On (signed) Takagi–Landsberg functions: $p^\text{th}$ variation, maximum, and modulus of continuity

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Abstract

We study a class $X^H$ of signed Takagi–Landsberg functions with Hurst parameter $H \in (0,1)$. We first show that the functions in $X^H$ admit a linear $p^\text{th}$ variation along the sequence of dyadic partitions of $[0,1]$, where $p = 1/H$. The slope of the linear increase can be represented as the $p^\text{th}$ absolute moment of the infinite Bernoulli convolution with parameter $2^{H-1}$. The existence of a continuous $p^\text{th}$ variation enables the use of the functions in $X^H$ as test integrators for higher-order pathwise Itô calculus. Our next results concern the maximum, the maximizers, and the modulus of continuity of the classical Takagi–Landsberg function for all $0 < H < 1$. Then we identify the uniform maximum, the uniform maximal oscillation, and a uniform modulus of continuity for the class $X^H$.

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1 Introduction

The purpose of this paper is twofold. Our first goal is to study geometric properties of (signed) Takagi–Landsberg functions with Hurst parameter $H \in (0,1)$. These are fractal functions that have been widely studied in the literature; see, e.g., the surveys [2, 16, 22]. The properties in

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which we are interested include size and location of the (uniform) maximum, the uniform maximal oscillation, and the (uniform) modulus of continuity. For instance, we derive an explicit formula for the maximum of the Takagi–Landsberg function $x^H$ and the location of its maximizers. We prove in particular that $t = \frac{1}{3}$ and $t = \frac{2}{3}$ are the unique points at which the function $x^H(t)$ attains its maximum. In particular, the maximizers are independent of $H$ as long as $H < 1$. This pattern changes at $H = 1$, where it was shown by Kahane [20] that the classical Takagi function attains its maximum at an uncountable Cantor-type set of Hausdorff dimension $\frac{1}{2}$. We then use our result on the maximum of $x^H$ to derive an exact modulus of continuity of the Takagi–Landsberg function. Here again, we observe that our result breaks down at $H = 1$ as can be seen from the work of Köno [21].

Our second goal is to establish the class of signed Takagi–Landsberg functions as a natural class of “rough test integrators” for a higher-order pathwise integration theory in the spirit of Föllmer’s pathwise Itô calculus [12]. Such an integration theory was recently developed by Cont and Perkowski [8]; see also Gradinaru et al. [19] and Errami and Russo [10] for related earlier work. To this end, we will argue that, for a given Hurst parameter $H \in (0, 1)$ and $p := 1/H$, the corresponding signed Takagi–Landsberg functions admit a linear $p^{th}$ variation along the dyadic partitions of $[0, 1]$ as defined in [8]. The slope of the linear increase of the $p^{th}$ variation is equal to the expectation $\mathbb{E}[|Z_H|^p]$, where $Z_H$ is a random variable whose law is the infinite Bernoulli convolution with parameter $2^{H-1}$.

A high-order pathwise integration theory as mentioned above makes the techniques of model-free finance available for tackling continuous-time phenomena that are rougher than the usual diffusion paths. One particularly exciting new development is provided by the observation by Gatheral et al. [18] that “volatility is rough”. That is, empirical volatility time series exhibit a Hurst parameter much smaller than $1/2$. So far, modeling of rough volatility has been based on fractional Brownian motion (see, e.g., [9] and the references therein). With higher-order pathwise Itô calculus at hand, model-free techniques in the sprit of Bick and Willinger [6], Föllmer [13], and the substantial follow-up literature become feasible. In analogy with our previous papers [24, 28], our contribution is to provide an explicit class of test integrators for such model-free approaches.

This paper is organized as follows. In Section 2.1, we state our result on the $p^{th}$ variation of functions in $\mathfrak{X}^H$. In Section 2.2, we present several properties of the Takagi–Landsberg function $x^H$ with Hurst parameter $H \in (0, 1)$ and related uniform properties of the class $\mathfrak{X}^H$. In particular, we derive the (uniform) maximum and its location, the maximal uniform oscillation, and (uniform) moduli of continuity for $x^H$ and the functions in $\mathfrak{X}^H$. All proofs are contained in Section 3.

2 Main results

Recall that the Faber–Schauder functions are defined as

$$e_0(t) := t, \quad e_{0,0}(t) := (\min\{t, 1-t\})^+, \quad e_{n,k}(t) := 2^{-n/2}e_{0,0}(2^nt - k)$$

for $t \in \mathbb{R}$, $n = 1, 2, \ldots$, and $k \in \mathbb{Z}$. It is well known that the restrictions of the Faber–Schauder functions to $[0, 1]$ form a Schauder basis for $C[0, 1]$. Conversely, the Faber–Schauder functions can be used to construct some interesting functions. A prominent example is the Takagi–Landsberg
Figure 1: Takagi–Landsberg functions \( x^H \) and their maxima for various choices of the Hurst parameter \( H \).

function with Hurst parameter \( H > 0 \),

\[
x^H(t) = \sum_{m=0}^{\infty} 2^{m(1/2-H)} \sum_{k=0}^{2^m-1} e_{m,k}(t), \quad 0 \leq t \leq 1.
\]  

(1)

See Figure 1 for a plot of \( x^H \) for various choices of \( H \). The case \( H = 1 \) corresponds to the celebrated Takagi function, which was introduced in [29], became rediscovered many times, and is sometimes also called the blancmange curve; see the surveys [2] and [22]. The extension of this function to a general Hurst parameter was attributed to Landsberg [23] by Mandelbrot [5, p. 246]. In this paper, we focus on the case \( 0 < H < 1 \).

We will also study the function class that arises if we allow for additional coefficients \( \theta_{m,k} \in \{-1,+1\} \) in front of the functions \( e_{m,k} \) in (1). We are thus interested in the class

\[
\mathcal{X}^H = \left\{ x \in C[0,1] \mid x = \sum_{m=0}^{\infty} 2^{m(1/2-H)} \sum_{k=0}^{2^m-1} \theta_{m,k} e_{m,k} \text{ for coefficients } \theta_{m,k} \in \{-1,+1\} \right\}
\]  

(2)

of signed Takagi–Landsberg functions with Hurst parameter \( H \in (0,1) \); see Figure 2 for an illustration. It can be checked easily that the Faber–Schauder series in the definition of the class \( \mathcal{X}^H \) converges uniformly for every \( H \) and all possible choices \( \theta_{m,k} \in \{-1,+1\} \). The class \( \mathcal{X}^H \) is a subset of the flexible Takagi class introduced by Allaart [1], and the special case \( \mathcal{X}^{1/2} \) was analyzed in [28].
Figure 2: The function \((t, H) \mapsto \sum_{m=0}^{10} 2^m (\frac{1}{2} - H) \sum_{k=0}^{2^m-1} \theta_{m,k} e_{m,k}(t)\) for randomly sampled coefficients \(\theta_{m,k} \in \{-1, +1\}\).

2.1 The \(p\)th variation of functions in \(X^H\).

The signed Takagi–Landsberg functions in the class \(X^H\) share many properties with the sample paths of the corresponding fractional Brownian bridge with the same Hurst parameter \(H \in (0, 1)\). For instance, it follows from [1, Theorem 3.1 (iii)] that the functions in \(X^H\) are nowhere differentiable. Moreover, [4] showed that the graph of \(x^H\) has the same Hausdorff dimension, \(2 - H\), as the trajectories of fractional Brownian motion [3]. In addition, for \(H = \frac{1}{2}\), the functions in \(X^H\) have linear quadratic variation \(\langle x \rangle_t = t\), just as the sample paths of standard Brownian motion or of the Brownian bridge; see [28]. Our Theorem 2.1 will extend the preceding result to all Hurst parameters \(H \in (0, 1)\). It hence enables us to use the functions in \(X^H\) as test integrators for higher-order pathwise integration theory in the spirit of Föllmer’s pathwise Itô calculus [12] as it was recently developed by Cont and Perkowski [8] (see also Gradinaru et al. [19] and Errami and Russo [10] for related earlier work).

This integration theory is based on the notion of continuous \(p\)th variation along a refining sequence of partitions, which we are going to recall next. Let

\[
T_n := \{ k2^{-n} | n \in \mathbb{N}, k = 0, \ldots, 2^n \}, \quad n = 0, 1, \ldots, \tag{3}
\]

denote the \(n\)th dyadic partition of \([0, 1]\). It will be convenient to denote by \(s'\) the successor of \(s\) in \(T_n\), i.e.,

\[
s' = \begin{cases} 
\min\{t \in T_n | t > s\} & \text{if } s < 1, \\
1 & \text{if } s = 1.
\end{cases}
\]

Now let \(x\) be a function in \(C[0, 1]\) and \(p \geq 1\). The function \(x\) admits the continuous \(p\)th variation
\( \langle x \rangle_t^{(p)} \) along the sequence \((T_n)\), if for each \(t \in [0,1] \)

\[
\langle x \rangle_t^{(p)} := \lim_{n \to \infty} \sum_{s \in T_n, s \leq t} |x(s') - x(s)|^p
\]

exists and the function \( t \mapsto \langle x \rangle_t^{(p)} \) is continuous (see, e.g., [8, Lemma 1.3]). According to Föllmer [12] in the case \( p = 2 \) and Cont and Perkowski [8] in the case of general even \( p \in \mathbb{N} \), this notion of \( p^{th} \) variation along a refining sequence of partitions is the key to a pathwise integration theory with integrator \( x \). Note that it is different from the usual concept of finite \( p \)-variation, which will be discussed at the end of this section.

**Theorem 2.1.** Let \( H \in (0,1) \), \( p > 0 \), and \( x \in \mathcal{X}^H \). Then, for all \( t \in (0,1] \),

\[
\lim_{n \to \infty} \sum_{s \in T_n, s \leq t} |x(s') - x(s)|^p = \begin{cases} 
0 & \text{if } p > 1/H, \\
+\infty & \text{if } p < 1/H, \\
t \cdot \mathbb{E}(|Z_H|^p) & \text{if } p = 1/H,
\end{cases}
\]

(4)

where

\[
Z_H := \sum_{m=0}^{\infty} 2^{m(H-1)}Y_m
\]

for an i.i.d. sequence \( Y_0, Y_1, \ldots \) of \(-1, +1\)-valued random variables with \( \mathbb{P}[Y_n = +1] = \frac{1}{2} \). In particular, for \( p = 1/H \), each \( x \in \mathcal{X}^H \) admits the continuous \( p^{th} \) variation \( \langle x \rangle_t^{(p)} = t \cdot \mathbb{E}(|Z_H|^p) \) along the sequence \((T_n)\) of dyadic partitions.

**Remark 2.2.** Note that the law of \( Z_H \) is the infinite Bernoulli convolution with parameter \( 2^{H-1} \).

These laws have been studied in their own right for many decades; see, e.g., [26]. According to the Itô-type formulas [8, Theorems 1.5 and 1.10], the most interesting case is the one in which \( p = 1/H \) is an even integer. For this case, Theorem 1 from Escribano et al. [11] provides an exact formula for \( \mathbb{E}(|Z_H|^p) \) in terms of Bernoulli numbers \( B_{2k} \) and partitions of \( n := p/2 \), namely,

\[
\mathbb{E}(|Z_H|^p) = (-1)^n \sum_{1 \leq n_1 + \cdots + n_k = n} \frac{(2n)!}{n_1! \cdots n_k!} \prod_{k=1}^{n} \left( \frac{1}{(2k)!} - \frac{(-1)^k}{2^k} \right) \frac{1}{2^k} (2^k - 1) B_{2k} \left( \frac{1 - 2^{H-1}}{2} \right)^{n_k}
\]

If \( 1/H \) is not an even integer, then (4) yields that any such \( x \in \mathcal{X}^H \) will have vanishing, and hence continuous, \( p^{th} \) variation along \((T_n)\) for any \( p > 1/H \), and so [8, Theorems 1.5 and 1.10] can be applied with any even integer \( p > 1/H \).

**Remark 2.3.** It is known that fractional Brownian motion \( B^H \) with Hurst parameter \( H \in (0,1) \) satisfies almost surely,

\[
\lim_{n \to \infty} \sum_{s \in T_n, s \leq t} |B^H(s') - B^H(s)|^p \begin{cases} 
0 & \text{if } p > 1/H, \\
+\infty & \text{if } p < 1/H, \\
t \cdot \mathbb{E}(|Z_H|^p) & \text{if } p = 1/H,
\end{cases}
\]

for \( t > 0 \); see [25, Section 1.18]. By Lemma 3.1, the same is true of a fractional Brownian bridge. Therefore, our Theorem 2.1 establishes yet another similarity between fractional Brownian sample paths and the signed Takagi–Landsberg functions in \( \mathcal{X}^H \).
Figure 3: The value \( \langle x^H \rangle^{(1/H)} = \mathbb{E}[|Z_H|^{1/H}] \) as a function of \( H \in [0, 1] \).

2.2 Maximum and modulus of continuity of (signed) Takagi–Landsberg functions

Our next result concerns the maximum of the Takagi–Landsberg function \( x^H \) for \( 0 < H < 1 \). The maximum of the classical Takagi function corresponding to \( H = 1 \) was obtained by Kahane [20], and the case \( H = \frac{1}{2} \) can be deduced from Lemma 5 in Galkina [17], which was later rediscovered by the second author in [28, Lemma 3.1]. The corresponding result for the maximum of \( x^{1/2} \) was stated independently in Galkin and Galkina [16] and in [28].

**Theorem 2.4.** For \( 0 < H < 1 \), we have

\[
\max_{t \in [0, 1]} x^H(t) = \frac{1}{3(1 - 2^{-H})}.
\]

Moreover, \( t = \frac{1}{3} \) and \( t = \frac{2}{3} \) are the unique points at which the function \( x^H(t) \) attains its maximum.

Note that the maximum points, \( t = \frac{1}{3} \) and \( t = \frac{2}{3} \), are independent of the Hurst parameter \( H \) as long as \( 0 < H < 1 \). This changes at \( H = 1 \), where the maximum is attained at an uncountable Cantor-type set of Hausdorff dimension \( \frac{1}{2} \) (see [20] or [2, Theorem 3.1]). Theorem 2.4 yields the following corollary, which extends [28, Theorem 2.2], where the particular case \( H = 1/2 \) was treated.

**Corollary 2.5.** The uniform maximum of functions in \( x \in \mathcal{X}^H \) is attained by \( x^H \) and equals

\[
\max_{x \in \mathcal{X}^H} \max_{t \in [0, 1]} x(t) = \max_{t \in [0, 1]} x^H(t) = \frac{1}{3(1 - 2^{-H})}.
\]

The maximal uniform oscillation of functions \( x \in \mathcal{X}^H \) is given by

\[
\max_{x \in \mathcal{X}^H} \max_{s,t \in [0, 1]} |x(t) - x(s)| = \frac{2^H + 3}{6(2^H - 1)}.
\]
It is attained for the points \( s = \frac{1}{3} \) and \( t = \frac{5}{6} \) and for the function

\[
\tilde{x}^H = e_{0,0} + \sum_{m=1}^{\infty} 2^{m(\frac{1}{2}-H)} \left( \sum_{k=0}^{2^m-1} e_{m,k} - \sum_{k=2^m-1}^{2^m-1} e_{m,k} \right) \in \mathcal{X}^H. \tag{5}
\]

Now we consider the modulus of continuity of \( x^H \). For \( h > 0 \), we let \( \nu(h) = \lceil -\log_2 h \rceil \), where \( [a] \) is the biggest integer not exceeding \( a \). Then we define

\[
\omega_H(h) = \frac{h2^{(\nu(h)-1)(1-H)}}{2^{1-H} - 1} + \frac{2^{(1-\nu(h))H}}{3(1-2^H)}.
\tag{6}
\]

Evidently, \( 2^{H-1}h^H \leq h^{\nu(h)(1-H)} \leq h^H \), and \( h^H \leq 2^{-\nu(h)H} \leq 2^H h^H \). Therefore, \( \omega_H(h) = O(h^H) \) as \( h \downarrow 0 \). For \( H = \frac{1}{2} \), the modulus of continuity of \( x^H \) was obtained in [28, Theorem 2.3 (a)]. The following theorem extends this result to all \( H \in (0, 1) \). Note also that our expression for \( \omega_H \) does not extend to the case \( H = 1 \). As a matter of fact, for \( H \geq 1 \), other effects occur and a different method must be used; see [21] and [1].

**Theorem 2.6.** The following assertions hold.

(a) For all \( h \in [0, 1) \) and \( t \in [0, 1-h] \), we have \( |x^H(t+h) - x^H(t)| \leq \omega_H(h) \).

(b) The inequality in (a) is sharp in the sense that

\[
\limsup_{h \downarrow 0} \max_{t \in [0, 1-h]} \frac{|x^H(t+h) - x^H(t)|}{\omega_H(h)} = 1. \tag{7}
\]

Our next result identifies the uniform modulus of continuity of the functions in the class \( \mathcal{X}^H \). It extends [28, Theorem 2.3 (b)], where the special case \( H = \frac{1}{2} \) was treated.

**Theorem 2.7.** The following assertions hold for \( \omega_H \) as in (6).

(a) For all \( h \in [0, 1) \) and \( t \in [0, 1-h] \),

\[
\max_{x \in \mathcal{X}^H} |x(t+h) - x(t)| \leq 2^{1-H} \omega_H(h).
\]

(b) The inequality from part (a) is sharp in the sense that

\[
\limsup_{h \downarrow 0} \max_{t \in [0, 1-h]} \frac{|\tilde{x}^H(t+h) - \tilde{x}^H(t)|}{\omega_H(h)} = 2^{1-H}, \tag{8}
\]

where \( \tilde{x}^H \in \mathcal{X}^H \) as in (5). In particular, the function \( 2^{1-H} \omega_H(h) \) is a uniform modulus of continuity for the class \( \mathcal{X}^H \),

\[
\limsup_{h \downarrow 0} \sup_{x \in \mathcal{X}^H} \max_{t \in [0, 1-h]} \frac{|x(t+h) - x(t)|}{2^{1-H} \omega_H(h)} = \limsup_{h \downarrow 0} \max_{t \in [0, 1-h]} \frac{|\tilde{x}^H(t+h) - \tilde{x}^H(t)|}{2^{1-H} \omega_H(h)} = 1. \tag{9}
\]
Remark 2.8. Theorems 2.6 and 2.7 imply in particular that the functions in $\mathcal{X}^H$ are Hölder continuous with exponent $H$ but not with any other exponent $\alpha > H$. This fact already follows from the work of Ciesielski [7], who showed that the space of Hölder continuous functions is isomorphic to $\ell^\infty$ and the isomorphism consists in a suitable weighting of the coefficients in the Faber–Schauder expansion. We refer to [16] for later results that also yield the Hölder continuity of the Takagi–Landsberg function. The Hölder continuity implies that functions in $\mathcal{X}^H$ are of finite $1/H$-variation in the sense of [15, Definition 5.1] and can thus be used as test integrators for Young integrals and rough-path calculus; see [14, 15] and the references therein. In this respect, the signed Takagi–Landsberg functions in $\mathcal{X}^H$ differ from the typical sample paths of fractional Brownian motion with Hurst parameter $H$: It was shown in [27] that these sample paths have almost surely infinite $1/H$-variation in the sense of [15, Definition 5.1].

3 Proofs

3.1 Proof of Theorem 2.1

We will need the following lemma.

Lemma 3.1. Suppose that $p \geq 1$ and $x, y \in C[0,1]$ are functions with continuous $p^{th}$ variation along $(\mathbb{T}_n)$. Then, if $\langle y \rangle^{(p)} = 0$, the function $x + y$ admits the continuous $p^{th}$ variation $\langle x + y \rangle^{(p)} = \langle x \rangle^{(p)}$.

Proof. For $z \in C[0,1]$, $t \in [0,1]$, and $n \in \mathbb{N}$, we define

$$S_{t,n}(z) := \left( \sum_{s \in \mathbb{T}_n, s \leq t} |z(s') - z(s)|^p \right)^{1/p}.$$  

Minkowski’s inequality yields that

$$S_{t,n}(x) - S_{t,n}(y) \leq S_{t,n}(x + y) \leq S_{t,n}(x) + S_{t,n}(y).$$

Passing to the limit $n \uparrow \infty$ thus yields the assertion. \hfill \square

Proof of Theorem 2.1. We start by proving the assertion for $p = 1/H$ and $t = 1$. For a given function

$$x = \sum_{m=0}^{\infty} 2^m(1 - H) \sum_{\ell=0}^{2^m-1} \theta_{m,\ell} e_{m,\ell} \in \mathcal{X}^H$$  

and $n \in \mathbb{N}$, the corresponding truncated function $x_n$ is defined as

$$x_n = \sum_{m=0}^{n-1} 2^m(1 - H) \sum_{\ell=0}^{2^m-1} e_{m,\ell}.$$  

Then $x(k 2^{-n}) = x_n(k 2^{-n})$ for $k = 0, \ldots, 2^n$. Moreover, the function $x_n$ is linear on $[k 2^{-n}, (k + 1) 2^{-n}]$ with slope

$$\sum_{m=0}^{n-1} 2^m(1 - H) 2^{m/2} \sigma_{m,k},$$
where $\sigma_{m,k} \in \{-1, +1\}$. It follows that

$$x((k + 1)2^{-n}) - x(k2^{-n}) = 2^{-n} \sum_{m=0}^{n-1} 2^{m(1-H)} \sigma_{m,k}. \quad (12)$$

Therefore,

$$S_n := \sum_{k=0}^{2^n-1} |x((k + 1)2^{-n}) - x(k2^{-n})|^p = \sum_{k=0}^{2^n-1} 2^{-n} \sum_{m=0}^{n-1} 2^{m(1-H)} \sigma_{m,k}^p. \quad (13)$$

Now we establish the convergence of $S_n$ as $n \uparrow \infty$. To this end, we claim that, when summing over $k$ in (13), the vector $(\sigma_{0,k}, \ldots, \sigma_{n-1,k})^\top$ runs through the entire set $\{-1, +1\}^n$, and each of its elements appears exactly once. This claim will be proved below. Once it has been established, we can represent $S_n$ by means of an expectation with respect to the uniform distribution on $\{-1, +1\}^n$ and, hence, with respect to the random variables $Y_0, \ldots, Y_{n-1}$. That is,

$$S_n = 2^n \mathbb{E} \left[ \left| 2^{-n} \sum_{m=0}^{n-1} 2^{m(H-1)} Y_m \right|^p \right] = \mathbb{E} \left[ \left| \sum_{m=0}^{n-1} 2^{(n-m)(H-1)} Y_m \right|^p \right],$$

where we have used our assumption $p = 1/H$ in the second step. Now

$$\left| \sum_{m=0}^{n-1} 2^{m(H-1)} Y_m \right| \leq \sum_{m=0}^{\infty} 2^{m(H-1)} = \frac{1}{1 - 2^{H-1}}.$$

Therefore, the infinite series

$$\sum_{m=0}^{\infty} 2^{m(H-1)} Y_m = Z_H$$

converges, and dominated convergence implies that

$$\lim_{n \uparrow \infty} S_n = \mathbb{E}[|Z_H|^p].$$

This proves the assertion (4) for $t = 1$.

Now we prove our auxiliary claim that when summing over $k$ in (13), the column vector $(\sigma_{0,k}, \ldots, \sigma_{n-1,k})^\top$ runs through the entire set $\{-1, +1\}^n$, and each of its elements appears exactly once. To this end, we consider first the particular case $x = x^H$. In this case, we write $\sigma_{m,k}$ for the signs in (12). The row vector $(\sigma_{m,0}, \ldots, \sigma_{m,2^n-1})$ consists of alternating blocks of length $2^{n-1-m}$ with entries all being $+1$ or $-1$. We thus obtain the matrix

$$\begin{pmatrix}
\sigma_{n-1,0}^H & \ldots & \sigma_{n-1,2^n-1}^H \\
\sigma_{n-2,0}^H & \ldots & \sigma_{n-2,2^n-1}^H \\
\vdots & \ddots & \vdots \\
\sigma_{0,0}^H & \ldots & \sigma_{0,2^n-1}^H
\end{pmatrix} =
\begin{pmatrix}
+1 & -1 & +1 & -1 & \cdots & -1 & +1 & -1 \\
+1 & +1 & -1 & -1 & \cdots & -1 & -1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
+1 & +1 & +1 & +1 & \cdots & +1 & -1 & -1 \\
\end{pmatrix} \quad (14).$$
When replacing all occurrences of \(-1\) with 0, the columns of this matrix run through the binary representations of all integers from \(2^n - 1\) to 0. This establishes our auxiliary claim for \(x = x^H\).

Next, we consider the case in which there is exactly one coefficient \(\theta_{m,\ell}\) in the representation (11) that is equal to \(-1\). In this case, only the signs \(\sigma_{m,2^{n-m}+1},\ldots,\sigma_{m,2^{n-m}(\ell+1)-1}\) are different from \((\sigma_{i,j}^H)\). More precisely, \(\sigma_{m,k} = -\sigma_{m,k}^H\) for \(k = 2^{n-m}\ell,\ldots,2^{n-m}(\ell+1)-1\). That is, the matrix \((\sigma_{i,j})\) is obtained from \((\sigma_{i,j}^H)\) by swapping the columns \(2^n-m\ell\) to \((\ell+1)2^n-m-1\) with the columns \((\ell+1\frac{1}{2})2^n-m\) to \((\ell+1)2^n-m-1\). In other words, when passing from \((\sigma_{i,j}^H)\) to \((\sigma_{i,j})\), only the order of columns changes and not the entire collection of all columns. Therefore, also for this \((t\in(15))\) implies that for \(x\) of columns changes and not the entire collection of all columns. Therefore, also for this \(x\in\mathcal{X}^H\). Note that our auxiliary claim implies in particular that \(S_n\) in (13) does not depend on \(x\in\mathcal{X}^H\).

In the next step, we establish the assertion for \(p = 1/H\) for arbitrary \(t \in [0,1]\). To this end, we first recall the following scaling properties of the Faber–Schauder functions:

\[
\sqrt{2}e_{m,k}(t) = e_{m-1,k}(2t) \quad \text{and} \quad e_{m,k}(t-\ell 2^{-m}) = e_{m,k+\ell}(t)
\]

(15)

for \(t \in \mathbb{R}, m \geq 0\), and \(k, \ell \in \mathbb{Z}\). Let \(x \in \mathcal{X}^H\) have the development (10). The first scaling property in (15) implies that for \(t \in [0,1/2]\),

\[
x(t) = \theta_{0,0}e_{0,0}(t) + \sum_{m=1}^{\infty} 2^m(\frac{1}{2}-H) \sum_{\ell=0}^{2^m-1} \theta_{m,\ell}e_{m,\ell}(t)
\]

\[
= \frac{1}{2} \theta_{0,0}2t + \sum_{m=1}^{\infty} 2^m(\frac{1}{2}-H) \sum_{\ell=0}^{2^m-1} \theta_{m,\ell}e_{m,\ell}(t)
\]

\[
= \frac{1}{2} \theta_{0,0}2t + 2^{-H} \sum_{m=0}^{\infty} 2^m(\frac{1}{2}-H) \sum_{\ell=0}^{2^m-1} \theta_{m+1,\ell}e_{m,\ell}(2t).
\]

That is, there exists \(y \in \mathcal{X}^H\) and a linear function \(f\) such that \(x(t) = f(2t) + 2^{-1/2}y(2t)\) for \(0 \leq t \leq 1/2\). It hence follows from Lemma 3.1 and the assumption \(p = 1/H\) that

\[
\langle x^{(p)} \rangle_{\frac{1}{2}} = \langle f + 2^{-H}y \rangle_{\frac{1}{2}} = \langle 2^{-H}y \rangle_{\frac{1}{2}} = \langle y \rangle_{\frac{1}{2}} = \frac{1}{2}\mathbb{E}[|Z_H|^p].
\]

Iteratively, we obtain \(\langle x^{(p)} \rangle_{\frac{1}{2}} = 2^{-k}\mathbb{E}[|Z_H|^p]\) for all \(k \in \mathbb{N}\). Using also the second scaling property in (15) gives in a similar way that \(\langle x \rangle_{(k+1)2^{-\ell}} - \langle x \rangle_{k2^{-\ell}} = 2^{-\ell}\cdot \mathbb{E}[|Z_H|^p]\) for \(k, \ell \in \mathbb{N}\) with \((k+1)2^{-\ell} \leq 1\). We therefore arrive at \(\langle x \rangle_{(p)} = \cdot \mathbb{E}[|Z_H|^p]\) for all dyadic rationals \(t\) in \([0,1]\). A sandwich argument extends this fact to all \(t \in [0,1]\).

If \(p > 1/H\), then

\[
\sum_{s \in \mathbb{T}_n, s \leq t} \langle x^{(p)} \rangle_{\frac{1}{2}} = \max_{u \in \mathbb{T}_n} |x(u') - x(u)|^{p-1/H} \sum_{s \in \mathbb{T}_n, s \leq t} |x(s') - x(s)|^{1/H}.
\]

The maximum tends to zero as \(n \uparrow \infty\), due to the uniform continuity of \(x\), whereas the sum on the right-hand side converges to \(\langle x \rangle_{(1/H)} = \cdot \mathbb{E}[|Z_H|^{1/H}] < \infty\). Hence, the assertion is proved for \(p > 1/H\).
If $t > 0$ and $p < 1/H$, we some $\alpha \in (0, p)$ and let $q := p/\alpha$ and $r := p/(p - \alpha)$. Then $\frac{1}{q} + \frac{1}{r} = 1$ and so

$$\sum_{s \in T_n, s \leq t} |x(s') - x(s)|^{1/H} = \sum_{s \in T_n, s \leq t} |x(s') - x(s)|^{\frac{1}{H} - \alpha}|x(s') - x(s)|^{\alpha} \leq \left( \sum_{s \in T_n, s \leq t} |x(s') - x(s)|^{r(H - \alpha)} \right)^{1/r} \left( \sum_{s \in T_n, s \leq t} |x(s') - x(s)|^{p} \right)^{1/q}.$$  

One easily checks that $r(H - \alpha) > \frac{1}{H}$, and so the first sum on the right-hand side tends to zero by the preceding step of the proof. Moreover, the sum on the left converges to the strictly positive value $t \cdot \mathbb{E}[|Z_H|^{1/H}]$. Therefore, the second sum on the right-hand side must converge to infinity. □

### 3.2 Proofs of Theorem 2.4 and Corollary 2.5

The following lemma extends Lemma 5 in Galkina [17], where the case $H = 1/2$ was treated. Lemma 5 in [17] was later rediscovered by the second author in [28, Lemma 3.1].

**Lemma 3.2.** Consider the sequence of functions

$$x_n^H(t) = \sum_{m=0}^{n-1} 2^{m(\frac{1}{2} - H)} \sum_{k=0}^{2^m - 1} e_{m,k}(t), \quad t \in [0, 1], n \geq 1.$$  

Define the sequence

$$M_n^H = \frac{1}{3(1 - 2^{-H})} + \frac{(-1)^{n-1}}{3(2^{1-H} + 1)2^n} - \frac{2^{-nH}}{(1 + 2^{1-H})(2^H - 1)},$$  

and let

$$J_n = \frac{1}{3}(2^n - (-1)^n)$$  

be the sequence of Jacobsthal numbers. Then the function $x_n^H$ has exactly two maximal points given by $t_n^- = 2^{-n}J_n \in [0, \frac{1}{2}]$ and $t_n^+ = 1 - t_n^- \in [\frac{1}{2}, 1]$, and the global maximum of $x_n^H$ is given by

$$\max_{t \in [0, 1]} x_n^H(t) = x_n^H(t_n^-) = x_n^H(t_n^+) = M_n^H.$$  

**Proof.** We follow the steps in the proof of [28, Lemma 3.1], modifying it for our purposes. Note that $x_n^H$ is symmetric with respect to $t = \frac{1}{2}$. Therefore, it is sufficient to consider the restriction of $x_n^H$ to $[0, \frac{1}{2}]$. In what follows, we let $t_n := t_n^-$. Now we proceed by induction on $n$. For $n = 1$ we have that

$$x_1^H(t) = e_{0,0}(t) = \min\{t, 1 - t\},$$  

and this function achieves its maximum at $t_1 = \frac{1}{2} = 2^{-1}J_1$, the maximal value being equal $x_1^H(t_1) = \frac{1}{2} = M_1$. Let $n = 2$. Then for $t \in [0, \frac{1}{2}]$

$$x_2^H(t) = e_{0,0}(t) + 2^{2-H}(e_{1,0}(t) + e_{1,1}(t)) = e_{0,0}(t) + 2^{2-H}e_{1,0}(t).$$
This function achieves its maximum at \( t_2 = \frac{1}{4} = 2^{-2}J_2 \) and it equals
\[
x^H_2(t_2) = \frac{1}{4} + 2^{-1-H} = M_2.
\]
The maximizers of \( x^H_1 \) and \( x^H_2 \) on \([0, \frac{1}{2}]\) are obviously unique. Now, let \( n \geq 2 \). Note that
\[
x^H_n(t) = x^H_{n-1}(t) + 2^{(n-1)(\frac{1}{2}-H)} \sum_{k=0}^{2^{n-1}-1} e_{n-1,k}(t),
\]
and
\[
x^H_{n+1}(t) = x^H_{n-1}(t) + 2^{(n-1)(\frac{1}{2}-H)} \sum_{k=0}^{2^{n-1}-1} f_{n-1,k}(t),
\]
where
\[
f_{m,k}(t) = e_{m,k}(t) + 2^{\frac{1}{2}-H} e_{m+1,2k}(t) + 2^{\frac{1}{2}-H} e_{m+1,2k+1}(t).
\]
According to the induction hypothesis, the maximal value of \( x^H_n \) is attained at the peak of some function \( e_{n-1,k} \). The support of \( f_{n-1,k} \) coincides with the support of \( e_{n-1,k} \), and \( x^H_{n-1} \) is linear on this support. The function \( f_{n-1,k} \) has two maxima at \( t_n - 2^{-n-1} \) and \( t_n + 2^{-n-1} \) and they are strictly larger than the maximum of \( e_{n-1,k} \). Therefore, either \( x^H_{n+1}(t_n - 2^{-n-1}) \) or \( x^H_{n+1}(t_n + 2^{-n-1}) \) is strictly larger than \( x^H_n(t_n) = \max_{t \in [0, \frac{1}{2}]} x^H_n(t) \). It means that the maximum of \( x^H_{n+1} \) is attained at the peak of some Faber–Schauder function \( e_{n,\ell} \) for some index \( \ell \). Let the support of \( e_{n,\ell} \) be an interval with endpoints \( s \) and \( s' \). Then its peak is at point \( s^* = \frac{s + s'}{2} \) and has height \( 2^{-\frac{5}{2}} \). The function \( x^H_n \) is linear on the support of \( e_{n,\ell} \), therefore,
\[
\max_{t \in [0, \frac{1}{2}]} x^H_{n+1}(t) = x^H_{n+1}(s^*) = x^H_n(s^*) + 2^{(\frac{1}{2}-H)} e_{n,\ell}(s^*)
\]
\[
= \frac{x^H_n(s) + x^H_n(s')}{2} + 2^{(\frac{1}{2}-H)} 2^{-\frac{5}{2}} = \frac{x^H_n(s) + x^H_n(s')}{2} + 2^{-nH-1}.
\]
At one of the points \( s \) or \( s' \), let it be \( s' \), the function \( x^H_n \) coincides with \( x^H_{n-1} \). Therefore, \( x^H_n(s') \leq M^H_{n-1} \). Obviously, \( x^H_n(s) \leq M^H_n \). therefore,
\[
x^H_{n+1}(s^*) \leq \frac{M^H_n + M^H_{n-1}}{2} + 2^{-nH-1} = M^H_{n+1}.
\]
(16)
We can take \( s^* \) as the midpoint between \( t_n \) and \( t_{n-1} \), because \( t_n \) and \( t_{n-1} \) enclose the interval of the support of some Faber–Schauder function of \( n \)th generation. In this case, we have equality in (16), and, moreover, \( s^* = \frac{t_n + t_{n-1}}{2} = t_{n+1} \). We immediately conclude that \( t_{n+1} \) is the unique maximizer of \( x^H_{n+1} \) in \([0, \frac{1}{2}]\) because by induction hypothesis, \( \frac{t_n + t_{n-1}}{2} \) is the only point at which we have equality in (16).

Proof of Theorem 2.4. By sending \( n \) to infinity in Lemma 3.2, one easily gets our formula for the maximum of \( x^H \) and the fact that \( x^H \) is maximized at \( \lim_t t_n^- = \frac{1}{3} \) and \( \lim_t t_n^+ = \frac{2}{3} \). It thus remains to show that these are the only maximum points. To this end, we assume by way of contradiction that there exists \( t_0 \in [0, \frac{1}{2}] \setminus \{\frac{1}{3}\} \) such that \( x^H(t_0) = x^H(\frac{1}{3}) \). Let \( t_n := t_n^- = 2^{-n}J_n \) be as in Lemma 3.2. Then, for all \( n \), the limit \( \frac{1}{3} \) is contained in the dyadic interval \( I_n \) with
endpoints \( t_n \) and \( t_{n-1} \). Let similarly \( S_n \) be that dyadic interval of the form \([k2^{-n}, (k + 1)2^{-n}]\) for \( k \in \{0, \ldots, 2^n - 1\} \) such that \( t_0 \in S_n \). Then there exists \( m \in \mathbb{N} \) such that \( I_n \) and \( S_n \) are disjoint for all \( n \geq m \). Note that \( x^H \) is linear on each of the two intervals \( I_m \) and \( S_m \).

Let \( f : [0, 2^{-m}] \to \mathbb{R}_+ \) be the function that increases linearly from \( x^H(t_{n-1}) \) to \( x^H(t_n) \). Next, we let \( s_0 \) and \( s_1 \) be the endpoints of \( S_m \) such that \( x^H(s_0) = x^H(s_1) \). Then we define \( g : [0, 2^{-m}] \to \mathbb{R}_+ \) as the function that increases linearly from \( x^H(s_0) \) to \( x^H(s_1) \). Since \( t_{m-1} \) and \( t_m \) are the unique respective maximizers of \( x^H \) and \( x^H \) in \([0, \frac{1}{2}]\) and \( t_{m-1}, t_m \notin S_m \), we must have that \( g(t) < f(t) \) for all \( t \in [0, 2^{-m}] \). Moreover, our assumption that both \( t_0 \) and \( \frac{1}{3} \) are maximizers of \( x^H \), together with the scaling properties \((15)\), the relation \( e_{m,k}(t) = 2^{-m/2} e_{0,0}(2^m t - k) \), and the fact that \( e_{0,0}(1 - t) = e_{0,0}(t) \) implies that for \( \tilde{t}_0 := t_0 - \min\{s_0, s_1\} \) and \( \tilde{t}_1 := \frac{1}{3} - \min\{t_m, t_{m-1}\} \),

\[
\max_{r \in [0, 1]} x^H(r) = x^H(t_0) = g(\tilde{t}_0) + \sum_{n=m}^{\infty} 2^{n(1/2 - H)} \sum_{k=0}^{2^n - 1} e_{n,k}(\tilde{t}_0) \\
< f(\tilde{t}_0) + \sum_{n=m}^{\infty} 2^{n(1/2 - H)} \sum_{k=0}^{2^n - 1} e_{n,k}(\tilde{t}_0) \\
\leq \max_{t \in I_n} x^H(t) = x^H(1/3) \\
= \max_{r \in [0, 1]} x^H(r).
\]

This is the desired contradiction. \( \square \)

**Proof of Corollary 2.5.** The inequality

\[
\max_{s, t \in [0, 1]} |x(s) - x(t)| \leq \max_{s \in [0, 1/2]} \max_{t \in [1/2, 1]} |\tilde{x}^H(s) - \tilde{x}^H(t)|,
\]

for any \( x \in \mathcal{X}^H \) can be proved in the same way as \((3.9)\) in [28]. Taking \( x = \tilde{x}^H \) thus yields that the right-hand side of \((17)\) is equal to \( \max_{s, t \in [0, 1]} |\tilde{x}^H(s) - \tilde{x}^H(t)| \). Therefore,

\[
\max_{x \in \mathcal{X}^H} \max_{s, t \in [0, 1]} |x(s) - x(t)| = \max_{s, t \in [0, 1]} |\tilde{x}^H(s) - \tilde{x}^H(t)|. 
\]

Furthermore, \( x^H = \tilde{x}^H \) on \([0, 1/2]\) so the maximal value of \( \tilde{x}^H \) is attained at \( t_1 = \frac{1}{3} \) and equals \( \frac{1}{3(1 - 2^{-m})} \). On the interval \([1/2, 1]\) we have that \( \tilde{x}^H(t) = \frac{1}{2} - x^*(t - 1/2) \), whence the minimal value of \( \tilde{x}^H \) is achieved at \( t_2 = \frac{5}{6} \) and equals \( \frac{1}{2} - \frac{1}{3(1 - 2^{-m})} \). From here, the result follows. \( \square \)

**3.3 Proofs of Theorem 2.6 and 2.7**

**Proof of Theorem 2.6.** The proof extends arguments from the proof of [28, Theorem 3.1 (a)]. Let \( h > 0 \) be given and \( n = \nu(h) \). Then \( 2^{-n-1} < h \leq 2^{-n} \). Since the Faber–Schauder functions \( e_{m,k} \) are linear with the slope \( \pm 2^{n/2} \) on the dyadic intervals of length \( 2^{-m-1} \), we get for \( m \leq n - 2 \) and fixed \( t \in [0, 1 - 2^{-n}] \) that

\[
|x^H(t + h) - x^H(t)| \leq \sum_{m=0}^{n-2} 2^{n(1/2 - H)} h 2^m + \sum_{m=n-1}^{\infty} 2^{m(1/2 - H)} \sum_{k=0}^{2^m - 1} |e_{m,k}(t + h) - e_{m,k}(t)|.
\]

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Evidently, the first sum on the right-hand side equals
\[
\frac{2^{(n-1)(1-H)} - 1}{2^{1-H} - 1}. \tag{19}
\]
To analyze the second sum, let the integer \(\ell\) be such that \(\ell 2^{-n} \leq t < (\ell + 1)2^{-n}\). Then \(s := t - \ell 2^{-n}\) is such that \(0 \leq 2^{n-1}s < 1/2\) and \(1/4 < 2^{n-1}(s+h) < 1\). Hence, the scaling properties (15) imply that for \(m \geq n\),
\[
e_m,k(t+h) - e_m,k(t) = 2^{\frac{1-n}{2}} (e_{m-(n-1),k-\ell 2^{m-n}}(2^{n-1}(s+h)) - e_{m-(n-1),k-\ell 2^{m-n}}(2^{n-1}s)). \tag{20}
\]
If \(\ell\) is even, the preceding identity also holds for \(m = n-1\), and we arrive at
\[
\left| \sum_{m=n-1}^{\infty} 2^m \left( \frac{1}{2} - e_{m,k}(t+h) - e_{m,k}(t) \right) \right| = 2^{(1-n)H} \left| x^H(2^{n-1}s + h) - x^H(2^{n-1}s) \right| \leq 2^{(1-n)H} \frac{1}{3(1 - 2^{-H})},
\]
where we have used Theorem 2.4 in the second step. It now follows from (18) and (19) that
\[
|x^H(t+h) - x^H(t)| \leq \frac{2^{(n-1)(1-H)} - 1}{2^{1-H} - 1} h + 2^{(1-n)H} \frac{1}{3(1 - 2^{-H})} \leq \omega_H(h). \tag{22}
\]
If \(\ell\) is odd, then the fact that
\[
e_{n-1,\ell/2}(r) + e_{n-1,\ell/2}(r) = 2^{\frac{1-n}{2}} \left( \frac{1}{2} - e_{0,0}(2^{n-1}r - \ell/2) \right)
\]
yields that
\[
\left| \sum_{m=n-1}^{\infty} 2^m \left( \frac{1}{2} - e_{m,k}(t+h) - e_{m,k}(t) \right) \right| = 2^{(1-n)H} \left| y(2^{n-1}(s+h) - y(2^{n-1}s) \right|,
\]
where the function \(y\) is given by
\[
y = -e_{0,0} + \sum_{m=1}^{\infty} 2^m \left( \frac{1}{2} - e_{m,k} \right) = x^H \circ \varphi - \frac{1}{2}, \tag{23}
\]
where \(\varphi(u) = |1/2 - u|\) for \(u \in [0,1]\); see Figure 3.3. Thus, (22) holds also in case \(\ell\) is odd. This concludes the proof of part (a).

To establish part (b), put \(t = 0\) and \(h_n = \frac{2}{\ell} 2^{-n}\). Then
\[
|x^H(t + h_n) - x^H(t)| = x^H(h_n) = \frac{2^{(n-1)(1-H)} - 1}{2^{1-H} - 1} h_n + 2^{(1-n)H} \frac{1}{3(1 - 2^{-H})}
\]
\[
= \omega_H(h_n) - \frac{h_n}{2^{1-H} - 1}.
\]
Since \(\omega_H(h_n) = O(h_n^H)\), it follows that
\[
\limsup_{n \uparrow \infty} \frac{|x^H(h_n) - x^H(0)|}{\omega_H(h_n)} \geq 1,
\]
and the proof of part (b) is complete. \(\square\)
Figure 4: The function $y(t)$ from (23) (black) and $x^H(t)$ (gray) for $H = \frac{3}{4}$.

Proof of Theorem 2.7. The proof extends arguments from the proof of [28, Theorem 3.1 (b)]. To prove part (a), let $h > 0$ be given, and $n = \nu(h)$. Arguing as in (18) and (19), we get for $m \leq n - 2$ and for $t \in [0, 1 - 2^{-n})$ that

$$|x(t + h) - x(t)| \leq \frac{2^{(n-1)(1-H)}}{2^{1-H} - 1} h + \left| \sum_{m=n-1}^{\infty} 2^{m(\frac{1}{2}-H)} \sum_{k=0}^{2^{m-1}} \theta_{m,k}(e_{m,k}(t + h) - e_{m,k}(t)) \right|$$

To analyze the sum on the right-hand side, let the integer $\ell$ be such that $\ell 2^{-n} \leq t < (\ell + 1)2^{-n}$. Then $s := t - \ell 2^{-n}$ is such that $0 \leq 2^{n-1}s < 1/2$ and $1/4 < 2^{n-1}(s + h) < 1$. We distinguish three cases.

(i) If $\ell$ is even, then (20) yields that

$$\sum_{m=n-1}^{\infty} 2^{m(\frac{1}{2}-H)} \sum_{k=0}^{2^{m-1}} \theta_{m,k}(e_{m,k}(t + h) - e_{m,k}(t)) = 2^{(1-n)H} (y(2^{n-1}s + h) - y(2^{n-1}s)),$$  

where

$$y = \sum_{m=0}^{\infty} 2^{m(\frac{1}{2}-H)} \sum_{k=0}^{2^{m-1}} \theta_{m+n-1,k+\ell 2^{m-1}} e_{m,k} \in \mathcal{X}^H.$$  

(ii) If $\ell$ is odd and, additionally, $\theta_{n-1,\frac{\ell-1}{2}} = \theta_{n-1,\frac{\ell+1}{2}}$, then (25) holds with

$$y = -\theta_{n-1,\frac{\ell-1}{2}} e_{0,0} + \sum_{m=1}^{\infty} 2^{m(\frac{1}{2}-H)} \sum_{k=0}^{2^{m-1}} \theta_{m+n-1,k+\ell 2^{m-1}} e_{m,k} \in \mathcal{X}^H.$$  

(iii) If $\ell$ is odd and $\theta_{n-1,\frac{\ell-1}{2}} = -\theta_{n-1,\frac{\ell+1}{2}}$, then, similarly to (24) and [28, Eq. (3.15)],

$$\left| \sum_{m=n}^{\infty} 2^{m(\frac{1}{2}-H)} \sum_{k=0}^{2^{m-1}} \theta_{m,k}(e_{m,k}(t + h) - e_{m,k}(t)) \right| = 2^{-nH} \left| g(2^n(s + h)) - y(2^n s) \right|.$$  

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where the function $y$ can take nonzero values on $[0,2]$ and can be decomposed as $y(r) = y_1(r)\mathbb{I}_{[0,1]}(r) + y_2(r-1)\mathbb{I}_{[1,2]}(r)$, for certain functions $y_1, y_2 \in \mathcal{X}^H$. The exact expressions of the functions $y_k$ are not important for our further considerations.

Now we analyze the right-hand side of the inequality (24) according to the cases (i)–(iii). In the cases (i) and (ii), we get as in (21) that

$$|x(t + h) - x(t)| \leq \frac{2^{(n-1)(1-H)} - 1}{2^{1-H} - 1} h + 2^{(1-n)H} \sup_{y \in \mathcal{X}^H} |y(2^{n-1}(s + h) - y(2^{n-1}s)|$$

$$\leq \frac{2^{(n-1)(1-H)} - 1}{2^{1-H} - 1} h + 2^{(1-n)H} \frac{2H + 3}{6(2H - 1)}$$

$$= \frac{2^{1-H}2^{(n-1)(1-H)}}{2^{1-H} - 1} h + (1 - 2^{1-H}) \frac{2^{(n-1)(1-H) + \log_2 h}}{2^{1-H} - 1} + 2^{(1-n)H} \frac{2H + 3}{6(2H - 1)}$$

$$\leq \frac{2^{1-H}2^{(n-1)(1-H)}}{2^{1-H} - 1} h + 2^{(1-n)H} \left( \frac{2H + 3}{6(2H - 1)} - \frac{1}{4} \right)$$

$$\leq 2^{1-H} \omega_H(h).$$

In case (iii), we get

$$|x(t + h) - x(t)| \leq \frac{2^{(n-1)(1-H)} - 1}{2^{1-H} - 1} h + 2^{-nH} \sup_{r,s \in [0,2]} |y(r) - y(s)|,$$

where $y(r) = y_1(r)\mathbb{I}_{[0,1]}(r) + y_2(r-1)\mathbb{I}_{[1,2]}(r)$, for certain functions $y_1, y_2 \in \mathcal{X}^H$. The supremum on the right-hand side realized when $y_1 = y_2 = x^H$ and, according to Theorem 2.4, given by $2/(2(1-2^{-H}))$. Therefore,

$$|x(t + h) - x(t)| \leq \frac{2^{(n-1)(1-H)} - 1}{2^{1-H} - 1} h + \frac{2^{1-nH}}{3(1-2^{-H})} \leq 2^{1-H} \omega_H(h).$$

This completes the proof of part (a). The proof of part (b) can be completed as the one of [28, Theorem 3.1 (b)].

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