Categories $\mathcal{O}$ for Root-Reductive Lie Algebras: II.
Translation Functors and Tilting Modules

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Abstract

This is the second paper of a series of papers on a version of categories $\mathcal{O}$ for root-reductive Lie algebras. Let $\mathfrak{g}$ be a root-reductive Lie algebra over an algebraically closed field $\mathbb{K}$ of characteristic 0 with a splitting Borel subalgebra $\mathfrak{b}$ containing a splitting maximal toral subalgebra $\mathfrak{h}$. For some pairs of blocks $\mathcal{O}[\lambda]$ and $\mathcal{O}[\mu]$, the subcategories whose objects have finite length are equivalence via functors obtained by the direct limits of translation functors. Tilting objects can also be defined in $\mathcal{O}$. There are also universal tilting objects $D(\lambda)$ in parallel to the finite-dimensional cases.

Key words: root-reductive Lie algebras, finitary Lie algebras, highest-weight modules, BGG categories $\mathcal{O}$, equivalences of categories, translation functors, tilting modules

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Introduction

The purpose of this paper is to further study a version of Bernstein-Gel’fand-Gel’fand (BGG) categories $\mathcal{O}$ for root-reductive Lie algebras with respect to Dynkin Borel subalgebras as defined in [9]. For a reductive Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic 0 whose derived algebra $[\mathfrak{g}, \mathfrak{g}]$ is finite-dimensional, if $\mathcal{O}_b^\mathfrak{g}[\lambda]$ denotes the BGG category $\mathcal{O}$ of $\mathfrak{g}$ with respect to a certain Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ as defined in [6, Chapter 1.1], then we know that some blocks of $\mathcal{O}_b^\mathfrak{g}[\lambda]$ are equivalent via translation functors (see [6, Chapter 1.13] and [6, Chapter 7]).

We denote by $\mathcal{O}_b^\mathfrak{g}[\lambda]$ for the block of $\mathcal{O}$ containing the simple object $\mathcal{L}(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$. Also, $W_{\mathfrak{g}, \mathfrak{b}}[\lambda]$ is the integral Weyl group for the weight $\lambda$ (see [6, Chapter 3.4], wherein the notation $W[\lambda]$ is used). The Borel subalgebra $\mathfrak{b}$ induces the set of simple reflections $S_{\mathfrak{g}, \mathfrak{b}}[\lambda]$, so that $(W_{\mathfrak{g}, \mathfrak{b}}[\lambda], S_{\mathfrak{g}, \mathfrak{b}}[\lambda])$ is a Coxeter system.

In fact, [10, Theorem 11] provides a stronger statement, as it a description of the categorical structure of $\mathcal{O}_b^\mathfrak{g}$ using the Weyl group of $\mathfrak{g}$. In other words, suppose that $\mathfrak{g}$ and $\mathfrak{g}'$ are two reductive Lie algebras with Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$; for $\lambda \in \mathfrak{h}'^*$ and $\lambda' \in (\mathfrak{h}')^*$, if the Coxeter systems $(W_{\mathfrak{g}, \mathfrak{b}}[\lambda], S_{\mathfrak{g}, \mathfrak{b}}[\lambda])$ and $(W_{\mathfrak{g}', \mathfrak{b}'}[\lambda'], S_{\mathfrak{g}', \mathfrak{b}'}[\lambda'])$ are isomorphic, then the blocks $\mathcal{O}_{b'}^{\mathfrak{g}'}[\lambda']$ and $\mathcal{O}_{b'}^{\mathfrak{g}'}[\lambda']$ are equivalent as categories.

The paper [4] studies Kac-Moody algebras and obtains a similar result to [10, Theorem 11]. If $\mathfrak{g}$ and $\mathfrak{g}'$ are complex symmetrizable Kac-Moody algebras with Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$ and Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$, where $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$ and $\mathfrak{h}' \subseteq \mathfrak{b}' \subseteq \mathfrak{g}'$. We denote by $\mathcal{O}_b^\mathfrak{g}$ and $\mathcal{O}_b^{\mathfrak{g}'}$ for the corresponding BGG categories $\mathcal{O}$ for the pairs $(\mathfrak{g}, \mathfrak{b})$ and $(\mathfrak{g}', \mathfrak{b}')$, respectively. Let $\Lambda \subseteq \mathfrak{h}^*$ be the set of highest weights of simple objects in a block of $\mathcal{O}_b^\mathfrak{g}$, and write $\mathcal{O}_\Lambda$ for the said block. The notations $\Lambda' \subseteq (\mathfrak{h}')^*$ and $\mathcal{O}_{\Lambda'}$ are defined similarly for $\mathcal{O}_b^{\mathfrak{g}'}$. For specific pairs $\Lambda$ and $\Lambda'$, [4, Theorem 4.1] establishes an equivalence between the categories $\mathcal{O}_\Lambda$ and $\mathcal{O}_{\Lambda'}$. One of the necessary conditions for the existence of an equivalence in [4, Theorem 4.1] is that there exists an isomorphism between relevant Coxeter systems.

In the present paper, we shall look at the subcategory $\mathcal{O}_b^\mathfrak{g}[\lambda]$ consisting of objects of finite length from the block $\mathcal{O}_b^\mathfrak{g}[\lambda]$, where $\mathcal{O}$ is an extended BGG category $\mathcal{O}$ for a root-reductive Lie algebra $\mathfrak{g}$.
with respect to a Dynkin Borel subalgebras \( \mathfrak{b} \), and \( \lambda \in \mathfrak{h}^\ast \). Here, \( \mathfrak{h} \) is the unique splitting maximal toral subalgebra of \( \mathfrak{g} \) contained in \( \mathfrak{b} \). We obtain a similar result to [10, Theorem 11] and [4, Theorem 4.1], with [10, Theorem 11] being the crucial ingredient for our proof.

Another main topic of this paper is tilting theory. In the case where \( \mathfrak{g} \) is a reductive Lie algebra with \( [\mathfrak{g}, \mathfrak{g}] \) being finite-dimensional, tilting modules in \( \mathcal{O}^\mathfrak{g} \) are objects with both standard and costandard filtrations (see [6, Chapter 11.1]). For a root-reductive Lie algebra \( \mathfrak{g} \), indecomposable objects in \( \mathcal{O}^\mathfrak{g} \) can potentially have infinite length, in which case the notion of filtrations may not apply to \( \mathcal{O} \). However, if we generalize the definition of filtrations, then it is possible to define tilting modules in \( \mathcal{O}^\mathfrak{g} \) in a similar manner.

This paper consists of three sections. The first section provides necessary foundations for other sections such as a brief recapitulation of the results from [9] and some relevant definitions such as generalized filtrations. This section also provides a characterization of integrable modules in our version of BGG categories \( \mathcal{O} \) for root-reductive Lie algebras, which are usually denoted by \( \bar{\mathcal{O}} \). The second section provides a visualization of the subcategory of each block of \( \bar{\mathcal{O}} \) consisting of modules of finite length, proving that each subcategory is a direct limit of subcategories of some categories \( \mathcal{O} \) for reductive Lie algebras with finite-dimensional derived algebras. The final section deals with the construction and the properties of tilting modules in \( \bar{\mathcal{O}} \).

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## 1 Preliminaries

All vector spaces and Lie algebras are defined over an algebraically closed field \( \mathbb{K} \) of characteristic 0. For a vector space \( V \), \( \dim V \) is the \( \mathbb{K} \)-dimension of \( V \) and \( V^\ast \) denotes its algebraic dual \( \text{Hom}_{\mathbb{K}}(V, \mathbb{K}) \). Unless otherwise specified, the tensor product \( \otimes \) is defined over \( \mathbb{K} \). For a Lie algebra \( \mathfrak{g} \), \( \mathfrak{u}(\mathfrak{g}) \) is its universal enveloping algebra.

### 1.1 Root-Reducive Lie Algebras and Categories \( \bar{\mathcal{O}} \)

Let \( \mathfrak{g} \) be a root-reductive Lie algebra in the sense of [9, Definition 1.1]. Suppose that \( \mathfrak{h} \) is a splitting maximal toral subalgebra of \( \mathfrak{g} \) in the sense of [9, Definition 1.2], and \( \mathfrak{b} \) is a Dynkin Borel subalgebra of \( \mathfrak{g} \) (see [9, Definition 1.5]) that contains \( \mathfrak{h} \). Let \( \mathfrak{n} := [\mathfrak{b}, \mathfrak{b}] \) (we sometimes write \( \mathfrak{b}^\pm \) and \( \mathfrak{n}^\pm \) for \( \mathfrak{b} \) and \( \mathfrak{n} \), respectively). If \( \mathfrak{b}^- \) is the unique Borel subalgebra of \( \mathfrak{g} \) such that \( \mathfrak{b}^+ \cap \mathfrak{b}^- = \mathfrak{h} \), then we have the following decompositions of vector spaces: \( \mathfrak{b}^\pm = \mathfrak{h} \oplus \mathfrak{n}^\pm \) and \( \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \).

For each \( \mathfrak{b} \)-root \( \alpha \) of \( \mathfrak{g} \), the \( \mathfrak{h} \)-root space of \( \mathfrak{g} \) associated to \( \alpha \) is given by \( \mathfrak{g}^\alpha \). With respect to \( \mathfrak{b} \), the set \( \Phi_{\mathfrak{g}, \mathfrak{h}} \) of \( \mathfrak{h} \)-roots of \( \mathfrak{g} \) can be partitioned into two disjoint subsets \( \Phi_{\mathfrak{g}, \mathfrak{b}}^+ \), consisting of positive \( \mathfrak{b} \)-roots and \( \Phi_{\mathfrak{g}, \mathfrak{b}}^- \) consisting of negative \( \mathfrak{b} \)-roots. Write \( W_{\mathfrak{g}, \mathfrak{h}} \) for the Weyl group of \( \Phi_{\mathfrak{g}, \mathfrak{b}} \). Also, \( \Lambda_{\mathfrak{g}, \mathfrak{b}} := \text{span}_\mathbb{Z} \Phi_{\mathfrak{g}, \mathfrak{b}} \). When there is no risk of confusion, we shall write \( \Phi, \Phi^+, \Phi^- \), \( W \), and \( \Lambda \) for \( \Phi_{\mathfrak{g}, \mathfrak{b}}, \Phi_{\mathfrak{g}, \mathfrak{b}}^+, \Phi_{\mathfrak{g}, \mathfrak{b}}^-, W_{\mathfrak{g}, \mathfrak{b}} \), and \( \Lambda_{\mathfrak{g}, \mathfrak{b}} \), respectively.

The set of (positive) \( \mathfrak{b} \)-simple roots is denoted by \( \Sigma_{\mathfrak{g}, \mathfrak{b}} \), or \( \Sigma_{\mathfrak{g}, \mathfrak{b}}^+ \). The set of negative \( \mathfrak{b} \)-simple roots is given by \( \Sigma_{\mathfrak{g}, \mathfrak{b}}^- \). For convenience, we also write \( \Sigma, \Sigma^+, \) and \( \Sigma^- \) for \( \Sigma_{\mathfrak{g}, \mathfrak{b}}, \Sigma_{\mathfrak{g}, \mathfrak{b}}^+, \) and \( \Sigma_{\mathfrak{g}, \mathfrak{b}}^- \). For each \( \alpha \in \Phi \), let \( x_\alpha \in \mathfrak{g}^+\alpha, x_\alpha \in \mathfrak{g}^-\alpha, \) and \( h_\alpha \in \lbrack \mathfrak{g}^\alpha, \mathfrak{g}^-\alpha \rbrack \) be such that \( h_\alpha = \lbrack x_\alpha, x_\alpha \rbrack \) and \( \alpha (h_\alpha) = 2 \) (that is, \( h_\alpha \) is the coroot of \( \alpha \)). Thus, \( \{ x_\alpha \mid \alpha \in \Phi \} \cup \{ h_\alpha \mid \alpha \in \Sigma \} \) is a Chevalley basis of \( [\mathfrak{g}, \mathfrak{g}] \). For convenience, we fix a filtration \( \mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \cdots \subseteq \mathfrak{g}_f \) of \( \mathfrak{g} \) such that each \( \mathfrak{g}_n \) is a finite-dimensional reductive Lie algebra with \( \mathfrak{b}_n := \mathfrak{b} \cap \mathfrak{g}_n \) and \( \mathfrak{n}_n := \mathfrak{h} \cap \mathfrak{g}_n \) as a Borel subalgebra and a Cartan subalgebra, respectively. We also define \( \mathfrak{g}_n := \mathfrak{g}_n + \mathfrak{h} \) and \( \mathfrak{b}_n := \mathfrak{b}_n + \mathfrak{h} \). The notations \( \mathfrak{b}_n^\pm, \mathfrak{n}_n^\pm, \) and \( \mathfrak{n}_n \) carry similar meanings.
Note that there exists $\rho \in \mathfrak{h}^*$ such that $\rho|_{\mathfrak{h}_n}$ is the half sum of $\mathfrak{h}_n$-positive roots of each $\mathfrak{g}_n$. Then, we define the dot action of $W$ on $\mathfrak{h}^*$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$ for each $\lambda \in \mathfrak{h}^*$. While the map $\rho$ may not be unique, the dot action is independent of the choice of $\rho$. For a fixed $\lambda \in \mathfrak{h}^*$, the subgroup $W^{\rho}_{\mathfrak{h},b}[\lambda]$ (also denoted by $W[\lambda]$) of $W_{\mathfrak{g},b}$ consists of $w \in W_{\mathfrak{g},b}$ such that $w \cdot \lambda - \lambda \in \Lambda_{\mathfrak{g},b}$.

Write $\bar{O}^\rho_{b}$ for the extended category $\bar{O}$ for the pair $(\mathfrak{g}, b)$ (see [9, Definition 2.1]). For simplicity, we also write $\bar{O}$ for $\bar{O}^\rho_{b}$. For each $M \in \bar{O}$ and $\lambda \in \mathfrak{h}^*$, $M^\lambda$ denotes the $\mathfrak{h}$-weight space with respect to the weight $\lambda$. Let $(\_)^\vee : \bar{O}^\rho_{b} \rightarrow \bar{O}^\rho_{b}$ be the duality functor as defined in [9, Definition 2.2.]. Then, $\Delta^\rho_{b}(\lambda)$, or simply $\Delta(\lambda)$, is defined to be the Verma module with highest weight $\lambda \in \mathfrak{h}^*$ (see [9, Definition 1.9]). We also write $\nabla^\rho_{b}(\lambda)$, or simply $\nabla(\lambda)$, for the co-Vera module with highest weight $\lambda \in \mathfrak{h}^*$, namely, $\nabla(\lambda) = (\Delta(\lambda))^\vee$. We denote $\mathfrak{g}_b^\rho(\lambda)$, or simply $\mathfrak{g}(\lambda)$, for the simple quotient of $\Delta(\lambda)$.

For each $\lambda \in \mathfrak{h}^*$, we shall denote by $[\lambda]$ the set of all weights $\mu \in W \cdot \lambda$ such that $\lambda - \mu \in \Lambda$. The definition of abstract blocks is given by [2, Definition 4.13]. A block in our consideration is the full subcategory of $\bar{O}$ consisting of all objects belonging in the same abstract block. Due to [9, Theorem 3.4], we can write each block of $\bar{O}$ as $\bar{O}[\lambda]$, where $\bar{O}[\lambda]$ is the unique block of $\bar{O}$ that contains $\Delta(\lambda)$. If $\Omega^\rho_{b}$, or simply $\Omega$, is the set of all $[\lambda]$, where $\lambda \in \mathfrak{h}^*$, then

$$\bar{O} = \bigoplus_{[\lambda] \in \Omega} \bar{O}[\lambda].$$

(1.1)

Note that, if $\mathfrak{g}$ is finite-dimensional, then $\bar{O}[\lambda] = \bar{O}[\lambda]$.

For a given $\lambda \in \mathfrak{h}^*$, let $\text{pr}_{\mathfrak{g}^\lambda, b} : \bar{O} \rightarrow \bar{O}[\lambda]$ denote the projection onto the $[\lambda]$-block. We also write $\text{inj}_{\mathfrak{g}^\lambda, b} : \bar{O}[\lambda] \rightarrow \bar{O}$ for the injection from the $[\lambda]$-block. When the context is clear, $\text{pr}^\lambda$ and $\text{inj}^\lambda$ are used instead. Note that both functors are exact, and are adjoint to one another.

For each $M \in \bar{O}$, $\Pi(M) \overset{\text{def}}{=} \{ \lambda \in \mathfrak{h}^* \mid M^\lambda \neq 0 \}$ and $\text{ch}_{b}(M) \overset{\text{def}}{=} \sum_{\lambda \in \mathfrak{h}^*} \text{dim}(M^\lambda) e^\lambda$ is the formal character of $M$ (with the standard multiplication rule given by $e^\lambda e^\mu \overset{\text{def}}{=} e^{\lambda+\mu}$ for all $\lambda, \mu \in \mathfrak{h}^*$). When the context is clear, $\text{ch}(M)$ denotes $\text{ch}_{b}(M)$. We note that $\text{ch}(M) = \sum_{\lambda \in \mathfrak{h}^*} [M : \mathfrak{g}(\lambda)] \text{ ch}(\mathfrak{g}(\lambda))$, where $[M : L]$ denotes the multiplicity of a simple object $L$ in $M$ (see [9, Corollary 2.10]). If $L$ is a simple object such that $[M : L] > 0$, then $L$ is called a composition factor of $M$.

## 1.2 Generalized Filtrations

Let $\mathcal{C}$ be an abelian category. Fix a family $\mathcal{F} \subseteq \mathcal{C}$.

**Definition 1.1** A generalized $\mathcal{F}$-filtration of $M \in \mathcal{C}$ is a collection $(M_j)_{j \in J}$ of subobjects $M_j$ of $M$, where $(J, \preceq)$ is a totally ordered set, such that

1. $M_j \subsetneq M_k$ for all $j, k \in J$ such that $j < k$,
2. $\bigcap_{j \in J} M_j = 0$,
3. $\bigcup_{j \in J} M_j = M$, and
4. for each $j \in J$, $M_j/\left( \bigcup_{k < j} M_k \right)$ is an object in $\mathcal{F}$.

**Definition 1.2** Two generalized $\mathcal{F}$-filtrations $(M_j)_{j \in J}$ and $(M'_j)_{j' \in J'}$ of $M \in \mathcal{C}$ (where $\preceq$ and $\preceq'$ are the respective total orders on $J$ and $J'$) are said to be $\mathcal{F}$-equivalent if there exists a bijection $f : J \rightarrow J'$ such that $M_j/\left( \bigcup_{k < j} M_k \right) \cong M'_{f(j)}/\left( \bigcup_{k' < f(j)} M'_{k'} \right)$ for every $j \in J$ that is not the least element of $J$. 

\[ \diamondsuit \]
Definition 1.3 We say that $F$ is a complete filter if, for any $M ∈ C$, $M$ has a generalized $F$-filtration. We say that $F$ is a good filter if any two filtrations $(M_j)_{j ∈ J}$ and $(M'_j)_{j' ∈ J'}$ of a single object $M ∈ C$ are $F$-equivalent.

Definition 1.4 We define $F(C)$ to be the full subcategory of $C$ whose objects are those with generalized $F$-filtrations.

Consider $C := O$. We have the following theorem.

Theorem 1.5 Define $Δ := \{Δ(λ) \mid λ ∈ h^+\}$, $∇ := \{∇(λ) \mid λ ∈ h^+\}$, and $L := \{L(λ) \mid λ ∈ h^+\}$ as subcollections of the category $O$.

(a) The collection $Δ$ is a good filter of $O$. (A generalized $Δ$-filtration is also called a generalized standard filtration.)

(b) The collection $∇$ is a good filter of $O$. (A generalized $∇$-filtration is also called a generalized standard filtration.)

(c) The collection $L$ is a good and complete filter of $O$. (A generalized $L$-filtration is also known as a generalized composition series.)

Proof Part (c) follows from [9, Corollary 2.10]. By employing duality, Part (b) is a trivial consequence of Part (a). We shall now prove Part (a).

Suppose that $M ∈ O$ has two generalized standard filtrations $(M_j)_{j ∈ J}$ and $(M'_j)_{j' ∈ J'}$ of $M ∈ C$ (where $≤$ and $≤'$ are the respective total orders on $J$ and $J'$). Without loss of generality, we may assume that $M$ lies a single block $O[λ]$ of $O$.

Let $p$ denote the formal character of $Δ(0)$. For $j ∈ J$ and $j' ∈ J'$ that are not the least elements of $J$ and $J'$, respectively, suppose that $μ(j)$ and $μ'(j')$ denote the highest weights of $M_j/\bigcup_{k < j} M_k$ and $M'_j/\bigcup_{k' < j'} M'_{k'}$, respectively. Write $a$ and $a'$ for the sum of all $e^μ(j)$ and the sum of all $e^{μ'(j')}$, respectively. It follows that $ch(M) = ap$ and $ch(M) = a'p$.

We now let $q$ to be the infinite product of $e^0 - e^{-α}$, where $α$ runs over all $b$-positive roots. Since $b$ is a Dynkin Borel subalgebra, $q$ is well defined. We can easily show that $pq = e^0$. Therefore,

$$a = ae^0 = a(pq) = (ap)q = ch(M)q = (a'p)q = a'(pq) = a'e^0 = a'. \quad (1.2)$$

The claim follows immediately.

Corollary 1.6 For each object $M ∈ Δ(O)$ and a given generalized standard filtration $(M_j)_{j ∈ J}$ of $M$, the number of times $Δ(λ)$ occurs as a quotient $M_j/\bigcup_{k < j} M_k$ is a finite nonnegative integer, which is independent of the choice of the generalized standard filtration $(M_j)_{j ∈ J}$. This number is denoted by $\{M : Δ(λ)\}$.

For each $M ∈ ∇(O)$ and a given generalized co-standard filtration $(M_j)_{j ∈ J}$ of $M$, the number of times $∇(λ)$ occurs as a quotient $M_j/\bigcup_{k < j} M_k$ is a finite nonnegative integer which is independent of the choice of the generalized co-standard filtration. This number is denoted by $\{M : ∇(λ)\}$.

Example 1.7 For fixed $λ, μ ∈ h^+$ such that $μ ≤ λ$, the truncated projective cover $P^{≤ λ}(μ)$ of $L(λ)$ lies in $Δ(O)$, whilst the truncated injective hull $I^{≤ λ}(μ)$ of $L(λ)$ lies in $∇(O)$. See [9, Section 4].

Proposition 1.8 Suppose that $M ∈ Δ(O)$. \

\[\]
(a) If $\lambda$ is a maximal weight of $M$, then $M$ has a submodule $N$ isomorphic to $\Delta(\lambda)$, and the factor module $M/N$ is in $\Delta(\hat{O})$.

(b) If $N$ is a direct summand of $M$, then $N \in \Delta(\hat{O})$.

(c) The module $M$ is a free $\mathfrak{U}(\mathfrak{n}^-)$-module.

Proof For Part (a), let $u$ be a maximal vector of $M$ with weight $\lambda$. There exists an index $j \in J$ such that $u \in M_j$ but $u \notin M_{<j} := \bigcup_{k<j} M_k$. Thus, the map $\psi : \Delta(\lambda) \to M_j/M_{<j}$ sending $g \cdot u' \mapsto g \cdot u + M_{<j}$ for each $g \in \mathfrak{U}(\mathfrak{g})$, where $u'$ is a maximal vector of $\Delta(\lambda)$, is a nonzero homomorphism of Verma modules (recalling that $M_j/M_{<j}$ is a Verma module). By [9, Theorem 1.1], $\psi$ must be injective. Therefore, $\text{im}(\psi)$ is a Verma submodule with highest weight $\lambda$ of the Verma module $M_j/M_{<j}$. Because $\lambda$ is a maximal weight of $M$, we conclude that $\text{im}(\psi) = M_j/M_{<j}$ and $\psi$ is an isomorphism of $\mathfrak{g}$-modules. Hence, the $\mathfrak{g}$-submodule $N$ of $M$ generated by $u$ is a Verma module isomorphic to $\Delta(\lambda)$. We now note that $M/M_j$ and $M_{<j}$ are both in $\Delta(\hat{O})$. Furthermore, because $N \cap M_{<j} \cong \ker(\psi) = 0$, we obtain a short exact sequence $0 \to M_{<j} \to M/N \to M/M_j \to 0$. Hence, $M/N$ has a generalized standard filtration given by patching the generalized standard filtration of $M_{<j}$ with the generalized standard filtration of $M/M_j$.

For Part (b), we may assume without loss of generality that $N$ is inductively decomposable direct summand of $M$. Now, define $N[0] := N$ and $\lambda[0] := \lambda$. We finish the proof using transfinite induction. For an ordinal $t$ with a predecessor $s$, suppose a module $N[s]$ and a weight $\lambda[s] \in \mathfrak{h}^*$ are given such that $\Delta(\lambda[s]) \subseteq N[s]$. Then, define $N[t] := N[s]/\Delta(\lambda[s])$. If $N[t] = 0$, then we are done. If $N[t] \neq 0$, then by taking $\lambda[t]$ to be a maximal weight of $N[t]$, using the same idea as the paragraph above, we conclude that $\Delta(\lambda[t]) \subseteq N[t]$. On the other hand, if $t$ is a limit ordinal, then we have a directed system of modules $(N[s])_{s \leq t}$. Define $N[t]$ to be the direct limit of the modules $N[s]$ for $s < t$. As before, if $N[t] = 0$, then we are done. If not, we then take $\lambda[t]$ to be a maximal weight of $N[t]$. Then, again, $\Delta(\lambda[t]) \subseteq N[t]$. Since the multiset of composition factors of $N$ is a countable multiset, this procedure must stop at some countable ordinal $\tau$, where $N[\tau] = 0$. Then we obtain a generalized standard filtration of $N$.

For Part (c), let $(M_j)_{j \in J}$ be a generalized standard filtration of $M$. For an element $j \in J$ that is not the minimum element of $J$, take $m_j \in M_j \setminus \bigcup_{k<j} M_k$ such that $m_j$ is a weight vector whose weight is the highest weight of $M_j/\bigcup_{k<j} M_k$. Then, $M$ is a free $\mathfrak{U}(\mathfrak{n}^-)$-module with basis $\{m_j \mid j \in J\}$. ■

Corollary 1.9 Suppose that $M \in \nabla(\hat{O})$.

(a) If $\lambda$ is a maximal weight of $M$, then $M$ has a submodule $N$ such that $M/N$ is isomorphic to $\nabla(\lambda)$, and the submodule $N$ is in $\nabla(\hat{O})$.

(b) If $N$ is a direct summand of $M$, then $N \in \nabla(\hat{O})$.

(c) The module $M$ is a free $\mathfrak{U}(\mathfrak{n}^-)$-module.

Definition 1.10 Let $\mathcal{D}^\mathfrak{g}_\mathfrak{o}$ (or simply, $\mathcal{D}$) denote the subcategory $\Delta(\hat{O}^g) \cap \nabla(\hat{O}^g)$. The objects in $\mathcal{D}$ are called tilting modules.

1.3 Integrable Modules

Definition 1.11 Let $\mathfrak{a}$ be an arbitrary Lie algebra. An $\mathfrak{a}$-module $M$ is said to be integrable (or $\mathfrak{a}$-integrable) if, for any $m \in M$ and $a \in \mathfrak{a}$, the elements $m, a \cdot m, a^2 \cdot m, \ldots$ span a finite-dimensional subspace of $M$. ♦
Definition 1.12 Let $\lambda \in \mathfrak{h}^*$. 

(a) We say that $\lambda \in \mathfrak{h}^*$ is integral (with respect to $\mathfrak{g}$ and $\mathfrak{h}$) if $h_\alpha(\lambda)$ is an integer for all $\alpha \in \Phi$.

(b) If $h_\alpha(\lambda) \in \mathbb{Z}_{\geq 0}$ for every $\alpha \in \Phi^+$, then $\lambda$ is said to be dominant-integral (with respect to $\mathfrak{g}$ and $\mathfrak{b}$).

(c) If $h_\alpha(\lambda) \notin \mathbb{Z}$ for all $\alpha \in \Phi$, then $\lambda$ is said to be nonintegral (with respect to $\mathfrak{g}$ and $\mathfrak{h}$).

(d) If $h_\alpha(\lambda) \notin \mathbb{Z}$ for all but finitely many $\alpha \in \Phi$, then $\lambda$ is said to be almost nonintegral (with respect to $\mathfrak{g}$ and $\mathfrak{h}$).

Theorem 1.13 A module $M \in \bar{O}$ is integrable if and only if it is a direct sum of simple integrable modules in $\bar{O}$. All simple integrable modules in $\bar{O}$ are of the form $\mathcal{L}(\lambda)$, where $\lambda \in \mathfrak{h}^*$ is dominant-integral. Nonisomorphic simple integrable modules belong in different blocks of $\bar{O}$.

Proof For the first statement, we may assume that $M$ is an indecomposable module (by means of [9, Corollary 2.6]). We shall prove that $M$ is simple. Let $u$ be a highest-weight vector of $M$ associated to the weight $\lambda$. We claim that $\lambda$ is dominant-integral and $M \cong \mathcal{L}(\lambda)$.

First, by considering $M$ as a $\mathfrak{g}_n$-module, we easily see that $M$ is a direct sum of simple finite-dimensional $\mathfrak{g}_n$-modules. In particular, the $\mathfrak{g}_n$-submodule $M_n$ generated by $u$ is a simple direct summand of $M$. Note that $M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$. If $M'$ is the union of $\bigcup_{n \in \mathbb{Z}_{>0}} M_n$ where each $M_n$ is a simple $\mathfrak{g}_n$-module, then $M'$ is a simple $\mathfrak{g}$-module isomorphic to $\mathcal{L}(\lambda)$. Clearly, $\lambda$ must be a dominant-integral weight with respect to the Lie algebra $\mathfrak{g}_n$ with the Borel subalgebra $\mathfrak{b}_n$. Therefore, $\lambda$ is dominant-integral with respect to $\mathfrak{g}$ and $\mathfrak{b}$.

If $M \neq M'$, then $M/M'$ has a highest-weight vector of the form $u' + M'$, where $v \in M$. Using the same argument, the submodule $M''$ of $M/M'$ generated by $u' + M'$ is isomorphic to $\mathcal{L}(\mu)$ for some dominant-integral weight $\mu \in \mathfrak{h}^*$ with respect to $\mathfrak{g}$ and $\mathfrak{b}$. Since $M$ is indecomposable, by [9, Proposition 3.2], we conclude that $\mu \in [\lambda]$. However, the only dominant-integral weight in $[\lambda]$ is $\lambda$ itself. Consequently, $\mu = \lambda$. This shows that the only possible composition factor of $M$ is $\mathcal{L}(\lambda)$. However, there are no nontrivial extensions of $\mathcal{L}(\lambda)$ by itself (see [9, Proposition 3.8(d)]).

Finally, we shall prove that every simple module of the form $\mathcal{L}(\lambda)$ is integrable if $\lambda \in \mathfrak{h}^*$ is dominant-integral. Let $v$ be a highest-weight vector of $L := \mathcal{L}(\lambda)$. Define $L_n := \mathcal{U}(\mathfrak{g}_n) \cdot v$ for every $n \in \mathbb{Z}_{>0}$. Clearly, each $L_n \cong \mathcal{L}_n^{\mathfrak{g}_n}(\lambda)$ is finite-dimensional with $L_1 \subseteq L_2 \subseteq L_3 \subseteq \ldots$, and $L = \bigcup_{n \in \mathbb{Z}_{>0}} L_n$.

Fix $k \in \mathbb{Z}_{>0}$. Now, for $n \geq k$, observe that each $L_n$ is a direct sum of simple finite-dimensional $\mathfrak{g}_k$-modules. Consequently, for each $n \geq k$, $L_{n+1} = L_n \oplus F^k_n$ with $F^k_n$ being a direct sum of simple finite-dimensional $\mathfrak{g}_k$-modules. That is, $L = L_k \oplus \bigoplus_{n \geq k} F^k_n$ is a direct sum of simple finite-dimensional $\mathfrak{g}_k$-modules. It follows immediately that $L$ is an integrable $\mathfrak{g}$-module.

Corollary 1.14 Let $\bar{O}_{\text{integrable}}$ denote the full subcategory of $\bar{O}$ consisting of integrable modules. Then, $\bar{O}_{\text{integrable}}$ is semisimple.

The theorem below gives another way to verify whether a module $M \in \bar{O}$ is $\mathfrak{g}$-integrable. Recall that $\Pi(M)$ is the set of $\mathfrak{h}$-weights of $M$.

Theorem 1.15 Let $M \in \bar{O}$. The following consitions on $M$ are equivalent:

(i) $M$ is $\mathfrak{g}$-integrable;

(ii) $M$ is $\mathfrak{n}^-$-integrable;

(iii) for all $w \in W$ and $\lambda \in \mathfrak{h}^*$, $\dim(M^w) = \dim(M^w)$. 


The set $\Pi(M)$ is stable under the natural action of $W$. 

**Proof** The direction (i)$\implies$(ii) is obvious. For (ii)$\implies$(iii), we note that $w \in W_n$ for any sufficiently large positive integer $n$. Let $M_n^{(\lambda)}$ be the $\mathfrak{g}_n$-submodule of $M$ given by

$$M_n^{(\lambda)} := \mathfrak{u}(\mathfrak{g}_n) \cdot M^{\lambda}.$$  

Because $M$ is $\mathfrak{n}^-$-integrable, $M^{\lambda}$ is finite-dimensional, $\mathfrak{u}(\mathfrak{g}_n) = \mathfrak{u}(\mathfrak{n}^-) \cdot \mathfrak{u}(\mathfrak{h}) \cdot \mathfrak{u}(\mathfrak{n}^+)$, and $M$ is locally $\mathfrak{u}(\mathfrak{n}^+)$-finite, we see that $M$ is locally $\mathfrak{u}(\mathfrak{n}^-)$-finite, whence $M_n^{(\lambda)}$ is a finite-dimensional $\mathfrak{g}_n$-submodule of $M$. For all sufficiently large $n$, we have $(M_n^{(\lambda)})^{\pi^\lambda} = M^{\pi^\lambda}$. Since the support of a finite-dimensional module is invariant under the action of the Weyl group, the claim follows.

The statement (iii)$\implies$(iv) is trivial. We now prove (iv)$\implies$(i). Let $M_n^{(\lambda)}$ be the module \((1.3)\). Because $M$ is locally $\mathfrak{u}(\mathfrak{n}^+)$-finite, $\Pi(M)$ is invariant under $W_{\mathfrak{g}_n, \mathfrak{h}_n}$, and $M$ has finite-dimensional $\mathfrak{h}$-weight spaces, we conclude that the weights $M_n^{(\lambda)}$ must lie in the orbit of $\lambda$ under $W_{\mathfrak{g}_n, \mathfrak{h}_n}$, making $M_n^{(\lambda)}$ a finite-dimensional $\mathfrak{g}_n$-module. Hence, $M = \bigcup_{n \in \mathbb{Z}_{>0}} \bigcup_{\lambda \in \Pi(M)} M_n^{(\lambda)}$ is an integrable \(\mathfrak{g}\)-module.\]

\section{Translation Functors}

\subsection{Some Functors between Extended Categories $\mathcal{O}$}

Fix a positive integer $n$. Let $M_n \in \widehat{\mathcal{O}}_{\mathfrak{b}_n}$. Define $\mathfrak{p}_{n+1}$ to be the parabolic subalgebra $\mathfrak{g}_n + \mathfrak{b}_{n+1}$ of $\mathfrak{g}_{n+1}$. It can be easily seen that $M_n$ is a $\mathfrak{p}_{n+1}$-module, where $\mathfrak{b}_{n+1}$ acts on $M_n$ via $x_m \cdot m = 0$ for all $m \in M_n$ and $\alpha \in \Phi_{\mathfrak{g}_{n+1, \mathfrak{b}_{n+1}}} \setminus \Phi_{\mathfrak{g}_n, \mathfrak{b}_n}$.

**Definition 2.1** Define $I_{n+1} : \widehat{\mathcal{O}}_{\mathfrak{b}_n} \rightarrow \widehat{\mathcal{O}}_{\mathfrak{b}_{n+1}}$ to be the parabolic induction functor

$$I_{n+1} M_n := \mathfrak{u}(\mathfrak{g}_{n+1}) \otimes_{\mathfrak{u}(\mathfrak{p}_{n+1})} M_n$$  \hspace{1cm} (2.1)

for all $M_n \in \widehat{\mathcal{O}}_{\mathfrak{b}_n}$.

**Proposition 2.2** The functor $I_{n+1} : \widehat{\mathcal{O}}_{\mathfrak{b}_n} \rightarrow \widehat{\mathcal{O}}_{\mathfrak{b}_{n+1}}$ is exact.

**Proof** The $\mathfrak{u}(\mathfrak{p}_{n+1})$-module $\mathfrak{u}(\mathfrak{g}_{n+1})$ is a free module due to the Poincaré-Birkhoff-Witt (PBW) Theorem. Thus, $\mathfrak{u}(\mathfrak{g}_{n+1})$ is a flat $\mathfrak{u}(\mathfrak{p}_{n+1})$-module.

**Remark 2.3** Observe that the functor $I_{n+1} : \widehat{\mathcal{O}}_{\mathfrak{b}_n} \rightarrow \widehat{\mathcal{O}}_{\mathfrak{b}_{n+1}}$ is right-adjoint to the forgetful functor $F_n : \widehat{\mathcal{O}}_{\mathfrak{b}_{n+1}} \rightarrow \widehat{\mathcal{O}}_{\mathfrak{b}_n}$, and left-adjoint to $G_n := \text{Hom}_{\mathfrak{u}(\mathfrak{p}_{n+1})}(\mathfrak{u}(\mathfrak{g}_{n+1}, \_)) : \widehat{\mathcal{O}}_{\mathfrak{b}_{n+1}} \rightarrow \widehat{\mathcal{O}}_{\mathfrak{b}_n}$. That is,

$$\text{Hom}_{\widehat{\mathcal{O}}_{\mathfrak{b}_{n+1}}}(M_{n+1}, I_{n+1} M_n) \cong \text{Hom}_{\widehat{\mathcal{O}}_{\mathfrak{b}_n}}(F_n M_{n+1}, M_n)$$  \hspace{1cm} (2.2)

and

$$\text{Hom}_{\widehat{\mathcal{O}}_{\mathfrak{b}_{n+1}}}(I_{n+1} M_n, M_{n+1}) \cong \text{Hom}_{\widehat{\mathcal{O}}_{\mathfrak{b}_n}}(M_n, G_n M_{n+1}),$$  \hspace{1cm} (2.3)

for all $M_n \in \widehat{\mathcal{O}}_{\mathfrak{b}_n}$ and $M_{n+1} \in \widehat{\mathcal{O}}_{\mathfrak{b}_{n+1}}$.

**Definition 2.4** Fix $\lambda \in \mathfrak{h}^*$. Define $\mathcal{R}_{n+1} : \widehat{\mathcal{O}}_{\mathfrak{b}_{n+1}} \rightarrow \widehat{\mathcal{O}}_{\mathfrak{b}_{n+1}}[\lambda]$ to be the truncation functor, where for each $M_{n+1} \in \widehat{\mathcal{O}}_{\mathfrak{b}_{n+1}}$, $\mathcal{R}_{n+1} M_{n+1}$ is the sum of all submodules $N_{n+1}$ of $M_{n+1}$ such that all composition factors of $N_{n+1}$ are not of the form $\mathfrak{g}_{\mathfrak{b}_{n+1}}^{(\mu)}$, with $\mu \in W_{\mathfrak{g}_n, \mathfrak{h}} \cdot \lambda$.

We have the following proposition. The proof is trivial.
Proposition 2.5 Fix $\lambda \in h^*$ and $n \in \mathbb{Z}_{>0}$.

(a) The functor $R^\lambda_{n+1} : \mathcal{O}_b^{n+1} \to \mathcal{O}_b^{n+1}[\lambda]$ is left-exact.

(b) The functor $R^\lambda_{n+1}I_{n+1} \inj_{b_n,b_n} : \mathcal{O}_b^n[\lambda] \to \mathcal{O}_b^{n+1}[\lambda]$ is left-exact.

(c) The functor $R^\lambda_{n+1}I_{n+1} : \mathcal{O}_b^n \to \mathcal{O}_b^{n+1}[\lambda]$ is left-exact.

Definition 2.6 Let $Q^\lambda_{n+1} : \mathcal{O}_b^n[\lambda] \to \mathcal{O}_b^{n+1}[\lambda]$ be given by

$$Q^\lambda_{n+1}M_n := I_{n+1} \inj_{b_n,b_n} M_n/R^\lambda_{n+1}I_{n+1} \inj_{b_n,b_n} M_n$$

for all $M_n \in \mathcal{O}_b^n[\lambda]$.

Theorem 2.7 The functor $Q^\lambda_{n+1} : \mathcal{O}_b^n[\lambda] \to \mathcal{O}_b^{n+1}[\lambda]$ is an exact functor.

Proof Fix an exact sequence $0 \to N_n \to M_n \to K_n \to 0$ of objects in $\mathcal{O}_b^n[\lambda]$. Because $W_{b_n, b} \cdot \lambda$ is finite, the length $k$ of the $g_n$-module $K_n$ is finite. We shall prove by induction on $k$.

For $k = 0$, there is nothing to prove. For $k = 1$, we see that $K_n$ is a simple $g_n$-module. Therefore, $K_n \cong \mathcal{O}_b^n(\mu)$ for some $\mu \in W_{b_n, b} \cdot \lambda$. It can be easily seen that $Q^\lambda_{n+1} \mathcal{O}_b^n = \mathcal{O}_b^{n+1}(\mu)$. Consider two exact sequences of $g_{n+1}$-modules:

$$0 \to I_{n+1}N_n \to I_{n+1}M_n \to I_{n+1}K_n \to 0$$

and

$$0 \to R^\lambda_{n+1}I_{n+1} \inj_{b_n,b_n} N_n \to R^\lambda_{n+1}I_{n+1} \inj_{b_n,b_n} M_n \to R^\lambda_{n+1}I_{n+1} \inj_{b_n,b_n} K_n.$$  

By definition of $Q^\lambda_{n+1}$, we obtain the following exact sequence

$$0 \to Q^\lambda_{n+1}N_n \to Q^\lambda_{n+1}M_n \to Q^\lambda_{n+1}K_n.$$  

Because $Q^\lambda_{n+1}K_n$ is simple, either $Q^\lambda_{n+1}N_n \cong Q^\lambda_{n+1}M_n$ or the sequence

$$0 \to Q^\lambda_{n+1}N_n \to Q^\lambda_{n+1}M_n \to Q^\lambda_{n+1}K_n \to 0$$

must be exact.

Write $L_n := \mathcal{O}_b^n(\mu)$ and $L_{n+1} := \mathcal{O}_b^{n+1}(\mu)$. Let $d$ denote the largest possible nonnegative integer such that there exists a quotient of $N_n$ isomorphic to $L_n^{\oplus d}$. Then, $d + 1$ is the largest possible nonnegative integer such that there exists a quotient of $M_n$ isomorphic to $L_n^{\oplus(d+1)}$. Then, there exists a $g_{n}$-submodule $Y_n$ of $N_n$ such that $N_n/Y_n \cong L_n^{\oplus t}$. We claim that $Q^\lambda_{n+1}N_n/Q^\lambda_{n+1}Y_n \cong L_n^{\oplus t}$.

As before, we have an exact sequence $0 \to Q^\lambda_{n+1}N_n \to Q^\lambda_{n+1}M_n \to Q^\lambda_{n+1}N_n \to L_n^{\oplus t}$, which implies that $Q^\lambda_{n+1}N_n/Q^\lambda_{n+1}Y_n \cong L_n^{\oplus t}$ for some integer $t$ such that $0 \leq t \leq d$. If $t < d$, then $Q^\lambda_{n+1}Y_n$ has length $d + 1$ as a quotient. Hence, $I_{n+1}Y_n$ has length $d + 1$. This contradicts the definition of $d$.

By a similar argument, if $X_n$ is a $g_{n}$-submodule of $M_n$ such that $M_n/X_n \cong L_n^{\oplus(d+1)}$, then $Q^\lambda_{n+1}M_n/Q^\lambda_{n+1}X_n \cong L_n^{\oplus(d+1)}$. Thus, $Q^\lambda_{n+1}N_n \cong Q^\lambda_{n+1}M_n$ cannot hold. Therefore, we must have an exact sequence (2.8).

Suppose now that $k > 1$. Consider two exact sequences of $g_{n}$-modules: $0 \to N_n \to M_n \to K_n \to 0$ and $0 \to Z_n \to K_n \to L_n \to 0$ for some simple object $L_n \in \mathcal{O}_b^n[\lambda]$ and for some $g_{n}$-submodule $X_n$ of $K_n$. Ergo, we can find exact sequences $0 \to U_n \to M_n \to L_n \to 0$, $0 \to N_n \to U_n \to Z_n \to 0$, where $U_n$ is a $g_{n}$-submodule of $M_n$. By induction hypothesis,

$$0 \to Q^\lambda_{n+1}Z_n \to Q^\lambda_{n+1}K_n \to Q^\lambda_{n+1}L_n \to 0.$$  


0 \to Q_{n+1}^\lambda U_n \to Q_{n+1}^\lambda M_n \to Q_{n+1}^\lambda L_n \to 0, \quad (2.10)
and

0 \to Q_{n+1}^\lambda N_n \to Q_{n+1}^\lambda U_n \to Q_{n+1}^\lambda Z_n \to 0 \quad (2.11)
are exact sequences. Therefore,

\begin{align*}
\text{ch} \left( Q_{n+1}^\lambda M_n \right) &= \text{ch} \left( Q_{n+1}^\lambda U_n \right) + \text{ch} \left( Q_{n+1}^\lambda L_n \right) \\
&= \left( \text{ch} \left( Q_{n+1}^\lambda N_n \right) + \text{ch} \left( Q_{n+1}^\lambda Z_n \right) \right) + \text{ch} \left( Q_{n+1}^\lambda L_n \right) \\
&= \text{ch} \left( Q_{n+1}^\lambda N_n \right) + \left( \text{ch} \left( Q_{n+1}^\lambda Z_n \right) + \text{ch} \left( Q_{n+1}^\lambda L_n \right) \right) \\
&= \text{ch} \left( Q_{n+1}^\lambda N_n \right) + \text{ch} \left( Q_{n+1}^\lambda K_n \right). \quad (2.12)
\end{align*}

By (2.7), we conclude that $0 \to Q_{n+1}^\lambda N_n \to Q_{n+1}^\lambda M_n \to Q_{n+1}^\lambda K_n \to 0$ must be an exact sequence. The proof is now complete. $\blacksquare$

**Corollary 2.8** The functor $Q_{n+1}^\lambda$ is an equivalence between $\mathcal{O}_{b_n}^{b_n}[\lambda]$ and the image $Q_{n+1}^\lambda \mathcal{O}_{b_n}^{b_n}[\lambda]$. More specifically, for any $\mu \in [\lambda]$ and $M_n \in \mathcal{O}_{b_n}^{b_n}[\lambda]$, we have

$$[M_n : \mathfrak{g}_{b_n}(\mu)] = [Q_{n+1}^\lambda M_n : \mathfrak{g}_{b_{n+1}}(\mu)]. \quad (2.13)$$

Furthermore, $Q_{n+1}^\lambda$ preserves the length of every object.

**Proposition 2.9** For every $M_n \in \mathcal{O}_{b_n}^{b_n}[\lambda]$, there exists an injective $\mathfrak{g}_n$-module homomorphism $\iota_{M_n} : M_n \to Q_{n+1}^\lambda M_n$ such that, for all objects $M_n, N_n \in \mathcal{O}_{b_n}^{b_n}[\lambda]$ along with a $\mathfrak{g}_n$-module homomorphism $f_n : M_n \to N_n$, the following diagram is commutative:

$$
\begin{array}{ccc}
M_n & \xrightarrow{f_n} & N_n \\
\downarrow \iota_{M_n} & & \downarrow \iota_{N_n} \\
Q_{n+1}^\lambda M_n & \xrightarrow{Q_{n+1}^\lambda f_n} & Q_{n+1}^\lambda N_n.
\end{array}
\quad (2.14)
$$

**Proof** For each $v \in M_n$, we define $\iota_{M_n}(v) := \left( 1_{\mathfrak{u}(g_{n+1})} \otimes v \right) + R_{n+1}^\lambda L_{n+1} \text{inj}_{\mathfrak{g}_{n+1}}^\lambda M_n \in Q_{n+1}^\lambda M_n$.

It is easy to see that $\iota_{M_n}$ satisfies the requirement. $\blacksquare$

**Proposition 2.10** Let $M_n \in \mathcal{O}_{b_n}^{b_n}[\lambda]$ and $N_{n+1} \in \mathcal{O}_{b_{n+1}}^{b_{n+1}}[\lambda]$. Suppose that all composition factors of $N_{n+1}$ take the form $\mathfrak{g}_{b_{n+1}}^{b_{n+1}}(\mu)$ with $\mu \in W_{n+1} \cdot \lambda$. If $f : M_n \to N_{n+1}$ is a $\mathfrak{g}_n$-module homomorphism, then there exists a unique $\mathfrak{g}_{n+1}$-module homomorphism $\tilde{f} : Q_{n+1}^\lambda M_n \to N_{n+1}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
M_n & \xrightarrow{f} & N_{n+1} \\
\downarrow \iota_{M_n} & & \downarrow \tilde{f} \\
Q_{n+1}^\lambda M_n.
\end{array}
\quad (2.15)
$$
Proof For each $u \in \mathcal{U}(g_{n+1})$ and $v \in M_n$, let the map $\tilde{f}$ send \((u \otimes v) + K_{n+1} \in Q_{n+1}^\lambda M_n, \)

\[ K_{n+1} := R_{n+1}^\lambda \lambda_n(\cdot)_{b_n,b_n} M_n, \text{ to } u \cdot f(v) \in N_{n+1}. \] Then, extend $\tilde{f}$ by linearity.

We claim that $\tilde{f}$ is a well defined homomorphism of $g_{n+1}$-modules. Suppose that $u_1, u_2, \ldots, u_k$ are elements of $\mathcal{U}(g_{n+1})$ and $v_1, v_2, \ldots, v_k$ are vectors in $M_n$ such that \[ \sum_{j=1}^k (u_j \otimes v_j) \in K_{n+1}. \]

We want to prove that \[ \sum_{j=1}^k u_j \cdot f(v_j) = 0. \] Write $z := \sum_{j=1}^k u_j \cdot f(v_j)$.

The $g_{n+1}$-submodule $Z_{n+1}$ of $N_{n+1}$ generated by $z$ cannot have a composition factor of the form $g_{b_n+1}(\xi)$ with $\xi \in W_{b_n+1} \cdot \lambda$. However, since all composition factors $N_{n+1}$ are of the form $\Phi_{b_n+1}(\mu)$ with $\mu \in W_{b_n+1} \cdot \lambda$, we conclude that $Z_{n+1} = 0$. Thus, $z = 0$.

Proposition 2.11 Let $M_n, N_n \in \mathcal{O}_{b_n}^\lambda [\lambda]$ and $f_{n+1} : Q_{n+1}^\lambda M_n \rightarrow Q_{n+1}^\lambda N_n$ be given. Then, there exists a $g_n$-module homomorphism $\tilde{Q}_n^\lambda f_{n+1} : M_n \rightarrow N_n$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
M_n & \xrightarrow{\tilde{Q}_n^\lambda f_{n+1}} & N_n \\
\downarrow{\iota_{M_n}} & & \downarrow{\iota_{N_n}} \\
Q_{n+1}^\lambda M_n & \xrightarrow{f_{n+1}} & Q_{n+1}^\lambda N_n.
\end{array}
\]

(2.16)

For $M_{n+1} \in \text{im} (Q_{n+1}^\lambda)$, suppose that $M_{n+1} = Q_{n+1}^\lambda M_n$ for some $M_n \in \mathcal{O}_{b_n}^\lambda [\lambda]$. Define the $g_n$-module $\tilde{Q}_n^\lambda M_{n+1} \in \mathcal{O}_{b_n}^\lambda [\lambda]$ to be $M_n$ itself. Then, $\tilde{Q}_n^\lambda : \text{im} (Q_{n+1}^\lambda) \leadsto \mathcal{O}_{b_n}^\lambda [\lambda]$ is the inverse equivalence of $Q_{n+1}^\lambda : \mathcal{O}_{b_n}^\lambda [\lambda] \leadsto \text{im} (Q_{n+1}^\lambda)$.

Proof Let $K_{n+1} := R_{n+1}^\lambda \lambda_n(\cdot)_{b_n,b_n} N_n$. For each $v \in M_n$, suppose that $u_1, u_2, \ldots, u_k$ are elements of $U(g_{n+1})$ and $v_1, v_2, \ldots, v_k$ are vectors in $N_n$ such that $f_{n+1}(\iota_{M_n}(v)) = \sum_{j=1}^k (u_j \otimes v_j) + K_{n+1}$. Denote by $u_j'$ the projection of $u_j$ onto $U(g_n)$ (in the PBW basis of $U(g_n)$). Set $f_n(v) := \sum_{j=1}^k u_j' \cdot v_j$. Then, $\tilde{Q}_n^\lambda f_{n+1} := f_n$ satisfies the required condition.

Corollary 2.12 Let $M_n, N_n \in \mathcal{O}_{b_n}^\lambda [\lambda]$. Then,

\[ \text{Hom}_{\mathcal{O}_{b_n}^\lambda [\lambda]} (M_n, N_n) \cong \text{Hom}_{\mathcal{O}_{b_{n+1}}^\lambda [\lambda]} (Q_{n+1}^\lambda M_n, Q_{n+1}^\lambda N_n). \]

(2.17)

Therefore, the image $Q_{n+1}^\lambda \mathcal{O}_{b_n}^\lambda [\lambda]$ is the full subcategory of $\mathcal{O}_{b_{n+1}}^\lambda$ whose objects have composition factors of the form $\Phi_{b_{n+1}}(\mu)$ with $\mu \in W_{b_{n+1}} \cdot \lambda$.

For each $n \in \mathbb{Z}_{\geq 0}$, fix a set $\mathcal{I}_n$ of representatives of $[\lambda] \in \Omega_{b_n}^\oplus$. We can further assume that $\mathcal{I}_n \supseteq \mathcal{I}_{n+1}$ for every positive integer $n$. Then, $\mathcal{I} := \bigcap_{n \in \mathbb{Z}_{\geq 0}} \mathcal{I}_n$ is a set of representatives of $[\lambda] \in \Omega_{b}^\oplus$.

Proposition 2.13 For each $\lambda \in \mathcal{I}$, the direct limit of \( \left( Q_{n+1}^\lambda : \mathcal{O}_{b_n}^\lambda [\lambda] \leadsto \mathcal{O}_{b_{n+1}}^\lambda [\lambda] \right)_{n \in \mathbb{Z}_{\geq 0}} \) is the full subcategory $\mathcal{O}_{b}^\oplus[\lambda]$ of $\mathcal{O}_{b_n}^\lambda[\lambda]$ consisting of $g$-modules of finite length.

Proof For convenience, write $\mathcal{O}[\lambda]$ for $\mathcal{O}_{b}^\oplus[\lambda]$. Fix $n \in \mathbb{Z}_{\geq 0}$. First, we define $q_{n+1}^\lambda : \mathcal{O}_{b_n}^\lambda [\lambda] \leadsto \mathcal{O}[\lambda]$ as follows. For a given $M_n \in \mathcal{O}_{b_n}^\lambda [\lambda]$ and $k \in \mathbb{Z}_{\geq 0}$, write $M_{n+k}$ for $Q_{n+1}^\lambda Q_{n+1+k}^\lambda \cdots Q_{n+k}^\lambda M_n \in \mathcal{O}_{b_{n+k}}^\lambda [\lambda]$. Using Proposition 2.9 above and noting that $\mathcal{O}_{b_n}^\lambda [\lambda] = \mathcal{O}_{b_n}^\lambda [\lambda]$ (whose objects are finitely generated),
we see that \((t_{M_{n+k}} : M_{n+k} \to M_{n+k+1})_{k \in \mathbb{Z}_{\geq 0}}\) is a directed system whose direct limit \(M\) is clearly in \(\mathcal{O}[\lambda]\). We set \(q_{n}^{\lambda}M_{n}\) to be the direct limit \(M\). It is easy to see that \(q_{n+1}^{\lambda}Q_{n+1}^{\lambda} = q_{n}^{\lambda}\).

Let \(p_{n}^{\lambda} : \mathcal{O}[\lambda] \twoheadrightarrow \mathcal{O}_{b_{n}}^{\alpha}[\lambda]\) be the functor defined as follows: \(p_{n}^{\lambda}M := \bigoplus_{\xi \in W_{b_{n}}^{\alpha}} u(g_{\xi}) \cdot M^{\xi}\) for every \(M \in \mathcal{O}[\lambda]\). We can easily show that \(p_{n}^{\lambda}q_{n}^{\lambda} = \text{Id}_{\mathcal{O}_{b_{n}}^{\alpha}[\lambda]}\) and \(p_{n+k}^{\lambda}q_{n}^{\lambda} = Q_{n+k}^{\lambda} \mathcal{O}_{b_{n+k-1}}^{\alpha}[\lambda] \cdots Q_{n+1}^{\lambda}\).

Suppose that there exists a category \(\tilde{\mathcal{O}}[\lambda]\) along with functors \(\tilde{q}_{n}^{\lambda} : \mathcal{O}_{b_{n}}^{\alpha}[\lambda] \twoheadrightarrow \tilde{\mathcal{O}}[\lambda]\) such that \(\tilde{q}_{n+1}^{\lambda}Q_{n+1}^{\lambda} = \tilde{q}_{n}^{\lambda}\) for all \(n = 1, 2, 3, \ldots\). Define \(t_{n}^{\lambda} : \mathcal{O}[\lambda] \twoheadrightarrow \tilde{\mathcal{O}}[\lambda]\) via \(t_{n}^{\lambda}M = \tilde{q}_{n}^{\lambda}p_{n}^{\lambda}M\) for all \(M \in \mathcal{O}[\lambda]\). Since \(M\) is a \(g\)-module of finite length, \(t_{n}^{\lambda}M = t_{n+1}^{\lambda}M = \cdots \) for some positive integer \(n_{0}(M)\). Let \(t^{\lambda} : \mathcal{O}[\lambda] \twoheadrightarrow \tilde{\mathcal{O}}[\lambda]\) be given by \(t^{\lambda}M = t_{n_{0}(M)}^{\lambda}M\) for every \(M \in \mathcal{O}[\lambda]\). We can easily see that \(t^{\lambda}q_{n}^{\lambda} = \tilde{q}_{n}^{\lambda}\) for every positive integer \(n\). Note that the functor \(t^{\lambda} : \mathcal{O}[\lambda] \twoheadrightarrow \tilde{\mathcal{O}}[\lambda]\) above is unique with the property that \(t^{\lambda}q_{n}^{\lambda} = \tilde{q}_{n}^{\lambda}\) for every positive integer \(n\). Therefore, \(\mathcal{O}[\lambda]\) is the required direct limit.

**Corollary 2.14** Let \(q_{n}^{\lambda} : \mathcal{O}_{b_{n}}^{\alpha}[\lambda] \twoheadrightarrow \mathcal{O}_{b_{n}}^{\alpha}[\lambda]\) be as given in the proof of the previous proposition. Then, \(q_{n}^{\lambda}\) is an exact functor.

**Corollary 2.15** Let \(Q_{n+1}^{\lambda} : \mathcal{O}_{b_{n}}^{\alpha} \twoheadrightarrow \mathcal{O}_{b_{n+1}}^{\alpha}\) be the functor defined by

\[
Q_{n+1}^{\lambda}M_{n} = \bigoplus_{\lambda \in \mathcal{X}} \text{inj}_{\lambda}^{\alpha}q_{n+1}^{\lambda}Q_{n+1}^{\lambda} \text{pr}_{\lambda}^{\alpha}M_{n}
\]

for all \(M_{n} \in \mathcal{O}_{b_{n}}^{\alpha}\). Then, the direct limit of \(\left(Q_{n+1}^{\lambda} : \mathcal{O}_{b_{n}}^{\alpha} \twoheadrightarrow \mathcal{O}_{b_{n+1}}^{\alpha}\right)_{n \in \mathbb{Z}_{\geq 0}}\) is the full subcategory \(\mathcal{O}_{b}^{\alpha}\) (or simply, \(\mathcal{O}\)) of \(\mathcal{O}_{b}^{\alpha}\) along with a family of exact functors \(\left(q_{n}^{\lambda} : \mathcal{O}_{b_{n}}^{\alpha} \twoheadrightarrow \mathcal{O}_{b}^{\alpha}\right)_{n \in \mathbb{Z}_{\geq 0}}\) where \(\mathcal{O}_{b}^{\alpha}\) is given by \(\mathcal{O}_{b}^{\alpha} = \bigoplus_{\lambda \in \mathcal{X}} \mathcal{O}_{b}^{\alpha}[\lambda]\).

### 2.2 Some Category Equivalences

Take \(\lambda \in \mathfrak{h}^{*}\). We define the following notations:

- \(\Phi_{\mathfrak{g}, \mathfrak{h}}[\lambda] := \left\{ \alpha \in \mathfrak{h} \mid \langle h_{\alpha} \rangle \in \mathbb{Z} \right\}\) (also denoted by \(\Phi[\lambda]\)),
- \(\Phi_{\pm}[\lambda] := \Phi_{\mathfrak{g}, \mathfrak{h}}[\lambda] \cap \Phi^{\pm}\) (also denoted by \(\Phi^{\pm}[\lambda]\)),
- \(\Sigma_{\mathfrak{g}, \mathfrak{h}}[\lambda]\) or \(\Sigma_{\mathfrak{g}, \mathfrak{h}}^{\pm}[\lambda]\) is the set of simple roots with respect to the set of positive roots \(\Phi_{\mathfrak{g}, \mathfrak{h}}^{\pm}[\lambda]\) of the root system \(\Phi_{\mathfrak{g}, \mathfrak{h}}[\lambda]\) (also denoted by \(\Sigma[\lambda]\) or \(\Sigma^{\pm}[\lambda]\)),
- \(\Sigma_{\mathfrak{g}, \mathfrak{h}}^{-}[\lambda] := -\Sigma_{\mathfrak{g}, \mathfrak{h}}^{+}[\lambda]\) (also denoted by \(\Sigma^{-}[\lambda]\)),
- \(\Lambda_{\mathfrak{g}, \mathfrak{h}}[\lambda] := \text{span}_{\mathbb{Z}} \Phi_{\mathfrak{g}, \mathfrak{h}}[\lambda]\) (also denoted by \(\Lambda[\lambda]\)),
- \(\mathfrak{h}[\lambda] := \text{span}_{\mathbb{R}} \left\{ h \in \mathfrak{h} \mid \langle h, h \rangle \in \mathbb{Z} \right\}\),
- \(\mathfrak{g}[\lambda] := \mathfrak{h}[\lambda] \oplus \bigoplus_{\alpha \in \Phi[\lambda]} \mathfrak{g}^{\alpha}\),
- \(\mathfrak{b}[\lambda] := \mathfrak{b}[\lambda] \oplus \bigoplus_{\alpha \in \Phi^{\pm}[\lambda]} \mathfrak{g}^{\alpha}\) (also denoted by \(\mathfrak{b}^{\pm}[\lambda]\)),
- \(\mathfrak{n}[\lambda] := \bigoplus_{\alpha \in \Phi^{\pm}[\lambda]} \mathfrak{g}^{\alpha}\) (also denoted by \(\mathfrak{n}^{\pm}[\lambda]\)),
- \(\mathfrak{n}^{-}[\lambda] := \bigoplus_{\alpha \in \Phi^{-}[\lambda]} \mathfrak{g}^{\alpha}\),
\[ \lambda^3 := \lambda|_{b[\lambda]} \in (b[\lambda])^*, \] and

\[ \overset{\phi}{W_{\mu,b}[\lambda]} \] (also denoted by \( W[\lambda] \)) is the subgroup of \( W_{\mu,b} \) consisting of elements \( w \) such that \( w \cdot \lambda = \lambda \).

**Proposition 2.16** Let \( n \) be a positive integer and \( \lambda \in b^* \). Then, there exists a categorical equivalence \( \mathcal{E}_n^\lambda : \mathcal{O}_{b,\mu}^n[\lambda] \rightarrow \mathcal{O}_{b,\mu}^n[\lambda]^2 \) which sends \( \mathcal{L}_{b,\mu}^n(\mu) \) to \( \mathcal{L}_{b,\mu}^n[\lambda](\mu^2) \) for all \( \mu \in [\lambda] \cap (W_{\mu,b} : \lambda) \).

**Proof** This proposition is a direct consequence of [10, Theorem 11].

**Corollary 2.17** For each \( \lambda \in b^* \), there exists a categorical equivalence \( \mathcal{E}_n^\lambda : \mathcal{O}_{b}^n[\lambda] \rightarrow \mathcal{O}_{b}^n[\lambda]^2 \) which sends \( \mathcal{L}_{b}^n(\mu) \) to \( \mathcal{L}_{b}^n[\lambda](\mu^2) \) for all \( \mu \in [\lambda] \).

**Proof** This corollary follows from the previous proposition, Corollary 2.8, Proposition 2.13, and Corollary 2.12.

However, as a result of [10, Theorem 11], and Corollary 2.17, we have the following theorem.

**Theorem 2.18** Let \( \mathfrak{g} \) and \( \mathfrak{g}' \) be root-reductive Lie algebras with Dynkin Borel subalgebras \( b \) and \( b' \), respectively. Suppose that \( S_{\mathfrak{g},b}[\lambda] \) is the set of simple reflections with respect to elements of \( \Sigma_{\mathfrak{g},b}[\lambda] \), and \( S_{\mathfrak{g}',b'}[\lambda'] \) is the set of simple reflections with respect to elements of \( \Sigma_{\mathfrak{g}',b'}[\lambda'] \). Suppose that there exists an isomorphism \( \varphi : W_{\mathfrak{g},b}[\lambda] \rightarrow W_{\mathfrak{g}',b'}[\lambda'] \) of Coxeter systems \( (W_{\mathfrak{g},b}[\lambda], S_{\mathfrak{g},b}[\lambda]) \) and \( (W_{\mathfrak{g}',b'}[\lambda], S_{\mathfrak{g}',b'}[\lambda]) \) such that

\[ \varphi(W_{\mathfrak{g},b}[\lambda]) = W_{\mathfrak{g}',b'}[\lambda']. \] (2.19)

Then, there exists an equivalence of categories \( \mathcal{O}_{b}^\lambda[\lambda] \cong \mathcal{O}_{b'}^\lambda[\lambda] \).

**Proof** By Corollary 2.17, we have \( \mathcal{O}_{b}^\lambda[\lambda] \cong \mathcal{O}_{b,b}^\lambda[\lambda]^2 \) and \( \mathcal{O}_{b'}^\lambda[\lambda] \cong \mathcal{O}_{b',b'}^\lambda[\lambda]^2 \). Therefore, it suffices to assume that \( \lambda \) and \( \lambda' \) are both integral weights; that is, \( \lambda = \lambda^2 \), \( \lambda' = (\lambda')^2 \), \( \mathfrak{g}[\lambda] = \mathfrak{g} \), \( b[\lambda] = b \), \( b'[\lambda'] = b' \), and \( b'[\lambda'] = b' \).

Let \( n \) be a positive integer. Define \( S_{\mathfrak{g},b,n} \) to be the set of simple reflections with respect to the elements of \( \Sigma_{\mathfrak{g},b,n} \). The notation \( S_{\mathfrak{g},b,n} \) is defined similarly. Set \( \Sigma_{\mathfrak{g},b,n} = \Sigma_{\mathfrak{g},b} \cup \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \), and for \( n > 1 \), let \( \Sigma_{\mathfrak{g},b,n} = \Sigma_{\mathfrak{g},b,n-1} \cup \{ \alpha_n \} \). Assume that \( \varphi \) sends the simple reflection with respect to \( \alpha_n \) to the simple reflection with respect to \( \alpha'_n \) for every positive integer \( n \).

Define \( g'_n \) to be the subalgebra of \( \mathfrak{g}' \) generated by \( b' \) and the root spaces corresponding to the roots \( \pm \alpha'_1, \pm \alpha'_2, \ldots, \pm \alpha'_n \). Take \( b'_n := g'_n \cap b' \). We note that the direct limits \( \mathfrak{g}' \) and \( b'' \) of the directed systems \( (g'_n)_{n \in \mathbb{Z}_{>0}} \) and \( (b''_n)_{n \in \mathbb{Z}_{>0}} \) are precisely \( g' \) and \( b' \), respectively. Hence, \( \mathcal{O}_{b'}^\lambda[\lambda] = \mathcal{O}_{b'}^\lambda[\lambda] \).

The existence of \( \varphi \) implies that, for each \( n \in \mathbb{Z}_{>0} \), the Coxeter systems \( (W_{\mathfrak{g},b,n}, S_{\mathfrak{g},b,n}) \) and \( (W_{\mathfrak{g}',b',n}, S_{\mathfrak{g}',b',n}) \) are isomorphic. Therefore, by [10, Theorem 11], there exists an equivalence of categories \( \varepsilon_n : \mathcal{O}_{b,n}^\lambda[\lambda] \rightarrow \mathcal{O}_{b',n}^\lambda[\lambda] \). Applying direct limit, we obtain an equivalence \( \varepsilon : \mathcal{O}_{b}^\lambda[\lambda] \rightarrow \mathcal{O}_{b'}^\lambda[\lambda] \).

**Open Question 2.19** Is the converse of Theorem 2.18 true? In other words, if \( \mathcal{O}_{b}^\lambda[\lambda] \cong \mathcal{O}_{b'}^\lambda[\lambda] \), then does there exist an isomorphism \( \varphi : W_{\mathfrak{g},b}[\lambda] \rightarrow W_{\mathfrak{g}',b'}[\lambda'] \) of Coxeter systems \( (W_{\mathfrak{g},b}[\lambda], S_{\mathfrak{g},b}[\lambda]) \) and \( (W_{\mathfrak{g}',b'}[\lambda'], S_{\mathfrak{g}',b'}[\lambda']) \) such that (2.19) is true?

### 2.3 Construction of Translation Functors

**Definition 2.20** For \( \lambda \in b^* \), we say that \( \lambda \) is restricted if \( \lambda(h_\alpha) = 0 \) for all but finitely many \( \alpha \in \Sigma_{\mathfrak{g},b}^+ \). For \( \lambda, \mu \in b^* \), we say that \( \lambda \) and \( \mu \) are compatible if \( \lambda - \mu \in \Lambda \) and \( \lambda - \mu \) is a restricted weight. The notation \( \lambda \parallel \mu \) means that \( \lambda \) and \( \mu \) are compatible.
Denote by \( h_Q \) the \( \mathbb{Q} \)-span of the coroots \( h_\alpha \in h \) with \( \alpha \in \Phi = \Phi_{g,h} \). Take \( h_R \) to be \( \mathbb{R} \otimes h_Q \). Let \( E := E^\theta_\mathbb{Q} \) denote the real vector space \( \text{Hom}_\mathbb{R}(h_R, \mathbb{R}) \). A root \( \alpha \in \Phi \) can be identified with the unique element (which we also denote by \( \alpha \)) of \( E \) which sends \( 1 \otimes \beta \in h_R \) to \( \alpha(\beta) \in \mathbb{Z} \) for all \( \beta \in \Phi \).

Similarly, if \( \lambda \in h^* \) satisfies \( \lambda(\beta) \in \mathbb{Q} \) for all \( \beta \in \Phi \), then we identify it with the unique element of \( E \) that sends \( \beta \mapsto \lambda(\beta) \) for all \( \beta \in \Phi \).

We decompose \( E \) into facets, where a facet \( F \) of \( E \) is a nonempty subset of \( E \) determined by the partition of \( \Phi \) into disjoint subsets \( \Phi^+(F) \), \( \Phi^0(F) \), and \( \Phi^-(F) \), where \( \lambda \in F \) if and only if all three conditions below are satisfied:

- \((\lambda + \rho)(h_\alpha) > 0 \) when \( \alpha \in \Phi^+(F) \),
- \((\lambda + \rho)(h_\alpha) = 0 \) when \( \alpha \in \Phi^0(F) \), and
- \((\lambda + \rho)(h_\alpha) < 0 \) when \( \alpha \in \Phi^-(F) \).

The closure \( \bar{F} \) of a facet \( F \) is defined to be the set of all \( \lambda \in h^* \) such that

- \((\lambda + \rho)(h_\alpha) \geq 0 \) when \( \alpha \in \Phi^+(F) \),
- \((\lambda + \rho)(h_\alpha) = 0 \) when \( \alpha \in \Phi^0(F) \), and
- \((\lambda + \rho)(h_\alpha) \leq 0 \) when \( \alpha \in \Phi^-(F) \).

Recall that \( \mathcal{S} \) is a (fixed) set of representatives of \([\lambda] \in \Omega_\theta^\mathbb{Q} \) (on which the definitions of \( Q_{n+1} \) and \( q_n \) depend). Now, for \( \lambda, \mu \in \mathcal{S} \) such that \( \lambda \parallel \mu \), there exists a unique dominant-integral weight \( \nu \in W_{g,h}(\lambda - \mu) \). Define for all sufficiently large \( n \) (i.e., for all positive integers \( n \) such that \( \nu \in W_{g,n,h}(\lambda - \mu) \)) the translation functor \((\theta^\mu_{b,n})_\lambda : \mathcal{O}^\mu_{b,n}[\lambda] \rightarrow \mathcal{O}^\mu_{b,n}[\mu] \). That is,

\[
(\theta^\mu_{b,n})_\lambda M_n \overset{\text{def}}{=} \text{pr}^\mu_{g_n,b_n}(\mathcal{O}^{\mu}_{b,n}(\nu) \otimes M_n)
\] (2.20)

for every \( M_n \in \mathcal{O}^\mu_{b,n}[\lambda] \). Recall that \((\theta^\mu_{b,n})_\lambda \) is an exact functor and it commutes with duality (see, for example, [6, Proposition 7.1]).

**Theorem 2.21** Let \( \lambda, \mu \in h^* \) be such that \( \lambda \parallel \mu \). If \( \lambda^\xi \) and \( \mu^\xi \) are in the same facet for the action of the integral Weyl group \( W_{g,h}[\lambda] = W_{g,h}[\mu] \) on \( E_\mathbb{Q}^\theta[\lambda] = E_\mathbb{Q}^\theta[\mu] \), then there exists an equivalence of categories \((\Theta^\mu_{\lambda})_\xi : \mathcal{O}_{b,n}[\lambda] \rightarrow \mathcal{O}_{b,n}[\mu] \).

**Proof** For convenience, write \( \mathcal{O} \) for \( \mathcal{O}^\mu_{b,n} \). For \( \zeta, \xi \in h^* \), we also denote by \((\theta_{n})_{\zeta}^\xi \) the functor \((\theta^\mu_{b,n})_\lambda \) in \( \mathcal{O}_{b,n}[\lambda] \). We also write \( W \) and \( W_n \) for \( W_{g,h} \) and \( W_{g,n,b_n} \), respectively.

Let \( n_0 \) be the smallest integer such that \( \nu \in W_{n_0}(\lambda - \mu) \). For each integer \( n \geq n_0 \), let \( \xi_n \in W_n[\lambda] \cdot \lambda \) be the antidominant weight that is linked to \( \lambda \), and let \( \zeta_n \in W_n[\mu] \cdot \mu \) be the antidominant weight that is linked to \( \mu \).

Since \( \lambda^\xi \) and \( \mu^\xi \) are in the same facet for the action of \( W_{g,n}[\lambda] = W_{g,n}[\mu] \) on \( E_\mathbb{Q}^\theta[\lambda] = E_\mathbb{Q}^\theta[\mu] \), there exists \( w_n \in W_{n}[\lambda] = W_{n}[\mu] \) such that \( w_n \cdot \lambda = \xi_n \) and \( w_n \cdot \mu = \zeta_n \). From [6, Theorem 7.8], we conclude that

\[
\Theta_n := (\theta_{n})_{\xi_n}^\zeta : \mathcal{O}_n[\lambda] \rightarrow \mathcal{O}_n[\mu]
\] (2.21)

is an equivalence of categories whose equivalence is

\[
\Theta'_n := (\theta_{n})_{\xi_n}^\zeta : \mathcal{O}_n[\mu] \rightarrow \mathcal{O}_n[\lambda].
\] (2.22)

Recall the functors \( \hat{Q}_{n+1}^\lambda : \text{im}(Q_{n+1}^\lambda) \rightarrow \mathcal{O}_n^\lambda \) and \( \hat{Q}'_{n+1}^\lambda : \text{im}(Q_{n+1}^\lambda) \rightarrow \mathcal{O}_n^\lambda \) from Proposition 2.11. Furthermore, due to [6, Proposition 7.8] any object in \( \text{im}(\Theta_{n+1}Q_{n+1}^\lambda) \) must
have composition factors of the form $\mathcal{L}_{b_{n+1}}(w \cdot \zeta_n)$, where $w \in W_n[\mu] \cdot \mu$. Therefore, $\im (\Theta_{n+1} Q_{n+1}^\lambda)$ is a subcategory of $\im (Q_{n+1}^\mu)$. Let

$$\mathcal{E}_n := \Theta_n^\lambda Q_{n+1}^\lambda : \tilde{O}_n[\lambda] \rightarrow \tilde{O}_n[\lambda].$$

Clearly, $\mathcal{E}_n$ is an auto-equivalence of $\tilde{O}_n[\lambda]$ (since $Q_{n+1}^\lambda$ is an equivalence on to its image, $\Theta_{n+1}$ and $\Theta_n$ are both equivalences, and $Q_n^\mu$ is an equivalence). Consequently,

$$\Theta_{n+1} Q_{n+1}^\lambda = Q_{n+1}^\mu \Theta_n \mathcal{E}_n \cong Q_{n+1}^\mu \Theta_n.$$  \hfill (2.24)

Write $\mathcal{O}$ for $\mathcal{O}_{\Theta}$. We can now let $\Theta : \mathcal{O}[\lambda] \rightarrow \mathcal{O}[\mu]$ be the direct limit of the directed system of functors $(\Theta_n : \tilde{O}_n[\lambda] \rightarrow \tilde{O}_n[\mu])_{n \in \mathbb{Z}_{>0}}$. Similarly, $\Theta' : \mathcal{O}[\mu] \rightarrow \mathcal{O}[\lambda]$ is the direct limit of the directed system of functors $(\Theta'_n : \tilde{O}_n[\mu] \rightarrow \tilde{O}_n[\lambda])_{n \in \mathbb{Z}_{>0}}$. As each $\Theta_n$ is an equivalence of categories with inverse $\Theta'_n$, we deduce that $\Theta$ is also an equivalence of categories with inverse $\Theta'$. We set $(\Theta_{\Theta}^\mu)^\lambda_\mathcal{O}$ to be the functor $\Theta$.

From the previous theorem, we have defined a "translation functor" $(\Theta_{\Theta}^\mu)^\lambda$ when $\mu$ and $\lambda$ are compatible and lie in the same facet for the action of $W_{b-b}[\lambda] = W_{b-b}[\mu]$. For arbitrary $\lambda, \mu \in \mathfrak{h}^*$ such that $\lambda \parallel \mu$, it is not clear whether the same construction yields a functor $(\Theta_{\Theta}^\mu)^\lambda : \mathcal{O}_{\Theta}[\lambda] \rightarrow \mathcal{O}_{\Theta}[\mu]$.

3 Tilting Modules

3.1 Extensions of Modules with Generalized Standard and Costandard Filtrations

In this subsection, we shall write Hom and Ext for $\text{Hom}_{\mathcal{O}}$ and $\text{Ext}_{\mathcal{O}}$. We shall first prove that any extension of a module with generalized costandard filtration by a module with generalized standard filtration is trivial.

**Theorem 3.1** Let $M \in \Delta(\mathcal{O})$ and $N \in \nabla(\mathcal{O})$. Then, $\text{Ext}^1(M, N) = 0$.

**Proof** If $M$ has a direct sum decomposition $M = \bigoplus_{\alpha \in A} M_\alpha$, then $\text{Ext}^1(M, N) \cong \prod_{\alpha \in A} \text{Ext}^1(M_\alpha, N)$. Thus, we may assume without loss of generality that $M$ is indecomposable.

We first prove the theorem when $N = \nabla(\mu)$ for some $\mu \in \mathfrak{h}^*$. Let $\Pi_{\mu}(M)$ denote the set of weights of $M$ that are greater than $\mu$. Note that $M$ is in the block $\mathcal{O}_{\lambda}[\lambda]$ for some $\lambda \in \mathfrak{h}^*$. If $\Pi_{\mu}(M)$ has infinitely many maximal elements, then we enumerate the maximal elements of $\Pi_{\mu}(M)$ by $\lambda_1, \lambda_2, \lambda_3, \ldots$. Note that $\lambda_i \in W[\lambda] \cdot \lambda$ for all $i = 1, 2, 3, \ldots$. This implies that $\lambda - \mu$ is not a finite integer combination of the simple roots. Therefore, $\lambda$ and $\mu$ are not in the same Weyl orbit. Thus, $\text{Ext}^1(M, N) = \text{Ext}^1(M, \nabla(\mu)) = 0$. From now on, we assume that $\Pi_{\mu}(M)$ has finitely many maximal elements.

We perform induction on the sum $m := \sum_{\xi \cdot \mu} \dim M^\xi$, which is finite due to the assumption in the previous paragraph. If $m = 0$, then by [9, Proposition 3.8(a)] and [9, Proposition 3.9(a)], we get $\text{Ext}^1(M, \nabla(\mu)) \cong \text{Ext}^1(\Delta(\mu), M^\xi) = 0$.

Let now $m$ be a positive integer. Fix a maximal weight $\xi \in \Pi_{\nu}(M)$. By Proposition 1.8(a), there exists a short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$, where $M'$ is isomorphic to $\Delta(\xi)$, and $M''$ has a generalized standard filtration. From the long exact sequence of $\text{Ext}^\bullet$, we get the following exact sequence

$$\text{Ext}^1(M'', \nabla(\mu)) \rightarrow \text{Ext}^1(M, \nabla(\mu)) \rightarrow \text{Ext}^1(M', \nabla(\mu)).$$  \hfill (3.1)

However, due to [9, Proposition 3.9(c)], we get $\text{Ext}^1(M', \nabla(\mu)) \cong \text{Ext}^1(\Delta(\xi), \nabla(\mu)) = 0$. By induction hypothesis, $\text{Ext}^1(M'', \nabla(\mu)) = 0$. Therefore, $\text{Ext}^1(M, \nabla(\mu)) = 0$ as well.
For each ordinal number $\gamma$, we shall define a submodule $X_\gamma \in \Delta(\mathcal{O})$ of $N^\vee$ such that $N^\vee/X_\gamma$ is also in $\Delta(\mathcal{O})$. First, we set $X_0 := 0$. If $\gamma$ is an ordinal with predecessor $\gamma'$, then we have by Proposition 1.8(a) that $N^\vee/X_{\gamma'}$ has a submodule $Y_\gamma$ such that $Y_\gamma \cong \Delta(\mu_\gamma)$ for some $\mu_\gamma \in \mathfrak{h}^*$ that is a maximal weight of $N^\vee/X_{\gamma'}$. Take $X_\gamma$ to be the preimage of $Y_\gamma$ under the canonical projection $N^\vee \twoheadrightarrow (N^\vee/X_{\gamma'})$. If $\gamma$ is a limit ordinal, then we set $X_\gamma$ to be $\bigcup_{\gamma' < \gamma} X_{\gamma'}$.

From the above construction, there exists the smallest ordinal $\kappa$ such that $N^\vee = X_\kappa$, and $N^\vee$ is in fact the direct limit of $(X_\gamma)_{\gamma < \kappa}$. Thus, $N$ is the inverse limit of $(N_\gamma)_{\gamma < \kappa}$, where $N_\gamma := (X_\gamma)^\vee$. We claim that $\text{Ext}^1(M, N_\gamma) = 0$ for all ordinals $\gamma \leq \kappa$.

If $\gamma$ has a predecessor $\gamma'$, then there exists a short exact sequence $0 \to \nabla(\mu_\gamma) \to N_\gamma \to N_{\gamma'} \to 0$. From the long exact sequence of $\text{Ext}^\bullet$, we have the following exact sequence

$$\text{Ext}^1(M, \nabla(\mu_\gamma)) \to \text{Ext}^1(M, N_\gamma) \to \text{Ext}^1(M, N_{\gamma'}).$$

(3.2)

By the induction hypothesis, $\text{Ext}^1(M, N_\gamma) = 0$. Since we have proven that $\text{Ext}^1(M, \nabla(\mu_\gamma)) = 0$, we can then conclude that $\text{Ext}^1(M, N_{\gamma'}) = 0$. If $\gamma$ is a limit ordinal, then $X_\gamma = \lim_{\gamma' < \gamma} X_{\gamma'}$; ergo,

$$\text{Ext}^1(M, N_{\gamma'}) \cong \text{Ext}^1((N_\gamma)^\vee, M^\vee) = \text{Ext}^1(X_\gamma, M^\vee).$$

(3.3)

Now, let $E$ be a $\mathfrak{g}$-module such that there exists an exact sequence $0 \to M^\vee \overset{i}{\to} E \overset{p}{\to} X_\gamma \to 0$. For each $\gamma' < \gamma$, we know that $\text{Ext}^1(X_{\gamma'}, M^\vee) \cong \text{Ext}^1(M, N_{\gamma'}) = 0$ by induction hypothesis, whence $p^{-1}(X_{\gamma'}) = i(M^\vee) \oplus \tilde{X}_{\gamma'}$, where $M_{\gamma'}$ and $\tilde{X}_{\gamma'}$ are submodules of $p^{-1}(X_{\gamma'})$ such that $M_{\gamma'} \cong M^\vee$ and $\tilde{X}_{\gamma'} \cong X_{\gamma'}$. We shall now define $v_{\gamma'} \in \mathcal{E}_{\gamma'}$ whenever $\gamma'$ has a predecessor $\gamma''$.

If $\gamma' = 1$, then $X_1 \cong \Delta(\mu_1)$ for some $\mu_1 \in \mathfrak{h}^*$. For each ordinal $\delta$ such that $\gamma' \leq \delta < \gamma$, define $\mathcal{E}_{\gamma'}^\delta$ to be the span of all possible $v_{\delta'} \in p^{-1}(X_{\delta'})$ such that $v_{\delta'}$ is a weight vector of weight $\mu_{\delta'}$ that lies in some submodule $\tilde{X}_{\delta} \cong X_{\delta}$ of $p^{-1}(X_{\delta})$ such that $p^{-1}(X_{\delta}) = i(M^\vee) \oplus \tilde{X}_{\delta}$ and that $v_{\delta'}$ generates $\tilde{X}_{\delta} \cong \Delta(\mu_{\delta'})$. Note that $\mathcal{E}_{\gamma'}^\delta$ is a finite-dimensional vector space with positive dimension, and $\mathcal{E}_{\gamma'}^{\delta'} \supseteq \mathcal{E}_{\gamma'}^{\delta}$ if $\gamma' < \delta' < \delta < \gamma$. Therefore, $\mathcal{E}_{\gamma'} := \bigcap_{\delta \in [\gamma', \gamma]} \mathcal{E}_{\gamma'}^{\delta} \neq 0$. We can choose $v_{\gamma'} \in \mathcal{E}_{\gamma'} \setminus \{0\}$ arbitrarily.

Let now $\gamma'$ be an ordinal such that $1 < \gamma' < \gamma$ and $\gamma'$ has a predecessor $\gamma''$. Suppose $v_{\tau}$ are all known for each $\tau < \gamma'$ such that $\tau$ has a predecessor. Set $Z_{\gamma'}$ to be the $\mathfrak{g}$-module generated by all such $v_{\tau}$. The choices of our vectors $v_{\tau}$ are to ensure that $Z_{\gamma'} \cong X_{\gamma'}$ is such that $p^{-1}(X_{\gamma'}) = i(M^\vee) \oplus Z_{\gamma'}$. Assume that $X_{\gamma'}/Z_{\gamma'} \cong \Delta(\mu_{\gamma'})$ for some $\mu_{\gamma'} \in \mathfrak{h}^*$. For each ordinal $\delta$ such that $\gamma' \leq \delta < \gamma$, define $\mathcal{E}_{\gamma'}^\delta$ to be the span of all possible $v_{\delta'} \in p^{-1}(X_{\delta'})$ such that $v_{\delta'}$ is a weight vector of weight $\mu_{\delta'}$ that lies in some submodule $\tilde{X}_{\delta} \cong X_{\delta}$ of $p^{-1}(X_{\delta})$ such that $\delta'(X_{\delta}) = i(M^\vee) \oplus \tilde{X}_{\delta}$ and that $v_{\delta'} + Z_{\gamma'}$ generates $\tilde{X}_{\delta}/Z_{\gamma'} \cong \Delta(\mu_{\delta'})$. We employ the same strategy as the previous paragraph by choosing $v_{\gamma'} \in \mathcal{E}_{\gamma'} \setminus Z_{\gamma'}$, where $\mathcal{E}_{\gamma'} := \bigcap_{\delta \in [\gamma', \gamma]} \mathcal{E}_{\gamma'}^\delta$.

Now, we let $\bar{X}_{\gamma}$ be the submodule of $E$ generated by $v_{\gamma'}$ for all ordinals $\gamma' < \gamma$ with predecessors. It follows immediately that $\bar{X}_{\gamma} \cong X_{\gamma}, i(M^\vee) \cap \bar{X}_{\gamma} = 0$, and $i(M^\vee) + \bar{X}_{\gamma} = E$. Thus, $\bar{E} = i(M^\vee) \oplus \bar{X}_{\gamma}$. Then, the projection $\varpi : E \to i(M^\vee)$ gives a retraction map $E \twoheadrightarrow M^\vee$. Therefore, the short exact sequence $0 \to M^\vee \to E \to \bar{X}_{\gamma} \to 0$ splits. That is, $\text{Ext}^1(X_{\gamma}, M^\vee) = 0$. Then, (3.3) implies that $\text{Ext}^1(M, N_\gamma) = 0$ as well. By transfinite induction, $\text{Ext}^1(M, N) = \text{Ext}^1(M, N_\kappa) = 0$.

**Conjecture 3.2** Let $M \in \Delta(\mathcal{O})$ and $N \in \nabla(\mathcal{O})$. Then, $\text{Ext}^k(M, N) = 0$ for every integer $k > 1$.

**Proposition 3.3** Let $M \in \mathcal{O}$ be a tilting module.

(a) The dual $M^\vee$ is also a tilting module.

(b) If $N \in \mathcal{O}$ is a tilting module, then $M \oplus N$ is also a tilting module.
(c) Any direct summand of $M$ is a tilting module.

**Proof** Parts (a) and (b) are trivial. Part (c) follows from Proposition 1.8(b) and Corollary 1.9. □

**Proposition 3.4** Let $M$ and $N$ be tilting modules in $\mathcal{O}$. Then, $\text{Ext}_0^1(M,N) = 0$. If Conjecture 3.2 is true, then we also have that $\text{Ext}_k^1(M,N) = 0$ for all integers $k > 1$.

### 3.2 Construction of the Tilting Modules $D(\lambda)$

**Proposition 3.5** Let $\lambda, \mu \in \mathfrak{h}^*$. Then, $\Delta(\lambda) \otimes \nabla(\mu)$ is a tilting module in $\mathcal{O}$.

**Proof** Let $M := \Delta(\lambda) \otimes \nabla(\mu)$. First, observe that, for all $\nu \leq \lambda + \mu$, we have

$$\dim(M^\nu) = \sum_{\xi \leq 0} \dim\left(\left(\Delta(\lambda)\right)^{\nu - \mu + \xi}\right) \cdot \dim\left(\left(\Delta(\mu)\right)^{\nu - \xi}\right)$$

(3.4)

Because there are only finitely many weights of $\Delta(\lambda)$ that is greater than or equal to $\nu - \mu$, we see that $\dim(M^\nu) < \infty$. Therefore, $M \in \mathcal{O}$.

Since $(\Delta(\lambda) \otimes \nabla(\mu))^\vee \cong (\Delta(\lambda))^\vee \otimes (\nabla(\mu))^\vee \cong \nabla(\lambda) \otimes \Delta(\mu) \cong \Delta(\mu) \otimes \nabla(\lambda)$, it suffices to show that $M := \Delta(\lambda) \otimes \nabla(\mu)$ has a generalized standard filtration.

Let $u$ be a maximal vector of $\Delta(\lambda)$. Pick a basis $v_1, v_2, v_3, \ldots$ of $\nabla(\mu)$ consisting of weight vectors. Then, define $w_i := u \otimes v_i$ for $i = 1, 2, 3, \ldots$. We first prove that $w_1, w_2, w_3, \ldots$ generate $M$ as a $\mathfrak{u}(\mathfrak{n}^-)$-module. Let $M'$ be the $\mathfrak{u}(\mathfrak{n}^-)$-submodule of $M$ generated by $w_1, w_2, w_3, \ldots$.

Fix a Chevalley basis of $\mathfrak{g}$ consisting of $x_{\pm \alpha}$ for positive roots $\alpha$, and $h_\alpha$ for simple positive roots $\alpha$. We then fix a PBW basis $B$ of $\mathfrak{u}(\mathfrak{n}^-)$. For each $t \in B$, the degree of $t$, denoted by $\deg(t)$, is defined to be $k$ if there exists positive roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $t = x_{-\alpha_1} x_{-\alpha_2} \cdots x_{-\alpha_k}$.

We shall prove that $(t \cdot u) \otimes v_i \in M'$. If $\deg(t) = 0$, then there is nothing to prove. Suppose now that $\deg(t) > 0$. Then, we can write

$$t = x_{-\alpha_1} x_{-\alpha_2} \cdots x_{-\alpha_k}$$

(3.5)

for some integer $k > 0$. By induction hypothesis, we know that $m := (x_{-\alpha_2} \cdots x_{-\alpha_k} \cdot u) \otimes v_i$ lies in $M'$. Using

$$x_{-\alpha_1} \cdot m = (t \cdot u) \otimes v_i + (x_{-\alpha_2} \cdots x_{-\alpha_k} \cdot u) \otimes (x_{-\alpha_1} \cdot v_i),$$

(3.6)

we conclude that $(t \cdot u) \otimes v_i$ is in $M'$, as both $x_{-\alpha_1} \cdot m$ and $(x_{-\alpha_2} \cdots x_{-\alpha_k} \cdot u) \otimes (x_{-\alpha_1} \cdot v_i)$ are in $M'$.

From the paragraph above, $M = M'$. We now need to show that $M$ is a free module over $\mathfrak{u}(\mathfrak{n}^-)$ generated by $w_1, w_2, w_3, \ldots$. Let now $M_k$ denote the $\mathfrak{u}(\mathfrak{n}^-)$-submodule of $M$ generated by $w_1, w_2, \ldots, w_k$, and $N_k$ the $\mathfrak{u}(\mathfrak{n}^-)$ submodule of $N$ generated by $w_k$ alone. Then, we can easily see that $M_k \cap N_{k+1} = 0$ for each $k = 1, 2, 3, \ldots$. Thus, $M_k = N_1 \oplus N_2 \oplus \ldots \oplus N_k$, making

$$M = N_1 \oplus N_2 \oplus N_3 \oplus \ldots$$

(3.7)

as a $\mathfrak{u}(\mathfrak{n}^-)$-module. Consequently, $M$ has a generalized standard filtration. □

**Theorem 3.6** Let $\lambda \in \mathfrak{h}^*$. There exists a unique, up to isomorphism, an indecomposable tilting module $D(\lambda) \in \mathcal{O}$, also denoted by $D(\lambda)$, such that $\dim\left(\left(D(\lambda)^\Lambda\right)^\lambda\right) = 1$ and all weights $\mu$ of $D(\lambda)$ satisfies $\mu \preceq \lambda$.

**Proof** Consider the $\mathfrak{g}$-module $M := \Delta(\lambda) \otimes \nabla(0)$. Define $D(\lambda)$ to be the indecomposable summand of $M$ that contains $M^\lambda$. By Proposition 3.3(c), we know that $D(\lambda)$ is a tilting module.

Suppose $T$ is another indecomposable tilting module such that $\dim(T^\lambda) = 1$ and every weight $\mu$ of $T$ satisfies $\mu \preceq \lambda$. Since $T$ has a generalized standard filtration and $\lambda$ is a maximal weight of $T$, by
Proposition 1.8(a), we know that $\Delta(\lambda)$ is a submodule of $T$ and $T/\Delta(\lambda)$ has a generalized standard filtration. From Proposition 3.4, we know that $\text{Ext}^1_\mathcal{O}(T/\Delta(\lambda), D(\lambda)) = 0$.

Now from the short exact sequence $0 \to \Delta(\lambda) \to T \to T/\Delta(\lambda) \to 0$ and from the long exact sequence of Ext-groups, we have the following exact sequence
\[
\text{Hom}_\mathcal{O}(T/\Delta(\lambda), D(\lambda)) \to \text{Hom}_\mathcal{O}(T, D(\lambda)) \to \text{Hom}_\mathcal{O}(\Delta(\lambda), D(\lambda)) \to \text{Ext}^1_\mathcal{O}(T/\Delta(\lambda), D(\lambda)) \cdot (3.8)
\]
Since $\text{Ext}^1_\mathcal{O}(T/\Delta(\lambda), D(\lambda)) = 0$, the map $\text{Hom}_\mathcal{O}(T, D(\lambda)) \to \text{Hom}_\mathcal{O}(\Delta(\lambda), D(\lambda))$ is surjective. Ergo, the embedding $\Delta(\lambda) \hookrightarrow D(\lambda)$ lifts to a homomorphism $\varphi : T \to D(\lambda)$.

Similarly, we also have a homomorphism $\psi : D(\lambda) \to T$ such that $\psi$ is an isomorphism on the copies of $\Delta(\lambda)$ in $D(\lambda)$ and $T$. Thus, the endomorphism $\varphi \circ \psi : D(\lambda) \to D(\lambda)$ is an isomorphism on $\Delta(\lambda) \subseteq D(\lambda)$. As $D(\lambda)$ is indecomposable, we know from [9, Theorem 2.5] that every endomorphism of $D(\lambda)$ is either an isomorphism or a locally nilpotent map. Since $\varphi \circ \psi$ preserves the weight space $(D(\lambda))^\lambda$, the map $\varphi \circ \psi$ is not locally nilpotent. Hence, $\varphi \circ \psi$ is an isomorphism. Consequently, both $\varphi$ and $\psi$ must be isomorphism, whence $T \cong D(\lambda)$. ■

Corollary 3.7 If $T \in \mathcal{O}$ is an indecomposable tilting module, then $T \cong D(\lambda)$ for some $\lambda \in \mathfrak{h}^*$. In particular, all tilting modules are self-dual.

Proof Let $\lambda$ be a maximal weight of $T$. Using the same argument as the theorem above, we can easily see that $T \cong D(\lambda)$.

For the second part of the corollary, we let $T$ be an arbitrary tilting module. We can then see from the paragraph above and [9, Corollary 2.6] that $T = \bigoplus \nolimits_\alpha D(\lambda_\alpha)$, where $\lambda_\alpha \in \mathfrak{h}^*$ for all $\alpha \in A$.

Since duality commutes with direct sum, it suffices to show that $D(\lambda)$ is self-dull for a fixed $\lambda \in \mathfrak{h}^*$. As $D(\lambda)$ is an indecomposable tilting module, $(D(\lambda))^\vee$ is also an indecomposable tilting module. By the theorem above, we conclude that $(D(\lambda))^\vee \cong D(\lambda)$. ■

3.3 Multiplicities of Verma Factors in a Tilting Module

In this subsection, we shall again write Hom and Ext for $\text{Hom}_\mathcal{O}$ and $\text{Ext}_\mathcal{O}$. We first need the following theorem.

Theorem 3.8 Suppose that $M \in \Delta(\mathcal{O})$. For every $\lambda \in \mathfrak{h}^*$, we have
\[
\{M, \Delta(\lambda)\} = \dim \text{Hom}_\mathcal{O}(M, \nabla(\lambda)) . (3.9)
\]

Proof Without loss of generality, assume that $M$ is indecomposable. We consider the set $\Pi_{\geq \lambda}(M)$ of weights of $M$ that is greater than or equal to $\lambda$. If this set is infinite, we can easily see that $M$ is not in the same block as $\Delta(\lambda)$. Therefore, $\{M, \Delta(\lambda)\} = 0$ and $\dim \text{Hom}_\mathcal{O}(M, \nabla(\lambda)) = 0$. Therefore, the assertion is true. From now on, we assume that $\Pi_{\geq \lambda}(M)$ is finite.

Define $m := \sum \nolimits_{\xi \geq \lambda} \dim M^\xi$. Then, $m$ is a nonnegative integer. We can then perform induction on $m$, the base case $m = 0$ being obvious. Let now $m > 0$. Suppose that $\mu \succeq \lambda$ is a maximal weight of $M$. By Proposition 1.8(a), $M$ has a submodule $\Delta(\mu)$ such that $M/\Delta(\mu)$ has a generalized standard filtration. From the short exact sequence $0 \to \Delta(\mu) \to M \to M/\Delta(\mu) \to 0$ and the long exact sequence of Ext-groups, we get the following exact sequence
\[
0 \to \text{Hom}(M/\Delta(\mu), \nabla(\lambda)) \to \text{Hom}(M, \nabla(\lambda)) \to \text{Hom}(\Delta(\mu), \nabla(\lambda)) \to \text{Ext}^1(M/\Delta(\mu), \nabla(\lambda)) \cdot (3.10)
\]
Because \( M/\Delta(\mu) \) has a generalized standard filtration and \( \nabla(\lambda) \) obviously has a generalized co-standard filtration, Thorem 3.1 ensures that \( \text{Ext}^1(M/\Delta(\mu), \nabla(\lambda)) = 0 \). Furthermore, because \( \dim \text{Hom}(\Delta(\mu), \nabla(\lambda)) = \delta_{\mu,\lambda} \), where \( \delta \) is the Kronecker delta, we conclude that

\[
\dim \text{Hom}(M, \nabla(\lambda)) = \dim \text{Hom}(M/\Delta(\mu), \nabla(\lambda)) + \delta_{\mu,\lambda}.
\]

(3.11)

On the other hand,

\[
\{M, \Delta(\lambda)\} = \{M/\Delta(\mu), \Delta(\lambda)\} + \{\Delta(\mu) : \Delta(\lambda)\} = \{M/\Delta(\mu), \Delta(\lambda)\} + \delta_{\mu,\lambda}.
\]

(3.12)

By induction hypothesis, \( \{M/\Delta(\mu), \Delta(\lambda)\} = \dim \text{Hom}(M/\Delta(\mu), \nabla(\lambda)) \), so \( \dim \text{Hom}(M, \nabla(\lambda)) \) and \( \{M, \Delta(\lambda)\} \) are equal.

**Corollary 3.9** Suppose that \( M \in \nabla(\mathcal{O}) \). For every \( \lambda \in \mathfrak{h}^* \), we have

\[
\{M, \nabla(\lambda)\} = \dim \text{Hom}_{\mathcal{O}}(M^\prime, \nabla(\lambda)).
\]

(3.13)

Fix \( \lambda \in \mathfrak{h}^* \). For each positive integer \( n \), we consider the restriction \( \tilde{D}_n(\lambda) := \text{Res}_{g_n}^\theta D(\lambda) \). Because

\[
\text{Res}_{g_n}^\theta \Delta^\theta_n(\lambda) \cong \bigoplus_{\nu \leq \lambda} \Delta^\theta_n(\nu),
\]

we can easily see that \( \tilde{D}_n(\lambda) \) is a \( g_n \)-module with generalized standard filtration. As the duality functor commutes with the restriction functor, we conclude that \( \tilde{D}_n(\lambda) \) is a tilting \( g_n \)-module.

Suppose that \( \mu \in \mathfrak{h}^* \) satisfies \( \mu \preceq \lambda \) and \( \mu \in W_{\mathfrak{g}_b}[\lambda] \cdot \lambda \). Let \( n_0(\mu, \lambda) \) be the smallest positive integer \( n \) such that \( \lambda - \mu \in \Lambda_{g_{n_0}, b_n} \). Due to Theorem 1.5(a), Equation (3.14) implies that

\[
\{D^\theta_n(\lambda) : \Delta^\theta_n(\mu)\} = \{\tilde{D}_{n_0(\mu, \lambda)}(\lambda) : \Delta^\theta_{n_0(\mu, \lambda)}(\mu)\}.
\]

(3.15)

Since \( D^\theta_{n_0(\mu, \lambda)}(\lambda) \) is the only indecomposable direct summand of \( \tilde{D}_{n_0(\mu, \lambda)}(\lambda) \) that can contribute to the multiplicity of \( \Delta^\theta_{n_0(\mu, \lambda)}(\mu) \) in \( \tilde{D}_{n_0(\mu, \lambda)}(\lambda) \), we get

\[
\{D^\theta_n(\lambda) : \Delta^\theta_n(\mu)\} = \{D^\theta_{n_0(\mu, \lambda)}(\lambda) : \Delta^\theta_{n_0(\mu, \lambda)}(\mu)\}.
\]

(3.16)

Recall from [6, Theorem 3.8] that, for each \( \lambda \in \mathfrak{h}^* \) and for each positive integer \( n \), \( \mathcal{O}^\theta_{b_n} \) has enough projectives. We let \( \mathcal{P}^\theta_{b_n}(\lambda) \) denote the projective cover of the module \( \mathcal{U}^\theta_{b_n}(\lambda) \) in \( \mathcal{O}^\theta_{b_n} \).

**Theorem 3.10** Let \( \lambda, \mu \in \mathfrak{h}^* \) with \( \mu \preceq \lambda \) and \( \mu \in W_{\mathfrak{g}_b}[\lambda] \cdot \lambda \). Write \( n := n_0(\mu, \lambda) \). Fix \( \xi \in \mathfrak{h}^* \) such that \( \xi \) is a \( \mathfrak{b}_n \)-antidominant weight in \( W_{\mathfrak{g}_{n}, b_n}[\lambda] \cdot \lambda \). If \( w^\theta_n \) is the longest element of \( W_{\mathfrak{g}_n, b_n} \), then

\[
\{D^\theta_n(\lambda) : \nabla^\theta_n(\mu)\} = \{D^\theta_n(\lambda) : \Delta^\theta_n(\mu)\} = \{\mathcal{P}^\theta_{b_n}(w^0_n \cdot \lambda), \Delta^\theta_{b_n}(w^0_n \cdot \mu)\}
\]

(3.17)

**Proof** Due to [1, Theorem 6.10], we have \( \{D^\theta_n(\lambda), \Delta^\theta_{b_n}(\mu)\} = \{\mathcal{P}^\theta_{b_n}(w^0_n \cdot \lambda), \Delta^\theta_{b_n}(w^0_n \cdot \mu)\} \). The theorem follows immediately from (3.16).

Write \( P^W_{x,y}(q) \in \mathbb{Z}[q] \) for the Kazhdan-Lusztig polynomial for elements \( x, y \) in a Coxeter group \( W \). Due to (3.16), we may assume without loss of generality that \( \lambda \) and \( \mu \) are integral weights of \( g_{n_0(\mu, \lambda)} \). We then have the following theorem.

**Theorem 3.11** Let \( \lambda, \mu \in \mathfrak{h}^* \) with \( \mu \preceq \lambda \) and \( \mu \in W_{\mathfrak{g}_b}[\lambda] \cdot \lambda \). Suppose that \( \lambda \) is a regular integral weight with respect to \( g_{n_0(\mu, \lambda)} \). Fix \( \xi \in \mathfrak{h}^* \) such that \( \xi \) is a \( g_{n_0(\mu, \lambda)} \)-antidominant weight in \( W_{g_{n_0(\mu, \lambda)}, b_{n_0(\mu, \lambda)}}[\lambda] \cdot \lambda \). If \( \lambda = x \cdot \xi \) and \( \mu = y \cdot \xi \) for some \( x, y \in W_{g_{n_0(\mu, \lambda)}, b_{n_0(\mu, \lambda)}} \), then

\[
\{D^\theta_n(\lambda) : \nabla^\theta_n(\mu)\} = \{D^\theta_n(\lambda) : \Delta^\theta_n(\mu)\} = P^W_{x,y}(1).
\]

(3.18)

**Proof** For simplicity, write \( n := n_0(\mu, \lambda) \). From [11, Theorem 4.4], we have

\[
\dim \text{Hom}_{\mathcal{O}^\theta_{b_n}}(\Delta^\theta_{b_n}(\mu), D^\theta_n(\lambda)) = \dim \text{Hom}_{\mathcal{O}^\theta_{b_n}}(\Delta^\theta_{b_n}(y \cdot \xi), D^\theta_n(x \cdot \xi)) = P^W_{x,y}(1).
\]

(3.19)

From Theorem 3.8, we have \( \dim \text{Hom}_{\mathcal{O}^\theta_{b_n}}(\Delta^\theta_{b_n}(\mu), D^\theta_n(\lambda)) = \{D^\theta_n(\lambda), \Delta^\theta_{b_n}(\mu)\} \). By (3.16), the claim follows immediately.
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