Probabilistic Backpropagation for Scalable Learning of Bayesian Neural Networks

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Abstract

Large multilayer neural networks trained with backpropagation have recently achieved state-of-the-art results in a wide range of problems. However, using backprop for neural net learning still has some disadvantages, e.g., having to tune a large number of hyperparameters to the data, lack of calibrated probabilistic predictions, and a tendency to overfit the training data. In principle, the Bayesian approach to learning neural networks does not have these problems. However, existing Bayesian techniques lack scalability to large dataset and network sizes. In this work we present a novel scalable method for learning Bayesian neural networks, called probabilistic backpropagation (PBP). Similar to classical backpropagation, PBP works by computing a forward propagation of probabilities through the network and then doing a backward computation of gradients. A series of experiments on ten real-world datasets show that PBP is significantly faster than other techniques, while offering competitive predictive abilities. Our experiments also show that PBP provides accurate estimates of the posterior variance on the network weights.

1. Introduction

Neural networks (NNs) have seen a recent resurgence of interest due to empirical achievements on a wide range of supervised learning problems. In their typical usage, neural networks are highly expressive models that can learn complex function approximations from input/output examples (Hornik et al., 1989). Part of the success of NNs is due to the ability to train them on massive data sets with stochastic optimization (Bottou, 2010) and the backpropagation (BP) algorithm (Rumelhart et al., 1986). These two features, along with faster machines, larger datasets, and innovations such as dropout (Srivastava et al., 2014) and rectified linear units (Nair & Hinton, 2010), have resulted in successes for NNs on tasks such as speech recognition (Hinton et al., 2012; Hannun et al., 2014), computer vision (Krizhevsky et al., 2012; Wu et al., 2015) and natural language processing (Collobert & Weston, 2008; Sutskever et al., 2014).

Despite all these successes, there are still some challenges in learning NNs from data with backpropagation (BP). First, there are many hyperparameters in BP-based stochastic optimization that require specific tuning to the data, e.g., learning rate, momentum, weight decay, etc., each of which may be layer-specific. With large data sets, finding the optimal hyperparameter values can take a large amount of time, even with an efficient procedure such as Bayesian optimization (Snoek et al., 2012). Second, in NNs trained with BP, we can only obtain point estimates of the weights in the network. As a result, these networks make predictions that do not account for uncertainty in the parameters. However, in many cases these weights may be poorly specified and it is desirable to produce uncertainty estimates along with predictions. Finally, it is common practice to use a very large NN to flexibly fit data, and then reign in overfitting using regularization terms, even if a smaller network would be cheaper and easier to train.

A Bayesian approach to neural networks can potentially avoid some of the pitfalls of stochastic optimization (MacKay, 1992b). Bayesian techniques, in principle, can automatically infer hyperparameter values by marginalizing them out of the posterior distribution or by determining them via type II maximum likelihood (empirical Bayes). Furthermore, Bayesian methods naturally account for uncertainty in parameter estimates and can propagate this uncertainty into predictions. Finally, Bayesian approaches are often more robust to overfitting, since they average over parameter values rather than choosing a single point estimate.

Several approaches have been proposed for Bayesian learn-
Probabilistic Backpropagation

In this paper, we describe a novel approach for learning Bayesian neural networks that is fast to fit and does not have the disadvantages of previous approaches. We call this method probabilistic backpropagation (PBP) as it works by propagating probabilities forward through the network to obtain the marginal likelihood, before propagating backward the gradients of the marginal likelihood with respect to the parameters of the posterior approximation. Our experiments on ten benchmark datasets show that PBP is fast, makes accurate predictions and also produces calibrated estimates of the posterior uncertainty in network weights.

2. Probabilistic neural network models

In this section we describe a probabilistic model for data based on a feedforward neural network. Given data \( D = \{ x_n, y_n \}_{n=1}^N \), made up of \( D \)-dimensional feature vectors \( x_n \in \mathbb{R}^D \) and corresponding scalar target variables \( y_n \in \mathbb{R} \), we assume that each \( y_n \) is obtained as \( y_n = f(x_n | W) + \epsilon_n \), where \( f(\cdot | W) \) is the output of a multi-layer neural network with connections between consecutive layers and weights given by \( W \). The evaluations of this NN are corrupted by additive noise variables \( \epsilon_n \) which are Gaussian distributed with zero mean and precision \( \gamma \), i.e., \( \epsilon_n \sim \mathcal{N}(0, \gamma^{-1}) \).

The NN has \( L \) layers, where \( V_l \) is the number of hidden units in layer \( l \), and \( W = \{ W_l \}_{l=1}^L \) is the collection of \( V_l \times (V_{l-1} + 1) \) weight matrices between the fully-connected layers. The \( +1 \) is introduced here to account for the additional per-layer biases. We denote the outputs of the layers by vectors \( \{ z_l \}_{l=0}^L \), where \( z_0 \) are the inputs, \( \{ z_l \}_{l=1} \) are the hidden units and \( z_L \) denotes the output layer which is one-dimensional since the \( y_n \) are scalars. We assume that the activation functions for each layer are rectified linear units (ReLUs) (Nair & Hinton, 2010), i.e., \( a(x) = \max(x, 0) \).

Let \( y \) be an \( N \)-dimensional vector with the targets \( y_n \) and \( X \) be an \( N \times D \) matrix of feature vectors \( x_n \). The likelihood for the network weights \( W \) and the noise precision \( \gamma \), given the data \( D \) is then

\[
p(y|W, X, \gamma) = \prod_{n=1}^N \mathcal{N}(y_n | f(x_n | W), \gamma^{-1}) . \tag{1}
\]

To complete our probabilistic model, we specify a Gaussian prior distribution for each entry in each of the weight matrices in \( W \). In particular,

\[
p(W | \lambda) = \prod_{l=1}^L \prod_{i=1}^{V_l} \prod_{j=1}^{V_l+1} \mathcal{N}(w_{ij,l} | 0, \lambda^{-1}) , \tag{2}
\]

where \( w_{ij,l} \) is the entry in the \( i \)-th row and \( j \)-th column of \( W_l \) and \( \lambda \) is a precision parameter. The hyper-prior for \( \lambda \) is chosen to be a gamma distribution, i.e.,

\[
p(\lambda) = \text{Gamma}(\lambda | \alpha^\lambda, \beta^\lambda) , \tag{3}
\]

with shape \( \alpha^\lambda = 6 \) and inverse scale \( \beta^\lambda = 6 \). The values chosen for \( \alpha^\lambda \) and \( \beta^\lambda \) make (3) equivalent to having observed \( \nu = 3 \) samples from \( \mathcal{N}(0, \lambda^{-1}) \) with empirical variance equal to 1. This low value for \( \nu \) makes (3) non-informative, while at the same time guarantees that the student’s \( t \) distributions that results from the marginalization of (2) with respect to (3) has finite variance. This will be useful for the inference procedure described in Section 3. The prior for the noise precision \( \gamma \) is Gamma:

\[
p(\gamma) = \text{Gamma}(\gamma | \alpha^\gamma, \beta^\gamma) . \tag{4}
\]

We assume that the \( y_n \) have been normalized to have unit variance and, as in (3), we fix \( \alpha^\gamma = 6 \) and \( \beta^\gamma = 6 \).

The posterior distribution for the parameters \( W, \gamma \) and \( \lambda \) can then be obtained by applying Bayes’ rule, that is, by multiplying the likelihood (1) and the priors (2), (3) and (4) and then normalizing:

\[
p(W, \gamma, \lambda | D) = \frac{p(y|W, X, \gamma)p(W | \lambda)p(\lambda)p(\gamma)}{p(y|X)} . \tag{5}
\]

where \( p(y|X) \) is a normalization constant. Given a new input vector \( x_\star \), we can then make predictions for its output \( y_\star \) using the predictive distribution given by

\[
p(y_\star | x_\star, D) = \int p(y_\star | x_\star, W, \gamma)p(W, \gamma, \lambda | D) d\gamma d\lambda dW , \tag{6}
\]

where \( p(y_\star | x_\star, W, \gamma) = \mathcal{N}(y_\star | f(x_\star, \gamma), \gamma) \). However, the exact computation of \( p(W, \gamma, \lambda | D) \) and \( p(y_\star | x_\star) \) is not tractable in most cases. Therefore, in practice we have to resort to approximate inference methods. In the following section we describe a technique for approximate Bayesian inference in NN models that is both fast and also offers excellent predictive performance.
3. Probabilistic backpropagation

Backpropagation (Rumelhart et al., 1986) is by far the most common method for training neural networks. This method operates in two phases to compute a gradient of the loss in terms of the network weights. In the first phase, the input features are propagated forward through the network to compute the function output and thereby the loss associated with the current parameters. In the second phase, the derivatives of the training loss with respect to the weights are propagated back from the output layer towards the input layer. These derivatives are finally used to update the network weights using, e.g., stochastic gradient descent with momentum.

In this section we describe a probabilistic alternative to the backpropagation algorithm, which we call probabilistic backpropagation (PBP). PBP does not use point estimates for the synaptic weights in the network. Instead, it uses a collection of one-dimensional Gaussians, each one approximating the marginal posterior distribution of a different weight. PBP also has two phases equivalent to the ones of BP. In the first phase, the input data is propagated forward through the network. However, since the weights are now random, the activations produced in each layer are also random and result in (intractable) distributions. PBP sequentially approximates each of these distributions with a collection of one-dimensional Gaussians that match their marginal means and variances. At the end of this phase, PBP computes, instead of the prediction error, the logarithm of the marginal probability of the target variable. In this case, the parameters of the new Gaussian beliefs regarding the scalar $w$ are updated according to Bayes’ rule:

$$ q(w) = N(w|m, v) .$$

where $Z$ is the normalization constant of $f(w)N(w|m, v)$. The updated beliefs $s(w)$ usually have a complex form and need to be approximated with a simpler distribution. A common choice is to approximate this posterior with a distribution that has the same form as $q$. In this case, the parameters of the new Gaussian beliefs $q^\text{new}(w)$ that minimize the Kullback-Leibler (KL) divergence between $s$ and $q^\text{new}$ can then be obtained as a function of $m$, $v$ and the gradient of $\log Z$ with respect to these quantities, namely

$$ m^\text{new} = m + v \frac{\partial \log Z}{\partial m}, \quad (8) $$

$$ v^\text{new} = v - v^2 \left[ \left( \frac{\partial \log Z}{\partial m} \right)^2 - 2 \frac{\partial \log Z}{\partial v} \right]. \quad (9) $$

These rules match moments between $q^\text{new}$ and $s$, guaranteeing that these two distributions have the same mean and variance. These are the main update equations used by PBP. The next section provides a detailed description of PBP, presenting it as an assumed density filtering (ADF) method (Opper & Winther, 1998) that uses some of the improvements on ADF given by expectation propagation (Minka, 2001).

3.1. PBP as an assumed density filtering method

Probabilistic backpropagation is an inference method that approximates the exact posterior of a neural network (5) with a factored distribution given by

$$ q(W, \gamma, \lambda) = \prod_{l=1}^{L-1} \prod_{i=1}^{V_l} \prod_{j=1}^{V_l} N(w_{ij,l}|m_{ij,l}, v_{ij,l}) $$

Gamma($\gamma|\alpha$,$\beta$)Gamma($\lambda|\alpha$,$\beta$). \quad (10)

The approximation parameters $m_{ij,l}$, $v_{ij,l}$, $\alpha$,$\beta$, $\alpha$,$\beta$ and $\beta$ are determined by applying an assumed density filtering method (Opper & Winther, 1998; Minka, 2001) on the posterior (5). For this, (10) is first initialized to be uniform, that is, $m_{ij,l} = 0$, $v_{ij,l} = \infty$, $\alpha$,$\beta$ = $\alpha$,$\beta$ = $\alpha$,$\beta$ = 0. After this, PBP iterates over the factors in the numerator of (5) and sequentially incorporates each of these factors into the approximation in (10). A general description of this operation is given in the following paragraph. Specific details on how to incorporate each type of factor in (5) are given in the following sections.

There are two factors for the priors on $\gamma$ and $\lambda$ given by (4) and (3), a total of $\prod_{l=1}^{L} V_l(V_l - 1)$ factors for the prior on $W$ given by (2) and finally, $N$ factors for the likelihood (1). Let $f(W, \lambda, \gamma)$ be one of these factors. PBP incorporates $f$ into the current posterior approximation (10) by minimizing the KL divergence between $s(W, \gamma, \lambda) = Z^{-1} f(W, \lambda, \gamma) q(W, \gamma, \lambda)$ and $q(W, \gamma, \lambda)$, with respect to the parameters of $q$, where $Z$ normalizes $s$ and the $q$ used to construct $s$ is kept constant during the minimization of the KL divergence. This makes the new $q$ approximate the product of the old $q$ and the factor $f$. The result is an approximation to an exact online learning algorithm within the Bayesian framework (Opper & Winther, 1998).
3.2. Incorporating the prior factors into \( q \)

The first factors to be incorporated are (4) and (3). Since these factors have the same functional form as (10), the resulting update for \( q \) is straightforward: \( \alpha_{\text{new}} = \alpha_{3}^{\text{new}} \), \( \beta_{\text{new}} = \beta_{3}^{\text{new}} \), \( \alpha_{\text{new}} = \alpha_{4}^{\text{new}} \) and \( \beta_{\text{new}} = \beta_{4}^{\text{new}} \).

After this, we sequentially incorporate the factors in (2). In this case, the updates for \( m_{ij,l} \) and \( v_{ij,l} \) in (10) are given by (8) and (9). Similar update rules can be obtained for the parameters \( \alpha^{\lambda} \) and \( \beta^{\lambda} \) in (10). In particular,

\[
\alpha_{\text{new}}^{\lambda} = [ZZ_{2}Z_{1}^{-2}(\alpha^{\lambda}+1)/\alpha^{\lambda} - 1.0]^{-1},
\]

\[
\beta_{\text{new}}^{\lambda} = [Z_{2}Z_{1}^{-1}(\alpha^{\lambda}+1)/\beta^{\lambda} - Z_{1}^{-1}\alpha^{\lambda}/\beta^{\lambda}]^{-1},
\]

where \( Z \) is the normalizer of \( s \), that is, the product of the factor that is being incorporated and \( q \) and \( Z_{1} \) and \( Z_{2} \) are the values of this normalizer when the parameter \( \alpha^{\lambda} \) in \( q \) is increased by one and two units, respectively. The update rules (11) and (12) do not exactly minimize the KL divergence since that would require matching the sufficient statistics for \( \lambda \) and \( s \), which does not have a closed form. Instead, the rules above match the first and second moments of \( \lambda \), which also produces good results (Minka, 2001; Cowell et al., 1996). One difficulty when applying the update rules just described is that the normalizer \( Z \) of \( s \) does not have a closed form. Nevertheless, we can approximate \( Z \) using

\[
Z = \int \mathcal{N}(w_{ij,l}|0, \lambda^{-1}) q(W|\lambda) dW d\gamma d\lambda
\]

\[
= \int \mathcal{N}(w_{ij,l}|0, \lambda^{-1}) \mathcal{N}(w_{ij,l}|m_{ij,l}, v_{ij,l})
\times \text{Gamma}(\lambda|\alpha^{\lambda}, \beta^{\lambda}) dw_{ij,l} d\lambda
\]

\[
= \int T(w_{ij,l}|0, \lambda^{-1}, 2\alpha^{\lambda}) \mathcal{N}(w_{ij,l}|m_{ij,l}, v_{ij,l}) dw_{ij,l}
\approx \int \mathcal{N}(w_{ij,l}|0, \lambda^{-1}) \mathcal{N}(w_{ij,l}|m_{ij,l}, v_{ij,l}) dw_{ij,l}
\]

\[
= \mathcal{N}(m_{ij,l}|0, \lambda^{-1} + v_{ij,l}).
\]

where \( T(\cdot|\mu, \beta, \nu) \) denotes a Student’s \( t \) distribution with mean \( \mu \), variance parameter \( \beta \) and degrees of freedom \( \nu \). In the next-to-last line we have approximated the Student’s \( t \) density with a Gaussian density that has the same mean and variance. Finally, \( Z_{1}, Z_{2} \) and the gradients of log \( Z \) with respect to \( m_{ij,l} \) and \( v_{ij,l} \) can be similarly approximated by incorporating this approximation of \( Z \) into their expressions. By plugging in the resulting quantities in (8), (9), (11) and (12) we obtain the new parameter values for \( q \).

3.3. Incorporating the likelihood factors into \( q \)

After incorporating all the factors in (2), PBP sequentially incorporates the \( N \) factors for the likelihood (1). As before, updates for all the \( m_{ij,l} \) and \( v_{ij,l} \) in (10) are given by (8) and (9), respectively. The updates for \( \alpha^{\gamma} \) and \( \beta^{\gamma} \) in (10) are given by (11) and (12), respectively. To compute all these updates we only require \( Z \), the normalization constant of \( s \). However, this is difficult to compute, as it requires integration of each likelihood factor with respect to the distribution of the network output, i.e., \( z_{L} = f(x_{n}|W) \), when the weights \( W \) are sampled from \( q \). Let us assume that we have an approximating Gaussian with mean \( m^{z_{L}} \) and variance \( v^{z_{L}} \) for the distribution of \( z_{L} \). We can then approximate \( Z \) as

\[
Z = \int \mathcal{N}(y_{n}|f(x_{n}|W), \gamma^{-1}) q(W, \gamma, \lambda) dW d\gamma d\lambda
\]

\[
= \int \mathcal{N}(y_{n}|z_{L}, \gamma^{-1}) \mathcal{N}(z_{L}|m^{z_{L}}, v^{z_{L}}) \text{Gamma}(\gamma|\alpha^{\gamma}, \beta^{\gamma}) dz_{L} d\gamma
\]

\[
= \int \mathcal{N}(y_{n}|m^{z_{L}}, \beta^{\gamma}/(\alpha^{\gamma} - 1) + v^{z_{L}})
\]

\[
\approx \mathcal{N}(y_{n}|m^{z_{L}}, \beta^{\gamma}/(\alpha^{\gamma} - 1) + v^{z_{L}})
\]

where the first approximation in (14) assumes that \( z_{L} = f(x_{n}|W) \sim \mathcal{N}(m^{z_{L}}, v^{z_{L}}) \) when \( W \) has distribution \( q \) and the second approximates the Student’s \( t \) density with a Gaussian density that has the same mean and variance. As before, this expression for \( Z \) can be substituted into (8), (9), (11) and (12) to obtain the new parameter values for \( q \).

However, it remains to compute the mean and variance parameters \( m^{z_{L}} \) and \( v^{z_{L}} \) in (14). This is done by propagating distributions forward through the network and, when necessary, approximating each new distribution with a Gaussian. For this, let us assume that, when \( W \) is distributed as \( q \), the output of the \( l \)-1 layer \( z_{l-1} \) is a diagonal Gaussian with means and variances given by the \( V_{l-1} \)-dimensional vectors \( m^{z_{l-1}} \) and \( v^{z_{l-1}} \), respectively. Furthermore, let \( a_{l} = W_{l}z_{l-1} \), so that the marginal means and variances of \( a_{l} \) (when \( W \) is distributed as \( q \)) are

\[
m^{a_{l}} = M_{l}m^{z_{l-1}}
\]

\[
v^{a_{l}} = (M_{l} \circ M_{l})v^{z_{l-1}} + V_{l}(m^{z_{l-1}} \circ m^{z_{l-1}}) + V_{l}v^{z_{l-1}}
\]

where \( M_{l} \) and \( V_{l} \) are \( V_{l} \times (V_{l-1}+1) \) matrices whose entries are given by \( m_{ij,l} \) and \( v_{ij,l} \), for \( i = 1, \ldots, V_{l} \) and \( j = 1, \ldots, V_{l-1} + 1 \), respectively, and \( \circ \) denotes the Hadamard elementwise product. We again assume that the entries in \( a_{l} \) are independent Gaussian with means and variances given by the equations above. The Central Limit Theorem states that \( a_{l} \) is approximately Gaussian when \( V_{l-1} \) is large (Soudry et al., 2014). Let \( b_{l} = a_{l}(a_{l}) \), where \( a \) is the rectifier linear activation function \( a(x) = \max(0, x) \). Then, the entries of \( b_{l} \) are a mixture of a point mass at 0 (when the rectifier is saturated) and a Gaussian truncated at 0 (when the rectifier is in the linear regime). The mean and variance of the \( i \)-th entry of \( b_{l} \) are then

\[
m^{b_{l}} = \Phi(\alpha_{l})v_{i}^{l},
\]

\[
v_{i}^{b_{l}} = m_{i}^{a_{l}}v_{i}^{l}\Phi(-\alpha_{l}) + \Phi(\alpha_{l})v_{i}^{a_{l}}(1 - \gamma_{i}(\alpha_{l} + \alpha_{l})),
\]

where \( v_{i}^{l} = m_{i}^{a_{l}} + \sqrt{v_{i}^{a_{l}}} \gamma_{i}, \alpha_{l} = m_{i}^{a_{l}}/\sqrt{v_{i}^{a_{l}}}, \gamma_{i} = \phi(-\alpha_{l})/\Phi(\alpha_{l}), \) and \( \Phi \) and \( \phi \) are respectively the CDF and the density function of a standard Gaussian. When \( \alpha_{l} \) is very large and negative the previous definition of \( \gamma_{l} \) is not numerically stable. Instead, when \( \alpha_{l} < -30 \), we use the
approximation $\gamma_i = -\alpha_i - \alpha_i^{-1} + 2\alpha_i^{-3}$ as recommended by Paquet et al. (2012). The output of the $l$-th layer, $z_l$, is obtained by concatenating $b_l$ with the constant 1 for the bias. We can therefore approximate the distribution of $z_l$ to be Gaussian with marginal means and variances

$$m^{z_l} = [m^{b_l}; 1], \quad v^{z_l} = [v^{b_l}; 0].$$

(19)

These additional concatenated means and variances reflect the lack on uncertainty in the “bias unit”. Finally, to compute the mean and variance parameters $m^L$ and $v^L$ in (14) we initialize $m^{z_L}$ to $[x_L; 1]$ and $v^{z_L}$ to 0 and then apply (15), (16), (17), (18) and (19) iteratively until we obtain $m^{z_L} = m^L$ and $v^{z_L} = v^L$. This resembles the forward pass of the standard backpropagation algorithm. In hand, we can evaluate the log of $Z$ as given by (14) and the gradients of that quantity that are required to apply rules (8) and (9). This is very similar to the reverse mode differentiation used in backpropagation. We used the automatic differentiation tool Theano for this task (Bergstra et al., 2010).

3.4. Expectation propagation

Expectation propagation (EP) (Minka, 2001) improves on assumed density filtering by iteratively incorporating each factor multiple times. On each pass over the list of factors, each factor is removed from the current posterior approximation, re-estimated, and then reincorporated. Each iteration improves the accuracy of the posterior approximation. The disadvantage of EP over ADF is that it needs to keep in memory all of the approximate factors, one for each exact factor in the numerator of the posterior. This is necessary, because each factor must be able to be removed and updated. With massive data sets, the number of likelihoods will be very large and it is not possible to store these factors in memory. Instead, we incorporate these factors multiple times, but without removing them from the current approximation. This is equivalent to doing multiple ADF passes through the data, treating each likelihood factor as a novel example. A disadvantage of this approach is that it can lead to underestimation of the variance parameters in (10) when too many passes are done over the data. Nevertheless, PBP is geared toward larger data sets, where only a reduced number of passes over the data (less than 100) are possible. Note that we can afford to keep in memory an approximate factor for each exact factor in the prior on the weights (2), since the number and size of these approximate factors are small. We therefore do one full EP update of these approximate factors for the prior after each ADF pass through the data. Details can be found in the code for PBP, included in the supplementary material. These approximate factors could also be updated more frequently, for example, each time we do an ADF pass through a small block of likelihood factors.

3.5. Implementation details

After incorporating the factors in (2) for the first time, we slightly perturb each mean parameter $m_{ij,l}$ in (10) from the original value of 0 to be $\epsilon_{ij,l}$, where $\epsilon_{ij,l} \sim \mathcal{N}(0, 10^{-4})$. This is similar to the random initialization of weights in NNs that is usually done before learning with backpropagation. This operation directs our inference method towards one of the multiple symmetric modes of the posterior distribution.

Because the computation of $Z$ in (14) is approximate, on rare occasions the variance parameters for some weights in (10) may be negative after incorporating one likelihood factor. When this happens, we undo the update for those weights and keep their previous mean and variance values. A similar operation is often done in EP when negative variances arise in Gaussian approximate factors (Minka, 2001). Python code for PBP along with a toy example can be found at http://jmhl.org.

4. Related Work

The gold standard method for Bayesian learning in neural networks is Hamilton Monte Carlo (HMC) (Neal, 1995). However, this is a batch method that can perform poorly on large data sets. HMC also requires problem-specific tuning parameters such as the length and number of symplectic integration steps. One alternative to MCMC inference in neural networks is the Laplace approximation (MacKay, 1992b). However, the Laplace approximation requires computation of the inverse Hessian of the log likelihood, which is infeasible to compute for large networks and data sets. Diagonal approximations to the Hessian are possible, but performance can deteriorate considerably. One alternative approach based on expectation propagation (EP) (Minka, 2001) is described by Jylänki et al. (2014). This is a batch method that is not expected to scale to large data sets and, unlike PBP, it requires numerical quadrature. Jylänki keeps in memory several approximate factors for each data point, which is not feasible in large scale settings. Furthermore, by using latent variables, this method breaks each likelihood factor into additional sub-factors that are incorporated into the posterior approximation in multiple disconnected steps. PBP incorporates each likelihood factor in a single step, which is expected to be more accurate.

A scalable variational inference (VI) method for neural networks is described by Graves (2011). This method maximizes a lower bound on the marginal likelihood of the NN. The computation of this bound requires computing the expectation of the log of the numerator of the exact posterior (5) under a factorized Gaussian approximation. This is intractable in general, and so Graves (2011) proposes a Monte Carlo approximation to compute the lower bound, which is then optimized using a second approximation for
stochastic gradient descent (SGD). While SGD is a common approach to optimization of neural networks, the initial approximation leads to inefficient use of data. As a result, the VI approach tends to generate poor solutions for larger data sets over which only a few passes are possible.

The technique that is most closely related to PBP is the expectation-backpropagation (EBP) method described by Soudry et al. (2014), which proposes an online EP technique for neural networks with sigmoid activation functions and binary weights, with an extension to continuous weights. As with PBP, EBP also includes a forward propagation of probabilities followed by a backward propagation of gradients. However, there are three important contributions of PBP with respect to EBP. First, EBP can only model data with binary targets and cannot be applied when the $y_n$ are continuous (as in regression), while PBP assumes continuous $y_n$ and can be extended to binary targets using the same method as in EBP. Second, and more importantly, EBP with continuous weights only updates the mean parameters of the Gaussian posterior approximations. In particular, the EBP update operation for each Gaussian approximation includes only equation (8) and does not perform the corresponding update for the variance given by (9). Therefore, EBP cannot produce accurate uncertainty estimates, as it keeps the posterior variances constant during the learning process. Note also that the “learning rate” in (8) is the variance of the Gaussian approximation. In effect, by not updating the variances, EBP makes inefficient updates for the means. Finally, unlike probabilistic backpropagation, EBP does not learn the hyperparameter for the prior variance $\lambda^{-1}$. Instead, EBP keeps $\lambda^{-1}$ fixed to a large initial value.

5. Experiments

We evaluate PBP in regression experiments with publicly available data sets and neural networks with one hidden layer. In PBP we make probabilistic predictions for the target variables by using (14), which approximates (6).

Table 1. Characteristics of the analyzed data sets.

| Dataset                  | $N$  | $d$ |
|--------------------------|------|-----|
| Boston Housing           | 506  | 13  |
| Concrete Compression     | 1030 | 8   |
| Strength                |      |     |
| Energy Efficiency        | 768  | 8   |
| Kin8nm                   | 8192 | 8   |
| Naval Propulsion         | 11,934 | 16 |
| Combined Cycle Power     | 9568 | 4   |
| Plant                   |      |     |
| Protein Structure        | 45,730 | 9  |
| Wine Quality Red         | 1599 | 11  |
| Yacht Hydrodynamics      | 308  | 6   |
| Year Prediction MSD      | 515,345 | 90 |

Table 2. Average test RMSE and standard errors.

| Dataset                  | VI      | BP      | PBP      |
|--------------------------|---------|---------|----------|
| Boston                   | 4.320±0.2914 | 3.247±0.2017 | 2.994±0.1817 |
| Concrete                 | 7.128±0.1230 | 5.787±0.1299 | 5.533±0.0990 |
| Energy                   | 2.646±0.0813 | 1.098±0.0738 | 1.695±0.0581 |
| Kin8nm                   | 0.099±0.0009 | 0.090±0.0013 | 0.092±0.0007 |
| Naval                    | 0.005±0.0005 | 0.001±0.0001 | 0.006±0.0000 |
| Power Plant              | 4.327±0.0352 | 4.126±0.0368 | 4.128±0.0334 |
| Protein                  | 4.842±0.0305 | 4.559±0.0217 | 4.733±0.0082 |
| Wine                     | 0.646±0.0081 | 0.636±0.0083 | 0.634±0.0078 |
| Yacht                    | 6.887±0.6749 | 1.298±0.0669 | 0.877±0.0495 |
| Year                     | 9.034±NA    | 8.929±NA  | 8.883±NA  |

Table 3. Average running time in seconds.

| Problem                  | VI      | BP      | PBP      |
|--------------------------|---------|---------|----------|
| Boston                   | 1035    | 677     | 13       |
| Concrete                 | 1085    | 758     | 24       |
| Energy                   | 2011    | 675     | 19       |
| Kin8nm                   | 5604    | 2001    | 156      |
| Naval                    | 8373    | 2351    | 220      |
| Power Plant              | 2955    | 2114    | 178      |
| Protein                  | 7691    | 4831    | 485      |
| Wine                     | 1195    | 917     | 50       |
| Yacht                    | 954     | 626     | 12       |
| Year                     | 142,077 | 65,131  | 6119     |

Table 4. Average test log-likelihood and standard errors.

| Dataset                  | VI      | PBP      |
|--------------------------|---------|----------|
| Boston                   | -2.903±0.071 | -2.550±0.089 |
| Concrete                 | -3.391±0.017 | -3.136±0.021 |
| Energy                   | -2.391±0.029 | -1.982±0.027 |
| Kin8nm                   | 0.897±0.010 | 0.964±0.007 |
| Naval                    | 3.734±0.116 | 3.653±0.004 |
| Power Plant              | -2.890±0.010 | -2.838±0.008 |
| Protein                  | -2.992±0.006 | -2.974±0.002 |
| Wine                     | -0.980±0.013 | -0.966±0.014 |
| Yacht                    | -3.439±0.163 | -1.483±0.018 |
| Year                     | -3.622±NA    | -3.603±NA  |

5.1. Predictive performance

We first evaluate the predictive accuracy of PBP. Table 1 lists the analyzed data sets and shows summary statistics. We use neural networks with 50 hidden units in all cases except in the two largest ones, i.e., Year Prediction MSD and Protein Structure, where we use 100 hidden units. We compare PBP with the variational inference (VI) approach described in Section 4 and with standard stochastic gradient descent via backpropagation (BP). These methods were coded using Theano (Bergstra et al., 2010).

The different methods, PBP, VI and BP, were run by performing 40 passes over the available training data, updating the model parameters after seeing each data point. The data sets are split into random training and test sets with 90% and 10% of the data, respectively. This splitting process is repeated 20 times and the average test performance of each method is reported. In the two largest data sets, Year Prediction MSD and Protein Structure, we do the train-test
Probabilistic Backpropagation

Figure 1. Predictions made by each method on the toy data set. The noisy observations are shown as black dots, the true data generating function is displayed as a black continuous line and the mean predictions are shown as a dark gray line. Credible intervals corresponding to ±3 standard deviations from the mean are shown as a light gray shaded area.

Table 5. Average test RMSE and standard errors in active learning.

| Dataset   | LA-R     | LA-A     | EP-R     | EP-A     | PBP-R    | PBP-A    | HMC-R    | HMC-A    |
|-----------|----------|----------|----------|----------|----------|----------|----------|----------|
| Boston    | 9.600±0.154 | 9.452±0.111 | 8.632±0.231 | 8.426±0.264 | 7.030±0.617 | 5.571±0.209 | 5.750±0.222 | 5.156±0.150 |
| Concrete  | 16.889±0.182 | 16.938±0.173 | 16.767±0.174 | 16.897±0.151 | 14.247±0.555 | 12.774±0.299 | 10.564±0.198 | 11.484±0.191 |
| Energy    | 10.110±0.075 | 10.135±0.070 | 3.616±0.101 | 3.634±0.159 | 3.965±0.151 | 3.564±0.086 | 3.246±0.067 | 3.118±0.062 |
| Kin8nm    | 0.271±0.003 | 0.270±0.002 | 0.272±0.002 | 0.271±0.002 | 0.259±0.006 | 0.254±0.004 | 0.226±0.004 | 0.223±0.003 |
| Naval     | 0.015±0.000 | 0.015±0.000 | 0.015±0.000 | 0.015±0.000 | 0.015±0.000 | 0.016±0.000 | 0.013±0.000 | 0.012±0.000 |
| Power Plant | 17.195±0.120 | 17.306±0.149 | 8.234±0.831 | 6.251±0.599 | 5.608±0.116 | 5.349±0.116 | 5.229±0.097 | 4.800±0.074 |
| Protein   | 6.165±0.073 | 6.227±0.088 | 6.118±0.074 | 6.151±0.077 | 6.265±0.185 | 6.059±0.090 | 5.613±0.089 | 5.727±0.090 |
| Wine      | 0.843±0.011 | 0.829±0.010 | 0.836±0.010 | 0.832±0.009 | 0.891±0.022 | 0.792±0.011 | 0.740±0.011 | 0.749±0.010 |
| Yacht     | 15.926±0.409 | 15.463±0.310 | 15.173±0.214 | 15.442±0.390 | 4.212±0.254 | 2.849±0.109 | 4.644±0.237 | 3.211±0.120 |

Table 4 shows the average test log-likelihood for VI and PBP. PBP is considerably better than VI, which performs rather poorly. Finally, Table 3 shows the average running time in seconds for each method. BP and VI are very slow since they have to be re-run 30 times to search for their optimal hyperparameter values. The BO search in these methods also has a considerable overhead in the smallest data sets. PBP is the fastest method since it does not have to select any hyperparameter values and is run only once.

5.2. Toy data set

We further evaluate the predictive distribution obtained by PBP in a toy data set generated by sampling 20 inputs \( x \) uniformly at random in the interval \([-4, 4]\). For each value of \( x \) obtained, the corresponding target \( y \) is computed as \( y = x^3 + \epsilon_n \), where \( \epsilon_n \sim \mathcal{N}(0, 0.9) \). We fitted a neural network with one layer and 100 hidden units to these data using PBP. We compare PBP with VI and BP, using 40 training epochs in all these methods. We also compare with a ground truth generated by Hamiltonian Monte Carlo (HMC). HMC is implemented by modifying the MCMC-stuff Matlab toolbox (Vanhatalo & Vehtari, 2006) to include rectified linear activation functions. We run HMC by drawing 30,000 samples from the posterior distribution.

Figure 1 shows the predictions generated by each method. PBP and BP are much closer to the ground truth HMC than VI. Furthermore, BP and PBP perform similarly, even though PBP automatically adjusts its hyperparameters while BP has to use BO methods for this task.

splitting only one and five times respectively. The data sets are normalized so that the input features and the targets have zero mean and unit variance in the training set. The normalization on the targets is removed for prediction.

BP and VI have several hyperparameters that have to be optimally adjusted to the data. These are learning rate and momentum in BP and VI and weight decay in BP. We select these hyperparameter values by maximizing the predictive performance of each method on a validation set with 20% of the training data. For this task we use Bayesian optimization (BO) techniques (Snoek et al., 2012). In particular, for each train-test split of the data, we use BO methods to sequentially evaluate the validation performance of 30 different hyper-parameter configurations. After that, an optimal value for the hyper-parameters is selected and used to fit the model on the training set. We used the implementation provided at https://www.whetlab.com to solve these BO problems.

Table 2 shows the average test root mean squared error (RMSE) for each method. On each data set, the results of the best method are shown in bold. Overall, PBP and BP perform best, both with the best results in 5 different data sets. However, unlike BP, PBP automatically adjusts its hyperparameters and does not require an expensive BO search. VI performs rather poorly, evidently due to the use of two stochastic approximations. First, VI approximates the lower bound on the model evidence by sampling from the variational approximation and second, VI further approximates that bound by subsampling the data. BP and PBP only perform the second type of approximation.
5.3. Active learning

We performed another series of experiments to evaluate the accuracy of the estimates of the posterior variance on the weights produced by PBP. For this, we use an active learning scenario (Settles, 2009) since in this type of problems it is necessary to produce accurate estimates of uncertainty for obtaining good performance.

In these experiments, we used a neural network with a single hidden layer and ten hidden units. We split each data set into training and test sets with 20 and 100 data instances, respectively, and pool sets with all the remaining data. PBP is fitted using the training data and then, its performance is evaluated on the test data. After this, one data point is collected from the pool set and then moved into the training set. The process repeats until 9 of these active additions to the training set have been completed, that is, until we have performed 10 evaluations on the test set. The entire process, including the random data set splitting, is repeated 40 times. The pool data is initially lacking the target variables and these become available only once the data is moved to the training set. As before, we run PBP for 40 epochs.

We compare PBP with a ground truth obtained by a HMC method in which we draw 500 samples from the posterior. We also compare with the batch EP algorithm for neural networks described by Jylänki et al. (2014). This method uses nonlinear activation functions given by the standard Gaussian CDF. We further compare with the Laplace approximation (LA) of MacKay (1992b) using the neural network toolbox from Matlab with tanh nonlinearities. In LA we approximate the Hessian of the unnormalized posterior distribution with the Levenberg-Marquardt approximation and assume a diagonal Hessian matrix. This allows LA to scale to large data sets and larger networks. We compare two versions of PBP, HMC, EP and LA. One in which the data from the pool set is collected actively (PBP-A, HMC-A, EP-A and LA-A) and another one in which the pool data is collected uniformly at random (PBP-R, HMC-R, EP-R and LA-R). We re-trained from scratch all the methods after each new addition to the training set from the pool set.

To actively collect data from the pool set we follow the information-based approach described by MacKay (1992a). The goal is to maximize the expected reduction in posterior entropy that is produced by adding data to the training set. This implies choosing the $x$ that maximizes

$$H(W, \gamma, \lambda|\mathcal{D}) - \mathbb{E}_{p(y|x, \mathcal{D})} H(W, \gamma, \lambda|\mathcal{D} \cup \{x, y\}),$$

where $H[\cdot]$ is the differential entropy. Following Houlsby et al. (2012), we can rewrite (20) by swapping the roles of $y$ and the model parameters $W, \gamma, \lambda$. We finally obtain

$$H(y|x, \mathcal{D}) - \mathbb{E}_{p(W, \gamma, \lambda|\mathcal{D})} \mathbb{E}_{p(y|x, \mathcal{D})} [H(W, \gamma, \lambda|x, \mathcal{D})].$$

Table 5 shows the average test RMSE for each method at the end of the data collection process. These results show that the active learning approach HMC-A is significantly better than the random approach HMC-R in the data sets Boston, Energy, Power Plant and Yacht. In these data sets we also see a significant improvement of PBP-A over PBP-R. This indicates that PBP produces useful estimates of posterior variance. In these experiments PBP is usually better than EP and LA. LA performs poorly because it cannot correctly select the hyperparameters $\lambda$ and $\gamma$, due to the diagonal Hessian approximation, as also observed by MacKay (1992c). PBP does not have this problem.

Finally, Figure 2 shows the evolution of the average test RMSE for each method during the data collection process in the problems Boston, Yacht and Energy. These plots indicate that the improvements of PBP-A over PBP-R are similar to those of HMC-A over HMC-R. Furthermore, we can see that the active learning strategy does not work as well in EP and LA as it does in PBP and HMC.

6. Conclusions and future work

We have presented probabilistic backpropagation (PBP), a new algorithm for scalable Bayesian learning of neural networks. PBP uses a product of Gaussians to approximate
the posterior over weights. The parameters of these Gaussians are updated in a two stage process similar to the one used by the backpropagation algorithm. First, probabilities are propagated forward through the network to obtain the marginal likelihood and second, the gradients of this quantity with respect to the Gaussian parameters are propagated backwards. These gradients are finally used to update the parameters of the approximation to the posterior distribution. Experiments on ten datasets show that PBP makes accurate predictions. Furthermore, we also show that PBP produces useful estimates of the posterior variance on the network weights. In summary, PBP is a fast method with state-of-the-art performance for Bayesian learning of neural networks. As future work we plan to evaluate PBP in multi-label and multi-class problems and experiment with networks that include more than one hidden layer.

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References

Bergstra, James, Breuleux, Olivier, Bastien, Frédéric, Lamblin, Pascal, Pascanu, Razvan, Desjardins, Guillaume, Turian, Joseph, Warde-Farley, David, and Bengio, Yoshua. Theano: a CPU and GPU math expression compiler. In Proceedings of the Python for Scientific Computing Conference (SciPy), June 2010. Oral Presentation.

Bottou, Léon. Large-scale machine learning with stochastic gradient descent. In Lechevallier, Yves and Saporta, Gilbert (eds.), Proceedings of COMPSTAT'2010, pp. 177–186. Physica-Verlag HD, 2010.

Collobert, Ronan and Weston, Jason. A unified architecture for natural language processing: Deep neural networks with multitask learning. In Proceedings of the 25th international conference on Machine learning, pp. 160–167. ACM, 2008.

Cowell, R. G., Dawid, P. A., and Sebastiani, P. A comparison of sequential learnin methods for incomplete data. Bayesian Statistics, 1996.

Graves, Alex. Practical variational inference for neural networks. In Shawe-Taylor, J., Zemel, R.S., Bartlett, P.L., Pereira, F., and Weinberger, K.Q. (eds.), Advances in Neural Information Processing Systems 24, pp. 2348–2356. Curran Associates, Inc., 2011.

Hannun, Awni, Case, Carl, Casper, Jared, Catanzaro, Bryan, Diamos, Greg, Elsen, Erich, Prenger, Ryan, Satheesh, Sainjeev, Sengupta, Shubho, Coates, Adam, and Ng, Andrew Y. Deep speech: Scaling up end-to-end speech recognition. arXiv:1412.5367 [cs.CL], 2014.

Hinton, Geoffrey and Camp, Drew Van. Keeping neural networks simple by minimizing the description length of the weights. In in Proc. of the 6th Ann. ACM Conf. on Computational Learning Theory, pp. 5–13. ACM Press, 1993.

Hinton, Geoffrey, Deng, Li, Yu, Dong, Dahl, George E, Mohamed, Abdel-rahman, Jaitly, Navdeep, Senior, Andrew, Vanhoucke, Vincent, Nguyen, Patrick, Sainath, Tara N, et al. Deep neural networks for acoustic modeling in speech recognition: The shared views of four research groups. Signal Processing Magazine, IEEE, 29(6):82–97, 2012.

Hornik, Kurt, Stinchcombe, Maxwell, and White, Halbert. Multi-layer feedforward networks are universal approximators. Neural networks, 2(5):359–366, 1989.

Houlsby, Neil, Hernández-lobato, Jose M., Huszar, Ferenc, and Ghahramani, Zoubin. Collaborative gaussian processes for preference learning. In Advances in Neural Information Processing Systems, pp. 2096–2104, 2012.

Jylänki, Pasi, Nummenmaa, Aapo, and Vehtari, Aki. Expectation propagation for neural networks with sparsity-promoting priors. The Journal of Machine Learning Research, 15(1):1849–1901, 2014.

Krizhevsky, Alex, Sutskever, Ilya, and Hinton, Geoffrey E. Imagenet classification with deep convolutional neural networks. In Advances in neural information processing systems, pp. 1097–1105, 2012.

MacKay, David J. C. Information-based objective functions for active data selection. Neural Comput., 4(4):590–604, July 1992a. ISSN 0899-7667. doi: 10.1162/neco.1992.4.4.590.

MacKay, David J. C. A practical bayesian framework for back-propagation networks. Neural computation, 4(3):448–472, 1992b.

MacKay, David J.C. Bayesian methods for adaptive models. PhD thesis, California Institute of Technology, 1992c.

Minka, Thomas P. A family of algorithms for approximate Bayesian inference. PhD thesis, Massachusetts Institute of Technology, 2001.

Nair, Vinod and Hinton, Geoffrey E. Rectified linear units improve restricted boltzmann machines. In Proceedings of the 27th International Conference on Machine Learning (ICML-10), pp. 807–814, 2010.

Neal, Radford M. Bayesian learning for neural networks. PhD thesis, University of Toronto, 1995.

Opper, Manfred and Winther, Ole. A Bayesian approach to online learning. On-line Learning in Neural Networks, ed. D. Saad, pp. 363–378, 1998.

Paquet, Ulrich, Thomson, Blaise, and Winther, Ole. A hierarchical model for ordinal matrix factorization. Statistics and Computing, 22(4):945–957, 2012. ISSN 0960-3174. doi: 10.1007/s11222-011-9264-x.

Rumelhart, D.E., Hinton, G.E., and Williams, R.J. Learning representations by back-propagating errors. Nature, 323(6088):533–536, 1986.

Settles, B. Active learning literature survey. Technical Report 1648, University of WisconsinMadison, 2009.

Snoek, Jasper, Larochelle, Hugo, and Adams, Ryan P. Practical bayesian optimization of machine learning algorithms. In Advances in Neural Information Processing Systems, pp. 2951–2959, 2012.
Soudry, Daniel, Hubara, Itay, and Meir, Ron. Expectation backpropagation: Parameter-free training of multilayer neural networks with continuous or discrete weights. In Ghahramani, Z., Welling, M., Cortes, C., Lawrence, N.D., and Weinberger, K.Q. (eds.), Advances in Neural Information Processing Systems 27, pp. 963–971. Curran Associates, Inc., 2014.

Srivastava, Nitish, Hinton, Geoffrey, Krizhevsky, Alex, Sutskever, Ilya, and Salakhutdinov, Ruslan. Dropout: A simple way to prevent neural networks from overfitting. The Journal of Machine Learning Research, 15(1):1929–1958, 2014.

Sutskever, Ilya, Vinyals, Oriol, and Le, Quoc V. V. Sequence to sequence learning with neural networks. In Ghahramani, Z., Welling, M., Cortes, C., Lawrence, N.D., and Weinberger, K.Q. (eds.), Advances in Neural Information Processing Systems 27, pp. 3104–3112. Curran Associates, Inc., 2014.

Vanhatalo, Jarno and Vehtari, Aki. Mcmc methods for mlp-network and gaussian process and stuff–a documentation for matlab toolbox mcmcstuff. Laboratory of Computational Engineering, Helsinki University of Technology, 2006.

Wu, Ren, Yan, Shengen, Shan, Yi, Dang, Qingqing, and Sun, Gang. Deep image: Scaling up image recognition. arXiv:1501.02876 [cs.CV], 2015.