Introduction.—The search for a condensed-matter realization of the Majorana fermion continues, motivated both by the underlying fundamental physics and potential technological applications. A promising route to this goal is to engineer a spinless superconducting state with nontrivial topology, e.g. a Kitaev chain, in a superconducting heterostructure. Such a phase is famously predicted to occur in a semiconducting nanowire in proximity contact with a superconductor and in an applied magnetic field. Transport signatures consistent with this were subsequently detected, although the definitive existence of the Majorana mode is still being debated.

Much attention has recently been directed at an alternative proposal, where a topological band arises from the overlapping Shiba states in a chain of magnetic impurities with helical spin order on the surface of a superconductor. The helical Shiba chain proposal has the significant advantage that it should be possible to directly image the Majorana end modes using scanning tunneling microscopy (STM). The helical spin texture, which here plays a critical role combining the effect of the spin-orbit coupling and external field in the nanowire proposal, represents the main experimental difficulty. Not only is it impossible to externally control, but the helical spin chain is generically unstable towards a ferromagnetic or antiferromagnetic configuration when the impurities are deposited on a normal metal. A pair-breaking effect in the superconducting state might nevertheless restore the stability.

These difficulties with the proposal could be avoided, however, if the topology of the Shiba chain relied upon properties of the superconducting host instead of the alignment of the impurity spins. For example, it has been proposed to use external magnetic fields and a supercurrent flow to tune a nontopological antiferromagnetic chain into a topological regime. In view of the importance of SOC in the nanowire proposal, it is also interesting to include SOC in the description of the superconducting host of the impurity chain. Indeed, one generically expects the presence of a Rashba SOC at the surface of the superconductor due to the broken inversion symmetry. We note that SOC intrinsic to the superconductor has been considered in other proposals for realizing topological systems, and the possible relevance of SOC in the context of Majorana modes in magnetic chains has been mentioned in Ref. This scenario has recently been invoked to explain STM measurements at Princeton University of zero-bias peaks at the ends of a ferromagnetic wire on a superconductor, although the physics of this system may be more similar to the nanowire proposal than a Shiba chain.

In this paper we show that the SOC indeed induces a topological state in a Shiba chain formed from ferromagnetically aligned impurity spins, and so demonstrate that the more delicate helical order is not essential to such proposals. To this end, we analytically construct a tight-binding model for the Shiba states valid in the limit of “deep” impurities, when the impurity band lies close to the middle of the superconducting gap. Although the SOC does not affect the Shiba states for an isolated impurity, it dramatically alters the results for the chain. Specifically, spin-flip correlations in the bulk superconductor, induced by the antisymmetric SOC, mix the two branches of the impurity band when the polarization of the impurity spins is transverse to the SOC along the chain. This can be interpreted as a triplet pairing amplitude in a Kitaev-like model, and is thus responsible for the topologically nontrivial state. A magnetic polarization parallel to the SOC, on the other hand, produces no such mixing but instead results in an asymmetric dispersion with trivial topology. We construct a phase diagram, demonstrating that a topological state is possible for infinitesimal SOC strength. Our analysis closely follows that of Ref. where a similar tight-binding model for the impurity band was obtained for a chain with spiral
magnetic texture embedded in a three-dimensional superconductor.

Model.—A bulk two-dimensional singlet s-wave superconductor with Rashba SOC is described by the Hamiltonian

$$H = \sum_k \Psi_k^\dagger (\xi_k \delta_0 + l_k \cdot \sigma) + \Delta \hat{\tau}_z \otimes \delta_0) \Psi_k,$$

where $\hat{\tau}_\mu$ are the Pauli matrices in Nambu (spin) space [24], and $\Psi_k = (c_{k+i}, c_{k,0}, c_{k-i}, -c_{k,0})^T$ is the spinor of creation and annihilation operators. The non-interacting dispersion is given by $\xi_k = \hbar^2 k^2 / 2m - \mu$ where $m$ is the effective mass and $\mu$ the chemical potential, the Rashba SOC is parametrized by $l_k = \lambda(k_y e_x - k_x e_y)$ where $\lambda$ is the SOC strength, and $\Delta$ is the superconducting gap.

The SOC lifts the spin degeneracy in the normal state, resulting in the dispersions $\xi_{k,\pm} = \xi_k \pm l_k$ where the plus (minus) sign corresponds to the positive (negative) helicity band. As time-reversal symmetry remains intact, however, in the superconducting phase there is only one pairing between states in the same helicity band. The bulk Green’s function can then be written as $G_k^\pm(\omega) = \frac{1}{2} \left( G_k^\uparrow(\omega) + G_k^\downarrow(\omega) \right)$, where

$$G_k^\pm(\omega) = \left( \omega \hat{\tau}_0 + \xi_{k,\pm} \hat{\tau}_z + \Delta \hat{\tau}_x \right) \otimes \left( \delta_0 \pm \sin \theta \hat{\sigma}_x \mp \cos \theta \hat{\sigma}_y \right) \times \left( \omega^2 - \xi_{k,\pm}^2 - \Delta^2 \right)^{-1},$$

is the Green’s function in each helicity sector. Note that the SOC produces normal spin-flip and triplet pairing terms in the Green’s function [22]. For clarity we suppress the momentum index in the dispersion of the helical bands, i.e. $\xi_{k,\pm} \equiv \xi_{\pm}$.

Single impurity.—We first consider a single (classical) magnetic impurity with spin $\mathbf{S}$ at the origin, interacting with the electron states with exchange strength $-J$. We include this in our model by adding $H_{\text{imp}} = -JS \cdot [\Psi^\dagger(0) \hat{\tau}_0 \otimes \delta \Psi(0)]$ to the bulk Hamiltonian Eq. [1]. We aim to solve the Bogoliubov-de Gennes equation $(H + H_{\text{imp}}) \psi(\mathbf{r}) = \omega \psi(\mathbf{r})$ for the impurity bound states, i.e. for energy $|\omega| < \Delta$. By straightforward manipulation [12], the spinor of the bound state at the impurity $\psi(0)$ satisfies the equation

$$\left\{ \mathbf{1} + \int \frac{d^2k}{(2\pi)^2} G_k^\downarrow(\omega) JS \cdot (\hat{\tau}_0 \otimes \delta) \right\} \psi(0) = 0. \tag{3}$$

To evaluate this equation, we split the Green’s function into positive and negative helicity components and then convert the integral over the momentum to an integral over the appropriate dispersion $\xi_{\pm}$ and the angle $\theta$

$$\int \frac{d^2k}{(2\pi)^2} G_k^\pm(\omega) \approx N_{\pm} \frac{1}{2\pi} \int_D d\xi_{\pm} \int_0^{2\pi} d\theta G_k^\pm(\omega), \tag{4}$$

where $N_{\pm} = (m/\pi \hbar^2)[1 \mp \lambda/(1 + \lambda^2)^{1/2}]$ is the density of states of the $\nu = \pm$ helicity band at the Fermi level, $\lambda = \lambda m / \hbar k_F$ is the dimensionless SOC strength with $k_F$ the Fermi wavevector in the absence of SOC, and $D \to \infty$ is a cutoff. The resulting integrals are presented in the supplemental material [23]. Due to the isotropic $\delta$-function structure of the potential, the integrals involving the spin-flip and triplet pairing terms in the Green’s function vanish, and Eq. [3] therefore has exactly the same form as a magnetic impurity in an s-wave superconductor without SOC [8][12], specifically

$$\left\{ \mathbf{1} - \frac{\alpha}{\sqrt{\Delta^2 - \omega^2}} \left[ \omega \hat{\tau}_0 + \Delta \hat{\tau}_x \right] \otimes (e_S \cdot \sigma) \right\} \psi(0) = 0, \tag{5}$$

where $\alpha = \frac{\pi}{2} (N_+ + N_-) J S, S = |S|$, and $e_S = S/S$. The solutions of this equation occur at $\omega = \pm \omega_0$, where $\omega_0 = \Delta(1 - \alpha^2)/(1 + \alpha^2)$. The form of the corresponding spinors $\psi_{\pm}(0)$ is dictated by the orientation of the impurity spin. Parametrizing $S = S(\cos \eta \sin \zeta, \sin \eta \sin \zeta, \cos \zeta)$, these spinors can then be written [12] up to unimportant normalization constant as

$$\psi_{\pm}(0) = \left( \frac{\chi_{\uparrow}}{\chi_{\downarrow}} \right), \quad \psi_{-}(0) = \left( \frac{\chi_{\downarrow}}{-\chi_{\uparrow}} \right), \tag{6}$$

where

$$\chi_{\uparrow} = (\cos \zeta / 2, e^{i \eta} \sin \zeta / 2)^T, \tag{7}$$

$$\chi_{\downarrow} = (e^{-i \eta} \sin \zeta / 2, -\cos \zeta / 2)^T. \tag{8}$$

Ferromagnetic chain.—The above analysis can be extended to a chain of magnetic impurities, with the impurity Hamiltonian now written as

$$H_{\text{imp}} = -J \sum_j \mathbf{S} \cdot [\Psi^\dagger(r_j) \hat{\tau}_0 \otimes \delta \Psi(r_j)], \tag{9}$$

where $r_j$ is the position of the $j$th impurity. We have suppressed the site index of the spins since they all point in the same direction. Without loss of generality, we assume that the chain runs along the $x$-axis, and so $r_j = x_j \mathbf{e}_x$. After similar manipulations as in the single impurity problem, the BdG equations for the subgap Shiba states on the chain can be written

$$\left\{ \mathbf{1} - \frac{\alpha}{\sqrt{\Delta^2 - \omega^2}} \left[ \omega \hat{\tau}_0 + \Delta \hat{\tau}_x \right] \otimes (e_S \cdot \sigma) \right\} \psi(x_i) = -\sum_{j \neq i} J(x_{ij}) e_S \cdot (\hat{\tau}_0 \otimes \delta) \psi(x_j), \tag{10}$$

where $x_{ij} = x_i - x_j$ and the matrix $J(x_{ij})$ is defined.
\[ J(x_{ij}) = JS \int \frac{d^2k}{(2\pi)^2} \tilde{G}_k(\omega) e^{ik_x x_{ij}} \]
\[ = \frac{JS}{2} \left\{ \left[ I^+_1(x_{ij}) + I^{-1}_1(x_{ij}) \right] \tilde{\sigma}_z \otimes \sigma_0 + \omega [I^+_3(x_{ij}) + I^-_3(x_{ij})] \tilde{\sigma}_z \otimes \sigma_0 + \Delta [I^+_4(x_{ij}) + I^-_4(x_{ij})] \tilde{\sigma}_z \otimes \sigma_y \right\} . \] (11)

We have expressed this in terms of the integrals
\[ I'_1(x) = \frac{N_v}{2\pi} \int_{-D}^{D} dk \int_{0}^{2\pi} d\theta \frac{\xi_k \cos \theta}{\omega^2 - \xi^2 - \Delta^2} , \] (12)
\[ I'_2(x) = \frac{N_v}{2\pi} \int_{-D}^{D} dk \int_{0}^{2\pi} d\theta \frac{\xi_k \cos \theta}{\omega^2 - \xi^2 - \Delta^2} , \] (13)
\[ I'_3(x) = \frac{N_v}{2\pi} \int_{-D}^{D} dk \int_{0}^{2\pi} d\theta \frac{\xi_k \cos \theta}{\omega^2 - \xi^2 - \Delta^2} , \] (14)
\[ I'_4(x) = \frac{N_v}{2\pi} \int_{-D}^{D} dk \int_{0}^{2\pi} d\theta \frac{\xi_k \cos \theta}{\omega^2 - \xi^2 - \Delta^2} , \] (15)

where \( k_{\nu}(\xi) = k_{F,\nu} / \xi / \hbar v_{F,\nu} \), while \( k_{F,\nu} = k_F [(1 + \tilde{\lambda}^2)^{1/2} - \nu \tilde{\lambda}] \) and \( v_{F,\nu} = (\hbar k_F / m) [(1 + \tilde{\lambda}^2)^{1/2} \) are the Fermi vector and velocity for the \( \nu \) helicity band, respectively. These integrals are explicitly evaluated in the supplemental information for \( D \to \infty \), where we also provide asymptotic expansions valid for \( k_{F,\nu} |x| \gg 1 \). Note that \( I'_1(x) \) and \( I'_2(x) \) are even functions of \( x \), whereas \( I'_3(x) \) and \( I'_4(x) \) are odd.

In contrast to the single-impurity system considered above, the presence of SOC makes a significant difference to the BdG equations for the multi-impurity problem: while the first line of Eq. (11) is identical to the result found in Ref. [12], the second line is only present for nonzero SOC. This line contains explicitly magnetic terms \( \times \tilde{\sigma}_y \), reflecting the orientation of the SOC vector \( \mathbf{n}_k || \mathbf{e}_y \) for \( k \) pointing along the magnetic chain.

**Tight-binding model.**–We do not attempt a general solution of Eq. (10), but instead consider the analytically tractable limit of dilute “deep” impurities, as discussed in Ref. [12]. Specifically, we assume that \( \alpha \approx 1 \), so that the energy \( \epsilon_0 \) of the isolated Shiba state lies close to the center of the gap, and that the spacing \( a \) between impurities is sufficiently large that the impurity band formed from the hybridized Shiba states lies entirely within the superconducting gap. Linearizing the BdG equations Eq. (10) in the energy \( \omega \) and the coupling between impurity sites, we obtain after straightforward manipulation

\[ \Delta [\mathbf{e}_x \cdot (\tilde{\sigma}_0 \otimes \tilde{\sigma}) - \alpha \tilde{\tau}_z \otimes \sigma_0] \psi(x) + \sum_{j \neq i} \mathbf{e}_x \cdot (\tilde{\sigma}_0 \otimes \tilde{\sigma}) \lim_{\omega \to 0} J(x_{ij}) \mathbf{e}_x \cdot (\tilde{\sigma}_0 \otimes \tilde{\sigma}) \psi(x_j) = \omega \psi(x_i) \] (16)

This equation is now projected into the Shiba states [Eq. (10)] at each site, to obtain a BdG-type equation for the impurity band

\[ \tilde{H}(i,j) \phi_j = \omega \phi_j \] (17)

where \( \phi_i = (u_{i,+}, u_{i,-})^T \) is the vector of the wavefunctions for the + and - Shiba states at site \( i \) and

\[ \tilde{H}(i,j) = \begin{pmatrix} A_{ij} + B_{ij} & C_{ij} \\ C_{ji} & -A_{ij} + B_{ij} \end{pmatrix} \] (18)

where

\[ A_{ij} = \epsilon_0 \delta_{ij} + \frac{1}{2} JS \Delta^2 \lim_{\omega \to 0} [I^+_4(x_{ij}) + I^-_4(x_{ij})] \] (19)
\[ B_{ij} = \frac{1}{2} JS \Delta^2 \sin \eta \sin \zeta \lim_{\omega \to 0} [I^+_4(x_{ij}) - I^-_4(x_{ij})] \] (20)
\[ C_{ij} = \frac{i}{2} JS \Delta \left( \cos^2 \frac{\xi}{2} + \sin^2 \frac{\xi}{2} e^{-2i\eta} \right) \times \lim_{\omega \to 0} [I^+_2(x_{ij}) - I^-_2(x_{ij})] . \] (21)

Note that the integrals in these expressions are to be regarded as vanishing for \( i = j \).

The effective tight-binding Hamiltonian Eq. (18) is the central result of this paper. Due to the antisymmetry of the integrals \( I'_2(x) \) in the off-diagonal terms, it can be interpreted as describing superconducting spinless fermions, recalling the Kitaev model [11], albeit with long-range hopping and pairing terms. The properties of this system depend crucially on the SOC in the bulk superconductor and the polarization of the impurity spins. Specifically, the pairing term \( C_{ij} \) is only present for nonvanishing SOC, and when the polarization of the ferromagnetic chain has a component perpendicular to the \( y \)-axis. Examining Eq. (11), we observe that the pairing term originates from the spin-flip correlations in the host superconductor induced by the SOC. A polarization component along the \( y \)-axis contributes an antisymmetric hopping \( B_{ij} \) in the presence of SOC. This echoes the asymmetric dispersion in a spin-orbit coupled electron gas in the direction of an applied magnetic field, and its appearance here is due to the triplet pairing correlations in the bulk Green’s function Eq. (2).
A similar tight-binding model was derived in Ref. [12], but there the odd-parity pairing term arose from the spiral magnetic texture of the impurity chain. This mechanism for generating a pairing term is still valid in the presence of the SOC considered here. Examining the interplay of spiral spin texture and SOC is an interesting topic which we leave to later work.

**Topology.**—To conclude we examine the topology of the impurity band. For an infinite chain with uniform spacing $a$ of the impurities, we define the Fourier transform of the Hamiltonian Eq. (18)

$$\tilde{H}(k) = \begin{pmatrix} A(k) + B(k) & C(k) \\ C^{*}(k) & -A(k) + B(k) \end{pmatrix}$$

(22)

where $A(k) = \sum_{j} A_{0j} e^{ikja}$, etc. Using the asymptotic forms for the integrals, it is possible to obtain analytical expressions for these quantities in the limit $k_{F}a \gg 1$, which are presented in the supplemental material [23]. The Hamiltonian Eq. (22) is in Altland-Zirnbauer symmetry class D, and for a fully-gapped system it is therefore characterized by the $\mathbb{Z}_2$ topological invariant [1]

$$Q = \text{sgn}\{A(0)A(\pi/a)\}.$$ (23)

The system is topologically nontrivial for $Q = -1$; conversely, $Q = 1$ indicates a trivial state.

To demonstrate that our model supports a topologically nontrivial state, in Fig. (1) we present a phase diagram as a function of the dimensionless SOC $\tilde{\lambda}$ and the parameter $k_{F}a$, which gives a measure of the Fermi surface volume or alternatively the spacing of the chain. We consider only a polarization in the $x$-$z$ plane. In the topologically non-trivial regions, we plot the minimum gap magnitude, demonstrating the existence of a fully gapped state; the non-topological regions are left white.

The most important aspect of this phase diagram is that a topological state is revealed to be possible even for infinitesimal SOC, although the gap closes as $\tilde{\lambda}$ is tuned to zero. We emphasize that our analysis is only valid for $\epsilon_{0}$ sufficiently close to zero, and so other methods are required to comprehensively survey the phase diagram.

**Summary.**—In this paper we have studied the appearance of a topological impurity band when a ferromagnetic chain of classical spins are embedded in a two-dimensional singlet $s$-wave superconductor with Rashba SOC. To this end, we have derived an effective tight-binding model for the overlapping Shiba states of the impurities. When the spins are polarized perpendicular to the SOC along the chain, an odd-parity pairing term is induced in the effective model, thus realizing a Kitaev-like model with generically non-trivial topology. Our work and recent others [17] explore alternative routes to a topological Shiba chain which do not rely upon helical spin texture [9,10]. This is a significant result, as the stability of the helical spin texture is debated [16,18]. In contrast, the SOC mechanism examined here is intrinsic to the superconductor surface. This implies that topological phases are possible for a much wider variety of impurity spin configurations than hitherto realized, which grants the Shiba-chain proposal additional robustness, leading strong theoretical support to experimental efforts to detect Majorana fermions in such a setting. In particular, we demonstrate explicitly the possibility of emergent Majorana modes in a ferromagnetic chain of impurities embedded on the surface of a bulk superconductor in the presence of SOC.

**Note added.**—It has been brought to our attention [25] that a topological state has been found in a similar model [26] at Princeton University.

**Acknowledgments.**—The authors thank B. A. Bernevig, P. Kotetes, and A. Yazdani for useful discussions. This work is supported by JQI-NSF-PFC.

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SUPPLEMENTAL MATERIAL

Important integrals

We encounter four integrals in our solution of the Shiba problem:

\[ I_1(x) = \frac{N_\nu}{2\pi} \int_{-D}^{D} d\xi \int_0^{2\pi} d\theta \frac{\xi e^{ik_\nu(\xi) x \cos \theta}}{\omega^2 - \xi^2 - \Delta^2}, \]  
(24)

\[ I_2(x) = \frac{N_\nu}{2\pi} \int_{-D}^{D} d\xi \int_0^{2\pi} d\theta \frac{\xi e^{ik_\nu(\xi) x \cos \theta}}{\omega^2 - \xi^2 - \Delta^2}, \]  
(25)

\[ I_3(x) = \frac{N_\nu}{2\pi} \int_{-D}^{D} d\xi \int_0^{2\pi} d\theta \frac{e^{ik_\nu(\xi) x \cos \theta}}{\omega^2 - \xi^2 - \Delta^2}, \]  
(26)

\[ I_4(x) = \frac{N_\nu}{2\pi} \int_{-D}^{D} d\xi \int_0^{2\pi} d\theta \frac{e^{ik_\nu(\xi) x \cos \theta}}{\omega^2 - \xi^2 - \Delta^2}, \]  
(27)

where \( k_\nu(\xi) = k_{F,\nu} + \xi/\hbar v_{F,\nu} \). These integrals are performed by extending the cutoff \( D \to \infty \). We distinguish two cases for the argument: \( x = 0 \) for the isolated Shiba impurity, and \( x \neq 0 \) for the Shiba chain.

**Isolated impurity: \( x = 0 \)**

In this case all the integrals except \( I_3^0(0) \) are vanishing, which evaluates to

\[ I_3^0(0) = \frac{\pi N_\nu}{\sqrt{\Delta^2 - \omega^2}}. \]  
(28)

**Impurity chain: \( x \neq 0 \)**

For \( x \neq 0 \), we first evaluate the integral over \( \xi \) using elementary contour integral methods, and then evaluate the angular integral. We hence find

\[ I_1^0(x) = \pi N_\nu \text{Im} \left\{ J_0((k_{F,\nu} + i\xi_{\nu}^{-1})|x|) + iH_0((k_{F,\nu} + i\xi_{\nu}^{-1})|x|) \right\}, \]  
(29)

\[ I_2^0(x) = -i\pi N_\nu \text{sgn}(x) \text{Re} \left\{ iJ_1((k_{F,\nu} + i\xi_{\nu}^{-1})|x|) + H_{-1}((k_{F,\nu} + i\xi_{\nu}^{-1})|x|) \right\}, \]  
(30)

\[ I_3^0(x) = -\frac{\pi N_\nu}{\sqrt{\Delta^2 - \omega^2}} \text{Re} \left\{ J_0((k_{F,\nu} + i\xi_{\nu}^{-1})|x|) + iH_0((k_{F,\nu} + i\xi_{\nu}^{-1})|x|) \right\}, \]  
(31)

\[ I_4^0(x) = -\text{sgn}(x) \frac{i\pi N_\nu}{\sqrt{\Delta^2 - \omega^2}} \text{Im} \left\{ iJ_1((k_{F,\nu} + i\xi_{\nu}^{-1})|x|) + H_{-1}((k_{F,\nu} + i\xi_{\nu}^{-1})|x|) \right\}, \]  
(32)

where \( J_n(z) \) and \( H_n(z) \) are Bessel and Struve functions of order \( n \), respectively, and \( \xi_{\nu} = \hbar v_{F,\nu}/\sqrt{\Delta^2 - \omega^2} \). Using asymptotic forms \( \text{valid for large values of the argument close to the positive real axis, we can approximate these as} \)

\[ I_1^0(x) \approx \pi N_\nu \sqrt{\frac{2}{\pi k_{F,\nu}|x|}} \sin \left( k_{F,\nu}|x| - \frac{\pi}{4} \right) e^{-|x|/\xi_{\nu}} + \frac{2N_\nu}{k_{F,\nu}|x|}, \]  
(33)

\[ I_2^0(x) \approx i\pi N_\nu \text{sgn}(x) \sqrt{\frac{2}{\pi k_{F,\nu}|x|}} \sin \left( k_{F,\nu}|x| - \frac{3\pi}{4} \right) e^{-|x|/\xi_{\nu}} + \text{sgn}(x) \frac{2iN_\nu}{(k_{F,\nu}|x|)^2}, \]  
(34)

\[ I_3^0(x) \approx -\frac{\pi N_\nu}{\sqrt{\Delta^2 - \omega^2}} \sqrt{\frac{2}{\pi k_{F,\nu}|x|}} \cos \left( k_{F,\nu}|x| - \frac{\pi}{4} \right) e^{-|x|/\xi_{\nu}}, \]  
(35)

\[ I_4^0(x) \approx -\text{sgn}(x) \frac{i\pi N_\nu}{\sqrt{\Delta^2 - \omega^2}} \sqrt{\frac{2}{\pi k_{F,\nu}|x|}} \cos \left( k_{F,\nu}|x| - \frac{3\pi}{4} \right) e^{-|x|/\xi_{\nu}}. \]  
(36)

These forms are valid up to \( O((k_{F,\nu}|x|)^{-3}) \).
Fourier transforms

The Fourier transform of the effective Hamiltonian Eq. (18) can be carried out analytically when we utilize the asymptotic expressions. Defining the Fourier transform as

$$A(k) = \sum_j A_{0j} e^{ikja},$$

we obtain

$$I_{\nu}^{\prime}(k) = N_{\nu} \sqrt{\frac{\pi}{2k F, \nu a}} \left\{ e^{-3\pi i/4} \text{Li}_{1/2} \left( e^{i(k F, \nu a + ka)/\xi_\nu} \right) - e^{3\pi i/4} \text{Li}_{1/2} \left( e^{i(-k F, \nu a + ka)/\xi_\nu} \right) ight. \\
- e^{-3\pi i/4} \text{Li}_{1/2} \left( e^{i(k F, \nu a - ka)/\xi_\nu} \right) - e^{3\pi i/4} \text{Li}_{1/2} \left( e^{i(-k F, \nu a - ka)/\xi_\nu} \right) \bigg\} + 2iN_{\nu} \left( \text{Li}_2 \left( e^{ika} \right) - \text{Li}_2 \left( e^{-ika} \right) \right),$$

$$I_{\nu}^{\prime}(k) = -N_{\nu} \sqrt{\frac{\pi}{\Delta^2 - \omega^2}} \left\{ e^{-\pi i/4} \text{Li}_{1/2} \left( e^{i(k F, \nu a + ka)/\xi_\nu} \right) + e^{\pi i/4} \text{Li}_{1/2} \left( e^{i(-k F, \nu a + ka)/\xi_\nu} \right) \\
+ e^{-\pi i/4} \text{Li}_{1/2} \left( e^{i(k F, \nu a - ka)/\xi_\nu} \right) + e^{\pi i/4} \text{Li}_{1/2} \left( e^{i(-k F, \nu a - ka)/\xi_\nu} \right) \bigg\},$$

$$I_{\nu}^{\prime}(k) = -iN_{\nu} \sqrt{\frac{\pi}{\Delta^2 - \omega^2}} \left\{ e^{-3\pi i/4} \text{Li}_{1/2} \left( e^{i(k F, \nu a + ka)/\xi_\nu} \right) + e^{3\pi i/4} \text{Li}_{1/2} \left( e^{i(-k F, \nu a + ka)/\xi_\nu} \right) \\
- e^{-3\pi i/4} \text{Li}_{1/2} \left( e^{i(k F, \nu a - ka)/\xi_\nu} \right) - e^{3\pi i/4} \text{Li}_{1/2} \left( e^{i(-k F, \nu a - ka)/\xi_\nu} \right) \bigg\},$$

where $\text{Li}_s(z)$ is the polylogarithm of order $s$.

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[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, (Dover, New York, 1964).