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CONCENTRATING SOLUTIONS FOR AN ANISOTROPIC ELLIPTIC PROBLEM WITH LARGE EXPONENT

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Abstract. We consider the following anisotropic boundary value problem
\[ \nabla (a(x)\nabla u) + a(x)u^p = 0, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \]
where \( \Omega \subset \mathbb{R}^2 \) is a bounded smooth domain, \( p \) is a large exponent and \( a(x) \) is a positive smooth function. We investigate the effect of anisotropic coefficient \( a(x) \) on the existence of concentrating solutions. We show that at a given strict local maximum point of \( a(x) \), there exist arbitrarily many concentrating solutions.

1. Introduction and Statement of the results. This paper is concerned with analysis of solutions to the boundary value problem
\[
\begin{cases}
\nabla (a(x)\nabla u) + a(x)u^p = 0 & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^2 \), \( p \) is a large exponent and \( a(x) \) is a smooth positive function over \( \overline{\Omega} \). Problem (1) was motivated by the study of the following equation
\[ \Delta u + u^p = 0, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \]
where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^2 \), \( p > 1 \). Problem (2) has been studied by many people in the last two decades. Standard variational methods have shown the existence of least energy solution. In [23, 24] the authors show that the least energy solution \( u_p \) has bounded \( L^\infty \)-norm and \( \|u_p\|_\infty \) is bounded away from zero uniformly in \( p \), for \( p \) large. In [1, 21] the authors give a further description of the asymptotic behavior of \( u_p \), as \( p \to \infty \), by identifying a limit profile problem of Liouville-type:
\[ \Delta u + e^u = 0 \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u < \infty, \]
and showing that \( \|u_p\|_\infty \to \sqrt{e} \) as \( p \to \infty \). For singular limits in Liouville-type equation, readers may refer to [10]. Conversely, many people are interested in
constructing the concentrating solutions to problem (2). In [12] the authors find
topological conditions on \( \Omega \) which ensure the existence of concentrating solutions.
More precisely, assume that \( \Omega \) is not simply connected, then given any \( m \geq 1 \)
there exists \( p_m > 0 \) such that for any \( p \geq p_m \) problem (2) has a solution \( u_p \) which
concentrates at \( m \) different points with simple bubbles in \( \Omega \), whose location is related
to the critical points of the function \( \varphi_m \) given by:
\[
\varphi_m(\xi_1, \cdots, \xi_m) = \sum_{j=1}^{m} H_D(\xi_j, \xi_j) + \sum_{i \neq j} G_D(\xi_i, \xi_j).
\]
Here \( G_D \) denotes the standard Green’s function of \( -\Delta \) with Dirichlet boundary
condition and \( H_D \) denotes the regular part of \( G_D \), i.e.
\[
H_D(x, y) = G_D(x, y) + \frac{1}{2\pi} \log |x - y|.
\]
Also the authors in [12] compared problem (2) with some widely studied problems
which have some analogies with it. For example,
\[
\Delta u + u^{s+\varepsilon} = 0, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\]
which has been studied by [3, 15, 17, 25, 26]. For more details readers may refer to
[12] and references therein. On the other hand, the case of sign changing solutions
to problem (2) has been considered in [13].

Our motivation in problem (1) are twofolds. First, since problem (1) is a natu-
ral generalization of equation (2), one may expect similar results in [12] hold. In
fact, this is true for general domain \( \Omega \) whether it is simply connected or not. Sec-
ondly, problem (1) is a special case of problem (2) in higher-dimension \( N \geq 3 \).
Actually, when we work with the cross-section of an \( N \)-dimensional torus having
axial symmetry, we can find that problem (2) is reduced to (1): let the torus be
\( T = \{(x', x_N) : (\|x'\|^2 + x_N^2)^{\frac{1}{2}} \leq r_0^2 \} \) with
\[
x' = (x_1, \ldots, x_{N-1}), \quad \|x'\| = \sqrt{x_1^2 + \cdots + x_{N-1}^2}, \quad r_0 < 1.
\]
If we look for solutions in the form \( u(x', x_N) = u(r, x_N) \) with \( r = \|x'\| \) for (2), a
direct calculus shows that the problem is transformed to \( \nabla (r^{N-2} \nabla u) + r^{N-2} u^p = 0 \)
in \( \Omega = \{(r, x_N) : (r-1)^2 + x_N^2 < r_0^2 \} \) with \( u = 0 \) on \( \partial \Omega \). This is just the problem
(1) with \( a(r, x_N) = r^{N-2} \).

When we consider the generalized problem (1), there are some natural questions:
Q1. Can we move the topological conditions on \( \Omega \) for non-constant function \( a(x) \)?
Q2. Is there any solution with concentrating points not simple?

In [14], for Hénon equation
\[
\Delta u + |x|^{2\alpha} u^p = 0 \text{ in } B(0, 1), \quad u = 0 \text{ on } \partial B(0, 1),
\]
where \( \alpha \notin \mathbb{N}, B(0, 1) \) is a unit ball in \( \mathbb{R}^2 \) with radius 1 and center 0, the authors
find many positive solutions and sign changing solutions for \( p \) large enough. Due to
the function \( |x|^{2\alpha} \), it is not necessary that the domain is not simply connected. But
the solutions they construct concentrate at simple symmetric points, hence there is
no clue to Q2.

In this paper we answer these two questions affirmatively. Our main result is the
following:
Theorem 1.1. Let $x_0 \in \Omega$ be a strict local maximum point of $a(x)$, i.e. there exists a neighborhood $B(x_0, \delta), \delta > 0$ such that
\[ a(x) < a(x_0), \quad \forall \ x \in B(x_0, \delta) \setminus \{x_0\}. \]
Then for any $m \in \mathbb{N}^*$, problem (1) has a family of solutions $u_p$ such that as $p \to +\infty$,
\[ \int_{\Omega} p u_p^{p+1} dx \to 8\pi e a(x_0). \]
Moreover, there exists $(\xi_1^p, \cdots, \xi_m^p) \in \Omega^m$ satisfying
\[ \xi_j^p \to x_0 \quad \text{and} \quad |\xi_i^p - \xi_j^p| \geq p^{-\frac{m^2 + 1}{2}}, \quad \forall \ i \neq j, \]
such that for any $\rho > 0$, $u_p \to 0$ uniformly in $\Omega \cup \bigcup_{j=1}^m B(\xi_j^p, \rho)$ and
\[ \sup_{x \in B(\xi_j^p, \rho)} u_p(x) \to \sqrt{e}, \quad \forall \ j = 1, \ldots, m. \]

Remark 1. In Theorem 1.1, if we have the following expansion of $a$ at $x_0$:
\[ a(x) \geq a(x_0) - c|x - x_0|^\alpha + o(|x - x_0|^\alpha), \quad c > 0, \ \alpha > 1, \]
in a neighborhood of $x_0$, then the distance between concentrating points satisfies
\[ |\xi_i^p - \xi_j^p| \geq p^{-\frac{m^2 + 1}{2}}, \quad i \neq j. \]
This implies that the flatter the anisotropic coefficient is, the bigger is the distance between the bubbles.

Since we will cite the results in [27] in the following proof, it is necessary to introduce the work in [27] quickly. Namely, in [27], Wei, Ye and Zhou have studied the anisotropic Emden-Fowler equation
\[ \nabla (a(x)\nabla u) + \varepsilon^2 a(x) e^u = 0 \quad \text{in} \ \Omega, \quad u = 0 \ \text{on} \ \partial \Omega, \]
where $a(x)$ is a smooth positive function in $\overline{\Omega}$. It is easy to see that problem (5) is a natural generalization of the following classical Emden-Fowler equation, or Gelfand’s equation
\[ \Delta u + \varepsilon e^u = 0 \quad \text{in} \ \Omega, \quad u = 0 \ \text{on} \ \partial \Omega, \]
which has been studied very widely, see [2, 16, 18, 20, 22, 28] and the references therein. They proved that if $a(x)$ has a local strict maximum point $x_0$, then for any $m \in \mathbb{N}^*$ problem (5) has a family of solutions $u_\varepsilon$ which makes an $m$-bubbles concentration at $x_0$.

Theorem 1.1 is proved via the so-called “localized energy method”-a combination of Liapunov-Schmidt reduction method and variational techniques. Namely, we first use Liapunov-Schmidt reduction method to reduce the problem to a finite dimensional one, with some reduced energy. Then, the solutions in Theorem 1.1 turn out to be generated by critical points of the reduced energy functional. Such an idea has been used in many other papers. See for instance [4, 8, 9, 11, 12, 27] and the references therein. Here we will follow those of [12] and [27].

Throughout the paper, the symbol $C$ always denotes a positive constant independent of $p$, which could be changed from one line to another and $|\cdot|$ is for Euclidean norm in $\mathbb{R}^2$. 
2. **Ansatz for the solutions.** The purpose of this section is to provide an ansatz for problem (1) and give some basic estimates for the error term.

It is well known that the solutions to the following Liouville-type equation (see [7])

\[
\Delta u + e^u = 0 \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u \text{d}x < +\infty
\]

can be all written in the following form

\[
U_{\delta,\xi}(x) = \log \frac{8\delta^2}{(\delta^2 + |x - \xi|^2)^2}, \quad \delta > 0, \; \xi \in \mathbb{R}^2. \tag{7}
\]

Let

\[
\Delta_a u = \frac{1}{a(x)} \nabla \cdot (a(x) \nabla u) = \Delta u + \nabla \log a \cdot \nabla u
\]

and \(G(x, y)\) be the Green's function satisfying

\[
\Delta_a G(x, y) + 8\pi \delta_y = 0 \quad \text{in } \Omega, \quad G(x, y) = 0 \quad \text{on } \partial \Omega. \tag{8}
\]

We decompose \(G(x, y)\) as

\[
G(x, y) = -4 \log |x - y| + H(x, y), \tag{9}
\]

where \(H(x, y)\) is the regular part of \(G(x, y)\). Then we have the following lemma proved in [27] and [19]:

**Lemma 2.1.** Let \(H_u(x) = H(x, y), \forall y \in \Omega\). Then \(y \mapsto H_u \in C(\Omega, C^\gamma(\overline{\Omega}))\) for any \(\gamma \in (0, 1)\). Let \(H_D\) be the regular part of Green's function defined by (4), then we have

\[
H(x, y) = 8\pi H_D(x, y) + \nabla \log a(y) \cdot \nabla x (|x - y|^2 \log |x - y|) + H_1(x, y) \tag{10}
\]

where \(x \mapsto H_1(x, y) \in C^{1,\gamma}(\overline{\Omega})\) for all \(\gamma \in (0, 1)\). Furthermore, the function \((x, y) \mapsto H_1(x, y) \in C^1(\Omega \times \Omega)\), in particular the corresponding Robin function \(x \mapsto H(x, x) \in C^1(\Omega)\). Moreover, since \(a(x)\) is smooth, Robin function \(H(x, x) \in C^\infty(\Omega)\).

Given now \(\xi_j \in \Omega\), with (7) we define

\[
U_{\delta_j,\xi_j}(x) = \log \frac{8\delta_j^2}{(\delta_j^2 + |x - \xi_j|^2)^2}, \quad j = 1, \ldots, m,
\]

where \(\delta_j = \mu_j e^{-\frac{\xi_j}{2}}\) and \(\mu_j\) is to be determined later.

The configuration space for \((\xi_1, \cdots, \xi_m)\) is chosen as follows

\[
\Lambda := \{ \xi = (\xi_1, \cdots, \xi_m) \in B(x_0, \delta) \times \cdots \times B(x_0, \delta) \mid \min_{i \neq j} |\xi_i - \xi_j| \geq p^{-M} \} \tag{11}
\]

where \(M\) is given by

\[
M = \frac{m^2 + 1}{2}. \tag{12}
\]

Note that by the choice of \(\xi_j\), we have if \((\xi_1, \cdots, \xi_m) \in \Lambda,\)

\[
|\log |\xi_i - \xi_j|| \leq C \log p, \quad \forall i \neq j. \tag{13}
\]

However, \(U_{\delta_j,\xi_j}\) is a good first approximation, but not enough for our approximation. We need to refine this first approximation.

Let us call \(v_\infty(y) = U_{1,0}(y)\) and radial functions \(w_0, w_1\) solving

\[
\Delta w_i + \frac{8}{(1 + |y|^2)^2} w_i = \frac{1}{(1 + |y|^2)^2} f_i(y) \quad \text{in } \mathbb{R}^2, \quad i = 0, 1,
\]
where
\[ f_0 = 4v_0^2, \quad f_1 = 8 \left( w_0v_\infty - \frac{1}{3}v_\infty^3 - \frac{1}{2}w_0^2 - \frac{1}{8}v_\infty^4 + \frac{1}{2}w_0v_\infty^2 \right). \tag{14} \]

According to [5], for a radial function \( f(y) = f(|y|) \) there exists a radial solution
\[ w(r) = \frac{1 - r^2}{1 + r^2} \left( \int_0^r \frac{\phi_f(s) - \phi_f(1)}{(s-1)^2} ds + \phi_f(1) \frac{r}{1-r} \right) \]
for the equation
\[ \Delta w + \frac{8}{(1 + |y|^2)^2} w = f(y), \]
where
\[ \phi_f(s) = \left( \frac{s^2 + 1}{s-1} \right)^2 \frac{(s-1)^2}{s} \int_0^s \frac{1 - t^2}{1 + t^2} f(t) dt \quad \text{for } s \neq 1 \]
and \( \phi_f(1) = \lim_{s \to 1} \phi_f(s) \).

We state the following lemma whose proof is given in [12] on page 37-38.

**Lemma 2.2.** Let \( r = |y| \), then
\[ w_0(r) = C_0 \log r + O \left( \frac{1}{r} \right), \quad \partial_r w_0(r) = \frac{C_0}{r} + O \left( \frac{1}{r^2} \right) \quad \text{as } r \to +\infty, \tag{15} \]
where \( C_0 = 12 - 12 \log 2 \). More precisely, we have the exact expression of \( w_0 \),
\[ w_0(r) = \frac{1}{2} v_\infty^2 (y) + 6 \log (r^2 + 1) + \frac{2 \log 8 - 10}{r^2 + 1} + \frac{2 \log 8 - 10}{r^2 + 1} \]
\[ \times \left[ 2 \log^2 (r^2 + 1) - \frac{1}{2} \log^2 8 + 4 \int_{r^2}^{+\infty} \frac{ds}{s+1} \log \frac{s+1}{s} - 8 \log r \log (r^2 + 1) \right] \tag{16} \]
and
\[ w_1(r) = C_1 \log r + O \left( \frac{1}{r} \right), \quad \partial_r w_1(r) = \frac{C_1}{r} + O \left( \frac{1}{r^2} \right) \quad \text{as } r \to +\infty, \tag{17} \]
for a suitable constant \( C_1 \).

Define now for any \( x \in \Omega \),
\[ U_j(x) = \frac{1}{\gamma \mu_j^{\frac{n-1}{2}}} \left[ U_{j,\xi_j}(x) + \frac{1}{p} w_0 \left( \frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2} w_1 \left( \frac{x - \xi_j}{\delta_j} \right) \right] \tag{18} \]
where
\[ \gamma = p^{\frac{n-1}{2}} e^{-\frac{p^2}{2(p-1)}}. \tag{19} \]
Let \( H_j^p \) solve
\[ \begin{cases} \Delta_a H_j^p + \nabla \log a(x) \cdot \nabla U_j(x) = 0 & \text{in } \Omega, \\ H_j^p = -U_j & \text{on } \partial \Omega. \end{cases} \tag{20} \]

Then

**Lemma 2.3.** For any \( \beta \in (0,1) \), \( p \) large enough, \( H_j^p(x) = \)
\[ \frac{1}{\gamma \mu_j^{\frac{n-1}{2}}} \left[ \left( 1 - \frac{C_0}{4p} - \frac{C_1}{4p^2} \right) H(x, \xi_j) - \log(8\delta_j^2) + \frac{C_0}{p} \log \delta_j + \frac{C_1}{p^2} \log \delta_j + O(\delta_j^\beta) \right]. \]
uniformly in \( \bar{\Omega} \), where \( H \) is the regular part of Green’s function defined by (9).
Proof. The boundary condition satisfied by $H^p_j(x)$ is

$$H^p_j(x) = \frac{1}{\gamma_{\mu_j}^{2\pi}} \left[ -\log(8\delta_j^2) + 4 \log |x - \xi_j| - \frac{C_0}{p} \log |x - \xi_j| ight.$$

$$\left. + \frac{C_0}{p} \log \delta_j - \frac{C_1}{p^2} \log |x - \xi_j| + \frac{C_1}{p^2} \log \delta_j + O(\delta_j) \right].$$

The regular part of Green’s function $H(x, y)$ satisfies

$$\begin{cases} -\Delta_a H(x, y) = 4\nabla \log a(x) \cdot \nabla \log \frac{1}{|x - y|} & \text{in } \Omega, \\ H(x, y) = 4\log |x - y| & \text{on } \partial \Omega. \end{cases} \quad (21)$$

For the difference, let $Z_p(x) :=$

$$H^p_j - \frac{1}{\gamma_{\mu_j}^{2\pi}} \left[ \left( 1 - \frac{C_0}{4p} - \frac{C_1}{4p^2} \right) H(x, \xi_j) - \log(8\delta_j^2) + \frac{C_0}{p} \log \delta_j + \frac{C_1}{p^2} \log \delta_j \right],$$

then $Z_p$ satisfies

$$-\Delta_a Z_p(x) = \nabla \log a(x) \cdot \left[ \nabla U_j + \frac{1}{\gamma_{\mu_j}^{2\pi}} \left( 1 - \frac{C_0}{4p} - \frac{C_1}{4p^2} \right) \nabla \log |x - \xi_j|^4 \right]$$

in $\Omega$ and

$$Z_p(x) = O \left( \delta_j^{-1} \mu_j^{\frac{2\pi}{|x - \xi_j|}} \right) \text{ on } \partial \Omega.$$

According to the definition of $U_j$ in (18), we get by direct computation that

$$I_p := \nabla U_j + \frac{1}{\gamma_{\mu_j}^{2\pi}} \left( 1 - \frac{C_0}{4p} - \frac{C_1}{4p^2} \right) \nabla \log |x - \xi_j|^4$$

$$= \frac{1}{\gamma_{\mu_j}^{2\pi}} \left[ \left( 1 - \frac{C_0}{4p} - \frac{C_1}{4p^2} \right) \nabla \log \frac{|x - \xi_j|^4}{(\delta_j^2 + |x - \xi_j|^2)^2} + O \left( \frac{\delta_j}{\delta_j^2 + |x - \xi_j|^2} \right) \right].$$

Note that

$$\left| \nabla \log \frac{|x - \xi_j|^4}{(\delta_j^2 + |x - \xi_j|^2)^2} \right| = \frac{4\delta_j^2}{|x - \xi_j|(\delta_j^2 + |x - \xi_j|^2)},$$

applying polar coordinates with center $\xi_j$, i.e. $r = |x - \xi_j|$, there holds

$$\|I_p\|_{L^q(\Omega)}^q \leq \frac{1}{(\gamma_{\mu_j}^{2\pi})^q} \left[ 2^{q+1} \pi \int_0^{+\infty} \left( \frac{\delta_j^2}{r(\delta_j^2 + r^2)} \right)^q r dr + 2\pi C \int_0^{+\infty} \left( \frac{\delta_j}{\delta_j^2 + r^2} \right)^q r dr \right]$$

$$= C\delta_j^{q-4} (\gamma_{\mu_j}^{2\pi})^{-q}.$$

In conclusion, for any $1 < q < 2$, we have

$$\|I_p\|_{L^q(\Omega)} \leq C\delta_j^{\frac{2-q}{q-2}} \gamma^{-1} \mu_j^{\frac{2\pi}{|x - \xi_j|}}.$$

Applying $L^q$ theory,

$$\|Z_p\|_{W^{2,q}(\Omega)} \leq C \left( \|I_p\|_{L^q(\Omega)} + \|Z_p\|_{C^2(\partial\Omega)} \right) \leq C\gamma^{-1} \mu_j^{\frac{2\pi}{|x - \xi_j|}} \left( \frac{4}{\delta_j^2 + 1} + \delta_j \right).$$
By Sobolev embedding theorem, we obtain
\[ \|Z_\alpha(x)\|_{C^{\bar{\alpha}}(\Omega)} \leq C \delta_j^{\frac{2}{q}} - \frac{1}{\gamma - 1} \mu_j^{\frac{2}{p^\alpha}} \]
for any \( \bar{\alpha} \in \left(0, 2 - \frac{2}{q}\right) \). Lemma 2.3 is proved since \( q \in (1, 2) \) is arbitrary. 

Denote
\[ u_j(x) = U_j(x) + H^p_j(x), \quad U_\xi(x) = \sum_{j=1}^m u_j(x). \quad (22) \]

Observe that \( |y| \leq \frac{1}{\delta_j p^2 M} \) where \( x = \delta_j y + \xi_j \), we have that
\[ u_j(\delta_j y + \xi_j) = \frac{1}{\gamma \mu_j^{\frac{2}{p^\alpha}}} \left[ v_\infty(y) + \frac{1}{p} w_0(y) + \frac{1}{p^2} w_1(y) + \left( 1 - \frac{C_0}{4p} - \frac{C_1}{4p^2} \right) H(\xi_j, \xi_j) - \log(8\delta_j^4) + \frac{C_0}{p} \log \delta_j + \frac{C_1}{p^2} \log \delta_j + O\left( \sqrt{\delta_j |y| + \delta_j^3} \right) \right] \]
and for any \( i \neq j \),
\[ u_i(\delta_j y + \xi_j) = \frac{1}{\gamma \mu_i^{\frac{2}{p^\alpha}}} \left[ U_{\delta_i, \xi_i}(x) + \frac{1}{p} w_0 \left( \frac{x - \xi_i}{\delta_i} \right) + \frac{1}{p^2} w_1 \left( \frac{x - \xi_i}{\delta_i} \right) + \left( 1 - \frac{C_0}{4p} - \frac{C_1}{4p^2} \right) H(x, \xi_i) - \log(8\delta_i^4) + \frac{C_0}{p} \log \delta_i + \frac{C_1}{p^2} \log \delta_i + O(\delta_i^3) \right] \]
\[ = \frac{1}{\gamma \mu_i^{\frac{2}{p^\alpha}}} \left[ \left( 1 - \frac{C_0}{4p} - \frac{C_1}{4p^2} \right) \left( -4 \log |\xi_i - \xi_j| + H(\xi_j, \xi_i) \right) + O\left( \sqrt{\delta_j |y| + \delta_j^3} \right) \right] \]
\[ = \frac{1}{\gamma \mu_i^{\frac{2}{p^\alpha}}} \left[ \left( 1 - \frac{C_0}{4p} - \frac{C_1}{4p^2} \right) G(\xi_j, \xi_i) + O\left( \sqrt{\delta_j |y| + \delta_j^3} \right) \right]. \]

Hence for \( |y| \leq \frac{1}{\delta_j p^2 M} \),
\[ U_\xi(\delta_j y + \xi_j) = \frac{1}{\gamma \mu_j^{\frac{2}{p^\alpha}}} \left[ p + v_\infty(y) + \frac{1}{p} w_0(y) + \frac{1}{p^2} w_1(y) + O\left( \sqrt{\delta_j |y| + \sum_{i=1}^m \delta_i^3} \right) \right], \quad (23) \]
is a good approximation for a solution to problem (1) provided that
\[ \log(8\delta_j^4) = \left( 1 - \frac{C_0}{4p} - \frac{C_1}{4p^2} \right) H(\xi_j, \xi_j) + \frac{C_0}{p} \log \delta_j + \frac{C_1}{p^2} \log \delta_j + \left( 1 - \frac{C_0}{4p} - \frac{C_1}{4p^2} \right) \mu_j^{\frac{2}{p^\alpha}} \sum_{i \neq j} \mu_i^{\frac{2}{p^\alpha}} G(\xi_j, \xi_i). \quad (24) \]

**Lemma 2.4.** Let
\[ \Sigma = \left\{ \mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m \mid \forall i = 1, \ldots, m, \quad \frac{1}{Cp^C} \leq \mu_i \leq C p^C \right\}, \]
where \( C > M \) is large enough but fixed independent of \( p \), then system (24) is solvable in \( \Sigma \).
Proof. Let us consider the following vector function
\[ g(t; \vec{\mu}) = \log \mu_j - \frac{1}{4} H(\xi_j, \xi_i) + \frac{1}{A} \left( 3 + \frac{C_1}{4p} \right) - \frac{t}{4} \sum_{i \neq j} \left( \frac{\mu_j}{\mu_i} \right)^{\frac{p}{2}} G(\xi_j, \xi_i) \]
where \( A = 4 \left( 1 - \frac{C_1}{4p} - \frac{C_1}{4p^2} \right) \), \( C_0 = 12 - 12 \log 2 \), \( \vec{\mu} = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m \) and \( t \in [0, 1] \), then system (24) is equivalent to \( g(1; \vec{\mu}) = 0 \).

Denote \( T := \{ t \ | \ g(t; \vec{\mu}) = 0 \text{ is solvable, } t \in [0, 1] \} \).

Obviously \( g(0; \vec{\mu}) = 0 \) is solvable for all \( C > 0 \), that is, \( T \neq \emptyset \). It’s easy to see that \( T \) is closed. Now we would prove that \( T \) is open. If so, \( T = [0, 1] \) which tells us that (24) is solvable. Indeed, for any \( \vec{\mu} \in \Sigma \), then
\[ \frac{\mu_i}{\mu_j} \in \left[ \frac{1}{C^2 p^{2C}}, C^2 p^{2C} \right]. \]
Using the expansion of exponential function, we can get that
\[ \left( C^2 p^{2C} \right)^{\frac{1}{p^2}} = e^{\frac{1}{p^2} \log(C p^C)} = 1 + O \left( \frac{\log p}{p} \right), \quad \text{as } p \to \infty. \]
Therefor \( \forall \ i, j, \)
\[ \left( \frac{\mu_i}{\mu_j} \right)^{\frac{1}{p^2}} = 1 + O \left( \frac{\log p}{p} \right), \quad \text{as } p \to \infty. \]
Direct computation gives out that
\[ (\nabla_{\vec{\mu}} g(t; \vec{\mu}))_{jj} = \frac{1}{\mu_j} - \frac{t}{2(p-1)\mu_j} \sum_{i \neq j} \left( \frac{\mu_j}{\mu_i} \right)^{\frac{p}{2}} G(\xi_j, \xi_i) = \frac{1}{\mu_j} \left[ 1 + O \left( \frac{\log p}{p} \right) \right] \]
and for any \( i \neq j, \)
\[ (\nabla_{\vec{\mu}} g(t; \vec{\mu}))_{ji} = \frac{t}{2(p-1)\mu_i} \left( \frac{\mu_j}{\mu_i} \right)^{\frac{p}{2}} G(\xi_j, \xi_i) = \frac{1}{\mu_i} O \left( \frac{\log p}{p} \right), \]
which lead to
\[ \det (\nabla_{\vec{\mu}} g(t; \vec{\mu})) = \frac{1}{\mu_1 \cdots \mu_m} \left[ 1 + O \left( \frac{\log p}{p} \right) \right] \neq 0. \]
So \( \nabla_{\vec{\mu}} g(t; \vec{\mu}) \) is not singular over \( \Sigma \) for any \( t \in [0, 1] \) and large \( p \).

For any \( t_0 \in T \) with \( g(t_0; \vec{\mu}_0) = 0 \) and \( \vec{\mu}_0 \in \Sigma \), using Implicit Function Theorem, we see that \( g(t; \vec{\mu}) = 0 \) is solvable in some open neighborhood of \( (t_0, \vec{\mu}_0) \), that is, for \( |t - t_0| \) small enough, we have \( g(t; \vec{\mu}) = 0 \) with
\[ \vec{\mu} \in \left[ \frac{1}{2C p^{2C}}, 2C p^{2C} \right]^m. \]

A direct computation shows that, for \( p \) large, \( \vec{\mu} = (\mu_1, \ldots, \mu_m) \) satisfies
\[ \mu_j = e^{-\frac{1}{2} \sum_{i \neq j} G(\xi_j, \xi_i) - \frac{t}{2} \mu(\xi_j, \xi_i)} \left[ 1 + O \left( \frac{\log^2 p}{p} \right) \right]. \]
Observe that \( \mu_j \) may not be \( O(1) \) since \( \xi_i \to x_0 \) for all \( i = 1, \ldots, m \), but we can derive that
\[ \frac{1}{C} \leq \mu_j \leq Cp^M \]
for some fixed positive number \( C \). To conclude, \( T \) is open in \([0, 1]\). \qed
Remark 2. For $p$ large, from the above computations, we can get easily that for any $j = 1, \ldots, m, |x - \xi_j| \leq p^{-2M}$, let $x = \delta_jy + \xi_j$, then

$$p + v_\infty(y) + \frac{1}{p}w_0(y) + \frac{1}{p^2}w_1(y) \geq p - 2\log(1 + |y|^2) + O(1)$$

with $p$ large enough. By (23) we obtain that $0 < U_\xi \leq 2\sqrt{e}$ and for any $\rho > 0$, sup$_{B(\xi, \rho)}U_\xi \to \sqrt{e}$ as $p \to \infty$. If for all $j = 1, \ldots, m, |x - \xi_j| \geq p^{-2M}$, from the definition of $u_j$

$$u_j(x) = \frac{1}{2\gamma \mu_j^{\frac{1}{2}}} \left[ \log \frac{8\delta_j^2}{(\delta_j^2 + |x - \xi_j|^2)^2} + \frac{1}{p}w_0\left(\frac{x - \xi_j}{\delta_j}\right) + \frac{1}{p^2}w_1\left(\frac{x - \xi_j}{\delta_j}\right) \right]$$

$$+ (1 - \frac{C_0}{4p} - \frac{C_1}{4p^2})H(x, \xi_j) - \log 8\delta_j^2 + \frac{C_0}{p}\log \delta_j + \frac{C_1}{p^2}\log \delta_j + O(\delta_j^\beta)$$

using the property of $\delta_j$ and $\mu_j$, we find that

$$\frac{1}{2\gamma \mu_j^{\frac{1}{2}}}G(x, \xi_j) \leq u_j(x) \leq \frac{C\log p}{p}, \quad \text{if} \quad |x - \xi_j| \geq \frac{1}{p^{2M}}. \quad (25)$$

By maximum principle, $G(x, \xi_j) > 0$ in $\Omega$. In conclusion,

$$0 < U_\xi \leq 2\sqrt{e} \quad \text{in} \quad \Omega.$$

Let us set

$$S_p[u] = \Delta u + w_+^p,$$

where $u_+ = \max\{u, 0\}$, \quad (26)

and we introduce the following functional defined in $H^1_0(\Omega)$:

$$J_p[u] = \frac{1}{2} \int_\Omega a(x)|\nabla u|^2 - \frac{1}{p + 1} \int_\Omega a(x)u_+^{p+1} \quad (27)$$

whose nontrivial critical points are solutions to (1).

It is easy to see that problem (1) is equivalent to

$$S_p[u] = 0, \quad u_+ \neq 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

by maximum principle. We will look for solutions $u$ of problem (1) in the form $u = U_\xi + \phi$, where $\phi$ will represent a higher-order term in the expansion of $u$. Observe that

$$S_p[U_\xi + \phi] = L[\phi] + R_\xi + N[\phi] = 0 \quad (28)$$

with

$$L[\phi] := \Delta \phi + W_\xi \phi, \quad W_\xi = pU_\xi^{p-1}(x)$$

and

$$R_\xi := \Delta u + U_\xi^{p}, \quad N[\phi] = (U_\xi + \phi)_+^p - U_\xi^p - W_\xi \phi. \quad (29)$$

In terms of $\phi$, problem (28) becomes

$$L[\phi] = -\left( R_\xi + N[\phi] \right) \quad \text{in} \quad \Omega, \quad \phi = 0 \quad \text{on} \quad \partial\Omega. \quad (30)$$

The main step in solving problem (31) is that of a solvability theory for the linear operator $L$ under a suitable choice of the points $\xi_i$. In developing this theory, we will take into account the invariance, under translations and dilations, of the problem
\[ \Delta v + e^v = 0 \text{ in } \mathbb{R}^2. \] We will perform the solvability theory for the linear operator \( L \) in weighted \( L^\infty \)-norm space. For any \( h \in L^\infty(\Omega) \), define

\[ \|h\|_* = \sup_{x \in \Omega} \left( \sum_{j=1}^{m} \frac{\delta_j}{(\delta_j^2 + |x - \xi_j|^2)^{\frac{3}{2}}} \right)^{-1} h(x). \] (32)

We conclude this section by showing an estimate of \( R_\xi \) in \( \| \cdot \|_* \).

**Proposition 1.** There exist \( C > 0 \) and \( p_0 > 0 \) such that for any \( \xi \in \Lambda \) and \( p \geq p_0 \) we have

\[ \| R_\xi \|_* \leq \frac{C}{p^2}. \]

**Proof.**

\[ \Delta_a U_\xi = \sum_{j=1}^{m} \Delta_a u_j = \sum_{j=1}^{m} \Delta U_j \]

\[ = \sum_{j=1}^{m} \frac{1}{\gamma \mu_j^2} \left[ \Delta U_{\xi_j} + \frac{1}{p \delta_j} \Delta w_0 \left( \frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2 \delta_j} \Delta w_1 \left( \frac{x - \xi_j}{\delta_j} \right) \right] \]

\[ = \sum_{j=1}^{m} \frac{1}{\gamma \mu_j^2} \left[ -e^{U_{\xi_j}} \frac{x - \xi_j}{\delta_j} + \frac{1}{p \delta_j} \tilde{f}_0 \left( \frac{x - \xi_j}{\delta_j} \right) + \frac{1}{p^2 \delta_j} \tilde{f}_1 \left( \frac{x - \xi_j}{\delta_j} \right) \right. \]

\[ - \frac{1}{p} e^{U_{\xi_j}} w_0 \left( \frac{x - \xi_j}{\delta_j} \right) - \frac{1}{p^2} e^{U_{\xi_j}} w_1 \left( \frac{x - \xi_j}{\delta_j} \right) \]

\[ \leq \frac{C}{p} e^{U_{\xi_k}} \left[ 1 + \frac{1}{p^2} \left( \frac{x - \xi_k}{\delta_k} \right) + \frac{1}{p^2} \left( \frac{x - \xi_k}{\delta_k} \right)^4 \right] \]

\[ \leq C p e^{U_{\xi_k}}. \] (34)

where for \( i = 0, 1 \), \( \tilde{f}_i(y) = \frac{1}{(1 + |y|^2)^2} f_i(y) \) with \( f_i \) given by (14).

If for some \( i = 1, \ldots, m, \) \( |x - \xi_i| \leq p^{-2M} \), then for any \( k \neq i, \) \( |x - \xi_k| \geq \frac{1}{2} p^{-M} \),

\[ |\Delta_a u_k| = \frac{e^{U_{\xi_k}}}{\gamma \mu_k^2} \left| -1 + \frac{1}{8p} f_0 \left( \frac{x - \xi_k}{\delta_k} \right) + \frac{1}{8p^2} f_1 \left( \frac{x - \xi_k}{\delta_k} \right) \right. \]

\[ - \frac{1}{p} w_0 \left( \frac{x - \xi_k}{\delta_k} \right) - \frac{1}{p^2} w_1 \left( \frac{x - \xi_k}{\delta_k} \right) \left. \right| \]

\[ \leq C \frac{p}{e^{U_{\xi_k}}} \left[ 1 + \frac{1}{p^2} \left( \frac{x - \xi_k}{\delta_k} \right) + \frac{1}{p^2} \left( \frac{x - \xi_k}{\delta_k} \right)^4 \right] \]

\[ \leq C p e^{U_{\xi_k}}, \] (35)

from which we can deduce that

\[ \sup_{|x - \xi_i| \leq p^{-2M}} \left| \sum_{j=1}^{m} \frac{\delta_j}{(\delta_j^2 + |x - \xi_j|^2)^{\frac{3}{2}}} \right|^{-1} \Delta_a u_k \]

\[ \leq \sup_{|x - \xi_i| \leq p^{-2M}} \frac{C p \delta_k}{(\delta_k^2 + |x - \xi_k|^2)^{\frac{3}{2}}} \leq C p^{M+1} \delta_k \leq \frac{C}{p^4}. \]
On the other hand,
\[
\Delta a u_i (\delta_i y + \xi_i) = \frac{1}{\gamma \mu_i^{\frac{p}{p+2}}} e^{U_{\xi, i}} \left[ -1 + \frac{1}{8p} f_0 \left( \frac{x - \xi_i}{\delta_i} \right) + \frac{1}{8p^2} f_1 \left( \frac{x - \xi_i}{\delta_i} \right) - \frac{1}{p} w_0 \left( \frac{x - \xi_i}{\delta_i} \right) - \frac{1}{p^2} w_1 \left( \frac{x - \xi_i}{\delta_i} \right) \right]
\]
\[
= \frac{1}{\gamma \mu_i^{\frac{p}{p+2}} \delta_i^2} e^{v_\infty} \left[ -1 + \frac{1}{8p} f_0 (y) + \frac{1}{8p^2} f_1 (y) - \frac{1}{p} w_0 (y) - \frac{1}{p^2} w_1 (y) \right].
\]

By (23), we have
\[
U_{\xi}^p (\delta_i y + \xi_i)
\]
\[
= \left( \frac{p}{\gamma \mu_i^{\frac{p}{p+2}} \delta_i^2} \right)^p \left[ 1 + \frac{1}{p} v_\infty (y) + \frac{1}{p^2} w_0 (y) + \frac{1}{p^3} w_1 (y) + O \left( \frac{\sqrt{|\delta_i|} |y|}{p} + \sum_{j=1}^{m} \frac{\delta_j^3}{p} \right) \right]^p
\]
\[
= \frac{1}{\gamma \mu_i^{\frac{p}{p+2}} \delta_i^2} \left[ 1 + \frac{1}{p} v_\infty (y) + \frac{1}{p^2} w_0 (y) + \frac{1}{p^3} \left( w_1 (y) + O \left( p^2 \sqrt{|\delta_i|} |y| + p^3 \sum_{j=1}^{m} \delta_j^3 \right) \right) \right]^p.
\]

To further computations, we first consider the region \(|x - \xi_i| \leq \sqrt{\delta_i} p^{-2M}\). By Taylor expansion of exponential and logarithmic function, we have that when \(p \to \infty\),
\[
\left( 1 + \frac{\eta(y)}{p} + \frac{\beta(y)}{p^2} + \frac{\gamma(y)}{p^3} \right)^p
\]
\[
eq e^{\eta(y)} \left[ 1 + \frac{1}{p} \left( \beta(y) - \frac{\eta^2(y)}{2} \right) + \frac{1}{p^2} \left( \gamma(y) - \eta(y) \beta(y) + \frac{\eta^3(y)}{3} \right) + \frac{\beta^2(y)}{2} + \frac{\eta^4(y)}{8} - \frac{\eta^2(y) \beta(y)}{2} \right] + O \left( \frac{\log^6 (|y| + 2)}{p^3} \right)
\]
provided \(-4 \log (|y| + 2) \leq \eta(y) \leq C\) and \(|\beta(y)| + |\gamma(y)| \leq C \log (|y| + 2)\). Thus
\[
U_{\xi}^p = \frac{1}{\gamma \mu_i^{\frac{p}{p+2}} \delta_i^2} e^{v_\infty} \left[ 1 + \frac{1}{p} \left( w_0 - \frac{1}{2} v_\infty \right) + \frac{1}{p^2} \left( w_1 - w_0 v_\infty + \frac{1}{3} v_\infty \right) + \frac{1}{2} w_0^2 + \frac{1}{8} v_\infty^2 - \frac{1}{2} v_\infty^2 w_0 + O \left( \frac{\log^6 (|y| + 2)}{p^3} \right) + \sqrt{|\delta_i|} |y| + \sum_{j=1}^{m} \delta_j^3 \right],
\]
(37)

Combining (33)–(37), it is easy to get that
\[
|\Delta a u_i + U_{\xi}^p| \leq \frac{8}{p^{\frac{1}{p+2}}} \left( 1 + |y|^2 \right)^2 O \left( \frac{\log^6 (|y| + 2)}{p^3} + \sqrt{|\delta_i|} |y| \right)
\]
\[
\leq \frac{\delta_i^2 \log^6 (|y| + 2) + p^3 \sqrt{|\delta_i|} |y|}{p^3 \left( \delta_i^2 + |x - \xi_i|^2 \right)^2},
\]
which implies that
\[
\sup_{|x - \xi_i| \leq \sqrt{\delta_i}p^{-2M}} \left( \sum_{j=1}^{m} \frac{\delta_j}{(\delta_j^2 + |x - \xi_j|^2)^{2}} \right)^{-1} \left| \Delta_{\alpha}u_{i} + U_{t}^{p} \right| 
\]
\[
\leq \frac{C}{p^4} \sup_{|y| \leq \delta_j^{1/2}p^{-2M}} \frac{1}{(1 + |y|^2)^{2}} \left( \log^6(|y| + 2) + \sqrt{|y|} \right) \leq \frac{C}{p^4}.
\]
If \(\sqrt{\delta_i}p^{-2M} \leq |x - \xi_i| \leq p^{-2M}\), by (36),
\[
|\Delta_{\alpha}u_{i}| = O \left( \frac{p}{\gamma_{i}^{2}} e^{\gamma_{i} |\xi_{i}|} \right) = O \left( \frac{1}{\delta_i^2 (1 + |y|^2)^{2}} \right)
\]
and due to \((1 + \frac{2}{p})^{p} \leq e^{p}\),
\[
U_{t}^{p}(x) = \frac{1}{\gamma_{i}^{2} \delta_i^{2}} \left[ 1 + \frac{1}{p} \left( v_{\infty}(y) + \frac{1}{p} w_{0}(y) + \frac{1}{p^2} w_{1}(y) + O\left( \sqrt{\delta_i} |y| + \sum_{j=1}^{m} \delta_j^2 \right) \right) \right]^{p}
\]
\[
\leq \frac{C}{\gamma_{i}^{2} \delta_i^2} e^{v_{\infty}(y) + \frac{1}{p} w_{0}(y) + \frac{1}{p^2} w_{1}(y) + O\left( \sqrt{\delta_i} |y| + \sum_{j=1}^{m} \delta_j^2 \right)}
\]
\[
\leq \frac{C}{\delta_i^{2} \gamma_{i} (1 + |y|^2)^2}.
\]
Thus in this region,
\[
\left| \left( \sum_{j=1}^{m} \frac{\delta_j}{(\delta_j^2 + |x - \xi_j|^2)^{2}} \right)^{-1} \left( \Delta_{\alpha}u_{i} + U_{t}^{p} \right)(x) \right| = O \left( \frac{1}{(1 + |y|^2)^{2}} \right) \leq C \delta_i^{2} \delta_i \leq C \frac{1}{p^4}.
\]
Finally, for any \(i = 1, \ldots, m\), \(|x - \xi_i| \geq p^{-2M}\), then as the computations in (34) and Remark 2,
\[
|\Delta_{\alpha}U_{\xi}| \leq C \sum_{i=1}^{m} e^{U_{\xi_{i}}}, \quad U_{t}^{p} \leq C \left( \log \frac{p}{p} \right)^{p},
\]
\[
\text{hence}
\sup_{\{x:|x - \xi_i| \geq p^{-2M}\}} \left| \left( \sum_{j=1}^{m} \frac{\delta_j}{(\delta_j^2 + |x - \xi_j|^2)^{2}} \right)^{-1} \left( \Delta_{\alpha}U_{\xi} + U_{t}^{p} \right) \right|
\]
\[
\leq C \sup_{\{x:|x - \xi_i| \geq p^{-2M}\}} \left( \sup_{1 \leq k \leq m} \left[ \frac{p \delta_{k}}{(\delta_k^2 + |x - \xi_k|^2)^{2}} \right] + \left( \log \frac{p}{p} \right)^{p} \delta_{k}^{-1} \delta_{k}^2 \right) \leq \frac{C}{p^4},
\]
which leads to the end of proof.

3. Analysis of the linearized operator. In this section, we prove bounded invertibility of the operator \(L\), uniformly on \(\xi \in \Lambda\), by using the weighted \(L^{\infty}\)–norm introduced in (32). Let us recall that \(L[\phi] = \Delta_{\alpha} \phi + W_{\xi} \phi\), where \(W_{\xi} = p U_{\xi}^{-1}\). For simplicity of notations, we will omit the dependence of \(W_{\xi}\) on \(\xi\). As in Proposition 1, we have for the potential \(W(x)\) the following expansion.
Lemma 3.1. Let \( p_0 \) be large enough, then there exists \( D_0 > 0 \) such that for any \( p > p_0 \),

\[
0 < W(x) \leq D_0 \sum_{j=1}^m e^{U_{j,\xi_j}(x)}
\]

for any \( \xi = (\xi_1, \ldots, \xi_m) \in \Lambda \). Furthermore, for \( |y| \leq \frac{1}{\sqrt{\delta_j p^{2M}}} \), \( j = 1, \ldots, m \),

\[
W(\delta_j y + \xi_j) = \frac{8}{\delta_j^2 (1 + |y|^2)^2} \left[ 1 + \frac{1}{p} \left( w_0 - v_\infty - \frac{1}{2} v_\infty^2 \right) + O \left( \frac{\log^4(|y| + 2)}{p^2} \right) \right]^{p}.
\]

Proof. If \( |x - \xi_j| \leq p^{-2M} \) for some \( j = 1, \ldots, m \), let \( x = \delta_j y + \xi_j \), with \( (1 + \frac{2}{p})^p \leq e^4 \),

\[
W(\delta_j y + \xi_j)
= \delta_j^{-2} \left[ 1 + \frac{1}{p} v_\infty(y) + \frac{1}{p^2} w_0(y) + \frac{1}{p^3} w_1(y) + O \left( \sum_{i=1}^m \frac{\delta_j^4}{p} + \frac{\sqrt{\delta_j |y|}}{p} \right) \right]^{p}
\]

\[
\leq C \delta_j^{-2} \left[ 1 + \frac{1}{p} v_\infty(y) + \frac{1}{p^2} w_0(y) + \frac{1}{p^3} w_1(y) + O \left( \sum_{i=1}^m \frac{\delta_j^4}{p} + \frac{\sqrt{\delta_j |y|}}{p} \right) \right]^{p} \quad (38)
\]

\[
\leq C \delta_j^{-2} e^{v_\infty(y) + \frac{1}{p} w_0(y) + \frac{1}{p^2} w_1(y) + O \left( \sum_{i=1}^m \delta_j^4 + \sqrt{\delta_j |y|} \right)}
\]

\[
\leq C \delta_j^{-2} e^{v_\infty(y) + \frac{1}{p} w_0(y) + \frac{1}{p^2} w_1(y) + O \left( \sum_{i=1}^m \delta_j^4 + \sqrt{\delta_j |y|} \right)}
\]

since \( v_\infty = \log \frac{8}{(1 + |y|^2)^2} > 4 \log(\delta_j p^{2M}) \geq -p - C \log p \geq -2p \) provided \( p \) large enough.

Indeed, if \( |y| \leq \delta_j^{- \frac{4}{p}} p^{-2M} \), by (23),

\[
W(\delta_j y + \xi_j)
= p \frac{p^p - 1}{\gamma^{p-1} \mu^2} \left[ 1 + \frac{1}{p} v_\infty(y) + \frac{1}{p^2} w_0(y) + \frac{1}{p^3} w_1(y) + O \left( \sum_{i=1}^m \frac{\delta_j^4}{p} + \frac{\sqrt{\delta_j |y|}}{p} \right) \right]^{p-1}
\]

\[
= \delta_j^{-2} \left[ 1 + \frac{1}{p} v_\infty(y) + \frac{1}{p^2} w_0(y) + \frac{1}{p^3} w_1(y) + O \left( \sum_{i=1}^m \frac{\delta_j^4}{p} + \frac{\sqrt{\delta_j |y|}}{p} \right) \right]^{p}
\]

\[
\times \left[ 1 + \frac{1}{p} v_\infty(y) + O \left( \frac{\log^4(|y| + 2)}{p^2} \right) \right]
\]

\[
= \frac{e^{v_\infty}}{\delta_j} \left[ 1 + \frac{1}{p} \left( w_0 - \frac{1}{2} v_\infty^2 \right) + O \left( \frac{\log^4(|y| + 2)}{p^2} \right) \right]
\]

\[
\times \left[ 1 + \frac{1}{p} v_\infty(y) + O \left( \frac{\log^4(|y| + 2)}{p^2} \right) \right]
\]

\[
= \frac{e^{v_\infty}}{\delta_j} \left[ 1 + \frac{1}{p} \left( w_0 - v_\infty - \frac{1}{2} v_\infty^2 \right) + O \left( \frac{\log^4(|y| + 2)}{p^2} \right) \right]
\]

(39)
where we have used Taylor expansion of exponential and logarithmic function
\[
\left[1 + \frac{\eta(y)}{p} + \frac{\beta(y)}{p^2}\right]^p \leq e^{\eta(y)} \left[1 + \frac{1}{p} \left(\beta(y) - \frac{\eta^2(y)}{2}\right) + O\left(\frac{\log^2(|y| + 2)}{p^2}\right)\right]
\]
provided \(-4\log(|y| + 2) \leq \eta(y) \leq C\) and \(|\beta(y)| \leq C\log(|y| + 2)\).

If \(|x - \xi_j| \geq p^{-2M}\) for any \(j = 1, \ldots, m\), then using Remark 2,
\[
W(x) = pU_{\xi}^{p-1} \leq C p \left(\frac{\log p}{p}\right)^{p-1} = o\left(e^{-\frac{C}{p}}\right) = o\left(e^{U_{\xi_j, \xi_j}(x)}\right).
\]
So we are done. \(\square\)

**Remark 3.** As for \(W\), let us point out that if \(|x - \xi_j| \leq p^{-2M}\) for some \(j = 1, \ldots, m\), there holds
\[
p \left[U_{\xi}(x) + O\left(\frac{1}{p}\right)\right]^p \leq C p \left(\frac{p}{\gamma_{\mu_j}^{\infty}}\right)^{p-2} e^{\gamma_{\mu_j}(\frac{z - \xi_j}{z_j})} = O\left(e^{U_{\xi_j, \xi_j}(x)}\right).
\]
Since this estimate is true if \(|x - \xi_j| \geq \frac{1}{p^{2M}}\) for any \(j = 1, \ldots, m\), we have
\[
p \left[U_{\xi}(x) + O\left(\frac{1}{p}\right)\right]^p \leq C \sum_{j=1}^m e^{U_{\xi_j, \xi_j}(x)}.
\] (40)

Set
\[
z_0(y) = \frac{|y|^2 - 1}{|y|^2 + 1}, \quad z_i(y) = \frac{4y_i}{|y|^2 + 1}, \quad i = 1, 2.
\]
It is well known that any bounded solution to
\[
\Delta \phi + \frac{8}{(1 + |y|^2)^2} \phi = 0 \quad \text{in} \quad \mathbb{R}^2
\]
is a linear combination of \(z_i, i = 0, 1, 2\). See Lemma 2.1 of [6]. Now we consider the following linear problem: given \(h \in C(\overline{\Omega})\), find a function \(\phi \in H^2(\Omega)\) such that
\[
\left\{
\begin{align*}
L[\phi] &= h + \frac{1}{\alpha(x)} \sum_{i=1}^2 \sum_{j=1}^m c_{ij} e^{U_{\xi_j, \xi_j}} Z_{ij} & \text{in} \ \Omega, \\
\phi &= 0, & \text{on} \ \partial\Omega, \\
\int_{\Omega} e^{U_{\xi_j, \xi_j}} Z_{ij} \phi &= 0, & i = 1, 2, 1 \leq j \leq m,
\end{align*}
\right.
\] (41)
for some coefficients \(c_{ij}, (i = 1, 2; 1 \leq j \leq m)\). Here and in the sequel, for any \(i = 0, 1, 2\) and \(j = 1, \ldots, m\), we denote
\[
Z_{ij}(x) := z_i \left(\frac{x - \xi_j}{\delta_j}\right) = \begin{cases} |x - \xi_j|^2 - \delta_j^2 & \text{if} \ i = 0, \\ \delta_j^2 + |x - \xi_j|^2 & \text{if} \ i = 1, 2, \end{cases}
\]
The main result of this section is the following:

**Proposition 2.** There exist \(p_0 > 0\) and \(C > 0\) such that for any \(p > p_0, \xi \in \Lambda, h \in C(\overline{\Omega})\) there is a unique solution to problem (41), which satisfies
\[
\|\phi\|_\infty \leq C p \|h\|_\ast.
\] (42)
Proof. The proof consists of six steps.

Step 1. There exists \( R > 1 \) large enough independent of \( p \), such that the operator \( L \) satisfies maximum principle in \( \Omega := \Omega \setminus \bigcup_{j=1}^{m} B(\xi_j, R\delta_j) \) provided \( p \) large. Namely,

\[
L[\psi] \leq 0 \quad \text{in} \quad \bar{\Omega} \quad \text{and} \quad \psi \geq 0 \quad \text{on} \quad \partial \Omega \quad \text{then} \quad \psi \geq 0 \quad \text{in} \quad \bar{\Omega}.
\]

In order to prove this fact, we show the existence of a positive function \( Z \) in \( \bar{\Omega} \) satisfying \( L[Z] < 0 \). We define \( Z \) to be

\[
Z(x) = \sum_{j=1}^{m} \left( \Phi_0 - \frac{\delta_j^k}{|x - \xi_j|^k} \right) \quad \text{in} \quad \bar{\Omega}, \quad k \in (0, 1) \quad \text{but fixed},
\]

where \( \Phi_0 \) satisfies

\[
-D_a \Phi_0 = 1 \quad \text{in} \quad \Omega, \quad \Phi_0 = 2 \quad \text{on} \quad \partial \Omega.
\]

Clearly \( \Phi_0 \geq 2 \) in \( \Omega \) and bounded. Thus in \( \bar{\Omega} \), \( 1 \leq Z(x) \leq C \) where \( C \) is independent of \( R \). On the other hand in \( \bar{\Omega} \),

\[
-D_a \left( \Phi_0 - \frac{\delta_j^k}{|x - \xi_j|^k} \right) = 1 + \frac{k^2 \delta_j^k}{|x - \xi_j|^{k+2}} - k \delta_j^k \frac{\nabla \log a(|x|)}{|x - \xi_j|^{k+1}}
\]

\[
\geq 1 + \frac{k^2 \delta_j^k}{|x - \xi_j|^{k+2}} (k - \| \nabla \log a(x) \|_\infty |x - \xi_j|)
\]

\[
\geq \frac{1}{2} + \frac{k^2 \delta_j^k}{2|x - \xi_j|^{k+2}},
\]

since either

\[
\frac{k^2 \delta_j^k}{|x - \xi_j|^{k+2}} (k - \| \nabla \log a(x) \|_\infty |x - \xi_j|) \geq \frac{k^2 \delta_j^k}{2|x - \xi_j|^{k+2}}, \quad \text{if} \quad |x - \xi_j| \leq \frac{k}{2\| a(x) \|_\infty}
\]

or

\[
1 - k \delta_j^k \| \nabla \log a(x) \|_\infty = 1 + O(\delta_j^k) \geq \frac{1}{2}, \quad \text{if} \quad |x - \xi_j| > \frac{k}{2\| a(x) \|_\infty}.
\]

Then for \( p \) large,

\[
L[Z] = \Delta_a Z + WZ(x) \leq -\frac{m}{2} - \sum_{j=1}^{m} \frac{k^2 \delta_j^k}{2|x - \xi_j|^{k+2}} + D_0 C \sum_{j=1}^{m} e^{U_{j, \xi_j}}
\]

\[
\leq -\frac{m}{2} - \sum_{j=1}^{m} \frac{\delta_j^k}{|x - \xi_j|^{k+2}} \left[ \frac{k^2}{2} - 8D_0 C \left( \frac{\delta_j}{|x - \xi_j|} \right)^{2-k} \right]
\]

\[
\leq -\frac{m}{2} - \frac{k^2}{4} \sum_{j=1}^{m} \frac{\delta_j}{|x - \xi_j|^{k+2}}
\]

\[
< -\frac{k^2}{4} \sum_{j=1}^{m} (\delta_j^2 + |x - \xi_j|^2)^{\frac{k}{2}}
\]

since \( \frac{\delta_j}{|x - \xi_j|} \leq \frac{k}{R}, 1 - k > 0, R \) large enough. Hence the function \( Z(x) \) is what we are looking for.

Step 2. Let \( k \) be defined in Step 1. Let us define the “inner norm” of \( \phi \) in the following way

\[
\| \phi \|_i = \sup_{x \in \bigcup_{j=1}^{m} B(\xi_j, \delta_j)} |\phi(x)|.
\]
We claim that there is a constant $C > 0$ such that if $L[\phi] = h$ in $\Omega$, $h \in C^{0,\alpha}(\overline{\Omega})$, then
\[
\|\phi\|_\infty \leq C(\|\phi\|_i + \|h\|_*)
\]
for any $h \in C^{0,\alpha}(\overline{\Omega})$. We will establish this estimate with the use of barriers. Indeed, set $\tilde{\phi} = \frac{4}{\pi^2}(\|\phi\|_i + \|h\|_*)Z(x)$, where $Z(x)$ was defined in the previous step, then on $\partial \Omega$, $\tilde{\phi} \geq |\phi|$ and the above computation shows that $L[\tilde{\phi}] \leq -|h|$. By maximum principle we get
\[
|\phi(x)| \leq C(\|\phi\|_i + \|h\|_*) , \quad \forall x \in \overline{\Omega}.
\]
By the definition of $\|\phi\|_i$, we obtain
\[
\|\phi\|_\infty \leq C(\|\phi\|_i + \|h\|_*)
\]
for some constant $C$ independent of $h$.

**Step 3.** We prove a uniform a-priori estimate for solutions $\phi$ of problem $L[\phi] = h$ in $\Omega$, $\phi = 0$ on $\partial \Omega$, where $h \in C^{0,\alpha}(\Omega)$ and in addition the orthogonality conditions:
\[
\int_{\Omega} e^{\varepsilon \xi_j} Z_{ij} \phi dx = 0 \quad \text{for} \quad i = 0, 1, 2, j = 1, \ldots, m. \tag{43}
\]
Namely, we prove that there exists a positive constant $C$ such that for any $\xi \in \Lambda$ and $h \in C^{0,\alpha}(\Omega)$,
\[
\|\phi\|_\infty \leq C\|h\|_*
\]
for $p$ sufficiently large. By contradiction, assume the existence of sequences $p_n \to \infty$, points $\xi^o \in \Lambda$, functions $h_n$ and associated solutions $\phi_n$ such that $\|h_n\|_* \to 0$ and $\|\phi_n\|_\infty = 1$. Thus Step 2 shows that $\lim \inf_{n \to \infty} \|\phi_n\|_i > 0$. Let us set
\[
\hat{\phi}_j^n(y) = \phi_n(\delta_j^n y + \xi^o) \quad \text{for} \quad j = 1, \ldots, m.
\]
By Lemma 3.1 and $a(x) \in C^1(\overline{\Omega})$, elliptic estimates readily imply that $\hat{\phi}_j^n(y)$ converges uniformly over compact sets to a bounded solution $\hat{\phi}_j^\infty$ of
\[
\Delta \phi + \frac{8}{(1 + |y|^2)^2} \phi = 0, \quad \text{in} \mathbb{R}^2.
\]
This implies that $\hat{\phi}_j^\infty$ is a linear combination of the functions $z_i, i = 0, 1, 2$. Since $\|\hat{\phi}_j^n(y)\|_\infty \leq 1$, by Lebesgue’s theorem, the orthogonality conditions on $\hat{\phi}_j^n$ pass to the limit and give
\[
\int_{\mathbb{R}^2} \frac{8}{(1 + |y|^2)^2} z_i(y)\hat{\phi}_j^\infty dy = 0 \quad \text{for} \quad i = 0, 1, 2.
\]
Hence $\hat{\phi}_j^\infty \equiv 0$ for any $j = 1, \ldots, m$ contradicting to $\lim \inf_{n \to \infty} \|\phi_n\|_i > 0$.

**Step 4.** We prove that there exists a positive constant $C$ such that any solution $\phi$ to $L[\phi] = h$ in $\Omega$, $\phi = 0$ on $\partial \Omega$ and in addition the orthogonality conditions:
\[
\int_{\Omega} e^{\varepsilon \xi_j} Z_{ij} \phi dx = 0 \quad \text{for} \quad i = 1, 2, j = 1, \ldots, m, \tag{44}
\]
satisfies
\[
\|\phi\|_\infty \leq C p \|h\|_*.
\]
Proceeding by contradiction as in Step 3, we can suppose further that
\[
p_n \|h_n\|_* \to 0 \quad \text{as} \quad n \to \infty. \tag{45}
\]
But here we lose the limit condition
\[
\int_{\mathbb{R}^2} \frac{8}{(1 + |y|^2)^2} z_0(y) \hat{\phi}_j^\infty dy = 0.
\]
Hence we have that
\[
\hat{\phi}_j^n \to \hat{\phi}_j^\infty = C_j \frac{|y|^2 - 1}{|y|^2 + 1} \text{ in } C^0_{\text{loc}}(\mathbb{R}^2)
\]
(46)
for some constant $C_j$. To reach a contradiction, we have to show that $C_j = 0$ for any $j = 1, \ldots, m$. We will obtain it from the stronger condition (45) on $h_n$. To this end, we perform the following construction. By Lemma 2.2, there exist radial solutions $w$ and $\zeta$ respectively of equations
\[
\Delta w + \frac{8}{(1 + |y|^2)^2} w = \frac{8}{(1 + |y|^2)^2} z_0(y) \text{ and } \Delta \zeta + \frac{8}{(1 + |y|^2)^2} \zeta = \frac{8}{(1 + |y|^2)^2} \text{ in } \mathbb{R}^2
\]
such that as $|y| \to \infty$,
\[
w(y) = \frac{4}{3} \log |y| + O \left( \frac{1}{|y|} \right), \quad \zeta(y) = O \left( \frac{1}{|y|} \right),
\]
due to
\[
8 \int_0^\infty (s^2 - 1)^2 s (s^2 + 1)^4 ds = \frac{4}{3} \quad \text{and} \quad 8 \int_0^\infty \frac{s^2 - 1}{(s^2 + 1)^2} ds = 0,
\]
see [12] on page 48.

For simplicity, from now on we will omit the dependence on $n$. For $j = 1, \ldots, m$, let
\[
v_j(x) = \frac{x - \xi_j}{\delta_j} + \frac{4}{3} \log(\delta_j) z_0 j(x) + \frac{1}{3} H(\xi_j, \xi_j) \zeta \left( \frac{x - \xi_j}{\delta_j} \right).
\]
Notice that $|v_j(x)| \leq C \log \left( \frac{|x - \xi_j|}{\delta_j} + 1 \right) + C |\log \delta_j|$. 

Suppose $h_j$ satisfy the equation
\[
\begin{aligned}
\Delta_n h_j(x) + \nabla \log a(x) \cdot \nabla v_j &= 0 \quad \text{in } \Omega,
\quad h_j(x) = -v_j(x) \quad \text{on } \partial \Omega.
\end{aligned}
\]
Thus $h_j(x) = -\frac{1}{4} H(x, \xi_j) + O(\delta_j)$, whose proof is very similar to Lemma 2.3, so we omit it.

Let $\omega_j(x) = v_j(x) + h_j(x)$, then
\[
\begin{aligned}
\omega_j(x) &= v_j(x) - \frac{1}{3} H(x, \xi_j) + O(\delta_j) \quad \text{in } C^1(\bar{\Omega}),
\omega_j(x) &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

\[
\omega_j(x) = -\frac{1}{3} G(\xi_i, \xi_j) + O \left( \delta_j p^M + |x - \xi_j| p^M \right), \quad \text{for } i \neq j, |x - \xi_i| \leq \sqrt{\delta_j p}^{-2M}.
\]

(47)

Also we can find that
\[
\Delta_n \omega_j(x) + W \omega_j(x) = e^{U_j, \xi_j} z_0 j + (W - e^{U_j, \xi_j}) \omega_j(x) + R_j
\]
where
\[
R_j = \left[ \omega_j(x) - v_j(x) + \frac{1}{3} H(\xi_j, \xi_j) \right] e^{U_j, \xi_j}(x)
= \left[ -\frac{1}{3} H(x, \xi_j) + \frac{1}{3} H(\xi_j, \xi_j) + O(\delta_j) \right] e^{U_j, \xi_j}(x)
= e^{U_j, \xi_j}(x) O \left( |x - \xi_j| + \delta_j \right).
\]

(49)
Multiply (48) by $a(x)\phi(x)$ and integrate by parts to obtain
\[
\int_\Omega e^{U_{\delta_j,\epsilon_j}} Z_{0j} a(x)\phi(x)dx + \int_\Omega (W - e^{U_{\delta_j,\epsilon_j}})\mathcal{P}_j a(x)\phi(x)dx = \int_\Omega a(x)h(x)\mathcal{P}_j dx - \int_\Omega a(x)R_j \phi(x)dx ,
\]
where we have used $L[\phi] = h$.

First of all, by Lebesgue's theorem and (46), we get
\[
\int_\Omega e^{U_{\delta_j,\epsilon_j}} Z_{0j} a(x)\phi(x)dx = \int_{(y: \delta_j y + \epsilon_j \in \Omega)} 8\frac{|y|^2 - 1}{(1 + |y|^2)^2 |y|^2 + 1} a(\delta_j y + \epsilon_j) \hat{\phi}_j(y)dy + 8a(\xi_j^\infty)C_j \int_{\mathbb{R}^2} (|y|^2 - 1)^2 dy = \frac{8\pi}{3} C_j a(\xi_j^\infty),
\]
where up to a subsequence, $\xi_j \to \xi_j^\infty, j = 1, \ldots, m$.

By Lemma 3.1 and (47), we have
\[
\int_\Omega (W - e^{U_{\delta_j,\epsilon_j}})\mathcal{P}_j a(x)\phi(x)dx = \int_{B(\delta_j, \sqrt{\delta_j p^{-2M}})} (W - e^{U_{\delta_j,\epsilon_j}})\mathcal{P}_j a(x)\phi(x)dx + \int_{\Omega \setminus B(\delta_j, \sqrt{\delta_j p^{-2M}})} W\mathcal{P}_j a(x)\phi(x)dx + o(1)
\]
\[
= \int_{B(0, \sqrt{\delta_j p^{-2M}})} \frac{8}{(1 + |y|^2)^2} \frac{1}{p} \left( w_0 - v_\infty - \frac{1}{2} v_\infty^2 \right) \frac{4}{3} \log(\delta_j) z_0(y) \hat{\phi}_j a(\delta_j y + \epsilon_j)dy
\]
\[
- \frac{1}{3} \sum_{i \neq j} G(\xi_i, \xi_j) \int_{B(\xi_i, \sqrt{\delta_i p^{-2M}})} W(x) a(x)\phi(x)dx + O\left( \frac{1}{p} \right)
\]
\[
= - \frac{8C_j}{3} a(\xi_j) \int_{\mathbb{R}^2} \frac{(|y|^2 - 1)^2}{(1 + |y|^2)^4} \left( w_0 - v_\infty - \frac{1}{2} v_\infty^2 \right) dy
\]
\[
- \frac{1}{3} \sum_{i \neq j} G(\xi_i, \xi_j) \int_{B(0, \sqrt{\delta_i p^{-2M}})} \frac{8}{(1 + |y|^2)^2} a(\delta_i y + \xi_i) \hat{\phi}_i dy + o(1)
\]
\[
= - \frac{8C_j}{3} a(\xi_j^\infty) \int_{\mathbb{R}^2} \frac{(|y|^2 - 1)^2}{(1 + |y|^2)^4} \left( w_0 - v_\infty - \frac{1}{2} v_\infty^2 \right) dy + o(1),
\]
since Lebesgue's theorem implies that
\[
\int_{B(0, \sqrt{\delta_i p^{-2M}})} \frac{8}{(1 + |y|^2)^2} a(\delta_i y + \xi_i) \hat{\phi}_i dy \to 8C_j a(\xi_j^\infty) \int_{\mathbb{R}^2} \frac{|y|^2 - 1}{(1 + |y|^2)^3} dy = 0.
\]
In a straightforward but tedious way, by (16) we can compute
\[
\int_{\mathbb{R}^2} \frac{(|y|^2 - 1)^2}{(1 + |y|^2)^4} \left( w_0 - v_\infty - \frac{1}{2} v_\infty^2 \right) dy = -\pi,
\]
Hence, inserting (51)–(55) in (50) we obtain that
\[ \int_{\Omega} (W - e^{U_j, \xi_j}) v_j a(x) \phi(x) dx = \frac{8\pi}{3} C_j a(\xi_j^\infty) + o(1). \] (53)

As far as the R.H.S in (50), we have that by (47)
\[ \left| \int_{\Omega} a(x) h v_j dx \right| \leq C \|h\|_* \int_{\Omega} \sum_{k=1}^m \frac{\delta_k}{(\delta_k^2 + |\xi_k|^2)^{\frac{3}{2}}} |v_j| dx \]
\[ = C \|h\|_* \sum_{k=1}^m \int_{\{y: \delta_k y + \xi_k \in \Omega\}} \frac{1}{(1 + |y|^2)^{\frac{3}{2}}} |v_j(\delta_k y + \xi_k)| dy \]
\[ \leq C \|h\|_* \int_{\mathbb{R}^3} \frac{\log(|y| + 2)}{(1 + |y|^2)^{\frac{3}{2}}} dy + C p \|h\|_* \int_{\mathbb{R}^3} \frac{1}{(1 + |y|^2)^{\frac{3}{2}}} dy \]
\[ = O(p \|h\|_*). \] (54)

Finally, with (49)
\[ \int_{\Omega} R_j a(x) \phi(x) dx = O \left( \int_{\Omega} (|x - \xi_j| + \delta_j) e^{U_j, \xi_j} dx \right) = O(\delta_j). \] (55)

Hence, inserting (51)–(55) in (50) we obtain that
\[ \frac{16\pi}{3} C_j a(\xi_j^\infty) = o(1) \] for any \( j = 1, \ldots, m. \)

Necessarily, \( C_j = 0 \) by contradiction and the claim is proved.

Step 5. We establish the validity of the following estimate:
\[ \|\phi\|_{\infty} \leq C p \|h\|_* \] (56)
for the solutions of problem (41) and \( h \in C^{0,\alpha}(\bar{\Omega}). \) Step 4 gives
\[ \|\phi\|_{\infty} \leq C p \left( \|h\|_* + \sum_{i=1}^m \sum_{j=1}^m |c_{ij}| \right) \]
since \( \|e^{U_j, \xi_j} Z_{ij}\|_* \leq 2 \|e^{U_j, \xi_j}\|_* \leq 16. \) Arguing by contradiction of (56), we can proceed as Step 3 and suppose further that
\[ p_n \|h_n\|_* \to 0, \quad \|\phi_n\|_{\infty} = 1, \quad p_n \sum_{i=1}^m \sum_{j=1}^m |c_{ij}^n| \geq \delta_0 > 0. \]

We omit the dependence on \( n. \) It suffices to estimate the values of constants \( c_{ij}. \)

For \( i = 1, 2 \) and \( j = 1, \ldots, m, \) now we define \( \Gamma_{ij} \) as the following
\[ \Delta \Gamma_{ij} = \Delta Z_{ij} \text{ in } \Omega, \quad \Gamma_{ij} = 0 \text{ on } \partial \Omega. \]

According to [12] on page 51, we have
\[ \Gamma_{ij} = Z_{ij} - \frac{8\pi}{3} \delta_j \frac{\partial H_D}{\partial (\xi_j)} (\cdot, \xi_j) + O(\delta_j^3) \text{ in } C^1(\bar{\Omega}) \]
and \( |T_{ij}| \leq |Z_{ij}| + C \delta_j \leq 2 + C \delta_j \leq 3. \)

Multiply the first equation of (41) by \( a(x) \Gamma_{ij} \) and integrate by part, we get
\[ \int_{\Omega} a(x) h \Gamma_{ij} dx + \sum_{i=1}^m c_{ih} \int_{\Omega} e^{U_k, \xi_k} Z_{ih} a(x) \Gamma_{ij} dx \]
\[ = \int_{\Omega} W(x) \phi(x) a(x) \Gamma_{ij} dx + \int_{\Omega} \nabla(a(x) \nabla \Gamma_{ij}) \phi(x) dx. \] (57)
Since $\Delta \Gamma_{ij} = \Delta Z_{ij} = -e^{U_{\delta_j, \xi_j}} Z_{ij}$, the above equality can be changed to
\begin{equation}
\int_{\Omega} a(x) h \Gamma_{ij} dx + \sum_{l=1}^{m} \sum_{h=1}^{\Omega} c_{ilh} \int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{lh} a(x) \Gamma_{ij} dx = \int_{\Omega} W(x) \phi(x) a(x) \Gamma_{ij} dx - \int_{\Omega} a(x) e^{U_{\delta_j, \xi_j}} Z_{ij} \phi dx + \int_{\Omega} \phi \nabla a(x) \cdot \nabla \Gamma_{ij} dx.
\end{equation}
(58)

First, let us deduce the following "orthogonality" relations: for $1 \leq i, l \leq 2$ and $1 \leq j, h \leq m$ with $j \neq h$,
\begin{align*}
\int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{lj} \Gamma_{ij} a(x) &= \frac{32}{3} \pi a(\xi_j) \delta_{il} + O(\delta_j), \\
\int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{lh} \Gamma_{ij} a(x) dx &= O(\delta_j p^{2M}) + o(\delta_h),
\end{align*}
(59)

where $\delta_{il}$ denotes the Kronecker's symbol.

\begin{align*}
\int_{\Omega} a(x) e^{U_{\delta_j, \xi_j}} Z_{ij} \Gamma_{ij} &\\
= & \int_{B(\xi_j, \rho^{-3M})} a(x) e^{U_{\delta_j, \xi_j}} Z_{ij} \left[ Z_{ij} - 8\pi \delta_j \frac{\partial H_D(x, \xi_j)}{\partial (\xi_j)_i} + O(\delta_j^3) \right] \\
+ & \left( \int_{\Omega \setminus B(\xi_j, \rho^{-3M})} e^{U_{\delta_j, \xi_j}} |Z_{ij}| \left[ |Z_{ij}| + O(\delta_j) \right] \right) \\
= & \int_{B(0, \frac{1}{\rho^{-3M}})} \frac{8a(\delta_j y + \xi_j)}{(1 + |y|^2)^2} \frac{4y_i}{1 + |y|^2} \left( \frac{4y_i}{1 + |y|^2} - 8\pi \delta_j \frac{\partial H_D(y, \xi_j)}{\partial (\xi_j)_i} + O(\delta_j |y| + \delta_j^3) \right) \\
+ & \left( \int_{\{|y, \delta_j y + \xi_j| \in \Omega, |y| \geq \frac{1}{\rho^{-3M}}\}} \frac{1}{(1 + |y|^2)^2} \delta_j^{2M} \right) \\
= & 128a(\xi_j) \int_{\mathbb{R}^2} \frac{y_i \delta_j}{(1 + |y|^2)^2} dy + O(\delta_j) \\
= & 64a(\xi_j) \delta_{il} \int_{\mathbb{R}^2} \frac{|y|^2}{(1 + |y|^2)^2} dy + O(\delta_j),
\end{align*}

and for $h \neq j, |x - \xi_h| \geq |\xi_j - \xi_h| - |x - \xi_j|,$
\begin{align*}
\int_{\Omega} e^{U_{\delta_j, \xi_j}} Z_{lh} \Gamma_{ij} a(x) dx \\
= & O \left[ \left( \int_{B(\xi_j, \rho^{-3M})} + \int_{\Omega \setminus B(\xi_j, \rho^{-3M})} \right) e^{U_{\delta_j, \xi_j}} |Z_{ij}| \left( |Z_{ij}| + O(\delta_j) \right) \right] \\
= & O \left( \int_{B(\xi_j, \rho^{-3M})} \delta_j^{3M} \right) \\
+ & O \left( \int_{\{|y, \delta_j y + \xi_h| \in \Omega, |\delta_j y + \xi_h - \xi_j| \geq \rho^{-2M}\}} \frac{1}{(1 + |y|^2)^2} \frac{|y|^2}{1 + |y|^2} \delta_j^{2M} \right) \\
= & o(\delta_h) + O(\delta_j p^{2M}).
\end{align*}
Now, since \( \int \Omega a(x) h \Gamma_{ij} dx = O(\|h\|_*), \) by (59),

\[
\text{L.H.S. of (58)} = \frac{32}{3} \pi a(\xi_j) c_{ij} + \frac{32}{3} \sum_{l=1}^{m} \sum_{h=1}^{m} |c_{lh}| + \|h\|_* . \quad (60)
\]

Moreover, by Lemma 3.1

\[
\text{R.H.S. of (58)} = \int_{B(\xi, \sqrt{3} \rho^{-2M})} W(x) \phi(x) a(x) \Gamma_{ij} dx - \int_{\Omega} a(x) e^{U_{\xi_j}} Z_{ij} \phi dx \\
+ O \left( \|\phi\|_* \|\nabla a(x)\|_* \int_\Omega |\nabla \Gamma_{ij}| + p^{2M} \sqrt{\delta_j} \|\phi\|_* \right) \\
= \int_{B(\xi, \sqrt{3} \rho^{-2M})} \phi(x) a(x) \Gamma_{ij} \left[ W(x) - e^{U_{\xi_j}} \right] dx \\
+ \int_{\Omega} a(x) e^{U_{\xi_j}} \phi (\Gamma_{ij} - Z_{ij}) dx + O \left( \frac{1}{p^2} \|\phi\|_* \right) \quad (61) \\
= O \left( \frac{1}{p} \|\phi\|_* \int_{\mathbb{R}^2} \frac{|\omega|}{(1 + |\xi|^2)^{\frac{3}{2}}} |w_0 - v_\infty - \frac{1}{2} v_\infty^2| \hat{\phi}_j dy + O \left( \frac{1}{p^2} \|\phi\|_* \right) \right) \\
= O \left( \frac{1}{p} \|\phi\|_* \right),
\]

where

\[
\int_\Omega |\nabla \Gamma_{ij}| \leq \int_\Omega |\nabla Z_{ij}| + C \delta_j \leq C \delta_j \int_\Omega \frac{1}{\delta_j^2 + |x - \xi_j|^2} dx + C \delta_j \leq C \delta_j |\log \delta_j| .
\]

Inserting the estimates (60) and (61) into (58), we deduce that

\[
\frac{32}{3} \pi a(\xi_j) c_{ij} + \frac{32}{3} \sum_{l=1}^{m} \sum_{h=1}^{m} |c_{lh}| = O \left( \|h\|_* + \frac{1}{p} \|\phi\|_* \right) .
\]

Hence we obtain that

\[
\sum_{l=1}^{m} \sum_{h=1}^{m} |c_{lh}| = O \left( \|h\|_* + \frac{1}{p} \|\phi\|_* \right) .
\]

Obviously we get

\[
\sum_{l=1}^{m} \sum_{h=1}^{m} |c_{lh}| = O(1) .
\]

As in Step 4, there holds

\[
\hat{\phi}_j \to C_j \frac{|\omega|^2 - 1}{|\omega|^2 + 1} \text{ in } C^0_{\text{loc}}(\mathbb{R}^2)
\]

for some \( j \) and constant \( C_j \). Hence, in (61) we have a better estimate

\[
\int_{B(0, \frac{1}{\sqrt{p^{2M}}})} \frac{32 y_i}{(1 + |\xi|^2)^{\frac{3}{2}}} \left( w_0 - v_\infty - \frac{1}{2} v_\infty^2 \right) \hat{\phi}_j dy
\]
converges to
\[
\int_{\mathbb{R}^2} \frac{32y_i(|y|^2 - 1)}{(1 + |y|^2)^4} \left( w_0 - v_\infty - \frac{1}{2} v_\infty^2 \right) dy = 0.
\]
Therefore, we get that the R.H.S. of (58) = \( o \left( p^{-1} \right) \), and in turn,
\[
\sum_{i=1}^{2} \sum_{h=1}^{m} |c_{ih}| = O(\|h\|_*) + o \left( \frac{1}{p} \right).
\]
This contradicts
\[
p \sum_{i=1}^{2} \sum_{h=1}^{m} |c_{ih}| \geq \delta_0 > 0
\]
and the claim is established.

**Step 6.** Now consider the following Hilbert space
\[
H = \left\{ \phi \in H^1_0(\Omega) : \int_{\Omega} e^{U_{\xi_{j,i}}} Z_{ij} \phi = 0, \quad \forall \ i = 1, 2, j = 1, \ldots, m \right\}
\]
with the norm \( \|\phi\|_{H^1_0(\Omega)} = \|\nabla \phi\|_{L^2(\Omega)} \). Problem (41) is equivalent to find \( \phi \in H \) such that
\[
\int_{\Omega} \left( a(x) \nabla \phi \nabla \psi - a(x) W \phi \psi \right) dx = \int_{\Omega} a(x) h \phi \psi dx, \quad \forall \ \psi \in H.
\]
By Fredholm’s alternative theorem, it is equivalent to the uniqueness of solutions to this problem, which is guaranteed by Proposition 2. Moreover, by elliptic regularity theory this solution is in \( H^2(\Omega) \). As \( p > p_0 \) fixed, by density of \( C_0^{\alpha, \alpha}(\Omega) \) in \( (C(\Omega), \|\cdot\|_{\infty}) \), we can approximate \( h \in C(\Omega) \) by smooth functions and, by elliptic regularity theory, we can show that (42) holds for any \( h \in C(\Omega) \). This ends the proof.

**Remark 4.** Given \( h \in C(\Omega) \), let \( \phi \) be the solution of (41) given by Proposition 2. Multiplying the first equation of (41) by \( a(x) \phi \) and integrating by parts, we get
\[
\int_{\Omega} a(x) \nabla \phi \nabla \psi dx = \int_{\Omega} a(x) W \phi \psi dx - \int_{\Omega} a(x) h \phi \psi dx.
\]
According to Lemma 3.1, we get
\[
\|\phi\|_{H^1_0(\Omega)} \leq C(\|\phi\|_{\infty} + \|h\|_*).
\]

4. **The nonlinear problem.** We want to solve here the nonlinear auxiliary problem
\[
\begin{cases}
\Delta_a (U_{\xi} + \phi) + (U_{\xi} + \phi)^p_+ = \frac{1}{a(x)} \sum_{i=1}^{2} \sum_{j=1}^{m} c_{ij} e^{U_{\xi_{j,i}}} Z_{ij}, & \text{in } \Omega, \\
\phi = 0, & \text{on } \partial \Omega,
\end{cases}
\]
for some coefficients \( c_{ij}, i = 1, 2 \) and \( j = 1, \ldots, m \), which depend on \( \xi \in \Lambda \). Recalling that \( N[\phi] = (U_{\xi} + \phi)^p_+ - U_{\xi}^p - pU_{\xi}^{p-1} \phi \), \( R_{\xi} = \Delta_a U_{\xi} + U_{\xi}^p \), we rewrite the first equation in (63) as the form
\[
L[\phi] = -R_{\xi} - N[\phi] + \frac{1}{a(x)} \sum_{i=1}^{2} \sum_{j=1}^{m} c_{ij} e^{U_{\xi_{j,i}}} Z_{ij}.
\]
Using the theory developed in the previous section for the linear operator \( L \), we have
Lemma 4.1. There exist $C > 0$ and $p_0 > 0$ such that, for any $p > p_0$ and $\xi \in \Lambda$, problem (63) has a unique solution $\phi_\xi$ satisfying

$$\| \phi_\xi \|_\infty \leq \frac{C}{p^3}, \quad \| \phi_\xi \|_{H^1_0(\Omega)} \leq \frac{C}{p^3}, \quad \sum_{i=1}^{m} \sum_{j=1}^{m} |c_{ij}(\xi)| \leq \frac{C}{p^4}. \quad (64)$$

Furthermore, the function $\xi \mapsto \phi_\xi$ is $C^1$.

Proof. The proof of this lemma can be done along the lines of that of Lemma 4.1 in [12]. We omit the details. \qed

5. The proof of Theorem 1.1. In view of Lemma 4.1, given $\xi = (\xi_1, \ldots, \xi_m) \in \Lambda$, we have $\phi_\xi$ and $c_{ij}(\xi)$ to be the unique solution to problem (63). Set

$$F_p(\xi) = J_p(U_\xi + \phi_\xi), \quad (65)$$

where

$$J_p[u] = \frac{1}{2} \int_\Omega a(x)|\nabla u|^2 - \frac{1}{p+1} \int_\Omega a(x)u^{p+1}_+, \quad \text{then we have the following}$$

Lemma 5.1. If $\xi = (\xi_1, \ldots, \xi_m) \in \Lambda$ is a critical point of $F_p(\xi)$, then $u = U_\xi + \phi_\xi$ is a critical point of $J_p$, that is, a solution to problem (1).

Proof. The proof is very similar to that of Lemma 5.1 in [12]. We omit it here. \qed

Next lemma shows the leading term of the function $F_p(\xi)$.

Lemma 5.2. Let $p_0$ be large enough and fixed. For any $p > p_0$, the following expansion holds:

$$F_p(\xi) = \frac{4\pi e}{p} \sum_{j=1}^{m} a(\xi_j) \left[ 1 - \frac{2}{p} \log p - \frac{1}{p} \sum_{i \neq j} G(\xi_j, \xi_i) \right] + O \left( \frac{1}{p^2} \right). \quad (66)$$

Proof. Since $u = U_\xi + \phi_\xi$ satisfies the equation (63), we have

$$\int_\Omega a(x)(U_\xi + \phi_\xi)^{p+1} dx$$

$$= \int_\Omega a(x)|\nabla (U_\xi + \phi_\xi)|^2 dx + \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} \int_\Omega e^{U_{\xi_j} + \phi_\xi} Z_{ij}(U_\xi + \phi_\xi) dx$$

$$= \int_\Omega a(x)|\nabla (U_\xi + \phi_\xi)|^2 dx + \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} \int_\Omega e^{U_{\xi_j} + \phi_\xi} Z_{ij} U_\xi dx$$

in view of the last equation in (63). Using $U_\xi = O(1)$ and (64) we get

$$\int_\Omega a(x)(U_\xi + \phi_\xi)^{p+1} dx = \int_\Omega a(x)|\nabla (U_\xi + \phi_\xi)|^2 dx + O \left( \frac{1}{p^4} \right)$$
uniformly for $\xi \in \Lambda$. Hence we can write
\[
\mathcal{F}_p(\xi) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\Omega} a(x)|\nabla (U_\xi + \phi_\xi)|^2 dx + O\left(\frac{1}{p^2}\right)
\]
\[
= \frac{p-1}{2(p+1)} \int_{\Omega} a(x) \left(|\nabla U_\xi|^2 + 2\nabla U_\xi \cdot \nabla \phi_\xi + |\nabla \phi_\xi|^2\right) dx + O\left(\frac{1}{p^2}\right)
\]
\[
= \frac{p-1}{2(p+1)} \int_{\Omega} a(x) |\nabla U_\xi|^2 + O\left(\frac{1}{p^2}\right) \left(\frac{1}{p^2} \int_{\Omega} a(x)|\nabla U_\xi|^2\right)^{1/2} + \frac{1}{p^2}\right).
\]

Recall that $U_\xi = \sum_{j=1}^{m} (U_j + H_j^p)$ defined in (22), we have
\[
\int_{\Omega} a(x)|\nabla U_\xi|^2 dx
\]
\[
= \int_{\Omega} a(x)(-\Delta U_\xi)U_\xi dx
\]
\[
= \sum_{j=1}^{m} \frac{1}{\gamma^{\mu_j^{\frac{1}{p^2}}}} \int_{B(\xi_j, \frac{1}{p^{1/2}})} a(x) e^{U_j} \Delta w_0 \left(\frac{x - \xi_j}{\delta_j}\right)
\]
\[
- \frac{1}{p^2} \Delta w_1 \left(\frac{x - \xi_j}{\delta_j}\right) U_\xi dx + O\left(\frac{1}{p^2}\right)
\]
\[
= \sum_{j=1}^{m} \frac{1}{\gamma^{\mu_j^{\frac{1}{p^2}}}} \int_{B(0, \frac{1}{p^{1/2}})} a(\delta_j y + \xi_j) \left[\frac{8}{(1 + |y|^2)^2} - \frac{1}{p} \Delta w_0(y) - \frac{1}{p^2} \Delta w_1(y)\right]
\]
\[
\times \left[p + v_\infty + \frac{1}{p} w_0(y) + \frac{1}{p^2} w_1(y) + O(\sqrt{\delta_j |y| + \delta_j^2})\right] dy + O\left(\frac{1}{p^2}\right)
\]
\[
= \sum_{j=1}^{m} \frac{1}{\gamma^{\mu_j^{\frac{1}{p^2}}}} \left[a(\xi_j) 8\pi p + O(1)\right] + O\left(\frac{1}{p^2}\right).
\]

Recall $\gamma = p^{\frac{p}{p-1}} e^{-\frac{\mu_j^{\frac{1}{p^2}}}{p-1}}$, then
\[
\gamma^{-2} = \frac{e}{p^2} \left[1 - \frac{2 \log p}{p - 1} + O\left(\frac{1}{p}\right)\right],
\]

and
\[
\mu_j^{\frac{1}{p^2}} = 1 - \frac{4}{p - 1} \log \mu_j + O\left(\frac{1}{p}\right)
\]

we get
\[
\gamma^{-2} \mu_j^{\frac{1}{p^2}} = \frac{e}{p^2} \left[1 - \frac{4}{p - 1} \log \mu_j - \frac{2 \log p}{p - 1} + O\left(\frac{1}{p}\right)\right].
\]

From (24) we get
\[
\log (\mu_j^\delta) = \sum_{i \neq j} G(\xi_j, \xi_i) + O(1).
\]

Thus
\[
\int_{\Omega} a(x)|\nabla U_\xi|^2 dx = \frac{8\pi e}{p} \sum_{j=1}^{m} a(\xi_j) \left[1 - \frac{4}{p} \log \mu_j - \frac{2 \log p}{p}\right] + O\left(\frac{1}{p^2}\right).
\]
Hence
\[ F_p(\xi) = \frac{4\pi e}{p} \sum_{j=1}^{m} a(\xi_j) \left[ 1 - \frac{2 \log p}{p} \sum_{i \neq j} \frac{1}{pG(\xi_j, \xi_i)} \right] + O \left( \frac{1}{p^2} \right) \]
where \( G(\xi_j, \xi_i) = -4 \log |\xi_i - \xi_j| + O(1). \)

\[ \square \]

**Lemma 5.3.** For \( p \) large enough, the following maximization problem
\[ \max_{(\xi_1, \ldots, \xi_m) \in \mathcal{X}} F_p(\xi_1, \ldots, \xi_m) \]
has a solution in the interior of \( \Lambda \).

**Proof.** Let \( (\xi_1^0, \ldots, \xi_m^0) \in \mathcal{X} \) be the maximizer of \( F_p \). We need to prove that \( (\xi_1^0, \ldots, \xi_m^0) \) belongs to the interior of \( \Lambda \). First, we obtain a lower bound. Let
\[ \hat{\xi}_j^0 = x_0 + \frac{1}{p^2} \xi_j^0 \]
where \( \xi_j^0, j = 1, \ldots, m \) form an \( m \)-regular polygon in \( \mathbb{R}^2 \). Then it is easy to see \( (\xi_1^0, \ldots, \xi_m^0) \in \Lambda \) since \( M = \frac{m^2 + 1}{2} \geq 1 \). From Lemma 5.2, using that \( x_0 \) is a strict maximum point of \( a(x) \), we obtain:
\[ \max_{(\xi_1, \ldots, \xi_m) \in \mathcal{X}} F_p(\xi_1, \ldots, \xi_m) \]
\[ \geq \frac{4\pi e}{p} \sum_{j=1}^{m} \left[ a(x_0) - \frac{C}{p} \right] \left[ 1 - \frac{2}{p} \log p - \log \frac{1}{|\xi_j^0 - \xi_i^0|} \right] \]
\[ = \frac{4\pi e}{p} a(x_0) \sum_{j=1}^{m} \left[ 1 - \frac{2}{p} \log p - \frac{1}{p} \log \frac{1}{|\xi_j^0 - \xi_i^0|} \right] + O \left( \frac{1}{p^2} \right) \]
\[ = \frac{4\pi e m}{p} a(x_0) \left( 1 - \frac{2}{p} \log p \right) - \frac{4\pi e}{p^2} a(x_0) \sum_{j=1}^{m} \log \frac{1}{|\xi_j^0 - \xi_i^0|} \]
\[ = \frac{4\pi e m}{p} a(x_0) \left( 1 - \frac{2}{p} \log p \right) \frac{2 \log p + O \left( \frac{1}{p^2} \right)}{p^2} \]
\[ \geq \frac{8\pi e m}{p} a(x_0) \log p + O \left( \frac{1}{p^2} \right). \]

Now suppose \( (\xi_1^p, \ldots, \xi_m^p) \in \partial \Lambda \). There are two possibilities: either there exists \( j_0 \) such that \( \xi_j^p \in \partial B_0(x_0) \), in which case, \( a(\xi_j^p) \leq a(x_0) - \delta_0 \) for some \( \delta_0 > 0 \); or there exists \( i_0 \neq j_0 \) such that \( |\xi_i^p - \xi_j^p| = \frac{1}{p^2} \).

In first case, we have
\[ \max_{(\xi_1, \ldots, \xi_m) \in \mathcal{X}} F_p(\xi_1, \ldots, \xi_m) \]
\[ \leq \frac{4\pi e}{p} a(x_0) \left( 1 - \frac{2}{p} \log p \right) \left[ (m-1)a(x_0) + a(x_0) \right] - \delta_0 + O \left( \frac{\log p}{p^2} \right) \]
\[ = \frac{4\pi e}{p} a(x_0) \left[ ma(x_0) - \delta_0 \right] + O \left( \frac{1}{p} \right), \]
which contradicts to (67). This also shows that \( a(\xi_j^p) \to a(x_0) \). By the condition over \( a \), we get \( \xi_j^p \to x_0 \).
In the second case, we have
\[
\max_{(\xi_1, \ldots, \xi_m) \in \mathcal{F}_p} \mathcal{F}_p(\xi_1, \ldots, \xi_m) \\
\leq \frac{4\pi em}{p} a(x_0) \left(1 - \frac{2}{p} \log p\right) + \frac{16\pi e}{p^2} a(x_0) \log |\xi^p_0 - \xi^p_j| + O \left(\frac{1}{p^2}\right) \tag{69}
\]
\[
= \frac{4\pi e}{p^m} a(x_0) \left(1 - \frac{2}{p} \log p\right) - \frac{16\pi e}{p^2} a(x_0) M \log p + O \left(\frac{1}{p^2}\right).
\]
Combining with (67) we have
\[
\frac{16\pi e}{p^2} a(x_0) M \log p + O \left(\frac{1}{p^2}\right) \leq \frac{8\pi e a(x_0)}{p^2} m(m-1) \log p + O \left(\frac{1}{p^2}\right)
\]
which is impossible by the choice of \(M\) in (12).

**Proof of Theorem 1.1:** According to Lemma 5.1, the function \(u_p = U_\xi + \phi_\xi\) where \(U_\xi\) and \(\phi_\xi\) are defined respectively by (22) and Lemma 4.1, is a solution of problem (1) if we adjust \(\xi\) so that it is a critical point of \(\mathcal{F}_p(\xi) = J_p(U_\xi + \phi_\xi)\) defined by (65). Lemma 5.3 then guarantees the existence of such critical point \(\xi^p = (\xi^p_1, \ldots, \xi^p_m)\) and thus a solution \(u_p\) for (1). Furthermore, from the ansatz (22), we get for any \(\rho > 0\), as \(p \to \infty\), \(u_p \to 0\) uniformly in \(\Omega \setminus B(x_0, \rho)\) and
\[
\sup_{x \in B(x_0, \rho)} u_p(x) \longrightarrow \sqrt{c}.
\]
The rest of the properties of \(u_p\) can be easily seen from the decomposition of \(u_p\).

**Proof of Remark 1:** We choose now \(M = \frac{\pi^2 + \alpha}{\alpha}\). We just need to change the lower bound estimate in the proof of Lemma 5.3.

Take \(\xi_j^0 = x_0 + p^{-\frac{1}{m}} \xi_j^0\) where \(\xi_j^0, j = 1, \ldots, m\) form an \(m\)-regular polygon in \(\mathbb{R}^2\), we get then
\[
\max_{(\xi_1, \ldots, \xi_m) \in \mathcal{F}_p} \mathcal{F}_p(\xi_1, \ldots, \xi_m) \\
\geq \frac{4\pi em}{p} a(x_0) \left(1 - \frac{2}{p} \log p\right) - \frac{16\pi em(m-1)}{\alpha p^2} a(x_0) \log p + O \left(\frac{1}{p^2}\right).
\]
Using (68) and (69), we prove again that \(\mathcal{F}_p(\xi)\) reaches its maximum in the interior of \(\Lambda\).

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