SOME QUANTITATIVE ASPECTS OF FRACTIONAL COMPUTABILITY

ILYA KAPOVICH AND PAUL SCHUPP

Abstract. In this article we apply the ideas of effective Baire category and effective measure theory to study complexity classes of functions which are “fractionally computable” by a partial algorithm. For this purpose it is crucial to specify an allowable effective density, $\delta$, of convergence for a partial algorithm. The set $\mathcal{FC}(\delta)$ consists of all total functions $f : \Sigma^* \to \{0, 1\}$ where $\Sigma$ is a finite alphabet with $|\Sigma| \geq 2$ which are “fractionally computable at density $\delta$”. The space $\mathcal{FC}(\delta)$ is effectively of the second category while any fractional complexity class, defined using $\delta$ and any computable bound $\beta$ with respect to an abstract Blum complexity measure, is effectively meager. A remarkable result of Kautz and Miltersen shows that relative to an algorithmically random oracle $A$, the relativized class $\mathcal{NP}^A$ does not have effective polynomial measure zero in $\mathcal{EA}$, the relativization of strict exponential time. We define the class $\mathcal{UFP}^A$ of all languages which are fractionally decidable in polynomial time at “a uniform rate” by algorithms with an oracle for $A$. We show that this class does have effective polynomial measure zero in $\mathcal{EA}$ for every oracle $A$. Thus relaxing the requirement of polynomial time decidability to hold only for a fraction of possible inputs does not compensate for the power of nondeterminism in the case of random oracles.

1. Introduction

We now know that “worst-case” complexity measures such as polynomial time do not necessarily give a good overall picture of a particular problem or algorithm since it depends on the difficulty of the hardest instances of the problem, and these may be very sparse. The famous classic example of this phenomenon is Dantzig’s Simplex Algorithm for linear programming. The examples of V. Klee and G. Minty [11] showing that the simplex algorithm can be made to take exponential time are very special. A “generic” or “random” linear programming problem is not “special”, and Dantzig’s algorithm works quickly. Indeed, later algorithms which are provably polynomial-time have not replaced the simplex algorithm in practice.

Observations of this type led to the development of average-case complexity by Gurevich [6] and Levin [12]. There are now different approaches to the average-case complexity, but they all require computing the expected

2000 Mathematics Subject Classification. Primary 68Q, Secondary 20P05.

Both authors were supported by the NSF grant DMS-0404991. The first author is also supported by the NSF grant DMS-0603921.
value of the running time of an algorithm with respect to some measure on
the set of inputs. It is often difficult to establish an average-case result since
a basic difficulty of worst-case complexity is still present: one needs a total
algorithm which solves the problem and some upper bound on its worst-case
difficulty.

Kapovich, Myasnikov, Schupp and Shpilrain [7] introduced the notion of
generic-case complexity, which deals with the performance of an algorithm
on “most” inputs and completely ignores what happens on the “sparse” set
of other inputs. They applied the idea to the classic decision problems of
group theory - the word and conjugacy problems - and found that the “linear
programming phenomenon” is extremely widespread there. An important
aspect of generic-case complexity is that it allows us to work with the entire
class of partial computable functions, which is the natural setting of the
general theory of computability, and one can often prove generic-case com-
plexity results about problems where the worst case complexity is unknown.

This paper grew out of our interest in generic-case complexity but here
we are interested in the more general concept of “fractionally computable
at an allowable density $\delta$. “ The basic idea is essentially the same as for
generic-case complexity. However, we do not demand that the fraction of
possible inputs on which a partial algorithm succeeds approaches one, but
only that the algorithm succeeds at the given density $\delta$.

Specific questions about fractional complexity are important in cryptog-
raphy, where one needs the assumption that problems such as calculating
the discrete logarithm are generically difficult. Proposition 6.3 in the book
by Talbot and Welsh [15] states that if there is a polynomial time algorithm
which solves the discrete logarithm problem for a subset $B_p \subseteq \mathbb{Z}_p^*$ where
$|B_p| \geq \epsilon |\mathbb{Z}_p^*|$ then there is a probabilistic algorithm that solves the discrete
logarithm problem in general with expected running time polynomial in $k$
and $1/\epsilon$.

In group theory, subgroups of finite index provide natural examples al-
gorithms with fractional complexity. Suppose that $G$ is a finitely generated
group and $N$ is a normal subgroup of finite index $j$. Let $\psi$ be the natural
homomorphism from $G$ onto $G/N$. Then the algorithm for the word prob-
lem of $G$ which simply consists of answering “no” on input $w$ if $\psi(w) \neq 1$
works on the fraction $1 - \frac{1}{j}$ of inputs. Note that there is no assumption
about the complexity of the word problem for $G$. (The same result holds for
subgroups $H$ which are not normal by using coset diagrams.) There are now
several suggestions for using problems about various groups for the purposes
of cryptography. Fractional computability issues, such as those coming from
subgroups of finite index, may pose difficulties for security.

The main results of this paper are (see Sections 2 and 3 below for precise
definitions):
**Theorem 1.1.** For every Blum complexity measure $\Phi$, for every allowable density $\delta$ and for every effective bound $\beta$, the fractional complexity class $\Phi[\beta, \delta]$ is effectively meager in the space $\mathcal{FC}(\delta)$.

**Theorem 1.2.** For every oracle $A$ the set $\mathcal{UFP}^A$ has effective polynomial-time measure zero with respect to $A$ in $\mathcal{E}^A$.

In Theorem 1.1, $\delta(n)$ is an effective density of convergence for a partial algorithm. Roughly speaking, $\delta(n)$ specifies the fraction of all inputs of length $n$ in which a partial function under consideration is required to be defined. The set $\mathcal{FC}(\delta)$ consists of all total functions $f : \Sigma^* \to \{0, 1\}$ where $\Sigma$ is a finite alphabet with $|\Sigma| \geq 2$ which are “fractionally computable at density $\delta$”. The function $\beta(n)$ is an effectively computable resource bound for some abstract Blum complexity measure (e.g. time). Informally, the class $\Phi[\beta, \delta]$ consists of all partial computable functions that can be computed on a fraction of the inputs of length $n$ which is at least $\delta(n)$ with a resource bound $\beta(n)$. The space $\mathcal{FC}(\delta)$ is effectively of the second category while any fractional complexity class, defined using $\delta$ and resource bound $\beta$, is effectively meager.

In Theorem 1.2, the space $\mathcal{E}^A$ consists of all total functions computable in strict exponential time with an oracle for $A$. The space $\mathcal{UFP}^A$ consists of those functions in $\mathcal{E}^A$ that are partially calculable by partial computable functions that are uniform and are computable in polynomial time. Here a partial function $\phi$ from $\Sigma^*$ to $\{0, 1\}$ is uniform if there exists a positive integer $k$ such that for every $w \in \Sigma^*$ with $|w| \geq k$ there is some $z$ with $|z| \leq k \log |w|$ such that $\phi(wz)$ is defined. Thus being uniform can be viewed as a version of “fractional computability”.

A remarkable result of Kautz and Miltersen [10] shows that for an algorithmically random set $A \subseteq \Sigma^*$ the class $\mathcal{NP}^A$ does not have effective polynomial-time measure zero in $\mathcal{E}^A$. Thus Theorem 1.2 above shows that fractional polynomial-time computability does compensate for the power of nondeterminism.

The main lines of our considerations are directly taken from known results in the theory of effective category and measure. The contribution of this paper consists in showing that such results apply to the study of fractional complexity classes. We are particularly indebted to the book “Computational Complexity: A Quantitative View” by Marius Zimand [18] and the articles by Calude [5] and by Kautz and Miltersen [10].

2. **Allowable Densities and Fractional Computability**

**Convention 2.1.** We fix a finite alphabet $\Sigma$ with $k \geq 2$ letters together with a linear ordering of the letters. As usual, $\Sigma^*$ denotes the set of all words over $\Sigma$. If $w \in \Sigma^*$ then the length, $|w|$, of $w$ is the number of letters in $w$. We denote the empty word by $\lambda$. The canonical or shortlex ordering of $\Sigma^*$ lists words in order of increasing length and within a given length by
the lexicographical order induced by the given alphabetical ordering of $\Sigma$. Thus for $\Sigma = \{a, b\}$ the list is
\[
\lambda, a, b, aa, ab, ba, bb, aaa, \ldots
\]
We take the listing
\[
w_1, w_2, w_3, \ldots
\]
as defining a bijection between $\Sigma^*$ and the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$. Using this bijection, we can consider functions from $\Sigma^*$ to $\{0, 1\}$ as functions from $\mathbb{N}$ to $\{0, 1\}$. In this article $\mathcal{F}$ denotes the set of total functions from $\Sigma^*$ to $\{0, 1\}$.

A language $L$ over $\Sigma$ is a subset of $\Sigma^*$. We can identify a language $L \subseteq \Sigma^*$ with its characteristic function $\chi_L$ where
\[
\chi_L(n) = \begin{cases} 1 & \text{if } w_n \in L, \\ 0 & \text{if } w_n \notin L. \end{cases}
\]
This identification gives a bijection between the set $\mathcal{L}$ of all languages over $\Sigma$ and the set $\mathcal{F}$ and we take these sets as being essentially the same.

A function $f$ from $\Sigma^* = \{w_1, w_2, \ldots, w_n, \ldots\}$ to $\{0, 1\}$ is an infinite sequence $(b_n)$ of 0’s and 1’s. If $f \in \mathcal{F}$ takes the value 1 infinitely often we can regard $f$ as the unique binary expansion of a real number in the half-open unit interval $(0, 1]$ which is not all 0’s from some point onwards.

Suppose that we have a partial algorithm $\Omega$ for a set $S \subseteq \Sigma^*$. In particular, this means that $\Omega$ is correct: If $\Omega$ converges on an input $w$ then $\Omega$ gives the correct answer as to whether or not $w \in S$. We again point out that we completely ignore the performance of $\Omega$ on words not in $S$ and the complexity classes we consider will generally contain functions $f$ which are not computable. Indeed, note that a single partial algorithm $\Omega$ generically computes uncountably many different functions if the set $D$ on which the partial algorithm converges is generic while its complement $\overline{D}$ is infinite. Let $f'$ be the partial function defined by $\Omega$. Then we can choose values on the set $\overline{D}$ in a totally arbitrary way to complete $f'$ to a total function $f$ which is generically computed by the given algorithm.

**Convention 2.2.** We want to fix an effective enumeration $(M_i)$ of all Turing machines with input alphabet $\Sigma$ and with a special output tape consisting of a single square in which a machine can print either 0 or 1. Let $\phi_i$ be the partial function from $\Sigma^*$ to $\{0, 1\}$ which is computed by $M_i$. We write $\phi_i(x) \downarrow$ if $\phi_i$ produces a value on input $x$.

A major concern of [7] was the rate of convergence of a given generic-case algorithm. It turns out that this is not an accident and that a general discussion of fractional complexity classes requires providing an effective density function which specifies a lower bound on how many values must
be defined at a given stage. We now regard the functions in \( \mathcal{F} \) as functions \( f : \mathbb{N} \to \{0,1\} \). We need “the acceptable density so far” to be defined at each input \( n \).

**Definition 2.3.** An *allowable density function* is a computable function \( \delta : \mathbb{N} \to \mathbb{Q} \cap [0,1] \) such that

\[
\liminf_{n \to \infty} \delta(n) > 0.
\]

Given an allowable \( \delta \), if \( \phi_i \) is a p.c. function, we write \( \Delta(\phi_i) \) if the condition

\[
\left| \left\{ m : m \leq n, \phi_i(w_m) \right\} \right| / n \geq \delta(n)
\]

holds for all \( n \geq 1 \).

Note that there is no claim that the predicate \( \Delta \) is computable. We can now precisely define the space \( \mathcal{FC}(\delta) \) of functions which are fractionally computable at density \( \delta \). We assume that an allowable density function \( \delta \) is now fixed.

**Notation 2.4.** If \( \phi \) is a partial function and \( f \in \mathcal{F} \) is a total function, we write \( \phi \sqsubseteq f \) if \( \phi(x) = f(x) \) at all arguments for which \( \phi \) is defined.

**Definition 2.5.** Let \( \delta \) be an allowable density function. We define \( \mathcal{FC}(\delta) \) to be the space of all functions \( f \in \mathcal{F} \) such that there exists a partial computable function \( \phi_i \) such that \( \Delta(\phi_i) \) and \( \phi_i \sqsubseteq f \).

In order to define the appropriate topology on \( \mathcal{FC}(\delta) \) we consider finite sequences \( \tau = (s_1, \ldots, s_n) \) where each \( s_i \) is from the three-letter alphabet \( \{0,1,\bot\} \). The symbol \( \bot \) represents an undefined value. If \( \tau \) has length \( n \) as a sequence we write \( |\tau| = n \). We also write \( \tau(j) \) for \( s_j \).

The set of positions for which \( \tau \) is defined is

\[
def(\tau) = \{ j : j \leq |\tau|, \tau(j) \neq \bot \}.
\]

As for partial functions we write \( \Delta(\tau) \) if \( |\{ j : j \leq l, j \in def(\tau) \}| / l \geq \delta(l) \) for all \( l \leq |\tau| \).

**Definition 2.6.** A finite sequence \( \tau \) is *\( \delta \)-allowable* if \( \tau \) contains at least one defined entry and \( \Delta(\tau) \). Let \( \mathcal{T} \) denote the set of all \( \delta \)-allowable finite sequences. If \( \tau_1 \) and \( \tau_2 \) are allowable sequences we write \( \tau_1 \sqsubseteq \tau_2 \) if \( \tau_2 \) agrees with \( \tau_1 \) at all positions for which \( \tau_1 \) is defined. If \( \tau \in \mathcal{T} \) and \( f \in \mathcal{F} \) we write \( \tau \sqsubseteq f \) if \( f \) agrees with \( \tau \) at all positions at which \( \tau \) is defined.

If \( \tau \in \mathcal{T} \) is an allowable sequence then the *basic neighborhood defined by \( \tau \)* is

\[
N(\tau) = \{ f : f \in \mathcal{FC}(\delta), \tau \sqsubseteq f \}.
\]

Note that if \( \tau_1 \sqsubseteq \tau_2 \) then \( N(\tau_2) \subseteq N(\tau_1) \) since \( \tau_2 \) specifies more information than \( \tau_1 \).
Regarding sequences as words over the three-letter alphabet \( \{ \bot, 0, 1 \} \) we can effectively enumerate all \( \delta \)-allowable sequences as 
\[ \tau_1, \tau_2, \ldots, \tau_n, \ldots \]
by considering all finite sequences over \( \{ \bot, 0, 1 \} \) in the canonical order and successively listing only those sequences which are \( \delta \)-allowable. It is easy to show that the collection \( \{ N(\tau) : \tau \in T \} \) is a system of basic neighborhoods and we use the topology generated by this system.

**Proposition 2.7.** For every \( \tau_i, \tau_j \in T \) with \( N(\tau_i) \cap N(\tau_j) \neq \emptyset \) there exists \( \tau \in T \) with \( N(\tau) \subseteq N(\tau_i) \cap N(\tau_j) \).

**Proof.** Since \( N(\tau_i) \cap N(\tau_j) \neq \emptyset \), it follows that the sequences \( \tau_i \) and \( \tau_j \) agree at all positions where both are defined. Thus the following sequence \( \tau \) of length \( r = \max\{|\tau_i|, |\tau_j|\} \) is well-defined. For \( x \leq r \) let

\[
\tau(x) = \begin{cases} 
\tau_i(x) & \text{if } x \in \text{def}(\tau_i), \\
\tau_j(x) & \text{if } x \in \text{def}(\tau_j), \\
\bot & \text{if } \tau_i(x) = \tau_j(x) = \bot.
\end{cases}
\]

Since \( \tau \) is defined where either of the \( \delta \)-allowable sequences \( \tau_i \) or \( \tau_j \) are defined, the sequence \( \tau \) is \( \delta \)-allowable and \( N(\tau) \subseteq N(\tau_i) \cap N(\tau_j) \) by definition. \( \square \)

Blum\(^3\) gave a very general definition of an abstract complexity measure and we work in that context since the specific nature of the complexity measure is not important.

**Definition 2.8.** A *Blum Complexity Measure* of partially computable functions is a partially computable function \( \Phi(i, x) \) satisfying the following two axioms:

1. \( \Phi(i, x) \downarrow \iff \phi_i(x) \downarrow. \)
2. The cost predicate

\[
\text{Cost}(i, x, y) = \begin{cases} 
1 & \text{if } \Phi(i, x) \leq y, \\
0 & \text{otherwise}.
\end{cases}
\]

is computable.

The standard measures of deterministic time or space are certainly Blum complexity measures. For the remainder of this section we assume that some Blum Complexity Measure \( \Phi \) is fixed.

We can now define fractional complexity classes using the complexity measure \( \Phi \) and the density \( \delta \). Recall that \( \{ \phi_i \}_i \) is an effective enumeration of partial computable functions from \( \Sigma^* \) to \( \{0, 1\} \) and that we think of such functions as being given by Turing machines which can print only the symbols 0 and 1 on their special output tape. We now need to consider functions from \( \Sigma^* \) to \( \{ \bot, 0, 1 \} \) and think that such functions are given by Turing machines which can print 0, 1 or \( \bot \) on their output tape.
Definition 2.9. Let $\beta$ be any total computable function, which we will refer to as the effective bound.

The function $\phi_i$ strictly bounded by $\beta$, which we denote by $\phi_i^{[\beta]}$, is defined as follows. We take the Turing machine $M$ for $\phi_i$ and obtain the Turing machine $M'$ by adding an initial subroutine which, on input $x$, calculates $\text{Cost}(i, x, \beta(x))$. If this value is 0, then either $\Phi(i, x)$ is undefined (and hence $\phi_i(x)$ is undefined) or $\Phi(i, x)$ is defined and the complexity $\Phi(i, x)$ on input $x$ exceeds $\beta(x)$. In either case, if $\text{Cost}(i, x, \beta(x)) = 0$, $M'$ prints the value $\perp$. If $\text{Cost}(i, x, \beta(x)) = 1$ (so that both $\phi_i(x)$ and $\Phi(i, x)$ are defined and, in addition, $\Phi(i, x)$ is bounded by $\beta(x)$), $M'$ prints the value calculated by $M$ on input $x$.

This construction gives us an effective enumeration of all the functions $\phi_i^{[\beta]}$. Note that if these are considered as functions from $\Sigma^*$ to $\{0, 1, \perp\}$ then they are total computable functions. Finally we have

Definition 2.10. The fractional complexity class, $\Phi[\beta, \delta]$, defined by $\Phi$, $\beta$ and $\delta$ is

$$\Phi[\beta, \delta] = \{ f \in FC(\delta) : \exists i[\Delta(\phi_i^{[\beta]}) \text{ and } \phi_i^{[\beta]} \sqsubseteq f] \}$$

We now turn to the notion of effective Baire category. The requirement for a set $S \subseteq FC(\delta)$ to be effectively nowhere dense is that there is a uniform effective method which, when given any basic open neighborhood $N$, produces another basic neighborhood $N' \subseteq N$ such that $S \cap N' = \emptyset$. For a meager set, that is, a countable union of nowhere dense sets, we require that the method be uniform over all the members of the union. Recall that $T$ denotes the set of all $\delta$-allowable finite sequences.

Definition 2.11. A set $X \subseteq FC(\delta)$ is effectively nowhere dense in $FC(\delta)$ if there exists a total computable witness function $\alpha : T \to T$ such that:

1. $\tau \sqsubseteq \alpha(\tau)$ for all $\tau \in T$.
2. $X \cap N(\alpha(\tau)) = \emptyset$.

A set $X \subseteq FC(\delta)$ is effectively meager if there exist a sequence of nowhere dense sets $(X_i)_i$ and a total computable witness function $\alpha : \mathbb{N} \times T \to T$ of two variables such that:

1. $X = \bigcup_{i=1}^{\infty} X_i$
2. $\tau \sqsubseteq \alpha(i, \tau)$ for all $(i, \tau) \in \mathbb{N} \times T$.
3. $X_i \cap N(\alpha(i, \tau)) = \emptyset$.

A set is effectively ample (effectively of the second category) if it is not effectively meager.

It is now easy to prove the desired result that any complexity class $\Phi[\beta, \delta]$ is effectively meager while the entire space $FC(\delta)$ is effectively of the second category. Indeed, we have the following result. (Compare [5].)

Lemma 2.12. For every meager set $X$ and for every $\tau \in T$, there is a total computable function $f \in N(\tau) - X$. 


Proof. Since $X$ is effectively meager we can write $X = \bigcup_{i=1}^{\infty} X_i$ where $X$ is effectively meager via the witness function $\alpha(i, \tau)$. We define a total computable function $f$ iteratively by a simple diagonalization argument. For a given $\tau$ let $\sigma_0$ be the sequence of length $|\tau| + 1$ agreeing with $\tau$ at all places where $\tau$ is defined and having 0 in all places where $\tau$ is undefined, and with 0 as the last entry of the sequence $\sigma_0$. Then $\tau \subseteq \sigma_0$.

Let $\eta_1 = \alpha(1, \sigma_0)$. Let $\sigma_1$ be the sequence of length $|\eta_1| + 1$ which agrees with $\eta_1$ in all places where $\eta_1$ is defined and which has 0 in all places where $\eta_1$ is undefined, and with 0 as the last entry of the sequence $\sigma_1$. So $|\sigma_1| > |\eta_1|$ and all entries in $\sigma_1$ are defined. Since $\eta_1 \subseteq \sigma_1$, $N(\sigma_1) \cap X_1 = \emptyset$.

We continue in the same fashion. Let $\eta_2 = \alpha(2, \sigma_1)$. Let $\sigma_2$ be the sequence of length $|\eta_2| + 1$ which agrees with $\eta_2$ in all places where $\eta_2$ is defined and which has 0 in all places where $\eta_2$ is undefined, and with one more defined position with entry 0 at the end of $\eta_2$. Thus $|\sigma_2| = |\eta_2| + 1$ and all entries in $\sigma_2$ are defined. Since $\sigma_2 \subseteq \eta_2$, we have $N(\sigma_2) \cap X_2 = \emptyset$.

By this process, we iteratively define a sequence $(\sigma_i)_i$ of $\delta$-allowable intervals $\sigma_i$ in which all entries are defined such that $N(\sigma_i) \cap X_i = \emptyset$ and such that $\sigma_i \subseteq \sigma_{i+1}$ and $|\sigma_i| < |\sigma_{i+1}|$ for every $i$. Let $\sigma = \sigma(1), \sigma(2), \ldots$ be the infinite binary sequence such that for every $i$ the initial segment of $\sigma$ of length $|\sigma_i|$ is $\sigma_i$. Note that every initial segment of $\sigma$ is a $\delta$-allowable sequence and that $|\sigma_i| \geq i$ for every $i$.

Consider the function $f$ defined as $f(j) = \sigma(j)$ for every $j$. Clearly, $f$ is a total computable function, since for every $n$ we have $n \leq |\sigma_n|$ and $f(n) = \sigma_n(n)$.

Now $f \in N(\tau)$, since it agrees with $\tau$ at all places where $\tau$ is defined, and $f \notin X_i$ for all $i$.

The theorem immediately yields the following corollary.

**Corollary 2.13.** The set $R$ of total effectively computable functions from $\Sigma^*$ to $\{0, 1\}$ is not meager in the space $FC(\delta)$.

**Theorem 2.14.** For every Blum complexity measure $\Phi$, for every allowable density $\delta$ and for every effective bound $\beta$, the fractional complexity class $\Phi[\beta, \delta]$ is effectively meager in the space $FC(\delta)$.

**Proof.** We have an effective enumeration $(\phi^{[\beta]}_i)_i$ of all strictly $\beta$-bounded partial functions. Let

$$C_i = \begin{cases} 
\{ f \in FC(\delta) : \phi^{[\beta]}_i \sqsubseteq f \} & \text{if } \Delta(\phi^{[\beta]}_i) \\
\emptyset & \text{otherwise.}
\end{cases}$$

It is clear that $\Phi[\beta, \delta] = \bigcup_i C_i$ so we need only specify an effective witness function $\alpha$. Given an index $i$ and a $\delta$-allowable sequence $\tau$ compute $\phi^{[\beta]}_i$ on the first $|\tau|$ inputs in the canonical order. If the computed sequence $\sigma$ of length $|\tau|$ is not $\delta$-allowable then $C_i = \emptyset$ and we set $\alpha(i, \tau) = \tau$. Suppose
now that $\sigma$ is $\delta$-allowable. If $\sigma$ has a defined value $v$ on an input $w_j$ with $v \neq \tau(j)$ again set $\alpha(i, \tau) = \tau$. Suppose now that $\sigma$ is allowable and that for all $j \leq |\tau|$ with a defined value $\sigma(j)$ we have $\sigma(j) = \tau(j)$.

We claim that there exists $r > |\tau|$ such that either $\phi_i^{[\beta]}$ has a defined value $\phi_i^{[\beta]}(r)$ or the sequence $\sigma_r$ of the values of $\phi_i^{[\beta]}$ on the first $r$ inputs is non-allowable. This follows from the assumption $\liminf_{n \to \infty} \delta(n) > 0$ in the definition of an allowable density function and from the definition of a $\delta$-allowable sequence. We continue computing values of $\phi_i^{[\beta]}$ until we find the smallest $r > |\tau|$ with the above property.

If the sequence $\sigma_r$ is not allowable then $C_i = \emptyset$ and we again set $\alpha(i, \tau) = \tau$. If $\sigma_r$ is allowable and the $r$-th entry of $\sigma_r$ is a defined value $v$, we set $\alpha(i, \tau)$ to be the sequence agreeing with $\sigma_r$ at all the positions $j < r$ and having value $1 - v$ at position $r$. In either case we have $\alpha(i, \tau) \cap C_i = \emptyset$. □

Note that, in general, a fractional complexity class $\Phi[\delta, \beta]$ contains uncountably many functions while the nonmeager set $\mathcal{R}$ is countable.

3. Nondeterminism versus fractional polynomial-time computability

It should be expected that partial computability at a fixed density cannot make great inroads into the power of nondeterminism. A nondeterministic machine can guess on every input, while in considering fractional complexity, we still have a deterministic machine which is required to actually do the desired calculation on a non-negligible set of inputs.

Turing himself [17] introduced the idea of Turing machines with an oracle. We think of an oracle Turing machine as a Turing machine with a special hardware slot and any set $A \subseteq \Sigma^*$ can be “plugged into” the slot. The machine has a special query tape and a “branching instruction” in addition to the standard Turing machine instructions. The branching instruction has the form $q_i, \sigma_l \rightarrow q_j, q_k$. It is crucial that all oracle machines are still specified by finite programs of instructions of the two types, so we still have an effective enumeration of all oracle Turing machines. In a Turing machine $M^A$ with an oracle for $A$, an instruction $q_i, \sigma_l \rightarrow q_j, q_k$ works as follows. If the machine $M^A$ is in state $q_i$ reading the symbol $\sigma_l$ on its work tape then the machine goes to state $q_j$ if the word written on the query tape belongs to the set $A$ and goes to state $q_k$ if the word on the query tape is not in the set $A$.

“Classical” results of computability theory “relativize” in the following strong sense. For example, take the proof of the unsolvability of the Halting Problem. Not only the statement of the theorem but the given proof remain correct if one everywhere replaces the words “Turing machine” by the words “Turing machine with an oracle for $A$”. One could take this relativization property as a definition of “classical”.
However, the well-known theorem of Baker, Gill and Solovay, [2] showed that the question of \( \mathcal{P} \) versus \( \mathcal{NP} \) does not relativize. It is easy to construct an oracle \( A \) such that \( \mathcal{P}^A = \mathcal{NP}^A \). Indeed, any set \( A \) which is complete for \( \mathcal{PSPACE} \) will do. But there are many oracles \( A \) for which \( \mathcal{P}^A \neq \mathcal{NP}^A \). Indeed, Bennett and Gill [1] showed that \( \mathcal{P}^A \neq \mathcal{NP}^A \) with respect to a “random” oracle. This means that the set of \( A \) such that \( \mathcal{P}^A \neq \mathcal{NP}^A \) has Lebesgue measure one in the space of all languages over \( \Sigma \).

Later results show that for a random oracle the separation between \( \mathcal{P}^A \) and \( \mathcal{NP}^A \) is indeed very strong. Our approach in this section is inspired by the remarkable result of Kautz and Miltersen [10] which we will explain below.

We use this approach to show that requiring polynomial time computation to succeed only on a “reasonable fraction” of the inputs does not significantly improve our computing power when compared to nondeterminism for “algorithmically random” oracles.

First of all, the ideas of generic-case computability, and indeed fractional computability at an allowable density \( \delta \) certainly relativize without any problem. All definitions are exactly the same except that we now consider Turing machines with an oracle for \( A \).

For this section we work inside the class of functions

\[
\mathcal{E}^A = \bigcup_c \text{DTIME}^A(2^{cn} + c),
\]

computable in strict exponential time by Turing machines with an oracle for \( A \). Note that if we are working with respect to an oracle \( A \) then the elements of \( \mathcal{E}^A \) are total functions \( \Sigma^* \rightarrow \{0, 1\} \).

Effective measure theory was formulated by Lutz [14] building on earlier work of Schnorr [16]. Recall that in discussing \( \mathcal{P} \) and \( \mathcal{NP} \) we are considering sets of languages over an alphabet \( \Sigma \). As mentioned earlier, we identify a language \( L \) with the infinite binary sequence specifying its characteristic function. We have the canonical enumeration of all words \( w_1, w_2, \ldots \) of all words in \( \Sigma^* \). We think of \( L \) as the infinite binary sequence \( L(0), L(1), \ldots \) where \( L(n) = 1 \) if \( w_n \in L \) and \( L(n) = 0 \) otherwise. We use the formulation of effective measure theory in terms of computable martingales, which are strategies for betting on the values of successive bits of an infinite binary sequence. Formally,

**Definition 3.1.** A **martingale** is a function \( d : \{0, 1\}^* \rightarrow \mathbb{R} \) such that for all \( \sigma \in \{0, 1\}^* \)

\[
(*) \quad d(\sigma) = \frac{d(\sigma0) + d(\sigma1)}{2}
\]

and the value \( d(\lambda) \) of \( d \) on the empty word is greater than 0.

The martingale \( d \) **succeeds** on a sequence \( \alpha \in \{0, 1\}^\infty \) if

\[
\limsup_{n \rightarrow \infty} d(\alpha[1, \ldots, n]) = \infty.
\]

where \( \alpha[1, \ldots, n] \) is the initial segment of \( \alpha \) of length \( n \). The martingale \( d \) **succeeds** on a set \( S \subseteq \{0, 1\}^\infty \) if it succeeds on all sequences in \( S \).
We can think that we start with one dollar and double the bet each time, splitting the bet between the two possible next values according to the strategy $d$. We succeed on the set $S$ if we win an infinite amount of money on every sequence in $S$. If we think of $\{0, 1\}^\infty$ as the unit interval one can show that a set $C \subseteq [0, 1]$ has Lebesgue measure 0 if and only there exists some martingale which succeeds on $C$.

For effective measure theory one imposes a condition on the difficulty of computing a martingale. We are interested in martingales which are computable in polynomial time with respect to a fixed oracle $A$.

**Definition 3.2.** An $A$-polynomial-time martingale is a function $d : \{0, 1\}^* \to \mathbb{Q}$ which satisfies the martingale equation (*) and which is computable in polynomial time by some Turing machine with an oracle for the set $A$.

A set $S \subseteq \{0, 1\}^\infty$ has effective polynomial-time measure zero with respect to $A$ if there exists an $A$-polynomial-time martingale $d$ which succeeds on all sequences in $S$. We write “$S$ has effective $P^A$ measure zero”.

Recall that we are working inside a space $E^A$ of functions computable in strict exponential time by Turing machines with an oracle for $A$. The argument given in Zimand [18] relativizes to give:

**Theorem 3.3.** [18] The set $E^A$ does not have effective $P^A$-measure zero.

**Proof.** For every $A$-polynomial time martingale $d$ we define a language $L \in E^A$ on which $d$ does not succeed. The martingale equation $2d(w) = d(w0) + d(w1)$ implies that either $d(w0) \leq d(w)$ or $d(w1) \leq d(w)$. We put the empty word $\lambda$ in $L$ and then iteratively define $L$. If $\sigma = L[1, ..., n]$ has already been defined, then $L[1, ..., n+1] = \sigma 1$ if $d(\sigma 1) \leq d(\sigma)$ and $L[1, ..., n+1] = \sigma 0$ otherwise. It is clear that $d$ does not succeed on $L$ since $d(L[1, ..., n]) \leq d(\lambda)$ for all $n$.

We need only check that $L \in E^A$. Given an arbitrary $w \in \{0, 1\}^*$, with $|w| = n$, we possibly need to calculate $d$ on all words of length of length $(n-1)$. There is a constant $c$ such that on inputs of length $r$, $d$ is calculable in time $r^c + c$ by a Turing machine with an oracle for $A$. Thus the entire calculation can be done in time $2^{n-1}[(n-1)^c]$ so $L \in E^A$. □

In their remarkable article, Kautz and Miltersen [10] use the concept of sets which are “algorithmically random” in the sense of Martin-Lof [13]. The precise details of that definition need not to be given here and the important point for us is that it yields a large class of sets for which the following theorem of Kautz and Miltersen holds.

**Theorem 3.4** (Kautz, Miltersen [10]). If $A \subseteq \Sigma^*$ is an algorithmically random set then the set $NP^A$ does not have effective $P^A$-measure zero in $E^A$.

In order to discuss fractional polynomial time computability we again need to impose a suitable effective density condition which now becomes “uniformity”. 
Definition 3.5. A partial function \( \phi \) from \( \Sigma^* \) to \( \{0, 1\} \) is k-uniform if for all \( w \in \Sigma^* \) with \( |w| \geq k \), there exists a \( z \) with \( |z| \leq k \log(|w|) \) such that \( \phi(wz) \downarrow \). Thus for every \( w \) there is a “reasonably short” \( z \) such that \( \phi_i \) converges on \( wz \).

A partial function \( \phi \) is uniform if it is k-uniform for some positive integer \( k \). We write \( U(\phi) \) if \( \phi \) is uniform.

Note that if we have an algorithm \( \Omega \) which generically solves a decision problem, then for every \( w \) there is some \( z \) such that \( \Omega \) converges on \( wz \). This is because any cylinder \( C = \{wu\} \) consisting of all words with prefix \( w \) is not a negligible set.

Convention 3.6. From now on, we will assume that \( \Sigma = \{0, 1\} \) although all the arguments below work for an arbitrary finite alphabet \( \Sigma \).

In general, a superscript \( A \) for a function, such as \( \phi^A \), indicates that \( \phi^A \) is a partial function computable by a Turing machine with an oracle for \( A \). Similarly, a superscript \( A \) for a Turing machine, such as \( M^A \), indicates that \( M^A \) is a Turing machine with an oracle for \( A \).

Definition 3.7. If \( \phi^A_i \) is a partial computable function, computed by the \( i \)-th Turing machine \( M^A_i \) with an oracle for \( A \), the function \( \phi^A_i[i] \) is the function computed as follows. We modify \( M^A_i \) to a Turing machine \( Q^A_i \) by adding a subroutine to force the the function obtained to be \( i \)-uniform with its computation time bounded by \( n^c \) on inputs \( |w| \) with \( |w| \geq i \), where \( c \) is a constant independent of \( i \) and \( w \).

In detail, on an input \( w \), \( Q^A_i \) prints \( \perp \) if \( |w| < i \).

Suppose now that \( |w| \geq i \). Then \( Q^A_i \) carries out the computation of \( M^A_i \) for \( n^c \) steps. If \( M^A_i \) calculates a value from \( \{0, 1\} \), then \( Q^A_i \) prints that value. If not, \( Q^A_i \) considers, in the canonical order, the extensions \( wz \) with \( |z| \leq i \log(|w|) \) and carries out the computation of \( M^A_i \) on \( wz \) for \( |w|^i \) steps. If \( M^A_i \) calculates a value on such an extension, then the condition that we are calculating a \( i \)-uniform function is verified for the input \( w \) and \( Q^A_i \) outputs the value \( \perp \) for input \( |w| \). If \( M^A_i \) does not calculate a value on any of these extensions, then \( Q^A_i \) outputs the value 0 for input \( w \), again ensuring that the calculated function is \( i \)-uniform.

The number of words \( w \) with \( |z| \leq i \log(|w|) \) is \( 2^i2^{\log(|w|)} = 2^i|w| \). It follows that for every \( w \) with \( |w| \geq i \) the machine \( Q^A_i \) prints a value 0, 1 or \( \perp \) in at most \( |w|^i + |w|^i2^i|w| \) steps. Recall, that if \( |w| \leq i - 1 \), then \( Q^A_i \) prints the value \( \perp \) in the input \( w \). Thus for every \( w \in \Sigma^* \) the machine \( Q^A_i \) computes a value from \( \{0, 1, \perp\} \) on the input \( w \) in \( \leq |w|^ci \) steps where \( c > 0 \) is independent of \( i \) and \( w \).

Since we uniformly effectively obtain \( Q^A_i \) from \( M^A_i \), there is an effective enumeration of all the functions \( \phi^A_i[i] \). Note that since any particular partial computable function has infinitely many indices, for any partial computable function \( \phi^A \) which is \( k \)-uniform for some \( k \) and whose computation time on
inputs for which it calculates a value is bounded by a polynomial, there is a large enough index \( i \) such that \( \phi^A_i[i](w) = \phi^A(w) \) for all inputs with \( |w| \geq i \).

Recall that we are identifying languages with their characteristic functions.

**Definition 3.8.** We consider the set \( \text{UFP}^A \) of all those languages (functions) in \( \mathcal{E}^A \) which are partially calculable by partial computable functions which are uniform with computation time strictly bounded by a polynomial \( n^j \) on some Turing machine with an oracle for \( A \). Formally,

\[
\text{UFP}^A = \{ f : f \in \mathcal{E}^A, \text{ and there is some } i \text{ such that } \phi^A_i[i] \sqsubseteq f \}
\]

For the next theorem we essentially use the proof in section 3.4 of Zimand [18] that polynomial time \( P \) has effective polynomial-time measure zero, noting that it applies to \( \text{UFP}^A \).

**Theorem 3.9.** For every oracle \( A \), the set \( \text{UFP}^A \) has effective \( P^A \)-measure zero in \( \mathcal{E}^A \).

**Proof.** First of all, as noted above, we can give an effective enumeration \( \{ Q^A_i \} \) of all Turing machines with an oracle for \( A \) such that \( Q^A_i \) calculates \( \phi^A_i[i] \). This means that we have one Turing machine \( Q^A_i(i, w) \) such that for every \( w \in \Sigma^* \) \( Q^A_i(i, w) \) simulates \( Q^A_i \) on input \( w \) in time bounded by \( (\log i)^{c_1} |w|^{c_2} \), where \( c_1 > 0, c_2 > 0 \) are constants independent of \( i \) and \( |w| \).

Let \( S_i \) be the set of functions

\[
S_i = \{ f \in \mathcal{E}^A : \phi^A_i[i] \sqsubseteq f \}.
\]

Then \( \text{UFP}^A = \bigcup_{i \in \mathbb{N}} S_i \). We define a martingale which succeeds on \( \text{UFP}^A \) in three stages.

First, we need to define a martingale \( d_i \) which succeeds on the set \( S_i \). We use the variable \( x \) to denote arguments to a martingale. Since a martingale is betting on characteristic sequences of languages, the position \( x(n) \) is supposed to tell us the value \( f(w_n) \) for the functions \( f \) in \( S_i \). It is important to keep in mind that \( |w_n| \leq \log n \). (By \( \log n \) we mean \( \lfloor \log_2 n \rfloor \).)

Let \( x \in \Sigma^* \) and let \( n = |x| \).

If \( |w_n| < i - 1 \), we put \( d_i(x) = 1 \).

Suppose that \( |w_n| \geq i - 1 \). We set:

\[
d_i(x0) := \begin{cases} 
2d_i(x) & \text{if } \phi^A_i[i](w_{n+1}) = 0, \\
0 & \text{if } \phi^A_i[i](w_{n+1}) = 1, \\
d_i(x) & \text{if } \phi^A_i[i](w_{n+1}) = \perp,
\end{cases}
\]

and

\[
d_i(x1) := \begin{cases} 
0 & \text{if } \phi^A_i[i](w_{n+1}) = 0, \\
2d_i(x) & \text{if } \phi^A_i[i](w_{n+1}) = 1, \\
d_i(x) & \text{if } \phi^A_i[i](w_{n+1}) = \perp.
\end{cases}
\]
It is easy to see that $d_i$ is a martingale.

Suppose that $j_1 < \cdots < j_m$ are indices such that $|w_{j_s}| \geq i$ and $\phi^A_i[i](w_{j_s})$ is defined for $s = 1, \ldots, m$. Then for any $x \in \Sigma^*$ with $|x| = j_m$ such that $\phi^A_i[i][1, \ldots, j_m] \subseteq x$ we have

$$d_i(x) = d_i(x(1) \ldots x(j_1) \ldots x(j_2) \ldots x(j_m)) \geq 2^m.$$  

Hence $d_i$ succeeds on $S_i$. There is a Turing machine $D^A(i, x)$ with an oracle for $A$, which, given $i$ and $x \in \{0, 1\}^*$ with $|x| = n$, computes $d_i(x)$ in time bounded by

$$n(\log i)^{c_1}(\log n)^{c_2 i}.$$  

If $|x| = n \geq i$, this time is at most

$$n(\log n)^{c_1}(\log n)^{c_2 i} \leq n(\log n)^{c_3 i}$$

where $c_3 > 0$ is independent of $i, n$. Similarly, if $n \geq |x|$ and $n \geq i$ then $d_i(x)$ is computed in time bounded by the estimate $(\dagger)$.

Second, in order to obtain a global martingale which is calculable in polynomial time we need to exponentially inflate indices. Let $\bar{S}_{2^{2i}} = S_i$, and let $\bar{S}_j = \emptyset$ if $j$ does not have the form $2^{2^i}$.

Let $\bar{d}_j$ be the constant martingale assigning 1 to all inputs if $S_j = \emptyset$. Let $\bar{d}_j = d_i$ if $j = 2^{2^i}$. In the this case, $\bar{d}_j(x)$ can be calculated in time

$$\leq n(\log n)^{c_3 \log(\log j)}$$

for $|x| = n \geq j$.

We now need the inequality

$$(\log n)^{\log \log j} \leq n$$

for $n \geq j \geq 2$.

Since log is an increasing function, if we fix $n$, it suffices to prove the inequality for $j = n$. Taking logs of both sides of the inequality and setting $j = n$ we need

$$(\log(\log n))^2 \leq \log n$$

which holds for $n \geq 2$.

The inequalities $(\dagger)$ and $(\ddagger)$ imply that $\bar{d}_j(x)$ can be calculated for $|x| = n \geq j$ in time $n^{c_4}$, where $c_4$ is independent of $j, n$. The same is true if $n \geq |x|$ and $n \geq j$.

Third, we now need to define another martingale $\tilde{d}_j$ which dampens $\bar{d}$.

If $x \in \{0, 1\}^*$ is nonempty, let $\text{pref}(x)$ denote the prefix of $x$ of length $|x| - 1$. Let $\delta_j(x)$ be defined for nonempty $x$ by $\tilde{d}_j(x) = \delta_j(x)\bar{d}_j(\text{pref}(x))$, provided $\bar{d}_j(\text{pref}(x)) \neq 0$. If $\bar{d}_j(\text{pref}(x)) = 0$, we put $\delta_j(x) = 1$.

Note that if $\tilde{d}_j(\text{pref}(x)) = 0$ then $\tilde{d}_j(x) = 0$ by the martingale equation for $\bar{d}_j$. Thus in this case we also have $\tilde{d}_j(x) = \delta_j(x)\tilde{d}_j(\text{pref}(x))$.

Note also that $\delta_j$ takes values in $\{0, 1, 2\}$.

We set

$$\tilde{d}_j(x) = \begin{cases} 
\text{the constant value } 2^{-j} & \text{if } |x| < j \\
\delta_j(x)\bar{d}_j(\text{pref}(x)) & \text{if } |x| \geq j 
\end{cases}$$
From the martingale equation for $\tilde{d}_j$ we have:

$$2\tilde{d}_j(x) = \tilde{d}_j(x_0) + \tilde{d}_j(x_1) = [\delta_j(x_0) + \delta_j(x_1)]\tilde{d}_j(x),$$

and so $\delta_j(x_0) + \delta_j(x_1) = 2$ for all $x$. Thus for $|x| \geq j - 1$ we have

$$\tilde{d}_j(x_0) + \tilde{d}_j(x_1) = \delta_j(x)\tilde{d}_j(x) + \delta_j(x)\tilde{d}_j(x) = 2\tilde{d}_j(x).$$

Similarly, it follows from the definition that for $|x| < j - 1$ we have $\tilde{d}_j(x_0) + \tilde{d}_j(x_1) = 2\tilde{d}_j(x)$. Thus $\tilde{d}_j$ satisfies the martingale equation.

It is easy to see that $\tilde{d}_j$ succeeds on $\tilde{S}_j$. To calculate $\tilde{d}_j$ we need to compute $\tilde{d}_j(y)$ and $\delta_j(y)$ on the prefixes $y$ of $x$ and this can be done in time $n^{c_5}$ on inputs $x$ with $|x| = n \geq j$.

We put these martingales together in the “global” martingale

$$\tilde{d}(x) = \sum_{j=1}^{\infty} \tilde{d}_j(x)$$

$$= \sum_{j=1}^{|x|} \tilde{d}_j(x) + \sum_{j=|x|+1}^{\infty} 2^{-j}$$

$$= \sum_{j=1}^{|x|} \tilde{d}_j(x) + 2^{-|x|}$$

Then $\tilde{d}$ is a martingale which is calculable in polynomial time by a Turing machine with an oracle for $A$. For each $j$ and $x$ with $|x| \geq j$ we have $\tilde{d}(x) > \tilde{d}_j(x)$, so since $\tilde{d}_j(x)$ succeeds on $\tilde{S}_j$ then $\tilde{d}$ succeeds on $\tilde{S}_j$. This implies that $\tilde{d}$ succeeds on $UFP^A = \cup_j \tilde{S}_j = \cup_i S_i$ and hence $UFP^A$ has effective $PA$-measure zero, as claimed.

\[\square\]

**Corollary 3.10.** We have:

$$NP^A - UFP^A \neq \emptyset.$$

Thus partial complexity cannot compensate for nondeterminism in the presence of a random oracle and it is reasonable to suppose that some similar separation remains true without an oracle. For example, let $GP$ be the class of languages which are generically decidable in polynomial time. The assumption that $NP - GP \neq \emptyset$, would say that there are languages in $NP$ which require nondeterminism on a nonnegligible set of inputs and is certainly a stronger hypothesis than just assuming that $NP \neq P$. It would be interesting to investigate the question of whether such a quantitative hypothesis yields stronger consequences.

**References**

[1] C. Bennett and J. Gill, *Relative to a random oracle $A$, $P^A \neq NP^A \neq co-NP^A$*, SIAM Journal on Computing, 10, (1981), 96-113.
[2] T. Baker, J. Gill and R. Solovay, *Relativizations of the P = NP question*, SIAM Journal on Computing, 4, (1975), 431-442.

[3] M. Blum, *A machine-independent theory of the complexity of recursive functions*, Journal of the ACM, 14, (1962), 322-336.

[4] W.W. Boone, *The word problem*, Annals of Math. 68, (1959), 207-265.

[5] C. Calude, *Topological size of sets of partial recursive functions*, Z. Math. Logik Grundlag. Math., 28. (1982), 455-462.

[6] Y. Gurevich, *Average case completeness*, Journal of Computer and Systems Sciences, 42, (1991), 346-398.

[7] I. Kapovich, A. Myasnikov, P. Schupp and V. Shpilrain, *Generic-case complexity*, *Decision problems in group theory and Random walks*, J. Algebra 264 (2003), no. 2, 665–694.

[8] I. Kapovich, A. Myasnikov, P. Schupp and V. Shpilrain, *Average-case complexity for the word and membership problems in group theory*, Advances in Mathematics 190 (2005), no. 2, 343–359.

[9] I. Kapovich, P. Schupp and V. Shpilrain, *Generic properties of Whitehead’s Algorithm and isomorphism rigidity of random one-relator groups*, Pacific J. Math. 223 (2006), no. 1, 113–140.

[10] S. Kautz and P. Miltersen, *Relative to a random oracle, NP is not small*, in Proceedings of the Nine Structure in Complexity Conference, IEEE Press, (1994), 162-174.

[11] V. Klee and G. Minty, *How good is the simplex algorithm?* Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin), pp. 159–175. Academic Press, New York, 1972.

[12] L. Levin, *Average case complete problems*, SIAM Journal of Computing, 15, (1986). 285-286.

[13] P. Martin-Löf, *The definition of random sequences*, Information and Control, 9, (1962), 602-619.

[14] J. Lutz, *Category and measure in complexity theory*, SIAM Journal on Computing, 19, (1990), 1100-1131.

[15] J. Talbot and D Welsh, *Complexity and Cryptography*, Cambridge University Press, 2006.

[16] C. Schnorr, *Zufälligkeit und Wahrscheinlichkeit*, Springer Lecture Notes in Computer Science, (218), 1971

[17] A. Turing, *Systems of logic based on ordinals*, Proc. London Math. Soc. 45 (1939), 161-228.

[18] M. Zimand, *Computational Complexity: A Quantitative Perspect*, North Holland Mathematics Studies series, 196. Elsevier, Amsterdam, New York.

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA

http://www.math.uiuc.edu/~kapovich/

E-mail address: kapovich@math.uiuc.edu

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA

http://www.math.uiuc.edu/People/schupp.html

E-mail address: schupp@math.uiuc.edu