Physical Fields in QED

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Dedicated to Jacques Bros, an esteemed colleague and friend

Abstract
The connection between the Gupta-Bleuler formulation and the Coulomb gauge formulation of QED is discussed. It is argued that the two formulations are not connected by a gauge transformation. Nor can the state space of the Coulomb gauge be identified with a subspace of the Gupta-Bleuler space. Instead a more indirect connection between the two formulations via a detour through the Wightman reconstruction theorem is proposed.

1 Introduction
This article is concerned with a major unsolved problem of QED, that of an exact formulation of the notion of gauge invariance, more especially the problem of an exact characterization of gauge transformations and their uses. Why are such seemingly disparate formulations as the Gupta-Bleuler (GB) formalism and the Coulomb gauge (C gauge) description physically equivalent, and how are they connected? Needless to say, this problem will not be solved here or even fully described. I will merely put forward a few possibly useful remarks and suggestions. Attention will be restricted to the two most widely used ‘gauges’ already mentioned, the GB and the C gauges. And since there still does not exist a rigorous formulation of QED in any gauge, these results will be based on the experience gained in perturbation theory (PT). Any result which is valid in every order of PT has a good chance of describing a feature present in a possibly existing exact theory. That this statement must be taken with a grain of salt will become apparent later on.

For the structural studies we have in mind, the Wightman functions are more convenient tools than the Green’s functions of the traditional formulations. The PT of the Wightman functions of QED has been developed in [1]. Our results are based on the rules derived there. But the reader’s acquaintance with these rules will not be assumed. The claims made will therefore in general be substantiated by somewhat heuristic arguments rather than full proofs. The emphasis is on

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statements of facts and the discussion of their significance, rather than on proofs. However, all the results claimed can be rigorously proved to all orders of PT. And they are usually plausible enough as they stand.

2 Formal Considerations

The basic fields of QED with charged particles of spin 1/2 are the electromagnetic potentials $A_\mu(x)$ and the Dirac spinors $\psi(x)$ and $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$.

The GB formalism has the advantage of working with local, covariant, fields $A_\mu$, $\psi$, as we like them. But it also has two grave drawbacks. First, its state space $V_{GB}$ is equipped with an indefinite scalar product, hence it is not a Hilbert space. This contradicts the basic rules of quantum mechanics, and it is mathematically inconvenient. Second, the Maxwell equations

$$\partial_\nu F^{\nu\mu}(x) = j^\mu(x),$$

(1)

with

$$F^{\nu\mu}(x) = \partial^\nu A^\mu(x) - \partial^\mu A^\nu(x), \quad j^\mu(x) = e \bar{\psi}(x)\gamma^\mu \psi(x),$$

(2)

are not satisfied as operator equations on $V_{GB}$, even after renormalization. This means that $V_{GB}$ contains unphysical states, that is vectors which do not describe states ever encountered in a laboratory. This raises the question of how to characterize the physical states, and the second question whether $V_{GB}$ can be defined such that it contains sufficiently many physical states to give a full description of reality. The standard textbook answer to the first question is this: the divergence

$$B(x) = \partial_\mu A^\mu(x)$$

(3)

solves the free field equation $\partial^\mu \partial_\mu B(x) = 0$, hence can be split into a creation part $B^-(x)$ and an annihilation part $B^+(x)$. And the physical subspace $V_{ph} \subset V_{GB}$ is defined to be the kernel of $B^+$:

$$B^+(x) V_{ph} = 0.$$ 

(4)

This condition ensures the validity of the Maxwell equations on $V_{ph}$. The second question is, as a rule, simply ignored. A clean definition of $V_{GB}$ and thus of $V_{ph}$ is hardly ever given.\footnote{There exist more thoughtful treatments of the problem outside of textbooks, for example \cite{2,3,4}. These approaches make essential use of the notion of asymptotic fields, which is not unproblematic in QED and will be avoided in the present work.}

The $C$ gauge has complementary advantages and drawbacks. Its state space $V_C$ is a Hilbert space and contains only physical states. The Maxwell equations are satisfied. But the basic fields $A_\mu$, $\psi$, $\bar{\psi}$, are neither local (i.e. they do not commute at spacelike distances) nor covariant. This is very inconvenient for a detailed \textit{ab ovo} elaboration of the theory, especially for a convincing formulation.
of renormalization. And the lack of manifest covariance also raises the question of how observational Lorentz invariance emerges from the theory.

The complementary aspects of the two methods suggest a joining of their forces. This may consist in first working out the theory of the GB-fields $A_\mu, \psi$, as fully as possible, and then identifying in this framework “physical fields” $A_\mu, \Psi$, which generate $V_{ph}$ out of the vacuum $\Omega$, yielding the C gauge formulation. Formally, such “C-fields” are obtained from the GB-fields by the Dirac ansatz

\[
A_\mu(x) = A_\mu(x) - \frac{\partial}{\partial x^\mu} \int dy \, r^\rho(x - y) A_\rho(y),
\]
\[
\Psi(x) = \exp \left\{ i \int dy \, r^\rho(x - y) A_\rho(y) \right\} \psi(x),
\]
\[
\bar{\Psi}(x) = \Psi^\ast(x) \gamma^0,
\]

with

\[
r^0 = 0, \quad r_j(x) = -r_j(x) = \delta(x^0) \frac{\partial_j}{|\vec{x}|}.
\]

for $j = 1, 2, 3$. The auxiliary functions $r^\mu$ satisfy

\[
\partial^\mu r^\mu(x) = \delta^4(x),
\]

or in momentum space

\[
p_\mu \tilde{r}^\mu(p) = i
\]

with $\tilde{r}_\mu(p) = \int dx \, \exp\{ipx\} \, r^\mu(x)$. Notice that formally the definitions have the form of a gauge transformation with the operator valued and field-dependent gauge function

\[
G(x) = - \int dy \, r^\rho(x - y) A_\rho(y) .
\]

That the fields really are the desired C-fields follows from the fact that they satisfy

\[
[B(x), \Psi(y)] = [B(x), \bar{\Psi}(y)] = [B(x), A_\mu(y)] = 0,
\]

which equations then also hold for the annihilation part $B^+$. And this together with $B^+ \Omega = 0$ shows that states of the form

\[
\Phi = \mathcal{P}(\Psi, \bar{\Psi}, A_\mu) \Omega,
\]

with $\mathcal{P}$ a polynomial – or a more general function, if definable – of fields averaged over test functions, are physical in the sense of. The property has been called “strict gauge invariance” by Symanzik and, following him, by Strocchi and Wightman. A proof of is found in the first of these references.

3 Problems, and a Solution

The contents of Sect. 2 were purely formal. In order to give the definition a rigorous meaning we must start from a rigorous description of the GB
formalism, especially of $\mathcal{V}_{GB}$. The use of a suitably adapted version of the Wightman formalism \[7\] suggests itself. Of course, the naturally defined scalar product being indefinite, $\mathcal{V}_{GB}$ cannot be a Hilbert space. But we assume it to be equipped with a non-degenerate scalar product. And we wish to retain all the other Wightman axioms. In particular, the fields $A_\mu$, $\psi$, $\bar{\psi}$, are to be operator valued tempered distributions. And we assume the subspace $\mathcal{V}_0$ spanned by tempered field polynomials applied to the vacuum $\Omega$ to be dense in $\mathcal{V}_{GB}$ in the weak topology induced by the scalar product. $\mathcal{V}_0$ is called the space of “local states”. The density of $\mathcal{V}_0$ is important because it allows to construct $\mathcal{V}_{GB}$ from the Wightman functions by the reconstruction theorem. The Wightman functions are easily calculated in PT and other schemes of approximation. And their properties, as properties of tempered distributions, are easier to investigate than those of unbounded operators in a space with indefinite metric.

Right at the start we are confronted with a slightly disturbing fact. Let the charge operator $Q$ be defined by

$$
[Q, \psi(x)] = -e \psi(x), \quad [Q, \bar{\psi}(x)] = e \bar{\psi}(x), \quad [Q, A_\mu(x)] = 0, \quad Q \Omega = 0,
$$

(12)

with $e$ the charge of the positron. Define the state $\Phi$ to be physical if the Gauss law

$$
Q \Phi = \int_{x^0 = t} d^3 x \nabla \vec{E}(x) \Phi
$$

holds on it\(^2\), with $\vec{E}$ the electric field strength. Then it is known \[8\] that $\mathcal{V}_0$ contains no charged physical states. This means that the construction of charged physical states in $\mathcal{V}_{GB}$ as limits of local states is a non-trivial task.\(^3\)

Let us now turn to the problem of giving a rigorous meaning to the equations defining the C-fields in terms of the GB-fields, supposing a rigorous theory of the latter to be at hand as just explained. In this attempt we encounter two problems, an ultraviolet (UV) one and an infrared (IR) one.

The UV problem is this: The factor $\psi(x)$ in $\Psi(x)$ is a distribution, not a function, and so is the exponential factor. Notice that the auxiliary ‘functions’ $r^j$ are not functions in the strict sense of the word, let alone test functions, so that the exponent does not exist as a function. This problem can be solved by standard renormalization procedures, most easily by subtraction at $p = 0$ in momentum space (‘intermediate renormalization’). Unfortunately such a subtraction destroys the product form of $\Psi$, because in $p$-space the product $a(x)b(x)$ becomes the convolution $\int dk \hat{a}(p-k) \hat{b}(k)$ (the tilde denotes the Fourier transform). This reads in its subtracted form

$$
\int dk \left[ \hat{a}(p-k) \hat{b}(k) - \hat{a}(-k) \hat{b}(k) \right],
$$

\(^2\)As is well known, in this crude version of the condition the right-hand side does not make sense. Suitable regularizations in space and time are necessary. But this problem is immaterial to our present purposes.

\(^3\)That it is not necessarily an unsolvable task has recently been re-emphasized by Morchio and Strocchi \[9\].
which is no longer a convolution. Hence the mapping \((\psi, A) \rightarrow (\Psi, A)\) has no longer the form of a gauge transformation. It is very unlikely that this problem can be solved by a more sophisticated method of renormalization. It is my opinion that we should not bemoan this fact, but accustom ourselves to the idea that the notion of gauge transformations is not as useful in QED as it is in classical electrodynamics, but is only of a heuristic value.\(^4\) The true problem is to find fields satisfying the condition \((\ref{eq:strict gauge invariance})\) of strict gauge invariance, no matter whether or not they can be derived from the GB-fields by a gauge transformation. That the renormalized Dirac ansatz yields a formulation of QED which satisfies all the necessary requirements, fully justifies its use.

The IR problem is this: Are \(\Psi, A_\mu\), after renormalization, defined as fields on \(\mathcal{V}_{GB}\)? In other words, is the space \(\mathcal{V}_C\) generated from the vacuum \(\Omega\) by the C-fields a subspace of \(\mathcal{V}_{GB}\)? At first, the answer gleaned from PT is a plain ‘no’! The scalar product \((\Phi_0, \Phi)\) of the physical state
\[
\Phi = \int dx f(x) \Psi(x) \Omega \quad \in \mathcal{V}_C
\]  
and the local state
\[
\Phi_0 = \int dy g(y) \bar{\psi}^* \Omega \quad \in \mathcal{V}_0, \tag{15}
\]  
with \(f\) and \(g\) tempered test functions, can be shown to diverge already in second order of PT (see end of Appendix). This divergence is caused by the divergence at large \(y\) of the exponent \(\int dy r^\rho(x-y) A_\rho(y)\) in \((\ref{eq:extension})\); it is an IR problem.

But this result is misleading. As is well known, the generic \(S\)-matrix element of QED calculated with the LSZ reduction formula is in general IR divergent in finite orders of PT. But these divergences can be isolated and summed over all orders, yielding a vanishing result. And this result is expected to be closer to the truth than the finite-order divergences, because it agrees with information obtained from other sources, especially the Bloch-Nordsieck model. Something similar might happen for the mixed 2-point function
\[
F(x, y) = (\Omega, \bar{\psi}(y) \Psi(x) \Omega) \tag{16}
\]  
occurring in \((\Phi_0, \Phi)\). And, indeed, it does happen! The IR divergences in \(F\) can be isolated in all orders of PT and summed to yield
\[
F(x, y) \equiv 0 . \tag{17}
\]
For a sketch of the proof we refer to the Appendix. Again, we expect the result \((\ref{eq:vanishing})\), rather than the finite-order divergences, to correspond to the true situation. But, as in the case of the \(S\)-matrix, this does not solve our problem. In the same way as \((\ref{eq:vanishing})\), it can be generally shown that \(\Phi\) is orthogonal to all local states. The state \(\Phi\) of charge \(-e\) is orthogonal to \(\mathcal{V}_0\), hence it cannot be

\(^4\)This scepticism does not extend to gauge transformations of the first kind, that is global transformations with an \(x\)-independent real gauge function.
the weak limit of a sequence of local states, hence \( V_C \) cannot be a subspace of \( V_{GB} \), in which we assumed \( V_0 \) to be dense.

As a result we obtain:

*The state space \( V_C \) of the Coulomb gauge cannot be obtained as a subspace of an extension \( V_{GB} \) of \( V_0 \), if the scalar product defined on \( V_0 \) can be extended to \( V_{GB} \) in such a way that \( V_0 \) is weakly dense in \( V_{GB} \).*

The problems discussed in this section lead to the following conclusions. The \( C \)-fields \( \Psi, A_\mu \), cannot be defined as fields acting on \( V_{GB} \). Nor are they related to the \( GB \)-fields \( \psi, A_\mu \), by a gauge transformation. The traditional explanations of the connection between the two formulations do not work. A working method of relating the two formalisms has been derived in Chap. 12 of [1]. It will be briefly described without giving proofs. The method makes essential use of the Wightman reconstruction theorem. At first it is demonstrated (in PT) that the Wightman functions of the \( C \)-fields can be obtained from those of the \( GB \)-fields by a limiting procedure, like this: Replace the auxiliary functions \( r^i(x) \) in (5) and (10) by the regularized version

\[
r^i_\xi(x) = \chi(\xi x) r^i(x), \quad 0 < \xi < \infty,
\]

with \( \chi(u) \) a test function with compact support, satisfying \( \chi(0) = 1 \). The resulting fields \( \Psi_\xi, A^\mu_\xi \) are definable as acting on a slight extension of \( V_0 \). Their Wightman functions can be computed. And the limits \( \xi \to 0 \) of the latter exist and define a field theory via the reconstruction theorem, which has all the desired properties of QED in the \( C \) gauge. It is QED in the \( C \) gauge! But the states and the fields of this theory have no discernible direct connection with the states and fields of the \( GB \) formalism.

Let us end this section with a few remarks on how observational Lorentz invariance emerges from the not manifestly covariant \( C \) gauge formalism. The problem has not yet been discussed in depth. I can therefore only state a program rather than results. This program is based on the spirit of the local observables approach [10] to quantum field theory, according to which only observables localized in bounded regions of space-time are true observables. The first problem facing us is finding an exact characterization of the operators representing observables. Since the \( C \)-fields should describe the theory fully, an observable in the bounded domain \( \mathcal{R} \) must be a function of the fields with arguments in \( \mathcal{R} \). Observables should be represented by hermitian operators (self-adjointness is hard to handle in PT), and they should satisfy the axioms of local quantum physics. In particular, two observables localized in relatively spacelike domains should commute. And the algebras of observables in the domain \( \mathcal{R} \) and its image \( \mathcal{R}_\Lambda \) under the Lorentz transformation \( \Lambda \) should be related by an automorphism \( \alpha_\Lambda \), which we can, of course, not expect to be unitarily implemented. The traditional requirement that observables should be gauge invariant is difficult or impossible to formulate in \( C \) gauge, on account of the doubtful status of gauge transformations. We propose to solve this conundrum like for the fields, by starting from the \( GB \) formalism. There, observables must
satisfy the same requirements as stated for the C gauge. But the additional requirement of gauge invariance can now be interpreted to denote strict gauge invariance. This means that observables must commute with $B(x)$.\footnote{On $\mathcal{V}_C$, $B$ vanishes identically.} The C gauge equivalent of such a GB observable $A$ can then be defined as a bilinear form by

$$\left(\Omega, \bar{\Psi}(x) A \Psi(y) \Omega\right) = \lim_{\xi \to 0} \left(\Omega, \bar{\Psi}_\xi(x) A \Psi_\xi(y) \Omega\right)$$

and obvious generalizations. Under exactly what conditions on $A$ this limit exists has not yet been investigated. It is clear that the observable fields $F_{\mu\nu}$ and $j_\mu$ belong to this class.

Given the local observables of the theory, the second problem is that of describing observational Lorentz invariance without using a non-existent unitary representation of the Lorentz group. The formulation proposed here is based on the important insights put forward by Haag and Kastler in\textsuperscript{11} concerning the relevance of Fell’s theorem (a purely mathematical statement) to quantum mechanics. This relevance rests on the observation that in any given experiment we can only measure a \textit{finite number} of local observables with a \textit{finite accuracy}. In view of this we can formulate observational invariance as follows.

\begin{align*}
\text{Let } A_1, \ldots, A_n \text{ be a finite set of local observables with measuring accuracies } \epsilon_i, A_i^\Lambda = \alpha_\Lambda(A_i) \text{ their images under the Lorentz transformation } \Lambda, \text{ and let } \Phi \text{ be a (physically preparable) state in } \mathcal{V}_C. \text{ Then there exists a state } \Phi^\Lambda \in \mathcal{V}_C \text{ such that}\end{align*}

$$\left| \langle \Phi^\Lambda, A_i^\Lambda \Phi^\Lambda \rangle - \langle \Phi, A_i \Phi \rangle \right| < \epsilon_i$$

(20)

for $i = 1, \ldots, n$.

If $\Lambda$ is a boost, then $\Phi^\Lambda$ represents the original state as seen by a moving observer, as far as the experiment in question is concerned. That this form of Lorentz invariance is satisfied in $\mathcal{V}_C$ is at the moment still a conjecture. But it has a good chance of being true. Partial results in this direction can be found at the end of Chap. 12 in\textsuperscript{1}.

\section*{Appendix: Proving Equation (17)}

Using the methods of\textsuperscript{1} a rigorous proof of the claimed result\textsuperscript{17} can be given by summing the IR relevant parts of $F$ in all orders of PT. In this appendix we will not give the full proof, but only sketch its essential ideas.

We denote the vacuum expectation value $\langle \Omega, \cdots \Omega \rangle$ by $\langle \cdots \rangle$. The expression of interest is

\begin{align*}
A &= \langle \bar{\psi}(y) \Psi(x) \rangle \\
&= \langle \bar{\psi}(y) \exp \{ i e \int du r^3(x - u) A_j(u) \} \psi(x) \rangle \tag{A.1}
\end{align*}
with
\[ r^j(v) = \delta(v^0) \frac{\partial^j}{|v|}. \]  
(A.2)

We are only interested in the IR problems connected with this expression, ignoring the UV problems which can, however, easily be taken into account. The reader is free to get rid of them by a suitable UV regularization. This means that our problem is the existence of the \( u \)-integral in (A.1) at large \( u \). Assume \( x, y \), to be restricted to bounded regions by integrating them over test functions with compact support. Again, this restriction is not essential. Then, because of the \( \delta \)-factor in (A.2), \( u \) can tend to infinity only in spacelike directions. For large \( u \) we can then neglect \( \vec{x} \) with respect to \( \vec{u} \) in \( r^j(x - u) \). Using the cluster property we find
\[
A \sim \langle \psi(x) \bar{\psi}(y) \rangle \langle \exp \{ ie \int du r^j(-u) A_j(u) \} \rangle, \quad (A.3)
\]
where the symbol \( \sim \) means that only the potentially IR dangerous contributions to \( A \) are considered. The first factor depending on \( x \) and \( y \) is IR harmless. We need therefore only discuss the second factor, which we call \( X \). Going over to momentum space we find
\[
X = \left\langle \exp \left\{ e \int dk \, d^3k \frac{k^j}{|k|^2} \tilde{A}_j(k) \right\} \right\rangle, \quad (A.4)
\]
with \( \tilde{A}_j \) the Fourier transform of \( A_j \) (the tilde will be omitted in the sequel). The existence of the \( k^0 \)-integral is part of the UV problem which we ignore. The same goes for for the existence of the \( \vec{k} \)-integral at \( |\vec{k}| \to \infty \). Our only concern is the possible divergence of the \( \vec{k} \)-integral at \( \vec{k} = 0 \). In PT the exponential in (A.4) is defined as a power series. (Notice the factor \( e \) in the exponent.) Only the even terms in this expansion survive because the Wightman functions of an odd number of \( A \)'s and no \( \psi \) or \( \bar{\psi} \) vanish. Hence we find
\[
X = \sum_{\sigma = 0}^{\infty} \frac{e^{2\sigma}}{(2\sigma)!} \left\langle \left( \int dk \frac{k^j}{|k|^2} A_j(k) \right)^{2\sigma} \right\rangle, \quad (A.5)
\]
This expression contains terms of the form
\[
\left\langle \prod_{\omega=1}^{2\sigma} A_{j_\omega}(k_\omega) \right\rangle
\]
which must be evaluated in PT, and its IR divergences isolated. The graph rules for these functions are similar to, but somewhat more complicated than, the familiar Feynman rules for Green’s functions. As in this case it is found that connected components with \( \geq 4 \) external lines of a graph give convergent contributions to \( X \), because they vanish sufficiently strongly at \( k_\omega = 0 \) to overcome the \( 1/|k_\omega| \) singularities in (A.5). This is a consequence of the Ward identities,
which for these fermion-less connected graphs take the form \( k^\mu A_\mu(k) = 0 \) in all orders except the 0th. These contributions can be factored out. The IR problem is concentrated in graphs consisting exclusively of 2-variable components. But such a 2-point function is assumed to satisfy the renormalization condition

\[
\langle A_\mu(k_1) A_\nu(k_2) \rangle = \delta^4(k_1 + k_2) \left( g_{\mu\nu}\delta_+(k_1) + \text{IR harmless terms} \right) \quad (A.6)
\]

with \( \delta_+(k) = \theta(k_0) \delta(k^2) \). The IR divergences are entirely due to the zero-order term \( g_{\mu\nu}\delta_+ \). Inserting this into the expansion (A.5) and doing the combinatorics right we find

\[
X \sim \text{(finite factor)} \cdot \exp \left\{ -e^2 \int \frac{d^3k}{2|\vec{k}|^3} \right\} . \quad (A.7)
\]

The \( k \)-integral diverges positively at \( \vec{k} = 0 \). The UV divergence at \( |\vec{k}| \to \infty \) is in a more detailed treatment removed by renormalization and does not concern us here. The result is, then:

\[
X \sim 0 \quad (A.8)
\]

which proves (17). Notice that the \( e^2 \) term in the power series expansion of (A.7) diverges, as was claimed in Sect. 3.

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