On perturbation theory and its application in solving ordinary differential equations using the asymptotic expansion method

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Abstract - The perturbation theory is one of the tricks or tools used mathematically to find approximate solutions to fluctuating problems for which no accurate solutions can be found. In this paper we will deal with a number of basic concepts related to perturbation theory, including regular perturbation and singular perturbation, and then apply these tricks or tools theoretically to ordinary first and second order differential equations for regular perturbation using the asymptotic expansion method, which is considered one of the most important methods used to find approximate solutions of perturbation equations.

Keywords
perturbation theory, singular perturbation, regular perturbation, asymptotic expansion method, approximate solution.

1. Introduction

When we studied many subjects in mathematics, physics, engineering and other fields, many mathematical problems arose, especially once we left the linear frame, where they cannot be solved precisely - either due to the impossibility of a solution, or due to the lack of insufficient skills. Hence, a resort to perturbation theory, which is often the method of choice for dealing with it for nonlinear systems. Where perturbation theory defined as a broad group of tricks or the methods used to obtain approximate solutions for problems that do not have an analytical solution. These approaches reduce the challenging problem to a relatively an infinite series of simple problems that can be solved analytically. The perturbation problems have a small parameter. This parameter is affected, it focuses about the problem so that a difference in the solution occurs quickly in some areas while it is related to other parts. We notice the development of a number of methods used, and one of these methods, which we will deal with in our research, is the method of asymptotic expansion to provide approximate solutions. For more details, it is possible to see [1], [2], [3], [5], and other references.

The main objective of this research is to give a general idea of the disturbance theory while explaining how it can be applied to ordinary differential equations of different ranks using one of the methods used to find an approximate solution to troubled problems, which is the method of convergent expansion with the aim of helping researchers to develop ideas and methods used to face the new problems that Face them and build the capabilities needed to solve them.
2. Types of perturbation problem

The perturbation problems are divided into two types in terms of the nature of the problem and not its type, the regular perturbation problem and the singular perturbation problem, where the regular problem is defined as a problem whose perturbated chain is an energy chain in $\varepsilon$ as it is characterized by the precise solution of the small solution $|\varepsilon|$ Small but not equal to zero approaches the non-perturbative otherwise zero-ordered solution as $\varepsilon \to 0$. As for the singular perturbation problem, it is defined as a problem in which the non-zero solution does not converge from the zero solution smoothly because the order of the equation changes when finding the zero solution, due to the parameter $\varepsilon$ being associated with the highest limit in the equation. Zero (the main term in the disorder chain) and solve the unperturbated problem, as it may be non-existent. There is no difference between the solutions as to whether it is disordered or not in the theory of regular perturbation, but in the theory of singular perturbation, the solution may depend zero and it may exist only for non-zero $\varepsilon$. This will be clarified through the following examples

Example

$$y'' + y = \varepsilon y^2 \quad y(0) = 1 \quad , \quad y'(0) = 1.$$  

Note that when $\varepsilon = 0$, there is no difference in the order of the differential equation and that the exact solution with respect to the small non-null parameter $|\varepsilon|$ It smoothly approaches the unperturbation solution so this problem is considered to be perturbation on a regular basis

Example

$$\varepsilon y'' + y' = 2x + 1 \quad y(0) = 1 \quad , \quad y'(1) = 4.$$  

is a singular perturbation problem because when we substitute for $\varepsilon = 0$, we notice the difference in order with respect to the equation, as an equation of the first order is obtained, and thus the resulting equation does not meet the two boundary conditions.

3. Method of asymptotic expansion for solving perturbation problem

The asymptotic expansion method is one of the methods used in finding an approximate solution to perturbation problems on a regular or singular basis. Where these problems are expressed in a power series with respect to the $\varepsilon$ in which the problem is close to the solution or not. We can get a good approximation when the chain is cut in the smallest term. The chain of forces, whether positive or negative, is one of The most used types of asymptotic expansion. Methods for creating such expansions include the Euler-Maclaurin aggregation equation and integral transformations such as Laplace and Myelin transformations. Often the repeated integration of parts leads to asymptotic expansion.

Example 1:- Determine the first order perturbation solution for the following intial value problem

$$\frac{dz}{dt} + e^xz = 0 \quad , \quad z(0) = 1 \quad (1)$$

Solution :-

we will resort to using perturbation theory to allow us to solve it precisely.

Step 1 :- Writing the equation in terms of the perturbation parameter $\varepsilon$ in this way
\[
\frac{dz}{dt} + \varepsilon e^x z = 0 \quad z(0) = 1 \quad (2)
\]

**Step 2** :- We find the unperturbed solution to the eq. by substituting in each \( \varepsilon = 0 \), we get
\[
\frac{dz}{dt} = 1, \text{ that we can solve exactly (} z = 1). \]

**Step 3** :- assume that the solution to the perturbed problem can be described by power series of \( \varepsilon \):
\[
z = z_0 + \varepsilon z_1 \quad (3)
\]

Substitute Equation (2) in (3)
\[
\frac{d}{dt} \left( z_0(t) + \varepsilon z_1(t) \right) + \varepsilon e^x \left( z_0(t) + \varepsilon z_1(t) \right) = 0
\]
\[
\frac{dz_0}{dt} + \varepsilon \left( \frac{dz_1}{dt} + e^x z_0 \right) + O(\varepsilon^2) = 0
\]

Equating the co-efficient of \( \varepsilon \), it become:

\( O(\varepsilon^0) \):
\[
\frac{dz_0}{dt} = 0 \quad z_0(0) = 1 \quad (4)
\]

\( O(\varepsilon^1) \):
\[
\frac{dz_1}{dt} + e^x z_0 = 0 \quad z_1(0) = 0 \quad (5)
\]

By solving the above equation we will get
\[
z_0(t) = 1
\]
\[
z_1(t) = -e^t
\]

Putting these values in equation (2), we have the solution
\[
z(t) = 1 - \varepsilon e^t.
\]

**Example 2** : consider the initial value problem
\[
\frac{d^2z}{dt^2} = -\varepsilon \frac{dz}{dt} - 2 \quad z(0) = 0 , \quad \frac{dz}{dt} (0) = 2 \quad \ldots \ldots (6)
\]

**Solution :-**

First, We find the unperturbed solution to the eq. by substituting in each \( \varepsilon = 0 \), we get
\[
\frac{d^2z_0}{dt^2} + 2 = 0 \quad , \quad z(0) = 0 \quad , \quad \frac{dz}{dt} (0) = 2
\]

that we can solve exactly
\[
z(t) = t - t^2
\]
Now, we assume that the solution to the perturbed problem can be described by power series of $\varepsilon$:

$$z(t) = z_0(t) + \varepsilon z_1(t) + \varepsilon^2 z_2(t) + O(\varepsilon^3) \quad (7)$$

Substitute Equation (7) in (6)

$$\frac{d^2 z_0}{dt^2} + \varepsilon \frac{dz}{dt} + 2 = 0$$

$$\frac{d^2}{dt^2} (z_0(t) + \varepsilon z_1(t) + \varepsilon^2 z_2(t) + O(\varepsilon^3)) + \varepsilon \frac{d}{dt} (z_0(t) + \varepsilon z_1(t) + \varepsilon^2 z_2(t) + O(\varepsilon^3)) + 2 = 0$$

$$\Rightarrow \frac{d^2 z_0}{dt^2} + 2 + \varepsilon \left( \frac{d^2 z_1}{dt^2} + \frac{dz_0}{dt} \right) + \varepsilon^2 \left( \frac{d^2 z_2}{dt^2} + \frac{dz_1}{dt} \right) + O(\varepsilon^3) = 0$$

Equating the co-efficient of $\varepsilon$, it become

$$O(\varepsilon^0): \frac{d^2 z_0}{dt^2} + 2 = 0 \quad z_0(0) = 0, \quad \frac{dz_0}{dt}(0) = 2 \quad (8)$$

$$O(\varepsilon^1): \frac{d^2 z_1}{dt^2} + \frac{dz_0}{dt} = 0 \quad z_1(0) = 0, \quad \frac{dz_1}{dt}(0) = 0 \quad (9)$$

$$O(\varepsilon^2): \frac{d^2 z_2}{dt^2} + \frac{dz_1}{dt} = 0 \quad z_2(0) = 0, \quad \frac{dz_2}{dt}(0) = 0 \quad (10)$$

By solving the above equations we will get

$$z_0(t) = t - t^2 \quad ............(11)$$

$$z_1(t) = -\frac{t^2}{2} + \frac{t^3}{6} \quad ............(12)$$

$$z_2(t) = \frac{t^3}{6} - \frac{t^4}{24} \quad ............(13)$$

Putting these values in equation (7), we have the solution

$$z(t) = 2t - t^2 + \varepsilon \left( -\frac{t^2}{2} + \frac{t^3}{6} \right) + \varepsilon^2 \left( \frac{t^3}{6} - \frac{t^4}{24} \right) + O(\varepsilon^3).$$

**Example 3**

Consider

$$z'' + z - \varepsilon z = 0 \quad , \quad z(0) = c \quad , \quad z'(0) = 0 \quad (14)$$

**Solution :-**

To find the unperturbated solution, we substitute for all of $\varepsilon = 0$ in equation (14), we get

$$z'' + z = 0 \quad , \quad z(0) = D \quad , \quad z'(0) = 0$$

that we can solve exactly

$$z(t) = D \cos t$$

Now, we assume that the solution to the perturbed problem can be described by power series of $\varepsilon$:
\[ z(t) = z_0(t) + \varepsilon z_1(t) + O(\varepsilon^2) \quad (15) \]

Substitute Equation (15) in (14)

\[ z''_0(t) + \varepsilon z''_1(t) + z_0(t) + \varepsilon z_1(t) - \varepsilon(z_0'(t) + \varepsilon z_1(t)) = 0 \]
\[ z''_0(t) + z_0(t)) + \varepsilon( z''_1(t) + z_1(t) - z_0(t)) + O(\varepsilon^2) = 0 \]

Equating the co-efficient of \( \varepsilon \), it becomesz

\[ O(\varepsilon^0): z'_0(t) + z_0(t) = 0 \quad , z_0(0) = D \quad , z'_0(0) = 0 \quad (16) \]
\[ O(\varepsilon^1): z''_1(t) + z_1(t) - z_0(t) = 0 \quad , z_1(0) = z'_1(0) = 0 \quad (17) \]

By solving the above equations we will get

\[ z_0(t) = D \cos t \]
\[ z_1(t) = \frac{1}{2} D z_0(t) \sin t \]

Putting these values in equation (15), we have the solution

\[ z(t) = D \cos t + \frac{1}{2} D \varepsilon z_0(t) \sin t. \]

4. Conclusion and recommendations

In this paper the concept of perturbation theory is discussed with the application of asymptotic expansion method for some models of regularly perturbed problems involving a small parameter \( \varepsilon \). As convergent solutions were obtained about whether the problem was confused or not when applying this method to the ordinary differential equations of the first and second order, which in turn gave us very good results resulting from regularly perturbed problems. Later researchers can apply this method to other types of higher-order equations.

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