Perfect Sampling of graph $k$-colorings for $k > 3\Delta$

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Abstract

We give an algorithm for perfect sampling from the uniform distribution on proper $k$-colorings of graphs of maximum degree $\Delta$ which terminates with a sample in expected $\text{poly}(k,n)$ time whenever $k > 3\Delta$ (here, $n$ is the number of vertices in the graph).

We provide a Coupling-from-the-Past based algorithm using the bounding chain approach. This approach was pioneered independently by Häggström & Nélandre (Scand. J. Statist., 1999) and Huber (STOC 1998) (who used the approach to give a perfect sampling algorithm requiring $k > \Delta^2 + 2\Delta$ for its expected running time to be a polynomial).

1 Introduction

In this paper, we make progress on the longstanding problem of improving the bound on the number of colors required (in terms of the maximum degree of the graph denoted as $\Delta$) to perfectly sample a proper coloring of a given graph in polynomial time on expectation. More precisely, we provide a Coupling-from-the-Past (CFTP) based sampling algorithm which evolves according to the Glauber Dynamics. Given as input a graph $G$ of size $n$ and maximum degree $\Delta$, and the set of colors $[k]$ where $k > 3\Delta$, our algorithm, which we call NEWBC, perfectly samples a uniformly random proper $k$-coloring of $G$ in time $\text{poly}(k,n)$ on expectation (Theorem 3.1). This improves on the previously known bounds, i.e., $k > \Delta^2 + 2\Delta$ (Huber [Hub98]) and more recently $k > \Delta^2 - \Delta + 3$ (Feng, Guo and Yin [FGY19]), by a factor of $\Omega(\Delta)$. We also mention that a recent result of Liu, Sinclair and Srivastava [LSS19] provides a deterministic approximate counting algorithm for $k$-colorings whenever $k > 2\Delta$ which runs in time $\left(\frac{n\Delta}{\varepsilon}\right)^{\exp(\Delta)}$. It has been pointed out to us that using the result of Jerrum, Valiant and Vazirani [JVV86] (see proof of Thm 3.3 in [JVV86] which shows how to obtain a perfect sampling algorithm using a deterministic approximate counter) when combined with the deterministic approximate counting algorithm of Liu et al. gives a perfect sampling algorithm whenever $k > 2\Delta$ and runs in time $(n\Delta/\varepsilon)^{\exp(\Delta)}$. In contrast to our algorithm (which runs in time poly$(k,n)$) this is not a poly$(k,n)$ time algorithm and requires $\Delta$ to be $O(1)$ to run in time poly$(n)$. Although we require $k > 3\Delta$ our algorithm can accommodate $k$ as large as poly$(n)$.

In their breakthrough paper in 1996, Propp and Wilson [PW96] presented a way of using coupling from the past to generate perfect samples from a target distribution; although the ideas behind CFTP can be traced back to the 60’s (see Levin, Peres and Wilmer [LPW06] section 22.1 for a detailed chronology). An issue while "efficiently" implementing CFTP based algorithms is that one needs to keep track of the evolution of the system’s state. A recent result of Liu, Sinclair and Srivastava [LSS19] provides a deterministic approximate counting algorithm for $k$-colorings whenever $k > 2\Delta$ which runs in time $\left(\frac{n\Delta}{\varepsilon}\right)^{\exp(\Delta)}$. It has been pointed out to us that using the result of Jerrum, Valiant and Vazirani [JVV86] (see proof of Thm 3.3 in [JVV86] which shows how to obtain a perfect sampling algorithm using a deterministic approximate counter) when combined with the deterministic approximate counting algorithm of Liu et al. gives a perfect sampling algorithm whenever $k > 2\Delta$ and runs in time $(n\Delta/\varepsilon)^{\exp(\Delta)}$. In contrast to our algorithm (which runs in time poly$(k,n)$) this is not a poly$(k,n)$ time algorithm and requires $\Delta$ to be $O(1)$ to run in time poly$(n)$. Although we require $k > 3\Delta$ our algorithm can accommodate $k$ as large as poly$(n)$.

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An earlier version of the paper using a different approach appears on arXiv.org as https://arxiv.org/abs/1909.10323v1 with the bound of $k > 2e\Delta^2/\ln(\Delta)$.
of chains starting from all possible states in the state-space (an exponential quantity for most interesting problems). If there is a partial order on the state-space and the chains can be coupled to evolve respecting this partial order (usually referred to as monotonicity), then we need to keep track of only the maximal and minimal elements which may be much fewer in cardinality. This idea was used in [PW96] to perfectly sample from the Ising Model. Häggström & Nelander [HN98, HN99] introduced the concept of Bounding Chains to deal with a more general class of Markov chains, e.g., anti-monotone systems. However, the situation for the Glauber Dynamics chain in case of Graph Coloring when \( k > 3 \) is neither monotone nor anti-monotone (see 3.2 [Hub98]). For perfect sampling of proper graph colorings we refer to the seminal work Huber [Hub98] (further elaborated in [Hub04]), in which Huber provides an algorithm using CFTP and proves that as long as \( k > \min \left( \Delta^2 + 2\Delta \frac{\ln n}{\ln \ln n} \right) \) the algorithm runs in polynomial time on expectation.

Another paradigm for perfect sampling, related to the Moser-Tardos framework for algorithmic versions of the Lovász local lemma, and also to the celebrated cycle-popping algorithm of Wilson for sampling uniformly random spanning trees, has recently been proposed by Guo, Jerrum and Liu [GJL17]. However, it turns out that that when this framework is applied to the problem of sampling \( k \)-colorings, it degenerates into usual rejection sampling: one samples a uniformly randomly coloring, accepts if the coloring is proper, and rejects the current coloring and repeats otherwise. The expected running time of such a procedure is proportional to the inverse of the fraction of proper colorings among all colorings, and hence cannot in general be bounded by a polynomial in the size of the graph.

We note that the problem of approximately sampling a proper coloring of \( G \) has also received much attention. Jerrum [Jer95] proved that the Glauber Dynamics mixes fast whenever \( k > 2\Delta \). Subsequently, Jerrum’s technique was simplified using the path coupling approach developed by Bubley and Dyer [BD97]. The best current fast-mixing results for the Glauber dynamics on general bounded degree graphs are based on the Vigoda’s [Vig00] approach based on an analysis of a different Markov chain called the flip dynamics. Vigoda showed, using this approach, that the Glauber dynamics is fast mixing (for general graphs of bounded maximum degree \( \Delta \)) whenever \( k \geq \frac{n}{6}\Delta \); his analysis has recently been tightened to allow \( k \geq \left( \frac{n}{6} - \varepsilon \right)\Delta \) (for a positive \( \varepsilon \sim 10^{-4} \)) by Chen et al. [CDM+19]. Much work has also been devoted to the analysis of the mixing time of the Glauber dynamics on special classes of graphs. Some early examples of work in this direction include the work of Hayes and Vigoda [HV03], who established fast mixing of the Glauber Dynamics for graphs with girth at least 9 whenever \( k > (1 + \varepsilon)\Delta \) (for all \( \varepsilon > 0 \) provided the maximum degree is at least \( c\varepsilon \ln n \) (where \( n \) is the number of the vertices in the graph); and that of Dyer, Frieze, Hayes and Vigoda [DFHV04] who showed fast mixing for graphs with girth \( \geq 6 \) whenever \( k > 1.49\Delta \), provided the maximum degree is at least as large as a fixed positive constant.

1.1 Overview of our approach

1.1.1 Huber’s algorithm for \( k > \Delta^2 + 2\Delta \)

We will briefly describe Huber’s algorithm for \( k > \Delta^2 + 2\Delta \) in the following paragraphs as it serves as inspiration for our algorithm. To this end we set up some notation. We have as input a graph \( G = (V, E) \) of size \( n \) and maximum degree \( \Delta \); let \([k]\) be the set of colors. We denote by \( S \subseteq [k]^V \) the set of all proper \( k \)-colorings of \( G \), and by \( \Pi \) the uniform distribution over \( S \). We shall consider the usual Glauber Dynamics on \( S \) where at each step a vertex \( v \in V \) is chosen at random and its color is updated by picking a color at random from the set of available colors, i.e., those not occupied by its neighbors. It is well known that \( \Pi \) is the stationary distribution of the \textsc{Glauber Dynamics} chain, when \( k \geq \Delta + 2 \).

As Huber’s algorithm is based on CFTP it will be convenient to recall the CFTP set-up for the particular case of sampling from \( \Pi \). We let \( 0 > N_1 > N_2 \ldots \) be a decreasing sequence of integers. From time \( t = N_1 \)
to $t = 0$, for every state $s \in S$ we run a markov chain starting at $s$ and at time $t = N_1$. Then, we evolve the $|S|$ chains in a coupled manner such that individually each chain follows the Glauber Dynamics. In the coupling, each chain uses the same $v \in V$ (picked uar, independently at each time $t$) to update at time $t$; further, at time $t$ we sample $\sigma \in \mathcal{S}[k]$ uar (again, independently at each time $t$) and each chain updates $v$ by using the first available color according to $\sigma$. If we observe that at $t = 0$ all the chains have coalesced to a unique state $s^*$, we output $s^*$.

Otherwise, we store the resulting states of the $|S|$ chains, which start at time $t = N_1$, at time $t = 0$. Then, we descend further back in the past and run the $|S|$ chains from time $t = N_2$ to $t = 0$. We evolve the $|S|$ chains using fresh randomness, in a coupled fashion as mentioned before, only from $t = N_2$ to $t = N_1$. For the remaining evolution from $t = N_1$ to $t = 0$ we use the resulting states we had stored previously. Again, if we observe coalescence at $t = 0$ we output the unique state $s^*$; otherwise, we store the resulting states of $|S|$ chains at time $t = 0$, and descend further back in the past to start the chains from $t = N_3$. Continuing in this manner we find the first index $i$ such that we observe coalescence of the $|S|$ chains from time $t = N_i$ to time $t = 0$ and we output the unique state $s^*$.

We now sketch why $s^*$ is distributed according to $\Pi$. First, it is not hard to see that $i$ is finite with prob. 1 whenever $k \geq \Delta + 2$, i.e., the chains eventually coalesce. Now, we pick an integer $j$ large enough such that with prob. at least $1 - 2\epsilon$ (where $\epsilon > 0$ is arbitrary) we observe coalescence at or before $t = N_i$. We consider an imaginary chain (subsequently, we refer to the imaginary chain as the Ghost Chain (GC)) running from time $t = N_j$ and with the starting state $s$ distributed according to $\Pi$. Clearly, the output of the GC, say $s'$, at time $t = 0$ is distributed according to $\Pi$ and with prob. at least $1 - \epsilon$ we have $s' = s^*$. Since, $\epsilon > 0$ is arbitrary we have $s^* \sim \Pi$. A more detailed description of CFTP appears in Section 2.3.

A drawback of the CFTP approach is that we need to track the evolution of the GC starting at all the states in $S$. A Bounding Chain (BC) is an efficient way of keeping track of the evolution of these chains. In particular, for the case of sampling colorings the BC we evolve will have $(2^{[k]})^V$ as its state-space; in other words, the BC at time $t$ assigns a list of colors to the vertices. Consider the GC evolving from time $t = N_1$ to $t = 0$. We wish to evolve the BC in a coupled manner with the GC while maintaining the invariant that at all times $t$ and for all $v \in V$ the list of colors assigned to $v$ by the BC contains the color assigned to $v$ by the GC. Hence, the state of the GC at time $t$ is contained in the state of the BC at time $t$.

Since, we do not know the state in which the GC is at $t = N_1$, we set $(BC)_{N_1} = [k]^V$. Now, if we observe that at $t = 0$ the BC produces lists of size 1 for all $v \in V$, i.e., $(BC)_0 = \{s^*\}$, then we have determined the state $s^*$ of the GC at time $t = 0$. (Of course, whether the BC coalesces w.h.p. or at all depends on the coupling.) If we do not observe coalescence of the BC at $t = 0$, we run a fresh BC for $t = N_2$ to $t = N_1$ and hope to observe coalescence of the BC at $t = N_1$; in particular, we find the first index $i$ such that the BC coalesces from $t = N_{i+1}$ to $t = N_i$. Hence, we now know the state of the GC at $t = N_i$, say $s$. However, we are supposed to output $s^*$, i.e., the state of the GC at $t = 0$, according to the CFTP procedure. Hence, we require that the BC stores all the randomness necessary to output the evolution of the GC, starting at $s$ and $t = N_i$, at $t = 0$.

Of course, we require the coupling between the GC and the BC to be such that the BC coalesces quickly. In particular, if we show that with prob. $\geq \frac{1}{2}$ the BC coalesces when run from $t = -T$ to $t = 0$ where $T$ is poly$(k,n)$, then we set $N_2 = -iT$ whence we see that on expectation the BC coalesces on or before $N_2$. Thus, if it takes poly$(k,n)$ times to iterate the BC for a single step the overall expected time to observe coalescence of the BC is poly$(k,n)$ and hence we output $s^* \sim \Pi$ on expectation in poly$(k,n)$-time.

Building on the aforementioned set-up we may now describe the BC proposed by Huber when $k > \Delta^2 + 2\Delta$ and its coupling with the GC. (Strictly speaking, we describe a slightly modified version of Huber’s BC that we call HuberBC; however, the running time analyses of the two chains are identical: a more detailed exposition is provided in Section 3.1.) Note that the algorithm only evolves the BC; the coupling
with the GC is described to prove the correctness of the output. We fix a time $T = \text{poly}(k,n)$ whose precise value is mentioned later. For time $t = -T$ to $t = -T + n$ we run a warm-up phase of the BC. During these $n$ steps the BC and hence the GC chooses the vertices in $V$ one by one, in some pre-decided order, and the BC assigns to the chosen vertex, say $v$, a list of size $\Delta + 1$ chosen from $\binom{[k]}{\Delta+1}$. The GC then assigns to $v$ the first available color, i.e., not blocked by the neighbors of $v$, from the list chosen by the BC. At the end of the warm-up we are in a situation where the BC has produced a list of size $\Delta + 1$ at every vertex.\footnote{We remark here that we are running the CFTP for time $-T$ to $t = 0$ with a round of the \textsc{Systematic Sweep Glauber Dynamics} followed by the \textsc{Glauber Dynamics}; however, it is easy to see that this doesn’t bias the output of the CFTP (see Section 2.3). We could have replaced the \textsc{Systematic Sweep Glauber Dynamics} with $n \ln n$ \textsc{Glauber Dynamics} updates as all we require from the warm-up phase is that each vertex is updated by the BC at least once which is ensured w.h.p. in $n \ln n$ \textsc{Glauber Dynamics} updates. However, for ease of analysis we use the \textsc{Systematic Sweep Glauber Dynamics}.}

To describe the subsequent behavior of the BC and its coupling with the GC it will be convenient to introduce more notation. Let $S_t(v)$ be the union of colors present in lists of the neighbors of $v$ at time $t$ according to the BC, let $G_t(v)$ be the union of colors present in the lists of those neighbors of $v$ which have list-size 1 at time $t$ according to the BC, and, finally, let $T_t(v)$ be the union of colors present at the neighbors of $v$ in the GC. It is clear that $G_t(v) \subseteq T_t(v) \subseteq S_t(v)$. Note also that while the BC, i.e., the algorithm, has access to $G_t(v)$ and $S_t(v)$ it may not know the set $T_t(v)$ in its entirety. For every time $t \geq -T + n$ the BC samples a vertex $v \in V$ and a list of $\Delta + 1$ colors from $\binom{[k]}{\Delta+1}$, say $\{c_1, \ldots, c_{\Delta+1}\}$ and checks if $c_1$ is outside $S_t(v)$ (see Figure 1 (a)). If so, then the BC updates the list at $v$ to be $\{c_1\}$ and the GC updates the color at $v$ to be $c_1$. Otherwise (see Figure 1 (b)), the BC updates the list at $v$ to be $\{c_1, \ldots, c_{\Delta+1}\}$ and the GC updates the color at $v$ to be the first color $c_i \in \{c_1, \ldots, c_{\Delta+1}\}$ such that $c_i \notin T_t(v)$; there must always exist such a color as $|T_t(v)| \leq \Delta$.

![Figure 1](image)

Figure 1: (a) $c_1 \notin S_t(v)$, hence the BC updates to $\{c_1\}$ and the GC updates to $c_1$
(b) $c_1 \in S_t(v)$, hence the BC updates to $\{c_1, \ldots, c_{\Delta+1}\}$ and the GC updates to first $c_i$ outside $T_t(v)$

It is easy to see that in this coupling the GC evolves according to the \textsc{Glauber Dynamics}. We hope to observe coalescence of the BC by $t = 0$. We now define a quantity which will help us track the BC’s progress to coalescence: $W_t := \{v \in V \mid \#(BC)_t(v) = 1\}$, i.e., the set of vertices where the BC has a list of size 1 at time $t$. Also, define $D_t = V \setminus W_t$. Hence, coalescence corresponds to $|W_0| = n$. Notice that if at some time $t < 0$ we...
have $|W_t| = n$, then $|W_0| = n$ as whenever $|W_t| = n$ then for all $v \in V$ we have $G_t(v) = T_t(v) = S_t(v)$ and hence the BC will only produce lists of size 1 from then on as it samples from $[k] \setminus G_t(v)$.

We will analyze how $|W_t|$ evolves from $t = -T + n$, i.e., after the warm-up to $t = 0$. In particular, we model the evolution as a random walk on $\{0, 1, \ldots, n\}$ with 0 as the reflecting state and $n$ as the absorbing state. Initially, we have $|W_{-T+n}| = 0$. To understand the expected time for the random walk to get absorbed we need to analyze the drift of this walk. To this end we will show that at all times the drift is positive, i.e., $\mathbb{E}[|W_{t+1}| - |W_t| | |W_t| > 0$; in fact a detailed calculation shows that $\mathbb{E}[|W_{t+1}| - |W_t| | |W_t| > \left(\frac{|D|}{n} (1 - \frac{\Delta^2 + 1}{k - \Delta})\right)$.

For this we need to consider how $|W_{t+1}|$ may increase by one over $|W_t|$. This happens when we choose $v \in D_t$ as the vertex to update and then choose the color $c_1 \in [k] \setminus S_t(v)$. Similarly, $|W_{t+1}|$ decreases by one when we choose the vertex $v$ to be updated from $W_t$ and $c_1$ from $S_t(v)$. Notice that whenever $k > \Delta^2 + 2\Delta$ the drift is positive and this is the reason Huber required $k > \Delta^2 + 2\Delta$ for the analysis to go through. Together with standard results about martingales (see Section 4.1) we conclude that the walk absorbs in polynomial time on expectation.

Hence, by setting $T$ to be twice the expected time of absorption of the walk we have that the BC coalesces from $t = -T$ to $t = 0$ with prob. at least $\frac{1}{2}$. From our previous discussions this means that HuberBC runs in $\text{poly}(k, n)$ time on expectation.

1.1.2 A bottleneck for HuberBC

A drawback of HuberBC is that irrespective of the time $t$ the algorithm may produce lists of size $\Delta + 1$ in the worst case, i.e., when the BC samples $c_1$ and it happens to be in $S_t(v)$ the BC prescribes a list of size $\Delta + 1$. (Although Huber’s algorithm, as stated above, is slightly different from HuberBC, its worst case behavior is similar.) This drawback prevents us from pushing down the value of $k$, i.e., the number of colors required to obtain a perfect sample: in particular, at any time $t$ we might have $|S_t(v)| = \Delta^2 + \Delta$, and for the BC to have a non-trivial probability to produce a list of size 1 at $v$ we need $k > \Delta^2 + \Delta$.

We improve upon HuberBC by addressing this issue. In particular, we demonstrate a different way of evolving the coupling between the BC and the GC, namely BoundedListsSampler_1, which after an initial warm-up phase will have a significantly milder worst case behavior: in fact, we will produce constant-sized lists after the warm-up phase even in the worst case, while maintaining a non-trivial probability of producing lists of size 1. However, in order to get the conditions required for BoundedListsSampler_1 to function we also need to modify the warm-up phase of HuberBC.

1.1.3 Our Algorithm for $k > 3\Delta$

Drawing on our previous discussion we will now illustrate an important primitive for NewBC called BoundedListsSampler_1 (a more detailed exposition appears in Section 3.2.1). Here we will introduce a different evolution of the BC and its coupling with the GC which will allow us to address the above-mentioned deficiency of HuberBC. In particular, the new evolution will produce a milder worst case behaviour of the BC, i.e., the list produced at vertex $v \in V$, the vertex to be updated, will be of size 2 in the worst case. We shall also maintain a non-trivial probability that the BC produces a list of size 1. However, for BoundedListsSampler_1 to function, we require that at time $t$, when the vertex $v$ is chosen for update, we have $|[k] \setminus S_t(v)| > \Delta$. This implies that the vertex $v$ sees at least $\Delta$ colors outside the union of the lists of its neighbors in the BC (we will strive to get the BC in such a scenario eventually). In this situation the BC first samples two colors $c_1 \in [k] \setminus S_t(v)$ and $c_2 \in S_t(v) \setminus G_t(v)$ uar (see Figure 2) and then tosses a coin with probability of heads as $p_{BC}(v)$. If the coin comes up heads the BC updates the list at $v$ as $\{c_1\}$ and the GC updates the color at $v$ as $c_1$ (this is fine as $c_1 \notin T_t(v)$). Otherwise the BC updates the list at
as \( \{c_1, c_2\} \) and the GC tosses a coin with probability of heads \( \frac{p_{GC}(c) - p_{BC}(c)}{1 - p_{BC}(c)} \). If this coin lands heads, the GC sets the color at \( v \) as \( c_1 \). If the coin lands tails and \( c_2 \notin T_r(v) \) (see Figure 2 (a)) the GC sets the color at \( v \) as \( c_2 \), otherwise the GC again uses \( c_1 \) (Figure 2 (b)).

![Figure 2: (a) \( c_2 \) outside \( T_r(v) \) and hence the GC may use \( c_1 \) or \( c_2 \) depending on the coin tosses (b) \( c_2 \) inside \( T_r(v) \) and hence the GC always uses \( c_1 \)](image)

It is clear that the BC will produce a list of size 2 in the worst case, and it has a probability \( p_{BC}(v) \) of producing a list of size 1 at every vertex \( v \) that is chosen to be updated. Of course we need to argue the existence of probabilities \( p_{BC}(v) \) and \( p_{GC}(v) \), so that the GC may maintain its evolution according to the Glauber Dynamics. A direct calculation (presented explicitly in Section 3.2.1) shows that we need \( p_{GC}(v) = 1 - \frac{|S_r(v)| - |G_r(v)|}{|k - |T_r(v)||} \), whence we see that for \( p_{GC}(v) > 0 \) we require that \( k - |T_r(v)| \geq |S_r(v)| - |G_r(v)| \). We ensure this by requiring that \( k > \Delta + |S_r(v)| \).

We would like to set the value of \( p_{BC}(v) \) as large as possible, however we require that \( p_{BC}(v) \leq p_{GC}(v) \) as the GC tosses a coin with probability \( \frac{p_{GC}(c) - p_{BC}(c)}{1 - p_{BC}(c)} \). Also, the BC may not have access to the entire set \( T_r(v) \), i.e., it may not know the value of \( |T_r(v)| \) and this prevents us from setting \( p_{BC}(v) = p_{GC}(v) = 1 - \frac{|S_r(v)| - |G_r(v)|}{|k - |T_r(v)||} \). Hence we settle on \( p_{BC}(v) = 1 - \frac{|S_r(v)| - |G_r(v)|}{\Delta} \). We mention here that \( p_{BC}(v) \) turns out to be exactly the underestimated probability of the BC producing a list of size 1 in the running time analysis of HUBERBC. Hence, we may replace the behaviour of the BC in HUBERBC after the warm-up stage with BOUNDEDLISTSAMPLER_1; the condition \( k > \Delta + |S_r(v)| \) is always satisfied when \( k > \Delta^2 + 2\Delta \) as after the warm-up phase we always have \( |S_r(v)| \leq \Delta^2 + \Delta \) due to the list sizes of all the vertices being \( \Delta + 1 \) smaller.

However, if we wish to push the bound on \( k \) further down say \( k > \Omega(\Delta) \), we may not be able to use BOUNDEDLISTSAMPLER_1 after the warm-up since the condition \( k > \Delta + |S_r(v)| \) may not be satisfied. This is because the list size at every vertex after the warm-up is \( \Delta + 1 \) and hence \( |S_r(v)| \) may be as large as \( [k] \) itself. Now, we shall describe a different way to warm-up the BC to a situation where for all \( v \) we have \( k > |S_r(v)| + \Delta \). During this stage of the algorithm, we shall not be interested in the probability of the BC producing lists of size 1. Rather, the BC will produce list of size \( \Delta + 1 \) at each vertex, similar to the warm-up phase of HUBERBC; however, after the warm-up we will be able to ensure that for all \( v \) we have \( |S_r(v)| \leq 2\Delta \). This saving in \( |S_r(v)| \) comes because we ensure that \( \forall w, w' \in N(v) \) we have \( |BC_r(w) \cap BC_r(w')| \geq \Delta \); hence, we are able to replace the weaker bound of \( |S_r(v)| \leq \sum_{w \in N(v)} |BC_r(w)| = \Delta^2 + \Delta \) with \( |S_r(v)| \leq 2\Delta \).
We now describe a primitive that is used to build the new warm-up phase of the BC. We call this primitive \texttt{WarmUpSampler}[A] where \( A \) is a subset of \([k]\) of size \( \Delta \) which serves as the input to \texttt{WarmUpSampler[]}\]. Suppose we choose to update the vertex \( v \) at time \( t \) using \texttt{WarmUpSampler}[A]. Then, the BC samples a color \( c_1 \in [k] \setminus A \) uar and updates its list at \( v \) to be \( A \cup \{c_1\} \). The GC couples itself with the BC in the following manner. The GC chooses a color \( c_2 \) uar from \( A \setminus T_r(v) \) if possible, and tosses a coin with probability of heads \( p \). If the coin lands heads the GC updates the color at \( v \) to \( c_2 \); if the coins lands tails the GC updates the color at \( v \) to \( c_1 \) if possible, i.e., \( c_1 \notin T_r(v) \), else the GC updates the color at \( v \) to \( c_2 \) (Fig. 3). We need to set a value of \( p \in [0,1] \) such that the GC is able to evolve according to the \texttt{Glauber Dynamics}. A direct calculation (detailed in section 3.2.2) shows that \( p \) exists as long as \( k \geq 2 \Delta \). In fact, if \(|A| = |T_r(v)|\) then \( p \) turns out to be 0, which is needed as \( A \setminus T_r(v) \) may be empty. A detailed analysis of \texttt{WarmUpSampler[]}\] is presented in Section 3.2.2.

![Figure 3: (a) GC uses \( c_2 \) if the coin lands heads; otherwise it uses \( c_1 \) (b) GC always uses \( c_2 \)](image)

Aided with the primitive \texttt{WarmUpSampler[]}, let us now describe the warm-up phase for an intermediate bound of \( k \geq 4 \Delta \). Let \( A \subset [k] \) be an arbitrary set of size \( \Delta \). For time \( t = -T \) to \( t = -T + n \) (where \( T \) will be fixed later as \( \text{poly}(k,n) \)), the BC picks the vertices in \( V \) one by one, in some pre-decided order, and runs \texttt{WarmUpSampler}[A] at the chosen vertex, say \( v \). Clearly, at \( t = -T + n \) for all \( v \in V \) and \( w, w' \in N(v) \) we have \(|BC_i(v)| = \Delta + 1\) and \(|BC_i(w) \cap BC_i(w')| \geq \Delta\). Hence, as promised at \( t = -T + n \) for all \( v \in V \) we have \(|S_i(v)| \leq 2 \Delta\). Now, from time \( t = -T + n \) to \( t = -T + 2n \) the BC again picks the vertices in \( V \) one by one and the BC runs \texttt{BoundedListSampler}_1 at the chosen vertex, say \( v \). Notice that the conditions required for \texttt{BoundedListSampler}_1 to function are ensured as the largest value of \( S_i(v) \) possible when \( t \in \{-T + n, -T + 2n\} \) is \( 3 \Delta - 1 \); this happens when \( \Delta - 1 \) neighbors of \( v \) have already been updated by \texttt{BoundedListSampler}_1 and its remaining neighbor still has the list prescribed by \texttt{WarmUpSampler}[A] in the previous round. Since, \( k \geq 4 \Delta \) we always have \( k > |S_i(v)| + \Delta\).

We are now in a position to describe the remaining algorithm for \( k \geq 4 \Delta \). After, the warm-up phase ends, i.e., at time \( t = -T + 2n \) we are in a stage where for all \( v \in V \) we have \(|BC_i(v)| \leq 2\). From here onward, i.e, \( t = -T + 2n \) to \( t = 0 \) we invoke \texttt{BoundedListSampler}_1 where the vertex \( v \in V \) is picked uar at each time step. It is easy to see that the condition required for \texttt{BoundedListSampler}_1, i.e., \( k > |S_i(v)| + \Delta \) is always maintained. A running time analysis similar to \texttt{HuberBC} shows that the BC coalesces on expectation in poly time. We will call this algorithm \texttt{NewBCAux}.

In section 3.2.3, we describe the algorithm \texttt{NewBC} where we show how to further modify the warm-up stage to obtain the bound \( k > 3 \Delta \). A running time analysis similar to \texttt{HuberBC} shows that \texttt{NewBC} runs in
poly time on expectation.

1.2 Organisation

The rest of the paper is organized as follows. Section 2 introduces some preliminary notions about Glauber Dynamics, Systematic Sweep Glauber Dynamics, Coupling from the Past and Bounding Chains. Section 3.1 has an exposition about HUB ER BC for \( k > \Delta^2 + 2\Delta \). Section 3.2 contains the descriptions of the new primitives that we describe i.e., BOUNDEDLISTSAMPLER_1, WARMUP SAMPLER[] and details of NEWBC.

2 Preliminaries

2.1 Glauber Dynamics

Consider a graph \( G = (V,E) \) of size \( n \) and let \([k]\) be the set of colors. Let \( S \subseteq [k]^V \) be the set of all proper \( k \)-colorings of \( G \). Let \( \Pi \) denote the uniform distribution over \( S \). The Glauber Dynamics chain with state space \( S \) evolves as follows: for any coloring \( s \in S \), pick a vertex \( v \in V \) uar and update its color by choosing uar from \( C(v,s) := [k] \setminus \bigcup_{u \in N(v)} \{s(u)\} \). It is well known that \( \Pi \) is a reversible distribution for this chain and hence is also stationary. Also, when \( k \geq \Delta + 2 \) the chain is ergodic.

2.2 Systematic Sweep Glauber Dynamics

This is another well know variant of the Glauber Dynamics chain. Here the vertex whose color is to be updated is not picked uar from \( V \), instead we iterate through the vertex set in a cyclic fashion, i.e., if \( V = \{n\} \) then at time \( t \mod n \) is updated. Again \( \Pi \) satisfies the detailed-balance condition and is a stationary distribution for this chain. If one thinks of each scan of this chain as one uniform step then, its easy to see that the chain is ergodic iff the original GLAUBER DYNAMICS is ergodic.

2.3 Coupling from the Past

Here we provide a short exposition, which is more general that the usual CFTP set-up, without any attempt to be comprehensive. For a detailed account the reader is directed to Häggström [Hä00], Levin, Peres and Wilmer [LPW06].

Given a finite set \( S \) and a target distribution, say \( \Gamma \), over a finite set \( S \), the CFTP algorithm will output a sample from \( S \) which is distributed according to \( \Gamma \). For describing the algorithm it is helpful to define the following quantities. Let \( F = \{ f : S \rightarrow S \} \). Let \( P \) be a distribution over \((F)^Z\) with the following properties:

(a) there is a distribution \( \Gamma' \) over \( S \) such that \( \lim_{t \rightarrow -\infty} P^0_t(\Gamma') = \Gamma \) in T.V.

(b) \( \lim_{t \rightarrow -\infty} |P^0_t(S)| = 1 \) a.s.

where \( P^0_t(\Gamma') := P(-1) \circ P(-2) \circ \ldots \circ P(t)(\Gamma') \) and \( P^0_t(S) := P(-1) \circ P(-2) \circ \ldots \circ P(t)(S) \). Here we interpret \( P(t)(\Gamma') \) to be the distribution of \( P(t)(s') \) where \( s' \in S \) is distributed according to \( \Gamma' \), and \( P(t)(S) \) is the random variable \( \{ P(t)(s) \mid s \in S \} \). Let \( 0, N_1, N_2, \ldots \) be a decreasing sequence of integers.

The CFTP algorithm produces a sample \( P \sim P \) and looks for the smallest index \( i \) such that \( P^0_i(S) \) has size 1 where \( P^0_{N_i}(S) = P(-1) \circ \ldots \circ P(N_i)(S) \). Then, it outputs this unique state \( s^* \). For the sake of completeness, let us prove that \( s^* \) is indeed distributed according to \( \Gamma \).
Proof. Consider \( \varepsilon > 0 \) and let \( j \in \mathbb{N} \) be large enough such that \( |\Gamma - \mathcal{P}_{N_j}^0(\Gamma')|_{TV} < \varepsilon \), and \( \Pr[|\mathcal{P}_{N_j}^0(S)| \neq 1] < \varepsilon \). The existence of such a \( j \) is guaranteed by (a) and (b). Now, consider a ghost chain (which we have referred to as GC from now on) starting at \( t = N_j \) with its starting state \( s^* \) distributed according to \( \Gamma' \). Clearly, with prob. at least \( 1 - \varepsilon \) the CFTP algorithm will output \( s^* = \mathcal{P}_{N_j}^0(s^*) \). Hence, the two random variables \( s^* \) and \( \mathcal{P}_{N_j}^0(\Gamma') \) are equal with prob. at least \( 1 - \varepsilon \) and therefore \( |\text{dist}(s^*) - \mathcal{P}_{N_j}^0(\Gamma')|_{TV} < \varepsilon \). Finally, by the triangle inequality we have \( |\text{dist}(s^*) - \Gamma|_{TV} < 2\varepsilon \). Since, this is true for all \( \varepsilon > 0 \), we conclude that \( |\text{dist}(s^*) - \Gamma|_{TV} = 0 \).



In the usual set-up for CFTP, \( \mathcal{P} \) is a product distribution and \( \mathcal{P}(t) \) corresponds to the transition of a homogeneous Markov chain with stationary distribution \( \Gamma \), in other words \( \Gamma' = \Gamma \) in (a). For us \( N_1 = -T, N_2 = -2T, \ldots, N_l = -iT, \ldots \) where \( T \) will be set later to be \( \text{poly}(k,n) \). We define the product distribution \( \mathcal{P} \) as follows: for all \( t < T \), \( \mathcal{P}(t) = \mathcal{P}(t + T) \) as distributions. In each \( T \)-chunk from \( N_l \) to \( N_{l-1} \) the first few steps will correspond to those of the coupled Systematic Sweep Glauber Dynamics chain and the remaining steps will be that of the usual coupled Glauber Dynamics chain as described in the introduction (see 1.1.1).

2.4 Bounding Chain (BC)

A drawback of the CFTP is that we need to keep track of the evolution of the GC starting at all states in the state-space \( S \). A BC is an efficient way of keeping track of the evolution of the chains in CFTP.

Recall that for Graph Coloring our state-space is \( S \), which was defined as the set of all proper \( k \)-colorings: \( \Pi \) is the uniform distribution over \( S \) and \( \mathcal{P} \) is a concatenation of the Systematic Sweep Glauber Dynamics and the usual Glauber Dynamics. We will keep track of the evolution of all the \( |S| \) chains required in the CFTP by evolving a chain, which will be our BC, whose state-space will be \( (2^{[k]})^V \). Suppose we evolve the CFTP from time \( t_1 \) to \( t_2 \) (\( t_1 < t_2 \leq 0 \)), using the coupled evolution rule \( \mathcal{P} \), then we define the state of the BC at time \( t_1 \), i.e., \( (BC)_{t_1} = [k]^V \) and we maintain the invariant that for any time \( t \in \{t_1, \ldots, t_2\} \) and for all \( s \in S, v \in V \) we have \( (\mathcal{P}^{t_1}_t(s))(v) \in (BC)_t(v) \), i.e., at all times \( t \) the value of the BC at any vertex \( v \) contains all the possible colors that \( v \) may have at time \( t \). Thus, if at \( t = t_2 \) we observe that \( \forall v \in V : \{(BC)_{t_2}(v)\} = 1 \), then we know that \( |\mathcal{P}^{t_1}_{t_2}(S)| = 1 \) along with the unique state \( s^* \). Of course, we need to couple the evolution of the BC with \( \mathcal{P} \) in such a way that the BC quickly coalesces.

In other words, we wish to establish a coupling between the evolution of the BC (which the algorithm runs) and the GC evolving from time \( t_1 \) with the state at \( t_1 \) being \( s \in S \) which is unknown to the BC (the evolution of the GC will be according to \( \mathcal{P} \)). We maintain the invariant that for all times \( t \in \{t_1, \ldots, t_2\} \) and for all \( v \in V \), \( (GC)_t(v) \in (BC)_t(v) \). Since, the BC does not know \( s \), we set \( (BC)_{t_1} = [k]^V \). As mentioned before we hope to observe coalescence of the BC by time \( t_2 \), and if this happens then we would know \( (GC)_{t_2} \).

Given such a BC, we will evolve the BC from time \(-T \) to 0 and hope to observe coalescence. If not, we will go further back in the past and run the BC from \(-2T \) to \(-T \) and hope to observe coalescence, and so on. In particular, we shall find the first integer \( i \) such that the BC from \(-iT \) to \(-(i-1)T \) coalesces: let us denote this state by \( s \) (strictly speaking we only need coalescence from \(-iT \) to 0). As dictated by the CFTP procedure the BC then outputs the state \( s^* \) that arises after evolving the GC starting at state \( s \) and running from \(-(i-1)T \) to 0 on the same randomness as used by the coupling. Hence, the BC must store enough information about the evolution of the chains from \(-(i-1)T \) to 0 to be able to output \( s^* \).

2.5 Basic Notation

For the remainder of the paper we follow the notations defined below, recalling them whenever necessary.
We define the state of the BC at time \( t \) as \( Y_t \) and that of the GC as \( X_t \). As mentioned in the introduction, let \( S_t(v) \) be the union of colors present in lists of the neighbors of \( v \) at time \( t \) according to the BC; let \( G_t(v) \) be the union of colors present in the lists of those neighbors of \( v \) which have list-size 1 at time \( t \) according to the BC; let \( T_t(v) \) be the union of colors present at the neighbors of \( v \) in the GC. Formally:

\[-\]

\[
\begin{align*}
\bullet & \quad G_t(v) := \{ Y_t(u) \mid u \in N(v), |Y_t(u)| = 1 \} \quad \text{(which we think of as the ‘good’ neighbors of \( v \))}
\bullet & \quad T_t(v) := \{ X_t(u) \mid u \in N(v) \} \quad \text{(which we think of as the set of blocked colors at \( v \))}
\bullet & \quad S_t(v) := \bigcup_{u \in N(v)} Y_t(u).
\end{align*}
\]

It is clear that \( G_t(v) \subseteq T_t(v) \subseteq S_t(v) \). Also, while the BC, i.e., the algorithm, has access to \( G_t(v) \) and \( S_t(v) \) it may not know the set \( T_t(v) \) in its entirety.

Recall that in the introduction we defined a quantity which will help us track the BC’s progress to coalescence: \( W_t := \{ v \in V : (BC)_{t}(v) = 1 \} \), i.e, the set of vertices where the BC has a list of size 1 at time \( t \). We also, defined \( D_t = V \setminus W_t \). Hence, coalescence corresponds to the condition \( |W_0| = n \).

## 3 Our Algorithm for \( k > 3\Delta \)

### 3.1 Huber’s algorithm for \( k > \Delta^2 + 2\Delta \)

Now we proceed to give an exposition of a slightly modified version of Huber’s Algorithm, as this serves as inspiration for our later algorithms. As mentioned in the previous section, we shall describe the algorithm in terms of the coupling between the BC and the GC.

The GC runs according to the transition rule \( \mathcal{P} \). To recall, let \( N_1 = -T, N_2 = -2T, \ldots, N_t = -iT, \ldots \) where \( T \) will be set later to be \( \text{poly}(k,n) \). We define the product distribution \( \mathcal{P} = \mathcal{P}(t + T) \) as distributions. In each \( T \)-chunk from \( N_t \) to \( N_{t-1} \) the first few steps will correspond to those of the Systematic Sweep Glauber Dynamics chain and the remaining steps will be that of the usual Glauber Dynamics chain. In particular, we define \( \mathcal{P}_{t + n} \) to correspond to one round of the Systematic Sweep Glauber Dynamics chain, and for \( t \geq -T + n \) we define \( \mathcal{P}(t) \) to be the usual Glauber Dynamics chain.

Recall that we defined the state of the BC at time \( t \) as \( Y_t \) and that of the GC as \( X_t \). From time \( t = -T \) to \( t = 0 \) Huber’s BC evolves as follows. (To explicitly define the coupling of the BC with the GC we mention updates of \( X_t \) below, however the algorithm only evolves the BC.)

**HUBERBC**

1. At time \(-T\) for all \( v \in V \) assign \( Y_{-T}(v) = [k] \) and \( X_{-T} \sim \Pi \)

2. Fix some ordering \( \{v_1, \ldots, v_n\} \) of the vertices; for \( j = 1 \) to \( j = n \): (this is the warm-up phase of the algorithm)

   a. Update vertex \( v_j \) at time \(-T + j\) as follows:

      i. For vertex \( v_j \) sample a block of \( \Delta + 1 \) colors uar; call this \( L_{-T+j} \)

      ii. Assign \( Y_{(-T+j)}(v_j) = L_{(-T+j)} \)

      iii. Set \( X_{(-T+j)}(v_j) \) to be the first color in \( L_{(-T+j)} \) which is outside \( T_t(v) \) (which exists due to \( \text{deg}(v) \leq \Delta \))

   b. For all \( w \neq v \) set \( Y_{(-T+j)}(w) = Y_{(-T+j-1)}(w) \) and \( X_{(-T+j)}(w) = X_{(-T+j-1)}(w) \)

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3. For time $t = -T + n$ to $t = -1$

(a) Select a vertex $v \in V$ uar

(b) Select a block of $\Delta + 1$ colors uar from $[k] \setminus G_t(v)$; call this $L_{t+1}$

(c) If $L_{t+1}[1] \notin S_t(v)$:
   i. Set $Y_{t+1}(v) = \{L_{t+1}[1]\}$ and for all $w \neq v$ set $Y_{t+1}(w) = Y_t(w)$
   ii. Set $X_{t+1}(v) = L_{t+1}[1]$ and for all $w \neq v$ set $X_{t+1}(w) = X_t(w)$

Else:
   i. Set $Y_{t+1}(v) = L_{t+1}$ and for all $w \neq v$ set $Y_{t+1}(w) = Y_t(w)$
   ii. Set $X_{t+1}(v)$ to be the first color in $L_{t+1}$ which is outside $T_t(v)$ and for all $w \neq v$ set $X_{t+1}(w) = X_t(w)$

The invariant that the BC always contains the state of the GC is clearly maintained. It is easy to verify that this procedure maintains the correct marginal on the GC. The GC accepts the first color in $L_{t+1}$ that is not in $T_t(v)$ and rejects all others. This means that colors in $[k] \setminus T_t(v)$ are uniformly distributed.

The above algorithm is somewhat different from the BC Huber described in his original paper [Hub98]. But Huber's analysis goes through even for this particular variant of his BC. Recall that we had defined $W_t = \{v \mid |Y_t(v)| = 1\}$ and $D_t := V \setminus W_t$. In one time step $|W_t|$ can change by at most 1. $|W_{t+1}|$ may increase by one over $|W_t|$ when we choose $v \in D_t$ as the vertex to update and then choose the color $c_1 \in [k] \setminus S_t(v)$. Similarly, $|W_{t+1}|$ decreases by one when we choose the vertex $v$ to be updated from $W_t$ and $c_1$ from $S_t(v)$. We also define: $\text{bad}_t(v) := \{|u \mid u \in N(v), u \in D_t\}$ and $\delta_t := |W_{t+1}| - |W_t|$. Then,

$$
\Pr[\delta_t = 1] = \frac{1}{n} \sum_{v \in D_t} \frac{k - |G_t(v)| - \text{bad}_t(v) \cdot (\Delta + 1)}{k - |G_t(v)|} \geq \frac{|D_t|}{n} - \frac{1}{n} \sum_{v \in D_t} \frac{\text{bad}_t(v) \cdot (\Delta + 1)}{k - \Delta}.
$$

Again,

$$
\Pr[\delta_t = -1] \leq \frac{1}{n} \sum_{v \in W_t} \frac{\text{bad}_t(v) \cdot (\Delta + 1)}{k - |G_t(v)|} \leq \frac{1}{n} \sum_{v \in W_t} \frac{\text{bad}_t(v) \cdot (\Delta + 1)}{k - \Delta}.
$$

Therefore,

$$
\mathbb{E}[\delta_t] = \Pr[\delta_t = 1] - \Pr[\delta_t = -1] \geq \frac{|D_t|}{n} - \frac{1}{n} \sum_{v \in V} \frac{\text{bad}_t(v) \cdot (\Delta + 1)}{k - \Delta} \geq \frac{|D_t|}{n} \left(1 - \frac{\Delta \cdot (\Delta + 1)}{k - \Delta}\right) \left(\sum_{v \in V} \text{bad}_t(v) \leq \Delta |D_t|\right).
$$

This implies that for $k - \Delta > \Delta \cdot (\Delta + 1)$ we have $\mathbb{E}[\delta_t] > 0$. The BC would have coalesced if $|W_0| = n$. Notice that if for any time $t \in [-T, 0]$ we have that $|W_t| = n$ then $|W_0| = n$. Hence the evolution of $|W_t|$ corresponds to that of a random walk on $[0, n]$ with unit steps and positive drift: at $i$ the drift is $\frac{(n-i)}{n} (1 - \frac{\Delta \cdot (\Delta + 1)}{k - \Delta})$. Here, $0$ is the reflecting state and $n$ is the absorbing state. The expected time for such a random walk to absorb is $\frac{n \ln n}{k - \Delta - 2\Delta}$. This follows from an application of Lemma 4.1.

Now, we set $T$ to be $\left(2 \cdot \frac{k - \Delta}{k - \Delta - 2\Delta} n \ln n + n\right)$: thus, we get that $\Pr[\text{BC from } N_t \text{ to } N_{t-1 \text{coalesces}}] \geq \frac{1}{2}$. Hence, the first integer $i$ such that the the BC from $-i T$ to $-(i - 1) T$ coalesces, is on expectation 2. Therefore, on expectation we need only $O(T)$ iterations of the BC and each iteration is poly$(k, n)$, and hence HUBERBC runs is expected poly$(k, n)$-time.
3.1.1 A bottleneck for HUBERBC

A drawback of HUBERBC is that irrespective of the time \( t \) our analysis does not rule out the algorithm producing lists of size \( \Delta + 1 \) in the worst case, i.e., when the BC samples \( c_1 \) and it happens to be in \( S_t(v) \) the BC prescribes a list of size \( \Delta + 1 \). Hence, at any time \( t \) we might have \( |S_t(v)| = \Delta^2 + \Delta \): and whenever this happens, for the BC to have a non-trivial probability to produce a list of size 1 we need \( k > \Delta^2 + \Delta \). As mentioned in the introduction, although Huber’s algorithm is slightly different from HUBERBC, its worst case behavior is similar. And in any case the analysis of both the algorithms are identical.

Recalling from the introduction, we obtain the improvement on HUBERBC by demonstrating a different way of evolving the coupling between the BC and the GC which after an initial warm-up phase will have a milder worst case behavior. More precisely, we will produce constant-sized lists after the warm-up phase even in the worst case, while maintaining a non-trivial probability of producing lists of size 1.

3.2 Our algorithm for \( k > 3\Delta \)

In this subsection we describe our BC (NEWBC) which produces a uniformly random proper coloring given \( k > 3\Delta \). The GC will evolve according to the following transition rule \( \mathcal{P} \). Let \( N_1 = -T, N_2 = -2T, \ldots, N_j = -iT \) where \( T \) will be set later to be \( \text{poly}(k,n) \); for all \( t < T, \mathcal{P}(t) = \mathcal{P}(t + T) \). Again, in each \( T \)-chunk from \( N_i \) to \( N_{i+1} \) the first few steps will correspond to those of the Systematic Sweep Glauber Dynamics chain and the remaining steps will be that of the usual Glauber Dynamics chain. In particular, for \( j \in \{0, \ldots, \Delta + 1 \} \) we define \( \mathcal{P}_{-T + i(j + 1)n - 1} \) to correspond to one round of the Systematic Sweep Glauber Dynamics chain, and for \( t \geq -T + (\Delta + 1)n \) we define \( \mathcal{P}(t) \) to be the usual Glauber Dynamics chain.

Let us recall some notation. We use \( X_t \) to denote the state of the GC and \( Y_t \) to denote the state of the BC at time \( t \). Further, let

- \( G_t(v) := \{ Y_t(u) \mid u \in N(v), |Y_t(u)| = 1 \} \)
- \( T_t(v) := \{ X_t(u) \mid u \in N(v) \} \)
- \( S_t(v) := \bigcup_{u \in N(v)} Y_t(u) \).

As per the discussion in sec. 3.1.1, we address the drawback of HUBERBC by presenting a coupling between the GC and the BC which after an initial warm-up phase will produce constant-sized lists even in the worst case, while maintaining a non-trivial probability of producing lists of size 1.

3.2.1 BOUNDEDLISTSAMPLER_1

Let us first describe a primitive that will help us build the final algorithm. We call this primitive BOUNDEDLISTSAMPLER_1. Suppose we have evolved the BC till time \( t \) and at time \( t \) vertex \( v \) is chosen for update (both for the BC and the GC). For BOUNDEDLISTSAMPLER_1 to function we need the following situation at time \( t \): \( |[k] \setminus S_t(v)| > \Delta \). This implies that the vertex \( v \) sees at least \( \Delta + 1 \) colors outside the union of the lists of its neighbors in the BC (we will strive to get the BC in such a situation eventually). Given this condition, we run BOUNDEDLISTSAMPLER_1 for vertex \( v \) as follows (again we are only concerned with evolving the BC; the values of \( p_{BC}(v) \) and \( p_{GC}(v) \) are defined after BOUNDEDLISTSAMPLER_1).

BOUNDEDLISTSAMPLER_1(v):

1. Pick a color \( c_1 \in [k] \setminus S_t(v) \) uar
2. Pick a color $c_2 \in S_t(v) \setminus G_t(v)$ uar

3. Accept $c_1$ by flipping a coin with probability of heads $p_{BC}(v)$

4. If HEADS:
   (a) Set $Y_{t+1}(v) = \{c_1\}$
   (b) Set $X_{t+1}(v) = c_1$

5. If TAILS:
   (a) Set $Y_{t+1}(v) = \{c_1, c_2\}$
   (b) Flip a coin with probability of heads $\frac{p_{BC}(v) - p_{GC}(v)}{1 - p_{BC}(v)}$
   (c) If HEADS:
      i. Set $X_{t+1}(v) = c_1$
   (d) If TAILS:
      i. Check if $c_2 \in T_t(v)$
      ii. If YES set $X_{t+1}(v) = c_1$
      iii. If NO set $X_{t+1}(v) = c_2$
   (e) For all $w \neq v$ set $Y_{t+1}(w) = Y_t(w)$ and $X_{t+1}(w) = Y_t(w)$

Clearly, the invariant that the BC contains the GC is maintained. Furthermore, BOUNDEDLISTSAMPLER_1 produces a list of size at most 2 and has a probability of $p_{BC}(v)$ (which we will set to be $> 0$) of producing a list of size 1.

We need to set the value of $p_{GC}(v) \in [0, 1]$ such that the marginal on the GC is correct, i.e., it samples a color uar from $[k] \setminus T_t(v)$. In other words, we require the equation

$$\frac{1}{k - |S_t(v)|}p_{GC}(v) + (1 - p_{GC}(v))\left(\frac{|T_t(v)| - |G_t(v)|}{|S_t(v)| - |G_t(v)|}\right) = \frac{1}{k - |T_t(v)|}$$

to hold. This can be seen by calculating the probability mass that the GC puts on the set $[k] \setminus S_t(v)$. A simple calculation shows that

$$p_{GC}(v) = 1 - \frac{(|S_t(v)| - |G_t(v)|)}{(k - |T_t(v)|)}.$$

To ensure that $p_{GC} > 0$ we impose the condition that $k > \Delta + |S_t(v)|$, which in turn implies that $k > |T_t(v)| + |S_t(v)| - |G_t(v)|$.

Notice that step 5-(b) of BOUNDEDLISTSAMPLER_1 forces $p_{BC} \leq p_{GC}$, i.e., $p_{BC}$ must be an underestimation of $p_{GC}$. Now, consider the following:

$$p_{GC}(v) = 1 - \frac{|S_t(v)| - |G_t(v)|}{(k - |T_t(v)|)}$$

$$(a) \geq 1 - \frac{\left|\bigcup_{w \in N(v) \setminus G_t(v)} Y_t(w)\right|}{k - \Delta}$$

where (a) is true as $|S_t(v)| - |G_t(v)| \leq |\bigcup_{w \in N(v) \setminus G_t(v)} Y_t(w)|$. Although the BC has no access to the $T_t(v)$ it does know the value of $|\bigcup_{w \in N(v) \setminus G_t(v)} Y_t(w)|$. Hence we set $p_{BC}(v) = 1 - \frac{\left|\bigcup_{w \in N(v) \setminus G_t(v)} Y_t(w)\right|}{k - \Delta}$, which is $> 0$ as we have assumed that $|[k] \setminus S_t(v)| > \Delta$. 

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Observe that in the setting of $k > \Delta^2 + 2\Delta$ the condition that $|[k] \setminus S_t(v)| > \Delta$ is satisfied after the systematic sweep step of HUBERBC for all $v \in V$: hence, we may use BOUNDEDLISTSAMPLER_1 in HUBERBC after the initial Systematic sweep step. An identical running time analysis as for HUBERBC shows that the modified algorithm runs in expected polynomial time.

### 3.2.2 WARMUP_SAMPLER[A]

In this section we describe in detail the primitive WARMUP_SAMPLER[A] (where $A$ is some fixed $\Delta$ sized subset of $[k]$), which allows us to push the bound on $k$ down to linear in $\Delta$. Fix some ordering on the vertices as $\{v_1, v_2, \ldots, v_n\}$. Then at the vertex $v$ the coupling between the BC and the GC is described as follows:

**WARMUP_SAMPLER[A](v)**

1. Pick a color $c_1$ uar from $[k] \setminus A$.
2. Set $Y_{t+1}(v) = \{A\} \cup \{c_1\}$.
3. Choose a color $c_2$ uar from $A \setminus T_t(v)$ (if $A \setminus T_t(v) = \emptyset$ choose nothing).
4. Flip a coin with probability of heads $p$.
5. If HEADS:
   (a) Set $X_{t+1}(v) = c_2$
6. If TAILS:
   (a) If $c_1 \notin T_t(v)$ set $X_{t+1}(v) = c_1$.
   (b) Else set $X_{t+1}(v) = c_2$.
7. For all $w \neq v$ set $Y_{t+1}(w) = Y_t(w)$ and $X_{t+1}(w) = X_t(w)$.

Observe that if $A = T_t(v)$ then the GC always uses the color $c_1$. Also, the invariant that the BC contains the state of the GC is maintained. We need to set the value of $p \in [0, 1]$ such that the GC maintains the correct marginal on the set $[k] \setminus T_t(v)$. To this end we define:

- $\ell_1 := |A \setminus T_t(v)|$.
- $\ell_2 := |A \cap T_t(v)|$.
- $\ell_3 := |T_t(v) \setminus A|$.

Consider a fixed color $red$ in the set $A \setminus T_t(v)$. $red$ can be chosen in the following way:

- $c_2$ is picked $red$.
- The coin with probability of heads $p$ lands heads.
- The coin lands tails, and $c_1 \in T_t(v)$. 

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Taking these arguments into account it is clear that \( p \) must satisfy the equation:

\[
p + (1-p) \frac{\ell_3}{k - |A|} = \frac{\ell_4}{k - |T_i(v)|}
\]

Since the L.H.S. is a convex combination of 1 and \( \frac{\ell_1}{k - |A|} \), it is easy to see that a correct value of \( p \) always exists whenever:

\[
(k - |T_i(v)|) \cdot \ell_3 \leq (k - |A|) \cdot \ell_1
\]

or

\[
(k - |T_i(v)|) \cdot (|T_i(v)| - \ell_2) \leq (k - |A|) \cdot (|A| - \ell_2)
\]

Recalling that \( |T_i(v)| \leq \Delta \) and \( |A| = \Delta \), it is easy to check that the L.H.S. is maximized when \( |T_i(v)| = \Delta \). Thus we conclude that the above condition is always satisfied whenever \( k \geq 2\Delta \).

We mention an important detail here. Earlier we had mentioned that the BC keeps track of all the randomness necessary to evolve the coupling between itself and the GC. As such, it must also keep track of all the ‘coins’ used by the GC in the various procedures, i.e., \textsc{BoundedListSampler}_1 and \textsc{WarmUpSampler}(). However, to do so the BC would need access to the set \( T_i(v) \) which may not be possible. So the BC needs to do the following:

- Since \textsc{WarmUpSampler}[A] requires the GC to pick a color unifomly from \( A \setminus T_i(v) \), the BC needs to maintain list of \( 2^\Delta \) colors, each entry in the list being a color picked uniformly from a subset of \( A \).
- With respect to \textsc{BoundedListSampler}._1, the BC needs to maintain a \( \Delta \) sized list of coin tosses, each coin having probability of heads \( p_{GC} \) as a function of \( |T_i(v)| \).

The second list maintained by the BC is linear in \( \Delta \) (since \( |T_i(v)| \) can have at most \( \Delta \) values) and thus requires only \( \text{poly}(k) \) time to generate. However, the first list is exponential in \( \Delta \) and for \( \Delta > O(\log n) \) needs time \( O(\exp(k)) \) to generate. To get around this, we modify the coupling slightly:

For the choice of \( c_2 \):

1. The BC samples a random permutation from the set of all permutations on \([\Delta]\). Call it \( \sigma_i \).
2. \( \sigma_i \) corresponds to an ordering on \( A \). The GC sets the first color in \( A \) with respect to \( \sigma_i \) which is not in \( T_i(v) \) as \( c_2 \).

It is easy to see that the above experiment simulates the step where the GC was required to sample a color uniformly from \( A \setminus T_i(v) \). The added benefit is that the BC no longer needs to store a list of size exponential in \( \Delta \), but may simply store the permutation \( \sigma \) (this requires only \( \text{poly}(k) \) time).

### 3.2.3 \textsc{NewBC}

In this section we elaborate on our final algorithm, which we call \textsc{NewBC}. Note that, following the discussion in Section 1.1.3, we already have all the ingredients necessary for the algorithm \textsc{NewBCaux} which outputs a uniformly random proper coloring for \( k > 4\Delta \).

Now suppose we wish to push the bound on \( k \) even further, i.e., \( k > 3\Delta \). A careful inspection of \textsc{NewBCaux} shows that during the second round of the warm-up, i.e., \( t = -T + n \) to \( t = -T + 2n \), we may not be able to guarantee the conditions needed for \textsc{BoundedListSampler}._1 if say \( k = 3\Delta + 1 \). In particular, let us consider the case when the vertex \( v \) that is chosen to be updated with \textsc{BoundedListSampler}._1 at time \( t \) sees \( \Delta - 1 \) of its neighbors already updated with \textsc{BoundedListSampler}._1 and its remaining neighbor
still has the list prescribed by $\text{WARMUpSAMPLER}[A]$ in the previous round (the update from $t = -T$ to $t = -T + n$). In such a case $|S_t(v)|$ may be as large as $3\Delta - 1$ and hence $k < |S_t(v)| + \Delta$. However, we now suggest a further modified warm-up to remedy this situation.

To this end, fix an ordering of $V$ say $v_1, v_2, \ldots, v_n$. It will be helpful to define the primitive $\text{SSGD}_i(A)$, where $A \subseteq [k]$ is a set of size $\Delta$, which iterates through the vertices $v_i, \ldots, v_n$ in order using $\text{WARMUpSAMPLER}[A]$. \footnote{We are running a variant of the usual \textsc{Systematic Sweep Glauber Dynamics}; however, it is easy to see that this doesn’t bias the output of the CFPT (see Section 2.3).}

Now, let $A_1 \subseteq [k]$ be a fixed set of size $\Delta$. We begin by applying $\text{SSGD}_2(A_1)$. At this stage we have $|S_t(v_1)| \leq 2\Delta$ and hence we are in a situation to apply $\text{BOUNDEDLISTSAMPLER}_1$ at $v_1$, which we do. After an application of $\text{BOUNDEDLISTSAMPLER}_1$ at $v_1$ the list size at $v_1$ is at most 2. However, if $v_1 \in N(v_2)$ then at this stage we may have $|S_t(v_2)| = 2\Delta + 1$, and hence we may be unable to use $\text{BOUNDEDLISTSAMPLER}_1$ at $v_2$. Therefore, we first run $\text{SSGD}_3(A_2)$ where $A_2 \subseteq [k]$ is a set of size $\Delta$ which contains the list of colors at $v_1$. After this update we have $|S_t(v_2)| \leq 2\Delta$. Hence, we may apply $\text{BOUNDEDLISTSAMPLER}_1$ at $v_2$. Hence, we have managed to bring the list sizes at $v_1$ and $v_2$ down to 2. In a similar fashion we will bring the list at $v_i$ down to 2 given that the list sizes at $v_1, \ldots, v_i-1$ are at most 2. We define $A_{i-1} \subseteq [k]$ to be a set of size $\Delta$ such that it has maximum intersection with $\bigcup_{j \in N(v) \cap \Delta} BC_i(v_j)$, i.e., the colors present in those neighbors of $v_i$ which are before it in the ordering and hence already have lists of size at most 2.

Now, we run $\text{SSGD}_{i+1}(A_i)$. Again, it is easy to see that at this stage $|S_t(v_i)| \leq 2\Delta$ and now we may apply $\text{BOUNDEDLISTSAMPLER}_1$ at $v_i$. Once the warm-up phase is over the list size at every vertex is at most 2. Similar to $\text{NEWBCAux}$, from here onward we invoke $\text{BOUNDEDLISTSAMPLER}_1$ where the vertex $v \in V$ is picked uar at each time step.

We now describe the above formally:

$\text{SSGD}_i(A)$:

1. For $(j = i : n)$
   - Run $\text{WARMUpSAMPLER}[A]$ on $v_j$.
   - $j++$

We are now ready to describe $\text{NEWBC}$:

1. At time $-T$ for all $v \in V$ assign $Y_{-T}(v) = [k]$ and $X_{-T} \sim \Pi$.
2. Fix $A_0 \subseteq [k]$ such that $|A_0| = \Delta$.
3. For $i = 0 : n - 1$
   a. Run $\text{SSGD}_{i+1}(A_i)$.
   b. Run $\text{BOUNDEDLISTSAMPLER}_1$ at $v_{i+1}$.
   c. Set $A_i \leftarrow B$, where $B$ is any subset of $[k]$ of size $\Delta$ which contains the set $\bigcup_{k \leq i+1, j \in N(v)} Y_t(v_k)$ at time $t = -T + \sum_{l=n-i}^{n} I + 2(i+1)$.
4. For time $t = -T + \frac{(n+1)(n+4)}{2} + 1$ to $t = 0$
   a. Pick uar and run $\text{BOUNDEDLISTSAMPLER}_1(v)$
Please note that we have assumed that updating the set $A_i$ requires unit time. An analysis similar to that in HubERBC shows that the this procedure runs on expectation in time $\text{poly}(k, n)$. A detailed proof is provided in Lemma 4.2. Hence we have justified the following theorem:

**Theorem 3.1.** Given a graph $G(V, E)$ of size $n$ with max. degree $\Delta$ and also given $k$ colors, NEWBC outputs a uniformly random proper coloring of $G$ in expected polynomial time in $k$ and $n$, whenever $k > 3\Delta$.

### 4 Lemmata and running time analysis

In this section we present the proofs of some of the lemmas and facts that we had deferred to the last section.

**Lemma 4.1.** Suppose that $M_t$ is a random walk on $\{0, 1, \ldots, n\}$ where $0$ is a reflecting state and $n$ is an absorbing state. Suppose:

- $e_i$ is the expected number of times the walk hits the state $i$.
- $|M_{t+1} - M_t| \leq 1$
- $\Pr[M_{t+1} \neq M_t] > 0$
- $\mathbb{E}[M_{t+1} - M_t | M_t = i] \geq \kappa_i > 0$ for all $M_t < n$.

Then

$$\sum_{i=0}^{n} e_i \leq \sum_{i=0}^{n} \frac{1}{\kappa_i}$$

For a proof of this lemma we refer the reader to Theorem 4 [Hub98].

**Lemma 4.2.** The expected time to coalesce of NEWBC is $\text{poly}(k, n)$.

**Proof.** It is clear from the analysis of Section 3.2.3 that the warm-up phase requires time $\text{poly}(k, n)$ because the number of steps taken by the chain is $O(n^2)$ and every step requires time $\text{poly}(k, n)$. What remains to be analysed is the expected time to coalescence once the list size of every vertex has reduced to 2. To that end, recall that BOUNDEDLISTSAMPLER_1 produces a list of size 1 at any vertex $v$ if and only if it accepts the first color in the list, which happens with probability $p_{BC}(v)$. We will use the notation of Section 1.5:

$$\Pr[\delta = 1] = \frac{1}{n} \sum_{v \in D_t} p_{BC}(v)$$

$$= \frac{1}{n} \sum_{v \in D_t} \left(1 - \frac{2 \text{bad}_t(v)}{k - \Delta}\right)$$

$$= \frac{|D_t|}{n} - \frac{1}{n} \sum_{v \in D_t} \frac{2 \text{bad}_t(v)}{k - \Delta}.$$  

Again,

$$\Pr[\delta = -1] = \frac{1}{n} \sum_{v \in W_t} (1 - p_{BC}(v)) = \frac{1}{n} \sum_{v \in W_t} \left(\frac{2 \text{bad}_t(v)}{k - \Delta}\right)$$
Therefore,

$$\mathbb{E}[\delta_i] = \Pr[\delta_i = 1] - \Pr[\delta_i = -1] = \frac{|D_i|}{n} - 1 \sum_{v \in V} \frac{2 \text{bad}_v(v)}{k - \Delta} \geq \frac{|D_i|}{n} \left(1 - \frac{2 \Delta}{k - \Delta}\right) \left(\sum_{v \in V} \text{bad}_v(v) = \Delta|D_i|\right).$$

Taking recourse to Lemma 2.1 we conclude that the expected time until every vertex has a list of size 1 is $\text{poly}(k, n)$, since the algorithm takes $\frac{k - \Delta}{k - 3\Delta} n \ln n$ steps, and each step requiring time $\text{poly}(k, n)$. Collecting all our results, we observe that the expected number of steps in which NEWBC coalesces is $n\Delta + 2n^{\frac{k - \Delta}{k - 3\Delta}} n \ln n$.

Thus setting $T = n + n\Delta + 2n + 2\left(\frac{k - \Delta}{k - 3\Delta} n \ln n\right)$ we get that $\Pr[\text{BC from } N_i \text{ to } N_i - 1 \text{ coalesces}] \geq \frac{1}{2}$. Hence, the first integer $i$ such that the BC from $-iT$ to $-(i-1)T$ coalesces, is on expectation 2. Therefore, on expectation we need only $O(T)$ iterations of the BC and each iteration is $\text{poly}(k, n)$, and hence NEWBC runs is expected $\text{poly}(k, n)$-time. This finishes the proof.

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