Reinforcing the Resilience of Complex Networks

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Given a connected network, it can be augmented by applying a growing strategy (e.g. random or scale-free rules) over the previously existing structure. Another approach for augmentation, recently introduced, involves incorporating a direct edge between any two nodes which are found to be connected through at least one self-avoiding path of length L. This work investigates the resilience of random and scale-free models augmented by using the three schemes identified above. Considering random and scale-free networks, their giant cluster are identified and reinforced, then the resilience of the resulting networks with respect to highest degree node attack is quantified through simulations. The results, which indicate that substantial reinforcement of the resilience of complex networks can be achieved by the expansions, also confirm the superior robustness of the random expansion.

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I. INTRODUCTION

A great deal has been learnt about several aspects of complex networks by looking at such models from different theoretical and practical points of view, such as network growth and critical phenomena (e.g. [1, 2]), node degree distribution (e.g. [3]), distance between nodes (e.g. [4]), diffusion (e.g. [5]), and resilience to attack, to name but a few. As each of these situations drives the researcher to focus attention on specific topological and functional aspects of the investigated networks, they contribute to a more comprehensive and integrated understanding of the many complexities of networks.

The current work addresses the resilience issue by taking into account three important issues frequently disregarded in the literature. First, we target the situation where one wants to enhance an already existing network with respect to attacks by adding a specific number of edges; second, we consider the abrupt change of rules during the network growth, producing hybrid models; and third, we investigate the potential of the recently introduced concept of L-expansion of a network for enhancing resilience.

Because of its immediate practical consequences to Internet and distributed systems, the problem of characterizing the resilience of complex networks has received growing attention, especially after the seminal papers by Albert et al. [6], who addressed node deletion in scale-free models of Internet, and Callaway et al.’s [7] investigation on random networks under attack. Other related works include Holme et al.’s [8] comprehensive comparative investigation of the resilience of several types of networks considering different schemes for attacking nodes and edges, and Cohen et al.’s analysis of internet breakdown [9]. Works targeting specific types of network include, but are not limited to, Newman’s investigation of e-mail networks [10], Jeong at al.’s study of metabolic systems [11], and Dunne’s analysis of food webs [12].

Recently, the concept of L-expansion of a complex network was suggested which, by enhancing the network connectivity, was believed to present good potential for increasing the resilience of existing networks.

This paper starts by reviewing the concept of L-expansions and augmentations of a network and follows by presenting the comparison framework, the obtained results, and respective discussion.

II. L-AUGMENTATIONS OF A NETWORK

Recently introduced, the concept of L-expansion of a given network (any type, directed or not) seems to provide good potential for reinforcing the connectivity regularity of existing networks, with implications for resilience. Given a graph $\Gamma$, its $L$-expansion consist of a graph where connections from node $i$ to $j$ are established whenever there exists a self-avoiding path (i.e. never passing by the same node twice) of length $L$ connecting $i$ to $j$ in $\Gamma$. Here we introduce the concept of L-augmented network in order to express networks obtained by the union of the original graph with its respective $L$-expanded model. This simple concept is illustrated in Figure $\text{1}$ which shows an original network (a) and its respective 2-, 3- and 4-augmented versions. Higher order expansions, which imply an exceedingly high number of additional edges, are not considered in the current work. It is interesting to observe that these augmentations reinforce the regularity of the network up to $L$ length. An important global measurement of the effect of the augmentation on the network connectivity is the ratio between the number of connections in the augmented and original networks, henceforth represented as $\rho$ and denominated augmentation ratio.
by an augmentation of the same type are equivalent to
\( \Pi \rightarrow L - \text{SF} \) (iv) scale-free followed by random — i.e. \( \Pi \rightarrow L - \text{SF} \), for the sake of a fair comparison of the models, all networks derived from the initial connected graph \( \Gamma \) have the same number of nodes and edges. More specifically, the procedure for generating the hybrid augmented models starts by growing a network of type \( \alpha \) up to \( P \) nodes, and the giant cluster is identified, containing \( N_0 \leq P \) nodes and \( E_0 \) connections. The number of nodes is determined as \( P = 2i/N_0/(N_0 - 1) \), for \( i = 1, 2, 3 \). This connected network acts as the original network \( \Gamma \), which is subsequently augmented by \( \Delta E \) new edges according to the model \( \beta \). Thus the resulting network contains \( N \) nodes and \( E = E_0 + \Delta E \) edges, so that \( \rho = E/E_0 \). The resilience of the hybrid models was quantified by considering the giant cluster of size \( M(n) \) obtained after removing an increasing number \( n \) of nodes. Although some analyses were performed with edge removal (see Section Results and Discussion), the present work concentrates on the highest degree node removal. All simulations adopted \( N_0 = 50 \) and were carried out for 100 realizations of each configuration.

### TABLE I:
The values of the ratio \( \rho \) for the six hybrid models considered in this work.

| Model       | \( i = 1 \)          | \( i = 2 \)          | \( i = 3 \)          |
|-------------|----------------------|----------------------|----------------------|
| Random      | \( L = 3 \)          | \( L = 4 \)          | \( L = 4 \)          |
|             | \( 4.24 \pm 0.89 \)   | \( 5.47 \pm 0.56 \)   | \( 7.72 \pm 0.42 \)   |
|             | \( 8.54 \pm 1.01 \)   | \( 11.82 \pm 0.53 \)  |
| Sc.-Free    | \( L = 3 \)          | \( L = 4 \)          | \( L = 4 \)          |
|             | \( 4.51 \pm 0.66 \)   | \( 6.91 \pm 0.58 \)   | \( 7.84 \pm 0.62 \)   |
|             | \( 5.97 \pm 1.13 \)   | \( 9.49 \pm 1.00 \)   | \( 9.50 \pm 1.02 \)   |

### III. HYBRID NETWORKS

The \( N \) nodes of the network of interest \( \Gamma \) are henceforth represented as \( A \) and the \( E \) edges as ordered pairs \((i, j)\), with the respective adjacency matrix being expressed as \( A \). No self-connections are allowed. Given a network \( \Gamma_\alpha \) of a specific type \( \alpha \) (e.g. random or scale-free), its augmentation (see, e.g., \( \Pi \rightarrow L - \text{SF} \)) \( \Pi_\alpha(\Gamma_\beta) \) can be obtained by applying the growing rules of any other model type \( \beta \) over the existing network \( \Gamma \), in order to add an additional number of edges \( \Delta E \) without changing the number of nodes. Thus, we can have a random model augmented by the scale-free, or a scale-free network followed by an \( L \)-expansion. Such combinations of growing schemes are henceforth called hybrid augmentation, of which the current work considers the six following situations:

(i) random followed by random — i.e. \( \Pi_R(\Gamma_R) \);
(ii) random followed by scale-free — i.e. \( \Pi_{SF}(\Gamma_R) \);
(iii) random followed by its \( L \)-expansion — i.e. \( \Pi_{L E}(\Gamma_R) \);
(iv) scale-free followed by random — i.e. \( \Pi_R(\Gamma_{SF}) \);
(v) scale-free followed by scale-free — i.e. \( \Pi_{SF}(\Gamma_{SF}) \);
(vi) scale-free followed by its \( L \)-expansion — i.e. \( \Pi_{L E}(\Gamma_{SF}) \).

Observe that the two cases where a model is followed by an augmentation of the same type are equivalent to considering a single network of the same type containing the same number of nodes and edges as in the other cases. It is interesting to observe that the augmentation of a network where \( \alpha \neq \beta \) typically is not commutative, i.e. generally \( \Pi_\alpha(\Gamma_\beta) \neq \Pi_\beta(\Gamma_\alpha) \). For the sake of a fair comparison of the models, all networks derived from the initial network \( \Gamma \) have the same number of nodes and edges. More specifically, the procedure for generating the hybrid augmented models starts by growing a network of type \( \alpha \) up to \( P \) nodes, and the giant cluster is identified, containing \( N_0 \leq P \) nodes and \( E_0 \) connections. The number of nodes is determined as \( P = 2i/N_0/(N_0 - 1) \), for \( i = 1, 2, 3 \). This connected network acts as the initial network \( \Gamma \), which is subsequently augmented by \( \Delta E \) new edges according to the model \( \beta \). Thus the resulting network contains \( N \) nodes and \( E = E_0 + \Delta E \) edges, so that \( \rho = E/E_0 \). The resilience of the hybrid models was quantified by considering the giant cluster of size \( M(n) \) obtained after removing an increasing number \( n \) of nodes. Although some analyses were performed with edge removal (see Section Results and Discussion), the present work concentrates on the highest degree node removal. All simulations adopted \( N_0 = 50 \) and were carried out for 100 realizations of each configuration.

### IV. RESULTS AND DISCUSSION

The numbers of remaining nodes in the network under attack, after \( n \) removals of the nodes with maximum degree, are shown in Figures 2 and 3 for random and scale-free initial networks, respectively, and the ratios \( \rho \) are shown in Table 1. The growing models are identified by the curve marks (see respective captions). The effect of increasing values of \( i \) and, to a lesser extent of \( L \), on the resilience is promptly observed from these two figures. In other words, the addition of \( \Delta E \) edges, quantified by the augmentation ratios in Table 1 contributed to substantially reinforcing the network structure and resilience to node attacks. The choice of scale-free model for the initial network tended to equalize the resilience provided by the three subsequent augmentation schemes. The best resilience was observed for the situation involving random initial network adopting \( i = 3 \) and \( L = 4 \) (see Figure 3(f)), with the networking breakdown occurring only after \( n/N_0 > 0.7 \). As expected, the random
augmentation allowed the best resilience reinforcement in all situations. The $L$-augmentation performed poorly, though presenting performance superior to the scale-free model at the very last stages of the attacks in some situations — i.e. $A$ b-e) and $A$ c). Except for these cases, the scale-free type of augmentation presented intermediate performance. Another interesting aspect is that the use of a scale-free network as the initial structure implies the slope of the decaying giant cluster size to be smaller (in absolute value) than that of the networks augmented from random kernels. An inclination of almost $-1$ (the highest possible, corresponding to a network where every node is connected to every node — i.e. complete graph) is obtained for large $i$ and $L$ after starting from the random model, irrespectively to the augmentation scheme. Investigations with larger values of $N_0$ tended to produce similar results, and simulations performed considering random edge attack seemed to indicate little difference between the augmentation schemes.

V. CONCLUDING REMARKS

This paper has investigated the resilience of six hybrid network models obtained through the process of augmenting an initial, connected network. This situation presents interest not only for its theoretical implications, but also because of practical concerns while trying to enhance the design of specific network systems in order to suit fault-tolerance specifications. The six network models included the random and scale-free traditional networks augmented by random, scale-free and $L$-augmentations (for $L = 3$ and 4). The obtained results revealed some interesting aspects. First, the augmentation of the initial giant cluster was observed to substantially enhance the resilience of the final network, at the expense of a larger number of edges. Second, the random model confirmed its superiority regarding the highest degree nodes attack, with the scale-free networks coming second, except at the very last stages of the attack sequence, where the $L$-augmentations tended to provide behaviour similar to that of the random networks. Another interesting result is the fact that the initial model (type and growth parameters) determines to a great extent the possibilities for reinforcing the initial network. The obtained results corroborated the resilience superiority of fully random connection when used both as the initial and as the augmentation model.

The problem of reinforcement can be understood as a specific situation of a broader class of problems where one wants to redesign or adapt a given network in order to obtain specific topological or functional properties. Such a situation could arise in several contexts, for instance in internetworking, electronic circuits (analogic or digital), and also biology. Regarding this latter situation, a particularly interesting case is the exposure of existing biological networks — including metabolic, protein, food chain and ecologic — to abrupt environmental variations of the geographical, environmental and meteorologic conditions that permeate the evolutionary process.

Future works may consider the evaluation of the performance of other hybrid systems, such as those obtained by union of two distinct models (e.g. $16$), progressive modification of the growing scheme (e.g. $17$), or even successive alternations of the augmentation schemes. Another interesting further work is to devise modifications of the $L$-augmentation scheme where augmentations are not applied indiscriminately over all nodes, but at random or selectively to specially critical nodes (e.g. those with low degree or betweeness centrality). Actually, such a line of reasoning inevitably leads to the following question:

* Given an existing network, how to identify the optimal augmentation scheme, i.e. that leading to the best overall resilience at the expense of the smallest number of additional edges?

Although this type of problem has been well-developed in the context of traditional graphs (e.g. $15$), it would be interesting to revisit it by considering the new concepts and results from complex network research. Another related question is:

* Given an augmented network, how to identify the initial and/or expanding models?

For instance, the concept of $L$-conditional expansion $\bar{R}$ can be used to identify the regular connections implied by $L$-augmentations. In addition, it is worth observing that, in addition to enhancing the connectivity of the initial network, $L$-augmentation are also likely to promote higher regularity of node degree at the scale defined by $L$. Possible means to identify augmentation schemes leading to high (or maximum) resilience is to use simulated annealing or the genetic algorithm.

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FIG. 2: The number of nodes for the cases (i)-(iii), \( i = 1, 2 \) and 3 and \( L = 3, 4 \), where filled \( \circ \) initial network, \( + = R \), \( \times = SF \) and \( \diamond = LE \) indicate the augmentation model.

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FIG. 3: The number of nodes for the cases (iv)-(vi), $i = 1, 2$ and $3$ and $L = 3, 4$, where filled $\diamond$ initial network, $\times$=SF and $\circ$=LE indicate the augmentation model.