VISCOSITY SOLUTIONS AND UNIQUENESS
FOR SYSTEMS OF INHOMOGENEOUS BALANCE LAWS

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Abstract. This paper is concerned with the Cauchy problem

\[(*) \quad u_t + [F(u)]_x = g(t, x, u), \quad u(0, x) = \bar{u}(x),\]

for a nonlinear 2 \times 2 hyperbolic system of inhomogeneous balance laws in one space dimension. As usual, we assume that the system is strictly hyperbolic and that each characteristic field is either linearly degenerate or genuinely nonlinear.

Under suitable assumptions on \(g\), we prove that there exists \(T > 0\) such that, for every \(\bar{u}\) with sufficiently small total variation, the Cauchy problem \((*)\) has a unique “viscosity solution”, defined for \(t \in [0, T]\), depending continuously on the initial data.

1. Introduction. This paper is concerned with the uniqueness and stability of solutions to the Cauchy problem

\[
\begin{align*}
    u_t + [F(u)]_x &= g(t, x, u), \\
    u(0, x) &= \bar{u}(x),
\end{align*}
\]

for a nonlinear \(n \times n\) hyperbolic system of inhomogeneous balance laws in one space dimension. As usual, we assume that the system is strictly hyperbolic and that each characteristic field is either linearly degenerate or genuinely nonlinear (see [12]). Here \(F: \mathbb{R}^n \to \mathbb{R}^n\) is a smooth function, and \(g: \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n\) is continuous w.r.t. \(t\), satisfying for every \(t\) and every \((x, u), (y, v)\),

\[
|g(t, x, u) - g(t, y, v)| \leq L_g(|x - y| + |u - v|), \quad |g_x(t, x, u)| \leq k(x),
\]

with \(k \in L^1(\mathbb{R})\) and \(L_g > 0\) (see [9]).

In [8] it was proved the existence of a semigroup \(S\) for the system of conservation laws

\[
u_t + [F(u)]_x = 0,
\]

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defined on some domain $D \subset L^1(\mathbb{R}; \mathbb{R}^n)$ containing all integrable functions with sufficiently small total variation, with the following properties.

(i) There exists a constant $L_S$ such that, for all $t \geq 0$ and $\bar{u}, \bar{v} \in D$, one has

$$\|S_t \bar{u} - S_t \bar{v}\|_{L^1} \leq L_S \cdot \|\bar{u} - \bar{v}\|_{L^1}.$$ 

(ii) If $\bar{u} \in D$ is piecewise constant, then for $t > 0$ sufficiently small $S_t \bar{u}$ coincides with the solution of (1.4) which is obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.

In [4] (see also [5]) a semigroup with these properties was called a Standard Riemann Semigroup (SRS). For uniqueness results related to smaller class of functions, we mention [1], [3], [6], [10], [13] and [15].

The main result of this paper is concerned with $2 \times 2$ systems of inhomogeneous balance laws.

**Theorem 1.1** Let $F$ be a smooth map from a neighborhood of the origin $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$. Assume that the system (1.1) is strictly hyperbolic and every characteristic field is either linearly degenerate or genuinely nonlinear. Then there exist a closed domain $D \subset L^1(\mathbb{R}, \mathbb{R}^2)$, constants $L_P, \delta_0, T > 0$ and a unique continuous operator $P : \mathcal{I} \times D \to \text{cl}\{\bar{u} \in L^1(\mathbb{R}, \mathbb{R}^2) | T.V.\{\bar{u}\} \leq \delta_0\}$, where $\mathcal{I} = \{(t, s) \in [0, T] \times [0, T] | t \geq s\}$ and “cl” denotes the closure in the $L^1$-norm, with the following properties.

i) $P(t, \tau)\bar{u} = \bar{u}$ for every $\bar{u} \in D$ and every $\tau \in [0, T]$. 

ii) $P(t, \tau)P(\tau, s)\bar{u} = P(t, s)\bar{u}$ for every $\bar{u} \in D$ and $0 \leq s \leq \tau \leq t \leq T$, such that $P(\tau, s)\bar{u} \in D$.

iii) Every function $\bar{u} \in L^1(\mathbb{R}, \mathbb{R}^2)$ with $T.V.\{\bar{u}\} \leq \delta_0$ lies in $D$.

iv) $\|P(t, \tau)\bar{u} - P(s, \tau)\bar{v}\|_{L^1} \leq L_P \|t - s\|^{\frac{1}{2}} \|\bar{u} - \bar{v}\|_{L^1}$, for every $(t, \tau), (s, \tau) \in \mathcal{I}$ and $\bar{u}, \bar{v} \in D$.

v) For every $\bar{u} \in D$, the map $u(t) = P(t, 0)\bar{u}$ yields a “viscosity solution” in the sense of Definition 7.1 to the Cauchy problem (1.1)–(1.2).

In section 6 it will be proved that a viscosity solution is a weak entropic solution of the Cauchy problem (1.1)–(1.2). In section 8 we outline the proof for large initial data, following [7].

From the proof of Theorem 1.1 and in particular from Proposition 4.4 we can make more explicit the dependence of $T$ from the total variation of the initial data. More precisely, the following result holds.

**Theorem 1.2** Under the same assumptions of Theorem 1.1, there exist positive constants $L_P, \delta, C'$ such that the following holds. For every $\delta \in [0, \delta/2]$ there exist a closed domain $D(\delta) \subset L^1(\mathbb{R}, \mathbb{R}^2)$, a time

$$T(\delta) \doteq \min \left\{ \frac{1}{C'} \log \frac{\delta}{2\delta}, \frac{\delta}{2C'\|k\|_{L^1}} \right\},$$

a unique continuous operator $P : \mathcal{E} \to \text{cl}\{\bar{u} \in L^1(\mathbb{R}, \mathbb{R}^2) | T.V.\{\bar{u}\} \leq \delta\}$, where $\mathcal{E} = \bigcup_{\delta < \delta} \{(t, s) \in [0, T(\delta)]^2 \ t \geq s\} \times D(\delta)$ with the following properties.

a) $P(t, \tau)\bar{u} = \bar{u}$ for every $(\tau, \tau, \bar{u}) \in \mathcal{E}$.

b) $P(t, \tau)P(\tau, s)\bar{u} = P(t, s)\bar{u}$ for every $(\tau, s, \bar{u}) \in \mathcal{E}$ such that $(t, \tau, P(\tau, s)\bar{u}) \in \mathcal{E}$. 

G. Crasta and B. Piccoli
c) Every function \( \bar{u} \in L^1(\mathbb{R}, \mathbb{R}^2) \) with T.V.(\( \bar{u} \)) \( \leq \delta/2 \) lies in \( \mathcal{D}(\delta) \).

d) \( \| P(t, \tau) \bar{u} - P(s, \tau) \bar{v} \|_{L^1} \leq L_P(\| t - s \| + \| \bar{u} - \bar{v} \|_{L^1}) \), for every \((t, \tau, \bar{u}), (s, \tau, \bar{v}) \in \mathcal{E} \).

e) For every \( \bar{u} \in \mathcal{D}(\delta) \), the map \( u(t) = P(t, 0) \bar{u} \) yields a “viscosity solution” in the sense of Definition 7.1 to the Cauchy problem (1.1)–(1.2).

f) For every \( \delta_1, \delta_2 \in [0, \delta/2] \), if \( \{ P(t, 0) \bar{u} : t \in [0, \tau(\delta_1)], \bar{u} \in \mathcal{D}(\delta_1) \} \subset \mathcal{D}(\delta_2) \), then the operator \( P \) can be uniquely extended to the domain \([0, \tau(\delta_1) + \tau(\delta_2)] \times \mathcal{D}(\delta_1) \).

We remark that the constant \( \delta \) is related to the domain of definition of the semi-group \( S \) for the homogeneous problem (1.4) constructed in [6]. The last property in Theorem 1.2 ensures the possibility of prolonging viscosity solutions as long as their total variation remains bounded by \( \delta/2 \).

A straightforward consequence of Theorem 1.1 is the global (in time) existence of viscosity solutions if in (1.3) we replace \( k(x) \) with a function \( k(t, x) \in L^1([0, +\infty) \times \mathbb{R}) \) (see for example [9]).

2. Preliminaries. Throughout this paper, the euclidean norm and inner product on \( \mathbb{R}^2 \) are denoted by \( | \cdot | \) and \( \langle \cdot, \cdot \rangle \). \( U \) is an open convex subset of \( \mathbb{R}^2 \) containing the origin, and the map \( F: U \rightarrow \mathbb{R}^2 \) is three times continuously differentiable. We denote by \( \mathcal{PC} \) the family of all piecewise constant functions \( u \in L^1(\mathbb{R}, \mathbb{R}^2) \), and T.V.(\( v; I \)) will denote the total variation of a function \( v: \mathbb{R} \rightarrow \mathbb{R}^2 \) over an open interval \( I \). By \( \| \cdot \| \) and \( \| \cdot \|_\infty \) we will denote the norms in \( L^1 \) and \( L^\infty \) respectively.

We assume that the system (1.1) is strictly hyperbolic. More precisely, we assume that each matrix \( A(u) = DF(u) \) has two distinct eigenvalues \( \lambda_1(u) < \lambda_2(u) \). It is not restrictive to assume \( \lambda_1(u) < 0 < \lambda_2(u) \) for every \( u \in U \), and \( 0 < \lambda < |\lambda| < \bar{\lambda} \) for some constants \( \bar{\lambda} \) and \( \lambda \).

It will be convenient to work with a set of Riemann coordinates \( v = (v_1, v_2) \). We can assume that the origin in the \( u \)-coordinates corresponds to the origin in the \( v \)-coordinates, and that the map \( v \mapsto u(v) \) is a local diffeomorphism. We define \( \Omega = u^{-1}(U) \). In these new variables, we can consider a family of right and left eigenvectors \( r_i(v), l_i(v), i = 1, 2 \), of the matrix \( A(u(v)) = DF(u(v)) \), depending smoothly on \( v \), normalized according to

\[
    r_1 = (1, 0), \quad r_2 = (0, 1), \quad \langle l_i(v), r_j(v) \rangle = \delta_{ij},
\]

where \( \delta_{ij} \) is the Kronecker symbol. If \( u^+ = u(v^+) \), by \( \lambda_i(v^-, v^+) \) we denote the eigenvalues of the averaged matrix \( A(u^-, u^+) = \int_{-1}^{1} DF(\theta u^+ + (1 - \theta) u^-) d\theta \).

Given a function \( \varphi \) defined on \( \Omega \), its directional derivative at the point \( v \) along the vector field \( r_i \) is denoted by

\[
    r_i \cdot \varphi(v) = \lim_{\varepsilon \to 0} \frac{\varphi(v + \varepsilon r_i(v)) - \varphi(v)}{\varepsilon}.
\]

As usual, we assume that, for each \( i = 1, 2 \), the \( i \)-th characteristic field is either linearly degenerate, so that \( r_i \cdot \lambda_i(v) \equiv 0 \), or genuinely nonlinear, satisfying \( r_i \cdot \lambda_i(v) > 0 \) for every \( v \in \Omega \).

Now let \( \varepsilon > 0 \) be given. Following [6], for \( i = 1, 2 \), through each point \( v \) we construct a parametrized curve \( \sigma = \psi^i(v, \sigma) \) which coincides with the rarefaction curve \( \psi^i_+ \) through \( v \) for \( \sigma \geq -\sqrt{\varepsilon} \) and with the shock curve \( \psi^i_- \) for \( \sigma \leq -2\sqrt{\varepsilon} \). This
is done by choosing a smooth function $\varphi : \mathbb{R} \to [0, 1]$ such that $\varphi(\sigma) = 1$ if $\sigma \leq -2$, $\varphi(\sigma) = 0$ if $\sigma \geq -1$, $\varphi'(\sigma) \in [-2, 0]$ if $\sigma \in [-2, -1]$, and by defining

$$
\psi_\varepsilon^r(v, \sigma) = \varphi(\sigma / \sqrt{\varepsilon}) \cdot \psi^-_1(v, \sigma) + (1 - \varphi(\sigma / \sqrt{\varepsilon})) \cdot \psi^+_1(v, \sigma).
$$

We define the map $E^\varepsilon = (E_1^\varepsilon, E_2^\varepsilon)$ by letting $E^\varepsilon(v^-, v^+)$ be $(\sigma_1, \sigma_2)$ if $v^+ = \psi^\varepsilon_2(\psi^\varepsilon_1(v^-, \sigma_1), \sigma_2)$, i.e. the strengths of waves in the approximate solution of the Riemann problem.

Let $v : \mathbb{R} \to \mathbb{R}^2$ be a piecewise constant function with bounded support. Call $x_1 < \ldots < x_N$ the points where $v$ has a jump. Let us define, at each point $x_\alpha$, the quantities $\sigma_{i\alpha}$, $i = 1, 2$, such that $v(x_\alpha^+) = \psi^\varepsilon_2(\psi^\varepsilon_1(v(x_\alpha^-), \sigma_{1\alpha}), \sigma_{2\alpha})$.

The total strength of wave in $v$ is then defined as

$$
V^\varepsilon(v) = \sum_{\alpha=1}^{N} \sum_{i=1}^{2} |\sigma_{i\alpha}|,
$$

while the interaction potential is

$$
Q^\varepsilon(v) = \sum_{(i\alpha)(j\beta) \in A} |\sigma_{i\alpha}\sigma_{j\beta}|.
$$

As usual, the sum ranges here over the set $A$ of all couples of approaching waves. We recall that, when $x_\alpha < x_\beta$, the two waves $\sigma_{i\alpha}$ approach if either $i = 2$, $j = 1$, or else $i = j$, the $i$-th family is genuinely nonlinear and $\min\{\sigma_{i\alpha}, \sigma_{j\beta}\} < 0$.

Remark that, since the functions $\psi^\varepsilon_i$ converge in the $C^2$–norm to $\psi_i$, then $V^\varepsilon$ and $Q^\varepsilon$ converge uniformly to the Glimm’s functionals $V$ and $Q$ respectively.

In order to prove the existence of a SRS for (1.4), in [6] it was constructed a family $(S^\varepsilon)$ of semigroups generating approximate solutions, in the sense of Definition 2.2. This construction is based on a modified wave–front tracking algorithm (see [2]).

Let $v^\flat, v^\sharp \in \Omega$, and assume that, for some $\sigma_1 < 0$,

$$
v^\sharp = (v^\sharp_1, v^\sharp_2) = (v^\flat_1 + \sigma_1, v^\flat_2),
$$

$$
h\varepsilon \leq v^\flat_1 < (h + 1)\varepsilon, \quad k\varepsilon \leq v^\flat_2 < (k + 1)\varepsilon,
$$

for some integers $h, k$. Let us define

$$
\hat{\omega}^1_j = \left(\frac{2j + 1}{2}\varepsilon, v^\flat_2\right), \quad \lambda^1_j(v^\flat, \sigma_1) = \lambda_1(v^\flat, \psi^-_1(v^\flat, \sigma_1)),
$$

and consider the averaged speed

$$
\lambda^1_1(v^\flat, \sigma_1) = \sum_j \left(\frac{\text{meas}([j\varepsilon, (j + 1)\varepsilon] \cap [v^\flat_1, v^\flat_2])}{|\sigma_1|}, \lambda_1(\hat{\omega}^1_j)\right).
$$
Finally, interpolate between the two previous speeds by setting
\[
\lambda^\varepsilon_t(v^\varepsilon, \sigma_1) = \varphi(\sigma_1/\sqrt{\varepsilon})\lambda^\varepsilon_t(v^\varepsilon, \sigma_1) + (1 - \varphi(\sigma_1/\sqrt{\varepsilon}))\lambda^\varepsilon_t(v^\varepsilon, \sigma_1). \tag{2.1}
\]

We are now in a position to give the following

**Definition 2.1** Let \( v = v(t,x) \) be a piecewise constant function. An \( \varepsilon \)-admissible wavefront of the first family is a line \( x = x(t) \) across which the function \( v \) has a jump, say with \( v^- = (v^-_1, v^-_2) \), \( v^+ = (v^+_1, v^+_2) \), satisfying the following conditions.

(i) If \( v^-_1 \geq v^+_1 \), then \( v^-_2 = v^+_2 \) and, for some integer \( k \), one has
\[
k\varepsilon \leq v^-_1 \leq v^+_1 \leq (k + 1)\varepsilon, \quad \hat{x} = \lambda_1(\hat{\omega}),
\]
where \( \hat{\omega} \) is the state with coordinates \( ((2k + 1)\varepsilon/2, v^+_2) \).

(ii) If \( v^-_1 < v^+_1 \), then \( v^+ = \psi(t)(v^-, \sigma_1) \), for some \( \sigma_1 < 0 \) and \( \hat{x} \) coincides with the speed \( \lambda^\varepsilon_t(v^-, \sigma_1) \), defined according to (2.1).

We regard (i)–(ii) as a set of approximate Rankine–Hugoniot conditions. The \( \varepsilon \)-admissible wavefronts of the second family are defined in an entirely similar way.

**Definition 2.2** A piecewise constant function \( v = v(t,x) \) is an \( \varepsilon \)-approximate solution if all of its lines of discontinuity are \( \varepsilon \)-admissible wavefronts.

For every \( \varepsilon > 0 \) let us define the domain
\[
\mathcal{D}^\varepsilon(\delta) = \{ \hat{v} \in \mathbb{L}^1(\mathbb{R}, \mathbb{R}^2) \mid \hat{v} \in \mathcal{PC}, \ V^\varepsilon(\hat{v}) + Q^\varepsilon(\hat{v}) < \delta \}.
\]

In [6] it was proved that there exists \( \delta > 0 \) such that the following proposition holds.

**Proposition 2.3** For every \( \varepsilon > 0 \) there exists a semigroup \( S^\varepsilon \colon [0, +\infty) \times \mathcal{D}^\varepsilon(\delta) \to \mathcal{D}^\varepsilon(\delta) \) with the following properties:

(i) \( S^\varepsilon_t \hat{v} = \hat{v} \), \( S^\varepsilon_t \hat{v} = S^\varepsilon_t S^\varepsilon_t \hat{v} \), for every \( t, s \geq 0 \), \( \hat{v} \in \mathcal{D}^\varepsilon(\delta) \);

(ii) \( \| S^\varepsilon_t \hat{v} - S^\varepsilon_s \hat{w} \| \leq L_\varepsilon(t - s + \| \hat{v} - \hat{w} \|) \), for every \( t, s \geq 0 \) and \( \hat{v}, \hat{w} \in \mathcal{D}^\varepsilon(\delta) \);

(iii) for every \( \hat{v} \in \mathcal{D}^\varepsilon(\delta) \), the map \( t \mapsto S^\varepsilon_t \hat{v} \) is an \( \varepsilon \)-approximate solution;

(iv) the map \( t \mapsto V^\varepsilon(S^\varepsilon_t \hat{v}) + Q^\varepsilon(S^\varepsilon_t \hat{v}) \) is nonincreasing for every \( \hat{v} \in \mathcal{D}^\varepsilon(\delta) \).

The SRS \( S \) is obtained as the limit of \( (S^\varepsilon) \) for \( \varepsilon \to 0 \).

In order to prove property (iii) above, it was introduced a Riemann–type metric on \( \mathcal{D}^\varepsilon(\delta) \), by defining a weighted length for a set of admissible paths.

**Definition 2.4** Let \([a, b] \) be an open interval. An elementary path is a map \( \gamma \colon [a, b] \to \mathbb{L}^1 \) of the form
\[
\gamma(\theta, x) = \sum_{\alpha=1}^N \omega_{\alpha} \chi_{[x_{\alpha-1}, x_{\alpha}]}(x), \quad x_{\alpha}^\alpha = \bar{x}_{\alpha} + \xi_{\alpha} \theta, \tag{2.2}
\]
with \( x_{\alpha-1}^\alpha < x_{\alpha}^\alpha \) for all \( \alpha = 1, \ldots, N \) and all \( \theta \).

**Definition 2.5** A continuous map \( \gamma \colon [a, b] \to \mathbb{L}^1 \) is a pseudopolygonal if there exist countably many open intervals \( J_\varepsilon \subset [a, b] \) such that
The restriction of \( \gamma \) to each \( J_h \) is an elementary path;
(ii) the set \( [a, b] \setminus \bigcup_h J_h \) is countable.

The weighted length of an elementary path \( \gamma \) is defined as
\[
||\gamma|| \doteq (b - a) \sum_{\alpha=1}^{N} \sum_{i=1}^{2} |\xi_\alpha| \phi(\sigma_{\alpha i}) \tilde{R}_{i\alpha},
\]
where
\[
\phi(\sigma) \doteq |\sigma|(2 + \text{sgn} \, \sigma), \quad \tilde{R}_{i\alpha} \doteq P_{\alpha i} e^{KQ}, \quad P_{i\alpha} \doteq 1 + K \sum_{(i\alpha)(j\beta) \in A} |\sigma_{j\beta}|,
\]
for some constant \( K > 0 \). In order to simplify notations, we dropped the dependence on \( \varepsilon \).

If \( \gamma \) is a pseudopolyg, we define its weighted length \( ||\gamma|| \) as the sum of the weighted lengths of its elementary paths.

On the domain \( D^\varepsilon(\hat{\delta}) \) we can define the Riemann–type metric
\[
d_{\varepsilon}(\bar{v}, \bar{w}) \doteq \inf \{||\gamma||; \gamma \text{ pseudopolyg, with values inside } D^\varepsilon, \text{ joining } \bar{v} \text{ with } \bar{w} \}.
\]
The property (ii) in Proposition 2.3 was proved in [6] showing that the distance \( d_{\varepsilon} \) is uniformly equivalent to the usual \( L^1 \) distance, and that \( S_{\varepsilon}^t \) is contractive w.r.t. the metric \( d_{\varepsilon} \), that is
\[
d_{\varepsilon}(S_{\varepsilon}^t \bar{v}, S_{\varepsilon}^t \bar{w}) \leq d_{\varepsilon}(\bar{v}, \bar{w}) \).
\]

For every \( \delta \in ]0, \hat{\delta}[, \) let us define the domain
\[
\tilde{D}(\delta) \doteq \{ \bar{v} \in L^1(\mathbb{R}, \mathbb{R}^2) \mid \bar{v} \in PC, V(\bar{v}) + Q(\bar{v}) < \delta \}.
\]

Since \( V^\varepsilon \rightarrow V \) and \( Q^\varepsilon \rightarrow Q \) uniformly, if \( \bar{v} \in \tilde{D}(\delta) \) then \( \bar{v} \in D^\varepsilon(\hat{\delta}) \) for every \( \varepsilon > 0 \) small enough. For every \( t \geq 0 \) and every \( \bar{v} \in \tilde{D} \) it can be shown that there exists \( S_t \bar{v} \doteq \lim_{\varepsilon \to 0+} S_{\varepsilon}^t \bar{v} \). Moreover \( S_t \) is a semigroup satisfying properties (i) and (ii) of Proposition 2.3, and property (iv) with \( V^\varepsilon \) and \( Q^\varepsilon \) replaced by \( V \) and \( Q \). By property (ii), we can extend \( S^\varepsilon \) and \( S \) respectively to \( [0, +\infty[ \times \text{cl } D^\varepsilon(\hat{\delta}) \) and \( [0, +\infty[ \times \text{cl } D(\delta) \), where “cl” denotes the closure in the \( L^1 \) topology.

3. Construction of approximate solutions. In this section we will construct a sequence \( (P^\nu) \) of operators, which will converge uniformly to an operator \( P \) satisfying the properties listed in Theorem 1.1.

For every fixed \( \nu \in \mathbb{N} \), let us define an approximating function \( g^\nu \) of \( g \), piecewise constant w.r.t. \( x \), in the following way. Let
\[
g_{\nu}(t, x, v) \doteq \sum_{j \in \mathbb{Z}} \chi_{[j\varepsilon^\nu,(j+1)\varepsilon^\nu]}(x)g_j(t, v),
\]
with \( \varepsilon_{\nu} \doteq 2^{-\nu} \) and
\[
g_j(t, v) \doteq \frac{1}{\varepsilon_{\nu}} \int_{j\varepsilon_{\nu}}^{(j+1)\varepsilon_{\nu}} g(t, x, v) \, dx.
\]
It is easily seen that every \( g_j \) is Lipschitz continuous w.r.t. \( v \), with the same Lipschitz constant \( L_g \) of \( g \).

In order to define the approximate evolution operators for the nonhomogeneous problem, the idea is to evolve from \( \tau = k\epsilon \) for a time interval of length \( \epsilon \), with the homogeneous approximate semigroup, and then to make the correction \( v \mapsto v + \epsilon g_{\nu}(\tau + \epsilon, v) \), which gives the contribution of the nonhomogeneous term.

Let us consider \( \bar{v} \in D^{\epsilon\nu}(\hat{\delta}) \). For every \( t \in [0, \epsilon] \) and \( m \in \mathbb{N} \) let us define

\[
P^\nu(m\epsilon\nu + t, m\epsilon\nu)\bar{v} \doteq S^\epsilon\nu_t \bar{v}, \quad P^\nu((m+1)\epsilon\nu, m\epsilon\nu)\bar{v} \doteq S^\epsilon\nu_t \bar{v} + \epsilon g_{\nu}((m+1)\epsilon\nu, \cdot, S^\epsilon\nu_t \bar{v}).
\]

Next, consider \( \tau = j\epsilon \), \( j \in \mathbb{N} \), and \( t \in [n\epsilon\nu, (n+1)\epsilon\nu[ \), \( n \geq 1 \). Define

\[
v_1 \doteq P^\nu(\tau + \epsilon\nu, \tau)\bar{v}.
\]

and then, by induction on \( k = 1, \ldots, n-1 \), if \( v_k \in D^{\epsilon\nu}(\hat{\delta}) \),

\[
v_{k+1} \doteq P^\nu(\tau + (k+1)\epsilon\nu, \tau + k\epsilon\nu)v_k.
\]

Finally, if \( v_n \in D^{\epsilon\nu}(\hat{\delta}) \), we set

\[
P^\nu(\tau + t, \tau)\bar{v} \doteq P^\nu(\tau + t, \tau + n\epsilon\nu)v_n = S^\epsilon\nu_{\tau + n\epsilon\nu}v_n.
\]

Hence, if \( v_1, \ldots, v_n \in D^{\epsilon\nu}(\hat{\delta}) \), we have

\[
P^\nu(\tau + t, \tau)\bar{v} = S^\epsilon\nu_{\tau + (n-1)\epsilon\nu}P^\nu(\tau + n\epsilon\nu, \tau + (n-1)\epsilon\nu) \cdots P^\nu(\tau + \epsilon\nu, \tau)\bar{v}.
\]

Clearly, if \( n \geq k \), and \( t \geq n\epsilon\nu \), then we have

\[
P^\nu(t, k\epsilon\nu)\bar{v} = P^\nu(t, n\epsilon\nu)P^\nu(n\epsilon\nu, k\epsilon\nu)\bar{v},
\]

provided that \( P^\nu(n\epsilon\nu, k\epsilon\nu)\bar{v} \in D^{\epsilon\nu}(\hat{\delta}) \).

In the following Lemma we shall prove some basic properties of \( g_{\nu} \).

**Lemma 3.1** For every fixed \( t \) and every \( \bar{v} \in PC \) with bounded total variation, the composed map \( x \mapsto g_{\nu}(t, x, \bar{v}(x)) \) has bounded total variation and

\[
T.V.\{g_{\nu}(t, \cdot, \bar{v}(.)); \mathbb{R}\} \leq L_g T.V.\{\bar{v}; \mathbb{R}\} + 2\|\bar{v}\|_{L^1(\mathbb{R})}.
\]

**Proof.** For every \( j \in \mathbb{Z} \) let us define \( J_j \doteq [j\epsilon\nu, (j+1)\epsilon\nu[ \). By hypothesis there exist points \( (x_i), x_i < x_{i+1} \) for every \( i \), and values \( (v_i) \) such that \( \bar{v}(x) = \sum v_i \chi_{[x_i, x_{i+1}[}(x) \). We have that

\[
T.V.\{g_{\nu}(t, \cdot, \bar{v}(.)); J_j\} = T.V.\{g_j(t, \bar{v}(\cdot)); J_j\} = \sum_{[x_i, x_{i+1}[ \subseteq J_j} |g_j(t, v_{i+1}) - g_j(t, v_i)| \leq L_g T.V.\{\bar{v}; J_j\}.
\]
and
\[
|g_\nu(t, j\varepsilon_\nu+, \tilde{v}(j\varepsilon_\nu+)) - g_\nu(t, j\varepsilon_\nu-, \tilde{v}(j\varepsilon_\nu-))| \leq \\
\leq |g_\nu(t, j\varepsilon_\nu+, \tilde{v}(j\varepsilon_\nu+)) - g_\nu(t, j\varepsilon_\nu+, \tilde{v}(j\varepsilon_\nu-))| + \\
+ |g_\nu(t, j\varepsilon_\nu+, \tilde{v}(j\varepsilon_\nu-)) - g_\nu(t, j\varepsilon_\nu-, \tilde{v}(j\varepsilon_\nu-))| \leq \\
\leq L_\nu|\tilde{v}(j\varepsilon_\nu+) - \tilde{v}(j\varepsilon_\nu-)| + \\
+ T.V.\{g(t, \cdot, \tilde{v}(j\varepsilon_\nu-))\}; (j-1)\varepsilon_\nu, (j+1)\varepsilon_\nu]\leq \\
\leq L_\nu|\tilde{v}(j\varepsilon_\nu+) - \tilde{v}(j\varepsilon_\nu-)| + \int_{(j-1)\varepsilon_\nu}^{(j+1)\varepsilon_\nu} k(x) dx.
\]

These estimates give
\[
T.V.\{g_\nu(t, \cdot, \tilde{v}(\cdot)); R\} = \\
= \sum \{T.V.\{g_\nu(t, \cdot, \tilde{v}(\cdot)); J_j\} + \\
+ |g_\nu(t, j\varepsilon_\nu+, \tilde{v}(j\varepsilon_\nu+)) - g_\nu(t, j\varepsilon_\nu-, \tilde{v}(j\varepsilon_\nu-))| \leq \\
\leq \sum \left[ L_\nu T.V.\{\tilde{v}; J_j\} + L_\nu|\tilde{v}(j\varepsilon_\nu+) - \tilde{v}(j\varepsilon_\nu-)| + \int_{(j-1)\varepsilon_\nu}^{(j+1)\varepsilon_\nu} k(x) dx \right] = \\
= L_\nu T.V.\{\tilde{v}; R\} + 2\|k\|_{L^1(R)},
\]
concluding the proof.

\[
\square
\]

4. Basic estimates. In the following, in order to simplify notations, we will omit the dependence on \(\varepsilon\). Since the maps \(\psi_i^\nu\) converge to \(\psi_i\) in the \(C^2\) norm, all the estimates will hold uniformly in \(\varepsilon\).

Let \(\tilde{g}(x, v)\) be a bounded function, Lipschitz continuous w.r.t. \(v\). Let \(x \in \mathbb{R}, v^+, v^- \in U, h > 0\), and let \(\sigma_i, \sigma'_i, i = 1, 2\), be respectively the strength of the waves generated by the Riemann problems with data \((v^-, v^+)\) and \((v^- + h\tilde{g}^-, v^+ + h\tilde{g}^+)\), where \(\tilde{g}^\pm \equiv \tilde{g}(x^\pm, v^\pm)\). With the notations of Section 2, we have that
\[
\sigma_i = E_i(v^-, v^+), \quad \sigma'_i = E_i(v^- + h\tilde{g}^-, v^+ + h\tilde{g}^+), \quad i = 1, 2.
\]
Since \(E_i(w, w) = 0\) for every \(w\), it follows
\[
\sigma_i = \int_0^1 \partial_2 E_i(v^-, v^- + t(v^+ - v^-)) \cdot (v^+ - v^-) dt, \\
\sigma'_i = \int_0^1 \partial_2 E_i(v^- + h\tilde{g}^-, v^- + h\tilde{g}^- + t(\Delta v + h\Delta g)) \cdot (\Delta v + h\Delta g) dt,
\]
where \(\Delta v \equiv v^+ - v^-, \Delta g \equiv \tilde{g}^+ - \tilde{g}^-\) and \(\partial_2\) denotes differentiation w.r.t. the second argument. Since \(E = (E_1, E_2)\) is of class \(C^2\) we get
\[
|\sigma_i - \sigma'_i| \leq |\Delta v| \int_0^1 \partial_2 E_i(v^-, v^- + t\Delta v) + \\
- \partial_2 E_i(v^- + h\tilde{g}^-, v^- + h\tilde{g}^- + t(\Delta v + h\Delta g)) dt + O(1)h|\Delta g| \leq \\
\leq O(1)h(|\Delta v| + |\Delta g|),
\]
where $O(1)$ denotes the Landau order symbol, i.e. a quantity which remains uniformly bounded (in absolute value) by some constant $C$, depending only on the system (1.1). Finally, from the Lipschitz continuity of $\tilde{g}$ w.r.t. $v$, we obtain

$$|\sigma_i - \sigma'_i| = O(1) h(|\sigma_1| + |\sigma_2|) + O(1) h|\tilde{g}(x^+, v^-) - \tilde{g}(x^-, v^-)|. \quad \text{(4.1)}$$

If $\tilde{g}$ is continuous at $x$, the last term at the right hand side of (4.1) vanishes and then

$$|\sigma_i - \sigma'_i| = O(1) h(|\sigma_1| + |\sigma_2|). \quad \text{(4.2)}$$

Let $\gamma : [a, b] \to \mathbb{L}^1$ be an elementary path as in (2.2), and let us assume that $\tilde{g}$ is piecewise constant w.r.t. $x$, continuous at every point $x^\alpha_\alpha$, $\alpha = 1, \ldots, N$, $\vartheta \in [a, b]$. Under this assumption, the map $\gamma : [a, b] \to \mathbb{L}^1$ defined by $\gamma(\vartheta, x) = \gamma(\vartheta, x) + h\tilde{g}(x, \gamma(\vartheta, x))$ is an elementary path. Let us denote by $\sigma'_i$ the strength of the waves of the Riemann problems associated to $\omega'_i = \omega + h\tilde{g}(x, \gamma(\vartheta, x))$, that is $\sigma'_i = E_i(\omega_{i-1}, \omega'_i)$. Let us denote by $\tilde{R}_{i\alpha}$, $P_{i\alpha}$ and $Q'$ the quantities corresponding respectively to $\tilde{R}_{i\alpha}$, $P_{i\alpha}$ and $Q$ for $\gamma'$.

**Lemma 4.1** With the above notations, if $\text{sgn} \sigma_{ia} = \text{sgn} \sigma'_{ia}$, then

$$|\tilde{R}_{i\alpha} - \tilde{R}'_{i\alpha}| = O(1) h(\text{T.V.}\{\gamma(\vartheta)\}) + \text{T.V.}\{\gamma'(\vartheta)\}. \quad \text{(4.3)}$$

**Proof.** We first estimate $|P_{i\alpha} - P'_{i\alpha}|$. If $(i\alpha)(j\beta) \in \mathcal{A} \Delta \mathcal{A}'$, then it must be $i = j$ and $\text{sgn} \sigma_{j\beta} \neq \text{sgn} \sigma'_{j\beta}$, so that $|\sigma_{j\beta}|, |\sigma'_{j\beta}| \leq |\sigma_{j\beta} - \sigma'_{j\beta}|$. Here $\mathcal{A} \Delta \mathcal{A}'$ denotes the symmetric difference between the two sets $\mathcal{A}$ and $\mathcal{A}'$. From (4.2) we obtain

$$|P_{i\alpha} - P'_{i\alpha}| \leq K \left( \sum_{(i\alpha)(j\beta) \in \mathcal{A} \Delta \mathcal{A}'} |\sigma_{j\beta} - \sigma'_{j\beta}| + \sum_{(i\alpha)(j\beta) \in \mathcal{A} \Delta \mathcal{A}'} |\sigma_{j\beta} - \sigma'_{j\beta}| \right) =$$

$$= K \sum_{(i\alpha)(j\beta) \in \mathcal{A} \Delta \mathcal{A}'} |\sigma_{j\beta} - \sigma'_{j\beta}| = O(1) h \sum_{j} (|\sigma_{j\beta}| + |\sigma_{2\beta}|) = \quad \text{(4.4)}$$

In order to estimate $|Q - Q'|$, observe that

$$|Q - Q'| = O(1) \left( \sum_{(j\beta)(j'\beta') \in \mathcal{A} \mathcal{A}'} |\sigma_{j\beta} \sigma_{j'\beta'}| - \sum_{(j\beta)(j'\beta') \in \mathcal{A} \mathcal{A}'} |\sigma'_{j\beta} \sigma'_{j'\beta'}| \right) =$$

$$= O(1) \left( \sum_{(j\beta)(j'\beta') \in \mathcal{A} \mathcal{A}'} |\sigma_{j\beta} \sigma_{j'\beta'}| + \sum_{(j\beta)(j'\beta') \in \mathcal{A} \mathcal{A}'} |\sigma_{j\beta} \sigma_{j'\beta'}| \right).$$

If $(j\beta)(j'\beta') \in \mathcal{A} \mathcal{A}'$, then either $|\sigma_{j\beta}| \leq |\sigma_{j\beta} - \sigma'_{j\beta}|$ or $|\sigma_{j'\beta'}| \leq |\sigma_{j'\beta'} - \sigma'_{j'\beta'}|$, and then

$$\sum_{(j\beta)(j'\beta') \in \mathcal{A} \mathcal{A}'} |\sigma_{j\beta} \sigma_{j'\beta'}| \leq \sum_{(j\beta)(j'\beta') \in \mathcal{A} \mathcal{A}'} (|\sigma_{j\beta}| \cdot |\sigma_{j'\beta'} - \sigma'_{j'\beta'}| + |\sigma_{j'\beta'}| \cdot |\sigma_{j\beta} - \sigma'_{j\beta}|) \leq$$

$$\leq 2 \text{T.V.}\{\gamma(\vartheta)\} \sum_{(j\beta)} |\sigma_{j\beta} - \sigma'_{j\beta}|.$$
In the same way one obtains
\[
\sum_{(j\beta)(j'\beta') \in A \setminus A} |\sigma_j \sigma_{j' \beta'} - \sigma_{j' \beta'} \sigma_{j' \beta'}| \leq 2 T.V.\{\gamma(\vartheta)\} \sum_{(j\beta)} |\sigma_j - \sigma_{j' \beta}|.
\]
Since
\[
\sum_{(j\beta)(j'\beta') \in A \cap A} |\sigma_j \sigma_{j' \beta'} - \sigma_{j' \beta'} \sigma_{j' \beta'}| \leq \sum_{(j\beta)(j'\beta') \in A \cap A} (|\sigma_j| \cdot |\sigma_{j' \beta'} - \sigma_{j' \beta'}| + |\sigma_{j' \beta'}| \cdot |\sigma_j - \sigma_{j' \beta}|) \leq [T.V.\{\gamma(\vartheta)\} + T.V.\{\gamma'(\vartheta)\}] \sum_{(j\beta)} |\sigma_j - \sigma_{j' \beta}|,
\]
from (4.2) we obtain
\[
|Q - Q'| = O(1) [T.V.\{\gamma(\vartheta)\} + T.V.\{\gamma'(\vartheta)\}] \sum_{(j\beta)} |\sigma_j - \sigma_{j' \beta}| = O(1) h[T.V.\{\gamma(\vartheta)\} + T.V.\{\gamma'(\vartheta)\}] .
\] (4.5)
From (4.4) and (4.5) it easily follows (4.3).

**Lemma 4.2** Let $B > 0$ be such that
\[
\max\{T.V.\{\gamma(\vartheta)\}, T.V.\{\gamma'(\vartheta)\}, |\tilde{R}_{i\alpha}|, |\tilde{R}'_{i\alpha}|\} \leq B.
\]
Then, with the above notations, one has, for every $\alpha = 1, \ldots, N$,
\[
\sum_{i=1,2} \phi(\sigma_{i\alpha})|\tilde{R}_{i\alpha} - \tilde{R}'_{i\alpha}| \leq O(1) Bh(|\sigma_{1\alpha}| + |\sigma_{2\alpha}|) .
\] (4.6)

**Proof.** If $\text{sgn} \sigma_{i\alpha} = \text{sgn} \sigma'_{i\alpha}$, by Lemma 4.1 and the estimate
\[
|\sigma'_{i\alpha}| \leq |\sigma_{i\alpha}| + |\sigma_{i\alpha} - \sigma'_{i\alpha}| = O(1) \sum_{i=1,2} |\sigma_{i\alpha}|,
\]
we obtain
\[
\phi(\sigma_{i\alpha})|\tilde{R}_{i\alpha} - \tilde{R}'_{i\alpha}| \leq 3|\sigma'_{i\alpha}| \cdot |\tilde{R}_{i\alpha} - \tilde{R}'_{i\alpha}| \leq O(1) Bh(|\sigma_{1\alpha}| + |\sigma_{2\alpha}|).
\]
If $\text{sgn} \sigma_{i\alpha} \neq \text{sgn} \sigma'_{i\alpha}$, the same estimate follows by the definition of $B$ and
\[
|\sigma'_{i\alpha}| \leq |\sigma_{i\alpha} - \sigma'_{i\alpha}| = O(1) h \sum_{i=1,2} |\sigma_{i\alpha}|,
\]
establishing (4.6).
Proposition 4.3 Let \( \bar{g}(x, v) \) be piecewise constant w.r.t. \( x \). Then, if \( h \) is sufficiently small, the map \( \bar{v} \mapsto \bar{v} + \bar{h}g(\cdot, \bar{v}) \), restricted to \( E : = \{ \bar{v} \in D^e(\hat{\delta}) \, | \, \bar{v} + \bar{h}g(\cdot, \bar{v}) \in D^e(\hat{\delta}) \} \), is Lipschitz continuous w.r.t. the metric \( d_\varepsilon \). More precisely, there exists a positive constant \( C \) such that

\[
d_\varepsilon(v + \bar{h}g(\cdot, \bar{v}), \bar{w} + \bar{h}g(\cdot, \bar{w})) \leq (1 + Ch)d_\varepsilon(\bar{v}, \bar{w}) , \quad \text{for every } \bar{v}, \bar{w} \in E . \tag{4.7}
\]

**Proof.** If \( \gamma : [a, b] \to E, \gamma(\bar{v}) = v^\bar{v} \), is a pseudopoligonal joining two elements \( \bar{v}, \bar{w} \in E \), then \( \gamma'(\bar{v}) = v^\bar{v} + \bar{h}g(\cdot, v^\bar{v}) \) is a pseudopoligonal joining \( \bar{v} + \bar{h}g(\cdot, \bar{v}) \) to \( \bar{w} + \bar{h}g(\cdot, \bar{w}) \). We remark that the constant \( B \) of Lemma 4.2 can be chosen independent of \( \gamma \). Moreover, since \( \bar{g} \) is piecewise constant w.r.t. \( x \), there exist finitely many values of \( \bar{v} \), \( \bar{v}_1, \ldots, \bar{v}_N \in [a, b] \), such that \( \bar{g} \) is constant in a neighborhood of every point of discontinuity of \( \gamma(\bar{v}) \), for every \( \bar{v} \neq \bar{v}_j, j = 1, \ldots, N \). Hence there exist countably many intervals \([a_j, b_j] \), pairwise disjoint, such that \( [a, b] \setminus \bigcup_j [a_j, b_j] \) is a countable set containing \( \bar{v}_1, \ldots, \bar{v}_N \), and the restriction of \( \gamma \) and \( \gamma' \) to each \( [a_j, b_j] \) is an elementary path. In particular, with the same notations of Lemmas 4.1 and 4.2, the tangent vectors to \( \gamma \) and \( \gamma' \) at \( \bar{v} \in [a_j, b_j] \) are respectively of the form

\[
\sum_{(\alpha)} |\xi_\alpha|\phi(\sigma_{\alpha})\bar{R}_{\alpha}, \quad \sum_{(\alpha)} |\xi_\alpha|\phi(\sigma'_{\alpha})\bar{R}'_{\alpha} .
\]

Using (4.2) and (4.6), we can estimate the length of the difference of two elementary paths \( \tilde{\gamma} \triangleq \gamma|_{[a_j, b_j]} \) and \( \tilde{\gamma}' \triangleq \gamma'|_{[a_j, b_j]} \):

\[
\|\tilde{\gamma} - \tilde{\gamma}'\|_\varepsilon \leq (b_j - a_j) \sum_{(\alpha)} |\xi_\alpha|\phi(\sigma_{\alpha})\bar{R}_{\alpha} - \phi(\sigma'_{\alpha})\bar{R}'_{\alpha}| \leq
\]

\[
\leq (b_j - a_j) \sum_{(\alpha)} |\xi_\alpha| \left[ |\phi(\sigma'_{\alpha})|\bar{R}_{\alpha} - \bar{R}'_{\alpha} + |\phi(\sigma_{\alpha}) - \phi(\sigma'_{\alpha})|\bar{R}_{\alpha} \right] \leq
\]

\[
\leq O(1)B(b_j - a_j) \sum_{(\alpha)} |\xi_\alpha||h|\sigma_{\alpha} + |\sigma_{\alpha} - \sigma'_{\alpha}| =
\]

\[
= O(1)Bh(b_j - a_j) \sum_{(\alpha)} |\xi_\alpha|\sigma_{\alpha} \leq
\]

\[
\leq O(1)Bh(b_j - a_j) \sum_{(\alpha)} |\xi_\alpha|\phi(\sigma_{\alpha})\bar{R}_{\alpha} = O(1)Bh\|\bar{\gamma}\|_\varepsilon .
\]

Hence there exists a constant \( C > 0 \) such that

\[
\|\tilde{\gamma}'\|_\varepsilon \leq (1 + Ch)\|\tilde{\gamma}\|_\varepsilon ,
\]

and then, by definition of \( d_\varepsilon \), (4.7) holds. \( \square \)

Proposition 4.4 Let \( \bar{v} \in D^e(\hat{\delta}) \), and let \( \tau > 0 \) be such that \( P^\nu(t, 0)\bar{v} \in D^e(\hat{\delta}) \) for every \( t \in [0, \tau] \). Then there exists a positive constant \( C' \), independent of \( \nu \) and \( \bar{v} \), such that

\[
V^{\varepsilon_{\nu}}(t) + Q^{\varepsilon_{\nu}}(t) \leq e^{C't}[V^{\varepsilon_{\nu}}(0) + Q^{\varepsilon_{\nu}}(0)] + C't\|k\|_{L^1(\mathbb{R})} , \quad \text{for every } t \in [0, \tau] ,
\]

\[
\text{for every } t \in [0, \tau] ,
\]
where $V(t) \equiv V(t, 0)\bar{v}$ and $Q^\nu(t) \equiv Q^\nu(t, 0)\bar{v}$.

**Proof.** Let $t = N_\nu\delta + \delta$, for some $N_\nu \in \mathbb{N}$ and $\delta \in [0, \nu]$. Let us define $t_k \equiv \delta_k, k = 0, \ldots, N_\nu$. By property (iv) of Proposition 2.3 we have that $(V(t) + Q^\nu)\delta_{t+k-1} \leq (V(t) + Q^\nu)\delta_{t+k}$. From (4.1) and reasoning as in Lemma 3.1 we have that

$$V^\nu(t_k) \leq (1 + O(1)\nu)V^\nu(t_k - ) + O(1)\nu\|k\|L^1(\mathbb{R}). \quad (4.8)$$

Moreover, from (4.5), one has

$$Q^\nu(t_k) - Q^\nu(t_k - ) = O(1)\nu\mathcal{V}\{P^\nu(t_k - , 0)\bar{v}\} = O(1)\nu V^\nu(t_k - ). \quad (4.9)$$

From (4.8) and (4.9) we obtain

$$(V^\nu + Q^\nu)(t_k) \leq (1 + C^\nu\nu)(V^\nu + Q^\nu)(t_k - ) + C^\nu\nu\|k\|L^1(\mathbb{R}),$$

for some constant $C^\nu > 0$, and finally, by induction,

$$(V^\nu + Q^\nu)(t) \leq (V^\nu + Q^\nu)(t_{N_k} - ) \leq \leq (1 + C^\nu\nu)(V^\nu + Q^\nu)(t_{N_k} - ) + C^\nu\nu\|k\|L^1(\mathbb{R}) \leq \leq (1 + C^\nu\nu)(V^\nu + Q^\nu)(0) + C^\nu N_\nu\nu\|k\|L^1(\mathbb{R}) \leq \leq e^{C^\nu t}(V^\nu + Q^\nu)(0) + C^\nu t\|k\|L^1(\mathbb{R}),$$

and the result is achieved. \hfill \Box

Let us define the times

$$T_\delta \equiv \min \left\{ \frac{\delta}{4C^\nu\|k\|L^1(\mathbb{R})}, \frac{\log 3/2}{C^\nu} \right\}, \quad T \equiv T_{\delta/2},$$

and let $T^\nu \equiv \{\delta_k | k \in \mathbb{N}\}$. By Proposition 4.4 it is clear that, if $\bar{v} \in D^\nu(\delta/2)$, then $P^\nu(t, \tau)\bar{v} \in D^\nu(\delta)$ for every $\tau \in [0, T_{\delta}] \cap T^\nu$ and $t \in [0, T_{\delta}]$, $t \geq \tau$.

**Proposition 4.5** There exists a constant $L^\nu$ such that, for every $\tau \in [0, T] \cap T^\nu$, for every $t, s \in [0, T], t, s \geq \tau$, and for every $\bar{v}, \bar{w} \in D^\nu(\delta/2)$,

$$\|P^\nu(t, \tau)\bar{v} - P^\nu(s, \tau)\bar{w}\| \leq L^\nu (|t - s| + \|\bar{v} - \bar{w}\| + \nu). \quad (4.10)$$

**Proof.** Let $\tau \in [0, T] \cap T^\nu$, and let $N \in \mathbb{N}$ be such that $\tau + N\delta \leq T$. Let us define $P_{n, \nu} = P^\nu(\tau + n\delta, \tau + n\delta)$. By Proposition 4.3 and the $d_{\nu}$-contractivity of $S^\nu$, we have that

$$d_{\nu}\left(P_{n, \nu}(\tau + n\delta, \tau)\bar{v}, P_{\nu}(\tau + N\delta, \tau)\bar{w}\right) = d_{\nu}(P_{N, 0, 0, 0} P_{N, 0, 1, 0}\bar{v}, P_{N, N-1, 0, 0}\bar{w}) = d_{\nu}(S_{N\nu, 0, 0, 0} P_{N, N-1, 0, 0}\bar{v}, S_{N\nu, 0, 0, 0} P_{N, N-1, 0, 0}\bar{w}) \leq (1 + C_{\nu}\nu)d_{\nu}(S_{\nu, 0, 0, 0} P_{N, N-1, 0, 0}\bar{v}, S_{\nu, 0, 0, 0} P_{N, N-1, 0, 0}\bar{w}) \leq \leq \leq (1 + C_{\nu}\nu)d_{\nu}(P_{N, 0, 0, 0} P_{N, N-1, 0, 0}\bar{v}, P_{N, N-1, 0, 0}\bar{w}),$$
where \( g^\nu \triangleq g_\nu(\tau + (N - 1)\varepsilon \nu, \cdot, S^\nu_{\varepsilon \nu} P_{N-1,0} \bar{v}) \), \( g^\nu \triangleq g_\nu(\tau + (N - 1)\varepsilon \nu, \cdot, S^\nu_{\varepsilon \nu} P_{N-1,0} \bar{w}) \).

By induction we obtain

\[
d_{\varepsilon \nu}(P^\nu(\tau + N\varepsilon \nu, \tau)\bar{v}, P^\nu(\tau + N\varepsilon \nu, \tau)\bar{w}) \leq (1 + C_{\varepsilon \nu})^N d_{\varepsilon \nu}(\bar{v}, \bar{w}) \leq C_{\varepsilon \nu} d_{\varepsilon \nu}(\bar{v}, \bar{w}).
\]

Let \( t \in [0, T] \), \( t \geq \tau \), and let \( N \in \mathbb{N} \) be such that \( t = \tau + N\varepsilon \nu + \delta, \delta \in [0, \varepsilon \nu] \). Using the previous estimate and the \( d_{\varepsilon \nu} \)-contractivity of \( S^\nu_{\varepsilon \nu} \) we obtain

\[
d_{\varepsilon \nu}(P^\nu(t, \tau)\bar{v}, P^\nu(t, \tau)\bar{w}) =
= d_{\varepsilon \nu}(P^\nu(t, \tau + N\varepsilon \nu)P^\nu(\tau + N\varepsilon \nu, \tau)\bar{v}, P^\nu(t, \tau + N\varepsilon \nu)P^\nu(\tau + N\varepsilon \nu, \tau)\bar{w}) \leq
= d_{\varepsilon \nu}(P^\nu(\tau + N\varepsilon \nu, \tau)\bar{v}) \leq e^{CN_{\varepsilon \nu}} d_{\varepsilon \nu}(\bar{v}, \bar{w}) \leq C_{\varepsilon \nu} d_{\varepsilon \nu}(\bar{v}, \bar{w}).
\]

(4.11)

Now let \( \tau \) and \( \tau \) as in the statement of proposition. There exist \( k, n \in \mathbb{N} \) such that \( t - \tau \in [k\varepsilon \nu, (k + 1)\varepsilon \nu] \), \( s - \tau \in [n\varepsilon \nu, (n + 1)\varepsilon \nu] \). By the Lipschitz property of the approximate semigroup \( S^\nu_{\varepsilon \nu} \), we have that

\[
\|P^\nu(t, \tau)\bar{v} - P^\nu(\tau + k\varepsilon \nu, \tau)\bar{v}\| =
= \|P^\nu(t, \tau + k\varepsilon \nu)P^\nu(\tau + k\varepsilon \nu, \tau)\bar{v} - P^\nu(\tau + k\varepsilon \nu, \tau)\bar{v}\|
\leq \|S^\nu_{\varepsilon \nu} P^\nu(\tau + k\varepsilon \nu, \tau)\bar{v} - P^\nu(\tau + k\varepsilon \nu, \tau)\bar{v}\|
\leq L_S |t - \tau - k\varepsilon \nu| \leq L_S \varepsilon \nu,
\]

(4.12)

and a similar estimate holds for \( \|P^\nu(s, \tau)\bar{v} - P^\nu(\tau + n\varepsilon \nu, \tau)\bar{v}\| \). Let us define \( \bar{z} \triangleq P^\nu(\tau + n\varepsilon \nu, \tau)\bar{v} \) and \( t_m \triangleq \tau + m\varepsilon \nu \). If \( r \triangleq k - n \geq 1 \), using again the Lipschitz property of \( S^\nu_{\varepsilon \nu} \) one has

\[
\|P^\nu(\tau + k\varepsilon \nu, \tau)\bar{v} - P^\nu(\tau + n\varepsilon \nu, \tau)\bar{v}\| = \|P^\nu(t_k, t_n)\bar{z} - \bar{z}\| \leq
\leq \sum_{j=0}^{r-1} \|P^\nu(t_{n+j+1}, t_n)\bar{z} - P^\nu(t_{n+j}, t_n)\bar{z}\| \leq
\leq \sum_{j=0}^{r-1} \|S^\nu_{\varepsilon \nu} P^\nu(t_{n+j+1}, t_n)\bar{z} + \varepsilon \nu g_\nu(t_{n+j+1}, \cdot, S^\nu_{\varepsilon \nu} P^\nu(t_{n+j+1}, t_n)\bar{z}) + P^\nu(t_{n+j+1}, t_n)\bar{z}\|
\leq \sum_{j=0}^{r-1} (L_S \varepsilon \nu + \varepsilon \nu \|g_\nu\|_{\infty}) = (L_S + \|g_\nu\|_{\infty}) r \varepsilon \nu \leq
\leq (L_S + \|g_\nu\|_{\infty})(|t - s| + \varepsilon \nu).
\]

(4.13)

By (4.12) and (4.13) we obtain

\[
\|P^\nu(t, \tau)\bar{v} - P^\nu(s, \tau)\bar{v}\| \leq (3L_S + \|g_\nu\|_{\infty}) \varepsilon \nu + (L_S + \|g_\nu\|_{\infty}) |t - s|,
\]

which, together with (4.11), gives (4.10) for a suitable constant \( L_P > 0 \).
5. Existence of a local evolution operator. Let us define the domain \( \hat{D} = \text{cl} \hat{D}(\delta/2) \). Remark that, if \( \bar{v} \in \hat{D}(\delta/2) \), then \( \bar{v} \in D^{\nu}(\delta/2) \) for every \( \nu \) large enough.

Let us denote by \( T \) the set of all times \( t \in [0, T] \) such that \( t = k2^{-n} \) for some \( k, n \in \mathbb{N} \), and let us define the set

\[
I \doteq \{ (t, s) \in [0, T] \times [0, T] \mid t \geq s \}.
\]

Let \( \{ t_j, \bar{v}_j \}_{j} \) be a dense subset of \([0, T] \times \hat{D}(\delta/2)\). By a diagonal argument, we can extract a subsequence of \( \{ P^\nu \} \), again denoted by \( \{ P^\nu \} \), such that for every \( t, \tau \in T \) and every \( j \) such that \( t_j \geq \tau \), there exists \( P(t_j, \tau)\bar{v}_j \doteq \lim_{\nu \to +\infty} P^\nu(t_j, \tau)\bar{v}_j \). If \( t_j, t_i \geq \tau \), passing to the limit for \( \nu \to +\infty \) in (4.10) with \( (t, \bar{v}) = (t_j, \bar{v}_j) \) and \( (s, \bar{w}) = (t_i, \bar{v}_i) \) we obtain

\[
\| P(t_j, \tau)\bar{v}_j - P(t_i, \tau)\bar{v}_i \| \leq L_P(\| t_{j} - t_{i} \| + \| \bar{v}_j - \bar{v}_i \|).
\]

By continuity, \( P \) can be extended to a Lipschitz function defined on \( I \times \hat{D} \) satisfying, for every \( t, \tau \in [0, T] \), \( (t, \bar{v}) \), \( (s, \bar{w}) \in [0, T] \times \hat{D} \), \( t, s \geq \tau \),

\[
\| P(t, \tau)\bar{v} - P(s, \tau)\bar{w} \| \leq L_P(\| t - s \| + \| \bar{v} - \bar{w} \|).
\]

Since the domain \( I \times \hat{D} \) is compact in the product topology of \( \mathbb{R} \times \mathbb{R} \times L^1(\mathbb{R}, \mathbb{R}^2) \), we have that \( P^\nu \) converges uniformly to \( P \) in this topology.

It is easy to see that \( P \) is a local evolution operator, that is

\[
P(t, \tau)P(\tau, s)\bar{v} = P(t, s)\bar{v}, \quad \text{for every } \bar{v} \in \hat{D}, \ 0 \leq s \leq \tau \leq t \leq T, \ P(\tau, s)\bar{v} \in \hat{D}.
\]

Indeed, if \( t, s, \tau \in T \), by the definition of \( P^\nu \) we have that for every \( \nu \) large enough and for every \( \bar{v} \in \hat{D}(\delta/2) \), \( P^\nu(t, \tau)P^\nu(\tau, s)\bar{v} = P^\nu(t, s)\bar{v} \). Passing to the limit for \( \nu \to +\infty \), we obtain \( P(t, \tau)P(\tau, s)\bar{v} = P(t, s)\bar{v} \). Since \( T \) is dense in \([0, T]\), by continuity we have that (5.2) holds.

Following the proof of Corollary 3 of [4], we have:

**Corollary 5.1** Let \( ]a, b[ \) be a (possibly unbounded) open interval, and let \( \lambda \) be an upper bound for all characteristic speeds. If \( \bar{v}, \bar{w} \in \hat{D} \), then for all \( t, \tau \in [0, T] \), \( t \geq \tau \), one has

\[
\int_{a + \lambda(t - \tau)}^{b - \lambda(t - \tau)} |P(t, \tau)\bar{v}(x) - P(t, \tau)\bar{w}(x)| \, dx \leq L_P \cdot \int_{a}^{b} |\bar{v}(x) - \bar{w}(x)| \, dx.
\]

6. Weak entropic solutions. In this section we will prove that the trajectories of \( P \) are weak entropic solutions of (1.1).

**Remark 6.1** From now on we will consider the domain \( D = \text{cl} \hat{D}(\delta/4) \), in such a way that \( P(t, 0)\bar{u} \in \hat{D} \) for every \( \bar{u} \in D \) and \( t \in [0, T] \).

**Proposition 6.2** For every \( \bar{u} \in D \), the map \( u(t) = P(t, 0)\bar{u}, \ t \in [0, T] \), is a weak entropic solution of (1.1).
Proof. We begin by showing that \( u(t) \) is a weak solution of (1.1), that is

\[
\int_{-\infty}^{+\infty} \phi(0, x) \dot{u}(x) \, dx + \int_{0}^{T} \int_{-\infty}^{+\infty} [\phi_t(t, x) u(t, x) + \phi_x(t, x) F(u(t, x))] \, dx \, dt + \\
+ \int_{0}^{T} \int_{-\infty}^{+\infty} \phi(t, x) g(t, x, u(t, x)) \, dx \, dt = 0, 
\]

for every \( \phi \in C_0^1([0, T[ \times \mathbb{R}). \) Without loss of generality we can assume that \( \bar{u} \) is piecewise constant. Let us define, for every \( t \in [0, T[ \),

\[
\phi_t(t, x) = \nu \int_{N_\nu} [\phi_t(t, x) u(t, x) + \phi_x(t, x) F(u(t, x))] \, dx \, dt = \\
= \int_{-\infty}^{+\infty} \phi((k + 1) \varepsilon, x) u((k + 1) \varepsilon, x) \, dx + \\
- \int_{-\infty}^{+\infty} \phi(k \varepsilon, x) u(k \varepsilon + , x) \, dx + \beta(t),
\]

with \( \lim_{t \to \infty} \beta(t) = 0. \) From this last identity we obtain

\[
\int_{0}^{T} \int_{-\infty}^{+\infty} \phi_t(t, x) u(t, x) + \phi_x(t, x) F(u(t, x)) \, dx \, dt = \\
= \left( \sum_{k=0}^{N_\nu - 1} \int_{k \varepsilon}^{(k+1) \varepsilon} + \int_{N_\nu \varepsilon}^{T} \right) \int_{-\infty}^{+\infty} \phi_t(t, x) u(t, x) + \\
+ \phi_x(t, x) F(u(t, x)) \, dx \, dt = \\
= \sum_{k=0}^{N_\nu - 1} \int_{-\infty}^{+\infty} \phi((k + 1) \varepsilon, x) u((k + 1) \varepsilon, x) \, dx + \\
- \sum_{k=0}^{N_\nu} \left[ \int_{-\infty}^{+\infty} \phi(k \varepsilon, x) u(k \varepsilon, x) \, dx + \varepsilon \beta(t) \right] = \\
= \sum_{k=0}^{N_\nu} \int_{-\infty}^{+\infty} \phi(k \varepsilon, x) g(t, x, u(t, x)) \, dx + \\
- \int_{-\infty}^{+\infty} \phi(0, x) u(0, x) \, dx + (T - \delta) \beta(t).
\]

Recalling that \( g(t) \) converges uniformly to \( g \) on compact subsets of \([0, T] \times \mathbb{R} \times \mathbb{R}^2\), and that \( u(t) \to u \) in \( L^1_{\text{loc}} \), it is easy to see that the sequence of functions \( \{ \phi(t) \} \), defined by

\[
\phi(t) = \sum_{k=0}^{N_\nu - 1} \chi_{[k \varepsilon, (k+1) \varepsilon]}(t) \int_{-\infty}^{+\infty} \phi(k \varepsilon, x) g(t, x, u(t, x)) \, dx,
\]
converges for almost every $t \in [0,T]$ to
\[
\varphi(t) \doteq \int_{-\infty}^{+\infty} \phi(t,x)g(t,x,u(t,x)) \, dx.
\]

By the Lebesgue dominated convergence theorem we have that
\[
\lim_{\nu \to +\infty} \int_{0}^{T} \varphi_{\nu}(t) \, dt = \int_{0}^{T} \varphi(t) \, dt.
\]

Hence, passing to the limit in (6.2) for $\nu \to +\infty$, we obtain (6.1).

It remains to show that $u(t)$ is an entropic solution. Let $\eta$ be a smooth convex entropy with flux $q$. We have to show that
\[
[\eta(u)]_{t} + [q(u)]_{x} - D\eta(u)g(t,x,u) \leq 0
\]
in the sense of distributions, that is
\[
\int_{0}^{T} \int_{-\infty}^{+\infty} \left[ \eta(u(t,x))\phi_{t}(t,x) + q(u(t,x))\phi_{x}(t,x) + D\eta(u(t,x))g(t,x,u(t,x))\phi(t,x) \right] \, dx \, dt - \int_{-\infty}^{+\infty} \phi(0,x)\eta(u(0,x)) \, dx \geq 0,
\]
for every $\phi \in C_{c}^{1}([0,T] \times \mathbb{R})$, $\phi \geq 0$. With the same notations of the first part of the proof, we have that
\[
\int_{0}^{T} \int_{-\infty}^{+\infty} \left[ \eta(u_{\nu}(t,x))\phi_{t}(t,x) + q(u_{\nu}(t,x))\phi_{x}(t,x) \right] \, dx \, dt =
\]
\[
= \sum_{k=0}^{N_{\nu} - 1} \int_{-\infty}^{+\infty} \eta(u_{\nu}((k + 1)\varepsilon_{\nu} - , x))\phi((k + 1)\varepsilon_{\nu}, x) \, dx +
\]
\[
- \sum_{k=0}^{N_{\nu}} \int_{-\infty}^{+\infty} \eta(u_{\nu}(k\varepsilon_{\nu} + , x))\phi(k\varepsilon_{\nu}, x) \, dx = \int_{-\infty}^{+\infty} \eta(u_{\nu}(0,x))\phi(0,x) \, dx +
\]
\[
- \sum_{k=1}^{N_{\nu}} \int_{-\infty}^{+\infty} \left[ \eta(u_{\nu}(k\varepsilon_{\nu} + , x)) - \eta(u_{\nu}(k\varepsilon_{\nu} - , x)) \right] \phi(k\varepsilon_{\nu}, x) \, dx \geq
\]
\[
\geq \int_{-\infty}^{+\infty} \eta(u_{\nu}(0,x))\phi(0,x) \, dx +
\]
\[
- \sum_{k=1}^{N_{\nu}} \varepsilon_{\nu} \int_{-\infty}^{+\infty} D\eta(u_{\nu}(k\varepsilon_{\nu} + , x))g_{\nu}(k\varepsilon_{\nu}, x,u_{\nu}(k\varepsilon_{\nu} - , x))\phi(k\varepsilon_{\nu}, x) \, dx,
\]
where in the last inequality we used the convexity of $\eta$. Reasoning as in (6.2), we can pass to the limit for $\nu \to +\infty$ obtaining (6.3). \qed
7. Viscosity solutions and uniqueness. We give now the definition of viscosity solution for a system of balance laws. For the equivalent definition in the homogeneous case, see [4]. We will prove the equivalence between viscosity solutions and trajectories of $P$. This in turn will imply that $P$ is independent of the subsequence $(P^\nu)$ constructed in section 5, and then it is the unique operator satisfying the properties listed in Theorem 1.1.

Consider a continuous function $u: [0, T] \to \mathcal{D}$ and fix $(\tau, \xi) \in [0, T] \times \mathbb{R}$. We define two functions $U^\rho, U^\xi$ that represent local parametrices for viscosity solutions. Let $\hat{U}^\rho$ be the solution to the linear problem

\[
\begin{cases}
 w_t + \hat{A} w_x = 0, \\
 w(\tau) = u(\tau),
\end{cases}
\]

where $\hat{A} = DF(u(\tau, \xi))$, and define

\[
U^\rho_{(u, \tau; \xi)}(t, x) \doteq \hat{U}^\rho(t, x) + (t - \tau)\hat{g}, \quad \hat{g} \doteq g(\tau, \xi, u(\tau, \xi)).
\] (7.1)

Consider now the Riemann problem associated to the initial condition $u(0, x) = u^-$ for $x < 0$, $u(0, x) = u^+$ for $x > 0$, where $u^\pm \doteq \lim_{x \to -\xi^\pm} u(\tau, x)$, and let $\omega(t, x)$ be the standard self-similar solution. Let us define

\[
U^\xi_{(u, \tau; \xi)}(t, x) \doteq \left\{ \begin{array}{ll}
\omega(t - \tau, x - \xi), & |x - \xi| \leq \hat{\lambda}(t - \tau), \\
u(\tau, x), & |x - \xi| > \hat{\lambda}(t - \tau).
\end{array} \right.
\] (7.2)

We can now give the following:

**Definition 7.1** Given $u: [0, T] \to \mathcal{D}$ continuous, we say that $u$ is a viscosity solution for the problem (1.1) if there exists $C > 0$ such that for all $(\tau, \xi) \in [0, T] \times \mathbb{R}$ and $\rho, \varepsilon > 0$ sufficiently small we have

\[
\frac{1}{\varepsilon} \int_{\xi - \rho + \varepsilon \lambda}^{\xi + \rho - \varepsilon \lambda} u(\tau + \varepsilon, x) - U^\rho_{(u, \tau; \xi)}(\tau + \varepsilon, x) \, dx \leq C \left[ (\text{T.V.} \{u(\tau) ; I_{\xi, \rho}\})^2 + \rho^2 + \rho \eta(\varepsilon) \right],
\] (7.3)

\[
\frac{1}{\varepsilon} \int_{\xi - \rho + \varepsilon \lambda}^{\xi + \rho - \varepsilon \lambda} \left| u(\tau + \varepsilon, x) - U^\xi_{(u, \tau; \xi)}(\tau + \varepsilon, x) \right| \, dx \leq C \left[ \text{T.V.} \{u(\tau) ; I_{\xi, \rho}'\} + \rho \right],
\] (7.4)

with $I_{\xi, \rho} \doteq |\xi - \rho, \xi + \rho|$, $I_{\xi, \rho}' \doteq |\xi - \rho, \xi + \rho| + \rho$ and $\lim_{\varepsilon \to 0^+} \eta(\varepsilon) = 0$.

**Theorem 7.2** Let $\bar{u} \in \mathcal{D}$. Then $u(t) \doteq P(t, 0)\bar{u}, \ t \in [0, T]$, is a viscosity solution of (1.1).

**Proof.** Fix $(\tau, \xi)$ and consider $\varepsilon, \rho > 0$ small. By continuity it suffices to prove (7.3) and (7.4) under the additional assumption $\varepsilon \in T$. This implies that, for every $\nu$ large enough, there exists $N_\nu \in \mathbb{N}$ such that $N_\nu \varepsilon_\nu = \varepsilon$.

Given $\varepsilon' > 0$ let $\bar{v}(x)$ be a piecewise constant function such that

\[
\bar{v}(\xi) = u(\tau, \xi), \quad \int_{I_{\xi, \rho}} |\bar{v} - u(\tau)| \, dx < \varepsilon',
\] (7.5)
Consider the function \( v(t, x) = \tilde{v}(t, x) + (t - \tau)\tilde{g} \), where \( \tilde{v} \) is the solution to the linear problem
\[
\begin{align*}
\left\{ v_t + \tilde{A}v_x &= 0, \\
v(\tau) &= \bar{v},
\end{align*}
\]
with \( \tilde{A}, \tilde{g} \) defined as above.

By Lemma 5.1 and (7.5) we obtain
\[
\int_{\xi + \rho - \epsilon \hat{\lambda}}^{\xi + \rho + \epsilon \hat{\lambda}} \left| P(\tau + \epsilon, \tau)u(\tau) - P(\tau + \epsilon, \tau)\bar{v} \right| dx \leq L_P \int_{I_{\xi, \rho}} \left| u(\tau) - \bar{v} \right| dx \leq L_P \epsilon'.
\]

Moreover, by the explicit form of \( U^\nu_{(u, \tau, \xi)}(t + \epsilon) \) and \( v(t + \epsilon) \), recalling (7.5), one has
\[
\int_{\xi + \rho - \epsilon \hat{\lambda}}^{\xi + \rho + \epsilon \hat{\lambda}} \left| U^\nu_{(u, \tau, \xi)}(t + \epsilon) - v(t + \epsilon) \right| dx \leq C_0 \epsilon',
\]
for some positive constant \( C_0 \).

For every \( k = 0, \ldots, \nu - 1 \) let us define
\[
J_k = [\xi - \rho + k\epsilon \hat{\lambda}, \xi + \rho - k\epsilon \hat{\lambda}],
\]
and let us denote by \( \tilde{v}_k \) the solution to the problem
\[
\begin{align*}
\left\{ w_t + \tilde{A}w_x &= 0, \\
w(\tau + k\epsilon) &= v(\tau + k\epsilon),
\end{align*}
\]
Using Lemma 27 of [6] one has
\[
\int_{J_{k+1}} \left| S_{\epsilon, v}(\tau + k\epsilon) - \tilde{v}_h(\epsilon) \right| dx = O(1)\epsilon(\text{T.V.}\{v(\tau + k\epsilon) + J_k\})^2 = O(1)\epsilon(\text{T.V.}\{\tilde{v}; I_{\xi, \rho}\})^2.
\]

From (10.1) and (10.12) of [6] we obtain
\[
\int_{J_{k+1}} \left| S^\nu_{\epsilon} v(\tau + k\epsilon) - S^\nu_{\epsilon} v(\tau + k\epsilon) \right| dx \leq \frac{LS}{h} \limsup_{h \to 0+} \left\| S^\nu_{\epsilon} v(\tau + k\epsilon) - S_h S^\nu_{\epsilon} v(\tau + k\epsilon) \right\|_{L^1} dt = O(1)\epsilon\sqrt{\epsilon}.
\]
Let us define \( \zeta_k(x) = (\tau + (k + 1)\epsilon, x) S^\nu_{\epsilon} v(\tau + k\epsilon)(x) \). Since \( g_\nu \) converges uniformly on compact sets to \( g \), the sequence
\[
a_\nu = \int_{J_{k+1}} \left| g_\nu(\zeta_k(x)) - g(\zeta_k(x)) \right| dx
\]
converges to 0 for $\nu \to +\infty$. Moreover, the continuity of $g$ w.r.t. $t$ implies that there exists $\eta' = \eta'(\epsilon)$ positive such that $\lim_{\epsilon \to 0^+} \eta'(\epsilon) = 0$, and

$$\int_{J_{k+1}} |g(\zeta_k(x)) - g(\tau, x, S^\varepsilon_{\nu} v(\tau + k\varepsilon_{\nu})(x))| \, dx \leq \rho \eta'(\epsilon). \tag{7.13}$$

By the Lipschitz continuity of $S^\varepsilon_{\nu}$ we have that

$$\int_{J_{k+1}} |S^\varepsilon_{\nu} v(\tau + k\varepsilon_{\nu})(x) - v(\tau, \xi)| \, dx \leq \int_{J_{k+1}} |S^\varepsilon_{\nu} v(\tau + k\varepsilon_{\nu}) - S^\varepsilon_{\nu} \bar{v}(\tau + k\varepsilon_{\nu})| \, dx +$$

$$+ \int_{J_{k+1}} |S^\varepsilon_{\nu} \bar{v}(\tau + k\varepsilon_{\nu})(x) - v(\tau, \xi)| \, dx \leq$$

$$\leq L_S \int_{J_k} |v(\tau + k\varepsilon_{\nu}) - \bar{v}(\tau + k\varepsilon_{\nu})| \, dx + O(1) \text{meas}(J_{k+1}) \mathit{T.V.}\{\bar{v}; I_{\xi, \rho}\} \leq O(1) \rho |k\varepsilon_{\nu}| \bar{g} + \mathit{T.V.}\{\bar{v}; I_{\xi, \rho}\}.$$  

By the Lipschitz continuity of $g$ and this last estimate one has

$$\int_{J_{k+1}} |g(\tau, x, S^\varepsilon_{\nu} v(\tau + k\varepsilon_{\nu})(x)) - g(\tau, \xi, v(\tau, \xi))| \, dx \leq$$

$$\leq L_g \int_{J_{k+1}} (|x - \xi| + |S^\varepsilon_{\nu} v(\tau + k\varepsilon_{\nu})(x) - v(\tau, \xi)|) \, dx \leq$$

$$\leq O(1) \left\{ \rho^2 + \rho |k\varepsilon_{\nu}| \bar{g} + \mathit{T.V.}\{\bar{v}; I_{\xi, \rho}\} \right\}. \tag{7.14}$$

From (7.12), (7.13) and (7.14) we obtain

$$\int_{J_{k+1}} |g_{\nu}(\tau + (k + 1)\varepsilon_{\nu}, x, S^\varepsilon_{\nu} v(\tau + k\varepsilon_{\nu})) - \bar{g}| \, dx \leq$$

$$\leq \int_{J_{k+1}} |g_{\nu}(\zeta_k(x)) - g(\zeta_k(x))| \, dx +$$

$$+ \int_{J_{k+1}} |g(\zeta_k(x)) - g(\tau, x, S^\varepsilon_{\nu} v(\tau + k\varepsilon_{\nu})(x))| \, dx +$$

$$+ \int_{J_{k+1}} |g(\tau, x, S^\varepsilon_{\nu} v(\tau + k\varepsilon_{\nu})(x)) - g(\tau, \xi, v(\tau, \xi))| \, dx \leq$$

$$\leq a_{\nu} + \rho \eta'(\epsilon) + O(1) \rho^2 + O(1) \rho |k\varepsilon_{\nu}| \bar{g} + \mathit{T.V.}\{\bar{v}; I_{\xi, \rho}\}. \tag{7.15}$$

From Corollary 5.1 and the definition of $P^\nu$, recalling (7.10), (7.11) and (7.15), one
We finally have, recalling $N_\nu \varepsilon_\nu = \varepsilon$, 

\[
I_k \leq \int_{\xi - \rho + \varepsilon}^{\xi + \rho - \varepsilon} \left| P^\nu(\tau + \varepsilon, \tau + k\varepsilon_\nu) v(\tau + k\varepsilon_\nu) + P^\nu(\tau + \varepsilon, \tau + (k + 1)\varepsilon_\nu) v(\tau + (k + 1)\varepsilon_\nu) \right| \, dx \leq \\
\leq L_P \int_{J_{k+1}} |P^\nu(\tau + (k + 1)\varepsilon_\nu, \tau + k\varepsilon_\nu) v(\tau + k\varepsilon_\nu) - v(\tau + (k + 1)\varepsilon_\nu)| \, dx \leq \\
\leq L_P \int_{J_{k+1}} |S_{\varepsilon_\nu} v(\tau + k\varepsilon_\nu) - \tilde{v}_k(\varepsilon_\nu)| \, dx + \\
+ L_P \int_{J_{k+1}} |\xi_{\varepsilon_\nu}^\nu v(\tau + k\varepsilon_\nu) - S_{\varepsilon_\nu} v(\tau + k\varepsilon_\nu)| \, dx + \\
+ L_P \int_{J_{k+1}} |g_\nu(\tau + (k + 1)\varepsilon_\nu, x, S_{\varepsilon_\nu}^\nu(\tau + k\varepsilon_\nu)) - \tilde{g} \, dx = \\
= O(1) \{\varepsilon_\nu (T.V.\{\tilde{v}; I_{\xi,\rho}\})^2 + \varepsilon_\nu \sqrt{\varepsilon_\nu} + \\
+ \varepsilon_\nu [a_\nu + \rho \eta'(\varepsilon) + \rho^2 + \rho T.V.\{\tilde{v}; I_{\xi,\rho}\} + \rho \varepsilon_\nu^2 \} .
\]

Using this estimate, we have that 

\[
\int_{\xi - \rho + \varepsilon}^{\xi + \rho - \varepsilon} |P^\nu(\tau + N_\nu \varepsilon_\nu, \tau) \tilde{v} - v(\tau + N_\nu \varepsilon_\nu)| \, dx \leq \sum_{k=0}^{N_\nu} I_k = \\
= O(1) (N_\nu \varepsilon_\nu) \{[(T.V.\{\tilde{v}; I_{\xi,\rho}\})^2 + \sqrt{\varepsilon_\nu} + a_\nu + \rho \eta'(\varepsilon) + \rho^2 + \\
+ \rho T.V.\{\tilde{v}; I_{\xi,\rho}\} + \rho (N_\nu \varepsilon_\nu)\} .
\]

We finally have, recalling $N_\nu \varepsilon_\nu = \varepsilon$, 

\[
\int_{\xi - \rho + \varepsilon}^{\xi + \rho - \varepsilon} |P(\tau + \varepsilon, \tau) \tilde{v} - v(\tau + \varepsilon)| \, dx \leq \\
\leq \int_{\xi - \rho + \varepsilon}^{\xi + \rho - \varepsilon} \left| P(\tau + \varepsilon, \tau) \tilde{v} - P^\nu(\tau + \varepsilon, \tau) \tilde{v} \right| \, dx + \\
+ \int_{\xi - \rho + \varepsilon}^{\xi + \rho - \varepsilon} \left| P^\nu(\tau + \varepsilon, \tau) \tilde{v} - v(\tau + \varepsilon) \right| \, dx \leq \\
\leq \int_{\xi - \rho + \varepsilon}^{\xi + \rho - \varepsilon} |P(\tau + \varepsilon, \tau) \tilde{v} - P^\nu(\tau + \varepsilon, \tau) \tilde{v} | \, dx + \\
+ O(1) \varepsilon \{[(T.V.\{\tilde{v}; I_{\xi,\rho}\})^2 + \sqrt{\varepsilon_\nu} + a_\nu + \rho \eta'(\varepsilon) + \rho^2 + \\
+ \rho T.V.\{\tilde{v}; I_{\xi,\rho}\} + \rho \varepsilon_\nu \} .
\]
By (7.7), (7.8) and (7.16) we obtain

$$
\frac{1}{\varepsilon} \int_{\xi - \rho + \varepsilon\lambda}^{\xi + \rho - \varepsilon\lambda} \left| P(\tau + \varepsilon, \tau)u(\tau) - U_{\rho(u, \tau, \xi)}(\tau + \varepsilon) \right| \, dx \leq \leq \frac{1}{\varepsilon} \int_{\xi - \rho + \varepsilon\lambda}^{\xi + \rho - \varepsilon\lambda} \left| P(\tau + \varepsilon, \tau)u(\tau) - P(\tau + \varepsilon, \tau)v \right| \, dx + \\
+ \frac{1}{\varepsilon} \int_{\xi - \rho + \varepsilon\lambda}^{\xi + \rho - \varepsilon\lambda} \left| U_{\rho(u, \tau, \xi)}(\tau + \varepsilon) - v(\tau + \varepsilon) \right| \, dx + \\
+ \frac{1}{\varepsilon} \int_{\xi - \rho + \varepsilon\lambda}^{\xi + \rho - \varepsilon\lambda} \left| P(\tau + \varepsilon, \tau)v - v(\tau + \varepsilon) \right| \, dx \leq \\
\leq \frac{Lp}{\varepsilon} \varepsilon' + \frac{C_0}{\varepsilon} \varepsilon' + \frac{1}{\varepsilon} \int_{\xi - \rho + \varepsilon\lambda}^{\xi + \rho - \varepsilon\lambda} \left| P(\tau + \varepsilon, \tau)v - P^N(\tau + \varepsilon, \tau)v \right| \, dx + \\
+ O(1) \left[ (T.V.\{\tilde{v}; I_{\varepsilon, \rho}\})^2 + \sqrt{\varepsilon\varepsilon'} + a_\rho(\varepsilon) + \rho^2 + \rho \text{TV.}\{\tilde{v}; I_{\varepsilon, \rho}\} + \rho \varepsilon \right].
$$

Passing to the limit in \( \nu \) and for the arbitrariness of \( \varepsilon' > 0 \) we obtain the estimate (7.3) with \( \eta(\varepsilon) \equiv \eta' (\varepsilon) + \varepsilon \).

We now turn to the estimate (7.4). Choose \( \bar{\lambda} \geq 4 \max \{DF(u) : u \in U \} \) and let \( \varepsilon' > 0 \) be given. Let \( \tilde{v} \) be a piecewise continuous function satisfying (7.5) together with \( \tilde{v}(\xi \pm) = u(\tau, \xi \pm) \) and \( T.V.\{\tilde{v}; I_{\varepsilon, \rho}\} \leq T.V.\{u(\tau); I_{\varepsilon, \rho}\} \). Let us define the function

$$
v(t, x) = \begin{cases} 
\omega(t - \tau, x - \xi), & |x - \xi| \leq \bar{\lambda}(t - \tau), \\
\tilde{v}(x), & |x - \xi| > \bar{\lambda}(t - \tau),
\end{cases}
$$

where, as before, \( \omega \) is the self–similar solution to the Riemann problem associated to \( u(\tau, \xi \pm) \). With this choice of \( \bar{\lambda} \), if we solve the Riemann problem in the two points \( (\tau + k\varepsilon, \xi \pm \bar{\lambda}k\varepsilon) \) for a time interval of length \( \varepsilon \), the corresponding waves do not enter the region \( |x - \xi| \leq \bar{\lambda}(k + 1)\varepsilon \), where the function \( v \) coincide with \( \omega \).

As in the first part of the proof, there exists a constant \( C_1 > 0 \) such that

$$
\int_{\xi - \rho + \varepsilon\lambda}^{\xi + \rho - \varepsilon\lambda} \left| u(\tau + \varepsilon) - P(\tau + \varepsilon, \tau)v \right| \, dx \leq C_1 \varepsilon', 
$$

and

$$
\int_{\xi - \rho + \varepsilon\lambda}^{\xi + \rho - \varepsilon\lambda} \left| U_{\rho(u, \tau, \xi)}(\tau + \varepsilon) - v(\tau + \varepsilon) \right| \, dx \leq C_1 \varepsilon'.
$$

Reasoning as in (7.11), we have that

$$
\int_{J_{k+1}} \left| S_{\varepsilon, v}(\tau + k\varepsilon) - S_{\varepsilon, v}(\tau + k\varepsilon) \right| \, dx = O(1)\varepsilon \sqrt{\varepsilon'}, 
$$

where \( J_k \) is defined in (7.9). Moreover,

$$
\int_{J_{k+1}} \left| S_{\varepsilon, v}(\tau + k\varepsilon) - v(\tau + (k + 1)\varepsilon) \right| \, dx = \\
= O(1)\varepsilon \text{TV.}\{\tilde{v}(\tau + k\varepsilon); J_k \cup [\xi - \bar{\lambda}k\varepsilon, \xi + \bar{\lambda}k\varepsilon]\} = \\
= O(1)\varepsilon \text{TV.}\{\tilde{v}; I_{\varepsilon, \rho}\}.
$$
Indeed, by the choice of $\hat{\lambda}$, the integrand is zero in the region $|x-\xi| \leq \hat{\lambda}(k+1)\varepsilon_{\nu}$, and, by property (iv) in Proposition 2.3,

$$\mathcal{T.V.}\{S_{\epsilon_{\nu}}v(\tau + k\varepsilon_{\nu}); J_{k+1}\} = O(1)\mathcal{T.V.}\{v(\tau + k\varepsilon_{\nu}); J_{k}\}.$$ 

By Corollary 5.1 and the definition of $P''$, recalling (7.19) and (7.20), one has

$$M_{k} \leq \int_{\frac{\xi+\rho-\varepsilon}{\xi-\rho+\varepsilon}}^{\xi+\rho-\varepsilon} \left| P''(\tau + \varepsilon, \tau + k\varepsilon_{\nu})v(\tau + k\varepsilon_{\nu}) + P''(\tau + \varepsilon, \tau + (k+1)\varepsilon_{\nu})v(\tau + (k+1)\varepsilon_{\nu}) \right| d\tau \leq \int_{J_{k+1}} \left| \frac{\partial}{\partial \tau} \right| \left| P''(\tau + \varepsilon, \tau + k\varepsilon_{\nu})v(\tau + k\varepsilon_{\nu}) - v(\tau + (k+1)\varepsilon_{\nu}) \right| d\tau + \int_{J_{k+1}} \left| \frac{\partial}{\partial \tau} \right| \left| P''(\tau + \varepsilon, \tau + (k+1)\varepsilon_{\nu})v(\tau + (k+1)\varepsilon_{\nu}) - v(\tau + (k+1)\varepsilon_{\nu}) \right| d\tau + \int_{J_{k+1}} \left| \frac{\partial}{\partial \tau} \right| \left| v(\tau + k\varepsilon_{\nu}) - v(\tau + (k+1)\varepsilon_{\nu}) \right| d\tau + O(1)\varepsilon_{\nu}\|v_{\nu}\|_{\infty} = O(1)\varepsilon_{\nu}[\mathcal{T.V.}\{v; I'_{\xi,\rho}\} + \sqrt{\varepsilon_{\nu} + \rho}] .$$

From this last estimate we obtain

$$\int_{\frac{\xi+\rho-\varepsilon}{\xi-\rho+\varepsilon}}^{\xi+\rho-\varepsilon} \left| P''(\tau + N_{\nu}\varepsilon_{\nu}, \tau)\bar{v} - v(\tau + N_{\nu}\varepsilon_{\nu}) \right| d\tau \leq \sum_{k=0}^{N_{\nu}-1} M_{k} = O(1)N_{\nu}\varepsilon_{\nu}[\mathcal{T.V.}\{v; I'_{\xi,\rho}\} + \sqrt{\varepsilon_{\nu} + \rho}] .$$

From (7.17), (7.18) and (7.21) we have

$$\frac{1}{\varepsilon} \int_{\frac{\xi+\rho-\varepsilon}{\xi-\rho+\varepsilon}}^{\xi+\rho-\varepsilon} \left| u(\tau + \varepsilon) - U_{(u,\xi,\xi)}^{1}(\tau + \varepsilon) \right| d\tau \leq \frac{1}{\varepsilon} \int_{\frac{\xi+\rho-\varepsilon}{\xi-\rho+\varepsilon}}^{\xi+\rho-\varepsilon} \left| u(\tau + \varepsilon) - P(\tau + \varepsilon, \tau)\bar{v} \right| d\tau + \frac{1}{\varepsilon} \int_{\frac{\xi+\rho-\varepsilon}{\xi-\rho+\varepsilon}}^{\xi+\rho-\varepsilon} \left| U_{(u,\xi,\xi)}^{2}(\tau + \varepsilon) - v(\tau + \varepsilon) \right| d\tau + \frac{1}{\varepsilon} \int_{\frac{\xi+\rho-\varepsilon}{\xi-\rho+\varepsilon}}^{\xi+\rho-\varepsilon} \left| P(\tau + \varepsilon, \tau)\bar{v} - P''(\tau + \varepsilon, \tau)\bar{v} \right| d\tau + \frac{1}{\varepsilon} \int_{\frac{\xi+\rho-\varepsilon}{\xi-\rho+\varepsilon}}^{\xi+\rho-\varepsilon} \left| P''(\tau + \varepsilon, \tau)\bar{v} - v(\tau + \varepsilon) \right| d\tau \leq \frac{2C_{1}}{\varepsilon} e^{1'} + O(1)\mathcal{T.V.}\{\bar{v}; I'_{\xi,\rho}\} + \sqrt{\varepsilon_{\nu} + \rho} + \frac{1}{\varepsilon} \int_{\frac{\xi+\rho-\varepsilon}{\xi-\rho+\varepsilon}}^{\xi+\rho-\varepsilon} \left| P(\tau + \varepsilon, \tau)\bar{v} - P''(\tau + \varepsilon, \tau)\bar{v} \right| d\tau .$$

Recalling that $P''$ converges uniformly to $P$, passing to the limit for $\nu \to +\infty$, and by the arbitrariness of $e'$, we have that (7.4) holds. 

\[\square\]
We pass now to prove the opposite implication.

**Theorem 7.3** Let \( u: [0, T] \to D \) be a viscosity solution of (1.1). Then \( u(t) = P(t, 0)u(0) \) for every \( t \in [0, T] \).

**Proof.** Clearly, it suffices to show that \( u(T) = P(T, 0)u(0) \) restricted to any interval of the form \( J = [-R + \lambda T, R - \lambda T] \), with \( R > \lambda T \) arbitrarily large. Let \( R \) and \( \delta_0 \) positive be given. We will prove that the inequality

\[
\int J |P(T, t)u(t) - P(T, 0)u(0)|dx \leq \delta_0 t
\]

(7.22)

holds for \( t = T \). Clearly, (7.22) is satisfied when \( t = 0 \). Let \( \tau \) be the supremum of the times for which (7.22) holds. By continuity we have that (7.22) holds for \( t = \tau \). If \( \tau < T \), we show that (7.22) holds for \( \tau + \varepsilon \) for some \( \varepsilon > 0 \), reaching a contradiction. Applying Theorem 7.2 to the map \( \varepsilon \to P(\tau + \varepsilon, \tau)u(\tau) \), it follows that, for every fixed \( \xi \) and for every \( \rho, \varepsilon > 0 \) sufficiently small,

\[
\frac{1}{\varepsilon} \int_{\xi - \rho + \varepsilon \lambda}^{\xi + \rho - \varepsilon \lambda} \left| P(\tau + \varepsilon, \tau)u(\tau) - U^\rho(u, \tau, \xi)(\tau + \varepsilon) \right| dx \leq 2 C \left[ (T.V.\{u(\tau); I_{\xi, \rho}\})^2 + \rho^2 + \rho \eta(\varepsilon) \right],
\]

(7.23)

\[
\frac{1}{\varepsilon} \int_{\xi - \rho + \varepsilon \lambda}^{\xi + \rho - \varepsilon \lambda} \left| P(\tau + \varepsilon, \tau)u(\tau) - U^\rho(u, \tau, \xi)(\tau + \varepsilon) \right| dx \leq C \left[ (T.V.\{u(\tau); I_{\xi, \rho}\})^2 + \rho \eta(\varepsilon) \right],
\]

(7.24)

with \( \lim_{\varepsilon \to 0} \eta(\varepsilon) = 0 \). Clearly the function \( \eta \) depends on the point \( \xi \).

Let us define the Borel measure \( \mu \) by \( \mu(I) = T.V.\{u(\tau); I\} \) for every open interval \( I \). Since the total variation of \( u(\tau) \) is bounded, there exists a finite set of points \( \mathcal{A} = \{\xi_1, \ldots, \xi_N\} \subset [-R, R] \) such that

\[
\mu(\{\xi_j\}) \geq C_1 \delta_0, \quad C_1 \geq \frac{1}{20NCL^p}.
\]

Let us define \( I_j = ]\xi_j - \rho_j, \xi_j + \rho_j[ \), \( I_j' = I_j \setminus \{\xi_j\} \) and \( \eta_j \) with \( \rho_j, \varepsilon_j > 0 \) such that

\[
\mu(I_j) \leq C_2 \delta_0, \quad \rho_j \leq C_2 \delta_0, \quad C_2 \geq \frac{1}{16NCL^p},
\]

and such that the estimates (7.3), (7.23) hold when \( \xi = \xi_j, \rho = \rho_j, \varepsilon \in [0, \varepsilon_j] \). Since the number of \( \xi_j \) is finite, the function \( \eta \) can be chosen independent of \( \xi_j \). By possibly shrinking the values \( \rho_j \), we can assume that the \( I_j \) are pairwise disjoint.

We can now cover the interval \( [-R + \lambda \tau, R - \lambda \tau] \setminus (\cup_j I_j) \) with a finite number of intervals \( J_l = ]\xi_l' - \rho_l', \xi_l' + \rho_l'[ \), \( l = 1, \ldots, M \) such that

\[
\mu(J_l) \leq C_3 \delta_0, \quad \rho_l' \leq C_3 \delta_0, \quad C_3 \geq \frac{1}{16CL^p[\mu([-R, R])] + 2R}.
\]

Moreover, we can assume that:

i) \( J_k \cap J_l \cap J_m = \emptyset \) for distinct indices \( k, l, m \);
ii) for some \( \varepsilon^*_i > 0 \) the viscosity estimates (7.4), (7.24) hold when \( \xi = \xi_i', \rho = \rho_i' \) and \( \varepsilon \in [0, \varepsilon^*_i] \).

Choose \( 0 < \varepsilon^* < \min\{\varepsilon_1, \ldots, \varepsilon_N, \varepsilon_1', \ldots, \varepsilon_M'\} \) such that
\[
\eta(\varepsilon) \leq C_3 \delta_0,
\]
for \( 0 \leq \varepsilon \leq \varepsilon^* \) and
\[
[-R + \tilde{\lambda}(\tau + \varepsilon^*), R - \tilde{\lambda}(\tau + \varepsilon^*)] \subset \left( \bigcup_{j=1}^{N} I_{j, \varepsilon^*} \right) \cup \left( \bigcup_{l=1}^{M} J_{l, \varepsilon^*} \right),
\]
\[
I_{j, \varepsilon} \ni [\rho_j - \tilde{\lambda} \varepsilon, \rho_j + \tilde{\lambda} \varepsilon], \quad J_{l, \varepsilon} \ni [\rho_l' - \tilde{\lambda} \varepsilon, \rho_l' + \tilde{\lambda} \varepsilon].
\]
For \( \varepsilon \in [0, \varepsilon^*] \), using the estimate (7.22) at \( t = \tau \) and Corollary 5.1, we have that
\[
\int_{\mathbb{R}} |P(T, \tau + \varepsilon)u(\tau + \varepsilon) - P(T, 0)u(0)| \, dx \leq
\]
\[
\leq \int_{\mathbb{R}} |P(T, \tau + \varepsilon)u(\tau + \varepsilon) - P(T, \tau + \varepsilon)P(\tau + \varepsilon, \tau)u(\tau)| \, dx +
\]
\[
+ \int_{\mathbb{R}} |P(T, \tau + \varepsilon)P(\tau + \varepsilon, \tau)u(\tau) - P(T, 0)u(0)| \, dx \leq
\]
\[
\leq L_p \int_{-R + \tilde{\lambda}(\tau + \varepsilon)}^{R - \lambda(\tau + \varepsilon)} |u(\tau + \varepsilon) - P(\tau + \varepsilon, \tau)u(\tau)| \, dx + \delta_0 \tau.
\]

(7.25)

Let us define, for every \( j = 1, \ldots, N \) and every \( l = 1, \ldots, M \),
\[
U_j^\varepsilon = U_j^\varepsilon(u, \tau, \xi_j), \quad U_l^\varepsilon = U_l^\varepsilon(u, \tau, \xi_l).
\]

Using the definition of viscosity solution and (7.23)–(7.24) we have
\[
\int_{-R + \tilde{\lambda}(\tau + \varepsilon)}^{R - \tilde{\lambda}(\tau + \varepsilon)} |u(\tau + \varepsilon) - P(\tau + \varepsilon, \tau)u(\tau)| \, dx \leq
\]
\[
\leq \sum_{j=1}^{N} \int_{I_{j, \varepsilon}} \left( |u(\tau + \varepsilon) - U_j^\varepsilon(\tau + \varepsilon)| +
\right.
\]
\[
+ \left. \left| U_j^\varepsilon(\tau + \varepsilon) - P(\tau + \varepsilon, \tau)u(\tau) \right| \right) \, dx +
\]
\[
+ \sum_{l=1}^{M} \int_{J_{l, \varepsilon}} \left( |u(\tau + \varepsilon)| +
\right.
\]
\[
- U_l^\varepsilon(\tau + \varepsilon) + \left| U_l^\varepsilon(\tau + \varepsilon) - P(\tau + \varepsilon, \tau)u(\tau) \right| \right) \, dx \leq
\]
\[
\leq \varepsilon \sum_{j=1}^{N} 2C \left[ \mu(J_j') + \rho_j \right] + \varepsilon \sum_{l=1}^{M} 2C \left[ (\mu(J_l'))^2 + \rho_l' (\rho_l' + \eta(\varepsilon)) \right] \leq
\]
\[
\leq 2C \varepsilon \left[ NC_2 \delta_0 + NC_2 \delta_0 + C_3 \delta_0 \sum_{l=1}^{M} \mu(J_l) + (\eta(\varepsilon) + C_3 \delta_0) \sum_{l=1}^{M} \rho_l' \right] \leq
\]
\[
\leq 2C \varepsilon \delta_0 \left[ 2NC_2 + 2C_3 \mu([-R, R]) + 8RC_3 \right] \leq \frac{\varepsilon \delta_0}{L_p}.
\]

(7.26)
Finally, using (7.26) in (7.25), we obtain
\[ \int_J |P(T, \tau + \varepsilon)u(\tau + \varepsilon) - P(T, 0)u(0)| \, dx \leq (\tau + \varepsilon)\delta_0, \]
in contradiction with the maximality of \( \tau \).

Since the estimate (7.22) holds for \( t = T \) and \( R, \delta_0 \) arbitrary, the proof is completed. \( \square \)

We now have that, given any semigroup \( P \) obtained by approximations as in Section 5, a continuous function \( u: [0, T] \to D \) is a viscosity solution if and only if \( u(t) = P(t, 0)u(0) \). Recalling (5.1) and (5.2), Theorem 1.1 is proved.

8. The case of large data. In [7] the authors consider the problem (1.4)–(1.2) where \( \bar{u} \) has bounded but possibly large total variation.

As a first result, they prove the existence of a Lipschitz semigroup whose domain contains all functions which are sufficiently close (in the norm of total variation) to a given Riemann data. It is assumed that the corresponding Riemann problem satisfies a linearized stability condition.

The approach is essentially the same of [6], but in this case three different metrics must be used, depending on the solution of the Riemann problem. Indeed this solution may contain two shock waves, or two rarefaction waves, or a shock and a rarefaction wave.

Another result obtained in [7] concerns the existence of a local solution for an arbitrary initial data \( \bar{u} \) with bounded variation, satisfying suitable stability and non-resonance conditions at every jump point. The uniqueness of this solution is proved within the class of viscosity solutions to (1.4)–(1.2) satisfying the additional condition (C) below.

More precisely, fix any point \((\tau, y)\) and consider the forward triangular neighborhood
\[ \triangle \triangleq \{ (t, x) \mid t \geq \tau, x \in [y - \rho + \lambda(t - \tau), y + \rho - \lambda(t - \tau)] \}, \]
for \( \rho > 0 \) small. Given any function \( u = u(t, x) \), defined on a domain containing \( \triangle \), we set
\[ u_{\triangle}(t, x) \triangleq \begin{cases} u(\tau, y-), & \text{if } (t, x) \notin \triangle, \; x < y, \\ u(t, x), & \text{if } (t, x) \in \triangle, \\ u(\tau, y+), & \text{if } (t, x) \notin \triangle, \; x \geq y. \end{cases} \]

It is assumed that
(C) at every point \((\tau, y)\), for every \( \varepsilon > 0 \), there exist \( \rho > 0 \), \( \rho' \in [0, \rho/\lambda] \) small enough, such that the corresponding function \( u_{\triangle} \) satisfies \( Q(u_{\triangle}(t, \cdot)) < \varepsilon \), for every \( t \in [\tau, \tau + \rho'] \).

We remark that the estimates given in sections 3 and 4 of the present paper are based on the Lipschitz continuity of the weights \( R_{i\alpha} \) with respect to the wave strengths. It is not difficult to check that these estimates continue to hold for more general metrics of the form
\[ R_{i\alpha} \triangleq c_{i\alpha} + \sum_{(i\alpha)(j\beta) \in A} |\sigma_{j\beta}| d_{i\alpha}^{j\beta} + Q, \quad (8.1) \]
on the domain $D$ considered in [7], containing all suitably small $BV$ perturbations of a Riemann data. In (8.1), the coefficients $c_{i\alpha}$ may depend on the relative position of $x_{\alpha}$ with respect to the (0, 1 or 2) large shocks present in the solution, while $d_{i\alpha}^{j\beta}$ may also depend on the relative position of $x_{\alpha}$ and $x_{\beta}$.

The local existence and uniqueness results stated in Theorems 2 and 3 of [7] thus admit a straightforward extension to the nonhomogeneous case, proved by the same arguments used here in Theorem 1.1.

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