STABILITY OF STANDING WAVES FOR A NONLINEAR SCHRÖDINGER EQUATION UNDER AN EXTERNAL MAGNETIC FIELD

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Abstract. In this paper we study the existence and orbital stability of ground states for logarithmic Schrödinger equation under a constant magnetic field. For this purpose we establish the well-posedness of the Cauchy Problem in a magnetic Sobolev space and an appropriate Orlicz space. Then we show the existence of ground state solutions via a constrained minimization on the Nehari manifold. We also show that the ground state is orbitally stable.

1. Introduction. The present paper is devoted to the analysis of existence and stability of the ground states for the following nonlinear Schrödinger equation

\[ i\partial_t u + \Delta u + 2iA \cdot \nabla u - |A|^2 u + i \text{div}(A)u + u \log |u|^2 = 0, \tag{1} \]

where \( u = u(x,t) \) is a complex-valued function of \((x,t) \in \mathbb{R}^{2N} \times \mathbb{R}, \ N \geq 1 \). Here \( A = (A_1, \ldots, A_{2N}) \) is a vector (or magnetic) potential modeling the effect of an external constant magnetic field. The magnetic field \( B \) is given by \( B = [B_{kj}] \in \mathcal{M}_{2N \times 2N} \), where \( B_{kj} := \partial_j A_k - \partial_k A_j \), i.e. it is the anti-symmetric gradient of the vector field \( A \); or, in geometrical terms, the differential \( dA \) of the 1-form which is standardly associated to \( A \). Numerical studies and explicit solutions describing linear and rotational internal oscillations of Eq. (1) with a uniform magnetic field were reported in [4].

Orbital stability of ground states solutions of (1) in the absence of the vector potential (i.e. when \( A(x) \equiv 0 \)) have been studied in various settings. Indeed, Cazenave [9]; Cazenave and Lions [13]; Blanchard and co. [6, 7]; Ardila [1]; research the orbital stability of stationary solutions of (1). The Cauchy problem for (1) with \( A(x) \equiv 0 \) was treated by Cazenave and Haraux (see [10, 12]) in a suitable functional framework. The study of logarithmic NLS equation has attracted considerable attention recently in the theoretical and the applied mathematical literature because of their significant role in nuclear physics, quantum optics and Bose-Einstein condensation; see e.g. [2, 14, 16, 20, 27] and the references therein.

Concerning the nonlinear Schrödinger equation with magnetic potential and power-type nonlinearities, to the best of our knowledge there are only a few papers dealing with the problem of existence and stability of ground states [17, 11, 26, 25]. Recently, a great attention has been focused on the study of problems involving fractional NLS with magnetic field. See [5, 15, 24] for more details.

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In this paper we assume that the magnetic field $B = [B_{kj}]$ is constant in $\mathbb{R}^{2N}$. More precisely, let us consider the $2 \times 2$ anti-symmetric matrix

$$\Lambda := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.\nonumber$$

The magnetic field $B$ is given by the $2N \times 2N$ anti-symmetric matrix generated by $N$-diagonal blocks of $\Lambda$, in the following way:

$$B := \mu \begin{bmatrix} \Lambda & 0 & \ldots & 0 \\ 0 & \Lambda & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \Lambda \end{bmatrix}.\nonumber$$

Here $\mu \in \mathbb{R}$. In this case, and up a gauge transform, there exists a unique magnetic potential satisfying $\partial_j A_k - \partial_k A_j = B_{kj}$ and $\text{div}(A) = 0$. This potential $A$ is defined for each $x \in \mathbb{R}^{2N}$ by

$$A(x) = \frac{\mu}{2} (-x_2, x_1, \ldots, -x_{2N}, x_{2N-1}), \quad x = (x_1, \ldots, x_{2N}).\nonumber$$

It is known that the magnetic Schrödinger operator $-\Delta_A := -\Delta u - 2iA \cdot \nabla u + |A|^2 u$ is a self-adjoint operator on $L^2(\mathbb{R}^{2N})$ with quadratic form domain

$$H^1_A(\mathbb{R}^{2N}, \mathbb{C}) = \{ u \in L^2(\mathbb{R}^{2N}) : \nabla A u \in L^2(\mathbb{R}^{2N}) \} \quad \text{and} \quad \nabla A u := (\nabla + iA)u.\nonumber$$

The choice of the vector potential $A(x)$ is not unique for a given magnetic field. More precisely, for an arbitrary real function $V \in C^1(\mathbb{R}^{2N})$, the magnetic potential $A^*(x) = A(x) + \nabla V(x)$ and $A(x)$ express the same magnetic field $B$. The fact that addition of an arbitrary gradient to the vector potential does not change the corresponding $B$-field is referred to as the gauge invariance to $B$. Moreover, it is easy to see that if $u^*(x) = e^{-iV(x)}u(x)$, then

$$\nabla A^* u^*(x) = e^{-iV(x)} \nabla A u(x) \quad \text{and} \quad \Delta A^* u^*(x) = e^{-iV(x)} \Delta A u(x). \quad (2)\nonumber$$

Notice that (1) conserves (at least formally) the energy functional $E_A$,

$$E_A(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} |\nabla A u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^{2N}} |u|^2 \log |u|^2 \, dx. \quad (3)\nonumber$$

However, due to the singularity of the logarithm at the origin, this functional takes the value $+\infty$ for some $u \in H^1_A(\mathbb{R}^{2N})$ and therefore it is not differentiable on $H^1_A(\mathbb{R}^{2N})$. Thus in order to study the existence and orbital stability of ground states of (1) from a variational point of view, it is convenient to define a Banach space endowed with a Luxemburg type norm; this norm allows to control the singularity at the origin. More precisely, if we consider the reflexive Banach space (see Section 2)

$$W_A(\mathbb{R}^{2N}) = \left\{ u \in H^1_A(\mathbb{R}^{2N}) : |u|^2 \log |u|^2 \in L^1(\mathbb{R}^{2N}) \right\}, \quad (4)\nonumber$$

then the functional $E_A$ is differentiable on $W_A(\mathbb{R}^{2N})$ (see Proposition 2) and in particular, one can see that if $u \in C(\mathbb{R}, W_A(\mathbb{R}^{2N})) \cap C^1(\mathbb{R}, W^1_A(\mathbb{R}^{2N}))$, then Eq. (1) makes sense in $W^1_A(\mathbb{R}^{2N})$. Here, $W^1_A(\mathbb{R}^{2N})$ is the dual of $W_A(\mathbb{R}^{2N})$.

We have the following proposition concerning the well-posedness of the Cauchy problem for (1). The proof is contained in Section 5.
Proposition 1. For every \( u_0 \in W_A(\mathbb{R}^{2N}) \), there exists a unique solution \( u \in C(\mathbb{R}, W_A(\mathbb{R}^{2N})) \cap C^1(\mathbb{R}, W_A^*(\mathbb{R}^{2N})) \) of (1) such that \( u(0) = u_0 \). In addition, we have both conservation of charge and the conservation of energy, that is

\[
\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 \quad \text{and} \quad E(u(t)) = E(u_0) \quad \text{for all} \ t \in \mathbb{R}.
\]

Standing waves of equation (1) are solutions of the form \( u(x, t) = e^{i\omega t} \phi(x) \) where \( \omega \in \mathbb{R} \). For solutions of this type, (1) is equivalent to

\[
-\Delta_A \phi + \omega \phi - \text{Log} \ |\phi|^2 = 0, \quad x \in \mathbb{R}^{2N}. \tag{5}
\]

In this paper we are interested in the existence of ground states (or least-energy solutions) of the problem (5) for every \( \omega \in \mathbb{R} \). In addition we study the stability of the corresponding solutions of (1).

For \( \omega \in \mathbb{R} \), we define the following functionals of class \( C^1 \) on \( W_A(\mathbb{R}^{2N}) \):

\[
S_{A,\omega}(u) = \frac{1}{2} \|\nabla_A u\|_{L^2}^2 + \frac{\omega + 1}{2} \|u\|_{L^2}^2 - \frac{1}{2} \int_{\mathbb{R}^{2N}} |u|^2 \text{Log} |u|^2 \, dx,
\]

\[
I_{A,\omega}(u) = \|\nabla_A u\|_{L^2}^2 + \omega \|u\|_{L^2}^2 - \int_{\mathbb{R}^{2N}} \|u\|^2 \text{Log} |u|^2 \, dx.
\]

Note that (5) is equivalent to \( S'_\omega(\phi) = 0 \), and \( I_\omega(u) = \langle S'_{\omega}(u), u \rangle \) is the so-called Nehari functional. The ground states can be obtained and characterized as minimizers of the variational problem

\[
d_A(\omega) = \inf \left\{ S_{A,\omega}(u) : u \in W_A(\mathbb{R}^{2N}) \setminus \{0\}, I_{A,\omega}(u) = 0 \right\}
\]

\[
= \frac{1}{2} \inf \left\{ \|u\|_{L^2}^2 : u \in W_A(\mathbb{R}^{2N}) \setminus \{0\}, I_{A,\omega}(u) = 0 \right\}. \tag{6}
\]

It is standard that the minimizers of (6) are positive for every \( \omega \in \mathbb{R} \). In fact, for every \( N \geq 1 \) and \( \omega \in \mathbb{R} \) one has that

\[
\pi^N e^{\omega+2N} \leq 2d_A(\omega) \leq \pi^N \gamma^{-N} e^{\omega+2\gamma N}, \quad \gamma = \frac{1 + \sqrt{1 + \mu^2}}{2}.
\]

The set of ground states is defined by

\[
G_{A,\omega} = \{ \phi \in W_A(\mathbb{R}^{2N}) \setminus \{0\} : S_{A,\omega}(\phi) = d_A(\omega), I_{A,\omega}(\phi) = 0 \}.
\]

Because of the presence of magnetic potential, equation (5) is not invariant under translations in \( \mathbb{R}^{2N} \). A translation operator commuting with \( \Delta_A \) can be constructed. Indeed, for \( y \in \mathbb{R}^{2N} \) we define \( T_y : H^1_A(\mathbb{R}^{2N}) \to H^1_A(\mathbb{R}^{2N}) \) by setting

\[
T_y u(x) := e^{iA(y)[x]} u(x + y), \quad A(y)[x] = \frac{1}{2} B[y, x].
\]

It is called the magnetic translation operator. Notice that \( \nabla_A \circ T_y = T_y \circ \nabla_A \). In particular if \( u \in G_{A,\omega} \), then \( T_y u \in G_{A,\omega} \).

The existence of minimizers for (6) will be obtained as a consequence of the stronger statement that any minimizing sequence for (6) is, up to magnetic translation, precompact in \( W_A(\mathbb{R}^{2N}) \).

Theorem 1.1. Let \( N \geq 1 \) and \( \omega, \mu \in \mathbb{R} \) with \( \mu \neq 0 \). Then the following hold.

(i) Any minimizing sequence \( \{u_n\} \) of \( d_A(\omega) \) is relatively compact in \( W_A(\mathbb{R}^{2N}) \) up to magnetic translation. More precisely, there exists \( \{y_n\} \subset \mathbb{R}^{2N} \) such that \( \{T_{y_n} u_n\} \) contains a convergent subsequence in \( W_A(\mathbb{R}^{2N}) \). In particular, there exists a minimizer for problem (6), which implies that \( G_{A,\omega} \) is not a empty set.
(ii) Let $v \in G_{A,\omega}$. If $|v| \in G_{A,\omega}$, then there exist $\theta \in \mathbb{R}$ and $y \in \mathbb{R}^{2N}$ such that $v(x) = e^{i\theta}T_y\phi_v(x)$, where

$$\phi_v(x) = e^{\frac{-i\gamma}{2\gamma} + \sqrt{\gamma^2 - 1}x^2}, \quad \gamma = \frac{1 + \sqrt{1 + \mu^2}}{2}. \tag{7}$$

Notice that if $v \in G_{A,\omega}$ and $(\nabla v, iAv)_{L^2} = 0$, then conclusion (ii) of the Theorem 1.1 follows. In fact, in this case it is easy to show that $I_{A,\omega}(|v|) \leq I_{A,\omega}(v)$ and $S_{A,\omega}(|v|) \leq S_{A,\omega}(v)$. Thus, $|v| \in G_{A,\omega}$. In particular, we remark that if $v \in G_{A,\omega}$ is real-valued (up to magnetic translation and rotation in $\mathbb{C}$), then there exist $\theta \in \mathbb{R}$ and $y \in \mathbb{R}^{2N}$ such that $v(x) = e^{i\theta}T_y\phi_v(x)$. We conjecture that the set of ground states is given by $G_{A,\omega} = \{e^{i\theta}T_y\phi_v : \theta \in \mathbb{R}, y \in \mathbb{R}^{2N}\}$, but presently we do not have a proof of this fact.

The following is our orbital stability result of solutions, which is a direct consequence of the result of relative compactness.

**Theorem 1.2.** Let $\omega \in \mathbb{R}$ and $\mu \neq 0$. Then the set $G_{A,\omega}$ is $W_A(\mathbb{R}^{2N})$-stable in the following sense. For arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that, if $u_0 \in W_A(\mathbb{R}^{2N})$ verifies

$$\inf_{\psi \in G_{A,\omega}} \|u_0 - \psi\|_{W_A(\mathbb{R}^{2N})} \leq \delta,$$

then the corresponding solution $u(x,t)$ of the Cauchy problem (1) with the initial data $u_0$ satisfies

$$\inf_{y \in \mathbb{R}^{2N}} \inf_{\psi \in G_{A,\omega}} \|T_y u(t) - \psi\|_{W_A(\mathbb{R}^{2N})} < \epsilon, \quad \text{for all } t \geq 0.$$

The paper is organized as follows. In Section 2 we analyse the structure of the energy space $W_A(\mathbb{R}^{2N})$. Furthermore, we recall several known results and introduce several notations. In Section 3 we prove, by variational techniques, the existence of a minimizer for $d_A(\omega)$ (Theorem 1.1) and the stability result is proved in Section 4. In Section 5 we give a sketch of proof of Proposition 1. In the Appendix we list some properties of the Orlicz space $L^B(\mathbb{R}^N)$ defined in Section 2 and that will be used constantly in this paper.

**Notation.** $(\cdot, \cdot)$ is the duality pairing between $B'$ and $B$, where $B$ is a Banach space and $B'$ is its dual. The Hilbert space $L^2(\mathbb{R}^N, \mathbb{C})$ will be denoted by $L^2(\mathbb{R}^N)$ and its norm by $\| \cdot \|_{L^2}$. Moreover, $2^*$ is defined by $2^* = 2N/(N-2)$ if $N \geq 3$, and $2^* = +\infty$ if $N = 1, 2$. Finally, throughout this paper, the letter $C$ will denote positive constants whose value may change form line to line.

**2. Preliminaries.** For the sake of self-containedness of this paper we review some basic properties of the magnetic Sobolev spaces $H_A^1(\mathbb{R}^{2N})$, which will be needed later. We write $\nabla_A u = (\nabla + iA)u$. The magnetic Sobolev space $H_A^1(\mathbb{R}^{2N})$, defined as

$$H_A^1(\mathbb{R}^{2N}) = \left\{ u \in L^2(\mathbb{R}^{2N}) : \nabla_A u \in L^2(\mathbb{R}^{2N}) \right\},$$

is a Hilbert space endowed with the norm

$$\|u\|_{H_A^1(\mathbb{R}^{2N})}^2 = \int_{\mathbb{R}^{2N}} |\nabla_A u|^2 \, dx + \int_{\mathbb{R}^{2N}} |u|^2 \, dx$$

$$= \int_{\mathbb{R}^{2N}} \left( |\nabla u|^2 + |A(x)|^2 |u|^2 + |u|^2 \right) \, dx - 2\Re \int_{\mathbb{R}^{2N}} i(\nabla u \cdot A)\pi \, dx.$$
By Theorem 7.22 of [22], $C_c^\infty(\mathbb{R}^{2N})$ is dense in $H_A^1(\mathbb{R}^{2N})$ and $H^1_A(\mathbb{R}^{2N}) \subset H^1_{loc}(\mathbb{R}^{2N})$. Moreover, the following diamagnetic inequality (see e.g. [22, Theorem 7.21])

$$|\nabla|u(x)|| \leq |\nabla A u(x)|,$$

holds pointwise for almost every $x \in \mathbb{R}^{2N}$. Notice that if $u \in H^1_A(\mathbb{R}^{2N})$, it is not necessarily true that $u \in H^1(\mathbb{R}^{2N})$. However, by the diamagnetic inequality $|u| \in H^1(\mathbb{R}^{2N})$ and therefore $u \in L^p(\mathbb{R}^{2N})$ for any $p \in [2, 2^*)$.

**Remark 1.** The spaces $H^1_A(\mathbb{R}^{2N})$ and the spaces $H^1(\mathbb{R}^{2N})$ are not comparable; more precisely, in general $H^1_A(\mathbb{R}^{2N}) \nsubseteq H^1(\mathbb{R}^{2N})$ and $H^1(\mathbb{R}^{2N}) \nsubseteq H^1_A(\mathbb{R}^{2N})$. However it is proved in [3, Lemma 2.3] that if $\Omega$ is an open bounded domain with regular boundary, then $u \in H^1_A(\Omega)$ if and only if $u \in H^1(\Omega)$.

Now we introduce some notation used throughout this paper. Define

$$F(z) = |z|^2 \log |z|^2 \quad \text{for every } z \in \mathbb{C}.$$

Following Cazenave [9], we define the functions $\Phi, \Psi$ on $[0, \infty)$ by

$$\Phi(s) = \begin{cases} -s^2 \log(s^2), & \text{if } 0 \leq s \leq e^{-3}; \\ 3s^2 + 4e^{-3}s - e^{-6}, & \text{if } s \geq e^{-3}; \end{cases} \quad \Psi(s) = F(s) + \Phi(s). \quad (9)$$

In addition, let $a, b$ be functions defined by

$$a(z) = \frac{2}{|z|^2} \Phi(|z|) \quad \text{and} \quad b(z) = \frac{2}{|z|^2} \Psi(|z|) \quad \text{for } z \in \mathbb{C}, \ z \neq 0. \quad (10)$$

We note that $b(z) - a(z) = z \log |z|^2$. One can easily show that $\Phi$ is a nonnegative convex and increasing function, and $\Phi \in C^1([0, +\infty)) \cap C^2((0, +\infty))$. In fact, it is proved in [9, Lemma 1.3] that $\Phi$ is a Young-function which is $\Delta_2$-regular. Then the Orlicz space $L^\Phi(\mathbb{R}^{2N})$ corresponding to $\Phi$ is defined by

$$L^\Phi(\mathbb{R}^{2N}) = \left\{ u \in L^1_{loc}(\mathbb{R}^{2N}) : \Phi(|u|) \in L^1(\mathbb{R}^{2N}) \right\},$$

equipped with the Luxemburg norm

$$\|u\|_{L^\Phi} = \inf \left\{ k > 0 : \int_{\mathbb{R}^{2N}} \Phi\left(k^{-1}|u(x)|\right) \, dx \leq 1 \right\}. \quad (11)$$

Notice that $(L^\Phi(\mathbb{R}^{2N}), \| \cdot \|_{L^\Phi})$ is a separable reflexive Banach space (see [9, Lemma 2.1]).

Finally we consider the reflexive Banach space $W_A(\mathbb{R}^{2N}) = H^1_A(\mathbb{R}^{2N}) \cap L^\Phi(\mathbb{R}^{2N})$ equipped with the usual norm $\|u\|_{W_A(\mathbb{R}^{2N})} = \|u\|_{H^1_A(\mathbb{R}^{2N})} + \|u\|_{L^\Phi}$. It is not hard to prove that (see [9, Proposition 2.2] for more details)

$$W_A(\mathbb{R}^{2N}) = \left\{ u \in H^1_A(\mathbb{R}^{2N}) : |u|^2 \log |u|^2 \in L^1(\mathbb{R}^{2N}) \right\}. \quad (12)$$

By Proposition 1.1.3 of [10] we see that the dual space

$$W'_A(\mathbb{R}^{2N}) = H^{-1}_A(\mathbb{R}^{2N}) + L^{\Phi'}(\mathbb{R}^{2N}),$$

where the Banach space $W'_A(\mathbb{R}^{2N})$ is equipped with its usual norm. One also can show the following chain of continuous embedding: $W_A(\mathbb{R}^{2N}) \hookrightarrow L^2(\mathbb{R}^{2N}) \hookrightarrow W'_A(\mathbb{R}^{2N})$. Here, $L^{\Phi'}(\mathbb{R}^{2N})$ is the dual space of $L^\Phi(\mathbb{R}^{2N})$ (see [9]).

One can follow the same argument as in the proof of Proposition 2.7 of [9] to prove the following proposition.
Proposition 2. The functional (3) is differentiable on $W_A(\mathbb{R}^{2N})$. Furthermore, for $u \in W_A(\mathbb{R}^{2N})$ the Fréchet derivative is given by

$$E'_A(u) = -\Delta_A u - u \log |u|^2 - u.$$  

The following inequality is quite useful.

Remark 2. It is not hard to prove that for every $\epsilon > 0$, there exists $C_{\epsilon} > 0$ such that $|\Psi(z) - \Psi(z_1)| \leq C_{\epsilon}(|z|^{1+\epsilon} + |z_1|^{1+\epsilon})|z - z_1|$ for all $z, z_1 \in \mathbb{C}$. Then integrating this inequality on $\mathbb{R}^{2N}$ and applying Hölder inequality and Sobolev embeddings gives, for all $u, v \in H^1_A(\mathbb{R}^{2N})$,

$$\int_{\mathbb{R}^{2N}} |\Psi(|u|) - \Psi(|v|)| \, dx \leq C \left(1 + \|u\|_{H^1_A(\mathbb{R}^{2N})} + \|v\|_{H^1_A(\mathbb{R}^{2N})}^2 \right) \|u - v\|_{L^2} \quad (11)$$

The next lemma is the logarithmic Sobolev inequality. For a proof we refer to [22, Theorem 8.14].

Lemma 2.1. If $f \in H^1(\mathbb{R}^N)$ and $\alpha > 0$, then

$$\int_{\mathbb{R}^N} |f(x)|^2 \log |f(x)|^2 \, dx \leq \frac{\alpha^2}{\pi} \|\nabla f\|_{L^2}^2 + \left(\log \|f\|_{L^2}^2 - N(1 + \log \alpha)\right) \|f\|_{L^2}^2. \quad (12)$$

There is equality if and only if $f$ is, up to translation, a multiple of $e^{-\pi|x|^2/(2\alpha^2)}$.

3. Existence of the ground state.

Lemma 3.1. Let $\omega, \mu \in \mathbb{R}$ with $\mu \neq 0$. Then the following assertions hold.

(i) If $u \in W_A(\mathbb{R}^N) \setminus \{0\}$ is a real-valued function and $I_{A,\omega}(u) = 0$, then $\|u\|_{L^2}^2 \geq \|\phi_\omega\|_{L^2}^2$, where $\phi_\omega$ is defined by (7).

(ii) The quantity $d_A(\omega)$ is positive and satisfies

$$\frac{1}{2} \pi^N e^{\omega/2N} \leq d_A(\omega) \leq \frac{1}{2} \pi^N \gamma^{-N} e^{\omega+2\gamma N}, \quad \gamma = \frac{1 + \sqrt{1 + \mu^2}}{2}. \quad (13)$$

Proof. We remark that $\gamma > 1$ and $\mu^2 = 4\gamma(\gamma - 1)$. By direct computations we see that $\|\phi_\omega\|_{L^2}^2 = \pi^N \gamma^{-N} e^{\omega+2\gamma N}$. Moreover, one can easily verify that

$$\inf \{\|\nabla u\|_{L^2}^2 + \gamma^2 \|xu\|_{L^2}^2 : v \in \Sigma(\mathbb{R}^{2N}), \|v\|_{L^2}^2 = 1\} = 2\gamma N. \quad (13)$$

Here $\Sigma(\mathbb{R}^{2N}) = \{v \in H^1(\mathbb{R}^{2N}) : \|v\|_{L^2} = 1\}$. Now we consider $u \in W_A(\mathbb{R}^{2N}) \setminus \{0\}$ real-valued function with $I_{A,\omega}(u) = 0$. It is clear that $\|\nabla A u\|_{L^2}^2 = \|\nabla u\|_{L^2}^2 + \gamma(\gamma - 1)\|xu\|_{L^2}^2$. Since $u \in \Sigma(\mathbb{R}^{2N})$, multiplying (13) by $\gamma^{-1}(\gamma - 1)$ we get

$$2(\gamma - 1)N \|u\|_{L^2}^2 \leq \gamma^{-1}(\gamma - 1) \|\nabla u\|_{L^2}^2 + \gamma(\gamma - 1)\|xu\|_{L^2}^2. \quad (14)$$

Now, recalling that $I_{A,\omega}(u) = 0$, using the logarithmic Sobolev inequality with $\alpha^2 = \pi/\gamma$ and (14) we see that

$$\left(\omega + 2\gamma N + N \log(\sqrt{\pi/\gamma})\right) \|u\|_{L^2}^2 \leq \left(\log \|u\|_{L^2}^2 \right) \|u\|_{L^2}^2,$$

which implies that $\|u\|_{L^2}^2 \geq \pi^N \gamma^{-N} e^{\omega+2\gamma N}$.

Next, we consider (ii). By direct computations we see that $I_{A,\omega}(\phi_\omega) = 0$ and, by the definition of $d_A(\omega)$, $d_A(\omega) \leq S_{A,\omega}(\phi_\omega) = \frac{1}{\pi} \|\phi_\omega\|_{L^2}^2 = \frac{1}{\pi} \pi^N \gamma^{-N} e^{\omega+2\gamma N}$. Let $u \in W_A(\mathbb{R}^{2N}) \setminus \{0\}$ be such that $I_u(u) = 0$. From the diamagnetic inequality
we have that $I_{0,\omega}(|u|) \leq I_{A,\omega}(u) = 0$. Thus, by using the logarithmic Sobolev inequality with $\alpha = \sqrt{\pi}$ we see that
\[
(\omega + 2N (1 + \log(\sqrt{\pi}))) \|u\|^2_{L^2} \leq \left( \log \|u\|^2_{L^2} \right) \|u\|^2_{L^2},
\]
which implies that $\|u\|^2_{L^2} \geq e^{\omega+2N\pi N}$. Then, by the definition of $d_A(\omega)$ given in (6), (ii) follow easily.

\begin{lemma}
If $\{u_n\}_{n \in \mathbb{N}}$ is a minimizing sequence of problem (6), then there exists a subsequence, which we still denote by $\{u_n\}_{n \in \mathbb{N}}$, and there exists a sequence $\{y_n\} \subset \mathbb{R}^{2N}$ such that $v_n(x) := T_{y_n} u_n(x)$ converges weakly in $W_A(\mathbb{R}^N)$ to a function $v \neq 0$. In addition, $\{v_n\}$ converges to $v$ a.e. and in $L^q_{\text{loc}}(\mathbb{R}^{2N})$ for every $q \in [2, 2^*)$.
\end{lemma}

\begin{proof}
From Lemma 2.1 one can easily proves that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $W_A(\mathbb{R}^{2N})$. Indeed, it is clear that the sequence $\|u_n\|_{L^2}$ is bounded. Furthermore, using the logarithmic Sobolev inequality, the diamagnetic inequality (8) and recalling that $I_{A,\omega}(u_n) = 0$, we see that
\[
\left(1 - \frac{\alpha^2}{\pi}\right) \|\nabla A u_n\|^2_{L^2} \leq \left( \log \left( \frac{e^{-(\omega+2N)}}{\alpha^2N} \right) \right) \|u_n\|^2_{L^2} \|u_n\|^2_{L^2},
\]
for $\alpha > 0$ sufficiently small, we have that $\{\nabla A u_n\}$ is bounded in $L^2(\mathbb{R}^{2N})$; that is, the sequence $\{u_n\}$ is bounded in $H_A^1(\mathbb{R}^{2N})$. Thus, using $I_{A,\omega}(u_n) = 0$ again, and (11) we have that
\[
\int_{\mathbb{R}^{2N}} |\nabla A u_n|^2 \, dx + \int_{\mathbb{R}^{2N}} \Phi (|u_n|) \, dx \leq \int_{\mathbb{R}^{2N}} \Psi (|u_n|) \, dx + \|\omega\|_2 \|u_n\|_{L^2}^2 \leq C,
\]
which implies, by (32) in the Appendix, that $\{u_n\}$ is bounded in $W_A(\mathbb{R}^{2N})$.

Let $2 < q < 2^*$ and $0 < \delta < 1$. We remark that there exists $M$, a positive constant depending only on $\omega$, such that $\int_{\mathbb{R}^{2N}} |u_n|^2 \log |u_n|^2 \, dx \geq -M$. Arguing as in the Lemma 3.3 in [9] it is immediate to verify that
\[
2d_A(\omega) \leq (\delta^{-2} - [(q - 2)\log\delta^2]^{-1}) \|u_n\|_{L^q}^2 - [\log\delta]^{-1} M.
\]
It is now time to choose $\delta$ such that $-(\log\delta^2)^{-1} M = d_A(\omega) ((q - 2)/(q + 2))$. Therefore, one can rewrite the preceding equation in the form
\[
\|u_n\|_{L^q}^2 \left( e^{M(q+2)/2d_A(\omega)} + \frac{d_A(\omega)}{M(q+1)} \right) \geq 2d_A(\omega) \left( \frac{q+6}{q+2} \right) \geq d_A(\omega).
\]
Making use of (15) and Lemma 3.9 in [21] we see that
\[
\sup_{y \in \mathbb{R}^{2N}} \int_{B_1(y)} |u_n|^2 \, dx \geq \epsilon > 0,
\]
in this case we can choose $\{y_n\} \subset \mathbb{R}^{2N}$ such that
\[
\int_{B_1(0)} |u_n(\cdot + y_n)|^2 \, dx \geq \epsilon > 0.
\]
Let $v_n(x) = T_{y_n} u_n(x)$. As it was observed above, $\|\nabla A v_n\|_{L^2}^2 = \|\nabla A u_n\|_{L^2}^2$ and $\|v_n(\cdot)\| = |u_n(\cdot + y_n)|$ a.e. in $\mathbb{R}^{2N}$. Thus $\{v_n\}$ is a bounded minimizing sequence of problem $d_A(\omega)$. Consequently, from (16), and due to the compactness of the embedding $H^1_{\text{loc}}(\mathbb{R}^{2N}) \hookrightarrow L^q_{\text{loc}}(\mathbb{R}^{2N})$, we deduce that the sequence $\{u_n\}$ has a weak limit, up magnetic translation, $v$ in $W_A(\mathbb{R}^{2N})$ that is not identically zero. Also, it follows that $\{v_n\}$ converges to $v$ strongly in $L^q_{\text{loc}}(\mathbb{R}^{2N})$ for any $q \in [2, 2^*)$. \qed
Lemma 3.3. Let \( \{u_n\} \) be a bounded sequence in \( W_A(\mathbb{R}^{2N}) \) such that \( u_n \to u \) a.e. in \( \mathbb{R}^{2N} \). Then \( u \in W_A(\mathbb{R}^{2N}) \) and
\[
\lim_{n \to \infty} \int_{\mathbb{R}^{2N}} \left\{ |u_n|^2 \log |u_n|^2 - |u_n - u|^2 \log |u_n - u|^2 \right\} \, dx = \int_{\mathbb{R}^{2N}} |u|^2 \log |u|^2 \, dx.
\]

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.2, we know that every minimizing sequence \( \{u_n\} \) for the problem (6) has a subsequence, again denoted by \( \{u_n\} \), satisfying that \( v_n(x) = T_{y_n} u_n(x) \) converges weakly in \( W_A(\mathbb{R}^{2N}) \) to a function \( v \neq 0 \) and a.e. in \( \mathbb{R}^{2N} \). We remark that \( \{v_n\} \) also is a minimizing sequence of problem \( d_A(\omega) \).

Now we claim that \( v \in \mathcal{G}_{A,\omega} \), that is, \( I_{A,\omega}(v) = 0 \) and \( S_{A,\omega}(v) = d_A(\omega) \). Suppose, by contradiction, that \( I_{A,\omega}(v) < 0 \). It is not hard to prove that there is \( 0 < \lambda < 1 \) such that \( I_{A,\omega}(\lambda v) = 0 \). Then from weak lower semicontinuity we have
\[
d_A(\omega) \leq \frac{1}{2} \|\lambda v\|_{L^2}^2 < \frac{1}{2} \|v\|_{L^2}^2 \leq \frac{1}{2} \liminf_{n \to \infty} \|v_n\|_{L^2}^2 = d_A(\omega),
\]
which is impossible. So \( I_{\omega}(v) \geq 0 \). Assume that \( I_{\omega}(v) > 0 \). Since the embedding \( W_A(\mathbb{R}^{2N}) \hookrightarrow H^1_A(\mathbb{R}^{2N}) \) is continuous, it follows that \( v_n \to v \) weakly in \( H^1_A(\mathbb{R}^{2N}) \). In particular,
\[
\begin{align*}
\|\nabla A v_n\|_{L^2}^2 - \|\nabla A v_n - \nabla A v\|_{L^2}^2 - \|\nabla A v\|_{L^2}^2 & \to 0, \quad (17) \\
\|v_n - v\|_{L^2}^2 - \|v_n - v\|_{L^2}^2 - \|v\|_{L^2}^2 & \to 0 \quad (18)
\end{align*}
\]
as \( n \to \infty \). From these observations and Lemma 3.3, it follows that
\[
\lim_{n \to \infty} I_{A,\omega}(v_n - v) = \lim_{n \to \infty} I_{A,\omega}(v_n) - I_{A,\omega}(v) = -I_{A,\omega}(v),
\]
Since \( I_{A,\omega}(v) > 0 \), one has that \( I_{A,\omega}(v_n - v) < 0 \) for sufficiently large \( n \). Then from (18) and applying the same argument as above, we have
\[
d_A(\omega) \leq \frac{1}{2} \lim_{n \to \infty} \|v_n - v\|_{L^2}^2 = d_A(\omega) - \frac{1}{2} \|v\|_{L^2}^2,
\]
which is a contradiction because \( \|v\|_{L^2}^2 > 0 \). Finally, we deduce that \( I_{A,\omega}(v) = 0 \) and applying the weak lower semicontinuity again, we have that \( d_A(\omega) = \frac{1}{2} \|v\|_{L^2}^2 \); that is, \( v \in \mathcal{G}_{A,\omega} \).

Next we prove that \( v_n \to v \) strongly in \( W_A(\mathbb{R}^{2N}) \). In fact, using (18), we infer that \( v_n \to v \) in \( L^2(\mathbb{R}^{2N}) \). Since the sequence \( \{v_n\} \) is bounded in \( H^1_A(\mathbb{R}^{2N}) \), from the inequality (11) we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^{2N}} \Psi (|v_n(x)|) \, dx = \int_{\mathbb{R}^{2N}} \Psi (|v(x)|) \, dx,
\]

together with the fact that \( I_{A,\omega}(v_n) = I_{A,\omega}(v) = 0 \) for any \( n \in \mathbb{N} \), gives
\[
\lim_{n \to \infty} \left[ \|\nabla A v_n\|_{L^2}^2 + \int_{\mathbb{R}^{2N}} \Phi (|v_n(x)|) \, dx \right] = \|\nabla A v\|_{L^2}^2 + \int_{\mathbb{R}^{2N}} \Phi (|v(x)|) \, dx. \quad (19)
\]
Since by (19), the weak lower semicontinuity and Fatou lemma, it follows easily that (see e.g. [19, Lemma 12 in chapter V])

\[
\lim_{n \to \infty} \| \nabla A v_n \|_{L^2}^2 = \| \nabla A v \|_{L^2}^2 \tag{20}
\]

\[
\lim_{n \to \infty} \int_{\mathbb{R}^{2N}} \Phi(|v_n(x)|) \, dx = \int_{\mathbb{R}^{2N}} \Phi(|v(x)|) \, dx. \tag{21}
\]

In particular, it follows from (20) that \( v_n \to v \) in \( H^1_A(\mathbb{R}^N) \). Finally, by Proposition 4 ii) in Appendix and (21) we know that \( v_n \to v \) in \( L^4(\mathbb{R}^{2N}) \). Then (i) follow easily.

Now we consider a minimizer \( v \in \mathcal{G}_{A,\omega} \). Assume that \( |v| \in \mathcal{G}_{A,\omega} \), then from Lemma 3.1 we deduce that \( \| v \|_{L^2}^2 = 2d_A(\omega) = \| \phi_\omega \|_{L^2}^2 \) and \( I_{A,\omega}(|v|) = 0 \). In more detail

\[
\| \nabla v \|_{L^2}^2 + \gamma (\gamma - 1)\| x v \|_{L^2}^2 + \omega \| v \|_{L^2}^2 = \int_{\mathbb{R}^{2N}} |v| \log |v| \, dx. \tag{22}
\]

We claim that \( |v| \) satisfies the equality in logarithmic Sobolev inequality (12) with \( \alpha^2 = \pi/\gamma \). Suppose, by contradiction, that we have the strict inequality in (12) with \( \alpha^2 = \pi/\gamma \). Since \( |v| \) satisfies (22), one can show easily that in this case \( \| v \|_{L^2}^2 > \| \phi_\omega \|_{L^2}^2 \) (see proof of Lemma 3.1(i)), which is impossible. It follows immediately from that observation and Lemma 2.1 that there exists \( z \in \mathbb{R}^{2N} \) such that

\[
|v(x)| = e^{-\frac{\pi + \gamma N}{2}} e^{-\frac{\gamma}{\gamma - 2} |x + z|^2}, \quad x \in \mathbb{R}^{2N}. \tag{23}
\]

Now since \( S'_{A,\omega}(v) = 0 \) and \( v \log |v| = (\omega + 2\gamma N)v - \gamma |x + z|^2 v \) we see that

\[
-\Delta A v + \gamma |x + z|^2 v = 2N\gamma v.
\]

Let \( \rho(x) := e^{-iA(z)[x]}v(x) \), then by change of gauge (see (2)) we infer that

\[
0 = \{-\Delta_A + \gamma |x + z|^2 - 2N\gamma\} v
\]

\[
= -\Delta_A [e^{iA(z)[x]}\rho(x)] + \gamma |x + z|^2 e^{iA(z)[x]}\rho(x) - 2N\gamma e^{iA(z)[x]}\rho(x)
\]

\[
e e^{iA(z)[x]} \{-\Delta - 2i A(x + z) \cdot \nabla + |A(x + z)|^2 + \gamma |x + z|^2 - 2N\gamma\} \rho
\]

\[
= e^{iA(z)[x]} \{-\Delta_A^* + \gamma |x + z|^2 - 2N\gamma\} \rho, \tag{24}
\]

where \( A^* = A + \nabla V \) with \( V(x) = A(z)[x] \). Since in addition it is well-known that (see [23, Theorem 3.2 and Propositions 5.3 and 5.4])

\[
\text{Ker}(-\Delta_A^* + \gamma |x + z|^2 - 2N\gamma) = \{ e^{i\theta} e^{-\frac{\pi}{\gamma - 2} |x + z|^2}; \theta \in \mathbb{R}, r > 0 \},
\]

it follows from (24) that there exist \( \theta_0 \in \mathbb{R} \) and \( r_0 > 0 \) such that

\[
\rho(x) = e^{i\theta_0} r_0 e^{-\frac{\pi}{\gamma - 2} |x + z|^2}. \quad \text{Finally, from (23) and remembering that}
\]

\[
\rho(x) = e^{-iA(z)[x]}v(x) \text{ we see that } v(x) = e^{i\theta_0} T_z \phi_\omega(x). \tag*{\Box}
\]

4. Stability of the ground state.

**Proof of Theorem 1.2.** This result can be proved by a classical argument, which we repeat for completeness. We arguing by way of contradiction. Suppose that \( \mathcal{G}_{A,\omega} \) is not \( W_A(\mathbb{R}^{2N}) \)-stable. Then there would exist \( \epsilon > 0 \), a sequence \( (u_{n,0})_{n \in \mathbb{N}} \) in \( W_A(\mathbb{R}^{2N}) \), and \( t_n \geq 0 \) such that for all \( n \),

\[
\inf_{\psi \in \mathcal{G}_{A,\omega}} \| u_{n,0} - \psi \|_{W_A(\mathbb{R}^{2N})} < \frac{1}{n}, \tag{25}
\]

and

\[
\inf_{y \in \mathbb{R}^{2N}} \inf_{\psi \in \mathcal{G}_{A,\omega}} \| T_y u_n(t_n) - \psi \|_{W_A(\mathbb{R}^{2N})} \geq \epsilon, \tag{26}
\]
where \( u_n \) denotes the solution of the Cauchy problem (1) with initial data \( u_{n,0} \). Set \( v_n(x) = u_n(x,t_n) \). Since \( \psi \in \mathcal{G}_{A,\omega} \), by (25) and conservation laws, one has that

\[
\|v_n\|_{L^2}^2 = \|u_n(t_n)\|_{L^2}^2 = \|u_{n,0}\|_{L^2}^2 \rightarrow 2d_A(\omega) \quad (27)
\]

\[
S_{A,\omega}(v_n) = S_{A,\omega}(u_n(t_n)) \rightarrow S_{A,\omega}(u_{n,0}) = d_A(\omega), \quad (28)
\]

as \( n \rightarrow \infty \). From (27) and (28), it follows that \( I_{A,\omega}(v_n) \rightarrow 0 \) as \( n \rightarrow \infty \). Now consider the sequence \( f_n(x) = \rho_n v_n(x) \) with

\[
\rho_n = \exp \left( \frac{I_{A,\omega}(v_n)}{2\|v_n\|_{L^2}^2} \right),
\]

where \( \exp(x) \) represent the exponential function. Then one can show easily that \( \lim_{n \rightarrow \infty} \rho_n = 1 \) and \( I_{\omega}(f_n) = 0 \) for any \( n \in \mathbb{N} \). Since the sequence \( \{v_n\} \) is bounded in \( W_A(\mathbb{R}^{2N}) \), we see that \( \|v_n - f_n\|_{W_A(\mathbb{R}^{2N})} \rightarrow 0 \) as \( n \rightarrow \infty \). Hence using (28), we know that \( \{f_n\} \) is a minimizing sequence for the problem (6). From Theorem 1.1(i), up to a subsequence, there exist \( (y_n) \subset \mathbb{R}^{2N} \) and \( \varphi \in \mathcal{G}_{A,\omega} \) such that

\[
\|\mathcal{T}_{y_n}f_n - \varphi\|_{W_A(\mathbb{R}^{2N})} \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty. \quad (29)
\]

Remembering that \( v_n = u_n(t_n) \), together with (29) implies that

\[
\|\mathcal{T}_{y_n}u_n(t_n) - \varphi\|_{W_A(\mathbb{R}^{2N})} \leq \|\mathcal{T}_{y_n}v_n - \mathcal{T}_{y_n}f_n\|_{W_A(\mathbb{R}^{2N})} + \|\mathcal{T}_{y_n}f_n - \varphi\|_{W_A(\mathbb{R}^{2N})} \rightarrow 0
\]

as \( n \rightarrow +\infty \), a contradiction. Hence the set \( \mathcal{G}_{A,\omega} \) is stable. \( \square \)

5. The Cauchy problem. The proof of Proposition 1 in the case \( A \equiv 0 \) can be found in Cazenave’s book [10, Section 9.3]. In the following we highlight the main points in the proof with nontrivial magnetic potential. The strategy of the proof consists in approximate the logarithmic nonlinearity by a smooth nonlinearity, and study the behaviour of these solutions by using standard compactness results. Thus we can pass the limit to show the existence of a weak global solution of the equation (1) in \( W_A(\mathbb{R}^{2N}) \).

To begin with, let us write two technical lemmas which will be used in what follows.

**Lemma 5.1.** Given \( k \in \mathbb{N} \), set \( \Omega_k = \{x \in \mathbb{R}^{2N} : |x| < k\} \). Let \( \{u^m\} \) be a bounded sequence in \( L^\infty(\mathbb{R}, H_A^1(\mathbb{R}^{2N})) \). If \( \{u^m|_{\Omega_k}\} \) is a bounded sequence of \( W^{1,\infty}(\mathbb{R}, H^{-1}(\Omega_k)) \) for \( k \in \mathbb{N} \), then there exists \( u \in L^\infty(\mathbb{R}, H_A^1(\mathbb{R}^{2N})) \) and there exists a subsequence, which we still denote by \( \{u^m\} \) such that following properties are true:

(i) \( u^m|_{\Omega_k} \in W^{1,\infty}(\mathbb{R}, H^{-1}(\Omega_k)) \) for all \( k \in \mathbb{N} \).
(ii) \( u^m(t) \rightharpoonup u(t) \) in \( H_A^1(\mathbb{R}^{2N}) \) as \( m \rightarrow \infty \) for every \( t \in \mathbb{R} \).
(iii) For every \( t \in \mathbb{R} \) there exists a subsequence \( m_j \) such that \( u^{m_j}(x,t) \rightarrow u(x,t) \) as \( j \rightarrow \infty \), for a.e. \( x \in \mathbb{R}^N \).
(iv) \( u^m(x,t) \rightarrow u(x,t) \) as \( m \rightarrow \infty \), for a.e. \( (x,t) \in \mathbb{R}^{2N} \times \mathbb{R} \).

**Proof.** Since the spaces \( H_A^1(\Omega_k) \) and \( H^1(\Omega_k) \) are equivalent (see Remark (1)), the proof follows along the same lines as the proof of Lemma 9.3.6 of [10] and we do not repeat here. \( \square \)
Now we regularize the logarithmic nonlinearity defining, for \( z \in \mathbb{C} \) and \( m \in \mathbb{N} \),
\[
a_m(z) = \begin{cases} \frac{a(z)}{m}, & \text{if } |z| \leq \frac{1}{m}; \\ a(z), & \text{if } |z| \geq \frac{1}{m}; \end{cases}
\]
and
\[
b_m(z) = \begin{cases} \frac{b(z)}{m}, & \text{if } |z| \leq m; \\ b(z), & \text{if } |z| \geq m, \end{cases}
\]
where \( a \) and \( b \) were defined in (10). For any fixed \( m \in \mathbb{N} \), we consider a family of regularized nonlinearities in the form \( g_m(z) = b_m(z) - a_m(z) \), for every \( z \in \mathbb{C} \). In order to construct a weak solution of (1), we solve first, for \( m \in \mathbb{N} \), the regularized problem
\[
i \partial_t u^m + \Delta A u^m + g_m(u^m) = 0. \quad (30)
\]
Notice that if \( m \geq 1 \), the function \( g_m \) is globally Lipschitz continuous \( \mathbb{C} \rightarrow \mathbb{C} \). The existence of a weak global solution of equation (30) and the conservation laws are known from the work of [18]. Indeed, we have the following result; see Theorem 3.1 in [18] for more details.

**Proposition 3.** For any \( u_0 \in H^1_0(\mathbb{R}^2N) \), there is a unique global solution \( u^m \in C(\mathbb{R}, H^1_0(\mathbb{R}^2N)) \cap C^1(\mathbb{R}, H^{-1}_0(\mathbb{R}^2N)) \) of (30) such that \( u^m(0) = u_0 \). In addition, the conservation of energy and charge hold; that is,
\[
\mathcal{E}_m(u^m(t)) = \mathcal{E}_m(u_0) \quad \text{and} \quad \|u^m(t)\|^2_{L^2(\mathbb{R}^N)} = \|u_0\|^2_{L^2(\mathbb{R}^N)} \quad \text{for all } t \in \mathbb{R},
\]
where
\[
\mathcal{E}_m(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla A u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} G_m(u) dx, \quad G_m(z) = \int_0^{|z|} g_m(s) ds.
\]

Now we give the proof of Proposition 1.

**Proof of Proposition 1.** The proof is an easy adaptation of the known proof for classical logarithmic NLS equation given in Cazenave’s book [10, Theorem 9.3.4], and we only give the sketch of the proof. Applying Proposition 3, it follows that exists a unique solution \( u^m \in C(\mathbb{R}, H^1_0(\mathbb{R}^2N)) \cap C^1(\mathbb{R}, H^{-1}_0(\mathbb{R}^2N)) \) of the regularized (30), with initial datum \( u_0 \). One can easily see from the conservation laws that the sequence \( u^m \) is bounded in the \( L^\infty(\mathbb{R}, H^1_0(\mathbb{R}^2N)) \) (see Step 2 of Theorem 9.3.4 in [10]). In particular, this implies from the NLS equation (30) that the sequence \( \partial_t u^m|_{\Omega_k} \) is bounded in the space \( L^\infty(\mathbb{R}, H^{-1}(\Omega_k)) \), where \( \Omega_k = \{ x \in \mathbb{R}^N : |x| < k \} \). Using these facts we may apply Lemma 5.1 to the sequence \( u^m \); let \( u \) be the limit of \( u^m \). From (30), for any test function \( \psi \in C_0^\infty(\mathbb{R}^2N) \) and \( \phi \in C_0^\infty(\mathbb{R}) \) we get
\[
\int_\mathbb{R} \left[ - \langle i u^m, \psi \rangle \phi'(t) + \langle u^m, R_A \psi \rangle \phi(t) \right] dt + \int_\mathbb{R} \int_{\mathbb{R}^N} g_m(u^m) \psi \phi dx dt = 0. \quad (31)
\]
Then by the properties (i)-(iv) of Lemma 5.1, together with integral formulation (31) we have that the sequence \( u^m \) converges to a weak solution \( u \) of (1) as \( n \) goes to \( +\infty \); furthermore, \( u(0) = u_0 \) and \( u \in L^\infty(\mathbb{R}, W_A^1(\mathbb{R}^2N)) \cap W^{1, \infty}(\mathbb{R}, W_A^1(\mathbb{R}^2N)) \) (see proof of Step 3 of [10, Theorem 9.3.4] for more details). Finally, by applying the arguments identical to those of Step 4 of Theorem 9.3.4 in [10] we obtain the uniqueness of the weak solution, the conservation of charge and energy, and the continuity of the solution \( u \in C(\mathbb{R}, W_A(\mathbb{R}^2N)) \cap C^1(\mathbb{R}, W_A^1(\mathbb{R}^2N)) \) in time \( t \). This completes the proof of Proposition 1. \( \square \)
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6. Appendix. We provide some basic results on the Orlicz space \( L^\Phi(\mathbb{R}^{2N}) \) that we have used above. For a proof of such statements we refer to [9, Lemma 2.1].

**Proposition 4.** Let \( \{u_m\} \) be a bounded sequence in \( L^\Phi(\mathbb{R}^{2N}) \), then the following properties are true:

i) If \( u_m \to u \) in \( L^\Phi(\mathbb{R}^{2N}) \), then \( \Phi(|u_m|) \to \Phi(|u|) \) in \( L^1(\mathbb{R}^{2N}) \) as \( n \to \infty \).

ii) If \( u_m \to u \) a.e. in \( \mathbb{R}^{2N} \) and if
\[
\lim_{n \to \infty} \int_{\mathbb{R}^{2N}} \Phi(|u_m(x)|) \, dx = \int_{\mathbb{R}^{2N}} \Phi(|u(x)|) \, dx,
\]
then \( u_m \to u \) in \( L^\Phi(\mathbb{R}^{2N}) \) as \( n \to \infty \).

iii) If \( u \in L^\Phi(\mathbb{R}^{2N}) \), then
\[
\min \left\{ \|u\|_{L^\Phi}, \|u\|_{L^\Phi}^2 \right\} \leq \int_{\mathbb{R}^{2N}} \Phi(|u(x)|) \, dx \leq \max \left\{ \|u\|_{L^\Phi}, \|u\|_{L^\Phi}^2 \right\}.
\]

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