A map between $q$-deformed and ordinary gauge theories

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Abstract. In complete analogy with the Seiberg–Witten map defined in noncommutative geometry we introduce a new map between a $q$-deformed gauge theory and an ordinary gauge theory. The construction of this map is elaborated in order to fit the Hopf algebra structure.

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1. Introduction

The concept of spacetime as a space of commuting coordinates is perhaps too naive and must possibly be modified at the Planck scale. In their seminal paper Connes et al [1] have found that noncommutative geometry [2] arises naturally in string theory. Quantum groups [3] provide another consistent mathematical framework to formulate physical theories on noncommutative spaces. They appeared first in the study of integrable systems [4] and are now applied to many branches of mathematics and physics. Although a tremendous amount of literature has been devoted to the study of quantum groups, we are still lacking a map which relates the quantum
deformed gauge theories and the ordinary ones. In the present paper, we introduce such a map. This map is a quantum analogue of the Seiberg–Witten map [5].

Let us first recall that the Seiberg–Witten map was discovered in the context of string theory where it emerged from the 2D-σ-model regularized in different ways. Seiberg and Witten have shown that the noncommutativity depends on the choice of the regularization procedure: it appears in point-splitting regularization whereas it is not present in the Pauli Villars regularization. This observation led them to argue that there exists a map connecting the noncommutative gauge fields and gauge transformation parameter to the ordinary gauge field and gauge parameter. This map can be interpreted as an expansion of the noncommutative gauge field in \( \theta \). Along similar lines, we introduce a new map between the \( q \)-deformed and undeformed gauge theories. This map can be seen as an infinitesimal shift in the parameter \( q \), and thus as an expansion of the deformed gauge field in \( q = e^{i\eta} = 1 + i\eta + o(\eta^2) \).

This paper is organized as follows. In section 2, we recall the Seiberg–Witten map. In section 3, we present the \( SU_q(2) \) quantum group techniques. In section 4, we recall the Woronowicz differential calculus using the adjoint representation \( M^q_b \) of the group \( SU_q(2) \). In section 5, we consider a hybrid structure consisting of a noncommutative base space defined by a Moyal product and a \( q \)-deformed nonabelian gauge group. This structure allows us to define a map which relates the \( q \)-deformed noncommutative and the ordinary gauge fields. We close this paper by constructing a new map relating the \( q \)-deformed and ordinary gauge fields.

2. The Seiberg–Witten map

Let us recall that in the noncommutative geometry the spacetime coordinates \( x^i \) are replaced by the Hermitian generators \( \hat{x}^i \) of a noncommutative \( C^* \)-algebra of functions on spacetime which obey the following algebra:

\[
[\hat{x}^i, \hat{x}^j] = i\theta^{ij},
\]

where \( \theta^{ij} \) is real and antisymmetric with the dimension of length squared. In this context the ordinary product of two functions is replaced by the Groenewold–Moyal star product [6]

\[
f(x) \star g(x) = f(x) \exp\left(\frac{i}{2} \partial_i \theta^{ij} \partial_j\right) g(x) = f(x)g(x) + \frac{i}{2} \theta^{ij} \partial_i f(x) \partial_j g(x) + o(\theta^2). \tag{2}
\]

For the ordinary Yang–Mills theory the infinitesimal gauge transformations and field strength are given by

\[
\delta_i A_i = \partial_i \lambda + i[\lambda, A_i]
\]

\[
F_{ij} = \partial_i A_j - \partial_j A_i - i[A_i, A_j]
\]

\[
\delta_i F_{ij} = i[\lambda, F_{ij}]
\]

where the symbols \([.,.\)] mean commutators.

For the noncommutative gauge theory we just replace the matrix multiplication by the \( \star \) product. The infinitesimal gauge transformations are given by

\[
\hat{\delta}_i \hat{A}_i = \partial_i \hat{\lambda} + i\hat{\lambda} \star \hat{A}_i - i\hat{A}_i \star \hat{\lambda}
\]

\[
\hat{\delta}_i \hat{F}_{ij} = i\hat{\lambda} \star \hat{F}_{ij} - i\hat{F}_{ij} \star \hat{\lambda}
\]

where

\[
\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i\hat{A}_i \star \hat{A}_j + i\hat{A}_j \star \hat{A}_i.
\]

New Journal of Physics 5 (2003) 7.1–7.9 (http://www.njp.org/)
Using first order expansion in $\theta$ these relations give

$$\hat{\delta} \hat{A}_i = \partial_i \hat{\lambda} + \hat{\lambda} \hat{A}_i - \hat{A}_i \hat{\lambda} - \frac{1}{2} \theta^{ij} \partial_j \hat{\lambda} \partial_i \hat{A}_i + \frac{1}{2} \theta^{jk} \partial_k \hat{A}_i \partial_j \hat{\lambda} + o(\theta^2)$$

(6)

$$\hat{\delta} \hat{F}_{ij} = i \hat{F}_{ij} \hat{\lambda} - i \hat{\lambda} \hat{F}_{ij} - \frac{1}{2} \theta^{kl} \partial_k \hat{\lambda} \partial_l \hat{F}_{ij} + \frac{1}{2} \theta^{kl} \partial_k \hat{F}_{ij} \partial_l \hat{\lambda} + o(\theta^2).$$

To ensure that an ordinary gauge transformation of $\hat{A}$ by $\lambda$ is equivalent to a noncommutative gauge transformation of $\hat{A}$ by $\hat{\lambda}$, Seiberg and Witten have proposed the following relation:

$$\hat{A}(A) + \hat{\delta} \hat{A}(A) = \hat{A}(A + \delta A).$$

(7)

They first worked the first order in $\theta$ and wrote

$$\hat{A}(\lambda, A) = A + \lambda'(\lambda, A).$$

(8)

Expanding (7) in powers of $\theta$ they found

$$\lambda'(A) - \delta \lambda' - i[\lambda', A_i] - i[\lambda, A'_i] = -\frac{1}{2} \theta^{ij} (\partial_j \lambda \partial_i A_i + \partial_i A_i \partial_j \lambda) + o(\theta^2),$$

(9)

where they used the expansion

$$f * g = fg + \frac{1}{2} i \theta^{ij} \partial_i f \partial_j g + o(\theta^2).$$

(10)

Equation (9) is solved by

$$\hat{A}_i(A) = A_i + A'_i(A) = A_i - \frac{1}{2} \theta^{ij} \partial_j A_i + o(\theta^2)$$

(11)

$$\hat{\lambda}(\lambda, A) = \lambda + \lambda'(\lambda, A) = \lambda + \frac{1}{4} \theta^{ij} \{\partial_i \lambda, A_j\} + o(\theta^2)$$

where the symbols $\{\ldots\}$ mean star anticommutators.

Equations (11) are called the Seiberg–Witten map.

3. The quantum group $SU_q(2)$

Let $A$ be the associative unital $C$-algebra generated by the linear transformations $M^n_m$ $(n, m = 1, 2)$

$$M^n_m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$  

(12)

the elements $a, b, c, d$ satisfying the relations

$$ab = qba \quad bc = cb \quad ac = qca,$$

$$bd = qdb \quad cd = qdc \quad ad - da = (q - q^{-1})bc$$

(13)

where $q$ is a deformation parameter. The classical case is obtained by setting $q$ equal to one.

$U_q(2)$ is obtained by requiring that the unitary condition hold for this $2 \times 2$ quantum matrix:

$$M^n_m^\dagger = M^n_m^{-1}.$$  

(14)

The $2 \times 2$ matrix belonging to $U_q(2)$ preserves the nondegenerate bilinear form $[7] B_{nm}$,

$$B_{nm} M^m_k M^n_l = D_q B_{kl}, \quad B^m_n M^l_m = D_q B^{kl}, \quad B_{kn} B_{nl} = \delta^l_k,$$

(15)

where

$$B_{nm} = \begin{pmatrix} 0 & -q^{-1/2} \\ q^{1/2} & 0 \end{pmatrix}, \quad B^{nm} = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}.$$  

(16)
and $D_q = ad - qbc$ is the quantum determinant. $SU_q(2)$ is obtained by taking the unimodularity condition $D_q = 1$.

Let us take $q = e^{i\eta} \simeq 1 + i\eta$. This gives

$$B_{nm} = \epsilon_{nm} + \frac{i}{2} b_{nm}, \quad B^*_{nm} = \epsilon'^{nm} + \frac{i}{2} b'^{nm}$$

where

$$\epsilon_{nm} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon'^{nm} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$b_{nm} = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}, \quad b'^{nm} = \begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix}.$$  \hfill (17)

The noncommutativity of the elements $M^m_n$ is controlled by the braiding matrix $R$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 1-q^2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  \hfill (19)

$R$ becomes the permutation operator $R^{nm}_{kl} = \delta^n_k \delta^m_l$ in the classical case $q = 1$.

The $R$ matrix satisfies the Yang–Baxter equation

$$R^{ij}_{pq} R^{pk}_{lr} R^{qr}_{mn} = R^{jk}_{pq} R^{ip}_{rm} R^{rq}_{lm}.$$  \hfill (20)

The noncommutativity of the elements $M^m_n$ is expressed as

$$R^{nm}_{kl} M^m_n M^k_l = M^m_n M^k_l R^{nm}_{kl}.$$  \hfill (21)

With the nondegenerate form $B$ the $R$ matrix has the form

$$R^{nm}_{kl} = \delta^n_k \delta^m_l + q B^{nm} B_{kl},$$

$$R^{nm}_{kl} = \delta^n_k \delta^m_l + q^{-1} B^{nm} B_{kl}.$$  \hfill (22)

The first equation, in terms of $\eta$, gives

$$R^{nm}_{kl} = \delta^n_k \delta^m_l + \epsilon^{nm} \epsilon_{kl} + \frac{i}{2} \epsilon^{nm} b_{kl} + \frac{1}{2} \eta^{nm} \epsilon_{kl} + o(\eta^2).$$  \hfill (23)

4. Woronowicz differential calculus

Now, we are going to consider the bicovariant bimodule [8] $\Gamma$ over $SU_q(2)$. Let $\theta^a$ be a left invariant basis of $\Gamma$, the linear subspace of all left-invariant elements of $\Gamma$, i.e. $\Delta_L (\theta^a) = I \otimes \theta^a$. For $q = 1$ the left coaction $\Delta_L$ coincides with the pullback for one-forms.

There exists an adjoint representation $M^a_b$ of the quantum group, defined by the right action on the left-invariant $\theta^a$:

$$\Delta_R (\theta^a) = \theta^b \otimes M^a_b, \quad M^a_b \in \mathcal{A}.$$  \hfill (24)

The adjoint representation is given in terms of the fundamental representation [9] as

$$(M^a_b) = \begin{pmatrix}
S(a)a & S(a)b & S(c)a & S(c)b \\
S(a)c & S(a)d & S(c)c & S(c)d \\
S(b)a & S(b)b & S(d)a & S(d)b \\
S(b)c & S(b)d & S(d)c & S(d)d
\end{pmatrix}.$$  \hfill (25)

where $S(\cdot)$ means the antipode.
In the quantum case we have $\theta^a M^m{}_n \neq M^m{}_n \theta^a$ in general, the bimodule structure of $\Gamma$ being non-trivial for $q \neq 1$. There exist linear functionals $f^a{}_b : \text{Fun}(SU_q(2)) \to \mathbb{C}$ for these left-invariant bases such that

$$\theta^a M^m{}_n = (f^a{}_b * M^m{}_n) \theta^b = (id \otimes f^a{}_b) \Delta(M^m{}_n) \theta^b = M^k{}_n f^a{}_b(M^k{}_m) \theta^b,$$

(26)

$$M^m{}_n \theta^a = \theta^b ((f^a{}_b \circ S^{-1}) * M^m{}_n) \theta^b,$$

(27)

where $\Delta$ refers to the coproduct. $\circ$ is the convolution product of an element $M^m{}_n \in \mathcal{A}$ and a functional $f^a{}_b$ [8].

Once we have the functionals $f^a{}_b$, we know how to commute elements of $\mathcal{A}$ through elements of $\Gamma$. These functionals are given by [10]

$$f^a{}_b(M^m{}_n) = q^{-\frac{1}{2}} R^{an}{}_mb.$$

(28)

The representation with the lower index of $\theta^a$ is defined by using the bilinear form $B$

$$\theta_b = \theta^a B_{ab}$$

(29)

which defines the new functional $\hat{f}^a{}_b$ corresponding to the basis $\theta_a$

$$\hat{f}^c{}_d = B_{ad} f^a{}_b B^{cb}$$

(30)

$$\theta_a M^m{}_n = (\hat{f}^b{}_a * M^m{}_n) \theta_b.$$

(31)

We can also define the conjugate basis $\tilde{\theta}^a = (\theta^a)^* \equiv \bar{\theta}_a$. Then the linear functionals $\tilde{f}^a{}_b$ are given by

$$\bar{\theta}_b M^m{}_n = (\tilde{f}^a{}_b * M^m{}_n) \bar{\theta}_a$$

(32)

and

$$\tilde{f}^a{}_b (S(M^m{}_n)) = q^{\frac{1}{2}} R^{-an}{}_mb.$$

(33)

We can easily find the transformation of the adjoint representation for the quantum group which acts on the generators $M^i{}_j$ as the right coaction $Ad_R$:

$$Ad_R(M^i{}_j) = M^k{}_l \otimes S(M^i{}_l) M^l{}_j.$$

(34)

As usual, in order to define the bicovariant differential calculus with the $*$-structure we have required that the $*$-operation is a bimodule antiautomorphism $(\Gamma Ad)^* = \Gamma Ad$. We found that the left-invariant bases containing the adjoint representation are obtained by taking the tensor product $\theta_i \tilde{\theta}^j \equiv \theta_i^j$ of two fundamental modules. The bimodule generated by these bases is closed under the $*$-operation. We found the right coaction on the basis $\theta_i^j$

$$\Delta_R(\theta_i^j) = \theta_i^k \otimes S(M^i{}_l) M^l{}_j.$$

(35)

We have also introduced the basis $\theta_{ij} = \theta_i \tilde{\theta}_j$

$$\theta_{ab} M^m{}_n = (f_{Ad}^{cd} ab * M^m{}_n) \theta_{cd} = f_{Ad}^{cd} ab (M^k{}_m) M^m{}_n \theta_{cd}$$

(36)

where

$$f_{Ad}^{cd} ab = \tilde{f}_b^d \ast \hat{f}_a^c.$$

(37)

The exterior derivative $d$ is defined as

$$d M^m{}_n = \frac{1}{N} [X, M^m{}_n]_\ast = (\chi^{ab} * M^m{}_n) \theta_{ab} = \chi^{ab} (M^k{}_m) M^m{}_n \theta_{ab}$$

(38)
where $X = B^{ab} \theta_{ab} = q^{-1/2} \theta_{12} - q^{1/2} \theta_{21}$ is the singlet representation of $\theta^{ab}$ and is both left and right co-invariant, $N \in C$ is the normalization constant which we take as purely imaginary, $N^* = -N$, and $\chi_{ab}$ are the quantum analogue of left-invariant vector fields.

Using equation (36)

$$dM^m_n = \frac{1}{N} (B^{ab} \theta_{ab} M^m_n - B^{ab} \delta^k_m M^k_n \delta_{ab}^\epsilon) = \frac{1}{N} (B^{ab} f_{Ad}^{cd} \theta_{cd} M^m_k - B^{ab} \delta^k_m M^k_n \delta_{ab}^\epsilon).$$

(39)

Then the left-invariant vector field is given by

$$\chi_{ab} = \frac{1}{N} (B^{cd} f_{Ad}^{ab} cd - B^{ab} \epsilon).$$

(40)

To construct the higher-order differential calculus an exterior product, compatible with left and right actions of the quantum group, was introduced. It can be defined by a bimodule automorphism $\Lambda$ in $\Gamma_{Ad} \otimes \Gamma_{Ad}$ that generalizes the ordinary permutation operator:

$$\Lambda(\eta_{ab} \otimes \theta_{cd}) = \theta_{cd} \otimes \eta_{ab}.$$  

(41)

We found [10]

$$\Lambda^{efgh}_{abcd} = \frac{1}{N} f_{Ad}^{ef} (M^a_h) M^b_k = R_i^{ef} R_j^{gh} R_k^{ij} R_l^{jg}.$$  

(42)

The external product is defined by

$$\theta_{ab} \wedge \theta_{cd} = (\theta_{ab} \otimes \theta_{cd}) \Lambda^{efgh}_{abcd} (\theta_{ef} \otimes \theta_{gh}).$$  

(43)

The quantum commutators of the quantum Lie algebra generators $\chi^{ab}$ are defined as

$$[\chi^{ab}, \chi^{cd}] = (1 - \Lambda)_{abcd}^{efgh} (\chi^{ef} \star \chi^{gh}),$$  

(44)

$$[\chi^{ab}, \chi^{cd}](M^{ij}) = (\chi^{ab} \otimes \chi^{cd}) \text{Ad}_g(M^{ij}) = \chi^{ab}(M^k_i) \otimes \chi^{cd}(S(M^j_i) M^k_j)$$  

(45)

where the convolution product of two functionals [8] is given by

$$\chi^{ab} \star \chi^{cd} = (\chi^{ab} \otimes \chi^{cd}) \Delta.$$  

(46)

The $\chi^{ab}$ functionals close on the quantum Lie algebra

$$[\chi^{ab}, \chi^{cd}](M^{km}) = (1 - \Lambda)_{abcd}^{efgh} (\chi^{ef} \star \chi^{gh})(M^{km}) = C_{ef}^{abcd} \chi^{ef}(M^{km}),$$

(47)

where $C_{ef}^{abcd}$ are the $q$-structure constants. They can also be expanded in terms of $\eta$ as

$$C_{ef}^{abcd} = C_{ef}^{abcd} + c_{ef}^{abcd},$$

(48)

where $C_{ef}^{abcd}$ is the classical matrix and $c_{ef}^{abcd}$ is the quantum correction linear in $\eta$.

The quantum Lie algebra generators $\chi^{ab}$ satisfy the quantum Jacobi identity

$$[\chi^{gh}, [\chi^{ab}, \chi^{cd}]] = [[\chi^{gh}, \chi^{ab}], \chi^{cd}] - \Lambda^{abcd}_{klmn}[[\chi^{gh}, \chi^{kl}], \chi^{mn}].$$

(49)

The quantum Killing metric is given by

$$g^{ab,cd} = \text{Tr}(\chi^{ab}(M^k_i) \chi^{cd}(M^k_i)).$$

(50)

Let us recall that the quantum gauge theory on a quantum group $SU_q(2)$ is constructed in such a way that the gauge transformations fit the Hopf algebra structures [11]. Given a left $\text{Fun}(SU_q(2))$-comodule algebra $V$ and a quantum algebra base $X_B$, a quantum vector bundle...
can be defined. The matter fields $\psi$ can be seen as sections: $X_B \to V$ and $V$ as a fibre of $E(X_B, V, Fun(SU_q(2)))$ with a quantum structure group $Fun(SU_q(2))$.

The quantum Lie-algebra-valued curvature $F : Fun(SU_q(2)) \to \Gamma^2(X_B)$ is given by

$$F_m^n = F(M_m^n) = \nabla_m \wedge \nabla_l^n = dA_m^n + A_m^l \wedge A_l^n.$$  \hfill (51)

We found the infinitesimal gauge transformations [10]

$$\delta_a A = -d\alpha + A \wedge \alpha = -d\alpha_{ab}\chi^{ab} + A_{ab}\alpha_{cd}[\chi^{ab}, \chi^{cd}]$$

$$\delta_a F = F \wedge \alpha = F_{ab}\alpha_{cd}[\chi^{ab}, \chi^{cd}] \text{Ad}_R.$$  \hfill (52)

$$\delta_a F = F \wedge \alpha = F_{ab}\alpha_{cd}[\chi^{ab}, \chi^{cd}] \text{Ad}_R = F_{ab}\alpha_{cd}[\chi^{ab}, \chi^{cd}]$$  \hfill (53)

$$\delta_a F = F \wedge \alpha = F_{ab}\alpha_{cd}[\chi^{ab}, \chi^{cd}] \text{Ad}_R = F_{ab}\alpha_{cd}[\chi^{ab}, \chi^{cd}]$$  \hfill (54)

where $\alpha = \alpha_{ab}\chi^{ab} : Fun(SU_q(2)) \to X_B$ and where the product $(\cdot)$ denotes the exterior product of two forms on the base $X_B$.

We can write the last relation in terms of components as

$$\delta_a F_m^n = \delta_a F_{ab}\chi^{ab}(M_m^n) = (F \otimes \alpha) \text{Ad}_R(M_m^n) = F_{cd}\chi^{cd}(M_m^n) \otimes \alpha_{ef}\chi^{ef}(S(M_m^l)M_k^n).$$  \hfill (55)

5. $q$-deformed noncommutative gauge symmetry versus ordinary gauge symmetry

Let us consider the quantum vector bundle $E(X_B, V, Fun(SU_q(2)))$ where the base space $X_B$ is the Moyal plane defined through the functions of operator-valued coordinates $\hat{x}^i$ satisfying (1) and where $V$ is a left $Fun(SU_q(2))$-comodule algebra. We define the $q$-deformed noncommutative gauge transformations as

$$\hat{\delta}_a \hat{A} = -d\hat{\alpha} + \hat{A} \wedge \hat{\alpha} = -d\hat{\alpha}_{ab}\chi^{ab} + \hat{A}_{ab}\hat{\alpha}_{cd}[\chi^{ab}, \chi^{cd}] = -d\hat{\alpha} + \hat{A} \hat{\alpha} - \hat{\alpha} \hat{A}$$  \hfill (56)

$$\hat{\delta}_b \hat{A} = \hat{A} \wedge \hat{\beta} = \hat{A}_{ab}\hat{\beta}_{cd}[\chi^{ab}, \chi^{cd}] \text{Ad}_R = \hat{\alpha}_{ab}\hat{\beta}_{cd}[\chi^{ab}, \chi^{cd}] = \hat{\alpha} \hat{\beta} - \hat{\beta} \hat{\alpha}$$  \hfill (57)

$$\hat{\delta}_a \hat{F} = \hat{F} \wedge \hat{\alpha} = \hat{F}_{ab}\hat{\alpha}_{cd}[\chi^{ab}, \chi^{cd}] \text{Ad}_R = \hat{\alpha}_{ab}\hat{\alpha}_{cd}[\chi^{ab}, \chi^{cd}]$$  \hfill (58)

where $\star$ is the Groenewold–Moyal star-product defined in (2), the convolution product $\star$ is defined in (46) and where the new product $\circ$ is defined as

$$\hat{\alpha} \circ \hat{\beta} = \hat{\alpha}_{ab}\hat{\beta}_{cd}[\chi^{ab}, \chi^{cd}].$$  \hfill (59)

The classical case is obtained setting $q = 1$ and $\theta = 0$.

Using first order expansion in $\theta$ and $q = 1 + i\eta$ these relations give

$$\hat{\delta}_a \hat{A}_{ijef} = \partial_i\hat{\alpha}_{ef} + ([\hat{\alpha}_{ab}, \hat{A}_{cd}] - \frac{1}{2}\theta^{jk}[\partial_j\hat{\alpha}_{ab}, \partial_k\hat{A}_{cd}])(C_{bcde}^{abcd} + c_{bcde}^{abcd})$$  \hfill (60)

$$\hat{\delta}_a \hat{F}_{ijef} = ([\hat{F}_{ijab}, \hat{\alpha}_{cd}] - \frac{1}{2}\theta^{kl}[\partial_k\hat{\alpha}_{ab}, \partial_l\hat{F}_{cdij}])(C_{bcde}^{abcd} + c_{bcde}^{abcd}).$$

The map between a $q$-deformed noncommutative gauge field and an ordinary gauge field is given by

$$\hat{A}_{ijef}(A) = A_{ijef} + A'_{ijef}(A) = A_{ijef} - \frac{1}{4}\theta^{kl}[A_{kab}, \partial_lA_{jcd}] + F_{jedc}(C_{bcde}^{abcd} + c_{bcde}^{abcd}) + o(\theta^2)$$

$$\hat{\alpha}_{ef}(A) = \alpha_{ef} + \alpha'_{ef}(\alpha, A) = \alpha_{ef} + \frac{1}{4}\theta^{ij}[\partial_i\alpha_{ab}, A_{jcd}](C_{bcde}^{abcd} + c_{bcde}^{abcd}) + o(\theta^2).$$  \hfill (61)
6. $q$-deformed gauge symmetry versus ordinary gauge symmetry

We consider now a Manin plane $x^i = (x, y)$, defined by $xy = qyx$, as a base space $X_B$ of the quantum vector bundle. Instead of the Groenewold–Moyal star product we use the Gerstenhaber \cite{12} star product which is defined by

$$f \star g = \mu e^{i\eta x \partial_x \otimes y \partial_y} f \otimes g$$

where

$$\mu(f \otimes g) = fg, \quad q = e^{i\eta}. \quad (62)$$

We can write this product as

$$f \star g = \sum_{r=0}^{\infty} \frac{(i\eta)^r}{r!} \left( x \frac{\partial}{\partial x} \right)^r f \left( y \frac{\partial}{\partial y} \right)^r g = fg + i\eta x \frac{\partial}{\partial x} fy \frac{\partial}{\partial y} g + o(\eta^2). \quad (64)$$

The quantum Lie-algebra-valued potential $\hat{A}$ and curvature $\hat{F}$ are given by

$$\hat{A} = \hat{A}_{ab} dx^i \chi^{ab}, \quad (65)$$

$$\hat{F}_{ij} = (\partial_i \hat{A}_{abj} - (q)P(ij) \partial_j \hat{A}_{abi}) \chi^{ab} + \frac{1}{2} \hat{A}_{lab} \star \hat{A}_{jcd} [\chi^{ab}, \chi^{cd}] \quad (66)$$

where the symbol $P(ij)$ is given by \cite{13}

$$P(ij) = \begin{cases} 1 & (i > j) \\ 0 & (i = j) \\ -1 & (i < j) \end{cases} \quad (67)$$

Using the same method we find

$$\hat{\alpha}_{ef}(\hat{A}) = \alpha_{ef} + \frac{1}{2} \eta \{ x \partial_{x} \alpha_{ab}, A_{2cd} \}(C_{ef}^{abcd} + c_{ef}^{abcd}) + o(\eta^2) \quad (69)$$

and the new map is given by

$$\hat{A}_{ef}(A) = A_{ef} + A_{ef}'(A) = A_{ef} - \frac{1}{2} \eta \{ x A_{lab}, y \partial_{x} A_{icd} + y F_{2icd} \}(C_{ef}^{abcd} + c_{ef}^{abcd}) + o(\eta^2), \quad (68)$$

$$\hat{\alpha}_{ef}(\alpha, A) = \alpha_{ef} + \alpha_{ef}'(\alpha, A) = \alpha_{ef} + \frac{1}{2} \eta \{ x \partial_{x} \alpha_{ab}, A_{2cd} \}(C_{ef}^{abcd} + c_{ef}^{abcd}) + o(\eta^2).$$

We can also take a Jordanian plane as a base space of the Jordanian vector bundle \cite{14, 15}. This gives a new map for the corresponding deformed structure.

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