A Concrete View of Rule 110 Computation

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Rule 110 is a cellular automaton that performs repeated simultaneous updates of an infinite row of binary values. The values are updated in the following way: 0s are changed to 1s at all positions where the value to the right is a 1, while 1s are changed to 0s at all positions where the values to the left and right are both 1. Though trivial to define, the behavior exhibited by Rule 110 is surprisingly intricate, and in [1] we showed that it is capable of emulating the activity of a Turing machine by encoding the Turing machine and its tape into a repeating left pattern, a central pattern, and a repeating right pattern, which Rule 110 then acts on. In this paper we provide an explicit compiler for converting a Turing machine into a Rule 110 initial state, and we present a general approach for proving that such constructions will work as intended. The simulation was originally assumed to require exponential time, but surprising results of Neary and Woods [2] have shown that in fact, only polynomial time is required. We use the methods of Neary and Woods to exhibit a direct simulation of a Turing machine by a tag system in polynomial time.

1 Compiling a Turing machine into a Rule 110 State

In this section we give a concrete algorithm for compiling a Turing machine and its tape into an initial state for Rule 110, following the construction given in [1]. We will create an initial state that will eventually produce the bit sequence 01101001101000 if and only if the corresponding Turing machine halts. Or, if one prefers time sequences to spatial sequences, it is also the case that the sequence 110100110101111 will be produced over time by a single cell if and only if the Turing machine halts. (These sequences are the shortest that occur for the first time in the collision that produces the \( F \) glider as in figure 12(z), which only occurs if the algorithm halts.) While based directly on the methods of [1], the presentation of the algorithm here is self-contained. This section can be viewed as heavily commented high level pseudocode, the intent being to explicitly provide all the details a program would use, regardless of programming language or input or output format.

Following [1], we will convert the Turing machine into a tag system, which we then convert into a cyclic tag system, and finally into a bit sequence for Rule 110. Of course, if one is starting with a tag system, or a cyclic tag system (whose appendants’ lengths are multiples of 6), then the unnecessary conversions may be omitted. For example, Paul Chapman’s tag system [3] for the 3x + 1 problem, given by \{A \leadsto C, B \leadsto D, C \leadsto AE, D \leadsto BF, E \leadsto CCD, F \leadsto DDD\}, with a starting tape of \( C[D]^{x-1} \), or Liesbeth De Mol’s more recent tag system for the same problem [4], \{A \leadsto CY, C \leadsto A, Y \leadsto AAA\}, can be implemented nicely in Rule 110 without going through a Turing machine representation. As another example, the cyclic tag system \{YYYYY,0,NNNNN,0\}, starting with a single Y on the tape, yields exactly the same behavior as the cyclic tag system shown in figure 2 of [1] with runs of Ys (after the initial Y) doubled and runs of Ns lengthened by a factor of six. This cyclic tag system is also shown on Page 96 of [5], which shows a random-looking graph of its behavior during the first million steps.

If we start with a cyclic tag system whose appendants are not each a multiple of six long, we can convert it into such a form as follows: Expand each appendant by adding 5 N symbols after every symbol, and expand the list of appendants by adding 5 empty appendants after every appendant. The tape should...
also be expanded just as each appendant was. This will make every appendant’s length be a multiple of six, while performing the same computation, on every sixth step, as the original cyclic tag system.

1.1 We start with a Turing machine

Suppose we are given a Turing machine with \( m \) states \( \Psi = \{ \psi_1, \psi_2, \ldots, \psi_m \} \) and \( t \) symbols \( \Sigma = \{ \sigma_1, \sigma_2, \ldots, \sigma_t \} \) and we are given its lookup tables for which symbol to write \( \Upsilon(\psi_i, \sigma_j) \in \Sigma \), which way to move \( \Delta(\psi_i, \sigma_j) \in \{ \text{left}, \text{right}, \text{halt} \} \), and what state to go into \( \Gamma(\psi_i, \sigma_j) \in \Psi \). (\( \Upsilon \) and \( \Gamma \) do not need to be defined for (state, symbol) pairs that cause the machine to halt, i.e. for which \( \Delta \) is \text{halt}.)

Suppose further that the Turing machine is currently in state \( \psi_\gamma \), and that the tape is of the form

\[
\sigma_{a_w} \sigma_{a_{w-1}} \cdots \sigma_{a_1} \sigma_{b_\gamma} \sigma_{b_{\gamma-1}} \cdots \sigma_{b_1} \sigma_{c_1} \sigma_{c_2} \cdots \sigma_{c_{\gamma}} \sigma_{d_1} \sigma_{d_2} \cdots \sigma_{d_y} \sigma_{e_1} \sigma_{e_2} \cdots \sigma_{e_z},
\]

where the box represents the current position of the Turing machine’s head, and the portions with lines over them are repeated into the distance of the two-way infinite tape.

Our task is to convert this complete description of the Turing machine and its infinite tape into an initial state for Rule 110.

1.2 We transform it into a tag system

Our first transformation will be to convert this Turing machine into a set of tag system rules for a tag system with deletion number \( s \), meaning that at each step, \( s \) symbols are removed from the front of the tape, and an appendant is appended according only to the first of the \( s \) removed symbols. There are two general approaches for doing this. The traditional (since 1964) approach of Cocke and Minsky \[6\] is simpler, but results in an exponential slowdown in simulation time. The modern (since 2006) approach of Neary and Woods \[2\] is more complicated, but amazingly solves what I call the “geometry problem” of cyclic-state tape processors such as tag systems and cyclic tag systems. The geometry problem in this instance is that the processing head is unaware, as it scans the tape, of which tape symbols are next to which other tape symbols. This is because it effectively has a fixed number of bits of memory, consisting of the phase of the head with respect to the tape, and these bits can only be read or written with difficulty. One difficulty is that the bits must be shared for an entire pass of processing the tape, meaning that the machine is unable to simply remember at each step what symbol it just saw, as a Turing machine might do. Indeed, everything the machine does remember is due to the sum total effect of the previous pass, rather than the current pass. This makes it nearly impossible for the system to know the ordering of the symbols on the tape. Without being able to detect the order of the symbols, encodings are limited to a unary representation, which makes the tape exponentially larger than a binary representation, thus making passes over the tape take exponentially longer.

In this section we will perform the transformation using the simpler method of Cocke and Minsky. For an explanation of why it is that a transformation like this results in a tag system that correctly emulates the Turing machine, see \[1\]. Here we just focus on the mechanics of the transformation itself. In section \[5\] we will show how a conversion like this can be done using the more complicated method of Neary and Woods.

To enable our transformation, we will first add two symbols to the alphabet, \( \sigma_{t+1} \) and \( \sigma_{t+2} \), for a total of \( s = t + 2 \) symbols. These two new symbols will be used to mark the left and right ends of the central nonperiodic portion of the tape. We will not need to define \( \Upsilon, \Delta, \) or \( \Gamma \) for \( \sigma_{t+1} \) or \( \sigma_{t+2} \).
Now we can transform the Turing machine into a set of tag system rules on an alphabet \( \Phi \) of \( 4m + 3ms \) symbols:

\[
\Phi = \begin{cases} 
H_{\psi_i}, L_{\psi_i}, R_{\psi_i}, R_{\psi_i^*} & \text{for each } i \in \{1 \ldots m\} \\
H_{\psi_i, \sigma_i}, L_{\psi_i, \sigma_i}, R_{\psi_i, \sigma_i} & \text{for each } i \in \{1 \ldots m\} \text{ and } j \in \{1 \ldots s\} 
\end{cases}
\]

The tag system rules, one for each symbol in \( \Phi \), are based on \( \Upsilon, \Delta, \) and \( \Gamma \) as follows:

- \( H_{\psi_i} \leadsto H_{\psi_1, \sigma_1} H_{\psi_2, \sigma_2} \ldots H_{\psi_i, \sigma_i} \)
- \( L_{\psi_i} \leadsto L_{\psi_1, \sigma_1} L_{\psi_2, \sigma_2} \ldots L_{\psi_i, \sigma_i} \)
- \( R_{\psi_i} \leadsto R_{\psi_1, \sigma_1} R_{\psi_2, \sigma_2} \ldots R_{\psi_i, \sigma_i} \)
- \( R_{\psi_i^*} \leadsto [R_{\psi_i}]^3 \)
- \( H_{\psi_i, \sigma_j, |\Delta(\psi_i, \sigma_j)|=left} \leadsto [R_{\Gamma(\psi_i, \sigma_j)}]^{s(\psi_i-\Upsilon(\psi_i, \sigma_j))} [H_{\Gamma(\psi_i, \sigma_j)}]^{j} \)
- \( H_{\psi_i, \sigma_j, |\Delta(\psi_i, \sigma_j)|=right} \leadsto [H_{\Gamma(\psi_i, \sigma_j)}]^{j} [L_{\Gamma(\psi_i, \sigma_j)}]^{s(\psi_i-\Upsilon(\psi_i, \sigma_j))} \)
- \( H_{\psi_i, \sigma_j, |\Delta(\psi_i, \sigma_j)|=halt} \leadsto \emptyset \)
- \( L_{\psi_i, \sigma_j, |\Delta(\psi_i, \sigma_j)|=left} \leadsto L_{\Gamma(\psi_i, \sigma_j)} \)
- \( L_{\psi_i, \sigma_j, |\Delta(\psi_i, \sigma_j)|=right} \leadsto [L_{\Gamma(\psi_i, \sigma_j)}]^{s^2} \)
- \( L_{\psi_i, \sigma_j, |\Delta(\psi_i, \sigma_j)|=halt} \leadsto \emptyset \)
- \( R_{\psi_i, \sigma_j, |\Delta(\psi_i, \sigma_j)|=left} \leadsto [R_{\Gamma(\psi_i, \sigma_j)}]^{s^2} \)
- \( R_{\psi_i, \sigma_j, |\Delta(\psi_i, \sigma_j)|=right} \leadsto R_{\Gamma(\psi_i, \sigma_j)} \)
- \( R_{\psi_i, \sigma_j, |\Delta(\psi_i, \sigma_j)|=halt} \leadsto \emptyset \)
- \( R_{\psi_i^*, |\Delta(\psi_i, \sigma_j)|=left} \leadsto [R_{\psi_i^*}]^3 \)

The notation \( [\text{symbol}]^n \) represents \( n \) consecutive copies of the symbol. Each symbol whose subscript is a pair \( (\psi_i, \sigma_j) \) that causes the Turing machine to halt leads to an empty right hand side, denoted by \( \emptyset \). If the Turing machine does not halt, these empty right hand sides will never be used.

Additionally, we transform the Turing machine tape into the following tag system tape:

\[
[H_{\psi_i}]^{1+s-c} [L_{\psi_i}]^{s+1} \sum_{k=2}^{m} (s-b_k) s^k [R_{\psi_i}]^{\sum_{k=2}^{m} (s-d_k) s^k}
\]

This completes our transformation of the Turing machine into a tag system with deletion number \( s \).

If for some reason one wants to avoid tag system rules that are exponentially long (in the length of the Turing machine’s tape’s periodicity), then instead of using one new symbol at the end of the tape to extend the tape by another period, one can use many new symbols, each of which extends the tape by a limited amount. Similarly, if one wants to avoid an initial tag system tape that is exponentially long (in the length of the initial Turing machine tape), new states can be added to the Turing machine for the sole purpose of writing the initial tape and positioning the Turing machine on it. Using these two methods, the entire compilation algorithm in this section takes only polynomial time in the size of the Turing machine’s initial configuration, and creates a Rule 110 initial state of polynomial size.
1.3 We further transform it into a cyclic tag system

Our next transformation will be to convert the tag system tape and rules into a tape and cyclic appendant sequence for a cyclic tag system.

Cyclic tag systems were invented as part of the proof of Rule 110’s universality, but they are also interesting in their own right. For example, Neary and Woods introduced their method in [2] by showing how a cyclic tag system can emulate a Turing machine. They simply cycle through a list of appendants as they read the tape, appending the current appendant to the end of the tape whenever a $Y$ is read.

To begin this transformation, we assign an ordering to the tag system alphabet $\Phi$: $\phi_1 = H_{\Psi_1}$, $\phi_2 = H_{\Psi_2}$, $\ldots$, $\phi_{4m+3ms} = R_{\Psi_m\sigma_s}$. Next, we extend $\Phi$ so that its size $|\Phi|$ becomes a multiple of 6, by adding anywhere from 0 to 5 dummy rules to the tag system, of the form: $\phi_{4m+3ms+1} \sim \emptyset$, $\phi_{4m+3ms+2} \sim \emptyset$, $\ldots$

Now we can create the cyclic tag system by listing the tag system’s rules in order, and converting each right hand side into a string of $Y$s and $N$s via a simple unary encoding where each $\phi_i$ becomes a string of $|\Phi|$ $N$s with the $i$th one changed to a $Y$: $\phi_i \mapsto [N]^{i-1} \ Y \ [N]^{|\Phi|-i}$

So for example the first rule, $H_{\Psi_1} \sim H_{\Psi_1\sigma_1}H_{\Psi_1\sigma_2} \ldots H_{\Psi_1\sigma_s}$ can be rewritten using the $\phi_i$s as $\phi_1 \sim \phi_{4m+1}\phi_{4m+2} \ldots \phi_{4m+s}$ whose right hand side gets converted to

$N^{4m} \ Y \ N^{|\Phi|-4m-1} \ N^{4m+1} \ Y \ N^{|\Phi|-4m-2} \ldots \ N^{4m+s-1} \ Y \ N^{|\Phi|-4m-s}$

which happens to simplify to

$N^{4m} \ Y \ [N^{|\Phi|} \ Y]^{s-1} \ N^{4m} \ Y^{s-2} \ldots \ Y^{s-4}$

and so this is the first appendant in the cyclic tag system’s cyclic list. The cyclic tag system’s cyclic list starts with the $|\Phi|$ appendants that can be generated in this way from the rules of the tag system, and then it is extended to a length of $s|\Phi|$ by simply adding $(s-1)|\Phi|$ empty appendants.

The initial tape for the cyclic tag system is simply the unary encoding of the tag system’s tape into $Y$s and $N$s. This completes our conversion of the tag system into a cyclic tag system with $s|\Phi|$ appendants.

1.4 We finally convert it into a Rule 110 state

Our final transformation will be to convert the cyclic tag system’s tape and appendants into an initial state for Rule 110. We will do this by simply gluing together bit sequences for the various glider clusters involved. We will start with the central tape region, and then we will specify the periodic sequences to its right and left.

We will start the central bit sequence with the row marked in figure [1] block $C$. We will first extend this row to the right by attaching other blocks. Each time we attach another block, we do it so that the zig-zag seam fits together perfectly, as if the two blocks were pieces of a large simple jigsaw puzzle, without worrying about whether the top or bottom edges of the blocks are aligned. Usually, the new block will not be able to be at exactly the same height as the previous block, but we simply need to make sure that the new block extends the $t = 0$ row marked in block $C$. This row will be the initial state for Rule 110. Programmatically, “attaching the blocks” is a very simple process: As we extend the initial state, we just keep track of which phase of the zig-zag the $t = 0$ row is at every time we cross from one block to another. This phase tells us which row to use from the new block, and then the phase of the zig-zag at the end of that same row in the new block becomes the phase for the next crossing.

The central region is formed fairly directly from the cyclic tag system’s tape: Each $N$ becomes $ED$, and each $Y$ becomes $FD$, but then the very last $D$ is changed to $G$, and $C$ is stuck on the front. So for example, a tape of $NNYN$ would become $CDED&DEG$.

Next, we form the periodic right hand side with a periodic sequence of blocks based on the cyclic tag system’s appendants. Each appendant from the cyclic tag system’s list is converted by changing each $Y$ to $\Pi$ and each $N$ to $\Xi$, but then the very first $\Pi$ is replaced with $KH$. If there is no first $\Xi$, due to
Figure 1: Blocks of bits used in generating the initial state for Rule 110. Two blocks are joined by simply fitting them together along the zig-zag edge so that the \( t = 0 \) row, defined in block \( C \), gets extended into the new block. Each block except for \( C \) is periodic: Blocks \( A \) and \( B \) repeat every 3 lines, and the other blocks repeat every 30 lines. The cyclic left hand side of Rule 110’s initial state is built with blocks \( A \) and \( B \). The central region is built with blocks \( C - G \).
Figure 2: More blocks of bits used in generating the initial state for Rule 110. The blocks in this figure are used to create the cyclic right hand side of Rule 110’s initial state based on the cyclic tag system’s appendant list. A program for creating the Rule 110 initial state would just store 30 strings of bits for each block, along with the vertical phase offset of the right hand zig-zag from the left hand one. Note that there is an $\bar{E}$ glider traveling right in the middle of every zig-zag region, so two blocks always join at an $\bar{E}$. 
the appendant being empty, then an \( L \) is used for that appendant. Once this is done for all appendants, the initial \( K \) of the first appendant is moved to the very end. So for example, the cyclic appendant list \( \{YN, NYYN, 0, 0\} \) would become \( HIIJKHIIIIIIJLLK \). This sequence of blocks gets repeated on the right hand side, and so the bit sequence for the \( t = 0 \) row must also be periodic, since after some number of repetitions of the sequence of blocks, the bit sequence will enter the same row of the initial block as it did at the very beginning, and so the bit sequence becomes periodic at that point.

The periodic left side has the form:

\[
[A]^{v}B[A]^{13}B[A]^{11}B[A]^{12}B
\]

where \( v = 76 \cdot (\text{the total number of } Y\text{s in all appendants}) + 80 \cdot (\text{the total number of } N\text{s in all appendants}) + 60 \cdot (\text{the number of nonempty appendants}) + 43 \cdot (\text{the number of empty appendants}) \)

Calculating the periodic sequence of bits for the \( t = 0 \) row on the left works just like calculating the sequence on the right, except that we work our way to the left from the \( t = 0 \) row in block \( C \), crossing over from block \( C \) to block \( B \), then twelve copies of block \( A \), and so on. The bit sequence will have to go through the \( [A]^{v}B[A]^{13}B[A]^{11}B[A]^{12}B \) block sequence three times before it starts repeating.

This completes our algorithm for transforming an arbitrary Turing machine into an initial state for Rule 110 consisting of a periodic sequence of bits on the left, followed by a central sequence, followed by a periodic sequence on the right.

### 1.5 Some comments on this algorithm

The blocks in figures \( \square \) and \( \square \) correspond to the conceptual clusters of gliders used in the construction in \( \square \).

For example, the \( \square \) assembly is an \textit{ossifier}, containing an \( A^4 \) in each \( B \), separated by pure \textit{ether} in the \( A \)s.

The \( E \) block, including the \( \bar{E} \)s at its seams, is a \textit{moving data} \( N \), while the \( F \) block is a \textit{moving data} \( Y \).

The \( D \) block glues adjacent elements of moving data together.

The remaining blocks each contain a glider cluster, extending from the first fully present glider to the \( \bar{E} \) in the right seam, so only the \( \bar{E} \) in the left seam is not a member of the cluster. The \( \bar{E} \) in the left seam is simply the last \( \bar{E} \) of the previous cluster, thus providing the position of the block’s cluster in relation to the previous cluster.

The \( H \) is a \textit{primary component}, while \( I \) and \( J \) are \textit{standard components}, differing only in the spacing from the previous component. The \( J \) uses more space, so that \( II \) encodes an \( N \) of \textit{table data}, while \( IJ \) encodes a \( Y \) of \textit{table data}.

The \( K \) block is a \textit{raw leader}, and the \( L \) block is a \textit{raw short leader}. The \( G \) block is a \textit{prepared leader}. There is no block shown for a \textit{prepared short leader}, since our algorithm assumed that the first appendant would not be empty. If the first appendant is empty, we would need a block for a prepared short leader, to be used in place of the \( G \). This block would be a modified form of \( L \) using exactly the same modification that turned the \( K \) into the \( G \).

The calculation of \( v \) for the periodic left hand side corresponds to a conservatively large rough estimate of twice the total vertical height of all the table data, including both components and leaders. This is used to set the vertical spacing between ossifiers so that ossifiers do not hit tape data by mistake. This assumes that at least one nonempty appendant will be appended on each cycle through the appendants if the system is not halting. This will indeed be the case if the system has been compiled from a Turing machine as described above, since the transformation to a tag system specifically ensures this property. If the system was compiled directly from a tag system or cyclic tag system, then an appropriate value of \( v \)
can be chosen if a bound can be placed on the number of consecutively rejected or empty appendants. If there is no such bound, then it will not be possible for this construction to work with a periodic left hand side. In this case, a more complicated left hand side will work, where the value of \(v\) increases linearly with each ossifier, since the length of the tape can only increase linearly with time, and clearly at any time the length of the tape constitutes a bound on the number of consecutively rejectable symbols. Note that if the entire tape is rejected before the next ossifier arrives, then there will be a collision between a prepared leader and an ossifier, as opposed to an ossifier hitting tape data as in figure 12(z).

The reader has probably noticed that we are encoding the tape as moving data. Although it might be more natural to encode it as tape data as described in [1], encoding it as moving data gives us some nice computational simplifications: Only two slopes of glider appear in the initial Rule 110 state, each glider only needs to be positioned relative to its immediate neighbors, and the moving data does not need to be reversed from the cyclic tag system tape.

The algorithm forced the tag system to have a size that is a multiple of six so that every appendant in the cyclic tag system would have a length that is a multiple of six. This is required so that the leader after a rejected appendant will hit the next symbol of tape data correctly (requiring the rejected appendant to have even length, as shown in figure 11(y)), producing invisibles that indeed pass through the ossifiers (requiring the rejected appendant to have length a multiple of three, as shown in figure 8(o)).

A cyclic tag system appendant in this construction may have a length that is not a multiple of six only if the cyclic tag system will always append that appendant. Otherwise the construction will not work properly. The safest approach is clearly to always use appendants whose length is a multiple of six.

### 1.6 Converting back to a Turing machine

We can continue with our series of conversions by converting the Rule 110 initial condition back into an initial state for a Turing machine tape. The benefit obtained from this big cycle of conversions is that the final Turing machine can be very small, since it only has to emulate Rule 110. The program, on the other hand, will have become much larger after going through the extensive compilation process in this cycle of conversions. Here we will show some Turing machines that can emulate Rule 110, and give an example for each of how it would emulate the evolution of the standard ether pattern. Encoding the Rule 110 initial state into an initial tape for these Turing machines is fairly direct and we will not discuss the details of these transformations.

Figures 3 and 4 show the lookup tables for the Turing machines of [1]. The captions give examples of initial tapes that allow the Turing machines to implement Rule 110 logic. Each machine emulates Rule 110 by sweeping back and forth over an ever wider stretch of tape, with each rightward sweep computing one more step of Rule 110’s activity.

More detailed initial tapes can be used to make the machines perform more detailed Rule 110 computations, for example by inserting gliders into a central ether region, or including gliders in the periodic portion. Note that the periodic patterns on the sides of the Turing machine’s tape correspond to periodic paths through both space and time on the sides of the Rule 110 evolution.

In contrast with the Turing machine of figure 3, the other three machines operate in such a way that the Rule 110 rows shift one cell to the right on the Turing machine tape with each simulated time step.

More recently, Neary and Woods [7] have achieved even smaller Turing machines that similarly emulate Rule 110.
Figure 3: This 2 state 5 symbol Turing machine can be started in state $S_0$ on the two-way infinite tape $[0^2 0^1 0^0]^\infty \neq [0^2 0^1 0^2]^\infty$, starting on the cell marked here with a box, and it will start computing Rule 110’s ether pattern. This Turing machine is slightly modified from the one in [1] so that fewer transitions are needed (the unused one is marked with “X”).

Figure 4: The 3 state 4 symbol Turing machine can be started in state $S_{00}$ on the tape $[0_L 1_R 0_R]^\infty [0_L 1_L 1_R 0_R 0_L]^\infty$ to compute the ether pattern. The 4 state 3 symbol Turing machine can be started in state $S_{00}$ on the tape $[B 0 1]^\infty [B 1 1 1 1 0 B]^\infty$ to compute the ether pattern. The 7 state 2 symbol Turing machine can be started in state $S_{11}$ on the tape $[1 1 0 0 1 1]^\infty [0 1 0 1 1 0 0 1 0]^\infty$, and it will compute the ether pattern on every second cell of the tape. This machine was compressed down to seven states by David Eppstein [8].
2 Towards a formal proof approach

The original exposition of the Rule 110 construction in [1] took a motivated approach to understanding how the construction could be made to work, by showing how some parts of the construction placed requirements on other parts of the construction, and then showing how the other parts had enough flexibility to satisfy those requirements. Only the inner workings of the components and leaders were treated as given without analysis. Of course, those could have been explained as well, as they were similarly designed by examining how to satisfy the relevant constraints, but this would have significantly lengthened the exposition.

In the present paper we take the converse approach, treating the entire construction as handed to us in a completely specified form, as given in section 1, for which all we have to do is check that as Rule 110 acts on the initial state, all of the ensuing collisions will be of the right form.

Each collision, whether between individual gliders or between clusters of gliders (which can be thought of as very large gliders), is completely determined by the spacing between its parts. In Rule 110’s two dimensional space-time, spacings can be completely specified in terms of over distance (“over distance”) and up distance (“up distance”), described in [1].

Our general approach is that each time there is a collision, we can determine the spacing between its parts by examining the spacings among all of the previous collisions bordering the ether region above the collision in question. If every such region can be shown to lead to the proper collision, then by induction, all the collisions will be correct.

All the collisions (at a glider cluster level) in this construction are between $\bar{E}$ material (leaders, table data, invisibles, moving data) on the right and either $C_2$ material (tape data) or $A$ material (ossifiers, acceptors, rejectors) on the left.

The slopes of the $A$ gliders and $C$ gliders, are directly related to the measurements that we need. When $C_2$ material hits $\bar{E}$ material, the correctness of the collision depends only on the $\parallel$ distance mod 4. When $A$ material hits $\bar{E}$ material, the correctness of the collision depends only on the $\vdash$ distance mod 6. The values are only important mod 4 and mod 6 because of the periodicity of the $\bar{E}$.

For most of the collisions, some participants in the collision have emerged from crossing collisions. In these cases, we measure the relevant distance back at the point when those gliders were first created, before all the crossing collisions, since the crossing collisions do not affect the relative distances. Only collisions with acceptors and rejectors are never preceded by crossing collisions.

This construction uses only the following collisions:

- ossifiers hitting moving data or invisibles
  - Measurements of these collisions are shown in figure 6.
  - The moving data is created with the correct spacing as shown in figure 7.
  - The invisibles are created with the correct spacing as shown in figure 8.

- tape data passing through moving data or invisibles
  - Measurements of these collisions are shown in figure 5.
  - The tape data is created with the correct spacing as shown in figure 6(e).
  - The moving data and invisibles are created with the correct spacing as shown in figure 10.

- an element of tape data hitting a prepared leader
  - Measurements of this collision are shown in figure 11(v).
  - The prepared leader is created with the correct spacing as shown in figure 11(w,x,y).
• an acceptor or rejector hitting table data
  – Measurements for these collisions are shown in figure 7.

• an acceptor or rejector hitting a raw leader
  – Measurements for these collisions are shown in figure 9.

We avoid an overly formal style in this presentation, but the general approach used here would be a good starting point for a complete formalization of the proof, a formalization of the sort that could be checked automatically by an automated proof checker. Most of the simpler spacing claims have essentially already been checked insofar as simulations of this construction appear to work with no problems. But of course most of the more general claims that appear cannot be verified in their full generality so trivially.

**Variations on the Construction**

Other than the $A^4 \to \bar{E}$ reaction used for ossification, shown in figure 6(e), there are two other reactions between an $A^4$ and an $\bar{E}$ that result in a $C_2$. One of them, in which the $\bar{E}$ is $\not\rightarrow$ from the $A^4$, is clearly unusable because it does not allow the ossification of figure 6(e) to work, since the $\sim$ distance between the new $C_2$ and the $\bar{E}$ is one, not zero. The other, in which the $\bar{E}$ is $\rightarrow$ from the $A^4$, does not have this problem. If it were used, then the leaders and primary components would need to have different internal arrangements. It turns out that, regardless of the details of these arrangements, the equation of invisibility after rejection, $4 + 1 + (2c - 1) \cdot 5 + 2 + 5 + 5 = 0$, from figure 8(m,o) and figure 6(g), which contains values that derive from nearly every aspect of the construction, would wind up being odd on one side and even on the other, and so the equation would be violated regardless of the value of $c$, and the construction would not work, regardless of any restrictions on the lengths of appendants. The construction as originally designed in February of 1994 suffered from this problem, which was discovered and fixed in the ensuing weeks. Note that [5] is mistaken when it says on page 1115 that some mistakes in the proof were corrected in 1998. In fact, the basic construction, completed in 1994, leaves a lot of flexibility in choosing many of the spacings (even within the leaders and components, whose internal design is not discussed in [1]), and what happened in 1998 was that a particular set of arrangements were chosen (for compactness) while writing a program to automatically generate a Rule 110 initial state corresponding to an arbitrary cyclic tag system (see [5], page 1116). Finally, in 1999, a standardized set of methods of measurement and analysis were chosen during the writing of the exposition of the 1994 construction, the publication of which (in [1]) was delayed until after the publication of [5].

The astute reader may notice that primary components (as well as short leaders) could be avoided if the acceptor and rejector could be produced $\not\rightarrow^2$ from where the leader currently produces them. The reader is invited to attempt this simplification, preferably without increasing the size of the construction.

3 A Polynomial Time Simulation

In [2], Neary and Woods solved the geometry problem described here on page 32 by using the following idea, which we present in a form adapted to tag systems. (Another adaptation to tag systems is given in Neary’s thesis [9].)

First of all, note that on a circular tape, it is sufficient for the Turing machine head to always move to the right. For example, the natural approach to simulating a cellular automaton works in this way: The
Figure 5: An $\bar{E}$ crossing a $C_2$. Vertical columns of dots represent the vertical columns of ether triangles used when measuring $\sim$ distance, which we treat as a value mod 4 due to the periodicity of the $\bar{E}$. Sequences of parallel short vertical columns of dots represent the marking of every fourth column of ether triangles for convenience of measurement. (a) Some basic measurements of the unique collision in which an $\bar{E}$ crosses a $C_2$. Note that the three measurements shown logically imply that the regenerated $C_2$ must be $\sim$ from the regenerated $\bar{E}$. (b) Putting together some measurements from (a), we see that two consecutive $\bar{E}$s can both cross a $C_2$ if and only if they are $\sim$ from each other. Note that the spacing between the $\bar{E}$s will be the same after the collisions as before the collisions, since each $\bar{E}$ undergoes an identical displacement in the collision. (c) Similarly, we see that consecutive $C_2$s can both cross an $\bar{E}$ if and only if they are $\sim_{1/3}$, which is $\sim_2$, from each other. Note that they will still be $\sim_2$ from each other after the collisions, since they each undergo the same displacement. (d) If we align several $C_2$s and $\bar{E}$s so that the first collisions are crossing collisions, then what will the remaining collisions be? After crossing the first $\bar{E}$, the $C_2$s are still $\sim_2$ from each other, so the second $\bar{E}$ will also cross them all, and indeed, all of the $\bar{E}$s will cross all of the $C_2$s.
Figure 6: How an $\bar{E}$ can hit an $A^4$. Vertical columns of dots are as in figure [5] Diagonal rows of dots represent the diagonal rows of ether triangles used when measuring $\nearrow$ distance, which we treat as a value mod 6 due to the periodicity of the $\bar{E}$. Sequences of parallel short diagonal rows of dots represent the marking of every sixth row of ether triangles for convenience of measurement. (e) There are three ways an $\bar{E}$ can hit an $A^4$ to produce a $C_2$, but only one works for the construction. It is the one where the $\bar{E}$ is $\nearrow 5$ from the $A^4$. The resulting $C_2$ is $\nearrow 0$ from the $\bar{E}$ that created it. (f) An $\bar{E}$ will cross an $A^4$ if and only if it is $\nearrow 0$ from the $A^4$. The crossing causes a visually striking but irrelevant displacement (not indicated here) to the $A^4$. The incoming $A^4$ is $\nearrow 5$ from the regenerated $\bar{E}$. We see that consecutive $A^4$'s must be $\nearrow$ from each other if they are both to cross an $\bar{E}$, and the construction always uses such a spacing between $A^4$'s. Every $\bar{E}$ will either pass through all the $A^4$'s (we call such an $\bar{E}$ an “invisible”), or else it is “moving data” and will be converted (“ossified”) into a $C_2$ (“tape data”) by the first $A^4$ (“ossifier”) it hits, as in (e). (g) After an invisible $\bar{E}$, the $\nearrow$ distance to the next $\bar{E}$ determines whether it is another invisible or ossifiable moving data. (h) After a moving data $\bar{E}$, the $\nearrow$ distance to the next $\bar{E}$ determines whether it is more moving data or an invisible.
Figure 7: Primary (pri) and standard (std) components will get processed correctly by either an acceptor (acc) or a rejector (rej). An acceptor converts components into moving data (md), whereas a rejector deletes components. The acceptor or rejector is produced by a prepared leader (prep) hitting a character of tape data (td), which produces a pair of invisibles (inv) as well as the acceptor or rejector. Note that lines in these diagrams represent clusters of parallel gliders, and the collisions between clusters are marked with a circle representing the many collisions that occur where the clusters meet. A measurement to or from a cluster is made to or from the closest glider in the cluster, namely the one that touches the ether in which the measurement is being made.
Figure 8: Invisibles are the result of a prepared leader hitting an element of tape data. (m) The alignment of the invisibles \((k + 5)\) depends on the alignment \((k)\) of the prepared leader. The incoming invisible or moving data was originally produced as in (o) or (n). (n) When invisibles follow moving data, the leader was prepared by an acceptor, and the prepared leader’s offset from the moving data is \(k = 0\). As shown in (m), this gives the invisibles an alignment of \(k + 5 = 5\), which will yield the correct interaction with the ossifiers, as shown in figure 6(h). (o) When invisibles follow previous invisibles, the leader was prepared by a rejector. If \(c\) characters of table data were rejected, then \(k = 4 + 1 + (2c - 1) \cdot 5 + 2 = 10c + 2\). Since \(c\) is a multiple of 6, we get \(k = 2\). In the special case \(c = 0\), the previous leader was a short leader, and this figure does not apply, but we still get \(k = 2\) as shown in figure 9(r). Either way, the invisibles are produced as in (m) with an alignment of \(k + 5 = 1\), yielding the correct collision with the ossifiers as shown in figure 6(g).
Figure 9: Raw leaders are placed \( \mathcal{A}_0 \) from the previous component, which yields the correct alignment when prepared by either an acceptor (p) or a rejector (q). If there were no previous components, then the raw leader is placed \( \mathcal{A}_0 \) from the previous short leader (r) (whose lower right gliders are untouched by the preparation), again yielding the correct alignment for preparation. Raw leaders that come after short leaders yield prepared leaders aligned as in (r), with \( k = 2 \). This is the zero-component version of figure 8(o). Note that raw short leaders, due to the different position of their initial \( E \), are always placed \( \mathcal{A}_{k+3} \) higher, as measured through the \( E^n \)'s, than the raw regular leaders shown here.
Figure 10: Moving data and invisibles are created with the correct alignment to pass through tape data. The top collision in (s) is also the top collision in (t), but (t) applies even when a rejector is produced. The lower collision in (s) is also the top collision in (u), but (u) applies also to later components, where the lower collision in (u) becomes the top collision in (u). (s) The first moving data after an invisible has correct alignment to pass through tape data, as in figure 5(b). (t) An invisible always has correct alignment to pass through tape data, as in figure 5(a,c). (u) Moving data after previous moving data is also aligned correctly for passing through tape data, as in figure 5(b).
Figure 11: A leader always gets prepared, as in (w), (x), or (y), so that it will be positioned correctly, as in (v), to read an element of tape data. In (y), since $c$ is a multiple of 6, the total distance mod 4 is $1 + 0 + 3 + (2c - 1) \cdot (0 + 3) + 0 = 6c + 1 = 1$, as needed in (v).
Figure 12: An $F$ is produced if and only if the Turing machine halts. The tag system in section 1 is set up so that when the Turing machine halts, no more appendants are produced for the tape. This will cause ossifier $A^4$s to eventually hit a character of tape data, which leads to the production of an $F$ as shown.

Turing machine can store the local configuration of the tape in its finite state, and can read in the old configuration and write out the new configuration in a single pass. Extra cells can be written onto the tape at the wrap-around point as necessary, allowing it to grow.

So, when the simulated Turing machine head is at a certain position on the tag system tape, what we would like to do is to find the next position on the tape to the right of the current position. The tag system can do this not in a single pass, but in a series of $\log(n)$ cycles, where each cycle consists of four passes. On the first cycle, it eliminates every second possibility. On the next cycle, it eliminates every second remaining possibility, and so on. After $\log(n)$ cycles, the only remaining possibility is the first one: the next position on the tape.

So that the system can know when $\log(n)$ cycles have occurred, it keeps on its tape a power of 2 that is larger than $n$, called the counter. It performs this same halving-of-possibilities process on the counter, where it is able to detect the end of the process since then for the first time there will be an odd number of possibilities remaining.

When the tape grows in length past a power of two, then the counter must be doubled in size. This condition is detected by keeping track of a “growth flag” which is set at the beginning of each series of cycles, and then if the tape is ever noticed to have a non-power-of-two length, then the flag is reset. If the flag is still set at the end of the series of cycles, and the tape is being extended, then the counter size is doubled.

In tag systems, one must take care that the counter behaves properly when the Turing machine “head” jumps across it, since then the counter gets processed twice before the tag system head is at the beginning of the tape again.

In the tag system, we keep all sorts of information in the symbols, using a large alphabet, since the communication bandwidth between symbols is very low (a single bit or less per pass). For example, every symbol on the tape knows the current state of the Turing machine and the current stage of the simulation algorithm. We use six stages, and each stage does one pass over the tape.
We will use the following symbols:

- For the head: H, h, P, Q
  
P is used for a head remembering “A”, Q for “B”.

- For the counter: U u X x V v Y y
  
X is an eliminated U, and V and Y are a form of U and X that represent the growth flag having been reset.

- For the tape: A a B b C c D d
  
A and B represent the two values used on the binary Turing machine tape. C and D are the eliminated forms of A and B.

The lower-case letters are used when we need to be able to detect which of the two positions is being read or ignored.

If the Turing machine has \( k \) states, then there are actually \( 6k \) symbols for each of the above symbols. We could write these for example as \( H_{4,7} \) being the \( H \) symbol that is used in stage 4 when the Turing machine is in state 7, but usually we will omit the indices, to avoid clutter. Not all of the \( 6k \) symbols are used, since not all of the letters are used in all stages.

For clarity, we will also use “-” as a symbol, to be used in positions where we know it will not be read, and we will use “0” as a symbol with an empty appendant. These two symbols do not need any subscripts.

### Stages

There is a cycle of four stages, with stage 4 going back to stage 1 until the first tape symbol has been isolated, at which point stage 4 goes on to stage 5 and then 6. Stage 6 performs the simulation of a step of the Turing machine and jumps back to stage 3 to start isolating the next tape symbol.

We will start in stage 2, because it has the simplest form. At the front is Hh, then the tape is some combination of AA and BB, and somewhere there is a power of 2 of Uu’s, at least as long as the A/B portion of the tape.

We will explain the stages by following an example.

### Stage 2

Example: H h A A U u U u U u U u B B A A (Turing machine tape is “A,B,A”)

Incoming parity: If reading second symbol of each pair, then the growth flag is unset.

Outgoing parity: Start with first symbol of each pair.

Main change: Cut number of U’s in half, and check whether counter size should be doubled.

Transfer to stage 3:

\[
\begin{align*}
H & \rightarrow H - \\
h & \rightarrow - H - \\
A & \rightarrow A A \\
B & \rightarrow B B \\
C & \rightarrow C C \\
D & \rightarrow D D \\
U & \rightarrow U \\
u & \rightarrow V \\
X & \rightarrow X X
\end{align*}
\]
Stage 3

Example: H - A A U U U B A A
Incoming parity: Start with first symbol of each pair.
Outgoing parity: If reading second symbols, then there was just one U.
Main change: Change odd U’s (first, third, etc.) to UuXx, erase even U’s.
Transfer to stage 4:

\[
\begin{align*}
H * &: H h \\
A * &: A a \\
B * &: B b \\
C * &: C c \\
D * &: D d \\
U * &: U u X x \\
V * &: V v Y y \\
X * &: X x \\
Y * &: Y y
\end{align*}
\]

Stage 4

Example: H h A a U u X x U u X x B b A a
Incoming parity: If reading small letters, then first tape symbol is isolated, and we will go on to stage 5.
Otherwise we go back to stage 1.
Outgoing parity: Start with the first symbol of each pair.
Main change to stage 1: Change A to Aa0, and B to Bb0.
Transfer to stage 1:

\[
\begin{align*}
H * &: H h \\
A * &: A a 0 \\
B * &: B b 0 \\
C * &: C C \\
D * &: D D \\
U * &: U U \\
V * &: V v \\
X * &: X X \\
Y * &: Y Y
\end{align*}
\]

Stage 1

Example: H - A a 0 U U X U X U X X B b 0 A a 0
Incoming parity: We read the H.
Outgoing parity: If there were an even (resp. odd) number of A’s and B’s, then we will read the first (resp. second) symbol of each pair.
Main change: Every second A or B is turned into a C or D.
Transfer to stage 2:

H * : H h
A * : A A
a * : C C
B * : B B
b * : D D
C * : C C
D * : D D
U * : U u
V * : V V
X * : X x
Y * : Y Y

Stage 2

Example: H h A A U u X x U u X x D D A A (read h)

Stage 3

Example: - H - A A V Y Y Y Y D D A A (read H)

Stage 4

Example: H h A a V v Y y Y y Y y Y y D d A a (read H)

Stage 1

Example: H - A a 0 V V Y Y Y Y Y Y D D A a 0 (read H)

Stage 2

Example: H h A A V V Y Y Y Y Y Y Y Y D D C C (read H)

Stage 3

Example: H - A A V Y Y Y Y Y Y Y Y D D C C (read H)

Stage 4

Example: H h A a V v Y y Y y Y y Y y Y y Y y D d C c (read h)

Main change to stage 5: H and U disappear. Isolated A/B becomes P/Q. C/D turn back into A/B. X/Y becomes a mixed-stage form UX/VY. The “head” has jumped from the H to the P/Q, so it has jumped onto the symbol it is reading. This may cause it to jump over the counter (the X’s/Y’s), which will be discussed below.
Transfer to stage 5:
Stage 5

Example: - P - V y V y V y V y V y B - A - (read P)
Incoming parity: We read the P or Q.
Outgoing parity: Read first symbols if P, read second symbols if Q.
Main change: x and y “stage 4” symbols disappear (more on them below), and parity is set to reflect P vs. Q.
Transfer to stage 6:

P * : P -
Q * : Q
A * : A a
B * : B b
U * : U u
V * : V v

Stage 6

Example: P - V v V v V v V v V v V v B b A a (read P)
Incoming parity: If P at front, will read first of each pair. If Q at front, will read second of each pair.
Outgoing parity: Will read first of each pair.
Main change: A step of the Turing machine is simulated. The state (invisible subscript) changes. The new symbols get written onto the tape. V’s turn back into U’s (i.e. the growth flag is reset). The counter size (number of U’s) doubles if necessary. The counter size is also halved for direct entry to stage 3.
Transfer to stage 3:

P * : [A a B b] H -
Q * : [A a B b] - H -
A * : A A (new state when reading an A)
a * : A A (new state when reading a B)
B * : B B (new state when reading an A)
b * : B B (new state when reading a B)
U * : U [U] (new state when reading an A)
u * : U [U] (new state when reading a B)
V * : U (new state when reading an A)
v * : U (new state when reading a B)
Notes regarding the transfer: The \([A \ a \ B \ b]\) portions depend on what symbol(s) are being written, and they are written with subscripts for stage 6, and the current (old) state (and they will be the last thing processed at the end of this transfer), while the H (and everything below) gets written with the new state and stage 3.

The \([U]\) portions are included only when the corresponding Turing machine transition writes 2 symbols. This is where the length of the counter gets doubled.

**Stage 3**

Example: \(H - U U U U B B A A B B B B\)

**Stage 4**

Example: \(H h U u X x U u X x B b A a B b B b\)

**Stage 1**

Example: \(H - U U X X U U X X B b 0 A a 0 B b 0 B b 0\)

**Stage 2**

Example: \(H h U u X x U u X x B B C C B B D D\)

**Stage 3**

Example: \(H - U X X U X B B C C B B D D\)

**Stage 4**

Example: \(H h U u X x X x X x B b C c B b D d\)

**Stage 1**

Example: \(H - U U X X X X X X B b 0 C C B b 0 D D\)

**Stage 2**

Example: \(H h U u X x X x X x B B C C D D D D\)

**Stage 3**

Example: \(H - U X X X X X X B B C C D D D D\)

**Stage 4**

Example: \(H h U u X x X x X x X x B b C c D d D d (\text{read} \ h)\)

Now we will see what the mixed-stage stuff is doing, where the stage-4 “x” turns into a “U x” pair with the U in stage 5 but the x in stage 4. Recall that h and u both disappear.
Stage 4.5
Example: U x U x U x U x - Q - A - B - B - (read x)

Now we see that that the stage-5 U’s will get ignored as stage 4 finishes by processing the counter a second time.

Stage 5
Example: - Q - A - B - B - U x U x U x U x

And we see that this time the stage-4 x’s will get ignored during the stage-5 processing. Stage 5 doesn’t do much because it is mostly serving as a signal to the counter regarding what stage is happening. The counter knows it is still stage 4 while the second symbols are getting read, and the counter knows stage 5 has arrived when the first symbols get read. So stage 5 has to have a fixed parity during its processing, making it a boring stage.

Stage 6
Example: Q A a B b B b U u U u U u U u U u

And so on... The counter will double in size on the next step if necessary. It works well in simulation.

As mentioned in [10], the fact that tag systems can compute efficiently means that all known small universal Turing machines work in polynomial time.

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