SUPERQUADRATIC FUNCTIONS AND REFINEMENTS OF INEQUALITIES BETWEEN AVERAGES

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Abstract. In this paper upper bounds are given for the successive differences $A_{n+1} - A_n$ and $B_n - B_{n-1}$ where $A_n = 1/(n-1) \sum_{r=1}^{n-1} f(r/n)$, $B_n = 1/(n+1) \sum_{r=0}^{n} f(r/n)$ and $f$ is superquadratic function. We obtain bounds for the successive differences of the more general sequence $1/c_n \sum_{r=1}^{n} f(a_r/b_n)$ when $f$ is superquadratic, which refine known results for convex functions. We also obtain bounds for various successive differences when $f$ is an increasing subquadratic function.

1. Introduction

We define the averages

$$A_n(f) = \frac{1}{n-1} \sum_{r=1}^{n-1} f\left(\frac{r}{n}\right), \quad n \geq 2$$

and

$$B_n(f) = \frac{1}{n+1} \sum_{r=0}^{n} f\left(\frac{r}{n}\right), \quad n \geq 1.$$ 

In [9] (see also [8]) it was shown that if $f$ is convex, then $A_n(f)$ increases with $n$ and $B_n(f)$ decreases. In [5], for the class of superquadratic functions, the theorems of [9] are generalized in the following Theorem A and Theorem B.

**Theorem A** [5] If $f$ is superquadratic on $[0,1]$ then for $n \geq 2$

$$A_{n+1}(f) - A_n(f) \geq f\left(\frac{1}{3n}\right) + \sum_{r=1}^{n-1} \lambda_r f\left(y_r\right),$$

where $y_r = \frac{\lfloor 2(n-1-r) \rfloor}{3n(n+1)}$. Moreover if $f$ is superquadratic and non-negative, then for $n \geq 3$

$$A_{n+1}(f) - A_n(f) \geq f\left(\frac{1}{3n}\right) + f\left(\frac{16}{81(n+3)}\right).$$

**Theorem B** [5] If $f$ is superquadratic on $[0,1]$, then for $n \geq 2$

$$B_{n-1}(f) - B_n(f) \geq f\left(\frac{1}{3n}\right) + \sum_{r=1}^{n} \lambda_r f\left(y_r\right),$$

where $y_r = \frac{\lfloor 2(n+1-3r) \rfloor}{3n(n-1)}$. Moreover, if $f$ is also non-negative, then for $n \geq 2$

$$B_{n-1}(f) - B_n(f) \geq f\left(\frac{1}{3n}\right) + f\left(\frac{16}{81n}\right).$$
In this article we find upper bounds for the difference \( A_{n+1}(f) - A_n(f) \) and for the difference \( B_{n-1}(f) - B_n(f) \). We also generalize the lower bounds obtained in \([10], [11] \) and \([14]\).

Now we present the class of superquadratic functions \( f \) that we use in this paper to get our results related to differences of averages. This was introduced in \([4] \) and \([5]\), and dealt with in \([1], [2], [3], [6], [7], [12], [13]\) and other papers.

**Definition 1.** \([4, 5]\) A function \( f \), defined on an interval \( I = [0, L] \) or \([0, \infty) \) is superquadratic, if for each \( x \) in \( I \), there exists a real number \( C(x) \) such that
\[
f(y) - f(x) \geq C(x) (y - x) + f(|y - x|) \tag{1.1}
\]
for all \( y \in I \). A function \( f \) is subquadratic if \(-f\) is superquadratic.

As stated in the following Lemma A, positive superquadratic functions are also convex, increasing and satisfy \( f(0) = 0 \) (like \( f(x) = x^m \), \( m \geq 2 \)). Therefore the results obtained in this paper leads to refinements of results in \([9], [10] \) and \([14]\).

**Lemma A** \([4]\) Let \( f \) be a superquadratic function with \( C(x) \) as in Definition \([7]\). Then
\[(i) \text{ if } f(0) \leq 0, \]
\[(ii) \text{ if } f(0) = f'(0) = 0, \]
\[(iii) \text{ if } f(x) \geq 0, x \in I, \]
\[\text{then } f \text{ is convex on } I \text{ and } f(0) = f'(0) = 0.\]

**Lemma B** \([4]\) Suppose that \( f \) is superquadratic. Let \( x_r \geq 0, 1 \leq r \leq n \) and let \( \overline{x} = \sum_{r=1}^{n} \lambda_r x_r \) where \( \lambda_r \geq 0 \), and \( \sum_{r=1}^{n} \lambda_r = 1. \) Then
\[
\sum_{r=1}^{n} \lambda_r f(x_r) \geq f(\overline{x}) + \sum_{r=1}^{n} \lambda_r f(|x_r - \overline{x}|). \tag{1.2}
\]
If \( f(x) \) is subquadratic the reverse inequality to \((1.2)\) holds.

From Lemma B we get the immediate result which we state in the following Lemma 1, by repeating \((1.2)\) \( t \) times.

**Lemma 1.** Let \( f \) be superquadratic on \([0, L] \) and let \( x, y \in [0, L], 0 \leq \lambda \leq 1. \) Then
\[
\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y) + \lambda f((1 - \lambda)|y - x|) + (1 - \lambda) f(\lambda|y - x|) \tag{1.3}
\]
If \( f \) is positive superquadratic we get from \((1.3)\) that
\[
\lambda f(x) + (1 - \lambda) f(y) \geq f(\lambda x + (1 - \lambda) y) + \sum_{k=0}^{t-1} \lambda f(2\lambda (1 - \lambda)|1 - 2\lambda|^{k}|x - y|) \tag{1.4}
\]
If \( f \) is subquadratic we get the reverse inequality of \((1.3)\).

From Lemma 1 we get the following result for a positive superquadratic functions.
Lemma 2. Let \( f \) be a positive superquadratic function on \([0, L]\). Let \( A_i \in [0, L], \ 0 \leq \lambda_i \leq 1, \ i = 1, ..., m \). Then
\[
\sum_{i=1}^{m} [\lambda_i f ((1 - \lambda_i) A_i) + (1 - \lambda_i) f (\lambda_i A_i)] \\
\geq \sum_{k=0}^{t} m f \left( \sum_{i=1}^{m} 2\lambda_i (1 - \lambda_i) (1 - \lambda_i)^k A_i \right), \quad t = 0, 1, 2, ...
\] (1.5)

If \( \lambda_i A_i \geq A, \ i = 1, ..., m, \) then
\[
\sum_{i=1}^{m} [\lambda_i f ((1 - \lambda_i) A_i) + (1 - \lambda_i) f (\lambda_i A_i)] \\
\geq \sum_{k=0}^{t} m f \left( \sum_{i=1}^{m} 2(1 - \lambda_i) (1 - \lambda_i)^k A \right).
\] (1.6)

2. Superquadracity, subquadracity and upper bounds of averages

In the following theorem, we establish an upper bounds for the differences \( A_{n+1}(f) - A_n(f) \) and \( B_{n-1}(f) - B_n(f) \) for superquadratic functions whereas in [6] and [2], lower bounds were established for convex functions and superquadratic functions respectively.

Theorem 1. Let \( f \) be a superquadratic function on \([0, 1]\). Then for \( 1 \leq r \leq n, \ n \geq 3, \) we get
\[
A_{n+1}(f) - A_n(f) \\
\leq \frac{1}{2} \left[ f \left( \frac{1}{n+1} \right) + f \left( \frac{n}{n+1} \right) \right] - \sum_{r=1}^{n-1} \left[ \frac{2r-1}{n(n-1)} f \left( \frac{n-r}{n+1} \right) + \frac{1}{n-1} f \left( \frac{r}{n} \right) \right].
\] (2.1)

Moreover, if \( f \) is also positive, then
\[
A_{n+1}(f) - A_n(f) \leq \frac{1}{2} \left[ f \left( \frac{1}{n+1} \right) + f \left( \frac{n}{n+1} \right) \right] - \left[ f \left( \frac{n-2}{n(n+1)} \right) + f \left( \frac{2}{n} \right) \right].
\] (2.2)

Proof: As \( f \) is a superquadratic function on \([0, 1]\), then by inserting in Lemma \ref{lemma2} \( x = \frac{1}{n+1}, \ y = \frac{n}{n+1}, \ \lambda = \frac{n}{n+1}, \ r = 1, ..., n \) we get
\[
f \left( \frac{r}{n+1} \right) \leq \frac{n-r}{n-1} f \left( \frac{1}{n+1} \right) + \frac{r-1}{n-1} f \left( \frac{n}{n+1} \right) - \left[ \frac{n-r}{n-1} f \left( \frac{r-1}{n+1} \right) + \frac{r-1}{n-1} f \left( \frac{n-r}{n+1} \right) \right].
\] (2.3)

Hence
\[
\sum_{r=1}^{n} f \left( \frac{r}{n+1} \right) \\
\leq \sum_{r=1}^{n} \left[ \frac{n-r}{n-1} f \left( \frac{1}{n+1} \right) + \frac{r-1}{n-1} f \left( \frac{n}{n+1} \right) \right] - \sum_{r=1}^{n} \left[ \frac{n-r}{n-1} f \left( \frac{r-1}{n+1} \right) + \frac{r-1}{n-1} f \left( \frac{n-r}{n+1} \right) \right] \\
= \frac{n}{2} \left[ f \left( \frac{1}{n+1} \right) + f \left( \frac{n}{n+1} \right) \right] - \sum_{r=1}^{n} \left[ \frac{n-r}{n-1} f \left( \frac{r-1}{n+1} \right) + \frac{r-1}{n-1} f \left( \frac{n-r}{n+1} \right) \right] \\
= \frac{n}{2} \left[ f \left( \frac{1}{n+1} \right) + f \left( \frac{n}{n+1} \right) \right] - \sum_{r=1}^{n-1} \frac{2r}{n-1} f \left( \frac{n-r-1}{n+1} \right).
\] (2.4)

From (2.4) we get (2.1).
If the superquadratic $f$ is also positive on $[0, 1]$ then $f$ is convex and from (2.1) we get
\[
A_{n+1}(f) - A_n(f) \leq \frac{1}{2} \left[ f\left(\frac{n+1}{n+1}\right) + f\left(\frac{n}{n+1}\right)\right] - \left[ f\left(\sum_{r=1}^{n-1} \frac{2(r-1)(n-r)}{n(n-1)(n+1)}\right) + f\left(\frac{1}{2}\right)\right] \\
= \frac{1}{2} \left[ f\left(\frac{n+1}{n+1}\right) + f\left(\frac{n}{n+1}\right)\right] - \left[ f\left(\frac{n-2}{n(n+1)}\right) + f\left(\frac{1}{2}\right)\right].
\]

Hence, (2.2) holds. This completes the proof of the theorem. \hfill \square

In the following theorem we establish an upper bound for the difference $B_{n-1}(f) - B_n(f)$ by similar reasoning to those used in Theorem 1. The proof is omitted here.

**Theorem 2.** Let $f$ be a positive superquadratic function on $[0, 1]$. Then
\[
B_{n-1}(f) - B_n(f) \leq \frac{n-1}{n} \left[ f\left(\frac{1}{n-1}\right) + f(1)\right] - \frac{n^2-3}{n^2} f\left(\frac{1}{3}\right) - f\left(\frac{1}{2}\right).
\]

**Remark 1.** The arguments in Theorems 1 and 2 can be generalized to an upper bound of $\sum_{i=1}^{m} f(\frac{a_i}{a_n})$ where $(a_i)_{i \geq 1}$ is positive increasing sequence and $f$ is superquadratic function. Therefore, putting in Lemma 7 that $\lambda = \frac{a_n-a_i}{a_n}$, $x = \frac{a_n}{a_n}$, $y = \frac{a_n}{a_n} = 1$, it follows
\[
f\left(\frac{a_n}{a_n}\right) \leq \frac{a_n-a_i}{a_n-a_1} f\left(\frac{a_1}{a_n}\right) + \frac{a_1-a_i}{a_n-a_1} f(1) - \frac{a_n-a_i}{a_n-a_1} f\left(\frac{a_1-a_i}{a_n-a_1}\right) - \frac{a_n-a_i}{a_n-a_1} f\left(\frac{a_1-a_i}{a_n-a_1}\right).
\]

Hence,
\[
\sum_{i=1}^{m} f\left(\frac{a_i}{a_n}\right) \leq f\left(\frac{a_1}{a_n}\right) \sum_{i=1}^{m} \frac{a_n-a_i}{a_n-a_1} + f(1) \sum_{i=1}^{m} \frac{a_1-a_i}{a_n-a_1} \\
- \sum_{i=1}^{m} \left[ \frac{a_n-a_i}{a_n-a_1} f\left(\frac{a_1-a_i}{a_n-a_1}\right) + \frac{a_1-a_i}{a_n-a_1} f\left(\frac{a_n-a_i}{a_n-a_1}\right)\right].
\]

If $f$ is also positive and therefore convex, we get from the last inequality that
\[
\sum_{i=1}^{m} f\left(\frac{a_i}{a_n}\right) \leq f\left(\frac{a_1}{a_n}\right) \sum_{i=1}^{m} \frac{a_n-a_i}{a_n-a_1} + f(1) \sum_{i=1}^{m} \frac{a_1-a_i}{a_n-a_1} \\
- mf \sum_{i=1}^{m} \frac{2(a_n-a_i)(a_1-a_i)}{(a_n-a_1)a_n a_m}.
\]

In theorems 1 and 2 we simplified the last inequality according to the specific $m$, $n$, and $(a_i)_{i \geq 1}$.

If $a_i$, $i = 1, \ldots, n$ is a general positive increasing sequence we get from (2.7), using $2(a_n-a_i)(a_i-a_1) > 2(a_n-a_{n-1})(a_2-a_1)$, that for $m = n$ the inequality
\[
\sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right) \leq f\left(\frac{a_1}{a_n}\right) \left(\frac{n a_n - \sum_{i=1}^{n} a_i}{a_n-a_1}\right) + f(1) \left(\frac{\sum_{i=1}^{n} a_i-a_n}{a_n-a_1}\right) \\
- nf \left(\frac{2(a_n-a_{n-1})(a_2-a_1)}{(a_n-a_1)a_n}\right)
\]
holds. Since for convex functions we have
\[
\sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right) \geq n \sum_{i=1}^{n} f\left(\frac{\sum_{i=1}^{n} a_i}{na_n}\right),
\]
from (2.6) and (2.7) the following results directly follows.
Theorem 4. Let \( f \) be a positive increasing sequence and \( a_i \). Then
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \frac{a_i}{a_n} \right) - \frac{1}{n+1} \sum_{i=1}^{n+1} \left( \frac{a_i}{a_{n+1}} \right)
\leq \frac{n a_n - \sum_{i=1}^{n} a_i}{n(a_n-a_{n-1})} f \left( \frac{a_n}{c_n} \right) + \sum_{i=1}^{n} \frac{a_i}{n(a_n-a_1)} f \left( \frac{1}{c_n} \right) - f \left( \frac{2(a_n-a_{n-1})(a_2-a_1)}{(a_n-a_1)a_n} \right) - f \left( \frac{\sum_{i=1}^{n+1} a_i}{(n+1)a_{n+1}} \right).
\]

Theorem 3. Let \( f \) be a positive superquadratic function on \([0,1]\) and let \((a_i)_{i \in \mathbb{N}}\) be a positive increasing sequence. Then
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \frac{a_i}{a_n} \right) - \frac{1}{n+1} \sum_{i=1}^{n+1} \left( \frac{a_i}{a_{n+1}} \right)
\leq \frac{n a_n - \sum_{i=1}^{n} a_i}{n(a_n-a_{n-1})} f \left( \frac{a_n}{c_n} \right) + \sum_{i=1}^{n} \frac{a_i}{n(a_n-a_1)} f \left( \frac{1}{c_n} \right) - f \left( \frac{2(a_n-a_{n-1})(a_2-a_1)}{(a_n-a_1)a_n} \right) - f \left( \frac{\sum_{i=1}^{n+1} a_i}{(n+1)a_{n+1}} \right).
\]

In the following theorem we prove some inequalities for \( A_n (f) \) and \( B_n (f) \) for increasing subquadratic functions.

First we state a lemma that follows immediately from Lemma B for subquadratic functions.

Lemma 3. Let \( f \) be increasing and subquadratic, and let
\[
x_r \leq 2 \sum_{i=1}^{n} \lambda_i x_i, \quad r = 1, \ldots, n
\]
where \( \lambda_i \geq 0, x_i \geq 0, i = 1, \ldots, n, \sum_{i=1}^{n} \lambda_i = 1 \). Then
\[
\sum_{r=1}^{n} \lambda_r f (x_r) \leq 2 f \left( \sum_{r=1}^{n} \lambda_r (x_r) \right).
\]
In particular, if \( \max \{ x_r : r = 1, \ldots, n \} \leq 2 \min \{ x_r : r = 1, \ldots, n \} \), then (2.9) holds and therefore (2.8) holds.

For subquadratic increasing functions (which are therefore also non-negative) the proofs of the following Theorem 5 for \( A_n (f) \) and \( B_n (f) \) show that (2.9) holds as (2.8) always holds. The bounds given here for subquadratic functions are not the best possible, but as they are easy to obtain we state these bounds and prove them.

Theorem 5. Let \( f \) be increasing subquadratic function on \([0,1]\). Then for \( n \geq 2 \)
\[
A_{n+1} (f) \leq 2 A_n (f)
\]
and
\[
B_{n-1} (f) \leq 2 B_n (f).
\]
If \( f \) is also convex, we get that
\[
A_n (f) \leq A_{n+1} (f) \leq 2 A_n (f)
\]
and
\[
B_n (f) \leq B_{n-1} (f) \leq 2 B_n (f).
\]
Proof. We use the same technique as in [5] and [9]. By some manipulations we get that
\[ A_{n+1}(f) = \frac{1}{n} \sum_{r=1}^{n} f \left( \frac{r}{n+r} \right) = \frac{1}{n} \sum_{r=1}^{n-1} \left[ \frac{r}{n} f \left( \frac{r}{n+r} \right) + \frac{n-r}{n} f \left( \frac{n-r}{n+r} \right) \right]. \]
Let us denote \( x_1(r) = \frac{r}{n+1}, \) \( x_2(r) = \frac{n-r}{n+1}, \) \( \lambda_1 = \frac{x_1}{n}, \) \( \lambda_2 = \frac{x_2}{n}, \) for \( 1 \leq r \leq n-1. \) Now we get that \( \lambda_1 x_1 + \lambda_2 x_2 = \frac{r}{n}. \)
As \( x_1 = \frac{r+1}{n+1} < 2 \left( \frac{x}{n} \right) = 2 (\lambda_1 x_1 + \lambda_2 x_2) \) and \( x_2 = \frac{r}{n+1} < 2 \left( \frac{x}{n} \right) = 2 (\lambda_1 x_1 + \lambda_2 x_2), \) we get that (2.8) is satisfied, and as \( f \) is subquadratic increasing we get by Lemma \[ \lambda \] that
\[ \frac{1}{n-1} \sum_{r=1}^{n-1} f \left( \frac{r}{n+r} \right) \leq 2 \frac{1}{n} \sum_{r=1}^{n-1} f \left( \frac{x}{n} \right), \]
and this is inequality (2.10). The same reasoning leads to (2.11). In [9] it was proved that \( A_n(f) \) increases with \( n \) and \( B_n(f) \) decreases when \( f \) is convex. Therefore, if \( f \) is convex increasing and subquadratic we get that (2.12) and (2.13) hold. \( \square \)

Remark 2. The same reasoning that lead to (2.10) also shows that \( A_n(f) \leq 2 f \left( \frac{1}{n} \right) \) for subquadratic increasing function, and if \( f \) is also convex, \( A_n(f) \geq f \left( \frac{1}{n} \right). \) Therefore, if \( f \) is convex increasing subquadratic function then \( f \left( \frac{1}{n} \right) \leq A_n(f) \leq A_{n+1}(f) \leq 2 A_n(f) \leq 4 f \left( \frac{1}{n} \right). \)

Remark 3. The following functions are examples of subquadratic increasing functions which are also convex (see [4]) and therefore satisfy Theorem [5]
\[
\begin{align*}
    f(x) &= x^p, \quad 1 \leq p \leq 2, \\
    f(x) &= (1 + x^p)^{1/p}, \quad 1 \leq p, \\
    f(x) &= (1 + x^p)^{1/p} - 1, \quad 1 \leq p \leq 2, \\
    f(x) &= 3x^2 - 2x^2 \log(x), \quad 0 \leq x \leq 1.
\end{align*}
\]

3. SUPERQUADRATICITY AND LOWER BOUNDS OF AVERAGES

The following theorem refines the results of [14] for convex functions which are also superquadratic (like \( f(x) = x^m, \) \( m \geq 2). \)

Theorem 6. Let \( f \) be a positive superquadratic function on \([0, 1]\) and let \( (a_i)_{i \in \mathbb{N}}, \) \( a_i > 0, \) and \( \left( i \left( 1 - \frac{a_i}{a_{i+1}} \right) \right)_{i \in \mathbb{N}} \) be increasing sequences. Then for \( n \geq 2 \) we get that
\[
\Delta := \frac{1}{n} \sum_{i=1}^{n} f \left( \frac{a_i}{a_n} \right) - \frac{1}{n+1} \sum_{i=1}^{n+1} f \left( \frac{a_i}{a_{n+1}} \right) 
\geq \frac{n-1}{n+1} \left[ f \left( \frac{a_n-a_{n-1}}{n a_n} \right) + f \left( \frac{(n-2)(a_2-a_1)}{2(n-1)n a_n} \right) + f \left( \frac{(n-2)(a_2-a_1)}{3n^2 a_n} \right) \right].
\]

Proof. From the conditions on \( a_i, \) \( i = 1, \ldots, n + 1, \) it is obvious that
\[
\frac{1}{n} \left( \frac{a_i}{a_n} - \frac{a_{i+1}}{a_n} \right) \geq \frac{a_i-a_{i+1}}{n a_n} =: A.
\]

By the same considerations as in [14] we get that
\[
\frac{(i-1)a_{i-1}+(n-i+1)a_i}{n a_n} \geq \frac{a_i}{a_{i+1}}
\]
and

\[
\Delta = \frac{1}{n+1} \sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right) - \frac{1}{n+1} \sum_{i=1}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right)
\]

\[= \frac{1}{n+1} \sum_{i=1}^{n} \left[ \frac{(i-1)}{n} f\left(\frac{a_i}{a_n}\right) + \frac{(n-i+1)}{n} f\left(\frac{a_i}{a_{n+1}}\right) \right]. \tag{3.4}
\]

From the superquadraticity of \(f\) we get from \((1.2)\) and \((3.4)\) that

\[
\Delta = \frac{1}{n+1} \sum_{i=1}^{n} \left[ \frac{(i-1)}{n} f\left(\frac{(n-i-1)(a_i-a_{i-1})}{n a_n}\right) + \frac{(n-i+1)}{n} f\left(\frac{(i-1)(a_i-a_{i-1})}{n a_n}\right) \right]
\]

\[+ \frac{1}{n+1} \sum_{i=1}^{n+1} f\left(\frac{(i-1)a_{i+1}+(n-i+1)a_i}{n a_n}\right) - f\left(\frac{a_i}{a_{n+1}}\right). \tag{3.5}
\]

where we let \(a_0 = 0\).

By Lemma A as \(f(x)\) is positive superquadratic it is also increasing. Therefore from \((3.3)\) we get that

\[
\sum_{i=1}^{n} \left[ f\left(\frac{(i-1)a_{i-1}+(n-i+1)a_i}{n a_n}\right) - f\left(\frac{a_i}{a_{n+1}}\right) \right] \geq 0, \tag{3.6}
\]

and from \((3.5)\) and \((3.6)\) we get

\[
\Delta \geq \frac{1}{n+1} \sum_{i=1}^{n} \left[ \frac{(i-1)}{n} f\left(\frac{(n-i+1)(a_i-a_{i-1})}{n a_n}\right) + \frac{(n-i+1)}{n} f\left(\frac{(i-1)(a_i-a_{i-1})}{n a_n}\right) \right]. \tag{3.7}
\]

As \(f(0) = 0\) it follows

\[
\sum_{i=1}^{n} \left[ \frac{(i-1)}{n} f\left(\frac{(n-i+1)(a_i-a_{i-1})}{n a_n}\right) + \frac{(n-i+1)}{n} f\left(\frac{(i-1)(a_i-a_{i-1})}{n a_n}\right) \right]
\]

\[= \sum_{i=1}^{n} \left[ \frac{i}{n} f\left(\frac{1}{n}\right) + \frac{n-i+1}{n} f\left(\frac{1}{n}\right) \right]. \tag{3.8}
\]

From \((3.7)\), \((3.8)\) and Lemma 2 we have

\[
\Delta \geq \frac{1}{n+1} \sum_{i=1}^{n-1} \left[ \frac{i}{n} f\left(\frac{1}{n}\right) + \left(1 - \frac{i}{n}\right) f\left(\frac{1}{n}\right) \right] \tag{3.9}
\]

\[
\geq \frac{n-1}{n+1} \sum_{k=0}^{n-2} f\left(\frac{\sum_{i=1}^{n-2} \left(\frac{1}{n}\right)^i \frac{2^{n-i}}{n} f\left(\frac{a_{i+1-a_i}}{a_n}\right)}{n-1}\right). \tag{3.10}
\]

holds by inserting in \((1.6)\) \(\lambda_i = \frac{i}{n}, A_i = \frac{a_{i+1-a_i}}{a_n}, t = 2\). We also get from \((1.0)\) that

\[
\Delta \geq \frac{n-1}{n+1} \sum_{k=0}^{n-2} f\left(\frac{\sum_{i=1}^{n-2} 2(n-i)\frac{2}{n} (a_2-a_1)}{n-1}\right). \tag{3.10}
\]

It is easy to verify that \(\sum_{i=1}^{n-1} 2(n-i) = n(n-1)\) and

\[
\sum_{i=1}^{n-1} 2(n-i) |n-2| = \begin{cases} \frac{(n-2)^2}{2} & \text{if } n \equiv 2 \pmod{2}, \\ \frac{n^2}{2} & \text{if } n \equiv 1 \pmod{2}, \end{cases}
\]

\[= \frac{n+1}{2}, \quad \frac{n^2}{2} < \frac{n(n-1)}{2}, n = 2, 3, \ldots.
\]
The following theorem deals with a lower bound for
\[
\frac{1}{cn} \sum_{i=1}^{n} f \left( \frac{a_i}{a_n} \right) - \frac{1}{cn+1} \sum_{i=1}^{n+1} f \left( \frac{a_i}{a_{n+1}} \right)
\]
under the same conditions as in [5, Theorem 5.6], where a different lower bound is obtained for positive superquadratic function \( f \).

**Theorem 7.** Let \((a_i)_{i \geq 0}\) and \((c_i)_{i \geq 0}\) be sequences such that \(a_i > 0, c_i > 0, \) for \(i \geq 1\), and

(I) \((c_i)_{i \in \mathbb{N}}\) is increasing and \(c_0 = 0\),

(II) \((c_i - c_{i-1})_{i \in \mathbb{N}}\) is increasing.

(III) \(c_1 \left( 1 - \frac{a_i}{a_j} \right) \leq c_{i-1} \left( 1 - \frac{a_i}{a_{j+1}} \right) \leq c_n \left( 1 - \frac{a_i}{a_{n+1}} \right)\), \(i = 1, \ldots, n, n \geq 1\),

(IV) \(a_0 = 0\) and \((a_i)_{i \in \mathbb{N}}\) is increasing.

If \(f\) is superquadratic and non-negative function on \([0,1]\), then

\[
D := \frac{1}{cn} \sum_{i=1}^{n} f \left( \frac{a_i}{a_n} \right) - \frac{1}{cn+1} \sum_{i=1}^{n+1} f \left( \frac{a_i}{a_{n+1}} \right) \geq \frac{n-1}{cn+1} f \left( \frac{2c_i(a_j-a_i)c_{i-1}a_n}{c_i(a_j-c_{i-1})a_n} \right), \quad n \geq 1.
\]

**Proof.** From the conditions (I)-(IV) it is clear that

\[
\frac{c_i}{c_{i-1}} \left( \frac{a_{i+1}}{a_i} - \frac{a_i}{a_n} \right) \geq c_i(a_j-a_i) =: A
\]

and that

\[
c_i - a_{i-1} + a_i(c_{i-1} - c_{i-2}) \geq \frac{a_i}{a_{n+1}}, \quad \text{for an arbitrary } i = 1, \ldots, n.
\]

From Lemma A we know that \(f(0) \leq 0\), therefore if \(f\) is positive then \(f(0) = 0\) and from (I) and (II) we get

\[
D = \frac{1}{cn+1} \left\{ \sum_{i=1}^{n} \left[ \frac{c_i}{cn} f \left( \frac{a_i}{a_n} \right) + \frac{c_{i+1}-c_i}{c_n} f \left( \frac{a_i}{a_n} \right) \right] - \sum_{i=1}^{n+1} f \left( \frac{a_i}{a_{n+1}} \right) \right\}
\]

\[
= \frac{1}{cn+1} \left\{ \sum_{i=1}^{n} \left[ \frac{c_i}{cn} f \left( \frac{a_i}{a_n} \right) + \frac{c_{i+1}-c_i}{c_n} f \left( \frac{a_i}{a_n} \right) \right] - \sum_{i=1}^{n} f \left( \frac{a_i}{a_{n+1}} \right) \right\}
\]

\[
\geq \frac{1}{cn+1} \left\{ \sum_{i=1}^{n} \left[ \frac{c_i-1}{c_n} f \left( \frac{a_i}{a_n} \right) + \frac{c_{i-1}}{c_n} f \left( \frac{a_i}{a_n} \right) \right] - \sum_{i=1}^{n} f \left( \frac{a_i}{a_{n+1}} \right) \right\} \quad (3.13)
\]

Using the superquadraticity of \(f\) from \([12]\) and \([31]3\) it follows

\[
D \geq \frac{1}{cn+1} \sum_{i=1}^{n} \left[ \frac{c_i}{cn} f \left( \frac{(c_{i-1} - c_{i-2})a_j}{a_n a_n} \right) + \frac{c_{i-1}}{c_n} f \left( \frac{a_i-a_{i-1}}{a_n a_n} \right) \right] + \frac{1}{cn+1} \left\{ \sum_{i=1}^{n} f \left( \frac{c_{i-1}a_i-a_i}{a_n a_n} \right) - \sum_{i=1}^{n} f \left( \frac{a_i}{a_{n+1}} \right) \right\} \quad (3.14)
\]
As \( f \) is increasing, according to (3.12), (3.13) implies

\[
D \geq \frac{1}{c_{n+1}} \sum_{i=1}^{n} f \left( \frac{c_{n}-c_{i-1}}{a_{n}c_{n}} \right) + \frac{c_{n}-c_{i-1}}{c_{n}} f \left( \frac{c_{i-1}(a_{i+1}-a_{i})}{a_{n}c_{n}} \right)
\]

\[
= \frac{1}{c_{n+1}} \sum_{i=1}^{n-1} \left[ \frac{c_{i}}{c_{n}} f \left( \frac{c_{i}}{a_{i}c_{n}} \right) + \frac{c_{i}}{c_{n}} f \left( \frac{c_{i}(a_{i+1}-a_{i})}{a_{n}c_{n}} \right) \right].
\]

(3.15)

The last equality follows from \( f(0) = 0 \) and \( c_{0} = 0 \).

Inserting in (1.5) that \( \lambda_{i} = \frac{c_{i}}{c_{n}} \) and \( A_{i} = \frac{a_{i+1}-a_{i}}{a_{n}} \), from (3.15), we get

\[
D \geq \frac{n-1}{c_{n+1}} \sum_{k=1}^{t} f \left( \sum_{i=1}^{n-1} \frac{2(\frac{c_{i}}{c_{n}})}{(n-1)a_{n}c_{n}} \left| 1-\frac{2c_{i}}{c_{n}} \right|^{k} \frac{a_{i+1}-a_{i}}{a_{n}} \right).
\]

(3.16)

Using (3.11) and (1.6) we get from (3.16) that

\[
D \geq \frac{n-1}{c_{n+1}} \sum_{k=0}^{t} f \left( \sum_{i=1}^{n-1} \frac{2(\frac{c_{i}}{c_{n}})}{(n-1)a_{n}c_{n}} \left| 1-\frac{2c_{i}}{c_{n}} \right|^{k} c_{i}(a_{2}-a_{1}) \right), \quad t = 0, 1, 2, \ldots
\]

(3.17)

As \( (c_{i})_{i \in \mathbb{N}} \) is increasing, from (3.17) we have

\[
D = \frac{1}{c_{n}} \sum_{i=1}^{n} f \left( \frac{a_{i}}{a_{n}} \right) - \frac{1}{c_{n+1}} \sum_{i=1}^{n+1} f \left( \frac{a_{i}}{a_{n+1}} \right)
\]

\[
\geq \frac{n-1}{c_{n+1}} \sum_{k=0}^{t} f \left( \sum_{i=1}^{n-1} \frac{2(\frac{c_{i}}{c_{n}})}{(n-1)a_{n}c_{n}} \left| 1-\frac{2c_{i}}{c_{n}} \right|^{k} c_{i}(a_{2}-a_{1}) \right)
\]

\[
\geq \frac{n-1}{c_{n+1}} f \left( \frac{2c_{1}(a_{2}-a_{1})}{c_{n}} c_{n-1} \right).
\]

Remark 4. If \( (c_{i})_{i \in \mathbb{N}} = (a_{i})_{i \in \mathbb{N}} \), from Theorem 7 we get a refinement of Theorem 2 in [14] for convex functions that are also superquadratic.

In the following theorem we extend our investigation to three sequences. Investigation with three sequences was also dealt in [3].

**Theorem 8.** Let \( f \) be a positive superquadratic function on \([0, L]\). Let \((a_{i})_{i \geq 0}, (b_{i})_{i \geq 0}, (c_{i})_{i \geq 0}\) be sequences such that \( a_{i} > 0, b_{i} > 0, c_{i} > 0 \) for \( i \geq 1 \) and

(a) \((a_{i})_{i \in \mathbb{N}}, (b_{i})_{i \in \mathbb{N}}, (c_{i})_{i \in \mathbb{N}}\) are increasing and \( a_{0} = c_{0} = 0 \),

(b) \((c_{i} - c_{i-1})_{i \in \mathbb{N}}\) is increasing,

(c) \( c_{n} \left( 1 - \frac{b_{n}}{b_{n+1}} \right) \geq c_{r} \left( 1 - \frac{a_{r}}{a_{r+1}} \right), \quad \text{for } r \leq n. \)

Then

\[
H := \frac{1}{c_{n}} \sum_{r=1}^{n} f \left( \frac{a_{r}}{a_{n}} \right) - \frac{1}{c_{n+1}} \sum_{r=1}^{n+1} f \left( \frac{b_{r}}{b_{n+1}} \right)
\]

\[
\geq \frac{n-1}{c_{n+1}} f \left( \frac{2c_{1}(c_{n}-c_{n-1})}{c_{n} b_{n}} A \right),
\]

(3.18)

where \( A := \min\{a_{i+1} - a_{i} : i = 1, \ldots, n\}. \)

**Proof.** The technique of the proof is analogue to the techniques that we used in proving Theorem 6 and Theorem 7. \( \square \)
4. Subquadracity and Averages

In this chapter we deal with functions that are increasing and subquadratic on \([0, 1]\), like are \(f(x) = x^n, 0 \leq m \leq 2\) and 

\[
f(x) = \begin{cases} 
x^2 - 2x^2 \log x, & 0 < x \leq 1 \\
0, & x = 0 \end{cases}
\]

The last function is not concave and not convex and therefore none of the results of [9, 10, 11, and 14] are applicable to this function.

**Theorem 9.** Let \(f\) be increasing subquadratic function on \([0, 1]\). Let \((a_i)_{i \geq 0}\) satisfy

(A) \((a_i)_{i \in \mathbb{N}}\) is increasing sequence and \(a_i > 0, i = 1, \ldots, n+1,
\)

(B) \(i \left( \frac{a_{i+1}}{a_i} - 1 \right) \leq n \left( \frac{a_{n+1}}{a_n} - 1 \right), \quad i = 1, \ldots, n.\)

Then

\[
E := \frac{1}{n+1} \sum_{i=1}^{n+1} f \left( \frac{a_i}{a_{n+1}} \right) - \frac{1}{n} \sum_{i=1}^{n} f \left( \frac{a_i}{a_n} \right)
\]

\[
\leq \sum_{i=1}^{n} \left[ \frac{i}{n(n+1)} f \left( \frac{n-i+1}{n+1} \cdot \frac{a_{i+1}-a_i}{a_n+1} \right) + \frac{n-i+1}{n(n+1)} f \left( \frac{i}{n+1} \cdot \frac{a_{i+1}-a_i}{a_n+1} \right) \right]. \tag{4.1}
\]

Moreover, if in addition

(C) \(\frac{a_{i+1}}{a_i} \leq 2, \quad i = 1, \ldots, n\)

holds, then

\[
\frac{1}{n+1} \sum_{i=1}^{n+1} f \left( \frac{a_i}{a_{n+1}} \right) \leq \frac{2}{n} \sum_{i=1}^{n} f \left( \frac{a_i}{a_n} \right). \tag{4.2}
\]

**Proof.** Since \((a_i)_{i \in \mathbb{N}}\) increases we get from (B) that

\[
(n+1) \left( \frac{a_{n+1}}{a_n} - 1 \right) \geq n \left( \frac{a_{n+1}}{a_n} - 1 \right) \geq i \left( \frac{a_{i+1}}{a_i} - 1 \right), \quad i = 1, \ldots, n,
\]

which is equivalent to

\[
\frac{i a_{i+1} + (n-i+1)a_i}{(n+1)a_{n+1}} \leq \frac{a_i}{a_n} \tag{4.3}
\]

Rewriting \(E\) we get

\[
E = \frac{1}{n} \sum_{i=1}^{n+1} \frac{n-i+1}{n+1} f \left( \frac{a_i}{a_{n+1}} \right) + \frac{n-i+1}{n+1} f \left( \frac{a_i}{a_{n+1}} \right) - \sum_{i=1}^{n} f \left( \frac{a_i}{a_n} \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n+1} \left[ \frac{n-i+1}{n+1} f \left( \frac{a_i}{a_{n+1}} \right) + \frac{n-i+1}{n+1} f \left( \frac{a_i}{a_{n+1}} \right) - f \left( \frac{a_i}{a_n} \right) \right]. \tag{4.4}
\]

As \(f\) is subquadratic, by using (4.3) we get from (4.4) that

\[
E \leq \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{i}{n+1} f \left( \frac{n-i+1}{n+1} \cdot \frac{a_{i+1}-a_i}{a_n+1} \right) + \frac{n-i+1}{n+1} f \left( \frac{i}{n+1} \cdot \frac{a_{i+1}-a_i}{a_n+1} \right) \right]
\]

\[
+ \frac{1}{n} \sum_{i=1}^{n} f \left( \frac{i a_{i+1} + (n-i+1)a_i}{(n+1)a_{n+1}} \right) - f \left( \frac{a_i}{a_n} \right). \tag{4.5}
\]
As \( f \) is increasing, using (4.3), we get from (4.5) that

\[
E \leq \sum_{i=1}^{n} \left[ \frac{n-i+1}{a_{n+1}} f \left( \frac{a_{n+1}}{a_{n+1}} \cdot \frac{a_{n+2} - a_i}{a_{n+1}} \right) + \frac{n-i+1}{a_{n+1}} f \left( \frac{1}{a_{n+1}} \cdot \frac{a_{n+2} - a_i}{a_{n+1}} \right) \right].
\]

Hence (4.1) is proved. If (C) is also satisfied, then it is easy to verify that

\[
\frac{n-i+1}{a_{n+1}} \cdot \frac{a_{n+1} - a_i}{a_{n+1}} \leq \frac{a_{i+1} + (n-i+1)a_i}{(n+1)a_{n+1}}, \quad i = 1, ..., n+1.
\]

(4.6)

As it is given that \( a_i > 0, \ i = 1, ..., n+1 \), it is obvious that also

\[
\frac{a_{i+1} - a_i}{(n+1)a_{n+1}} \leq \frac{a_{i+1} + (n-i+1)a_i}{(n+1)a_{n+1}}, \quad i = 1, ..., n+1.
\]

(4.7)

By (4.6) and (4.7) we get from (4.1) and (4.3) that \( E \leq \frac{1}{n} \sum_{i=1}^{n} f \left( \frac{a_i}{a_n} \right) \) which is the same as (4.2). Hence the theorem is proved.

**Theorem 10.** Let \( f \) be an increasing subquadratic function on \([0, 1]\). Let \((a_i)_{i \geq 0}\) satisfy

(i) \( a_i > 0, \ i = 1, ..., n+1 \) and \( a_0 = 0 \),

(ii) \((a_i)_{i \in \mathbb{N}}\) and \((a_i - a_{i-1})_{i \in \mathbb{N}}\) are increasing sequences.

Then

\[
R := \frac{1}{a_{n+1}} \sum_{i=1}^{n+1} f \left( \frac{a_i}{a_{n+1}} \right) - \frac{1}{a_n} \sum_{i=1}^{n} f \left( \frac{a_i}{a_n} \right)
\]

\[
\leq \frac{1}{a_n} \sum_{i=1}^{n} \left[ \frac{a_{i+1} - a_i}{a_{n+1}} f \left( \frac{a_i}{a_n} \cdot \frac{a_{i+1} - a_i}{a_{n+1}} \right) + \frac{a_i - a_{i-1}}{a_{n+1}} f \left( \frac{a_i - a_{i-1}}{a_{n+1}} \cdot \frac{a_{n+1} - a_{i-1}}{a_{n+1}} \right) \right].
\]

(4.8)

Moreover, if an addition

(iii) \( \frac{a_{i+1}}{a_i} \leq 2, \ i = 1, ..., n \)

then

\[
\frac{1}{a_{n+1}} \sum_{i=1}^{n+1} f \left( \frac{a_i}{a_{n+1}} \right) \leq \frac{2}{a_n} \sum_{i=1}^{n} f \left( \frac{a_i}{a_n} \right).
\]

(4.9)

**Proof.** The steps of the proof are analogue to the steps we made in proving Theorem 9 \( \square \)

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