Hardy inequality and Pohozaev identity for operators with boundary singularities: some applications

Abstract

We consider the Schrödinger operator \( A_\lambda := -\Delta - \lambda/|x|^2, \lambda \in \mathbb{R}, \) when the singularity is located on the boundary of a smooth domain \( \Omega \subset \mathbb{R}^N, \ N \geq 1. \)

The aim of this Note is two folded. Firstly, we justify the extension of the classical Pohozaev identity for the Laplacian to this case. The problem we address is very much related to Hardy-Poincaré inequalities with boundary singularities. Secondly, the new Pohozaev identity allows to develop the multiplier method for the wave and the Schrödinger equations. In this way we extend to the case of boundary singularities well known observability and control properties for the classical wave and Schrödinger equations when the singularity is placed in the interior of the domain (Vanconstenoble and Zuazua [17]).

Résumé

Nous allons considérer l’opérateur de Schrödinger \( A_\lambda := -\Delta - \lambda/|x|^2, \lambda \in \mathbb{R}, \) lorsque l’origine est située sur la frontière d’un domaine borné et régulière \( \Omega \subset \mathbb{R}^N, \ N \geq 1. \)

Cette Note a deux objectifs. Premièrement, nous montrons l’extension de l’identité classique de Pohozaev pour le Laplacien dans ce cas. Le problème que nous abordons est très lié aux inégalités de Hardy-Poincaré avec des singularités sur la frontière. En second lieu, la nouvelle identité de Pohozaev permet de dériver le méthode de multiplicateurs pour les équations des ondes et de Schrödinger. De cette façon, nous étendons au cas de la singularité frontalière propriétés d’observabilité et contrôle pour les équations des ondes classique et de Schrödinger bien connues dans le cas d’une singularité à l’intérieur (Vancostenoble et Zuazua [17]).

Version française abrégée

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Preprint submitted to Elsevier Science 26 octobre 2011
Dans cette Note nous nous intéressons à l’opérateur $A_\lambda := -\Delta - \lambda/|x|^2$, $\lambda \in \mathbb{R}$, lorsque l’origine est située sur la frontière d’un domaine régulier $\Omega \subset \mathbb{R}^N$, $N \geq 1$. Il est connu que la valeur $\lambda(N) := N^2/4$, qui est la meilleure constante dans les inégalités de Hardy ci-après, est critique lorsque l’on étudie les propriétés qualitatives de $A_\lambda$. Dans la première partie de cette Note nous montrons que pour tout $\lambda \leq \lambda(N)$, l’identité de Pohozaev (voir Théorème 2) est vérifiée dans le domaine de $A_\lambda$, défini par $D(A_\lambda) := \{u \in H_\lambda \mid A_\lambda u \in L^2(\Omega)\}$. Nous allons définir plus tard l’espace $H_\lambda$ et quelques unes de ses propriétés. Formellement, le Théorème 2 peut être obtenu par intégration directe. Cependant, la singularité $x = 0$ engendre une perte de régularité de l’opérateur $(A_\lambda, D(A_\lambda))$ et les intégrations par parties ne sont plus justifiées rigoureusement. De plus, la régularité $L^2$ de la dérivée normale n’a plus lieu car les estimations elliptiques standards ne s’appliquent plus puisque la singularité est localisée sur le bord. Néanmoins, la trace d’un élément de $D(A_\lambda)$ existe dans un espace $L^2$ à poids, dont le poids est généré à l’origine, comme il est montré dans le Théorème 1. Dans la deuxième partie de cette Note, nous montrons plusieurs applications des Théorèmes 1, 2. D’abord des solutions non-triviales d’une EDP singulière sont traitées dans le Théorème 3. Ensuite, nous répondons à la question concernant la controllabilité des systèmes conservatifs. Le résultat principal est donné par le Théorème 5 et est dû à l’identité des multiplicateurs (10), en combination avec une inégalité forte de Hardy, formulée dans le Théorème 6. Pour plus de clarté dans la présentation, nous allons traiter notamment le cas $C_1$ de la figure 1. Cependant, les mêmes résultats peuvent être étendus aux cas $C_2$, $C_3$, $C_4$ dans un cadre fonctionnel plus faible, dû à l’inégalité plus faible de Hardy (2).

1. Introduction

Let us consider $\Omega$ to be a smooth subset of $\mathbb{R}^N$, $N \geq 1$, with the origin $x = 0$ placed on its boundary $\Gamma$. Without losing the generality we distinguish the following geometrical configurations for $\Omega$ as in Figure 1: $C_1$ - $\Omega$ is a subset of $\mathbb{R}_+^N := \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0\}$ (Fig. 1, top left). $C_2$ - Close to $x = 0$, the points $x \in \Gamma$ satisfy $x \cdot \nu \geq 0$. Nevertheless, $\Omega$ crosses the hyperplane $x_N = 0$ far from origin (Fig. 1, top right). $C_3$ - Close to $x = 0$, the points $x \in \Gamma$ verify $x \cdot \nu \leq 0$ (Fig. 1, bottom left). $C_4$ - For $x \in \Gamma$ the sign of $x \cdot \nu$ changes at origin (Fig. 1, bottom right).

The following Hardy inequalities are well-known: if $\Omega$ verifies the case $C_1$ in Fig. 1, then (e.g. [4]) for any $u \in C_0^\infty(\Omega)$ it holds that

$$\int_{\Omega} |\nabla u|^2 \, dx \geq \frac{N^2}{4} \int_{\Omega} u^2 \, dx + \frac{1}{4} \int_{\Omega} \frac{u^2}{|x|^2 \log^2(R_0/|x|)} \, dx. \quad (1)$$

where $R_0 = \sup_{x \in \Omega} |x|$. If $\Omega$ satisfies the cases $C_2$, $C_3$, $C_4$ as in Fig. 1 then (e.g. [8], [9]) there exist $C_2 = C_2(\Omega) \in \mathbb{R}$ and $C_3 = C_3(\Omega, N) > 0$ such that any $u \in C_0^\infty(\Omega)$ satisfies

$$C_2 \int_{\Omega} u^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx \geq \frac{N^2}{4} \int_{\Omega} u^2 \, dx + C_3 \int_{\Omega} \frac{u^2}{|x|^2 \log^2(R_0/|x|)} \, dx. \quad (2)$$

In both situations above, the constant $\lambda(N) = N^2/4$ is optimal. For the sake of clarity and because similar results can be obtained in the other cases in an other functional framework, next we analyze the situation when $\Omega$ verifies the case $C_1$ in Fig. 1. In the sequel let us consider the operator $(A_\lambda, D(A_\lambda))$, $\lambda \leq \lambda(N)$, acting on such $\Omega$.

Firstly, this Note is aimed to justify the Pohozaev identity (5) in the functional setting $(A_\lambda, D(A_\lambda))$ in which Theorem 1 plays a crucial role. Pohozaev-type identities (e.g. pp. 515, [7]) have been widely used to show non-existence results to nonlinear elliptic equations. In particular, we point out that this issue has been also studied for nonlinear equations with singular potentials (see e.g. [5], [10]). In those cases, due to the regularizing effect of the nonlinearity, the solutions become regular enough to obtain the corresponding Pohozaev identity by direct computations. This is not precisely our case. To the best of
our knowledge, the regularity of the operator \((A_\lambda, D(A_\lambda))\) in terms of the Sobolev spaces is yet unknown and this affects the direct justification of (5). To do this we proceed by approximations arguments near the singularity. We analyze two situations when discussing the values of \(\lambda\): the subcritical case \(\lambda \leq \lambda(N)\) respectively the critical case \(\lambda = \lambda(N)\). However, the main novelty appears for the critical value \(\lambda = \lambda(N)\) in which case \(H_\lambda\) is strictly larger than \(H^1_0(\Omega)\). This case requires a better understanding of the norm of \(H_\lambda(N)\) as discussed in Section 2.

The controllability properties and stabilization of the heat and wave equation corresponding to \(A_\lambda\) have been analyzed in [16], [6], [17] in the case of interior singularity. Roughly speaking, they showed that the parameter \(\lambda^* = (N-2)^2/4\), which is the optimal constant in the Hardy inequality with interior singularity, is critical when asking the well-posedness and control properties of such systems. In the second part of this Note we address the question of controllability for the wave and Schrodinger equations corresponding to \(A_\lambda\), in the case of boundary singularity. Our main result asserts that we can increase the range of values \(\lambda\) (from \(\lambda^*\) to \(\lambda(N)\)) for which the exact controllability holds.

2. The space \(H_\lambda\), proper norm and main elliptic results. Following the idea in [18], thanks to inequality (1) we consider the Hardy functional \(B_\lambda[u] = \int_\Omega \left[|\nabla u|^2 - \lambda u^2/|x|^2\right]dx\), which is positive and finite for all \(u \in C_0^\infty(\Omega)\). We define the Hilbert space \(H_\lambda\) to be the completion of \(C_0^\infty(\Omega)\) functions in the norm induced by \(B_\lambda[u]\). If \(\lambda < \lambda(N)\), it holds that \(H^1_0(\Omega) = H_\lambda\) due to Hardy inequality which ensures the equivalence of the norms. Similar to the case of interior singularity emphasized in [19], an interesting phenomena appears in the critical space \(H_{\lambda(N)}\). Assume \(\Omega = \{x \in \mathbb{R}^N \mid |x| \leq 1, x_N > 0\}\) and let \(e_1 := x_N|x|^{-N/2}J(z_{0,1}|x|)\) where \(z_{0,1}\) is the first positive zero of the Bessel function \(J_0\). Then there exists \(\lim_{\epsilon \to 0} \int_{x \in \Omega, |x| > \epsilon} \left[|\nabla e_1|^2 - \lambda(N)e_1^2/|x|^2\right]dx < \infty\) although \(e_1 \notin H^1_0(\Omega)\). Surprisingly, the meaning of \(B_{\lambda(N)}[e_1 - \phi]\) is not well-defined in the sense of principle value. Indeed, if it were one can check that \(B_{\lambda(N)}[e_1 - \phi] \geq C_0 > 0\), for all \(\phi \in C_0^\infty(\Omega)\) for some universal constant \(C_0 > 0\). This is in contradiction with the definition of \(H_{\lambda(N)}\) ! The remedy for this is to consider the functional
I does not have non trivial solutions in \( D^1 \). If \( \epsilon \) Wave-like process result is proved by combining Theorem 2 and unique continuation results as in [12]. □

Therefore, \( B_{\lambda, 1} \) induces a new norm in the space \( H_\lambda \) which is well understood in the sense of principal value. In the sequel, we denote by \( || . ||_{H_\lambda} \) the norm induced by \( B_{\lambda, 1} \).

Notations: For any \( \epsilon > 0, \theta_\epsilon \) is a smooth cut-off function which satisfies \( \theta_\epsilon = 0 \) for \( |x| \leq \epsilon \) respectively \( \theta_\epsilon = 1 \) for \( |x| \geq 2\epsilon \). Besides, \( q \in (C^2(\Omega))^N \) denotes a vector field such that \( q = \nu \) on \( \Gamma \). \( H_\lambda \) denotes the dual space of \( H_\lambda \). Next we state our main elliptic results.

**Theorem 1 (trace regularity)** Assume \( \Omega \subset \mathbb{R}^N \), \( N \geq 1 \), verifies the case C1. Let us consider \( \lambda \leq \lambda(N) \) and \( u \in D(A_\lambda) \). Then \( (\partial u / \partial \nu)|x| \in L^2(\Gamma) \) and there exists \( C = C(\Omega) > 0 \) such that

\[
\int_{\Gamma} (\frac{\partial u}{\partial \nu})^2 (x)^2 d\sigma \leq C(||u||^2_{H_\lambda} + ||A_\lambda u||^2_{L^2(\Omega)}).
\]

Sketch of the proof. In order to avoid the singularity, we multiply \( A_\lambda u \) by \( |x|^2 (q \cdot \nabla u) \theta_\epsilon \), and we integrate by parts. Then we obtain an identity which, combined with Cauchy-Schwartz inequality, allows to get uniform upper bounds for the boundary term. Then, by Fatou Lemma we can pass to the limit as \( \epsilon \) tends to zero to end up the proof. □

**Theorem 2 (Pohozaev identity)** Assume \( \Omega \subset \mathbb{R}^N \), \( N \geq 1 \), verifies the case C1 and let \( \lambda \leq \lambda(N) \). Then for all \( u \in D(A_\lambda) \) it holds that

\[
\frac{1}{2} \int_{\Gamma} (x \cdot \nu) (\frac{\partial u}{\partial \nu})^2 d\sigma = - \int_{\Omega} A_\lambda u (x \cdot \nabla u) dx - \frac{N-2}{2} ||u||^2_{H_\lambda}.
\]

Sketch of the proof. Note firstly that all terms in (5) are finite. Indeed, thanks to the fact that \( x \cdot \nu = O(|x|^2) \) and Theorem 1 we obtain the integrability of the boundary term. Moreover, \( x \cdot \nabla u \in L^2(\Omega) \) as shown in Theorem 6. We proceed by multiplier technique. If \( \lambda < \lambda(N) \) the multiplier \( (x \cdot \nabla u) \theta_\epsilon \) is used to obtain (5) in the limit process. If \( \lambda = \lambda(N) \) we apply the subcritical result for \( \lambda(N) - \delta \) and we argue we can pass to the limit as \( \delta \) tends to zero. Indeed, following approximation lemma which is proved in generality in [1], for a given \( u \in D(A_\lambda) \), the solution \( u_\delta \) of \( A_{\lambda(N) - \delta} u_\delta = A_{\lambda(N)} u \), converges strongly to \( u \) in \( H_\lambda(N) \). Secondly, by comparison arguments we show the asymptotic behavior of \( u_\delta, \nabla u_\delta \) when \( A_\lambda u \) is smooth. This is done by constructing super solutions and using rescaling arguments in [3] to get rid of the singularity. A density argument together with Theorem 1 concludes the proof. □

**Theorem 3** Assume \( \lambda \leq \lambda(N) \) and \( \Omega \subset \mathbb{R}^N \), \( N \geq 3 \), satisfying the case C1. Let us consider the problem

\[
-\Delta u - \frac{\lambda}{|x|^2} u = |u|^{\alpha-1} u, \quad x \in \Omega; \quad u = 0, \quad x \in \Gamma,
\]

1). If \( 1 < \alpha < (N+2)/(N-2) \), problem (6) has non trivial solutions in \( H_\lambda \). Moreover, if \( 1 < \alpha < N/(N-2) \) it has non trivial solutions in \( D(A_\lambda) \).

2). Assume \( \Omega \) is a star-shaped domain (i.e. \( x \cdot \nu \geq 0 \), for all \( x \in \Gamma \)). If \( \alpha \geq (N+2)/(N-2) \), problem (6) does not have non trivial solutions in \( D(A_\lambda) \).

Sketch of the proof. The existence of non-trivial solutions reduces to finding minimizers for the problem

\[
I = \inf \{ ||u||_{L^{\alpha+1}(\Omega)} | ||u||^2_{H_\lambda} \}
\]

and is due to the compact embedding \( H_\lambda \hookrightarrow L^{\alpha+1}(\Omega) \). The non-existence result is proved by combining Theorem 2 and unique continuation results as in [12]. □

**3. Applications to controllability**

Let us set \( \Gamma_0 := \{ x \in \Gamma \ | \ x \cdot \nu \geq 0 \} \), a non-empty part of the boundary \( \Gamma \). Next we consider the Wave-like process

\[
B_{\lambda, 1}[u] = \int_{\Omega} \left[ \nabla u + \frac{N}{2} \frac{x}{|x|^2} u - \frac{e_N}{x^N} u \right]^2 dx + (\lambda(N) - \lambda) \int_{\Omega} u^2 dx,
\]

where \( e_N \) denotes the \( N \)-th canonical vector of \( \mathbb{R}^N \). We observe that \( B_{\lambda}[u] = B_{\lambda, 1}[u] \), for all \( u \in C_c^\infty(\Omega) \). Therefore, \( B_{\lambda, 1}[u] \) induces a new norm in the space \( H_\lambda \) which is well understood in the sense of principal value. In the sequel, we denote by \( || . ||_{H_\lambda} \) the norm induced by \( B_{\lambda, 1} \).
The solution of (7) is defined in weak sense by the transposition method (J.L. Lions [13]). In this section we address the question of exact controllability of system (7), i.e. whether for any initial data \((u_0, u_1) \in L^2(\Omega) \times H'_\lambda\) and any target \((\overline{u}_0, \overline{u}_1) \in L^2(\Omega) \times H'_\lambda\), there exists a finite time \(T > 0\) and a control \(h \in L^2((0, T) \times \Gamma_0)\) such that the solution of (7) satisfies \((u_1(T, x), u(T, x)) = (\overline{u}_1(x), \overline{u}_0(x))\) for all \(x \in \Omega\). In view of the time-reversibility of the equation it is enough to consider the null-controllability problem, i.e. the case where the target \((\overline{u}_0, \overline{u}_1) = (0, 0)\). By now classical Hilbert Uniqueness Method (HUM) (see J. L. Lions [13]) the null-controllability of system (7) is characterized through the adjoint system

\[
\begin{cases}
u_{tt} - \Delta v - \lambda \frac{|v|^2}{|x|^2} = 0, & (t, x) \in Q_T, \\
\nu(t, x) = h(t, x)\chi_{\Gamma_0}, & (t, x) \in \Sigma_T, \\
(\nu(0, x), \nu_0(0, x)) = (\nu_0(x), \nu_1(x)), & x \in \Omega,
\end{cases}
\]

(8)

where \(Q_T = (0, T) \times \Omega\), \(\Sigma_T = (0, T) \times \Gamma\) and \(\chi_{\Gamma_0}\) denotes the characteristic function of \(\Gamma_0\).

Theorem 4 (Hidden regularity) Assume \(\lambda \leq \lambda(N)\) and \(v\) is the solution of (8) corresponding to the initial data \((\nu_0, \nu_1) \in H_\lambda \times L^2(\Omega)\). Then \(v\) satisfies

\[
\int_0^T \int_\Omega \left| \frac{\partial v}{\partial t} \right|^2 |x|^2 d\sigma dt \leq C(||\nu_0||_{H_\lambda}^2 + ||\nu_1||_{L^2(\Omega)}^2).
\]

(9)

for some universal constant \(C > 0\). Moreover, \(v\) verifies the identity

\[
\frac{1}{2} \int_0^T \int_\Omega (x \cdot \nu) \left( \frac{\partial v}{\partial t} \right)^2 d\sigma dt = \frac{T}{2} (||\nu_0||_{H_\lambda}^2 + ||\nu_1||_{L^2(\Omega)}^2) + \int_\Omega \nu_1 (x \cdot \nabla v + \frac{N - 1}{2} v)^T dx.
\]

(10)

Sketch of the proof. By density, it suffices to prove Theorem 4, for initial data \((\nu_0, \nu_1) \in D(A_\lambda)\). For the proof of (9) we multiply \(A_\lambda v\) by \(|x|^2(q \cdot \nabla v)\theta_{\epsilon}\) and integrate. The integration in time and the conservation of energy allow to obtain uniform bounds for the boundary term in the energy space. Then by Fatou Lemma we pass to the limit as \(\epsilon\) tends to zero and the proof finishes.

For the proof of (10) we proceed straightforward from Theorem 2. Indeed, for a fixed time \(t \in [0, T]\) we apply Theorem 2 for \(A_\lambda v = -v_\epsilon\). Then we integrate in time, and due to the equipartition of energy we can finish the proof. \(\square\)

Due to Theorem 4 the operator \((\nu_0, \nu_1) \mapsto (\int_0^T \int_\Omega (x \cdot \nu) (\partial v/\partial t)^2 d\sigma dt)^{1/2}\) is a linear continuous map in \(H_\lambda \times L^2(\Omega)\). Let \(\mathcal{H}\) be the completion of this norm in \(H_\lambda \times L^2(\Omega)\). We consider the functional \(J : \mathcal{H} \to \mathbb{R}\) defined by

\[
J(\nu_0, \nu_1)(v) := \frac{1}{2} \int_0^T \int_\Omega (x \cdot \nu) \left( \frac{\partial v}{\partial t} \right)^2 d\sigma dt - \langle u_1, \nu_0 \rangle_{H'_\lambda} + (u_0, \nu_1)_{L^2(\Omega), L^2(\Omega)},
\]

(11)

where \(v\) is the solution of (8) corresponding to initial data \((\nu_0, \nu_1)\). Of course, \(\langle \cdot, \cdot \rangle_{H'_\lambda, H_\lambda}\) denotes the duality product. The control \(h \in L^2((0, T) \times \Gamma_0)\) for (7) could be chosen as \(h = (x \cdot \nu)\epsilon_{\min}\) where
For all $\lambda \leq \lambda(N)$, there exists a positive constant $D_1 = D_1(\Omega, \lambda, T) > 0$ such that for all $T \geq 2R_\Omega$, and any initial data $(v_0, v_1) \in H^1_\lambda \times L^2(\Omega)$ the solution of (8) verifies

$$||v_0||^2_{H^1_\lambda} + ||v_1||^2_{L^2(\Omega)} \leq D_1 \int_0^T \int_{\Gamma_0} (\nabla v \cdot \nu) \frac{\partial v}{\partial \nu}^2 \, d\sigma \, dt. \quad (12)$$

**Sketch of the proof.** The proof of Theorem 5 relies mainly on Theorem 4, combining compactness uniqueness argument (cf. [14]) and the sharp Hardy inequality stated in Theorem 6.

**Theorem 6** Assume $\Omega$ satisfies one of the cases C1-C4. Then, there exists a constant $C = C(\Omega) \in \mathbb{R}$ such that

$$\int_\Omega |x|^2 \nabla v^2 \, dx \leq R^2_\Omega \left[ \int_\Omega \nabla v^2 \, dx - \frac{N^2}{4} \int_\Omega \frac{v^2}{|x|^2} \, dx \right] + C \int_\Omega v^2 \, dx \quad \forall v \in C^\infty_0(\Omega). \quad (13)$$

**Remark 1** The proof of Theorem 6 is quite technical and we omit it here. The constant $R^2_\Omega$ which appears in inequality (13), helps to obtain the control time $T > T_0 = 2R_\Omega$, which is sharp from the Geometric Control Condition considerations, see [2].

The results above guarantee the exact boundary controllability of (7). More precisely, we obtain

**Theorem 7** (Controllability) Assume that $\Omega$ satisfies C1 and $\lambda \leq \lambda(N)$. For any time $T > 2R_\Omega$, $(u_0, u_1) \in L^2(\Omega) \times H^1_\lambda$ and $(\overline{u}_0, \overline{u}_T) \in L^2(\Omega) \times H^1_\lambda$ there exists $h \in L^2((0, T) \times \Gamma_0)$ such that the solution of (7) satisfies $(u(T, x), u(T, x)) = (\overline{u}_T(x), \overline{u}_0(x))$ for all $x \in \Omega$.

**Schrödinger equation.** In the above geometrical settings, we consider the Schrödinger equation

$$\begin{cases}
i u_t - \Delta u - \lambda \frac{u}{|x|^2} = 0, \quad (t, x) \in QT, \\u(t, x) = h(t, x)\chi_{\Gamma_0}, \quad (t, x) \in \Sigma_T, \\
u(0, x) = u_0(x), \quad x \in \Omega,
\end{cases} \quad (14)$$

For all $\lambda \leq \lambda(N)$, we define the Hilbert space $H^1_\lambda(\Omega; \mathbb{C})$ as the completion of $H^1_\lambda(\Omega; \mathbb{C})$ with respect to the norm induced by the inner product $\langle u, v \rangle_{H^1_\lambda(\Omega; \mathbb{C})} := \text{Re} \int_\Omega (\nabla u(x) \cdot \nabla \overline{v}(x) - \lambda u(x)\overline{v}(x)/|x|^2) \, dx$. Then

**Theorem 8** (Controllability) For any $\lambda \leq \lambda(N)$, $u_0 \in H^1_\lambda$, $\overline{u}_0 \in H^1_\lambda$ and any time $T > 0$ there exists $h \in L^2((0, T) \times \Gamma_0)$ such that the solution of (14) satisfies $u(T, x) = \overline{u}_0(x)$ for all $x \in \Omega$.

This result holds true due to the result valid for the wave equation. Indeed, the general theory presented in an abstract form in [15], assure the observability of systems like $\ddot{z} = -A_0 \dot{z}$ using results available for systems of the form $\ddot{\bar{z}} = -A_0 \bar{z}$.

**Acknowledgements.** The author thanks Enrique Zuazua and Adi Adimurthi for fruitful discussions and suggestions.

Partially supported by the Grant MTM2008-03541 of the MICINN (Spain), project PI2010-04 of the Basque Government, the ERC Advanced Grant FP7-246775 NUMERIWAVES, the ESF Research Networking Program OPTPDE, the grant PN-II-ID-PCE-2011-3-0075 of CNCS-UEFISCDI Romania and a doctoral fellowship from UAM (Universidad Autónoma de Madrid).

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