ON THE CONFORMAL BENDING OF A CLOSED RIEMANNIAN MANIFOLD

RIRONG YUAN

Abstract. In this paper, we bend a closed Riemannian manifold in the conformal class, through solving a fully nonlinear equation. As a result, we prove that each metric of quasi-negative Ricci curvature is conformal to a metric with negative Ricci curvature.

1. Introduction

In 1980s, Gao-Yau [5] proved that any closed three-manifold admits a metric with negative Ricci curvature. The existence of negatively Ricci curved metric was extended by Lohkamp [12] to higher dimensional closed manifolds.

One natural question to ask is: given a conformal class \([g] = \{e^{2u}g : u \in C^\infty(M)\}\), is there a metric with negative Ricci curvature. When \(M\) is an open manifold, this problem has been studied by Lohkamp [13], who proved that every Riemannian metric on an open manifold is conformal to a complete metric of negative Ricci curvature. A new proof was given by the author in [20]. Nevertheless, the problem is still open when \(M\) is closed. In this paper, we answer the problem when the Ricci curvature of \(g\) is quasi-negative.

First we summarize some notations and notions. Let \(Ric_g\) be the Ricci curvature of Riemannian metric \(g\). The \(Ric_g\) is quasi-negative means that \(Ric_g\) is nonpositive everywhere but strictly negative somewhere.

**Theorem 1.1.** Let \((M, g)\) be a closed connected Riemannian manifold of dimension \(n \geq 3\) with quasi-negative Ricci curvature. Then there is a unique smooth Riemannian metric \(\tilde{g} \in [g]\) with

\[
\det(\lambda(-\tilde{g}^{-1}Ric_{\tilde{g}})) = 1.
\]

In fact we prove a more general theorem for a modified Schouten tensor

\[
A^\tau_\alpha = \frac{\alpha}{n-2} \left( Ric_g - \frac{\tau}{2(n-1)} R_g \cdot g \right), \quad \alpha = \pm 1, \ \tau \in \mathbb{R},
\]

where \(R_g\) is the scalar curvature of \(g\). More precisely, we can construct a metric \(\tilde{g} = e^{2u}g\) satisfying \(\lambda(g^{-1}A^\tau_\alpha) \in \Gamma\) in \(M\), through solving a fully nonlinear equation

\[
f(\lambda(\tilde{g}^{-1}A^\tau_\alpha)) = \psi,
\]

where \(f\) is a given function.
In [3], $f$ is a smooth, symmetric and concave function defined in an open, symmetric and convex cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and boundary $\partial \Gamma \neq \emptyset$, containing the positive cone $\Gamma_\alpha := \{ \lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0 \}$. We denote the closure of $\Gamma$ by $\overline{\Gamma} = \Gamma \cup \partial \Gamma$.

Our main result can be stated as follows.

**Theorem 1.2.** Let $(\alpha, \tau)$ satisfy (1.2). Let $M$ be a closed connected manifold of dimension $n \geq 3$ and suppose a Riemannian metric $g$ with

$$\lambda(g^{-1}A^{\tau,\alpha}_g) \in \overline{\Gamma} \text{ in } M,$$

(1.3)

$$\lambda(g^{-1}A^{\tau,\alpha}_g) \in \Gamma \text{ at some } p_0 \in M.$$  

Assume in addition that

$$f(t\lambda) = t^s f(\lambda), \quad \forall \lambda \in \Gamma, \forall t > 0, \text{ for some constant } 0 < \varsigma \leq 1,$$

(1.5)

$$f > 0 \text{ in } \Gamma, \quad f = 0 \text{ on } \partial \Gamma.$$  

Then there is a unique smooth admissible metric $\tilde{g} \in [g]$ satisfying (1.1).

**Remark 1.3.** For the equation (1.1), we call $g$ an admissible metric if $\lambda(g^{-1}A^{\tau,\alpha}_g) \in \Gamma$ in $M$. Meanwhile, we say $g$ is weakly admissible if $g \in C^2$ and $\lambda(g^{-1}A^{\tau,\alpha}_g) \in \overline{\Gamma}$ in $M$.

The equation (1.1) include many important equations as special cases. When $f = \sigma_1$, $\tau = 0$ and $\psi$ is a proper constant, it is closely related to the well-known Yamabe problem, proved by Schoen [17] with important contributions from Aubin [1] and Trudinger [18]. For $f = \sigma^{1/k}_k$, $\psi = 1$ and $\tau = \alpha = 1$, it was proposed by Viaclovsky [19], referred to $k$-Yamabe problem. The $k$-Yamabe problem has been much studied in recent years [4, 6, 7, 11, 10, 15].

We shall mention more known work [9, 16] on prescribed $\sigma_k$ curvature equation

$$\sigma_k^{1/k}(\lambda(\tilde{g}^{-1}A_g^{\tau,\alpha})) = \psi$$

(1.7)

on a closed manifold, in which

$$\begin{cases}
\tau < 1 & \text{if } \alpha = -1, \\
\tau > n - 1 & \text{if } \alpha = 1.
\end{cases}$$

(1.8)

In [9] Gursky-Viaclovsky solved the equation for $\alpha = -1$ and $\tau < 1$, while the case $\tau > n - 1$ and $\alpha = 1$ was considered by Sheng-Zhang [16] via a fully nonlinear flow. Notice that their approach relies crucially upon two assumptions: one is the existence of admissible metric, the other is the assumption (1.8). This assumption automatically ensures the uniform ellipticity (parabolicity) of the equation.

As a contrast our result and strategy are different. We want to stress that in Theorem 1.2 the given metric allows to be weakly admissible. To achieve this, we
construct in Section 3 an admissible metric, based on Morse theory and differential topology. In addition, the assumption (1.2) is much more broader than (1.8) and it allows the critical case \( \tau = n - 1 \) (the Einstein tensor \( G_g = Ric_g - \frac{R}{2} \cdot g \)) when \( \Gamma \neq \Gamma_n \). To overcome the difficulty, we use Theorem 2.1 to explore the structure of (1.1). The critical case is fairly interesting in dimension three, since the Einstein tensor is closely related to sectional curvature. See a formula in [8, Section 2].

2. Preliminaries and Notations

2.1. Some formulas and reduction of equation. By the formula under the conformal change \( \tilde{g} = e^{2u} g \) (see e.g. [2]),

\[
Ric_{\tilde{g}} = Ric_g - \Delta u g - (n - 2)\nabla^2 u - (n - 2)|\nabla u|^2 g + (n - 2)du \otimes du.
\]

Thus

\[
A_{\tilde{g}}^{\tau\alpha} = A_g^{\tau\alpha} + \frac{\alpha(\tau - 1)}{n - 2}\Delta u g - \alpha\nabla^2 u + \frac{\alpha(\tau - 2)}{2}|\nabla u|^2 g + \alpha du \otimes du.
\]

We denote

\[
V[u] = \Delta u g - \varphi\nabla^2 u + \gamma|\nabla u|^2 g + \varphi du \otimes du + A,
\]

\[
\varphi = \frac{n - 2}{\tau - 1}, \quad \gamma = \frac{(\tau - 2)(n - 2)}{2(\tau - 1)}, \quad A = \frac{n - 2}{\alpha(\tau - 1)} A_{\tilde{g}}^{\tau\alpha}.
\]

Notice that \( V[u] = \frac{\varphi^2 - \varphi}{\alpha^2(\tau - 1)} A_{\tilde{g}}^{\tau\alpha} \). The equation (1.1) reads as follows

\[
f(\lambda(n^{-1} V[u])) = \frac{(n - 2)^2\psi}{\alpha^2(\tau - 1)^2} e^{2n\mu}.
\]

2.2. Structure of operators. In prequel [20] the author explored the structure of nonlinear operator subject to

\[
\lim_{t \to +\infty} f(t\lambda) > f(\mu) \text{ for any } \lambda, \mu \in \Gamma.
\]

We denote \( f_t(\lambda) = \frac{df}{d\lambda}(\lambda) \).

**Theorem 2.1** ([20]). Suppose \((f, \Gamma)\) satisfies (2.5). Then

\[
f_{t}(\lambda) \geq 0 \text{ in } \Gamma, \quad \forall 1 \leq i \leq n,
\]

and for any \( \lambda \in \Gamma \) with \( \lambda_1 \leq \cdots \leq \lambda_n \),

\[
f_{t}(\lambda) \geq \vartheta_{t} \sum_{j=1}^{n} f_j(\lambda) > 0, \quad \forall 1 \leq i \leq \kappa_{\Gamma} + 1,
\]

where \( \kappa_{\Gamma} = \max \left\{ k : (0, \cdots , 0, 1, \cdots , 1) \in \Gamma \right\} \), as well as \( \vartheta_{\Gamma} = \frac{1}{n} \) for \( \Gamma = \Gamma_n \), and

\[
\vartheta_{t} = \sup \left\{ \frac{a_1/n}{\alpha_1/n} - \sum_{j=1}^{n} \frac{a_j}{\alpha_j} : (\alpha_1, \cdots , \alpha_k) \in \Gamma; \alpha_i > 0, \text{ and } \sum_{j=1}^{n} \frac{\alpha_j}{\gamma_j} \text{ for } \Gamma \neq \Gamma_n. \right\}
\]
Based on Theorem 2.1, the author confirmed the uniform ellipticity of (2.4). Firstly by (1.2),

\[ q < \frac{1}{1 - \kappa \vartheta_{G}} \quad \text{and} \quad q \neq 0. \]

From [20, Proposition 3.2], we know

**Theorem 2.2** ([20]). Suppose (1.2), (1.5) and (1.6) hold. Then the equation (2.4) is of uniform ellipticity at any solution \( u \) with \( \lambda(g^{-1}V[u]) \in \Gamma. \)

The following lemma plays a key role in the construction of admissible metrics.

**Lemma 2.3** ([20]). For \( q \) satisfying (2.6), \((1, \ldots, 1, 1 - q) \in \Gamma.\)

### 3. Construction of admissible metrics and proof of main results

In this section, we construct admissible metric in \([g]\) under the assumptions (1.3) and (1.4).

**Theorem 3.1.** Let \((a, \tau)\) satisfy (1.2). Suppose \( M \) is a closed connected manifold of dimension \( n \geq 3 \) and suppose a Riemannian metric \( g \) satisfying (1.3) and (1.4). Then there exists an admissible metric in \([g]\).

**Proof.** We use the notation denoted in (2.2) and (2.3). For a \( C^2 \)-smooth function \( w \) on \( M \), we denote the critical set by

\[ C(w) = \{ x \in M : dw(x) = 0 \}. \]

By the assumption (1.4) and the openness of \( \Gamma \), there is a uniform positive constant \( r_0 \) such that

\[ \lambda(g^{-1}A) \in \Gamma \text{ in } B_{r_0}(p_0). \]

Take a smooth Morse function \( w \) with the critical set

\[ C(w) = \{ p_1, \ldots, p_m, p_{m+1} \ldots p_{m+k} \} \]

among which \( p_1, \ldots, p_m \) are all the critical points being in \( M \setminus B_{r_0/2}(p_0) \). Pick \( q_1, \ldots, q_m \in B_{r_0/2}(p_0) \) but not the critical point of \( w \). By the homogeneity lemma (see e.g., [14]), one can find a diffeomorphism \( h : M \to M \), which is smoothly isotopic to the identity, such that

- \( h(p_i) = q_i, 1 \leq i \leq m. \)
- \( h(p_i) = p_i, m+1 \leq i \leq m+k. \)

Then we obtain a Morse function

\[ v = w \circ h^{-1}. \]

One can check that

\[ C(v) = \{ q_1, \ldots, q_m, p_{m+1} \ldots p_{m+k} \} \subset \overline{B_{r_0/2}(p_0)}. \]

Next we complete the proof. Assume \( v \leq -1. \) Take \( \tilde{u} = e^{Nv}, \tilde{g} = e^{2\mu}g, \) then

\[ V[\tilde{u}] = A + N^2e^{Nv}\overline{\left(\Delta v - \varrho\nabla^2 v\right)/N + \left(1 + \gamma e^{Nv}\right)|\nabla v|^2g + \varrho(e^{Nv} - 1)dv \otimes dv}\right). \]
Notice that
\[
\lambda(g^{-1}((1 + \gamma e^{Nv})|\nabla v|^2 g + g(e^{Nv} - 1)dv \otimes dv))
= |\nabla v|^2(1, \cdots, 1, 1 - \varrho) + e^{Nv}|\nabla v|^2(\gamma, \cdots, \gamma, \gamma + \varrho).
\]

By Lemma 2.3 and the openness of $\Gamma$,
\[
(3.5) \quad (1, \cdots, 1, 1 - \varrho) + e^{Nv}(\gamma, \cdots, \gamma, \gamma + \varrho) \in \Gamma \text{ for } N \gg 1.
\]

**Case 1:** $x \in B_{r_0}(p_0)$. By (3.1) and the openness of $\Gamma$,
\[
\lambda(g^{-1}(A + Ne^{Nv}(\Delta g - \varrho \nabla v))) \in \Gamma.
\]
Combining (3.4) and (3.5),
\[
\lambda(g^{-1}V[u]) \in \Gamma \text{ in } B_{r_0}(p_0).
\]

**Case 2:** $x \notin B_{r_0}(p_0)$. By (3.3) there is a uniform positive constant $m_0$ such that
\[
|\nabla| \geq m_0 \text{ in } M \setminus B_{r_0}(p_0). \text{ By (3.4), (3.5), (1.3), and the openness of } \Gamma,
\]
\[
\lambda(g^{-1}V[u]) \in \Gamma \text{ in } M \setminus B_{r_0}(p_0).
\]
This completes the proof. \qed

Using the admissible metric constructed above, together with Theorem 2.2, we immediately derive the $C^0$-estimate using maximum principle.

Building on Theorem 2.2, the author derived interior estimates for (2.4).

**Theorem 3.2** ([20]). *Let $B_r \subset M$ be a geodesic ball of radius $r$. Let $u \in C^4(B_r)$ be a solution with $\lambda(g^{-1}V[u]) \in \Gamma$ to the equation (2.4) in $B_r$. Assume (1.2), (1.5) and (1.6) hold. Then
\[
\sup_{B_{r/2}}(|\nabla^2 u| + |\nabla u|^2) \leq C,
\]
where $C$ depends on $|u|_{C^0(B_r)}$, $r^{-1}$, and other known data.*

In conclusion, we obtain
\[
|\nabla^2 u| \leq C.
\]
Then we can prove Theorem 1.2 via a standard continuity method.

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School of Mathematics, South China University of Technology, Guangzhou 510641, China

Email address: yuanrr@scut.edu.cn