The Signaling Role of Leaders in Global Games*  
(Preliminary and Incomplete)  

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September 27, 2022  

Abstract  

How important are leaders’ actions in facilitating coordination? In this paper, we investigate their signaling role in a global games framework. A perfectly informed leader and a team of followers face a coordination problem. Despite the endogenous information generated by the leader’s action, we provide a necessary and sufficient condition that makes the monotone equilibrium strategy profile uniquely $\Delta$-rationalizable and hence guarantees equilibrium uniqueness. Moreover, the unique equilibrium is fully efficient. This result remains valid when the leader observes a noisy signal about the true state except full efficiency may not be obtained. We discuss the implications of our results for a broad class of phenomena such as adoption of green technology, currency attacks and revolutions.  

* We thank Sandeep Baliga, Alessandro Pavan, Wojciech Olszewski, Harry Pei, and Marciano Siniscalchi for invaluable support throughout the writing process.  
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1 Introduction

Coordination problems arise in a diverse variety of socio-economic phenomena that range from bank runs and currency crises to political change and revolutions. The global games approach has gained tremendous popularity in studying such issues. This is in part due to the fact that this methodology often leads to unique equilibrium selection in coordination problems. These predictions, however, crucially depend on the exogeneity of information and the static nature of the game. Relaxing any of these two assumptions has been shown to lead to multiplicity of equilibria, each of which may yield divergent predictions. A relaxation of these assumptions naturally arises, though, if one wishes to study cases that involve a leader, that is, a player whose visibility endows her with a special role. Often, this special role is granted to the leader by the fact that she has to act first, with her actions providing information to her followers about a relevant state of nature.

In this paper, we study precisely this signaling role of a leader and we show how both the two aforementioned standard assumptions of the global games framework can be relaxed and, yet, under a necessary and sufficient condition, a unique rationalizable behavior can be pinned down. In particular, we consider a two-stage game where a leader and a team of followers have binary, irreversible actions and face a coordination problem. The perfectly informed leader moves first and her action is observable to the followers, which implies that endogenous information is present due to signaling. The followers, then, simultaneously make their choices. Even though the continuation game (that is, the game after the leader has moved) may feature multiple monotone equilibria, we prove that the game as a whole has a unique monotone equilibrium. The equilibrium strategy profile is, moreover, uniquely rationalizable under a specific condition on the magnitude of the noise of the followers’ signals. Furthermore, the monotone equilibrium strategy profile is fully efficient.

The result extends to the case where the leader is not perfectly informed but rather observes a noisy signal about the state similarly to the followers. If a sufficient condition that relates the magnitudes of the noises in the leader’s and the followers’ signals is satisfied, the game admits a unique rationalizable (and thus, equilibrium) strategy profile which is in monotone strategies. In this case, however, a "mild" inefficiency arises even when this condition is satisfied and equilibrium is unique.

We relate our results on uniqueness of rationalizable behavior to the properties of the conditional rank belief function, a generalization of rank belief functions introduced in Morris et al. (2016) and Morris and Yildiz (2019), which accommodates the fact that the probability that a follower assigns to the event that another follower has received a signal less than his own, depends on the threshold used in the strategy of the leader. Specifically, we show that our con-

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1 That is, there exists a unique equilibrium in the class of monotone strategies.
ditions that ensure unique rationalizable play, essentially bound the derivative of the conditional rank belief function so that the the expected payoff from investing of the threshold follower in the subgame, never crosses zero in the case of the perfectly informed leader, and has a single-crossing property in the case where the leader observes a noisy signal of the state.

The visibility as well as the signaling role of the leader play a prominent part in our model. She moves first and, therefore, leads by example since her action is observable by the followers and irreversible. The choice of action by the leader conveys information to the followers about the state. Contrary to existing works where the leader is a player with different incentives than the followers, in our model, their incentives are aligned. Thus, our model features strategic complementarities within and across stages. Moreover, the leader may be perfectly informed about the state, or have better or worse information than the followers. Contrary to followers, however, our leader faces not only the uncertainty about the true state (when he is uninformed about it) but also the risk of mis-coordination by the followers in the second stage of the game. What our results imply is that this risk is mitigated by the ability of the leader to move first. In the case where the leader knows the state, this risk disappears if the noise of the followers’ signals is sufficiently high. However, in the limit where noise vanishes this risk is maximised, as is the strategic uncertainty that followers face about the behavior of each other.

Our model plays the role of a metaphor that can interpreted in a way that fits a variety of situations. The leader can be a nation that initiates an environmentally friendly policy in the hope that more countries will follow or the manager of a firm that wishes to induce her employees to exert costly effort when her only instrument is her own choice to work or shirk. It can also be a prominent investor contemplating whether to attack a currency or not or whether to roll-over debt or not. Or, the leader can be a vanguard in revolution. By attacking the regime first, she can inspire other citizens to do so. As we point out, the (application specific) "desired” play of the followers will not happen for sure, but only when the conditions we identify are satisfied. The surprising fact is that it is in the case when the leader is arbitrarily better informed than the followers that the cardinality of the set of rationalizable strategy profiles is maximized.

Methodologically, we contribute to the literature by establishing the uniqueness of rationalizable behavior in a two-stage game with endogenous information under some conditions. We relax in a specific way the timing of a standard static global game while introducing signaling by the leader and, thus, learning by the followers. Our framework is not fully flexible: each actor in our model makes an irreversible choice at an exogenously fixed time period. However, by making this choice we are able to balance the trade-off between generality and sharpness of results. It also highlights how strong the requirement of uniqueness is when we move away from a static game even in this inflexible way. On the contrary, existing models usually abstain from the analysis of rationalizable behavior and establish uniqueness of monotone equilibria, without conditions that guarantee that it is without loss to focus on these types of strategies.
While establishing conditions that guarantee the non-existence of complicated, non monotone equilibria is in principle a difficult task, we are able to do so by proving that the monotone equilibrium is the only strategy profile that survives an elimination procedure characterizing $\Delta$-rationalizability—a solution concept that extends extensive-form rationalizability à la Pearce (1984) to games with incomplete information (Battigalli and Siniscalchi, 2003). This concept allows for the presence of explicit restrictions on beliefs and behavior. Our choice of this specific rationalizability notion for a multi-stage game allows us to bypass some technicalities but is without consequence. In particular, our results obtain under alternative solution concepts such as interim sequential rationalizability (Penta, 2012).

1.1 Related Literature

Our paper is related to several different strands of literature. First and foremost our framework is a global game, pioneered by Carlsson and van Damme (1993) and popularized by Morris and Shin (1998, 2003). Ever since, this literature has vastly expanded with the papers most closely related to ours being Xue (2003), Corsetti et al. (2004), Angeletos et al. (2006), Dasgupta (2007), and Kovác and Steiner (2013). The papers by Corsetti et al. (2004) and Dasgupta (2007) feature two-stage games where a large player in the former or a continuum of players in the latter have the option to move first or delay their choice until the second period. Both these papers feature signaling. In Corsetti et al. (2004) the action of a large player is observable by the followers while in Dasgupta (2007), followers observe a noisy signal of the aggregate first stage action. The papers by Xue (2003) and Kovác and Steiner (2013) study dynamic coordination and the role of reversibility of actions. Similarly to Corsetti et al. (2004) and Dasgupta (2007) they focus on monotone strategies but remain silent about the rationalizable behavior of the game.

In Angeletos et al. (2006) there is a perfectly informed policy maker whose action is observable by the players before the coordination stage. This generates, as is the case in our model, endogenous information due to signaling. In their framework, this leads to equilibrium multiplicity. There are two important differences between the two papers. First, the leader and the followers have perfectly aligned incentives in our model, while in Angeletos et al. (2006) there is conflict of interest. Second, in our extended model, the leader is not perfectly informed about the true state but rather observes a noisy signal. Nevertheless, under some conditions, uniqueness of equilibrium can be obtained. Our results, therefore, complement the results of Angeletos et al. (2006) in that we show that signaling does not necessarily imply multiplicity of equilibria.

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2 Noisy signaling is necessary, since with a continuum of first stage players, perfect observation of the aggregate action makes the followers able to infer perfectly the true state which directly leads to equilibrium multiplicity.
On the methodological side, our model is an extensive-form game with incomplete information that features strategic complementarities within and across stages. It therefore shares elements with Echenique (2004) and Van Zandt and Vives (2007). It is interesting to see how our model differs from the latter. The models nested in the general framework developed in Van Zandt and Vives (2007) are static monotone supermodular games. As this paper illustrates, their results do not immediately extend to multistage games. For example, even though the continuation game after the leader moves may have multiple Bayesian Nash equilibria, the whole game admits a unique equilibrium for sufficiently large noise. Moreover, if the conditions we identify are not satisfied, both actions become rationalizable for an interval of player types. However, it is not the case that the set of $\Delta$-rationalizable actions is bounded by monotone equilibria as is the case in Van Zandt and Vives (2007).

1.2 Organization of the Paper

The remainder of the paper is organized as follows: Section 2 presents the main model. Section 3 derives and discusses the main results. Section 4 examines an alternative information structure for the leader and generalizes the results of Section 3 in this situation. Section 5 concludes. All the proofs are in the Appendix.

2 The Model

Consider a two-period game played by $n + 1$ players, $i \in N = \{L, 1, \ldots, n\}, n \geq 2$. Each player has to decide whether to invest ($d_i = I$) or not to invest ($d_i = N$). The cost of investing is $c > 0$. Not investing is a safe action (i.e., staying with the existing technology) with benefits normalized to zero, while investing involves an adoption of a new technology (or good management practices). Let $D_i = \{I, N\}$ be the action space for player $i \in N$. The player, $L$, is a leader who moves first in period 1. All other players, or followers $j \in F = \{1, \ldots, n\}$, make investment decisions in period 2 after observing the leader’s action. Our game thus constitutes a multi-stage game with observable actions. We assume that the leader’s action is irreversible; that is, the leader cannot “exit” after choosing to invest or “delay” after choosing not to invest, which may be understood as the commitment made by the leader or the consequence of a high reputation cost of exit or delay.

We consider the situation that the adoption of the new technology exhibits network externalities; that is, the more people adopt the technology, the more valuable it is. In other words,

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3 We borrow the terms from Kováč and Steiner (2013).

4 See, for example, Farrell and Saloner (1986) and Katz and Shapiro (1986) for a discussion of possible sources of network externalities.
players’ investment decisions are strategic complements. Let \( \tilde{b}(\theta, a_{-i}) \) be an additively separable benefit (from investing) function for player \( i \)'s, where \( \theta \in \Theta = \mathbb{R} \) is a payoff-relevant state (“fundamentals”) that affects the gross return from investment, and \( a_{-i} = (a_k)_{k \in N; k \neq i} \) is the vector of other players’ actions that generates positive network benefit when at least one other player invests. Thus, player \( i \)'s payoff to investing is given by \( u_i = \tilde{b}(\theta, a_{-i}) - c \). We further assume that \( \tilde{b} \) is increasing in both \( \theta \) and \( a_{-i} \) and is symmetric in \( a_{-i} \).\(^5\) Let \( \mathbb{I}(\cdot) \) be the indicator function and \( A_{-i} = n^{-1} \sum_{k \neq i} \mathbb{I}(a_k = I) \) be the proportion of other players choosing to invest. Then we could write player \( i \)'s payoff to investing as \(^6\)

\[
 u_i = u(\theta, A_{-i}) = \theta + A_{-i} - 1. 
\]

This payoff function is familiar in the global games literature, for example, see [Morris and Shin (2003)] and [Morris and Yildiz (2019)]. Figure 1 illustrates an example with two followers \((n = 2)\).

\[
\begin{array}{c|c|c|c|c|c}
 & I & N & & I & N \\
\hline
I & \theta, \theta, \theta & \theta - \frac{1}{2}, \theta - \frac{1}{2}, 0 & & & \\
N & \theta - \frac{1}{2}, 0, \theta - \frac{1}{2} & \theta - 1, 0, 0 & & & \\
\end{array}
\]

\[\begin{array}{c|c|c|c|c|c}
 & I & N & & I & N \\
\hline
I & 0, \theta - \frac{1}{2}, \theta - \frac{1}{2} & 0, \theta - 1, 0 & & & \\
N & 0, 0, \theta - 1 & 0, 0, 0 & & & \\
\end{array}\]

**Figure 1** A two-follower example, with the leader’s payoff listed first, the row follower’s second, and the column follower’s third.

Assume that the leader observes the realization of \( \theta \) (henceforth her type). We explain the signaling role in the Intro] Followers, on the other hand, have fundamental uncertainty about the fundamentals, \( \theta \). The initial common prior is an improper uniform distribution over the real line. Each follower \( j \in F \) receives a signal \( x_j = \theta + \sigma_F \epsilon_j \), where \( \sigma_F > 0 \) measures the quality of private information and \( \epsilon_j \) is an idiosyncratic standard Gaussian noise that is independent of \( \theta \) and independently and identically distributed (IID) across all followers. In an Online Appendix we show that our results hold for a set of noise distributions with densities that are positive, continuously differentiable, symmetric (about zero), and log-concave over \( \mathbb{R} \). We refer to \( x_j \) as follower \( j \)'s type and let \( X_j = \mathbb{R} \) be the corresponding type space.

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\(^5\) By “increasing” we mean “strictly increasing”. In this paper, we ignore the word “strictly” when stating a strict order.

\(^6\) Suppose that \( \tilde{b}(\theta, a_{-i}) = v(\theta) + \sum_{k \neq i} w(a_k) \), where \( v \) is an increasing function and \( w \) is such that \( w(0) = 0 \) and \( w(a_k) = \omega > 0 \). Then player \( i \)'s payoff is \( \tilde{u}(\theta, a_{-i}) = \tilde{b}(\theta, a_{-i}) - c = v(\theta) + \omega \sum_{k \neq i} \mathbb{I}(a_k = I) - c \). A monotone transformation, \( u = a \tilde{u} + \beta \), gives that \( u(\theta, A_{-i}) = a v(\theta) + \beta + A_{-i} - 1 \) by letting \( a = 1/(\omega n) \) and \( \beta = 1 - ac \). We may then assume, without loss of generality, that \( a v + \beta = id \), the identity map from \( \mathbb{R} \) to \( \mathbb{R} \).
From a global game perspective, the assumption that there is common knowledge that the leader knows $\theta$ is critical. In particular, under this assumption, the subgame does not have the feature of dominance regions. In Section 4, we investigate an alternative information structure where the leader also observes a noisy private signal about $\theta$. As we will show, the current model can be understood as the limiting case when the noise of leader’s information approaches zero while keeping $\sigma_F$ fixed. The spirit of the main result is the same in both cases though there will be differences in terms of efficiency of the outcomes of the respective games.

Note that, under complete information, it is straightforward to see that the model admits multiple subgame perfect equilibria when $\theta \in (0, 1/n)$. In the two-follower case, for example, we have two subgame perfect equilibria $((I, I, N, I, N)$ and $(N, N, I, N)$ with the former being the fully efficient one.

We will show later that our results have useful implications for efficient selection.

3 Analysis and Main Results

In this section, we present the main results of our analysis. We first show the existence of a unique perfect Bayesian equilibrium in monotone strategies. We then identify a necessary and sufficient condition under which the model exhibits a unique rationalizable behavior and hence a unique equilibrium. Finally, we discuss implications of the analysis for equilibrium selection.

3.1 Monotone Equilibrium

A (pure) strategy for the leader is a mapping $s_L(\theta) : \Theta \rightarrow D_L$ from her type space into an investment decision. A strategy for a follower $j \in F$, is a mapping $s_j(x_j, h) : X_j \times D_L \rightarrow D_j$; from private information and history (i.e., the leader’s action $d_L$) into a decision of whether to invest.

We first confine attention to symmetric perfect Bayesian equilibria with thresholds $(\hat{\theta}_L, \hat{x}_{I}, \hat{x}_{N})$ in which (i) the leader chooses $s_L(\theta) = I$ if and only if $\theta > \theta_L^*$, and (ii) all followers use the same strategy $s_F$ such that $s_F(x_j, h) = I$ if and only if $x_j > x_h^*$. We call any strategy defined as in (i) or (ii) a monotone strategy and any symmetric perfect Bayesian equilibrium satisfying both (i) and (ii) a monotone equilibrium. We also write $\hat{x}_h = -\infty$ if all types of a follower invest under history $h$ and $\hat{x}_h = \infty$ if no type of a follower invests.

Let $\hat{\theta}_L, \hat{x}_I, \hat{x}_N$ be candidates for the equilibrium thresholds.

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7 By $\mathcal{I}, \mathcal{N}$ we mean a follower invests when the leader invests and does not invest when the leader does not.
8 We use the two notions, $h$ and $d_L$, interchangeably throughout the paper.
9 We break ties by requiring a type not invest when indifferent.
Follower Problem

Consider type $x$ of follower $j \in F$. Since the leader uses a monotone strategy, any follower can infer from history $h = I$ that $\theta > \hat{\theta}_L$. Therefore, type $x$’s posterior belief about $\theta$ has a truncated Gaussian distribution with density

$$
\psi^h(\theta; x, \hat{\theta}_L) = \begin{cases} 
\frac{1}{\sigma_F} \phi\left( \frac{\theta - x}{\sigma_F} \right) \mathbb{1}(\theta > \hat{\theta}_L) & \text{if } h = I \\
\frac{1}{\sigma_F} \phi\left( \frac{\theta - x}{\sigma_F} \right) \Phi\left( \frac{x - \hat{\theta}_L}{\sigma_F} \right) & \text{if } h = N
\end{cases}
$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cumulative distribution function (CDF) and the density of the standard Gaussian distribution, respectively. Let $\Psi^h(\theta; x, \hat{\theta}_L)$ be the corresponding CDF for history $h$. We denote by $\lambda(x) = \phi(x)/\Phi(x)$ the reverse hazard rate. Then type $x$’s expectation of $\theta$ can be written as

$$
\mathbb{E}_{\theta \sim \Psi^h(\cdot; x, \hat{\theta}_L)}[\theta] = \begin{cases} 
x + \sigma_F \lambda \left( \frac{x - \hat{\theta}_L}{\sigma_F} \right) & \text{if } h = I \\
x - \sigma_F \lambda \left( \frac{\hat{\theta}_L - x}{\sigma_F} \right) & \text{if } h = N
\end{cases}
$$

(1)

The next lemma is a direct consequence of $\Psi^h(\theta; x, \hat{\theta}_L)$ being log-concave.

**Lemma 1.** $\mathbb{E}_{\theta \sim \Psi^h(\cdot; x, \hat{\theta}_L)}[\theta]$ is increasing in $x$ and $\hat{\theta}_L$. Moreover,

$$
\lim_{x \to -\infty} \mathbb{E}_{\theta \sim \Psi^h(\cdot; x, \hat{\theta}_L)}[\theta] = \begin{cases} 
\hat{\theta}_L & \text{if } h = I \\
-\infty & \text{if } h = N
\end{cases}
$$

and

$$
\lim_{x \to \infty} \mathbb{E}_{\theta \sim \Psi^h(\cdot; x, \hat{\theta}_L)}[\theta] = \begin{cases} 
\infty & \text{if } h = I \\
\hat{\theta}_L & \text{if } h = N
\end{cases}
$$

Given history $h$, a follower will invest if and only if her type is greater than $\hat{x}_h$. This means that, at a given state $\theta$, the probability that follower $j$ assigns to $k$ other followers investing equals $[1 - \Phi((\hat{x}_h - \theta)/\sigma_F)]^k$, $k \in \{0, 1, \ldots, n - 1\}$. It follows that follower $j$’s expected proportion of other players investing at state $\theta$ is given by

$$
A_{-j}^n(\theta) = \sum_{k=0}^{n-1} \binom{n-1}{k} \left( \frac{k + \chi_T}{n} \right) \left[ 1 - \Phi\left( \frac{\hat{x}_h - \theta}{\sigma_F} \right) \right]^k \Phi\left( \frac{\hat{x}_h - \theta}{\sigma_F} \right)^{n-1-k} = \frac{n-1}{n} \left[ 1 - \Phi\left( \frac{\hat{x}_h - \theta}{\sigma_F} \right) \right] + \frac{\chi_T}{n},
$$
where we write \( \chi_{\mathcal{I}} = 1(h = \mathcal{I}) \) for ease of notation\(^\text{10}\). The second equality follows from the binomial identity \( \sum_{k=0}^{n-1} \binom{n-1}{k} (1 - q)^k q^{n-1-k} = (n - 1)(1 - q) \). Since the leader’s action is observable, follower \( j \) has certainty about receiving the network benefit \( \chi_{\mathcal{I}} / n \). Thus, we may write the payoff to investing for type \( x \), under history \( h \), as

\[
\pi^h_F(x; \widehat{\theta}_L, \widehat{x}_h) = \mathbb{E}_{\theta \sim \Psi^h(\cdot; x, \widehat{\theta}_L)} \left[ \theta - \frac{n - 1}{n} \Phi \left( \frac{\widehat{x}_h - \theta}{\sigma_F} \right) \right] - \frac{\chi_{\mathcal{N}}}{n}.
\]

The next lemma guarantees that follower \( j \)’s best response to monotone strategies used by the leader and other followers is also a monotone strategy.

**Lemma 2.** \( \pi^h_F(x; \widehat{\theta}_L, \widehat{x}_h) \) is increasing in \( x \) and \( \widehat{\theta}_L \), and is decreasing in \( \widehat{x}_h \). Moreover,

\[
\lim_{x \to -\infty} \pi^h_F(x; \widehat{\theta}_L, \widehat{x}_h) = \begin{cases} \widehat{\theta}_L & \text{if } h = \mathcal{I} \\ -\infty & \text{if } h = \mathcal{N} \end{cases}
\]

and

\[
\lim_{x \to \infty} \pi^h_F(x; \widehat{\theta}_L, \widehat{x}_h) = \begin{cases} \infty & \text{if } h = \mathcal{I} \\ \widehat{\theta}_L - \frac{1}{n} & \text{if } h = \mathcal{N} \end{cases}.
\]

We define follower \( j \)’s *conditional rank belief* as the probability he assigns to the event that another follower’s type \( x_k \) is lower than his own \( (x_j = x) \) conditional on history \( h \):

\[
R^h(x; \widehat{\theta}_L) = \Pr (x_k < x_j \mid x_j = x, h) = \frac{1}{2} \left[ \Phi \left( \frac{x - \widehat{\theta}_L}{\sigma_F} \right) + \frac{\chi_{\mathcal{N}}}{n} \right].
\]

This definition is a direct extension of the rank belief function introduced in [Morris et al. (2016)](https://www.jstor.org/stable/26664895) and [Morris and Yildiz (2019)](https://www.jstor.org/stable/26664895). A Bayesian Nash equilibrium in the subgame following history \( h \) obtains when the threshold type \( x = \widehat{x}_h \) for each follower is indifferent between investing and not investing; i.e., \( \pi^h_F(\widehat{x}_h; \widehat{\theta}_L, \widehat{x}_h) = 0 \), where

\[
\pi^h_F(\widehat{x}_h; \widehat{\theta}_L, \widehat{x}_h) = \mathbb{E}_{\theta \sim \Psi^h(\cdot; x, \widehat{\theta}_L)} [\theta] + \frac{n - 1}{n} \left[ 1 - R^h(\widehat{x}_h; \widehat{\theta}_L) \right] + \frac{\chi_{\mathcal{I}}}{n} - 1.
\]

Thus, \( \widehat{x}_h \) must solve for all \( h \)

\[
\mathbb{E}_{\theta \sim \Psi^h(\cdot; x, \widehat{\theta}_L)} [\theta] = \frac{n - 1}{n} R^h(\widehat{x}_h; \widehat{\theta}_L) + \frac{\chi_{\mathcal{N}}}{n}.
\]

**Leader Problem**

If type \( \theta \) of the leader invests, her expected network externalities are

\[
A_{-L}(\theta) = 1 - \Phi \left( \frac{\widehat{x}_I - \theta}{\sigma_F} \right).
\]

\(^{10}\) Likewise, \( \chi_{\mathcal{N}} = 1(h = \mathcal{N}). \)
Note that the behavior of followers matters to the leader only when $a_L = I$; otherwise the safe action $a_L = N$ gives a payoff of zero. The payoff to investing for type $\theta$ is therefore

$$\pi_L(\theta; \tilde{x}_I) = \theta - \Phi \left( \frac{\tilde{x}_I - \theta}{\sigma_F} \right).$$

It is straightforward to see that $\pi_L(\theta; \tilde{x}_I)$ is increasing in $\theta$ and crosses zero only once from below. In other words, the leader will best respond by using a monotone strategy with threshold $\hat{\theta}_L$ being the value of $\theta$ that solves $\pi_L(\hat{\theta}_L; \tilde{x}_I) = 0$; that is,

$$\hat{\theta}_L = \Phi \left( \frac{\tilde{x}_I - \hat{\theta}_L}{\sigma_F} \right). \quad (3)$$

**Equilibrium Characterization**

A monotone equilibrium obtains when the thresholds, $\hat{\theta}_L$, $\tilde{x}_I$, and $\tilde{x}_N$, solve Equations (2) and (3) simultaneously. We point out next that our model admits no standard monotone equilibrium.

**Lemma 3.** There is no monotone equilibrium that solves Equations (2) and (3).

Our next result shows that the only monotone equilibrium features that all types of each follower invest if the leader invests and do not invest if the leader does not. This equilibrium is in the same spirit as the strong herding equilibrium in Dasgupta (2000).

**Proposition 1.** There exists a unique monotone equilibrium with thresholds $\theta^*_L = 0$, $x^*_I = -\infty$, and $x^*_N = \infty$.

### 3.2 Rationalizable Behavior

We now turn to the question of which actions are rationalizable. Since our model is a two-stage game with observation actions, we resort to $\Delta$-rationalizability of Battigalli and Siniscalchi (2003), which extends Pearce’s (1984) notion of extensive-form rationalizability to games with incomplete information. The “$\Delta$” in $\Delta$-rationalizability indicates a specific set of restrictions on beliefs that is required to be satisfied at each round of the iterative procedure. In our case, it is the signal structure commonly known to all players. We will show that the unique monotone equilibrium presented in Proposition 1 is the unique $\Delta$-rationalizable (or, simply, rationalizable) strategy profile (and hence the unique equilibrium) when followers’ information is sufficiently imprecise; i.e., $\sigma_F$ is sufficiently large ($\sigma_F > 0.17$ for the case $n = 2$, for example). Otherwise, there are multiple rationalizable actions for some types of the leader and followers, implying the possibility of existing other equilibria in more complicated, non-monotone strategies.
Theorem 1. Consider the following procedure.

Let $\pi_L = \Theta \times D_L$ and $\pi_{F_j} = X_j \times S_j$ for each $j \in F$, where $S_j$ is the set of strategies $s_j(h)$ that maps each history $h \in D_L$ into an action $d_j \in D_j$.

(i) $\pi_L(\theta, d_L) \in R^k_L$ if and only if $(\theta, d_L) \in R^{k-1}_L$ and there exists a belief $\mu_L$ over $R^k_L$ such that $\mu_L(R^{k-1}_L) = 1$ and $d_L$ is a best response with respect to $\mu_L$ for type $\theta$ of the leader.

(ii) For every follower $j \in F$, $(x_j, s_j) \in R^{k-1}_F$ if and only if $(x_j, s_j) \in R^{k-1}_F$ and for each history $h$ there exists a belief $\mu_j(h)$ over $R^0_L \times R^0_{-j}$ such that $\mu_j(R^k_L \times R^{k-1}_{F,-j}|h) = 1$ and $s_j(h)$ is a best response with respect to $\mu_j(h)$ for type $x_j$.

Finally, let $R^0_F = \bigcap_{k=0}^\infty R^k_F$ and $R^0_{F,j} = \bigcap_{k=0}^\infty R^k_{F,j}$. Then an action $d_L$ is $\Delta$-rationalizable for type $\theta$ of the leader if $(\theta, d_L) \in R^0_F$. Analogously, a strategy $s_j$ is $\Delta$-rationalizable for type $x_j$ of follower $j$ if $(x_j, s_j) \in R^0_F$. We now illustrate intuitively how $\Delta$-rationalizability proceeds. First note that the payoff to investing for the leader satisfies the standard two-sided “limit dominance” property of global games (Morris and Shin, 2003) with the dominance regions being $(-\infty, 0)$ and $(1, \infty)$. That is, not investing is dominant for all types $\theta < 0$, and investing is dominant for all types $\theta > 1$. This implies that the leader will eliminate, in Round 1, all type-action pairs $(\theta, T)$ with $\theta < \theta^1_L = 0$ and $(\theta, N)$ with $\theta > \theta^1_L = 1$.

By knowing the leader’s lower dominance bound, $\theta^1_L$, and upper dominance bound, $\theta^1_L$, each follower can infer from $h = T$ that this decision must be made by a type $\theta > \theta^1_L$. This, in turn, determines each follower’s dominance regions. For type $x$ of a follower, not investing is never a best response for $\mathbb{E}_{\theta \sim \pi_T}(x, d_\theta^1) \leq 0$, and not investing is never a best response if $\mathbb{E}_{\theta \sim \pi_T}(x, d_\theta^1) > (n - 1)/n$. But since $\mathbb{E}_{\theta \sim \pi_T}(x, d_\theta^1) = \theta^1_L = 0$ (by Lemma 1), the subgame violates the two-sided limit dominance property because investing is not dominated for any type $x^1_L$. Set $x^1_L = -\infty$ and $x^1_T$ to be the unique solution to $\mathbb{E}_{\theta \sim \pi_T}(x, d_\theta^1) = (n - 1)/n$. Thus, they are the dominance bounds for all followers under history $h = T$ in Round 1.

Analogously, under history $h = N$, followers learn that it is a type $\theta \leq \theta^1_L$ that has made the decision. Since $\mathbb{E}_{\theta \sim \pi_N}(x, d_\theta^1) < 1$ for any type $x$, the corresponding subgame exhibits no upper dominance region and hence $x^1_N = \infty$. The lower dominance bound is given by the unique solution to $\mathbb{E}_{\theta \sim \pi_N}(x, d_\theta^1) = 1/n$. In sum, each follower $j$ will delete type-strategy pairs $(x, s_j)$ such that (i) $x > x^1_T$ and $s_j(T) = N$, and (ii) $x < x^1_N$ and $s_j(N) = I$. In Round 2, with the knowledge of $x^1_L$ and $x^1_N$, the payoff to investing for type $\theta$ of the leader is bounded above by $\pi_L(\theta; x^1_T)$ and bounded below by $\pi_L(\theta; x^1_N)$. The upper bound obtains

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11 See Baliga and Sjöström (2004) and Bueno de Mesquita (2010) for applications with one-sided limit dominance but different signal structures.
when type $\theta$ believes that all follower-types $x > x^1_I$ will choose to invest while the lower bound is associated with the belief that only type $x > x^1_I$ will invest. Note that $\pi_L(\theta; x^1_I) = \theta$; therefore it implies that the lower dominance bound is Round 2 is again $\theta^2_L = 0$. The upper dominance bound $\theta^2_L$ is the unique value of $\theta$ that solves $\pi_L(\theta; x^1_I) = 0$. Moreover, $\theta^2_L < \theta^1_L$ because $\pi_L(\theta; \hat{x}_I)$ is decreasing in $\hat{x}_I$. Now followers know that only types $\theta > \theta^2_L$ may invest and only types $\theta \leq \theta^2_L$ may not invest. Therefore, by a similar argument as detailed in Round 1, the dominance bounds are $x^2_h$ and $\hat{x}_h$.

The iteration procedure yields six sequences defined on the extended real line $\mathbb{R} = [-\infty, \infty]$. We summarize them in the following lemma.

**Lemma 4.** The following statements are true.

(a) $(\theta^k_L)_{k=0}^\infty$ is such that $\theta^k_L = \theta^*_L = 0$ for all $k \geq 1$;
(b) $(\theta^k_L)_{k=0}^\infty$ is a decreasing sequence;
(c) $(x^k_I)_{k=0}^\infty$ is such that $x^k_I = x^*_I = -\infty$ for all $k \geq 0$;
(d) $(\hat{x}^k_N)_{k=0}^\infty$ is a decreasing sequence;
(e) $(\hat{x}^k_N)_{k=0}^\infty$ is an increasing sequence;
(f) $(x^k_N)_{k=0}^\infty$ is such that $x^k_N = x^{**}_N = \infty$ for all $k \geq 0$.

Since we consider the sequences on $\mathbb{R}$, they always converge. We denote the limits in (b), (d), and (e) by $\theta^*_L$, $x^{**}_I$, and $x^{**}_N$, respectively. We now state the main result of the paper.

**Proposition 2.** There exists $\hat{\sigma}_F$ such that the unique monotone equilibrium is uniquely $\Delta$-rationalizable if and only if $\sigma_F > \hat{\sigma}_F$, where $\hat{\sigma}_F$ is the unique value of $\sigma_F$ that makes the following two equations

\[
\pi_F(x; 0, x) = 0 \\
\frac{d}{dx} \pi_F(x; 0, x) = 0
\]

have a unique solution.

Figure 2 illustrates how the (numerical) value of $\hat{\sigma}_F$ changes with the number of followers. In words, Proposition 2 means that when the leader has sufficient information advantage, imitating her action becomes the unique $\Delta$-rationalizable choice for all followers no matter what types they are. But when followers have precise information, multiple rationalizable strategy profiles occur. This requirement for uniqueness is surprising in the sense that, under history $h = I$, we do not obtain “limit uniqueness” but rather “limit multiplicity” of Bayesian Nash equilibria, contrary to the standard results in the global games literature.

To see why limit multiplicity presents itself, recall that we can write Equation (2) under $h = I$ as

\[
E_F \left( \frac{x}{\sigma_F} \right) = \frac{n - 1}{n} \left[ \frac{1}{2} \Phi \left( \frac{x}{\sigma_F} \right) \right]
\]

In the standard sense, this means either (i) the sequence converges to a real number, or (ii) it diverges to $-\infty$. 

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given $\theta^*_L = 0$. If there exists one Bayesian Nash equilibria other than the one stated in Proposition 1, Equation (5) must have at least one solution. Observe that the noise, $\sigma_F$, affects both the expected gross return and the conditional rank beliefs. Moreover, the former is increasing in $\sigma_F$. In contrast, the latter does not change monotonically with $\sigma_F$ but features rapid slope change around $\theta^*_L$. For large values of $\sigma_F$, the expected gross return dominates the conditional rank belief, implying that Equation (5) has no solution. This means that $x^*_I \to x^*_L$ as $k \to \infty$ and hence investing the unique rationalizable action in the subgame following $h = I$. See Figure 3 for an illustration. As a consequence, $\theta^*_L = \theta^{**}_L$, $x^*_N = x^{**}_N$, and the monotone equilibrium becomes the unique $\Delta$-rationalizable strategy profile.

On the contrary, for small values of $\sigma_F$ (i.e., $\sigma_F \leq \hat{\sigma}_F$), the expected gross return stops to play a dominant role and hence Equation (5) has multiple solutions (see Figure 4). In particular, the largest solution corresponds to the limit $x^{**}_I$ to which the sequence $(\tilde{x}^k_I)$ will converge. In fact, the monotone strategy profiles with thresholds $x^*_I$ and $x^{**}_I$, respectively, are the least and greatest Bayesian Nash equilibria that bound all rationalizable strategies in the sub-game that follows action $I$ of the leader (Van Zandt and Vives, 2007). An interesting aspect of the model is that $R^I(x^{**}_I, \theta^*_L) \to 1/2$ as $\sigma_F \to 0^+$. This “uniform limit rank belief” (Morris and Yildiz, 2019) guarantees that the monotone strategy profile with threshold $x^{**}_I$ is a “risk dominant” equilibrium (Harsanyi and Selten, 1988). That is, a follower will only invest if investing is a best response to a uniform belief over other followers choosing each action. However, it is not supported in a monotone equilibrium of the whole game, as proved in Lemma 5. When followers play according to the threshold $x^{**}_I$, the leader will best respond by using a monotone strategy with threshold $\theta^{**}_L > \theta^*_L$, which is the limit of $\theta^k_L$ as $k \to \infty$. Thus any leader type

**Figure 2** $\hat{\sigma}_F$ increases with $n$. 

![Figure 2](image.png)
Figure 3 Unique Bayesian Nash equilibrium when $\sigma_F > \hat{\sigma}_F$ ($n = 4$).

$\theta \in (\theta_L^*, \theta_L^{**})$ will find both actions rationalizable.

Figure 4 Multiple Bayesian Nash equilibria when $\sigma_F \leq \hat{\sigma}_F$ ($n = 4$).

3.3 Discussion

Equilibrium Selection

When the unique $\Delta$-rationalizable strategy profile obtains, all followers resolve their strategic uncertainty and assign probability one to others investing if the leader invests and not investing
if the leader chooses not to invest. That is, \( R^I(x, \theta^*_L) = 0 \) and \( R^N(x; \theta^*_L) = 1 \) for all \( x \). This result has a direct implication for equilibrium selection, referred to as Stackelberg selection. Note that the unique equilibrium features that all players invest when \( \theta > \theta^*_L \) = 0 and no one invests when \( \theta \leq \theta^*_L \), and therefore it is tantamount to the fully efficient subgame perfect equilibrium of the corresponding complete information game. Let \( \Gamma(\sigma_F) \) denote the model described in Section 2.

**Corollary 1** (Stackelberg Selection). Fix \( \eta > 0 \). For any sequence of games \( \Gamma(\sigma^k_F) \) where \( \sigma^k_F \to \hat{\sigma}_F + \eta \), any sequence of equilibria in those games converge to the fully efficient subgame perfect equilibrium of the complete information game.

**Peer-confirming Equilibrium**

Although our model features incomplete information about a fundamental state, the result is consistent with the prediction delivered by the peer-confirming equilibrium of Lipnowski and Sadler (2019). In a leader-centered star network where all followers observe the leader’s strategy, Lipnowski and Sadler (2019) argue that “imitation” may arise as the unique extensive-form peer-confirming equilibrium since followers do not observe any information that could contradict the leader’s rationality. The leader, thus, have the advantage of inducing others to imitate her behavior. In this regard, Proposition 2 deals with the interplay between strategic uncertainty and fundamental uncertainty by identifying a necessary and sufficient condition under which the unique peer-confirming equilibrium highlighting imitation arises.

### 4 Alternative Information Structure

Suppose, now, that instead of perfectly learning the state \( \theta \), the leader observes a noisy signal \( x_L = \theta + \sigma_L \varepsilon_L \) with \( \varepsilon_L \) being a standard Gaussian noise independent of \( \theta \) and \( \varepsilon_j \) for any \( j \in F \). This generalization is important from a global games perspective, since perturbations that remove common knowledge of the fact that the leader is perfectly informed may affect the results of the previous analysis.

Following the structure of Section 3, we focus on monotone equilibria before diving into the discussion of rationalizable behavior. Call \( x_L \in X_L = \mathbb{R} \) the leader’s type. A monotone strategy for the leader is now a mapping \( s_L : X_L \to D_L \) such that \( s_L(x_L) = I \) if \( x_L > \tilde{x}_L \) and \( s_L(x_L) = N \) if \( x_L \leq \tilde{x}_L \). Since there is no learning for the leader, type \( x_L \)’s posterior belief about \( \theta \) follows a Gaussian distribution with mean \( x_L \) and variance \( \sigma^2_L \).

Let \( \hat{x}_L, \tilde{x}_L, \text{ and } \hat{x}_N \) be candidates for the equilibrium thresholds and consider type \( x \) of
follower \( j \in F \). Upon observing \( h = I \), type \( x \)'s posterior belief has a CDF

\[
G^I(\theta; x, \hat{x}_L) = \frac{1}{\Phi\left(\frac{x - \hat{x}_L}{\sigma_F}\right)} \int_0^1 \frac{1}{\sigma_F} \phi\left(\frac{t - x}{\sigma_F}\right) \Phi\left(\frac{t - \hat{x}_L}{\sigma_L}\right) dt, \tag{6}
\]

where \( \sigma^2 = \sigma_F^2 + \sigma_L^2 \). Similarly, type \( x \)'s posterior CDF under history \( h = N \) is

\[
G^N(\theta; x, \hat{x}_L) = \frac{1}{\Phi\left(\frac{\hat{x}_L - x}{\sigma_F}\right)} \int_0^1 \frac{1}{\sigma_F} \phi\left(\frac{t - x}{\sigma_F}\right) \Phi\left(\frac{\hat{x}_L - t}{\sigma_L}\right) dt. \tag{7}
\]

Unlike the posterior beliefs in the main model, \( G^h(\theta; x, \hat{x}_L) \) has support over the entire real line.

At a given state \( \theta \), each follower invests with probability \( 1 - \Phi((\theta - \hat{x}_h)/\sigma_F) \) under history \( h \). This implies that type \( x \)'s payoff to investing yields

\[
\pi^h_L(x_j; \hat{x}_L, \hat{x}_h) = \mathbb{E}_{\theta \sim G^h(\cdot, x, \hat{x}_L)} \left[ \theta - \frac{n - 1}{n} \Phi\left(\frac{\hat{x}_h - \theta}{\sigma_F}\right) \right] = \frac{x_N}{n}.
\]

We will show in the Appendix that \( \pi^h_L(x; \hat{x}_L, \hat{x}_h) \) is increasing in \( x \) and crosses zero once from below. Furthermore, it is increasing in \( \hat{x}_L \) while decreasing in \( \hat{x}_h \). This ensures that follower \( j \)'s best response is a monotone strategy. We now define type \( x \)'s conditional rank belief in a similar fashion as in Section 3

\[
R^I(x; \hat{x}_L) = \Pr(x_k < x_j \mid x_j = x, x_L > \hat{x}_L) = \frac{1}{2} - \frac{T\left(\frac{x - \hat{x}_L}{\sigma_L}, \alpha\right)}{\Phi\left(\frac{x - \hat{x}_L}{\sigma_L}\right)},
\]

\[
R^N(x; \hat{x}_L) = \Pr(x_k < x_j \mid x_j = x, x_L \leq \hat{x}_L) = \frac{1}{2} + \frac{T\left(\frac{x - \hat{x}_L}{\sigma_L}, \alpha\right)}{\Phi\left(\frac{x - \hat{x}_L}{\sigma_L}\right)},
\]

where \( \alpha = \sigma_F / \sqrt{2\sigma_L^2 + \sigma_F^2} \) and \( T(y, a) \) is the Owen's T-function.\(^\text{13}\) Note that \( T(y, a) \) belongs to the family of bell-shaped functions and, accordingly, \( 1/2 - T(y, a) / \Phi(y) \) is a sigmoid function with horizontal asymptotes at 0 and 1/2. In the Bayesian Nash equilibria of the subgame following history \( h \), we must have

\[
\mathbb{E}_{\theta \sim G^h(\cdot, x, \hat{x}_L)} [\theta] = \frac{n - 1}{n} R^h(\hat{x}_h; \hat{x}_L) + \frac{x_N}{n}. \tag{8}
\]

Now we turn to the leader. The payoff to investing for type \( x_L \) is

\[
\pi^h_L(x_L; \hat{x}_L) = x_L - \Phi\left(\frac{\hat{x}_L - x_L}{\sigma}\right).
\]

\(^\text{13}\) Owen’s T-function, first introduced by Owen (1956), is defined by

\[
T(y, a) = \frac{1}{2\pi} \int_0^a e^{-(1+t^2)y^2/2} \frac{1 + t^2}{1 + t^2} dt.
\]

It gives the probability of the event \( \{ X > y, 0 < Y < aX \} \) when \( X \) and \( Y \) are independent standard Gaussian random variables. See Savischenko (2014) and Brychkov and Savischenko (2016) for an overview of the function.
It is increasing in $x_L$ and is decreasing in $\hat{x}_I$. It follows that the leader’s best response is also a monotone strategy. In equilibrium, the threshold $\hat{x}_L$ is such that

$$\hat{x}_L = \Phi \left( \frac{\hat{x}_I - \hat{x}_L}{\sigma} \right).$$

(9)

**Proposition 3.** There exists a unique monotone equilibrium with thresholds $x^*_L$, $x^*_I$, and $x^*_N$ that simultaneously solve (8) and (9). Moreover, this equilibrium converges to that of Proposition 1 when $\sigma_L \to 0^+$ (while keep $\sigma_F$ fixed), or when $\sigma_L \to 0^+$, $\sigma_F \to 0^+$ and $\frac{\sigma_L}{\sigma_F} \to 0^+$.

The monotone equilibrium is trivially a sequential equilibrium à la Kreps and Wilson (1982) because all players deem every state possible. For a given pair of $(\sigma_L, \sigma_F)$, note that we can view it as a point on the ray from the origin with slope $\sigma_F/\sigma_L$. To understand which pair induces a unique rationalizable behavior, we establish a sufficient condition on each fixed ray $\sigma_F = \gamma \sigma_L$, $\gamma \geq 0$, along which the slope parameter of the Owen’s T-function is $\alpha = \gamma / \sqrt{2 + \gamma^2}$.

**Proposition 4.** The unique monotone equilibrium is uniquely $\Delta$-rationalizable if

$$\sigma_L \geq \bar{\sigma}_L(\gamma) = \left( \frac{n-1}{n} \right) \sqrt{(1 + \gamma^2)\Lambda(\gamma)^2},$$

where

$$\Lambda(\gamma) = \arg \max_w \frac{S'(w)}{1 + \gamma^2(1 + \lambda'(w))}.$$

Figure 5 shows the curve $\bar{\sigma}_L(\gamma)$ for different $n$. In each case, any pair of $(\sigma_F, \sigma_L)$ that lies outside $\bar{\sigma}_L(\gamma)$ begets the uniqueness of rationalizable behavior. Proposition 4 is an analog of Proposition 2 when the leader observes a noisy signal of the state. In this case, both subgames feature dominance regions, which now depend on the threshold $\hat{x}_L$ used by the leader. This, in turn, ensures the existence of a monotone equilibrium with finite thresholds. The spirit of the condition we identified is similar to the one in Section 3. Specifically, in order to guarantee uniqueness of rationalizable behavior, one must ensure that the investment subgame has a unique Bayesian Nash equilibrium.

### 4.1 Discussion

Proposition 4 is an analog of Proposition 2 when the leader observes a noisy signal of the state. In this case, both subgames feature dominance regions, which now depend on the threshold $\hat{x}_L$ used by the leader. This, in turn, ensures the existence of a monotone equilibrium with finite thresholds. The spirit of the condition we identified is similar to the one in Section 3. Specifically, in order to guarantee uniqueness of rationalizable behavior, one must ensure that the investment subgame has a unique Bayesian Nash equilibrium, as illustrated in Figures 6 and 7. For this to happen, it must be the case that the derivative of the conditional rank belief
Figure 5 Curve $\sigma_\gamma L$ in the $(\sigma_L, \sigma_F)$-space.

function is sufficiently bounded so that the expected payoff of the threshold type of a follower has a unique solution. Like in the previous section, this is not generally the case, since around $\hat{x}_L$, for certain values of $\sigma_L$ and $\sigma_F$, the rank belief function abruptly changes from 0 to 1/2, which means that the expected payoff of the threshold type of a follower $j$ rapidly decreases. The condition of Proposition 4 ensures that this rapid decrease is not enough to make the expected payoff of follower $j$ change sign more than once, or, equivalently, Equation 8 has a unique solution (Figure 7).

Figure 6 Multiple Bayesian Nash equilibria ($n = 4, \sigma_L = 0.05, \sigma_F = 0.07$)
Figure 7 Unique Bayesian Nash equilibrium ($n = 4, \sigma_L = 0.2, \sigma_F = 0.25$)

Inefficiency of the Unique Outcome

Contrary to the main model, the more general model features an inefficient outcome irrespective of the uniqueness of rationalizable play\textsuperscript{14} In this case, the most (but not fully) efficient outcome is the monotone equilibrium derived in Proposition\textsuperscript{3}. It is worth noting, though, that in all cases, the extensive form game leads to outcomes at least as efficient as the ones that would obtain if the game was a simultaneous move game, a prediction consistent with existing literature.

5 Conclusion

In this paper we have investigated the signaling role that a leader’s actions may have in a coordination game modeled in a similar spirit to the global games framework. Due to the departure from the standard static framework and the endogenous information structure that arises, we have illustrated that rationalizable behavior is generally not unique. We have been able to identify conditions that guarantee such uniqueness in the case where the leader is perfectly informed about the underlying state of uncertainty, as well as in the case where the leader observes the state with noise. Our results illustrate that even though signaling can lead to multiplicity, this is not always the case. Moreover, when uniqueness can be attained, the outcome is fully efficient, when the leader knows the state, and “almost efficient”, when the leader observes a noisy signal about the state.

\textsuperscript{14} This result obtains whenever $\sigma_L$ is bounded away from zero. In the limiting case where $\sigma_L \to 0^+$ and for $\sigma_F$ sufficiently large we recover the unique efficient $\Delta$-rationalizable profile of the main model.
Our results call into question the robustness of the global games approach when one moves away from the static benchmark as well as in the case where information is endogenous. Such cases may naturally arise in applications. While it has been shown in the literature that both cases may lead to multiplicity of equilibria or rationalizable play, it is not well understood when this will happen or whether there exist conditions (and what they look like) that exclude these scenarios. We have been able to shed some light on these questions. Our framework, though, is not fully flexible since we assume exogenous timing and a specific time when each player can act. Thus, we believe that deriving similar results in more general situations is a fruitful avenue for future research. Moreover, from a more theoretical perspective, our analysis challenges the extent to which known results about static monotone Bayesian supermodular games extend to multistage settings. We think that more work is needed into this direction.

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**Appendix**

**Proof of Lemma**

First note that, conditional on $x$, type $x$’s belief about $\theta$ has a Gaussian distribution with mean $x$ and variance $\sigma_F^2$. Recall that $\lambda(x) = \phi(x)/\Phi(x)$ is the reverse hazard rate of a standard Gaussian random variable. Rewrite Equation (1) as follows:

$$
E_{\theta \sim \Psi^h (\cdot; x, \hat{\theta}_L)} [\theta] = \begin{cases} 
\sigma_F [z + \lambda(z)] + \hat{\theta}_L & \text{if } h = \mathcal{I} \\
-\sigma_F [-z + \lambda(-z)] + \hat{\theta}_L & \text{if } h = \mathcal{N}
\end{cases}
$$

where $z = (x - \hat{\theta}_L)/\sigma_F$. Since Gaussian distributions are log-concave, $x + \lambda(x)$ is increasing in $x$ (Bagnoli and Bergstrom 2005, Theorem 5). It is now straightforward to see that $E_{\theta \sim \Psi^h (\cdot; x, \hat{\theta}_L)} [\theta]$ is increasing in $x$ and $\hat{\theta}_L$ because $\lambda(\cdot)$ is a decreasing function and $\sigma_F > 0$.

Using the derivative formula $\phi'(x) = -x\phi(x)$ and L’Hôpital’s rule, we have $\lim_{x \to -\infty} x + \lambda(x) = \lim_{x \to -\infty} \phi(x)/\phi'(x) = 0$ and hence $x + \lambda(x) > 0$ for all $x \in \mathbb{R}$. We also have $\lim_{x \to \infty} x + \lambda(x) = \infty$ because $\lim_{x \to \infty} \lambda(x) = 0$. Thus,

$$
E_{\theta \sim \Psi^\mathcal{I} (\cdot; x, \hat{\theta}_L)} [\theta] \to \begin{cases} 
\hat{\theta}_L & \text{as } x \to -\infty \\
\infty & \text{as } x \to \infty
\end{cases}
$$

and

$$
E_{\theta \sim \Psi^\mathcal{N} (\cdot; x, \hat{\theta}_L)} [\theta] \to \begin{cases} 
-\infty & \text{as } x \to -\infty \\
\hat{\theta}_L & \text{as } x \to \infty
\end{cases}
$$

This completes the proof. \qed
Proof of Lemma 2

We first prove that \( \Psi^h(\theta; x, \hat{\theta}_L) \) is increasing in \( x \) and \( \hat{\theta}_L \) in the sense of first-order stochastic dominance. Given \( h = I \). For \( x' > x \), we have

\[
\frac{\psi^I(\theta; x', \hat{\theta}_L)}{\psi^I(\theta; x, \hat{\theta}_L)} = \frac{\Phi\left(\frac{\theta - x'}{\sigma_F}\right)}{\Phi\left(\frac{\theta - x}{\sigma_F}\right)} \cdot \frac{\Phi\left(\frac{x - \hat{\theta}_L}{\sigma_F}\right)}{\Phi\left(\frac{x' - \hat{\theta}_L}{\sigma_F}\right)}.
\]

Since \( \phi(\cdot) \) is log-concave, \( \phi((\theta - x')/\sigma_F)/\phi((\theta - x)/\sigma_F) \) is increasing in \( \theta \) (Saumard and Wellner, 2014, Proposition 2.3) and so is \( \psi^I(\theta; x', \hat{\theta}_L)/\psi^I(\theta; x, \hat{\theta}_L) \). This implies that \( \psi^I(\theta; x, \hat{\theta}_L) \) is log-supermodular in \((\theta, x)\) or, equivalently, \( \psi^I(\theta; x', \hat{\theta}_L) \) dominates \( \psi^I(\theta; x, \hat{\theta}_L) \) in the monotone likelihood ratio order. Thus, \( \Psi^I(\theta; x', \hat{\theta}_L) \) first-order stochastically dominates \( \Psi^I(\theta; x, \hat{\theta}_L) \).

By definition, we have

\[
\Psi^I(\theta; x, \hat{\theta}_L) = 1 - \frac{\Phi\left(\frac{x - \theta}{\sigma_F}\right)}{\Phi\left(\frac{x - \hat{\theta}_L}{\sigma_F}\right)}.
\]

It is decreasing in \( \hat{\theta}_L \); that is, \( \Psi^I(\theta; x, \hat{\theta}'_L) < \Psi^I(\theta; x, \hat{\theta}_L) \) for \( \hat{\theta}'_L > \hat{\theta}_L \) and for all \( \theta \in (\hat{\theta}_L, \infty) \), where \( \Psi^I(\theta; x, \hat{\theta}'_L) = 0 \) when \( \theta \in (\hat{\theta}_L, \hat{\theta}'_L] \). This proves that \( \Psi(\theta; x, \hat{\theta}_L) \) dominates \( \Psi(\theta; x, \hat{\theta}_L) \) in the first-order stochastic dominance sense. The proof for \( \Psi^N(\theta; x, \hat{\theta}_L) \) is analogous. It is now immediate that \( \pi^h_F(x; \hat{\theta}_L, \hat{x}_h) \) is increasing in \( x \) and \( \hat{\theta}_L \) because \( \theta - ((n - 1)/n)\Phi(\hat{x}_h - \theta)/\sigma_F \) is increasing in \( \theta \).

The limiting results follow directly from \( \psi^h(\theta; x, \hat{\theta}_L) \to 0 \) as \( x \to -\infty \) and \( \psi^h(\theta; x, \hat{\theta}_L) \to 0 \) as \( x \to \infty \). \( \square \)

Proof of Lemma 3

We prove by contradiction. Suppose that \( \hat{\theta}_L \) and \( \hat{x}_I \) solve Equations (2) and (3). By Equation (1) we can write Equation (2) as

\[
\hat{x}_I + \sigma_F \lambda \left(\frac{\hat{x}_I - \hat{\theta}_L}{\sigma_F}\right) = \frac{n - 1}{2n} \Phi\left(\frac{\hat{x}_I - \hat{\theta}_L}{\sigma_F}\right). \tag{A1}
\]

Subtracting Equation (3) from Equation (A1) yields

\[
\hat{x}_I - \hat{\theta}_L + \sigma_F \lambda \left(\frac{\hat{x}_I - \hat{\theta}_L}{\sigma_F}\right) = -\frac{n + 1}{2n} \Phi\left(\frac{\hat{x}_I - \hat{\theta}_L}{\sigma_F}\right). \tag{A2}
\]

Note that \( x + \lambda(x) \) is increasing in \( x \) with \( \lim_{x \to -\infty} x + \lambda(x) = 0 \), and hence \( x + \lambda(x) > 0 \) for all \( x \). This implies that the left-hand side of (A2) is positive. But since the right-hand side of (A2) is negative, this gives a contradiction. \( \square \)
Proof of Proposition

Fix type $x$ of follower $j \in F$. Suppose that the leader uses threshold $\theta^*_L = 0$ and other followers use thresholds $x^*_I = -\infty$ and $x^*_N = \infty$. If the leader invests, then type $x$'s payoff to investing is

$$\pi^*_F(x; \theta^*_L, x^*_I) = \mathbb{E}_{\theta \sim \Psi^I(x; \theta^*_L)}[\theta] > 0.$$  

This means that type $x$ will always invest under history $h = I$. Since $x$ is chosen arbitrarily, follower $j$'s best response is a monotone strategy with threshold $x^*_I = -\infty$. In contrast, if the leader does not invest, then the payoff to investing for type $x$ yields

$$\pi^*_F(x; \theta^*_L, x^*_N) = \mathbb{E}_{\theta \sim \Psi^I(x; \theta^*_L)}\left[\theta - \frac{n-1}{n}\right] < 0.$$  

Thus, under history $h = N$, follower $j$ will best respond by using a monotone strategy with threshold $x^*_N = \infty$.

Consider now type $\theta$ of the leader. Since all followers will invest if they see the leader invests, investing generates a payoff of $\theta$ for type $\theta$. Therefore, type $\theta$ invests if and only if $\theta > 0$. In other words, the leader will best response by choosing threshold $\theta^*_L = 0$. The proof is complete.

Proof of Lemma

Define iteratively six sequences on $\bar{\mathbb{R}}$ as follows. Let $\theta^0_L = x^0_I = x^0_N = -\infty$ and $\theta^0_I = \bar{x}^0_I = \bar{x}^0_N = \infty$, and for $k \geq 1$,

$$\begin{align*}
\theta^k_L &= \text{BR}_L(\theta^{k-1}_L) \\
\theta^k_I &= \text{BR}_I(\theta^{k-1}_I) \\
x^k_I &= \text{BR}^F_I(\theta^k_L, \theta^{k-1}_I) \\
x^k_N &= \text{BR}^N_I(\theta^k_L, \theta^{k-1}_I) \\
\Delta^k_N &= \text{BR}^N_F(\theta^k_L, x^{k-1}_I) \\
\bar{x}^k_N &= \text{BR}^N_F(\theta^k_L, x^{k-1}_I)
\end{align*}$$

where $\text{BR}_L(x)$, $x \in \bar{\mathbb{R}}$, is the unique solution of $\theta$ to

$$\pi_L(\theta; x) = \theta - \Phi\left(\frac{x - \theta}{\sigma_F}\right) = 0,$$

and, for each $h$, $\text{BR}^h_F(\theta', x')$, $(\theta', x') \in \bar{\mathbb{R}}^2$, is the unique value of $x$, if exists, that solves

$$\pi^h_F(x; \theta', x') = \mathbb{E}_{\theta \sim \Psi^h(x; \theta')}\left[\theta - \frac{n-1}{n} \Phi\left(\frac{x' - \theta}{\sigma_F}\right) - \frac{\chi_N}{n}\right] = 0;$$
otherwise
\[ \text{BR}_F^k(\theta', x') = \begin{cases} -\infty & \text{if } h = \mathcal{I} \\ \infty & \text{if } h = \mathcal{N}. \end{cases} \]

Now we prove statements (a)-(f) as follows.

(a) \& (c): Since \( x_0^i = -\infty \), \( \theta_0^i = 0 \). It follows that \( x_1^i = -\infty \) because \( \mathbb{E}_{\theta \sim \Psi F(x; \theta_1^i)}[\theta] > 0 \) for all \( x \). Thus, it is immediate that \( \theta_1^i = 0 \) and \( x_2^i = \infty \) for all \( k \geq 1 \).

(d): Given \( \theta_1^i = 0 \), we have \( x_1^i < x_2^i \) because the former solves
\[ \mathbb{E}_{\theta \sim \Psi F(x; \theta_1^i)}[\theta] = \frac{n - 1}{n}. \]
Since Lemma 2 implies that \( \pi_F^n(x; \theta', x') \) is decreasing \( x' \), we can conclude that \((x_2^i)^\infty_{k=0} \) is a decreasing sequence.

(b): The proof immediately follows from (d) and \( \pi_L(x; \theta') \) being decreasing in \( x \).

(e): By Lemma 2, \( \pi_F^n(x; \theta', x') \) is increasing in \( \theta' \) and decreasing in \( x' \). Then (b) and (d) together give the proof.

(f): We prove by induction. Note that \( x_0^i = \infty \) implies that \( \theta_1^i = 1 \). But
\[ \pi_F^n(x; \theta_1^i, x_0^i) = \mathbb{E}_{\theta \sim \Psi F(x; \theta_1^i)}[\theta] - 1 < 0; \]
so \( x_1^i = \infty \). Suppose that \( x_k^i = \infty \) for \( k \geq 2 \). Since \( \theta_L^k < \theta_L^k \),
\[ \pi_F^n(x; \theta_L^k, \theta_0^i) < \theta_L^k - 1 < \theta_L^1 - 1 = 0. \]
This implies that \( x_0^i = \infty \).

\[ \square \]

**Proof of Proposition 2**

Let \( \theta_L = 0 \) and consider history \( h = \mathcal{I} \). Then we can write Equation (2) as
\[ x = \frac{n - 1}{2n} \Phi \left( \frac{x}{\sigma_F} \right) - \sigma_F \lambda \left( \frac{x}{\sigma_F} \right). \tag{A.3} \]
Define \( \rho(x, \sigma_F) \) to be the right-hand side of Equation (A.3). Observe that \( \partial \rho(x, \sigma_F)/\partial x > 0 \), \( \lim_{x \to \infty} \rho(x, \sigma_F) = (n - 1)/(2n) \), \( \lim_{x \to -\infty} \rho(x, \sigma_F) = 1 \), and \( x > \rho(x, \sigma_F) \) for \( x < 0 \) but \( |x| \) sufficiently large. Let \( \bar{x} = (n - 1)/(8n\phi(0)) \). It suffices to consider the following two cases.

**Case 1:** Suppose that \( \sigma_F \leq \bar{x} \). It is equivalent to \( \rho(0, \sigma_F) \geq 0 \). Since
\[ \frac{\partial \rho(0, \sigma_F)}{\partial x} = \frac{n - 1}{2n\sigma_F} \phi(0) - \lambda'(0) \geq 8\phi(0)^2 > 1 \]
by the derivative formula \( \lambda'(x) = -\lambda(x) [x + \lambda(x)] \), Equation (A.3) must have at least two solutions.
Case 2: Suppose that $\sigma_F > \bar{\sigma}_F$. For $x > 0$,

$$
\frac{\partial \rho(x, \sigma_F)}{\partial \sigma_F} = -\frac{x}{\sigma_F} \left( \frac{\partial \rho(x, \sigma_F)}{\partial x} \right) < 0,
$$

(A.4)

and

$$
\frac{\partial^2 \rho(x, \sigma_F)}{\partial x^2} = \frac{n - 1}{2n \sigma_F^2} \phi' \left( \frac{x}{\sigma_F} \right) - \frac{1}{\sigma_F} \lambda'' \left( \frac{x}{\sigma_F} \right) < 0
$$

(A.5)

because $\lambda''(x) > 0$. By (A.4) and (A.5), there exists a unique $\tilde{\sigma}_F$ such that $x$ and $\rho(x, \tilde{\sigma}_F)$ are tangent to each other at $x = x^{**} > 0$. Moreover, for $x > 0$, $x > \rho(x, \sigma_F)$ if and only if $\sigma_F > \tilde{\sigma}_F$. If we can prove that $x > \rho(x, \sigma_F)$ for $x \leq 0$ whenever $\sigma_F > \tilde{\sigma}_F$, then we are done. Note that, at $\sigma_F = \tilde{\sigma}_F$,

$$
\rho(x, \tilde{\sigma}_F) - x = \bar{\sigma}_F \left[ 2\lambda(0)\Phi(z) - \lambda(z) - z \right] < 0,
$$

where $z = x / \bar{\sigma}_F$ because $\lambda(0) = 2\phi(0)$ and $2\lambda(0)\Phi(x) - \lambda(x) - x < 0$ for $x < 0$. It follows that $x > \rho(x, \sigma_F)$ for all $x \leq 0$. Thus, (A.3) can only have solutions if $\sigma_F \leq \tilde{\sigma}_F$.

Taking Case 1 and Case 2 together, we can conclude that (A.3) has no solution if and only if $\sigma > \tilde{\sigma}_F$. □