Adjoint difference equation for the
Nikiforov–Uvarov–Suslov difference equation of
hypergeometric type on non-uniform lattices

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Abstract
In this article, we obtain the adjoint difference equation for the Nikiforov–Uvarov–
Suslov difference equation of hypergeometric type on non-uniform lattices, and prove
it to be a difference equation of hypergeometric type on non-uniform lattices as well.
The particular solutions of the adjoint difference equation are then obtained. As an
application of these particular solutions, we use them to obtain the particular solutions
for the original difference equation of hypergeometric type on non-uniform lattices.
In addition, we give another kind of fundamental theorems for the Nikiforov–Uvarov–
Suslov difference equation of hypergeometric type, which are essentially new results
and their expressions are different from the Suslov Theorem. Finally, we give an
example to illustrate the application of the new fundamental theorems.

Keywords Special function · Orthogonal polynomials · Adjoint equation ·
Difference equation of hypergeometric type · Non-uniform lattice

Mathematics Subject Classification 33D20 · 33D45 · 33C45

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1 Introduction

The differential equation of hypergeometric type:

\[ \sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0, \]  

(1)

where \( \sigma(z) \) and \( \tau(z) \) are polynomials of degrees at most two and one, respectively, and \( \lambda \) is a constant, has attracted great attention, since its solutions are certain types of special functions of mathematical physics, such as the classical orthogonal polynomials, the hypergeometric and cylindrical functions. In particular, for some positive integer \( n \) such that \( \lambda = -\frac{n(n-1)}{2} \sigma''(z) - n \tau'(z) \) and \( \lambda_m \neq \lambda_n \) for \( m = 0, 1, \ldots, n - 1 \), Eq. (1) has a polynomial solution \( y_n(z) \) of degree \( n \), which can be expressed by the Rodrigues formula \(^{[1–6]}\) as

\[ y_n(z) = \frac{1}{\rho(z)} \frac{d^n}{dz^n} (\rho(z) \sigma^n(z)), \]  

(2)

where \( \rho(z) \) satisfies the Pearson equation

\[ (\sigma(z) \rho(z))' = \tau(z) \rho(z). \]  

(3)

These solution functions are useful in quantum mechanics, the theory of group representations, and computational mathematics. Because of this, the classical theory of hypergeometric type equations has been greatly developed by Andrews and Askey \(^{[5,6]}\), Wilson and Ismail \(^{[7–10]}\); Nikiforov et al. \(^{[1–3,11–13]}\); George and Rahman \(^{[14]}\); Koornwinder \(^{[15]}\); and many other researchers \(^{[16–27]}\). On the other hand, many researchers like Álvarez-Nodarse et al. \(^{[28–30]}\) studied particular solutions for the adjoint differential equation of Eq. (1) as

\[ \sigma(z)w''(z) + (2\sigma'(z) - \tau(z))w'(z) + (\lambda + \sigma''(z) - \tau'(z))w(z) = 0, \]  

(4)

or the alternative one as

\[ \sigma(z)y''(z) - \tau_{-2}(z)y'(z) + (\lambda - \kappa_{-1})y(z) = 0, \]  

(5)

where

\[ \tau_v(z) = \tau(z) + v \sigma'(z), \kappa_v = \tau' + \frac{1}{2} (v - 1) \sigma''. \]  

(6)

Nikiforov et al. \(^{[1,2]}\) generalized Eq. (1) to a difference equation of hypergeometric type case and studied the Nikiforov–Uvarov–Suslov difference equation on a lattice \( x(s) \) with variable step size \( \nabla x(s) = x(s) - x(s - 1) \) as

\[ \tilde{\sigma}[x(s)] \frac{\Delta}{\Delta x(s - 1/2)} \left[ \frac{\nabla y(s)}{\nabla x(s)} \right] + \frac{1}{2} \tau[x(s)] \left[ \frac{\Delta y(s)}{\Delta x(s)} + \frac{\nabla y(s)}{\nabla x(s)} \right] + \lambda y(s) = 0, \]  

(7)
where $\tilde{\sigma}(x)$ and $\tilde{\tau}(x)$ are polynomials of degrees at most two and one in $x(s)$, respectively, $\lambda$ is a constant, $\Delta y(s) = y(s + 1) - y(s)$, $\nabla y(s) = y(s) - y(s - 1)$, and $x(s)$ is a lattice function that satisfies

$$\frac{x(s + 1) + x(s)}{2} = \alpha x \left( s + \frac{1}{2} \right) + \beta, \alpha, \beta \text{ are constants,} \tag{8}$$

$$x^2(s + 1) + x^2(s) \text{ is a polynomial of degree at most two w.r.t. } x \left( s + \frac{1}{2} \right). \tag{9}$$

It should be pointed out that Eq. (7) was obtained as a result of approximating Eq. (1) on a non-uniform lattice $x(s)$. Here, two kinds of lattice functions $x(s)$ called *non-uniform lattices* which satisfy the conditions in Eqs. (8) and (9) are

$$x(s) = c_1 q^s + c_2 q^{-s} + c_3, \tag{10}$$

$$x(s) = \tilde{c}_1 s^2 + \tilde{c}_2 s + \tilde{c}_3, \tag{11}$$

where $c_i, \tilde{c}_i$ are arbitrary constants and $c_1 c_2 \neq 0, \tilde{c}_1 \tilde{c}_2 \neq 0$.

It was found that Eq. (7) is of independent importance and the equation raises many interesting questions. Its solutions essentially generalize the solutions of the original differential equation and are of interest in their own selves. Some of its solutions have been used in quantum mechanics, the theory of group representations, and computational mathematics [1,2]. In particular, Suslov [3] established an analogous fundamental result for the difference equation on non-uniform lattices, which generalizes the Rodrigues formula for polynomial solutions on non-uniform lattices.

We should mention that the adjoint difference equation of Eq. (7) for the case of non-uniform lattices is also of independent importance itself and the adjoint equation may raise certain other interesting questions. For example, it could help us to obtain the particular solutions for the difference equation of hypergeometric type in Eq. (7) on non-uniform lattices, or obtain an extension of the Rodrigues formula in the non-uniform lattice case, etc. To our best knowledge, the adjoint difference equation of Eq. (7) for the case of uniform lattices such as $x(s) = s$ and $x(s) = q^s$ has already been obtained in [28–30]. However, for the case of non-uniform lattices where $x(s)$ is defined in Eqs. (10) or (11), it is more complicated and difficult to establish and simplify and then solve the adjoint equations on the non-uniform lattice, and as a result, the related study has not been done yet.

Following all the work surveyed in [3], our purpose in this article is to establish an adjoint difference equation for the difference equation in Eq. (7) on non-uniform lattices where $x(s)$ is given in Eqs. (10) or (11). We will prove that the adjoint difference equation is still a difference equation of hypergeometric type on non-uniform lattices, and then obtain the particular solutions of the new adjoint equation. In addition, we will prove other kinds of fundamental theorems for the Nikiforov–Uvarov–Suslov difference equation of hypergeometric type in Eq. (7), which are essentially new results. Their expressions are different from the Suslov theorem (as seen in the comparison...
between Theorems 5.1 and 5.2 and Corollaries 5.1 and 5.2 in this article). As an application of the new fundamental theorems, we use them to obtain the form of particular solutions of the original difference equation of hypergeometric type in Eq. (7) on non-uniform lattices.

The rest of the paper is organized as follows. In Sect. 2, we introduce some preliminary information about the difference equation of hypergeometric type on non-uniform lattices and give some related propositions and lemmas. In Sect. 3, we first consider a more general equation than Eq. (7) on non-uniform lattices and derive its adjoint equation. We then prove that the adjoint equation is also a difference equation of hypergeometric type on non-uniform lattices. As a special case of these results, we obtain the adjoint equation of hypergeometric type for Eq. (7) on non-uniform lattices. In Sect. 4, we derive the forms of particular solutions for the general adjoint equation and its special adjoint equation for Eq. (7), respectively. In Sect. 5, we use these particular solutions to obtain particular solutions for both Eq. (7) and its more general difference equation of hypergeometric type on non-uniform lattices. In addition, we give another two new fundamental theorems for both Eq. (7) and its more general difference equation of hypergeometric type on non-uniform lattices. Finally, we give an example to illustrate the application of the new fundamental theorems. The conclusion is then given in Sect. 6.

Throughout this paper, we follow those notations used in [1–3] which now have become standard. Furthermore, the properties listed below will be used in our study:

\[
\frac{x(z + \mu) + x(z)}{2} = \alpha(\mu)x\left(z + \frac{\mu}{2}\right) + \beta(\mu),
\]

\[
x(z + \mu) - x(z) = \gamma(\mu)\nabla x\left(z + \frac{\mu + 1}{2}\right),
\]

where

\[
\alpha(\mu) = \begin{cases} 
\frac{q^{\frac{\mu}{2}} - q^{-\frac{\mu}{2}}}{2} & \text{if } \mu \in \mathbb{C} \\
1 & \text{otherwise}
\end{cases}, \quad 
\beta(\mu) = \begin{cases} 
\frac{1 - \alpha(\mu)}{\mu^2} & \text{if } \mu \in \mathbb{C} \\
\frac{1 - \alpha}{\mu} & \text{otherwise}
\end{cases}, \quad 
\gamma(\mu) = \begin{cases} 
\frac{q^{\frac{\mu}{2}} - q^{-\frac{\mu}{2}}}{q^2 - q^{-2}} & \text{if } \mu \in \mathbb{C} \\
\frac{1}{\mu} & \text{otherwise}
\end{cases}.
\]

2 Preliminary information and lemmas

In this section, we give some preliminary information on the difference equation of hypergeometric type on non-uniform lattices. Let \(x(s)\) be a lattice, where \(s \in \mathbb{C}\) (complex numbers). It can be seen that for any real \(v\), \(x_v(s) = x(s + \frac{v}{2})\) is also a lattice. Given a function \(f(s)\), we define two difference operators with respect to \(x_v(s)\) as

\[
\Delta_v f(s) = \frac{\Delta f(s)}{\Delta x_v(s)}, \quad \nabla_v f(s) = \frac{\nabla f(s)}{\nabla x_v(s)}.
\]
Moreover, for any non-negative integer \( n \), we define

\[
\Delta_v^{(n)} f(s) = \begin{cases} 
  f(s) & \text{if } n = 0, \\
  \frac{\Delta}{\Delta x_v n f(s)} & \text{if } n \geq 1.
\end{cases}
\]

\[
\nabla_v^{(n)} f(s) = \begin{cases} 
  f(s) & \text{if } n = 0, \\
  \frac{\nabla}{\nabla x_v n f(s)} & \text{if } n \geq 1.
\end{cases}
\]

The following proposition can be verified straightforwardly.

**Proposition 2.1** Given two functions \( f(s), g(s) \) with complex variable \( s \), the following difference equalities hold:

\[
\Delta_v (f(s)g(s)) = f(s + 1)\Delta_v g(s) + g(s)\Delta_v f(s) \\
= g(s + 1)\Delta_v f(s) + f(s)\Delta_v g(s),
\]

\[
\Delta_v \left( \frac{f(s)}{g(s)} \right) = \frac{g(s + 1)\Delta_v f(s) - f(s + 1)\Delta_v g(s)}{g(s)g(s + 1)} \\
= \frac{g(s)\Delta_v f(s) - f(s)\Delta_v g(s)}{g(s)g(s + 1)},
\]

\[
\nabla_v (f(s)g(s)) = f(s - 1)\nabla_v g(s) + g(s)\nabla_v f(s) \\
= g(s - 1)\nabla_v f(s) + f(s)\nabla_v g(s),
\]

\[
\nabla_v \left( \frac{f(s)}{g(s)} \right) = \frac{g(s - 1)\nabla_v f(s) - f(s - 1)\nabla_v g(s)}{g(s)g(s - 1)} \\
= \frac{g(s)\nabla_v f(s) - f(s)\nabla_v g(s)}{g(s)g(s - 1)}.
\]

It can be seen that the difference equation of hypergeometric type in Eq. (7) can be written as

\[
\tilde{\sigma}[x(s)]\Delta_{-1} y(s) + \frac{\tilde{\tau}[x(s)]}{2} [\Delta_0 y(s) + \nabla_0 y(s)] + \lambda y(s) = 0. \tag{16}
\]

Let

\[
z_k(s) = \Delta_0^{(k)} y(s) = \Delta_{k-1} \Delta_{k-2} \cdots \Delta_0 y(s), \quad k = 1, 2, \ldots.
\]

Then, for any non-negative integer \( k \), \( z_k(s) \) satisfies an equation that has the same type of equation as Eq. (16) as [1,2]:

\[
\tilde{\sigma}_k[x_k(s)]\Delta_{-1} z_k(s) + \frac{\tilde{\tau}_k[x_k(s)]}{2} [\Delta_k z_k(s) + \nabla_k z_k(s)] + \mu_k z_k(s) = 0, \tag{17}
\]
where \( \mu_k \) is a constant, and \( \ddot{\sigma}_k(x_k) \) and \( \ddot{\tau}_k(x_k) \) are polynomials of degrees at most two and one in \( x_k \), respectively, which are given as

\[
\ddot{\sigma}_k[x_k(s)] = \frac{\ddot{\sigma}_{k-1}[x_{k-1}(s+1)] + \ddot{\sigma}_{k-1}[x_{k-1}(s)]}{2} \\
+ \frac{1}{4} \Delta_{k-1} \ddot{\tau}_{k-1}(s) \frac{\Delta x_k(s) + \nabla x_k(s) \left[ \Delta x_{k-1}(s) \right]^2}{2 \Delta x_{k-1}(s)} \\
+ \frac{\ddot{\tau}_{k-1}[x_{k-1}(s+1)] + \ddot{\tau}_{k-1}[x_{k-1}(s)]}{2} \frac{\Delta x_k(s) - \nabla x_k(s)}{4},
\]

\[
\ddot{\sigma}_0[x_0(s)] = \ddot{\sigma}[x(s)];
\]

\[
\ddot{\tau}_k[x_k(s)] = \Delta_{k-1} \ddot{\sigma}_{k-1} \left[ x_{k-1}(s) \right] + \Delta_{k-1} \ddot{\tau}_{k-1} \left[ x_{k-1}(s) \right] \frac{\Delta x_k(s) - \nabla x_k(s)}{4} \\
+ \frac{\ddot{\tau}_{k-1}[x_{k-1}(s+1)] + \ddot{\tau}_{k-1}[x_{k-1}(s)]}{2} \frac{\Delta x_k(s) + \nabla x_k(s)}{2 \Delta x_{k-1}(s)},
\]

\[
\ddot{\tau}_0[x_0(s)] = \ddot{\tau}[x(s)];
\]

\[
\mu_k = \mu_{k-1} + \Delta_{k-1} \ddot{\tau}_{k-1} \left[ x_{k-1}(s) \right], \quad \mu_0 = \lambda.
\]

To analyze additional properties of solutions of Eq. (17), it is convenient to use the equality

\[
\frac{1}{2} \left[ \Delta_k z_k(s) + \nabla_k z_k(s) \right] = \Delta_k z_k(s) - \frac{1}{2} \Delta \left[ \nabla_k z_k(s) \right]
\]

and rewrite Eq. (17) in an equivalent expression as

\[
\sigma_k(s) \Delta_{k-1} \nabla_k z_k(s) + \tau_k(s) \Delta_k z_k(s) + \mu_k z_k(s) = 0, \quad (18)
\]

where

\[
\sigma_k(s) = \ddot{\sigma}_k[x_k(s)] - \frac{1}{2} \ddot{\tau}_k[x_k(s)] \nabla x_{k+1}(s), \quad (19)
\]

\[
\sigma_k(s) = \sigma_{k-1}(s) = \sigma(s); \quad (20)
\]

\[
\tau_k(s) \nabla x_{k+1}(s) = \sigma(s + k) - \sigma(s) + \tau(s + k) \nabla x_1(s + k), \quad (21)
\]

\[
\tau_k(s) = \ddot{\tau}_k[x_k(s)]; \quad (22)
\]

\[
\mu_k = \lambda + \sum_{j=0}^{k-1} \Delta_j \tau_j(s). \quad (23)
\]

Equation (17) can be further rewritten into a self-adjoint form as

\[
\Delta_{k-1} \left[ \sigma(s) \rho_k(s) \nabla_k z_k(s) \right] + \mu_k \rho_k(s) z_k(s) = 0,
\]

where \( \rho_k(s) \) satisfies the Pearson-type difference equation as

\[
\Delta_{k-1} \left[ \sigma(s) \rho_k(s) \right] = \tau_k(s) \rho_k(s).
\]
Letting $\rho(s) = \rho_0(s)$, then we have

$$\rho_k(s) = \rho(s + k) \prod_{i=1}^{k} \sigma(s + i).$$

Thus, if for a positive integer $n$ such that $\mu_n = 0$ and $\lambda_n = \lambda_n$, then we have

$$\rho(s + k) = \rho(s + k) \prod_{i=1}^{k} \sigma(s + i).$$

Thus, if for a positive integer $n$ such that $\mu_n = 0$ and $\lambda_n = \lambda_n$, then we have

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$$\rho(s + k) = \rho(s + k) \prod_{i=1}^{k} \sigma(s + i).$$
where

\[
A = v(\mu)\sigma(z), \quad B = -\tau_{v-\mu}(z), \quad C = -\kappa_{\mu-2\nu+1}.
\]

The following identities about the explicit form of \(\tau_v(s), \mu_k, \) and \(\lambda_n\) are not difficult to check when the non-uniform lattice is either \(x(s) = c_1q^s + c_2q^{-s} + c_3\) or \(x(s) = \tilde{c}_1s^2 + \tilde{c}_2s + \tilde{c}_3\).

**Proposition 2.2** [3]. Given any real \(v\), let \(\alpha(v)\) and \(v(v)\) be defined in Eq. (14), \(\kappa_v\) be defined in Eq. (6), and if \(x(s) = c_1q^s + c_2q^{-s} + c_3\), then

\[
\tau_v(s) = \kappa_{2\nu+1}x_v(s) + c(v),
\]

where \(c(v)\) is a function with respect to \(v\) as

\[
c(v) = c_3(1 - q^{-v})(q^2 - q^{-v}) + c_3(2 - q^{-v})v(v) + 2\tau(0)\alpha(v) + \tilde{\sigma}(0)v(v).
\]

On the other hand, if \(x(s) = \tilde{c}_1s^2 + \tilde{c}_2s + \tilde{c}_3\), then

\[
\tau_v(s) = \kappa_{2\nu+1}x_v(s) + \tilde{c}(v),
\]

where \(\tilde{c}(v)\) is a function with respect to \(v\) as

\[
\tilde{c}(v) = \frac{\tilde{\sigma}''}{4}c_1v^3 + \frac{3\tilde{\tau}'}{4}c_1v^2 + \tilde{\sigma}(0)v + 2\tilde{\tau}(0).
\]

**Proposition 2.3** [3] For \(\alpha(\mu)\) and \(v(\mu)\), it holds that

\[
\sum_{j=0}^{k-1} \alpha(2j) = \alpha(k-1)v(k) - \sum_{j=0}^{k-1} v(2j) = v(k-1)v(k).
\]

**Proposition 2.4** [1,2] For \(x(s) = c_1q^s + c_2q^{-s} + c_3\) or \(x(s) = \tilde{c}_1s^2 + \tilde{c}_2s + \tilde{c}_3\), it holds that

\[
\mu_k = \lambda + \kappa_k v(k), \quad k = 1, 2, \ldots, n, \tag{28}
\]

\[
\lambda_n = -n\kappa_n. \tag{29}
\]

3 Adjoint difference equation

We now seek the second-order adjoint difference equation corresponding to Eq. (7). To this end, we first consider the operator

\[
L[y(z)] \equiv \sigma(z) \frac{\Delta}{\Delta x_{v-\mu-1}}(z) \left( \nabla y(z) \frac{\nabla y(z)}{\nabla x_{v-\mu}(z)} \right) + \tau_{v-\mu}(z) \frac{\Delta y(z)}{\Delta x_{v-\mu}(z)} + \lambda y(z). \tag{30}
\]
One may see that the equation
\[ L[y(z)] \equiv \sigma(z) \frac{\Delta}{\Delta x_{\nu-\mu-1}(z)} \left( \frac{\nabla y(z)}{\nabla x_{\nu-\mu}(z)} \right) + \tau_{\nu-\mu}(z) \frac{\Delta y(z)}{\Delta x_{\nu-\mu}(z)} + \lambda y(z) = 0 \]
(31)
is a more generalized Nikiforov–Uvarov–Suslov equation than Eq. (7), and it can be reduced to Eq. (7) by letting \( \mu = \nu \). If we rewrite Eq. (31) as
\[
\tilde{\sigma}_{\nu-\mu}(z) \frac{\Delta}{\Delta x_{\nu-\mu-1}(z)} \left( \frac{\nabla y(z)}{\nabla x_{\nu-\mu}(z)} \right) + \frac{\tau_{\nu-\mu}(z)}{2} \left[ \frac{\Delta y(z)}{\Delta x_{\nu-\mu}(z)} + \frac{\nabla y(z)}{\nabla x_{\nu-\mu}(z)} \right] + \lambda y(z) = 0, \tag{32}
\]
where
\[
\tilde{\sigma}_{\nu-\mu}(z) = \sigma(z) + \frac{1}{2} \tau_{\nu-\mu}(z) \nabla x_{\nu-\mu+1}(z),
\]
by Lemma 2.1, one may see that \( \tilde{\sigma}_{\nu-\mu}(z) \) and \( \tau_{\nu-\mu}(z) \) are polynomials of degrees at most two and one, respectively, in the variable \( x_{\nu-\mu}(s) \), and hence, Eq. (32) is a difference equation of hypergeometric type.

**Definition 3.1** For \( y(z) \) and \( w(z) \), the scalar product \( \langle w(z), y(z) \rangle \) with respect to \( \Delta x_{\nu-\mu-1}(z) \) is defined as
\[
\langle w(z), y(z) \rangle = \sum_{z=a}^{b-1} w(z) y(z) \Delta x_{\nu-\mu-1}(z),
\]
where \( a, b \) are complex with the same imaginary parts, and \( b - a \in \mathbb{N} \).

**Definition 3.2** For \( w(z) \) and operator \( L[y(z)] \), assume that the boundary conditions \( w(a) = w(b) = 0, y(a) = y(b) = 0 \) are satisfied. If the scalar product
\[
\langle w(z), L[y(z)] \rangle = \langle y(z), L^*[w(z)] \rangle
\]
holds, then the operator \( L^*[w(z)] \) is called the adjoint operator of \( L[y(z)] \), and \( L^*[w(z)] = 0 \) is called the adjoint equation of \( L[y(z)] = 0 \).

We now find the operator \( L^*[w(z)] \). Since
\[
\langle w(z), L[y(z)] \rangle = \sum_{z=a}^{b-1} w(z) L[y(z)] \Delta x_{\nu-\mu-1}(z)
\]
using the summation by parts and the boundary conditions, we obtain

\[
\sum_{z=a}^{b-1} w(z) \sigma(z) \Delta \left( \frac{\nabla y(z)}{\nabla x_{v-\mu} (z)} \right) \Delta x_{v-\mu -1} (z) = \sum_{z=a}^{b-1} \nabla y(z) \frac{\nabla y(z)}{\nabla x_{v-\mu} (z)} - \sum_{z=a}^{b-1} \nabla y(z) \frac{\nabla (w(z) \sigma(z))}{\nabla x_{v-\mu} (z)}
\]

and

\[
\sum_{z=a}^{b-1} w(z) \tau_{v-\mu} (z) \Delta y(z) \Delta x_{v-\mu -1} (z) = \sum_{z=a}^{b-1} w(z) \tau_{v-\mu} (z) \Delta x_{v-\mu -1} (z) \Delta y(z) \nabla \left[ w(z) \tau_{v-\mu} (z) \Delta x_{v-\mu -1} (z) \right] \Delta x_{v-\mu -1} (z)
\]

Thus, we let

\[
\langle w(z), L[y(z)] \rangle
\]

\[
= \sum_{z=a}^{b-1} y(z) \left\{ \frac{\Delta}{\Delta x_{v-\mu -1} (z)} \left[ \nabla (w(z) \sigma(z)) \right] \right. \\
- \frac{\nabla}{\Delta x_{v-\mu -1} (z)} \left[ w(z) \tau_{v-\mu} (z) \Delta x_{v-\mu -1} (z) \right] - \lambda w(z) \Delta x_{v-\mu -1} (z)
\]
which gives

\[ L^*[w(z)] = \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left[ \frac{\nabla (w(z)\sigma(z))}{\nabla x_{v-\mu}(z)} \right] \]

\[ - \frac{\nabla}{\Delta x_{v-\mu-1}(z)} \left[ w(z)\tau_{v-\mu}(z) \frac{\Delta x_{v-\mu-1}(z)}{\Delta x_{v-\mu}(z)} \right] - \lambda w(z). \quad (33) \]

Therefore, we define Eq. (33) as the adjoint operator of Eq. (30).

It can be seen that

\[ \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left[ \frac{\nabla (w(z)\sigma(z))}{\nabla x_{v-\mu}(z)} \right] \]

\[ = \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left[ \sigma(z-1) \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} + w(z) \frac{\nabla \sigma(z)}{\nabla x_{v-\mu}(z)} \right] \]

\[ = \sigma(z-1) \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left[ \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} \right] + \frac{\Delta w(z)}{\Delta x_{v-\mu}(z)} \frac{\Delta \sigma(z-1)}{\Delta x_{v-\mu}(z)} \]

\[ + \frac{\Delta \sigma(z)}{\Delta x_{v-\mu}(z)} \frac{\Delta w(z)}{\Delta x_{v-\mu-1}(z)} + w(z) \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left[ \frac{\nabla \sigma(z)}{\nabla x_{v-\mu}(z)} \right], \quad (34) \]

\[ \frac{\nabla}{\Delta x_{v-\mu-1}(z)} \left[ w(z)\tau_{v-\mu}(z) \frac{\Delta x_{v-\mu-1}(z)}{\Delta x_{v-\mu}(z)} \right] \]

\[ = \tau_{v-\mu}(z-1) \frac{\Delta x_{v-\mu-1}(z)}{\Delta x_{v-\mu}(z)} \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} \]

\[ + w(z) \frac{\nabla}{\Delta x_{v-\mu-1}(z)} \left[ \tau_{v-\mu}(z) \frac{\Delta x_{v-\mu-1}(z)}{\Delta x_{v-\mu}(z)} \right], \quad (35) \]

\[ \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} = \frac{\Delta w(z)}{\Delta x_{v-\mu}(z)} - \Delta \left[ \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} \right] \]

\[ = \frac{\Delta w(z)}{\Delta x_{v-\mu}(z)} - \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left[ \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} \right] \Delta x_{v-\mu-1}(z). \quad (36) \]

Substituting Eqs. (34–36) into Eq. (33), we obtain another expression of \( L^*[w(z)] \) as

\[ L^*[w(z)] = \sigma^*(z) \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left( \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} \right) \]

\[ + \tau_{v-\mu}^*(z) \frac{\Delta w(z)}{\Delta x_{v-\mu}(z)} + \lambda_{v-\mu}^* w(z), \quad (37) \]
where

$$\sigma^*(z) = \sigma(z - 1) + \tau_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z),$$  \hspace{1cm} (38)

$$\tau^*_{v-\mu}(z) = \frac{\sigma(z + 1) - \sigma(z - 1) - \tau_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z)}{\Delta x_{v-\mu-1}(z)},$$  \hspace{1cm} (39)

$$\lambda^*_{v-\mu} = \lambda - \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left( \frac{\tau_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z) - \nabla \sigma(z)}{\nabla x_{v-\mu}(z)} \right).$$  \hspace{1cm} (40)

We now seek the relationship between $\rho_{v-\mu}(z) L[y(z)]$ and $L^*[\rho_{v-\mu}(z) y(z)]$. It can be seen that $\rho_{v-\mu}(z) L[y(z)]$ has a self-adjoint form as

$$\rho_{v-\mu}(z) L[y(z)] = \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left( \sigma(z) \rho_{v-\mu}(z) \frac{\nabla y(z)}{\nabla x_{v-\mu}(z)} \right) + \lambda \rho_{v-\mu}(z) y(z),$$  \hspace{1cm} (41)

where $\rho_{v-\mu}(z)$ satisfies the Pearson-type equation

$$\frac{\Delta(\sigma(z) \rho_{v-\mu}(z))}{\Delta x_{v-\mu-1}(z)} = \tau_{v-\mu}(z) \rho_{v-\mu}(z).$$  \hspace{1cm} (42)

Introducing $w(z) = \rho_{v-\mu}(z) y(z)$, we obtain

$$\frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} = \rho_{v-\mu}(z) \frac{\nabla y(z)}{\nabla x_{v-\mu}(z)} + \frac{\nabla \rho_{v-\mu}(z)}{\nabla x_{v-\mu}(z)} y(z - 1),$$

that is,

$$\rho_{v-\mu}(z) \frac{\nabla y(z)}{\nabla x_{v-\mu}(z)} = \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} - \frac{\nabla \rho_{v-\mu}(z)}{\nabla x_{v-\mu}(z)} y(z - 1).$$  \hspace{1cm} (43)

From Eq. (42), we have

$$\Delta(\sigma(z) \rho_{v-\mu}(z)) = \tau_{v-\mu}(z) \rho_{v-\mu}(z) \Delta x_{v-\mu-1}(z),$$

implying

$$\rho_{v-\mu}(z) \Delta \sigma(z) + \Delta \rho_{v-\mu}(z) \sigma(z + 1) = \tau_{v-\mu}(z) \rho_{v-\mu}(z) \Delta x_{v-\mu-1}(z),$$

$$\Delta \rho_{v-\mu}(z) = \frac{\tau_{v-\mu}(z) \Delta x_{v-\mu-1}(z) - \Delta \sigma(z)}{\sigma(z + 1)} \rho_{v-\mu}(z),$$
and hence
\[
\frac{\nabla \rho_{v-\mu}(z)}{\nabla x_{v-\mu}(z)} y(z - 1) = \frac{\Delta \rho_{v-\mu}(z - 1)y(z - 1)}{\nabla x_{v-\mu}(z)} = \frac{\tau_{v-\mu}(z - 1)\nabla x_{v-\mu-1}(z) - \nabla \sigma(z)}{\sigma(z)\nabla x_{v-\mu}(z)} w(z - 1). \quad (44)
\]
Substituting Eqs. (43)–(44) into Eq. (41), we obtain
\[
\rho_{v-\mu}(z) L[y(z)] = \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left[ \sigma(z) \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} \right] - \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left[ \tau_{v-\mu}(z - 1)\nabla x_{v-\mu-1}(z) - \nabla \sigma(z) \right] \frac{\Delta w(z)}{\Delta x_{v-\mu}(z)} w(z) + \lambda w(z).
\]
This gives
\[
\rho_{v-\mu}(z) L[y(z)] = \sigma(z) \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left( \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} \right) + \frac{\Delta \sigma(z)}{\Delta x_{v-\mu-1}(z)} \frac{\Delta w(z)}{\Delta x_{v-\mu}(z)} - \frac{\tau_{v-\mu}(z - 1)\nabla x_{v-\mu-1}(z) - \nabla \sigma(z) \nabla w(z)}{\Delta x_{v-\mu-1}(z)} \frac{\Delta w(z)}{\Delta x_{v-\mu}(z)} w(z) + \lambda w(z),
\]
which implies that
\[
\rho_{v-\mu}(z) L[y(z)] = \sigma(z) \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left( \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} \right) + \frac{\Delta \sigma(z)}{\Delta x_{v-\mu-1}(z)} \frac{\Delta w(z)}{\Delta x_{v-\mu}(z)} - \frac{\tau_{v-\mu}(z - 1)\nabla x_{v-\mu-1}(z) - \nabla \sigma(z) \nabla w(z)}{\Delta x_{v-\mu-1}(z)} \frac{\Delta w(z)}{\Delta x_{v-\mu}(z)} w(z) + \lambda w(z). \quad (45)
\]
Using the following difference equalities
\[
\frac{\Delta w(z)}{\Delta x_{v-\mu}(z)} - \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} = \Delta \left( \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} \right), \quad (46)
\]
\[
\frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} = \frac{\Delta w(z)}{\Delta x_{v-\mu}(z)} - \Delta \left( \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} \right), \quad (47)
\]
we can simplify Eq. (45) to

\[
\rho_{v-\mu}(z) L[y(z)] = \sigma^*(z) \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left( \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} \right) + \tau^*_{v-\mu}(z) \frac{\Delta w(z)}{\Delta x_{v-\mu}(z)} + \lambda^*_{v-\mu} w(z),
\]

where

\[
\sigma^*(z) = \sigma(z - 1) + \tau_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z),
\]

\[
\tau^*_{v-\mu}(z) = \frac{\sigma(z + 1) - \sigma(z - 1) - \tau_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z)}{\Delta x_{v-\mu-1}(z)},
\]

\[
\lambda^*_{v-\mu} = \lambda - \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left( \frac{\tau_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z) - \nabla \sigma(z)}{\nabla x_{v-\mu}(z)} \right).
\]

Comparing with Eq. (37), we see that the right-hand side of Eq. (48) is \( L^*[w(z)] \). Hence, we have obtained an important relationship between the adjoint difference operator and the original difference operator as described in Proposition 3.1.

**Proposition 3.1** For \( y(z) \), it holds

\[
L^*[\rho_{v-\mu}(z) y(z)] = \rho_{v-\mu}(z) L[y(z)].
\]

Thus, we define

\[
L^*[w(z)] = \sigma^*(z) \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left( \frac{\nabla w(z)}{\nabla x_{v-\mu}(z)} \right) + \tau^*_{v-\mu}(z) \frac{\Delta w(z)}{\Delta x_{v-\mu}(z)} + \lambda^*_{v-\mu} w(z) = 0
\]

as the **adjoint difference equation** corresponding to Eq. (31). In particular, when \( \mu = \nu \), Eq. (53) gives the **adjoint difference equation** corresponding to Eq. (7).

Based on Definition 2.1 and Proposition 2.2, it is not difficult to obtain the following corollary.

**Corollary 3.1** Eqs. (50) and (51) can be simplified to

\[
\tau^*_{v-\mu}(z) = -\tau_{v-\mu-2}(z + 1) = -\kappa_{2v-2\mu-3} x_{v-\mu}(z) + c(v - \mu),
\]

\[
\lambda^*_{v-\mu} = \lambda - \Delta_{v-\mu-1} \tau_{v-\mu-1}(z) = \lambda - \kappa_{2v-2\mu-1}.
\]

**Proof** Since

\[
\sigma(z + 1) - \sigma(z - 1) - \tau_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z)
\]
For the adjoint difference equation in Eq. (50) and Proposition 2.2, we obtain from Eqs. (49) and (59), we find some interesting dual properties as described in the following proposition.

**Proposition 3.2**  For the adjoint difference equation in Eq. (53), it holds

\[
\begin{align*}
\sigma(z) &= \sigma^*(z - 1) + \tau^*_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z), \\
\tau_{v-\mu}(z) &= \frac{\sigma^*(z + 1) - \sigma^*_{v-\mu}(z - 1) - \tau^*_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z)}{\Delta x_{v-\mu-1}(z)}
\end{align*}
\]

Regarding to the adjoint difference equation in Eq. (53), we find some interesting dual properties as described in the following proposition.

Proposition 3.2  For the adjoint difference equation in Eq. (53), it holds

\[
\begin{align*}
\sigma(z) &= \sigma^*(z - 1) - \tau_{v-\mu}(z - 1) \nabla x_{v-\mu+1}(z - 1) \\
&= \sigma(z + 1) - \tau_{v-\mu}(z - 1 + v - \mu) \nabla x_1(z - 1 + v - \mu) \\
&= -\tau_{v-\mu-2}(z + 1) \nabla x_{v-\mu-1}(z + 1) \\
&= -\tau_{v-\mu-2}(z + 1) \Delta x_{v-\mu-1}(z),
\end{align*}
\]

we obtain from Eq. (50) and Proposition 2.2 that

\[
\tau^*_{v-\mu}(z) = -\tau_{v-\mu-2}(z + 1) \\
= -\kappa_{2v-2\mu-3} x_{v-\mu-2}(z + 1) + c(v - \mu) \\
= -\kappa_{2v-2\mu-3} x_{v-\mu}(z) + c(v - \mu).
\]

Using a similar argument, we have

\[
\begin{align*}
\tau_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z) &= -\nabla \sigma(z) \\
&= \tau_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z) + \sigma(z - 1) - \sigma(z), \\
\tau(z - 1 + v - \mu) \nabla x_1(z - 1 + v - \mu) &= \tau_{v-\mu-1}(z) \nabla x_{v-\mu}(z).
\end{align*}
\]

Hence, we obtain

\[
\lambda^*_{v-\mu} = \lambda - \Delta_{v-\mu-1} \tau_{v-\mu-1}(z) \\
= \lambda - \Delta_{v-\mu-1} \{\kappa_{2v-2\mu-1} x_{v-\mu-1}(z)\} \\
= \lambda - \kappa_{2v-2\mu-1},
\]

and complete the proof. □

Based on Eqs. (49) and (59), we obtain

\[
\sigma(z + 1) = \sigma^*(z) + \tau^*_{v-\mu}(z) \Delta x_{v-\mu-1}(z),
\]

and complete the proof. □

Regarding to the adjoint difference equation in Eq. (53), we find some interesting dual properties as described in the following proposition.

Proposition 3.2  For the adjoint difference equation in Eq. (53), it holds

\[
\begin{align*}
\sigma(z) &= \sigma^*(z - 1) + \tau^*_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z), \\
\tau_{v-\mu}(z) &= \frac{\sigma^*(z + 1) - \sigma^*_{v-\mu}(z - 1) - \tau^*_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z)}{\Delta x_{v-\mu-1}(z)}
\end{align*}
\]

Proof  From Eq. (50), we have

\[
\tau^*_{v-\mu}(z) \Delta x_{v-\mu-1}(z) = \sigma(z + 1) - \sigma(z - 1) - \tau_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z).
\]

Based on Eqs. (49) and (59), we obtain

\[
\sigma(z + 1) = \sigma^*(z) + \tau^*_{v-\mu}(z) \Delta x_{v-\mu-1}(z),
\]
implying that
\[ \sigma(z) = \sigma^*(z - 1) + \tau^*_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z). \]

Thus, we obtain from Eq. (49) that
\[
\tau_{v-\mu}(z - 1) = \frac{\sigma^*(z) - \sigma(z - 1)}{\nabla x_{v-\mu-1}(z)} = \frac{\sigma^*(z) - \sigma^*(z - 2) - \tau^*_{v-\mu}(z - 2) \nabla x_{v-\mu-1}(z - 1)}{\nabla x_{v-\mu-1}(z)},
\]

which is Eq. (57). Moreover, we obtain
\[
\tau_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z) - \nabla \sigma(z) = \sigma^*(z) - \sigma(z) - \nabla \sigma(z) = \sigma^*(z) - [\sigma^*(z - 1) + \tau^*_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z)] = \nabla \sigma^*(z) - \tau^*_{v-\mu}(z - 1) \nabla x_{v-\mu-1}(z). \quad (60)
\]

Using Eq. (51) together with Eq. (60) gives Eq. (58) and hence the proof is completed.

Parallel to Corollary 3.1, one may obtain the following corollary.

**Corollary 3.2** Equations (57) and (58) can be simplified to, respectively,
\[
\tau_{v-\mu}(z) = -\tau^*_{v-\mu-2}(z + 1), \quad (61)
\]
\[
\lambda = \lambda^*_{v-\mu} - \kappa^*_{v-2\mu-1}. \quad (62)
\]

**Proposition 3.3** The adjoint difference equation Eq. (53) can be rewritten as
\[
sigma(z + 1) \Delta_{v-\mu-1} v_{v-\mu} w(z) - \tau_{v-\mu-2}(z + 1) \nabla v_{v-\mu} w(z) + (\lambda - \kappa_{v-2\mu-1}) w(z) = 0. \quad (63)
\]

**Proof** Since
\[
\Delta_{v-\mu} w(z) - \nabla v_{v-\mu} w(z) = \Delta \left( \frac{\nabla w(z)}{v_{v-\mu}(z)} \right),
\]
we have
\[
\tau^*_{v-\mu}(z) \Delta_{v-\mu} w(z) = \tau^*_{v-\mu}(z) \nabla v_{v-\mu} w(z) + \tau^*_{v-\mu}(z) \Delta \left( \frac{\nabla w(z)}{v_{v-\mu}(s)} \right)
\]
\[
= \tau^*_{v-\mu}(z) \nabla v_{v-\mu} w(z) + \tau^*_{v-\mu}(z) \Delta x_{v-\mu-1}(z) \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left( \frac{\nabla w(z)}{v_{v-\mu}(z)} \right). \quad (64)
\]
Substituting Eq. (64) into Eq. (53), we obtain

\[
\[\sigma^* (z) + \tau^*_{v-\mu} (z) \Delta x_{v-\mu-1} (z) \] \frac{\Delta}{\Delta x_{v-\mu-1} (z)} \left( \frac{\nabla w(z)}{\nabla_{v-\mu} x(z)} \right) \\
+ \tau^*_{v-\mu} (z) \nabla_{v-\mu} w(z) + \lambda^*_{v-\mu} w(z) \\
= 0. \tag{65}
\]

From Eq. (54), one may have

\[
\sigma^* (z) + \tau^*_{v-\mu} (z) \Delta x_{v-\mu-1} (z) = \sigma (z + 1) . \tag{66}
\]

Substituting it into Eq. (65) and then using Eq. (54), we obtain Eq. (63) and hence complete the proof.

In the end of this section, we would like to prove that the adjoint difference equation in Eqs. (53) or (63) is also a difference equation of hypergeometric type on non-uniform lattices. To this end, we need only to prove that

\[
\tilde{\sigma}^* (z) = \sigma^* (z) + \frac{1}{2} \tau^*_{v-\mu} (z) \Delta x_{v-\mu-1} (z) \\
= \sigma (z + 1) + \frac{1}{2} \tau_{v-\mu-2} (z + 1) \Delta x_{v-\mu-1} (z)
\]

is a polynomial of degree at most two in the variable \( x_{v-\mu} (z) \). In fact, from Eq. (25) and Lemma 2.1, we see that

\[
\tilde{\sigma}^* (z) = \sigma (z + 1) + \frac{1}{2} \tau_{v-\mu-2} (z + 1) \nabla x_{v-\mu-1} (z + 1) = \tilde{\sigma}_{v-\mu-2} (z + 1)
\]

is a polynomial of degree at most two in the variable \( x_{v-\mu-2} (z + 1) = x_{v-\mu} (z) \). Thus, we obtain the following theorem.

**Theorem 3.1** The adjoint equation Eq. (63) or

\[
\tilde{\sigma}_{v-\mu-2} (z + 1) \Delta x_{v-\mu-1} \nabla_{v-\mu} w(z) \\
- \frac{1}{2} \tau_{v-\mu-2} (z + 1) [\Delta_{v-\mu} w(z) + \nabla_{v-\mu} w(z)] + (\lambda - \kappa_{2v-2\mu-1}) w(z) \\
= 0 \tag{67}
\]

is also a difference equation of hypergeometric type on non-uniform lattices.

By letting \( \mu = \nu \) in the above equation, we immediately obtain the following corollary.
Corollary 3.3 The adjoint equation of Eq. (7) or
\[
\tilde{\sigma}_{-2}(z + 1)\Delta_{-1} \nabla_0 w(z)
- \frac{1}{2} \tau_{-2}(z + 1)[\Delta_0 w(z) + \nabla_0 w(z)] + (\lambda - \kappa_{-1}) w(z)
= 0
\]
(68)
is also a difference equation of hypergeometric type on non-uniform lattices.

4 Particular solutions for adjoint difference equations

In this section, we first derive the forms of particular solutions for a difference equation of hypergeometric type on non-uniform lattices (see Proposition 4.1) and then use it to obtain the forms of particular solutions for the adjoint difference equation given in Eq. (53) or an alternative equation in Eq. (63) (see Theorem 4.1). By letting \( \mu = \nu \), one may obtain the forms of particular solutions for the adjoint difference equation given in Eq. (68) (see Theorem 4.2).

Proposition 4.1 On those classes of non-uniform lattices \( x = x(z) \), the difference equation of hypergeometric type on non-uniform lattices
\[
\sigma(z) \frac{\Delta}{\Delta x_{\nu-\mu-1}(z)} \left( \frac{\nabla y(z)}{\nabla x_{\nu-\mu}(z)} \right) - \tau_{\nu-\mu}(z) \frac{\nabla y(z)}{\nabla x_{\nu-\mu}(z)} + \tilde{\lambda} y(z) = 0 \quad (\tilde{\lambda} \in R)
\]
(69)
has particular solutions in the form of
\[
y(z) = \sum_{s=a}^{b-1} \frac{\rho_{\nu}(s) \nabla x_{\nu+1}(s)}{[x_{\nu}(s) - x_{\nu}(z - 1)](\mu - 1)},
\]
(70)
and also in the form of
\[
y(z) = \oint_{C} \frac{\rho_{\nu}(s) \nabla x_{\nu+1}(s) ds}{[x_{\nu}(s) - x_{\nu}(z - 1)](\mu - 1)},
\]
(71)
where \( a, b \) are complex with the same imaginary parts, \( C \) is a contour in the complex \( s \)-plane, and \( x_{\nu}(s) = x(s + \frac{1}{2}) \), if

(i) functions \( \rho(z) \) and \( \rho_{\nu}(z) \) satisfy
\[
\frac{\Delta}{\nabla x_{1}(z)} [\sigma(z) \rho(z)] = \tau(z) \rho(z), \quad \frac{\Delta}{\nabla x_{\nu+1}(z)} [\sigma(z) \rho_{\nu}(z)] = \tau_{\nu}(z) \rho_{\nu}(z);
\]
(72)
(ii) \( \mu, \nu \) satisfy
\[
\tilde{\lambda} + \kappa_{2\nu-(\mu-1)} \gamma(\mu - 1) = 0;
\]
(73)
(iii) difference derivatives of the functions calculated by

\[ \phi_{\nu \mu}(z) = \sum_{s=a}^{b-1} \frac{\rho_{\nu}(s) \nabla x_{\nu-1}(s)}{[x_{\nu}(s) - x_{\nu}(z-1)]^{(\mu-1)}}, \]

or

\[ \phi_{\nu \mu}(z) = \oint_{C} \frac{\rho_{\nu}(s) \nabla x_{\nu-1}(s) ds}{[x_{\nu}(s) - x_{\nu}(z-1)]^{(\mu-1)}} \]

can be carried out by means of the formula

\[ \frac{\nabla \phi_{\nu \mu}(z)}{\nabla x_{\nu-\mu}(z)} = \gamma(\mu - 1) \phi_{\nu,\mu+1}(z); \]

(iv) the following equalities hold:

\[ \psi_{\nu \mu}(a, z) = \psi_{\nu \mu}(b, z), \oint_{C} \Delta_{s} \psi_{\nu \mu}(s, z) dz = 0, \]

where

\[ \psi_{\nu \mu}(s, z) = \frac{\sigma(s) \rho_{\nu}(s)}{[x_{\nu-1}(s) - x_{\nu-1}(z)]^{(\mu)}}. \]

**Proof** To establish the relationship among \( \Delta_{\nu-\mu-1} \nabla_{\nu-\mu} y(z) \), \( \nabla_{\nu-\mu} y(z) \), and \( y(z) \), we need to find non-zero functions \( A_{i}(z), i = 1, 2, 3 \), such that

\[ A_{1}(z) \Delta_{\nu-\mu-1} \nabla_{\nu-\mu} y(z) + A_{2}(z) \nabla_{\nu-\mu} y(z) + A_{3}(z) y(z) = 0. \]

Note that

\[ \nabla_{\nu-\mu} y(z) = \gamma(\mu - 1) \sum_{s=a}^{b-1} \frac{\rho_{\nu}(s) \nabla x_{\nu+1}(s)}{[x_{\nu}(s) - x_{\nu}(z-1)]^{(\mu)}}, \]

\[ \Delta_{\nu-\mu-1} \nabla_{\nu-\mu} y(z) = \gamma(\mu) \gamma(\mu - 1) \sum_{s=a}^{b-1} \frac{\rho_{\nu}(s) \nabla x_{\nu+1}(s)}{[x_{\nu}(s) - x_{\nu}(z)]^{(\mu-1)}}. \]

Substituting them into Eq. (79) gives

\[
\sum_{s=a}^{b-1} \frac{\rho_{\nu}(s) \nabla x_{\nu+1}(s)}{[x_{\nu}(s) - x_{\nu}(z)]^{(\mu+1)}} [\gamma(\mu) \gamma(\mu - 1) A_{1}(z) + \gamma(\mu - 1) A_{2}(z) [x_{\nu}(s) - x_{\nu}(z)] \\
+ A_{3}(z) [x_{\nu}(s) - x_{\nu}(z)][x_{\nu}(s) - x_{\nu}(z - \mu)]] \\
= \sum_{s=a}^{b-1} \frac{\rho_{\nu}(s) \nabla x_{\nu+1}(s)}{[x_{\nu}(s) - x_{\nu}(z)]^{(\mu+1)}} P(s),
\]
where
\[
P(s) = \gamma(\mu)\gamma(\mu - 1)A_1(z) + \gamma(\mu - 1)A_2(z)[x_v(s) - x_v(z)] \\
+ A_3(z)[x_v(s) - x_v(z)][x_v(s) - x_v(z - \mu)].
\]

On the other hand, we set
\[
\sum_{s=a}^{b-1} \rho_v(s) \nabla x_{v+1}(s) = \sum_{s=a}^{b-1} \Delta_v \left[ \frac{\sigma(s)\rho_v(s)}{[x_{v-1}(s) - x_{v-1}(z)]^{(\mu+1)}} \right] \\
= \sum_{s=a}^{b-1} \frac{\tau_v(s)\rho_v(s)\nabla x_{v+1}(s)}{[x_{v-1}(s + 1) - x_{v-1}(z)]^{(\mu+1)}} - \sum_{s=a}^{b-1} \frac{\gamma(\mu)\sigma(s)\rho_v(s)\nabla x_{v+1}(s)}{[x_v(s) - x_v(z)]^{(\mu+1)}} \\
= - \sum_{s=a}^{b-1} \frac{\rho_v(s)\nabla x_{v+1}(s)}{[x_v(s) - x_v(z)]^{(\mu+1)}} \{ \gamma(\mu)\sigma(s) - \tau_v(s)[x_{v-\mu}(s) - x_{v-\mu}(z)] \} \\
= - \sum_{s=a}^{b-1} \frac{\rho_v(s)\nabla x_{v+1}(s)}{[x_v(s) - x_v(z)]^{(\mu+1)}} Q(s),
\]

where
\[
Q(s) = \gamma(\mu)\sigma(s) - \tau_v(s)[x_{v-\mu}(s) - x_{v-\mu}(z)].
\]

By Lemma 2.2, we have
\[
Q(s) = \gamma(\mu)\sigma(z) - \tau_{v-\mu}(z)[x_v(s) - x_v(z)] \\
- \kappa_{2v-(\mu-1)}[x_v(s) - x_v(z)][x_v(s) - x_v(z - \mu)].
\]

Comparing \(P(s)\) with \(Q(s)\) gives
\[
A_1(z) = \frac{1}{\gamma(\mu - 1)}\sigma(z), \quad A_2(z) = -\frac{\tau_{v-\mu}(z)}{\gamma(\mu - 1)}, \quad A_3(z) = -\kappa_{2v-(\mu-1)},
\]
and hence the proof is completed. \(\Box\)

**Theorem 4.1** On those classes of non-uniform lattices \(x = x(z)\), the adjoint difference equation given in Eq. (53) or an alternative equation in Eq. (63) as
\[
\sigma(z + 1) \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left( \frac{\nabla y(z)}{\nabla x_{v-\mu}(z)} \right) - \tau_{v-\mu-2}(z + 1) \frac{\nabla y(z)}{\nabla x_{v-\mu}(z)} \\
+ \lambda_{v-\mu}^{\kappa} y(z) = 0,
\]
(80)
has particular solutions in the form of

\[ y(z) = \sum_{s=a}^{b-1} \rho_v(s) \nabla x_{v+1}(s) \frac{\nabla x_{v+1}(s)}{[x_v(s) - x_v(z)]^{\mu+1}}, \]

and also in the form of

\[ y(z) = \oint_C \rho_v(s) \nabla x_{v+1}(s) ds \frac{\nabla x_{v+1}(s)}{[x_v(s) - x_v(z)]^{\mu+1}}, \]

where \( C \) is a contour in the complex \( s \)-plane, and \( x_v(s) = x(s + \frac{1}{2}) \), if

(i) functions \( \rho(z) \) and \( \rho_v(z) \) satisfy

\[
\frac{\Delta}{\nabla x_1(z)} [\sigma(z) \rho(z)] = \tau(z) \rho(z), \quad \frac{\Delta}{\nabla x_{v+1}(z)} [\sigma(z) \rho_v(z)] = \tau_v(z) \rho_v(z);
\]

(ii) \( \mu, \nu \) satisfy

\[
\lambda_{\nu-\mu}^* + \kappa_{2v-(\mu+1)} \gamma (\mu + 1) = 0; \tag{82}
\]

(iii) difference derivatives of the functions calculated by

\[ \phi_{\nu\mu}(z) = \sum_{s=a}^{b-1} \rho_v(s) \nabla x_{v-1}(s) \frac{\nabla x_{v-1}(s)}{[x_v(s) - x_v(z)]^{\mu+1}}, \tag{83} \]

or

\[ \phi_{\nu\mu}(z) = \oint_C \rho_v(s) \nabla x_{v-1}(s) ds \frac{\nabla x_{v-1}(s)}{[x_v(s) - x_v(z)]^{\mu+1}} \tag{84} \]

can be carried out by means of the formula

\[ \frac{\nabla \phi_{\nu\mu}(z)}{\nabla x_{v-\mu}(z)} = \gamma (\mu + 1) \phi_{\nu, \mu+1}(z); \tag{85} \]

(iv) the following equalities hold:

\[ \psi_{\nu\mu}(a, z) = \psi_{\nu\mu}(b, z), \oint_C \Delta s \psi_{\nu\mu}(s, z) dz = 0 \tag{86} \]

where

\[ \psi_{\nu\mu}(s, z) = \frac{\sigma(s) \rho_v(s)}{[x_{v-1}(s) - x_{v-1}(z+1)]^{\mu+1}}. \tag{87} \]
\textbf{Proof} Eq. (81) can be written as

\[
\sigma(z + 1) \frac{\Delta}{\Delta x_{v-\mu-3}(z + 1)} \left( \frac{\nabla y(z)}{\nabla x_{v-\mu-2}(z + 1)} \right) - \tau_{v-\mu-2}(z + 1) \frac{\nabla y(z)}{\nabla x_{v-\mu-1}(z + 1)} - \kappa_{2v-(\mu+1)} (\mu + 1) y(z) = 0.
\]

Letting \( y(z) = \tilde{y}(z + 1) \), we obtain

\[
\sigma(z) \frac{\Delta}{\Delta x_{v-\mu-3}(z)} \left( \frac{\nabla \tilde{y}(z)}{\nabla x_{v-\mu-2}(z)} \right) - \tau_{v-\mu-2}(z) \frac{\nabla \tilde{y}(z)}{\nabla x_{v-\mu-2}(z)} - \kappa_{2v-(\mu+1)} (\mu + 1) \tilde{y}(z) = 0.
\]

Letting \( \mu + 2 = \bar{\mu} \), we further obtain

\[
\sigma(z) \frac{\Delta}{\Delta x_{v-\bar{\mu}-1}(z)} \left( \frac{\nabla \tilde{y}(z)}{\nabla x_{v-\bar{\mu}}(z)} \right) - \tau_{v-\bar{\mu}}(z) \frac{\nabla \tilde{y}(z)}{\nabla x_{v-\bar{\mu}}(z)} = 0.
\]

By Proposition 4.1, we have

\[
y(z) = \tilde{y}(z + 1) = \sum_{s=a}^{b-1} \frac{\rho_v(s) \nabla x_{v+1}(s)}{[x_v(s) - x_v(z)]^{\bar{\mu}-1}} = \sum_{s=a}^{b-1} \frac{\rho_v(s) \nabla x_{v+1}(s)}{[x_v(s) - x_v(z)]^{(\mu+1)}},
\]

and

\[
y(z) = \tilde{y}(z + 1) = \int_{C} \frac{\rho_v(s) \nabla x_{v+1}(s) ds}{[x_v(s) - x_v(z)]^{(\bar{\mu}-1)}} = \int_{C} \frac{\rho_v(s) \nabla x_{v+1}(s) ds}{[x_v(s) - x_v(z)]^{(\mu+1)}},
\]

and hence complete the proof. \( \square \)

Finally, by letting \( \mu = \nu \), then from Eq. (55), \( \lambda^* = \lambda - \kappa_{-1} \), one may obtain the forms of particular solutions for the adjoint difference equation in Eq. (68).

\textbf{Theorem 4.2} \textit{On those classes of non-uniform lattices} \( x = x(z) \), \textit{the adjoint difference equation given in Eq. (68) as}

\[
\sigma(z + 1) \frac{\Delta}{\Delta x_{-1}(z)} \left( \frac{\nabla y(z)}{\nabla x(z)} \right) - \tau_{-2}(z + 1) \frac{\nabla y(z)}{\nabla x(z)} + \lambda^* y(z) = 0,
\]

\textit{has particular solutions in the form of}

\[
y(z) = \sum_{s=a}^{b-1} \frac{\rho_v(s) \nabla x_{v+1}(s)}{[x_v(s) - x_v(z)]^{(\nu+1)}},
\]

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and also in the form of

\[ y(z) = \oint_{C} \frac{\rho_v(s) \nabla x_{v+1}(s)}{[x_v(s) - x_v(z)]^{(v+1)}} ds, \]

where \( C \) is a contour in the complex \( s \)-plane, and \( x_v(s) = x(s + \frac{1}{2}) \), if

(i) functions \( \rho(z) \) and \( \rho_v(z) \) satisfy

\[ \frac{\Delta}{\nabla x_1(z)}[\sigma(z)\rho(z)] = \tau(z)\rho(z), \quad \frac{\Delta}{\nabla x_{v+1}(z)}[\sigma(z)\rho_v(z)] = \tau_v(z)\rho_v(z); \quad (89) \]

(ii) \( \mu, \nu \) satisfy

\[ \lambda^* + \kappa_{v-1} \gamma(v + 1) = 0; \quad (90) \]

(iii) difference derivatives of the functions calculated by

\[ \phi_{\nu
u}(z) = \sum_{s=a}^{b-1} \frac{\rho_v(s) \nabla x_{v-1}(s)}{[x_v(s) - x_v(z)]^{(v+1)}}, \quad (91) \]

or

\[ \phi_{\nu
u}(z) = \oint_{C} \frac{\rho_v(s) \nabla x_{v-1}(s)}{[x_v(s) - x_v(z)]^{(v+1)}} ds \quad (92) \]

can be carried out by means of the formula

\[ \frac{\nabla \phi_{\nu
u}(z)}{\nabla x(z)} = \gamma(v + 1)\phi_{\nu\nu+1}(z); \quad (93) \]

(iv) the following equalities hold:

\[ \psi_{\nu
u}(a, z) = \psi_{\nu
u}(b, z), \oint_{C} \Delta_x \psi_{\nu
u}(s, z) dz = 0 \quad (94) \]

where

\[ \psi_{\nu
u}(s, z) = \frac{\sigma(s)\rho_v(s)}{[x_{v-1}(s) - x_{v-1}(z + 1)]^{(v+1)}}. \quad (95) \]

5 Application and new fundamental theorems

Based on Proposition 3.1 and Theorem 4.1, one may obtain the following corollary.
Corollary 5.1 Under the hypotheses of Theorem 4.1, the equation

\[ \sigma(z) \frac{\Delta}{\Delta x_{v-\mu-1}(z)} \left( \frac{\nabla y(z)}{\nabla x_{v-\mu}(z)} \right) + \tau_{v-\mu}(z) \frac{\Delta y(z)}{\Delta x_{v-\mu}(z)} + \lambda y(z) = 0 \]  

has particular solutions in the form of

\[ y(z) = \frac{1}{\rho_{v-\mu}(z)} \sum_{s=a}^{b-1} \frac{\rho_v(s) \nabla x_{v+1}(s)}{[x_v(s) - x_v(z)]^{(\nu+1)}} \]  

and also the form of

\[ y(z) = \frac{1}{\rho_{v-\mu}(z)} \oint_C \frac{\rho_v(s) \nabla x_{v+1}(s) ds}{[x_v(s) - x_v(z)]^{(\nu+1)}} \]  

where \( \rho(z), \rho_v(z) \) satisfy

\[ \frac{\Delta(\sigma(z) \rho(z))}{\nabla x_1(z)} = \tau(z) \rho(z), \quad \frac{\Delta(\sigma(z) \rho_v(z))}{\nabla x_{v+1}(z)} = \tau_v(z) \rho_v(z), \]

and \( v, \mu \) are roots of the equation

\[ \lambda + \kappa_{2v-\mu} \gamma(\mu) = 0. \]

Note that by letting \( \mu = v \) in Corollary 5.1, Eq. (96) can be reduced to Eq. (7). Thus, we obtain the following well-known theorem given in [3].

Corollary 5.2 (Theorem 2.2 in [3]). Under the hypotheses of Corollary 5.1 with \( \mu = v \), the equation

\[ \sigma(z) \frac{\Delta}{\Delta x_{v-1}(z)} \left( \frac{\nabla y(z)}{\nabla x_0(z)} \right) + \tau(z) \frac{\Delta y(z)}{\Delta x_0(z)} + \lambda y(z) = 0 \]  

has particular solutions in the form of

\[ y(z) = \frac{1}{\rho(z)} \sum_{s=a}^{b-1} \frac{\rho_v(s) \nabla x_{v+1}(s)}{[x_v(s) - x_v(z)]^{(v+1)}} \]  

and also in the form of

\[ y(z) = \frac{1}{\rho(z)} \oint_C \frac{\rho_v(s) \nabla x_{v+1}(s) ds}{[x_v(s) - x_v(z)]^{(v+1)}} \]  

where \( \rho(z), \rho_v(z) \) satisfy

\[ \frac{\Delta(\sigma(z) \rho(z))}{\nabla x_1(z)} = \tau(z) \rho(z), \quad \frac{\Delta(\sigma(z) \rho_v(z))}{\nabla x_{v+1}(z)} = \tau_v(z) \rho_v(z), \]
and $v$ is the root of the equation

$$\lambda + \kappa_v \gamma(v) = 0.$$ 

**Remark 5.1** It should be pointed out that Theorem 4.1 may be obtained based on the Suslov theorem coupled with Proposition 3.1. However, without using Proposition 3.1, Theorem 4.1 seems cannot be obtained simply using the Suslov theorem without coupling with Proposition 3.1. We consider Theorem 4.1 to be a new result because we prove it directly and have not seen it as well as Proposition 3.1 from the other literatures. Reversely, the Suslov theorem (Corollary 5.2) can be obtained based on Theorem 4.1 and Proposition 3.1. This new proof not only gives another way to prove the Suslov theorem, but also is our purpose showing the important application of the obtained adjoint equations and their solutions in this study.

**Remark 5.2** One of interests for the adjoint equation in [29,30] is because it can be used to find the general solution of the hypergeometric and $q$–hypergeometric equation. Using our results obtained in this study, it is possible to do something similar. Indeed, we have done some work where the results can be seen in our recent manuscript [31].

We now prove another kind of fundamental theorems for Eqs. (7) and (96), respectively, which are essentially new results and their expressions are different from the Suslov theorem (as seen in Corollaries 5.1 and 5.2).

**Theorem 5.1** On those classes of non-uniform lattices $x = x(z)$, the difference equation of hypergeometric type on non-uniform lattices

$$\sigma(z) \frac{\Delta}{\Delta x_{\nu-\mu-1}(z)} \left( \frac{\nabla y(z)}{\nabla x_{\nu-\mu}(z)} \right) + \tau_{\nu-\mu}(z) \frac{\Delta y(z)}{\Delta x_{\nu-\mu}(z)} + \lambda y(z) = 0 \quad (103)$$

has particular solutions in the form of

$$y(z) = \sum_{s=a}^{b-1} (x_{\nu}(s) - x_{\nu}(z))^{(\mu+1)} \rho_{\nu}(s) \nabla x_{\nu+1}(s), \quad (104)$$

and also in the form of

$$y(z) = \oint_{C} (x_{\nu}(s) - x_{\nu}(z))^{(\mu+1)} \rho_{\nu}(s) \nabla x_{\nu+1}(s) \text{d}s, \quad (105)$$

where $C$ is a contour in the complex $s$-plane, and $x_{\nu}(s) = x(s + \frac{1}{2})$, if

(i) functions $\rho(z)$ and $\rho_{\nu}(z)$ satisfy

$$\frac{\nabla}{\nabla x_{\nu+1}(z)} [\sigma(z) \rho_{\nu}(z)] + \tau_{\nu}(z) \rho_{\nu}(z) = 0; \quad (106)$$

(ii) $\mu, \nu$ satisfy the equation

$$\lambda + \kappa_{\nu-\mu} \gamma(\mu + 1) = 0; \quad (107)$$
(iii) difference derivatives of the functions calculated by

\[ \phi_{v\mu}(z) = \sum_{s=a}^{b-1} [x_{v}(s) - x_{v}(z)]^{(\mu+1)} \rho_{v}(s) \nabla x_{v-1}(s), \]  

or

\[ \phi_{v\mu}(z) = \oint_{C} [x_{v}(s) - x_{v}(z)]^{(\mu+1)} \rho_{v}(s) \nabla x_{v-1}(s)ds \]  

can be carried out by means of the formula

\[ \frac{\Delta \phi_{v\mu}(z)}{\Delta x_{v-\mu}(z)} = -\gamma(\mu + 1)\phi_{v,\mu-1}(z); \]  

(iv) the following equalities hold:

\[ \psi_{v\mu}(a, z) = \psi_{v\mu}(b, z), \oint_{C} \nabla \psi_{v\mu}(s, z) dz = 0, \]  

where

\[ \psi_{v\mu}(s, z) = \sigma(s) \rho_{v}(s)[x_{v+1}(s) - x_{v+1}(z - 1)]^{(\mu)}. \]  

**Proof** To establish the relationship among \( \Delta_{v-\mu-1} \nabla_{v-\mu} y(z) \), \( \Delta_{v-\mu} y(z) \), and \( y(z) \), we need to find non-zero functions \( A_{i}(z), i = 1, 2, 3 \), such that

\[ A_{1}(z)\Delta_{v-\mu-1} \nabla_{v-\mu} y(z) + A_{2}(z)\nabla_{v-\mu} y(z) + A_{3}(z)y(z) = 0. \]  

Substituting

\[ \Delta_{v-\mu} y(z) = -\gamma(\mu + 1) \sum_{s=a}^{b-1} [x_{v}(s) - x_{v}(z)]^{(\mu)} \rho_{v}(s) \nabla x_{v+1}(s), \]  

\[ \Delta_{v-\mu-1} \nabla_{v-\mu} y(z) = \gamma(\mu + 1)\gamma(\mu) \sum_{s=a}^{b-1} [x_{v}(s) - x_{v}(z - 1)]^{(\mu-1)} \rho_{v}(s) \nabla x_{v+1}(s) \]  

into Eq. (103), we obtain

\[ \sum_{s=a}^{b-1} \rho_{v}(s) \nabla x_{v+1}(s)[x_{v}(s) - x_{v}(z - 1)]^{(\mu-1)}[\gamma(\mu + 1)\gamma(\mu)A_{1}(z) \]  

\[ - \gamma(\mu + 1)A_{2}(z)[x_{v}(s) - x_{v}(z)] + A_{3}(z)[x_{v}(s) - x_{v}(z)][x_{v}(s) - x_{v}(z - \mu)] \]
where
\[ P(s) = \{ \gamma(\mu + 1)\gamma(\mu)A_1(z) - \gamma(\mu + 1)A_2(z)[x_v(s) - x_v(z)] + A_3(z)[x_v(s) - x_v(z)][x_v(s) - x_v(z - \mu)] \}. \]

On the other hand, we let
\[
\sum_{s=a}^{b-1} \rho_v(s) \nabla x_{v+1}(s)[x_v(s) - x_v(z - 1)]^{(\mu-1)} P(s)
\]
\[ = \sum_{s=a}^{b-1} \nabla z[\sigma(s)\rho_v(s)[x_{v+1}(s) - x_{v+1}(z - 1)]^{(\mu)}] \]
\[ = - \sum_{s=a}^{b-1} \tau_v(s)\rho_v(s) \nabla x_{v+1}(s)[x_{v+1}(s - 1) - x_{v+1}(z - 1)]^{(\mu)} \]
\[ + \sum_{s=a}^{b-1} \gamma(\mu)\sigma(s)\rho_v(s) \nabla x_{v+1}(s)[x_v(s) - x_v(z - 1)]^{(\mu-1)} \]
\[ = \sum_{s=a}^{b-1} \rho_v(s) \nabla x_{v+1}(s)[x_v(s) - x_v(z - 1)]^{(\mu-1)} \]
\[ \times \{ \gamma(\mu)\sigma(s) - \tau_v(s)[x_{v-\mu}(s) - x_{v-\mu}(z)] \} \]
\[ = \sum_{s=a}^{b-1} \rho_v(s) \nabla x_{v+1}(s)[x_v(s) - x_v(z - 1)]^{(\mu-1)} Q(s), \]

where
\[ Q(s) = \gamma(\mu)\sigma(s) - \tau_v(s)[x_{v-\mu}(s) - x_{v-\mu}(z)]. \]

By Lemma 2.2, we have
\[ Q(s) = \gamma(\mu)\sigma(z) - \tau_v(s)[x_v(s) - x_v(z)] - \kappa_{2v-(\mu-1)}[x_v(s) - x_v(z)][x_v(s) - x_v(z - \mu)] \]

Comparing \( P(s) \) with \( Q(s) \) gives
\[ A_1(z) = \frac{1}{\gamma(\mu + 1)}\sigma(z), \quad A_2(z) = \frac{\tau_v - \mu(z)}{\gamma(\mu + 1)}, \quad A_3(z) = -\kappa_{2v-(\mu-1)}, \]

and hence we have completed the proof. \( \square \)
Letting $\mu = \nu$ in Theorem 5.1 gives the following theorem.

**Theorem 5.2** Under the hypotheses of Theorem 5.1 with $\mu = \nu$, the equation

$$
\sigma(z) \frac{\Delta}{\Delta x_{-1}(z)} \left( \frac{\nabla y(z)}{\nabla x_0(z)} \right) + \tau(z) \frac{\Delta y(z)}{\Delta x_0(z)} + \lambda y(z) = 0
$$

(113)

has particular solutions in the form of

$$
y(z) = \sum_{s=a}^{b-1} \left[ x_{\nu}(s) - x_{\nu}(z) \right]^{(\nu+1)} \rho_{\nu}(s) \nabla x_{\nu+1}(s)
$$

(114)

and also in the form of

$$
y(z) = \oint_{C} \left[ x_{\nu}(s) - x_{\nu}(z) \right]^{(\nu+1)} \rho_{\nu}(s) \nabla x_{\nu+1}(s) ds,
$$

(115)

where $\rho(z), \rho_{\nu}(z)$ satisfy

$$
\nabla \left( \sigma(z) \rho_{\nu}(z) \right) + \tau_{\nu}(z) \rho_{\nu}(z) = 0,
$$

(116)

and $\nu$ is the root of the equation

$$
\lambda + \kappa_{\nu+1} \gamma(\nu + 1) = 0.
$$

(117)

In contrast with obtaining the solution $\rho_{\nu}(z)$ from the well-known Pearson equation in Eq. (102), it seems more difficult to obtain $\rho_{\nu}(z)$ directly from Eq. (116). However, coupling Eq. (116) with Eq. (102), we may build up a useful relationship between them as described in the following lemma.

**Lemma 5.1** Let $\tilde{\rho}_{\nu}(z), \rho_{\nu}(z)$ satisfy the Pearson equation

$$
\frac{\Delta \left( \sigma(z) \tilde{\rho}_{\nu}(z) \right)}{\nabla x_{\nu+1}(z)} = \tau_{\nu}(z) \tilde{\rho}_{\nu}(z),
$$

(118)

and

$$
\frac{\nabla \left( \sigma(z) \rho_{\nu}(z) \right)}{\nabla x_{\nu+1}(z)} + \tau_{\nu}(z) \rho_{\nu}(z) = 0,
$$

(119)

then it holds

$$
\rho_{\nu}(z) = \frac{\text{const}}{\sigma(z) \sigma(z + 1) \tilde{\rho}_{\nu}(z + 1)}.
$$

(120)
**Proof** Note that

\[
\Delta[\sigma(z)\tilde{\rho}_v(z)\sigma(z-1)\rho_v(z-1)] \\
= \sigma(z)\tilde{\rho}_v(z)\Delta[\sigma(z-1)\rho_v(z-1)] + \sigma(z)\rho_v(z)\Delta[\sigma(z)\tilde{\rho}_v(z)] \\
= \sigma(z)\tilde{\rho}_v(z)\Delta[\sigma(z)\rho_v(z)] + \sigma(z)\rho_v(z)\Delta[\sigma(z)\tilde{\rho}_v(z)].
\]

Based on Eq. (118) and Eq. (119), one may obtain

\[
\Delta[\sigma(z)\tilde{\rho}_v(z)\sigma(z-1)\rho_v(z-1)] \\
= -\sigma(z)\tilde{\rho}_v(z)\tau_v(z)\rho_v(z)\nabla x_{v+1}(z) + \sigma(z)\rho_v(z)\tau_v(z)\tilde{\rho}_v(z)\nabla x_{v+1}(z) \\
= 0,
\]

which yields

\[
\sigma(z)\tilde{\rho}_v(z)\sigma(z-1)\rho_v(z-1) = \text{const}.
\]

Hence, Eq. (120) is obtained. \(\square\)

In particular, when the quadratic lattice \(x(z) = z^2\), we have the following lemma.

**Lemma 5.2** For \(x(z) = z^2\), let \(\rho_v(z)\) satisfy Eq. (119), then

\[
\frac{\rho_v(z+1)}{\rho_v(z)} = \frac{\sigma(z)}{\sigma(-z-1-v)}.
\]

**Proof** For \(x(z) = z^2\), we have \(\nabla x_1(z) = 2z\) and the property (also seen in Eq. (3.10.4) on page 123 in [1])

\[
\sigma(z) + \tau(z)\nabla x_1(z) = \sigma(-z).
\]

Let \(\tilde{\rho}_v(z)\) satisfy Eq. (118). From Eqs. (26) and (122), we obtain

\[
\frac{\tilde{\rho}_v(z+1)}{\tilde{\rho}_v(z)} = \frac{\sigma(z) + \tau_v(z)\nabla x_{v+1}(z)}{\sigma(z+1)} \\
= \frac{\sigma(z + \nu) + \tau(z + \nu)\nabla x_1(z + \nu)}{\sigma(z + 1)} \\
= \frac{\sigma(-z - \nu)}{\sigma(z + 1)}.
\]

By Lemma 5.1 and Eq. (123), we have

\[
\frac{\rho_v(z+1)}{\rho_v(z)} = \frac{\sigma(z)\sigma(z+1)\tilde{\rho}_v(z+1)}{\sigma(z+1)\sigma(z+2)\tilde{\rho}_v(z+2)} \\
= \frac{\sigma(z)}{\sigma(z+2)} \tilde{\rho}_v(z+2)
\]
\[ \frac{\sigma(z)}{\sigma(z + 2) \sigma(-z - 1 - \nu)} = \frac{\sigma(z + 2)}{\sigma(z + 2) \sigma(-z - 1 - \nu)} \]

and complete the proof. \[\square\]

Finally, we give an example to illustrate the application of Theorem 5.2 for the case of the quadratic lattice \( x(z) = z^2 \).

**Example 5.1** Consider the equation

\[ \sigma(z) \frac{\Delta}{\Delta x_{-1}(z)} \left( \nabla y(z) \right) + \tau(z) \frac{\Delta y(z)}{\Delta x_0(z)} + \lambda y(z) = 0, \]

where the lattice \( x(s) = s^2 \), \( \sigma(s) = \prod_{k=1}^{4} (s - s_k) \), and \( s_k, k = 1, 2, 3, 4 \), are arbitrary complex numbers, which give \( \sigma(-s - 1 - \nu) = \prod_{k=1}^{4} (s_k - s - \nu - 1) \). We would like to find its solution.

**Solution.** From Eq. (121), we have

\[ \frac{\rho_\nu(s + 1)}{\rho_\nu(s)} = \frac{\sigma(s)}{\sigma(-s - 1 - \nu)} = \prod_{k=1}^{4} \frac{(s - s_k)}{(-s_k - s - \nu - 1)}. \quad (124) \]

Since

\[ \frac{(s - s_k)}{(-s_k - s - \nu - 1)} = \frac{\Gamma(s + 1 - s_k) \Gamma(-s_k - s - \nu - 1)}{\Gamma(s - s_k) \Gamma(-s_k - s - \nu)}, \]

we choose a solution of Eq. (124) in the form

\[ \rho_\nu(s) = C_0 \prod_{k=1}^{4} \Gamma(s - s_k) \Gamma(-s_k - s - \nu) \sin 2\pi \left( s + \frac{\nu + 1}{2} \right), \]

\[ C_0^{-1} = \frac{\sin \pi(s - z + \nu + 1)}{\sin \pi(s - z)}. \]

Using the “generalized power” in the form given in [3]

\[ [x_\nu(s) - x_\nu(z)]^{(\nu + 1)} = \frac{\Gamma(s - z + \nu + 1) \Gamma(s + z + \nu + 1)}{\Gamma(s - z) \Gamma(s + z)}, \]

\[ x(z) = z^2, \]

and

\[ \frac{\Gamma(s - z + \nu + 1)}{\Gamma(s - z)} = \frac{\pi/[\sin \pi(s - z + \nu + 1) \Gamma(z - s - \nu)]}{\pi/[\sin \pi(s - z) \Gamma(1 + z - s)]}. \]
Adjoint difference equation

\[
\frac{\sin \pi(s - z) \Gamma(1 + z - s)}{\sin \pi(s - z + \nu + 1) \Gamma(z - s - \nu)} = C_0^{-1} \frac{\Gamma(1 + z - s)}{\Gamma(z - s - \nu)},
\]

we obtain

\[
[x_\nu(s) - x_\nu(z)]^{(\nu + 1)} = C_0^{-1} \frac{\Gamma(1 + z - s) \Gamma(s + z + \nu + 1)}{\Gamma(z - s - \nu) \Gamma(s + z)},
\]

\[x(z) = z^2.\]

Based on Eq. (115) in Theorem 5.2, we obtain

\[
y_\nu(z) = \oint_C [x_\nu(s) - x_\nu(z)]^{(\nu + 1)} \rho_\nu(s) \nabla x_{\nu+1}(s) ds
\]

\[
= \oint_C \frac{\Gamma(1 + z - s) \Gamma(s + z + \nu + 1)}{\Gamma(z - s - \nu) \Gamma(s + z)}
\]

\[
\times \prod_{k=1}^{4} \frac{\Gamma(s - s_k) \Gamma(-s_k - s - \nu)(2s + \nu)}{\Gamma(z - s - \nu) \Gamma(s - s_k)}.\]

Setting \(2s + \nu = 2t\), we obtain

\[
2t = \frac{\Gamma(1 + 2t)}{\Gamma(2t)}
\]

\[
= \frac{\pi}{\Gamma(2t) \Gamma(-2t) \sin \pi(-2t)}
\]

\[
= \frac{\pi}{\Gamma(2t) \Gamma(-2t) \sin 2\pi(t + \frac{1}{2})}.
\]

Thus, we obtain a solution

\[
y_\nu(z) = \pi \int_{-\infty}^{i\infty} \frac{\Gamma(1 + z + \frac{\nu}{2} - t) \Gamma(1 + z + \frac{\nu}{2} + t)}{\Gamma(2t) \Gamma(-2t) \Gamma(z - \frac{\nu}{2} - t) \Gamma(z - \frac{\nu}{2} + t)}
\]

\[
\times \prod_{k=1}^{4} \frac{\Gamma(-s_k - \frac{\nu}{2} + t) \Gamma(-s_k - \frac{\nu}{2} - t)}{\Gamma(-s_k - \frac{\nu}{2} - i\nu) \Gamma(-s_k - \frac{\nu}{2} + i\nu)} dt
\]

\[
= \pi \int_{-\infty}^{\infty} \frac{\Gamma(1 + z + \frac{\nu}{2} - t) \Gamma(1 + z + \frac{\nu}{2} + t)}{\Gamma(2t) \Gamma(-2t) \Gamma(z - \frac{\nu}{2} - t) \Gamma(z - \frac{\nu}{2} + t)}
\]

\[
\times \prod_{k=1}^{4} \frac{\Gamma(-s_k - \frac{\nu}{2} + i\nu) \Gamma(-s_k - \frac{\nu}{2} - i\nu)}{\Gamma(-s_k - \frac{\nu}{2} - i\nu) \Gamma(-s_k - \frac{\nu}{2} + i\nu)} dx.
\]
Using the integral representation given in [32] as

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\lambda + i x) \Gamma(\lambda - i x) \Gamma(\mu + i x) \Gamma(\mu - i x)}{\Gamma(2ix) \Gamma(-2ix)} \frac{\Gamma(\gamma + i x) \Gamma(\gamma - i x) \Gamma(\rho + i x) \Gamma(\rho - i x) \Gamma(\sigma + i x) \Gamma(\sigma - i x)}{\Gamma(\tau + i x) \Gamma(\tau - i x)} \, dx
\]

\[
= \frac{2\Gamma(\lambda + \mu) \Gamma(\lambda + \gamma) \Gamma(\lambda + \rho) \Gamma(\lambda + \sigma)}{\Gamma(\lambda + \tau) \Gamma(\mu + \tau) \Gamma(\gamma + \tau) \Gamma(\rho + \tau) \Gamma(\sigma + \tau) \Gamma(\lambda + \mu + \gamma + \tau)}
\times 7 F_6 \left[ \begin{array}{c}
\lambda + \mu + \gamma + \tau - 1, \frac{\lambda + \mu + \gamma + \tau + 1}{2}, \\
\lambda + \tau, \\
\lambda + \mu, \lambda + \gamma, \lambda + \rho, \lambda + \tau - \sigma, \lambda - \rho \\
\end{array} \right] \right]
\]

\(\lambda, \mu, \gamma, \rho, \sigma, \tau \) in Eq. (125), we simplify the solution \(y_\nu(z)\) as

\[
y_\nu(z) = \frac{4\pi^2 \Gamma(1 + z - s_1) \Gamma(1 + z - s_2) \Gamma(1 + z - s_3) \Gamma(1 + z - s_4) \Gamma( - s_1 - s_2 - v)}{\Gamma(1 + 2z) \Gamma(z - s_1 - v) \Gamma(z - s_2 - v) \Gamma(1 + z - s_1 - s_2 - s_3 - v) \Gamma(1 + z - s_3 - s_4 - v) \Gamma(1 + z - s_3 - s_4 - v)}
\times 7 F_6 \left[ \begin{array}{c}
2z - s_1 - s_2 - v, \frac{2z - s_1 - s_2 - v + 2}{2}, \\
\frac{2z - s_1 - s_2 - v}{2}, 1 + 2z, \\
1 + z - s_1, 1 + z - s_2, -s_1 - s_2 - v, z + s_4, z + s_3 \\
z - s_2 - v, z - s_1 - v, 1 + z - s_1 - s_2 - s_4 - v, 1 + z - s_1 - s_2 - s_3 - v \\
\end{array} \right] \right]
\]

Ignoring a constant factor, the solution can be further written as

\[
y_\nu(z) = \frac{\Gamma(1 + 2z - s_1 - s_2 - v) \Gamma(1 + z - s_1) \Gamma(1 + z - s_3) \Gamma(1 + z - s_4) \Gamma(s_1 - s_2 - v)}{\Gamma(1 + 2z) \Gamma(z - s_1 - v) \Gamma(z - s_2 - v) \Gamma(1 + z - s_1 - s_2 - s_3 - v) \Gamma(1 + z - s_3 - s_4 - v) \Gamma(1 + z - s_3 - s_4 - v)}
\times 7 F_6 \left[ \begin{array}{c}
2z - s_1 - s_2 - v, \frac{2z - s_1 - s_2 - v + 2}{2}, -s_1 - s_2 - v, \\
\frac{2z - s_1 - s_2 - v}{2}, 1 + 2z, \\
1 + z - s_1, 1 + z - s_2, -s_1 - s_2 - v, z + s_4, z + s_3 \\
z - s_2 - v, z - s_1 - v, 1 + z - s_1 - s_2 - s_4 - v, 1 + z - s_1 - s_2 - s_3 - v \\
\end{array} \right] \right]
\]

\[
= \frac{2 \prod_{k=1}^{\nu} (z - s_k - v)_{\nu + 1} \prod_{k=3}^{4} (1 + z - s_1 - s_2 - s_k - v)_{s_1 + s_2 + v}}{(1 + 2z - s_1 - s_2 - v)_{s_1 + s_2 + v}}
\]
$\times \, \begin{array}{c} 7 \\ 6 \end{array} F_6 \left[ \begin{array}{c} 2z - s_1 - s_2 - \nu, \frac{2z - s_1 - s_2 - \nu + 2}{2}, -s_1 - s_2 - \nu, \\
\frac{2z - s_1 - s_2 - \nu}{2}, 1 + 2z, \\
1 + z - s_1, 1 + z - s_2, z + s_4, z + s_3 \\
z - s_2 - \nu, z - s_1 - \nu, 1 + z - s_1 - s_2 - s_4 - \nu, 1 + z - s_1 - s_2 - s_3 - \nu \end{array} ; 1 \right]. \quad (126)$

In particular, if we choose $s_1 + s_2 + \nu = n$ and $\nu + 1 = n$ in Eq. (126), then $s_1 + s_2 = 1$. For this case, we can obtain a polynomial solution for Eq. (113) as

$$y_n(z) = \frac{\prod_{k=1}^{4} (1 + z - s_k - n)_n}{(2z + 1 - n)_n} \cdot \begin{array}{c} 7 \\ 6 \end{array} F_6 \left[ \begin{array}{c} 2z - n, z - \frac{n}{2} + 1, -n, \\
z - \frac{n}{2}, 1 + 2z, \\
z + s_2, z + s_1, z + s_4, z + s_3 \\
1 + z - s_2 - n, 1 + z - s_1 - n, 1 + z - s_4 - n, 1 + z - s_3 - n \end{array} ; 1 \right]. \quad (127)$$

which is the same as the well-known formula obtained in [1], (seen in Eq. (3.11.6) on page 134 in [1]). This indicates that Theorem 5.2 gives a more general solution form which includes the well-known polynomial solution as its particular solution.

### 6 Conclusion

We have obtained the adjoint difference equation for the Nikiforov–Uvarov–Suslov difference equation of hypergeometric type on non-uniform lattices given as $x(s) = c_1 q^s + c_2 q^{-s} + c_3$ or $x(s) = \tilde{c}_1 s^2 + \tilde{c}_2 s + \tilde{c}_3$ and proved it to be a difference equation of hypergeometric type on non-uniform lattices as well. The particular solutions of the adjoint difference equation have then been obtained. By applying these particular solutions for the adjoint equation, we can obtain the particular solutions of the original difference equation of hypergeometric type on non-uniform lattices. Finally, we have obtained new fundamental theorems for the Nikiforov–Uvarov–Suslov difference equation of hypergeometric type and illustrated their [33] applications by an example.

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