Weyl Invariance and Spurious Black Hole
in
Two-Dimensional Dilaton Gravity

Takanori Fujiwara\(^{(1),(a)}\), Yuji Igarashi\(^{(2),(b)}\) and Jisuke Kubo\(^{(3),(c)}\)

\(^{(1)}\) Department of Physics, Ibaraki University, Mito 310, Japan
\(^{(2)}\) Faculty of General Education, Niigata University, Niigata 950-21, Japan
\(^{(3)}\) College of Liberal Arts, Kanazawa University, Kanazawa 920, Japan

ABSTRACT

In two-dimensional dilaton gravity theories, there may exist a global Weyl invariance which makes black hole spurious. If the global invariance and the local Weyl invariance of the matter coupling are intact at the quantum level, there is no Hawking radiation. We explicitly verify the absence of anomalies in these symmetries for the model proposed by Callan, Giddings, Harvey and Strominger. The crucial observation is that the conformal anomaly can be cohomologically trivial and so not truly anomalous in such dilaton gravity models.

\(^{(a)}\) E-mail address: tfjiwa@tansei.cc.u-tokyo.ac.jp
\(^{(b)}\) E-mail address: igarashi@ed.niigata-u.ac.jp
\(^{(c)}\) E-mail address: jik@hep.s.kanazawa-u.ac.jp
1 Introduction

The two-dimensional cosmological model of Callan, Giddings, Harvey and Strominger (CGHS) \[1\] has two candidates, \( g_{\alpha\beta} \) and \( \hat{g}_{\alpha\beta} \), for the metric. In the world described by \( g_{\alpha\beta} \), there exists a black hole, but it disappears in the \( \hat{g}_{\alpha\beta} \)-world \[1\]. It seems at first sight that one can choose one of them for one’s own purpose. However, there is a global symmetry \[3\] with the transformation acting on \( g_{\alpha\beta} \) and \( \hat{g}_{\alpha\beta} \) differently, and one finds that the curvature scalar \( R(g) \) experiences a change under the global symmetry transformation while \( R(\hat{g}) \) does not. So one may be naturally led to choose \( \hat{g}_{\alpha\beta} \) as the metric because \( R(\hat{g}) \) has the absolute meaning with respect to the symmetry transformation. As we will see in section 2, the global symmetry is a Weyl symmetry, and is responsible also for decoupling of the matter fields from the conformal factor of \( g_{\alpha\beta} \), implying that the matter does not feel the existence of the black hole. This suggests that physics remains the same whether one chooses \( g_{\alpha\beta} \) or \( \hat{g}_{\alpha\beta} \) as the metric.

Quantum theoretically things may be different because Weyl symmetry is anomalous in general \[4, 5\]. In ref. \[4\] (hereafter referred to as I), we have investigated the algebraic structure of possible anomalies in the CGHS model, by considering the consistency conditions on anomalies. Under a certain set of reasonable assumptions, we have shown that there exist no cohomologically non-trivial anomalies. This result of the pure algebraic investigation has motivated us to confirm the absence of cohomologically non-trivial anomalies by explicit calculation of anomalies.

There may be many physically inequivalent quantum theories to a single classical theory. We will be assuming in this paper that they have at least following two properties:

(i): In the limit \( \hbar \to 0 \), they should approach to the classical theory in some definite sense.

(ii): All the symmetries of the classical theory have to be realized in the quantum theories in a corresponding fashion, except for those which can not be realized because of anomaly, but does not influence the consistency of the theories (scale invariance in QCD

\[1\] See ref. \[2\] and references therein for review and recent developments.

\[2\] \( \hat{g}_{\alpha\beta} \) is defined in eq. (10) in the next section.
for instance). What we would like to attempt to explicitly show in subsequent sections is that, to the classical CGHS model, there are no quantum theories in which the CGHS black holes and the corresponding Hawking radiations are physically real, rather than that there is one quantum theory in which the CGHS black holes are spurious. Obviously, this finding is in a sharp contrast to the results of the previous investigations on the CGHS black holes.

In section 3 we will briefly review the canonical setting which is based on the extended-phase-space method of Batalin, Fradkin and Vilkovisky (BFV) [7]. The conformal-gauge equivalent of the algebraic proof on the absence of anomalies of I will be presented in section 4.

We will start, in section 5, to calculate the contributions to anomalous Schwinger terms. We would like to apply the results of Polyakov [5] and Fujikawa [8] on the conformal anomaly to determine the contribution of the matter fields. To use their results appropriately, we will be assuming in that section that the metric which couples to the matter \( g_{\alpha\beta} \) differs from \( \bar{g}_{\alpha\beta} \) in the conformal factor. Classically, there is no difference, because the conformal factor of \( \bar{g}_{\alpha\beta} \) does not appear in the action thanks to the local Weyl invariance of the matter coupling. We will show that this local Weyl symmetry is intact at the quantum level. This implies, on one hand, that the theory with \( \bar{g}_{\alpha\beta} \) in the matter coupling is gauge-equivalent to the original theory, and on the other hand, that the conformal factor of \( g_{\alpha\beta} \) really decouples from the matter.

Furthermore, the matter does not contribute to the anomaly in the global Weyl invariance in this modified system because it is the invariance under the change of \( g_{\alpha\beta} \) and not of \( \bar{g}_{\alpha\beta} \). So, the matter sector has nothing to do with the anomalies in the CGHS model. This is why we will consider the model without the matter fields in the last part of section 5. We will go to the modified light-cone gauge, the light-cone gauge for \( \hat{g}_{\alpha\beta} \) [9], in which the theory takes the most simple form, and find no anomalies, conforming the previous, algebraic result. We will also discuss how the result can be modified if the system in the modified light-cone gauge couples to conformal matter. We will find again that the matter contribution to anomalies is cohomologically trivial and so not really anomalous.

Section 6 is devoted to discussion and appendix contains an algebraic proof which is
needed in section 4.

2 Spurious black hole in the classical approximation

The CGHS action is given by

\[ S_{\text{cl}} = \int d^2 \sigma \sqrt{-g} \left\{ \exp(-2\phi) \left[ R(g) + 4 g^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi + 4 \mu^2 \right] - \frac{1}{2} \sum_{i=1}^{n} g^{\alpha \beta} \partial_\alpha f_i \partial_\beta f_i \right\} , \quad (\alpha, \beta = 0, 1) , \]  

(1)

where \( f' \)s stand for matter fields, \( \phi \) for the dilaton field, \( \mu^2 \) for a cosmological constant, and the curvature scalar for the metric \( g_{\alpha \beta} \) is denoted by \( R(g) \). The action is invariant under 2D general coordinate transformations, and under the global, non-linear transformation

\[ \phi \rightarrow \phi' = \phi - \frac{1}{2} \ln(1 + \Lambda \exp(2\phi)) , \]

\[ g_{\alpha \beta} \rightarrow g'_{\alpha \beta} = g_{\alpha \beta}(1 + \Lambda \exp(2\phi))^{-1} , \]  

(2)

where \( \Lambda \) is a constant parameter. This global symmetry manifests itself in degeneracy of classical solutions, and is responsible for the unrecognizability of the black hole in the model as we have stated in I.

To see this, let us work in the conformal gauge,

\[ -g_{00} = g_{11} = \exp 2\rho , \quad g_{01} = 0 \]  

(3)

and write the action (1) in this gauge:

\[ S_{\text{CG}}^{\text{cl}} = \int d^2 \sigma \left\{ -\partial_+ \psi \partial_-(\rho - \phi) - \partial_- \psi \partial_+ (\rho - \phi) + 4 \mu^2 \exp 2(\rho - \phi) + \frac{1}{2} \sum_{i=1}^{n} \partial_+ f_i \partial_- f_i \right\} , \]  

(4)

with \( \psi \equiv \exp(-2\phi) \),

where we have used \( \partial_\pm = \partial_0 \pm \partial_1 \) so that \( \partial_\pm \sigma^\pm = 2 \), and we will also use the abbreviations, \( \dot{h} = \partial_0 h = \partial_+ h \) , \( h' = \partial_1 h = \partial_- h \). The action yields the equations of motion

\[ \partial_+ \partial_- (\rho - \phi) = 0 , \quad \partial_+ \partial_- \psi + 4 \mu^2 \exp 2(\rho - \phi) = 0 . \]  

(5)

\( ^3 \)The Lagrangian for the action (4) differs from that of (1) by a total derivative. But the canonical structure derived from (4) agrees with that of the next section.
Under the transformation defined in (2), \( \rho \) transforms as

\[
\rho' \to \rho - \frac{1}{2} \ln(1 + \Lambda \exp(2\phi)),
\]

so that the black hole solution [1],

\[
\rho = \phi = \rho_{\text{BH}}, \quad \exp(-2\rho_{\text{BH}}) = \frac{M}{\mu} - \mu^2 \sigma^+ \sigma^-,
\]

undergoes the change

\[
\exp -2\rho'_{\text{BH}} = \exp -2\rho_{\text{BH}} + \Lambda.
\]

By applying the symmetry transformation appropriately, we can arrive at

\[
\exp -2\rho''_{\text{BH}} = -\mu^2 \sigma^+ \sigma^-, \quad \text{with } R(\rho''_{\text{BH}}) = 0.
\]

This sounds strange because the curvature has no longer the absolute meaning. In fact, the curvature scalar \( R(g) \) non-trivially transforms under the transformation (2):

\[
\sqrt{-g'} R(g') = \sqrt{-g} R(g) + \partial_{\alpha} [\sqrt{-g} g^{\alpha\beta} \partial_{\beta} \ln(1 + \Lambda \exp 2\phi)].
\]

In this connection, we would like to mention that the re-scaled quantity

\[
\hat{g}_{\alpha\beta} \equiv g_{\alpha\beta} \exp(-2\phi)
\]

can equally well be used as the two-dimensional metric. So, at the classical level, there are at least two candidates for the metric. But the discussion above suggests that we should presumably regard \( \hat{g}_{\alpha\beta} \) as the true metric because it does not change under the global transformation (2) and hence has the absolute meaning. Needless to say that in that case the classical background (7) does not exhibit a black hole any more. In the following discussions, we nevertheless consider \( g_{\alpha\beta} \) as the metric and quantize the theory. Our result however suggests that physics remains the same whether we regard \( g_{\alpha\beta} \) or \( \hat{g}_{\alpha\beta} \) as the metric.

3 The Batalin-Fradkin-Vilkovisky quantization
3.1 Parametrization and constraints

To quantize the theory, we employ the generalized Hamiltonian method of BFV [7], and use the Arnowitt-Deser-Misner parametrization of the metric $g_{\alpha\beta}$:

$$g_{\alpha\beta} \equiv \begin{pmatrix} -\lambda^+\lambda^- & (\lambda^+ - \lambda^-)/2 \\ (\lambda^+ - \lambda^-)/2 & 1 \end{pmatrix} \exp 2\rho .$$  

(11)

In terms of these new variables, the CGHS action becomes

$$S^\text{cl} = \int d^2\sigma \left( \mathcal{L}_\text{d}^\text{cl} + \mathcal{L}_\mu^\text{cl} + \mathcal{L}_f^\text{cl} \right) ,$$  

(12)

where

$$\mathcal{L}_\text{d}^\text{cl} = \frac{\psi'}{\lambda^+ + \lambda^-} [2 (\lambda^+ - \lambda^-) (\dot{\rho} - \dot{\phi}) + 4 \lambda^+\lambda^- (\rho' - \phi') + 2 (\lambda^+\lambda^-)']$$

$$+ \frac{\bar{\psi}}{\lambda^+ + \lambda^-} [4 (\dot{\rho} - \dot{\phi}) + 2 (\lambda^+ - \lambda^-) (\rho' - \phi') + 2 (\lambda^+ - \lambda^-)'] ,$$

$$\mathcal{L}_\mu^\text{cl} = 2\mu^2 (\lambda^+ + \lambda^-) \exp 2(\rho - \phi) ,$$

$$\mathcal{L}_f^\text{cl} = \frac{1}{\lambda^+ + \lambda^-} \sum_{i=1}^n [\dot{f}_i f_i - (\lambda^+ - \lambda^-) \dot{f}_i f_i' - (\lambda^+\lambda^-) f_i f_i'] .$$

The conjugate momenta to $\lambda^\pm$, $\rho$, $\phi$ and $f_i$ are respectively given by

$$\pi_+^\lambda = 0 , \pi_-^\lambda = 0 ,$$

$$\pi_\rho = (\lambda^+ + \lambda^-)^{-1} [2 (\lambda^+ - \lambda^-) \psi' - 4 \bar{\psi}] ,$$

$$\pi_\phi = -\pi_\rho - 2 (\lambda^+ + \lambda^-)^{-1} \psi [-4 (\dot{\rho} - \dot{\phi})$$

$$+ 2 (\lambda^+ - \lambda^-) (\rho' - \phi') + 2 (\lambda^+ - \lambda^-)'] ,$$

$$\pi_f^i = (\lambda^+ + \lambda^-)^{-1} [2 \dot{f}_i - (\lambda^+ - \lambda^-) f_i'] .$$

(13)

So, $\pi_\pm^\lambda$ are primary constraints, and the Dirac algorithm further leads to the secondary constraints, the Virasoro constraints

$$\varphi_\pm = -2\mu^2 \exp 2(\rho - \phi) + (\partial_{\sigma} - Y_\pm) (\psi' \mp \frac{1}{2} \pi_\rho) + \frac{1}{4} \sum_{i=1}^n (\pi_f^i \pm f_i')^2 ,$$

(14)

with

$$Y_\pm \equiv (\rho - \phi)' \pm \frac{1}{4\psi} (\pi_\rho + \pi_\phi) ,$$

(15)

which satisfy under Poisson bracket the Virasoro algebra

$$\{ \varphi_\pm(\sigma) , \varphi_\pm(\sigma') \}_{\text{PB}} = \mp (\varphi_\pm(\sigma) \partial_{\sigma'} - \varphi_\pm(\sigma') \partial_{\sigma}) \delta(\sigma - \sigma') ,$$

$$\{ \varphi_\pm(\sigma) , \varphi_\mp(\sigma') \}_{\text{PB}} = 0 .$$

(16)
Since there are four first-class constraints, $\pi_\lambda^\pm$ and $\varphi_\pm$, the theory without the matter would have no physical degree of freedom.

### 3.2 Extended phase space and BRST charge

According to BFV [4], we define the extended phase space by including to the classical phase space the ghost-auxiliary field sector

\[(C^A, \overline{P}_A), (P^A, \overline{C}_A), (N^A, B_A), \quad (17)\]

where $A(=\lambda^\pm, \pm)$ labels the first-class constraints, $\pi_\pm$ and $\varphi_\pm$. $C^A$ and $P^A$ are the BFV ghost fields carrying one unite of ghost number, $\text{gh}(C^A) = \text{gh}(P^A) = 1$, while $\text{gh}(\overline{P}_A) = \text{gh}(\overline{C}_A) = -1$ for their canonical momenta, $\overline{P}_A$ and $\overline{C}_A$. The last canonical pairs in (17) are auxiliary fields and carry no ghost number. We assign 0 to the canonical dimension of $\phi, \lambda^\pm$ and $\rho$, and correspondingly +1 to $\pi_\phi, \pi_\lambda^\pm$, and $\pi_\rho$. The canonical dimensions of $C^\pm$, $\overline{P}_\pm$ and $\overline{P}_\pm^A$ are fixed only relative to that of $C^\pm$. For $c \equiv \text{dim}(C^\pm)$, we have:

\[\text{dim}(C_+^\pm) = 1 + c, \quad \text{dim}(\overline{P}_\pm) = 1 - c, \quad \text{dim}(\overline{P}_\pm^A) = -c. \quad (18)\]

Given the first-class constraints with the corresponding algebra (16), we can construct a BRST charge

\[Q = \int d\sigma [ C_+^\lambda \pi_+^\lambda + C_-^\lambda \pi_-^\lambda + C^+(\varphi_+ + \overline{P}_+C^+) \\
+ C^-(\varphi_- - \overline{P}_-C^-) + B_A P^A ], \quad (A = \lambda^\pm, \pm) \quad (19)\]

which satisfies under Poisson bracket the nilpotency condition

\[\{Q, Q\}_{PB} = 0. \quad (20)\]

At this stage, we would like to emphasize that the BRST charge (19) is given prior to gauge fixing. The BRST transformation of $A$ is then defined as

\[\delta A = -\{Q, A\}_{PB}, \quad (21)\]
and one finds that

\[
\begin{align*}
\delta f_i &= \frac{1}{2} \left[ C^i (\pi_f^i + f_i) + C^- (\pi_f^i - f_i) \right], \\
\delta \pi_f^i &= \frac{1}{2} \left[ C^i (\pi_f^i + f_i) - C^- (\pi_f^i - f_i) \right], \\
\delta \rho &= \frac{1}{2} \left[ C^{i'} - C^{i'} + (C^+ + C^-) \rho' + (C^+ + C^-) \frac{1}{4 \psi} (2 \pi_\rho + \pi_\phi) \right], \\
\delta \pi_\rho &= 4 \mu^2 (C^+ + C^-) \exp 2 (\rho - \phi) - \frac{1}{2} \left[ (C^+ - C^-) \pi_\rho \right]' - \left( (C^+ + C^-) \psi \right)' \\
\delta \phi &= \frac{1}{2} \left[ (C^+ + C^-) \phi' + (C^+ + C^-) \frac{1}{4 \psi} \pi_\rho \right], \\
\delta \pi_\phi &= -4 \mu^2 (C^+ + C^-) \exp 2 (\rho - \phi) + \frac{1}{2} \left[ (C^+ - C^-) \pi_\phi \right]' + \left[ (C^+ + C^-) \psi \right]' + 2 \left[ (C^+ + C^-) (\rho' - \phi') \right]' \psi + 2 (C^{+''} + C^{''-}) \psi \\
&- (C^+ + C^-) \frac{1}{4 \psi} (\pi_\rho^2 + \pi_\rho \pi_\phi), \\
\delta C^\pm &= \pm C^\pm C^{\mp'} + C^\mp C^\pm, \quad \delta C^\mp &= C^\pm, \\
\delta \Phi^\pm &= - \Phi^\pm = - \varphi^\pm \mp (2 \Phi^\pm C^{\pm'} + \Phi^\mp C^\pm), \\
\delta \Phi^\mp &= - \pi^\mp, \quad \delta N^A = \Phi^A, \quad \delta C_A = - B_A, \\
\delta \pi^\lambda &= \delta \Phi^A = \delta B_A = 0.
\end{align*}
\]

\[ (22) \]

3.3 The fundamental brackets

Gauge fixing appears in defining the total Hamiltonian \( H_T \). Since the canonical Hamiltonian vanishes in the present case, it is given by the BRST variation of gauge fermion \( \Psi \), i.e.

\[
H_T = \{ Q, \Psi \}_{PB},
\]

where the standard gauge fermion has the form

\[
\Psi = \int d\sigma \left[ \slashed{C}_{A\lambda}^A + \slashed{\Phi}_A N^A \right], \quad (A = \lambda^\pm, \pm).
\]

Eq. (23) immediately leads to

\[
\{ Q, H_T \}_{PB} = \frac{d}{dt} Q = 0.
\]

\[ (25) \]
In terms of the phase space variables, the charge $W$, which generates the non-linear symmetry transformation (2), can be written as

$$W = -\frac{1}{2} \int d\sigma (\pi_\phi + \pi_\rho) \psi^{-1} = - \int d\sigma (Y_+ - Y_-),$$

where $Y_\pm$ are defined in (15), and its Poisson brackets with $Q$ and $H_T$ are

$$\{Q, W\}_\text{PB} = 0,$$

$$\{H_T, W\}_\text{PB} = 0. \tag{28}$$

Eqs. (20), (25), (27) and (28) are the basic bracket relations which exhibit the 2D general covariance and the global invariance at the classical level. Note however that because of eq. (23), the bracket relations (25) and (28) are consequences of (20) and (27). Therefore, only (20) and (27) must be regarded as the fundamental bracket relations.

### 3.4 Gauge-fixed action

The BRST invariant master action of the theory (1) is given by

$$S_\Psi = \int d^2\sigma \left\{ \sum_{i=1}^n \pi_i^i \dot{f}_i + \pi_\phi^+ \dot{\lambda}^+ + \pi_\phi^- \dot{\lambda}^- + \pi_\phi \dot{\phi} + \pi_\rho \dot{\rho} + \mathcal{P}_A \dot{C}^A - \mathcal{H}^{\text{cl}} - \mathcal{H}^{\text{gf}} - \mathcal{H}^{\text{FP}} \right\}, \ (A = \lambda^\pm, \pm), \tag{29}$$

where

$$\mathcal{H}^{\text{cl}} = \varphi_+ N^+ + \varphi_- N^- + \pi_\phi^+ \lambda^+ + \pi_\phi^- \lambda^-,$$

$$\mathcal{H}^{\text{gf}} = B_A \chi^A,$$

$$\mathcal{H}^{\text{FP}} = \mathcal{C}_A \delta \dot{\chi}^A + \mathcal{P}_A \dot{\mathcal{C}}^A + (2\mathcal{P}_+ \mathcal{C}^+ + \mathcal{P}_- \mathcal{C}^-) N^+ - (2\mathcal{P}_- \mathcal{C}^- + \mathcal{P}_+ \mathcal{C}^+) N^-.$$

To illustrate how to proceed to obtain a gauge-fixed action, let us discuss the conformal gauge-fixing for definiteness. Terms in $\mathcal{H}^{\text{cl}}$ suggest that $N^\pm$ play the role for $\lambda^\pm = (\sqrt{-g} \pm g_{01})/g_{11}$. So it is natural to impose $N^\pm = \lambda^\pm$. The conformal gauge then corresponds to

$$\lambda_\pm^\lambda = \lambda^\pm - N^\pm, \ \chi_\pm^\pm = N^\pm - 1. \tag{30}$$

We have suppressed here the Legendre terms, $\mathcal{C}_A \dot{N}^A = -\delta (\mathcal{C}_A \dot{N}^A)$, by shifting the gauge fermion, $\Psi \rightarrow \Psi + (\mathcal{C}_A \dot{N}^A)$. 

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Many variables such as $\lambda^\pm, \pi^\pm, N^\pm, B^\pm, P^\pm, \overline{P}^\pm$, etc. become non-dynamical in the gauge, and we can eliminate them by means of the equations of motion like

$$\overline{P}^\pm = -\overline{C}^\pm.$$  \hfill (31)

In doing so, we arrive at the conformal-gauge action

$$S_{CG} = S^{cl}_{CG} - \int d^2\sigma \left( \overline{C}^+ \partial^- C^+ + \overline{C}^- \partial^+ C^- \right), \hfill (32)$$

where $S^{cl}_{CG}$ is given in (4). The gauge-fixed form of the BRST charge can be similarly obtained from the gauge-independent one (19):

$$Q_{CG} = Q^{(+)}_{CG} + Q^{(-)}_{CG},$$  \hfill (33)

$$Q^{(+)}_{CG} = \pm \int d\sigma^+ C^+ \left\{ -\partial^+ (\rho - \phi) \partial^\pm \psi + \frac{1}{2} \partial^\pm \partial^\pm \psi \right\} + \frac{i}{4} \sum^\infty = 1 \left( \partial^\pm f_i \right)^2 - \frac{1}{2} \overline{C}^\pm \partial^\pm C^\pm.$$

In a similar way, one can find the gauge-fixed action in different gauges. In section 6, we will work in the light-cone gauge to perform explicit calculations.

## 4 Anomalies

### 4.1 Consistency conditions

The “nilpotency of $Q$” expressed in eq. (20) means that the underlying constraints in the theory are first-class while eq. (27) exhibits the presence of the global symmetry. At the quantum level, $Q$ and $W$ must be suitably regularized to become well-defined operators because they are composite operators. We mean by an anomaly that the algebraic structure which is to be traced back to a symmetry of a classical theory can not be maintained upon quantization. So, anomaly arises if $Q$ and $W$ do no satisfy the fundamental algebra under commutator. In I, we have assumed that anomalous Schwinger terms, which exhibit an anomaly in the algebra, may be expanded in $\hbar$ as

$$[Q, Q] = i \sum_{m=2}^\infty \hbar^m \Omega^{(m)},$$  \hfill (34)
\[ [Q, W] = \frac{i}{2} \sum_{m=2} h^m \Xi^{(m)}, \quad (35) \]

([ , ] denotes super-commutator), and that the super-commutation relations between \( Q \) and \( W \) satisfy:

(i) (anti-) symmetry,

\[ [A, B] = \pm (-1)^{|A||B|} [B, A], \]

(ii) bilinearity

\[ [A, B + C] = [A, B] + [A, C], \]

(vi) derivative property (super-Jacobi identity)

\[ [A, [B, C]] + (-1)^{|C|(|A|+|B|)} [C, [A, B]] + (-1)^{|A|(|B|+|C|)} [B, [C, A]] = 0, \quad (36) \]

and (iv)

\[ [A, B] = i \hbar \{A, B\}_{PB} + O(h^2), \quad (37) \]

where \(|A|\) is the grassmann parity of the operator \( A \) and can be either even (= 0 mod 2) or odd (= 1 mod 2). One easily observes that the outer commutators in the super-Jacobi identities for \( Q \) and \( W \),

\[ [Q, [Q, Q]] = 0, \quad (38) \]
\[ 2 [Q, [Q, W]] + [W, [Q, Q]] = 0, \quad (39) \]

define a set of consistency conditions at each order in \( h \) in terms of Poisson brackets. In the lowest order, they are \[ 10, 3 \]

\[ \delta \Omega^{(2)} = 0, \quad (40) \]
\[ \delta \Xi^{(2)} = \{W, \Omega^{(2)}\}_{PB}. \quad (41) \]

The true anomalies \( \Omega^{(2)} \) and \( \Xi^{(2)} \) should be cohomologically non-trivial. To see this, we first observe that, if \( \Omega^{(2)} \) and \( \Xi^{(2)} \) are solutions, then \( \Omega^{(2)} + \delta X \) and \( \Xi^{(2)} + \{W, X\}_{PB} + \delta Y \)
are solutions too. It is then easy to find that $X$ and $Y$ can be absorbed to order $\hbar^2$ into a re-definition of $Q$ and $W$, defined by $Q \to Q - (\hbar X/2)$ and $W \to W - (\hbar Y/2)$. If furthermore there is no non-trivial solution to (40) (this will be the case in the CGHS theory), the non-trivial $\Xi^{(2)}$ is a non-trivial solution to the homogeneous part of (41):

$$\delta \Xi^{(2)}_{\hbar} = 0 . \quad (42)$$

One can also convince oneself that the consistency conditions of $O(\hbar^3)$ are exactly the same as those of $O(\hbar^2)$ if there is no non-trivial solution at $O(\hbar^2)$. Therefore, the non-existence of the non-trivial solutions to the consistency conditions (40) and (42) ensures the non-existence of those to the higher order consistency conditions. This is the non-renormalization theorem \[11\] in our language.

### 4.2 Solutions in the conformal gauge

To find possible anomalies, we thus have to solve the classical, algebraic problem defined by (40) and (41) (or (42) if there is no non-trivial solution to (40)). In I we have looked for solutions $\Omega^{(2)}$ and $\Xi^{(2)}$ in the form

$$\Omega^{(2)} = \int d\sigma \omega , \quad (43)$$

$$\Xi^{(2)} (\Xi^{(2)}_{\hbar}) = \int d\sigma \xi (\xi_{\hbar}) , \quad (44)$$

where $\omega$ and $\xi (\xi_{\hbar})$ are polynomials of local operators with $\text{gh}(\omega) = 2$, $\text{gh}(\xi) = \text{gh}(\xi_{\hbar}) = 1$, $\dim(\omega) = 3 + 2c$, and $\dim(\xi) = \dim(\xi_{\hbar}) = 2 + c$. We have first divided the total phase space, according to the action of $\delta$, into two sectors,

$$S_1 \text{ consisting of } (f_i , \pi^i_f) , (\rho , \pi_\rho) , (\phi , \pi_\phi) \text{ and } (C^\pm , \overline{P}_\pm) ,$$

$$S_2 \text{ consisting of all the other fields} , \quad (45)$$

such that the $\delta$ operation closes on each sector. The $S_2$-sector is BRST trivial because it is made of pairs $(U^a , V^a)$ with $\delta_2 U^a = \pm V^a$. As shown in ref. \[10\], there exists no non-trivial solution to (40) ((42)) if $\omega (\xi_{\hbar})$ contains the $S_2$-variables. We then have used the linear independence of the generalized Virasoro constraints, $\Phi_{\pm} \equiv -\delta \overline{P}_\pm$, and the fact that $\overline{P}_\pm$ is the only one which produces $\overline{P}_\pm$ under $\delta$, to conclude that $\overline{P}_\pm$ can not
be involved in the non-trivial part of $\omega$ and $\xi_h$. Therefore, $\omega$ and $\xi_h$ are functions only of $f_i, \pi^i_f, \rho, \pi_\rho, \phi, \pi_\phi, C^\pm$ and their spatial derivatives.

The solutions to (40) and (41) are gauge-independent because all the quantities involved there are defined prior to gauge-fixing. To proceed, we have made the basic assumption in I which is based on the time-independent reparametrization invariance. Here we would like to present the conformal-gauge equivalent of the proof that there are no non-trivial solutions to (40) and (42).

Using the equation of motion (5) and those for the ghost fields in the conformal gauge, one easily obtains the conformal fields, e.g.,

\[
Y_\pm = \pm \partial_\pm (\rho - \phi) , \quad (\dim(Y_\pm) = 1) ,
\]

\[
F^i_\pm = \partial_\pm f_i , \quad (\dim(F_\pm) = 1) ,
\]

\[
G_\pm = \frac{1}{2} \partial_\pm \partial_\pm \psi - \partial_\pm (\rho - \phi) \partial_\pm \psi , \quad (\dim(G_\pm) = 2) ,
\]

\[
C^\pm , \quad (\dim(C^\pm) = c) .
\]

The BRST transformations of these quantities are closed:

\[
\delta Y_\pm = \pm \frac{1}{2} \partial_\pm [ (\frac{1}{2} \partial_\pm \pm Y_\pm) C^\pm] ,
\]

\[
\delta F^i_\pm = \frac{1}{2} \partial_\pm (C^\pm F^i_\pm) ,
\]

\[
\delta G_\pm = \frac{1}{2} C^\pm \partial_\pm G_\pm + \partial_\pm C^\pm G_\pm ,
\]

\[
\delta C^\pm = \frac{1}{2} C^\pm \partial_\pm C^\pm ,
\]

The assumption of I is equivalent to that the non-trivial solutions to (40) and (42) are functions only of the conformal fields listed above \footnote{\(C^\pm\), which are \(\mathcal{P}^\pm\) in the conformal gauge (see eq. (31), are also conformal fields, but they are not involved in the non-trivial solutions as we have argued above.} and that they respect the discrete symmetry defined by \(C^\pm \rightarrow C^\mp, \partial_\sigma \rightarrow -\partial_\sigma\).

We first consider the \(Q^2\)-anomaly in the conformal gauge. In that gauge, the BRST charge is split into its chiral components, \(Q^{(+)\text{CG}}\) and \(Q^{(-)\text{CG}}\) (see eq. (33)), which simplifies the analyses because we may further assume \footnote{As a matter of fact, (48) is not an assumption; we justify it in appendix.} that

\[
[Q^{(+)\text{CG}}, Q^{(-)\text{CG}}] = 0 .
\]

\[
eq (48)
\]
It therefore follows
\[
\Omega_{CG}^{(m)} = \Omega_{CG}^{(+m)} + \Omega_{LG}^{(-m)}, \quad (m = 2, \ldots),
\] (49)
where
\[
[Q_{CG}^{(\pm)}, Q_{CG}^{(\pm)}] = i \sum_{m=2} \hbar^m \Omega_{CG}^{(\pm m)} = i \sum_{m=2} \hbar^m \int d\sigma \omega_{CG}^{(\pm m)}. \quad (50)
\]
With this in mind, we write the most general form for \(\omega_{CG}^{(+2)}\):
\[
\omega_{CG}^{(+2)} = C^+ \partial_+ C^+ \{ \kappa_1 G_+ + \kappa_2 (Y_+)^2 + \kappa_3 \partial_+ Y_+ + \sum_{i=1}^n \kappa^{(i)}_4 (F_i^+)^2 \} + \kappa_{KO} \partial_+ C^+ (\partial_+)^2 C^+, \quad (51)
\]
where \(\kappa\)'s are constants independent of the fields. (For \(\omega_{CG}^{(-2)}\), we have to do the same analysis.) By requiring that \(\delta \omega_{CG}^{(+2)}\) be at most a total derivative, we find the conditions on \(\kappa\)’s
\[
\kappa_2 = -\kappa_3. \quad (52)
\]
But if eq. (52) is satisfied, \(\omega_{CG}^{(+2)}\) is a trivial term because
\[
\omega_{CG}^{(+2)} = -2\delta \left[ \kappa_1 C^+ G_+ + \kappa_2 C^+ (Y_+)^2 + \sum_{i=1}^n \kappa^{(i)}_4 C^+ (F_i^+)^2 + 2\kappa_{KO} \partial_+ C^+ Y_+ \right] + \partial_+ \left[ (-\kappa_2 + 2\kappa_{KO}) Y_+ C^+ \partial_+ C^+ \right], \quad (53)
\]
implying that there is no non-trivial solution to (40) to \(O(\hbar^2)\) and hence to any finite order of \(\hbar\). Note that the Kato-Ogawa anomaly term [12], the last term on the r.h.s. of eq. (51), is trivial. As for anomaly in the non-linear global symmetry, we would like to slightly modify the treatment of I. This is because the global symmetry appears as a residual, local symmetry in the conformal gauge: One can easily verify that the gauge-fixed action, \(S_{CG}\) given in (32), is invariant under the bosonic transformation
\[
\delta_B^{(\pm)} \rho = \delta_B^{(\pm)} \phi = -\frac{1}{2} \xi^\pm \exp 2\phi, \quad (54)
\]
That is, the central extension of the Virasoro algebra is cohomologically trivial. This is our language to the observation of refs. [17, 18].
as well as under the fermionic transformation

\[ \delta_F(\pm) \rho = \delta_F(\pm) \phi = \mp \frac{1}{4} \partial_\pm \eta^\pm \exp 2\phi, \]
\[ \delta_F(\pm) C_\pm = \pm \frac{1}{2} (\partial_\pm)^2 \eta^\pm + \partial_\pm \eta^\pm \partial_\pm (\rho - \phi), \]
\[ \delta_F(\pm) C^\pm = 0, \]

(55)

where \( \eta^+ (\eta^-) \) is a fermionic function of \( \sigma^+ (\sigma^-) \) with the ghost number \(-1\) while \( \zeta^+ (\zeta^-) \) is a bosonic function of \( \sigma^+ (\sigma^-) \). The generators for the bosonic and fermionic transformations are

\[ W_B^{(\pm)}[\zeta] = \mp \int d\sigma^\pm \zeta_\pm \partial_\pm (\rho - \phi), \]
\[ W_F^{(\pm)}[\eta] = \pm \int d\sigma^\pm \partial_\pm \eta^\pm \left[ \frac{1}{2} \partial_\pm + \partial_\pm (\rho - \phi) \right] C^\pm. \]

(56)

(57)

Together with \( Q_{CG}^{(\pm)} \) they form a closed algebra under Poisson bracket:

\[ \{ Q_{CG}^{(\pm)}, W_B^{(\pm)}[\zeta] \}_{PB} = W_F^{(\pm)}[\zeta], \]

(58)

where \( W \)'s commute with each other. Since the original non-linear transformation (2) corresponds to the bosonic one (56) with \( \zeta^+ = \zeta^- = \Lambda/2 \), the closure of the algebra (58) under commutator implies the absence of anomaly in the non-linear symmetry (2) automatically.

Appendix is devoted to give an algebraic proof for its absence. So our main conclusion in this section is that there are no non-trivial solutions to the consistency conditions (40) and (42) to any finite order in \( \hbar \).

5 Matter contribution to anomalies

5.1 Basic idea

In the previous section, we have performed an algebraic analysis on possible anomalies, which is based on the consistency conditions (40) and (42). They have been derived from the basic assumptions (i)-(v). Of those, the Jacobi identity (36) is undoubtedly the most crucial one. Although there are no examples of Jacobi-identity violating commutator
known in two dimensions (at least to our knowledge), there exist some examples in four
dimension \([13]\). So, it is certainly desirable to explicitly show the absence of anomalies
in CGHS theory.

So far, the single, global symmetry defined in (2) has been made responsible for two
independent, physical consequences; degeneracy of the black hole background (7) and
the fact that the conformal factor of \(g_{\alpha\beta}\) does not couple to the matter fields. It is more
convenient if we can treat these two things independently and individually. To this end,
we consider an action similar to \(S^{\text{cl}}\) given in (1) but with the matter coupling replaced by
\[
-\frac{1}{2} \sqrt{-g} \sum_{i=1}^{n} \sqrt{g^{\alpha\beta}} \partial_\alpha f_i \partial_\beta f_i ,
\]
where \(\sqrt{g_{\alpha\beta}}\) is supposed to differ from \(g_{\alpha\beta}\) only in the conformal factor. We parametrize
\(g_{\alpha\beta}\) in a similar way as we did for \(g_{\alpha\beta}\) in (11):
\[
\sqrt{g_{\alpha\beta}} \equiv \begin{pmatrix} -\lambda^+\lambda^- & (\lambda^+-\lambda^-)/2 \\
(\lambda^+-\lambda^-)/2 & 1 \end{pmatrix} \exp 2\varphi .
\]
Because of the local Weyl invariance of the matter coupling (59), \(\varphi\) does not show up in
the action at the classical level, and so the term (59) transforms like a density under a
two-dimensional general coordinate transformation even if \(g_{\alpha\beta}\) is regarded as the metric.
Therefore, two theories are equivalent at the classical level. In this way, we have intro-
duced two independent Weyl symmetries, global and local, and achieved to separate the
theoretical origin for degeneracy of the black hole background from that for decoupling of
the matter from the conformal factor of \(g_{\alpha\beta}\). If the local Weyl symmetry turns out to be
good at the quantum level, we may impose different gauge conditions. In the \(\varphi = \rho\)-gauge,
for instance, we recover the original theory with \(g_{ab}\) in the matter coupling, and we can
manifestly see the decoupling of conformal factor of \(g_{ab}\) from the matter in the \(\varphi\)-gauge.

### 5.2 BRST symmetrization of the local Weyl symmetry

We denote the classical action with the matter coupling (59) by \(S^{\text{cl}}\), and follow the BFV
procedure as we have done in section 3. Every thing is the same, except for the fact that
there is an additional first-class constraint
\[
\pi_{\varphi} \sim 0 ,
\]
where $\pi_\bar{\rho}$ is the conjugate momentum of $\bar{\rho}$. Therefore, the BRST charge can be easily obtained from $Q$ (given in (19)):

$$\sqrt{Q} = Q + \int d\sigma \left( \frac{1}{2} C^\rho \pi_\rho + B_\rho P_\rho \right).$$

To determine the matter contribution to the anomalous Schwinger term $\Omega$, we will use the results of refs. [5] and [8] on the path-integration of the matter fields $f_i$. Their results are written in a covariant form, and so we first have to covariantize our canonical formulation. (Since $g_{\alpha\beta}$ is supposed to couple to the matter, the covariantization has to be made with respect $g_{\alpha\beta}$ of course.) This requires elimination of various phase space variables by means of the equations of motion, and for that we have to specify a gauge. In the standard form of the gauge fermion defined in eq. (24), we have five gauge conditions $\chi^A$ with $A = \lambda^\pm, \pm, \bar{\rho}$. To identify $g_{\alpha\beta}$ and $\bar{g}_{\alpha\beta}$ as well as the reparametrization ghosts and the Weyl ghost with the extended-phase-space variables, we use two of them to impose the geometrization conditions [14]

$$\chi^\lambda = \chi^\pm - N^\pm,$$

while making an (inessential) assumption that $\chi^\pm$ and $\chi^\bar{\rho}$ do not contain

$$\pi^\lambda, \pi^\bar{\rho}, \bar{C}^\lambda, \bar{P}^\pm, N_\lambda^\pm, N_{\bar{\rho}}, \bar{P}_{\lambda}^\pm, \text{and } B_\lambda^\pm.$$

One finds that

$$\lambda^\pm = N^\pm, N_\lambda^\pm = \dot{\lambda}^\pm, N_{\bar{\rho}}^\pm = 2\dot{\bar{\rho}},$$

$$P^\pm = C^\lambda = \dot{C}^\pm \pm C^\pm N^{\pm'} \mp C^{\pm'} N^\pm$$

(64)

can be still unambiguously derived. Then one can verify that the covariant variables defined by

$$C^0 \equiv C^0/N^0, C^1 = C^1 - N^1 C^0/N^0,$$

$$C^W \equiv C^\bar{\rho} - C^0 N_{\bar{\rho}} - 2 C^1 \bar{\rho} - 2 C^{0'} N^1 - 2 C^{1'},$$

$$g_{\alpha\beta} \equiv \begin{pmatrix} -N^+ N^- & (N^+ - N^-)/2 \\ (N^+ - N^-)/2 & 1 \end{pmatrix} \exp 2\rho,$$

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta} \exp 2(\bar{\rho} - \rho),$$

(65)
obey the covariant BRST transformation rules \[8\]

\[
\delta \gamma_{\alpha \beta} = C^\gamma \partial_\gamma \gamma_{\alpha \beta} + \partial_\alpha C^\gamma \gamma_{\gamma \beta} + \partial_\beta C^\gamma \gamma_{\alpha \gamma} + C_W \gamma_{\alpha \beta} ,
\]

\[
\delta g_{\alpha \beta} = C^\gamma \partial_\gamma g_{\alpha \beta} + \partial_\alpha C^\gamma g_{\gamma \beta} + \partial_\beta C^\gamma g_{\alpha \gamma} ,
\]

\[
\delta C^\alpha = C^\gamma \partial_\gamma C^\alpha , \quad \delta C_W = C^\gamma \partial_\gamma C_W ,
\]

(66)

where \(C^\pm = C^0 \pm C^1\) and similarly for other quantities. Not that the \(\tau\)-derivatives in eq. (66) are exactly those which appear in eq. (64).

5.3 Triviality of the matter contribution

We are now in position to apply the results of refs. \[5\] and \[8\], and find that the matter contribution to the anomalous Schwinger term in \([Q, \bar{Q}]\) is given by (see also ref. \[15, 16\])

\[
i \Omega_m = i \kappa \int d\sigma \sqrt{-g} \left[ C^0 C_W R(\bar{g}) + \bar{g}^{\alpha \beta} C_W \partial_\beta C_W + MC^0 C_W \right] ,
\]

(67)

\[
\kappa = -\frac{n}{24\pi} ,
\]

where \(n\) is the number of the matter fields. In string theory, the \(\Omega_m\) is BRST non-trivial (as well known), but it is not the case here, because

\[
\Omega_m = \Omega_{KO} - \kappa \delta \int d\sigma \vartheta ,
\]

(68)

\[
\Omega_{KO} = -\kappa \int d\sigma \left[ (C^+/C^+) - (+ \rightarrow -) \right]
\]

\[
= \kappa \delta \int d\sigma \left[ C^+ Y_+ + (+ \rightarrow -) \right] ,
\]

(69)

where \(\Omega_{KO}\) is the Kato-Ogawa anomaly term \[12\]. \(Y_\pm\) is given in (15), and \(\vartheta\) is given by \[14\]

\[
\vartheta = \frac{1}{4} \frac{G^2 + C^+ G^+ C^+ + \frac{1}{4} G^2 C^- + G^- C^-}{2(G^+ G^-)} - \frac{M}{2} (C^+ C^-) \exp 2\bar{\rho} ,
\]

(70)

\[
G_\pm = \frac{1}{N_0} \left[ \pm N\bar{\sigma} + 2(N^0 \mp N^1) \bar{\sigma} \mp 2 N^{1\prime} \right] .
\]

Therefore, the matter contribution can be completely absorbed into a re-definition of \(\bar{Q}\) so that its nilpotency can be maintained \[8\]. This implies that the local Weyl symmetry

---

\[8\] A similar observation has been made for the Jackiw-Teitelboim model in refs. \[17, 18\].
of the matter coupling (59) is quantum theoretically intact. (The dilaton sector can not contribute to the anomaly in that local Weyl symmetry.) So we explicitly have shown that two theories, one with only $g_{\alpha\beta}$ and the other with $\overline{g}_{\alpha\beta}$ in the matter coupling, are equivalent. In other words, the quantum theory corresponding to the action $\overline{S}^{cl}$ is gauge-equivalent to that corresponding to $S^{cl}$, and the decoupling of the conformal factor of $g_{\alpha\beta}$ from the matter is manifest.

6 Light-cone gauge fixing

6.1 Gauge-fixed form of $Q$ and $W$

It is clear that the matter does not contribute to the anomaly in the global Weyl symmetry because the path-integration of the matter fields are supposed to be covariantized with respect $\overline{g}_{\alpha\beta}$. This fact, together with the results we have obtained in the previous section, leads to the observation that the matter fields can never contribute non-trivially to the anomalies we are concerned with.

Therefore, we may completely switch off the matter coupling to compute the possible anomalies. So we consider the original CGHS theory without the matter fields in the modified light-cone gauge, in which the theory takes the most simple form, and calculate the anomalies in that gauge. It is the light-cone gauge for the scaled metric $\hat{g}_{\alpha\beta}$ defined in eq. (10), as proposed in ref. [9].

The modified light-cone gauge,

$$\hat{g}_{+} = \frac{1}{4} ( \hat{g}_{00} - \hat{g}_{11} ) = - 1/2 ,$$
$$\hat{g}_{-} = \frac{1}{4} ( \hat{g}_{00} + \hat{g}_{11} - 2 \hat{g}_{01} ) = 0 ,$$

is realized by the gauge-fixing functions (in the gauge fermion $\Psi$)

$$\chi^{+} = N^{+} - 1 , \quad \chi^{-} = [ (N^{-} + 1) \exp 2 (\rho - \phi) ] - 2 ,$$

along with those given in (63).

---

9Some of the results we will obtain below have already been found in ref. [9], but we would like to present our results in some detail for completeness.
To obtain the gauge-fixed action, we substitute the gauge fermion (24) with \( \chi \)'s given in (63) and (72) into the extended-phase-space action (29), and then integrate out all the non-dynamical fields by using the equations of motion as we did to obtain the conformal gauge action (32); for instance,

\[
\begin{align*}
\pi_\phi &= 2 \partial_- (\psi \exp 2(\rho - \phi)) - 4 \psi \rho' \\
&\quad - 2 (C^+ \overline{C}_+ + C^- \overline{C}_-) \exp 2(\rho - \phi), \\
\pi_\rho &= -2 (\partial_\psi) \exp 2(\rho - \phi) + 4 \psi \phi' \\
&\quad + 2 (C^+ \overline{C}_+ + C^- \overline{C}_-) \exp 2(\rho - \phi), \\
\overline{P}_+ &= -\overline{C}_+, \overline{P}_- = -\exp 2(\rho - \phi) \overline{C}_-.
\end{align*}
\]

The gauge-fixed action is then given by

\[
S_{LG} = S_{LG}^{cl} + S_{LG}^{gh},
\]

\[
S_{LG}^{cl} = \int d^2 \sigma \left[ 4 \mu^2 - \partial_- \psi \partial_- (\exp 2(\rho - \phi)) \right], \quad (\psi = \exp -2\phi),
\]

\[
S_{LG}^{gh} = -\int d^2 \sigma \left[ b_{++} \partial_- c^+ + b \partial_- c_+ \right],
\]

where we have re-defined the ghost fields as \[19\]

\[
\begin{align*}
c^+ &= C^+, \quad b = \overline{C}_-, \\
c_+ &= C^- + (\exp 2(\rho - \phi) - 1) (C^+ + C^-) + \frac{\sigma^-}{2} \partial_+ C^+, \\
b_{++} &= \overline{C}_+ - (\exp 2(\rho - \phi) - 1) \overline{C}_- + \frac{\sigma^-}{2} \partial_+ \overline{C}_-,
\end{align*}
\]

in order to simplify the ghost sector. The re-defined ghost fields satisfy the free equations of motion

\[
\begin{align*}
\partial_- c^+ &= \partial_- c_+ = 0, \\
\partial_- b_{++} &= \partial_- b = 0.
\end{align*}
\]

Next we derive the gauge-fixed form of the generator for the non-linear transformation (26), \( W_{LG} \), and that of the BRST charge, \( Q_{LG} \), by using equations of motions (73) and (75), and

\[
(\partial_-)^2 \psi = (\partial_-)^2 \exp 2(\rho - \phi) = 0.
\]

20
Suppressing the total spatial derivative term, we can easily obtain

\[ W_{LG} = - \int d\sigma \partial_-(\exp 2(\rho - \phi)) . \]  

(78)

After some algebraic calculations, we also find that \( Q_{LG} \) takes the form (see also refs. \[9, 19\])

\[ Q_{LG} = \int d\sigma \{ c^{+}( T^{d}_{++} + \frac{1}{2} T_{b++} + T_{b} ) + c_{+} T \} , \]  

(79)

\[ T^{d}_{++} = \frac{1}{2} \{ (\partial_{+})^{2} \psi - 2 \partial_{-}\partial_{+}\psi + 2 \exp 2(\rho - \phi) \partial_{-}\partial_{+}\psi + \partial_{+}(\exp 2(\rho - \phi)) \partial_{-}\psi - \partial_{-}(\exp 2(\rho - \phi)) \partial_{+}\psi + \sigma^{-}\partial_{+}T \} , \]  

\[ T_{b++} = -\frac{1}{2} \partial_{+}b_{++} c^{+} - b_{++} \partial_{+}c^{+} , \]  

\[ T = \frac{1}{2} \{ -4 \mu^{2} - \partial_{+}\partial_{-}\psi - \partial_{-}(\exp 2(\rho - \phi)) \partial_{-}\psi \} . \]

We further use the equations of motion (77) to expand \( \psi \) and \( \exp 2(\rho - \phi) \) according to

\[ \psi = \exp -2\phi = \xi(\sigma^{+}) + \frac{\sigma^{-}}{2} \xi_{-}(\sigma^{+}) , \]  

\[ \exp 2(\rho - \phi) = h_{++}(\sigma^{+}) + \frac{\sigma^{-}}{2} h_{+}(\sigma^{+}) . \]  

(80)

In terms of \( \xi \)'s and \( h \)'s, \( T^{d}_{++} \) and \( T \) can be written as \[10\]

\[ T^{d}_{++} = \frac{1}{2} \{ (\partial_{+})^{2} \xi - 2 \partial_{+}\xi_{-} + 2 h_{++} \partial_{+}\xi_{-} + \partial_{+}h_{++} \xi_{-} - h_{+} \partial_{+}\xi \} , \]  

(81)

\[ T = \frac{1}{2} \{ -4 \mu^{2} - \partial_{+}\xi_{-} - h_{+}\xi_{-} \} , \]

and also \( W_{LG} \) becomes

\[ W_{LG} = - \int d\sigma^{+} h_{+}(\sigma^{+}) . \]  

(82)

We now have the facility to explicitly compute the anomalous Schwinger terms \( \Omega \) and \( \Xi \) defined in eqs. (34) and (35).

### 6.2 Explicit calculation of \( \Omega \) and \( \Xi \)

Until now, we have worked on the basis of the Poisson bracket. To compute the anomalous Schwinger terms, we of course have to go to the commutator and specify the ordering of

\[10\] See also ref. \[9\].
operators. To this end, let us first derive the commutation relations among \( \xi \)'s and \( h \)'s (defined in (80)) from their original Poisson brackets. Using eq. (73), one finds that

\[
\{ \exp(2(\rho - \phi))(\tau, \sigma), (\partial_-(\psi))(\tau, \sigma') \}_{\text{PB}} = -\delta(\sigma - \sigma') ,
\]

\[
\{ \psi(\tau, \sigma), [\partial_-(\exp(2(\rho - \phi)))(\tau, \sigma')]_{\text{PB}} = -\delta(\sigma - \sigma') ,
\]

\[
\{ \exp(2(\rho - \phi))(\tau, \sigma), \psi(\tau, \sigma') \}_{\text{PB}} = 0 ,
\]

\[
[\partial_-(\exp(2(\rho - \phi)))(\tau, \sigma), (\partial_-(\psi))(\tau, \sigma') \}_{\text{PB}} = 0 ,
\]

from which we deduce that

\[
[h_+^+(\sigma^+), \xi_-(\sigma'^+)] = [\xi(\sigma^+), h_+(\sigma'^+)] = -i\hbar \delta(\sigma^+ - \sigma'^+) ,
\]

and other commutators vanish identically.

We define the positive and negative frequency parts of \( J(\sigma^+)(=h_+^+(\sigma^+)\cdots) \) by

\[
J^{(\pm)}(\sigma^+) = \int d\sigma'^+ \delta^{(\pm)}(\sigma'^+ - \sigma^+) J(\sigma'^+) ,
\]

\[
\delta^{(\pm)}(\sigma^+) = \frac{\pm i}{2\pi(\sigma^+ \pm i0^+)} ,
\]

and employ the normal ordering prescription with respect to \( J^{(\pm)} \) to define the products of operators \( J^{(+)} \) stands to the right of \( J^{(-)} \). We then split the commutation relation (84) into

\[
[h_+^{(+)}(\sigma^+), \xi_-^{(-)}(\sigma'^+)] = -i\hbar \delta^{(+)}(\sigma^+ - \sigma'^+) ,
\]

\[
[h_+^{(-)}(\sigma^+), \xi_-^{(+)}(\sigma'^+)] = -i\hbar \delta^{(-)}(\sigma^+ - \sigma'^+) ,
\]

and similarly for \( \xi \) and \( h_+ \), and also for \( c^+, c_+ \cdots \).

As for the \( Q^2 \)-anomaly, we can verify the result of ref. \[9\] that \( Q_{\text{LG}} \) given in (79) with (81) satisfies the nilpotency condition. The main reason of this is that the central charge, \( c^d \), for the algebra

\[
[ T^{d}_{++}(\sigma^+), T^{d}_{++}(\sigma'^+) ] = i\hbar (T^{d}_{++}(\sigma^+) + T^{d}_{++}(\sigma'^+)) \delta'(\sigma^+ - \sigma'^+)
\]

\[
- i\hbar^2 c^d \frac{\delta''(\sigma^+ - \sigma'^+)}{24\pi} ,
\]

\[11\]The step function is given by \( \theta(\pm \omega) = \int dx \delta^{(\pm)}(x) e^{i\omega x} \).
is exactly $+28$ which is canceled by the ghost contribution \cite{20,21}. ($T_{++}^d$ is normal-ordered.)

The second commutator, $[Q_{\text{LG}}, W_{\text{LG}}]$, can be trivially calculated because $W_{\text{LG}}$ depends on $h_+$ only linearly; the commutator vanishes identically. So $\Omega$ and $\Xi$ are zero in the (modified) light-cone gauge, as we have expected from the algebraic consideration in section 4.

Triviality of the contribution of the conformal matter can be seen in the modified light-cone gauge too. Suppose that the matter contribution to the energy-momentum tensor is denoted by $T_{++}^m$ in that gauge, which satisfies the Virasoro algebra (87) with the central charge $c^m$. The introduction of the matter modifies the BRST charge (79) to

$$Q_{\text{LG}}^m = Q_{\text{LG}} + \int d\sigma^+ c^+ T_{++}^m , \quad (88)$$

which is no longer nilpotent, i.e.,

$$[Q_{\text{LG}}^m, Q_{\text{LG}}^m] = \frac{i\hbar^2}{24\pi} \left(\frac{1}{8}\right) c^m \int d\sigma^+ \partial_+ c^+ (\partial_+)^2 c^+ , \quad (89)$$

where all the operator products are normal-ordered according to (85). But the r.h.s. of (89) is BRST trivial, because it can be written as

$$-\frac{i\hbar^2}{12\pi} \left(\frac{1}{8}\right) c^m \delta \int d\sigma^+ \partial_+ c^+ \partial_+ h_+ , \quad (90)$$

where $h_+$ is defined in (80). Therefore, we can absorb this trivial anomaly into a re-definition of $Q_{\text{LG}}^m$, and one finds that the re-defined one is given by

$$Q_{\text{LG}}^{m'} = Q_{\text{LG}}^m - \frac{\hbar}{24\pi} \left(\frac{1}{8}\right) c^m \int d\sigma^+ \partial_+ c^+ h_+ . \quad (91)$$

One can easily verify that $Q_{\text{LG}}^{m'}$ given above exactly satisfies the quantum-nilpotency condition.

Before we close this section, we summarize its content. We considered the CGHS action with the matter coupling switched off, because we observed that the matter fields $f_i$ can not non-trivially contribute to the anomalous Schwinger terms $\Omega$ and $\Xi$. Then we went to the (modified) light-cone gauge in which the theory takes the most simple form so that we have to deal with only free fields. The BRST charge and $W$ were expressed in
terms of those free fields. Having defined the normal ordering prescription, we computed the relevant commutators and found that $\Omega$ and $\Xi$ vanish identically in the (modified) light-cone gauge.

The general result of ref. [22] on the relation between the trace anomaly and the Hawking radiation in two-dimensions remains of course correct. But our analysis suggests that one should first investigate whether Weyl anomalies are cohomologically trivial or not in a given model before one applies the general result.

7 Discussion

There are some shortcomings in our investigation on anomalies in the cosmological model of CGHS. We find the following two may be serious:

(i) We have not explicitly taken into account the black hole background in computing anomalies. That is, we have assumed that the invariance property of the theory is not influenced by a space-time singularity. This assumption is certainly a strong assumption, and should be carefully checked in future.

(ii) We have neglected the infrared problem in two-dimensions [23] (see also ref. [24] and references therein). If this problem is real, we have to introduce an infrared cut-off which might violate the Weyl invariance. In this connection, it may be interesting to introduce massive matter fields and then the mass-zero limit. Some non-trivial interplay between the mass of the matter and the infrared cut-off in the limit could be observed. The message of this paper is therefore that Weyl invariance can be anomaly-free in two-dimensional dilaton gravity models and the scale of its violation is presumably related to the rate of the Hawking radiation in two-dimensions.

It might be possible that the classical CGHS theory can not be consistently quantized. At least for the case of a compact space-time, this problem seems to be real [25]. We nevertheless believe that the CGHS model still remains as a good theoretical laboratory in understanding the real nature of Hawking radiation.

We thank Kazuo Fujikawa and Hikaru Kawai for useful discussions.
Appendix

Before we present the algebraic proof for the closure of the algebra (58) under commutator, we would like to justify the chiral split of the $Q^2$-anomaly, which is expressed by eqs. (48) and (49).

The split is by no means trivial because $\partial_+ \psi (\partial_- \psi)$ in $Q_{LG}^{(+)} (Q_{LG}^{(-)})$ contains left and right movers, as one can easily see from the equation of motion (5). One can also see that $\partial_+ \psi$ and $\partial_- \psi$ become chiral if $\mu^2 = 0$. Therefore, the chiral split (48) is absolutely correct if $\mu^2 = 0$, and one may write the violation of the chiral split as

$$[Q_{\mathcal{CG}}^{(+)}, Q_{\mathcal{CG}}^{(-)}] = i \hbar^2 \mu^2 \Omega_{\mathcal{CG}}^{(+,-2)} + O(\hbar^3), \quad (A.1)$$

with the most general form for $\Omega_{\mathcal{CG}}^{(+,-2)}$ (consistent with the assumptions specified in section 4)

$$\Omega_{\mathcal{CG}}^{(+,-2)} = \int d\sigma \left[ \alpha_1 (C^+ \partial_+ C^+ + C^- \partial_- C^-) + \alpha_2 (C^+ \partial_- C^- + C^- \partial_+ C^+) \right. \left. + \alpha_3 C^+ C^- (Y_+ - Y_-) \right]. \quad (A.2)$$

The term in the first parenthesis is BRST trivial as one can easily find from the BRST transformations (47), while the term in second parenthesis is a spatial, total derivative which follows from the equations of motion for $C^\pm$. The consistency condition on $\Omega_{\mathcal{CG}}^{(+,-2)}$ then requires that $\alpha_3 = 0$. This justifies the assumption on the chiral split (48).

Now we come to the original task. We would like to finish the proof along the line of section 4. To this end, we have to BRST symmetrize the residual symmetries. Introducing a new fermionic ghost, $C_t^{(+)}$, and as well as a new bosonic ghost, $C_b^{(+)}$, we first construct a BRST charge for the residual symmetries:

$$Q_{R}^{(+)} = \int d\sigma^+ \left\{ C_t^{(+)} \partial_+ (\rho - \phi) + C_b^{(+)} \partial_+ [(\frac{1}{2} \partial_+ + \partial_+ (\rho - \phi)) C^+] \right\}, \quad (A.3)$$

and similarly for $Q_{R}^{(-)}$, but we discuss only the left-moving sector below (because the right-moving sector can be treated in the exactly same way). One, however, easily observes that the above BRST charge does not commute with $Q_{\mathcal{CG}}^{(+)}$ (33) under Poisson bracket. So we modify $Q_{\mathcal{CG}}^{(+)}$ as

$$\tilde{Q}_{\mathcal{CG}}^{(+)} = Q_{\mathcal{CG}}^{(+)} + \int d\sigma^+ C_t^{(+)} P_b^{(+)} , \quad (A.4)$$
where $\mathbf{P}^{+}_{b(f)}$ is the conjugate momentum of $C^{+}_{b(i)}$. The algebra (58) is thus transferred to

$$
\{ \tilde{Q}^{(+)}_{CG}, Q^{(+)}_{R} \}_{PB} = \{ Q^{(+)}_{R}, Q^{(+)}_{R} \}_{PB} = 0 .
$$

(A.5)

The new term in the BRST charge (A.4) can not influence the quantum-nilpotency property, i.e. $[ \tilde{Q}^{(+)}_{CG}, \tilde{Q}^{(+)}_{CG} ] = 0$. Because of this, there are two possible Schwinger terms:

$[ \tilde{Q}^{(+)}_{CG}, Q^{(+)}_{R} ] = i\hbar^2 \tilde{\Omega}_{CGR}^{(+2)} + O(\hbar^3) ,

[ Q^{(+)}_{R}, Q^{(+)}_{R} ] = i\hbar^2 \Omega_{R}^{(+2)} + O(\hbar^3) .

(A.6)

First we consider $\Omega_{CGR}^{(+2)}$ which has to have the form

$$
\Omega_{CGR}^{(+2)} = \int d\sigma^{+} \{ c_{i}^{(+)} C^{+} Y_{i} ( \partial_{+}, Y_{+}, G_{+}, F_{+}^{i} ) + c_{b}^{(+)} C^{+} \partial_{+} Y_{b} ( \partial_{+}, Y_{+}, G_{+}, F_{+}^{i} ) \},
$$

(A.7)

where $Y_{+}, G_{+}, F_{+}^{i}$ are defined in (46). The consistency condition on $\Omega_{CGR}^{(+2)}$, which can be derived from the jacobi identity

$$
2 \{ \tilde{Q}^{(+)}_{CG}, [ \tilde{Q}^{(+)}_{CG}, Q^{(+)}_{R} ] \} + \{ Q^{(+)}_{R}, [ \tilde{Q}^{(+)}_{CG}, \tilde{Q}^{(+)}_{CG} ] \} = 0 ,
$$

requires at $O(\hbar^3)$ that

$$
\delta_{CG} \Omega_{CGR}^{(+2)} = - \{ \tilde{Q}^{(+)}_{CG}, \Omega_{CGR}^{(+2)} \}_{PB} = 0 .
$$

(A.8)

After some algebraic calculations, we find that the solution to (A.8) is given by

$$
\Omega_{CGR}^{(+2)} = \delta_{CG} \int d\sigma^{+} \{ c_{i}^{(+)} \left[ \beta_{1} (\partial_{+})^{2} C^{+} + \beta_{2} \partial_{+} Y_{+} C^{+} + \beta_{3} Y_{+}^{2} C^{+} \right.

+ \beta_{4} G_{+} C^{+} + \beta_{5} \sum_{i=1}^{n} (F_{+}^{i})^{2} C^{+} \left. \right] + \beta_{6} c_{i}^{(+)} Y_{+} \} ,
$$

(A.9)

which is BRST trivial.

As for the Schwinger term $\Omega_{R}^{(+2)}$, it has to satisfy two independent consistency conditions

$$
\delta_{CG} \Omega_{R}^{(+2)} = - \{ \tilde{Q}^{(+)}_{CG}, \Omega_{R}^{(+2)} \}_{PB} = 0 ,

\delta_{R} \Omega_{R}^{(+2)} = - \{ Q^{(+)}_{R}, \Omega_{R}^{(+2)} \}_{PB} = 0 .
$$

(A.10)

We again write the most general form for $\Omega_{R}^{(+2)}$, and go through algebraic calculations similar to the previous ones. One finally finds that there is no no-trivial solution to (A.10).

Together with the result on $\Omega_{CGR}^{(+2)}$, we thus conclude that the algebra (A.5) and hence (58) closes under commutator too.
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