otimes \) there is a commutative diagram
\[
\begin{array}{ccc}
K_0(D) & \xrightarrow{c_t} & 1 + K_0(D)[[t]] \\
\downarrow{f^*} & & \downarrow{1 \otimes f^*} \\
K_0(C) & \xrightarrow{c_t} & 1 + K_0(C)[[t]].
\end{array}
\]
Hence the Chern power series define a natural transformation 
\[
    c_t : K_0(-) \to 1 + K_0(-)[[t]]
\]
of functors. Here we view \( K_0(-) \) and \( 1 + K_0(-)[[t]] \) as functors 
\[
    K_0(-), 1 + K_0(-)[[t]] : \text{Cat}^\otimes \to Ab
\]
where \( Ab \) is the category of abelian groups.

Definition 2.8. We say that a natural transformation 
\[
    u : K_0(-) \to 1 + K_0(-)[[t]]
\]
of functors is a theory of characteristic classes with values in \( K_0 \). Let \( \text{Class}(K_0) \) denote the set of theories of characteristic classes with values in \( K_0 \).

It follows the Chern power series define a theory of characteristic classes \( c_t \in \text{Class}(K_0) \).

Since the Chern power series \( c_t(x) \) is a unit for the multiplication in \( K_0(C)[[t]] \) there exists a power series \( s_t(x) \in 1 + K_0(C)[[t]] \) - the Segre power series - defined by \( s_t(x) = c_t(x)^{-1} \). Let 
\[
    s_t(x) = \sum_{k \geq 0} s_k(x)t^k.
\]
We get
\[
    s_t(x + y) = c_t(x + y)^{-1} = (c_t(x)c_t(y))^{-1} = c_t(x)^{-1}c_t(y)^{-1} = s_t(x)s_t(y)
\]
and
\[
    f^*(s_t(x)) = f^*(c_t(x)^{-1}) = (f^*(c_t(x)))^{-1} = c_t(f^*(x))^{-1} = s_t(f^*(x)).
\]

Definition 2.9. We let the classes \( s_k(x) \in K_0(C) \) for \( k \geq 0 \) be the virtual Segre classes of the element \( x \in K_0(C) \).

It follows the Segre classes satisfy the following formulas: Let \( x, y \in K_0(C) \) be arbitrary elements and let \( f : D \to C \) be a morphism in \( \text{Cat}^\otimes \).

\[
\begin{align}
    s_k(x + y) &= \sum_{i+j=k} s_i(x)s_j(y) \quad (2.9.1) \\
    f^*(s_k(x)) &= s_k(f^*(x)) \quad (2.9.2)
\end{align}
\]
Hence the Segre power series define a natural transformation 
\[
    s_t(-) : K_0(-) \to 1 + K_0(-)[[t]]
\]
of functors. It follows \( s_t \in \text{Class}(K_0) \) It is not true in general that \( s_t(x) = 0 \) for \( x \) effective and \( i > e(x) \).

**Example 2.10.** Characteristic classes in algebraic and topological K-theory.

We include a discussion of existence of characteristic classes with values in algebraic and topological Grothendieck groups using the constructions above.

Following [4], let \( X \) be an arbitrary scheme and let \( vb(X) \) be the category of locally free finite rank \( \mathcal{O}_X \)-modules. It follows that \( vb(X) \) is an ACTEP.

It is a standard fact that the Grothendieck ring \( K_0(X) = K_0(vb(X)) \) is a commutative ring with unit. Direct sum induce the addition operation and tensor product induce the multiplication.

Given any morphism \( f : Y \to X \) the pull-back \( f^* \) defines a map of rings

\[
 f^* : K_0(X) \to K_0(Y).
\]

Let

\[
 0 \to E' \to E \to E'' \to 0
\]

be an exact sequence of locally free sheaves of ranks \( e', e \) and \( e'' \). From [5], Exercise II.5.16, there is for all \( 2 \leq l \leq e \) a filtration

\[
 F_l = F_{l+1} \subseteq \cdots \subseteq F_0 = \wedge^l E
\]

where

\[
 F_i / F_{i+1} \cong \wedge^i (E') \otimes \wedge^{l-i} (E'')
\]

for all \( 0 \leq i \leq l \). Let \( Op(K_0) \) be the set of natural transformations of the underlying set-valued functor of the functor \( K_0 \). The operation \( \lambda^k[E] = [\wedge^k E] \) extends to give a natural transformation \( \lambda^k \in Op(K_0) \) for all \( k \geq 0 \). Let \( \text{Sch} \) denote the category of schemes. The construction \( K_0 \) and \( \lambda^k \) defines a contravariant functor

\[
 K_0^\lambda : \text{Sch} \to \lambda\text{-Rings}
\]

by

\[
 K_0^\lambda(X) = \{ K_0(vb(X)), \{ \lambda^n \}_{n \geq 0} \}.
\]

Let \( u = t/1 - t \) and define the powerseries

\[
 \gamma_t(x) = \lambda_u(x) = \sum_{l \geq 0} \lambda^l(x)u^l = \sum_{l \geq 0} \gamma^l(x)t^l \in 1 + K_0(X)[[t]].
\]

Following [4] (and [6]) we get operations - gamma operations - \( \gamma^l \in Op(K_0) \) for all \( l \geq 0 \).

**Definition 2.11.** Let \( x = \sum_i n_i [E_i] \in K_0(X) \) be an element. Define the following map

\[
 e : K_0(X) \to \mathbb{Z}
\]

by

\[
 e(x) = \sum_i n_i \text{rk}(E_i).
\]

Define furthermore

\[
 d : K_0(X) \to K_0(X)
\]

by

\[
 d(x) = e(x)[\mathcal{O}_X].
\]

For any integer \( l \geq 0 \) define the \( l' \)th characteristic class of \( x \) to be

\[
 c_l(x) = (-1)^l \gamma^l(x - d(x)) \in K_0(X).
\]
Proposition 2.12. Let \( f : Y \to X \) be a map of schemes and let \( x, y \in K_0(X) \) be arbitrary elements. The following formulas hold:

\[
\begin{align*}
\text{(2.12.1)} & \quad c_i(x + y) = \sum_{i+j=l} c_i(x)c_j(y) \\
\text{(2.12.2)} & \quad c_i(x) = 0 \text{ if } l > e(x) \\
\text{(2.12.3)} & \quad f^*(c_i(x)) = c_i(f^*(x))
\end{align*}
\]

Proof. The proof is similar to the proof of Theorem [276] and is left to the reader as an exercise. \( \square \)

Let \( c_i(x) = \sum k \geq 0 c_k(x)t^k \in 1 + K_0(X)[[t]] \) be the Chern power series of \( x \in K_0(X) \). Define \( s_i(x) = c_i(x)^{-1} \). It follows we get Segre classes \( s_k(x) \in K_0(X) \) with \( k \geq 0 \) for any element \( x = \sum_i n_i[E_i] \).

If one considers the Grothendieck ring \( K_0(B) \) of finite rank continuous vector bundles on a topological space \( B \) a similar construction using tensor operations defines characteristic classes \( c_i(E) \in K_0(B) \) where \( E \) is a complex continuous vector bundle on a topological space \( B \). If \( E \) is a rank \( n \) vector bundle on \( B \) it follows we get a total characteristic class \( c(E) = \sum_{i=0}^n c_i(E) \in K_0(B) \).

Let \( \pi : P(E^*) \to B \)

be the associated projective bundle of \( E \). Its fiber over a point \( x \in B \) is the projectivization \( P(E^*_x) \) of the dual vector space \( E^*_x \) where \( E_x \) is the fiber of \( E \) at \( x \). By [6] Proposition IV.7.4 the following holds: The map

\( \pi^* : K_0(B) \to K_0(P(E^*)) \)

is injective and the ring \( K_0(P(E^*)) \) is free of rank \( n \) on the element \( h = 1 - [O(-1)] \)

where \( O(-1) \) is the tautological bundle on \( P(E^*) \). It follows we get an equation

\[
\text{(2.12.4)} \quad h^n - c_1(E)h^{n-2} + c_2(E)h^{n-2} + \cdots + (-1)^nc_n(E) = 0
\]

where the classes \( c_i(E) \) in Equation (2.12.4) are the ones defined by operations on \( K_0 \). The classes defined by Equation 2.12.4 are the Chern-classes of the vector bundle \( E \) with values in \( K_0(B) \). Defining the Chern-classes of a vector bundle using the projective bundle \( P(E^*) \) is usually referred to as the projective bundle formula.

The characteristic class \( c_i(E) \in K_0(B) \) is related to the exterior product of bundles in the following way (see [6] Section IV.2.18)

\[
c_i(E) = \binom{n}{i}[\wedge^i E] - \binom{n-1}{i-1}[\wedge^{i-1} E] + \cdots + (-1)^i \binom{n-i}{0} \wedge^i E,
\]

hence we may use the exterior product to define well behaved characteristic classes with values in \( K_0 \)-theory.

Defining the characteristic classes \( c(x) \) directly using operations is an alternative approach to the theory of characteristic classes going “around” the projective bundle formula. As we have seen: In the case of the category \( \text{Cat}^\wedge \) there is no replacement for the projective bundle formula. Hence operations give a direct, intrinsic and elementary approach to the construction of characteristic classes.
3. Characteristic classes of connections

In this section we define characteristic classes with values in $K_0(\text{conn-}g)$ where $g$ is a restricted Lie-Rinehart algebra using exterior products. We introduce $\lambda$-operations on the growthendieck ring $K_0(\text{conn-}g)$ using techniques similar to the ones found in [6] and [12].

Let $k \subseteq K$ be fields of characteristic $p > 0$.

**Definition 3.1.** A sub $k$-Lie-algebra and $K$-vector space $g \subseteq \text{Der}_k(K)$ is a $p$-$(k,K)$-Lie algebra if for any $\partial \in g$ it follows that $\partial^{[p]} = \partial \circ \cdots \circ \partial \in g$. Given two $p$-$(k,K)$-Lie algebras $g$ and $h$ a morphism of $p$-$(k,K)$-Lie algebras is given by an an inclusion $I : g \to h$. Let $\text{Lie}_{K/k}$ denote the category of $p$-$(k,K)$-Lie algebras and morphisms. A connection $\rho$ is a map of $K$-vector spaces

$$\rho : g \to \text{End}_k(V)$$

where $V$ is a finite dimensional $K$-vector space satisfying the following formula:

$$\rho(\partial)(ax) = a\rho(\partial)(x) + \partial(a)x$$

for all $a \in K$ and $x \in V$. The curvature $R_\rho$ of $\rho$ is the map

$$R_\rho(\partial,\partial') = \rho([\partial,\partial']) - [\rho(\partial),\rho(\partial')]$$

The map

$$\psi_\rho(\partial) = \rho(\partial^{[p]}) - \rho(\partial)^{[p]}$$

is the $p$-curvature of $\rho$. The connection is flat if $R_\rho = 0$. It is $p$-flat if $R_\rho = \psi_\rho = 0$. The connection $(V,\rho)$ is nilpotent of exponent $\leq n$ if there is a filtration of $g$-connections

$$0 = F^n \subseteq F^{n-1} \subseteq \cdots \subseteq F^1 \subseteq F^0 = V$$

where the induced connection $F^i/F^{i+1}$ has $p$-curvature zero for all $i$. Let $(V,\rho)$ and $(W,\eta)$ be $g$-connections. A $K$-linear map

$$\phi : V \to W$$

is a map of $g$-connections if for all $x \in g$ there is a commutative diagram

$$\begin{array}{ccc}
V & \xrightarrow{\phi} & W \\
\downarrow{\rho(x)} & & \downarrow{\eta(x)} \\
V & \xrightarrow{\phi} & W.
\end{array}$$

Let $\text{conn-}g$ (resp. $\text{flat-}g$, $\text{mod-}g$) denote the category of $g$-connections (resp flat $g$-connections, $p$-flat $g$-connections) of finite dimension over $K$ and morphisms of $g$-connections.

**Theorem 3.2.** There is a contravariant functor

$$K_0 : \text{Lie}_{K/k} \to \text{Rings}$$

where for all $g \in \text{Lie}_{K/k}$ $K_0(g)$ is the growthendieck ring of the category $\text{conn-}g$.

**Proof.** The proof is left to the reader as an exercise. □
Note: A $p$-($k, K$)-Lie algebra is also referred to as a restricted Lie-Rinehart algebra. A $p$-flat $g$-connection is also referred to as a restricted $g$-module. A connection $\rho : g \to \text{End}_k(V)$ is flat if and only if $\rho$ is a morphism of $k$-Lie algebras. There exists a restricted enveloping algebra $U^{[p]}(g)$ of the restricted Lie-Rinehart algebra $g$ with the following property: There is an equivalence of categories

$$\text{mod-}g \cong \text{mod-}U^{[p]}(g),$$

where $\text{mod-}U^{[p]}(g)$ is the category of finite dimensional left $U^{[p]}(g)$-modules.

**Lemma 3.3.** Let $(V, \rho)$ be a $g$-connection, where $g \subseteq \text{Der}_k(K)$ is a restricted Lie-Rinehart algebra. It follows $V^{\otimes l}, \wedge^l V$ and $\text{Sym}^l(V)$ are $g$-connections.

**Proof.** The proof is left to the reader as an exercise. \(\square\)

Let $0 \to U \to V \to W \to 0$ be an exact sequence in $\text{conn-}g$, where $g \subseteq \text{Der}_k(K)$ is a restricted Lie-Rinehart algebra.

There is the following Lemma:

**Lemma 3.4.** There is for all $l \geq 2$ a $g$-stable filtration

$$0 = F_{l+1} \subseteq F_l \subseteq \cdots \subseteq F_1 \subseteq F_0 = \wedge^l V$$

where $F_i$ is a $g$-connection, with the following property:

$$F_i/F_{i+1} \cong \wedge^i U \otimes \wedge^{(l-i)}W$$

is an isomorphism of $g$-connections.

**Proof.** Let $\pi : V^{\otimes l} \to \wedge^l V$ be the canonical projection map. It follows $\pi$ is a map of $g$-connections. There is a filtration of $g$-connections

$$0 \subseteq U^{\otimes l} \subseteq U^{\otimes (l-1)} \otimes V \subseteq \cdots \subseteq U^{\otimes (l-i)} \otimes V^{\otimes i} \subseteq \cdots \subseteq V^{\otimes l}.$$ 

Make the following definition:

$$F_i = \pi(U^{\otimes i} \otimes V^{\otimes (l-i)}) \subseteq \wedge^l V.$$

Since $\pi$ is a map of $g$-connections, it follows that the filtration

$$0 = F_{l+1} \subseteq F_l \subseteq \cdots \subseteq F_1 \subseteq F_0 = \wedge^l V$$

is a filtration of $g$-connections. There is a commutative diagram of $g$-connections

$$\begin{array}{ccc}
U^{\otimes i} \otimes V^{\otimes (l-i)} & \xrightarrow{\pi} & F_i \\
\downarrow{1 \otimes \rho} & & \downarrow \\
U^{\otimes i} \otimes W^{\otimes (l-i)} & \xrightarrow{\pi} & F_i/F_{i+1} .
\end{array}$$

The claim is that the bottom horizontal map

$$g : \wedge^i U \otimes \wedge^{(l-i)}W \to F_i/F_{i+1}$$

is an isomorphism of $g$-connections. It is enough to prove it is an isomorphism of $K$-vector spaces.
The sequence
\[ 0 \to U \to V \to W \to 0 \]
is split as sequence of \(K\)-vector spaces, hence \(V = U \oplus W\) as \(K\)-vector space.
Consider the diagram
\[ U \otimes i \otimes V \otimes (l-i) \to V \otimes l \]
\[ F_i \to \wedge^l V \]
Since \(V = U \oplus W\) as \(K\)-vector space the following holds:
\[ F_i = f(U \otimes i \otimes V \otimes (l-i)) = \wedge^i U \wedge (\wedge^{l-i}(U \oplus W)) = \]
\[ \wedge^i U \wedge \oplus_{j=0}^{l-i} \wedge^j U \otimes \wedge^{l-i-j} W = \]
\[ \oplus_{j=0}^{l-i} \wedge^{i+j} U \otimes \wedge^{l-(i+j)} W = \oplus_{s=0}^{l} \wedge^s U \otimes \wedge^{l-s} W. \]
From this it follows that
\[ F_i / F_{i+1} = \wedge^i U \otimes \wedge^{l-i} W \]
as \(K\)-vector space, hence \(g\) is an isomorphism of \(\mathfrak{g}\)-connections, and the claim of the Lemma follows.

Let \(Op(K_0)\) denote the set of all natural transformations of the underlying set valued functor of \(K_0\). It follows that \(Op(K_0)\) is an associative ring. The exterior product \(\wedge^l\) from Lemma 3.4 defines operations \(\lambda^l \in Op(K_0)\).

**Theorem 3.5.** For any field extension \(k \subseteq K\) of fields of characteristic \(p\) there is a contravariant functor
\[ K_0^\lambda : \text{Lie}_K/k \to \lambda\text{-Rings} \]
defined by
\[ K_0^\lambda(g) = (K_0(\text{conn-}g), \lambda^l_{\geq 0}). \]
Here \(K_0(\text{conn-}g)\) is the grothendieck ring of the category \(\text{conn-}g\) and \(\lambda^l[V, \rho] = [\wedge^l V, \lambda^l \rho] \in K_0(\text{conn-}g)\).

**Proof.** The proof uses Lemma 3.4 and is left to the reader as an exercise.

**Example 3.6.** Radicial descent.

There is an isomorphism of \(\lambda\)-rings
\[ K_0(\text{mod-}g) \cong \mathbb{Z}. \]
There is by Theorem 2.3 in [1] an equivalence between the category \(\text{mod-}g\) of \(p\)-flat \(g\)-connections of finite dimension and the category \(\text{Vect}_{K^*}\) of \(K^*\)-vectorspaces of finite dimension, hence there is an isomorphism of grothendieck groups
\[ e : K_0(\text{mod-}g) \cong K_0(\text{Vect}_{K^*}) \cong \mathbb{Z}. \]

In the following we use the operations defined above to define characteristic classes of arbitrary virtual connections in the grothendieck ring \(K_0(\text{conn-}g)\) as done in Section 2. Consider the functor
\[ K_0 : \text{Lie}_K/k \to \text{Rings} \]
where for any \(g \in \text{Lie}_K/k\) \(K_0(\text{conn-}g)\) is the grothendieck ring of the category \(\text{conn-}g\).
We aim to show that $K_0$ has a theory of characteristic classes $c$ using operations defined in the previous section.

Let $u = t/1 - t$ and consider the formal power series

$$\gamma_t(x) = \lambda_u(x) = \sum_{l \geq 0} \lambda^l(x)u^l = \sum_{l \geq 0} \gamma^l(x)t^l.$$ 

It has the property that for any exact sequence

$$0 \to U \to V \to W \to 0$$

in $\text{conn-g}$ there is an equality of formal power series in $1 + K_0(\text{conn-g})[[t]]$:

$$\gamma_t(V) = \gamma_t(U)\gamma_t(W).$$

For all $l \geq 0$ it follows $\gamma^l \in Op(K_0)$. We get well defined operations - *gamma operations* -

$$\gamma^l : K_0(\text{conn-g}) \to K_0(\text{conn-g}).$$

**Definition 3.7.** Let $x = \sum n_i[V_i, \rho_i] = \sum n_i[V_i] \in K_0(\text{conn-g})$ be an element. Define the following map

$$e : K_0(\text{conn-g}) \to \mathbb{Z}$$

by

$$e(x) = \sum n_i \dim_K(V_i).$$

Define furthermore

$$d : K_0(\text{conn-g}) \to K_0(\text{conn-g})$$

by

$$d(x) = e(x)[\theta_K]$$

where $\theta_K$ is the trivial rank one $\mathfrak{g}$-connection. For any integer $l \geq 0$ define the $l$'th *characteristic class* of $x$ to be

$$c_l(x) = (-1)^l \gamma^l(x - d(x)) \in K_0(\text{conn-g}).$$

One checks immediately that the notions above are well defined.

Let $F : \mathfrak{h} \to \mathfrak{g}$ be a morphism in $\text{Lie}_{K/k}$. There is for every $\mathfrak{g}$-connection $(W, \rho)$ a canonical $\mathfrak{h}$-connection $F^*W = K \otimes W$. This construction commutes with exterior product: $F^*(\wedge^l W) = \wedge^l(F^*W)$. We say that an element $x = \sum n_i[V_i] \in K_0(\text{conn-g})$ is *effective* if $n_i > 0$ for all $i$. We get the following result for the group $K_0(\text{conn-g})$:

**Theorem 3.8.** Let $x, y \in K_0(\text{conn-g})$ be arbitrary elements with $x$ effective. The following formulas hold:

\begin{align*}
(3.8.1) & \quad c_l(x + y) = \sum_{i+j=l} c_i(x)c_j(y) \\
(3.8.2) & \quad c_l(x) = 0 \text{ if } l > e(x) \\
(3.8.3) & \quad F^*c_l(x) = c_l(F^*x)
\end{align*}

**Proof.** The proof is similar to the proof of Theorem 2.6 and is left to the reader as an exercise. \(\square\)
**Definition 3.9.** Let for any effective \( x \in K_0(\text{conn}-\mathfrak{g}) \)
\[
c(x) = \sum_{l \geq 0} c_l(x) \in K_0(\text{conn}-\mathfrak{g})
\]
be its virtual characteristic class.

By Lemma 3.8 Formula 3.8.2 it follows \( c(x) \) is well defined for any \( x \in K_0(\text{conn}-\mathfrak{g}) \).

### 4. Differential forms, curves and Ore-extensions

In this section we make the constructions in Section 2 explicit and construct characteristic classes for connections on curves and connections defined in terms of differential forms on fields.

First we introduce the Cartier operator (following the presentation in [11] and [8]) and relate it to the \( p \)-curvature of a connection defined on the field \( K \).

Let \( K^p \subseteq L \subseteq K \) be fields of characteristic \( p > 0 \) and let \( \mathfrak{g} = \text{Der}_L(K) \) be the \( p \)-Lie algebra of derivations of \( K \) over \( L \). Let \( K^{1/p} \) be the field of \( p \)'th roots of elements of \( K \) \( e^{K^{1/p}} \) is the splitting field of all polynomials \( T^p - a \) with \( a \in K \). It has the property that for all elements \( a \in K \) there is a unique element \( x = a^{1/p} \in K^{1/p} \) with \( x^p = a \). Furthermore one has that \((a + b)^{1/p} = a^{1/p} + b^{1/p}\). Let \( \omega = xdy \in \Omega^1_{K/L} \) be a differential form and define the following map:
\[
C\omega : \mathfrak{g} \to K^{1/p}
\]
by
\[
C\omega(\partial) = (\omega(\partial^p) - \partial^{p-1}(\omega(\partial)))^{1/p}.
\]

**Definition 4.1.** The map \( C \) is the Cartier operator for the field extension \( L \subseteq K \).

**Proposition 4.2.** The following holds:

\[
\begin{align*}
(4.2.1) \quad & C(\omega + \omega') = C\omega + C\omega' \\
(4.2.2) \quad & C(x\omega) = x^{1/p}C\omega, \ x \in L \\
(4.2.3) \quad & C(dx) = 0
\end{align*}
\]

**Proof.** We prove (4.2.1)
\[
C(\omega + \omega')(\partial) = ((\omega + \omega')(\partial) - \partial^{p-1}((\omega + \omega')(\partial)))^{1/p} =
(\omega(\partial^p) - \partial^{p-1}(\omega(\partial)))^{1/p} + \omega'(\partial) - \partial^{p-1}(\omega'(\partial))^{1/p} =
(\omega(\partial) - \partial^{p-1}(\omega(\partial)))^{1/p} + (\omega'(\partial) - \partial^{p-1}(\omega'(\partial)))^{1/p} = C\omega(\partial) + C\omega'(\partial).
\]

We prove (4.2.2) Let \( x \in L \). We get
\[
C(x\omega)(\partial) = (x\omega(\partial) - \partial^{p-1}(x\omega(\partial)))^{1/p} =
(x(\omega(\partial) - \partial^{p-1}(\omega(\partial))))^{1/p} = x^{1/p}C\omega(\partial).
\]

(4.2.3) is obvious. \( \square \)

Define the following connection
\[
r : \text{Der}_L(K) \to \text{End}_L(K)
\]
by
\[
r(\partial)(x) = \partial(x) + \omega(\partial)x.
\]

Let \( R_r(\partial, \partial') = [r(\partial), r(\partial')] - r([\partial, \partial']) \) be the curvature of \( r \) and \( \psi_r(\partial) = r(\partial^p) - r(\partial)^p \) the \( p \)-curvature.
Theorem 4.3. The following holds:

\[ R_r(\partial, \partial') = d\omega(\partial, \partial') \]
\[ \psi_r(\partial) = (-1)^p(C\omega(\partial) - \omega(\partial))^p. \]

Proof. Assume \( \omega = xdy \). We first prove \ref{4.3.1}. It follows

\[ d\omega(\partial, \partial') = \partial(x)\partial'(y) - \partial'(x)\partial(y). \]

It follows that

\[ \partial(\omega(\partial')) - \partial'(\omega(\partial)) - \omega([\partial, \partial']) = \]
\[ \partial(x)\partial'(y) - \partial'(x)\partial(y) = x[\partial, \partial'](y) = \]
\[ \partial(x)\partial'(y) + x\partial\partial'(y) - \partial'(x)\partial(y) - x\partial\partial'(y) = x[\partial, \partial'](y) = \]
\[ \partial(x)\partial'(y) - \partial'(x)\partial(y) = d\omega(\partial, \partial'). \]

We get thus the formula

\[ \partial(\omega(\partial')) - \partial'(\omega(\partial)) = d\omega(\partial, \partial') + \omega([\partial, \partial']). \]

We get

\[ [r\partial, r\partial'](u) = (\partial + \omega(a))(\partial' + \omega(\partial')(u) - (\partial' + \omega(\partial'))(\partial + \omega(\partial))(u) = \]
\[ \partial\partial'(u) + \partial(\omega(\partial'))u + \omega(\partial)\partial'(u) + \omega(\partial')\omega(\partial')u = \]
\[ -\partial'(\partial(u) - \partial'(\omega(\partial))u - \omega(\partial')\partial(u) - \omega(\partial')\omega(\partial)u = \]
\[ [\partial, \partial'](u) + \partial(\omega(\partial'))u + \omega(\partial)\partial'(u) - \partial'(\omega(\partial))u - \omega(\partial')\partial(u) = \]
\[ [\partial, \partial'](u) + \partial(\omega(\partial'))u - \partial'(\omega(\partial))u = \]
\[ [\partial, \partial'](u) + d\omega(\partial, \partial')(u) + \omega([\partial, \partial'])(u). \]

It follows that

\[ R_r(\partial, \partial') = [r\partial, r\partial'] - r([\partial, \partial']) = d\omega(\partial, \partial') \]

hence \ref{4.3.1} is proved. In the ring \( \text{End}_L(K) \) the following holds:

\[ (a + \partial)^p = a^p + \partial^p + \partial^{p-1}(a). \]

This is proved using induction. We prove \ref{4.3.2}

\[ \psi_r(\partial) = r(\partial^p) - r(\partial)^p = \partial^p + \omega(\partial^p) - (\partial + \omega(\partial))^p = \]
\[ \partial^p + \omega(\partial^p) - \omega(\partial)^p - \partial^p - \partial^{p-1}(\omega(\partial)) = \]
\[ \omega(\partial^p) - \partial^{p-1}(\omega(\partial)) = \omega(\partial)^p = \]
\[ C\omega(\partial)^p - \omega(\partial)^p = (C\omega(\partial) - \omega(\partial))^p, \]

and the claim follows.

\( \square \)

Example 4.4. Differential forms.
Let $\omega \in \Omega = \Omega^1_{K/L}$ and let $\mathfrak{g} = \text{Der}_L(K)$ Define the following map

$$\rho_\omega : \mathfrak{g} \rightarrow \text{End}_L(K)$$

by

$$\rho_\omega(\partial)(x) = \partial(x) + \omega(\partial)(x).$$

It follows $\rho_\omega$ is a connection and from Proposition 4.3 one gets

$$R_{\rho_\omega}(\partial, \partial') = d\omega(\partial, \partial'),$$

and

$$\psi_{\rho_\omega}(\partial) = (-1)^p(C\omega(\partial) - \omega(\partial))^p,$$

where $C\omega$ is the Cartier operator (see [1] and [8]). Note: By Proposition 7 in [1] it follows that $R_{\rho_\omega} = \psi_{\rho_\omega} = 0$ if and only if $\omega = d\log(x) = x^{-1}dx$. We get thus a map

$$\phi : \Omega \rightarrow K_0(\text{conn-}g)$$

defined by

$$\phi(\omega) = c(K, \rho_\omega)$$

detecting if a differential form $\omega$ is a logarithmic derivative.

**Example 4.5. Adjoint representation.**

Let $\mathfrak{g} \subseteq \text{Der}_k(K)$ be a restricted Lie-Rinehart algebra. There is a representation

$$ad : \mathfrak{g} \rightarrow \text{End}_k(\text{Der}_k(K))$$

defined by

$$ad(x)(y) = [x,y].$$

One sees

$$ad(x)(\alpha y) = [x, \alpha y] = x(\alpha y) - \alpha xy = x(\alpha)y + \alpha xy = \alpha x + x(\alpha)y =

\alpha ad(x)(y) + x(\alpha)y,$$

hence $ad$ is a connection. The Jacobi-identity shows $R_{ad} = 0$ hence the map $ad$ is a representation of Lie algebras. In $\text{End}_k(\text{Der}_k(K))$ the following formula holds:

$$[\partial[\partial[\cdots[\partial, \eta] \cdots]] = \sum_{i=0}^{n} \binom{n}{i} \partial^{n-i} \eta \partial^i$$

for all $n \geq 1$. We get the formula

$$ad(\partial^p)(\eta) = [\partial^p, \eta] = \partial^p \eta - \eta \partial^p = \sum_{i=0}^{p} \binom{p}{i} \partial^{p-i} \eta \partial^i =$$

$$[\partial[\partial[\cdots[\partial, \eta] \cdots]] = ad(\partial)^{p}(\eta)$$

hence it follows

$$\psi_{p}(\partial) = ad(\partial^p) - ad(\partial)^p = 0$$

and it follows $ad$ is a flat $g$-connection on $\text{Der}_k(K)$ with zero $p$-curvature.

One may also check that for any ideal $I \subseteq \text{Der}_k(K)$ it follows $I$ is closed under $p$-powers, hence $I$ is a restricted Lie-Rinehart algebra. The map

$$ad : \text{Der}_k(K) \rightarrow \text{End}_k(I)$$

and

$$ad : \text{Der}_k(K) \rightarrow \text{End}_k(\text{Der}_k(K)/I)$$
makes \( I \) and \( \text{Der}_k(K)/I \) into \( p \)-representations and the sequence
\[
0 \to I \to \text{Der}_k(K) \to \text{Der}_k(K)/I \to 0
\]
is an exact sequence of \( p \)-flat \( g \)-connections. We get a characteristic class
\[
c(\text{Der}_k(K)/I, \text{ad}) \in K_0(\text{conn}\cdot g)
\]
for each ideal \( I \subseteq \text{Der}_k(K) \).

There is a \( p \)-flat connection
\[
\text{ad} : g \to \text{End}_k(U[p](g))
\]
where \( U[p](g) \) is the restricted enveloping algebra of \( g \), hence we get a characteristic class
\[
c(U[p](g), \text{ad}) \in K_0(\text{conn}\cdot g).
\]

**Example 4.6. Curves and Ore extensions.**

If \( \pi : C \to C' \) is a finite morphism of projective curves over an arbitrary field \( k \) and \( K' \subseteq K \) is the corresponding finite extension of function fields, we get an exact sequence of Lie-Rinehart algebras
\[
0 \to \mathfrak{h} \to \text{Der}_k(K) \to d \pi K \otimes K' \text{Der}_k(K') \to 0.
\]
The Lie-Rinehart algebra \( \mathfrak{h} = \text{Der}_{K'}(K) \) is a finite dimensional \( K \)-vector space. Let \( g = \text{Der}_k(K) \) and \( \mathfrak{l} = \text{Der}_k(K') \). The Lie-Rinehart algebras \( g \) and \( \mathfrak{l} \) are infinite dimensional \( k \)-Lie algebras. We get for any flat \( g \)-connection \( (V, \rho) \) and \( i \geq 0 \) canonical flat connections - Gauss-Manin connections (see [14]) -
\[
\nabla_{GM}^{i, \rho} : \mathfrak{l} \to \text{End}_k(H^i(\mathfrak{h}, V))
\]
where \( H^i(\mathfrak{h}, V) \) is the Chevalley-Hochschild cohomology of \( V \) as \( \mathfrak{h} \)-module. Let \( K_0(\text{flat}\cdot l) \) denote the grothendieck ring of the category \( \text{flat}\cdot l \). We get a well defined cohomology class
\[
d\pi!(V, \rho) = \sum_{i \geq 0} (-1)^i [H^i(\mathfrak{h}, V)] \in K_0(\text{flat}\cdot l).
\]
The connection \( d\pi! (\rho) \) is not \( p \)-flat in general. (See [7] for examples where the \( p \)-curvature of \( d\pi! (\rho) \) is related to the Kodaira-Spencer class).

Let \( \rho = \text{ad} \) from Example 4.5 with
\[
\rho : \text{Der}_k(K) \to \text{End}_k(\text{Der}_k(K))
\]
and make the following definition:

**Definition 4.7.** Let the cohomology class
\[
\Delta(\pi) = d\pi!(\text{Der}_k(K), \rho) = \sum_{i \geq 0} (-1)^i [H^i(\mathfrak{h}, \text{Der}_k(K))] \in K_0(\text{flat}\cdot l)
\]
be the *ramification class* of the morphism \( \pi \).

If \( \pi \) is a separable morphism, it follows \( \mathfrak{h} = 0 \). In this case
\[
\Delta(\pi) = [\text{Der}_k(K)] \in K_0(\text{flat}\cdot l).
\]

Hence the cohomology class
\[
\Delta(\pi) \in K_0(\text{flat}\cdot l)
\]
is related to the ramification of the morphism $\pi$. We get a characteristic class

$$c(\pi) = \sum_{i \geq 0} c_i(\Delta(\pi)) \in K_0(\text{flat-l})$$

defined for an arbitrary finite morphism $\pi : C \to C'$ of projective curves.

Assume there is a separable morphism of curves $C' \to P^1_k$ where $k$ is a field of characteristic zero and let $(V, \rho)$ be a flat $g$-connection. The function field $K(P^1_k)$ equals $k(t)$ where $t$ is a transcendental variable over $k$. We get an isomorphism

$$l \cong K'\partial$$

where $\partial$ is partial derivative with respect to $t$. It follows that the category of flat finite dimensional $l$-connections is equivalent to the category of $K'\{T\}$-modules of finite dimension over $K'$. Here $K'\{T\}$ is the Ore extension of $K'$ by $T$. It is the twisted polynomial ring on the variable $T$ with multiplication defined as follows:

$$Ta = aT + \partial(a)$$

for any $a \in K'$. In this case the cohomology group $H^i(\mathfrak{h}, V)$ is a finite dimensional $K'$-vector space. It follows $H^i(\mathfrak{h}, V)$ is a finite dimensional $K'$-vector space. The Gauss-Manin connection

$$\nabla_{GM}^{i, \rho} : l \to \text{End}_k(H^i(\mathfrak{h}, V))$$

makes $H^i(\mathfrak{h}, V)$ into a finite dimensional left $K'\{T\}$-module. By the cyclic vector theorem (see [9] Theorem 5.4.2) any finite dimensional left $K'\{T\}$-module is of the form

$$K'\{T\}/P(T)K'\{T\}$$

where $P(T) \in K'\{T\}$ is a non-commutative polynomial. It follows there is a non-commutative polynomial

$$P_i(T) \in K'\{T\}$$

such that

$$H^i(\mathfrak{h}, V) \cong K'\{T\}/P_i(T)K'\{T\}$$

as $K'\{T\}$-module. In this case we get a cohomology class

$$\Delta(V) = \sum_{i \geq 0} (-1)^i[H^i(\mathfrak{h}, V)] = \sum_{i \geq 0} (-1)^i[K'\{T\}/P_i(T)K'\{T\}] \in K_0(\text{flat-l}).$$

We get a characteristic class

$$c(V) = \sum_{i \geq 0} c_i(\Delta(V)) \in K_0(\text{flat-l}).$$

By a Theorem of Quillen (see Theorem 1 in [15]) it follows

$$K_n(\text{flat-l}) \cong K_n(K'\{T\}) \cong K_n(K') \cong \mathbb{Z}$$

for all $n \geq 0$. It follows the class

$$[H^i(\mathfrak{h}, V), \nabla_{GM}^{i, \rho}] \in K_0(\text{flat-l})$$

is independent of choice of connection whenever $C'$ is a separable cover of $P^1_k$. It follows that the grothendieck group $K_0(\text{flat-l})$ does not contain enough information to detect invariants of the connection $\nabla_{GM}^{i, \rho}$. 
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