Models of some cardinal invariants with large continuum

Diego Alejandro Mejía Guzmán

Graduate School of System Informatics
Kobe University

Forcing extensions and large cardinals
RIMS Set Theory Workshop
December 5th, 2012
Some cardinal invariants

Let $X$ be one of the Polish spaces between $2^\omega$, $\omega^\omega$, $\mathbb{R}$ and $[0,1]$ with the Lebesgue measure. $\mathcal{M}$ denotes the $\sigma$-ideal of meager sets of $X$, $\mathcal{N}$ the $\sigma$-ideal of null sets of $X$. For $\mathcal{I} \in \{\mathcal{M},\mathcal{N}\}$, let

$\text{add}(\mathcal{I})$ The additivity of the ideal $\mathcal{I}$ is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union is not in $\mathcal{I}$.

$\text{cov}(\mathcal{I})$ The covering of the ideal $\mathcal{I}$ is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union covers all the reals, i.e., $\bigcup \mathcal{F} = X$.

$\text{non}(\mathcal{I})$ The uniformity of the ideal $\mathcal{I}$ is the least size of a subset of $X$ that is not in $\mathcal{I}$.

$\text{cof}(\mathcal{I})$ The cofinality of the ideal $\mathcal{I}$ is the least size of a cofinal subfamily of $\langle \mathcal{I}, \subseteq \rangle$. 
Some cardinal invariants

Let $X$ be one of the Polish spaces between $2^\omega$, $\omega^n$, $\mathbb{R}$ and $[0,1]$ with the Lebesgue measure. $\mathcal{M}$ denotes the $\sigma$-ideal of meager sets of $X$, $\mathcal{N}$ the $\sigma$-ideal of null sets of $X$. For $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$, let

- $\text{add}(\mathcal{I})$ The *additivity of the ideal $\mathcal{I}$* is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union is not in $\mathcal{I}$.

- $\text{cov}(\mathcal{I})$ The *covering of the ideal $\mathcal{I}$* is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union covers all the reals, i.e., $\bigcup \mathcal{F} = X$.

- $\text{non}(\mathcal{I})$ The *uniformity of the ideal $\mathcal{I}$* is the least size of a subset of $X$ that is not in $\mathcal{I}$.

- $\text{cof}(\mathcal{I})$ The *cofinality of the ideal $\mathcal{I}$* is the least size of a cofinal subfamily of $\langle \mathcal{I}, \subseteq \rangle$. 
Let $X$ be one of the Polish spaces between $2^\omega$, $\omega^\omega$, $\mathbb{R}$ and $[0,1]$ with the Lebesgue measure. $\mathcal{M}$ denotes the $\sigma$-ideal of meager sets of $X$, $\mathcal{N}$ the $\sigma$-ideal of null sets of $X$. For $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$, let

- $\text{add}(\mathcal{I})$ The *additivity of the ideal* $\mathcal{I}$ is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union is not in $\mathcal{I}$.
- $\text{cov}(\mathcal{I})$ The *covering of the ideal* $\mathcal{I}$ is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union covers all the reals, i.e., $\bigcup \mathcal{F} = X$.
- $\text{non}(\mathcal{I})$ The *uniformity of the ideal* $\mathcal{I}$ is the least size of a subset of $X$ that is not in $\mathcal{I}$.
- $\text{cof}(\mathcal{I})$ The *cofinality of the ideal* $\mathcal{I}$ is the least size of a cofinal subfamily of $\langle \mathcal{I}, \subseteq \rangle$.
Some cardinal invariants

Let $X$ be one of the Polish spaces between $2^\omega$, $\omega^\omega$, $\mathbb{R}$ and $[0, 1]$ with the Lebesgue measure. $\mathcal{M}$ denotes the $\sigma$-ideal of meager sets of $X$, $\mathcal{N}$ the $\sigma$-ideal of null sets of $X$. For $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$, let

- \text{add}(\mathcal{I}) The \textit{additivity of the ideal \mathcal{I}} is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union is not in $\mathcal{I}$.
- \text{cov}(\mathcal{I}) The \textit{covering of the ideal \mathcal{I}} is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union covers all the reals, i.e., $\bigcup \mathcal{F} = X$.
- \text{non}(\mathcal{I}) The \textit{uniformity of the ideal \mathcal{I}} is the least size of a subset of $X$ that is not in $\mathcal{I}$.
- \text{cof}(\mathcal{I}) The \textit{cofinality of the ideal \mathcal{I}} is the least size of a cofinal subfamily of $\langle \mathcal{I}, \subseteq \rangle$. 

Diego Alejandro Mejía Guzmán
Some cardinal invariants

Let $X$ be one of the Polish spaces between $2^\omega$, $\omega^\omega$, $\mathbb{R}$ and $[0,1]$ with the Lebesgue measure. $\mathcal{M}$ denotes the $\sigma$-ideal of meager sets of $X$, $\mathcal{N}$ the $\sigma$-ideal of null sets of $X$. For $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$, let

- $\text{add}(\mathcal{I})$ The *additivity of the ideal* $\mathcal{I}$ is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union is not in $\mathcal{I}$.

- $\text{cov}(\mathcal{I})$ The *covering of the ideal* $\mathcal{I}$ is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union covers all the reals, i.e., $\bigcup \mathcal{F} = X$.

- $\text{non}(\mathcal{I})$ The *uniformity of the ideal* $\mathcal{I}$ is the least size of a subset of $X$ that is not in $\mathcal{I}$.

- $\text{cof}(\mathcal{I})$ The *cofinality of the ideal* $\mathcal{I}$ is the least size of a cofinal subfamily of $\langle \mathcal{I}, \subseteq \rangle$.
Some cardinal invariants

Let $X$ be one of the Polish spaces between $2^\omega$, $\omega^\omega$, $\mathbb{R}$ and $[0,1]$ with the Lebesgue measure. $\mathcal{M}$ denotes the $\sigma$-ideal of meager sets of $X$, $\mathcal{N}$ the $\sigma$-ideal of null sets of $X$. For $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$, let

- $\text{add}(\mathcal{I})$ The *additivity of the ideal* $\mathcal{I}$ is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union is not in $\mathcal{I}$.

- $\text{cov}(\mathcal{I})$ The *covering of the ideal* $\mathcal{I}$ is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union covers all the reals, i.e., $\bigcup \mathcal{F} = X$.

- $\text{non}(\mathcal{I})$ The *uniformity of the ideal* $\mathcal{I}$ is the least size of a subset of $X$ that is not in $\mathcal{I}$.

- $\text{cof}(\mathcal{I})$ The *cofinality of the ideal* $\mathcal{I}$ is the least size of a cofinal subfamily of $\langle \mathcal{I}, \subseteq \rangle$. 
Some cardinal invariants

Let $X$ be one of the Polish spaces between $2^\omega$, $\omega^\omega$, $\mathbb{R}$ and $[0,1]$ with the Lebesgue measure. $\mathcal{M}$ denotes the $\sigma$-ideal of meager sets of $X$, $\mathcal{N}$ the $\sigma$-ideal of null sets of $X$. For $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$, let

- **add($\mathcal{I}$)** The additivity of the ideal $\mathcal{I}$ is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union is not in $\mathcal{I}$.

- **cov($\mathcal{I}$)** The covering of the ideal $\mathcal{I}$ is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union covers all the reals, i.e., $\bigcup \mathcal{F} = X$.

- **non($\mathcal{I}$)** The uniformity of the ideal $\mathcal{I}$ is the least size of a subset of $X$ that is not in $\mathcal{I}$.

- **cof($\mathcal{I}$)** The cofinality of the ideal $\mathcal{I}$ is the least size of a cofinal subfamily of $\langle \mathcal{I}, \subseteq \rangle$. 
Consider an increasing sequence $\langle \square_n \rangle_{n<\omega}$ of closed relations in $\omega^\omega$ and $\square = \bigcup_{n<\omega} \square_n$ such that, for every $g \in \omega^\omega$, $\square^g = \{ f \in \omega^\omega \mid f \not\sqsubset g \}$ is meager.

- For a set $Y$ and a real $f \in \omega^\omega$, $f$ is $\square$-unbounded over $Y$ means that $f \not\sqsubset g$ for all $g \in Y \cap \omega^\omega$.
- $b_\square$ is the least size of a $\square$-unbounded family.
- $d_\square$ is the least size of a $\square$-dominating family.
General context

Context

Consider an increasing sequence \( \langle \Box_n \rangle_{n<\omega} \) of closed relations in \( \omega^\omega \) and \( \Box = \bigcup_{n<\omega} \Box_n \) such that, for every \( g \in \omega^\omega \),
\[
\Box^g = \{ f \in \omega^\omega \mid f \not\subset g \}
\]
is meager.

- For a set \( Y \) and a real \( f \in \omega^\omega \), \( f \) is \( \Box \)-unbounded over \( Y \) means that \( f \not\subset g \) for all \( g \in Y \cap \omega^\omega \).
- \( \mathfrak{b}_\Box \) is the least size of a \( \Box \)-unbounded family.
- \( \mathfrak{d}_\Box \) is the least size of a \( \Box \)-dominating family.
General context

Consider an increasing sequence $\langle \sqsubset_n \rangle_{n<\omega}$ of closed relations in $\omega^\omega$ and $\sqsubset = \bigcup_{n<\omega} \sqsubset_n$ such that, for every $g \in \omega^\omega$, $\sqsubset^g = \{ f \in \omega^\omega \mid f \sqsubset g \}$ is meager.

- For a set $Y$ and a real $f \in \omega^\omega$, $f$ is $\sqsubset$-unbounded over $Y$ means that $f \not\sqsubset g$ for all $g \in Y \cap \omega^\omega$.
- $b_{\sqsubset}$ is the least size of a $\sqsubset$-unbounded family.
- $d_{\sqsubset}$ is the least size of a $\sqsubset$-dominating family.
Consider an increasing sequence $\langle \sqsubset n \rangle_{n<\omega}$ of closed relations in $\omega^\omega$ and $\sqsubset = \bigcup_{n<\omega} \sqsubset n$ such that, for every $g \in \omega^\omega$, $\sqsubset^g = \{f \in \omega^\omega \mid f \not\sqsubset g\}$ is meager.

- For a set $Y$ and a real $f \in \omega^\omega$, $f$ is $\sqsubset$-unbounded over $Y$ means that $f \not\sqsubset g$ for all $g \in Y \cap \omega^\omega$.
- $b_\sqsubset$ is the least size of a $\sqsubset$-unbounded family.
- $d_\sqsubset$ is the least size of a $\sqsubset$-dominating family.
Examples

- Define the relation (in $\omega^\omega$) $f \equiv g$ as $f(n) \neq g(n)$ for all but finitely many $n \in \omega$. Here $b_\equiv = \text{non}(\mathcal{M})$ and $d_\equiv = \text{cov}(\mathcal{M})$.

- In $\omega^\omega$, define $f <^* g$ as $f(n) < g(n)$ for all but finitely many $n \in \omega$. Here, $b_{<^*} = b$ and $d_{<^*} = d$ (the well known unbounding and dominating numbers).

- For $f \in \omega^\omega$ and $\varphi : \omega \to [\omega]^{<\omega}$ slalom (i.e., exists $l < \omega$ such that $|\varphi(n)| \leq (n + 1)^l$ for all $n < \omega$), define $f \subseteq^* \varphi$ iff $f(n) \in \varphi(n)$ except for finitely many $n$. Here, $b_{\subseteq^*} = \text{add}(\mathcal{N})$, $d_{\subseteq^*} = \text{cof}(\mathcal{N})$.

- Fix $\langle I_n \rangle_{n<\omega}$ an interval partition of $\omega$ such that $|I_n| = 2^{n+1}$ for every $n < \omega$. For $f, g \in 2^\omega$, define $f \pitchfork g$ iff $f|I_n \neq g|I_n$ for all but finitely many $n < \omega$. 
Examples

- Define the relation (in $\omega^\omega$) $f \equiv g$ as $f(n) \neq g(n)$ for all but finitely many $n \in \omega$. Here $b_\equiv = \text{non}(\mathcal{M})$ and $\delta_\equiv = \text{cov}(\mathcal{M})$.

- In $\omega^\omega$, define $f <^* g$ as $f(n) < g(n)$ for all but finitely many $n \in \omega$. Here, $b_{<^*} = b$ and $\delta_{<^*} = \delta$ (the well known unbounding and dominating numbers).

- For $f \in \omega^\omega$ and $\varphi : \omega \to [\omega]^{<\omega}$ slalom (i.e., exists $l < \omega$ such that $|\varphi(n)| \leq (n + 1)^l$ for all $n < \omega$), define $f \subseteq^* \varphi$ iff $f(n) \in \varphi(n)$ except for finitely many $n$. Here, $b_{\subseteq^*} = \text{add}(\mathcal{N})$, $\delta_{\subseteq^*} = \text{cof}(\mathcal{N})$.

- Fix $\langle I_n \rangle_{n<\omega}$ an interval partition of $\omega$ such that $|I_n| = 2^{n+1}$ for every $n < \omega$. For $f, g \in 2^\omega$, define $f \triangleleft g$ iff $f|I_n \neq g|I_n$ for all but finitely many $n < \omega$. 

Diego Alejandro Mejía Guzmán

Models of some cardinal invariants with large continuum
Examples

- Define the relation (in $\omega^\omega$) $f \equiv g$ as $f(n) \neq g(n)$ for all but finitely many $n \in \omega$. Here $b_\omega = \text{non}(\mathcal{M})$ and $d_\omega = \text{cov}(\mathcal{M})$.

- In $\omega^\omega$, define $f <^* g$ as $f(n) < g(n)$ for all but finitely many $n \in \omega$. Here, $b_{<^*} = b$ and $d_{<^*} = d$ (the well known unbounding and dominating numbers).

- For $f \in \omega^\omega$ and $\varphi : \omega \to [\omega]^{<\omega}$ slalom (i.e., exists $l < \omega$ such that $|\varphi(n)| \leq (n + 1)^l$ for all $n < \omega$), define $f \subseteq^* \varphi$ iff $f(n) \in \varphi(n)$ except for finitely many $n$. Here, $b_{\subseteq^*} = \text{add}(\aleph_1)$, $d_{\subseteq^*} = \text{cof}(\aleph_1)$.

- Fix $\langle I_n \rangle_{n<\omega}$ an interval partition of $\omega$ such that $|I_n| = 2^{n+1}$ for every $n < \omega$. For $f, g \in 2^\omega$, define $f \triangle g$ iff $f|I_n \neq g|I_n$ for all but finitely many $n < \omega$. 

Diego Alejandro Mejía Guzmán

Models of some cardinal invariants with large continuum
Examples

- Define the relation (in $\omega^\omega$) $f \equiv g$ as $f(n) \neq g(n)$ for all but finitely many $n \in \omega$. Here $b_\omega = \text{non}(\mathcal{M})$ and $d_\omega = \text{cov}(\mathcal{M})$.

- In $\omega^\omega$, define $f <^* g$ as $f(n) < g(n)$ for all but finitely many $n \in \omega$. Here, $b_{<^*} = b$ and $d_{<^*} = d$ (the well known unbounding and dominating numbers).

- For $f \in \omega^\omega$ and $\varphi : \omega \to [\omega]^{<\omega}$ slalom (i.e., exists $l < \omega$ such that $|\varphi(n)| \leq (n + 1)^l$ for all $n < \omega$), define $f \subseteq^* \varphi$ iff $f(n) \in \varphi(n)$ except for finitely many $n$. Here, $b_{\subseteq^*} = \text{add}(\mathcal{N})$, $d_{\subseteq^*} = \text{cof}(\mathcal{N})$.

- Fix $\langle I_n \rangle_{n<\omega}$ an interval partition of $\omega$ such that $|I_n| = 2^{n+1}$ for every $n < \omega$. For $f, g \in 2^\omega$, define $f \sqsupseteq g$ iff $f|I_n \neq g|I_n$ for all but finitely many $n < \omega$. 

Diego Alejandro Mejía Guzmán  
Models of some cardinal invariants with large continuum
Examples

- Define the relation (in $\omega^\omega$) $f \equiv g$ as $f(n) \neq g(n)$ for all but finitely many $n \in \omega$. Here $b_\omega = \text{non}(\mathcal{M})$ and $d_\omega = \text{cov}(\mathcal{M})$.

- In $\omega^\omega$, define $f <^* g$ as $f(n) < g(n)$ for all but finitely many $n \in \omega$. Here, $b_{<^*} = b$ and $d_{<^*} = d$ (the well known unbounding and dominating numbers).

- For $f \in \omega^\omega$ and $\varphi : \omega \to [\omega]^{<\omega}$ slalom (i.e., exists $l < \omega$ such that $|\varphi(n)| \leq (n + 1)^l$ for all $n < \omega$), define $f \subseteq^* \varphi$ iff $f(n) \in \varphi(n)$ except for finitely many $n$. Here, $b_{\subseteq^*} = \text{add}(\mathcal{N})$, $d_{\subseteq^*} = \text{cof}(\mathcal{N})$.

- Fix $\langle I_n \rangle_{n<\omega}$ an interval partition of $\omega$ such that $|I_n| = 2^{n+1}$ for every $n < \omega$. For $f, g \in 2^\omega$, define $f \mathrel{\triangleleft} g$ iff $f|I_n \neq g|I_n$ for all but finitely many $n < \omega$. 

Diego Alejandro Mejía Guzmán  
Models of some cardinal invariants with large continuum
Examples

- Define the relation (in $\omega^\omega$) $f =^* g$ as $f(n) \neq g(n)$ for all but finitely many $n \in \omega$. Here $b_\omega = \text{non}(\mathcal{M})$ and $d_\omega = \text{cov}(\mathcal{M})$.

- In $\omega^\omega$, define $f <^* g$ as $f(n) < g(n)$ for all but finitely many $n \in \omega$. Here, $b_{<^*} = b$ and $d_{<^*} = d$ (the well known unbounding and dominating numbers).

- For $f \in \omega^\omega$ and $\varphi : \omega \to [\omega]^{<\omega}$ slalom (i.e., exists $l < \omega$ such that $|\varphi(n)| \leq (n + 1)^l$ for all $n < \omega$), define $f \subseteq^* \varphi$ iff $f(n) \in \varphi(n)$ except for finitely many $n$. Here, $b_{\subseteq^*} = \text{add}(\mathcal{N})$, $d_{\subseteq^*} = \text{cof}(\mathcal{N})$.

- Fix $\langle I_n \rangle_{n<\omega}$ an interval partition of $\omega$ such that $|I_n| = 2^{n+1}$ for every $n < \omega$. For $f, g \in 2^\omega$, define $f \upharpoonright I_n \neq g\upharpoonright I_n$ for all but finitely many $n < \omega$. 
Examples

- Define the relation (in $\omega^\omega$) $f \equiv g$ as $f(n) \neq g(n)$ for all but finitely many $n \in \omega$. Here $b_\equiv = \text{non}(\mathcal{M})$ and $d_\equiv = \text{cov}(\mathcal{M})$.

- In $\omega^\omega$, define $f \prec^* g$ as $f(n) < g(n)$ for all but finitely many $n \in \omega$. Here, $b_{\prec^*} = b$ and $d_{\prec^*} = d$ (the well known unbounding and dominating numbers).

- For $f \in \omega^\omega$ and $\varphi : \omega \to [\omega]^{<\omega}$ slalom (i.e., exists $l < \omega$ such that $|\varphi(n)| \leq (n + 1)^l$ for all $n < \omega$), define $f \subseteq^* \varphi$ iff $f(n) \in \varphi(n)$ except for finitely many $n$. Here, $b_{\subseteq^*} = \text{add}(\mathcal{N})$, $d_{\subseteq^*} = \text{cof}(\mathcal{N})$.

- Fix $\langle I_n \rangle_{n<\omega}$ an interval partition of $\omega$ such that $|I_n| = 2^{n+1}$ for every $n < \omega$. For $f, g \in 2^\omega$, define $f \pitchfork g$ iff $f|I_n \neq g|I_n$ for all but finitely many $n < \omega$. 
Examples

- Define the relation (in $\omega^\omega$) $f \equiv g$ as $f(n) \neq g(n)$ for all but finitely many $n \in \omega$. Here $b_\equiv = \non(M)$ and $d_\equiv = \cov(M)$.

- In $\omega^\omega$, define $f <^* g$ as $f(n) < g(n)$ for all but finitely many $n \in \omega$. Here, $b_{<^*} = b$ and $d_{<^*} = d$ (the well known unbounding and dominating numbers).

- For $f \in \omega^\omega$ and $\varphi : \omega \to [\omega]^{< \omega}$ slalom (i.e., exists $l < \omega$ such that $|\varphi(n)| \leq (n + 1)^l$ for all $n < \omega$), define $f \subseteq^* \varphi$ iff $f(n) \in \varphi(n)$ except for finitely many $n$. Here, $b_{\subseteq^*} = \add(N)$, $d_{\subseteq^*} = \cof(N)$.

- Fix $\langle I_n \rangle_{n < \omega}$ an interval partition of $\omega$ such that $|I_n| = 2^{n+1}$ for every $n < \omega$. For $f, g \in 2^\omega$, define $f \uplus g$ iff $f|I_n \neq g|I_n$ for all but finitely many $n < \omega$. 
Examples

Lemma

\[ \text{cov}(\mathcal{N}) \leq b \leq \text{non}(\mathcal{M}) \text{ and } \text{cov}(\mathcal{M}) \leq d \leq \text{non}(\mathcal{N}). \]

For \( X, A \in [\omega]^{\omega} \), define

- \( X \) splits \( A \) iff \( X \cap A \) and \( A \setminus X \) are infinite.
- \( X \subseteq^* A \) iff \( X \setminus A \) is finite.

Define \( A \subseteq X \) as “\( X \subseteq^* A \) or \( X \subseteq^* \omega \setminus A \)” (i.e. \( A \) does not split \( X \)). Then, \( b \subseteq s \) and \( d \subseteq r \) (the so called splitting and reaping numbers).
**Introduction**

Preserving $\square$-unbounded families
Preserving $\square$-unbounded reals
Matrix Iterations
Questions

**Some cardinal invariants**

**General context**

More cardinal invariants

---

**Examples**

---

**Lemma**

\[
\text{cov}(\mathcal{N}) \leq b \leq \text{non}(\mathcal{M}) \text{ and } \text{cov}(\mathcal{M}) \leq d \leq \text{non}(\mathcal{N}).
\]

For $X, A \in [\omega]^{\omega}$, define

- $X$ splits $A$ iff $X \cap A$ and $A \setminus X$ are infinite.
- $X \subseteq^* A$ iff $X \setminus A$ is finite.

Define $A \subseteq X$ as “$X \subseteq^* A$ or $X \subseteq^* \omega \setminus A$” (i.e. $A$ does not split $X$). Then, $b \subseteq = s$ and $d \subseteq = r$ (the so called splitting and reaping numbers).
Examples

Lemma

\[ \text{cov}(\mathcal{N}) \leq b_\mathfrak{f} \leq \text{non}(\mathcal{M}) \text{ and } \text{cov}(\mathcal{M}) \leq d_\mathfrak{f} \leq \text{non}(\mathcal{N}). \]

For \( X, A \in [\omega]^{\omega} \), define

- \( X \text{ splits } A \) iff \( X \cap A \) and \( A \setminus X \) are infinite.
- \( X \subseteq^* A \) iff \( X \setminus A \) is finite.

Define \( A \subseteq X \) as “\( X \subseteq^* A \) or \( X \subseteq^* \omega \setminus A \)” (i.e. \( A \) does not split \( X \)). Then, \( b_\mathfrak{f} = s \) and \( d_\mathfrak{f} = r \) (the so called splitting and reaping numbers).
Lemma

\[ \text{cov}(\mathcal{N}) \leq b_{\mathfrak{f}} \leq \text{non}(\mathcal{M}) \text{ and } \text{cov}(\mathcal{M}) \leq d_{\mathfrak{f}} \leq \text{non}(\mathcal{N}). \]

For \( X, A \in [\omega]^\omega \), define

- *\( X \) splits \( A \) iff \( X \cap A \) and \( A \setminus X \) are infinite.
- *\( X \) \( \subseteq^* \) \( A \) iff \( X \setminus A \) is finite.

Define \( A \in X \) as \( \text{“} X \subseteq^* A \text{ or } X \subseteq^* \omega \setminus A \text{”} \) (i.e. \( A \) does not split \( X \)). Then, \( b_{\mathfrak{c}} = s \) and \( d_{\mathfrak{c}} = r \) (the so called splitting and reaping numbers).
Lemma

\[ \text{cov}(\mathcal{N}) \leq b_{\beth} \leq \text{non}(\mathcal{M}) \text{ and } \text{cov}(\mathcal{M}) \leq d_{\beth} \leq \text{non}(\mathcal{N}). \]

For \( X, A \in [\omega]^{\omega} \), define

- \( X \) splits \( A \) iff \( X \cap A \) and \( A \setminus X \) are infinite.
- \( X \subseteq^* A \) iff \( X \setminus A \) is finite.

Define \( A \subseteq X \) as “\( X \subseteq^* A \) or \( X \subseteq^* \omega \setminus A \)” (i.e. \( A \) does not split \( X \)). Then, \( b_{\beth} = s \) and \( d_{\beth} = r \) (the so-called splitting and reaping numbers).
Examples

Lemma

\[ \text{cov}(\mathcal{N}) \leq b \leq \text{non}(\mathcal{M}) \text{ and } \text{cov}(\mathcal{M}) \leq d \leq \text{non}(\mathcal{N}). \]

For \( X, A \in [\omega]^{\omega} \), define

- \( X \) splits \( A \) iff \( X \cap A \) and \( A \setminus X \) are infinite.
- \( X \) \( \subseteq^* \) \( A \) iff \( X \setminus A \) is finite.

Define \( A \in X \) as “\( X \subseteq^* A \) or \( X \subseteq^* \omega \setminus A \)” (i.e. \( A \) does not split \( X \)). Then, \( b = s \) and \( d = r \) (the so called splitting and reaping numbers).
More cardinal invariants

Say that $\mathcal{F} \subseteq [\omega]^\omega$ is a *filter base* if it is closed under finite intersections and contains all the cofinite subsets of $\omega$. $A \in [\omega]^\omega$ is a *pseudo-intersection* of $\mathcal{F}$ if $A \subseteq^* X$ for every $X \in \mathcal{F}$. Define
- $p$ (pseudo-intersection number): the least size of a filter base without pseudo-intersection.
- $u$ (ultrafilter number): the least size of a filter base that generates a (non-principal) ultrafilter.
Say that $\mathcal{F} \subseteq [\omega]^\omega$ is a filter base if it is closed under finite intersections and contains all the coinfinite subsets of $\omega$. $A \in [\omega]^\omega$ is a pseudo-interesection of $\mathcal{F}$ if $A \subseteq^* X$ for every $X \in \mathcal{F}$. Define

- $\text{p}$ (pseudo-intersection number): the least size of a filter base without pseudo-intersection.
- $\text{u}$ (ultrafilter number): the least size of a filter base that generates a (non-principal) ultrafilter.
More cardinal invariants

Say that $\mathcal{F} \subseteq [\omega]^\omega$ is a filter base if it is closed under finite intersections and contains all the coinfinites of $\omega$. $A \in [\omega]^\omega$ is a pseudo-interesection of $\mathcal{F}$ if $A \subseteq^* X$ for every $X \in \mathcal{F}$. Define

- $p$ (pseudo-intersection number): the least size of a filter base without pseudo-intersection.
- $u$ (ultrafilter number): the least size of a filter base that generates a (non-principal) ultrafilter.
More cardinal invariants

Say that $\mathcal{F} \subseteq [\omega]^\omega$ is a filter base if it is closed under finite intersections and contains all the coinfinite subsets of $\omega$. $A \in [\omega]^\omega$ is a pseudo-interesection of $\mathcal{F}$ if $A \subseteq^* X$ for every $X \in \mathcal{F}$. Define

- $p$ (pseudo-intersection number): the least size of a filter base without pseudo-intersection.
- $u$ (ultrafilter number): the least size of a filter base that generates a (non-principal) ultrafilter.
Some forcing notions

1. Trivial forcing $1 = \{0\}$.

A. Amoeba forcing.
B. Random forcing.
C. Cohen forcing.
D. Hechler forcing.
E. Eventually different real forcing.

All these are Suslin c.c.c. forcing notions.

$\mathbb{M}_\mathcal{F}$ Mathias forcing with a filter base $\mathcal{F}$. 
Some forcing notions

1. Trivial forcing $\mathbb{1} = \{0\}$.

A. Amoeba forcing.

B. Random forcing.

C. Cohen forcing.

D. Hechler forcing.

E. Eventually different real forcing.

All these are Suslin c.c.c. forcing notions.

M$\mathcal{F}$ Mathias forcing with a filter base $\mathcal{F}$. 
Some forcing notions

1. Trivial forcing $\mathbb{1} = \{0\}$.

A. Amoeba forcing.

B. Random forcing.

C. Cohen forcing.

D. Hechler forcing.

E. Eventually different real forcing.

All these are Suslin c.c.c. forcing notions.

$\mathbb{M}_\mathcal{F}$ Mathias forcing with a filter base $\mathcal{F}$.
Some forcing notions

1. Trivial forcing $\mathbb{1} = \{0\}$.
2. Amoeba forcing.
3. Random forcing.
4. Cohen forcing.
5. Hechler forcing.
6. Eventually different real forcing.

All these are Suslin c.c.c. forcing notions.

$\mathbb{M}_{\mathcal{F}}$ Mathias forcing with a filter base $\mathcal{F}$. 
Some forcing notions

1. Trivial forcing $\mathbb{1} = \{0\}$.

A. Amoeba forcing.

B. Random forcing.

C. Cohen forcing.

D. Hechler forcing.

E. Eventually different real forcing.

All these are Suslin c.c.c. forcing notions.

$M_{\mathcal{F}}$ Mathias forcing with a filter base $\mathcal{F}$. 
Some forcing notions

1. Trivial forcing $\mathbb{1} = \{0\}$.
2. Amoeba forcing.
3. Random forcing.
4. Cohen forcing.
5. Hechler forcing.
6. Eventually different real forcing.

All these are Suslin c.c.c. forcing notions.

$\mathbb{M}_\mathcal{F}$ Mathias forcing with a filter base $\mathcal{F}$. 
Some forcing notions

1. Trivial forcing $\mathbb{1} = \{0\}$.

A. Amoeba forcing.

B. Random forcing.

C. Cohen forcing.

D. Hechler forcing.

E. Eventually different real forcing.

All these are Suslin c.c.c. forcing notions.

$M_{\mathcal{F}}$ Mathias forcing with a filter base $\mathcal{F}$. 
Some forcing notions

1. Trivial forcing $1 = \{0\}$.

A. Amoeba forcing.

B. Random forcing.

C. Cohen forcing.

D. Hechler forcing.

E. Eventually different real forcing.

All these are Suslin c.c.c. forcing notions.

$M_{\mathcal{F}}$ Mathias forcing with a filter base $\mathcal{F}$.

Diego Alejandro Mejía Guzmán

Models of some cardinal invariants with large continuum
Fix $\kappa$ an uncountable regular cardinal.

For $F \subseteq \omega^\omega$ consider the property

$$(\triangle, \square, F, \kappa)$$

For all $X \subseteq \omega^\omega$, if $|X| < \kappa$, then there exists an $f \in F$ which is $\square$-unbounded over $X$.

For a forcing notion $\mathbb{P}$, consider the property

$$(+_{\mathbb{P}, \square}^\kappa)$$

$\mathbb{P}$ is $\kappa$-c.c. and, for every $\dot{h}$ $\mathbb{P}$-name for a real in $\omega^\omega$, there exists a $Y \subseteq \omega^\omega$, $|Y| < \kappa$ such that, for every real $f$ that is $\square$-unbounded over $Y$, $\not\models f \not\subseteq \dot{h}$.

For $\kappa = \aleph_1$ the previous properties are denoted by $(\triangle, \square, F)$ and $(+_{\mathbb{P}, \square})$, respectively.
Fix $\kappa$ an uncountable regular cardinal. For $F \subseteq \omega^\omega$ consider the property

$$(\triangle, \square, F, \kappa)$$

For all $X \subseteq \omega^\omega$, if $|X| < \kappa$, then there exists an $f \in F$ which is $\square$-unbounded over $X$.

For a forcing notion $\mathbb{P}$, consider the property

$$(+_{\mathbb{P}, \square}^{\kappa})$$

$\mathbb{P}$ is $\kappa$-c.c. and, for every $\dot{h}$ $\mathbb{P}$-name for a real in $\omega^\omega$, there exists a $Y \subseteq \omega^\omega$, $|Y| < \kappa$ such that, for every real $f$ that is $\square$-unbounded over $Y$, $\not\vDash f \not\subseteq \dot{h}$.

For $\kappa = \aleph_1$ the previous properties are denoted by $(\triangle, \square, F)$ and $(+_{\mathbb{P}, \square})$, respectively.
Fix $\kappa$ an uncountable regular cardinal. For $F \subseteq \omega^\omega$ consider the property

$$(\Delta, \sqsubset, F, \kappa) \text{ For all } X \subseteq \omega^\omega, \text{ if } |X| < \kappa, \text{ then there exists an } f \in F \text{ which is } \sqsubset \text{-unbounded over } X.$$ 

For a forcing notion $\mathbb{P}$, consider the property

$$(+^{\kappa}_{\mathbb{P}}, \sqsubset) \text{ } \mathbb{P} \text{ is } \kappa\text{-c.c. and, for every } \dot{h} \mathbb{P}\text{-name for a real in } \omega^\omega, \text{ there exists a } Y \subseteq \omega^\omega, |Y| < \kappa \text{ such that, for every real } f \text{ that is } \sqsubset \text{-unbounded over } Y, \models f \nsubseteq \dot{h}.$$ 

For $\kappa = \aleph_1$ the previous properties are denoted by $(\Delta, \sqsubset, F)$ and $(+_{\mathbb{P}, \sqsubset})$, respectively.
Fix $\kappa$ an uncountable regular cardinal. For $F \subseteq \omega^\omega$ consider the property

$$(\triangleleft, \sqsubseteq, F, \kappa)$$

For all $X \subseteq \omega^\omega$, if $|X| < \kappa$, then there exists an $f \in F$ which is $\sqsubseteq$-unbounded over $X$.

For a forcing notion $\mathbb{P}$, consider the property

$$(+^\kappa_{\mathbb{P}, \sqsubseteq})$$

$\mathbb{P}$ is $\kappa$-c.c. and, for every $\dot{h}$ $\mathbb{P}$-name for a real in $\omega^\omega$, there exists a $Y \subseteq \omega^\omega$, $|Y| < \kappa$ such that, for every real $f$ that is $\sqsubseteq$-unbounded over $Y$, $\Vdash f \not\in \dot{h}$.

For $\kappa = \aleph_1$ the previous properties are denoted by $(\triangleleft, \sqsubseteq, F)$ and $(+_{\mathbb{P}, \sqsubseteq})$, respectively.
Preservation properties

Fix $\kappa$ an uncountable regular cardinal. For $F \subseteq \omega^\omega$ consider the property

$$(\bullet, \sqsubset, F, \kappa)$$
For all $X \subseteq \omega^\omega$, if $|X| < \kappa$, then there exists an $f \in F$ which is $\sqsubset$-unbounded over $X$.

For a forcing notion $\mathbb{P}$, consider the property

$$(+_{\mathbb{P}, \sqsubset})$$
$\mathbb{P}$ is $\kappa$-c.c. and, for every $\dot{h}$ $\mathbb{P}$-name for a real in $\omega^\omega$, there exists a $Y \subseteq \omega^\omega$, $|Y| < \kappa$ such that, for every real $f$ that is $\sqsubset$-unbounded over $Y$, $\forces f \not\in \dot{h}$.

For $\kappa = \aleph_1$ the previous properties are denoted by $(\bullet, \sqsubset, F)$ and $(+_{\mathbb{P}, \sqsubset})$, respectively.
Fix $\kappa$ an uncountable regular cardinal.

For $F \subseteq \omega^\omega$ consider the property
\[
(\blacktriangle, \square, F, \kappa)
\]
For all $X \subseteq \omega^\omega$, if $|X| < \kappa$, then there exists an $f \in F$ which is $\square$-unbounded over $X$.

For a forcing notion $\mathbb{P}$, consider the property
\[
(+^\kappa_{\mathbb{P}}, \square)
\]
$\mathbb{P}$ is $\kappa$-c.c. and, for every $\dot{h} \mathbb{P}$-name for a real in $\omega^\omega$, there exists a $Y \subseteq \omega^\omega$, $|Y| < \kappa$ such that, for every real $f$ that is $\square$-unbounded over $Y$, $\vDash f \not\subset \dot{h}$.

For $\kappa = \aleph_1$ the previous properties are denoted by $(\blacktriangle, \square, F)$ and $(+_{\mathbb{P}}, \square)$, respectively.
Preservation properties

Lemma

\((\Delta, \square, F, \kappa)\) implies \(b_{\square} \leq |F|\) and \(\kappa \leq d_{\square}\).

Theorem (Judah and Shelah, 1990, Brendle, 1991)

Forcing notions satisfying \((+^{\kappa}_{\square})\) preserve \((\Delta, \square, F, \kappa)\) and \(\lambda \leq d_{\square}\) for any \(\lambda \geq \kappa\).

Theorem (Judah and Shelah, 1990, Brendle, 1991)

\((+^{\kappa}_{\square})\) is preserved in f.s.i. of \(\kappa\)-c.c. notions.
Preservation properties

Lemma

$$(\triangleleft, \sqsubseteq, F, \kappa) \text{ implies } b_{\mathbb{C}} \leq |F| \text{ and } \kappa \leq \mathfrak{d}_{\mathbb{C}}.$$ 

Theorem (Judah and Shelah, 1990, Brendle, 1991)

Forcing notions satisfying $(+, \kappa, \subseteq)$ preserve $(\triangleleft, \sqsubseteq, F, \kappa)$ and $\lambda \leq \mathfrak{d}_{\mathbb{C}}$ for any $\lambda \geq \kappa$.

Theorem (Judah and Shelah, 1990, Brendle, 1991)

$(+, \kappa, \subseteq)$ is preserved in f.s.i. of $\kappa$-c.c. notions.
Preservation properties

Lemma

\((\blacktriangle, \Box, F, \kappa)\) implies \(b_{\Box} \leq |F|\) and \(\kappa \leq \mathfrak{d}_{\Box}\).

Theorem (Judah and Shelah, 1990, Brendle, 1991)

Forcing notions satisfying \((+^\kappa, \Box)\) preserve \((\blacktriangle, \Box, F, \kappa)\) and \(\lambda \leq \mathfrak{d}_{\Box}\) for any \(\lambda \geq \kappa\).

Theorem (Judah and Shelah, 1990, Brendle, 1991)

\((+^\kappa, \Box)\) is preserved in f.s.i. of \(\kappa\)-c.c. notions.
Particular cases

- Every forcing notion of size $< \kappa$ satisfies $(+^{\kappa}, \subseteq)$. In particular, $(+^{\mathcal{C}}, \subseteq)$ holds.
- $(+^{\mathcal{B}}, <^*)$ and $(+^{\mathcal{E}}, <^*)$ hold (last by Miller, 1981).
- (Brendle, 1991) Given $\mu < \kappa$, $\mu$-centered forcing notions satisfies $(+^{\kappa}, \subseteq)$.
- (Judah and Shelah, 1990, Brendle, 1991) Given $\mu < \kappa$, every $\mu$-centered forcing notion satisfies $(+^{\kappa}, \subseteq^*)$.
- (Kamburelis, 1989) Every subalgebra of $\mathcal{B}$ satisfies $(+^{\cdot}, \subseteq^*)$.
- (Baumgartner, Dordal, 1985) $(+^{\mathcal{D}}, \in)$ holds.
Particular cases

- Every forcing notion of size $< \kappa$ satisfies $(+_\kappa \subseteq)$. In particular, $(+_C \subseteq)$ holds.
- $(+_B, <^*)$ and $(+_E, <^*)$ hold (last by Miller, 1981).
- (Brendle, 1991) Given $\mu < \kappa$, $\mu$-centered forcing notions satisfies $(+_{\kappa, \mu})$.
- (Judah and Shelah, 1990, Brendle, 1991) Given $\mu < \kappa$, every $\mu$-centered forcing notion satisfies $(+_{\kappa, \subseteq^*})$.
- (Kamburelis, 1989) Every subalgebra of $B$ satisfies $(+_{\kappa, \subseteq^*})$.
- (Baumgartner, Dordal, 1985) $(+_D, \subseteq)$ holds.
Particular cases

- Every forcing notion of size $< \kappa$ satisfies $(+^{\kappa}, \subseteq)$. In particular, $(+^{\mathcal{C}}, \subseteq)$ holds.
- $(+^{\mathcal{B}}, <^*)$ and $(+^{\mathcal{E}}, <^*)$ hold (last by Miller, 1981).
- (Brendle, 1991) Given $\mu < \kappa$, $\mu$-centered forcing notions satisfies $(+^{\kappa}, \subseteq^*)$.
- (Judah and Shelah, 1990, Brendle, 1991) Given $\mu < \kappa$, every $\mu$-centered forcing notion satisfies $(+^{\kappa}, \subseteq^*)$.
- (Kamburelis, 1989) Every subalgebra of $\mathcal{B}$ satisfies $(+^{\cdot}, \subseteq^*)$.
- (Baumgartner, Dordal, 1985) $(+^{\mathcal{D}}, \in)$ holds.
Particular cases

- Every forcing notion of size $< \kappa$ satisfies $(+\kappa, \subseteq)$. In particular, $(+\mathcal{C}, \subseteq)$ holds.
- $(+\mathcal{B}, <^*)$ and $(+\mathcal{E}, <^*)$ hold (last by Miller, 1981).
- (Brendle, 1991) Given $\mu < \kappa$, $\mu$-centered forcing notions satisfies $(+\kappa, \mathfrak{m})$.
- (Judah and Shelah, 1990, Brendle, 1991) Given $\mu < \kappa$, every $\mu$-centered forcing notion satisfies $(+\kappa, \subseteq^*)$.
- (Kamburelis, 1989) Every subalgebra of $\mathcal{B}$ satisfies $(+\kappa, \subseteq^*)$.
- (Baumgartner, Dordal, 1985) $(+\mathcal{D}, \subseteq)$ holds.
Particular cases

- Every forcing notion of size $< \kappa$ satisfies $(\kappa, \mathcal{C})$. In particular, $(\kappa, \mathcal{C})$ holds.
- $(\mathcal{B}, <^*)$ and $(\mathcal{E}, <^*)$ hold (last by Miller, 1981).
- (Brendle, 1991) Given $\mu < \kappa$, $\mu$-centered forcing notions satisfies $(\kappa, \mathfrak{m})$.
- (Judah and Shelah, 1990, Brendle, 1991) Given $\mu < \kappa$, every $\mu$-centered forcing notion satisfies $(\kappa, \subseteq^*)$.
- (Kamburelis, 1989) Every subalgebra of $\mathcal{B}$ satisfies $(\kappa, \subseteq^*)$.
- (Baumgartner, Dordal, 1985) $(\mathcal{D}, \subseteq)$ holds.
Particular cases

- Every forcing notion of size $< \kappa$ satisfies $(+^{\kappa}, \subseteq)$. In particular, $(+^{\kappa}, \subseteq)$ holds.
- $(+_{\mathcal{B}}, <^*)$ and $(+_{\mathcal{E}}, <^*)$ hold (last by Miller, 1981).
- (Brendle, 1991) Given $\mu < \kappa$, $\mu$-centered forcing notions satisfies $(+^{\kappa}, \subseteq)$.
- (Judah and Shelah, 1990, Brendle, 1991) Given $\mu < \kappa$, every $\mu$-centered forcing notion satisfies $(+^{\kappa}, \subseteq^*)$.
- (Kamburelis, 1989) Every subalgebra of $\mathcal{B}$ satisfies $(+^{\kappa}, \subseteq^*)$.
- (Baumgartner, Dordal, 1985) $(+_{\mathcal{D}}, \subseteq)$ holds.
Theorem

Let $\mu_1 \leq \mu_2 \leq \kappa$ be uncountable regular cardinals, $\lambda \geq \kappa$ a cardinal such that $\text{cf}(\lambda) \geq \kappa$. Then, it is consistent that $\text{add}(\mathcal{N}) = \mu_1$, $\text{cov}(\mathcal{N}) = \mu_2$, $\mathfrak{p} = \text{non}(\mathcal{M}) = \kappa$ and $\text{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$.

Here, $\mathfrak{s} = \kappa$ and $\mathfrak{r} = \mathfrak{u} = \lambda$. 

Diego Alejandro Mejía Guzmán
Models of some cardinal invariants with large continuum
If $\mu_1 \leq \mu_2 \leq \mu_3 \leq \kappa$ are regular uncountable, $\lambda \geq \kappa$ and $\text{cf}(\lambda) \geq \mu_3$, we can get models of ZFC plus:

$p = s = \mu_3$ and $r = u = c = \lambda$. 
Applications

Question 1

(1) Does \((+_{\mathcal{E},\subseteq})\) hold?

(2) Which conditions do we require for a suborder \(\mathbb{P}\) of \(\mathbb{D}\) so that \((+_{\mathbb{P},\subseteq})\) holds?

(3) In general, if \(\$\) is a suslin ccc poset, does \((+_{\$},\subseteq)\) hold?
Applications

Question 1

(1) Does \((+_{E,\infty})\) hold?

(2) Which conditions do we require for a suborder \(P\) of \(D\) so that \((+_{P,\infty})\) holds?

(3) In general, if \(S\) is a suslin ccc poset, does \((+_{S,\infty})\) hold?
Applications

Question 1

(1) Does $(+_{E,\infty})$ hold?

(2) Which conditions do we require for a suborder $P$ of $D$ so that $(+_{P,\infty})$ holds?

(3) In general, if $S$ is a suslin ccc poset, does $(+_{S,\infty})$ hold?
Applications

Question 1

(1) Does $(+_{\mathcal{E}, \infty})$ hold?

(2) Which conditions do we require for a suborder $\mathbb{P}$ of $\mathbb{D}$ so that $(+_{\mathbb{P}, \infty})$ holds?

(3) In general, if $\mathbb{S}$ is a suslin ccc poset, does $(+_{\mathbb{S}, \infty})$ hold?
Fix $M \subseteq N$ transitive standard models of ZFC.

- If $P \in M$ and $Q$ are p.o., we say that $P$ is a complete suborder of $Q$ respect to $M$, denoted by $P \leq_M Q$, iff $P \subseteq Q$ and every maximal antichain of $P$ in $M$ is a maximal antichain of $Q$.

**Theorem (Brendle, Fischer, 2011)**

Let $\delta$ be an ordinal, $P_{0,\delta} = \langle P_{0,\alpha}, Q_{0,\alpha} \rangle_{\alpha < \delta}$ a f.s.i. of c.c.c. forcing defined in $M$ and $P_{1,\delta} = \langle P_{1,\alpha}, Q_{1,\alpha} \rangle_{\alpha < \delta}$ a f.s.i. of c.c.c. forcing defined in $N$. Then, $P_{0,\delta} \leq_M P_{1,\delta}$ iff, for every $\alpha < \delta$, $P_{0,\alpha} \leq_M P_{1,\alpha}$ and $\models_{P_{1,\alpha}, N} Q_{0,\alpha} \leq_M P_{0,\alpha} Q_{1,\alpha}$. 
Relative complete suborder

Fix $M \subseteq N$ transitive standard models of ZFC.

- If $P \in M$ and $Q$ are p.o., we say that $P$ is a complete suborder of $Q$ respect to $M$, denoted by $P \leq_M Q$, iff $P \subseteq Q$ and every maximal antichain of $P$ in $M$ is a maximal antichain of $Q$.

**Theorem (Brendle, Fischer, 2011)**

Let $\delta$ be an ordinal, $P_{0,\delta} = \langle P_{0,\alpha}, \dot{Q}_{0,\alpha} \rangle_{\alpha < \delta}$ a f.s.i. of c.c.c. forcing defined in $M$ and $P_{1,\delta} = \langle P_{1,\alpha}, \dot{Q}_{1,\alpha} \rangle_{\alpha < \delta}$ a f.s.i. of c.c.c. forcing defined in $N$. Then, $P_{0,\delta} \leq_M P_{1,\delta}$ iff, for every $\alpha < \delta$, $P_{0,\alpha} \leq_M P_{1,\alpha}$ and $\Vdash_{P_{1,\alpha}, N} \dot{Q}_{0,\alpha} \leq_M P_{0,\alpha} \dot{Q}_{1,\alpha}$. 

Diego Alejandro Mejía Guzmán

Models of some cardinal invariants with large continuum
Fix $M \subseteq N$ transitive standard models of ZFC.

- If $P \in M$ and $Q$ are p.o., we say that $P$ is a complete suborder of $Q$ respect to $M$, denoted by $P \preceq_M Q$, iff $P \subseteq Q$ and every maximal antichain of $P$ in $M$ is a maximal antichain of $Q$.

**Theorem (Brendle, Fischer, 2011)**

Let $\delta$ be an ordinal, $P_{0,\delta} = \langle P_{0,\alpha}, Q_{0,\alpha} \rangle_{\alpha < \delta}$ a f.s.i. of c.c.c. forcing defined in $M$ and $P_{1,\delta} = \langle P_{1,\alpha}, Q_{1,\alpha} \rangle_{\alpha < \delta}$ a f.s.i. of c.c.c. forcing defined in $N$. Then, $P_{0,\delta} \preceq_M P_{1,\delta}$ iff, for every $\alpha < \delta$, $P_{0,\alpha} \preceq_M P_{1,\alpha}$ and $\models_{P_{1,\alpha}, N} Q_{0,\alpha} \preceq_M P_{0,\alpha} Q_{1,\alpha}$. 
Preservation of □-unbounded reals

Consider the context for the relation □. If \( P \in M, Q \in N, \)
\( P \succeq_M Q \) and \( c \in N \cap \omega^\omega \) is a □-unbounded real over \( M \), define
the property

\[ (∗, P, Q, M, N, □, c) \]

For every \( \dot{h} \in M \) \( P \)-name for a real in \( \omega^\omega \),
\( \models_{Q,N} c \not\triangleleft \dot{h} \). This is equivalent to say that \( \models_{Q,N} \text{“} c \) is
□-unbounded over \( M^P \)”, i.e., \( c \) is □-unbounded over
\( M[G \cap P] \) for every \( G \) \( Q \)-generic over \( N \).

\[ \begin{array}{c}
  c \in N \\
  \hline
  N[G] \\
  \hline
  M \\
  \hline
  M[G \cap P]
\end{array} \]
Theorem (Brendle, Fischer, 2011)

With the hypothesis of the previous theorem, if $\mathbb{P}_{0,\delta} \preceq_M \mathbb{P}_{1,\delta}$,

$(\star, \mathbb{P}_{0,\delta}, \mathbb{P}_{1,\delta}, M, N, \square, c)$ iff, for every $\alpha < \delta$,

$(\star, \mathbb{P}_{0,\alpha}, \mathbb{P}_{1,\alpha}, M, N, \square, c)$ and

$\models \mathbb{P}_{1,\alpha}, N (\star, \mathcal{Q}_{0,\alpha}, \mathcal{Q}_{1,\alpha}, M^{\mathbb{P}_{0,\alpha}}, N^{\mathbb{P}_{1,\alpha}}, \square, c)$. 

\[ c \in N = N_0 \]

\[ M = M_0 \]
Cases of preservation of □-unbounded reals

**Theorem**

Let $c \in N$ be a □-unbounded real over $M$.

(a) If $P$ is a Suslin c.c.c. forcing notion with parameters in $M$ and $(+P,\Box)$ holds in $M$, then $(\ast, P^M, P^N, M, N, \Box, c)$.

(b) (Brendle, Fischer, 2011) If $P \in M$ is a p.o., then $(\ast, P, P, M, N, \Box, c)$.

Note also that every Cohen real over $M$ is □-unbounded over $M$. 
Cases of preservation of $\square$-unbounded reals

**Theorem**

Let $c \in N$ be a $\square$-unbounded real over $M$.

(a) If $P$ is a Suslin c.c.c. forcing notion with parameters in $M$ and $(+_{P}, \square)$ holds in $M$, then $(\ast, P^{M}, P^{N}, M, N, \square, c)$.

(b) (Brendle, Fischer, 2011) If $P \in M$ is a p.o., then $(\ast, P, P, M, N, \square, c)$.

Note also that every Cohen real over $M$ is $\square$-unbounded over $M$. 
Cases of preservation of □-unbounded reals

**Theorem**

Let \( c \in N \) be a □-unbounded real over \( M \).

(a) If \( P \) is a Suslin c.c.c. forcing notion with parameters in \( M \) and \((+_{P,\Box})\) holds in \( M \), then \((\ast, P^M, P^N, M, N, \Box, c)\).

(b) (Brendle, Fischer, 2011) If \( P \in M \) is a p.o., then \((\ast, P, P, M, N, \Box, c)\).

Note also that every Cohen real over \( M \) is □-unbounded over \( M \).
A case of preservation of unbounded reals

Theorem (Blass, Shelah, 1984)

In $M$, let $\mathcal{U}$ be an ultrafilter. If $c \in N$ is a $<^*$-unbounded real over $M$, then there exists an ultrafilter $\mathcal{V}$ in $N$ extending $\mathcal{U}$ such that $(\star, \mathbb{M}_\mathcal{U}, \mathbb{M}_\mathcal{V}, M, N, <^*, c)$ holds.

The same holds if we consider $\subseteq^*$ instead of $<^*$. 
Matrix iterations of c.c.c. forcing notions

For $\delta, \gamma$ ordinals, in a ground model $V$ we consider a matrix iteration $\langle \langle P_{\alpha,\xi}, Q_{\alpha,\xi} \rangle_{\xi<\gamma} \rangle_{\alpha \leq \delta}$ defined by the following conditions.

1. $P_{\delta,0} = \langle P_{\alpha,0}, R_{\alpha} \rangle_{\alpha<\delta}$ is a f.s.i. of c.c.c. notions.
2. For a fixed $\alpha \leq \delta$, $P_{\alpha,\gamma} = \langle P_{\alpha,\xi}, Q_{\alpha,\xi} \rangle_{\xi<\gamma}$ is a f.s.i. of c.c.c forcing notions.
3. For $\alpha < \beta \leq \delta, \xi < \gamma$, $P_{\alpha,\xi} \leq_V P_{\beta,\xi}$.
4. For $\alpha < \beta \leq \delta, \xi < \gamma$, $\forces_{\beta,\xi} Q_{\alpha,\xi} \leq_V P_{\alpha,\xi} Q_{\beta,\xi}$.

(3)+(4) is equivalent to $P_{\alpha,\xi} \leq_V P_{\beta,\xi}$ for every $\alpha \leq \beta \leq \delta, \xi \leq \gamma$. 
Matrix iterations of c.c.c. forcing notions

For $\delta$, $\gamma$ ordinals, in a ground model $V$ we consider a matrix iteration $\langle \langle P_{\alpha, \xi}, Q_{\alpha, \xi} \rangle_{\xi < \gamma} \rangle_{\alpha \leq \delta}$ defined by the following conditions.

(1) $P_{\delta, 0} = \langle P_{\alpha, 0}, Q_{\alpha} \rangle_{\alpha < \delta}$ is a f.s.i. of c.c.c. notions.

(2) For a fixed $\alpha \leq \delta$, $P_{\alpha, \gamma} = \langle P_{\alpha, \xi}, Q_{\alpha, \xi} \rangle_{\xi < \gamma}$ is a f.s.i. of c.c.c. forcing notions.

(3) For $\alpha < \beta \leq \delta$, $\xi < \gamma$, $P_{\alpha, \xi} \preceq_V P_{\beta, \xi}$.

(4) For $\alpha < \beta \leq \delta$, $\xi < \gamma$, $\models_{\beta, \xi} Q_{\alpha, \xi} \preceq_V P_{\alpha, \xi} Q_{\beta, \xi}$.

(3)+(4) is equivalent to $P_{\alpha, \xi} \preceq_V P_{\beta, \xi}$ for every $\alpha \leq \beta \leq \delta$, $\xi \leq \gamma$. 
Matrix iterations of c.c.c. forcing notions

For $\delta, \gamma$ ordinals, in a ground model $V$ we consider a matrix iteration $\langle \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \gamma} \rangle_{\alpha \leq \delta}$ defined by the following conditions.

(1) $P_{\delta, 0} = \langle P_{\alpha, 0}, \dot{R}_\alpha \rangle_{\alpha < \delta}$ is a f.s.i. of c.c.c. notions.

(2) For a fixed $\alpha \leq \delta$, $P_{\alpha, \gamma} = \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \gamma}$ is a f.s.i. of c.c.c forcing notions.

(3) For $\alpha < \beta \leq \delta, \xi < \gamma$, $P_{\alpha, \xi} \leq_V P_{\beta, \xi}$.

(4) For $\alpha < \beta \leq \delta, \xi < \gamma$, $\models_{\beta, \xi} \dot{Q}_{\alpha, \xi} \leq_V P_{\alpha, \xi} \dot{Q}_{\beta, \xi}$.

(3)+(4) is equivalent to $P_{\alpha, \xi} \leq_V P_{\beta, \xi}$ for every $\alpha \leq \beta \leq \delta$, $\xi \leq \gamma$. 
Matrix iterations of c.c.c. forcing notions

For δ, γ ordinals, in a ground model V we consider a matrix iteration \( \langle\langle P_{\alpha,\xi}, Q_{\alpha,\xi}\rangle\xi<\gamma\rangle_{\alpha\leq\delta} \) defined by the following conditions.

1. \( P_{\delta,0} = \langle\langle P_{\alpha,0}, R_{\alpha}\rangle\alpha<\delta \rangle \) is a f.s.i. of c.c.c. notions.
2. For a fixed \( \alpha \leq \delta \), \( P_{\alpha,\gamma} = \langle\langle P_{\alpha,\xi}, Q_{\alpha,\xi}\rangle\xi<\gamma \rangle \) is a f.s.i. of c.c.c forcing notions.
3. For \( \alpha < \beta \leq \delta, \xi < \gamma \), \( P_{\alpha,\xi} \preceq V P_{\beta,\xi} \).
4. For \( \alpha < \beta \leq \delta, \xi < \gamma \), \( V P_{\alpha,\xi} \preceq P_{\beta,\xi} \).

(3)+(4) is equivalent to \( P_{\alpha,\xi} \preceq V P_{\beta,\xi} \) for every \( \alpha \leq \beta \leq \delta, \xi \leq \gamma \).
For δ, γ ordinals, in a ground model V we consider a matrix iteration \( \langle \langle P_{\alpha, \xi}, Q_{\alpha, \xi} \rangle_{\xi < \gamma} \rangle_{\alpha \leq \delta} \) defined by the following conditions.

(1) \( P_{\delta, 0} = \langle P_{\alpha, 0}, R_\alpha \rangle_{\alpha < \delta} \) is a f.s.i. of c.c.c. notions.
(2) For a fixed \( \alpha \leq \delta \), \( P_{\alpha, \gamma} = \langle P_{\alpha, \xi}, Q_{\alpha, \xi} \rangle_{\xi < \gamma} \) is a f.s.i. of c.c.c forcing notions.
(3) For \( \alpha < \beta \leq \delta, \xi < \gamma \), \( P_{\alpha, \xi} \leq_V P_{\beta, \xi} \).
(4) For \( \alpha < \beta \leq \delta, \xi < \gamma \), \( \Vdash_{\beta, \xi} Q_{\alpha, \xi} \leq_V P_{\alpha, \xi} Q_{\beta, \xi} \).

(3)+(4) is equivalent to \( P_{\alpha, \xi} \leq_V P_{\beta, \xi} \) for every \( \alpha \leq \beta \leq \delta, \xi \leq \gamma \).
Matrix iterations of c.c.c. forcing notions

For $\delta, \gamma$ ordinals, in a ground model $V$ we consider a matrix iteration $\langle \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \gamma} \rangle_{\alpha \leq \delta}$ defined by the following conditions.

1. $P_{\delta, 0} = \langle P_{\alpha, 0}, \dot{R}_{\alpha} \rangle_{\alpha < \delta}$ is a f.s.i. of c.c.c. notions.
2. For a fixed $\alpha \leq \delta$, $P_{\alpha, \gamma} = \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \gamma}$ is a f.s.i. of c.c.c forcing notions.
3. For $\alpha < \beta \leq \delta$, $\xi < \gamma$, $P_{\alpha, \xi} \leq_V P_{\beta, \xi}$.
4. For $\alpha < \beta \leq \delta$, $\xi < \gamma$, $\models_{\beta, \xi} \dot{Q}_{\alpha, \xi} \leq_V P_{\alpha, \xi} \dot{Q}_{\beta, \xi}$.

(3)+(4) is equivalent to $P_{\alpha, \xi} \leq_V P_{\beta, \xi}$ for every $\alpha \leq \beta \leq \delta$, $\xi \leq \gamma$. 
Matrix iterations of c.c.c. forcing notions

Like in the case of “linear” iterations, $V_{\alpha,\xi}$ denotes a $P_{\alpha,\xi}$-extension for $\alpha \leq \delta, \xi \leq \gamma$. Here, $V_{0,0} = V$ and the generic extensions can be seen as in the figure.
An application

Theorem

Let $\mu_1 \leq \mu_2 \leq \kappa$ be uncountable regular cardinals, $\lambda \geq \kappa$ a cardinal such that $\text{cf}(\lambda) \geq \mu_1$. Then, it is consistent with ZFC that $\text{add}(\mathcal{N}) = \mu_1$, $\text{cov}(\mathcal{N}) = p = \text{cof}(\mathcal{M}) = \mu_2$, $\text{non}(\mathcal{N}) = r = \kappa$ and $\text{cof}(\mathcal{N}) = c = \lambda$. 
Sketched proof

Start with $V$ a model of ZFC plus $\text{add}(\mathcal{N}) = \text{non}(\mathcal{M}) = \mu_1$ and $\text{cov}(\mathcal{M}) = c = \lambda$. Also, there exists an $A$ of size $\mu_1$ such that $(\triangle, A, \subseteq^*, \mu_1)$. Let $t : \kappa \mu_2 \to \kappa$ such that, for each $\alpha < \kappa$ and $\eta < \kappa \mu_2$, there exists a $\delta$ such that $\eta < \delta < \kappa \mu_2$ and $t(\delta) = \alpha$. Also, fix a bijection $g : \lambda \to \kappa \times \lambda$. Perform a matrix iteration $\langle \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \lambda \kappa \mu_2} \rangle_{\alpha \leq \kappa}$ (dimensions $\kappa \times (\lambda \kappa \mu_2)$) as follows: let $P_{\alpha, 0}$ be the $\alpha$-iteration of Cohen forcing, $\dot{c}_\alpha$ the $P_{\alpha+1, 0}$-name of the Cohen real added in the step $\alpha + 1$. We proceed to define the horizontal iterations in the interval $[\lambda \rho, \lambda (\rho + 1))$ for each $\rho < \kappa \mu_2$.

(a) If $\xi = \lambda \rho$, let

$$\dot{Q}_{\alpha, \xi} = \begin{cases} 1, & \text{if } \alpha \leq t(\rho), \\ \dot{B}_\rho, & \text{if } \alpha > t(\rho), \end{cases}$$

where $\dot{B}_\rho$ is a $P_{t(\rho), \xi}$-name for $\dot{B}$.

(b) If $\xi = \lambda \rho + 1$, $\dot{Q}_{\alpha, \xi}$ is a $P_{\alpha, \xi}$-name for $\dot{D}$.
Sketched proof

Start with $V$ a model of ZFC plus $\text{add}(\mathcal{N}) = \text{non}(\mathcal{M}) = \mu_1$ and $\text{cov}(\mathcal{M}) = c = \lambda$. Also, there exists an $A$ of size $\mu_1$ such that $(\triangle, A, \subseteq^*, \mu_1)$. Let $t : \kappa \mu_2 \to \kappa$ such that, for each $\alpha < \kappa$ and $\eta < \kappa \mu_2$, there exists a $\delta$ such that $\eta < \delta < \kappa \mu_2$ and $t(\delta) = \alpha$. Also, fix a bijection $g : \lambda \to \kappa \times \lambda$. Perform a matrix iteration $\langle \langle P_\alpha, \xi, \dot{Q}_\alpha, \xi \rangle_{\xi \leq \lambda \mu_2} \rangle_{\alpha \leq \kappa}$ (dimensions $\kappa \times (\lambda \kappa \mu_2)$) as follows: let $P_{\alpha, 0}$ be the $\alpha$-iteration of Cohen forcing, $\dot{c}_\alpha$ the $P_{\alpha + 1, 0}$-name of the Cohen real added in the step $\alpha + 1$. We proceed to define the horizontal iterations in the interval $[\lambda \rho, \lambda (\rho + 1))$ for each $\rho < \kappa \mu_2$.

(a) If $\xi = \lambda \rho$, let

$$\dot{Q}_\alpha, \xi = \begin{cases} 1, & \text{if } \alpha \leq t(\rho), \\ \dot{B}_\rho, & \text{if } \alpha > t(\rho), \end{cases}$$

where $\dot{B}_\rho$ is a $P_{t(\rho), \xi}$-name for $\mathbb{B}$.

(b) If $\xi = \lambda \rho + 1$, $\dot{Q}_\alpha, \xi$ is a $P_{\alpha, \xi}$-name for $\mathbb{D}$.
Start with $V$ a model of ZFC plus $\text{add}(\mathcal{N}) = \text{non}(\mathcal{M}) = \mu_1$ and $\text{cov}(\mathcal{M}) = c = \lambda$. Also, there exists an $A$ of size $\mu_1$ such that $(\Delta, A, \subseteq^*, \mu_1)$. Let $t : \kappa\mu_2 \to \kappa$ such that, for each $\alpha < \kappa$ and $\eta < \kappa\mu_2$, there exists a $\delta$ such that $\eta < \delta < \kappa\mu_2$ and $t(\delta) = \alpha$. Also, fix a bijection $g : \lambda \to \kappa \times \lambda$. Perform a matrix iteration $\langle \langle P_{\alpha, \xi}, Q_{\alpha, \xi} \rangle_{\xi < \lambda \kappa \mu_2} \rangle_{\alpha \leq \kappa}$ (dimensions $\kappa \times (\lambda \kappa \mu_2)$) as follows: let $P_{\alpha, 0}$ be the $\alpha$-iteration of Cohen forcing, $\dot{c}_{\alpha}$ the $P_{\alpha+1,0}$-name of the Cohen real added in the step $\alpha + 1$. We proceed to define the horizontal iterations in the interval $[\lambda \rho, \lambda(\rho + 1))$ for each $\rho < \kappa\mu_2$.

(a) If $\xi = \lambda \rho$, let

$$
\dot{Q}_{\alpha, \xi} = \left\{ \begin{array}{ll}
1, & \text{if } \alpha \leq t(\rho), \\
\dot{B}_\rho, & \text{if } \alpha > t(\rho),
\end{array} \right.
$$

where $\dot{B}_\rho$ is a $P_{t(\rho), \xi}$-name for $B$.

(b) If $\xi = \lambda \rho + 1$, $\dot{Q}_{\alpha, \xi}$ is a $P_{\alpha, \xi}$-name for $\dot{D}$.
Start with $V$ a model of ZFC plus $\text{add}(\mathcal{N}) = \text{non}(\mathcal{M}) = \mu_1$ and $\text{cov}(\mathcal{M}) = c = \lambda$. Also, there exists an $A$ of size $\mu_1$ such that $(\triangle, A, \subseteq^*, \mu_1)$. Let $t : \kappa \mu_2 \to \kappa$ such that, for each $\alpha < \kappa$ and $\eta < \kappa \mu_2$, there exists a $\delta$ such that $\eta < \delta < \kappa \mu_2$ and $t(\delta) = \alpha$. Also, fix a bijection $g : \lambda \to \kappa \times \lambda$. Perform a matrix iteration $\langle (\mathcal{P}_\alpha, \delta, \dot{\mathcal{Q}}_{\alpha, \delta})_{\delta < \lambda \kappa \mu_2} \rangle_{\alpha \leq \kappa}$ (dimensions $\kappa \times (\lambda \kappa \mu_2)$) as follows: let $\mathcal{P}_{\alpha, 0}$ be the $\alpha$-iteration of Cohen forcing, $\dot{c}_\alpha$ the $\mathcal{P}_{\alpha+1, 0}$-name of the Cohen real added in the step $\alpha + 1$. We proceed to define the horizontal iterations in the interval $[\lambda \rho, \lambda(\rho + 1))$ for each $\rho < \kappa \mu_2$.

(a) If $\xi = \lambda \rho$, let

$$\dot{Q}_{\alpha, \xi} = \begin{cases} 1, & \text{if } \alpha \leq t(\rho), \\ \dot{B}_\rho, & \text{if } \alpha > t(\rho), \end{cases}$$

where $\dot{B}_\rho$ is a $\mathcal{P}_{t(\rho), \xi}$-name for $B$.

(b) If $\xi = \lambda \rho + 1$, $\dot{Q}_{\alpha, \xi}$ is a $\mathcal{P}_{\alpha, \xi}$-name for $\dot{D}$. 
Sketched proof

Start with \( V \) a model of ZFC plus \( \text{add}(\mathcal{N}) = \text{non}(\mathcal{M}) = \mu_1 \) and \( \text{cov}(\mathcal{M}) = c = \lambda \). Also, there exists an \( A \) of size \( \mu_1 \) such that \( (\blacksquare, A, \subseteq^*, \mu_1) \).

Let \( t: \kappa \mu_2 \to \kappa \) such that, for each \( \alpha < \kappa \) and \( \eta < \kappa \mu_2 \), there exists a \( \delta \) such that \( \eta < \delta < \kappa \mu_2 \) and \( t(\delta) = \alpha \). Also, fix a bijection \( g: \lambda \to \kappa \times \lambda \).

Perform a matrix iteration \( \langle \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \lambda \kappa \mu_2} \rangle_{\alpha \leq \kappa} \) (dimensions \( \kappa \times (\lambda \kappa \mu_2) \)) as follows: let \( P_{\alpha, 0} \) be the \( \alpha \)-iteration of Cohen forcing, \( \dot{c}_{\alpha} \) the \( P_{\alpha + 1, 0} \)-name of the Cohen real added in the step \( \alpha + 1 \). We proceed to define the horizontal iterations in the interval \([\lambda \rho, \lambda (\rho + 1)]\) for each \( \rho < \kappa \mu_2 \).

(a) If \( \xi = \lambda \rho \), let

\[
\dot{Q}_{\alpha, \xi} = \begin{cases} 
1, & \text{if } \alpha \leq t(\rho), \\
\dot{B}_{\rho}, & \text{if } \alpha > t(\rho),
\end{cases}
\]

where \( \dot{B}_{\rho} \) is a \( P_{t(\rho), \xi} \)-name for \( B \).

(b) If \( \xi = \lambda \rho + 1 \), \( \dot{Q}_{\alpha, \xi} \) is a \( P_{\alpha, \xi} \)-name for \( \dot{D} \).
Sketched proof

Start with $V$ a model of $\text{ZFC}$ plus $\text{add}(\mathcal{N}) = \text{non}(\mathcal{M}) = \mu_1$ and $\text{cov}(\mathcal{M}) = c = \lambda$. Also, there exists an $A$ of size $\mu_1$ such that $(\triangledown, A, \subseteq^*, \mu_1)$. Let $t : \kappa\mu_2 \to \kappa$ such that, for each $\alpha < \kappa$ and $\eta < \kappa\mu_2$, there exists a $\delta$ such that $\eta < \delta < \kappa\mu_2$ and $t(\delta) = \alpha$. Also, fix a bijection $g : \lambda \to \kappa \times \lambda$. Perform a matrix iteration $\langle \langle P_{\alpha, \xi}, \dot{Q}_{\alpha, \xi} \rangle_{\xi < \lambda\kappa\mu_2} \rangle_{\alpha \leq \kappa}$ (dimensions $\kappa \times (\lambda\kappa\mu_2)$) as follows: let $P_{\alpha,0}$ be the $\alpha$-iteration of Cohen forcing, $\dot{c}_\alpha$ the $P_{\alpha+1,0}$-name of the Cohen real added in the step $\alpha + 1$. We proceed to define the horizontal iterations in the interval $[\lambda\rho, \lambda(\rho + 1))$ for each $\rho < \kappa\mu_2$.

(a) If $\xi = \lambda\rho$, let

$$\dot{Q}_{\alpha, \xi} = \begin{cases} 1, & \text{if } \alpha \leq t(\rho), \\ \dot{B}_\rho, & \text{if } \alpha > t(\rho), \end{cases}$$

where $\dot{B}_\rho$ is a $P_{t(\rho), \xi}$-name for $B$.

(b) If $\xi = \lambda\rho + 1$, $\dot{Q}_{\alpha, \xi}$ is a $P_{\alpha, \xi}$-name for $\dot{D}$. 
Sketched proof

Start with $V$ a model of ZFC plus $\text{add}(\mathcal{N}) = \text{non}(\mathcal{M}) = \mu_1$ and $\text{cov}(\mathcal{M}) = c = \lambda$. Also, there exists an $A$ of size $\mu_1$ such that $(\Box, A, \subseteq^*, \mu_1)$. Let $t : \kappa \mu_2 \to \kappa$ such that, for each $\alpha < \kappa$ and $\eta < \kappa \mu_2$, there exists a $\delta$ such that $\eta < \delta < \kappa \mu_2$ and $t(\delta) = \alpha$. Also, fix a bijection $g : \lambda \to \kappa \times \lambda$. Perform a matrix iteration $\langle \langle P_\alpha, \xi, \dot{Q}_\alpha, \xi \rangle_{\xi < \lambda \kappa \mu_2} \rangle_{\alpha \leq \kappa}$ (dimensions $\kappa \times (\lambda \kappa \mu_2)$) as follows: let $P_\alpha, 0$ be the $\alpha$-iteration of Cohen forcing, $\dot{c}_\alpha$ the $P_{\alpha + 1, 0}$-name of the Cohen real added in the step $\alpha + 1$. We proceed to define the horizontal iterations in the interval $[\lambda \rho, \lambda(\rho + 1))$ for each $\rho < \kappa \mu_2$.

(a) If $\xi = \lambda \rho$, let

$$
\dot{Q}_\alpha, \xi = \begin{cases} 
1, & \text{if } \alpha \leq t(\rho), \\
\dot{B}_\rho, & \text{if } \alpha > t(\rho), 
\end{cases}
$$

where $\dot{B}_\rho$ is a $P_{t(\rho), \xi}$-name for $\mathcal{B}$.

(b) If $\xi = \lambda \rho + 1$, $\dot{Q}_\alpha, \xi$ is a $P_{\alpha, \xi}$-name for $\mathcal{D}$. 
Start with $V$ a model of ZFC plus $\text{add}(\mathcal{N}) = \text{non}(\mathcal{M}) = \mu_1$ and $\text{cov}(\mathcal{M}) = c = \lambda$. Also, there exists an $A$ of size $\mu_1$ such that $(\triangle, A, \subseteq^*, \mu_1)$. Let $t : \kappa\mu_2 \to \kappa$ such that, for each $\alpha < \kappa$ and $\eta < \kappa\mu_2$, there exists a $\delta$ such that $\eta < \delta < \kappa\mu_2$ and $t(\delta) = \alpha$. Also, fix a bijection $g : \lambda \to \kappa \times \lambda$. Perform a matrix iteration $\langle \langle \mathbb{P}_{\alpha, \xi}, \check{Q}_{\alpha, \xi} \rangle_{\xi < \lambda\kappa\mu_2} \rangle_{\alpha \leq \kappa}$ (dimensions $\kappa \times (\lambda\kappa\mu_2)$) as follows: let $\mathbb{P}_{\alpha, 0}$ be the $\alpha$-iteration of Cohen forcing, $\check{c}_{\alpha}$ the $\mathbb{P}_{\alpha+1, 0}$-name of the Cohen real added in the step $\alpha + 1$. We proceed to define the horizontal iterations in the interval $[\lambda \rho, \lambda(\rho + 1))$ for each $\rho < \kappa\mu_2$.

(a) If $\xi = \lambda \rho$, let

$$\check{Q}_{\alpha, \xi} = \begin{cases} 1, & \text{if } \alpha \leq t(\rho), \\ \check{B}_\rho, & \text{if } \alpha > t(\rho), \end{cases}$$

where $\check{B}_\rho$ is a $\mathbb{P}_{t(\rho), \xi}$-name for $\mathcal{B}$.

(b) If $\xi = \lambda \rho + 1$, $\check{Q}_{\alpha, \xi}$ is a $\mathbb{P}_{\alpha, \xi}$-name for $\check{\mathcal{D}}$. 
(c) If $\xi = \lambda \rho + 2$, let

$$\dot{Q}_{\alpha,\xi} = \begin{cases} 1, & \text{if } \alpha \leq t(\rho), \\ M\dot{U}_\rho, & \text{if } \alpha > t(\rho), \end{cases}$$

where $\dot{U}_\rho$ is a $P_{t(\rho),\xi}$-name for a non-principal ultrafilter on $\omega$.

Now, for $\alpha < \kappa$, consider, $\langle \dot{A}_\rho^\alpha,\gamma \rangle_{\gamma < \lambda}$ and $\langle \dot{F}_\rho^\alpha,\gamma \rangle_{\gamma < \lambda}$ the $P_{\alpha,\lambda \rho + 3}$-names for all suborders of $A_{V_{\alpha,\lambda \rho + 3}}$ of size $< \mu_1$ and all filter basis in $V_{\alpha,\lambda \rho + 3}$ of size $< \mu_2$, respectively. For $\epsilon < \lambda$,

(d) If $\xi = \lambda \rho + 3 + 2\epsilon$, put

$$\dot{Q}_{\alpha,\xi} = \begin{cases} 1, & \text{if } \alpha \leq (g(\epsilon))_0, \\ A_{g(\epsilon)}^\rho, & \text{if } \alpha > (g(\epsilon))_0. \end{cases}$$

(e) If $\xi = \lambda \rho + 3 + 2\epsilon + 1$, put

$$\dot{Q}_{\alpha,\xi} = \begin{cases} 1, & \text{if } \alpha \leq (g(\epsilon))_0, \\ M\dot{F}_{g(\epsilon)}^\rho, & \text{if } \alpha > (g(\epsilon))_0. \end{cases}$$
(c) If $\xi = \lambda \rho + 2$, let

$$Q_{\alpha, \xi} = \begin{cases} 1, & \text{if } \alpha \leq t(\rho), \\ \mathbb{M}_{U_{\rho}}, & \text{if } \alpha > t(\rho), \end{cases}$$

where $U_{\rho}$ is a $\mathbb{P}_{t(\rho), \xi}$-name for a non-principal ultrafilter on $\omega$.

Now, for $\alpha < \kappa$, consider, $\langle \hat{A}^\rho_{\alpha, \gamma} \rangle_{\gamma < \lambda}$ and $\langle \hat{F}^\rho_{\alpha, \gamma} \rangle_{\gamma < \lambda}$ the $\mathbb{P}_{\alpha, \lambda \rho + 3}$-names for all suborders of $A^{V_{\alpha, \lambda \rho + 3}}$ of size $< \mu_1$ and all filter basis in $V_{\alpha, \lambda \rho + 3}$ of size $< \mu_2$, respectively. For $\epsilon < \lambda$,

(d) If $\xi = \lambda \rho + 3 + 2\epsilon$, put

$$Q_{\alpha, \xi} = \begin{cases} 1, & \text{if } \alpha \leq (g(\epsilon))_0, \\ \hat{A}^\rho_{g(\epsilon)}, & \text{if } \alpha > (g(\epsilon))_0. \end{cases}$$

(e) If $\xi = \lambda \rho + 3 + 2\epsilon + 1$, put

$$Q_{\alpha, \xi} = \begin{cases} 1, & \text{if } \alpha \leq (g(\epsilon))_0, \\ \mathbb{M}_{\hat{F}^\rho_{g(\epsilon)}}, & \text{if } \alpha > (g(\epsilon))_0. \end{cases}$$
(c) If $\xi = \lambda \rho + 2$, let

$$
\dot{Q}_{\alpha, \xi} = \begin{cases} 
1, & \text{if } \alpha \leq t(\rho), \\
M_{\dot{U}_\rho}, & \text{if } \alpha > t(\rho),
\end{cases}
$$

where $\dot{U}_\rho$ is a $\mathbb{P}_{t(\rho), \xi}$-name for a non-principal ultrafilter on $\omega$.

Now, for $\alpha < \kappa$, consider, $\langle \dot{A}_\alpha, \gamma \rangle_{\gamma < \lambda}$ and $\langle \dot{F}_\alpha, \gamma \rangle_{\gamma < \lambda}$ the $\mathbb{P}_{\alpha, \lambda \rho + 3}$-names for all suborders of $A_{V_{\alpha, \lambda \rho + 3}}$ of size $< \mu_1$ and all filter basis in $V_{\alpha, \lambda \rho + 3}$ of size $< \mu_2$, respectively. For $\epsilon < \lambda$,

(d) If $\xi = \lambda \rho + 3 + 2\epsilon$, put

$$
\dot{Q}_{\alpha, \xi} = \begin{cases} 
1, & \text{if } \alpha \leq (g(\epsilon))_0, \\
\dot{A}_\rho, & \text{if } \alpha > (g(\epsilon))_0.
\end{cases}
$$

(e) If $\xi = \lambda \rho + 3 + 2\epsilon + 1$, put

$$
\dot{Q}_{\alpha, \xi} = \begin{cases} 
1, & \text{if } \alpha \leq (g(\epsilon))_0, \\
M_{\dot{F}_\rho}, & \text{if } \alpha > (g(\epsilon))_0.
\end{cases}
$$
Sketched proof

**Theorem (Brendle, Fischer, 2011)**

If $\xi \leq \lambda \kappa \mu_2$ and $x$ is a real in $V_{\kappa, \xi}$, then $x \in V_{\alpha, \xi}$ for some $\alpha < \kappa$.

In the iterations for each $\rho < \kappa \mu_2$, $B^{V_{t(\rho), \lambda \rho}}$ adds a random real $r_\rho \in V_{t(\rho)+1, \lambda \rho+1}$ over $V_{t(\rho), \lambda \rho}$ and $M^{U_{\rho}}$ adds a Mathias real $m_\rho \in V_{t(\rho)+1, \lambda \rho+3}$ over $V_{t(\rho), \lambda \rho+2}$.

**Claim**

For every family of Borel non-null sets coded in $V_{\kappa, \lambda \kappa \mu_2}$ of size $< \mu_2$, there is a $r_\rho$ that is not in its union. Thus, $\mu_2 \leq \text{cov}(\mathcal{N})$ and $\text{non}(\mathcal{N}) \leq \kappa$.

**Claim**

For every family of size $< \mu_2$ of infinite subsets of $\omega$ in $V_{\kappa, \lambda \kappa \mu_2}$ there is some $m_\rho$ which cannot be splitted by any member of the family. Thus, $\tau \leq \kappa$. 
Sketched proof

**Theorem (Brendle, Fischer, 2011)**

*If* $\xi \leq \lambda \kappa \mu_2$ *and* $x$ *is a real in* $V_{\kappa, \xi}$, *then* $x \in V_{\alpha, \xi}$ *for some* $\alpha < \kappa$.

In the iterations for each $\rho < \kappa \mu_2$, $B^{V_{t(\rho)}, \lambda \rho}$ *adds a random real $r_\rho \in V_{t(\rho)+1, \lambda \rho+1}$ over* $V_{t(\rho), \lambda \rho}$ *and* $M_{\mathcal{U}_\rho}$ *adds a Mathias real $m_\rho \in V_{t(\rho)+1, \lambda \rho+3}$ over* $V_{t(\rho), \lambda \rho+2}$.

**Claim**

For every family of Borel non-null sets coded in $V_{\kappa, \lambda \kappa \mu_2}$ of size $< \mu_2$, there is a $r_\rho$ that is not in its union. Thus, $\mu_2 \leq \text{cov}(\mathcal{N})$ and $\text{non}(\mathcal{N}) \leq \kappa$.

**Claim**

For every family of size $< \mu_2$ of infinite subsets of $\omega$ in $V_{\kappa, \lambda \kappa \mu_2}$ there is some $m_\rho$ which cannot be splitted by any member of the family. Thus, $\tau \leq \kappa$. 
**Theorem (Brendle, Fischer, 2011)**

If $\xi \leq \lambda \kappa \mu_2$ and $x$ is a real in $V_{\kappa, \xi}$, then $x \in V_{\alpha, \xi}$ for some $\alpha < \kappa$.

In the iterations for each $\rho < \kappa \mu_2$, $B^{V_{t(\rho)}, \lambda \rho}$ adds a random real $r_{\rho} \in V_{t(\rho)+1, \lambda \rho+1}$ over $V_{t(\rho), \lambda \rho}$ and $M_{U_{\rho}}$ adds a Mathias real $m_{\rho} \in V_{t(\rho)+1, \lambda \rho+3}$ over $V_{t(\rho), \lambda \rho+2}$.

**Claim**

For every family of Borel non-null sets coded in $V_{\kappa, \lambda \kappa \mu_2}$ of size $< \mu_2$, there is a $r_{\rho}$ that is not in its union. Thus, $\mu_2 \leq \text{cov}(\mathcal{N})$ and $\text{non}(\mathcal{N}) \leq \kappa$.

**Claim**

For every family of size $< \mu_2$ of infinite subsets of $\omega$ in $V_{\kappa, \lambda \kappa \mu_2}$ there is some $m_{\rho}$ which cannot be splitted by any member of the family. Thus, $\tau \leq \kappa$. 
Sketched proof

**Theorem (Brendle, Fischer, 2011)**

If $\xi \leq \lambda\kappa\mu_2$ and $x$ is a real in $V_{\kappa,\xi}$, then $x \in V_{\alpha,\xi}$ for some $\alpha < \kappa$.

In the iterations for each $\rho < \kappa\mu_2$, $B^{V_{t(\rho),\lambda\rho}}$ adds a random real $r_\rho \in V_{t(\rho)+1,\lambda\rho+1}$ over $V_{t(\rho),\lambda\rho}$ and $M_{U_\rho}$ adds a Mathias real $m_\rho \in V_{t(\rho)+1,\lambda\rho+3}$ over $V_{t(\rho),\lambda\rho+2}$.

**Claim**

For every family of Borel non-null sets coded in $V_{\kappa,\lambda\kappa\mu_2}$ of size $< \mu_2$, there is a $r_\rho$ that is not in its union. Thus, $\mu_2 \leq \text{cov}(\mathcal{N})$ and $\text{non}(\mathcal{N}) \leq \kappa$.

**Claim**

For every family of size $< \mu_2$ of infinite subsets of $\omega$ in $V_{\kappa,\lambda\kappa\mu_2}$ there is some $m_\rho$ which cannot be splitted by any member of the family. Thus, $\tau \leq \kappa$. 
**Theorem (Brendle, Fischer, 2011)**

If $\xi \leq \lambda \kappa \mu_2$ and $x$ is a real in $V_{\kappa, \xi}$, then $x \in V_{\alpha, \xi}$ for some $\alpha < \kappa$.

In the iterations for each $\rho < \kappa \mu_2$, $B^{V_{t(\rho), \lambda \rho}}$ adds a random real $r_\rho \in V_{t(\rho)+1, \lambda \rho+1}$ over $V_{t(\rho), \lambda \rho}$ and $M_{U_\rho}$ adds a Mathias real $m_\rho \in V_{t(\rho)+1, \lambda \rho+3}$ over $V_{t(\rho), \lambda \rho+2}$.

**Claim**

For every family of Borel non-null sets coded in $V_{\kappa, \lambda \kappa \mu_2}$ of size $< \mu_2$, there is a $r_\rho$ that is not in its union. Thus, $\mu_2 \leq \text{cov}(\mathcal{N})$ and $\text{non}(\mathcal{N}) \leq \kappa$.

**Claim**

For every family of size $< \mu_2$ of infinite subsets of $\omega$ in $V_{\kappa, \lambda \kappa \mu_2}$ there is some $m_\rho$ which cannot be splitted by any member of the family. Thus, $\tau \leq \kappa$. 
Sketched proof

**Theorem (Brendle, Fischer, 2011)**

If $\xi \leq \lambda \kappa \mu_2$ and $x$ is a real in $V_\kappa,\xi$, then $x \in V_\alpha,\xi$ for some $\alpha < \kappa$.

In the iterations for each $\rho < \kappa \mu_2$, $B^{V_{t(\rho)},\lambda_\rho}$ adds a random real $r_\rho \in V_{t(\rho)+1,\lambda_\rho+1}$ over $V_{t(\rho),\lambda_\rho}$ and $M_{U_\rho}$ adds a Mathias real $m_\rho \in V_{t(\rho)+1,\lambda_\rho+3}$ over $V_{t(\rho),\lambda_\rho+2}$.

**Claim**

For every family of Borel non-null sets coded in $V_\kappa,\lambda \kappa \mu_2$ of size $< \mu_2$, there is a $r_\rho$ that is not in its union. Thus, $\mu_2 \leq \text{cov}(\mathcal{N})$ and $\text{non}(\mathcal{N}) \leq \kappa$.

**Claim**

For every family of size $< \mu_2$ of infinite subsets of $\omega$ in $V_\kappa,\lambda \kappa \mu_2$ there is some $m_\rho$ which cannot be splitted by any member of the family. Thus, $\tau \leq \kappa$. 
Similarly, with $\mu_1 \leq \mu_2 \leq \mu_3 \leq \kappa$ uncountable regular cardinals, $\lambda \geq \kappa$, we can get models of ZFC plus:
When $\text{cf}(\lambda) \geq \aleph_1$, 

Here, $p = s = \mu_1$ and $r = \kappa$. 
When $\text{cf}(\lambda) \geq \mu_1$, 

Here, $p = s = \mu_2$ and $\tau = \kappa$. 
More applications

When \( \text{cf}(\lambda) \geq \mu_2 \),

Here, \( p = s = \mu_3 \) and \( r = \kappa \).
When $\text{cf}(\lambda) \geq \mu_2$,

Here, $p = s = r = u = \mu_3$. 

Diego Alejandro Mejía Guzmán
Models of some cardinal invariants with large continuum
Question 2

Does Blass-Shelah Theorem hold for $\mathfrak{m}$ instead of $<^*$?

A positive answer to this will lead to a model of ZFC plus $u < \text{non}(\mathcal{N}) < \text{cof}(\mathcal{N}) = c$. 
Question 3

If $\aleph_1 < \kappa_0 < \kappa_1 < \kappa_2$ for $\kappa_0, \kappa_1, \kappa_2$ regular cardinals, is it consistent with ZFC that $\aleph_1 = \text{non}(\mathcal{M}) = \text{cov}(\mathcal{M}) < \kappa_0 = \mathfrak{d} = \text{cof}(\mathcal{M}) < \kappa_1 = \text{non}(\mathcal{N}) < \kappa_2 = \text{cof}(\mathcal{N}) = \mathfrak{c}$?