Abstract. Let $X$ be a finite connected simplicial complex, and let $\delta$ be a persistence (i.e., some function from integers to integers). One can consider two categories: (1) the category of perverse sheaves cohomologically constructible with respect to the triangulation, and (2) the category of sheaves constant along the perverse simplices ($\delta$-sheaves). We interpret the categories (1) and (2) as categories of modules over certain quadratic (and even Koszul) algebras $A(X, \delta)$ and $B(X, \delta)$ respectively, and we prove that $A(X, \delta)$ and $B(X, \delta)$ are Koszul dual to each other. We define the $\delta$-perverse topology on $X$ and prove that the category of sheaves on perverse topology is equivalent to the category of $\delta$-sheaves. Finally, we study the relationship between the Koszul duality functor and the Verdier duality functor for simplicial sheaves and cosheaves.

Introduction.

1. The study of constructible sheaves on a cell complex $X$ leads to the notion of a cellular sheaf, which was developed by W. Fulton, M. Goresky, R. MacPherson, and C. McCrory in a seminar at Brown University in 1977-78. The systematic exposition of the theory of cellular sheaves has been presented in the A. Shepard’s Doctoral Thesis [Shep] (cf. [Kash]).

A cellular sheaf is a gadget which assigns vector spaces to cells in $X$ and linear maps to pairs of incident cells. (Cellular sheaves can also be interpreted as sheaves on the finite topology generated by open stars of cells.) It is easy to interpret such linear algebra gadgets as modules over an associative algebra $B(X)$.

In this paper we will work with a finite connected simplicial complex $X$. We can consider simplicial complexes without the loss of generality since any reasonable stratified space can be triangulated [Gor]. The category of constructible sheaves of $\mathbb{F}$-vector spaces on $X$ is denoted by $\mathcal{SH}_c(X)$. We formulate here the basic result of the cellular sheaf theory.

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**Theorem A.** The following categories

\[ \mathcal{SH}_c(X) \simeq \text{mod-}B(X) \]

are equivalent.

2. A perversity function \( \overline{p} : \mathbb{Z} \to \mathbb{Z} \) is a key ingredient in the definition of the intersection homology groups introduced by M. Goresky and R. MacPherson in [GM1]. This seminal paper also contains the construction of the basic sets. The later version of basic sets called perverse skeleta, and their building blocks called perverse cells were introduced in [Mac2], as well as the cellular version \( \delta : \mathbb{Z} \to \mathbb{Z} \) of the perversity function. (The classical and cellular perversity functions are in one-to-one correspondence.) An extremely important application of the idea of perverse cell complex appears in the work of I. Mirkovic and K. Vilonen [MkVi]. They use perverse cells in loop Grassmannians associated to complex algebraic groups to give a proof of the geometric Satake isomorphism theorem announced by V. Ginzburg in [Ginz].

Given a cellular perversity function \( \delta \) we construct the \( \delta \)-perverse finite topology on \( X \). The idea of the existence of perverse topology appeared in the discussions surrounding the lectures [Mac2], and it was communicated to us by M. Goresky and R. MacPherson. One recovers the topology generated by open stars for bottom (i.e. \( \delta(k) = -k, k \geq 0 \)) cellular perversity. Sheaves on perverse topology can be interpreted as modules over an associative algebra \( B(X, \delta) \). In other words, we have

**Theorem B1.** The following categories

\[ \mathcal{SH}(\mathcal{T}(X, \delta)) \simeq \text{mod-}B(X, \delta) \]

are equivalent.

Let us consider the category \( \mathcal{SH}_c(X, \delta) \) of sheaves on \( X \) constant along \((\delta)-\)perverse simplices (\( \delta \)-sheaves). We give a linear algebra description of the category of \( \delta \)-sheaves, generalizing Theorem A. Our linear algebra gadgets assign vector spaces to perverse simplices and linear maps to pairs of “incident” perverse simplices. As in the classical case, it is easy to interpret such linear algebra gadgets as modules over the algebra \( B(X, \delta) \). In other words, we have

**Theorem B2.** The following categories

\[ \mathcal{SH}_c(X, \delta) \simeq \text{mod-}B(X, \delta) \]

are equivalent.

We believe that it is very important to study \( \delta \)-sheaves for more general stratified spaces. The construction of \( \delta \)-sheaves on flag varieties should yield new geometric realization of various representation theoretic constructions. In particular, for a parabolic flag variety this approach could lead to a geometric realization of a singular block of the Bernstein-Gelfand-Gelfand category \( \mathcal{O} \), which would be useful for a geometric categorification of the Temperley-Lieb algebra (see [BFK]).
3. Let $X$ be a finite connected simplicial complex, and let $\delta$ be a perversity. The category of cohomologically constructible with respect to the triangulation perverse sheaves on $X$ [BBD, Mac 1] is denoted by $\mathcal{M}_c(X, \delta)$. A lot of work has been done to represent perverse sheaves in terms of linear algebra data (e.g. [MaVi], [BrGr]). The linear algebra data description of the category $\mathcal{M}_c(X, \delta)$ was obtained by R. MacPherson in [Mac2, Mac3]. MacPherson’s linear algebra gadgets are called cellular perverse sheaves. It is easy to interpret cellular perverse sheaves as modules over an associative algebra $A(X, \delta)$. In other words, we have

**Theorem C.** The following categories

$$\mathcal{M}_c(X, \delta) \simeq \text{mod-}A(X, \delta)$$

are equivalent.

4. In the present and the following sections we describe our main new results: Koszul duality patterns for sheaves and applications. The proof of the Theorem D2 was sketched in [Vyb2]. Theorem D1, Corollary D3, and Theorem E are new.

The notion of a Koszul algebra was introduced by S. Priddy [Prid]. In many important cases the algebras underlying the categories of perverse sheaves turn out to be Koszul (cf. [BGSc, BGSo, PS]). The Koszulity of $A(X, \delta)$ and $B(X, \delta)$ has been proved by A. Polishchuk in [Pol]. Given a Koszul category, it seems to be very important to investigate the Koszul dual category and the duality functor. We establish the following claims:

**Theorem D1.** Perverse sheaves on the ordinary topology of $X$ are Koszul dual to the ordinary sheaves on the perverse topology. (For the precise formulation see 4.2.11.)

**Theorem D2.** Perverse sheaves constant along ordinary simplices of $X$ are Koszul dual to ordinary sheaves constant along perverse simplices. (For the precise formulation see 4.2.11.)

The basic notion underlying the definition of the Intersection Homology is that of the allowable set (see [GM1] and 4.2.12).

**Corollary D3.** The Koszul duality functor transforms a stratification-constructible perverse sheaf on a pseudomanifold into a complex of sheaves with allowable support. (For the precise formulation see 4.2.12.)

One of the properties of the Koszul duality functor is that it preserves the hypercohomology of a complex of sheaves.

If one considers a perverse sheaf as a “generalized homology theory” (Fáry functor [Mac4, GMMV]), then Theorem D2 gives a way to see “geometric chains” of this homology theory.
5. Let $D$ be the Verdier duality functor acting in the derived category of simplicial sheaves. We introduce another functor $\tilde{D}$, which we call the “Verdier duality for cosheaves.” We prove that the Koszul duality functor $L$ intertwines $D$ and $\tilde{D}$. More precisely, we have

**Theorem E.** The following functors

$$L \circ D \simeq \tilde{D} \circ L$$

are isomorphic.

The question of how Koszul duality commutes with Verdier duality motivating Theorem E was formulated by V. Ginzburg in connection with Beilinson-Ginzburg conjecture [BG, 5.18, 5.24].

6. We try to give the explicit reference in the text each time we borrow some material. However, we would like to emphasize that our main sources are [BBD], [BGSo], [CPS], [GM1], [KS], [Mac2, Mac3], [Pol], and [Shep]. Some standard fact listed in the text without reference is probably lifted from one of the above.

7. The paper is organized as follows. In Chapter 1 we list some standard facts mostly lifted from classical books and papers [BBD, GeMa, GM1, GM2, KS, Mas] and other sources, in particular [Mac2], and more recent papers [Pol, Vyb1, Vyb2]. In Chapter 2 we study the relationship between sheaves and presheaves on a base of topology. We prove Theorem A, Theorem B1, and list some properties of simplicial sheaves. Chapter 3 is devoted to sheaves constant along perverse simplices. We prove Theorem B2 and study some properties of $\delta$-sheaves. In Chapter 4 we are initially concerned with the perverse algebras. Then we study the Koszul duality functors for sheaves and their properties. We complete the proof of Theorem C, Theorem D1, Theorem D2, Corollary D3, and Theorem E.

8. We list here some notational conventions adopted in the paper. Unless specified otherwise, $X$ is assumed to be a finite connected simplicial complex. Other conventions: $\mathbb{Z}$ integer numbers; $\mathbb{R}$ real numbers; $\mathbb{F}$ commutative field of char = 0; $\mathbb{F}$ constant sheaf associated to $\mathbb{F}$; Id identity operator (morphism, functor). All vector spaces are assumed to be over $\mathbb{F}$ unless specified otherwise.

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Chapter 1. Preliminaries

1.1. Simplicial complexes and perversities. The bulk of this section is standard algebraic topology lifted from [GeMa, KS, Mas]. The material on perversities is borrowed from [BBD, GM1, Mac2, Pol, Vyb2].

1.1.1. Let $\Delta = \{v_0, v_1, \ldots, v_r\}$ be the $r$-dimensional nondegenerate simplex spanned by vertices $\{v_0, v_1, \ldots, v_r\}$. The barycenter of $\Delta$ will usually be denoted by $c$.

1.1.2. Let $K$ be a simplicial set with the set of vertices $\text{Vert}$. Let $\mathbb{R}^\text{Vert}$ denote the set of maps from $\text{Vert}$ to $\mathbb{R}$. An element $x \in \mathbb{R}^\text{Vert}$ is nothing but a family $x(v) \in \mathbb{R}$, indexed by $v \in \text{Vert}$. We equip $\mathbb{R}^\text{Vert}$ with the usual Euclidean topology. To a simplex of $K$ we associate the geometric simplex $\Delta$ in $\mathbb{R}^\text{Vert}$ by:

$$\Delta = \{x \in \mathbb{R}^\text{Vert} : x(v) = 0 \text{ for } v \notin \Delta, x(v) > 0 \text{ for } v \in \Delta \text{ and } \sum_v x(v) = 1\}.$$ 

The geometric realization of $K$ denoted by $|K|$ is called a flat (or Euclidean) simplicial complex. A finite simplicial complex $X$ is a topological space homeomorphic to $|K|$ for some simplicial set $K$. The topology induced from $|K|$ will sometimes be referred to as usual topology. A simplex $\Delta$ of $X$ is a homeomorphic image of the simplex $\Delta$ of $|K|$. We say that two simplices $\Delta$ and $\Delta'$ are incident if either $\Delta$ is a face of $\Delta'$ or $\Delta'$ is a face of $\Delta$. If $\Delta$ and $\Delta'$ are incident, we write $\Delta \leftrightarrow \Delta'$. Unless specified otherwise, we will always assume that $X$ is a finite connected simplicial complex. We assume that $\dim X = n$ (i.e. $X$ has some $n$-dimensional simplices and does not have any $(n+1)$-dimensional simplices).
1.1.3. Since $X$ is a (regular) cell complex we can choose an orientation for each simplex (see [Mas, IX.5]) and define the incidence numbers $[\Delta : \Delta']$ for any pair of simplices such that $\Delta'$ is a codim 1 face of $\Delta$. We record the following

**Lemma** [Mas, IX.7.1]. Let $\Delta', \Delta$ be as above. Then $[\Delta : \Delta'] = \pm 1$.

1.1.4. Let $K$ be a finite simplicial set. The first barycentric subdivision of $K$ is denoted by $\hat{K}$. If $X$ is a simplicial complex then its first barycentric subdivision is denoted by $\hat{X}$.

1.1.5. **Definition.**

1. (Mac2) A **cellular perversity** $\delta: \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ is a function from the non-negative integers $\mathbb{Z}_{\geq 0}$ to the integers such that $\delta(0) = 0$ and $\delta$ takes every interval $\{0, 1, \ldots, k\} \subset \mathbb{Z}_{\geq 0}$ bijectively to an interval $\{a, a + 1, \ldots, a + k\} \subset \mathbb{Z}$ for some $a \in \mathbb{Z}_{\leq 0}$. In other words, a perversity is such a function $\delta$ that $\delta(0) = 0$ and for $k \in \mathbb{Z}_{\geq 0}$,

$$\delta(k) = \begin{cases} \max_{i \in [0,k-1]} \delta(i) + 1 & \text{for } k \text{ of type } \ast \\ \min_{i \in [0,k-1]} \delta(i) - 1 & \text{for } k \text{ of type } ! \end{cases}.$$

2. (cf. [GM]) A **classical (or Goresky-MacPherson) perversity** $\overline{p}: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is a monotonously non-decreasing function from the set of non-negative integers $\mathbb{Z}_{\geq 0}$ to itself such that $\overline{p}(0) = 0$ and $\overline{p}(k) - \overline{p}(k-1)$ is either 0 or 1.

3. ([BBD]) A **BBDG perversity** $p: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\leq 0}$ is a function such that $p(0) = 0$, $0 \leq p(m) - p(n) \leq n - m$ for $m \leq n$. We define the **dual perversity** by $p^*(k) = -p(k) - k$.

**Example.** We will often work with two extreme cellular perversities, top perversity $\delta(k) = k$, $k \geq 0$, and bottom perversity $\delta(k) = -k$, $k \geq 0$.

Let $p$ be a BBDG perversity. An integer $k \in \mathbb{Z}_{\geq 0}$ is of type $\ast$ (resp. type $!$) if $p(k) = p(k - 1) - 1$ (resp. $p(k) = p(k - 1)$). It is convenient to assume that 0 is of both types (see [Pol, 1]).

1.1.6. In this subsection we study the relationship between different versions of perversity. It is easy to see that if $\overline{p}$ is a classical perversity, then $p = -\overline{p}$ is a BBDG perversity and vice versa.

**Lemma.** There is a one-to-one correspondence between cellular perversities and BBDG perversities.

**Proof.** Let $p$ be a BBDG perversity. The corresponding cellular perversity $\delta$ is defined by:

$$\delta(k) = \begin{cases} -p(k), & k \text{ of type } \ast \\ p^*(k), & k \text{ of type } ! \end{cases}.$$
Below we work mostly with the cellular version of perversity, but sometimes we have to refer to the associated classical and BBDG perversity since their use is standard in the context of Intersection Homology and perverse sheaves respectively.

1.1.7. Until the end of this section \( X \) is assumed to be a finite connected simplicial complex, \( \dim X = n \). In this subsection we recall the definition of the basic sets given in [GM1]. Let us fix a classical perversity \( \overline{p} \). For an integer \( i \geq 0 \) define the function \( L_{\overline{p}}^i \) as follows:

\[
L_{\overline{p}}^i(0) = i, \quad L_{\overline{p}}^i(n + 1) = -1,
\]

and if \( 1 \leq c \leq n \) set:

\[
L_{\overline{p}}^i(c) = \begin{cases} 
-1 & \text{if } i - c + \overline{p}(c) \leq -1 \\
 n - c & \text{if } i - c + \overline{p}(c) \geq n - c \\
 i - c + \overline{p}(c) & \text{otherwise}.
\end{cases}
\]

Define \( dL_{\overline{p}}^i(c) = L_{\overline{p}}^i(c) - L_{\overline{p}}^i(c + 1) \) (which is either 0 or 1). Define \( Q_{\overline{p}}^i \) to be the subcomplex of \( \hat{X} \) spanned by the set of barycenters of such simplices \( \Delta \) of \( X \) that:

\[
dL_{\overline{p}}^i(n - \dim \Delta) = 1.
\]

1.1.8. Definition [Mac2]. Given a perversity \( \delta \), we define \( \delta(\Delta) = \delta(\dim \Delta) \), where \( \Delta \) is a simplex of \( X \). Given a simplex \( \hat{\Delta} = \{c_0, c_1, \ldots, c_s\} \) of \( \hat{X} \) (where \( \{c_0, c_1, \ldots, c_s\} \) are barycenters of \( \Delta_0, \Delta_1, \ldots, \Delta_s \)) we denote by \( \max \hat{\Delta} \) such vertex \( c_i \) that \( \delta(\Delta_i) = \max\{\delta(\Delta_0), \delta(\Delta_1), \ldots, \delta(\Delta_s)\} \). Given a simplex \( \Delta \) with the barycenter \( c \) we define the corresponding perverse simplex:

\[
\delta \Delta = \bigsqcup_{\max \hat{\Delta} = c} \hat{\Delta}.
\]

We define the \( k \)-th perverse skeleton \( X_k^\delta \), \( \min_{[0,n]} \delta \leq k \leq \max_{[0,n]} \delta \), as follows:

\[
X_k^\delta = \bigsqcup_{\delta(\Delta) \leq k} \delta \Delta \subset X.
\]

Thus, we have a filtration:

\[
X_i^\delta \subset X_{i+1}^\delta \subset \cdots \subset X_{i+n-1}^\delta \subset X_{i+n}^\delta = X,
\]

where \( i = \min_{[0,n]} \delta \). It is easy to see that perverse simplices are connected components of \( X_k^\delta - X_{k-1}^\delta \), \( \min_{[0,n]} \delta \leq k \leq \max_{[0,n]} \delta \). The decomposition of \( X \) into a disjoint union of perverse simplices is called \( \delta \)-perverse triangulation of \( X \).
1.1.9. Lemma. Let $\delta$ be the cellular perversity corresponding to a classical perversity $\overline{p}$. Let $\overline{q}(k) = \overline{p}(n) - \overline{p}(n - k)$ be another perversity. Then

$$Q_i^\delta = X^\delta_{\overline{p}(n) - n + i}.$$  

Proof. The cellular perversity $\delta$ corresponding to a classical perversity $\overline{p}$ is given by:

$$\delta(k) = \begin{cases} \overline{p}(k), & k \text{ of type } * \\ \overline{p}(k) - k, & k \text{ of type } !. \end{cases}$$

Here $k$ is of type $*$ (resp. type $!$) if $\overline{p}(k) = \overline{p}(k - 1) + 1$ (resp. $\overline{p}(k) = \overline{p}(k - 1)$). Let $m$ be an integer $m \in \mathbb{Z}_{\geq 0}$. It is easy to see that the following conditions are equivalent:

1. $dL_i^\overline{q}(n - m) = 1.$
2. $\delta(m) \leq \overline{p}(n) - n + i.$

1.2. Partially ordered sets, presheaves, and algebras. In this section we introduce the notion of a presheaf on a partially ordered set, and the notion of a simplicial perverse sheaf. We notice that presheaves and simplicial perverse sheaves can be interpreted as modules over certain quadratic algebras.

1.2.1. Let $\Lambda$ be a partially ordered set with the relation $\alpha \leq \beta$. We can associate to $\Lambda$ a small category whose objects are elements of $\Lambda$ and the set of morphisms $\text{Hom}(\alpha, \beta)$ contains one element if $\alpha \leq \beta$, and is empty otherwise. By abuse of notation we will also denote this category by $\Lambda$.

Definition. A presheaf $S$ on $\Lambda$ consists of a vector space $S(\alpha)$ associated to every $\alpha \in \Lambda$ (stalk of $S$ at $\alpha$), and a linear map $s(\alpha, \beta) : S(\alpha) \to S(\beta)$, for $\alpha \geq \beta$ (restriction map) associated to every $\alpha \geq \beta$ in such a way that $S$ is a functor from the category opposite to $\Lambda$ to the category of vector spaces. A morphism $f$ between two presheaves $S$ and $S'$ is a collection of stalkwise linear maps commuting with the restriction maps. The category of presheaves on $\Lambda$ is denoted by $\text{Pre } \Lambda$.

1.2.2. Until the end of this section $X$ is assumed to be a finite connected simplicial complex. Let us fix a perversity $\delta$. We will introduce a partial order on the set of simplices of $X$ as follows (cf. [Pol, 2]). We say that $\Delta \geq \Delta'$ if there exists a sequence of simplices $\Delta = \Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_r = \Delta'$ such that:

1. $\Delta_i \leftrightarrow \Delta_{i+1} \leftrightarrow \Delta_{i+2}, 0 \leq i \leq r - 2,$
2. $\delta(\Delta_i) = \delta(\Delta_{i+1}) + 1, 0 \leq i \leq r - 1.$

The set of all simplices of $X$ with this partial order is denoted by $\Lambda(X, \delta)$, and from now on $\Delta \geq \Delta'$ will refer to the relation in $\Lambda(X, \delta)$.

1.2.3. Lemma. (a) Let $\Delta \leftrightarrow \Delta', \delta(\Delta) \geq \delta(\Delta')$. Then $\Delta \geq \Delta'$. Moreover, we can choose a sequence of simplices such that $\Delta \leftrightarrow \Delta_i \leftrightarrow \Delta'$ for all $i$.

(b) If $\Delta \geq \Delta'$ and $\delta(\Delta) \geq \delta(\Delta') \geq 0$ or $0 \geq \delta(\Delta) \geq \delta(\Delta')$, then $\Delta \leftrightarrow \Delta'$.

(c) If $\Delta \geq \Delta'$ and $\Delta \not\leftrightarrow \Delta'$, then $\delta(\Delta) > 0$, $\delta(\Delta') < 0$. If in addition $\delta(\Delta) - \delta(\Delta') = 2$, then $\delta(\Delta) = 1$, $\delta(\Delta') = -1$.

Proof. The proof is elementary and is left to the reader.
1.2.4. We define a category $\mathcal{R}(X, \delta)$ of linear algebra data.

**Definition.** An object $S$ of $\mathcal{R}(X, \delta)$ is the following data:

1. (Stalks) For every simplex $\Delta$ in $X$, a finite dimensional vector space $S(\Delta)$ called the *stalk* of $S$ at $\Delta$,
2. (Restriction maps) For every pair of simplices $\Delta$ and $\Delta'$ in $X$ such that $\delta(\Delta) = \delta(\Delta') + 1$, and $\Delta \leftrightarrow \Delta'$, a linear map $s(\Delta, \Delta') : S(\Delta) \to S(\Delta')$ called the *restriction map*,

subject to the following “equivalence” axiom:

Suppose that $\Delta'$, $\Delta''$, $\Delta_1$ and $\Delta_2$ are simplices of $X$ such that:

1. $\delta(\Delta') = k + 1$, $\delta(\Delta'') = k - 1$, and $\delta(\Delta_1) = \delta(\Delta_2) = k$,
2. $\Delta' \leftrightarrow \Delta_1 \leftrightarrow \Delta''$ and $\Delta' \leftrightarrow \Delta_2 \leftrightarrow \Delta''$.

Then

$$s(\Delta_1, \Delta'') \circ s(\Delta', \Delta_1) = s(\Delta_2, \Delta'') \circ s(\Delta', \Delta_2).$$

A morphism $h$ in this category is the collection of stalkwise linear maps $h(\Delta)$ commuting with the restriction maps (cf. 1.2.1).

1.2.5. In this subsection $n$ is not supposed to be the dimension of $X$.

**Lemma-Definition.** Let $\Delta_I \geq \Delta_T$ be two simplices, and let $S \in \mathcal{R}(X, \delta)$. We define the linear map $s(\Delta_I, \Delta_T) : S(\Delta_I) \to S(\Delta_T)$ as follows. If $\Delta_I = \Delta_T$, then $s(\Delta_I, \Delta_T) = \text{Id}_{S(\Delta_I)}$. Let $\Delta_I > \Delta_T$, and let

\[ \Delta_I = \Delta_0 > \Delta_1 > \Delta_2 > \cdots > \Delta_n = \Delta_T, \]
\[ \delta(\Delta_i) = \delta(\Delta_{i+1}) + 1, \quad i = 0, \ldots, n-1; \quad \text{and} \]
\[ \Delta_I = \Delta'_0 > \Delta'_1 > \Delta'_2 > \cdots > \Delta'_n = \Delta_T, \]
\[ \delta(\Delta'_i) = \delta(\Delta'_{i+1}) + 1, \quad i = 0, \ldots, n-1 \]

be two sequences of simplices of $X$. Then

\[ s(\Delta_I, \Delta_T) := s(\Delta_{n-1}, \Delta_n) \circ \cdots \circ s(\Delta_0, \Delta_1) = s(\Delta'_{n-1}, \Delta'_n) \circ \cdots \circ s(\Delta'_0, \Delta'_1). \]

**Proof.** The proof is elementary and is left to the reader.

1.2.6. **Lemma.** The following categories

$$\text{Pre } \Lambda(X, \delta) = \mathcal{R}(X, \delta)$$

are isomorphic.

**Proof.** We leave it to the reader to define the functor $F : \text{Pre } \Lambda(X, \delta) \to \mathcal{R}(X, \delta)$, the functor $G : \mathcal{R}(X, \delta) \to \text{Pre } \Lambda(X, \delta)$, and verify that $F \circ G = \text{Id}$, and $G \circ F = \text{Id}$.

If $\delta = \text{bottom perversity}$ (resp. $\delta = \text{top perversity}$), then objects of $\text{Pre } \Lambda(X, \delta) = \mathcal{R}(X, \delta)$ are called simplicial sheaves (resp. simplicial cosheaves).
1.2.7. We will introduce another category $\mathcal{P}(X, \delta)$ of linear algebra data.

**Definition [Mac2].** An object $S$ of $\mathcal{P}(X, \delta)$ is the following data:

1. (Stalks) For every simplex $\Delta$ in $X$, a finite dimensional vector space $S(\Delta)$ called the *stalk* of $S$ at $\Delta$,
2. (Boundary maps) For every pair of simplices $\Delta$ and $\Delta'$ in $X$ such that $\delta(\Delta) = \delta(\Delta') + 1$ and $\Delta \leftrightarrow \Delta'$, a linear map $s(\Delta, \Delta') : S(\Delta) \to S(\Delta')$ called the *boundary map*,

subject to the following “chain complex” axiom:

If $\Delta'$ is any simplex such that $\delta(\Delta') = k + 1$ and $\Delta''$ is any simplex such that $\delta(\Delta'') = k - 1$, then

$$\sum_{\Delta : \delta(\Delta) = k, \Delta' \leftrightarrow \Delta \leftrightarrow \Delta''} s(\Delta, \Delta'') \circ s(\Delta', \Delta) = 0.$$ 

The morphisms in this category are stalkwise linear maps commuting with the boundary maps. Objects of $\mathcal{P}(X, \delta)$ are called simplicial perverse sheaves.

1.2.8. A quiver $Q$ is a finite simple oriented tree. We will denote the set of vertices of $Q$ by $V(Q)$ and the set of arrows by $E(Q)$.

We associate a quiver $Q(X, \delta)$ to the set $\Lambda(X, \delta)$ as follows. The vertices of $Q(X, \delta)$ are indexed by the elements of $\Lambda(X, \delta)$ (i.e. $V(Q(X, \delta)) = \Lambda(X, \delta)$), and there is an arrow from $\Delta$ to $\Delta'$ if $\delta(\Delta) = \delta(\Delta') + 1$, and $\Delta \leftrightarrow \Delta'$.

Let $Q$ be a quiver. One can consider the quiver algebra $\mathbb{F}Q$ (see e.g. [Vyb1]).

1.2.9. We are especially interested in the following quotients of the quiver algebra $\mathbb{F}Q(X, \delta)$.

**Definition A.** The algebra $A(X, \delta)$ is the quotient of $\mathbb{F}Q(X, \delta)$ by the “chain complex” relations:

If $\Delta'$ is any simplex such that $\delta(\Delta') = k + 1$ and $\Delta''$ is any simplex such that $\delta(\Delta'') = k - 1$, then

$$\sum_{\Delta : \delta(\Delta) = k, \Delta' \leftrightarrow \Delta \leftrightarrow \Delta''} a(\Delta, \Delta'')a(\Delta', \Delta) = 0,$$

where $a(\Delta, \Delta''), a(\Delta', \Delta)$ are the generators of $A(X, \delta)$.

**Definition B.** The algebra $B(X, \delta)$ is the quotient of $\mathbb{F}Q(X, \delta)$ by the “equivalence” relations:

Suppose that $\Delta', \Delta'', \Delta_1$ and $\Delta_2$ are simplices of $X$ such that:

1. $\delta(\Delta') = k + 1, \delta(\Delta'') = k - 1$ and $\delta(\Delta_1) = \delta(\Delta_2) = k$,
2. $\Delta' \leftrightarrow \Delta_1 \leftrightarrow \Delta''$ and $\Delta' \leftrightarrow \Delta_2 \leftrightarrow \Delta''$. 


Then

\[ b(\Delta_1, \Delta'') b(\Delta', \Delta_1) = b(\Delta_2, \Delta'') b(\Delta', \Delta_2), \]

where \( b(\Delta_1, \Delta''), b(\Delta', \Delta_1), b(\Delta_2, \Delta''), \) and \( b(\Delta', \Delta_2) \) are the generators of \( B(X, \delta) \).

1.2.10. We will present now an algebraic interpretation of the category \( \mathcal{P}(X, \delta) \) and the category \( \mathcal{R}(X, \delta) \).

Let us construct a functor \( \Xi_A(X, \delta) : \mathcal{P}(X, \delta) \to \text{mod-}A(X, \delta) \) (resp. \( \Xi_B(X, \delta) : \mathcal{R}(X, \delta) \to \text{mod-}B(X, \delta) \)). Let \( S \in \mathcal{P}(X, \delta) \) (resp. \( R \in \mathcal{R}(X, \delta) \)). Let \( M = \Xi_A(X, \delta)(S) \) (resp. \( \Xi_B(X, \delta)(R) \)). As a vector space \( M = \bigoplus S(\Delta) \), where the sum is taken over all simplices of \( X \). The action of the respective algebras is defined as follows:

1. \( e(\Delta)m = m \) for \( m \in S(\Delta) \), and \( e(\Delta)m = 0 \) for \( m \in S(\Delta') \), \( \Delta \neq \Delta' \),
2. \( a(\Delta, \Delta')m = s(\Delta, \Delta')(m) \) for \( m \in S(\Delta) \), and \( a(\Delta, \Delta')m = 0 \) for \( m \in S(\Delta'') \), \( \Delta \neq \Delta'' \)
   (resp. \( b(\Delta, \Delta')m = s(\Delta, \Delta')(m) \) for \( m \in S(\Delta) \), and \( b(\Delta, \Delta')m = 0 \) for \( m \in S(\Delta'') \), \( \Delta \neq \Delta'' \)).

Lemma.

1. The functor \( \Xi_A(X, \delta) : \mathcal{P}(X, \delta) \to \text{mod-}A(X, \delta) \)
   is an isomorphism of categories.
2. The functor \( \Xi_B(X, \delta) : \mathcal{R}(X, \delta) \to \text{mod-}B(X, \delta) \)
   is an isomorphism of categories.

Proof. The proof is left to the reader.

1.2.11. If \( \delta = \text{bottom perversity} \), then the symbol \( \delta \) will just be omitted from the notation, for example \( \Lambda(X, \delta) \) will be denoted by \( \Lambda(X) \).

1.3. Elements of Sheaf Theory. In this section we list some sheaf theoretic definitions and constructions mostly lifted from [BBD, Iver, KS, Pol], where we refer the reader for many more details.

1.3.1. Let \( (X, \mathcal{T}) \) be a topological space. The set \( \mathcal{T} \) has a natural partial order: for \( U, V \in \mathcal{T} \) we say that \( U \leq V \) if \( U \subseteq V \). Let \( S \) be a presheaf on \( \mathcal{T} \), and let \( U \subseteq V \). The restriction map \( S(V) \to S(U) \) is sometimes denoted by \( r^V_U \), and if \( s \in S(V) \), then \( r^V_U(s) \) is sometimes denoted by \( s|_U \).

We record the following standard definition here only to be able to refer to axioms S1 and S2 later in the text.

Definition. A presheaf \( S \) on \( \mathcal{T} \) is called a sheaf if it satisfies conditions S1 and S2 below.

S1. For any open set \( U \subseteq X \), any open covering \( U = \cup_{\alpha} U_{\alpha} \), any section \( s \in S(U) \), \( s|_{U_{\alpha}} = 0 \) for all \( \alpha \) implies \( s = 0 \).
For any open set $U \subseteq X$, any open covering $U = \bigcup \alpha U_\alpha$, any family $s_\alpha \in S(U_\alpha)$ satisfying $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ for all pairs $(\alpha, \beta)$, there exists $s \in S(U)$ such that $s|_{U_\alpha} = s_\alpha$ for all $\alpha$.

The category of sheaves $\mathcal{SH}(\mathcal{T})$ on $(X, \mathcal{T})$ is a full subcategory in the category of presheaves $\text{Pre}\mathcal{T}$.

1.3.2. Let $S$ be a presheaf on $\mathcal{T}$. Let $S_x$ be the stalk of $S$ at a point $x \in X$. The image of $s \in S(U)$ in $S_x$ is denoted by $s_x$. For $s \in S(U)$ its support is denoted by $\text{supp}(s)$.

1.3.3. Lemma [KS, 2.2.3]. Given a presheaf $S$ on $(X, \mathcal{T})$, there exists a sheaf $S^+$ and a morphism $\theta : S \to S^+$. Moreover, $(S^+, \theta)$ is unique up to an isomorphism, and for any $x \in X$, $\theta_x : S_x \to S^+_x$ is an isomorphism.

Sometimes $S^+$ is called the sheafification of $S$. The sheafification functor is denoted by $+: \text{Pre}\mathcal{T} \to \mathcal{SH}(\mathcal{T})$.

1.3.4. Let $A$ be a vector space. The constant sheaf associated to $A$ is denoted by $A$.

1.3.5 [KS, 2.3.1, 2.5], [Iver, II.6.6]. Let $X$ and $Y$ be two topological spaces, $f : Y \to X$ a continuous map. Let $T$ be a sheaf on $Y$. The direct image of $T$ by $f$ (resp. direct image with proper supports) is denoted by $f_*T$ (resp. $f!T$). Let $S$ be a sheaf on $X$. The inverse image of $S$ by $f$ (resp. inverse image with proper supports) is denoted by $f^*S$ (resp. $f^!S$). Let $f : Y \hookrightarrow X$ be an inclusion of a subset and let $S$ be a sheaf on $X$. Then $f^*S$ is also denoted by $S|_Y$.

1.3.6. Definition. Let $X$ be a finite connected simplicial complex equipped with the usual topology (cf. 1.1), and let $\Delta \subseteq X$ be a simplex in $X$. A sheaf $S$ of $\mathbb{F}$-vector spaces on $X$ is called constructible with respect to the triangulation, if $S|_{\Delta}$ is the constant sheaf associated to a finite dimensional $\mathbb{F}$-vector space for all $\Delta$.

The category of constructible sheaves (denoted by $\mathcal{SH}_c(X)$) is a full subcategory of $\mathcal{SH}(\mathcal{T})$, where $\mathcal{T}$ is the usual topology on $X$.

1.3.7. Let $\mathcal{A}$ be an abelian category. The bounded homotopic (resp. derived) category of $\mathcal{A}$ will be denoted by $\mathcal{C}^b(\mathcal{A})$ (resp. $\mathcal{D}^b(\mathcal{A})$).

Definition. Let $X$ be a finite connected simplicial complex equipped with the usual topology $\mathcal{T}$. Let $(S^i, d^i) \in \mathcal{D}^b(\mathcal{SH}(\mathcal{T}))$. Then $(S^i, d^i)$ is called cohomologically constructible with respect to the triangulation if the sheaves $\mathcal{H}^i(S^\bullet) = \ker d^i/\text{im} d^{i-1}$ are constructible with respect to the triangulation. The category of cohomologically constructible complexes of sheaves (denoted by $\mathcal{D}^b_c(X)$) is a full subcategory of $\mathcal{D}^b(\mathcal{SH}(\mathcal{T}))$.

The following theorem is due to M. Kashiwara [Kash] and A. Shepard [Shep]. We refer the reader to [KS, 8.1.11].
Theorem. The following categories

\[ \mathcal{D}^b(\mathcal{S} \mathcal{H}_c(X)) \simeq \mathcal{D}^b_c(X) \]

are equivalent.

1.3.8 [BBD, 1.3.1]. Let \( \mathcal{D} \) be a triangulated category, and let \( (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) \) be a \( t \)-structure on \( \mathcal{D} \) [BBD, 1.3.1]. The core of the \( t \)-structure \( (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) \) is the full subcategory \( \mathcal{A} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} \). It is known [BBD, 1.3.6] that the core of a \( t \)-structure is an abelian category stable under extensions.

1.3.9. This subsection is mostly lifted from [BBD]. Let \( X \) be a topological space, and \( X = \bigcup_{S \in \mathcal{S}} S \) be a finite decomposition (stratification) of \( X \) into the disjoint union of locally closed subspaces (strata). Let \( p \) be a BBDG perversity. Let \( \mathcal{D}^b(X) \) be the derived category of (cohomologically constructible with respect to some triangulation refining the stratification) sheaves of \( \mathbb{F} \)-vector spaces on \( X \).

**Definition [BBD, 2.1.2]**. The subcategory \( p\mathcal{D}^{\leq 0}(X) \) (resp. \( p\mathcal{D}^{\geq 0}(X) \)) of \( \mathcal{D}^b(X) \) is the subcategory formed by the complexes \( \mathcal{S} \) such that for every inclusion of a stratum \( i_S : S \hookrightarrow X \), we have \( H^n(i_S^* \mathcal{S}) = 0 \) for \( n > p(\dim S) \) (resp. \( H^n(i_S^! \mathcal{S}) = 0 \) for \( n < p(\dim S) \)).

**Proposition [BBD, 2.1.4]**. \((p\mathcal{D}^{\leq 0}(X), p\mathcal{D}^{\geq 0}(X))\) is a \( t \)-structure on \( \mathcal{D}^b(X) \).

The core of this \( t \)-structure, denoted by \( \mathcal{M}_\mathcal{S}(X, p) := p\mathcal{D}^{\leq 0}(X) \cap p\mathcal{D}^{\geq 0}(X) \) is called the category of cohomologically constructible with respect to the stratification \( p \)-perverse sheaves on \( X \).

In particular, if the stratification \( \mathcal{S} \) is a triangulation \( \mathcal{T} \) of \( X \), we sometimes denote the category \( \mathcal{M}_\mathcal{T}(X, p) \) of the \( p \)-perverse sheaves cohomologically constructible with respect to the triangulation simply by \( \mathcal{M}_c(X, p) \).

Let \( \mathcal{T} \) be a triangulation refining a stratification \( \mathcal{S} \). Then there is an exact embedding functor \( \text{ref} : \mathcal{M}_\mathcal{S}(X, p) \hookrightarrow \mathcal{M}_\mathcal{T}(X, p) \) (see [BBD, 2.1.14, 2.1.15]), i.e. \( \mathcal{M}_\mathcal{S}(X, p) \) may be considered as a subcategory of \( \mathcal{M}_\mathcal{T}(X, p) \).

Let \( \delta \) be the cellular perversity corresponding to \( p \) (see 1.1). Below we will often denote the category \( \mathcal{M}_\mathcal{S}(X, p) \) by \( \mathcal{M}_\mathcal{S}(X, \delta) \).

1.3.10. It is known that the simple objects of \( \mathcal{M}_c(X, \delta) \) are in one-to-one correspondence with simplices \( \Delta \) in \( X \). In this subsection we will give an explicit description of the simple objects \( \mathcal{S}_\Delta \) in \( \mathcal{M}_c(X, \delta) = \mathcal{M}_c(X, p) \), following [Pol].

**Lemma [Pol, 1.1]**. For a simplex \( \Delta \) in \( X \) we have:

\[
\mathcal{S}_\Delta = \begin{cases} 
  i_\Delta^* \mathcal{F}_\Delta[-p(\dim \Delta)], & \text{dim } \Delta \text{ is of type } * \\
  i_\Delta^! \mathcal{O}_\Delta[-p(\dim \Delta)], & \text{dim } \Delta \text{ is of type } !, 
\end{cases}
\]

where \( \mathcal{F}_\Delta \) is the constant sheaf on \( \Delta \) associated to \( \mathbb{F} \), and \( \mathcal{O}_\Delta \) is the orientation sheaf on \( \Delta \) (see [KS, 3.3]).
Chapter 2. Constructible sheaves and perverse topology

2.1. Sheaves and a base for topology. In this section we discuss the relationship between sheaves on a topological space \((X, \mathcal{T})\) and presheaves on a base for topology. Many technical details are lifted from [KS].

2.1.1. Let \((X, \mathcal{T})\) be a topological space. A subfamily \(\mathcal{B}\) of \(\mathcal{T}\) is said to be a base for \(\mathcal{T}\) if every member of \(\mathcal{T}\) can be expressed as the union of some members of \(\mathcal{B}\).

Lemma. Let \(X\) be a set and \(\mathcal{B}\) a family of its subsets covering \(X\). Then the following statements are equivalent:

1. there exists a topology on \(X\) with \(\mathcal{B}\) as a base,
2. for any \(B_1, B_2 \in \mathcal{B}\), \(B_1 \cap B_2\) can be expressed as the union of some members of \(\mathcal{B}\),
3. for any \(B_1, B_2 \in \mathcal{B}\) and \(x \in B_1 \cap B_2\), there exists \(B_3 \in \mathcal{B}\) such that \(x \in B_3\) and \(B_3 \subset B_1 \cap B_2\).

2.1.2. Let \((X, \mathcal{T})\) be a topological space and let \(\mathcal{B}\) be a base for \(\mathcal{T}\). The set \(\mathcal{B}\) has a natural partial order: for \(U, V \in \mathcal{B}\) we say that \(U \preceq V\) if \(U \subseteq V\). Let \(W \in \mathcal{T}\) be an open set. Define

\[ \mathcal{B}|_W = \{ U \subset W : U \in \mathcal{B} \}. \]

It is sometimes convenient to assign some indices to the elements of \(\mathcal{B}|_W\), i.e. we regard \(\mathcal{B}|_W = \{ U_\lambda \}\) as a family of subsets parametrized by \(\lambda\).

2.1.3. In this subsection we construct a functor \(\tilde{\mathcal{S}} : \text{Pre} \mathcal{B} \rightarrow \text{Pre} \mathcal{T}\).

On objects: let \(S \in \text{Pre} \mathcal{B}\) be a presheaf on \(\mathcal{B}\). Let \(W \in \mathcal{T}\) be an open set and let \(\mathcal{B}|_W = \{ U_\lambda \}\). We define

\[ \tilde{S}(W) = \lim_{\lambda} S(U_\lambda) = \{ s \in \prod_\lambda S(U_\lambda) : s(\lambda)|_{U_\mu} = s(\mu) \text{ for } U_\mu \subseteq U_\lambda \}. \]

Sometimes we write \(s_\lambda\) instead of \(s(\lambda)\).

We have a natural restriction map \(\tilde{S}(W) \rightarrow S(U_\lambda)\): for \(s \in \tilde{S}(W)\), \(r^W_{U_\lambda}(s) = s(\lambda)\).

Let \(W_1, W_2 \in \mathcal{T}\) be such open sets that \(W_2 \subseteq W_1\), and let \(\mathcal{B}|_{W_1} = \{ U_\lambda \}\) and \(\mathcal{B}|_{W_2} = \{ U_\mu \}\). For \(s \in \tilde{S}(W_1)\) we define:

\[ r^W_{W_2}(s) = \prod_\mu r^W_{U_\mu}(s) = \prod_\mu s(\mu). \]

We have \(s(\mu)|_{U_\nu} = s(\nu)\) for \(U_\nu \subseteq U_\mu\) since \(s \in \lim_{\lambda} S(U_\lambda)\). Therefore \(r^W_{W_2}(s) \in \lim_{\mu} S(U_\mu) = \tilde{S}(W_2)\). Clearly \(r^W = \text{Id}\).
Let $W_1, W_2, W_3 \in \mathcal{T}$ be such open sets that $W_3 \subseteq W_2 \subseteq W_1$. Let $\mathcal{B}|_{W_1} = \{U_\lambda\}$, $\mathcal{B}|_{W_2} = \{U_\mu\}$, and $\mathcal{B}|_{W_3} = \{U_\nu\}$. Let $s \in \tilde{S}(W_1) = \varprojlim_{\lambda} S(U_\lambda)$. Then $r^{W_3}_{W_2}(s) = \prod_\nu s(\nu)$ and $r^{W_3}_{W_2} \circ r^{W_2}_{W_1}(s) = r^{W_3}_{W_2}(\prod_\mu s(\mu)) = \prod_\nu s(\nu)$. Thus

$$r^{W_3}_{W_2} = r^{W_3}_{W_2} \circ r^{W_2}_{W_1}.$$ 

We leave it to the reader to construct the functor on morphisms.

2.1.4. Let $|_B : \text{Pre} \mathcal{T} \to \text{Pre} \mathcal{B}$ be the obvious “restriction” functor.

**Lemma.** $|_B \circ \sim \simeq \text{Id}.$

The proof is left to the reader.

2.1.5. Let $+ : \text{Pre} \mathcal{T} \to \mathcal{SH}(\mathcal{T})$ be the sheafification functor and $\imath : \mathcal{SH}(\mathcal{T}) \to \text{Pre} \mathcal{T}$ be the inclusion functor. We define $\Phi^+ : \text{Pre} \mathcal{B} \to \mathcal{SH}(\mathcal{T})$, $\Phi^+ = + \circ \sim$, and $\Phi^- : \mathcal{SH}(\mathcal{T}) \to \text{Pre} \mathcal{B}$, $\Phi^- = |_B \circ \imath$.

2.1.6. **Lemma.** $\Phi^+ \circ \Phi^- \simeq \text{Id}.$

The proof is left to the reader.

2.2. **Proof of the Theorem A.** In this section we give the sequence of elementary lemmas leading to the proof of Theorem A. This theorem is well known [Kash, Shep, KS]. The proofs of the lemmas are left to the reader.

2.2.1. Let $X$ be a finite connected simplicial complex. Let us consider the partially ordered set $\Lambda(X, \delta)$ introduced in 1.2, when $\delta =$ bottom perversity. In this case we will denote it by $\Lambda(X)$. Clearly $\Delta \geq \Delta'$ if and only if $\Delta \subseteq \Delta'$.

2.2.2. This subsection is mostly lifted from [KS, 8.1.4]. Let $K$ be a finite simplicial set with a set of vertices Vert (cf. 1.1). Let $\Delta$ be a simplex of $|K|$. Let $x \in \Delta$ be a point. For $0 < \varepsilon \leq 1$, set $I_{\varepsilon} = \{\alpha \in \mathbb{R} : \varepsilon \leq \alpha \leq 1\}$ and define the map $\pi_{\varepsilon} : I_{\varepsilon} \times \Delta^* \to \Delta^*$ by:

$$\pi_{\varepsilon}(\alpha, y)(v) = \alpha y(v) + (1 - \alpha)x(v) \quad \text{for} \ v \in \text{Vert}.$$ 

The map $\pi_{\varepsilon}$ is continuous and surjective, $\pi_{\varepsilon}(1, \cdot)$ is the identity $\{1\} \times \Delta^* \simeq \Delta^*$, and $\pi_{\varepsilon}(\varepsilon, \cdot)$ is a homeomorphism $\{\varepsilon\} \times \Delta^* \simeq \pi_{\varepsilon}(\{\varepsilon\} \times \Delta^*)$. We denote $\Delta^*(x, \varepsilon) := \pi_{\varepsilon}(\{\varepsilon\} \times \Delta^*)$. In other words:

$$\Delta^*(x, \varepsilon) = \{\varepsilon y + (1 - \varepsilon)x : y \in \Delta^*\}$$

Clearly, $\Delta^*(x, \varepsilon) \subset \Delta^*$ for $0 < \varepsilon < 1$. A set $\Delta^*(x, \varepsilon)$ will be called the $\varepsilon$-neighborhood of a point $x \in \Delta$. It is clear that the $\varepsilon$-neighborhoods form a base for topology of $|K|$.

Let $X$ be a finite connected simplicial complex. For a point $x \in \Delta$, (and $0 < \varepsilon \leq 1$) its $\varepsilon$-neighborhood $\Delta^*(x, \varepsilon)$ is the homeomorphic image of the corresponding $\varepsilon$-neighborhood in $|K|$. Certainly the $\varepsilon$-neighborhoods form a base $\mathcal{B}$ for topology $\mathcal{T}$ of $X$. 

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2.2.3. Let $\mathcal{T}$ be a presheaf on $\Lambda(X)$. Let $\Delta^*(x, \varepsilon)$ be some $\varepsilon$-neighborhood of a point $x \in \Delta$. Notice that if $\Delta'$ is another simplex, then $\Delta' \cap \Delta^*(x, \varepsilon) \neq \emptyset$ if and only if $\Delta' \leq \Delta$.

**Definition.** (a) A function $\varphi : \Delta^*(x, \varepsilon) \to \bigoplus \Delta T(\Delta)$ is called $\mathcal{T}$-constructible if:

1. $\varphi|_{\Delta' \cap \Delta^*(x, \varepsilon)} \equiv \varphi(\Delta') \in T(\Delta')$, is a constant function for $\Delta' \leq \Delta$,
2. $\varphi(\Delta'') = t(\Delta', \Delta'') \varphi(\Delta')$ for $\Delta'' \leq \Delta' \leq \Delta$.

(b) Let $U \subseteq X$ be an open set. A function $\varphi : U \to \bigoplus \Delta T(\Delta)$ is called locally $\mathcal{T}$-constructible if every point $x \in U$ has an $\varepsilon$-neighborhood such that $\varphi$ becomes $\mathcal{T}$-constructible when restricted to this neighborhood. Notice that a locally $\mathcal{T}$-constructible function becomes constructible upon restriction to any $\varepsilon$-neighborhood of any point.

**Lemma.** Let $x \in \Delta \subset X$ be a point and let $0 < \varepsilon \leq 1$. Then

$$\{\text{the space of } \mathcal{T}\text{-constructible functions on } \Delta^*(x, \varepsilon)\} = T(\Delta).$$

Moreover, if $\Delta_1^*(x, \varepsilon_1) \supseteq \Delta_2^*(y, \varepsilon_2)$ then the map from the space of $\mathcal{T}$-constructible functions on $\Delta_1^*(x, \varepsilon_1)$ to the space of $\mathcal{T}$-constructible functions on $\Delta_2^*(y, \varepsilon_2)$ given by restriction is precisely $t(\Delta_1, \Delta_2)$.

2.2.4. Let $\mathcal{T}$ be a presheaf on $\Lambda(X)$. We will construct a presheaf (denoted by the same symbol $\mathcal{T}$) on $\mathcal{B}$ assigning to each $\Delta^*(x, \varepsilon)$ the space of $\mathcal{T}$-constructible functions on it, with restriction maps given by restricting functions. Such a presheaf will be called constructible. (In other words, $\mathcal{T} \in \text{Pre} \mathcal{B}$ is constructible if $\text{Id} = t(\Delta^*, \Delta^*(x, \varepsilon)) : T(\Delta^*) \to T(\Delta^*(x, \varepsilon))$ for all appropriate $\varepsilon$ and $x \in \Delta$.) The full subcategory of $\text{Pre} \mathcal{B}$ consisting of constructible presheaves will be denoted $\text{Pre}_c \mathcal{B}$. Clearly, $\text{Pre} \Lambda(X) = \text{Pre}_c \mathcal{B}$. This explains the same notation for the objects of the two categories. Sometimes we will use one instead of the other.

2.2.5. **Lemma.** Let $\mathcal{T}$ be a presheaf on $\Lambda(X)$. Let $\check{\mathcal{T}}$ be a presheaf on $\mathcal{T}$ corresponding to $\mathcal{T} \in \text{Pre}_c \mathcal{B}$ (see 2.1.3). Let $U \in \mathcal{T}$ be an open set. Then

$$\check{T}(U) = \{\text{the space of locally } \mathcal{T}\text{-constructible functions on } U\}.$$

Moreover, the restriction map $\check{T}(U_1) \to \check{T}(U_2)$, for $U_2 \subseteq U_1$ applied to a function $\varphi \in T(U_1)$ is the restriction of $\varphi$ to $U_2$.

2.2.6. **Lemma.** (a) $\check{T}$ is a sheaf.

(b) $\check{T}$ is constructible with respect to the triangulation.

2.2.7. In this subsection we will slightly generalize the argument of 2.2.6(b). Let $Y \subseteq X$ be a closed union of simplices. Let $\mathcal{T}$ be a presheaf on $\Lambda(X)$ and let $\check{T}$ be the corresponding constructible sheaf. We can restrict the presheaf $\mathcal{T}$ to a presheaf $\mathcal{T}|_Y$ on the partially ordered set $\Lambda(Y)$ in the obvious way.
Lemma. $\widetilde{T}|_Y = \widetilde{T}|_Y$.

Let $U \subseteq X$ be an open union of simplices. It is clear that $\widetilde{T}|_U = \widetilde{T}|_U$. (Here $\widetilde{T}|_U$ is the obvious subsheaf of $\widetilde{T}|_U$.) Then the statement of the Lemma holds for any locally closed $Y \subseteq X$.

2.2.8. Lemma 2.2.6 implies that the functor $\widetilde{\Psi}$ (see 2.1) restricts to a functor $\Psi^+ : \text{Pre} \Lambda(X) = \text{Pre}_c B \to \mathcal{S}\mathcal{H}_c(X)$. Notice that if $\Psi^+$ is considered as a functor from $\text{Pre}_c B$, then it coincides with the restriction of the functor $\Phi^+$ defined in 2.1. The functor $\Psi^- : \mathcal{S}\mathcal{H}_c(X) \to \text{Pre} \Lambda(X)$ is constructed as follows. Let $S \in \mathcal{S}\mathcal{H}_c(X)$ and let $T = \Psi^-(S)$. Then

1. $T(\Delta) = S(\Delta^*)$,
2. $t(\Delta_1, \Delta_2) = t_{\Delta_1^*}^{\Delta_2^*}$ (restriction maps of $S$).

It is clear from our construction that $\Psi^- \circ \Psi^+ \simeq \text{Id}$. (Alternatively, we can use Lemma 2.1.4.)

2.2.9. Lemma. $\Psi^+ \circ \Psi^- \simeq \text{Id}$.

2.2.10. Summarizing, we have constructed the functor $\Psi^+ : \text{Pre} \Lambda(X) \to \mathcal{S}\mathcal{H}_c(X)$ which to a presheaf $T$ assigns the sheaf of locally $T$-constructible functions, and the functor $\Psi^- : \mathcal{S}\mathcal{H}_c(X) \to \text{Pre} \Lambda(X)$ which to a constructible sheaf $S$ assigns the presheaf of sections on open stars of simplices. We have proved that these functors are quasi-inverse to each other. In 1.2 we constructed the equivalence functors:

$$\text{Pre} \Lambda(X) \xrightarrow{F} \mathcal{R}(X) \xrightarrow{\Xi_B(X)} \text{mod-} B(X).$$

Let $\Theta(X) : \mathcal{S}\mathcal{H}_c(X) \to \text{mod-} B(X)$ be the equivalence functor defined as $\Theta(X) = \Xi_B(X) \circ F \circ \Psi^-$. In other words, we have

Theorem A. The functor

$$\Theta(X) : \mathcal{S}\mathcal{H}_c(X) \xrightarrow{\sim} \text{mod-} B(X)$$

is an equivalence of categories.

2.3. Perverse topology. In this section we introduce perverse topology and prove Theorem B1. For notational conventions on simplices, simplicial sets, and simplicial complexes see 1.1.

2.3.1. Let $K$ be a finite simplicial set. Let $\Delta \in K$ be a simplex and $|\Delta|$ be the corresponding geometric simplex. Let $|\Delta|^*$ be the open star of $|\Delta|$ (i.e. the union of all simplices having $|\Delta|$ as a face). If $\Delta_1$ and $\Delta_2$ are two simplices, then

$$|\Delta_1|^* \cap |\Delta_2|^* = \begin{cases} |\Delta_1 \cup \Delta_2|^*, & \text{if } \Delta_1 \cup \Delta_2 \in K \\ \emptyset, & \text{otherwise.} \end{cases}$$
2.3.2. Let $\Delta \in K$ be a simplex and let $\overline{|\Delta|}$ be the closure of the corresponding geometric simplex. Notice that

$$\overline{|\Delta|} = \bigcup_{\Delta' \subseteq \Delta} |\Delta'|.$$ 

If $\Delta_1$ and $\Delta_2$ are two simplices, then $|\Delta_1| \cap |\Delta_2| = |\Delta_1 \cap \Delta_2|$. (We assume that $|\emptyset| = \emptyset$.)

2.3.3. Until the end of this section $X$ is assumed to be a finite connected simplicial complex. Let us fix a perversity $\delta$. Let $\Lambda(X, \delta)$ be the partially ordered set introduced in 1.2.2. For a simplex $\Delta \subset X$ we define the perverse star $U(\Delta, \delta)$ of $\Delta$ to be the following union of simplices:

$$U(\Delta, \delta) = \bigcup_{\Delta' \leq \Delta} \Delta'.$$

If $\delta = \text{bottom perversity}$, then $U(\Delta, \delta) = \Delta^*$, and if $\delta = \text{top perversity}$, then $U(\Delta, \delta) = \overline{|\Delta|}$. Let $\mathcal{B}(X, \delta)$ be the family of sets $U(\Delta, \delta)$ parametrized by all simplices of $X$.

**Lemma.** There exists a topology on $X$ with $\mathcal{B}(X, \delta)$ as a base.

**Proof.** First of all the family of subsets $\mathcal{B}(X, \delta)$ covers $X$. Indeed, if $x \in X$ is a point, then $x \in \Delta$ for some $\Delta$, and $x \in U(\Delta, \delta)$.

Let $\Delta$ and $\Delta'$ be two simplices. Then

$$U(\Delta, \delta) \cap U(\Delta', \delta) = \bigcup_{\Delta'' \leq \Delta, \Delta'' \leq \Delta'} U(\Delta'', \delta).$$

Indeed, if $\Delta'' \subseteq U(\Delta, \delta) \cap U(\Delta', \delta)$ (i.e. $\Delta'' \leq \Delta$ and $\Delta'' \leq \Delta'$), then $\Delta'' \subseteq U(\Delta'', \delta)$. Vice versa if $\Delta'' \leq \Delta$ and $\Delta'' \leq \Delta'$, then $U(\Delta'', \delta) \subseteq U(\Delta, \delta) \cap U(\Delta', \delta)$ since if $\tilde{\Delta} \subseteq U(\Delta'', \delta)$, then $\tilde{\Delta} \leq \Delta'' \leq \Delta$, and $\tilde{\Delta} \leq \Delta'' \leq \Delta'$.

Thus, the claim follows from Lemma 2.1.1.

The topology generated by $\mathcal{B}(X, \delta)$ will be denoted by $\mathcal{T}(X, \delta)$. An element $U \in \mathcal{T}(X, \delta)$ will be called a perverse set.

2.3.4. The partially ordered set $\Lambda(X, \delta)$ can be interpreted now as a set of perverse stars $U(\Delta, \delta)$ with the partial order given by inclusion. (Indeed, $U(\Delta', \delta) \subseteq U(\Delta, \delta)$ if and only if $\Delta' \leq \Delta$.) In other words, $\Lambda(X, \delta) = \mathcal{B}(X, \delta)$ as partially ordered sets and $\text{Pre} \Lambda(X, \delta) = \text{Pre} \mathcal{B}(X, \delta)$. Let $\mathcal{S} \in \text{Pre} \mathcal{B}(X, \delta)$ and let $\tilde{\mathcal{S}}$ be a presheaf on $\mathcal{T}(X, \delta)$ associated to $\mathcal{S}$ (see 2.1).
Lemma. \( \tilde{S} \) is a sheaf.

Proof. Let \( U \) be a perverse set, and let \( U = \bigcup_{\alpha=1}^{l} U_{\alpha} \) be a covering of \( U \) by perverse sets. We claim that each \( U(\Delta, \delta) \subseteq U \) is contained in some \( U_{\alpha} \). Indeed, if \( U(\Delta, \delta) \subseteq \bigcup_{\alpha} U_{\alpha} = \bigcup_{\lambda} U(\Delta_{\lambda}, \delta) \), then \( \Delta \subseteq \bigcup_{\lambda} U(\Delta_{\lambda}, \delta) \). Then there exists such \( \lambda_{0} \) that \( \Delta \subseteq U(\Delta_{\lambda_{0}}, \delta) \). Then \( U(\Delta, \delta) \subseteq U(\Delta_{\lambda_{0}}, \delta) \subseteq U_{\alpha} \) for some \( \alpha \).

We have to check axioms 1.3.1.S1 and 1.3.1.S2.

1.3.1.S1. Let \( s \in \tilde{S}(U) \). This means that \( s = \prod_{\Delta \subseteq U} s(\Delta) \) with \( s(\Delta) \in S(U(\Delta, \delta)) \) and \( s(\Delta)|_{U(\Delta', \delta)} = s(\Delta') \) for \( U(\Delta', \delta) \subseteq U(\Delta, \delta) \). Let \( s|_{U_{\alpha}} = 0 \) for all \( \alpha \). This implies that \( s(\Delta) = 0 \) whenever \( U(\Delta, \delta) \subseteq U_{\alpha} \) for some \( \alpha \), but since every perverse star is a subset of some \( U_{\alpha} \), \( s(\Delta) = 0 \) for all \( \Delta \subseteq U \) and \( s = 0 \).

1.3.1.S2. Let \( s_{\alpha} \in \tilde{S}(U_{\alpha}) \), and let \( s_{\alpha}|_{U_{\alpha} \cap U_{\beta}} = s_{\beta}|_{U_{\alpha} \cap U_{\beta}} \) for all \( \alpha, \beta \). Each \( s_{\alpha} = \prod_{\Delta \subseteq U_{\alpha}} s_{\alpha}(\Delta) \) with \( s_{\alpha}(\Delta) \in S(U(\Delta, \delta)) \). Let \( U(\Delta, \delta) \subseteq U \). Then there exists such \( \alpha \) that \( U(\Delta, \delta) \subseteq U_{\alpha} \). We define \( s(\Delta) = s_{\alpha}|_{U(\Delta, \delta)} = s_{\alpha}(\Delta) \). This definition does not depend on \( \alpha \) due to compatibility on intersections. We set \( s = \prod_{\Delta \subseteq U} s(\Delta) \). Clearly, \( s|_{U_{\alpha}} = s_{\alpha} \).

2.3.5. Recall that in 2.1 we constructed functors \( \sim : \text{Pre}B(X, \delta) \to \text{Pre}T(X, \delta) \), \( \Phi^{+} : \text{Pre}B(X, \delta) \to \text{SH}(T(X, \delta)) \), and \( \Phi^{-} : \text{SH}(T(X, \delta)) \to \text{Pre}B(X, \delta) \). Lemma 2.3.4 implies that in the case of perverse topology the functors \( \sim = \Phi^{+} \) are isomorphic.

Lemma. The functors \( \Phi^{+} \) and \( \Phi^{-} \) are quasi-inverse to each other.

Proof. Follows from Lemmas 2.1.4, 2.1.6.

2.3.6. Recall that in 1.2 we constructed the equivalence functors:

\[
\begin{align*}
\text{Pre} \Lambda(X, \delta) & \xrightarrow{F} \mathcal{R}(X, \delta) \\
& \xrightarrow{\Xi_{B}(X, \delta)} \text{mod-}B(X, \delta).
\end{align*}
\]

Let \( \Omega(X, \delta) : \text{SH}(T(X, \delta)) \to \text{mod-}B(X, \delta) \) be the equivalence functor defined as follows: \( \Omega(X, \delta) = \Xi_{B}(X, \delta) \circ F \circ \Phi^{-} \). Summarizing the discussion we formulate:

Theorem B1. The functor

\[
\Omega(X, \delta) : \text{SH}(T(X, \delta)) \xrightarrow{\sim} \text{mod-}B(X, \delta)
\]

is an equivalence of categories.

2.4. Borel-Moore-Verdier duality for simplicial sheaves and cosheaves. In this section we list some basic facts about simplicial sheaves and cosheaves. The material is essentially borrowed from [Shep] and [GMMV], where we refer the reader for more details.

In this section \( X \) is assumed be a finite connected simplicial complex. We denote by \( V^{*} = \text{Hom}_{\mathbb{F}}(V, \mathbb{F}) \), where \( V \) is a vector space. If \( f \) is a linear map, then \( f^{*} \) denotes the adjoint map.
2.4.1. Let $S_1$ and $S_2$ be two simplicial sheaves. Then the sheaf $T = \text{Hom}(S_1, S_2)$ is defined as follows:

$$T(\Delta) = \text{Hom}(S_1|_{\Delta^*}, S_2|_{\Delta^*}).$$

If $\Delta_1 \subseteq \Delta_2$, then the map $t(\Delta_1, \Delta_2) : T(\Delta_1) \to T(\Delta_2)$ is given by restriction of a map $S_1|_{\Delta_1^1} \to S_2|_{\Delta_1^1}$ to a map $S_1|_{\Delta_2^2} \to S_2|_{\Delta_2^2}$ (note that $\Delta_2^2 \subseteq \Delta_1^1$).

2.4.2. Definition. Let $\Delta \subset X$ be a simplex and let $V$ be a vector space. We define the simplicial sheaf $[\Delta]^V$ as follows:

$$[\Delta]^V(\Delta') = \begin{cases} V, & \Delta' \subseteq \Delta \\ 0, & \Delta' \not\subseteq \Delta, \end{cases}$$

where the restriction maps between copies of $V$ are all identity maps. We will also write $[\Delta] = [\Delta]^F$. It is easy to see that for any $\Delta$ and $V$, $[\Delta]^V$ is an injective simplicial sheaf.

Lemma. Let $S$ be a simplicial sheaf, and let $[\Delta]^V$ be an injective sheaf. Then

1. $\text{Hom}(S, [\Delta]^V) = \text{Hom}_F(S(\Delta), V)$.
2. $\text{Hom}(S, [\Delta]^V) = [\Delta]^\text{Hom}_F(S(\Delta), V)$.
3. In particular, if $\Delta'' \subseteq \Delta'$, then $\text{Hom}([\Delta']^V, [\Delta'']^W) = \text{Hom}_F(V, W)$.

Proof. The proof is left to the reader.

2.4.3. Let $S$ be a simplicial sheaf. There is a canonical map $S \to [\Delta]^S(\Delta)$ induced by $\text{Id} : S(\Delta) \to S(\Delta)$. Taking the sum over all simplices we obtain a canonical map $S \to \oplus_{\Delta} [\Delta]^{S(\Delta)} = \mathbf{I}^0$ which is clearly injective. Moreover,

Lemma. For any simplicial sheaf $S$ there exists the canonical injective resolution, $S \xrightarrow{q.i.} \mathbf{I}^\bullet$.

Proof. See [Shep, 1.3].

Remark. Let $F$ be a constant sheaf on $X$. The canonical injective resolution of $F$ is as follows:

$$F \to \oplus_{\Delta_0}[\Delta_0] \to \oplus_{\Delta_1 \subseteq \Delta_0}[\Delta_1] \to \oplus_{\Delta_2 \subseteq \Delta_1 \subseteq \Delta_0}[\Delta_2] \to \ldots$$

Taking the global sections (see [Shep, 1.4]) we obtain the complex of vector spaces computing the simplicial cohomology (with coefficients in $\mathbb{F}$) of the first barycentric subdivision $\hat{X}$.

2.4.4. Let $C^b(\text{Pre}\Lambda(X))$ (resp. $D^b(\text{Pre}\Lambda(X))$) be the homotopic (resp. derived) category of the category of simplicial sheaves.
Definition. The dualizing complex \( \omega_X = \omega_X^* \in \mathcal{D}^b(\text{Pre } \Lambda(X)) \) has in degree \(-i\) the sheaf:

\[
\omega_X^{-i} = \bigoplus_{\dim \Delta = i} [\Delta].
\]

The boundary map \( \omega_X^{-i} \to \omega_X^{-i+1} \) is the zero map between components \([\Delta]\) and \([\Delta']\) if \( \Delta' \not\subset \Delta \), and is induced by multiplication by \([\Delta : \Delta']\) if \( \Delta' \) is a codim 1 face of \( \Delta \).

2.4.5. In 1.2 we have seen that \( \text{Pre } \Lambda(X) = \mathcal{R}(X) \). Thus, \( \mathcal{C}^b(\text{Pre } \Lambda(X)) = \mathcal{C}^b(\mathcal{R}(X)) \) and \( \mathcal{D}^b(\text{Pre } \Lambda(X)) = \mathcal{D}^b(\mathcal{R}(X)) \). We transport the injective objects \([\Delta]\) and the dualizing complex \( \omega_X \) to the categories \( \mathcal{R}(X) \) and \( \mathcal{D}^b(\mathcal{R}(X)) \) respectively, preserving the notation.

Let \( (S^i, d^i_S) \in \mathcal{C}^b(\mathcal{R}(X)) \), and let us consider a bicomplex:

\[
DS^{i,j} = \text{Hom}(S^i, \omega_X^j) = \bigoplus_{\dim \Delta = -j} [\Delta]^{S^i(\Delta)^*}.
\]

The differential

\[
DS^{i,j} \xrightarrow{d^{i,j}_I} DS^{i,j+1},
\]

\[
\bigoplus_{\dim \Delta = -j} [\Delta]^{S^i(\Delta)^*} \xrightarrow{d^{i,j}_I} \bigoplus_{\dim \Delta = -j-1} [\Delta]^{S^i(\Delta)^*}
\]

is the zero map between components \([\Delta]^{S^i(\Delta)^*}\) and \([\Delta']^{S^i(\Delta')^*}\) if \( \Delta' \not\subset \Delta \) and is induced by \([\Delta : \Delta']s^i(\Delta', \Delta)^* : S^i(\Delta)^* \to S^i(\Delta')^*\), if \( \Delta' \) is a codim 1 face of \( \Delta \).

The differential

\[
DS^{i,j} \xrightarrow{d^{i,j}_I} DS^{i-1,j},
\]

\[
\bigoplus_{\dim \Delta = -j} [\Delta]^{S^i(\Delta)^*} \xrightarrow{d^{i,j}_I} \bigoplus_{\dim \Delta = -j-1} [\Delta]^{S^{i-1}(\Delta)^*}
\]

is the sum of the morphisms \([\Delta]^{S^i(\Delta)^*} \to [\Delta]^{S^{i-1}(\Delta)^*}\) induced by the linear maps \((-1)^{j-i+1}d^{i-1}_S(\Delta)^* : S^i(\Delta)^* \to S^{i-1}(\Delta)^*\).

Let us consider the complex

\[
DS^p = \bigoplus_{p=j-i} DS^{i,j}
\]

with the differential \( d = d_I + d_{II} \). The complex \( DS^* \) is called the Verdier dual of \( S^* \). Notice that:

\[
DS^* = \text{Hom}^*(S^*, \omega_X^*),
\]

where we consider \( \omega_X^* \) as an object in \( \mathcal{C}^b(\mathcal{R}(X)) \). Since \( \omega_X^* \) is a complex of injective objects, the functor \( D \) induces the functor \( D^b(D) : \mathcal{D}^b(\mathcal{R}(X)) \to \mathcal{D}^b(\mathcal{R}(X)) \).
2.4.6. Recall that $\mathcal{R}(X) = \mathcal{R}(X, \text{bottom perversity})$, and let us denote $\mathcal{R}'(X) = \mathcal{R}(X, \text{top perversity})$.

In this subsection we construct the duality functor $\ast : \text{C}^b(\mathcal{R}(X)) \to \text{C}^b(\mathcal{R}'(X))$ as follows. Let $(S^\bullet, d_S^\bullet) \in \text{C}^b(\mathcal{R}(X))$. We define:

1. $\ast S^i(\Delta) = S^{\ast i}(\Delta)$,
2. if $\Delta'$ is a codim 1 face of $\Delta$, then $\ast s^{\ast i}(\Delta, \Delta') = s^{-i}(\Delta', \Delta) : S^{-i}(\Delta)^* \to S^{-i}(\Delta')^*$,
3. $\ast d^i(\Delta) = (-1)^{i+1} s^{-i-1}(\Delta)^* : S^{-i}(\Delta)^* \to S^{-i-1}(\Delta)^*$.

It is clear that the contravariant functor $\ast$ transforms quasi-isomorphisms to quasi-isomorphisms, and hence induces the functor $\mathcal{D}^b(\ast) : \mathcal{D}^b(\mathcal{R}(X)) \to \mathcal{D}^b(\mathcal{R}'(X))$. It is easy to see that $\mathcal{D}^b(\ast)$ is a contravariant equivalence of categories, $\mathcal{D}^b(\mathcal{R}(X))^\text{opp} \cong \mathcal{D}^b(\mathcal{R}'(X))$.

2.4.7. Recall that $\mathcal{P}(X) = \mathcal{P}(X, \text{bottom perversity})$, and let us denote $\mathcal{P}'(X) = \mathcal{P}(X, \text{top perversity})$.

**Lemma.** The following categories

\[ \mathcal{R}(X) = \mathcal{P}(X), \]
\[ \mathcal{R}'(X) = \mathcal{P}'(X) \]

are isomorphic.

The proof is left to the reader.

The derived functor $\mathcal{D}^b(\ast) : \mathcal{D}^b(\mathcal{R}'(X)) \to \mathcal{D}^b(\mathcal{P}'(X))$ is an equivalence of categories.

**Chapter 3. Sheaves constant along perverse simplices**

3.1. $\mathcal{S}\mathcal{H}(X, \delta) \simeq \mathcal{S}(X, \delta)$. Let $X$ be a finite connected simplicial complex and let $\hat{X}$ be its first barycentric subdivision. We fix a perversity $\delta$. For definitions and notation on perverse simplices see 1.1.

3.1.1. **Definition.** A sheaf $S$ on $X$ is called constant along perverse simplices if for any perverse simplex $Y$ the restriction $S|_Y$ is a constant sheaf associated to a finite dimensional $\mathbb{F}$-vector space.

The category of sheaves constant along $(-\delta)$-perverse simplices is denoted by $\mathcal{S}\mathcal{H}_c(X, \delta)$. It is a full subcategory of the category $\mathcal{S}\mathcal{H}_c(\hat{X})$.

3.1.2. **Definition.** The category $\mathcal{S}(X, \delta)$ is a full subcategory of the abelian category $\text{Pre} \Lambda(\hat{X})$. An object $T$ of $\text{Pre} \Lambda(\hat{X})$ belongs to $\mathcal{S}(X, \delta)$ if for any $\hat{\Delta}, \hat{\Delta}' \subseteq ^{-\delta}\Delta$ we have:

1. $T(\hat{\Delta}) = T(\hat{\Delta}')$,
2. if $\hat{\Delta} \subseteq \hat{\Delta}'$, then $t(\hat{\Delta}, \hat{\Delta}') = \text{Id}_{T(\hat{\Delta})}$.
3.1.3. Let \( i : Y \hookrightarrow \hat{X} \) be the inclusion of a locally closed union of simplices into \( \hat{X} \). By Lemma 2.2.7 we have \( \psi^+ \circ i^* = i^* \circ \psi^+ \). Hence, the following diagram commutes (here \( i^* : \text{Pre} \Lambda(\hat{X}) \to \text{Pre} \Lambda(Y) \) is defined in the obvious way):

\[
\begin{array}{ccc}
\text{SH}_c(\hat{X}) & \xleftarrow{\psi^+} & \text{Pre} \Lambda(\hat{X}) \\
\downarrow i^* & & \downarrow i^* \\
\text{SH}_c(Y) & \xleftarrow{\psi^+} & \text{Pre} \Lambda(Y).
\end{array}
\]

(\*)

Notice that \( Y \) is an open union of simplices in the simplicial complex \( \overline{Y} \), hence the functor \( \psi^+ : \text{Pre} \Lambda(Y) \to \text{SH}_c(Y) \) is obtained in the obvious way from the functor \( \psi^+ : \text{Pre} \Lambda(\overline{Y}) \to \text{SH}_c(\overline{Y}) \).

Notice that each perverse simplex is a locally closed union of simplices of \( \hat{X} \). Thus for each \( T \in \mathcal{S}(X, \delta) \) we have \( \psi^+(T) \in \text{SH}_c(X, \delta) \).

The restriction of \( \psi^+ \) to \( \mathcal{S}(X, \delta) \) will be denoted by \( \mathcal{T}^+ : \mathcal{S}(X, \delta) \to \text{SH}_c(X, \delta) \).

The commutativity of the diagram (\*) is equivalent to the commutativity of the following diagram:

\[
\begin{array}{ccc}
\text{SH}_c(\hat{X}) & \xrightarrow{\psi^-} & \text{Pre} \Lambda(\hat{X}) \\
\downarrow i^* & & \downarrow i^* \\
\text{SH}_c(Y) & \xrightarrow{\psi^-} & \text{Pre} \Lambda(Y),
\end{array}
\]

(**)

which means that there exists an isomorphism of functors \( \psi^- \circ i^* \sim i^* \circ \psi^- \).

3.1.4. Let \( S \in \text{SH}_c(X, \delta) \). We will construct a functor \( \mathcal{T}^- : \text{SH}_c(X, \delta) \to \mathcal{S}(X, \delta) \) and an isomorphism \( f : \mathcal{T}^- \to \psi^- \) (here \( \psi^- \) is restricted to \( \text{SH}_c(X, \delta) \)). Let \( \mathcal{T}^-(S) = T \), \( \psi^-(S) = T' \) and let \( Y = -\delta \Delta \) be a perverse simplex. We set

\[
T|_Y = \psi^-(S|_Y).
\]

(Here \( \psi^-(S|_Y) := \psi^-(S|_{\overline{Y}})|_Y \).) The isomorphism of functors \( \psi^- \circ i^* \sim i^* \circ \psi^- \) provided by the diagram 3.1.3(**) gives us the isomorphism for all \( Y \)

\[
(*) \quad f|_Y : T|_Y \sim T'|_Y.
\]

This is equivalent to providing isomorphisms \( f(\Delta) : T(\Delta) \to T'(\Delta) \) for all \( \Delta \subseteq \hat{X} \), commuting with restriction maps for all \( Y \). Let us complete the definition of \( T \). Let \( \Delta_1 \subseteq \Delta_2 \). We define

\[
t(\Delta_1, \Delta_2) = f^{-1}(\Delta_2)t'(\Delta_1, \Delta_2)f(\Delta_1).
\]

Notice that if \( \Delta_1, \Delta_2 \subseteq Y \) for some perverse simplex \( Y \), then \( t(\Delta_1, \Delta_2) = \text{Id} \) due to (\*). The definition of \( \mathcal{T}^- \) on morphisms is left to the reader. The stalkwise isomorphisms \( f(\Delta) : T(\Delta) \to T'(\Delta) \) commuting with restriction maps provide the isomorphism \( T \to T' \). Therefore the functors \( \mathcal{T}^- \sim \psi^- \) are isomorphic.
3.1.5. Lemma.

(1) Υ− ⋯ Υ+ ≃ Id.
(2) Υ+ ⋯ Υ− ≃ Id.

Proof.

(1) Clear, cf. 2.1.4, 2.2.8.
(2) Id ≃ Ψ+ ⋯ Ψ− ⋯ Υ− ⋯ Υ+ ≃ Ψ+ ⋯ Υ−, the first equivalence is Lemma 2.2.9.

3.2. Proof of the Theorem B2.

3.2.1. Let F : Pre Λ(X) → R(ˆX) be the isomorphism of 1.2. By abuse of notation we denote the full subcategory of R(ˆX), corresponding to S(X, δ) ⊂ Pre Λ(ˆX) under F, by S(X, δ). In other words,

Definition. The category S(X, δ) is a full subcategory of the abelian category R(ˆX). An object T of R(ˆX) belongs to S(X, δ) if for any ˆ∆, ˆ∆′ ⊆ −δ∆ we have:

(1) T( ˆ∆) = T( ˆ∆′),
(2) if ˆ∆ is a codim 1 face of ˆ∆′, then t( ˆ∆, ˆ∆′) = Id_{T( ˆ∆)}.

Until the end of this section ≥ stands for the relation in Λ(X, δ).

3.2.2. Lemma. Let ˆ∆, ˆ∆′ of ˆX be such that ˆ∆ is a codim 1 face of ˆ∆′. Let ˆ∆ ⊆ −δ∆, ˆ∆′ ⊆ −δ∆′. Then ∆ ↔ ∆′ and ∆ ≥ ∆′.

Proof. Let ˆ∆ = {v1, . . . , vk}, ˆ∆′ = {v1, . . . , vk, e}. We will consider two cases.

Case 1. e = max( ˆ∆′). Then e is the barycenter of ˆ∆′. By definitions −δ( ˆ∆′) > −δ(∆) i.e. δ(∆) > δ( ˆ∆′). Let vi = max( ˆ∆). Then vi is the barycenter of ∆. The 1-simplex {vi, e} is a simplex in ˆX, thus ∆ ↔ ˆ∆′. Moreover, ∆ > ˆ∆′ by Lemma 1.2.3.

Case 2. e ̸= max( ˆ∆′). Then vi = max( ˆ∆) = max( ˆ∆′), and vi is the barycenter of both ∆ and ˆ∆′, thus ∆ = ˆ∆′.

3.2.3. Theorem. The following categories

R(X, δ) = S(X, δ)

are isomorphic.

Proof. The functor Φ : Pre Λ(X, δ) → S(X, δ). If S is an object of Pre Λ(X, δ), then T = Φ(S) is constructed as follows:

T( ˆ∆) = S(∆) for ˆ∆ ⊆ −δ∆.

Let ˆ∆′ be a codim 1 face of ˆ∆″ and let ˆ∆′ ⊆ −δ∆′ and ˆ∆″ ⊆ −δ∆″. By Lemma 3.2.2 ∆′ ≥ ∆″. We set:

t( ˆ∆′, ˆ∆″) = s(∆′, ∆″).
Now let $\hat{\Delta}' \subseteq -\delta\Delta'$, $\hat{\Delta}_1 \subseteq -\delta\Delta_1$, $\hat{\Delta}_2 \subseteq -\delta\Delta_2$, $\hat{\Delta}'' \subseteq -\delta\Delta''$ be such a quadruple for which we have to check the equivalence axiom. Then by Lemma 3.2.2 we have:

$$\Delta' \geq \Delta_1 \geq \Delta'', \quad \Delta' \geq \Delta_2 \geq \Delta''.$$ 

By definitions we have:

$$t(\hat{\Delta}_1, \hat{\Delta}'') \circ t(\hat{\Delta}', \hat{\Delta}_1) = s(\Delta_1, \Delta'') \circ s(\Delta', \Delta_1) = \cdots = s(\Delta_2, \Delta'') \circ s(\Delta', \Delta_2) = t(\hat{\Delta}_2, \hat{\Delta}'') \circ t(\hat{\Delta}', \hat{\Delta}_2).$$ 

Let $h : S_1 \to S_2$ be a morphism in $\text{Pre} \Lambda(X, \delta)$. We set:

$$\Phi(h)(\hat{\Delta}) = h(\Delta)$$

for $\hat{\Delta} \subseteq -\delta\Delta$. Let $T_1 = \Phi(S_1)$ and $T_2 = \Phi(S_2)$. Let $\hat{\Delta}'$ is a codim 1 face of $\hat{\Delta}''$ and let $\hat{\Delta}' \subseteq -\delta\Delta'$ and $\hat{\Delta}'' \subseteq -\delta\Delta''$. The commutativity of the diagram

$$\begin{array}{ccc}
T_1(\hat{\Delta}') & \xrightarrow{\Phi(h)(\hat{\Delta}')} & T_2(\hat{\Delta}') \\
t_1(\hat{\Delta}', \hat{\Delta}'') & & t_2(\hat{\Delta}', \hat{\Delta}'') \\
T_1(\hat{\Delta}'') & \xrightarrow{\Phi(h)(\hat{\Delta}'')} & T_2(\hat{\Delta}'')
\end{array}$$

follows from the commutativity of the diagram

$$\begin{array}{ccc}
S_1(\Delta') & \xrightarrow{h(\Delta') \circ \delta} & S_2(\Delta') \\
s_1(\Delta', \Delta'') & & s_2(\Delta', \Delta'') \\
S_1(\Delta'') & \xrightarrow{h(\Delta'')} & S_2(\Delta'').
\end{array}$$

The functor $\Psi : S(X, \delta) \to R(X, \delta)$. If $T$ is an object of $S(X, \delta)$, then $S = \Psi(T)$ is constructed as follows:

$$S(\Delta) = T(\hat{\Delta})$$

for $\hat{\Delta} \subseteq -\delta\Delta$. $S(\Delta)$ is well defined since $T$ is constant along $-\delta\Delta$. Let $\Delta'$ and $\Delta''$ be two incident simplices of $X$ such that $\delta(\Delta') = \delta(\Delta'') + 1$. We have to construct the map $s(\Delta', \Delta'')$. Let $c'$ be the barycenter of $\Delta'$ and $c''$ be the barycenter of $\Delta''$. We set:

$$s(\Delta', \Delta'') = t(\{c'\}, \{c', c''\}).$$
where \( \{c', c''\} \) is a simplex in \( \tilde{X} \).

We have to check the equivalence axiom. Let \( \Delta' > \Delta_b > \Delta'', \delta(\Delta') = \delta(\Delta_b) + 1 = \delta(\Delta'') + 2 \). We will consider two cases.

Case 1. \( \Delta' \) and \( \Delta'' \) are incident. Let \( c', b \) and \( c'' \) be the barycenters of \( \Delta', \Delta_b \) and \( \Delta'' \) respectively. For the purposes of this proof we set \( s(\Delta', \Delta'') = t(\{c', \{c', c''\}) \).

Let us assign special names to the following four simplices of \( \hat{\Delta} \):

- \( \Delta_0 \)
- \( \Delta' \)
- \( \Delta'' \)
- \( \Delta \)

Let \( \psi \) follow from the commutativity of the diagram

\[
\begin{array}{ccc}
S_1(\Delta) & \xrightarrow{\Psi(\Delta)} & S_2(\Delta) \\
S_1(\Delta') & \xrightarrow{\Psi(\Delta')} & S_2(\Delta') \\
S_1(\Delta) & \xrightarrow{\Psi(\Delta)} & S_2(\Delta') \\
S_1(\Delta') & \xrightarrow{\Psi(\Delta')} & S_2(\Delta')
\end{array}
\]

The equivalence axiom follows.

Case 2. \( \Delta' \) and \( \Delta'' \) are not incident. By Lemma 1.2.3 \( \delta(\Delta') = 1, \delta(\Delta'') = -1 \). Then there is exactly one \( \Delta_0 \) such that \( \Delta' \leftrightarrow \Delta_0 \leftrightarrow \Delta'', \delta(\Delta_0) = 0 \), and the equivalence axiom is vacuous.

Let \( h : T_1 \to T_2 \) be a morphism in \( S(X, \delta) \). We set:

\[
\Psi(h)(\Delta) = h(\hat{\Delta}) \text{ for } \hat{\Delta} \subseteq -\delta\Delta.
\]

\( \Psi(h)(\Delta) \) is well defined since \( T_1 \) and \( T_2 \) are constant along \( -\delta\Delta \). Let \( S_1 = \Psi(T_1) \) and \( S_2 = \Psi(T_2) \), and let \( \delta(\Delta) = \delta(\Delta') + 1 \), and \( \Delta \leftrightarrow \Delta' \). The commutativity of the diagram

\[
\begin{array}{ccc}
S_1(\Delta) & \xrightarrow{\Psi(\Delta)} & S_2(\Delta) \\
S_1(\Delta') & \xrightarrow{\Psi(\Delta')} & S_2(\Delta') \\
S_1(\Delta) & \xrightarrow{\Psi(\Delta)} & S_2(\Delta') \\
S_1(\Delta') & \xrightarrow{\Psi(\Delta')} & S_2(\Delta')
\end{array}
\]

follows from the commutativity of the diagram

\[
\begin{array}{ccc}
T_1(\{c\}) & \xrightarrow{h(\{c\})} & T_2(\{c\}) \\
T_1(\{c, c'\}) & \xrightarrow{h(\{c, c'\})} & T_2(\{c, c'\})
\end{array}
\]

Let \( G : \mathcal{R}(X, \delta) \to \text{Pre}\Delta(X, \delta) \) be the isomorphism of 1.2. It follows from our explicit construction that \( \Psi \circ \Phi \circ G = \text{Id} \).
We claim that:

\[ \Phi \circ G \circ \Psi = \text{Id}. \]

Let \( T \in \mathcal{S}(X, \delta) \), and let \( \Phi \circ G \circ \Psi(T) = \bar{T} \). For a simplex \( \hat{\Delta} \) in \( \hat{X} \):

\[ \bar{T}(\hat{\Delta}) = T(\hat{\Delta}). \]

Let \( \hat{\Delta}' \) be a codim 1 face of \( \hat{\Delta}'' \) and \( \hat{\Delta}' \subseteq -^\delta \hat{\Delta}' \) and \( \hat{\Delta}'' \subseteq -^\delta \hat{\Delta}'' \). By Lemma 3.2.2 \( \hat{\Delta}' \leftrightarrow \hat{\Delta}'' \) and \( \hat{\Delta}' \ni \hat{\Delta}'' \). If \( \hat{\Delta}' = \hat{\Delta}'' \), then \( \tilde{i}(\hat{\Delta}', \hat{\Delta}'') = \text{Id} = t(\hat{\Delta}', \hat{\Delta}'') \). If \( \hat{\Delta}' > \hat{\Delta}'' \), then there exists a sequence \( \hat{\Delta}' = \Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_m = \Delta'' \) such that:

1. \( \Delta_i \ni \Delta_{i+1} \ni \Delta_{i+2}, \ 0 \leq i \leq m - 2 \),
2. \( \delta(\Delta_i) = \delta(\Delta_{i+1}) + 1, \ 0 \leq i \leq m - 1 \),
3. \( \Delta' \ni \Delta_i \ni \Delta'' , \ 0 \leq i \leq m \) (Lemma 1.2.3).

Let \( c_i \) be the barycenter of \( \Delta_i \). If \( \mathcal{S} = G \circ \Psi(T) \), then by definitions \( s(\Delta_i, \Delta_{i+1}) = t(\{c_i\}, \{c_i, c_{i+1}\}) \). Then

\[
\tilde{i}(\hat{\Delta}', \hat{\Delta}'') = s(\Delta', \Delta'')
\]

\[
= s(\Delta_m, \Delta_m) \circ \cdots \circ s(\Delta_0, \Delta_1)
\]

\[
= t(\{c_{m-1}\}, \{c_{m-1}, c_m\}) \circ \cdots \circ t(\{c_0\}, \{c_0, c_1\})
\]

We claim that:

\[
t(\{c_0\}, \{c_0, c_m\}) = t(\{c_{m-1}\}, \{c_{m-1}, c_m\}) \circ \cdots \circ t(\{c_0\}, \{c_0, c_1\}) = \tilde{i}(\hat{\Delta}', \hat{\Delta}'').
\]

If \( m = 1 \) then there is nothing to prove. In general the claim is proved by induction. The step of induction is an argument similar to (*). Since \( \hat{\Delta}' \subseteq -^\delta \hat{\Delta}' \) and \( \hat{\Delta}'' \subseteq -^\delta \hat{\Delta}'' \), \( c' = \max \hat{\Delta}' \), and \( c'' = \max \hat{\Delta}'' \) (\( c' \) and \( c'' \) are barycenters of \( \Delta' \) and \( \Delta'' \) respectively). We have:

\[
\{c'\} \subseteq \hat{\Delta}' \subseteq \hat{\Delta}''
\]

\[
\{c'\} \subset \{c', c''\} \subseteq \hat{\Delta}''
\]

Notice that \( c' \subseteq -^\delta \hat{\Delta}' \), \( \{c', c''\} \subseteq -^\delta \hat{\Delta}'' \). Let \( T' \in \mathcal{S}(X, \delta) \subset \text{Pre} \Lambda(\hat{X}) \) be the presheaf corresponding to \( T \in \mathcal{S}(X, \delta) \subset \mathcal{R}(\hat{X}) \). We have:

\[
t(\{c'\}, \{c', c''\}) = t'(\{c'\}, \{c', c''\})
\]

\[
= t'(\{c', c''\}, \hat{\Delta}'') \circ t'(\{c'\}, \{c', c''\})
\]

\[
= t'(\{c'\}, \hat{\Delta}'')
\]

\[
= t'(\{c'\}, \hat{\Delta}')
\]

\[
= t(\hat{\Delta}', \hat{\Delta}'')
\]

Therefore \( \tilde{i}(\hat{\Delta}', \hat{\Delta}'') = t(\hat{\Delta}', \hat{\Delta}'') \) and \( \bar{T} = T \).
3.2.4. Summarizing, we have constructed the equivalence functors:

\[ \mathcal{SH}_c(X, \delta) \xrightarrow{\Upsilon^\sim} \mathcal{S}(X, \delta) = \mathcal{S}(X, \delta) \xrightarrow{\Psi^\sim} \mathcal{R}(X, \delta). \]

Recall that in 1.2 we constructed the isomorphism functor:

\[ \Xi_B : \mathcal{R}(X, \delta) \to \text{mod-}B(X, \delta). \]

Let \( \Theta(X, \delta) : \mathcal{SH}_c(X, \delta) \to \text{mod-}B(X, \delta) \) be the equivalence functor defined as \( \Theta(X, \delta) = \Xi_B \circ \Psi \circ \Upsilon^\sim \). In other words, we have

**Theorem B2.** The functor \( \Theta(X, \delta) : \mathcal{SH}_c(X, \delta) \sim \to \text{mod-}B(X, \delta) \)

is an equivalence of categories.

### 3.3. Injective objects in \( \mathcal{R}(X, -\delta) \).

In this section we list some simple facts concerning injective objects in the category \( \mathcal{R}(X, -\delta) \). The material of this section is a generalization of some of the facts presented in section 2.4.

#### 3.3.1. In this subsection we construct injective objects in the category Pre \( \Lambda(X, -\delta) \).

Our construction generalizes injective simplicial sheaves (see 2.4). Let \( \preceq \) denote the relation in \( \Lambda(X, \delta) \).

**Definition.** Let \( \Delta \subset X \) be a simplex and let \( V \) be a vector space. We define the object \( [\delta \Delta]^V \) as follows:

\[ [\delta \Delta]^V(\Delta') = \begin{cases} V, & \Delta' \preceq \Delta \\ 0, & \Delta' \not\preceq \Delta, \end{cases} \]

where the restriction maps between copies of \( V \) are all identity maps. We will also write \( [\delta \Delta] = [\delta \Delta]^F \). It is easy to see that for any \( \Delta \) and \( V \), \( [\delta \Delta]^V \) is an injective object in \( \text{Pre} \Lambda(X, -\delta) \).

**Lemma.** Let \( S \in \text{Pre} \Lambda(X, -\delta) \), and let \( [\delta \Delta]^V \) be an injective object. Then

1. \( \text{Hom}(S, [\delta \Delta]^V) = \text{Hom}_F(S(\Delta), V) \).

2. In particular, if \( \Delta'' \preceq \Delta' \), then \( \text{Hom}([\delta \Delta'^V, [\delta \Delta'']^W] = \text{Hom}_F(V, W) \).

**Proof.** The proof is left to the reader.

#### 3.3.3. In 1.2 we have seen that \( \text{Pre} \Lambda(X, -\delta) = \mathcal{R}(X, -\delta) \). We transport the injective objects \( [\delta \Delta]^V \) to the category \( \mathcal{R}(X, -\delta) \) preserving the notation.

Let \( F \) be the “constant sheaf” in \( \mathcal{R}(X, -\delta) \), i.e.:

\[ F(\Delta) = F \quad \text{for all} \ \Delta \in X, \]

where the restriction maps between copies of \( F \) are all identity maps.
Definition. Let $S \in R(X, -\delta)$. The vector space $\Gamma(X; S)$ of global sections of $S$ is defined as follows:

$$\Gamma(X; S) = \text{Hom}_{R(X, -\delta)}(F, S).$$

Lemma. Let $\Delta \subset X$ be a simplex and let $V$ be a vector space. Then

$$\Gamma(X; [^0\Delta]^V) = V.$$  

Proof. Indeed, by definitions and Lemma 3.3.2:

$$\Gamma(X; [^0\Delta]^V) := \text{Hom}_{R(X, -\delta)}(F, [^0\Delta]^V) = \text{Hom}_{F}(F, V) = V.$$  

Chapter 4. Koszul duality and perverse sheaves

4.1. Mixed quiver algebras.

4.1.1. Definition. (cf. [BGSo, 4.1]) A quiver $Q$ is called mixed if it is equipped with a function $w : V(Q) \to \mathbb{Z}$ (called a weight) such that for any two vertices $\alpha, \beta \in V(Q)$ there is no arrow $(\alpha \to \beta) \in E(Q)$ if $w(\alpha) \leq w(\beta)$.

Notice that this definition is different from the one given in [Vyb1], where we also assumed that:

(*) For $\alpha, \beta \in V(Q)$ there is no arrow $(\alpha \to \beta) \in E(Q)$ unless $w(\alpha) = w(\beta) + 1$.

In the rest of this section we will consider only mixed quivers satisfying (*).

Note that the algebra $FQ$ has a natural grading:

$$FQ_m = \{\text{vector space spanned by all paths of length } m\}$$

We denote by $FQ_+ = \bigoplus_{m > 0} FQ_m$.

4.1.2. Definition. A mixed quiver algebra $C = FQ/J$ is the quotient of $FQ$ by a homogeneous ideal $J \subset FQ_+$.

A mixed quiver algebra $C$ inherits its grading from $FQ$ since $J$ is a homogeneous ideal. Let $I$ be the set of local idempotents $e(\alpha)$ with $\alpha \in V(Q)$. Since there is a one-to-one correspondence $I = V(Q)$ we may assume that $w$ is a function on $I$, $w : I \to \mathbb{Z}$. We define $I_l := w^{-1}(-l)$.

4.1.3. In this subsection we consider the category $\text{mod-}C$ of left finitely generated modules over a mixed quiver algebra $C$. If $M$ is such a module, $M$ admits a natural (standard) grading:

$$M_l = \sum_{e \in I_l} eM, \quad \text{and} \quad M = \bigoplus_l M_l.$$  

We can shift the standard grading by any integer $k$: $(M(k))_l = M_{l-k}$ ($k$-grading). In this terminology standard grading is 0-grading.

Now let us consider the category $\text{modgr-}C$ of finitely generated graded modules over the graded algebra $C$. If $M$ is an object of $\text{modgr-}C$ and $x \in M_l$ is a homogeneous element of $M$, then $\text{grdeg}(x) = i$ denotes the homogeneous degree of $x$. Note that both $\text{mod-}C$ and $\text{modgr-}C$ are abelian categories since $C$ is Noetherian.
4.1.4. Proposition [Vyb1]. The abelian category $\text{modgr-}C$ is the direct sum:

$$\text{modgr-}C = \bigoplus_{k \in \mathbb{Z}} (\text{mod-}C)_k,$$

where $(\text{mod-}C)_k$ is (isomorphic to) the category $\text{mod-}C$ with $k$-grading.

**Corollary.** The bounded derived category $\mathcal{D}^b(\text{modgr-}C)$ is the direct sum:

$$\mathcal{D}^b(\text{modgr-}C) = \bigoplus_{k \in \mathbb{Z}} \mathcal{D}^b((\text{mod-}C)_k).$$

**Remark.** A similar situation is discussed in [PP].

4.1.5. Let $X$ be a connected finite simplicial complex, and let $\delta$ be a perversity. Notice that the quiver $Q(X, \delta)$ is mixed with weight function $w = \delta$. Hence the algebras $A(X, \delta)$ and $B(X, \delta)$ introduced in 1.2 are mixed quiver algebras. The algebras $A(X, \delta)$ and $B(X, \delta)$ are also quasi-hereditary and of finite global dimension [CPS, Vyb1].

4.2. Koszul duality. In this section we complete the proof of Theorem C, Theorem D1, and Theorem D2. The material is mostly borrowed from [BGSc, BGS0, Mac2, Mac3, Pol, Vyb 1, Vyb2].

4.2.1. Definition [BGS0]. A positively graded ring $C = \bigoplus_{j \geq 0} C_j$ with semisimple $C_0$ is called Koszul if $C_0$ considered as a graded left $C$-module admits a graded projective resolution

$$\ldots \to P^2 \to P^1 \to P^0 \to C_0 \to 0$$

such that $P^i$ is generated by its component of degree $i$, $P^i = CP^i_0$.

**Proposition [BGS0, 2.1.3].** Let $C = \bigoplus_{j \geq 0} C_j$ be a positively graded ring and suppose $C_0$ is semisimple. The following conditions are equivalent:

1. $C$ is Koszul.
2. For any two pure $C$-modules $M$, $N$ of weights $m$, $n$ respectively we have $\text{Ext}^i_{\text{modgr}-C}(M, N) = 0$ unless $i = m - n$.
3. $\text{Ext}^i_{\text{modgr}-C}(C_0, C_0(n)) = 0$ unless $i = n$.

4.2.2. Until the end of this section $X$ is a finite connected simplicial complex, and $\delta$ is a perversity.

**Lemma.** The following conditions are equivalent:

1. $B(X, \delta)$ is Koszul.
2. For any two simple $B(X, \delta)$-modules $S_e$ and $S_{e'}$ we have $\text{Ext}^i_{B(X, \delta)}(S_e, S_{e'}) = 0$ unless $i = \delta(e) - \delta(e')$. 
There is an obvious analogous statement for $A(X, \delta)$.

**Proof.** By Corollary 4.1.4 we have

$$\text{Ext}^i_{\text{modgr-} B(X, \delta)}(S_e, S_{e'}) = \text{Ext}^i_{B(X, \delta)}(S_e, S_{e'}),$$

where we consider $S_e$ and $S_{e'}$ as graded modules equipped with standard grading. Then it is easy to see that the condition (2) is equivalent to the condition (2) of Proposition 4.2.1.

**4.2.3. Lemma.**

1. The (left) quadratic dual of the quadratic algebra $A(X, \delta)$ is isomorphic to $B(X, -\delta)$, $A(X, \delta) \cong B(X, -\delta)$.
2. The (left) quadratic dual of the quadratic algebra $B(X, \delta)$ is isomorphic to $A(X, -\delta)$, $B(X, \delta) \cong A(X, -\delta)$.

**Proof.** The proof is straightforward since the chain complex relations, and the equivalence relations are quadratic dual to each other.

**4.2.4.** Let $\mathcal{M}_c(X, \delta)$ be the category of constructible perverse sheaves, and let $S_\Delta \in \mathcal{M}_c(X, \delta)$ be the simple perverse sheaf associated to a simplex $\Delta$ (see 1.3). Lemma A and Lemma C below are borrowed from [Pol].

**Lemma A [Pol, 1.2].** Let $\Delta$ and $\Delta'$ be two simplices in $X$. Then

$$\text{Ext}^i_{D^b_c(X)}(S_\Delta, S_{\Delta'}) = \begin{cases} F, & \Delta \geq \Delta' \text{ and } i = \delta(\Delta) - \delta(\Delta') \\ 0, & \text{otherwise.} \end{cases}$$

**Lemma B.** The algebra $B(X)$ is Koszul.

**Proof.** The following categories $D^b_c(X) \simeq D^b(S\mathcal{H}_c(X)) \simeq D^b(\text{mod-} B(X))$ are equivalent (see 1.3 and 2.2). Let $e$ and $e'$ be the local idempotents of $B(X)$ corresponding to simplices $\Delta$ and $\Delta'$. Then

$$\text{Ext}^i_{B(X)}(S_e, S_{e'}) = \text{Ext}^i_{D^b_c(X)}(S_\Delta, S_{\Delta'}) = \text{Ext}^i_{D^b_c(X)}(S_\Delta, S_{\Delta'}) = 0$$

unless $i = \delta(\Delta) - \delta(\Delta') = \dim \Delta' - \dim \Delta = \delta(e) - \delta(e')$. (In this case $\delta = \text{bottom perversity.}$) Then $B(X)$ is Koszul by Lemma 4.2.2.

**Lemma C [Pol, 2].** The following two algebras

$$\bigoplus_{\Delta, \Delta'} \text{Ext}^\ast_{D^b_c(X)}(S_\Delta, S_{\Delta'}) = B(X, \delta)$$

are canonically isomorphic.
4.2.5. In this subsection we sketch some material mostly borrowed from [BGSo, Pol, Vyb2]. Theorem C below (in a slightly different form) is due to R. MacPherson [Mac2, Mac3].

Any object $S \in \mathcal{M}_c(X, \delta)$ has a canonical finite increasing filtration $W_\bullet = W_\bullet S$ such that $\text{gr}^W_i S = W_i S / W_{i-1} S$ is a direct sum of simple objects $S_\Delta$ with $\delta(\Delta) = i$ (see [BBD, 5.3.6], [BGSo, 4.1.2], [Pol, 4.1]). Let us define a functor:

$$
\Sigma^i(X, \delta) : \mathcal{M}_c(X, \delta) \to \text{Vect}_{F},
$$

$$
\Sigma^i(X, \delta)(S) = \text{Hom}_{\mathcal{M}_c(X, \delta)}(\bigoplus \Delta S_\Delta, \text{gr}^W_i S).
$$

The functor

$$
\Sigma(X, \delta) = \bigoplus_i \Sigma^i(X, \delta) : \mathcal{M}_c(X, \delta) \to \text{Vect}_{\text{gr} F},
$$

$$
\Sigma(X, \delta)(S)_i = \Sigma^i(X, \delta)(S)
$$

assigns a graded vector space to each $S \in \mathcal{M}_c(X, \delta)$. Let us also define:

$$
\tilde{A}_i = \bigoplus_a \text{Hom}(\Sigma^a(X, \delta), \Sigma^{a+i}(X, \delta)).
$$

The composition of morphisms defines a product $\tilde{A}_i \times \tilde{A}_j \to \tilde{A}_{i+j}$. Note that $\tilde{A}_0 = \text{End}(\bigoplus \Delta S_\Delta)$. The graded algebra $\tilde{A}$ acts canonically on the graded vector space $\Sigma(X, \delta)(S)$ for any $S \in \mathcal{M}_c(X, \delta)$. One can show that the functor $\Sigma(X, \delta) : \mathcal{M}_c(X, \delta) \to \text{mod-} \tilde{A}$ is an equivalence of categories, and that the algebras, $\tilde{A} = B(X, -\delta)_! = A(X, \delta)$ are isomorphic (see [Pol, 4]). In other words, we have

**Theorem C.** The functor

$$
\Sigma(X, \delta) : \mathcal{M}_c(X, \delta) \sim \to \text{mod-} A(X, \delta)
$$

is an equivalence of categories.

4.2.6. The following theorem is borrowed from [Pol]. The statement (3) was obtained by R. MacPherson [Mac3].

**Theorem.**

1. The algebra $B(X, \delta)$ is Koszul.
2. The algebra $A(X, \delta)$ is Koszul.
3. The following triangulated categories

$$
\mathcal{D}^b(\mathcal{M}_c(X, \delta)) \simeq \mathcal{D}^b_{c}(X)
$$

are equivalent.

**Remark.** A. Polishchuk shows in [Pol] that (3) follows from (2). Using Lemma 4.2.2, Lemma 4.2.4.A, and Theorem C, it is also possible to show that (2) follows from (3).
4.2.7. Here we adapt the construction of the Koszul duality functors from [BGS0, 2.12] to our situation. We denote $k = A(X, δ)_0 = B(X, −δ)_0$.

The functor $K : C^b(\text{modgr}-A(X, δ)) \to C^b(\text{modgr}-B(X, −δ))$ is constructed as follows. Let $M = M^• \in C^b(\text{modgr}-A(X, δ))$. Then

$$KM = B(X, −δ) \otimes_k M,$$

$$(KM)^p_q = \bigoplus_{p=i+j, q=l−j} B(X, −δ)_l \otimes_k M^i_j.$$

Similarly, the functor $K' : C^b(\text{modgr}-A(X, δ)) \to C^b(\text{modgr}-B(X, −δ))$ is constructed as follows. Let $N = N^• \in C^b(\text{modgr}-A(X, δ))$. Then

$$K'N = \text{Hom}_k(B(X, −δ), M),$$

$$(K'N)^p_q = \bigoplus_{p=i+j, q=l−j} \text{Hom}_k(B(X, −δ)_l, M^i_j).$$

The differentials of $KM^•$ and $K'N^•$ are defined in [BGS0, 2.12.1]. Since the functors $K$ and $K'$ are exact they induce the functors $D^b(K), D^b(K') : D^b(\text{modgr}-A(X, δ)) \to D^b(\text{modgr}-B(X, −δ))$.

Theorem [BGS0, 2.12.6].

(1) The functor $D^b(K) : D^b(\text{modgr}-A(X, δ)) \sim \to D^b(\text{modgr}-B(X, −δ))$

is an equivalence of triangulated categories.

(2) The functor $D^b(K') : D^b(\text{modgr}-A(X, δ)) \sim \to D^b(\text{modgr}-B(X, −δ))$

is an equivalence of triangulated categories.

Recall that by Corollary 4.1.4:

$$D^b(\text{modgr}-A(X, δ)) = \bigoplus_{m \in \mathbb{Z}} D^b((\text{mod}-A(X, δ))_m),$$

$$D^b(\text{modgr}-B(X, δ)) = \bigoplus_{m \in \mathbb{Z}} D^b((\text{mod}-B(X, δ))_m).$$

Lemma.

(1) Let $M \in D^b((\text{mod}-A(X, δ))_0)$. Then $D^b(K)(M) \in D^b((\text{mod}-B(X, −δ))_0)$.

(2) Let $N \in D^b((\text{mod}-A(X, δ))_0)$. Then $D^b(K')(N) \in D^b((\text{mod}-B(X, −δ))_0)$.

Proof. The proof is left to the reader.
Corollary. The functors
\[ D^b(K), D^b(K') : D^b(\text{mod-}A(X, \delta)) \xrightarrow{\sim} D^b(\text{mod-}B(X, -\delta)) \]
are equivalences of triangulated categories.

4.2.8. The category \( D^b(\text{mod-}A(X, \delta)) \) has the standard \( t \)-structure with the core \( \text{mod-}A(X, \delta) \). The Koszul duality functors \( K \) and \( K' \) transform this standard \( t \)-structure to non-standard \( t \)-structures in \( D^b(\text{mod-}B(X, -\delta)) \) (see [BGSo, 2.13]). We will give a description of the cores of these non-standard \( t \)-structures as follows.

Lemma. (cf. [BGSo, 2.13.3])
\begin{enumerate}
\item Let \( M \in \text{mod-}A(X, \delta) \). Then \( KM \in D^b(\text{mod-}B(X, -\delta)) \) is a complex of projective \( B(X, -\delta) \)-modules
\[ \ldots \rightarrow P^i \rightarrow P^{i+1} \rightarrow \ldots \]
such that for any \( i \), \( P^i \) is generated by its component of degree \( -i \), \( P^i = B(X, -\delta)P^i_{-i} \).
\item Let \( N \in \text{mod-}A(X, \delta) \). Then \( K'N \in D^b(\text{mod-}B(X, -\delta)) \) is a complex of injective \( B(X, -\delta) \)-modules
\[ \ldots \rightarrow I^i \rightarrow I^{i+1} \rightarrow \ldots \]
such that for any \( i \), \( I^i \) is cogenerated by its component of degree \( -i \), \( I^i = \text{Hom}_k(B(X, -\delta), I^i_{-i}) \).
\end{enumerate}

Proof.
\begin{enumerate}
\item By construction we have \( P^j = B(X, -\delta) \otimes_k M_j \), which is a projective module generated by \( M_j \). (\( M_j \) is considered as a semisimple \( B(X, -\delta) \)-module concentrated in degree \( -j \).)
\item By construction we have \( I^j = \text{Hom}_k(B(X, -\delta), N_j) \), which is an injective module cogenerated by \( N_j \).
\end{enumerate}

4.2.9. Since the categories \( \text{mod-}A(X, \delta) = \mathcal{P}(X, \delta) \) (resp. \( \text{mod-}B(X, \delta) = \mathcal{R}(X, \delta) \)) are isomorphic, we can transport the Koszul duality functor \( K' : C^b(\text{mod-}A(X, \delta)) \rightarrow C^b(\text{mod-}B(X, -\delta)) \) to the functor \( K' : C^b(\mathcal{P}(X, \delta)) \rightarrow C^b(\mathcal{R}(X, -\delta)) \) (cf. 4.3). Let \( S \in \mathcal{P}(X, \delta) \), and let us consider \( \mathcal{P}(X, \delta) \) as the core of the standard \( t \)-structure on \( D^b(\mathcal{P}(X, \delta)) \). We will describe \( K'S \in C^b(\mathcal{R}(X, -\delta)) \) explicitly as follows:
\[ K'S^i = \bigoplus_{\delta(\Delta) = -i} [\delta \Delta]^{S(\Delta)} . \]
The differential
\[ K'S^i \xrightarrow{d^i} K'S^{i+1} \]
\[ \bigoplus_{\delta(\Delta) = -i} [\delta \Delta]^{S(\Delta)} \xrightarrow{d^i} \bigoplus_{\delta(\Delta) = -i-1} [\delta \Delta]^{S(\Delta)} \]
is the zero map between components \([\delta \Delta]^{S(\Delta)}\) and \([\delta \Delta']^{S(\Delta')}\) if \(\Delta' \not\leq \Delta\), and is induced by \(s(\Delta, \Delta') : S(\Delta) \to S(\Delta')\) if \(\Delta' < \Delta\), and \(\delta(\Delta) = \delta(\Delta') + 1\). (Here \(\leq\) refers to the relation in \(\Lambda(X, \delta)\).)

Let us consider the complex \(\Gamma(X; (K'\mathcal{S})^\bullet)\) of global sections (see 3.3) of \((K'\mathcal{S})^\bullet\):

\[
\Gamma(X; K'\mathcal{S}^i) = \bigoplus_{\delta(\Delta) = -i} S(\Delta)
\]

The differential \(d_i : \Gamma(X; K'\mathcal{S}^i) \to \Gamma(X; K'\mathcal{S}^{i+1})\) is the zero map between \(S(\Delta)\) and \(S(\Delta')\) if \(\Delta' \not\leq \Delta\), and is given by \(s(\Delta, \Delta') : S(\Delta) \to S(\Delta')\) if \(\Delta' < \Delta\), and \(\delta(\Delta) = \delta(\Delta') + 1\). We will denote the cohomology spaces of this complex by \(H^i(X; K'\mathcal{S})\).

### 4.2.10. Let \(S \in \mathcal{P}(X, \delta)\). The total chain complex \(C^\bullet(X; S)\) associated to \(S\) is constructed as follows [Mac2]:

\[
C_i(X; S) = \bigoplus_{\delta(\Delta) = i} S(\Delta).
\]

The differential \(d_i : C_i(X; S) \to C_{i-1}(X; S)\) is the map whose matrix elements are \(s(\Delta, \Delta') : S(\Delta) \to S(\Delta')\) if \(\Delta' < \Delta\) and \(\delta(\Delta) = \delta(\Delta') + 1\), and zero otherwise. The homology spaces of \(C^\bullet(X; S)\) are denoted by \(H_i(X; S)\).

**Lemma.** Let \(S \in \mathcal{P}(X, \delta)\). Then the complexes

\[(C_i(X; S), d_i) = (\Gamma(X; K'\mathcal{S}^{-i}), d^{-i})\]

coincide. Hence:

\[H_i(X; S) = H^{-i}(X; K'\mathcal{S}).\]

**Proof.** This is clear by inspection.

### 4.2.11. We have constructed the equivalence functors:

\[
\begin{array}{ccc}
\mathcal{D}^b(\mathcal{M}_c(X, \delta)) & \mathcal{D}^b(\mathcal{SH}(\mathcal{T}(X, \delta))) \\
\mathcal{D}^b(\Sigma(X, \delta)) & \mathcal{D}^b(\Omega(X, -\delta))
\end{array}
\]

\[
\begin{array}{c}
\mathcal{D}^b(\text{mod-}A(X, \delta)) \\
\mathcal{D}^b(\text{mod-}B(X, \delta))
\end{array}
\]

\[
\begin{array}{c}
\mathcal{D}^b(K) \\
\mathcal{D}^b(\Omega(X, -\delta))^{-1} \circ \mathcal{D}^b(K) \circ \mathcal{D}^b(\Sigma(X, \delta))
\end{array}
\]

Let \(\tilde{K}_1 : \mathcal{D}^b(\mathcal{M}_c(X, \delta)) \to \mathcal{D}^b(\mathcal{SH}((\mathcal{T}(X, -\delta))))\) be the equivalence functor defined as \(\tilde{K}_1 = \mathcal{D}^b(\Omega(X, -\delta))^{-1} \circ \mathcal{D}^b(K) \circ \mathcal{D}^b(\Sigma(X, \delta))\). In other words, we have
**Theorem D1.** The functor

\[ \tilde{K}_1 : D^b(M_c(X, \delta)) \sim \to D^b(SH(T(X, -\delta))) \]

is an equivalence of triangulated categories.

Similarly, we have:

\[ D^b(M_c(X, \delta)) \rightarrow D^b(SH_c(X, -\delta)) \]
\[ D^b(\Theta(X, -\delta)) \rightarrow D^b(\Theta(X, -\delta)) \]

Let \( \tilde{K}_2 : D^b(M_c(X, \delta)) \rightarrow D^b(SH_c(X, -\delta)) \) be the equivalence functor defined as \( \tilde{K}_2 = D^b(\Theta(X, -\delta))^{-1} \circ D^b(K) \circ D^b(\Sigma(X, \delta)) \). In other words, we have

**Theorem D2.** The functor

\[ \tilde{K}_2 : D^b(M_c(X, \delta)) \sim \to D^b(SH_c(X, -\delta)) \]

is an equivalence of triangulated categories.

Clearly, one could also state a version of Theorem D1 and Theorem D2 for the functor \( K' \) simply by substituting \( K' \) instead of \( K \) everywhere. The corresponding equivalence functors are denoted by \( \tilde{K}'_1 \) and \( \tilde{K}'_2 \) respectively. The latter functor is used in the following section.

**4.2.12. A corollary of the Theorem D2: perverse sheaves as complexes of sheaves with allowable support.**

Let us fix a perversity \( p \) as in [GM1], and the associated cellular perversity \( \delta \).

**Definition [GM1].**

1. A pseudomanifold of dimension \( n \) is a compact space \( X \) such that \( X \) is the closure of the union of the \( n \)-simplices in any triangulation of \( X \), and each \( n-1 \) simplex is a face of exactly two \( n \)-simplices.
2. A stratification of a pseudomanifold is a filtration by closed subspaces

\[ X = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset \ldots \supset X_0 \]

satisfying certain conditions (see [GM1, 1.1]). We can represent \( X \) as a disjoint union \( X = \bigsqcup_{S \in S} S \) where the strata \( S \) are connected components of \( X_i - X_{i-1} \).
3. If \( i \) is an integer, a subspace \( Y \subset X \) is called \( (\mathcal{P}, i) \)-allowable if \( \dim(Y) \leq i \) and \( \dim(Y \cap X_{n-k}) \leq i - k + \overline{p}(k) \) for all \( k \geq 2 \).
Let $\mathcal{T}$ be a triangulation of a pseudomanifold $X$ refining a stratification $S$.

Let $\mathcal{M}_S(X, \delta)$ be category of perverse sheaves constructible with respect to the stratification $S$. Let $\mathcal{M}_c(X, \delta)$ be our usual category of perverse sheaves constructible with respect to the triangulation $\mathcal{T}$.

The refinement functor (see 1.3.9) provides the embedding $\text{ref} : \mathcal{M}_S(X, \delta) \hookrightarrow \mathcal{M}_c(X, \delta)$. Let us take $A \in \mathcal{M}_S(X, \delta)$ and consider $\tilde{K}_2'(\text{ref}A)$ as the complex of sheaves constant along $\delta$-perverse simplices.

**Corollary D3.** Let $A \in \mathcal{M}_S(X, \delta)$. For $i \in \mathbb{Z}$, $-\overline{p}(n) \leq i \leq n - \overline{p}(n)$, let $j = n - \overline{p}(n) - i$, and let $\overline{q}$ be another perversity defined by $\overline{q}(k) = \overline{p}(n) - \overline{p}(n - k)$. Then $\tilde{K}_2'(\text{ref}A)$ is (quasi-isomorphic to) a complex of sheaves

$$\ldots \rightarrow B^{i-1} \rightarrow B^i \rightarrow B^{i+1} \rightarrow \ldots$$

on $X$ such that $\text{supp} B^i$ is $(\overline{q}, j)$-allowable with respect to the stratification $S$.

**Proof.** We will show that in fact the claim is true for any $A \in \mathcal{M}_c(X, \delta)$. It is clear from our construction (in particular 4.2.8.(2)) that $\text{supp} B^i \subset X_{\delta-i}$, and $\text{supp} B^i$ is a union of perverse simplices, and therefore a union of simplices of the first barycentric subdivision of $\mathcal{T}$. By Lemma 1.1.9 we have $Q^\overline{q}_j = X_{\overline{p}(n) - n + j}$. Then $Q^\overline{q}_{j=n-\overline{p}(n)-i} = X_{\delta-i}$. Since $Q^\overline{q}_j$ is $(\overline{q}, j)$ allowable with respect to the stratification $S$ [GM1, 3.2], and $\text{supp} B^i \subset Q^\overline{q}_j$ we conclude that $\dim \text{supp} B^i \leq j$ and $\dim (\text{supp} B^i \cap X_{n-k}) \leq j - k + \overline{q}(k)$, i.e. $\text{supp} B^i$ is $(\overline{q}, j)$-allowable.

**Remark.** By the above Corollary the category of stratification-constructible perverse sheaves $\mathcal{M}_S(X, \delta)$ is embedded by the duality functor into the category of complexes of sheaves with the allowable support. One may ask how to characterize $S$-constructible perverse sheaves inside this larger category. This very interesting question will be addressed in [GMMV].

### 4.3. Verdier and Koszul duality for simplicial sheaves and cosheaves.

In this section we study the relationship between the two dualities. $X$ is assumed to be a finite connected simplicial complex.

#### 4.3.1.
Recall that $B(X) = B(X, \text{bottom perversity})$, and let us denote $A'(X) = A(X, \text{top perversity})$, and $k = A'(X)_0 = B(X)_0$. In 4.2 we have seen that $A'(X)$ is the (left) quadratic dual of $B(X)$, $A'(X) = B(X)^!$. We have also seen that $A'(X)$ and $B(X)$ are Koszul algebras. We will consider the quasi-inverse Koszul duality functors (which we denote by $L$ and $L^{-1}$ in this section) for $A'(X)$ and $B(X)$, from a point of view somewhat different from 4.2. The rest of this subsection is essentially lifted from [BGSo, 2.12].

The functor $L : \mathcal{C}^b(\text{modgr-}B(X)) \to \mathcal{C}^b(\text{modgr-}A'(X))$ is defined as follows. Let $M^\bullet \in \text{modgr-}B(X)$. Then

$$LM^\bullet = A'(X) \otimes_k M^\bullet.$$
The differential of $LM^\bullet$ is (up to signs) the sum of the differential coming from the complex $M^\bullet$ and the Koszul complex, see [BGSo, 2.12.1].

The functor $L^{-1} : C^b(\text{modgr-}A'(X)) \to C^b(\text{modgr-}B(X))$ is defined as follows. Let $N^\bullet \in C^b(\text{modgr-}A'(X))$. We define a bicomplex:

\[(*) \quad (L^{-1}N)^{i,j} = \text{Hom}_k(B(X), N^j_i).\]

The differentials of this bicomplex are constructed in [BGSo, 2.12.1] (also see above).

The diagonal complex of the bicomplex $(*)$ is $L^{-1}N^\bullet$.

If $M^\bullet$ is equipped with the standard grading, then $LM^\bullet$ is also equipped with the standard grading, and likewise, if $N^\bullet$ is equipped with the standard grading, then $L^{-1}N^\bullet$ is also equipped with the standard grading (cf. 4.2). Hence, the functors $L$ and $L^{-1}$ induce the functors:

\[D^b(L) : D^b(\text{modgr-}B(X)) \to D^b(\text{modgr-}A'(X)) \quad \text{and} \quad D^b(L^{-1}) : D^b(\text{modgr-}A'(X)) \to D^b(\text{modgr-}B(X))\]

which are quasi-inverse equivalences of triangulated categories.

4.3.2. Recall that $R(X) = R(X, \text{bottom perversity}), P'(X) = P(X, \text{top perversity})$. We will construct a functor $L^{-1} : C^b(P'(X)) \to C^b(R(X))$. Let $T^\bullet \in C^b(P'(X))$. We define a bicomplex $U^{\bullet\bullet}$ as follows:

\[(*) \quad U^{i,j} = \bigoplus_{\text{dim } \Delta = -j} [\Delta]^{T^i(\Delta)}.\]

The differential

\[U^{i,j} \xrightarrow{d_{i,j}} U^{i,j+1}\]

is the zero map between components $[\Delta]^{T^i(\Delta)}$ and $[\Delta']^{T^i(\Delta')}$ if $\Delta' \nsubseteq \Delta$, and is induced by $(-1)^it^i(\Delta, \Delta') : T^i(\Delta) \to T^i(\Delta')$ if $\Delta'$ is a codim 1 face of $\Delta$.

The differential

\[U^{i,j} \xrightarrow{d_{i,j}^I} U^{i+1,j}\]

is the sum of the maps $[\Delta]^{T^i(\Delta)} \to [\Delta]^{T^i+1(\Delta)}$ induced by $d_T^i(\Delta) : T^i(\Delta) \to T^{i+1}(\Delta)$.

The diagonal complex of the bicomplex $(*)$ is $(L^{-1}T)^\bullet$ with the differential $d = d_I + d_{II}$.

4.3.3. Lemma. The following functorial diagram

$$
\begin{array}{ccc}
C^b(P'(X)) & \xrightarrow{L^{-1}} & C^b(R(X)) \\
\downarrow C^b(\Xi_{A'}(X)) & & \downarrow C^b(\Xi_{B}(X)) \\
C^b(\text{modgr-}A'(X)) & \xrightarrow{L^{-1}} & C^b(\text{modgr-}B(X))
\end{array}
$$
commutes. (Here $\Xi_A(X) = \Xi_A(X, \text{top perversity}).$)

**Proof.** The proof follows easily from the fact that:

$$\text{Hom}_k(B(X), N^i_j) = \bigoplus_{e \in I_{-j}} \mathcal{I}(eN^i)$$

where $\mathcal{I}(eN^i)$ is the sum of dim $eN^i$ copies of the indecomposable injective module $\text{Hom}_k(B(X), \mathbb{F}e)$.

**Corollary.** The functor $L^{-1} : \mathcal{C}^b(P'(X)) \to \mathcal{C}^b(\mathcal{R}(X))$ induces a functor $D^b(L^{-1}) : \mathcal{D}^b(P'(X)) \to \mathcal{D}^b(\mathcal{R}(X))$.

Let us also define a functor $L : \mathcal{C}^b(\mathcal{R}(X)) \to \mathcal{C}^b(P'(X))$ as follows:

$$L = \mathcal{C}^b(\Xi_A(X))^{-1} \circ L \circ \mathcal{C}^b(\Xi_B(X)).$$

**4.3.4.** Let $D : \mathcal{C}^b(\mathcal{R}(X)) \to \mathcal{C}^b(\mathcal{R}(X))$ be the Verdier duality functor defined in 2.4. Recall that in 2.4. we also defined the functors $*$ and $\top$.

**Lemma.** The following two functors

$$D \simeq L^{-1} \circ \mathcal{C}^b(\top) \circ *$$

are isomorphic.

**Proof.** Let $(S^\bullet, d^\bullet_S) \in \mathcal{C}^b(\mathcal{R}(X))$. Let $T^\bullet = (\mathcal{C}^b(\top) \circ *)(S^\bullet)^\bullet$, and let $U^{\bullet\bullet}$ be a bicomplex defined as in 4.3.2. Let us also define a bicomplex $V^{i,j} := DS^{-i,j}$ (see 2.4 for the definition of $DS^{\bullet\bullet}$). Then the diagonal complex of $V^{\bullet\bullet}$ is precisely $DS^\bullet$.

Observe that $V^{i,j} = U^{i,j}$, and $d^{i,j}_{U,I} = (-1)^i d^{i,j}_{V,I}$, and $(-1)^j d^{i,j}_{U,II} = d^{i,j}_{V,II}$.

Let $q^{i,j} : V^{i,j} \to U^{i,j}$ be defined as $q^{i,j} = (-1)^{ij} \text{Id}$. Then $q^{\bullet\bullet}$ is an isomorphism of bicomplexes inducing the isomorphism of the corresponding diagonal complexes.

**4.3.5. Corollary.** The following two functors

$$D^b(D) \simeq D^b(L^{-1}) \circ D^b(\top) \circ D^b(\bullet)$$

are isomorphic.

**4.3.6.** Let us simplify the notation for the purposes of this subsection. We denote $D = D^b(D)$, $L = D^b(L)$, $L^{-1} = D^b(L^{-1})$, $\top = D^b(\top)$, and $\bullet = D^b(\bullet)$. We also denote $\tilde{D} = \top \circ \bullet \circ L^{-1}$. We call $\tilde{D}$ the “Verdier duality for cosheaves.”

**Theorem E.** The following functorial diagram

$$\mathcal{D}^b(\mathcal{R}(X)) \xrightarrow{D} \mathcal{D}^b(\mathcal{R}(X))$$

$$\downarrow L \quad \downarrow L$$

$$\mathcal{D}^b(P'(X)) \xrightarrow{\tilde{D}} \mathcal{D}^b(P'(X))$$

commutes. In other words, $L \circ D \simeq \tilde{D} \circ L$.

**Proof.** By Corollary 4.3.5:

$$L \circ D \simeq L \circ L^{-1} \circ \top \circ \bullet \simeq \top \circ \bullet \simeq \top \circ \bullet \circ L^{-1} \circ L \simeq \tilde{D} \circ L.$$
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