Dual Quantum Mechanics

W. Chagas-Filho
Physics Department, Federal University of Sergipe, Brazil

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Abstract

We point out a possible complementation of the basic equations of quantum mechanics in the presence of gravity. This complementation is suggested by the well-known fact that quantum mechanics can be equivalently formulated in the position or in the momentum representation. As a way to support this complementation, starting from the action that describes conformal gravity in the world-line formalism, we show that there are duality transformations that relate the dynamics in the presence of position dependent vector and tensor fields to the dynamics in the presence of momentum dependent vector and tensor fields.

1 Introduction

The wave-particle duality of matter and energy at the quantum level is one of the most fundamental aspects of physics. The far reaching theoretical implications of the existence of the wave-particle duality are not completely understood until now. The best known implication of this duality is the fact that quantum mechanics can be equivalently formulated in the position representation or in the momentum representation. While the position representation emphasizes the particle aspect by assuming a defined position, the momentum representation is related to the wave aspect because the magnitude $p$ of the momentum of a particle is directly related to the wave length $\lambda$ of the associated wave by the de Broglie relation $p = \frac{h}{\lambda}$, where $h$ is Planck’s constant.

Some years ago, it was discovered [1] that this duality of the descriptions in terms of position and momentum in quantum mechanics has a symmetric version as a local $Sp(2, R)$ symmetry of a classical action describing conformal gravity on the world-line. Local $Sp(2, R)$ symmetry treats position and momentum as indistinguishable variables and, for this, it requires conformal gravity in a space-time with an extra space-like dimension and an extra time-like dimension [1]-[18]. Extra space-like dimensions had previously been found in string theory, but this was the first time that an extra time-like dimension was explicitly found. For this reason, this area of research is sometimes referred to as two-time (2T) physics.
The important aspect of 2T physics is that we can always use the local invariance of the action to eliminate the extra dimensions and work with the emergent gauge-fixed theory, containing only the physical degrees of freedom, and therefore avoiding the ghost problem in the quantized theory. Using this approach, it was demonstrated [14], [19] that the Standard Model of Particles and Forces and General Relativity as we know them are only holographic shadows of a more symmetric theory with one extra space-like dimension and one extra time-like dimension. For this reason, it is important to try to understand other aspects of the gravitational physics with two time-like dimensions. Following this point of view, in this paper we present a number of \((d + 2)\)-dimensional constrained Hamiltonian formalisms starting from the first order conformal gravity action in the world-line formalism. The new aspect of these Hamiltonian formalisms is that they are connected by duality transformations that interchange position and momentum. Dualities of this kind play a significant role in M-theory, and because of the wave-particle duality one may expect this same kind of duality to appear in quantum gravity. We use the dual classical Hamiltonian formalisms we present in this paper as a basis to suggest a complementation of the basic equations of quantum mechanics in the presence of gravity.

2T physics is usually considered as an approach that provides a new perspective for understanding one-time dynamics (1T physics) from a higher dimensional, more unified point of view. In this paper we are not interested in this 2T to 1T holographic property [7] of 2T physics. What we have in mind in this paper is the fact that all the fundamental interactions are described by gauge theories, and the fact that all gauge theories can be described as constrained Hamiltonian systems with first class constraints [20], [21].

In papers [5], [7], [19] it was required that the local invariance of the conformal gravity action must be the \(Sp(2, R)\) invariance even when space-time fields are present. To satisfy this requirement, in these papers, the space-time gravitational and vector fields must satisfy certain conditions that cannot be derived from the action. This uncomfortable situation is avoided later by performing a transition to the field theory formalism [14], [19], where an action is introduced from which the conditions leading to local \(Sp(2, R)\) invariance can be derived. However, finding a natural way of introducing space-time tensor and vector fields in conformal gravity in the world-line formalism remained an open problem until now. In this paper we present an initial attempt to solve this problem. Although our attempt does not require local \(Sp(2, R)\) symmetry, being instead based on the presence of another local symmetry of the action, it brings with it interesting new insights into the internal structure of quantum mechanics.

The \(Sp(2, R)\) symmetry is the local symmetry of conformal gravity, discovered for the world-line action in the absence of interactions in the \((d + 2)\)-dimensional flat space-time. However, \(Sp(2, R)\) is no longer the local symmetry when fermions are introduced in the formalism. It is substituted [2], [5] by local \(OSp(n | 2)\), which contains \(Sp(2, R)\) and reduces to it when the fermions are removed from the formalism. This result may be viewed as suggesting a possible solution to the problem of introducing space-time fields in the world-
line formalism. Instead of requiring local $Sp(2, R)$ symmetry of the action with space-time fields as a starting point, we can try to find another local symmetry that can be used to eliminate the same number of unphysical degrees of freedom that are eliminated using the local $Sp(2, R)$ symmetry, thus also avoiding the ghost problem in the quantized theory. A consistency condition is that this new local symmetry must reproduce the local $Sp(2, R)$ symmetry when the space-time fields are absent. We will describe in this paper how, starting from the first order conformal gravity action in the world-line formalism, we can construct a natural constrained Hamiltonian formalism, with first class constraints, containing space-time tensor and vector fields. In this Hamiltonian formalism, the first class constraints generate local symmetries that reproduce the $Sp(2, R)$ local symmetry when the space-time fields are absent. The formalism displays duality transformations that change the dynamics with position dependent tensor and vector fields into the dynamics with momentum dependent tensor and vector fields, and vice versa.

In the usual 1T physics, position dependent tensor fields play an important role in the most general position space formulation of quantum mechanics in the presence of gravity [22]. In this general formulation, these tensor fields appear in the spectral decomposition of the unity, define the correct integration measure for the inner product and are present in the most general expression of the position matrix elements for self adjoint momentum operators in position space [22]

$$
\langle x | \hat{p}_\alpha | x \rangle = \frac{i\hbar}{g^\pi(x)} \frac{\partial}{\partial x^\alpha} \left[ \frac{1}{g^\pi(x)} \delta^n(x - x) \right] + \frac{1}{\sqrt{g(x)}} A_\alpha(x) \delta^n(x - x) 
$$  (1.1)

where $g(x) = \det g_{\alpha\beta}(x)$ and $\alpha, \beta = 1, ..., n$. Since quantum mechanics can be equivalently formulated in the position or in the momentum representation, the appearance of a momentum dependent tensor field in conformal gravity can be interpreted as an indication that the momentum space versions of quantum mechanical equations such as (1.1) and others are still lacking. The construction and the justification of these momentum space equations are the motivations for this paper.

As can be seen in (1.1), the other central object in the general position space formulation of quantum mechanics described in [22] is the vector field $A_\alpha(x)$. It has a vanishing strength tensor,

$$
F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} = 0 
$$  (1.2)

and because of this condition it defines a section of a flat U(1) bundle over the position space. The vector field is present only if the position space has a non-trivial topology. In position spaces with trivial topology $A_\alpha(x)$ can always be gauged away [22].

In this paper we describe how one can extend to momentum space the general position space formulation of quantum mechanics described in [22].
paper is organized as follows. In section two we describe how duality trans-
formations relate position dependent and momentum dependent tensor fields in
d-dimensional relativistic massless particle theory in a constrained Hamiltonian
framework.

In section three we review the global and local symmetries of the conformal
gravity action.

Section four presents the basic equations of a formulation of quantum me-
chanics that completely incorporates the wave-particle duality in the presence
of gravity. This is done by introducing the corresponding momentum space
expressions of the position space expressions obtained in [22].

In section five we present the developments that suggest our construction
in section four. The starting point is the action describing conformal grav-
ity on the world-line. In section 5.1 we present an action in a position de-
pendent tensor background and compute the conserved Hamiltonian Noether
charge corresponding to the local invariance of the action. We find that the
conserved Noether charge and the equations of motion reproduce the conserved
charge and the equations of motion of conformal gravity in a transition to flat
space. In section 5.2 we study the same situation for an action in a momentum
dependent tensor background and find exactly the same behavior of the con-
served Noether charge and of the equations of motion. In addition, a duality
transformation changes the equations of motion in the momentum dependent
background into the equations of motion in the position dependent background
obtained in section 5.1. In section 5.3 we consider the case of flat space with a
position dependent vector field. In section 5.4 we consider the case of flat space
with a momentum dependent vector field. Again we discover that the conserved
charges and the equations of motion reproduce those of conformal gravity when
the vector field vanishes, and that the equations of motion in the presence of the
vector field are turned into one another by a duality transformation. Section 5.5
considers the case when both the tensor and vector fields are present, and we
are lead to identical conclusions about the conserved charges and equations of
motion. All the actions we compute in this section describe the correct number
of physical degrees of freedom, thus avoiding the ghost problem in the quantized
theory. Concluding remarks appear in section six.

2 Massless Relativistic Particles

In this section we consider massless relativistic particle theory. We describe
how a duality transformation relates the local symmetry and the equations of
motion in a position dependent background to the local symmetry and equations
of motion in a momentum dependent background.

A massless scalar relativistic particle in a $d$-dimensional Minkowski space-
time with signature $(d - 1, 1)$ is described by the action

$$S = \frac{1}{2} \int d\tau \lambda^{-1} \dot{x}^2$$  \hspace{1cm} (2.1)
where $\lambda(\tau)$ is an auxiliary variable, $x^\mu = x^\mu(\tau)$, $\dot{x}^2 = \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}$ and $\eta_{\mu\nu}$, with $\mu, \nu = 0, 1, \ldots, d - 1$, is the flat Minkowski metric. A dot denotes derivatives with respect to the arbitrary parameter $\tau$. Action (2.1) is invariant under the local infinitesimal reparametrizations

$$\delta x^\mu = \alpha(\tau) \dot{x}^\mu \quad \delta \lambda = \frac{d}{d\tau} [\alpha(\tau) \lambda] \quad (2.2)$$

where $\alpha(\tau)$ is an arbitrary function. Due to the presence of this local invariance action (2.1) can be looked at as describing gravity on the world-line. It is well known that action (2.1) also has a global $d$ dimensional conformal invariance. Global conformal invariance in $d$ dimensions is isomorphic to global Lorentz invariance in $d + 2$ dimensions [23], [24]. Therefore, there is a $d + 2$ dimensional Lorentz invariant extension of action (2.1). This higher dimensional action is the subject of the next section.

In the transition to the Hamiltonian formalism action (2.1) gives the canonical momenta

$$p_\lambda = 0 \quad (2.3)$$
$$p_\mu = \frac{\dot{x}_\mu}{\lambda} \quad (2.4)$$

and the Hamiltonian

$$H = \frac{1}{2} \lambda p^2 \quad (2.5)$$

Equation (2.3) is a primary constraint [20]. Introducing the Lagrange multiplier $\xi(\tau)$ for this constraint we can write the total Hamiltonian [20]

$$H_T = H + \xi p_\lambda = \frac{1}{2} \lambda p^2 + \xi p_\lambda \quad (2.6)$$

Introducing the Poisson bracket $\{\lambda, p_\lambda\} = 1$ and requiring the dynamical stability [20] of constraint (2.3)

$$\dot{p}_\lambda = \{p_\lambda, H_T\} = 0 \quad (2.7)$$

we obtain the secondary constraint

$$\phi = \frac{1}{2} p^2 \approx 0 \quad (2.8)$$

Constraint (2.8) needs not be incorporated into the formalism because it already appears in the Hamiltonian. Requiring its dynamical stability

$$\dot{\phi} = \{\phi, H_T\} = 0 \quad (2.9)$$

we find that it is automatically satisfied. Constraints (2.3) and (2.8) have vanishing Poisson bracket and are therefore first class constraints [20]. Constraint (2.3) generates translations in the arbitrary variable $\lambda(\tau)$ and can be dropped from the formalism. The notation $\approx 0$ means that $\phi$ weakly vanishes [20], [21].
Action (2.1) can be rewritten in Hamiltonian form as

\[ S = \int \! d\tau (\dot{x} \cdot p - \frac{1}{2} \lambda p^2) \]  

(2.10)

By introducing the Poisson brackets

\[ \{x_\mu, x_\nu\} = 0 \quad \{p_\mu, p_\nu\} = 0 \quad \{x_\mu, p_\nu\} = \eta_{\mu\nu} \]  

(2.11)

we can check that the first class constraint (2.8) generates the local transformations with arbitrary parameter \( \epsilon(\tau) \)

\[ \delta x_\mu = \epsilon(\tau) \{x_\mu, \phi\} = \epsilon p_\mu \]  

(2.12a)

\[ \delta p_\mu = \epsilon(\tau) \{p_\mu, \phi\} = 0 \]  

(2.12b)

under which action (2.10) transforms as

\[ \delta S = \int \! d\tau \left[ \frac{d}{d\tau} (\epsilon \phi) + i \phi - \phi \delta \lambda \right] \]  

(2.12c)

If we choose \( \delta \lambda = \dot{\epsilon} \) the variation (2.12c) becomes

\[ \delta S = \int \! d\tau \frac{d}{d\tau} (\epsilon \phi) \]  

(2.12d)

In this case the quantity

\[ Q = \epsilon \phi = \frac{1}{2} \epsilon p^2 \]  

(2.13)

can be interpreted as the conserved Hamiltonian Noether charge, or as the generator of the local symmetry transformations (2.12), depending on whether the equations of motion are satisfied or not [25]. Using the local invariance generated by the charge (2.13), it is possible [21] to eliminate the time-like degree of freedom by a gauge-fixing. This leaves us with \( d - 1 \) canonical pairs describing the physical degrees of freedom. In this case there will be no negative norm states (ghosts) in the quantized theory. As we will see in the following, all the actions we discuss in this paper describe this same number of physical canonical pairs.

Hamiltonian (2.5) generates the equations of motion.

\[ \dot{x}_\mu = \{x_\mu, H\} = \lambda p_\mu \]  

(2.14a)

\[ \dot{p}_\mu = \{p_\mu, H\} = 0 \]  

(2.14b)

\subsection*{2.1 Position dependent tensor fields}

Action (2.10) has an extension in curved space given by [19]

\[ S = \int \! d\tau [\dot{x}^\mu p_\mu - \frac{1}{2} \lambda g_{\mu\nu}(x)p^\mu p^\nu] \]  

(2.15)
where the Hamiltonian is

\[ H = \frac{1}{2} \lambda g_{\mu\nu}(x)p^\mu p^\nu \]  \hspace{1cm} (2.16)

The position dependent tensor field \( g_{\mu\nu}(x) \) is defined over phase space. The equation of motion for \( \lambda(\tau) \) gives the constraint

\[ \dot{\phi} = \frac{1}{2} g_{\mu\nu}(x)p^\mu p^\nu \approx 0 \]  \hspace{1cm} (2.17)

Requiring the dynamical stability [20] condition \( \dot{\phi} = \{\phi, H_T\} = 0 \) for constraint (2.17), we find that it is automatically satisfied. This implies that constraint (2.17) is a first class constraint [20].

Constraint (2.17) generates the local transformations

\[ \delta x_\mu = \epsilon(\tau)\{x_\mu, \phi\} = \epsilon g_{\mu\nu}(x)p^\nu \]  \hspace{1cm} (2.18a)

\[ \delta p_\mu = \epsilon(\tau)\{p_\mu, \phi\} = -\frac{1}{2} \epsilon \partial g_{\alpha\beta}(x) \frac{\partial p^\alpha}{\partial x^\mu} p^\beta \]  \hspace{1cm} (2.18b)

under which

\[ \delta S = \int d\tau \left[ \frac{d}{d\tau}(\epsilon \phi) + \dot{\epsilon} \phi - \phi \delta \lambda \right] \]  \hspace{1cm} (2.18c)

If we then choose \( \delta \lambda = \dot{\epsilon} \) we get

\[ \delta S = \int d\tau \left[ \frac{d}{d\tau}(\epsilon \phi) \right] \]  \hspace{1cm} (2.18d)

This shows that the conserved Noether charge corresponding to the local symmetry transformations (2.18) in the background \( g_{\mu\nu}(x) \) is the quantity

\[ Q = \epsilon \phi = \frac{1}{2} \epsilon g_{\mu\nu}(x)p^\mu p^\nu \]  \hspace{1cm} (2.19)

Using the local symmetry generated by the conserved charge (2.19) it is possible [21] to eliminate the time-like degrees of freedom. This leaves only \( d - 1 \) physical canonical pairs. In this case the physical components of the tensor field will depend only on the physical components of the position variable. There will be no ghosts in the quantized theory.

Hamiltonian (2.16) generates the equations of motion

\[ \dot{x}_\mu = \{x_\mu, H\} = \lambda g_{\mu\nu}(x)p^\nu \]  \hspace{1cm} (2.20a)

\[ \dot{p}_\mu = \{p_\mu, H\} = -\frac{1}{2} \lambda \frac{\partial g_{\alpha\beta}(x)}{\partial x^\mu} p^\alpha p^\beta \]  \hspace{1cm} (2.20b)

The equations of motion (2.20) reproduce the equations of motion (2.14) when \( g_{\mu\nu}(x) = \eta_{\mu\nu} \).
2.2 Momentum dependent tensor fields

Now we apply to action (2.15) the duality transformation

\[ x_\mu(\tau) \rightarrow p_\mu(\tau) \quad p_\mu(\tau) \rightarrow -x_\mu(\tau) \] (2.21)

This duality transformation leaves invariant the definition \( \{ A, B \} = \frac{\partial A}{\partial x_\mu} \frac{\partial B}{\partial p_\mu} - \frac{\partial A}{\partial p_\mu} \frac{\partial B}{\partial x_\mu} \) of the classical Poisson bracket between two functions \( A \) and \( B \) of the canonical variables, and also the fundamental commutator \([x_\mu, p_\nu] = x_\mu p_\nu - p_\nu x_\mu = i\hbar \eta_{\mu\nu}\) of quantum mechanics. We obtain the action

\[ S = \int d\tau \left[ -x^\mu \dot{p}_\mu - \frac{1}{2} \lambda g_{\mu\nu}(p)x^\mu x^\nu \right] \] (2.22)

where the Hamiltonian is

\[ H = \frac{1}{2} \lambda g_{\mu\nu}(p)x^\mu x^\nu \] (2.23)

and \( g_{\mu\nu}(p) \) describes a momentum dependent tensor field defined over phase space. The equation of motion for the variable \( \lambda(\tau) \) gives the constraint

\[ \phi = \frac{1}{2} g_{\mu\nu}(p)x^\mu x^\nu \approx 0 \] (2.24)

Requiring the dynamical stability [20] of constraint (2.24), we find that it is automatically satisfied and that constraint (2.24) is a first class constraint [20]. It generates the local transformations

\[ \delta x_\mu = \epsilon(\tau) \{ x_\mu, \phi \} = \frac{1}{2} \epsilon \frac{\partial g_{\alpha\beta}(p)}{\partial p^\mu} x^\alpha x^\beta \] (2.25a)

\[ \delta p_\mu = \epsilon(\tau) \{ p_\mu, \phi \} = -\epsilon g_{\mu\alpha}(p)x^\alpha \] (2.25b)

under which

\[ \delta S = \int d\tau \left[ \frac{d}{d\tau}(\epsilon \phi) + i\phi - \phi \delta \lambda \right] \] (2.25c)

If we choose \( \delta \lambda = \dot{\epsilon} \) the variation (2.25c) becomes

\[ \delta S = \int d\tau \frac{d}{d\tau}(\epsilon \phi) \] (2.25d)

This demonstrates that the conserved Noether charge in the background \( g_{\mu\nu}(p) \) is the quantity

\[ Q = \epsilon \phi = \frac{1}{2} \epsilon g_{\mu\nu}(p)x^\mu x^\nu \] (2.26)

Using the local symmetry generated by the conserved charge (2.26) it is possible [21] to eliminate the time-like degrees of freedom. This again leaves only \( d - 1 \) physical canonical pairs. The physical components of the tensor field will depend only on the physical components of the momentum variable, and there will be
no ghosts in the quantized theory. The $d$-dimensional massless particle actions (2.10), (2.15) and (2.22) therefore describe the dynamics of the same number of physical canonical pairs.

Notice that the local symmetry transformations (2.25) and the corresponding conserved charge (2.26) in the background $g_{\mu\nu}(p)$ are turned by the duality transformation (2.21) into the local symmetry transformations (2.18) and the corresponding conserved charge (2.19) in the background $g_{\mu\nu}(x)$. Although the duality transformation (2.21) is not a symmetry of actions (2.15) and (2.22), it relates the local symmetries of these two actions.

Hamiltonian (2.23) generates the equations of motion
\begin{align*}
\dot{x}_\mu &= \{x_\mu, H\} = \frac{1}{\lambda} \frac{\partial g_{\alpha\beta}(p)}{\partial p^\mu} x^\alpha x^\beta \\
\dot{p}_\mu &= \{p_\mu, H\} = -\lambda g_{\mu\alpha}(p)x^\alpha
\end{align*}
(2.27a)
(2.27b)
Notice that the equations of motion (2.27) are turned by the duality transformation (2.21) into the equations of motion (2.20). The transformation (2.21) relates two possible Hamiltonian descriptions of the dynamics in the presence of tensor backgrounds. It is this dual behavior of the local symmetries and of the dynamical evolutions in the presence of space-time fields that we want to reproduce in $d + 2$ dimensions. For this purpose we must first show that our $(d + 2)$-dimensional actions have a local symmetry that can be used to eliminate three unphysical degrees of freedom from each of the canonical variables, leaving only $d − 1$ physical canonical pairs. Local $Sp(2, R)$ is not an acceptable symmetry in this case because it treats position and momentum as indistinguishable variables. To search for another local symmetry, we must first understand how local $Sp(2, R)$ works. This is the subject of the next section.

## 3 Conformal Gravity and 2T Physics

The construction of 2T physics [1]-[18] is based on the introduction of a new gauge invariance in phase space by gauging the duality (2.21) for the quantum commutator $[X_M, P_N] = i\hbar \eta_{MN}$ with $M, N = 0, 1, ..., d + 1$. This procedure leads to a symplectic $Sp(2, R)$ gauge theory. To remove the distinction between position and momentum we rename them $X^M_1 = X^M(\tau)$ and $X^M_2 = P^M(\tau)$ and define the doublet $X^M_i(\tau) = (X^M_1, X^M_2)$. The local $Sp(2, R)$ symmetry acts as [1]
\begin{equation}
\delta X^M_i(\tau) = \epsilon_{ik} \omega^{jl}(\tau) X^M_l(\tau)
\end{equation}
(3.1)
$\omega^{ij}(\tau)$ is a symmetric matrix containing three local parameters and $\epsilon_{ij}$ is the Levi-Civita symbol that serves to raise or lower indices. The $Sp(2, R)$ gauge field $A^{ij}$ is symmetric in $(i, j)$ and transforms as [1]
\begin{equation}
\delta A^{ij} = \partial_\tau \omega^{ij} + \omega^{ij} \epsilon_{kl} A^{kl} + \omega^{jk} \epsilon_{kl} A^{il}
\end{equation}
(3.2)
The covariant derivative is [1]
\begin{equation}
D_\tau X^M_i = \partial_\tau X^M_i - \epsilon_{ik} A^{kl} X^M_l
\end{equation}
(3.3)
An action invariant under the local $Sp(2,R)$ symmetry is [1]

$$S = \frac{1}{2} \int d\tau (D_\tau X_i^M) \epsilon^{ij} X_j^N \eta_{MN}$$

$$= \int d\tau \left[ \frac{1}{2} (\partial_\tau X_i^M X_j^N - X_i^M \partial_\tau X_j^N) - \frac{1}{2} A^{ij} X_i^M X_j^N \right] \eta_{MN}$$

$$= \int d\tau \left[ \frac{1}{2} (\dot{X}^M P_M - X^M \dot{P}_M) - \left( \frac{1}{2} \lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2} \lambda_3 X^2 \right) \right]$$ (3.4a)

where $A^{11} = \lambda_3$, $A^{12} = A^{21} = \lambda_2$, $A^{22} = \lambda_1$. After an integration by parts this action can be written as

$$S = \int d\tau \left[ \frac{1}{2} (\dot{X}^M P_M - \left( \frac{1}{2} \lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2} \lambda_3 X^2 \right) \right]$$ (3.4b)

From the form (3.4b) for the action we can identify the Hamiltonian as

$$H = \frac{1}{2} \lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2} \lambda_3 X^2$$ (3.5)

The first-order Hamiltonian action (3.4b) describes conformal gravity on the world-line [1], [26], [27]. We can obtain the usual second-order Lagrangian action for conformal gravity by solving the equation of motion for $P_M$ that follows from action (3.4b) and inserting the solution back into it.

The equations of motion for the variables $\lambda_\alpha(\tau)$, $\alpha = 1, 2, 3$ give the constraints

$$\phi_1 = \frac{1}{2} P^2 \approx 0$$ (3.6a)

$$\phi_2 = X.P \approx 0$$ (3.6b)

$$\phi_3 = \frac{1}{2} X^2 \approx 0$$ (3.6c)

and therefore the $\lambda_\alpha$ are arbitrary variables. Constraints (3.6) can only be simultaneously satisfied in a flat space-time with two time-like dimensions [1]-[18]. Constraints (3.6) were independently obtained in [23].

We now introduce the Poisson brackets

$$\{P_M, P_N\} = 0 \quad \{X_M, X_N\} = 0 \quad \{X_M, P_N\} = \eta_{MN}$$ (3.7)

and require the dynamical stability [20] of constraints (3.6)

$$\dot{\phi}_\alpha = \{\phi_\alpha, H\} = \lambda_\beta \{\phi_\alpha, \phi_\beta\} = 0$$ (3.8)

We then obtain the bracket relations

$$\{\phi_1, \phi_2\} = -2\phi_1$$ (3.9a)

$$\{\phi_1, \phi_3\} = -\phi_2$$ (3.9b)
\[ \{ \phi_2, \phi_3 \} = -2\phi_3 \quad (3.9c) \]
The bracket relations (3.9) weakly vanish, indicating that the stability conditions (3.8) are automatically satisfied and that constraints (3.6) are first class constraints. The algebra (3.9) is the \( Sp(2,R) \) gauge algebra.

Action (3.4b) is invariant under global Lorentz \( SO(d,2) \) transformations with generator \( L_{MN} = X_M P_N - X_N P_M \)

\[ \delta X_M = \frac{1}{2} \omega_{RS} \{ L_{RS}, X_M \} = \omega_{MR} X_R \quad (3.10a) \]
\[ \delta P_M = \frac{1}{2} \omega_{RS} \{ L_{RS}, P_M \} = \omega_{MR} P_R \quad (3.10b) \]
\[ \delta \lambda_\alpha = 0, \quad \alpha = 1, 2, 3 \quad (3.10c) \]
under which \( \delta S = 0 \). The \( L_{MN} \) are gauge invariant because they have vanishing brackets with constraints (3.6). The remarkable consequence of this is that the global Lorentz \( SO(d,2) \) symmetry survives the gauge-fixing process and is present in all the gauge-fixed systems. Perhaps the most striking example of this is the hidden Lorentz \( SO(d,2) \) symmetry of the non-relativistic massive particle (see ref. [4] for details).

The first class constraints (3.6) generate local infinitesimal \( Sp(2,R) \) transformations with arbitrary parameters \( \epsilon_\alpha(\tau) \)

\[ \delta X_M = \epsilon_\alpha(\tau) \{ X_M, \phi_\alpha \} = \epsilon_1 P_M + \epsilon_2 X_M \quad (3.11a) \]
\[ \delta P_M = \epsilon_\alpha(\tau) \{ P_M, \phi_\alpha \} = -\epsilon_2 P_M - \epsilon_3 X_M \quad (3.11b) \]
under which

\[ \delta S = \int d\tau \left[ (\epsilon_2 \lambda_1 - \epsilon_1 \lambda_2) 2\phi_1 + (\epsilon_3 \lambda_2 - \epsilon_1 \lambda_3) 2\phi_2 \right. \]
\[ + (\epsilon_3 \lambda_2 - \epsilon_2 \lambda_3) 2\phi_3 + \frac{d}{d\tau}(\epsilon_\alpha \phi_\alpha) + \epsilon_\alpha \phi_\alpha - \phi_\alpha \delta \lambda_\alpha \] \quad (3.11c) \]

Using the constraint equations (3.6) we can write

\[ \delta S \approx \int d\tau \left[ \frac{d}{d\tau}(\epsilon_\alpha \phi_\alpha) + \epsilon_\alpha \phi_\alpha - \phi_\alpha \delta \lambda_\alpha \right] \quad (3.11d) \]
We can see from (3.11d) that if we now choose \( \delta \lambda_\alpha = \dot{\epsilon}_\alpha \) the variation of the action becomes

\[ \delta S \approx \int d\tau \left[ \frac{d}{d\tau}(\epsilon_\alpha \phi_\alpha) \right] \quad (3.11e) \]

Equation (3.11e) indicates that, in conformal gravity in the world-line formalism, the quantity

\[ Q = \epsilon_\alpha \phi_\alpha = \frac{1}{2} \epsilon_1 P^2 + \epsilon_2 X.P + \frac{1}{2} \epsilon_3 X^2 \quad (3.12) \]
can be interpreted as the conserved Hamiltonian Noether charge or as the generator of the local infinitesimal transformations (3.11), depending on whether the equations of motion are satisfied or not [25]. Using the local symmetry generated by the conserved charge (3.12) we can eliminate one space-like degree of freedom and two time-like degrees of freedom of each of the canonical variables [1]-[19]. In this case we are left with \( d - 1 \) physical canonical pairs, which is the same number of physical canonical pairs we found for the \( d \)-dimensional massless particle in the previous section. There will be no ghosts in the quantized theory.

Action (3.4a) is invariant under the duality transformation

\[
X_M(\tau) \to P_M(\tau) \quad P_M(\tau) \to -X_M(\tau) \quad (3.13a)
\]

\[
\lambda_1 \to \lambda_3 \quad \lambda_2 \to -\lambda_2 \quad \lambda_3 \to \lambda_1 \quad (3.13b)
\]

If we perform the duality transformation (3.13) with the \( \epsilon_\alpha \) in the place of the \( \lambda_\alpha \) we find that the local transformation equations (3.11a), (3.11b) and the conserved charge (3.12) remain invariant.

Hamiltonian (3.5) generates the equations of motion

\[
\dot{X}_M = \{X_M, H\} = \lambda_1 P_M + \lambda_2 X_M \quad (3.14a)
\]

\[
\dot{P}_M = \{P_M, H\} = -\lambda_2 P_M - \lambda_3 X_M \quad (3.14b)
\]

The equations of motion (3.14) are invariant under the duality transformation (3.13). Now we clearly see what are the consequences of local \( Sp(2,R) \) symmetry: it leaves the local transformation equations and the equations of motion invariant under dualities of the type (3.13).

But this invariance of the equations of motion under the duality transformation (3.13a) does not exist in quantum mechanics. The quantum operators \( \hat{x}_\mu = x_\mu \), \( \hat{p}_\mu = i\hbar \frac{\partial}{\partial x_\mu} \) are turned into the operators \( \hat{p}_\mu = p_\mu \), \( \hat{x}_\mu = -i\hbar \frac{\partial}{\partial p_\mu} \), and the equations of motion in the position representation are turned into the equations of motion in the momentum representation under (3.13). For this reason, we will not require local \( Sp(2,R) \) symmetry for our actions in the presence of vector and tensor fields. For our purposes here we need to find another local symmetry. This other symmetry must lead to the correct number of physical canonical pairs and must reproduce the local \( Sp(2,R) \) symmetry in the absence of space-time fields. This will be the subject of section five. Let us now consider how we can introduce a complementation of the basic equations of quantum mechanics in the presence of gravity.

4 Dual Quantum Mechanics

The results in this paper bring with them the possibility of a deeper insight into the structure of quantum mechanics. The idea is to further incorporate into quantum mechanics the duality of the descriptions in terms of position and momentum. To this end we must introduce an additional assumption between
assumptions A1 and A2 of reference [22]. The complementation of quantum mechanics we present in this section is based on three assumptions, of which the first and the third ones are identical to A1 and A2 in [22]. Our assumptions are

1) There exists a basis \( | x \rangle \) of the position space which is spanned by the eigenvalues of the position operators \( \hat{x}^\alpha (\alpha = 1, 2, \ldots, n) \), whose domain of eigenvalues coincides with all the possible values of the coordinates \( x^\alpha \) parameterizing the position space \( M(x) \),

\[
\hat{x}^\alpha | x \rangle = x^\alpha | x \rangle , \quad \{ x^\alpha \} \in M(x)
\]

2) There exists a basis \( | p \rangle \) of the momentum space which is spanned by the eigenvalues of the momentum operators \( \hat{p}^\alpha (\alpha = 1, 2, \ldots, n) \), whose domain of eigenvalues coincides with all the possible values of the momenta \( p^\alpha \) parameterizing the momentum space \( D(p) \),

\[
\hat{p}^\alpha | p \rangle = p^\alpha | p \rangle , \quad \{ p^\alpha \} \in D(p)
\]

3) The representation spaces of the algebra are endowed with a Hermitian positive definite inner product \( \langle . | . \rangle \) for which the operators \( \hat{x}^\alpha \) and \( \hat{p}^\alpha \) are self-adjoint.

Now consider quantum mechanics in the position representation. The construction of quantum mechanics describing the diffeomorphism-covariant representations of the Heisenberg algebra in terms of topological classes of a flat U(1) bundle over position space has the parameterization [22] of the inner product \( \langle x | x' \rangle \)

\[
\langle x | x' \rangle = \frac{1}{\sqrt{g(x)}} \delta^n(x - x')
\]  

(4.1)

where \( g(x) \) is an arbitrary positive definite function defined over the position space \( M \). For a Riemannian manifold the natural choice [22] for \( g \) is the determinant of the metric tensor, \( g = \det g_{\alpha\beta}(x) \). Position dependent metric tensors naturally appear in this formulation of quantum mechanics.

Equation (4.1) implies the spectral decomposition [22] of the identity operator in the position eigenbasis \( | x \rangle \)

\[
1 = \int_M d^n x \sqrt{g(x)} | x \rangle \langle x |
\]

(4.2)

which in turn leads to the position space wave function representations \( \psi(x) = \langle x | \psi \rangle \) and \( \langle \psi | x \rangle = \langle x | \psi \rangle^* = \psi^*(x) \) of any state \( | \psi \rangle \) belonging to the Heisenberg algebra representation space,

\[
| \psi \rangle = \int_M d^n x \sqrt{g(x)} \psi(x) | x \rangle
\]

(4.3)

\[
\langle \psi | = \int_M d^n x \sqrt{g(x)} \psi^*(x) \langle x |
\]

(4.4)
The inner product of two states \( | \psi \rangle \) and \( | \varphi \rangle \) is then given in terms of their position space wave functions \( \psi(x) \) and \( \varphi(x) \) as

\[
\langle \psi | \varphi \rangle = \int_M d^n x \sqrt{g(x)} \psi^*(x) \varphi(x) \quad (4.5)
\]

The most general position space wave function representations of the position and momentum operators are [22]

\[
\langle x | \hat{x}_\alpha | \psi \rangle = x_\alpha \langle x | \psi \rangle = x_\alpha \psi(x) \quad (4.6a)
\]

\[
\langle x | \hat{p}_\alpha | \psi \rangle = \frac{-i \hbar}{g^{1/4}(x)} \left[ \frac{\partial}{\partial x^\alpha} + \frac{i}{\hbar} A_\alpha(x) \right] g^{1/4}(x) \psi(x) \quad (4.6b)
\]

The vector field \( A_\alpha(x) \) is present only in the case of topologically non-trivial position spaces [22]. It has a vanishing strength tensor \( F_{\alpha\beta} \) as given by (1.2) and is related to arbitrary local phase transformations of the position eigenvectors

\[
| x' \rangle = e^{i \hbar \chi(x)} | x \rangle \quad (4.7a)
\]

when

\[
A'_\alpha(x) = A_\alpha(x) + \frac{\partial \chi(x)}{\partial x^\alpha} \quad (4.7b)
\]

where \( \chi(x) \) is an arbitrary scalar function. From the above equations we see that position dependent tensors play a central role in this formulation of quantum mechanics. The other central object in this formulation is the vector field \( A_\alpha(x) \) of vanishing strength tensor.

Now consider quantum mechanics in the momentum representation. The normalization of the momentum eigenstates is parameterized according to [22]

\[
\langle p | p' \rangle = \frac{1}{\sqrt{h(p)}} \delta^n(p - p') \quad (4.8)
\]

where \( h(p) \) is an arbitrary positive definite function defined over the domain \( D(p) \) of the momentum eigenvalues. The authors in [22] do not go beyond this point and do not consider the possible forms of the function \( h(p) \). However, as a consequence of the wave-particle duality, we may expect the form \( h(p) = \det g_{\alpha\beta}(p) \) to be a possible one.

As a consequence of (4.8) and of our second assumption, we have the spectral decomposition of the identity operator in the momentum eigenbasis \( | p \rangle \)

\[
1 = \int_{D(p)} d^n p \sqrt{h(p)} | p \rangle \langle p | \quad (4.9)
\]

This leads to the momentum space wave functions \( \psi(p) = \langle p | \psi \rangle \) and \( \langle \psi | p \rangle = \langle p | \psi \rangle^* = \psi^*(p) \) of any state \( | \psi \rangle \) belonging to the Heisenberg algebra representation space

\[
| \psi \rangle = \int_{D(p)} d^n p \sqrt{h(p)} \psi(p) | p \rangle \quad (4.10a)
\]
\[ \langle \psi | = \int_{D(p)} d^n p \sqrt{h(p)} \psi^*(p) | p \rangle \] (4.10b)

The inner product of two states \(| \psi \rangle \) and \(| \varphi \rangle \) is given in terms of their momentum space wave functions \(\psi(p)\) and \(\varphi(p)\) as

\[ \langle \psi | \varphi \rangle = \int_{D(p)} d^n p \sqrt{h(p)} \psi^*(p) \varphi(p) \] (4.11)

The most general wave function \(\langle x | p \rangle\) is given by [22]

\[ \langle x | p \rangle = e^{i \varphi(x_0, p)} \frac{\Omega[P(x_0 \rightarrow x)]}{(2\pi\hbar)^{\frac{3}{4}}} \frac{e^{i \bar{h}(x-x_0) \cdot p}}{g^\frac{1}{4}(x) h^\frac{1}{4}(p)} \] (4.12)

\(\varphi(x_0, p)\) is a specific but otherwise arbitrary real function and \(\Omega[P(x_0 \rightarrow x)]\) is the path ordered U(1) holonomy along the path \(P(x_0 \rightarrow x)\). Notice that \(g(x)\) and \(h(p)\) are both necessary because they simultaneously appear in the most general wave function (4.12). The wave function (4.12) generalizes in a transparent manner the usual plane wave solutions of application to the trivial representation of the Heisenberg algebra with \(A_\alpha(x) = 0\) and with the choices \(g(x) = 1\) and \(h(p) = 1\).

Now we point out that the wave-particle duality can be made explicit in quantum mechanics if we further introduce the equations

\[ \langle p | \hat{p}_\alpha | \psi \rangle = p_\alpha \langle p | \psi \rangle = p_\alpha \psi(p) \] (4.13a)

\[ \langle p | \hat{x}_\alpha | \psi \rangle = i \hbar \frac{\partial}{\partial p^\alpha} + \frac{i \hbar}{\bar{h} A_\alpha(p)} h^{1/4}(p) \psi(p) \] (4.13b)

which are the momentum representation correspondents of equations (4.6). The vector field \(A_\alpha(p)\) has a vanishing strength tensor in momentum space,

\[ F_{\alpha\beta} = \frac{\partial A_\beta}{\partial p^\alpha} - \frac{\partial A_\alpha}{\partial p^\beta} = 0 \] (4.14)

and is related to arbitrary local phase transformations of the momentum eigenvectors

\[ | p' \rangle = e^{i \gamma(p)} | p \rangle \] (4.15a)

when

\[ A'_\alpha(p) = A_\alpha(p) + \frac{\partial \gamma(p)}{\partial p} \] (4.15b)

where \(\gamma(p)\) is an arbitrary scalar function. Equations (4.15) are the momentum representation correspondents of equations (4.7).

Now we need further evidence that the complementation of the basic equations of quantum mechanics we suggested in this section really makes sense. Part of this evidence comes from the results we derived for the massless particle in section two. We describe more evidence in the next section,
5 Dual Fields in Conformal Gravity

In this section we describe world-line constrained Hamiltonian formalisms containing tensor and vector fields. These formalisms are constructed starting from action (3.4b). We will use these formalisms to support our constructions in section four. These formalisms are based on new local symmetries in the presence of tensor and vector fields. For the actions we display here, local $Sp(2, \mathbb{R})$ is a broken symmetry. But this does not cause any problem in the transition to the physical sector of the theory. This is because these new local symmetry can also be used to arrive at the correct number of physical degrees of freedom, and reproduce the local $Sp(2, \mathbb{R})$ symmetry when the fields are removed.

5.1 Position dependent tensor fields

Action (3.4b) can be extended to an action in $d + 2$ dimensions where the geometry is described by a position dependent tensor $G_{MN}(X)$. This action is given by

$$S = \int d\tau \left\{ \frac{1}{2} (\dot{X}^M P_M - X^M \dot{P}_M) - \left[ \frac{1}{2} \lambda_1 G_{MN}(X) P^M P^N + \frac{1}{2} \lambda_2 G_{MN}(X) X^M X^N \right] \right\}$$

$$= \int d\tau \left\{ \dot{X}^M P_M - \left[ \frac{1}{2} \lambda_1 G_{MN}(X) P^M P^N + \frac{1}{2} \lambda_2 G_{MN}(X) X^M X^N \right] \right\}$$

(5.1)

where the Hamiltonian is

$$H = \frac{1}{2} \lambda_1 G_{MN}(X) P^M P^N + \lambda_2 G_{MN}(X) X^M P^N + \frac{1}{2} \lambda_3 G_{MN}(X) X^M X^N$$

(5.2)

The duality transformation (3.13) is not a symmetry of action (5.1) because $G_{MN}(X)$ becomes $G_{MN}(P)$ under (3.13).

The equations of motion for the variables $\lambda_\alpha$ give the constraints

$$\phi_1 = \frac{1}{2} G_{MN}(X) P^M P^N \approx 0$$

(5.3a)

$$\phi_2 = G_{MN}(X) X^M P^N \approx 0$$

(5.3b)

$$\phi_3 = \frac{1}{2} G_{MN}(X) X^M X^N \approx 0$$

(5.3c)

Following Dirac’s algorithm for constrained systems [20], we must now require the dynamical stability of constraints (5.3), which is the requirement that

$$\dot{\phi}_\alpha = \{ \phi_\alpha, H \} = \lambda_\beta \{ \phi_\alpha, \phi_\beta \} = 0$$

(5.4)
Conditions (5.4) lead to the Poisson brackets

\[ \{ \phi_1, \phi_2 \} = -G_{MN} \frac{\partial \phi_1}{\partial X^M} X^N - G_{MN} \frac{\partial \phi_2}{\partial X^M} P^N \] (5.5a)

\[ \{ \phi_1, \phi_3 \} = -G_{MN} \frac{\partial \phi_3}{\partial X^M} P^N \] (5.5b)

\[ \{ \phi_2, \phi_3 \} = -G_{MN} \frac{\partial \phi_3}{\partial X^M} X^N \] (5.5c)

The bracket relations (5.5) reproduce the $Sp(2, R)$ gauge algebra (3.9) when $G_{MN} = \eta_{MN}$.

We see from (5.5) that we can achieve dynamical stability of constraints (5.3) if we impose the conditions

\[ G_{MN}(X) \frac{\partial \phi_1}{\partial X^M} X^N \approx 0 \] (5.6a)

\[ G_{MN}(X) \frac{\partial \phi_2}{\partial X^M} P^N \approx 0 \] (5.6b)

If we interpret conditions (5.6) as new independent secondary [20] constraints and require their dynamical stability we get new conditions involving second order derivatives of the constraints (5.3) with respect to $X_M$. These new conditions are direct consequences of conditions (5.6). We therefore retain only conditions (5.6) as the necessary conditions for the dynamical stability of constraints (5.3). In a transition to flat space $G_{MN} = \eta_{MN}$ the conditions (5.6) reproduce the first class constraints (3.6). When conditions (5.6) hold, constraints (5.3) become first class constraints.

Constraints (5.3) generate the local transformations

\[ \delta X_M = \epsilon_1(\tau) \{ X_M, \phi_1 \} = \epsilon_1 G_{MR} P^R + \epsilon_2 G_{MR} X^R \] (5.7a)

\[ \delta P_M = \epsilon_2(\tau) \{ P_M, \phi_1 \} = -\frac{1}{2} \epsilon_1 \frac{\partial G_{RS}}{\partial X^M} P^R P^S - \epsilon_2 \frac{\partial G_{RS}}{\partial X^M} X^R P^S \]

\[ - \epsilon_2 G_{MR} P^R - \frac{1}{2} \epsilon_3 \frac{\partial G_{RS}}{\partial X^M} X^R X^S - \epsilon_3 G_{MR} X^R \] (5.7b)

with arbitrary parameters $\epsilon_\alpha(\tau)$. Under transformations (5.7) action (5.1) transforms as

\[ \delta S = \int d\tau [(\epsilon_1 \lambda_2 - \epsilon_2 \lambda_1) G_{MN} \frac{\partial \phi_1}{\partial X^M} X^N + (\epsilon_2 \lambda_1 - \epsilon_1 \lambda_2) G_{MN} \frac{\partial \phi_2}{\partial X^M} P^N \]

\[ + (\epsilon_3 \lambda_1 - \epsilon_1 \lambda_3) G_{MN} \frac{\partial \phi_3}{\partial X^M} P^N + (\epsilon_3 \lambda_2 - \epsilon_2 \lambda_3) G_{MN} \frac{\partial \phi_3}{\partial X^M} X^N \]

\[ + \frac{d}{dt}(\epsilon_\alpha \phi_\alpha) + \dot{\epsilon}_\alpha \phi_\alpha - \phi_\alpha \delta \lambda_\alpha] \] (5.7c)
Using the conditions (5.6) we can write the variation (5.7c) as
\[
\delta S \approx \int d\tau \left[ \frac{d}{d\tau} (\epsilon_\alpha \phi_\alpha) + \dot{\epsilon}_\alpha \phi_\alpha - \dot{\phi}_\alpha \delta \lambda_\alpha \right] \quad (5.7d)
\]
If we now choose \( \delta \lambda_\alpha = \dot{\epsilon}_\alpha \) we obtain
\[
\delta S \approx \int d\tau \frac{d}{d\tau} (\epsilon_\alpha \phi_\alpha) \quad (5.7e)
\]
This confirms that when conditions (5.6) hold action (5.1) has a local invariance generated by the first class constraints (5.3). The conserved charge corresponding to this local invariance is the quantity
\[
Q = \epsilon_\alpha \phi_\alpha = \frac{1}{2} \epsilon_1 G_{MN}(X) P^M P^N + \epsilon_2 G_{MN}(X) X^M P^N
\]
\[
+ \frac{1}{2} \epsilon_3 G_{MN}(X) X^M X^N \quad (5.8)
\]
Since we have here three first class constraints, which is the same number of first class constraints associated with the local \( Sp(2, R) \) symmetry, it is possible [21] to use the local symmetry generated by the conserved charge (5.8) to eliminate one space-like degree of freedom and two time-like degrees of freedom from each of the canonical variables. This leaves us with \( d-1 \) physical canonical pairs. The physical components of the tensor field are described in terms of the physical components of the position variable. There will be no ghosts in the quantized theory. As we will see in the next section, a duality transformation of the type (3.13) changes the local transformation equations (5.7) and the corresponding conserved charge (5.8) in the background \( G_{MN}(X) \) into the local transformation equations and the corresponding conserved charge in a background \( G_{MN}(P) \). In a transition to the flat \( d+2 \) dimensional space-time the transformation equations (5.7) and the corresponding conserved charge (5.8) reproduce the local \( Sp(2, R) \) transformation equations (3.11) and the corresponding conserved charge (3.12).

Hamiltonian (5.2) generates the equations of motion
\[
\dot{X}_M = \{ X_M, H \} = \lambda_1 G_{MR} P^R + \lambda_2 G_{MR} X^R \quad (5.9a)
\]
\[
\dot{P}_M = \{ P_M, H \} = -\frac{1}{2} \lambda_1 \frac{\partial G_{RS}}{\partial X^M} P^R P^S - \lambda_2 \frac{\partial G_{RS}}{\partial X^M} X^R P^S - \lambda_2 G_{MR} P^R \frac{1}{2} \lambda_3 \frac{\partial G_{RS}}{\partial X^M} X^R X^S - \lambda_3 G_{MR} X^R \quad (5.9b)
\]
The equations of motion (5.9) reproduce the equations of motion (3.14) when \( G_{MN} = \eta_{MN} \). As we will see in the next section, the duality transformation (3.13) changes the equations of motion (5.9) in the background \( G_{MN}(X) \) into the equations of motion in a background \( G_{MN}(P) \) and vice versa. It is this dual behavior of the local symmetries and of the equations of motion under the duality transformation (3.13) that we expect to exist also at the quantum level.
5.2 Momentum dependent tensor fields

Now we apply the duality transformation (3.13) to action (5.1) and obtain the action

\[ S = \int d\tau \left\{ \frac{1}{2} (\dot{X}^M P_M - X^M \dot{P}_M) - \frac{1}{2} \lambda_1 G_{MN}(P) P^M P^N \\ + \lambda_2 G_{MN}(P) X^M P^N + \frac{1}{2} \lambda_3 G_{MN}(P) X^M X^N \right\} \]

\[ = \int d\tau \left\{ -X^M \dot{P}_M - \frac{1}{2} \lambda_1 G_{MN}(P) P^M P^N \right. \]

\[ + \left. \lambda_2 G_{MN}(P) X^M P^N + \frac{1}{2} \lambda_3 G_{MN}(P) X^M X^N \right\} \]  

(5.10)

where the Hamiltonian is

\[ H = \frac{1}{2} \lambda_1 G_{MN}(P) P^M P^N + \lambda_2 G_{MN}(P) X^M P^N \]

\[ + \frac{1}{2} \lambda_3 G_{MN}(P) X^M X^N \]  

(5.11)

The equations of motion for the variables \( \lambda_\alpha(\tau) \) give the constraints

\[ \phi_1 = \frac{1}{2} G_{MN}(P) P^M P^N \approx 0 \]  

(5.12a)

\[ \phi_2 = G_{MN}(P) X^M P^N \approx 0 \]  

(5.12b)

\[ \phi_3 = \frac{1}{2} G_{MN}(P) X^M X^N \approx 0 \]  

(5.12c)

Requiring the dynamical stability of constraints (5.12)

\[ \dot{\phi}_\alpha = \{ \phi_\alpha, H \} = \lambda_\beta \{ \phi_\alpha, \phi_\beta \} = 0 \]  

(5.13)

we arrive at the bracket relations

\[ \{ \phi_1, \phi_2 \} = -G_{MN} \frac{\partial \phi_1}{\partial P_M} P^N \]  

(5.14a)

\[ \{ \phi_1, \phi_3 \} = -G_{MN} \frac{\partial \phi_1}{\partial P_M} X^N \]  

(5.14b)

\[ \{ \phi_2, \phi_3 \} = G_{MN} \frac{\partial \phi_3}{\partial P_M} P^N - G_{MN} \frac{\partial \phi_2}{\partial P_M} X^N \]  

(5.14c)

The bracket relations (5.14) reproduce the \( Sp(2,R) \) gauge algebra (3.9) when \( G_{MN} = \eta_{MN} \). Therefore, for consistency, we see from equations (5.14) that to achieve dynamical stability of constraints (5.12) we must impose the conditions

\[ G_{MN}(P) \frac{\partial \phi_\alpha}{\partial P_M} X^N \approx 0 \]  

(5.15a)
When conditions (5.15) are satisfied, the bracket relations (5.14) weakly vanish and the constraints (5.12) become first class constraints. Conditions (5.15) reproduce the $Sp(2,R)$ constraints (3.6) when $G_{MN} = \eta_{MN}$.

Constraints (5.12) generate the local transformations

$$
\delta X^M = \epsilon_\alpha(\tau)\{X^M, \phi_\alpha\} = \frac{1}{2} \epsilon_1 \frac{\partial G_{RS}}{\partial P^M} X^R P^S + \epsilon_1 G_{MR} P^R \\
+ \epsilon_2 \frac{\partial G_{RS}}{\partial P^M} X^R P^S + \epsilon_2 G_{MR} X^R + \frac{1}{2} \epsilon_3 \frac{\partial G_{RS}}{\partial P^M} X^R X^S \\
\delta P^M = \epsilon_\alpha(\tau)\{P^M, \phi_\alpha\} = -\epsilon_2 G_{MR} P^R - \epsilon_3 G_{MR} X^R 
$$

(5.16a)

under which

$$
\delta S = \int d\tau [(\epsilon_2 \lambda_1 - \epsilon_1 \lambda_2) G_{MN} \frac{\partial \phi_1}{\partial P^M} P^N + (\epsilon_3 \lambda_1 - \epsilon_1 \lambda_3) G_{MN} \frac{\partial \phi_2}{\partial P^M} P^N \\
+ (\epsilon_3 \lambda_2 - \epsilon_2 \lambda_3) G_{MN} \frac{\partial \phi_2}{\partial P^M} X^N + (\epsilon_2 \lambda_3 - \epsilon_3 \lambda_2) G_{MN} \frac{\partial \phi_3}{\partial P^M} X^N \\
+ \frac{d}{d\tau}(\epsilon_\alpha \phi_\alpha) + \dot{\epsilon}_\alpha \phi_\alpha - \dot{\phi}_\alpha \delta \lambda_\alpha] 
$$

(5.16c)

When conditions (5.15) hold, we can write the variation of the action as

$$
\delta S \approx \int d\tau \frac{d}{d\tau}(\epsilon_\alpha \phi_\alpha) + \dot{\epsilon}_\alpha \phi_\alpha - \dot{\phi}_\alpha \delta \lambda_\alpha 
$$

(5.16d)

If we now choose $\delta \lambda_\alpha = \dot{\epsilon}_\alpha$ the variation of the action becomes

$$
\delta S \approx \int d\tau \frac{d}{d\tau}(\epsilon_\alpha \phi_\alpha) 
$$

(5.16e)

This confirms that, when conditions (5.15) are valid, action (5.10) has a local invariance generated by the first class constraints (5.12). The conserved Hamiltonian Noether charge corresponding to this local invariance is the quantity

$$
Q = \epsilon_\alpha \phi_\alpha = \frac{1}{2} \epsilon_1 G_{MN}(P) P^M P^N + \epsilon_2 G_{MN}(P) X^M P^N \\
+ \frac{1}{2} \epsilon_3 G_{MN}(P) X^M X^N 
$$

(5.17)

Again it is in principle possible to use the local symmetry generated by the conserved charge (5.17) to eliminate one space-like degree of freedom and two time-like degrees of freedom from each of the canonical variables. Again, we are left with $d-1$ physical canonical pairs and with a tensor field that depends only on the physical components of the momentum variable. There will be no ghosts in the quantized theory. If we apply the duality transformation (3.13a), together with the transformations $\epsilon_1 \rightarrow \epsilon_3$, $\epsilon_2 \rightarrow -\epsilon_2$, $\epsilon_3 \rightarrow \epsilon_1$ to the local
transformation equations (5.16) and to the conserved charge (5.17), they are turned into the local transformation equations (5.7) and conserved charge (5.8).

In a transition to flat space $G_{MN} = \eta_{MN}$ the transformation equations (5.16) and the conserved charge (5.17) reproduce the local $Sp(2, R)$ transformations (3.11) and the corresponding conserved Noether charge (3.12).

Hamiltonian (5.11) generates the equations of motion

$$\dot{X}_M = \{X_M, H\} = \frac{1}{2}\lambda_1 \frac{\partial G_{RS}}{\partial P^M} P^R P^S + \lambda_1 G_{MR} P^R$$

$$+ \lambda_2 \frac{\partial G_{RS}}{\partial P^M} X^R P^S + \lambda_2 G_{MR} X^R + \frac{1}{2}\lambda_3 \frac{\partial G_{RS}}{\partial P^M} X^R X^S$$

(5.18a)

$$\dot{P}_M = \{P_M, H\} = -\lambda_2 G_{MR} P^R - \lambda_3 G_{MR} X^R$$

(5.18b)

Equations (5.18) reproduce the equations of motion (3.14) when $G_{MN} = \eta_{MN}$. Under the duality transformation (3.13), the equations of motion (5.18) in the background $G_{MN}(P)$ are transformed into the equations of motion (5.9) in the background $G_{MN}(X)$.

### 5.3 Position dependent vector fields

Starting from action (3.4b) we can construct the following action [28] in the presence of a vector field $A_M(X)$

$$S = \int d\tau \left\{ \frac{1}{2} (\dot{X}^M P_M - X^M \dot{P}_M) - \left( \frac{1}{2}\lambda_1 P^2 + \lambda_2 X.P ight) 
+ \frac{1}{2}\lambda_3 X^2 + \lambda_4 X.A + \lambda_5 P.A + \frac{1}{2}\lambda_6 A^2 \right\}$$

$$= \int d\tau [\dot{X}.P - \left( \frac{1}{2}\lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2}\lambda_3 X^2 
+ \lambda_4 X.A + \lambda_5 P.A + \frac{1}{2}\lambda_6 A^2 \right)]$$

(5.19)

where the Hamiltonian is

$$H = \frac{1}{2}\lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2}\lambda_3 X^2$$

$$+ \lambda_4 X.A(X) + \lambda_5 P.A(X) + \frac{1}{2}\lambda_6 A^2(X)$$

(5.20)

The duality transformation (3.13) is not a symmetry of action (3.19) because $A_M(X)$ becomes $A_M(P)$ under (3.13).

The equations of motion for the variables $\lambda_\varrho(\tau) \ (\varrho = 1, 2, ..., 6)$ give the constraints

$$\phi_1 = \frac{1}{2} P^2 \approx 0 \quad \phi_2 = X.P \approx 0 \quad \phi_3 = \frac{1}{2} X^2 \approx 0$$

(5.21a)
\[ \phi_4 = X.A(X) \approx 0 \quad \phi_5 = P.A(X) \approx 0 \quad \phi_6 = \frac{1}{2} A^2(X) \approx 0 \quad (5.21b) \]

Requiring the dynamical stability of constraints (5.21)
\[ \dot{\phi}_e = \{\phi_e, H\} = \lambda_\gamma \{\phi_e, \phi_\gamma\} = 0 \quad (5.22) \]

we arrive at the \( Sp(2, R) \) brackets (3.9), together with the new brackets
\[
\begin{align*}
&\{\phi_1, \phi_4\} = -P^M \frac{\partial \phi_4}{\partial X^M} & &\{\phi_1, \phi_5\} = -P^M \frac{\partial \phi_5}{\partial X^M} \\
&\{\phi_1, \phi_6\} = -P^M \frac{\partial \phi_6}{\partial X^M} & &\{\phi_2, \phi_4\} = -X^M \frac{\partial \phi_4}{\partial X^M} \\
&\{\phi_2, \phi_5\} = \phi_5 - X^M \frac{\partial \phi_5}{\partial X^M} & &\{\phi_2, \phi_6\} = -X^M \frac{\partial \phi_6}{\partial X^M} \\
&\{\phi_3, \phi_4\} = 0 & &\{\phi_3, \phi_5\} = \phi_4 \\
&\{\phi_3, \phi_6\} = 0 & &\{\phi_4, \phi_5\} = A^M \frac{\partial \phi_4}{\partial X^M} \\
&\{\phi_4, \phi_6\} = 0 & &\{\phi_5, \phi_6\} = -A^M \frac{\partial \phi_5}{\partial X^M} \quad (5.23)
\end{align*}
\]

To satisfy the dynamical stability requirement (5.22) we then impose the conditions
\[
\eta_{MN} \frac{\partial \phi_\gamma}{\partial X^M} X^N \approx 0 \quad \eta_{MN} \frac{\partial \phi_\gamma}{\partial X^M} P^N \approx 0 \quad \eta_{MN} \frac{\partial \phi_\gamma}{\partial X^M} A^N \approx 0 \quad (5.24)
\]

As in the case when only a tensor field is present, conditions (5.24) should not be interpreted as secondary constraints. They are interpreted here as restrictions on the motion of the massless particle arising from the presence of constraints (5.21). When these conditions are valid the constraints (5.21) become first class constraints.

Action (5.19) is invariant under global Lorentz \( SO(d, 2) \) transformation with generator \( L_{MN} = X_M P_N - X_N P_M \)
\[
\begin{align*}
&\delta X_M = \frac{1}{2} \omega_{RS} \{L_{RS}, X_M\} = \omega_{MR} X_R \quad (5.25a) \\
&\delta P_M = \frac{1}{2} \omega_{RS} \{L_{RS}, P_M\} = \omega_{MR} P_R \quad (5.25b) \\
&\delta A_M = \frac{\partial A_M}{\partial X_R} \delta X_R \\
&\delta \lambda_\gamma = 0, \quad \gamma = 1, 2, \ldots, 6 \quad (5.25d)
\end{align*}
\]

under which \( \delta S = 0 \). It can be checked that \( L_{MN} \) has weakly vanishing Poisson brackets with constraints (5.21), being therefore gauge invariant in the presence of the vector field \( A_M(X) \).
Constraints (5.21) generate the local transformations
\[ \delta X_M = \epsilon_\phi \{X_M, \phi_\phi\} = \epsilon_1 P_M + \epsilon_2 X_M + \epsilon_5 A_M \]  
\[ \delta P_M = \epsilon_\phi \{P_M, \phi_\phi\} = -\epsilon_2 P_M - \epsilon_3 X_M - \epsilon_4 A_M \]
\[ - \epsilon_4 X^N \frac{\partial A_N}{\partial X^M} - \epsilon_5 P^N \frac{\partial A_N}{\partial X^M} - \epsilon_6 A_N \frac{\partial A_N}{\partial X^M} \]  
Using conditions (5.24), the variation of the action becomes
\[ \delta S \approx \int d\tau \left[ (\epsilon_2 \lambda_1 - \epsilon_1 \lambda_2) 2\phi_1 + (\epsilon_3 \lambda_1 - \epsilon_1 \lambda_3) \phi_2 \right. \]
\[ + (\epsilon_3 \lambda_2 - \epsilon_2 \lambda_3) 2\phi_3 + (\epsilon_3 \lambda_3 - \epsilon_5 \lambda_3) \phi_4 \]
\[ + (\epsilon_2 \lambda_5 - \epsilon_5 \lambda_2) \phi_5 + \frac{d}{d\tau} (\epsilon_\phi \phi_\phi) + \dot{\epsilon}_\phi \phi_\phi - \phi_\phi \delta \lambda_\phi \]  
Using now the constraint equations (5.21), we find that this variation reduces to
\[ \delta S \approx \int d\tau \left[ \frac{d}{d\tau} (\epsilon_\phi \phi_\phi) \right] \]  
If we then finally choose \( \delta \lambda_\phi = \dot{\epsilon}_\phi \) we are left with
\[ \delta S \approx \int d\tau \left[ \frac{d}{d\tau} (\epsilon_\phi \phi_\phi) \right] \]  
Equation (5.26e) shows that the conserved Hamiltonian Noether charge in a flat \( d + 2 \) dimensional space-time with the presence of a background vector field \( A_M(X) \) is the quantity
\[ Q = \epsilon_\phi \phi_\phi = \frac{1}{2} \epsilon_1 P^2 + \epsilon_2 X.P + \frac{1}{2} \epsilon_5 X^2 \]
\[ + \epsilon_4 X.A(X) + \epsilon_5 P.A(X) + \frac{1}{2} \epsilon_6 A^2(X) \]  
We can use the local symmetry generated by the conserved charge (5.27) to eliminate one space-like and two time-like degrees of freedom from each of the canonical variables and from the vector field. We are then left with a physical phase space with \( d - 1 \) canonical pairs over which is defined a vector field with \( d - 1 \) physical components. The vector field depends only on the physical components of the position variable. Again, there will be no ghosts in the quantized theory. The transformation equations (5.26) and the corresponding conserved charge (5.27) reproduce the local \( Sp(2,R) \) transformation equations (3.11) and the corresponding conserved charge (3.12) when \( A_M(X) = 0 \).

The Hamiltonian equations of motion in the presence of the vector field \( A_M(X) \) are
\[ \dot{X}_M = \{X_M, H\} = \lambda_1 P_M + \lambda_2 X_M + \lambda_5 A_M \]  
\[ \dot{P}_M = \{P_M, H\} = -\lambda_2 P_M - \lambda_3 X_M - \lambda_4 A_M \]
\[ - \lambda_4 X^N \frac{\partial A_N}{\partial X^M} - \lambda_5 P^N \frac{\partial A_N}{\partial X^M} - \lambda_6 A_N \frac{\partial A_N}{\partial X^M} \]  
Clearly these equations of motion reproduce the equations of motion (3.14) when \( A_M(X) = 0 \).
5.4 Momentum dependent vector fields

Now we perform in action (3.19) the duality transformation

\[ X_M(\tau) \to P_M(\tau) \quad P_M(\tau) \to -X_M(\tau) \] (5.29a)
\[ \lambda_1 \to \lambda_3 \quad \lambda_2 \to -\lambda_2 \quad \lambda_3 \to \lambda_1 \] (5.29b)
\[ \lambda_4 \to \lambda_5 \quad \lambda_5 \to -\lambda_4 \quad \lambda_6 \to \lambda_6 \] (5.29c)

and obtain the following action with a background vector field \( A_M(P) \)

\[
S = \int d\tau \left[ \frac{1}{2} (\dot{X}^M P_M - \dot{X}^M P_M) - \left( \frac{1}{2} \lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2} \lambda_3 X^2 + \lambda_4 X.A(P) + \lambda_5 P.A(P) + \frac{1}{2} \lambda_6 A^2(P) \right) \right]
\] (5.30)

where the Hamiltonian is

\[
H = \frac{1}{2} \lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2} \lambda_3 X^2 + \lambda_4 X.A(P) + \lambda_5 P.A(P) + \frac{1}{2} \lambda_6 A^2(P)
\] (5.31)

The equations of motion for the variables \( \lambda_\xi(\tau) \) give the constraints

\[
\phi_1 = \frac{1}{2} P^2 \approx 0 \quad \phi_2 = X.P \approx 0 \quad \phi_3 = \frac{1}{2} X^2 \approx 0 \] (5.32a)
\[
\phi_4 = X.A(P) \approx 0 \quad \phi_5 = P.A(P) \approx 0 \quad \phi_6 = \frac{1}{2} A^2(P) \approx 0 \] (5.32b)

Requiring the dynamical stability condition (5.22) for constraints (5.32) we arrive at the new bracket relations

\[
\{ \phi_1, \phi_4 \} = -\phi_5 \quad \{ \phi_1, \phi_5 \} = 0
\]
\[
\{ \phi_1, \phi_6 \} = 0 \quad \{ \phi_2, \phi_4 \} = -\phi_4 + P_M \frac{\partial \phi_4}{\partial P_M}
\]
\[
\{ \phi_2, \phi_5 \} = P_M \frac{\partial \phi_5}{\partial P_M} \quad \{ \phi_2, \phi_6 \} = P_M \frac{\partial \phi_6}{\partial P_M}
\]
\[
\{ \phi_3, \phi_4 \} = X_M \frac{\partial \phi_4}{\partial P_M} \quad \{ \phi_3, \phi_5 \} = X_M \frac{\partial \phi_5}{\partial P_M}
\]
\[
\{ \phi_3, \phi_6 \} = X_M \frac{\partial \phi_6}{\partial P_M} \quad \{ \phi_4, \phi_5 \} = A_M \frac{\partial \phi_5}{\partial P_M}
\]
\[ \{ \phi_4, \phi_6 \} = A^M \frac{\partial \phi_6}{\partial P_M} \qquad \{ \phi_5, \phi_6 \} = 0 \quad (5.33) \]

We see from (5.33) that to obtain dynamical stability for constraints (5.32) we must impose the conditions
\[ \eta_{MN} \frac{\partial \phi_6}{\partial P_M} X^N \approx 0 \quad \eta_{MN} \frac{\partial \phi_6}{\partial P_M} P^N \approx 0 \quad \eta_{MN} \frac{\partial \phi_6}{\partial P_M} A^N \approx 0 \quad (5.34) \]

When conditions (5.34) hold, constraints (5.32) become first class constraints.

Action (5.30) has a global Lorentz \( SO(d,2) \) invariance with generator
\[ L_{MN} = X^M \frac{\partial}{\partial P_N} - X^N \frac{\partial}{\partial P_M} \]
under which \( \delta S = 0 \). It can be verified that \( L_{MN} \) has weakly vanishing Poisson brackets with constraints (5.32) being therefore also gauge invariant in the presence of the vector field \( A_M(P) \).

Constraints (5.32) generate the local transformations
\[ \delta X_M = \epsilon_6(\tau) \{ X_M, \phi_4 \} = \epsilon_1 P_M + \epsilon_2 X_M + \epsilon_4 X^S \frac{\partial A_S}{\partial P_M} + \epsilon_5 A_M + \epsilon_3 P_M - \epsilon_4 X_M - \epsilon_4 A_M \quad (5.36a) \]
\[ \delta P_M = \epsilon_6(\tau) \{ P_M, \phi_4 \} = -\epsilon_2 P_M - \epsilon_3 X_M - \epsilon_4 A_M \quad (5.36b) \]
\[ \delta A_M = \frac{\partial A_M}{\partial P_N} \delta P_N \quad (5.36c) \]
under which, after using the conditions (5.34), action (5.30) varies as
\[ \delta S \approx \int d\tau [ (\epsilon_2 \lambda_1 - \epsilon_1 \lambda_2) 2\phi_1 + (\epsilon_3 \lambda_1 - \epsilon_1 \lambda_3) \phi_2 \\
+ (\epsilon_3 \lambda_2 - \epsilon_2 \lambda_3) 2\phi_3 + (\epsilon_4 \lambda_2 - \epsilon_2 \lambda_4) \phi_4 \\
+ (\epsilon_4 \lambda_1 - \epsilon_1 \lambda_4) \phi_5 + \frac{d}{d\tau} (\epsilon_6 \phi_6) + \dot{\epsilon}_6 \phi_6 - \phi_6 \delta \lambda_6 ] \quad (5.36d) \]
Using the constraint equations (5.32), the above variation becomes
\[ \delta S \approx \int d\tau \left[ \frac{d}{d\tau} (\epsilon_6 \phi_6) + \dot{\epsilon}_6 \phi_6 - \phi_6 \delta \lambda_6 \right] \quad (5.36e) \]
If we now choose $\delta \lambda_\phi = \dot{\epsilon}_\phi$ we obtain

$$\delta S \approx \int d\tau \frac{d}{d\tau}(\epsilon_\phi \dot{\phi}_\phi)$$  \hspace{1cm} (5.36d)

The conserved charge is then the quantity

$$Q = \epsilon_\phi \dot{\phi}_\phi = \frac{1}{2} \epsilon_1 P^2 + \epsilon_2 X.P + \frac{1}{2} \epsilon_3 X^2 + \epsilon_4 X.A(P) + \epsilon_5 P.A(P) + \frac{1}{2} \epsilon_6 A^2(P)$$  \hspace{1cm} (5.37)

It is possible to use the local symmetry generated by the conserved charge (5.37) to eliminate the unphysical degrees of freedom. At the end we are left with a physical phase space with $d - 1$ canonical pairs over which is defined a vector field with $d - 1$ physical components. The vector field depends only on the physical components of the momentum variable. If we perform the duality transformation (5.29a), complemented with the transformations $\epsilon_1 \rightarrow \epsilon_3$, $\epsilon_2 \rightarrow -\epsilon_2$, $\epsilon_3 \rightarrow \epsilon_1$, $\epsilon_4 \rightarrow \epsilon_5$, $\epsilon_5 \rightarrow -\epsilon_4$, $\epsilon_6 \rightarrow \epsilon_6$, in the local transformation equations (5.36) and in the conserved charge (5.37), they are turned into the local transformation equations (5.26) and conserved charge (5.27). Therefore, there are duality transformations that relate the local symmetry of the action in the presence of the vector field $A_M(P)$ to the local symmetry of the action in the presence of the vector field $A_M(X)$ and vice versa. The transformation equations (5.36) and the corresponding conserved charge (5.37) reproduce the local $Sp(2,R)$ transformation equations (3.11) and the corresponding conserved charge (3.12) when $A_M(P) = 0$.

Hamiltonian (5.31) generates the equations of motion

$$\dot{X}_M = \{ X_M, H \} = \lambda_1 P_M + \lambda_2 X_M + \lambda_3 A_M$$

$$+ \lambda_4 X^N \frac{\partial A_N}{\partial P_M} + \lambda_5 P_N \frac{\partial A_N}{\partial P_M} + \lambda_6 A^N \frac{\partial A_N}{\partial P_M}$$  \hspace{1cm} (5.38a)

$$\dot{P}_M = \{ P_M, H \} = -\lambda_2 P_M - \lambda_3 X_M - \lambda_4 A_M$$  \hspace{1cm} (5.38b)

Equations of motion (5.38) reproduce the equations of motion (3.14) when $A_M(P) = 0$. If we perform the duality transformation (5.29) in the equations of motion (5.38), they are turned into the equations of motion (5.28). The duality transformation (5.29) therefore relates the dynamical evolution of the massless particle in the vector field $A_M(P)$ to its dynamical evolution in the vector field $A_M(X)$ and vice versa.

### 5.5 Tensor and vector fields

As a final way to give evidence that the complementation of quantum mechanics we suggested in section four makes sense, we now construct two dual actions
with tensor and vector fields. The first of these actions describes the case when \( G_{MN} = G_{MN}(X) \) and \( A_M = A_M(X) \). This action is

\[
S = \int d\tau \left\{ \dot{X}^M P_M - \frac{1}{2} \lambda_1 G_{MN}(X) P^M P^N + \lambda_2 G_{MN}(X) X^M P^N \\
+ \frac{1}{2} \lambda_3 G_{MN}(X) X^M X^N + \lambda_4 G_{MN}(X) X^M A^N \\
+ \lambda_5 G_{MN}(X) P^M A^N + \frac{1}{2} \lambda_6 G_{MN}(X) A^M A^N \right\} \quad (5.39)
\]

where the Hamiltonian is

\[
H = \frac{1}{2} \lambda_1 G_{MN}(X) P^M P^N + \lambda_2 G_{MN}(X) X^M P^N \\
+ \frac{1}{2} \lambda_3 G_{MN}(X) X^M X^N + \lambda_4 G_{MN}(X) X^M A^N \\
+ \lambda_5 G_{MN}(X) P^M A^N + \frac{1}{2} \lambda_6 G_{MN}(X) A^M A^N \quad (5.40)
\]

The equations of motion for the variables \( \lambda_\phi \) give the constraints

\[
\phi_1 = \frac{1}{2} G_{MN}(X) P^M P^N \approx 0 \quad \phi_2 = G_{MN}(X) X^M P^N \approx 0 \quad (5.41a)
\]
\[
\phi_3 = \frac{1}{2} G_{MN}(X) X^M X^N \approx 0 \quad \phi_4 = G_{MN}(X) X^M A^N(X) \approx 0 \quad (5.41b)
\]
\[
\phi_5 = G_{MN}(X) P^M A^N(X) \approx 0 \quad (5.41c)
\]
\[
\phi_6 = \frac{1}{2} G_{MN}(X) A^M(X) A^N(X) \approx 0 \quad (5.41d)
\]

Requiring dynamical stability of constraints (5.41) we arrive at the brackets

\[
\{\phi_1, \phi_2\} = -G_{MN} \frac{\partial \phi_1}{\partial X_M} X^N - G_{MN} \frac{\partial \phi_2}{\partial X_M} P^N \\
\{\phi_1, \phi_3\} = -G_{MN} \frac{\partial \phi_3}{\partial X_M} P^N \\
\{\phi_2, \phi_3\} = -G_{MN} \frac{\partial \phi_3}{\partial X_M} X^N \\
\{\phi_1, \phi_4\} = -G_{MN} \frac{\partial \phi_4}{\partial X_M} P^N \\
\{\phi_1, \phi_5\} = G_{MN} \frac{\partial \phi_1}{\partial X_M} A^N - G_{MN} \frac{\partial \phi_5}{\partial X_M} P^N \\
\{\phi_1, \phi_6\} = -G_{MN} \frac{\partial \phi_6}{\partial X_M} P^N
\]
\[
\{\phi_2, \phi_4\} = -G_{MN} \frac{\partial \phi_4}{\partial X_M} X^N \\
\{\phi_2, \phi_5\} = G_{MN} \frac{\partial \phi_5}{\partial X_M} A^N - G_{MN} \frac{\partial \phi_5}{\partial X_M} X^N \\
\{\phi_2, \phi_6\} = -G_{MN} \frac{\partial \phi_6}{\partial X_M} X^N \\
\{\phi_3, \phi_5\} = G_{MN} \frac{\partial \phi_3}{\partial X_M} A^N \\
\{\phi_4, \phi_5\} = G_{MN} \frac{\partial \phi_4}{\partial X_M} A^N \\
\{\phi_5, \phi_6\} = -G_{MN} \frac{\partial \phi_6}{\partial X_M} A^N
\] (5.42)

To achieve dynamical stability of constraints (5.51) we then impose the conditions

\[
G_{MN}(X) \frac{\partial \phi_\varrho}{\partial X_M} X^N \approx 0 \quad (5.43a) \\
G_{MN}(X) \frac{\partial \phi_\varrho}{\partial X_M} P^N \approx 0 \quad (5.43b) \\
G_{MN}(X) \frac{\partial \phi_\varrho}{\partial X_M} A^N \approx 0 \quad (5.43c)
\]

If we make a transition to flat space \(G_{MN} = \eta_{MN}\) equations (5.42) reproduce the bracket relations (3.9) and (5.23) for the constraints (5.21) in the presence of the vector field \(A_M(X)\) only. If we remove the vector field by setting \(A_M(X) = 0\) the brackets (5.42) turn into the bracket relations (5.5) for the constraints (5.3) in the presence of the tensor field \(G_{MN}(X)\) only. If both the tensor and vector fields are removed then equations (5.42) turn into the \(Sp(2, R)\) bracket relations (3.9) for the first class constraints (3.6).

Constraints (5.41) generate the local transformations

\[
\delta X_M = \epsilon_\varrho(\tau) \{X_M, \phi_\varrho\} = \epsilon_1 G_{MR} P^R + \epsilon_2 G_{MR} X^R \\
+ \epsilon_5 G_{MR} A^R
\]

\[
\delta P_M = \epsilon_\varrho(\tau) \{P_M, \phi_\varrho\} = -\frac{1}{2} \epsilon_1 \frac{\partial G_{RS}}{\partial X_M} P^R P^S \\
- \epsilon_2 \frac{\partial G_{RS}}{\partial X_M} X^R P^S - \epsilon_2 G_{MR} P^R - \frac{1}{2} \epsilon_3 \frac{\partial G_{RS}}{\partial X_M} X^R X^S \\
- \epsilon_3 G_{MR} X^R - \epsilon_4 \frac{\partial G_{RS}}{\partial X_M} X^R A^S - \epsilon_4 G_{MR} A^R \\
- \epsilon_4 G_{RS} X^R \frac{\partial A^S}{\partial X_M} - \epsilon_5 \frac{\partial G_{RS}}{\partial X_M} P^R A^S - \epsilon_5 G_{RS} P^R \frac{\partial A^S}{\partial X_M} \\
- \frac{1}{2} \epsilon_6 \frac{\partial G_{RS}}{\partial X_M} A^R A^S - \epsilon_6 G_{RS} A^R \frac{\partial A^S}{\partial X_M}
\] (5.44b)
\[ \delta A_M = \frac{\partial A_M}{\partial X^R} \delta X^R \]  
\[ \delta G_{MN} = \frac{\partial G_{MN}}{\partial X^R} \delta X^R \]  

under which, after using conditions (5.43), action (5.39) varies as

\[ \delta S \approx \int d\tau \left[ \frac{d}{d\tau}(\epsilon \phi) + \epsilon \phi - \phi \delta \lambda \right] \]  

By choosing \( \delta \lambda = \dot{\epsilon} \), the variation (5.44e) becomes

\[ \delta S \approx \int d\tau \frac{d}{d\tau}(\epsilon \phi) \]  

This shows that the quantity

\[ Q = \epsilon \phi = \frac{1}{2} \epsilon_1 G_{MN}(X) P^M P^N + \epsilon_2 G_{MN}(X) X^M P^N + \frac{1}{2} \epsilon_3 G_{MN}(X) X^M P^N + \epsilon_4 G_{MN}(X) X^M X^N + \frac{1}{2} \epsilon_6 G_{MN}(X) A^M(X) A^N(X) \]  

is the conserved Hamiltonian Noether charge in the presence of the tensor field \( G_{MN}(X) \) and the vector field \( A_M(X) \). It is possible to use the local symmetry generated by the conserved charge (5.45) to reach a physical phase space with \( d - 1 \) canonical pairs over which a vector field with \( d - 1 \) physical components is defined. The physical components of the tensor and vector fields depend only on the physical components of the position variable. There will be no ghosts in the quantized theory.

Hamiltonian (5.40) generates the equations of motion

\[ \dot{X}_M = \{ X_M, H \} = \lambda_1 G_{MR} P^R + \lambda_2 G_{MR} X^R + \lambda_3 G_{MR} A^R \]  
\[ \dot{P}_M = \{ P_M, H \} = -\frac{1}{2} \lambda_1 \frac{\partial G_{RS}}{\partial X^M} P^R P^S - \lambda_2 \frac{\partial G_{RS}}{\partial X^M} X^R P^S - \lambda_3 \frac{\partial G_{RS}}{\partial X^M} X^R X^S - \lambda_4 \frac{\partial G_{RS}}{\partial X^M} X^R A^S - \lambda_4 \frac{\partial G_{RS}}{\partial X^M} R^S - \lambda_5 \frac{\partial G_{RS}}{\partial X^M} \frac{\partial A^S}{\partial X^M} - \lambda_6 \frac{\partial G_{RS}}{\partial X^M} A^R A^S - \lambda_6 G_{RS} A^R \frac{\partial A^S}{\partial X^M} \]  

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If we make a transition to flat space-time, the equations of motion (5.46) reduce to the equations of motion (5.28) in the presence of the vector field $A_M(X)$ only. If we set $A_M(X) = 0$ the equations of motion (5.46) reduce to the equations of motion (5.9) in the presence of the tensor field $G_{MN}(X)$ only. If we set $G_{MN} = \eta_{MN}$ and $A_M(X) = 0$ the equations of motion (5.46) reduce to the conformal gravity equations of motion (3.14).

Performing the duality transformation (5.29) in action (5.39), we turn to the case when $G_{MN}(P) = G_{MN}(P)$ and $A_M = A_M(P)$. The action is

$$S = \int d\tau \{-X^M P_M - \frac{1}{2} \lambda_1 G_{MN}(P) P^M P^N + \lambda_2 G_{MN}(P) X^M P^N$$

$$+ \frac{1}{2} \lambda_3 G_{MN}(P) X^M X^N + \lambda_4 G_{MN}(P) X^M A^N$$

$$+ \lambda_5 G_{MN}(P) P^M A^N + \frac{1}{2} \lambda_6 G_{MN}(P) A^M A^N\} \tag{5.47}$$

where the Hamiltonian is

$$H = \frac{1}{2} \lambda_1 G_{MN}(P) P^M P^N + \lambda_2 G_{MN}(P) X^M P^N$$

$$+ \frac{1}{2} \lambda_3 G_{MN}(P) X^M X^N + \lambda_4 G_{MN}(P) X^M A^N$$

$$+ \lambda_5 G_{MN}(P) P^M A^N + \frac{1}{2} \lambda_6 G_{MN}(P) A^M A^N \tag{5.48}$$

The equations of motion for the $\lambda_0(\tau)$ give the constraints

$$\phi_1 = \frac{1}{2} G_{MN}(P) P^M P^N \approx 0 \quad \phi_2 = G_{MN}(P) X^M P^N \approx 0 \quad \tag{5.49a}$$

$$\phi_3 = \frac{1}{2} G_{MN}(P) X^M X^N \approx 0 \quad \phi_4 = G_{MN}(P) X^M A^N(P) \approx 0 \quad \tag{5.49b}$$

$$\phi_5 = G_{MN}(P) P^M A^N(P) \approx 0 \quad \tag{5.49c}$$

$$\phi_6 = \frac{1}{2} G_{MN}(P) A^M(P) A^N \approx 0 \quad \tag{5.49d}$$

It can be verified that constraints (5.49) become dynamically stable first class constraints when the conditions

$$G_{MN}(P) \frac{\partial \phi_1}{\partial P_M} X^N \approx 0 \quad \tag{5.50a}$$

$$G_{MN}(P) \frac{\partial \phi_3}{\partial P_M} P^N \approx 0 \quad \tag{5.50b}$$

$$G_{MN}(P) \frac{\partial \phi_6}{\partial P_M} A^N \approx 0 \quad \tag{5.50c}$$
Constraints (5.49) generate local transformations that can be obtained from the transformation equations (5.44) by using the duality transformation (5.29) with the $\epsilon_\rho$ in the place of the $\lambda_\rho$. In this case the conserved Hamiltonian Noether charge is the quantity

$$Q = \epsilon_\rho \phi_\rho = \frac{1}{2} \epsilon_1 G_{MN}(P) P^M P^N + \epsilon_2 G_{MN}(P) X^M P^N$$

$$+ \frac{1}{2} \epsilon_3 G_{MN}(P) X^M X^N + \epsilon_4 G_{MN}(P) X^M A^N(P)$$

$$+ \epsilon_5 G_{MN}(P) P^M A^N(P) + \frac{1}{2} \epsilon_6 G_{MN}(P) A^M(P) A^N(P)$$

(5.51)

We can again in principle use the local symmetry generated by the conserved charge (5.51) to reach the physical sector of the theory, with $d - 1$ canonical pairs and $d - 1$ components of the vector field. The physical components of the tensor and vector fields will depend only on the physical components of the momentum variable. If we perform the duality transformation (5.29) with the $\epsilon_\rho$ in the place of the $\lambda_\rho$ we find that the conserved charge (5.51) is turned into the conserved charge (5.45).

Hamiltonian (5.48) generates the equations of motion

$$\dot{X}_M = \{X_M, H\} = \frac{1}{2} \lambda_1 \frac{\partial G_{RS}}{\partial P^M} P^R P^S + \lambda_1 G_{MR} P^R$$

$$+ \lambda_2 \frac{\partial G_{RS}}{\partial P^M} X^R P^S + \lambda_2 G_{MR} X^R + \frac{1}{2} \lambda_3 \frac{\partial G_{RS}}{\partial P^M} X^R X^S$$

$$+ \lambda_4 \frac{\partial G_{RS}}{\partial P^M} X^R A^S + \lambda_4 G_{RS} X^R \frac{\partial A^S}{\partial P^M} + \lambda_5 \frac{\partial G_{RS}}{\partial P^M} P^R A^S$$

$$+ \lambda_5 G_{MR} A^R + \lambda_5 G_{RS} P^R \frac{\partial A^S}{\partial P^M} + \frac{1}{2} \lambda_6 \frac{\partial G_{RS}}{\partial P^M} A^R A^S$$

$$+ \lambda_6 G_{RS} A^R \frac{\partial A^S}{\partial P^M}$$

(5.52a)

$$\dot{P}_M = \{P_M, H\} = -\lambda_2 G_{MR} P^R - \lambda_3 G_{MR} X^R$$

$$- \lambda_4 G_{MR} A^R$$

(5.52b)

If we perform a transition to flat space-time, the equations of motion (5.52) reduce to the equations of motion (5.38) in the presence of the vector field $A_M(P)$ only. If we set $A_M(P) = 0$ the equations of motion (5.52) reduce to the equations of motion (5.18) in the presence of the tensor field $G_{MN}(P)$ only. If we set $G_{MN} = \eta_{MN}$ and $A_M(P) = 0$ the equations of motion (5.52) reduce to the conformal gravity equations of motion (3.14). The equations of motion (5.52) in the presence of the tensor field $G_{MN}(P)$ and vector field $A_M(P)$ are transformed by the duality transformation (5.29) into the equations of motion (5.46) in the presence of the tensor field $G_{MN}(X)$ and vector field $A_M(X)$.

These results indicate that there are two dual descriptions of the local symmetry and of the dynamical evolution of the massless relativistic particle in the
presence of tensor and vector fields. These two dual descriptions are related by
duality transformations that interchange position and momentum. These two
dual descriptions reduce to conformal gravity on the world-line when the tensor
and vector fields are removed. It is the existence of these two dual descrip-
tions that suggests the complementation of quantum mechanics we described in
section four.

6 Concluding remarks

In this paper we suggested a complementation of the basic equations of quantum
mechanics in the presence of gravity. The new equations we suggested here are
based on a duality transformation that interchanges position and momentum
at the classical level. At the quantum level, this duality transformation leaves
invariant the definition of the fundamental commutator \([X_M, P_N] = i\hbar\eta_{MN}\). It
is then reasonable to expect that this duality transformation at the quantum
level will change position dependent tensor and vector fields into momentum
dependent tensor and vector fields without any modification of the fundamental
aspects of the quantum dynamics. In this paper we made an attempt to describe
a particular classical limit of this dual behavior of the dynamics in the presence
of tensor and vector fields, starting from the conformal gravity action in the
world-line formalism, and using constrained Hamiltonian methods.

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