A STRUCTURE RESULT FOR BRICKS IN HEISENBERG GROUPS

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Abstract. We show that for a sufficiently big brick $B$ of the $(2n+1)$-dimensional Heisenberg group $H_n$ over the finite field $\mathbb{F}_p$, the product set $B \cdot B$ contains at least $|B|/p$ many cosets of some non trivial subgroup of $H_n$.

1. Introduction

The notion of additive bases occupies a central position in Combinatorial Number Theory. In an additive semigroup $G$, a basis means a subset $A$ of $G$ such that there exists an integer $h$, depending only on $A$, for which any element of $x \in G$ can be written as a sum of $h$ (or at most $h$) members of $A$. The idea has been widely investigated in different structures in which a number of results have been shown. One can quote the celebrated Lagrange theorem in the set of nonnegative integers but also such results in $\sigma$-finite abelian groups [HR].

In an additive structure we will use the notation $A + B = \{a + b : a \in A, b \in B\}$, its extension $hA = A + A + \cdots + A$ ($h$ times) and also their counterparts $A \cdot B$, $A^h$ in a multiplicative structure. In a group we also denote $-A$ (resp. $A^{-1}$) for the set of the inverses of elements of $A$. With this notation, $A$ is a basis in $G$ whenever for some integer $h$ one has $hA = G$ or $A^h = G$ according to the underlying structure. One also defines the notion of doubling constant (resp. squaring constant) of a finite set $A$ that is $|A + A|/|A|$ (resp. $|A \cdot A|/|A|$).

Another aspect concerns inverse results in number theory in which the Freiman theorem has a central place. It asserts that a finite set $A$ with a small doubling constant in an abelian (additive) group $G$ has a sharp structure, namely it is included, as a rather dense subset, in a (generalized) arithmetic progression of cosets of some subgroup of $G$ (cf. [GR]). An important tool for the proof, known as the Bogolyubov-Ruzsa Lemma, is the fact that $2A - 2A$ contains a dense substructure.

According to the preceding discussion, one may consider the general problem of investigating in which conditions on a finite $A$, the sumset $A + A$ (or the product set $A \cdot A$)
contains a rich substructure. In this paper we will focus on Heisenberg groups which give an interesting counterpoint to the commutative case.

Let $p$ be a prime number and $\mathbb{F}$ the field with $p$ elements. We denote by $H_n$ the $(2n+1)$-dimensional Heisenberg linear group over $\mathbb{F}$ formed with the upper triangular square matrices of size $n+2$ of the following kind

$$[\mathbf{x}, \mathbf{y}, z] = \begin{pmatrix}
1 & x & z \\
0 & I_n & y' \\
0 & 0 & 1
\end{pmatrix},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, $x_i, y_i, z \in \mathbb{F}$, $i = 1, 2, \ldots, n$, and $I_n$ is the $n \times n$ identity matrix. We have $|H_n| = p^{2n+1}$. and we recall the product rule in $H_n$:

$$[\mathbf{x}, \mathbf{y}, z][\mathbf{x}', \mathbf{y}', z'] = [\mathbf{x} + \mathbf{x}', \mathbf{y} + \mathbf{y}', \langle \mathbf{x}, \mathbf{y}' \rangle + z + z'],$$

where $\langle \cdot, \cdot \rangle$ is the inner product, that is $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$.

So this set of $(n+2) \times (n+2)$ matrices form a group whose unit is $e = [0, 0, 0]$. As group-theoretical properties of $H_n$, we recall that $H_n$ is non abelian and two-step nilpotent, that is the double commutator satisfies $aba^{-1}b^{-1}cabc^{-1}a^{-1} = e$ for any $a, b, c \in H_n$, where the commutator of $a$ and $b$ is defined as $aba^{-1}b^{-1}$.

The Heisenberg group possesses an interesting structure in which we can prove that in general there is no good model for a subset $A$ with a small squaring constant $|A \cdot A|/|A|$ (see [G], [HH] for more details), unlike for subsets of abelian groups. We should add that the situation is less unusual if we assume that $A$ has a small cubing constant $|A \cdot A \cdot A|/|A|$ (see [T2]).

We now quote the following well-known results.

**Lemma 1.1.** Let $X$ and $Y$ be subsets of a finite (multiplicative) group $G$. If $|X| + |Y| > |G|$ then $G = X \cdot Y$.

The proof follows from the simplest case of the sieve formula:

$$|X \cap \{x\} \cdot Y^{-1}| = |X| + |\{x\} \cdot Y^{-1}| - |X \cup \{x\} \cdot Y^{-1}| \geq |X| + |Y| - |G| > 0.$$

**Lemma 1.2.** Let $X$ and $Y$ be subsets of $\mathbb{F}$. If $X + Y \neq \mathbb{F}$ then $|X + Y| \geq |X| + |Y| - 1$.

This general lower bound for the cardinality of sumsets in $\mathbb{F}$ is known as the Cauchy-Davenport Theorem (see e.g. [TV]).

We deduce from Lemma 1.1 that a sufficient condition ensuring that a subset $A \subseteq H_n$ is a basis is $|A| > |H_n|/2$. Moreover this condition is sharp if $p = 2$ since in that case $H_n$ has
a subgroup of index 2. For $p > 2$, any subset of $H_n$ with cardinality bigger than $|H_n|/p$ is not contained in a coset of a proper subgroup of $H_n$, hence it is a basis for some order $h$ bounded by a function depending only on $p$: indeed by a theorem of Freiman in arbitrary finite groups (see [H], paragraph 4.9), it is known that if $A$ is not included in some coset of some proper subgroup of $H_n$ then $|A \cdot A| \geq 3|A|/2$. From this we deduce by iteration that the $2^j$-fold product set $A^{2^j}$ satisfies $|A^{2^j}| > |H_n|/2$ for $j \geq \ln(p/2)/\ln(3/2)$, hence the result by Lemma [1].

The above discussion shows that any sufficiently dense subset of $H_n$ is a basis. This does not hold true in general for sparse subsets. Another main consideration is that the squaring constant $|A \cdot A|/|A|$ of a $A \subset H_n$ is not necessarily big. So we can ask the following: is it true that for any subset $A$ of $H_n$ which is large enough, the product set $A \cdot A$ always contains some non-trivial substructure of $H_n$? A dual question emerged in $\text{SL}_2(\mathbb{F})$ in [H] (see also [GH]) where it is proved that for any generating subset $A$ of $\text{SL}_2(\mathbb{F})$ such that $|A| < p^{3-\delta}$, one has $|A \cdot A \cdot A| > |A|^{1+\epsilon}$ with $\epsilon$ depending only on $\delta > 0$.

Nevertheless, the structure cannot be handled in general. We will restrict our attention to subsets that will be called bricks.

Let $B \subseteq H_n$, and write the projections of $B$ onto each coordinates by $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n$ and $Z$, i.e. one has $[x, y, z] \in B$, $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$, if and only if $x_i \in X_i$ or $y_i \in Y_i$ for some $i$, or $z \in Z$.

A subset $B$ of $H_n$ is said to be a brick if

$$B = [X, Y, Z] := \{[x, y, z] \text{ such that } x \in X, \ y \in Y, \ z \in Z\}$$

where $X = X_1 \times \cdots \times X_n$ and $Y = Y_1 \times \cdots \times Y_n$ with non empty-subsets $X_i, Y_i \subset \mathbb{F}^*$. If $|Z| > p/2$ then $2Z = \mathbb{F}$ by Lemma [1], hence $B \cdot B = [2X, 2Y, \mathbb{F}]$. This shows that $B \cdot B$ is a union of $|B \cdot B|/p \geq |B|/p$ cosets of the subgroup $[0, 0, \mathbb{F}]$ of $H_n$. Our aim is to partially extend this result under an appropriate assumption on the size of $B$.

**Theorem 1.3.** For every $\epsilon > 0$, there exists a positive integer $n_0$ such that if $n \geq n_0$, $B \subseteq H_n$ is a brick and

$$|B| > |H_n|^{3/4+\epsilon}$$

then there exists a non trivial subgroup $G$ of $H_n$, namely its center $[0, 0, \mathbb{F}]$, such that $B \cdot B$ contains a union of at least $|B|/p$ many cosets of $G$.

We stress the fact that $n_0$ depends only on $\epsilon$ and that this result is valid uniformly in $p$. 
In the case when $B$ has a small squaring constant, namely $|B \cdot B| < 3|B|^2 / 2$, we already observed that $B \cdot B$ is a coset modulo some subgroup of $H_n$ (see [1]). Theorem 1.3 thus provides a partial extension of this fact by giving a substructure result for sufficiently big bricks $B$ in $H_n$.

The statement in Theorem 1.3 can be plainly extended to any subset $B' \subset H_n$ which derives from a brick $B$ by conjugation:

$$B' = P^{-1}BP$$

where $P$ is a given element of $H_n$.

As a remark also in connection with the above-mentioned Helfgott’s result, we notice that for a brick $B$ of the Heisenberg group $H_n$ its squaring set $B \cdot B$ could have a small period, namely the maximal subgroup $P$ such that $B \cdot B \cdot P = B \cdot B$ and at the same time its squaring constant $|B \cdot B| / |B|$ could be small. To see this take the brick $B = \{[x, y, z], 0 \leq x, y < \sqrt{p}/2, z \in \mathbb{F}\}$ in $H_1$ (the general case $n \geq 1$ could be easily derived from this discussion). Then $B$ generates the full group $H_1$ and its cardinality satisfies $|B| \asymp p^2$. Furthermore $B \cdot B = \{[x, y, z], 0 \leq x, y < \sqrt{p}, z \in \mathbb{F}\}$ has cardinality less than $4|B|$ and its period is the subgroup $[0, 0, \mathbb{F}]$. In the opposite direction one can show:

**Theorem 1.4.** There exist an absolute constant $c > 0$ and a constant $\alpha = \alpha(p)$ such that for any brick $B$ there exists a subgroup $G$ of $H_n$ satisfying

$$B \cdot B \cdot G = B \cdot B$$

and

$$\frac{|B \cdot B|}{|B|} \geq c \left( \frac{|B|}{|G|} \right)^\alpha.$$

This lower bound gives either a growing property for $B \cdot B$ in the case where $G$ is small or a rather regular structure for $B \cdot B$ if $G$ is big.

Finally we will show that the exponent $3/4 + \varepsilon$ in Theorem 1.3 cannot be essentially reduced to less than $1/2$:

**Proposition 1.5.** For any $n$ and $p$ there exists a brick $B \subseteq H_n$ such that

$$|B| \geq \frac{\sqrt{p}}{4(2^{1/2}n)^n}|H_n|^{1/2}$$

and the only cosets contained in $B \cdot B$ are cosets of the trivial subgroup of $H_n$.

Choosing $p$ large relative to $n$ in this result implies the desired effect.

2. A big period or a big squaring constant

**Proof of Theorem 1.4.** If $B = [X, Y, Z]$ is a brick in $H_n$ then $|B \cdot B| \geq |2X| |2Y|$ since for any $x \in 2X$ and any $y \in 2Y$ there clearly exists $z \in \mathbb{F}$ such that $[x, y, z] \in B \cdot B$. Thus if $k$ is
the number of components $X_i, Y_j$ such that $1 < |X_i|, |Y_j| \leq p/\sqrt{2}$, then $|B \cdot B| \geq (\sqrt{2})^k |B|$ since for such component $X_i, Y_j$ we have

$$|2X_i| \geq \min(p, 2|X_i| - 1) \geq \min(p, 3|X_i|/2) \geq \sqrt{2}|X_i|$$

and similarly $|2Y_j| \geq \sqrt{2}|Y_j|$ by the Cauchy–Davenport Theorem (cf. Lemma 1.1). If there exist at least two components $X_i$ and $X_j$ with cardinality bigger than $p/\sqrt{2}$ then there are two elements $w_i, w_j$ in $\mathbb{F}$ having at least $p/2$ solutions to the equations $w_i = x_i + x'_i$, $x_i, x'_i \in X_i$, and $w_j = x_j + x'_j$, $x_j, x'_j \in X_j$. Let $X'_i = X_i \cap (w_i - X_i)$ and $X'_j = X_j \cap (w_j - X_j)$. We now observe that $B \cdot B$ contains $[X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_{j-1}, X'_j, X_{j+1}, \ldots, X_n, Y]$. Since $|X'_i|, |X'_j| > p/2$ we have by the additive analogue of Lemma 1.1 $y_iX'_i + y_jX'_j = \mathbb{F}$ for any $y_i, y_j \in \mathbb{F}^*$. It follows that $B \cdot B$ fully covers the union $2X_1 \times \cdots \times \{w_i\} \times \cdots \times \{w_j\} \times \cdots \times 2X_n, 2Y, F$ of cosets of the center $[0, 0, 0, \mathbb{F}]$ of $H_n$. Note also that for the indices $h$ such that $|Y_h| > p/2$ (or $|X_h| > p/2$ for $h \neq i, j$) we have $2Y_h = \mathbb{F}$ (respectively $2X_h = \mathbb{F}$).

We get a similar conclusion in the same way if we assume $|Y_i|, |Y_j| > p/\sqrt{2}$ for some $i \neq j$. Thus if we denote by $\ell$ the numbers of components $X_h$ and $Y_h$ with cardinality bigger than $p/\sqrt{2}$ and if we assume $\ell \geq 3$, then $B \cdot B$ contains at least $(\sqrt{2})^k$ cosets of a big period $G$ of cardinality $p^{\ell-1}$, namely

$$G = [K_1, \ldots, K_n, L_1, \ldots, L_n, \mathbb{F}]$$

where $K_i, L_j = \{0\}$ or $\mathbb{F}$. By the facts that $|B| \leq p^{k+\ell+1}$ and $\ell = \ln |G|/\ln p + 2$, we get

$$\frac{|B \cdot B|}{|B|} \geq (\sqrt{2})^k \geq \frac{1}{4} \left(\frac{|B|}{|G|}\right)^{\frac{\ln 3}{2\ln p}}.$$

If $\ell \leq 2$ this bound still holds with $G = \{0\}$. \hfill \square

3. Fourier analysis for a sum-product estimate

We will use the following sum-product estimate:

**Proposition 3.1.** Let $n, m \in \mathbb{N}$, $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots Y_n \subseteq \mathbb{F}^* = \mathbb{F} \setminus \{0\}$, $Z \subseteq \mathbb{F}$. We have

$$mZ + \sum_{j=1}^{n} X_j \cdot Y_j := \{z_1 + \cdots + z_m + \sum_{j=1}^{n} x_j y_j, \ z_i \in Z, \ x_j \in X_j, \ y_j \in Y_j\} = \mathbb{F},$$

provided

$$|Z|^2 \prod_{i=1}^{n} |X_i||Y_i| > p^{n+2}.$$
Proof. Let $X_i(t)$ (resp. $Y_i(t)$ and $Z(t)$) be the indicator of the set $X_i$ (resp $Y_i$ and $Z$). One defines
\[ f_i(t) = \frac{1}{|X_i|} \sum_{a \in X_i} Y_i(t/a), \quad i = 1, 2, \ldots, n. \]
Notice that $0 \leq f_i(t) \leq 1$, and $f_i(t) > 0$ if and only if $t \in X_i \cdot Y_i$. The Fourier transform of $f_i$ is
\[ \hat{f}_i(r) = \sum_x f_i(x)e(xr) \]
where $e(x) = \exp(2\pi ix/p)$ as usual.

An easy calculation shows that for every $i = 1, 2, \ldots, n$
\[ \hat{f}_i(0) = \frac{1}{|X_i|} \sum_{a \in X_i} \hat{Y}_i(0) = |Y_i|, \]
since $\hat{Y}_i(0) = \sum_x Y_i(x) = |Y_i|$. Using the Cauchy inequality and the Parseval equality we get if $p \nmid r$
\[ |\hat{f}_i(r)| \leq \frac{1}{\sqrt{|X_i|}} \sqrt{\sum_x |\hat{Y}_i(x)|^2} = \sqrt{\frac{|Y_i|}{|X_i|}}. \]

Let $u \in \mathbb{F}$. Let $S$ be the number of solutions of the equation
\[ u = z_1 + z_2 + \cdots + z_m + \sum_{j=1}^n x_j y_j, \quad z_i \in Z, \ x_j \in X_j, \ y_j \in Y_j. \]
We can express $S$ by the mean of the Fourier transforms of $Z$ and $f_i$ as follows:
\[ pS = \sum_{r \in \mathbb{F}_p} \hat{Z}(r)^m \prod_{i=1}^n \hat{f}_i(r)e(-ru). \]
Our task is to show that this exponential sum is positive if the desired bound for the cardinalities (1) holds. Separating $r = 0$ and using (2) we can bound $S$ as
\[ pS \geq |Z|^m \prod_{i=1}^n |Y_i| - \sum_{r \neq 0} |\hat{Z}(r)|^m \prod_{i=1}^n |\hat{f}_i(r)| \]
\[ \geq |Z|^m \prod_{i=1}^n |Y_i| - |Z|^{m-2} \prod_{i=1}^n \sqrt{\frac{|Y_i|}{|X_i|}} \sum_{r \neq 0} |\hat{Z}(r)|^2 \]
\[ \geq |Z|^m \prod_{i=1}^n |Y_i| - p|Z|^{m-1} \prod_{i=1}^n \sqrt{\frac{|Y_i|}{|X_i|}}. \]
by the Parseval equality and (3). Hence $S > 0$ whenever

$$|Z|^2 \prod_{i=1}^{n} |X_i||Y_i| > p^{n+2}. $$

This completes the proof.  

\[
\square
\]

4. Proofs of Theorem 1.3 and Proposition 1.5

Proof of Theorem 1.3. By the remark preceding Theorem 1.3 we may plainly assume that $|Z| < p/2$.

By the assumption on the brick $B$ we have

\[(4) \quad |B| = |Z| \left( \prod_{i=1}^{n} |X_i||Y_i| \right) > |H_n|^{3/4+\varepsilon} = p^{3n/2+3/4+\varepsilon(2n+1)}. \]

For each $i$, there exists an element $a_i \in \mathbb{F}$ such that the number of solutions to the equation $a_i = x_i + x_i'$, $x_i, x_i' \in X_i$, is at least $|X_i|^2/p$. We denote by $\tilde{X}_i = X_i \cap (a_i - X_i)$ the set of the elements $x_i \in X_i$ such that $a_i - x_i \in X_i$. We thus have $|\tilde{X}_i| \geq |X_i|^2/p$. We similarly define $\tilde{Y}_i = Y_i \cap (b_i - Y_i)$ for some appropriate $b_i$ and also have $|\tilde{Y}_i| \geq |Y_i|^2/p$. It follows by (1) that

$$|Z|^2 \left( \prod_{i=1}^{n} |\tilde{X}_i||\tilde{Y}_i| \right) \geq \frac{(|Z| \prod_{i=1}^{n} |X_i||Y_i|)^2}{p^{2n}} > p^{n+3/2+\varepsilon(4n+2)}. $$

Hence for $n > 1/8\varepsilon$ we obtain from Proposition 3.1 that $2Z + \sum_{i=1}^{n} \tilde{X}_i \cdot \tilde{Y}_i = \mathbb{F}$ and consequently

$$B \cdot B \supseteq [(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n), \mathbb{F}],$$

that is $B \cdot B$ contains at least one coset of the non trivial subgroup $G = \{0, 0, \mathbb{F}\}$ of $H_n$.

In fact we may derive from the preceding argument a little bit more: for any index $i$ we have

$$\sum_{a_i \in \mathbb{F}} |X_i \cap (a_i - X_i)| = |X_i|^2, \quad \sum_{b_i \in \mathbb{F}} |Y_i \cap (b_i - Y_i)| = |Y_i|^2,$$

hence

$$\prod_{i=1}^{n} \left( \sum_{a_i \in \mathbb{F}} |X_i \cap (a_i - X_i)| \right) \left( \sum_{b_i \in \mathbb{F}} |Y_i \cap (b_i - Y_i)| \right) = \prod_{i=1}^{n} |X_i|^2|Y_i|^2,$$

or equivalently by developing the product

\[(5) \quad \sum_{a, b \in \mathbb{F}^n} \prod_{i=1}^{n} |X_i \cap (a_i - X_i)||Y_i \cap (b_i - Y_i)| = \prod_{i=1}^{n} |X_i|^2|Y_i|^2. \]

We denote by $E$ the set of all pairs $(a, b) \in \mathbb{F}^n \times \mathbb{F}^n$ such that

$$|Z|^2 \prod_{i=1}^{n} |X_i \cap (a_i - X_i)||Y_i \cap (b_i - Y_i)| > p^{n+2}. $$
For such a pair \((a, b)\), the coset \([a, b, F]\) is contained in \(B \cdot B\) by the above argument. Then by (5)
\[
\left|\prod_{i=1}^n |X_i||Y_i|\right|\left|E\right| + p^{n+2}(p^{2n} - \left|E\right|) > \left(\prod_{i=1}^n |X_i||Y_i|\right)^2
\]
hence
\[
\left|E\right| > \frac{p^{n-1}n\prod_{i=1}^n |X_i|^2|Y_i|^2}{\prod_{i=1}^n |X_i||Y_i| - p^{n+2}}.
\]
For \(n > 1/\epsilon\), we have by (4) and the fact that \(|Z| \leq p\)
\[
\prod_{i=1}^n |X_i||Y_i| > p^{3n/2+7/4},
\]
hence
\[
\left|E\right| \geq (1 - p^{-3/2}) \prod_{i=1}^n |X_i||Y_i| = (1 - p^{-3/2})\frac{|B|}{|Z|}.
\]
Since \(|Z| \leq p/2\), we thus have shown that \(B \cdot B\) contains at least \(2(1 - p^{-3/2})|B|/p \geq |B|/p\) cosets \([a, b, F] = [a, b, 0][0, 0, F]\), as we wanted.

**Proof of Proposition 1.3** Since \(B\) is a brick, \(B \cdot B\) is contained in a brick which takes the form \([U, V, W]\) where \(U, V \subset \mathbb{F}^n\) are direct products of subsets of \(\mathbb{F}\) and \(W \subset \mathbb{F}\). Since any non trivial subgroup of \(H_n\) has at least one of his \((2n + 1)\) coordinate projections equals to \(\mathbb{F}\), it suffices to prove that neither \(W\) is equal to \(\mathbb{F}\), nor \(U\), nor \(V\) contains a subset of the type \(\{x_1\} \times \cdots \times \mathbb{F} \times \cdots \times \{x_n\}\).

Let \(B = [R, R, Z]\) where
\[
R = \left\{(r_1, r_2, \ldots, r_n) \in \mathbb{F}^n \mid 0 \leq r_i < \sqrt{(p - 1)/2n}\right\} \text{ and } Z = \left\{z \in \mathbb{F} \mid 0 \leq z < p/4\right\}.
\]
We have \(|B| \geq p^{n+1}/(2n)^n\) and \(B \cdot B \subseteq [R + R, R + R, Z + Z + \langle R, R\rangle]\). Clearly \(R + R \subseteq [0, \sqrt{2p/n}]^n\), \(Z + Z \subseteq [0, (p - 1)/2]\) and \(\langle R, R\rangle \subseteq [0, (p - 1)/2]\). Hence the statement.

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