THE MAXIMAL ORDER OF SEMIDISCRETE SCHEMES FOR QUASILINEAR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We prove that a semidiscrete \((2r+1)\)-point scheme for quasilinear first order PDE cannot attain an order higher than \(2r\). Moreover, if the forward Euler fully discrete scheme obtained from the linearization about any constant state of the semidiscrete scheme is stable, then the upper bound for the order of the scheme is \(2r - 1\). This bound is attained for a wide range of schemes and equations.

1. INTRODUCTION

A basic strategy for obtaining high order numerical methods for quasilinear first order partial differential equations \(u_t = a(u)u_x\) consists in the method of lines: the spatial term \(a(u)u_x\) is approximated by means of high order finite differences

\[
(a(u)u_x)(x_j, t) \approx \frac{H(v_{j-r}(t), \ldots, v_{j+r}(t))}{h},
\]

for approximations \(v_j(t) \approx u(x_j, t)\), where \(h\) is the grid step size and \(x_j = x_0 + jh\). The resulting ODE

\[
(v_j'(t) = \frac{H(v_{j-r}(t), \ldots, v_{j+r}(t))}{h})
\]

is then solved by some ODE solver and the fully discrete numerical method thus obtained has an order which is the minimum of the orders of the finite difference formula and the ODE solver. Strong Stability Preserving Runge-Kutta (SSPRK) ODE solvers (see [4] and references therein) are widely used for the solution of (1) due to its nonlinear stability features. Since the forward Euler scheme is the basic building block for SSPRK solvers, the stability of that scheme is crucial for the stability of high order fully discrete schemes obtained from the method of lines.

The problem of the maximal order that can be attained by a semidiscrete scheme for first order partial differential equations is considered in [5] for linear schemes for the linear advection equation. The authors of this work conclude that the maximal order of linear schemes for which [1] is stable is \(2r\). In [2], the author derives conditions for linear, explicit time-marching methods approximating the \(m\)-th order linear equation with constant coefficients to any order.

In this work we deal with the maximal order that can be attained with a general semidiscrete scheme for the first-order quasilinear partial differential equation. We prove that the order of any such scheme is bounded above by \(2r\) and that if the order attains this bound then the forward Euler scheme is unconditionally unstable. We

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also point out that finite difference \((2r+1)\)-point WENO schemes \([8,6,1]\) obtained from maximal order reconstructions do attain order \(2r−1\).

2. THE MAXIMAL ORDER OF A SEMIDISCRETE SCHEME

We start with two lemmas that establish the invertibility of some Vandermonde-like matrices that appear in the analysis of the schemes. We then introduce some notation and establish a generalization of the chain rule known as Faà di Bruno’s formula \([3]\) in Theorem \([4]\) whose proof is included in an appendix, and we finally state our main result in Theorem \([2]\).

Lemma 1. Given \(a_1, \ldots, a_n\), then

\[
\det(a_j^n)_{i,j=1} = \prod_{j=1}^{n} \prod_{k=j+1}^{n} (a_k - a_j) \prod_{j=1}^{n} a_j.
\]

If \(a_1, \ldots, a_n\) are pairwise distinct and not null then \(\det(a_j^n)_{i,j=1} \neq 0\).

Proof. This is easily obtained from the fact that the determinant of a Vandermonde matrix \((a_i−1)_{i,j=1}^{n}\) is given by \(\det(a_i−1)_{i,j=1}^{n} = \prod_{k=1}^{n} (a_k - a_j)\).

Lemma 2. The determinant of the matrix given by the entries

\[
A(a_1, \ldots, a_n)_{i,j} = \begin{cases} 1 & \text{if } i = 1 \\ a_j^i & \text{if } i \geq 2 \end{cases},
\]

for any \(a_1, \ldots, a_n, n \geq 2\), is given by:

\[
\det A(a_1, \ldots, a_n) = \prod_{j=1}^{n} \prod_{k=j+1}^{n} (a_k - a_j) \sum_{j=1}^{n} \prod_{k=1,k\neq j}^{n} a_k.
\]

If \(a_1 = 0, a_2, \ldots, a_n\) are pairwise distinct then \(\det A(a_1, \ldots, a_n) \neq 0\).

Proof. The result is proved by induction on \(n\). The result for \(n = 2\) is

\[
\det \begin{bmatrix} 1 & 1 \\ a_1^2 & a_2^2 \end{bmatrix} = (a_2 - a_1)(a_1 + a_2),
\]

which clearly holds true. Assume now \(n > 2\) and the result to be true for \(n − 1\). We do elimination in the first column, by subtracting from row \(i + 1\) the first row multiplied by \(a_1^{i+1}, i = 1, \ldots, n−1\). This yields

\[
\det A(a_1, \ldots, a_n) = \det(a_j^{i+1} - a_1^{i+1})_{i,j=1}^{n-1} = \det \left( a_{j+1} - a_1 \left( \sum_{k=0}^{i} a_k^{j+1}a_1^{i-k} \right) \right)_{i,j=1}^{n-1}
\]

\[
= \prod_{j=2}^{n} (a_j - a_1) \det \left( \sum_{k=0}^{i} a_k^{j+1}a_1^{i-k} \right)_{i,j=1}^{n-1}.
\]

Subtracting to row \(i\) of the matrix

\[
\left( \sum_{k=0}^{i} a_k^{j+1}a_1^{i-k} \right)_{i,j=1}^{n-1}.
\]
its row $i - 1$ multiplied by $a_1$, since

$$
\sum_{k=0}^{i} a_{j+1}^{k}a_{i-1}^{i-k} - a_1 \sum_{k=0}^{i-1} a_{j+1}^{k}a_{i-1}^{i-1-k} = \sum_{k=0}^{i} a_{j+1}^{k}a_{i-1}^{i-k} - \sum_{k=0}^{i-1} a_{j+1}^{k}a_{i-1}^{i-k} = a_{j+1}^{i},
$$

we get

$$
\det \left( \sum_{k=0}^{i} a_{j+1}^{k}a_{i-1}^{i-k} \right)_{i,j=1}^{n-1} = \det \left( b_{i,j} \right)_{i,j=1}^{n-1},
$$

where

$$
b_{i,j} = \begin{cases} a_1 + a_{j+1} & \text{if } i = 1 \\ a_{j+1} & \text{if } 2 \leq i \leq n - 1. \end{cases}
$$

Therefore, we can split this determinant into the sum of two determinants:

$$
\det \left( b_{i,j} \right)_{i,j=1}^{n-1} = a_1 \det(A(a_2, \ldots, a_{n-1})) + \det((a_{j+1})_{i,j=1}^{n-1}).
$$

Now, using the induction hypothesis for the determinant of the first summand and applying Lemma 1 for the determinant of the second one:

$$
\det A(a_1, \ldots, a_n) = \prod_{j=2}^{n} (a_j - a_1) \left[ a_1 \prod_{j=2}^{n} \prod_{k=2, k \neq j}^{n} (a_k - a_j) \sum_{j=2}^{n} \prod_{k=2, k \neq j}^{n} a_k \\
+ \prod_{j=2}^{n} \prod_{k=2, k \neq j}^{n} (a_k - a_j) \prod_{j=2}^{n} a_j \right] = \prod_{j=1}^{n} \prod_{k=1, k \neq j}^{n} (a_k - a_j) \sum_{j=1}^{n} \prod_{k=1, k \neq j}^{n} a_k.
$$

Finally, if $a_1 = 0, a_2, \ldots, a_n$ are pairwise distinct, then:

$$
\det A(a_1, \ldots, a_n) = \prod_{j=2}^{n} \prod_{k=2}^{n} (a_k - a_j) \prod_{j=2}^{n} a_j^2 \neq 0.
$$

\[ \square \]

**Notation 1.** Denote by $\mathcal{M}(s, n)$ the vector space of multilinear functions ($s$-order covariant tensors)

$$
T: (\mathbb{R}^n)^s \rightarrow \mathbb{R}, \quad (\mathbb{R}^n)^s = \mathbb{R}^n \times \cdots \times \mathbb{R}^n.
$$

Since $(\mathbb{R}^n)^s$ is isomorphic to the vector space of $n \times s$ matrices, we can regard $s$-order tensors as acting on the columns of $n \times s$ matrices. Tensors can be characterized as $n \times \cdots \times n$ matrices $(T_i_{1 \ldots i_s})$, i.e., their action on an $n \times s$ matrix $A$ is given by

$$
T(A) = \sum_{i_1 \ldots i_s=1}^{n} T_{i_1 \ldots i_s} A_{i_1,1} \ldots A_{i_s,s}.
$$

Assume $T: \mathbb{R}^n \rightarrow \mathcal{M}(s, n)$ is differentiable (equivalently, $T_{i_1 \ldots i_s}$ are differentiable). Then $T'(u) \in \mathcal{M}(s+1, n)$ is given by:

$$
T'(u)_{i_0,i_1,\ldots,i_s} = \frac{\partial T_{i_1 \ldots i_s}}{\partial u_{i_0}}(u).
$$
With this notation the derivatives of real functions can be mapped to tensors in the following way: If \( f: \mathbb{R}^n \to \mathbb{R} \) is in \( \mathcal{C}^s \), then we define \( f^{(k)} \in \mathcal{M}(k,n) \), \( 1 \leq k \leq s \) by

\[
f^{(k)}_{i_1,\ldots,i_k}(u) = \frac{\partial^k}{\partial u_{i_1}\cdots\partial u_{i_k}} f(u).
\]

**Notation 2.** For \( s \in \mathbb{N} \), we denote

\[
\mathcal{P}_s = \{ m \in \mathbb{N}^s / \sum_{j=1}^s jm_j = s \},
\]

and for \( m \in \mathbb{N}^s \), \( |m| = \sum_{j=1}^s m_j \). For \( m \in \mathcal{P}_s \), we denote

\[
\left[ \begin{array}{c} s \\ m \end{array} \right] = \frac{s!}{m_1! \cdots m_s!}
\]

and for \( m \in \mathbb{N}^s \) and a function \( u: \mathbb{R} \to \mathbb{R}^n \), the \( n \times |m| \) matrix \( D^m u(x) \) is given column-wise by

\[
(D^m u(x))_{i_1+k} = \frac{1}{j!} \frac{\partial^j u(x)}{\partial x^j}, \quad k = 1, \ldots, m_j, \quad j = 1, \ldots, s,
\]

(3)

\[
D^m u(x) = \begin{bmatrix}
\frac{\partial u(x)}{dx} & \cdots & \frac{\partial^m u(x)}{dx^m} \\
\frac{\partial u(x)}{dx} & \cdots & \frac{\partial^m u(x)}{dx^m} \\
\vdots & \ddots & \vdots \\
\frac{\partial u(x)}{dx} & \cdots & \frac{\partial^m u(x)}{dx^m}
\end{bmatrix},
\]

where it is to be understood here that \( j \)-th order derivatives do not appear in \( D^m u \) if \( m_j = 0 \).

We state the generalized chain rule for high order derivatives of compositions of functions due to Faà di Bruno [3].

**Theorem 1** (Faà di Bruno’s formula). Let \( f: \mathbb{R}^n \to R, u: \mathbb{R} \to \mathbb{R}^n \) be \( s \) times continuously differentiable. Then

\[
\frac{d^s f(u(x))}{dx^s} = \sum_{m \in \mathcal{P}_s} \left[ \begin{array}{c} s \\ m \end{array} \right] f(|m|)(u(x)) D^m u(x)
\]

(4)

**Theorem 2.** Consider the semidiscrete scheme \( (\ref{1}) \), with \( H \) a smooth function, for the approximate solution of \( u_t = a(u)u_x \), with \( v_j(t) \approx u(x_j, t) \). Then the order \( p \) of \( (\ref{1}) \) satisfies \( p \leq 2r \) and if \( p = 2r \) then the forward Euler scheme applied to the linearized scheme about any constant state is unconditionally unstable.

**Proof.** We drop the dependence on \( t \) until it is required. By the definition of the order of \( (\ref{1}) \), for any smooth function \( u \) and any \( x \) we get:

\[
\frac{H(u(x-\delta x), \ldots, u(x+\delta x))}{\delta x} = a(u(x))u'(x) + \mathcal{O}(\delta x^p).
\]

Let us fix \( u, x \) and denote \( u_j(h) = u(x + jh) \), \( U(h) = (u_{r-1}(h), \ldots, u_r(h)) \) and \( \Phi(h) = H(U(h)) \). Then \( (\ref{5}) \) and the Taylor development of \( \Phi \) about \( 0 \) yield:

\[
\Phi(s)(0) = \sum_{m \in \mathcal{P}_s} \left[ \begin{array}{c} s \\ m \end{array} \right] H(|m|)(U(0)) D^m U(0), \quad U = U_{u,x},
\]

(7)

Faà di Bruno’s formula \( (\ref{4}) \) yields

\[
\Phi(s)(0) = \sum_{m \in \mathcal{P}_s} \left[ \begin{array}{c} s \\ m \end{array} \right] H(|m|)(U(0)) D^m U(0), \quad U = U_{u,x}.
\]
for any \( u \) and \( x \)

For fixed \( s \in \mathbb{N} \) and \( v \in \mathbb{R} \), we consider in \( x = 0, u(y) = v + Cy' \), which verifies
\( u^{(\ell)}(0) = 0, \ell = 1, \ldots, s - 1, u^{(s)}(0) = 1. \) Then, for \( m \in \mathcal{P}_s \) such that \( m_x = 0 \),
one has \( D^mU(0) = 0 \), since the columns of this matrix are \( 0 = u^{(j)}(0)/j! \) for some
\( j \in \{1, \ldots, s - 1\} \). On the other hand, for \( m = (0, \ldots, 0, 1) \in \mathbb{N}^s, D^mU(0) \) is a one
column matrix given by
\[
D^{(0, \ldots, 0, 1)}U(0) = \frac{1}{s!}U^{(s)}(0) = \frac{1}{s!} \begin{bmatrix} a_{r-r}^{(s)}(0) \\ \vdots \\ a_{r-r}^{(s)}(0) \end{bmatrix} = \frac{1}{s!} u^{(s)}(x) \begin{bmatrix} (-r)^s \\ \vdots \\ r^s \end{bmatrix} = \frac{1}{s!} \begin{bmatrix} (-r)^s \\ \vdots \\ r^s \end{bmatrix}
\]

Therefore,
\[
\Phi^{(s)}(0) = \left[ \sum_{e_s} \frac{s}{e_s} \right] H^{(e_s)}(U(0))D^{e_s}U(0) = \sum_{l=-r}^r \frac{\partial H}{\partial u_l}(v, \ldots, v)l^s.
\]

From (8) for \( s = 1 \) and (6) applied to \( u(y) = v + y \) we obtain
\[
\Phi'(0) = \sum_{l=-r}^r l \frac{\partial H}{\partial u_l}(v, \ldots, v) = a(v),
\]
for any \( v \in \mathbb{R} \).

On the other hand, equation (10) for \( s = 0 \) reads as \( 0 = \Phi(0) = H(v, \ldots, v) \) for
any \( v(= u(x)) \), which in turn yields that
\[
\sum_{l=-r}^r \frac{\partial H}{\partial u_l}(v, \ldots, v) = 0,
\]
for any \( v \).

Recapping, we have obtained the following equations:
\[
\delta_{k,1}a(v) = \sum_{l=-r}^r l^k \frac{\partial H}{\partial u_l}(v, \ldots, v), \quad k = 0, \ldots, p, \quad \forall v.
\]

Now, if \( p > 2r \), then
\[
0 = \sum_{l=-r}^r l^k \frac{\partial H}{\partial u_l}(v, \ldots, v), \quad k = 0, \ldots, 2r + 1, k \neq 1,
\]
i.e., \( \left( \frac{\partial H}{\partial u_r}(v, \ldots, v), \ldots, \frac{\partial H}{\partial u_0}(v, \ldots, v) \right) \) is a solution of system (11), whose matrix,
with the notation of Lemma 2 is \( A(-r, \ldots, 0, \ldots, r) \) and satisfies
\[
\det A(-r, \ldots, 0, \ldots, r) \neq 0,
\]
so the unique solution of system (11) is the trivial one, i.e.,
\[
\frac{\partial H}{\partial u_l}(v, \ldots, v) = 0, \quad l = -r, \ldots, r, \forall v,
\]
which contradicts the fact that, from (10) for \( s = 1, \sum_{l=-r}^r l \frac{\partial H}{\partial u_l}(v, \ldots, v) = a(v) \)
for generic \( a \) and \( v \). Therefore \( p \leq 2r \).

If \( p = 2r \) then
\[
\sum_{l=1}^r l^{2k} \left( \frac{\partial H}{\partial u_l}(v, \ldots, v) + \frac{\partial H}{\partial u_{-l}}(v, \ldots, v) \right) = 0, \quad k = 1, \ldots, r,
\]
and this, after Lemma 1, gives

\[ \frac{\partial H}{\partial u_l}(v, \ldots, v) + \frac{\partial H}{\partial u_{-l}}(v, \ldots, v) = 0, \quad l = 1, \ldots, r. \]

On the other hand, since \( \sum_{l=-r}^{r} \frac{\partial H}{\partial u_l}(v, \ldots, v) = 0 \) by (9), we get from (12)

\[ \frac{\partial H}{\partial u_0}(v, \ldots, v) = 0. \]

If \( \bar{v} \) is any constant, since \( H(\bar{v}, \ldots, \bar{v}) = 0 \), then the linearized scheme for \( v_j(t) = \bar{v} + w_j(t) \) reads as:

\[ w_j' = \frac{\partial H}{\partial u_{-r}}(v, \ldots, \bar{v})w_j^- + \cdots + \frac{\partial H}{\partial u_r}(v, \ldots, v)w_j^+ - h, \]

which, by (2) and (13) can be written as

\[ w_j' = \sum_{l=1}^{r} \alpha_l (w_j^{+l} - w_j^{-l}), \]

where \( \alpha_l = \frac{\partial H}{\partial u_l}(\bar{v}, \ldots, \bar{v}) \). The forward Euler scheme applied to the system of ODE (14) is:

\[ w_j^{n+1} = w_j^n + \frac{k}{h} \sum_{l=1}^{r} \alpha_l (w_j^{n+l} - w_j^{n-l}), \]

or, in linear operator form, \( w^{n+1} = \Psi_{k/h} w^n \). The Fourier transform of \( \Psi_{k/h} \) is:

\[ \hat{\Psi}_{k/h}(\theta) = 1 + 2i \frac{k}{h} \sum_{l=1}^{r} \alpha_l \sin(\theta l), i = \sqrt{-1}, \]

which satisfies \( |\hat{\Psi}_{k/h}(\theta)| > 1 \) for any \( k/h \) and some \( \theta \), that is, the forward Euler scheme for the system of ODE (13) is unstable for any \( k/h \).

\[ \square \]

**Remark 1.** The Lax-Wendroff scheme is a second order 3-point scheme for conservation laws. Nevertheless, this is not a contradiction to Theorem 2, for this scheme is fully discrete. On the other hand, \( (2r+1) \)-point smooth and stable schemes of order \( 2r - 1 \) for conservation laws can be obtained, for example, via finite difference WENO schemes with global Lax-Friedrichs flux splittings [9, 1].

**Appendix A.**

For the sake of completeness, we include in this appendix a proof of Theorem 1 for we have not found satisfactory references for its proof.

The following result is easily established.

**Lemma 3.** Assume \( T: \mathbb{R}^n \to \mathcal{M}(s, n) \) is differentiable (equivalently, \( T_{i_1, \ldots, i_s} \) are differentiable) and that \( A: \mathbb{R} \to \mathbb{R}^{n \times s}, \ u: \mathbb{R} \to \mathbb{R}^n \) are also differentiable. Then, \( \forall x \in \mathbb{R} \)

\[ \frac{d}{dx} T(u(x))A(x) = T'(u(x))[u'(x) A(x)] + T(u(x)) \sum_{j=1}^{s} d_j A(x), \]
where we have used the notation $d_j A(x)$ for the $n \times s$ matrix given by the columns:

$$(d_j A(x))_k = \begin{cases} A_k(x) & k \neq j \\ A_j'(x) & k = j \end{cases}$$

**Notation 3.** We introduce some further notation for the proof of Theorem 1. For $s \in \mathbb{N}$, we denote

$$\mathcal{P}_{s,j} = \{ m \in \mathcal{P} / m_j \neq 0 \}.$$

We denote also

$$S_0 : \mathcal{P}_s \to \mathcal{P}_{s+1,1}, \quad S_0(m) = \begin{cases} 0 & k = s + 1 \\ m_k & s \geq k \neq 1 \\ m_1 + 1 & k = 1, \end{cases}$$

$$S_j : \mathcal{P}_{s,j} \to \mathcal{P}_{s+1,j+1}, \quad S_j(m) = \begin{cases} 0 & k = s + 1 \\ m_k & s \geq k \neq j, j + 1 \\ m_j - 1 & s \geq k = j \\ m_{j+1} + 1 & s \geq k = j + 1. \end{cases}$$

for $1 \leq j < s$, and $S_s$ that maps $(0, \ldots, 0, 1) \in \mathbb{N}^s$ to $(0, \ldots, 0, 1) \in \mathbb{N}^{s+1}$.

**Proof.** (of Theorem 1) We use induction on $s$, the case $s = 1$ being the chain rule. By the induction hypothesis for $s$ and Lemma 3 we deduce:

$$\frac{d^{s+1} f(u(x))}{dx^{s+1}} = \sum_{m \in \mathcal{P}_s} \left[ \binom{s}{m} \frac{d}{dx} \left( f^{(|m|)}(u(x)) D^m u(x) \right) \right]$$

$$= \sum_{m \in \mathcal{P}_s} \left[ \binom{s}{m} \left( f^{(|m|)}(u(x))[u'(x) D^m u(x)] + f^{(|m|)}(u(x)) \sum_{j=1}^n d_j D^m u(x) \right) \right]$$

$$= \sum_{m \in \mathcal{P}_s} \left[ \binom{s}{m} \left( f^{(|m|+1)}(u(x))[u'(x) D^m u(x)] + f^{(|m|)}(u(x)) \sum_{j=1}^n d_j D^m u(x) \right) \right].$$

Now,

$$d_j D^m u(x) = D^{S_j(m)} u(x) PE,$$

where $P$ is a permutation matrix corresponding to the transposition of $j$ and $\sum_{l \leq k} m_l$, with $\sum_{l \leq k} m_l < j \leq \sum_{l \leq k} m_l$ and $E$ is a diagonal matrix with $k + 1$ in the $\sum_{l \leq k} m_l$ entry and 1 in the rest.

By the symmetry of $f^{(|m|)}$, if $\sum_{l \leq k} m_l < j \leq \sum_{l \leq k} m_l$

$$f^{(|m|)}(u(x))d_j D^m u(x) = (k + 1)f^{(|S_k(m)|)}(u(x))D^{S_k(m)} u(x),$$

therefore, collecting identical terms,

$$\frac{d^{s+1} f(u(x))}{dx^{s+1}} = \sum_{m \in \mathcal{P}_s} \left[ \binom{s}{m} \left( f^{(|S_0(m)|)}(u(x)) D^{S_0(m)} u(x) \right) \right]$$

$$+ \sum_{j=1}^n f^{(|m|)}(u(x))d_j D^{S_j(m)} u(x).$$
can be written as
\[
\frac{d^{s+1}f(u(x))}{dx^{s+1}} = \sum_{m \in \mathcal{P}_s} \left[ \frac{s}{m} \right] (f^{(\lfloor S_0(m) \rfloor)}(u(x))) D^{S_0(m)} u(x) \\
+ \sum_{k=1}^n m_k (k+1) f^{(\lfloor S_k(m) \rfloor)}(u(x))) D^{S_k(m)} u(x),
\]
where we point out that in the last expression the only terms that actually appear are those for which \(m_k > 0\). Since \(m_k - 1 = (S_k(m))_k\), by collecting the terms for \(m, k\) such that \(S_k(m) = \hat{m}\), (15) can be written as
\[
\frac{d^{s+1}f(u(x))}{dx^{s+1}} = \sum_{\hat{m} \in \mathcal{P}_{s+1}} a_{\hat{m}} f^{(\lfloor \hat{m} \rfloor)}(u(x)) D^{\hat{m}} u(x),
\]
where
\[
a_{\hat{m}} = \left\{ \begin{array}{ll}
\tilde{a}_{\hat{m}} & \text{if } \hat{m}_1 = 0 \\
\tilde{a}_{\hat{m}} + \left[ \frac{s}{S_0^{-1}(\hat{m})} \right] & \text{if } \hat{m}_1 \neq 0,
\end{array} \right.
\]
\[
\hat{m} = S_k(m), \quad k \in \{1, \ldots, s\}, \quad m \in \mathcal{P}_{s,k}
\]
For \(k \in \{1, \ldots, s\}\), and \(m \in \mathcal{P}_{s,k}\), such that \(\hat{m} = S_k(m)\), i.e., \(\hat{m}_1 = m_i, i \neq k, k+1, \hat{m}_k = m_k - 1, \hat{m}_{k+1} = m_{k+1} + 1\), we deduce:
\[
\left[ \frac{s}{m} \right] m_k(k+1) = \frac{s!}{m_1! \cdots (m_k - 1)! m_{k+1}! \cdots m_s!} (k+1)
\]
\[
= \frac{s!}{\hat{m}_1! \cdots \hat{m}_k! \cdots \hat{m}_{k+1}! \cdots \hat{m}_s!} (k+1)
\]
\[
= \frac{s!}{\hat{m}_1! \cdots \hat{m}_k! \cdots \hat{m}_{k+1}! \cdots \hat{m}_s!} \hat{m}_{k+1}(k+1)
\]
Let \(\hat{m} = S_k(m)\) with \(k < s\), then one has \(\hat{m}_{s+1} = 0\). The only element \(m \in \mathcal{P}_{s,s}\) is \((0, \ldots, 0, 1) \in \mathbb{N}^s\) and \(S_s(m) = (0, \ldots, 0, 1) \in \mathbb{N}^{s+1}\). Therefore
\[
\tilde{a}_{\hat{m}} = \frac{s!}{\hat{m}_1! \cdots \hat{m}_s+1!} \sum_{\hat{m} \in \mathcal{P}_{s,s}, k \in \{1, \ldots, s\}, m \in \mathcal{P}_{s,k}} \hat{m}_{k+1}(k+1)
\]
\[
\tilde{a}_{\hat{m}} = \frac{s!}{\hat{m}_1! \cdots \hat{m}_s+1!} \sum_{k=1}^s \hat{m}_{k+1}(k+1) = \frac{s!}{\hat{m}_1! \cdots \hat{m}_s+1!} \sum_{k=2}^{s+1} \hat{m}_k k.
\]
On the other hand, if \(\hat{m}_1 \neq 0\), then:
\[
\left[ \frac{s}{S_0^{-1}(\hat{m})} \right] = \frac{s!}{(\hat{m}_1 - 1)! \hat{m}_2! \cdots \hat{m}_s!} = \frac{s!}{\hat{m}_1! \hat{m}_2! \cdots \hat{m}_s! \hat{m}_s+1!} \hat{m}_1,
\]
where the last equality holds since, as before, we have \(\hat{m}_{s+1} = 0\). Then, regardless of \(\hat{m}_1\), (18) and (19) yield for \(\hat{m} \in \mathcal{P}_{s+1}\)
\[
a_{\hat{m}} = \frac{s!}{\hat{m}_1! \cdots \hat{m}_{s+1}!} \sum_{k=1}^{s+1} \hat{m}_k k = \frac{s!}{\hat{m}_1! \cdots \hat{m}_{s+1}!} (s+1) = \left[ \frac{s+1}{\hat{m}} \right],
\]
since $\hat{m} \in P_{s+1}$ means $\sum_{k=1}^{s+1} \hat{m}_k k = s + 1$. We deduce from (16), (17) and (20) that

$\frac{d^{s+1}f(u(x))}{dx^{s+1}} = \sum_{\hat{m} \in P_{s+1}} \left[ s + 1 \hat{m} \right] f(|\hat{m}|)(u(x))D^{\hat{m}}u(x),$

which concludes the proof by induction. \hfill \Box

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