Analytical Description of Classical Trajectories and Deflection Function in Scattering

S. K. Gupta, Arun K. Jain and B. M. Jyrwa*
Nuclear Physics Division,
Bhabha Atomic Research Centre,
Mumbai 400 085, India
*Physics Department, North-Eastern Hill University,
Shillong 793 022, India

Abstract
Analytical expressions are derived for classical trajectories in repulsive Coulomb plus multi-step attractive potentials. Thereafter the closed form expressions are obtained for the classical deflection functions. The expressions are expected to be of use in heavy ion interactions in nuclear physics.

1 Introduction
Ford and Wheeler [1] demonstrated that under the conditions of applicability of semiclassical analysis of quantal scattering the quantum-mechanical scattering amplitude can be simply related to the classical deflection function. Many of the interesting characteristics of scattering are related to various features of the classical deflection function. In literature usually the closed-form expression for the deflection function is available only for the Coulomb potential see e.g.[1-4]. In this paper, the closed-form results for attractive staircase potential having several steps plus the repulsive Coulomb potential are being presented. Though the formalism described here is general and is applicable to all cases of scattering, it is especially suited to describe heavy ion scattering in nuclear physics [3,4]. We also give expressions for the classical trajectories because many classical and semiclassical formulations use the trajectories in the description of scattering and other phenomena such as fusion or deep inelastic scattering.

2 Trajectories and deflection function
In the centre of mass system, the scattering between two charged particles reduces to that of a point particle at energy $E$ being scattered by a central
potential $V(r)$, where $r$ is the inter-particle distance. For a given impact parameter $b$, the trajectory is reflection-symmetric about the bisector of the angle between the initial and final directions. The bisector intersects the trajectory at a distance $r = a$, where $a$ is the distance of closest approach measured from the origin. Here the angle $\phi$ is measured with respect to the bisector. For impact parameter $b$, the trajectory, i.e. a relationship between angle $\phi$ and $r$ is given by

$$\phi(b, r) = \int_a^r dr \frac{b}{r^2 \sqrt{1 - \frac{V(r)}{E} - \frac{b^2}{r^2}}}.$$

where $a$ is given by the solution of the following equation

$$a^2 - \frac{V(a)}{E} a^2 - b^2 = 0.$$

The deflection function, $\Theta(b)$, a function of the impact parameter $b$, is calculated as

$$\Theta(b) = \pi - 2\phi(b, \infty).$$

where $d$ is the distance of closest approach for head-on collisions,

$$d = \frac{Z_1 Z_2 e^2}{E},$$

where particle charges are $Z_1 e$ and $Z_2 e$, the Coulomb interaction can be written as

$$V_C(r) = \frac{E d}{r}.$$  

In this work the potential $V(r)$ is taken as the sum of the repulsive Coulomb potential $V_C(r)$ and an attractive potential $V_N(r)$, given by

$$V(r) = V_C(r) + V_N(r).$$

For the Coulomb potential alone, the distance of closest approach is denoted by $a_C$, given by

$$a_C = \frac{d}{2} + \sqrt{\frac{d^2}{4} + b^2}.$$
The Coulomb trajectory is given by

\[ \phi_C(b, r) = \int_a^r dr \frac{b}{r^2 \sqrt{1 - \frac{d}{r} - \frac{b^2}{r^2}}} \]  \hspace{1cm} (8)

or \[ \phi_C(b, r) = -W(b, r, 0) \] \[ \bigg|_a^r \] \hspace{1cm} (9)

or \[ \phi_C(b, r) = \frac{\pi}{2} - W(b, r, 0), \] \hspace{1cm} (10)

where we define

\[ W(b, r, y) = \tan^{-1} \frac{\frac{d}{2b} + \frac{b}{r}}{\sqrt{1 + \frac{y}{2} - \frac{d}{r} - \frac{b^2}{r^2}}} \] \hspace{1cm} (11)

The lower limit, corresponding to \(-W(b, a, 0)\), yields \(-\frac{\pi}{2}\).

The deflection function is then given by

\[ \Theta_C(b) = 2W(b, \infty, 0), \] \hspace{1cm} (12)

or \[ \Theta_C(b) = 2 \tan^{-1} \frac{d}{2b} \] \hspace{1cm} (13)

These are standard results available in literature [1-4]. Here the details for evaluating the integral have been provided, because adding a single square well potential or a sum of many square well potentials, similar integrals arise and can be evaluated.

3 Trajectories for Coulomb plus attractive square-well potential

We apply the ideas of the previous section by choosing \(V_N(r)\) to be an attractive square well potential written as

\[ V_N(r) = -V_0 \ U(r - R_0), \] \hspace{1cm} (14)

of strength \(V_0\) and range \(R_0\). The function \(U\) is the unit step function given by

\[ U(x - x_0) = 1, \quad x \leq x_0 \] \hspace{1cm} (15)

and \[ U(x - x_0) = 0, \quad x > x_0. \] \hspace{1cm} (16)

The effective potential consisting of the Coulomb, the attractive well and the centrifugal potentials can have an outer barrier maximum at \(r < R_0\).
For large impact parameters, the energy $E$ is not sufficient to overcome the barrier, the distance of closest approach remains $a_C$. As the impact parameter is reduced, the centrifugal barrier $\frac{b^2}{r^2}$ decreases, the energy $E$ is above the barrier maximum and $a$ becomes less than $R_0$. In this case $a_C$ will be larger than $a$, however it is also less than $R_0$ though $V_N(r)$ is not included in its computation. So the distance of closest approach, $a$ can be written as

$$a = a_C U(R_0 - a_C) + a_0 U(a_C - R_0),$$  \hspace{1cm} (17)

where

$$a_0 = \frac{\frac{d}{2} + \sqrt{\frac{d^2}{4} + b^2(1 + \frac{V_0}{E})}}{1 + \frac{V_0}{E}}.$$ \hspace{1cm} (18)

The discontinuity in the potential arises due to a discontinuity in the distance of closest approach as a function of impact parameter.

The first term of $a$ occurs when the attractive potential is not seen while the second term arises when the top of the barrier is overcome. In the latter case, the integral splits into two, the first one is between $r$ and $R_0$ while the second one is between $R_0$ and $a$. The expression for the trajectory with $r < R_0$ is given by

$$\phi(b, r) = \frac{\pi}{2} - W(b, r, V_0).$$ \hspace{1cm} (19)

The equation for the trajectory with $r > R_0$ is given by

$$\phi(b, r) = \phi_C(b, r) + \phi_0(b, R_0) U(a_C - R_0),$$ \hspace{1cm} (20)

where $\phi_0$ is given by

$$\phi_0(b, R_0) = W(b, R_0, 0) - W(b, R_0, V_0).$$ \hspace{1cm} (21)

It is worth noting that $\phi_0$ is not a function of $r$.

The deflection function for this case is given by

$$\Theta(b) = \Theta_C(b) - 2\phi_0(b, R_0) U(a_C - R_0).$$ \hspace{1cm} (22)

The first term can be interpreted as the Coulomb term while the second term can be interpreted to arise due to the attractive potential and is effective only when the Coulomb barrier is overcome. This expression is similar to that of the quantal scattering amplitude $f(\theta)$ where the Coulomb scattering amplitude gets added to the nuclear amplitude in the presence of the Coulomb interaction.
4 Trajectories for Coulomb plus attractive staircase potential

The attractive staircase potential is written as

\[ V_N(r) = -\sum_{i=0}^{m} v_i U(r - R_i), \]  

as a sum of \( m + 1 \) square wells each of strength \( v_i \) and range \( R_i \) for \( i = 0 \) to \( m \). We choose \( R_m < R_{m-1} < R_{m-2} \ldots < R_0 \). In this case there is a local maximum of the barrier arising at every step and the distance of closest approach \( a \) as a function of impact parameter has \( m + 1 \) discontinuities. These are then reflected in the equations of the trajectory and the deflection function.

First the expression for \( a \), the distance of closest approach is to be determined as it is the lower limit of the integral of the equation for the trajectory. If \( a \) lies between \( R_s \) and \( R_{s+1} \), we denote it as \( a_s \). Let \( V_s = \sum_{i=0}^{s} v_i \), then \( a_s \) can be obtained by solving the equation

\[ (1 + \frac{V_s}{E})a_s^2 - da_s - b^2 = 0 \]  

yielding

\[ a_s = \frac{\frac{d}{2} + \sqrt{\frac{d^2}{4} + b^2(1 + \frac{V_s}{E})}}{1 + \frac{V_s}{E}}. \]  

To determine \( a_s \), first we calculate all the \( a_i \)'s as, \( a_m, a_{m-1}, a_{m-2}, \ldots, a_0, a_C \) and compare the ratios \( \frac{R_m}{a_{m-1}}, \frac{R_{m-1}}{a_{m-2}}, \ldots, \frac{R_0}{a_C} \) with one. If the first ratio is greater than one, \( a = a_m \). If it is less than one, the second ratio is compared with one, if it is found greater than one, \( a = a_{m-1} \), otherwise the procedure is continued till \( a \) is determined as \( a_s \). This implies that the potential between \( R_s \) and \( R_{s+1} \) becomes effective in defining the trajectory.

The expression of trajectory for \( a_s < r < R_s \) is given by

\[ \phi(b, r) = \frac{\pi}{2} - W(b, r, V_s). \]  

The expression of trajectory for \( R_j < r < R_{j-1} < R_0 \) is given by

\[ \phi(b, r) = \frac{\pi}{2} - W(b, r, V_{j-1}) + \sum_{i=j}^{s} (W(b, R_i, V_{i-1}) - W(b, R_i, V_i)). \]  

Defining \( V_{-1} = 0 \), we write the expression of trajectory for \( r > R_0 \) as

\[ \phi(b, r) = \phi_C(b, r) + \phi_N(b, R_s)U(a_C - R_0), \]
where
\[ \phi_N(b, R_s) = \sum_{i=0}^{s} (W(b, R_i, V_{i-1}) - W(b, R_i, V_i)). \] (29)

In the expression of \( \phi(b, r) \), the first term can be considered to arise in pure Coulomb field while the second term represents the effect of the attractive potential.

The deflection function, \( \Theta(b) \) is then calculated by taking \( r \) going to \( \infty \) as
\[ \Theta(b) = \Theta_C(b) - 2\phi_N(b, R_s)U(a_C - R_0). \] (30)

5 Conclusions

If usually encountered nuclear potential of the Woods-Saxon form is approximated by a few discrete steps, analytical expressions of the present paper for trajectories and deflection function can be used to gain more insight into the scattering processes. Expressions for trajectories in the case of repulsive Coulomb plus attractive multistep potentials can also be utilised for many classical and semiclassical formulations in heavy-ion nuclear physics to describe scattering and other phenomena such as fusion and deep inelastic scattering.

Trajectories and deflection function for any potential, \( V(r) \) which can be parametrized as \( V(r) = \alpha - \frac{\beta}{r} - \frac{\gamma}{r^2} \) piecewise can be calculated by the method described in this paper. The present formulation holds even when the Coulomb potential is absent.

Acknowledgments

We thank Sudhir Jain for going through the manuscript and for making useful comments.

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