Radial sign-changing solutions to biharmonic nonlinear Schrödinger equations

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Abstract

In this work we obtain three radial solutions of a biharmonic stationary Schrödinger equation, being one positive, one negative and one that changes sign. The Dual Decomposition Method is used to split the natural second order Sobolev space considered in order to apply the appropriate variational approach.

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1 Introduction

In the last two decades, the stationary nonlinear Schrödinger equation given by

\[ \begin{cases} -\epsilon^2 \Delta u + V(x)u = f(x, u) & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.1} \]

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has been widely studied by several authors in works dealing with existence, multiplicity and concentration of solutions when the parameter $\epsilon \to 0$. In order to highlight some of the most influential works about this subject, we could quote the pioneering work of Floer and Weinstein \[10\], in which they used the Lyapunov-Schmidt reduction to obtain positive solutions exhibiting concentration behavior for the unidimensional case. In \[17\], Rabinowitz used a variational approach to obtain under global hypotheses in the potential $V$, positive solutions to \( (1.1) \). Later, Wang in \[18\] proved that the solutions obtained by Rabinowitz concentrate around the global infimum of the potential. In the celebrated paper \[7\], del Pino and Felmer used the so-called penalization technique to prove the same kind of concentration behavior of solutions of \( (1.1) \), considering the potential $V$ under a local version of the Rabinowitz condition.

In all of the above mentioned works, the authors worked with potentials $V$ bounded away from zero. Stationary NLS problems with vanishing potentials were treated for instance by Bonheure et al in \[3, 4\], where the authors obtained concentration of positive solutions around global minimum points of an auxiliary function and even around some lower dimensional spheres in $\mathbb{R}^N$.

Although this amount of results treating with the second order case, just few works can be found dealing with similar questions involving the fourth-order equation

\[
\begin{align*}
\Delta^2 u + V(x)u &= f(x, u) \quad \text{in} \quad \mathbb{R}^N \\
u &\in H^2(\mathbb{R}^N).
\end{align*}
\]

In \[13, 14\], the author and Soares proved the existence of a concentrating sequence of solutions of the singular perturbed version of \( (1.2) \), considering, respectively, a global and a local condition in the potential $V$, and a subcritical power-type nonlinearity $f$. The main arguments of these works were strongly inspired by Rabinowitz \[17\], Wang \[18\] and Del Pino and Felmer \[7\], in which other arguments to overcome the lack of a maximum principle of the biharmonic operator were required. In \[9\] the author and Figueiredo using Ljusternik-Schnirelmann theory, deal with the same problem that in \[14\], obtaining multiple solutions exhibiting the concentration phenomenon.

Another relevant question about both problems \( (1.1) \) and \( (1.2) \) is the existence of sign-changing solutions, sometimes called nodal solutions. For the second-order problem there are several papers dealing with this subject.
In a pioneering work [3], Castro, Cossio and Neuberger have obtained three solutions, including a nodal one, for a second order problem in a bounded domain and with Dirichlet boundary conditions. In [1, 2], Alves and Soares use the penalization technique to get nodal solutions concentrating around extremal points of the potential $V$. In this approach, they use arguments based in minimization of the energy functional in some Nehari sets, considering $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$, respectively, the positive and negative parts of a function $u \in H^1(\mathbb{R}^N)$. At a first sight, one could think that these arguments are trivially adaptable to the fourth order case, however, as long in $H^1(\mathbb{R}^N)$ the decomposition $u = u^+ + u^-$ is trivially allowed, in $H^2(\mathbb{R}^N)$ this factorization is no longer to be available.

In this work, we consider the problem (1.2) where $N \geq 5$ and $f$ and $V$ satisfies the following assumption set:

$$(V_1) \ 0 < V_0 := \inf_{\mathbb{R}^N} V \text{ and } V(x) = V(|x|), \text{ for all } x \in \mathbb{R}^N.$$  

$$(f_1) \ f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function;}$$

$$(f_2) \ f(x, s) = o(|s|) \text{ as } s \to 0, \text{ a.e. in } \mathbb{R}^N;$$

$$(f_3) \text{ there are constants } c_1, c_2 > 0 \text{ and } 0 < p < 2^*_s - 2, \text{ where } 2^*_s = 2N/(N-4), \text{ such that }$$

$$|f(x, s) - f(x, t)| \leq (c_1 + c_2(|s|^p + |t|^p))|s - t| \text{ for a.e. } x \in \mathbb{R}^N \text{ and } s, t \in \mathbb{R};$$

$$(f_4) \lim_{|s| \to \infty} \frac{F(x, s)}{s^2} = +\infty \text{ a.e. in } \mathbb{R}^N, \text{ where } F(x, s) = \int_0^s f(x, t)dt;$$

$$(f_5) \frac{f(x, s)}{s} \text{ is nondecreasing for } s > 0 \text{ and nonincreasing for } s < 0, \text{ for a.e. } x \in \mathbb{R}^N.$$ 

**Remark 1.1.** Note that $(f_5)$ implies that $f(x, s)$ is nondecreasing in $\mathbb{R}$ and that $(x, s) \mapsto f(x, s)s - 2F(x, s)$ is also nondecreasing for $s > 0$ and nonincreasing for $s < 0$, for a.e. $x \in \mathbb{R}^N$.

**Remark 1.2.** An example of nonlinearity satisfying $(f_1) - (f_5)$ is $f(x, s) = \sum_{i=1}^k a_i(x)|s|^{p_i}s$, where $a_i \in L^\infty(\mathbb{R}^N)$, $a_i \geq 0$ in a positive measure set of $\mathbb{R}^N$, and $0 < p_i < 2_s - 2$, for all $i \in \{1, ..., k\}$. 

In order to overcome the lack of the decomposition of $H^2(\mathbb{R}^N)$ in terms of positive and negative parts of their functions, we use an alternative method developed by Moreau in [12] called Dual Cones Decomposition Method, which consists in split a function $u$ in a Hilbert space $H$ as $u = u_1 + u_2$, in such a way that $u \in K$ and $u_2 \in K^*$, where $K$ is a cone in $H$ and $K^*$ is called his dual cone. Considering $H = H^2(\mathbb{R}^N)$ and $K = \{ u \in H^2(\mathbb{R}^N); \ u \geq 0 \}$, if it is possible to prove that $K^* \subset -K$, then we have a decomposition of a function $u \in H^2(\mathbb{R}^N)$ in term of a non-negative and a non-positive functions such that, in many times, can substitute the trivial decomposition not-available in our case. Our approach follows closely the work of Weth [19], in which the author takes advantage of the Moreau’s method to obtain signed and sign-changing solutions of the problem

$$\Delta^2 u = f(x, u) \quad \text{in } \Omega$$

(1.3)

under Dirichlet or Navier boundary conditions in a bounded domain $\Omega$. It is worth pointing out that to prove that $K^* \subset -K$, it is necessary some kind of a maximum principle to $\Delta^2$. In fact, this is the main reason why this method becomes so restrictive when dealing with fourth order problems. In [19], Weth uses the fact that under Navier boundary conditions, one can use twice the strong maximum principle to $-\Delta$ in $\Omega$, in order to obtain a version of this result to $\Delta^2$. Considering Dirichlet boundary conditions, the same is true at least to some domains like balls and limacons. In this sense, we use some arguments of Chabrowski and Yang in [6], to prove that there exists positive solutions of a linear version of (1.2), which in particular implies that $K^* \subset -K$ (see Lemma 2.10).

Another difficulty that deserves to be highlighted is the lack of compactness, since the problem is in $\mathbb{R}^N$. In order to overcome this difficulty, we consider the problem restricted to $H^2_{rad}(\mathbb{R}^N)$ consisting in the radial functions belonging to $H^2(\mathbb{R}^N)$. This is interesting because of a version of the Strauss Lemma to higher order Sobolev spaces proved by Ebihara and Schonbek in [8]. At the end, once critical points of the restricted energy functional are in our hand we obtain critical points of the functional using the principle of symmetric criticality of Palais.

Our main result is the following.

**Theorem 1.3.** Assume that conditions $(V_1)$ and $(f_1)-(f_5)$ hold. Then there exist at least one positive, one negative and one nodal classical radial solution of (1.2).
The proof involves variational arguments consisting in searching for critical points of the energy functional, looking for stationary points of a Cauchy problem in Banach spaces. In this sense, some results of Liu and Sun in the invariance of some sets will be necessary.

Finally, we note that the nonlinearity \( f \) do not satisfy the well known Ambrosetti-Rabinowitz super-linearity condition, given by

(\( AR \)) there exists \( \mu > 2 \) such that \( 0 < \mu F(x,s) \leq f(x,s)s \), for all \( s \neq 0 \) and a.e. in \( \mathbb{R}^N \).

Instead of this condition, in order to increase the range of admissible nonlinearities, we consider the weaker assumption \((f_4)\). This requires some arguments of Miyagaki and Souto in to prove the boundedness of a certain sequence.

In the first section we describe the variational framework. In the second one we introduce the Cauchy problem and prove the invariance of some sets. The last section is left for the proof of the main result.

To save notation in all of this paper we denote \( \int_{\mathbb{R}^N} gdx \) just by \( \int g \). The norm \( \| \cdot \|_{L^p(\mathbb{R}^N)} \) will be simple denoted by \( \| \cdot \|_p \).

\section{The variational framework}

As mentioned in the introduction, in order to overcome the lack of compactness, let us consider \( H = H^2_{rad}(\mathbb{R}^N) \) which is a Hilbert space when endowed with the following inner product

\[
\langle u, v \rangle = \int (\Delta u \Delta v + V(x)uv),
\]

which gives rise to the following norm

\[
\| u \| = \left( \int (|\Delta u|^2 + V(x)u^2) \right)^{\frac{1}{2}}.
\]

By \((V_1)\), it follows easily that \( \| \cdot \| \) is equivalent to the usual norm in \( H^2(\mathbb{R}^N) \).

Before introducing the energy functional associated to \((1.2)\), let us remember some results proved by Ebihara in \([8]\) that will be used along this text.
Lemma 2.1 (Corollary 2 in [8]). The following embeddings are compact
\[ H^2_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), \quad \text{for all } 2 < q < 2_\ast \text{ if } N \geq 5 \]
\[ H^2_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), \quad \text{for all } 2 < q \text{ if } N = 1, 2, 3, 4. \]

Lemma 2.2 (Theorem 2.1 in [8]). \( H^2_{\text{rad}}(\mathbb{R}^N) \) is not compacted embedded into \( L^2(\mathbb{R}^N) \).

Let us consider the restriction to \( H \) of the energy functional whose Euler-Lagrange equation is (1.2),
\[ I : H \rightarrow \mathbb{R}, \quad I(u) = \frac{1}{2} \int (|\Delta u|^2 + V(x)u^2) - \int F(x, u) = \frac{1}{2} \|u\|^2 - \int F(x, u). \]

Note that by \((f_3)\) and Sobolev embeddings, \( I \) is well defined.
Since for each \( u \in H, v \mapsto \int f(x, u)v \) is a continuous linear functional in \( H \), it is well defined \( A(u) \in H \) such that
\[ \langle A(u), v \rangle = \int f(x, u)v, \quad \text{for all } v \in H. \]

The following result states some interesting properties of the operator \( A : H \rightarrow H \) defined above.

Lemma 2.3. \( A \) is a compact, locally Lipschitz operator, such that, \( A(u) = \nabla \Psi \) where \( \Psi : H \rightarrow \mathbb{R} \) is given by \( \Psi(u) = \int F(x, u) \). Moreover, for each \( \epsilon > 0 \), there exists \( A(\epsilon) > 0 \) such that
\[ |\langle A(u), v \rangle| \leq (\epsilon \|u\| + A(\epsilon)\|u\|^{p+1}) \|v\|, \quad \text{for all } u, v \in H, \]
where \( p \) is given in \((f_3)\).

Proof. Let us prove the estimate. Note that from \((f_2)\) and \((f_3)\) it follows that for each \( \epsilon < 0 \), there exists \( A(\epsilon) > 0 \) such that
\[ |f(x, s)| \leq \epsilon |s| + A(\epsilon)|s|^{p+1}, \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and for all } s \in \mathbb{R}. \]

Using Hölder with the conjugated exponents \( \frac{2}{p+1} \) and \( \frac{2}{2, -(p+1)} \), we have
\[ |\langle A(u), v \rangle| \leq \int (\epsilon |u| + A(\epsilon)|u|^{p+1}) |v| \leq \epsilon \|u\||v\| + A(\epsilon)\|u\|^{p+1} \|v\|. \]
Just by definition and Lebesgue Dominated Convergence Theorem, it follows that $A = \nabla \Psi$, where $\Psi(u) = \int F(u)$. On the other hand, $(f_2)$ implies that $A$ is a locally Lipschitz operator.

What is left to show is that $A$ is a compact operator. Although this follows by straightforward calculations, we describe all the details, since this was the reason why we had to consider the space $H^{2}_{rad}(\mathbb{R}^N)$ rather than $H^2(\mathbb{R}^N)$.

Let $(u_n) \subset H^{2}_{rad}(\mathbb{R}^N)$ a bounded sequence. Along a subsequence, we have that $u_n \rightharpoonup u$ in $H^{2}_{rad}(\mathbb{R}^N)$, $u_n \rightarrow u$ in $L^q(\mathbb{R}^N)$, for $2 < q < \frac{2N}{N-4}$.

For each $v \in C_0^\infty(\mathbb{R}^N)$, it follows by $(f_3)$ and Hölder inequality with $2 + \epsilon$ and $(2 + \epsilon)'$, for small enough $\epsilon > 0$, that

$$ |\langle A(u_n) - A(u), v \rangle| \leq \int |f(x, u_n) - f(x, u)||v| $$

$$ \leq \int_{\text{supp}(v)} c_1 |u_n - u||v| + \int_{\text{supp}(v)} c_2 (|u_n|^p + |u|^p)|u_n - u||v| $$

$$ \leq C(v)\|u_n - u\|_{2+\epsilon} + C(v) \left( \int_{\text{supp}(v)} |u_n|^p|u_n - u| + \int_{\text{supp}(v)} |u|^p|u_n - u| \right). $$

By taking $2 < r < \min\{2, 2s/p\}$, using Hölder inequality with $r$ and $r'$ and by (2.1), we obtain

$$ |\langle A(u_n) - A(u), v \rangle| \leq C(v)\rho_n(1) + C(v)\|u_n\|^p_{p,r}\|u_n - u\|_{r'} + C(v)\|u\|^p_{p,r}\|u_n - u\|_{r'} $$

$$ \leq \rho_n(1). $$

Therefore, for all $v \in C_0^\infty(\mathbb{R}^N)$ it follows that

$$ \lim_{n \rightarrow \infty} \langle A(u_n) - A(u), v \rangle = 0. $$

(2.2)

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^{2}_{rad}(\mathbb{R}^N)$, it follows that (2.2) holds for all $v \in H^{2}_{rad}(\mathbb{R}^N)$ and this implies that

$$ \lim_{n \rightarrow \infty} A(u_n) = A(u). $$

□
It is straightforward to prove that critical points of $I$ correspond to fixed points of $A$.

Let us consider the following Cauchy problem in the Hilbert space $H$

$$
\begin{align*}
\frac{\partial}{\partial t} \varphi(t, u) &= -\nabla I(\varphi(t, u)) = A(\varphi(t, u)) - \varphi(t, u) \\
\varphi(0, u) &= u,
\end{align*}
$$

where $\varphi : \mathcal{G} \to H$ and $\mathcal{G} = \{(t, u) \in \mathbb{R} \times H; \ u \in H \text{ and } t \in [0, T(u))\}$ and $[0, T(u))$ is the maximal interval of existence of the trajectory $t \mapsto \varphi(t, u)$. Note that since $A$ is a Lipschitz continuous operator, the flow $\varphi$ is well defined.

The follow is a key point in our approach.

**Proposition 2.4.** If for some $u \in H$, $\{I(\varphi(t, u)); 0 \leq t < T(u)\}$ is bounded from below, then:

i) $T(u) = \infty$,

ii) there exists $t_n \to \infty$ such that $\{\varphi(t_n, u), n \in \mathbb{N}\}$ is bounded in $H$ and the $\omega$-limit set of $u$

$$
\omega(u) = \bigcap_{0 \leq t < \infty} \bigcup_{t \leq s < \infty} \varphi(s, u)
$$

is a non-empty set formed by critical points of $I$.

**Proof.**

i) Note that

$$
\|\varphi(t, u) - \varphi(s, u)\| \leq \int_s^t \|\nabla I(\varphi(\tau, u))\|d\tau \\
\leq \sqrt{t-s} \left( \int_s^t \|\nabla I(\varphi(\tau, u))\|^2d\tau \right)^{\frac{1}{2}} \\
= \sqrt{t-s} (I(\varphi(s, u)) - I(\varphi(t, u)))^{\frac{1}{2}},
$$

where we have used Hölder and the equation (2.3). Suppose the assertion of the item is false. Then by last estimate, the trajectory $\{\varphi(t, u); t \in [0, T(u))\}$ would be bounded, which implies that $T(u) = \infty$, which gives rise to a contradiction.
\(\text{ii)}\) First of all, let us note that there exists a sequence \(t_n \to \infty\) such that \(\|\nabla I(\varphi(t_n, u))\| \to 0\), as \(n \to \infty\). This follows just by noting that

\[
\int_0^\infty \|\nabla I(\varphi(\tau, u))\|^2 d\tau = \lim_{t \to \infty} \int_0^t \|\nabla I(\varphi(\tau, u))\|^2 d\tau
= \lim_{t \to \infty} |I(\varphi(t, u)) - I(u)| < \infty.
\]

We claim that \(\{\varphi(t_n, u)\}_{n \in \mathbb{N}}\) is uniformly bounded in \(H\) with respect to \(n \in \mathbb{N}\).

Suppose, contrary to our claim, that \(\|\varphi(t_n, u)\| \to \infty\) as \(n \to \infty\). Let us define

\(w_n = \varphi(t_n, u)\|\varphi(t_n, u)\|\).

Since \((w_n)\) is a bounded sequence in \(H\), it follows that there exists \(w \in H\) such that \(w_n \rightharpoonup w\) in \(H\), up to a subsequence. Then Lemma 2.1 implies that \(w_n \to w\) in \(L^q(\mathbb{R}^N), 2 < q < 2^*\), and also \(w_n \to w\) a.e. in \(\mathbb{R}^N\).

In order to prove that \(w = 0\), let us consider \(\Gamma = \{x \in \mathbb{R}^N; \varphi(x) \neq 0\}\) and prove that \(\Gamma\) has zero Lebesgue measure. For all \(x \in \Gamma\), we have that \(\lim_{n \to \infty} \varphi(t_n, u)(x) = \infty\). By \((f_4)\), for each \(M > 0\), there exists \(r > 0\) such that

\(F(x, s) \geq Ms^2\), for all \(s \geq r\) and for a.e. \(x \in \mathbb{R}^N\).

Since \(\{I(\varphi(t_n, u))\}_{n \to \infty}\) is bounded from below, then

\[
\frac{1}{2} + o_n(1) \geq \int \frac{F(x, \varphi(t_n, u))}{\|\varphi(t_n, u)\|^2} d\tau
\geq \int_{\{|\varphi(t_n, u)| > r\} \cap \Gamma} \frac{F(x, \varphi(t_n, u))}{\varphi(t_n, u)^2} w_n^2 dx
\geq M \int_{\{|\varphi(t_n, u)| > r\} \cap \Gamma} w_n^2 dx.
\]

Observing that \(\varphi(t_n, u)(x) \to +\infty\) as \(n \to \infty\) for all \(x \in \Gamma\) and by Fatou’s Lemma, it follow that

\[
\frac{1}{2} \geq M \int_\Gamma w^2 dx,
\]
which is a contradiction since $\int_{\Gamma} w^2 > 0$ and $M$ is arbitrarily. Hence, $|\Gamma| = 0$.

Note that the function $t \mapsto I(t \varphi(t_n, u))$ is smooth in $(0, 1)$. Let $s_n \in [0, 1]$ such that

$$I(s_n \varphi(t_n, u)) = \max_{t \in [0,1]} I(t \varphi(t_n, u)).$$

For each $R > 0$, and for all $n$ large enough, we have

$$I(s_n \varphi(t_n, u)) \geq I \left( \frac{R}{\| \varphi(t_n, u) \|} \varphi(t_n, u) \right) = \frac{R^2}{2} - \int F(x, Rw_n) = \frac{R^2}{2} + o_n(1).$$

Hence $\lim_{n \to \infty} I(s_n \varphi(t_n, u)) = +\infty$ which implies that $s_n \in (0, 1)$. Then, for $n$ large enough $I'(s_n \varphi(t_n, u)) s_n \varphi(t_n, u) = 0$. By (f4), for all $t \in [0, 1]$ we have that

$$2I(t \varphi(t_n, u)) \leq 2I(s_n \varphi(t_n, u)) - I'(s_n \varphi(t_n, u)) s_n \varphi(t_n, u)$$

$$= \int (f(x, s_n \varphi(t_n, u)) s_n \varphi(t_n, u) - 2F(x, s_n \varphi(t_n, u)))$$

$$\leq \int (f(x, \varphi(t_n, u)) \varphi(t_n, u) - 2F(x, \varphi(t_n, u)))$$

$$= 2I(\varphi(t_n, u)) + o_n(1) \leq C_1$$

For a given $R_0 > 0$, for $n$ large enough, it follows that $\frac{R_0}{\| \varphi(t_n, u) \|} < 1$. Then,

$$2I(R_0 w_n) = 2I \left( \frac{R_0}{\| \varphi(t_n, u) \|} \varphi(t_n, u) \right) \leq C_1. \quad (2.4)$$

On the other hand, for all $R_0 > 0$,

$$2I(R_0 w_n) = R_0^2 - 2 \int F(x, R_0 w_n) = R_0^2 + o_n(1). \quad (2.5)$$

Since (2.4) contradicts (2.5), we have that $\{ \varphi(t_n, u) \}_{n \in \mathbb{N}}$ is uniformly bounded in $H$ with respect to $n \in \mathbb{N}$ and the claim is proved.
In order to show that \( \omega(u) \neq \emptyset \), let us consider the bounded sequence \( \{\varphi(t_n, u)\} \). Since \( A \) is compact, there exists \( u_0 \in H \) such that \( A(\varphi(t_n, u)) \to u_0 \) along a subsequence. Hence

\[
0 = \lim_{n \to \infty} \| \nabla I(\varphi(t_n, u)) \|
= \lim_{n \to \infty} (\varphi(t_n, u) - A(\varphi(t_n, u)))
= \lim_{n \to \infty} (\varphi(t_n, u) - u_0).
\]

Then \( \lim_{n \to \infty} \varphi(t_n, u) = u_0 \) and \( u_0 \in \omega(u) \).

Concerning the proof that every point in \( \omega(u) \) is a critical point of \( I \), note that \( I(\varphi(t, u)) \to d \) as \( t \to +\infty \). Then, if \( v \in \omega(u) \), there exists \( t_n \to \infty \) such that \( \varphi(t_n, u) \to v \) in \( H \) as \( n \to \infty \). Hence

\[
I(\varphi(t, v)) = \lim_{n \to \infty} I(\varphi(t, \varphi(t_n, u))) = \lim_{n \to \infty} I(\varphi(t + t_n, u)) = d,
\]

for all \( t \geq 0 \). Then for all \( t \geq 0 \),

\[
0 = \frac{\partial}{\partial t} I(\varphi(t, v)) = \| \nabla I(\varphi(t, v)) \|^2,
\]

which implies that \( \nabla I(v) = 0 \) and \( v \) is a critical point of \( I \).

\[ \square \]

**Definition 2.5.** We call \( D \subset H \) a positive invariant set if \( \varphi(t, u) \in D \) for all \( t \in [0, T(u)] \) and \( u \in D \). If \( D \) is a positive invariant set we define its absorption domain by

\[
A(D) = \{ u \in H; \exists t_0 \in [0, T(u)) \text{ such that } \varphi(t_0, u) \in D \}.
\]

Let us define the following set

\[
A_0 = \{ u \in H; T(u) = \infty \text{ and } \varphi(t, u) \to 0 \text{ as } t \to \infty \}.
\]

**Lemma 2.6.** \( A_0 \) is an open subset of \( H \) and there exists \( r > 0 \) such that \( B_r(0) \subset A_0 \).
Proof.

Let $0 < \epsilon < 1$ and $A(\epsilon)$ given in Lemma 2.3. Let us consider $\alpha_0 = \left(\frac{1 - \epsilon}{2A(\epsilon)}\right)^{\frac{1}{p}}$. For each $u \in B_{\alpha_0}(0)$, by Lemma 2.3 we have that

$$
\Psi(u) = \int_0^1 \frac{\partial}{\partial t} \Psi(tu) dt = \int_0^1 \langle A(tu), u \rangle dt \leq \int_0^1 \left( \epsilon \|tu\| + A(\epsilon) \|tu\|^{p+1} \right) \|u\| dt \leq \int_0^1 t \|u\|^2 \left( \epsilon + A(\epsilon) \|u\|^p \right) dt = \frac{\|u\|^2}{2} \left( \epsilon + A(\epsilon) \|u\|^p \right) \leq \frac{\|u\|^2}{2} \left( \epsilon + A(\epsilon) \alpha_0^p \right) = \frac{\epsilon + 1}{4} \|u\|^2.
$$

Then,

$$
I(u) = \frac{1}{2} \|u\|^2 - \Psi(u) \geq \left( \frac{1}{2} - \frac{\epsilon + 1}{4} \right) \|u\|^2 \geq 0,
$$

since $\epsilon \in (0, 1)$.

Moreover, for all $u \in \partial B_{\alpha_0}(0)$,

$$
I(u) \geq \left( \frac{1}{2} - \frac{\epsilon + 1}{4} \right) \alpha_0^2 =: \beta_0 > 0.
$$

(2.6)

By continuity, there exists $r \in (0, \alpha_0)$ such that

$$
I(u) < \beta_0 \quad \text{for all} \quad u \in B_r(0).
$$

Since the energy do not grow along any trajectory, if $u \in B_r(0)$, then $I(\varphi(t, u)) < \beta_0$, for all $t \in [0, T(u))$. Then, by (2.6), $\varphi(t, u) \in B_{\alpha_0}(0)$ for all $t \in [0, T(u))$. Hence $I(\varphi(t, u)) \geq 0$ for all $t \in [0, T(u))$, which implies that $T(u) = \infty$. Moreover $\omega(u) \subseteq B_{\alpha_0}(0)$ is a compact nonempty set formed of
critical points of \( I \). On the other hand, if \( v \in B_{\alpha_0}(0) \) is a critical point of \( I \), then

\[
\|v\|^2 = \langle A(v), v \rangle \leq \|v\|^2(\epsilon + A(\epsilon)\|v\|^p) \leq \|v\|^2(\epsilon + A(\epsilon)\alpha_0^p) = \frac{\epsilon + 1}{2}\|v\|^2,
\]

which only is possible in the case where \( v = 0 \). Therefore, \( \omega(u) = \{0\} \) for all \( u \in B_r(0) \) and then \( B_r(0) \subset \mathcal{A}_0 \), which implies that \( \mathcal{A}_0 = \mathcal{A}(B_r(0)) \). Since \( B_r(0) \) is an open set, then so is \( \mathcal{A}(B_r(0)) \).

The following is a key result in our argument and in particular imply that \( \partial \mathcal{A}_0 \) is a great place to look for nontrivial critical points of \( I \).

**Proposition 2.7.** \( \partial \mathcal{A}_0 \) is a closed positively invariant set of \( H \) and \( \inf_{u \in \partial \mathcal{A}_0} I(u) \geq 0 \). In particular, for all \( u \in \partial \mathcal{A}_0 \), \( \omega(u) \) is a non-empty set consisting in nontrivial critical points of \( I \).

The proof of the positively invariance can be found in [11] while the other results are straightforward to see.

Although \( \partial \mathcal{A}_0 \) is a great set to look for nontrivial critical points of \( I \), once found, nothing can be said about its signal. Let us introduce the concept of dual cone and state the Dual Cone Decompositions Theorem which is given by Moreau in [12].

**Definition 2.8.** Given a cone \( \mathcal{K} \) in a Hilbert space \( H \), its dual cone is defined by

\[
\mathcal{K}^* = \{ v \in H; \langle u, v \rangle \leq 0, \ \forall u \in \mathcal{K} \}.
\]

**Theorem 2.9.** Let \( \mathcal{K} \subset H \) a closed convex cone. Then for all \( x \in H \), there exist \( y \in \mathcal{K} \) and \( z \in \mathcal{K}^* \) such that

\[
x = y + z \quad \text{and} \quad \langle y, z \rangle = 0.
\]

Let us define the following cones and afterwards prove some invariance properties of them.

Let

\[
\mathcal{K} = \{ u \in H; u \geq 0 \ \text{a.e. in} \ \mathbb{R}^N \}
\]

\[
-\mathcal{K} = \{ u \in H; u \leq 0 \ \text{a.e. in} \ \mathbb{R}^N \}.
\]  

which are closed convex cones.
Let us denote by $P$ and $Q$ the orthogonal projections of $H$ in $K$ and $-K$, respectively. Denoting by $P^* = Id - P$ and $Q^* = Id - Q$, note that

$$\langle Pu, P^* u \rangle = 0, \quad \text{for all } u \in H$$

and

$$P^* u \in K^*,$$

where $K^*$ is the dual cone associated to $K$. We have analogous results involving $Q$, $-K$ and $(-K)^*$.

In order to prove the invariance of $K$ and $-K$, as we will see, it will be necessary to prove that $K^* \subset -K$ and $(-K)^* \subset K$. In the classical argument developed by Weth in [19], the maximum principle to the operator $\Delta^2$ under certain boundary conditions and in certain domains is absolutely useful. Since this is not an option for us, let us prove some result that in some sense will substitute the lack of this result.

**Lemma 2.10.** For each $h \in C^\infty_0(\mathbb{R}^N)$, $h \geq 0$ and $h \not\equiv 0$, there exists a positive continuous radial solution $v \in H^2_{rad}(\mathbb{R}^N)$ of the linear problem

$$\Delta^2 v + V(x)v = h(x) \quad \text{in } \mathbb{R}^N. \quad (2.8)$$

**Proof.** The existence of a solution $v \in H^2_{rad}(\mathbb{R}^N)$ follows straightforwardly just by applying Riesz Theorem. Regularity is a simple matter just by calling Proposition 2.5 in [16].

To the positiveness we apply some arguments of Chabrowski and Yang in [6] which we describe below.

Since $v$ is smooth, $\Omega = \{x \in \mathbb{R}^N; \quad v(x) < 0\}$ is an open subset of $\mathbb{R}^N$. Supposing by contradiction that there exists $x_0 \in \Omega$, let

$$R = \frac{1}{2} \sup \{ r > 0; B_r(x_0) \subset \Omega \}.$$ 

Assuming by simplicity that $R = 1$, let us denote

$$\bar{v}(x) = \begin{cases} v(x), & \text{if } |x - x_0| \leq 1 \\ 0, & \text{if } |x - x_0| > 1 \end{cases}$$

and $\rho \in C^\infty_0(\mathbb{R}^N)$ with $supp(\rho) \subset B_1(0)$ and $\int \rho = 1$. Let us define

$$\tilde{v}(x) = \int r^{-N} \rho \left( \frac{x - y}{r} \right) \bar{v}(y)dy = \int_{B_1(x_0)} r^{-N} \rho \left( \frac{x - y}{r} \right) v(y)dy,$$
where $0 < r \leq 1$.

Note that $\text{supp}(\tilde{v}) \subset B_{r+1}(x_0)$. Then, for $x \in B_{r+1}(x_0)$, $\tilde{v}$ satisfies

$$
\Delta^2 \tilde{v}(x) = \int_{B_1(x_0)} r^{-N} \rho \left( \frac{x-y}{r} \right) \Delta^2 v(y) dy \\
= \int_{B_1(x_0)} r^{-N} \rho \left( \frac{x-y}{r} \right) (h(y) - V(y)v(y)) dy =: \tilde{h}(x),
$$

where $\tilde{h} \geq 0$ as for each $y \in B_{r+1}(0)$, $h(y) - V(y)v(y) \geq 0$. Then $\tilde{v}$ satisfies the following problem

$$
\begin{aligned}
\Delta^2 \tilde{v} &= \tilde{h} & \text{in } B_{r+1}(0) \\
\tilde{v} &= \Delta \tilde{v} = 0 & \text{on } \partial B_{r+1}(0)
\end{aligned}
\tag{2.9}
$$

Let us remember that from standard minimization arguments and elliptic regularity theory, as proved in \cite{6}[Lemma 3], for all bounded smooth domain $\Omega \subset \mathbb{R}^N$ and for all $g \in L^{2N/(N-4)}(\Omega) \cap C^{0,\alpha}(\Omega)$, $0 < \alpha < 1$, there exists a classical solution $w > 0$ of

$$
\begin{aligned}
\Delta^2 w &= g & \text{in } \Omega \\
w &= \Delta w = 0 & \text{on } \partial \Omega
\end{aligned}
\tag{2.10}
$$

For any $L^{2N/(N-4)}(B_{r+1}(x_0)) \cap C^{0,\alpha}(B_{r+1}(x_0))$ positive function $g$, let $w$ be the positive smooth solution of (2.10) with $\Omega = B_{r+1}(x_0)$. Using $w$ as test function in (2.9) we obtain

$$
\int_{B_{r+1}(x_0)} \tilde{h} w dx = \int_{B_{r+1}(x_0)} \Delta^2 \tilde{v} w dx \\
= \int_{B_{r+1}(x_0)} \tilde{v} \Delta^2 w dx \\
= \int_{B_{r+1}(x_0)} \tilde{v} g dx.
$$

However, while the last integral is negative, the first one is positive, which give us a contradiction. □

Using the last lemma it is possible to prove the following claim.

**Claim.** $\mathcal{K}^* \subset -\mathcal{K}$.
Proof. Let $u \in \mathcal{K}^*$. For each $h \in L^2(\mathbb{R}^N)$, $h \geq 0$ a.e. in $\mathbb{R}^N$ let $(h_n)$ in $C_0^\infty(\mathbb{R}^N)$, $h_n \geq 0$, such that $h_n \to h$ in $L^2(\mathbb{R}^N)$. For each $n \in \mathbb{N}$, let $v_n$ be the positive solution of the linear problem (2.8) with $h = h_n$, given by Lemma 2.10. Then $v_n \in \mathcal{K}$ and,

$$0 \geq \langle u, v_n \rangle = \int (\Delta u \Delta v_n + V(x)uv_n) = \int uh_n. \quad (2.11)$$

Hence, by (2.11) and Fatou’s Lemma,

$$\int uh \leq \liminf_{n \to \infty} \int u(x)h_n(x) \leq 0.$$

Then it follows that $u \leq 0$ a.e. in $\mathbb{R}^N$ and therefore $u \in -\mathcal{K}$. □

Remark 2.11. It is worth pointing out that if $u \in H$, then $u = Pu + P^*u$ where $Pu \geq 0$ and $P^*u \leq 0$ a.e. in $\mathbb{R}^N$. Then $u \leq Pu$, and consequently, $u^+ \leq Pu$ a.e. in $\mathbb{R}^N$. In the same way one can prove that $P^*u \leq u^-$, $Qu \leq u^-$ and $u^+ \leq Q^*u$ a.e. in $\mathbb{R}^N$.

The following is a very important result to prove the invariance of $\mathcal{K}$ and $-\mathcal{K}$ under the flux $\varphi$.

Lemma 2.12. The operator $A$ satisfies the following conditions

i) $\langle A(u), v \rangle \leq \langle A(P^*u), v \rangle$, for all $u \in H$ and $v \in \mathcal{K}^*$;

ii) $\langle A(u), v \rangle \leq \langle A(Q^*u), v \rangle$, for all $u \in H$ and $v \in (-\mathcal{K})^*$.

Proof. Since ii) can be proved in the same way, we just prove i). Let $u \in H$ and $v \in \mathcal{K}^*$. By Remark 2.11 $P^*u \leq u^-$ and by $(f_5)$

$$f(x, P^*u(x)) \leq f(x, u^-(x)).$$

Since $v \leq 0$, once more by $(f_5)$ it follows that

$$\langle A(u), v \rangle = \int f(x, u)v \leq \int f(x, u^-)v \leq \int f(x, P^*u)v = \langle A(P^*u), v \rangle.$$

□
Lemma 2.13.  

i) \( A(K) \subset K \) and \( A(-K) \subset -K \).

ii) For sufficiently small \( \alpha > 0 \), the \( \alpha \)-neighborhood of \( K \), \( B_\alpha(K) \) is positively invariant under \( \varphi \). Moreover, all critical point of \( I \) in \( B_\alpha(K) \) belong to \( K \). The same holds to the cone \(-K\).

iii) \( K \) and \(-K\) are positively invariant under \( \varphi \).

Proof. Let us prove the results just to \( K \), since for \(-K\) the arguments are the same.

i) By Lemma 2.12 i), for \( u \in K \),

\[
\| P^*(A(u)) \|^2 = \langle A(u) - P(A(u)), A(u) - P(A(u)) \rangle \\
= \langle A(u), P^*(A(u)) \rangle \\
\leq \langle A(P^*u), P^*(A(u)) \rangle = 0,
\]

which implies that \( A(u) \in K \).

ii) Let \( u \in H \), by Lemma 2.3 and Lemma 2.12 i), for \( 0 < \epsilon < 1 \), we have that

\[
\| P^*(A(u)) \|^2 = \langle A(u), P^*(A(u)) \rangle \\
\leq \langle A(P^*u), P^*(A(u)) \rangle \\
\leq \| P^*(A(u)) \| \left( \epsilon \| P^*u \| + A(\epsilon) \| P^*u \|^{p+1} \right),
\]

which implies that

\[
\| P^*(A(u)) \| \leq \epsilon \| P^*u \| + A(\epsilon) \| P^*u \|^{p+1} = \| P^*u \| \left( \epsilon + A(\epsilon) \| P^*u \|^{p} \right).
\]

(2.12)

since \( \| P^*(A(u)) \| \neq 0 \). Then, if \( 0 < \| P^*u \| < \left( \frac{1-\epsilon}{2A(\epsilon)} \right)^\frac{1}{p} =: \alpha_0 \),

\[
\| P^*(A(u)) \| < \| P^*u \|.
\]

(2.13)

Hence, for all \( \alpha < \alpha_0 \), every fixed point of \( A \) in \( \overline{B_\alpha(K)} \) belongs to \( K \). In fact, if \( u \in \overline{B_\alpha(K)} \setminus K \) and \( A(u) = u \) then \( 0 < \| P^*u \| \leq \alpha < \alpha_0 \), which implies by (2.13) that

\[
\| P^*u \| = \| P^*(A(u)) \| < \| P^*u \|,
\]
which is a contradiction.

Now let us prove that $B_{\alpha}(\mathcal{K})$ is positively invariant.

Note that by (2.13)

$$A(\partial B_{\alpha}(\mathcal{K})) \subset \text{int}(B_{\alpha}(\mathcal{K})). \quad (2.14)$$

Suppose, contrary to our claim, that there exists $u_0 \in B_{\alpha}(\mathcal{K})$ such that $\varphi(t_0, u_0) \in \partial B_{\alpha}(\mathcal{K})$ where $t_0 \in [0, T(u_0))$ is the least positive real with this property. As $B_{\alpha}(\mathcal{K})$ is a convex open set and $\{\varphi(t_0, u_0)\}$ is compact, by Mazur Separation Theorem, there exists a linear functional $\rho \in H^*$ and a real number $\beta$ such that $\rho(\varphi(t_0, u_0)) = \beta$ and $\rho(u) > \beta$ for all $u \in B_{\alpha}(\mathcal{K})$.

By (2.14) it follows that

$$\left. \frac{\partial}{\partial t} \rho(\varphi(t, u_0)) \right|_{t=t_0} = \rho(-\nabla I(\varphi(t_0, u_0)))$$

$$= \rho(A(\varphi(t_0, u_0))) - \beta > 0.$$

Therefore, there exists $\eta > 0$ such that $\rho(\varphi(t, u_0)) < \beta$ as long as $t \in (t_0 - \eta, t_0)$. Then $\varphi(t, u_0) \notin B_{\alpha}(\mathcal{K})$ for all $t \in (t_0 - \eta, t_0)$, which contradicts the minimality of $t_0$.

\begin{itemize}
  \item[iii)] This item follows straightforwardly observing that $\mathcal{K} = \bigcap_{\alpha > 0} B_{\alpha}(\mathcal{K})$.
\end{itemize}

\[\square\]

From now on let us consider $\alpha > 0$ such that the statement of Lemma 2.13 holds for $\mathcal{K}$ and $-\mathcal{K}$.

To obtain the signed solutions we will use the following result.

**Proposition 2.14.** Assume that there exists $u_0 \in \mathcal{K}$ such that $I(u_0) < 0$, then there exists a nontrivial critical point of $I$ in $\mathcal{K}$. The same holds for $-\mathcal{K}$.

**Proof.** First note that by definition of $\mathcal{A}_0$, $I(u) \geq 0$ for all $u \in \mathcal{A}_0$. Then by continuity, $I(u) \geq 0$ for all $u \in \overline{\mathcal{A}_0}$.

Since $I(u_0) < 0$ then $u_0 \notin \mathcal{A}_0$. As $\mathcal{A}_0$ is an open neighborhood of the origin, there exists $s \in (0, 1)$ such that $su_0 \in \partial \mathcal{A}_0 \cap \mathcal{K}$. Since $\partial \mathcal{A}_0 \cap \mathcal{K}$ is
a closed positively invariant set, by Proposition 2.4 \( \omega(su_0) \subset \partial A_0 \cap K \) is nonempty and any of its points are critical points of \( I \).

Let us denote by

\[ A_+ = A(B_\alpha(K)) \cap \partial A_0 \quad \text{and} \quad A_- = A(B_\alpha(-K)) \cap \partial A_0. \]

**Lemma 2.15.** \( A_+ \) and \( A_- \) are disjoint relatively open sets of \( \partial A_0 \).

**Proof.** Since \( B_\alpha(K) \) and \( B_\alpha(-K) \) are open sets, then so are \( A(B_\alpha(K)) \cap \partial A_0 \) and \( A(B_\alpha(-K)) \cap \partial A_0 \).

Suppose, contrary to our claim, that there exists \( u \in A_+ \cap A_- \). Since \( u \in \partial A_0 \), then \( T(u) = \infty \) and \( \omega(u) \neq \emptyset \). Further, since \( u \in A(B_\alpha(K)) \cap \partial A_0 \), \( \omega(u) \subset B_\alpha(K) \cap B_\alpha(-K). \) But since \( \omega(u) \) consists of critical points of \( I \), by Lemma 2.13, \( \omega(u) \in K \cap -K = \{0\} \), which contradicts the fact that \( u \in \partial A_0 \). \( \square \)

**Proposition 2.16.** Suppose that there exists a continuous path \( h : [0,1] \to H \), such that \( h(0) \in K \), \( h(1) \in -K \) and \( I(h(t)) < 0 \) for all \( t \in [0,1] \). Then \( I \) has at least three nontrivial critical points, being \( u_1 \in K \), \( u_2 \in -K \) and \( u_3 \in H \setminus (K \cup -K). \)

**Proof.** Proposition 2.14 gives the existence of the signed critical points \( u_1 \in K \) and \( u_2 \in -K \).

To get the nodal one, let us first highlight that \( h([0,1]) \cap \overline{A_0} = \emptyset \).

Let \( Q = [0,1]^2 \) and \( B \subset Q \) be defined by

\[ B = \{(s_1, s_2) \in Q; s_1 h(s_2) \in A_0\}. \]

Note that \( B \) is relatively open in \( Q \) and the following hold

- \( \{0\} \times [0,1] \subset B \), since \( 0h(s) = 0 \in A_0, \forall s \in [0,1] \);
- \( \{1\} \times [0,1] \cap \overline{B} = \emptyset \), since \( I(h(s)) < 0, \forall s \in [0,1] \).

By calling the Leray-Schauder continuation principle, it follows that there exists a connected component \( \Gamma \) of \( \partial B \), such that

\[ \Gamma \cap ([0,1] \times \{0\}) \neq \emptyset \quad \text{and} \quad \Gamma \cap ([0,1] \times \{1\}) \neq \emptyset. \]

Now, let us consider \( \Sigma \) the closure of the connected component of \( \Gamma \setminus \partial Q \) that intersects \([0,1] \times \{0\}\) and \([0,1] \times \{1\}\).
Denoting by $\Gamma_0 = \{s_1 h(s_2); (s_1, s_2) \in \Sigma\}$, we have that $\Gamma_0$ is a connected subset of $\partial A_0$ such that $\Gamma_0 \cap \pm K \neq \emptyset$. Since by Lemma 2.15 $A_{\pm}$ are disjoint open subsets of $\partial A_0$, then $\Gamma_0 \cap \pm K \neq \emptyset$. Since by Lemma 2.15, $A_{\pm}$ are disjoint open subsets of $\partial A_0$, then $\Gamma_0 \cap A_{\pm}$ are open disjoints subsets of $\Gamma_0$. By connectedness of $\Gamma_0$, there exists $u \in \Gamma_0 \setminus (A_+ \cup A_-)$ and once $\partial A_0 \setminus (A_+ \cup A_-)$ is positively invariant, then $\{\varphi(t, u); t \geq 0\} \subset \partial A_0 \setminus (A_+ \cup A_-)$. Using the fact that this is a closed subset in $\partial A_0$, it follows that $\omega(u) \subset \partial A_0 \setminus (A_+ \cup A_-)$. In particular, $\omega(u) \cap (K \cup -K) \neq \emptyset$ and any of his points are nodal critical points of $I$. □

The next result will be useful to put the energy functional $I$ in the context of the last proposition.

**Lemma 2.17.** If $S \subset H \setminus \{0\}$ is a compact subset and $\tilde{S} = \{tu; u \in S \text{ and } t \geq 0\}$, then

$$I(u) \to -\infty, \text{ as } u \in \tilde{S} \text{ and } \|u\| \to \infty.$$ 

**Proof.** Let $(u_n) \subset \tilde{S}$ be a sequence such that $\|u_n\| \to \infty$. Then there exist a sequence in $\mathbb{R}_+ (t_n)$ such that $u_n = t_n v_n$ and $(v_n) \subset S$. Since $S$ is compact, we can suppose that along a subsequence $v_n \to v$, as $n \to \infty$, for some $v \in S$. As $\|u_n\| \to \infty$, then one trivially see that $t_n \to \infty$.

Now let us prove that

$$\lim_{n \to \infty} \int \frac{F(x, t_n v_n)}{t_n^2} = +\infty.$$ 

In fact, let $\Gamma = \{x \in \mathbb{R}^N; v(x) \neq 0\}$, then by Fatou lemma and $(f_4)$, it follows that

$$\lim_{n \to \infty} \int \frac{F(x, t_n v_n)}{t_n^2} = \lim_{n \to \infty} \int \frac{F(x, t_n v_n)}{t_n^2 v_n^2} v_n^2 \geq \lim_{n \to \infty} \int_{\Gamma} \frac{F(x, ts_n v_n)}{t_n^2 v_n^2} v_n^2 dx = +\infty.$$ (2.15)
Let $M > 0$ be such that $\|v_n\| \leq M$, for all $n \in \mathbb{N}$. Then we have

\[
I(u_n) = I(t_n v_n) \\
= t_n^2 \left( \frac{\|v_n\|^2}{2} - \int \frac{F(x, t_n v_n)}{t_n^2} \right) \\
\leq t_n^2 \left( \frac{M^2}{2} - \int \frac{F(x, t_n v_n)}{t_n^2} \right) \\
\to -\infty
\]

as $n \to \infty$. \hfill \Box

Now, let us prove the main result of this work

**Proof of Theorem 1.3** Let $u \in K$ and $v \in K^*$, such that $u, v \neq 0$ and $u$ and $v$ are linearly independent in $H$. For each $s > 0$, let us define $h_s : [0, 1] \to H$ as $h_s(t) = s(tu + (1 - t)v)$.

For each $s > 0$,

- $h_s(1) = su \in K \setminus \{0\}$,
- $h_s(0) = sv \in K^* \setminus \{0\}$.

By Lemma 2.17 applied to the compact set $S = \{tu + (1 - t)v; t \in [0, 1]\}$, we see that if $s$ is large enough, then $I(h_s(t)) < 0$, $\forall t \in [0, 1]$. Hence, Proposition 2.16 give us the existence of $u_1, u_2$ and $u_3$, respectively, a positive, a negative and a nodal critical point of $I$ restricted to $H$. The existence of the critical points of $I$ in all space $H^2(\mathbb{R}^N)$ follows just by applying the Principle of symmetric criticality of Palais to the functional $I$, once observed that $I$ is invariant by the action of the group $O(N)$, and

$$
H^2_{rad}(\mathbb{R}^N) = \{u \in H^2(\mathbb{R}^N); \ u(g(x)) = u(x), \forall g \in O(N)\}.
$$

Hence the theorem follows. \hfill \Box

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**References**

[1] C.O. Alves, S.H.M. Soares  *On the location and profile of spike-layer nodal solutions to nonlinear Schrödinger equations.* Journal of Mathematical Analysis and Applications 296 (2004), 563 - 577.
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[2] C.O. Alves, S.H.M. Soares, *Nodal solutions for singularly perturbed equations with critical exponential growth*, Journal of Differential Equations 234 (2007), 464 - 484.

[3] D. Bonheure, J.V. Schaftingen, *Bound state solutions for a class of nonlinear Schrödinger equations*, Rev. Mat. Iberoam. 24 (2008), 297 - 351.

[4] D. Bonheure, J. Di Cosmo, J.V. Schaftingen, *Nonlinear Schrödinger equation with unbounded or vanishing potentials: solutions concentrating on lower dimensional spheres*, Journal of Differential Equations. 252 (2012), 941 - 968.

[5] A. Castro, J. Cossio, J. Neuberger, *A sign-changing solution for a superlinear Dirichlet problem*, Rocky Mountain Journal of Mathematics. 27, 4 (1997), 1041 - 1053.

[6] J. Chabrowski, J. Yang, *Nonnegative solutions for semilinear biharmonic equations in \( \mathbb{R}^N \)*, Analysis, 27 (1997), 35 - 59.

[7] M. del Pino, P. Felmer, *Local mountain pass for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differential Equations 4 (1996), 121 - 137.

[8] Y. Ebihara, T. Schonbek, *On the (non)compactness of the radial Sobolev spaces*, Hiroshima Math. J. 16 (1986), 665 - 669.

[9] G.J.M. Figueiredo, M.T.O. Pimenta, *Multiplicity of solutions for a biharmonic equation with subcritical or critical growth*, to appear in Bull. Belgian Math. Soc.

[10] A. Floer, A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equations with a bounded potential*, Journal of Functional Analysis 69 (1986), 397 - 408.

[11] LIU, Z., SUN, J. *Invariant sets of descending flow in critical point theory with applications to nonlinear differential equations*, Journal of differential Equations, 172 (2001), 257 - 299.

[12] J.J. Moreau, *Décomposition orthogonale d’un espace hilbertien selon deux cônes mutuellement polaires*, C. R. Acad. Sci. Paris. 255 (1962), 238 - 240.
[13] M.T.O. Pimenta, S.H.M. Soares *Existence and concentration of solutions for a class of biharmonic equations*, J. Math. Anal. Appl., 390 (2012), 274 - 289.

[14] M.T.O. Pimenta, S.H.M. Soares *Singularly perturbed biharmonic problems with superlinear nonlinearities*, preprint.

[15] Miyagaki, O., Souto, M. *Superlinear problems without Ambrosetti and Rabinowitz growth condition*, Journal of Differential Equations, 245 (2008), 3628 - 3638.

[16] M. Ramos, *Uniform estimates for the biharmonic operator in $\mathbb{R}^N$ and applications*, Commun. Appl. Analysis, 8 (2009) 435 - 457.

[17] P.H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, ZAMP 43 (1992), 270 - 291.

[18] X. Wang, *On concentration of positive bound states of nonlinear Schrödinger equations*, Commun. Math. Phys. 153 (1993), 229 - 244.

[19] Szulkin, A., Weth, T. *The method of Nehari manifold*, Handbook of Nonconvex Analysis and Applications, D.Y. Gao and D. Motreanu eds., International Press, Boston (2010), 597-632.