Chordality and hyperbolicity of a graph*

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Abstract

Let $G$ be a connected graph with the usual shortest-path metric $d$. The graph $G$ is $\delta$-hyperbolic provided for any vertices $x, y, u, v$ in it, the two larger of the three sums $d(u, v) + d(x, y)$, $d(u, x) + d(v, y)$ and $d(u, y) + d(v, x)$ differ by at most $2\delta$. The graph $G$ is $k$-chordal provided it has no induced cycle of length greater than $k$. Brinkmann, Koolen and Moulton find that every 3-chordal graph is 1-hyperbolic and is not $\frac{1}{2}$-hyperbolic if and only if it contains one of two special graphs as an isometric subgraph. For every $k \geq 4$, we show that a $k$-chordal graph must be $\left\lfloor \frac{k}{2} \right\rfloor$-hyperbolic and there does exist a $k$-chordal graph which is not $\left\lfloor \frac{k-2}{2} \right\rfloor$-hyperbolic. Moreover, we prove that a 5-chordal graph is $\frac{1}{2}$-hyperbolic if and only if it does not contain any of a list of six special graphs (See Fig. 3) as an isometric subgraph.

Keywords: Chordality; hyperbolicity.

1 Introduction

1.1 Tree-likeness

Trees are graphs with some very distinctive and fundamental properties and it is legitimate to ask to what degree those properties can be transferred to more general structures that are tree-like in some sense [32, p. 253]. Roughly speaking, tree-likeness stands for something related to low dimensionality, low complexity, efficient information deduction (from local to global), information-lossless decomposition (from global into simple pieces) and nice shape for efficient implementation of divide-and-conquer strategy. For the very basic interconnection structures like a graph or a hypergraph, tree-likeness is naturally reflected by the strength of interconnection, namely its connectivity/homotopy type or cyclicity/acyclicity, or just the degree of derivation from some characterizing conditions of a tree/hypertree and its various associated structures and generalizations.

In vast applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with [21]. A support to this from the fixed-parameter complexity point of view is the observation that on various tree-structures we can design very good algorithms for many purposes and these algorithms can somehow be lifted to tree-like structures [1, 36, 37, 75]. It is thus very useful to get information on approximating general structures by tractable structures, namely tree-like structures. On the other hand, one not only finds it natural that tree-like structures appear extensively in many fields, say biology [45], structured programs [90] and database theory [47], as graphical representations of various types of hierarchical relationships, but also notice surprisingly that many practical structures we encounter are just tree-like, say the internet [1, 73, 88] and chemical compounds [95]. This prompts in many areas the very active study of tree-like structures. Especially, lots of ways to define/measure a tree-like structure have been proposed in the literature from many different considerations, just to name a few, say tree-width [55, 56], tree-length [55, 92], combinatorial

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1.2 Chordality

We only consider simple, unweighted, connected, but not necessarily finite graphs. Any graph $G$ together with the usual shortest-path metric on it, $d_G : V(G) \times V(G) \to \{0, 1, 2, \ldots \}$, gives rise to a metric space. We often suppress the subscript and write $d(x, y)$ instead of $d_G(x, y)$ when the graph is known by context. Moreover, we may use the shorthand $xy$ for $d(x, y)$ to further simplify the notation. Note that a pair of vertices $x$ and $y$ form an edge if and only if $xy = 1$. For $S, T \subseteq V(G)$, we write $d(S, T)$ for $\min_{x \in S, y \in T} d(x, y)$. We often omit the brackets and adopt the convention that $x$ stands for the singleton set \{x\} when no confusion can be caused.

Let $G$ be a graph. A walk of length $n$ in $G$ is a sequence of vertices $x_0, x_1, x_2, \ldots, x_n$ such that $x_{i-1}x_i = 1$ for $i = 1, \ldots, n$. If these $n+1$ vertices are pairwise different, we call the sequence a path of length $n$. A pseudo-cycle of length $n$ in $G$ is a cyclic sequence of $n$ vertices $x_1, \ldots, x_n \in V(G)$ such that $x_ix_j = 1$ whenever $j = i + 1 \pmod n$: we will reserve the notation $[x_1x_2 \cdots x_n]$ for this pseudo-cycle. We call this pseudo-cycle an $n$-cycle, or a cycle of length $n$, if $x_1, \ldots, x_n$ are $n$ different vertices. A chord of a path or cycle is an edge joining nonconsecutive vertices on the path or cycle. An odd chord of a cycle of even length is a chord connecting different vertices the distance between which in the cycle is odd. A cycle without chord is called an induced cycle, or a chordless cycle. For any $n \ge 3$, the $n$-cycle graph is the graph with $n$ vertices which has a chordless $n$-cycle and we denote this graph by $C_n$. A subgraph $H$ of a graph $G$ is isometric if for any $u, v \in V(H)$ it holds $d_H(u, v) = d_G(u, v)$. A 4-cycle of a graph $G$ is
an isometric 4-cycle provided the subgraph of $G$ induced by the vertices of this cycle is isometric and the subgraph has only those four edges which are displayed in the cycle. Indeed, this amounts to saying that this cycle is an induced/chordless cycle; c.f. Lemma 39.

We say that a graph is $k$-chordal if it does not contain any induced $n$-cycle for $n > k$. Clearly, trees are nothing but 2-chordal graphs. A 3-chordal graph is usually termed as a chordal graph and a 4-chordal graph is often called a hole-free graph. The class of $k$-chordal graphs is also discussed under the name $k$-bounded-hole graphs [59].

The chordality of a graph $G$ is the smallest integer $k \geq 2$ such that $G$ is $k$-chordal [11]. Following [11], we use the notation $\mathcal{I}(G)$ for this parameter as it is merely the length of the longest chordless cycle in $G$ when $G$ is not a tree. Note that our use of the concept of chordality is basically the same as that used in [18, 19] but is very different from the usage of this term in [81].

The recognition of $k$-chordal graphs is coNP-complete for $k = \Theta(n^\epsilon)$ for any constant $\epsilon > 0$ [91]. Especially, to determine the chordality of the hypercube is attracting much attention under the name of the snake-in-the-box problem due to its connection with some error-checking codes problem [72]. Just like the famous snake-in-the-box problem, it looks hard to determine the exact value of the chordality of general grid graphs – it is only easy to see that $\mathcal{I}(C_{m,n})$ should be roughly proportional to $nm$ when $\min(n,m) > 2$. Nevertheless, just like many other tree-likeness parameters, quite a few natural graph classes are known to have small chordality [11]. We review some 5-chordal (4-chordal) graphs in the remainder of this subsection.

An asteroidal triple (AT) of a graph $G$ is a set of three vertices of $G$ such that for any pair of them there is a path connecting the two vertices whose distance to the remaining vertex is at least two. A graph is AT-free if no three vertices form an AT [15, p. 114]. Obviously, all AT-free graphs are 5-chordal. A graph is an interval graph exactly when it is both chordal and AT-free [15, Theorem 7.2.6]. AT-free graphs also include cocomparability graphs [15, Theorem 7.2.7]; moreover, all bounded tolerance graphs are cocomparability graphs [56, 57, Theorem 2.8] and a graph is a permutation graph if and only if itself and its complement are cocomparability graphs [15, Theorem 4.7.1]. An important subclass of cocomparability graphs is the class of threshold graphs, which are those graphs without any induced subgraph isomorphic to the 4-cycle, the complement of the 4-cycle or the path of length 3 [57, p. 23].

A graph is weakly chordal [56, 63] when both itself and its complement are 4-chordal. Note that all tolerance graphs [57] are domination graphs [57] and all domination graphs are weakly chordal [29]. A graph is strongly chordal if it is chordal and if every even cycle of length at least 6 in this graph has an odd chord [56, p. 21]. A graph is distance-hereditary if each of its induced paths, and hence each of its connected induced subgraphs, is isometric [55]. We call a graph a cograph provided it does not contain any induced path of length 3 [15, Theorem 11.3.3]. It is easy to see that each cograph is distance-hereditary and all distance-hereditary graphs form a proper subclass of 4-chordal graphs. It is also known that cocomparability graphs are all 4-chordal [11, 50].

1.3 Hyperbolicity

1.3.1 Definition and background

For any vertices $x, y, u, v$ of a graph $G$, put $\delta_G(x,y,u,v)$, which we often abbreviate to $\delta(x,y,u,v)$, to be the difference between the largest and the second largest of the following three terms:

$$\frac{uv + xy}{2}, \frac{ux + vy}{2}, \text{and} \frac{uy + vx}{2}.$$ 

Clearly, $\delta(x,y,u,v) = 0$ if $x, y, u, v$ are not four different vertices. A graph $G$, viewed as a metric space as mentioned above, is $\delta$-hyperbolic (or tree-like with defect at most $\delta$) provided for any vertices $x, y, u, v$ in $G$ it holds $\delta(x,y,u,v) \leq \delta$ and the (Gromov) hyperbolicity of $G$, denoted $\delta^*(G)$, is the minimum half integer $\delta$ such that $G$ is $\delta$-hyperbolic [13, 16, 25, 26, 31, 61]. Note that it may happen $\delta^*(G) = \infty$. But for a finite graph $G$, $\delta^*(G)$ is clearly finite and polynomial time computable.

Note that in some earlier literature the concept of Gromov hyperbolicity is used a little bit different from what we adopt here; what we call $\delta$-hyperbolic here is called $2\delta$-hyperbolic in [11, 16, 17, 25, 42, 45].
and hence the hyperbolicity of a graph is always an integer according to their definition. We also refer to \[2, 13, 16, 93\] for some equivalent and very accessible definitions of Gromov hyperbolicity which involve some other comparable parameters.

The concept of hyperbolicity comes from the work of Gromov in geometric group theory which encapsulates many of the global features of the geometry of complete, simply connected manifolds of negative curvature \[16, \text{p. } 398\]. This concept not only turns out to be strikingly useful in coarse geometry but also becomes more and more important in many applied fields like networking and phylogenetics \[24, 25, 26, 27, 29, 11, 12, 13, 15, 53, 65, 66, 70, 73, 78\]. The hyperbolicity of a graph is a way to measure the additive distortion with which every four-points sub-meter of the given graph metric embeds into a tree metric \[1\]. Indeed, it is not hard to check that the hyperbolicity of a tree is zero – the corresponding condition for this is known as the four-point condition (4PC) and is a characterization of general tree-like metric spaces \[41, 45, 67\]. Moreover, the fact that hyperbolicity is a tree-likeness parameter is reflected in the easy fact that the hyperbolicity of a graph is the maximum hyperbolicity of its 2-connected components – This observation implies the classical result that 0-hyperbolic graphs are exactly block graphs, namely those graphs in which every 2-connected subgraph is complete, which are also known to be those diamond-free chordal graphs \[9, 43, 65\]. More results on bounding hyperbolicity of graphs and characterizing low hyperbolicity graphs can be found in \[7, 8, 17, 24, 25, 35, 74\]; we will only report in Section \[2\] some work most closely related to ours and refer the readers to corresponding references for many other interesting unaddressed work.

For any vertex \(u \in V(G)\), the Gromov product, also known as the overlap function, of any two vertices \(x, y\) of \(G\) with respect to \(u\) is equal to \(\frac{1}{2}(xu + yu - xy)\) and is denoted by \(\langle x, y \rangle_u\) \[16, \text{p. } 410\]. As an important context in phylogenetics \[42, 43, 49\], for any real number \(\rho\), the Farris transform based at \(u\), denoted \(D_{\rho, u}\), is the transformation which sends \(d_G\) to the map

\[
D_{\rho, u}(d_G) : V(G) \times V(G) \rightarrow \mathbb{R} : (x, y) \mapsto \rho - \langle x, y \rangle_u.
\]

We say that \(G\) is \(\delta\)-hyperbolic with respect to \(u \in V(G)\) if the following inequality

\[
\langle x, y \rangle_u \geq \min((x \cdot v)_u, (y \cdot v)_u) - \delta
\]

holds for any vertices \(x, y, v\) of \(G\). It is easy to check that the inequality (1) can be rewritten as

\[
xy + uv \leq \max(xu + yu, xv + yu) + 2\delta
\]

and so we see that \(G\) is \(\delta\)-hyperbolic if and only if \(G\) is \(\delta\)-hyperbolic with respect to every vertex of \(G\). By a simple but nice argument, Gromov shows that \(G\) is \(2\delta\)-hyperbolic provided it is \(\delta\)-hyperbolic with respect to any given vertex \[2, \text{Proposition } 2.2\] \[61, 1.1B\].

The tree-length \[34, 35, 80, 92\] of a graph \(G\), denoted \(t(G)\), is the minimum integer \(k\) such that there is a chordal graph \(G'\) satisfying \(V(G) = V(G')\), \(E(G) \subseteq E(G')\) and \(\max(d_G(u, v) : d_{G'}(u, v) = 1) = k\). We use the convention that the tree-length of a graph without any edge is 1. It is straightforward from the definition that chordal graphs are exactly the graphs of tree-length 1. It is also known that \(AT\)-free graphs and distance-hereditary graphs have tree-length at most \[2, \text{p. } 367\]; a way to see this is to use the forthcoming result relating chordality and tree-length as well as the fact that \(AT\)-free graphs are 5-chordal and distance-hereditary graphs are 4-chordal.

**Theorem 1** \[51, \text{Lemma } 6\] \[52, \text{Theorem } 3.3\] \textit{If } \(G\) \textit{is a } \(k\)-chordal graph, \textit{then } \(t(G) \leq \lfloor \frac{k}{2} \rfloor\). \textit{Proof: [Outline]} To obtain a minimal triangulation of \(G\), it suffices to select a maximal set of pairwise parallel minimal separators of \(G\) and add edges to make each of them a clique \[83, \text{Theorem } 4.6\]. It is easy to check that each such new edge connects two points of distance at most \(\lfloor \frac{k}{2} \rfloor\) apart in \(G\).

The following is an interesting extension of the classical result that trees are 0-hyperbolic and its proof can be given in a way generalizing the well-known proof of the latter fact.

**Theorem 2** \[24, \text{Proposition } 13\] \textit{A graph } \(G\) \textit{is } \(k\)-hyperbolic \textit{provided its tree-length is no greater than } \(k\).
It is noteworthy that a converse of Theorem 2 has also been established, which means that hyperbolicity and tree-length are comparable parameters of tree-likeness.

**Theorem 3** [24, Proposition 14] The inequality $D(G) \leq 12k + 8k \log_2 n + 17$ holds for any $k$-hyperbolic graph $G$ with $n$ vertices.

### 1.3.2 Three examples

Let us try our hand at three examples to get a feeling of the concept of hyperbolicity. The first example says that graphs with small diameter, hence those so-called small-world networks, must have low hyperbolicity. Note that additionally similar simple results will be reported as Lemmas 43 and 50.

**Example 4** [74, p. 683] The hyperbolicity of a graph $G$ with diameter $D$ is at most $\lfloor D^2 \rfloor$.  

**Proof:** Take $x, y, u, v \in V(G)$. Our goal is to show that $\delta(x, y, u, v) \leq D$. Without loss of generality, assume that $xy + vu \geq xu + yv \geq xv + yu$. (2) and hence $\delta(x, y, u, v) = \frac{1}{2}((xy + uv) - (xu + yv))$. (3) In the first place, we have $xu + yv \geq xy, xu + vx \geq uv, xv + yv \geq xy, vy + uy + uv$. Summing up these inequalities yields $(xu + yv) + (xv + yu) \geq xy + uv$, which, according to Eq. (2), implies that $xu + yv \geq \frac{1}{2}(xy + uv)$. This along with Eq. (3) gives $\delta(x, y, u, v) \leq \frac{1}{4}(xy + uv) \leq \frac{D}{4}$. Moreover, if $\delta(x, y, u, v) = \frac{D}{4}$, then we have $xu + yv = D, xv + yv = xy = D, xu + xv = uv = D$. (4) By adding the equalities in Eq. (4) together, we see that $3D = 2(xu + xv + yv)$ and so $D$ must be even. □

The bound asserted by Example 4 is clearly not tight when $D = 1$. But, as can be seen from the next example, the bound given in Example 4 in terms of the diameter $D$ is best possible for every $D \geq 2$. Note that this forthcoming example can also be seen directly via Example 4 as indicated in [74, p. 683].

**Example 5** [74, p. 683] For any $n \geq 3$, the chordality of the $n$-cycle is $n$ while the hyperbolicity of the $n$-cycle is $\delta^*(C_n) = \begin{cases} \lfloor \frac{n}{4} \rfloor - \frac{1}{2}, & \text{if } n \equiv 1 \pmod{4}; \\ \lfloor \frac{n}{4} \rfloor, & \text{else}. \end{cases}$ (5) Note that the diameter of $C_n$ is $\lfloor \frac{n}{2} \rfloor$ and $\delta^*(C_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor, & \text{if } n \equiv 0 \pmod{4}; \\ \frac{\lfloor \frac{n}{2} \rfloor}{2} - \frac{1}{2}, & \text{else}. \end{cases}$
At the moment, we see that there are only two possibilities, either
\[ xy \] 
\[ xu \]

It follows that
\[ xu + vυ \geq |xυ - uy| \]

This implies \( xu + vy \geq xv + vy \) and \( vx + xu \geq vy + yu \). According to the geometric distribution of the four points, we then come to
\[ xy = xv + vy \quad \text{and} \quad vu = vυ + yu. \]

It follows that
\[ xy + vu = (xυ + yυ) + 2vy \]

and
\[ xy + vu = xv + vy + vy + yu = (xυ + vy + yυ) + vy \geq xu + vy. \]

At the moment, we see that there are only two possibilities, either \( xy + vu \geq xu + vy > xv + yυ \) or \( xy + vu \geq xv + yυ \geq xu + vy. \)

If the first case happens, we have
\[
\delta(x, y, u, v) = \frac{1}{2}(xy + vu - xu - vy) = \frac{n}{2} - xu. \quad \text{(By Eqs. 10 and 11)}
\]

By Eqs. 13 and 7 and \( xu + vy > xv + yυ \), we see that \( xu \geq \left\{ \left\lceil \frac{n}{4} \right\rceil + 1, \quad \text{if } n \equiv 0, 1, 2 \pmod{4} \right\} \)

and hence Eq. 13 forces
\[
\delta(x, y, u, v) = \frac{n}{2} - xu \leq \begin{cases} 
\left\lceil \frac{n}{4} \right\rceil - 1, & \text{if } n \equiv 0 \pmod{4}; \\
\left\lceil \frac{n}{4} \right\rceil, & \text{if } n \equiv 2 \pmod{4}; \\
\left\lceil \frac{n}{4} \right\rceil - \frac{1}{2}, & \text{if } n \equiv 1, 3 \pmod{4}. 
\end{cases}
\] (10)

For the second case, we have
\[
\delta(x, y, u, v) = \frac{1}{2}(xy + vu - xv - yυ) = vy, \quad \text{(By Eq. 5)}
\]

and hence by Eqs. 10 and 7 and \( xv + yυ \geq vy + xu \), we further obtain
\[
\delta(x, y, u, v) = vy \leq \begin{cases} 
\left\lceil \frac{n}{4} \right\rceil, & \text{if } n \equiv 0, 2, 3 \pmod{4}; \\
\left\lceil \frac{n}{4} \right\rceil - 1, & \text{if } n \equiv 1 \pmod{4}. 
\end{cases}
\] (11)

Combining Eqs. 10 and 11 yields
\[
\delta(x, y, u, v) \leq \begin{cases} 
\left\lceil \frac{n}{4} \right\rceil, & \text{if } n \equiv 0, 2, 3 \pmod{4}; \\
\left\lceil \frac{n}{4} \right\rceil - \frac{1}{2}, & \text{if } n \equiv 1 \pmod{4}. 
\end{cases}
\] (12)

Taking
\[
(x, y, u, v) = \begin{cases} 
(c_0, c_k, c_{2k}, c_{3k}), & \text{if } n \equiv 0 \pmod{4}, \\
(c_0, c_k, c_{2k}, c_{3k}), & \text{if } n \equiv 1 \pmod{4}, \\
(c_0, c_k, c_{2k}, c_{3k+1}), & \text{if } n \equiv 2 \pmod{4}, \\
(c_0, c_{k+1}, c_{2k+1}, c_{3k+2}), & \text{if } n \equiv 3 \pmod{4}, 
\end{cases}
\]
we see that Eq. (12) is tight and hence Eq. (5) is established. \hfill \Box

For any two graphs $G_1$ and $G_2$, we define its Cartesian product $G_1 \square G_2$ to be the graph satisfying $V(G_1 \square G_2) = V(G_1) \times V(G_2)$ and $d_{G_1 \square G_2}((u_1, u_2), (v_1, v_2)) = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)$ [58 §1.4].

**Example 6** Let $G_1$ and $G_2$ be two graphs satisfying $\delta^*(G_1) = \delta^*(G_2) = 0$. Then $\delta^*(G_1 \square G_2) = \min(D_1, D_2)$ where $D_1$ and $D_2$ are the diameters of $G_1$ and $G_2$, respectively.

**Proof:** For any $v \in V(G_1 \square G_2)$, we often use the convention that $v = (v_1, v_2)$ for $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. For any $u, v \in V(G_1 \square G_2)$, we write $uv$ for $d_{G_1 \square G_2}(u, v)$, $(uv)_1$ for $d_{G_1}(u, v_1)$, $(uv)_2$ for $d_{G_2}(u_2, v_2)$ and we use $\delta$ for $d_{G_1 \square G_2}$.

Take $a, b \in V(G_1)$ such that $d_{G_1}(a, b) = D_1$ and take $c, d \in V(G_2)$ such that $d_{G_2}(c, d) = D_2$. Let $x = (a, c), y = (a, d), u = (b, c), v = (b, d)$. It is straightforward that $\delta(x, y, u, v) = \min(D_1, D_2)$.

To complete the proof, we pick any four vertices $x, y, u, v$ of $G_1 \square G_2$ and aim to show that

$$\delta(x, y, u, v) \leq \min(D_1, D_2).$$  \hspace{1cm} (13)

Let $A = xy + uv, A_1 = (xy)_1 + (uv)_1, A_2 = (xy)_2 + (uv)_2, B = xu + yv, B_1 = (xu)_1 + (yv)_1, B_2 = (xu)_2 + (yv)_2, C = xv + yu, C_1 = (xv)_1 + (yu)_1, C_2 = (xv)_2 + (yu)_2$. Because $\delta^*(G_1) = \delta^*(G_2) = 0$, we can suppose $A_1 = \max(B_1, C_1)$ and $A_2 = \max(B_2, C_2)$.

If it happens either $(A_1, A_2) = (B_1, B_2)$ or $(A_1, A_2) = (C_1, C_2)$, we can immediately conclude that $\delta(x, y, u, v) = 0$. By symmetry between $B$ and $C$ and between $G_1$ and $G_2$, it thus remains to deduce Eq. (13) under the condition that

$$B \geq C, A_1 = B_1 > C_1, \text{ and } A_2 = C_2 < B_2.$$  

Since $A_1 = B_1$, we have $\delta(x, y, u, v) = \frac{A_1B_2 - A_2B_1}{2}$. We proceed with a direct computation and find

$$\delta(x, y, u, v) = \frac{((xy)_2 - (xu)_2) + ((uv)_2 - (yu)_2)}{2} \leq (yu)_2 \leq D_2.$$  \hspace{1cm} (14)

Making use of $A_2 = C_2$ and $B \geq C$, we can obtain instead

$$\delta(x, y, u, v) \leq \frac{A_2-B_2+B-C}{2} = \frac{A_2-C_2+B_1-B_2}{2} = \frac{B_1-C_1}{(xu)_1 + (yu)_1} \leq (uv)_1 \leq D_1.$$  \hspace{1cm} (15)

Combining Eqs. (14) and (15) we now get Eq. (13), as desired. \hfill \Box

**Remark 7** For any $t$ natural numbers $m_1, \ldots, m_t$, the $t$-dimensional grid graph $G_{m_1, \ldots, m_t}$ is the graph with vertex set $\{1, 2, \ldots, m_1\} \times \cdots \times \{1, 2, \ldots, m_t\}$ and $(i_1, \ldots, i_t)$ and $(j_1, \ldots, j_t)$ are adjacent in $G_{m_1, \ldots, m_t}$ if any only if $\sum_{p=1}^t (i_p - j_p)^2 = 1$. Example 6 implies that $\delta^*(G_{m_1, m_2}) = \min(m_1, m_2) - 1$ and hence $G_{m, m}$ provides another example that the bound reported in Example 4 is tight. It might be interesting to determine the hyperbolicity of $t$-dimensional grid graphs for $t \geq 3$.

**Remark 8** Dourisboure and Gavoille show that the tree-length of $G_{n, m}$ is $\min(n, m)$ if $n \neq m$ or $n = m$ is even and is $n - 1$ if $n = m$ is odd [52, Theorem 3]. Remark 7 tells us that $\delta^*(G_{n, m}) = \min(m, n) - 1$. This says that Theorem 4 is tight.

**2 Chordality vs. hyperbolicity**

**2.1 Main results**

Firstly, we point out that a graph with low hyperbolicity may have large chordality. Indeed, take any graph $G$ and form the new graph $G'$ by adding an additional vertex and connecting this new vertex with
every vertex of $G$. It is obvious that $\delta^*(G') \leq 1$ while $\text{Ic}(G') = \text{Ic}(G)$ if $G$ is not a tree. Moreover, it is
equally easy to see that $G'$ is even $\frac{1}{2}$-hyperbolic if $G$ does not have any induced 4-cycle [74, p. 695]. Surely,
this example does not preclude the possibility that for many important graph classes we can bound their
chordality in terms of their hyperbolicity.

One of our main results says that hyperbolicity can be bounded from above in terms of chordality.

**Theorem 9** For each $k \geq 4$, all $k$-chordal graphs are $\frac{k}{2}$-hyperbolic.

**Remark 10** A graph is bridged [33, 77] if it does not contain any finite isometric cycles of length at least
four, or equivalently, if it is cop-win and has no chordless cycle of length 4 or 5. In contrast to Theorem
§5.2 make the remark that “a characterization of all 1-hyperbolic graphs by forbidden isometric subgraphs is not in sight, in as much as isometric cycles of lengths up to 7 may occur, thus complicating the picture”. Note that our Theorem says that all 5-chordal graphs are
1-hyperbolic and hence the appearance of those chordless 6-cycles and chordless 7-cycles may be a real
headache to deal with in pursuing a characterization of all 1-hyperbolic graphs.

**Example 12** For any $t \geq 2$ we set $F_t$ to be the graph obtained from the $4t$-cycle $[v_1 v_2 \cdots v_{4t}]$ by adding the
two edges $\{v_1, v_3\}$ and $\{v_{2t+1}, v_{2t+3}\}$; see Fig. 2 for an illustration of $F_2$. Clearly, $\delta(v_2, v_{t+2}, v_{2t+2}, v_{3t+2}) = t-\frac{1}{2}$. Furthermore, we can check that $\text{Ic}(F_t) = 4t-2$ and $\delta^*(F_t) = t-\frac{1}{2} = \delta(v_2, v_{t+2}, v_{2t+2}, v_{3t+2}) = \frac{\text{Ic}(F_t)}{4}$.

$F_t$ is clearly an outerplanar graph. Thus, applying the result that $\text{Ic}(G) = \lceil \frac{\text{Ic}(G)}{4} \rceil$ for every outerplanar
graph $G$ [33, Theorem 1], we even know that $\text{Ic}(F_t) = \lceil \frac{4t-2}{4} \rceil$.

It is clear that if the bound claimed by Theorem is tight for $k = 4t$ ($k = 4t - 2$) then it is tight
for $k = 4t + 1$ ($k = 4t - 1$). Consequently, Examples 5 and 12 indeed mean that the bound reported in
Theorem is tight for every $k \geq 4$. Surely, the logical next step would be to characterize all those extremal
graphs $G$ satisfying

$$\delta^*(G) = \frac{\text{Ic}(G)}{2}. \quad (16)$$

However, there seems to be still a long haul ahead in this direction.

**Remark 13** For any graph $G$ and any positive number $t$, we put $S^t(G)$ to be a subdivision graph of
$G$, which is obtained from $G$ by replacing each edge $\{u, v\}$ of $G$ by a path $u, n_{u,v}^1, \ldots, n_{u,v}^{t-1}, v$ of length $t$
connecting $u$ and $v$ through a sequence of new vertices $n_{u,v}^1, \ldots, n_{u,v}^{t-1}$ (we surely require that $n_{u,u}^{t-g} = n_{u,v}^{t-g}$). For any four vertices $x, y, u, v \in V(G)$, we obviously have $\delta_{S^t(G)}(x, y, u, v) = t\delta_G(x, y, u, v)$ and so
$$\delta^*(S^t(G)) \geq t\delta^*(G).$$

Instead of the trivial fact $\text{Ic}(S^t(G)) \geq t\text{Ic}(G)$, if the good shape of $G$ permits us to deduce a good upper bound of $\text{Ic}(S^t(G))$ in terms of $\text{Ic}(G)$, we will see that $\delta^*(S^t(G))$ is high relative to $\text{Ic}(S^t(G))$ provided so is $G$. Recall that the cycles whose lengths are divisible by 4 as discussed in Example are used to demonstrate the tightness of the bound given in Theorem also observe that the graphs
suggested by Example 12 is nothing but a slight “perturbation” of cycles of length divisible by 4. Since
$C_4 = S^1(C_4)$, these examples can be said to be generated by the “seed” $C_4$. It might deserve to look for
some other good “seeds” from which we can use the above subdivision operation or its variant to produce
graphs satisfying $\text{Eq. (16)}$. Returning to Remark 13, it is natural to investigate if some graphs mentioned in Theorem 14 besides
$C_4$ can be used as “good seeds”. The next example comes from Gavoille [54].

Example 15 [54] Let $t, q$ be two positive integers with $q < t$ and let $H_2$ be the graph shown in the
upper-right corner of Fig. 3. We construct a planar graph $G_{4t}^q$ from $S^1(H_2)$ as follows: let $u_a = n_{u,a}^{q-1}$, $u_c = n_{u,c}^{q-1}$, $y_c = n_{y,c}^{q-1}$, $y_d = n_{y,d}^{q-1}$, $v_d = n_{v,d}^{q-1}$, $v_b = n_{v,b}^{q-1}$, $x_b = n_{x,b}^{q-1}$, $x_a = n_{x,a}^{q-1}$, and then add the
new edges $\{u_a, u_c\}$, $\{x_a, x_b\}$, $\{y_c, y_d\}$, $\{v_b, v_d\}$ to $S^1(H_2)$; see Fig. 4. It can be checked that $C = [u_a \cdots a \cdots x_a x_b \cdots b \cdots v_b v_d \cdots d \cdots y_d y_c \cdots c \cdots u_c]$ is an isometric 4t-cycle of $G_{4t}^q$ and that $\text{lc}(G_{4t}^q) = 4t$. It is also easy to see that $\delta^*(G_{4t}^q) = t$ and thus Theorem 2 tells us that $\delta^*(G_{4t}^q) = t$.

Motivated by the above construction of Gavoille, we construct the next graph family whose chordality
parameters are 1 modulo 4.

Example 16 By deleting the edge $\{y_c, n_{d,y}^{q-1}\}$ and adding a new edge $\{y_c, n_{a,y}^{q-1}\}$, we obtain from $G_{4t}^q$ a graph $G_{4t+1}^q$. Using similar analysis like Example 15 we find that $\text{lc}(G_{4t+1}^q) = 4t + 1$ and $\delta^*(G_{4t+1}^q) = t = \left\lceil \frac{t+1}{2} \right\rceil$. 

Figure 3: Six 5-chordal graphs with hyperbolicity 1.
Figure 4: $G^q_{6t}$.

Figure 5: $G^q_{6(2t+1)}$.

Similar constructions using $F_2$ (see Fig. 2) as the “seed” will lead to corresponding extremal graphs whose chordality parameters are 2 or 3 modulo 4.

Example 17 Let $t > q$ be two positive integers. We construct an outerplanar graph $G^q_{6(2t+1)}$ by adding two new edges $\{v_{21}, v_{23}\}$ and $\{v_{65}, v_{67}\}$ to the graph $S^t(F_2)$ where $v_{21} = n^q_{v_{22}, v_1}$, $v_{23} = n^q_{v_{23}, v_2}$, $v_{65} = n^q_{v_{66}, v_5}$, $v_{67} = n^q_{v_{67}, v_6}$; see Fig. 5 for an illustration. It is not hard to check that $\delta^+(G^q_{6(2t+1)}) = 6(2t + 1)$ and $\delta^*(G^q_{6(2t+1)}) = 3t + \frac{3}{2}$. Moreover, if we replace the edge $\{v_{21}, v_{23}\}$ by the edge $\{v_{21}, v^q_{32}, v_2\}$, then we obtain from $G^q_{6(2t+1)}$ another outerplanar graph $G^q_{6(2t+1)+1}$ for which we have $\delta^+(G^q_{6(2t+1)+1}) = 6(2t + 1) + 1$ and $\delta^*(G^q_{6(2t+1)+1}) = 3t + \frac{3}{2}$.

Let $C_6, G_1, G_2, G_3$ be the graphs depicted in Fig. 6. It is clear that $G_1, G_2, G_3, C_4, C_6, H_i, i = 1, \ldots, 5$, are 6-chordal graphs with hyperbolicity 1.

Conjecture 18 A 6-chordal graph is $\frac{1}{2}$-hyperbolic if and only if it does not contain any of a list of ten special graphs $G_1, G_2, G_3, C_4, C_6, H_i, i = 1, \ldots, 5$, as an isometric subgraph.

Let $E_1$ and $E_2$ be the graphs depicted in Fig. 7. In comparison with Conjecture 18 when we remove the 6-chordal restriction, we can present the following characterization of all $\frac{1}{2}$-hyperbolic graphs obtained by Bandelt and Chepoi [7]. We refer to [7, Fact 1] for two other characterizations; also see [18, 59].

Theorem 19 [7, p. 325] A graph $G$ is $\frac{1}{2}$-hyperbolic if and only if $G$ does not contain isometric $n$-cycles for any $n > 5$, for any two vertices $x$ and $y$ of $G$ one cannot find two non-adjacent neighbors of $x$ which are both closer to $y$ in $G$ than $x$, and none of the six graphs $H_1, H_2, G_1, G_2, E_1, E_2$ occurs as an isometric subgraph of $G$.

Remark 20 Instead of Theorem 19, it would be interesting to determine, if possible, a finite list of graphs such that a graph is $\frac{1}{2}$-hyperbolic if and only if it does not include any graph from that list as an isometric subgraph. Koolen and Moulton point out a possible approach to deduce such kind of a characterization in [74, p. 696].
Figure 6: Four graphs with hyperbolicity 1 and chordality 6.

Figure 7: Two bridged graphs with hyperbolicity 1.
Note that a 5-chordal graph cannot contain any isometric $n$-cycle for $n > 5$. It is also easy to see that $\kappa(G_1) = \kappa(G_2) = 6, \kappa(E_1) = 7, \kappa(E_2) = 8$. Therefore, we obtain the following easy consequence of Theorem [13]. It is interesting to compare it with Theorems [9] and [14].

**Corollary 21** A 5-chordal graph $G$ is $\frac{1}{2}$-hyperbolic if and only if it does not contain the graph $H_1$ and $H_2$ as isometric subgraphs and for any two vertices $x$ and $y$ of $G$ the neighbors of $x$ which are closer to $y$ than $x$ are pairwise adjacent.

### 2.2 Some consequences

Note that $\kappa(C_4) = 4, \kappa(H_1) = \kappa(H_2) = 3, \kappa(H_3) = \kappa(H_4) = \kappa(H_5) = 5$. The next two results follow immediately from Theorem [14].

**Corollary 22** Every 4-chordal graph must be 1-hyperbolic and it has hyperbolicity one if and only if it contains one of $C_4, H_1$ and $H_2$ as an isometric subgraph.

**Corollary 23** [17, Theorem 1.1] Every chordal graph is 1-hyperbolic and it has hyperbolicity one if and only if it contains either $H_1$ or $H_2$ as an isometric subgraph.

We remark that as long as every 4-chordal graph is 1-hyperbolic is known, Corollary 22 also immediately follows from Corollary 23. In addition, it is noteworthy that the first part of Corollary 23, namely every chordal graph is 1-hyperbolic is immediate from Theorem 2 as chordal graphs have tree-length 1.

**Corollary 24** Each weakly chordal graph is $\frac{1}{2}$-hyperbolic and has hyperbolicity one if and only if it contains one of $C_4, H_1, H_2$ as an isometric subgraph.

**Proof:** By definition, each weakly chordal graph is 4-chordal. It is also easy to check that that $C_4, H_1$ and $H_2$ are all weakly chordal. Hence, the result follows from Corollary 22.

**Corollary 25** All strongly chordal graphs are $\frac{1}{2}$-hyperbolic.

**Proof:** Note that the cycle $C = [x, a, u, c, y, d, v, b]$ in $H_1$ and $H_2$ does not have any odd chord and hence neither $H_1$ nor $H_2$ can appear as an isometric subgraph of a strongly chordal graph. Since strongly chordal graphs must be chordal graphs, this result holds by Corollary 23.

**Corollary 26** All threshold graphs are $\frac{1}{2}$-hyperbolic.

**Proof:** It is obvious that threshold graphs are chordal as they contain neither 4-cycle nor path of length 3 as induced subgraph. Since the subgraph induced by $x, u, b, c$ in either $H_1$ or $H_2$ is just the complement of $C_4$, the result follows from Corollary 23 and the definition of a threshold graph.

**Corollary 27** Every $AT$-free graph is 1-hyperbolic and it has hyperbolicity one if and only if it contains $C_4$ as an isometric subgraph.

**Proof:** First observe that an $AT$-free graph must be 5-chordal. Further notice that the triple $u, y, v$ is an $AT$ in any of the graphs $H_1, \ldots, H_5$. Now, an application of Theorem 14 concludes the proof.

**Corollary 28** A cocomparability graph is 1-hyperbolic and has hyperbolicity one if and only if it contains $C_4$ as an isometric subgraph.
Proof: We know that cocomparability graphs are $AT$-free and $C_4$ is a cocomparability graph. Thus the result comes directly from Corollary 27. The deduction of this result can also be made via Corollary 22 and the fact that cocomparability graphs are 4-chordal [11, 50]. □

Corollary 29 A permutation graph is 1-hyperbolic and has hyperbolicity one if and only if it contains $C_4$ as an isometric subgraph.

Proof: Every permutation graph is a cocomparability graph and $C_4$ is a permutation graph. So, the result follows from Corollary 28. □

Corollary 30 [8, p. 16] A distance-hereditary graph is always 1-hyperbolic and is $\frac{1}{2}$-hyperbolic exactly when it is chordal, or equivalently, when it contains no induced 4-cycle.

Proof: It is easy to see that distance-hereditary graphs must be 4-chordal and can contain neither $H_1$ nor $H_2$ as an isometric subgraph. The result now follows from Corollary 22. □

Corollary 31 A cograph is 1-hyperbolic and has hyperbolicity one if and only if it contains $C_4$ as an isometric subgraph.

Proof: We know that $C_4$ is a cograph and every cograph is distance-hereditary. Applying Corollary 30 yields the required result. □

3 Relevant tree-likeness parameters

3.1 Tree-length

It turns out that tree-length is a very useful concept for connecting chordality and hyperbolicity. Indeed, the following theorem, which can be read from Theorem 9 (Corollary 23), comes directly from Theorems 1 and 2. This result is firstly notified to us by Dragan [38] and is presumably in the folklore.

Theorem 32 For any $k \geq 3$, every $k$-chordal graph is $\lfloor \frac{k}{2} \rfloor$-hyperbolic.

In view of Remark 8 to get better estimate than Theorem 32 along the same approach one may try to beef up Theorem 1. We point out that Dourisboure and Gavoille [35, Question 1] posed as an open problem that whether or not

$$\sharp I(G) \leq \left\lfloor \frac{\text{ ecx}(G)}{3} \right\rfloor$$

is true. The $k$th-power of a graph $G$, denoted $G^k$, is the graph with $V(G)$ as vertex set and there is an edge connecting two vertices $u$ and $v$ if and only if $d_G(u, v) \leq k$. Let us interpret the problem of Dourisboure and Gavoille as a Chordal Graph Sandwich Problem:

Question 33 For any graph $G$, is there always a chordal graph $H$ such that $V(H) = V(G) = V(G^{\left\lfloor \frac{\text{ ecx}(G)}{3} \right\rfloor})$ and $E(G) \subseteq E(H) \subseteq G^{\left\lfloor \frac{\text{ ecx}(G)}{3} \right\rfloor}$ ?

If (17) can be established, it will be the best we can expect in the sense that $\sharp I(G) = \left\lfloor \frac{\text{ ecx}(G)}{3} \right\rfloor$ for every outerplanar graph $G$ [35, Theorem 1].
3.2 Approximating trees, slimness and thinness

We introduce in this subsection two general approaches to connect chordality with hyperbolicity. A result is given together with a proof only when that proof is short and when we do not find it appear very explicitly elsewhere. This section also aims to provide the reader a warm-up before entering the longer proof in the main part of this paper.

A result weaker than Theorem 9 (Theorem 32) and reported in [27, p. 64] as well as [26, p. 3] is that each \( k \)-chordal graph is \( k \)-hyperbolic. The two approaches to be reported below by far basically only lead to this weaker result. Despite of this, it might be interesting to see different ways of bounding hyperbolicity in terms of chordality via the use of some other intermediate tree-likeness parameters.

The first approach is to look at distance approximating trees. A tree \( T \) is a distance \( t \)-approximating tree of a graph \( G \) provided \( V(T) = V(G) \) and \( |d_G(u, v) - d_T(u, v)| \leq t \) for any \( u, v \in V(G) \) [6, 14, 23, 40]. It is well-known that a graph with a good distance approximating tree will have low hyperbolicity, which is briefly mentioned in [25, p. 3] and [27, p. 64] and is in the same spirit of a general result on hyperbolic geodesic metric spaces [16, p. 402, Theorem 1.9]. We make this point clear in the following simple lemma.

**Lemma 34** Let \( G \) be a graph and \( t \) be a nonnegative integer. If \( G \) has a distance \( t \)-approximating tree \( T \), then \( G \) is \( 2t \)-hyperbolic.

**Proof:** For any \( x, y, u, v \in V(G) \), our aim is to show that \( \delta_G(x, y, u, v) \leq 2t \). Assume, as we may, that \( d_G(x, y) + d_G(u, v) \leq d_G(x, u) + d_G(y, v) \geq d_G(x, v) + d_G(y, u) \). Since the tree metric \( d_T \) is a four-point inequality metric (or additive metric) [31], we know that \( \delta^*(T) = 0 \) and so the following three cases are exhaustive.

**Case 1:** \( d_T(x, y) + d_T(u, v) = d_T(x, u) + d_T(y, v) \geq d_T(x, v) + d_T(y, u) \).

\[
\delta_G(x, y, u, v) = \frac{1}{2}(d_G(x, y) + d_G(u, v)) - \frac{1}{2}(d_G(x, u) + d_G(y, v)) \leq \frac{1}{2}(d_T(x, y) + d_T(u, v) + 2t) - \frac{1}{2}(d_T(x, u) + d_T(y, v) - 2t) = 2t.
\]

**Case 2:** \( d_T(x, y) + d_T(u, v) = d_T(x, v) + d_T(y, u) \geq d_T(x, u) + d_T(y, v) \).

\[
\delta_G(x, y, u, v) = \frac{1}{2}(d_G(x, y) + d_G(u, v)) - \frac{1}{2}(d_G(x, u) + d_G(y, v)) \leq \frac{1}{2}(d_T(x, y) + d_T(u, v) + 2t) - \frac{1}{2}(d_T(x, v) + d_T(y, u) - 2t) = 2t.
\]

**Case 3:** \( d_T(x, v) + d_T(y, u) = d_T(x, u) + d_T(y, v) \).

\[
\delta_G(x, y, u, v) = \frac{1}{2}(d_G(x, y) + d_G(u, v)) - \frac{1}{2}(d_G(x, u) + d_G(y, v)) \leq \frac{1}{2}(d_T(x, y) + d_T(u, v) + 2t) - \frac{1}{2}(d_T(x, u) + d_T(y, v) - 2t) = 2t.
\]

After showing that the existence of good distance approximating tree guarantees low hyperbolicity, in order to connect chordality with hyperbolicity, we need to make sure that low chordality graphs have good distance approximating trees [14, 23]. Here is an exact result.

**Theorem 35** [23] Let \( G \) be a \( k \)-chordal graph. Then, there is a tree \( T \) with \( V(T) = V(G) \) such that for any \( u, v \in V(G) \) it holds

\[
|d_G(u, v) - d_T(u, v)| \leq \begin{cases} 
\frac{k}{2} + 2, & \text{if } k = 4, 5, \\
\frac{k}{2} + 1, & \text{else}.
\end{cases}
\]

The other possible approach to connect hyperbolicity and chordality is via the concept of the thinness/slimness of geodesic triangles. This approach also consists of two parts, one is to show that a graph with low thinness/slimness has low hyperbolicity, as summarized in [25, Proposition 1], and the other part is to show that low chordality implies low thinness/slimness.
Given a graph \( G \), we can put an orientation on it by choosing two maps \( \partial_0 \) and \( \partial_1 \) from \( E(G) \) to \( V(G) \) such that each edge \( e \) just have \( \partial_0(e) \) and \( \partial_1(e) \) as its two endpoints. The discrete metric space \((V(G), d_G)\) can then be naturally embedded into the metric graph \([\mathbf{16}] \text{ p. 7}\) \((X_G, \tilde{d}_G)\), where \( X_G \) is the quotient space of \( E(G) \times [0,1] \) under the identification of \((e, i)\) and \((e', i')\) whenever \( \partial_i(e) = \partial_{i'}(e') \) for any \( e, e' \in E(G) \) and \( i, i' \in \{0,1\} \), and \( \tilde{d}_G \) is the metric on \( X_G \) satisfying \( \tilde{d}_G((e, t), (e, t')) = |t - t'| \) if \( e = e' \) and \( \tilde{d}_G((e, t), (e, t')) = \min(d_G(\partial_0(e), \partial_0(e')) + t + t', d_G(\partial_1(e), \partial_1(e')) + t + 1 - t', d_G(\partial_1(e), \partial_0(e')) + 1 - t + t', d_G(\partial_1(e), \partial_1(e')) + 2 - t - t') \) else. It is easy to see that the definition of \((X_G, \tilde{d}_G)\) is indeed independent of the orientation of \( G \). Also, any cycle of \( G \) naturally corresponds to a circle, namely one-dimensional sphere, embedded in \( X_G \). For any two points \( x, y \in X_G \), there is a not necessarily unique geodesic connecting them in \((X_G, \tilde{d}_G)\), which we will use the notation \([x, y]\) if no ambiguity arises. We say that \((x, y, z)\) is \((\delta_1, \delta_2)\)-thin provided for any choice of the geodesics \([x, y], [y, z], [z, x]\) and \( m^y_z \in [y, z], m^z_x \in [z, x], m^x_y \in [x, y] \) satisfying

\[
\begin{cases}
\tilde{d}_G(m^y_z, x) = \tilde{d}_G(m^x_z, x) = (y \cdot z)_x, \\
\tilde{d}_G(m^y_z, y) = \tilde{d}_G(m^x_z, y) = (z \cdot x)_y, \\
\tilde{d}_G(m^z_x, z) = \tilde{d}_G(m^y_x, z) = (x \cdot y)_z,
\end{cases}
\]

(Figure 8 is an illustration of \([13] \text{ as well as a widely-used geometric interpretation of the Gromov product.})

the following two conditions hold:

(A) \( \delta_1 \geq \min(\tilde{d}_G(m^y_z, m^z_x), \tilde{d}_G(m^z_x, m^y_z)) \);

(B) \( \{ p \in X_G : \tilde{d}_G(p, [y, z] \cup [y, x]) \leq \delta_2 \} \supseteq [x, z] \).

Modifying the original definition of Gromov slightly \([\mathbf{22}] \text{ p. 8, Definition 1.5}\) \([\mathbf{16}] \text{ p. 408}\) \([\mathbf{61}]\), we say that a graph \( G \) is \((\delta_1, \delta_2)\)-thin provided every triple \((x, y, z)\) of its vertices is \((\delta_1, \delta_2)\)-thin.

**Lemma 36** Let \( G \) be a graph. If \( G \) is \((\delta_1, \delta_2)\)-thin, then it is \((\delta_1 + \delta_2)\)-hyperbolic.

**Proof:** The proof is taken from \([\mathbf{24}] \text{ p. 15, (2) implies (5)}\). It suffices to establish \([\mathbf{11}] \text{ for any } x, y, u, v \in V(G) \). By Eq. \([\mathbf{13}] \text{ and Condition (A) for the } (\delta_1, \delta_2)\)-thinness of \((x, u, y)\), we have

\[
(x \cdot y)_u + \delta_1 \geq \min(\tilde{d}_G(u, m^y_x), \tilde{d}_G(m^y_x, m^x_u), \tilde{d}_G(u, m^x_u), \tilde{d}_G(m^x_u, m^u_y)) \geq \tilde{d}_G(u, m^y_u). \tag{19}
\]

By Condition (B) for the \((\delta_1, \delta_2)\)-thinness of \((x, v, y)\), we can suppose, without loss of generality, that there is \( q \in [y, v] \) such that

\[
\delta_2 \geq \tilde{d}_G(m^y_u, q). \tag{20}
\]

It follows from \( \tilde{d}_G(q, v) + \tilde{d}_G(q, y) = \tilde{d}_G(y, v), \tilde{d}_G(u, q) + \tilde{d}_G(q, v) \geq \tilde{d}_G(u, v), \text{ and } \tilde{d}_G(u, q) + \tilde{d}_G(q, y) \geq \tilde{d}_G(u, y) \) that

\[
\tilde{d}_G(u, q) \geq (y \cdot v)_u. \tag{21}
\]

We surely have

\[
\tilde{d}_G(u, m^y_u) + \tilde{d}_G(m^y_u, q) \geq \tilde{d}_G(u, q). \tag{22}
\]

To complete the proof, we just need to add together \([\mathbf{13}], [\mathbf{20}], [\mathbf{21}], \text{ and } [\mathbf{22}]\). \(\square\)
According to Gromov \[61\], Rips invents the concept of slimness: For any real number \(\delta\), we say that a graph \(G\) is \(\delta\)-slim if for every triple \((x, y, z)\) of vertices of \(G\), we have

\[
\{p \in X_G : \tilde{d}_G(p, [y, z] \cup [y, x]) \leq \delta\} \supseteq [x, z].
\]

(23)

An easy observation is that a \((\delta_1, \delta_2)\)-thin graph is \(\delta_2\)-slim. It is mentioned in \[25\] Proposition 1 that every \(\delta\)-slim graph is \(8\delta\)-hyperbolic. The next lemma gives a better bound.

**Lemma 37** If a graph is \(\delta\)-slim, it must be \((2\delta, \delta)\)-thin and hence \(3\delta\)-hyperbolic.

**Proof:** By Lemma \[36\] our task is to prove that any \(\delta\)-slim graph \(G\) is \((2\delta, \delta)\)-thin. For this purpose, it suffices to deduce \(2\delta \geq \min(\tilde{d}_G(m_{y,x}^z), \tilde{d}_G(m_{x,y}^z), \tilde{d}_G(m_{x,y}^z))\) for any triple \((x, y, z)\) of vertices of \(G\). The following argument is almost word-for-word the same as that given in \[2\] p. 13, (1) implies (3)]. By (23), we suppose, as we may, that there is \(w \in [y, x]\) such that \(\tilde{d}_G(m_{y,x}^z, w) \leq \delta\). Observe that

\[
\tilde{d}_G(x, w) \geq \tilde{d}_G(m_{y,x}^z, x) - \tilde{d}_G(m_{y,x}^z, w) \geq \tilde{d}_G(m_{y,x}^z, x) - \delta = \tilde{d}_G(m_{y,x}^z, x) - \delta
\]

and that

\[
\tilde{d}_G(x, w) \leq \tilde{d}_G(m_{y,x}^z, x) + \tilde{d}_G(m_{y,x}^z, w) \leq \tilde{d}_G(m_{y,x}^z, x) + \delta = \tilde{d}_C(m_{y,x}^z, x) + \delta.
\]

It then follows \(\tilde{d}_G(m_{y,x}^z, w) \leq \delta\) and henceforth \(\tilde{d}_G(m_{y,x}^z, m_{y,x}^z) \leq \tilde{d}_G(m_{y,x}^z, w) + \tilde{d}_G(w, m_{y,x}^z) \leq 2\delta\), as desired. \(\square\)

**Lemma 38** Every \(k\)-chordal graph is \((\frac{k}{2}, \frac{k}{2})\)-thin.

**Proof:** Consider any triple \((x, y, z)\) of vertices of \(G\). By an abuse of notation as usual, denote by \([x, y], [y, z]\) and \([z, x]\) three geodesic segments joining the corresponding endpoints and put \(m_{y,x}^z \in [y, z]\) and \(m_{y,x}^z \in [x, y]\) be three points of \(X_G\) satisfying Eq. (18). For any nonnegative number \(t \leq (y \cdot z)_x\), there is a unique point \(p\) in \([x, y]\) such that \(\tilde{d}_G(u, x) = t\); we use the notation \((z; x)_t\) for this point \(u\). Similarly, we define \((y; x)_t\) for any \(0 \leq t \leq (y \cdot z)_x\) and so on. By symmetry, it suffices to show that \(\tilde{d}_G((z; x)_t, (y; x)_t) \leq \frac{k}{4}\) for any \(0 \leq t \leq (y \cdot z)_x\). Take the maximum \(t' \leq t\) such that \((z; x)_t = (y; x)_t\). The case of \(t' = t\) is trivial and so we assume that \(t' < t\).

**Case 1:** There exists \(t''\) such that \((z; x)_t'' = (y; x)_t''\) and \((y \cdot z)_x \geq t'' > t\). We can assume that \((z; x)_t'' \neq (y; x)_t''\) for any \(t' < t'' < t\).

Clearly, walking along \([x, y]\) from \((z; x)_t\) to \((z; x)_t''\) and then go back to \((y; x)_t''\) along \([x, z]\) gives rise to a cycle \(C\) in \(G\). This cycle might contain chords. But, surely \(C\) has no chord which connects one point whose distance to \(x\) is less than \(t\) to another point whose distance to \(x\) is larger than \(t\). This means that \((z; x)_t\) and \((y; x)_t\) must appear in a circle in \(X_G\) corresponding to a chordless cycle of \(G\). Since \(G\) is \(k\)-chordal, we arrive at \(d_{\gamma}((z; x)_t, (y; x)_t) \leq \frac{k}{2}\).

**Case 2:** There exists no \(t''\) such that \((z; x)_t'' = (y; x)_t''\) and \((y \cdot z)_x \geq t'' > t\).

Let \(\Lambda = \{(z; x)_s, (y; x)_s : 0 \leq s \leq (y \cdot z)_x\}\) and \(\Upsilon = \{(z; y)_s : 0 \leq s < (z \cdot x)_y\} \cup \{(y; z)_s : 0 \leq s < (x \cdot y)_z\}\). For any \(y \in \Upsilon\), \(d_G(x, y) > (y \cdot z)_x\) holds and for any \(y \in \Lambda\), \(d_G(x, y) \leq (y \cdot z)_x\) holds. This says that

\[
\Lambda \cap \Upsilon = \emptyset.
\]

(24)

Analogously, by considering both the distance to \(y\) and the distance to \(z\), we have

\[
\Lambda \cap [y, z] = \emptyset.
\]

(25)

Combining Eqs. (24) and (25), we get that there is a geodesic \(P\) connecting \((z; x)_t\) and \((y; x)_t\) whose internal points fall inside \([y, z] \cup \Upsilon\). We produce a circle in \(X_G\) as follows: Walk along \(P\) from \((z; x)_t\) to
(y; x)_t and then go along [x, z] from (y; x)_t to (y; x)_v and finally return to (z; x)_t by following [x, y]. This circle naturally corresponds to a cycle of G. This cycle might have chords. But for each chord which splits the circle into two smaller circles, our assumption guarantees that the two vertices (z; x)_t and (y; x)_t will still appear in one of them simultaneously. This means that there is a circle of $X_G$ corresponding to a chordless cycle of G and passing from both (z; x)_t and (y; x)_t. As G is k-chordal, $d_G((z; x)_t, (y; x)_t) \leq \frac{k}{2}$ follows, as expected.

4 Proofs

4.1 Lemmas

The proof of our main results, namely Theorems 9 and 14, is divided into a sequence of lemmas/corollaries.

In the course of our proof, we will frequently make use of the triangle inequality for the shortest-path metric, namely $ab + bc \geq ac$, without any claim. Besides this, we will also freely apply the ensuing simple observation, which is so simple that we need not bother to give any proof here.

**Lemma 39** Let H be a vertex induced subgraph of a graph G. Then H is an isometric subgraph of G if and only if $d_H(u, v) = d_G(u, v)$ for each pair of vertices $(u, v) \in V(G) \times V(G)$ satisfying $d_H(u, v) \geq 3$. In particular, H must be isometric if its diameter is at most 2.

One small matter of convention here and in what follows. When we refer to a graph, say a graph depicted in Fig. 3, we sometimes indeed mean that graph together with the special labeling of its vertices as indicated when it is introduced and sometimes we mean a graph which is isomorphic to it. We just leave it to readers to decide from the context which usage it is. Two immediate corollaries of Lemma 39 are given subsequently. We state them with the above convention and omit their routine proofs.

**Corollary 40** Let G be a graph. Let $H \in \{H_1, H_2, H_4\}$ be an induced subgraph of G such that $d_G(x, y) = d_G(u, v) = 3$. Then H is an isometric subgraph of G.

**Corollary 41** Let G be a graph and $H_3$ be an induced subgraph of G. If $d_G(x, y) = 3$, then $H_3$ is an isometric subgraph of G.

It is time to deliver some formal proofs.

**Corollary 42** Let G be a graph and $H_5$ be an induced subgraph of G. If $d_G(x, y) = d_G(u, v) = 3$ and $d_G(b, c) = 4$. Then $H_5$ is an isometric subgraph of G.

**Proof:** Based on the fact that $d_G(b, c) = 4$, we can derive from the triangle inequality that $d_G(u, b) = d_G(y, b) = d_G(c, x) = d_G(c, v) = 3$. The result then follows from Lemma 39 as

$$\{x, y\}, \{u, v\}, \{u, b\}, \{y, b\}, \{c, x\}, \{c, v\}, \{b, c\}$$

are all pairs inside $\binom{V(H_5)}{2}$ which are of distance at least 3 apart in $H_5$. □

**Lemma 43** Let G be a graph and let $x, y, u, v \in V(G)$. Then $\delta_G(x, y, u, v) \leq \min(uv, xy, ux, yv, uy, xv)$.

**Proof:** Suppose that $d_G(x, S) = d_1, d_G(y, S) = d_2$, where $S = \{u, v\}$. We can check the following:

\[
\begin{align*}
xy + uv &\leq (d_1 + d_2 + uv) + uv = d_1 + d_2 + 2uv, \\
d_1 + d_2 &\leq xv + yu \leq (d_1 + uv) + (d_2 + uv) = d_1 + d_2 + 2uv, \\
d_1 + d_2 &\leq x + yu \leq (d_1 + uv) + (d_2 + uv) = d_1 + d_2 + 2uv,
\end{align*}
\]

from which we get $\delta_G(x, y, u, v) \leq uv$ and hence our claim follows by symmetry. □

The next two simple lemmas concern the graph $H_6$ as given in Fig. 3 which is obviously a 5-chordal graph with hyperbolicity 1.
Lemma 44 Let $H$ be a graph satisfying $V(H) = V(H_2) = V(H_5) = \{x, y, u, v, a, b, c, d\}$ and $E(H_5) \subseteq E(H) \subseteq E(H_2)$. Let $t$ be the size of $E(H) \cap \{\{a, b\}, \{b, d\}, \{d, c\}, \{c, a\}\}$. If $t \in \{1, 2, 3\}$, then either $H$ contains an induced $C_4$ or there is an isomorphism from $H$ to $H_6$.

Proof: For any $v_1, v_2 \in V(H)$, $v_1 v_2$ always refers to $d_H(v_1, v_2)$ in the following.

Case 1: $bc = 1$.

Let us show that $H$ contains an induced $C_4$ in this case. Since $t \in \{1, 2, 3\}$, by symmetry, we can assume that either $ac = 1, cd \neq 1$ or $cd = 1, bd \neq 1$. In the former case, $[acyd]$ is an induced 4-cycle of $H$ and in the latter case $[cdvb]$ is an induced 4-cycle of $H$.

Case 2: $bc \neq 1$.

First observe that replacing the two edges $\{a, c\}$ and $\{d, c\}$ by the two new edges $\{a, b\}$ and $\{d, b\}$ will transform $H_6$ into another graph which is isomorphic to $H_6$. Thus, by symmetry, the condition that $t \in \{1, 2, 3\}$ means it is sufficient to consider the case that $ac = 1, cd > 1$ and the case that $ac = cd = 1, ab = bd = 2$. For the first case, $[acyd]$ is an induced 4-cycle of $H$; for the second case, $H$ itself is exactly $H_6$ after identifying vertices of the same labels. \hfill \Box

Lemma 45 Suppose that $G$ is a 5-chordal graph which has $H_6$ as an induced subgraph. If $d_G(x, y) = d_G(u, v) = 3$, then $G$ contains either $C_4$ or $H_2$ or $H_3$ as an isometric subgraph.

Proof: We can check that the subgraph of $H_6$, and hence of $G$, induced by $x, a, c, d, v, b$ is isomorphic to $H_3$. If $d_G(b, c) = 3$, Corollary [11] shows that $G$ contains $H_3$ as an isometric subgraph. Thus, in the remaining discussions we will assume that

$$d_G(b, c) = 2.$$  \hfill (26)

Case 1: $\min(d_G(b, u), d_G(b, y)) = 2$.

Assume, as we may, that $d_G(b, u) = 2$. Take, accordingly, $w \in V(G)$ satisfying $d_G(b, w) = d_G(w, u) = 1$. As $d_{H_6}(b, u) = 3$, we see that $w \not\in V(H_6)$. Observe that

$$2 = 3 - 1 = d_G(u, v) - d_G(u, w) \leq d_G(v, w) \leq d_G(v, b) + d_G(b, w) = 2,$$

which gives

$$d_G(v, w) = 2.$$  \hfill (27)

Case 1.1: $d_G(w, d) = 1$.

In this case, it follows from Eq. (27) that $[wbcd]$ is an isometric $C_4$ of $G$. 

Figure 9: A 5-chordal graph with hyperbolicity 1.
Figure 10: Case 2.1 in the proof of Lemma 45.

CASE 1.2: \( d_G(w, d) \geq 2 \).

Since \( G \) is 5-chordal, we know that the 6-cycle \([wbvdau]\) cannot be chordless in \( G \). By Eq. (27) and the current assumption that \( d_G(w, d) \geq 2 \), we can draw the conclusion that \( d_G(w, a) = 1 \) and hence find that the subgraph of \( G \) induced by \( w, u, a, d, v, b \) is isomorphic to \( H_3 \). As we already assumed that \( d_G(u, v) = 3 \), this induced \( H_3 \) is even an isometric subgraph of \( G \), taking into account Corollary 41.

CASE 2:

\( d_G(b, u) = d_G(b, y) = 3 \).

By Eq. (26), we can choose \( w \in V(G) \) such that \( d_G(b, w) = d_G(w, c) = 1 \). Since \( d_H_6(b, c) = 3 \), we know that \( w \not\in V(H) \). In addition, we have

\[
d_G(u, w) \geq d_G(u, b) - d_G(b, w) = 3 - 1 = 2, \quad \text{and} \quad d_G(y, w) \geq d_G(y, b) - d_G(b, w) = 3 - 1 = 2. \quad (28)
\]

Clearly, the map which swaps \( u \) and \( y \), \( a \) and \( d \) and \( x \) and \( v \) is an automorphism of \( H_6 \) and the requirement to specify our Case 2 will not be affected after applying this automorphism of \( H_6 \). Therefore, noting that \( d_G(w, v), d_G(w, d), d_G(w, x), d_G(w, a) \in \{1, 2\} \), we may take advantage of this symmetry of \( H_6 \) and merely consider the following situations.

CASE 2.1: \( d_G(w, v) = d_G(w, d) = d_G(w, x) = d_G(w, a) = 1 \).

From Eq. (28) and our assumption it follows that the subgraph of \( G \) induced by \( x, a, u, c, y, d, v, w \) is isomorphic to \( H_2 \); see Fig. 10. Because \( d_G(x, y) = d_G(u, v) = 3 \), Corollary 40 now tells us that \( G \) contains \( H_2 \) as an isometric subgraph.

CASE 2.2: \( d_G(w, v) = d_G(w, d) = 2 \).

It is not difficult to check that the subgraph of \( G \) induced by \( w, b, v, d, c, y \) is isomorphic to \( H_3 \). The condition that \( d_G(b, y) = 3 \) then enables us to appeal to Corollary 41 and conclude that \( G \) contains the graph \( H_3 \) as an isometric subgraph.

CASE 2.3: \( d_G(w, v) = 2 \) and \( d_G(w, d) = 1 \).

\( G \) contains the induced 4-cycle \([wbvd]\).

CASE 2.4: \( d_G(w, v) = 1 \) and \( d_G(w, d) = 2 \).

\([wvdc]\) is a required induced \( C_4 \) of \( G \).

\( \square \)

Lemma 46 Let \( G \) be a 5-chordal graph which has \( H_5 \) as an induced subgraph. If \( d_G(x, y) = d_G(u, v) = 3 \), then \( G \) contains at least one of the subgraphs \( C_4, H_2, H_3 \) and \( H_5 \) as an isometric subgraph.
Proof: Because $H_3$ is an induced subgraph of $G$, it is clear that $d_G(b,u), d_G(b,y), d_G(c,x), d_G(c,v) \in \{2,3\}$. There are thus two cases to consider.

Case 1: $\min(d_G(b,y), d_G(b,u), d_G(c,x), d_G(c,v)) = 2$.

Without loss of generality, let us assume that $d_G(b,y) = 2$. There is then a vertex $w$ of $G$ such that $d_G(b,w) = d_G(w,y) = 1$. Observe that

$$d_G(w,x) \geq d_G(x,y) - d_G(b,w) = 3 - 1 = 2;$$

(29)

Case 1.1: $d_G(w,a) = 1$.

By Eq. (29), $[wbxa]$ is an induced 4-cycle of $G$.

Case 1.2: $d_G(w,a) > 1$.

Consider the 6-cycle $[wbxyda]$. As $G$ is 5-chordal, this cycle has a chord in $G$. According to Eq. (29) and our assumption that $d_G(w,a) > 1$, the only possibility is that such a chord connects $w$ and $d$. We now examine the subgraph of $G$ induced by $w,b,x,a,d,y$ and realize that it is isomorphic with $H_3$; see Fig. 11. Armed with Corollary 41, our assumption that $d_G(x,y) = 3$ shows that this $H_3$ is even an isometric subgraph of $G$, as wanted.

Case 2: $d_G(b,u) = d_G(b,y) = d_G(c,x) = d_G(c,v) = 3$.

By Corollary 42, $H_5$ is an isometric subgraph of $G$ provided $d_G(b,c) = 4$. Thus, we shall restrict our attention to the cases that $d_G(b,c) \in \{2,3\}$.

Case 2.1: $d_G(b,c) = 2$.

Pick a $w \in V(G)$ such that $d_G(b,w) = d_G(w,c) = 1$. It is clear that $[bwycvd]$ is a 6-cycle in the 5-chordal graph $G$ and hence must have a chord. We contend that this chord can be nothing but $\{w,d\}$. To see this, one simply needs to notice the following:

$$\begin{align*}
\{ & d_G(w,y) \geq d_G(b,y) - d_G(b,w) = 3 - 1 = 2; \\
& d_G(w,v) \geq d_G(c,v) - d_G(c,w) = 3 - 1 = 2.
\end{align*}$$

From the structure of the subgraph of $G$ induced by $b,w,c,y,d,v$ we deduce that both $[cwdy]$ and $[bwvd]$ are isometric 4-cycles in $G$, establishing our claim in this case.
Case 2.2: \( d_G(b, c) = 3 \).
We choose \( p, q \in V(G) \) such that \( d_G(b, p) = d_G(p, q) = d_G(q, c) = 1 \). We first note that
\[
d_G(q, v) \geq d_G(c, v) - d_G(c, q) = 3 - 1 = 2.
\]
Due to the symmetry of \( H_5 \), it is manifest then that
\[
\min(d_G(q, v), d_G(p, y), d_G(q, x), d_G(p, u)) \geq 2. \tag{30}
\]
We also have \( d_G(q, y) \in \{1, 2\} \) as it holds
\[
2 = 1 + 1 = d_G(q, c) + d_G(c, y) \geq d_G(q, y) \geq d_G(b, y) - d_G(b, q) = 3 - 2 = 1.
\]
Arguing by analogy, we indeed have
\[
d_G(q, y), d_G(q, u), d_G(p, v), d_G(p, x) \in \{1, 2\}. \tag{31}
\]
Eq. (30) along with Eq. (31) shows that
\[
\{p, q\} \cap V(H_5) = \emptyset. \tag{32}
\]
Assume, as we may, that
\[
d_G(q, y) + d_G(p, v) \geq d_G(q, u) + d_G(p, x). \tag{33}
\]
Case 2.2.1: \( d_G(q, y) = d_G(p, v) = 2 \).
We start with two observations: Thanks to Eq. (30), we have \( d_G(p, y) \geq 2, d_G(q, v) \geq 2 \) while as \( b, p, q, c \) is a geodesic, we obtain \( d_G(p, c) = d_G(q, b) = 2 \). Now, let us take a look at the 7-cycle \([bpqcydv]\) of the 5-chordal graph \( G \). The cycle must have a chord, which, according to our previous observations and our assumption that \( d_G(q, y) = d_G(p, v) = 2 \), can only be the one connecting \( d \) to \( p \) or to \( q \). Without loss of generality, let \( d_G(q, d) = 1 \). Then, we can find a 4-cycle \([cqdy]\), as desired.

Case 2.2.2: \( d_G(q, y) = 2, d_G(p, v) = 1 \).
In this case, the 5-chordal graph \( G \) possesses the 6-cycle \([pqcydv]\), which must have a chord. We already assume that \( d_G(q, y) = 2 \); as \( b, p, q, c \) is a geodesic, we get \( d_G(p, c) = 2 \); finally, we have
\[
\begin{align*}
d_G(p, y) &\geq d_G(b, y) - d_G(b, p) = 3 - 1 = 2; \\
d_G(q, v) &\geq d_G(c, v) - d_G(c, q) = 3 - 1 = 2.
\end{align*}
\]
Consequently, it happens either \( d_G(q, d) = 1 \) or \( d_G(p, d) = 1 \). If \( d_G(q, d) = 1 \), we will come to an isometric 4-cycle \([cqdy]\). When \( d_G(p, d) = 1 \) and \( d_G(q, d) \geq 2 \), the subgraph of \( G \) induced by \( p, q, c, y, d, v \) is isomorphic to \( H_3 \) as shown by Fig. 12 which is even an isometric subgraph in view of Corollary 41 as well as the assumption that \( d_G(c, v) = 3 \).
Figure 13: Case 2.2.4.1 in the proof of Lemma 46

Figure 14: Case 2.2.4.3 in the proof of Lemma 46

Case 2.2.3 \( d_G(q, y) = 1, d_G(p, v) = 2 \).
This case can be disposed of as Case 2.2.2.

Case 2.2.4: \( d_G(q, y) = d_G(p, v) = 1 \).

By Eqs. (31) and (33), we obtain \( d_G(q, u) = d_G(p, x) = 1 \). Noting Eq. (32), we further get
\[
\begin{align*}
1 & \leq d_G(p, d) \leq d_G(p, v) + d_G(v, d) = 1 + 1 = 2; \\
1 & \leq d_G(q, d) \leq d_G(q, y) + d_G(y, d) = 1 + 1 = 2; \\
1 & \leq d_G(p, a) \leq d_G(p, x) + d_G(x, a) = 1 + 1 = 2; \\
1 & \leq d_G(q, a) \leq d_G(q, u) + d_G(u, a) = 1 + 1 = 2.
\end{align*}
\] (34)

Because of Eq. (34), it is only necessary to consider the following three cases, since all others would follow by symmetry.

Case 2.2.4.1: \( d_G(p, d) = d_G(q, d) = 2 \).
The subgraph of \( G \) induced by \( c, y, q, p, v, d \) is isomorphic to \( H_3 \); see Fig. 13. But \( d_G(c, v) = 3 \) is among the standing assumptions for Case 2 and hence Corollary 41 demonstrates that \( G \) has this \( H_3 \) as an isometric subgraph.

Case 2.2.4.2: \( \{d_G(p, d), d_G(q, d)\} = \{1, 2\} \).
There is no loss of generality in assuming that \( d_G(p, d) = 1 \) and \( d_G(q, d) = 2 \). In such a situation, by recalling from Eq. (30) that \( d_G(p, y) \geq 2 \), we find that \( [pdyq] \) is an induced 4-cycle of \( G \), as wanted.

Case 2.2.4.3: \( d_G(p, d) = d_G(q, d) = d_G(p, a) = d_G(q, a) = 1 \).

After checking all those existing assumptions on pairs of adjacent vertices as well as the fact that \( \{q, v\}, \{p, y\} \notin E(G) \) as guaranteed by Eq. (30), we are led to the conclusion that the subgraph of \( G \)
induced by \( x, a, u, q, y, d, v, p \) is just \( H_2 \); see Fig. 14. Noting further our governing assumption in the lemma that \( \delta_G(x, y) = \delta_G(u, v) = 3 \), Corollary 10 then enables us reach the conclusion that this \( H_2 \) is indeed an isometric subgraph of \( G \), as was to be shown. \( \square \)

The next simple result resembles [17, Lemma 2.2] closely.

**Lemma 47** Let \( G \) be a \( k \)-chordal graph and let \( C = [x_1 \cdots x_k x_{k+1} \cdots x_{k+t}] \) be a cycle of \( G \). If no chord of \( C \) has an endpoint in \( \{x_2, \cdots, x_{k-1}\} \), then \( x_1 x_k = 1 \).

**Proof:** We consider the induced subgraph \( H = G[x_1, x_k, x_{k+1}, \ldots, x_{k+t}] \). There must exist a shortest path in \( H \) connecting \( x_1 \) and \( x_k \), say \( P \). If the length of \( P \) is greater than \( 1 \), then we walk along \( P \) from \( x_k \) to \( x_1 \) and then continue with \( x_2, x_3, \ldots \), and finally get back to \( x_k \), creating a chordless cycle of length at least \( k + 1 \), which is absurd as \( G \) is \( k \)-chordal. This proves that \( x_1 x_k = 1 \), as desired. \( \square \)

Let \( G \) be a graph. When studying \( \delta_G(x, y, u, v) \) for some vertices \( x, y, u, v \) of \( G \), it is natural to look at a *geodesic quadrangle* \( Q(x, u, y, v) \) with corners \( x, y, u, v \), which is just the subgraph of \( G \) induced by the union of all those vertices on four geodesics connecting \( x \) and \( u \) and \( y \) and \( v \), and \( v \) and \( x \), respectively. Let us fix some notation to be used throughout the paper.

**Assumption I:** Let us assume that \( x, u, y, v \) are four different vertices of a graph \( G \) and the four geodesics corresponding to the geodesic quadrangle \( Q(x, u, y, v) \) are

\[
\begin{align*}
  P_a : x & = a_0, a_1, \ldots, a_{xu} = u; \\
  P_b : x & = b_0, b_1, \ldots, b_{xv} = v; \\
  P_c : y & = c_0, c_1, \ldots, c_{yu} = u; \\
  P_d : y & = d_0, d_1, \ldots, d_{yw} = v.
\end{align*}
\]

We call \( P_a, P_b, P_c \) and \( P_d \) the four *sides* of \( Q(x, u, y, v) \) and often just refer to them as vertex subsets of \( V(G) \) rather than vertex sequences. We write \( P(x, u, y, v) \) for the pseudo-cycle

\[
[x, a_1, \ldots, a_{xu-1}, u, c_{yu-1}, \ldots, c_1, y, d_1, \ldots, d_{yw-1}, v, b_{xv-1}, \ldots, b_1].
\]

Note that \( P(x, u, y, v) \) is not necessarily a cycle as the vertices appearing in the sequence may not be all different. Let us say that \( x \) is *opposite* to \( P_x \) and \( P_d \), say that \( x \) and \( y \) are *opposite corners*, say that \( x \) and \( v \) are *adjacent corners*, say that \( x \) is the *common peak* of \( P_a \) and \( P_b \), say that \( P_a \) and \( P_b \) are *adjacent* to each other, say that \( P_a \) and \( P_d \) are *opposite* to each other, and say that those vertices inside \( P_a \setminus \{x, u\} \)
are ordinary vertices of $P_a$, etc. An edge of $Q(x, u, y, v)$ which intersects with two adjacent sides but do not lie in any single side is called an $A$-edge and an edge of $Q(x, u, y, v)$ which intersect with two opposite sides but does not lie in any single side is called an $H$-edge. Suppose that $a_i = v_0, v_1, \ldots, v_{a_i, d_j} = d_j$ is a geodesic connecting $a_i$ and $d_j$ in $G$. We call the two walks

$$x, a_1, \ldots, a_i, v_1, \ldots, v_{a_i, d_j}, d_j, d_{j-1}, \ldots, d_1, y$$

and

$$u, a_{xu}, \ldots, a_i, v_1, \ldots, v_{a_i, d_j}, d_j, d_{j+1}, \ldots, d_{yu}, v$$

$Z$-walks of $Q(x, u, y, v)$ through $\{a_i, d_j\}$ or just $Z$-walks of $Q(x, u, y, v)$ between $P_a$ and $P_d$. In an apparent way, we define similar concepts for $Z$-walks of $Q(x, u, y, v)$ between $P_b$ and $P_c$.

**Lemma 48** Let $G$ be a graph and let $Q(x, u, y, v)$ be one of its geodesic quadrangles for which Assumption I holds. Suppose any two adjacent sides of $Q(x, u, y, v)$ has only one common vertex and that vertex is their common peak. Then $Q(x, u, y, v)$ contains a cycle on which $b_1, x, a_1$ appear in that order consecutively. Moreover, if $\min(d(P_a, P_d), d(P_b, P_c)) \geq t$ for some $t$, then we may even require that the length of the cycle is no shorter than $4t$.

**Proof:** If $\min(d(P_a, P_d), d(P_b, P_c)) \geq t$ for $t > 0$, then $P(x, u, y, v)$ itself gives rise to a required cycle. Otherwise, without loss of generality, assume that $d(P_a, P_d) = 0$. Take the minimum $i$ such that $d(a_i, P_d) = 0$. There is a unique $j > 0$ such that $a_i = d_j$. It is plain that $i > 0$ and $j < yv$. This then shows that $[d_j \cdots, yv, xb, \cdots, b_1xa \cdots a_{i-1}]$ is a required cycle. \hfill $\square$

**Lemma 49** [17], p. 67, Claim 2] We make Assumption I. Further assume that

$$ub_i = 1$$

for some $i \geq 1$ and that

$$xy + uv \geq xv + yu + 2.$$  \hspace{1cm} (36)

Then, $b_1u < xu$.

**Proof:** We first check the following:

$$xu + uv - 2 \geq (xy - yu) + uv - 2$$

$$\geq xv \quad \text{(By Eq. (36))}$$

$$= xb_i + b_1u$$

$$= (xb_i + b_iu) + (ub_i + b_1v) - 2 \quad \text{(By Eq. (33))}$$

$$\geq xu + uv - 2.$$ 

Clearly, equalities have to hold throughout all the above inequalities. In particular, we have $xu = xb_i + b_1u$. This implies that there is a geodesic between $x$ and $u$ passing through $b_1$ and hence it is straightforward to see $b_1u < xu$, as wanted. \hfill $\square$

The next lemma is some variation of Lemma 43 and will play an important role in our short proof of Theorem 3 as to be presented in Section 4.2.

**Lemma 50** Let $G$ be a graph and let $Q(x, u, y, v)$ be one of its geodesic quadrangles for which Assumptions I holds. If

$$2\delta_G(x, y, u, v) = (xy + uv) - \max(xu + yv, xv + yu),$$  \hspace{1cm} (37)

then $\delta_G(x, y, u, v) \leq \min(d(P_a, P_d), d(P_b, P_c))$. 

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Proof: Without loss of generality, we assume that there exist \(i\) and \(j\) such that
\[
a_i d_j = \min(d(P_a, P_d), d(P_b, P_c)).
\] (38)

Focusing on a \(Z\)-walk of \(Q(x, u, y, v)\) through \(\{a_i, d_j\}\) connecting \(x\) and \(y\), we find that
\[
xy \leq xa_i + a_i d_j + d_j y = i + a_i d_j + j;
\] (39)

see Fig. 16. Analogously, we have
\[
uv \leq (xu - i) + a_i d_j + (yv - j).
\] (40)

Henceforth, we arrive at the following:
\[
2\delta(x, y, u, v) = (xy + uv) - \max(xu + yv, xv + yu) \quad \text{(By Eq. (37))}
\leq (xy + uv) - (xu + yv)
\leq (i + a_i d_j + j) + ((xu - i) + a_i d_j + (yv - j)) - (xu + yv) \quad \text{(By Eqs. (39) and (40))}
= 2a_i d_j.
\]

Combining this with Eq. (38), we finish the proof of the lemma.

As regards the inequality asserted in Lemma 50, we need to say something more for the purpose of deriving Theorem 14.

Lemma 51 Let \(G\) be a graph and let \(Q(x, u, y, v)\) be one of its geodesic quadrangles for which both Assumption I and Eq. (37) hold.

(i) If \(\delta(x, y, u, v) = d(P_a, P_d) = a_i d_j\), then
- any \(Z\)-walk of \(Q(x, u, y, v)\) through \(\{a_i, d_j\}\) between \(P_a\) and \(P_d\) must be a geodesic, and
- under the additional assumption that \(a_i d_j = 1\) and \(G\) is 5-chordal, either \(G\) has an isometric 4-cycle or \(\{a_i, d_j\}\) is the only edge intersecting both \(P_a\) and \(P_d\).

(ii) If \(\delta(x, y, u, v) = d(P_b, P_c) = b_p c_q\), then
- any \(Z\)-walk of \(Q(x, u, y, v)\) through \(\{b_p, c_q\}\) between \(P_b\) and \(P_c\) must be a geodesic, and
- under the additional assumption that \(b_p c_q = 1\) and \(G\) is 5-chordal, either \(G\) has an isometric 4-cycle or \(\{b_p, c_q\}\) is the only edge intersecting both \(P_b\) and \(P_c\).
Proof: (i) Let us continue our discussion launched in the proof of Lemma 50.

In the event that \( \delta(x, y, u, v) = d(P_a, P_d) = a_i d_j \), the equalities in both Eq. (39) and Eq. (40) must occur, which clearly shows that any \( Q \)-walk of \( Q(x, y, u, v) \) through \( \{a_i, d_j\} \) between \( P_a \) and \( P_d \) must be a geodesic.

We now further assume that \( a_i d_j = d(P_a, P_d) = 1 \) and \( G \) is 5-chordal. Set
\[
I = \{(k, \ell) : a_k d_{\ell} = 1\}.
\]
For any \((k, \ell) \in I\), by considering each \( Q \)-walk through \( \{a_k, d_{\ell}\} \) connecting \( x \) and \( y \), which is a geodesic as we already know, we come to
\[
I = \{(k, \ell) : a_k d_{\ell} = 1\} \subseteq \{(k, \ell) : k + \ell = xy - 1\}.
\]
Eq. (41) means that there exists \((i', j')\) and \( 0 = t_0 < t_1 < \cdots < t_m \) such that
\[
I = \{(i' - t_{\alpha}, j' + t_{\alpha}) : \alpha = 0, 1, \ldots, m\}.
\]
Suppose that \( \{a_i, d_j\} \) is not the only edge intersecting both \( P_a \) and \( P_d \). This means that \( |I| - 1 = m \geq 1 \) and then we see that
\[
[a_{i_1} a_{i_1 - 1} \cdots a_{i_1 - t_1} b_{j' + t_1} b_{j' + t_1 - 1} \cdots b_{j'}]
\]
is a chordless cycle of length \( 2t_1 + 2 \). Since \( G \) is 5-chordal and \( t_1 \) is a positive integer, this cycle can only be an isometric 4-cycle of \( G \), finishing the proof of (i).

(ii) The proof can be carried out in the same way as that of (i).

\[ \square \]

Lemma 52 Let \( G \) be a graph and we will adopt Assumption I. We choose \( j \) to be the maximum number such that \( a_j b_j \leq 1 \), \( i \) the minimum number such that \( b_i d_{yv - xu + i} \leq 1 \), \( \ell \) the maximum number such that \( c_\ell d_\ell \leq 1 \), and \( m \) the minimum number such that \( a_m c_{yv - xu + m} \leq 1 \). Put
\[
\begin{cases}
\pi(b) = i - j + \frac{a_j b_j + b_i d_{yv - xu + i}}{2}, \\
\pi(d) = (yv - xu + i) - \ell + \frac{b_i d_{yv - xu + i} + c_\ell d_\ell}{2}, \\
\pi(c) = (yu - xu + m) - \ell + \frac{a_m c_{yv - xu + m} + c_\ell d_\ell}{2}, \\
\pi(a) = m - j + \frac{a_j b_j + a_m c_{yv - xu + m}}{2}.
\end{cases}
\]

If Eq. (37) is valid, then \( \delta_G(x, y, u, v) \leq \min(\pi(a), \pi(b), \pi(c), \pi(d)) \leq 1 + \min(i - j, (yv - xu + i) - \ell, (yu - xu + m) - \ell, m - j) \).

Proof: By symmetry, we only need to show that \( \delta_G(x, y, u, v) \leq \pi(b) \).

Taking into account the fact that we can walk from \( x \) to \( y \) in \( G \) by following \( x, b_1, \ldots, b_i \) and then moving in at most one step from \( b_i \) to \( d_{yv - xu + i} \) and finally traversing from \( d_{yv - xu + i} \) to \( y \) along \( P_d \), we get that
\[
xy \leq i + b_i d_{yv - xu + i} + (yv - (xv - i)).
\]
Similarly, starting from \( u \), we can first walk along \( P_a \) and then jump from \( a_j \) to \( b_j \) in at most one step and then walk along \( P_b \) to arrive at \( v \). This gives us
\[
uv \leq (xu - j) + a_j b_j + (xv - j).
\]
See Fig. 17 Accordingly, we have
\[
2\delta_G(x, y, u, v) = (xy + uv) - \max(xu + yv, xv + yu) \quad \text{(By Eq. (37))}
\leq (xy + uv) - (xu + yv)
\leq (i + b_i d_{yv - xu + i} + (yv - (xv - i))) + ((xu - j) + a_j b_j + (xv - j)) - (xu + yv)
\leq 2\pi(b),
\]
which is exactly what we want.

\[ \square \]

Brinkmann, Koolen and Moulton [17] introduced an extremality argument to deduce upper bounds of hyperbolicity of graphs. We follow their approach and make the following standing assumption in the main step of proving Theorems 9 and 14 and thus in several subsequent lemmas.
Figure 17: $xy \leq i + b_i d_{yv - xv + i} + (yv - (xv - i))$, $uv \leq (xu - j) + a_j b_j + (xv - j)$.

**Assumption II:** We assume $x, y, u, v$ are four different vertices of $G$ such that the sum $xy + uv$ is minimal subject to the condition

$$xy + uv = \max(xu + yv, xv + yu) + 2\delta^*(G).$$  \hfill (46)

**Lemma 53** [17, p. 67, Claim 1] [74, p. 690, Claim 1] Let $G$ be any graph and $u, v, x, y \in V(G)$. Under the Assumptions I and II, we have $a_1 v \geq xv$, $a_{xv - 1} y \geq vy$, $b_1 u \geq xu$, $b_{xv - 1} y \geq vy$, $c_1 v \geq yv$, $c_{yu - 1} x \geq ux$, $d_1 u \geq yu$, $d_{yv - 1} x \geq vx$.

**Proof:** By symmetry, we only need to show that $a_1 v \geq xv$. If $a_1 v < xv$, then, as a result of $a_1 v \geq xv - xa_1 = xv - 1$, we have

$$a_1 v = xv - 1.$$ \hfill (47)

Notice the obvious fact that

$$a_1 u = xu - 1.$$ \hfill (48)

We then come to the following:

$$a_1 y + uv \geq (xy - xa_1) + uv$$
$$= (xy - 1) + uv$$
$$= (xy + uv) - 1$$
$$= \max(xu + yv - 1, xv + yu - 1) + 2\delta^*(G) \quad \text{(By Eq. (46))}$$
$$= \max(a_1 u + yv, a_1 v + yu) + 2\delta^*(G). \quad \text{(By Eqs. (47) and (48))}$$ \hfill (49)

According to the definition of $\delta^*(G)$, we read from Eq. (49) that $a_1 y + uv = \max(a_1 u + yv, a_1 v + yu) + 2\delta^*(G)$ and hence that $a_1 y + uv = xy + uv - 1$. This contrasts with the minimality of the sum $xy + uv$ (Assumption II), completing the proof. \hfill □

**Corollary 54** [17, p. 67, Claim 2] Under Assumptions I and II and stipulating that $\delta^*(G) \geq 1$, we have that each corner of $Q(x, u, y, v)$ is not adjacent to its opposite corner and any ordinary vertex of its opposite sides and hence has degree 2 in $Q(x, u, y, v)$.

**Proof:** By symmetry, we only need to prove the claim for the corner $u$, which directly follows from Lemmas 49 and 53. \hfill □
Lemma 55  Suppose that Assumptions I and II are met. (i) Any two adjacent sides of $Q(x, u, y, v)$ only intersect at their common peak. In particular, no corner of $Q(x, u, y, v)$ can be an ordinary vertex of some side of $Q(x, u, y, v)$. (ii) For any geodesic connecting two adjacent corners of $Q(x, u, y, v)$, say $P = w_0, w_1, \ldots, w_m$, no corner of $Q(x, u, y, v)$ can be found among $\{w_1, \ldots, w_{m-1}\}$. (iii) Let $w$ be the common peak of two adjacent sides $P$ and $P'$ of $Q(x, u, y, v)$. Let $\alpha \in P \setminus \{w\}$ and $\alpha' \in P' \setminus \{w\}$ be two vertices of $Q(x, u, y, v)$ such that $\alpha \alpha' = 1$, then $\alpha \omega = \alpha' \omega$.

Proof:  (i) By symmetry, it suffices to prove that $a_p \neq b_q$ for any $p \geq q > 0$. Suppose otherwise, it then follows that $b_1, b_2, \ldots, b_i = a_p, a_{p+1}, \ldots, a_{xu} = u$ is a path connecting $b_1$ and $u$ and so $b_1 u < xu$, violating Lemma 53.

(ii) Assume the contrary, we can replace the side of $Q(x, u, y, v)$ connecting the two asserted adjacent corners by the geodesic $\delta(P)$ and get to a new geodesic quadrangle for which Assumptions I and II still hold but for which a corner appears as an ordinary vertex in the side $P$, yielding a contradiction to (i).

(iii) It is no loss to merely prove that if $i, j > 0$ and $a_i b_j = 1$ then $i = j$. In the case of $i > j$, $b_1, b_2, \ldots, b_j, a_i, a_{i+1}, \ldots, a_{xu} = u$ is a path connecting $b_1$ and $u$ of length smaller than $xu$, contrary to Lemma 53. Similarly, $i < j$ is impossible as well. □

Corollary 56  Let $G$ be a graph with $\delta^s(G) > 0$ and let $Q(x, u, y, v)$ be a geodesic quadrangle for which Assumptions I and II hold. Then $P(x, u, y, v)$ is a cycle. Moreover, if $\delta^s(G) > \frac{1}{2}$, then all chords of $P(x, u, y, v)$ must be either $\kappa$-edges or $\mathbb{H}$-edges.

Proof:  This follows from Lemma 50, Lemma 55 (i) and Corollary 54 in a straightforward fashion. □

The next result is very essential to our proof of Theorem 14 and both its statement and its proof have their origin in [17, p. 65, Prop. 3.1] and [74, p. 691, Claim 2].

Lemma 57  Suppose that $G$ is a graph for which Assumptions I and II are met and $Q(x, u, y, v)$ has at least one $\kappa$-edge. Then we have $xu + yv = xv + yu$.

Proof:  If the claim were false, without loss of generality, we suppose that

$$xu + yv > xv + yu.$$  \hspace{1cm} (50)

By symmetry and because of Lemma 55 (iii), let us work under the assumption that $a_i b_i = 1$. It clearly holds

$$a_i \neq x.$$  \hspace{1cm} (51)

Before moving on, let us prove that

$$a_i \neq u.$$  \hspace{1cm} (52)

Suppose for a contradiction that $a_i = u$, we find that

$$yv \geq yu + xv - xu + 1 \quad \text{(By Eq. (50))}$$

$$= yu + xv - xa_i + 1$$

$$= yu + xv - i + 1$$

$$= yu + b_i v + 1.$$  \hspace{1cm} (53)

But we surely have $yv \leq yu + ub_i + b_iv = yu + b_iv + 1$ and so we conclude that we can get a geodesic $P$ connecting $y$ to $v$ in $G$ by first walking along $P_i$ to go from $y$ to $u$, then moving from $u$ to $b_i$ in one step and finally traversing along $P_{b_i}$ from $b_i$ to $v$. Since this geodesic passes through $u$ in the middle, we obtain a contradiction to Lemma 55 (ii) and hence establish Eq. (52).

To go one step further, let us check the following:

$$xv + 1 = xb_i + b_i v + 1 = x b_i + b_i v + a_i b_i$$

$$\geq x a_i + a_i v = xa_i + a_i v$$

$$= 1 + a_i a_i + a_i v \quad \text{(By Eq. (51))}$$

$$\geq 1 + xv.$$  \hspace{1cm} (53)
Clearly, equalities hold throughout Eq. (53). In particular, we have
\[ b_i v + 1 = a_i v. \]  
(54)
From Eq. (54) and \( x_i v = b_i v + i \) we deduce that
\[ a_i v = x_i v - (i - 1). \]  
(55)
Here comes the punch line of the proof:
\[ a_i y + u_i v \geq (x_i y - a_i y) + u_i v \]
\[ = (x_i y - i) + u_i v \]
\[ = \max(x_i y + u_i v, x_i y + y) + 2\delta^*(G) - i \quad \text{(By Eq. (50))} \]  
(56)
\[ = \max((x_i y - i) + y_i a_i v + y_i v) + 2\delta^*(G) \quad \text{(By Eq. (55))} \]
\[ = \max((x_i y - i) + y_i a_i v + y_i v) + 2\delta^*(G). \]
According to Eqs. (51) and (52), we can apply Lemma 55 (i) to find that \( a_i, y, u, v \) are four different vertices. We further conclude from the definition of \( \delta^*(G) \) that Eq. (56) should hold equalities throughout, hence that \( xy + u_i v \leq a_i y + u_i v \) as a result of the minimality of \( xy + u_i v \) as indicated in our Assumption II, and finally that the first inequality in Eq. (56) must be strict in light of Eq. (51), getting a contradiction with the assertion that all equalities in Eq. (56) hold. This is the end of the proof. □

Lemma 58 Let \( G \) be a graph for which Assumptions I and II are required. Suppose that \( Q(x, u, y, v) \) has an \( A \)-edge. If there is \( 1 \leq i \leq x_i u - 1 \) and \( 0 \leq j \leq y_i v \) such that \( a_i d_j \leq 1 \), then \( a_i d_j = 1, a_i u + d_j y = y_i v \) and \( a_i x + d_j v = x_i v \).

Proof: We start with an easy observation:
\[
\begin{cases}
  a_i u + d_j y = a_i x - 1 a_i + 1 + d_j y \geq a_i x - 1 a_i + a_i d_j + d_j y \geq a_i x - 1 y, \\
  a_i x + d_j v = a_i a_i + 1 + d_j v \geq a_i a_i + a_i d_j + d_j v \geq a_i v.
\end{cases}
\]  
(57)
In view of Lemma 53 this says that
\[ a_i u + d_j y \geq uy, \quad a_i x + d_j v \geq xv. \]  
(58)
Adding together the two inequalities in Eq. (58), we obtain
\[ xu + yv \geq x_i v + y_i u. \]  
(59)
But, it follows from Lemma 57 and the existence of an \( A \)-edge of \( Q(x, u, y, v) \) that the equality in Eq. (59) must occur. Consequently, none of the inequalities in Eqs. (57) and (58) can be strict, which is exactly what we want to prove. □

With a little bit of luck, the forthcoming lemma contributes the number \( \left\lfloor \frac{k}{2} \right\rfloor \), which is just the mysterious one we find in Theorem 9. Note that \( \left\lfloor \frac{k}{2} \right\rfloor \) is the smallest half integer that is greater than \( \frac{k}{2} - \frac{2}{4} \).

Lemma 59 Let \( G \) be a \( k \)-chordal graph for some \( k \geq 4 \) and let \( Q(x, u, y, v) \) be a geodesic quadrangle for which Assumptions I and II hold. Then we have \( \delta^*(G) \leq \frac{k}{2} \) provided
\[ \min(d(P_a, P_d), d(P_b, P_e)) > 1. \]  
(60)
Lemma 55 (i) additionally, this says that $G$ is $k$-chordal. Moreover, by Lemma 55 (iii), Eq. (60) and the choice of $i, j, \ell, m$, we know that $\min(a, b, c, d) \leq \frac{|k|}{2}$. (61)

Suppose, for a contradiction, that the inequality (61) does not hold. In this event, as $\frac{|k|}{2} \geq 1$, we know that $\min(i - j, (yv - xv + i) - \ell, (yu - xu + m) - \ell, m - j) \geq \min(a, b, c, d) - 1 > 0$. By virtue of Lemma 55 (i) and Eq. (60), this implies that $C = [a, b]_{ij} c_{ij} d_{ij} a_{ij} \cdots$ is a cycle, where the redundant $a_j$ should be deleted from the above notation when $a_j = b_j = x$, the redundant $b_i$ should be deleted from the above notation when $b_i = d_{iy - xv + i} = v$, etc.; see Fig. 18. Moreover, by Lemma 55 (iii), Eq. (64) and the choice of $i, j, \ell, m$, we know that $C$ is even a chordless cycle. But the length of $C$ is just $\pi(a) + \pi(b) + \pi(c) + \pi(d)$, which, as the assumption is that (61) is violated, is no smaller than $4(\frac{1}{2} + \frac{|k|}{2})$ and hence is at least $k + 1$. This contradicts the assumption that $G$ is $k$-chordal, finishing the proof.

Lemma 60 Let $G$ be a 5-chordal graph and we demand that Assumptions I and II hold. Take $i, j, \ell, m$ to be the numbers as specified in Lemma 52. Suppose that $Q(x, u, y, v)$ has no $H$-edges and $\min(xu, xv, yu, yv, 2\delta^*(G)) \geq 2$. Then we have

$$a_j b_j + b_i d_{iy - xv + i} + c_{ij} d_{ij} + a_{ij} c_{uy - xu + m} \geq 2.$$ (63)

Furthermore, we have the following conclusions: if $a_j b_j = 1$, then $(i, m) \in \{(j, j), (j, xu), (xv, j)\}$; if $b_i d_{iy - xv + i} = 1$, then $(j, \ell) \in \{(i, yv - xv + i), (i, 0), (0, yv - xv + i)\}$; if $c_{ij} d_{ij} = 1$, then $(yu - xu + m, yv - xv + i) \in \{(\ell, \ell, \ell, y\ell), (yv, y), (yu, \ell)\}$; if $a_{ij} c_{uy - xu + m} = 1$, then $(j, \ell) \in \{(m, yu - xu + m), (m, 0), (0, yu - xu + m)\}$.

Proof: Since $\delta^*(G) \geq 1$, it follows from Lemma 52 that $\min(i - j, (yv - xv + i) - \ell, (yu - xu + m) - \ell, m - j) \geq 0$. (64)

Using Lemma 52 instead, we obtain from $\delta^*(G) \geq 1$ that $\min(d(P_a, P_d), d(P_b, P_c)) \geq 1$. Considering Lemma 55 (i) additionally, this says that $G$ has a cycle $C$ as displayed in Eq. (62) whose length is
\[\pi(a) + \pi(b) + \pi(c) + \pi(d),\] where \(\pi(a), \pi(b), \pi(c), \pi(d)\) stand for the numbers introduced in Eq. \((42)\). As \(Q(x, u, y, v)\) has no \(\mathbb{H}\)-edges, by the choice of \(i, j, \ell, m\) and by Lemma \([55] (iii)\), we see that \(C\) is chordless and are thus led to \(\pi(a) + \pi(b) + \pi(c) + \pi(d) \leq 5\).

We first observe that \(a_i b_j + b_i d_y - x_i + 1 + c_i d_i a_m c_y - u + m > 0\); as otherwise \(C\) will be a chordless cycle of length \(2x + xu + y + y + y + y + y = 1\), contradicting \(\mathcal{I}(G) \leq 5\). Let us proceed to consider the case that \((a_ib_i b_i d_y - x_i + 1 + c_i d_i a_m c_y - u + m) = (1, 0, 0, 0)\). Note that Corollary \([54]\) guarantees that \(0 < j < \min(xu, xv)\). Evoking our assumption \(\min(yv, yu) \geq 2\), it is then obvious that the cycle \(C\) contains at least 6 different vertices \(a_i, b_j, v, d_i, y, c_i, u\), which is absurd as \(G\) is 5-chordal. By symmetry, Eq. \((63)\) is thus established.

Among the four conclusions, let us now only deal with the one accompanied with the assumption that \(a_i b_j = 1\). If \(a_m c_y - u + m = b_i d_y - x_i + 1 = 0\), Eq. \((63)\) implies \(c_i d_i = 1\) and so \(C\) will have at least 6 different vertices \(a_i, b_j, v, d_i, c_i, u\), contrary to \(\mathcal{I}(G) \leq 5\). To this point, we can conclude that \(\max(b_i d_y - x_i + 1, a_m c_y - u + m) = 1\) and so it suffices to prove that \(i \in \{j, xv\}\) and \(m \in \{j, xv\}\). We only prove the first claim and the second one will follow by symmetry. Since we already have \(i \geq j\) as guaranteed by Eq. \((61)\), our task is now to get from \(i > j\) to \(i = xv\). If this is not true, the chordless cycle \(C\) will already have four different vertices \(a_i, b_j, b_i, d_y - x_i + 1\), which are all outside of \(P_i\) according to Corollary \([52]\). Consequently, due to Corollary \([52]\) and \(\mathcal{I}(G) \leq 5\), we find that \(C\) must have the fifth vertex \(c \in P_i \setminus \{u, y\}\) such that \(a_i c_i = c_i d_i - x_i + 1\). In view of Lemma \([55] (iii)\), we then see that

\[cu = a_j u, cy = d_y - x_i + 1 y, a_j x = b_j x, b_i v = d_y - x_i + 1 v.\]

We sum them up and yield \(xv + yu = xu + yv + b_j > xu + yv\), which is a contradiction with Lemma \([57]\).

This completes the proof of the lemma. \(\square\)

**Lemma 61** Let \(G\) be a graph for which we will make Assumptions I and II. Let \(P\) and \(P'\) be two adjacent sides of \(Q(x, u, y, v)\) whose common peak is \(w\). Let \(\alpha, \beta \in P \setminus \{w\}\) and \(\alpha', \beta' \in P' \setminus \{w\}\) be four vertices of \(Q(x, u, y, v)\) such that \(\alpha \alpha' = 1\) and \(\beta \beta' = 5\). Then it holds \(\beta \beta' = 1\) in the case that \(G\) is 4-chordal as well as in the case that \(G\) is 5-chordal and \(\beta w = \beta' w > 5\).

**Proof:** By symmetry, we only need to show that for any \(i \geq 3\) \((i \geq 2)\) we can obtain from \(a_i b_i = 1\) that \(a_{i-1} b_{i-1} = 1\) provided \(G\) is 5-chordal (4-chordal). But Lemma \([55] (i)\) states that \(C = [a_{i-1} a_i b_i b_{i-1} \cdots b_1 a_1 a_1 \cdots a_{i-2}]\) is a cycle of length at least 7 \((5)\). Thus, Lemma \([57] (iii)\) in conjunction with Lemma \([55] (iii)\) applies to give \(a_{i-1} b_{i-1} = 1\), as wanted. \(\square\)

**Corollary 62** Let \(G\) be a 5-chordal graph without isometric \(C_4\) for which we will make Assumptions I and II. If there is an \(\mathbb{H}\)-edge connecting \(\alpha\) and \(\alpha'\) lying in two adjacent sides \(P\) and \(P'\) with common peak \(w\), respectively, then this is the only \(\mathbb{H}\)-edge between \(P\) and \(P'\) and \(\alpha w = \alpha' w \leq 2\).

**Proof:** This follows directly from Lemmas \([55] (iii)\) and \([61]\). \(\square\)

**Lemma 63** Let \(G\) be a 5-chordal graph with \(\delta^*(G) \geq 1\) and let Assumptions I and II hold. Assume that \(Q(x, u, y, v)\) has no \(\mathbb{H}\)-edges. (i) If there is an \(\mathbb{H}\)-edge between \(P_b\) and \(P_c\), then \(\max(xu, yv) = 2\). (ii) If there is an \(\mathbb{H}\)-edge between \(P_a\) and \(P_d\), then \(\max(xu, yu) \leq 2\).

**Proof:** We only prove \(xu \leq 2\) under the assumption that there is an \(\mathbb{H}\)-edge between \(P_b\) and \(P_c\) and all the other claims follow similarly. Take the minimum \(i\) such that \(b_i\) is incident with an \(\mathbb{H}\)-edge and then pick the maximum \(j\) such that \(b_j c_j\) is an \(\mathbb{H}\)-edge. By Corollary \([54]\) we have \(\min(i, yu - j) \geq 1\). Since \(Q(x, u, y, v)\) has no \(\mathbb{H}\)-edges, we find that

\[\begin{align*}
[b_1 \cdots b_j c_{j+1} \cdots c_{yv - 1} a_{yu - 1} \cdots a_1 x]
\end{align*}\]

is a chordless cycle of length \(xu + 1 + i + (yu - j) \geq xu + 3\). Finally, because \(G\) is 5-chordal, we conclude that \(xu \leq 2\), as desired. \(\square\)
Lemma 64 Let $G$ be a 5-chordal graph with $\delta^*(G) \geq 1$. We keep Assumptions I and II. In addition, assume that $Q(x,u,y,v)$ has a side of length one. Then, $G$ contains at least one graph among $C_4, H_3$ and $H_5$ as an isometric subgraph.

Proof: It involves no restriction of generality in assuming that $xv = 1$. Owning to Lemma 55 (i), the walk along $P_a,P_c$ and $P_d$ will connect $x$ and $v$ without passing through $x$ or $v$ in the middle and hence there is a shortest path connecting $x$ and $v$ in the graph obtained from $Q(x,u,y,v)$ by deleting the edge $\{x,v\}$. This says that $Q(x,u,y,v)$ has an induced cycle passing through $x$ and $v$ contiguously, say $C = [w_1w_2 \cdots w_n]$, where $w_1 = x$ and $w_2 = v$. From Corollary 54 we know that $w_3 = d_{yv-1} \neq a_1 = w_n$ and hence $n > 3$. Since $G$ is 5-chordal, our task is to derive that if $n = 5$ then $G$ contains an isometric $C_4$, $H_3$ or $H_5$.

Case 1: $w_3$ is a corner of $Q(x,u,y,v)$, namely $yu = 1$.

In light of Corollary 53, we have $w_3 = c_1$. If $c_1 = u$ or $w_5 = u$ occurs, then $Q(x,u,y,v)$ turns out to be a 5-cycle and hence has hyperbolicity $\frac{1}{2}$. This is impossible as Assumption II means that this hyperbolicity can be no smaller than $\delta^*(G) \geq 1$. Accordingly, by Lemma 55 (iii) we know that $w_4u$ and $w_5u$ have a common value, say $m$.

If $m > 3$ or there are two $A$-edges between $P_a$ and $P_c$, Corollary 62 says that $G$ contains an isometric $C_4$.

When $m = 2$ and there are no two $A$-edges between $P_a$ and $P_c$, the graph $H_5$ as depicted on the right of Fig. 19 is an induced graph of $G$. Utilizing Eq. (16) and the assumption that $\delta^*(G) \geq 1$, we find that

$$4 = 3 + 1 = ux + xv \geq uv \geq xv + uy + 2\delta^*(G) - xy = 2 + 2\delta^*(G) \geq 4.$$  

This illustrates that $uv = 4$. It follows from Lemma 53 that $a_2y \geq uy = 3$. In addition, we have $a_2y \leq a_2a_1 + a_4c_1 + c_1y = 3$ and so we see that $a_2y = 3$. Similarly, we have $c_2x = 3$. Getting that $a_2y = c_2x = 3$ and $uv = 4$, we apply Corollary 42 and conclude that the above-mentioned $H_5$ must be an isometric subgraph of $G$.

When $m = 1$, the graph $H_3$ as depicted on the left of Fig. 19 is an induced graph of $G$. As in the case of $m = 2$, we make use of Eq. (16) and $\delta^*(G) \geq 1$ to get an important information:

$$3 = uw_5 + w_5x + xv \geq uw \geq xv + uy + 2\delta^*(G) - xy = 1 + 2\delta^*(G) \geq 3.$$  

This implies $uv = 3$ and hence we deduce from Corollary 44 that this $H_3$ is even an isometric subgraph of $G$.

Case 2: $w_5$ is a corner of $Q(x,u,y,v)$, namely $xu = 1$.

The analysis is symmetric to that of Case 1.

Case 3: Neither $w_3$ nor $w_5$ is a corner. In this case, Corollary 64 ensures that $w_3$ is not a corner as well. We proceed to show that this case indeed cannot happen.
Case 3.1: \( w_4 \in P_a \).

Since \( P_a \) is a geodesic, we get that \( w_4 = a_2 \). It is easy to see that \( xy \leq vy + vx = vy + 1 \) and that \( uv = uw_2 \leq w_2w_3 + v_3w_4 + w_4u = 2 + a_2u = xu \). Adding together, we obtain by Assumption II that

\[
2\delta^*(G) = (xy + uv) - \max(xv + yu, vy + xu) \leq (xy + uv) - (vy + xu) \leq 1,
\]

violating the assumption that \( \delta^*(G) \geq 1 \).

Case 3.2: \( w_4 \in P_d \).

Reasoning as in Case 3.1 rules out the possibility that this case may happen.

Case 3.3: \( w_4 \in P_c \).

In this case, \( Q(x, u, y, v) \) contains \( \delta \)-edges and hence Lemma 57 tells us

\[
xu + yv = xv + yu.
\]

But, by Lemma 54 (iii) we have \( xu - 1 = uw_5 = uw_2 \) and \( yv - 1 = w_3y = w_4y \). We therefore get that \( xu + yv = 2 + uw_2 + w_4y = 2 + yu = 1 + xv + yu \), which contradicts Eq. (65) and finishes the proof. \( \square \)

Lemma 65 Let \( G \) be a 5-chordal graph and let Assumptions I and II hold. If \( \min(ux, xv, uy, yv, 2\delta^*(G)) \geq 2 \), and \( Q(x, u, y, v) \) has no \( \delta \)-edges, then \( \delta^*(G) = 1 \) and either \( ux = xv = uy = yv = 2 \) or \( G \) has an isometric 4-cycle.

Proof: By Corollary \( \delta \) \( P(x, u, y, v) \) is a cycle of length at least 8. As \( G \) is 5-chordal, this cycle must have chords, which, by Corollary \( \delta \) again and by the fact that \( Q(x, u, y, v) \) has no \( \delta \)-edges, must be \( \delta \)-edges. So, without loss of generality, suppose that \( Q(x, u, y, v) \) has an \( \delta \)-edge between \( P_a \) and \( P_d \). On the one hand, we can thus go to Lemma 65 and get

\[
xv = yu = 2.
\]

On the other hand, this allows us to apply Lemmas \( \delta \) and \( \delta \) to deduce that \( \delta^*(G) = 1 \) and that either \( G \) has an isometric \( C_4 \) or has exactly one \( \delta \)-edge between \( P_a \) and \( P_d \). If the latter case happens, say we have an \( \delta \)-edge connecting \( a_i \) and \( d_j \), we will get two chordless cycles of \( G \), \( [a_i a_{i-1} \cdots x b_1 \cdots b_{yv-1} d_{yv-1} \cdots d_j] \) and \( [a_i a_{i+1} \cdots w_{yv-1} \cdots c_{yd} \cdots d_j] \). Since neither of these two chordless cycles can be longer than 5, it follows from Eq. (66) that \( a_i x + d_j v \leq 2 \) and \( u a_i + yd_{j} \leq 2 \). Taking into account additionally that \( 2 \leq ux = u a_i + a_i x \) and \( 2 \leq yv = yd_{j} + d_j v \), we thus have \( xu = yv = 2 \). This is the end of the proof. \( \square \)

Lemma 66 We take a 5-chordal graph \( G \) satisfying \( \delta^*(G) = 1 \) and require Assumptions I and II. Suppose that \( Q(x, u, y, v) \) has no \( \delta \)-edge and \( [ua_{xu-1}b_{xu-1}d_{yu-1}c_{yu-1}] \) is an induced 5-cycle of \( G \); see Fig. 20. Then \( G \) has at least one graph among \( C_4 \), \( H_3 \) and \( H_5 \) as an isometric subgraph.

Proof: By Corollary \( \delta \) and Lemma 54 (iii), it will be enough to consider the following cases, \( b_{xu-1}v = d_{yu-1}v > 3 \) or \( b_{xu-1}v = d_{yu-1}v \in \{1, 2\} \).

Case 1: \( b_{xu-1}v = d_{yu-1}v > 3 \).

Corollary \( \delta \) implies that \( G \) contains an isometric \( C_4 \).
Case 2: $b_{xu-1}v = d_{yu-1}v \in \{1, 2\}$.

Before jumping into the analysis of two separate subcases, we make some general observations. Note that

$$xu + yv + 2 = (xa_{xu-1} + ua_{xu-1}) + (yd_{yu-1} + d_{yu-1}v) + 2$$

$$= xb_{xu-1} + ua_{xu-1} + yd_{yu-1} + b_{xu-1}v + (a_{xu-1}b_{xu-1} + b_{xu-1}d_{yu-1})$$

$$= (xb_{xu-1} + b_{xu-1}d_{yu-1} + yd_{yu-1}) + (ua_{xu-1} + a_{xu-1}b_{xu-1} + b_{xu-1}v)$$

$$\geq xy + (ua_{xu-1} + a_{xu-1}v)$$

$$\geq xy + uv$$

$$= \max(xu + yv, xu + yu) + 2\delta^*(G) \quad \text{(By Assumption II)}$$

$$\geq xu + yv + 2. \quad \text{(By } \delta^*(G) = 1\text{)}$$

It follows that all inequalities in Eq. (67) are best possible and hence we have

$$uv = ua_{xu-1} + a_{xu-1}b_{xu-1} + b_{xu-1}v = 2 + b_{xu-1}v \quad \text{(68)}$$

and

$$a_{xu-1}v = a_{xu-1}b_{xu-1} + b_{xu-1}v = 1 + b_{xu-1}v. \quad \text{(69)}$$

Case 2.1: $b_{xu-1}v = d_{yu-1}v = 1$.

We derive from Corollary 54 that the subgraph of $G$ induced by $u, a_{xu-1}, b_{xu-1}, v, d_{yu-1}, c_{yu-1}$ is isomorphic to $H_3$ in an obvious way. Thanks to Corollary 41 in order to check that this $H_3$ is isometric, our task is to show that $uv = 3$. But $uv = 3$ is an immediate result of Eq. (68), proving the claim in this case.

Case 2.2: $b_{xu-1}v = d_{yu-1}v = 2$.

To start things off we look at the following:

$$b_{xu-1}v + 1 = d_{yu-1}v + 1 = d_{yu-1}d_{yu-1} + 2$$

$$= d_{yu-1}d_{yu-1} + a_{xu-1}b_{xu-1} + b_{xu-1}d_{yu-1}$$

$$\geq a_{xu-1}d_{yu-1} \quad \text{(By the triangle inequality)}$$

$$\geq xd_{yu-1} - xa_{xu-1} \quad \text{(By the triangle inequality)}$$

$$\geq xv - xa_{xu-1} \quad \text{(By Lemma 53)}$$

$$= (xb_{xu-1} + b_{xu-1}v) - xa_{xu-1}$$

$$= b_{xu-1}v.$$

A consequence of Eq. (70) is that

$$b_{xu-1}v + 1 \geq a_{xu-1}d_{yu-1} \geq b_{xu-1}v. \quad \text{(71)}$$
By symmetry, we also have
\[ b_{xu-1}v + 1 = d_{yu-1}v + 1 \geq c_{yu-1}b_{xv-1} \geq d_{yu-1}v = b_{xu-1}v. \] (72)

As a result of Eqs. (71) and (72) we obtain
\[ 3 \geq \max(a_{xu-1}d_{yu-1}, c_{yu-1}b_{xv-1}) \geq \min(a_{xu-1}d_{yu-1}, c_{yu-1}b_{xv-1}) \geq 2. \] (73)

Finally, let us remark that \( b_{xu-1}v = 2 \) implies \( xu - 1 = xv - 2 \) and hence \( b_{xu-1}b_{xv-1} = d_{xu-1}d_{xv-1} = 1. \)

According to Eq. (73), the following two subcases are exhaustive.

**Case 2.2.1:** \( \min(a_{xu-1}d_{yu-1}, c_{yu-1}b_{xv-1}) = 2. \)

Without loss of generality, we suppose that there is a vertex \( w \in V(G) \) such that \( a_{xu-1}w = wd_{yu-1} = 1. \) Note that Lemma 55(iii) says that \( w \notin \{a_{xu-1}, b_{xu-1}, b_{xv-1}, v, d_{yu-1}\} \) and hence \( C = [a_{xu-1}b_{xu-1}b_{xv-1}vd_{yu-1}w] \) is a 6-cycle in \( G. \) Because \( G \) is 5-chordal, \( C \) has at least one chord. Observe that Eq. (69) says that
\[ a_{xu-1}v = 1 + 2 = 3 \] (74)

and so
\[ wv \geq a_{xu-1}v - a_{xu-1}w = 3 - 1 = 2. \]

In consequence, by virtue of Lemma 55(iii), we have \( \min(wb_{xu-1}, wb_{xv-1}, b_{xv-1}d_{yu-1}) = 1. \) There are three cases to dwell on.

**Case 2.2.1.1:** If \( b_{xu-1}d_{yu-1} = 1 \), then \( [b_{xu-1}b_{xv-1}d_{yu-1}] \) is a required isometric 4-cycle.

**Case 2.2.1.2:** If \( wb_{xv-1} = 1 \) and \( b_{xu-1}d_{yu-1} > 1 \), we find that \( [b_{xv-1}vd_{yu-1}] \) is an isometric 4-cycle, as desired.

**Case 2.2.1.3:** If \( \min(wb_{xv-1}, b_{xv-1}d_{yu-1}) > wb_{xu-1} = 1 \), as a result of Eq. (74), we can make use of Corollary 41 to yield that the subgraph induced by \( a_{xu-1}, b_{xu-1}, b_{xv-1}, v, d_{yu-1}, w \) is an isometric \( H_3 \) in \( G; \) see Fig. 21.

**Case 2.2.2:** \( a_{xu-1}d_{yu-1} = c_{yu-1}b_{xu-1} = 3. \)

From Eq. (68) we obtain \( uv = 4. \) This, together with the standing assumption of Case 2.2, enables us to deduce from Corollary 42 that the subgraph induced by
\[ u, a_{xu-1}, b_{xu-1}, b_{xv-1}, v, d_{yu-1}, d_{yu-1}, c_{yu-1} \]
is an isometric \( H_5. \)

□
4.2 Proofs of Theorems 9 and 14

We now have all necessary tools to prove our main results.

**Proof of Theorem 9:** Using typical compactness argument, it suffices to prove that every connected finite induced subgraph of a $k$-chordal graph $G$ is $\lfloor \frac{k}{2} \rfloor$-hyperbolic. If $G$ has less than 4 vertices, the result is trivial. Thus, we can simply assume that $4 \leq |V(G)| < \infty$ and henceforth there surely exists a geodesic quadrangle $Q(x, u, y, v)$ in $G$ fulfilling Assumptions I and II. When $\min(d(P_a, P_d), d(P_b, P_c)) \leq 1$, the result is direct from Lemma 50 and the fact that $1 \leq \lfloor \frac{k}{2} \rfloor$ while when $\min(d(P_a, P_d), d(P_b, P_c)) > 1$ we are done by Lemma 59. □

**Proof of Theorem 14:** Consider a 5-chordal graph $G$ with $\delta^*(G) = 1$. We surely can get a geodesic quadrangle $Q(x, u, y, v)$ in $G$ for which Assumption I and Assumption II hold. Passing to the proof that $G$ contains one graph from Fig. 3 as an isometric subgraph, we have to distinguish four main cases.

**Case 1:** $\min(xu, xv, yu, yv) = 1$.

Lemma 64 tells us that $G$ has either an isometric $C_4$ or an isometric $H_3$ or an isometric $H_5$.

**Case 2:** $\min(xu, xv, yu, yv) \geq 2$ and there exist no $A$-edges.

**Case 2.1:** $\max(xu, xv, yu, yv) > 2$.

By Lemma 65, $G$ must have an isometric $C_4$.

**Case 2.2:** $xu = xv = yu = yv = 2$.

By Corollary 54, $Q(x, u, y, v)$ must have an $H$-edge. By Corollary 54, we may assume, without loss of generality, that $a_1 d_1 = 1$. It then follows from Lemma 51 (i) that $xy = wv = 3$.

**Case 2.2.1:** $Q(x, u, y, v)$ has only one $H$-edge and hence the subgraph of $G$ induced by its vertices is isomorphic to $H_5$.

By Lemma 46, $G$ has one of $C_4, H_2, H_3$ and $H_5$ as an isometric subgraph.

**Case 2.2.2:** $Q(x, u, y, v)$ has two $H$-edges and hence the subgraph of $G$ induced by its vertices is isomorphic to $H_4$.

By Corollary 40, $G$ contains $H_4$ as an isometric subgraph.

**Case 3:** $\min(xu, xv, yu, yv) \geq 2$ and there exist no $H$-edges.

Take $i, j, \ell, m$ to be the numbers as specified in Lemma 52. By Lemma 60, Eq. (63) holds. So, without loss of generality, we can assume that $i = j$, $a_j b_j = 1$ and $b_j d_{yv-xv+j} = 1$.

**Case 3.1:** $d_\ell c_\ell = a_m c_{yu-xu+m} = 1$.

By Lemma 61, the chordless cycle displayed in Eq. (62) is an isometric $C_4$.

**Case 3.2:** $(d_\ell c_\ell, a_m c_{yu-xu+m}) = (0, 1)$ or $(1, 0)$.

We only consider the case that $(d_\ell c_\ell, a_m c_{yu-xu+m}) = (0, 1)$. For now, the chordless cycle shown in Eq. (62) is just the 5-cycle $[a_j b_j d_{yv-xv+j} c_{yu-xu+j}]$; see Fig. 22. Lemma 66 demonstrates that $G$ contains one graph among $C_4, H_3$ and $H_5$ as an isometric subgraph.

**Case 3.3:** $d_\ell c_\ell = a_m c_{yu-xu+m} = 0$.

This case is impossible as the chordless cycle demonstrated in Eq. (62) will contain 6 different vertices $a_j, b_j, d_{yv-xv+j}, y, c_1, u$. 

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Case 4: \( \min(xu, xv, yu, yv) \geq 2 \) and there exist both \( H \)-edges and \( A \)-edges.

Before delving into the case by case analysis, here are some general observations. First note that Lemma 57 can be applied to give
\[
xu + yv = xv + yu.
\] (75)

Secondly, according to Corollary 54, we can suppose that there are
\[
1 \leq i \leq xu - 1 \quad \text{and} \quad 1 \leq j \leq yv - 1
\] (76)
such that \( a_id_j = 1 \) and, by Lemma 58, hence that
\[
a_ix + d_jy = xu \quad \text{and} \quad a_iu + d_jy = xv.
\] (77)

Thirdly, as \( \delta^*(G) = a_id_j = 1 \), Lemma 50 gives
\[
d(P_a, P_d) = 1.
\] (78)

Finally, Lemma 51 (i) says that the \( Z \)-walks of \( Q(x, u, y, v) \) through the \( H \)-edge \( \{a_i, d_j\} \) must be geodesics. Since any subpath of a geodesic is still a geodesic, we come to
\[
udj = ua_i + a_id_j = ua_i + 1 \quad \text{and} \quad a_iy = a_id_j + d_jy = 1 + d_jy.
\] (79)

CASE 4.1: \( yu = xv = 2 \).

In this case, Eq. (75) forces \( xu = xv = yu = yv = 2 \) and so Eq. (76) tells us that \( i = j = 1 \). It follows that \( \max(xy, uv) \leq 3 \) due to the existence of the path \( x, a_1, d_1, y \) and the path \( u, a_1, d_1, v \). For the moment, in view of Eq. (16), we can get
\[
xy = uv = 3.
\] (80)

Identifying \( a_1, b_1, c_1, d_1 \) with \( a, b, c, d \), respectively, Corollary 54 says that \( Q(x, u, y, v) \) is obtained from the graph \( H_5 \) as depicted in Fig. 3 by adding \( t \) additional edges among \( \{a, b\}, \{b, d\}, \{d, c\}, \{c, a\} \), where \( t \in \{1, 2, 3, 4\} \), and adding possibly the edge \( \{b, c\} \).

If \( t = 4 \) and \( bc = 1 \), we easily infer from Eq. (80) and Corollary 40 that \( Q(x, u, y, v) \) is an isometric subgraph of \( G \) which is isomorphic to \( H_2 \).

If \( t = 4 \) and \( bc > 1 \), we can check that \( Q(x, u, y, v) \) is an induced subgraph of \( G \) isomorphic to \( H_1 \) and then, again by Eq. (80) and Corollary 40 \( G \) contains an isometric \( H_1 \).

If \( t < 4 \), as a consequence of Lemma 44, either \( C_4 \) is an induced subgraph of \( G \) or \( Q(x, u, y, v) \) is isomorphic with \( H_6 \). Accordingly, Eq. (80) together with Lemma 45 implies that \( G \) has an isometric subgraph which is isomorphic to either \( C_4 \) or \( H_2 \) or \( H_3 \).
Figure 23: Case 4.2.1 in the proof of Theorem 14.

**CASE 4.2**: \( \max(yu, xv) > 2 \).

We will show that \( G \) contains an isometric subgraph which is isomorphic to \( H_3 \), under the assumption that \( G \) has no isometric \( C_4 \). Note that the nonexistence of an isometric \( C_4 \) in \( G \) together with Eq. (78) yields that there exists exactly one \( \mathbb{H} \)-edge between \( P_a \) and \( P_d \), namely \( \{a_i, d_j\} \), as a result of Lemma 51 (i).

It is no loss of generality in setting \( yu > 2 \). (81)

By Lemma 55 (i) and Eq. (78), the following is a set of pairwise different vertices:

\[
y, c_1, \ldots, c_{yu-1}, u, a_{xu-1}, \ldots, a_i, d_j, d_{j-1}, \ldots, d_1.
\]

In the subgraph \( F \) induced by these vertices in \( G \), \( a_i \) and \( d_j \) are connected by a path disjoint from the edge \( \{a_i, d_j\} \). This means that there is a chordless cycle \( [w_1w_2 \cdots w_n] \) in \( F \) where \( n \geq 3 \) and \( w_1 = a_i, w_2 = d_j \). Recall that it is already stipulated that the 5-chordal graph \( G \) has no isometric 4-cycle and hence \( n \) can only take value either 3 or 5.

**CASE 4.2.1**: \( n = 3 \).

Since there is exactly one \( \mathbb{H} \)-edge between \( P_a \) and \( P_d \), \( w_3 \) is neither on \( P_a \) nor on \( P_d \). Hence, there is \( 0 < q < yu \) such that \( w_3 = c_q \). It follows from Lemma 55 (iii) that \( uw_i = uc_q \) and \( yc_q = yd_j \). From Eq. (81), we obtain \( \max(yc_q, c_qu) \geq 2 \). Without loss of generality, assume that \( yc_q = \max(yc_q, c_qu) \geq 2 \). Since \( G \) contains no isometric \( C_4 \), we infer from Corollary 62 that \( q = j = 2 \) and \( c_1d_1 = 2 \). This then demonstrates that the subgraph induced by the vertices \( a_i, d_2, d_1, y, c_1, c_2 \) is isomorphic to \( H_3 \); see Fig. 23. Granting that \( a_iy = 3 \), Corollary 41 will bring to us that \( G \) contains \( H_3 \) as an isometric subgraph. But \( a_iy = 3 \) follows from Eq. (79) and \( d_2j = j = 2 \).

**CASE 4.2.2**: \( n = 5 \).

We aim to prove that this case will never happen by deducing contradictions in all the following subcases.

**CASE 4.2.2.1**: Both \( w_3 \) and \( w_5 \) belong to \( P_e \).

First consider the case that both \( w_3 \) and \( w_5 \) are ordinary vertices of \( P_e \). From Lemma 55 (iii) we obtain \( a_iu = uw_5 \) and \( d_jy = w_3y \). It then follows \( uy = uw_5 + w_3y \) by means of Eq. (77). Since \( w_3 \) and \( w_5 \) are on the same geodesic connecting \( u \) and \( y \), this is possible only when \( w_3 = w_5 \), yielding a contradiction.

Next the case that at least one of \( w_3 \) and \( w_5 \) is a corner. We could assume that \( w_3 \) is a corner, and then, in view of Corollary 51, it holds \( w_3 = y \). This implies that \( w_5 \neq u \), as otherwise we obtain \( yu = w_3w_5 = 2 \), contradicting Eq. (81). Accordingly, it follows from Lemma 55 (iii) that \( a_iu = uw_5 \). But,
we surely have \( 2 = w_5w_3 = w_5y \) and \( yd_j = w_3w_2 = 1 \). Putting together, we get \( a_i u + yd_j = uw_5 + 1 < uw_5 + 2 = uw_5 + w_3y = uy \), contradicting Eq. (77).

**Case 4.2.2.2:** Neither \( w_3 \) nor \( w_5 \) belongs to \( P_c \).

Because there is just one \( \mathbb{H} \)-edge between \( P_a \) and \( P_d \), we see that \( w_3, w_4, w_5 \) are ordinary vertices of \( P_d, P_c \) and \( P_a \), respectively. By Lemma 55 (iii), it occurs that \( a_i u = 1 + w_5u = 1 + w_4u \) and \( d_j y = 1 + w_3y = 1 + w_3y \). Consequently, we arrive at \( a_i u + d_j y = 2 + uy \), which is contrary to Eq. (77).

**Case 4.2.2.3:** One of \( w_3 \) and \( w_5 \) is outside of \( P \), and the other lies inside \( P \).

Incurred no loss of generality, we make the assumption that \( w_5 \notin P_c \) and \( w_3 \in P_c \). As there exists only one \( \mathbb{H} \)-edge between \( P_a \) and \( P_d \), we know that \( w_5 = a_{i+1} \) and either \( w_4 = a_{i+2}, w_3 \neq y \) or \( w_4 \in P_c \setminus \{u\} \). By Lemma 55 (iii), the former implies that \( uy = uw_3 + w_3y = uw_4 + w_3y = uw_4 + yd_j = ua_i + yd_j - 2 \), violating Eq. (77). In consequence, we must have \( w_4 \in P_c \setminus \{u\} \) and hence it holds either \( w_3 = c_t, w_4 = c_{t+1} \) or \( w_3 = c_{t+1}, w_4 = c_t \) for some \( t < yu - 1 \). If it happens the latter case, we deduce from Lemma 55 (iii) that \( uy = uw_5 + w_3y - 1 = uw_5 + d_jy - 1 = ua_i + d_jy - 2 \), yielding a contradiction with the first part of Eq. (77). At this point, our object is to exclude the first possibility as well. By way of contradiction, let us assume that this case happens and turn to the quartet \((w_1, w_2, w_3, u)\). The following calculation can be trivially verified:

\[
\begin{align*}
uw_1 + w_2w_3 &= uw_1 + 1; \\
uw_2 + w_1w_3 &= (uw_1 + 1) + 2 \quad \text{(By the first part of Eq. (79))} \\
&= uw_1 + 3; \\
uw_3 + w_1w_2 &= (uw_1 + 1) + 1 \\
&= (uw_5 + 1) + 1 \quad \text{(By Lemma 55 (iii))} \\
&= uw_1 + 1.
\end{align*}
\]

This gives \( \delta(u, w_1, w_2, w_3) = 1 = \delta^*(G) \) and \( \max(uw_2 + w_1w_3, uw_1 + w_2w_3, uw_3 + w_1w_2) = uw_1 + 3 = (uw_1 + 1) + 2 \leq ux + yv \leq xy + uw - 2\delta^*(G) = xy + uv - 2 \), which is the desired contradiction to Assumption II on \( Q(x, u, y, v) \).

\[\square\]

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