On character sums with determinants

On the 50th Anniversary of Chen’s Theorem

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Abstract We estimate weighted character sums with determinants \(ad - bc\) of \(2 \times 2\) matrices modulo a prime \(p\) with entries \(a, b, c\) and \(d\) varying over the interval \([1, N]\). Our goal is to obtain non-trivial bounds for values of \(N\) as small as possible. In particular, we achieve this goal, with a power saving, for \(N \geq p^{1/8+\varepsilon}\) with any fixed \(\varepsilon > 0\), which is very likely to be the best possible unless the celebrated Burgess bound is improved. By other techniques, we also treat more general sums but sometimes for larger values of \(N\).

Keywords character sum, determinant, Burgess bound

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1 Introduction and statement of the results

1.1 The original question

In the spring of 2022, Satadal Ganguly turned the attention of the first named author to the question of obtaining cancellations in the quadruple sum of Legendre symbols

\[
\sum_{|a| \leq x} \sum_{|b| \leq x} \sum_{|c| \leq x} \sum_{|d| \leq x} \left( \frac{ad - bc}{p} \right),
\]

where \(p\) is a large prime, \(x\) is as small as possible, with the view of counting the analogues of quadratic non-residues in the context of matrices over the finite field \(\mathbb{F}_p\) of \(p\) elements.

For a prime \(p\) and an integer \(N \geq 2\), we identify \(\mathbb{F}_p\) with the set of integers \(\{0, 1, \ldots, p - 1\}\) and introduce the following notations:

- \(\mathcal{M}_2(p)\) is the set of \(2 \times 2\) matrices, with coefficients in \(\mathbb{F}_p\);
- \(\mathcal{M}_2^\star(p) (= GL(2, \mathbb{F}_p))\) is the subset of \(\mathcal{M}_2(p)\) containing all the non-singular matrices;
- \(\mathcal{M}_2^\square(p)\) is the subset of \(\mathcal{M}_2^\star(p)\) defined by

\[
\mathcal{M}_2^\square(p) := \{A \in \mathcal{M}_2^\star(p) : A \neq B^2 \text{ for all } B \in \mathcal{M}_2(p)\},
\]

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and the elements of \( \mathcal{M}_2^{\square}(p) \) are called matrices without square root;
- \( M(N, p) \) is the subset of \( \mathcal{M}_2(p) \) defined by
  \[
  M(N, p) := \left\{ \begin{pmatrix} a \mod p & b \mod p \\ c \mod p & d \mod p \end{pmatrix} : 0 \leq a, b, c, d \leq N \right\}.
  \]

The question now is:

**Question 1.1.** Estimate the size of the set
\[
M(N, p) \cap \mathcal{M}_2^{\square}(p),
\]
and in particular, find a real number \( \kappa_2 \) (\( 0 < \kappa_2 < 1 \)), as small as possible, such that for sufficiently large \( p \), one has
\[
M(p^{\kappa_2}, p) \cap \mathcal{M}_2^{\square}(p) \neq \emptyset. \tag{1.2}
\]

The restriction of this question to invertible matrices of \( \mathcal{M}_2^*(p) \) comes from the observation that for every \( b \mod p \) with \( b \neq 0 \), there exists no matrix \( B \in \mathcal{M}_2(p) \) such that
\[
B^2 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.
\]
In other words, it is easy to find a matrix \( A \in \mathcal{M}_2(p) \) which only has small entries and which is not a square. However, the above method produces a singular matrix.

Now recall the situation in the context of matrices with dimension one, which means \( \mathbb{F}_p \). Let \( \mathfrak{f}_p \) be the least integer \( n \geq 2 \) which is not a quadratic residue modulo \( p \). A well-known question in number theory is to give a bound for \( \mathfrak{f}_p \), in the following sense:

**Find a real number \( \kappa_1 \) (\( 0 < \kappa_1 < 1 \)), as small as possible, such that, for sufficiently large \( p \), one has**
\[
2 \leq \mathfrak{f}_p \leq p^{\kappa_1}. \tag{1.3}
\]

It is universally believed that (1.3) is satisfied by any \( \kappa_1 \) (\( 0 < \kappa_1 < 1 \)). This conjecture is considered as very difficult. It is time to recall that we know that (1.3) holds for any
\[
\kappa_1 > 1/(4\sqrt{e}). \tag{1.4}
\]
This result is the content of [3, Theorem 2]. It is a consequence of the famous Burgess bound [3] for short sums of characters that we recall in Lemma 2.2 below, together with a sieving idea of the work of Vinogradov [15].

A first way to the answer to Question 1.1 is to use the properties of the determinant and of the Legendre symbol to deduce the inclusion
\[
\left\{ A \in \mathcal{M}_2^*(p) : \left( \frac{\det A}{p} \right) = -1 \right\} \subseteq \mathcal{M}_2^{\square}(p), \tag{1.5}
\]
but we lose information by this inclusion (see Appendix A below). We are led to the study of the least element of the left-hand side of (1.5). If \( \kappa_1 \) satisfies (1.3), then any \( \kappa_2 > \kappa_1/2 \) satisfies (1.2); this implication is elementary but we present it here for completeness.

**Theorem 1.2.** Let \( 0 < \kappa_1 < 1 \) and \( C \) satisfy that, for all \( p \geq 3 \), one has the inequality
\[
2 \leq \mathfrak{f}_p \leq Cp^{\kappa_1}. \tag{1.6}
\]

Then, for all \( p \geq 3 \) one has
\[
M(\sqrt{C}p^{\kappa_1/2} + 1, p) \cap \mathcal{M}_2^{\square}(p) \neq \emptyset. \tag{1.7}
\]
Proof. We want to express the least quadratic non-residue $z_p$ as

$$z_p = ad - bc,$$

with positive $a$, $b$, $c$ and $d$ of suitable sizes. So we choose

$$a = \left\lceil \frac{z_1}{2p} \right\rceil \quad (\leq \frac{z_1}{2p} + 1),$$

$$b \equiv -z_p \pmod{a} \quad \text{with} \quad 1 \leq b \leq a \text{ and } c = 1.$$

By construction, the number

$$d = \frac{z_p + bc}{a}$$

is an integer and satisfies the inequalities

$$0 < d \leq \frac{z_1}{2p} \leq \frac{z_1}{2p} + 1.$$

Then, by (1.6), we have the inequalities

$$1 \leq a, b, c, d \leq \sqrt{Cp^\frac{1}{2} + 1},$$

and the result follows.

Therefore, in view of (1.4), we have proved that (1.2) is satisfied with any

$$\kappa_2 > 1/(8\sqrt{e}).$$

Our purpose now is to use more sophisticated tools to describe the cardinality of the set appearing on the left-hand side of (1.7). The first of these tools is a lower bound for the number of quadratic non-residues in some interval beginning at $z_p$ (see Lemma 2.4). The second one concerns with the number of solutions to the determinant equation $\Delta = ad - bc$ (see Lemma 2.8). We have the following theorem.

**Theorem 1.3.** For every $\varepsilon > 0$ there exist some $C > 0$ and $p_0$, which depend only on $\varepsilon$, such that for every $p \geq p_0$, for every $N \geq p^{1/8\sqrt{e} + \varepsilon}$ one has the inequality

$$\sharp (M(N, p) \cap M_2^{\not\equiv_{\square}}(p)) \geq CN^4.$$

Certainly, for every $p$ and every $N$ with $1 \leq N < p$ one has the inequality

$$\sharp (M(N, p) \cap M_2^{\not\equiv_{\square}}(p)) \leq \sharp M(N, p) \leq (N + 1)^4,$$

and so Theorem 1.3 gives the correct order of magnitude. We give its proof in Section 3 and it is obvious that its counting process can be generalized in several directions. For example, applying Lemma 2.8 with $\alpha_n$ and $\beta_n$ to be the characteristic functions of the set of primes, our argument yields the lower bound

$$\sharp (M_1(N, p) \cap M_2^{\not\equiv_{\square}}(p)) \geq CN^4 \log p^{-2},$$

where $M_1(N, p)$ is the subset of $M(N, p)$ obtained by restricting the entries of the second row of the matrix to prime values.

The above questions and results make it natural to ask about the size of $M_2^{\not\equiv_{\square}}(p)$. Quite surprisingly, as far as we know, this question has never been addressed in the literature. In Appendix A we present some elementary arguments which give the asymptotic formula

$$\sharp M_2^{\not\equiv_{\square}}(p) = \frac{5}{8} p^4 + O(p^3),$$

and in fact with somewhat tedious dealing with several discarded cases (absorbed in the error term $O(p^3)$), one can easily derive an explicit closed form formula for $\sharp M_2^{\not\equiv_{\square}}(p)$.

Finally, one may also consider the GL(2,$F_p$) analogue of primitive roots modulo a prime and ask questions similar to the ones considered above. A study of such questions has been initiated in [9]. See, in particular, [9, Subsection 1.5].
1.2 Oscillation of characters

When studying the least element of the left-hand side of (1.5), we are naturally led to search for cancellations in the sum of Legendre symbols introduced in (1.1) and more generally in the sum

\[ S(N, \chi) := \sum_{1 \leq a, b \leq N} \sum_{1 \leq c, d \leq N} \chi(ad - bc), \]

where \( \chi \) is a non-principal multiplicative character modulo \( p \); we refer to [11, Section 3] for a background on characters.

Before stating the other results, we recall the following convention: As usual, the notations \( F \ll G \) and \( F = O(G) \), are equivalent to \( |F| \leq cG \) for some constant \( c > 0 \), which throughout the paper may depend on the real \( \varepsilon > 0 \) (with or without subscripts).

The results that we present can be extended to the larger set of summation \( 0 \leq a, b, c, d \leq N \), since the contribution of the terms with \( abcd = 0 \) is easily treated. Keeping the variables \( a, b, c, d \) fixed, and applying the Burgess bound [3] (see Lemma 2.2 below) to the sum in \( d \), we deduce that, for every \( \varepsilon > 0 \) there exists a \( \delta \), such that, for \( N \geq p^{1+\varepsilon} \) one has the bound \( S(N, \chi) \ll N^{4-\delta} \). By this trivial approach, we did not benefit from cancellations coming from the other variables. Our next purpose is to efficiently take advantage of the other three variables.

Actually we do not benefit from the smoothness of each of these variables, some of which can be attached with general coefficients. We also work with the sum of the modulus of some interior sums.

Clearly, all these cases contain \( S(N, \chi) \) as a particular case.

The first sum that we study appealing to the properties of the determinant equation is

\[ U_{\chi}(\alpha, \beta, N) := \sum_{1 \leq a, b \leq N} \alpha_a \sum_{1 \leq c, d \leq N} \beta_b \sum_{1 \leq e, f \leq N} \sum_{1 \leq g, h \leq N} \chi(ad - bc), \]

where \( \alpha = (\alpha_a) \) and \( \beta = (\beta_b) \) are quite general sequences.

**Theorem 1.4.** For every \( \varepsilon > 0 \) there exists a \( \delta_0 > 0 \) such that the inequality

\[ U_{\chi}(\alpha, \beta, N) \ll N^{4-\delta_0} \]

holds uniformly for bounded weights \( \alpha \) and \( \beta \), for all \( a \) and \( b \geq 1 \) and uniformly for \( p^{1/2} > N \geq p^{1/8+\varepsilon} \).

Since, with the above conditions, we have \( |ad - bc| \leq N^2 \), the exponent \( 1/8 \) somehow is optimal by comparison with the critical exponent \( 1/4 \) appearing in Burgess bound mentioned in Lemma 2.2 below.

The second sum that we study is

\[ T_{\chi}(A, B, C, D; \alpha) := \sum_{1 \leq a, b \leq A} \sum_{1 \leq c, d \leq B} \sum_{1 \leq e, f \leq C} \sum_{d \in D} \alpha_d \chi(ab - cd), \]

where \( A, B, C \) and \( D \) are integers greater than or equal to \( 1 \), \( D \subseteq \mathbb{F}_p^* \) has cardinality \( D \), and \( \chi \) is a non-principal character modulo \( p \). As we have mentioned, we identify \( \mathbb{F}_p^* \) with the set of integers \( \{1, 2, \ldots, p-1\} \).

For this sum, where no variable of summation is smooth, we prove the following result.

**Theorem 1.5.** For every integer \( \nu \geq 1 \), there exists a constant \( C(\nu) \) such that, for all prime \( p \), for all non-principal character \( \chi \) modulo \( p \), for all positive integers \( A, B, C \) and \( D \) satisfying

\[ ABC < p \quad \text{and} \quad D < p, \]

for all set \( D \subseteq \mathbb{F}_p^* \) of cardinality \( \sharp D = D \), and for all arbitrary complex weights \( \alpha \) satisfying

\[ |\alpha_d| \leq 1 \text{ if } d \in D, \]

we have the inequality

\[ T_{\chi}(A, B, C, D; \alpha) \leq C(\nu)ABC \left( \frac{p}{ABC} \right)^{1/(2\nu)} D^{-1/2} + \left( \frac{p^{1/2}}{ABC} \right)^{1/(2\nu)} (\log p)^{4/\nu}. \]
Returning to Theorem 1.5 and choosing $\nu$ sufficiently large (depending on $\varepsilon$), we deduce the following upper bound.

**Corollary 1.6.** There exists a constant $c > 0$, such that, for every $\varepsilon > 0$, for every prime $p$, for every non-principal character $\chi$, for any positive integers $A, B, C$ and $D$ with

$$\frac{p}{ABC} \geq p^{1/2+\varepsilon} \quad \text{and} \quad D \geq p^\varepsilon,$$

for every $\mathcal{D} \subseteq \mathbb{F}_p^*$ of cardinality $|\mathcal{D}| = D$, and for every complex weight $\alpha$ satisfying (1.9), we have the inequality

$$T_\chi(A, B, C, D; \alpha) \ll ABCDp^{-c\varepsilon^2}.$$ 

Corollary 1.6 immediately leads to a bound for the original sum $S(N, \chi)$, by choosing $A = B = C = D = N$, but the obtained bound is interesting for $N > p^{1/6+\varepsilon}$ which is worse than what we obtain by Theorem 1.4.

The last sum we are concerned with is

$$T_\chi(N) := \sum_{1 \leq a \leq N} \sum_{1 \leq b \leq N} \sum_{1 \leq c \leq N} \sum_{1 \leq d \leq N} |\chi(ab - cd)|,$$

where $N \geq 1$. This sum is more general than $U_\chi(\alpha, \beta, N)$. It is treated by different methods to obtain the following upper bound.

**Theorem 1.7.** For any $\varepsilon > 0$, one has the inequality

$$T_\chi(N) \ll N^4 \log \log p \log p,$$

for every prime $p$, for every non-principal character $\chi$ modulo $p$, and for every

$$p > N \geq p^{1/8+\varepsilon}.$$  \hspace{1cm} (1.10)

It is worth noticing that the condition (1.10) coincides with the condition appearing in Theorem 1.4.

**Remark 1.8.** We note that our use of absolute values in the above character sums is equivalent to attaching weights (to the variables outside of these absolute values) which are bounded in the $L_\infty$-norm. Simple modifications of our arguments also allow to introduce weights for which we control $L_1$- and $L_2$-norms.

## 2 Conventions and classical tools

### 2.1 Conventions

The additive character with period 1 is denoted by

$$\xi \in \mathbb{R} \mapsto e(\xi) = \exp(2\pi i \xi).$$

For a finite set $S$ we use $|S|$ to denote its cardinality. In summation ranges we sometimes write the Legendre symbol modulo of some bulky expression $\mathfrak{F}$ as $(\mathfrak{F}/p)$. For $Y \geq 1$, the characteristic function of the interval $(Y/2, Y]$ is denoted by $1_Y$. The letter $p$ is reserved for prime numbers.

### 2.2 Around Burgess bound

The Burgess bound [3] concerns short sums of values of characters over consecutive integers. A typical modern version of this bound is of the following form (see, for example, [11, Theorem 12.6 and Equation (12.58)]).
Lemma 2.1. For every \( r \geq 2 \), there exist two constants \( c(r) \) and \( C(r) \), such that for every prime \( p \), for every non-principal character \( \chi \) (mod \( p \)), for every \( M \) and for every \( N \geq 1 \), one has the inequality
\[
\left| \sum_{M \leq n \leq M+N} \chi(n) \right| \leq C(r)N^{1-1/r}p^{(r+1)/4r^2}(\log p)^{c(r)}.
\]

The value of the constant \( c(r) \) has been the subject of improvements (see [5, 12]). However, the improvement of the exponent \( 1/4 \) in the following statement remains a famous challenging problem.

Choosing \( r \) in an optimal way, we obtain from Lemma 2.1 the following version, which is well suited for applications:

Lemma 2.2. For every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) (depending on \( \varepsilon \)), such that for every prime \( p \), for every non-principal character \( \chi \) modulo \( p \), for every \( M \) and for every \( N \geq p^{1/4+\varepsilon} \), one has the inequality
\[
\sum_{M \leq n \leq M+N} \chi(n) \ll Np^{-\delta}.
\]

The important tool in Burgess proof is the Riemann Hypothesis for curves proved by Weil [16, 17]. It leads to the following important step, which we call the Davenport-Erdős Lemma to refer to the seminal result in [4, Lemma 3 and the footnote on page 262].

Lemma 2.3. For any prime \( p \), for every non-principal multiplicative character \( \chi \) modulo \( p \), for any set \( D \subseteq \mathbb{F}_p^* \) with \( |D| = D \), for any complex weight \( \alpha \) satisfying \(|\alpha_d| \leq 1 \) for \( d \in D \) and for every integer \( \nu \geq 1 \), we have the inequality
\[
\sum_{\lambda=1}^{p-1} \left| \sum_{d \in D} \alpha_d \chi(\lambda + d) \right|^{2\nu} \leq (2\nu D)^{\nu} p + 2\nu D^{2\nu} p^{1/2}.
\]

One easily checks that the proof given in [11, p. 329], which concerns the case \( D = \{1, \ldots, D\} \) and \( \alpha_d = 1 \), actually extends to arbitrary sets \( D \subseteq \mathbb{F}_p^* \) and any bounded weight \( \alpha = (\alpha_d)_{d \in D} \) without any changes.

2.3 Counting quadratic non-residues with small size

In (1.4) we recalled the size of \( s_p \), the least quadratic non-residue modulo \( p \). The following result gives a lower bound for the cardinality of the set of quadratic non-residues which satisfy this inequality.

Lemma 2.4. For every \( \varepsilon > 0 \), there exist \( \delta > 0 \) and \( p_0 \) such that, for every \( p > p_0 \), one has the inequality
\[
\sharp\{n : 1 \leq n \leq p^{1/(\sqrt{\varepsilon}+\varepsilon)}, (n/p) = -1\} \geq \delta p^{1/(4\sqrt{\varepsilon}+\varepsilon)}.
\]

This is the content of [1, Theorem 2.1]. For an explicit value of \( \delta \), see [10, Corollary 1.8].

2.4 Bounds on some arithmetic functions

For an integer \( s \geq 1 \), let \( \tau_s(m) \) denote the \( s \)-fold divisor function, i.e., the number of ordered factorizations \( m = k_1 \cdots k_s \) with non-negative integers \( k_1, \ldots, k_s \).

In particular \( \tau_2(m) = \tau(m) \), the usual divisor function. On average the function \( \tau_s(m) \) is bounded by a power of \( \log m \) since we have the following estimate (see, for example, [11, Equation (1.80)])..

Lemma 2.5. Let \( s \geq 1 \) be a fixed integer. Then for every \( M \geq 1 \), one has the inequality
\[
\sum_{1 \leq m \leq M} \tau_s(m)^2 \ll M(\log 2M)^{s-1}.
\]

Finally, the following estimate is an easy consequence of the Prime Number Theorem (or in fact of Chebychev’s inequality).

Lemma 2.6. Uniformly for \( x \geq 1 \), one has the inequality
\[
\sum_{p \leq x} \frac{1}{p^2} \ll \frac{1}{x \log 2x}.
\]
2.5 Sifted integers

Given two real numbers \( y \geq x > 2 \) and an integer \( N \geq 1 \), we denote by \( \mathcal{A}_0(N; x, y) \) the set of positive integers \( n \leq N \) that do not have a prime divisor in the half-open interval \( (x, y] \). We need the following upper bound on the cardinality \( \# \mathcal{A}_0(N; x, y) \):

**Lemma 2.7.** Uniformly over the integer \( N \) and real numbers \( x \) and \( y \) with \( N \geq y \geq x \geq 2 \), we have

\[
\# \mathcal{A}_0(N; x, y) \ll N \frac{\log x}{\log y}.
\]

**Proof.** A classical method from the sieve theory leads to the upper bound

\[
\#(1 < m \leq M : p \mid m \Rightarrow p \geq y) = O\left( \frac{M}{\log 2y} \right),
\]

uniformly for \( M \) and \( y \geq 1 \). We write every \( n \in \mathcal{A}_0(N; x, y) \) under the unique form

\[
n = dm,
\]

where \( d \) has all its prime factors less than \( x \) and \( m \) has all its prime factors greater than \( y \). With Lemma 2.7, we obtain the inequality

\[
\# \mathcal{A}_0(N; x, y) \ll 1 + \sum_{d \mid \mathcal{M}} \frac{N/d}{\log 2y} \ll 1 + \frac{N}{\log 2y} \prod_{p \leq x} \left( 1 + \frac{1}{p} \right) \ll N \frac{\log 2x}{\log 2y},
\]

by the Mertens formula (see, for example, [11, Equation (2.15)]).

\[\square\]

2.6 The determinant equation

Due to the structure of the problem, we need some information on the so-called determinant equation

\[
\Delta = ad - bc,
\]

where \( \Delta \) is a given integer, and where the unknowns \( a, b, c \) and \( d \) lie in some interval. The following result is an improvement of [6, Theorem 1], which is based on bounds of sums of Kloosterman fractions. However, we remark that the original result from [6] is also sufficient for our purposes.

We now recall the following asymptotic formula due to Bettin and Chandee [2, Corollary 1].

**Lemma 2.8.** Let \( \Delta \neq 0 \) be an integer and let

\[
\mathcal{T}_\Delta(M_1, M_2, N_1, N_2) := \sum_{m_1 \in M_1} \sum_{m_2 \in M_2} \sum_{n_1 \in N_1} \sum_{n_2 \in N_2} f(m_1)g(m_2)\alpha_{n_1, n_2},
\]

where the functions \( f(m_1), g(m_2) \) and the weights \( \alpha = \{\alpha_{n_1}\} \) and \( \beta = \{\beta_{n_2}\} \) are supported on \( M_1 := [M_1/2, M_1], M_2 := [M_2/2, M_2], N_1 := [N_1/2, N_1] \) and \( N_2 := [N_2/2, N_2] \), respectively. Moreover, assume that the functions \( f \) and \( g \) are of \( C^\infty \)-class and satisfy the inequalities

\[
f(j)(t) \ll \eta^j M_1^{-j}, \quad t \in M_1, \quad \text{and} \quad g(j)(t) \ll \eta^j M_2^{-j}, \quad t \in M_2
\]

for all fixed \( j \geq 0 \) and some \( \eta > 1 \). Then, for any \( \varepsilon > 0 \), we have

\[
\mathcal{T}_\Delta(M_1, M_2, N_1, N_2) = \sum_{n_1 \in N_1} \sum_{n_2 \in N_2} \frac{\gcd(n_1, n_2)}{\gcd(n_1, n_2)\Delta} \alpha_{n_1, n_2} \beta_{n_2} \int f \left( \frac{x + \Delta}{n_2} \right) g \left( \frac{x}{n_1} \right) dx
\]

\[
+ O((\eta R)^2 \|\alpha\|_2 \|\beta\|_2 (N_1 N_2)^{1+\varepsilon} (M_1 M_2)^\varepsilon),
\]

where \( \|\alpha\|_2 \) and \( \|\beta\|_2 \) are \( L_2 \)-norms of the weights \( \alpha \) and \( \beta \), respectively, and

\[
R := \frac{M_1 N_2}{M_2 N_1} + \frac{M_2 N_1}{M_1 N_2}.
\]

(2.1)
In our application, we have that $M_2$, $M_1$, $N_1$ and $N_2$ are approximately of the same size $X$, and so $R$ is of moderate size. Hence, if the size of $\Delta$ is about $M_1N_2$, and if $\alpha_{n_1}$ and $\beta_{n_2}$ are the characteristic functions of the intervals $N_1$ and $N_2$, the main term of the above formula is approximately $X^2$ and the error term has size $O(\eta^{3}X^{\frac{23}{3}+\varepsilon})$.

In the case where $\beta_{n_2} = 1$ we can obtain an upper bound of quality comparable with (2.1) but with simpler tools. Indeed, the determinant equation $m_1n_2 - m_2n_1 = \Delta$ is transformed into the congruence $m_1 \equiv \Delta n_2^2 \pmod{n_1}$. It suffices to apply classical Fourier techniques and bounds for short Kloosterman sums to obtain the desired result.

2.7 A smooth partition of unity

The use of Lemma 2.8 requires smooth functions. To fulfil this condition, we appeal to the following smooth partitioning of unity (see, for example, [7, Lemme 2]).

Lemma 2.9. Let $\Xi > 1$ be a real number. There exists a sequence $b_{t,\Xi}$ ($t \geq 0$) of functions belonging to $C^\infty([\Xi^{-1}, \Xi^{1}], \mathbb{R})$, with the following two properties:

- for all $\xi \geq 1$, one has
  \[ \sum_{t=0}^{\infty} b_{t,\Xi}(\xi) = 1; \]

- for all $\xi \geq 1$, for all fixed $\nu \geq 0$, the derivatives satisfy
  \[ b_{t,\Xi}^{(\nu)}(\xi) \ll \xi^{-\nu} \Xi^\nu (\Xi - 1)^{-\nu}. \]

Notice that the above conditions imply the equality $b_{t,\Xi}(\Xi^{\ell}) = 1$ and the inequality $0 \leq b_{t,\Xi}(t) \leq 1$ for all $t \geq 1$.

3 Proof of Theorem 1.3

Assume that $N$ satisfies the inequalities $p^{1/(8\sqrt{\varepsilon} + \varepsilon)} < N < \sqrt{\varepsilon}$ (as otherwise the result follows easily from classical tools).

We start from the inequality
\[
\hat{\varphi}(M(N, p) \cap \mathcal{M}_2^{\neq}(p)) \geq \sum_{1 \leq a, b, c, d \leq N} \sum_{((ad-bc)/p) = -1} \sum_{-N^2 \leq \Delta \leq N^2 - 1} \sum_{(\Delta/p) = -1} 1 \geq \sum_{1 < \Delta \leq N^2/1000} T_{\Delta}(N, N, N, N), \tag{3.1}
\]

where $T_{\Delta}(M_1, M_2, N_1, N_2)$ is defined in Lemma 2.8, with which we use the following parameters:

- $M_1 = M_2 = N_1 = N_2 = N$;
- $\alpha_{n_1}$ is the characteristic function of $N_1$ and $\beta_{n_2}$ is the characteristic function of $N_2$;
- $f(\xi) = g(\xi) = \psi(\frac{\xi}{2\sqrt{N}/4})$, where $\psi$ is a fixed positive function belonging to $C^\infty([2/3, 4/3], \mathbb{R})$ such that $\psi(t) = 1$ for $5/6 \leq t \leq 7/6$ and such that $0 \leq \psi(t) \leq 1$ for all real $t$.

In particular, we see from the last condition that for all real number $\xi$ and all integer $j \geq 0$ we have the inequality
\[
|f^{(j)}(\xi)| \leq (4/3)^{j} N^{-j} \sup_{t} |\psi^{(j)}(t)|.
\]

With these choices, we have $R = 2$ and $\eta = 4/3$. The error term in (2.1) is $O(N^{\frac{23}{3}+\varepsilon_{0}})$, where $\varepsilon_{0} > 0$ is arbitrary. To deal with the main term, we first consider the integral. We have the following lemma.
Lemma 3.1. We adopt the definitions of $f$ and $g$ given above. We also assume that $2 \leq \Delta \leq N^2/1000$ and that $n_1 \in \mathcal{N}_1$ and $n_2 \in \mathcal{N}_2$. Furthermore, we suppose that

\[ \frac{9}{10} \leq \frac{n_1}{n_2} \leq \frac{10}{9}. \]  

Then the Lebesgue measure of the set of real number $x$ such that

\[ f(x + \Delta/n_2)g(x/n_1) = 1 \]

is at least $cN^2$, where $c > 0$ is independent from $\Delta$, $N$, $n_1$ and $n_2$ as above.

Proof. By the definition of the functions $f$ and $g$, we see that the set of $x$ such that $f(x + \Delta/n_2)g(x/n_1) = 1$ is included in the set of $x$ satisfying the two conditions

\[ \begin{cases} \frac{5}{8}n_2N - \Delta \leq x \leq \frac{7}{8}n_2N - \Delta, \\ \frac{5}{8}n_1N \leq x \leq \frac{7}{8}n_1N. \end{cases} \]  

(3.3)

By (3.2), the second condition of (3.3) is satisfied when one has the inequalities

\[ \frac{5}{8} \cdot \frac{10}{9} n_2N \leq x \leq \frac{7}{8} \cdot \frac{9}{10} n_2N. \]

This formula defines a non-empty segment, which is included in the segment corresponding to the first condition of (3.3).

Combining Lemma 3.1 with Lemma 2.8, and recalling our above observation that the error term in (2.1) is $O(N^{2/3} + \varepsilon_0)$, we deduce that if $2 \leq \Delta \leq N^2/1000$, then one has the inequality

\[ T_\Delta(N, N, N, N) \gg N^2 \sum_{n_1 \in \mathcal{N}_1, n_2 \in \mathcal{N}_2} \frac{\gcd(n_1, n_2)}{n_1n_2} + O(N^{2/3} + \varepsilon_0) \gg N^2. \]

We now return to (3.1) to insert this lower bound. We sum over $\Delta$ with $(\Delta/p) = -1$ to complete the proof by appealing to Lemma 2.4.

4 Proof of Theorem 1.4

4.1 Preliminaries

Let $\varepsilon > 0$ be fixed and $p^{1/2} > N \geq p^{1/8+\varepsilon}$. We introduce the parameters $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Their values are small and are given at the end of the proof in terms of $\varepsilon$ and the constant $\delta$ (depending on $\varepsilon$), which appears in Lemma 2.2.

Clearly, without loss of generality, we can assume that $|\alpha_a| \leq 1$ and $|\beta_b| \leq 1$ for admissible values of $a$ and $b$.

4.2 Splitting the summations

The proof of Theorem 1.4 uses Lemma 2.8 in a crucial way. In order to apply it, our first task is to decompose $U_\gamma(\alpha, \beta, N)$ into subsums where the orders of magnitude of the variables $a$, $b$, $c$ and $d$ are controlled. Let

\[ Y := N^{\varepsilon_1} \quad \text{and} \quad \Xi := 1 + N^{-\varepsilon_1} = 1 + Y^{-1}. \]
We introduce the following finite set of real numbers:

$$\mathcal{N} := \{N/2^n : N/2^n \geq 1, \, \nu = 0, 1, \ldots\}$$

and the counting function

$$T_\Delta(N) := \sum_{1 \leq a \leq N, 1 \leq b \leq N, 1 \leq c \leq N, 1 \leq d \leq N} \alpha_a \beta_b.$$ 

We decompose $U_\chi(\alpha, \beta, N)$ as

$$U_\chi(\alpha, \beta, N) = U_\chi^+(\alpha, \beta, N) + U_\chi^-(\alpha, \beta, N), \quad (4.1)$$

where

$$U_\chi^+(\alpha, \beta, N) := \sum_{0 < \Delta < N^2} \chi(\Delta) T_\Delta^+(N),$$

$$U_\chi^-(\alpha, \beta, N) := \sum_{-N^2 < \Delta < 0} \chi(\Delta) T_\Delta^-(N).$$

The proof of Theorem 1.4 reduces to the existence of a positive $\delta_0$ such that

$$U_\chi^+(\alpha, \beta, N) = O(N^{4 - \delta_0}), \quad (4.2)$$

since the study of the sum $U_\chi^-(\alpha, \beta, N)$ is similar.

Let us impose the following restriction

$$\varepsilon_1 \geq \delta_0. \quad (4.3)$$

We decompose the characteristic functions of the intervals of variations of $a$ and $b$ as a sum of characteristic functions $1_K$ of intervals $[K/2, K]$ (see Subsection 2.1), where $K$ belongs to $\mathcal{N}$. For the variables $c$ and $d$, we use the functions $b_{\ell, \Xi}$ introduced in Lemma 2.9 with $1 \leq \ell \leq L$, where $L$ is the unique integer satisfying

$$\Xi^{L+1} \leq N < \Xi^{L+2}. \quad (4.4)$$

For a positive $\Delta$, we have the equality

$$T_\Delta(N) = \sum_{A \in \mathbb{N}} \sum_{B \in \mathbb{N}} \sum_{1 \leq \ell \leq L} \sum_{1 \leq r \leq L} \sum_{1 \leq c \leq N, 1 \leq d \leq N} 1_A(a) \alpha_a \cdot 1_B(b) \beta_b \cdot b_{\ell, \Xi}(c) \cdot b_{r, \Xi}(d) + E(\Delta). \quad (4.5)$$

The complementary term $E(\Delta)$ corresponds to the contribution of quadruples $(a, b, c, d)$ with $ad - bc = \Delta$, with $1 \leq a, b \leq N$ and at least one of the variables $c$ or $d$ is in the interval $[\Xi^L, N]$. We consider $E(\Delta)$ as an error term since, when summing over $\Delta$ and appealing to the definition of $L$ (see (4.4)) we obtain the bound

$$\sum_{1 \leq \Delta < N^2} |\chi(\Delta) E(\Delta)| \leq N^3 (\Xi^{L+2} - \Xi^L) \ll N^4 Y^{-1}. \quad (4.6)$$

This error term is acceptable by the restriction (4.3).

We write (4.5) as

$$T_\Delta(N) = \sum_{A \in \mathbb{N}} \sum_{B \in \mathbb{N}} \sum_{1 \leq \ell \leq L} \sum_{1 \leq r \leq L} T_\Delta(\Xi^\ell, \Xi^r, A, B) + E(\Delta), \quad (4.7)$$

with obvious notations.
4.3 Restricting the summation over $\Xi^\ell, \Xi^r, A$ and $B$

Decompose the sum in (4.7) into

$$T_\Delta^*(N, \leq N^4Y^{-1}) + T_\Delta^*(N, > N^4Y^{-1}) + E(\Delta),$$

where

- $T_\Delta^*(N, \leq N^4Y^{-1})$ is the quadruple sum (over $\Xi^\ell, \Xi^r, A$ and $B$) restricted by the extra condition $\Xi^\ell\Xi^rAB \leq N^4/Y$, and
- $T_\Delta^*(N, > N^4Y^{-1})$ corresponds to the restriction $\Xi^\ell\Xi^rAB > N^4Y^{-1}$.

Changing the order of summation, similarly to (4.6), we deduce the obvious inequality

$$\sum_{1 \leq \Delta < \sqrt{N^2}} \chi(\Delta) T_\Delta^*(N, \leq N^4Y^{-1}) = O(N^4Y^{-1}).$$

(4.8)

This error term is acceptable under the restriction (4.3).

We now concentrate on the sum $T_\Delta^*(N, > N^4Y^{-1})$. Actually, the inequalities $\Xi^\ell \leq N, \Xi^r \leq N, A \leq N$ and $B \leq N$ combined with $\Xi^\ell \Xi^rAB > N^4Y^{-1}$ imply that $\Xi^\ell, \Xi^r, A$ and $B$ are of comparable sizes, which means

$$NY^{-1} \leq \Xi^\ell, \Xi^r, A, B \leq N.$$ (4.9)

Furthermore, the number $Q$ of such quadruples $(\Xi^\ell, \Xi^r, A, B)$ is bounded by

$$Q \ll (\log N)^2 \left( \frac{\log N}{\log(1 + Y^{-1})} \right)^2 \ll Y^2(\log N)^4.$$ (4.10)

4.4 Bounding the error term

In order to prove (4.2), we are now led to proving the inequality

$$\sum_{1 \leq \Delta < \sqrt{N^2}} \chi(\Delta) \sum_{1 \leq a \leq N} \sum_{1 \leq b \leq N} \sum_{1 \leq c \leq N} \sum_{1 \leq d \leq N} 1_A(a)1_B(b)1_{\Xi^\ell}(c)1_{\Xi^r}(d) \ll N^{4-\delta_0-3\varepsilon_1}$$ (4.11)

for all $(\Xi^\ell, \Xi^r, A, B)$ satisfying (4.9). This is a consequence of (4.1), (4.6), (4.7), (4.8) and (4.10).

In (4.11), the quadruple sum is exactly $T_\Delta(\Xi^\ell, \Xi^r, A, B)$ which is treated in Lemma 2.8 and with which we apply the following values,

$$\begin{cases}
M_1 := \Xi^\ell + 1, \\
M_2 := \Xi^r + 1, \\
N_1 := A, \\
N_2 := B, \\
f(\xi) := b_{\Xi^\ell}(\xi), \\
g(\xi) := b_{\Xi^r}(\xi).
\end{cases}$$

The corresponding value of $R$, defined in (2.2), satisfies the inequality

$$R \leq 2Y^2,$$

as a consequence of (4.9). The corresponding value of $\eta$ is

$$\eta = \frac{\Xi}{\Xi - 1} \leq 2Y.$$

We now fix one of the quadruples $(\Xi^\ell, \Xi^r, A, B)$ satisfying (4.9) for which we need to estimate the sum $T_\Delta(\Xi^\ell, \Xi^r, A, B)$.

Next, we appeal to the formula (2.1) that we write under the form

$$T_\Delta(\Xi^\ell, \Xi^r, A, B) := MT(\Delta) + Err(\Delta).$$ (4.12)
With the above values of the parameters, we see that the contribution of the error term to the left-hand side of (4.11) is

$$\sum_{1 \leq \Delta < N^2} \chi(\Delta) \text{Err}(\Delta) \ll \sum_{1 \leq \Delta < N^2} (Y^3)^{3/2}(A^{1/2}B^{1/2}(AB)^{7/20}(A + B)^{1/4 + \varepsilon_1} \left(\Xi \Xi'\right)^{\varepsilon_1})$$

$$\ll N^{79/20 + 9\varepsilon_1/2 + \varepsilon_2} \ll N^{79/20 + 8\varepsilon_1}.$$  

This bound fulfills the bound (4.11), provided that we have

$$\delta_0 + 11\varepsilon_1 \leq \frac{1}{20}. \quad (4.13)$$

### 4.5 Application of the Burgess bound

Consider now the contribution of the main term $MT(\Delta)$ (see (4.12)) to the left-hand side of (4.11). After inverting summations and integration, we see that this contribution is

$$\sum_{1 \leq \Delta < N^2} \chi(\Delta) MT(\Delta) = \sum_{1 \leq a \leq N} \sum_{1 \leq b \leq N} 1_A(a) \alpha a \cdot 1_B(b) \beta b \cdot \frac{\gcd(a, b)}{ab}$$

$$\times \int \text{d}r, \Xi \left(\frac{x}{a} + \frac{\Delta}{b}\right) \chi(\Delta) \text{Err}(\Delta) \left(\frac{x + \Delta}{b}\right) \text{d}x. \quad (4.14)$$

We want to exploit the oscillations of the character $\chi(\Delta)$ by using Lemma 2.2. However, this requires several technical preparations. First of all, in the integral in (4.14), we can reduce the variable $x$ to the interval

$$\Xi^{r-1} \leq x/a \leq \Xi^{r+1},$$

which implies that

$$x \leq N^2. \quad (4.15)$$

We now establish the following estimate.

**Lemma 4.1.** Let $\chi$ be a non-principal character modulo $p$. Let $\varepsilon > 0$ and let $\delta$ be defined by Lemma 2.2. We have the bound

$$\sum_{1 \leq \Delta < N^2} \chi(\Delta) \text{Err}(\Delta) \ll \min\left\{N^{2t-1}, tY^3N^{-2}p^{1/2 + 2\varepsilon} + t^{-1}Y^3N^2p^{-\delta}\right\}$$

under the conditions (4.9), uniformly for $t \geq 1$, for $x$ satisfying (4.15) and for $N \geq p^{1/8+\varepsilon}$.

**Proof.** The first part of the above bound is trivial. For the second one, we put $\Delta = t\Delta^*$. By multiplicativity we are led to bound the sum

$$S := \sum_{1 \leq \Delta^* < N^{2t-1}} \chi(\Delta^*) b_{t,\Xi} \left(\frac{x + t\Delta^*}{b}\right).$$

We plan to use Abel summation. First we observe that by Lemma 2.9 and since the quadruple $(\Xi', \Xi, A, B)$ satisfies (4.9), the derivative of the function

$$\Delta^* \mapsto b_{t,\Xi} \left(\frac{x + t\Delta^*}{b}\right)$$

is bounded by

$$O\left(\frac{t}{b} \cdot Y^{\Xi - t}\right) = O(tY^3N^{-2}).$$
So we have the inequality
\[ S \ll tY^3N^{-2}\left(\int_{1}^{p^{1/4+\varepsilon}} + \int_{p^{1/4+\varepsilon}}^{N^2t^{-1}}\right)\left(\sum_{1 \leq \Delta \leq u} \chi(\Delta^*)\right)du \]
\[ \ll tY^3N^{-2}\left\{p^{1/2+2\varepsilon} + N^4t^{-2}p^{-\delta}\right\}, \]
where \( \delta > 0 \) is defined in Lemma 2.2 and depends on \( \varepsilon \).

We return to (4.14) and insert the bound given by Lemma 4.1. Let
\[ T = N^{\varepsilon_2}. \]
Write \( t = \gcd(a, b) \), \( a = a^*t \) and \( b = b^*t \) to remark that the contribution to \( \sum_{1 \leq \Delta \leq N^2} \chi(\Delta)MT(\Delta) \) of the \((t, a^*, b^*)\) with \( t \geq T \) is negligible, by using the bound \( N^2t^{-1} \) for \( S \) and (4.15) for the length of the interval of integration. Indeed this contribution \( \mathfrak{A}_1 \) can be bounded as follows,
\[ \mathfrak{A}_1 \ll \sum_{t \geq T} \sum_{A/(2t) \leq a^* \leq A/t} \sum_{B/(2t) \leq b^* \leq B/t} \frac{1}{la^*b^*}N^2(N^2t^{-1}) \ll N^4T^{-1}. \] (4.16)

The bound (4.16) on \( \mathfrak{A}_1 \) is compatible with the desired bound (4.11) as soon as we have the inequality
\[ \delta_0 + 3\varepsilon_1 \ll \varepsilon_2. \] (4.17)

To deal with the contribution of the \((t, a^*, b^*)\) with \( 1 \leq t \leq T \), we use the second part of the upper bound given by Lemma 4.1. More precisely, this contribution \( \mathfrak{A}_2 \) can be bounded as follows,
\[ \mathfrak{A}_2 \ll \sum_{t \leq T} \sum_{A/(2t) \leq a^* \leq A/t} \sum_{B/(2t) \leq b^* \leq B/t} \frac{1}{la^*b^*}N^2(tY^3N^{-2}p^{1/2+2\varepsilon} + t^{-1}Y^3N^2p^{-\delta}). \]
Therefore,
\[ \mathfrak{A}_2 \ll TY^3p^{1/2+2\varepsilon} + N^4Y^3p^{-\delta} \ll TY^3N^4p^{-2\varepsilon} + N^4Y^3p^{-\delta} \ll TY^3N^4-2\delta, \] (4.18)
by the assumption \( p^{1/2} > N \geq p^{1/8+\varepsilon} \). The bound (4.18) on \( \mathfrak{A}_2 \) is compatible with the desired bound (4.11) as soon as one has the two inequalities
\[ \delta_0 + 6\varepsilon_1 + 4\varepsilon_2 \ll 4\varepsilon \quad \text{and} \quad \delta_0 + 6\varepsilon_1 \ll 2\delta. \] (4.19)

### 4.6 Choices of the parameters
Recall that when \( \varepsilon > 0 \) is given, the parameter \( \delta > 0 \) has a fixed value. To complete the proof of Theorem 1.4, it remains to find positive values for \( \varepsilon_1 \) and \( \varepsilon_2 \) and \( \delta_0 \) such that the inequalities (4.3), (4.13), (4.17) and (4.19) are satisfied. By choosing
\[ \begin{cases} \varepsilon_1 = \min\{1/240, 4\varepsilon/11, 2\delta/7\}, \\ \varepsilon_2 = 4\varepsilon_1, \\ \delta_0 = \varepsilon_1, \end{cases} \]
we complete the proof of Theorem 1.4.

### 5 Proof of Theorem 1.5
Factoring out \( \chi(c) \), we write the equality
\[ T_\chi(A, B, C, D; \alpha) = \sum_{1 \leq a \leq A} \sum_{1 \leq b \leq B} \sum_{1 \leq c \leq C} \sum_{d \in D} a_d \chi(ab/c - d). \] (5.1)
Let $I$ be the counting function defined, for any integer $1 \leq \lambda \leq p - 1$, by the formula

$$I(\lambda) = \sharp\{(a, b, c) \in [1, A] \times [1, B] \times [1, C] : ab/c \equiv \lambda \pmod{p}\}.$$ 

Therefore, we can write (5.1) as

$$T\chi(A, B, C, D; \alpha) = \left|\sum_{\lambda=1}^{p-1} I(\lambda) \sum_{d \in D} \alpha_d \chi(\lambda - d)\right|$$

$$\leq \sum_{\lambda=1}^{p-1} \left|\sum_{d \in D} \alpha_d \chi(\lambda - d)\right| \cdot \left(\sum_{\lambda=1}^{p-1} I(\lambda)\right)^{-1/\nu} \cdot \left(\sum_{\lambda=1}^{p-1} I(\lambda)^{1/\nu}\right),$$

where $\nu \geq 1$ is an integer.

We apply Hölder’s inequality to this trilinear form with the choice of exponents $(1/2\nu, 1\nu, 1/2\nu)$,

leading to the inequality

$$T\chi(A, B, C, D; \alpha)^{2\nu} \leq \sum_{\lambda=1}^{p-1} \left|\sum_{d \in D} \alpha_d \chi(\lambda - d)\right|^{2\nu} \cdot \left(\sum_{\lambda=1}^{p-1} I(\lambda)\right)^{2\nu-2} \cdot \left(\sum_{\lambda=1}^{p-1} I(\lambda)^2\right)^{1/2\nu},$$

with an obvious definition of $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$.

Lemma 2.3 gives the following bound for $\Sigma_1$,

$$\Sigma_1 \ll (2\nu D)^\nu p + 2\nu D^{2\nu} p^{1/2}. \quad (5.3)$$

For $\Sigma_2$ we use the trivial remark that

$$\Sigma_2 = (ABC)^{2\nu - 2}. \quad (5.4)$$

For $\Sigma_3$, we benefit from the inequality $ABC < p$, and write

$$\Sigma_3 = \sharp\{(a_1, a_2, b_1, b_2, c_1, c_2) \in [1, A]^2 \times [1, B]^2 \times [1, C]^2 : a_1 b_1 c_2 = a_2 b_2 c_1\}$$

$$\leq \sum_{1 \leq m \leq ABC} \tau_3^2(m),$$

which finally gives

$$\Sigma_3 \ll ABC(\log p)^8 \quad (5.5)$$

by Lemma 2.5. Combining (5.2)–(5.5), we complete the proof of Theorem 1.5.

6 Proof of Theorem 1.7

6.1 Preliminary transformations

As an immediate consequence of Corollary 1.6, we know that the statement of Theorem 1.7 is correct if one imposes the extra condition $p > N > p^{1/6 + \varepsilon}$. So, without loss of generality we can assume that

$$N \leq 0.5 p^{1/4}. \quad (6.1)$$

We fix some $\varepsilon > 0$ and observe that by Corollary 1.6 we can assume that $\varepsilon < 1/20$. We also set

$$\eta = \frac{4\varepsilon^{-1} \log \log p}{\log p}. \quad (6.2)$$
Then we define

\[ x = p^\eta = (\log p)^{4/\varepsilon} \quad \text{and} \quad y = p^\varepsilon. \]  

(6.3)

Next, for an integer \( r \geq 0 \) we denote by \( A_r(N; x, y) \) the set of positive integers \( d \leq N \) which have exactly \( r \) prime divisors (counted with multiplicities) in the half-open interval \( I = (x, y] \). In particular, the cardinality of \( A_0(N; x, y) \) has been estimated in Lemma 2.7. Let \( R \) be the largest value of \( r \) for which \( A_r(N; x, y) \neq \emptyset \). In particular, we have the trivial bound

\[ R \ll \frac{\log N}{\log x}. \]  

(6.4)

We now write the formula of decomposition

\[ T_\chi(N) = \sum_{1 \leq a, b, c \leq N} \left| \sum_{1 \leq d \leq N} \chi(ab/c - d) \right| \leq \sum_{r=0}^{R} U_r, \]

where

\[ U_r = \sum_{1 \leq a, b, c \leq N} \left| \sum_{d \in A_r(N; x, y)} \chi(ab/c - d) \right|. \]

Estimating \( U_0 \) trivially as

\[ U_0 \leq N^3 \cdot (\#A_0(N; x, y)), \]

and using Lemma 2.7, we obtain the inequality

\[ T_\chi(N) \ll \sum_{r=1}^{R} U_r + N^4 \frac{\log x}{\log y} \ll \sum_{r=1}^{R} U_r + \eta N^4. \]  

(6.5)

Clearly, there are at most \( N/(x \log 2x) \) integers

\[ d \in \bigcup_{r=1}^{R} A_r(N; x, y), \]

which are divisible by a square of a prime \( p \in I \). This is a consequence of Lemma 2.6. Hence we can rewrite (6.5) as

\[ T_\chi(N) \ll \sum_{r=1}^{R} U_r^* + \eta N^4 + N^4/(x \log 2x), \]

(6.6)

with

\[ U_r^* = \sum_{1 \leq a, b, c \leq N} \left| \sum_{d \in A_r^*(N; x, y)} \chi(ab/c - d) \right|. \]

where the set \( A_r^*(N; x, y) \subseteq A_r(N; x, y) \) is defined as the set of positive integers \( d \leq N \) which have exactly \( r \) distinct prime divisors \( p \in I \).

Clearly, every integer \( d \in A_r^*(N; x, y) \) has exactly \( r \) representations as \( d = \ell m \) with a prime \( \ell \in I \) and integer \( m \in A_{r-1}^*(N/\ell; x, y) \). Hence, for \( r = 1, \ldots, R \), we have

\[ U_r^* = \frac{1}{r} \sum_{1 \leq a, b, c \leq N} \left| \sum_{\ell \in I \atop \gcd(\ell, m) = 1} \sum_{m \in A_{r-1}^*(N/\ell; x, y)} \chi(ab/c - \ell m) \right| \]

\[ = \frac{1}{r} \sum_{1 \leq a, b, c \leq N} \left| \sum_{\ell \in I \atop \gcd(\ell, m) = 1} \sum_{m \in A_{r-1}^*(N/\ell; x, y)} \chi(m) \chi(ab/(cm) - \ell) \right|. \]

(6.7)

where throughout the proof, \( \ell \) always denotes a prime number. Changing the order of summation, we now write the inequality

\[ U_r^* \leq \frac{1}{r} V_r, \]  

(6.7)
where
\[
V_r := \sum_{1 \leq a,b,c \leq N} \sum_{m \in A_{r-1}(N/x,x,y)} \left| \sum_{\ell \in \mathcal{I}[1,N/m]} \chi(ab/(cm) - \ell) \right|.
\]
Let \(V_{r}^\dagger\) be the following modification of \(V_r\):
\[
V_{r}^\dagger := \sum_{1 \leq a,b,c \leq N} \sum_{m \in A_{r-1}(N/x,x,y)} \left| \sum_{\ell \in \mathcal{I}[1,N/m]} \chi(ab/(cm) - \ell) \right|.
\]
To control the gap between \(V_r\) and \(V_{r}^\dagger\), we introduce
\[
E_r := |V_r - V_{r}^\dagger| \ll N^3 \sum_{x < \ell \leq y} \sum_{m \in A_r(N/x,y)} \ell^2 |m|^{1}.
\]
Summing over all possible \(r \leq R\), we have the following upper bound
\[
\sum_{r \leq R} E_r \ll N^4/(x \log x) \quad (6.8)
\]
by Lemma 2.6.

Furthermore, in the definition of \(V_{r}^\dagger\), we also replace the condition \(m \in A_{r-1}(N/x,x,y)\) by \(m \leq N/x\). We split this interval into \(m \leq N/y\) and \(N/y < m \leq N/x\), which of course simply increases the sum.

Thus, from the triangular inequality \(V_r \ll V_{r}^\dagger + E_r\), we derive
\[
V_r \ll W_1 + W_2 + E_r, \quad (6.9)
\]
where
\[
\begin{align*}
W_1 &= \sum_{1 \leq a,b,c \leq N} \sum_{m \leq N/y} \left| \sum_{\ell \in \mathcal{I}} \chi(ab/(cm) - \ell) \right|, \\
W_2 &= \sum_{1 \leq a,b,c \leq N} \sum_{N/y < m \leq N/x} \left| \sum_{\ell \in \mathcal{I}[1,N/m]} \chi(ab/(cm) - \ell) \right|.
\end{align*}
\]
(6.10) (6.11)

Note that we dropped the subscript \(r\) in the sums \(W_1\) and \(W_2\) as they no longer depend on \(r\). Gathering (6.6)–(6.9), we prove the inequality
\[
T\chi(N) \ll (W_1 + W_2) \left( \sum_{r \leq R} \frac{1}{r} \right) + \eta N^4 + N^4/(x \log 2x). \quad (6.12)
\]
The purpose of the next two subsections is to give suitable bounds for \(W_1\) and \(W_2\).

### 6.2 Study of the sum \(W_1\)
To estimate \(W_1\) defined in (6.10), we collect together the ratios \(ab/(cm)\) which fall in the same residue class modulo \(p\) and define
\[
J(\lambda) = \sharp\{(a,b,c,m) \in [1,N]^3 \times [1,N/y] : ab/(cm) \equiv \lambda \pmod{p}\}. \quad (6.13)
\]
Hence
\[
W_1 = \sum_{\lambda=1}^{p-1} J(\lambda) \left| \sum_{\ell \in \mathcal{I}} \chi(\lambda - \ell) \right|.
\]
We now see that the Hölder inequality (with the same exponents as in Section 5) and Lemma 2.3 (after discarding the primality condition on \(\ell\) to simplify the final expression, which is inconsequential for the final result) yields the inequality
\[
W_1^{2v} \leq ((2\nu y)^vp+2\nu y^{2v}p^{1/2}) \left( \sum_{\lambda=1}^{p-1} J(\lambda) \right)^{2v-2} \sum_{\lambda=1}^{p-1} J(\lambda)^2.
\] (6.14)
Recalling the choice of \( y \) in (6.3), we see that with
\[
\nu = \lceil \epsilon^{-1} \rceil,
\]
we have \( y^\nu p \ll y^{2\nu} p^{1/2} \) and (6.14) becomes
\[
W_1^{2\nu} \ll y^{2\nu} p^{1/2} \left( \sum_{\lambda=1}^{p-1} J(\lambda) \right)^{2
nu-2} \sum_{\lambda=1}^{p-1} J(\lambda)^2.
\]

We clearly have
\[
\sum_{\lambda=1}^{p-1} J(\lambda) \ll N^4/y,
\]
while using the condition (6.1) we derive that
\[
\sum_{\lambda=1}^{p-1} J(\lambda)^2 = \sharp \{(a_1, a_2, b_1, b_2, c_1, c_2, m_1, m_2) \in [1, N]^6 \times [1, N/y]^2 : a_1 b_1 c_2 m_2 = a_2 b_2 c_1 m_1\}.
\]

Lemma 2.5 with \( s = 4 \) now implies the inequality
\[
\sum_{\lambda=1}^{p-1} J(\lambda)^2 \leq (N^4/y) \log^{15} N.
\]

Hence, substituting the bounds (6.17) and (6.18) into (6.16), we obtain
\[
W_1^{2\nu} \ll N^{8\nu-4} p^{1/2} y (\log p)^{15} \ll N^{8\nu} p^{-3\epsilon} (\log p)^{15},
\]
by the hypothesis (1.10). Therefore
\[
W_1 \ll N^4 p^{-3\epsilon/2\nu} (\log p)^{15/2\nu}.
\]

Recall that \( \nu \) is defined in (6.15), so this bound has the shape
\[
W_1 \ll N^4 p^{-c_0 \epsilon^2},
\]
for some absolute positive \( c_0 \). The bound (6.19) is sufficient for our purpose, as it is shown by (6.12) and the value of \( R \).

6.3 Study of the sum \( W_2 \)

Clearly, the sum \( W_2 \) defined in (6.11) is less than the sum of \( O(\log p) \) sums of the form
\[
W_2(z) = \sum_{1 \leq a, b, c, m \leq N} \sum_{N \leq m < 2N/z} \left| \sum_{\ell \in [1, N/m]} \chi(ab/(cm) - \ell) \right|
\]
with some integer \( z \in [x, y] \). This implies that there exists some integer \( z \in [x, y] \) for which one has the inequality
\[
W_2 \ll W_2(z) \log p.
\]

Using the standard completing technique (see [11, Subsection 12.2]), we can make the range of summation over \( \ell \) independent of \( m \), by appealing to the additive character \( e \). More precisely, for every \( N/z < m \leq 2N/z \), we have the equalities
\[
\sum_{\ell \in [1, N/m]} \chi(ab/(cm) - \ell) = \sum_{\ell \in [1, z]} \chi(ab/(cm) - \ell) \left( \frac{1}{z} \sum_{h=0}^{z-1} \sum_{r \in [1, N/m]} e(h(\ell - r)/z) \right)
\]
\begin{align*}
&= \frac{1}{z} \sum_{h=0}^{z-1} \sum_{r \in [1, N/m]} e(-hr/z) \sum_{\ell \in [1, z]} \chi(ab/(cm) - \ell) e(h\ell/z).
\end{align*}

Using [11, Bound (8.6)], we now derive
\begin{equation}
W_2(z) \ll \frac{1}{|h| + 1} W_2(h; z),
\end{equation}
where
\begin{equation}
W_2(h; z) = \sum_{1 \leq a, b, c \leq N} \sum_{N/z \leq m \leq 2N/z} \left| \sum_{\ell \in [1, z]} \chi(ab/(cm) - \ell) e(h\ell/z) \right|.
\end{equation}

We now proceed as in Subsection 6.2. However, instead of (6.13) we now define
\begin{equation}
J(z, \lambda) = \# \{ (a, b, c, m) \in [1, N]^4 \times [1, 2N/z] : ab/(cm) \equiv \lambda \pmod{p} \}
\end{equation}
(note that at this point we discard the lower bound \( m \geq N/z \)). With this notation, we have
\begin{equation}
W_2(h; z) \leq p \sum_{\lambda = 1}^{p} J(z, \lambda) \left| \sum_{\ell \in [1, z]} \chi(\lambda - \ell) e(h\ell/z) \right|.
\end{equation}

We have the following full analogues of (6.17) and (6.18) (with \( z \) instead of \( y \)):
\begin{equation}
\sum_{\lambda = 1}^{p-1} J(z, \lambda) \ll N^4/z
\end{equation}
and
\begin{equation}
\sum_{\lambda = 1}^{p-1} J(z, \lambda)^2 \ll \left( N^4/z \right)^2 \log^{15} N.
\end{equation}

Using Lemma 2.3 and the bounds (6.22) and (6.23), similarly to (6.14) (with \( z \) instead of \( y \)) we derive
\begin{equation}
W_2(h; z)^{2\nu} \ll (N^4/z)^{2\nu-1}((2\nu z)^{\nu}p + 2\nu z^{2\nu}p^{1/2}) \log^{15} p.
\end{equation}

However, we emphasize that we now take the parameter \( \nu \) to be a function of \( z \) and thus it is not uniformly bounded anymore. Hence we include the dependence on \( \nu \) in our estimates.

Using the crude bound
\begin{equation}
((2\nu z)^{\nu}p + 2\nu z^{2\nu}p^{1/2}) \leq (2\nu)^{\nu}(z^{\nu}p + z^{2\nu}p^{1/2}),
\end{equation}
and choosing
\begin{equation}
\nu = \left| \frac{\log p}{\log z} \right|,
\end{equation}
so the last term in (6.25) dominates. Returning to (6.24), we obtain the inequality
\begin{align*}
W_2(h; z)^{2\nu} &\ll (2\nu)^{\nu}(N^4/z)^{2\nu-1} z^{2\nu} p^{1/2} \log^{15} p \\
&= (2\nu)^{\nu} N^{8\nu-4p^{1/2} z} \log^{15} p.
\end{align*}

Using that \( z \leq y \) and recalling (1.10) and (6.3) (and then replacing \( \log^{15} p \) with \( p^\epsilon \)) we obtain
\begin{equation}
W_2(h; z)^{2\nu} \ll (2\nu)^{\nu} N^{8\nu} y p^{-4\epsilon} \log^{15} p \ll (2\nu)^{\nu} N^{8\nu} p^{-2\epsilon}.
\end{equation}

Therefore we have the inequality
\begin{equation}
W_2(h; z) \ll \nu N^4 p^{-\epsilon/\nu}.
\end{equation}
Since $z \geq x$ we see that
\[ \nu \lesssim \frac{\log p}{\log x} = \frac{1}{\eta}, \]
and recalling the choice of $\eta$ in (6.2) we obtain the inequalities
\[ \nu \ll \log p \quad \text{and} \quad p^{\xi/\nu} \geq p^{\delta_0} = (\log p)^4. \]
Thus the bound (6.26) implies that, for any admissible $h$ and $z$, we have
\[ W_2(h; z) \ll N^4 (\log p)^{-3}. \]
Substituting (6.27) into (6.21), we obtain
\[ W_2(z) \ll N^4 (\log p)^{-2}, \]
which together with (6.20) gives
\[ W_2 \ll N^4 (\log p)^{-1}. \]

6.4 Concluding the proof

Substituting the bounds on $W_1$ and $W_2$ given by (6.19) and (6.28), respectively, into (6.12), we obtain the bound
\[ T\chi(N) \ll \log R(N^4 p^{-\omega} + N^4 (\log p)^{-1}) + \eta N^4 + N^4/(x \log 2x). \]
It suffices to recall the value of $R$ (see (6.4)) and $\eta$ and $x$ (see (6.2) and (6.3)) to complete the proof of Theorem 1.7.

7 Comments

It is easy to know that in our decomposition in Subsection 6.1 we can discard integers $d \in A_r(N; x, y)$ for an abnormally large $r$ and at the cost of a small error term, truncate the summation in (6.6) for $R$ replaced with a smaller value:
\[ R_0 := 100 \log \log N. \]
Indeed, the contribution of $n$ which has at least $R_0$ prime factors is negligible since if $\omega(n) \geq R_0$, then $\tau(n) \geq 2^{R_0}$ and from $\sum_{n \leq N} \tau(n) \sim N \log N$, we deduce that the number of less than or equal to $N$ with $\omega(n) \geq R_0$ is bounded by the order of magnitude by
\[ N 2^{-R_0} (\log N) \ll N (\log N)^{-2}. \]
Unfortunately, this does not improve our final result since the bottleneck comes from the choices of $\eta$ and $x$ in our optimization of (6.29).

We also note that the bounds of Theorem 1.7 also applies to more general sums with $a^{\pm 1} b^{\pm 1} - u c^{\pm 1} d^{\pm 1}$ for a fixed integer $u \not\equiv 0 \pmod{p}$ and any fixed choice of signs in the exponents.

There are two natural generalizations of our question concerning matrices without square root. For one direction, one may think about the non-solvability of the equation $A = X d$ (instead of $A = X^2$), with $X \in \mathcal{M}_{n}^*(p)$, where $d \geq 2$ is a divisor of $p - 1$. Although our present method applies, we remark that for large $d$ several other effects take place in the distribution of $d$-th power non-residues as it has been known since the work of Vinogradov [15]; see also [4,14]. Another interesting direction is towards multi-dimensional analogues of our work, i.e., for $n \times n$ invertible matrices modulo $p$ with small entries, which are not squares in $\mathcal{M}_{n}^*(p)$ (defined fully analogously to $\mathcal{M}_{2}^*(p)$).

Finally, we notice that the “decomposition” idea, used in the proof of Theorem 1.7, takes its roots in the work of Korolev [13]. Of course, the literature contains other examples of decomposition techniques to create bilinear forms by force (for example, [8, p. 560]), but we prefer to follow Korolev’s idea [13] because of its elegance.
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Appendix A Matrices with square roots

We now present some simple arguments which allow to evaluate $\sharp M^\square_2(p)$ explicitly. In fact, just to show the ideas, we compress the details related to some exceptional case into the error term instead of calculating their precise contribution (which can be easily accomplished). It is easier to work with the complementary set

$$M^\square_2(p) : = M_2(p) \setminus M^\square_2(p).$$

We have the decomposition of $M^\square_2(p)$ into two disjoint subsets

$$M^\square_2(p) = \{A \in M_2(p) : A = B^2\text{ for some } B \in M_2(p)\} \cup \mathcal{E}(p), \quad (A.1)$$

where $\mathcal{E}(p) \subseteq M_2(p)$ is such that $\sharp \mathcal{E}(p) = O(p^3)$. This bound directly follows from the classical one:

$$\sharp M_2(p) - \sharp M^\square_2(p) = O(p^3).$$

Now write (A.1) as

$$M^\square_2(p) : = M^\square_2(p) \cup \mathcal{E}(p). \quad (A.2)$$

Consider the equation

$$A = B^2, \quad (A.3)$$
where $A \in \mathcal{M}_2(p)$ is given and where the unknown is $B \in \mathcal{M}_2(p)$. Clearly, 

$$(\det B)^2 = \det A,$$

hence for some $s \in \mathbb{F}_p$ we must have $\det A = s^2$ and thus $\det B = \pm s$. From the characteristic equation of $B$,

$$B^2 - tB + \det BI_2 = 0,$$

where $t = \text{Tr}B$ and $I_2 \in \mathcal{M}_2(p)$ is the identity matrix, using (A.3), we derive $A - tB \pm sI_2 = 0$ and hence

$$tB = A \pm sI_2.$$  \hfill (A.4)

Taking the trace again we arrive at

$$t^2 = \text{Tr}A \pm 2s.$$

We now discard $O(p)$ diagonal matrices $A$ of the form $A = uI_2$, $u \in \mathbb{F}_p$. For the remaining matrices we see from (A.4) that $t \neq 0$. On the other hand, when $s$ and $t$ are fixed then the equation (A.4) defines $B$ uniquely and $B^2 = A$. Hence we now conclude that for all but $O(p)$ quadruples $(a, b, c, d) \in \mathbb{F}_p^4$, we have

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathcal{M}_2^O(p)$$

if and only if for some $(s, t) \in \mathbb{F}_p \times \mathbb{F}_p^*$,

$$ad - bc = s^2 \quad \text{and} \quad a + d + 2s = t^2.$$

We obviously have $p^2/2 + O(p)$ choices for the pair $(s, t^2)$. Denoting

$$u := s^2 \quad \text{and} \quad v := t^2 - 2s,$$

we obtain

$$ad - bc = u \quad \text{and} \quad a + d = v.$$ \hfill (A.5)

We now count the number of pairs $(u, v)$ obtained this way, i.e., the image size of the map

$$(s, t^2) \mapsto (s^2, t^2 - 2s).$$

We now consider the following two types of pairs $(s, t^2)$:

**A.** Pairs $(s, t^2)$ such that $4s = t^2 - w^2$ for some $w \in \mathbb{F}_p^*$. Note that if $(s, t^2)$ is of this type then so is $(-s, w^2)$ since $-4s = w^2 - t^2$. Furthermore, $(s, t^2)$ and $(-s, w^2)$ are mapped to the same pair

$$(u, v) = (s^2, t^2 - 2s) = ((-s)^2, w^2 - 2(-s))$$

and in fact they are the only pre-images of this pair $(u, v)$.

**B.** Pairs $(s, t^2)$ such that $4s \neq t^2 - w^2$ for any $w \in \mathbb{F}_p^*$. Then one easily checks that in this case the pair $(s, t^2)$ is the only one which is mapped to $(u, v) = (s^2, t^2 - 2s)$.

Clearly, each of the above cases A and B contains $p^2/4 + O(p)$ pairs, thus they contribute $p^2/8 + O(p)$ and $p^2/4 + O(p)$ pairwise distinct pairs $(u, v)$, respectively. Hence, in total we obtain $3p^2/8 + O(p)$ pairwise distinct pairs $(u, v)$. Therefore, the elements of $\mathcal{M}_2^O(p)$ correspond to $3p^2/8 + O(p)$ distinct systems of equations (A.5). Forming a quadratic equation

$$x^2 - vx + u + bc = 0$$

for $(a, d)$ we see that it has a solution if its discriminant $\Delta = v^2 - 4(u + bc)$ is a square in $\mathbb{F}_p$. This gives $(p + 1)/2$ possibilities for the product $bc$ and hence $p^2/2 + O(p)$ values for the pair $(b, c) \in \mathbb{F}_p^2$. There are $O(p)$ pairs corresponding $\Delta = 0$ for which (A.5) has one solution. Otherwise, we obtain two distinct
pairs \((a, d)\) corresponding to the permutation of the roots of this equation. Hence in total, for each of the \(3p^2/8 + O(p)\) pairs \((u, v) = (s^2, t^2 - 2s)\), we obtain

\[
2\left(\frac{p^2}{2} + O(p)\right) = p^2 + O(p)
\]

distinct quadruples \((a, b, c, d)\) satisfying (A.5). Therefore, we have

\[
\sharp M_2^2(p) = \frac{3}{8}p^4 + O(p^3).
\]

It remains to recall (A.1) and (A.2) to complete the proof of (1.8).