Plasma equilibrium and stability in a current-carrying conductor vicinity

K V Brushlinskii1,2,4 and E V Stepin1,2,3,5

1 Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Miusskaya sq. 4, 125047 Moscow, Russia
2 National Research Nuclear University MEPhI, Kashirskoe shosse, 31, 115409 Moscow, Russia
3 Author to whom any correspondence should be addressed
4 brush@keldysh.ru
5 eugene.v.stepin@gmail.com

Abstract. Our paper is connected with a cycle of works on mathematical modeling of plasma confinement processes in the magnetic traps – Galateas. A magnetic field confining plasma is created in these traps by current-carrying conductors which are immersed in the plasma volume but not in contact with the hot dense plasma. The idealized models of strictly equilibrium configurations of infinitely conductive plasma in traps possessing symmetry are based on two-dimensional boundary value problems with the scalar Grad-Shafranov equation for the magnetic flux function. Magnetic traps have a perspective in the controlled thermonuclear fusion problem if plasma configurations in them are stable. In our former publications, a question about stability of the configurations around a straight conductor was raised and particularly solved. Here these studies are continued. An area of maximal pressure and a section of its decrease near the outer boundary are included into the considered vicinity. The main result is that the instabilities observed here can be weakened if the plasma pressure slowly decreases near the trap outer boundary.

1. Introduction
Magnetic traps for plasma confinement by a magnetic field are one of the main objects of a research program in the field of controlled thermonuclear fusion. The promising trap class is represented by traps in which current-carrying conductors creating a magnetic field are immersed in the plasma volume but at the same time not in contact with the hot dense plasma. Attention to them was drawn by A.I. Morozov who called these traps “Galateas” and initiated a meaningful cycle of their first development and studies presented, for example, in the review [1] with an extensive bibliography. A significant role in such investigations is played by mathematical models and calculations of equilibrium magnetoplasma configurations in traps. The apparatus of modeling in terms of the continuum mechanics approach uses the magnetohydrodynamics (MHD) equations. Computation of basic properties of toroidal traps often is carried out, for simplicity, for their straightened cylindrical counterparts and then clarified by amendment imposition [2–5]. Traps and plasma configurations in them possessing symmetry are studied, as a rule, by using two-dimensional MHD models. Strictly equilibrium two-dimensional
configurations are investigated in terms of boundary value problems with the Grad-Shafranov equation [6, 7] for the magnetic flux function $\psi$. Their strict statement requires additional determination of the functions $p(\psi)$ and $I(\psi)$ describing a desired distribution of plasma pressure and electric current density between magnetic surfaces. Formation of particular quasi-equilibrium configurations is studied in terms of time-dependent MHD problems taking into account a finite plasma conductivity. Isolation of conductors from the plasma is provided by the electric current increase in them at the initial stage of the formation process [8–10].

Equilibrium plasma configurations in magnetic traps are of interest primarily in connection with the prospect to realize a controlled thermonuclear fusion reaction. Hence the requirement for their stability which became a subject of numerous works in modern plasma physics follows from it. It is generally accepted that investigation of strictly equilibrium configurations is carried out in the linear approximation [11–13]. It generally deals with cumbersome spectral analysis of three-dimensional differential operators. In problems allowing symmetry of the configurations, this analysis becomes simpler since coefficients of linear equations for three-dimensional perturbations depend on a lower number of variables. Interim information about the conditions promoting stability can be associated with an iterative method of boundary value problems with the Grad-Shafranov equation solving. If the iterations converge, the solution is stable relative to magnetic field perturbations of the same dimension. This stability type called “diffusion stability” [14] is necessary but not sufficient in the general case. However, it poses uncomplicated quantitative information connected with a tendency to stability and therefore can be of interest.

The above-mentioned approaches to the stability study are relatively simple realized in one-dimensional problems about cylindrical plasma configurations surrounding a single straight current conductor but not in contact with it. On the one hand, they must be considered as the simplest element common to any Galatea. On the other hand, it is a generalization of the well-known problem about Z-pinches, in which an electric current is present in the plasma cylinder but a current conductor is absent [5, 12].

In the paper [15], the analytical and numerical studies of configurations with defined plasma pressure distribution in a conductor vicinity determined a dependence of a magnetic field and plasma current density on the radius. The “diffusion stability” condition is pointed out, which is almost satisfied in all calculation variants. MHD instability is discovered in evolution of the first azimuthal harmonic of small perturbations corrugated along the axial coordinate. In the paper [16], instability of perturbations not depending on the azimuth is demonstrated in the configurations where the plasma pressure turns to zero on the outer boundary of the considered conductor neighborhood.

In continuation and development of the results [15, 16], the present paper focuses on the influence of the plasma pressure distribution in the outer part of the ring vicinity on stability of equilibrium configurations. The obtained results demonstrate that instability intensity is reduced if the plasma pressure decreases with the radius more smoothly at the ring configuration periphery. Qualitatively, this result coincides with the instability tendency of Z-pinches [12], and the provided calculations allow to estimate this tendency quantitatively.

2. Ring shape plasma configuration around a current-carrying conductor

The considered cylindrically symmetrical plasma configuration has a ring shape form surrounding a current-carrying conductor

$$1 < r < R$$

The equilibrium state of a configuration is determined by three functions describing plasma pressure $p(r)$, magnetic field intensity $H_\varphi = H(r)$, and plasma electric current density $j_z = j(r)$. They must satisfy two differential equations

$$\frac{dp}{dr} = -jH; \quad j = \frac{1}{r} \frac{dHr}{dr}$$

(2)
following from the MHD equation of equilibrium and Maxwell’s ones [12, 15–17]. Hereinafter all the values are dimensionless, i.e. assigned to the units of measurement composed of the dimensional constants of the problem: the conductor radius $r_c$ and electric current in it $J_c$. In order for the mathematical model to become fully determined, one of three functions participating in it should be defined in accordance with a certain requirement. As in [15, 16], let us specify a distribution of plasma pressure $p(r)$ to be vanished on the conductor boundary ($r = 1$). As part of the further development of the mentioned studies, we will make this distribution more complicated for $r > 1$: we will include to the area (1) a section of maximal pressure and a section of its smoothly decreasing with radius and vanishing at the outer boundary $r = R$. Namely, let’s construct the area (1) from four sections with parabolic (for simplicity) dependence of $p(r)$ presented on the figure 1:

$$p(r) = \begin{cases} 
  p_0 \left(1 - \left(\frac{r-r_1}{r_1-1}\right)^2\right), & 1 \leq r \leq r_1 \\
  p_0, & r_1 < r < r_2 \\
  p_0 - \left(p_0 - p_1\right) \frac{(r-r_2)^2}{(r_1-r_2)(R-r_2)}, & r_2 \leq r \leq r_3 \\
  p_1 + \left(p_0 - p_1\right) \frac{(R-r)^2}{(r_1-r_2)(R-r_2)}, & r_3 < r \leq R 
\end{cases}$$

(3)

The equations (2) allow to determine distributions of the magnetic field and electric current [15, 16]:

$$H(r) = \frac{\sqrt{G(r)}}{r}, \quad j(r) = \frac{1}{r} \frac{d\sqrt{G(r)}}{dr},$$

(4)

where $G(r) \equiv (Hr)^2 = 1 - \frac{4p_0}{(r_1-1)^2} \left(\frac{r^3-1}{3} - \frac{r^4-1}{4}\right)$

presented by the graphics on the figure 2.

![Figure 1. Distribution of plasma pressure p(r) in configuration with $p_0 = 0.2$, $p_1 = 0.1$, $r_1 = 2.5$, $r_2 = 3.5$, $r_3 = 4.5$, $R = 5.5$.](image)

Therefore, the mathematical model of the considered equilibrium configuration is fully determined by the formulas (3), (4). It contains the dimensionless parameters $p_0, p_1, r_1, r_2, r_3, R$, which can be chosen arbitrarily under the natural restrictions.
The specificity of this configuration type is connected with questions about its stability. MHD instability is simply detected on the periphery of the ring shape area and manifests itself stronger when pressure decreases harder up to zero near the outer boundary. Therefore here $p(r) > 0$ under $p_i > 0$ and decrease character of $p(r)$ is weaken at the section $r_5 < r < R$.

Figure 2. Corresponding distributions of magnetic field intensity $H(r)$ and electric current density $j(r)$.

3. On stability of equilibrium configurations

Equilibrium configurations of plasma, magnetic field, and electric current in magnetic traps are of interest in the controlled thermonuclear fusion problem only if they are stable for a long time. In two-dimensional problems about equilibrium configurations that are uniform on one of the coordinates (plane, axial or helical symmetry), their mathematical models widely use the scalar Grad-Shafranov equation for the magnetic flux function [6, 7, 15–17]. The boundary value problems are numerically solved by using the iterative relaxation method, and convergence of the iterations allows us to speak about so-called “diffusion stability” relative to the magnetic field perturbations of the same dimension [14, 17]. It doesn’t give a full answer about stability, but contains some quantitative information: for example, it requires the necessary restriction on a relation of characteristic gas pressure and magnetic one in a trap [3, 17]. In the considered one-dimensional case, a boundary value problem with the Grad-Shafranov equation

$$1 < r_1 < r_2 < r_3 < R; \quad p_0 \leq p_0^{cr} = \frac{3}{r_1^2 + 2r_1 + 3}; \quad 0 \leq p_1 \leq p_0$$

$$1 < r_1 < r_2 < r_3 < R; \quad p_0 \leq p_0^{cr} = \frac{3}{r_1^2 + 2r_1 + 3}; \quad 0 \leq p_1 \leq p_0$$

where

$$g(\psi) = \frac{d\psi}{d\psi} = \frac{dp}{dr} / \frac{d\psi}{dr} = j, \quad \frac{d\psi}{dr} = -H$$

is formulated on the interval $1 < r < R$ with the boundary conditions

$$\frac{d\psi}{dr}(1) = -1, \quad \psi(R) = 0.$$  

Convergence of the solving iterative process is connected with the spectral property of the operator

$$L[u] = -\frac{1}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) - Q(r)u = 0; \quad \frac{du}{dr}(1) = 0; \quad u(R) = 0, \quad (5)$$

where

$$Q(r) = \frac{dg}{d\psi} = -\frac{1}{H} \frac{dg}{dr} = \frac{r^2}{G} \left( \frac{d^2 p}{dr^2} + \frac{1}{r} \frac{dp}{dr} + \frac{r^2}{G} \left( \frac{dp}{dr} \right)^2 \right).$$
If $Q(r) = 0$, its eigenvalues $\mu_n$ are strictly positive and can be simply calculated using cylinder functions. Hence the sufficient criterion of $L[u]_+$ positivity follows:

$$\max Q(r) \leq \mu_1,$$

where $\mu_1$ is the first, i.e. minimal eigenvalue $\mu_n$ [15–17].

In the trap configurations considered in [15, 16], the condition (6) is satisfied under almost all of the parameter values. In the configuration presented here, it is still achieved in the configuration main part $1 < r \leq r_3$. At the same time, the inequality (6) is strongly violated in the section $r_3 < r < R$ near the outer boundary, where $d^2 p / dr^2 > 0$ and the pressure decreases slower with radius. Here $\max Q(r) = Q(r_3) \approx 0.75 > \mu_1 \approx 0.21$ and almost doesn’t depend on the parameter $p_1$. However, convergence of iterations is taking place in calculations even under violation of the condition (6) which is sufficient but not necessary for diffusion stability. The last one is taking place only under the restriction $p_1 \leq 0.5 p_0$.

A strict study of MHD stability of equilibrium magnetoplasma configurations consists of the following. The system of MHD equations is linearized relative to arbitrary small three-dimensional perturbations of the equilibrium state. The vector equation for a velocity of perturbations with a second-order on spatial coordinates differential operator can be derived from it [12, 16, 17]

$$\rho \frac{\partial^2 \mathbf{v}}{\partial t^2} = -\mathbf{K}[\mathbf{v}]$$

The coefficients of this linear uniform equation don’t depend on time $t$, therefore its solutions have the form

$$\mathbf{v}(t, r, \varphi, z) = e^{i\omega t} \mathbf{v}_1(r, \varphi, z),$$

and the problem with the equation (7) and uniform boundary conditions transforms into the eigenvalue problem

$$\rho \omega^2 \mathbf{v}_1 = \mathbf{K}[\mathbf{v}_1]$$

The operator $\mathbf{K}[\mathbf{v}]$ is self-adjoint, therefore the eigenvalues $\omega^2$ are real. Hence the perturbations don’t increase with time if $\omega^2 > 0$, and in this sense, the configurations should be considered as stable. The contrary case $\omega^2 < 0$ corresponds to instability because the perturbations exponentially increase in time.

In the one-dimensional problems about cylindrically symmetrical configurations considered here, the coefficients of the operator $\mathbf{K}$ don’t depend on $\varphi$ and $z$, and solutions of the equations (7), (9) depend on these variables also exponentially

$$\mathbf{v}_1(r, \varphi, z) = e^{im_{\varphi} - \imath \omega z} \mathbf{u}_1(r).$$

The three-dimensional vector problem (9) transforms into the series of one-dimensional problems for each of the velocity harmonics. Each of them allows to exclude two velocity components $\nu_\varphi$, $\nu_z$ and deals with the scalar equation for $\nu_r$

$$\frac{d}{dr} \left( F \frac{d \nu_r}{dr} \right) + G \nu_r = 0,$$

where the coefficients $F$ and $G$ nonlinearly depend on $r$, $\omega^2$ and the parameters $m$ and $k$ [18]. A cumbersome numerical solving of the boundary value problems with it allows to find the eigenvalues $\omega^2$, but, in questions about stability, it is sufficient to determine a sign of the minimal one. This allows to significantly simplify the solving and to confine to determination of “stability boundary” in the area of the problem parameters on which a senior eigenvalue passes through zero [14, 17]. The equation (10) under $\omega^2 = 0$ has the form
where \( u = r \dot{r}, \ \alpha = k/m, \ \eta(r) = 1 + \alpha^2 r^2 \). It is required to find conditions under which a boundary value problem with this equation has a nontrivial solution. The equation (11) degenerates at \( m = 0 \), but stability relative to perturbations not depending on \( \varphi \) is determined using the energy principle [12, 19] in this case: a quadratic form with the operator \( K[u] \) is positive-defined under the condition

\[
\frac{r}{p} \frac{dp}{dr} < \frac{4\gamma}{2 - \gamma \beta^2},
\]

where \( \gamma = 5/3 \) is an adiabatic index, \( \beta = 2 p/H^2 \) is a ratio of a gas dynamic pressure and a magnetic one expressed in the dimensionless variables. This condition is always satisfied in the section \( 1 < r < r_2 \) where the pressure doesn’t decrease with radius and is nontrivial at \( r_2 < r < R \). Instability here is provoked by the intensive pressure decrease at the trap outer boundary. In the considered configurations, it depends on the maximal pressure \( p_m \), the length of this area \( r_2 \), and the outer boundary pressure \( p_1 \).

The calculations demonstrated that the stability condition under \( m = 0 \) is satisfaction of the inequality (12) under \( r = r_2 \) which is achieved in the considered variants of configurations under any value \( p_1 \leq p_0 \). This result highlights only necessity but not sufficiency of the diffusion stability conditions determined above. The questions about stability of the considered configurations relative to the perturbations with positive values \( m \geq 1 \) resulted in the tendencies determined in [15, 16]. Namely, an area of stable parameters expands if \( m \) increases, but under each fixed \( m \) it narrows if the parameter \( k \), i.e. the oscillation frequency along the \( z \)-direction, increases.

References
[1] Morozov A I and Savel’ev V V 1998 PHYS-USP 41 (11) 1049–89
[2] Brushlinskii K V and Savel’ev V V 1999 Matem. modelirovanie 11 (5) 3–36
[3] Brushlinskii K V and Ignatov P A 2010 Comp. Math. and Math. Phys. 50 (12) 2071–81
[4] Brushlinskii K V and Goldich A S 2016 Diff. Eq. 52 (7) 845–54
[5] Brushlinskii K V and Kondratyev I A 2019 Math. models and Comp. Simul. 11 (1) 121–32
[6] Shafranov V D 1958 Sov. Phys. JETP 6 545–54
[7] Grad H and Rubin H 1959 Proc. 2–nd UN Int. Conf. on the Peaceful Uses of Atomic Energy Geneva 31 (NY: Columbia Univ. Press) p 190–97
[8] Dudnikova G I, Morozov A I, Fedoruk M P 1997 Plasma Phys. Reports 23 (5) 357–66
[9] Berezin Yu A, Dudnikova G I, Liseikina, Fedoruk M P T V 2018 Modeling of time-dependent plasma processes (Novosibirsk: IPC NSU) [in Russian]
[10] Brushlinskii K V and Chmykhova N A 2014 Vestn. Nats. Issled. Yad. Univ. MIFI 3 (1) 40–52 [in Russian]
[11] Shafranov V D 1966 Rev. of Plasma Phys. ed by M A Leontovich (NY: Consultants Bureau) vol 2 pp 103–52
[12] Kadomtsev B B 1966 Ibid. pp 153–206
[13] Bateman G 1979 MHD Instability (London, Cambridge MA: The MIT Press)
[14] Brushlinskii K V 2001 J. Appl. Math. Mech. 65 (2) 229–36
[15] Brushlinskii K V, Krivtsov S A, Stepin E V 2020 Comp. Math. and Math. Phys. 60 (4) 686–96
[16] Brushlinskii K V, Stepin E V 2020 Diff. Eq. 56 (7) 872–81
[17] Brushlinskii K V 2009 Mathematical and Computational Problems in Magnetohydrodynamics (Moscow: Binom) [in Russian]
[18] Solov’ev L S 1967 Rev. of Plasma Phys. ed by M A Leontovich (Berlin: Springer) vol 3 pp 277–325
[19] Bernstein I B, Frieman E A, Kruskal M D, Kelzrud R M 1958 Proc. Roy. Soc. 244 17–40