A HIGHER SPEED TYPE II BLOW-UP FOR THE 4-D FUJITA EQUATION

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Abstract. This paper is concerned with blow-up solutions of the 4-dimensional energy critical heat equation $u_t = \Delta u + u^3$. Our main result is to show that the existence of type II blowup solutions, and
$$\|u(\cdot, t)\|_\infty \sim (T - t)^{-k} |ln(T - t)|^{\frac{2k}{2k - 1}}, (k = 1, 2, 3 \cdots).$$

The inner-outer gluing method has been employed.

1. INTRODUCTION

There are many studies on the following Fujita type equation
$$u_t = \Delta u + |u|^{p-1}u \quad \text{in} \quad \Omega \times (0, T). \quad (1.1)$$

In 1966, Fujita [1] started the research of equation (1). Since it is a classical model of the semilinear equation, the blowup phenomena of equation (1) have been studied. The smooth solution of equation (1.1) blows up at time $T$ if
$$\lim_{t \to T} \|u(\cdot, t)\|_\infty = \infty$$

If we only consider the solution in time variable, (1.1) becomes an ordinary differential equation
$$\dot{g} = |g|^{p-1}g,$$

and it’s solution
$$g(t) = (p - 1)^{-\frac{1}{p-1}}(T - t)^{-\frac{1}{p-1}}$$

blows up at time $T$, that is the most simple blowup solution of (1.1), more precisely, we call a solution of equation (1.1) a type I solution, if
$$\lim_{t \to T} (T - t)^{-\frac{1}{p}} \|u(\cdot, t)\|_\infty < \infty$$

And if a blow-up solution satisfies
$$\lim_{t \to T} (T - t)^{-\frac{1}{p-1}} \|u(\cdot, t)\|_\infty = \infty$$

then the solution with is said of type II blow-up solution. One can obviously see type II blow-up growth faster than type I blow-up. The result about type I blow-up can be seen [2–6].

There are two important critical value, $p_S = \frac{n+2}{n-2}$ denote the critical Sobolev exponent and let $p_{JL}$ denote the Joseph-Lundgreen exponent

$$p_{JL} = \begin{cases} \infty, & 3 \leq n \leq 10, \\ \frac{4}{n-2\sqrt{n-1}}, & \text{n} \geq 11. \end{cases}$$

In the subcritical case $p < p_S$, Giga and Kohn [7] show the only possible blowup solution is type I blow-up. In the supercritical case, it is possible occur type II phenomena, the first example of type II solution is given by M.A. Herrero and Velázquez [8, 9], they constructed a radial positive solution in the case $p > p_{JL}$. Later, Collot [10] constructed another type II solution. If $p$ is between $p_S$ and $p_{JL}$, Matano and Merle [11] excluded radially type II blow-up solutions, but in nonradial case there are examples for type II blow-up solutions [12]. For the critical case $p = p_{JL}$, Seki constructed a II blow-up solution in [13].

For the Sobolev critical case $p = p_S$, Filipas, Herrero, and Velázquez [14] proved there is no type II blow-up in radial positive class, and they have formally obtained type II solutions with the following...
blow-up rates
\[
\|u(\cdot, t)\|_{\infty} = \begin{cases} 
(T - t)^{-k}, & n = 3, \\
(T - t)^{-k}|\ln(T - t)|^{\frac{2k}{n-4}}, & n = 4, \\
(T - t)^{-3k}, & n = 5, \\
(T - t)^{\frac{n}{2}}|\ln(T - t)|^{\frac{2n}{n+4}}, & n = 6,
\end{cases}
\]
where \( k = 1, 2, \ldots \). When \( n \geq 7 \), Collot, Merle and Raphael [15] proved the type II blow-up solution can't be around the Aubin-Talenti bubble. And in [], Wang and Wei show there is no II blow-up if \( n \geq 7 \) or \( n \geq 5 \) and in radial case. The first rigorous type II in Sobolev critical case is constructed by Schweyer [16] in 2012 for \( n = 4 \) in radial case, and type II solutions has been constructed by Del.Pino, Musso, J.Wei, Zhang, and Zhou [17], where result include multi-points blow-up. In the case \( n = 5 \) type II blow-up solution is first constructed by Del.Pino, Musso, J.Wei [18], but only with the first blow-up rate of \( k = 1 \) in (1.2), Junichi Harada follows their work [19] and complete the case of \( k > 1 \) in (1.2) for \( n = 5 \). Junichi Harada [20] also gives the construction of II blow-up solution of \( n = 6 \). Del.Pino, Musso, J.Wei, Zhang, and Zhou [21] finished the proof of \( n = 3 \).

As for the energy-critical case, the Aubin-Talenti bubbles are the solutions to the stationary equation
\[
\Delta_y u(y) + |u|^{\frac{4n}{n-2}} u = 0, \quad y \in \mathbb{R}^n.
\]
where \( U(y) \) is
\[
U(y) = \alpha_n \left( \frac{1}{1 + |y|^2} \right)^{\frac{n+2}{2}}, \quad \alpha_n = (n(n-2))^{\frac{1}{n-2}},
\]
all positive bounded solution is a family of Aubin-Talenti bubbles
\[
U_{\lambda, \xi} = \lambda^{-\frac{n-2}{2}} U\left( \frac{x - \xi}{\lambda} \right),
\]
we expect
\[
u(x, t) \approx U_{\lambda(t)}
\]
for the 4-dimensional energy critical equation,
\[
\begin{cases}
{u_t} = \Delta u + u^3, \quad &\text{in } \mathbb{R}^4 \times (0, T), \\
u(x, 0) = u_0.
\end{cases}
\]
our main result is

**Theorem 1.** Given integer \( l \), for each \( T > 0 \) small enough there exists an initial condition \( u_0 \) such that the solution of problem (1.3) blows up at time \( T \) at point zero. It looks at the main order like
\[
u(x, t) = U_{\lambda(t)} - Z_0^*(x, t) \eta\left( \frac{x}{(T - t)^{\frac{n}{n-4}}} \right) + \theta(x, t),
\]
where
\[ \lambda(t) \rightarrow 0, \quad as \quad t \rightarrow T, \]
and for any \( \epsilon > 0, \) \( \| \theta(x, t) \|_{L^\infty(\mathbb{R}^4 \setminus B_\epsilon)} < \infty. \) The scaling parameter satisfies
\[ \lambda(t) \sim \frac{(T - t)^{\frac{1}{2} + 1}}{|\ln(T - t)|^{\frac{1}{2} + 1}}. \] (1.5)

2. CONSTRUCTION OF THE APPROXIMATE SOLUTION

In this section, we are going to choose an approximate solution to equation (1.4), and give the error estimate.

2.1. CONSTRUCTION OF THE FIRST ERROR. We define the error operator
\[ S(u) = -u_t + \Delta u + u^3. \] (2.1)

Find a solution \( u \) of equation (4) is equivalent to finding a function \( u \) such that \( S(u) = 0, \) as we start to choose an approximate solution, recall the Aubin-Talenti bubble
\[ U(y) = \frac{\alpha_0}{1 + |y|^2}, \] (2.2)
where \( \alpha_0 = 2\sqrt{2}, \) is a steady solution of the equation (1.3)
\[ \Delta_y U(y) + U_3(y) = 0, \quad y \in \mathbb{R}^4. \] (2.3)

Notice that for \( \lambda \neq 0, \lambda \in \mathbb{R}, \xi \in \mathbb{R}^4, \) \( U_{\lambda, \xi} = \lambda^{-1} U(\frac{\xi}{\lambda}) \) is also a solution of (2.3), we choose
\[ U_{\lambda} = \lambda^{-1} U(\frac{\lambda t}{\alpha}) \] as our first approximate solution, where \( \lambda = \lambda(t) \) is been approximately choosen as
\[ \lambda(t) \approx \frac{(T - t)^{\frac{1}{2} + 1}}{|\ln(T - t)|^{\frac{1}{2} + 1}}, \] (2.4)
then
\[ S(U_{\lambda}) = \lambda^{-2} \lambda^{(l+1)}(\frac{C_l}{1 + |y|^2} + Q(y, t)), \] (2.5)
where \( y = \frac{\lambda t}{\alpha} \), let us take \( l \)-th derivative of \( S(U_{\lambda}) \) in \( t \) variable, direct computation shows
\[ \partial_t l S(U_{\lambda}) = \lambda^{-2} \lambda^{(l+1)}(-\frac{C_l}{1 + |y|^2} + Q(y, t)), \] (2.6)
where
\[ Q(y, t) = Q_1(y) + Q_2(y, t), \]
\[ Q_1(y) \sim \frac{1}{(1 + |y|^2)} \] as \( |y| \rightarrow \infty \) is fast decay in space, and \( |Q_2(y, t)| \lesssim \frac{1}{|\ln(T - t)|^{\frac{1}{2} + 1} + 1} \) is fast decay in time, where \( C_l = \alpha_0^l \times (l - 1) \times \cdot \cdot \cdot \times 2 \times 1 = \alpha_0! \).

Since the error does not have enough decay rate as \( y \rightarrow \infty, \) then we use the following function to cancel the slow decay part \(-\frac{C_l \lambda^{(l+1)}\rho^2}{\rho^2} \).

Consider
\[ \partial_t u^{(l)} - \Delta u^{(l)} = -\frac{C_l \lambda^{(l+1)}}{\rho^2}, \]
a solution is explicitly by
\[ u^{(l)} = -C_l \int_{-T}^{t} \lambda^{(l+1)}(\rho, t - s)ds, \] (2.7)
where \( k(\rho, t) = \frac{1 - e^{-\rho^2}}{\rho^2} \), to avoid the singularity at original, we slightly modify \( u^{(l)} \), let
\[ \Psi_0^{(l)} = -C_l \int_{-T}^{t} \lambda^{(l+1)}(\zeta, t - s)ds, \] (2.8)
where \( \zeta(\rho, t) = \sqrt{\rho^2 + \lambda^2(t)} \), and we impose \( \Psi^{(k)}_0(x, T) = 0 \), then

\[
\frac{\partial}{\partial t} \Psi^{(l)}_0 - \Delta \Psi^{(l)}_0 - \lambda^{2(l)}(\Psi^{(k)}_0 - \frac{\alpha_0}{1 + |y|^2})
\]

\[
= -\frac{\alpha_0 \lambda}{\zeta} \int_{-T}^t \lambda(s) k_\zeta(\zeta, t - s) ds + \alpha_0 \int_{-T}^t \lambda^{(l)}(s) [-k_t(\zeta, t - s) + \rho^2 k_{\zeta}(\zeta, t - s) + \frac{3}{2} k_\zeta(\zeta, t - s)]
\]

\[
= R_1[\lambda],
\]

denote \( K_\zeta[\lambda] = \lambda^{-2} \lambda^{(l+1)} Q(y) - \cal R_1[\lambda] \), with direct computation, we know \( k(\zeta, t) \) satisfies \(-k_t + k_{\zeta \zeta} + \frac{2}{\xi} k_\zeta = 0 \), then

\[
R_1[\lambda] = -\frac{\alpha_0 \lambda}{\zeta} \int_{-T}^t \lambda^{(l+1)}(s) k_\zeta(\zeta, t - s) ds + \frac{\alpha_0}{\lambda(t) (1 + |y|^2)^{3/2}} \int_{-T}^t \lambda^{(l)}(s) [-\zeta k_{\zeta \zeta}(\zeta, t - s) + k_\zeta(\zeta, t - s)] ds,
\]

let \( \Upsilon = \frac{2}{\tau} \), and \( K(\Upsilon) = \frac{1}{\tau} e^{-\frac{\Upsilon^2}{2}} \), then \( k(\zeta, \tau) = \frac{1}{\tau} e^{-\frac{\Upsilon^2}{2}} = \frac{1}{\tau} e^{-\frac{2}{\tau^2}} = \frac{1}{\tau} K(\Upsilon) \), then

\[
k_\zeta(\zeta, \tau) = \frac{1}{\tau} K(\Upsilon) \frac{\partial \Upsilon}{\partial \zeta} = K(\Upsilon) \frac{2 \zeta}{\tau^2} = \Upsilon K(\Upsilon) \frac{2}{\tau^2},
\]

since \( \Upsilon K(\Upsilon) \sim \Upsilon \) near 0, and \( \Upsilon K(\Upsilon) \sim \frac{1}{\tau} \) at \( \infty \), then

\[
-\frac{\alpha_0 \lambda}{\zeta} \int_{-T}^t \lambda^{(l+1)}(s) k_\zeta(\zeta, t - s) ds \lesssim \lambda \hat{\lambda}(t) || \lambda^{(l)}(\cdot) ||_{\infty} \int_0^{t+T} \Upsilon K(\Upsilon) \frac{1}{\tau} d\tau
\]

\[
\lesssim \frac{\lambda \hat{\lambda}(t)}{\zeta^2} \left( \int_0^{\tau^2} \Upsilon K(\Upsilon) \frac{1}{\tau} d\tau + \int_{\tau^2}^{t+T} \Upsilon K(\Upsilon) \frac{1}{\tau} d\tau \right)
\]

\[
\lesssim \frac{\lambda \hat{\lambda}(t)}{\zeta^2} \left( \int_0^{\tau^2} \frac{1}{\Upsilon K(\Upsilon)} d\tau + \int_{\tau^2}^{t+T} \frac{\Upsilon}{\tau} d\tau \right)
\]

\[
\lesssim \frac{\lambda \hat{\lambda}(t)}{\zeta^2} \frac{\lambda^2(t)}{1 + |y|^2}.
\]
notice that $-\zeta k\zeta(k, t - s) + k\zeta(k, t - s) = -4 \left(\frac{\zeta}{t - s}\right)^{3/2} K_{TT}(\zeta)$, therefore the second term

$$\frac{\alpha_0}{\lambda(t) (1 + |y|^2)^{3/2}} \int_{-T}^{t} \chi^{(l+1)}(s) [-\zeta k\zeta(k, t - s) + k\zeta(k, t - s)] ds$$

$$= \frac{\alpha_0}{\lambda(t) (1 + |y|^2)^{3/2}} \int_{-T}^{t} \chi^{(l+1)}(s) \left[-4 \left(\frac{\zeta}{t - s}\right)^{3/2} K_{TT}(\zeta)\right] ds$$

$$\lesssim \frac{\alpha_0\|\chi^{(l+1)}(\cdot)\|_{\infty}}{\lambda(t) (1 + |y|^2)^{3/2}} \left(\int_{0}^{T} -4 \left(\frac{\zeta}{t - s}\right)^{3/2} K_{TT}(\zeta) ds + \int_{t}^{t+T} -4 \left(\frac{\zeta}{t - s}\right)^{3/2} K_{TT}(\zeta) ds\right)$$

(2.12) \(\dagger\)

the estimate of $\Psi_0$

Recall that

$$\Psi_0^{(l)} = -C_t \int_{-T}^{t} \chi^{(l+1)}(s) k(k, \rho, t, t - s) ds,$$

To deal with the integral above, we have

For $T - t > \frac{\zeta^2}{2\log T}$, we decompose

$$\int_{-T}^{t} \chi^{(l+1)}(s) 1 - e^{-\frac{2\zeta^2}{2(T - s)}} ds = \left(\int_{-T}^{t-(T-t)} + \int_{t-(T-t)}^{t-\frac{\zeta^2}{2}}\right) \chi^{(l+1)}(s) 1 - e^{-\frac{2\zeta^2}{2(T - s)}} ds.$$

Since $T - s < 2(t - s)$ and $\frac{\zeta^2}{2(T-s)} < 1$, we have

$$\left|\int_{-T}^{t-(T-t)} \chi^{(l+1)}(s) 1 - e^{-\frac{2\zeta^2}{2(T - s)}} ds\right| \lesssim \int_{-T}^{t-(T-t)} \frac{|\chi^{(l+1)}(s)|}{T - s} ds$$

$$\lesssim |\log T|^{\frac{1}{2\pi + 1}} \int_{-T}^{t-(T-t)} \frac{1}{(T - s)|\log(T - s)|^{\frac{1}{2\pi + 1}}} ds$$

$$\lesssim |\log T|^{\frac{1}{2\pi + 1}} \left|\frac{1}{(\log(T - t))^{\frac{1}{2\pi + 1}}} - \frac{1}{(\log(T))^{\frac{1}{2\pi + 1}}}\right| \lesssim 1.$$
next, we have
\[ | \int_{-\infty}^t \lambda^{(l+1)}(s) \frac{1 - e^{-\frac{2}{\lambda^2} s}}{\lambda^2} ds | \lesssim \frac{1}{\lambda^2} \int_{-\infty}^t |\lambda^{(l+1)}(s)| ds \lesssim 1, \]
and when \( |x| = 1 \),
\[ |\Psi_0^{(l)}| \lesssim \| \lambda^{(l+1)}(\cdot) \|_\infty \lesssim |\ln T|^{-1} \lesssim |\ln T|^{-1} (1 + t) = \tilde{\Psi}_0, \]
use comparison principle, we have the estimate of \( \Psi_0 \) for \( |x| \geq 1 \)
\[ |\Psi_0^{(l)}| \lesssim \| \ln T \|^{-1} \frac{1}{1 + |x|^2}. \]

2.2. local behavior of the heat equation. To clarify the behavior of the solution in the self-similar variable, we need some preparation of the heat equation, we start to consider
\[ u_t = \Delta u \quad \text{for} \quad (x, t) \in \mathbb{R}^n \times (0, \infty), \]
let us make use of self-similar variables
\[ w(z, \tau) = u(x, t), \quad z = \frac{x}{\sqrt{T-t}}, \quad T - t = e^{-\tau}. \]
The function \( w(z, \tau) \) solves
\[ w_{\tau} = A_z w \quad \text{for} \quad (z, \tau) \in \mathbb{R}^n \times (0, \infty), \]
where \( A_z = \Delta_z - \frac{z}{2} \cdot \nabla_z \). We define the weighted \( L^2 \) space
\[ L^2_\rho(\mathbb{R}^n) := \{ f \in L^2_{\text{loc}}(\mathbb{R}^n), \| f \|_\rho < \infty \}, \quad \| f \|_\rho^2 = \int_{\mathbb{R}^n} f^2(z) \rho(z) dz, \]
where \( \rho(z) = e^{-\frac{|z|^2}{4}} \). And the inner product is denoted by
\[ (f_1, f_2) _\rho := \int_{\mathbb{R}^n} f_1(z) f_2(z) \rho(z) dz. \]
Consider the eigenvalue problem of \( A_z \) in \( L^2_\rho(\mathbb{R}^n) \). Let \( e_i \) be the eigenfunction
\[ -A_z e_i = \lambda_i e_i, \]
where \( i \in \mathbb{N} \) is a positive integer, the corresponding eigenvalue is
\[ \lambda_i = i, \]
The eigenfunction \( e_i(z) \) is the \( i \)-th Hermite polynomial. Then \( (T - t)^i e_i \) is a solution of the heat equation, we
\[ -\Delta e_i + \frac{z}{2} e_i = le_i, \]
an orthonormal basis of \( L^2_\rho \) space is \{\( e_0, e_1, e_2 \cdot \cdot \cdot \)\}. 

2.3. the inner-outer gluing scheme. Let $Z_0^*(z, t) = (T - t)^{e_l(z)}$, where $e_l = a_0 + a_1|z|^2 + \cdots + a_l|z|^{2l}$ is the 2l-th degree polynomial which solves the eigenvalue problem, and $||e_l||_p = 1$, $a_0 > 0$. Direct computation shows $Z_0^*(z, t) = (T - t)^{e_l(z)}$ is a solution of heat equation. Our goal is to find a solution to the form

$$u(x, t) = U_{\lambda}(t) + \Psi_0 - Z_0^*(x, t)\eta(z(T - t)^{\frac{1}{2} + \tau}) + \theta(x, t), \tag{2.16}$$

let

$$\theta(x, t) = \lambda^{-2}\phi(y, t)\eta_R + \psi(x, t) \tag{2.17}$$

here $\eta_R = \eta(\frac{r}{\lambda^{\frac{1}{2}}}),$ and substitute it into equation, we easily deduce the equation of $\phi$ and $\psi$, let us decouple the equation of $\theta$ into a system so-called inner-outer gluing system, where $\phi(y, t)$ solves the inner problem

$$\begin{cases}
\lambda^2\partial_t\phi = \Delta_y\phi + pU^{p-1}(y) + H(y, t) \\
\phi(y, 0) = \phi_0
\end{cases} \quad (y, t) \in B_{2R}(0) \times (0, T), \quad y \in B_{2R}. \tag{2.18}$$

where

$$H(y, t) := 3\lambda^2U_1(y)(-Z_0^* + \Psi_0 + \psi), \tag{2.19}$$

and $\psi$ solves the outer problem

$$\psi_t = \Delta\psi + G, \tag{2.20}$$

where

$$G = 3\lambda^{-2}(1 - \eta_R)U(y)^2(-Z_0^*\eta + \psi) + \lambda^{-3}(\Delta_y\eta_R\phi + 2\nabla_y\eta_R \cdot \nabla\phi - \lambda^2\phi\partial_t\eta_R)
+ \lambda^{-2}\phi + y \cdot \nabla\phi + h_{out} + K + N, \tag{2.21}$$

where

$$N := (U_{\lambda}(t) + \Psi_0 - Z_0^*(x, t)\eta(x\sqrt{T - t}^\frac{1}{2}) + \theta(x, t))^3 - U_{\lambda}^3 - 3U_{\lambda}^2(\Psi_0 - Z_0^*(x, t)\eta(x\sqrt{T - t}^\frac{1}{2})) + \theta(x, t), \tag{2.22}$$

and

$$h_{out} = -Z_0^*(x, t)(\partial_t - \Delta)\eta(x\sqrt{T - t}^\frac{1}{2}) - 2\nabla Z_0^*(x, t) \cdot \nabla\eta(x\sqrt{T - t}^\frac{1}{2}). \tag{2.23}$$

Throughout this paper, the symbol \( \lesssim \) denotes \( \lesssim C \) for a positive constant independent of $t$ and $T$, $C$ might be different from line to line.

3. INNER PROBLEM

We solve the inner problem in this section, firstly we choose a proper parameter $\lambda$ in subsection 3.1, the scaling parameter $\lambda$ should be determined by an orthogonality condition $\int HZ_5 = 0$, however, this orthogonality condition is equal to a non-local ODE, which leads to some difficulty, we can only gain an approximate solution to $\int HZ_5 = 0$ with a small remainder. Secondly, in subsection 3.2 we list a linear theorem for the inner problem, which gives a good estimate of the inner solution $\phi$ at the neck part $|y| \sim 2R$.

3.1. parameter problem. Recall the inner problem,

$$\lambda^2\partial_t\phi = \Delta_y\phi + pU^{p-1}(y) + H(y, t),$$

let us take some formal observation, since we are seeking a type II blow-up solution, then it is reasonable for us to suppose

$$|\lambda| << \sqrt{T - t},$$

if we additionally suppose $\phi_t \sim (T - t)^{-1}\phi$, we have $\lambda^2\partial_t\phi = o(1)\phi$, from this observation, it is reasonable for us to ignore $\lambda^2\partial_t\phi$ in the inner equation, then in our very formally computation, the inner-problem has becomes the following elliptic equation

$$\Delta_y\phi + pU^{p-1}(y) + H(y, t) = 0.$$
The Fredholm theorem implies the solvability condition of this equation is $H(\psi, \lambda)$ orthogonal to the kernel of the operator $\Delta + pU^{p-1}(y)$ since $R$ is chosen to be large, the orthogonal conditions are approximate as

$$
\int_{R^4} H(y, t)Z_j(y)dy = 0 \quad j = 1, \ldots, 5,
$$

where $Z_j$ are the kernel of $\Delta + pU^{p-1}(y)$, however, the inner problem is a parabolic equation, it always has a solution, but can not make sure it has a proper estimate, to guarantee the existence of an inner solution in a suitable space.

In this section, we approximately see the outer solution as $\partial_t^j \psi \sim \partial_t^j \psi(0, T) + o(1)$, we will put the definition of the workspace of $\psi$ in next section, let us denote $\int_{B_{2R}} H(y, t)Z_5(y)dy = \lambda(t)q(t)$, to simplify the computation, let us take $l$-th derivative of $q(t)$, let

$$
\lambda^{(l+1)}(t) = k_1 \frac{\ln T}{\sqrt[l]{\ln(T-t)}} + k_2 \frac{\ln T}{\sqrt[l]{\ln(T-t)^{2^{l+1}}} + 1} + k_3 \frac{\ln T}{\sqrt[l]{\ln(T-t)^{2^{l+2}}} + 1} + k_4 \frac{\ln T}{\sqrt[l]{\ln(T-t)^{2^{l+3}}} + 1} + k_5 \frac{\ln T}{\sqrt[l]{\ln(T-t)^{2^{l+4}}} + 1} + k_6 \frac{\ln T}{\sqrt[l]{\ln(T-t)^{2^{l+5}}} + 1} + k_7 \frac{\ln T}{\sqrt[l]{\ln(T-t)^{2^{l+6}}} + 1} + k_8 \frac{\ln T}{\sqrt[l]{\ln(T-t)^{2^{l+7}}} + 1} + k_9 \frac{\ln T}{\sqrt[l]{\ln(T-t)^{2^{l+8}}} + 1}
$$

(3.1) (?)

where $k_i, i = 1, \ldots, 9$ are constant been determined, then

$$
\partial_t^1 q(t) = \int_{B_{2R}} \partial_t^1 [3U^2(y)(-Z_5^1(\frac{\lambda}{\sqrt{T-t}}), t)] Z_5(\lambda, y, t)dy + r_1
$$

$$
= 3 \int_{B_{2R}} U^2(y)Z_5(\lambda, \Psi_0 + \psi(\lambda, t))dy + \int_{B_{2R}} 3U^2(y)Z_5(\lambda, \Psi_0)dy + r_1 + r_2
$$

$$
= 3 \int_{B_{2R}} U^2(y)Z_5(\lambda, \Psi_0)dy + \int_{B_{2R}} 3U^2(y)Z_5(\lambda, \Psi_0)dy + r_1 + r_2 + r_3.
$$

where $r_1$ is the term we take derivative on the boundary $\partial B_{2R}$, notice

$$
q(t) = \int_{B_{2R}} 3U^2(y)(-Z_5^1(\frac{\lambda}{\sqrt{T-t}}), t)] Z_5 dy
$$

$$
= \int_0^{2R} \Omega_4 ||y||^3 3U^2(|y|)(-Z_5^1(\frac{\lambda}{\sqrt{T-t}}), t)] Z_5 dy,
$$

and we have

$$
r_1 \sim (T-t)^{-(l-1)} |2R|^3 U^2(2R)(-Z_5^1(\frac{\lambda}{\sqrt{T-t}}), t)] Z_5(2R)
$$

$$
\sim \frac{T-t}{R^3},
$$

(3.2) (?)

thus $r_1$ is a small term that can be neglected, similarly,
\[ r_2 = \int_{B_{2R}} [3U^2(y)((-\partial_t^2 Z_0(\frac{\lambda}{\sqrt{T-t}}, y, t) + \lambda l e_1(0)) + (\psi(\lambda y, t) - \psi^{(l)}(0, T))] Z_5(y)dy \]
\[ \lesssim \int_{B_{2R}} [3U^2(y)Z_5(y)] |z|^2dy \]
\[ \lesssim | \frac{\lambda}{\sqrt{T-t}} |^2 \int_{B_{2R}} |3U^2(y)Z_5(y)| |y|^2dy \]
\[ \lesssim | \frac{\lambda}{\sqrt{T-t}} |^2, \] (3.3) \( ? \)

and
\[ |r_3| = -3 \int_{R^4 \cup B_{2R}} U^2(y)Z_5(y)(-l l e_1(0) + \psi^{(l)}(0, T) + \Psi_0^{(l)}(0, T))dy \lesssim \frac{1}{R^2}, \] (3.4) \( ? \)

are also neglectable, denote \( r_0 = r_1 + r_2 + r_3 \), on the other hand
\[ \int_{R^4} 3U^2(y)Z_5(y)\Psi_0(\rho, t)dy = -C_1 \int_{R^4} 3U^2(y)Z_5(y) \int_{-T}^{t} \lambda^{(l+1)}(s) k(\zeta(\rho, t), t - s)dsdy \]
\[ = -3C_1 \int_{R^4} U^2(y)Z_5(y) \left( \int_{-T}^{t} \frac{\lambda^{(l+1)}(s)}{t-s} K(\zeta)ds \right) dy, \] (3.5) \( ? \)

combine this computation, we can deduce that
\[ \partial_t^l q(t) = -3C_0 \int_{R^4} U^2(y)Z_5(y) \left( \int_{-T}^{t} \frac{\lambda^{(l+1)}(s)}{t-s} K(\zeta)ds \right) dy + 3 \left[ -l l e_1(0) + \psi^{(l)}(0, T) \right] \int_{R^4} U^2(y)Z_5(y)dy + r_0, \]
define
\[ \Gamma(\tau) = C_0 \int_{-\infty}^{\infty} -3U^2(y)Z_5(y)|y|^3 K(\zeta)dy \big|_{\tau = \tau(1 + |y|^2)} d|y|, \] (3.6) \( ? \)

then
\[ \partial_t^l q(t) = \int_{-T}^{t} \frac{\lambda^{(l+1)}(t)}{t-s} \lambda^{(l+1)}(t) \Gamma(\frac{\lambda^{(l+1)}(t)}{t-s}) ds + 3 \left[ -l l e_1(0) + \psi^{(l)}(0, T) \right] \int_{-T}^{t} U^2(y)Z_5(y)dy + r_0, \]
we first compute \( f_0^{(T-t)} \lambda^{(l+1)}(t) \Gamma(\frac{\lambda^{(l+1)}(t)}{t-s}) ds = \int_{-T}^{t} \frac{\lambda^{(l+1)}(t)}{t-s} \lambda^{(l+1)}(t) \Gamma(\frac{\lambda^{(l+1)}(t)}{t-s}) ds, \)
\[ \int_{-T}^{t} \frac{\lambda^{(l+1)}(t)}{t-s} \lambda^{(l+1)}(t) \Gamma(\frac{\lambda^{(l+1)}(t)}{t-s}) ds = \int_{-T}^{t} \frac{\lambda^{(l+1)}(t)}{t-s} \Gamma(\frac{\lambda^{(l+1)}(t)}{t-s}) ds + \int_{t-(T-t)}^{t} \frac{\lambda^{(l+1)}(s)}{t-s} \Gamma(\frac{\lambda^{(l+1)}(s)}{t-s}) ds, \] (3.7) \( ? \)

we decompose the first part
\[ \int_{-T}^{t} \frac{\lambda^{(l+1)}(t)}{t-s} \Gamma(\frac{\lambda^{(l+1)}(t)}{t-s}) ds = \int_{-T}^{t} \frac{\lambda^{(l+1)}(s)}{t-s} \Gamma(\frac{\lambda^{(l+1)}(s)}{t-s}) ds + \int_{t-(T-t)}^{t} \frac{\lambda^{(l+1)}(s)}{t-s} \Gamma(\frac{\lambda^{(l+1)}(s)}{t-s}) ds, \] (3.8) \( ? \)

and the second part
\[ \int_{t-(T-t)}^{t} \frac{\lambda^{(l+1)}(s)}{t-s} \Gamma(\frac{\lambda^{(l+1)}(s)}{t-s}) ds \]
\[ = \int_{t-(T-t)}^{t} \frac{\lambda^{(l+1)}(s)}{t-s} \Gamma(\frac{\lambda^{(l+1)}(s)}{t-s}) ds + \int_{t-(T-t)}^{t} \frac{\lambda^{(l+1)}(s)}{t-s} \Gamma(\frac{\lambda^{(l+1)}(s)}{t-s}) ds, \] (3.9) \( ? \)

then we start to compute these terms
• The first one is

\[
\left| \int_{-T}^{t-(T-t)} \frac{\lambda^{(l+1)}(s)}{t-s} (\Gamma(s) - \Gamma(0)) ds \right| \leq \|\Gamma'(\cdot)\| \infty \lambda^2(t) \int_{-T}^{t-(T-t)} \frac{\lambda^{(l+1)}(s)}{(t-s)^2} ds
\]

\[
\lesssim \frac{(T-t)^{2l+2}}{|\ln(T-t)|^{\frac{2l+2}{2l+1}}} \cdot \frac{1}{T-t}
\]

\[
\lesssim T - t,
\]

which is small enough,

• and as for term \( \Gamma(0) \int_{-T}^{t-(T-t)} \frac{\lambda^{(l+1)}(s)}{t-s} ds \), we use L’Hospital’s rule to estimate it, denote

\[
\sum_{i=1}^{9} A_{1,i} \kappa_i = \Gamma(0) \int_{-T}^{T} \frac{\lambda^{(l+1)}(s)}{T-s} ds
\]

then, we have

\[
\lim_{t \to T} \Gamma(0) \int_{-T}^{t-(T-t)} \frac{\lambda^{(l+1)}(s)}{t-s} ds - \sum_{i=1}^{6} A_{1,i} \kappa_i
\]

\[
= \Gamma(0) \lim_{t \to T} \left[ \int_{-T}^{t-(T-t)} \frac{\lambda^{(l+1)}(s)}{t-s} ds + 2 \lambda^{(l+1)}(t-(T-t)) \frac{\lambda^{(l+1)}(t-(T-t))}{t-(T-t)} \right]
\]

\[
- \int_{-T}^{t-(T-t)} \frac{\lambda^{(l+2)}(s)}{t-s} ds - \lambda^{(l+1)}(t-(T-t)) \frac{\lambda^{(l+1)}(t-(T-t))}{t-(T-t)}
\]

\[
= \Gamma(0) \lim_{t \to T} \left[ \int_{-T}^{t-(T-t)} \frac{\lambda^{(l+2)}(s)}{t-s} ds + \lambda^{(l+1)}(t-(T-t)) \frac{\lambda^{(l+1)}(t-(T-t))}{t-(T-t)} \right]
\]

\[
\int_{-T}^{t-(T-t)} \frac{\lambda^{(l+2)}(s)}{t-s} ds + \lambda^{(l+1)}(t-(T-t)) \frac{\lambda^{(l+1)}(t-(T-t))}{t-(T-t)}
\]

\[
= \Gamma(0) \lim_{t \to T} \left[ \int_{-T}^{t-(T-t)} \frac{\lambda^{(l+2)}(s)}{t-s} ds + \lambda^{(l+1)}(t-(T-t)) \frac{\lambda^{(l+1)}(t-(T-t))}{t-(T-t)} \right]
\]

\[
= \frac{1}{2l+1} (T-t)^{-1} |\ln(T-t)|^{-\frac{2l+2}{2l+1}} + (2l+1) \Gamma(0) \kappa_1 |\ln T|^{\frac{1}{2l+1}}
\]

since

\[
|\Gamma(0) \lim_{t \to T} \int_{-T}^{t-(T-t)} \frac{\lambda^{(l+2)}(s)}{t-s} ds
\]

\[
\leq \lim_{t \to T} \left[ \int_{-T}^{t-(T-t)} \frac{\lambda^{(l+2)}(s)}{t-s} ds \right]
\]

\[
\leq \lim_{t \to T} \left[ \frac{\lambda^{(l+2)}(t-(T-t))}{2(t-(T-t))} ds \right]
\]

\[
= \lim_{t \to T} \left[ \frac{1}{2l+1} (T-t)^{-1} |\ln(T-t)|^{-\frac{2l+2}{2l+1}} \right]
\]

\[
\sim \lim_{t \to T} \left[ |\ln(T-t)|^{\frac{1}{2l+1}} \right]
\]

\[
= 0
\]

if we continue with almost the same computation to get a higher order expansion, we have

\[
\Gamma(0) \int_{-T}^{t-(T-t)} \frac{\lambda^{(l+1)}(s)}{t-s} ds = \sum_{i=1}^{9} A_{1,i} \kappa_i + \frac{(2l+1) \Gamma(0) \kappa_1 |\ln T|^{\frac{1}{2l+1}}}{|\ln(T-t)|^{\frac{2l+2}{2l+1}}} + g_1
\]

\[
+ O\left( \frac{|\ln T|^{\frac{1}{2l+1}}}{|\ln(T-t)|^{\frac{2l+2}{2l+1}}} \right)
\]
where \( g_1 \) is a linear combination of \( \frac{\ln T}{\ln(T-t)} \), \( \frac{\ln(T-t)}{\ln(T)} \), \( \frac{\ln(T)}{\ln(T-t)} \), and \( \frac{\ln(\ln(T-t))}{\ln(T-t)} \),
\[
\frac{\ln\left(\frac{\ln(T-t)}{\ln(T)}\right)}{\ln(T-t)^{1+\frac{3}{2}}} + \frac{\ln\left(\frac{\ln(T)}{\ln(T-t)}\right)}{\ln(T-t)^{2+\frac{3}{2}}} + \frac{\ln\left(\frac{\ln(T-t)}{\ln(T)}\right)}{\ln(T-t)^{2+\frac{3}{2}}} + \frac{\ln\left(\frac{\ln(T-t)}{\ln(T-t)}\right)}{\ln(T-t)^{2+\frac{3}{2}}}
\]
and the coefficients of these terms are linearly dependent of \( \kappa_i, i = 1, \ldots, 9 \).

We can see here occurs a constant and lower order term \((2l+1)\Gamma(0)\kappa_1\frac{\ln T}{\ln(T-t)}\), we are going to cancel the lower order term in the later computation.

As for the term \( \int_{t-(T-t)}^t \frac{\lambda^{(l+1)}(s)-\lambda^{(l+1)}(t)}{s} \Gamma(\frac{s^2(t)}{t-s}) ds \), recall that
\[
\lambda^{(l+1)}(t) = \kappa_1 \frac{\ln T}{\ln(T-t)^{\frac{3}{2}}} + \kappa_2 \frac{\ln T}{\ln(T-t)^{\frac{3}{2}+1}} + \kappa_3 \frac{\ln T}{\ln(T-t)^{\frac{3}{2}+2}} + \kappa_4 \frac{\ln T}{\ln(T-t)^{\frac{3}{2}+3}} + \kappa_5 \frac{\ln(\ln(T-t))}{\ln(T-t)^{\frac{3}{2}+1}} + \kappa_6 \frac{\ln(\ln(T-t))}{\ln(T-t)^{\frac{3}{2}+2}} + \kappa_7 \frac{\ln(\ln(T-t))}{\ln(T-t)^{\frac{3}{2}+3}} + \kappa_8 \frac{\ln(T)}{\ln(T-t)^{\frac{3}{2}+4}}
\]

let us take \( \frac{\ln T}{\ln(T-t)^{\frac{3}{2}}} \int_{t-(T-t)}^t \frac{\ln(T-s)}{\ln(T-t)^{\frac{3}{2}}} \Gamma(\frac{s^2(t)}{t-s}) ds \) as an example,
\[
\int_{t-(T-t)}^t \frac{\ln(T-s)}{\ln(T-t)^{\frac{3}{2}}} \frac{1}{t-s} \Gamma(\frac{s^2(t)}{t-s}) ds = \int_{t-(T-t)}^t \frac{\ln(T-s)}{\ln(T-t)^{\frac{3}{2}}} \frac{1}{t-s} \Gamma(\frac{s^2(t)}{t-s}) ds = \int_{t-(T-t)}^t \frac{\ln(T-s)}{\ln(T-t)^{\frac{3}{2}}} \frac{1}{t-s} \Gamma(\frac{s^2(t)}{t-s}) ds
\]
\[
= \int_{t-(T-t)}^t \frac{\ln(T-s)}{\ln(T-t)^{\frac{3}{2}}} \frac{1}{t-s} \Gamma(\frac{s^2(t)}{t-s}) ds
\]
\[
= \int_{t-(T-t)}^t \frac{\ln(T-s)}{\ln(T-t)^{\frac{3}{2}}} \frac{1}{t-s} \Gamma(\frac{s^2(t)}{t-s}) ds
\]
\[
= \int_{t-(T-t)}^t \frac{1}{t-s} \Gamma(\frac{s^2(t)}{t-s}) ds
\]
\[
= \int_{t-(T-t)}^t \frac{1}{t-s} \Gamma(\frac{s^2(t)}{t-s}) ds
\]
\[
= \int_{t-(T-t)}^t \sum_{i=1}^\infty (-1)^i \frac{1}{t-s} \Gamma(\frac{s^2(t)}{t-s}) ds
\]
\[
= \int_{t-(T-t)}^t \sum_{i=1}^\infty (-1)^i \frac{1}{t-s} \Gamma(\frac{s^2(t)}{t-s}) ds
\]
\[
+ O(\sqrt{T-t})\int_{t-(T-t)}^t (-1)^i \frac{1}{t-s} \Gamma(\frac{s^2(t)}{t-s}) ds
\]
\[
+ A_{21} \frac{\ln T}{\ln(T-t)^{1+\frac{3}{2}}} + A_{22} \frac{\ln(T-t)^{1+\frac{3}{2}}}{\ln(T-t)^{2+\frac{3}{2}}} + O(\frac{\ln(T-t)^{3+\frac{3}{2}}}{\ln(T-t)^{3+\frac{3}{2}}})
\]
we did not put many details here, the reader can check this computation easily by expanding \( \ln(1 + \frac{T-t}{t-s}) \), and using the definition of \( \Gamma \), then a very similar computation of the rest terms. We deduce that
\[
\int_{t-(T-t)}^t \frac{\lambda^{(l+1)}(s)-\lambda^{(l+1)}(t)}{s} \Gamma(\frac{s^2(t)}{t-s}) ds = g_2 + O(\frac{\ln T}{\ln(T-t)^{3+\frac{3}{2}}})
\]
where \( g_2 \) is a linear combination of \( \frac{|\ln T|^{\frac{1}{2}}}{|\ln(T-t)|^{\frac{2}{2}+\delta}} \), \( \frac{|\ln T|^{\frac{1}{2}}}{|\ln(T-t)|^{\frac{1}{2}+\delta}} \), and \( \frac{|\ln T|^{\frac{1}{2}}}{|\ln(T-t)|^{\frac{1}{2}+\delta}} \), and the coefficients of these terms are linearly dependent of \( \kappa_i \), \( i = 1, \cdots, 9 \).

Notice that compared with the previous computation, the order of these terms has been added \( \frac{1}{|\ln(T-t)|} \), and thus there will not appear term \( \frac{|\ln T|^{\frac{1}{2}}}{|\ln(T-t)|^{\frac{1}{2}+\delta}} \).

- Recall that

\[
\Gamma(\tau) = \alpha_0 |S^3| \int_0^{\infty} -3U^2(y)Z_5(y)|y|^3K(Y) \mid Y=\tau(1+|y|^2) \mid d[y],
\]

(3.13) \( \Box \)

let us start compute \( \int_{-T}^{T-T} \frac{\lambda^{(t+1)}(t)}{t-s} \Gamma(\lambda^{(t)}(t))d[\lambda^{(t)}(t)]ds \), substitute (41) into \( \int_{-T}^{T-T} \frac{\lambda^{(t+1)}(t)}{t-s} \Gamma(\lambda^{(t)}(t))d[\lambda^{(t)}(t)]ds \)

and let us change variable of \( (s, |y|) \) in to \( (Y, |y|) \), where \( s = t - \frac{\lambda^{(t)}(t)}{Y^2} \), \( ds = \frac{\lambda^{(t)}(t)}{Y^2}dY \), then

\[
\int_{-T}^{T-T} \frac{\lambda^{(t+1)}(t)}{t-s} \Gamma(\lambda^{(t)}(t))d[\lambda^{(t)}(t)]ds
\]

\[
= \lambda^{(t+1)}(t) \int_{-T}^{T-T} \frac{1}{t-s} \alpha_0 |S^3| \int_0^{\infty} (-3U^2(y)Z_5(y)|y|^3K(Y))d[y]ds
\]

\[
= \alpha_0 |S^3| \lambda^{(t+1)}(t) \int_0^{\infty} \frac{1}{\int_{-T}^{T-T} \frac{\lambda^{(t+1)}(t)}{t-s} \Gamma(\lambda^{(t)}(t))d[\lambda^{(t)}(t)]ds}
\]

(3.14) \( \Box \)

and the second integral of variable \( Y \) with more details

\[
\int_{-T}^{T-T} K(Y) dY = \int_{-T}^{T-T} \frac{1}{Y^2} dY
\]

\[
= \frac{1}{\delta} + \int_{-T}^{T} \frac{1}{Y^2} dY
\]

\[
= \frac{1}{\delta} + \int_{-T}^{T} \frac{1}{Y^2} dY
\]

\[
= \frac{1}{\delta} + \int_{-T}^{T} \frac{1}{Y^2} dY
\]

(3.15) \( \Box \)
substitute \( \delta = \frac{\lambda^2(t)(1+|y|^2)}{(T-t)} = (T-t)^{2+1}(1+|y|^2)(\frac{\kappa_1|\ln T|}{\ln|\ln(T-t)|})^2(1+\tilde{\delta}) \), we get

\[
\int_{t}^{t-\delta(T-t)} \frac{\lambda^{(t+1)}(t)}{t-s} \Gamma(t-s)ds
\]

\[
= \alpha_0|s^3|\lambda^{(t+1)}(t) \int_{0}^{\infty} \int_{\delta}^{\infty} [-3U^2(y)Z_5(y)|y|^3K(Y)] d\tau d|y|
\]

\[
= -\alpha_0|s^3|\lambda^{(t+1)}(t) \int_{0}^{\infty} \int_{\delta}^{\infty} [-3U^2(y)Z_5(y)|y|^3] \left[\frac{ln(T-t)-/(t+1)^2}{4} \right] - \frac{ln(1+|y|^2)}{4} + C
\]

(3.16) \( ? \)

\[
= -\frac{1}{2} \ln(\frac{\kappa_1|\ln T|}{\ln(\ln(T-t))}) + \frac{ln(1+\tilde{\delta})}{4} + O(\frac{1}{\ln(T-t)})^3) d|y|
\]

(3.17) \( ? \)

where \( \tilde{\delta} = \frac{f_1}{\ln(\ln(T-t))} + \frac{f_2}{\ln(\ln(T-t))} + O(\frac{1}{\ln(T-t)}) \), for some constants \( f_1 \), \( f_2 \), and \( g_3 \) is a linear combination of \( \frac{ln(T-t)}{\ln|\ln(T-t)|} \) and \( \frac{ln(T-t)}{\ln|\ln(T-t)|\ln(T-t)} \), and the coefficients of these terms are linearly dependent of \( \kappa_i \), \( i = 1, \cdots, 9 \).

here the coefficient of the first term in last equation is because

\[
\Gamma(0) = \alpha_0|s^3| \int_{0}^{\infty} [-3U^2(y)Z_5(y)|y|^3K(Y)] d|y|
\]

notice in the previous computation, the exponential \( \frac{1}{\ln(T-t)} \) help us to cancel the order of \( \frac{1}{\ln(T-t)} \), with other exponential \( \nu \neq \frac{2+2}{2+1} \) term \( \frac{1}{\ln(T-t)} \) will not be canceled, therefore our choice

\[
\lambda^{(t+1)}(t) = \kappa_1 \frac{|\ln T|}{\ln(T-t)} + \kappa_2 \frac{|\ln T|}{\ln(T-t)^{2+1}} + \kappa_3 \frac{|\ln T|}{\ln(T-t)^{2+2}} + \kappa_4 \frac{|\ln T|}{\ln(T-t)^{2+3}}
\]

\[
+ \kappa_5 \frac{|\ln(T-t)|^{2+1}}{|\ln(T-t)|^{2+2}} + \kappa_6 \frac{|\ln(T-t)|^{2+2}}{|\ln(T-t)|^{2+3}} + \kappa_7 \frac{|\ln(T-t)|^{2+2}}{|\ln(T-t)|^{2+3}}
\]

\[
+ \kappa_8 \frac{|\ln(T-t)|^{2+2}}{|\ln(T-t)|^{2+3}} + \kappa_9 \frac{|\ln(T-t)|^{2+3}}{|\ln(T-t)|^{2+3}}
\]

(3.18) \( ? \)

is reasonable, adjust \( \kappa_i \) \( i = 1, \cdots, 9 \), such that the coefficient of \( \frac{|\ln T|}{\ln|\ln(T-t)|} \), \( \frac{|\ln T|}{\ln|\ln(T-t)|\ln(T-t)} \), \( \frac{|\ln T|}{\ln|\ln(T-t)|\ln(T-t)^{2+1}} \), \( \frac{|\ln T|}{\ln|\ln(T-t)|\ln(T-t)^{2+2}} \), \( \frac{|\ln T|^2}{\ln|\ln(T-t)|\ln(T-t)^{2+2}} \), and \( \frac{|\ln(T-t)|^2}{\ln(T-t)^{2+3}} \) and the constant \( \sum_{i=1}^{9} A_1, \kappa_i \) in \( g(t) \) are 0.
We claim that it is possible since the coefficients of these terms are linearly dependent on \( \kappa_i \), \( i = 1, \cdots, 9 \), and the coefficient matrix is a triangular matrix, the terms that appear in secondary-diagonal are non-zero, we omit the details. After that, we know
\[
\partial_t \bar{q}(t) = O\left( \frac{|\ln T|^{1/4}}{|\ln(T-t)|^{1/2+\delta}} \right),
\]
(3.19) since our choice of \( \lambda \), in our computation, we can easily verify that \( \lambda \sim \frac{1}{T} \).

**Remark 3.1.** In fact, in our computation, we can see that if we add higher-order log terms, we can gain an arbitrary higher-order decay of \( \partial_t \bar{q}(t) \), i.e.
\[
\partial_t \bar{q}(t) = O\left( \frac{1}{|\ln(T-t)|^{1/2+m}} \right),
\]
for any \( m \in \mathbb{N} \).

**Remark 3.2.** The term \( -\frac{\ln \lambda}{T} \) in (3.15) is essential, it helps cancel the lower order term \( \frac{|\ln T|^{1/4}}{|\ln(T-t)|^{1/2+\delta}} \), and it also brings many trouble on our computation, that is the double log terms, that is the reason we put \( \lambda \) as a complex form.

3.2. linear inner problem. In the rest of this section, we list a linear theory for the associated linear problem of the inner problem, and we use this theorem to solve the inner problem.

Let us consider the corresponding linear problem of (2.18).
\[
\begin{aligned}
\lambda^2 \phi_t &= \Delta_y \phi + pU(y)\phi + h(y,t) \quad \text{for } (y,t) \in B_{2R} \times [0,T), \\
\phi(y,0) &= l_0 Z_0(y) + l_1 Z_5(y) \quad \text{for } y \in B_{2R},
\end{aligned}
\]
(3.20) where \( l_0 \) and \( l_1 \) are constants be choosing later, \( Z_0 \) is the eigenfunction of \( L_0 := \Delta + pU \) satisfies the equation
\[ L_0(Z_0) = \mu_0 Z_0, \quad Z_0 \in L^\infty(\mathbb{R}^4), \]
where the eigenfunction \( Z_0 \) is radially symmetric and its behavior as \( Z_0(y) \sim |y|^{-2} e^{-\sqrt{\mu_0}|y|} \), as \( y \to \infty \), where \( \mu_0 \) is the only positive eigenvalue of \( L_0 \).

The kernels of \( L_0 \) are
\[ Z_j(x) = \frac{\partial U(x)}{\partial x_j} \quad j = 1, 2, 3, 4, \]
and
\[ Z_5(x) = \frac{\partial U_\lambda(x)}{\partial \lambda} \bigg|_{\lambda=1}, \]

since we are dealing with the radially symmetry case, then
\[
\int_{B_{2R}} \phi(y,t) Z_j(y) dy = 0 \quad j = 1, 2, 3, 4,
\]

multiplying equation (3.20) by \( Z_0 \) for both sides and integrating over \( \mathbb{R}^4 \), we have
\[ \lambda^2 \int_{\mathbb{R}^4} \phi(y,t) Z_0(y) dy - \mu_0 \int_{\mathbb{R}^4} \phi(y,t) Z_0(y) dy = \int_{\mathbb{R}^4} h(y,t) Z_0(y) dy, \]
denote \( p_{\mu_0}(t) = \int_{\mathbb{R}^4} \phi(y,t) Z_0(y) dy \) and \( q_{\mu_0}(t) = \int_{\mathbb{R}^4} h(y,t) Z_0(y) dy \), we have
\[ p_{\mu_0}(t) = e^{\int_0^r \mu_0 \lambda^{-2}(r) dr} \left( p_{\mu_0}(0) + \int_0^t q_{\mu_0}(s) \lambda^{-2}(s) e^{-\int_0^s \mu_0 \lambda^{-2}(r) dr} ds \right). \]

choose
\[ p_{\mu_0}(0) = -\int_0^T q_{\mu_0}(s) \lambda^{-2}(s) e^{-\int_0^s \mu_0 \lambda^{-2}(r) dr} ds, \]
since \(\mu_0\) is positive, then \(p_{\mu_0}(t)\) decay at \(T\), and using L'Hospital's rule, we have

\[ p_{\mu_0}(t) \sim q_{\mu_0}(t) \sim \lambda(t), \quad t \to T, \]

similarly, let us multiply (3.20) by \(Z_5\) and integrate over \(\mathbb{R}^4\), we have

\[ p_0(t) = p_0(0) + \int_0^t q_0(s)\lambda^{-2}(s)ds. \]

where \(p_0(t) = \int_{\mathbb{R}^4} \phi(y,t)Z_5(y)dy\) and \(q_0(t) = \int_{\mathbb{R}^4} h(y,t)Z_5(y)dy\). Since we expect \(p_0(t)\) to decay at \(T\), it is necessary to fulfill the integrable condition

\[ \int_0^t q_0(s)\lambda^{-2}(s)ds < \infty, \]

thanks to the choice of \(\lambda(t)\) in subsection 3.1, from (3.19) we deduce that

\[ q_0(t) \sim \frac{\lambda(T-t)\ln T}{\ln(T-t)^{3+\frac{2s}{2s+1}}}, \quad (3.21) \]

then

\[ \int_0^t q_0(s)\lambda^{-2}(s)ds \sim \int_0^t \frac{1}{(T-s)\ln^3(T-s)}ds, \]

is integrable, and we choose

\[ p_1(0) = -\int_0^t q_1(s)\lambda^{-2}(s)ds, \]

then

\[ p_0(t) = -\int_t^T q_0(s)\lambda^{-2}(s)ds \]

\[ \sim -\int_t^T \frac{1}{(T-s)\ln^3(T-s)}ds \]

\[ = -\frac{1}{2|\ln(T-t)|^2}, \]

direct computation shows

\[ \partial_t^k [p_0(t)Z_5(y)] \sim \frac{1}{(T-t)^k\ln(T-t)^2}Z_5(t). \quad (3.22) \]

Now we decompose \(\phi = \phi_{\mu_0} + \phi_0 + \phi_1\), our discuss before has shown the estimate of \(\phi_{\mu_0}\) and \(\phi_0\), where

\[ \phi_{\mu_0}(y,t) = p_{\mu_0}(t)Z_0(y) \lesssim \frac{\lambda(T-t)^4}{1 + |y|^{2+a}}, \quad (3.23) \]

and

\[ \phi_0(y,t) = p_0(t)Z_5(y) \sim -\frac{1}{\ln(T-t)^2}\frac{1}{1 + |y|^2}, \quad (3.24) \]

the term \(\phi_1\) is the solution to the equation

\[ \lambda^2 \phi_1 = \Delta_y \phi_2 + pU(y)^{p-1}\phi_2 + h_1(y,t), \quad (3.25) \]

where

\[ h_1(y,t) = h(y,t) - q_{\mu_0}(t)Z_0(y) - q_0(t)Z_5(y) \]

and we have \(h_1(y,t)\) orthogonal to \(Z_5(y)\) for \(t \in [0,T]\).

Similar discuss if we formally treat equation (3.25) approximately as an elliptic equation

\[ \Delta_y \phi_1 + pU(y)^{p-1}\phi_1 + h_1(y,t) = 0 \quad (3.26) \]
if \( h_1(y, t) \) satisfies the orthogonality condition and behaves like \( h_1 \sim \frac{e(t)}{1 + |y|^a} \), from the Fredholm theorem, we expect \( \phi_1 \sim \frac{e(t)}{1 + |y|^a} \), and the decay rate can help us get a good estimate in neck part \( |y| \sim 2R \), however (3.25) is a parabolic equation hence there is no Fredholm theorem, thanks the linear theory in [24] which is so-called secondary inner-outer gluing, can give us a better estimate around the original and the same at \( |y| \sim R \), we copy this lemma and use it in our inner problem.

**Lemma 3.1.** Let constant \( a, \nu > 0 \) and \( \sigma \in (0, 1) \). For \( T > 0 \) sufficiently small and any \( h(y, t) = h(|y|, t) \) satisfying the orthogonality condition \( \int_{B_{2R}} h(y, t) Z(y) dy = 0 \) and \( \|h\|_{\nu, 2+a} < \infty \), there exists a solution \( \phi \) solving (53), and \( \phi \) satisfies the estimate

- For \( |y| \leq 2R^\sigma \)
  \[
  |\phi(y, t)| + (1 + |y|)|\nabla \phi(y, t)| \lesssim \frac{\lambda_1^{\nu}(t) R^{\sigma(4-a)} \log R}{1 + |y|^4} \|h\|_{\nu, 2+a}.
  \]

- For \( 2R^\sigma \leq |y| \leq 2R \)
  \[
  |\phi(y, t)| + (1 + |y|)|\nabla \phi(y, t)| \lesssim \frac{\lambda_1^{\nu}(t) \log R}{1 + |y|^4} \|h\|_{\nu, 2+a}.
  \]

where
  \[
  \|h\|_{\nu, a} := \sup_{y \in B_{2R}, t \in [0, T]} \lambda_1^{\nu}(1 + |y|^a) |h(y, t)|,
  \]
we omit the proof, the readers can read it in [24], now we start to estimate the derivative of \( \phi_1(y, t) \), recall

\[
\begin{align*}
  h_1(y, t) &= h(y, t) - q_{\mu_0}(t) Z_0(y) - q_0(t) Z_5(y) \\
  &= 3\lambda U^2(y)(-Z_0^* + \Psi_0 + \psi) - q_{\mu_0}(t) Z_0(y) - q_0(t) Z_5(y),
\end{align*}
\]

define

\[
\tau = \int_0^t \frac{1}{\lambda^2(s)} ds,
\]

(3.27) \( ? \)

we have

\[
\partial_\tau = \lambda^2 \partial_t,
\]

denote \( \tilde{\phi}_1(y, \tau) = \phi_1(y, t) \), and \( \tilde{\phi}_1^{(k)} = \partial_{\tau}^k \tilde{\phi}_1 \), take the \( l \)-th derivative \( \partial_\tau^l \) of both sides of the inner problem (3.25)

\[
\frac{\partial_\tau^l \tilde{\phi}_1^{(l)}}{\partial_\tau} = \Delta_y \tilde{\phi}_1^{(l)} + pU(y)\tilde{\phi}_1^{(l-1)} + \partial_\tau^l h_1(y, t),
\]

(3.28) \( ? \)

then it’s easy to see

\[
\partial_\tau^l h(y, t) = \partial_\tau^l [3\lambda U^2(y)(-Z_0^*(\lambda_i y_i) + \Psi_0 + \psi(\lambda_i y_i))]
\]

\[
\sim \frac{\lambda^{2l+1}}{(1 + |y|^2)}^l,
\]

(3.29) \( ? \)

and

\[
\partial_\tau^l (q_{\mu_0}(t) Z_0(y)) \sim \lambda^{2l+1} Z_0(y),
\]

(3.30) \( ? \)

also from (3.24), we have

\[
|\partial_\tau^l [q_0(t) Z_5(y)]| \lesssim \frac{\lambda^{2l+1} R^a}{|\ln(T-t)|^2 (1 + |y|^{2+a})},
\]

(3.31) \( ? \)

with the above estimate, choose \( \sigma \) very small and use the lemma we have

\[
|\partial_\tau^l \phi_1(y, t)| + (1 + |y|)|\nabla \partial_\tau^l \phi_1(y, t)| \lesssim \begin{cases} \frac{\lambda^{2l+1} R^a \log R}{\ln(T-t)^2 (1 + |y|^2)} \|h_1\|_{\nu, 2+a}, & |y| \leq 2R^\sigma \\ \frac{\lambda^{2l+1} R^a \log R}{|\ln(T-t)|^2 (1 + |y|^2)} \|h_1\|_{\nu, 2+a}, & 2R^\sigma \leq |y| \leq 2R. \end{cases}
\]

(3.32) \( ? \)
then, after integral $\partial_t^k \phi_1$ we easily get

$$
|\partial_t^k \phi_1(y, t)| + (1 + |y|)|\nabla \partial_t^k \phi_1(y, t)| \lesssim \begin{cases}
\frac{(T-t)^{4 \lambda - \nu + 2} \log R}{|\ln (|1 + |y|^2)|} \|h_1\|_{\nu, 2+a}, & |y| \leq 2R^\sigma \\
\frac{|\ln (|1 + |y|^2)|}{|\ln |T-t||} \|h_1\|_{\nu, 2+a}, & 2R^\sigma \leq |y| \leq 2R.
\end{cases}
$$

which is the estimate of the inner problem.

4. OUTER PROBLEM

In this section we shall solve the outer problem, our goal is to find a solution $|\psi| \leq \delta(T-t)^{3}$ near 0, and has a proper decay rate at the infinity, in our proof, the critical choice of $R = R(t)$ is the key to solving the outer problem, recall the restrain of $R$

$$
\lambda R(t) \ll \sqrt{T-t},
$$

and in the last section, we choose $\sigma$ and $a$ be small, hence the choice of $R$ will not influence the solvation of the inner problem, and for the outer problem, we hope $R$ to be large, throughout this paper, we take

$$
R(t) = \frac{\sqrt{T-t}}{|\ln (T-t)|^{\frac{4}{\gamma}}}.
$$

(4.1)

And we put

$$
W(x, t) = \begin{cases}
(1 + |x|^{2 + 2}) & |x| \leq 2(T-t)^{-\frac{\gamma}{\nu+\sigma}} \\
|\frac{1}{|x|^{2 + 2}}| & 2(T-t)^{\frac{\gamma}{\nu+\sigma}} \leq |x| \leq 1, \\
& |x| \geq 1,
\end{cases}
$$

(4.2)

fixed small positive constant $\delta_0$, let $X$ be the space of continuous functions on $\mathbb{R}^4 \times [0, T]$ satisfying

$$
|\psi(x, t)| \leq \delta_0 W(x, t) \text{ for } t \in [0, T].
$$

(4.3)

Throughout this section, we assume the outer solution $\partial_t^l \psi \in X$.

To solve the outer problem, the following lemma is useful.

**Lemma 4.1.** Consider nonhomogeneous heat equation

$$
\begin{cases}
\partial_t \Phi - \Delta \Phi = g(x, t) & \text{for } (x, t) \in \mathbb{R}^n \times (0, T), \\
\Phi |_{t=0} = 0 & \text{for } x \in \mathbb{R}^n.
\end{cases}
$$

Suppose

$$
|g(x, t)| \leq \frac{1}{\lambda^2(t)(1 + |y|^{2 + a})}, \quad (x, t) \in \mathbb{R}^n \times (0, T),
$$

where $y = \frac{x}{\lambda(t)}$, $0 < a < 1$ be fixed, and $\lambda(t) \sim \frac{\lambda(t)}{(T-t)} \to 0$, as $t \to T$, then there exist constants $C_1$, $\gamma > 0$ such that

$$
|\Phi| \leq C_1 (T^\gamma + \frac{1}{1 + |y|^{2a}}).
$$

4.1. The estimate of $G$. Recall the outer problem

$$
\psi_t = \Delta \psi + G,
$$

where

$$
G = 3\lambda^{-2}(1 - \eta_R)U(y)^2(-Z_0^* \eta + \Psi_0 + \psi) + \lambda^{-3}(\Delta_y \eta_R \phi + 2\nabla_y \eta_R \cdot \nabla \phi - \lambda^2 \phi \partial_t \eta_R)
$$

$$
+ \lambda^{-2}(\phi + y \cdot \nabla \phi) + h_{out} + \mathcal{K} + \mathcal{N},
$$

we are going to estimate the right-hand side term $G$ in 3 ways, the form we list in the previous lemma

$$
\frac{1}{\lambda^2(t)(1 + |y|^{2 + a})},
$$


we give an estimate of the terms in
\[ \partial_t^l \psi_t = \Delta \partial_t^l \psi + \partial_t^l G, \]
we give an estimate of the terms in \( \partial_t^l G \)

\[ |\partial_t^l \lambda^2(1-\eta)U(y)^2(-Z_0^\eta + \Psi_0 + \psi)| \lesssim 3\lambda^{-2} \frac{(1-\eta)}{(1+|y|^2)^2}, \]  
(4.4)

one of the difficult term is \( \partial_t^l [\lambda^{-3}(\Delta_y \eta R \phi + 2\nabla_y \eta R \cdot \nabla \phi - \lambda^2 \phi \partial_t \eta R)], \) thanks the estimate (3.33) in last section, we have

\[ \partial_t^k \phi_1(y,t)|_{y\sim 2R} \sim \lambda \frac{(T-t)^{l-k}}{|\ln(T-t)|^2} |_{y\sim 2R} \sim \lambda \frac{(T-t)^{l-k}}{|\ln(T-t)|^2}, \]
(4.5)

then

\[ |\partial_t^l [\lambda^{-3}(\Delta_y \eta R \phi_1 + 2\nabla_y \eta R \cdot \nabla \phi_1 - \lambda^2 \phi_1 \partial_t \eta R)]| \lesssim \lambda^{-2} \frac{1}{|\ln(T-t)|^3 R^2}, \]
(4.6)

and since \( \phi_0 = q_0(t) Z_5(y) \), by the choice of \( R \)

\[ |\partial_t^l [\lambda^{-3}(\Delta_y \eta R \phi_0 + 2\nabla_y \eta R \cdot \nabla \phi_0 - \lambda^2 \phi_0 \partial_t \eta R)]| \lesssim (T-t)^{-l} \lambda^{-3} \frac{1}{R^2 (T-t)^2 |\ln(T-t)|^2 R^2}, \]
(4.7)

\[ \lambda^{-2} \hat{\lambda} \hat{\lambda} = \lambda^{-2} \hat{\lambda} (\phi_{\mu_0} + y \cdot \nabla \phi_{\mu_0}) + \lambda^{-2} \hat{\lambda} (\phi_0 + y \cdot \nabla \phi_0) + \lambda^{-2} \hat{\lambda} (\phi_1 + y \cdot \nabla \phi_1), \]
(4.8)

firstly

\[ |\partial_t^l (\lambda^{-2} \hat{\lambda} (\phi_{\mu_0} + y \cdot \nabla \phi_{\mu_0}))| \lesssim \lambda^{-2} \frac{\hat{\lambda}}{1+|y|^{2+a}}, \]
(4.9)

and

\[ |\partial_t^l (\lambda^{-2} \hat{\lambda} (\phi_0 + y \cdot \nabla \phi_0))| \lesssim |\lambda^{-2} \hat{\lambda} (\phi_0 + y \cdot \nabla \phi_0)| \]
\[ \lesssim \lambda^{-2} \frac{\hat{\lambda}(T-t)^4 R^{a+\sigma(4-a)}}{|\ln(T-t)|^2 (1+|y|^2)}, \]
(4.10)

let us estimate \( \partial_t^l (\lambda^{-2} \hat{\lambda} (\phi_1 + y \cdot \nabla \phi_1)) \) in two region, if \( |y| \leq 2R^a \), since \( \sigma \) is small enough,

\[ |\partial_t^l (\lambda^{-2} \hat{\lambda} (\phi_1 + y \cdot \nabla \phi_1))| \lesssim |\lambda^{-2} \hat{\lambda} (\phi_1 + y \cdot \nabla \phi_1)| \]
\[ \lesssim \lambda^{-2} \frac{\lambda(t)(T-t)^1 R^{a+\sigma(4-a)}}{|\ln(T-t)|^4 (1+|y|^4)}, \]
(4.11)
if $2R^\sigma \leq |y| \leq 2R$,  
\[ |\partial_t^l (\lambda^{-2} \Phi_1 + y \cdot \nabla \Phi_1)| \lesssim |\lambda^{-2} \lambda^{(l+1)} (T-t)^l \lambda(t) R^\sigma \log R | \frac{1}{\ln(T-t)} \left( 1 + \frac{|y|^4}{|y|^2} \right) \]
\[ \lesssim \lambda^{-2} \lambda^{(l+1)} (T-t)^l \lambda(t) R^2 \log R | \frac{1}{\ln(T-t)} \left( 1 + |y|^2 \right) \]
\[ \lesssim \frac{1}{|\ln T|} \lambda^{-2} \lambda^{(l+1)} \left( 1 + |y|^2 \right), \]  

\begin{align*}
\text{the nonlinear term} \\
&\partial_t^l N = \partial_t^l [(U\lambda + Z_0^* + \Psi_0 + \Psi)^3 - U^3 \lambda - 3U^2 Z_0^*(Z_0^* + \Psi_0)] \\
&= \partial_t^l [3U\lambda (Z_0^* + \Psi_0 + \Psi)^2 + (Z_0^* + \Psi_0 + \Psi)^3],
\end{align*}

is troublesome, if $|z| \leq 2(T-t)^{-\frac{1}{4l+6}}$  
\[ |\partial_t^l [3U\lambda (Z_0^* + \Psi_0 + \Psi)^2]| \lesssim \lambda^{-1} (T-t)^l \left( 1 + \frac{|z|^{4l}}{1 + |y|^2} \right) \]
\[ \lesssim \lambda^{-1} (T-t)^l \left( 1 + \frac{|z|^{4l+4}}{1 + |y|^2} \right) \]
\[ \lesssim \lambda^{-2} \lambda^{(l+1)} \left( 1 + |y|^2 \right), \]

if $2(T-t)^{\frac{1}{4l+6}} \leq |x| \leq 1$, notice $U\lambda(y) \sim \frac{1}{|x|^2}$, and $|\psi| \lesssim (T-t)^{-\frac{1}{2}}$, since $l \geq 1$, then we have  
\[ |\partial_t^l [3U\lambda (Z_0^* + \Psi_0 + \Psi)^2]| \lesssim \lambda^{-1} (T-t)^l \frac{1}{1 + |y|^2} \]
\[ \lesssim \lambda^{-2} \lambda^{(l+1)} \left( 1 + |y|^2 \right), \]

if $|x| \geq 1$,  
\[ |\partial_t^l [3U\lambda (Z_0^* + \Psi_0 + \Psi)^2]| \lesssim T \frac{1}{|x|^2}, \]  

and there is another nonlinear term, if $|z| \leq 2(T-t)^{-\frac{1}{4l+6}}$, since $l \geq 1$ 
\[ |\partial_t^l (Z_0^* + \Psi_0 + \Psi)^3| \lesssim (T-t)^{2l} |1 + \frac{|z|^{6l} + \delta^3_3 |z|^{6l+6}}{(T-t)^{\frac{6l+6}{4l+6}}} | \left( 1 + \frac{|z|^2}{|y|^2} \right) \]
\[ \lesssim (T-t)^{2l} |(T-t)^{-\frac{6l}{4l+6}} + \delta^3_3 (T-t)^{-\frac{6l+6}{4l+6}} | \left( 1 + \frac{|z|^2}{|y|^2} \right) \]
\[ \lesssim (T-t)^{\frac{6l+6}{4l+6} + \frac{2l}{4l+6}} \left( 1 + \frac{|z|^2}{|y|^2} \right) \]
\[ \lesssim \sqrt{T} (1 + |z|^2) \]

if $2(T-t)^{\frac{2l+1}{4l+6}} \leq |x| \leq 1$, since $l \geq 1$  
\[ |\partial_t^l (Z_0^* + \Psi_0 + \Psi)^3| \lesssim (T-t)^{2l} \left( 1 + \frac{\delta^3_3 (T-t)^{-1}}{|x|^\frac{2l}{4l+6}} \right) \]
\[ \lesssim \frac{(T-t)^{2l}}{|x|^\frac{2l}{4l+6}} \]
\[ \lesssim \frac{T}{|x|^\frac{2l}{4l+6}}, \]
if $|x| \geq 1$

$$\partial_t^2 (Z^*_0 + \Psi_0 + \Psi)^3 \lesssim \delta^3_0 (T-t)^2$$

$$|x|^{\frac{3m}{2n}}$$

$$\lesssim \frac{T}{|x|^3},$$

(4.19) (7)

• Recall in chapter 2, we have

$$-\alpha_0 \lambda \zeta \int_{-T}^{t} \lambda^{(l+1)}(s) \kappa_\zeta(\zeta, t-s) ds \lesssim \lambda^{-2}(t) \frac{\lambda \dot{\lambda}(t)}{1+|y|^2}$$

and

$$\frac{\alpha_0}{\lambda(t) (1+|y|^2)^{3/2}} \int_{-T}^{t} \lambda^{(l+1)}(s) [-\zeta \kappa_\zeta (\zeta, t-s) + \kappa_\zeta (\zeta, t-s)] ds \lesssim \frac{\alpha_0 \lambda^{(l+1)}(\cdot) \|_{\infty}}{\lambda^2(t) (1+|y|^2)^2}$$

(4.20) (7)

(4.21) (7)

In summary, we have

$$|\partial_t^2 G| \lesssim \frac{1}{|\ln T|} \lambda^{-2} \lambda^{(l+1)} 1 + |y|^2 + \lambda^{-2} \frac{1}{|\ln T| (1+|y|^{2+a})} + \frac{T}{|x|^{\frac{3m}{2n}}} 2(T-t)^{\frac{2l+1}{2l+1}} \lesssim |x|^{1}$$

(4.22) (7)

4.2. Solve the outer problem.

• From the similar estimate of (2.13) and (2.14), we know that

$$\int_{0}^{t} e^{(t-s)\Delta} \left( \frac{1}{|\ln T|} \lambda^{-2}(s) \frac{\lambda^{(l+1)}(s)}{1+|y|^2} \right) ds \lesssim \frac{1}{|\ln T|},$$

• From lemma 4.1, we have

$$\int_{0}^{t} e^{(t-s)\Delta} \left( \frac{1}{|\ln T|} \lambda^{-2}(s) \frac{1}{|\ln T| (1+|y|^{2+a})} \right) ds \lesssim \frac{1}{|\ln T|},$$

• we use the comparison principle to estimate $\int_{0}^{t} e^{(t-s)\Delta} \frac{T}{|x|^2} 1_{|x| \geq 1} ds$, we choose $\frac{t}{|x|^2}$ as a comparison function, the initial condition is obviously, then we verify

$$(\partial_t - \Delta) \frac{t}{|x|^2} = \frac{1}{|x|^2} > \frac{T}{|x|^2},$$

thus by comparison principle in $|x| \geq 1$,

$$\int_{0}^{t} e^{(t-s)\Delta} \frac{T}{|x|^2} 1_{|x| \geq 1} ds \lesssim \frac{t}{|x|^2},$$

• we will treat the term $\frac{T}{|x|^2} 1_{2(T-t)^{\frac{2l+1}{2l+1}}} \lesssim |x| \leq 1$, similar as $\frac{T}{|x|^2} 1_{|x| \geq 1}$, we choose $\frac{t}{|x|^2}$ as a comparison function, the initial condition is easy to check, then due to $\frac{2l+2}{2l+1} < 2$, for $l \geq 1$, then

$$(\partial_t - \Delta) \frac{t}{|x|^2} \geq \frac{1}{|x|^2} \geq \frac{T}{|x|^2},$$

(4.23) (7)

by comparison principle in $2(T-t)^{\frac{2l+1}{2l+1}} \leq |x| \leq 1$, $2(T-t)^{\frac{2l+1}{2l+1}} \leq |x| \leq 1$,

$$\int_{0}^{t} e^{(t-s)\Delta} \frac{T}{|x|^2} 1_{2(T-t)^{\frac{2l+1}{2l+1}}} ds \lesssim \frac{t}{|x|^2}. $$
then, we deal with the term $\sqrt{T}(1 + |z|^2)1_{|z| \leq (T-t) - \frac{4}{\sqrt{T}} + h_{out}$, obviously

$$|h_{out}| = |(T-t)[Z_0(x,t)(\partial_t - \Delta)\eta(z(T-t)\frac{\partial}{\partial t}) - 2\nabla Z_0(x,t)\nabla\eta(z(T-t)\frac{\partial}{\partial t})]|$$

$$\lesssim |z|^21_{(T-t) - \frac{4}{\sqrt{T}} \lesssim |z| \leq 2(T-t) - \frac{4}{\sqrt{T}}}, \quad \text{(4.24) (7)}$$

we use the comparison principle, let $\tilde{\Psi} = \Psi_2(z, \tau')$ solve

$$\begin{cases}
\partial_{\tau'}\Psi_2 - A_z\Psi_2 = e^{-\tau}\sqrt{T}(1 + |z|^2)1_{|z| \leq (T-t) - \frac{4}{\sqrt{T}}} + e^{-\tau}h_{out} := H_s, \\
\Psi_2|_{\tau' = \tau_0} = 0,
\end{cases} \quad \text{(4.25) (7)}$$

and we decompose $\Psi_2$ into

$$\Psi_2 = \Psi_{2,0} + \Psi_{2,1}$$

where $\Psi_{2,0} = (\Psi_2, e_0)e_0$, and $\Psi_{2,1} = \Psi_2 - \Psi_{2,0}$, from equation (4.25), we have

$$|(\Psi_2, e_0)| = e_0\int_{\tau_0}^{\tau'}\int_{\mathbb{R}^4} e^{-s}[|e^{-\tau}\sqrt{T}(1 + |z|^2)1_{|z| \leq (T-t) - \frac{4}{\sqrt{T}}} + h_{out}(z, s)]e^{-\frac{|z|^2}{4}}dzds$$

$$\leq e_0\int_{\tau_0}^{\tau'}\int_{\mathbb{R}^4} (T\frac{2}{\sqrt{T}}(1 + |z|^2)) + |z|^21_{(T-t) - \frac{4}{\sqrt{T}} \lesssim |z| \leq 2(T-t) - \frac{4}{\sqrt{T}}})e^{-\frac{|z|^2}{4}}dzds$$

$$\lesssim T^2 + e_0\int_{\tau_0}^{\tau'}e^{-\frac{|z|^2}{4}}dzds < T,$$

then $\Psi_{2,1}$ satisfies

$$\begin{cases}
\partial_{\tau'}\Psi_{2,1} - A_z\Psi_{2,1} = H_s - (H_s, e_0)e_0 \\
\Psi_{2,1}|_{\tau' = \tau_0} = 0
\end{cases} \quad \text{(4.27) (7)}$$

times $\Psi_{2,1}e^{-\frac{|z|^2}{4}}$ both sides of (4.27) and integral over $\mathbb{R}^4$, since $(\Psi_{2,1}, e_0) = 0$, we deduce that

$$(H_s, \Psi_{2,1}) = \frac{1}{2}\partial_{\tau'}||\Psi_{2,1}(z, \tau')||^2_{\rho} - (A_z\Psi_{2,1}, \Psi_{2,1})$$

$$\geq \frac{1}{2}\partial_{\tau'}||\Psi_{2,1}||^2_{\rho} + ||\Psi_{2,1}||^2_{\rho}$$

then we easily get

$$\frac{1}{2}\partial_{\tau'}||\Psi_{2,1}||^2_{\rho} + \frac{1}{2}||\Psi_{2,1}||^2_{\rho} \leq ||H_s||^2_{\rho}$$

and then

$$||\Psi_{2,1}||_{\rho} \lesssim e^{-\frac{\tau}{T}} \leq \sqrt{T}$$

for large fixed constant $0 < 2L < T^{-\frac{4}{\sqrt{T}}}$, we have $||\Psi_{2,1}||_{L^2} \leq 2\sqrt{T}$, and from parabolic $L^2$ estimate, we deduce that

$$||\Psi_{2,1}||_{L^\infty(B_L)} \leq C||\Psi_{2,1}||_{\rho} \lesssim 2\sqrt{T}$$

since we choose $L$ largely enough, we have $|e_{l+1}(z)| = |al_0 + \cdots + a_l + |z|^{2l+2}| > \frac{|a_{l+1}|}{2}z^{2l+2} > 0$ in $\mathbb{R}^4 \setminus B_L$

we use the comparison principle, let $\tilde{\Psi}_2(z, \tau') = T\frac{4}{\sqrt{T}}|e_{l+1}(z)|$ be a comparison function, we verify that

$$\partial_{\tau'}\tilde{\Psi}_2(z, \tau') - A_z\tilde{\Psi}_2(z, \tau') \lesssim |z|^21_{(T-t) - \frac{4}{\sqrt{T}} \lesssim |z| \leq 2(T-t) - \frac{4}{\sqrt{T}}})$$

$$\lesssim T^{2l+2}|z|^{2l+2}$$

$$\lesssim T^{\frac{4}{\sqrt{T}}}(l+1)|e_{l+1}(z)|$$

$$= \partial_{\tau'}\tilde{\Psi}_2(z, \tau') - A_z\tilde{\Psi}_2(z, \tau')$$
We finish the proof of theorem 1.1 by fixed point argument, we first give a proof of Theorem 1.1 since $T$ from the estimate (4.35) of section 4, we know the inner solution $\phi = \phi(|x|)$ and on the boundary that the scale parameter $\lambda = \lambda(|x|)$ is chosen by comparison function is chosen by $\psi = \frac{(t + \delta_0)}{|x|^{\frac{2l+1}{2l+4}}}$, we use comparison principle again, the $\partial_t \tilde{\Psi}_2 - \Delta \tilde{\Psi}_2 (z, \tau') > \frac{1}{|x|^{\frac{2l+1}{2l+4}}} > 0 = \partial_t \Psi_2 - \Delta \Psi_2$ and the boundary$\psi\big|_{|x|=2(T-t)^{\frac{2l+1}{2l+4}}} = \frac{(t + \delta_0)}{|x|^{\frac{2l+1}{2l+4}}} \big|_{|x|=2(T-t)^{\frac{2l+1}{2l+4}}} = \frac{(t + \delta_0)(T-t)^{-\frac{1}{2l+4}}}{2\frac{2l+1}{2l+4}} \geq 2T^{\frac{1}{2l+4}}(T-t)^{-\frac{1}{2l+4}}.
$then we have $|\Psi_2| \leq \frac{(t + \delta_0)}{|x|^{\frac{2l+1}{2l+4}}}$, and in region $|x| \geq 1$, we use $\frac{3\delta_0}{4|x|^2}$ as a comparison function, we verify that

$$(\partial_t - \Delta) \frac{3\delta_0}{4|x|^2} = 0 = (\partial_t - \Delta) \Psi_2 \quad (4.34)$$

and on the boundary $\frac{1}{|x|^2} = \frac{3\delta_0}{4} \geq t + \frac{\delta_0}{2} = \Psi_2 |_{|x|=1}.$

That is $|\phi| = \begin{cases} \frac{\delta_0(1 + |x|^{2l+2})}{2(1 + |x|^{2l+2})} & |x| \leq 2(T-t)^{-\frac{1}{2l+4}} \\ \frac{\delta_0}{|x|^{\frac{2l+1}{2l+4}}} & 2(T-t)^{\frac{2l+1}{2l+4}} \leq |x| \leq 1, \\ \frac{1}{|x|^{\frac{2l+1}{2l+4}}} & |x| \geq 1, \end{cases} \quad (4.35)$

5. The proof of theorem 1.1

We finish the proof of theorem 1.1 by fixed point argument, we first give a $\psi \in X$, then we choose the scale parameter $\lambda = \lambda(|x|)$ as in section 4, after choosing $\lambda$, we solve the inner problem and gain the inner solution $\phi = \phi(|x|)$, and finally, we solve the outer problem and get $\psi^* = \psi^*[\psi, \lambda, \phi] = \psi^*|\psi|$, we define $T[\psi] = \psi^*$, \quad (5.1)

from the estimate (4.35) of section 4, we know $T : X \to X$.

By Schauder’s fixed-point theorem, we know that $T$ has a fixed point $\psi$ in $X$. Which finished the proof of Theorem 1.1
References

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