Halfspace depth for general measures: the ray basis theorem and its consequences

Petra Laketa · Stanislav Nagy

Received: 20 December 2020 / Revised: 27 July 2021 / Accepted: 9 August 2021 / Published online: 19 August 2021
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract
The halfspace depth is a prominent tool of nonparametric multivariate analysis. The upper level sets of the depth, termed the trimmed regions of a measure, serve as a natural generalization of the quantiles and inter-quantile regions to higher-dimensional spaces. The smallest non-empty trimmed region, coined the halfspace median of a measure, generalizes the median. We focus on the (inverse) ray basis theorem for the halfspace depth, a crucial theoretical result that characterizes the halfspace median by a covering property. First, a novel elementary proof of that statement is provided, under minimal assumptions on the underlying measure. The proof applies not only to the median, but also to other trimmed regions. Motivated by the technical development of the amended ray basis theorem, we specify connections between the trimmed regions, floating bodies, and additional equi-affine convex sets related to the depth. As a consequence, minimal conditions for the strict monotonicity of the depth are obtained. Applications to the computation of the depth and robust estimation are outlined.

Keywords Halfspace depth · Tukey depth · Multivariate median · Ray basis theorem · Floating body

1 Introduction: characterization of the halfspace median

The lack of natural ordering of the points in multidimensional spaces makes the nonparametric analysis of multivariate data challenging. The depth introduces a data-dependent ordering of the sample points, in the direction from the most central observations being those that attain high depth values, to the peripheral ones with low depth (Chernozhukov et al. 2017; Liu et al. 1999; Zuo and Serfling 2000a, c). A definition of depth-based central regions of the data, which are the regions where
the depth exceeds given thresholds, ensues naturally. The smallest non-empty depth region is often termed the depth median set of the data. The depth medians provide convenient robust location estimators, well studied in the literature.

We consider the seminal halfspace (or Tukey) depth, and the general setup of finite Borel (not necessarily probability) measures. Write $\mathbb{R}^d$ for the $d$-dimensional Euclidean space equipped with the inner product $\langle \cdot, \cdot \rangle$, and $S^{d-1}$ for the unit sphere in $\mathbb{R}^d$. The set of all closed halfspaces in $\mathbb{R}^d$ is denoted by $\mathcal{H}$. Elements of $\mathcal{H}$ can be represented as

$$H_{x,u} = \{ y \in \mathbb{R}^d : \langle x, u \rangle \leq \langle y, u \rangle \}$$

for $x \in \mathbb{R}^d$ a point in the boundary hyperplane and $u \in S^{d-1}$ the inner unit normal of $H_{x,u}$. We write $H_{x} = \{ H_{x,u} : u \in S^{d-1} \}$ for all halfspaces whose boundary passes through $x \in \mathbb{R}^d$. The collection of all finite Borel measures on $\mathbb{R}^d$ is denoted by $\mathcal{M}(\mathbb{R}^d)$. The halfspace depth was first considered by Tukey (1975) and later substantially elaborated on by Donoho and Gasko (1992). The halfspace depth of a point $x \in \mathbb{R}^d$ with respect to (w.r.t.) a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is defined as

$$D(x; \mu) = \inf_{H \in H(x)} \mu(H) = \inf_{H \in \mathcal{H}, x \in H} \mu(H).$$

(1)

The two expressions for the depth are easily seen to be equivalent. Our principal interest lies in the concept of the depth-trimmed regions (also called central regions) of $\mu$, defined for $\alpha \in \mathbb{R}$ by $D_\alpha(\mu) = \{ x \in \mathbb{R}^d : D(x; \mu) \geq \alpha \}$. It is a simple observation that the regions $D_\alpha(\mu)$ are always closed convex sets, non-increasing in $\alpha \in \mathbb{R}$ in the sense of set inclusion. For $\alpha \leq 0$ we have $D_\alpha(\mu) = \mathbb{R}^d$; for all $\alpha$ large enough $D_\alpha(\mu) = \emptyset$. The supremum of all $\alpha$ such that $D_\alpha(\mu) \neq \emptyset$ is denoted by $\alpha^*(\mu)$.

We call the set $D_{\alpha^*}(\mu)$ the (halfspace) median set of $\mu$, and denote it by $D^*(\mu)$. The median set is always non-empty and compact. Elements of $D^*(\mu)$ are called (halfspace) medians of $\mu$.\footnote{Frequently, in the literature only the barycentre of the median set $D^*(\mu)$ is called the halfspace median of $\mu$. To simplify our notation we do not follow that convention.} Any halfspace $H \in \mathcal{H}(x)$ that satisfies $D(x; \mu) = \mu(H)$ is called a minimizing halfspace of $\mu$ at $x \in \mathbb{R}^d$.

The so-called ray basis theorem provides a convenient characterization of a halfspace depth median of a measure $\mu$ in terms of its minimizing halfspaces. In its simplest form, the theorem asserts that for measures $\mu \in \mathcal{M}(\mathbb{R}^d)$ with continuous and positive density, a point $x \in \mathbb{R}^d$ is a halfspace median of $\mu$ if and only if the collection of its minimizing halfspaces covers the whole space $\mathbb{R}^d$. That result was first proved by Donoho and Gasko (1992, Claim on p. 1818), and later extended in Rousseeuw and Ruts (1999, Propositions 8 and 12). For the special case of uniform measures on convex bodies, such a characterization of the deepest point relates to an early observation of Grünbaum (1963, p. 251) from convex geometry. For details and additional discussion about the ray basis theorem, its history and relevance in both statistics and geometry see Nagy et al. (2019, Section 4.3.1) and Patáková et al. (2020).
Our initial goal is to revisit the ray basis theorem, and consolidate its statement by extending it to any depth region $D_\alpha(\mu)$ of a general measure $\mu \in \mathcal{M}(\mathbb{R}^d)$, under minimal assumptions.\(^2\) We do so in Sect. 2, where tools from convex geometry are employed to devise an elementary proof of a generalized version of the theorem. We provide conditions under which it is possible to cover the complement to $D_\alpha(\mu)$ by halfspaces of $\mu$-mass $\alpha$. As a special case, we obtain connections between the median set, and the set of all points that allow covering $\mathbb{R}^d$ by their minimizing halfspaces. An important part of our contribution are the examples, that throughout the paper demonstrate that the conditions stated in our main results cannot be avoided.

In Sect. 3 we thoroughly discuss the links of our general ray basis theorem with the properties of the central regions $D_\alpha(\mu)$. It is known from the literature (Rousseeuw and Ruts 1999, Proposition 6) that, writing $A^c$ for the complement to $A \subseteq \mathbb{R}^d$, we have

$$D_\alpha(\mu) = \bigcap \{ H \in \mathcal{H} : \mu(H^c) < \alpha \}.$$

Each $D_\alpha(\mu)$ is thus an intersection of closed convex sets, and must be closed and convex itself. We begin from (2), and specify relations between the central regions $D_\alpha(\mu)$, the trimmed regions as considered in Nolan (1992), Massé and Theodorescu (1994) and van der Vaart and Wellner (1996, Section 3.9.4.6), and the floating body known from convex geometry (Bobkov 2010; Nagy et al. 2019).

Our paper is concluded with three applications of our results in Sect. 4. The set of depth medians $D^*(\mu)$ is not necessarily a single point set. Especially for empirical measures $\mu$, that is measures corresponding to datasets, the median set is frequently full-dimensional. In Sect. 4.1 we propose to single out the collection of those medians that satisfy an additional covering property. We obtain a smaller collection of covering medians of $\mu$, which share qualitatively better properties than the general elements of $D^*(\mu)$. An algorithm for finding covering medians is given. In Sect. 4.2 we obtain a consequence of the general ray basis theorem regarding the structure of the central regions $D_\alpha(\mu)$ for $\mu$ an empirical measure. In that case, each facet $F$ of the convex polytope $D_\alpha(\mu)$ is shown to lie in a hyperplane determined by data points. This observation promises applications in the computation of the trimmed regions $D_\alpha(\mu)$ also in the case when they are not full-dimensional, in a spirit similar to that lately employed in Liu et al. (2019). Finally, in Sect. 4.3 we provide, as an interesting by-product of our study, the minimal set of assumptions that guarantees the depth to be strictly monotone. As argued by Dyckerhoff (2017, Example 4.2), strict monotonicity is one of the most important properties a depth can have. For a depth w.r.t. a probability measure $\mu$, it ensures the almost sure uniform convergence of the depth upper level sets estimated from the data towards their population counterparts. As such, sufficient conditions for strict monotonicity find applications in the estimation of the depth-trimmed regions $D_\alpha(\mu)$ from data in multivariate statistics. The proofs of all theoretical results are provided in an extensive appendix accompanying the paper.

\(^2\) The extension of these results from Borel probability measures to finite Borel measures on $\mathbb{R}^d$ is minor. By our treatment of general measures we mean generalizations of the ray basis theorem and related results to measures from $\mathcal{M}(\mathbb{R}^d)$ that do not have to possess positive densities, or satisfy other simplifying conditions concerning, e.g., their support.
1.1 Preliminaries and notation

In the proofs of our results, we use tools from measure theory, as well as from the theory of convex sets. Our general reference to the concepts used from convexity theory is Schneider (2014). We now set the most important notations used in the paper, and state a preliminary observation about the halfspace depth that will be useful later. For a set $A \subseteq \mathbb{R}^d$ its interior, closure and boundary are denoted by $\text{int}(A)$, $\text{cl}(A)$ and $\text{bd}(A)$, respectively. Denote by $\text{aff}(A)$ and $\text{conv}(A)$ the affine hull and the convex hull of $A$. The sets $\text{relint}(A)$, $\text{relcl}(A)$ and $\text{relbd}(A)$ represent the (relative) interior, closure and boundary of $A$ in the space $\text{aff}(A)$, and $\dim A = \dim(\text{aff}(A))$ is the dimension of $A$. The complement to $A$ is $A^c = \mathbb{R}^d \setminus A$. For sets $A$ and $B$ we write $A \subseteq B$ if $A \subseteq B$ and $A \neq B$.

We say that $H$ is a touching halfspace of a non-empty convex set $A$ if $H \cap \text{cl}(A) \neq \emptyset$ and $\text{int}(H) \cap A = \emptyset$. The collection of all touching halfspaces to $A$ is denoted by $\mathcal{H}(A)$. We also define $\mathcal{H}(\emptyset) = \emptyset$. In this notation, $\mathcal{H}(x)$ is the same as $\mathcal{H}((x))$ for $x \in \mathbb{R}^d$.

**Well-behaved measures** We say that $\mu \in \mathcal{M}(\mathbb{R}^d)$ is smooth if $\mu(\text{bd}(H)) = 0$ for each $H \in \mathcal{H}$. We call $\mu$ smooth at a convex set $A \subseteq \mathbb{R}^d$ if $\mu(\text{bd}(H)) = 0$ for all $H \in \mathcal{H}(A)$, and smooth at a point $x \in \mathbb{R}^d$ if it is smooth at $\{x\}$. Smoothness of $\mu$ at a point is a condition stronger than $\mu$ being atom-less; yet, it is still weaker than smoothness in the whole $\mathbb{R}^d$. A measure $\mu$ is said to have contiguous support if the support of $\mu$ cannot be separated by a slab between two parallel hyperplanes of non-empty interior with zero $\mu$-mass. Finally, $\mu$ is said to have contiguous support at a convex set $A \subseteq \mathbb{R}^d$ if

$$\text{for each } H' \in \mathcal{H}(A) \text{ and } H' \subseteq H \in \mathcal{H}, \text{ bd}(H) \cap \text{int}(A) \neq \emptyset \text{ implies } \mu(H') < \mu(H).$$

Note that the condition (3) is void if $\text{int}(A) = \emptyset$, and therefore it is enough to consider $A$ full-dimensional. In that case, (3) means that any shift $H \supset H'$ of a touching halfspace $H'$ of $A$ has $\mu$-mass larger than $H'$. It is satisfied if, for instance, the set $A$ is a subset of the support of $\mu$. An absolutely continuous measure is smooth (at any $A$ convex). A measure with connected support has contiguous support (at any convex subset $A$ of the support of $\mu$).

**Minimizing halfspaces** For $x \in \mathbb{R}^d$ recall that $H \in \mathcal{H}(x)$ is a minimizing halfspace of $\mu \in \mathcal{M}(\mathbb{R}^d)$ at $x$ if $D(x; \mu) = \mu(H)$. In general, the set of minimizing halfspaces of $\mu$ at a point may be empty. It is guaranteed to be non-empty if, for instance, the measure $\mu$ is smooth at $x$. Our first observation that will prove to be useful in the sequel is that it is always possible to find $H \in \mathcal{H}(x)$ with the property $\mu(\text{int}(H)) \leq D(x; \mu)$. We call such a halfspace $H$ a generalized minimizing halfspace of $\mu$ at $x$.

**Lemma 1** For any $\mu \in \mathcal{M}(\mathbb{R}^d)$ there exists a generalized minimizing halfspace of $\mu$ at any point $x \in \mathbb{R}^d$. If $\mu$ is smooth at $x$, then there exists a minimizing halfspace of $\mu$ at $x$. 

Springer
Additional notations In addition to the upper level set of the halfspace depth \( D_\alpha(\mu) = \{ x \in \mathbb{R}^d : D(x; \mu) \geq \alpha \} \), we also consider the set

\[
U_\alpha(\mu) = \left\{ x \in \mathbb{R}^d : D(x; \mu) > \alpha \right\} = \bigcup_{\beta > \alpha} D_\beta(\mu).
\]

It is convex, but not always closed. As will be seen later in the paper, the set \( U_\alpha(\mu) \) is also not open in general. Both \( D_\alpha(\mu) \) and \( U_\alpha(\mu) \) are bounded for \( \alpha > 0 \).

A face of a convex set \( A \) is a convex subset \( F \subseteq A \) such that \( x, y \in A \) and \( (x + y)/2 \in F \) implies \( x, y \in F \). Denote by \( \mathcal{F}(A) \) the set of all faces of \( \text{cl}(A) \). A facet of \( A \) is a face of \( A \) of dimension \( \text{dim}(A) - 1 \). For \( F \in \mathcal{F}(A) \) and \( A \) convex, denote by \( \mathcal{H}(A, F) = \{ H \in \mathcal{H}(A) : H \subseteq \text{bd}(A) \} \) the collection of all halfspaces touching \( A \) at its face \( F \). We say that a sequence of halfspaces \( \{ H_{n} \} \in \mathcal{H} \) converges to \( H_{x,u} \in \mathcal{H} \) if \( x_n \to x \) in \( \mathbb{R}^d \) and \( u_n \to u \) in \( S^{d-1} \).

We write \( L(x, y) \) and \( L[x, y] \) for the relatively open (that is, not containing \( x \) and \( y \)) and the relatively closed (containing \( x \) and \( y \)) line segment between distinct point \( x, y \in \mathbb{R}^d \), respectively, and \( l(x, y) \) for the infinite line determined by \( x \) and \( y \).

2 The general ray basis theorem

The standard ray basis theorem, as dubbed by Rousseeuw and Ruts (1999), asserts that under certain conditions on a measure \( \mu \in \mathcal{M}(\mathbb{R}^d) \), the halfspace median \( x \) of \( \mu \) is characterized by the fact that \( \mathbb{R}^d \) can be covered by minimizing halfspaces of \( x \).

The formal statement of the theorem is given below.

**Theorem 2** (Rousseeuw and Ruts 1999, Propositions 8 and 12) For \( \mu \in \mathcal{M}(\mathbb{R}^d) \), the following holds true:

(i) If for some \( x \in \mathbb{R}^d \)

\[
\mathbb{R}^d = \bigcup \{ H \in \mathcal{H}(x) : \mu(H) = D(x; \mu) \}, \tag{4}
\]

then \( x \) is a median of \( \mu \), i.e. \( D(x; \mu) = \alpha^*(\mu) \).

(ii) If \( x \) is a median of \( \mu \) and, in addition, \( \mu \) is absolutely continuous with a density that is continuous and positive in an open convex set, then (4) is true for \( x \).

The proof of the direct part (i) of Theorem 2 is simple: since \( \mathbb{R}^d \) is covered by halfspaces of \( \mu \)-mass exactly \( D(x; \mu) \), each \( y \in \mathbb{R}^d \) must be contained in such a halfspace \( H_y \), and consequently \( D(y; \mu) \leq \mu(H_y) = D(x; \mu) \), as follows directly from the definition of the halfspace depth (1).
condition turns out to be $\mu(\text{int}(H)) \leq \alpha$. In the following lemma we introduce a condition that will play an important role in the formulation of the extended ray basis theorem. We treat not only the depth-trimmed regions $D_{\alpha}(\mu)$, but also $U_{\alpha}(\mu)$, as well as their interiors.

**Lemma 3** Let $\mu \in \mathcal{M}(\mathbb{R}^d), \alpha \in \mathbb{R}$ and $S \in \{D_{\alpha}(\mu), U_{\alpha}(\mu), \text{int}(D_{\alpha}(\mu)), \text{int}(U_{\alpha}(\mu))\}$. Then

$$S^c \subseteq (\text{int}(S))^c \subseteq \bigcup \{H \in \mathcal{H} : \mu(\text{int}(H)) \leq \alpha \text{ and } \text{int}(H) \cap S = \emptyset\}. \tag{5}$$

Application of Lemma 3 to the median set $S = \text{int}(D^*(\mu))$ gives a slightly weaker version of the standard inverse ray basis theorem. Suppose that the measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ satisfies the conditions of part (ii) of Theorem 2. First, it is not difficult to observe that for such $\mu$ our auxiliary Lemma 1 about the existence of a generalized minimizing halfspace ensures that the median set $D^*(\mu)$ cannot be full-dimensional.\(^3\)

Because the median set is always convex, it is not full-dimensional if and only if its interior is empty. Therefore, for any $\mu$ with a positive continuous density, Lemma 3 gives that the whole space can be covered by halfspaces (i) of $\mu$-mass at most $\alpha^*(\mu)$; and (ii) without a particular connection to any fixed point $x \in D^*(\mu)$.

Our intention is now to extend the results of Lemma 3 by restricting the halfspaces on the right hand side of (5) to only those that touch $S$, i.e. to the collection $\mathcal{H}(S)$. For that purpose, we introduce additional notation.

**Definition** For a convex set $A \subset \mathbb{R}^d$ and a point $x \notin \text{int}(A)$ define

$$\mathcal{F}(x, A) = \begin{cases} \{F \in \mathcal{F}(A) : \text{relint}(\text{conv}(F \cup \{x\})) \cap \text{cl}(A) = \emptyset\} & \text{if } x \notin \text{cl}(A), \\ \{F \in \mathcal{F}(A) : x \in \text{relint}(F)\} & \text{if } x \in \text{bd}(A). \end{cases}$$

The collection $\mathcal{F}(x, A)$ consists of those faces of the closed convex set $\text{cl}(A)$ that are completely visible from a point $x \notin \text{int}(A)$. Note that $F \in \mathcal{F}(x, A)$ implies $\dim(F) < d$, and that $\mathcal{F}(x, \emptyset) = \{\emptyset\}$. The concept of visible faces relates to the theory of illumination of convex bodies by external light sources (Bezdek and Khan 2018); for its application to the statistics of the depth see Nagy and Dvořák (2021). Note that any point $x \notin \text{int}(A)$ illuminates at least one non-empty face of a non-empty convex set $A$; for details see Lemma 17 presented in the Appendix.

Now we are able to extend Lemma 3 to the touching halfspaces. Starting from (5), our intention is to find $H \in \mathcal{H}(S)$ that covers $x \notin S \setminus \mathcal{F}(x, A)$, with the properties as on the right hand side of (5). The main idea is to approach any point $y \in F \in \mathcal{F}(x, S)$ by a sequence $\{y_n\}_{n=1}^{\infty} \subset (\text{cl}(S))^c$, that converges to $y$ from the outside of $S$. Lemma 3 gives that for each $y_n$ there is a halfspace $H_n \ni y_n$ whose interior has empty intersection with $S$ and $\mu(\text{int}(H_n)) \leq \alpha$, see the left hand panel of Fig. 1. Since $y_n \to y$, there exists a convergent subsequence of $\{H_n\}_{n=1}^{\infty}$ whose limiting halfspace $H$ can be shown to be touching $S$, containing $x$, and satisfying $\mu(\text{int}(H)) \leq \alpha$, as we wanted to show. The formal proof of Lemma 4 is postponed to the Appendix.

\(^3\) This claim is proved under weaker conditions as Corollary 7 in Sect. A.6 in the Appendix.
**Lemma 4** Consider \( \mu \in \mathcal{M}(\mathbb{R}^d) \) and a bounded convex set \( S \subset \mathbb{R}^d \) satisfying (5) with \( \alpha \in \mathbb{R} \). For any \( x \not\in S \) and \( F \in \mathcal{F}(x, S) \) there exists \( H(x, F) \in \mathcal{H}(S, F) \) such that \( x \in H(x, F) \) and \( \mu(\text{int}(H(x, F))) \leq \alpha \). In particular,

\[
S^c \subseteq (\text{int}(S))^c = \bigcup \{ H \in \mathcal{H}(S) : \mu(\text{int}(H)) \leq \alpha \}.
\]

Lemma 4 is technical, but presents an important intermediate result. It provides multiple consequences that we explore in the sequel. Lemma 3 allows us to apply Lemma 4 to the sets \( D_{\alpha}(\mu), U_{\alpha}(\mu) \) or their interiors, for any \( \mu \in \mathcal{M}(\mathbb{R}^d) \). This is the foundation for the most general statement of the ray basis theorem that can be devised for general measures. To obtain finer results, it is necessary to impose additional assumptions. In the literature on the halfspace depth, two typical assumptions are the smoothness of \( \mu \), and the contiguity of its support. We require these conditions only locally, at the considered depth regions. Application of Lemma 4 to \( S = D_{\alpha}(\mu) \) yields the following generalization of the inverse ray basis theorem.

**Theorem 5** (General inverse ray basis theorem) For any \( \mu \in \mathcal{M}(\mathbb{R}^d), \alpha \in \mathbb{R}, x \not\in \text{int}(D_{\alpha}(\mu)) \) and \( F \in \mathcal{F}(x, D_{\alpha}(\mu)) \) there exists \( H(x, F) \in \mathcal{H}(D_{\alpha}(\mu), F) \) such that \( x \in H(x, F) \) and \( \mu(\text{int}(H(x, F))) \leq \alpha \). In particular,

\[
(\text{int}(D_{\alpha}(\mu)))^c = \bigcup \{ H \in \mathcal{H}(D_{\alpha}(\mu)) : \mu(\text{int}(H)) \leq \alpha \}.
\]  

Additionally, if

1. \( \mu \) is smooth at \( D_{\alpha}(\mu) \), then \( \mu(H(x, F)) = \alpha \) and

\[
(\text{int}(D_{\alpha}(\mu)))^c = \bigcup \{ H \in \mathcal{H}(D_{\alpha}(\mu)) : \mu(H) = \alpha \}.
\]
(ii) \( \mu \) has contiguous support at \( D_\alpha(\mu) \), then

\[
(int(D_\alpha(\mu)))^c = \bigcup \{ H \in \mathcal{H}: \mu(int(H)) \leq \alpha \}.
\]

(iii) \( \mu \) is smooth at \( D_\alpha(\mu) \) and has contiguous support at \( D_\alpha(\mu) \), then

\[
(int(D_\alpha(\mu)))^c = \bigcup \{ H \in \mathcal{H}: \mu(H) = \alpha \}.
\]

Note that if \( D_\alpha(\mu) \) fails to be full-dimensional, the left hand sides in the formulas in Theorem 5 are all \( \mathbb{R}^d \), and the theorem therefore gives conditions under which the whole sample space can be covered by halfspaces of limited \( \mu \)-mass. For \( D_\alpha(\mu) \) contained in a hyperplane, also the condition from part (ii) of Theorem 5 is trivially satisfied.

Before moving to the discussion about the relevance of Theorem 5, we state an analogous result for \( U_\alpha(\mu) \) as another corollary of Lemma 4. Since obviously \( U_\alpha(\mu) \subseteq D_\alpha(\mu) \), its general statement is a refinement of the first claim (6) of Theorem 5.

**Theorem 6** For any \( \mu \in \mathcal{M}(\mathbb{R}^d) \), \( \alpha \in \mathbb{R} \), \( x \not\in int(U_\alpha(\mu)) \) and \( F \in \mathcal{F}(x,U_\alpha(\mu)) \) there exists \( H(x,F) \in \mathcal{H}(U_\alpha(\mu),F) \) such that \( x \in H(x,F) \) and \( \mu(int(H(x,F))) \leq \alpha \). In particular,

\[
(int(U_\alpha(\mu)))^c = \bigcup \{ H \in \mathcal{H}: \mu(int(H)) \leq \alpha \}.
\] (7)

Additionally, if \( \mu \) is smooth at \( U_\alpha(\mu) \), then \( U_\alpha(\mu) = int(U_\alpha(\mu)) \), so \( U_\alpha(\mu) \) is open.

Comparison of Theorem 6 and part (ii) of Theorem 5 draws connections between the depth regions \( D_\alpha(\mu) \) and \( U_\alpha(\mu) \) — for \( \mu \) with contiguous support at \( D_\alpha(\mu) \), we have \( int(D_\alpha(\mu)) = int(U_\alpha(\mu)) \). We postpone this discussion into Sect. 3, where connections between upper level sets of the depth and related convex constructions are explored thoroughly. In that section, also further applications of these observations are found.

If \( \mu \) is not smooth, the last statement of Theorem 6 certainly cannot be claimed. This was observed already in Struyf and Rousseeuw (1999, Lemma 6) where it was noted that for \( \mu \) an atomic measure on points with unit weights in general position, \( U_\alpha(\mu) \) equals \( D_{\alpha+1}(\mu) \) for any \( \alpha \in \mathbb{R} \), and as such, \( U_\alpha(\mu) \) is always a closed set. Our first consequence of the general ray basis theorem is the following observation concerning the dimensionality of the median set. It presents a refinement of Small (1987, Proposition 3.4).

**Corollary 7** Let \( \mu \in \mathcal{M}(\mathbb{R}^d) \).

(i) If \( \mu \) has contiguous support at \( D^*(\mu) \), then \( \dim(D^*(\mu)) \neq d \).

(ii) If \( \mu \) is smooth at \( D^*(\mu) \) and \( d > 1 \), then \( \dim(D^*(\mu)) \neq d - 1 \).

Without the assumptions of smoothness and contiguous support, the median set may be of any dimension. Consider, for instance \( \mu \in \mathcal{M}(\mathbb{R}^2) \) that gives mass 1 to each of the points \((-1, -1)^T, (-1, 1)^T, (1, 1)^T\), and mass 2 to \((1, -1)^T\). It is easy to
see that $\alpha^*(\mu) = 2$, and $D^*(\mu) = L\left[(0, 0)\top, (1, -1)\top\right]$. An example of a smooth measure with a full-dimensional median set is presented below in Example 2. For additional discussion on the dimensionality of the median set for empirical measures we refer to Liu et al. (2020).

We are now ready to reformulate the inverse ray basis theorem from part (ii) of Theorem 2, under minimal assumptions.

**Corollary 8 (Inverse ray basis theorem for the median)** Suppose that $\mu \in \mathcal{M}(\mathbb{R}^d)$ is smooth at $D^*(\mu)$. Then

$$\left(\text{int} \left(D^*(\mu)\right)\right)^c = \bigcup \{H \in \mathcal{H}(D^*(\mu)) : \mu(H) = \alpha^*(\mu)\}. $$

If, in addition, $\mu$ has contiguous support at $D^*(\mu)$, then the covering condition (4) holds true for any $x \in D^*(\mu)$.

The assumptions of Corollary 8 are weaker than those in Theorem 2: (i) instead of the existence of the density $f$ of $\mu$ we require only local smoothness of $\mu$, and (ii) instead of the strict positivity and continuity of $f$ in a neighbourhood of the median we need a weaker condition of locally contiguous support at the median set.

**Remark (Uniqueness of the median)** In Corollary 7 we show that under the assumptions of both contiguous support and smoothness of $\mu$ at $D^*(\mu)$, the median set cannot be of dimension $d$, or $d - 1$. In particular, for $d = 1$ and 2 it follows that the median must be unique. It is tempting to claim that for a smooth measure $\mu$ with convex support (part (ii) of Theorem 2), the median set must consist of a single point, compare with Mizera and Volauš (2002, Proposition 7) and the discussion in Small (1987, Section 3). Surprisingly, it turns out that there exist probability distributions with a density that is smooth and positive in a convex set in dimension $d > 2$ that fail to possess a unique halfspace median. The appropriate conditions for the uniqueness of the halfspace median in higher dimensions turn out to be not only smoothness and contiguous support, but, quite surprisingly, also a certain integrability assumption. For a detailed discussion on the problem of the uniqueness of the halfspace median we refer to Pokorný et al. (2021). For that reason, one has to be careful when interpreting the inverse ray basis theorem in Corollary 8. Suppose that a measure $\mu$ is smooth with contiguous support. Then for any point $x \in D^*(\mu)$ it is true that the sample space $\mathbb{R}^d$ is covered by minimizing halfspaces of $\mu$ at $x$. That, however, does not mean that the median of $\mu$ must be unique. Especially in higher dimension, the non-trivial median set may still lie in the boundary of all the halfspaces from the covering system.

We continue by giving examples that demonstrate that the assumptions of Corollary 8 are difficult to be weakened. In our first example we show that without local smoothness, even in the case when $x$ is the unique median of $\mu$, it may fail to satisfy the covering condition (4).

**Example 1** Consider $\mu \in \mathcal{M}(\mathbb{R}^2)$ whose support is plotted in the right hand panel of Fig. 1. It is given as a mixture of the uniform distributions on line segments $L_u = L\left[(0, 0)\top, (1, 1)\top\right]$, $L_d = L\left[(0, 0)\top, (0, -1)\top\right]$, $L_l = L\left[(0, 0)\top, (-1, 0)\top\right]$. 

\[ Springer \]
and \( L_r = L \begin{bmatrix} (0, 0)^T, (1, 0)^T \end{bmatrix} \), such that \( \mu(L_l) = \mu(L_r) = \mu(L_d) = 1 \) and \( \mu(L_u) = 2 \). The origin \( o = (0, 0)^T \) is the unique median of \( \mu \) with \( D(o; \mu) = 2 \), since for any other point \( x \in \mathbb{R}^2 \) there is a halfspace \( H \in \mathcal{H}(x) \) that is parallel with one of the axes such that \( \mu(H) < 2 \). At the same time, each closed halfspace containing \( L_u \) has \( \mu \)-mass at least 3, implying that it is impossible to cover (any point from) \( L_u \) by \( H \in \mathcal{H}(o) \) such that \( \mu(H) \leq \omega^*(\mu) = 2 \). On the other hand, observe that part (ii) of Theorem 5 is valid, as we easily find open halfspaces whose boundary passes through the origin with mass at most \( \alpha^*(\mu) \) whose closures cover \( \mathbb{R}^2 \).

It is known that for a uniform distribution \( \mu \) on a triangle \( \Delta \) in the plane, the barycentre \( o \in \Delta \) of \( \Delta \) is the unique median of \( \mu \) (Rousseeuw and Ruts 1999, Section 5.3). We construct an example of a uniform distribution on the set obtained by removing a narrow strip containing \( o \) from \( \Delta \). We show that our measure does not satisfy the assumption of contiguous support at \( D^*(\mu) \), at the same time its median set is full-dimensional, and contains points that fail to cover the plane by their minimizing halfspaces as in (4).

**Example 2** Consider the equilateral triangle in \( \mathbb{R}^2 \) centred at the origin \( o = (0, 0)^T \) determined by points \( a = (0, 2)^T \), \( b = (-\sqrt{3}, -1)^T \) and \( c = (\sqrt{3}, -1)^T \) as displayed in Fig. 2. For \( x \in (0, 2) \) and \( y \in (0, 1) \) denote \( x_c = (0, x)^T \) and \( y_c = (0, -y)^T \), and let \( x_l \) and \( x_r \) be the points where the horizontal line containing \( x_c \) intersects \( L(a, b) \) and \( L(a, c) \), respectively. Analogously, define points \( y_l \) and \( y_r \) for the horizontal line that contains \( y_c \). Let \( \mu \in \mathcal{M}(\mathbb{R}^2) \) be the uniform distribution on the set \( S = \operatorname{conv} (\{(a, x_l, x_r)\}) \cup \operatorname{conv} (\{(y_l, b, c, y_r)\}) \) with total mass being the area of \( S \). It is possible to choose \( x \) and \( y \) positive, but small enough so that the set \( D^*(\mu) \) is full-dimensional, while \( y_c \in D^*(\mu) \) is the only point in \( \mathbb{R}^2 \) that satisfies the covering property (4). For a detailed technical proof of this claim, we refer to Sect. A.8 in the Appendix.

### 3 Depth regions and floating body

Formula (2) allows us to write any depth region \( D_\alpha(\mu) \) as an intersection of closed halfspaces whose complements have \( \mu \)-mass smaller than \( \alpha \). Another important affine equivariant set is the floating body of \( \mu \) corresponding to \( \alpha \in \mathbb{R} \)

\[
U^F_B(\mu) = \bigcap \left\{ H \in \mathcal{H} : \mu(H^C) \leq \alpha \right\},
\]

as defined in Bobkov (2010, Section 5). According to the discussion in Nagy et al. (2019), the floating bodies are of great interest in both geometry and probability theory. For the special case of \( \mu \) a uniform distribution on a full-dimensional convex set \( K \subset \mathbb{R}^3 \), the floating body has a compelling mechanical interpretation — the set \( U^F_B(\mu) \) can be described as the part of the convex solid \( K \) of (volumetric mass) density \( \alpha \in (0, \alpha^*(\mu)) \) that never submerges beneath the surface of water of unit density, when fully rotated on the surface. The history of the research on floating bodies goes well into the 19th century. In statistics, a construction equivalent to the floating body of a measure is much more recent, and sometimes referred to as the multivariate trimming.
Fig. 2 Example 2: For the inverse ray basis theorem, it is not enough to assume only the smoothness of \( \mu \), without the property of contiguous support. For the uniform measure \( \mu \) on the coloured region, the median is a full-dimensional set, yet the point \( y_c \) displayed in the left hand panel is the only point in \( \mathbb{R}^2 \) that satisfies the covering condition (4). On the right hand panel we see several numerically computed depth regions \( D_\alpha(\mu) \), with the median set \( D^*(\mu) \) being the smallest region, located in the closed strip removed from the triangle

(Massé and Theodorescu 1994; Nolan 1992). Being intersections of closed sets, both \( D_\alpha(\mu) \) and \( U^{FB}_\alpha(\mu) \) are closed. In what follows we use the results of Sect. 2, and precise the connections between the depth regions \( D_\alpha(\mu), U_\alpha(\mu) \), the floating body \( U^{FB}_\alpha(\mu) \), and a further set that turns out to be of interest in our analysis

\[
U^\circ_\alpha(\mu) = \bigcap \left\{ \text{int}(H) : H \in \mathcal{H}, \mu(H^c) \leq \alpha \right\}.
\]

We demonstrate that this last region is an upper level set of a function closely related to the halfspace depth

\[
D^\circ(x; \mu) = \inf_{x \in H} \mu(\text{int}(H)),
\]

considered in, e.g., Mizera and Volauf (2002, Lemma 1). Note also that \( (U^\circ_\alpha(\mu))^c = \bigcup \{H \in \mathcal{H} : \mu(\text{int}(H)) \leq \alpha \} \) already appeared in Theorem 6, meaning that \( U^\circ_\alpha(\mu) = \text{int}(U_\alpha(\mu)) \) must be an open set. The following theorem comprehensively covers the inter-relations between all these affine constructions, and generalizes several results that can be found scattered in the relevant literature on multivariate trimming concepts (Brunel 2019; Nagy et al. 2019; Massé and Theodorescu 1994; Mizera and Volauf 2002; Small 1987).

**Theorem 9** For \( \mu \in \mathcal{M}(\mathbb{R}^d) \) and \( \alpha \in \mathbb{R} \)

\[
U^\circ_\alpha(\mu) = \text{int}(U_\alpha(\mu)) \subseteq \text{cl}(U_\alpha(\mu)) \subseteq U^{FB}_\alpha(\mu) \subseteq D_\alpha(\mu).
\]  

(8)

Additionally,
(i) $U_a^\circ (\mu) = \{ x \in \mathbb{R}^d : D^\circ (x; \mu) > \alpha \}$.

(ii) if $\text{int} (U_a(\mu)) \neq \emptyset$, then $\text{cl} (U_a(\mu)) = U_a^{FB} (\mu)$.

(iii) if $\mu$ has contiguous support at $D_\alpha (\mu)$, then $\text{int} (D_\alpha (\mu)) = \text{int} (U_a(\mu))$.

(iv) if for each $H' \in \mathcal{H}(D_\alpha(\mu))$ we have that $H' \subset H \in \mathcal{H}$ implies $\mu (H') < \mu (H)$, then $U_a^{FB} (\mu) = D_\alpha (\mu)$.

(v) if $\text{int} (U_a(\mu)) \neq \emptyset$ and $\mu$ has contiguous support at $D_\alpha (\mu)$, then $\text{cl} (U_a(\mu)) = D_\alpha (\mu)$.

An application of Theorem 9 to the estimation of the depth regions $D_\alpha (\mu)$ from datasets is given in Sect. 4.3. The condition of non-empty interior of $U_a(\mu)$ that figures in parts (ii) and (v) of the previous theorem is not restrictive, as shown in the next lemma.

**Lemma 10** If $\mu \in \mathcal{M} (\mathbb{R}^d)$ and $\alpha \in \mathbb{R}$ there exists a point $x \in U_a (\mu)$ such that $\mu$ is smooth at $x$, then $\text{int} (U_a(\mu)) \neq \emptyset$.

Combining Lemma 10 and Theorems 6 and 9 we get that for $\mu \in \mathcal{M} (\mathbb{R}^d)$ with a density and $\alpha \in (0, \alpha^*(\mu))$ the set $U_a (\mu)$ is open and $U_a (\mu) \subset \text{cl} (U_0 (\mu)) = U_a^{FB} (\mu)$. If the support of $\mu$ is, in addition, contiguous, we can also write $U_a^{FB} (\mu) = D_\alpha (\mu)$. In the simplest situation of $\mu \in \mathcal{M} (\mathbb{R}^d)$ with a density that is positive in the convex support of $\mu$, we can therefore write $\text{cl} (U_\mu (\mu)) = U_a^{FB} (\mu) = D_\alpha (\mu)$, and the floating bodies completely coincide with the central regions of the depth. The last situation is common in the literature on floating bodies in geometry, where all the above definitions are used interchangeably. In the general setup of the depth and (probability) measures, it is however necessary to differentiate between them.

Our statement of Theorem 9 is strict — none of the inclusions can be reversed, in general. Each strict upper level set $U_a (\mu)$ of a smooth measure $\mu$ is open (Theorem 6), and thus strictly smaller than $\text{cl} (U_a (\mu))$. It is easy to construct a measure without contiguous support that violates $U_a^{FB} (\mu) = D_\alpha (\mu)$, see e.g. Nagy et al. (2019, Fig. 7). Even for measures with contiguous support, equality $U_a^{FB} (\mu) = D_\alpha (\mu)$ from part (iv) of Theorem 9 can fail; consider $\mu$ being the Dirac measure at the origin $o \in \mathbb{R}^d$, and $\alpha = 1$. In that case, $U_a^{FB} (\mu) = \emptyset$, while $D_\alpha (\mu) = \{ o \}$. In the following example we show that in the case when $\text{int} (U_a (\mu)) = \emptyset$, the halfspace depth may fail to satisfy $\text{cl} (U_a (\mu)) = D_\alpha (\mu)$ (the so-called strict monotonicity property), even under the assumption of contiguous support. The example demonstrates that also the remaining set inclusions in Theorem 9 cannot be reversed for general measures without local assumptions.

**Example 3** Consider $\mu \in \mathcal{M} (\mathbb{R}^2)$ given by a mixture of the uniform distribution on the unit disc $C = \{ x \in \mathbb{R}^2 : \| x \| \leq 1 \}$ with mass 1/4, and two atoms located at $m = (0, 1)^T$ and $z = (0, 1/2)^T$ with masses 1/2 and 1/4, respectively. Point $m$ is certainly the unique halfspace median of $\mu$, with $D (m; \mu) = \alpha^* (\mu) = 1/2$. It is also easy to see that the halfspace depth of $\mu$ at the point $o = (0, 0)^T$ equals 1/8. Because of the two large atoms of $\mu$, all points in the open line segment $L(o, z)$ have also depth 1/8. Similarly, all points in $L(z, m)$ have the same depth as $z$, that is $D (z; \mu) = 1/8 + 1/4 = 3/8$. Because for any $y \notin L[o, m]$ certainly $D (y; \mu) < 1/8$, we have for $\alpha = 1/8$ that $D_\alpha (\mu) = L[o, m]$, while $\text{cl} (U_\alpha (\mu)) = L[z, m]$, meaning...
that the strict monotonicity property of the halfspace depth is violated, see also the left hand panel of Fig. 3. In addition, it is easy to verify that $U_{FB}^\alpha(\mu)$ is equal to $D_\alpha(\mu) \supset \text{cl}(U_\alpha(\mu))$, while the set $U_\alpha^\circ(\mu)$ from (8) is empty. We conclude that, in general, one cannot write $U_\alpha(\mu)$ as a simple intersection of halfspaces, as possible for $D_\alpha(\mu)$ in (2).

4 Applications: refined medians, computation, and consistency

4.1 Covering halfspace median

Our first application of the ray basis theorems from Sect. 2 is motivated by another refinement of the main equality (6) applied to $\alpha = \alpha^*(\mu)$: For any measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ there exists a point $x \in D^*(\mu)$ that allows us to cover the whole space with halfspaces $H \in \mathcal{H}(x)$ whose interior has $\mu$-mass at most

$$\gamma^*(\mu) = \inf\{\alpha > 0 : \text{int}(D_\alpha(\mu)) = \emptyset\} = \sup\{\alpha > 0 : \text{int}(D_\alpha(\mu)) \neq \emptyset\}.$$ 

Observe that $\gamma^*(\mu) \leq \alpha^*(\mu)$. Therefore, the following claim is stronger than the inverse ray basis theorem not only (i) because it asserts the existence of such a special point in $D^*(\mu)$, but also in the sense that (ii) the masses of the halfspaces that cover $\mathbb{R}^d$ are ensured to be at most $\gamma^*(\mu)$, and not just $\alpha^*(\mu)$.

**Theorem 11** For any $\mu \in \mathcal{M}(\mathbb{R}^d)$ there exists a point $x \in D^*(\mu)$, such that

$$\mathbb{R}^d = \bigcup \{H \in \mathcal{H}(x) : \mu(\text{int}(H)) \leq \gamma^*(\mu)\}.$$ 

(9)
Example 5: The set of covering medians $C^* (\mu)$ may fail to be convex. For the measure $\mu$, whose support is displayed in this plot in colour, the points $x$ and $y$ belong to $C^* (\mu)$. But, their midpoint $o$ is not contained in $C^* (\mu)$.

Suppose that $D^* (\mu)$ is not a single point set. It is certainly of interest to single out the subset $C^* (\mu)$ of $D^* (\mu)$ of those points that satisfy the additional covering property (9) characteristic to centrally located points. We call such medians the covering halfspace medians (or simply covering medians) of the measure $\mu$. The concept of covering medians is interesting especially in the situation when the median set is full-dimensional, as frequently happens with data generated from a smooth distribution. In that situation, the relatively large median set typically reduces to a smaller subset of the most centrally located points being the covering medians.

Example 4 For an empirical measure $\mu \in \mathcal{M} (\mathbb{R}^2)$ with atoms at points $a = (3, 0)^T$, $b = (0, 5)^T$, $c = (-5, 0)^T$, $d = (1, 3)^T$, $e = (-1, 3)^T$ and $f = (-1, 1)^T$, the median set $D^* (\mu)$ is equal to the triangle determined by points $d$, $e$ and $f$, while $C^* (\mu)$ is a smaller triangle determined by the lines $l(a, e)$, $l(b, f)$ and $l(c, d)$, see the right hand panel of Fig. 3.

The covering medians present a genuine refinement of the halfspace medians. They can be shown to satisfy an array of properties expected from well-behaved location estimators, such as (i) existence as proved in Theorem 11; (ii) affine equivariance; or (iii) plausible continuity properties when considered as a set-valued function of the measure $\mu$. The covering medians are also intimately connected with the robustness properties of the halfspace median and the depth. All these results will be presented elsewhere in an appropriate context. Here we mention only several basic observations closely linked to Theorem 11. The first one demonstrates that unlike the standard halfspace median set, the set of the covering medians does not have to be convex.

Example 5 Let $\mu \in \mathcal{M} (\mathbb{R}^2)$ be the mixture of uniform distributions on the triangles A–K displayed in Fig. 4, such that $\mu(A) = \mu(F) = \mu(G) = 2$ and $\mu(B) = \mu(C) = \mu(D) = \mu(E) = \mu(H) = \mu(I) = \mu(J) = \mu(K) = 1$. It can be shown that the set $C^* (\mu)$ contains points $x$ and $y$ displayed in Fig. 4, but it does not contain the origin.
Halfspace depth for general measures: the ray basis theorem...

\[ o = \frac{(x + y)}{2}. \] Therefore, the set of the covering halfspace medians of \( \mu \) is not convex. For a detailed proof see the Appendix, Sect. A.12.

The proof of Theorem 11 allows us to devise a simple algorithm for finding the covering medians of \( \mu \) listed as Algorithm 1. Although this program is applicable to any measure including empirical measures of random samples, its main purpose is not the computation of the sample covering medians of large datasets. Rather, it is intended to guide a quick manual procedure for restricting the location of possible covering medians in visual examples such as those presented throughout this paper.

**Algorithm 1:** Search for a covering halfspace median of a measure \( \mu \).

| input | the full-dimensional median region \( D^*(\mu) \) of a measure \( \mu \) |
| input | a small positive constant \( \varepsilon \) determining desired precision |
| output | a covering median of \( \mu \) |

\[
S_0 \leftarrow D^*(\mu); \\
x_0 \leftarrow \text{barycentre of } S_0; \\
k \leftarrow 0; \\
\text{while } x_k \notin C^*(\mu) \text{ and volume of } S_k \text{ exceeds } \varepsilon \text{ do} \\
\text{ } \\
\text{ } \\
| H_k \leftarrow \text{generalized minimizing halfspace of } \mu \text{ at } x_k; \\
// \text{ such a halfspace exists by Lemma 1} \\
S_{k+1} \leftarrow \text{cl}(S_k \setminus H_k); \\
k \leftarrow k + 1; \\
x_k \leftarrow \text{barycentre of } S_k |
\]

end

return \( x_k \);

Once a covering median of \( \mu \) is found, it is of interest to determine whether it is unique. Supposing that a covering median of \( \mu \) and its collection of generalized minimizing halfspaces are available as the output of Algorithm 1, the following theorem gives a sufficient condition for the uniqueness of this covering median.

**Theorem 12** Let \( x \in \mathbb{R}^d \) be a covering median of \( \mu \in \mathcal{M} (\mathbb{R}^d) \). Denote by \( \mathcal{H}_{\text{min}} \) the collection of halfspaces on the right hand side of the following display

\[
\mathbb{R}^d = \bigcup \{ H \in \mathcal{H}(x): \mu(\text{int} (H)) \leq \gamma^*(\mu) \}.
\]

If for each \( H \in \mathcal{H}_{\text{min}} \) and \( H' \supset H \) it follows that \( \mu(H') > \mu(H) \), and if there is no subset \( \mathcal{H}' \subseteq \mathcal{H}_{\text{min}} \) such that \( \bigcup \mathcal{H}' = \mathbb{R}^d \) and \( \bigcap \mathcal{H}' \neq \{x\} \), then \( C^*(\mu) = \{x\} \).

Note that the point \( y_c \) from Example 2 given in Sect. 3 satisfies the conditions of Theorem 12 and \( C^*(\mu) = \{y_c\} \), while the median of that measure \( \mu \) is full-dimensional. On the other hand, Theorem 12 does not apply to any of the two covering medians \( x, y \) found in Example 5, as the condition regarding contiguous support at minimizing halfspaces is not satisfied.
4.2 Trimmed regions for atomic measures

Our second application of the generalized ray basis theorem is a necessary condition on the depth regions $D_\alpha(\mu)$ of atomic measures.

**Corollary 13** For $\mu$ an atomic measure with finitely many atoms, $\alpha \in \mathbb{R}$, $x \notin \text{int}(D_\alpha(\mu))$ and $F \in \mathcal{F}(x, D_\alpha(\mu))$ there exists $H(x, F) \in \mathcal{H}(D_\alpha(\mu), F)$ such that $x \in H(x, F)$ and $\mu(\text{int}(H(x, F))) < \alpha$. Moreover, each face $F$ of the convex polytope $D_\alpha(\mu)$ of dimension $\dim(F) < d$ is contained in the convex hull of at least $\min\{\dim(D_\alpha(\mu)) + 1, d\}$ atoms located in a hyperplane in $\mathbb{R}^d$.

A special case of this result in the situation when $D_\alpha(\mu)$ is full-dimensional was used in the derivation of a fast algorithm for the computation of the depth regions of datasets in Liu et al. (2019). The present general version may find applications in the computation of the halfspace depth of datasets in the situation when $\dim(D_\alpha(\mu)) < d$.

For example, suppose that for given $\alpha \in \mathbb{R}$ and $\mu$ corresponding to a dataset the algorithm from Liu et al. (2019) fails to find an interior point of the region $D_\alpha(\mu)$. The reason may be twofold: either (i) $D_\alpha(\mu)$ is an empty set, or (ii) it is less than full-dimensional. Corollary 13 asserts that in the latter case, the region $D_\alpha(\mu)$ must be contained in a hyperplane spanned by $d$ data points. Thus, the search for $D_\alpha(\mu)$ may continue in the intersection of data-determined hyperplanes, and the last known non-empty region $D_\beta(\mu)$ for $\beta < \alpha$.

4.3 Estimation of central regions

Part (v) of Theorem 9 yields an important equality. According to Dyckerhoff (2017, Definition 3.1) the halfspace depth is strictly monotone at $\mu \in \mathcal{M}(\mathbb{R}^d)$ if for all $\alpha \in (0, \alpha^*(\mu))$ we can write $D_\alpha(\mu) = \text{cl}(U_\alpha(\mu))$. Strict monotonicity is a crucial assumption that ensures the sample version consistency of the halfspace depth-trimmed regions (Dyckerhoff 2017, Example 4.2). As a consequence of Lemma 10 we obtain a condition ensuring the strict monotonicity of the halfspace depth—for a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ to have a strictly monotone depth it is enough to assume contiguous support and smoothness at a single point of the median set. In particular, under these very mild conditions, it is possible to guarantee the almost sure uniform convergence of the sample depth trimmed regions to their population counterparts. To precise this, we need to consider a topology on the space of compact subsets of $\mathbb{R}^d$. A natural choice is that given by the Hausdorff distance from Schneider (2014, Section 1.8). For $K, L \subset \mathbb{R}^d$ compact the Hausdorff distance of $K$ and $L$ is given by

$$\delta_H(K, L) = \max\left\{\sup_{x \in K} \inf_{y \in L} |x - y|, \sup_{x \in L} \inf_{y \in K} |x - y|\right\}.$$  

**Corollary 14** Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a probability measure with contiguous support that is smooth at some $x \in D^*(\mu)$. Let $X_1, \ldots, X_n$ be a random sample from $\mu$ defined on the probability space $(\Omega, \mathcal{A}, P)$, and denote by $\tilde{\mu}_n \equiv \tilde{\mu}_n(\omega) \in \mathcal{M}(\mathbb{R}^d)$ the empirical
measure of $X_1, \ldots, X_n$. Then for any closed interval $A \subset (0, \alpha^*(\mu))$

$$
P\left( \left\{ \omega \in \Omega : \lim_{n \to \infty} \sup_{\alpha \in A} \delta_H(D_\alpha(\mu_n), D_\alpha(\mu)) = 0 \right\} \right) = 1. $$

Further, let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be any measure that is smooth with contiguous support. Then for any closed interval $A \subset (0, \alpha^*(\mu))$ and any $\mu_n$ converging weakly to $\mu$ we have

$$
\lim_{n \to \infty} \sup_{\alpha \in A} \delta_H(D_\alpha(\mu_n), D_\alpha(\mu)) = 0.
$$

This result follows directly from Dyckerhoff (2017, Theorem 4.5) and our previous discussion. The General assumption in Dyckerhoff (2017, p. 9) is satisfied thanks to our Lemma 10; the assumption of compact convergence of the depth from Dyckerhoff (2017) follows, e.g., from Nagy et al. (2019, Section 3.2.7). Corollary 14 should be compared with earlier contributions regarding the consistency of the trimmed regions and derived quantities (Kim 2000; He and Wang 1997; Nolan 1992; Massé 2002; Wang and Serfling 2006; Wang 2019; Zuo and Serfling 2000c). In those references analogous consistency results are proved under more restrictive conditions.

Acknowledgements We gratefully acknowledge the helpful suggestions of anonymous referees. This research was supported by the grant 19-16097Y of the Czech Science Foundation, and by the PRIMUS/17/SCI/3 project of Charles University.

Appendix A: Proofs of the theoretical results

We begin with a lemma collecting several properties of the convergence of halfspaces.

**Lemma 15** Consider $\mu \in \mathcal{M}(\mathbb{R}^d)$, a sequence of closed halfspaces $\{H_n\}_{n=1}^\infty \subset \mathcal{H}$, and $\alpha \in \mathbb{R}$. The following claims hold true:

(i) If there is a sequence $\{x_n\}_{n=1}^\infty$ such that $x_n \in \text{bd}(H_n)$ and $x_n \to x$, then there exists a subsequence $\{H_{n_k}\}_{k=1}^\infty$ converging to a closed halfspace $H \in \mathcal{H}(x)$.

(ii) If $H_n \to H$ and $\mu(\text{int}(H_n)) \leq \alpha$ for each $n$, then $\mu(\text{int}(H)) \leq \alpha$.

(iii) If $H_n \to H$, $\mu(\text{int}(H_n)) \leq \alpha$ for each $n$, $H \cap D_\alpha(\mu) \neq \emptyset$ and $\mu(\text{bd}(H)) = 0$, then $\mu(H) = \alpha$.

(iv) If $H_n \to H$ and $x \notin \text{int}(H_n)$ for each $n$, then $x \notin \text{int}(H)$.

**Proof** For part (i), denote $H_n = H_{x_n,v_n}$. The set of unit vectors $S^{d-1}$ is bounded, meaning that $\{v_n\}_{n=1}^\infty \subset S^{d-1}$ contains a convergent subsequence $\{v_{n_k}\}_{k=1}^\infty$, whose limit point we denote by $v \in S^{d-1}$. We obtain that $H_{n_k} \to H_{x,v} = H \in \mathcal{H}(x)$ as $k \to \infty$ as desired. Part (ii) is a consequence of the Fatou lemma (Dudley 2002, Lemma 4.3.3) and the fact that $\text{int}(H) \subseteq \liminf_{k \to \infty} \text{int}(H_{n_k})$. Under the assumptions of part (iii) there exists $x \in D_\alpha(\mu) \cap H$. For this point we can directly write by part (ii) of this lemma $\alpha \leq D(x; \mu) \leq \mu(H) = \mu(\text{int}(H)) \leq \alpha$. To prove (iv), assume that $x \in \text{int}(H)$. Then $x \in \text{int}(H_n)$ starting from some index, a contradiction. \qed
Lemma 16 If $H \in \mathcal{H}$ and $S \subset \mathbb{R}^d$ is a convex set such that $\text{int} (H) \cap S = \emptyset$ and $\text{relint} (S) \cap H \neq \emptyset$, then $\text{aff} (S) \subseteq \text{bd} (H)$.

**Proof** Consider $x \in \text{relint} (S) \cap H$. Since $x \in \text{relint} (S)$, there exists a ball $B_x$ centred at $x$ in space $\text{aff} (S)$ so that $B_x \subset S$. Since $\text{int} (H) \cap S = \emptyset$ we get $x \in \text{bd} (H)$ and $B_x \subset \text{bd} (H)$, meaning that $\text{aff} (S) = \text{aff} (B_x) \subseteq \text{bd} (H)$.

Lemma 17 For $A \subset \mathbb{R}^d$ non-empty convex and $x \notin \text{int} (A)$ the collection $\bar{\mathcal{S}}(x, A)$ contains a non-empty set.

**Proof** In the case that $x \in \text{bd} (A)$, there is a face $F \neq \emptyset$ of $\text{bd} (A)$ containing $x$ in its relative interior (Schneider 2014, Theorem 2.1.2), so $\bar{\mathcal{S}}(x, A)$ contains a non-empty set. Otherwise, let $x \notin \text{cl} (A)$. The Hahn-Banach theorem (Schneider 2014, Theorem 1.3.4) guarantees that there exists a touching halfspace $H \in \mathcal{H}(A)$ of $\text{cl} (A)$ such that $x \in \text{int} (H)$. Denote $F = \text{cl} (A) \cap \text{bd} (H)$. Then, $F$ is a non-empty face of $\text{cl} (A)$, and certainly also $\text{cl} (A) \setminus F \subseteq H^c$, $F \subseteq \text{bd} (H)$, and $x \in \text{int} (H)$, meaning that $F \in \bar{\mathcal{S}}(x, A)$.

A.1 Proof of Lemma 1

Let $\alpha = D (x; \mu)$. The definition of the halfspace depth ensures that for each $n = 1, 2, \ldots$ there exists $H_n \in \mathcal{H}(x)$ such that $\mu (H_n) < \alpha + 1/n$. Applying part (i) of Lemma 15 we obtain a subsequence $\{H_{n_k}\}_{k=1}^\infty$ converging to $H \in \mathcal{H}(x)$. From the Fatou lemma (Dudley 2002, Lemma 4.3.3) we get $\mu (\text{int} (H)) \leq \liminf_{k \to \infty} \mu (\text{int} (H_{n_k})) \leq \alpha$.

Under the additional assumption of smoothness of $\mu$ at $x$ we know that $\mu (\text{bd} (H)) = \mu (\text{bd} (H_n)) = 0$ for each $n = 1, 2, \ldots$. In that case, the Fatou lemma guarantees $\mu (H) = \lim_{k \to \infty} \mu (H_{n,k}) \leq \alpha$. Since also $\alpha = D (x; \mu) \leq \mu (H) \leq \alpha$, we obtain the desired result.

A.2 Proof of Lemma 3

For any $S \subset \mathbb{R}^d$ convex and $H \in \mathcal{H}$, the condition $\text{int} (H) \cap S = \emptyset$ implies $\text{int} (H) \cap \text{int} (S) = \emptyset$. Furthermore $\text{int} (S) \subseteq S$. Thus, it is enough to show the claim (5) with the smaller set $\text{int} (S)$ on the left hand side, and the condition $\text{int} (H) \cap S = \emptyset$ in the expression on the right hand side.

We start with $D_\alpha (\mu)$. First note that by the very definition of $D_\alpha (\mu)$

$$
\mu (\text{int} (H)) < \alpha \text{ implies } \text{int} (H) \cap D_\alpha (\mu) = \emptyset. \tag{10}
$$

Consider any $x \notin \text{int} (D_\alpha (\mu))$. Suppose first that $x \notin D_\alpha (\mu)$, meaning that $D (x; \mu) < \alpha$. The auxiliary Lemma 1 implies that there exists $H \in \mathcal{H}(x)$ such that $\mu (\text{int} (H)) < \alpha$, and (10) ensures that $x$ is covered by the union on the right hand side of (5). If $x \in \text{bd} (D_\alpha (\mu))$, we can approach $x$ by a sequence $\{x_n\}_{n=1}^\infty \subset (D_\alpha (\mu))^c$ such that $x_n \to x$. We apply Lemma 1 to each $x_n$ to obtain halfspaces $H_n \in \mathcal{H}(x_n)$ such that $\mu (\text{int} (H_n)) < \alpha$, and consequently $\text{int} (H_n) \cap D_\alpha (\mu) = \emptyset$ for each $n$ by (10). From
Lemma 15, parts (i), (ii) and (iv) we conclude that there exists \( H \in \mathcal{H}(x) \) such that 
\[
\mu(\text{int}(H)) \leq \alpha \quad \text{and} \quad \text{int}(H) \cap D_\alpha(\mu) = \emptyset,
\]
as desired.

The proof for \( U_\alpha(\mu) \) is analogous. First, observe that \( \mu(\text{int}(H)) \leq \alpha \) implies 
\[
\text{int}(H) \cap U_\alpha(\mu) = \emptyset
\]
by the definition of \( U_\alpha(\mu) \). Consider any \( x \notin \text{int}(U_\alpha(\mu)) \). If \( x \notin U_\alpha(\mu) \), then 
\[
D(x; \mu) \leq \alpha,
\]
so there is \( H \in \mathcal{H}(x) \) such that \( \mu(\text{int}(H)) \leq \alpha \) by Lemma 1. Otherwise, \( x \in \text{bd}(U_\alpha(\mu)) \) may be approached by a sequence \( \{x_n\}_{n=1}^\infty \subset (U_\alpha(\mu))^c \) such that \( x_n \to x \), and Lemma 15 again gives the desired result.

### A.3 Proof of Lemma 4

Since each \( H \in \mathcal{H}(S) \) has empty intersection with \( \text{int}(S) \), it is only left to prove 
\[
(\text{int}(S))^c \subset \bigcup \{H \in \mathcal{H}(S) : \mu(\text{int}(H)) \leq \alpha\}.
\]
We take any \( x \notin \text{int}(S) \) and consider three cases.

**Case (i):** \( \text{int}(S) = \emptyset \). By definition, \( \mathcal{H}(\emptyset) = \mathcal{H} \). Therefore, the statement reduces directly to condition (5).

**Case (ii):** \( x \notin \text{cl}(S) \) and \( \text{int}(S) \neq \emptyset \). We consider only the situation when \( F \) is non-empty, which is possible by Lemma 17, as the other case follows trivially from condition (5). Denote \( C = \text{relint}(\text{conv}(F \cup \{x\})) \). Take \( y \in \text{relint}(F) \), and the sequence \( y_n = (1 - 1/n) y + x/n \in C \), for \( n = 1, 2, \ldots \), that converges to \( y \). Because \( x \notin S \), the choice of \( F \) ensures that \( C \cap \text{cl}(S) = \emptyset \). Thus, for each \( n \) we have \( y_n \notin \text{cl}(S) \) and condition (5) implies the existence of \( H_n \in \mathcal{H} \) such that \( y_n \in H_n \), 
\[
\text{int}(H_n) \cap S = \emptyset \quad \text{and} \quad \mu(\text{int}(H_n)) \leq \alpha.
\]
Applying Lemma 15, parts (i), (ii) and (iv), we conclude that there is a subsequence \( \{H_{n_k}\}_{k=1}^\infty \) converging to \( H \in \mathcal{H}(y) \), such that \( \mu(\text{int}(H)) \leq \alpha \) and \( \text{int}(H) \cap S = \emptyset \). Then \( y \in \text{relint}(F) \cap H \). From Lemma 16 it follows that \( F \subset \text{bd}(H) \), so \( H \in \mathcal{H}(S, F) \).

**Case (iii):** \( x \in \text{bd}(S) \) and \( \text{int}(S) \neq \emptyset \). The face \( F \) was chosen so that \( x \in \text{relint}(F) \). It is enough to follow the lines of Case (ii) of this proof with \( y \) replaced by \( x \), and \( x \) replaced by \( z = x + u \notin S \) for \( u \in \mathbb{S}^{d-1} \) (any) outer normal to \( S \) at \( x \). The latter choice assures that \( F \in \mathcal{H}(y, S) \), which allows us to proceed exactly as in Case (ii).

### A.4 Proof of Theorem 5

The general statement follows from Lemmas 3 and 4. Thus, it is enough to prove only the results under additional assumptions. We take \( x \in \mathbb{R}^d \) and consider the construction of the halfspace \( H = H(x, F) \in \mathcal{H}(D_\alpha(\mu), F) \) from the proof of Lemma 4. Only the situation when both \( D_\alpha(\mu) \) and \( F \) are non-empty is considered; the other case is straightforward and trivial.

**Part (i):** Under the assumption of smoothness at \( D_\alpha(\mu) \) we have 
\[
\mu(H(x, F)) = \mu(\text{int}(H(x, F)))
\]
and \( \emptyset \neq F \subset D_\alpha(\mu) \) we also have that 
\[
H(x, F) \cap D_\alpha(\mu) = \emptyset,
\]
which implies 
\[
\mu(H(x, F)) = \alpha.
\]

**Part (ii):** It is enough to show that for any \( H \in \mathcal{H} \) the condition \( \mu(\text{int}(H)) \leq \alpha \) implies 
\[
(\text{int}(D_\alpha(\mu))) \cap H = \emptyset.
\]
Suppose \( x \in H \cap \text{int}(D_\alpha(\mu)) \) and denote by \( v \) the unit inner normal of \( H \). Then 
\[
H_{x,v} \subseteq H \quad \text{and} \quad \text{bd}(H_{x,v}) \cap D_\alpha(\mu) \neq \emptyset.
\]
Since \( D_\alpha(\mu) \) is closed and convex, there exists \( y \in \text{int}(H_{x,v}) \cap D_\alpha(\mu) \), such that 
\[
H_{y,v} \in \mathcal{H}(S, F).
\]
\( \mathcal{H}(D_\alpha(\mu)) \), and \( H_{y,v} \subset H_{x,v} \). Because \( \mu \) has contiguous support at \( D_\alpha(\mu) \) it follows that \( \mu(H_{y,v}) < \mu(H_{x,v}) \leq \mu(H) \leq \alpha \), which is in contradiction with \( y \in D_\alpha(\mu) \).

**Part (iii):** The result follows from parts (i) and (ii).

### A.5 Proof of Theorem 6

The inclusion "\( \subseteq \)" in (7) is a direct consequence of Lemmas 3 and 4. For the other inclusion, suppose for contradiction that \( x \in \text{int} \;(U_\alpha(\mu)) \) is contained in the right hand side of (7). Then we can find \( H \in \mathcal{H} \) with \( x \in H \) and \( \mu(\text{int} \;(H)) \leq \alpha \), and since both \( \text{int} \;(H) \) and \( \text{int} \;(U_\alpha(\mu)) \) are open, there must exist \( y \in \text{int} \;(U_\alpha(\mu)) \cap \text{int} \;(H) \). Denote by \( v \in \mathbb{S}^{d-1} \) the inner unit normal of \( H \). We obtain that \( H_{y,v} \subset \text{int} \;(H) \), meaning that \( \alpha < D \;(y; \mu) \leq \mu(H_{y,v}) \leq \mu(H) \) \( \leq \alpha \), a contradiction. We have verified (7).

If \( \mu \) is smooth at \( U_\alpha(\mu) \), then for \( H(x, F) \in \mathcal{H}(U_\alpha(\mu), F) \) from the proof of Lemma 4 we have \( \mu(H(x, F)) = \mu(\text{int} \;(H(x, F))) \leq \alpha \) and consequently \( H(x, F) \cap U_\alpha(\mu) = \emptyset \), meaning that \( F \cap U_\alpha(\mu) = \emptyset \). This holds true for any \( x \not\in \text{int} \;(U_\alpha(\mu)) \) and any face \( F \) from \( \mathcal{F}(x, U_\alpha(\mu)) \). In particular, it must be true for any face \( F \) of \( U_\alpha(\mu) \) that is not full-dimensional, by considering \( x \in \text{relint} \;(F) \subset \text{bd} \;(U_\alpha(\mu)) \). That implies \( U_\alpha(\mu) \cap \text{bd} \;(U_\alpha(\mu)) = \emptyset \), meaning that \( U_\alpha(\mu) = \text{int} \;(U_\alpha(\mu)) \) and \( U_\alpha(\mu) \) is open.

### A.6 Proof of Corollary 7

**Part (i):** Suppose for contradiction that \( x \in \text{int} \;(D^*(\mu)) \). Lemma 1 gives \( H \in \mathcal{H}(x) \) with \( \mu(\text{int} \;(H)) \leq \alpha^*(\mu) \). We can shift \( H \) in the direction of its inner normal to obtain \( H' \in \mathcal{H}(D^*(\mu)) \) such that \( H' \subset \text{int} \;(H) \). Since \( D^*(\mu) \) is closed, there exists \( y \in D^*(\mu) \cap H' \). The assumption of contiguous support at \( D^*(\mu) \) then gives \( D \; (y; \mu) \leq \mu(H') < \mu(\text{int} \;(H)) = \alpha^*(\mu) \), which is impossible as \( y \in D^*(\mu) \).

**Part (ii):** If \( \dim \;(D^*(\mu)) = d-1 \), for any \( x \) not lying in the hyperplane \( \text{aff} \;(D^*(\mu)) \) we have \( D^*(\mu) \in \mathcal{F}(x, D^*(\mu)) \). Theorem 5 gives that there exists a halfspace \( H \in \mathcal{H} \) that contains \( D^*(\mu) \) in its boundary hyperplane, and \( x \) in its interior, with \( \mu(H) = \mu(\text{int} \;(H)) \leq \alpha^*(\mu) \) due to the assumption of the smoothness of \( \mu \) at \( D^*(\mu) \). Take any \( y \not\in H \). Again, by Theorem 5 we obtain \( G \in \mathcal{H} \) such that \( D^*(\mu) \subset \text{bd} \;(G) \), \( y \in \text{int} \;(G) \), and \( \mu(G) \leq \alpha^*(\mu) \). The conditions \( \dim \;(D^*(\mu)) = d-1 \), \( D^*(\mu) \subset \text{bd} \;(H) \cap \text{bd} \;(G) \) and \( H \neq G \) determine that \( H \) and \( G \) must be complementary, i.e. \( H \cup G = \mathbb{R}^d \). Necessarily, \( \mu(\mathbb{R}^d) = \mu(H) + \mu(G) \leq 2\alpha^*(\mu) \), meaning that the measure \( \mu \) is halfspace symmetric by Zuo and Serfling (2000b). But, by Zuo and Serfling (2000b, Theorem 2.1) we know that the median of a halfspace symmetric measure is either unique, or \( \mu \) is concentrated in an infinite line \( L \in \mathbb{R}^d \). The former case contradicts \( \dim \;(D^*(\mu)) = d-1 \). In the latter case, also the median set of \( \mu \) is contained in \( L \), which contradicts our assumption of smoothness of \( \mu \).
A.7 Proof of Corollary 8

The first part of the statement follows directly from part (i) of Theorem 5 with \( \alpha = \alpha^*(\mu) \). For the second part, note that due to the contiguous support of \( \mu \) at \( D^*(\mu) \), \( \dim (D^*(\mu)) < d \) by Corollary 7. Therefore, \( \operatorname{int}(D^*(\mu)) = \emptyset \), so \( \operatorname{int}(D^*(\mu))^{c} = \mathbb{R}^d \). To obtain the result, apply Theorem 5 with \( x \) replaced by \( y \notin \operatorname{aff}(D^*(\mu)) \).

That theorem allows to choose \( F \) to be \( D^*(\mu) \in \mathcal{S}(y, D^*(\mu)) \). We obtain that \( \mathbb{R}^d \setminus \operatorname{aff}(D^*(\mu)) \) can be covered by halfspaces \( H \) from \( \mathcal{H}(D^*(\mu), D^*(\mu)) \) with \( \mu(H) = \alpha^*(\mu) \), but at the same time \( H \in \mathcal{H}(D^*(\mu), D^*(\mu)) \) implies \( D^*(\mu) \subseteq \operatorname{aff}(D^*(\mu)) \subseteq \partial(H) \). In particular, we see that also \( \operatorname{aff}(D^*(\mu)) \) is covered by our collection of halfspaces, and any point in the median set \( x \in D^*(\mu) \) is contained in the boundary of \( H \) for each such \( H \). We conclude that \( \mathbb{R}^d \) can be covered by halfspaces \( H \) from \( \mathcal{H}(x) \) with \( \mu(H) = D(x; \mu) \), as desired.

A.8 Proof of Example 2

Throughout the proof we adopt the following notation. For \( m = 1, 2, \ldots \) and points \( x_1, \ldots, x_m \in \mathbb{R}^2 \), by \( x_1 \ldots x_m \) we mean the polygon with vertices \( x_1, \ldots, x_m \). For \( m = 2 \) we obtain a line segment \( x_1x_2 \). Where no confusion arises, by \( x_1x_2 \) we also denote the length of that line segment. For polygons with non-empty interior, or more generally, for measurable sets \( V \subseteq \mathbb{R}^2 \) we denote by \( \lambda(V) \) the two-dimensional Lebesgue measure (the area) of \( V \). Recall that for \( a, b \in \mathbb{R}^2 \) we write \( [a, b] \) for the closed line segment between points \( a \) and \( b \) (that is, \( ab \)), \( (a, b) = [a, b] \setminus \{a, b\} \) is the corresponding open line segment, and \( (a, b) \) the infinite line spanned by \( a \) and \( b \).

We suppose the distances between the origin \( o = (0, 0)^T \) and the points \( x_c \) and \( y_c \) take values in ranges \( x \in (0, 1/4) \) and \( y \in (0, 1/5) \), respectively. Set \( d = (0, -1)^T \). Because \( \mu \) is a uniform distribution on \( S \) with total mass \( \lambda(S) \), the \( \mu \)-mass of any measurable set \( V \) is proportional to \( \lambda(S \cap V) \). For \( z \in (0, y) \), \( z_c = (0, -z)^T \), and \( \theta \in (-\pi/2, \pi/2] \) denote by \( l_{z,\theta} \) the infinite line \( l(z_c, z_c + (\cos(\theta - \pi/2), \sin(\theta - \pi/2))^T) \) that passes through \( z_c \) and determines the angle \( \theta \) with line \( (a, d) \). In Fig. 6 the line \( l_{z,\theta} \) is drawn for \( \theta = \pi/4 - 3/10 \). Any \( l_{z,\theta} \) determines two closed halfplanes. We write \( H_{z,\theta}^+ \) for the one that contains \( d \), and write \( H_{z,\theta}^- \) for the complementary closed halfplane. Denote

\[
\lambda_{z}^{\theta+} = \lambda(S \cap H_{z,\theta}^+) \quad \text{and} \quad \lambda_{z}^{\theta-} = \lambda(S \cap H_{z,\theta}^-).
\]

By Lemma 1 a minimizing halfplane of \( \mu \) at \( z_c \) exists. In order to determine one, we find \( \theta \) that minimizes \( \min \{ \lambda_{z}^{\theta+}, \lambda_{z}^{\theta-} \} \). The condition \( \lambda_{z}^{\theta+} + \lambda_{z}^{\theta-} = \lambda(S) \) reduces the problem to the search for the minimum and the maximum of \( \lambda_{z}^{\theta-} \). Because of the symmetry of the figure \( S \), it is enough to consider \( \theta \in [0, \pi/2] \).

As depicted in Fig. 5, denote by \( o_l \) and \( o_r \) the points of intersection of the line \( l(o, (0, 1)^T) \) with \( L(a, b) \) and \( L(a, c) \), respectively. Computing the lengths of the corresponding line segments we obtain \( oo_r = 2dc/3 = 2/\sqrt{3} \). Using the similarity of
triangles, \( axc : ao = xxr : oor, \) which together with \( ao = 2 \) and \( axc = 2 - x \) gives

\[
x_{c}x_{r} = x_{c}x_{l} = \frac{2 - x}{\sqrt{3}}.
\]

Further, we obtain \( x : y = (oor - x_{c}x_{r}) : (y_{c}y_{r} - oor), \) and consequently

\[
y_{c}y_{r} = y_{c}y_{l} = \frac{y + 2}{\sqrt{3}}.
\]

**Part (I):** \( y_{c} \in D^{*} (\mu) \) and \( y_{c} \) **satisfies the covering property** (4)

We start by considering those values of \( \theta \in [0, \pi/2] \) that correspond to the change of the behaviour in the formula for the area \( \lambda_{y}^{\theta+}. \) In the considered range for the angle \( \theta, \) this happens twice: (i) at \( \theta_{1} \) when the line \( l_{y,\theta_{1}} \) passes through the vertex \( c, \) see the left hand panel of Fig. 5; and (ii) at \( \theta_{2} \) when the line \( l_{y,\theta_{2}} \) passes through the point \( x_{l}, \) see the right hand panel of Fig. 5. Denote \( e = l_{y,\theta_{1}} \cap l(a, b) \) and \( f = l_{y,\theta_{2}} \cap l(b, c). \) Note that \( \theta_{1} \) and \( \theta_{2} \) correspond to \( f = c \) and \( e = x_{l}, \) respectively. A simple computation yields

\[
\tan \theta_{1} = \frac{dc/y_{c}d}{\sqrt{3}/(1-y)},
\]

\[
\tan \theta_{2} = \frac{x_{c}x_{l}/y_{c}x_{c}}{(2-x)/(\sqrt{3}(x+y))).
\]

We see that \( \theta_{1} \to \pi/3 \) and \( \theta_{2} \to \pi/2 \) as \( x \) and \( y \) both decrease to 0. Thus, for \( x \) and \( y \) small enough, \( \theta_{1} < \theta_{2}; \) in particular, it is straightforward to establish that for our choices of \( x \in (0, 1/4) \) and \( y \in (0, 1/5), \) we have \( \theta_{1} < \theta_{2}. \) We distinguish three cases, according to the value of \( \theta, \) and evaluate the function \( \lambda_{y}^{\theta+} \) in each situation.
Case (i): $0 \leq \theta < \theta_1$.

For $x'$ the point of intersection of $L(e, f)$ and $L(x_l, x_r)$, we derive $\lambda_y^{\theta+}$ by computing

$$\lambda_y^{\theta+} = \lambda(ebf) - \lambda(x'x_ly_ly_c). \quad (11)$$

Denote $\lambda_x = \lambda(oo_rx_rx_c)$ and $\lambda_y = \lambda(y_cy_or_o)$. Then $\lambda_x = (x_cx_r + oor_o) \cdot x/2 = x(4 - x)/(2\sqrt{3})$ and $\lambda_y = (y_cy_r + oor_o) \cdot y/2 = y(4 + y)/(2\sqrt{3})$. Denote by $h$ the length of the normal from $e$ to line $l(y_l, y_c)$. We compute $h : (h + 1 - y) = y_cy_l : bf$. On the other hand, $bf = bd + df = \sqrt{3} + (y_cd) \tan \theta = \sqrt{3} + (1 - y) \tan \theta$. From here, we get $h = (y + 2)/(1 + \sqrt{3} \tan \theta)$, and consequently

$$\lambda(ebf) = (h + 1 - y) \frac{bf}{2} = \frac{(\sqrt{3} + (1 - y) \tan \theta)^2}{2\left(\frac{1}{\sqrt{3}} + \tan \theta\right)}. \quad (12)$$

Since $x_cy_c = x + y$ and the angle between $l(y_c, x')$ and $l(y_c, x_c)$ equals $\theta$, it holds true that $\lambda(x_cx'y_c) = (x + y)^2 \tan \theta/2$ and

$$\lambda(x'x_ly_ly_c) = \lambda_x + \lambda_y - \lambda(x_cx'y_c) = \frac{(4 - x)x}{2\sqrt{3}} + \frac{(4 + y)y}{2\sqrt{3}} - \frac{\tan \theta}{2}(x + y)^2. \quad (13)$$

Finally, substituting (12) and (13) into (11) gives

$$\lambda_y^{\theta+} = \frac{(\sqrt{3} + (1 - y) \tan \theta)^2}{2\left(\frac{1}{\sqrt{3}} + \tan \theta\right)} - \frac{2}{\sqrt{3}}(x + y) - \frac{1}{2\sqrt{3}}(y^2 - x^2) + \frac{\tan \theta}{2}(x + y)^2.$$
Now we take $t = 1 / \sqrt{3} + \tan \theta$, and substitute $\tan \theta$ by $t - 1 / \sqrt{3}$ in the last equation. Consolidating the terms with $t$, we get

$$\lambda_y^{\theta^+} = \frac{1}{\sqrt{3}}(2 + y)(1 - x - 2y) + F_1(t) \quad (14)$$

where

$$F_1(t) = \frac{(2 + y)^2}{6} t + \left(\frac{1}{2} + (1 - y)^2 + (x + y)^2\right) \frac{t}{2}.$$ 

The derivative of $F(t)$ is given by

$$F'_1(t) = \frac{1}{2} - \frac{(2 + y)^2}{6} \frac{1}{t^2}.$$ 

Condition $F'_1(t) = 0$ gives $t^2 = (2 + y) / \left(3 \left((1 - y)^2 + (x + y)^2\right)\right)$. Denote

$$t_m = \frac{2 + y}{\sqrt{3} \left((1 - y)^2 + (x + y)^2\right)}.$$ 

The derivative $F'_1(t)$ is negative for $t < t_m$ and positive for $t > t_m$. Therefore, the function $F_1(t)$ decreases with $t$ until $t = t_m$, and then increases again, so it is only left to check whether $t_m < \tan \theta_1 + 1 / \sqrt{3}$. In order to do so, we should verify

$$\frac{2 + y}{\sqrt{3} \left((1 - y)^2 + (x + y)^2\right)} < \frac{\sqrt{3}}{1 - y} + \frac{1}{\sqrt{3}} = \frac{4 - y}{\sqrt{3}(1 - y)},$$

which can be rewritten as

$$\frac{2 + y}{\sqrt{1 + \left(\frac{x + y}{1 - y}\right)^2}} < 4 - y. \quad (15)$$

Since the left hand side of the last equation is smaller than $2 + y$, it is enough to show $2 + y < 4 - y$, which is satisfied because we started with $y < 1 / 5$. We conclude that $F_1(t)$ attains its minimum value at $t = t_m$, or equivalently, at $\tan \theta_{\min} = t_m - 1 / \sqrt{3}$. The appropriate minimum area is given by

$$\lambda_y^{\theta_{\min}^+} = \frac{1}{\sqrt{3}}(2 + y)(1 - x - 2y) + \frac{1}{\sqrt{3}}(2 + y)\sqrt{(1 - y)^2 + (x + y)^2}. \quad (16)$$

**Case (ii): $\theta_1 \leq \theta < \theta_2$.**

Now $e \in L[a, x_1]$ and $f \notin L(d, c)$, see the right hand panel of Fig. 6. Denote $g = L[y_c, f] \cap L[a, c]$. In order to get the formula for $\lambda_y^{\theta^+}$ in this case, one simply
subtracts $\lambda(gcf)$ from formula (14). Denote by $h$ the orthogonal projection of $g$ onto $l(d, f)$. Then $\tan \theta = fd/y, d = (fc + \sqrt{3})/(1 - y)$, so $fc = (1 - y)\tan \theta - \sqrt{3}$, and $hc = hg/\sqrt{3}$. At the same time, $\tan \theta = hf/hg = (hc + cf)/hg = cf/hg + 1/\sqrt{3}$, implying $hg = cf/(\tan \theta - 1/\sqrt{3})$. Now we may calculate $\lambda(gcf)$ as $cf \cdot hg/2$, which, after the substitution $t = 1/\sqrt{3} + \tan \theta$, gives

$$
\lambda(gcf) = \frac{(1 - y)^2}{2} + \frac{(2 + y)^2}{6} \frac{1}{t - \frac{2}{\sqrt{3}}} - \sqrt{3}(1 - y).
$$

For $\theta = \theta_1$ one gets $\lambda(gcf) = 0$, as expected. Finally, we derive

$$
\lambda_y^{\theta+} = \frac{1}{\sqrt{3}} (5 - 6y - 2y^2 - 2x - xy) + F_2(t),
$$

where

$$
F_2(t) = (x + y) \frac{t}{2} + \frac{(2 + y)^2}{6} \frac{1}{t - \frac{2}{\sqrt{3}}}.
$$

The above formula holds true for $\tan \theta_1 + 1/\sqrt{3} \leq t < \tan \theta_2 + 1/\sqrt{3}$. By introducing $s = t - \tan \theta_1 - 1/\sqrt{3} \geq 0$, we may write

$$
F_2(t) = \frac{(x + y)^2}{2} - \frac{2 - y}{2\sqrt{3}} + G(s),
$$

where

$$
G(s) = \frac{(x + y)^2}{2} - \frac{(2 + y)^2}{3\sqrt{3}} \frac{1}{(s + \sqrt{3})^2 - \frac{1}{3}}.
$$

One may investigate $G(s)$ instead of $F_2(t)$, because $t \mapsto s$ is a simple shift by a constant. The derivative of $G(s)$,

$$
G'(s) = \frac{(x + y)^2}{2} + \frac{2(2 + y)^2}{3\sqrt{3}} \frac{1}{\left((s + \sqrt{3})^2 - \frac{1}{3}\right)^2} \left(s + \sqrt{3} \frac{1 - y}{1 - y}\right),
$$

is always positive for $s \geq 0$, so $\lambda_y^{\theta+}$ grows with $\theta$ for $\theta_1 \leq \theta < \theta_1$.

Case (iii): $\theta_1 \leq \theta \leq \pi/2$.

As $e \in L[x_l, y_l]$, $\lambda_y^{\theta+}$ obviously grows with increasing $\theta$.

Summary of Part (I)

Now that we established the behaviour of the mapping $\theta \mapsto \lambda_y^{\theta+}$, we are able to find its extreme values. The value of $\lambda_y^{\theta+}$ decreases for $0 \leq \theta < \theta_{\text{min}}$ and increases
In particular, it is easy to show that for \( x = \frac{\lambda \pi}{123} \), \( \mu \) must be a median of \( \lambda \). Consider to conclude the existence of \( x \) and \( y \), such that

\[
\lambda = \frac{\lambda \pi}{123} = \frac{\lambda \pi}{2} = \frac{\lambda \pi}{2}
\]

Note that

\[
\lambda = \frac{\lambda \pi}{2} = \frac{\lambda \pi}{2} = \frac{\lambda \pi}{2}
\]

and there are three minimizing halfspaces of \( \lambda \). As \( \lambda \pi/2 < \lambda \), the intermediate value theorem (Dudley 2002, Problem 2.2.14(d)) allows us to conclude the existence of \( y \) in \( (0, 1/5) \) such that \( f(x, y) = 0 \) for each \( x \in (0, 1/4) \).

Finally, from all the considerations above, we conclude that for each \( x < 1/4 \), there exists \( y \) such that \( D(y_c; \mu) = \lambda(ax_1x_r) \) and there are three minimizing halfspaces of \( \mu \) at \( y_c \), being \( H_{\theta_{\min}^+} \), \( H_{\theta_{\min}^-} \), and \( H_{\theta_{\min}^{1/2}} \). Obviously, those halfspaces cover the whole \( \mathbb{R}^2 \), so \( y_c \) satisfies the covering property (4), and by part (i) of Theorem 2, \( y_c \) must be a median of \( \mu \). The course of the function \( \lambda_{\theta_{\min}} \) can be observed in Fig. 7.

![Fig. 7](image-url)

**Fig. 7** Example 2: Functions \( \lambda_{\theta_{\min}} \) (blue curve) and \( \lambda_{\theta_{\min}} \) (thick orange curve) on their full domain \( [0, 2\pi] \) (left panel) and the same functions zoomed into the interval \([0, \pi/2]\) (right panel). Both functions attain the same minimum value. On the right hand panel, the point \( \theta_{\min} \) where \( \lambda_{\theta_{\min}} \) attains its minimum, along with the points \( \theta_1 \) and \( \theta_2 \) of non-differentiability of function \( \lambda_{\theta_{\min}} \) are marked by dashed vertical lines.
Halfspace depth for general measures: the ray basis theorem...

Fig. 8 Proof of Example 2: A scheme of the configuration that corresponds to Part (II) of the proof, with point $z_c \in D^*(\mu)$ displayed (left panel), and the region where the additional median $w$ is searched for (right panel).

Part (II): $\dim (D^*(\mu)) = 2$.

We first show that there exists a point $z_c = (0, -z)^T \neq y_c$ in $\mathbb{R}^2$ that is also a halfspace median of $\mu$. For that reason, we use the results from Part (I) to calculate $\lambda_z^{\theta+}$. We consider only $x$ and $y$ from Part (I) that satisfy $\lambda^{\pi/2-} = \lambda_{\theta+}^{\text{min}} = D(y_c; \mu) = \alpha^*(\mu)$.

Denote $q = x_l y_l \cap x_l y_r$ and consider point $z_c = (0, -z)^T$ for $z \in (0, y)$ on the line segment $L(q, y_c)$, see the left panel of Fig. 8. Because $z_c \in L(q, y_c)$, the line $l(x_l, z_c)$ intersects $L[y_c, y_r]$. Denote $\delta = y - z > 0$ and $y' = l(x_l, z_c) \cap L[y_c, y_r]$.

Denote $x_z = l_z,\theta \cap l(x_l, x_r)$, $y_z = l_z,\theta \cap l(y_l, y_r)$, $e_z = l_z,\theta \cap l(a, b)$ and $f_z = l_z,\theta \cap l(b, c)$. Consider angles $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\theta}_3$ corresponding to cases $f_z = c$, $e_z = x_l$ and $y_z = y_r$, respectively. It is not difficult to calculate

\[
\tan \hat{\theta}_1 = \frac{dc}{zd} = \sqrt{3}/(1 - z),
\]

\[
\tan \hat{\theta}_2 = \frac{x_c x_l}{z_c x_c} = (2 - x)/(\sqrt{3}(x + z)),
\]

\[
\tan \hat{\theta}_3 = \frac{y_c y_r}{y_z z_c} = (y + 2)/(\sqrt{3}\delta).
\]

If $\delta$ is small enough, then $\hat{\theta}_1 < \hat{\theta}_2 < \hat{\theta}_3$. We consider four different cases and calculate $\lambda_z^{\theta+}$.

For $0 \leq \theta < \hat{\theta}_1$, one obtains $\lambda_z^{\theta+}$ by substituting $y$ by $z$ in (14), and subsequently subtracting $\lambda(y_l y_z z_c z_l)$, where $z_l = l_z,\pi/2 \cap l(a, b)$. Note that $\lambda(z_c y_c y_z) = \delta^2 \tan \theta/2$ and $\lambda(y_l y_c y_z z_l) = (4 + y + z)\delta/(2\sqrt{3})$, so

\[
\lambda(y_l y_z z_c z_l) = \frac{\delta^2}{2} t + \frac{4 + y + z}{2\sqrt{3}} \delta - \frac{1}{2\sqrt{3}} \delta^2,
\]

where $t = 1/\sqrt{3} + \tan \theta$. 

Springer
**Case (i):** $0 \leq \theta < \tilde{\theta}_1$.

The previous calculation leads to

\[
\lambda^\theta_z = \frac{1}{\sqrt{3}} \left( 2 - 3z - 2z^2 - 2x - xz - 2\delta - \frac{y + z}{2}\delta + \frac{\delta^2}{2} \right) + \tilde{F}_1(t),
\]

where

\[
\tilde{F}_1(t) = \frac{(2 + z)^2}{6} \frac{1}{t} + \left( (1 - z)^2 + (x + z)^2 - \delta^2 \right) \frac{t}{2}.
\]

For $\delta$ small, $\tilde{F}_1(t)$ behaves similarly to $F_1(t)$. In that case $\lambda^\theta_z$ decreases with $\theta$ for small positive values of $\theta$, reaches its minimum value at

\[
\tan \tilde{\theta}_{\min} = \frac{2 + z}{\sqrt{3} \left( (1 - z)^2 + (x + z)^2 - \delta^2 \right)} - \frac{1}{\sqrt{3}},
\]

and then increases again as $\theta$ grows. We also check $\tan \tilde{\theta}_{\min} < \tan \tilde{\theta}_1$ in the same way as we did in Part (I) of this proof — analogously to (15), we should verify

\[
\frac{2 + z}{\sqrt{1 + \left( \frac{x + z}{1 - z} \right)^2 - \frac{\delta^2}{(1 - z)^2}}} < 4 - z.
\]

For $\delta$ positive small enough the left hand side of the previous display is smaller than $2 + z$, and we may conclude $\tilde{\theta}_{\min} < \tilde{\theta}_1$.

**Case (ii):** $\tilde{\theta}_1 \leq \theta < \tilde{\theta}_2$.

In this situation, analogously to Case (i) we substitute $y$ by $z$ in (17) and subtract (21) to obtain

\[
\lambda^\theta_z = \frac{1}{\sqrt{3}} \left( 5 - 6z - 2z^2 - 2x - xz - 2\delta - \frac{z + y}{2}\delta + \frac{\delta^2}{2} \right) + \tilde{F}_2(t),
\]

where

\[
\tilde{F}_2(t) = \left( (x + z)^2 - \delta^2 \right) \frac{t}{2} + \frac{(2 + z)^2}{6} \left( \frac{1}{t} - \frac{1}{t - \frac{2}{\sqrt{3}}} \right).
\]

We introduce again $s = t - \tan \tilde{\theta}_1 - 1/\sqrt{3} \geq 0$ and investigate $\tilde{G}(s)$, analogously as in (18) and (19), instead of $\tilde{F}_2(t)$. As in (20), we obtain the derivative of $\tilde{G}(s)$ of the form

\[
\tilde{G}'(s) = \frac{(x + z)^2 - \delta^2}{2} + \frac{2(2 + z)^2}{3\sqrt{3}} \left( \left( s + \frac{\sqrt{3}}{1 - z} \right)^2 - \frac{1}{3} \right)^{-2} \left( s + \frac{\sqrt{3}}{1 - z} \right).
\]
which is positive for small δ, meaning that λzθ+ increases as a function of θ ∈ [θ1, θ2).

**Case (iii):** \( \widetilde{θ}_2 \leq θ < \widetilde{θ}_3 \).

For these values of the parameter, it is easy to see that λzθ+ grows with increasing θ, for the same reason as in Part (I) of this proof.

**Case (iv):** \( \widetilde{θ}_3 \leq θ \leq \frac{π}{2} \).

In this range for θ, the function λzθ+ is clearly constant, and equal to λ(y, y)bc.

**Summary of Part (II)**

We conclude our analysis by observing that λzθ+ attains its minimum at \( θ = \widetilde{θ}_{\min} \), and its maximum at \( θ = π/2 \). Because \( \widetilde{θ}_{\min} \in [0, \widetilde{θ}_1) \), \( y_c \) lies in the interior of \( H_z^{\theta_{\min}+} \) and therefore \( λ_z^{\theta_{\min}+} > λ_y^{\theta_{\min}+} = λ_y^{π/2−} = α^*(\mu) \). On the other hand, \( λ_y^{π/2−} = λ_c π/2− = α^*(\mu) \). Because by Part (I) we know that \( y_c \) is a median of \( μ \), this means that \( z_c \) and \( y_c \) have the same depth, and the set \( D^*(\mu) \) contains at least two distinct points \( z_c \) and \( y_c \). The minimizing halfspaces of \( μ \) at \( z_c \) are those determined by the angle \( \theta \in [-π/2, -\widetilde{θ}_3) \cup [\widetilde{θ}_3, π/2] \). Therefore, it is impossible to cover \( \mathbb{R}^2 \) with minimizing halfspaces of \( μ \) at point \( z_c \), meaning that \( z_c \in D^*(\mu) \) is a point that fails to satisfy the covering property (4).

Due to the convexity of the median set \( D^*(\mu) \) we already know that the line segment \( L[y_c, z_c] \) is contained in \( D^*(\mu) \). To see that the set \( D^*(\mu) \) is in fact two-dimensional, now we find an additional median of \( μ \) of the form \( w = (−ε, \tau)^T \in ((−∞, 0) \times (−y, −z)) \setminus H_z^{θ_{\min}+} \), see the right hand panel of Fig. 8. The smoothness of \( μ \) implies that for \( \tau \) fixed, \( \inf \{ μ(H) : H \in \mathcal{H}(w) \text{ and } \{ y_c, z_c \} \subset H^C \} \rightarrow 1/2 \) as \( ε \rightarrow 0 \). Since \( D(w; μ) ≤ α^*(μ) < 1/2 \) we obtain that for any \( \tau \) there exists \( ε \) small enough so that each minimizing halfspace of \( μ \) at \( w \) has to contain at least one of the points \( y_c \) and \( z_c \). Each halfspace \( H \in \mathcal{H}(w) \), whose boundary passes through \( w \) that contains either \( y_c \) or \( z_c \), however, must have \( μ \)-mass at least \( α^*(μ) \), since both \( y_c \) and \( z_c \) belong to \( D^*(μ) \). Therefore, for any \( \tau \in (−y, −z) \) and all \( ε \) small enough, we conclude that the point \( w = (−ε, \tau)^T \) is contained in the median set \( D^*(μ) \), and the latter set must therefore be two-dimensional.

**Part (III): The single point that satisfies the covering property (4) is \( y_c \).**

In Part (I) of this proof we demonstrated that \( y_c \) satisfies (4). The fact that no other point in \( \mathbb{R}^2 \) shares that property is a consequence of Theorem 12 proved in Sect. 4.1, whose assumptions are satisfied by \( y_c \).

**A.9 Proof of Theorem 9**

First note that all the considered sets are convex, meaning that each of them is of dimension \( d \) if and only if its interior is non-empty (Schneider 2014, Theorem 1.1.13). The equality \( U^\circ_a(μ) = \text{int}(U_a(μ)) \) follows from Theorem 6; the interior of any set is a subset of its closure, which implies \( \text{int}(U_a(μ)) \subseteq \text{cl}(U_a(μ)) \); the floating body \( U_a^{FB}(μ) \) is an intersection of a larger collection of halfspaces than the depth region \( U_a(μ) \), meaning that \( U_a^{FB}(μ) \subseteq U_a(μ) \). We prove the remaining non-trivial statements of the proposition in several steps.

**Inclusion** \( \text{cl}(U_a(μ)) \subseteq U_a^{FB}(μ) \). Suppose that there exists \( x \in U_a(μ) \setminus U_a^{FB}(μ) \). Since \( x \notin U_a^{FB}(μ) \), there must exist \( H_{y, v} \in \mathcal{H} \) such that \( x \notin H_{y, v} \) and \( \mu(H_{y, v}) ≤ α \).
Its complement $H_{y,v}^C$ is an open halfspace whose boundary passes through $y$ and has inner normal $-v$, i.e. $H_{y,v}^C = \text{int}(H_{y,-v})$. We know that $x \in \text{int}(H_{y,-v})$ and $\mu(\text{int}(H_{y,-v})) = \mu(H_{y,v}^C) \leq \alpha$. Therefore, $H_{x,-v} \subset \text{int}(H_{y,-v})$ and consequently $\mu(H_{x,-v}) \leq \mu(\text{int}(H_{y,-v})) \leq \alpha$, which is in contradiction with $D(x; \mu) > \alpha$, as well as $x \in U_\alpha(\mu)$. We conclude that $U_\alpha(\mu) \subseteq U_\alpha^{FB}(\mu)$ and consequently $\text{cl}(U_\alpha(\mu)) \subseteq U_\alpha^{FB}(\mu)$, because $U_\alpha^{FB}(\mu)$ is closed.

**Part (i):** *Inclusion* $U_\alpha^0(\mu) \subseteq \{ x \in \mathbb{R}^d : D^0(x; \mu) > \alpha \}$. Pick $x \in U_\alpha^0(\mu)$, and suppose that $D^0(x; \mu) = \beta \leq \alpha$. Analogously to Lemma 1 it is possible to show that if $D^0(x; \mu) = \beta$, there must exist $G \in \mathcal{H}(x)$ such that $\mu(\text{int}(G)) = \beta$. Define $H = (\text{int}(G))^\mathcal{C} \in \mathcal{H}$ and note that $x \in \partial(H)$ and $\mu(H^C) = \mu(\text{int}(G)) = \beta \leq \alpha$. We reached a contradiction, since for such a halfspace $H$ the point $x$ must be contained in $\text{int}(H)$.

**Part (ii):** *Inclusion* $U_\alpha^{FB}(\mu) \subseteq \text{cl}(U_\alpha(\mu))$ if $\text{int}(U_\alpha(\mu)) \neq \emptyset$. Note that $\text{int}(U_\alpha(\mu)) \subseteq \text{cl}(U_\alpha(\mu)) \subseteq U_\alpha^{FB}(\mu)$, meaning that $\text{int}(U_\alpha^{FB}(\mu)) \neq \emptyset$. Therefore, if $U_\alpha^{FB}(\mu) \setminus \text{cl}(U_\alpha(\mu))$ is non-empty, there must exist $x \in \text{int}(U_\alpha^{FB}(\mu)) \setminus \text{cl}(U_\alpha(\mu))$. To see this, denote by $y$ any point in $U_\alpha^{FB}(\mu) \setminus \text{cl}(U_\alpha(\mu))$. Since $U_\alpha^{FB}(\mu)$ is full-dimensional and convex, there exists an open ball $B_1 \subset U_\alpha^{FB}(\mu)$ (Schneider 2014, Theorem 1.1.13), and $\text{conv}(B_1 \cup \{y\}) \subset U_\alpha^{FB}(\mu)$. Since $\text{cl}(U_\alpha(\mu))$ is a closed set that does not contain $y$, there exists an open ball $B_2$ containing $y$ that does not intersect $\text{cl}(U_\alpha(\mu))$. The set $\text{conv}(B_1 \cup \{y\}) \cap B_2$ is then a convex, full-dimensional subset of $U_\alpha^{FB}(\mu) \setminus \text{cl}(U_\alpha(\mu))$, and as such has to contain a point in its interior $x \in \text{int}(U_\alpha^{FB}(\mu)) \setminus \text{cl}(U_\alpha(\mu))$, as we needed to show. Now, because $x \notin U_\alpha(\mu)$, Lemma 1 gives that there exists $H \in \mathcal{H}(x)$ such that $\mu(\text{int}(H)) \leq D(x; \mu) \leq \alpha$. Denote by $H_x = (\text{int}(H))^\mathcal{C}$ the closed halfspace that satisfies $\mu(H_x^C) = \mu(\text{int}(H)) \leq \alpha$. We obtain $U_\alpha^{FB}(\mu) \subseteq H_x$, which is in contradiction with $x \in \text{int}(U_\alpha^{FB}(\mu))$.

**Part (iii):** $\text{int}(D_\alpha(\mu)) = \text{int}(U_\alpha(\mu))$. Under the considered condition of $\mu$ having contiguous support at $D_\alpha(\mu)$ the result follows by Theorem 6 and part (ii) of Theorem 5.

**Part (iv):** $U_\alpha^{FB}(\mu) = D_\alpha(\mu)$. In dimension $d = 1$, the equality of the two expressions is straightforward to verify by rewriting the conditions defining both $U_\alpha^{FB}(\mu)$ and $D_\alpha(\mu)$ in terms of the function $F(t) = \mu([t, \infty))$, for $t \in \mathbb{R}$. The general statement follows by projecting $\mu$ via $\pi_u : \mathbb{R}^d \to \mathbb{R} : y \mapsto (y, u)$, for $u \in \mathbb{S}^{d-1}$, into its pushforward measure $\mu_u \in \mathcal{M}(\mathbb{R})$. The image of a halfspace $H_{x,u}$ by $\pi_u$ is the interval $[x, u, \infty)$. Knowing the equality of the two concepts of central regions after projecting into $\mathbb{R}$, it is enough to realise that $D_\alpha(\mu) = \bigcap_{u \in \mathbb{S}^{d-1}} \pi_u^{-1}(D_\alpha(\mu_u))$ for $\pi_u^{-1}$ the inverse map to $\pi_u$, and analogously for the floating body $U_\alpha^{FB}(\mu)$. For details see Dyckerhoff (2004, Theorem 2).

**Part (v):** $\text{cl}(U_\alpha(\mu)) = D_\alpha(\mu)$. Using part (iii) and the assumption of non-empty interior of both involved sets, Schneider (2014, Theorem 1.1.15) ensures that $D_\alpha(\mu) = \text{cl}(\text{int}(D_\alpha(\mu))) = \text{cl}(\text{int}(U_\alpha(\mu))) = \text{cl}(U_\alpha(\mu))$ as needed to verify.
A.10 Proof of Lemma 10

Suppose that \( \text{int} (U_a(\mu)) = \emptyset \). We may apply Theorem 6 with \( F = U_a(\mu) \) and conclude that there exists \( H \in \mathcal{H} \) such that \( U_a(\mu) \subset \text{bd} (H) \) and \( \mu(\text{int} (H)) \leq \alpha \). At the same time, existence of \( x \in U_a(\mu) \) implies \( \mu(H) \geq D(x; \mu) > \alpha \) and consequently \( \mu(\text{bd} (H)) > 0 \). Therefore, for any \( x \in U_a(\mu), \mu \) is not smooth at \( x \).

A.11 Proof of Theorem 11

We distinguish two cases according to whether the median set \( D^* (\mu) \) is full-dimensional, or not.

**Case (i):** \( \text{int} (D^* (\mu)) = \emptyset \). We write \( a = \gamma^*(\mu) \) and start with \( U_a(\mu) = \bigcup_{\beta > a} D^*_\beta(\mu) \). Since \( \text{int} (D^*_\beta(\mu)) = \emptyset \) for each \( \beta > a \) and sets \( D^*_\beta(\mu) \) are nested, we conclude that also \( \text{int} (U_a(\mu)) = \emptyset \). Choose any \( x \in \text{aff} (U_a(\mu)) \). From Theorem 6 applied with \( F = U_a(\mu) \) it follows that \( \mathbb{R}^d \) can be covered by halfspaces \( H \in \mathcal{H}(U_a(\mu)) \subset \mathcal{H}(x) \) such that \( \mu(\text{int} (H)) \leq a = \gamma^*(\mu) \) for each such \( H \). Since \( \text{aff} (D^* (\mu)) \subset \text{aff} (U_a(\mu)) \), we obtain that also all points of \( D^* (\mu) \) are possible to be used for such a covering, as we needed to show.

**Case (ii):** \( \text{int} (D^* (\mu)) \neq \emptyset \). In this case, \( \gamma^*(\mu) = \alpha^*(\mu) \). Denote \( S_0 = D^* (\mu) \). This is a full-dimensional convex set, and for its barycentre \( x_0 \in S_0 \) and \( \nu_0 \in \mathcal{M} (\mathbb{R}^d) \) the uniform probability distribution on \( S_0 \) we know that \( D(x_0; \nu_0) > e^{-1} \), due to a result of Grünbaum, see Nagy et al. (2019, Theorem 3). Let \( H_0 \in \mathcal{H}(x_0) \) be a generalized minimizing halfspace of \( \mu \) at \( x_0 \) in the sense of Lemma 1, i.e. \( \mu(\text{int} (H_0)) \leq \alpha^*(\mu) \). Writing \( \text{vol} (A) \) for the \( d \)-dimensional volume of a measurable set \( A \), this implies \( \nu_0(H_0) = \text{vol}(S_0 \cap H_0)/\text{vol}(S_0) \geq D(x_0; \nu_0) > e^{-1} \). Denote \( S_1 = \text{cl} (S_0 \setminus H_0) \) with \( \text{vol}(S_1) = \text{vol}(S_0)(1 - \nu_0(H_0)) < \text{vol}(S_0)(1 - e^{-1}) \). We iterate the previous procedure and for \( k \geq 1 \) define \( x_k \in S_k \) to be the barycentre of \( S_k \). In each step, we can find a generalized minimizing halfspace \( H_k \in \mathcal{H}(x_k) \) of \( \mu \) at \( x_k \), put \( S_{k+1} = \text{cl} (S_k \setminus H_k) \), and again conclude

\[
\text{vol}(S_k) < \text{vol}(S_0) \left(1 - e^{-1}\right)^k.
\]

Necessarily, \( \text{vol}(S_k) \to 0 \) as \( k \to \infty \). Denote \( S = \bigcap_{k=0}^{\infty} S_k \). Then \( S \subset D^* (\mu) \) is a convex set, \( \text{vol}(S) = 0 \), and consequently \( \text{int} (S) = \emptyset \).

Note that \( D^* (\mu) = S_0 \) satisfies the assumption (5) of our main Lemma 4. Recursively, if \( S_k \) satisfies (5) for some \( k = 0, 1, 2, \ldots \), then the same holds true for \( S_{k+1} \), because

\[
S_{k+1}^c = S_k^c \cup \text{int} (H_k).
\]

By the induction step, \( S_k^c \) is covered by halfspaces \( H \in \mathcal{H} \) with \( \mu(\text{int} (H)) \leq b^*(\mu) \) and \( \text{int} (H) \cap S_{k+1} \subseteq \text{int} (H) \cap S_k = \emptyset \). Adding to that collection the halfspace \( H_k \) that also satisfies \( \mu(\text{int} (H_k)) \leq b^*(\mu) \) and \( \text{int} (H_k) \cap S_{k+1} = \emptyset \), we obtain a covering of \( S_{k+1}^c \) with the desired property, i.e. also \( S_{k+1} \) satisfies (5). We finally show that also \( S \) satisfies (5). Take any \( x \notin S \) and the first index \( k = 0, 1, 2, \ldots \) such that \( x \notin S_k \).
There exists $H \in \mathcal{H}$ with $x \in H, \mu(\text{int}(H)) \leq \alpha$ and $\text{int}(H) \cap S_k = \emptyset$. Since $S \subseteq S_k$, it follows that $\text{int}(H) \cap S = \emptyset$. Lemma 4 can now be applied to the set $S$ to give the desired result, as follows by $\text{int}(S) = \emptyset$.

### A.12 Proof of Example 5

Each point in $\mathbb{R}^2$ is contained in one of the four halfspaces of $\mu$-mass 6 determined by the dashed lines presented in Fig. 9. Therefore $\alpha^*(\mu) \leq 6$. To prove that both points $x$ and $y$ are medians, it is enough to show $D(x; \mu) = D(y; \mu) = 6$. Because of the symmetry of $\mu$, the problem reduces to showing $D(y; \mu) = 6$.

Consider an infinite line $l$ passing through $y$ and denote by $H^+$ and $H^-$ the two closed halfspaces determined by $l$. We need to show that neither $\mu(H^+)$ nor $\mu(H^-)$ decreases below 6 as we rotate $l$ around $y$. Equivalently, as $\mu(\mathbb{R}^2) = 14$, it is enough to show that $6 \leq \mu(H^+) \leq 8$ when rotating $l$ by an angle of $\pi$. Several specific positions of $l$ are shown in the left hand panel of Fig. 10; the corresponding halfspace $H^+$ for line $l_1$ is the one that does not contain the origin $o$. We start from $l = l_1$ when $\mu(H^+) = 6$. As we rotate $l$ around $y$ counter-clockwise, $\mu(H^+)$ remains the same for $l$ being between $l_1$ and $l_2$, since the triangles $A$ and $G$ in Fig. 10 are symmetric with respect to $y$. Continuing the rotation, $\mu(H^+)$ remains constant and then increases from $l = l_3$ to $l = l_4$, when $\mu(H^+) = 7$. For $l$ between $l_4$ and $l_5$, $H^+$ contains the triangles $A, B, I, J, K$, so $\mu(H^+) \geq 6$ and $H^-$ contains $D, E, F, G$, meaning that $\mu(H^-) \geq 6$. For $l = l_5$, $\mu(H^+)$ equals 7 and then decreases until $l = l_6$, when $\mu(H^+) = 6$. Afterwards $\mu(H^+)$ increases again and reaches value 8 for $l = l_7$. Finally, past $l = l_7$ the $\mu$-mass decreases and becomes equal to 6 when $l$ is horizontal and then increases again and becomes 8 for $l = -l_1$, where “−” in front of $l_1$ means that the orientation of the halfspace $H^+$ is the opposite from that in the beginning of
Consider the covering medians of $\mu$. Certainly $\gamma^*(\mu) = 6$. We are able to cover the whole space by halfspaces whose boundary contains $y$. To see this, consider the halfspaces determined by the horizontal and the vertical line containing $y$, respectively, and the one determined by line $q$ in Fig. 9. For $x$, the halfspaces are determined by the line $p$, and the horizontal and vertical line. Therefore, $x, y \in C^* (\mu)$. In the right hand panel of Fig. 10 we see that each halfspace whose boundary passes through the origin $o$ and contains point $a$ has $\mu$-mass greater than 6. Therefore, it is not possible to cover $a$ by minimizing halfspaces of $o$, and consequently $o \notin C^* (\mu)$.

A.13 Proof of Theorem 12

For any $y \in \mathbb{R}^d$ denote by $\mathcal{H}_{\text{min}}(y)$ the collection of those halfspaces $H \in \mathcal{H}(y)$ that satisfy $\mu(\text{int}(H)) \leq \gamma^*(\mu)$. In particular, $\mathcal{H}_{\text{min}} = \mathcal{H}_{\text{min}}(x)$. Suppose there is $y \in C^* (\mu)$ such that $y \neq x$. Denote $z = (x + y)/2$. Since $\mathbb{R}^d = \bigcup \{ H \in \mathcal{H}_{\text{min}}(y) \}$, there exists $v \in S^{d-1}$ and $H_{y,v} \in \mathcal{H}_{\text{min}}(y)$ such that $x \in H_{y,v}$. Then $H_{x,v} \subseteq H_{y,v}$ and $\mu(\text{int}(H_{x,v})) \leq \mu(\text{int}(H_{y,v})) \leq \gamma^*(\mu)$, meaning that $H_{x,v} \in \mathcal{H}_{\text{min}}(x)$. If $x \in \text{int}(H_{y,v})$, then $H_{x,v} \subset H_{z,v} \subset \text{int}(H_{y,v})$. Our contiguity assumption then implies $\mu(H_{x,v}) < \mu(H_{z,v}) \leq \mu(\text{int}(H_{y,v})) \leq \gamma^*(\mu) \leq \alpha^*(\mu)$, which contradicts $x \in D^* (\mu)$. Thus, $x \in \text{bd}(H_{y,v})$ and $H_{y,v} \in \mathcal{H}_{\text{min}}(x)$.

Consider any $w \in \mathbb{R}^d$. If $w \in H_{y,v}$, choose $H_w = H_{y,v}$. Otherwise, the sequence of points $z_n = (1 - 1/n)z + w/n, n = 1, 2, \ldots$ converges to $z$ and $z_n \notin H_{y,v}$. For each $z_n$ there is $H_n \in \mathcal{H}_{\text{min}}(y)$ such that $z_n \in H_n$ and $\mu(\text{int}(H_n)) \leq \gamma^*(\mu)$. Using the same argument as for the halfspace $H_{y,v}$ in the first part of the proof, we conclude that $x \notin \text{int}(H_n)$, because of $H_n \in \mathcal{H}_{\text{min}}(y)$ and because of our contiguity assumption. Lemma 15 implies the existence of a convergent subsequence $H_{n_k} \to H_w \in \mathcal{H}(z)$ such...
that $\mu(\text{int}(H_w)) \leq \gamma^*(\mu)$. Note that also $y \notin \text{int}(H_n)$ because $y \in \text{bd}(H_n)$. Using Lemma 15, part (iv), we conclude that $\{x, y\} \cap \text{int}(H_w) = \emptyset$. Because $z \in L(x, y)$ and $z \in \text{bd}(H_w)$, it has to be $\{x, y\} \subset \text{bd}(H_w)$, meaning that $H_w \in \mathcal{H}_{\min}(x) \cap \mathcal{H}_{\min}(y)$. Finally, we conclude that for each $w \in \mathbb{R}^d$, there is $H_w \in \mathcal{H}_{\min}(x) \cap \mathcal{H}_{\min}(y)$ such that $w \in H_w$. Then $\mathcal{H}' = \{H_w: w \in \mathbb{R}^d\} \subseteq \mathcal{H}_{\min}(x)$ covers $\mathbb{R}^d$ by halfspaces whose interior has mass at most $\gamma^*(\mu)$, that at the same time all contain $y \neq x$, which violates the assumption of our theorem.

### A.14 Proof of Corollary 13

The first part of the corollary is a direct consequence of Theorem 5. Because $\mu$ is atomic with finitely many atoms, there are only finitely many unique $\mu$-masses of halfspaces $\mathcal{H}$. Therefore, when applying the Fatou lemma of Lemma 15 in the proof of Lemma 4, one obtains a strict inequality $\mu(\text{int}(H(x, F))) < \alpha$.

As for the second claim, first note that $D_\alpha(\mu)$ is a convex polytope by Laketa and Nagy (2021, Lemma 1). If $D_\alpha(\mu)$ is full-dimensional, each face $F$ of $D_\alpha(\mu)$ is a subset of a $(d - 1)$-dimensional face $\tilde{F}$ of $D_\alpha(\mu)$; if $\dim(D_\alpha(\mu)) < d$, $F \subseteq \tilde{F} = D_\alpha(\mu)$. If we prove our claim for $\tilde{F}$, it is necessarily true also for $F$. Without loss of generality, we may therefore suppose that $F$ is of dimension $\dim(\tilde{F}) = \min\{\dim(D_\alpha(\mu)), d - 1\}$. Let $H(x, F)$ be the halfspace from the first part of the proof. Denote by $\mu_1 \in \mathcal{M}(\mathbb{R}^d)$ the restriction of $\mu$ to $\text{bd}(H(x, F))$ for $H(x, F) \in \mathcal{H}(D_\alpha(\mu), F)$ from the first part of the proof, and consider any $x \in F$. If $D(x; \mu_1) = 0$, then there exists a closed $(d - 1)$-dimensional halfspace $H_1$ in the hyperplane $\text{bd}(H(x, F))$ such that $x \in H_1$ and $\mu_1(H_1) = 0$. The existence of that minimizing halfspace follows because $\mu$ contains only finitely many atoms, thus there are only finitely many possible values of $\mu_1(H)$ for $H \in \mathcal{H}$. Consider now a slight perturbation of the halfspace $H(x, F)$, in the sense that the unit normal $v \in S^{d-1}$ of $H(x, F)$ is perturbed, but (i) the $(d - 2)$-dimensional affine space $\text{relbd}(H_1)$ remains in the boundary of the perturbed halfspace $H' \in \mathcal{H}(x)$, and (ii) $H_1 \subset H'$. Because there are only finitely many atoms of $\mu$, it is certainly possible to obtain $H'$ such that $\mu(H') = \mu(\text{int}(H(x, F))) + \mu(H_1) = \mu(\text{int}(H(x, F))) < \alpha$, which contradicts $x \in F \subseteq D_\alpha(\mu)$. We obtain that $F \subseteq U_\beta(\mu_1)$ for $\beta = 0$. Since $U_0(\mu_1)$ is, again by the assumption of only finitely many atoms of $\mu$, a polytope whose vertices are atoms of $\mu_1$, $F$ has to be contained in a convex hull of at least $\dim(F) + 1$ atoms of $\mu$, all lying in the hyperplane $\text{bd}(H(x, F))$.

### References

Bezdek K, Khan MA (2018) The geometry of homothetic covering and illumination. In: Discrete geometry and symmetry, volume 234 of Springer Proc. Math. Stat. Springer, Cham, pp 1–30

Bobkov SG (2010) Convex bodies and norms associated to convex measures. Probab. Theory Relat. Fields 147(1–2):303–332

Brunel V-E (2019) Concentration of the empirical level sets of Tukey’s halfspace depth. Probab. Theory Relat. Fields 173(3–4):1165–1196

Chernozhukov V, Galichon A, Hallin M, Henry M (2017) Monge-Kantorovich depth, quantiles, ranks and signs. Ann Stat 45(1):223–256
Donoho DL, Gasko M (1992) Breakdown properties of location estimates based on halfspace depth and projected outlyingness. Ann Stat 20(4):1803–1827
Dudley RM (2002) Real analysis and probability, volume 74 of Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge (Revised reprint of the 1989 original)
Dyckerhoff R (2004) Data depths satisfying the projection property. Allg Stat Arch 88(2):163–190
Dyckerhoff R (2017) Convergence of depths and depth-trimmed regions. arXiv preprint arXiv:1611.08721
Grünebaum B (1963) Measures of symmetry for convex sets. In: Proc. sympos. pure math., vol. VII. Amer. Math. Soc., Providence, RI, pp 233–270
He X, Wang G (1997) Convergence of depth contours for multivariate datasets. Ann Stat 25(2):495–504
Kim J (2000) Rate of convergence of depth contours: with application to a multivariate metrically trimmed mean. Stat Probab Lett 49(4):393–400
Laketa P, Nagy S (2021) Reconstruction of atomic measures from their halfspace depth. J Multivar Anal 183:104727
Liu RY, Parelius JM, Singh K (1999) Multivariate analysis by data depth: descriptive statistics, graphics and inference. Ann Stat 27(3):783–858
Liu X, Mosler K, Mozharovskyi P (2019) Fast computation of Tukey trimmed regions and median in dimension $p > 2$. J Comput Graph Stat 28(3):682–697
Liu X, Luo S, Zuo Y (2020) Some results on the computing of Tukey’s halfspace median. Stat Pap 61(1):303–316
Massé J-C (2002) Asymptotics for the Tukey median. J Multivar Anal 81(2):286–300
Massé J-C, Theodorescu R (1994) Halfplane trimming for bivariate distributions. J Multivar Anal 48(2):188–202
Mizera I, Volauf M (2002) Continuity of halfspace depth contours and maximum depth estimators: diagnostics of depth-related methods. J Multivar Anal 83(2):365–388
Nagy S, Dvořák J (2021) Illumination depth. J Comput Graph Stat 30(1):78–90
Nagy S, Schütt C, Werner EM (2019) Halfspace depth and floating body. Stat. Surv. 13:52–118
Nolan D (1992) Asymptotics for multivariate trimming. Stochastic Process. Appl. 42(1):157–169
Patáková Z, Tancer M, Wagner U (2020) Barycentric cuts through a convex body. In: Cabello S, Chen EDZ (eds) 36th international symposium on computational geometry (SoCG 2020), volume 164 of Leibniz international proceedings in informatics (LIPIcs). Dagstuhl, Germany, Schloss Dagstuhl-Leibniz-Zentrum für Informatik, pp 62:1–62:16
Pokorný D, Laketa P, Nagy S (2021) Halfspace depth for general measures: Dupin’s theorem and the uniqueness of the halfspace median (in preparation)
Rousseeuw PJ, Ruts I (1999) The depth function of a population distribution. Metrika 49(3):213–244
Schneider R (2014) Convex bodies: the Brunn-Minkowski theory, volume 151 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge expanded edition
Small CG (1987) Measures of centrality for multivariate and directional distributions. Can J Stat 15(1):31–39
Struyf A, Rousseeuw PJ (1999) Halfspace depth and regression depth characterize the empirical distribution. J Multivar Anal 69(1):135–153
Tukey JW (1975) Mathematics and the picturing of data. In: Proceedings of the international congress of mathematicians (Vancouver, B. C., 1974), vol. 2, Canad. Math. Congress, Montreal, Que, pp 523–531
van der Vaart AW, Wellner JA (1996) Weak convergence and empirical processes. Springer series in statistics. Springer, New York
Wang J (2019) Asymptotics of generalized depth-based spread processes and applications. J Multivar Anal 169:363–380
Wang J, Serfling R (2006) On scale curves for nonparametric description of dispersion. In: Data depth: robust multivariate analysis, computational geometry and applications, vol. 72 of DIMACS Ser Discrete Math Theoret Comput Sci. Amer Math Soc, Providence, RI, pp 37–48
Zuo Y, Serfling R (2000a) General notions of statistical depth function. Ann Stat 28(2):461–482
Zuo Y, Serfling R (2000b) On the performance of some robust nonparametric location measures relative to a general notion of multivariate symmetry. J Stat Plan Inference 84(1–2):55–79
Zuo Y, Serfling R (2000c) Structural properties and convergence results for contours of sample statistical depth functions. Ann Stat 28(2):483–499