A CRITERION FOR RINGS WHICH ARE LOCALLY VALUATION

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ABSTRACT. Using the notion of cyclically pure injective modules, a characterization for rings which are locally valuation is established. As applications, new characterizations for Prüfer domains and pure semi-simple rings are provided. Namely, we show that a domain $R$ is Prüfer if and only if two of the three classes of pure injective, cyclically pure injective and RD-injective modules are equal. Also, we prove that a commutative ring $R$ is pure semi-simple if and only if every $R$-module is cyclically pure injective.

1. INTRODUCTION

Throughout this paper, $R$ denotes a commutative ring with identity, and all modules are assumed to be left unitary. The notion of pure injective modules has a substantial role in commutative algebra and model theory. Among various generalizations of this notion, the notion of cyclically pure injective modules has been extensively studied by M. Hochster [9] and L. Melkersson [12]. Recall that an exact sequence $0 \to A \to B \to C \to 0$ of $R$-modules and $R$-homomorphisms is said to be cyclically pure if the induced map $R/a \otimes_R A \to R/a \otimes_R B$ is injective for all (finitely generated) ideals $a$ of $R$. Also, an $R$-module $D$ is said to be cyclically pure injective if for any cyclically pure exact sequence $0 \to A \to B \to C \to 0$, the induced homomorphism $\text{Hom}_R(B, D) \to \text{Hom}_R(A, D)$ is surjective. In the sequel, we use the abbreviation CP for the term “cyclically pure”.

More generally, let $\mathcal{S}$ be a class of $R$-modules. An exact sequence $0 \to A \to B \to C \to 0$ of $R$-modules and $R$-homomorphisms is said to be $\mathcal{S}$-pure if for all $M \in \mathcal{S}$, the induced homomorphism $\text{Hom}_R(M, B) \to \text{Hom}_R(M, C)$ is surjective. An $R$-monomorphism $f : A \to B$ is said to be $\mathcal{S}$-pure if the exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{\text{nat}} B/f(A) \to 0$ is $\mathcal{S}$-pure. An $R$-module $D$ is said to be $\mathcal{S}$-pure injective if for any $\mathcal{S}$-pure exact sequence $0 \to A \to B \to C \to 0$, the induced homomorphism

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\( \text{Hom}_R(B,D) \to \text{Hom}_R(A,D) \) is surjective, see [13]. When \( S \) is the class of finitely presented \( R \)-modules, \( S \)-pure exact sequences and \( S \)-pure injective modules are called pure exact sequences and pure injective modules, respectively. If \( S \) denotes the class of all \( R \)-modules of the form \( R/Rr, r \in R \), then \( S \)-pure exact sequences and \( S \)-pure injective modules are called RD-exact sequences and RD-injective modules, respectively. For a survey on the notions of pure injective and RD-injective modules, we refer the reader to [6].

Let \( S \) be the class of all \( R \)-modules \( M \) for which there is a cyclic submodule \( G \) of \( R^n \), for some \( n \in \mathbb{N} \), such that \( M \) is isomorphic to \( R^n/G \). In [3], we showed that CP-exact sequences and CP-injective modules coincide with \( S \)-pure exact sequences and \( S \)-pure injective modules, respectively. In the same paper we have systematically investigated the structure of CP-injective modules and presented several characterizations of this class of modules. Our aim in this paper is the following:

i) Classifying the commutative rings that over which the two notions of “RD-injective” and “cyclically pure injective” coincide.

ii) Classifying the commutative rings that over which the two notions of “pure injective” and “cyclically pure injective” coincide.

In Section 2, we show that \( R_p \) is a valuation ring for all prime ideals \( p \) of \( R \) if and only if every CP-injective \( R \)-module is RD-injective, if and only if every pure injective \( R \)-module is CP-injective. From this we obtain a characterization for semi-hereditary rings and also one for Prüfer domains. In the literature, there are several characterizations for Prüfer domains. In particular, by [6, Chapter XIII, Theorem 2.8], it is known that a domain \( R \) is Prüfer if and only if every pure injective \( R \)-module is RD-injective. Also, it is known by [6, Chapter IX, Proposition 3.4] that a domain \( R \) is Prüfer if and only if every divisible \( R \)-module is absolutely pure. Here we show that a domain \( R \) is Prüfer if and only if every CP-injective \( R \)-module is RD-injective, if and only if every pure injective \( R \)-module is CP-injective. Also, we show that a domain \( R \) is Prüfer if and only if every absolutely CP-module is absolutely pure. Finally, a new characterization for pure semi-simple rings is given. We show that a ring \( R \) is pure semi-simple if and only if every \( R \)-module is CP-injective, if and only if every \( R \)-module is RD-pure injective.

The first example of a CP-exact sequence which is not pure was presented in [1]. Our first characterization for Prüfer domains mentioned above shows that over a non-Prüfer domain \( R \) the class of CP-injective \( R \)-modules is strictly larger than that of RD-injective \( R \)-modules and strictly smaller than that of pure injective \( R \)-modules. However, these may be viewed as kind of implicit strict inclusions. In Section 3, we provide some examples for which we can explicitly show proper containments in this regard. In [3], we proved that
in many aspects CP-injective modules behave similar to pure injective and RD-injective modules. But Remark 2.2 and Example 3.5 below display some differences between the former class and the later two.

2. A CHARACTERIZATION FOR Prüfer rings

In the remainder of this paper, let \( S_1 \) denote the class of all \( R \)-modules of the form \( R/\mathfrak{r} \), \( \mathfrak{r} \in R \). Also, let \( S_4 \) (resp. \( S_2 \)) denote the class of all finitely presented (resp. finitely presented cyclic) \( R \)-modules. Finally, we let \( S_3 \) denote the class of all \( R \)-modules \( M \) for which there are an integer \( n \in \mathbb{N} \) and a cyclic submodule \( G \) of \( R^n \) such that \( M \) is isomorphic to \( R^n/G \).

**Definition 2.1.** Let \( S \) be a class of \( R \)-modules. An exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) of \( R \)-modules and \( R \)-homomorphisms is called \( S \)-flat if for all \( M \in S \) the induced map \( A \otimes_R M \rightarrow B \otimes_R M \) is injective.

**Remark 2.2.** Let \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) be an exact sequence of \( R \)-modules and \( R \)-homomorphisms.

i) For \( i = 1, 4 \), the above exact sequence is \( S_i \)-pure if and only if it is \( S_i \)-flat, see [13, Propositions 2 and 3].

ii) By [3, Proposition 2.2], the above exact sequence is \( S_3 \)-pure if and only if it is \( S_2 \)-flat.

Example 3.5 in the next section, shows that there exists an \( S_2 \)-flat exact sequence which is not \( S_2 \)-pure.

**Definition 2.3.** Let \( S \) be a class of \( R \)-modules. An \( R \)-module \( P \) is said to be \( S \)-pure projective if for any \( S \)-pure exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \), the induced homomorphism \( \text{Hom}_R(P, B) \rightarrow \text{Hom}_R(P, C) \) is surjective.

**Lemma 2.4.** Let \( S \) and \( T \) be two classes of \( R \)-modules. The following are equivalent:

i) Every \( T \)-pure exact sequence is \( S \)-pure exact.

ii) Every \( S \)-pure projective \( R \)-module is \( T \)-pure projective.

iii) Every element of \( S \) is a direct summand of a direct sum of modules in \( T \).

Moreover, if \( S \) and \( T \) are both contained in \( S_4 \), then the above conditions are equivalent to the following

iv) Every \( S \)-pure injective \( R \)-module is \( T \)-pure injective.

**Proof.** Let \( U \) be a class of \( R \)-modules. By the definition every element of \( U \) is \( U \)-pure projective. In general, by [13, Proposition 1], it turns out that an \( R \)-module \( M \) is \( U \)-pure projective if and only if \( M \) is a direct summand of a direct sum of modules in \( U \). Hence the equivalence of i), ii) and iii) is immediate.
Next, assume that $S$ and $T$ are both contained in $S_4$. Let $\mathcal{U} \subseteq S_4$ be a class of $R$-modules and $E$ an injective cogenerator of $R$. By [5, Lemma 1.2], there is a class $\mathcal{U}^*$ of $R$-modules such that an exact sequence $0 \to A \to B \to C \to 0$ of $R$-modules and $R$-homomorphisms is $\mathcal{U}$-pure if and only if

$$0 \to A \otimes_R M^* \to B \otimes_R M^* \to C \otimes_R M^* \to 0$$

is exact for all $M^* \in \mathcal{U}^*$. Thus by using adjoint property, it follows that $\text{Hom}_R(M^*, E)$ is a $\mathcal{U}$-pure injective $R$-module for all $M^* \in \mathcal{U}^*$.

$iv) \Rightarrow i)$ Let $0 \to A \to B \to C \to 0(*)$ be a $T$-pure exact sequence and $M^* \in S^*$ an arbitrary element. Since $\text{Hom}_R(M^*, E)$ is $S$-pure injective, it is also $T$-pure injective, by our assumption. Thus, by applying the functor $\text{Hom}_R(\cdot, \text{Hom}_R(M^*, E))$ on $(*)$ and using adjoint property, we deduce the following exact sequence

$$0 \to \text{Hom}_R(C \otimes_R M^*, E) \to \text{Hom}_R(B \otimes_R M^*, E) \to \text{Hom}_R(A \otimes_R M^*, E) \to 0.$$ 

Thus, it turns out that the sequence

$$0 \to A \otimes_R M^* \to B \otimes_R M^* \to C \otimes_R M^* \to 0$$

is exact. Therefore $(*)$ is $S$-pure exact.

Now, since the implication $i) \Rightarrow iv)$ clearly holds, the proof is finished. $\square$

**Lemma 2.5.** Assume that every pure injective $R$-module is CP-injective. Then an exact sequence $l : 0 \to A \to B \to C \to 0$ is $S_2$-pure exact if and only if it is CP-exact.

**Proof.** Assume that $l$ is a CP-exact sequence. Then, by Lemma 2.4, it is pure exact. Hence it is clearly $S_2$-pure, because $S_2 \subseteq S_4$.

Now, assume that $l$ is $S_2$-pure exact. Let $E$ be an injective cogenerator of $R$ and $(\cdot)^\lor$ denote the faithfully exact functor $\text{Hom}_R(\cdot, E)$. Let $l^\lor$ denote the induced exact sequence $0 \to C^\lor \to B^\lor \to A^\lor \to 0$. Let $I$ be a finitely generated ideal of $R$. Since $R/I$ is finitely presented, the two $R$-modules $R/I \otimes_R M^\lor$ and $\text{Hom}_R(R/I, M)^\lor$ are naturally isomorphic for all $R$-modules $M$. So the exact sequence $l^\lor$ is a CP-exact. Hence $l^\lor$ is pure exact, by Lemma 2.4. Let $N \in S_3$. Then by Remark 2.2 i), the sequence $N \otimes_R l^\lor$ is exact. The exact sequences

$$0 \to N \otimes_R C^\lor \to N \otimes_R B^\lor \to N \otimes_R A^\lor \to 0$$

and

$$0 \to \text{Hom}_R(N, C)^\lor \to \text{Hom}_R(N, B)^\lor \to \text{Hom}_R(N, A)^\lor \to 0$$

are naturally isomorphic. Thus the second sequence is also exact, and so

$$0 \to \text{Hom}_R(N, A) \to \text{Hom}_R(N, B) \to \text{Hom}_R(N, C) \to 0$$
is an exact sequence, because \((\cdot)^V\) is a faithfully exact functor. Therefore \(l\) is a CP-exact sequence. \(\square\)

**Lemma 2.6.** Let \(a\) be an ideal of \(R\). Assume that every CP-injective \(R\)-module is RD-injective. Then every CP-injective \(R/a\)-module is an RD-injective \(R/a\)-module.

**Proof.** Set \(T = R/a\). Let \(M = T^n/V\), where \(n \in \mathbb{N}\) and \(V\) is a cyclic \(T\)-submodule of \(T^n\). So, there are \(b_1, \ldots, b_n \in R\) such that \(V = T(b_1 + a, \ldots, b_n + a)\). Let \(N = R^n/U\), where \(U = R(b_1, \ldots, b_n)\). We show that \(M\) and \(N \otimes_R T\) are naturally isomorphic as \(T\)-modules. To this end, let \(\phi : M \rightarrow N \otimes_R T\) be the map defined by

\[(x_1 + a, \ldots, x_n + a) + V \mapsto ((x_1, \ldots, x_n) + U) \otimes (1 + a)\]

for all \((x_1 + a, \ldots, x_n + a) + V \in M\). Also, we define \(\psi : N \otimes_R T \rightarrow M\) by

\[((x_1, \ldots, x_n) + U) \otimes (r + a) \mapsto (rx_1 + a, \ldots, rx_n + a) + V\]

It is a routine check to see that \(\phi\) and \(\psi\) are well defined \(T\)-homomorphisms and that \(\psi \phi = id_M\) and \(\phi \psi = id_{N \otimes_R T}\). Now, as \(- \otimes_R T\) commutes with direct sums, the conclusion is immediate by Lemma 2.4 iii) \(\iff\) iv). \(\square\)

Recall that a valuation ring (not necessarily a domain) is a commutative ring whose ideals are linearly ordered under inclusion.

**Theorem 2.7.** The following are equivalent:

i) \(R_p\) is a valuation ring for all prime ideals \(p\) of \(R\).

ii) Every pure injective \(R\)-module is RD-injective.

iii) Every CP-injective \(R\)-module is RD-injective.

iv) Every pure injective \(R\)-module is CP-injective.

v) Every pure projective \(R\)-module is RD-projective.

vi) Every CP-projective \(R\)-module is RD-projective.

vii) Every pure projective \(R\)-module is CP-projective.

**Proof.** By Lemma 2.4, the equivalences ii) \(\iff\) v), iii) \(\iff\) vi) and iv) \(\iff\) vii) are obvious. Also, the implications ii) \(\Rightarrow\) iii) and ii) \(\Rightarrow\) iv) are clear.

i) \(\Rightarrow\) vi) As we have mentioned in the proof Lemma 2.4, for a given class \(U\) of \(R\)-modules, an \(R\)-module \(M\) is \(U\)-pure projective if and only if \(M\) is a direct summand of a direct sum of modules in \(U\). So, to deduce vi), it is enough to show that every finitely presented \(R\)-module is RD-projective. By [6, Proposition 4], a finitely presented \(R\)-module \(M\) is RD-projective if and only if \(M_m\) is an RD-projective \(R_m\)-module for all maximal ideals \(m\) of \(R\). Hence vi) follows by [15, Theorem 1].

vi) \(\Rightarrow\) i) follows by [13, Proposition 1] and [15, Theorem 3].
Assume that there exists a prime ideal $p$ of $R$ so that $R_p$ is not a valuation ring. Let $N = (R_p)^n / G$, where $n \in \mathbb{N}$ and $G$ is a cyclic $R_p$-submodule of $(R_p)^n$. Clearly $N$ is equal to the localization at $p$ of an element of $S_3$. Hence, as localization at $p$ commutes with direct sums, by Lemma 2.4, we may and do assume that $R$ is a local ring which is not a valuation ring. Denote by $m$ the maximal ideal of $R$. Since $R$ is not a valuation ring, there are two elements $a, b \in R$ such that $Ra \nsubseteq Rb$ and $Rb \nsubseteq Ra$. Set $\mathcal{I} := ma + mb$. Lemma 2.6 yields that every CP-injective $R/\mathcal{I}$-module is an RD-injective $R/\mathcal{I}$-module. Replace $R, a$ and $b$ by $R/\mathcal{I}, a + \mathcal{I}$ and $b + \mathcal{I}$, respectively. So we can assume that $R$ is a local ring which is not a valuation ring and that there are two elements $a, b \in R$ such that $Ra \nsubseteq Rb$, $Rb \nsubseteq Ra$, $ma = mb = 0$ and $Ra \cap Rb = 0$. In view of the proof of [15, Theorem 2], it becomes clear that $M := (R \oplus R)/R(a, -b)$ is a non-cyclic indecomposable $R$-module. Lemma 2.4 implies that $M$ is a direct summand of a direct sum of cyclic modules. Now, by [14, Proposition 3], over a commutative local ring, any indecomposable direct summand of a direct sum of cyclic modules is cyclic. We achieved at a contradiction.

By Lemmas 2.4 and 2.5, it follows that every finitely presented $R$-module is a direct summand of a direct sum of cyclic modules. Now, we assume that i) does not hold and search for a contradiction. Then there is a prime ideal $p$ of $R$ so that $R_p$ is not a valuation ring. Hence, by [15, Theorem 2], there exists an indecomposable finitely presented $R_p$-module $M$ which is not cyclic. Since every finitely presented $R_p$-module is the localization at $p$ of a finitely presented $R$-module, we deduce that $M$ is a direct summand of a direct sum of cyclic $R_p$-module. But then by [14, Proposition 3], $M$ should be a cyclic $R_p$-module. □

**Definition 2.8.**

i) (See [4]) A ring $R$ is said to be projective principal ring (P.P.R.) if every principal ideal of $R$ is projective.

ii) A ring $R$ is said to be semi-hereditary if every finitely generated ideal of $R$ is projective.

iii) (See [10]) An $R$-module $M$ is said to be absolutely pure (resp. absolutely cyclically pure) if it is pure (resp. cyclically pure) as a submodule in every extension of $M$.

iv) (See [5]) An $R$-module $M$ is said to be divisible if for every $r \in R$ and $x \in M$, $\text{Ann}_R r \subseteq \text{Ann}_R x$ implies that $x \in rM$. (This is equivalent to the usual definition where $R$ is domain.)

In the proof of the following lemma we use the methods of the proofs of [10, Proposition 1 and Corollary 2].

**Lemma 2.9.** Let $M$ be an $R$-module.

i) $M$ is absolutely cyclically pure if and only if $\text{Ext}_k(N, M) = 0$ for all $N \in S_3$. 
ii) $M$ is absolutely cyclically pure if and only if any diagram

$$
\begin{array}{ccc}
P' & \xrightarrow{\alpha} & P \\
\downarrow{\delta} & & \\
M & &
\end{array}
$$

with $P'$ cyclic, $\alpha$ monic and $P$ projective, there exists a homomorphism $\gamma : P \to M$ such that $\gamma \alpha = \beta$.

**Proof.** i) Let $L$ be an extension of $M$ and $N \in S_3$. From the exact sequence $0 \to M \to L \to L/M \to 0$, we deduce the following exact sequence

$$
0 \to \Hom_R(N, M) \to \Hom_R(N, L) \to \Hom_R(N, L/M) \to \Ext^1_R(N, M) \to \Ext^1_R(N, L)(*)
$$

Assume that $M$ is an absolutely CP-module and let $L$ be an injective extension of $M$. Then by Remark 2.2 ii) and (*), we conclude that $\Ext^1_R(N, M) = 0$ for all $N \in S_3$.

Now, assume that $\Ext^1_R(N, M) = 0$ for all $N \in S_3$. Let $L$ be an extension of $M$. Then Remark 2.2 ii) and (*) imply that the exact sequence $0 \to M \to L \to L/M \to 0$ is CP-exact.

ii) We may assume that $P$ is a finitely generated free $R$-module. Thus the result follows by using i) and the following exact sequence

$$
\Hom_R(P, M) \to \Hom_R(P', M) \to \Ext^1_R(P/\alpha(P'), M) \to 0.
$$

**Lemma 2.10.** The following are equivalent:

i) $R$ is a P.P.R.

ii) Every cyclic submodule of a projective $R$-module is projective.

iii) Every quotient of an absolutely CP-module is also an absolutely CP-module.

**Proof.** i) $\leftrightarrow$ ii) follows by [4, Theorem 3.2].

ii) $\leftrightarrow$ iii) In view of Lemma 2.9, the proof is immediate by adapting the argument of [10, Theorem 2] and replacing the phrases “absolutely pure” and “finitely generated submodule” with “absolutely cyclically pure” and “cyclic submodule”, respectively.

**Corollary 2.11.** Assume that $R$ is a P.P.R. The following are equivalent:

i) $R$ is a semi-hereditary ring.

ii) Every pure injective $R$-module is RD-injective.

iii) Every CP-injective $R$-module is RD-injective.

iv) Every pure injective $R$-module is CP-injective.

v) Every divisible $R$-module is absolutely pure.

vi) Every absolutely CP-module is absolutely pure.

vii) Every pure projective $R$-module is RD-projective.
viii) Every CP-projective R-module is RD-projective.
ix) Every pure projective R-module is CP-projective.

Proof. As, we have mentioned in the proof of Theorem 2.7, by Lemma 2.4, the equivalences ii) \(\iff\) vii) \(\iff\) viii) and iv) \(\iff\) ix) are obvious.

Now, assume that \(R\) is semi-hereditary. Let \(p\) be a prime ideal of \(R\). Then \(R_p\) is also a semi-hereditary ring. Hence for each nonzero element \(a\) of \(R_p\), the \(R_p\)-module \(aR_p\) is a nonzero free \(R_p\)-module. Thus, we conclude that \(R_p\) is a domain. But, it is known that a domain is semi-hereditary if and only if it is Prüfer. So \(R_p\) is a valuation domain for all prime ideals \(p\) of \(R\). Therefore the implication i) \(\Rightarrow\) ii) and the equivalences ii) \(\iff\) iii) and iii) \(\iff\) iv) are immediate by Theorem 2.7.

ii) \(\Rightarrow\) v) Let \(M\) be a divisible \(R\)-module and \(E\) denote the injective envelop of \(M\). Then [5, Lemma 2.2] implies that the sequence \(0 \to M \to E \to E/M \to 0\), is RD-exact. Hence, by Lemma 2.4, it is pure and so \(\text{Ext}_R^1(N, M) = 0\) for all \(N \in S_4\). Thus, by [10, Proposition 1], \(M\) is absolutely pure.

v) \(\Rightarrow\) vi) Let \(M\) be an absolutely CP-module. Then, by Lemma 2.9 i), \(\text{Ext}_R^1(N, M) = 0\) for all \(N \in S_3\). In particular, \(\text{Ext}_R^1(R/R_r, M) = 0\) for all \(r \in R\), and so \(M\) is a divisible \(R\)-module by [5, Lemma 2.2]. Thus \(M\) is absolutely pure, as required.

Finally, we prove vi) \(\Rightarrow\) i). Since \(R\) is a P.P.R., Lemma 2.10 yields that every quotient of an absolutely CP-module is again an absolutely CP-module. So, if vi) holds, then every quotient of an absolutely pure module is again absolutely pure. Thus i) follows by [10, Theorem 2]. \(\square\)

Now, since a domain \(R\) is Prüfer if and only if it is semi-hereditary, we can obtain the main result of this paper. Note that every domain is a P.P.R.

Corollary 2.12. Assume that \(R\) is a domain. The following are equivalent:
i) \(R\) is Prüfer.
ii) Every pure injective \(R\)-module is RD-injective.
iii) Every CP-injective \(R\)-module is RD-injective.
iv) Every pure injective \(R\)-module is CP-injective.
v) Every divisible \(R\)-module is absolutely pure.
vi) Every absolutely CP-module is absolutely pure.
vii) Every pure projective \(R\)-module is RD-projective.
viii) Every CP-projective \(R\)-module is RD-projective.
ix) Every pure projective \(R\)-module is CP-projective.
Let $C_{RDR}$ denote the class of all RD-injective $R$-modules. Also, let $C_{CPR}$ and $C_{PR}$ denote the class of all CP-injective $R$-modules and that of all pure injective $R$-modules, respectively. It follows, by Theorem 2.7 that if two of three classes $C_{RDR}$, $C_{CPR}$ and $C_{PR}$ are equal, then all three classes are equal. The following result shows that if each of these three classes is equal to the class of all $R$-modules, then the two other classes are also equal to the class of all $R$-modules. First, we bring a definition.

**Definition 2.13.** A ring $R$ is said to be pure-semi simple if every $R$-module is a direct sum of finitely generated $R$-modules.

**Theorem 2.14.** The following are equivalent:
i) Every $R$-module is RD-pure injective.
ii) Every $R$-module is CP-injective.
iii) Every $R$-module is pure injective.
iv) $R$ is pure-semi simple.

**Proof.** The implications $i) \Rightarrow ii)$ and $ii) \Rightarrow iii)$ are clear.

Assume that $iii)$ holds. Then every pure exact sequence of $R$-modules splits, and so it follows from [8] that every $R$-module is a direct sum of finitely generated $R$-modules. Thus $iii)$ implies $iv)$.

Now, we prove the implication $iv) \Rightarrow i)$. By [7, Theorem 4.3], $R$ is an Artinian principal ideal ring and every $R$-module is a direct sum of cyclic $R$-modules. Hence, since every ideal of $R$ is principal, it follows that every $R$-module is a direct sum of modules of the form $R/\mathfrak{r}R, \mathfrak{r} \in R$. From this we can conclude that every RD-exact sequence splits. Therefore, every $R$-module is RD-injective. □

3. SOME EXAMPLES

Theorem 2.7 shows that there exists a ring $R$ such that $C_{RDR} \subsetneq C_{CPR} \subsetneq C_{PR}$. In this section, we present some explicit examples for these strict containments.

**Example 3.1.** i) Let $\mathbb{Z}$ be the ring of integers and $p$ a prime integer. Since every ideal of $\mathbb{Z}$ is principal, the two notions of RD-injectivity and of CP-injectivity are coincide for $\mathbb{Z}$-modules. Hence by [3, Theorem 3.6], $D = \mathbb{Z}/p\mathbb{Z}$ is an RD-injective $\mathbb{Z}$-module, while it is not an injective $\mathbb{Z}$-module.

ii) By [1, Example 1], there are an Artinian local ring $R$ and an $R$-algebra $S$ containing $R$, such that the inclusion map $R \hookrightarrow S$ is cyclically pure, but it is not pure. It is known that every Artinian $R$-module is pure injective (see e.g. [11, Corollary 4.2]). Hence $R$ is a pure injective $R$-module. But $R$ is not CP-injective, because otherwise by [3, Theorem 3.4], the inclusion map $R \hookrightarrow S$ splits.
Lemma 3.2. Let $R$ be a domain, $B$ a torsion free $R$-module and $0 \rightarrow K \hookrightarrow B \rightarrow M \rightarrow 0$ an exact sequence of $R$-modules. The following are equivalent:

i) $M$ is torsion-free.

ii) The inclusion map $K \hookrightarrow B$ is RD-pure.

Proof. It is easy to see that an $R$-module $L$ is torsion free if and only if $\text{Tor}_1^R(R/\mathfrak{r}, L) = 0$ for all $\mathfrak{r} \in R$. Since $B$ is torsion-free for any $\mathfrak{r} \in R$, from the exact sequence $0 \rightarrow K \hookrightarrow B \rightarrow M \rightarrow 0$, we deduce the exact sequence

$$0 \rightarrow \text{Tor}_1^R(R/\mathfrak{r}, M) \rightarrow (R/\mathfrak{r}) \otimes_R K \rightarrow (R/\mathfrak{r}) \otimes_R B \rightarrow (R/\mathfrak{r}) \otimes_R M \rightarrow 0.$$ 

Therefore, the assertion follows by Remark 2.2 i). □

Lemma 3.3. Let $R$ be a domain and $D$ an RD-injective $R$-module. Then $\text{Ext}_1^R(M, D) = 0$ for all torsion-free $R$-modules $M$.

Proof. Let $M$ be a torsion-free $R$-module. Consider an exact sequence $0 \rightarrow K \hookrightarrow F \rightarrow M \rightarrow 0$, in which $F$ is a free $R$-module. Then, by Lemma 3.2, the inclusion map $i$ is RD-pure. Now, from the exact sequence

$$0 \rightarrow \text{Hom}_R(M, D) \rightarrow \text{Hom}_R(F, D) \rightarrow \text{Hom}_R(K, D) \rightarrow \text{Ext}_1^R(M, D) \rightarrow 0,$$

we deduce that $\text{Ext}_1^R(M, D) = 0$. Note that since $D$ is RD-injective, the map $\text{Hom}_R(i, id_D)$ is surjective. □

Example 3.4. Let $(R, \mathfrak{m})$ be a local Noetherian domain with $\text{dim } R > 1$. Since $R$ is not a Prüfer domain, it turns out that $R$ possesses an ideal $\mathfrak{a}$ which is not projective. Thus $\text{Ext}_1^R(\mathfrak{a}, R/\mathfrak{m}) \neq 0$, by [2, Proposition 1.3.1]. Now, by [3, Theorem 3.6], $R/\mathfrak{m}$ is a CP-injective $R$-module, while by Lemma 3.3, $R/\mathfrak{m}$ is not RD-injective.

The following example shows that the two notions of $S_2$-flatness and $S_2$-pureness are not the same.

Example 3.5. Assume that $R$ is a Noetherian domain such that $\text{dim } R > 1$. Hence $R$ is not Prüfer, and so by Corollary 2.12, there exists an absolutely CP-module $M$ which is not injective. So, there is an ideal $\mathfrak{a}$ such that $\text{Ext}_1^R(R/\mathfrak{a}, M) \neq 0$. Let $E$ denote the injective envelope of $M$. Then from the exact sequence $0 \rightarrow M \hookrightarrow E \overset{\pi}{\rightarrow} E/M \rightarrow 0$ (†), we deduce the following exact sequence

$$0 \rightarrow \text{Hom}_R(R/\mathfrak{a}, M) \rightarrow \text{Hom}_R(R/\mathfrak{a}, E) \rightarrow \text{Hom}_R(R/\mathfrak{a}, E/M) \rightarrow \text{Ext}_1^R(R/\mathfrak{a}, M) \rightarrow 0.$$ 

Hence the map $\text{Hom}_R(id_{R/\mathfrak{a}}, \pi)$ is not surjective. Thus (†) is a $S_2$-flat sequence which is not $S_2$-pure.
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