Differential operator realizations of superalgebras and free field representations of corresponding current algebras

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Abstract

Based on the particular orderings introduced for the positive roots of finite dimensional basic Lie superalgebras, we construct the explicit differential operator representations of the $osp(2r|2n)$ and $osp(2r + 1|2n)$ superalgebras and the explicit free field realizations of the corresponding current superalgebras $osp(2r|2n)_k$ and $osp(2r+1|2n)_k$ at an arbitrary level $k$. The free field representations of the corresponding energy-momentum tensors and screening currents of the first kind are also presented.

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1 Introduction

The interest in two-dimensional non-linear $\sigma$-models with supergroups or their cosets as target spaces has grown drastically over the last ten years because of their applications ranging from string theory \cite{1,2} and logarithmic conformal field theories (CFTs) \cite{3,4} (for a review, see e.g. \cite{5,6}, and references therein) to modern condensed matter physics \cite{7,8,9,10,11,12,13,14}. The Wess-Zumino-Novikov-Witten (WZNW) models associated with supergroups stand out as an important class of such $\sigma$-models. This is due to the fact that, besides their own importance, the WZNW models are also the “building blocks” for other coset models which can be obtained by gauging or coset constructions \cite{15,16,17,18}. In these models, current or affine (super)algebras \cite{19} are the underlying symmetry algebras and are relevant to integrability of the model.

In contrast to the bosonic versions, the WZNW models on supergroups are far from being understood \cite{20} and references therein), although some progress has been made \cite{21} recently for the models related to type I supergroups \cite{22}. This is largely due to technical reasons (such as indecomposability of the operator product expansion (OPE) \cite{23,24}, appearance of logarithms in correlation functions and continuous modular transformations of the irreducible characters \cite{25}), combined with the lack of “physical intuition”.

On the other hand, the Wakimoto free field realizations of current algebras \cite{26} have been proved very powerful in the study of the WZNW models on bosonic groups \cite{27,28,29,30,31,32}. Since the work of Wakimoto on the $sl(2)$ current algebra, much effort has been made to obtain similar results for the general case \cite{33,34,35,36,37,38,39}. In these constructions, the explicit differential operator realizations of the corresponding finite dimensional (super)algebras play a key role. However, explicit differential operator expressions heavily depend on the choice of local coordinate systems in the so-called big cell $\mathcal{U}$ \cite{40}. Thus it is at least very involved, if not impossible, to obtain explicit differential operator expressions for higher-rank (super)algebras in the usual coordinate systems \cite{36,37,38,39,42,43}. Recently it was shown in \cite{44,45,46} that there exists a certain coordinate system in $\mathcal{U}$, which drastically simplifies the computation involved in the construction of explicit differential operator expressions for higher-rank (super)algebras. We call such a coordinate system the “good coordinate system”.

This paper will show how to establish a “good coordinate system” of the big cell $\mathcal{U}$ for
an arbitrary finite-dimensional basic Lie superalgebra \[22\]. It will be seen that the “good coordinate system” indeed exits and is related to a particular ordering for the positive roots of the superalgebra. Based on such an ordering of the positive roots, we construct the “good coordinate system” for the superalgebras \(osp(2r|2n)\) and \(osp(2r+1|2n)\) and derive their explicit differential operator representations. We then apply these differential operators to construct explicit free field representations of the \(osp(2r|2n)\) and \(osp(2r+1|2n)\) current algebras.

This paper is organized as follows. In section 2, we briefly review finite-dimensional simple basic Lie superalgebras and their corresponding current algebras, which also introduces our notation and some basic ingredients. In section 3, we introduce the particular orderings for the positive roots of the superalgebras \(osp(2r|2n)\) and \(osp(2r+1|2n)\). Based on the orderings, we construct the explicit differential operator representations of \(osp(2r|2n)\) and \(osp(2r+1|2n)\). In section 4 we apply these differential operator expressions to construct the explicit free field realizations of the \(osp(2r|2n)\) and \(osp(2r+1|2n)\) currents, the energy-momentum tensors and the screening currents. Section 5 provides some discussions. In the Appendix A, we give the matrix forms of the defining representations of superalgebras \(osp(2r|2n)\) and \(osp(2r+1|2n)\).

2 Notation and preliminaries

Let \(\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_1\) be a finite dimensional simple basic Lie superalgebra \[22, 47\] with a \(\mathbb{Z}_2\)-grading:

\[
[a] = \begin{cases} 
0 & \text{if } a \in \mathcal{G}_0, \\
1 & \text{if } a \in \mathcal{G}_1.
\end{cases}
\]

The superdimension of \(\mathcal{G}\), denoted by \(sdim\), is defined by

\[
sdim(\mathcal{G}) = \dim(\mathcal{G}_0) - \dim(\mathcal{G}_1). \tag{2.1}
\]

For any two homogenous elements (i.e. elements with definite \(\mathbb{Z}_2\)-gradings) \(a, b \in \mathcal{G}\), the Lie bracket is defined by

\[
[a, b] = a b - (-1)^{[a][b]} b a.
\]
This (anti)commutator extends to inhomogenous elements through linearity. Let \( \{ E_i | i = 1, \ldots, d \} \), where \( d = \dim(G) \), be the basis of \( G \), which satisfy (anti)commutation relations,

\[
[E_i, E_j] = \sum_{l=1}^{d} f_{ij}^{l} E_l.
\] (2.2)

The coefficients \( f_{ij}^{l} \) are the structure constants of \( G \). Alternatively, one can use the associated root system [17] to label the generators of \( G \) as follows. Let \( H \) be the Cartan subalgebra of \( G \). A root \( \alpha \) of \( G \) (\( \alpha \neq 0 \)) will be an element in \( H^* \), the dual of \( H \), such that:

\[
G_{\alpha} = \{ a \in G | [h, a] = \alpha(h) a, \quad \forall h \in H \} \neq 0.
\] (2.3)

The set of roots is denoted by \( \Delta \). Let \( \Pi = \{ \alpha_i | i = 1, \ldots, r \} \) be the simple roots of \( G \), where the rank of \( G \) is equal to \( r = \dim(H) \). With respect to \( \Pi \), the set of positive roots is denoted by \( \Delta_+ \), and we write \( \alpha > 0 \) if \( \alpha \in \Delta_+ \). A root \( \alpha \) is called even or bosonic (odd or fermionic) if \( G_\alpha \in G_0 \) (\( G_\alpha \in G_1 \)). The set of even roots is denoted by \( \Delta_0 \), while the set of odd roots is denoted by \( \Delta_1 \). Associated with each positive root \( \alpha \), there is a raising operator \( E_\alpha \), a lowering operator \( F_\alpha \) and a Cartan generator \( H_\alpha \). These operators have definite \( \mathbb{Z}_2 \)-gradings:

\[
[H_\alpha] = 0, \quad [E_\alpha] = [F_\alpha] = \begin{cases} 
0, & \alpha \in \Delta_0 \cap \Delta_+ \\
1, & \alpha \in \Delta_1 \cap \Delta_+.
\end{cases}
\]

Moreover, one has the Cartan-Weyl decomposition of \( G \)

\[
G = G_- \oplus H \oplus G_+,
\] (2.4)

where \( G_- \) is a span of lowering operators \( \{ F_\alpha \} \) and \( G_+ \) is a span of raising operators \( \{ E_\alpha \} \), and \( G_\pm \) respectively generates an nilpotent subalgebra of \( G \).

One can introduce a nondegenerate and invariant supersymmetric metric or bilinear form for \( G \), which is denoted by \( (E_i, E_j) \) (e.g. see [A.20] for \( osp(2r|2n) \) and [A.42] for \( osp(2r + 1|2n) \)). Then the affine Lie superalgebra \( G_k \) (or \( G \) current algebra) associated to \( G \) is generated by \( \{ E_i^n | i = 1, \ldots, d; n \in \mathbb{Z} \} \) satisfying (anti)commutation relation:

\[
[E_i^n, E_j^m] = \sum_{l=1}^{d} f_{ij}^{l} E_i^{n+m} + nk(E_i, E_j) \delta_{n+m,0}.
\] (2.5)

Introduce currents

\[
E_i(z) = \sum_{n \in \mathbb{Z}} E_i^n z^{-n-1}, \quad i = 1, \ldots, d.
\]
Then the (anti)commutation relations (2.5) can be re-expressed in terms of the OPEs [24] of the currents,

\[ E_i(z)E_j(w) = k \frac{(E_i, E_j)}{(z - w)^2} + \sum_{m=1}^{d} f_{ij}^m E_m(w), \quad i, j = 1, \ldots, d, \] (2.6)

where \( f_{ij}^m \) are the structure constants [27,22]. The aim of this paper is to construct explicit free field realizations of the current algebras associated with the unitary series \( sl(r|n) \) (or \( gl(r|n) \)) and the orthosymplectic series \( osp(2r|2n) \) and \( osp(2r + 1|2n) \) at an arbitrary level \( k \).

### 3 Differential operator realizations of superalgebras

Let \( G \) be a Lie supergroup with \( G \) being its Lie superalgebra, and \( X \) be the flag manifold \( G/B_- \), where \( B_- \) is the Borel subgroup corresponding to the subalgebra \( G_- \oplus H \). The differential operator realization of \( G \) can be obtained from the infinitesimal action of the corresponding group element on sections of a line bundle over \( X \) [18] or an \( \eta \)-invariant lifting of the vector fields on \( X \) which form a representation of \( G \) [40]. As an open set of \( X \), we will take the big cell \( U \), which is the orbit of the unit coset under the action of subgroup \( N_+ \) with Lie superalgebra \( G_+ \). After choosing some local coordinates of \( U \), all the generators of \( G \) in principle can be realized by first-order differential operators of the coordinates. In this section we show that there are “good coordinate systems” which enable us to obtain the explicit differential operator realizations of all basic Lie superalgebras. We shall construct such coordinate systems for the three infinite series of basic superalgebras \( sl(r|n) \), \( osp(2r|2n) \) and \( osp(2r + 1|2n) \) with generic \( r \) and \( n \). Our coordinate system in \( U \) is based on a particular ordering introduced for positive roots \( \Delta_+ \) of the corresponding superalgebra. We call this ordering the normal ordering [49] of \( \Delta_+ \).

**Definition 1** The roots of \( \Delta_+ \) are in normal ordering if all roots are ordered in such a way that: (i) for any pairwise non-colinear roots \( \alpha, \beta, \gamma \in \Delta_+ \) such that \( \gamma = \alpha + \beta \), \( \gamma \) is between \( \alpha \) and \( \beta \); (ii) for \( \alpha, 2\alpha \in \Delta_+ \), \( 2\alpha \) is located on the nearest right of \( \alpha \).

Such an ordering was constructed explicitly for all (super)algebras with rank less than 3 in [49]. In the following, we shall give the normal ordering of positive roots for each of the three infinite series superalgebras \( sl(r|n) \), \( osp(2r|2n) \) and \( osp(2r + 1|2n) \).
3.1 Differential operator realization of $sl(r|n)$

Hereafter, let us fix two non-negative integers $n$ and $r$ such that $2 \leq n + r$. Let us introduce $n + r$ linear-independent vectors: $\{\delta_i| i = 1, \ldots, n\}$ and $\{\epsilon_i| i = 1, \ldots, r\}$. These vectors are endowed with a symmetric inner product such that

$$(\delta_m, \delta_l) = \delta_{ml}, \quad (\delta_m, \epsilon_i) = 0, \quad (\epsilon_i, \epsilon_j) = -\delta_{ij}, \quad i, j = 1, \ldots, r, \quad m, l = 1, \ldots, n. \quad (3.1)$$

The root system $\Delta$ of $sl(r|n)$ (or $A(r-1, n-1)$) can be expressed in terms of the vectors:

$$\Delta = \{\epsilon_i - \epsilon_j, \delta_m - \delta_l, \delta_m - \epsilon_i, \epsilon_i - \delta_m\}, \quad 1 \leq i \neq j \leq r, \quad 1 \leq m \neq l \leq n,$$

while the even roots $\Delta_0$ and the odd roots $\Delta_1$ are given respectively by

$$\Delta_0 = \{\epsilon_i - \epsilon_j, \delta_m - \delta_l\}, \quad \Delta_1 = \{\pm(\delta_m - \epsilon_i)\}, \quad 1 \leq i \neq j \leq r, \quad 1 \leq m \neq l \leq n.$$

The distinguished simple roots are

$$\alpha_1 = \delta_1 - \delta_2, \ldots, \alpha_{n-1} = \delta_{n-1} - \delta_n, \quad \alpha_n = \delta_n - \epsilon_1,$$

$$\alpha_{n+1} = \epsilon_1 - \epsilon_2, \ldots, \alpha_{n+r-1} = \epsilon_{r-1} - \epsilon_r.$$

With regard to the simple roots, the corresponding positive roots $\Delta_+$ are

$$\delta_m - \delta_l, \quad \epsilon_i - \epsilon_j, \quad 1 \leq i < j \leq r, \quad 1 \leq m < l \leq n, \quad (3.2)$$

$$\delta_m - \epsilon_i, \quad 1 \leq m \leq n, \quad 1 \leq i \leq r. \quad (3.3)$$

Among these positive roots, $\{\delta_m - \epsilon_i| i = 1, \ldots, r, \quad m = 1, \ldots, n\}$ are odd and the others are even. Then we construct the normal ordering of the corresponding positive roots.

**Proposition 1** A normal ordering of $\Delta_+$ for $sl(r|n)$ is given by

$$\epsilon_{r-1} - \epsilon_r, \ldots, \epsilon_1 - \epsilon_r, \ldots, \epsilon_1 - \epsilon_2; \delta_n - \epsilon_r, \ldots, \delta_n - \epsilon_1; \delta_l - \epsilon_r, \ldots, \delta_l - \delta_n, \ldots, \delta_1 - \delta_2. \quad (3.4)$$

**Proof.** One can directly verify that the above ordering of the positive roots (3.2)-(3.3) of $sl(r|n)$ fulfills all requirements of Definition 1. □
It is well-known that the big cell $U$ is isomorphic to the subgroup $N_+$ and hence to the subalgebra $G_+$ via the exponential map. Therefore we can choose the following coordinate system $G_+(x, \theta)$ for the associated big cell $U$:

$$G_+(x, \theta) = (G_{n+r-1,n+r}) \cdots (G_{j,n+r} \cdots G_{j,j+1}) (G_{1,n+r} \cdots G_{1,2}).$$

(3.5)

Here, for $i < j$, $G_{i,j}$ is given by

$$G_{i,j} = \begin{cases} e^{x_{n+i,n+j}E_{\epsilon_i - \epsilon_j}}, & \text{if } 1 \leq i < j \leq r, \\ e^{\theta_{i,n+j}E_{\epsilon_i - \epsilon_j}}, & \text{if } 1 \leq i \leq n, 1 \leq j \leq r, \\ e^{x_{i,j}E_{\epsilon_i - \delta_j}}, & \text{if } 1 \leq i < j \leq n. \end{cases}$$

(3.6)

In the above equations, $\{x_{i,j}\}$ are bosonic coordinates while $\{\theta_{i,n+j}\}, 1 \leq i \leq n, 1 \leq j \leq r$ are fermionic ones. The coordinate system (3.5)-(3.6) enabled us to obtain the explicit differential operator realization of $sl(r|n)$ (or $gl(r|n)$). In the following, we shall show how a similar normal ordering of the positive roots allows us to construct "good coordinate systems" in the associated big cell $U$ of superalgebras $osp(2r|2n)$ and $osp(2r + 1|2n)$.

### 3.2 Differential operator realization of $osp(2r|2n)$

The root system $\Delta$ of $osp(2r|2n)$ (or $D(r, n)$) can be expressed in terms of the vectors $\{\delta_i\}$ and $\{\epsilon_i\}$ (3.1) as follows:

$$\Delta = \{ \pm \epsilon_i \pm \epsilon_j, \pm \delta_m \pm \delta_l, \pm 2\delta_l, \pm \delta_l \pm \epsilon_i \}, \quad 1 \leq i \neq j \leq r, 1 \leq m \neq l \leq n,$$

while the even roots $\Delta_0$ and the odd roots $\Delta_1$ are given by

$$\Delta_0 = \{ \pm \epsilon_i \pm \epsilon_j, \pm \delta_m \pm \delta_l, \pm 2\delta_l \}, \quad \Delta_1 = \{ \pm \delta_l \pm \epsilon_i \},$$

$$1 \leq i \neq j \leq r, 1 \leq m \neq l \leq n.$$  

The distinguished simple roots are

$$\alpha_1 = \delta_1 - \delta_2, \ldots, \alpha_{n-1} = \delta_{n-1} - \delta_n, \alpha_n = \delta_n - \epsilon_1,$$

$$\alpha_{n+1} = \epsilon_1 - \epsilon_2, \ldots, \alpha_{n+r-1} = \epsilon_{r-1} - \epsilon_r, \alpha_{n+r} = \epsilon_{r-1} + \epsilon_r.$$  

(3.7)

With regard to the simple roots, the corresponding positive roots $\Delta_+$ are

$$\delta_m - \delta_l, \quad 2\delta_l, \quad \delta_m + \delta_l, \quad 1 \leq m < l \leq n,$$

$$\delta_l - \epsilon_i, \quad \delta_l + \epsilon_i, \quad 1 \leq i \leq r, 1 \leq l \leq n,$$

$$\epsilon_i - \epsilon_j, \quad \epsilon_i + \epsilon_j, \quad 1 \leq i < j \leq r.$$  

(3.8)
Among these positive roots, \( \{ \delta_l \pm \epsilon_i | i, l = 1 \ldots n \} \) are odd and the others are even. Associated with each positive root \( \alpha \), there is a raising generator \( E_\alpha \), a lowering generator \( F_\alpha \) and a Cartan generator \( H_\alpha \), giving rise to the Cartan-Weyl decomposition (2.4) of \( osp(2r|2n) \):

\[
osp(2r|2n) = osp(2r|2n)_- \oplus H_{osp(2r|2n)} \oplus osp(2r|2n)_+.
\] (3.11)

In the defining representation of \( osp(2r|2n) \), the matrix realization of the generators associated with all roots is given in Appendix A.1, from which one may derive the structure constants \( f_{ij}^k \) in (2.2) of the algebra for this particular choice of the basis.

In order to obtain an explicit differential operator realization of \( osp(2r|2n) \), let us introduce the normal ordering of its positive roots.

**Proposition 2** A normal ordering of \( \Delta_+ \) for \( osp(2r|2n) \) is given by

\[
\epsilon_{r-1} + \epsilon_r, \epsilon_{r-1} - \epsilon_r; \ldots; \epsilon_1 + \epsilon_2, \ldots, \epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_r, \ldots, \epsilon_1 - \epsilon_2; \\
\delta_n + \epsilon_1, \ldots, \delta_n + \epsilon_r, 2\delta_n, \delta_n - \epsilon_r, \ldots, \delta_n - \epsilon_1; \ldots; \\
\delta_1 + \delta_2, \ldots, \delta_1 + \delta_n, \delta_1 + \epsilon_1, \ldots, \delta_1 + \epsilon_r, 2\delta_1, \\
\delta_1 - \epsilon_r, \ldots, \delta_1 - \epsilon_1, \delta_1 - \delta_n, \ldots, \delta_1 - \delta_2.
\] (3.12)

**Proof.** One can directly verify that the above ordering of the positive roots (3.8)-(3.10) of \( osp(2r|2n) \) obeys all requirements of Definition 1. \( \square \)

For the case \( r = 0 \), the ordering (3.12) gives rise to the normal ordering of the positive roots of \( sp(2n) \), while for the case \( n = 0 \) it yields the normal ordering of the positive roots of \( so(2r) \). Based on these orderings, a “good coordinate system” in each of the associated big cells for \( so(2n) \) and \( sp(2n) \) was constructed in [15]. Here we use the ordering (3.12) to construct the “good coordinate system” in the associated big cell \( \mathcal{U} \) and the explicit differential operator realization of \( osp(2r|2n) \).

Let us introduce a bosonic coordinate \( (x_{m,l}, \bar{x}_{m,l}, x_l, y_{i,j} \text{ or } \bar{y}_{i,j} \text{ for } m < l \text{ and } i < j) \) with a \( \mathbb{Z}_2 \)-grading zero: \( [x] = [\bar{x}] = [y] = [\bar{y}] = 0 \) associated with each positive even root (resp. \( \delta_m - \delta_l, \delta_m + \delta_l, 2\delta_l, \epsilon_i - \epsilon_j \text{ or } \epsilon_i + \epsilon_j \text{ for } m < l \text{ and } i < j \)), and a fermionic coordinate \( (\theta_{l,i} \text{ or } \bar{\theta}_{l,i}) \) with a \( \mathbb{Z}_2 \)-grading one: \( [\theta] = [\bar{\theta}] = 1 \) associated with each positive odd root (resp. \( \delta_l - \epsilon_i \text{ or } \delta_l + \epsilon_i \)). These coordinates satisfy the following (anti)commutation relations:

\[
[x_{i,j}, x_{m,l}] = 0, \ [\partial_{x_{i,j}}, \partial_{x_{m,l}}] = 0, \ [\partial_{x_{i,j}}, x_{m,l}] = \delta_{im}\delta_{jl}, \quad (3.13)
\]
the following coordinate system $G$ and the other (anti)commutation relations vanish.

Thus all generators of operators of the coordinates $x, \bar{x}, \theta, \bar{\theta}$ be the highest weight vector of the representation of $\mathfrak{osp}(2r|2n)$, we may introduce the following coordinate system $G_+(x, \bar{x}; y, \bar{y}; \theta, \bar{\theta})$ for the associated big cell $U$:

$$G_+(x, \bar{x}; y, \bar{y}; \theta, \bar{\theta}) = (G_{n+r-1,n+r} G_{n+r-1,n+r}) \cdots$$

$$\times (G_{n+1,n+2} \cdots G_{n+1,n+r} G_{n+1,n+r} \cdots G_{n+1,n+2})$$

$$\times (G_{n,n+1} \cdots G_{n,n+r} G_{n,n+r} \cdots G_{n,n+1}) \cdots$$

$$\times (\bar{G}_{1,2} \cdots \bar{G}_{1,n+r} G_1 G_1, n+r \cdots G_{1,2}).$$

Here $G_{i,j}, \bar{G}_{i,j}$ and $G_i$ are given by

$$G_{m,l} = e^{x_{m,l} E_{\delta_{m,l}}}, \quad G_{m,l} = e^{\bar{x}_{m,l} E_{\delta_{m,l}}}, \quad 1 \leq m < l \leq n,$$

$$G_l = e^{x_l E_{\delta_l}}, \quad G_l = e^{\bar{x}_l E_{\delta_l}}, \quad 1 \leq l \leq n, 1 \leq i \leq r,$$

$$G_{n+i,n+j} = e^{y_{i,j} E_{\epsilon_{i,j}}}, \quad G_{n+i,n+j} = e^{\bar{y}_{i,j} E_{\epsilon_{i,j}}}, \quad 1 \leq i < j \leq r.$$

Thus all generators of $\mathfrak{osp}(2r|2n)$ can be realized in terms of the first order differential operators of the coordinates $\{x, \bar{x}; y, \bar{y}; \theta, \bar{\theta}\}$ as follows.

Hereafter, let us adopt the convention that

$$E_i \equiv E_{\alpha_i}, \quad F_i \equiv F_{\alpha_i}, \quad i = 1, \ldots, n + r.$$

Let $\langle \Lambda \rangle$ be the highest weight vector of the representation of $\mathfrak{osp}(2r|2n)$ with highest weights $\{\lambda_i\}$, satisfying the following conditions:

$$\langle \Lambda | F_i = 0, \quad 1 \leq i \leq n + r,$$

$$\langle \Lambda | H_i = \lambda_i \langle \Lambda |, \quad 1 \leq i \leq n + r.$$
Here the generators \( H_i \) are expressed in terms of some linear combinations of \( H_\alpha \) \( (A.15)-(A.17) \). An arbitrary vector in the corresponding Verma module \( \mathcal{V} \) is parametrized by \( \langle \Lambda | \) and the corresponding bosonic and fermionic coordinates as

\[
\langle \Lambda; x, \bar{x}; y, \bar{y}; \theta, \bar{\theta} | = \langle \Lambda| G_+ (x, \bar{x}; y, \bar{y}; \theta, \bar{\theta}), \tag{3.27}
\]

where \( G_+ (x, \bar{x}; y, \bar{y}; \theta, \bar{\theta}) \) is given by \( (3.20)-(3.23) \).

One can define a differential operator realization \( \rho^{(d)} \) of the generators of \( osp(2r|2n) \) by

\[
\rho^{(d)}(g) \langle \Lambda; x, \bar{x}; y, \bar{y}; \theta, \bar{\theta} | \equiv \langle \Lambda; x, \bar{x}; y, \bar{y}; \theta, \bar{\theta} | g, \quad \forall g \in osp(2r|2n). \tag{3.28}
\]

Here \( \rho^{(d)}(g) \) is a differential operator of the coordinates \( \{x, \bar{x}; y, \bar{y}; \theta, \bar{\theta}\} \) associated with the generator \( g \), which can be obtained from the defining relation \( (3.28) \). The defining relation also assures that the differential operator realization is actually a representation of \( osp(2r|2n) \). Therefore it is sufficient to give the differential operators related to the simple roots, as the others can be constructed through the simple ones by the (anti)commutation relations. Using the relation \( (3.28) \) and the Baker-Campbel-Hausdorff formula, after some algebraic manipulations, we obtain the differential operator representation of the simple generators.

**Proposition 3** The differential operator representations of the generators associated with the simple roots of \( osp(2r|2n) \) are given by

\[
\rho^{(d)}(E_l) = \sum_{m=1}^{l-1} (x_{m,l} \partial x_{m,l+1} - \bar{x}_{m,l+1} \partial \bar{x}_{m,l}), \quad 1 \leq l \leq n-1, \tag{3.29}
\]

\[
\rho^{(d)}(E_n) = \sum_{m=1}^{n-1} (x_{m,n} \partial \theta_{\bar{m},n} + \bar{\theta}_{m,n} \partial x_{m,n}) + \partial \theta_{\bar{m},n}, \tag{3.30}
\]

\[
\rho^{(d)}(E_{n+i}) = \sum_{m=1}^{n} (\theta_{m,i} \partial \theta_{\bar{m},i+1} - \bar{\theta}_{m,i+1} \partial \bar{\theta}_{m,i})
+ \sum_{m=1}^{i-1} (y_{m,i} \partial y_{m,i+1} - \bar{\theta}_{m,i+1} \partial \bar{\theta}_{m,i}) + \partial y_{n,i+1}, \quad 1 \leq i \leq r-1, \tag{3.31}
\]

\[
\rho^{(d)}(E_{n+r}) = \sum_{m=1}^{n} (2\theta_{m,r-1} \partial x_{m} + \theta_{m,r-1} \partial \theta_{\bar{m},r} - \theta_{m,r} \partial \theta_{\bar{m},r-1})
\]

\[^1\text{The irreducible highest weight representation can be obtained from the Verma module through the cohomology procedure \cite{35} with the help of screening operators (e.g. \( (4.32)-(4.35) \) below).} \]
\[ \rho^{(d)}(F_l) = \sum_{m=1}^{l-1} (x_{m,l+1} \partial_{\bar{x}_{m,l}} - \bar{x}_{m,l} \partial_{x_{m,l+1}}) - x_l \partial_{\bar{x}_{l,l+1}} - 2\bar{x}_{l,l+1} \partial_{x_{l+1}} + \sum_{m=1}^{l+2} (x_{l,m} \bar{x}_{l,m} \partial_{\bar{x}_{l,l+1}} - \bar{x}_{l,m} \partial_{x_{l+1,m}} - 2\bar{x}_{l,m} \partial_{x_{l+1,m}} - \bar{x}_{l,m} \partial_{\bar{x}_{l+1,m}}) - \sum_{m=1}^{r} \left( \theta_{l,m} \bar{\theta}_{l,m} \partial_{\bar{x}_{l,l+1}} + \theta_{l,m} \partial_{\bar{\theta}_{l+1,m}} + 2\bar{\theta}_{l,m} \theta_{l+1,m} \partial_{x_{l+1}} + \bar{\theta}_{l,m} \partial_{\bar{\theta}_{l+1,m}} \right) - x_{l,l+1}^2 \partial_{x_{l+1}} + 2x_{l,l+1} \partial_{x_{l+1}} - 2x_{l,l+1}x_l \partial_{x_{l}} + x_{l,l+1} \left[ \sum_{m=1}^{l+2} (x_{l+1,m} \partial_{x_{l+1,m}} + \bar{x}_{l+1,m} \partial_{\bar{x}_{l+1,m}} - x_{l+1,m} \partial_{\bar{x}_{l+1,m}} - \bar{x}_{l+1,m} \partial_{x_{l+1,m}}) \right] + x_{l,l+1} \left[ \sum_{m=1}^{r} \left( \theta_{l+1,m} \partial_{\bar{\theta}_{l+1,m}} + \bar{\theta}_{l+1,m} \partial_{\bar{x}_{l+1,m}} - \theta_{l+1,m} \partial_{\bar{x}_{l+1,m}} - \bar{\theta}_{l+1,m} \partial_{\bar{\theta}_{l+1,m}} \right) \right] + x_{l,l+1}(\lambda_l - \lambda_{l+1}), \quad 1 \leq l \leq n - 1, \quad (3.33) \]

\[ \rho^{(d)}(F_n) = \sum_{m=1}^{n-1} \left( \theta_{m,1} \partial_{x_{m,n}} - \bar{x}_{m,n} \partial_{\bar{g}_{m,1}} \right) - x_n \partial_{\bar{g}_{n,1}} + \sum_{m=2}^{r} \left( \theta_{n,m} \partial_{y_{1,m}} - \theta_{n,m} \partial_{\bar{y}_{n,1}} + \bar{\theta}_{n,m} \partial_{y_{1,m}} \right) - \theta_{n,1} \sum_{m=2}^{r} \left( \theta_{n,m} \partial_{\bar{g}_{n,m}} + \bar{\theta}_{n,m} \partial_{\bar{g}_{n,m}} + y_{1,m} \partial_{y_{1,m}} + \bar{y}_{1,m} \partial_{\bar{y}_{1,m}} \right) - 2\theta_{n,1} x_n \partial_{x_{n}} - 2\theta_{n,1} \bar{\theta}_{n,1} \theta_{n,1} + \theta_{n,1}(\lambda_n + \lambda_{n+1}), \quad (3.34) \]

\[ \rho^{(d)}(F_{n+i}) = \sum_{m=1}^{n} \left( \theta_{m,i+1} \partial_{\bar{g}_{m,i}} - \bar{\theta}_{m,i} \partial_{\bar{g}_{m,i+1}} \right) + \sum_{m=1}^{i-1} \left( y_{m,i+1} \partial_{\bar{y}_{m,i}} - \bar{y}_{m,i} \partial_{\bar{y}_{m,i+1}} \right) + \sum_{m=i+2}^{r} \left( y_{i,m} \bar{y}_{i,m} \partial_{\bar{y}_{i+1,m}} - y_{i,m} \partial_{\bar{y}_{i+1,m}} - \bar{y}_{i,m} \partial_{\bar{y}_{i+1,m}} \right) + y_{i,i+1} \sum_{m=i+2}^{r} \left( y_{i+1,m} \partial_{\bar{y}_{i+1,m}} + \bar{y}_{i+1,m} \partial_{\bar{y}_{i+1,m}} - y_{i,m} \partial_{\bar{y}_{i+1,m}} - \bar{y}_{i,m} \partial_{\bar{y}_{i+1,m}} \right) - y_{i,i+1}^2 \partial_{y_{i,i+1}} + y_{i,i+1}(\lambda_{n+i} - \lambda_{n+i+1}), \quad 1 \leq i \leq r - 1, \quad (3.35) \]

\[ \rho^{(d)}(F_{n+r}) = \sum_{m=1}^{n} \left( \theta_{m,r} \partial_{\bar{g}_{m,r-1}} + 2\bar{\theta}_{m,r-1} \theta_{m,r} \partial_{x_{m}} - \bar{\theta}_{m,r-1} \partial_{\bar{g}_{m,r}} \right) + \sum_{m=1}^{r-2} \left( y_{m,r} \partial_{\bar{y}_{m,r-1}} - \bar{y}_{m,r} \partial_{\bar{y}_{m,r-1}} \right) - \bar{y}_{r-1,r}^2 \partial_{g_{r-1,r}} \quad (3.36) \]
The root system $\Delta$ of $osp_{3.3|2n}$ differential realization of $osp_{differential realization of}$ Serre relations. This implies that the differential representation of non-simple generators $osp$

With regard to the simple roots, the corresponding positive roots $\Delta = \{\pm \delta_i, \pm \epsilon_i, \pm \delta_m \pm \delta_l, \pm \delta_l \pm \epsilon_i\}$, $1 \leq i \neq j \leq r, 1 \leq m \neq l \leq n$, while the even roots $\Delta_0$ and the odd roots $\Delta_1$ are given respectively by

$$\Delta_0 = \{\pm \delta_i, \pm \epsilon_i, \pm \delta_m \pm \delta_l, \pm 2\delta_l\}, \quad \Delta_1 = \{\pm \delta_l \pm \epsilon_i, \pm \delta_l\},$$

$1 \leq i \neq j \leq r, 1 \leq m \neq l \leq n.$

The distinguished simple roots are

$$\alpha_1 = \delta_1 - \delta_2, \ldots, \alpha_{n-1} = \delta_{n-1} - \delta_n, \alpha_n = \delta_n - \epsilon_1,$$

$$\alpha_{n+1} = \epsilon_1 - \epsilon_2, \ldots, \alpha_{n+r-1} = \epsilon_{r-1} - \epsilon_r, \alpha_{n+r} = \epsilon_r.$$  \hfill (3.39)

With regard to the simple roots, the corresponding positive roots $\Delta_+$ are

$$\delta_m - \delta_l, \quad 2\delta_l, \quad \delta_m + \delta_l, \quad 1 \leq m < l \leq n,$$

$$\delta_l - \epsilon_i, \quad \delta_l + \epsilon_i, \quad \delta_l, \quad 1 \leq i \leq r, 1 \leq l \leq n,$$

$$\epsilon_i - \epsilon_j, \quad \epsilon_i + \epsilon_j, \quad \epsilon_i, \quad 1 \leq i < j \leq r.$$  \hfill (3.42)
Among these positive roots, \( \{\delta_l, \delta_l \pm \epsilon_i \mid i = 1, \ldots, r, l = 1 \ldots, n\} \) are odd and the others are even. Associated with each positive root \( \alpha \), there is a raising generator \( E_\alpha \), a lowering generator \( F_\alpha \) and a Cartan generator \( H_\alpha \), giving rise to the Cartan-Weyl decomposition (2.4) of \( osp(2r + 1|2n) \):

\[
osp(2r + 1|2n) = osp(2r + 1|2n)_- \oplus H_{osp(2r+1|2n)} \oplus osp(2r + 1|2n)_+.
\] (3.43)

In the defining representation of \( osp(2r + 1|2n) \), the matrix realization of the generators associated with all roots is given in Appendix A.2, from which one may derive the structure constants \( f_{ij}^l \) in (2.2) of the algebra for this particular choice of the basis.

To obtain an explicit expression of the differential operator realization of \( osp(2r + 1|2n) \), let us introduce the normal ordering of its positive roots.

**Proposition 4** A normal ordering of \( \Delta_+ \) for \( osp(2r + 1|2n) \) is given by

\[
\begin{align*}
\epsilon_r; & \; \epsilon_{r-1} + \epsilon_r, \; \epsilon_{r-1}, \; \epsilon_r - \epsilon_r; \ldots; \; \epsilon_1 + \epsilon_2, \ldots, \; \epsilon_1 + \epsilon_r, \; \epsilon_1 - \epsilon_r, \ldots, \; \epsilon_1 - \epsilon_2; \\
\delta_n + \epsilon_1, \ldots, & \; \delta_n + \epsilon_r, \; 2\delta_n, \; \delta_n - \epsilon_r, \ldots, \; \delta_n - \epsilon_1; \ldots; \\
\delta_1 + & \; \delta_2, \ldots, \; \delta_1 + \delta_n, \; \delta_1 + \epsilon_1, \ldots, \; \delta_1 + \epsilon_r, \; 2\delta_1, \; \delta_1, \\
\delta_1 - & \; \epsilon_r, \ldots, \; \delta_1 - \epsilon_1, \; \delta_1 - \delta_n, \ldots, \; \delta_1 - \delta_2.
\end{align*}
\] (3.44)

**Proof.** One can directly verify that the above ordering of the positive roots (3.40)-(3.42) of \( osp(2r + 1|2n) \) satisfies all requirements of Definition 1. □

For the case \( n = 0 \), the ordering (3.44) gives rise to the normal ordering of the positive roots of \( so(2r + 1) \). Based on this ordering a “good coordinate system” in the associated big cell of \( so(2r + 1) \) was constructed in [45]. Here we use the ordering (3.44) to construct a “good coordinate system” in the associated big cell \( U \) and the explicit differential operator realization of \( osp(2r + 1|2n) \).

In addition to the coordinates \( \{x_{m,l}, \bar{x}_{m,l}; x_m; y_{i,j}, \bar{y}_{i,j}; \theta_{l,i}, \bar{\theta}_{l,i}\} \), which are associated with the positive roots \( \{\delta_m - \delta_l, \delta_m + \delta_l; 2\delta_l; \epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j; \delta_l - \epsilon_i, \delta_l + \epsilon_i\} \), we also need to introduce \( n + r \) extra coordinates \( \{\theta_l \mid l = 1, \ldots, n\} \) and \( \{\epsilon_i \mid i = 1, \ldots, r\} \) associated with the positive roots \( \{\delta_l \mid l = 1, \ldots, n\} \) and \( \{\epsilon_i \mid i = 1, \ldots, r\} \) respectively. The coordinates \( \{x_{m,l}, \bar{x}_{m,l}; x_m; y_{i,j}, \bar{y}_{i,j}; \theta_{l,i}, \bar{\theta}_{l,i}\} \) and their differentials satisfy the same (anti)commutation relations as (3.13)-(3.19). The other non-trivial relations are

\[
[y_i, y_j] = [\partial_{y_i}, \partial_{y_j}] = 0, \quad [\partial_{y_i}, y_j] = \delta_{ij}, \quad i, j = 1, \ldots, r.
\] (3.45)
\[ [\theta_m, \theta_l] = [\partial_{\theta_m}, \partial_{\theta_l}] = 0, \quad [\partial_{\theta_m}, \theta_l] = \delta_{ml}, \quad m, l = 1, \ldots, n. \]  

(3.46)

Based on the very ordering (3.44) of the positive roots of \( osp(2r + 1|2n) \), we introduce the following coordinate system \( G_+(x, \bar{x}; y, \bar{y}; \theta, \bar{\theta}) \) for the associated big cell \( U \):

\[
G_+(x, \bar{x}; y, \bar{y}; \theta, \bar{\theta}) = (G_{n+r})(G_{n+r-1,n+r}G_{n+r-1}G_{n+r-1,n+r+1}) \ldots \times (G_{n+1,n+2} \ldots G_{n+1,n+r}G_{n+1,n+r+1} \ldots G_{n+1,n+2}) \times (G_{n,n+1} \ldots G_{n,n+r}G_nG_{n,n+1} \ldots G_{n,n+1}) \ldots \times (G_{1,2} \ldots G_{1,n+r}G_1G_{1,n+r} \ldots G_{1,2}).
\]

(3.47)

Here \( G_{i,j}, \bar{G}_{i,j}, G_i \) and \( \bar{G}_i \) are given by

\[
G_{m,l} = e^{x_{m,l}E_{h_{m,l}}}, \quad \bar{G}_{m,l} = e^{\bar{x}_{m,l}E_{\bar{h}_{m,l}}}, \quad 1 \leq m < l \leq n,
\]

(3.48)

\[
\bar{G}_l = e^{\bar{x}_lE_{\bar{h}_l}}, \quad G_l = e^{x_lE_{h_l}}, \quad 1 \leq l \leq n, \quad 1 \leq i \leq r,
\]

(3.49)

\[
G_{l,n+i} = e^{x_lE_{h_{n+i}}}, \quad \bar{G}_{l,n+i} = e^{\bar{x}_lE_{\bar{h}_{n+i}}}, \quad 1 \leq l \leq n, \quad 1 \leq i \leq r,
\]

(3.50)

\[
G_{n+i,n+j} = e^{x_{n+i}E_{h_{n+j}}}, \quad \bar{G}_{n+i,n+j} = e^{\bar{x}_{n+i}E_{\bar{h}_{n+j}}}, \quad 1 \leq i < j \leq r.
\]

(3.51)

Then the first order differential operator realization of the generators of \( osp(2r + 1|2n) \) can be obtained as follows.

Similarly to the \( osp(2r|2n) \) case, we adopt the convention (3.24) for the raising/lowering generators associated with the simple roots. Let \( \langle \Lambda \rangle \) be the highest weight vector of the representation of \( osp(2r+1|2n) \) with highest weights \( \{ \lambda_i \} \), satisfying the following conditions:

\[
\langle \Lambda \rangle | F_i = 0, \quad 1 \leq i \leq n + r,
\]

(3.52)

\[
\langle \Lambda \rangle | H_i = \lambda_i \langle \Lambda \rangle, \quad 1 \leq i \leq n + r.
\]

(3.53)

Here the generators \( H_i \) are expressed in terms of some linear combinations of \( H_a \) (A.37)-(A.39). An arbitrary vector in the corresponding Verma module is parametrized by \( \langle \Lambda \rangle \) and the corresponding bosonic and fermionic coordinates as

\[
\langle \Lambda; x, \bar{x}; y, \bar{y}; \theta, \bar{\theta} \rangle = \langle \Lambda | G_+(x, \bar{x}; y, \bar{y}; \theta, \bar{\theta}),
\]

(3.54)

where \( G_+(x, \bar{x}; y, \bar{y}; \theta, \bar{\theta}) \) is given by (3.47)-(3.51).

One can define a differential operator realization \( \rho^{(d)} \) of the generators of \( osp(2r + 1|2n) \) by

\[
\rho^{(d)}(g) \langle \Lambda; x, \bar{x}; y, \bar{y}; \theta, \bar{\theta} \rangle = \langle \Lambda; x, \bar{x}; y, \bar{y}; \theta, \bar{\theta} \rangle g, \quad \forall g \in osp(2r + 1|2n).
\]

(3.55)
Here $\rho^{(d)}(g)$ is a differential operator of the coordinates $\{x, \bar{x}; y, \bar{y}; \theta, \bar{\theta}\}$ associated with the generator $g$, which can be obtained from the defining relation (3.55). The defining relation also assures that the differential operator realization is actually a representation of $osp(2r+1|2n)$. Therefore it is sufficient to give the differential operators related to the simple roots, as the others can be constructed through the simple ones by the (anti)commutation relations.

Using the relation (3.55) and the Baker-Campbell-Hausdorff formula, after some algebraic manipulations, we obtain the following differential operator representation of the simple generators.

**Proposition 5** The differential operator representation of the generators associated with the simple roots of $osp(2r+1|2n)$ are given by

$$\rho^{(d)}(E_l) = \sum_{m=1}^{l-1} \left( x_{m,i} \partial x_{m,i+1} - \bar{x}_{m,i+1} \partial x_{m,i} \right) + \partial x_{l,i+1}, \quad 1 \leq l \leq n - 1, \quad (3.56)$$

$$\rho^{(d)}(E_n) = \sum_{m=1}^{n-1} \left( x_{m,n} \partial \theta_{m,1} + \bar{\theta}_{m,1} \partial x_{m,n} \right) + \partial \theta_{n,1}, \quad (3.57)$$

$$\rho^{(d)}(E_{n+i}) = \sum_{m=1}^{n-i} \left( \theta_{m,i} \partial \theta_{m,i+1} - \bar{\theta}_{m,i+1} \partial \theta_{m,i} \right) \right)$$

$$\rho^{(d)}(E_{n+r}) = \sum_{m=1}^{n} \left( \theta_{m,r} \partial \theta_{m,r} - \theta_{m,r} \partial \theta_{m} - \theta_{m,r} \theta_m \partial x_m \right)$$

$$+ \sum_{m=1}^{r-1} \left( \theta_{m,r} \partial \theta_{m,r} - \theta_{m,r} \partial \theta_{m,r} \right) + \partial y_r, \quad (3.58)$$

$$\rho^{(d)}(F_l) = \sum_{m=1}^{l-1} \left( x_{m,l+1} \partial x_{m,l+1} - \bar{x}_{m,l+1} \partial x_{m,l+1} \right) - x_l \partial x_{l,l+1} - 2 \bar{x}_{l,l+1} \partial x_{l,l+1}$$

$$+ \sum_{m=l+2}^{n} \left( x_{l,m} \bar{x}_{l,m} \partial x_{l+m} - x_{l,l} \partial x_{l+m} - 2 \bar{x}_{l,m} x_{l+1,m} \partial x_{l+1} - \bar{x}_{l,m} \partial x_{l+1,m} \right)$$

$$- \sum_{m=1}^{l} \left( \theta_{l,m} \bar{\theta}_{l,m} \partial x_{l,m} + \theta_{l,m} \partial \theta_{l+1,m} + 2 \bar{\theta}_{l,m} \theta_{l+1,m} \partial x_{l+1} + \bar{\theta}_{l,m} \partial \theta_{l+1,m} \right)$$

$$- \theta_l \partial \theta_{l+1} - \theta_l \partial \theta_{l+1} \partial x_{l+1} + x_{l+1,l+1} \partial \theta_{l+1} - x_{l+1,l+1} \partial \theta_l$$

$$- \bar{x}_{l,l+1} \partial x_{l+1} + 2 x_{l+1,l+1} \partial x_{l+1} - 2 x_{l+1,l} \partial x_{l+1}$$

$$\rho^{(d)}(F_{l+1}) = \sum_{m=1}^{l} \left( x_{m,l+1} \partial x_{m,l+1} - \bar{x}_{m,l+1} \partial x_{m,l+1} \right) - x_l \partial x_{l,l+1} - 2 \bar{x}_{l,l+1} \partial x_{l,l+1}$$

$$+ \sum_{m=l+2}^{n} \left( x_{l,m} \bar{x}_{l,m} \partial x_{l+m} - x_{l,l} \partial x_{l+m} - 2 \bar{x}_{l,m} x_{l+1,m} \partial x_{l+1} - \bar{x}_{l,m} \partial x_{l+1,m} \right)$$

$$- \sum_{m=1}^{l} \left( \theta_{l,m} \bar{\theta}_{l,m} \partial x_{l,m} + \theta_{l,m} \partial \theta_{l+1,m} + 2 \bar{\theta}_{l,m} \theta_{l+1,m} \partial x_{l+1} + \bar{\theta}_{l,m} \partial \theta_{l+1,m} \right)$$

$$- \theta_l \partial \theta_{l+1} - \theta_l \partial \theta_{l+1} \partial x_{l+1} + x_{l+1,l+1} \partial \theta_{l+1} - x_{l+1,l+1} \partial \theta_l$$

$$- \bar{x}_{l,l+1} \partial x_{l+1} + 2 x_{l+1,l+1} \partial x_{l+1} - 2 x_{l+1,l} \partial x_{l+1}$$

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\[
\rho^{(d)}(F_n) = \sum_{m=1}^{n-1} (\theta_{m,1} \partial x_{m,n} - \bar{x}_{m,n} \partial \theta_{m,1}) - x_n \partial \theta_{n,1}
\]
\[
+ \sum_{m=2}^{r} \left( \theta_{n,m} \partial y_{1,m} - \theta_{n,m} \bar{\theta}_{n,m} \partial \theta_{n,1} + \bar{\theta}_{n,m} \partial y_{1,m} \right) - \theta_n \partial y_1
\]
\[-\theta_{n,1} \sum_{m=2}^{r} \left( \theta_{n,m} \partial \theta_{n,m} + \bar{\theta}_{n,m} \partial \theta_{n,m} + y_{1,m} \partial y_{1,m} + \bar{y}_{1,m} \partial y_{1,m} \right)
\]
\[-2 \theta_{n,1} x_n \partial x_n - 2 \theta_{n,1} \bar{\theta}_{n,1} \partial \theta_{n,1} - \theta_{n,1} \theta_n \partial \theta_n - \theta_{n,1} y_{1,1} \partial y_1\]
\[+ \theta_{n,1} (\lambda_n + \lambda_{n+1}), \quad (3.60)\]

\[
\rho^{(d)}(F_{n+i}) = \sum_{m=1}^{n} \left( \theta_{m,i+1} \partial \theta_{m,i} - \bar{\theta}_{m,i} \partial \theta_{m,i+1} \right) + \sum_{m=1}^{i-1} \left( y_{m,i+1} \partial y_{m,i} - \bar{y}_{m,i} \partial y_{m,i+1} \right)
\]
\[+ \sum_{m=i+2}^{r} \left( y_{i,m} \partial y_{i,m} - y_{i,m} \partial y_{i,m+1} - \bar{y}_{i,m} \partial y_{i,m+1} \right)
\]
\[-y_{i,i+1} \partial y_{i,i+1} + \frac{y_{i,i+1}^2}{2} \partial y_{i,i+1} + y_{i,i+1} y_{i,i+1} \partial y_{i,i+1} - y_{i,i+1} y_{i,i+1} \partial y_{i}
\]
\[+ y_{i,i+1} \sum_{m=i+2}^{r} \left( y_{i+1,m} \partial y_{i+1,m} + \bar{y}_{i+1,m} \partial y_{i+1,m} - y_{i+1,m} \partial y_{i+1,m} - \bar{y}_{i+1,m} \partial y_{i+1,m} \right)
\]
\[-y_{i+1,i+1} \partial y_{i+1,i+1} + y_{i+1,i+1} (\lambda_{n+i} - \lambda_{n+i+1}), \quad 1 \leq i \leq r - 1, \quad (3.62)\]

\[
\rho^{(d)}(F_{n+r}) = \sum_{m=1}^{n} \left( \bar{\theta}_{m,r} \partial \theta_{m} - \bar{\theta}_{m,r} \theta_{m} \partial x_{m} - \theta_{m} \partial \theta_{m,r} \right)
\]
\[+ \sum_{m=1}^{r-1} \left( y_{m,r} \partial y_{m,r} - \bar{y}_{m,r} \partial y_{m,r} \right) - \frac{y_r^2}{2} \partial y_r + y_r \lambda_{n+r}, \quad (3.63)\]

\[
\rho^{(d)}(H_l) = \sum_{m=1}^{l-1} \left( x_{m,l} \partial x_{m,l} - \bar{x}_{m,l} \partial x_{m,l} \right) - \sum_{m=l+1}^{n} \left( x_{l,m} \partial x_{l,m} + \bar{x}_{l,m} \partial x_{l,m} \right)
\]
\[- \sum_{m=1}^{r} \left( \theta_{l,m} \partial \theta_{l,m} + \bar{\theta}_{l,m} \partial \theta_{l,m} \right) - 2 x_l \partial x_l - \theta_l \partial \theta_l + \lambda_l, \quad 1 \leq l \leq n, \quad (3.64)\]

\[
\rho^{(d)}(H_{n+i}) = \sum_{m=1}^{n} \left( \theta_{m,i} \partial \theta_{m,i} - \bar{\theta}_{m,i} \partial \theta_{m,i} \right) + \sum_{m=1}^{i-1} \left( y_{m,i} \partial y_{m,i} - \bar{y}_{m,i} \partial y_{m,i} \right)
\]
A direct computation shows that these differential operators (3.56)-(3.65) satisfy the $osp(2r + 1|2n)$ (anti)commutation relations corresponding to the simple roots and the associated Serre relations. This implies that the differential representation of non-simple generators can be consistently constructed from the simple ones. Hence, we have obtained an explicit differential realization of $osp(2r + 1|2n)$.

4 Free field realization of current superalgebras

4.1 Current superalgebra $osp(2r|2n)_k$

4.1.1 Free field realization of the currents

With the help of the explicit differential operator expressions of $osp(2r|2n)$ given by (3.29)-(3.38) we can construct the explicit free field representation of the $osp(2r|2n)$ current algebra at arbitrary level $k$ in terms of $n^2 + r^2 - r$ bosonic $\beta$-$\gamma$ pairs $\{(\beta_{i,j}, \gamma_{i,j}), (\bar{\beta}_{i,j}, \bar{\gamma}_{i,j}), (\beta_i, \gamma_i)$, $(\beta'_{i,j}, \gamma'_{i,j}), (\bar{\beta}'_{i,j}, \bar{\gamma}'_{i,j}), 1 \leq i < j \leq n, 1 \leq i' < j' \leq r\}$, $2nr$ fermionic $b$-$c$ pairs $\{((\Psi^+_{i,j}, \Psi_{i,j}), (\bar{\Psi}^+_{i,j}, \bar{\Psi}_{i,j}), 1 \leq i \leq n, 1 \leq j \leq r\}$ and $n + r$ free scalar fields $\phi_i, i = 1, \ldots, n + r$. These free fields obey the following OPEs:

$$\beta_{i,j}(z) \gamma_{m,l}(w) = -\gamma_{m,l}(z) \beta_{i,j}(w) = \frac{\delta_{im}\delta_{jl}}{(z-w)}, \quad 1 \leq i < j \leq n, 1 \leq m < l \leq n, \quad (4.1)$$

$$\bar{\beta}_{i,j}(z) \bar{\gamma}_{m,l}(w) = -\bar{\gamma}_{m,l}(z) \bar{\beta}_{i,j}(w) = \frac{\delta_{im}\delta_{jl}}{(z-w)}, \quad 1 \leq i < j \leq n, 1 \leq m < l \leq n, \quad (4.2)$$

$$\beta_m(z) \gamma_l(w) = -\gamma_m(z) \beta_l(w) = \frac{\delta_{ml}}{(z-w)}, \quad 1 \leq m, l \leq n, \quad (4.3)$$

$$\beta'_{i,j}(z) \gamma'_{m,l}(w) = -\gamma'_{m,l}(z) \beta'_{i,j}(w) = \frac{\delta_{im}\delta_{jl}}{(z-w)}, \quad 1 \leq i < j \leq r, 1 \leq m < l \leq r, \quad (4.4)$$

$$\bar{\beta}'_{i,j}(z) \bar{\gamma}'_{m,l}(w) = -\bar{\gamma}'_{m,l}(z) \bar{\beta}'_{i,j}(w) = \frac{\delta_{im}\delta_{jl}}{(z-w)}, \quad 1 \leq i < j \leq r, 1 \leq m < l \leq r, \quad (4.5)$$

$$\Psi^+_{m,i}(z) \Psi_{i,j}(w) = \Psi_{i,j}(z) \Psi^+_{m,i}(w) = \frac{\delta_{ml}\delta_{ij}}{(z-w)}, \quad 1 \leq m, l \leq n, 1 \leq i, j \leq r, \quad (4.6)$$

$$\Psi^+_{m,i}(z) \Psi_{i,j}(w) = \Psi_{i,j}(z) \Psi^+_{m,i}(w) = \frac{\delta_{ml}\delta_{ij}}{(z-w)}, \quad 1 \leq m, l \leq n, 1 \leq i, j \leq r, \quad (4.7)$$

$$\phi_m(z) \phi_l(w) = -\delta_{ml} \ln(z-w), \quad 1 \leq m, l \leq n, \quad (4.8)$$

$$\phi_{n+i}(z) \phi_{n+j}(w) = \delta_{ij} \ln(z-w), \quad 1 \leq i, j \leq r, \quad (4.9)$$
and the other OPEs are trivial.

The free field realization of the $osp(2r|2n)$ current algebra is obtained by the following substitutions in the differential operator realization (3.29)-(3.38) of $osp(2r|2n)$:

\[
x_{m,l} \longrightarrow \gamma_{m,l}(z), \quad \partial_{x_{m,l}} \longrightarrow \beta_{m,l}(z), \quad 1 \leq m < l \leq n, \quad (4.10)
\]

\[
x_{l} \longrightarrow \gamma_{l}(z), \quad \partial_{x_{l}} \longrightarrow \beta_{l}(z), \quad 1 \leq l \leq n, \quad (4.12)
\]

\[
y_{i,j} \longrightarrow \gamma'_{i,j}(z), \quad \partial_{y_{i,j}} \longrightarrow \beta'_{i,j}(z), \quad 1 \leq i < j \leq r, \quad (4.13)
\]

Moreover, in order that the resulting free field realization satisfies the desirable OPEs for $osp(2r|2n)$ currents, one needs to add certain extra (anomalous) terms which are linear in $\partial \gamma(z), \partial \bar{\gamma}(z), \partial \gamma'(z), \partial \bar{\gamma}'(z)$, $\partial \Psi(z)$ and $\partial \bar{\Psi}(z)$ in the expressions of the currents associated with negative roots (e.g. the last term in the expressions of $F_{i}(z)$, see (4.22)-(4.25) below).

Here we present the results for the currents associated with the simple roots.

**Theorem 1** The currents associated with the simple roots of the $osp(2r|2n)$ current algebra at a generic level $k$ are given in terms of the free fields (4.1)-(4.9) as

\[
E_{l}(z) = \sum_{m=1}^{l-1} \left( \gamma_{m,l}(z) \beta_{m,l+1}(z) - \bar{\gamma}_{m,l+1}(z) \bar{\beta}_{m,l}(z) \right) + \beta_{l,l+1}(z), \quad 1 \leq l \leq n - 1, \quad (4.18)
\]

\[
E_{n}(z) = \sum_{m=1}^{n-1} \left( \gamma_{m,n}(z) \Psi_{m,1}(z) + \bar{\Psi}_{m,1}(z) \bar{\Psi}_{m,n}(z) \right) + \Psi_{n,1}(z), \quad (4.19)
\]

\[
E_{n+i}(z) = \sum_{m=1}^{n-1} \left( \Psi_{m,i}(z) \Psi_{m,i+1}(z) - \bar{\Psi}_{m,i+1}(z) \bar{\Psi}_{m,i}(z) \right)
+ \sum_{m=1}^{i-1} \left( \gamma'_{m,i}(z) \beta'_{m,i+1}(z) - \bar{\gamma}'_{m,i+1}(z) \bar{\beta}'_{m,i}(z) \right) + \beta'_{i,i+1}(z), \quad 1 \leq i \leq r - 1, \quad (4.20)
\]

\[
E_{n+r}(z) = \sum_{m=1}^{n} \left( 2 \Psi_{m,r-1}(z) \Psi_{m,r}(z) \beta_{m}(z) + \bar{\Psi}_{m,r}(z) \bar{\Psi}_{m,r-1}(z) \right)
+ \sum_{m=1}^{r-2} \left( \gamma'_{m,r-1}(z) \beta'_{m,r}(z) - \gamma'_{m,r}(z) \beta'_{m,r-1}(z) \right) + \beta'_{r-1,r}(z), \quad (4.21)
\]
\[ F_l(z) = \sum_{m=1}^{l-1} (\gamma_{m,l+1}(z)\beta_{m,l}(z) - \tilde{\gamma}_{m,l}(z)\tilde{\beta}_{m,l+1}(z)) - \gamma_l(z)\tilde{\beta}_{l,l+1}(z) \]
\[-2\gamma_l(z)\tilde{\beta}_{l+1}(z) + \sum_{m=l+2}^{n} (\gamma_{l,m}(z)\tilde{\gamma}_{l,m}(z)\tilde{\beta}_{l,l+1}(z) - \gamma_{l,m}(z)\beta_{l+1,m}(z)) \]
\[-\sum_{m=l+2}^{n} (2\gamma_{l,m}(z)\gamma_{l+1,m}(z)\beta_{l+1}(z) + \gamma_{l,m}(z)\tilde{\beta}_{l+1,m}(z)) \]
\[-\sum_{m=1}^{r} (\Psi_{l,m}^{+}(z)\tilde{\Psi}_{l,m}^{+}(z)\tilde{\beta}_{l+1}(z) + \Psi_{l,m}(z)\Psi_{l+1,m}(z)) \]
\[-\sum_{m=1}^{r} (2\tilde{\Psi}_{l,m}(z)\Psi_{l+1,m}(z)\beta_{l+1}(z) + \tilde{\Psi}_{l,m}(z)\tilde{\Psi}_{l+1,m}(z)) \]
\[-\gamma_{l,l+1}(z)\beta_{l,l+1}(z) - \gamma_{l,l+1}(z) \sum_{m=l+2}^{n} (\gamma_{l,m}(z)\beta_{l,m}(z) + \tilde{\gamma}_{l,m}(z)\tilde{\beta}_{l,m}(z)) \]
\[+\gamma_{l,l+1}(z) \sum_{m=l+2}^{n} (\gamma_{l+1,m}(z)\beta_{l+1,m}(z) + \tilde{\gamma}_{l+1,m}(z)\tilde{\beta}_{l+1,m}(z)) \]
\[-\gamma_{l,l+1}(z) \sum_{m=1}^{r} (\Psi_{l,m}^{+}(z)\Psi_{l,m}(z) + \tilde{\Psi}_{l,m}^{+}(z)\tilde{\Psi}_{l,m}(z)) \]
\[+\gamma_{l,l+1}(z) \sum_{m=1}^{r} (\Psi_{l+1,m}^{+}(z)\Psi_{l+1,m}(z) + \tilde{\Psi}_{l+1,m}^{+}(z)\tilde{\Psi}_{l+1,m}(z)) \]
\[+2\gamma_{l,l+1}(z)\gamma_{l+1}(z)\beta_{l+1}(z) - 2\gamma_{l,l+1}(z)\gamma_{l}(z)\beta_{l}(z) \]
\[+\sqrt{k + 2(r - n - 1)\gamma_{l,l+1}(z) (\partial\phi_{l}(z) - \partial\phi_{l+1}(z)) \]
\[+(-k + 2(l - 1))\partial\gamma_{l,l+1}(z), \quad 1 \leq l \leq n - 1, \quad (4.22) \]
\[ F_n(z) = \sum_{m=1}^{n-1} (\Psi_{m,1}^{+}(z)\beta_{m,n}(z) - \tilde{\gamma}_{m,n}(z)\tilde{\Psi}_{m,1}(z)) - \gamma_n(z)\tilde{\Psi}_{n,1}(z) \]
\[+\sum_{m=2}^{r} (\Psi_{n,m}^{+}(z)\beta_{1,m}(z) - \Psi_{n,m}(z)\tilde{\Psi}_{n,m}(z)\tilde{\Psi}_{n,1}(z) + \Psi_{n,m}(z)\tilde{\beta}_{1,m}(z)) \]
\[-\Psi_{n,1}^{+}(z) \sum_{m=2}^{r} (\Psi_{n,m}^{+}(z)\Psi_{n,m}(z) + \tilde{\Psi}_{n,m}(z)\tilde{\Psi}_{n,m}(z)) - 2\Psi_{n,1}^{+}(z)\tilde{\Psi}_{n,1}(z)\tilde{\Psi}_{n,1}(z) \]
\[-\Psi_{n,1}^{+}(z) \sum_{m=2}^{r} (\gamma_{1,m}(z)\beta_{1,m}(z) + \gamma_{1,m}(z)\tilde{\beta}_{1,m}(z)) - 2\Psi_{n,1}^{+}(z)\gamma_n(z)\beta_n(z) \]
\[+\sqrt{k + 2(r - n - 1)\Psi_{n,1}^{+}(z) (\partial\phi_{n}(z) + \partial\phi_{n+1}(z)) \]
\[+(-k + 2(n - 1))\partial\Psi_{n,1}^{+}(z), \quad (4.23) \]
\[ F_{n+i}(z) = \sum_{m=1}^{n} (\Psi_{m,i+1}^{+}(z)\Psi_{m,i}(z) - \bar{\Psi}_{m,i}^{+}(z)\bar{\Psi}_{m,i+1}(z)) \]
\[ + \sum_{m=1}^{i-1} (\gamma_{m,i+1}(z)\beta_{m,i}(z) - \bar{\gamma}_{m,i}(z)\bar{\beta}_{m,i+1}(z)) \]
\[ + \sum_{m=i+2}^{r} (\gamma_{i,m}(z)\beta_{i,i+1}(z) - \gamma_{i,m}(z)\beta_{i+1,m}(z)) \]
\[ + \gamma_{i,i+1}(z) \sum_{m=i+2}^{r} (\gamma_{i+1,m}(z)\beta_{i+1,m}(z) + \gamma_{i+1,m}(z)\bar{\beta}_{i+1,m}(z)) \]
\[ - \gamma_{i,i+1}(z) \sum_{m=i+2}^{r} (\gamma_{i,m}(z)\beta_{i,m}(z) + \gamma_{i,m}(z)\beta_{i,m}(z)) \]
\[ \sqrt{k + 2(r - n - 1)} \gamma_{i,i+1}(z) (\partial \phi_{n+i}(z) - \partial \phi_{n+i+1}(z)) \]
\[ + (k + 2(i - n - 1)) \partial \gamma_{i,i+1}(z), \quad 1 \leq i \leq r - 1, \quad (4.24) \]

\[ F_{n+r}(z) = \sum_{m=1}^{n} (\bar{\Psi}_{m,r}^{+}(z)\Psi_{m,r-1}(z) + 2\bar{\Psi}_{m,r-1}(z)\bar{\Psi}_{m,r}(z)\beta_{m}(z) - \bar{\Psi}_{m,r-1}(z)\Psi_{m,r}(z)) \]
\[ + \sum_{m=1}^{r-2} (\gamma_{m,r}(z)\beta_{m,r-1}(z) - \gamma_{m,r-1}(z)\beta_{m,r}(z) \]
\[ - \gamma_{r-1,r}(z)\beta_{r-1,r}(z) \]
\[ + \sqrt{k + 2(r - n - 1)} \gamma_{r-1,r}(z) (\partial \phi_{n+r-1}(z) + \partial \phi_{n+r}(z)) \]
\[ + (k + 2(r - n - 2)) \partial \gamma_{r-1,r}(z), \quad (4.25) \]

\[ H_l(z) = \sum_{m=1}^{l-1} (\gamma_{m,l}(z)\beta_{m,l}(z) - \gamma_{m,l}(z)\bar{\beta}_{m,l}(z)) - \sum_{m=l+1}^{n} (\gamma_{l,m}(z)\beta_{l,m}(z) + \gamma_{l,m}(z)\bar{\beta}_{l,m}(z)) \]
\[ - 2\gamma_{l}(z)\beta_{l}(z) - \sum_{m=1}^{r} (\Psi_{l,m}^{+}(z)\Psi_{l,m}(z) + \bar{\Psi}_{l,m}^{+}(z)\bar{\Psi}_{l,m}(z)) \]
\[ + \sqrt{k + 2(r - n - 1)} \partial \phi_{l}(z), \quad 1 \leq l \leq n, \quad (4.26) \]

\[ H_{n+i}(z) = \sum_{m=1}^{n} (\Psi_{m,i}^{+}(z)\Psi_{m,i}(z) - \bar{\Psi}_{m,i}^{+}(z)\bar{\Psi}_{m,i}(z)) + \sum_{m=1}^{i-1} (\gamma_{m,i}(z)\beta_{m,i}(z) - \gamma_{m,i}(z)\bar{\beta}_{m,i}(z)) \]
\[ - \sum_{m=i+1}^{r} (\gamma_{i,m}(z)\beta_{i,m}(z) + \gamma_{i,m}(z)\beta_{i,m}(z)) \]
\[ + \sqrt{k + 2(r - n - 1)} \partial \phi_{n+i}(z), \quad 1 \leq i \leq r. \quad (4.27) \]

Here normal ordering of free fields is implied.
Proof. It is straightforward to check that the above free field realization of the currents satisfies the OPEs of the \( osp(2r|2n) \) current algebra: Direct calculation shows that there are at most second order singularities (e.g. \( \frac{1}{(z-w)^2} \)) in the OPEs of the currents. Comparing with the definition of the current algebra (2.6), terms with first order singularity (e.g. the coefficients of \( \frac{1}{(z-w)} \)) are fulfilled due to the fact that the differential operator realization (3.29)-(3.38) is a representation of the corresponding finite-dimensional superalgebra \( osp(2r|2n) \); terms with second order singularity \( \frac{1}{(z-w)^2} \) also match those in the definition (2.6) after the suitable choice we made for the anomalous terms in the expressions of the currents associated with negative roots. □

Some remarks are in order. The free field realization of the currents associated with the non-simple roots can be obtained from the OPEs of the simple ones. For \( n = r \), our result reduces to the free field realization of the \( osp(2n|2n) \) current algebra [46]. When \( n = 0 \) (or \( r = 0 \)), our result recovers the free field realization of \( so(2r) \) (or \( sp(2n) \)) current algebra proposed in [45].

The free field realization of the \( osp(2r|2n) \) current algebra (4.18)-(4.27) gives rise to the Fock representations of the current algebra in terms of the free fields (4.1)-(4.9). These representations are in general not irreducible. In order to obtain irreducible ones, one needs certain screening charges, which are the integrals of screening currents (see [436]-[439] below), and performs the cohomology procedure as in [28, 33, 35, 34]. We shall construct the associated screening currents in subsection 4.1.3.

4.1.2 Energy-momentum tensor

In this subsection we construct the free field realization of the Sugawara energy-momentum tensor \( T(z) \) of the \( osp(2r|2n) \) current algebra. The energy-momentum tensor \( T(z) \) can be constructed by means of the second-order Casimir element of \( osp(2r|2n) \), namely,

\[
T(z) = \frac{1}{2(k + 2(r - n - 1))} \left\{ - \sum_{m < l}^{n} (E_{\delta_{m} - \delta_{l}}(z)F_{\delta_{m} - \delta_{l}}(z) + F_{\delta_{m} - \delta_{l}}(z)E_{\delta_{m} - \delta_{l}}(z)) \\
- \sum_{m < l}^{n} (E_{\delta_{m} + \delta_{l}}(z)F_{\delta_{m} + \delta_{l}}(z) + F_{\delta_{m} + \delta_{l}}(z)E_{\delta_{m} + \delta_{l}}(z)) \\
- \sum_{l=1}^{n} \left[ 2(E_{2\delta_{l}}(z)F_{2\delta_{l}}(z) + F_{2\delta_{l}}(z)E_{2\delta_{l}}(z)) + H_{l}(z)H_{l}(z) \right] \right\}
\]
expressed in terms of the free fields (4.1)-(4.9) as
\[ T(z) = \sum_{i=1}^{n} \sum_{l=1}^{r} (E_{b_i}(z) F_{\bar{b}_i}(z) - F_{b_i}(z) E_{\bar{b}_i}(z)) \]
\[ + \sum_{i=1}^{n} \sum_{l=1}^{r} (E_{\bar{b}_i}(z) F_{b_i}(z) - F_{\bar{b}_i}(z) E_{b_i}(z)) \]
\[ + \sum_{i<j} (E_{b_i}(z) F_{b_j}(z) + F_{b_i}(z) E_{b_j}(z)) \]
\[ + \sum_{i<j} (E_{\bar{b}_i}(z) F_{\bar{b}_j}(z) + F_{\bar{b}_i}(z) E_{\bar{b}_j}(z)) + \sum_{i=1}^{r} H_{n+i}(z) H_{n+i}(z) \].

After a tedious calculation, we have

**Proposition 6** The energy-momentum tensor \( T(z) \) of the \( osp(2r|2n) \) current algebra can be expressed in terms of the free fields (4.1)-(4.9) as

\[
T(z) = - \sum_{i=1}^{n} \left( \frac{1}{2} \partial \phi_i(z) \partial \phi_i(z) - \frac{n+1-r-l}{\sqrt{k+2(r-n-1)}} \partial^2 \phi_i(z) \right) \]
\[ + \sum_{i=1}^{r} \left( \frac{1}{2} \partial \phi_{n+i}(z) \partial \phi_{n+i}(z) - \frac{r-l}{\sqrt{k+2(r-n-1)}} \partial^2 \phi_{n+i}(z) \right) \]
\[ + \sum_{m<l}^{n} \left( \beta_{m,l}(z) \partial \gamma_{m,l}(z) + \bar{\beta}_{m,l}(z) \partial \bar{\gamma}_{m,l}(z) \right) \]
\[ + \sum_{i<j}^{r} \left( \beta'_{i,j}(z) \partial \gamma'_{i,j}(z) + \bar{\beta}'_{i,j}(z) \partial \bar{\gamma}'_{i,j}(z) \right) \]
\[ - \sum_{i=1}^{n} \sum_{l=1}^{r} \left( \Psi_{i,l}(z) \partial \Psi_{i,l}^+(z) + \bar{\Psi}_{i,l}(z) \partial \bar{\Psi}_{i,l}^+(z) \right), \tag{4.28} \]

where normal ordering of free fields is implied. \( T(z) \) satisfies the following OPE:

\[
T(z) T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)^3}, \tag{4.29} \]

with the central charge \( c \) given by

\[
c = \frac{(n-r)(2n-2r+1)k}{k+2(r-n-1)} = k \times \text{sdim}(osp(2r|2n)). \tag{4.30} \]

It is remarked that for the special case of \( n = r \) the central charge (4.30) vanishes. This makes the WZNW model associated with \( OSP(2n|2n) \) supergroup an important class of
CFTs \[8, 9, 13, 14, 20\]. Moreover, we find that the \(osp(2r|2n)\) currents associated with the simple roots (4.18)-(4.27) are indeed primary fields with conformal dimensional one, namely,

\[
T(z)E_i(w) = \frac{E_i(w)}{(z-w)^2} + \frac{\partial E_i(w)}{(z-w)}, \quad 1 \leq i \leq n + r,
\]

\[
T(z)F_i(w) = \frac{F_i(w)}{(z-w)^2} + \frac{\partial F_i(w)}{(z-w)}, \quad 1 \leq i \leq n + r,
\]

\[
T(z)H_i(w) = \frac{H_i(w)}{(z-w)^2} + \frac{\partial H_i(w)}{(z-w)}, \quad 1 \leq i \leq n + r.
\]

It is expected that the \(osp(2r|2n)\) currents associated with non-simple roots, which can be constructed through the simple ones, are also primary fields with conformal dimensional one. Therefore, \(T(z)\) is the energy-momentum tensor of the \(osp(2r|2n)\) current algebra.

### 4.1.3 Screening currents

Important objects in the application of free field realizations to the computation of correlation functions of CFTs are screening currents. A screening current is a primary field with conformal dimension one and has the property that the singular part of its OPE with the affine currents is a total derivative. These properties ensure that the integrated screening currents (screening charges) may be inserted into correlators while the conformal or affine Ward identities remain intact \[27, 30\].

Free field realization of screening currents may be constructed from certain differential operators \[35, 39\] defined by the relation,

\[
\rho^{(d)}(s_\alpha) \langle \Lambda; x, \bar{x}; y, \bar{y}; \theta, \bar{\theta} \rangle \equiv \langle \Lambda | E_\alpha \ G_+(x, \bar{x}; y, \bar{y}; \theta, \bar{\theta}) \rangle, \quad \text{for} \ \alpha \in \Delta_+,
\]

where \(\langle \Lambda \rangle\) is given by (3.25)\((3.26)\) and \(G_+(x, \bar{x}; y, \bar{y}; \theta, \bar{\theta})\) is given by (3.20)-(3.23). The operators \(\rho^{(d)}(s_\alpha)\) \((\alpha \in \Delta_+)\) give a differential operator realization of the subalgebra \(osp(2r|2n)_+\). Again it is sufficient to construct \(s_l \equiv \rho^{(d)}(s_\alpha_l)\) related to the simple roots. Using (4.31) and the Baker-Campbell-Hausdorff formula, after some algebraic manipulations, we obtain the following explicit expressions for \(s_l\):

\[
s_l = \sum_{m=l+2}^{n} \left( -x_{l+1,m}x_{l+1,m} \partial_{x_{l+1,l+1}} + \bar{x}_{l+1,m} \partial_{\bar{x}_{l+1,l+1}} + 2x_{l+1,m} \bar{x}_{l,m} \partial_{x_l} + x_{l+1,m} \partial_{x_{l,l,m}} \right)
\]

\[
+ \sum_{m=1}^{r} \left( -\bar{\theta}_{l+1,m} \partial_{x_{l+1,l+1}} + \bar{\theta}_{l+1,m} \partial_{\bar{x}_{l+1,l+1}} - 2\theta_{l+1,m} \bar{\theta}_{l,m} \partial_{x_l} + \theta_{l+1,m} \partial_{\bar{x}_l} + \theta_{l+1,m} \partial_{\bar{x}_{l,l,m}} \right)
\]

\[
+ x_{l+1} \partial_{x_{l,l+1}} + 2\bar{x}_{l+1} \partial_{\bar{x}_l} + \partial_{x_{l,l+1}}, \quad 1 \leq l \leq n - 1,
\]

\[23\]
\[ s_n = \sum_{m=2}^{r} (\bar{y}_{1,m} \partial_{\bar{\theta}_{1,m}} - \bar{y}_{1,m} y_{1,m} \partial_{\bar{\theta}_{1,1}} + y_{1,m} \partial_{\bar{\theta}_{1,m}} - 2 y_{1,m} \tilde{\theta}_{1,m} \partial x_u) \]
\[ - 2 \theta_{n,1} \partial x_n + \partial \theta_{n,1}, \quad \text{for} \quad i \leq r - 1, \]
\[ s_{n+i} = \sum_{m=i+2}^{r} (\bar{y}_{i+1,m} \partial_{\bar{y}_{i,m}} - \bar{y}_{i+1,m} y_{i+1,m} \partial_{\bar{y}_{i,i+1}} + y_{i+1,m} \partial_{y_{i,m}}) \]
\[ + \partial y_{i,i+1}, \quad 1 \leq i \leq r - 1, \]
\[ s_{n+r} = \partial y_{r-1,r}, \quad \text{for} \quad i \leq r - 1, \]

One may obtain the differential operators \( s_{\alpha} \) associated with the non-simple generators from the above simple ones. Following the procedure similar to \([35, 39]\), we find the free field realization of the screening currents \( S_i(z) \) corresponding to the differential operators \( s_i \).

**Proposition 7** The screening currents associated with the simple roots of the \( \text{osp}(2r|2n) \) current algebra at a generic level \( k \) are given by

\[
S_l(z) = \left\{ \sum_{m=l+2}^{n} \left( - \gamma_{l+1,m}(z) \gamma_{l+1,m}(z) \tilde{b}_{l,l+1}(z) + \gamma_{l+1,m}(z) \tilde{b}_{l,m}(z) \right) \right. \\
+ \sum_{m=l+2}^{n} \left( 2 \gamma_{l+1,m}(z) \gamma_{l,m}(z) \tilde{b}_{l}(z) + \gamma_{l+1,m}(z) \tilde{b}_{l,m}(z) + \gamma_{l+1}(z) \tilde{b}_{l,l+1}(z) \right) \right. \\
+ 2 \gamma_{l+1}(z) \tilde{b}_{l}(z) - \sum_{m=1}^{r} \left( \bar{\Psi}_{l+1,m}^+(z) \Psi_{l+1,m}^+(z) \tilde{b}_{l,l+1}(z) - \tilde{b}_{l,m}(z) \Psi_{l,m}(z) \right) \right. \\
- \sum_{m=1}^{r} \left( 2 \Psi_{l+1,m}^+(z) \bar{\Psi}_{l,m}^+(z) \tilde{b}_{l}(z) - \Psi_{l+1,m}(z) \bar{\Psi}_{l,m}(z) + \beta_{l,l+1}(z) \right) \right\} \frac{a_l(z)}{e^{\sqrt{k+2(r-n-1)}},}
\]
\[ 1 \leq l \leq n - 1, \tag{4.36} \]

\[
S_n(z) = \left\{ \sum_{m=2}^{r} \left( \gamma_{1,m}(z) \bar{\Psi}_{n,m}(z) - \gamma_{1,m}(z) \gamma_{1,m}(z) \bar{\Psi}_{n,1}(z) - 2 \gamma_{1,m}(z) \bar{\Psi}_{n,m}(z) \beta_n(z) \right) \right. \\
+ \sum_{m=2}^{r} \gamma_{1,m}(z) \Psi_{n,m}(z) - 2 \bar{\Psi}_{n,1}(z) \beta_n(z) + \Psi_{n,1}(z) \right\} \frac{a_n(z)}{e^{\sqrt{k+2(r-n-1)}},} \tag{4.37} \]

\[
S_{n+i}(z) = \left\{ \sum_{m=i+2}^{r} \left( \gamma_{l+1,m}(z) \tilde{b}_{l,m}(z) - \gamma_{l+1,m}(z) \gamma_{l+1,m}(z) \tilde{b}_{l,i+1}(z) \right) \right. \\
+ \sum_{m=i+2}^{r} \gamma_{l+1,m}(z) \tilde{b}_{l,m}(z) + \beta_{l,i+1}(z) \right\} \frac{a_{n+i}(z)}{e^{\sqrt{k+2(r-n-1)}},} \quad 1 \leq i \leq r - 1, \tag{4.38} \]

\[
S_{n+r}(z) = \frac{a_{n+r}(z)}{e^{\sqrt{k+2(r-n-1)}}}. \tag{4.39} \]

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Here normal ordering of free fields is implied and \( \phi(z) \) is
\[
\tilde{\phi}(z) = \sum_{i=1}^{n} \phi_i(z) \delta_i + \sum_{j=1}^{r} \phi_{n+j}(z) \epsilon_j. \tag{4.40}
\]

From a direct calculation, one may find that the OPEs of the screening currents with the energy-momentum tensor and the \( \mathfrak{osp}(2r|2n) \) currents \((4.18)-(4.27)\) are
\[
T(z)S_i(w) = \frac{S_i(w)}{(z-w)^2} + \frac{\partial S_i(w)}{(z-w)} = \partial_w \left\{ \frac{S_i(w)}{(z-w)} \right\}, \quad i = 1, \ldots, n + r, \tag{4.41}
\]
\[
E_i(z)S_j(w) = 0, \quad i, j = 1, \ldots, n + r, \tag{4.42}
\]
\[
H_i(z)S_j(w) = 0, \quad i, j = 1, \ldots, n + r, \tag{4.43}
\]
\[
F_i(z)S_j(w) = (-1)^{|[i]|+|F_i|} \delta_{ij} \partial_w \left\{ \frac{(k + 2(r - n - 1)) e^{\alpha z \partial_w (z-w)}}{(z-w)} \right\}, \tag{4.44}
\]
\[
i, j = 1, \ldots, n + r.
\]

Here \([i]\) is defined by
\[
[i] = \begin{cases} 1, & i = 1, \ldots, n, \\ 0, & i = n + 1, \ldots, n + r. \end{cases} \tag{4.45}
\]

The screening currents obtained this way are called screening currents of the first kind \(34\).

Moreover, the screening current \(S_n(z)\) is fermionic and the others are bosonic.

### 4.2 Current superalgebra \( \mathfrak{osp}(2r + 1|2n)_k \)

#### 4.2.1 Free field realization of the currents

With the help of the explicit differential operator expressions of \( \mathfrak{osp}(2r + 1|2n) \) given by \((3.56)-(3.63)\) we can construct the explicit free field representation of the \( \mathfrak{osp}(2r + 1|2n) \) current algebra at an arbitrary level \(k\) in terms of \(n^2 + r^2\) bosonic \(\beta - \gamma\) pairs \(\{(\beta_{i,j}, \gamma_{i,j}), (\tilde{\beta}_{i,j}, \tilde{\gamma}_{i,j}), (\tilde{\beta}_i, \gamma_i), (\beta'_i, \gamma'_i), (\tilde{\beta}'_{i,j}, \tilde{\gamma}'_{i,j}), (\beta'_i, \gamma'_i), 1 \leq i < j \leq n, 1 \leq i' < j' \leq r\}\), \(n(2r + 1)\) fermionic \(b - c\) pairs \(\{(\Psi^+_{i,j}, \Psi_{i,j}), (\tilde{\Psi}^+_{i,j}, \tilde{\Psi}_{i,j}), (\psi^+_{i,j}, \psi_{i,j}), 1 \leq i \leq n, 1 \leq j \leq r\}\) and \(n + r\) free scalar fields \(\phi_i, i = 1, \ldots, n + r\). The free fields \(\{(\beta_{i,j}, \gamma_{i,j}), (\tilde{\beta}_{i,j}, \tilde{\gamma}_{i,j}), (\tilde{\beta}_i, \gamma_i), (\beta'_i, \gamma'_i), (\tilde{\beta}'_{i,j}, \tilde{\gamma}'_{i,j}), (\beta'_i, \gamma'_i)\}\), \(\{(\Psi^+_{i,j}, \Psi_{i,j}), (\tilde{\Psi}^+_{i,j}, \tilde{\Psi}_{i,j})\}\) and \(\{(\phi_i)\}\) obey the same OPEs as \((4.1)-(4.9)\).

The other non-trivial OPEs are
\[
\beta_i'(z) \gamma_j'(w) = -\gamma_i'(z) \beta_j'(w) = \frac{\delta_{ij}}{(z-w)}, \quad 1 \leq i, j \leq r, \tag{4.46}
\]
\[
\Psi^+_{m}(z) \Psi_{l}(w) = \Psi_{l}(z) \Psi^+_{m}(w) = \frac{\delta_{ml}}{(z-w)}, \quad 1 \leq m, l \leq n. \tag{4.47}
\]
The free field realization of the \(osp(2r + 1|2n)\) current algebra is obtained by the following substitutions in the differential operator realization \((3.56)-(3.65)\) of \(osp(2r + 1|2n)\):

\[
x_{m,l} \rightarrow \gamma_{m,l}(z), \quad \partial_{x_{m,l}} \rightarrow \beta_{m,l}(z), \quad 1 \leq m < l \leq n, \quad (4.48)
\]
\[
x_{i} \rightarrow \gamma_{i}(z), \quad \partial_{x_{i}} \rightarrow \beta_{i}(z), \quad 1 \leq l \leq n, \quad (4.49)
\]
\[
y_{i,j} \rightarrow \gamma'_{i,j}(z), \quad \partial_{y_{i,j}} \rightarrow \beta'_{i,j}(z), \quad 1 \leq i < j \leq r, \quad (4.50)
\]
\[
y_{i} \rightarrow \gamma'_{i}(z), \quad \partial_{y_{i}} \rightarrow \beta'_{i}(z), \quad 1 \leq i \leq r, \quad (4.51)
\]
\[
\theta_{l} \rightarrow \Psi_{l}^{\dagger}(z), \quad \partial_{\theta_{l}} \rightarrow \Psi_{l}(z), \quad 1 \leq l \leq n, \quad (4.52)
\]
\[
\lambda_{j} \rightarrow \sqrt{k + 2r - 2n - 1} \partial \phi_{j}(z) \quad 1 \leq j \leq n + r, \quad (4.53)
\]

followed with by the addition of anomalous terms linear in \(\partial \gamma(z), \partial \gamma'(z), \partial \gamma''(z), \partial \Psi^{+}(z)\) and \(\partial \bar{\Psi}^{+}(z)\) in the expressions of the currents. Here we present the results for the currents associated with the simple roots.

**Theorem 2** The currents associated with the simple roots of the \(osp(2r + 1|2n)\) current algebra at a generic level \(k\) are given in terms of the free fields \((4.1)-(4.3)\) and \((4.40)-(4.41)\) as

\[
E_{l}(z) = \sum_{m=1}^{l-1} \left( \gamma_{m,l}(z) \beta_{m,l+1}(z) - \gamma_{m,l+1}(z) \tilde{\beta}_{m,l}(z) \right) + \beta_{l,l+1}(z), \quad 1 \leq l \leq n - 1, \quad (4.58)
\]
\[
E_{n}(z) = \sum_{m=1}^{n-1} \left( \gamma_{m,n}(z) \Psi_{m,1}(z) + \bar{\Psi}_{m,1}^{+}(z) \tilde{\beta}_{m,n}(z) \right) + \Psi_{n,1}(z), \quad (4.59)
\]
\[
E_{n+i}(z) = \sum_{m=1}^{n} \left( \Psi_{m,i}^{+}(z) \Psi_{m,i+1}(z) - \Psi_{m,i+1}^{+}(z) \tilde{\Psi}_{m,i}(z) \right) + \bar{\Psi}_{m,i+1}^{+}(z), \quad 1 \leq i \leq r - 1, \quad (4.60)
\]
\[
E_{n+r}(z) = \sum_{m=1}^{n} \left( \Psi_{m}^{+}(z) \bar{\Psi}_{m+r}(z) - \Psi_{m+r}^{+}(z) \Psi_{m}(z) - \Psi_{m+r}^{+}(z) \Psi_{m}(z) \beta_{m}(z) \right) + \beta_{r+r}(z), \quad (4.61)
\]
\[ F_l(z) = \sum_{m=1}^{l-1} (\gamma_{m,l+1}(z)\beta_{m,l}(z) - \tilde{\gamma}_{m,l}(z)\tilde{\beta}_{m,l+1}(z)) - \gamma_l(z)\tilde{\beta}_{l+1}(z) \]

\[ -2\tilde{\gamma}_{l,l+1}(z)\beta_{l+1}(z) + \sum_{m=l+2}^{n} (\gamma_{l,m}(z)\tilde{\gamma}_{l,m}(z)\tilde{\beta}_{l,l+1}(z) - \gamma_{l,m}(z)\beta_{l+1,m}(z)) \]

\[ - \sum_{m=l+2}^{n} \left( 2\tilde{\gamma}_{l,m}(z)\gamma_{l+1,m}(z)\beta_{l+1}(z) + \tilde{\gamma}_{l,m}(z)\tilde{\beta}_{l+1,m}(z) \right) \]

\[ - \sum_{m=1}^{r} \left( (\Psi_{l,m}^+(z)\bar{\Psi}_{l,m}^+(z)\tilde{\beta}_{l,l+1}(z) + \Psi_{l,m}^+(z)\bar{\Psi}_{l+1,m}(z)) \right) \]

\[ - \Psi_{l}^+(z)\Psi_{l+1}(z) - \Psi_{l}^+(z)\Psi_{l+1}(z)\beta_{l+1}(z) \]

\[ -\gamma_{l,l+1}(z)\beta_{l,l+1}(z) - \gamma_{l+1}(z)\sum_{m=l+2}^{n} (\gamma_{l,m}(z)\beta_{l,m}(z) + \tilde{\gamma}_{l,m}(z)\tilde{\beta}_{l,m}(z)) \]

\[ + \gamma_{l,l+1}(z)\sum_{m=l+2}^{n} (\gamma_{l+1,m}(z)\beta_{l+1,m}(z) + \tilde{\gamma}_{l+1,m}(z)\tilde{\beta}_{l+1,m}(z)) \]

\[ - \gamma_{l,l+1}(z)\sum_{m=1}^{r} (\Psi_{l,m}^+(z)\Psi_{l,m}(z) + \bar{\Psi}_{l,m}^+(z)\bar{\Psi}_{l,m}(z)) \]

\[ + \gamma_{l,l+1}(z)\sum_{m=1}^{r} (\Psi_{l+1,m}^+(z)\Psi_{l+1,m}(z) + \bar{\Psi}_{l+1,m}^+(z)\bar{\Psi}_{l+1,m}(z)) \]

\[ + 2\gamma_{l,l+1}(z)\gamma_{l+1}(z)\beta_{l+1}(z) - 2\gamma_{l,l+1}(z)\gamma_{l}(z)\beta_{l}(z) \]

\[ + \gamma_{l,l+1}(z)\Psi_{l+1}(z)\Psi_{l+1}(z) - \gamma_{l,l+1}(z)\Psi_{l}^+(z)\Psi_{l}(z) \]

\[ + \sqrt{k + 2r - 2n - 1}\gamma_{l,l+1}(z)(\partial\phi_{l}(z) - \partial\phi_{l+1}(z)) \]

\[ + (-k + 2(l - 1))\partial\gamma_{l,l+1}(z), \quad 1 \leq l \leq n - 1, \quad (4.62) \]

\[ F_n(z) = \sum_{m=1}^{n-1} (\Psi_{m,1}^+(z)\beta_{m,n}(z) - \tilde{\gamma}_{m,n}(z)\tilde{\Psi}_{m,1}(z)) - \gamma_n(z)\tilde{\Psi}_{n,1}(z) \]

\[ + \sum_{m=2}^{r} (\Psi_{n,m}^+(z)\beta_{n,m}(z) - \Psi_{n,m}^+(z)\tilde{\Psi}_{n,m}(z)\tilde{\Psi}_{n,1}(z) + \bar{\Psi}_{n,m}^+(z)\tilde{\beta}_{n,m}(z)) \]

\[ - \Psi_{n,1}^+(z)\sum_{m=2}^{r} (\Psi_{n,m}^+(z)\Psi_{n,m}(z) + \bar{\Psi}_{n,m}^+(z)\bar{\Psi}_{n,m}(z)) - 2\Psi_{n,1}^+(z)\bar{\Psi}_{n,1}^+(z)\bar{\Psi}_{n,1}(z) \]

\[ - \Psi_{n,1}^+(z)\sum_{m=2}^{r} (\gamma_{n,m}^+(z)\beta_{n,m}(z) + \tilde{\gamma}_{n,m}(z)\tilde{\beta}_{n,m}(z)) - 2\Psi_{n,1}^+(z)\gamma_n(z)\beta_n(z) \]

\[ - \Psi_{n}^+(z)\beta_{n}^+(z) - \Psi_{n,1}^+(z)\Psi_{n}(z)\Psi_{n}(z) - \Psi_{n,1}^+(z)\gamma_{n}^+(z)\beta_{n}^+(z) \]
\[
 F_{n+i}(z) = \sum_{m=1}^{n} \left( \Psi_{m,i+1}^+(z) \Psi_{m,i}(z) - \bar{\Psi}_{m,i}(z) \bar{\Psi}_{m,i+1}(z) \right) \\
 + \sum_{m=1}^{i-1} \left( \gamma'_{m,i+1}(z) \bar{\beta}_{m,i}(z) - \bar{\gamma}'_{m,i}(z) \beta'_{m,i+1}(z) \right) \\
 + \sum_{m=i+2}^{r} \left( \gamma'_{i,m}(z) \bar{\beta}_{i,m}(z)_{i+1}(z) - \gamma'_{i,m}(z) \beta'_{i+1,m}(z) - \bar{\gamma}'_{i,m}(z) \beta'_{i+1,m}(z) \right) \\
 - \gamma'_{i}(z) \beta'_{i+1}(z) + \frac{1}{2} \gamma'_{i}(z) \beta'_{i+1}(z) \\
 + \gamma'_{i,i+1}(z) \sum_{m=i+2}^{r} \left( \gamma'_{i+1,m}(z) \beta'_{i+1,m}(z) + \bar{\gamma}'_{i+1,m}(z) \bar{\beta}'_{i+1,m}(z) \right) \\
 - \gamma'_{i,i+1}(z) \sum_{m=i+2}^{r} \left( \gamma'_{i,m}(z) \beta'_{i,m}(z) + \bar{\gamma}'_{i,m}(z) \bar{\beta}'_{i,m}(z) \right) \\
 - \gamma'_{i,i+1}(z) \left( \gamma'_{i,i+1}(z) \beta'_{i,i+1}(z) - \gamma'_{i+1}(z) \beta'_{i+1}(z) + \gamma'_{i}(z) \beta'_{i}(z) \right) \\
 + \sqrt{k + 2r - 2n - 1} \gamma'_{i,i+1}(z) \left( \partial \phi_{n+i}(z) - \partial \phi_{n+i+1}(z) \right) \\
 + (k + 2(i - n - 1)) \partial \gamma'_{i,i+1}(z), \quad 1 \leq i \leq r - 1, \tag{4.63}
\]

\[
 F_{n+r}(z) = \sum_{m=1}^{n} \left( \Psi_{m,r}^+(z) \Psi_{m}(z) - \bar{\Psi}_{m,r}(z) \bar{\Psi}_{m}(z) \right) \\
 + \sum_{m=1}^{r-1} \left( \gamma'_{m}(z) \beta'_{m,r}(z) - \bar{\gamma}'_{m,r}(z) \beta'_{m}(z) \right) - \frac{1}{2} \gamma'_{r}(z) \gamma'_{r}(z) \beta'_{r}(z) \\
 + \sqrt{k + 2r - 2n - 1} \gamma'_{r}(z) \partial \phi_{n+r}(z) + (k + 2(r - n - 1)) \partial \gamma'_{r}(z), \tag{4.64}
\]

\[
 H_l(z) = \sum_{m=1}^{l-1} \left( \gamma_{m,l}(z) \beta_{m,l}(z) - \bar{\gamma}_{m,l}(z) \bar{\beta}_{m,l}(z) \right) - \sum_{m=l+1}^{n} \left( \gamma_{l,m}(z) \beta_{l,m}(z) + \bar{\gamma}_{l,m}(z) \bar{\beta}_{l,m}(z) \right) \\
 - 2\gamma_{l}(z) \beta_{l}(z) + \sum_{m=1}^{r} \left( \Psi_{m,l}^+(z) \Psi_{l,m}(z) + \bar{\Psi}_{m,l}(z) \bar{\Psi}_{l,m}(z) \right) - \Psi_{l}^+(z) \Psi_{l}(z) \\
 + \sqrt{k + 2r - 2n - 1} \partial \phi_{l}(z), \quad 1 \leq l \leq n, \tag{4.65}
\]

\[
 H_{n+i}(z) = \sum_{m=1}^{n} \left( \Psi_{m,i}^+(z) \Psi_{m,i}(z) - \bar{\Psi}_{m,i}(z) \bar{\Psi}_{m,i}(z) \right) + \sum_{m=1}^{i-1} \left( \gamma'_{m,i}(z) \beta'_{m,i}(z) - \bar{\gamma}'_{m,i}(z) \bar{\beta}'_{m,i}(z) \right) \\
 - \sum_{m=i+1}^{r} \left( \gamma'_{i,m}(z) \beta'_{i,m}(z) + \bar{\gamma}'_{i,m}(z) \bar{\beta}'_{i,m}(z) \right) - \gamma'_{i}(z) \beta'_{i}(z) \\
 + \sqrt{k + 2r - 2n - 1} \partial \phi_{n+i}(z), \quad 1 \leq i \leq r. \tag{4.66}
\]
Here normal ordering of free fields is implied.

Proof. The proof of the theorem is similar to that of Theorem 1. □

The free field realization of the currents associated with the non-simple roots can be obtained from the OPEs of the simple ones. For the case of \( n = 0 \), our result recovers the free field realization proposed in [45] for \( \mathfrak{so}(2r + 1) \) current algebra.

4.2.2 Energy-momentum tensor

The energy-momentum tensor \( T(z) \) associated with the \( \mathfrak{osp}(2r + 1|2n) \) current algebra can be expressed in terms of the free fields through the Sugawara construction,

\[
T(z) = \frac{1}{2 (k + 2r - 2n - 1)} \left\{ - \sum_{m<l}^{n} (E_{\delta_m-\delta_l}(z) F_{\delta_m-\delta_l}(z) + F_{\delta_m-\delta_l}(z) E_{\delta_m-\delta_l}(z)) \\
- \sum_{m<l}^{n} (E_{\delta_m+\delta_l}(z) F_{\delta_m+\delta_l}(z) + F_{\delta_m+\delta_l}(z) E_{\delta_m+\delta_l}(z)) \\
- \sum_{l=1}^{n} \left[ 2 (E_{2\delta_l}(z) F_{2\delta_l}(z) + F_{2\delta_l}(z) E_{2\delta_l}(z)) + H_l(z) H_l(z) \right] \\
+ \sum_{l=1}^{n} (E_{\delta_l}(z) F_{\delta_l}(z) - F_{\delta_l}(z) E_{\delta_l}(z)) \\
+ \sum_{l=1}^{n} \sum_{i=1}^{r} (E_{\delta_l-\epsilon_i}(z) F_{\delta_l-\epsilon_i}(z) - F_{\delta_l-\epsilon_i}(z) E_{\delta_l-\epsilon_i}(z)) \\
+ \sum_{l=1}^{n} \sum_{i=1}^{r} (E_{\delta_l+\epsilon_i}(z) F_{\delta_l+\epsilon_i}(z) - F_{\delta_l+\epsilon_i}(z) E_{\delta_l+\epsilon_i}(z)) \\
+ \sum_{i<j} (E_{\epsilon_i-\epsilon_j}(z) F_{\epsilon_i-\epsilon_j}(z) + F_{\epsilon_i-\epsilon_j}(z) E_{\epsilon_i-\epsilon_j}(z)) \\
+ \sum_{i<j} (E_{\epsilon_i+\epsilon_j}(z) F_{\epsilon_i+\epsilon_j}(z) + F_{\epsilon_i+\epsilon_j}(z) E_{\epsilon_i+\epsilon_j}(z)) \\
+ \sum_{i=1}^{r} \left[ (E_{\epsilon_i}(z) F_{\epsilon_i}(z) + F_{\epsilon_i}(z) E_{\epsilon_i}(z)) + H_{n+i}(z) H_{n+i}(z) \right] \right\}.
\]

After a tedious calculation, we have

Proposition 8 The energy-momentum tensor \( T(z) \) of the \( \mathfrak{osp}(2r + 1|2n) \) current algebra
can be expressed in terms of the free fields (4.1)-(4.9) and (4.46)-(4.47) as

\[
T(z) = - \sum_{i=1}^{n} \left( \frac{1}{2} \partial \phi_i(z) \partial \phi_i(z) - \frac{2n + 1 - 2r - 2l}{2\sqrt{k + 2r - 2n - 1}} \partial^2 \phi_i(z) \right) 
+ \sum_{i=1}^{r} \left( \frac{1}{2} \partial \phi_{n+i}(z) \partial \phi_{n+i}(z) - \frac{2r - 2i + 1}{2\sqrt{k + 2r - 2n - 1}} \partial^2 \phi_{n+i}(z) \right) 
+ \sum_{m<l} \beta_m(z) \beta_m(z) + \sum_{l=1}^{n} \beta_i(z) \beta_i(z) 
+ \sum_{i<j} \beta'_{i,j}(z) \beta'_{i,j}(z) + \sum_{i=1}^{r} \beta'_i(z) \beta'_i(z) 
- \sum_{l=1}^{n} \sum_{i=1}^{r} \left( \Psi_{i,l}(z) \partial \Psi_{i,l}(z) + \bar{\Psi}_{i,l}(z) \partial \bar{\Psi}_{i,l}(z) \right) - \sum_{l=1}^{n} \Psi_i(z) \partial \bar{\Psi}_{i,l}(z), 
\]

where normal ordering of the free fields is implied. \( T(z) \) satisfies the OPE of the Virasoro algebra,

\[
T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)},
\]

with the central charge \( c \) given by

\[
c = \frac{(r-n)(2r-2n+1)k}{k + 2r - 2n - 1} \equiv \frac{k \times \text{sdim}(osp(2r+1|2n))}{k + 2r - 2n - 1}.
\]

Note that when \( n = r \), i.e. for the \( osp(2n+1|2n) \) case, the central charge (4.70) vanishes. It can be easily checked that the \( osp(2r+1|2n) \) currents (4.58)-(4.67) are primary fields with conformal dimensional one, namely,

\[
T(z)E_i(w) = \frac{E_i(w)}{(z-w)^2} + \frac{\partial E_i(w)}{(z-w)}, \quad 1 \leq i \leq n + r,
\]

\[
T(z)F_i(w) = \frac{F_i(w)}{(z-w)^2} + \frac{\partial F_i(w)}{(z-w)}, \quad 1 \leq i \leq n + r,
\]

\[
T(z)H_i(w) = \frac{H_i(w)}{(z-w)^2} + \frac{\partial H_i(w)}{(z-w)}, \quad 1 \leq i \leq n + r.
\]

### 4.2.3 Screening currents

Similarly to the \( osp(2r|2n) \) case, the free field realization of the screening currents can be constructed from certain differential operators defined by the relation,

\[
\rho^{(d)}(s_\alpha) \langle \Lambda; x, \bar{x}; y, \bar{y}; \theta, \bar{\theta} | \rangle \equiv \langle \Lambda | E_\alpha G_+(x, \bar{x}; y, \bar{y}; \theta, \bar{\theta}), \quad \text{for } \alpha \in \Delta_+,
\]

(4.71)
where $\langle \Lambda \rangle$ is given by (3.52)-(3.53) and $G_+ (x, \bar{x}; y, \bar{y}, \theta, \bar{\theta})$ is given by (3.47)-(3.51). The operators $\rho^{(d)} (s_\alpha)$ ($\alpha \in \Delta_+$) give a differential operator realization of the subalgebra $osp(2r + 1|2n)_+$. After some algebraic manipulations, we obtain the following explicit expressions for $s_i \equiv \rho^{(d)} (s_\alpha)$:

$$
s_l = \sum_{m=l+2}^n \left( -\bar{x}_{l+1, m}\partial_{\bar{x}_{l,t+1}} + \bar{x}_{l+1, m}\partial_{x_{l,m}} + 2x_{l+1, m}\bar{x}_{l,m}\partial_{x_l} + x_{l+1, m}\partial_{x_{l,m}} \right)
+ \sum_{m=1}^r \left( -\bar{\theta}_{l+1, m}\partial_{\bar{\theta}_{l,t+1}} + \bar{\theta}_{l+1, m}\partial_{\theta_{l,m}} - 2\theta_{l+1, m}\bar{\theta}_{l,m}\partial_{x_l} + \theta_{l+1, m}\partial_{\theta_{l,m}} \right)
+ x_{l+1}\partial_{x_{l,t+1}} + \theta_{l+1}\partial_{\theta_l} - \theta_{l+1}\partial_{x_l} + 2x_{l+1,1}\partial_{x_1} + \partial_{x_{l,t+1}}, \quad 1 \leq l \leq n-1, \quad (4.72)
$$

$$
s_n = \sum_{m=2}^r \left( \bar{y}_{l,m}\partial_{\bar{y}_{n,m}} - \bar{y}_{l,m}y_{l+1,m}\partial_{\bar{y}_{n,1}} + y_{l,m}\partial_{\theta_{n,m}} - 2y_{l,m}\bar{\theta}_{n,m}\partial_{x_n} \right)
+ y_{l+1}\partial_{\bar{y}_n} - y_l\partial_{\theta_n} - \frac{y_l^2}{2}\partial_{\bar{\theta}_n} + 2\bar{\theta}_{n,1}\partial_{x_n} + \partial_{\theta_{n,1}}, \quad (4.73)
$$

$$
s_{n+i} = \sum_{m=i+2}^r \left( \bar{y}_{l+1,m}\partial_{\bar{y}_{i,m}} - \bar{y}_{l+1,m}y_{l+1+m}\partial_{\bar{y}_{i+1,1}} + y_{l+1,m}\partial_{y_{i,m}} \right)
+ y_{l+1}\partial_{\bar{y}_{i+1}} - y_l\partial_{y_{i+1}} + \partial_{y_{i+1}}, \quad 1 \leq i \leq r-1, \quad (4.74)
$$

$$
s_{n+r} = \partial_{y_r}. \quad (4.75)
$$

Then we have

**Proposition 9** The free field realization of the screening currents $S_l(z)$ of the $osp(2r+1|2n)$ current algebra corresponding to the above differential operators $s_i$ is given by

$$
S_l(z) = \left\{ \sum_{m=l+2}^n \left( -\gamma_{l+1,m}(z)\gamma_{l+1,m}(z)\tilde{\beta}_{l,t+1}(z) + \tilde{\gamma}_{l+1,m}(z)\tilde{\beta}_{l,m}(z) \right) \right.
+ \sum_{m=l+2}^n \left( 2\gamma_{l+1,m}(z)\gamma_{l,m}(z)\beta_l(z) + \gamma_{l+1,m}(z)\beta_{l,m}(z) \right) + \gamma_{l+1}(z)\beta_{l,t+1}(z)
+ 2\gamma_{l,t+1}(z)\beta_t(z) - \sum_{m=1}^r \left( \Psi^+_{l+1,m}(z)\Psi^+_{l+1,m}(z)\tilde{\beta}_{l,t+1}(z) - \Psi^+_{l+1,m}(z)\tilde{\Psi}_{l,m}(z) \right)
+ \tilde{\Psi}_{l+1}(z)\Psi_l(z) - \Psi^+_{l+1}(z)\Psi_l(z)\beta_l(z) + \beta_{l,t+1}(z)
- \sum_{m=1}^r \left( 2\Psi^+_{l+1,m}(z)\Psi^+_{l+1,m}(z)\beta_t(z) - \Psi^+_{l+1,m}(z)\Psi_{l,m}(z) \right) \right\} e^{a_1(z)\tilde{\beta}(z)},
\quad 1 \leq l \leq n-1, \quad (4.76)
$$

$$
S_n(z) = \left\{ \sum_{m=2}^r \left( \tilde{\gamma}_{l,m}(z)\tilde{\Psi}_{n,m}(z) - \gamma_{l+m}(z)\gamma_{l,m}(z)\Psi_{n,1}(z) - 2\gamma_{l,m}(z)\tilde{\Psi}_{n,m}(z)\beta_n(z) \right) \right\}
$$
the energy-momentum tensor (4.68) and the osp\(\vec{\phi}(4.36)-(4.39)\) and (4.76)-(4.79) of the associated screening currents of the first kind. We have also found the free field representations of the corresponding basic Lie superalgebras. The corresponding energy-momentum tensors are given in terms of the explicit free field representations (4.18)-(4.27) and (4.58)-(4.67) of their corresponding cur-\(\vec{\phi}(z)\) is given by (4.40).}

From direct calculation we find that the screening currents satisfy the required OPEs with the energy-momentum tensor (4.68) and the osp(2r + 1|2n) currents (4.58)-(4.67), namely,

\[
T(z)S_i(w) = \frac{S_i(w)}{(z - w)^2} + \frac{\partial S_i(w)}{(z - w)} = \partial_w \left\{ \frac{S_i(w)}{(z - w)} \right\}, \quad i = 1, \ldots, n + r, \tag{4.80}
\]
\[
E_i(z)S_j(w) = 0, \quad i, j = 1, \ldots, n + r, \tag{4.81}
\]
\[
H_i(z)S_j(w) = 0, \quad i, j = 1, \ldots, n + r, \tag{4.82}
\]
\[
F_i(z)S_j(w) = (-1)^{[i][j]} + |F_i| \delta_{ij} \partial_w \left\{ \frac{(k + 2r - 2n - 1) e^{-\frac{\alpha_i \vec{\phi}(w)}{\sqrt{k + 2r - 2n - 1}}}}{(z - w)} \right\}, \quad i, j = 1, \ldots, n + r, \tag{4.83}
\]

where \([i]\) is given by (4.45).

5 Discussions

Based on the particular orderings (3.12) and (3.44) for the positive roots of the finite dimensional basic Lie superalgebras, we have constructed the explicit differential operator realiza-tions (3.29)-(3.38) and (3.56)-(3.65) for the osp(2r|2n) and osp(2r + 1|2n) superalgebras and explicit free field representations (4.18)-(4.27) and (4.58)-(4.67) of their corresponding current superalgebras. The corresponding energy-momentum tensors are given in terms of the free fields by (4.28) and (4.68) respectively. We have also found the free field representations (4.36)-(4.39) and (4.76)-(4.79) of the associated screening currents of the first kind.
These free field realizations of the $osp(2r|2n)$ and $osp(2r+1|2n)$ current algebras give rise to the Fock representations of the current algebras. They provide explicit realizations of the vertex operator construction [50, 51] of representations for affine superalgebras $osp(2r|2n)_k$ and $osp(2r+1|2n)_k$. These representations are in general not irreducible. To obtain irreducible representations, one needs the associated screening charges, which are the integrals of the corresponding screening currents (4.36)-(4.39) and (4.76)-(4.79), and one then performs the cohomology analysis as in [28, 33, 35, 34].

An important open problem is to construct the free field representations of the primary fields for the current superalgebras studied in this paper. It is well-known that there exist two types of representations for the underlying finite-dimensional superalgebras: typical and atypical representations. Atypical representations have no counterpart in the bosonic algebra setting and our understanding of such representations is still very much incomplete. Although the construction of the primary fields associated with typical representations is similar to the bosonic algebra cases, it is a highly nontrivial task to construct the primary fields associated with atypical representations even for the relatively simple $gl(2|2)$ current algebra [52].

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Appendix A: Defining representation of $osp(m|2n)$

Let $V$ be a $\mathbb{Z}_2$-graded $(m+2n)$-dimensional vector space with the orthonormal basis $\{|i\}, i = 1, \dotsc, m+2n\}$. The $\mathbb{Z}_2$-grading is chosen as: $[1] = \cdots = [m] = 0$, $[m+1] = \cdots = [m+2n] = 1$. For any $(m+2n) \times (m+2n)$ matrix $A$, one can define the supertrace,

$$str(A) = \sum_{l=1}^{m+2n} (-1)^{[l]} A_{ll} = \sum_{l=1}^{m} A_{ll} - \sum_{l=m+1}^{m+2n} A_{ll}. \quad (A.1)$$

Let $e_{ij}, i, j = 1, \dotsc, n$, be an $n \times n$ matrix with entry 1 at the $i$th row and the $j$th column and zero elsewhere. Let $e_i, i = 1, \dotsc, n$, be an $n$-dimensional row vector with the
ith component being 1 and all others being zero, and \( e_i^T \) be the transpose of \( e_i \), namely,

\[
e_i = (0, \ldots, 0, 1, 0, \ldots, 0), \quad e_i^T = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.
\]

Similarly, one can introduce \( r \times r \) matrices \( \bar{e}_{ij} \), \( i, j = 1, \ldots, r \), and \( r \)-dimensional row vectors \( \bar{e}_i \). With help of these matrices \( \{e_{ij} | i, j = 1, \ldots, n\} \) and \( \{\bar{e}_{ij} | i, j = 1, \ldots, r\} \) and row vectors \( \{e_i | i = 1, \ldots, n\} \) and \( \{\bar{e}_i | i = 1, \ldots, r\} \), one can realize the defining representations of \( osp(2r|2n) \) and \( osp(2r+1|2n) \) as follows.

**A1. The \( osp(2r|2n) \) case**

Let \( m = 2r \), i.e. \( \dim(V) = 2(r+n) \). The defining representation of \( osp(2r|2n) \) on \( V \), denoted by \( \rho_0 \), is given by the following \( 2(r+n) \times 2(r+n) \) matrices,

\[
\rho_0(E_{\delta_l-\delta_l}) = \begin{pmatrix} e_{ml} \\ -e_{lm} \end{pmatrix}, \quad \rho_0(F_{\delta_l-\delta_l}) = \begin{pmatrix} e_{lm} \\ -e_{ml} \end{pmatrix}, \quad m < l, \quad \tag{A.2}
\]

\[
\rho_0(E_{2\delta_l}) = \begin{pmatrix} 0 \\ e_{ll} \\ 0 \end{pmatrix}, \quad \rho_0(F_{2\delta_l}) = \begin{pmatrix} 0 \\ 0 \\ e_{ll} \end{pmatrix}, \quad \tag{A.3}
\]

\[
\rho_0(E_{\delta_l+\delta_l}) = \begin{pmatrix} 0 \\ e_{ml} + e_{lm} \end{pmatrix}, \quad \rho_0(F_{\delta_l+\delta_l}) = \begin{pmatrix} 0 \\ e_{ml} + e_{lm} \end{pmatrix}, \quad m < l, \quad \tag{A.4}
\]

\[
\rho_0(E_{\delta_l-\epsilon_l}) = \begin{pmatrix} e_i^T e_l \\ 0 \\ 0 \end{pmatrix}, \quad \rho_0(F_{\delta_l-\epsilon_l}) = \begin{pmatrix} e_i^T e_l \\ 0 \\ 0 \end{pmatrix}, \quad \tag{A.5}
\]

\[
\rho_0(E_{\delta_l+\epsilon_l}) = \begin{pmatrix} 0 \\ e_l^T e_i \\ 0 \end{pmatrix}, \quad \rho_0(F_{\delta_l+\epsilon_l}) = \begin{pmatrix} 0 \\ e_l^T e_i \\ 0 \end{pmatrix}, \quad \tag{A.6}
\]
\[ \rho_0(E_{e_i-e_j}) = \begin{pmatrix} \bar{e}_{ij} & -\bar{e}_{ji} \\ -\bar{e}_{ji} & \bar{e}_{ij} \end{pmatrix}, \quad \rho_0(F_{e_i-e_j}) = \begin{pmatrix} \bar{e}_{ji} & -\bar{e}_{ij} \\ -\bar{e}_{ij} & \bar{e}_{ji} \end{pmatrix}, \quad i < j, \quad (A.7) \]

\[ \rho_0(E_{e_i+e_j}) = \begin{pmatrix} 0 & \bar{e}_{ij} - \bar{e}_{ji} \\ \bar{e}_{ji} & 0 \end{pmatrix}, \quad \rho_0(F_{e_i+e_j}) = \begin{pmatrix} 0 & 0 \\ -\bar{e}_{ij} + \bar{e}_{ji} & 0 \end{pmatrix}, \quad i < j, \quad (A.8) \]

\[ \rho_0(H_{\delta_m-\delta_l}) = \begin{pmatrix} e_{mm} - e_{ll} & -e_{ll} \\ e_{ll} & e_{mm} \end{pmatrix}, \quad m < l, \quad (A.9) \]

\[ \rho_0(H_{\delta_m+\delta_l}) = \begin{pmatrix} e_{mm} + e_{ll} & -e_{mm} - e_{ll} \\ -e_{mm} - e_{ll} & e_{mm} + e_{ll} \end{pmatrix}, \quad m < l, \quad (A.10) \]

\[ \rho_0(H_{2\delta_l}) = \begin{pmatrix} e_{ll} \\ -e_{ll} \end{pmatrix}, \quad (A.11) \]

\[ \rho_0(H_{\delta_i-e_i}) = \begin{pmatrix} \bar{e}_{ii} & -\bar{e}_{ii} \\ -\bar{e}_{ii} & \bar{e}_{ii} \end{pmatrix}, \quad \rho_0(H_{\delta_i+e_i}) = \begin{pmatrix} -\bar{e}_{ii} & \bar{e}_{ii} \\ \bar{e}_{ii} & -\bar{e}_{ii} \end{pmatrix}, \quad (A.12) \]

\[ \rho_0(H_{e_i-e_j}) = \begin{pmatrix} \bar{e}_{ii} - \bar{e}_{jj} & \bar{e}_{jj} - \bar{e}_{ii} \\ \bar{e}_{jj} - \bar{e}_{ii} & \bar{e}_{ii} - \bar{e}_{jj} \end{pmatrix}, \quad i < j, \quad (A.13) \]

\[ \rho_0(H_{e_i+e_j}) = \begin{pmatrix} \bar{e}_{ii} + \bar{e}_{jj} & \bar{e}_{jj} - \bar{e}_{ii} \\ \bar{e}_{jj} - \bar{e}_{ii} & \bar{e}_{ii} + \bar{e}_{jj} \end{pmatrix}, \quad i < j. \quad (A.14) \]

Then we introduce \( r + n \) linear-independent generators \( H_i \) \((i = 1, \ldots, r + n)\),

\[ H_l = H_{2\delta_l}, \quad 1 \leq l \leq n, \quad (A.15) \]

\[ H_{n+i} = \frac{1}{2}(H_{e_i-e_j} + H_{e_i+e_j}), \quad i = 1, \ldots, r - 1, \text{ and } i < j, \quad (A.16) \]

\[ H_{n+r} = \frac{1}{2}(H_{e_i+e_r} - H_{e_i-e_r}), \quad i \leq r - 1. \quad (A.17) \]
Actually, the above generators \( \{H_i\} \) span the Cartan subalgebra of \( \mathfrak{osp}(2r|2n) \). In the defining representation, these generators can be realized by

\[
\rho_0(H_l) = \begin{pmatrix} e_{ll} & 0 \\ 0 & -e_{ll} \end{pmatrix}, \quad l = 1, \ldots, n, \tag{A.18}
\]

\[
\rho_0(H_{n+i}) = \begin{pmatrix} \bar{e}_{ii} & 0 \\ 0 & -\bar{e}_{ii} \end{pmatrix}, \quad i = 1, \ldots, r. \tag{A.19}
\]

The corresponding nondegenerate invariant bilinear supersymmetric form of \( \mathfrak{osp}(2r|2n) \) is given by

\[
(x, y) = \frac{1}{2} \text{str} (\rho_0(x)\rho_0(y)), \quad \forall x, y \in \mathfrak{osp}(2r|2n). \tag{A.20}
\]

**A2. The \( \mathfrak{osp}(2r+1|2n) \) case**

Let \( m = 2r + 1 \), i.e. \( \dim(V) = 2(r + n) + 1 \). The defining representation of \( \mathfrak{osp}(2r+1|2n) \) on \( V \), denoted by \( \rho_0 \), is given by the following \((2(r + n) + 1) \times (2(r + n) + 1)\) matrices,

\[
\rho_0(E_{\delta_m - \delta_l}) = \begin{pmatrix} e_{ml} & 0 \\ 0 & -e_{lm} \end{pmatrix}, \quad \rho_0(F_{\delta_m - \delta_l}) = \begin{pmatrix} e_{lm} & 0 \\ 0 & -e_{ml} \end{pmatrix}, \quad m < l, \tag{A.21}
\]

\[
\rho_0(E_{2\delta_i}) = \begin{pmatrix} e_{ll} & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_0(F_{2\delta_i}) = \begin{pmatrix} 0 & e_{ll} \\ 0 & 0 \end{pmatrix}, \tag{A.22}
\]

\[
\rho_0(E_{\delta_m + \delta_l}) = \begin{pmatrix} 0 & e_{ml} + e_{lm} \\ 0 & 0 \end{pmatrix}, \quad \rho_0(F_{\delta_m + \delta_l}) = \begin{pmatrix} 0 & 0 \\ e_{ml} + e_{lm} & 0 \end{pmatrix}, \quad m < l, \tag{A.23}
\]

\[
\rho_0(E_{\delta_l - \epsilon_i}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ e_i^T e_i & 0 \\ 0 & e_i^T e_i \\ 0 & 0 \end{pmatrix}, \quad \rho_0(F_{\delta_l - \epsilon_i}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & e_i^T e_i \\ 0 & 0 \end{pmatrix}, \tag{A.24}
\]

\[
\rho_0(E_{\delta_l + \epsilon_i}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ e_i^T e_i & 0 \\ 0 & e_i^T e_i \\ 0 & 0 \end{pmatrix}, \quad \rho_0(F_{\delta_l + \epsilon_i}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & e_i^T e_i \\ 0 & 0 \end{pmatrix}, \tag{A.25}
\]
\[
\rho_0(E_{\delta_l}) = \begin{pmatrix} 0 & e_l \\ e_l^T & 0 \end{pmatrix},
\rho_0(F_{\delta_l}) = \begin{pmatrix} e_l & 0 \\ 0 & 0 \end{pmatrix},
\]  
(A.26)

\[
\rho_0(E_{\epsilon_i - \epsilon_j}) = \begin{pmatrix} 0 & \bar{\epsilon}_{ij} \\ -\bar{\epsilon}_{ji} & 0 \end{pmatrix},
\rho_0(F_{\epsilon_i - \epsilon_j}) = \begin{pmatrix} 0 & \bar{\epsilon}_{ji} \\ -\bar{\epsilon}_{ij} & 0 \end{pmatrix},
i < j,
\]  
(A.27)

\[
\rho_0(E_{\epsilon_i + \epsilon_j}) = \begin{pmatrix} 0 & 0 & \bar{\epsilon}_{ij} \\ \bar{\epsilon}_{ij} & 0 \\ 0 & 0 \end{pmatrix},
\rho_0(F_{\epsilon_i + \epsilon_j}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \bar{\epsilon}_{ij} + \bar{\epsilon}_{ji} \end{pmatrix},
i < j,
\]  
(A.28)

\[
\rho_0(E_{\epsilon_i}) = \begin{pmatrix} 0 & \bar{\epsilon}_i \\ \bar{\epsilon}_i^T & 0 \end{pmatrix},
\rho_0(F_{\epsilon_i}) = \begin{pmatrix} 0 & -\bar{\epsilon}_i \\ \bar{\epsilon}_i & 0 \end{pmatrix},
\]  
(A.29)

\[
\rho_0(H_{\delta_m - \delta_l}) = \begin{pmatrix} e_{mm} - e_{ll} \\ e_{ll} - e_{mm} \end{pmatrix},
m < l,
\]  
(A.30)

\[
\rho_0(H_{\delta_m + \delta_l}) = \begin{pmatrix} e_{mm} + e_{ll} \\ -e_{mm} - e_{ll} \end{pmatrix},
m < l,
\]  
(A.31)

\[
\rho_0(H_{2\delta_l}) = \begin{pmatrix} e_{ll} \\ -e_{ll} \end{pmatrix},
\rho_0(H_{\delta_i - \epsilon_i}) = \begin{pmatrix} 0 \\ -\bar{\epsilon}_{ii} \end{pmatrix},
\]  
(A.32)

\[
\rho_0(H_{\delta_i + \epsilon_i}) = \begin{pmatrix} 0 \\ \bar{\epsilon}_{ii} \end{pmatrix},
\rho_0(H_{\delta_l}) = \begin{pmatrix} e_{ll} \\ -e_{ll} \end{pmatrix},
\]  
(A.33)

\[
\rho_0(H_{\epsilon_i - \epsilon_j}) = \begin{pmatrix} 0 & \bar{\epsilon}_{ii} - \bar{\epsilon}_{jj} \\ \bar{\epsilon}_{jj} - \bar{\epsilon}_{ii} \end{pmatrix},
i < j,
\]  
(A.34)
\[ \rho_0 (H_{\epsilon_1 + \epsilon_j}) = \begin{pmatrix} 0 & \bar{e}_{ii} + \bar{e}_{jj} \\ -\bar{e}_{ii} - \bar{e}_{jj} \end{pmatrix}, \quad i < j, \quad (A.35) \]

\[ \rho_0 (H_{\epsilon_i}) = \begin{pmatrix} 0 & \bar{e}_{ii} \\ -\bar{e}_{ii} \end{pmatrix}. \quad (A.36) \]

We introduce \( r + n \) linear-independent generators \( H_i \) \((i = 1, \ldots r + n)\),
\[
H_l = H_{2\delta_l}, \quad 1 \leq l \leq n, \quad (A.37)
\]
\[
H_{n+i} = \frac{1}{2}(H_{\epsilon_i - \epsilon_j} + H_{\epsilon_i + \epsilon_j}), \quad i = 1, \ldots, r - 1, \text{ and } i < j, \quad (A.38)
\]
\[
H_{n+r} = \frac{1}{2}(H_{\epsilon_i + \epsilon_r} - H_{\epsilon_i - \epsilon_r}), \quad i \leq r - 1. \quad (A.39)
\]

Actually, the above generators \( \{H_i\} \) span the Cartan subalgebra of \( osp(2r + 1|2n) \). In the defining representation, these generators can be realized by
\[
\rho_0 (H_l) = \begin{pmatrix} 0 \\ e_{ll} - e_{ll} \end{pmatrix}, \quad l = 1, \ldots, n, \quad (A.40)
\]
\[
\rho_0 (H_{n+i}) = \begin{pmatrix} 0 & \bar{e}_{ii} \\ -\bar{e}_{ii} \end{pmatrix}, \quad i = 1, \ldots, r. \quad (A.41)
\]

The corresponding nondegenerate invariant bilinear supersymmetric form of \( osp(2r + 1|2n) \) is given by
\[
(x, y) = \frac{1}{2} str (\rho_0 (x) \rho_0 (y)), \quad \forall x, y \in osp(2r + 1|2n). \quad (A.42)
\]

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