Global well-posedness and scattering for nonlinear Schrödinger equations with algebraic nonlinearity when $d = 2, 3, u_0$ radial

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Abstract: In this paper we discuss global well - posedness and scattering for some initial value problems that are $L^2$ supercritical and $\dot{H}^1$ subcritical, with radial data. We prove global well - posedness and scattering for radial data in $H^s$, $s > s_c$, where the problem is $\dot{H}^{s_c}$ - critical. We make use of the long time Strichartz estimates of [15] to do this.

1 Introduction

In this paper we examine the initial value problem

$$(i\partial_t + \Delta)u = |u|^2u, \quad u(0, x) = u_0 \in H^s(\mathbb{R}^3),$$

(1.1)
as well as the initial value problem

$$(i\partial_t + \Delta)u = |u|^{2k}u, \quad u(0, x) = u_0 \in H^s(\mathbb{R}^2),$$

(1.2)
and in each case $u_0$ is radial.

(1.1) is $\dot{H}^{1/2}$ - critical. That is, a solution to (1.1) gives rise to a family of solutions via the scaling,

$$u(t, x) \mapsto u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x).$$

(1.3)

Under this scaling,

$$\|u_\lambda(0, x)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} = \|u(0, x)\|_{\dot{H}^{1/2}(\mathbb{R}^3)},$$

(1.4)
\[ \|u_\lambda(0, x)\|_{H^1(\mathbb{R}^3)} = \lambda^{1/2}\|u(0, x)\|_{H^1(\mathbb{R}^3)}, \]  

(1.5)

\[ \|u_\lambda(0, x)\|_{L^2(\mathbb{R}^3)} = \lambda^{-1/2}\|u(0, x)\|_{L^2(\mathbb{R}^3)}. \]  

(1.6)

Meanwhile, (1.2) is \( \dot{H}^{1-\frac{1}{2}}(\mathbb{R}^2) \) - critical, since a solution to (1.2) gives rise to a family of solutions under the mapping

\[ u(t, x) \mapsto u_\lambda(t, x) = \lambda^{\frac{1}{2}}u(\lambda^2 t, \lambda x). \]  

(1.7)

This scaling is crucial to local well - posedness.

**Theorem 1.1** (1.1) is locally well - posed for any \( u_0 \in H^s(\mathbb{R}^3), \ s > \frac{1}{2} \) on some interval \([-T, T]\), \( T(\|u_0\|_{H^s}, s) > 0 \). If \( u_0 \in \dot{H}^{1/2}(\mathbb{R}^3) \) then (1.1) is locally well - posed on some interval \([-T, T]\), \( T(u_0) > 0 \), where \( T(u_0) \) depends on the profile of the initial data and not just its size. Moreover, for \( \|u_0\|_{H^{1/2}(\mathbb{R}^3)} \) small, (1.1) is globally well - posed and asymptotically complete.

The corresponding results also hold for (1.2) and the critical space \( \dot{H}^{1-\frac{1}{2}}(\mathbb{R}^2) \).

**Proof:** See [5], [7]. □

**Remark:** [8] and [9] proved that theorem 1.1 is sharp.

**Definition 1.1 (Well - posedness)** The initial value problem (1.1) is well - posed on an open interval \( I \subset \mathbb{R} \), \( 0 \in I \), for \( u_0 \in X \), \( X \) is a function space, if

1. (1.1) has a unique solution on \( I \times \mathbb{R}^3 \),
2. The solution satisfies the Duhamel formula

\[ u(t) = e^{it\Delta}u_0 - i \int_0^t e^{i(t-\tau)\Delta}(\|u\|^2 u)(\tau)d\tau, \]  

(1.8)

3. For any compact \( J \subset I \), the map \( u_0 \mapsto L^5_t L^k_x(J \times \mathbb{R}^3) \) is continuous.

The definition for global well - posedness and asymptotic completeness for (1.2) corresponds to (1) – (3) above. Replace \( L^5_t L^k_x \) with \( L^{\frac{4k}{k+1}}_t L^{\frac{4k}{k+1}}_x \).
Definition 1.2 (Asymptotic completeness) The initial value problem (1.1) is asymptotically complete if for any $u_0 \in X$ there exist $u_+ \in H^s(\mathbb{R}^3)$ such that

$$\lim_{t \to +\infty} \|u(t) - e^{it\Delta} u_+\|_{H^s(\mathbb{R}^3)} = 0, \quad \lim_{t \to -\infty} \|u(t) - e^{it\Delta} u_-\|_{H^s(\mathbb{R}^3)} = 0. \quad (1.9)$$

Remark: [22] proved that (1.1) is globally well-posed and asymptotically complete if and only if $\|u(t)\|_{H^{1/2}(\mathbb{R}^3)}$ is uniformly bounded on its interval of existence. To the author’s knowledge there is no corresponding result for (1.2), $k > 1$.

(1.2) with $k = 1$ is now completely solved. [25] proved that (1.2) is globally well-posed and scattering for any $u_0 \in L^2(\mathbb{R}^2)$, $u_0$ radial. [16] extended this to nonradial data.

In this paper we show that (1.1) and (1.2) are globally well posed for $u_0 \in H^s$, $s > \frac{1}{2}$, and $s > 1 - \frac{1}{k}$ respectively.

Theorem 1.2 The initial value problem (1.1) is globally well-posed for any $s > \frac{1}{2}$ for $u_0$ radial. Moreover, the problem is asymptotically complete.

Theorem 1.3 The initial value problem (1.2) is globally well-posed and asymptotically complete for any $s > 1 - \frac{1}{k}$, $u_0$ radial.

The I-method is used to prove theorems 1.2 and 1.3. A solution to (1.1) conserves the quantities mass, $M(u(t)) = \int |u(t,x)|^2 dx = M(u(0))$, and energy, $E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 dx + \frac{1}{4} \int |u(t,x)|^4 dx$. (1.10)

A solution to (1.2) has the conserved energy

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 dx + \frac{1}{2k+2} \int |u(t,x)|^{2k+2} dx. \quad (1.11)$$

A solution to (1.2) has the conserved energy

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 dx + \frac{1}{2k+2} \int |u(t,x)|^{2k+2} dx. \quad (1.12)$$
combined with theorem 1 prove that (1.2) are globally well-posed. See [19], [29] for a proof of scattering in the radial case, [13], [11], [33] for a proof of scattering in the nonradial case for \( u_0 \in H^1 \).

The reason for the gap between the local well posedness result of theorem 1 and the regularity needed to prove a global result in [19] \( s = 1 \) is due to an absence of a conserved quantity that controls \( \|u(t)\|_{L^s} \) for \( 0 < s < 1 \). It is true that the momentum, a \( \dot{H}^{1/2} \) - critical quantity, is conserved, but this quantity does not control the \( \dot{H}^{1/2} \) norm.

The first progress in extending the global well-posedness results for data in \( H^1 \) to \( H^s \), \( s < 1 \) came from the Fourier truncation method. [3] proved that the cubic nonlinear initial value problem is globally well-posed in two dimensions for data in \( H^s \), \( s > \frac{4}{7} \) when \( d = 2 \). In three dimensions \([4]\) proved global well-posedness for \( s > \frac{11}{13} \) and global well-posedness and scattering for \( s > \frac{5}{6} \) for \( u_0 \) is radial. In fact, [3], [4] proved something more, namely that for \( s \) in the appropriate interval

\[
u(t) - e^{it\Delta} u_0 \in H^1(\mathbb{R}^d).
\]

It was precisely (1.13) that lead to the development of the \( I \) - method since (1.13) is false for many dispersive partial differential equations. See [24] for example. Instead, [12] defined an operator \( I : H^s(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d) \). Tracking the change of \( E(Iu(t)) \), [12] proved global well-posedness for the cubic nonlinear Schrödinger equation when \( d = 2 \), \( s > \frac{4}{7} \), and when \( d = 3 \), \( s > \frac{5}{6} \). [13] extended the \( d = 3 \) result to \( s > \frac{5}{6} \). [14] extended this to \( s > \frac{5}{7} \), and then [39] extended this result to \( s > \frac{5}{3} \).

Both [14] and [39] utilized the linear - nonlinear decomposition. See also [37] for this method in the context of the wave equation. Here we will use the long time Strichartz estimates of [15]. We show that for radial data, the long time Strichartz estimates decay rapidly, and thus can beat any polynomial power of \( N \) arising from the \( I \) - operator.

In §2 we will recall some linear estimates needed in the proof. In §3 we will describe the \( I \) - method and outline the proof of theorems 1.2 and 1.3. In §4 we will make an induction on frequency argument and prove long time Strichartz estimates for \( d = 3 \). In §5 we will prove the energy increment in \( d = 3 \), yielding theorem 1.2. Then in §6 we will make an induction on frequency argument and prove long - time Strichartz estimates for \( d = 2 \).
At this point it is necessary to mention some notation used in the paper. The expression $A \lesssim B$ indicates $A \leq C(B)D$, where $C(B)$ is some constant. When we say $A \lesssim \|u_0\|_{H^s}$ or $A \lesssim \|u_0\|_{H^s,k}$ we mean that $C$ depends on $\|u_0\|_{H^s}$ and the value of $s$ itself.

We will also use the notation $A \lesssim B^{a+}$. This means that for any $\epsilon > 0$, there exists $C(\epsilon)$ such that $A \leq C(\epsilon)B^{a+\epsilon}$. We will also use expressions like $\|u\|_{L^p} \lesssim A$, which means that $\|u\|_{L^p} \leq C(\epsilon)A$.

Throughout the paper it is unnecessary to distinguish between $u$ and $\bar{u}$. Therefore, we will often write expressions like $|u|^2u$ as $u^3$ for convenience.

## 2 Linear estimates

### 2.1 Sobolev spaces

**Definition 2.1 (Littlewood - Paley decomposition)** Take $\psi \in C_0^\infty(\mathbb{R}^d)$, $\psi(x) = 1$ for $|x| \leq 1$, $\psi = 0$ for $|x| > 2$, $\psi(x)$ is radial and decreasing. Then for any $j$ let

$$\phi_j(x) = \psi(2^{-j}x) - \psi(2^{-j+1}x).$$

Then let $P_j$ be the Fourier multiplier given by

$$\hat{P}_j f(\xi) = \phi_j(\xi) \hat{f}(\xi).$$

The Littlewood - Paley decomposition is quite useful since

**Theorem 2.1 (Littlewood - Paley theorem)** For any $1 < p < \infty$,

$$\|f\|_{L^p(\mathbb{R}^d)} \sim_{p,d} \left( \sum_{j=-\infty}^{\infty} |P_j f|^2 \right)^{1/2} \|L^p(\mathbb{R}^d)\).$$

**Definition 2.2 (Sobolev spaces)** For $s \in \mathbb{R}$ the Sobolev space $\dot{H}^s(\mathbb{R}^d)$ is the space of functions whose Fourier transform has finite weighted $L^2$ norm,

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} = \|\xi|^{s} \hat{f}(\xi)\|_{L^2(\mathbb{R}^d)},$$

where
\[ \hat{f}(\xi) = (2\pi)^{-d/2} \int e^{-ix\cdot\xi} f(x) dx. \]  
(2.5)

We define the inhomogeneous space
\[ \|f\|_{H^s(\mathbb{R}^d)} = \|(1 + |\xi|^2)^{s/2} \hat{f}(\xi)\|_{L^2(\mathbb{R}^d)}. \]  
(2.6)

Notice that
\[ \|P_j f\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-js} \|f\|_{H^s(\mathbb{R}^d)}, \quad \|P_j f\|_{L^2(\mathbb{R}^d)} \lesssim \inf(2^{-j\delta}, 1) \|f\|_{H^s(\mathbb{R}^d)}. \]  
(2.7)

Remark: (2.7) is called Bernstein’s inequality.

It follows from Hölder’s inequality that for \(2 \leq p \leq \infty\),
\[ \|P_j f\|_{L^p(\mathbb{R}^d)} \lesssim d 2^{jd(\frac{1}{2} - \frac{1}{p})} \|P_j f\|_{L^2(\mathbb{R}^d)}. \]  
(2.8)

Then for \(1 < p < \infty\), \(s = d(\frac{1}{2} - \frac{1}{p})\),
\[ \|f\|_{L^p(\mathbb{R}^d)} \lesssim_{s,d} \|f\|_{H^s(\mathbb{R}^d)}. \]  
(2.9)

We also have the radial Sobolev embedding
\[ \||x| P_j f\|_{L^\infty(\mathbb{R}^3)} \lesssim \|P_j f\|_{H^{1/2}(\mathbb{R}^3)}. \]  
(2.10)

See [36], [37], [38], [39], and many other sources for more details on Sobolev spaces.

### 2.2 Strichartz estimates

**Theorem 2.2** Let \(e^{it\Delta} \) be the solution operator to the linear evolution equation \((i\partial_t + \Delta)u = \). That is, \(u = e^{it\Delta}u_0 \) solves
\[ (i\partial_t + \Delta)u = 0, \quad u(0,x) = u_0. \]  
(2.11)

When \(d = 3\) let
\[ (p,q) \in \mathcal{A}_3 \iff 2 \leq p \leq \infty, \quad \frac{2}{p} = 3\left(\frac{1}{2} - \frac{1}{p}\right). \]  
(2.12)

When \(d = 2\) let
\[(p, q) \in A_2 \iff 2 < p \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}. \quad (2.13)\]

Let \(p'\) denote the Lebesgue dual, \(\frac{1}{p} + \frac{1}{p'} = 1\). Then if \((p, q), (\tilde{p}, \tilde{q}) \in A_d\), \(d = 2, 3\), in other words \((p, q), (\tilde{p}, \tilde{q})\) are admissible pairs,

\[
\|e^{it\Delta} u_0\|_{L^p_t L^q_x (\mathbb{R} \times \mathbb{R}^2)} \lesssim_p \|u_0\|_{L^2(\mathbb{R}^2)},
\]

\[
\| \int_0^t e^{i(t-\tau)\Delta} F(\tau) d\tau \|_{L^p_t L^q_x (I \times \mathbb{R}^2)} \lesssim_{p,q,\tilde{p},\tilde{q}} \|F\|_{L^p_t L^q_x (I \times \mathbb{R}^2)},
\]

and

\[
\|e^{it\Delta} u_0\|_{L^p_t L^q_x (\mathbb{R} \times \mathbb{R}^3)} \lesssim \|u_0\|_{L^2(\mathbb{R}^3)},
\]

\[
\| \int_0^t e^{i(t-\tau)\Delta} F(\tau) d\tau \|_{L^p_t L^q_x (I \times \mathbb{R}^3)} \lesssim \|F\|_{L^p_t L^q_x (I \times \mathbb{R}^3)}.
\]

**Proof:** \[38\] proved this theorem in the case \(p = q, \tilde{p} = \tilde{q}\). See \[10\], \[20\], and \[46\] for a proof of the general result, \(p > 2\). \[23\] proved the endpoint result \(p = 2\) when \(d = 3\). \[40\] gives a nice description of the overall theory. □

It is convenient, especially in three dimensions, to work with the Strichartz space and the dual Strichartz space. Let

**Definition 2.3 (Strichartz space)** Let \(S^0\) be the Strichartz space

\[
S^0(I \times \mathbb{R}^3) = L^\infty_t L^2_x (I \times \mathbb{R}^3) \cap L^2_t L^6_x (I \times \mathbb{R}^3).
\]

(2.16)

Let \(N^0\) be the dual

\[
N^0(I \times \mathbb{R}^3) = L^2_t L^2_x (I \times \mathbb{R}^3) + L^2_t L^{6/5}_x (I \times \mathbb{R}^3).
\]

(2.17)

Then theorem \(2.2\) implies

\[
\|e^{it\Delta} u_0\|_{S^0(\mathbb{R} \times \mathbb{R}^3)} \lesssim \|u_0\|_{L^2(\mathbb{R}^3)},
\]

\[
\| \int_0^t e^{i(t-\tau)\Delta} F(\tau) d\tau \|_{S^0(I \times \mathbb{R}^3)} \lesssim \|F\|_{N^0(I \times \mathbb{R}^3)}.
\]

(2.18)

We will also utilize the local smoothing estimate of \[34\]. \(\psi\) is the same \(\psi\) as in definition \(2.1\)

\[
\|e^{it\Delta} (P_j u_0)\|_{L^2_t L^2_x (I \times \{x: |x| \leq R\})} \lesssim 2^{-j/2} R^{1/2} \|P_j u_0\|_{L^2(\mathbb{R}^d)},
\]

(2.19)
as well as its dual,

\[ \| \int e^{-it\Delta} \psi \left( \frac{x}{R} \right) (P_j F(\tau)) \|_{L^2_\tau L^2_x} \leq 2^{-j/2} R^{1/2} \| \psi \left( \frac{x}{R} \right) P_j F \|_{L^2_\tau L^2_x}, \]  

(2.20)

### 2.3 \( U^2_\Delta \) and \( V^2_\Delta \) spaces

We will also make use of a function space adapted to the long time Strichartz estimates. This is a class of function spaces first introduced in [42] to study wave maps. [27], [28] applied these spaces to nonlinear Schrödinger problems. See [21] for a general description of these spaces. These spaces are quite useful to critical problems since the \( X^{s,b} \) spaces of [1], [2] (see also [18]) are not scale invariant except at \( b = \frac{1}{2} \), which has the same difficulty as the failure of the embedding \( \dot{H}^{1/2}(\mathbb{R}) \subset L^\infty(\mathbb{R}) \). Since we take \( u_0 \in H^s \) for any \( s > \frac{1}{2} \) they will prove to be useful here as well.

**Definition 2.4 (\( U^p_\Delta \) spaces)** Let \( 1 \leq p < \infty \). Let \( U^p_\Delta \) be an atomic space whose atoms are piecewise solutions to the linear equation,

\[ u_\lambda = \sum_k 1_{[t_k, t_{k+1})} e^{it\Delta} u_k, \quad \sum_k \| u_k \|_{L^2}^p = 1. \]  

(2.21)

Then for any \( 1 \leq p < \infty \),

\[ \| u \|_{U^p_\Delta} = \inf \{ \sum_\lambda |c_\lambda| : u = \sum_\lambda c_\lambda u_\lambda, u_\lambda \text{ are } U^p_\Delta \text{ atoms} \}. \]  

(2.22)

For any \( 1 \leq p < \infty \), \( U^p_\Delta \subset L^\infty L^2 \). Additionally, \( U^p_\Delta \) functions are continuous except at countably many points and right continuous everywhere.

**Definition 2.5 (\( V^p_\Delta \) spaces)** Let \( 1 \leq p < \infty \). Then \( V^p_\Delta \) is the space of right continuous functions \( u \in L^\infty(L^2) \) such that

\[ \| v \|_{V^p_\Delta} = \| v \|_{L^\infty(L^2)}^p + \sup_{\{t_k\}} \sum_k \| e^{-it_k \Delta} v(t_k) - e^{-it_{k+1} \Delta} v(t_{k+1}) \|_{L^2}^p. \]  

(2.23)

The supremum is taken over increasing sequences \( t_k \).

**Theorem 2.3** The function spaces \( U^p_\Delta \) and \( V^p_\Delta \) obey the embeddings

\[ U^p_\Delta \subset V^p_\Delta \subset U^q_\Delta \subset L^\infty(L^2), \quad p < q. \]  

(2.24)
Let $DU^p_\Delta$ be the space of functions

$$DU^p_\Delta = \{(i\partial_t + \Delta)u; u \in U^p_\Delta\}. \quad (2.25)$$

By Duhamel’s formula

$$\|u\|_{U^p_\Delta} \lesssim \|u(0)\|_{L^2} + \|(i\partial_t + \partial^2_x)u\|_{DU^p_\Delta}. \quad (2.26)$$

Finally, there is the duality relation

$$(DU^p_\Delta)^* = V^p_\Delta. \quad (2.27)$$

These spaces are also closed under truncation in time.

$$\chi_I: U^p_\Delta \to U^p_\Delta,$$
$$\chi_I: V^p_\Delta \to V^p_\Delta. \quad (2.28)$$

Proof: See [21]. □

### 3 Description of the I - method and outline of the proof

Since there are no known conserved quantities that control $\|u\|_{H^s}$ for $0 < s < 1$, we utilize the by now well known modified energy of [12].

**Definition 3.1 (I - operator)** Let $I : H^s(\mathbb{R}^d) \to H^1(\mathbb{R}^d)$ be the Fourier multiplier

$$\hat{If}(\xi) = m_N(\xi)\hat{f}(\xi), \quad (3.1)$$

where

$$m_N(\xi) = 1, \text{ when } |\xi| \leq N, \quad m_N(\xi) = \frac{N^{1-s}}{|\xi|^{1-s}}, \text{ when } |\xi| \geq 2N. \quad (3.2)$$

We suppress the $N$ for the rest of the paper. There is an obvious tradeoff here. Taking $N = \infty$, we see that $\frac{d}{dt}E(Iu(t)) = 0$, however, for $s < 1$, $E(Iu(0)) = \infty$. Moreover, as $N$ increases $\frac{d}{dt}E(Iu(t))$ decreases and $E(Iu(t))$ increases. Therefore, the question of global well - posedness centers on which side will win this tug of war. More precisely, by Sobolev embedding, when $d = 3$, 

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\[ E(Iu(t)) \lesssim \|Iu\|^2_{H^1_1(\mathbb{R}^d)} + \|Iu\|^2_{H^{1/2}_1(\mathbb{R}^d)} \|u\|^2_{H^{1/2}_1(\mathbb{R}^d)}. \] (3.3)

When \( d = 2 \),
\[ E(Iu(t)) \lesssim \|Iu\|^2_{H^1_1(\mathbb{R}^2)} + \|Iu\|^2_{H^{1/2}_1(\mathbb{R}^2)} \|u\|^2_{H^{1/2}_1(\mathbb{R}^2)}. \] (3.4)

Therefore,
\[ E(Iu(0)) \lesssim \|u(0)\|_{H^s_2} N^{2(1-s)}. \] (3.5)

Meanwhile,
\[ \|u(t)\|^2_{H^s(\mathbb{R}^d)} \lesssim E(Iu(t)) + M(Iu(t)). \] (3.6)

Since \( M(Iu(t)) \leq M(u(t)) = M(u(0)) \), a uniform bound on \( E(Iu(t)) \) for all \( t \) yields a uniform bound on \( \|u(t)\|_{H^s(\mathbb{R}^d)} \). Now rescale. Choose \( \lambda \) in \((1.3), (1.7)\) satisfying \( \lambda^{s-\frac{1}{2}} \sim \|u(0)\|_{H^s(\mathbb{R}^d)} N^{s-1} \), respectively \( \lambda^{s-\frac{1}{2} + \frac{1}{k}} \sim \|u(0)\|_{H^s(\mathbb{R}^2)} N^{s-1} \) such that
\[ E(Iu(0)) \leq \frac{1}{2^2}. \] (3.7)

Then by \((1.3), (1.7)\),
\[ \|u_\lambda(0)\|_{L^2_2(\mathbb{R}^3)} \lesssim \|u(0)\|_{H^s(\mathbb{R}^3)} N^{\frac{1}{2s-1}} \|u(0)\|_{L^2_2(\mathbb{R}^3)}, \] (3.8)
\[ \|u_\lambda(0)\|_{L^2_2(\mathbb{R}^2)} \lesssim \|u(0)\|_{H^s(\mathbb{R}^2)} N^{\frac{k-1}{s-1+\frac{1}{k}}} \|u(0)\|_{L^2_2(\mathbb{R}^2)}. \]

We suppress \( \lambda \) for the rest of the paper. Next recall the interaction Morawetz estimate.

**Theorem 3.1 (Interaction Morawetz estimate)** Suppose \( u \) is a solution to \((1.1), (1.2)\), on some interval \( J \). Then
\[ \|\nabla^{\frac{s-d}{2}} |u|^2\|_{L^2_{t,x}(J \times \mathbb{R}^d)} \lesssim \|u\|^2_{L^\infty_t L^2_x(J \times \mathbb{R}^d)} \|u\|^2_{L^\infty_{t} H^{1/2}_1(J \times \mathbb{R}^d)}. \] (3.9)

**Proof:** This was proved in three dimensions by \[13\]. \[33\] and \[11\] independently proved \((3.10)\) in dimensions one and two. \[11\] proved the interaction Morawetz estimate in dimensions \( d \geq 4 \), a result that will not be needed here. \(\square\)

\((3.10)\) is extremely useful due to a local well-posedness result of \[13\].
Lemma 3.2 If $E(Iu(a_i)) \leq 1$, $J_t = [a_t, b_t]$, $\|u\|_{L_t^4(J_t \times \mathbb{R}^3)} \leq \epsilon$ for some $\epsilon > 0$ sufficiently small, then

$$\|\nabla Iu\|_{S^0(J_t \times \mathbb{R}^3)} \lesssim 1.$$  \hspace{1cm} (3.10)

Proof: See \cite{13} or \cite{14}. \hfill \Box

We will need a similar result in dimension two.

Lemma 3.3 If $E(Iu(a_i)) \leq 1$, $J_t = [a_t, b_t]$, $\|\nabla |u|^{1/2} |u|^2\|_{L_t^4(J_t \times \mathbb{R}^3)} \leq \epsilon$ for some $\epsilon(k) > 0$ sufficiently small, then for $(p,q) \in \mathcal{A}_2$,

$$\|\nabla Iu\|_{L_t^p L_x^q(J_t \times \mathbb{R}^2)} \lesssim k.$$  \hspace{1cm} (3.11)

Proof: By Sobolev embedding

$$\|u\|_{L_t^4 L_x^8(J_t \times \mathbb{R}^2)} \lesssim \|\nabla |u|^{1/2} |u|^2\|_{L_t^4(J_t \times \mathbb{R}^3)} \leq \epsilon.$$  \hspace{1cm} (3.12)

Interpolating (3.12) with $\|P_j u\|_{L_t^\infty L_x^2(J_t \times \mathbb{R}^2)} \lesssim 2^j \|P_j u\|_{L_t^2(J_t \times \mathbb{R}^2)}$, combined with theorem 2.1 proves

$$\|Iu\|_{L_t^1 L_x^6(J_t \times \mathbb{R}^2)} \lesssim \epsilon^{\frac{4}{k}} \|\nabla Iu\|_{L_t^\infty L_x^2(J_t \times \mathbb{R}^2)}^{1-\frac{4}{k}}.$$  \hspace{1cm} (3.13)

By Bernstein’s inequality and (3.2),

$$\|(1 - I)u\|_{L_t^3 L_x^{6k}(J_t \times \mathbb{R}^2)} \lesssim N^{-\frac{1}{2}} \|\nabla Iu\|_{L_t^3 L_x^{6k}(J_t \times \mathbb{R}^2)}.$$  \hspace{1cm} (3.14)

Then by theorem 2.2

$$\|\nabla Iu\|_{L_t^3 L_x^{6k} \cap L_t^\infty L_x^2(J_t \times \mathbb{R}^2)} \lesssim \|\nabla Iu(a_i)\|_{L_t^2(\mathbb{R}^2)} + \|\nabla Iu\|_{L_t^\infty L_x^2(J_t \times \mathbb{R}^2)} \hspace{1cm} (\epsilon^{\frac{4}{k}} \|\nabla Iu\|_{L_t^\infty L_x^2(J_t \times \mathbb{R}^2)} + N^{-\frac{1}{2}} \|\nabla Iu\|_{L_t^3 L_x^{6k}(J_t \times \mathbb{R}^2)})^{2k}.$$  \hspace{1cm} (3.15)

Since $N$ is large and $\epsilon > 0$ is small the proof is complete. \hfill \Box

(3.15) also implies

$$\|\nabla Iu\|_{U_2^i(J_t \times \mathbb{R}^2)} \lesssim k.$$  \hspace{1cm} (3.16)
We also have
\[ \| \nabla Iu \|_{U^2_\Delta(J \times \mathbb{R}^3)} \lesssim 1. \] (3.17)

Theorems 1.2 and 1.3 are then proved by a bootstrapping estimate. Let
\[ J = \{ t : E(Iu(t)) \leq 1 \}. \] (3.18)

Assume that \( J \) is an interval. \( J \) is clearly nonempty since \( 0 \in J \). Moreover, \( J \) is closed by standard local well-posedness theory. Therefore, to prove \( J = [0, \infty) \) it suffices to show that \( J \) is open. By (3.12), interpolation, and Bernstein's inequality,
\[ \| P \lesssim N u(t) \|_{\dot{H}^{1/2} \mathbb{R}^d} \lesssim \| Iu(t) \|_{\dot{H}^1_0 \mathbb{R}^d}^{1/2} \| P \lesssim N u(t) \|_{L^2_\mathbb{R}^d}^{1/2}, \] (3.19)

and
\[ \| P \gtrsim N u(t) \|_{\dot{H}^{1/2} \mathbb{R}^d} \lesssim N^{-1/2} \| Iu \|_{\dot{H}^1_0 \mathbb{R}^d}. \] (3.20)

Therefore if \( J \) is an interval such that \( E(Iu(t)) \leq 1 \) on \( J \), then (3.18), (3.10), (3.19), (3.20), and the conservation of mass imply that
\[ \| u \|_{L^4_{t,x} \mathbb{R}^3} \lesssim \| u(0) \|_{H^s} N^{\frac{3(1-s)}{2s-1}}, \] (3.21)

\[ \| u \|_{L^4_t L^8_x \mathbb{R}^2} \lesssim \| u(0) \|_{H^s} N^{\frac{k-1}{k-3(1-s)}+\frac{1}{k}}. \]

When \( d = 3 \) partition \( J \) into \( \sim \| u(0) \|_{H^s} N \frac{3(1-s)}{2s-1} \) subintervals such that on each subinterval \( J_t \),
\[ \| u \|_{L^4_{t,x} \mathbb{R}^3} \leq \varepsilon. \] (3.22)

When \( d = 2 \) partition \( J \) into \( \sim \| u(0) \|_{H^s}, k N^{\frac{k-1}{k-3(1-s)}+\frac{1}{k}} \) subintervals such that
\[ \| u \|_{L^4_t L^8_x \mathbb{R}^2} \leq \varepsilon. \] (3.23)

Then we show that for \( N(d,k,\| u(0) \|_{H^s}) \) sufficiently large,
\[ \int_J \frac{d}{dt} E(Iu(t)) dt \leq \frac{1}{10}. \] (3.24)

To do this we use long term Strichartz estimates. [15] utilized the long-time Strichartz estimates within the context of the mass-critical nonlinear Schrödinger initial value problem. The long-time Strichartz estimates have
been utilized in subsequent papers ([16], [17], [26], [30], [31], [32], [45]). By (3.24) and theorem 1, for any $T > 0$ there exists $\delta(T) > 0$ such that if $[0,T] \subset J$, $[0, T + \delta) \subset J$. Therefore $J$ is open and the proof is complete. Finally, we can recover the $\|u(t)\|_{H^s}$ bound by rescaling back and then computing the $\|u(t)\|_{H^s}$ norm from the bounds on $M(u(t))$ and $E(Iu(t))$ after rescaling.

4 Induction on frequency and long time Strichartz estimates in three dimensions

Take $1 \leq M \leq N$. As in [15] we make an induction on energy argument. Notice that, for $P > M = \sum_{j > M} P_j$,

$$P > M(|u|_x^2 u) \equiv 0.$$  \hfill (4.1)

**Remark:** This is why this method does not immediately carry over to a non-algebraic nonlinearity, $p \neq 2k$ for some positive integer $k$. By Duhamel’s principle, and $E(Iu(t)) \leq 1$ on $J$,

$$\|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} \lesssim \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} + \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} \lesssim 1 + \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)}.$$  \hfill (4.2)

Recalling theorem 2.3, $V^2_{\Delta}(J \times \mathbb{R}^3) \subset U^p_{\Delta}(J \times \mathbb{R}^3)$ for any $q > 2$. Combining this with $(DU^2_{\Delta})^* = V^2_{\Delta}$ implies that $L^p_{\Delta} L^q_{\Delta}(J \times \mathbb{R}^3) \subset DU^2_{\Delta}(J \times \mathbb{R}^3)$ for any $p > 2$, where $(p, q)$ is an admissible pair. Namely,

$$\|\nabla I((P > M u)^2 u)\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} \lesssim \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} \lesssim \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} + \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} \lesssim 1 + \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)}.$$  \hfill (4.3)

$$\lesssim \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} \lesssim \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} + \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} \lesssim 1 + \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)}.$$  \hfill (4.4)

$$\lesssim \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} + \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} \lesssim 1 + \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)}.$$  \hfill (4.5)

$$\lesssim \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} + \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)} \lesssim 1 + \|\nabla I(P > M u)^2 u\|_{DU^2_{\Delta}(J \times \mathbb{R}^3)}.$$  \hfill (4.6)

Interpolating (3.16) and (3.21) with $E(Iu(t)) \leq 1$ on $J$, combined with Bernstein’s inequality implies
\[ \| Iu \|_{L_t^\infty L_x^6(J \times \mathbb{R}^3)} + \| \nabla Iu \|_{L_t^\infty L_x^2(J \times \mathbb{R}^3)} \lesssim \| u_0 \|_{H^s(\mathbb{R}^3)} N^+ , \]  

(4.7)

and

\[ \| P_{> N} u \|_{L_t^\infty L_x^6(J \times \mathbb{R}^3)} \lesssim N^{-1/2+} . \]  

(4.8)

Interpolation, Sobolev embedding, Bernstein’s inequality, \( E(Iu(t)) \leq 1 \), and (3.2) imply that

\[ \| P_{> M} u \|_{L_t^4 L_x^6(J \times \mathbb{R}^3)} \lesssim \| \nabla I \|_{U^2 (J \times \mathbb{R}^3)} . \]  

(4.10)

\[ \| P_{> M} u \|_{L_t^4 L_x^6(J \times \mathbb{R}^3)} \lesssim M^{-1/2} \| \nabla IP_{> M} u \|_{L_t^2 L_x^2(J \times \mathbb{R}^3)} . \]  

(4.9)

Therefore, (4.9), (4.10), and (4.11) imply that

\[ \text{(4.10)} \]

It only remains to analyze

\[ \| \nabla I(P_{> M} u)(P_{\leq M} u)^2 \|_{U^2 (J \times \mathbb{R}^3)} . \]  

(4.12)

Interpolating (2.14) and (2.19), for any \( q > 2 \)

\[ \| \nabla \|_{q} \lesssim R^{1/q} \| u_0 \|_{L_x^2} . \]  

(4.13)

Let \( \chi \in C_0^\infty(\mathbb{R}^3) \), \( \chi \equiv 1 \) on \( |x| \leq 1 \), \( \chi \) supported on \( |x| \leq 2 \). Take \( \| v \|_{U^2(J \times \mathbb{R}^3)} = 1 \), \( \hat{v}(\xi) \) supported on \( |\xi| \geq M \). Then

\[ \int_j \langle v, \chi^2 \nabla I((P_{> M} u)(P_{\leq M} u)^2) \rangle dt \lesssim \| \chi v \|_{L_t^6 L_x^6(\mathbb{R}^3)} \cdot \| \nabla I \|_{U^2(J \times \mathbb{R}^3)} \]  

(4.14)

Now take the cutoff supported on the annulus \( |x| \sim 2^j \), \( \psi_j(x) = \chi^2(2^{-j+1}x) - \chi^2(2^{-j}x) \), where \( j \geq 0 \). Here we use the radial symmetry of the solution. Combining radial Sobolev embedding with the standard Sobolev estimate and Bernstein’s inequality,
\[ \|x\| u^2 \|_{L^\infty_2(\mathbb{R}^3)} \lesssim \sum_{N_1 \leq N_2} \|x\|_{L^\infty_2(\mathbb{R}^3)} \|P_{N_2} u\|_{L^\infty_2(\mathbb{R}^3)} \|P_{N_1} u\|_{L^\infty_2(\mathbb{R}^3)} \]

\[ \lesssim \sum_{N_1 \leq N_2} \left( \frac{N_1}{N_2} \right)^{1/2} \|P_{N_1} u\|_{\dot{H}^1(\mathbb{R}^3)} \|P_{N_2} u\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \|u\|^2_{\dot{H}^1(\mathbb{R}^3)}. \] (4.15)

Interpolating this with

\[ \|Iu\|_{L^1_t L^\infty_2(J \times \mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}^s(\mathbb{R}^3)} N^{\frac{3(1-s)}{4s-1}}, \] (4.16)

which is a consequence of (3.16) and Strichartz estimates implies that

\[ \|x\|^{1/2} Iu\|_{L^\infty_2(L^\infty_2(J \times \mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}^s(\mathbb{R}^3)} N^+. \] (4.17)

Combining (4.13), (4.17), \(E(Iu(t)) \leq 1\), and \(\hat{v}(t, \xi)\) is supported on \(|\xi| \geq M\),

\[ \int_J \langle v, \psi_j(x) \nabla(I((P_{\geq M} u)(P_{\leq M} u)^2))dt \]

\[ \lesssim \|u_0\|_{\dot{H}^s(\mathbb{R}^3)} M^{-1} N^+ (2^j)^{4} \|\nabla IP_{\geq M} u\|_{\dot{H}^s_2(J \times \mathbb{R}^3)}. \] (4.18)

Finally, \(E(Iu(t)) \leq 1\) combined with \(\|u(t)\|_{L^2_t L^2_2(\mathbb{R}^3)} \lesssim N^{\frac{1-s}{2s-1}}\) implies that

\[ \int_J \langle v, (1 - \chi(2^{-j} x)) \nabla IP_{\geq M} u)(P_{\leq M} u)^2)dt \]

\[ \lesssim \|v\|_{L^1_t L^2_2(J \times \mathbb{R}^3)} \|\nabla IP_{\geq M} u\|_{L^2_t L^2_2(J \times \mathbb{R}^3)} \|Iu\|_{L^1_t L^2_2(J \times \mathbb{R}^3)} \]

\[ \times \|Iu\|_{L^1_t L^2_2(J \times \mathbb{R}^3)} \|1 - \chi(2^{-j} x)\|_{L^1_t L^2_2(J \times \mathbb{R}^3)} \]

\[ \lesssim 2^{-j/2} N^{-\frac{1}{2s-1}} \|\nabla IP_{\geq M} u\|_{\dot{H}^s_2(J \times \mathbb{R}^3)}. \] (4.19)

Combining (4.11), (4.14), (4.18), and (4.19), for \(j \approx \ln(N)\) sufficiently large,

\[ \|\nabla IP_{\geq M} (|u|^2 u)\|_{DU^2_3(J \times \mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}^s(\mathbb{R}^3)} M^{-1} N^+ \|\nabla IP_{\geq M} u\|_{\dot{H}^s_2(J \times \mathbb{R}^3)}. \] (4.20)

We have the base case (3.16). Starting the induction at \(C(s, \|u_0\|_{\dot{H}^s}) N^{1/2}\) for \(C(s, \|u_0\|_{\dot{H}^s})\) sufficiently large,

\[ \|\nabla IP_{\geq \frac{N}{100}} u\|_{\dot{H}^s_2(J \times \mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}^s, s} 1 + N^{-\frac{3(1-s)}{4s-1}} N^{-\frac{4}{8} \ln(N)} + 1, \] (4.21)
for some constant \( c > 0 \).

**Remark:** We could replace \( \frac{c}{q} \) with \( \frac{c}{q'} \) for any \( q' > 2 \). Therefore, choosing \( N \) sufficiently large, say \( \ln(N) = \frac{18(1-s)}{4s-2} \ln(8) + \ln(C(s, \|u_0\|_{H^s})) \),

\[
\|\nabla IP_{\frac{N}{100}} u\|_{U^2_{\Delta}(J \times \mathbb{R}^3)} \lesssim \|u_0\|_{H^s(\mathbb{R}^3)} \tag{4.22}
\]

### 5 Energy Increment in three dimensions

Now we show a bound on the modified energy increment.

**Lemma 5.1** For \( \ln(N) = \frac{18(1-s)}{4s-2} \ln(8) + \ln(C(s, \|u_0\|_{H^s})) \),

\[
\int_J \left| \frac{d}{dt} E(Iu(t)) \right| dt \lesssim \frac{1}{N^{1-}}. \tag{5.1}
\]

**Proof:** \( (1.1) \) implies

\[
iIu_t + \Delta u = |Iu|^2(Iu) + I(|u|^2u) - |Iu|^2(Iu). \tag{5.2}
\]

Therefore,

\[
\frac{d}{dt} E(Iu(t)) = \langle Iu_t, |Iu|^2(Iu) - I(|u|^2u) \rangle. \tag{5.3}
\]

By \( (5.2) \) and integrating by parts,

\[
\frac{d}{dt} E(Iu(t)) = -\langle i\nabla Iu, \nabla((Iu)^3 - I(u^3)) \rangle - \langle iI(u^3), ((Iu)^3 - I(u^3)) \rangle. \tag{5.4}
\]

We start with \( \langle i\nabla Iu, \nabla((Iu)^3 - I(u^3)) \rangle \).

\[
(IP_{\leq \frac{N}{4}} u)^3 - I((P_{\leq \frac{N}{4}} u)^3) \equiv 0. \tag{5.5}
\]

Next,

\[
(IP_{> \frac{N}{4}} u)(IP_{\leq \frac{N}{4}} u)^2 - I((P_{> \frac{N}{4}} u)(P_{\leq \frac{N}{4}} u)^2)
\]

\[
= (IP_{> \frac{N}{4}} u)(P_{\leq \frac{N}{4}} u)^2 - I((P_{> \frac{N}{4}} u)(P_{\leq \frac{N}{4}} u)^2). \tag{5.6}
\]

By the fundamental theorem of calculus,

\[
|m(\xi_2 + \xi_3 + \xi_4) - m(\xi_2)| \lesssim \frac{|\xi_3 + \xi_4|}{|\xi_2|}. \tag{5.7}
\]
Moreover,
\[ I((P_{> \frac{N}{8}} u)(P_{\leq \frac{N}{8}} u)^2) - (IP_{> \frac{N}{8}} u)(P_{\leq \frac{N}{8}} u)^2 \]  
(5.8)
has Fourier transform supported on \(|\xi| \geq \frac{N}{8}\). By (5.7),
\[ - \int_J (i \nabla u, \nabla((IP_{> \frac{N}{8}} u)(P_{\leq \frac{N}{8}} u)^2 - I((P_{> \frac{N}{8}} u)(P_{\leq \frac{N}{8}} u)^2)))dt \]  
(5.9)
\[ \lesssim \frac{1}{N} \|\nabla IP_{> \frac{N}{8}} u\|^2_{L_t^2 L_x^2(J \times \mathbb{R}^3)} \|\nabla u\|_{L_t^\infty L_x^2(J \times \mathbb{R}^3)} \|Iu\|_{L_t^\infty L_x^2(J \times \mathbb{R}^3)} \lesssim \frac{1}{N}. \]  
(5.10)
Next, since \(E(Iu(t)) \leq 1\),
\[ \int_J (i \nabla u, \nabla((IP_{> \frac{N}{8}} u)^2(P_{\leq \frac{N}{8}} u) - I((P_{> \frac{N}{8}} u)^2(P_{\leq \frac{N}{8}} u))))dt \]  
(5.11)
\[ \lesssim \|\nabla u\|_{L_t^\infty L_x^2} \|\nabla IP_{> \frac{N}{8}} u\|_{L_t^2 L_x^6} \|IP_{> \frac{N}{8}} u\|_{L_t^2 L_x^6} \|P_{\leq \frac{N}{8}} u\|_{L_t^\infty L_x^6} \lesssim \frac{1}{N}. \]  
(5.12)
Finally,
\[ \int_J (i \nabla u, \nabla((IP_{> \frac{N}{8}} u)^3 - I((P_{> \frac{N}{8}} u)^3)))dt \]  
(5.13)
\[ \lesssim \|\nabla u\|_{L_t^\infty L_x^2} \|\nabla IP_{> \frac{N}{8}} u\|_{L_t^2 L_x^6} \|P_{> \frac{N}{8}} u\|^2_{L_t^2 L_x^6} \lesssim \frac{1}{N}. \]  
(5.14)
This takes care of the first term in (5.4). Now we consider the term
\[ \int_J (I(u^3), I(u^3) - (Iu)^3)dt. \]  
(5.15)
(5.5), (5.6) imply that this six-linear term must have at least two \(P_{> \frac{N}{8}} u\) terms. By Sobolev embedding and Bernstein’s inequality and (3.2),
\[ \|I((P_{> \frac{N}{8}} u)^3)\|_{L_t^2 L_x^2} \lesssim \|\nabla IP_{> \frac{N}{8}} u\|_{L_t^2 L_x^6} \|P_{> \frac{N}{8}} u\|^2_{L_t^\infty L_x^6} \lesssim \frac{1}{N}. \]  
(5.16)
Therefore,
\[
\int J \langle I((P_{\geq N_8} u)u)^3), (IP_{\geq N_8} u)^3) + (IP_{\geq N_8} u)^3 \rangle dt \lesssim \frac{1}{N_8^2}. \quad (5.17)
\]

Next,
\[
\int J \langle (P_{\geq N_8} u)^3), (P_{\geq N_8} u)^3 \rangle dt \lesssim \|I(P_{\geq N_8} u)^3)\|_{L^2_t L^6_x} \|P_{\geq N_8} u\|_{L^\infty_t L^6_x} \lesssim \frac{1}{N_8^2}. \quad (5.18)
\]

Finally,
\[
\int J \int (P_{\geq N_8} u)^2(P_{\leq N_8} u)^2 u^2 dx dt \lesssim \|P_{\geq N_8} u\|_{L^2_t L^6_x} \|P_{\leq N_8} u\|_{L^\infty_t L^6_x} \lesssim \frac{1}{N_8^2}. \quad (5.19)
\]

This proves lemma 5.1 \(\square\)

Rescaling back, we have proved
\[
\|u(t)\|_{L^2_x(R^3)} = \|u(0)\|_{L^2_x(R^3)}, \quad (5.21)
\]

and
\[
\|u(t)\|_{\dot{H}^s_x(R^3)} \lesssim \|u(0)\|_{L^2_x(R^3)} + N^{\frac{1-s}{2-s}} \|u(0)\|_{\dot{H}^s_x(R^3)}. \quad (5.22)
\]

Therefore, by (4.21),
\[
\|u(t)\|_{H^s_x(R^3)} \lesssim C(s, \|u_0\|_{H^s_x(R^3)}) \|u_0\|_{H^s_x(R^3)}, \quad (5.23)
\]

where \(C\) behaves like \(e^{C_1 \frac{1-s}{2-s-1}}\) for some constant \(C_1\) as \(s \searrow \frac{1}{2}\). This completes the proof of theorem 1.2 since (5.23) gives a bound on \(\|u\|_{L^t_x L^6_x}\) by theorem 3.1.

Interpolating this with the uniform bound on \(\|u(t)\|_{H^s}\) implies a bound on \(L^p_t L^q_x\), where \((p, q)\) is a \(\frac{1}{2}\) - admissible pair, that is \(\frac{2}{p} = 3(\frac{1}{2} - \frac{1}{q} - \frac{1}{6})\). Since \(s > \frac{1}{2}\), \(p < \infty\). Partitioning \(R\) into finitely many pieces with \(\|u\|_{L^p_t L^q_x(J, \times R^3)} < \epsilon\) and making a perturbation argument, \(\|u\|_{L^p_{t,x}(R \times R^3)} < \infty\), which implies scattering. \(\square\)
6 Induction on frequency in two dimensions

We turn now to the two dimensional problem (12), \( k > 1, k \in \mathbb{Z} \). In this case the critical space is \( \dot{H}^{s_c}, s_c = \frac{k-1}{k} \). Once again take the \( I \) operator as defined in (3.2). Then,

\[
E(Iu(0)) \lesssim k \|u_0\|_{\dot{H}^s(\mathbb{R}^2)} N^{2(1-s)}.
\] (6.1)

Rescale with \( \lambda \sim \|u_0\|_{\dot{H}^s,k} N^{\frac{1-s}{s-c}} \) so that \( E(Iu(0)) = \frac{1}{2} \). Suppose \( J \) is an interval with \( E(Iu(t)) \leq 1 \) for all \( t \in J \). Recalling (3.21),

\[
\|u\|_{L_t^4 L_x^8(J \times \mathbb{R}^2)} \lesssim \|\nabla^{1/2} |u|^2\|_{L_t^2 L_x^\infty} \lesssim \|u(0)\|_{\dot{H}^s(\mathbb{R}^2),k} N^{s_c \frac{3(1-s)}{s-c}}.
\] (6.2)

Then, by lemma [5.3]

\[
\|\nabla Iu\|_{L_t^2 L_x^\infty(J \times \mathbb{R}^2)} \lesssim \|u(0)\|_{\dot{H}^s(\mathbb{R}^2),k} N^{s_c \frac{3(1-s)}{s-c}}.
\] (6.3)

Once again make an induction on frequency argument.

\[
\|\nabla IP_{M}u(t)\|_{L_t^2 L_x^{2k}(J \times \mathbb{R}^2)} \lesssim \|\nabla IP_{M}u(0)\|_{L_t^2 L_x^{2k}(\mathbb{R}^2)} + \|\nabla IP_{M}(\|u\|^{2k}\|u\|)\|_{L_t^2 L_x^{2k}(J \times \mathbb{R}^2)}
\lesssim 1 + \|\nabla IP_{M}(\|u\|^{2k}\|u\|)\|_{L_t^2 L_x^{2k}(J \times \mathbb{R}^2)}.
\] (6.4)

Since the nonlinearity is algebraic there exists \( c(k) \) such that

\[
P_{M}(\|u \leq c(k)M\|^{2k}\|u \leq c(k)M\|) \equiv 0.
\] (6.5)

\[
\|\nabla I((P_{c(k)M}u)^{2k-1}\|u\|^{2k-1})\|_{L_t^2 L_x^{2k}(J \times \mathbb{R}^2)}
\]

\[
\lesssim \|\nabla IP_{c(k)M}u\|_{L_t^{2k+1} L_x^\infty} \|P_{c(k)M}u\|_{L_t^{2k} L_x^\infty} \|Iu\|_{L_t^{2k} L_x^\infty} \|u\|_{L_t^{2k+2} L_x^{2k+2}}
\]

\[
+ \|\nabla IP_{c(k)M}u\|_{L_t^{2k+1} L_x^\infty} \|P_{c(k)M}u\|_{L_t^{2k} L_x^\infty} \|P_{N}u\|_{L_t^{2k} L_x^{2k+2}}
\]

\[
+ \|\nabla Iu\|_{L_t^{\infty} L_x^{2k}} \|P_{c(k)M}u\|_{L_t^{2k} L_x^{2k}} \|P_{N}u\|_{L_t^{2k+1} L_x^{2k}}
\] (6.7)
\[ + \| \nabla Iu \|_{L_t^\infty - L_t^\infty} \| P_{> c(k)} M u \|_{L_t^2}^2 \| Iu \|_{L_t^\infty - L_t^\infty}^{2k-1}. \tag{6.10} \]

Making an argument almost identical to the estimates when \( d = 3, \)
\[ \| \nabla I(\varphi c(k)M u)(P_{\leq c(k)} M u) \|_{DU^2_\Delta(J \times \mathbb{R}^2)}. \tag{6.11} \]

Then we can make an induction argument provided we have a good estimate on
\[ \| \nabla I(P_{> c(k)} M u)(P_{\leq c(k)} M u) \|_{DU^2_\Delta(J \times \mathbb{R}^2)}. \tag{6.12} \]

We once again utilize the local smoothing estimate
\[ \| |\nabla|^{1/4} e^{it\Delta} u \|_{L_t^q L_x^2(\mathbb{R} \times \{|x| \leq R\})} \lesssim R^{1/q} \| u \|_{L^2}. \tag{6.13} \]

Let \( \chi \in C_0^\infty(\mathbb{R}^2), \chi \equiv 1 \) on \( |x| \leq 1, \chi \) supported on \( |x| \leq 2. \) Take \( \| v \|_{V^2_\Delta(J \times \mathbb{R}^2)} = 1, \hat{v}(\xi) \) supported on \( |\xi| \geq M. \) Then when \( k \geq 2, \)
\[ \int_J (v, \chi x^2 \nabla I((P_{> c(k)} M u)(P_{\leq c(k)} M u)^{2k}))dt \lesssim \| \chi x \|_{L_t^2 L_x^\infty(\mathbb{R} \times \mathbb{R}^2)} \| \nabla I P_{> c(k)} M u \|_{L_t^2 L_x^4(J \times \mathbb{R}^2)} \| Iu \|_{L_t^\infty - L_x^{4k+1}(J \times \mathbb{R}^2)} \tag{6.14} \]
\[ \lesssim N^+ M^{-1/2} \| \nabla I P_{> c(k)} M u \|_{DU^2_\Delta(J \times \mathbb{R}^2)}. \]

Now take the cutoff supported on the annulus \( |x| \sim 2^j, \psi_j(x) = \chi^2(2^{-j} x) - \chi^2(2^{-j+1} x), \) where \( j \geq 0. \) When \( k = 2, \)
\[ \| |\nabla|^{1/2} (Iu)^2 \|_{L^2_x(\mathbb{R}^2)} \lesssim \| |\nabla|^{1/2} Iu \|_{L^2_x(\mathbb{R}^2)} \| Iu \|_{L^6_x(\mathbb{R}^2)} \lesssim E(Iu)^{2/3} \leq 1. \tag{6.15} \]

When \( k \geq 3, \)
\[ \| |\nabla|^{1/2} (Iu)^k \|_{L^2_x(\mathbb{R}^3)} \lesssim \| |\nabla|^{1/2} Iu \|_{L^2_x(\mathbb{R}^2)} \| Iu \|_{L^k_x(\mathbb{R}^2)}^{k-1} \lesssim 1. \tag{6.16} \]

The last inequality follows by interpolating \( \| Iu \|_{\dot{H}^1} \leq E(Iu(t)) \leq 1 \) with \( \| Iu \|_{L_t^{2k+2}} \leq E(Iu(t)) \leq 1. \) Taking \( \| v \|_{V^2_\Delta} = 1 \) supported on \( |\xi| > M, \) combined with (5.21),
\[
\int_J \langle v, \psi_j (\nabla IP_{>c(k)} M u)(Iu)^{2k} \rangle dt \\
\lesssim \|v\|_{L^2_t L^2_x(J \times |x|^{-2j})} \|\psi_j (Iu)^{2k}\|_{L^\infty_t L^\infty_x} \|\nabla IP_{>c(k)} M u\|_{L^2_t L^2_x(J \times |x|^{-2j})} \\
\lesssim 2^{j(0-)} N^+ M^{-1+} \|\nabla IP_{>c(k)} M u\|_{U^2_\Delta(J \times \mathbb{R}^2)}.
\]

(6.17)

Since $u$ also has finite mass, see (3.8), by (6.15), (6.16),

$$
\|x|^{1/2} |(Iu)^k\|_{L^\infty_x} \leq \|x|^{1/2} P_{>N^0} (Iu)^k\|_{L^\infty_x} + \|x|^{1/2} P_{\leq N^0} (Iu)^k\|_{L^\infty_x} \lesssim \ln(N) + 1.
$$

(6.18)

Next, for each $J_l \subset J$, by (3.8), Strichartz estimates, and a computation similar to (6.15) and (6.16),

$$
\|\nabla |x|^{1/2} (Iu)^k\|_{L^\infty_t L^2_x(J_l \times \mathbb{R}^2)} \lesssim N^+.
$$

(6.19)

Since there is a bound on the number of $J_l \subset J$, (3.21), we have

$$
\|\nabla IP_{>M} |u|^{2k} u\|_{DU^2_\Delta(J \times \mathbb{R}^2)} \lesssim \|u_0\|_{H^s(\mathbb{R}^2), k} M^{-1+} N^+ \|\nabla IP_{>c(k)} M u\|_{U^2_\Delta(J \times \mathbb{R}^2)}.
$$

(6.20)

This time we starting the induction at $C(s, \|u_0\|_{H^s, k}) N^{3/4}$ for $C(s, \|u_0\|_{H^s, k})$ sufficiently large,

$$
\|\nabla IP_{\gtrsim \frac{N}{100}} u\|_{U^2_\Delta(J \times \mathbb{R}^2)} \lesssim \|u_0\|_{H^s, k} 1 + N^{\frac{3(1-s)}{4(2-2c)}} s c N^{-\frac{c}{6}} \ln(N) + 1,
$$

(6.21)

for some constant $c > 0$.

**Remark:** We could replace $\frac{c}{6}$ with $\frac{c}{q}$ for any $q > 4$. Therefore, choosing $N$ sufficiently large,

$$
\|\nabla IP_{\gtrsim \frac{N}{100}} u\|_{U^2_\Delta(J \times \mathbb{R}^2)} \lesssim \|u_0\|_{H^s, k} 1.
$$

(6.22)

### 7 Energy Increment

To complete the proof of theorem 1.3 it remains to prove
Lemma 7.1 If $J$ is an interval with $E(Iu(t)) \leq 1$ on $J$,

$$
\int_J \left| \frac{d}{dt} E(Iu(t)) \right| dt \lesssim_k \frac{1}{N^{1-}}. 
$$

(7.1)

Proof: We compute

$$
\frac{d}{dt} E(Iu(t)) = \langle |Iu_t|, |Iu|^{2k} \rangle - I(||u|^{2k} u) \rangle
$$

$$
- \langle i \nabla Iu, \nabla ((Iu)^{2k+1} - I(u^{2k+1})) \rangle - \langle iI(u^{2k+1}), ((Iu)^{2k+1} - I(u^{2k+1})) \rangle.
$$

(7.2)

Once again,

$$
(IP_{\leq c(k)N}u)^{2k+1} - I((P_{\leq c(k)N}u)^{2k+1}) \equiv 0.
$$

(7.3)

Also,

$$
(IP_{> c(k)N}u)(IP_{\leq c(k)N}u)^{2k} - I((P_{> c(k)N}u)(P_{\leq c(k)N}u)^{2k})
$$

$$
= (IP_{> \frac{N}{2}}u)(P_{\leq c(k)N}u)^{2k} - I((P_{> \frac{N}{2}}u)(P_{\leq c(k)N}u)^{2k}).
$$

(7.4)

As before by (5.4),

$$
- \int_J \langle i \nabla Iu, \nabla ((IP_{> c(k)N}u)(P_{\leq c(k)N}u)^{2k} - I((P_{> c(k)N}u)(P_{\leq c(k)N}u)^{2k})) \rangle dt
$$

$$
\lesssim \frac{1}{N} \| \nabla IP_{> c(k)N}u \|_{L_t^2 L_x^k(J \times \mathbb{R}^2)}^{2} \| \nabla Iu \|_{L_t^{\infty} L_x^{k+1}(J \times \mathbb{R}^2)} \| Iu \|_{L_t^{\infty} L_x^{k+1}(J \times \mathbb{R}^2)}^{2k-1} \lesssim \frac{1}{N^{1-}}.
$$

(7.5)

The estimate $\| Iu \|_{L_t^{4k-2}} \lesssim 1$ follows from interpolating $\| Iu \|_{H^1} \lesssim 1$ with $\| Iu \|_{L_t^{2k+2}} \lesssim 1$, along with the fact that $k > 1$. We also use (6.22) to estimate $\| \nabla IP_{> c(k)N}u \|_{L_t^{2} L_x^{\infty}(J \times \mathbb{R}^2)}$ along with lemma 3.3 and (3.21).

Next, since $E(Iu(t)) \leq 1$,

$$
\int_J \langle i \nabla Iu, \nabla ((IP_{> c(k)N}u)^{2}(P_{\leq c(k)N}u)^{2k-1} - I((P_{> c(k)N}u)^{2}(P_{\leq c(k)N}u)^{2k-1})) \rangle dt
$$

(7.7)
\[
\|\nabla I u\|_{L_t^\infty L_x^2} \lesssim \|\nabla P_{c(k)N} u\|_{L_t^2 L_x^\infty} \|P_{c(k)N} u\|_{L_t^2 L_x^\infty} \| P_{\leq c(k)N} u\|_{L_t^2 L_x^{2k-2}} \lesssim \frac{1}{N^{1/2}}. 
\] (7.8)

Finally, we skip ahead to

\[
\int \langle i \nabla I u, \nabla ((IP_{c(k)N} u)^{2k+1} - I((P_{c(k)N} u)^{2k+1})) \rangle \, dt 
\] (7.9)

\[
\lesssim \|\nabla I u\|_{L_t^\infty L_x^2} \|\nabla P_{c(k)N} u\|_{L_t^2 L_x^\infty} \|P_{c(k)N} u\|_{L_t^2 L_x^\infty} \| P_{\leq c(k)N} u\|_{L_t^2 L_x^{2k}} \lesssim \frac{1}{N^{1/2}}. 
\] (7.10)

Remark: The other terms can be handled in a similar manner.

Remark: \( L_t^{4k} \) is \( \dot{H}^{k-1} \) - critical. Interpolation shows that

\[
\|P_{c(k)N} u\|_{L_t^2 L_x^{2k}} \lesssim E(I u(t))^{1/2} (2^{k-1})^{-1/2} \|\nabla P_{c(k)N} u\|_{L_t^1 L_x^\infty}^{1/2} \| P_{c(k)N} u\|_{L_t^2 L_x^\infty} \lesssim \frac{1}{N^{1/2}}. 
\] (7.11)

Then by (6.22), \( E(I u(t)) \leq 1 \), we are done with the first term in (7.2). Now we consider the term

\[
\int \langle I(u^{2k+1}), I(u^{2k+1}) - (I u)^{2k+1} \rangle \, dt. 
\] (7.12)

Once again this term must have at least two \( P_{c(k)N} u \) terms. By Sobolev embedding, Bernstein’s inequality, and (6.22),

\[
\|I((P_{c(k)N} u)^{2k+1})\|_{L_t^2 L_x^2} \lesssim \|\nabla P_{c(k)N} u\|_{L_t^2 L_x^\infty} \|P_{c(k)N} u\|_{L_t^2 L_x^\infty} \| P_{\leq c(k)N} u\|_{L_t^2 L_x^{2k}} \lesssim \frac{1}{N}. 
\] (7.13)

Therefore,

\[
\int \langle I((P_{c(k)N} u)^{2k+1}), I((P_{c(k)N} u)^{2k+1}) - (I P_{c(k)N} u)^{2k+1} \rangle \, dt \lesssim \frac{1}{N^2}. 
\] (7.14)

Next,

\[
\int \langle I((P_{c(k)N} u)^{2k+1}), (P_{c(k)N} u)^{2k}(P_{\leq c(k)N} u) \rangle \, dt \lesssim \|I(P_{c(k)N} u)^{2k+1}\|_{L_t^2 L_x^2} \|P_{c(k)N} u\|_{L_t^2 L_x^{2k}} \|P_{\leq c(k)N} u\|_{L_t^\infty L_x^2} \lesssim \frac{1}{N^2}. 
\] (7.15)

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Finally,

\[
\int_J \int (P_{>c(k)N}u)^2 (P_{\leq c(k)N}u)^2 u^{4k-2} \, dx \, dt \\
\lesssim \int_J \int (P_{>c(k)N}u)^{4k+2} \, dx \, dt + \int_J \int (P_{>c(k)N}u)^2 (P_{\leq c(k)N}u)^4 \, dx \, dt.
\] (7.16)

Interpolating the \(L^2_x\) and \(\dot{H}^1\) norms, since \(E(Iu(t)) \leq 1\),

\[
\|Iu\|_{L^\infty_t L^4_x} \lesssim 1.
\] (7.17)

This proves lemma 7.1. □

Rescaling back, we have proved

\[
\|u(t)\|_{L^2_x(\mathbb{R}^2)} = \|u(0)\|_{L^2_x(\mathbb{R}^2)},
\] (7.18)

and

\[
\|u(t)\|_{\dot{H}^s(\mathbb{R}^2)} \lesssim \|u(0)\|_{L^2(\mathbb{R}^2)} + N^{\delta_0} \frac{1}{k-\epsilon_c} \|u(0)\|_{\dot{H}^s(\mathbb{R}^2)}.
\] (7.19)

Therefore,

\[
\|u(t)\|_{\dot{H}^s(\mathbb{R}^2)} \lesssim C(\|u_0\|_{\dot{H}^s(\mathbb{R}^2)}, k) \|u_0\|_{\dot{H}^s(\mathbb{R}^2)},
\] (7.20)

where \(C\) behaves like \(e^{C_1 \frac{1}{k-s}}\) for some constant \(C_1\).

References

[1] J. Bourgain, “Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations”, Geom. Funct. Anal., 3 no. 2 (1993) 107 – 156.

[2] J. Bourgain, “Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II. The KdV-equation”, Geom. Funct. Anal., 3 no. 3 (1993) 209 – 262.

[3] J. Bourgain, “Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity”, International Mathematical Research Notices, 5 (1998) 253 – 283.
[4] J. Bourgain, “Scattering in the energy space and below for 3D NLS”, 
*Journal d’Analyse Mathématique*, **75** (1998) 267 – 297.

[5] T. Cazenave and F. B. Weissler, “The Cauchy problem for the nonlinear Schrödinger equation in \( H^1 \)”, *Manuscripta Math.*, **61** (1988) 477 – 494.

[6] T. Cazenave and F. B. Weissler, *Some remarks on the nonlinear Schrödinger equation in the subcritical case (New methods and results in nonlinear field equations (Bielefeld, 1987))*), Lecture Notes in Physics **347**, Springer, Berlin, 1989.

[7] T. Cazenave and F. B. Weissler, “The Cauchy problem for the critical nonlinear Schrödinger equation in \( H^s \)”, *Nonlinear Anal.*, **14** (1990) 807 – 836.

[8] M. Christ, J. Colliander, and T. Tao, “A priori bounds and weak solutions for the nonlinear Schrödinger equation in Sobolev spaces of negative order”, *Journal of Functional Analysis* **254** no. 2 (2008) 368 – 395.

[9] M. Christ, J. Colliander, and T. Tao, “Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations”, *American Journal of Mathematics* **125** no. 6 (2003) 1235 – 1293.

[10] M. Christ and A. Kiselev, “Maximal functions associated to filtrations”, *Journal of Functional Analysis*, **179** no. 2 (2001) 409 – 425.

[11] J. Colliander, M. Grillakis, and N. Tzirakis. “Tensor products and correlation estimates with applications to nonlinear Schrödinger equations”, *Communications on Pure and Applied Mathematics*, **62** no. 7 (2009) 920 – 968.

[12] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, “Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation”, *Mathematical Research Letters*, **9** no. 5 - 6 (2002) 659 – 682.

[13] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, “Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on \( \mathbb{R}^3 \)”, *Communications on Pure and Applied Mathematics*, **21** (2004) 987 - 1014.
[14] B. Dodson, “Global well-posedness and scattering for the defocusing, cubic nonlinear Schrödinger equation when $n = 3$ via a linear-nonlinear decomposition”, *Discrete and Continuous Dynamical Systems. Series A*, 33 no. 5 (2013) 1905 – 1926.

[15] B. Dodson, “Global well-posedness and scattering for the defocusing $L^2$-critical nonlinear Schrödinger equation when $d \geq 3$”, *Journal of the American Mathematical Society*, 25 no. 2 (2012) 429 – 463.

[16] B. Dodson, “Global well-posedness and scattering for the defocusing $L^2$-critical nonlinear Schrödinger equation when $d = 2$”, preprint, arXiv:1006.1375v2.

[17] B. Dodson, “Global well-posedness and scattering for the defocusing $L^2$-critical nonlinear Schrödinger equation when $d = 1$”, preprint, arXiv:1010.0040v2.

[18] J. Ginibre, “Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d’espace (d’après Bourgain)” (Séminaire Bourbaki 1994/1995), *Astérisque*, 237 (1996) Exp. No. 796, 163 – 187.

[19] J. Ginibre and G. Velo, “Scattering theory in the energy space for a class of nonlinear Schrödinger equations”, *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, 64 no. 4 (1985) 363 – 401.

[20] J. Ginibre and G. Velo, “Smoothing properties and retarded estimates for some dispersive evolution equations”, *Communications in Mathematical Physics*, 144 no. 1 (1992) 163 – 188.

[21] M. Hadac and S. Herr and H. Koch, “Well-posedness and scattering for the KP-II equation in a critical space”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26 no. 3 (2009) 917 – 941.

[22] C. Kenig and F. Merle, “Scattering for $\dot{H}^{1/2}$ bounded solutions to the cubic, defocusing NLS in 3 dimensions”, *Transactions of the American Mathematical Society* 362 no. 4 (2010) 1937 – 1962.

[23] M. Keel and T. Tao, “Endpoint Strichartz Estimates” *American Journal of Mathematics* 120 no. 4 - 6 (1998) 945 – 957.

[24] M. Keel and T. Tao “Local and global well posedness of wave maps on $\mathbb{R}^{1+1}$ for rough data”, *International Mathematics Research Notices* 21 (1998) 1117 – 1156.
[25] R. Killip, T. Tao, and M. Visan, “The cubic nonlinear Schrödinger equation in two dimensions with radial data”, *Journal of the European Mathematical Society* **11** no. 6 (2009) 1203 – 1258.

[26] R. Killip and M. Visan, “Global well - posedness and scattering for the defocusing quintic NLS in three dimensions”, *Analysis and PDE* **5** no. 4 (2012) 855 – 885.

[27] H. Koch and D. Tataru, “A priori bounds for the 1D cubic NLS in negative Sobolev spaces”, *International Mathematics Research Notices IMRN* **16** (2007) Art. ID rnm053, 36 pp.

[28] H. Koch and D. Tataru, “Energy and local energy bounds for the 1-D cubic NLS equation in $H^{-1/4}$”, *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire* **29** no. 6 (2012) 955 – 988.

[29] J. Lin and W. Strauss, “Decay and Scattering of solutions of a non-linear Schrödinger equation”, *Journal of Functional Analysis* **30** no. 2 (1978) 245 – 263.

[30] J. Murphy, “Inter - critical NLS: critical $\dot{H}^s$ bounds imply scattering”, preprint arxiv 1209.4582.

[31] J. Murphy, “The defocusing $\dot{H}^{1/2}$ - critical NLS in high dimensions”, *Discrete and Continuous Dynamical Systems. Series A* **34** no. 2 (2014) 733 – 748.

[32] J. Murphy, “The radial, defocusing, nonlinear Schrödinger equation in three dimensions”, arXiv:1401.4766.

[33] F. Planchon and L. Vega, “Bilinear virial identities and applications” *Annales Scientifiques de l’École Normale Supérieure Quatrième Série* **42** no. 2 (2009) 261 - 290.

[34] A. Ruiz and L. Vega, “On local regularity of Schrödinger equations”, *International Mathematics Research Notices IMRN* **1** (1993) 13–27.

[35] T. Roy, “Adapted linear - nonlinear decomposition and global well - posedness for solutions to the defocusing wave equation on $\mathbb{R}^3$”, *Discrete and Continuous Dynamical Systems. Series A* **24** no. 4 (2009) 1307 – 1323.

[36] E. M. Stein, *Singular Integrals and Differentiability Properties of functions*, Princeton University Press, Princeton, NJ, 1970.
[37] E. M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.

[38] R. S. Strichartz, “Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations”, *Duke Mathematical Journal* **44** no. 3 (1977) 705 - 714.

[39] Q. Su, “Global well - posedness and scattering for the defocusing, cubic NLS in $\mathbb{R}^3$”, Mathematical Research Letters, **19** (2012), 431 - 451.

[40] T. Tao, *Nonlinear Dispersive Equations. Local and Global Analysis*, CBMS Regional Conference Series in Mathematics **104** Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006.

[41] T. Tao, M. Visan, and X. Zhang, “The nonlinear Schrödinger equation with combined power-type nonlinearities”, *Communications in Partial Differential Equations*, **32** no. 7-9 (2007) 1281–1343.

[42] D. Tataru, “Local and global results for wave maps I”, *Communications in PDE*, **23** no. 9 - 10 (1998) 1781 - 1793.

[43] M. E. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Birkhäuser, Boston, 1991.

[44] M. E. Taylor, *Partial Differential Equations I - III*, Second Edition, Applied Mathematical Sciences **115** Springer-Verlag, New York, 2011.

[45] M. Visan, “Global well - posedness and scattering for the defocusing cubic nonlinear Schrödinger equation in four dimensions”, *International Mathematics Research Notices. IMRN* **5** (2012) 1037 – 1067.

[46] K. Yajima, “Existence of solutions for Schrödinger evolution equations”, *Communications in Mathematical Physics* **110** no. 3 (1987) 415 - 426.