Mechanics of elastic networks

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Abstract

We consider a periodic lattice structure in \(d = 2\) or 3 dimensions with unit cell comprising \(Z\) thin elastic members emanating from a similarly situated central node. A general theoretical approach provides an algebraic formula for the effective elasticity of such frameworks. The method yields the effective cubic elastic constants for 3D space-filling lattices with \(Z = 4, 6, 8, 12\) and 14, the latter being the "stiffest" lattice proposed by [1]. The analytical expressions provide explicit formulas for the effective properties of pentamode materials, both isotropic and anisotropic, obtained from the general formulation in the stretch dominated limit for \(Z = d + 1\).

1. Introduction

Space frames, or periodic lattice structures of rods and joints, have long been of interest to engineers, architects, materials scientists and others. The octet truss, for instance, which is common in modern large scale structures because of its load bearing capacity may be attributed to Alexander Graham Bell’s interest in tetrahedral cells for building man-carrying kites [2]. Recent fabrication of micro-architected materials have used the octet truss tetrahedral cell design to achieve ultralight and ultrastiff structures [3]. An even stiffer structure comprising tetrakaidecahedral unit cells was proposed by [4, 1], see Fig. 1. Unlike the octet truss which has cubic elastic symmetry, the tetrakaidecahedral structure can display isotropic effective elastic properties. At the other end of the stiffness spectrum for elastic lattice structures are pentamode materials (PMs) with five easy modes of deformation [5] (see also page 666 of [6]). The range of such material properties, including high stiffness, strength and fracture toughness, exhibited by low density micro-architected materials is reviewed in [7].

The response of low density lattice structures depends on whether the deformation under load is dominated by stretching versus bending. This in turn depends upon the coordination number, \(Z\), the number of nearest neighboring joints in the unit cell, see Fig. 1 for several examples ranging from pentamodal \((Z = 4)\) to stiffest \((Z = 14)\). Maxwell [8] described the necessary although not sufficient condition for a \(d\)-dimensional \((d = 2, 3)\) space frame of \(b\) struts and \(j\) pin joints to be just rigid: \(b - 3 = (j - 3)d\). For an infinite periodic structure, \(b \approx jZ/2\), Maxwell’s condition becomes \(Z = 2d\). Structures with \(Z = 2d\), known as isostatic lattices, are at the threshold of mechanical stability [9]. A closer examination of the issue taking into account the degrees of freedom in the applied strain field, \(d(d + 1)/2\), leads to the conclusion that the necessary and sufficient condition for rigidity of frameworks with coordination number \(Z\) is \(Z \geq d(d + 1)/2\) [10]. The octet truss lattice \((Z = 12)\) is an example of a 3D lattice which satisfies the rigidity condition [11]. Three dimensional frameworks with \(Z < 12\) admit soft modes; thus, as we will see in §44.6 a cubic framework with \(Z = 6\) has 3 soft modes. Zero frequency modes, "floppy" modes, that occur for \(Z < 2d\) correspond to collapse mechanisms, a topic also examined by [12] for truss-like 2D lattices.

Three dimensional elasticity is characterized by 6 positive eigenvalues [13]. A pentamode material (PM) in 3D is the special case of elasticity with five zero eigenvalues, hence "penta". An inviscid compressible fluid like water serves as a useful reference material for PMs since it has a single bulk modulus but zero shear rigidity, the elastic stiffness tensor is \(C = K_0 I \otimes I \iff C_{ijkl} = K_0 \delta_{ij} \delta_{kl}\), where \(K_0\) is the bulk modulus and \(C_{ijkl}\) are the components of \(C\). This form of the elastic moduli corresponds to a rank one \(6 \times 6\) matrix Voigt matrix \([C_{IJ}]\) with single non-zero eigenvalue \(3K_0\). PMs can therefore be thought of as...
elastic generalizations of water but without the ability to flow; however, unlike water, for which the stress is isotropic, PMs can display anisotropy. Recent interest in PMs has increased after the observation that they provide the potential for realizing transformation acoustics [14]. Pentamode materials can be realized from specific microstructures with tetrahedral-like unit cells [5, 15]. These types of PM lattice structures are related to low density materials such as foams in which the low density is a consequence of the low filling fraction of the solid phase, see [10] for a review of mechanical properties of low density materials. Here we consider specific microstructures and find explicit values for the elastic moduli for isotropic and anisotropic PMs.

The purpose of the present paper is two-fold. First, we fill the need for a general theoretical approach that provides a simple means to estimate the effective elasticity of frameworks with nodes which are all similarly situated. Nodes are similarly situated if the framework appears the same when viewed from any one of the nodes [17]: the unit cell must therefore be space-filling, as are the cases in Fig. 1. Specific homogenization methods have been proposed for lattice structures: e.g. [18] use a mix of analytical and finite element methods, while [14, 20] provide a general mathematical scheme that is not easy to implement in practice. More general micropolar elasticity theories have also been considered for two-dimensional frameworks, e.g. by applying force and moment balances on the unit cell [21], or alternatively, using energy based methods [22]. The method proposed here derives the elastic tensor relating the symmetric stress to the strain. It does not assume micropolar theories, although the solution involves a local rotation within the unit cell required for balancing the moments, see [3].

In contrast to prior works, the present method is explicit and practical; it provides for instance, the effective cubic elastic constants for all the examples in Fig. 1 see [44]. The second objective is to provide analytical expressions for the effective properties of pentamode materials, both isotropic and anisotropic. The general theory derived here is perfectly suited to this goal. We show in §4.1 that the minimal coordination number necessary for a fully positive definite elasticity tensor is \(Z = d + 1\) (\(d = 2\) or \(3\)), the pentamode limit therefore follows by taking the stretch dominated limit for \(Z = d + 1\).

The paper proceeds as follows: The lattice model is introduced and the main results for the effective properties are summarized in §2. The detailed derivation is presented in §3. In §4 some properties of the effective moduli are described, including the stretch dominated limit, and examples of 5 different lattice structures are given. Pentamode materials, which arise as a special case of the stretch dominated limit when the coordination number is \(d + 1\), are discussed in §5. The two dimensional case is presented in §6 and conclusions are given in §7.

2. Lattice model

The structural unit cell in \(d\)-dimensions (\(d = 2, 3\)) comprises \(Z \geq d + 1\) rods and has volume \(V\). Let \(\mathbf{0}\) denote the position of the single junction in the unit cell with the cell edges at the midpoints of the rods, located at \(\mathbf{R}_i\) for \(i = 1, \ldots, Z\). Under the action of a static loading the relative position of the vertex initially located at \(\mathbf{R}_i\) moves to \(\mathbf{r}_i\). The angle between members \(i\) and \(j\) before and after deformation is \(\psi_{ij} = \cos^{-1}\left(\mathbf{R}_i \cdot \mathbf{R}_j / (|\mathbf{R}_i||\mathbf{R}_j|)\right)\) and \(\psi_{ij} = \cos^{-1}\left(\mathbf{r}_i \cdot \mathbf{r}_j / (|\mathbf{r}_i||\mathbf{r}_j|)\right)\), respectively, where \(\mathbf{R}_i = [\mathbf{R}_i|, \mathbf{r}_i = [\mathbf{r}_i|\). The end displacement \(\Delta \mathbf{r}_i = \mathbf{r}_i - \mathbf{R}_i\) is decomposed as \(\Delta \mathbf{r}_i = \Delta \mathbf{r}_i^\parallel + \Delta \mathbf{r}_i^\perp\). In the linear approximation assumed here \(\Delta \mathbf{r}_i^\parallel \approx \Delta \mathbf{r}_i\mathbf{e}_i\), where \(\Delta \mathbf{r}_i = \mathbf{r}_i - \mathbf{R}_i\) and the unit axial vector is \(\mathbf{e}_i = \mathbf{R}_i / |\mathbf{r}_i|\) (\(|\mathbf{e}_i| = 1\)). The change in angle between members \(i\) and \(j\) is \(\Delta \psi_{ij} = \psi_{ij} - \psi_{ij}, j \neq i\). The transverse displacement \(\Delta \mathbf{r}_i^\perp\) can include a contribution \(\Delta \mathbf{r}_i^\perp = \Delta \mathbf{r}_i^\perp\) (\(\mathbf{e}_i \cdot \Delta \mathbf{r}_i^\perp = 0\)) caused by rigid body rotation of the unit cell. We therefore define \(\Delta \mathbf{r}_i^\perp = \Delta \mathbf{r}_i^\perp\) as the transverse displacement associated with flexural bending. Vectors perpendicular to \(\mathbf{e}_i\) are used to define transverse bending forces. The unit vector \(\mathbf{e}_{ij}\) is perpendicular to \(\mathbf{e}_i\) and lies in the plane spanned by \(\mathbf{e}_i\) and \(\mathbf{e}_j\) with \(\mathbf{e}_{ij} \cdot \mathbf{e}_j < 0\), that is: \(\mathbf{e}_{ij} = \mathbf{e}_i \times (\mathbf{e}_i \times \mathbf{e}_j) / |\mathbf{e}_i \times \mathbf{e}_j|^\perp = (\cos \psi_{ij} \mathbf{e}_i - \mathbf{e}_j) / \sin \psi_{ij}, i \neq j \in \mathbb{Z}\). The unit vector(s) \(\mathbf{e}_i^\alpha, \alpha = 1 : d - 1\), are such that \(\{\mathbf{e}_i, \mathbf{e}_i^\alpha\}\) form an orthonormal set of \(d\)-vectors. Summation on repeated lower case Greek superscripts is understood (and only relevant for 3D). Define

\[
P_i^\parallel = \mathbf{e}_i \otimes \mathbf{e}_i, \quad P_i^\perp = \mathbf{e}_i^\alpha \otimes \mathbf{e}_i^\alpha, \quad (1)
\]

If \(\mathbf{e}_i = -\mathbf{e}_j\) we consider a slight perturbation so that \(\psi_{ij} \neq \pi\).
so that $\mathbf{P}_i^\parallel + \mathbf{P}_i^\perp = \mathbf{I}$, the unit matrix in $d$-dimensions. The axial tensor of a vector $\mathbf{v}$ is defined by its action on a vector $\mathbf{w}$ as $\text{ax}(\mathbf{v})\mathbf{w} = \mathbf{v} \times \mathbf{w}$. Finally, although the derivation will be mostly coordinate free, for the purpose of defining examples and the components of the effective stiffness tensor, we will use the orthonormal basis $\{\mathbf{a}_q\}$ ($q = 1 : d$).

![Unit cell for some lattices considered, see Table 1. The stretch dominated $Z = 14$ lattice with node at the center of the tetrakaidecahedral unit cell has maximal stiffness [1]. Colours are used to illustrate the structure for $Z = 12$ and to differentiate the shorter (blue) members from the longer ones (green) for $Z = 14$.](image)

2.1. Forces on individual members

The members interact in the static limit via combined axial forces directed along the members, and bending moments, associated with axial deformation and transverse flexure, respectively. We also include the possibility of nodal bending stiffness, associated with torsional spring effects at the junction. The strain energy can then be represented

$$ \mathcal{H} = \mathcal{H}^s + \mathcal{H}^b + \mathcal{H}^n $$

for stretch, bending and nodal deformation, respectively. Later, we consider the limit in which the contributions from bending, $\mathcal{H}^b$ and $\mathcal{H}^n$, are small, and the deformation may be approximated by axial forces only,
which is the stretch-dominated limit. Physically, this corresponds to slender members with small thickness to length ratio.

We assume strain energy of the form

\[
H^s = \sum_{i=1}^{Z} \frac{(\Delta r_i)^2}{2M_i}, \quad H^b = \sum_{i=1}^{Z} \frac{(\Delta r^b_i)^2}{2N_i}, \quad H^n = \sum_{i=1}^{Z} \sum_{j \neq i} R_i R_j (\Delta \psi_{ij})^2
\]

where \(M_i\) are the axial compliances, \(N_i\) the bending compliances and \(N_{ij}\) are the nodal bending compliances.

The force acting at the end of member \(i (i = 1, \ldots, Z)\) is

\[
f_i = f_i^s + f_i^b + f_i^n
\]

where \(f_i^s = \Delta r_i^s / M_i\), acting parallel to the member, is associated with stretching. The perpendicular component of the force acting on the member’s end is comprised of a shear force \(f_i^b = \Delta r^b_i / N_i\) caused by the bending of the member, plus a shear force \(f_i^n = R_j \Delta \psi_{ij} e_{ij} / N_{ij}\) associated with the node compliance.

The axial and bending compliances can be related to the member properties via

\[
M_i = \int_0^{R_i} \frac{dx}{E_i A_i}, \quad N_i = \int_0^{R_i} \frac{x^2 dx}{E_i I_i}, \quad i \in 1Z
\]

where \(E_i(x), A_i(x), I_i(x)\) are the Young’s modulus, cross-sectional area and moment of inertia, with \(x = 0\) at the nodal junction. We assume circular or square cross-section in 3D so that only a single bending compliance is required for each member, otherwise the results below involving \(N_i\) are not generally valid although they could be amended with necessary analytical complication. The nodal bending compliances \(N_{ij} \geq 0\) are arbitrary and satisfy the symmetry \(N_{ij} = N_{ji}\) which ensures that the sum of the moments of the node bending forces are zero.

### 2.2 Effective stress and moduli

We consider the forces on the members of the unit cell responding to an applied macroscopic loading. The forces acting at the node of the unit cell are equilibrated, as are the moments,

\[
\sum_{i=1}^{Z} f_i = 0, \quad \sum_{i=1}^{Z} R_i \times f_i = 0.
\]

Treating the volume of the cell as a continuum with equilibrated stress \(\sigma\), integrating \(\text{div} x \otimes \sigma = \sigma\) over \(V\) and identifying the tractions as the point forces \(f_i\) acting on the cell boundary, implies the well-known connection

\[
\sigma = V^{-1} \sum_{i=1}^{Z} R_i \otimes f_i.
\]

The symmetry of the stress, \(\sigma = \sigma^T\), is guaranteed by the moment balance \(\mathbb{M}\). Our aim is to derive the effective elastic moduli defined by the fourth order tensor \(C\) which relates the stress to the macroscopic strain \(\epsilon\) according to

\[
\sigma = C \epsilon.
\]

The elements of the elastic stiffness \(C\) when expressed in an orthonormal basis possess the symmetries \(C_{ijkl} = C_{jikl}\) and \(C_{ijkl} = C_{klij}\), and the elements can also be represented in terms of the Voigt notation via \(C_{ijkl} \rightarrow C_{IJ} = C_{JI}\).
2.3. Summary of the main result for the effective elastic stiffness

We first introduce the vectors $d_i$, $d_i^\alpha$, $d_{ij}$ ($= d_{ij}$), the second order symmetric tensors $D_i$, $D_i^\alpha$, $D_{ij}$ ($= D_{ji}$) and the $L \times L$ matrix with elements $P_{ij}$, where $L = Zd + Z(Z-1)/2$:

$$d_i = \frac{e_i}{\sqrt{M_i}},$$
$$d_i^\alpha = \frac{e_i^\alpha}{\sqrt{N_i}},$$
$$\alpha = 1 : d - 1$$
$$\text{d}_{ij} = \frac{R_i R_j}{N_{ij}} \left( \frac{e_{ij}}{R_i} + \frac{e_{ji}}{R_j} \right).$$

These results are derived in the next section and implications are discussed in the discussion of the elastic moduli represented in $6 \times 6$ Voigt notation. The general structure of eqs. (9) holds for $d = 2$ without requiring the zero rotation conditions of eq. (21), as discussed in [43].

3. Derivation of the effective elasticity tensor

3.1. Affine deformation

Strain is introduced through the affine kinematic assumption that the effect of deformation is to cause the cell edges to displace in a linear manner proportional to the (local) deformation gradient $F$. Edge points originally located at $R_i$ are translated to $FR_i$. In addition to the affine motion, we include two $d$-vectors, introduced to satisfy the equilibrium conditions [40]. Following [28], we assume that the junction moves from the origin to $\chi$. An additional rotation $Q \in \text{SO}(d)$ is introduced, so that the vector defining the edge relative to the vertex is

$$r_i = QFR_i - \chi.$$

The linear approximation for the deformation is $F = I + \epsilon + \omega$ with $\epsilon = \epsilon^T$ and $\omega = -\omega^T$. We take $Q = e^T = I + \Gamma + O(\Gamma)^2$ where the skew symmetric matrix $\Gamma$ is defined by the $d$-vector $\gamma$ as $\Gamma = ax(\gamma)$. Hence,

$$\Delta r_i = r_i - R_i = (\epsilon + \omega + \Gamma)R_i - \chi.$$  

In the linear approximation $r_i$ can equally well be taken along $R_i$ as far as second order terms are concerned. Thus,

$$\Delta r_i^\parallel = (R_iP_i^\parallel : e_i : \chi) e_i,$$
$$\Delta r_i^\perp = P_i^\perp (R_i e_i - \chi) + R_i (\omega + \Gamma) e_i,$$
$$\Delta \phi_{ij} = e_i : e_{ij} + e_j : e_{ji} - (R_i^{-1} e_{ij} + R_j^{-1} e_{ji}) \cdot \chi.$$

The tangential displacement governing the shear bending force is, after removing the affine rigid body rotation,

$$\Delta r_i^b = \Delta r_i^\perp - R_i \omega e_i.$$  

These results are derived in the next section and implications are discussed in [43] including a simple expression [28] for the elastic moduli represented in $6 \times 6$ Voigt notation. The general structure of eqs. (9) holds for $d = 2$ without requiring the zero rotation conditions of eq. (21), as discussed in [43].
therefore reduces to only the bending shear forces  

\[ f = M_{ii}^{-1} R_i (P_i^\parallel : e) e_i + N_i^{-1} R_i (P_i^\perp e e_i + \gamma \times e_i) + \sum_{j \neq i} N_{ij}^{-1} R_j (e_i \cdot e e_{ij} + e_j \cdot e e_{ji}) e_{ij} \]

\[- \left( M_{ii}^{-1} P_i^\parallel + N_i^{-1} P_i^\perp + \sum_{j \neq i} N_{ij}^{-1} R_j e_{ij} \otimes \left( R_i^{-1} e_{ij} + R_j^{-1} e_{ji} \right) \right) \chi. \]  

(15)

This explicit expression for the forces allows us to determine the vectors  \( \chi \)  and  \( \gamma \), next.

3.2. Solution of the equilibrium equations

Consider first the moment balance condition (6). Of the three terms comprising the force in eq. (4) only the bending shear forces  \( f^\parallel \) does not automatically yield zero moment. Equilibrium of the moments therefore reduces to

\[ \sum_{i=1}^{Z} R_i \times f_i^\parallel = 0. \]  

(16)

Substituting  \( f_i^\parallel = \Delta P_i^\parallel / N_i \) and using eqs. (14) and (16) allows us to find  \( \gamma \) in the form

\[ \gamma = B \left( g \times \chi - \sum_{j=1}^{Z} R_j^2 N_j e_j \times e e_j \right) \]  

where  \( B = \left( \sum_{i=1}^{Z} R_i^2 N_i \right)^{-1} \)  

\[ g = \sum_{i=1}^{Z} R_i e_i. \]  

(17)

The force on member  \( i \) becomes, using eq. (15),

\[ f_i = R_i e_i (P_i^\parallel : e) e_i + R_i N_i \left( P_i^\perp e e_i + ax(e_i)B \sum_{j=1}^{Z} R_j^2 N_j ax(e_j) e e_j \right) + \sum_{j \neq i} R_j N_j (e_i \cdot e e_{ij} + e_j \cdot e e_{ji}) e_{ij} \]

\[- \left( P_i^\parallel + P_i^\perp / N_i + R_i N_i ax(e_i)B ax(g) + \sum_{j \neq i} R_j N_j e_{ij} \otimes (e_{ij} / R_i + e_{ji} / R_j) \right) \chi. \]  

(18)

The equilibrium condition (6) can then be solved for  \( \chi \) as

\[ \chi = A^{-1} \sum_{i=1}^{Z} \left( R_i / M_i (P_i^\parallel : e) e_i + R_i / N_i (P_i^\perp + ax(g)B ax(e_i)) \cdot e_i \right) \]

\[ + \sum_{j \neq i} R_j N_j (e_i \cdot e e_{ij} + e_j \cdot e e_{ji}) e_{ij} \]  

(19)

where

\[ A = \sum_{i=1}^{Z} \left( P_i^\parallel / M_i + P_i^\perp / N_i + R_i R_j e_{ij} / R_i \otimes (e_{ij} / R_i + e_{ji} / R_j) + ax(g)B ax(g) \right). \]  

(20)

Equations (17), (18), and (19) provide the desired linear relation between the strain and the stress from which one can derive the effective elastic moduli.

3.3. A simplification

While eqs. (17), (18), (13)–(20) provide all of the necessary ingredients for the most general situation we assume for the remainder of the paper that the unit cell rotation vanishes, implying  \( \gamma = 0 \). Hence, the vector  \( g \) and the second term in the expression for  \( \gamma \) in eq. (17) vanish. The latter identity is equivalent to
\((Dv) \times v = 0 \ \forall v\) where \(D = \sum_{i=1}^{Z} \frac{R_i^2}{N_i} e_i \otimes e_i\). This implies that \(D\) must be proportional to the identity, hence the zero rotation condition may be written

\[
\sum_{i=1}^{Z} \frac{R_i^2}{N_i} e_i = 0 \quad \text{and} \quad \sum_{i=1}^{Z} \frac{R_i^2}{N_i}(e_i \otimes e_i - \frac{1}{d} I) = 0 \quad \Leftrightarrow \quad \text{zero cell rotation.} \quad (21)
\]

The identities \((21)\) hold for the examples considered later. Note that the assumption of zero rotation is not necessary for stretch dominated lattices in which bending effects are negligible.

### 3.4. Effective stiffness

In order to arrive at an explicit expression for the elastic stiffness tensor we first write the stress in terms of strain, using eqs. \((7), (15)-(19)\).

\[
\sigma = \frac{1}{V} \sum_{i=1}^{Z} \left( \frac{R_i^2}{M_i} \mathbf{p}_i^i (\mathbf{p}_i^i : \epsilon) + \frac{R_i^2}{N_i} \mathbf{e}_i \otimes \mathbf{e}_i^o (\mathbf{e}_i^o \cdot \mathbf{e}_i) \right) + \sum_{j \neq i} \frac{R_i R_j}{N_{ij}} \mathbf{e}_i \otimes \mathbf{e}_{ij} (\mathbf{e}_i \cdot \mathbf{e}_{ij} + \mathbf{e}_j \cdot \mathbf{e}_{ij})
\]

\[
- \frac{1}{V} \left( \sum_{i=1}^{Z} \left( \frac{R_i}{M_i} \mathbf{d}_i + \frac{R_i}{N_i} (\mathbf{e}_i \otimes \mathbf{e}_i^o) \mathbf{d}_i^o \right) + \sum_{j \neq i} \frac{R_i R_j}{N_{ij}} \mathbf{e}_i \otimes \mathbf{e}_{ij} \mathbf{d}_{ij} \right) \cdot A^{-1}
\]

\[
\cdot \left( \sum_{k=1}^{Z} \left( \mathbf{d}_k R_k \sqrt{M_k} \mathbf{p}_k^k : \epsilon + \mathbf{d}_k^o R_k \sqrt{N_k} \mathbf{e}_k^o \cdot \mathbf{e}_k \right) + \frac{1}{2} \sum_{j \neq k} \mathbf{d}_{kl} \frac{R_i R_l}{N_{kl}} (\mathbf{e}_k \cdot \mathbf{e}_{kl} + \mathbf{e}_l \cdot \mathbf{e}_{lk}) \right) \quad (22)
\]

It follows from \((22)\), the symmetry of the stress and strain and from the definition of the second order symmetric tensors \(D_i, D_{ij}\) in \((9b)\) that the elastic moduli can be expressed

\[
C = \sum_{i=1}^{Z} \left( D_i \otimes D_i + D_i^o \otimes D_i^o \right) + \frac{1}{2} \sum_{i=1}^{Z} D_{ij} \otimes D_{ij}
\]

\[
- \left( \sum_{i=1}^{Z} (D_i \mathbf{d}_i + D_i^o \otimes \mathbf{d}_i^o) + \frac{1}{2} \sum_{j \neq i} D_{ij} \mathbf{d}_{ij} \right) \cdot A^{-1}
\]

\[
\cdot \left( \sum_{k=1}^{Z} (D_k \mathbf{d}_k + D_k^o \otimes \mathbf{d}_k^o) + \frac{1}{2} \sum_{j \neq k} \mathbf{d}_{kl} \mathbf{D}_{kl} \right) \quad (23)
\]

Finally, we note, based on the definitions of the vectors in \((9b)\), that

\[
A = \sum_{i=1}^{Z} \left( \mathbf{d}_i \otimes \mathbf{d}_i + \mathbf{d}_i^o \otimes \mathbf{d}_i^o \right) + \frac{1}{2} \sum_{i=1}^{Z} \mathbf{d}_{ij} \otimes \mathbf{d}_{ij} = \sum_{i=1}^{L} \mathbf{u}_i \otimes \mathbf{u}_i \quad (24)
\]

The sets \(\{u_k\}\) and \(\{U_k\}\) defined in \((9c)\) combine the \(Z\) vectors/tensors associated with stretch, the \((d - 1)Z\) vectors/tensors associated with shear, and the \(Z(Z - 1)/2\) vectors/tensors associated with nodal bending into sets of \(L = dZ + Z(Z - 1)/2\) elements in terms of which \((23)\) becomes

\[
C = \sum_{i=1}^{L} \mathbf{U}_i \otimes \mathbf{U}_i - \left( \sum_{i=1}^{L} \mathbf{U}_i \mathbf{u}_i \right) \cdot \left( \sum_{j=1}^{L} \mathbf{u}_j \otimes \mathbf{u}_j \right)^{-1} \cdot \left( \sum_{k=1}^{L} \mathbf{u}_k \mathbf{U}_k \right) \quad (25)
\]

It then follows from the definition of \(P\) in \((9c)\) that \(C\) can be expressed in the form \((10)\).
4. Properties of the effective moduli

4.1. Generalized Kelvin form

The $L \times L$ symmetric matrix $P$ with elements $P_{ij}$ defined in eq. (24) has the crucial properties

$$P^2 = P, \quad \text{rank} P = L - d,$$

i.e. $P$ is a projector, and the dimension of its projection space is $\text{tr} P = L - d$. Hence, the summation in (10) is essentially the sum of $L - d$ tensor products of second order tensors. This is to be compared with the Kelvin form for the elasticity tensor $\mathbf{C}$

$$\mathbf{C} = \sum_{i=1}^{3d-3} \lambda_i \mathbf{S}_i \otimes \mathbf{S}_i \quad \text{where} \quad \lambda_i > 0, \quad \text{tr} \mathbf{S}_i \mathbf{S}_j = \delta_{ij}. \quad (27)$$

The second order symmetric tensors are eigenvectors $\{\mathbf{S}_i\}$ that diagonalize the elasticity tensor, with eigenvectors $\lambda_i$ known as the Kelvin stiffnesses. Equation (10) provides a non-diagonal representation for $\mathbf{C}$.

Note that $L \equiv L_s + L_b + L_n$ where $L_s = Z$ is associated with stretch, $L_b = (d - 1)Z$ with bending shear and $L_n = Z(Z - 1)/2$ with nodal bending. A necessary although not sufficient condition for positive definiteness of $\mathbf{C}$ is that the rank of $\mathbf{P}$, which is $L - d$, exceed $3d - 3$. Ignoring nodal bending ($L = L_s + L_b$) this is satisfied if $Z \geq d + 1$ for $d = 2$ and 3. The requirement is stricter in the stretch dominated limit ($L = L_s$): $Z \geq 6$ in 2D and $Z \geq 10$ in 3D.

4.2. $6 \times 6$ matrix in 3 dimensions

The main result of eq. (10) implies a simple representation for the $6 \times 6$ matrix of elastic moduli $[C_{ij}]$ based on the compact Voigt notation ($C_{ijkl} \rightarrow C_{ij}$) in the orthonormal basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Let $[\mathbf{u}]_{3 \times L}$ denote the $L$ vectors $\{\mathbf{u}_k\}$ and let $[\mathbf{U}]_{6 \times L}$ denote the $L$ second order tensors $\{\mathbf{U}_k\}$ according to $U_{ik} = \mathbf{a}_i \cdot \mathbf{U}_k \cdot \mathbf{a}_j$ with the standard correspondence $i \in \{1, 2, 3, 4, 5, 6\} \rightarrow ij \in \{11, 22, 33, 23, 31, 12\}$. Then eq. (10) becomes

$$\begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\
C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\
C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}
\end{pmatrix} = [\mathbf{U}]^T [\mathbf{U}] - [\mathbf{U}]^T \mathbf{u} \mathbf{u}^T (\mathbf{u} \mathbf{u}^T)^{-1} \mathbf{u} [\mathbf{U}]^T. \quad (28)$$

4.3. Bulk modulus

If the effective medium has isotropic or cubic symmetry then a strain $\epsilon = \epsilon \mathbf{I}$ produces strain $\sigma = dK \epsilon \mathbf{I}$ where $K$ is the $d$–dimensional bulk modulus. More generally, whether or not the symmetry is cubic or isotropic, we can define $K = d^{-2}C_{iijj}$. The bulk modulus follows from eqs. (1) and (11) as

$$K = \frac{1}{d^2V} \sum_{i,j=1}^Z P_{ij} \frac{R_i R_j}{\sqrt{M_i M_j}}. \quad (29)$$

This simplifies further under the broad assumption that

$$\sum_{i=1}^Z \frac{\epsilon_i}{\sqrt{M_i}} = 0, \quad (30)$$

certainly true of all the examples of Fig. 1 considered in [11, 12] so that

$$K = \frac{1}{d^2V} \sum_{i=1}^Z \frac{R_i^2}{M_i}. \quad (31)$$
Note that the bulk modulus depends only on the axial stiffness of the members. Assume the members are the same material \((E_i = E)\), and each has constant cross-section (area or width) \(A_i\), then according to eqs. [51] and [31],

\[
K = \frac{\phi}{d^2 E} \text{ where } \phi = \frac{1}{V} \sum_{i=1}^{Z} A_i R_i
\]

is the volume fraction of solid material in the lattice. The scaling of bulk modulus with volume fraction, \(K \propto \phi E\), is well known, e.g. [24, eq. (2.2)] for \(d = 2\), [25], [26] for tetrakaidecahedral unit cells (see below), and [1].

4.4. Model simplification

While the model considered is quite general, in practice there is little information on the form of the nodal compliances for practical situations. For the remainder of the paper we concentrate on just the stretch and shear bending effects, so that \(L = L_s + L_h + L_n \rightarrow L_s + L_h = dZ\). The stress-strain relation is then

\[
\sigma = \sum_{i=1}^{Z} \left( R_i \otimes X_i [I \otimes R_i] - \sum_{k=1}^{Z} X_k \otimes R_j \right) : \epsilon \text{ where } X_i = \frac{P_i}{M_i} + \frac{P_i^+}{N_i}.
\]

A further simplifications is obtained by ignoring shear bending effects, i.e. \(L \rightarrow L_s = Z\), the stretch dominated limit, considered next.

4.5. Stretch dominated limit

In this limit the forces \(f\), have no transverse components. Physically, this corresponds to infinite bending compliances, \(1/N_i = 0\), \(1/N_{ij} = 0\), and may be achieved approximately by long slender members. By ignoring shear and nodal bending the expression for \(C\) reduces to

\[
C = \frac{1}{V} \sum_{i,j=1}^{Z} \frac{R_i R_j}{M_i M_j} P_{ij} \frac{P_i^\parallel \otimes P_j^\parallel}{P_i^\parallel M_j} \cdot \left( \sum_{k=1}^{Z} \frac{P_k}{M_k} \right)^{-1} \cdot \frac{\epsilon_j}{\sqrt{M_j}}.
\]

It follows from eq. (26) that the \(Z \times Z\) projection matrix \(P\) with elements \(P_{ij}\) has rank \(Z - d\).

4.6. Examples in 3D: \(Z = 4, 6, 8, 12, 14\)

All examples display cubic symmetry, with three independent elastic moduli: \(C_{11}, C_{12}\) and \(C_{44}\). Introduce the fourth order tensors \(\Pi, J\) and \(D\) with components \(\Pi_{ijkl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), J_{ijkl} = \frac{1}{2} \delta_{ij} \delta_{kl},\) and \(D_{ijkl} = \delta_{i1} \delta_{j1} \delta_{k1} \delta_{l1} + \delta_{i2} \delta_{j2} \delta_{k2} \delta_{l2} + \delta_{i3} \delta_{j3} \delta_{k3} \delta_{l3}\). A solid of cubic symmetry has elasticity of the form

\[
C = 3K J + 2\mu_1 (I - D) + 2\mu_2 (D - J).
\]

The isotropic tensor \(J\) and the tensors of cubic symmetry \((I - D)\) and \((D - J)\) are positive definite [27], so the requirement of positive strain energy is that \(K, \mu_1\) and \(\mu_2\) are positive. These three parameters, called the “principal elasticities” by Kelvin [13], can be related to the standard Voigt stiffness notation: \(K = (C_{11} + 2C_{12})/3, \mu_1 = C_{44}\) and \(\mu_2 = (C_{11} - C_{12})/2\). The bulk modulus follows from eq. [52] as

\[
K = \frac{\phi}{9} E \forall Z; \quad K = \frac{Z R^2}{9 V M}, \quad Z \neq 14; \quad K = \frac{4a^2}{3V} \left( \frac{1}{M_1} + \frac{1}{M_2} \right), \quad Z = 14
\]

where for \(Z = 14\) \(M_1, M_2\) are the axial compliances of the two different types of members. It may be checked that \(K_{14} = K_6 + K_8\) where \(K_6\) denotes the bulk modulus for coordination number \(Z\). The shear moduli are given in Table [11]. Note that the effective compliance, relating strain to stress by \(\epsilon = C^{-1} \sigma\) is simply
The volume fraction $\phi$ = 14

M compliances are

Hence, Table 1 indicates that $\mu_\phi = C \mu R$.

Rods extend from the center to all faces of the Kelvin cell as shown in Fig. 1. Note the functional dependence

Table 1: 3D lattice structures considered. They display cubic elastic symmetry with $C_{11} = K + \frac{4}{3} \mu_0$, $C_{12} = K - \frac{2}{3} \mu_0$ and $C_{44} = \mu_1$ where $K$ is given by eq. (33). All cases except $Z = 14$ have uniform rod length $R$ and compliances $M, N$. The boundary of the tetrakaidecahedral ($Z = 14$) unit cell has 36 edges each of length $a$ and the cell comprises members of two types: 6 of length $R_1 = \sqrt{3}a$ and 8 of length $R_2 = \sqrt{2}a$, the average length of the members is $\bar{R} = 1.306a$. The associated compliances are $M_1, N_1$ and $M_2, N_2$. The volume fraction $\phi$ in all cases is based on cylindrical rods of uniform radius $b$. Note that the volume fraction increases with coordination number $Z$.

| $Z$ cell | $\phi$ | $(e_i)$ (not normalized) | $\frac{M_1}{N_1}$ | $\frac{M_2}{N_2}$ |
|----------|--------|-------------------------|-------------------|-------------------|
| 4 diamond | $\frac{6}{\sqrt{2}}R^3$ | 1.02 $\frac{5}{2}$ | (−1) | (−1) | 1 | (−1) | (−1) | $\frac{9M}{2N}$ | $\frac{3M}{2N}$ |
| 6 simple cubic | $8R^3$ | 2.36 $\frac{5}{2}$ | (±1) | (±1) | 0 | (±1) | (±1) | $\frac{3M}{2N}$ | $\frac{3}{2}$ |
| 8 BCC | $\frac{32}{12}R^3$ | 4.08 $\frac{5}{2}$ | (±1) | (±1) | 0 | (±1) | (±1) | $\frac{1}{4} + M_{N}^{2}$ | $\frac{3}{2} + \frac{9M}{2N}$ |
| 12 FCC octet truss | $4\sqrt{3}a^3$ | 6.66 $\frac{5}{2}$ | (±1) | (±1) | 0 | (±1) | (±1) | $\frac{3}{2} + \frac{M_{1}^{2}}{N_{1}^{2}} + \frac{M_{2}^{2}}{N_{2}^{2}} + \frac{\phi_{6}^{2}}{N_{1}^{2}}$ | $\frac{3}{2} + \frac{M_{1}^{2}}{N_{1}^{2}} + \frac{M_{2}^{2}}{N_{2}^{2}}$ |
| 14 tetrakaidecahedral | $8\sqrt{2}a^3$ | 8.66 $\frac{5}{2}$ | (±1) | (±1) | 0 | (±1) | (±1) | $\frac{3}{2} + \frac{M_{1}^{2}}{N_{1}^{2}} + \frac{M_{2}^{2}}{N_{2}^{2}} + \frac{\phi_{6}^{2}}{N_{1}^{2}}$ | $\frac{3}{2} + \frac{M_{1}^{2}}{N_{1}^{2}} + \frac{M_{2}^{2}}{N_{2}^{2}}$ |

$C^{-1} = (3k)^{-1} \mathbb{J} + 2(\mu_1)^{-1} (\mathbb{I} - \mathbb{D}) + 2(\mu_2)^{-1} (\mathbb{D} - \mathbb{J})$. The ratio $M/N$ may also be expressed in terms of the volume fraction $\phi$ since the rods are assumed to be solid circular so that

$$\frac{M}{N} = \frac{3b^2}{4R^2}. \quad (37)$$

Hence, Table 1 indicates that $\mu_1 = O(\phi^2)$ for $Z = 4, 6$ and $\mu_2 = O(\phi^2)$ for $Z = 4, 8$; otherwise $\mu_1, \mu_2 = O(\phi)$.

$Z = 14$: the tetrakaidecahedral unit cell

The tetrakaidecahedron is a truncated octahedron with all edges of the same length $a \Rightarrow V = 8\sqrt{2}a^3$. Rods extend from the center to all faces of the Kelvin cell as shown in Fig. 1. Note the functional dependence $\mu_1 = \mu_1(M_1, N_2)$, $\mu_2 = \mu_2(M_2, N_1, N_2)$. Isotropy ($\mu_1 = \mu_2$) is achieved if

$$\frac{3}{M_1} - \frac{3}{N_1} = \frac{2}{M_2} - \frac{2}{N_2} \quad (38)$$

in which case the effective Poisson’s ratio is

$$\nu = \frac{M_1^{-1} - N_1^{-1}}{4M_1^{-1} - 2N_1^{-1} + 2N_2^{-1}}. \quad (39)$$

In the stretch dominated limit $1/N_1, 1/N_2 \to 0$ the $6 \times 6$ Voigt matrix of effective elastic moduli is

$$C_{14} = C_6 + C_8 \quad \text{where} \quad C_6 = \frac{\phi_6}{3} E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_8 = \frac{\phi_6}{9} E \begin{pmatrix} J & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{all elements of} \ 3 \times 3 \text{ matrix} \ J \text{ are unity, and the volume fractions} \ \phi_6 = 6R_1A_1/V, \ \phi_8 = 8R_2A_2/V \text{ satisfy} \ \phi = \phi_6 + \phi_8. \ \text{The three moduli follow from} \ (10) \text{ as}$$

$$K = \frac{\phi}{9} E, \ \mu_1 = \frac{\phi_6}{9} E, \ \mu_2 = \frac{\phi_6}{6} E. \quad (41)$$
Isotropy, \( \mu_1 = \mu_2 \equiv \mu \), is achieved if \( 3\phi_0 = 2\phi_8 \), i.e.,

\[
A_1 = \frac{4}{3\sqrt{3}} A_2 \quad \Rightarrow \quad \mu = \frac{\phi}{15} E
\]

(42)
in which case the effective Poisson’s ratio is \( \nu = \frac{1}{3} \), in agreement with (39). This effective solid is the 3D isotropic “optimal” material introduced by [1].

5. Pentamode lattices

5.1. \( Z = d + 1 \) and the pentamode limit

As discussed in [44], \( Z = d + 1 \) is the minimal coordination number necessary for a fully positive definite elasticity tensor. We now examine this case in particular in the limit of stretch dominant deformation.

Given that a PM is an elastic solid with a single Kelvin modulus the elastic stiffness must be of the form

\[
C = \lambda S \otimes S, \quad \lambda > 0, \quad S \in \text{Sym.}
\]

(43)

Note that the parameter \( \lambda \) is somewhat arbitrary since it can be replaced by unity by subsuming it into the definition of \( S \). Since \( \text{rank} \ P = Z - d \) it follows that the single non-zero eigenvalue of \( P \) of (33) is unity, i.e. there exists a \((d + 1)\)-vector \( b \) such that

\[
P = bb^T \quad \text{where} \quad b^T b = 1.
\]

(44)

Hence, (44),1 yields the moduli explicitly in the form (43) with

\[
\lambda = 1, \quad S = V^{-1/2} \sum_{i=1}^{d+1} R_i M_i^{1/2} b_i P_i^T.
\]

(45)

The eigenvalue property \( Pb = b \) implies that \( b \) satisfies \( \sum_{i=1}^{d+1} b_i u_i = 0 \), i.e. it is closely related with the fact that the \( d + 1 \) vectors \( u_i \) are necessarily linearly dependent. Alternatively, \( b \) follows by assuming that \( C \) of eq. (33) has PM form \( C = S \otimes S \), then use \( CI = S \text{tr} S \) and \( I : CI = (\text{tr} S)^2 \), from which we deduce that the moduli have the form (43), with

\[
\lambda = (V \sum_{k=1}^{d+1} \gamma_k)^{-1}, \quad S = \sum_{i=1}^{d+1} \gamma_i P_i^T \quad \text{where} \quad \gamma_i = \frac{R_i^2}{M_i} - \frac{R_e}{M_e} \left( \sum_{k=1}^{d+1} \frac{M_k}{M_i} \right)^{-1} \sum_{j=1}^{d+1} \frac{R_j}{M_j}.
\]

(46)

Equations (43), (45) and (46) provide two alternative and explicit formulas for the PM moduli.

It is interesting to note that either of the above formulas for \( C \) leads to an expression for the axial force in member \( i \) based on equations (1) and (S). Thus, using eq. (45) gives \( f_i = V\lambda(S : e)R_i^{-1}\gamma_i e_i \). It may be checked from the definition of \( \gamma_i \) that the forces are equilibrated, since

\[
\sum_{i=1}^{d+1} R_i^{-1} \gamma_i e_i = 0.
\]

(47)

This identity implies that \( \gamma_i = 0 \) for some member \( i \) only if (but not iff) the remaining \( d \) members are linearly dependent. When this unusual circumstance occurs the member \( i \) bears no load since \( f_i = 0 \) for any applied strain. For instance, if two members are collinear in 2D, say members 1 and 2, then the third member is not load bearing only if \( R_1^{-1} \gamma_1 = R_2^{-1} \gamma_2 \). When \( d \) of the members span a \((d - 1)\)-plane the remaining member is non-load bearing if it is orthogonal to the plane.

Writing \( S \) in terms of its principal directions and eigenvalues, \( S = s_1 q_1 q_1 + s_2 q_2 q_2 + s_3 q_3 q_3 \), where \( \{q_1, q_2, q_3\} \) is an orthonormal triad, it follows that the elastic moduli in this basis are

\[
C_{IJ} = \lambda s_I s_J \quad \text{if} \quad I, J \in \{1, 2, 3\}, \quad 0 \quad \text{otherwise}.
\]

(48)
with vectors \( n \) depends upon the soft moduli is the Poisson’s ratio: for a given pair of directions defined by the orthonormal generalized shear moduli must be determined as part of the full elasticity tensor. A measurable quantity that structure of Fig. 1(a) with shear moduli given by Table 1, \( Z = M \langle \nu \rangle \in \mathbb{R} \) in the \( n \) direction for a uniaxial applied stress along \( n \), i.e. \( \nu_{nm} = -\langle m n : M n m \rangle / \langle m n : M n n \rangle \) where \( M = C^{-1} \) is the fourth order tensor of elastic compliance. As an example, consider the diamond-like structure of Fig. 1(a) with shear moduli given by Table 1 \( Z = 4 \). In the pentamode limit \( K \gg \mu_1 = 3\mu_2 \), with \( n_i, m_i \) as the components in the principal axes, we obtain (see e.g. [28])

\[
\nu_{nm} = \frac{\frac{1}{2} - n_i^2 m_j^2 - n_i^2 m_k^2 - n_j^2 m_k^2}{n_i^2 + n_j^2 + n_k^2} \in [0, \frac{1}{2}],
\]

(49)

The actual values of the soft moduli \( \{ \mu_i, i = 1, \ldots, 5 \} \) are sensitive to features such as junction strength and might not be easily calculated in comparison with the pentamode stiffness. An estimate of the Poisson effect can be obtained by assuming the five soft moduli equal, in which case \( C(0) \equiv C \) of eq. [43] is modified to

\[
C(\mu) = C(0) + 2\mu(\mathbb{I} - (\lambda tr (S^2))^{-1} C(0)),
\]

(50)

which is invertible (and positive definite) for \( \mu > 0 \). Using \( M = C^{-1}(\mu) \) define \( \nu_{nm}(\mu) \), then the limit exists as the shear modulus is reduced to zero: \( \nu_{nm}(0) \equiv \nu_{nm} \) where

\[
\nu_{nm} = \frac{\langle m \cdot S m \rangle \langle n \cdot S n \rangle}{S^2 - \langle n \cdot S n \rangle^2}.
\]

(51)

For the example of Figure 1(a) \( S = I \) and eq. [31] gives \( \nu_{nm} = 1/2 \). Generally, the values of \( \nu_{nm} \) from eq. [43] associated with the principal axes of \( S \) (see [28]) are \( \nu_{ij} = s_i s_j / (s_i^2 + s_j^2) \), \( i \neq j \neq k \neq i \). If \( s_1 > s_2 > s_3 > 0 \) then the largest and smallest values are \( \nu_{12} > \frac{1}{2} \) and \( \nu_{32} < \frac{1}{2} \), respectively. Compare this

Figure 2: The elastic moduli for 2D and 3D PM lattices with rods of equal length \( (R_1 = R_2) \) and stiffness \( (M_1 = M_2) \) as a function of the junction angle \( \theta \). Note that the 2D (3D) moduli are identical at the isotropy angle 60° (70.53°). The axial stiffness \( C_{11} \) vanishes at \( \theta = \frac{\pi}{2} \). Since \( C_{12} = \sqrt{C_{11}C_{22}} \) it follows that \( C_{12} \) also vanishes at \( \theta = \frac{\pi}{2} \).
with the Poisson’s ratio of an incompressible isotropic elastic material: \( \nu = \frac{1}{2} \). Negative values of Poisson’s ratio occur if the principal values of \( S \) are simultaneously positive and negative.

\[

\nu_{ij} = \frac{\nu_{12}}{\frac{1}{2} + 2\nu_{12}}

\]

\( \theta = 50^\circ \) \hspace{1cm} \( \theta = 60^\circ \) \hspace{1cm} \( \theta = 70^\circ \)

Figure 3: Each of these two-dimensional PM lattices have isotropic quasi-static properties. The ratio of the \( R_1 \) (red) to \( R_2 \) (blue) is determined by (55). The pure honeycomb structure is \( \theta = 60^\circ \).

5.3. Transversely isotropic PM lattice

Assume the unit cell has symmetry consistent with macroscopic transverse isotropy. It comprises two types of rods: \( i = 1 \) with \( R_1, M_1 \) in direction \( e = e_1 \), and \( i = 2, \ldots, d + 1 \) with \( R_2, M_2 \) in directions \( e_i \) symmetrically situated about \( -e \) with \( -e \cdot e_i = \cos \theta \). Let \( c = \cos \theta, s = \sin \theta \). We find after some simplification, that (43) and (46) give the PM elastic stiffness as

\[

C = \frac{ds^4R_2^2}{V(d-1)^2(2c^2M_1 + M_2)}(I + (\beta - 1)e \otimes e) \otimes (I + (\beta - 1)e \otimes e)

\]

(52)
where the non-dimensional parameter \( \beta \) and the unit cell volume \( V \) are

\[
\beta = \frac{(d - 1)c(R_1 + cR_2)}{s^2R_2}, \quad V = (sR_2)^{d-1}(R_1 + cR_2) \times \begin{cases} 4, & d = 2, \\ 6\sqrt{3}, & d = 3. \end{cases}
\] (53)

Note that the elasticity of the rods enters only through the combination \( dc^2M_1 + M_2 \).

The nondimensional geometrical parameter \( \beta \) defines the anisotropy of the pentamode material, with isotropy iff \( \beta = 1 \). If \( \beta > 1 \) the PM is stiffer along the axial or preferred direction \( \mathbf{e} \) than in the orthogonal plane, and conversely it is stiffer in the plane if \( 0 < \beta < 1 \). The axial stiffness vanishes if \( \beta = 0 \) which is possible if \( \theta = \frac{\pi}{2} \). The unit cell becomes re-entrant if \( \theta > \frac{\pi}{2} \Leftrightarrow c < 0 \). If \( c < 0 \) then \( \beta < 0 \) and the principal values of \( \mathbf{S} \) are simultaneously positive and negative with the negative value associated with the axial direction. Note that \( R_1 + cR_2 \) must be positive since the unit cell volume \( V \) is positive. As \( R_1 + cR_2 \to 0 \) the members criss-cross and the infinite lattice becomes stacked in a slab of unit thickness, hence the volume per cell tends to zero \( (V \to 0) \).

Let \( \mathbf{e} \), the axis of transverse isotropy, be in the 1-direction. A transversely isotropic elastic \( (d = 3) \) has 5 independent moduli: \( C_{11}, C_{22} (= C_{33}), C_{12} (= C_{13}), C_{23} \) and \( C_{66} (= C_{55}) \) with \( C_{44} = \frac{1}{2}(C_{22} - C_{23}) \). The PM has \( C_{66} = 0 \) and \( C_{23} = C_{22} \Rightarrow C_{44} = 0 \) and \( C_{11}C_{22} = C_{12}^2 \), which are consistent with rank \( C = 1 \). The 2D version, technically of orthotropic symmetry, is defined by 4 independent moduli \( C_{11}, C_{22}, C_{12} \) and \( C_{66} \), which in the PM limit satisfy \( C_{66} = 0 \) and \( C_{11}C_{22} = C_{12}^2 \). In either case the non-zero moduli are

\[
\begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix} = K_0 \begin{pmatrix} \beta & 1 \\ 1 & \beta^{-1} \end{pmatrix} \quad \text{where} \quad K_0 = \frac{d}{(d - 1)} \frac{c^2R_2(R_1 + cR_2)}{V(M_2 + dc^2M_1)}. \] (54)

The PM is isotropic for \( \beta = 1 \), i.e. when the angle \( \theta \) and \( R_1/R_2 \) are related by

\[
\frac{R_1}{R_2} = \frac{1 - d \cos \theta^2}{(d - 1) \cos \theta} \Leftrightarrow \text{isotropy} \quad (\beta = 1). \] (55)

Hence, isotropy can be obtained if \( \theta \in [\cos^{-1}\frac{1}{\sqrt{2}}, \frac{\pi}{2}] \) with the proper ratio of lengths, see Fig. 3 At the limiting angles \( R_1 \to 0 \) \( (R_2 \to 0) \) as \( \theta \to \cos^{-1}\frac{1}{\sqrt{2}} \) \( (\theta \to \frac{\pi}{2}) \). If the lengths are equal \( (R_1 = R_2) \) isotropy is obtained for \( \cos \theta = \frac{1}{2} \), i.e. \( \theta = 60^\circ, 70.53^\circ \), for \( d = 2, 3 \), corresponding to hexagonal and tetrahedral unit cells, respectively. Some examples of isotropic PMs and their properties are illustrated in Fig. 3 Transversely isotropic PMs are considered in Figs. 2 - 5.

The stiffness parameter \( K_0 \) of (54) is the bulk modulus of the isotropic PM. Note that \( K_0 \) is not equivalent to \( K \) of (32) since the latter is consequent upon the condition (30) which is not assumed here. Instead, eqs.
imply that the isotropic PM bulk modulus for uniform members is

\[ K_0 = K f, \quad f = d^2 s^4 \left[ d - 1 + \frac{A_1}{A_2 dc(1 - dc^2)} \right]^{-1} \left[ d - 1 + \frac{A_2}{A_1 dc(1 - dc^2)} \right]^{-1} \]  

(56)

where \( A_1, A_2 \) are the cross-sectional areas (strut thicknesses for \( d = 2 \)). For a given \( \theta \) and \( d \), \( f \leq 1 \) with equality iff \( \frac{A_1}{A_2} = dc \). Hence the maximum possible isotropic effective bulk modulus for a given volume fraction \( \phi \) is precisely \( K_0 \) of (32). This result agrees with [24, eq. (2.2)] for \( d = 2 \), and with the bulk modulus for a regular lattice with tetrakaidecahedral unit cells [25, 26], i.e. an open Kelvin foam, see Fig. 6. The latter structure, comprising joints with 4 struts and a unit cell of 14 faces (6 squares and 8 hexagons), has cubic symmetry; however the two shear moduli are almost equal so that the structure is almost isotropic. In fact, if the struts are circular and have Poisson’s ratio equal to zero then the effective material is precisely isotropic with shear modulus \( \mu = \frac{4\sqrt{2}}{9\pi} \phi E \) [25].

Figure 6: The tetrakaidecahedral open foam unit cell [25] has low density PM behavior similar to the diamond lattice.

Note that [29] considered a tetrahedral unit cell of four identical half-struts that join at equal angles and found \( K = \frac{d}{8} (not \frac{d}{E}) \); the difference arises from taking the cell volume for the tetrahedron, but since the tetrahedron is not a space filling polyhedron, this is not the correct unit volume to use.

6. Two dimensions: a special case

6.1. Shear force as a nodal bending force

For \( d = 2 \) the total force \( f_i \) on member \( i \) can be simplified as

\[ f_i = M_i^{-1} \Delta \mathbf{r}_i \mathbf{e}_i + \sum_{j \neq i} N_{ij}^{(0)} R_j \Delta \psi_{ij} \mathbf{e}_{ij} \]  

(57)

with generalized nodal compliance \( N_{ij} \) given by

\[ \frac{1}{N_{ij}} = \frac{1}{N_{ij}} + \frac{1}{N_{ij}^{(0)}} \text{ where } N_{ij}^{(0)} = \frac{N_i N_j}{R_i R_j} \sum_k \frac{R_k^2}{N_k}. \]  

(58)

Hence, the shear force can be considered as an equivalent nodal bending force. Significantly, the moments of the shear forces are now automatically equilibrated due to the symmetry \( N_{ij} = N_{ji} \).

Equation (57) follows by first noting that the vector moment of the shear force is in the direction perpendicular to the plane of the lattice, say \( \mathbf{a}_3 \). Define the angle of deflection associated with flexural bending: \( \theta_i \equiv \mathbf{a}_3 \cdot (\mathbf{e}_i \times \Delta \mathbf{r}_i^0)/R_i \). The moment of the shear force is \( \mathbf{R}_i \times f_i = (R_i^2/N_i)\theta_i \mathbf{a}_3 \), and the moment equilibrium condition (17) becomes

\[ \sum_i \frac{R_i^2}{N_i} \theta_i = 0 \quad \Rightarrow \quad \theta_i = \left( \sum_k \frac{R_k^2}{N_k} \right)^{-1} \sum_{j \neq i} \frac{R_j^2}{N_j} (\theta_i - \theta_j). \]  

(59)
However, $\theta_i - \theta_j = \pm \Delta \psi_{ij}$ (more precisely $\theta_i - \theta_j = \Delta \psi_{ij} \mathbf{a}_3 \cdot (\mathbf{e}_j \times \mathbf{e}_i)/|\mathbf{e}_j \times \mathbf{e}_i|$), therefore eq. (57) allows us to express the single shear force acting on member $i$ as the sum of nodal bending forces with compliances $N_{ij}^{(b)}$, from which eq. (57) follows.

The significance of eq. (57) is that it allows us to express the effective moduli for $d = 2$ as follows: Define

$$\mathbf{d}_i = \frac{\mathbf{e}_i}{\sqrt{M_i}}, \quad \mathbf{D}_i = \frac{R_i \mathbf{e}_i \otimes \mathbf{e}_i}{\sqrt{V M_i}}, \quad \mathbf{d}_{ij} = \sqrt{\frac{R_i R_j}{N_{ij}^d}} \left( \frac{\mathbf{e}_i}{R_i} + \frac{\mathbf{e}_j}{R_j} \right),$$

(60a)

$$\mathbf{D}_{ij} = \frac{R_i R_j}{V N_{ij}^d} \left( \mathbf{e}_i \otimes \mathbf{e}_i + \mathbf{e}_j \otimes \mathbf{e}_j \right)$$

where $\frac{1}{N_{ij}^d} = \frac{1}{N_i} + \frac{R_i R_j}{N_i N_j} \left( \sum_{k=1}^Z \frac{R_k^2}{N_k} \right)^{-1}$,

(60b)

$$\{ \mathbf{u}_k \}_{k=1}^L = \{ \mathbf{d}_i, \mathbf{d}_j \}, \quad \{ \mathbf{U}_k \}_{k=1}^L = \{ \mathbf{D}_i, \mathbf{D}_{ij} \}, \quad L = Z(Z + 1)/2,$$

(60c)

$$P_{ij} = \delta_{ij} - \mathbf{u}_i \cdot \left( \sum_{k=1}^L \mathbf{u}_k \otimes \mathbf{u}_k \right)^{-1} \cdot \mathbf{u}_j \Rightarrow \mathbf{C} = \sum_{i,j=1}^L P_{ij} \mathbf{U}_i \otimes \mathbf{U}_j.$$  

(60d)

Note that this result is valid for any similarly situated 2D lattice structure; in particular it does not require the zero rotation assumption (21).

6.2. Example: honeycomb lattice

As an application of eq. (60b) we consider the transversely isotropic lattice of \[\text{[46.3]}\] in 2-dimensions ($Z = 3$), now including the effects of the bending compliances of the individual members, $N_1$ and $N_2$. Using the same notation as in [46.3] we find

$$C_{11} = \frac{1}{2} c s \frac{1}{(2 c^2 M_1 + M_2) N_2 + 2 s^2 M_1 M_2} \left( \frac{\beta(N_2 + s^2-2M_2)}{1/2(N_2 + s^2-2M_2) (N_2 - M_2)} \right),$$

$$C_{12} = \frac{1}{s} R_2 (R_1 + c R_2) \frac{1}{s^2(2 R_1^2 N_1 + R_2^2 N_2) + (c R_1 + R_2)^2 M_2}.$$  

(61)

These are in agreement with the in-plane moduli found by \[\text{[40]}\]. Note that the moduli of eq. (61) reduce to the PM moduli \[\text{[47]}\] as the bending compliance $N_2 \rightarrow \infty$, independent of the bending compliance $N_1$.

7. Conclusions

Our main result, eq. (9), is that the effective moduli of the lattice structure can be expressed $\mathbf{C} = \sum_{i,j=1}^L P_{ij} \mathbf{U}_i \otimes \mathbf{U}_j$ where $L = Zd + Z(Z - 1)/2$, $\mathbf{U}_i$ are second order tensors, and $P_{ij}$ are elements of a $L \times L$ projection matrix $\mathbf{P}$ of rank $L - d$. Explicit forms for the parameters $\{ \mathbf{U}_i, P_{ij} \}$ have been derived in terms of the cell volume, and the length, orientation, axial and bending stiffness of each of the $Z$ rods. This Kelvin-like representation for the elasticity tensor implies as a necessary although not sufficient condition for positive definiteness of $\mathbf{C}$ that the rank of $\mathbf{P}$ exceed $3d - 3$, which is satisfied if the coordination number satisfies $Z \geq d + 1$. The $L$ second order tensors $\{ \mathbf{U}_i \}$ are split into $Z$ stretch dominated and $Z(Z - 1)/2$ bending dominated elements. The latter contribute little to the stiffness in the limit of very thin members, in which case the elastic stiffness is stretch dominated and, at most, of rank $Z - d$. The formulation developed here is applicable to the entire range of stiffness possible in similarly situated lattice frameworks, from the $Z = 14$ structure proposed by \[\text{[1]}\] with full rank $\mathbf{C}$ to pentamode materials corresponding to coordination number $Z = d + 1$, with $\mathbf{C}$ of rank one.

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