Self-duality in Super D3-brane Action

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Abstract

We establish self-duality of super D3-brane theory as an exact symmetry of the action both in the Lagrangian and Hamiltonian formalism. In the Lagrangian formalism, the action is shown to satisfy the Gaillard-Zumino condition. This algebraic relation is recognized in our previous paper to be a necessary and sufficient condition for generic action of U(1) gauge field strength coupled with gravity and matters to be self-dual. For the super D3-brane action, SO(2) duality transformation of a world-volume gauge field should be associated with SO(2) rotation of fermionic brane coordinates in N=2 SUSY multiplet. This SO(2) duality symmetry is lifted to SL(2,R) symmetry in the presence of a dilaton and an axion background fields. In the canonical formalism, we show that the duality rotation is described by a canonical transformation, and the Hamiltonian of the D3-brane action is invariant under the transformation.

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1 Introduction

SL(2,Z) symmetry has been recognized to play an important role in understanding S duality. The type IIB D-branes appear in multiplets under this SL(2,Z) symmetry. Among others the D3-brane has a special position to be a singlet, thus often referred to be self-dual. Our main purpose in this paper is to show the self-duality of the super D3-brane based on our earlier works\cite{1},\cite{2}. Actually, there have appeared some papers claiming this self-duality as Tseytlin’s for bosonic action\cite{3} and Aganagic et al.’s for super action\cite{4}; however those works are based on semi-classical treatments. We would like to emphasize that our result to be reported here does not depend on any semi-classical approximations.

The precise statement of the self-duality is that the super D3 action\cite{1} is invariant under a combined transformation of vector duality on the world-volume gauge field and the SL(2,Z) transformation of the external supergravity backgrounds. Since the world-volume is four-dimensional, the vector duality for D3-brane is nothing but the well-known electric-magnetic duality. Usually the electric-magnetic duality is regarded as a symmetry at the level of equations of motion (EOM)\cite{2}. This makes the self-duality of the D3 action highly non-trivial.

In the following we discuss our strategy to show the self-duality. The proof is given both in the Lagrangian and the Hamiltonian approaches.

In the Lagrangian formalism, our proof is based on an alternative description of electric-magnetic duality rotation in terms of the gauge potential\cite{6}, which was given in our previous paper\cite{2}. (This rotation is referred to as \textit{A-transformation}, while the conventional one for the field strength as \textit{F-transformation}.) It enables us to formulate the duality as a symmetry of generic actions of U(1) gauge field strength coupled with gravity and matters. In the same paper we also emphasized that a necessary and sufficient condition for the invariance\cite{6} of the action may be expressed as an algebraic relation, the Gaillard-Zumino (GZ) condition \cite{7},\cite{8},\cite{9}. Here we show that the D3 action obeys this condition, thereby establishing its self-duality as an exact symmetry without resort to any semi-classical approximations. It should be stressed that, in order for the action to be invariant, SO(2) duality rotation of the gauge field should be associated with SO(2) rotation of the fermionic brane coordinates, while the bosonic coordinates yet remain unchanged.

The proof for self-duality is also discussed in the Hamiltonian formalism, where the duality symmetry is realized as an invariance of the Hamiltonian. Based on general analysis of type IIB super D-branes given in ref.\cite{10}, we investigate transformation properties of the constraints in the D3 action: the bosonic constraints are shown to be invariant under the duality transformation, and fermionic ones to transform by the SO(2) rotation

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1Hereafter the D3-brane means \textit{super} D3-brane if not stated otherwise.
2Self-duality of bosonic D3-brane was discussed in ref.\cite{5} as a symmetry of the EOM.
3Strictly speaking, the actions are not exactly invariant but pseudo-invariant, which means that the actions remain invariant only up to surface terms.
mentioned above. These give a proof of the invariance of the Hamiltonian for the D3 action. The covariance of the fermionic constraint is a consequence of the transformation properties of the fermionic coordinates, which is determined to satisfy the GZ condition from the viewpoint of the Lagrangian formalism.

We also confirm the idea that the vector duality transformations can be essentially identified with canonical transformations. Based on a canonical analysis of the D-string\cite{11}, we gave in a previous paper\cite{1} the canonical transformation that relates the D-string action with the type IIB superstring action. We construct here, as a natural extension of that work, the canonical transformation to generate the A-transformation for the D3 action.

To write down D3 action explicitly, we need an integrated expression of the Wess-Zumino term. It was given by Cederwall et al.\cite{12} and by Kamimura and Hatsuda\cite{10}. Two Lagrangian densities differ only by a total derivative and they essentially give the same action.\footnote{Here expected is some cohomology argument similar to the one given in \cite{13}} Here we will take the action of ref.\cite{12} below. The action contains a dilaton and an axion as scalar SUGRA backgrounds, which lift the duality symmetry from SO(2) to SL(2,R)\cite{14}: they become the variables parametrizing the coset space SL(2,R)/SO(2), and give a non-linear realization of the SL(2,R) symmetry. For simplicity, we assume these scalars to be constant fields, though extension to the on-shell SUGRA multiplet is possible\cite{12}.

This paper is organized as follows: the next section describes the duality condition for generic action of an interacting U(1) gauge field strength. In section 3, we show that the D3 action obeys the SO(2) duality condition. The SO(2) duality is shown to be lifted to the SL(2,R) duality in the presence of the dilaton and the axion. The proof of invariance or covariance of the constraints in the Hamiltonian formalism is given in section 4. The final section is devoted to summary and discussion. Here some comments will also be made on other approaches to implement the duality symmetry at the action level and further to relate it to a possible non-perturbative definition of string theory.

2 The Gaillard-Zumino condition

We begin with a brief summary of the GZ condition\cite{7,8,9}, which is discussed in detail in ref.\cite{2}. Consider a generic Lagrangian density $L(F_{\mu\nu}, g_{\mu\nu}, \Phi^A) = \sqrt{g}L(F_{\mu\nu}, g_{\mu\nu}, \Phi^A)$ in D=4, which depends on a U(1) gauge field strength $F_{\mu\nu}$, metric $g_{\mu\nu}$, and matter fields $\Phi^A$. The constitutive relation is given by

$$\tilde{K}^{\mu\nu} = \frac{\partial L}{\partial F_{\mu\nu}}, \quad \frac{\partial F_{\alpha\beta}}{\partial F_{\mu\nu}} = (\delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu),$$

(2.1)
where the Hodge dual components\footnote{We use the following convention: $\eta^\rho_\mu \sigma$ denotes the covariantly constant anti-symmetric tensor with indices raised and lowered using the metric $g_{\mu \nu}$ whose signature is $(+-++)$. We also use the tensor densities $\epsilon^{\mu \nu \rho \sigma}$ and $\epsilon_{\mu \nu \rho \sigma}$ with weight $-1$ and $1$. They are defined by $\epsilon^{\mu \nu \rho \sigma} = \sqrt{-g} \eta^{\rho \mu \sigma}$ and $\eta_{\mu \nu \rho \sigma} = \sqrt{-g} \epsilon_{\mu \nu \rho \sigma}$ with $g = \det g_{\mu \nu}$, normalized as $\epsilon^{0123} = - \epsilon_{0123} = 1.$} for the anti-symmetric tensor $K_{\mu \nu}$ are defined by

$$\tilde{K}_{\mu \nu} = \frac{1}{2} \eta_{\mu \rho \sigma} K_{\rho \sigma}, \quad \tilde{\tilde{K}}_{\mu \nu} = - K_{\mu \nu}. \quad (2.2)$$

Gaillard and Zumino considered an infinitesimal duality transformation which consists of the most general linear transformation on $F$ and $K$, and a transformation of matter fields,

$$\delta \left( \begin{array}{c} F \\ K \end{array} \right) = \alpha \begin{array}{c} \beta \\ \gamma \delta \end{array} \begin{array}{c} F \\ K \end{array}, \quad \delta \Phi^A = \xi^A(\Phi), \quad \delta g_{\mu \nu} = 0, \quad (2.3)$$

and required invariance of stationary surfaces of the system under (2.3). It was shown\footnote{We use the following convention: $\eta^\rho_\mu \sigma$ denotes the covariantly constant anti-symmetric tensor with indices raised and lowered using the metric $g_{\mu \nu}$ whose signature is $(+-++)$. We also use the tensor densities $\epsilon^{\mu \nu \rho \sigma}$ and $\epsilon_{\mu \nu \rho \sigma}$ with weight $-1$ and $1$. They are defined by $\epsilon^{\mu \nu \rho \sigma} = \sqrt{-g} \eta^{\rho \mu \sigma}$ and $\eta_{\mu \nu \rho \sigma} = \sqrt{-g} \epsilon_{\mu \nu \rho \sigma}$ with $g = \det g_{\mu \nu}$, normalized as $\epsilon^{0123} = - \epsilon_{0123} = 1.$} that

1. the F-transformation in (2.3) is an element of SL(2,R) given by $\delta = -\alpha$;
2. the Lagrangian should transform as

$$\delta L = \frac{1}{4} \left( \gamma F \tilde{F} + \beta K \tilde{K} \right). \quad (2.4)$$

As to be seen later, the non-compact SL(2,R) duality is possible only when there are scalar fields in the theory. In their absence, the relevant duality group becomes SO(2); the transformation described by the compact maximal subgroup, $U(1) \sim SO(2)$, where the parameters satisfy the conditions $\alpha = -\delta = 0$, $\beta = -\gamma \equiv \lambda$. The SO(2) transformation is given by

$$\delta F = \lambda K, \quad \delta K = - \lambda F. \quad (2.5)$$

Since the Lagrangian changes by

$$\delta L = \frac{1}{2} \frac{\partial L}{\partial F_{\mu \nu}} \delta F_{\mu \nu} + \frac{\partial L}{\partial \Phi^A} \delta \Phi^A = \frac{\lambda}{2} \tilde{K}^{\mu \nu} K_{\mu \nu} + \delta \Phi L, \quad (2.6)$$

the duality condition (2.4) reduces to

$$\frac{\lambda}{4} \left( F \tilde{F} + K \tilde{K} \right) + \delta \Phi L = 0, \quad (2.7)$$

where $F \tilde{F} = F_{\mu \nu} \tilde{F}^{\mu \nu}$.

As shown in \cite{2}, eq. (2.7) is the crucial condition for a generic action for an interacting U(1) field to be self-dual: this algebraic relation is not an on-shell relation but sensible even for off-shell fields; furthermore, in the formulation based on the A-transformation, the GZ condition (2.7) ensures the duality as a symmetry of the action.

A comment is in order. In the GZ condition, the first two terms may be obtained with the definition for $K$ once an action is specified. So the question of the duality reduced to a problem to find an appropriate matter transformation so that the GZ condition is satisfied. In this sense, it may be regarded as a condition on the matter transformations.
3 D3 action and the Gaillard-Zumino condition

In this section we first consider D3 action without scalar SUGRA backgrounds and show that it satisfies the SO(2) duality condition (2.7). Let $X^M$ be bosonic brane coordinates in $\text{D}=10$ flat target space ($M=0,...,9$), and $\theta_a$ be its fermionic partners described by the Majorana-Weyl spinor with spinor index $\alpha$ and N=2 SUSY index $A$. We shall use the same conventions for the Dirac matrices as those given in ref.[15]. These indices for spinors are suppressed below. The D3 action for the brane coordinates $(X, \theta)$ and world-volume gauge field $A_{\mu}$ is required to have the kappa symmetry and N=2 SUSY. It takes the form

$$ S = \int d^4 \sigma \, L^{DBI} + \int d^4 \sigma \, L^{WZ} \tag{3.1} $$

where

$$ L^{DBI} = - \sqrt{- \det(G_{\mu\nu} + F_{\mu\nu})}, \quad G_{\mu\nu} = \Pi^M \Pi_{\nu M}, $$

$$ F_{\mu\nu} = \partial_\mu A_\nu + \Omega_{\mu\nu}^3, \quad \Omega_{\mu\nu}^j = \theta \hat{\Pi}_{\mu j} \partial_\nu \theta \quad (j=1,3). \tag{3.2} $$

The Pauli matrices $\tau_i$ act on N=2 SUSY indices. The basic one-form is defined by

$$ \Pi^M \equiv dX^M + \bar{\theta} \Gamma^M d\theta \equiv d\sigma^\mu \Pi^M_\mu, \quad \Pi^M_\mu = \partial_\mu X^M - \bar{\theta} \Gamma^M \partial_\mu \theta. \tag{3.3} $$

and

$$ \hat{\Pi}^M = \Pi^M - \frac{1}{2} \bar{\theta} \Gamma^M d\theta = dX^M + \frac{1}{2} \bar{\theta} \Gamma^M d\theta. \tag{3.4} $$

For the Wess-Zumino (WZ) action, we take the one given in ref.[12]. Using differential forms, it is given by the 2-form $\mathcal{F}$, a pullback of Ramond-Ramond 2-form $C^{(2)}$ and a 4-form $C^{(4)}$:

$$ L^{WZ} = C^{(2)} \mathcal{F} + C^{(4)} \tag{3.5} $$

$$ C^{(2)} = \theta \hat{\Pi} \tau_1 d\theta = \Omega_1, \tag{3.6} $$

$$ C^{(4)} = \Xi - \frac{1}{2} \Omega_1 \Omega_3, \tag{3.7} $$

where

$$ \Xi = \frac{1}{6} \theta \hat{\Pi}^3 \tau_3 \tau_1 d\theta $$

$$ - \frac{1}{12} \theta (\hat{\Pi}^2 \beta_0 + \hat{\Pi} \beta_0 \hat{\Pi} + \beta_0 \hat{\Pi}^2) \tau_3 \tau_1 d\theta $$

$$ + \frac{1}{18} \theta (\hat{\Pi} \beta_0^2 + \beta_0 \hat{\Pi} \beta_0 + \beta_0^2 \hat{\Pi}) \tau_3 \tau_1 d\theta $$

$$ - \frac{1}{12} \theta \hat{\Pi} \tau_1 d\theta \theta \beta_0 \tau_3 d\theta $$

$$ - \frac{1}{24} \theta \beta_0^3 \tau_3 \tau_1 d\theta, \quad (\beta_0 \equiv \theta \Gamma d\theta). \tag{3.8} $$
Now let us see whether the GZ condition is satisfied for the above action. First of all, we calculate the first two terms of the condition. From the definition in (2.1) the $\tilde{K}$ is obtained as,

$$\tilde{K}^{\mu\nu} = \frac{\partial L}{\partial F_{\mu\nu}} = \frac{\sqrt{-G}}{\sqrt{-G_F}} (\mathcal{F}^{\rho\sigma} + \mathcal{T} \tilde{F}^{\rho\sigma}) + \tilde{C}^{(2)\mu\nu},$$

where use has been made of the determinant formula for the four-by-four matrix;

$$G_F \equiv \det(G + F) = G \left(1 + \frac{1}{2} F^{\mu\nu} \mathcal{F}_{\mu\nu} + \mathcal{T}^2\right), \quad \mathcal{T} \equiv \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}. \quad (3.10)$$

Taking the Hodge dual of (3.9), we find the $K$ as,

$$K_{\mu\nu} = -\frac{1}{2} \eta_{\mu\nu\rho\sigma} \tilde{K}^{\rho\sigma} = \frac{\sqrt{-G}}{\sqrt{-G_F}} (\tilde{F}_{\mu\nu} + \mathcal{T} F_{\mu\nu}) + C^{(2)\mu\nu}, \quad (3.11)$$

The last terms in (3.9) and (3.11) arise from the first term in the WZ term in (3.2),

$$C^{(2)} F = \frac{1}{4} d^4 \sigma \epsilon^{\mu\nu\rho\sigma} C^{(2)\mu\nu} F_{\rho\sigma}. \quad (3.12)$$

The product of (3.9) and (3.11) gives

$$K_{\mu\nu} \tilde{K}^{\mu\nu} + F_{\mu\nu} \tilde{F}^{\mu\nu} = -2 \tilde{F}^{\mu\nu} \Omega^3_{\mu\nu} - \Omega^3_{\mu\nu} \tilde{\Omega}^3_{\mu\nu} + 2 \tilde{K}^{\mu\nu} C^{(2)\mu\nu} - \tilde{C}^{(2)\mu\nu} C^{(2)\mu\nu}. \quad (3.13)$$

It may be appropriate to make a few remarks on the bosonic truncation of the D3 action, $L_B$. Obviously the r.h.s. of (3.13) vanishes in this case. Substituting the relation (3.13) into (2.4) with $\beta = -\gamma = \lambda$ for $SO(2)$, we obtain the variation of the Lagrangian as

$$\delta L = -\frac{1}{2} F \tilde{F}; \quad \text{the bosonic DBI Lagrangian density transforms into a total derivative.}$$

Eq.(3.13) also implies from the GZ condition that $\delta X L_B = 0$: so $\delta X = 0$ is a right assignment for the matter transformation.

Let us turn to the supersymmetric case and discuss the matter contribution in the GZ condition. It is used to find an appropriate transformation for the matter fields in such a way that it makes the action invariant under the dual transformation. For our present case of the D3 brane, we will find that the following transformation for the matter fields, $X$ and $\theta$, suites our purpose:

$$\delta \theta = \lambda \frac{i \tau_3}{2} \theta, \quad \delta X = 0, \quad (3.14)$$

which gives

$$\delta \Pi^M_{\mu} = \delta G_{\mu\nu} = \delta \Xi = 0, \quad \delta \Omega^3_{\mu\nu} = -\lambda \Omega^1_{\mu\nu}, \quad \delta \Omega^1_{\mu\nu} = \lambda \Omega^3_{\mu\nu}. \quad (3.15)$$

Tseytlin discussed the pseudo-invariance of this bosonic action for the flat metric case.
Note that the Majorana-Weyl fermions (θ₁, θ₂) and (Ω₁, Ω₃) transform as SO(2) doublets. Presently we will find the invariance of Ξ under duality rotation is crucial to satisfy the duality condition.

We turn to the variation of the total Lagrangian in (3.1) with respect to the matter transformation,

$$\delta_{\Phi} L = \delta_{\theta} L = \frac{1}{2} \frac{\partial L}{\partial F_{\mu\nu}} \delta \Omega^3_{\mu\nu} + \frac{1}{2} \tilde{F}^{\mu\nu} \delta C^{(2)}_{\mu\nu} + \delta \tilde{C}^{(4)}, \quad (3.16)$$

where $\tilde{C}^{(4)}$ is the Hodge dual of $C^{(4)}$: $C^{(4)} = d^4\sigma \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} C^{(4)}_{\mu\nu\rho\sigma} \equiv d^4\sigma \sqrt{-G} \tilde{C}^{(4)}$. $C^{(2)} = \Omega_1$ and the invariance of Ξ give rise to a relation of the differential forms

$$\delta \Xi = \delta \left[ \frac{1}{2} C^{(2)} \Omega_3 + C^{(4)} \right] = \frac{\lambda}{2} \left( - \left(C^{(2)}\right)^2 + \Omega_3^2 \right) + \delta \tilde{C}^{(4)} = 0. \quad (3.17)$$

Combining the results in (3.13), (3.16) and (3.17), we find

$$\frac{\lambda}{4} \left( F \tilde{F} + K \tilde{K} \right) + \delta_{\Phi} L = \frac{\lambda}{4} \left( - C^{(2)}_{\mu\nu} \tilde{C}^{(2)}_{\mu\nu} + \Omega^3_{\mu\nu} \tilde{\Omega}^{\mu\nu}_3 \right) + \delta \tilde{C}^{(4)} = 0. \quad (3.18)$$

Therefore, the duality condition is satisfied.

It has been recognized that the SO(2) duality may be lifted to the SL(2,R) duality by introducing a dilaton φ and an axion χ[14], [7]. They are assumed to be constant background fields. According to the general method, one defines a new Lagrangian using the D3 Lagrangian $L(F, X, \theta)$ which obeys the SO(2) duality

$$\hat{L}(F, X, \theta; \phi, \chi) = L(e^{-\phi/2}F, X, \theta) - \frac{1}{4} \chi F \tilde{F}. \quad (3.19)$$

If one introduces $\hat{F} = e^{-\phi/2}F$ and $\hat{K}$ by taking the dual of $(-)\partial L(\hat{F}, X, \theta)/\partial \hat{F}$, the background dependence is absorbed in the rescaled variable $(\hat{F}, \hat{K})$. These are related with the background dependent $(F, K)$ by

$$\left( \begin{array}{c} F \\ K \end{array} \right) = V \left( \begin{array}{c} \hat{F} \\ \hat{K} \end{array} \right), \quad V = e^{\phi/2} \left( \begin{array}{cc} 1 & 0 \\ -\chi & e^{-\phi} \end{array} \right). \quad (3.20)$$

Here $V$ is a non-linear realization of SL(2,R)/SO(2) transforming as

$$V \rightarrow \Lambda V O(\Lambda)^{-1}. \quad (3.21)$$

Here $\Lambda$ is a global SL(2,R) matrix

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R), \quad ad - bc = 1 \quad (3.22)$$

and $O(\Lambda)$ is an SO(2) transformation

$$O(\Lambda)^{-1} = \begin{pmatrix} \cos \lambda & \sin \lambda \\ -\sin \lambda & \cos \lambda \end{pmatrix} \in SO(2). \quad (3.23)$$
This “compensating” transformation is induced so that the form of $V$ is unchanged:
\[
\cos \lambda = \frac{a - b \chi}{\sqrt{(a - b \chi)^2 + b^2 e^{-2\phi}}}, \quad \sin \lambda = \frac{-b e^{-\phi}}{\sqrt{(a - b \chi)^2 + b^2 e^{-2\phi}}}. \tag{3.24}
\]

This procedure enables us to make the SO(2) dual theory discussed above to an SL(2,R) dual theory.

4 Hamiltonian formalism of D3 action

In this section, we give a proof of self-duality of the D3 action in the Hamiltonian formalism, using a general analysis of constraints[11] for super D-brane actions in type IIB theory. Let us include dilaton and axion, $\phi$ and $\chi$, from the beginning. From (3.19) the action is given by
\[
S = \int d^4 \sigma \mathcal{L}^{DBI} + \int L^{WZ} - \int \frac{1}{2} \chi F^2 = \int d^4 \sigma \mathcal{L}^{total}, \tag{4.1}
\]
where the 2-form component $\mathcal{F}_{\mu\nu}$ appeared in $\mathcal{L}^{DBI}$ and $L^{WZ}$ is replaced by
\[
\mathcal{F}_{\mu\nu} = e^{-\frac{\phi}{2}} F_{\mu\nu} + \Omega^3_{\mu\nu}. \tag{4.2}
\]

Let $(X^M, P_M), (\theta, \pi_\theta)$, and $(A_\mu, E^\mu)$ be canonically conjugate pairs of the phase space variables, and define the three-dimensional anti-symmetric tensor by $\epsilon^{ijk} = \epsilon_0^{ijk}$. We will soon find it useful to define the following new variables,
\[
\mathcal{B}^i = \frac{1}{2} \epsilon^{ijk} \mathcal{F}_{jk} = e^{-\frac{\phi}{2}} B^i + \frac{1}{2} \epsilon^{ijk} \Omega^3_{jk}, \quad B^i = \frac{1}{2} \epsilon^{ijk} \mathcal{F}_{jk}, \quad (i,j,k = 1,2,3) \quad \tag{4.3}
\]
and
\[
\mathcal{P}_M = \frac{\partial \mathcal{L}^{DBI}}{\partial \Pi^M_0} = P_M - e^\frac{\phi}{2} (E + \chi B)^i \frac{\partial \mathcal{F}_{0i}(\Pi, \dot{\theta})}{\partial \Pi^M_0} - \frac{\partial \mathcal{L}^{WZ}(\Pi, \mathcal{F}, \dot{\theta})}{\partial \Pi^M_0}, \tag{4.4}
\]
\[
\mathcal{E}^i = \frac{\partial \mathcal{L}^{DBI}}{\partial \mathcal{F}_{0i}} = e^\frac{\phi}{2} (E + \chi B)^i - \frac{\partial \mathcal{L}^{WZ}(\Pi, \mathcal{F}, \dot{\theta})}{\partial \mathcal{F}_{0i}}. \tag{4.5}
\]

In the last equation use has been made of the equation: $E^i = \partial \mathcal{L}^{total} / \partial F_{0i}$. We find constraints of the system to be given by:

(1) the $U(1)$ constraints,
\[
E^0 = 0, \quad \partial_i E^i = 0; \tag{4.6}
\]

(2) the $p + 1$ diffeomorphism constraints,
\[
\varphi_i \equiv \mathcal{P} \cdot \Pi_i + \mathcal{E}^j \mathcal{F}_{ij} = \mathcal{P} \cdot \Pi_i + \epsilon_{ijk} \mathcal{E}^j \mathcal{B}^k = 0, \quad (i = 1,2,3) \quad \tag{4.7}
\]
\[
\varphi_0 \equiv \frac{1}{2} \left[ \mathcal{P}^2 + \gamma + \gamma_{ij} (\mathcal{E}^i \mathcal{E}^j + \mathcal{B}^i \mathcal{B}^j) \right] = 0; \]
the fermionic constraints,
\[
\psi \equiv \pi_\theta - P_M \frac{\partial \Pi^M_0}{\partial \theta} - e^\hat{\phi}(E + \chi B)^i \frac{\partial F_{0i}(\Pi, \dot{\theta})}{\partial \theta} - \frac{\partial L^{WZ}(\Pi, F, \dot{\theta})}{\partial \theta} = 0. \quad (4.8)
\]
Here \(\gamma_{ij}\) is the spatial part of the induced metric and \(\gamma\) is its determinant.

We now show the invariance of the bosonic constraints (\(\varphi_0\) and \(\varphi_i\)) and the covariance of the fermionic constraints (\(\psi\)) under SL(2,R) transformation of (\(B, E\)) and (\(\phi, \chi\)) associated with the SO(2) rotation of the fermionic fields \(\theta\). To this end, we rewrite (4.3) and (4.5) as
\[
\begin{pmatrix} B \\ E \end{pmatrix} = V^{-1} \begin{pmatrix} B \\ E \end{pmatrix} + \begin{pmatrix} \Omega^3 \\ -\Omega^1 \end{pmatrix}, \quad (\Omega^f)_i^j \equiv \epsilon^{ijk} \hat{\Pi}_j \tau_k \partial_i \theta, \quad (4.9)
\]
where \(V\) is an SL(2,R)/SO(2) matrix given in (3.20). Under SL(2,R) transformation, \(\begin{pmatrix} B \\ E \end{pmatrix}\) rotates into \(O(\Lambda) \begin{pmatrix} B \\ E \end{pmatrix}\) as an SO(2) vector: each element of the first term in (4.9) transforms by
\[
\begin{pmatrix} B \\ E \end{pmatrix} \rightarrow \Lambda \begin{pmatrix} B \\ E \end{pmatrix}, \quad \Lambda \in \text{SL}(2,R), \quad (4.10)
\]
Likewise, the second term transforms by
\[
\begin{pmatrix} \Omega^3 \\ -\Omega^1 \end{pmatrix} \rightarrow O(\Lambda) \begin{pmatrix} \Omega^3 \\ -\Omega^1 \end{pmatrix} \quad (4.12)
\]
under the \(\tau_2\) rotation of spinors \(\theta\),
\[
\theta \rightarrow O(\Lambda) \theta. \quad (4.13)
\]
Here \(O(\Lambda)\) corresponds to the fundamental (spin 1/2) representation of SO(2).

Next we consider \(P\) given by
\[
P_M = P_M - \left( \mathcal{E}^i \frac{\partial F_{0i}(\Pi, \dot{\theta})}{\partial \Pi^M_0} + B^i \frac{\partial C^{(2)}_{0i}(\Pi, \dot{\theta})}{\partial \Pi^M_0} \right) - \left( \frac{1}{2} \epsilon^{ijk} C^{(2)}_{jk} \frac{\partial F_{0i}(\Pi, \dot{\theta})}{\partial \Pi^M_0} + \frac{\partial}{\partial \Pi^M_0} \sqrt{-G} \tilde{C}^{(4)}(\Pi, \dot{\theta}) \right). \quad (4.14)
\]
We will see that each term in the above expression, and thus \(P\) itself, is invariant under the duality transformation. The invariance of \(X^M\) implies that \(P_M\), the conjugate variable, is also invariant.[7] The second term on the r.h.s. of (4.14) may be rewritten as
\[
\begin{pmatrix} \mathcal{E}^i \frac{\partial F_{0i}(\Pi, \dot{\theta})}{\partial \Pi^M_0} \right) + \right) \cdot \right) = \mathcal{E}_i \partial_\tau \right) + \right) \cdot \right) = (B^i, \mathcal{E}^i) \left( \Theta M^\tau_3 \partial_i \dot{\theta} \right) = \Theta M^\tau_3 \partial_i \dot{\theta}, \quad (4.15)
\]

[7] We will make a more accurate statement on the invariance of \(P_M\) at the end of this section.
where
\[ \tilde{\tau}^i_0 = \mathcal{E}^i \tau_3 + \mathcal{B}^i \tau_1. \] (4.16)

When sandwiched between \( \theta \)'s, \( \tilde{\tau}^a_0 \) is invariant under simultaneous rotation of \( \theta \) and \((\mathcal{B}, \mathcal{E})\). So the second term is invariant. In terms of the differential form, the last term on the r.h.s. of (4.14) is expressed as
\[ C^{(2)0i} \frac{\partial \mathcal{F}_{0i}}{\partial \Pi^M_0} + \frac{\partial}{\partial \Pi^M_0} \sqrt{-G} \tilde{C}^{(4)} \]
\[ \quad \rightarrow \frac{1}{2} \left[ (\bar{\theta} \Gamma_M \tau_3 d\theta)(\bar{\theta} \hat{\Pi} \tau_1) d\theta \right] + \left[ \frac{\partial C^{(4)}}{\partial \Pi^M_0} \right]_3 \]
\[ = \frac{1}{2} \left[ (\bar{\theta} \Gamma_M \tau_3 d\theta)(\bar{\theta} \hat{\Pi} \tau_1) d\theta \right]_3 + \left[ \frac{\partial}{\partial \Pi^M_0} \left( C^{(4)} + \frac{1}{2} \Omega_1 \Omega_3 \right) \right]_3, \] (4.17)

where \([ \ ]_3 \) denotes a spatial 3-form coefficient of \([ \ ] \). In the last expression we observe that two terms are invariant separately: the first is written with an anti-symmetrization of \( \tau_1 \) and \( \tau_3 \) and is invariant under the \( \theta \) rotation; the second term is obviously related to the invariant quantity \( \Xi = [C^{(4)} + \frac{1}{2} \Omega_1 \Omega_3] \). This completes our proof of the invariance of \( \mathcal{P} \).

In the diffeomorphism constraints, \( \mathcal{B} \) and \( \mathcal{E} \) appear only in SO(2) invariant combinations. Therefore, one concludes that \( \varphi_0 \) and \( \varphi_i \) are invariant.

We may see the covariance of the fermionic constraint in parallel with the above discussion on \( \mathcal{P} \). The constraint is expressed as
\[ \psi = \pi_\theta + P_M (\bar{\theta} \Gamma^M) - \left( \mathcal{E}_i \frac{\partial \mathcal{F}_{0i}}{\partial \theta} + \frac{1}{2} \epsilon_{ijk} \mathcal{B}^j \frac{\partial C^{(2)jk}}{\partial \theta} \right) \]
\[ - \left( C^{(2)0i} \frac{\partial \mathcal{F}_{0i}}{\partial \theta} + \frac{\partial}{\partial \theta} \sqrt{-G} \tilde{C}^{(4)} \right), \] (4.18)

where one finds
\[ \left( \mathcal{E}_i \frac{\partial \mathcal{F}_{0i}}{\partial \theta} + \frac{1}{2} \epsilon_{ijk} \mathcal{B}^j \frac{\partial C^{(2)jk}}{\partial \theta} \right) = \frac{1}{2} \bar{\theta} \Gamma \tau^i_0 \partial_\theta - \bar{\theta} \Gamma \tau^i_0 \hat{\Pi}_i, \] (4.19)

and
\[ \left( C^{(2)0i} \frac{\partial \mathcal{F}_{0i}}{\partial \theta} + \frac{\partial}{\partial \theta} \sqrt{-G} \tilde{C}^{(4)} \right) = \left[ \frac{1}{2} (\bar{\theta} \hat{\Pi} \tau_1 d\theta)(\bar{\theta} \Gamma_M \tau_3 d\theta) \right]_3 + \left[ \frac{\partial}{\partial \theta} \Xi \right]_3. \] (4.20)

These expressions and the fact that \( \pi_\theta \) transforms as \( \pi_\theta \rightarrow \pi_\theta O(\Lambda)^T \) imply that the fermionic constraint \( \psi \) transforms covariantly: \( \psi \rightarrow \psi O(\Lambda)^T \).
We now show that the A-duality transformation is described as a canonical transformation, using our results [2] on the curved space extension of the A-transformation. It should be remarked that, in the case at hand, the intrinsic $D=3$ metric $g_{ij}$ used in ref. [2] is replaced by the induced metric $\gamma_{ij}$ expressed in terms of brane coordinates, $(X, \theta)$: the invariance of $D=3$ metric puts a non-trivial condition on the induced metric, which is satisfied for the present case as shown in (3.13).

For any function on the phase space $R(p, q)$, the transformation

$$\delta R = -i[R, \mathcal{W}],$$

is defined via the generator

$$\mathcal{W} = \lambda \int d^3\sigma \sqrt{\gamma} \left[ \frac{1}{2} \frac{E^i}{\sqrt{\gamma}} D^{-1}_{i\ell} \frac{E^\ell}{\sqrt{\gamma}} + \frac{1}{2} A_i D^{ij} A_j + \pi_{0} \frac{i\tau_2}{2} \theta \right].$$

Here we have used the following operators for $D=3$ covariant formulation: $D^{-1}_{i\ell}$ is a tensor operator acting on a vector,

$$D^{-1}_{i\ell} \equiv (\tilde{\Delta}^{-1})^{k}_{i} \nabla_{j} \eta_{j\ell k} = \eta_{ijk} \nabla^{k}(\tilde{\Delta}^{-1})_{i}^{j},$$

where $\eta_{j\ell k} = \epsilon_{j\ell k} \sqrt{\gamma}$ is the covariantly constant anti-symmetric tensor. It is the inverse of

$$D^{jk} = \eta^{jk} \nabla_{\ell} = \nabla_{\ell} \eta^{jk},$$

in a projected space

$$D^{-1}_{im} D^{mk} = O^i_{k}(\nabla), \quad D^{im} D^{-1}_{mk} = O^i_{k}(\nabla),$$

$$O^i_{k}(\nabla) = \delta^i_{k} - \nabla_i(\Delta^{-1})\nabla^k.$$ (4.25)

Note that the operator $O^i_{k}(\nabla)$ projects out any longitudinal component defined with the covariant derivative $\nabla_i$.

The curved space extension of the Laplacian operator $(\tilde{\Delta})^{i}_{j}$, which maps a vector $T_i$ into a vector $(\tilde{\Delta})^{i}_{j}T_i$, is given by

$$(\tilde{\Delta})^{i}_{j} = \Delta \delta^{i}_{j} - R^{i}_{j},$$

where $\Delta = \nabla^{i} \nabla_{i}$ and $R^{i}_{j}$ is the Ricci tensor. We assume that boundary conditions can be arranged so that the Laplacian operator has no non-trivial kernel, and its inverse, $(\tilde{\Delta}^{-1})^{i}_{j}$, is well-defined.

One finds that $\mathcal{W}$ in (4.22) generates the desired A-duality transformation for the gauge field as well as the SO(2) rotation of $\theta$:

$$\delta A_\ell = \lambda (\tilde{\Delta}^{-1})^{k}_{\ell} \nabla^{j} \epsilon_{jkm} E^{m} = D^{-1}_{\ell m} \left( \lambda \frac{E^{m}}{\sqrt{\gamma}} \right).$$ (4.27)
\[ \delta E^i = -\lambda \epsilon^{ijk} \partial_j A_k; \quad (4.28) \]
\[ \delta \theta = \lambda \frac{i\tau_2}{2} \theta, \quad \delta \pi_\theta = -\pi_\theta \lambda \frac{i\tau_2}{2}; \quad (4.29) \]
\[ \delta X = 0, \quad \delta P = 0. \quad (4.30) \]

It follows from the above expressions that

\[ \delta B^i = \lambda E^i, \quad \delta E^i_\perp = -\lambda B^i, \quad (4.31) \]

where

\[ E^k_\perp = E^k - \sqrt{\gamma} \nabla^k (\Delta^{-1}) \left( \nabla_m \frac{E^m}{\sqrt{\gamma}} \right). \quad (4.32) \]

This describes duality exchange between the electric and magnetic fields.

In (4.29) and (4.30), although not described explicitly, there appear some additional terms in the transformations of momenta, \((\delta P, \delta \pi_\theta)\). It is because the relevant metric is the induced one given via \((X, \theta)\), and the metric dependent term in (4.22) generates new contributions to \((\delta P, \delta \pi_\theta)\). These terms, however, are shown to be proportional to the Gauss law constraint, \(\partial_i E^i = 0\), and therefore do not affect the transformation rule (4.21) on the constraint surfaces.

In summary we have shown that the constraint equations of D3 action are invariant or covariant under:

1) linear SL(2,R) transformation of \(\begin{pmatrix} B \\ E \end{pmatrix} \rightarrow \Lambda \begin{pmatrix} B \\ E \end{pmatrix}\);

2) rotation of \((\theta, \pi_\theta)\) by \(\begin{pmatrix} O(\Lambda) \theta \\ \pi_\theta O(\Lambda)^T \end{pmatrix}\), induced by the SL(2,R) transformation;

3) non-linear transformation of the backgrounds \(\phi\) and \(\chi\) as (3.21).

Note that \((X, P)\) are left invariant (up to the Gauss law constraint).

This completes the proof of the invariance of the Hamiltonian, the self-duality in the canonical formalism. It is worth mentioning that the duality transformation does not commute with global SUSY transformation. This is suggested by the fact that Majorana-Weyl spinors \(\theta\) transform under the duality transformation, while the bosonic counterpart \(X^M\) is left invariant. The SUSY charge \(Q\) undergoes the same transformation as \(\pi_\theta\) under the duality rotation: \(Q \rightarrow Q O(\Lambda)^T\). In this connection, note that the NS-NS two form \(B^{(2)} = -\Omega^2\) and R-R two form \(C^{(2)} = \Omega^1\) mixed as an SO(2) vector.

## 5 Summary and Discussion

In D=4 spaces, irrespective of being target space or world-volume, the vector duality transformation is special in the sense that it is nothing but the electric-magnetic duality
rotation. Gaillard and Zumino showed that the maximal group of the duality transformation allowed for an interacting gauge field strength is the SL(2,R). They also found the duality condition: the Lagrangian needs to transform in a particular way under the duality rotation in order for EOM to remain invariant or covariant. It turns out, however, that the GZ condition is more than that: this algebraic relation is the necessary and sufficient condition for invariance of the action, and may serve therefore a guiding principle of constructing D=4 actions of U(1) gauge field strength coupled with gravity and matters in string and field theories. Our proof of the self-duality of the D3 action may be the first non-trivial application of this idea.

Obviously, the existence of the criterion for duality symmetry such as the GZ condition in the Lagrangian formalism is only possible in D=4 theories including D3-action. As for the other D-branes in the effective action approach to string and M-theory, we believe that the Hamiltonian formalism suits better for establishing exact symmetries or relations. Actually, we showed in a previous article the canonical equivalence between D-string action and IIB string action. The previous work and the present one for D3-action strongly support the idea that the vector dualities in type IIB theory can be understood as canonical transformations. We expect furthermore that the relationships between the D-brane actions in type IIA theory and the dimensionally reduced M-brane actions may be understood similarly in the canonical formalism.

It would be appropriate to make some comments on other approaches to implement the duality symmetry at the action level. In the DBI action is reformulated in a duality manifest way by introducing another world-volume gauge field. This approach is an extension of the Schwarz-Sen model with two gauge fields, and its covariant version, the PST model. Possible relations among higher dimensional theories, two gauge field formulations and a manifestly dual invariant formulation of string (effective) theories are extremely interesting, though there must be many points to be clarified to find them in concrete. In particular, to formulate the supersymmetric version of two gauge models, the knowledge of fermion transformations would be crucial. It is very interesting to imagine that there is a condition in two gauge models, an extension of the GZ condition, which puts some restrictions on matter transformations and helps us to find yet unknown supersymmetric extensions.

Our A-transformation approach has its own drawbacks: non-locality and the sacrifice of the manifest D=4 covariance. The two gauge field formulation and its extensions have been introduced to overcome these difficulties. An extension of our argument given in this paper may be extended to those approaches. However we did not take those point of views because of the following reasons: our present approach is enough to show the self-duality; and we believe that there are much more to be done to figure out a real relation of those approaches to the non-perturbative string theory.

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A super D3-brane action with two gauge fields was given in ref. in a different context(see also ref.). It would be interesting to investigate how duality symmetry is realized there.
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