Weyl composition of symbols in large dimension

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Abstract

This paper is concerned with the Weyl composition of symbols in large dimension. We specify a class of symbols in order to estimate the Weyl symbol of the product of two Weyl $h$--pseudodifferential operators, with constants independent of the dimension. The proof includes a regularized and a hybrid compositions together with a decomposition formula. We also analyze in this context the remainder term of the semiclassical expansion of the Weyl composition.

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1. Statement of results.

Composition of Weyl pseudodifferential operators is a largely studied area in the literature, giving numerous classical results, depending on the class where the symbols of the operators belong.

The purpose here is to establish estimates for the Weyl composition of symbols, independently of the dimension, allowing in particular the dimension to go to infinity. To this aim, the two composed symbols are chosen in a simple class, defined in such a way that the constants appearing in the inequalities are also independent of the dimension.

In a recent work with L. Jager [A-J-N-2], we obtain an upper bound in the $L^2$ norm of operators with a symbol belonging to this class. The constants involved in the inequality estimating this norm are also independent of the dimension.

When $A$ and $B$ are two functions on $\mathbb{R}^{2n}$, bounded together with their derivatives, their Weyl composition, depending on the parameter $h > 0$ (the Weyl symbol of the product of the two Weyl $h$--pseudodifferential operators with symbols $A$ and $B$) is the function $C_h(A, B)$ formally defined on $\mathbb{R}^{2n}$ by:

\[
C_h^{\text{weyl}}(A, B)(X) = (\pi h)^{-2n} \int_{\mathbb{R}^{4n}} A(X + Y)B(X + Z)e^{-\frac{\pi i}{h} \sigma(Y, Z)}dYdZ
\]

where $\sigma$ is the symplectic form $\sigma(X, Y) = y \cdot \xi - x \cdot \eta$ for $X = (x, \xi)$ and $Y = (y, \eta)$ in $\mathbb{R}^{2n}$. The theory may be found in Hörmander [HO] Chapter 18, (also see [LER], [S] and, in the semiclassical setting, see e.g. [M], [R]). If $A$ and $B$ are bounded continuous functions on $\mathbb{R}^{2n}$, one notes that equality (1.1) formally defines a tempered distribution on $\mathbb{R}^{2n}$. If $A$ and $B$ are in the class $C^m$ ($m$ being sufficiently large) with bounded derivatives up to order $m$ then $C_h^{\text{weyl}}(A, B)$ is a bounded continuous function.

The objective of this work is to derive estimates for $C_h^{\text{weyl}}(A, B)$ where all the constants are independent of the dimension $n$. In order to do that, we shall first define a class of symbols where all the constants are also specified.
**Definition 1.1.** Let \((\rho_j)_{j \geq 1}\) and \((\delta_j)_{j \geq 1}\) be two sequences of real numbers \(\geq 0\). Fix \(M \geq 0\) and an integer \(m \geq 0\). Define \(S_m(M, \rho, \delta)\) as the set of functions \(F\) continuous on \(\mathbb{R}^{2n}\) \((n \geq 1)\) such that, for each multi-index \((\alpha, \beta)\) verifying \(\alpha_j \leq m\) and \(\beta_j \leq m\) for all \(j \leq n\), the derivative \(\partial_x^\alpha \partial_\xi^\beta F\) exists, is continuous and bounded, and satisfies:

\[
(1.2) \quad \sup_{X \in \mathbb{R}^{2n}} \left| \partial_x^\alpha \partial_\xi^\beta F(X) \right| \leq M \prod_{j \leq n} \rho_j^{\alpha_j} \delta_j^{\beta_j}
\]

In [A-J-N-2] we give a precise upper bound of the \(L^2\) norm of Weyl \(h\)–pseudodifferential operators associated with a symbol \(A\) in \(S_2(M, \rho, \delta)\) when \(h \rho_j \delta_j \leq 1\) for all \(j \leq n\). In the works of Bernard Lascar [LA-1] to [LA-4] one finds an extensive analysis of pseudodifferential operators in large and infinite dimension.

**Theorem 1.2.** There exists a universal constant \(K > 0\) such that, for all \(n \geq 1\), for any \(A\) in \(S_6(M, \rho, \delta)\) and \(B\) in \(S_6(M', \rho, \delta)\), the Weyl composition \(C_h(A, B)\) is a bounded function on \(\mathbb{R}^{2n}\) and satisfies, if \(h \rho_j \delta_j \leq 1\) for all \(j \leq n\):

\[
(1.3) \quad \sup_{X \in \mathbb{R}^{2n}} |C_h(A, B)(X)| \leq M''
\]

where \(M'' = MM' \prod_{j \leq n} (1 + Kh \rho_j \delta_j)\)

If \(A\) is in \(S_m(M, \rho, \delta)\) and \(B\) in \(S_m(M', \rho, \delta)\) \((m \geq 6)\) then \(C_h(A, B)\) belongs to \(S_{m-6}(M'', 2\rho, 2\delta)\), with \(M''\) defined in (1.3).

Next, we give the asymptotic expansion of the Weyl composition with constants again independent of the dimension.

**Theorem 1.3.** For every \(N \geq 1\), let \(R_N\) be the function defined by:

\[
(1.4) \quad C_h^{\text{weyl}}(A, B)(X) = \sum_{k=0}^{N-1} \frac{h^k}{(2i)^k k!} \sigma(\nabla Y, \nabla Z)^k |A(X+Y)B(X+Z)| \bigg|_{Y=Z=0} + R_N(X, h)
\]

Then we have, for all \(A\) in \(S_m(M, \rho, \delta)\) and \(B\) in \(S_m(M', \rho, \delta)\) \((m \geq N + 6)\):

\[
(1.5) \quad R_N(\cdot, h) \in S_{m-N-6} \left( M'' \frac{h^N}{N!} \left[ \sum_{j=1}^{n} \rho_j \delta_j \right]^N, 2\rho, 2\delta \right)
\]

where \(M''\) is defined in (1.3).

The idea of the proof is to first introduce a regularized composition \(C_h^{\text{reg}}(A, B)\) for any functions \(A\) and \(B\) bounded on \(\mathbb{R}^{2n}\). Namely, it is defined as the Wick symbol of the product of the two operators with anti-Wick symbols \(A\) and \(B\) respectively. The \(L^\infty(\mathbb{R}^{2n})\) norm of this regularized composition is bounded by the product of the \(L^\infty(\mathbb{R}^{2n})\) norms of \(A\) and \(B\) (see Section 2). Immediately thereafter, we define for all subset \(I\) in \(\{1, ..., n\}\), a hybrid composition \(C_h^{\text{hyb}}(A, B)\) behaving as a Weyl composition with respect to the variables \(x_j\) with \(j \in I\), and behaving as a regularized composition with respect to the variables \(x_j\) with \(j \in \{1, ..., n\} \setminus I\). In the next step, on the basis of a decomposition of the identity, Proposition 3.1 provides a decomposition of the Weyl composition \(C_h^{\text{weyl}}(A, B)\) as a sum, where each term in the sum is related to a hybrid composed symbol associated with some subset \(I\) of \(\{1, ..., n\}\), the sum being taken over all these subsets. As a further step, the hybrid composition is written in Proposition 2.1 as an integral expression, then, integrations by parts combined with other techniques allow to bound these hybrid compositions (Section 4). In the last step, it remains to take the sum of the bounds associated to each subset \(I\) of \(\{1, ..., n\}\) to derive the estimate (1.3). The other claim in Theorem 1.2 and Theorem 1.3 are then deduced relying on standard arguments (Section 5).
2. Regularized and hybrid compositions of symbols.

We shall first study a composition law on $L^\infty(\mathbb{R}^{2n})$ which shall be a continuous bilinear map. For that purpose, we define for all $A$ and $B$ in $L^\infty(\mathbb{R}^{2n})$, for all $X$ in $\mathbb{R}^{2n}$:

\[
C_h^{reg}(A,B)(X) = e^{\frac{i}{\hbar} \Delta} C_h^{wckl}(e^{\frac{i}{\hbar} \Delta} A, e^{\frac{i}{\hbar} \Delta} B)(X)
\]

Then, for all subsets $I$ in $\{1, \ldots, n\}$, we also define:

\[
C_h^{hyb,I}(A,B)(X) = e^{\frac{i}{\hbar} \Delta_{I^c}} C_h^{wckl}(e^{\frac{i}{\hbar} \Delta_{I^c}} A, e^{\frac{i}{\hbar} \Delta_{I^c}} B)(X)
\]

where $I^c$ is the complement of $I$ in $\{1, \ldots, n\}$ and

\[
\Delta_{I^c} = \sum_{j \in I^c} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial \xi_j^2}
\]

Thus, if $I = \emptyset$ then $C_h^{hyb,I}(A,B) = C_h^{reg}(A,B)$ and if $I = \{1, \ldots, n\}$ then $C_h^{hyb,I}(A,B) = C_h^{wckl}(A,B)$.

For all subsets $I$ of $\{1, \ldots, n\}$ and for every functions $A$ and $B$ on $((\mathbb{R}^2)^I$, $C_h^{reg,I}(A,B)$ denotes the function on $((\mathbb{R}^2)^I$ defined as in (2.1), when replacing $\{1, \ldots, n\}$ by $I$. For all functions $A$ on $\mathbb{R}^{2n}$ and for all $X_I$ in $((\mathbb{R}^2)^I$, we define a function $A_{X_I}$ on $((\mathbb{R}^2)^I$ setting $A_{X_I}(X_{I^c}) = A(X_I, X_{I^c})$. With these notations, we may write:

\[
C_h^{hyb,I}(A,B)(X_I, X_{I^c}) = (\pi \hbar)^{-2|I|} \int_{((\mathbb{R}^2)^I} C_h^{reg,I}(A_{X_I+Y_I}, B_{X_I+Z_I})(X_{I^c}) e^{-\frac{\hbar}{2|I|} (Y_I-Z_I)Y_I} dY_I dZ_I
\]

We shall now express $C_h^{reg,I}(A,B)$ under an integral form.

**Proposition 2.1.** For each subset $I$ of $\{1, \ldots, n\}$ we have:

\[
C_h^{hyb,I}(A,B)(X) = \int_{\mathbb{R}^{2n}} A(X+Y)B(X+Z) K_{I,h}(Y,Z)d\lambda(Y,Z)
\]

\[
K_{I,h}(Y,Z) = (\pi \hbar)^{-2|I|} (2\pi \hbar)^{-4} e^{-\frac{2}{\hbar} (Y_I-Z_I) e^{-\frac{2}{\hbar} (Y_I-Z_I) Y_I e^{-\frac{2}{\hbar} (Y_I-Z_I) Z_I} - \frac{2}{\hbar} (Y_{I^c}^2+|Z_{I^c}|^2)}
\]

**Proof.** We first prove the proposition for $I = \emptyset$, that is to say, for the function $C_h^{reg}(A,B)$. Let us recall the coherent states:

\[
\Psi_X = \Psi_{a,b}(u) = (\pi \hbar)^{-n/4} e^{-\frac{|y-a|^2}{2\hbar}} e^{\frac{i}{\hbar} (y-a)\cdot u} u \in \mathbb{R}^n
\]

We denote by $Op_{\hbar}^{AW}(A)$ the anti-Wick operator associated with the symbol $A$, that is to say, the operator defined for all $f$ and $g$ in $L^2(\mathbb{R}^n)$ by:

\[
< Op_{\hbar}^{AW}(A)f, g > = (2\pi \hbar)^{-n} \int_{\mathbb{R}^{2n}} A(X)f(X) < f, \Psi_X, g > dX
\]

We know that the Weyl symbol of this operator is $e^{\frac{i}{\hbar} \Delta} A$. Consequently, $C_h(e^{\frac{i}{\hbar} \Delta} A, e^{\frac{i}{\hbar} \Delta} B)$ is the Weyl symbol of the product $Op_{\hbar}^{AW}(A) \circ Op_{\hbar}^{AW}(B)$. Moreover, we call Wick symbol of an operator $C$ bounded in $L^2(\mathbb{R}^n)$, the function defined on $\mathbb{R}^{2n}$ by:

\[
\sigma_h^{wick}(C)(X) = < C \Psi_X, \Psi_X > X \in \mathbb{R}^{2n}
\]
If \( C \) is written under the form \( C = \text{Op}_h^{\text{reg}}(F) \), we know that its Wick symbol \( \sigma_h^{\text{wick}}(C) = e^{\frac{i}{\hbar} \Delta} F \). These points imply that:

\[
(2.10) \quad C_h^{\text{reg}}(A, B) = \sigma_h^{\text{wick}}(\text{Op}_h^{\text{AW}}(A) \circ \text{Op}_h^{\text{AW}}(B))
\]

Taking these considerations into account, it appears:

\[
C_h^{\text{reg}}(A, B)(X) = (2\pi\hbar)^{-2n} \int_{\mathbb{R}^{4n}} A(Z_1)B(Z_2) < \Psi_{X, h}, \Psi_{Z_2, h} > < \Psi_{Z_1, h}, \Psi_{X, h} > dZ_1 dZ_2
\]

Then recalling:

\[
(2.11) \quad < \Psi_{X, h}, \Psi_{Y, h} > = e^{-\frac{i}{\hbar}|X-Y|^2} e^{\frac{i}{\hbar} \sigma(X, Y)}
\]

we express \( C_h^{\text{reg}}(A, B) \) as:

\[
C_h^{\text{reg}}(A, B)(X) = (2\pi\hbar)^{-2n} \int_{\mathbb{R}^{4n}} A(X + Y_1)B(X + Y_2) e^{\frac{i}{\hbar} Y_2 \cdot \overline{Y}_1} e^{-\frac{i}{\hbar} (|Y_1|^2 + |Y_2|^2)} dY_1 dY_2
\]

We similarly derive an analogous equality for \( C_h^{\text{reg}, I^c} \) substituting \( \{1, ..., n\} \) to \( I^c \). We then deduce (2.5)-(2.6) using (2.4).

**Proposition 2.2.** For all \( A \) and \( B \) measurable bounded functions on \((\mathbb{R}^2)^I\), the function \( C_h^{\text{reg}, I^c}(A, B) \) is bounded on \((\mathbb{R}^2)^I\) and satisfies:

\[
(2.12) \quad \| C_h^{\text{reg}, I^c}(A, B) \|_{\infty} \leq \| A \|_{\infty} \| B \|_{\infty}
\]

**Proof.** It suffices to give the proof for \( I = \{1, ..., n\} \). In view of (2.8) -(2.10), it is clear that:

\[
|C_h^{\text{reg}}(A, B)(X)| \leq \| \text{Op}_h^{\text{AW}}(A) \|_{\mathcal{L}(L^2(\mathbb{R}^n))} \| \text{Op}_h^{\text{AW}}(B) \|_{\mathcal{L}(L^2(\mathbb{R}^n))}
\]

We also know that \( \| \text{Op}_h^{\text{AW}}(A) \|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \| A \|_{\infty} \). We then deduce (2.12).

**Proposition 2.3.** The following inequality holds:

\[
\| C_h^{\text{hyb}, I^c}(A, B) \|_{\infty} \leq (\pi\hbar)^{-2|I|} N_{I, h}(A) N_{I, h}(B)
\]

where

\[
N_{I, h}(A) = \int_{(\mathbb{R}^2)^I} \| A Y_I \|_{\infty} dY_I
\]

**Proof.** This proposition directly follows from (2.4) and Proposition 2.2.

**3. Decomposition formula.**

For all subsets \( I \) of \( \{1, ..., n\} \), set:

\[
(3.1) \quad T_{I, h} = \prod_{j \in I} (1 - e^{\frac{i}{\hbar} \Delta_j})
\]

\[
\Delta_I = \sum_{j \in I} \Delta_j \quad \Delta_j = \partial^2_{x_j} + \partial^2_{\xi_j}
\]
For every finite subset $E$ of $\{1, \ldots, n\}$, $P_3(E)$ denotes the set of partitions of $I$ into three disjoint subsets. More precisely, an element $(I, J, K)$ of $P_3(E)$ is an ordered sequence of three disjoint subsets of $E$ constituting a partition of $E$, one of them or two of them possibly being empty, or even the three of them if $E$ is itself empty.

**Proposition 3.1.** For all $A$ in $S(M, \varepsilon, \varepsilon)$ and $B$ in $S(M', \varepsilon, \varepsilon)$, we have the following expression

$$
C_h^{weyl}(A, B) = \sum_{E \subseteq \{1, \ldots, n\}} \sum_{(I, J, L) \in P_3(E)} e^{\frac{h}{2}(\Delta_I + \Delta_J + \Delta_L)} T_{I, h} C_h^{hyb, E}(e^{\frac{h}{2}\Delta_J} T_{J, h} A, T_{L, h} B))
$$

**Proof.** We see, similarly to the paper concerning norms ([A-J-N-2]):

$$
C_h^{weyl}(A, B) = \sum_{I \subseteq \{1, \ldots, n\}} e^{\frac{h}{2}\Delta_I} T_{I, h} C_h^{weyl}(A, B)
$$

For each subset $I$ of $\{1, \ldots, n\}$, we also get replacing $\{1, \ldots, n\}$ by $I^c$:

$$
A = \sum_{J \subseteq I^c} e^{\frac{h}{2}\Delta_{(I \cup J)^c}} T_{J, h} A
$$

For every finite subsets $I$ and $J$, we similarly see replacing $\{1, \ldots, n\}$ by $(I \cup J)^c$ that:

$$
B = \sum_{L \subseteq (I \cup J)^c} e^{\frac{h}{2}\Delta_{(I \cup J \cup L)^c}} T_{L, h} B
$$

Combining these three equalities yields:

$$
C_h^{weyl}(A, B) = \sum_{E \subseteq \{1, \ldots, n\}} \sum_{(I, J, L) \in P_3(E)} e^{\frac{h}{2}\Delta_I} T_{I, h} C_h^{weyl}(e^{\frac{h}{2}\Delta_J} T_{J, h} A, e^{\frac{h}{2}\Delta_{(I \cup J \cup L)^c}} T_{L, h} B)
$$

Applying definition (2.2) to each term, where $I$ is replaced by $E = I \cup J \cup L$, we obtain (3.2). \qed

### 4. Proof of Theorem 1.2.

We first begin with the proof of (1.3) when $\rho_j = \delta_j$ for all $j$. These common values are denoted by $\varepsilon_j$.

For every subset $I$ of $\{1, \ldots, n\}$, $M_m(I)$ denotes the set of multi-indices $(\alpha, \beta)$ such that:

$$
\alpha_j = \beta_j = 0 \quad \text{if} \quad j \notin I \quad \quad \alpha_j \leq m \quad \beta_j \leq m \quad \text{if} \quad j \in I
$$

For all functions $F$ in $S(M, \varepsilon, \varepsilon)$, set:

$$
N_{I, h}^{(m)}(F) = \sum_{(\alpha, \beta) \in M_m(I)} h^{(|\alpha| + |\beta|)/2} \left\| \partial_x^\alpha \partial_\xi^\beta F \right\|_{L^\infty(\mathbb{R}^{2n})}
$$

**Lemma 4.1.** There exists $K_0 > 0$ such that, for all $A$ in $S_d(M, \varepsilon, \varepsilon)$ and $B$ in $S_d(M', \varepsilon, \varepsilon)$, for every finite subset $E$ of $\{1, \ldots, n\}$, for all $X$ in $\mathbb{R}^{2n}$:

$$
|C_h^{hyb, E}(A, B)(X)| \leq K_0^{E|N_{E, h}^{(4)}(A) N_{E, h}^{(4)}(B)}
$$

**Proof.** We start from the expressions (2.5)-(2.6). Notice that, for any $j \in E$:

$$
L_j K_{E, h}(Y, Z) = L_j' K_{E, h}(Y, Z) = K_{E, h}(Y, Z)
$$

5
L_j = \left(1 + \frac{1}{h}(\delta_j^2 + \eta_j^2)\right)^{-1} \left(I - \frac{h}{4}(\partial_j^2 + \partial_j^2)\right)
L_j' = \left(1 + \frac{1}{h}(\ell_j^2 + \zeta_j^2)\right)^{-1} \left(I - \frac{h}{4}(\partial_j^2 + \partial_j^2)\right)

This provides:

\[ C_{h}^{\text{hyb}, E}(A, B)(X) = \int_{\mathbb{R}^n} K_{E, h}(Y, Z) \left(\prod_{j \in E} (L_j) (L_j')^2 A(X + Y) B(X + Z)\right) dY dZ \]

We may write:

\[ \left(\prod_{j \in E} (L_j) (L_j')^2 A(X + Y) B(X + Z)\right) = \sum_{(\alpha, \beta, \gamma, \delta) \in M_4(E)} \Phi_{\alpha, \beta, \gamma, \delta} \left(\frac{Y_1}{\sqrt{h}}\right) \Psi_{\alpha, \beta, \gamma, \delta} \left(\frac{Z_1}{\sqrt{h}}\right) \ldots \]

\[ \ldots h^{(|\alpha| + |\beta| + |\gamma| + |\delta|)/2} \left(\partial_\alpha \partial_\beta \partial_\gamma \partial_\delta A\right)(X + Y) \left(\partial_\alpha \partial_\beta \partial_\gamma \partial_\delta B\right)(X + Z) \]

where all the \(\Phi_{\alpha, \beta, \gamma, \delta}\) and \(\Psi_{\alpha, \beta, \gamma, \delta}\) are functions on \((\mathbb{R}^2)^I\) satisfying for some universal constant \(K > 0\):

\[ \int_{(\mathbb{R}^2)^E} |\Phi_{\alpha, \beta, \gamma, \delta}(Y_E)| dY_E \leq K^{|E|} \]

and likewise for the \(\Psi_{\alpha, \beta, \gamma, \delta}\). In particular, for \(X\) fixed:

\[ C_{h}^{\text{hyb}, E}(A, B)(X) = \sum_{(\alpha, \beta, \gamma, \delta) \in M_4(E)} h^{(|\alpha| + |\beta| + |\gamma| + |\delta|)/2} C_{h}^{\text{hyb}, E}(A_{\alpha\beta\gamma\delta,X}, B_{\alpha\beta\gamma\delta,X})(X) \]

with

\[ A_{\alpha\beta\gamma\delta,X} = \Phi_{\alpha, \beta, \gamma, \delta} \left(\frac{Y_1 - X_1}{\sqrt{h}}\right) \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta A \]

and similarly:

\[ B_{\alpha\beta\gamma\delta,X} = \Psi_{\alpha, \beta, \gamma, \delta} \left(\frac{Z_1 - X_1}{\sqrt{h}}\right) \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta B \]

Taking Proposition 2.3 into account, this implies

\[ \| C_{h}^{\text{hyb}, E}(A_{\alpha\beta\gamma\delta,X}, B_{\alpha\beta\gamma\delta,X})\|_\infty \leq (\pi h)^{-2|E|} N_{E, h}(A_{\alpha\beta\gamma\delta,X}) N_{E, h}(B_{\alpha\beta\gamma\delta,X}) \]

Besides,

\[ N_{E, h}(A_{\alpha\beta\gamma\delta,X}) \leq \| \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta A\|_\infty \int_{(\mathbb{R}^2)^E} \left|\Phi_{\alpha, \beta, \gamma, \delta} \left(\frac{Y_1}{\sqrt{h}}\right)\right| dY_1 \]

and similarly for \(B_{\alpha\beta\gamma\delta,X}\). Therefore the proof of (4.2) is completed.

**Proposition 4.2.** There exists a universal constant \(K > 0\) such that, for any \(A\) in \(S_6(M, \varepsilon, \varepsilon)\) and \(B\) in \(S_6(M', \varepsilon, \varepsilon)\), for all \(E \subset \{1, \ldots, n\}\), for all \((I, J, L) \in \mathcal{P}_3(E)\),

\[ \| e^{\frac{4}{h}(\Delta_{j+\Delta_{L}})} T_{I,h} C_{h}^{\text{hyb}, E}(e^{\frac{4}{h}\Delta_{L}} T_{J,h} A, T_{L,h} B))\|_\infty \leq MM'(Kh)|E| \prod_{j \in E} \varepsilon_j^2 \]
Proof. From the definition (3.1) of \(T_{I,h}\) and heat kernel properties, we get:

\[
\| e^{\frac{h}{2}(\Delta + \Delta_L)} T_{I,h} C_h^{\text{hyb},E} \left( e^{\frac{h}{2} \Delta_L} T_{I,h} A, T_{L,h} B \right) \|_\infty \leq (h/4)^{|I|} \left\| \prod_{\sigma \in I} \Delta_{\sigma} \right\| C_h^{\text{hyb},E} \left( e^{\frac{h}{2} \Delta_L} T_{I,h} A, T_{L,h} B \right) \|_\infty
\]

Using (2.5), it is clear that, for every \(i \in I\), for all \(F\) and \(G\):

\[
\Delta_i C_h^{\text{hyb},E} (F, G) = C_h^{\text{hyb},E} (\Delta_i F, G) + 2 C_h^{\text{hyb},E} (\partial_{\xi_i} F, \partial_{\xi_i} G) + 2 C_h^{\text{hyb},E} (\partial_{\xi_i} F, \partial_{\xi_i} G) + C_h^{\text{hyb},E} (F, \Delta_i G)
\]

This yields:

\[(4.4) \prod_{\sigma \in I} \Delta_{\sigma} C_h^{\text{hyb},E} (F, G) = \sum_{(\lambda, \mu, \lambda', \mu') \in \tilde{M}(I)} c_{\lambda, \mu, \lambda', \mu'} C_h^{\text{hyb},E} (\partial^\lambda_{\xi} \partial^\mu_{\xi} F, \partial^\lambda_{\xi} \partial^\mu_{\xi} G)
\]

where \(\tilde{M}(I)\) denotes the set of multi-indices \((\lambda, \mu, \lambda', \mu')\) such that:

\[
\lambda_i = \mu_i = \lambda_i' = \mu_i' = 0 \quad \text{if} \quad i \notin I \\
\lambda_i + \mu_i + \lambda_i' + \mu_i' = 2 \quad \text{if} \quad i \in I
\]

In (4.4), the \(c_{\lambda, \mu, \lambda', \mu'}\) are constants with absolute values smaller or equal than \(2^{|I|}\). According to Lemma 4.1 we deduce that:

\[
\| e^{\frac{h}{2}(\Delta + \Delta_L)} T_{I,h} C_h^{\text{hyb},E} \left( e^{\frac{h}{2} \Delta_L} T_{I,h} A, T_{L,h} B \right) \|_\infty \leq ...
\]

\[
\leq 2^{|I|} K_{1}|E| \sum_{(\lambda, \mu, \lambda', \mu') \in \tilde{M}(I)} N_{\gamma, \delta}^{(4)} (\partial^\lambda_{\xi} \partial^\mu_{\xi} e^{\frac{h}{2} \Delta_L} T_{I,h} A) N_{\gamma, \delta}^{(4)} (\partial^\lambda_{\xi} \partial^\mu_{\xi} T_{I,h} B)
\]

Besides, for all \(A \in S_0(M, \varepsilon, \varepsilon)\), for every disjoint indices \(I, J\) and \(L\) of \(E \subset \{1, ..., n\}\), for each multi-index \((\lambda, \mu, \lambda', \mu')\) in \(\tilde{M}(I)\) and for each multi-index \((\alpha, \beta)\) in \(M_4(E)\), if \(h \varepsilon_j^2 \leq 1\) for all \(j \leq n:\)

\[
j \left( |\lambda| + |\beta| \right) \| \partial_{\xi}^\lambda \partial_{\xi}^\mu \partial_{\xi}^\gamma \partial_{\xi}^\delta e^{\frac{h}{2} \Delta_L} T_{I,h} A \|_\infty \leq M(h/4)^{|J|} \prod_{j \in J} \varepsilon_j^2 \prod_{i \in I} \varepsilon_i^{|\lambda|+|\mu|}
\]

Similarly, if \(B\) is in \(S_0(M', \varepsilon, \varepsilon)\), for each multi-index \((\lambda, \mu, \lambda', \mu')\) in \(\tilde{M}(I)\) and for each multi-index \((\gamma, \delta)\) in \(M_4(E)\)

\[
\| \partial_{\xi}^\gamma \partial_{\xi}^\delta T_{L,h} B \|_\infty \leq M'(h/4)^{|L|} \prod_{j \in J} \varepsilon_j^2 \prod_{i \in I} \varepsilon_i^{|\lambda|+|\mu|}
\]

The numbers of elements of \(\tilde{M}(I)\) (for \(I \subset E\)) and those of \(M_4(E)\) are both being bounded by \(K_{1}|E|\). Thus, we indeed deduce (4.3).

End of the proof of Theorem 1.2. In view of Propositions 3.1 and 4.2, if \(A\) belongs to \(S_0(M', \varepsilon, \varepsilon)\) and \(B\) lies in \(S_6(M', \varepsilon, \varepsilon)\) then:

\[
\| C_h^{\text{wyl}} (A, B) \|_\infty \leq \sum_{E \subset \{1, ..., n\}} \sum_{(I, J, L) \in P_3(E)} MM'(K_4)^{|E|} \prod_{j \in E} \varepsilon_j^2
\]

For every finite subset \(E\) of \(\{1, ..., n\}\), the number of elements of \(P_3(E)\) is \(3!(1 + \sigma_p^2 + \sigma_p^3)\) where \(p = |E|\) and the \(\sigma_p^k\) are the Stirling numbers of second kind. This number of elements is bounded by \(K_{1}|E|\). Consequently:

\[
\| C_h^{\text{wyl}} (A, B) \|_\infty \leq MM' \sum_{E \subset \{1, ..., n\}} (KK_1)^{|E|} \prod_{j \in E} \varepsilon_j^2 = MM' \prod_{1 \leq j \leq n} (1 + KK_1 \varepsilon_j^2) = (1 + KK_1 \varepsilon_j^2)^n
\]
proving the first claim of Theorem 1.2 when \( \rho_j = \delta_j = \varepsilon_j \) for all \( j \). In the general case, we set, for any function \( F \) on \( \mathbb{R}^{2n} \) and for any sequence \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of positive real numbers:

\[
(\delta_{\lambda} F)(x, \xi) = F \left( \frac{\lambda_1 x_1, \ldots, \lambda_n x_n}{\lambda_1, \ldots, \lambda_n} \right)
\]

In particular,

\[
\delta_{\lambda} C^\text{weyl}_n(A, B) = C^\text{weyl}_n(\delta_{\lambda} A, \delta_{\lambda} B)
\]

If \( A \) belongs to \( S_0(M, \rho, \delta) \) and \( B \) is in \( S_6(M', \rho, \delta) \), the two sequences \( (\rho_j) \) and \( (\delta_j) \) being positive, then \( \delta_{\lambda} A \) lies in \( S_0(M, \varepsilon, \varepsilon) \) and \( \delta_{\lambda} B \) in \( S_6(M', \varepsilon, \varepsilon) \), when setting \( \varepsilon_j = \sqrt{\rho_j \delta_j} \) and \( \lambda_j = \sqrt{\delta_j / \rho_j} \). The preceding result applied to \( \delta_{\lambda} A \) and \( \delta_{\lambda} B \) allows to deduce a bound in the supremum norm of \( \delta_{\lambda} C^\text{weyl}_n(A, B) \), which is the same as the one of \( C^\text{weyl}_n(A, B) \). The first claim in Theorem 1.2 is therefore derived if all the \( \rho_j \) and \( \delta_j \) are positive and also proved by continuity if some of them are vanishing. For the second claim in Theorem 1.2, we remark that, if \((\alpha, \beta)\) is a multi-index such that \( \alpha_j \leq m \) and \( \beta_j \leq m \) for all \( j \):

\[
\partial_{\alpha} \partial_{\beta}^2 C^\text{weyl}_n(A, B) = \sum_{\alpha', \alpha'' \leq \alpha} \frac{\alpha!}{\alpha'! \alpha''!} \beta! C^\text{weyl}_n((\partial_{\alpha'} \partial_{\beta'} A, \partial_{\alpha''} \partial_{\beta''} B)
\]

If \( A \) is in \( S_{m+6}(M, \rho, \delta) \) and \( B \) in \( S_{m+6}(M', \rho, \delta) \) then \( \partial_{\alpha'} \partial_{\beta'} A \) is in \( S_6(M \prod \rho_j, \delta_j, \rho, \delta) \) and similarly for the other factor. Inequality (1.3) gives:

\[
\|\partial_{\alpha} \partial_{\beta}^2 C^\text{weyl}_n(A, B)\|_{\infty} \leq \left[ \sum_{\alpha', \alpha'' \leq \alpha} \frac{\alpha!}{\alpha'! \alpha''!} \beta! \right] M\prod \rho_j^\alpha \delta_j^\beta
\]

The above sum is bounded by \( 2^{|S(\alpha)| + |S(\beta)|} \) where \( S(\alpha) \) is the support of \( \alpha \), i.e., the set of indices \( j \) satisfying \( \alpha_j \neq 0 \). We therefore deduce the second claim in Theorem 1.2.

5. Proof of Theorem 1.3.

In this last section, the remainder term of the semiclassical expansion of the Weyl composition is considered.

**Proposition 5.1.** Under the hypotheses of Theorem 1.3, the function \( R_N(\cdot, h) \) defined in (1.4) may be written as:

\[
R_N(X, h) = \frac{Nh^N}{(2i)^N} \sum_{|\alpha|+|\beta|=N} \frac{(-1)^{|\beta|}}{\alpha!\beta!} \int_0^1 (1 - \theta)^{N-1} C^\text{weyl}_n(\partial_{\alpha} \partial_{\beta} A, \partial_{\alpha} \partial_{\beta} B)(X) d\theta
\]

**Proof.** Set:

\[
F_h(A, B, X, \theta) = C^\text{weyl}_n(A, B)(X)
\]

From (1.1), it is clear that:

\[
F_h(A, B, X, \theta) = \int_{\mathbb{R}^{4n}} A(X+Y)B(X+Z)K_h(Y, Z, \theta) dY dZ
\]

\[
K_h(Y, Z, \theta) = (\pi \theta h)^{-2n} e^{-\frac{h}{\theta} \sigma(Y, Z)}
\]

We then verify that:

\[
\frac{\partial K_h}{\partial \theta} = L K_h \quad \frac{L}{2i} = \frac{h}{2i} \sigma(\nabla Y, \nabla Z)
\]
In particular:
\[ \partial^m_{\theta} F_h(A, B, X, \theta) = \int_{\mathbb{R}^4} K_h(Y, Z, \theta) L^m |A(X + Y)B(X + Z)|dYdZ \]
We have \( K_h(\cdot, 0) = \delta_{(0,0)} \). Then, we may write:
\[ C^\text{weyl}(A, B)(X) = \sum_{k=0}^{N-1} \frac{h^k}{(2\pi h)^k k!} \sigma(\nabla Y, \nabla Z)^k [A(X + Y)B(X + Z)] \bigg|_{Y=Z=0} + R_N(X, h) \]
\[ R_N(X, h) = \frac{h^N}{(2\pi h)^N(N-1)!} \int_{\mathbb{R}^4 \times [0,1]} (1 - \theta)^{N-1} K_h(Y, Z, \theta) \sigma(\nabla Y, \nabla Z)^N [A(X + Y)B(X + Z)]d\lambda(Y, Z)d\theta \]
Besides:
\[ \frac{1}{N!} \sigma(\nabla Y, \nabla Z)^N [A(X + Y)B(X + Z)] = \sum_{|\alpha|+|\beta|=N} \frac{(-1)^{|\beta|}}{\alpha!\beta!} [\partial^{\beta}_Y \partial^{\alpha}_Z A(X + Y)] \left[ \partial^{\beta}_Y \partial^{\alpha}_Z B(X + Z) \right] \]
From these considerations, we then deduce (5.1).

**Proof of Theorem 1.3.** If \( A \) belongs to \( S_m(M, \rho, \delta) \), if \( B \) is in \( S_m(M', \rho, \delta) \) \((m \geq 6)\), and if \(|\alpha|+|\beta|=N\), then
\[ \partial^{\beta}_Y \partial^{\alpha}_Z A \in S_{m-N} \left( M \prod_{j \leq n} \rho_j^{\delta_j} \delta^{\alpha_j}, \rho, \delta \right) \]
According to Theorem 1.2, we then deduce that:
\[ C^\text{weyl}_{\theta h} (\partial^{\beta}_Y \partial^{\alpha}_Z A, \partial^{\beta}_Y \partial^{\alpha}_Z B) \in S_{m-N-6} \left( M' \prod_{j \leq n} (\rho_j \delta_j)^{\alpha_j+\beta_j}, 2\rho, 2\delta \right) \]
Noticing that
\[ N \int_0^1 (1 - \theta)^{N-1} d\theta = 1 \]
we obtain:
\[ R_N(\cdot, h) \in S_{m-N-6} \left( M' \frac{h^N}{2N} \sum_{|\alpha|+|\beta|=N} \frac{1}{\alpha!\beta!} \prod_{j \leq n} (\rho_j \delta_j)^{\alpha_j+\beta_j}, 2\rho, 2\delta \right) \]
\[ = S_{m-N-6} \left( M' \frac{h^N}{N!} \left[ \sum_{j=1}^{n} \rho_j \delta_j \right]^N, 2\rho, 2\delta \right) \]
and the proof of Theorem 1.3 is completed. \( \square \)

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