World-line approach to the Bern-Kosower formalism
in two-loop Yang-Mills theory

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Abstract

Based on the world-line formalism with a sewing method, we derive the Yang-Mills effective action in a form useful to generate the Bern-Kosower-type master formulae for gluon scattering amplitudes at the two-loop level. It is shown that four-gluon ($\Phi^4$ type sewing) contributions can be encapsulated in the action with three-gluon ($\Phi^3$ type) vertices only, the total action thus becoming a simple expression. We then derive a general formula for a two-loop Euler-Heisenberg type action in a pseudo-abelian $su(2)$ background. The ghost loop and fermion loop cases are also studied.

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1 Introduction

More than a decade ago, the analogy between first quantized approach in string theory and world-line representation in field theory was pointed out in the $\phi^3$ theory effective action \cite{1}, and a similar approach was considered for Yang-Mills theory \cite{2}. Since then, the relation of string theory to quantum field theory has been studied intensively for the particular purpose of obtaining field theory scattering amplitudes in a remarkably simple way \cite{3}-\cite{8}. String theory organizes scattering amplitudes in a compact form (by virtue of the conformal symmetry on the world-sheet), and field theory, as a singular limit of string theory, inherits this useful feature, by which the summation of Feynman diagrams is already installed without need of performing loop integrals and the Dirac traces. In particular, Bern and Kosower derived a set of simple rules for one-loop gluon scattering amplitudes through analyzing the field theory limit of a heterotic string theory \cite{3}. The rules turned out to correspond to a subtle combination of the background and Gervais-Neveu gauges \cite{1}. They are applied to five gluon amplitudes \cite{3, 4}, quantum gravity \cite{1} and super Yang-Mills theories \cite{1}. The Bern-Kosower (BK) rules are also derivable from bosonic string theory \cite{1}, and the concrete identification of a corner of moduli space with each Feynman diagram and its divergence is verified \cite{10}.

The completion of Feynman diagram summation means gauge invariance. It is well-known that the Feynman diagram calculation splits a gauge invariant amplitude into non-invariant terms, and this causes a cancellation between divergent diagrams (gauge cancellation), which brings a serious problem especially with numerical computation. In the Bern-Kosower formalism we do not have this problem, since the only divergence appears from the final integration of a universal master formula. The master formula does not depend on the simplicity of specific scatterings with small number of external legs. (We want to keep this universality as much as possible when considering multi-loop generalization.) Hence the BK formalism has a great deal of potential to renovate the computational technique and efficiency in quantum field theory.

The BK rules for one-loop cases are also attainable directly (without making use of string theory) in terms of the world-line method in quantum field theory \cite{11, 12}. In this case, we have to evaluate an effective action in some particular form, which is a path integral for a one-dimensional quantum mechanical action (world-line action), using the proper time and
background field methods as well. Then expanding the background field as a sum of Fourier plane wave modes, we get the same kind of objects that are called the vertex operators in string theory. One particle irreducible (1PI) Green functions can be obtained as multi-integrals of the master formula, which is a correlation function evaluated by Wick’s contraction with the two-point correlator (world-line Green function) determined from the world-line action. It is very interesting that this kind of vertex operator technique resembles string theory calculations, and that all Feynman diagrams are consequently contained in a single master formula like string theory amplitudes. In fact, various field theory examples can be understood from this viewpoint: Photon splitting [13], axion decay in a constant magnetic field [14], and Yukawa interactions [15] up to some finite values of $N$ (the number of external legs) are explicitly verified; for photon scattering and $\phi^3$ theory, the equivalence is formally proven up to the two-loop order with an arbitrary value of $N$ [18]. This formalism is also useful for a manifestly covariant calculation of the effective action [16], and for decompositions into gauge invariant partial amplitudes [17].

Furthermore, since the first proposal of a multi-loop generalization of the Bern-Kosower formalism [19], various steps in this direction have been made [18]-[29] — mostly investigated in $\phi^3$ theory [18]-[21] and in spinor/scalar QED [24]-[27]. A few preliminary studies for QCD have also been performed [28, 30, 31]. Generally speaking, we need some new types of master formulae, depending on the places where external legs are inserted. The multi-loop combinatorial problem, which is how to combine the master formulae of different types, is solved in the cases of neutral $\phi^3$ theory and scalar/spinor QED at the two-loop order [18]. The other new participants are the multi-loop world-line Green functions [19]-[21, 27] and path integral normalizations. In $\phi^3$ theory, they are determined from the string theory side as well [21]-[23].

Now, all ingredients seem ready to be generalized to the gluon scattering case at the two-loop level within the world-line framework, since basic features are almost in common with the above cases, up to the four-gluon interaction. On first thoughts, we might have only to insert a path integral representation of a gluon propagator [27] in a one-loop gluon diagram. However, things are not really straightforward. The insertion of a propagator destroys the simple trace structure, and we hence have to find out an alternative expression of the trace-log formula. In addition, in order to determine precise multi-loop combinatorics, we have to construct the two-loop analogue of the trace-log formula in a systematic way. Fortunately we already have a suitable technique
for these purposes. It all can be done by introducing an auxiliary field representing a quadratic term of the internal quantum field $[18]$. Roughly speaking, the auxiliary field plays the role of an adhesive to glue the inserted propagator. We shall show more details how to exactly realize this idea later on. Once having a precise formula for the two-loop effective action, we expect that the substitution by a sum of all plane waves

$$A^a_\mu \hat{\lambda}^a \rightarrow g \sum_{i=1}^{N} \hat{\lambda}^{a i} \epsilon^i_\mu \exp[i k_i \cdot x] \quad (1.1)$$

will yield correct combinatorics even in the multi-loop cases. This is actually verified in the $\phi^3$ theory case up to the two-loop order $[18]$. Regarding this point, we shall confine ourselves to discuss a general prescription at the present stage.

In this paper, we present a derivation of the world-line formulae for the two-loop effective action mainly in pure Yang-Mills theory. In Section 2, starting with the background field Lagrangian together with the auxiliary field method mentioned above, we set up the sewing between a loop and a propagator so as to generate the two-loop analogue of a trace-log formula, which consists of four types of sewing. We also address more general sewing rules for a multi-loop construction. In Section 3, we explicitly perform the sewing procedures at the level of world-line path integral representations. Briefly observing a conjecture in Section 4, we then unify three of the four types into a single expression in Section 5. In Section 6, we verify this fact in the su(2) pseudo-abelian case, and derive a general formula for the effective action in a constant (pseudo-abelian) background field (Euler-Heisenberg type action). We also include short remarks on the ghost loop case in Section 7 and on the fermion loop case in Section 8. Conclusions and discussions are in Section 9. In Appendix A, we attach two kinds of non-world-line calculations for comparison. One is solely based on the auxiliary field method (without the use of world-line representations), and the other is based on the usual field theory method. It is shown that the results of the main text perfectly coincide with those obtained by these two different methods. In Appendix B we show an outline of the method how to obtain (pure Yang-Mills) $N$-point amplitudes, and in Appendix C computational details of two-loop gluon world-line Green function are presented.
2 The two-loop analogue of the trace-log formula

In this section, we derive the two-loop analogue of the trace-log formula for the pure Yang-Mills effective action, starting with the following background field Lagrangian [33] (for the moment we include the fermion and ghost Lagrangians as well):

\[ L = -\frac{1}{4}(F_{\mu\nu}^a)^2 - \frac{1}{2}Q_\mu^a(\Delta^{-1})_{\mu\nu}^{ab}Q_\nu^b - F_{\mu\nu}^aD_{\mu}Q_{\nu}^c - f^{abc}(D_{\mu}Q_{\nu})^aQ_\nu^bQ_\mu^c - \frac{1}{4}(f^{abc}Q_\mu^bQ_\nu^c)^2 + \bar{c}^a(D^2)^{ab}c^b + f^{abc}(D_{\mu}\bar{c})^a\bar{c}^bQ_\mu^c + \bar{\psi}_i\gamma^\mu(\hat{D}_\mu - iQ_\mu^a\hat{\lambda}^a)q_j^i , \] (2.1)

where \( Q_\mu^a \) is the quantum gauge field, and \( D_\mu \) is the covariant derivative w.r.t. a background gauge field \( A_\mu^a \), whose field strength is denoted by \( F_{\mu\nu}^a \). The (non-) hat notations stand for the (adjoint) fundamental representations, and \( [\hat{\lambda}^a, \hat{\lambda}^b] = i f^{abc}\hat{\lambda}^c \). The propagator \( \Delta \) in a general background gauge is given by

\[ \Delta_{\mu\nu}^{ab} = [-g_{\mu\nu}D^2 + (1 - \frac{1}{\xi})D_\mu D_\nu + 2F_{\mu\nu}f^{abc}]^{-1} , \] (2.2)

however we choose the Feynman gauge \( \xi = 1 \) in this paper. Here are some notational remarks. We often put the Lorentz indices upside down just by reason of typography, and abbreviate the indices and space-time integrals which can be understood from the same/corresponding terms written beforehand. We also follow the convention \( (X_{\mu\nu})^2 \equiv X_{\mu\nu}X^{\mu\nu} \) for an arbitrary tensor, and the Lorentz and color summations are implicit in doubly appearing indices as usual.

Basically, we follow the same method as we developed in \( \phi^3 \) theory (The details can be found in Section 3 of Ref. [18]). We introduce the three sets of auxiliary fields regarding the quantum gauge field, ghost and quark parts. Inserting the following identities in the Lagrangian (2.1):

\[ \delta(B - fQQ) \equiv \int D\alpha_{\mu\nu}^a \exp\left[i \int d^Dx (B_{\mu\nu}^a - f^{abc}Q_{\mu\nu}^bQ_{\nu}^c)\alpha_{\mu\nu}^a \right] , \] (2.3)
\[ \delta(C - f(Dc)c) \equiv \int D\beta_{\mu}^a \exp\left[i \int d^Dx (C_{\mu}^a - f^{abc}(D_{\mu}\bar{c})^b\bar{c})\beta_{\mu}^a \right] , \] (2.4)
\[ \delta(E - \bar{\psi}\gamma^\mu\hat{\lambda}\psi) \equiv \int D\epsilon_{\mu}^a \exp\left[i \int d^Dx (E_{\mu}^a - \bar{\psi}\gamma^\mu(\hat{\lambda})q_j^i)\epsilon_{\mu}^a \right] , \] (2.5)

we obtain the generating functional in the form

\[ Z[A] = Z_0 \int DB_{\mu\nu}^aDC_{\mu}^aDC_{\mu}^aDE_{\mu}^aDE_{\mu}^a [\text{Det} \Delta^{-1}]^{-1/2} \text{Det}[(D^2)^{ac} - D_{\mu}^a\beta_{\mu}^d\beta_{\mu}^d] \]
\[
\times \det \{ i\gamma^\mu (\hat{D}_\mu + i\epsilon_\mu^a \lambda^a)_{ij} \} \exp \left[ \frac{i}{2} \int d^D y_1 d^D y_2 J_{\mu}^a(y_1) \Delta_{\mu\nu}^{ab}(y_1, y_2) J_{\nu}^b(y_2) \right] \\
\times \exp \left[ i \int d^D x \left( -\frac{1}{4} (B_{\mu\nu}^a)^2 + \alpha_\mu^a B_{\mu\nu}^a + \beta_\mu^a C_{\mu\nu}^a + \epsilon_\mu^a E_{\mu\nu}^a \right) \right],
\]

(2.6)

where

\[
J_{\nu}^a[B, C, E, F] = D_{\mu}^c F_{\nu}^c + D_{\mu}^c B_{\nu}^c + C_{\nu}^a + E_{\nu}^a,
\]

(2.7)

\[
(\hat{\Delta}^{-1})_{\mu\nu}^{ab} = (\Delta^{-1})_{\mu\nu}^{ab} + 2 J_{\mu\nu}^{abc} \alpha_{\nu}^c,
\]

(2.8)

\[
Z_0 = \exp \left[ i \int \frac{-1}{4} (F_{\mu\nu}^a)^2 \right].
\]

(2.9)

To perform the \(B, C,\) and \(E\) integrals, we apply the following (general) formula for a function of \(B\) and \(\alpha\)

\[
\int D\alpha DB f(iB, \alpha)e^{i\alpha B} = \int D\alpha DB f \left( \frac{\delta}{\delta\alpha}, \alpha \right) e^{i\alpha B} = \int D\alpha f \left( \frac{\delta}{\delta\alpha}, \alpha \right) \delta(\alpha),
\]

(2.10)

where the \(\alpha\) differentiation acts on the \(\delta\)-function \(\delta(\alpha)\). Keeping the partial integrations for the \(\delta\)-function in mind, one can see that the integrations of these \(\delta\)-functions then lead to the following replacements in the functional (2.6):

\[
B_{\mu\nu}^a \rightarrow i \frac{\delta}{\delta\alpha_{\mu\nu}^a} \bigg|_{\alpha=0}, \quad C_{\mu}^a \rightarrow i \frac{\delta}{\delta\beta_{\mu}^a} \bigg|_{\beta=0}, \quad E_{\mu}^a \rightarrow i \frac{\delta}{\delta\epsilon_{\mu}^a} \bigg|_{\epsilon=0},
\]

(2.11)

where \(\frac{1}{2}\) derives from the anti-symmetric nature of the \(B\) field. Then the functional differentiations become to act on all \(\alpha, \beta\) and \(\epsilon\) fields. Removing irrelevant parts of the effective action, we obtain

\[
Z[A] \approx \exp \left[ -\frac{1}{2} \text{Tr} \ln \hat{\Delta}^{-1} \right] \exp [\text{Tr} \ln (D^2 - D/\beta f)] \exp [\text{Tr} \ln (\gamma^\mu (i\hat{D}_\mu - \epsilon_\mu^a \hat{\lambda}^a))]
\times \exp \left[ \frac{i}{2} \left( -i \frac{\delta}{\delta\beta} D + i \frac{\delta}{\delta\epsilon} \hat{\Delta} \right) \left( -i \frac{\delta}{\delta\alpha} D + i \frac{\delta}{\delta\beta} + i \frac{\delta}{\delta\epsilon} \right) \right]
\times \exp \left[ \frac{i}{16} \frac{\delta}{\delta\alpha} \frac{\delta}{\delta\alpha} \bigg|_{\alpha=\beta=\epsilon=0} \right].
\]

(2.12)

Since all the functional differentiations act on every \(\alpha\) field etc., the ordering of the exponential objects is not important. The covariant derivatives appearing with the \(\alpha\) differentiation should be understood as partially integrated ones (thus acting on the propagator) in order to get rid of acting on the functional derivatives. Thus the pure Yang-Mills part of \(Z[A]\) reads

\[
iZ_{\text{gluon}} = \exp \left[ \frac{i}{2} D^3 \Delta \delta \alpha \right] \exp \left[ \frac{i}{16} \delta \alpha \delta \alpha \right] \exp \left[ -\frac{1}{2} \text{Tr} \ln \hat{\Delta}^{-1} \right] \bigg|_{\alpha=0},
\]

(2.13)
where $\delta_\alpha$ is the abbreviation of $\delta/\delta \alpha$. The third exponential object in Eq.(2.13) generates loops including the one-loop effective action, and the first one the propagator insertions to produce multi-loop diagrams by three-gluon interactions. The second one corresponds to four-gluon interactions. One can see the similarity of this gluon action (2.13) to the following $\phi^3$ theory action (massless, Euclidean) [18]:

\[
Z[\bar{\phi}] = \exp\left(-\frac{g^2}{2(3!)} \delta_\alpha (-\partial^2 + g\bar{\phi} + 2i\alpha)^{-1} \delta_\alpha \right) \exp\left[-\frac{1}{2} \text{Tr} \ln(-\partial^2 + g\bar{\phi} + 2i\alpha)\right] \bigg|_{\alpha=0},
\]

(2.14)

Now we extract the two-loop (1PI) parts from the pure Yang-Mills generating functional (2.13) as shown in Fig. 1. The two-loop effective action comprises the following four types:

\[
i\Gamma_1 = -\frac{1}{2} \int d^Dy_1d^Dy_2 (\frac{i}{2})^3 \delta_\alpha D\Delta \delta_\alpha D\text{Tr} \ln \Delta^{-1} \bigg|_{\alpha=0},
\]

(2.15)

\[
i\Gamma_2 = \int d^Dy_1d^Dy_2 (\frac{i}{2})^3 \delta_\alpha D\Delta \delta_\alpha D^{1\text{PI}} \bigg|_{\alpha=0},
\]

(2.16)

\[
i\Gamma_3^{(1)} = -\frac{1}{2} \int d^Dy_1 (\frac{i}{16}) \delta_\alpha \delta_\alpha \text{Tr} \ln \Delta^{-1} \bigg|_{\alpha=0},
\]

(2.17)

\[
i\Gamma_3^{(2)} = \frac{1}{2} \int d^Dy_1 (\frac{i}{16}) \delta_\alpha \delta_\alpha (-\frac{1}{2} \text{Tr} \ln \Delta^{-1})^2 \bigg|_{\alpha=0}^{\text{connected}},
\]

(2.18)

where we have revived the omitted space-time integrations, and one may of course insert $1 = \int d^Dy_2 \delta(y_1 - y_2)$ into the $\delta_\alpha$ square terms (acting on the same point) in Eqs.(2.17) and (2.18). In $\Gamma_2$ and $\Gamma_3^{(2)}$, we have to extract the 1PI pieces from the naive $\alpha$ differentiations. These will be explained in Section 3. We shall refer to the diagram (a) in Figure 1 as the propagator insertion type, (b) the double folding type, and to the rests as the shrinkage types or the eight figure diagrams.

**Figure 1:** The sewing diagrams. We call (a) the propagator insertion type, (b) the double folding type. The diagrams are made of the loops, the lines, and the black/white dots. See the text for details.
Here we rather explain the method how to write down these necessary pieces for the pure Yang-Mills effective action, starting from graphical representations. These quantities (2.15)-(2.18) can be expressed by the graphical representations (sewing diagrams) in Figure 1(a)-(d). In the following, we present a general procedure to obtain the desired expressions in terms of our sewing technique. We shall follow the three steps explained below. (1) The first step: The basic parts to construct the sewing diagrams are the loop $\text{Tr} \ln \Delta^{-1}$, the line $\Delta$, the white dot (identity propagator), and the cross $\delta_a$. The edges of the $\Delta$ line are expressed by the black dots (the covariant derivatives $D$’s), and the white dots themselves can formally be regarded as another kind of propagator edges as well. All these propagator edges should be joined with the $\delta_a$ crosses, which are put on a loop or a line. The way of joining dots and crosses is that one has to connect a black dot with a single cross, and a white dot with a pair of two crosses. In this way, one can draw all possible sewing diagrams for a given topology of vacuum Feynman diagrams. In fact, Figure 1(a)-(d) are all the possibilities to construct the two-loop vacuum topologies.

(2) The second step: After listing up the sewing diagrams, we then assign the integration variables $y_j$ for all crosses. It is enough to do this labeling just once, because they are just dummy variables. Then we perform the following identifications: If a white dot is attached to the crosses at $y_i$ and $y_j$, the identity propagator should be replaced by $\delta(y_i - y_j)$, and the pair of crosses becomes $(\delta/\delta\alpha^a_{\mu\nu}(y_i))^2$. If the edges of a line propagator are attached to the crosses at $y_i$ and $y_j$, then it becomes $(\delta_a D)(y_i)\Delta(y_i, y_j)(D\delta_a)(y_j)$. The color and Lorentz indices are to be read from the $J\Delta J$ term in Eq.(2.6) (taking account of the partial integrations):

$$\frac{\delta}{\delta\alpha^a_{\lambda\mu}} D^\alpha_{\lambda\mu} \Delta^{ij}_{\mu\nu} D^{ij}_{\rho\nu} \frac{\delta}{\delta\alpha^e_{\rho\nu}} .$$

(2.19)

(3) The third step: For a given sewing diagram $s_n$, which possesses $q$ crosses, $L$ loops, $k$ propagators, and $p$ white dots, we write down the following formal integral

$$\Gamma[s_n] = C_n \int \prod_{j=1}^q d^D y_j \delta_a(y_j) \left( \prod \delta_a \right) \left( \prod \delta_a D \Delta D \delta_a \right) \left( \text{Tr} \ln \Delta^{-1} \right)^L_{|_{\alpha=0}} ,$$

(2.20)
and the numerical coefficient $C_n$ is determined by the following rules:

\[
\begin{align*}
    i/2 & \quad \text{for} \quad \delta_\alpha \\
    (\frac{i}{2})^n/n! & \quad \text{for} \quad \text{n propagators} \quad (D\Delta D)^n \\
    (\frac{-i}{4})^n/n! & \quad \text{for} \quad \text{n identity propagators} \\
    ((-2)^L L!)^{-1} & \quad \text{for} \quad L \text{ loops} \quad (\text{Trln}\Delta^{-1})^L .
\end{align*}
\] (2.21)

Note that we may not attach an additional factor 2 in Fig. 1(a) (upside-down attachment of the $D\Delta D$ propagator to the loop), since it is already included in the color summation, which can easily be understood from the color contraction in the second term on the r.h.s. of Eq. (2.7).

Actually the cross term contributions from $J\Delta J$ correspond to the factor 2 of the $\phi^3$ case. In this way, this point superficially differs from the $\phi^3$ case [18] (however is basically the same).

Hereafter we deal with the Euclidean formulation for the world-line (path integral) representations. Applying the path integral representations of gluon loop and propagator (in terms of only the bosonic world-line field) [27]

\[
\text{Trln}\Delta^{-1} = - \int_0^\infty \frac{dS}{S} \int D\!x \exp\left[ - \int_0^s \frac{1}{4} \dot{x}^2(\tau)d\tau \right] (\text{Pexp} \int_0^s M[x(\tau)]d\tau)^{\mu\sigma}_{\mu\nu},
\] (2.22)

\[
\Delta_{\mu\nu}^{ab}(y_1,y_2) = \int_0^\infty d(\tau_2 - \tau_1) \int_{x(\tau_2)=y_2}^{x(\tau_1)=y_1} D\!x e^{-\int_{\tau_1}^{\tau_2} \frac{1}{4} \dot{x}^2(\tau)d\tau} (\text{Pexp} \int_{\tau_1}^{\tau_2} M[x(\tau)]d\tau)^{ab}_{\mu\nu},
\] (2.23)

\[
\equiv \int_0^\infty d(\tau_2 - \tau_1) \int_{x(\tau_2)=y_2}^{x(\tau_1)=y_1} D\!x K^{ab}_{\mu\nu}(x|y_1,y_2; \tau_1,\tau_2),
\] (2.24)

where

\[
M_{ab}[x(\tau)] = 2i(F_{\mu\nu}^c + \alpha_{\mu\nu}^c - \delta_{\mu\nu}^c \frac{1}{2} A_\mu^c \dot{x}^\mu)(\lambda^c)_{ab},
\] (2.25)

we derive the following path integral representations of the $\Gamma_i$ (= $\Gamma_1, \Gamma_2, \Gamma_3^{(1)}, \Gamma_3^{(2)}$) after some calculation. (These will be derived in Section [3]):

\[
\begin{align*}
    \Gamma_i &= \delta^{\mu\nu'} \delta^{\rho\sigma'} \int_0^\infty \frac{dS}{S} \int_0^s d\tau_\beta \int_0^s d\tau_\alpha \int_0^\infty dT_3 \int d^Dy_1d^Dy_2 \\
    & \times \delta(y_1 - x(\tau_\beta))\delta(y_2 - x(\tau_\alpha))\mathcal{V}_i(y_1, y_1', y_2) \int_{x(s)=x(\tau_\beta)} D\!x[S] \int_{y_1(0)=y_1'}^{y_2(0)=y_2} [D\!w][w] \bigg|_{y_i'=y_i} \\
    & \times \left[ (\text{Pexp} \int_{\tau_\beta}^{\tau_\alpha} M(x))_{\delta_\rho\lambda}^{km} \right] (\text{Pexp} \int_{\tau_\alpha}^{\tau_\beta} M(x))_{\mu\lambda}^a \left[ (\text{Pexp} \int_{\tau_\beta}^{\tau_3} M(w))_{\rho j}^{\gamma} \right],
\end{align*}
\] (2.26)

where $y_i'$ should be set to $y_i$ after $\mathcal{V}_i$ operating on the boundaries of path integrals, whose free parts are denoted by

\[
[D\!x]_S = D\!x \exp\left[ - \int_0^s \frac{1}{4} \dot{x}^2(\tau)d\tau \right] \quad \text{etc.}
\] (2.27)

8
Because of the anti-symmetric nature of $\alpha_{\mu \nu}^a$, we have introduced the symbol
\begin{equation}
\delta_{\mu \nu}^{\mu' \nu'} \equiv \delta_{\mu}^{\mu'} \delta_{\nu}^{\nu'} - \delta_{\mu}^{\nu'} \delta_{\nu}^{\mu'},
\end{equation}
and setting $\alpha_{\mu \nu}^a = 0$, we have
\begin{equation}
M_{ab}(x) = \bar{M}_{ab}(x(\tau)) \bigg|_{\alpha=0}.
\end{equation}

The differential operators $\mathcal{V}_i$ are listed as follows:
\begin{align}
\mathcal{V}_1 &= \frac{1}{4} D_{\mu'}^a(y_1) D_{\rho'}^c(y_2) \delta^{mg} \delta^{kl} \delta^{ni} g_{\delta \epsilon \sigma} g_{\gamma \nu}, \\
\mathcal{V}_2 &= \frac{1}{2} D_{\mu'}^a(y'_1) D_{\rho'}^c(y_2) \delta^{mg} \delta^{kl} \delta^{in} g_{\delta \epsilon \sigma} g_{\gamma \nu}, \\
\mathcal{V}_3^{(1)} &= -\frac{1}{8N} \delta^{ae} \delta^{ij} g_{\mu' \nu'} \delta(y_1 - y_2) \delta(T_3) \delta^{mg} \delta^{kl} \delta^{ni} g_{\delta \epsilon \sigma} g_{\gamma \nu}, \\
\mathcal{V}_3^{(2)} &= -\frac{1}{16N} \delta^{ae} \delta^{ij} g_{\mu' \nu'} \delta(y_1 - y_2) \delta(T_3) \delta^{mk} \delta^{lo} \delta^{ni} g_{\delta \epsilon \sigma} g_{\gamma \nu}.
\end{align}

Finally, some remarks are in order. (i) In deriving the above representation (2.26), we have employed the anti-path ordering in (2.24). Since the orderings of world-line paths are not related to the time orderings, this choice does not cause any trouble. Thus the direction of a matrix ordering and its proper time direction coincide in our case. One can easily transform them into the normal path ordering formalism by exchanging $y_1$ and $y_2$, or equivalently, $K_{\mu \nu}^{ab}(y_1, y_2) \to K_{\mu \nu}^{ba}(y_1, y_2)$. (ii) We have artificially introduced $y'_i$ variables only for $\mathcal{V}_2$ in order to keep track of the original covariant derivative positions, which shall be essential only for the analyses (Section 5) of embeddings of 8-figure diagrams into $\Phi^3$ type diagrams. Since the embeddings are not a necessity but a matter of conveniences, one may put $y'_i = y_i$ as assumed in the above sewing rules. In the next section, we briefly view some details of how we obtain the expressions (2.30). (iii) Remember that these are still intermediate results since non-world-line objects (i.e. covariant derivatives) still remain in the representations (2.30). However, at this level, we can see that the Lorentz and color structures in Eq. (2.26) with (2.30) coincide with those obtained by other methods in Appendix A. For a reference, the graphical representations of the Lorentz and color indices for the $\mathcal{V}_i$, $i = 1, 2$ are shown in Fig. 2. (iv) In order to see in detail the way how to obtain amplitudes in a Bern-Kosower form, one has to perform the substitution (1.1), expanding the background field as mentioned before. The general procedures to do this are explained in Appendix B for the general action formula (2.26).
Figure 2: The graphical representations of (a) $V_1$ and (b) $V_2$. The cross lines stand for the contractions defined by Eq.(2.28). The dashed lines express the color indices contractions.

3 Derivation of the $\Gamma_i$

In this section, we explain how we further proceed with the computations of Eqs.(2.15)-(2.18), in particular how to perform the extraction of 1PI parts. Basically, a second derivative of $K$ (defined in Eq.(2.24)) gives rise to two kinds of terms composed of triple $K$ products due to the successive applications of the following formula:

$$
\frac{\delta}{\delta \alpha_{\mu \nu} (y)} K_{\rho \sigma}^{ij} (x|y_1, y_2; \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} d\tau K_{\rho \gamma}^{ib} (x|y_1, y_1; \tau_1, \tau) \left( \frac{\delta}{\delta \alpha_{\mu \nu} (y)} \bar{M}[x(\tau)] \right)_{\gamma \delta}^{bc} K_{\delta \alpha}^{cj} (x|y, y_2; \tau, \tau_2). 
$$

In $\Gamma_2$, one of these two kinds corresponds to the desired 1PI parts, and the other kind to the one particle reducible (1PR) parts. On the other hand in $\Gamma_1$, both kinds contribute to the 1PI parts. We show the computational details of these facts case by case in the following subsections. These may rather sound like meticulous technical details, however as we shall see later, identifying 1PR parts is certainly useful in order to investigate the way how shrinking propagator limits form the $\Phi^4$ type “eight-figure” diagrams $\Gamma_3^{(i)}$, $i = 1, 2$. For example, in Section 3.3, we point out interesting relations between $\Gamma_3^{(1)}$ and $\Gamma_1$ as well as between $\Gamma_3^{(2)}$ and
\[ \Gamma_1^R, \text{ which is a 1PR type of } \Gamma_1 \text{ in a sense. The } \Gamma_1 \text{ calculation is also useful for comparing with the ghost loop in Section 6.} \]

### 3.1 The double folding types \( \Gamma_2 \) and \( \Gamma_2^R \)

Let us start with the most non-trivial part, the separation of 1PR contribution in \( \Gamma_2 \). Originally the double folding \( \Gamma_2 \) is a part of the following quantity (c.f. Eq.(2.16) and Figure 1(b)):

\[
z_2 \equiv -\frac{1}{8} \int d^Dy_1d^Dy_2 D\delta_\alpha \bar{\Delta}D\delta_\alpha \bigg|_{\alpha=0} = \Gamma_2 + \Gamma_2^R. \tag{3.2}
\]

Let us explain the method how to separate the 1PI part from \( z_2 \). Operating a differentiation \( \delta_\alpha \) on a propagator \( \bar{\Delta} \) (\( \sim K \) of length \( L \)), two segments of 'propagator' \( K \) are created:

\[
\frac{\delta}{\delta\alpha_{\mu\nu}(y_1)} \int_0^\infty dL K_{\nu_\sigma}^{ij}(x|y_1, y_2; 0, L) \tag{3.3}
\]

\[
= \int_0^\infty d\tau_\beta \int_0^{\tau_\beta} dT_3 K_{\nu_\mu}^{jb}(x|y_1, y_2; 0, \tau_\beta)[2i\delta_{\mu\nu}^{\prime}(\lambda^a)_{bc}\delta(y_1 - x(\tau_\beta))]K_{\nu_\sigma}^{cj}(w|y_1, y_2; 0, T_3),
\]

where \( L \) is shifted by the relation \( T_3 = L - \tau_\beta \), and \( w \) is a redefined field (see Eq.(3.6)).

The graphical representation of this formula is shown in Fig. 3(a). A second differentiation applies to each of these two propagators by the Leibniz rule. One is (un-shifted integral)

\[
\frac{\delta}{\delta\alpha_{\mu\nu}(y_2)} \int_0^\infty dT_3 K_{\nu_\mu}^{jb}(w|y_1, y_2; 0, T_3) \tag{3.4}
\]

\[
= \int_0^\infty d\tau_\beta \int_0^{\tau_\beta} dT_3 \alpha_{\mu\rho}(x|y_1, y_2; 0, T_3)[2i\delta_{\rho\sigma}^{\prime}(\lambda^e)_{fg}\delta(y_2 - w(0))]K_{\sigma_\mu}^{cj}(z|y_2, y_2; 0, T_3).
\]

And the other is (with shifting \( T = T_3 - T_3' \))

\[
\frac{\delta}{\delta\alpha_{\mu\nu}(y_1)} \int_0^\infty dT_3 K_{\nu_\sigma}^{cj}(w|y_1, y_2; 0, T_3) \tag{3.5}
\]

\[
= \int_0^{\tau_\beta} dT_3' \int_0^{\tau_\beta} dT K_{\nu_\rho}^{cf}(w|y_1, y_2; 0, T_3')[2i\delta_{\rho\sigma}^{\prime}(\lambda^e)_{fg}\delta(y_2 - w(0))]K_{\sigma_\sigma}^{cj}(z|y_2, y_2; 0, T).
\]

The first one (3.4) corresponds to the 1PI diagram \( \Gamma_2 \), and the latter one (3.5) to the 1PR diagram \( \Gamma_2^R \). Their graphical representations are shown in Fig. 3. Having separately three propagators, we have introduced the following splittings of the world-line field:

\[
w(\tau) = x(\tau_\beta + \tau), \quad (0 \leq \tau \leq T_3) \quad \text{for 1PI,} \tag{3.6}
\]
and for the 1PR case

\[
\begin{aligned}
    z(\tau) &= x(\tau_\alpha + \tau), \\
    w(\tau) &= x(\tau_\beta + \tau),
\end{aligned}
\]

\(0 \leq \tau \leq T = L - \tau_\alpha\)

(3.7)

\[
\begin{aligned}
    z(\tau) &= x(\tau_\alpha + \tau), \\
    w(\tau) &= x(\tau_\beta + \tau), \\
    (0 \leq \tau \leq T' = \tau_\alpha - \tau_\beta),
\end{aligned}
\]

where the shifted length parameters \(T_3, T'_3\) and \(T\) all run from zero to infinity.

\[\begin{array}{c}
\text{(a)} \quad x(0) \quad w(0) \quad w(T_3) \\
\times \quad x(\tau_\beta) \\
\end{array}\]

\[\begin{array}{c}
\text{(b)} \quad x(\tau) \quad w(\tau) \quad z(\tau) \\
\times \quad T'_3 \quad \tau_\beta \quad \tau_\alpha
\end{array}\]

**Figure 3:** The sewing procedures for \(\Gamma_2\) and \(\Gamma_2^R\). The 1PI diagram is obtained if \(\tau_\alpha\) is inserted in the region \(0 < \tau_\alpha < \tau_\beta\) in the diagram (a). Otherwise the 1PR diagram follows from the case (b).

The final results of the \(\alpha\)-differentiations are therefore

\[
\Gamma_2 = \frac{1}{2} \delta_{\mu \nu} \delta_{\rho \sigma} \int_0^\infty d\tau_3 \int_0^\infty dT_3 \int_0^{\tau_\beta} d\tau_\alpha \int d^D y' \times \nonumber
\]

\[
\times D_{\mu \nu}^{(\prime)}(y') D_{\rho \sigma}^{(\prime)}(x(\tau_\alpha)) \left[ \int x(x(\tau_\beta) = y, x(0) = y' \right] D_x \left[ w(T_3) = x(\tau_\alpha) \right] D_x \left[ w(0) = x(\tau_\beta) \right] \right] d^D y' \nonumber
\]

\[
\times \left[ (P \epsilon_{\mu} \tau_\alpha M(x))_{\nu \rho} \lambda_\alpha (P \epsilon_{\rho} \tau_3 M(w))_{\nu \rho \sigma} \right]_{\nu \rho \sigma} \]

\(3.8\)

\[
\Gamma_2^R = \frac{1}{2} \delta_{\mu \nu} \delta_{\rho \sigma} \int_0^\infty d\tau_3 \int_0^\infty dT_3 \int_0^\infty dT \int d^D y_1 d^D y_2 \times \nonumber
\]

\[
\times D_{\mu \nu}^{(\prime)}(y_1) D_{\rho \sigma}^{(\prime)}(y_2) \left[ \int x(\tau_\beta) = y_1, x(0) = y_1' \right] D_x \left[ z(T) = y_2, z(0) = y_2' \right] D_z \left[ w(T'_3) = z(0) \right] D_w \left[ w(0) = x(\tau_\beta) \right] \right] d^D y_1 \times \nonumber
\]

\[
\times \left[ (P \epsilon_{\mu} \tau_3 M(x))_{\nu \rho} \lambda_\alpha (P \epsilon_{\rho} \tau_3 M(w))_{\nu \rho \sigma} \right]_{\nu \rho \sigma} \]

\(3.9\)

Fixing \(\tau_\beta\) to \(S\) in Eq. (2.26), thereby getting a factor \(S\) from the \(\tau_\beta\) integral \(1\), one can see that the above result \((3.8)\) coincides with the one given in Eq. (2.26) with \(V_2\).

\(1\) This is because of the manifest rotational invariance in Eq. (2.26). We call this procedure **fixing** hereafter.
3.2 The propagator insertion types $\Gamma_1$ and $\Gamma_1^R$

First let us consider the $\Gamma_1$ case. One can formally express the path integrand of (2.22) in terms of $K$ defined (2.24); that is nothing but $K_{a\mu}^{aa}(x|x, x; 0, S)$ with imposing the condition $x(0) = x(S)$ (world-line length $S$). Note that this is definitely different from the $K_{a\mu}^{aa}(x|x, \tau, \tau)$ which are living on a point (world-line length zero). Now, the Leibniz rule on a second derivative of the $K$ creates the following three segments of the propagator:

$$
\frac{\delta}{\delta \alpha_{\mu \nu}(y_1)} \frac{\delta}{\delta \alpha_{\rho \sigma}(y_2)} \int_0^\infty \frac{dS}{S} K^{ii}_{\alpha \alpha}(x|x, x; 0, S) = \int_0^\infty \frac{dS}{S} \int_0^S d\tau_\beta \int_0^\tau d\tau_\alpha (2i)^2 \delta(y_1 - x(\tau_\beta)) \delta(y_2 - x(\tau_\alpha)) \delta_{\mu \nu} \delta_{\rho \sigma}
\times \left\{ \theta(\tau_\beta - \tau_\alpha) \right\}
\times \left\{ \theta(\tau_\alpha - \tau_\beta) \right\}.
$$

(3.10)

Here we have two terms with step functions, and the first step-function term becomes

$$
\theta(\tau_\beta - \tau_\alpha) \left\{ (Pe^{\int_{\tau_\beta}^{\tau_\alpha} M(x)})_{\nu \rho} \lambda^\nu \lambda_a (Pe^{\int_{\tau_\alpha}^{\tau_\beta} M(x)})_{\sigma \mu} \right\}^{ii},
$$

(3.11)

owing to the formula

$$
(Pe^{\int_{\tau_\beta}^{\tau_\alpha} M(x)})_{\nu \rho} (Pe^{\int_{\tau_\alpha}^{\tau_\beta} M(x)})_{\sigma \mu} = (Pe^{\int_{\tau_\alpha}^{\tau_\beta} M(x)})_{\nu \rho}.
$$

(3.12)

where $P'$ follows the path from $\tau_\beta$ to $\tau_\alpha$ via $S$. The second step-function term in Eq.(3.10) amounts to the same quantity, and we hence drop the step functions because of the property $\theta(x) + \theta(-x) = 1$. Therefore we arrive at Eq.(2.23) with $\nu_1$ given in Eq.(2.30):

$$
\Gamma_1 = \frac{1}{4} \delta_{\mu \nu} \delta_{\rho \sigma} \int_0^\infty \frac{dS}{S} \int_0^S d\tau_\beta \int_0^\tau d\tau_\alpha \int_0^\infty dT_3 \oint [Dx] S \times \left\{ D^{ai}_{\mu}(x(\tau_\beta)) D^{ej}_{\rho}(x(\tau_\alpha)) \right\}
\times \left\{ \int_{T_3 = x(\tau_\beta)}^{T_3 = x(\tau_\alpha)} [Dw] T_3 \right\}
\times \left\{ \Tr[(Pe^{\int_{\tau_\beta}^{\tau_\alpha} M(x)})_{\nu \rho} \lambda^\nu \lambda_a (Pe^{\int_{\tau_\alpha}^{\tau_\beta} M(x)})_{\sigma \mu} \lambda^\sigma] (Pe^{\int_{T_3}^{T_3} M(w)})_{ij} \right\}.
$$

(3.13)

For later convenience, we also write down the 1PR part made of two loops and one propagator (see Fig. 4(a)). According to the rules explained in Section 2, this can be extracted from the
quantity (\ref{eq:2.13}) as follows:

\begin{align}
\Gamma_1^R & \equiv \frac{1}{2}(-i)\int dy_1dy_2\left(-\frac{1}{2}\text{Tr}\ln\Delta^{-1}\right)^2\left(\frac{i}{2}\right)^3 \delta_\alpha D\delta_\alpha D|_{\alpha=0}^{\text{connected}} \\
&= \frac{1}{8}\delta_{\mu'}^{\mu''}\delta_{\rho'}^{\rho''}\int_0^\infty \frac{dS}{S}\int_0^\infty \frac{dT}{T}\int_0^S d\tau_3 \int_0^T d\tau_1 \int_0^T d\tau_2 \int_{\mathbb{D}} [Dx]_{S} \int_{[Dz]}_{T} \\
&\times D_{\mu'}^{\mu''}(x(\tau_3))D_{\rho'}^{\rho''}(z(\tau_3)) \int_{w(\tau_3)=z(\tau_\alpha)} [Dw]_{\tau_3} \\
&\times \text{Tr}[(P\epsilon f_0^{M(x)})_{\nu\mu\lambda}^\alpha]\text{Tr}[(P\epsilon f_0^{M(x)})_{\sigma\rho\lambda}^\alpha]\text{Tr}[(P\epsilon f_0^{M(x)})_{\nu\sigma'}^\alpha].
\end{align}

(3.15)

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig4.png}
\caption{\textbf{(a)} The sewing diagram for $\Gamma_1^R$. \textbf{(b)} Another 1PR sewing diagram which we do not consider.}
\end{figure}

### 3.3 The shrinkage types $\Gamma_3^{(1)}$ and $\Gamma_3^{(2)}$

Comparing the defining equations of $\Gamma_1$, $\Gamma_1^R$ and $\Gamma_3^{(i)}$ (q.v. Eqs.\ref{eq:2.13},\ref{eq:2.15},\ref{eq:2.17} and \ref{eq:3.14}), we notice that $\Gamma_3^{(1)}$ and $\Gamma_3^{(2)}$ can be obtained from $\Gamma_1$ and $\Gamma_1^R$ respectively in terms of the following replacement:

\begin{align}
\left(\frac{D_{\mu'}^{\mu''}}{\delta\alpha_{\mu'}^{\alpha_{\mu''}}}(y_1)\Delta_{\sigma'}^{\nu'}(y_1, y_2)\left(\frac{D_{\rho'}^{\rho''}}{\delta\alpha_{\rho'}^{\alpha_{\rho''}}}(y_2)\right)\right) & \rightarrow \frac{1}{2}\delta_{\alpha\beta} g_{\mu'\mu''} g_{\nu'\nu''} \delta(y_1-y_2) \frac{\delta}{\delta\alpha_{\mu'}^{\alpha_{\mu''}}(y_1)} \frac{\delta}{\delta\alpha_{\rho'}^{\alpha_{\rho''}}(y_2)}. \\
\end{align}

(3.16)

In view of this fact, we have only to make the following replacement in Eqs.\ref{eq:3.13} and \ref{eq:3.15}:

\begin{align}
D_{\mu'}^{\mu''}(x(\tau_3))D_{\rho'}^{\rho''}(z(\tau_\alpha))(P\epsilon f_0^{M(w)})_{\nu\sigma'}^\alpha \rightarrow \frac{1}{2}\delta_{\alpha\beta} g_{\mu'\mu''} g_{\nu'\nu''} \delta(T_3) \delta(x(\tau_3)-x(\tau_\alpha)),
\end{align}

(3.17)

and we immediately obtain

\begin{align}
\Gamma_3^{(1)} &= \frac{1}{8}\delta_{\mu'}^{\mu''}\delta_{\rho'}^{\rho''}\int_0^\infty \frac{dS}{S}\int_0^S \frac{dT}{T}\int_0^T d\tau_3 \int_0^T d\tau_1 \int_{\mathbb{D}} [Dx]_{S} \\
&\times g_{\mu'\mu''} g_{\nu'\nu''} \delta(x(\tau_3)-x(\tau_\alpha))\text{Tr}[(P'\epsilon f_{\tau_3}^{M(x)})_{\nu\mu\lambda}^\alpha]\text{Tr}[(P\epsilon f_{\tau_3}^{M(w)})_{\sigma\mu\lambda}^\alpha].
\end{align}

(3.18)
\[
\Gamma_3^{(2)} = -\frac{1}{16} \delta^{\mu' \nu'} \delta^{\rho' \sigma'} \int_0^\infty \frac{dS}{S} \int_0^\infty \frac{dT}{T} \int_0^{\tau_\beta} d\tau_\alpha \oint [Dx]_S \oint [Dz]_T \\
\times g_{\mu' \rho'} g_{\nu' \sigma'} \delta(x(\tau_\beta) - z(\tau_\alpha)) \text{Tr}[(Pe_\int_0^S M(x))_{\nu \mu} \lambda^a] \text{Tr}[(Pe_\int_0^T M(z))_{\rho \sigma} \lambda^a]. \quad (3.19)
\]

Eq. (3.18) coincides with the action (2.26) with \( V_3^{(1)} \) given in Eq. (2.30). The coincidence of Eq. (3.19) with (2.26) can be confirmed as follows. Fixing \( \tau_\beta = S \) and \( \tau_\alpha = 0 \), and shifting \( T = U - S \), we see that the quantity (3.19) behaves as (up to the overall constant and metric symbols etc.)

\[
\Gamma_3^{(2)} \sim \int_0^\infty dU \int_0^\infty dS \oint [Dx]_U \oint [Dz]_{U-S} \delta(x(S) - z(0)) (Pe_\int_0^S M(x))_{ab} (Pe_\int_0^U - S M(z))_{cd}. \quad (3.20)
\]

We then merge the \( z(\tau) \) integration into the \( x(\tau) \) integration through the relation

\[
z(\tau) = x(S + \tau); \quad 0 \leq \tau \leq T, \quad (3.21)
\]

thereby having

\[
\Gamma_3^{(2)} \sim \int_0^\infty dU \int_0^\infty dS \oint [Dx]_U \delta(x(S) - x(U)) (Pe_\int_0^S M(x))_{ab} (Pe_\int_0^U M(x))_{cd}. \quad (3.22)
\]

Renaming \( S \to U \) and \( \tau_\beta \to S \) after fixing \( \tau_\alpha = S \) in Eq. (2.26), we therefore prove that Eq. (3.22) produces Eq. (2.26) with \( V_3^{(2)} \) given in Eq. (2.30).

### 4 The shrinking limit conjecture

In this section, we make a conjecture emerging from the analysis of Section 3.3. We have reproduced the shrinkage types \( \Gamma_3^{(1)} \) and \( \Gamma_3^{(2)} \) from \( \Gamma_1 \) and \( \Gamma_R \), and we infer that this fact might also apply to the double folding types by taking an appropriate shrinking limit of a propagator with two covariant derivatives.

Let us consider the replacement (3.17), in which \( \delta^{ae} \) should be understood as \( \delta^{ai} \delta^{ej} \delta^{ij} \). It may be expressed as the following limit operation \( \delta_R \):

\[
\int [D]_T K^{ij}_{\rho \sigma}(x|y_1, y_2; 0, T_i) \xrightarrow{T_i \to 0} 2 \delta^{ij} g_{\rho \sigma}. \quad (4.1)
\]

with

\[
D_{\mu}^{ai}(y_1) D_{\nu}^{ej}(y_2) \to -\frac{1}{4} \delta^{ai} \delta^{ej} g_{\mu \nu} \delta(y_1 - y_2). \quad (4.2)
\]

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The origins of the numerical factors will be clarified later in Section 5. In this case, we seem to have

\[
\begin{align*}
\delta_R(T_3) & : \Gamma_1 \xrightarrow{T_3 \to 0} \Gamma_3^{(1)}, \\
\delta_R(T_3) & : \Gamma_1^R \xrightarrow{T_3 \to 0} \Gamma_3^{(2)}, \\
\delta_R(T'_3) & : \Gamma_2^R \xrightarrow{T'_3 \to 0} \Gamma_3^{(2)}, \\
\delta_R(T_3) & : \Gamma_2 \xrightarrow{T_3 \to 0} -\Gamma_3^{(1)}. \\
\end{align*}
\]

(4.3)

However, there should be something wrong with the limit of \( \Gamma_2 \), since the sum of \( \Gamma_1 \) and \( \Gamma_2 \) vanishes in the limit (4.3).

In fact, \( \delta_R \) can not apply to \( \Gamma_2 \) and \( \Gamma_2^R \), since the position of \( D_{\text{ai}} \) is not exactly the same point as the starting point \( y_1 \) of the shrinking propagator, but rather the point \( y'_1 \) before setting \( y'_1 = y_1 \) (q.v. Eqs. (2.26), (2.30), (3.8), (3.9)). The graphical situations in sewing diagrams are shown in Fig. 5. The limit \( \delta_R \) is only relevant to the diagram Fig. 5(a). While in the diagram Fig. 5(b), we assume the following limit operation \( \delta_0 \):

\[
\begin{align*}
D_{\mu}(y'_1)K_{\rho \sigma}(y_1, y_2; 0, T_i)D_{\nu}^{\epsilon}(y_2) & \xrightarrow{T_i \to 0} -\frac{1}{2}\delta_{\epsilon}^{\alpha}\delta_{\rho}^{\beta}\delta_{\sigma}^{\gamma}g_{\mu \rho}g_{\nu \sigma}\delta(y'_1 - y_2),
\end{align*}
\]

which leads to

\[
\begin{align*}
\delta_0(T_3) & : \Gamma_2 \xrightarrow{T_3 \to 0} 0, \\
\delta_0(T'_3) & : \Gamma_2^R \xrightarrow{T'_3 \to 0} 0.
\end{align*}
\]

(4.5)

\begin{figure}[h]
\centering
\begin{minipage}[c]{0.45\textwidth}
\centering
\begin{tikzpicture}
  \draw[thick] (0,0) -- (2,0) node[midway, below] {$\rho$};
  \draw[thick] (2,0) -- (4,0) node[midway, below] {$\sigma$};
  \node at (1,0) {$D_{\mu}(y_1)$};
  \node at (3,0) {$D_{\nu}(y_2)$};
\end{tikzpicture}
\caption{(a)}
\end{minipage}\hspace{0.5cm}
\begin{minipage}[c]{0.45\textwidth}
\centering
\begin{tikzpicture}
  \draw[thick] (0,0) -- (2,0) node[midway, below] {$\rho$};
  \draw[thick] (2,0) -- (4,0) node[midway, below] {$\sigma$};
  \node at (1,0) {$D_{\mu}(y'_1)$};
  \node at (3,0) {$D_{\nu}(y_2)$};
  \node at (1.5,0) {$\times$};
  \node at (3.5,0) {$\times$};
\end{tikzpicture}
\caption{(b)}
\end{minipage}
\caption{The positions of shrinking points.}
\end{figure}

For later convenience, we here arrange the 1PR parts \( \Gamma_i^R; \ i = 1, 2 \), in a compact form similar to Eq. (2.26). Fixing \( \tau_\beta = S \) and \( \tau_\alpha = 0 \) in Eq. (3.15), and renaming some integration

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variables in Eq. (3.9), we have

\[
\Gamma_i^R = \delta^{\mu \nu'} \delta^{\rho \sigma'} \int_0^\infty dS \int_0^\infty dT_3 \int_0^\infty dT
\times V^{1PR}_i \int_{x(S)=y_1}^{x(0)=y_0'} [Dx]_S \int_{z(T)=y_2'}^{x(0)=y_2} [Dz]_T \int_{w(T_3)=z(0)}^{x(S)} [Dw]_T \bigg|_{y_i'=y_i}
\times \left[(P e^{\int_0^S M(x)}(z)_{a\mu} \Lambda^\alpha)_{a\mu} \lambda^\alpha \right. \\
\left. \left[(P e^{\int_0^{T_3} M(w)}(z)_{a\mu} \Lambda^\alpha)_{a\mu} \lambda^\alpha \right]ight]^g_l
\quad (4.6)
\]

with

\[
V^{1PR}_1 = \frac{1}{8} \delta^{km} \delta^{g' \gamma} \delta^{f \sigma} g_{a\alpha} g_{b\beta} g_{c\xi} \delta_{c\xi} D^{a\alpha \beta}(x(S)) D^{\beta \gamma}(z(0)) , \quad (4.7)
\]
\[
V^{1PR}_2 = \frac{1}{2} \delta^{km} \delta^{g' \gamma} \delta^{f \sigma} g_{a\alpha} g_{b\beta} g_{c\xi} \delta_{c\xi} D^{a\alpha \beta}(y_1') D^{\beta \gamma}(y_2') , \quad (4.8)
\]

where \(T_3\) is assigned to be the parameter (length of the propagator) which is taken to be zero in the limits \(\delta_R\) and \(\delta_0\). The conjecture is then written in the form:

\[
\delta_R \Gamma_1 + \delta_0 \Gamma_2 \xrightarrow{T_3 \to 0} \Gamma_3^{(1)} , \quad (4.9)
\]
\[
\delta_R \Gamma_1^R + \delta_0 \Gamma_2^R \xrightarrow{T_3 \to 0} \Gamma_3^{(2)} . \quad (4.10)
\]

In the next section, we shall present a piece of evidence for the shrinking limits (1.9) and (1.10) in view of the full world-line representation, thus clarifying the origin of the factors attached in the rules (4.1), (4.3) and (4.4).

5 The pure Yang-Mills world-line formulae

In this section, we simplify all the previous results including the 1PR parts \(\Gamma_i^R\). Actually we show that the quantities (2.6) and (1.6) can be contained in a few concise expressions by analyzing the shrinking limits in the full world-line picture.

Let us start with the following observation. If a covariant derivative is acting on an edge of the gauge particle propagator, the following formula holds (on the propagator):

\[
D^{ab}(x_1) \int_{w(T)=x_1}^{w(0)=x_2} [Dw]_T = \frac{1}{2} \delta^{ab} \int_{w(T)=x_1}^{w(0)=x_2} [Dw]_T \dot{w}_\mu(T) . \quad (5.1)
\]

When one applies the above formula twice, one needs an extra minus sign on the r.h.s. of the formula; i.e. first getting the factor \(\frac{1}{2}\), and then \(-\frac{1}{2}\). This sign is consistent with the Minkowski
and we obtain

\[ (A\lambda^e B\lambda^a C)^{ae} = -\text{Tr}[(A\lambda^e B\lambda^a)]C^{ae} , \]

with

\[ \left( P e^{r_\tau M(\tau) d\tau} \right)^{ab}_{\mu\mu} = \left( P e^{r_\tau M(-\tau + \tau_a + \tau_\beta) d\tau} \right)^{ba}_{\mu\nu} , \]

one can see that \( \Gamma_2 \) consists of two parts, one of which has the same Lorentz index structure as \( \Gamma_1 \), and the other has a different one. We then transform \( \Gamma_1 + \Gamma_2 \) into the following simplified combinations (classified by the types of Lorentz index structures).

\[ \Gamma_1 + \Gamma_2 = \Gamma_3 + \Gamma_4 , \]

with

\[
\begin{align*}
\Gamma_3 &= -\frac{1}{8} \int_0^\infty dS \int_0^S d\tau_0 \int_0^\infty dT_3 \int [Dx] \int_{w(T_3) = x(\tau_0)} [Dw] T_3 \left( \hat{w}_\mu(0) - \hat{x}_\mu(0) \right) \hat{w}_\rho(T_3) \\
&\quad \times \text{Tr} \left[ (Pe^{\int_{\tau_0}^S M(x)} e^0)_{\nu\rho} \lambda^e (Pe^{\int_{\tau_0}^S M(x)} e^0)_{\sigma} \lambda^a \right] \left( Pe^{\int_{	au_0}^T M(w)} e^0 \right)^{ae} , \hspace{1cm} (5.5)
\end{align*}
\]

\[
\begin{align*}
\Gamma_4 &= \frac{1}{8} \int_0^\infty dS \int_0^S d\tau_0 \int_0^\infty dT_3 \int [Dx] \int_{w(T_3) = x(\tau_0)} [Dw] T_3 \hat{x}_\nu(0) \hat{w}_\rho(T_3) \\
&\quad \times \text{Tr} \left[ (Pe^{\int_{\tau_0}^S M(x)} e^0)_{\mu} \lambda^e (Pe^{\int_{\tau_0}^S M(x)} e^0)_{\nu} \lambda^a \right] \left( Pe^{\int_{	au_0}^T M(w)} e^0 \right)^{ae} , \hspace{1cm} (5.6)
\end{align*}
\]

where the indices \( \rho \) and \( \sigma \) in \( \Gamma_3 \) are anti-symmetrized by the lowercased symbol \([\cdot\cdot]\) (Note that we used \( x(0) = x(S) \)). For the 1PR parts, we can proceed similarly with

\[ (F\lambda^e G\lambda^b H)^{ab} = \text{Tr}[(F\lambda^e) G^{ab}] \text{Tr}[\lambda^b H] , \]

\[ \text{Tr}[\Delta_{\mu\nu}(y_1, y_2)\lambda^a] = -\text{Tr}[\Delta_{\nu\mu}(y_2, y_1)\lambda^a] = 0 , \]

and we obtain

\[
\begin{align*}
\Gamma_R &= \Gamma_1^R + \Gamma_2^R \\
&= \frac{1}{8} \int_0^\infty dS \int_0^S dT_3 \int [Dx] \int [Dz] T_3 \int_{w(T_3) = z(T)} [Dw] T_3 \\
&\quad \times \left( \hat{w}_\mu(0) \hat{w}_\rho(T_3) + \hat{x}_\mu(0) \hat{z}_\sigma(T) \right) \\
&\quad \times \text{Tr} \left[ (Pe^{\int_{\tau_0}^S M(x)} e^0)_{\nu\mu} \lambda^e \right] \left( Pe^{\int_{\tau_0}^T M(w)} e^0 \right)^{ae} \text{Tr} \left[ (Pe^{\int_{\tau_0}^T M(z)} e^0)_{\nu\rho} \lambda^a \right] . \hspace{1cm} (5.9)
\end{align*}
\]
We shall verify the conjecture in this way: we show that $\Gamma_3^{(1)}$ and $\Gamma_3^{(2)}$ survive as singular integrands of $\Gamma_1$ and $\Gamma_3^R$ in the limit $T_3 \to 0$, while $\Gamma_2$ and $\Gamma_2^R$ do not contribute.

Let us first consider the 1PI case (4.9). For the purpose of understanding the shrinking limit, we have only to analyze a free propagator; both edges of a shrinking propagator already involve four gluon lines, hence there is no need to take external gluon lines into account. The vacuum diagrams suffice since the short distance behavior in the vicinities of the two vertices is important. Actually the effect of external lines can be evaluated by insertions of vertex operators afterward. On these grounds it is sufficient to observe the $\phi^3$ world-line Green functions [13, 20, 21]. The necessary two-loop world-line Green functions are as follows [20]:

\[
G_{ww}^{(1)}(\tau_1, \tau_2) = \langle w(\tau_1)w(\tau_2) \rangle = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T_3 + G_B(\tau_\alpha, \tau_\beta)},
\]

\[
G_{wx}^{(1)}(\tau_1, \tau_2) = \langle w(\tau_1)\tau(\tau_2) \rangle = G_{xx}^{(1)}(\tau_\beta, \tau_2) + \frac{1}{T_3 + G_B(\tau_\alpha, \tau_\beta)}(T_3\tau_1 - \tau_1^2 + \tau_1[G_B(\tau_2, \tau_\alpha) - G_B(\tau_2, \tau_\beta)]),
\]

where $G_B$ and $G_{xx}^{(1)}$ are the one-loop [11, 12] and two-loop [13] world-line Green functions on the $x(\tau)$ loop (length $S$):

\[
G_B(\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{S},
\]

\[
G_{xx}^{(1)}(\tau_1, \tau_2) = G_B(\tau_1, \tau_2) - \frac{1}{4} \left[ G_B(\tau_1, \tau_\alpha) - G_B(\tau_1, \tau_\beta) - G_B(\tau_2, \tau_\alpha) + G_B(\tau_2, \tau_\beta) \right]^2.
\]

We evaluate

\[
\langle \dot{w}(0)\dot{w}(T_3) \rangle = \partial_1 \partial_2 G_{ww}^{(1)}(\tau_1, \tau_2) \bigg|_{\tau_1=0,\tau_2=T_3} = 2\delta(T_3) + \frac{2}{T_3 + G_B(\tau_\alpha, \tau_\beta)},
\]

\[
\langle \dot{w}(0)\dot{w}(T_3) \rangle = \partial_1 \partial_2 G_{ww}^{(1)}(\tau_1, \tau_2) \bigg|_{\tau_1=T_3,\tau_2=0} = \frac{-2|\tau_\beta - \tau_\alpha|}{S(T_3 + G_B(\tau_\alpha, \tau_\beta))},
\]

where we have taken account of the absolute value of $\tau_\beta - \tau_\alpha$ in the second quantity (5.13), since the Green function (5.11) is in fact defined for the ordering $\tau_\alpha < \tau_\beta$ [20]. These yield the following relation:

\[
\langle (\dot{w}_\mu(0) - \dot{x}_\mu(0))\dot{w}_\rho(T_3) \rangle = 2g_{\mu\rho} \left[ \delta(T_3) + \frac{S + |\tau_\alpha - \tau_\beta|}{S(T_3 + G_B(\tau_\alpha, \tau_\beta))} \right].
\]

One can equally well work with another representation of these Green functions [13]. In that case, one should pay attention to the sign (direction) of each $\tau$ parameter [21]. The $\tau_\beta$ is revived here for convenience of presentation.
The first term on the r.h.s. in Eq.(5.16) provides a singular term in the $T_3$ integrand, while the second term contributes to a regular quantity in the $T_3$ integrand, since we can not take both of $T_3 \to 0$ and $\tau_\beta \to \tau_\alpha$ simultaneously. (If one takes both of them, one of the loops shrinks to a point). The singularity $\delta(T_3)$ comes only from the quantity (5.14), and the shrinking conjecture (4.9) clearly corresponds to this singularity in the limit $T_3 \to 0$ (with $|\tau_\beta - \tau_\alpha|$ kept finite). Apparently $\Gamma_4$ does not contribute the singularity (because of Eq.(5.15)).

The meaning of the other term in Eq.(5.16) is the following. Let us take $\tau_\beta \to \tau_\alpha$ in the r.h.s. of Eq.(5.16). It reduces to

$$\text{R.H.S. of (5.16)} \to 2g_{\mu\rho} \left( \delta(T_3) + \frac{1}{T_3} \right).$$

Note that this is the same form as the second derivative $\partial_1 \partial_2 G_B(\tau_1, \tau_2)$ of the one-loop Green function on a loop of length $T_3$. The $\delta$-function is the singular term relevant to the shrinking limit $T_3 \to 0$ as mentioned above. The second term also seems to be singular, however it is not (Remember $T_3 \neq 0$). In the situation of $\tau_\beta \to \tau_\alpha$, the $w(\tau)$ path integral becomes a loop integral (q.v. (5.5)) and this $T_3^{-1}$ factor simply describes a part of the usual loop integral measure:

$$\lim_{\epsilon \to 0} \int_\epsilon^{\infty} \frac{dT_3}{T_3} \oint [Dw]_{T_3}. \quad (5.18)$$

Therefore we can embed the $\Gamma_3^{(1)}$ into $\Gamma_3$ as the integrable edge singularity at $T_3 = 0$ ($\epsilon = 0$) produced by the Wick contraction of two vertices (5.14).

In the case of the 1PR function (5.9), we expect the similar shrinking (4.10) takes place. Since $x(\tau)$ and $z(\tau)$ can not approach each other, there is no singular term created from $\Gamma_2^R$. This non-singular situation itself is common in $\Gamma_2$ and $\Gamma_2^R$, and the position of $\tau_\alpha$ (whether $\tau_\alpha < \tau_\beta$ or $\tau_\beta < \tau_\alpha$) is not an important issue (see Fig. 3(b)). Hence the folding diagram piece Fig. 3(a) does not yield any singularity, and neither does the diagram Fig. 4(b). The only singular part comes from the $\dot{w}(0)\dot{w}(T_3)$ contraction originated in $\Gamma_1^R$. To see this, the free (open) bosonic two-point function suffices:

$$\partial_1 \partial_2 <w_\mu(\tau_1)w_\sigma(\tau_2)>_{\text{tree}}^{\text{tree}} |_{\tau_1=0,\tau_2=T_3} = g_{\mu\sigma} \partial_1 \partial_2 |_{\tau_1=0,\tau_2=T_3} = 2g_{\mu\sigma} \delta(T_3). \quad (5.19)$$

Thus one can embed $\Gamma_3^{(2)}$ into the edge singularity of $\Gamma_R$, with obtaining the relation (4.10). It is also easy to understand Eqs.(3.17), (4.1), (4.2) due to Eqs.(5.1) and (5.19):

$$\lim_{T_3 \to 0} D^a_{\mu}(y_1)D^b_{\nu}(y_2)K^c_{\rho\sigma}(y_1, y_2; 0, T_3) = 2 \left( \frac{i}{2} \right)^2 \delta^{ai} \delta^{bj} \delta_{\rho\sigma} g_{\mu\nu} g_{\rho\sigma} \delta(y_1 - y_2). \quad (5.20)$$
The factor $\frac{1}{4}$ in Eq. (4.2) is derived from the normalization $\frac{1}{2}$ in the formula (5.1), and the factor 2 in Eq. (4.1) is the coefficient 2 in front of the $\delta(T_3)$ singularity.

In this section, we have derived the shrinking relations (4.9) and (4.10) in paraphrase by world-line language: in other words we showed that $\Gamma^{(i)}_3$ can be embedded into $\Gamma_3$ and $\Gamma_R$ as the edge singularities of the $T_3$ integrals. We hence have only three compact representations (5.5), (5.6) and (5.9). However, remember that a simpler alternative set of compact representations is of course to have $\Gamma_3$, $\Gamma_4$ and $\Gamma^{(2)}_3$ (see Eqs. (3.19) or (A.6)).

### 6 The pseudo-abelian case

We examine how the results obtained in Section 5 can be simplified in the $su(2)$ pseudo-abelian case with constant field strength. We then derive a general formula for the two-loop Euler-Heisenberg type action in this case. The setting is the following [27]: We assume the particular decomposition

$$A^a_{\mu} = A_{\mu} n^a, \quad F^a_{\mu\nu} = F_{\mu\nu} n^a,$$

$$M_{ij}(x) = M(x) (\lambda^a n^a)_{ij},$$

where all the color vectors are chosen to be proportional to a unit color vector; for instance, $\vec{n} = (n^1, n^2, n^3)$ for $su(2)$. The quantities $A_{\mu}$, $F_{\mu\nu}$, $M$ are all commuting quantities in color space, and we further assume the following relation for the commuting gauge field:

$$A_{\mu} \dot{x}^{\mu} = \frac{1}{2} x^{\mu} F_{\mu\nu} \dot{x}^{\nu}.$$  \hspace{1cm} (6.3)

In the following we also use the brief notation

$$\mathcal{P}_{\mu\nu}(i) = \left( P \exp \left( \int M(x_i) d\tau \right) \right)_{\mu\nu}, \quad i = 1, 2, 3,$$

where $i$ stands for the $i$th path ordered exponential as they appear from left to right in Eqs. (2.26) and (4.6). Also the integration ranges should be understood in the same way. Let us consider the $su(2)$ case. Defining the following bases:

$$\mathcal{T}_- \equiv \lambda^a n^a, \quad \mathcal{T}_+ \equiv (\mathcal{T}_-)^2, \quad \mathcal{I} = \text{diag}(1, 1, 1) - \mathcal{T}_+,$$

$$\mathcal{P}_{\mu\nu}(i) = \left( P \exp \left( \int M(x_i) d\tau \right) \right)_{\mu\nu}, \quad i = 1, 2, 3.$$  \hspace{1cm} (6.4)

where $i$ stands for the $i$th path ordered exponential as they appear from left to right in Eqs. (2.26) and (4.6). Also the integration ranges should be understood in the same way. Let us consider the $su(2)$ case. Defining the following bases:

$$\mathcal{T}_- \equiv \lambda^a n^a, \quad \mathcal{T}_+ \equiv (\mathcal{T}_-)^2, \quad \mathcal{I} = \text{diag}(1, 1, 1) - \mathcal{T}_+,$$

$$\mathcal{P}_{\mu\nu}(i) = \left( P \exp \left( \int M(x_i) d\tau \right) \right)_{\mu\nu}, \quad i = 1, 2, 3.$$  \hspace{1cm} (6.5)
we can expand the path ordered exponential \( (6.4) \) on these:

\[
\mathcal{P}_{\mu\nu}(i) = \mathcal{P}_{\mu\nu}^+(i)\mathcal{T}_+ + \mathcal{P}_{\mu\nu}^-(i)\mathcal{T}_- + \delta_{\mu\nu}\mathcal{I}
\]

where the coefficients are given by

\[
\mathcal{P}^+(i) = \cosh(\int \mathcal{M}(x_i)d\tau) \quad \mathcal{P}^-(i) = \sinh(\int \mathcal{M}(x_i)d\tau)
\]

We also define the following quantities just for compactness of presentation:

\[
\mathcal{P}_{\pm}(i,j)_{\mu\nu} \equiv \left( \mathcal{P}_{\mu\nu}^+(i)\mathcal{P}_{\mu\nu}^+(j) \pm \mathcal{P}_{\mu\nu}^-(i)\mathcal{P}_{\mu\nu}^-(j) \right)
\]

After some calculation by using the formulae

\[
\text{Tr}[\mathcal{I}\mathcal{L}^a\mathcal{I}^a] (\mathcal{T}_\pm)^{ae} = 2, \quad \text{Tr}[\mathcal{T}_\pm\mathcal{L}^a\mathcal{I}^a] (\mathcal{I})^{ae} = 2 \quad \text{etc.} \quad (6.9)
\]

we obtain

\[
\Gamma_3 = -\frac{1}{4} \int dSdT_3d\tau_3 \int [Dx]S \int_{w(T_3)=x(\tau_0)} \left[ Dw \right]_{T_3} \left( \delta_{\mu\nu}(0) - \dot{x}_{\mu}(0) \right) \dot{w}_{\nu}(T_3)
\]

\[
\times \left\{ \left( \mathcal{P}^-(1,2) + \mathcal{P}^+(1,3) + \mathcal{P}^+(2,3) \right)_{\mu\nu} - \mathcal{P}^+_{\mu\nu}(2) \text{Tr}_L[\mathcal{P}^+(1) + \mathcal{P}^+(3)]
\]

\[
- \mathcal{P}^+_{\mu\nu}(2) \text{Tr}_L[\mathcal{P}^-(1) + \mathcal{P}^-(3)] - \delta_{\mu\nu} \text{Tr}_L\mathcal{P}^+(-1,3) \right\}, \quad (6.10)
\]

where \( \text{Tr}_L \) means the trace w.r.t. the Lorentz indices. The minus symbols of the first arguments in \( \mathcal{P}^-(i,j) \) mean that the direction of their paths are reverted by the changes of \( \tau \) directions.

Also, applying the following formulae:

\[
\text{Tr}[\mathcal{P}_{\mu\nu}(1)\lambda^a\mathcal{P}_{\rho\sigma}(2)\lambda^a] = 2\left( \mathcal{P}_{\mu\nu}^+(1)\delta_{\rho\sigma} + \delta_{\mu\nu}\mathcal{P}_{\rho\sigma}^+(2) + \sum_{\kappa=\pm} \mathcal{P}_{\mu\nu}^\kappa(1)\mathcal{P}_{\rho\sigma}^\kappa(2) \right), \quad (6.11)
\]

\[
\text{Tr}[\mathcal{P}_{\mu\nu}(1)\lambda^a]\text{Tr}[\mathcal{P}_{\rho\sigma}(2)\lambda^a] = 4\mathcal{P}_{\mu\nu}^-(1)\mathcal{P}_{\rho\sigma}^-(2), \quad (6.12)
\]

\[
\text{Tr}[\mathcal{P}_{\mu\nu}(1)\lambda^a][\mathcal{P}_{\nu\rho}(2)] = 4\mathcal{P}_{\mu\nu}^-(1)\mathcal{P}_{\nu\sigma}^-(2), \quad (6.13)
\]

we obtain \( \Gamma_3^{(i)} \) and the reducible function \( (5.9) \) as follows:

\[
\Gamma_3^{(1)} = -\frac{1}{2} \int dSdT_3 \int [Dx]S \left\{ -\text{Tr}_L\mathcal{P}^+(1)(\text{Tr}_L\mathcal{P}^+(2)) - (\text{Tr}_L\mathcal{P}^-(1))(\text{Tr}_L\mathcal{P}^-(2))
\]

\[
+ \text{Tr}_L[\mathcal{P}^+(1) + \mathcal{P}^+(2) + \mathcal{P}^-(1,2)] \right\}, \quad (6.14)
\]
\[
\Gamma_3^{(2)} = \int dSdT \int [\mathcal{D}x]_S [\mathcal{D}z]_T \operatorname{Tr}_L \left[ \mathcal{P}^- (1) \mathcal{P}^- (2) \right], \\
\Gamma_R = \frac{1}{2} \int dSdT dT_3 \int [\mathcal{D}x]_S [\mathcal{D}z]_T \int_{u(T_3) = z(T)} [\mathcal{D}w]_{T_3} w(0) = \mathcal{R}(S) \times (\dot{w}_\mu (0) \dot{w}_\nu (T_3) + \dot{x}_\mu (0) \dot{z}_\nu (T_3)) (\mathcal{P}^- (2) \mathcal{P}^- (1))_{\nu \mu}. 
\]

With the replacements
\[
\dot{w}_\mu (0) \dot{w}_\nu (T_3) \to \frac{2}{g_{\mu \nu}} \delta (T_3), \quad \text{otherwise} \to 0, 
\]
it is easy to see that the previous shrinking limits hold:
\[
\Gamma_R \xrightarrow{T_3 \to 0} \Gamma_3^{(2)}, \quad \Gamma_3 \xrightarrow{T_3 \to 0} \Gamma_3^{(1)}. 
\]

Here is a remark. The r.h.s. of (6.10) is not a symmetric expression in 1 ↔ 2 exchange, just because of the fixing \(\tau_\beta = 0\). One may of course take an average to have the symmetric expression if any strong reason exists. The other quantities are symmetric as seen in Eqs. (6.14)-(6.16).

Now, let us derive a general formula for an Euler-Heisenberg type action. We concentrate on the \(\Gamma_3\) part, since we intend to discuss a general strategy only, and the \(\Gamma_3^{(2)}\) part is straightforward from the one-loop action \([27]\) (see Eq.(6.15)). We assume \(F_{\mu \nu}\) to be a constant matrix, and introduce the “symmetric” type world-line representation with splitting \(S = T_1 + T_2\):}

\[
\int \frac{dS}{S} dT_3 d\tau_\alpha d\tau_\beta \int [\mathcal{D}x]_S \int_{u(T_3) = x(\tau_\alpha)} [\mathcal{D}w]_{T_3} = \int d^D y_1 d^D y_2 \int dT_1 dT_2 dT_3 \prod_{a=1}^{3} \int_{x_a(0) = y_1}^{x_a(T_a) = y_2} [\mathcal{D}x_a]_{T_a}. 
\]

Every term in (6.10) has one free internal line (without background field) out of three lines \(a = 1, 2, 3\). Denoting the free line label as \(b\), we consider the following general term \(\Gamma_{EH}^{(b)}\) for the linear combination \(\Gamma_{EH}^{(b)}\):

\[
\Gamma_{EH}^{(b)} = \int \prod_{a=1}^{3} dT_a [\mathcal{D}x_a]_{T_a} V_{\mu \nu} \left( \exp \left[ \int_0^{T_a} \mathcal{M}(x_a) d\tau_a \right] \right)_{\nu \mu} 
\]

\[
= \int \prod_{a=1}^{3} dT_a [\mathcal{D}x_a] (e^{2iFT_a})_{\nu \mu} V_{\mu \nu} \left[ - \frac{1}{4} \sum_{a=1}^{3} \int_0^{T_a} (\dot{x}_a^2 + 2i \kappa_a x_a^\mu F_{\mu \nu} \dot{x}_a^\nu) d\tau_a \right], 
\]

where \(\kappa_a (a \neq b)\) is either of \(\pm 1\),

\[
V_{\mu \nu} = \left( \dot{x}_3^\mu (0) - \dot{x}_1^\mu (0) \right) x_3^\nu (T_3), 
\]
and the primes on $\prod$ and $\sum$ denote to set $F_{\mu\nu} = 0$ in the $x_b$ and $\tau_b$ integrals. In this paper, we omit the sign factor $\kappa^a$ for simplicity, since it is not difficult to revive it by rescaling $F \rightarrow \kappa F$. After performing the path integrals (see Appendix C for details), the action $\Gamma^{(b)}_{EH}$ takes the following form:

$$\Gamma^{(b)}_{EH} = \left( \prod_{a=1}^{3} \int dT_a(e^{2iFT_a}) \right) \nu_{\mu\nu} \mathcal{N}_b < V_{\mu\nu} >' ,$$

(6.22)

where $< V_{\mu\nu} >'$ is an expectation value in the action

$$S^{(b)} = -\frac{1}{4} \sum_{a=1}^{3} \int_0^{T_a} (x^2_a + 2ix_a F x_a) d\tau_a ,$$

(6.23)

and $\mathcal{N}_b$ is the path integral determinant factor

$$\mathcal{N}_b = (4\pi)^{\frac{D}{2}} \left( \prod_{a=1}^{3} (4\pi T_a)^{-\frac{D}{2}} \right) \det^{-1/2} \det \frac{\sin(FT_a)}{FT_a} .$$

(6.24)

The quantity $< V_{\mu\nu} >'$ can be evaluated by the following world-line two-point correlator:

$$G_{\mu\nu}(\tau_a, \tau'_c) = -\delta_{ac} \tilde{G}_{\mu\nu}(\tau_a, \tau'_c) + 2 \left( \left[ \sum ' F \cot(FT_a) \right]^{-1} \right)_{\rho\sigma} \left( e^{2iF \tau_a} - 1 - \frac{1}{2} \right)_{\rho\mu} \left( e^{2iF \tau'_c} - 1 - \frac{1}{2} \right)_{\sigma\nu} ,$$

(6.25)

where

$$\tilde{G}_{\mu\nu}(\tau, \tau') = G^a_{\mu\nu}(\tau, \tau') - G^a_{\mu\nu}(\tau, 0) - G^a_{\mu\nu}(0, \tau')$$

(6.26)

with

$$G^a_{\mu\nu}(\tau, \tau') = \begin{cases} \delta_{\mu\nu} G^a_B(\tau, \tau') = \delta_{\mu\nu} \left[ |\tau - \tau'| - \frac{(\tau - \tau')^2}{T_a} \right] & \text{(for } a \text{ on a free line)} \\ \left[ \frac{2\pi}{\sin(FT_a)} e^{-iFT_a \partial_\tau G^a_B(\tau, \tau')} + iF \partial_\tau G^a_B(\tau, \tau') - \frac{T_a}{2} \right]_{\mu\nu} & \text{(otherwise)}. \end{cases}$$

(6.27)

7 The ghost loop

The two-loop contribution from the ghost part can be easily extracted from the generating functional (2.12):

$$\Gamma_5 = -i \int dy_1 dy_2 \text{Tr} \ln(D^2 - D\beta f) \left[ \frac{\delta^3}{2} \delta \Delta \delta \beta \right]_{\alpha=0, \beta=0} .$$

(7.1)

It is convenient to represent the $D\beta f$ term as

$$(D^2 - D\beta f)^{ac} = (D^2)^{ac} - iD^b_{\mu} \beta^d_{\mu} (\lambda^d)_{bc} ,$$

(7.2)
and also to have the formal analogies of Eqs. (2.24) and (2.23) without representing the right derivative \( \hat{D}_\mu \) parts in terms of a world-line field:

\[
\text{Tr}_{\ln(\hat{D}^2 - D\beta f)} = -\int_0^\infty \frac{dS}{S} \int_0^S D\mathcal{X} \exp[- \int_0^S \frac{1}{4} \hat{x}^2(\tau)d\tau] (\mathcal{P} \exp \int_0^S \hat{N}[x(\tau)]d\tau)^{a\alpha} , \tag{7.3}
\]

\[
\omega^{ab}(y_1, y_2) = \int_0^\infty d(\tau_2 - \tau_1) \int_{x(\tau_2) = y_2} \int_{x(\tau_1) = y_1} D\mathcal{X} \exp[- \int_{\tau_1}^{\tau_2} \hat{x}^2(\tau)d\tau] (\mathcal{P} \exp \int_{\tau_1}^{\tau_2} \hat{N}[x(\tau)]d\tau)^{a\alpha} , \tag{7.4}
\]

where

\[
\hat{N}_{ab}[x(\tau)] = -iA^c_\mu \delta^{ic}_a (\lambda^c)_{ab} - i\hat{D}^{ca}_\mu (\lambda^c)^d_{\beta \mu} . \tag{7.5}
\]

Here we have a definite reason of not expressing the right derivative \( \hat{D}_\mu \) in terms of the world-line field at this stage. This kind of derivative is understood to be artificially exponentiated as a consequence of the introduction of auxiliary field \( \beta \), since the \( \beta \)-derivatives in Eq. (7.1) get the right derivatives \( \hat{D} \) down from the exponent. Thus the \( \hat{D} \) acts on a boundary of the path integral, and it will be replaced by a world-line field according to formula (5.2) after all.

Substituting the expressions (2.23), (7.3) and (7.4) in Eq. (7.1), we can perform the \( \delta \beta \) differentiations in the same way as done in Section 3.2 with

\[
\frac{\delta \hat{N}^{ic}_a[x(\tau)]}{\delta \beta^{\mu}_{\beta \mu}(y)} = -i\hat{D}^{ci}_\mu (\lambda^c)_{bc} \delta(y - x(\tau)) . \tag{7.6}
\]

The only difference is that we have to cut open the loop integral to manifest the boundaries where the \( \hat{D} \)'s operate:

\[
\int_0^\infty \frac{dS}{S} \int_0^S d\tau_\alpha \int_0^S d\tau_\beta \int_0^S [D\mathcal{X}]_S \hat{D}_\alpha(y_2) \hat{D}_\beta(y_1) \delta(y_1 - x(\tau_\beta)) \delta(y_2 - x(\tau_\alpha))
\]

\[
= \int d^2y'_1 d^2y'_2 \int_0^\infty dT_1 \int_0^\infty dT_2 \int_{x(T_1) = y'_2} \int_{x(T_2) = y'_1} [D\mathcal{X}]_{T_1} \int_{x(T_2) = y'_1} [D\mathcal{X}]_{T_2}
\]

\[
\times \hat{D}_\alpha(y'_2) \hat{D}_\beta(y'_1) \delta(y_1 - \bar{x}(T_2)) \delta(y_2 - \bar{x}(T_1)) \bigg|_{y'_1 = y_1} . \tag{7.7}
\]

The result is thereby

\[
\Gamma_5 = -\frac{1}{2} \int_0^\infty dT_1 dT_2 dT_3 \int_{x(T_3) = y'_2} \int_{x(T_3) = y'_1} [D\mathcal{X}]_{T_1} \int_{x(T_2) = y'_1} [D\mathcal{X}]_{T_2} \int_{x(T_2) = y'_2} [D\mathcal{X}]_{T_3}
\]

\[
\times \left( \mathcal{P} \int_0^{T_3} M(w) \delta^{ic}_{\alpha \mu} (\mathcal{P} \int_0^{T_3} N(x) \delta^{ji}_{\alpha \mu} f_j(y'_2) (\lambda^c)^f_{ji} (\mathcal{P} \int_0^{T_3} N(x) \delta^{ji}_{\alpha \mu} f_j(y'_2) (\lambda^c)^f_{ji} \right) \bigg|_{y'_1 = y_1} , \tag{7.8}
\]

where the \( y'_i \) integrations are implicit for simplicity, and

\[
N(x) = \hat{N}(x) \bigg|_{\beta = 0} = -iA^a_\mu x_\mu \lambda^a . \tag{7.9}
\]
Then applying the formula (5.1), we finally reach the expression
\[ \Gamma_5 = \frac{1}{8} \int_0^\infty dT_1 dT_2 dT_3 \int_{\gamma(T_1) = \tilde{x}(0)} |Dx| T_1 |D\tilde{x}| T_2 \int_{\gamma(T_3) = \tilde{x}(T_2)} |Dw| T_3 \]
\[ \times \dot{x}_\mu(T_1) \dot{x}_\mu(T_2) (P e^{\int_0^{T_3} M(w)_{\mu \nu}}) \Tr \left[ (P e^{\int_0^{T_1} N(x)}) \lambda^e (P e^{\int_0^{T_2} N(\tilde{x})}) \lambda^a \right], \quad (7.10) \]

In closing this section, several remarks are in order. (i) One may further replace \( \dot{x}_\mu(T_1) \dot{x}_\mu(T_2) \) with \( \dot{x}(\tau_\alpha) \dot{x}(\tau_\beta) \) to have a closed path integral for \( x \) like \( \Gamma_1 \). However, before doing this, one should keep in mind that no (pinching) singularity is caused by \( \tau_\alpha \to \tau_\beta \), as expected from the ordinary Feynman rule. (ii) In the same way as the argument of Section 5, this fact can easily be justified either from the symmetric world-line Green function [19] or from the following loop type Green function [20]:
\[ G_{xx}(\tau, \bar{\tau}) = G_B(\tau, \bar{\tau}) - \frac{1}{T_3 + G_B(\tau_\alpha, \tau_\beta)} (\tau_\beta - \bar{\tau} - |\tau_\alpha - \tau_\beta| S)^2, \quad (7.11) \]
where \( \bar{\tau} \) is the world-line parameter for the \( \bar{x} \) field, and the ordering \( \tau_\alpha < \tau_\beta \) is assumed by definition. Note that the first term \( G_B(\tau, \bar{\tau}) \) does not generate the singularity because of the ordering constraint \( \tau < \bar{\tau} \), and
\[ < \dot{x}(T_1) \dot{x}(T_2) = \partial_\tau \partial_{\bar{\tau}} G_{xx}(\tau, \bar{\tau}) \bigg|_{\tau = T_1, \bar{\tau} = T_2} = \frac{2T_3}{S(T_3 + G_B(\tau_\alpha, \tau_\beta))}. \quad (7.12) \]
(iii) On the contrary, in the case of scalar loop, the Green function (5.13) generates the singularity, which is a realization of embedding contact interaction.

### 8 The fermion loop

In this section, we make a short remark on the fermion loop case. The fermion two-loop part can be extracted from the action (2.6) as
\[ \Gamma_6 = -i \int dy_1 dy_2 \Tr \ln(\gamma^\mu (i\hat{D}_\mu - \epsilon^a \hat{\lambda}^a)) \frac{\delta^3}{2} \delta_e \Delta \delta_e \bigg|_{\alpha = 0, \epsilon = 0}. \quad (8.1) \]

Regarding the auxiliary field \( \epsilon^a_\mu \) in the determinant in Eq. (2.6) as a counterpart of classical background field in the covariant derivative, we have only to perform the shift \( A^a_\mu \to A^a_\mu - \epsilon^a_\mu \) in the usual world-line fermion loop formula, thus
\[ \Tr \ln(\gamma^\mu (i\hat{D} - \epsilon^a \hat{\lambda}^a)) = -2 \int_0^\infty dS \int DxD\psi \Tr P \exp[- \int_0^S (L_0 + \bar{M}_F) d\tau], \quad (8.2) \]
\[ \bar{\Delta}^{ij}(y_1, y_2) \overset{\text{def.}}{=} \int_0^\infty d(\tau_2 - \tau_1) \int_{x(\tau_2) = y_2} x(\tau_1) = y_1 \right. \] 
\[ Dx D\psi (\text{Pexp} - \int_{\tau_1}^{\tau_2} (L_0 + \bar{M}_F) d\tau)^{ij}, \quad (8.3) \]

where
\[ L_0 = \frac{1}{4} \dot{x}^\mu \dot{x}_\mu + \frac{1}{2} \psi^\mu \dot{\psi}_\mu + i A_\mu \dot{x}_\mu - i \psi^\mu F_{\mu\nu} \psi^\nu, \quad (8.4) \]
\[ \bar{M}_F = -i \epsilon_\mu \dot{x}_\mu - \psi^\mu [\epsilon_\mu, \epsilon_\nu] \psi^\nu + 2i \psi^\mu (D^a_\mu \psi^a)\psi^\nu \hat{\lambda}^c, \quad (8.5) \]

and \( \epsilon_\mu = \epsilon^a_\mu \hat{\lambda}^a \) etc. If we apply the derivative \( \delta / \delta \epsilon^a_\mu \) to the loop (8.2), we get a vertex
\[ \delta \bar{M}_F(\tau) \delta \epsilon^a_\mu(y_1) = \delta(y_1 - x(\tau)) \left( i(\dot{x}_\mu - 2\psi^\nu D^a_\mu \psi^a)\hat{\lambda}^c + \frac{\delta}{\delta \epsilon^a_\mu} \psi \cdot [\epsilon, \epsilon] \psi \right). \quad (8.6) \]

The first term on the r.h.s. of (8.6) is pointed out in the abelian context in [19]. The second term (the commutator term in (8.6)) further contributes in a second derivative, and it becomes
\[ \frac{\delta^2 \bar{M}_F(\tau)}{\delta \epsilon^a_\mu(y_2) \epsilon^a_\mu(y_1)} = \delta(y_1 - x(\tau))\delta(y_2 - x(\tau))2\psi^\mu(\tau)[\hat{\lambda}^a, \hat{\lambda}^b] \psi^\nu(\tau). \quad (8.7) \]

It is worthwhile noting that this is equivalent to the pinching terms prescribed at the one-loop level by Strassler [11] (see also [28]):
\[ \text{Eq.}(8.7) \sim O_{ji} + O_{ij}, \quad (8.8) \]

where \( \sim \) means to remove the plane wave modes \( \epsilon_n \exp[ik_n x(\tau_n)]; n = i, j \) in order to form a second loop by joining the two external lines, and
\[ O_{ji} = (-ig)^2 \hat{\lambda}^a_j \hat{\lambda}^a_i \int_0^S d\tau_j d\tau_i \delta(\tau_j - \tau_i)2\epsilon_j \cdot \psi(\tau_j)\epsilon_i \cdot \psi(\tau_i) e^{i(k_i + k_j) \cdot x(\tau_i)}. \quad (8.9) \]

As in the case of QED [24], the two-loop effective action can be formulated very elegantly using world-line supersymmetry on the fermion loop: One has to substitute supervariables \( X_\mu(\tilde{\tau}) = x_\mu(\tau) + \sqrt{2}\theta \psi_\mu(\tau) \) with \( \tilde{\tau} = (\tau, \theta) \) everywhere for \( x_\mu(\tau) \) on the loop in the corresponding scalar case; in particular the superaction only contains the interaction term \( DX_\mu(\tilde{\tau})A_\mu(X) \) \( (D = \partial_\theta - \theta \partial_\tau) \), and the inserted gauge field propagator has a simple form in supervariables at its endpoints. The non abelian commutator pieces are then obtained automatically by a supersymmetric generalization of the ordering \( \theta \)-function [32].

Given our formalism for pure Yang-Mills theory, it is also an interesting question if a world-line supersymmetric formulation of the interaction part of the action together with a (necessarily) supersymmetry breaking kinetic part can be used for a compact formulation.
9 Conclusions

In this paper, we presented a method how to construct the two-loop effective action in terms of the bosonic world-line path integral representation in pure Yang-Mills theory, and we discussed the way how one of the “eight figure” sewing diagrams can be unified in the $\Phi^3$ type sewing diagram. The effective action is then summarized as

$$\Gamma^{2-\text{loop}} = \Gamma_3 + \Gamma_4 + \Gamma_3^{(2)},$$

(9.1)

and the additional term $\Gamma_3^{(2)}$ is related to one of the 1PR sewing contributions (vid. Fig. 4(a)).

In Section 2, we explained the method how to obtain the two-loop analogue of the trace-log formula in the case of (full) Yang-Mills theory, using the background field and auxiliary field methods. The pure Yang-Mills part of the generating functional (2.13) resembles the $\phi^3$ theory case; i.e., the propagator $\Delta$ consists of the free propagator, background field and auxiliary field terms. The whole generating functional (2.12) contains the full information of how to join the trace-log loops and the gluon propagators at any loop order. The sewing method does not require the computation of a number of Wick contractions w.r.t. space-time fields, and organizes all such terms into a compact expression automatically (as seen in Appendix A). The world-line calculations are thus expected to be easier than those in the space-time Wick contraction method.

In Section 3, we performed this sewing procedure in the language of the world-line representation. We realized that the bosonic field representations (2.22) and (2.23) fit well with the sewing method. All the $\Gamma_i$ including 1PR parts coincide with the $\Gamma$’s (in Eq.(A.2)) evaluated without world-line (path integral) representation. It is worth noting that the emerging ways of the triple products of propagators are totally different from each other. One comes from the cutting rule (3.1) of the path integral, and the other is from power series expansions. This is certainly a non-trivial observation showing how the purely bosonic world-line representations are integrated in the entire framework (with the help of the sewing method).

In Section 4, we derived the compact world-line formulae (5.5), (5.6) and (3.19) for the two-loop pure Yang-Mills theory. We showed that Eq.(5.5) includes the $\Gamma_3^{(1)}$ part as the edge singularity of the integrand of $\Gamma_3$, and as a result, the full effective action (9.1) is simply the sum of (5.5), (5.6) and (3.19). We also found the similar relation between $\Gamma_3^{(2)}$ and $\Gamma_R$. The
latter relation itself seems not to be useful, since having the crude expression of $\Gamma_3^{(2)}$ (q.v. Eq. (3.19)) is much simpler than the embedding into $\Gamma_R$. However it was certainly helpful to check the validity of the shrinking limit. In the $su(2)$ pseudo-abelian case with much simpler expressions, we observed these results more clearly in Section 3. It is also interesting to note that the addition of the 1PR related part $\Gamma_3^{(2)}$ resembles the one-loop observation that we have to add and shrink 1PR tree vertices in the Bern-Kosower rules [3].

We also wrote down the ghost part for the two-loop effective action in Section 4. An interesting question is how to realize gauge independence [14] by gathering the ghost loop $\Gamma_5$ and the pure Yang-Mills part (1.1). We still have a problem to obtain a path integral representation of the gluon propagator in an arbitrary gauge, however this could be avoided formally separating the propagator into Feynman gauge part plus others [15]. This strategy works in the one-loop case, however the present status between $\Gamma_5$ and $\Gamma_3 + \Gamma_4$ (plus $\Gamma_3^{(2)}$) seems still far away from the goal.

Nevertheless, we now have the effective action, and we will be able to compute amplitudes with Fourier expanding the background fields (i.e. substituting plane wave modes as explained in Appendix B). This substitution determines a combinatorial factor, and the whole procedure is straightforward. After that, we will be able to compare our results with conventional calculations. Studying a connection to string theory would also be of interest [30, 31]. A non-abelian version of the Euler-Heisenberg Lagrangian in two-loop order [27, 29] is a possible further interest as well. In this case we should proceed to the linear combination of $\Gamma_{EH}^{(b)}$ to get a more detailed expression. We hope to address these issues in a future publication.

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Appendix A. Equivalence to Feynman diagram technique

We present an equivalence of the world-line path-integral formulae to the results with the conventional technique. First of all, without introducing the path-integral representations (2.22) and (2.23), one can of course perform the functional $\delta_\alpha$ differentiations directly: with using the
cyclicity of color trace, the anti-symmetry of \( f^{abc} = i(\lambda^a)_{bc} \), and the following relation

\[
\Delta_{\mu\nu}^{ab}(1, 2) \equiv \Delta_{\mu\nu}^{ab}(x_1, x_2) = \Delta_{\nu\mu}^{ba}(x_2, x_1) .
\]  

(A.1)

The results are organized as follows:

\[
\begin{align*}
\Gamma_1 &= \frac{1}{4} \delta_{\mu\nu}^{\rho\sigma} \int dx_1dx_2(D_{\mu}^{ai}(x_1)D_{\nu}^{aj}(x_2)\Delta_{i,j}(1, 2))\text{Tr}[\Delta_{\rho\sigma}(1, 2)\lambda^a\Delta_{\sigma\mu}(2, 1)\lambda^a] , \\
\Gamma_R^1 &= \frac{1}{8} \delta_{\mu\nu}^{\rho\sigma} \int dx_1dx_2(D_{\mu}^{ai}(x_1)D_{\nu}^{aj}(x_2)\Delta_{i,j}(1, 1))\text{Tr}[\Delta_{\rho\sigma}(1, 2)\lambda^a]\text{Tr}[\Delta_{\sigma\rho}(2, 1)\lambda^a] , \\
\Gamma_2 &= \frac{1}{2} \delta_{\mu\nu}^{\rho\sigma} \int dx_1dx_2(D_{\mu}^{ai}(x_1)D_{\nu}^{aj}(x_2)\Delta_{i,j}(1, 1))\text{Tr}[\Delta_{\rho\sigma}(2, 2)\lambda^a] , \\
\Gamma_R^2 &= \frac{1}{2} \delta_{\mu\nu}^{\rho\sigma} \int dx_1dx_2(D_{\mu}^{ai}(x_1)D_{\nu}^{aj}(x_2)\Delta_{i,j}(1, 1))\text{Tr}[\Delta_{\rho\sigma}(1, 2)\lambda^a] , \\
\Gamma_3^{(1)} &= -\frac{1}{8} \delta_{\mu\nu}^{\rho\sigma} \delta_{\rho\sigma}^{\rho\sigma} \int dx_1dx_2g_{\mu\rho}g_{\nu\sigma}\text{Tr}[\Delta_{\rho\sigma}(1, 2)\lambda^a] , \\
\Gamma_3^{(2)} &= -\frac{1}{16} \delta_{\mu\nu}^{\rho\sigma} \delta_{\rho\sigma}^{\rho\sigma} \int dx_1dx_2g_{\mu\rho}g_{\nu\sigma}\text{Tr}[\Delta_{\rho\sigma}(1, 1)\lambda^a] , \\
\end{align*}
\]

where \( x'_i ; i = 1, 2 \), are introduced to express derivative’s positions where to operate, and they should be set to be \( x_i \) after the differentiations. Here a few remarks are in order. (i) One can see the consistency of the world-line representation (2.28) with these results if identifying the quantity (2.23) (with \( \alpha_{\mu\nu}^a = 0 \)) to be \( \Delta_{\mu\nu}^{ab} \). (ii) The above results are not unique expressions. For example, noticing the formulae

\[
\text{Tr}[\Delta_{\mu\nu}(1, 2)\lambda^a] = -\text{Tr}[\Delta_{\nu\mu}(2, 1)\lambda^a] ,
\]

(A.3)

\[
(\lambda^a)_{ij}(\lambda^a)_{kl} = (\lambda^a)_{ik}(\lambda^a)_{jl} - (\lambda^a)_{jk}(\lambda^a)_{il} ,
\]

(A.4)

\[
\delta_{\mu\nu}^{\rho\sigma} \delta_{\rho\sigma}^{\rho\sigma} \delta_{\mu\nu}^{\rho\sigma} \delta_{\rho\sigma}^{\rho\sigma} = 2\delta_{\rho\sigma}^{\rho\sigma} ,
\]

(A.5)

we can also have the following expressions:

\[
\Gamma_3^{(2)} = -\frac{1}{4} \text{Tr}[\Delta_{\mu\nu}(1, 1)\lambda^a] \text{Tr}[\Delta_{\mu\nu}(2, 2)\lambda^a] ,
\]

(A.6)

\[
= \frac{1}{8} \delta_{\mu\nu}^{\rho\sigma} \delta_{\rho\sigma}^{\rho\sigma} \text{Tr}[\Delta_{\rho\sigma}(1, 1)\lambda^a] \Delta_{\mu\nu}(2, 2)\lambda^a] .
\]

(A.7)

Let us derive the 1PI parts of the above \( \Gamma_i \) in another method; i.e. by Wick contracting quantum fields \( Q_{\mu}^a \). We have the pure Yang-Mills action

\[
\mathcal{L}_{YM} = \mathcal{L}_0 + \mathcal{L}_3 + \mathcal{L}_4 ,
\]

(A.8)
where $L_0$ stands for the one-loop (kinetic) term of quantum gauge field, and the other terms are

\[
L_3 = -f^{abc}Q_{\mu\nu}^a Q_{\mu\nu}^b Q_{\mu\nu}^c ; \quad Q_{\mu\nu}^a = (D_{\mu}Q_{\nu})^a ,
\]

\[
L_4 = -\frac{1}{4}(f^{abc}Q_{\mu\nu}^a Q_{\mu\nu}^c)^2 .
\]

The two-loop effective action can thus be obtained as

\[
i\Gamma_{\text{2-loop}} = <iL_4> >_0 + \frac{1}{2} <iL_3(x_1)iL_3(x_2) >_0 \\
\equiv iS_4 + iS_3 ,
\]

where $< X > >_0$ means a correlation function evaluated in the one-loop action $L_0$ (The space-time integrations are implicit). First let us consider the $S_3 \sim <L_3L_3>_0$ part. After Wick contracting by using the gluon propagator

\[
< Q_{\mu\nu}^a(x_1)Q_{\mu\nu}^b(x_j) >_{ij} = -i\Delta_{ab}^{\mu\nu}(i,j) ,
\]

we split $S_3$ into the following two quantities specified by single and double contractions w.r.t. $Q_{\mu\nu}^a$:

\[
S_3 = \frac{i}{2} <L_3(x_1)L_3(x_2) >_0 = W_1 + W_2 ,
\]

where

\[
W_1 = -\frac{i}{2} f^{abc} f^{efg} \left[ <Q_{\mu\nu}^a Q_{\rho\sigma}^c >_{12} - (\rho \leftrightarrow \sigma) \right] \Delta_{\mu\nu}^{bf}(1,2) \Delta_{\rho\sigma}^{cg}(1,2) ,
\]

\[
W_2 = \frac{1}{2} f^{abc} f^{efg} <Q_{\mu\nu}^a Q_{\rho\sigma}^c >_{12} \left( \Delta_{\mu\nu}^{bg}(1,2) \left[ <Q_{\rho\sigma}^b Q_{\mu\nu}^c >_{12} - (\rho \leftrightarrow \sigma) \right] \right. + \Delta_{\rho\sigma}^{bg}(1,2) \left. \left[ <Q_{\mu\nu}^c Q_{\rho\sigma}^b >_{12} - (\rho \leftrightarrow \sigma) \right] \right) .
\]

After some rearrangement to have the antisymmetric symbols $\delta_{\mu\nu}^{\mu'\nu'} \delta_{\rho\sigma}^{\rho'\sigma'}$, we verify the equalities

\[
W_1 = \Gamma_1 , \quad W_2 = \Gamma_2 .
\]

Next, let us consider the $L_4$ part in the same way. We first divide $S_4$ into the following quantities (self and mutual contractions w.r.t. space-time coordinates):

\[
S_4 = -\frac{1}{4} f^{abc} f^{efg} <Q_{\mu\nu}^a(x_1)Q_{\rho\sigma}^b(x_2) >_0 \delta_{\mu\nu}^{\rho\sigma} g^{\mu\nu} g^{\rho\sigma} \delta(x_1 - x_2) \\
= W_3^{(1)} + W_3^{(2)} ,
\]
where the self-contraction part is given by
\[ W^{(2)}_3 = -\frac{1}{4} f^{abc} f^{efg} \delta^{ae} g^{\mu \rho} g^{\nu \sigma} \delta(x_1 - x_2) < Q^b_\mu Q^c_\nu >_{11} < Q^f_\rho Q^g_\sigma >_{22} , \] (A.18)
and the mutual-contraction part is given by
\[ W^{(1)}_3 = -\frac{1}{4} f^{abc} f^{efg} \delta^{ae} g^{\mu \rho} g^{\nu \sigma} \delta(x_1 - x_2)
\times ( < Q^b_\mu Q^f_\rho >_{12} < Q^c_\nu Q^g_\sigma >_{12} + < Q^b_\mu Q^g_\sigma >_{12} < Q^c_\nu Q^f_\rho >_{12} ) . \] (A.19)
We thus verify that
\[ W^{(1)}_3 = \Gamma^{(1)}_3 , \quad W^{(2)}_3 = \Gamma^{(2)}_3 . \] (A.20)

The ghost part is evaluated as follows. The action is
\[ L_{FP} = \bar{c} D^2 c + f^{abc} c^a (D_\mu \bar{c})^b Q^c_\mu , \] (A.21)
and the two-loop contribution is given by
\[ W_5 = \frac{i}{2} f^{abc} f^{efg} < c^a (D_\mu \bar{c})^b Q^c_\mu (x_1) c^e (D_\nu \bar{c})^f Q^g_\nu (x_2) > , \] (A.22)
which can be evaluated with the propagators \([A.12]\) and
\[ < c^a c^b >_{12} = -i \Xi^{ab}(1,2) . \] (A.23)
We thus obtain
\[ W_5 = \frac{1}{2} \Delta^{ae}_{\mu \nu}(1,2) (D_\mu (x_1) \Xi(1,2))^{bg} (\lambda^e)_{gf} D^f_{\mu}(x_2) \Xi^{cj}(1,2)(\lambda^a)_{cb} , \] (A.24)
and this coincides with Eq.(7.8) identifying \( \Xi^{ab} \) (with \( \beta^a_\mu = 0 \) to be \( \Xi^{ab} \):
\[ W_5 = \Gamma_5 . \] (A.25)

**Appendix B. General structure of gluon N-point functions**

In this appendix, we explain a general prescription how to calculate the N-point proper functions in the (two-loop) pure Yang-Mills case. Let us start with the following function:
\[ \Gamma^{(0) \text{def}} = \int d^D y_1 d^D y_2 \prod_{a=1}^3 \int dT_a \int_{x_a(x_0) = y_2}^{T_a} [Dx_a] T_a \text{Pexp} \left[ \int_{x_a(0) = y_1}^{T_a} M(x_a) dT^{(a)} \right] \lambda(a) \] (B.1)
or equivalently

\[
\Gamma^{(0)} = \int_0^{\infty} dS T_3 \int_0^S d\tau_0 \int [\mathcal{D}x]S \int_{w(T_3)=x(\tau_0)}^{w(0)=x(S)} [\mathcal{D}w] T_3 \mathcal{P}_{\mu_1\nu_1}(x) \lambda^{\alpha_1} \mathcal{P}_{\mu_2\nu_2}(x) \lambda^{\alpha_2} \mathcal{P}_{\mu_3\nu_3}(w) \tag{B.2}
\]

where we have defined \( \lambda(1) = \lambda^{\alpha_1}, \lambda(2) = \lambda^{\alpha_2}, \) and \( \lambda(3) = 1 \), and the notation \( \mathcal{P}_{\mu\nu} \) is defined in Eq.(6.4).

To calculate the corresponding \( N \)-point proper Green functions (\( \equiv \Gamma^{(0)}_N \)), we expand the background (interaction) terms

\[
\left( \mathcal{P}_e \int_0^{T_{a}} M\mathcal{D}^{(a)} \right)_{\mu\nu} = \sum_{N_a=0}^{\infty} (P_{N_a})_{\mu\nu}^N \quad \tag{B.3}
\]

where

\[
P_{N_a} = \int_0^{T_a} d\tau_{N_a} \int_0^{\tau_{N_a}} d\tau_{N_{a-1}} \cdots \int_0^{\tau_2} d\tau_1 \mathcal{M}(\tau_{N_a}) \cdots \mathcal{M}(\tau_1), \quad \tag{B.4}
\]

and insert the plane wave modes \( \tilde{\mathcal{V}}_j(\tau) \) in this expression. Then \( P_{N_a} \) becomes multiple (ordered) integrals of the following quantity:

\[
\tilde{V}_j(\tau) = -ig\lambda^{(j)} \left[ e^{i\tau \delta_{\mu\nu}} - 2i(k_\mu^j e_\nu^j - k_\nu^j e_\mu^j) \right] e^{i k^{(j)} \cdot x(\tau)} + \mathcal{O}(g^2). \quad \tag{B.5}
\]

Here we neglect the higher order term, since we expect that it will be evaluated later with the pinching techniques (vid. Refs.\[[11, 28, 32]\]).

As usual \([11, 18]\), we are allowed to perform the following replacement by virtue of the total momentum conservation law (concerning the external legs):

\[
(P_{N_a})_{\mu\nu}^N \rightarrow \sum_{\sigma(N_a)} \int_0^{T_a} d\tau_{N_a}^{(a)} \int_0^{\tau_{N_a}^{(a)}} d\tau_{N_{a-1}}^{(a)} \cdots \int_0^{\tau_2^{(a)}} d\tau_1^{(a)} \left( \tilde{V}_{i_1}(\tau_{i_1}^{(a)}) \cdots \tilde{V}_{i_{N_a}}(\tau_{i_{N_a}}^{(a)}) \right)_{\mu\nu}^N, \quad \tag{B.6}
\]

where \( \sigma(N_a) \equiv (i_1, i_2, \cdots, i_{N_a}) \) counts all permutations of the \( N_a \) leg labels. Thus

\[
\Gamma^{(0)}_N = \int dy_1 dy_2 \prod_{a=1}^{3} \int dT_a \int_{x(T_a)=y_1}^{x(T_a)=y_2} \mathcal{D}x_a T_a \sum_{\sigma(N_a)} \int_0^{T_a} d\tau_{N_a}^{(a)} \int_0^{\tau_{N_a}^{(a)}} d\tau_{N_{a-1}}^{(a)} \cdots \int_0^{\tau_2^{(a)}} d\tau_1^{(a)} \times \left( \tilde{V}_{i_1}(\tau_{i_1}^{(a)}) \cdots \tilde{V}_{i_{N_a}}(\tau_{i_{N_a}}^{(a)}) \right)_{\mu\nu}^N \lambda^{(a)}. \tag{B.7}
\]

The remaining tasks to obtain the full gluon amplitudes are rather straightforward: insert the operators defined by \( \mathcal{O}_{\mu\nu} \), which are at most products of two world-line fields as explained in Section \( \Box \), and then compute correlations among these world-line fields by using the world-line correlator discussed in Appendix C (setting the extra constant background fields to be zero).
Appendix C. Derivation of the background Green function

In this appendix, we derive the two-loop world-line Green function in a constant background with demonstrating how to perform the path integrals in Eq. (3.20). Let us introduce an external (\(\tau\) dependent) source term as usual in field theory:

\[
z[J] \equiv \int d^D y_1 d^D y_2 \left( \prod_{a=1}^{3} \int_{x_a(0)=y_1} x_a(T_a)=y_2} D x_a \right) \exp \left[ S^{(b)} + \sum_a \int_0^{T_a} J_a^\mu(\tau) x_a^\mu(\tau) d\tau \right]. \tag{C.1}
\]

To perform the path integrals, we decompose \(x_a\) into classical and ‘quantum’ fields:

\[
x_a(\tau_a) = x_a^c(\tau_a) + \tilde{x}_a(\tau_a), \tag{C.2}
\]

with the boundary condition \(\tilde{x}_a(0) = \tilde{x}_a(T_a) = 0\), and the classical field is given by [27]

\[
x_a^c(\tau_a) = \left( \frac{y_1 + y_2}{2} \right)_\mu + R^a_{\mu\nu}(y_2 - y_1)_\nu, \tag{C.3}
\]

where

\[
R^a_{\mu\nu} = \left( e^{2i\mathcal{F} \tau_a} - 1 \right) \frac{1}{2}_{\mu\nu}. \tag{C.4}
\]

Then Eq. (C.1) is rewritten in the form

\[
z[J] = \int d^D y_1 d^D y_2 \prod_{a=1}^{3} \int_{\tilde{x}_a(0)=0}^{\tilde{x}_a(T_a)=0} D \tilde{x}_a \exp \left[ \int_0^{T_a} \left\{ \frac{1}{2} \tilde{x}_a^\mu (\frac{1}{2} \partial^2 - i \mathcal{F} \partial_\tau) \tilde{x}_a + J_a \cdot \tilde{x}_a \right\} d\tau \right] \times \exp \left[ -\frac{1}{4} (y_2 - y_1)^\mu A^a_{\mu\nu}(y_2 - y_1)_\nu + \int_0^{T_a} (\mathcal{F} \cot(\mathcal{F} T_a)) \right] R^a_{\mu\nu} d\tau \right] \times \exp \left[ \left( \frac{y_1 + y_2}{2} \right)_\mu \int_0^{T_a} J_a(\tau) d\tau \right], \tag{C.5}
\]

where

\[
A^a_{\mu\nu} = \left( \mathcal{F} \cot(\mathcal{F} T_a) \right)_{\mu\nu}. \tag{C.6}
\]

The path integrals (of 'quantum' fields) yield \(\exp[-\frac{1}{2} J(\frac{1}{2} \partial^2 - i \mathcal{F} \partial_\tau)^{-1} J]\), where the inverse operator is given [27] by Eq. (6.27), and the combination (6.26) should be taken for the boundary condition \(\tilde{x}_a(T_a) = \tilde{x}_a(0) = 0\) as well. The (classical part) integrals provide the zero mode divergence (corresponding to the momentum conservation factor in the sense of Minkowski formulation)

\[
\int d\left( \frac{y_1 + y_2}{2} \right) \prod_{a=1}^{3} e^{\frac{y_1 + y_2}{2}} \int J_a = i\delta^D \left( \sum_{a=1}^{3} \int_0^{T_a} J_a^\mu(\tau) d\tau \right), \tag{C.7}
\]

34
and determine the path integral normalization factor

\[ N_b = (4\pi)^{D/2} \left( \prod_{a=1}^{3} (4\pi T_a)^{-D/2} \right) \text{Det}_L^{-\frac{1}{2}} \left( \sum_a A^a \right) \prod_a \text{Det}_L^{-\frac{1}{2}} \left( \frac{\sin (FT_a)}{FT_a} \right) . \tag{C.8} \]

The final expression of \( z[J] \) is therefore

\[
z[J] = i\delta^D \left( \sum_{a=1}^{3} \int_0^{T_a} J^\mu_a(\tau) d\tau \right) (4\pi)^{D/2} \left( \prod_{a=1}^{3} (4\pi T_a)^{-D/2} \right) \text{Det}_L^{-\frac{1}{2}} \left( \sum_a A^a \right) \prod_a \text{Det}_L^{-\frac{1}{2}} \left( \frac{\sin (FT_a)}{FT_a} \right) \\
\times \exp \left[ -\frac{1}{2} \sum_a \int_0^{T_a} \int_0^{T_a} J^\mu_a(\tau_a) G_{\mu\nu}(\tau_a, \tau_a') J^\nu_a(\tau_a') d\tau_a d\tau_a' \right] \\
\times \exp \left[ \left( \sum_a A^a_{\rho\sigma} \right)^{-1} \left( \sum_a \int_0^{T_a} R^a_{\rho\mu}(\tau) J^\mu_a(\tau) \right) \left( \sum_c \int_0^{T_c} R^c_{\sigma\nu}(\tau) J^\nu_c(\tau) \right) \right] , \tag{C.9} \]

and the two-point correlator \((6.25)\) is derived as

\[ G_{\mu\nu}(\tau_a, \tau_c') = \frac{\delta}{\delta J^\mu_a(\tau_a)} \frac{\delta}{\delta J^\nu_c(\tau_c')} \ln z[J] \big|_{J=0} . \tag{C.10} \]

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