Classification of Asymptotic Behavior in A Stochastic SIR Model

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Abstract

This paper investigates asymptotic behavior of a stochastic SIR epidemic model, which is a system with degenerate diffusion. It gives sufficient conditions that are very close to the necessary conditions for the permanence. In addition, this paper develops ergodicity of the underlying system. It is proved that the transition probabilities converge in total variation norm to the invariant measure. Our result gives a precise characterization of the support of the invariant measure. Rates of convergence are also ascertained. It is shown that the rate is not too far from exponential in that the convergence speed is of the form of a polynomial of any degree.

Keywords. SIR model; Extinction; Permanence; Stationary Distribution; Ergodicity.

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1 Introduction

Since epidemic models were first introduced by Kermack and McKendrick in [15, 16], the study on mathematical models has been flourished. Much attention has been devoted to analyzing, predicting the spread, and designing controls of infectious diseases in host populations; see [1, 2, 4, 6, 8, 18, 19, 15, 16, 26, 27] and the references therein. One of classic epidemic models is the SIR (Susceptible-Infected-Removed) model that is suitable for modeling some diseases with permanent immunity such as rubella, whooping cough, measles, smallpox, etc. In the SIR model, a homogeneous host population is subdivided into three epidemiologically distinct types of individuals:

- (S): The susceptible class, the class of those individuals who are capable of contracting the disease and becoming infective,
- (I): the infective class, the class of those individuals who are capable of transmitting the disease to others,
- (R): the removed class, the class of infected individuals who are dead, or have recovered, and are permanently immune, or are isolated.

If we denote by $S(t), I(t), R(t)$ the number of individuals at time $t$ in classes (S), (I), and (R), respectively, the spread of infection can be formulated by the following deterministic system of differential equations:

$$
\begin{align*}
\frac{dS(t)}{dt} &= (\alpha - \beta S(t)I(t) - \mu S(t))dt \\
\frac{dI(t)}{dt} &= (\beta S(t)I(t) - (\mu + \rho + \gamma)I(t))dt \\
\frac{dR(t)}{dt} &= (\gamma I(t) - \mu R(t))dt,
\end{align*}
$$

where $\alpha$ is the per capita birth rate of the population, $\mu$ is the per capita disease-free death rate and $\rho$ is the excess per capita death rate of infective class, $\beta$ is the effective per capita contact rate, and $\gamma$ is per capita recovery rate of the infective individuals. On the other hand, it is well recognized that the population is always subject to random disturbances and it is desirable to learn how randomness effects the models. Thus, it is important to investigate stochastic epidemic models. Jiang et al. [13] investigated the asymptotic behavior of global positive solution for the non-degenerate stochastic SIR model

$$
\begin{align*}
\frac{dS(t)}{dt} &= (\alpha - \beta S(t)I(t) - \mu S(t))dt + \sigma_1 S(t)dB_1(t) \\
\frac{dI(t)}{dt} &= (\beta S(t)I(t) - (\mu + \rho + \gamma)I(t))dt + \sigma_2 I(t)dB_2(t) \\
\frac{dR(t)}{dt} &= (\gamma I(t) - \mu R(t))dt + \sigma_3 R(t)dB_3(t),
\end{align*}
$$

(1.2)
where $B_1(t)$, $B_2(t)$, and $B_3(t)$ are mutually independent Brownian motions, $\sigma_1, \sigma_2, \sigma_3$ are the intensities of the white noises. However, in reality, the classes (S), (I), and (R) are usually subject to the same random factors such as temperature, humidity, pollution and other extrinsic influences. As a result, it is more plausible to assume that the random noise perturbing the three classes is correlated. If we assume that the Brownian motions $B_1(t)$, $B_2(t)$, and $B_3(t)$ are the same, we obtain the following model

$$
\begin{align*}
\frac{dS(t)}{dt} &= (\alpha - \beta S(t)I(t) - \mu S(t))dt + \sigma_1 S(t)dB(t) \\
\frac{dI(t)}{dt} &= (\beta S(t)I(t) - (\mu + \rho + \gamma)I(t))dt + \sigma_2 I(t)dB(t) \\
\frac{dR(t)}{dt} &= (\gamma I(t) - \mu R(t))dt + \sigma_3 R(t)dB(t),
\end{align*}
$$

which has been considered in [20]. Compared to (1.2), (1.3) is more difficult to deal with due to the degeneracy of the diffusion. One of the important questions is concerned with whether the transition to a disease free state or the disease state will survive permanently. For the deterministic model (1.1), the asymptotic behavior has been classified completely as follows. If $\lambda_d = \frac{\beta \alpha}{\mu} - (\mu + \rho + \gamma) \leq 0$, then the population tends to the disease-free equilibrium $(\alpha \mu, 0, 0)$ while the population approaches an endemic equilibrium in case $\lambda_d > 0$. In [27], similar results are given for a general epidemic model with reaction-diffusion in terms of basic reproduction numbers. In [20], the authors attempted to answer the aforementioned question for (1.3) in case $\sigma_1 > 0$ and $\sigma_2 > 0$. By using Lyapunov-type functions, they provided some sufficient conditions for extinction or permanence as well as ergodicity for the solution of system (1.3). Using the same methods, the extinction and permanence in some different stochastic SIR models have been studied in [14, 28] etc. In practice, because of the randomness and the degeneracy of the diffusion, the model is much more difficult to deal with compared in contrast to the deterministic counter part. Moreover, although one may assume the existence of appropriate Lyapunov function, it is fairly difficult to find an effective Lyapunov function in practice. In other words, there has been no decisive classification for stochastic SIR models that is similar to the deterministic case.

Our main goal in this paper is to provide such a classification. We shall derive a sufficient and almost necessary condition for permanence (as well as ergodicity) and extinction of the disease for the stochastic SIR model (1.3) by using a value $\lambda$, which is similar to $\lambda_d$ in the deterministic model. Note that such kind of results are obtained for a stochastic SIS model in [9]. However, the model studied there can be reduced to one-dimensional equation that is much easier to investigate. The method used in [9] cannot treat the stochastic SIR model.
(1.3). Estimation for the convergence rate is also not given in [9]. A more general method therefore need to be introduced. The new method can remove most assumptions in [20] as well as can treat the case $\sigma_1 > 0, \sigma_2 < 0$, which has not been taken into consideration in [20]. Note that the case $\sigma_1 > 0$ and $\sigma_2 < 0$ indicates the random factors have opposite effects to healthy individuals and infected ones. For instance, patients with tuberculosis or some other pulmonary disease do not endure well in cool and humid weather while healthy people may be fine in such kind of weather. In addition, individuals with a disease, usually have weaker resistance to some other kinds of disease. Our new method is also suitable to deal with other stochastic variants of (1.1) such as models introduced in [5, 14, 28], etc.

The rest of the paper is arranged as follows. Section 2 derives a threshold that is used to classify the extinction and permanence of the disease. To establish the desired result, by considering the dynamics on the boundary, we obtain a threshold $\lambda$ that enables us to determine the asymptotic behavior of the solution. In particular, it is shown that if $\lambda < 0$, the disease will decay in an exponential rate. In case $\lambda > 0$, the solution converges to a stationary distribution in total variation. It means that the disease is permanent. The rate of convergence is proved to be bounded above by any polynomial decay. The ergodicity of the solution process is also proved. Finally, Section 3 is devoted to some discussion and comparison to existing results in the literature. Some numerical examples are provided to illustrate our results.

2 Threshold Between Extinction and Permanence

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual condition, i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $B(t)$ be an $\mathcal{F}_t$-adapted, Brownian motions. Because the dynamics of class of recover has no effect on the disease transmission dynamics, we only consider the following system:

\[
\begin{align*}
    dS(t) &= [\alpha - \beta S(t)I(t) - \mu S(t)]dt + \sigma_1 S(t)dB(t), \\
    dI(t) &= [\beta S(t)I(t) - (\mu + \rho + \gamma)I(t)]dt + \sigma_2 I(t)dB(t).
\end{align*}
\]

(2.1)

Assume that $\sigma_1, \sigma_2 \neq 0$. By the symmetry of Brownian motions, without loss of generality, we suppose throughout this paper that $\sigma_1 > 0$. Using standard arguments, it can be easily shown that for any positive initial value $(u, v) \in \mathbb{R}^2_+ := \{(u', v') : u' > 0, v' > 0\}$, there exists uniquely a global solution $(S_{u,v}(t), I_{u,v}(t)), t \geq 0$ that remains in $\mathbb{R}^2_+$ with probability
To obtain further properties of the solution, we first consider the equation on the boundary,
\[ d\hat{S}(t) = (\alpha - \mu \hat{S}(t))dt + \sigma_1 \hat{S}(t)dB(t). \] (2.2)

Let \( \hat{S}_u(t) \) be the solution to (2.2) with initial value \( u \). It follows from the comparison theorem [11, Theorem 1.1, p.437] that \( S_{u,v}(t) \leq \hat{S}_u(t) \) \( \forall t \geq 0 \) a.s. By solving the Fokker-Planck equation, the process \( \hat{S}_u(t) \) has a unique stationary distribution with density
\[ f^*(x) = \frac{b^a}{\Gamma(a)} x^{-(a+1)} e^{-\frac{b}{x}}, x > 0 \] (2.3)

where \( c_1 = \mu + \frac{\sigma_1^2}{2}, a = \frac{2\alpha_1}{\sigma_1^2}, b = \frac{2\alpha}{\sigma_1^2} \) and \( \Gamma(\cdot) \) is the Gamma function. By the strong law of large number we deduce that
\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \hat{S}_u(s)ds = \int_0^\infty x f^*(x)dx := \frac{\alpha}{\mu} \text{ a.s.} \] (2.4)

To proceed, we define the threshold as follows:
\[ \lambda := \frac{\alpha \beta}{\mu} - \left( \mu + \rho + \gamma + \frac{\sigma_2^2}{2} \right). \] (2.5)

### 2.1 Case 1: \( \lambda < 0 \)

**Theorem 2.1.** If \( \lambda < 0 \), then for any initial value \( (S(0), I(0)) = (u, v) \in \mathbb{R}_+^2 \) we have
\[ \limsup_{t \to \infty} \frac{1}{t} \ln I_{u,v}(t) \leq \lambda \text{ a.s.} \] and the distribution of \( S_{u,v}(t) \) converges weakly to the unique invariant probability measure \( \mu^* \) with the density \( f^* \).

**Proof.** Let \( \hat{I}_v(t) \) be the solution to
\[ d\hat{I}(t) = \hat{I}(t) \left( - (\mu + \rho + \gamma) + \beta \hat{S}_u(t) \right) dt + \sigma_2 \hat{I}(t) dB(t), \quad \hat{I}(0) = v. \]

where \( \hat{S}(t) \) is the solution to (2.2). By comparison theorem, \( I_{u,v}(t) \leq \hat{I}_v(t) \) a.s. given that \( \hat{S}(0) = S(0) = u, I(0) = \hat{I}(0) = v \). In view of the Itô formula and the ergodicity of \( \hat{S}_u(t) \),
\[ \limsup_{t \to \infty} \frac{1}{t} \ln \hat{I}_v(t) = \limsup_{t \to \infty} \left( \frac{1}{t} \int_0^t \left( - (\mu + \rho + \gamma + \frac{\sigma_2^2}{2}) + \beta \hat{S}_u(\tau) \right) d\tau + \frac{\sigma_2 B(t)}{t} \right) \]
\[ = \frac{\alpha \beta}{\mu} - \left( \mu + \rho + \gamma + \frac{\sigma_2^2}{2} \right) = \lambda < 0 \text{ a.s.} \] (2.6)

That is, \( I_{u,v}(t) \) converges almost surely to 0 at an exponential rate.

For any \( \varepsilon > 0 \), it follows from (2.6) that there exists \( t_0 > 0 \) such that \( \mathbb{P}(\Omega_\varepsilon) > 1 - \varepsilon \) where
\[ \Omega_\varepsilon := \left\{ I_{u,v}(t) \leq \exp \left\{ \frac{\lambda t}{2} \right\} \forall t \geq t_0 \right\} = \left\{ \ln I_{u,v}(t) \leq \frac{\lambda t}{2} \forall t \geq t_0 \right\}. \]
Clearly, we can choose \( t_0 \) satisfying \(-\frac{2\beta}{\lambda} \exp\left\{ \frac{\lambda t_0}{2} \right\} < \varepsilon \). Let \( \hat{S}_u(t) \), \( t \geq t_0 \) be the solution to (2.2) given that \( \hat{S}(t_0) = S(t_0) \). We have from the comparison theorem that \( \mathbb{P}\{S_{u,v}(t) \leq \hat{S}_u(t) \; \forall t \geq t_0\} = 1 \). In view of the Itô formula, for almost all \( \omega \in \Omega_{\varepsilon} \) we have

\[
0 \leq \ln \hat{S}_u(t) - \ln S_{u,v}(t) = \alpha \int_{t_0}^{t} \left[ \frac{1}{\hat{S}_u(\tau)} - \frac{1}{S_{u,v}(\tau)} \right] d\tau + \beta \int_{t_0}^{t} I_{u,v}(\tau) d\tau.
\]

\[
\leq \beta \int_{t_0}^{t} \exp\left\{ \frac{\lambda \tau}{2} \right\} d\tau = -\frac{2\beta}{\lambda} \left( \exp\left\{ \frac{\lambda t_0}{2} \right\} - \exp\left\{ \frac{\lambda t}{2} \right\} \right) < \varepsilon.
\]

As a result,

\[
\mathbb{P}\{|\ln S_{u,v}(t) - \ln \hat{S}_u(t)| > \varepsilon\} \leq 1 - \mathbb{P}(\Omega_{\varepsilon}) < \varepsilon \; \forall t \geq t_0.
\] (2.7)

Let \( \nu^* \) be the distribution of a random variable \( \ln X \) provided that \( X \) admits \( \mu^* \) as its distribution. In lieu of proving that the distribution of \( S(t) \) converges weakly to \( \mu^* \), we claim an equivalent statement that the distribution of \( \ln S(t) \) converges weakly to \( \nu^* \). By the Portmanteau theorem (see [3, Theorem 1]), it is sufficient to prove that for any \( g(\cdot) : \mathbb{R} \mapsto \mathbb{R} \) satisfying \( |g(x) - g(y)| \leq |x - y| \) and \( |g(x)| < 1 \; \forall x, y \in \mathbb{R} \), we have

\[
\mathbb{E}g(\ln S_u(t)) \to \overline{g} := \int_{\mathbb{R}} g(x) \nu^*(dx) = \int_{0}^{\infty} g(\ln x) \mu^*(dx).
\]

Since the diffusion given by (2.2) is non-degenerate, it is well known that the distribution of \( \hat{S}_u(t) \) weakly converges to \( \mu^* \) as \( t \to \infty \) (see e.g., [10]). Thus

\[
\lim_{t \to \infty} \mathbb{E}g(\ln \hat{S}_u(t)) = \overline{g}.
\] (2.8)

Note that

\[
|\mathbb{E}g(\ln S_{u,v}(t)) - \overline{g}| \leq \mathbb{E}|g(\ln S_{u,v}(t)) - g(\ln \hat{S}_u(t))| + \mathbb{E}|g(\ln \hat{S}_u(t)) - \overline{g}|
\]

\[
\leq \varepsilon \mathbb{P}\{|\ln S_{u,v}(t) - \ln \hat{S}_u(t)| \leq \varepsilon\} + 2\mathbb{P}\{|\ln S_{u,v}(t) - \ln \hat{S}_u(t)| > \varepsilon\}
\]

\[
+ \mathbb{E}|g(\ln \hat{S}_u(t)) - \overline{g}|.
\] (2.9)

Applying (2.7) and (2.8) to (2.9) yields

\[
\limsup_{t \to \infty} \mathbb{E}|g(\ln S_{u,v}(t)) - \overline{g}| \leq 3\varepsilon.
\]

Since \( \varepsilon \) is taken arbitrarily, we obtain the desired conclusion. The proof is complete. □

### 2.2 Case 2: \( \lambda > 0 \)

We now focus on the case \( \lambda > 0 \). Let \( P(t,(u,v),\cdot) \) be the transition probability of \( (S_{u,v}(t),I_{u,v}(t)) \). To obtain properties of \( P(t,(u,v),\cdot) \), we first rewrite equation (2.1) in Stratonovich’s form

\[
\begin{cases}
    dS(t) = [\alpha - c_1 S(t) - \beta S(t) I(t)] dt + \sigma_1 S(t) \circ dB(t), \\
    dI(t) = [-c_2 I(t) + \beta S(t) I(t)] dt + \sigma_2 I(t) \circ dB(t).
\end{cases}
\] (2.10)
where $c_1 = \mu + \sigma_1^2/2$, $c_2 = \mu + \rho + \gamma + \sigma_2^2/2$. Put

$$A(x, y) = \begin{pmatrix} \alpha - c_1x - \beta xy \\ -c_2y + \beta xy \end{pmatrix} \quad \text{and} \quad B(x, y) = \begin{pmatrix} \sigma_1x \\ \sigma_2y \end{pmatrix}. $$

To proceed, we first recall the notion of Lie bracket. If $\Phi(x, y) = (\Phi_1, \Phi_2)^\top$ and $\Psi(x, y) = (\Psi_1, \Psi_2)^\top$ are vector fields on $\mathbb{R}^2$ then the Lie bracket $[\Phi, \Psi]$ is a vector field given by

$$[\Phi, \Psi](x, y) = \left( \Phi_1 \frac{\partial \Psi_j}{\partial x}(x, y) \right) - \left( \Psi_1 \frac{\partial \Phi_j}{\partial x}(x, y) \right), \quad j = 1, 2. $$

Denote by $\mathcal{L}(x, y)$ the Lie algebra generated by $A(x, y), B(x, y)$ and $\mathcal{L}_0(x, y)$ the ideal in $\mathcal{L}(x, y)$ generated by $B$. We have the following lemma.

**Lemma 2.1.** For $\sigma_1 > 0, \sigma_2 \neq 0$, the Hörmander condition holds for the diffusion (2.10). To be more precise, we have $\dim \mathcal{L}_0(x, y) = 2$ at every $(x, y) \in \mathbb{R}_+^2$ or equivalently, the set of vectors $B, [A, B], [A, [A, B]], [B, [A, B]], \ldots$ spans $\mathbb{R}^2$ at every $(x, y) \in \mathbb{R}_+^2$. As a result, the transition probability $P(t, (u, v), \cdot)$ has smooth density $p(t, u, v, u', v')$.

**Proof.** This lemma has been proved in [20] for the case $\sigma_2 > 0$. Assume that $r = -\frac{\sigma_2}{\sigma_1} > 0$. It is easy to obtain

$$C := \frac{1}{\sigma_1} B(x, y) = \begin{pmatrix} x \\ -ry \end{pmatrix}, $$

$$D := [A, C](x, y) = \begin{pmatrix} \alpha - r\beta xy \\ -\beta xy \end{pmatrix}, $$

$$E := [C, D](x, y) = \begin{pmatrix} -\alpha + r^2\beta xy \\ -\beta xy \end{pmatrix}, $$

$$F := [C, E](x, y) = \begin{pmatrix} \alpha - r^3\beta xy \\ -\beta xy \end{pmatrix}. $$

Since $\det(D, F) = 0$ only if $r^2 = 1$ or $r = 1$ (since $r > 0$). When $r = 1$, solving $\det(D, E) = 0$ obtains $\beta xy = \alpha$ which implies

$$\det(C, D) = \begin{vmatrix} x & 0 \\ -y & -\alpha \end{vmatrix} \neq 0. $$

As a result, $B, D, E, F$ span $\mathbb{R}^2$ for all $(x, y) \in \mathbb{R}_+^2$. The lemma is proved. 

In order to describe the support of the invariant measure $\pi^*$ (if it exists) and to prove the ergodicity of (2.1), we need to investigate the following control system on $\mathbb{R}^2_+$

$$\begin{align*}
\dot{u}_\phi(t) &= \sigma_1 u_\phi(t) \phi(t) + \alpha - \beta u_\phi(t) v_\phi(t) - c_1 u_\phi(t), \\
\dot{v}_\phi(t) &= \sigma_2 v_\phi(t) \phi(t) + \beta u_\phi(t) v_\phi(t) - c_2 v_\phi(t),
\end{align*}$$

(2.11)
where \( \phi \) is taken from the set of piecewise continuous real-valued functions defined on \( \mathbb{R}_+ \).

Let \((u_\phi(t, u, v), v_\phi(t, u, v))\) be the solution to equation (2.11) with control \( \phi \) and initial value \((u, v)\). Denote by \( \mathcal{O}^+(u, v) \) the reachable set from \((u, v) \in \mathbb{R}_{+}^2 \), that is the set of \((u', v') \in \mathbb{R}_{+}^2 \) such that there exists a \( t \geq 0 \) and a control \( \phi(\cdot) \) satisfying \( u_\phi(t, u, v) = u', v_\phi(t, u, v) = v' \).

We now recall some concepts introduced in [17]. Let \( X \) be a subset of \( \mathbb{R}_{+}^2 \) satisfying the property that for any \( w_1, w_2 \in X, w_2 \in \bar{\mathcal{O}}^+(w_1) \). Then there is a unique maximal set \( Y \supset X \) such that this property still holds for \( Y \). Such a \( Y \) is called a control set. A control set \( W \) is said to be invariant if \( \bar{\mathcal{O}}^+(w) \subset \bar{W} \) for all \( w \in W \).

Putting \( r := \frac{-\sigma_2}{\sigma_1} \) and \( z_\phi(t) = u'_\phi(t)v_\phi(t) \), we have an equivalent system
\[
\begin{align*}
\dot{u}_\phi(t) &= \sigma_1\phi(t)u_\phi(t) + g(u_\phi(t), z_\phi(t)), \\
\dot{z}_\phi(t) &= h(u_\phi(t), z_\phi(t)),
\end{align*}
\]
(2.12)
where
\[
g(u, z) = -c_1\epsilon u + \alpha - \beta zu^{1-r},
\]
and
\[
h(u, z) = u^{-r}z\left[-(c_1r + c_2)u^r + \beta u^{1+r} + \alpha ru^{r-1} - \beta rz\right].
\]

**Lemma 2.2.** For the control system (2.12), the following claims hold

1. For any \( u_0, u_1, z_0 > 0 \) and \( \epsilon > 0 \), there exists a control \( \phi \) and \( T > 0 \) such that \( u_\phi(T, u_0, z_0) = u_1, |z_\phi(T, u_0, z_0) - z_0| < \epsilon \).

2. For any \( 0 < z_0 < z_1 \), there is a \( u_0 > 0 \), a control \( \phi \), and \( T > 0 \) such that \( z_\phi(T, u_0, z_0) = z_1 \) and that \( u_\phi(t, u_0, z_0) = u_0 \) \( \forall 0 \leq t \leq T \).

3. Let \( d^* = \inf_{u > 0} \{- (c_1r + c_2)u^r + \beta u^{1+r} + \alpha ru^{r-1}\} \).

(a) If \( d^* \leq 0 \) then for any \( z_0 > z_1 \), there is \( u_0 > 0 \), a control \( \phi \), and \( T > 0 \) such that \( z_\phi(T, u_0, z_0) = z_1 \) and that \( u_\phi(t, u_0, z_0) = u_0 \) \( \forall 0 \leq t \leq T \).

(b) Suppose that \( d^* > 0 \) and \( z_0 > c^* := \frac{d^*}{\beta r} \). If \( c^* < z_1 < z_0 \), there is \( u_0 > 0 \) and a control \( \phi \) and \( T > 0 \) such that \( z_\phi(T, u_0, z_0) = z_1 \) and that \( u_\phi(t, u_0, z_0) = u_0 \) \( \forall 0 \leq t \leq T \). However, there is no control \( \phi \) and \( T > 0 \) such that \( z_\phi(T, u_0, z_0) < c^* \).

**Proof.** Suppose that \( u_0 < u_1 \) and let \( \rho_1 = \sup\{|g(u, z)|, |h(u, z)| : u_0 \leq u \leq u_1, |z - z_0| \leq \epsilon\} \).

We choose \( \phi(t) \equiv \rho_2 \) with \( (\frac{\sigma_1\rho_2 u_0}{\rho_1} - 1)\epsilon \geq u_1 - u_0 \). It is easy to check that with this control,
there is $0 \leq T \leq \frac{1}{\rho_1}$ such that $u_0(T, u_0, z_0) = u_1$, $|z_0(T, u_0, z_0) - z_0| < \varepsilon$. If $u_0 > u_1$, we can construct $\phi(t)$ similarly. Then the claim 1 is proved.

By choosing $u_0$ to be sufficiently large, there is a $\rho_3 > 0$ such that $h(u_0, z) > \rho_3 \forall z_0 \leq z \leq z_1$. This property, combining with (2.12), implies the existence of a feedback control $\phi$ and $T > 0$ satisfying that $z_0(T, u_0, z_0) = z_1$ and that $u_0(t, u_0, z_0) = u_0$, $\forall 0 \leq t \leq T$.

We now prove claim 3. If $r < 0$ then
\[
\lim_{u \to 0} \left[ - (c_1 r + c_2)u^r + \beta u^{1+r} + \alpha ru^{r-1} \right] = -\infty
\]
and
\[
\lim_{u \to 0} \left[ - (c_1 r + c_2)u^r + \beta u^{1+r} + \alpha ru^{r-1} \right] = 0 \quad \text{if} \quad r > 1.
\]
As a result, $d^* \leq 0$ if $r \notin (0, 1]$ which implies that for any $z_0 > z_1$, we choose $u_0$ such that $\sup_{z \in [z_1, z_0]} h(u_0, z) < 0$, which implies that there is a feedback control $\phi$ and $T > 0$ satisfying $z_\phi(T, u_0, z_0) = z_1$ and $u_\phi(t, u_0, z_0) = u_0 \forall 0 \leq t \leq T$.

If $r \in (0, 1]$ there exists $u_0$ such that $- (c_1 r + c_2)u_0^r + \beta u_0^{1+r} + \alpha u_0^{r-1} = d^*$. If $d^* \leq 0$, then for any $z_0 > z_1 > 0$ we have $\sup_{z \in [z_1, z_0]} h(u_0, z) \leq u_0^{-r} \sup_{z \in [z_1, z_0]} \left\{ - \beta r z^2 \right\} < 0$ which implies the desired claim.

Consider the remaining case when $r \in (0, 1]$ and $d^* > 0$. First, assume $c^* < z_1 < z_0$. Let $u_0$ satisfy $- (c_1 r + c_2)u_0^r + \beta u_0^{1+r} + \alpha u_0^{r-1} = d^* = \beta r c^*$. Hence
\[
\sup_{z \in [z_1, z_0]} \left\{ h(u_0, z) \right\} = u_0^{-r} \sup_{z \in [z_1, z_0]} \left\{ z \left( - (c_1 r + c_2)u_0^r + \beta u_0^{1+r} + \alpha u_0^{r-1} - \beta r z \right) \right\} = - \beta ru_0^{-r} z_1 (c^* - z_1) < 0.
\]
Thus, there is a feedback control $\phi$ and $T > 0$ satisfying $z_\phi(T, u_0, z_0) = z_1$ and $u_\phi(t, u_0, z_0) = u_0 \forall 0 \leq t \leq T$. The final assertion follows from the fact that $h(u, c^*) \geq 0$ for all $u \in \mathbb{R}$.

To obtain the convergence in total variation norm and to estimate the convergence rate, we aim to apply [12, Theorem 3.6, p. 235]. In order to do that, we construct a function $V: \mathbb{R}_+^{2,0} \to [1, \infty)$ satisfying that
\[
\mathbb{E} V(S_{u,v}(t^*), I_{u,v}(t^*)) \leq V(u, v) - \kappa_1 V^\gamma(u, v) + \kappa_2 \mathbf{1}_{\{u,v\in K\}}
\]
for some petite set $K$ and some $\gamma \in (0, 1), \kappa_1, \kappa_2 > 0$, $t^* > 1$. Recall that a set $K$ is said to be petite with respect to the Markov chain $S_{u,v}(nt^*), I_{u,v}(nt^*)$, $n \in \mathbb{N}$ if there exists a measure $\psi$ with $\psi(\mathbb{R}_+^{2,0}) > 0$ and a probability distribution $\nu(\cdot)$ concentrated on $\mathbb{N}$ such that
\[
K(u, v, Q) := \sum_{n=1}^{\infty} P(n t^*, u, v, Q) \nu(n) \geq \psi(Q) \forall (u, v) \in K, Q \in \mathcal{B}(\mathbb{R}_+^{2,0}).
\]
We also have to prove that the skeleton Markov chain \((S_{u,v}(nt^*), I_{u,v}(nt^*))\), \(n \in \mathbb{N}\) is irreducible and aperiodic. We refer to [25] or [22] for the definitions and properties of irreducibility, aperiodicity, as well as petite sets. The estimation of the convergence rate is divided into some lemmas and propositions.

**Lemma 2.3.** For any \(0 < p^* < \min\{\frac{2\mu}{\sigma_1^2}, \frac{2(\mu + \rho + \gamma)}{\sigma_2^2}\}\). Let \(U(u, v) = (u + v)^{1+p^*} + u^{-\frac{p^*}{2}}\). There exists positive constants \(K_1, K_2\) such that

\[
e^{K_1t} \mathbb{E}(U(S_{u,v}(t), I_{u,v}(t))) \leq U(u, v) + \frac{K_2(e^{K_1t} - 1)}{K_1}. \quad (2.13)
\]

**Proof.** Consider the Lyapunov function \(U(u, v) = (u + v)^{1+p^*} + u^{-\frac{p^*}{2}}\). By directly calculating the differential operator \(LU(u, v)\) associated with equation (2.1), we have

\[
LU(u, v) = (1 + p)(u + v)^{p^*}(\alpha - \mu u - (\mu + \rho + \gamma)v) + \frac{(1 + p)p^*}{2}(u + v)^{p-1}(\sigma_1 u + \sigma_2 v)^2
\]

\[
- \frac{p^*}{2} u^{-\frac{p^*}{2} - 1}(\alpha - \beta uv - \mu u) + \frac{p^*(2 + p^*)}{8} \sigma_2^2 u^{-\frac{p^*}{2}}
\]

\[
= (1 + p^*)\alpha (u + v)^{p^*} - (1 + p^*)(u + v)^{p^* - 1}\left[ \left( \mu - \frac{p^*}{2}\sigma_1^2 \right) u^2 + (\mu + \rho + \gamma - \frac{p^*}{2}\sigma_2^2) v^2 \right]
\]

\[
+ (2\mu + \rho + \gamma - p^*\sigma_1\sigma_2)uv - p^*\alpha u^{-\frac{2 + p^*}{2}} + \frac{\beta p^*}{2} u^{-\frac{p^*}{2}} v + \frac{p^*}{2}\left[ \frac{(2 + p^*)\sigma_1^2}{4} + \mu \right] u^{-\frac{p^*}{2}}. \quad (2.14)
\]

By Young’s inequality, we have

\[
u^{-\frac{p^*}{2}} v \leq \frac{3p^*}{4 + 3p^*} u^{-\frac{4 + 3p^*}{6}} + \frac{4}{4 + 3p^*} v^{-\frac{4 + 3p^*}{4}}. \quad (2.15)
\]

Choose a number \(K_1\) satisfying \(0 < K_1 < \min\{\mu - \frac{p^*}{2}\sigma_1^2, \mu + \rho + \gamma - \frac{p^*}{2}\sigma_2^2\}\). From (2.15) (2.14), we obtain \(K_2 = \sup_{u,v \in \mathbb{R}_+} \{LU(u, v) + K_1 U(u, v)\} < \infty\). As a result,

\[
LU(u, v) \leq K_2 - K_1 U(u, v) \forall (u, v) \in \mathbb{R}_+^2. \quad (2.16)
\]

For \(n \in \mathbb{N}\), define the stopping time

\[
\eta_n = \inf\{t \geq 0 : U(S_{u,v}(t), I_{u,v}(t)) \geq n\}.
\]

Then Itô’s formula and (2.16) yield that

\[
\mathbb{E}(e^{K_1(t \land \eta_n)} U(S_{u,v}(t \land \eta_n), I_{u,v}(t \land \eta_n)))
\]

\[
\leq U(u, v) + \mathbb{E} \int_0^{t \land \eta_n} e^{K_1\tau} \left( LU(S_{u,v}(\tau), I_{u,v}(\tau)) + K_1 U(S_{u,v}(\tau), I_{u,v}(\tau)) \right) d\tau
\]

\[
\leq U(u, v) + \frac{K_2(e^{K_1t} - 1)}{K_1}.
\]
By letting $n \to \infty$ we obtain from Fatou’s lemma that
\[
\mathbb{E} e^{K_1 t}(U(S_{u,v}(t), I_{u,v}(t))) \leq U(u, v) + \frac{K_2(e^{K_1 t} - 1)}{K_1}. \tag{2.17}
\]
The lemma is proved.

**Lemma 2.4.** There are positive constants $K_3, K_4$ such that, for any $t \geq 1$ and $A \in \mathcal{F}$
\[
\mathbb{E} \left( [\ln I_{u,v}(t)]^2 \mathbf{1}_A \right) \leq [\ln v]^2 \mathbb{P}(A) + K_3 \sqrt{\mathbb{P}(A)t[\ln v]_+} + K_4 t^2 \sqrt{\mathbb{P}(A)}, \tag{2.18}
\]
where $[\ln x]_+ = \max\{0, -\ln x\}$.

**Proof.** We have
\[
-\ln I_{u,v}(t) = -\ln I_{u,v}(0) - \beta \int_0^t S_{u,v}(\tau)d\tau + c_2 t + \sigma_2 B(t) \\
\leq -\ln v + c_2 t + \sigma_2 |B(t)|,
\]
where $c_2 = \mu + \rho + \gamma + \frac{\sigma_2^2}{2}$. Therefore,
\[
[\ln I_{u,v}(t)]_- \leq [\ln v]_+ + c_2 t + \sigma_2 |B(t)|.
\]
This implies that
\[
[\ln I_{u,v}(t)]^2 \mathbf{1}_A \leq [\ln v]^2 \mathbf{1}_A + (c_2^2 t^2 + \sigma_2^2 B^2(t)) \mathbf{1}_A + 2c_2 t[\ln v]_- \mathbf{1}_A \\
+ 2\sigma_2 |B(t)| \mathbf{1}_A[\ln v]_- + 2c_1 t\sigma_2 |B(t)| \mathbf{1}_A.
\]
By using Hölder inequality, we obtain
\[
\mathbb{E}|B(t)| \mathbf{1}_A \leq \sqrt{\mathbb{E} B^2(t) \mathbb{P}(A)} \leq \sqrt{t \mathbb{P}(A)} \leq t \sqrt{\mathbb{P}(A)}.
\]
Taking expectation both sides and using the estimate above, we have
\[
\mathbb{E}[\ln I_{u,v}(t)]^2 \mathbf{1}_A \leq [\ln v]^2 \mathbb{P}(A) + K_3 t \sqrt{\mathbb{P}(A)[\ln v]_-} + K_4 t^2 \sqrt{\mathbb{P}(A)},
\]
for some positive constants $K_3, K_4$.

We now, choose $\varepsilon \in (0, 1)$ satisfying
\[
-\frac{3\lambda}{2}(1 - \varepsilon) + K_3 \sqrt{\varepsilon} < -\lambda \quad \text{and} \quad -\frac{3\lambda}{4}(1 - \varepsilon) + 2K_3 \sqrt{\varepsilon} < -\frac{\lambda}{2}. \tag{2.19}
\]
Choose $H$ so large that
\[
\beta H - 2c_2 \geq 2 + \lambda; \quad \exp \left\{ -\frac{\beta H - 2c_2}{2\sigma_2^2} \right\} < \frac{\varepsilon}{2} \quad \text{and} \quad \exp \left\{ -\frac{\lambda(\beta H - c_2)}{4\sigma_2^2} \right\} < \frac{\varepsilon}{2}. \tag{2.20}
\]
Lemma 2.5. For \( \varepsilon \) and \( H \) chosen as above, there is \( \delta \in (0,1) \) and \( T^* > 1 \) such that

\[
P\{ \ln v + \frac{3\lambda t}{4} \leq \ln I_{u,v}(t) < 0 \text{ for all } t \in [T^*, 2T^*] \} \geq 1 - \varepsilon \quad (2.21)
\]

for all \( u \in [0,H], v \in (0,\delta] \).

Proof. Let \( \theta \in (0,1) \) such that

\[
\frac{\beta\alpha}{\mu + \beta\theta} - c_2 \geq \frac{11\lambda}{12}.
\]

(2.22)

Let \( \tilde{S}_u(t) \) be the solution with initial value \( u \) to

\[
d\tilde{S}(t) = [\alpha - (\beta\theta + \mu)\tilde{S}(t)]dt + \sigma_1\tilde{S}(t)dB(t)
\]

(2.23)

Similar to (2.4),

\[
P\left\{ \lim_{t \to \infty} \frac{1}{t} \int_0^t \tilde{S}_u(\tau)d\tau = \frac{\alpha}{\mu + \beta\theta} \right\} = 1 \quad \forall u \in [0,\infty).
\]

In view of the strong law of large numbers for martingales, \( P\{\lim_{t \to \infty} \frac{B(t)}{t} = 0\} = 1 \). Hence, there exists \( T^* > 1 \) such that

\[
P\left\{ \frac{\sigma_1B(t)}{t} \geq -\frac{\lambda}{12} \text{ for all } t \geq T^* \right\} \geq 1 - \frac{\varepsilon}{3} \quad (2.24)
\]

and

\[
P\left\{ \frac{1}{t} \int_0^t \tilde{S}_0(\tau)d\tau \geq \frac{\alpha}{\mu + \beta\theta} - \frac{\lambda}{12\beta} \text{ for all } t \geq T^* \right\} \geq 1 - \frac{\varepsilon}{3}.
\]

By the uniqueness of solutions to (2.23),

\[
P\left\{ \tilde{S}_0(t) \leq \tilde{S}_u(t) \text{ for all } t \geq 0 \right\} = 1 \quad \forall u \geq 0.
\]

Hence,

\[
P\left\{ \frac{1}{t} \int_0^t \tilde{S}_u(\tau)d\tau \geq \frac{\alpha}{\mu + \beta\theta} - \frac{\lambda}{12\beta} \text{ for all } t \geq T^* \right\} \geq 1 - \frac{\varepsilon}{3}.
\]

(2.25)

Similar to [24, Lemmas 3.1, 3.2], it can be shown that there exists \( \delta \in (0,\theta) \),

\[
P\{ \zeta_{u,v} \leq 2T^* \} \leq \frac{\varepsilon}{3}, \quad \forall v \leq \delta, \quad u \in [0,H] \text{ where } \zeta_{u,v} = \inf\{t \geq 0 : I_{u,v}(t) \geq \theta\}.
\]

(2.26)

Observe also that

\[
P\left\{ S_{u,v}(t) \geq \tilde{S}_u(t) \text{ for all } t \leq \zeta_{u,v} \right\} = 1
\]

(2.27)
which we have from the comparison theorem. From (2.22), (2.24),(2.25), (2.26) and (2.27) we can show that with probability greater than \(1 - \varepsilon\), for all \(t \in [T^*, 2T^*]\),

\[
\ln \theta \geq \ln(I_{u,v}(t)) = \ln v + \beta \int_0^t S_{u,v}(\tau)d\tau - c_2 t + \sigma_1 B(t)
\]

\[
\geq \ln v + \frac{\beta \alpha t}{(\mu + \beta \theta)} - \frac{\lambda t}{12} - c_2 t - \frac{\lambda t}{12} \geq \ln v + \frac{3\lambda}{4} t.
\]

The proof is complete. \(\square\)

**Proposition 2.1.** Assume \(\lambda > 0\). Let \(\delta, H\) and \(T^*\) be as in Lemma 2.5. There exists \(K_5\), independent of \(T^*\) such that

\[
\mathbb{E}[\ln I_{u,v}(t)]^2_\leq \leq \mathbb{E}[\ln v]^2_\leq - \lambda t[\ln v]_\leq + K_5 t^2
\]

for any \(v \in (0, \infty), \ 0 \leq u \leq H, \ t \in [T^*, 2T^*]\).

**Proof.** First, consider \(v \in (0,\delta], \ 0 \leq u \leq H\). We have \(\mathbb{P}(\Omega_{u,v}) \geq 1 - \varepsilon\) where

\[
\Omega_{u,v} = \{ \ln v + \frac{3\lambda t}{4} \leq \ln I_{u,v}(t) < 0 \ \forall t \in [T^*, 2T^*]\}.
\]

In \(\Omega_{u,v}\) we have

\[-\ln v - \frac{3\lambda t}{4} \geq - \ln I_{u,v}(t) > 0.
\]

Hence,

\[0 \leq \ln I_{u,v}(t)_\leq \leq \ln v - \frac{3\lambda t}{4} \forall t \in [T^*, 2T^*].\]

As a result

\[\ln I_{u,v}(t)^2_\leq \leq \ln v^2_\leq - \frac{3\lambda t}{2} \ln v_\leq + \frac{9\lambda^2 t^2}{16} \forall t \in [T^*, 2T^*].\]

Which implies that

\[
\mathbb{E}[\mathbf{1}_{\Omega_{u,v}} \ln I_{u,v}(t)_\leq^2] \leq \mathbb{P}(\Omega_{u,v})[\ln v]^2_\leq - \frac{3\lambda t}{2} \mathbb{P}(\Omega_{u,v})[\ln v]_\leq + \frac{9\lambda^2 t^2}{16} \mathbb{P}(\Omega_{u,v}).
\]

In \(\Omega_{u,v}^c = \Omega \setminus \Omega_{u,v}\), we have from Lemma 2.4 that

\[
\mathbb{E}[\mathbf{1}_{\Omega_{u,v}^c} \ln I_{u,v}(t)_\leq^2] \leq \mathbb{P}(\Omega_{u,v}^c)[\ln v]^2_\leq + K_3 t\sqrt{\mathbb{P}(\Omega_{u,v}^c)[\ln v]_\leq} + K_4 t^2 \sqrt{\mathbb{P}(\Omega_{u,v}^c)}.
\]

Adding (2.29) and (2.30) side by side, we obtain

\[
\mathbb{E}[\ln I_{u,v}(t)^2_\leq] \leq \ln v^2_\leq + \left( - \frac{3\lambda}{2}(1 - \varepsilon) + K_3 \sqrt{\varepsilon} \right) t[\ln v]_\leq + \left( \frac{9\lambda^2}{16} + K_4 \right) t^2.
\]
In view of (2.19) we deduce
\[
\mathbb{E}[\ln I_{u,v}(t)]_+^2 \leq [\ln v]_+^2 - \lambda t[\ln v]_+ + \left(\frac{9\lambda^2}{16} + K_4\right)t^2.
\]

Now, for \(v \in [\delta, \infty)\) and \(0 \leq u \leq H\), we have from Lemma 2.4 that
\[
\mathbb{E}[\ln I_{u,v}(t)]_+^2 \leq [\ln v]_+^2 + K_3 t[\ln v]_+ + K_4 t^2
\leq [\ln \delta]_+^2 + K_3 t[\ln \delta]_+ + K_4 t^2.
\]

Letting \(K_5\) sufficiently large such that \(K_5 > \frac{9\lambda^2}{16} + K_4\) and \(|\ln \delta|_+^2 + K_3 t|\ln \delta|_+ + K_4 t^2 \leq K_5 t^2 \quad \forall t \in [T^*, 2T^*]\), we obtain the desired result. \(\square\)

**Proposition 2.2.** Assume \(\lambda > 0\). There exists \(K_6 > 0\) such that
\[
\mathbb{E}[\ln I_{u,v}(2T^*)]_+^2 \leq [\ln v]_+^2 - \frac{\lambda}{2}[\ln v]_+ + K_6 T^* t^2
\]
for \(v \in (0, \infty), u > H\).

**Proof.** First, consider \(v \leq \exp\{-\frac{\lambda T^*}{2}\}\). Defined the stopping time
\[
\xi_{u,v} = T^* \wedge \inf\{t > 0 : S_{u,v}(t) \leq H\}.
\]

Let
\[
\Omega_1 = \\left\{ \sigma_2 B(2T^*) - \frac{(\beta H - 2c_2)T^*}{2} \leq 1 \right\}
\]
and
\[
\Omega_2 = \left\{ \sigma_2 B(t) - (\beta H - c_2)t \leq \frac{\lambda}{8} \quad \forall t \in [0, 2T^*] \right\}.
\]

By the exponential martingale inequality [21, Theorem 7.4, p. 44],
\[
\mathbb{P}(\Omega_1) \geq 1 - \exp\left\{ - \frac{\beta H - 2c_2}{2\sigma_2^2} \right\} \geq 1 - \frac{\varepsilon}{2}. \tag{2.32}
\]

and
\[
\mathbb{P}(\Omega_2) \geq 1 - \exp\left\{ - \frac{\lambda(\beta H - c_2)}{4\sigma_2^2} \right\} \geq 1 - \frac{\varepsilon}{2}.
\]

Let
\[
\Omega_3 = \Omega_1 \cap \{\xi_{u,v} = T^*\}; \quad \Omega_4 = \{- \ln I_{u,v}(\xi_{u,v}) \leq - \ln v + \frac{\lambda}{8}\} \cap \{\xi_{u,v} < T^*\}; \quad \Omega_5 = \Omega \backslash (\Omega_3 \cup \Omega_4).
\]

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If $\omega \in \Omega_3$, we have

$$- \ln I_{u,v}(2T^*) = - \ln v - \int_0^{2T^*} (\beta S_{u,v}(\tau) - c_2) d\tau + \sigma_2 B(2T^*)$$

$$\leq - \ln v - \int_0^{2T^*} (\beta S_{u,v}(\tau) - c_2) d\tau + \int_{T^*}^{2T^*} c_2 d\tau + \sigma_2 B(2T^*)$$

$$\leq - \ln v - T^*(\beta H - c_2) + T^* c_2 + \sigma_2 B(2T^*)$$

$$\leq - \ln v - T^*(\beta H - 2c_2) + \sigma_2 B(2T^*)$$

$$\leq - \ln v - \frac{T^*(\beta H - 2c_2)}{2} + 1 \quad \text{(in view of (2.32))}$$

$$\leq - \ln v - \frac{\lambda T^*}{2} \quad \text{(by (2.20)).}$$

If $v \leq \exp\{ - \frac{\lambda T^*}{2} \}$ then $- \ln v - \frac{\lambda T^*}{2} > 0$. Therefore

$$[\ln I_{u,v}(2T^*)]_0 \leq \frac{\lambda T^*}{2} + [\ln v]_0.$$

Squaring and then multiplying by $1_{\Omega_3}$ and then taking expectation both sides, we yield

$$\mathbb{E} \left( [\ln I_{u,v}(2T^*)]^2 1_{\Omega_3} \right) \leq [\ln v]^2 \mathbb{P}(\Omega_3) - \lambda T^* [\ln v]_0 \mathbb{P}(\Omega_3) + \frac{\lambda^2 T^{*2}}{4}. \quad \text{(2.33)}$$

If $\omega \in \Omega_2$,

$$- \ln I_{u,v}(\xi_{u,v}) = - \ln v - \int_0^{\xi_{u,v}} [\beta S_{u,v}(\tau) - c_2] d\tau + \sigma_2 B(\xi_{u,v})$$

$$\leq - \ln v - (\beta H - c_2) \xi_{u,v} + \sigma_2 B(\xi_{u,v}) \leq - \ln v + \frac{\lambda}{8}.$$

As a result, $\Omega_2 \cap \{ \xi_{u,v} < T^* \} \subset \Omega_4$. Hence

$$\mathbb{P}(\Omega_5) = \mathbb{P}(\Omega_5 \cap \{ \xi_{u,v} < T^* \}) + \mathbb{P}(\Omega_5 \cap \{ \xi_{u,v} = T^* \}) \leq \mathbb{P}(\Omega_1) + \mathbb{P}(\Omega_2) \leq \varepsilon.$$

Let $t < T^*$, $u' > 0$ and $v'$ satisfy $- \ln v' \leq - \ln v + \frac{\lambda}{8} \leq 0$. In view of Proposition 2.1 and the strong Markov property, we can estimate the conditional expectation

$$\mathbb{E} \left[ [\ln I_{u,v}(2T^*)]^2 | \xi_{u,v} = t, I_{u,v}(\xi_{u,v}) = v', S_{u,v}(\xi_{u,v}) = u' \right]$$

$$\leq [\ln v']_0^2 - \lambda (2T^* - t)[\ln v']_0 + K_5 (2T^* - t)^2$$

$$\leq [\ln v']_0^2 - \lambda T^* [\ln v']_0 + 4K_5 T^{*2}$$

$$\leq (- \ln v + \frac{\lambda}{8})^2 - \lambda T^* (- \ln v) + 4K_5 T^{*2}$$

$$\leq (- \ln v)^2 - (\lambda T^* - \frac{\lambda}{4})(- \ln v) + 4K_5 T^{*2} + \frac{\lambda^2}{64}$$

$$\leq [\ln v]_0^2 - \frac{3\lambda T^*}{4} [\ln v]_0 + 4K_5 T^{*2} + \frac{\lambda^2}{64}.$$
As a result,

\[ E \left( 1_{\Omega_5} \cdot \left[ \ln I_{u,v}(2T^*) \right]^2 \right) \leq \left[ \ln v \right]_+^2 P(\Omega_5) - \frac{3\lambda T^*}{4} \left[ \ln v \right]_+ P(\Omega_4) + 4K_5 T^{*2} + \frac{\lambda^2}{64}. \]  

(2.34)

In view of Lemma 2.4,

\[ E \left( 1_{\Omega_5} \cdot \left[ \ln I_{u,v}(2T^*) \right]^- \right) \leq \left[ \ln v \right]^-^2 P(\Omega_5) + K_3 \sqrt{P(\Omega_5)} 2T^* \left[ \ln v \right]^- + 4K_4 T^{*2}. \]  

(2.35)

Adding side by side (2.33), (2.34), (2.35), we have

\[ E \left( \left[ \ln I_{u,v}(2T^*) \right]^- \right) \leq \left[ \ln v \right]^-_+ - T^* \left( \frac{3\lambda}{4} (1 - \varepsilon) + 2K_3 \sqrt{\varepsilon} \right) + K_7 T^{*2} \]

\[ \leq \left[ \ln v \right]^-_+ - \frac{\lambda T^*}{2} + K_7 T^{*2} \]  

(2.36)

for some \( K_7 > 0 \). We note that, if \( v \geq \exp \left\{-\frac{\lambda T^*}{2} \right\} \) then \( -\ln v \leq \frac{\lambda T^*}{2} \). Therefore, it follows from Lemma 2.4 that

\[ E \left( \left[ \ln I_{u,v}(2T^*) \right]^- \right) \leq \left( \frac{\lambda^2}{4} + K_3 \frac{\lambda}{2} + 4K_4 \right) T^{*2}. \]

Let \( K_6 = \max\{K_7, \frac{\lambda^2}{4} + K_3 \frac{\lambda}{2} + 4K_4\} \), we have

\[ E \left( \left[ \ln I_{u,v}(2T^*) \right]^- \right) \leq \left[ \ln v \right]^-_+ - \frac{\lambda T^*}{2} \left[ \ln v \right]^- + K_6 T^{*2}, \forall u \geq H, v \in (0, \infty). \]

The proof is complete. \( \square \)

**Lemma 2.6.** Any compact set \( K \subset \mathbb{R}^2_{+} \) is petite for the Markov chain \( (S_{u,v}(2nT^*), I_{u,v}(2nT^*)) \) \((n \in \mathbb{N}) \). The irreducibility and aperiodicity of \((S_{u,v}(2nT^*), I_{u,v}(2nT^*)) \) \((n \in \mathbb{N}) \) is a byproduct (see [25, 22]).

**Proof.** Note that, we can always choose \( \phi_* \in \mathbb{R} \) such that \((c_1 - \sigma_1 \phi_*) > 0, (c_2 - \sigma_2 \phi_*) > 0 \) and \( \alpha \beta - (c_1 - \sigma_1 \phi_*)(c_2 - \sigma_2 \phi_*) > 0 \). Hence, with the constant control \( \phi_* \) we can show that the solution to (2.11) with control \( \phi_* \), \((u_{\phi_*}(t, u, v), v_{\phi_*}(t, u, v))\), converges to a positive equilibrium \((u_*, v_*)\) for all \((u, v) \in \mathbb{R}^2_{+}\). Let \((u_0, v_0) \in \mathbb{R}^2_{+}\) such that \( p(2T^*, u_*, v_*, u_0, v_0) > 0 \).

By the smoothness of \( p(2T^*, \cdot, \cdot, \cdot) \), there exists a neighborhood \( W_\delta = (u_* - \delta, u_* + \delta) \times (v_* - \delta, v_* + \delta) \), that is invariant under (2.11) and a open set \( G \ni (u_0, v_0) \) such that

\[ p(1, u, v, u', v') \geq m' > 0 \forall (u, v) \in W_\delta, (u', v') \in G. \]  

(2.37)

Since \((u_{\phi_*}(t, u, v), v_{\phi_*}(t, u, v))\) converges to a positive equilibrium \((u_*, v_*)\), in view of the support theorem (see [11, Theorem 8.1, p. 518]), there is \( n_{u,v} \in \mathbb{N} \) such that

\[ P(2n_{u,v}T^*, u, v, W_\delta) := 2\rho_{u,v} > 0. \]
Since \((S_{u,v}(t), I_{u,v}(t))\) is a Markov-Feller process, there exists an open set \(V_{u,v} \ni (u, v)\) such that \(P(n_{u,v}, u', v', W_\delta) \geq \rho_{u,v} \forall (u', v') \in V_{u,v}\). Since \(K\) is a compact set, there is a finite number of \(V_{u,v_i}, i = 1, \ldots, l\) satisfying \(K \subset \bigcup_{i=1}^l V_{u,v_i}\). Let \(\rho_K = \min\{\rho_{u,v_i}, i = 1, \ldots, l\}\). For each \((u, v) \in K\), there exists \(n_{u,v_i}\) such that

\[
P(n_{u,v_i}, u, v, W_\delta) \geq \rho_K. \quad (2.38)
\]

From (2.37) (2.38), for all \((u, v) \in K\) there exists \(n_{u,v_i}\) such that

\[
p((2n_{u,v_i} + 2)T^*, u, v, u', v') \geq \rho_Km' \forall (u', v') \in G. \quad (2.39)
\]

Define the kernel

\[
\mathcal{K}(u, v, Q) := \frac{1}{l} \sum_{i=1}^l P((2n_{u,v_i} + 2)T^*, u, v, Q) \forall Q \in \mathcal{B}(\mathbb{R}_+^{2\circ}).
\]

We derive from (2.39) that

\[
\mathcal{K}(u, v, Q) \geq \frac{1}{l} \rho_Km' \mu(G \cap Q) \forall Q \in \mathcal{B}(\mathbb{R}_+^{2\circ}), \quad (2.40)
\]

where \(\mu(\cdot)\) is the Lebesgue measure on \(\mathbb{R}_+^{2\circ}\). (2.40) means that every compact set \(K \subset \mathbb{R}_+^{2\circ}\) is petite for the Markov chain \((S_{u,v}(2T^*n), I_{u,v}(2T^*n))\).

**Theorem 2.2.** Let \(\lambda > 0, d^*\) as in Lemma 2.2. There exists an invariant probability measure \(\pi^*\) such that

\[
\lim_{t \to \infty} t^{q^*} \|P(t, (u, v), \cdot) - \pi^*(\cdot)\| = 0 \forall (u, v) \in \mathbb{R}_+^{2\circ}, \quad (2.41)
\]

where \(\| \cdot \|\) is the total variation norm and \(q^*\) is any positive number. The support of \(\pi^*\) is \(\mathbb{R}_+^{2\circ}\) if \(d^* \leq 0\) and is \(\{(u, v) \in \mathbb{R}_+^{2\circ} : u^*v \geq d^*\}\) if \(d^* > 0\). Moreover, for any initial value \((u, v) \in \mathbb{R}_+^{2\circ}\) and a \(\pi^*\)-integrable function \(f\) we have

\[
P\left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T f(S_{u,v}(t), I_{u,v}(t)) dt = \int_{\mathbb{R}_+^{2\circ}} f(u', v') \pi^*(du', dv') \right\} = 1. \quad (2.42)
\]

**Proof.** By virtue of Lemma 2.3, there are \(h_1 > 0, H_1 > 0\) satisfying

\[
\mathbb{E}U(S_{u,v}(2T^*)), I_{u,v}(2T^*)) \leq (1 - h_1)U(u, v) + H_1 \forall (u, v) \in \mathbb{R}_+^{2\circ}. \quad (2.43)
\]

Let \(V(u, v) = U(u, v) + [\ln v]^2\). In view of Propositions 2.1, 2.2 and (2.43), there is a compact set \(K \subset \mathbb{R}_+^{2\circ}, h_2 > 0, H_2 > 0\) satisfying

\[
\mathbb{E}V(S_{u,v}(2T^*), I_{u,v}(2T^*)) \leq V(u, v) - h_2\sqrt{V(u, v)} + H_2 1_{\{(u, v) \in K\}} \forall (u, v) \in \mathbb{R}_+^{2\circ}. \quad (2.44)
\]
Applying (2.44) and Lemma 2.6 to [12, Theorem 3.6] we obtain that
\[ n\|P(2nT^*, (u,v), \cdot) - \pi^*\| \to 0 \text{ as } n \to \infty, \tag{2.45} \]
for some invariant probability measure \( \pi^* \) of the Markov chain \((S_{u,v}(2nT^*), I_{u,v}(2nT^*))\). It is shown in the proof of [12, Theorem 3.6] that (2.44) implies \( \mathbb{E}\tau_K < \infty \) where \( \tau_K = \inf\{n \in \mathbb{N} : (S_{u,v}(2nT^*), I_{u,v}(2nT^*)) \in K\} \). In view of [17, Theorem 4.1], the Markov process \((S_{u,v}(t), I_{u,v}(t))\) has an invariant probability measure \( \pi_* \). As a result, \( \pi_* \) is also an invariant probability measure of the Markov chain \((S_{u,v}(2nT^*), I_{u,v}(2nT^*))\). In light of (2.45), we must have \( \pi_* = \pi^* \), that is, \( \pi^* \) is an invariant measure of the Markov process \((S_{u,v}(t), I_{u,v}(t))\).

In the proofs, we use the function \( [\ln v]_{\cdot}^{2} \) for the sake of simplicity. In fact, we can treat \( [\ln v]_{\cdot}^{1+q} \) for any small \( q \in (0, 1) \) in the same manner. We can show that there is \( h_q, H_q > 0 \), a compact set \( K_q \) satisfying
\[ \mathbb{E}V_q(S_{u,v}(2T^*), I_{u,v}(2T^*)) \leq V_q(u,v) - \frac{h_q[V_q(u,v)]^{1+q}}{1+q} + H_q1_{\{(u,v)\in K_q\}} \quad \forall (u,v) \in \mathbb{R}^2_+, \tag{2.46} \]
where \( V_q(u,v) = U(u,v) + [\ln v]_{\cdot}^{1+q} \). Then applying [12, Theorem 3.6], we can obtain
\[ n^{1/q}\|P(2nT^*, (u,v), \cdot) - \pi\| \to 0 \text{ as } n \to \infty. \]
Since \( \|P(t, (u,v), \cdot) - \pi^*\| \) is decreasing in \( t \), we easily deduce
\[ t^{q^*}\|P(t, (u,v), \cdot) - \pi^*\| \to 0 \text{ as } t \to \infty \]
where \( q^* = 1/q \in (1, \infty) \).

On the one hand, in view of Lemma 2.2, the invariant control set of (2.11), says \( C \), is \( \mathbb{R}^2_+ \) if \( d^* \leq 0 \) and \( \{(u,v) \in \mathbb{R}^2_+ : u'v \geq d\} \) if \( d^* > 0 \). By [17, Lemma 4.1], \( C \) is exactly the support of the unique invariant measure \( \pi^* \). The strong law of large number can be obtained by using [23, Theorem 8.1] or [17].

3 Discussion and Numerical Examples

3.1 Discussion

We have shown that the extinction and permanence of the disease in a stochastic SIR model can be determined by the sign of a threshold value \( \lambda \). Only the critical case \( \lambda = 0 \) is not studied in this paper. To illustrate the significance of our results, let us compare our results with those in [20].
Theorem 3.1. [20, Theorem 3.1] Assume that $\sigma_1 > 0, \sigma_2 > 0$. Let $(S_{u,v}(t), I_{u,v}(t))$ be a solution of system (2.10). If $\mu > \sigma_1, \mu + \rho + \gamma > \sigma_2^2, R_0 > 1$ and
\[ \delta < \min \left\{ \frac{\mu^2}{\mu - \sigma_1^2} S^*_{\sigma_2^2}, \frac{(\mu + \rho + \gamma)^2}{\mu + \rho + \gamma - \sigma_2^2} I^*_{\sigma_2^2} \right\}. \]
Then there exists a stationary distribution $\pi^*$ for the Markov process $(S_{u,v}(t), I_{u,v}(t))$ which is the limit in total variation of transition probability $P(t, (u, v), \cdot)$. Here
\[ \delta = \frac{\mu \sigma_1^2}{\mu - \sigma_1^2} S^*_{\sigma_2^2} + \frac{(\mu + \rho + \gamma) \sigma_2^2}{\mu + \rho + \gamma - \sigma_2^2} I^*_{\sigma_2^2} + \frac{(\mu + \rho + \gamma) I^*_{\sigma_2^2}}{2\beta}, \]
\[ S^* = \frac{\mu + \rho + \gamma}{\beta}, \quad I^* = \frac{\alpha}{\mu + \rho + \gamma} - \frac{\mu}{\beta}; \quad R_0 = \frac{\beta \alpha}{\mu(\mu + \rho + \gamma)}. \]

By straightforward calculations or by arguments in Section 4 of [7] we can show that their conditions are much more restrictive than the condition $\lambda > 0$. Moreover, it should be noted that Theorem 2.1 is the same as Lemma 3.5 in [20]. In contrast to the aforementioned paper, we provide a rigorous proof of Theorem 2.1 here. Moreover, the conclusions in Theorems 2.1 and 2.2 still hold for the non-degenerate model (1.2). As a result we have the following theorem for model (1.2).

**Theorem 3.2.** Let $(S_{u,v}(t), I_{u,v}(t))$ be the solution to (1.2) with initial value $(S(0), I(0)) = (u, v) \in \mathbb{R}_+^{2,0}$. Define $\lambda$ as (2.5). If $\lambda < 0$, then $\lim_{t \to \infty} I_{u,v}(t) = 0$ a.s. and the distribution of $S_{u,v}(t)$ converges weakly to $\mu^*$, which has the density (2.3). If $\lambda > 0$, the solution process $(S_{u,v}(t), I_{u,v}(t))$ has a unique invariant probability measure $\varphi^*$ whose support is $\mathbb{R}_+^{2,0}$. Moreover, the transition probability $P(t, (u, v), \cdot)$ of $(S_{u,v}(t), I_{u,v}(t))$ converges to $\varphi^*(\cdot)$ in total variation. The rate of convergence is bounded above by any polynomial rate. Moreover, for any $\varphi^*$-integrable function $f$, we have
\[ \mathbb{P} \left\{ \lim_{t \to \infty} \frac{1}{t} \int_0^t f(S_{u,v}(\tau), I_{u,v}(\tau)) d\tau = \int_{\mathbb{R}_+^{2,0}} f(u', v') \varphi^*(du', dv') \right\} = 1 \quad \forall (u, v) \in \mathbb{R}_+^{2,0}. \]

It should be emphasized that our techniques can be also used to improve results in [5, 14, 28].

**3.2 Example**

Let us finish this paper by providing some numerical examples.
Example 3.1. Consider (2.1) with parameters $\alpha = 20$, $\beta = 4$, $\mu = 1$, $\rho = 10$, $\gamma = 1$, $\sigma_1 = 1$, and $\sigma_2 = -1$. Direct calculation shows that $\lambda = 67.5 > 0$, $d^* = 7.75 > 0$, and $c^* = 1.9375$. By virtue of Theorem 2.2, (2.1) has a unique invariant probability measure $\pi^*$ whose support is $\{(S, I) : S \geq \frac{1.9375}{T}\}$. Consequently, the strong law of large numbers and the convergence in total variation norm of the transition probability hold. A sample path of solution to (2.1) is illustrated by Figures 1, while the phase portrait in Figure 2 demonstrates that the support of $\pi^*$ lies above and includes the curve $S = \frac{c^*}{T} = \frac{1.9375}{T}$ as well as the empirical density of $\pi^*$. In non-degenerate case (Eq. (1.2)), with this same set of parameters the empirical density of $\pi^*$ is illustrated by Figure 3.

Example 3.2. Consider (2.1) with parameters $\alpha = 7$, $\beta = 3$, $\mu = 1$, $\rho = 1$, $\gamma = 2$, $\sigma_1 = 1$, and $\sigma_2 = 1$. For these parameters, the conditions in Theorem 3.1 are not satisfied. We obtain $\lambda = 16.5 > 0$, $d^* = -\infty$. As a result of Theorem 2.2, (2.1) has a unique invariant probability measure $\pi^*$ whose support is $\mathbb{R}_+^2$. Consequently, the strong law of large numbers and the convergence in total variation norm of the transition probability hold. A sample path of solution to (2.1) is depicted in Figures 4, while the phase portrait in Figure 5 demonstrates that the support of $\pi^*$ and the empirical density of $\pi^*$.

Example 3.3. Consider (2.1) with parameters $\alpha = 5$, $\beta = 5$, $\mu = 4$, $\rho = 1$, $\gamma = 1$, $\sigma_1 = 2$, and $\sigma_2 = -1$. It can be shown that $\lambda = -1.75 < 0$. As a result of Theorem 2.1, $I_{u,v}(t) \to 0$ a.s. as $t \to \infty$. This claim is supported by Figures 6. That is, the population will eventually have no disease. The distribution of $S_{u,v}(t)$ convergence to $f^*(x)$ as $t \to \infty$. The graphs of $f^*(x)$ and empirical density of $S_{u,v}(t)$ at $t = 50$ are illustrated by Figure 7.

![Figure 1: Trajectories of $S_{u,v}(t), I_{u,v}(t)$ in Example 3.1.](image-url)
Figure 2: Phase portrait of (2.1); the boundary $S = \frac{1.9375}{T}$ of the support of $\pi^*$ and the empirical density of $\pi^*$ in Example 3.1.

Figure 3: The empirical density of $\varphi^*$ in Example 3.1 for the non-degenerate equation (1.2).

Figure 4: Trajectories of $S_{u,v}(t), I_{u,v}(t)$ in Example 3.2.
Figure 5: Phase portrait of (2.1) and the empirical density of $\pi^*$ in Example 3.2.

Figure 6: Trajectories of $S_{u,v}(t), I_{u,v}(t)$ in Example 3.3.

Figure 7: The graph of the stationary density $f^*$ (in blue) and the graph of the empirical density of $S(50)$ (in red) in Example 3.3.
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