The local Langlands conjecture for $GL(n)$ over a $p$-adic field, $n < p$.

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Introduction

Let $F$ be a $p$-adic field and $n$ a positive integer. Let $A(n, F)$ denote the set of equivalence classes of irreducible admissible representations of $GL(n, F)$, and let $A_0(n, F)$ be its subset of supercuspidal representations. Similarly, let $G(n, F)$ denote the set of equivalence classes of $n$-dimensional complex representations of the Weil-Deligne group $WD(F)$ on which Frobenius acts semisimply, and let $G_0(n, F)$ denote its subset of irreducible representations. In [H1] a map was constructed, $\pi \mapsto \sigma(\pi)$ from $A_0(n, F)$ to $G(n, F)$. (The map was incorrectly normalized, however; see the Erratum at the end of this introduction.) This map enjoys a number of natural properties, some of which are recalled below. Using these properties, Henniart showed that this map is a bijection with $G_0(n, F)$, and that this bijection preserves Artin conductors ([He4], see [BHK]). Bijections with most of these properties had already been constructed by Henniart [He2]. These properties do not suffice to determine the bijection uniquely, but Henniart proved there is at most one such bijection compatible with the local epsilon factors of pairs [He3].

More precisely, Langlands and Deligne have defined local epsilon factors [L1,D2] for complex representations of the Weil group (more generally, of the Weil-Deligne group); these local constants are compatible with the functional equations of complex representations of the global Weil group. The Langlands and Deligne constructions, applied to the tensor product of two representations $\sigma_1 \otimes \sigma_2$ of the Weil group, yields the local epsilon factor of the pair $(\sigma_1, \sigma_2)$. On the other hand, Jacquet, Piatetski-Shapiro, and Shalika have defined local epsilon factors for pairs $(\pi_1, \pi_2)$, with $\pi_1$ an admissible irreducible representation of $GL(n_i, F)$, $i = 1, 2$; these local constants are compatible with the functional equations of automorphic $L$-functions of $GL(n_1, F) \times GL(n_2, F)$. Henniart’s theorem is that, for any $n$, there is at most one family of bijections $A_0(m, F) \leftrightarrow G_0(m, F)$, with $m \leq n$, preserving local epsilon factors of pairs. The local Langlands conjecture for $GL(n, F)$ can thus be formulated as the assertion that a family of bijections preserving local epsilon factors does indeed exist. The main theorem of the present article (Theorem 3.2) states that this is the case for $n < p$.

The correspondence of [H1] is constructed globally. One realizes the supercuspidal representation $\pi$ as the local component at a place $v$ of a cohomological
automorphic representation $\Pi$ of $GL(n, E)$, where $E$ is a CM field of a certain type, with $E_v \cong F$. In particular, one can arrange that the global $L$-function of $\Pi$ is at almost all places the $L$-function of a compatible family $\sigma(\Pi)$ of $\lambda$-adic representations of $Gal(\overline{E}/E)$, and one takes $\sigma(\pi)$ to be the representation of a decomposition group at $v$. Note that it is not obvious that this correspondence is independent of $\ell$. However, it is compatible with the Langlands correspondence at unramified places, by construction. It is therefore natural to expect that $\pi \mapsto \sigma(\pi)$ “is” the local Langlands correspondence.

Using a somewhat similar global approach, Laumon, Rapoport, and Stuhler proved the local Langlands conjectures for local fields of positive characteristic. To compare local epsilon factors, they followed an approach introduced by Henniart [He1] and used the fact, observed by Deligne [D2], that, in the case of function fields, the functional equation of the $L$-function of a compatible family $\sigma(\Pi)$ of $\lambda$-adic representations is consistent with the local epsilon factors of Langlands and Deligne. This approach breaks down for number fields, since in general one doesn’t know how to prove the functional equation of the $L$-function of a compatible $\lambda$-adic family, except by showing that the $L$-function is automorphic. The exception, of course, is when the compatible family is associated to a complex representation of the Weil group.

In general, an irreducible representation $\sigma$ of the local Weil group $W_F$ cannot be globalized to a complex representation of the Weil group of a number field whose associated compatible family of $\lambda$-adic representations can be obtained from cohomological automorphic representations of $GL(n)$. Specifically, as Henniart pointed out to me, a primitive representation (one not induced from a proper subgroup) cannot be so realized. However, when $\sigma$ is monomial – i.e., induced from a character $\chi$ of a subgroup $H \subset W_F$ of finite index – there is no obstacle in principle to such a globalization. The present paper constructs globalizations of the desired sort when the fixed field $F'$ of $H$ is tamely ramified over $F$. In other words, we construct an extension $E'/E$ of global CM fields, a place $v$ of $E$ with $E_v \cong F$, $E'_v \cong F'$, and an algebraic Hecke character $\chi$ of $E'$ with local component isomorphic to $\chi$, (up to unramified twist) such that $L(s, \chi)$ is equal to the $L$-function of a cohomological cuspidal automorphic representation of $GL(n, E)$, where $n = [F' : F] = [E' : E]$. Here and below equality of $L$-functions is understood to mean equality of Euler products, in this case over the primes of $E$.

When $F'$ is a Galois extension of $F$ this is provided by the global automorphic induction constructed by Arthur and Clozel in [AC] (in the case of interest, the article [K] of Kazhdan suffices). In this case, our main theorem is due to Henniart.
The difficulty is thus to construct automorphic induction for non-Galois extensions, in enough cases to cover the local data of interest (Lemma 1.6, Proposition 2.4). Roughly speaking, we do this by working with a family of automorphic representations to which, as Clozel has shown [C2], one can associate compatible families of $\lambda$-adic Galois representations. The Galois representations are used to “rigidify” the automorphic data.

It has been known for some time that every $\sigma \in \mathcal{G}_0(n, F)$ is monomial when $n$ and $p$ are relatively prime (cf. [KZ], [CH]; we use the reference [KZ] for convenience). In the literature, this is referred to as the tame case. Using difficult calculations with Gauss sums, Bushnell and Fröhlich constructed bijections from $\mathcal{G}_0(n, F)$ to $\mathcal{A}_0(n, F)$ in the tame case, preserving $\epsilon$-factors for the standard $L$-functions of $GL(n) \times GL(1)$ [BF]. [These bijections were based on the internal structure of the multiplicative group of the division algebra, whose ; see also [M]).] They also observed that this property does not characterize such bijections uniquely. By adapting the global methods of [He1] to the case of non-Galois automorphic induction, we are able to show that the bijections $\pi \mapsto \sigma(\pi)$ preserve $\epsilon$-factors for pairs of irreducible Weil-group representations of degree prime to $p$. When $n < p$, this suffices, by [He3], to characterize the local correspondence uniquely.

It should be pointed out that the correspondence in [H1] is constructed on $\ell$-adic cohomology, for $\ell \neq p$. In principle it is possible that the correspondence depends on $\ell$. At least for $n < p$, the independence of $\ell$ is a consequence of our main theorem.

More generally, we can realize any monomial representation of a Weil group, up to unramified twist, as the local constituent of the compatible family of $\lambda$-adic representations attached to a cuspidal automorphic representation obtained (by non-Galois automorphic induction) from an appropriate Hecke character. By Brauer’s theorem, the local epsilon factors of monomial representations suffice to determine the local factors in general. It is thus likely that the local Langlands conjecture can be obtained for all $n$ by a generalization of the methods of the present paper. In §4 we reduce the local Langlands conjecture in general to a partial generalization to $GL(n)$ of Carayol’s work [Ca] on the local Galois correspondence for Hilbert modular forms at places of bad reduction. Generalizing Carayol’s results will require a more complete understanding of the bad reduction of the Shimura varieties used by Clozel to construct his compatible $\lambda$-adic families, extending the results of [H1] and [H2] recalled in Theorem 1.7.

In the recent article [BHK] of Bushnell, Henniart, and Kutzko, it is proved that any correspondence with the properties of the one constructed in [H1] preserves
conductors of pairs. Their approach, valid in all degrees, uses the relation between the Plancherel formula and the fine structure theory of representations of $GL(n, F)$ – the theory of types, due to Bushnell and Kutzko.

The present paper makes extensive use of Henniart’s ideas on the local Langlands correspondence, especially those summarized in the letter [He4] (incorporated in [BHK]). Henniart’s generous advice helped me to clarify my ideas and spared me the consequences of a number of potentially embarrassing errors. I also thank Clozel and Rohrlich for helpful comments.

**Erratum to [H1]**

The normalization of the correspondence $\pi \mapsto \sigma(\pi)$ in [H1] was incorrect in two ways. In the first place, the conventions for the Shimura variety in [H1] were inconsistent with the conventions of Rapoport and Zink for $p$-adic uniformization. The $h_\Phi$ defined on p. 89 of [loc. cit.] is in fact the complex conjugate of the Shimura datum $\overline{h}_\Phi$ determined by the conditions on the bottom of p. 302 of [RZ].

In the second place, the calculation of the twisting character $\nu$ on pp. 100-101 was inconsistent with the definition of $h_\Phi$. For the sake of completeness, and for future reference, here is the correct formula for the twisting character for the Shimura datum determined by $h_\Phi$. As in [H1], we let $Z$ be the connected center of $G\mathbb{G}$. Define $r_\mu$ to be the algebraic character $R_{K/Q} \mathbb{G}_m, K \to Z$ which on $Q$-rational points is the map

$$K^\times \to \mathbb{Z}(Q); \ a \mapsto N_{K/K_0} a.$$

With this definition of $r_\mu$, the formula in [H1]:

(E.1) \[ \nu(G\pi) = \xi^{-1} \circ r_\mu \cdot \left| \frac{a}{\mathbb{A}} \right|^2 \]

makes Theorem 2 of [loc. cit.] correct. Recall that Theorem 2 was a restatement of results on the $L$-function of the Shimura variety, due to Kottwitz, Clozel, and Taylor. Similarly, the twisting character $\tilde{\nu}$ for the Shimura datum $\overline{h}_\Phi$ is given by formula (E.1), with $r_\mu$ replaced by $\tilde{r}_\mu$, where

(E.2) \[ \tilde{r}_\mu(a) = r_\mu(\iota(a)); \]

here $\iota$ denotes complex conjugation.

The replacement of $\overline{h}_\Phi$ by $h_\Phi$ means we have inadvertently calculated the local Galois representation, not at the prime $v$ where the Shimura variety admits $p$-adic uniformization, but rather at its complex conjugate $\overline{v}$. To continue, it is simplest...
(for reasons that will become clear in the Appendix) to return to the conventions of Rapoport and Zink. Define $\nu$ as above. With the Shimura datum $(G, \overline{X}_{n-1})$, where $\overline{X}_{n-1}$ is defined in terms of $\overline{h}_F$, Theorem 2 of [loc. cit.] becomes

$$L^T(s, H^{n-1}(\overline{S}_N, \overline{Q}_\ell)[\pi]) = L^T(s - \frac{n - 1}{2}, \Pi_{\overline{K}^c}, St, \nu_0(G\pi))^m$$

$$= L^T(s, \Pi_{\overline{K}^c}, St, \nu(G\pi))^m.$$  

Here $m$ is a multiplicity that plays no role in the present discussion. The presence of $\Pi_{\overline{K}^c}$ rather than $\Pi_{\overline{K}}$ in the formula differs from the conventions of much of the literature but is consistent with the considerations discussed on pp. 82-83 of [H3].

Reviewing the constructions in §3 of [H1], we see that the local correspondence $\pi \mapsto \sigma(\pi)$ is compatible with the global correspondence on cohomology (cf. Theorem 1.7, below) if we define $\sigma(\pi_v)$ by

$$(E.4) \quad \sigma(\pi_v) = [\tilde{\sigma}(\pi_v) \otimes \nu(G\pi_v^{-1})]^*$$

Here $*$ denotes contragredient and $\tilde{\sigma}(\pi_v)$ is defined as in [H1]:

$$(E.5) \quad \tilde{\sigma}(\pi_v) = [\text{Hom}_{GJ}(H^{n-1}_c(\overline{M}_N, \overline{Q}_\ell)_{SS(F)}, \pi_p) \otimes GJL(\pi_v)^*]^{GG}.$$ 

With this definition, the global arguments in [H1] that show compatibility of the correspondence $\pi \mapsto \sigma(\pi)$ with cyclic base change, automorphic induction, local abelian class field theory, and so on, remain (or become) correct.

**Notation**

Let $G$ be a reductive group over the number field $E$. For any place $v$ of $E$, we write $G_v = G(E_v)$. Let $G_\infty = \prod G(E_\sigma)$, the product taken over the set of archimedean places of $G$; thus $E_\infty$ denotes the product of the archimedean completions of $E$. We also define $E_{\mathbb{A}}$ (resp. $E_{\mathbb{A}^f}$) to be the adeles (resp. finite adeles) of $E$; if $S$ is a finite set of finite primes of $E$ we let $E_{\mathbb{A}^f,S}$ denote the ring of finite adeles with entry 0 at all places of $S$. We let $\mathcal{A}(G)$ denote the space of automorphic forms on $G(E) \setminus G(E_{\mathbb{A}})$, (briefly: automorphic forms on $G$), relative to an implicit choice of maximal compact subgroup $K \subset G_\infty$. We let $\mathcal{A}_0(G) \subset \mathcal{A}(G)$ be the space of cusp forms. Let $Z_G$ denote the center of $G$, and let $\xi$ be a Hecke character of $Z_G(E) \setminus Z_G(E_{\mathbb{A}})$. We let $\mathcal{A}_0(G, \xi) \subset \mathcal{A}_0(G)$ denote the subspace of forms $f$ such that

$$f(\xi g) = \xi(\gamma) f(\gamma), \quad \gamma \in Z_G(E_{\mathbb{A}}), \quad g \in G(E_{\mathbb{A}}).$$
We let $\mathcal{T}(G)$ denote the set of automorphic representations of $G$, which we take in the sense of irreducible admissible representations of $G(\mathbb{A})$ that occur as subspaces of $\mathcal{A}(G)$. We define $\mathcal{T}_0(G)$ to be the set of cuspidal automorphic representations, and $\mathcal{T}_0(G, \xi)$ the set of cuspidal automorphic representations with central character $\xi$. If $S$ is a finite set of places of $E$, let $\mathcal{T}(G, S)$ denote the set of automorphic representations of $G$ unramified outside $S$, and define $\mathcal{T}_0(G, S)$ and $\mathcal{T}_0(G, S, \xi)$ in the obvious way.

If $\pi \in \mathcal{T}(G)$, we let $\tilde{\pi}$ denote its contragredient.

Most frequently, we regard $G$ as a group over $\mathbb{Q}$ by restriction of scalars, and write $G(\mathbb{Q})$ and $G(\mathbb{A})$ instead of $G(E)$ and $G(E_A)$.

By a Hecke character of the number field $E$ we mean a continuous complex character of the idele class group $\mathbb{E} \times \mathbb{A}/\mathbb{E} \times \mathbb{A}$. If $E'/E$ is a finite extension of local or global fields, we let $N_{E'/E}$ and $Tr_{E'/E}$ denote the norm and trace maps, respectively, from $E'$ to $E$.

Let $\Gamma$ be a group and let $H$ be a subgroup of $\Gamma$ of finite index. If $\sigma$ is a finite-dimensional representation of $H$ over the coefficient field $k$, we let $Ind_{H}^{\Gamma} \sigma$ denote the representation of $\Gamma$ induced from $\sigma$. If $\mathcal{L}/L$ is a (possibly infinite) Galois extension of fields, $\Gamma$ is the Galois group $Gal(\mathcal{L}/L)$, and $L' \subset \mathcal{L}$ is the fixed field of the (open and closed) subgroup $H$, we also write $Ind_{L'/L} \sigma$ for $Ind_{H}^{\Gamma} \sigma$. When $L$ is a local field and $\Gamma = W_L$ is its Weil group, $H = W_{L'}$ the Weil group of $L'$, then we again write $Ind_{L'/L} \sigma$ for the induced representation of $\Gamma$. Let $K \subset G$ be a second subgroup of finite index, and let $A \subset G$ be a set of representatives of the double cosets $H \backslash G/K$. The Mackey constituents of $Ind_{H}^{G} \sigma$, restricted to $K$, are then the representations $Ind_{aH_{a^{-1}} \cap K}^{K} a(\sigma)$, for $a \in A$. Here $a(\sigma)$ denotes the representation of $aH_{a^{-1}}$ obtained from $\sigma$ via the canonical isomorphism $aH_{a^{-1}} \sim H$. Up to isomorphism, the Mackey constituents do not depend on the set $A$ of double coset representatives, and Mackey’s Theorem is the isomorphism

$$Ind_{H}^{G} \sigma \sim \bigoplus_{a \in A} Ind_{aH_{a^{-1}} \cap K}^{K} a(\sigma).$$

1. Representations of $GL(n)$ and $\ell$-adic representations.

In the present section we let $E$ be an arbitrary number field and $G = GL(n)_E$, viewed alternatively as a reductive group over $E$ or over $\mathbb{Q}$, by restriction of scalars. For $\pi \in \mathcal{T}(G)$ we let $L(s, \pi)$ denote the standard (principal) $L$-function of $\pi$, as in [GJ], with the archimedean (Gamma) factors excluded. If $v$ is a finite place of $E$ we let $L_v(\sigma, \pi)$ be the local Euler factor at $v$ of $L(s, \pi)$, and for any finite set $S$
of places of $E$ we write $L^S(s, \pi)$ for the partial Euler product $\prod_{v \not\in S} L_v(s, \pi)$, the product being taken over finite places of $E$ not in $S$.

Let $\sigma = \{(\sigma_\lambda, W_\lambda)\}$ be a compatible family of $\lambda$-adic representations of $\text{Gal}(\overline{E}/E)$, where $\lambda$ runs through the set of finite places of some number field $L$, possibly with a finite set of $\lambda$ excluded, and where $W_\lambda$ is a finite-dimensional vector space over $L_\lambda$. We let $\ell(\lambda)$ denote the residue characteristic of $\lambda$. Here by “compatible” we mean that there is a finite set $S$ of finite places of $E$ such that $\sigma_\lambda$ is unramified for all $v$ outside $S \cup$ primes of residue characteristic $\ell(\lambda)$ and that the characteristic polynomials $P_{v, \lambda}(T) = \det(1 - \sigma_\lambda(\text{Frob}_v)T)$ of geometric Frobenius $\text{Frob}_v$ at $v$ have coefficients in $L_i$; it is assumed that $P_{v, \lambda}(T) = P_{v, \lambda'}(T)$ as polynomials in $L(T)$ for distinct primes $\lambda$ and $\lambda'$ of $L$, of residue characteristic different from that of $v$, when $v \not\in S$. In practice we will assume $\sigma$ to be the sum of representations $\sigma_i$, with each $\sigma_i$ pure of some fixed weight $w$. Thus for $v \not\in S$ the eigenvalues of $\sigma_i(\text{Frob}_v)$ have complex absolute values $(Nv)^{\frac{w}{2}}$, for any complex embedding of the number field the eigenvalues generate, with $Nv$ the cardinality of the residue field of $v$. Let $L^S(s, \sigma) = \prod_{v \not\in S} L_v(s, \sigma)$ be the partial $L$-function attached to the compatible family $\sigma$. We give $L^S(s, \sigma)$ the unitary (Langlands) normalization:

$$L_v(s, \sigma) = \prod_{\beta}(1 - \frac{\beta}{|\beta|}Nv^{-s})^{-1},$$

where $\beta$ runs over the set of reciprocal roots of $P_{v, \lambda}$ (any $\lambda$). By our purity hypothesis the absolute values $|\beta|$ are well defined.

It should be borne in mind that $L^S(s, \sigma)$ in the unitary normalization is not itself the (partial) $L$-function attached to a Galois representation unless $w$ is even. However, the unitary normalization is convenient when applying base change. In any case, $L^S(s - \frac{w}{2}, \sigma)$ is the $L$-function attached to $\sigma$ in the arithmetic normalization.

**Definition 1.1.** The compatible family $\sigma$ is weakly associated to $\pi \in \mathcal{T}(G)$ if $L^{S'}(s, \sigma) = L^{S'}(s, \pi)$, as Euler products over places of $E$, for some finite set $S'$ containing $S$. A compatible family $\sigma$ of dimension $n$ is called automorphic if it is weakly associated to some $\pi \in \mathcal{T}(G)$. We let $\mathcal{C}(n, E) \subset \mathcal{T}(G)$ denote the set of $\pi \in \mathcal{T}(G)$ for which there exists a compatible family $\sigma$ as above, weakly associated to $\pi$.

Let $F$ be a $p$-adic field. There is a one-to-one correspondence between spherical representations $\tau$ of $GL(n, F)$ and unramified $n$-dimensional completely decomposable complex representations of the Weil group $W_F$. We denote this correspondence $\tau \mapsto \sigma(\tau)$. The relation $L^S(s, \sigma) = L^{S'}(s, \pi)$ of the preceding definition is equivalent to the assertion that $\sigma_v$ is equivalent to $\pi_v$ for all finite $v \not\in S'$. Here
is the restriction of the compatible system \( \sigma \) to a decomposition group of \( v \) in \( \text{Gal}(\overline{E}/E) \); \( \pi_v \) is the \( v \)-component of \( \pi \), an irreducible admissible representation of \( GL(n, E_v) \); and the compatible system \( \sigma_v \) of unramified \( \lambda \)-adic representations of \( \text{Gal}(\overline{E_v}/E_v) \) is identified with a complex representation of \( W_{E_v} \) in the usual way.

Of course, the local representations \( \sigma(\pi_v) \) can be attached to \( \pi \) at all unramified places.

To each \( \pi \in \mathcal{T}(G) \) we may associate its cuspidal spectrum \( Cusp(\pi) \). This is an unordered set of pairs \( (n_i, \pi_i) \), \( i = i, \ldots, r \), where \( n_i \) is a positive integer and \( \pi_i \in \mathcal{T}_0(GL(n_i, E)) \), such that \( n = \sum n_i \) and \( \pi \) is a subquotient of the induced representation \( \text{Ind}_P^G(\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_r) \); here \( P \subset G \) is the standard parabolic subgroup attached to the partition \( n = \sum n_i \). If \( \sigma \) and \( \pi \) are weakly associated, as above, then \( \sigma \) and \( Cusp(\pi) \) determine each other uniquely, by Chebotarev density and the classification theorem of Jacquet-Shalika [JS], respectively. Let \( \text{Isob}(n, E) \subset \mathcal{T}(G) \) denote the set of isobaric automorphic representations of \( G \) [L2, C1]. If \( \pi \in \text{Isob}(n, E) \) then it is uniquely determined by \( Cusp(\pi) \), hence by a weakly associated \( \sigma \).

We let \( \text{Reg}(n, E) \subset \text{Isob}(n, E) \) denote the set of isobaric representations \( \pi \) such that, for each \( \pi_i \in Cusp(\pi) \), the archimedean component \( \pi_{i,\infty} \) is of cohomological type [C1, Def. 3.12]; equivalently, \( \pi_{i,\infty} \) has regular infinitesimal character. The following theorem is mostly due to Clozel [C2]:

**Theorem 1.2.** Let \( E \) be a number field and let \( \pi \in \text{Reg}(n, E) \). Let \( (n_i, \pi_i) \) be its cuspidal spectrum. Suppose

(i) For each \( i \) there exists a finite place \( v(i) \) of \( E \) such that the local constituent \( \pi_{i, v(i)} \) is square-integrable; let \( p_i \in \mathbb{Q} \) be the rational prime divisible by \( v(i) \), for each \( i \).

(ii) \( E \) is of the form \( E_0 \cdot K_0 \), with \( E_0 \) totally real and \( K_0 \) imaginary quadratic and, for each \( i \), \( \pi_i \simeq \pi_i \circ c \), where \( c : E \rightarrow E \) is the non-trivial element of \( \text{Gal}(E/E_0) \).

We also assume that each of the primes \( p_i \) splits in \( K_0 \). Then \( \pi \in \mathcal{C}(n, E) \).

We let \( \mathcal{CU}(n, E) \subset \mathcal{C}(n, E) \) denote the set of \( \pi \) verifying the conditions of Theorem 1.2.

Obviously it suffices to prove the theorem when \( \pi \) is cuspidal. Under a slightly strengthened version of assumption (i) (depending on \( n \ (\text{mod} \ 4) \)), and (ii), the existence of a compatible system \( \sigma \) weakly associated to cuspidal \( \pi \) was essentially proved by Clozel, using the determination by Kottwitz of the zeta-functions of the Shimura varieties to be discussed below, and Clozel’s work on stable base change. However, Clozel’s argument only yielded \( \sigma \) such that \( L_S(\sigma) = L_S(\pi) \), for
some undetermined multiplicity $m$. The existence of $\sigma$ weakly associated to $\pi$ was then deduced by Taylor ([T], unpublished; see [H1, §3]). The sufficiency of (i) in the cases of bad parity was observed by Blasius, using an argument based on quadratic base change (unpublished, but see [Bl, 4.7]; an alternative argument can be found in [C3], Theorem 2.6).

In (ii) one can take $E$ to be an arbitrary CM field, but then one only obtains Galois representations over a certain reflex field containing $E$ as a proper subfield, in general; cf. [C2, Théorème 5.3]. For $E$ as in (ii) one verifies immediately, as in [BR, p. 66], that the reflex field is just $E$ itself.

We let $\mathcal{C}U(n, E) \subset \mathcal{C}(n, E)$ denote the set of $\pi$ verifying the conditions of Theorem 1.2; it is only defined for $E$ as in (ii).

**Definition 1.3.** Let $E'$ be a finite extension of $E$. Let $\pi \in Isob(n, E)$ and $\pi' \in Isob(m, E')$, for some positive integers $n$ and $m$. Let $S$ be a finite set of finite places of $E$ and let $S'$ be the set of places of $E'$ above $S$. We assume $S$ (resp. $S'$) contains all places at which $\pi$ (resp. $\pi'$) is ramified.

(a) Suppose $n = m$. We say $\pi'$ is a **weak base change** of $\pi$, or that $\pi$ is a **weak descent** of $\pi'$, relative to $S$, if, for any finite place $v$ of $E$, $v \notin S$, and for any place $v'$ of $E'$ dividing $v$, we have

$$\sigma(\pi'_{v'}) = \sigma(\pi_v)|_{Gal(E_v'/E_v')}.$$

(b) Suppose $n = m[E':E]$. We say $\pi$ is a **weak automorphic induction** of $\pi'$, relative to $S$, if, for any finite place $v$ of $E$, $v \notin S$, and for any place $v'$ of $E'$ dividing $v$, we have

$$\sigma(\pi_v) = Ind_{E_v'/E_v} \sigma(\pi'_{v'}).$$

Here $Ind_{E_v'/E_v}$ denotes induction from $Gal(E_v'/E_v')$ to $Gal(E_v/E_v)$, as in the notation section. Condition (b) is equivalent to the condition that $L^S(s, \pi) = L^{S'}(s, \pi')$ (as partial Euler products). We say that $\pi'$ is a weak base change of $\pi$ if it is a weak base change relative to some $S$, and similarly for descent and automorphic induction.

If $E'$ is a solvable extension of $E$, Arthur and Clozel have proved the existence of the weak base change map $BC_{E'/E} : Isob(n, E) \rightarrow Isob(n, E')$ and the weak automorphic induction map $AI_{E'/E} : Isob(m, E') \rightarrow Isob(m[E':E], E)$, relative to the sets $S$ and $S'$ of ramified places [AC, Theorems III.4.2, III.5.1, III.6.2]. In fact, Arthur and Clozel construct canonical local base change maps and show that their global base change is compatible with local base change at all places. (For our
applications, the results of Kazhdan on cyclic automorphic induction of characters are largely sufficient [K].

(1.4) If \( E'/E \) is cyclic of prime degree \( \ell \), then Arthur and Clozel determine the image and fiber of the base change map [AC, Chapter 3]. Let \( \alpha \) be a generator of \( \text{Gal}(E'/E) \). We denote the action of \( \text{Gal}(E'/E) \) on \( \text{Isob}(n, E') \) by \( \pi' \mapsto g \pi' \). Then \( \pi' \in \text{Isob}(n, E') \) is in the image of \( BC_{E'/E} \) if and only if \( \pi' \cong \alpha \pi' \). The fibers of \( BC_{E'/E} \) are completely determined by the following rules:

(a) If \( \pi' \) is cuspidal and \( \pi' \cong \alpha \pi' \) then \( \pi' \) is the base change of precisely \( [E' : E] \) representations \( \pi \in \mathcal{C}(GL(n)_E) \), all twists of one of them by powers of the class field character associated to \( E'/E \).

(b) Suppose \( m = \ell^{-1} \cdot n \) and suppose \( \text{Cusp}(\pi') = \{(m, \pi_1), (m, \pi_2), \ldots, (m, \pi_\ell)\} \), with \( \pi_{i+1} \cong \alpha \pi_i \) for \( i = 1, \ldots, \ell - 1 \). If \( \pi_1 \not\cong \alpha \pi_1 \) then \( \pi' \) is the base change of exactly one \( \pi \in \text{Isob}(n, E) \); moreover, \( \pi \) is cuspidal.

The following lemma is an easy consequence of this description.

**Lemma 1.5.** Let \( \pi' \in \text{Isob}(n, E') \), and suppose \( \pi' = BC_{E'/E}(\pi) \), for some \( \pi \in \text{Isob}(n, E) \). Suppose \( \pi \in \mathcal{C}(n, E) \). Then \( \pi' \in \mathcal{C}(n, E') \). Moreover, any other weak descent \( \pi_0 \in \text{Isob}(n, E) \) of \( \pi' \) is also in \( \mathcal{C}(n, E) \).

**Proof.** If \( \sigma \) is weakly associated to \( \pi \), then \( \sigma_{\text{Gal}(E'/E)} \) is weakly associated to \( \pi' \). The second assertion easily reduces to the case in which \( \pi \) is cuspidal, and then it follows from (a) and (b) above, since \( \mathcal{C}(n, E) \) is stable with respect to twists by characters of finite order.

Henniart and Herb have constructed canonical local automorphic induction maps (for cyclic extensions of prime degree, hence for solvable extensions) and shown that they are compatible at all places with the map \( AI_{E'/E} \) of Arthur and Clozel [HH]. The local automorphic induction is determined by certain character relations. We are interested in extending the map \( AI_{E'/E} \) to the case of an extension \( E'/E \) which is not Galois but whose Galois closure \( \tilde{E}' \) is solvable over \( E \). Assuming \( \pi' \in \mathcal{CU}(1, E') \) – in particular, \( \pi' \) is an algebraic Hecke character – then under additional regularity hypotheses relative to the extension \( E'/E \) one can construct a representation \( AI_{E'/E}(\pi') \in \mathcal{CU}([E'/E], E) \) whose weakly associated compatible family is obtained by usual induction from the compatible family of one-dimensional representations associated to \( \pi' \). In other words, \( AI_{E'/E}(\pi') \) is a weak automorphic induction of \( \pi' \).

This is a vague formulation of a general principle that will be discussed further in §4. At present, we restrict our attention to the simplest case, in which \( \text{Gal}(\tilde{E}'/E) \)
is a semi-direct product of cyclic groups:
\[ \Gamma = Gal(\tilde{E}'/E) \cong A \rtimes T, \]
with \(E'\) the fixed field of the non-normal subgroup \(T\). Moreover, we assume we are in the situation of (ii) of Theorem 1.2: there is an extension \(E'_0/E_0\) of totally real fields, with Galois closure \(\tilde{E}'_0\), such that \(Gal(\tilde{E}'_0/E_0) \cong A \rtimes T\), with \(E'_0\) the fixed field of \(T\), and an imaginary quadratic field \(K_0\), such that \(E = E_0 \cdot K_0, E' = E'_0 \cdot K_0\), etc. We let \(E^u \subset \tilde{E}'\) denote the fixed field of \(A\); it is a cyclic extension of \(E\) with Galois group \(T\). We let \(c\) denote complex conjugation on any subfield of \(\tilde{E}'\), and let \(n = [E' : E] = |A|\).

**Lemma 1.6.** Let \(\chi\) be an algebraic Hecke character of \(E'\) such that \(\chi = \chi^{-1} \circ c\). Let \(\tilde{\chi} = \chi \circ N_{\tilde{E}'/E'}\) denote the base change of \(\chi\) to \(\tilde{E}'\). Suppose \(\pi^u = AI_{\tilde{E}'/E^u}\tilde{\chi} \in CU(n, E^u)\) and is cuspidal, and let \(v^u\) be a finite place of \(E^u\) at which \(\pi^u\) is square-integrable; the existence of such a place is provided by condition (i) of Theorem 1.2. Suppose \(v^u\) divides the place \(v\) of \(E\) and is the only divisor of \(v\). Finally, suppose the compatible family of \(\lambda\)-adic representations of \(Gal(\overline{Q}/E^u)\) weakly associated to \(\pi^u\) is irreducible. Then there is a cuspidal automorphic representation \(\pi \in CU(n, E)\) which is a weak automorphic induction of \(\chi\).

**Proof.** We let \(\sigma'\) denote the compatible family of \(\lambda\)-adic representations associated to \(\chi\), and let \(\tilde{\sigma}\) and \(\sigma^u\) be the compatible families (weakly) associated to \(\tilde{\chi}\) and \(\pi^u\), respectively. Then \(\tilde{\sigma}\) is the restriction of \(\sigma'\) to \(Gal(\overline{Q}/\tilde{E}')\) and \(\sigma^u\) is the induction of \(\tilde{\sigma}\) to \(Gal(\overline{Q}/E^u)\). It follows that \(\sigma^u\) is isomorphic to its conjugates with respect to \(T = Gal(E^u/E)\). Thus \(\pi^u\) is isomorphic at almost all places to \(t(\pi^u)\) for any \(t \in T\). By strong multiplicity one, \(\pi^u\) is thus isomorphic to its \(T\)-conjugates, hence descends to a cuspidal automorphic representation \(\pi\) of \(GL(n, E)\). More precisely, as in (1.4.a) and letting \(m = |T|\), there is a set \(\mathcal{P}\) of \(m\) distinct cuspidal automorphic representations \(\pi\) of \(GL(n, E)\) that base change to \(\pi^u\). Let \(\xi\) be a faithful character of \(T\). Fixing \(\pi \in \mathcal{P}\), the elements of \(\mathcal{P}\) are all of the form \(\pi \otimes \xi^i \circ \det, i = 1, \ldots, m\), where \(\xi\) is identified via class field theory with a finite Hecke character of \(E^\times\) trivial on the norms from \(E^u\).

In order to complete the proof, it suffices to show that one, hence every \(\pi \in \mathcal{P}\), belongs to \(\mathcal{C}(n, E)\). Indeed, assume \(\pi \in \mathcal{C}(n, E)\), and let \(\sigma\) denote the weakly associated compatible family. By the properties of base change, \(\sigma\) is an extension of \(\sigma^u\) to \(Gal(\overline{Q}/E)\). By hypothesis \(\sigma^u\) is irreducible, hence there are precisely \(m\) such extensions, each obtained by twisting \(\sigma\) by a power of \(\xi\). Thus every such extension is weakly associated to exactly one element of \(\mathcal{P}\). On the other hand, let
\( \sigma_1 \) denote the compatible family of \( \lambda \)-adic representations of \( \text{Gal}(\overline{\mathbb{Q}}/E) \) obtained by inducing \( \sigma' \). It is elementary to verify that \( \sigma_1 \) is an extension of \( \sigma^u \). Without loss of generality, we may thus assume \( \sigma_1 = \sigma \), and \( \pi \) is then a weak automorphic induction of \( \chi \). Since \( \chi \) satisfies \( \chi = \chi^{-1} \circ c \), it follows that \( \sigma' \circ c \) is isomorphic to the contragredient of \( \sigma' \). Thus \( \sigma \) has the same property. Since \( \sigma \) is weakly associated to \( \pi \), it follows from strong multiplicity one that \( \pi \simeq \hat{\pi} \circ c \). Thus \( \pi \) actually belongs to \( \mathcal{CU}(n, E) \).

We first verify that \( \pi \in \text{Reg}(n, E) \) and satisfies condition (i) of Theorem 1.2. First, \( \pi \in \text{Reg}(n, E) \) because \( \pi^u \in \text{Reg}(n, E^u) \), since for any complex place \( w \) of \( E \) we have \( \pi_w \simeq \pi_w^u \) for any place \( w^u \) of \( E^u \) dividing \( w \). Next, condition (i) at the place \( v \) is automatic since \( v^u \) is the only divisor of \( v \) in \( E^u \) and by the hypothesis on \( \pi^u \) at \( v^u \).

Finally, we verify a weak analogue of (ii) of Theorem 1.2 that will be sufficient for our purposes. We recall that base change from \( E \) to \( E^u \) commutes with complex conjugation and with the contragredient. Thus \( BC_{E^u/E}(\hat{\pi} \circ c) = \hat{\pi}^u \circ c = \pi^u \). It then follows that \( \hat{\pi} \circ c \cong \pi \otimes \eta \circ det \), where \( \eta \) is a power of \( \xi \). The abelian extension \( E^u/E \) is a lift of a totally real abelian extension of \( E_0 \); thus \( \eta = \eta_0 \circ N_{E/E_0} \), for some finite Hecke character \( \eta_0 \) of the ideles of \( E_0 \), trivial at the archimedean places of \( E_0 \). Let \( \alpha \) be a finite Hecke character of \( E^\times_\mathbb{A}/E^\times \) that restricts to \( \eta_0 \) on the ideles of \( E_0 \). Such an \( \alpha \) exists: indeed, \( \eta_0 \) is trivial at infinity, hence it suffices to construct a continuous character of the compact group \( E^\times_\mathbb{A}/E^\times \cdot E^\times_\infty \) that restricts to \( \eta_0 \) on the closed subgroup \( E^\times_0,\mathbb{A}/E^\times_0 \cdot E^\times_0,\infty \), and this is certainly possible. Let \( \pi_1 = \pi \otimes \alpha \), where for brevity we write \( \otimes \alpha \) instead of \( \otimes \alpha \circ det \). Then \( \pi_1 \) is still regular and still satisfies (i); moreover, we have

\[
\hat{\pi}_1 \circ c = \hat{\pi} \circ c \otimes \alpha^{-1} \circ c \\
= \pi \otimes \eta_0 \circ N_{E/E_0} \cdot \alpha^{-1} \circ c \\
= \pi_1;
\]

the last equality follows from the identity

\[
\alpha(a) \cdot \alpha(a^c) = \alpha(N_{E/E_0}(a)) = \eta_0(N_{E/E_0}(a)).
\]

Thus \( \pi_1 \in \mathcal{CU}(n, E) \subset \mathcal{C}(n, E) \). But \( \mathcal{C}(n, E) \) is obviously invariant under twists by finite Hecke characters, and we are done.

**Remark.** Under appropriate hypotheses, one can use similar arguments to construct base change for the non-Galois extension \( E'/E \), or more generally in the setting of Proposition 4.8, below.
Let $F$ be a finite extension of $\mathbb{Q}_p$. In [H1] a map $\pi \mapsto \sigma(\pi)$ is constructed from the set of equivalence classes of supercuspidal representations of $GL(n, F)$ to the set of $n$-dimensional representations of the Weil group $W_F$ of $F$. It is shown that this map is compatible with isomorphisms $F \sim F'$, commutes with twists by characters of $F^\times \cong W_F^{ab}$, and takes base change and local automorphic induction to restriction and induction of Weil group representations, respectively. Using the results and techniques of [He2], Henniart showed that these conditions implied that $\pi \mapsto \sigma(\pi)$ is a bijection with the set of irreducible representations and preserves Artin conductors [He 4, BHK]. It also commutes with taking contragredients, though this was not stated explicitly.

We will need the following property of this bijection.

**Theorem 1.7.** Let $E$ be a number field and let $\pi \in \text{CU}(n, E)$. Let $v$ be a finite place of $E$ and suppose the local component $\pi_v$ at $v$ is supercuspidal. Let $\sigma$ be the compatible family of $\lambda$-adic representations of $\text{Gal}(\overline{E}/E)$ weakly associated to $\pi$, and let $\sigma_v$ denote the restriction of $\sigma$ to a decomposition group $\text{Gal}(\overline{E}_v/E_v)$ of $v$. Then $\sigma_v$ is equivalent to $\sigma(\pi_v)$.

As noted in the introduction, $\sigma(\pi_v)$ is really a family $\{\sigma(\pi_v)_\ell\}$ of representations with coefficients in $\overline{\mathbb{Q}}_\ell$, for all $\ell \neq p$. The theorem should be understood to mean that $\sigma_{v, \ell}$ is equivalent to $\sigma(\pi_v)_\ell$ for each $\ell$.

The complete proof of this theorem will be given elsewhere [H 2]. However, when $\pi_\infty$ has cohomology with coefficients in the trivial representation (i.e., has the same infinitesimal character as the trivial representation), then this theorem is essentially proved in [H1]. Indeed, $\sigma(\pi_v)$ is defined to make Theorem 1.7 true in this case. The work in [H1] is to prove, using $p$-adic uniformization, that $\sigma(\pi_v)$ thus defined depends only on the supercuspidal representation $\pi_v$ and not on the automorphic representation $\pi$. The case of cohomology with trivial coefficients suffices for the applications in §3. The arguments in §4 require more general coefficient systems. The reader may therefore prefer to consider the results of §4 conditional, pending appearance of [H2].

It should also be noted that the choice of level subgroup in [H1] (especially in the appendix) forces the local constituent of $\pi$ at every place $w \neq v$ dividing $p$ to be a twist of the Steinberg representation. However, this hypothesis was only made in order to simplify notation and is irrelevant to the final result. In particular, the $p$-adic uniformization of [H1,(A.11)] is valid with minor modifications for general level subgroups at the primes $w$ as above. At the referee’s request, we explain how to remove this hypothesis in the appendix.
Finally, the local correspondence in [H1] is constructed on \( \ell \)-adic cohomology and therefore associates \( n \)-dimensional \( W_F \)-modules over \( \overline{\mathbb{Q}}_\ell \) to supercuspidal representations of \( GL(n, F) \) with coefficients in \( \overline{\mathbb{Q}}_\ell \). To obtain a candidate for the local Langlands correspondence we need to choose an isomorphism between \( \overline{\mathbb{Q}}_\ell \) and \( \mathbb{C} \). On the other hand, in the present article we work with local components of induced representations from algebraic Hecke characters \( \chi \). These take their values a priori in \( \mathbb{C} \), but the local components can all be defined over the subfield \( L(\chi) \subset \mathbb{C} \) generated by the values of \( \chi \) on the finite ideles. Then \( L(\chi) \) is a number field and it is easy to see that the representations constructed by automorphic induction are also defined over \( L(\chi) \), up to twisting by a half-integer power of the absolute value character (which may introduce a square root of \( p \)). Of course the association of compatible families of \( \ell \)-adic representations to complex-valued Hecke \( L \)-functions goes back to Weil. The relation between complex and \( \ell \)-adic epsilon factors is worked out in [D2, §6].

In particular, the number field \( L \) that appears above Definition 1.1 will be of the form \( L(\chi) \), and in particular is given as a subfield of \( \mathbb{C} \). We could just as well have adopted this point of view in [H1]. Indeed, the global cohomological automorphic representation \( G\pi \) used on p. 95 ff of [H1] to construct the local correspondence can be realized over a number field \( L(\pi) \) which is also a subfield of \( \mathbb{C} \). By varying \( G\pi \) while keeping the local component \( \pi_v \) fixed it should be possible to define \( \sigma(\pi_v) \) directly over a fixed finite extension of the (number) field of definition of \( \pi_v \) (with the square root of \( p \) adjoined, as above). This finite extension should be given by the field of definition of the representation of the representation \( JL(\pi_v) \) of the multiplicative group of the division algebra over \( F \) with invariant \( \frac{1}{n} \), where the notation \( JL(\bullet) \) denotes the Jacquet-Langlands correspondence, as in [H1]. In this way it may be possible to verify that the correspondence of [H1] is in fact independent of \( \ell \); however, I have not carried this out in detail. Note that this argument would depend on multiplicity one theorems for automorphic representations of unitary similitude groups that are not presently in the literature.

2. Tame representations of local Galois groups.

For any finite extension \( \Phi \) of \( \mathbb{Q}_p \) we let \( \Gamma_\Phi \) denote \( \text{Gal}(\overline{\mathbb{Q}}_p/\Phi) \); \( G^0_\Phi \) denotes the inertia subgroup. By a representation of \( \Gamma_\Phi \) we will always mean a finite-dimensional representation over an algebraically closed field of characteristic zero that factors through a finite quotient.

Fix a finite extension \( F \) of \( \mathbb{Q}_p \). A representation of \( \Gamma_F \) is tame if its restriction to the wild ramification subgroup \( G_1 \) decomposes as the sum of one-dimensional representations.
representations; in other words, the restriction factors through the abelianization of $G^1_F$. If $(n, p) = 1$ then it is easy to see that any irreducible $n$-dimensional representation of $G_F$ is tame (see, e.g. [KZ, 1.8]). Only slightly more difficult to show is the fact that any irreducible tame representation of $\Gamma_F$ is monomial, i.e. induced from a character of a subgroup of the form $\Gamma_{F'}$, for $F'$ a finite tamely ramified extension of $F$ [KZ, p. 104] (cf. also [CH] and [D1], Proposition 3.1.4).

The characters of $\Gamma_{F'}$ can be identified via class field theory with characters $\chi$ of $(F')^\times$ of finite order; we will use the same notation to designate the two associated characters. Koch and Zink determine the pairs $(F'/F, \chi)$, with $F'/F$ tamely ramified and $\chi$ a character of $\Gamma_{F'}$, such that $\text{Ind}_{F'/F}\chi$ is irreducible, and construct a bijection between the set of classes of such pairs, under a natural equivalence relation, and the set of equivalence classes of tame representations of $\Gamma_F$ [KZ, Theorem 1.8 and 3.1]. Here the notation $\text{Ind}_{F'/F}$ is as in the notation section. For our purposes, it suffices to note the following lemma:

**Lemma 2.1.** (Koch, Zink). Suppose $F'/F$ is a tamely and totally ramified finite extension and $\chi$ is a character of $\Gamma_{F'}$ such that $\text{Ind}_{F'/F}\chi$ is irreducible. Let $\chi^1$ denote the restriction of $\chi$ to the wild ramification subgroup $\Gamma^1_F$. Then $\Gamma_{F'}$ is the stabilizer of $\chi^1$ in $\Gamma_F$.

A complete proof is given on p. 104 of [KZ], though the lemma is not stated as such. Specifically, Koch and Zink begin by letting $\chi^1$ be any character of $\Gamma^1_F$ contained in the restriction of $\text{Ind}_{F'/F}\chi$ to $\Gamma^1_F$; obviously our chosen $\chi^1$ fits that description. They then define $K_1$ to be the fixed field of the stabilizer of $\chi^1$. Thus $F' \supset K_1 \supset F$. Finally, they deduce that $F'$ is unramified over $K_1$, hence equals $K_1$ under our hypotheses.

**Corollary 2.2.** Under the hypotheses of Lemma 2.1, let $\tilde{F}$ denote the Galois closure of $F'$ over $F$ and let $F^u \subset \tilde{F}$ denote the maximal subextension unramified over $F$. Let $\tilde{\chi}$ denote the restriction of $\chi$ to $\Gamma_{\tilde{F}}$. Then the stabilizer of $\tilde{\chi}$ in $\text{Gal}(\tilde{F}/F^u)$ is trivial, and $\text{Ind}_{\tilde{F}/F^u}\tilde{\chi}$ is irreducible.

**Proof.** The extension $F'/F$ is tame, hence $\Gamma_{\tilde{F}}$ contains $\Gamma^1_F$. It thus follows from Lemma 2.1 that the stabilizer of $\tilde{\chi}$ in $\text{Gal}(\tilde{F}/F^u)$ is trivial. The final assertion follows immediately, since $\text{Gal}(\tilde{F}/F^u)$ is cyclic.

We now return to the global setting of §1. Let $E'/E$ be an extension of number fields of degree $n$, as in the discussion preceding Lemma 1.6; in particular, it is assumed that $E = E_0 \cdot K_0$ and $E' = E'_0 \cdot K_0$. Define the fields $\tilde{E}$ and $E^u$, so that $E = \text{Gal}(\tilde{E}/E) \cong A \rtimes T$, with $E^u$ the fixed field of $A$ and $E'$ the fixed field of $T$. We will denote by $\text{Ind}_{E'/E}$ the restriction functor; in particular, it is straightforward to verify that $\text{Ind}_{E'/E} = \text{Ind}_{\tilde{E}/E}$. We have written $A \rtimes T$ in place of $A \times T$, for the following reason: the extension $E'/E$ may be non-tame, and we may wish to distinguish the wild and tame parts of the Galois group of $E'/E$. This can be done by observing that $E'/E$ is tame if and only if $\text{Ind}_{E'/E} = \text{Ind}_{\tilde{E}/E}$. We will thus distinguish the wild and tame parts of the Galois group of $E'/E$ by writing $A \rtimes T$ in place of $A \times T$.
We assume there is a finite place \( v \) of \( E_0 \) such that \( E_{0,v} \) is the \( p \)-adic field \( F \) discussed above, and such that \( v \) is divisible by exactly one place \( \tilde{v} \) of \( \tilde{E}_0 \). Let \( v' \) and \( v'' \) be the corresponding places of \( E'_0 \) and \( E''_0 \), respectively. We assume the rational prime \( p \) splits in \( K_0 \), so that \( v \) splits in \( E \) as the product of two primes \( v_1 \) and \( v_2 \); we define the primes \( v'_1 \) and \( v'_2 \) of \( E' \), \( v''_1 \) and \( v''_2 \) of \( E'' \), and \( \tilde{v}_1 \) and \( \tilde{v}_2 \) of \( \tilde{E} \) analogously, and identify \( F \) with \( E_{v_1} \). We let \( \tilde{F} \) denote the completion of \( \tilde{E} \) at \( \tilde{v}_1 \) and define \( F' \) and \( F'' \) analogously; thus the decomposition group \( \text{Gal}(\tilde{F}/F) \) is isomorphic to \( \Gamma \). We assume \( \tilde{F} \) is tamely ramified over \( F \). Under our hypotheses, \( \tilde{F} \) is necessarily unramified over \( F' \).

**Definition 2.3.** An algebraic Hecke character \( \chi \) of \( E' \) is called \( E'/E \)-regular if, for any archimedean place \( \tau \) of \( E \), the components of \( \chi \) at the distinct places of \( E' \) dividing \( \tau \) are distinct.

**Proposition 2.4.** Let \( \chi_v \) be a character of \( (F')^\times \cong W_{F'} \) with the property that \( \text{Ind}_{F'/F} \chi_v \) is an irreducible representation of \( W_F \). Let \( \chi \) be an algebraic Hecke character of \( E' \) with component \( \chi_v \) at \( v'_1 \). Suppose \( \chi = \chi^{-1} \circ c \). Suppose \( \chi \) is \( E'/E \)-regular. Then there is a cuspidal automorphic representation \( \pi \in \text{CU}(n, E) \) which is a weak automorphic induction of \( \chi \). Moreover, the local component \( \pi_{v_1} \) at \( v_1 \) is supercuspidal, and \( \sigma(\pi_{v_1}) \) is equivalent to \( \text{Ind}_{F'/F} \chi_v \), where \( \sigma(\bullet) \) is the correspondence of \([H1]\), discussed in \( \S 1 \). Finally, at every archimedean place \( \tau \) of \( E \) we have

\[
(2.4.1) \quad \pi_\tau \simeq \chi_{\tau_1} \oplus \cdots \oplus \chi_{\tau_n},
\]

where \( \oplus \) denotes Langlands sum and \( \{ \tau_1, \ldots, \tau_n \} \) is the set of places of \( E' \) above \( \tau \).

**Proof.** We first assume \( F'/F \) is totally ramified, and verify the conditions of Lemma 1.6 in this case. Define \( \pi^u = \text{AI}_{\tilde{E}/E_u} \tilde{\chi} \) as in the statement of Lemma 1.6. I claim that the local component of \( \pi^u \) at \( v''_1 \) is supercuspidal. Indeed, there is an unramified character \( \beta \) of \( (F')^\times \) such that \( \chi^*_v = \chi_v \otimes \beta \) is a character of finite order. Since

\[
\text{Ind}_{F'/F}(\chi_v \otimes \beta_v) \cong \text{Ind}_{F'/F}(\chi_v) \otimes \beta_v
\]

we see that \( \text{Ind}_{F'/F} \chi_v^* \) is irreducible. By Corollary 2.2 the stabilizer of \( \tilde{\chi}_v \) in the cyclic group \( \text{Gal}(\tilde{F}/F_u) \cong A \), which is also the stabilizer of \( \tilde{\chi}^*_v \), is trivial; thus \( \text{AI}_{\tilde{F}/F_u} \tilde{\chi}_v \) is supercuspidal \([HH, \text{Proposition 5.5}]\). It follows that \( \pi^u \) is cuspidal. Moreover, it follows as in the proof of Proposition 5 of \([H1]\) that \( \pi^u \cong \pi^\circ c. \) Finally, we verify that \( \pi^u \in \text{Reg}(n, E') \). Note that the hypothesis of \( E'/E \) regularity implies that \( \tilde{\chi} \) is \( \tilde{E}/E' \) regular, in the obvious sense. Let \( \tau \) be an archimedean place of
$E^u$ and let $\nu_j$, $j = 1, \ldots, n$, $n = [\bar{E}/E^u] = |A|$, denote the places of $\bar{E}$ dividing $\tau$. Then the local component of $\pi^u$ at $\tau$ is

\begin{equation}
\tilde{\chi}_{\nu_1} \boxplus \tilde{\chi}_{\nu_2} \boxplus \cdots \boxplus \tilde{\chi}_{\nu_n}
\end{equation}

where $\boxplus$ denotes Langlands sum (cf [H1,(22)]). Thus the regularity of $\pi^u$ is a consequence of the $\bar{E}/E^u$-regularity of $\tilde{\chi}$.

It follows that $\pi^u \in \mathcal{CU}(n,E^u)$, and that its component at $v^u_i$, which we denote $\pi^u_{v_i}$, is supercuspidal. Let $\sigma^u$ denote the compatible family of $\lambda$-adic representations of $Gal(\bar{\mathbb{Q}}/E^u)$ weakly associated to $\pi^u$. By Theorem 1.7 the restriction of $\sigma^u$ to a decomposition group of $v^u_i$ is equivalent to $\sigma(\pi_v)$. By Henniart [He4] this is an irreducible representation, hence $\sigma^u$ is a fortiori irreducible. Of course $\sigma^u$ is equivalent to $\text{Ind}_{E^u/E^u}\tilde{\chi}$.

We have thus verified all the conditions of Lemma 1.6, from which the existence of the weak automorphic induction $\pi$ of $\chi$ follows. Moreover, by the construction in Lemma 1.6, the base change to $F^u$ of the local component $\pi_{v_1}$ is the supercuspidal representation $\pi^u_{v_1}$; hence $\pi_{v_1}$ is itself supercuspidal [AC, I.6; this fact is proved but not stated during the proof of Lemma I.6.10]. Next, letting $\sigma$ denote the compatible family weakly associated to $\pi$, it follows from Definition 1.3 (b) that $\sigma(\pi_w) = \text{Ind}_{E^u_{w}/E^u}\sigma(\pi_{v_w}^u)$ for almost all unramified places $w$, where the notation is as in Definition 1.3. The assertion regarding $\sigma(\pi_{v_1})$ thus follows from Chebotarev density, as in the proof of Proposition 5 of [H1]. Finally, (2.4.1) is an immediate consequence of (2.4.2).

This completes the proof of the proposition, provided $F'/F$ is totally ramified. In general, let $F_1/F$ be the maximal unramified extension contained in $F'$, and let $E_1$ be the corresponding extension of $E$ contained in $E'$. The irreducibility of $\text{Ind}_{F'/E}\chi_v$ obviously implies irreducibility of $\text{Ind}_{F'/F_1}\chi_v$, and $E'/E_1$-regularity of $\chi$ follows from $E'/E$-regularity. Thus the above argument shows the existence of the weak automorphic induction $\pi_1 \in \mathcal{CU}(n_1,E_1)$, with $n_1 = [F_1:F] = [E_1:F]$. But $F_1/F$ is unramified, hence cyclic. It follows that $E_1$ is a cyclic extension of $E$, and thus the automorphic induction $\pi = AI_{E_1/E}(\pi_1)$ is defined. Obviously, $\pi$ is a weak automorphic induction of $\chi$, and it remains to show that $\pi_{v_1}$ is supercuspidal, the final assertion following from Chebotarev density as before. Let $v_1$ denote the prime of $E_1$ dividing $v_1$ and let $\pi_{v_1}$ denote the local component of $\pi_1$ at $v_1$. We know from the totally ramified case that $\pi_{v_1}$ is supercuspidal, and that $\sigma(\pi_{v_1})$ is equivalent to $\text{Ind}_{F'/F_1}\chi_v$. It remains to show that $\pi_{v_1}$ has trivial stabilizer in the cyclic group $Gal(E_1/E)$. But the local correspondence $\bullet \mapsto \sigma(\bullet)$ is...
Gal(F_1/F)-equivariant, so it suffices to show that Ind_{F'/F,\chi_v} has trivial stabilizer in Gal(F_1/F). This follows immediately from the irreducibility of Ind_{F'/F,\chi_v}.

**Remark.** The proof of Proposition 2.4 makes use of Theorem 1.7, whose proof has not been published in general. For the purpose of comparing epsilon factors of pairs in the tame case, it will suffice to consider \chi for which the Langlands sum in (2.4.1) is a cohomological representation with the infinitesimal character of the trivial representation, as in [H1, (22)]. In that case, as we have already mentioned, Theorem 1.7 is proved in [H1].

The representations constructed in Proposition 2.4 will be called of monomial type. More general monomial type representations (with more general solvable Galois groups) will be considered in §4.

### 3. Comparison of epsilon factors of pairs

The existence of a global extension \bar{E}/E adapted to \bar{F}/F, as in §2, is guaranteed by [D2, Lemma 4.13]. The fact that E can be taken in the form \E_0 \cdot K_0 and so on follows from an easy approximation argument.

For any finite place w of E' we let U_w denote the unit subgroup of the multiplicative group of the completion E'_w. We let \chi_0^w denote the restriction of \chi_v to the unit subgroup U_{F'} of (F')^\times. Obviously, there exists a Hecke character \gamma of F' of finite order, unramified at \nu'_2 and equal to \chi_v on U_{\nu'_1} \cong U_{F'}. We let \chi(1) = \gamma \cdot \gamma^{-1} \circ c. Then \chi(1) restricts to \chi_v on U_{F'}.

Next, for each archimedean (complex) place \nu of E' we fix a local character \chi_\nu of \C^\times, trivial on \R^\times, such that \chi_{\nu_1} \neq \chi_{\nu_2} whenever \nu_1 and \nu_2 restrict to the same place on E. Let

$$\chi_\infty = \prod_\nu \chi_\nu : (E'_\infty)^\times/(E_{0,\infty})^\times \to \C^\times$$

denote the corresponding infinity type. Extend the character \chi_\infty trivially to a character \delta of (E'_\infty)^\times \times U_{\nu'_1} \times U_{\nu'_2} \cdot (E'_{0,A})^\times \cdot (E')^\times, viewed as a subgroup of the ideles of E'. Let \delta(1) denote any extension of \delta to a Hecke character of E', and let \chi = \chi(1) \cdot \delta(1). Then \chi is an E'/E-regular Hecke character of E', whose restriction to (F')^\times equals \chi_v, up to an unramified twist. Moreover, \chi is trivial on (E'_{0,A})^\times, by construction, hence satisfies \chi = \chi^{-1} \circ c.

Finally, we assume that every local component \chi_\nu is of the form \(z/\bar{z})^{a(\nu)}/, where the a(\nu) are all integers of the same parity. Then if a is any integer congruent to a(\nu) (mod 2) (any \nu), the product \chi \cdot || \cdot ||^a is a Hecke character of type A_0, in Weil’s terminology (also called a motivic Hecke character).
In order to obtain cohomological representations of $GL(n)$ by automorphic induction we will take $a(\nu) \equiv n - 1 \pmod{2}$ We note that any $E'/E$-regular infinity type with the given parity condition can be used in this construction. If the infinity type is chosen as in [H1,(23)], where the present $n$ replaces the $d$ of [H1] and $n$ of [loc. cit.] is taken to equal 1 – the hypothesis of [loc. cit.] that $E'/E$ is cyclic is irrelevant to the present construction – we obtain automorphic representations contributing to cohomology with trivial coefficients.

More generally, suppose $F_1/F$ and $F_2/F$ are two finite tame extensions; let $F'$ denote their compositum and $\tilde{F}'$ its Galois closure. Then we can find totally real global fields $	ilde{E}'_0 \supset E'_0 = E_{1,0} \cdot E_{2,0} \supset E_0$ with $Gal(\tilde{E}'/E) \cong Gal(\tilde{F}/F)$, as before. We choose an imaginary quadratic field $K_0$ in which $p$ splits and define $E = E_0 \cdot K_0$, $E_1 = E_{1,0} \cdot K_0$, and so on. The above argument yields:

**Lemma 3.1.** Let $F$ be a $p$-adic field and let $F_1$ and $F_2$ be finite tame extensions of $F$ of degree $n_1$ and $n_2$, respectively. Let $\chi_i$ be characters of $G_{F_i}$, such that $\text{Ind}_{F_i/F} \chi_i$ is irreducible, $i = 1, 2$. Then there exists a global CM field $E = E_0 \cdot K_0$, with $K_0$ imaginary quadratic and $E_0$ totally real, and cuspidal automorphic representations $\Pi_i \in \mathcal{CU}(n_i, E)$, $i = 1, 2$, with the following properties:

(a) Both $\Pi_1$ and $\Pi_2$ are of monomial type;

(b) There is a place $v_1$ of $E$ with $E_{v_1} \cong F$, and the local constituents of $\Pi_1$ and $\Pi_2$ at $v_1$ are supercuspidal.

(c) Let $\Sigma_i$ be the compatible family of $\lambda$-adic representations weakly associated to $\Pi_i$, $i = 1, 2$. There are unramified characters $\beta_i$ of $F_i^\times$, $i = 1, 2$, so that the restriction of $\Sigma_i$ to a decomposition group of $v$ is equivalent to $(\text{Ind}_{F_i/F} \chi_i) \otimes \beta_i$, $i = 1, 2$.

We now come to the main theorem of this paper.

**Theorem 3.2.** Let $n_1$ and $n_2$ be two positive integers prime to $p$. Let $\pi_1$ and $\pi_2$ be supercuspidal representations of $GL(n_1, F)$ and $GL(n_2, F)$, respectively. Let $\sigma_i = \sigma(\pi_i)$, $i = 1, 2$, be the corresponding irreducible representations of $W_F$. Then we have an equality of local factors:

$$(3.2.1) \quad \epsilon(\pi_1 \times \pi_2, s, \psi, dx) = \epsilon(\sigma_1 \otimes \sigma_2, s, \psi, dx).$$

Here $\psi$ is an arbitrary additive character of $F$, $dx$ is a Haar measure on $F$, self-dual relative to $\psi$, the local factor on the right is that defined in [JPS], and that on the left is the one defined in [D9].
In particular, if \( m \leq n < p \), and if \( \pi_1 \) and \( \pi_2 \) are irreducible admissible representations of \( GL(n, F) \) and \( GL(m, F) \), respectively, then the equality (3.2.1) holds, where \( \sigma \) is extended to general irreducible admissible representations by the procedure of [BZ]. The map \( \pi \mapsto \sigma(\pi) \) is the only bijection \( A_0(n, F) \leftrightarrow G_0(n, F) \) with this property for \( n < p \).

**Proof.** By the results of Koch and Zink recalled in §2, we may assume \( \sigma_i \) to be of the form \( \text{Ind}_{F_i/F} \chi_i \), \( i = 1, 2 \) with \( \chi_i \) as in Lemma 3.1. Let \( \Pi_1 \) and \( \Pi_2 \) be the cuspidal automorphic representations whose existence is asserted in Lemma 3.1, and let \( \Sigma_1 \) and \( \Sigma_2 \) be the weakly associated compatible \( \lambda \)-adic families. Then we have an identity of partial \( L \)-functions

\[
L^S(\Pi_1 \times \Pi_2, s) = L^S(\Sigma_1 \times \Sigma_2, s),
\]

where \( S \) is a finite set of places. Now Lemma 3.1 (a) implies that the right-hand side of (3.2.2) is the partial \( L \)-function of a complex representation of the Weil group of \( E \). Thus the completed \( L \)-function \( L(\Sigma_1 \times \Sigma_2, s) \) satisfies a functional equation of the form

\[
L(\Sigma_1 \times \Sigma_2, s) = \epsilon(\Sigma_1 \times \Sigma_2, s)L(\tilde{\Sigma}_1 \times \tilde{\Sigma}_2, 1 - s),
\]

where \( \epsilon(\Sigma_1 \times \Sigma_2, s) \) is the product of the local factors of Langlands and Deligne. On the other hand, the left-hand side satisfies

\[
L(\Pi_1 \times \Pi_2, s) = \epsilon(\Pi_1 \times \Pi_2, s)L(\tilde{\Pi}_1 \times \tilde{\Pi}_2, 1 - s),
\]

where \( \epsilon(\Sigma_1 \times \Sigma_2, s) \) is the product of the local factors of [JPS]. Moreover, (2.4.1) implies that the archimedean \( L \) and \( \epsilon \)-factors of \( \Sigma_1 \times \Sigma_2 \) and \( \Pi_1 \times \Pi_2 \) are equal. It then follows from [He1, Theorem 4.1] that, for any non-trivial additive character \( \psi \) of \( F \), we have

\[
\gamma(\Sigma_{1,v_1} \times \Sigma_{2,v_1}, s, \psi) = \gamma(\Pi_{1,v_1} \times \Pi_{2,v_1}, s, \psi).
\]

Here the subscript \( v_1 \) designates the local factor at \( v_1 \), and \( \gamma(\bullet, s, \psi) \) is the local “gamma” factor

\[
\gamma(\Pi_{1,v_1} \times \Pi_{2,v_1}, s, \psi) = \frac{\epsilon(\Pi_{1,v_1} \times \Pi_{2,v_1}, s, \psi) \cdot L(\tilde{\Pi}_{1,v_1} \times \tilde{\Pi}_{2,v_1}, 1 - s)}{L(\Pi_{1,v_1} \times \Pi_{2,v_1}, s)},
\]

with the analogous formula when the \( \Pi \)'s are replaced by \( \Sigma \)'s.

Now it follows from Lemma 3.1 (b) and Theorem 1.7 that \( \Sigma_{i,v_1} = \sigma(\Pi_{i,v_1}) \). Lemma 3.1(c) implies that \( \Sigma_{i,v_1} = \sigma_i \otimes \beta_i \), so that \( \Pi_{i,v_1} = \sigma_i \otimes \beta_i \), since the
correspondence \( \sigma(\bullet) \) commutes with twists by characters. Let \( s_0 \) be a complex number such that \( \beta_1 \cdot \beta_2 = \lvert \bullet \rvert^{s_0} \), where \( \lvert \bullet \rvert \) is the absolute value character. Thus we have the identity

\[
\gamma(\sigma_1 \times \sigma_2, s + s_0, \psi) = \gamma(\pi_1 \times \pi_2, s + s_0, \psi).
\]

Since \( \pi_1 \) and \( \pi_2 \) are taken to be supercuspidal, the \( L \)-factors in the numerator and denominator of the right-hand side of (3.2.4) have no common poles (and indeed are both trivial unless \( n_1 = n_2 \) and \( \pi_1 = \pi_2 \)). The corresponding fact holds for the left-hand side, because \( \sigma_1 \) and \( \sigma_2 \) are irreducible. The equality (3.2.1) then follows as in [He1, §4] and [LRS, p. 318]. The final assertions are then immediate from the additivity properties of the local factors and from [He3].

4. Remarks on the general case

We have already noted that [BZ] provides an extension of \( \pi \mapsto \sigma(\pi) \) to a bijection \( \mathcal{A}(n, F) \leftrightarrow \mathcal{G}(n, F) \), for all \( n \). We again denote this bijection \( \pi \mapsto \sigma(\pi) \). The inverse bijection is denoted \( \sigma \mapsto \pi(\sigma) \). If \( n = n_1 + \cdots + n_r \) is a partition of \( n \), and if \( \pi_i \in \mathcal{A}_0(n_i, F) \), for \( 1 \leq i \leq r \), then we write \( \pi_1 \boxplus \cdots \boxplus \pi_r \in \mathcal{A}(n, F) \) (Langlands sum) for the inverse image under \( \sigma \) of \( \sigma(\pi_1) \oplus \cdots \oplus \sigma(\pi_r) \):

\[
\pi_1 \boxplus \cdots \boxplus \pi_r = \pi(\sigma(\pi_1) \oplus \cdots \oplus \sigma(\pi_r)).
\]

Let \( \mathcal{G}_{ss}(n, F) \subset \mathcal{G}(n, F) \) denote the subset of representations of the Weil group (i.e., representations without monodromy operator), and let \( \mathcal{A}_{ss}(n, F) \subset \mathcal{A}(n, F) \) denote the corresponding subset. Then \( \mathcal{A}_{ss}(n, F) \) can be described as the set of irreducible admissible representations of \( GL(n, F) \) obtained as Langlands sums of supercuspidals, and \( \bigcup_n \mathcal{A}_{ss}(n, F) \) (disjoint union) becomes a monoid under Langlands sum. We let \( RG(F) \) denote the Grothendieck group of representations of \( W_F \). Then \( \sigma \) places \( RG(F) \) in bijection with the group completion \( RA(F) \) of the monoid \( \bigcup_n \mathcal{A}_{ss}(n, F) \).

The local \( \epsilon \)-factors \( \epsilon(\sigma \times \sigma', s, \psi) \) attached to pairs of Weil group representations are additive in each of the two factors \( \sigma, \sigma' \), with respect to direct sums [D2, Theorem 4.1]. The same is true of the local \( L \)-factors, hence of the local gamma-factor of pairs discussed in the proof of Theorem 3.2. Similarly, and by design, the local \( \epsilon \)-factors \( \epsilon(\pi \times \pi', s, \psi) \) of [JPS], attached to \( \pi \in \mathcal{G}(n, F) \) and \( \pi' \in \mathcal{G}(m, F) \), are additive in \( \pi \) and \( \pi' \) with respect to Langlands sums; the same is true of the local gamma factors. It thus follows that \( \epsilon(\pi \times \pi', s, \psi) \) (resp. \( \gamma(\pi \times \pi', s, \psi) \)) extends...
to a function on $RA(F) \times RA(F)$ with values in the multiplicative group of entire (resp. meromorphic) functions on the complex line; here $\psi$ is viewed as fixed.

Let $\sigma \in G(n, F)$, $\sigma' \in G(m, F)$, and define

\begin{align*}
(4.1) \quad \epsilon_A(\sigma \times \sigma', s, \psi) &= \epsilon(\pi(\sigma) \times \pi(\sigma'), s, \psi),
\end{align*}

with the right-hand side defined as in [JPS]. In this way, the automorphic local $\epsilon$-factor of pairs defines a function on pairs of representations of $W_F$, and indeed on $RG(F) \times RG(F)$. Thus the correspondence $\pi \mapsto \sigma(\pi)$ qualifies as the conjectured local Langlands correspondence provided

\begin{align*}
(4.2) \quad \epsilon_A(\sigma \times \sigma', s, \psi) &= \epsilon(\sigma \times \sigma', s, \psi)
\end{align*}

for all pairs $(\sigma, \sigma')$ as above.

With $F$ fixed, we let $G^0(F)$ denote the set of pairs $(F', \chi)$, where $F'$ is a finite extension of $F$ and $\chi$ is a character of $W_{F'}$, or, equivalently, of $(F')^\times$. Let $R^0(F)$ denote the free abelian group on the elements of $G^0(F)$. There is a natural homomorphism

\begin{align*}
\phi : R^0(F) &\to RG(F)
\end{align*}

declared on generators by

\begin{align*}
(4.3) \quad \phi((F', \chi)) &= \text{Ind}_{F'/F} \chi \in G_{ss}([F': F], F).
\end{align*}

By Brauer’s theorem, $\phi$ is surjective. The following lemma is thus an immediate consequence of our earlier remarks on multiplicativity of $\epsilon$-factors.

**Lemma 4.4.** Suppose, for every pair $((F', \chi_1), (F'', \chi_2)) \in G^0(F) \times G^0(F)$ we have

\begin{align*}
\epsilon_A(\text{Ind}_{F'/F} \chi_1 \times \text{Ind}_{F''/F} \chi_2, s, \psi) &= \epsilon(\text{Ind}_{F'/F} \chi_1 \times \text{Ind}_{F''/F} \chi_2, s, \psi),
\end{align*}

Then (4.2) holds for all pairs $(\sigma, \sigma').$

**Lemma 4.4.1.** Suppose, for every pair $((F', \chi_1), (F'', \chi_2)) \in G^0(F) \times G^0(F)$ we have

\begin{align*}
(4.4.2) \quad \gamma(\pi(\text{Ind}_{F'/F} \chi_1) \times \pi(\text{Ind}_{F''/F} \chi_2), s, \psi) &= \gamma(\text{Ind}_{F'/F} \chi_1 \times \text{Ind}_{F''/F} \chi_2, s, \psi),
\end{align*}

where the left-hand side is defined by (3.2.4) and the right-hand side is the Galois-theoretic analogue. Then (4.2) holds for all pairs $(\sigma, \sigma')$.

**Proof.** We have to show that (4.4.2) implies (4.2). It suffices to show (4.2) when $\sigma$ and $\sigma'$ are irreducible, and thus $\pi(\sigma)$ and $\pi(\sigma')$ are supercuspidal. As in the
proof of Theorem 3.2, (4.2) then follows from the identity of gamma-factors in the irreducible/supercuspidal case. But (4.4.2) implies the analogue of (4.2) for gamma-factors, by the same argument used to prove Lemma 4.4.

Let \((F', \chi) \in G^0(F)\), with \([F' : F] = n\), and let \(\tilde{F}\) be the Galois closure of \(F'\) over \(F\). Let \(E'/E\) be an extension of CM fields as in Theorem 1.2 (ii), with \(E\) containing the imaginary quadratic field \(K_0\), in which \(p\) splits, and let \(\tilde{E}\) be the Galois closure of \(E'\) over \(E\). We choose complex conjugate places \(v_1\) and \(v_2\) of \(E\) dividing \(p\), as in \(\S 2\), so that \(E_{v_1} \cong F\). We assume that there is exactly one prime \(\tilde{v}_1\) (resp. \(\tilde{v}_2\)) of \(\tilde{E}\) dividing \(v_1\) (resp. \(v_2\)), and that the isomorphism \(E_{v_1} \sim \to F\) extends to an isomorphism \(\tilde{E}_{\tilde{v}_1} \sim \to \tilde{F}\). Let \(v'_i\), \(i = 1, 2\), denote the corresponding primes of \(E'\), so that \(E'_{v'_i} \cong F'\).

The strategy for proving (4.4.2) is analogous to that used above in the tame case. Our goal is to embed \(\chi\) as the local component of an \(E'/E\)-regular Hecke character \(\chi\) (note change in notation) for which we can construct a weak automorphic induction \(\pi \in \mathcal{CU}(n, E)\). This is more involved than in the tame case, since the local Galois groups are not usually so simple, but the general principle is the same. What is missing to complete the argument is the analogue of Theorem 1.7. More precisely, it will generally not be the case that the local component \(\pi_{v_1}\) is supercuspidal. Thus we have no information about the relation between \(\sigma(\pi_{v_1})\), defined by extension of the supercuspidal/irreducible correspondence, and \(\text{Ind}_{F'/F} \chi\).

Since \(\text{Ind}_{F'/F} \chi\) is no longer assumed to be irreducible, our techniques have to be modified. We assume there is a second pair \(v(*)_1\) and \(v(*)_2\) of complex conjugate places of \(E\) dividing \(p\), and that each \(v(*)_i\), \(i = 1, 2\), is divisible by exactly one prime \(\tilde{v}(*)_i\) of \(\tilde{E}\), with compatible isomorphisms

\[ E_{v(*), 1} \sim \to F, \quad \tilde{E}_{\tilde{v}(*)_1} \sim \to \tilde{F}. \]

This can be arranged by replacing \(E\) by its compositum with a real quadratic field in which \(p\) splits. Define \(v(*)'_i, i = 1, 2\), as before, so that \(E'_{v(*)'_i} \cong F'\).

**Definition 4.5.** Let \(\chi(*)\) be a complex-valued character of \(W_{F'}\), or equivalently a character of \(F'^{\times}\). We say that \(\chi(*)\) is in general position (relative to \(F\)) if there is a sequence \(\tilde{F} = F_0 \supset F_1 \supset \cdots \supset F_{t-1} \supset F_t = F\) such that

(i) For all \(i\), \(F_i/F_{i+1}\) is a cyclic extension of prime degree;

(ii) The Mackey constituents of \(\text{Ind}_{F'/F} \chi(*)\) restricted to \(W_{F_i}\) are all irreducible, for \(i = 0, \ldots, t\).

The Mackey constituents have been defined in the Notation section. For \(a \in A_i = W_i \setminus W_i / W_i\), we let \(I(a, i, \chi(*)_i) = \text{Ind}_{W_i} W_i a \chi(*)_i\) denote the corresponding Mackey constituent.
Mackey constituent, where \(a(F')\) is viewed as a subfield of \(\bar{F}\). Let \(n(a, i) = [F_i : F_i \cap a(F)]\) and let \(\pi(a, i, \chi(\ast)) \in \mathcal{A}_0(n(a, i), F_i)\) denote the supercuspidal representation that corresponds to \(I(a, i, \chi(\ast))\).

Let \(\sigma(\ast) = \text{Ind}_{F'/F} \chi(\ast) = I(1, t, \chi(\ast))\). If \(\chi(\ast)\) is in general position then \(\sigma(\ast)\) is irreducible and corresponds to a supercuspidal representation \(\pi(\ast) = \pi(\sigma(\ast))\) of \(GL(n, F)\). Moreover, for each \(i\) we have

\[
(4.6) \quad BC_{F_i/F_{i+1}} \circ BC_{F_{i+1}/F_{i+2}} \circ \cdots \circ BC_{F_{t-1}/F} \pi(\ast) \cong \bigoplus_{a \in A_i} \pi(a, i, \chi(\ast)).
\]

The proof of the following lemma, which is probably well known to experts, is postponed to the end of this section.

**Lemma 4.7.** Let \(F'/F\) be an extension of local fields. There exist characters of \(F'^{\times}\) of arbitrarily large conductor that are in general position relative to \(F\).

Now we can prove the following extension of Proposition 2.4:

**Proposition 4.8.** Let \(\chi\) and \(\chi(\ast)\) be characters of \(W_{F'}\), with \(\chi(\ast)\) in general position relative to \(F\). Let \(\chi\) be an \(E'/E\)-regular algebraic Hecke character of \(E'\) with components \(\chi\) and \(\chi(\ast)\) at \(v'_1\) and \(v(\ast)'_1\), respectively, and such that \(\chi = \chi^{-1} \circ c\). Then there is a cuspidal automorphic representation \(\Pi \in \mathcal{C}U(n, E)\) which is a weak automorphic induction of \(\chi\). Moreover, the local component \(\Pi_{v(\ast)_1}\) at \(v(\ast)_1\) is supercuspidal.

**Proof.** As before, when referring to representations of global Weil groups we use the language of complex representations and their associated \(\lambda\)-adic families interchangeably.

The isomorphism \(\text{Gal}(\bar{E}/E) \sim \text{Gal}(\bar{F}/F)\) provides a collection of subfields \(E_i \subset \bar{E}\) with \(\text{Gal}(ar{E}/E_i) = \text{Gal}(\bar{F}/F_i), i = 0, \ldots, t\). We prove for each \(i\) the existence of an isobaric representation \(\Pi_i \in \mathcal{C}U(n, E_i)\) whose weakly associated \(\lambda\)-adic family is given by the restriction to \(\text{Gal}(\bar{E}_i/E_i)\) of \(\text{Ind}_{E'/E} \chi\). Note that \(A_i\) gives a set of representatives of \(\text{Gal}(ar{E}/E') \backslash \text{Gal}(ar{E}/E)/\text{Gal}(ar{E}/E_i)\). Thus the Mackey constituents of \(\text{Ind}_{E'/E} \chi\), restricted to \(\text{Gal}(ar{E}_i/E_i)\), are parametrized by \(A_i\). For \(a \in A_i\) we let \(I(a, i, \chi)\) denote the corresponding Mackey constituent.

I claim that, by induction on \(t\), I may assume that \(I(a, i, \chi)\) is weakly associated to a cuspidal automorphic representation \(\Pi(a, i) \in \mathcal{C}U(n(a, i), E_i)\), for \(i = 0, \ldots, t - 1\) and all \(a \in A_i\), with local component supercuspidal at the prime dividing \(v(\ast)_1\). Indeed, this is clear for \(i = 0\), since in this case \(n(a, i) = 1\) for all \(a \in A_i\). Moreover, the degree \([E_0 : E_i]\) is a proper divisor of \([E_0 : E]\) for \(i < t\). To prove the claim, it thus suffices to show that, for all \(a \in A_i\), the Hecke character \(c(\chi) \circ N\).
of $a(E') \cdot E_i$ satisfies the same hypotheses relative to $E_i$ as did $\chi$ relative to $E$, namely:

(i) $a(\chi) \circ N_{a(E') \cdot E_i/a(E')} = a(E') \cdot E_i/E_i$-regular;

(ii) $a(\chi) \circ N_{a(E') \cdot E_i/a(E')} = [a(\chi) \circ N_{a(E') \cdot E_i/a(E')}]^{-1} \circ c$;

(iii) The local constituent of $a(\chi) \circ N_{a(E') \cdot E_i/a(E')}$ at the prime dividing $v(*)_1$ is in general position relative to $F_i$.

Of these hypotheses, (i) and (ii) are obvious, and (iii) follows from the hypothesis of general position for $\chi(*s)$, since the Mackey constituents of the restriction of $I(a, i, \chi)$ to $\text{Gal}(\overline{E_j}/E_j)$, for $j < i$, are among the Mackey constituents of $\text{Ind}_{E'/E} \chi$, restricted to $\text{Gal}(\overline{E_j}/E_j)$ (“transitivity of restriction”).

**Remark.** The preceding argument makes use of the full strength of Theorem 1.7, in that the infinite component of $\Pi(a, i)$ is a priori an arbitrary representation of cohomological type. For any archimedean place $\tau$ of $E$ it is possible to choose the infinity type of $\chi$ so that, for a specific choice of place $\tau_i$ dividing $\tau$, each $\Pi(a, i)_{\tau_i}$ is a cohomological representation with the infinitesimal character of a one-dimensional representation; i.e., an abelian twist of the Langlands sum considered in [H1,(22)]. However, I see no way to control the local components of $\Pi(a, i)$ at the other primes of $E_i$ dividing $\tau$.

Thus by induction we may define

$$\Pi_{t-1} = \bigoplus_{a \in A_{t-1}} \Pi(a, t-1) \in \mathcal{CU}(n, E_{t-1}),$$

whose weakly associated $\lambda$-adic family $\Sigma_{t-1}$ is given by the restriction to $\text{Gal}(\overline{E_{t-1}}/E_{t-1})$ of $\text{Ind}_{E'/E} \chi$. Now $E_{t-1}/E$ is cyclic of prime degree $q$, say, and $\text{Gal}(\overline{E}/E)$ acts transitively on the right on $A_{t-1}$ ($\text{Gal}(\overline{E}/E_{t-1})$ is a normal subgroup of $\text{Gal}(\overline{E}/E)$). Moreover, $\Sigma_{t-1}$ is the restriction of a $\lambda$-adic family of representations of $\text{Gal}(\overline{E}/E)$, hence is $\text{Gal}(E_{t-1}/E)$-invariant. The weakly associated isobaric representation $\Pi_{t-1}$ is thus also $\text{Gal}(E_{t-1}/E)$-invariant, by strong multiplicity one.

Thus there are two possibilities. If $A_{t-1}$ has $q$ elements then it is a principal homogeneous space under $\text{Gal}(E_{t-1}/E)$. Choosing a basepoint $e \in A_{t-1}$ and denoting by $\alpha$ a generator of $\text{Gal}(E_{t-1}/E)$, we thus have

$$\Pi_{t-1} = \bigoplus_{a \in A_{t-1}} \Pi(e, t-1) = \Pi(e, t-1) \boxplus \alpha(\Pi(e, t-1)) \boxplus \cdots \boxplus \alpha^{q-1}(\Pi(e, t-1)).$$

Then $\Pi = \text{Al}_{E_{t-1}/E} \Pi(e, t-1)$ is the unique automorphic representation of $GL(n, E)$ that base changes to $\Pi_{t-1}$ [AC, Lemma III.6.4]. Moreover, it follows from the irreducibility of $\text{Ind}_{E'/E} \chi(*)$ that the supercuspidal representations $\alpha_j(\Pi(e, t-1))$, among
are all distinct, hence [HH, Proposition 5.5] that the local component $\Pi_{v(*)_1}$ is supercuspidal. It is thus clear that $\Pi$ is a weak automorphic induction of $\chi$.

It remains to consider the case in which $A_{t-1}$ is a singleton $e$. Then $\Pi_{t-1} = \Pi(e, t-1)$ is already cuspidal and has $q$ distinct descents to cuspidal automorphic representations of $GL(n, E)$. On the other hand, $I(e, t-1, \chi)$ has $q$ distinct extensions to irreducible $\lambda$-adic families of representations of $\text{Gal}(\overline{E}/E)$. Thus we conclude by applying the argument used to prove Lemma 1.6.

To continue, we need to state a conjecture.

**Conjecture 4.9.** Let $E$ be a CM field as in Theorem 1.2 (ii) and let $\Pi \in \mathcal{CU}(n, E)$. Let $\Sigma$ be the $\lambda$-adic family weakly associated to $\Pi$, and let $w$ be a finite place of $E$. Let $\Sigma_w$ denote the restriction of $\Sigma$ to a decomposition group at $w$. Then

$$\Sigma_w \cong \sigma(\Pi_w).$$

This conjecture was proved by Carayol when $n = 2$ [Ca], with $\sigma(\Pi_w)$ given by the local Langlands correspondence. In the general case, it can be translated into a problem about bad reduction of the Shimura varieties considered in [C2] and [H1]. Translation into a precise problem is probably the main step in proving the conjecture. Theorem 1.2 states that the conjecture is true for almost all unramified places $w$, and Theorem 1.7 states that it is true at supercuspidal places. A weaker version, probably much easier to prove, is sufficient for our purposes:

**Conjecture 4.9 bis.** Under the hypotheses of Conjecture 4.9, let $\Sigma_{w, ss} \in G_{ss}(n, F)$ denote the restriction of $\Sigma_w$ to $W_F$ (i.e., forget about monodromy), and define $\sigma(\Pi_w)_{ss}$ analogously. Then

$$\Sigma_{w, ss} \cong \sigma(\Pi_w)_{ss}.$$ 

**Proposition 4.10.** Suppose Conjecture 4.9 bis. Then the identity of $\epsilon$-factors (4.2) holds for all pairs $(\sigma, \sigma')$.

In other words, the local Langlands conjecture would follow from Conjecture 4.9 bis. We will see from the proof that it suffices to know Conjecture 4.9 bis when $\Sigma_w$ is a *monomial* representation.

**Proof.** It follows from Lemma 4.4.1 that it suffices to prove the identity of local gamma-factors (4.4.2) for pairs of monomial representations $(\phi(F_1, \chi_1), \phi(F_2, \chi_2))$. Now consider the analogue of Lemma 3.1 in our situation: there exists a global CM field $E = E_0 \cdot K_0$ as in Lemma 3.1, CM extensions $E_1$ and $E_2$ of $E$, $E_i/E$-regular Hecke characters $\chi_i$, and cuspidal automorphic representations $\Pi_i \in \mathcal{CU}(n, F)$ such...
that $\Pi_i$ is a weak automorphic induction of $\chi_i$, $i = 1, 2$, satisfying the hypothesis of Lemma 3.1 (c). This generalization of Lemma 3.1 follows from Proposition 4.8, by the argument preceding Lemma 3.1. Then the argument used to prove Theorem 3.2 yields the identity

$$\gamma(\Pi_{1,v_1} \times \Pi_{2,v_1}, s, \psi) = \gamma(\phi(F_1, \chi_1) \times \phi(F_2, \chi_2), s, \psi),$$

By Conjecture 4.9 bis we can replace $\Pi_{1,v_1} \times \Pi_{2,v_1}$ by $\pi(\phi(F_1, \chi_1)) \times \pi(\phi(F_2, \chi_2))$ on the left-hand side. This has the effect of transforming (4.11) into (4.4.2) and thus completes the proof.

It remains to prove Lemma 4.7. We will construct characters $\chi(\ast)$ of $F^{\times,\times}$ of finite order that are in general position relative to $F$, with arbitrarily large conductor. In particular, in the Galois-theoretic notation of §2, we write $G = \Gamma_F$, $H = \Gamma_{F'}$, $\tilde{G} = \Gamma_{\tilde{F}}$; $\chi(\ast)$ will be viewed equivalently as a character of $F^{\times,\times}$ or of $H$. We first describe a sufficient condition for $\text{Ind}_{H}^{G} \chi(\ast)$ to be irreducible. Let $A \subset G$ be a set of representatives for $H \backslash G/H$, with $e$ the representative for the identity double coset. By Clifford-Mackey theory, for $\text{Ind}_{H}^{G} \chi(\ast)$ to be irreducible it is necessary and sufficient that, for all $a \in A$, $a \neq e$, $\chi(\ast)$ and $a(\chi(\ast))$ have distinct restrictions to $H \cap aHa^{-1}$. In particular, letting $\tilde{\chi}(\ast) = \chi(\ast) \circ N_{F/F'}$, a sufficient condition for irreducibility of $\text{Ind}_{H}^{G} \chi(\ast)$ is that $H$ is the stabilizer in $G$ of $\tilde{\chi}(\ast)$:

$$H = \{ g \in G | g(\tilde{\chi}(\ast)) = \tilde{\chi}(\ast) \}.$$ 

On the other hand, let $K$ be any subgroup of $G$ containing $\tilde{G}$. Recall the explicit description of the Mackey constituents of $\text{Ind}_{H}^{G} \chi(\ast)$, restricted to $K$:

$$\text{Ind}_{aHa^{-1}}^{K} a(\chi(\ast)), \ a \in H \backslash G/K.$$ 

It follows easily from Clifford-Mackey theory that (4.12) is a sufficient condition for $\chi(\ast)$ to be in general position relative to $F$.

So it suffices to construct characters $\chi(\ast)$ with arbitrarily large conductor satisfying (4.12). Let $X(F')$, resp. $X(\tilde{F})$ denote the groups of characters of $F^{\times,\times}$, resp. of $\tilde{F}^{\times}$, with values in $\mathbb{C}^{\times}$, and let $\nu : X(F') \to X(\tilde{F})$ denote pullback via $N_{\tilde{F}/F'}$. Similarly, let $X_p(F')$ and $X_p(\tilde{F})$ denote the groups of $\mathbb{Z}_p^{\times}$-valued characters, and $\nu_p : X_p(F') \to X_p(\tilde{F})$ the pullback. It follows from class field theory that the image of $\nu$ is of finite index in $X(\tilde{F})^H$. Thus it suffices to show that $X(\tilde{F})^H$ contains characters $\xi$ of arbitrarily large conductor such that

$$H = \{ g \in G | g(\xi) = \xi \}.$$
In fact, it is enough to find characters \( \xi_p \) in \( X_p(\bar{F})^H \) of infinite conductor satisfying (4.13). Indeed, if \( \xi_p \) satisfies (4.13), then so does its reduction modulo \( p^N \):

\[
\xi_p \pmod{p^N} : \bar{F}^\times \rightarrow (\mathbb{Z}/p^N\mathbb{Z})^\times
\]

for sufficiently large \( N \). By further increasing \( N \), we can guarantee that the conductor of \( \xi_p \pmod{p^N} \) is arbitrarily large; then composing with an embedding \( (\mathbb{Z}/p^N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times \), we obtain a complex character with the desired property.

Let \( M \subset \tilde{F} \) denote the image of \( \tilde{F}^\times \) under the \( p \)-adic logarithm map, and let \( \mathcal{O}_F \) denote the ring of integers in \( F \). For any subgroup \( M' \subset M \) we let \( X_p(M') \) denote the group of \( \mathbb{Z}_p^\times \)-valued characters of \( M' \), and \( X^1_p(M') \subset X_p(M') \) the subgroup of characters with values in \( 1 + p\mathbb{Z}_p \). It suffices to find a \( G \)-invariant subgroup \( M' \) of \( M \) of finite index and a character \( \xi \in X^1_p(M') \) satisfying (4.13). Indeed, such a character necessitates has infinite conductor. Multiplication by \( p^h \) for sufficiently large \( h \) defines a \( G \)-equivariant embedding of \( M \) in \( M' \), hence a restriction \( r_h : X^1_p(M')^H \rightarrow X^1_p(M)^H \). The \( X^1_p \) groups are free \( \mathbb{Z}_p \)-modules. Hence \( r_h \) is injective and \( r_h(\xi) \) again satisfies (4.13). The same is therefore true of its inflation to \( X_p(\bar{F})^H \).

But now, by using the normal basis theorem for the Galois extension \( \bar{F}/F \), we see easily that \( M \) contains a subgroup \( M' \) of finite index, isomorphic as \( \mathbb{Z}_p[G/\tilde{G}] \)-module to \( \mathcal{O}_F[G/\tilde{G}] \). By duality,

\[
X^1_p(M') \simeq \text{Hom}(\mathcal{O}_F[G/\tilde{G}], \mathbb{Z}_p)
\]

also contains a subgroup isomorphic as \( \mathbb{Z}_p[G/\tilde{G}] \)-module to \( \mathcal{O}_F[G/\tilde{G}] \). The condition (4.13) is dense in \( \mathcal{O}_F[G/\tilde{G}]^H \), being the complement of a finite union of proper \( \mathcal{O}_F \)-submodules. Thus there are \( p \)-adic characters of \( M' \) of infinite conductor satisfying (4.13). This completes the proof of Lemma 4.7.

**Appendix. Remarks on Theorem 1.7.**

As promised at the end of §1, we sketch how to extend the results of [H1] on the compatibility of the local Galois correspondence with the global correspondence realized on the cohomology of certain Shimura varieties, as needed for the applications of the present paper. We freely make use of the notation and techniques of [H1] and [RZ].

In [H1] we work with a Shimura variety denoted \( S(GG, X_{n-1})_{C(N)} \) attached to the \( \mathbb{Q} \)-group \( GG = GU(B, \alpha) \), the unitary similitude group of a division algebra \( B \) over the CM field \( E \) with involution \( \alpha \) of the second kind. Here \( C(N) \) is a level subgroup, depending on a positive integer \( N \), and assumed to factor as the product
\( \prod_q C(N)_q \) over the rational primes \( q \), and the similitude factor is assumed to be rational. The field \( E \), denoted \( K \) in [H1], is assumed to be of the form \( E_0 \cdot K_0 \) as in Theorem 1.2 (ii), and \( p \) is assumed to split in \( K_0 \) as the product \( p_1 \cdot p_2 \). The primes of \( E \) dividing \( p_i \) are denoted \( v_i^{(j)} \), \( i = 1, 2, j = 1, \ldots, s \), with \( v = v_1^{(1)} \) the distinguished place for which \( E_v = F \). In general, we let \( F_j \) denote the completion of \( E \) at \( v_i^{(j)} \). Then the \( p \)-adic points of \( G_E \) are given by

\[
(A.1) \quad G_E(\mathbb{Q}_p) \cong \prod_j B(F_j)^\times \times \mathbb{Q}_p^\times
\]

(cf. [H1,(2)]).

Starting in \$2\$ of [H1] \( B \) was assumed to be a division algebra at every \( v_i^{(j)} \). Moreover, the \( p \)-level subgroup \( C(N)_p \) was assumed to have a factorization

\[
(A.2) \quad C(N)_p \cong C_{v,0}^{N,0} \times \prod_{j > 1} C^{(j)}_p \times \mathbb{Z}_p^\times,
\]

where \( C_{v,0}^{N,0} \) is a principal congruence subgroup defined in [loc. cit.] but \( C^{(j)}_p \) is assumed maximal for \( j > 1 \). These assumptions were made exclusively to simplify notation, and to be able to refer freely to the discussion in Chapter 6 of [RZ]. Upon closer inspection, [RZ] turns out to assume that \( B \) is split at every \( v_i^{(j)} \) for \( j > 1 \), but the corresponding \( C^{(j)}_p \) are still assumed maximal. In any case, this restriction is irrelevant, and the general case can be found in the literature: implicitly in the earlier chapters of [RZ] and explicitly in [BoZ] and [V]. The non-expert will be bewildered to discover that no two of these three references work with quite the same Shimura datum \((G_E, X_{n-1})\). In [H1] and [RZ] the Shimura variety parametrizes weight \(-1\) Hodge structures; in [BoZ] and [V] the weights are 1 and 0, respectively. Moreover, [BoZ] and [V] both use the full similitude group rather than the group with rational similitude factor. Passage between these points of view is standard for specialists in Shimura varieties and we will say no more on this point.

Now suppose we are given a supercuspidal representation \( \pi_v \) of \( GL(n, F) \) and an automorphic representation \( \pi \) of \( G_E \) of cohomological type whose \( p \)-adic constituent \( \pi_p \) factors with respect to (A.1) as

\[
(A.3) \quad \pi_v \otimes (\bigotimes_{j > 1} \pi_j) \otimes \eta_p.
\]

The reader will verify that, as in [H1, p. 95 ff.], we may assume the character \( \eta_p \) of \( \mathbb{Q}_p^\times \) to be trivial. However, in \$4\$ above we allowed at least one \( \pi_j \) to be non-trivial, and indeed quite general. It needs to be established that the local Galois
representation $\sigma(\pi_v)$ attached to $\pi$ by the recipe of [H1, §3] depends only on $\pi_v$ and not on the other local factors of $\pi$. The argument from $p$-adic uniformization (as in [H1, Proposition 2] and recalled below) shows easily that $\sigma(\pi_v)$ is independent of $\pi_q$ for $q \neq p$. Thus it is a matter of showing that $\sigma(\pi_v)$ depends neither on the invariants of $B(F_j)$ nor on $\pi_j$ for $j > 1$.

The definition of $\sigma(\pi_v)$ has been recalled in the Erratum to [H1]. Replacing the notation $G\pi$ used there by the current notation $\pi$, we have

$$\sigma(\pi_v) = [\hat{\sigma}(\pi_v) \otimes \tilde{\nu}(\pi_v)^{-1}]^*, \quad (A.4)$$

where $*$ denotes contragredient and where

$$\sigma(\pi_v) = [\text{Hom}_{GJ}(H_{c}^{n-1}(\tilde{\mathcal{M}}_N, \overline{Q}_\ell)_{SS(F)}, \pi_p) \otimes GJL(\pi_v)]^{GG}. \quad (A.5)$$

An elementary calculation shows that $\tilde{\nu}(\pi_v)$ may well depend on the character $\eta_p$ (which we have assumed trivial) but is independent of the $\pi_j$ for $j > 1$. Moreover, the factor $GJL(\pi_v)^*$ in (A.5) does not affect the Galois action. It thus remains to show that the dependence of the $Gal(E_v/E_v)$ module

$$\text{Hom}_{GJ}(H_{c}^{n-1}(\tilde{\mathcal{M}}_N, \overline{Q}_\ell), \pi_p) \otimes \nu(\pi_v) \quad (A.6)$$

depends only on $\pi_v$. Here $GJ$ is the group of $p$-adic points of the inner twist $GJ$ of $GG$ used in [H1]:

$$GJ = GL(n, F) \times \prod_{j > 1} B(F_j)^{\times} \times Q_p^{\times}. \quad (A.7)$$

The rigid parameter space $\tilde{\mathcal{M}}_N$ actually depends on the level subgroup $C_p$, which in the present paper we are allowing to vary. More precisely, we take $\tilde{\mathcal{M}}_N$ to be the inverse limit as $\prod_{j > 1} C_p^{(j)}$ shrinks to the trivial group, while $C_{v,0}^{N,0}$ remains fixed. Just as in the appendix to [H1], $\tilde{\mathcal{M}}_N$ is contained in a product $\prod_j \tilde{\mathcal{M}}_j$. Here $\tilde{\mathcal{M}}_1 = \tilde{\mathcal{M}}_{v,N}$, is Drinfeld’s rigid space of level $N$ attached to $F$, normalized as in [H1] (following [RZ]) to include a morphism to $Q_p^{\times}/Z_p^{\times}$ for the similitude factor (polarization). This morphism splits (non-canonically) to yield an isomorphism

$$\tilde{\mathcal{M}}_1 \sim \tilde{\mathcal{M}}_{v,N}^{+} \times Q_p^{\times}/Z_p^{\times}. \quad (A.8)$$

For $j > 1$ there is a non-canonical identification $\tilde{\mathcal{M}}_j \sim B(F_j)^{\times} \times Q_p^{\times}/Z_p^{\times}$, and $\tilde{\mathcal{M}}_N \subset \prod_j \tilde{\mathcal{M}}_j$ can be defined as the subset of the product on which the natural maps to $Q_p^{\times}/Z_p^{\times}$ agree. Thus we can identify

$$\tilde{\mathcal{M}}_N \sim \prod_j \tilde{\mathcal{M}}_{v,N}^{+} \times \prod_{j > 1} B(F_j)^{\times} \times Q_p^{\times}/Z_p^{\times}. \quad (A.9)$$

Here $GJ$ is the group of $p$-adic points of the inner twist $GJ$ of $GG$ used in [H1]:

$$GJ = GL(n, F) \times \prod_{j > 1} B(F_j)^{\times} \times Q_p^{\times}. \quad (A.7)$$

The rigid parameter space $\tilde{\mathcal{M}}_N$ actually depends on the level subgroup $C_p$, which in the present paper we are allowing to vary. More precisely, we take $\tilde{\mathcal{M}}_N$ to be the inverse limit as $\prod_{j > 1} C_p^{(j)}$ shrinks to the trivial group, while $C_{v,0}^{N,0}$ remains fixed. Just as in the appendix to [H1], $\tilde{\mathcal{M}}_N$ is contained in a product $\prod_j \tilde{\mathcal{M}}_j$. Here $\tilde{\mathcal{M}}_1 = \tilde{\mathcal{M}}_{v,N}$, is Drinfeld’s rigid space of level $N$ attached to $F$, normalized as in [H1] (following [RZ]) to include a morphism to $Q_p^{\times}/Z_p^{\times}$ for the similitude factor (polarization). This morphism splits (non-canonically) to yield an isomorphism

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$$\tilde{\mathcal{M}}_N \sim \prod_j \tilde{\mathcal{M}}_{v,N}^{+} \times \prod_{j > 1} B(F_j)^{\times} \times Q_p^{\times}/Z_p^{\times}. \quad (A.9)$$

Here $GJ$ is the group of $p$-adic points of the inner twist $GJ$ of $GG$ used in [H1]:
The first factorization is compatible with the action of $GJ$ via the factorization (A.7); the second factorization groups together the first and last factors of the first. All this is proved just as in the last chapter of [RZ], or as in [BoZ, §1].

Now we recall that $\tilde{\mathcal{M}}_N$ is naturally defined over the field $F^{nr}$, the maximal unramified extension of $F$, but that it is given with a Weil descent datum [RZ, p. 99]

$$\zeta : \tilde{\mathcal{M}}_N \to \phi^* \tilde{\mathcal{M}}_N$$

([RZ, Proposition 6.49]; [BoZ, pp. 33-34]). Here $\phi$ is induced from Frobenius acting on $F^{nr}$. Now $\text{Gal}(F/F^{nr})$ acts trivially on the last two factors of (A.8). Moreover, the explicit calculation of the Weil descent data in [RZ] and [BoZ] shows that, under the second factorization of (A.8), the last term splits off:

(A.9)  
$$\zeta = \zeta_v \times 1 : \tilde{\mathcal{M}}_{v,N} \times \prod_{j>1} B(F_j)^\times \to \phi^*(\tilde{\mathcal{M}}_{v,N}) \times \prod_{j>1} B(F_j)^\times.$$  

In other words, the factor $\prod_{j>1} B(F_j)^\times$ is irrelevant to the Galois representation on $H_{n-1}^c(\tilde{\mathcal{M}}_N, \Omega_{\ell})$. Thus $\sigma(\pi_v)$ really does depend only on $\pi_v$.

The calculations that lead to (A.9) are based on the following considerations. Suppose for definiteness that $B(F_j) \cong GL(n,F_j)$; the more general case is analogous. The moduli space $\tilde{\mathcal{M}}_j$, for $j > 1$, parametrizes pairs consisting of $\mathbb{Q}_p$-homogeneous polarizations and compatible $p$-adic level structures on the $p$-divisible $O_{F_j}$-module $X \times \hat{X}$, where $X$ is the trivial étale $p$-divisible $O_{F_j}$-module of rank $n$ and $\hat{\cdot}$ denotes Cartier dual. Since the polarization is assumed to be compatible with the level structure, this boils down to a pair consisting of an $O_{F_j}$-linear level structure on $X$ and a quasi-isogeny from $\mathbb{G}_m$ to itself. The Galois group $\text{Gal}(F/F_j)$ obviously acts trivially on the level structure on $X$ and by a character on the quasi-isogeny. (The descent datum only becomes effective when the factor $\mathbb{Q}_p^\times / \mathbb{Z}_p^\times$ in (A.8) is replaced by a finite quotient, cf. [H1], pp. 114-115.) This translates directly into (A.9).

REFERENCES

[AC] Arthur, J. and L. Clozel: Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula, Annals of Math Studies 120 (1989).

[BZ] Bernstein, J. and A.V. Zelevinski: Representations of the group $GL(n,F)$, where $F$ is a non-archimedean local field, Russian Math. Surveys 31 1-68 (1976).

[B] Blasius, D: Automorphic forms and Galois representations: some examples, in L. Clozel and J. S. Milne (eds.), Automorphic Forms, Shimura Varieties, and $L$-functions, Perspectives in Mathematics, 10, Vol. II, 1-13 (1999).
[BR] Blasius, D. and J. Rogawski: Motives for Hilbert modular forms, \textit{Invent. Math.}, 114, 55-87 (1993).

[BoZ] Boutot, J.-F. and T. Zink: The \(p\)-adic uniformization of Shimura curves, Preprint 95-107, Universität Bielefeld (1995).

[BF] Bushnell, C. and A. Fröhlich: Gauss sums and \(p\)-adic division algebras, \textit{Lecture Notes in Math.} 987 (1983).

[BHK] Bushnell, C., G. Henniart, and P. Kutzko: Correspondance de Langlands locale pour \(GL_n\) et conducteurs de paires, manuscript (1997).

[Ca] Carayol, H.: Sur les représentations \(\ell\)-adiques associées aux formes modulaires de Hilbert, \textit{Ann. scient. Ec. Norm. Sup}, 19, 409-468 (1986).

[C1] Clozel, L.: Motifs et formes automorphes: applications du principe de fonctorialité, in L. Clozel and J. S. Milne (eds.) \textit{Automorphic Forms, Shimura Varieties, and \(L\)-functions, Perspectives in Mathematics}, 10, Vol. I, 77-160 (1990).

[C2] Clozel, L.: Représentations Galoisiennes associées aux représentations automorphes autoduales de \(GL(n)\), \textit{Publ. Math. I.H.E.S.}, 73, 97-145 (1991).

[C3] Clozel, L.: On the cohomology of Kottwitz’s arithmetic varieties, \textit{Duke Math. J.}, 72, 757-795 (1993).

[CH] Corwin, L. and R. Howe: Computing characters of tamely ramified \(p\)-adic division algebras, \textit{Pacific J. Math.}, 73, 461-477 (1977).

[D1] Deligne, P.: Formes modulaires et représentations de \(GL(2)\), in P. Deligne and W. Kuyk (eds.), Modular functions of one variable, II, \textit{Lecture Notes in Math.} 349, 55-105 (1973).

[D2] Deligne, P.: Les constantes des équations fonctionnelles des fonctions \(L\), in P. Deligne and W. Kuyk (eds.), Modular functions of one variable, II, \textit{Lecture Notes in Math.} 349, 501-597 (1973).

[H1] Harris, M.: Supercuspidal representations in the cohomology of Drinfel’d upper half spaces; elaboration of Carayol’s program, \textit{Invent. Math.}, 129, 75-119 (1997).

[H2] Harris, M.: Galois properties of automorphic representations of \(GL(n)\), in preparation.

[H3] Harris, M.: \(L\)-functions and periods of polarized regular motives, \textit{J. Reine Angew. Math.}, 483, 75-161 (1997).

[He1] Henniart, G.: On the local Langlands conjecture for \(GL(n)\): the cyclic case, \textit{Ann. of Math.}, 123, 145-203 (1986).

[He2] Henniart, G.: La conjecture de Langlands locale numérique pour \(GL(n)\), \textit{Ann. scient. Ec. Norm. Sup}, 21, 497-544 (1988).
[He3] Henniart, G.: Caractérisation de la correspondance de Langlands locale par les facteurs $\epsilon$ de paires, *Invent. Math.*, **113**, 339-350 (1993).

[He4] Henniart, G.: Letter, January 1994.

[HH] Henniart, G. and R. Herb: Automorphic induction for $GL(n)$ (over local non-archimedean fields), to appear.

[JPS] Jacquet, H., I. I. Piatetski-Shapiro, and J. Shalika: Rankin-Selberg convolutions, *Am. J. Math.*, **105**, 367-483 (1983).

[K] Kazhdan, D.: On lifting, in *Lie Group Representations, Lecture Notes in Math.*, **1041**, 209-249 (1984).

[KZ] Koch, H. and E.-W. Zink: Zur Korrespondenz von Darstellungen der Galoisgruppen und der zentralen Divisionsalgebren über lokalen Körpern (Der zahme Fall), *Math. Nachr.*, **98**, 83-119 (1980).

[L1] Langlands, R.P.: On the functional equation of Artin’s L-functions, unpublished manuscript.

[L2] Langlands, R.P.: Automorphic representations, Shimura varieties, and motives. Ein Märchen, in *Automorphic forms, representations, and L-functions, Proc. Symp. Pure Math. AMS*, **33**, part 2, 205-246 (1979).

[LRS] Laumon, G., M. Rapoport, and U. Stuhler, $\mathcal{D}$-elliptic sheaves and the Langlands correspondence, *Invent. Math.*, **113**, 217-338 (1993).

[M] Moy, A.: Local constants and the tame Langlands correspondence, *Am. J. Math.*, **108**, 863-930 (1986).

[RZ] Rapoport, M. and T. Zink: Period spaces for $p$-divisible groups, *Annals of Math. Studies*, **141** (1996).

[T] Taylor, R.: Letter to L. Clozel, December 1991.

[V] Varshavsky, Y. $P$-adic uniformization of unitary Shimura varieties, Manuscript (1995).