M-coextensive objects and the strict refinement property

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Abstract

The notion of an M-coextensive object is introduced in an arbitrary category C, where M is a distinguished class of morphisms from C. This notion allows for a categorical treatment of the strict refinement property in universal algebra, and highlights its connection with extensivity in the sense of Carboni, Lack and Walters. If M is the class of all product projections in a variety of algebras C, then the M-coextensive (or projection-coextensive) objects in C turn out to be precisely those algebras which have the strict refinement property. If M is the class of surjective homomorphisms in the variety, then the M-coextensive objects are precisely those algebras which have directly-decomposable (or factorable) congruences. In exact Mal'tsev categories, every centerless object with global support has the strict refinement property. We will also show that in every exact majority category, every object with global support has the strict refinement property.

1 Introduction

In universal algebra, various refinement properties have been defined for direct-product decompositions of structures, all of which give information about the uniqueness of such decompositions. The so-called strict refinement property was first defined in [8], and implies that any isomorphism between a product of irreducible structures is uniquely determined by a family of isomorphisms between each factor. If A is a universal algebra which has the strict refinement property, then it was proved in [8] that the factor-congruences of A (see Definition 7.1) form a Boolean sublattice of Eq(X) -
the lattice of congruences on $X$. Moreover, this is a characteristic property of algebras which have the strict refinement property. Examples of structures which possess the strict refinement property include any unitary ring, any centerless or perfect group, any connected poset or digraph [10, 16], any lattice, or more generally, any congruence distributive algebra. Many geometric structures possess the dual property, which is to say that they have the costrict refinement property. As we will show in this paper, this is mainly due to the fact that categories of geometric structures tend to be extensive in the sense of [5]. The main aim of this paper is to investigate the relationship between the strict refinement property and coextensivity. In particular, we introduce the notion of an $\mathcal{M}$-coextensive object in an arbitrary category $\mathcal{C}$, where $\mathcal{M}$ is a distinguished class of morphisms from $\mathcal{C}$. When the base category is regular, the strict refinement property is captured as $\mathcal{M}$-coextensivity where $\mathcal{M}$ is the class of all product projections in $\mathcal{C}$. We will also show that if $\mathcal{M}$ is the class of all regular epimorphisms in $\mathcal{C}$, then $\mathcal{M}$-coextensive objects are precisely those objects which have factorable congruences in the sense of [14] (see also Definition 7.1).

1.1 The notion of $\mathcal{M}$-coextensivity

Recall that a category $\mathcal{C}$ with finite products is coextensive if the canonical functor

$$(A \downarrow \mathcal{C}) \times (B \downarrow \mathcal{C}) \to (A \times B \downarrow \mathcal{C}),$$

is an equivalence of categories. The following proposition provides an equivalent definition of coextensivity, which is what the notion of $\mathcal{M}$-coextensivity is based on.

**Proposition** (Dual of Proposition 2.2 in [5]). A category $\mathcal{C}$ with finite products is coextensive if and only it admits pushouts of arbitrary morphisms along product projections, and in every commutative diagram

$$
\begin{array}{ccc}
A_1 & \xrightarrow{X} & A_2 \\
| & \downarrow & | \\
B_1 & \xleftarrow{Y} & B_2
\end{array}
$$

where the top row is a product diagram, the bottom row is a product diagram if and only if both squares are pushouts.
Definition 1.1. Let $\mathcal{C}$ be a category and $\mathcal{M}$ a class of morphisms from $\mathcal{C}$. A commutative square

$$
\begin{array}{c}
X & \longrightarrow & A \\
\downarrow & & \downarrow a \\
B & \underset{b}{\longrightarrow} & P
\end{array}
$$

in $\mathcal{C}$ is called an $\mathcal{M}$-pushout if it is a pushout in $\mathcal{C}$, and $a, b$ are morphisms in $\mathcal{M}$.

Definition 1.2. Let $\mathcal{C}$ be a category and $\mathcal{M}$ a class of morphisms in $\mathcal{C}$. An object $X$ is said to be $\mathcal{M}$-coextensive if every morphism in $\mathcal{M}$ with domain $X$ admits an $\mathcal{M}$-pushout along every product projection of $X$, and in each commutative diagram

$$
\begin{array}{c}
A_1 & \leftarrow & X & \longrightarrow & A_2 \\
\downarrow & & \downarrow & & \downarrow \\
B_1 & \leftarrow & Y & \longrightarrow & B_2
\end{array}
$$

where the top row is a product diagram and the vertical morphisms belong to $\mathcal{M}$, the bottom row is a product diagram if and only if both squares are $\mathcal{M}$-pushouts.

If $\mathcal{M}$ is the class of all morphisms in a category $\mathcal{C}$ with finite products, then $\mathcal{C}$ is coextensive if and only if every object is $\mathcal{M}$-coextensive. Of particular importance to the current paper is when $\mathcal{M}$ is the class of all product projections in $\mathcal{C}$, and in this case we call an object projection-coextensive if it is $\mathcal{M}$-coextensive. We show that every projection-coextensive object in a category with (finite) products has the (finite) strict refinement property, and when the base category is regular [1], the converse also holds. Every projection-coextensive object has epimorphic product projections, so that the full subcategory of $(X \downarrow \mathcal{C})$ consisting of product projections is a preorder. We show that the posetal-reflection of this preorder $\text{Proj}(X)$ is a Boolean-lattice, and we also show how this is a characteristic property of projection-coextensivity (Theorem 3.1). Pre-exact categories are defined in Definition 4.4 as an intermediate between regular and exact categories. In the pre-exact context we establish a characterization of the strict refinement property (Theorem 4.2) which allows us to show that centerless objects in a Mal’tsev category have the strict refinement property, as well as that every object (with global support) in a pre-exact majority category [13] has the strict refinement property.
When $\mathcal{M}$ is chosen to be the class of regular epimorphisms in $\mathcal{C}$, then we say that $X$ is regularly-coextensive if it is $\mathcal{M}$-coextensive. We will show that if $\mathcal{C}$ is then a variety of universal algebras, then any non-empty algebra $X$ is regularly-coextensive if and only if it has factorable congruences in the sense of [14]. Moreover, this result extends to regular categories. It is then immediate that any object $X$ in a regular category with global support which has factorable congruences necessarily has the strict refinement property. This generalizes a result of [14].

**Convention 1.** Throughout this paper, we will assume that categories have finite products, so that by “a category $\mathcal{C}$”, we mean “a category $\mathcal{C}$ with finite products”.

## 2 $\mathcal{M}$-coextensive objects have strict refinements

Throughout this section, suppose that $\mathcal{C}$ is a category and that $\mathcal{M}$ is a class of morphisms in $\mathcal{C}$. Unless stated otherwise, we will assume that $\mathcal{M}$ contains all isomorphisms in $\mathcal{C}$, and is closed under composition and products in $\mathcal{C}$.

**Remark 2.1.** Under these assumptions on $\mathcal{M}$, any product projection of an $\mathcal{M}$-coextensive object is contained in $\mathcal{M}$. This is because we may take the vertical morphisms in the diagram of Definition 1.2 to be the identity morphisms. Note also that if $X$ is $\mathcal{M}$-coextensive, then for any product projection $X \xrightarrow{p} A$ we have that any terminal morphism $A \to 1$ is in $\mathcal{M}$. This is because in the diagram:

\[
\begin{array}{cccc}
X & \xrightarrow{1_X} & X & \xrightarrow{1} & 1 \\
\downarrow{p} & & \downarrow{p} & & \\
A & \xleftarrow{i_A} & A & \xrightarrow{1} & 1
\end{array}
\]

the bottom row is a product diagram, all the vertical morphisms are in $\mathcal{M}$, and hence the right-hand square is an $\mathcal{M}$-pushout by $\mathcal{M}$-coextensivity of $X$.

If $X \xrightarrow{\pi_1} X_1$ is a product projection, then a morphism $X \xrightarrow{\pi_2} X_2$ is called a complement of $\pi_1$ if the diagram

\[
\begin{array}{cccc}
X_1 & \xleftarrow{\pi_1} & X & \xrightarrow{\pi_2} & X_2
\end{array}
\]
is a product diagram in \( \mathcal{C} \). The following proposition is essentially just the dual of Proposition 2.6 in [5], when coextensivity is restricted to single objects.

**Proposition 2.1.** Let \( X \) be an \( \mathcal{M} \)-coextensive object in \( \mathcal{C} \), then the pushout of any product projection of \( X \) along a complementary product projection is the terminal object, and moreover any product projection of \( X \) is an epimorphism.

**Proof.** Consider the diagram below, where the top row is a product diagram.

\[
\begin{array}{ccc}
A & \xrightarrow{\pi_1} & X & \xrightarrow{\pi_2} & B \\
\downarrow & & \downarrow & & \downarrow \\
1 & \xleftarrow{\pi_2} & B & \xleftarrow{1_B} & B
\end{array}
\]

By Remark 2.1, all the vertical morphisms are in \( \mathcal{M} \). Since the bottom row is a product diagram, it follows that both squares are pushouts. This implies that \( \pi_2 \) is an epimorphism, and that the pushout of \( \pi_1 \) along \( \pi_2 \) is a terminal object.

**Proposition 2.2.** Let \( X \) be an \( \mathcal{M} \)-coextensive object in \( \mathcal{C} \), and suppose that \( X \xrightarrow{p} A \) is any product projection. Then for any morphism \( f : A \rightarrow B \) in \( \mathcal{C} \), if \( f \circ p \in \mathcal{M} \) then \( f \in \mathcal{M} \).

**Proof.** There exists an \( \mathcal{M} \)-pushout diagram of \( p \) along \( p \circ f \) isomorphic to the pushout diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{p} & X \\
\downarrow & & \downarrow \quad \quad \downarrow \\
B & \xleftarrow{1_B} & B
\end{array}
\]

Since \( \mathcal{M} \) contains all isomorphisms and is closed under composition, it follows that \( f \) is contained in \( \mathcal{M} \).

**Definition 2.1.** An object \( X \) in a category \( \mathcal{C} \) is called projection-coextensive if it is \( \mathcal{M} \)-coextensive with \( \mathcal{M} \) the class of all product projections in \( \mathcal{C} \).

**Remark 2.2.** Note that Remark 2.1 implies that every object in \( \mathcal{C} \) that is \( \mathcal{M} \)-coextensive, is necessarily projection-coextensive.
The strict refinement property mentioned in the introduction may be formulated as follows:

**Definition 2.2.** An object $X$ in a category $\mathcal{C}$ is said to have the (finite) strict refinement property if for any two (finite) product diagrams $(X \xrightarrow{f_i} A_i)_{i \in I}$ and $(X \xrightarrow{g_j} B_j)_{j \in J}$, there exist families of morphisms $(A_i \xrightarrow{\alpha_{i,j}} C_{i,j})_{i \in I, j \in J}$ and $(B_j \xrightarrow{\beta_{i,j}} C_{i,j})_{i \in I, j \in J}$ such that $\alpha_{i,j} f_i = \beta_{i,j} g_j$ and the diagrams $(A_i \xrightarrow{\alpha_{i,s}} C_{i,s})_{s \in J}$ and $(B_j \xrightarrow{\beta_{t,j}} C_{t,j})_{t \in I}$ are product diagrams for any $i \in I$ and $j \in J$.

**Theorem 2.1.** If $X$ is a projection-coextensive object in a category $\mathcal{C}$ which admits (finite) products, then $X$ has the (finite) strict refinement property.

**Proof.** Suppose that $(X \xrightarrow{f_i} A_i)_{i \in I}$ and $(X \xrightarrow{g_j} B_j)_{j \in J}$ are any two product diagrams for $X$. Let $\overline{A_n}$ be the product of the $A_i$'s where $n \neq i$ and let $\overline{f_n} : X \to \overline{A_n}$ be the induced morphism $(f_i)_{i \neq n}$, and similarly let $\overline{B_m}$ be the product of the $B_j$'s where $j \neq m$. Since $X$ is projection-coextensive, for each $n \in I$ and $m \in J$ there is a diagram

$$
\begin{array}{ccc}
A_n & \xleftarrow{f_n} & X \\
\downarrow{\alpha_{n,m}} & \nearrow{\overline{f_n}} & \downarrow{\overline{g_m}} \\
C_{n,m} & \xrightarrow{\beta_{n,m}} & \overline{A_n}
\end{array}
\begin{array}{ccc}
B_m & \xrightarrow{\overline{g_m}} & \overline{B_m} \\
\downarrow{\beta_{n,m}} & \nearrow{\overline{f_n}} & \downarrow{\alpha_{n,m}} \\
\overline{C}_{n,m} & \xleftarrow{\alpha_{n,m}} & A_n
\end{array}
$$

where each square is a pushout, and the bottom row is a product diagram. We show that the diagrams $(A_n \xrightarrow{\alpha_{n,j}} C_{n,j})_{j \in J}$ and $(B_m \xrightarrow{\beta_{m,j}} C_{m,j})_{i \in I}$ are product diagrams for any $n \in I$ and $m \in J$. In the diagram

$$
\begin{array}{ccc}
\prod_{j \in J} C_{n,j} & \xleftarrow{(\alpha_{n,j})_{j \in J}} & A_n \\
\downarrow{\prod_{j \in J} \beta_{n,j}} & \nearrow{(\overline{f_n})_{j \in J}} & \downarrow{(\overline{g_m})_{j \in J}} \\
\prod_{j \in J} B_j & \xrightarrow{(\overline{g_m})_{j \in J}} & \prod_{j \in J} \overline{B}_{n,j} \\
\downarrow{\prod_{j \in J} \beta_{n,j}} & \nearrow{(\overline{f_n})_{j \in J}} & \downarrow{(\overline{g_m})_{j \in J}} \\
\prod_{j \in J} C_{n,j} & \xleftarrow{(\alpha_{n,j})_{j \in J}} & A_n
\end{array}
$$

the bottom row is a product diagram. Moreover, the outer vertical morphisms are product projections by Proposition 2.2, and therefore by projection-coextensivity of $X$ the two squares above are pushouts. Since the central
vertical morphism in the diagram is an isomorphism, it follows that the morphism \((\alpha_{n,j})_{j \in J}\) is an isomorphism, and we can similarly obtain \((\beta_{i,m})_{i \in I}\) as an isomorphism.

Proposition 2.3. If \(X\) is an \(\mathcal{M}\)-coextensive object in \(\mathcal{C}\) and \(p : X \to A\) any product projection of \(X\), then \(A\) is \(\mathcal{M}\)-coextensive.

Proof. Suppose that \(A' \xleftarrow{\rho'} X \xrightarrow{\rho} A\) and \(A_1 \xleftarrow{\pi_1} A \xrightarrow{\pi_2} A_2\) are both product diagrams, where \(X\) is an \(\mathcal{M}\)-coextensive object. Consider the following diagram

\[
\begin{array}{ccc}
A_1 \times A_1^{(\pi_1 p, p')} & \xrightarrow{\pi_{2p}} & A_2 \\
p_1 \downarrow & & \downarrow 1_{A_2} \\
A_1 & \xleftarrow{s_1} & A \xrightarrow{s_2} A_2 \\
f_1 \downarrow & & \downarrow f_2 \\
B_1 & \xleftarrow{s_3} & B \xrightarrow{s_4} B_2
\end{array}
\]

where \((\pi_1 p, p')\) is the morphism induced into the product \(A_1 \times A'\). Note that both \((\pi_1 p, p')\) and \(\pi_1 p\) are product projections of \(X\), and hence both are members of \(\mathcal{M}\) by Remark 2.1. By Proposition 2.2, \(p_1\) is a member of \(\mathcal{M}\) since \(p_1 (\pi_1 p, p') = \pi_1 p \in \mathcal{M}\). Since the top and middle rows are product diagrams and \(p_1, p, 1_A\) are morphisms in \(\mathcal{M}\), both the squares \(S_1\) and \(S_2\) are \(\mathcal{M}\)-pushout diagrams, since \(X\) is \(\mathcal{M}\)-coextensive. We now show that \(S_3\) and \(S_4\) are \(\mathcal{M}\)-pushouts if and only if the bottom row is a product diagram. If the bottom row is a product diagram, then both \(S_1 + S_3\) and \(S_2 + S_3\) are pushout diagrams by \(\mathcal{M}\)-coextensivity. It then follows from a general fact that since \(S_1 + S_3\) and \(S_1\) are pushouts, that \(S_3\) is a pushout. Similarly, \(S_4\) is a pushout. Conversely, if \(S_3\) and \(S_4\) are pushouts, then both \(S_1 + S_3\) and \(S_2 + S_4\) are pushouts, which implies that the bottom row is a product diagram by projection-coextensivity of \(X\). Finally, the pushout of a morphism in \(\mathcal{M}\) with domain \(A\) along a product projection of \(A\) always exists because the right-hand vertical morphism in \(S_2\) is the identity and \(f\) and \(\pi_2\) were arbitrary. \(\square\)
3 A characterization of projection-coextensivity

Definition 3.1. Let $X$ be an object in a category $\mathcal{C}$, then by $\Proj_{\mathcal{C}}(X)$ we mean the full subcategory of $X \downarrow \mathcal{C}$ consisting of product projections of $X$. If $X$ has epimorphic product projections, then $\Proj_{\mathcal{C}}(X)$ is a preorder, so that in particular, if $X$ is projection-coextensive then $\Proj_{\mathcal{C}}(X)$ is a preorder (see Proposition 2.1). In what follows we shall write $\Proj(X)$ for the posetal-reflection of $\Proj_{\mathcal{C}}(X)$ when $X$ has epimorphic product projections.

Remark 3.1. Let $X$ be a projection-coextensive object in a category $\mathcal{C}$. If $X \xrightarrow{\alpha} A$ is a product projection, then we shall write $[\alpha]$ for the element of $\Proj(X)$ that $\alpha$ represents. For product projections $p, q$ of $X$, note that $[p] \leq [q]$ if and only if $q$ factors through $p$. Note that the join $[p] \lor [q]$ exists, and is represented by the diagonal morphism in any $\mathcal{M}$-pushout of $p$ along $q$. Then $\Proj(X)$ is bounded with top element $[X \to 1]$ and bottom element $[1_X]$.

Proposition 3.1. Let $X$ be a projection-coextensive object in a category $\mathcal{C}$, and let $[\alpha] \in \Proj(X)$ be any product projection. Then there exists unique $[\beta] \in \Proj(X)$ such that $[\alpha] \lor [\beta] = [X \to 1]$.

Proof. Suppose that $X \xrightarrow{\alpha} A$ is any product projection with complements $X \xrightarrow{\beta} B$ and $X \xrightarrow{\alpha'} B'$. By Proposition 2.1 we may form the following diagram where every square is a pushout, and every edge a product diagram.

It then follows that $\alpha$ and $\beta$ are isomorphisms so that $[\alpha] = [\beta]$.

Corollary 3.1. Given any projection-coextensive object $X$ in a category $\mathcal{C}$, and any product diagram $A \xleftarrow{a} X \xrightarrow{b} B$ in $\mathcal{C}$. If $a$ is an isomorphism then $B$ is a terminal object.

Let $X$ be a projection coextensive object in a category $\mathcal{C}$, then for any $[p] \in \Proj(X)$ there exists unique $[p] \perp \in \Proj(X)$ with $[p] \lor [p] \perp = [X \to 1]$ by Proposition 3.1. We always denote the complement of $[p]$ by $[p] \perp$ in what follows.
**Proposition 3.2.** For a projection-coextensive object \( X \) in a category \( C \), the map \([p] \mapsto [p]^{-}\) is order-reversing.

**Proof.** The proof amounts to showing that in the diagram

\[
\begin{array}{c}
C_1 \xrightarrow{\gamma} B \xrightarrow{} C_2 \\
\uparrow \pi_1 \quad \pi_1' \downarrow \quad \pi_2' \downarrow \\
A \xrightarrow{\pi_2} X \xrightarrow{\pi_2'} A' \\
\downarrow \pi_1' \quad \pi_2' \downarrow \quad \downarrow \pi_2 \\
C_3 \xleftarrow{\alpha} B' \xleftarrow{} C_4
\end{array}
\]

where the central column and central row are product diagrams and each square is a pushout, if the dotted arrow exists making the upper right-hand triangle commute, then \( \pi_2 \) factors through \( \pi_2' \). Suppose that the dotted arrow exists, so that \( \pi_1' \) factors through \( \pi_1 \). Then since \( \pi_1 \) is an epimorphism (Proposition 2.1), the upper left-hand triangle commutes, and therefore \( \gamma \) is an isomorphism. The object \( B \) is projection-coextensive by Proposition 2.3, so that by Corollary 3.1 it follows that that \( C_2 \) is terminal. Finally, this implies that \( \beta \) is an isomorphism, since the right-hand edge is a product diagram, and hence \( \pi_2 \) factors through \( \pi_2' \) via \( \beta^{-1} \).

**Corollary 3.2.** Let \( C \) be a category and \( X \) a projection-coextensive object in \( C \), then \( \text{Proj}(X) \) is a Boolean lattice.

**Proof.** By Proposition 3.2 and Proposition 3.1, the map \([p] \mapsto [p]^{-}\) is idempotent and order reversing. This turns \( \text{Proj}(X) \) into a lattice, where meets are given by:

\[ [p] \land [q] = ([p]^{-} \lor [q]^{-})^{-}. \]

By Theorem 6.5 in [2], it follows that \( \text{Proj}(X) \) is distributive. Since every element of \( \text{Proj}(X) \) admits a complement, \( \text{Proj}(X) \) is Boolean.

**Proposition 3.3.** Suppose that \( C \) is a category, and let \( X \) be an object with epimorphic product projections, where the pushout of two product projections of \( X \) are again product projections. If \( \text{Proj}(X) \) is a Boolean lattice, then \( X \) is projection-coextensive.
Proof. Consider the diagram:

\[
\begin{array}{c}
A_1 \xrightarrow{a_1} X \xrightarrow{a_2} A_2 \\
\downarrow p_1 \quad \quad \downarrow p_2 \\
B_1 \xleftarrow{b_1} B \xrightarrow{b_2} B_2
\end{array}
\]

where \(a_1, a_2\) are complementary product projections, and \(p_1, p, p_2\) are product projections. Suppose that the two squares are pushouts. Since \(A_1 \xrightarrow{p_1} B_1\) and \(A_2 \xrightarrow{p_2} B_2\) are product projections, the morphism \(X \xrightarrow{(c_1, c_2)} B_1 \times B_2\) is a product projection. By the universal property of product, it follows that \([p] \leq [(c_1, c_2)]\) in \(\text{Proj}(X)\). Also, we have \([(c_1, c_2)] \leq [c_1]\) and \([(c_1, c_2)] \leq [c_2]\), but since \([c_1] = [a_1] \lor [p]\) and \([c_2] = [a_2] \lor [p]\), it follows that

\[[(c_1, c_2)] \leq ([a_1] \lor [p]) \land ([a_2] \lor [p]) = [p],\]

and thus \([(c_1, c_2)] = [p]\) so that the bottom row is a product diagram.

Suppose now that the bottom row is a product diagram, then we must have that \([c_1] \land [c_2] = [p]\). Then the sublattice of \(\text{Proj}(X)\) generated by \([a_1], [a_2], [p], [c_1], [c_2]\) is given by:

\[
\begin{array}{c}
[X \to 1] \\
\downarrow \quad \downarrow \\
[c_1] \quad [c_2] \\
\downarrow \quad \downarrow \\
[a_1] \lor [p] \quad [a_2] \lor [p] \\
\downarrow \quad \downarrow \\
[p]
\end{array}
\]

By Birkhoff’s characterization of distributive lattices, \(\text{Proj}(X)\) cannot contain sublattices isomorphic to the pentagon. This forces \([c_1] = [a_1] \lor [p]\) and \([c_2] = [a_2] \lor [p]\).

As a consequence of Corollary 3.2 and Proposition 3.3 we have the following characterization of projection-coextensivity.

**Theorem 3.1.** Suppose that \(\mathcal{C}\) is a category and let \(X\) be any object with epimorphic product projections. Then the following are equivalent:
(i) $X$ is projection-coextensive.

(ii) The pushout of any two product projections of $X$ are product projections, and $\text{Proj}(X)$ is a Boolean lattice.

4 Projection-coextensivity in regular categories

A category $\mathcal{C}$ is regular \[1\] if it has finite limits, coequalizers of kernel-pairs, and the pullback of a regular epimorphism along any morphism is again a regular epimorphism. Listed below are some elementary facts about morphisms in a regular category $\mathcal{C}$.

- Every morphism in $\mathcal{C}$ factors a regular epimorphism followed by a monomorphism.
- If $f, g$ are regular epimorphisms in $\mathcal{C}$, then their product $f \times g$ is a regular epimorphism.
- For any two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, if $g \circ f$ is a regular epimorphism, then $g$ is a regular epimorphism. Also, if $f$ and $g$ are regular epimorphisms, then $g \circ f$ is a regular epimorphism.

Definition 4.1. An object $X$ in a regular category $\mathcal{C}$ is said to have global support if any terminal morphism $X \to 1$ is a regular epimorphism.

Remark 4.1. If $X$ is an object with global support in a regular category $\mathcal{C}$, then any product projection of $X$ is a regular epimorphism. This is because any factor of an object with global support itself has global support, and any product projection of $X$ can be obtained as a pullback of a terminal morphism of one of its factors.

4.1 Subobjects and relations in regular categories

Given any object $X$ in a regular category $\mathcal{C}$, consider the preorder of all monomorphisms with codomain $X$. The posetal-reflection of this preorder is $\text{Sub}(X)$ — the poset of subobjects of $X$. For any morphism $f : X \to Y$ in $\mathcal{C}$ there is an induced Galois connection

$$\text{Sub}(X) \xrightarrow{f^*} \text{Sub}(Y) \leftarrow \text{Sub}(X)$$
which is defined as follows. Given a subobject $A$ of $X$ represented by a monomorphism $A_0 \xrightarrow{a} X$, $f^*(A)$ is defined to be the subobject represented by the monomorphism part of a regular epi, mono factorization of $f \circ a$. Given a subobject $B$ of $Y$ represented by a monomorphism $B_0 \xrightarrow{b} Y$, we define $f_*(B)$ to be the subobject of $X$ represented by the monomorphism obtained from pulling back $b$ along $f$. A relation from $X$ to $Y$ is a subobject of $X \times Y$, i.e., an isomorphism class of monomorphisms with codomain $X \times Y$.

In regular categories we can compose relations as follows: given a relation $R$ from $X$ to $Y$ and a relation $S$ from $Y$ to $Z$, and two representatives $(r_1, r_2) : R_0 \to X \times Y$ and $(s_1, s_2) : S_0 \to Y \times Z$ of $R$ and $S$ respectively, form the pullback of $s_1$ along $r_2$:

\[
\begin{array}{ccc}
P & \xrightarrow{p_2} & S_0 \\
p_1 & \downarrow & \downarrow s_1 \\
R_0 & \xrightarrow{r_2} & Y 
\end{array}
\]

Then $R \circ S$ is defined to be the relation represented by the mono-part of any regular-image factorization of $(r_1p_1, r_2p_2) : P \to X \times Z$.

**Definition 4.2.** In what follows we will write $\text{Eq}(X)$ for poset of all equivalence relations on an object $X$, and $\text{Ef}(X)$ for the poset of all effective equivalence relations on $X$.

**Definition 4.3.** A factor relation on an object $X$ is an equivalence relation $F$ represented by the kernel equivalence relation of a product projection $X \twoheadrightarrow A$ of $X$. A factor relation represented by the kernel equivalence relation of a complement of $p$ is correspondingly called a complement of $F$, and will often be denoted by $F'$. The poset $F(X)$ of factor relations on $X$ is bounded with top element $\nabla_X$ and bottom element $\Delta_X$.

For any effective equivalence relations $F, F'$ on an object $X$ in a regular category $\mathcal{C}$, we have $F \circ F' = \nabla_X$ if and only if the canonical morphism

\[
X \xrightarrow{(q_F, q_{F'})} X/F \times X/F'
\]

is a regular epimorphism (see the Proposition below). The kernel equivalence relation of $(q_F, q_{F'})$ is given by $F \cap F'$, so that $(q_F, q_{F'})$ is mono if and only if $F \cap F' = \Delta_X$. These remarks are summarized in the following proposition, whose proof is omitted.
Proposition 4.1 (Proposition 1.44. in [11]). A pair of effective equivalence relations $F, F'$ on an object $X$ in a regular category $C$ are complementary factor relations if and only if $F \cap F' = \Delta_X$ and $F \circ F' = \nabla_X$.

Remark 4.2. Suppose that $X$ is an object with global support in a regular category $C$. Every element of $F(X)$ maps to an element of $\text{Proj}(X)$ (by taking coequalizers), and likewise each element of $\text{Proj}(X)$ maps to an element of $F(X)$ (by taking kernel pairs). These maps are inverse poset isomorphisms.

Proposition 4.2. Let $C$ be a regular category, and suppose that in the diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\pi_1} & X & \xrightarrow{\pi_2} & A_2 \\
p_1 \downarrow & & \downarrow & & \downarrow p_2 \\
B_1 & \xleftarrow{\pi'_1} & Y & \xrightarrow{\pi'_2} & B_2
\end{array}
\]

the vertical morphisms are regular epimorphisms, the top and bottom rows are product diagrams, and that $X$ has global support, then the squares are pushouts.

Proof. Consider the diagram

\[
\begin{array}{ccc}
K_1 & \xrightarrow{u} & K & \xleftarrow{v} & K_2 \\
A_1 & \xrightarrow{\pi_1} & X & \xleftarrow{\pi_2} & A_2 \\
p_1 \downarrow & & \downarrow & & \downarrow p_2 \\
B_1 & \xleftarrow{\pi'_1} & Y & \xrightarrow{\pi'_2} & B_2
\end{array}
\]

where $K_1, K$ and $K_2$ are the kernel-pairs of $p_1, p$ and $p_2$ respectively, and $u, v$ are the induced morphisms. The bottom row being a product diagram implies that the top row is a product diagram, and $X$ having global support implies that $K$ has global support, so that by Remark 4.1 both $u$ and $v$ are (regular) epimorphisms. Then $u$ and $v$ being epimorphisms implies that the bottom squares are pushouts.

Theorem 4.1. Suppose that $X$ is an object with global support in a regular category $C$, and that $X$ admits pushouts of any two of its product projections. Then $X$ has the finite strict refinement property if and only if it is projection-coextensive.
Proof. If $X$ is projection-coextensive, then by Theorem 2.1 it follows that $X$ has the strict refinement property. Suppose that $A_1 \xleftarrow{\pi_1} X \xrightarrow{\pi_2} A_2$ is a product diagram, and that $X \xrightarrow{p} B$ is any product projection. If $X$ has the finite strict refinement property, then there exists product projections $p_1, p_2, b_1, b_2$ making the diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\pi_1} & X & \xrightarrow{\pi_2} & A_2 \\
\downarrow{p_1} & & \downarrow{p} & & \downarrow{p_2} \\
B_1 & \xleftarrow{b_1} & Y & \xrightarrow{b_2} & B_2
\end{array}
\]

commute, and the bottom row a product diagram. Note that since $X$ has global support and $\mathcal{C}$ is regular, every morphism in the diagram above is a regular epimorphism. By Proposition 4.2 the two squares are pushouts. Therefore, if we are given the diagram above, where the top row is a product diagram and the vertical morphisms are product projections, if the squares are pushouts then the bottom row is a product diagram. On the other hand, if the bottom row is a product diagram, then by Proposition 4.2 it follows that the two squares are pushouts.

Corollary 4.1. Given an object $X$ with global support in a regular category $\mathcal{C}$, then $X$ is projection-coextensive if and only if the pushout of a product diagram along a product projection exists, and is a product diagram.

Corollary 4.2. Let $\mathcal{C}$ be a complete regular category and let $X$ be an object with global support in $\mathcal{C}$. If $X$ has the finite strict refinement property, then $\mathcal{C}$ has the strict refinement property.

Proof. This follows from the fact that if $X$ has the finite strict refinement property, then it is projection-coextensive. Then $X$ being projection-coextensive, it has the strict refinement property by Theorem 2.1.

4.2 Pre-exact categories and strict refinement

Given a category $\mathcal{C}$ with kernel pairs and coequalizers of equivalence relations, any equivalence relation $E$ on any object $A$ in $\mathcal{C}$ admits an effective closure $\overline{E}$, namely, the kernel equivalence relation of any representative of $E$.

Definition 4.4. A regular category $\mathcal{C}$ is said to be pre-exact if it admits coequalizers of equivalence relations, and for any two equivalence relations $E_1$ and $E_2$, we have $E_1 \cap \overline{E_2} = \overline{E_1} \cap \overline{E_2}$.
Proposition 4.3. Let $\mathbb{C}$ and $\mathbb{D}$ be regular categories which have coequalizers of equivalence relations, and let $F : \mathbb{C} \to \mathbb{D}$ be a functor which preserves finite limits, coequalizers of equivalence relations, and reflects epimorphisms. If $\mathbb{D}$ is pre-exact, then so is $\mathbb{C}$.

The proof of the above proposition is a standard preservation and reflection argument, the details of which can be in the proof of Proposition 4.1 in [11]. We include a sketch of that argument here:

Proof Sketch. The functor $F$ preserves finite limits and effective closures of equivalence relations, and reflects equality of effective equivalence relation. The latter is due to the fact that any morphism in $\mathbb{C}$ between kernel pairs, which is an epimorphism, is necessarily an isomorphism. □

Example 4.1. The dual categories categories $\text{Top}^\text{op}$, $\text{Ord}^\text{op}$, $\text{Grph}^\text{op}$, $\text{Rel}^\text{op}$ of topological spaces, ordered sets, graphs and binary relations, all admit forgetful functors to $\text{Set}^\text{op}$; these forgetful functors satisfy the conditions of Proposition 4.3, and since $\text{Set}^\text{op}$ is pre-exact (since it is exact), it follows that each of the categories above are pre-exact (since they are all regular categories which have coequalizers of equivalence relations). The category of topological groups $\text{Grp}(\text{Top})$ is regular, but not exact. It has all small limits and colimits, and the forgetful functor $\text{Grp}(\text{Top}) \to \text{Grp}$ has both a left and a right adjoint, and therefore it preserves all limits and colimits which exist in $\text{Grp}(\text{Top})$. This functor reflects epimorphisms, and therefore since $\text{Grp}$ is exact, $\text{Grp}(\text{Top})$ is pre-exact.

The next result is an analogue of Theorem 4.5 in [8], for regular categories (see also Theorem 5.15 in [15]). Note that, given two factor relations $F$ and $G$ on an object $X$ in a pre-exact category $\mathbb{C}$ such that $F \circ G = G \circ F$, the composite $F \circ G$ is an equivalence relation which is the join of $F$ and $G$ in $\text{Eq}(X)$. Moreover, the join of $F$ and $G$ in the lattice $\text{Ef}(X)$ of effective equivalence relations exists, and is given by the effective closure $\overline{F \circ G}$ of $F \circ E$. In what follows we will write $F \od G$ for $\overline{F \circ G}$.

Proposition 4.4. The following are equivalent for an object $X$ with global support in a pre-exact category $\mathbb{C}$.

(i) $X$ is projection-coextensive.

(ii) $F(X)$ is a sublattice of $\text{Ef}(X)$ which is Boolean.
(iii) $F(X)$ forms a Boolean lattice under the operations $\lor$ and $\land$.

Proof. $(i) \implies (ii)$: By Corollary 3.2, if $X$ is projection-coextensive, then $\text{Proj}(X)$ is a Boolean lattice where joins are given by pushout. Note that $F(X)$ is isomorphic to $\text{Proj}(X)$ (see Remark 4.2), and the join of two factor relations $F$ and $G$ is given by their join in $\text{Ef}(X)$. We next show that the meet of $F$ and $G$ in $F(X)$ is given by their meet $F \cap G$ in the lattice $\text{Ef}(X)$ of effective equivalence relations on $X$. Suppose that $F', G'$ are the complementary factor relations of $F$ and $G$ respectively. Since $X$ is projection-coextensive, every edge in the outer square:

\[
\begin{array}{c}
\text{X} \\
\downarrow \\
\text{F} \\
\downarrow \\
\text{F} \lor \text{G} \\
\downarrow \\
\text{X} \\
\downarrow \\
\end{array}
\quad\begin{array}{c}
\text{X} \\
\downarrow \\
\text{F} \\
\downarrow \\
\text{F} \lor \text{G} \\
\downarrow \\
\text{X} \\
\downarrow \\
\end{array}
\quad\begin{array}{c}
\text{X} \\
\downarrow \\
\text{F} \\
\downarrow \\
\text{F} \lor \text{G} \\
\downarrow \\
\text{X} \\
\downarrow \\
\end{array}
\quad\begin{array}{c}
\text{X} \\
\downarrow \\
\text{F} \\
\downarrow \\
\text{F} \lor \text{G} \\
\downarrow \\
\text{X} \\
\downarrow \\
\end{array}
\]

is a product diagram. This implies that

\[(F \lor G) \cap (F' \lor G) = G \quad \text{and} \quad (F \lor G) \cap (F \lor G') = F.\]

Since the canonical morphism

\[X \longrightarrow \frac{X}{F \lor G} \times \frac{X}{F' \lor G} \times \frac{X}{F \lor G'}\]

is a complementary product projection of $X \rightarrow \frac{X}{F \lor G'}$, and by Proposition 3.1 complements are unique, it follows that the meet $F \land G$ of two factor relations in $F(X)$ is given by:

\[F \land G = (F' \lor G')' = (F \lor G) \cap (F' \lor G) \cap (F \lor G') = F \cap G.\]

Therefore $F(X)$ is a sublattice of $\text{Ef}(X)$ which is Boolean. $(ii) \implies (iii)$: Suppose that $F, G \in F(X)$, and that $F'$ is the complement of $F$ and $G'$ the complement of $G$. Then

\[((F \circ G \circ F) \cap F') \subseteq (F \lor G) \cap F' \subseteq G,\]

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which implies that

\[ F \circ G \circ F \subseteq (F \circ G \circ F) \cap (F' \circ F) = ((F \circ G \circ F) \cap F') \circ F \subseteq G \circ F. \]

This implies that \( F \circ G \) is an equivalence relation, so that \( F \circ G = F \lor G \) in \( F(X) \), and therefore \( F(X) \) is a Boolean lattice under \( \lor \) and \( \cap \). For (iii) \( \implies (i) \), just note that \( F(X) \) being Boolean under \( \lor \) and \( \cap \) implies that \( \text{Proj}(X) \) is a Boolean lattice, and that the pushout of any two product projections of \( X \) are product projections. Therefore, by Theorem 4.4 it follows that \( X \) is projection-coextensive.

In [3], the author characterized the congruence distributivity property for regular Goursat categories in terms a preservation of binary meets of equivalence relations by regular-epimorphisms. One of the basic observations is that if \( f : X \to Y \) is any regular epimorphism in a regular category \( \mathcal{C} \) and \( E \) any equivalence relation on \( X \), then in the notation of Section 4.1 we have

\[(f \times f)_*(f \times f)^*(E) = K \circ E \circ K,
\]

where \( K \) is the kernel equivalence relation of \( f \). In what follows we will denote \( f^{-1}(E) = (f \times f)_*(E) \) and \( f(E) = (f \times f)^*(E) \), so that the above equation reduces to:

\[ f^{-1}(f(E)) = K \circ E \circ K. \]

**Proposition 4.5.** The following are equivalent for an object \( X \) with global support in a pre-exact category \( \mathcal{C} \).

(i) \( X \) is projection-coextensive.

(ii) For any \( F, G \in F(X) \), \( F \circ G = G \circ F \) and we have

\[ q(G) \cap q(G') = \Delta_{X/F}, \]

where \( q : X \to X/F \) is a canonical quotient.

**Proof.** Suppose that \( X \) is projection-coextensive, so that by Proposition 4.4, \( F(X) \) is a Boolean lattice under \( \lor \) and \( \cap \). Let \( q : X \to X/F \) be a canonical quotient, then:

\[
q(G) \cap q(G') = q(q^{-1}(q(G) \cap q(G'))) = q(q^{-1}(q(G))) \cap q^{-1}(q(G'))
= q((F \circ G) \cap (F \circ G'))
\subseteq q((F \lor G) \cap (F \lor G'))
= q(F) = \Delta_{X/F}.
\]
For the converse, suppose $F, G \in F(X)$ and that $q : X \to X/F$ is a canonical quotient, then since $q(G) \cap q(G') = \Delta_{X/F}$ it follows that

$$F = q^{-1}(q(G) \cap q(G')) = q^{-1}q(G) \cap q^{-1}q(G') \implies F = (F \circ G) \cap (F \circ G').$$

Since

$$\nabla_X = F \circ G \circ F' \circ G \subseteq \overline{F \circ G \circ F' \circ G},$$

the canonical morphism

$$X \to (X/F \circ G) \times (X/F' \circ G),$$

is a regular epimorphism. Its kernel equivalence relation is given by

$$\overline{F' \circ G} \cap \overline{F \circ G} = (\overline{F' \circ G}) \cap (\overline{F \circ G}) = G,$$

since $C$ is pre-exact. This implies that pushing out the product diagram $X/F \leftarrow X \to X/F'$ along $X \to X/G$ is a product diagram, and thus by Corollary 4.1 it follows that $X$ is projection-coextensive.

As a summary of Theorem 2.1, Theorem 4.1, Propositions 4.4 and 4.5, we have the following theorem.

**Theorem 4.2.** The following are equivalent for an object $X$ with global support in a small complete pre-exact category $C$.

(i) $X$ has the strict refinement property.

(ii) $X$ has the finite strict refinement property.

(iii) $X$ is projection-coextensive.

(iv) $F(X)$ is a sublattice of $\text{Ef}(X)$ which is Boolean.

(v) $F(X)$ is a Boolean lattice under the operations $\overline{\circ}$ and $\cap$.

(vi) For any $F, G \in F(X)$, $F \circ G = G \circ F$ and we have

$$q(G) \cap q(G') = \Delta_{X/F}$$

where $q : X \to X/F$ is a canonical quotient, and $G'$ is the complement of $G$. 

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5 Majority categories have strict refinements

The notion of a majority category was first introduced and studied in [13], and in [12] appears the following characterization of them.

Theorem 5.1 ([12]). For a regular category $\mathbb{C}$, the following are equivalent.

(i) For any three reflexive relations $R, S, T$ on any object $X$ in $\mathbb{C}$ we have $R \cap (S \circ T) \leq (R \cap S) \circ (R \cap T)$.

(ii) For any three reflexive relations $R, S, T$ on any object $X$ in $\mathbb{C}$ we have $R \circ (S \cap T) \geq (R \circ S) \cap (R \circ T)$.

(iii) For any three equivalence relations $A, B, C$ on any object $X$ in $\mathbb{C}$ we have $A \cap (B \circ C) = (A \cap B) \circ (A \cap C)$.

(iv) For any three equivalence relations $A, B, C$ on any object $X$ in $\mathbb{C}$ we have $A \circ (B \cap C) = (A \circ B) \cap (A \circ C)$.

(v) $\mathbb{C}$ is a majority category.

Corollary 5.1. For any three effective equivalence relations $A, B, C$ on any object $X$ in a regular majority category, we have $A \cap (B \circ C)$.

In [9] the notion of a factor permutable category was introduced, and as we will see, every regular majority category is factor permutable.

Definition 5.1 ([9]). A regular category $\mathbb{C}$ is said to be factor permutable if for any equivalence relation $E$ on any object $A$ in $\mathbb{C}$ we have $F \circ E = E \circ F$ for any factor relation $F \in F(A)$.

Proposition 5.1. Every regular majority category $\mathbb{C}$ is factor-permutable.

Proof. Let $E$ be any equivalence relation on an object $A$ in $\mathbb{C}$, and let $F$ be a factor relation on $A$ with complement $F'$. Then $C \circ F = (C \circ F) \cap (F \circ F') \leq ((C \circ F) \cap F') \circ ((C \circ F) \cap F') \leq F \circ F \cap F' \circ C \cap F' \leq F \circ C$ by Theorem 5.1. $\square$
Proposition 5.2. Every object $X$ with global support in a pre-exact majority category is projection-coextensive.

Proof. Given any two factor relations $F, G$ with complements $F', G'$ respectively, the two equivalence relations $F$ and $G$ permute by Proposition 5.1 so that the composite $F \circ G$ is an equivalence relation. Moreover we have

$$F \circ G \cap (F' \cap G') = (F \circ G) \cap (F' \cap G') = \Delta_X,$$

but also, we have

$$\nabla_X = (F \circ G) \cap (F' \cap G') \subseteq F \circ G \cap (F' \cap G').$$

Therefore, $F \circ G$ and $F' \cap G'$ are complementary factor relations, so that $F(X)$ is closed under the operations of $\circ$ and $\cap$. It then follows by Theorem 5.1 that $F(X)$ is distributive, so that by Theorem 4.2, $X$ is projection-coextensive. \qed

6 Centerless objects in a Mal’tsev category have strict refinements

Recall the notion of a Mal’tsev category:

Definition 6.1 ([6, 7]). A finitely complete category $\mathbb{C}$ is Mal’tsev if any reflexive relation is an equivalence relation.

In [4], the authors develop a categorical approach to centrality of equivalence relations. The central notion, that of a connector of equivalence relations, is defined in any finitely-complete category. When the base category is Mal’tsev, it reduces to the following:

Definition 6.2 ([4]). Let $\mathbb{C}$ be a Mal’tsev category, and let $(r_1, r_2) : R_0 \to X \times X$ and $(s_1, s_2) : S_0 \to X \times X$ represent two equivalence relations $R$ and $S$ on an object $X$, respectively. Consider the pullback below:

$$\begin{array}{ccc}
R_0 \times_X S_0 & \xrightarrow{p_1} & S_0 \\
\downarrow p_2 & & \downarrow s_1 \\
R_0 & \xrightarrow{r_2} & X
\end{array}$$
A connector between $R_0$ and $S_0$ is a morphism $p : R_0 \times_X S_0 \to X$ such that $xR_0 p(x, y, z)S_0 z$ and

$$p(x, y, y) = x = p(y, y, x).$$

If there exists a representative of $R$ which admits a connector with a representative of $S$, then we say that $R$ and $S$ are connected, or that the pair $(R, S)$ are commuting.

Let $C$ be a regular Mal’tsev category, then for any equivalence relations $E, F, G, H$ on any object $X$ in $C$, the following properties hold:

- If $E \leq F$ and $(F, G)$ are commuting, then $(E, G)$ are commuting (Proposition 3.10 in [4]).
- If $(E, F)$ are commuting and $f : X \to Y$ is any regular epimorphism, then $(f(E), f(F))$ are commuting (Proposition 4.2 in [4]).
- If $(E, F)$ are commuting, and $(E, G)$ are commuting, then $(E, F \lor G)$ are commuting (Proposition 4.3 in [4]).
- If $(E, G)$ are commuting, and $(F, H)$ are commuting, then $(E \times F, G \times H)$ are commuting (Proposition 3.12).
- If $E \cap F = \Delta_X$, then $(E, G)$ is commuting (Lemma 3.9 in [4]).

**Definition 6.3.** An object $X$ in a regular Mal’tsev category $C$ is said to be centerless if for any equivalence relation $E$ on $X$, we have that if $(E, \nabla_X)$ are commuting then $E = \Delta_X$.

**Proposition 6.1.** Let $X$ and $Y$ be objects in a regular Mal’tsev category $C$, then $X \times Y$ is centerless if and only if both $X$ and $Y$ are centerless.

**Proof.** Suppose that $X \times Y$ is centerless, and let $E$ be any equivalence relation on $A$ such that $(E, \nabla_A)$ is commuting. Trivially, we have that $(\nabla_A, \nabla_B)$ is commuting, so that $(E \times \Delta_B, \Delta_A \times \nabla_A)$ is commuting. Therefore $E \times \Delta_B = \Delta_{A \times B}$, which implies $E = \Delta_A$. Conversely, if $A$ and $B$ are centerless then given any equivalence relation $E$ on $A \times B$ such that $(E, \nabla_{A \times B})$ is commuting, then $(\pi_1(E), \nabla_A)$ and $(\pi_2(E), \nabla_B)$ are commuting, so that $\pi_1(E) = \Delta_A$ and $\pi_2(E) = \Delta_B$. This implies that $E = \Delta_{A \times B}$.

**Proposition 6.2.** If $X$ is a centerless object with global support in a pre-exact Mal’tsev category, then $X$ is projection-coextensive.
Proof. We will show that $X$ satisfies the conditions of $(vi)$ in Theorem 4.2.
Suppose that $F, G \in F(X)$ are any factor relations, and let $G'$ be the corresponding complement of $G$. Then we have that $F \circ G = G \circ F$, since regular Mal’tsev categories are congruence permutable [7]. Let $q : X \to X/F$ be a canonical quotient morphism, then by Proposition 6.1, it follows that $X/F$ is centerless. Since $G \cap G' = \Delta_X$ it follows that the pair $(G, G')$ are commuting, and therefore $(q(G), q(G'))$ are commuting, so that both $(q(G) \cap q(G'), q(G))$ and $(q(G) \cap q(G'), q(G'))$ are commuting, and consequently $(q(G) \cap q(G'), q(G') \vee q(G))$ are commuting. Since $q(G') \vee q(G) = \nabla_{X/F}$, we have that $(q(G) \cap q(G'), \nabla_{X/F})$ is commuting. This implies $q(G) \cap q(G') = \Delta_{X/F}$ since $X/F$ is centerless.

7 Regular-coextensivity and factorable congruences

Suppose that $X$ is any object in a category $C$ with products. Given any product diagram $A \xleftarrow{\pi_1} X \xrightarrow{\pi_2} B$, and any effective equivalence relations $E$ and $F$ represented by $e : E_0 \to A^2$ and $f : F_0 \to B^2$ then the composite monomorphism

$$E_0 \times F_0 \xrightarrow{e \times f} A^2 \times B^2 \xrightarrow{\tau} X^2$$

represents an effective equivalence relation on $X$ which will always be denoted by $E \times F$ in what follows.

**Definition 7.1.** An object $X$ in a category with finite products is said to have factorable congruences if for any effective equivalence relation $E$ on $X$, and any product diagram $A \xleftarrow{\pi_1} X \xrightarrow{\pi_2} B$, there exist effective equivalence relations $E_1$ and $E_2$ such that $E_1 \times E_2 = E$.

**Definition 7.2.** An object $X$ in a category $C$ is said to be regularly-coextensive if it is $\mathcal{M}$-coextensive with $\mathcal{M}$ the class of all regular epimorphisms in $C$.

**Proposition 7.1.** Let $C$ be a regular category, then any object $X$ with global support is regularly-coextensive if and only if for any regular epimorphism $X \xrightarrow{q} Q$, and any binary product diagram $A \xleftarrow{\pi_1} X \xrightarrow{\pi_2} B$ there exists morphisms $q_1 : A \to Q_1$ and $q_2 : B \to Q_2$ and morphisms $p_1 : Q \to Q_1$ and
$p_2 : Q \rightarrow Q_2$ such that in the commutative diagram

$$
\begin{array}{c}
A \xrightarrow{\pi_1} X \xrightarrow{\pi_2} B \\
\downarrow q_1 \quad \downarrow q \quad \downarrow q_2 \\
Q_1 \xrightarrow{p_1} Q \xrightarrow{p_2} Q_2
\end{array}
$$

the bottom row is a product diagram.

The proof below is similar to the proof of Theorem 4.1.

Proof. The “only if” part is immediate. Consider the diagram in the statement of the proposition, then by Proposition 4.2 it follows that the two squares are pushouts. This also implies that pushing out the top row along $q$ produces a product diagram. On the other hand, if the bottom row is a product diagram, then by Proposition 4.2 both squares are pushouts.

Proposition 7.2. Let $\mathcal{C}$ be a regular category, then any object $X$ with global support has factorable congruences if and only if it is regularly-coextensive.

Proof. Let $X$ have factorable congruences, and suppose that $E$ is any effective equivalence relation, and that $X \xrightarrow{f} Q$ is a quotient of $X$ by $E$. Suppose also that $A \xleftarrow{\pi_1} X \xrightarrow{\pi_2} B$ is any product diagram. By assumption, there exists effective equivalence relations $E_1$ and $E_2$ on $A$ and $B$ respectively with $E_1 \times E_2 = E$. Let $f_1 : A \rightarrow Q_1$ and $f_2 : B \rightarrow Q_2$ be the respective quotients of $E_1$ and $E_2$, then there exists $Q \xrightarrow{q_1} Q_1$ and $Q \xrightarrow{q_2} Q_2$ such that the bottom row in the diagram

$$
\begin{array}{c}
A \xrightarrow{\pi_1} X \xrightarrow{\pi_2} B \\
\downarrow f_1 \quad \downarrow f \quad \downarrow f_2 \\
Q_1 \xrightarrow{q_1} Q \xrightarrow{q_2} Q_2
\end{array}
$$

is a product diagram. Thus by Proposition 7.1 it follows that $X$ is regularly-coextensive.

Recall from Remark 4.1 that in a regular category $\mathcal{C}$, every object with global support has regular-epimorphic product projections. This immediately gives the following corollary.

Corollary 7.1. If $X$ is an object with global support in a regular category $\mathcal{C}$, which is regularly-coextensive then $X$ is projection-coextensive.
Remark 7.1. In [14] the author shows, amongst other things, that if a non-empty universal algebra has factorable congruences, then it has the strict refinement property. Corollary 7.1 generalizes this result.

8 Concluding remarks

If $\mathcal{M}$ is a class of morphisms closed under products in $\mathcal{C}$, then we could have defined $X$ to be $\mathcal{M}$-coextensive when for any product diagram $A \xleftarrow{\pi_1} X \xrightarrow{\pi_2} B$ the canonical functor

$$(\mathcal{M} \downarrow A) \times (\mathcal{M} \downarrow B) \rightarrow (\mathcal{M} \downarrow X)$$

is an equivalence (where $\mathcal{M} \downarrow Y$ is the full subcategory of $\mathcal{C} \downarrow Y$ consisting of morphisms from $\mathcal{M}$). Under suitable conditions on $\mathcal{M}$ this notion could be equivalently reformulated in a similar way as Definition 1.2, where “$\mathcal{M}$-pushout” would have a different meaning. It is possible to show that, under suitable conditions on $\mathcal{M}$, this notion of $\mathcal{M}$-coextensivity would still imply the strict refinement property.

Still further variants of coextensivity are possible, and it is not known if other refinement and factorization properties, such as the intermediate refinement property, are captured by some variant of coextensivity.

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