Evaluation of thin square bending plate using IGA Galerkin

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Abstract. This paper presents an evaluation of thin bending plate problems using IGA Galerkin. It focuses on square plates under uniform load with different boundary conditions so that B-Spline is sufficient to generate basis functions. To demonstrate the performance of IGA Galerkin formulation, two numerical tests are presented.

1. Introduction
Researchers in worldwide have developed many kinds of numerical methods for structural engineering application. One of these numerical methods, Finite Element Analysis (FEA) has become the most popular and powerful numerical method that is widely used in many engineering and scientific disciplines. FEA now has been implemented in many commercial programs such as Computer Aided Engineering (CAE) for structural analysis. In FEA polynomial function is adopted as shape function to build discretization of structure. The results of our research in Finite Element Method have been published in [1-13].

In 2005, Hughes [14] introduced a new method called Isogeometric Analysis (IGA). Isogeometric Analysis (IGA) is a new development of FEA but it is not based on polynomial function as FEA. It uses Non-Uniform Rational B-Splines (called NURBS) as basis function. NURBS is a technology which has been used in Computer Aided Design (CAD) system to approximate the geometric shape with high accuracy. With the idea of using the same basis function NURBS in CAD and IGA, hopefully this can simplify mesh refinement. IGA also has flexibility in increasing of element number and polynomial degree independently, which FEA doesn’t have.

Many researches about IGA and its applications have been developed, especially for beam, plate and shell. For plate bending problem, Kiendl [15] developed the solution of Reissner-Mindlin plate problem using Collocation Method in IGA, and Reali [16] developed the solution of Kirchhoff plate problem, also using Collocation Method in IGA. In 2017, the development of Isogeometric Galerkin based on UI approach have been proposed by Katili [17].

This paper will discuss evaluation of thin square bending plate problem under uniform load with different boundary conditions using IGA. The presentation will focus on square plate, and the problems will be solved by using Galerkin Method. Numerical test will be conducted to see the performance of IGA formulation for thin plate with different boundary conditions. The reference solution from [18] and [19], will be used to validate the output of this numerical test.

2. IGA discretization for rectangular thin plate
As described in previous section, the idea of Isogeometric Analysis (IGA) is to implement NURBS function as the replacement of shape function used in FEA. Equation (1) is NURBS equation in two
dimension. For rectangular plate problem, \( \omega_{ij} = 1 \), so \( \sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\xi)M_{j,q}(\eta)\omega_{i,j} = 1 \) [20]. Regarding this simplification, for rectangular plate problem, NURBS is as simple as B-Spline, which is the tensor product of two basis functions in one dimension, \( N_{i,p}(\xi) \) and \( M_{j,q}(\eta) \).

\[
R_{i,j}^{p,q}(\xi, \eta) = \frac{N_{i,p}(\xi)M_{j,q}(\eta)\omega_{i,j}}{\sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\xi)M_{j,q}(\eta)\omega_{i,j}} = N_{i,p}(\xi)M_{j,q}(\eta)
\]

(1)

B-Spline in one dimension is defined in equation (2), where \( \xi_i \) is knot-\( i \).

\[
N_{i,0}(\xi) = \begin{cases} 1, & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0, & \text{otherwise} \end{cases} \quad \text{with} \quad N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi)
\]

(2)

As can be seen, knot is a component to build B-Spline. Knot depends on polynomial degree and number of element. Knot value is in parametric space [0,1]. Polynomial degree will give multiplicity to knot value 0 and 1, which amounts \( p+1 \). Number of element depends on the knot span number between the interval 0 and 1. The set of knot is called knot vector (\( \Xi \)) expressed in equation (3).

\[
\Xi_{\xi} = \left[ \xi_1, \xi_2, \cdots, \xi_{n+p+1} \right]
\]

(3)

where \( n+p+1 \) is number of knot in knot vector, and \( n \) is number of basis function, which is determined after number of knot is known. Further explanation about knot vector and basis function can be seen in references. Analogous with shape function and degree of freedom \( \{u_n\} \) in FEA are basis function and control variable in IGA. In IGA for two dimensional problems, displacement function can be expressed in equation (4), where \( \{R(\xi, \eta)\} \) is basis function and \( \{\hat{w}_n\} \) is control variable.

\[
w(\xi, \eta) = \{R(\xi, \eta)\}\{\hat{w}_n\}
\]

(4)

Control variable (CV) is not a real displacement as degree of freedom in FEA. Control variable number is equal to basis function number in parametric space. Example is given for \( p = q = 3 \) and NELT = 2\( \times 2 \), which gives \( r = m \times n = 25 \) basis functions, each element will have the same 25 control variables, \( \hat{w}_1, \hat{w}_2, \hat{w}_3, \ldots, \hat{w}_{25} \), as can be seen in figure 1. The transverse shear deformation is neglected in Kirchhoff plate theory, where \( \gamma_x = \gamma_y = 0 \). Then, the vertical displacement will have a straight relation with rotation with \( \beta_x = -w_{,x} \) and \( \beta_y = -w_{,y} \). Based on equation (4), curvature \( \{\chi\} \) can be discretized as equation (5).

\[
\begin{bmatrix} \chi_x(\xi, \eta) \\ \chi_y(\xi, \eta) \\ \chi_{xy}(\xi, \eta) \end{bmatrix} = \begin{bmatrix} w(\xi, \eta)_{,xx} \\ w(\xi, \eta)_{,yy} \\ 2w(\xi, \eta)_{,xy} \end{bmatrix} = \begin{bmatrix} -\sum_{i=1}^{r} R_{i,xx}(\xi, \eta)\hat{w}_i \\ -\sum_{i=1}^{r} R_{i,yy}(\xi, \eta)\hat{w}_i \\ -2\sum_{i=1}^{r} R_{i,xy}(\xi, \eta)\hat{w}_i \end{bmatrix} = \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \end{bmatrix}
\]

(5)

And then, curvature can be expressed in bending strain matrix \( [B_b] \) in equation (6).

\[
\{\chi(\xi, \eta)\} = [B_b]\{\hat{w}_n\} \quad \text{with} \quad [B_b]_{3\times r} = \begin{bmatrix} R_{1,xx} & R_{2,xx} & \cdots & R_{r,xx} \\ R_{1,yy} & R_{2,yy} & \cdots & R_{r,yy} \\ 2R_{1,xy} & 2R_{2,xy} & \cdots & 2R_{r,xy} \end{bmatrix}_{3\times r}
\]

(6)
Galerkin Method adopted to solve the problem in this paper is developed by the principle of virtual work. Equation (7) represents the virtual work, which is based only on bending strain energy and distributed load as external load.

\[ \int_A \left( \chi \right) \left[ H_b \right] \left[ \chi \right]^T dA = \int_A w^* f dA \]  

Applying the discretization in weak form equation (Galerkin) will yield stiffness matrix equation as shown in equation (8).

\[ [k]\{\hat{w}_n\} = \{f_n\} \quad \text{with} \quad [k] = \int_A \left[ B_h \right]^T [H_b] [B_h] dA \]  

Isotropic elasticity matrix \([H_b]\) are calculated using equation (9).

\[ [H_b] = D_b \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} ; \quad D_b = \frac{Eh^3}{12(1-\nu^2)} \]

\(D_b\) is the generalized bending constitutive matrix, \(E\) is the Young’s Modulus, \(h\) is the thickness of plate and \(\nu\) is the Poisson’s ratio. Equivalent external load for distributed load can be expressed in equation (10).

\[ \{f_n\} = \{f_n\} ; \quad f_n = \int_A f z R dA \]  

\[ \{f_n\} = \begin{bmatrix} f_{n1} \\ \vdots \\ f_{nr} \end{bmatrix} ; \quad f_n = \int_A f_z R dA \]  

\[ \{f_n\} = \begin{bmatrix} f_{n1} \\ \vdots \\ f_{nr} \end{bmatrix} ; \quad f_n = \int_A f_z R dA \]  

**Figure 1.** Control Variable in Parametric Space for \(p = q = 3\) and \(NELT = 2 \times 2\).

In Isogeometric analysis using Galerkin Method, as in FEA, boundary conditions will reduce stiffness matrix and equivalent external load matrix. The difference between IGA and FEA is, in IGA, applying boundary conditions will yield the relations between control variables. These relations of control variables will be used to develop reduction matrix. The boundary conditions in Galerkin Method are vertical displacement and rotation as shown in equation (11).

\[ w(\xi, \eta) = \left\langle R(\xi, \eta) \right\rangle \{\hat{w}_n\} ; \quad \beta_x(\xi, \eta) = -\left\langle R_{x, \xi} (\xi, \eta) \right\rangle \{\hat{w}_n\} ; \quad \beta_y(\xi, \eta) = -\left\langle R_{y, \eta} (\xi, \eta) \right\rangle \{\hat{w}_n\} \]

Analytical integration will face complexity issue and will take much longer time in computation. Numerical integration is employed to solve the stiffness matrix and equivalent load matrix for plate bending problem. Numerical integration is also used to solve the equivalent load matrix. Gauss integration will be used as numerical integration method. In IGA, the determination of Gauss point (Gauss Quadrature) is different with FEA. Nguyen [21] gives an alternative to use this method which is similar with FEA. According to Nguyen, the determination of number of Gauss point can be calculated using equation (12).
\[ NPG_{\xi} = p + 1 \quad \text{and} \quad NPG_{\eta} = q + 1 \]  

Gauss points are in parent space (analogous with parametric space in FEA), while the basis functions are defined in parametric space, so the transformation from parent space to parametric space can be calculated using equation (13).

\[
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} \left( (\xi_{i+1} - \xi_i) \xi + (\xi_{i+1} + \xi_i) \right) \\
\frac{1}{2} \left( (\eta_{i+1} - \eta_i) \eta + (\eta_{i+1} + \eta_i) \right)
\end{pmatrix}
\]

(13)

Numerical integration must be done in parent space, so using equation (13), we have transformation of Jacobian matrix from parent space to parametric space as shown in equation (14).

\[
\left[ J \right] = \begin{pmatrix}
\frac{\partial \xi}{\partial \xi} & \frac{\partial \xi}{\partial \eta} \\
\frac{\partial \eta}{\partial \xi} & \frac{\partial \eta}{\partial \eta}
\end{pmatrix} \quad \text{and} \quad |J| = \frac{1}{4} (\xi_{i+1} - \xi_i)(\eta_{i+1} - \eta_i)
\]

(14)

The stiffness matrix is expressed in equation (15).

\[
[k] = [k_b] = \sum_{I} \omega_I \left| J \right| J [B_{\xi} (\xi_I, \eta_I)]^T [H_b] [B_{\eta} (\xi_I, \eta_I)]
\]

(15)

Based on the same principle, equivalent load matrix can be expressed in equation (16).

\[
f_{\eta} = \sum_{I} \omega_I \left| J \right| f_z R_{zI} (\xi_I, \eta_I)
\]

(16)

3. Numerical Test

The conducted numerical test is a convergence test in Isogeometric Analysis for square thin plate under uniform load using Galerkin method. Parameters used in this convergence test are vertical displacement and bending moment. Refinements applied in IGA for convergence test are:

- Increasing in number of element (number of element is increased from 1×1, 2×2, 4×4 up to 8×8 in constant polynomial degree \( p = 5 \)).
- Increasing in degree of polynomial (degree of polynomial is increased with \( p = 4, 6, 8, 10 \) in constant number of element \( \text{NELT} = 2 \times 2 \)).

Properties of geometry are presented in figure 2, with length \( L = 1000 \), thickness \( h = 10 \), material properties (\( E = 10.92 \) and \( \nu = 0.3 \)) and uniform distributed load \( f_z = -1 \). Square thin plate bending problems with two boundary conditions are presented: (a) full simply supported (SSSS) and (b) full clamped (CCCC).

![Figure 2. SSSS (a) and CCCC (b) square plates with uniform load \( f_z \)](image)

3.1 Full Simply Supported (SSSS)

In full simply supported problem boundary conditions \( w = \beta_x = 0 \) are applied on all plate edges. The convergence of vertical displacement at the center of plate is shown in table 1 and figure 3. Table 2 and figure 3 present the convergence of bending moment at the center of plate. We found a good results converge to the reference solution.
Table 1. Convergence of displacement for SSSS problem.

| $p$ Escalation (using NELT = 2×2) | NELT Addition (using $p = 5$) |
|-------------------------------|------------------------------|
| $p$ | CV | $w$ | Error (%) | NELT | CV | $w$ | Error (%) |
| 4 | 6 | -4140588 | 1.93 | 1×1 | 6 | -4140586 | 1.93 |
| 6 | 8 | -4066534 | 0.11 | 2×2 | 7 | -4060986 | 0.02 |
| 8 | 10 | -4066713 | 0.12 | 4×4 | 9 | -4064282 | 0.06 |
| 10 | 12 | -4066959 | 0.12 | 8×8 | 13 | -4069792 | 0.19 |

Ref. -4062000 (Timoshenko & Wonowsky-Krieger [19])

Figure 3. Convergence of Displacement and Bending moments $M_x = M_y$ for SSSS problem.

Table 2. Convergence of Bending Moment for SSSS problem.

| $p$ Escalation (using NELT = 2×2) | NELT Addition (using $p = 5$) |
|-------------------------------|------------------------------|
| $p$ | CV | $M_x = M_y$ | Error (%) | NELT | CV | $M_x = M_y$ | Error (%) |
| 4 | 6 | -51722 | 7.98 | 1×1 | 6 | -51722 | 7.98 |
| 6 | 8 | -47934 | 0.07 | 2×2 | 7 | -47463 | 0.91 |
| 8 | 10 | -47902 | 0.00 | 4×4 | 9 | -47824 | 0.16 |
| 10 | 12 | -47946 | 0.10 | 8×8 | 13 | -47935 | 0.07 |

Ref. -47900 (Timoshenko & Wonowsky-Krieger [19])

3.2 Full Clamped (CCCC)

In full clamped problem boundary conditions $w = \beta_x = \beta_y = 0$ applies on all plate edges. The convergence of vertical displacement at the center of plate is shown in table 3 and figure 4. Table 4 and figure 4 present the convergence of bending moment at the center of plate. We found a good results converge to the reference solution.

Table 3. Convergence of displacement for CCCC problem.

| $p$ Escalation (using NELT = 2×2) | NELT Addition (using $p = 5$) |
|-------------------------------|------------------------------|
| $p$ | CV | $w$ | Error (%) | NELT | CV | $w$ | Error (%) |
| 4 | 6 | -1329210 | 5.08 | 1×1 | 6 | -1329209 | 5.08 |
| 6 | 8 | -1266006 | 0.08 | 2×2 | 7 | -1259755 | 0.41 |
| 8 | 10 | -1271989 | 0.55 | 4×4 | 9 | -1264963 | 0.00 |
| 10 | 12 | -1268337 | 0.26 | 8×8 | 13 | -1265304 | 0.02 |

Ref. -1265000 (Timoshenko & Wonowsky-Krieger [19])
Figure 4. Convergence of Displacement and Bending moments \( M_x = M_y \) for CCCC problem.

Table 4. Convergence of Displacement for CCCC problem.

|          | \( p \) Escalation (using NELT = 2x2) | NELT Addition (using \( p = 5 \)) |
|----------|--------------------------------------|-----------------------------------|
| \( p \)  | CV  | \( M_x = M_y \) | Error (%) | CV  | \( M_x = M_y \) | Error (%) |
| 4        | 6   | -22110       | 4.28      | 1x1 | -27648       | 19.69     |
| 6        | 8   | -23064       | 0.16      | 2x2 | -22088       | 4.38      |
| 8        | 10  | -22656       | 1.92      | 4x4 | -22772       | 1.42      |
| 10       | 12  | -23254       | 0.67      | 8x8 | -22889       | 0.92      |

Ref. -23100 (Timoshenko & Wonowsky-Krieger [19])

4. Conclusion

Numerical test has been conducted in Isogeometric Analysis for square thin plate bending problem under uniform load with different boundary conditions using Galerkin method. The convergence tests show good correlation between the results and the reference solution for both \( p \) escalation and NELT addition, which means that Isogeometric Analysis using Galerkin method has good prospect to be an alternative computational method in solving the structural analysis problems.

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