Phase space analysis of the $F(X) - V(\phi)$ scalar field Lagrangian and scaling solutions in flat cosmology

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Abstract. We review a system of autonomous differential equations developed in our previous work [1] describing a flat cosmology filled with a barotropic fluid and a scalar field with a modified kinetic term of the form $\mathcal{L} = F(X) - V(\phi)$. We analyze the critical points and summarize the conditions to obtain scaling solutions. We consider a set of transformations and show that they leave invariant the equations of motion for the systems in which the scaling solution is obtained, allowing to reduce the number of degrees of freedom.

1. Introduction
In cosmology, the scalar fields play an important role due to its broad phenomenology, which can be used in order to describe different phenomena. For that reason they have been used to model phenomena like inflation [2], dark energy [3, 4], dark matter [5], bounce scenarios [6], and unification models [7].

The canonical Lagrangian for a scalar field is given by $\mathcal{L} = X - V(\phi)$, where the first term $X = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi$ is the kinetic term, and the second is the potential that can have different functional forms depending on the model. In the last years, a generalization has been a studied in which the Lagrangian is a general function of $X$ and $\phi$. These Lagrangians were used initially in the study of inflation [8] and dark energy [9], and in a previous work we used this type of Lagrangian to unify inflation, dark matter, and dark energy [10].

One particular form of this kind of Lagrangian is the sum-separable, in which

$$\mathcal{L} = F(X) - V(\phi),$$

and there is a clear separation between the kinetic term $F(X)$ and the potential term $V(\phi)$. This class of Lagrangians has the advantage of being more easily studied than the more general case while still conserving some of the rich phenomenology that is not present in the canonical case. In a previous work we studied the dynamical system of this class of Lagrangians in a flat Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmology filled with the field and an additional barotropic fluid [1]. This analysis allowed us to find the critical points of the system and its stability, from which we drew conclusions about the general behaviour and evolution of the system.

In the present work we will review this analysis and we will add an extra analysis for the case in which the system has non-trivial critical points. We will show that in that case there is a
symmetry that allows the degrees of freedom of the system to reduce. This behaviour is similar to what happens in the canonical Lagrangian with exponential potential. This symmetry allows the existence of scaling solutions even when the Lagrangian is not of the form $L = X g(X e^{\lambda \phi})$, which in the literature [11] is considered to be the most general form of Lagrangians with this behaviour.

2. Autonomous system

Along this work we will consider a flat FLRW cosmology filled with a scalar field with Lagrangian of the form (1) and a barotropic fluid with equation of state $P_m = (\gamma_m - 1) \rho_m$ where $\gamma_m$ is constant. In this system, if no interaction between the field and the fluid is considered, the equations of motion correspond to the two independent Einstein equations, also called Friedmann equations and the two continuity equations as follows:

$$H^2 = \frac{1}{3 M_{Pl}^2}[2 X F_X - F + V + \rho_m],$$

$$\frac{dH}{dt} = -\frac{1}{2 M_{Pl}^2}[2 X F_X + \gamma_m \rho_m],$$

$$\frac{d\rho_m}{dt} = -3H \gamma_m \rho_m,$$

$$\frac{d}{dt}(2 X F_X - F + V) = -6 H X F_X,$$

where the subscript $F_X$ means differentiation of $F$ with respect to $X$. Here only three of the equations are independent from each other, as the system has only three degrees of freedom. We can choose an initial $\phi$, $\dot{\phi}$ and $\rho_m$, which in turn will define the initial Hubble parameter with the equation (2) and the initial kinetic term $X$ with the relation

$$X = \frac{1}{2} \dot{\phi}^2.$$

In order to obtain an autonomous system of differential equations we define as independent variable the dimensionless $dN = d \log a$, and then define the new dynamical variables

$$x = \sqrt{\frac{\rho_k}{\rho_c}},$$

$$y = \sqrt{\frac{V}{\rho_c}},$$

$$\sigma = -\frac{M_{Pl}}{\sqrt{3 \rho_k}} \frac{d \log V}{dt},$$

where $\rho_k = 2 X F_X - F$ is the density associated to the kinetic part of the Lagrangian, and $\rho_c = 3 M_{Pl}^2 H^2$ is the critical density for a FLRW cosmology. The evolution equation (2) is written in terms of the new defined dynamical variables as $x^2 + y^2 + \Omega_m = 1$, where $\Omega_m = \rho_m/\rho_c$ is positive, which implies that both variables $x$ and $y$ are bounded between 0 and 1.

Using the previous relation and the evolution equations (2-5), we find the autonomous differential equations in terms of the dynamical variables as

$$\frac{dx}{dN} = \frac{3}{2} \sigma y^2 + \frac{3}{2} x \left[ \gamma_k (x^2 - 1) + \gamma_m (1 - x^2 - y^2) \right],$$

$$\frac{dy}{dN} = \frac{3}{2} y \left[ -\sigma x + \gamma_m (1 - y^2) + x^2 (\gamma_k - \gamma_m) \right],$$

$$\frac{d\sigma}{dN} = -3 \sigma^2 x (\Gamma - 1) + \frac{3 \sigma (2 \Xi \gamma_k + \gamma_k - 2)}{2 \gamma_k (2 \Xi + 1)} \left( \gamma_k - \frac{\sigma y^2}{x} \right),$$

where $\Xi = \gamma_m (1 - x^2) / (\gamma_k - \gamma_m)$ and $\gamma_k = X g' g = \phi'$. These equations allow to find the attractor solutions as $x = y = \sigma = 0$.
where we required the introduction of the auxiliary variables

\[ \gamma_k = \frac{\rho_k + P_k}{\rho_k} = \frac{2X F_X}{2X F_X - F}, \]  

(13)

\[ \Xi = \frac{X F_{XX}}{F_X}, \]  

(14)

\[ \Gamma = \frac{V V_{\phi\phi}}{V_{\phi}^2}. \]  

(15)

These auxiliary variables depend on derivatives of the potential and kinetic terms in the Lagrangian and in general are not constant, which implies that in order to solve the system of equations (10-12) it is necessary to consider evolution equations for the extra three variables. These equations in general will depend on higher derivatives of the Lagrangian terms which in turn will have evolution equations.

In order to cut the succession of equations it is possible to demand the Lagrangian to satisfy that both \( \gamma_k \) and \( \Gamma \) be constant, which implies that

\[ F(X) = AX^n, \]  

(16)

and

\[ V(\phi) = B(\phi - \phi_0)^n, \]  

or

\[ V(\phi) = V_0 e^{-\lambda \phi}, \]  

(18)

where \( A, B, V_0, \eta, n, \) and \( \lambda \) are constants. In this case \( \gamma_k = 2\eta/(2\eta - 1), \Xi = \eta - 1, \) and \( \Gamma \) is one if the potential is exponential or \( (n - 1)/n \) if it is a power-law. In this particular Lagrangian the system of equations (10-12) is closed and we can study its solutions without concern about other equations, as we will do in the next section.

3. Critical points

The critical points of the system are obtained when the equations (10-12) are equal to zero. Besides analyzing the conditions for the existence of the critical points we determine their stability. Equating only the equations (10) and (11) to zero, we obtain the critical points (a), (b), (\( \alpha \)), (\( \beta \)) and (\( \gamma \)) summarized in table 1, where the first two are present in the canonical case and have been analyzed in previous works like [12, 13] and the last three are new. Another two critical points are found for a particular case of the Lagrangian, when it can be written either as a canonical Lagrangian with exponential potential (18) or as the sum of two power-law terms (16, 17), where their exponents satisfy

\[ \eta = \frac{n}{2 + n}. \]  

(19)

These critical points are named (c) and (d), and written in table 1 too. As the values for \( x \) and \( y \) in these points are dependent on \( \sigma \), for them to be fixed it is also necessary for the evolution equation (12) to be zero, in [1] we showed that this is the case when (19) is satisfied or the Lagrangian is a canonical with exponential potential. In the next section we will add more elements to this analysis and we will show that in fact \( \sigma \) is a function of \( x \) and \( y \) for these Lagrangians.
Table 1. Stability and existence of the critical points. The points labeled with Latin letters can be reduced in the canonical case to the ones already studied in the literature, the points with Greek letters are new. The points (c) and (d) exist only when (19) holds or when the Lagrangian is canonical with exponential potential.

|   |   |   | Existence | Stability | Ωφ | γφ |
|---|---|---|-----------|----------|----|----|
| (a) | 0 | 0 | Always | Unstable node for $\gamma_k < \gamma_m$ | 0 | - |
|     |   |   |         | Saddle point for $\gamma_m < \gamma_k$ |   |    |
| (α) | 0 | 1 | $\sigma = 0$ | Saddle point for $\gamma_k < 0$ | 1 | 0 |
|     |   |   |   | Stable node for $\gamma_k > 0$ |   |    |
| (b) | 1 | 0 | Always | Unstable node for $\{\gamma_m, \sigma\} \{\gamma_k\}$ | 1 | $\gamma_k$ |
|     |   |   |   | Stable node for $\gamma_k < \{\gamma_m, \sigma\}$ |   |    |
|     |   |   |   | Otherwise saddle point |   |    |
| (β) | 0 | Arbitrary | $\gamma_m = 0$ | Stable line for $\omega_k > -1$ | $\frac{y^2}{6}$ | 0 |
|     |   |   |   | Unstable otherwise |   |    |
| (γ) | Arbitrary | 0 | $\gamma_k = \gamma_m$ | Stable line for $x \sigma > \gamma_k$ | $\frac{x^2}{6}$ | $\omega_m$ |
|     |   |   |   | Unstable otherwise |   |    |
| (c) | $\frac{\sigma}{\gamma_k}$ | $\sqrt{1 - \frac{\sigma^2}{\gamma_k^2}}$ | $\sigma^2 > \gamma_k^2$ | Saddle point for $\sigma^2 > \gamma_k \gamma_m$ | 1 | $\frac{\sigma^2}{\gamma_k}$ |
|     |   |   | $\sigma \gamma_k > 0$ | Otherwise stable node |   |    |
| (d) | $\frac{\gamma_m}{\sigma}$ | $\sqrt{\gamma_m (\gamma_k - \gamma_m)}$ | $\sigma > \frac{\gamma_m}{\gamma_k}$ | Stable node for $\frac{\sigma^2 (9 \gamma_m - \gamma_k)}{8 \gamma_k^2 \gamma_m} < 1$ | $\frac{\gamma_k \gamma_m}{\sigma^2}$ | $\gamma_m$ |
|     |   |   | $\gamma_m < \gamma_k$ | Stable spiral otherwise |   |    |

4. Symmetry
As we showed in the previous section, when the Lagrangian of the scalar field is the sum of two power-law terms and the relation (19) holds, two extra critical points are present in the system. In this section we will see that for this system exists a symmetry in the equations of motion (2-5). This symmetry permits the number of degrees of freedom to be reduced to only two.

The equations of motion (2-5) written for a Lagrangian with power-law terms (16, 17) and with $dN$ as independent variable instead of $dt$ are given by

$$H^2 = \frac{1}{3M_{Pl}^2}[(2\eta - 1)AX^n + B\phi^n + \rho_m], \quad (20)$$

$$H \frac{dH}{dN} = -\frac{1}{2M_{Pl}^2}[2\eta AX^n + \gamma_m \rho_m], \quad (21)$$

$$\frac{d\rho_m}{dN} = -3\gamma_m \rho_m, \quad (22)$$

$$\frac{d}{dN}((2\eta - 1)AX^n + B\phi^n) = -6\eta AX^n. \quad (23)$$

If we consider $\phi$, $X$, and $\rho_m$ as the independent variables we can apply the transformation

$$\phi \rightarrow \xi^{2\eta} \phi, \quad X \rightarrow \xi^{2n} X, \quad \rho_m \rightarrow \xi^{2\eta} \rho_m, \quad (24)$$

where if we leave $N$ invariant, from the definition of $X$

$$X = \frac{1}{2} \left( H \frac{d\phi}{dN} \right)^2 \quad (25)$$
the Hubble parameter transforms as $H \rightarrow \xi^{n-2\eta}H$. This transformation will leave invariant the equations of motion (20-23) as long as $n$ and $\eta$ satisfy the relation (19), which determines the transformation of the Hubble parameter to be $H \rightarrow \xi^nH$.

The fact that the equations of motion are invariant under the transformations defined in (24) for arbitrary $\xi$ means that this set of transformations is a symmetry of the system, and allows to reduce the number of degrees of freedom to two. The only thing needed is to define a set of variables that are invariant under the transformation, but $x$, $y$ and $\sigma$ are already invariant. Moreover, there are only two invariant degrees of freedom, which implies that one of them is dependent of the other. In fact we can see that the relation

$$\sigma = -M_{Pl}nB^{1/n} \sqrt{\frac{2}{3}}(A(2\eta - 1))^{-1/2\eta} \left( \frac{x}{y} \right)^{2/n},$$

holds when (19) is satisfied. If we define

$$s \equiv -M_{Pl}nB^{1/n} \sqrt{\frac{2}{3}}(A(2\eta - 1))^{-1/2\eta},$$

the dynamical system can be written from (10, 11) as

$$\frac{dx}{dN} = \frac{3}{2} x \left[ sy \left( \frac{y}{x} \right)^{2/\gamma_k} + \gamma_k(x^2 - 1) + \gamma_m(1 - x^2 - y^2) \right],$$

$$\frac{dy}{dN} = -\frac{3}{2} s x^2 \left( \frac{y}{x} \right)^{2/\gamma_k} + 3 y \left[ \gamma_m(1 - y^2) + x^2(\gamma_k - \gamma_m) \right],$$

corresponding to only two equations for two variables. Here we used the fact that $\gamma_k = 2n/(n-2)$ when the relation (19) is satisfied.

4.1. Exponential potential

The behaviour stated in the this section is similar to what happens for the canonical Lagrangian with exponential potential (18), and studied in Ref. [14]. In that case a similar procedure to the one used in this section can tell us that the equations of motion are invariant under the transformations

$$\phi \rightarrow \phi + \xi, \quad X \rightarrow e^{-\lambda \xi}X, \quad \rho_m \rightarrow e^{-\lambda \xi} \rho_m. \quad (30)$$

The fact that the system is invariant under the transformations above, allows it to be described by only two variables as in the original work of this type of cosmological analysis [12]. In that case, unlike the relation (26), $\sigma$ is a constant given by $\sqrt{2/3M_{Pl}\lambda}$.

5. Phase space

The critical points found in section 3 are still valid for the 2-dimensional system (28, 29), as can be seen by equating the right-hand side of both equations to zero. The critical points (c) and (d) depend on $\sigma$ that is function of $x$ and $y$ so we should be able to drop de dependence on $\sigma$ using the relation (26). For (c) it cannot be made analytically while for (d) we obtain:

$$x_d = \sqrt{\gamma_m(\gamma_k - \gamma_m)} \left( \frac{\gamma_m}{\gamma_k - \gamma_m} \right)^{1/\gamma_k}, \quad y_d = \frac{\gamma_k - \gamma_m}{s} \left( \frac{\gamma_m}{\gamma_k - \gamma_m} \right)^{1/\gamma_k}. \quad (31)$$

The behaviour of the critical points stated in the table 1 can be seen more graphically in the phase space plot in figure 1. There we see the phase space of a system with Lagrangian

$$\mathcal{L} = AX^2 + B\phi^{-4}, \quad (32)$$

plus a dust type barotropic fluid with $\gamma_m = 1$, where we chose $s = 4$ meaning that $AB = 4M_{Pl}^4/27$. In the figure we can observe the critical points:
Figure 1. Phase space for the dynamical system corresponding to $\mathcal{L} = AX^2 + B\phi^{-4}$. The critical points (a), (b), (c) and (d) can be observed.

- (a) The (0,0) point that in this case is a saddle point.
- (b) The (1,0) point that in this case is an unstable node.
- (c) Corresponding to a saddle point, with domination of field density over the barotropic fluid.
- (d) The point $(3^{1/4}/4, 3^{-1/4}/4)$ corresponding to a stable spiral, and whose value can be computed from expression (31) for the values of this particular example.

The other critical points are not present in this example.

6. Conclusions

In this work we made an analysis of the dynamical system associated to a flat FLRW cosmology containing a barotropic fluid and a scalar field with Lagrangian of the form $\mathcal{L} = F(X) - V(\phi)$. We observed that this system can be described by three variables and its equations of motion, from those we defined $x$, $y$ and $\sigma$ and obtained their evolution equations. For Lagrangians with constant values for $\gamma_k$ and $\Gamma$, the three autonomous differential equations (10 - 12) are enough to describe the system. For more general Lagrangians however, it is necessary to obtain dynamical equations for the variables $\gamma_k$, $\Gamma$, and $\Xi$, that in turn will include new variables in terms of higher derivatives of the Lagrangian, that will require extra equations of motion. In order to have only the minimal three equation system, in our work [1] we used power-law kinetic terms that produce $\gamma_k$ constant, and power-law or exponential potentials that correspond to a constant value of $\Gamma$.

We obtained the critical points of the system, and studied their existence and stability conditions, which are summarized in table 1. The critical points (a), (b), (c), (d) can be reduced for the canonical Lagrangian to those studied in previous works [13, 15], meanwhile (α), (β) and (γ) are new.

In our previous work [1] we showed that for the existence of the critical points (c) and (d), the Lagrangian needs to be either canonical with exponential potential or the sum of two power-law terms that satisfy the relation (19). In the present work we showed that when this relation
holds, the equations of motion (20 - 23) are invariant under the set of transformations defined by (24). This symmetry allows to reduce the number of degrees of freedom of the system to two. We showed that \( x \) and \( y \) are invariant under the same transformation (24) and hence are suitable to describe the system. Therefore, it was possible to obtain \( \sigma \) as function of \( x \) and \( y \) in equation (26). Using this equation we found the two dimensional autonomous system (28, 29).

We showed that for the two dimensional system the critical points are the same as the ones obtained in table 1, but with \( \sigma \) as a function of \( x \) and \( y \). The critical point (c) can be obtained then as a function of the auxiliary variables \( \gamma_k \) and \( \gamma_m \), eliminating the dependence on \( \sigma \), however it is not possible to do it analytically. For the critical point (d) the analytical expression can be obtained as we have done in (31).

The symmetry transformations of (24) are similar to those of (30), corresponding to canonical exponential Lagrangians and studied in [14]. Both transformations allow the systems to be treated with a two dimensional set of dynamical equations. For the canonical exponential Lagrangian the equations get reduced to (10, 11) with a constant value of \( \sigma \), while for the Lagrangian whose terms satisfy (19), they get reduced to equations (28, 29).

**Acknowledgments**

JDS is supported by ININ Grant.

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