A PROOF OF THE GROTHENDIECK-SERRE CONJECTURE ON PRINCIPAL BUNDLES OVER REGULAR LOCAL RINGS CONTAINING INFINITE FIELDS

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Abstract. Let $R$ be a regular local ring containing an infinite field. Let $G$ be a reductive group scheme over $R$. We prove that a principal $G$-bundle over $R$ is trivial if it is trivial over the fraction field of $R$. In other words, if $K$ is the fraction field of $R$, then the map of non-abelian cohomology pointed sets

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G)$$

induced by the inclusion of $R$ into $K$ has a trivial kernel.

1. Introduction

Assume that $U$ is a regular scheme. Let $G$ be a reductive $U$-group scheme, that is, $G$ is affine and smooth as a $U$-scheme and, moreover, the geometric fibers of $G$ are connected reductive algebraic groups (see [DG Exp. XIX, Def. 2.7]).

Recall that a $U$-scheme $G$ with an action of $G$ is called a principal $G$-bundle over $U$, if $G$ is faithfully flat and quasi-compact over $U$ and the action is simply transitive, that is, the natural morphism $G \times_U G \to G$ is an isomorphism (see [Gro3] Sect. 6)). It is well known that such a bundle is trivial locally in the étale topology but in general not in the Zariski topology. Grothendieck and Serre conjectured that if $G$ is generically trivial, then it is locally trivial in the Zariski topology (see [Ser] Remarque, p.31, [Gro2] Remarque 3, p.26-27, and [Gro4] Remarque 1.11.a]). More precisely, the following conjecture is widely attributed to them.

**Conjecture 1.** Let $R$ be a regular local ring, let $K$ be its field of fractions. Let $G$ be a reductive group scheme over $U := \text{Spec } R$, let $G$ be a principal $G$-bundle. If $G$ is trivial over $\text{Spec } K$, then it is trivial. Equivalently, the map of non-abelian cohomology pointed sets

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G)$$

induced by the inclusion of $R$ into $K$ has a trivial kernel.

The main result of this paper is the following theorem.

**Theorem.** The above conjecture holds if $R$ is a regular local ring containing an infinite field.

The theorem has the following corollary.

**Corollary.** Notation as in the conjecture, two principal $G$-bundles over $U$ that become isomorphic upon restriction to $\text{Spec } K$ are isomorphic.

This result is new even for constant group schemes (that is, for group schemes coming from the ground field).

Key words and phrases. Reductive group schemes; Principal bundles.
1.1. **History of the topic.** In his 1958 paper Jean–Pierre Serre asked whether a principal bundle is Zariski locally trivial, once it has a rational section (see [Ser, Remarque, p.31]). In his setup the group is any algebraic group over an algebraically closed field. He gave an affirmative answer to the question when the group is $PGL(n)$ (see [Ser, Prop. 18]) and when the group is an abelian variety (see [Ser, Lemme 4]). In the same year, Alexander Grothendieck asked a similar question (see [Gro1, Remarque 3, p.26-27]).

A few years later, Grothendieck conjectured that the statement is true for any semi-simple group scheme over any regular scheme (see [Gro4, Remarque 1.11.a]). Now by the Grothendieck–Serre conjecture we mean Conjecture I though this may be slightly inaccurate from historical perspective. Many results corroborating the conjecture are known.

- For some simple group schemes of classical series the conjecture is solved in works of the second author, A. Suslin, M. Ojanguren, and K. Zainouline; see [Oja1], [Oja2], [PS1], [OP], [Zai], [OPZ].
- The case of an arbitrary reductive group scheme over a discrete valuation ring or over a Henselian ring is completely solved by Y. Nisnevich in [Nis1]. He also proved the conjecture for two-dimensional local rings in the case when $G$ is quasi-split in [Nis2].
- The case where $G$ is an arbitrary torus over a regular local ring was settled by J.-L. Colliot-Thélène and J.-J. Sansuc in [CTS].
- The case where the group scheme $G$ comes from an infinite ground field is completely solved by J.-L. Colliot-Thélène, M. Ojanguren, and M. S. Raghunathan in [CTO] and [Rag1, Rag2]; O. Gabber announced a proof for group schemes coming from arbitrary ground fields.
- Under an isotropy condition on $G$ the conjecture is proved in a series of preprints [PSV] and [Pan].
- The case of strongly inner simple adjoint group schemes of types $E_6$ and $E_7$ is done by the second author, V. Petrov, and A. Stavrova in [PPS]. No isotropy condition is imposed there.
- The case when $G$ is of type $F_4$ with trivial $g_3$-invariant and the field is of characteristic zero is settled by V. Chernousov in [Che]; the case when $G$ is of type $F_4$ with trivial $f_3$-invariant and the field is infinite and perfect is settled by V. Petrov and A. Stavrova in [PS2].

In the case of anisotropic group schemes the conjecture remained wide open in many cases, in particular, for group schemes of types $D_n$, $F_4$, and $E_8$. We will present a uniform proof.

After the first version of the current preprint was posted, the conjecture was solved for regular local rings containing finite fields by the second author [Pan2]. Thus, the conjecture holds for regular local rings containing a field. In the mixed characteristic case the conjecture was solved by the first author provided that $G$ is split and some strong conditions on the local ring are satisfied [Fed1].

1.2. **Overview of the proof.** Very roughly, the idea of the proof is to relate the problem of triviality of the original principal bundle to the triviality of a principal bundle over the affine line over $U$ (see Theorem 2) and then to triviality of a principal bundle over the projective line over $U$ (see Theorem 3). The first reduction is based on the geometric part of the paper [PSV] by the second author with...
A. Stavrova and N. Vavilov. We also use results of the second author [Pan1] to reduce our problem to the case when $G$ is simple and simply-connected (at a price of replacing a local ring by semi-local). Also, by a result of Popescu [Pop, Swa, Spi] we may assume that $U$ is of geometric origin.

The proof of Theorem 3 is inspired by the theory of affine Grassmannians. We do not use the affine Grassmannians explicitly in this paper, however, the interested reader is invited to look at [Fed2], where an alternative proof of our Theorem 3 is sketched.

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2. Main results

Theorem 1. Let $R$ be a regular semi-local domain containing an infinite field, and let $K$ be its field of fractions. If $G$ is a reductive group scheme over $R$, then the map

$$H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G)$$

induced by the inclusion of $R$ into $K$ has a trivial kernel. In other words, under the above assumptions on $R$ and $G$, each principal $G$-bundle over $R$ having a $K$-rational point is trivial.

Theorem 1 has the following corollary.

Corollary 1. Under the same hypothesis as in Theorem 1, the map

$$H^2_{\text{ét}}(R, G) \to H^2_{\text{ét}}(K, G)$$

induced by the inclusion of $R$ into $K$ is injective. Equivalently, two principal $G$-bundles over $R$ that become isomorphic upon restriction to $K$ are isomorphic.

Proof. Let $G_1$ and $G_2$ be two principal $G$-bundles over $U := \text{Spec } R$. Assume that $G_1$ and $G_2$ are isomorphic over $\text{Spec } K$. Recall that the functor sending a $U$-scheme $T$ to the set of isomorphisms of principal $G$-bundles $G_1 \times_U T \to G_2 \times_U T$ is represented by an affine $U$-scheme $\text{Iso}(G_1, G_2)$. Consider also the scheme $\text{Aut } G_2 := \text{Iso}(G_2, G_2)$ of $G$-bundle automorphisms of $G_2$. It is a reductive group scheme because it is étale locally over $R$ isomorphic to $G$. It is easy to see that $\text{Iso}(G_1, G_2)$ is a principal $\text{Aut } G_2$-bundle. By Theorem 1 it is trivial, and we see that $G_1 \cong G_2$. □
While Theorem 1 was previously known for reductive group schemes $G$ coming from the ground field (see [CTO, Rag1, Rag2]), in certain cases the corollary is a new result even for such group schemes. For example, it was not known for split group schemes $G$ of type $E_8$. Also, the corollary was not known for Spin$(A,\sigma)$, where $A$ is a skew-field over a field $k$ (char $k \neq 2$) and $\sigma$ is an involution of orthogonal type on $A$.

For a scheme $U$ we denote by $A^1_U$ the affine line over $U$ and by $P^1_U$ the projective line over $U$. If $T$ is a $U$-scheme, we will use the term “principal $G$-bundle over $T$” to mean a principal $G \times_U T$-bundle over $T$.

In Section 3 we deduce Theorem 1 from the following result of independent interest (cf. [PSV, Thm. 1.3]).

**Theorem 2.** Let $R$ be the semi-local ring of finitely many closed points on an irreducible smooth affine variety over an infinite field $k$ and set $U = \text{Spec} R$. Let $G$ be a simple, simply-connected group scheme over $U$ (see [DG, Exp. XXIV, Sect. 5.3] for the definition). Let $E_t$ be a principal $G$-bundle over the affine line $A^1_U = \text{Spec} R[t]$, and let $h(t) \in R[t]$ be a monic polynomial. Denote by $(A^1_U)_h$ the open subscheme in $A^1_U$ given by $h(t) \neq 0$ and assume that the restriction of $E_t$ to $(A^1_U)_h$ is a trivial principal $G$-bundle. Then for each section $s : U \to A^1_U$ of the projection $A^1_U \to U$ the $G$-bundle $s^*E_t$ over $U$ is trivial.

The derivation of Theorem 1 from Theorem 2 is based on results of the second author, A. Stavrova, and N. Vavilov, namely, on [Pan1] and [PSV, Thm. 1.2].

Let $Y$ be a semi-local scheme. We will call a simple $Y$-group scheme isotropic if its restriction to each connected component of $Y$ contains a proper parabolic subgroup scheme. (Note that by [DG, Exp. XXVI, Cor. 6.14] this is equivalent to the usual definition, that is, to the requirement that the group scheme contains a torus isomorphic to $\mathbb{G}_m,Y$.) Theorem 2 is, in turn, derived from the following statement.

**Theorem 3.** Let $R$ be the semi-local ring of finitely many closed points on an irreducible smooth affine variety over an infinite field $k$ and set $U = \text{Spec} R$. Let $G$ be a simple, simply-connected group scheme over $U$.

Let $Z \subset P^1_U$ be a closed subscheme finite over $U$. Let $Y \subset P^1_U$ be a closed subscheme étale over $U$. Assume that $Y \cap Z = \emptyset$, and $G_Y := G \times_U Y$ is isotropic. Assume also that for every closed point $u \in U$ such that the algebraic group $G_u := G|_u$ is isotropic, there is a $k(u)$-rational point in $Y_u := P^1_u \cap Y$. (Here $k(u)$ is the residue field of $u$.)

Let $G$ be a principal $G$-bundle over $P^1_U$ such that its restriction to $P^1_U - Z$ is trivial. Then the restriction of $G$ to $P^1_U - Y$ is also trivial.

The proof of this result was inspired by the theory of affine Grassmannians (see [Fed2] for a proof using affine Grassmannians explicitly).

**Remarks.** 1. Assume that for every closed point $u \in U$ the algebraic group $G_u$ is anisotropic. Then we can take $Y = \emptyset$.

2. It is not necessary to assume that $Y \cap Z = \emptyset$. Indeed, let $Y$ satisfy the conditions of the theorem except that it may intersect $Z$. Since $U$ is semi-local, there is a projective transformation $\theta : P^1_U \to P^1_U$ such that $\theta(Y) \cap Y = \emptyset$. By the above theorem the restriction of $G$ to $P^1_U - \theta(Y)$ is trivial. Now we can apply the theorem again with $Z = \theta(Y)$ to show that the restriction of $G$ to $P^1_U - Y$ is trivial.
3. In the situation of Theorem 3 let \( G \) be isotropic. Then it follows from the theorem that one can take \( Y = \{ \infty \} \times \mathbb{P}^1 \), that is, the restriction of \( G \) to \( \mathbb{A}^1 \) is trivial. In fact, this is a partial case of [PSV, Thm. 1.3]. On the other hand, if \( G \) is anisotropic, this restriction is not in general trivial. For an example see [Fed2].

2.1. Organization of the paper. In Section 3 we reduce Theorem 1 to Theorem 2. This reduction is based on [Pan1], [PSV, Thm. 1.2], and a theorem of D. Popescu [Pop, Swa, Spi]. In Section 4 we reduce Theorem 2 to Theorem 3.

In Section 5 we prove Theorem 3. The main idea is to modify the principal bundle \( G \) in a neighborhood of \( Y \) so that \( G \) becomes trivial. We use the technique of Henselization. One can give an essentially equivalent proof based on formal loops, see [Fed2, Sect. 6.2].

In Section 6 we give an application of Theorem 1.

3. Reducing Theorem 1 to Theorem 2

In what follows “\( G \)-bundle” always means “principal \( G \)-bundle”. Now we assume that Theorem 2 holds. We start with the following particular case of Theorem 1.

Proposition 3.1. Let \( R \) be the semi-local ring of finitely many closed points on an irreducible smooth affine variety over an infinite field \( k \) and set \( U = \text{Spec} \, R \). Let \( G \) be a simple, simply-connected group scheme over \( U \). Let \( E \) be a principal \( G \)-bundle over \( U \), trivial at the generic point of \( U \). Then \( E \) is trivial.

**Proof.** Under the hypothesis of the proposition, a particular case of [PSV, Thm. 1.2] reads as follows: there exist

(a) a principal \( G \)-bundle \( E_t \) over \( \mathbb{A}^1 \); 
(b) a monic polynomial \( h(t) \in R[t] \).

Moreover, these data satisfy the following conditions:

1. the restriction of \( E_t \) to \( (\mathbb{A}^1)_h \) is a trivial principal \( \mathbb{G}_m \)-bundle;
2. there is a section \( s : U \to \mathbb{A}^1 \) such that \( s^* E_t = E \).

Now it follows from Theorem 2 that \( E \) is trivial. \( \square \)

Proposition 3.2. Let \( R \) be the semi-local ring of finitely many closed points on an irreducible smooth affine variety over an infinite field \( k \) and set \( U = \text{Spec} \, R \). Let \( G \) be a reductive group scheme over \( U \). Let \( E \) be a principal \( G \)-bundle over \( U \) trivial at the generic point of \( U \). Then \( E \) is trivial.

**Proof.** The following is proved in [Pan1]:

- Denote by \( G_{\text{der}} \) the derived group scheme of \( G \). If the Grothendieck–Serre conjecture holds for any inner form of \( G_{\text{der}} \), then it holds for \( G \). (Recall that an inner forms of a group scheme \( H \) is a group scheme isomorphic to \( \text{Aut}(H) \), where \( H \) is an \( H \)-bundle.)
- If the Grothendieck–Serre conjecture holds for any inner form of the simply-connected cover of a semi-simple \( U \)-group scheme \( H \), then it holds for \( H \).

Thus, we may assume that \( G \) is semi-simple and simply-connected. By [DG, Exp. XXIV, Prop. 5.10] (which is valid for simply-connected group schemes as well, see the beginning of [DG, Exp. XXIV, Sect. 5]) there is a sequence \( U_1, \ldots, U_r \) of finite étale \( U \)-schemes, and for each \( i = 1, \ldots, r \) a simple simply-connected \( U_i \)-group scheme \( G_i \) such that

\[
G \cong \prod_{i=1}^r R_{U_i/U}(G_i).
\]
where \( R_{U_i/U} \) is the Weil restriction functor. Now the Faddeev–Shapiro Lemma (see [DG Exp. XXIV, Prop. 8.4]) shows that the Grothendieck–Serre conjecture for \( G \) holds, if for each \( i \) the conjecture holds for \( G_i \). For more details, see [PSV] Thm. 11.1. Thus, we may assume that \( G \) is simple and simply-connected. Now the proposition is reduced to Proposition 3.3. \( \square \)

**Remark 3.3.** Even if we start with a local scheme \( U \), the schemes \( U_i \) are only semi-local in general. This is why we have to work with semi-local schemes from the beginning.

**Proof of Theorem 3 assuming Theorem 2.** Let us prove a general statement first. Let \( k' \) be an infinite field, \( X \) be a \( k' \)-smooth irreducible affine variety, \( H \) be a reductive group scheme over \( X \). Denote by \( k'[X] \) the ring of regular functions on \( X \) and by \( k'(X) \) the field of rational functions on \( X \). Let \( H \) be a principal \( H \)-bundle over \( X \) trivial over \( k'(X) \). Let \( p_1, \ldots, p_n \) be prime ideals in \( k'[X] \), and let \( O_{p_1, \ldots, p_n} \) be the corresponding semi-local ring.

**Lemma 3.4.** The principal \( H \)-bundle \( H \) is trivial over \( O_{p_1, \ldots, p_n} \).

**Proof.** For each \( i = 1, 2, \ldots, n \) choose a maximal ideal \( m_i \subset k'[X] \) containing \( p_i \). One has inclusions of \( k' \)-algebras

\[
O_{m_1, \ldots, m_n} \subset O_{p_1, \ldots, p_n} \subset k'(X).
\]

By Proposition 3.2 the principal \( H \)-bundle \( H \) is trivial over \( O_{m_1, \ldots, m_n} \). Thus it is trivial over \( O_{p_1, \ldots, p_n} \). \( \square \)

Let us return to our situation. Let \( m_1, \ldots, m_n \) be all the maximal ideals of \( R \). Let \( E \) be a \( G \)-bundle over \( R \) trivial over the fraction field of \( R \). Clearly, there is a non-zero \( f \in R \) such that \( E \) is trivial over \( Rf \). Let \( k' \) be the algebraic closure of the prime field of \( R \) in \( k \). Note that \( k' \) is perfect. It follows from Popescu’s theorem ([Pop Swa Spil]) that \( R \) is a filtered inductive limit of smooth \( k' \)-algebras \( R_\alpha \). Modifying the inductive system \( R_\alpha \) if necessary, we can assume that each \( R_\alpha \) is integral. There exist an index \( \alpha \), a reductive group scheme \( G_\alpha \) over \( R_\alpha \), a principal \( G_\alpha \)-bundle \( E_\alpha \) over \( R_\alpha \), and an element \( f_\alpha \in R_\alpha \) such that \( G = G_\alpha \times_{\text{Spec } R_\alpha} \text{Spec } R \), \( E \) is isomorphic to \( E_\alpha \times_{\text{Spec } R_\alpha} \text{Spec } R \) as principal \( G \)-bundle, \( f \) is the image of \( f_\alpha \) under the homomorphism \( \varphi_\alpha : R_\alpha \to R \), and \( E_\alpha \) is trivial over \( (R_\alpha)_{f_\alpha} \).

If the field \( k' \) is infinite, then for each maximal ideal \( m_i \in R \) \( (i = 1, \ldots, n) \) set \( p_i = \varphi_\alpha^{-1}(m_i) \). The homomorphism \( \varphi_\alpha \) induces a homomorphism of semi-local rings \( (R_\alpha)_{p_1, \ldots, p_n} \to R \). By Lemma 3.4 the principal \( G_\alpha \)-bundle \( E_\alpha \) is trivial over \( (R_\alpha)_{p_1, \ldots, p_n} \). Whence the \( G \)-bundle \( E \) is trivial over \( R \).

If the field \( k' \) is finite, then \( k \) contains an element \( t \) transcendental over \( k' \). Thus \( R \) contains the subfield \( k'(t) \) of rational functions in the variable \( t \). So, if \( R'_\alpha := R_\alpha \otimes_{k'} k'(t) \), then \( \varphi_\alpha \) can be decomposed as follows

\[
R_\alpha \to R_\alpha \otimes_{k'} k'(t) = R'_\alpha \xrightarrow{\psi_\alpha} R.
\]

Let \( G'_\alpha = G_\alpha \times_{\text{Spec } R_\alpha} \text{Spec } R'_\alpha, E'_\alpha = E_\alpha \times_{\text{Spec } R_\alpha} \text{Spec } R'_\alpha, f'_\alpha = f_\alpha \otimes 1 \in R'_\alpha \), then the \( G'_\alpha \)-bundle \( E'_\alpha \) is trivial over \( (R'_\alpha)_{f'_\alpha} \).

Let \( q_i = \psi_\alpha^{-1}(m_i) \) for \( i = 1, \ldots, n \). The ring \( R'_\alpha \) is a \( k'(t) \)-smooth algebra over the infinite field \( k'(t) \), and the \( G'_\alpha \)-bundle \( E'_\alpha \) is trivial over \( (R'_\alpha)_{f'_\alpha} \). By Lemma 3.4 the \( G'_\alpha \)-bundle \( E'_\alpha \) is trivial over \( (R'_\alpha)_{q_1, \ldots, q_n} \). The homomorphism \( \psi_\alpha \) can be factored as

\[
R'_\alpha \to (R'_\alpha)_{q_1, \ldots, q_n} \to R.
\]
Thus the $G$-bundle $\mathcal{E}$ is trivial over $R$. □

**Remark.** If $k$ is perfect, we can use it instead of $k'$, and the above proof simplifies.

### 4. Reducing Theorem 2 to Theorem 3

Now we assume that Theorem 3 is true. Let $U$ and $G$ be as in Theorem 2. Let $u_1, \ldots, u_n$ be all the closed points of $U$. Let $k(u_i)$ be the residue field of $u_i$. Consider the reduced closed subscheme $u$ of $U$, whose points are $u_1, \ldots, u_n$. Thus

$$u \cong \coprod_i \text{Spec } k(u_i).$$

Set $G_u = G \times_U u$. By $G_{u_i}$ we denote the fiber of $G$ over $u_i$; it is a simple simply-connected algebraic group over $k(u_i)$. Let $u' \subset u$ be the subscheme of all closed points $u_i$ such that the group $G_{u_i}$ is isotropic. Set $u'' = u - u'$. It is possible that $u'$ or $u''$ is empty.

**Proposition 4.1.** There is a closed subscheme $Y \subset \mathbb{P}_U^1$ such that $Y$ is étale over $U$, $G_Y = G \times_U Y$ is isotropic, and for all $u_i \in u'$ there is a $k(u_i)$-rational point $y_i \in Y$ lying over $u_i$.

**Proof.** If $u'$ is empty, we just take $Y = \emptyset$.

Otherwise, for every $u_i$ in $u'$ choose a proper parabolic subgroup $P_{u_i}$ in $G_{u_i}$. Let $\mathcal{P}_i$ be the $U$-scheme of parabolic subgroup schemes of $G$ of the same type as $P_{u_i}$. It is a smooth projective $U$-scheme (see [DG, Cor. 3.5, Exp. XXVI]). The subgroup $P_{u_i}$ in $G_{u_i}$ is a $k(u_i)$-rational point $p_i$ in the fibre of $\mathcal{P}_i$ over the point $u_i$.

We claim that there is a closed subscheme $Y_i$ of $\mathcal{P}_i$ such that $Y_i$ is étale over $U$ and $p_i \in Y_i$. Indeed, let $r$ be the dimension of $\mathcal{P}_i$ over $U$ and take an embedding of $\mathcal{P}_i$ into the projective space $\mathbb{P}^N_U = \text{Proj}(R[x_0, \ldots, x_N])$. Let $m_j$ be the maximal ideal in $R$ corresponding to $u_j \in u$. Since $k$ is infinite, by a variant of Bertini’s theorem (see [SGA, Exp. XI, Thm. 2.1]), for each $j$ there is a sequence of homogeneous quadratic polynomials $H_1, \ldots, H_r \in (R/m_j)[x_0, \ldots, x_N]$ such that the subscheme $T_j$ of $\mathbb{P}^N_{k(u_j)}$ given by the equations $H_1 = \ldots = H_r = 0$ intersects the fibre of $\mathcal{P}_i$ over $u_j$ transversally. Moreover, we may assume that $p_i \in T_i$. By the Chinese Remainder Theorem for each $m \in \{1, \ldots, r\}$ there is a common lift of polynomials $H_m^i$ to a quadratic polynomial $H_m \in R[x_0, \ldots, x_N]$. Let $T$ be the scheme given by $H_1 = \ldots = H_r = 0$. Then $Y_i := T \cap \mathcal{P}_i$ is the required subscheme. Indeed, we only need to check that $Y_i$ is étale over $U$. However, for every closed point of $U$ the fiber of $Y_i$ over this point is étale by construction. Hence, it is enough to check that $Y_i$ is flat over $U$. The flatness follows immediately from [Mat, Thm. 23.1].

Now consider $Y_i$ just as a $U$-scheme and set $Y = \coprod_{u_i \in u} Y_i$. Next, $G_Y$ is isotropic by the choice of $Y_i$. Thus $G_Y$ is isotropic as well. Since the field $k$ is infinite and $Y$ is finite étale over $U$, we can choose a closed $U$-embedding of $Y$ in $\mathbb{A}_U^1$. We will identify $Y$ with the image of this closed embedding. Since $Y$ is finite over $U$, it is closed in $\mathbb{P}_U^1$.

**Proof of Theorem 2 assuming Theorem 3.** Set $Z := \{h = 0\} \cup s(U) \subset \mathbb{A}_U^1$. Note that $\{h = 0\}$ is closed in $\mathbb{P}_U^1$ and finite over $U$ because $h$ is monic. Further, $s(U)$ is also closed in $\mathbb{P}_U^1$ and finite over $U$ because it is a zero set of a degree one monic polynomial. Thus $Z \subset \mathbb{P}_U^1$ is closed and finite over $U$. 
Let $Y$ be as in Proposition 4.1. Since $U$ is semi-local, there exists a projective transformation $\theta : \mathbb{P}^1_U \to \mathbb{P}^1_Y$ such that $Z \cap \theta(Y) = \emptyset$. Thus, replacing $Y$ by $\theta(Y)$ we may assume that $Z \cap Y = \emptyset$.

Since the principal $G$-bundle $E_t$ is trivial over $(\mathbb{A}^1_U)_h$, and $G$-bundles can be glued in the Zariski topology, there exists a principal $G$-bundle $G$ over $\mathbb{P}^1_U$ such that

(i) its restriction to $\mathbb{A}^1_U$ coincides with $E_t$;

(ii) its restriction to $\mathbb{P}^1_U - Z$ is trivial.

Applying Theorem 3 with the above choice of $Y$ and $Z$, we see that the restriction of $G$ to $\mathbb{P}^1_U - Y$ is a trivial $G$-bundle. Since $s(U)$ is in $(\mathbb{P}^1_U - Y) \cap \mathbb{A}^1_U$, and $G|_{\mathbb{A}^1_U}$ coincides with $E_t$, we conclude that $s^* E_t$ is a trivial principal $G$-bundle over $U$. □

5. PROOF OF THEOREM 3

We will be using notation from Theorem 3. Let $u, u', u''$ be as in Section 4. For $u \in u$ set $G_u = G|_u$.

Proposition 5.1. Let $E$ be a $G$-bundle over $\mathbb{P}^1_U$ such that $E|_{\mathbb{P}^1_U}$ is a trivial $G_u$-bundle for all $u \in u$. Assume that there exists a closed subscheme $T$ of $\mathbb{P}^1_U$ finite over $U$ such that the restriction of $E$ to $\mathbb{P}^1_U - T$ is trivial. Then $E$ is trivial.

Proof. This follows from Proposition 9.6 of [PSV]. □

Remark 5.2. The same proof goes through for any semi-simple $U$-group scheme $G$.

5.1. An outline of a proof of Theorem 3

A detailed proof will be given in the present text below. Firstly, we give an outline of the proof.

Denote by $Y^h$ the Henselization of the pair $(\mathbb{A}^1_U, Y)$; it is a scheme over $\mathbb{A}^1_U$. We review some facts about Henselization of pairs in Section 5.3. In particular, there exists a canonical closed embedding $s^h : Y \to Y^h$, and we set $Y^h := Y^h - s^h(Y)$. We have a natural Cartesian square (see Section 5.4 for more details)

\[
\begin{array}{ccc}
Y^h & \longrightarrow & Y^h \\
\downarrow & & \downarrow \\
\mathbb{P}^1_U - Y & \longrightarrow & \mathbb{P}^1_U.
\end{array}
\]

This square can be used to glue principal bundles. In particular, if $G'$ is a $G$-bundle over $\mathbb{P}^1_U - Y$, then by $\text{Gl}(G', \varphi)$ we denote the $G$-bundle over $\mathbb{P}^1_U$ obtained by gluing $G'$ with the trivial $G$-bundle $G \times_U Y^h$ via a $G$-bundle isomorphism $\varphi : G \times_U Y^h \to G'|_{Y^h}$.

Similarly, set $Y_u := Y \times_U u$ and denote by $Y_u \times U$ the Henselization of the pair $(\mathbb{A}^1_u, Y_u)$, let $s^h_u : Y_u \to Y^h_u$ be the closed embedding. Set $Y^h_u := Y^h_u - s_u(Y_u)$. Let $G_u'$ be a $G_u$-bundle over $\mathbb{P}^1_U - Y_u$, where $G_u := G \times_U u$. Denote by $\text{Gl}_u(G_u', \varphi_u)$ the $G_u$-bundle over $\mathbb{P}^1_u$ obtained by gluing $G_u'$ with the trivial bundle $G_u \times_u Y^h_u$ via a $G_u$-bundle isomorphism $\varphi_u : G_u \times_u Y^h_u \to G_u'|_{Y^h_u}$.

We will prove in Section 5.2 that the restriction of the $G$-bundle $G$ to $Y^h$ is trivial, so $G$ can be presented in the form $\text{Gl}(G', \varphi)$, where $G' = G|_{\mathbb{P}^1_U - Y}$. The idea is to show that...
There is an element $α ∈ G(\hat{Y}^h)$ such that the $G_u$-bundle $Gl(G', ϕ ∘ α)|_{P^1_u}$ is (⋆) trivial (here $α$ is regarded as an automorphism of the $G$-bundle $G × U \hat{Y}^h$ given by right translation action of $α$).

If we find $α$ satisfying condition (⋆), then Proposition 5.1 applied to $T = Y ∪ Z$, shows that the $G$-bundle $Gl(G', ϕ ∘ α)$ is trivial over $P^1_{t_u}$. On the other hand, its restriction to $P^1_{t_u} − Y$ coincides with the $G$-bundle $G' = G|_{P^1_{t_u} − Y}$. Thus $G|_{P^1_{t_u} − Y}$ is a trivial $G$-bundle.

To prove (⋆), one should show that
(i) the bundle $G|_{P^1_{t_u} − Y_u}$ is trivial;
(ii) each element $γ_u ∈ G_u(Y^h_u)$ can be written in the form
$$\alpha|_{Y^h_u} · β_u|_{Y^h_u}$$
for certain elements $α ∈ G(\hat{Y}^h)$ and $β_u ∈ G_u(Y^h_u)$.

If we succeed in showing that (i) and (ii) above hold, then we proceed as follows. Present the $G$-bundle $G$ in the form $Gl(G', ϕ)$ as above. Observe that
$$Gl(G', ϕ)|_{P^1_u} ≅ Gl_u(G'_{u}, ϕ_u),$$
where $G'_{u} := G'|_{P^1_{u} − Y_u}$, $ϕ_u := ϕ|_{G_u × u \hat{Y}^h_u}$.

Using property (i), find an element $γ_u ∈ G_u(Y^h_u)$ such that the $G_u$-bundle $Gl_u(G'_u, ϕ_u ∘ γ_u)$ is trivial. For this $γ_u$ find elements $α$ and $β_u$ as in (ii). Finally take the $G$-bundle $Gl(G', ϕ ∘ α)$. Then its restriction to $P^1_u$ is trivial. Indeed, one has a chain of $G_u$-bundle isomorphisms
$$Gl(G', ϕ ∘ α)|_{P^1_u} ≅ Gl_u(G'_u, ϕ_u ∘ α|_{Y^h_u}) ≅ Gl_u(G'_u, ϕ_u ∘ α|_{Y^h_u} ∩ β_u|_{Y^h_u}) = Gl_u(G'_u, ϕ_u ∘ γ_u),$$
which is trivial by the very choice of $γ_u$. Thus, (⋆) will be achieved.

Let us prove (i) and (ii). If $u ∈ u'$, then there is a $k(u)$-rational point in $Y_u := P^1_u ∩ Y$. Hence the $G_u$-bundle $G'_u := G|_{P^1_u}$ is trivial over $P^1_u − Y_u$ (see [Gil11] Cor. 3.10(a)). If $u ∈ u''$, then $G_u$ is anisotropic and $G_u$ is trivial even over $P^1_u$ (again, by [Gil11] Cor. 3.10(a)). Thus $G|_{P^1_{t_u} − Y_u}$ is trivial. So, (i) is achieved.

To achieve (ii) recall that for a domain $A$, its fraction field $L$, and a simple group scheme $H$ over $A$, having a parabolic subgroup scheme $P$, one can form a subgroup $E(L)$ of ‘elementary matrices’ in $H(L)$. It is known (see [Gil3] Fait 4.3, Lemma 4.5) that if $A$ is a Henselian discrete valuation ring and $H$ is simply-connected, then every element $γ ∈ H(L)$ can be written in the form $γ = α · β$, where $α ∈ E(L)$ and $β ∈ H(A)$. Applying this observation in our context, we see that $γ_u ∈ G_u(Y^h_u)$ can be written in the form $γ_u = α_u · β_u|_{Y^h_u}$, where $β_u ∈ G_u(Y^h_u)$ and $α_u ∈ E(Y^h_u)$. It remains to observe that the natural homomorphism $E(Y^h) → E(Y^h_u)$ is surjective, since $Y^h_u$ is a closed subscheme of the affine scheme $Y^h$, and so (ii) is achieved.

A realization of this plan in details is given below in the paper.

5.2. Henselization of commutative rings. For a commutative ring $A$ we denote by $Rad(A)$ its Jacobson ideal. One can find the following definition in [Gabl Sect. 0] (see also [Ray] Chapter 11).
Definition 5.3. If $I$ is an ideal in a commutative ring $A$, then the pair $(A, I)$ is called Henselian, if $I \subseteq \text{Rad}(A)$ and for every two relatively prime monic polynomials $\overline{g}, \overline{h} \in A[t]$, where $A = A/I$, and monic lifting $f \in A[t]$ of $\overline{g}\overline{h}$, there exist monic liftings $g, h \in A[t]$ such that $f = gh$. (Two polynomials are called relatively prime, if they generate the unit ideal.)

Lemma 5.4. A pair $(A, I)$ is Henselian if and only if for every étale $A$-algebra $A'$ and every $\sigma \in \text{Hom}_{A-\text{alg}}(A', A/I)$ there is a unique $\overline{\sigma} \in \text{Hom}_{A-\text{alg}}(A', A)$ that lifts $\sigma$.

Proof. See [Gab] Sect. 0. □

Lemma 5.5. Let $(A, I)$ be a Henselian pair with a semi-local ring $A$ and $J \subseteq A$ be an ideal. Then the pair $(A/J, (I + J)/J)$ is Henselian.

Proof. Clearly $(I + J)/J \subseteq \text{Rad}(A/J)$. Now let $\overline{g}, \overline{h} \in (A/(I + J))[t]$ be two relatively prime monic polynomials and let $f \in (A/J)[t]$ be a monic polynomial such that $f \mod (I + J)/J = \overline{g}\overline{h} \in (A/(I + J))[t]$.

We claim that there exist relatively prime monic liftings of $\overline{g}$ and $\overline{h}$ to $(A/I)[t]$. Indeed, let $m_1, \ldots, m_n$ be all the maximal ideals of $A/I$ not containing $(I + J)/I$ (recall that $A$ is semi-local). By the Chinese remainder theorem we can find monic $\overline{G}, \overline{H} \subseteq (A/I)[t]$ such that

$$\overline{G} \mod (I + J)/I = \overline{g}, \quad \overline{G} \mod m_i = t^{\deg \overline{g}} \quad \text{for } i = 1, \ldots, n,$$

$$\overline{H} \mod (I + J)/I = \overline{h}, \quad \overline{H} \mod m_i = t^{\deg \overline{h}} - 1 \quad \text{for } i = 1, \ldots, n.$$

Then $\overline{G}$ and $\overline{H}$ are relatively prime. The ring homomorphism

$$A \to (A/I) \times_{A/(I + J)} (A/J)$$

is surjective. Thus there exists a monic polynomial $F \in A[t]$ such that $F \mod I = GH$ and $F \mod J = f$.

The pair $(A, I)$ is Henselian. Thus there exist monic liftings $G, H \in A[t]$ of $\overline{G}, \overline{H}$ such that $F = GH$. Let $g = G \mod J \in (A/J)[t]$ and $h = H \mod J \in (A/J)[t]$. Clearly, $g$ and $h$ are monic polynomials in $(A/J)[t]$, $f = gh \in (A/J)[t]$. And finally, $g \mod (I + J)/J = \overline{g}, h \mod (I + J)/J = \overline{h}$ in $(A/(I + J))[t]$. Whence the Lemma. □

One can find the following definition in [Gab] Sect. 0.

Definition 5.6. The Henselization of a pair $(A, I)$ is the pair $(A^h_I, I^h)$ (over $(A, I)$) defined as follows

$$(A^h_I, I^h) := \text{the filtered inductive limit over the category } \mathcal{N} \text{ of } (A', \text{Ker}(\sigma)),$$

where $\mathcal{N}$ is the filtered category of pairs $(A', \sigma)$ such that $A'$ is an étale $A$-algebra and $\sigma \in \text{Hom}_{A-\text{alg}}(A', A/I)$.

Note that the category $\mathcal{N}$ is filtered because finite direct limits preserve étaleness.

5.3. Henselization of affine pairs. Let us translate the previous section in the geometric language. Let $S = \text{Spec } A$ be a scheme and $T = \text{Spec } (A/I)$ be a closed subscheme. Then we define a category $\text{Neib}(S, T)$ whose objects are triples $(W, \pi : W \to S, s : T \to W)$ satisfying the following conditions:

(i) $W$ is affine;
of the previous section we have $\pi$ that is, is Henselian, which means that for any affine étale morphism $A \to T$ such that $\pi' \circ \rho = \pi$ and $\rho \circ s = s'$. Note that such $\rho$ is automatically étale by [Gro3, Cor. 17.3.5].

Consider the functor from $\tilde{\text{Neib}}(S,T)$ to the category of $S$-schemes, sending $(W,\pi,s)$ to $(W,\pi)$. This functor has a projective limit $(T^h,\pi^h)$. In the notation of the previous section we have $T^h = \text{Spec } A^h_I$ and $\pi^h : T^h \to S$ is induced by the structure of an $A$-algebra on $A^h_I$. We also get a closed $S$-embedding $s^h : T \to T^h$, that is, $\pi^h \circ s^h$ coincides with the inclusion $T \hookrightarrow S$. We call $(T^h,\pi^h,s^h)$ the Henselization of the pair $(S,T)$ (cf. Definition 5.6). Note that the pair $(T^h,s^h(T))$ is Henselian, which means that for any affine étale morphism $\pi : Z \to T^h$, any section $s$ of $\pi$ over $s^h(T)$ uniquely extends to a section of $\pi$ over $T^h$; this follows from Lemma 5.4.

Denote by $\text{Neib}(S,T)$ the full subcategory of $\tilde{\text{Neib}}(S,T)$ consisting of triples $(W,\pi,s)$ such that

(iii) $\pi \circ s$ coincides with the inclusion $T \to S$ (thus $s$ is a closed embedding).

A morphism from $(W,\pi,s)$ to $(W',\pi',s')$ in this category is a morphism $\rho : W \to W'$ such that $\pi' \circ \rho = \pi$ and $\rho \circ s = s'$. Note that such $\rho$ is automatically étale by [Gro3, Cor. 17.3.5].

Consider the functor from $\tilde{\text{Neib}}(S,T)$ to the category of $S$-schemes, sending $(W,\pi,s)$ to $(W,\pi)$. This functor has a projective limit $(T^h,\pi^h)$. In the notation of the previous section we have $T^h = \text{Spec } A^h_I$ and $\pi^h : T^h \to S$ is induced by the structure of an $A$-algebra on $A^h_I$. We also get a closed $S$-embedding $s^h : T \to T^h$, that is, $\pi^h \circ s^h$ coincides with the inclusion $T \hookrightarrow S$. We call $(T^h,\pi^h,s^h)$ the Henselization of the pair $(S,T)$ (cf. Definition 5.6). Note that the pair $(T^h,s^h(T))$ is Henselian, which means that for any affine étale morphism $\pi : Z \to T^h$, any section $s$ of $\pi$ over $s^h(T)$ uniquely extends to a section of $\pi$ over $T^h$; this follows from Lemma 5.4.

Denote by $\text{Neib}(S,T)$ the full subcategory of $\tilde{\text{Neib}}(S,T)$ consisting of triples $(W,\pi,s)$ such that

(iv) the schemes $(\pi)^{-1}(T)$ and $s(T)$ coincide.

Remark. Let $(W,\pi,s)$ and $(W',\pi',s')$ be objects of $\text{Neib}(S,T)$. Let $\rho : W \to W'$ be a morphism such that $\pi' \circ \rho = \pi$. Then it is easy to see that $\rho \circ s = s'$ so that $\rho$ is a morphism in $\text{Neib}(S,T)$. (Again, $\rho$ is automatically étale.)

**Lemma 5.7.** $\text{Neib}(S,T)$ is co-final in $\tilde{\text{Neib}}(S,T)$.

**Proof.** We need to check that for an object $(W,\pi,s)$ of $\tilde{\text{Neib}}(S,T)$ there is an object $(W',\pi',s')$ of $\text{Neib}(S,T)$ and a morphism $(W',\pi',s') \to (W,\pi,s)$. Let $\pi_T : (\pi)^{-1}(T) \to T$ be the base-changed morphism, which is étale. It follows from (iii) that $s$ is a section of $\pi_T$. As was already mentioned above, a section of an étale morphism is étale by [Gro3, Cor. 17.3.5]. Thus $s$ is both an open and a closed embedding, and we have a disjoint union decomposition $(\pi)^{-1}(T) = s(T) \coprod T_0$ for a scheme $T_0$. All our schemes are affine, so there is a regular function $f$ on $W$ such that $f = 0$ on $T_0$ and $f = 1$ on $s(T)$.

Set $W' = W - \{f = 0\}$; $\pi' = \pi|_{W'}$, $s' = s$. Then $W'$ is affine; thus $(W',\pi',s') \in \text{Neib}(S,T)$, and we have an obvious morphism $(W',\pi',s') \to (W,\pi,s)$. 

The lemma implies that the category $\text{Neib}(S,T)$ is co-filtered, and that the Henselization can be computed by taking the limit over $\text{Neib}(S,T)$, instead of $\tilde{\text{Neib}}(S,T)$. It is now easy to check that if $(T^h,\pi^h,s^h)$ is the Henselization of $(S,T)$, then $(\pi^h)^{-1}(T) = s^h(T)$.

Note the two following properties of Henselization of affine pairs.

**Lemma 5.8.** Let $T$ be a semi-local scheme. Then the Henselization commutes with restriction to closed subschemes. In more detail, if $S' \subset S$ is a closed subscheme, then we get a base change functor $\tilde{\text{Neib}}(S,T) \to \tilde{\text{Neib}}(S',T \times_S S')$. This functor yields a morphism $(T \times_S S')^h \to T^h \times_S S'$. This morphism is an isomorphism and the canonical section $s' : T \times_S S' \to (T \times_S S')^h$ coincides under this identification with $s \times_S \text{Id}_{S'} : T \times_S S' \to T^h \times_S S'$.

**Sketch of proof.** Let us construct a morphism in the opposite direction. Since $T$ is semi-local, $T^h$ is also semi-local (the proof is straightforward). Therefore by Lemma 5.6 the pair $(T^h \times_S S',s(T) \times_S S')$ is Henselian.
Let \((W, \pi, s) \in \widehat{\mathrm{Neib}}(S', T \times_S S')\). From \(\pi\) by a base change we get an étale morphism \(\tilde{\pi} : (T'^h \times_S S') \times_S W \to T'^h \times_S S'\). This morphism has an obvious section over \(s(T) \times S\). Since the pair \((T'^h \times_S S', s(T) \times_S S')\) is Henselian, this section extends uniquely to a section of \(\tilde{\pi}\) over \(T'^h \times_S S'\), which, in turn, gives a morphism \(T'^h \times_S S' \to W\). These morphisms give the desired morphism \(T'^h \times_S S' \to (T \times_S S')^h\).

**Lemma 5.9.** If \(T = \bigsqcup T_i\) is a disjoint union, then \(T'^h = \bigsqcup T_i'^h\).

*Sketch of a proof.* Note that the functor from \(\prod_i \widehat{\mathrm{Neib}}(S, T_i)\) to \(\widehat{\mathrm{Neib}}(S, T)\), sending a collection of schemes to their disjoint union, is co-final. \(\square\)

### 5.4. Gluing principal \(G\)-bundles.

Recall that \(U = \mathrm{Spec} \, R\), where \(R\) is the semi-local ring of finitely many closed points on an irreducible smooth affine variety over an infinite field \(k\). Also, \(G\) is a simple simply-connected group scheme over \(U\), and \(Y\) is a closed subscheme of \(\mathbb{P}^1_U\), étale over \(U\). We may assume that \(Y \subset A^1_U\) (otherwise, just change the coordinate on \(\mathbb{P}^1_U\)). We will apply the Henselization discussed above to \(S = A^1_U\), \(T = Y\). Thus we have an affine scheme \(Y^h\) with a projection \(\pi^h : Y^h \to Y\) and a section \(s^h : Y \to Y^h\). Set \(Y^h = Y^h - s(Y)\).

**Lemma 5.10.** If \((W, \pi, s) \in \mathrm{Neib}(A^1_U, Y)\), then \(s(Y)\) is a principal divisor in \(W\) and therefore \(W - s(Y)\) is affine.

*Proof.* Since \(U\) is a regular semi-local ring, \(Y\) is a principal divisor in \(A^1_U\). Thus \(s(Y) = (\pi)^{-1}(Y)\) is also a principal divisor in the affine scheme \(W\). \(\square\)

Let us make a general remark. Let \(\mathcal{F}\) be a \(G\)-bundle over a \(U\)-scheme \(T\). By definition, a trivialization of \(\mathcal{F}\) is a \(G\)-equivariant isomorphism \(G \times_U T \to \mathcal{F}\). Equivalently, it is a section of the projection \(\mathcal{F} \to T\). If \(\varphi\) is such a trivialization and \(f : T' \to T\) is a \(U\)-morphism, we get a trivialization \(f^*\varphi\) of \(f^* \mathcal{F}\). Sometimes we denote this trivialization by \(\varphi|_{T'}\). We also sometimes call a trivialization of \(f^* \mathcal{F}\) a trivialization of \(\mathcal{F}\) on \(T'\).

We will recall some consequences of Nisnevich descent. Let \(\text{in} : A^1_U \hookrightarrow \mathbb{P}^1_U\) be the standard inclusion. For each object \((W, \pi, s)\) in \(\mathrm{Neib}(A^1_U, Y)\) there is an elementary distinguished square (see \cite[Def. 2.1]{Voe})

\[
\begin{array}{ccc}
W - s(Y) & \longrightarrow & W \\
\downarrow & & \downarrow \text{in} \circ \pi \\
\mathbb{P}^1_U - Y & \longrightarrow & \mathbb{P}^1_U.
\end{array}
\]

(1)

It is used here that \(Y\) is closed in \(\mathbb{P}^1_U\).

The elementary distinguished square (1) can be used to construct principal \(G\)-bundles over \(\mathbb{P}^1_U\) via Nisnevich descent. In particular, one can glue a principal bundle over \(\mathbb{P}^1_U - Y\) with a trivial principal bundle over \(W\) via an isomorphism on \(W - s(Y)\). More precisely, let \(\mathcal{A}(W, \pi, s)\) be the category of pairs \((E, \varphi)\), where \(E\) is a \(G\)-bundle over \(\mathbb{P}^1_U\), \(\varphi\) is a trivialization of \(E|_{W} := (\text{in} \circ \pi)^* E\). A morphism between \((E, \varphi)\) and \((E', \varphi')\) is an isomorphism \(E \to E'\) compatible with trivializations.

Similarly, let \(\mathcal{B}(W, \pi, s)\) be the category of pairs \((E, \varphi)\), where \(E\) is a \(G\)-bundle over \(\mathbb{P}^1_U - Y\), \(\varphi\) is a trivialization of \(E|_{W - s(Y)}\).

**Lemma 5.11.** The categories \(\mathcal{A}(W, \pi, s)\) and \(\mathcal{B}(W, \pi, s)\) are groupoids whose objects have no non-trivial automorphisms.
Consider the restriction functor $\Phi : \mathcal{A}(W, \pi, s) \to \mathcal{B}(W, \pi, s)$. The following proposition is a version of Nisnevich descent.

**Proposition 5.12.** The functor $\Phi$ is an equivalence of categories.

**Proof.** Let us prove that $\Phi$ is essentially surjective. Let $(\mathcal{E}, \varphi)$ be an object of $\mathcal{B}(W, \pi, s)$, set $\mathcal{E}' = \mathcal{E}|_{\mathbb{A}^1_Y - Y}$. By Lemma 5.10 and [CTQ, Prop. 2.6(iv)] there is a $\mathbb{G}$-bundle $\mathcal{E}''$ over $\mathbb{A}^1_Y$, a trivialization $\varphi''$ of $\mathcal{E}''$ on $W$, and an isomorphism $\mathcal{E}''|_{\mathbb{A}^1_Y - Y} \to \mathcal{E}' = \mathcal{E}|_{\mathbb{A}^1_Y - Y}$ compatible with the trivializations on $W - s(Y)$. We can use this isomorphism to glue $\mathcal{E}$ with $\mathcal{E}''$ to make a $\mathbb{G}$-bundle $\mathcal{E}$ over $\mathbb{P}^1_U$ (gluing in the Zariski topology). The trivialization $\varphi''$ gives rise to a trivialization $\tilde{\varphi}$ of $\mathcal{E}$ on $W$. Clearly, $\Phi(\mathcal{E}, \tilde{\varphi}) \cong (\mathcal{E}, \varphi)$.

It follows immediately from Lemma 5.11 that $\Phi$ is faithful. It remains to show that $\Phi$ is full. Let $(\mathcal{E}, \varphi)$ and $(\mathcal{E}', \varphi')$ be objects of $\mathcal{A}(W, \pi, s)$. Let $\alpha$ be a morphism from $\Phi(\mathcal{E}, \varphi)$ to $\Phi(\mathcal{E}', \varphi')$. We need to show that $\alpha$ is of the form $\Phi(\beta)$.

Recall that the presheaf $\text{Iso}(\mathcal{E}, \mathcal{E}')$ is represented by a $\mathbb{P}^1_U$-scheme (see the proof of Corollary 4), so, in particular, it is a sheaf in the Nisnevich topology. Thus, since $\mathbb{P}^1_U$ is an elementary distinguished square, to give a section of $\text{Iso}(\mathcal{E}, \mathcal{E}')$ over $\mathbb{P}^1_U$ is the same as to give sections over $\mathbb{P}^1_U - Y$ and over $W$ that coincide over $W - s(Y)$ (see [MY, Sect. 3, Prop. 1.3]).

Note that $\alpha$ gives a section of $\text{Iso}(\mathcal{E}, \mathcal{E}')$ over $\mathbb{P}^1_U - Y$, while $\varphi' \circ \varphi^{-1}$ is a section over $W$. By definition of $\mathcal{B}(W, \pi, s)$ these sections coincide on $W - s(Y)$, so we obtain a section $\beta$ of $\text{Iso}(\mathcal{E}, \mathcal{E}')$ over $\mathbb{P}^1_U$. By construction $\beta$ is a morphism in $\mathcal{A}(W, \pi, s)$ and $\Phi(\beta) = \alpha$. □

The main Cartesian square we will work with is

\[
\begin{array}{ccc}
Y^h & \longrightarrow & Y^h \\
\downarrow & & \downarrow \text{iso}^h \\
\mathbb{P}^1_U - Y & \longrightarrow & \mathbb{P}^1_U.
\end{array}
\]

**Proposition 5.13.** (a) $Y^h$ is the projective limit of $W - s(Y)$ over $\text{Neib}(\mathbb{A}^1_U, Y)$. (b) $Y^h$ is an affine scheme.

**Proof.** Part (a) follows from the definition of projective limit and the equality $s^h(Y) = (\pi^h)^{-1}(Y)$. Part (b) follows from Lemma 5.10 part (a), and [Gro2, Prop. 8.2.3]. □

Let $\mathcal{A}$ be the category of pairs $(\mathcal{E}, \psi)$, where $\mathcal{E}$ is a $\mathbb{G}$-bundle over $\mathbb{P}^1_U$, $\psi$ is a trivialization of $\mathcal{E}|_{Y^h} := (\text{iso} \circ \pi^h)^* \mathcal{E}$. A morphism between $(\mathcal{E}, \psi)$ and $(\mathcal{E}', \psi')$ is an isomorphism $\mathcal{E} \to \mathcal{E}'$ compatible with trivializations.

Similarly, let $\mathcal{B}$ be the category of pairs $(\mathcal{E}, \psi)$, where $\mathcal{E}$ is a $\mathbb{G}$-bundle over $\mathbb{P}^1_U - Y$, $\psi$ is a trivialization of $\mathcal{E}|_{Y^h}$. 
**Lemma 5.14.** The categories $\mathcal{A}$ and $\mathcal{B}$ are groupoids whose objects have no non-trivial automorphisms.

**Proof.** It is obvious that the categories are groupoids. Note that for a $G$-bundle $\mathcal{E}$ we have

$$(\text{Aut}(\mathcal{E}))(Y^h) = \lim_{(W,\pi,s)\in \text{Neib}(\Delta^h_1, Y)} (\text{Aut}(\mathcal{E}))(W).$$

Thus an automorphism of $\mathcal{E}$ that is equal to the identity on $Y^h$ is equal to the identity on some $W$ with $(W,\pi,s)\in \text{Neib}(\Delta^h_1, Y)$. Now Lemma 5.11 shows that such an automorphism is equal to the identity. The statement for objects of $\mathcal{B}$ is proved similarly in view of Proposition 5.13. $\square$

Consider the restriction functor $\Psi : \mathcal{A} \to \mathcal{B}$.

**Proposition 5.15.** The functor $\Psi$ is an equivalence of categories.

**Proof.** Let us prove that $\Psi$ is essentially surjective; let $(\mathcal{E},\psi) \in \mathcal{B}$. Then using Lemma 5.10 and Proposition 5.13, we can find $(W,\pi,s)\in \text{Neib}(\Delta^h_1, Y)$ and a trivialization $\varphi$ of $\mathcal{E}$ on $W - s(Y)$ such that $\varphi|_{Y^h} = \psi$. By proposition 5.12 there is $(\tilde{\mathcal{E}},\tilde{\varphi}) \in \mathcal{A}(W,\pi,s)$ such that $\Phi(\tilde{\mathcal{E}},\tilde{\varphi}) \cong (\mathcal{E},\varphi)$. Then

$$\Psi(\tilde{\mathcal{E}},\tilde{\varphi}|_{Y^h}) = (\tilde{\mathcal{E}}|_{\mathbb{P}^1 - Y}, \tilde{\varphi}|_{Y^h}) \cong (\mathcal{E},\varphi|_{Y^h}) = (\mathcal{E},\psi).$$

It follows immediately from Lemma 5.14 that $\Psi$ is faithful. It remains to show that $\Psi$ is full. Let $(\mathcal{E},\psi)$ and $(\mathcal{E}',\psi')$ be objects of $\mathcal{A}$. Let $\alpha$ be a morphism from $\Psi(\mathcal{E},\psi)$ to $\Psi(\mathcal{E}',\psi')$. We need to show that $\alpha$ is of the form $\Psi(\beta)$.

We can find $(W,\pi,s)\in \text{Neib}(\Delta^h_1, Y)$ and trivializations $\varphi$ and $\varphi'$ of $\mathcal{E}$ and $\mathcal{E}'$ respectively on $W$ such that $\varphi|_{Y^h} = \psi$, $\varphi'|_{Y^h} = \psi'$. Using Proposition 5.13, it is easy to check that the restriction morphism $\text{Iso}(\mathcal{E},\mathcal{E}')(W - s(Y)) \to \text{Iso}(\mathcal{E},\mathcal{E}')(Y^h)$ is injective. Thus $\alpha$ is a morphism in $\mathcal{B}(W,\pi,s)$ from $\Phi(\mathcal{E},\varphi)$ to $\Phi(\mathcal{E}',\varphi')$. By Proposition 5.12 there is a morphism $\beta$ from $(\mathcal{E},\varphi)$ to $(\mathcal{E}',\varphi')$ such that $\Phi(\beta) = \alpha$. Then $\beta$ is also a morphism in $\mathcal{A}$ from $(\mathcal{E},\psi)$ to $(\mathcal{E}',\psi')$ and $\Psi(\beta) = \alpha$. $\square$

**Construction 5.16.** By Proposition 5.15 we can choose a functor quasi-inverse to $\Psi$. Fix such a functor $\Theta$. Let $\Lambda$ be the forgetful functor from $\mathcal{A}$ to the category of $G$-bundles over $\mathbb{P}^1_U$. For $(\mathcal{E},\psi) \in \mathcal{B}$ set

$$\text{Gl}(\mathcal{E},\psi) = \Lambda(\Theta(\mathcal{E},\psi)).$$

By construction $\text{Gl}(\mathcal{E},\psi)$ comes with a prescribed trivialization over $Y^h$.

Conversely, if $\mathcal{E}$ is a principal $G$-bundle over $\mathbb{P}^1_U$ such that its restriction to $Y^h$ is trivial, then $\mathcal{E}$ can be represented as $\text{Gl}(\mathcal{E}',\psi)$, where $\mathcal{E}' = \mathcal{E}|_{\mathbb{P}^1 - Y}$, $\psi$ is a trivialization of $\mathcal{E}'$ on $Y^h$.

Let $u$ be as in Section 4. $Y_u := Y \times_U u$. Let $(Y^h_u, \pi^h_u, s^h_u)$ be the Henselization of $(\Delta^h_1, Y_u)$. Using Lemma 5.8 we get an identification $Y^h_u = Y^h \times_U u$. Thus we have a closed embedding $Y^h_u \rightarrow Y^h$. Set $\tilde{Y}^h = Y^h - s_u(Y_u)$. We get a closed embedding $Y^h_u \rightarrow \tilde{Y}^h$. Thus the pull-back of the Cartesian square (2) by means of the closed embedding $u \rightarrow U$ has the form

$$
\begin{array}{ccc}
\tilde{Y}^h & \longrightarrow & \tilde{Y}^h \\
\downarrow & & \downarrow_{\text{in}_u \circ \pi^h_u} \\
\mathbb{P}^1 - Y_u & \longrightarrow & \mathbb{P}^1_u.
\end{array}
$$
where \( \text{in}_u : A^1_u \rightarrow \mathbb{P}^1_u \) is the standard embedding. Similarly to the above, let \( A_u \) be the category of pairs \((E_u, \psi_u)\), where \( E_u \) is a \( G_u \)-bundle over \( \mathbb{P}^1_u \), \( \psi_u \) is a trivialization of \( E_u|_{Y^h_u} \). A morphism between \((E_u, \psi_u)\) and \((E'_u, \psi'_u)\) is an isomorphism \( E_u \rightarrow E'_u \) compatible with trivializations. Let \( B_u \) be the category of pairs \((E_u, \psi_u)\), where \( E_u \) is a \( G_u \)-bundle over \( \mathbb{P}^1_u - Y_u \), \( \psi_u \) is a trivialization of \( E_u|_{Y^h_u} \). We have an obvious restriction functor \( \Psi_u : A_u \rightarrow B_u \), and similarly to Proposition 5.15 we show that \( \Psi_u \) is an equivalence of categories.

Next, we have obvious restriction functors \( R_A : A \rightarrow A_u \) and \( R_B : B \rightarrow B_u \) and the diagram

\[
\begin{align*}
\xymatrix{ A \ar[r]^{R_A} & A_u \ar[d]^{\Psi_u} \\
B \ar[r]_{R_B} & B_u }
\end{align*}
\]

commutes in the sense that the functors \( \Psi_u \circ R_A \) and \( R_B \circ \Psi \) are isomorphic.

Let \( \Theta_u \) be a functor quasi-inverse to \( \Psi_u \) and \( \Lambda_u \) be the forgetful functor from \( A_u \) to the category of \( G_u \)-bundles over \( \mathbb{P}^1_u \). Let \( E_u \) be a principal \( G_u \)-bundle over \( \mathbb{P}^1_u - Y_u \) and \( \psi_u \) be a trivialization of \( G_u \) on \( Y^h_u \). Set \( \text{Gl}_u(E_u, \psi_u) = \Lambda_u(\Theta_u(E_u, \psi_u)) \).

**Lemma 5.17.** Let \((E, \psi) \in B\), and let \( \text{Gl}(E, \psi) \) be the \( G_u \)-bundle obtained by Construction 5.16 Then

\[
\text{Gl}_u(E|_{\mathbb{P}^1_u - Y_u}, \psi|_{Y^h_u}) \quad \text{and} \quad \text{Gl}(E, \psi)|_{\mathbb{P}^1_u}
\]

are isomorphic as \( G_u \)-bundles over \( \mathbb{P}^1_u \).

**Proof.** By definition of \( \text{Gl} \) we have

\[
\Theta(E, \psi) = \left( \text{Gl}(E, \psi), \sigma \right),
\]

where \( \sigma \) is the canonical trivialization of \( \text{Gl}(E, \psi) \) on \( Y^h \). Similarly,

\[
\Theta_u(E|_{\mathbb{P}^1_u - Y_u}, \psi|_{Y^h_u}) = \left( \text{Gl}_u(E|_{\mathbb{P}^1_u - Y_u}, \psi|_{Y^h_u}), \sigma_u \right),
\]

where \( \sigma_u \) is the canonical trivialization of \( \text{Gl}_u(E|_{\mathbb{P}^1_u - Y_u}, \psi|_{Y^h_u}) \) on \( Y^h_u \). Thus (since \( \Psi_u \) is an equivalence of categories) it suffices to check that

\[
\Psi_u(R_A(\Theta(E, \psi))) \cong \Psi_u(\Theta_u(E|_{\mathbb{P}^1_u - Y_u}, \psi|_{Y^h_u})).
\]

In fact, both sides are isomorphic to \((E|_{\mathbb{P}^1_u - Y_u}, \psi|_{Y^h_u})\) because diagram (3) is commutative. \( \square \)

**Lemma 5.18.** For any \((E_u, \psi_u) \in B_u \) and any \( \beta_u \in G_u(Y^h_u) \) the \( G_u \)-bundles

\[
\text{Gl}_u(E_u, \psi_u) \quad \text{and} \quad \text{Gl}_u(E_u, \psi_u \circ \beta_u|_{Y^h_u})
\]

are isomorphic (here \( \beta_u|_{Y^h_u} \) is regarded as an automorphism of the \( G_u \)-bundle \( G_u \times_u Y^h_u \) given by the right translation action).

**Proof.** Denote by \( \sigma_u \) and \( \tau_u \) the canonical trivializations on \( Y^h_u \) of \( \text{Gl}_u(E_u, \psi_u) \) and \( \text{Gl}_u(E_u, \psi_u \circ \beta_u|_{Y^h_u}) \) respectively. It is straightforward to check that \((E_u, \psi_u)\) is isomorphic in \( B_u \) to both \( \Psi_u(\text{Gl}_u(E_u, \psi_u), \sigma_u) \) and \( \Psi_u(\text{Gl}_u(E_u, \psi_u \circ \beta_u|_{Y^h_u}), \tau_u \circ \beta_u^{-1}) \).

Since \( \Psi_u \) is an equivalence of categories, we conclude that \((\text{Gl}_u(E_u, \psi_u), \sigma_u)\) and \((\text{Gl}_u(E_u, \psi_u \circ \beta_u|_{Y^h_u}), \tau_u \circ \beta_u^{-1})\) are isomorphic in \( A_u \). Applying the functor \( \Lambda_u \), we see that the \( G_u \)-bundles \( \text{Gl}_u(E_u, \psi_u) \) and \( \text{Gl}_u(E_u, \psi_u \circ \beta_u|_{Y^h_u}) \) are isomorphic. \( \square \)
5.5. **Proof of Theorem 3:** presentation of \( \mathcal{G} \) in the form \( \text{Gl}(\mathcal{G}', \varphi) \). Let \( U, G, Z, \) and \( \mathcal{G} \) be as in Theorem 3. We may assume that \( Z \subset \mathbb{A}^1_U \).

**Proposition 5.19.** The \( G \)-bundle \( \mathcal{G} \) over \( \mathbb{P}^1_U \) is of the form \( \text{Gl}(\mathcal{G}', \varphi) \) for the \( G \)-bundle \( \mathcal{G}' := \mathcal{G}|_{\mathbb{P}^1_U - Y} \) and a trivialization \( \varphi \) of \( \mathcal{G}' \) over \( Y^h \).

**Proof.** In view of Construction 5.16, it is enough to prove that the restriction of the principal \( G \)-bundle \( \mathcal{G} \) to \( Y^h \) is trivial. Let us choose a closed subscheme \( Z' \subset \mathbb{A}^1_U \) such that \( Z' \) contains \( Z \), \( Z' \cap Y = \emptyset \), and \( \mathbb{A}^1_U - Z' \) is affine. Then \( \mathbb{A}^1_U - Z' \) is an affine neighborhood of \( Y \). Thus, the Henselization of the pair \( (\mathbb{A}^1_U - Z', Y) \) coincides with the Henselization of the pair \( (\mathbb{A}^1_U, Y) \). Since \( \mathcal{G} \) is trivial over \( \mathbb{A}^1_U - Z' \), its pull-back to \( Y^h \) is trivial too. The proposition is proved.

Our aim is to modify the trivialization \( \varphi \) via an element

\[ \alpha \in G(Y^h) \]

so that the \( G \)-bundle \( \text{Gl}(\mathcal{G}', \varphi \circ \alpha) \) becomes trivial over \( \mathbb{P}^1_U \).

5.6. **Principal bundles over open subsets of projective lines.** We will recall some results from [Gil1]. In this section \( k \) denotes any field, \( V \) denotes an open subscheme of \( \mathbb{P}^1_k \), \( G \) is a connected reductive group over \( k \).

**Lemma 5.20.** (a) A \( G \)-bundle over \( V \) is locally trivial in the Zariski topology on \( V \) if it is trivial at the generic point of \( V \);

(b) Let \( T \) be a maximal split torus of \( G \), let \( \hat{T} \) be its lattice of co-characters, and let \( \text{Pic}(V) \) denotes the group of isomorphism classes of line bundles over \( V \). Then there is a natural surjection

\[ \hat{T} \otimes \mathbb{Z} \text{Pic}(V) \to H^1_{\text{Zar}}(V, G) \]

(Here \( H^1_{\text{Zar}} \) stands for the set of isomorphism classes of Zariski locally trivial \( G \)-bundles.)

**Proof.** It is a reformulation of [Gil1] Cor. 3.10(a), see also [Gil2].

Note that part (a) of the lemma is a particular case of the Grothendieck-Serre conjecture. Note also, that the map in part (b) is given as follows: given a co-character of \( T \), we get a homomorphism \( \mathbb{G}_m : \to G \). Then every line bundle over \( V \) yields a principal \( G \)-bundle via pushforward.

5.7. **Proof of Theorem 3:** proof of property (i) from the outline. Now we are able to prove property (i) from the outline of the proof. In fact, we will prove the following

**Lemma 5.21.** Let \( \text{Gl}(\mathcal{G}', \varphi) \) be the presentation of the \( G \)-bundle \( \mathcal{G} \) over \( \mathbb{P}^1_U \) given in Proposition 5.19. Set \( \varphi_u := \varphi|_{Y_u} \). Then there is \( \gamma_u \in G_u(Y^h) \) such that the \( G_u \)-bundle \( \text{Gl}_u(\mathcal{G}'|_{\mathbb{P}^1_U - Y_u}, \varphi_u \circ \gamma_u) \) is trivial.

**Proof.** We show first that \( \mathcal{G}'|_{\mathbb{P}^1_U - Y_u} \) is trivial. Recall that \( u' \subset u \) is the subscheme of all closed points \( u' \) such that the group \( G_u' \) is isotropic, and \( u'' := u - u' \). We can write

\[ \mathbb{P}^1_u = \left( \coprod_{u \in u'} \mathbb{P}^1_u \right) \coprod \left( \coprod_{u \in u''} \mathbb{P}^1_u \right) \]

For \( u \in u \) set \( Y_u := Y \times_U u \), \( G_u := G \times_U u \), and \( \mathcal{G}_u := \mathcal{G} \times_U u \).
For \( u \in u' \) the algebraic \( k(u) \)-group \( G_u \) is anisotropic. Since \( G_u \) is trivial over an open subset of \( P_u^1 \), Lemma 5.20\( \text{[u]} \) shows that \( G_u \) is locally trivial in the Zariski topology. Now Lemma 5.20\( \text{[u]} \) shows that \( G_u \) is trivial. Thus \( G|_{P_u^1 - Y_u} \) is trivial.

Take \( u \in u' \). By our assumption on \( Y \), there is a \( k(u) \)-rational point \( p_u \in Y_u \). Set \( A_u^1 = P_u^1 - p_u \). Then we can write \( Y_u = p_u \coprod T_u \) and \( P_u^1 - Y_u \cong A_u^1 - T_u \). The \( G_u \)-bundle \( G_u \) is trivial over \( A_u^1 - Z \). Thus, again by Lemma 5.20\( \text{[u]} \) it is trivial over \( A_u^1 \). Whence it is trivial over \( P_u^1 - Y_u \).

We see that \( G'|_{P_u^1 - Y_u} = G|_{P_u^1 - Y_u} \) is trivial. Choosing a trivialization, we may identify \( \varphi_u \) with an element of \( G_u(Y^h_u) \). Set \( \gamma_u = \varphi_u^{-1} \). By the very choice of \( \gamma_u \) the \( G_u \)-bundle \( G_u(G'|_{P_u^1 - Y_u}, \varphi_u \circ \gamma_u) \) is trivial. \( \square \)

5.8. **Proof of Theorem 3**. reduction to property (ii) from the outline. The aim of this section is to deduce Theorem 3 from the following

**Proposition 5.22.** Each element \( \gamma_u \in G_u(Y^h_u) \) can be written in the form

\[
\alpha|_{Y^h_u} \cdot \beta_u|_{Y^h_u}
\]

for certain elements \( \alpha \in G(Y^h) \) and \( \beta_u \in G_u(Y^h_u) \).

**Deduction of Theorem 3 from Proposition 5.22.** Let \( G'(\varphi, \beta) \) be the presentation of the \( G \)-bundle \( G \) from Proposition 5.19. Let \( \gamma_u \in G_u(Y^h_u) \) be the element from Lemma 5.21. Let \( \alpha \in G(Y^h) \) and \( \beta_u \in G_u(Y^h_u) \) be the elements from Proposition 5.22. Set

\[
G^\text{new} = G(G', \varphi \circ \alpha).
\]

**Claim.** The \( G \)-bundle \( G^\text{new} \) is trivial over \( P_u^1 \).

Indeed, by Lemmas 5.17 and 5.18 one has a chain of isomorphisms of \( G_u \)-bundles

\[
G^\text{new}|_{P_u^1} \cong G_u(G'|_{P_u^1 - Y_u}, \varphi_u \circ \alpha|_{Y^h_u}) \cong G_u(G'|_{P_u^1 - Y_u}, \varphi_u \circ \alpha|_{Y^h_u} \circ \beta_u|_{Y^h_u}) = G_u(G'|_{P_u^1 - Y_u}, \varphi_u \circ \gamma_u);
\]

the bundle \( G_u(G'|_{P_u^1 - Y_u}, \varphi_u \circ \gamma_u) \) is trivial by the choice of \( \gamma_u \). The \( G \)-bundles \( G|_{P_u^1 - Y} \) and \( G^\text{new}|_{P_u^1 - Y} \) coincide by the very construction of \( G^\text{new} \). By Proposition 5.1 applied to \( T = Z \cup Y \) the \( G \)-bundle \( G^\text{new} \) is trivial. Whence the claim.

The claim above implies that the \( G \)-bundle \( G|_{P_u^1 - Y} \) is trivial. Theorem 3 is proved. \( \square \)

5.9. **End of proof of Theorem 3.** proof of property (ii) from the outline. In the remaining part of Section 5 we will prove Proposition 5.22. This will complete the proof of Theorem 3.

By assumption, the group scheme \( G_Y = G \times U Y \) is isotropic. Thus we may choose a parabolic subgroup scheme \( P^+ \) in \( G_Y \) such that the restriction of \( P^+ \) to each connected component of \( Y \) is a proper subgroup scheme in the restriction of \( G_Y \) to this component of \( Y \).

Since \( Y \) is an affine scheme, by [DG, Exp. XXVI, Cor. 2.3, Thm. 4.3.2(a)] there is an opposite to \( P^+ \) parabolic subgroup scheme \( P^- \) in \( G_Y \). Let \( U^+ \) be the unipotent radical of \( P^+ \), and let \( U^- \) be the unipotent radical of \( P^- \).

**Definition 5.23.** If \( T \) is a \( Y \)-scheme, we write \( E(T) \) for the subgroup of \( G_Y(T) = G(T) \) generated by the unipotent subgroups \( U^+(T) \) and \( U^-(T) \). Thus \( E \) is a functor from the category of \( Y \)-schemes to the category of groups.
Lemma 5.24. The functor $E$ has the property that for every closed subscheme $S$ in an affine $Y$-scheme $T$ the induced map $E(T) \to E(S)$ is surjective.

Proof. The restriction maps $U^\pm(T) \to U^\pm(S)$ are surjective, since $U^\pm$ are isomorphic to vector bundles as $Y$-schemes (see [DG, Exp. XXVI, Cor. 2.5]).

Recall that $(Y^h, \pi^h, s^h)$ is the Henselization of the pair $(\mathbb{A}^1_Y, Y)$. Recall that $in : \mathbb{A}^1_Y \to \mathbb{P}^1_Y$ is the standard embedding. Denote the projection $\mathbb{A}^1_Y \to U$ by $pr_Y$.

Lemma 5.25. There is a morphism $r : Y^h \to Y$ making the following diagram commutative

$$Y^h \xrightarrow{r} Y$$

\[\begin{array}{c c}
\text{in} \circ \pi^h & \xrightarrow{pr_Y} U \\
\mathbb{P}^1_Y & \xrightarrow{pr} U
\end{array}\]

and such that $r \circ s^h = Id_Y$.

Proof. As before, we may assume that $Y \subset \mathbb{A}^1_Y$. Note that the morphism

$$\pi := \text{Id} \times (pr|_Y) : \mathbb{A}^1_Y \to \mathbb{A}^1_Y$$

is étale. Let $s : Y \to \mathbb{A}^1_Y \times_U Y = \mathbb{A}^1_Y$ be the morphism induced by the embedding $Y \to \mathbb{A}^1_Y$ and $Id_Y$. Then $(\mathbb{A}^1_Y, \pi, s) \in \text{Neib}(\mathbb{A}^1_Y, Y)$. Thus there is a canonical morphism $\text{can} : Y^h \to \mathbb{A}^1_Y$ such that $(\text{Id} \times (pr|_Y)) \circ \text{can} = \pi^h$. Set

$$r := pr_Y \circ \text{can} : Y^h \to Y.$$

With this $r$ diagram commutes, and $r \circ s^h = Id_Y$. □

We view $Y^h$ as a $Y$-scheme via $r$. Thus various subschemes of $Y^h$ also become $Y$-schemes. In particular, $\hat{Y}^h$ and $\hat{Y}^h_u$ are $Y$-schemes, and we can consider

$$E(\hat{Y}^h) \subset G(\hat{Y}^h) \quad \text{and} \quad E(\hat{Y}^h_u) \subset G(\hat{Y}^h_u) = G_u(\hat{Y}^h_u).$$

Lemma 5.26.

$$G_u(\hat{Y}^h_u) = E(\hat{Y}^h_u) G_u(\hat{Y}^h_u).$$

Proof. Firstly, one has $Y_u = \prod_{u \in u} \prod_{y \in Y_u} y$. (Note that $Y_u$ is a finite scheme.) Thus by Lemma 5.9 we have

$$Y^h_u = \prod_{u \in u} \prod_{y \in Y_u} y^h, \quad \hat{Y}^h_u = \prod_{u \in u} \prod_{y \in Y_u} \hat{y}^h,$$

where $(y^h, \pi^h, s^h)$ is the Henselization of the pair $(\mathbb{A}^1_u, y)$, $\hat{y}^h := y^h - s^h(y)$. We see that $y^h$ and $\hat{y}^h$ are subschemes of $Y^h$, so we can view them as $Y$-schemes, and $G_{y^h} := G_Y \times_Y y^h$ is isotropic. Also, $E(y^h)$ makes sense as a subgroup of $G(y^h) = G_u(y^h) = G_{y^h}(y^h)$.

There are equalities of the form

$$G_u(\hat{Y}^h_u) = \prod_{u \in u} \prod_{y \in Y_u} G_u(y^h) = \prod_{u \in u} \prod_{y \in Y_u} G_{y^h}(y^h),$$

$$E(\hat{Y}^h_u) = \prod_{u \in u} \prod_{y \in Y_u} E(y^h),$$

$$G_u(Y^h_u) = \prod_{u \in u} \prod_{y \in Y_u} G_u(y^h) = \prod_{u \in u} \prod_{y \in Y_u} G_{y^h}(y^h).$$
Thus, to prove the lemma it suffices for each $u \in \mathfrak{u}$ and each $y \in Y_u$ to check the equality

$$G_y(y^h) = E(y^h)G_{y^h}(y^h).$$

Note that $y^h = \text{Spec } \mathcal{O}$, where $\mathcal{O} = k(u)[t]_{m_y}$ is a Henselian discrete valuation ring, and $m_y \subset k(u)[t]$ is the maximal ideal defining the point $y \in \mathbb{A}_x^1$. (Without loss of generality we can assume that $y$ is not the infinite point of $\mathbb{P}^1_\mathbb{A}_x$.) Further, $\hat{y}^h = \text{Spec } L$, where $L$ is the fraction field of $\mathcal{O}$. Also, $G_{y^h}$ is isotropic. Thus, the equality follows from [Gil3, Lemma 4.5(1)] in view of our definition of $E$ and [Gil3 Fait 4.3(2)].

By Lemma 4.24 and Proposition 5.13[3] the restriction map $E(Y^h) \to E(Y^h_u)$ is surjective. Since $E(Y^h) \subset G(Y^h)$, the proposition follows. This completes the proof of Theorem 3.

6. An Application

The following result is a straightforward consequence of Theorem 3 and an exact sequence for étale cohomology. Recall that by our definition a reductive group scheme has geometrically connected fibres.

**Theorem 4.** Let $R$ be a regular local ring containing an infinite field and $G$ be a reductive $R$-group scheme. Let $\mu : G \to T$ be a group scheme morphism to an $R$-torus $T$ such that $\mu$ is locally in the étale topology on $\text{Spec } R$ surjective. Assume further that the $R$-group scheme $H := \text{Ker}(\mu)$ is reductive. Let $K$ be the fraction field of $R$. Then the group homomorphism

$$T(R)/\mu(G(R)) \to T(K)/\mu(G(K))$$

is injective.

**Proof.** We have a commutative diagram whose rows are exact in the sense that in each row the image of $\mu$ coincides with the kernel of $\nu$.

$$\begin{array}{ccc}
G(R) & \xrightarrow{\mu} & T(R) \\
\downarrow & & \downarrow \\
G(K) & \xrightarrow{\mu} & T(K)
\end{array}$$

By Theorem 3 the right vertical arrow has trivial kernel. Now a simple diagram chase completes the proof. \qed

This theorem extends all the known results of this form proved in [CTO], [PS1], [Zai]. [OPZ]. Theorem 4 has the following corollary.

**Corollary.** Under the hypothesis of Theorem 4 let additionally the $K$-algebraic group $G_K$ be $K$-rational as a $K$-variety and let the ring $R$ be of characteristic 0. Then the norm principle holds for all finite flat $R$-domains $S \supset R$. That is, if $S \supset R$ is such a domain, and $a \in T(S)$ belongs to $\mu(G(S))$, then the element $N_{S/R}(a) \in T(R)$ belongs to $\mu(G(R))$.

**Proof.** Let $L$ be the fraction field of $S$. Let $a \in G(S)$ be such that $\mu(a) = a \in T(S)$. Then $\mu(\alpha_L) = a_L \in T(L)$, where $\alpha_L$ is the image of $\alpha$ in $G(L)$, $a_L$ is the image of $a$ in $T(L)$. The hypothesis on the algebraic $K$-group $G_K$ implies that there exists an element $\beta \in G(K)$ such that $\mu(\beta) = N_{L/K}(a_L) \in T(K)$ (see [Mer]). Note
that \( N_{L/K}(a_L) = (N_{S/R}(a))_K \in T(K) \). By Theorem 4 there exists an element \( \gamma \in G(R) \) such that \( \mu(\gamma) = N_{S/R}(a) \in T(R) \). Whence the corollary. \( \square \)

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