A note on exponential Rosenbrock-Euler method for the finite element discretization of a semilinear parabolic partial differential equation

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\textbf{Abstract}

In this paper we consider the numerical approximation of a general second order semilinear parabolic partial differential equation. Equations of this type arise in many contexts, such as transport in porous media. Using finite element method for space discretization and the exponential Rosenbrock-Euler method for time discretization, we provide a rigorous convergence proof in space and time under only the standard Lipschitz condition of the nonlinear part for both smooth and nonsmooth initial solution. This is in contrast to very restrictive assumptions made in the literature, where the authors have considered only approximation in time so far in their convergence proofs. The optimal orders of convergence in space and in time are achieved for smooth and nonsmooth initial solution.

\textit{Keywords:} Parabolic partial differential equation, Exponential Rosenbrock-type methods, Nonsmooth initial data, Finite element method, Errors estimate.

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1. Introduction

We consider the following abstract Cauchy problem with boundary conditions

\[
\frac{du(t)}{dt} = Au(t) + F(u(t)), \quad u(0) = u_0, \quad t \in (0, T], \quad T > 0, \quad (1)
\]

on the Hilbert space \( H = L^2(\Omega) \), where \( \Omega \) is an open bounded subset of \( \mathbb{R}^d \) \( (d = 1, 2, 3) \). The linear operator \( A : D(A) \subset H \rightarrow H \) is negative, not necessarily self adjoint and is the generator of a strongly continuous semigroup \( S(t) := e^{At}, t \geq 0 \). The nonlinear function \( F : H \rightarrow H \) is assumed to be autonomous without loss of generality. Our main focus will be the case where the operator \( A \) is a general second order elliptic operator. Under some technical conditions (see for example [7, 23]), it is well known that the mild solution of (1) is given by

\[
u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds, \quad t \in [0, T]. \quad (2)
\]

In general it is hard to find the exact solution of many PDEs. Numerical approximations are currently the only important tool to approximate the solution. Approximations are done at two levels, spatial approximation and temporal approximation. The finite element [28], finite volume [26], finite difference methods are mostly used for space discretization of the problem (1) while explicit, semi implicit and fully implicit methods are usually used for time discretization. References about standard discretization methods for (1) can be found in [26]. Due to time step size constraints, fully implicit schemes are more popular for time discretization for quite a long time compared to explicit Euler schemes. However, implicit schemes need at each time step a solution of large systems of nonlinear equations. This can be the bottleneck in computations when dealing with realistic problems. Recent years, exponential integrators have become an attractive alternative in many evolutions equations [2, 8, 9, 22, 26, 27]. Most exponential integrators analyzed early in the literature [2, 8, 22] were bounded on the nonlinear problem as in (1) where the linear part \( A \) and the nonlinear function \( F \) are explicitly known a priori. Such approach is justified in situations where the nonlinear function \( F \) is small. Due to the fact that in more realistic applications the nonlinear function \( F \) can be stronger\(^1\), Exponential Rosenbrock-Type methods have been proposed in

\(^1\)Typical examples are semi linear advection diffusion reaction equations with stiff reaction term.
where at every time step, the Jacobian of $F$ is added to the linear operator $A$. The lower order of them called Exponential Rosenbrock-Euler method (EREM) has been proved to be efficient in various applications \cite{5, 27}. For smooth initial solutions, this method is well known to be second order convergence in time \cite{1, 10}. However in many applications initial solutions are not always smooth. Typical examples are option pricing in finance or reaction diffusion advection with discontinuous initial solution. We refer to \cite{3, 6, 14, 16, 20, 19, 21} for standard numerical technique with nonsmooth initial data. Recently exponential Rosenbrock-Euler with nonsmooth initial solution was analysed in \cite{24, 25} under severe commutativity assumption in \cite[Assumption 1]{24, 25} also used in \cite[Assumption 2.4]{11}. Although \cite[Assumption 1]{24, 25} is fulfilled for a linear functions such as $F(u) = u$, $u \in H$, it is quite restrictive and excludes many nonlinear Nemytskii operators such as $F(u) = \frac{1 - u}{1 + u^2}$, $u \in H$ (see the discussion in the introduction of \cite{12}). Furthermore, only convergence in time is investigated for smooth or nonsmooth initial solution in all existing Exponential Rosenbrock-Type methods to the best of our knowledge.

The goal of this paper is to provide a rigorous convergence proof of EREM in space and time for smooth or nonsmooth initial solution under more relaxed conditions than those used in \cite{24, 25}. Indeed only the standard Lipschitz condition of the nonlinear part is used for both smooth and nonsmooth initial solution and optimal convergence orders in space and time are achieved. The space discretization is performed using finite element method. Recently work in \cite{26} can be used to obtain the similar convergence proof for finite volume method.

The paper is organized as follows. In Section 2 results about the well posedness are provided along with EREM scheme and the main result. The proof of our main result is presented in Section 3.

2. Mathematical setting and numerical method

Let us start by presenting briefly the notation of the main function spaces and norms that we will use in this paper. We denote by $\| \cdot \|$ the norm associated to the inner product $(\cdot, \cdot)$ of the Hilbert space $H = L^2(\Omega)$. The norms in the Sobolev spaces $H^m(\Omega)$, $m \geq 0$ will be denoted by $\| \cdot \|_m$. For a Hilbert space $U$ we denote by $\| \cdot \|_U$ the norm of $U$, $L(U, H)$ the set of bounded functionals from $U$ to $H$. We denote by $\mathcal{S}$ the set of smooth functions in $C^\infty$ and by $\mathcal{S}'$ the set of distributions in $\mathcal{S}'$. The notation $A = O(\varepsilon)$ means that $A$ is a function of $\varepsilon$ bounded uniformly in $\varepsilon$, i.e., $|A(\varepsilon)| \leq C(1+|\varepsilon|)$.

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linear operators from $U$ to $H$. For ease of notation $L(U, U) =: L(U)$.

2.1. Assumptions and well posedness

Throughout this paper, we make the following assumptions, which are less restrictive than current assumptions used in [24, 25].

For the linear operator $A$, we make the following standard assumption.

**Assumption 2.1.** The linear operator $A$ is negative and is the generator of a strongly continuous semigroup $S(t) := e^{At}$ on the Hilbert space $H = L^2(\Omega)$.

As we are dealing with non-smooth initial data, we assume the following.

**Assumption 2.2.** The initial value $u_0$ is such that $u_0 \in D((-A)^{\beta/2})$, $\beta \in [0, 2]$.

The following condition is assumed to hold for the nonlinear function $F$.

**Assumption 2.3.** We assume that the function $F : H \to H$ is Lipschitz continuous and twice Fréchet differentiable along the strip of the exact solution, i.e. there exists a positive constant $L$ such that

\[
\|F(u) - F(v)\| \leq L\|u - v\|, \quad \forall u, v \in H,
\]

\[
\|F_v(v)\|_{L(H)} \leq L, \quad \text{and} \quad \|F_{vv}(v)\|_{L(H \times H; H)} \leq L, \quad \forall v \in H,
\]

where $F_v(v) = D_vF(v) := \frac{\partial F}{\partial v}(v)$ and $F_{vv}(v) = D_{vv}F(v) := \frac{\partial^2 F}{\partial v^2}(v)$.

The well posedness result is given by the following theorem.

**Theorem 2.1.** Under Assumptions 2.1, 2.2 and 2.3 the initial value problem (1) has a unique mild solution $u \in C([0, T], H)$. Moreover for all $t \in [0, T]$ and for all $\gamma \in [0, 2)$ the following inequalities hold

\[
\|u(t)\| \leq C(1 + \|u_0\|), \quad (3)
\]

\[
\|u(t)\|_\gamma \leq C(1 + \|u_0\|_\gamma), \quad (4)
\]

where $C$ is a positive constant.

**Proof.** For the existence and the uniqueness, see [23, Chapter 6, Theorem 1.2] or [17, Theorem 3.29]. The proof of the estimation (3) can be found in [17, Theorem 3.29] while the one of
estimation of (4) is done in the same manner.

2.2. Numerical scheme

In the rest of this paper, we assume that the linear operator $A$ is defined by

$$Au = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left( q_{ij}(x) \frac{\partial u}{\partial x_j} \right) - \sum_{i=1}^{d} q_i(x) \frac{\partial u}{\partial x_i},$$

(5)

where $q_{ij} \in L^\infty(\Omega)$, $q_i \in L^\infty(\Omega)$. We assume that there is a positive constant $c_1 > 0$ such that

$$\sum_{i,j=1}^{d} q_{ij}(x) \xi_i \xi_j \geq c_1 |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad x \in \overline{\Omega}.$$

As in [4, 18], we introduce two spaces $\mathbb{H}$ and $V$, such that $\mathbb{H} \subset V$, that depend on the boundary conditions for the domain of the operator $A$ and the corresponding bilinear form. For example, for Dirichlet (or first-type) boundary conditions we take

$$V = \mathbb{H} = H^1_0(\Omega) = \{ v \in H^1(\Omega) : v = 0 \quad \text{on} \quad \partial \Omega \}.$$

For Robin (third-type) boundary condition and Neumann (second-type) boundary condition, which is a special case of Robin boundary condition ($\alpha_0 = 0$), we take $V = H^1(\Omega)$

$$\mathbb{H} = \{ v \in H^2(\Omega) : \frac{\partial v}{\partial n_A} + \alpha_0 v = 0, \quad \text{on} \quad \partial \Omega \}, \quad \alpha_0 \in \mathbb{R}.$$

Using Green’s formula and the boundary conditions, we obtain the corresponding bilinear form associated to $-A$ given by

$$a(u, v) = \int_{\Omega} \left( \sum_{i,j=1}^{d} q_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{d} q_i \frac{\partial u}{\partial x_i} v \right) dx, \quad u, v \in V,$$

for Dirichlet and Neumann boundary conditions and

$$a(u, v) = \int_{\Omega} \left( \sum_{i,j=1}^{d} q_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{d} q_i \frac{\partial u}{\partial x_i} v \right) dx + \int_{\partial \Omega} \alpha_0 uv dx, \quad u, v \in V.$$

for Robin boundary conditions. Using Gårding’s inequality we obtain

$$a(v, v) \geq \lambda_0 \|v\|^2_1 - c_0 \|v\|^2, \quad \forall v \in V.$$
By adding and subtracting $c_0u$ on the right hand side of (1), we have a new operator that we still call $A$ corresponding to the new bilinear form that we still call $a$ such that the following coercivity property holds

$$a(v, v) \geq \lambda_0 \|v\|^2, \quad \forall v \in V.$$  (6)

Note that the expression of the nonlinear term $F$ has changed as we include the term $-c_0u$ in a new nonlinear term that we still denote by $F$.

The coercivity property (6) implies that $A$ is sectorial on $L^2(\Omega)$ i.e. there exists $C_1, \theta \in (\frac{1}{2}\pi, \pi)$ such that

$$\| (\lambda I - A)^{-1} \|_{L(L^2(\Omega))} \leq \frac{C_1}{|\lambda|} \quad \lambda \in S_{\theta},$$  (7)

where $S_{\theta} = \{ \lambda \in \mathbb{C} : \lambda = \rho e^{i\phi}, \rho > 0, 0 \leq |\phi| \leq \theta \}$ (see [7]). Then $A$ is the infinitesimal generator of a bounded analytic semigroup $S(t) := e^{tA}$ on $L^2(\Omega)$ such that

$$S(t) := e^{tA} = \frac{1}{2\pi i} \int_{C} e^{t\lambda}(\lambda I - A)^{-1} d\lambda, \quad t > 0,$$  (8)

where $C$ denotes a path that surrounds the spectrum of $A$. The coercivity property (6) also implies that $-A$ is a positive operator and its fractional powers are well defined for any $\alpha > 0$, by

$$\begin{cases}
(-A)^{-\alpha} &= \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1} e^{tA} dt, \\
(-A)^{\alpha} &= ((-A)^{-\alpha})^{-1},
\end{cases}$$  (9)

where $\Gamma(\alpha)$ is the Gamma function (see [7]).

Let us first perform the space approximation of problem (1). We start by discretizing our domain $\Omega$ by a finite triangulation. Let $\mathcal{T}_h$ be a triangulation with maximal length $h$. Let $V_h \subset V$ denote the space of continuous functions that are piecewise linear over the triangulation $\mathcal{T}_h$. We consider the projection $P_h$ defined from $H = L^2(\Omega)$ to $V_h$ by

$$(P_h u, \chi) = (u, \chi), \quad \forall \chi \in V_h, u \in H.$$  (10)

The discrete operator $A_h : V_h \to V_h$ is defined by

$$(A_h \phi, \chi) = -a(\phi, \chi), \quad \forall \phi, \chi \in V_h.$$  (11)
The discrete operator $A_h$ of $A$ is also a generator of a semigroup $S_h(t) := e^{tA_h}$. As in \cite{4, 14, 18}, we characterize the domain of the operator $(-A)\beta/2$, $\beta \in \{1, 2\}$ as follow

$$\mathcal{D}((-A)^{\beta/2}) = \mathbb{H} \cap H^\beta(\Omega), \quad \text{for Dirichlet boundary conditions}.\]

$$\mathcal{D}(-A) = \mathbb{H}, \quad \mathcal{D}((-A)^{1/2}) = H^1(\Omega), \quad \text{for Robin boundary conditions}.\]

The semi-discrete in space version of problem (11) consists to find $u^h(t) \in V_h$, $t \in (0, T]$ such that

$$\frac{du^h(t)}{dt} = A_h u^h(t) + P_h F(u^h(t)), \quad u^h(0) = P_h u_0. \tag{12}$$

The operators $A_h$ and $P_h F$ satisfy the same assumptions as $A$ and $F$ respectively. Therefore, Theorem 2.1 ensures the existence of the unique mild solution $u^h(t)$ of (12) such that

$$\|u^h(t)\| \leq C(1 + \|P_h u_0\|) \leq C(1 + \|u_0\|), \quad \forall t \in [0, T].$$

The mild solution of (12) is represented by

$$u^h(t) = S_h(t)u^h(0) + \int_0^t S_h(t-s)P_h F(u^h(s))ds. \tag{13}$$

For the time discretization, we consider the exponential Rosenbrock-Euler method to compute the numerical approximation $u^h_n$ of $u^h(t_n)$ at discrete time $t_n = n \Delta t \in (0, T]$, $\Delta t > 0$. The method is based on the following linearisation of equation (12) at each step

$$\frac{du^h(t)}{dt} = A_h u^h(t) + J^h_n u^h(t) + G^h_n(u^h(t)), \quad t_n \leq t \leq t_{n+1}, \tag{14}$$

where $J^h_n$ is the Fréchet derivative of $P_h F$ at $u^h_n$ and $G^h_n$ the remainder given by

$$J^h_n := D_n P_h F(u^h_n), \quad G^h_n(u^h(t)) := P_h F(u^h(t)) - J^h_n u^h(t). \tag{15}$$

Before continuing with the discretization, let us provide the following important remarks and lemma.

\textbf{Remark 2.1.} Using the properties of the inner product $(\cdot, \cdot)$ and the definition of $P_h$, one can easily check that $P_h$ is a linear map from $H$ to $V_h$. Therefore, $D_v P_h v = P_h v$ for all $v \in H$, where $D_v$ is the differential operator (Fréchet derivative at $v$). Then it follows that for all $v \in H$ we have

$$D_v P_h F(v) = D_v (P_h \circ F)(v) = D_F (P_h(F(v))) \circ D_v F(v) = P_h D_v F(v),$$

$$D_{vv}(P_h F)(v) = D_v (D_v P_h F(v)) = D_v (P_h D_v F(v)) = P_h D_{vv} F(v).$$
and therefore \( J^h_n := D_u P_h F(u^h_n) = P_h D_u F(u^h_n) \). Note that \( f \circ g \) also stands for composition of mappings \( f \) and \( g \).

**Remark 2.2.** Under Assumption 2.3, using the fact that the derivatives of \( F \) and \( P_h \) can be swapped and the fact that \( P_h \) is bounded, it follows that the Jacobian satisfies the global Lipschitz condition. Then there exists a positive constant \( C > 0 \) such that

\[
\|J^h(u) - J^h(v)\|_L(H) \leq C\|u - v\|, \quad \forall u, v \in H.
\]

**Lemma 2.1.** For all \( n \in \mathbb{N} \), \( A_h + J^h_n \) is a generator of a strongly continuous semigroup \( S^h_n(t) := e^{(A_h+J^h_n)t} \), called perturbed semigroup. Furthermore, \( (S^h_n)_{n \in \mathbb{N}} \) is uniformly bounded (independent of \( n \) and \( h \)).

**Proof.** Since \( S_h \) is a strongly continuous semigroup, there exist \( M > 0 \) and \( w \in \mathbb{R} \) such that

\[
\|S_h(t)\|_{L(H)} \leq Me^{wt}, \quad \forall t \in [0, T].
\]

Using Assumption 2.1 and the fact that \( P_h \) is uniformly bounded, it follows that \( J^h_n \) is a uniformly bounded linear operator, i.e. there exists \( C > 0 \) such that

\[
\|J^h_n\|_{L(H)} \leq C, \quad \forall n \in \mathbb{N}.
\]

Therefore applying [23, Chapter 3, Theorem 1.1, page 76] ends the proof.

Giving the solution \( u^h(t_n) \) at \( t_n \) and applying the variation of constants formula to (14) with initial value \( u^h(t_n) \), we obtain the solution at \( t_{n+1} \) in the following mild representation

\[
u^h(t_{n+1}) = e^{(A_h+J^h_n)t}u^h(t_n) + \int_{t_n}^{t_{n+1}} e^{(A_h+J^h_n)(t_{n+1}-s)}G^h_n(u^h(s))ds. \tag{16}\]

We note that (16) is the exact solution of (12) at \( t_{n+1} \). To establish our numerical method, we use the following approximation

\[
G^h_n(u^h(t_n + s)) \approx G^h_n(u^h_n). \tag{17}\]

Therefore the integral part of (16) can be approximated as follow

\[
\int_{t_n}^{t_{n+1}} e^{(A_h+J^h_n)(t_{n+1}-s)}G^h_n(u^h(s))ds \approx \int_0^{\Delta t} e^{(A_h+J^h_n)(\Delta t-s)}G^h_n(u^h(t_n + s))ds \approx G^h_n(u^h_n)(A_h + J^h_n)^{-1}(e^{(A_h+J^h_n)\Delta t} - I). \tag{18}\]
Inserting (18) in (16) and using the approximation $u^h(t_n) \approx u^h_n$ gives the following approximation $u^h_{n+1}$ of $u^h(t_{n+1})$

$$u^h_{n+1} = e^{(A_h + J^h_n)\Delta t}u^h_n + (A_h + J^h_n)^{-1}(e^{(A_h + J^h_n)\Delta t} - I)G^h_n(u^h_n), \quad n = 0, \ldots, N. \quad (19)$$

The scheme (19) is called Exponential Rosenbrock-Euler method (EREM). The numerical scheme (19) can be written in the following equivalent form, which is efficient for implementation

$$u^h_{n+1} = u^h_n + \varphi_1(\Delta t(A_h + J^h_n))[(A_h + J^h_n)u^h_n + G^h_n(u^h_n)],$$

where

$$\varphi_1(\Delta t(A_h + J^h_n)) := (A_h + J^h_n)^{-1}(e^{(A_h + J^h_n)\Delta t} - I) = \int_0^{\Delta t} e^{(A_h + J^h_n)(\Delta t-s)}ds.$$ 

We note that $\varphi_1(\Delta t(A_h + J^h_n))$ is a uniformly bounded operator (see [8, Lemma 2.4]).

Having the numerical method in hand, our goal is to prove its convergence toward the exact solution in the $L^2(\Omega)$ norm.

### 2.3. Main result

Throughout this paper, we use a fixed time step $\Delta t = T/N, \ N \in \mathbb{N}$ without loss of generality, and set $t_n = n\Delta t \in (0, T], \ n \in \mathbb{N}$. We denote by $C$ any generic constant independent of $h, n$ and $\Delta t$, which may change from one place to another. The main result of this paper is formulated in the following theorem.

**Theorem 2.2.** Let $u$ be the mild solution of problem (1) and $u^h_n$ the approximated solution at time $t_n$ by EREM scheme (19). Assume that Assumption 2.1, Assumption 2.2 (with the corresponding $\beta \in [0, 2]$) and Assumption 2.3 are fulfilled. Then there exists a positive constant $C$ independent of $h, n$ and $\Delta t$ such that:

If $\beta \in [0, 1)$, then for $n = 1, \ldots, N$ we have

$$\|u(t_n) - u^h_n\| \leq C(h^{1+\beta}t_n^{(-1+\beta)/2} + \Delta t^2).$$

If $\beta \in [1, 2)$, then for $n = 1, \ldots, N$ we have

$$\|u(t_n) - u^h_n\| \leq C(h^\beta + \Delta t^2).$$
If $\beta = 2$, then for $n = 1, \cdots, N$ we have
$$
\|u(t_n) - u_n^h\| \leq C(h^2(1 + \ln(t_n/h^2)) + \Delta t^2).
$$

If $\beta = 2$ and if in addition there exists $c > 0$ and $\gamma \in (0, 1/10]$ small enough such that
$$
\|(-A)^{\gamma}F(v)\| \leq c(1 + \|v\|_{\gamma}), \quad \forall v \in \mathcal{D}((-A)^{\gamma}),
$$
then for $n = 1, \cdots, N$ we have
$$
\|u(t_n) - u_n^h\| \leq C(h^2 + \Delta t^2).
$$

\textbf{Remark 2.3.} We note from Theorem 2.2 that we have achieved uniform convergence in time with order 2. It is important to note that although order 2 is achieved in [24, 25, Theorem 1], their result presents singularities at the origin and their upper bound of the error depends on a positive parameter defined in [24, 25, Assumption 1]. Theorem 2.2 covers the worst case ($\beta = 0$) highlighted in [24, 25], where there is a logarithmic reduction of the order. Note in Theorem 2.2 that the initial solution $u_0$ is less smooth than the one in [24, 25, Theorem 1]. Note also that if the space discretization is performed using finite volume method, recent work in [26] can be used to obtain similar error estimates with optimal order 1 in space.

\section*{3. Proof of the main result}

The proof of the main result need some preparatory results.

\subsection*{3.1. Preparatory results}

In the convergence proof of our main results, the following lemmas will be important.

\textbf{Lemma 3.1.} [18, Lemma 3.1]

Consider the linear parabolic problem $u' = Au, \ u(0) = v, \ t \in (0, T]$. Assume that $v \in \mathcal{D}((-A)^{\alpha/2})$, then the following inequality holds
$$
\|S(t)v - S_h(t)P_hv\| = \|T_h(t)v\| \leq Ch^r t^{-(r-\alpha)/2}\|v\|_{\alpha}, \quad r \in [1, 2], \quad \alpha \leq r.
$$

\clearpage
**Lemma 3.2.** The function $G^h_n$ defined by (15) satisfies the global Lipschitz condition with a uniform constant. i.e. there exists a positive constant $C > 0$ such that

\[
\|G^h_n(u^h) - G^h_n(v^h)\| \leq C\|u^h - v^h\|, \quad \forall n \in \mathbb{N}, \quad \forall u^h, v^h \in V_h.
\]

**Proof.** Using Assumption 2.3 the proof is straightforward. □

Following closely [14, Theorem 1.1], [28, Theorem 14.3] and [15, Proposition 3.3] we have the following result.

**Lemma 3.3.** \([\text{Local error in space}]\) Let $u(t)$ and $u^h(t)$ be the mild solutions of (1) and (12) respectively. Assume that Assumption 2.1, Assumption 2.2 (with the corresponding $\beta \in [0, 2]$) and Assumption 2.3 are fulfilled, then there exists a positive constant $K = K(u_0, T, \beta)$ independent of $h$ such that for $0 < t \leq T$:

If $0 \leq \beta < 1$, then

\[
\|u(t) - u^h(t)\| \leq Kh^{(-1+\beta)/2}.
\]

If $1 \leq \beta < 2$, then

\[
\|u(t) - u^h(t)\| \leq Kh^\beta.
\]

If $\beta = 2$, then

\[
\|u(t) - u^h(t)\| \leq Kh^2(1 + \ln(t/h^2)).
\]

If $\beta = 2$, and if in addition there exist $c > 0$ and $\gamma \in (0, 1/10]$ small enough such that

\[
\|(-A)^\gamma F(v)\| \leq c(1 + \|v\|_\gamma), \quad \forall v \in \mathcal{D}((-A)^\gamma).
\]

(22)

then there exist a positive constant $K = K(u_0, T, \beta, \gamma)$ independent of $h$ such that for all $0 < t \leq T$

\[
\|u(t) - u^h(t)\| \leq Kh^2.
\]

**Proof.** The proofs of the first three estimates follow the same lines as that of [14, Theorem 1.1] or [28, Theorem 14.3] or [15, Proposition 3.3] using Lemma 3.1. For the last estimate,
the proof uses the mild solutions (2) and (13). Indeed
\[
e(t) := \left\| u(t) - u^h(t) \right\| \\& \leq \left\| S(t)u_0 - S_h(t)P_hu_0 \right\| + \left\| \int_0^t S(t-s)F(u(s))ds - \int_0^t S_h(t-s)P_hF(u^h(s))ds \right\| \\
=: e_1(t) + e_2(t).
\]
(23)

Using Lemma 3.1 with \( r = \alpha = 2 \) we have
\[
e_1(t) := \left\| (S(t) - S_h(t)P_h)u_0 \right\| \leq Ch^2\left\| u_0 \right\|. \tag{24}
\]

For the estimation of \( e_2(t) \), we have
\[
e_2(t) := \left\| \int_0^t S(t-s)F(u(s))ds - \int_0^t S_h(t-s)P_hF(u^h(s))ds \right\| \\
\leq \int_0^t \left\| S(t-s)F(u(s)) - S_h(t-s)P_hF(u^h(s)) \right\| ds. \tag{25}
\]

By adding and subtracting \( S_h(t-s)P_hF(u(s)) \) in (25) and using the triangle inequality yields
\[
e_2(t) \leq \int_0^t \left\| S_h(t-s)P_h(F(u(s)) - F(u^h(s))) \right\| ds + \int_0^t \left\| (S(t-s) - S_h(t-s)P_h)F(u(s)) \right\| ds.
\]

Using the fact that \( S_h \) and \( P_h \) are bounded, and \( F \) satisfies the global Lipschitz condition yields
\[
e_2(t) \leq C \int_0^t e(s)ds + \int_0^t \left\| (S(t-s) - S_h(t-s)P_h)(-A)^{-\gamma}(-A)^\gamma F(u(s)) \right\| ds \\
\leq C \int_0^t e(s)ds + \int_0^t \left\| (S(t-s) - S_h(t-s)P_h)(-A)^{-\gamma} \right\|_{L(H)}\left\| (-A)^\gamma F(u(s)) \right\| ds \\
\leq C \int_0^t e(s)ds \\
+ \sup_{0 \leq s \leq T} \left\| (-A)^\gamma F(u(s)) \right\| \int_0^t \left\| (S(t-s) - S_h(t-s)P_h)(-A)^{-\gamma} \right\|_{L(H)} ds. \tag{26}
\]

Using the definition of the norm of the operator and following closely [18, Page 21] we have
\[
\left\| (S(t-s) - S_h(t-s)P_h)(-A)^{-\gamma} \right\|_{L(H)} = \sup_{v \neq 0, v \in H} \frac{\left\| (S(t-s) - S_h(t-s)P_h)(-A)^\gamma v \right\|}{\left\| v \right\|}.
\]
It is clear that for all $v \in \mathcal{L}^2(\Omega)$, $(-A)^{-\gamma}v \in \mathcal{D}((-A)^{-\gamma})$. Then applying Lemma 3.1 with $r = 2$ and $\alpha = \gamma$ yields

$$
\| (S(t-s) - S_h(t-s)P_h)(-A)^{-\gamma} \|_{L(H)} = \sup_{v \neq 0, v \in H} \| (S(t-s) - S_h(t-s)P_h)(-A)^{-\gamma}v \| \| v \|
\leq Ch^2 (t-s)^{-1+\gamma/2} \sup_{v \neq 0, v \in H} \| (-A)^{-\gamma}v \| \| v \|
\leq Ch^2 (t-s)^{-1+\gamma/2}.
$$

(27)

Substituting (27) in (26), using the additional condition (22) and inequality (4) yields

$$
e_2(t) \leq C \int_0^t e(s)ds + Ch^2 \int_0^t (t-s)^{-1+\gamma/2}ds
\leq Ch^2 + C \int_0^t e(s)ds.
$$

(28)

Substituting (21) and (28) in (23) yields

$$
e(t) \leq Ch^2 + C \int_0^t e(s)ds.
$$

(29)

Applying Gronwall’s inequality to (29) gives the desired result.

Remark 3.1. Lemma 3.3 generalizes [14, Theorem 1.1] and [28, Theorem 14.3] for a general non-self adjoint operator $A$. Condition (22) is made to avoid logarithmic reduction of order when $\beta = 2$.

Lemma 3.4. Let $u^h(t)$ be the mild solution of (12). If Assumption 2.1, Assumption 2.2 and Assumption 2.3 are fulfilled, then the following estimation holds for all $t \in (0, T]$

$$
\| D^l_t u^h(t) \| \leq C, \quad l = 1, 2.
$$

(30)

Here $C$ is a positive constant independent of $t$ and $h$.

Proof. We recall that the mild solution $u^h(t)$ satisfies the following semi-discrete problem:

$$
D_t u^h(t) = A_h u^h(t) + P_h F(u^h(t)), \quad u^h(0) = P_h u(0).
$$

(31)

Therefore $u^h(t)$ is differentiable and its derivative is given by (31). Since $A_h$ is linear, it follows that $A_h u^h(t)$ is differentiable. The function $P_h F(u^h(t))$ is differentiable as a composition of
differentiable mappings. Hence $D_t u^h(t)$ is differentiable, i.e. $u^h(t)$ is twice differentiable in time. Using the chain rule we have

$$D_t^2 u^h(t) = A_h D_t u^h(t) + P_h D_u F(u^h(t)) D_t u^h(t),$$  \tag{32}$$

where we have used Remark 2.1 to swap $P_h$ and $D_u$. Using the same argument as above, it follows that the right hand side of (32) is differentiable in time. Hence $D_t^3 u^h(t)$ exists. As in the proof of [16, Theorem 5.2], we set $v^h(t) := t D_t u^h(t)$. Using the fact that $D_u P_h F(u^h(t)) = P_h D_u F(u^h(t))$ (see Remark 2.1) it follows that $v^h(t)$ satisfies the following equation

$$D_t v^h(t) = A_h v^h(t) + D_t^2 u^h(t) + P_h D_u F(u^h(t)) v^h(t), \quad v^h(0) = 0.$$  

Therefore by Duhamel’s principle we have

$$v^h(t) = \int_0^t S_h(t - s)[D_s u^h(s) + P_h D_u F(u^h(s)) v^h(s)]ds. \tag{33}$$

Taking the norm in both sides of (33) and using the fact that $S_h$, $P_h$ and the first derivative of $F$ are uniformly bounded yields

$$\|v^h(t)\| = \|t D_t u^h(t)\| \leq C \int_0^t \|D_s u^h(s)\|ds + C \int_0^t \|s D_s u^h(s)\|ds \leq C \int_0^t \|D_s u^h(s)\|ds + CT \int_0^t \|D_s u^h(s)\|ds \leq C \int_0^t \|D_s u^h(s)\|ds. \tag{34}$$

Therefore it follows from (34) that for all $t \in (0, T]$ we have

$$\|D_t u^h(t)\| \leq C t^{-1} \int_0^t \|D_s u^h(s)\|ds. \tag{35}$$

Using the fact that $t > 0$, we can add $C$ in the right hand side of (35) and obtain

$$\|D_t u^h(t)\| \leq C + C t^{-1} \int_0^t \|D_s u^h(s)\|ds. \tag{36}$$

Applying the continuous Gronwall’s lemma (see [7, Section 1.2.1, page 6]) to (36) yields

$$\|D_t u^h(t)\| \leq C \exp \left(t^{-1} \int_0^t ds \right) \leq C. \tag{37}$$

To prove (30) in the case $l = 2$, we set $w^h(t) := t D_t^2 u^h(t)$. Then it follows that

$$D_t w^h(t) = D_t^2 u^h(t) + t D_t^3 u^h(t). \tag{38}$$
Taking the second derivative in time in the both sides of (31) gives

\[ D^3_t u^h(t) = A_h D^2_t u^h(t) + D^2_t (P_h F(u^h(t))). \]  

(39)

Using the Faà di Bruno’s formula (see [13]) we have

\[ D^2_t (P_h F(u^h(t))) = D_u P_h F(u^h(t)) D^2_t u^h(t) + D_{uu} P_h F(u^h(t)) (D_t u^h(t))^2. \]  

(40)

Substituting (40) in (39) yields:

\[ D^3_t u^h(t) = A_h D^2_t u^h(t) + D_u P_h F(u^h(t)) D^2_t u^h(t) + D_{uu} P_h F(u^h(t)) (D_t u^h(t))^2. \]  

(41)

Substituting (41) in (38) and using Remark 2.1, it follows that \( w^h(t) \) satisfies the following equation

\[ D_t w^h(t) = A_h w^h(t) + D^2_t u^h(t) + P_h D_u F(u^h(t)) w^h(t) + t P_h D_{uu} F(u^h(t)) (D_t u^h(t))^2, \quad w^h(0) = 0. \]

Therefore, by Duhamel’s principle, we have:

\[ w^h(t) = \int_0^t S_h (t - s) [D^2_s u^h(s) + P_h D_u F(u^h(s)) w^h(s) + s P_h D_{uu} F(u^h(s)) (D_s u^h(s))^2] ds. \]  

(42)

Taking the norm in both sides of (42) and using the fact that \( S_h, P_h \) and all derivatives of \( F \) up to order 2 are uniformly bounded yields

\[ \| w^h(t) \| \leq C \int_0^t [\| D^2_s u^h(s) \| + s \| D^2_s u^h(s) \| + s \| D_s u^h(s) \|^2] ds. \]  

(43)

Using inequality (37) we have

\[ \int_0^t s \| D_s u^h(s) \|^2 ds \leq Ct^2. \]  

(44)

Substituting (41) in (43) yields

\[ \| w^h(t) \| = t \| D^2_t u^h(t) \| \leq Ct^2 + C \int_0^t \| D^2_s u^h(s) \| ds. \]  

(45)

So from (45) the following inequality holds

\[ \| D^2_t u^h(t) \| \leq Ct + C \int_0^t t^{-1} \| D^2_s u^h(s) \| ds. \]  

(46)

Applying the continuous Gronwall’s lemma to (46) yields

\[ \| D^2_t u^h(t) \| \leq Ct \exp \left( \int_0^t t^{-1} ds \right) \leq C. \]
Lemma 3.5. Let \( e_n^h := u(t_n) - u_n^h \) and

\[
g_n(t) := G_n^h(u^h(t)) = P_h F(u^h(t)) - J_n^h u^h(t).
\] (47)

If Assumption 2.1, Assumption 2.2 and Assumption 2.3 are fulfilled, then there exists a positive constant \( C \) independent of \( h, n \) and \( \Delta t \) such that for all \( t_n, t \in (0, T) \)

\[
\|g_n'(0)\| = 0, \quad \|g_n'(t_n)\| \leq C\|e_n^h\|, \quad \|g_n'(t)\| \leq C, \quad \text{and} \quad \|g_n''(t)\| \leq C.
\]

Proof. We recall that \( J_n^h = D_u P_h F(u_n^h) \) is a linear map. Hence the time derivative of \( J_n^h u^h(t) \) at \( t_n \) is given by \( J_n^h D_t u^h(t_n) = D_u P_h F(u_n^h) D_t u^h(t_n) \). Taking the time derivative in (47) and using the chain rule we have

\[
g_n'(t) = D_u P_h F(u^h(t)) D_t u^h(t) - D_u P_h F(u_n^h) D_t u^h(t)
\]

(48)

If \( n = t = 0 \), then using the fact that \( u_0^h = u^h(0) \) it follows from (48) that \( g_0'(0) = 0 \).

If \( n = 0 \) and \( t > 0 \), then it follows from (48) and Remark 2.1 that

\[
\|g_0'(t)\| \leq \|P_h[D_u(F(u^h(t)) - F(u_0^h))]\| \|D_t u(t)\|.
\] (49)

Using the fact that the projection \( P_h \) is bounded, together with Assumption 2.3 and Lemma 3.4 it follows from (49) that \( \|g_0'(t)\| \leq C \).

If \( t_n \neq 0 \) then using the relation \( D_u P_h F = P_h D_u F \) (see Remark 2.1), the fact the projection \( P_h \) is bounded and the fact that the Jacobian satisfies the global Lipschitz condition (see Remark 2.2), it follows from (48) that

\[
\|g_n'(t_n)\| \leq \|J^h(u^h(t_n)) - J^h(u_n^h)\|_{L(H)} \|D_t u^h(t_n)\|
\]

\[
\leq C\|u^h(t_n) - u_n^h\| \|D_t u^h(t_n)\| = C\|e_n^h\| \|D_t u^h(t_n)\|.
\]

Using Lemma 3.4 gives the desired estimation of \( \|g_n'(t_n)\| \). Here the advantage of the linearisation allows to keep \( \|e_n^h\| \) in the upper bound of \( \|g_n'(t_n)\| \) which will be useful in the convergence proof to reach the convergence order 2 in time.

Taking the second derivative in (47), using the Faà di Bruno’s formula (see [13]) and using Remark 2.1 yields

\[
g_n''(t) = P_h D_{uu} F(u^h(t))(D_t u^h(t))^2 + P_h D_{uu} F(u^h(t)) D_t^2 u^h(t) - P_h D_{uu} F(u_n^h) D_t^2 u^h(t).
\] (50)
Since the projection \( P \), the first and the second derivative of \( F \) are uniformly bounded, it follows from (50) that

\[
\|g''_n(t)\| \leq C\|D_t u^h(t)\|^2 + C\|D^2_t u^h(t)\|.
\]

Using Lemma 3.4 completes the proof.

Lemma 3.6. Under Assumption 2.1 and Assumption 2.3, the following bound holds for the perturbed semigroup \( S^h_n \)

\[
\left\| \exp((A_h + J^h_j)\Delta t \cdots \exp((A_h + J^h_{n-1})\Delta t) \exp((A_h + J^h_n)\Delta t) - \exp(A_h\Delta t)S^h_{j-1,k}) \right\|_{L(H)} \leq C, \quad \forall \ 0 \leq k \leq n,
\]

where \( C \) is a positive constant independent of \( h, n, k \) and \( \Delta t \).

Proof. Let us provide a new proof, simpler than the one in [24]. Set

\[
S^h_{n,k} := \begin{cases} 
\exp((A_h + J^h_k)\Delta t \cdots \exp((A_h + J^h_{n-1})\Delta t) \exp((A_h + J^h_n)\Delta t) - \exp(A_h\Delta t)S^h_{j-1,k}), & \text{if } n \geq k \\
I, & \text{if } n < k.
\end{cases}
\]

The composition of the perturbed semigroup can be expanded into a telescopic sum as follow

\[
S^h_n = \exp(A_h t_{n+1-k}) + \sum_{j=k}^{n} \exp(A_h(t_{n+1-j+1}) \exp((A_h + J^h_j)\Delta t - \exp(A_h\Delta t)S^h_{j-1,k}).
\]  

(51)

Taking the norm in both sides of (51) and using the stability properties of \( \exp(tA_h) \) yields

\[
\|S^h_{n,k}\|_{L(H)} \leq C + C\sum_{j=k}^{n} \|\exp((A_h + J^h_j)\Delta t - \exp(A_h\Delta t)S^h_{j-1,k}\|_{L(H)}\|S^h_{j-1,k}\|_{L(H)}.
\]  

(52)

Using the variation of parameter formula (see [23, (1.2), Page 77, Chapter 3]), it holds that

\[
\left( \exp((A_h + J^h_j)\Delta t - \exp(A_h\Delta t) \right) x = \int_{0}^{\Delta t} \exp(A_h(\Delta t-s)J^h_j) \exp((A_h + J^h_j)s) x ds, \quad \forall x \in D(-A).
\]  

(53)

Taking the norm in both sides of (53) and using the stability properties of \( \exp(A_h t) \) and \( \exp(A_h + J^h_j) t \) together with the uniformly boundedness of \( J^h_j \) gives

\[
\left\| \left( \exp((A_h + J^h_j)\Delta t - \exp(A_h\Delta t) \right) x \right\| \leq \int_{0}^{\Delta t} C\|x\| ds \leq C\Delta t \|x\|.
\]  

(54)
Therefore from (54) we have
\[
\left\| e^{(A_h + J_h^0)\Delta t} - e^{A_h\Delta t} \right\|_{L(H)} \leq C\Delta t. \tag{55}
\]
Inserting (55) in (52) gives
\[
\| S_{n,k}^h \|_{L(H)} \leq C + C\Delta t \sum_{j=k}^{n} \| S_{j-1,k}^h \|_{L(H)}.
\]
Applying the discrete Gronwall’s lemma completes the proof of Lemma 3.6. ■

3.2. Main proof

Let us now prove Theorem 2.2, which is our main result in this work. Following the standard technique in the error estimate, we use the triangle inequality to split up the error in two parts
\[
\| u(t_n) - u_n^h \| \leq \| u(t_n) - u_n^h(t_n) \| + \| u_n^h(t_n) - u_n^h \| =: I + II.
\]

The space error $I$ is estimated by Lemma 3.3. It remains to estimate the time error $II$. To start, we recall that the exact solution at $t_n$ is given by
\[
u_n^h = e^{A_h + J_h^0}u_n^h + \int_{t_{n-1}}^{t_n} e^{(A_h + J_h^0)(t-s)}G_{n-1}^h(u(s))ds.
\]

We also recall that the numerical solution (19) at $t_n$ can be rewritten as follow
\[
u_n^h = e^{A_h + J_h^0}u_{n-1}^h + \int_{t_{n-1}}^{t_n} e^{(A_h + J_h^0)(t-s)}G_{n-1}^h(u_{n-1}^h)ds.
\]

If $n = 1$, then it follows from (56) and (57) that
\[
II := \| u_n^h(t_1) - u_n^h \| = \left\| \int_0^{\Delta t} e^{(A_h + J_h^0)(\Delta t - s)}[G_0^h(u^h(s)) - G_0^h(u_0^h)]ds \right\|.
\]

Using the uniformly boundedness of $e^{(A_h + J_h^0)t}$ (see Lemma 2.1), it follows from (58) that
\[
II \leq \int_0^{\Delta t} \| G_0^h(u^h(s)) - G_0^h(u_0^h) \| ds \leq C\int_0^{\Delta t} \| G_0^h(u^h(s)) - G_0^h(u^h(t_0)) \| ds.
\]
Let $f : [a, b] \rightarrow H$ be a continuously differentiable function, the following fundamental theorem of Analysis holds

$$f(b) - f(a) = \int_a^b f'(t)dt.$$  \hfill (60)

Using (60) and Lemma 8.3.5 it follows from (59) that

$$II \leq C \int_0^{\Delta t} \|G^h_0(u^h(s)) - G^h_0(u^h(0))\|ds = C \int_0^{\Delta t} \|g^0(s) - g^0(0)\|ds$$

$$= C \int_0^{\Delta t} \left\| \int_0^s g^0(r)dr \right\| ds \leq C \int_0^{\Delta t} \int_0^s \|g^0(r)\|drds$$

$$\leq C \Delta t^2.$$  \hfill (61)

If $n \geq 2$, then iterating the exact solution (56) gives

$$u^h(t_n) = e^{(A_h + J_h^n_1)\Delta t}e^{(A_h + J_h^n_2)\Delta t} \cdots e^{(A_h + J_h^n_2)\Delta t} u^h_0$$

$$+ \int_{t_{n-1}}^{t_n} e^{(A_h + J_h^n_1)(t_n-s)}G^h_{n-1}(u^h(s))ds$$

$$+ \sum_{k=0}^{n-2} \int_{t_{n-k-2}}^{t_{n-k-1}} e^{(A_h + J_h^n_1)\Delta t} \cdots e^{(A_h + J_h^n_2)\Delta t} e^{(A_h + J_h^n_{n-k-2})(t_{n-k-1}-s)}G^h_{n-k-2}(u^h(s))ds.$$  \hfill (62)

For $n \geq 2$, iterating the numerical solution (57) gives

$$u^h_n = e^{(A_h + J_h^n_1)\Delta t}e^{(A_h + J_h^n_2)\Delta t} \cdots e^{(A_h + J_h^n_2)\Delta t} u^h_0$$

$$+ \int_{t_{n-1}}^{t_n} e^{(A_h + J_h^n_1)(t_n-s)}G^h_{n-1}(u^h_n)ds$$

$$+ \sum_{k=0}^{n-2} \int_{t_{n-k-2}}^{t_{n-k-1}} e^{(A_h + J_h^n_1)\Delta t} \cdots e^{(A_h + J_h^n_2)\Delta t} e^{(A_h + J_h^n_{n-k-2})(t_{n-k-1}-s)}G^h_{n-k-2}(u^h_{n-k-2})ds.$$  \hfill (63)

Therefore, it follows from (62), (63) and the triangle inequality that

$$II := \|u^h(t_n) - u^h_n\|$$

$$\leq \sum_{k=0}^{n-2} \int_{t_{n-k-2}}^{t_{n-k-1}} \left\| e^{(A_h + J_h^n_1)\Delta t} \cdots e^{(A_h + J_h^n_2)(t_{n-k-1}-s)} \right\| ds$$

$$\times \left\| G^h_{n-k-2}(u^h(s)) - G^h_{n-k-2}(u^h_{n-k-2}) \right\| ds$$

$$+ \int_{t_{n-1}}^{t_n} \left\| e^{(A_h + J_h^n_1)(t_n-s)} \left[ G^h_{n-1}(u^h(s)) - G^h_{n-1}(u^h_{n-1}) \right] \right\| ds$$

$$\leq \sum_{k=0}^{n-2} \int_{t_{n-k-2}}^{t_{n-k-1}} \left\| e^{(A_h + J_h^n_1)\Delta t} \cdots e^{(A_h + J_h^n_2)(t_{n-k-1}-s)} \right\| L(H) \left\| e^{(A_h + J_h^n_{n-k-2})(t_{n-k-1}-s)} \right\| L(H)$$

$$\times \left\| G^h_{n-k-2}(u^h(s)) - G^h_{n-k-2}(u^h_{n-k-2}) \right\| ds$$

$$+ \int_{t_{n-1}}^{t_n} \left\| e^{(A_h + J_h^n_1)(t_{n-1}-s)} \right\| L(H) \left\| G^h_{n-1}(u^h(s)) - G^h_{n-1}(u^h_{n-1}) \right\| ds.$$
Using lemmas 3.6 and 2.1 together with the triangle inequality yields

\[
II \leq C \sum_{k=0}^{n-2} \int_{t_{n-k}}^{t_{n-k-1}} \|G_{n-k-2}^h(u^h(s) - G_{n-k-2}^h(u^h_{n-k-2})\|ds + C \int_{t_{n-1}}^{t_n} \|G_{n-1}^h(u^h(s)) - G_{n-1}^h(u^h_{n-1})\|ds \\
\leq C \sum_{k=0}^{n-2} \int_{t_{n-k}}^{t_{n-k-1}} \|G_{n-k-2}^h(u^h(s) - G_{n-k-2}^h(u^h_{n-k-2})\|ds + C \int_{t_{n-1}}^{t_n} \|G_{n-1}^h(u^h(s)) - G_{n-1}^h(u^h_{n-1})\|ds \\
+ C \int_{t_{n-1}}^{t_n} \|G_{n-1}^h(u^h(s)) - G_{n-1}^h(u^h_{n-1})\|ds
\]

Using (65), triangle inequality and Lemma 3.5 allows to have

\[
II_1 + II_2 = C \sum_{k=0}^{n-2} \int_{t_{n-k}}^{t_{n-k-1}} \|g_{n-k-2}(s) - g_{n-k-2}(t_{n-k-2})\|ds + C \int_{t_{n-1}}^{t_n} \|g_{n-1}(s) - g_{n-1}(t_{n-1})\|ds \\
= C \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \|g_k(s) - g_k(t_k)\|ds \\
= C \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \|\int_{t_k}^{s} g'_k(r)dr\|ds \\
\leq C \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \|g'_k(r)\|drds \\
\leq \int_0^{\Delta t} \int_0^s \|g'_0(r)\|drds + \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \|g'_k(r)\|drds \\
\leq C \Delta t^2 + C \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \|g'_k(r) - g'_k(t_k)\|drds + C \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \|g'_k(t_k)\|drds \\
\leq C \Delta t^2 + C \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \|g''_k(l)\|dl|drds + C \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \|g'_k(t_k)\|drds \\
\leq C \Delta t^2 + C \Delta t^2 + C \Delta t^2 \sum_{k=1}^{n-1} \|e_k^h\|.
\]
Using Lemma 3.2 we obtain the following estimate for $II_3 + II_4$

\[
II_3 + II_4 \leq C \sum_{k=0}^{n-2} \int_{t_{n-k-2}}^{t_{n-k-1}} \|u^h(t_{n-k-2}) - u^h_{n-k-2}\| ds + C \int_{t_{n-1}}^{t_n} \|u^h(t_{n-1}) - u^h_{n-1}\| ds \\
\leq C \sum_{k=0}^{n-2} \Delta t \|u^h(t_{n-k-2}) - u^h_{n-k-2}\| + C \Delta t \|u^h(t_{n-1}) - u^h_{n-1}\| \\
\leq C \Delta t \sum_{k=0}^{n-1} \|u^h(t_k) - u^h_k\| 
\]

Inserting (68) and (66) in (64) yields

\[
II = \|u^h(t_n) - u^h_n\| \leq C \Delta t^2 + C \sum_{k=0}^{n-1} \Delta t \|u^h(t_k) - u^h_k\|. 
\]

Applying the discrete Gronwall’s lemma to (69) yields

\[
II = \|u^h(t_n) - u^h_n\| \leq C \Delta t^2. 
\]

Combining the estimate of $I$ and $II$ completes the proof of Theorem 2.2.

3.3. Comments on similar works in the literature

In the current literature (see for example [24, 25, 8, 10]), the time error $e^h = \|u^h(t_n) - u^h_n\|$ is usually estimated using the following decomposition of the exact solution

\[
u^h(t_{n+1}) = e^{K^h_n \Delta t} u^h(t_n) + \Delta t \varphi_1(\Delta t K^h_n) (K^h_n u^h(t_n) + g_n(t_n)) + \delta^h_{n+1}, \quad K^h_n := A_h + J^h_n, 
\]

where $\delta^h_{n+1}$ is called the defect error. Subtracting the numerical solution (19) from (71) yields the following error representation

\[
e^h_{n+1} = e^{K^h_n \Delta t} e^h_n + \Delta t \varphi_1(\Delta t K^h_n) e^h_n + \Delta t \varphi_1(\Delta t K^h_n) (g_n(t_n) - G^h_n(u^h_n)) + \delta^h_{n+1}. 
\]

So by recursion, (72) implies

\[
e^h_n = \Delta t \sum_{j=0}^{n-1} \left[ e^{\Delta t K^h_{n-j}} \cdots e^{\Delta t K^h_{j+1}} \varphi_1(\Delta t K^h_{j+1}) e^h_j \right], 
\]

where $e^h_n = u^h(t_n) - u^h_n$ and $\rho^h_j = \varphi_1(\Delta t K^h_j) (g_j(t_j) - G^h_j(u^h_j))$.

To estimate the term involving the defect in (73), namely $\sum_{j=0}^{n-1} e^{\Delta t K^h_{n-j}} \cdots e^{\Delta t K^h_{j+1}} \delta^h_{j+1}$, the author of [24, 25] used the very restrictive assumption (24, 25, Assumption 1]), while the authors
of [8, 10] used the smoothness of the exact solution since they are dealing with smooth initial solutions. Our analysis enjoys the fact that the time error (64) does not involve that defect term, and therefore any further assumption is needed. Our analysis can be easily extended to the case of non autonomous problem (i.e when the nonlinear function depends on $t$ and $u$) by linearizing as in [8, 5]. For the implementation of the Exponential Rosenbrock-Euler method, we refer to [1, 10, 5, 27, 24].

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