Abstract

A trick to obtain a systematic solution to the set-theoretical reflection equation is presented from a known one to the Yang-Baxter equation. Examples are given from crystals and geometric crystals associated to the quantum affine algebra of type $A_{n-1}^{(1)}$.

1. Introduction

Principal features of integrable systems in the bulk and at the boundary are the Yang-Baxter equation \[1\] and the reflection equation \[2, 20, 26\], respectively. In their original formulation in quantum field theory or statistical mechanics, they are cubic and quartic relations of the form \(RRR = RRR\) and \(RRK = KRR\), where \(R\) and \(K\) are matrices encoding the interactions in the bulk and at the boundary of the system. By now extensive results and knowledge on these equations and solutions have been accumulated especially via their connection to the theory of quantum groups \[6, 15\].

The Yang-Baxter and reflection equations can also be formulated in a wider context where the linear operators \(R\) and \(K\) are replaced by transformations of various kind such as bijection between sets, birational maps between varieties and so forth. Such extensions, originally suggested for the Yang-Baxter equation \[7\], are often called the set-theoretical ones. See for instance \[29\] for a guide to the set-theoretical \(R\)'s, and \[22, 3, 4, 27\] for some concrete examples of set-theoretical \(K\)'s. By definition a set-theoretical solution to the reflection equation means a pair of set-theoretical \(R\) and \(K\).

In this paper we present new set-theoretical solutions to the reflection equation. They are most natural and systematic examples associated with the Drinfeld-Jimbo quantum affine algebra \(U_q(A_{n-1}^{(1)})\). There are two versions \((R_{B,B'}, K_B)\) and \((\mathcal{R}, \mathcal{K})\) which originate in the crystals \[18\] and the geometric crystals \[2\] related to the Kirillov-Reshetikhin modules of \(U_q(A_{n-1}^{(1)})\), respectively. For Kirillov-Reshetikhin modules of quantum affine algebras and their rich background, see e.g., \[12\].

In the first solution, \(R_{B,B'} : B \times B' \to B' \times B\) and \(K_B : B \to B'\) are bijections among the indicated finite sets called crystals. Here \(\mathcal{V}\) denotes the dual whose detail is given in Section 2. On the other hand, \(\mathcal{R}\) and \(\mathcal{K}\) treated in Section 3 are birational maps between varieties. The solution \((R_{B,B'}, K_B)\) can be recovered from \((\mathcal{R}, \mathcal{K})\) by the procedure called tropicalization (cf. \[28\]) or ultra-discretization in another terminology frequently used for integrable cellular automata (cf. \[13, 22\]).

The set-theoretical \(R_{B,B'}\) is a combinatorial \(R\)-matrix\(^1\) in the sense of \[16\] Def.4.2.1] and corresponds to a quantum \(R\)-matrix at \(q = 0\). It preserves the crystal structure (sort of colored oriented graph) on \(B \times B'\) and \(B' \times B\) inheriting the quantum group symmetry of the relevant quantum \(R\)-matrices. It is a highly nontrivial bijection described by an elegant tableau combinatorics \[25\]. The set-theoretical \(\mathcal{R}\) is called the geometric \(R\)-matrix \[10\], which covers a few special cases known earlier \[22\] sec. A.3]. A contribution of this paper is the construction of \(K_B\) and \(K\) from \(R_{B,B'}\) and \(\mathcal{R}\), respectively, in a systematic manner. As already mentioned, \(K_B\) is deducible in principle from \(\mathcal{K}\) by ultra-discretization. However this procedure just provides the former with a formidable complicated piecewise linear formula which can be handled effectively only on computers. Thus it is worth while to describe \(K_B\) independently by a neat combinatorial algorithm. This will be done in Section 2 by attributing \(K_B\) to the combinatorial \(R\)-matrix \(R_{B',B}\).

Let us digest our approach to the set-theoretical reflection equation along the example \((R_{B,B'}, K_B)\). Recall that the Yang-Baxter equation is depicted by the world lines of three particles on a plane. Now

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\(^1\) Although \(R_{B,B'}, K_B, \mathcal{R}, \mathcal{K}\) are all set-theoretical, we refer to them with “matrix” following the terminology in \[15, 10\].
we consider the Yang-Baxter equality involving two pairs of particles, hence four world lines in total, in the vicinity of a virtual boundary. Arrange two incoming particles within a pair so that their motion become mirror image of each other with respect to the boundary. See Figure 1 where the boundary is depicted by broken lines. The mirror images are realized by switching to the dual crystals/tableaux signified by the superscript \( \vee \). Then it can be shown that the reflection symmetry of the incoming state via \( \vee \) persists throughout the whole scattering event. This claim is formulated as Proposition 3 and Corollary 3. The resulting Yang-Baxter equality restricted on either side of the mirror is nothing but a set-theoretical reflection equation. This trick of using the virtual boundary works efficiently in a set-theoretical situation like here and \([3, 4]\). However in general, it does not extend naively to the original (quantum) version of the reflection equation where \( R \) and \( K \) are linear operators.

We remark that the three dimensional analogue of the reflection equation known as the tetrahedron reflection equation \( R_{456}R_{489}K_{3579}R_{269}R_{258}K_{1678}K_{1234} = K_{1234}K_{1678}R_{258}R_{269}K_{3579}R_{489}R_{456} \) also admits a similar triad of quantum, combinatorial and birational versions of solutions in which the latter two are set-theoretical \([21]\) Tab.1.

2. Kirillov-Reshetikhin Crystal for \( A_{n-1}^{(1)} \)

We recall the Kirillov-Reshetikhin crystal, KR crystal for short, \( B^{k,l} \) for type \( A_{n-1}^{(1)} \), where \( 1 \leq k \leq n-1, l \geq 1 \). It is a crystal basis, in the sense of Kashiwara \([18]\), of a certain finite-dimensional module, called Kirillov-Reshetikhin module over the quantized enveloping algebra of affine type \( A_{n-1}^{(1)} \). See e.g. \([12]\). As a set \( B^{k,l} \) consists of semi-standard tableaux of \( k \times l \) rectangular shape with letters from \( \{1, 2, \ldots, n\} \). On \( B^{k,l} \) applications of Kashiwara operators \( \epsilon_i, \tilde{f}_i \) for \( 0 \leq i \leq n-1 \) are defined \([17, 25]\). A few examples follow.

**Example 1.**

1. \( n = 4, B^{1,5} \)

\[ \epsilon_2 \begin{vmatrix} 1 & 2 & 3 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 & 3 & 4 \end{vmatrix}, \quad \tilde{f}_0 \begin{vmatrix} 1 & 2 & 3 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & 3 & 3 \end{vmatrix} \]

2. \( n = 6, B^{4,1} \)

\[ \epsilon_5 \begin{vmatrix} 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 3 \end{vmatrix}, \quad \tilde{f}_0 \begin{vmatrix} 1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \end{vmatrix} \]

3. \( n = 6, B^{4,3} \)

\[ \epsilon_3 \begin{vmatrix} 1 & 1 & 3 & 2 & 2 & 4 & 3 & 4 & 4 & 5 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 3 & 4 & 4 & 5 & 5 & 5 \end{vmatrix} \]

For details see e.g. \([24]\). For an element b of a KR crystal, we set

\[ \epsilon_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \epsilon_i b \neq 0\}, \quad \varphi_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i b \neq 0\}. \]

Let \( B_1, B_2 \) be KR crystals. The Cartesian product of \( B_1 \) and \( B_2 \) denoted by \( B_1 \otimes B_2 = \{ b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2 \} \) is also endowed with the crystal structure by

\[
\begin{align*}
\epsilon_i(b_1 \otimes b_2) &:= \begin{cases} 
\epsilon_i b_1 \otimes b_2 & \text{if } \epsilon_i(b_1) > \varphi_i(b_2), \\
 b_1 \otimes \tilde{f}_i b_2 & \text{if } \epsilon_i(b_1) \leq \varphi_i(b_2),
\end{cases} \\
\tilde{f}_i(b_1 \otimes b_2) &:= \begin{cases} 
\tilde{f}_i b_1 \otimes b_2 & \text{if } \epsilon_i(b_1) \geq \varphi_i(b_2), \\
 b_1 \otimes \tilde{f}_i b_2 & \text{if } \epsilon_i(b_1) < \varphi_i(b_2),
\end{cases}
\end{align*}
\]

We use this convention of the tensor product, which is opposite from the one in \([18]\).

For KR crystals \( B_1, B_2 \) there exists a unique bijection \( R_{B_1,B_2} : B_1 \otimes B_2 \rightarrow B_2 \otimes B_1 \), called the combinatorial \( R \)-matrix, that commutes with \( \epsilon_i, \tilde{f}_i \) for any \( i \) \([16]\). Uniqueness follows from the fact that \( B_1 \otimes B_2 \) is connected, namely, any element of \( B_1 \otimes B_2 \) can be reached from a fixed element by applying \( \epsilon_i \)'s or \( \tilde{f}_i \)'s. Explicitly, the combinatorial \( R \)-matrix for type \( A_{n-1}^{(1)} \) can be calculated by the so-called tableau product \([11]\). Since the tensor product of two representations corresponding to rectangular shapes is multiplicity free and the operators \( \epsilon_i, \tilde{f}_i \) \((i \neq 0)\) commute with the operations to construct the tableau product, for the tableau product \( b_1 \cdot b_2 \) \((b_1 \in B^{k_1,l_1}, b_2 \in B^{k_2,l_2})\) there is a unique pair
$(\tilde{b}_2, \tilde{b}_1) \in B^{k_2, l_2} \times B^{k_1, l_1}$ such that $b_1 \cdot b_2 = \tilde{b}_2 \cdot \tilde{b}_1$. In this case $R(b_1 \otimes b_2) = \tilde{b}_2 \otimes \tilde{b}_1$. Combinatorial $R$-matrices satisfy the Yang-Baxter equation as a map from $B_1 \otimes B_2 \otimes B_3$ to $B_3 \otimes B_2 \otimes B_1$:

\[
(R_{B_2, b_3} \otimes 1)(1 \otimes R_{B_1, b_3})(R_{B_1, b_2} \otimes 1) = (1 \otimes R_{B_3, b_2})(R_{B_1, b_3} \otimes 1)(1 \otimes R_{B_2, b_3}).
\]

**Example 2.** Let $n = 6$, and $b_1 = \begin{bmatrix} 1 & 3 & 4 & 1 \ 2 & 6 & 6 \ 1 & 1 & 3 \ 2 & 2 & 4 \ 3 & 4 & 5 \ 5 & 5 & 6 \end{bmatrix} \in B^{2,3}$, $b_2 = \begin{bmatrix} 1 & 1 & 3 \ 2 & 4 \ 3 & 3 \ 5 & 4 \ 6 & 6 \end{bmatrix} \in B^{4,3}$. Then the tableau product is given by

\[
b_1 \cdot b_2 = \begin{bmatrix} 1 & 1 & 1 & 3 & 4 & 5 \ 2 & 2 & 2 & 4 \ 3 & 3 & 5 \ 4 & 5 & 6 \ 6 & 6 \end{bmatrix}.
\]

The only pair of tableaux $(\tilde{b}_2, \tilde{b}_1) \in B^{4,3} \times B^{2,3}$ such that $b_1 \cdot b_2 = \tilde{b}_2 \cdot \tilde{b}_1$ is given by $\tilde{b}_2 = \begin{bmatrix} 1 & 2 & 2 \ 2 & 3 & 3 \ 4 & 5 & 5 \ 6 & 6 & 6 \end{bmatrix}$, $\tilde{b}_1 = \begin{bmatrix} 1 & 1 & 3 \ 2 & 4 & 5 \ 4 & 4 & 5 \ 6 & 6 & 6 \end{bmatrix}$.

Hence we have $R(b_1 \otimes b_2) = \tilde{b}_2 \otimes \tilde{b}_1$. See Example 3.7 and the explanations above for more details. (Note however that the convention for the tensor product there is opposite.)

The notion of dual crystal is given in [23, Section 7.4]. Let $B$ be a crystal. Then there is a crystal denoted $B^\vee$ obtained from $B$ defined by $B^\vee = \{ b^\vee \mid b \in B \}$ with

\[
\tilde{e}_i b^\vee = (\tilde{f}_i b)^\vee, \quad \tilde{f}_i b^\vee = (\tilde{e}_i b)^\vee.
\]

Then there is an isomorphism $(B_1 \otimes B_2)^\vee \cong B_2^\vee \otimes B_1^\vee$ given by $(b_1 \otimes b_2)^\vee \mapsto b_2^\vee \otimes b_1^\vee$. For a KR crystal $B^{k,l}$ of type $A_{n-1}^{(1)}$ we have $(B^{k,l})^\vee = B^{n-k,l}$ and $b^\vee$ is obtained from $b$ by replacing each column of a rectangular tableau with its compliment in $\{1, \ldots, n\}$ and reversing the order of the columns.

**Example 3.**

1. When $B = B^{1,1}$ one can use a sequence of nonnegative integers $x(b) = (x_1, \ldots, x_n)$ to parametrize a crystal element $b$, where $x_i$ stands for the number of $i$ in the one-row tableau $b$.

Then $b^\vee \in B^{n-1,1}$ is given by the $(n - 1) \times l$ tableau such that the number of columns missing $i$ is $x_i$.

2. Let $n = 6$. For $b = \begin{bmatrix} 1 & 3 & 4 \ 2 & 2 & 4 \ 3 & 4 & 5 \ 5 & 5 & 6 \end{bmatrix} \in B^{4,3}$ we have $b^\vee = \begin{bmatrix} 1 & 3 & 4 \ 2 & 6 & 6 \ 5 & 5 & 6 \end{bmatrix} \in B^{2,3}$. \(\tilde{f}_3 b^\vee\) is given by $\tilde{e}_3 b^\vee$ where \(\tilde{e}_3 b\) is in Example 1(3).

**Proposition 4.** If $R_{B_1, B_2}(b_1 \otimes b_2) = c_2 \otimes c_1$, then $R_{B_2^\vee, B_1^\vee}(b_2^\vee \otimes b_1^\vee) = c_1^\vee \otimes c_2^\vee$.

**Proof.** For any isomorphism of crystals $\psi : B \to B'$, the map $\psi^\vee$ defined by $\psi^\vee = \vee \circ \psi \circ \vee$ becomes an isomorphism from $B^\vee$ to $(B')^\vee$. For instance, $\psi^\vee(\tilde{e}_i b^\vee) = \psi((\tilde{f}_i b)^\vee) = (\psi(\tilde{f}_i b))^\vee = (\tilde{f}_i \psi(b))^\vee = \tilde{e}_i \psi^\vee(b^\vee)$.

Combining the isomorphism $\iota_{i,j} : (B_1 \otimes B_2)^\vee \to B_2^\vee \otimes B_1^\vee$, one finds

\[
(\iota_{2,1} \circ \vee \circ R_{B_1, B_2} \circ \vee \circ \iota_{1,2}^{-1})(b_2^\vee \otimes b_1^\vee) = c_1^\vee \otimes c_2^\vee
\]

also gives an isomorphism. Since $B_2^\vee \otimes B_1^\vee$ is connected, this map agrees with $R_{B_2^\vee, B_1^\vee}$. \(\square\)

**Corollary 5.** We have:

1. If $R_{B_1^\vee, B_2^\vee}(b_1^\vee \otimes b_2^\vee) = c_2 \otimes c_1^\vee$, then $R_{B_2, B_1}(b_2 \otimes b_1) = c_1 \otimes c_2^\vee$.

2. $R_{B_1^\vee, B_2^\vee}(b^\vee \otimes b) = c \otimes c^\vee$.

**Proof.** For (1) set $B_1 = B_1^\vee, B_2 = B, b_1 = b_2^\vee, c_1 = c_2^\vee$ in Proposition 4. For (2) set $b_1$ and $b_2$ to be $b$ in (1). Since $c_1 = c_2$, we have the desired result. \(\square\)

Using $b$ and $c^\vee$ in Corollary 5(2), we define the combinatorial $K$-matrix $K_B$ by

$$K_B : B \to B^\vee, \quad b \mapsto c^\vee.$$
**Example 6.** (1) We continue considering an example in Example 3(1). Suppose $R_{(B^{i+1})^\vee,B^{i+1}}$ sends $b^\vee \otimes c$ to $\tilde{c} \otimes \tilde{b}^\vee$. Set $x(b) = (x_i)_{i=1}^n$, $x(c) = (y_i)_{i=1}^n$. Then we have

$$x(\tilde{b})_i = x_i + p_{i+1} - p_i, \quad y(\tilde{c})_i = y_i + p_{i+1} - p_i,$$

where $p_i = \min(x_i,y_i)$ and the index $i$ should be considered modulo $n$. See [22] (2.2)]. Note that our convention of the tensor product is opposite from there. Set $l = m, x_i = y_i$ for all $i$. Then we find $x(\tilde{b})_i = x(\tilde{c})_i = x_{i+1}$. Hence, $K_{B^{i+1}}(b) = \tilde{b}$ where $x(\tilde{b})_i = x_{i+1}$.

(2) We take an example from Example 2. Namely, we set $n = 6, B = B^{4,3}$, then $B^\vee = B^{3,2}$. The image of

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 6 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

by $R$ is

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 4 & 5 & 5 \\ 6 & 6 & 6 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 3 \\ 4 & 4 & 5 \end{bmatrix}$$

Corollary 5 is confirmed in this example.

Hence, the image of

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 4 \\ 3 & 3 & 5 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 3 \\ 4 & 4 & 5 \end{bmatrix}$$

by $K_B$ is defined to be

$$\begin{bmatrix} 1 & 1 & 3 \\ 5 & 5 & 6 \end{bmatrix}$$

**Theorem 7.** The combinatorial $R$-matrix and $K$-matrix satisfy the set-theoretical reflection equation.

$$R_{B^{i+1}}(K_{B^{i+1}} \otimes 1) R_{B^{i+1}}(K_{B^{i+1}} \otimes 1) = (K_{B^{i+1}} \otimes 1) R_{B^{i+1}}(K_{B^{i+1}} \otimes 1) R_{B^{i+1}}(K_{B^{i+1}} \otimes 1)$$

**Proof.** Figure 1 illustrates the proof. Each side represents the application of each side of

$$R_{B^{i+1}}^{3,4} R_{B^{i+1}}^{1,2} R_{B^{i+1}}^{2,3} R_{B^{i+1}}^{3,4} R_{B^{i+1}}^{1,2} R_{B^{i+1}}^{2,3} R_{B^{i+1}}^{3,4} = R_{B^{i+1}}^{2,3} R_{B^{i+1}}^{1,2} R_{B^{i+1}}^{3,4} R_{B^{i+1}}^{1,2} R_{B^{i+1}}^{2,3} R_{B^{i+1}}^{3,4} R_{B^{i+1}}^{1,2} R_{B^{i+1}}^{3,4}$$

(5)

on $c^\vee \otimes b^\vee \otimes b \otimes c \in B^{i+1}_1 \otimes B^{i+1}_1 \otimes B^{i+1}_1 \otimes B^{i+1}_2$. Here superscripts in the $R$-matrices stands for the positions of the components on which it acts. Equation (5) itself is verified by successive use of the Yang-Baxter equation, where the transitions of elements upon applications of $R$-matrices are depicted in Figure 1 using Proposition 4 and Corollary 5. Eventually, we obtain $c_3 = c_6, b_3 = b_6, b_4^\vee = b_5^\vee, c_3^\vee = c_6^\vee$. Just viewing the right parts of both sides of Figure 1 and recalling the definition of the $K$-matrix, we finish the proof of the reflection equation.

\[\square\]

**Figure 1.** Proof of the reflection equation.

A similar argument has also been given in [4].
Example 8. \( n = 5, B_1 = B^{1,2}, B_2 = B^{2,1} \), which implies \( B_1^\vee = B^{4,2}, B_2^\vee = B^{3,1} \).

For a KR crystal \( B \) we can define its affinization \( \text{Aff}(B) = \{ z^d b \mid b \in B, d \in \mathbb{Z} \} \). On \( \text{Aff}(B) \) Kashiwara operators act as \( \tilde{e}_i(z^d b) = z^{d+\delta_{0i}}(\tilde{e}_i b), \tilde{f}_i(z^d b) = z^{d-\delta_{0i}}(\tilde{f}_i b) \). Combinatorial \( R_{B_1,B_2} \)-matrix is also upgraded by introducing the energy function \( H \) \[10\] as

\[
R_{B_1,B_2} : \text{Aff}(B_1) \otimes \text{Aff}(B_2) \rightarrow \text{Aff}(B_2) \otimes \text{Aff}(B_1)
\]

\[
z^{d_1 b_1} \otimes z^{d_2 b_2} \rightarrow z^{d_2 + H(b_1 \otimes b_2)}(c_2 \otimes z^{d_1 - H(b_1 \otimes b_2)} c_1).
\]

This version of the combinatorial \( R \)-matrices also satisfy the Yang-Baxter equation. Hence, by defining \( K_B(z^d b) = z^{-d - H(b') \otimes b)}c \) when \( R_{B',B}(z^{-d}b' \otimes z^d b) = z^{d + H(b' \otimes b)}c \otimes z^{-d - H(b' \otimes b)}c \), upgraded combinatorial \( R \)-matrices and \( K \)-matrices satisfy the reflection equation. This generalizes \[22, \text{sec}.2.3].

3. Geometric crystal for \( A^{(1)}_{n-1} \)

Geometric crystal is a notion introduced by Berenstein and Kazhdan in \[2\] as an algebro-geometric analogue of crystal. A geometric crystal of type \( A^{(1)}_{n-1} \) is a pentad \( (X, \gamma, \varphi_i, \varepsilon_i, e_i) \) where \( X \) is an irreducible complex algebraic variety, \( \gamma : X \rightarrow (\mathbb{C}^\times)^n \) is a rational map, for \( i \in \mathbb{Z}/n \mathbb{Z} \), \( \varphi_i, \varepsilon_i : X \rightarrow \mathbb{C}^\times \) are rational functions and \( e_i : \mathbb{C}^\times \times X \rightarrow X \) is a rational action. These data must satisfy further relations. For instance, denoting the image \( e_i(c, x) \) by \( e_i^c(x) \), the actions \( e_i^c, e_j^c \) should satisfy \( e_i^c e_j^c = e_j^c e_i^c \) if \( |i - j| > 1 \), \( e_i^c e_i^{c_1} e_j^{c_2} = e_j^{c_2} e_i^{c_1} e_j^{c_2} \) if \( |i - j| = 1 \).

For \( 1 \leq k \leq n - 1 \), Frieden \[9\] gave a geometric crystal structure of type \( A^{(1)}_{n-1} \) on \( \text{Gr}(n - k, n) \times \mathbb{C}^\times \) where \( \text{Gr}(m, n) \) is the Grassmannian of \( m \)-dimensional subspaces in \( \mathbb{C}^n \). Introduce the space of “rational \( k \)-rectangle” by \( T_k = (\mathbb{C}^\times)^{R_k} \times \mathbb{C}^\times \) where

\[
R_k = \{(i,j) \mid 1 \leq i \leq k, 1 \leq j \leq i + n - k - 1\}.
\]

An element of \( T_k \) is denoted by \((X, s)\). There is an open embedding of \( T_k \) into \( \text{Gr}(n - k, n) \times \mathbb{C}^\times \). In the construction in \[10\] it is also important that there exists an injection \( g \) from \( T_k \) to \( G\text{L}_n(\mathbb{C}(\lambda)) \) satisfying

\[
g(e_i^c(X, s)) = \tilde{x}_i \left( \frac{c - 1}{\varphi_i(X, s)} \right) g(X, s) \tilde{x}_i \left( \frac{c^{-1} - 1}{\varepsilon_i(X, s)} \right),
\]

where \( \tilde{x}_i(a) = I + a \lambda^{-1} E_{i,i+1} \) for \( i \in \mathbb{Z}/n \mathbb{Z}, I \) is the identity matrix and \( E_{ij} \) stands for the \((i,j)\) matrix unit.

Like the tensor product in crystals, there is a product of \((X, s)\) and \((Y, t)\) denoted by \((X, s) \times (Y, t)\) such that \( g((X, s) \times (Y, t)) = g(X, s)g(Y, t) \). Thanks to this map \( g \), the product \((X, s) \times (Y, t)\) acquires the structure of the geometric crystal.

**Theorem 9** \([10]\). There exists a birational map \( R : T_{k_1} \times T_{k_2} \rightarrow T_{k_2} \times T_{k_1} \) that commutes with \( e_i^c \). Moreover, on \( T_{k_1} \times T_{k_2} \times T_{k_3} \) the Yang-Baxter equation

\[
R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}
\]

is satisfied. Here \( R_{ij} \) means that \( R \) acts on the \( i \)-th and \( j \)-th components and the other one trivially.
Theorem 10 (10). There exists a duality map \( D : T_k \rightarrow T_{n-k} \) satisfying \( e_i^c \circ D = D \circ e_i^{c-1} \) and
\[
D((X,s) \times (Y,t)) = D(Y,t) \times D(X,s).
\]

In terms of the Grassmannian, the duality map \( D \) corresponds to the orthogonal compliment with respect to a certain bilinear form of the ambient vector space. In the image of \( g, g(D(X,s)) \) is closely related to the matrix inverse of \( g(X,s) \).

From these two theorems, one obtains similar claims to Proposition [4] and Corollary [3]. Hence, by defining \( K : T_k \rightarrow T_{n-k}, (X,s) \mapsto (\tilde{X}^\lor, s) \) when \( R : T_{n-k} \times T_k \rightarrow T_k \times T_{n-k}, (X^\lor, s) \times (X,s) \mapsto (\tilde{X}, s) \times (\tilde{X}^\lor, s) \) where \( D(X,s) = (X^\lor, s) \), one can show

Theorem 11. The geometric \( R \)-matrix and \( K \)-matrix satisfy the set-theoretical reflection equation.

\[
RK_1RK_1 = K_1RK_1R
\]

In literature, there is a procedure called ultra-discretization among mathematical physicists or tropicalization among mathematicians, turning positive rational maps into piecewise linear maps. See, e.g., [13, sec.4.1]. It is known in [10] that \( R \) and \( D \) are positive maps. Upon use of ultra-discretization Theorem [7] is reconfirmed.

Example 12. In this example we use coordinate variables \( (x_{ij}) \) \((i,j) \in R_k\) to represent a point in \( T_k \). Upon ultra-discretization, \( x_{ij} \) counts the number of \( j \) in the \( i \)-th row and \( t \) the length of a tableau.

1. Let \( k_1 = n - 1, k_2 = 1 \). The geometric \( R \)-matrix \( R : T_{n-1} \times T_1 \rightarrow T_1 \times T_{n-1}, (X^\lor, s) \times (Y,t) \mapsto (\tilde{Y}, t) \times (\tilde{X}^\lor, s) \) is given by
\[
\tilde{x}_i = x_i \frac{P_{i+1}}{P_i}, \quad \tilde{y}_i = y_i \frac{P_{i+1}}{P_i}
\]
where \( P_i = x_i + y_i \).

Here \( x_i = x_{1i} \) are for \( X \in T_1 \), etc. By ultra-discretization: \( \times \rightarrow +, + \rightarrow \min \), this formula agrees with the one in Example [6]. Setting \( x_i = y_i, s = t \), the geometric \( K \)-matrix for this case is obtained as \( K(X,s) = (\tilde{X}^\lor, s) \) where \( \tilde{X} = (x_{i+1}), \) if \( X = (x_i) \).

2. Let \( n = 5, k = 2 \). In this case \( K(X,s) = (\tilde{X}^\lor, s) \) is given by the following formulas.
\[
\begin{align*}
\tilde{x}_{11} & = x_{12}Q_1, & \tilde{x}_{12} & = \frac{x_{13}x_{14}x_{23}}{Q_1Q_2}, & \tilde{x}_{13} & = \frac{x_{11}Q_2}{x_{22}x_{23}}, \\
\tilde{x}_{22} & = Q_1, & \tilde{x}_{23} & = \frac{x_{14}Q_3}{Q_1Q_2}, & \tilde{x}_{24} & = \frac{x_{25}}{x_{14}}, \\
Q_1 & = x_{13} + x_{24}, & Q_2 & = x_{12} + x_{24}, & Q_3 & = x_{12}x_{13} + x_{12}x_{24} + x_{23}x_{24}.
\end{align*}
\]

Examples of the combinatorial \( K \)-matrix for \( B^{2,1} \):

\[
\begin{array}{c|c|c|c}
1 & 3 & 2 & 4 \ \Rightarrow \ 1^\lor/2^\lor/3^\lor/4^\lor \\
5 & & & \\
\end{array}
\]

are checked by ultra-discretization of the above formulas.

4. Discussions

In [22] the case when the KR crystal is \( B^{1,1} \) is treated and the corresponding solution is denoted by \textbf{Rotatelseleft}. Imitating the construction there, one can generalize box-ball systems with a boundary using the solution of the reflection equation obtained in this note. It should also be noted that the \( K \)-matrix in this case is the \( q \rightarrow 0 \) limit of the one constructed as an intertwiner of certain coideal subalgebra of \( U_q(A^{(1)}_{n-1}) \) [23] §6.3.

One might wonder if the trick explained in this note could be applied to the case when the \( R \)-matrix is a linear operator on a certain vector space, such as an intertwiner of the quantized enveloping algebra \( U_q(g) \). However, as far as the authors try, a naive extension does not work.

KR crystals have been constructed for all non-exceptional types of quantum affine algebras [8]. Hence, it is natural to ask whether the technique in this note is generalized to other types than \( A^{(1)}_{n-1} \). Unfortunately, in those cases, the dual crystal \( B^\lor \) coincides \( B \) itself, except \( B^{n,1} \) and \( B^{n-1,1} \) for \( D^{(1)}_n \) and \( n \) is odd. If \( B^\lor = B \), then from Corollary [8](2), we have \( R_{B^\lor,B}(b^\lor \otimes b) = R_{B^\lor,B}(b^\lor \otimes b) = b^\lor \otimes b \), since \( R_{B^\lor,B} \) is the identity. Therefore, \( K_B \) also turns out the identity. So, if either \( B_1 \) or \( B_2 \) is self-dual, then the reflection equation turns out trivial or reduces to the set-theoretical inversion relation \( R_{B_2,B_1}R_{B_1,B_2} = id \).
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