On the quantisation of gravity by embedding
spacetime in a higher dimensional space

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Abstract. Certain difficulties of quantum gravity can be avoided if we embed the spacetime \( V_4 \) into a higher dimensional space \( V_N \); then our spacetime is merely a 4-surface in \( V_N \). What remains is conceptually not so difficult: just to quantise this 4-surface. Our formal procedure generalises our version of Stueckelberg’s proper time method of worldline quantisation. We write the equations of \( V_4 \) in the covariant canonical form starting from a model Lagrangian which contains the classical Einstein gravity as a particular case. Then we perform quantisation in the Schrödinger picture by using the concepts of a phase functional and wave functional. As a result we obtain the uncertainty relations which imply that an observer is ‘aware’ either of a particular spacetime surface and has no information about other spacetime surfaces (which represent alternative histories); or conversely, he loses information about a particular \( V_4 \) whilst he obtains some information about other spacetimes (and histories). Equivalently, one cannot measure to an arbitrary precision both the metric on \( V_4 \) and matter distribution on various alternative spacetime surfaces. We show how this special case in the ‘coordinate’ representations can be generalised to an arbitrary vector in an abstract Hilbert space.

1 Introduction

All attempts at the quantisation of general relativity have met so far with considerable difficulties (see Anderson 1964, Brill and Gowdy 1970, Ashtekar and Geach 1974). The problems are of various types, from the conceptual to the technical ones. So in spite of the tremendous amount of work done on the subject we still do not possess a generally accepted theory of quantum gravity. Forced with this undesirable situation I am becoming inclined to the view that we have missed some important points in the development of our concepts about the relation between quantum theory and general relativity.

Let me develop here my own view of the starting points of quantum gravity. Relativistic quantum mechanics in itself is a paradox, as it unites ‘fatalism’ (everything is...
written in spacetime once and for all) with quantal probabilism (de Beauregard 1979). Since in the spacetime of general (or special) relativity all worldlines are ‘frozen’ and strict determinism is valid, there is no room for an observer’s free decisions. This is still true even if we postulate that an observer’s ‘now’ (by this we mean a three-dimensional hypersurface Σ of ‘simultaneous’ events) moves forward along a certain time-like direction in the spacetime $V_4$; the motions on Σ are determined in advance. In the moment when we postulate that a conscious observer has a free will (whatever is meant by this) which is not an illusion but real we already have an inconsistency. Instead of free will it is enough to take into account the validity of the quantum probability principle and we find again that quantum mechanics is incompatible with the existence of an objective reality in spacetime (Pavšič 1981a, b). In the case of special relativity this incompatibility was not so obvious and we were still able to construct more or less satisfactory relativistic quantum field theories. On the other hand, in the case of general relativity this incompatibility is so fatal that, in my opinion, it practically prevents any consistent construction of a quantum theory of gravity in four-dimensional spacetime.

According to us this problem could be resolved—at least conceptually—if we embed the spacetime $V_4$ into a certain higher dimensional space $V_N$; then our spacetime is merely a 4-surface in a higher space. What remains for us to do is conceptually not so difficult: just to quantise this surface. We already have some experience as to how to quantise two-dimensional surfaces in spacetime (which are strings from a three-dimensional point of view) (Polyakov 1981a, b, Tataru-Mihai 1982, Henneaux 1983, Gervais and Neveu 1982, Fradkin and Tsetlin 1982). These methods or any other analogous ones we have to generalise for the case of a four-dimensional surface.

Let me explain briefly how in higher space we avoid the incompatibility of the concept of spacetime with the concept of ‘free will’ or ‘free decision’. We may postulate that higher space is pseudo Euclidean (essentially flat) and has a certain structure of events (which are ‘physical’ in a broader sense); this higher dimensional world is deterministic, all events in it are frozen (these are either points, lines or a continuous distribution of a certain ‘matter’ density (see the following sections). Next, we can postulate that what a conscious observer observes is a succession of certain three-dimensional surfaces (also called the simultaneity surface or surface of ‘now’) (Pavšič 1981a, b); let us call this succession the motion of simultaneity 3-surface or Σ motion. Some parts of this Σ motion are under the direct conscious control of a certain observer: he can move his arms and legs and thus influence the course of events on Σ. Other parts of Σ are out of an observer’s direct control: he cannot influence the motion of rivers, planets, etc. On a sufficiently small scale, the motion of Σ, even if outside an observer’s conscious or direct control, is due to quantum fluctuations, and therefore outside the predictive power of our observer. In all three cases, nevertheless, the Σ motion in higher space can have an arbitrary direction (within the constraints imposed by the theory to be described later), and describes a four-
dimensional surface $V_4$. To different possible sequences of an observer’s decisions\(^1\) (or to different sequences of quantum decisions about the outcome of experiments) there correspond different surfaces $V_4$ which intersect different sets of events in higher space, thus bringing about different possible histories. All these various histories actually coexist in the higher space, but only one of them is followed by a given ‘stream of consciousness’\(^2\).

If $V_4$ is a curved 4-surface, then there is present a certain gravitational field on $V_4$. Since $\Sigma$ motion fluctuates quantum mechanically at a certain microscopic scale, so does the corresponding gravitational field.

In the following sections we shall develop a model theory based on the general assumptions outlined here. The approach adopted seems to be free of the conceptual and of some technical difficulties occurring in various previous approaches to quantum gravity. So the difficulty with the non-arbitrariness of Cauchy data, related to the fact that there are more variables in the theory than there are physical degrees of freedom (see Brill and Gowdy 1970), does not occur anymore. Moreover, this approach also resolves the interpretational difficulties and paradoxes of the conventional quantum theory\(^3\) (objective reality, EPR paradox, Schrödinger’s cat, measurement problem, etc). All these difficulties apparently reduce to a single ‘difficulty’, namely the acceptance of a higher dimensional space with a given ‘matter’ distribution, in which there exists a conscious observer whose three-dimensional simultaneity 3-surface moves forward in any direction according to a certain quantum law of motion\(^4\).

2 The postulates of quantum gravity

The theory that we are going to develop is based on the following postulates.

(i) There exists an $N$-dimensional space $V_N$ parametrised by the coordinates $\eta^a$\(^5\).

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\(^1\)Various (in fact, many) possible streams of consciousness are flowing in a certain higher dimensional structure $B$; to a particular stream of consciousness there corresponds a particular detailed motion of $\Sigma$ through the structure $B$. $\Sigma$ motion defines within $B$ a four-dimensional structure which we call the brain. To a certain $\Sigma_B$ within $B$ there corresponds a certain outside or external $\Sigma_{out}$, the correspondence being due to the coupling of the external and the internal world through the sensory organs. This is closely related to Wigner’s point of view that the collapse of the wavefunction occurs in an observer’s consciousness. See also the next three footnotes.

\(^2\)This is true only for particular experiments in which positions of events are measured. In general (e.g. when energy and momentum are measured) a surface $V_4$ is not determined at all, as we shall see later, and, a particular stream of consciousness chooses instead among the eigenstates (or ‘universes’ according to the Everett-Wheeler interpretation) in an abstract Hilbert space.

\(^3\)This is implicit in the fact that the quantum theory of a spacetime surface in an embedding space is just one particular representation of the Everett (1957)-Wheeler (1957,1973) interpretation of quantum mechanics (see also DeWitt 1967b) in which the above mentioned interpretational paradoxes do not exist.

\(^4\)In our approach we actually combine Wignerian dualism (Wigner 1967) with Everett-Wheeler-DeWitt plurimundialism. Further details will exceed the scope of this paper and will be given elsewhere. See also Pavšič (1981a, b).
(a = 1, 2, 3, ..., N). In this space there exist static ‘material’ or physical events described by the matter density $\omega(\eta)$.

(ii) There exists a three-dimensional surface $\Sigma$ moving in the space $V_N$. For an observer, associated with a particular $\Sigma$ motion, of all material events in $V_N$ only those that lie on successive $\Sigma$ are observable.

(iii) A simultaneity surface $\Sigma$ moves according to quantum laws. In the classical limit its motion describes a four-dimensional continuum $V_4$, parametrised by the coordinates $x^\mu (\mu = 0, 1, 2, 3)$ with the metric $g_{\mu\nu}$ and the matter density $\omega(x)) \equiv \rho(x)$ satisfying the Einstein equations.

As a working hypothesis we shall assume that the dimension $N$ of the embedding space $V_N$ is 10; namely, according to the general theorem (Fronsdal 1959, Eisenhart,1926) every $n$-dimensional Riemannian space can be embedded locally in a $N$-dimensional pseudo-Euclidean space $M_N$ with $N = n(n + 1)/2$, so that for $n = 4$ it is $N = 10$. As stated already by Fronsdal (1959), all spacetimes which have been tested experimentally so far, like the Schwarzschild solution, the Friedmann cosmological solution, etc, can be embedded in $M_6$. So it seems reasonable to assume that a ten-dimensional embedding space will suffice.

We shall not consider the complications which result from Clarke’s (1970) work which deals with global embedding of a generic spacetime (which moreover is not necessarily a solution of Einstein’s equations). In our approach we do not worry about an embedding of a given spacetime; in other words, we do not start from the intrinsic geometry of a $V_4$ and then search for its embedding, but on the contrary, we start from the embedding space $M_N$—with a given dimension (say 10)—in which there exists a 4-surface $V_4$. The latter satisfies a certain variational principle with respect to $M_N$. Moreover, we consider $M_N$ as a physical space and not merely as an auxiliary space; all events (with the coordinates $\eta^a$) of $M_N$ are physical, though classically an observer is directly aware only of those events which belong to a certain spacetime 4-surface $V_4$. A given 4-surface $V_4$ is chosen by initial conditions and by the equation of motion for $\eta^a(x)$.

In the following we shall first write the classical equations of motion of the surface $\Sigma$; in other words, we shall write the equation of a four-dimensional surface $V_4$, embedded in $V_N$. In principle the metric and curvature of $V_N$ can be arbitrary, but we shall take it as a flat one. The metric on $V_4$ so obtained will satisfy the Einstein equations.

Next we shall quantise the motion of the 3-surface $\Sigma$. One possibility would be to extend the procedure of the string quantisation (Bohr and Nielsen 1983, Kato and Ogawa 1983): instead of a one-dimensional string in four-dimensional spacetime we now have a three-dimensional surface in an $N$-dimensional space $V_N$. Instead of using this approach, we shall rather follow a different though probably equivalent procedure by formulating the theory in the Schrödinger picture.

\footnote{We can hardly consider the Penrose plane wave spacetime (Penrose 1965) as a physical one, just because no spacelike hypersurface exists for the global specification of Cauchy data.}
3 Equation of a classical \( \Sigma \) motion: equation of a spacetime surface embedded in a higher dimensional space

We shall derive the equations of a surface \( V_4 \) from the following action:

\[
W = \int \mathcal{L} \, d^4x
\]  

(3.1)

with the Lagrangian density

\[
\mathcal{L} = \sqrt{-g} \left( \frac{R}{8\pi} + L_m \right)
\]

(3.2)

where \( R \) is the curvature scalar of \( V_4 \) and \( L_m \) is the matter Lagrangian (to be specified latter).

If the surface \( V_4 \) embedded in a pseudo-Euclidean space \( M_N \) is described by the parametric equation

\[
\eta^a = \eta^a(x) \quad (a = 1, 2, 3, ..., N)
\]

(3.3)

then the metric tensor on \( V_4 \) is given by

\[
g_{\mu\nu} = \partial_\mu \eta^a \partial_\nu \eta_a \quad (\mu, \nu = 0, 1, 2, 3).
\]

(3.5)

The Riemann tensor takes the form

\[
R_{\mu\alpha\nu\beta} = D_\mu D_\nu \eta^a D_\alpha D_\beta \eta_a - D_\alpha D_\beta \eta^a D_\mu D_\nu \eta_a
\]

(3.5)

where \( D_\mu \) means the covariant derivative, so that (Eisenhart 1926)

\[
D_\mu D_\nu \eta_a = b_{\mu a} \partial_\mu \partial_\nu \eta_a \quad b_{\mu a} = \delta_{\mu a} - \partial_\rho \eta_a \partial_\rho \eta_N
\]

(3.6)

and where \( \delta_{\mu a} \) is the diagonal pseudo-Euclidean metric tensor of the flat space \( V_N \).

The curvature scalar is then

\[
R = D^\mu D^\nu \eta^a D_\mu D_\nu \eta_a - D^\mu D_\mu \eta^a D^\nu D_\nu \eta_a.
\]

(3.7)

We see that the Lagrangian density \( \mathcal{L} \) given by (3.2) depends on the variables \( \eta^a \), its first derivatives \( \partial_\mu \eta^a \) and second derivatives \( \partial_\mu \partial_\nu \eta^a \):

\[
\mathcal{L} = \mathcal{L}(\eta^a, \partial_\mu \eta^a, \partial_\mu \partial_\nu \eta^a).
\]

(3.2a)

The variation principle \( \delta W = 0 \) gives the following equation for \( \eta^a(x) \):

\[
E_a \equiv \frac{\partial \mathcal{L}}{\partial \eta^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \eta^a} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \eta^a} = 0.
\]

(3.8)
In deriving equation (3.8) it was assumed that the surface $V_4$ is bounded by a certain three-dimensional surface on which the variations $\delta \eta^a$ and $\delta \partial_\mu \eta^a$ are fixed and set to zero.

For the Lagrangian density (3.2) equations (3.8) assume the explicit form

$$\partial_\mu \left[ \sqrt{-g} \left( \frac{G^{\mu\nu}}{8\pi} + T^{\mu\nu} \right) \partial_\nu \eta_a \right] = 0 \tag{3.9}$$

where $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ and $T^{\mu\nu} = (1/\sqrt{-g}) \left( \partial L_m/\partial g^{\mu\nu} - \partial_\alpha \partial_\beta g^{\mu\nu}/\partial \partial_\alpha g_{\mu\nu} \right)$.

In order to further explore equation (3.9), let us introduce new variables $C^\mu_a$ and replace (3.9) by the system of equations

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} C^\mu_a) = 0 \tag{3.10a}$$

$$\left( \frac{G^{\mu\nu}}{8\pi} + T^{\mu\nu} \right) \partial_\nu \eta_a = C^\mu_a. \tag{3.10b}$$

Multiplying (3.10b) by $\partial^\alpha \eta_a$ (and sum over $a$) one obtains

$$G^{\mu\nu} = -8\pi G(T^{\mu\nu} - C^{\mu\nu}) \tag{3.11}$$

with

$$C^{\mu\nu} \equiv C^\mu_a \partial^\nu \eta_a. \tag{3.12}$$

One can easily prove the relations

$$C^\nu_{\ a;\nu} = C^{\mu\nu} D_\mu D_\nu \eta_a + C^{\mu\nu}_{\ ;\nu} \partial_\mu \eta_a$$

$$C^{\mu\nu}_{\ ;\nu} = C^\mu_a \partial^\nu \eta_a$$

From (3.10a) it then follows $C^{\mu\nu}_{\ ;\nu} = 0$. Since $G^{\mu\nu}_{\ ;\nu} = 0$ identically, we have also $T^{\mu\nu}_{\ ;\nu} = 0$. Equations (3.11) are the Einstein equations, apart from the term $C^{\mu\nu}$ which can be included in the redefinition of the stress-energy tensor $T^{\mu\nu}$.

By the way, since $G^{\mu\nu}$ and $T^{\mu\nu}$ are symmetric, it follows also that $C^{\mu\nu}$ is symmetric.

We see that the Lagrangian (3.2) which gives the Einstein equations when considered as a function of the metric $g_{\mu\nu}$ and the derivatives $\partial_\alpha g_{\mu\nu}$ gives essentially the same Einstein equations (apart from $C^{\mu\nu}$) also if it is considered as a function of the embedding coordinates $\eta^a$ and the derivatives $\partial_\alpha \eta^a$, $\partial_\mu \partial_\nu \eta^a$.

Let us now further investigate the properties of equation (3.9). By using the well known relation

$$\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} A^{\mu\nu}) + \Gamma^n_{\mu\nu} A^{\mu\nu} = A^{\mu\nu}_{\ ;\nu}$$

for a generic tensor $A^{\mu\nu}$ and the expression $\Gamma^n_{\alpha\beta} = \partial^n \eta^a \partial_\alpha \partial_\beta \eta_a$ for the affinity (Eisenhart 1926), we can write (3.9) in the equivalent forms

$$\left( \frac{G^{\mu\nu}}{8\pi} + T^{\mu\nu} \right) D_\mu D_\nu \eta_a = 0 \tag{3.9a}$$
\[
\frac{G^{\mu\nu}}{8\pi} D_\mu D_\nu \eta_a + \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^{\mu\nu} \partial_\nu \eta_a) = 0. \tag{3.9b}
\]

Since our Lagrangian (3.2) is invariant with respect to reparametrisation of the coordinates \( x^\mu \) on the subspace \( V_4 \) we expect four constraints satisfied by the equation of motion; indeed using the identification (3.8) we have as a consequence of (3.9a) and (3.6):

\[
E_a \partial^a \eta^a = 0. \tag{3.13}
\]

This identity is analogous to the well known identity \( u^\mu du_\mu / ds = 0 \) which holds for a free particle’s worldline.

Let us assume for the moment that \( \mathcal{L}_m \equiv \sqrt{-g} L_m = \sqrt{-g} \omega (1 - g_{\mu\nu} u^\mu u^\nu) \), where \( \omega (\eta) \) is an arbitrary function of position in \( M_N \) (see later) and \( u^\nu \) is a certain unit 4-vector field on \( V_4 \). Then from (3.8) it follows that \( T^{\mu\nu} = \omega (\eta(x)) u^\mu u^\nu \) which can be identified with the dust stress-energy tensor, since we will consider \( \omega (\eta(x)) \equiv \rho(x) \) as the mass density in \( V_4 \), and \( u^\mu \) as the 4-velocity field. Then the second term in (3.9b) assumes the form

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\rho u^\mu u^\nu \partial_\nu \eta_a \sqrt{-g}) = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \rho u^\mu U_a). \tag{3.14a}
\]

Since (3.9b) implies \( \partial^a \eta_a \partial_\mu (\omega U^\mu) = 0 \) which is identical to the conservation of rest mass \( (1/\sqrt{-g}) \partial_\mu (\sqrt{-g} \rho u^\mu) = 0 \), it can be written as

\[
\frac{G^{\mu\nu}}{8\pi} D_\mu D_\nu \eta_a + \omega \, dU_a / ds = 0 \tag{3.14b}
\]

where \( U_a = \partial^a \eta_a u_\nu \) is the velocity \( U_a = d\eta_a / ds \) with respect to \( M_N \) and \( dU_a / ds = u^\mu \partial_\mu (u^\nu \partial_\nu \eta_a) \). By the way, \( u^\mu = dx^\mu / ds = U_a \partial_a \eta_a \) and \( ds^2 = dx^\mu dx_\mu = d\eta^a d\eta_a \). Let us multiply (3.14b) by \( \partial^a \eta^a \) and sum over \( a \); one immediately observes that \( D_\mu D_\nu \eta_a \partial^a \eta^a = 0 \) so that it remains

\[
u^\mu \partial_\mu (u^\nu \partial_\nu \eta_a) \partial^a \eta^a = \frac{dU_a}{ds} \partial^a \eta^a = \frac{du^a}{ds} + \partial^a \eta^a \partial_\mu \partial_\nu \eta_a u^\mu u^\nu = 0
\]

which is the geodesic equation.

We have seen that from our second-order Lagrangian \( \mathcal{L} = \mathcal{L}(\eta^a, \partial_\mu \eta^a, \partial_\mu \partial_\nu \eta^a) \) we obtain equation (3.9a) which is not a fourth-order equation (as expected) but merely a second-order equation. This is in agreement with the result obtained by Rund (1971) who extensively studied variational problems on subspaces of a Riemannian manifold. The occurrence of a second-order equation of motion from a second-order Lagrangian indicates that our special Lagrangian entails some kind of degeneracy (Rund 1971) which results in the non-uniqueness of the solution \( \eta^a(x) \) to the preceding variational procedure. In order to fix a solution \( \eta^a(x) \) we need some additional equation. There are certainly various possible ways of completing the equations (3.9) or (3.9a). Here I shall tentatively adopt the following procedure.
First, let us observe that our Lagrangian (3.2) for dust, \( L = \sqrt{-g}(R/8\pi + \omega(1 - g_{\mu\nu}u^\mu u^\nu)) \), in the case when the Einstein equations are satisfied, becomes \( L = \sqrt{-g} \omega \). Let us denote the latter Lagrangian by \( L_1 = L_1(\eta^a, \partial_\mu \eta^a) \) and the former one by \( L_2 = L_2(\eta^a, \partial_\mu \eta^a, \partial_\mu \partial_\nu \eta^a) \). From the Lagrangians \( L_1 \) and \( L_2 \) we obtain the following system of equations:

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \omega \eta^a) = \frac{\partial \omega}{\partial \eta^a}, \tag{3.14}
\]

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \omega u^\mu U_a) = -\frac{1}{8\pi} G^\mu\nu D_\mu D_\nu \eta_a. \tag{3.15}
\]

The quantity \( \omega(\eta) \) is the matter density in the higher space and is a given function of \( \eta^a \). The choice of \( \omega(\eta^a) \) depends on the model of the universe that we adopt and the scale which we are interested in. For given initial and boundary values of \( \eta^a \) and for a chosen parametrisation \( x^\mu \) we can calculate \( \eta^a(x) \) from (3.14). Once \( \eta^a(x) \) is known, we know also \( g_{\mu\nu} \) and \( G_{\mu\nu} \). Therefore in (3.15) only the velocity \( U_a(x) \) of a point on \( V_4 \) is unknown (remember that \( u_\mu = \partial_\mu \eta^a U_a \)) and can be calculated from the equation. In other words, though \( V_4 \) is known from (3.14) and is represented by the equation \( \eta^a = \eta^a(x) \), we still do not know the direction (i.e. \( U^a \) or \( u_\mu = \partial_\mu \eta^a U_a \)) into which the mass flows, unless we solve (3.15). We have also seen that (3.15) automatically implies the conservation of rest mass \( D_\mu (\rho u^\mu) = 0 \) and the validity of the Einstein equations (3.11) (with addition of the term \( C_{\mu\nu} \)) together with \( D_\nu (\rho u^\mu u^\nu) = 0 \) and \( D_\nu C_{\mu\nu} = 0 \).

However, it may happen that \( C_{\mu\nu} = 0 \). In another paper (Pavšič: 1985) we have an example of \( \omega(\eta), \eta^a(x) \) and \( U^a \) which solve our system (3.14) and (3.15).

From the preceding we can conclude that our starting Lagrangian for the ‘field’ \( \eta^a(x) \) can be taken to be \( L = \sqrt{-g} \omega(\eta) \) with the corresponding equation of motion \( D_\mu (\omega \partial^\mu \eta_a) = \partial \omega / \partial \eta_a \) (equation (3.14)). The ‘true’ dynamical variables in our theory are the coordinates \( \eta^a(x) \) of the spacetime surface \( V_4 \). The metric tensor components \( g_{\mu\nu} \) are not ‘true’ dynamical variables. In the usual approaches to quantum gravity they caused troubles, since \( g_{\mu\nu} \) are not all independent but are related through \( G_{\mu\nu} = 0 \). Moreover, one cannot arbitrarily specify Cauchy data, namely \( g_{ij}(x) \) and \( \pi^{kl}(x) \), since they must satisfy the Einstein equations \( G^{0\nu} = 0 \). On the other hand, \( \eta^a(x) \) and \( \pi^a(x) \) can be specified at will on a given 3-surface \( \Sigma \). Though the variables \( \eta^a(x) \) and \( \pi^a(x) \) are not all independent, but obey the four identities (3.13), this does not cause any trouble in setting the theory into a canonical form. Thus, this is analogous to the situation which occurs in special relativity, where a worldline is described by \( x^\mu = x^\mu(\lambda) \), \( \lambda \) being an arbitrary parameter. In the following we shall make the canonical formulation of the theory of a spacetime sheet satisfying this specific first-order Lagrangian.
4 Generator for infinitesimal transformations, momentum and stress-energy tensor of the field

We may consider \( \eta_a(x) \) as a field entering the Lagrangian density \( \mathcal{L} \). Let us confine us to a first-order Lagrangian, say

\[ \mathcal{L} = \sqrt{-g} \omega(\eta) \]  

(4.1)

and let \( W = \int \mathcal{L} \, d^4x \) be the action. The variation of \( W \) is

\[ \delta W = \int \left( \frac{\partial \mathcal{L}}{\partial \eta^a} \delta \eta^a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \eta^a} \delta \partial_\mu \eta^a \right) \, d^4x. \]  

(4.2)

This can be rearranged so that after taking into account the equations of motion \( \partial \mathcal{L} / \partial \eta^a - \partial_\mu \partial \mathcal{L} / \partial \partial_\mu \eta^a = 0 \) we obtain

\[ \delta W = \int \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \eta^a} \delta \eta^a \right) \, d^4x. \]  

(4.3)

This represents the variation of the action \( W \) when going from a spacetime surface \( V_4 \) to another spacetime surface \( V_4 + \delta V_4 \) (see figure 1a), where both \( V_4 \) and \( V_4 + \delta V_4 \) are solutions of the field equations, explicitly of equation (3.14).

![Figure 1](a) The variation of the action \( W \) between two successive spacetime slices \( V_4 \) and \( V_4 + \delta V_4 \). (b) A family of spacetime slices and the path \( C \) of \( \Sigma \).

Let us introduce the notation

\[ \pi^\mu_a = \frac{\partial \mathcal{L}}{\partial \partial_\mu \eta^a} \]  

(4.4)

and let us express \( \delta W \) in (4.3) as the integral over a closed three-dimensional surface \( B \) confining the segment \( \Omega \) of \( V_4 \):

\[ \delta W = \oint d\Sigma_\mu \pi^\mu_a \delta \eta^a \]  

(4.5)
where \( d\Sigma_\mu \) is a 3-surface element. Now we assume that the derivatives \( \partial_\mu \eta^a \) and \( \partial \mathcal{L} / \partial \partial_\mu \eta^a \) tend to zero at spatial infinity (Barut and Mullen 1962, 1964). We can then assume that \( \mathbf{B} \) consists of two spacelike 3-surfaces \( \Sigma_1 \) and \( \Sigma_2 \) alone (figure 1a). Equation (4.5) becomes

\[
\delta W = \int_{\Sigma_1}^{\Sigma_2} d\Sigma_\mu \pi^\mu_a \delta \eta^a. \tag{4.6}
\]

The expression

\[
\delta S_1 = \int d\Sigma_\mu \pi^\mu_a \delta \eta^a = G_1(\Sigma) \tag{4.7}
\]

we shall call the generator of variations of \( V_4 \) satisfying the equations of motion.

Let \( \xi_r \ (r = 1, 2, 3) \) be the coordinates which parametrise the 3-surface \( \Sigma \). By writing the 3-surface element \( d\Sigma \) as the product of the three-dimensional (spatial) volume \( d^3\xi \) and the normal \( n^\mu \)

\[
d\Sigma_\mu = d^3\xi n_\mu = d\Sigma n_\mu \tag{4.8}
\]

and by identifying

\[
\pi_a \equiv \pi^\mu_a n_\mu \tag{4.9}
\]

equation (4.7) becomes

\[
\delta S_1 = \int_{\Sigma} d\Sigma \pi_a \delta \eta^a. \tag{4.10}
\]

In the case of our Lagrangian (4.1) we obtain from (4.4) and (4.9)

\[
\pi^\mu_a = \sqrt{-g} \omega \partial^\mu \eta_a \tag{4.11}
\]

\[
\pi_a = \sqrt{-g} \omega \partial^\mu \eta_a n_\mu. \tag{4.12}
\]

Further, we can choose \( \Sigma \) such that the normal \( n_\mu \) is equal to the 4-velocity \( u_\mu \). Then we obtain

\[
\pi_a = \sqrt{-g} \omega U_a \tag{4.13}
\]

where \( U_a = \partial_\mu \eta_a u^\mu \) is the velocity of a point on \( V_4 \) with respect to the embedding space.

So far we have considered the variation of the action \( W \) such that the boundary of integration on a given \( V_4 \) was fixed but the variations on the boundary were different from zero. Now we extend the variation of \( W \) so that the boundary is also subjected to variation. This induces the variation of the coordinates \( x^\mu \) of the boundary into the coordinates \( x^\mu = x^\mu + \delta x^\mu \) of a new boundary. The contribution of this variation is

\[
\delta_B \int \mathcal{L} \, d^4 x = \int_{\Omega - \Omega'} \mathcal{L} \, d^4 x = \oint_{\mathbf{B}} \mathcal{L} \, d\Sigma_\mu \, \delta x^\mu = \int_{\Sigma_1}^{\Sigma_2} \mathcal{L} \, d\Sigma_\mu \, \delta x^\mu \tag{4.14}
\]

\(^6\) Alternatively, we may assume that spacetime \( V_4 \) is spatially closed.
where in the last step we assumed that the boundary surface \( B \) consists of the initial and the final surfaces \( \Sigma_1 \) and \( \Sigma_2 \) alone. The generator of this variation is then

\[
\delta S_2 = \int_{\Sigma} d\Sigma_\mu \mathcal{L} \delta x^\mu = G_2(\Sigma). \tag{4.15}
\]

The total generator is then the sum of (4.6a) and (4.15):

\[
\delta S = \delta S_1 + \delta S_2 \equiv G(\Sigma). \tag{4.16}
\]

This last expression can be written as

\[
\delta S = \int d\Sigma (\pi_\mu \delta \eta^\mu(x) + \mathcal{L} \delta s) \tag{4.17}
\]

where \( d\Sigma_\mu = d^3 \xi n_\mu \), \( \delta x^\mu = n^\mu \delta s(\xi) \), \( n^\mu \) being the normal vector to \( \Sigma \), and \( \xi \equiv \xi^r \) the 3-coordinates on \( \Sigma \). Occasionally we shall call \( s(\xi) = \int \delta s(\xi) \) the proper time function; this name is justified by the fact that if we choose \( \Sigma \) such that the normal \( n^\mu \) coincides with the 4-vector \( u^\mu \), then the vector \( \delta x^\mu(\xi) = u^\mu \delta s(\xi) \) is tangential to a geodesic characterised by the parameters \( \xi^r \) \( (r = 1, 2, 3) \). Here \( \delta \eta^\mu \) is the following variation of \( V_4 \):

\[
\delta \eta^\mu = \eta^\mu(x) - \eta^\mu(0). \quad \tag{4.18}
\]

If \( \eta^\mu(x) \) is considered as a field, then this variation is identical to the intrinsic variation of the field.

Now we can consider the total variation

\[
\bar{\delta} \eta^\mu(x) = \eta^\mu(x) - \eta^\mu(x) = \delta \eta^\mu(x) + \partial_\nu \eta^\mu \delta x^\nu. \quad \tag{4.19}
\]

Then the generator can be written as

\[
\delta S = \int d\Sigma_\mu (\pi_\mu \bar{\delta} \eta^\mu(x) - \mathcal{T}_\mu^\nu \delta x^\nu) \tag{4.20}
\]

where

\[
\mathcal{T}_\mu^\nu = \pi_\mu^\nu \partial_\nu \eta^\rho - \mathcal{L} \delta^\mu^\rho \quad \tag{4.21}
\]

is the stress-energy tensor. This formal stress-energy tensor should not be confused with the one in the Einstein equations. If the Lagrangian is given by (3.2) or (4.1) then the stress-energy tensor is explicitly

\[
\mathcal{T}_\mu^\nu = \sqrt{-g} (\omega \partial_\mu \eta_a \partial_\nu \eta^a - \omega g_{\mu \nu}) \quad \tag{4.22}
\]

which is equal to zero. Therefore (4.20), as a consequence of \( \delta \eta^\mu(x)|_\Sigma = \delta \eta^\mu(\xi) \) (see appendix 1) is simply

\[
\delta S = \int d\Sigma_\mu \pi_\mu^\rho \delta \eta^\rho(x) = \int d\Sigma \pi_\rho \delta \eta^\rho(x) = \int d\Sigma \pi_\rho \delta \eta^\rho(\xi). \quad \tag{4.23}
\]
This can be considered as a variation of the so-called phase functional $S[\eta(\xi)]$ defined in appendix 1. The functional derivative of $S$ with respect to $\eta^a(\xi)$ is the canonical momentum

$$\pi_a(\xi) = \frac{\delta S}{\delta \eta^a(\xi)} = \pi_a[\eta(\xi)].$$

From equation (4.21) we obtain the following Hamiltonian functional:

$$H = \int d\Sigma \mu_n \mathcal{F}^{\mu\nu} = \int d\Sigma (\pi_a U^a - \mathcal{L}) \equiv \int d\Sigma \mathcal{H}$$

where $n^\mu \delta s = \delta x^\mu$.

Though the Hamiltonian is zero we can still use its functional dependence on $\pi_a$ and $U^a$ to derive the Hamiltonian equations of motion. Namely, by separating the derivative $\partial_\mu$ into a normal (to the 3-surface element $d\Sigma$) directional derivative $\hat{\partial}_\mu$ and a tangential derivative $\tilde{\partial}_\mu$ as follows (Barut and Mullen 1962, 1964)

$$\partial_\mu = n_\mu \hat{\partial}_\mu + \tilde{\partial}_\mu \hat{\partial}_\mu$$

one can show that a variation of the Hamiltonian is

$$\delta H = \int d\Sigma \left( \hat{\partial}\eta^a \delta \pi_a - \hat{\partial}\pi_a \delta \eta^a(\xi) \right).$$

From (4.26) we obtain the following equations of motion in the Hamiltonian form\(^7\) for the canonical variables $\pi_a(\xi)$ and $\eta^a(\xi)$:

$$\hat{\partial} \pi_a = -\frac{\delta H}{\delta \eta^a(\xi)} = \{\pi_a, H\} \quad \hat{\partial} \eta^a = \frac{\delta H}{\delta \pi_a} = \{\eta^a, H\}$$

where

$$\{u, v\} = \frac{\delta u}{\delta \eta^c} \frac{\delta v}{\delta \pi_c} - \frac{\delta u}{\delta \pi_c} \frac{\delta v}{\delta \eta^c}$$

is the Poisson bracket. These equations (4.27) are equivalent to the field equation (3.14).

Incidentally, we observe that the generator (4.20) can be written as

$$\delta S = \int d\Sigma (\pi_a \hat{\partial}\eta^a(x) - \mathcal{H} \delta s) = \int d\Sigma (\pi_a \delta \eta^a(\xi) - \mathcal{H} \delta s)$$

The partial functional derivative of $S$ with respect to $s$ is

$$\frac{\delta S}{\delta s} = \frac{\delta P S}{\delta s} = -\mathcal{H} = 0.$$  \(4.28\)

\(^7\) It is the 3-surface $\Sigma$ which moves.
This derivative I call partial, because the functional dependence on \( s \) is also included in the total variation \( \delta \eta^a(x) \). The total and the partial functional derivative are related according to

\[
\frac{\delta_T S}{\delta s} = \frac{\delta_F S}{\delta s} + \frac{\delta S}{\delta \eta^a(\xi)} \frac{d\eta^a}{ds} = -\mathcal{H} + \pi_a \dot{\eta}^a = \mathcal{L}
\] (4.29)

which is consistent with the definition of \( \mathcal{H} \) given by (4.24).

Having defined the generator for infinitesimal transformations, momentum and the stress-energy tensor of the field \( \eta^a(x) \) (which is actually a spacetime surface \( V_4 \)), we are already prepared to perform the formal quantisation of the theory.

5 Quantisation of a spacetime surface embedded in a higher dimensional space

We could quantise the motion of a three-dimensional surface \( \Sigma \) in a higher dimensional (more than four) space by using techniques analogous to those used in the quantisation of strings (Polyakov 1981a, b, Fradkin and Tsetlin 1982, Horwitz and Piron 1973, Horwitz and Arshanski 1982, Menski 1976, Aghassi et al 1970). String is a one-dimensional continuum moving in a (e.g.) four-dimensional space thus describing a two-dimensional continuum \( V_2 \). Here we wish to demonstrate the basis of another method which appears suitable to a direct understanding of quantum gravity. We shall generalise our version (Pavšič 1984) of the Stueckelberg (1941a, b, c) proper time method of worldline quantisation, i.e. the quantisation of the motion of a zero-dimensional 'continuum' — a point which describes a one-dimensional continuum, a worldline. Instead of a one-dimensional classical continuum, a worldline, we now have a four-dimensional classical continuum, a spacetime surface.

We shall assume that a quantum state corresponding to a surface \( V_4 \) can be represented by a wave functional \( \psi(\Sigma, s) \equiv \psi[\eta(\xi), s(\xi)] \) where \( \eta^a(\xi) \) is the parametric equation (in \( V_N \)) of the simultaneity surface \( \Sigma \), and \( s = s(\xi) \) is the proper time function defined in §3.

In order to find the equation for a wave functional let us proceed as follows. First, let us replace the Hamiltonian (4.24) by the Hamiltonian operator

\[
\hat{H} = \int d\Sigma(\hat{\pi}_a \hat{U}^a - \hat{\mathcal{L}}) \equiv \int \hat{\mathcal{H}} d\Sigma.
\] (5.1)

In a suitable representation we can set

\[
\hat{\pi}_a = -i \frac{\delta}{\delta \eta^a(\xi)} \quad \hat{U}^a = \gamma^a \quad \hat{\mathcal{L}} = -i \frac{\delta_T}{\delta s} \quad \hat{\mathcal{H}} = i \frac{\delta}{\delta s}
\] (5.2)

where it will turn out (see equation (5.7)) that the velocity operator \( \hat{U}^a \) can be represented by the Dirac matrices satisfying \( (\delta_{ab} \text{ is a pseudo-Euclidean metric of} \)
Then we observe that classically $\delta S/\delta s = H = 0$, and we set the analogous quantum equation

$$i \frac{\delta \psi}{\delta s} = \hat{\mathcal{H}} \psi[\eta(\xi), s(\xi)] = (\hat{\pi} a \gamma^a - \hat{\mathcal{L}}) \psi = 0.$$  \hfill (5.4)

This equation is a generalisation of Dirac’s equation. We also observe that (5.4) is a kind of Tomonoga-Schwinger equation (see Blokintsev 1973) which in turn is a generalisation of the Schrödinger equation. It is now understood that $\psi[\eta(\xi), s(\xi)]$ is a functional multiplied by a suitable spinor.

Equation (5.4) implies that there is no evolution along the proper time function $s(\xi)$:

$$\psi[\eta(\xi), s(\xi)] = \exp \left( -i \int \hat{H} \delta s \right) \psi[\eta(\xi), 0] = \psi[\eta(\xi), 0]$$ \hfill (5.5)

so that matrix elements or expectation values of operators remain constant with $s(\xi)$ (see also DeWitt 1967a, b). However they do change with $\eta^a(\xi)$; and since the totality of $\eta^a(\xi)$ fill all the space $V_N$, it still holds that matrix elements change with ‘time’. Time is now not the proper time $s(\xi)$ but $\Sigma$ (i.e. $\eta^a(\xi)$) (see also Pavšič 1984 and §6).

A particular solution—with definite $\pi_a$ and $\mathcal{L}$—of the wave functional equation (5.4) is

$$\psi_\pi[\eta(\xi)] = \psi_0 \exp \left( i \int \pi_a \delta \eta^a(\xi) d\Sigma \right)$$ \hfill (5.6)

under the condition

$$(\pi_a \gamma^a)^2 = \pi_a \pi^a = \mathcal{L}^2$$ \hfill (5.7)

where $\mathcal{L} \psi_\pi = \mathcal{L} \psi_\pi$. In equations (5.6) and (5.7) $\pi^a(\xi) = \pi^a(\xi)[\eta(\xi)]$ and $\mathcal{L}(\xi) = \mathcal{L}(\xi)[\eta(\xi)]$ are the eigenvalue fields corresponding to the operators (5.2). These (functional) fields are defined over the set $\{\eta^a(\xi) : \eta^a(\xi) \subset V_4, V_4 \in \{V_4\}\}$, i.e. the set of $\eta^a(\xi)$ that belong to the family of $V_4$ defining the particular $\pi_a$.

Each $\eta^a(\xi)$ represents a certain geometry. Wheeler (1967) represented a given 3-geometry by a point in the so-called superspace, let us call it $g$ superspace. Analogously, we can consider a given 3-surface as a point in the corresponding $\Sigma$ superspace which is analogous to the $g$ superspace, but not identical.

A general solution of the wave functional equation (5.5) is a linear superposition of the particular solutions (5.6):

$$\psi = \sum_\pi C_\pi \psi_\pi$$ \hfill (5.8)

where the sum runs over various families $\{V_4\}_1$, $\{V_4\}_2$,...; defining $\pi^a_1$, $\pi^a_1$,.... The momenta are arbitrary, not restricted by a fixed $\mathcal{L}[\eta(\xi)]$.

---

8 In a general case the eigenvalues are not constants but fields (see Pavšič 1982).
The solution (5.8) represents a quantum state with indefinite momentum $\pi^a[\eta(\xi)]$ and indefinite density $\mathcal{L}[\eta(\xi)]$; it is the projection of a state $|a\rangle$ into the state $|\eta(\xi)\rangle$ with definite $\eta^a(\xi)$:

$$\psi[\eta(\xi), s(\xi)] \equiv \langle \eta(\xi) | a[s(\xi)] \rangle \equiv \psi(\Sigma, s). \quad (5.9)$$

Here we are dealing with what I shall name the generalised Schrödinger representation in which a state depends on the proper time function $s(\xi)$. However, because $\hat{H} = 0$, a state is constant at all values of $s(\xi)$. We have called $\psi[\eta(\xi), s(\xi)]$ the wave functional; it is a generalisation of the concept of a wavefunction.

The wave functional has an analogous meaning as a wavefunction: if we measure the 3-surface $\Sigma$ then the probability density that we obtain as a result of measurement the values $\eta^a(\xi)$ is given by:

$$\psi^\dagger[\eta(\xi)]\psi[\eta(\xi)]$$

= probability density of finding surface $\Sigma$ with $\eta^a(\xi)$ within the ‘volume’ element $\mathcal{D}\eta(\xi)$

where $\mathcal{D}\eta(\xi) = \prod_{\xi} d^N\eta(\xi)$ is the volume element of the $\Sigma$ superspace. Normalisation of $\psi[\eta(\xi)]$ is such that

$$\int \psi^\dagger \psi \mathcal{D}\eta = 1 \quad (5.10)$$

where the integration runs over a certain chosen volume of $\Sigma$ superspace. An explicit meaning of the integral (5.10) is given in appendix 2. The expression $\psi^\dagger \psi$ is the expectation value

$$\langle \hat{Q} \rangle \equiv \int \psi^\dagger[\eta'(\xi)]\hat{Q}\psi[\eta'(\xi)] \mathcal{D}\eta'(\xi) \quad (5.11)$$

of the—let it be called—localisation operator

$$\hat{Q} = \delta[\eta(\xi) - \eta'(\xi)] = \prod_{\xi, a} \delta(\eta^a(\xi) - \eta'^a(\xi)) \quad (5.12)$$

where $\delta[\ ]$ stands here for a generalised (or functional) $\delta$ function defined by

$$\int F[y'(x)]\delta[y(x) - y'(x)] \mathcal{D}y'(x) = F[y(x)] \quad (5.13)$$

where $F[y(x)]$ is an arbitrary functional whilst $y(x)$ and $y'(x)$ arbitrary functions.

In general, an observable $\hat{A}$ is a functional operator and its expectation value is given by (5.11) in which $\hat{Q}$ is replaced by $\hat{A}$. These concepts are most easily visualised if we consider the 3-surface $\Sigma$ as a point in the ‘superspace’, as already mentioned.
6 More about the physical interpretation of the formalism

In the previous section we wrote down an expression for the wave functional $\psi[\eta(\xi)] \equiv \langle \eta(\xi)|a\rangle$, where $|a\rangle$ is a state. In general (Wheeler 1962) it is a superposition of wave functionals with definite momenta $\pi^a(\xi)$. Now let us illustrate what we mean by a state with definite momentum. By this we mean a family (figure 2) of spacetime surfaces $V_4$ (or $\eta^a(x)$) each $V_4$ being a solution of the classical field equations (3.14) for a given matter density $\omega(\eta)$. On each surface $V_4$ one can choose a three-dimensional surface $\Sigma$ (with the parametric equation $\eta^a = \eta^a(\xi)$), and the normal $n^\mu$ to $\Sigma$. Let us choose a particular $V_4$ and $\Sigma$. Then we can calculate the quantity $\pi^a(\xi) = \pi^a n^\mu = \omega \sqrt{-g} \partial_\mu \eta^a n^\mu(\eta(\xi))$ that is the momentum density of $\Sigma$ (strictly, the momentum conjugate to $\eta^a(\xi)$). Since it functionally depends on the chosen $\eta^a(\xi)$, we can write $\pi^a(\xi) = \pi^a(\xi)\eta(\xi))$. Here $\pi^a(\xi)$ and $\eta^a(\xi)$ are vectors with the discrete index $a$ and the continuous index $\xi$. If we now vary $\eta^a(\xi)$ over all spacetime surfaces $V_4$ belonging to a given family, we can consider $\pi^a[\eta(\xi)]$ as a (generalised) $N$-vector (functional) field ($N$ dimensions of the higher space) defined over such a set of $\eta^a(\xi)$ (see figure 2\[9\]). For a state with a definite momentum it is the field $\pi^a(\xi)[\eta^a(\xi)]$ which is known to an observer, whilst the surface $\Sigma$ is not known. It could be any $\Sigma$ of the family $\{V_4\}$ defining $\pi^a$. Just the opposite is the situation in which the observer gets as a result of measurement the definite surface $\Sigma$ (and the matter distribution $\rho(x) \equiv \omega(\eta(x))$ on this 3-surface) whilst he has no information about the family of surfaces $V_4$, i.e. about ‘alternative histories’. There are also situations between these two extremes:

\[9\] An analogous definition of momentum field for the case of a family of worldlines in a fixed background metric is given by Pavšič (1982).
they are represented by a certain superposition of the states with definite momentum field \( \pi^a[\eta(\xi)] \) (figure 2).

The examples described above are in fact manifestations of (generalised) Heisenberg uncertainty relations. They result from the following commutation relations:

\[
[\hat{\eta}^a, \hat{\pi}_b] = i\delta^a_b \quad (6.1)
\]
\[
[\hat{s}, \hat{\mathcal{L}}] = i \quad (6.2)
\]

One can easily prove these relations in the ‘coordinate’ representation in which the operators \( \hat{\eta}^a \) are c-number fields \( \eta^a(\xi) \) and \( \hat{\pi}_b \) are the functional derivatives

\[
\hat{\pi}_a = -i \frac{\delta}{\delta \eta^a(\xi)} \quad \hat{\mathcal{L}} = -i \frac{\delta_T}{\delta s(\xi)}
\]

Actually

\[
[\eta^a, -i \frac{\delta}{\delta \eta^b}] \psi = \eta^a \left( -i \frac{\delta\psi}{\delta \eta^b} \right) + i \frac{\delta(\eta^a\psi)}{\delta \eta^b} = i\delta^a_b \psi
\]

and similarly for the relation (6.2). Once verified in one representation they must be true in any representation.

The physical meaning of the commutation relations (6.1) and (6.2) is the following.

(a) Definite \( \pi^a[\eta(\xi)] \); then also \( \mathcal{L}^2 = \pi^a[\eta(\xi)]\pi_a[\eta(\xi)] \) and the family of spacetime surfaces \( \eta^a(x) \) are definite. On the other hand, the individual 3-surface \( \eta^a(\xi) \) and the proper time field \( s(\xi) \) are indefinite.

(b) Definite individual \( \eta^a(\xi) \); then \( \pi^a[\eta(\xi)] \) is indefinite—an observer has no information about other spacetime surfaces. On the other hand, the matter distribution field \( \mathcal{L}[\eta(\xi)] \) can be: (i) either definite; then the proper time field \( s(\xi) \) is indefinite, or (ii) indefinite; then \( s(\xi) \) can be definite, as suggested by (6.2).

A 3-surface \( \eta^a(\xi) \) and the proper time field \( s(\xi) \) define a set of various spacetime surfaces \( V_4 \), passing through that particular 3-surface \( \Sigma \), all \( V_4 \) having the same metric \( g_{\mu\nu}(x) \) (apart from a general coordinate transformation).

The commutation relations (6.2) can also be interpreted in the following way: if an observer obtains as a result of measurement a definite metric, then he has no knowledge about the matter distribution field \( \mathcal{L}[\eta(\xi)] \), i.e. he has no information about the matter distribution on an alternative three-dimensional surface \( \Sigma \); therefore he does not know alternative worlds. To state it differently: if an observer measures the proper time \( s(\xi) \) in each point \( \xi \) of his 3-space, then he cannot also measure with an arbitrary precision the matter distribution field \( \mathcal{L}(\xi) = \mathcal{L}(\xi)[\eta(\xi)] = \rho[\eta(\xi)]\sqrt{-g} \).

In a normal, awake, state an observer is continuously measuring (at least with his sense organs) the proper time; he has no idea of ‘other worlds’. Besides this situation, our theory also predicts a situation in which an observer does not measure the proper time, and as a compensation he can then have some knowledge about the distribution of matter through the higher space, i.e. he can experience in some way
the existence of alternative worlds. Has this last prediction any relation to reality? My conjecture is that it has. Remember that an observer is not always in an awake state of consciousness. He can be under the influence of hypnosis, drugs, etc, or he can simply sleep and dream. When in such a state of consciousness, he is no longer precisely ‘measuring’ the proper time; he is not aware of the usual three-dimensional space or world, but nevertheless he is aware of something: he experiences various hallucinations, dreams, etc. Is this related to some certain extent with the existence of higher dimensional space and alternative worlds? If so, then the fact that these experiences are often not strictly logical or causal can be a consequence of the fact that in such a state the observer is not measuring (at least not precisely enough) his proper time.

7 Transition to the one-particle theory

So far we have been concerned with the classical and quantum motion of a 3-surface \( \Sigma \). Classically, a motion of \( \Sigma \) gives an observer the impression that three-dimensional objects—the sections of higher dimensional objects with \( \Sigma \)—are moving in 3-space. Classical motion of \( \Sigma \) is only a limiting case of a more general, quantum motion, for which it is characteristic that one cannot determine at once both \( \Sigma \) itself and the momentum \( \pi^a = \omega \partial_\mu \eta^a n^\mu \sqrt{-g} \). Since the motion of \( \Sigma \) is due to quantum uncertainty the motion of material 3-objects is also quantum mechanically uncertain.

Let us derive the wavefunction of a 3-space material particle from the wave functional \( \psi(\Sigma) \). The latter is a superposition of the wave functionals with definite momentum \( \pi^a \). Let the matter distribution \( \omega(\eta) \) in higher space be

\[
\omega(\eta) = m \delta^N(\eta - \eta_0(\alpha))
\]  

(7.1) i.e. the matter in higher space is distributed on a given four-dimensional surface \( V_4^* \), \( \eta^a = \eta_0^a(\alpha) \) (\( \alpha \) are the coordinates on \( V_4^* \)), and is zero elsewhere. The state with a definite momentum field \( \pi^a \) can be represented by a family of spacetime surfaces \( V_4 \) (each being a solution of the field equations (3.14) for a given \( \omega(\eta) \)). Each \( V_4 \) intersects with \( V_4^* \) in a one-dimensional continuum—a worldline \( \tilde{C} \). If we project all those worldlines on a certain spacetime slice \( V_4^0 \), we obtain a family of possible worldlines \( P \) in a given spacetime (in our case \( V_4^0 \)).

Without loss of generality we can assume that \( n^\mu = u^\mu \) (choice of \( \Sigma \) on \( V_4^0 \)) and write

\[
\pi_a \delta \eta^a(\xi) = \pi_a d\eta^a = \sqrt{-g} \omega U_a \partial_\mu \eta^a dx^\mu = \sqrt{-g} \omega u_\mu d x^\mu
\]  

(7.2) where \( d\eta^a \) is the projection of \( \delta \eta^a(\xi) \) into \( V_4^0 \). Here \( \delta \eta^a(\xi) \) is taken between those points on \( V_4 \) and \( V_4 + \delta V_4 \) which lie on the corresponding worldlines (figure 3). From (7.2) one obtains

\[
\int \int \pi_a \delta \eta^a(\xi) d\Sigma = \int p_\mu dx^\mu \quad p_\mu = \int \sqrt{-g} \omega u_\mu d\Sigma.
\]  

(7.3)
Figure 3: (a) A family of spacetime slices $V_4$ intersected by a surface $V_4^*$ of non-vanishing matter distribution $\omega(\eta)$. (b) The intersections $V_4 \cap V_4^*$ are projected on a chosen spacetime surface $V_4^0$, thus giving a family of possible worldlines $P$.

Therefore the wave functional (5.6) actually becomes in this special case the wavefunction

$$\psi_p(x) = \psi_0 \exp \left( i \int p_\mu dx^\mu \right).$$

(7.4)

A general wavefunction is a superposition of $\psi_p$, and obeys the wave equation

$$\hat{L}\psi = \gamma^\mu \hat{p}_\mu \psi \quad (\hat{p}_\mu = -i \partial_\mu, \quad \hat{L} = -i d/ds).$$

(7.5)

For a state with definite mass $m$ it is $\hat{L}\psi = m\psi$, and equation (7.5) becomes

$$(\gamma^\mu \hat{p}_\mu - m)\psi = 0$$

(7.6)

which is the well known Dirac equation in a fixed background metric.

These results essentially mean that starting from a quantum theory of a spacetime surface embedded in a higher dimensional space with a given matter distribution $\omega(\eta)$, one obtains in the special case (given by (7.1)) the one-particle quantum theory. For an arbitrary (in general complicated) $\omega(\eta)$ one would obtain many-particle quantum theory. Both quantum gravity (at least the model theory presented here) and quantum theory of matter are thus intimately related.

Note added in 2014: In a curved space $V_4^0$, the spin connection should also occur in eqs. (7.5) and (7.6). This can be brought into the game, if $\psi$ is a Clifford algebra valued (geometric) spinor, expanded according to $\psi = \psi^\alpha \xi_\alpha$, $\alpha = 1, 2, 3, 4$, where $\xi_\alpha$ are basis spinors. The derivative $\partial_\mu$ can be understood in the generalised sense, such that if acting on a scalar component it gives the partial derivative $\partial_\mu \psi^\alpha$, and if acting on a basis spinor it gives $\partial_\mu \xi_\alpha = \Gamma^\beta_{\mu \alpha} \xi_\beta$, where $\Gamma^\beta_{\mu \alpha}$ is the spin connection. Then $\partial_\mu \psi = \partial_\mu (\psi^\alpha \xi_\alpha) = \partial_\mu \psi^\alpha \xi_\alpha + \psi^\alpha \partial_\mu \xi_\alpha = (\partial_\mu \psi^\alpha + \Gamma^\alpha_{\mu \beta} \psi^\beta) \xi_\alpha \equiv (D_\mu \psi^\alpha) \xi_\alpha$ (see M. Pavšič 2006 Int. J. Mod. Phys. A 21 5905–56).

Here we neglect other forces, like the electromagnetic, strong, etc. The simultaneity surface $\Sigma$
8 Conclusion

We have developed what appears to be a consistent formalism of a model theory for quantum gravity, based on the quantisation of a spacetime surface embedded in a higher dimensional space \( V_N \); namely, we consider the space \( \mathcal{S} = \{\Sigma\} \) of the simultaneity 3-surfaces \( \Sigma \subset V_N \), and suitably fix its measure. Each \( \Sigma \) can be considered as a point in the superspace \( \mathcal{S} \). This superspace then represents the basis for the Hilbert space of functions \( f(\Sigma) \). A quantum state is represented by a vector in this Hilbert space. In the classical approximation an observer experiences that \( \Sigma \) proceeds forward in his proper time and describes a four-dimensional spacetime continuum \( V_4 \).

In our theory various spacetime surfaces coexist in \( V_N \) and they represent different classical histories of events. Because of quantum effects there is a certain interference between different histories, as described in §6. Roughly speaking, one has the uncertainty principle between metric (or proper time) and matter distribution in the higher space (or alternative histories). So an observer may face a state in which he has complete knowledge about the metric, proper time and matter distribution in his simultaneity 3-surface, but absolutely no knowledge about alternative histories or spacetimes and the matter distribution on them. This is the state we are all familiar with. On the other hand, the uncertainty principle of §6 implies that an observer can experience a state in which he has no knowledge (or not precise enough knowledge) of the proper time and matter distribution on a certain simultaneity 3-surface, but instead he has some knowledge about alternative histories situated on different spacetime sheets. In particular, in the usual quantum mechanical experiments, an observer, who is not continuously measuring a particle’s position, obtains certain knowledge about the particle’s alternative positions (or histories); namely, the very existence of an interference pattern in the double slit experiment is then a manifestation of alternative possible positions of the particle, formally represented by its wavefunction. These alternative positions lie on different spacetime slices and belong to the corresponding different (classical) histories; the usual quantum mechanical situations (e.g. one-particle motion) are limiting cases of our generalised quantum theory of a spacetime sheet, such that the metric tensor is not fluctuating but remains the same on all spacetime slices (see §7).

In our theory we have a very interesting link between determinism and indeterminism. Deterministic or given is the matter distribution in higher space; this higher dimensional world is fixed and timeless—this is the physical reality. On the other hand, the world as perceived by an observer is indeterministic: the path of an observer’s three-dimensional ‘now’ is unpredictable and obeys the quantum laws as described in §§5–7. A certain path is only one of many possible paths through the higher space. The world as perceived by an observer on a certain spacetime sheet is three-dimensional only as a working hypothesis, in order to reproduce the usual, four-dimensional Einstein gravity. If instead, we add some more dimensions to \( \Sigma \), we could reproduce the higher dimensional gravity which, according to Kaluza (1921) and Klein (1926, 1928) and the modern elaborations (Luciani 1978), includes other forces.
is not the only possible world. There are other worlds (and other observers)—all belonging to the higher dimensional world—which are not directly perceived or measured by an observer. Their existence manifests themselves to an observer through quantum phenomena.

The well known quantum phenomena are just a subset of phenomena belonging to the proposed theory of spacetime $V_4$. We have not fully explored the experimental consequences and predictions of the proposed theory but only set its conceptual and formal foundations, trying to demonstrate its self-consistency and usefulness in unifying various branches of physics. What we present here should be regarded for the moment only as a model theory for quantum gravity. We hope that physically relevant solutions could be found non-perturbatively, thus avoiding the problem of renormalisability. We suggest future work in two directions: (i) to evaluate the experimental consequences of the theory regarding the gravity itself and eventually find a more realistic Lagrangian and (ii) to generalise the theory to more than the four-dimensional spacetime surface, thus bringing into play—via the Kaluza-Klein mechanism—other interactions besides gravity.

Appendix 1. Definition of the phase functional

In § 4 we have defined the generator of infinitesimal variations of $V_4$ 

$$
\delta S = \delta S_1 + \delta S_2 = \int (\pi_a \delta \eta^a(x) + \mathcal{L} \delta s) d\Sigma = \int \pi_a \tilde{\delta} \eta^a(x) d\Sigma
$$

(A1.1)

where $\tilde{\delta} \eta^a = \eta^a(x') - \eta^a(x)$, $\delta \eta^a(x) = \eta^a(x) - \eta^a(x)$ and $\mathcal{H} = 0$. We shall define the phase functional by the integral

$$
S(V_4, s) = S_1 + S_2 = \int_{C, \Sigma} \pi_a \tilde{\delta} \eta^a(x) d\Sigma.
$$

(A1.2)

Here $V_4$ is a spacetime surface also denoted by $V_4(\alpha) = \eta^a(x, \alpha)$, where $\alpha$ is a set of parameters (two parameters) which, when fixed, determine a particular $V_4$; the latter is a solution of the second-order equation (3.14). Now we let $\alpha$ vary. Thus we obtain a family $\mathcal{F} = \{V_4(\alpha)\}$ (figure 1b) and the set

$$
\mathcal{F} = \{\eta^a(\xi)\}_{\mathcal{F}} = \{\eta^a(\xi) : \eta^a(\xi) \subset V_4(\alpha), V_4(\alpha) \in \mathcal{F}\}.
$$

(A1.3)

Next we choose a path $C = \{\eta^a(\xi)\}_C$ (used in (A1.2)) which is a subset of $\mathcal{F}$ (figure 1b), consisting of $\eta^a(\xi)$ such that each $\eta^a(\xi) \in C$ belongs to different $V_4(\alpha) \in \mathcal{F}$. The integral (A1.2) is the limit

$$
S = \lim_{h \to 0} \left( \int_{\Sigma} \pi_a \tilde{\delta} \eta^a(x) d\Sigma |_{V_4(\alpha)} + \int_{\Sigma} \pi_a \delta \eta^a(x) d\Sigma |_{V_4(\alpha + h)} + \ldots \right)
$$

(A1.4)

where $\tilde{\delta} \eta^a(x)$ is chosen along the path $C$ crossing various $V_4(\alpha)$ (figure 1b). The definition of $\tilde{\delta} \eta^a(x)$ is already given in (A1.1) and § 4, but it is instructive to clarify it as follows.
some other non-equivalent basis, say Q represent a basis in a function space—which we call Σ superspace translated 3-surfaces (determined by the choice of α). Let P be a certain orthonormal set of functions P. We are interested in the set of 3-surfaces Σ momenta π. The phase functional (A1.2) is the sum of the term S due to a normal and the term S' due to a tangent variation.

The phase functional can also be understood as the limiting case—for continuum ξi—of the equation

\[ S(\eta_A(\xi_i), \eta_B(\xi_i)) = \sum_k \int_A^B \pi_a d\eta^a(\xi_k) \Delta \Sigma^k \]  

(A1.7)

where A denotes an initial 3-surface and B a final one. Before the limiting procedure is performed, η(ξ) is approximated by a discrete set of points ηa(ξi) ≡ (ηa(ξ1), ηa(ξ2),...) which can be considered as a vector in a finite (or infinite but countable) dimensional space \( \mathcal{S}_{nF} \).

From (A1.7) it is obvious that \( \pi_a(\eta(\xi_i)) = \partial S/\partial \eta^a(\xi_i)(\Delta \Sigma_i)^{-1} \). Therefore S(ηA(ξi), ηB(ξi)) is independent of the path between the points A and B in \( \mathcal{S}_{nF} \), and—for a fixed ηA—it is a unique function S(η(ξi)) (we omit the subscript B).

In the limit of continuum ξi we obtain the unique phase functional S[η(ξ)] . The momentum is the functional derivative \( \pi_a[\eta(\xi)] = \delta S/\delta \eta^a(\xi) \). For different families F = \{V(α)\} we obtain different S_F[η(ξ)], different spaces \( \mathcal{S}_F = \{\eta(\xi)\}_F \) and different momenta \( \pi_a[\eta(\xi)]_F \).

Appendix 2. Fixation of the measure in Σ superspace

We are interested in the set of 3-surfaces Σ ≡ ηa(ξ) which can be expanded over a certain orthonormal set of functions \( P^a_n(\xi) \):

\[ \eta^a(\xi) = \sum_{n=0}^{\infty} \alpha_n^a P^a_n(\xi) \text{ (no sum over a).} \]  

(A2.1)

Let \( P^a_0(\xi) = 1 \). We do not fix the ‘end points’ so that (A2.1) also includes the translated 3-surfaces (determined by the choice of \( \alpha^a_0 \)). \( P^a_n(\xi) \) for \( n = 1, 2, ..., \infty \) represent a basis in a function space—which we call Σ superspace \( \mathcal{S}_P \). A choice of some other non-equivalent basis, say \( Q^a_n \), would span a different superspace \( \mathcal{S}_Q \).
A variation of a 3-surface belonging to \( \mathcal{S}_P \) is

\[
\delta \eta^a(\xi) = \sum_{n=0}^{\infty} \frac{\partial \eta^a}{\partial \alpha_n} d\alpha_n^a = \sum_{n=0}^{\infty} P_n^a d\alpha_n^a \quad (\text{no sum over } a). \tag{A2.2}
\]

The measure \( \mathcal{D} \eta(\xi) \) in the space of \( \eta^a(\xi) \in \mathcal{S}_P \) is defined as

\[
\mathcal{D} \eta(\xi) = \prod_{\xi,a} d\eta^a(\xi_i) = \frac{\partial(\eta^a)}{\partial(\alpha)} \prod_{n=0}^{\infty} \prod_{\alpha=1}^{N} d\alpha_n^a \tag{A2.3}
\]

where \( \frac{\partial(\eta^a)}{\partial(\alpha)} = \det P_n^a(\xi_i) \) is the Jacobian of the transformation (A2.1) from the variables \( \eta^a(\xi_i) \) variables \( \alpha_n^a \).

For the expectation value of a generic operator \( \hat{A} \) we have

\[
\langle \hat{A} \rangle = \int \psi^\dagger \eta(\xi) \hat{A} \psi \eta(\xi) \mathcal{D} \eta
\]

\[
= \int \psi^\dagger(\alpha_1, \alpha_2, \ldots) \hat{A} \psi(\alpha_1, \alpha_2, \ldots) \frac{\partial(\eta^a)}{\partial(\alpha)} \prod_{n=0}^{\infty} \prod_{\alpha=1}^{N} d\alpha_n^a. \tag{A2.4}
\]

As an example let us calculate the probability \( w \) of finding as a result of measurement a certain \( \eta^a(\xi) \) represented by (A2.1) within a given volume of the superspace \( \mathcal{S}_P \). Let the wave functional be a superposition of \( \psi_\pi \) (equation (5.6)):

\[
\psi[\eta(\xi)] = \sum_k c_k N \exp \left( i \int \pi^{(k)}_a \delta \eta^a d\Sigma \right)
\]

\[
= \sum_k c_k N \exp \left( i \sum_n \mathcal{P}^{(k)}(\alpha_n^a - \alpha_n^a(0)) \right) \tag{A2.5}
\]

where \( \mathcal{P}^{(k)} \equiv \int \pi_a P_n(\xi) d\Sigma \) (no sum over \( a \)). Then using (A2.3), identifying \( \bar{N}^\dagger \bar{N} \equiv N^\dagger N \partial(\eta)/\partial(\alpha) \) and setting \( \alpha_n^a(0) = 0 \) it is

\[
w = \int \psi^\dagger \psi \mathcal{D} \eta = \int \sum_{k,k'} c^*_k c_{k'} \prod_n \bar{N}^\dagger \bar{N} \exp \left( i \alpha_n^a \left( \mathcal{P}^{(k)}(\alpha_n^a - \mathcal{P}^{(k')}_{na}) \right) \right) d^N \alpha
\]

\[
= \mu \sum_{k,k'} c^*_k c_{k'} \prod_n \delta^N \left( \mathcal{P}^{(k)}_{na} - \mathcal{P}^{(k')}_{na} \right) = \sum_{kk'} c^*_k c_{k'} \delta_{kk'} = \sum_k c^*_k c_k. \tag{A2.6}
\]

where \( \mu \) is a suitable constant which absorbs various normalisation factors.

The question is what determines the superspace \( \mathcal{S}_P \). The latter is a subspace of the space of all possible configurations \( \eta^a(\xi) \). We argue that the choice of the basis \( P \) and hence of \( \mathcal{S}_P \) is implicit in the construction of the measuring apparatus. By the latter we understand a structure in \( V_N \) consisting of the chain: sensor-brain, where
sensor is a chain: artificial sensor-instrument-sense organ. This is in a sense analogous to the usual quantum mechanics where a set of eigenvalues is determined by a kind of measuring device (not necessarily identical with the measuring apparatus defined above). In other words, as a measuring device determines which set of eigenvalues we shall measure, so the measuring apparatus, by fixing $P_n^a(\xi)$, determines which set \{\eta(\xi)\}_P \equiv \mathcal{F}_P$ is disposable.

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