Two upper bounds for the Erdős-Hooley Delta-function

On the 50th Anniversary of Chen’s Theorem

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Abstract

For integer \(n \geq 1\) and real \(u\), let \(\Delta(n, u) := |\{d : d \mid n, e^u < d \leq e^{u+1}\}|\). The Erdős-Hooley Delta-function is then defined by \(\Delta(n) := \max_{u \in \mathbb{R}} \Delta(n, u)\). We improve the current upper bounds for the average and normal orders of this arithmetic function.

Keywords

concentration of divisors, average order, normal order, Waring’s problem

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1 Introduction and statement of results

For integer \(n \geq 1\) and real \(u\), put

\[\Delta(n, u) := |\{d : d \mid n, e^u < d \leq e^{u+1}\}|, \quad \Delta(n) := \max_{u \in \mathbb{R}} \Delta(n, u).\]

Introduced by Erdős [1] (see also [2]) and studied by Hooley [6], the \(\Delta\)-function has proved very useful in several branches of number theory—see, e.g., [5] and [13] for further references. If \(\tau(n)\) denotes the total number of divisors of \(n\), then \(\Delta(n)/\tau(n)\) coincides with the concentration of the numbers \(\log d, d \mid n\).

In this work, we aim at improving the current upper bounds for the average and normal orders. In the former case, we consider weighted versions.

For \(A > 0, y \geq 1, c > 0\) and \(\eta \in [0, 1]\), we define the class \(\mathcal{M}_A(y, c, \eta)\) comprising those arithmetic functions \(g\) that are multiplicative, non-negative, and satisfy the conditions

\[g(p^\nu) \leq A^\nu \quad (\nu \geq 1),\]

\[\forall \varepsilon > 0 \quad g(n) \ll_{\varepsilon} n^{\varepsilon} \quad (n \geq 1),\]

\[\sum_{p \leq x} g(p) = y\log(x) + O(xe^{-c(\log x)^\eta}) \quad (x \geq 2).\]

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Here and in the sequel we reserve the letter $p$ to designate a prime number. Note for the sake of further reference that by a theorem of Shiu [11, Theorem 1], we have

$$\sum_{n \leq x} g(n) \ll x (\log x)^{y-1}$$

(1.4)

for any $g$ in $\mathcal{M}_A(y, c, \eta)$.

Regarding average values of the $\Delta$-function, we consider the weighted sum

$$S(x; g) := \sum_{n \leq x} g(n) \Delta(n).$$

Here and throughout, we let $\log_k$ denote the $k$-fold iterated logarithm.

**Theorem 1.1.** Let $A > 0$, $y \geq 1$, $c > 0$, $\eta \in ]0, 1[$, $g \in \mathcal{M}_A(y, c, \eta)$ and $a > \sqrt{2}\log 2 \approx 0.980258$. We have

$$S(x; g) \ll x (\log x)^{2y-2} a^{\log \log x} \quad (x \geq 3).$$

(1.5)

We note that by a different approach Koukoulopoulos (private communication) obtained a similar estimate for $g = 1$ with $a = 2.1$.

When $y \geq 1$, Theorem 1.1 provides a small improvement over known estimates, for example [5, Theorem 70] stating that, for any $\varepsilon > 0$,

$$S(x; 1) \ll xe^{(1+\varepsilon)\sqrt{2\log_2 x \log_3 x}} \quad (x \to \infty).$$

(1.6)

When $y \leq \frac{1}{2}$, the proof of [5, Theorem 64] may be readily generalized to any $g \in \mathcal{M}_A(y, c, \eta)$ to yield

$$S(x; g) \ll x (\log x)^{y-1} (\log_2 x)^{\delta(y,1/2)},$$

(1.7)

where $\delta(u,v) := 1$ if $u = v$, and $\delta(u,v) := 0$ otherwise.

For $y \geq 1 + \frac{5}{4} \sqrt{2}$, we may adapt mutatis mutandis [5, Theorem 67] and derive

$$S(x; g) \ll x (\log x)^{2y-2} (\log_2 x)^{\delta(y,1+\sqrt{2}/2)}.$$  

(1.8)

Finally, we note that the proof of [5, Theorem 71] may be extended to get, for any fixed $y < 1$,

$$S(x; g) \ll x (\log x)^{y-1} \exp \left\{ \frac{4\log 3}{1-y} (\log_3 x)^2 \right\}.$$  

We omit further details since the relevant approaches are straightforward.

As put forward by Hooley [6], among other applications, average bounds such as (1.5) may be employed to count solutions of certain Diophantine equations. For given $k \in \mathbb{N}^*$ and positive integers $c_j$, $\ell_j$ ($0 \leq j \leq k$) with $\ell_0 = \min \ell_j = 2$, we consider as in [10] the number $N(k)$ of solutions $(m, n) = (m_0, \ldots, m_k, n_0, \ldots, n_k) \in \mathbb{N}^{2k+2}$ of the system

$$\sum_{0 \leq j \leq k} c_j m_j^{\ell_j} = \sum_{0 \leq j \leq k} c_j n_j^{\ell_j} \leq x, \quad m_0 \neq n_0,$$

and let $V(x)$ denote the number of integers $n \leq x$ that are representable in the form

$$n = \sum_{0 \leq j \leq k} c_j n_j^{\ell_j}.$$

The case $k = 2$, $c_0 = c_1 = c_2 = 1$ and $\ell_1 = \ell_2 = 4$ has been studied by the second author [13].

Applying Theorem 1.1 with $y = 1$ and following the approach displayed in [10], we derive the next corollary.
Corollary 1.2. Assume $\sum_{1 \leq j \leq k} 1/\ell_j = \frac{1}{2}$ and let $a > \sqrt{2}\log 2$. Then, as $x \to \infty$, we have
\[
N(x) \ll x^{a\sqrt{\log_2 x}}, \quad (1.9)
\]
\[
V(x) \gg x^{-a\sqrt{\log_2 x}}. \quad (1.10)
\]
We omit the details of the proof, since they are almost identical to those in [10].

We next turn our attention to the normal order of the $\Delta$-function. We employ the mention pp to indicate that a formula holds on a sequence of natural density 1. Improving on estimates of Maier and Tenenbaum [7, 9], Ford, Green and Koukoulopoulos [3] recently claimed
\[
\Delta(n) > (\log_2 n)^{\gamma_1} \quad \text{pp}
\]
for any $\gamma_1 < 0.35332$. Regarding upper bounds, Maier and Tenenbaum [8, 9] proved that, given any $\gamma_2 > \log 2 \approx 0.693147$, we have
\[
\Delta(n) \leq (\log_2 n)^{\gamma_2} \quad \text{pp}. \quad (1.11)
\]
We are now able to improve on this result by reducing the exponent further.

Theorem 1.3. Let $\gamma_3 > (\log 2)/(\log 2 + 1/\log 2 - 1) \approx 0.6102495$. We have
\[
\Delta(n) \leq (\log_2 n)^{\gamma_3} \quad \text{pp}.
\]

2 Average order: Proof of Theorem 1.1

2.1 Reductions

Let us start with a technical reduction similar to that of the proof of [12, (25)] and enabling to substitute the evaluation of a logarithmic mean to that of a Cesàro mean. Considering the inequality (see [5, Lemma 61.1])
\[
\Delta(mn) \leq \tau(m)\Delta(n) \quad (m \geq 1, \ n \geq 1), \quad (2.1)
\]
we may write, for any function $g$ in $\mathcal{M}_A(y, c, \eta)$,
\[
\sum_{n \leq x} g(n)\Delta(n) \log n \leq \sum_{m \leq x} g(m)\Delta(m) \left( \sum_{p^\nu \leq x/m} (\nu + 1)g(p^\nu) \log p^\nu \right) \ll x \sum_{m \leq x} \frac{g(m)\Delta(m)}{m},
\]
where the second bound is obtained by invoking (1.2) in the form $g(p^\nu) \ll_A (3/2)^\nu$ for $p \leq 2A$. Since we trivially have
\[
\sum_{n \leq x} g(n)\Delta(n) \log \left( \frac{x}{n} \right) \ll x \sum_{n \leq x} \frac{g(n)\Delta(n)}{n},
\]
it follows that
\[
\sum_{n \leq x} g(n)\Delta(n) \ll \frac{x}{\log x} \sum_{n \leq x} \frac{g(n)\Delta(n)}{n}. \quad (2.2)
\]
Moreover, the canonical representation $n = md$ where $m$ is squarefree and $d$ is squarefull implies
\[
\sum_{n \leq x} \frac{g(n)\Delta(n)}{n} \ll \sum_{n \leq x} \frac{\mu(n)^2 g(n)\Delta(n)}{n}. \quad (2.3)
\]
Next, we observe that proving (1.5) for $y = 1$ implies the required bound for $y \geq 1$. Indeed, any $g$ in $\mathcal{M}_A(y, c, \eta)$ is representable as $g(n) = y^{r(n)}h(n)$ with $h \in \mathcal{M}_A(1, c, \eta)$. The identity
\[
y^{r(n)} = \sum_{d \mid n} \mu(d)^2 (y - 1)^{\omega(d)},
\]
already used in [4], and the inequality (2.1) hence imply
\[
\sum_{n \leq x} \frac{\mu(n)^2 g(n) \Delta(n)}{n} \leq \sum_{d \leq x} \frac{\mu(d)^2 h(d) (2y - 2)^{\omega(d)}}{d} \sum_{m \leq x/d} \frac{\Delta(m) h(m)}{m} \ll e^{\sqrt{\log x}} (\log x)^{2y - 1},
\]
by applying (1.5) to \(h\). The required estimate follows by (2.2).

In the sequel, we hence consider a function \(g \in \mathcal{M}_A(1, c, \eta)\) and aim at estimating the right-hand side of (2.3).

Let \(\{p_j(n) : 1 \leq j \leq \omega(n)\}\) denote the increasing sequence of distinct prime factors of a generic integer \(n\) and define \(n_k := \prod_{j=1}^k (\omega(n) - j + 1)\) if \(\omega(n) \geq k\); \(n_k = n\) otherwise. Our final estimate will be derived from a bound for
\[
D_k(x; g) := \sum_{n \leq x} \frac{\mu(n)^2 g(n) \Delta(n_k)}{n}
\]
obtained by induction on \(k\). To determine a suitable size for \(k\), we appeal to a straightforward variant of [5, Theorem 72] providing
\[
\sum_{n \leq x} \Delta(n) g(n) y^{\omega(n)} = x (\log x)^{2y - 2 + o(1)} \quad (y \geq 1, \ x \to \infty),
\]
and hence, for any fixed \(y > 1,\)
\[
\sum_{\omega(n) > 2y \log_2 x} n \frac{g(n) \Delta(n)}{n} \ll \sum_{n \leq x} n \frac{g(n) \Delta(n) y^{\omega(n) - 2y \log_2 x}}{n} \ll x (\log x)^{-2(y \log y - y + 1) + o(1)} = o(x).
\]
Therefore, we see that it will be sufficient to bound \(D_k(x; g)\) for \(k \leq K_x := (2 + \varepsilon) \log_2 x\).

A last reduction is described as follows. Given \(\xi(x)\) tending to infinity arbitrarily slowly, let \(A_x\) denote the set of those integers \(n \geq 1\) that are squarefree and satisfy
\[
\omega(n) \geq k \Rightarrow \log_2 p_k(n) > k/5 \quad (\xi(x) \leq k \leq K_x).
\]
(2.4)

Let
\[
D(x; g) := \sum_{n \in [1, x]} \frac{\mu(n)^2 g(n) \Delta(n)}{n}, \quad D^-(x; g) := \sum_{n \in [1, x] \setminus A_x} \frac{\mu(n)^2 g(n) \Delta(n)}{n}.
\]

Put \(\omega(n, t) := \sum_{p | n, p \leq t} 1\) and \(r_k := \exp(\exp(k/5))\). Letting \(h_{v, k}\) denote the multiplicative function supported on squarefree integers and defined by \(h_{v, k}(p) := (v - 1)1_{[2, r_k]}(p)\), we have, for any \(v \geq 1,\)
\[
D^-(x; g) \leq \sum_{\xi(x) \leq k \leq K_x} v^{-k} \sum_{n \leq x} \frac{\mu(n)^2 g(n) y^{\omega(n, r_k)} \Delta(n)}{n} \leq \sum_{\xi(x) \leq k \leq K_x} v^{-k} \sum_{d \leq x} \frac{\mu(d)^2 g(d) h_{v, k}(d) \Delta(d) 2^{\omega(d)}}{d} \sum_{n \leq x} \frac{\mu(n)^2 g(n) \Delta(n)}{n} \ll D(x; g) \sum_{\xi(x) \leq k \leq K_x} e^{2(v - 1)k/5 - k \log v} = o(D(x; g))
\]
by selecting \(v = \frac{3}{2}\) since \(1/5 - \log(3/2) < 0\).

Thus we obtain that suitable averages over \(A_x\) will imply (1.5) as stated.

### 2.2 A lemma

We write
\[
M_q(n) := \int_{\mathbb{R}} \Delta(n, u)^q \, du \quad (n \geq 1, \ q \geq 1).
\]
(2.5)
The following estimate will play a crucial role in the proof.
Lemma 2.1. Let $A > 0$, $c > 0$, $\eta > 0$ and $g \in \mathcal{M}_A(1, c, \eta)$. We have
\[
\sum_{\omega(n) \geq k} \frac{\mu(n)^2 g(n) M_2(n)}{n^\sigma} \ll \frac{k^{2k}}{\sigma - 1} \quad (k \geq 1, 1 < \sigma \leq 2).
\] (2.6)

Proof. Let $S_2(\sigma)$ denote the left-hand side of (2.6). Put $\tau(n, \vartheta) := \sum_{d \mid n} d^{i\vartheta} (n \geq 1, \vartheta \in \mathbb{R})$. Plainly,
\[
S_2(\sigma) \ll \frac{1}{\sigma - 1} \sum_{m \leq 1 \atop \omega(m) = k} \frac{\mu(m)^2 g(m) M_2(m)}{m \log P^+(m)}
\]
\[
\ll \frac{1}{\sigma - 1} \sum_p \frac{g(p)}{p \log p} \sum_{m \leq 1 \atop \omega(m) = k-1} \frac{\mu(m)^2 g(m)}{m} \int_\mathbb{R} \frac{|\tau(m, \vartheta)|^2}{1 + \vartheta^2} \, d\vartheta
\]
by Parseval’s formula in view of [5, (3.2)]. The last integral is classically dominated by the contribution of the interval $[-1, 1]$—see the Montgomery-Wirsing lemma as stated, e.g., in [14, Lemma III.4.10]. For $|\vartheta| \leq 1$ and $t \geq 2$, we have
\[
\sum_{r \leq t \atop r \in \mathbb{P}} \frac{g(r) |\tau(r, \vartheta)|^2}{r} = 4 \log_2 t - 2 \log (1 + |\vartheta| \log t) + O(1),
\]
whence
\[
\sum_{m = k-1}^{P^+(m) < p} \frac{\mu(m)^2 g(m) |\tau(m, \vartheta)|^2}{m} \ll \frac{4 \log_2 p - 2 \log (1 + |\vartheta| \log p) + O(1)}{(k-1)!}.
\]
We therefore get
\[
\sum_p \frac{g(p)}{p \log p} \sum_{m = k-1}^{P^+(m) < p} \frac{\mu(m)^2 g(m)}{m^\sigma} \int_\mathbb{R} \frac{|\tau(m, \vartheta)|^2}{1 + \vartheta^2} \, d\vartheta \ll \frac{2^k}{(k-1)!} \int_0^1 \{T_1(\vartheta) + T_2(\vartheta)\} \, d\vartheta,
\]
with, for a suitable absolute constant $c_0$,
\[
T_1(\vartheta) := \sum_{p \leq \exp(1/\vartheta)} \frac{g(p)(2 \log_2 p + c_0)^{k-1}}{p \log p},
\]
\[
T_2(\vartheta) := \sum_{p > \exp(1/\vartheta)} \frac{g(p)(\log_2 p + \log(1/\vartheta) + c_0)^{k-1}}{p \log p}.
\]
It follows that
\[
\int_0^1 T_1(\vartheta) \, d\vartheta \ll \sum_p \frac{g(p)(2 \log_2 p + c_0)^{k-1}}{p \log p} \ll (k-1)!,
\]
\[
\int_0^1 T_2(\vartheta) \, d\vartheta \ll \int_0^1 \sum_{p > \exp(1/\vartheta)} \frac{g(p)(\log_2 p + \log(1/\vartheta) + c_0)^{k-1}}{p \log p} \, d\vartheta
\]
\[
\ll \int_0^1 \frac{1}{\vartheta} \int_2^{\log(1/\vartheta)} (v + c_0)^{k-1} e^{-v} \, dv \, d\vartheta \ll k!.
\]
This yields (2.6) as required. \(\square\)
2.3 Completion of the proof

With notation (2.5) and

\[ L(n) := \text{meas}\{ u \in \mathbb{R} : \Delta(n, u) > 0 \}, \]

we introduce the series

\[ F_{k,q}(\sigma) := \sum_{\omega(n) \geq k}^* \frac{g(n)M_q(nk)L(nk)^{(q-1)/2}}{2kM_2(nk)^{(q-1)/2}n^\sigma}, \quad G_{k,q}(\sigma) := \sum_{\omega(n) \geq k}^* \frac{g(n)M_q(nk)^{1/q}}{n^\sigma} \]

for \( \sigma > 1, k \geq 1 \). Here and throughout, the asterisk indicates that the summation domain is restricted to \( A_x \).

Since by [5, Theorem 72] we have

\[ \Delta(n) \leq 2M_q(nk)^{1/q} \quad (n \geq 1, q \geq 1), \quad (2.7) \]

the validity of the bound

\[ \sum_{k \leq K_x} G_{k,q(k)}(1 + 1/ \log x) \ll a e\sqrt{\log x} \log x \quad (2.8) \]

for any \( a > \sqrt{2} \log 2 \) and suitable \( q(k) \) implies the same estimate for the right-hand side of (2.2). An appropriate choice \( q(k) \) will be given later.

Put

\[ N_j,q(n,p) := \int_{\mathbb{R}} \Delta(n, u)j^j \Delta(n, u - \log p)^{q-j} \, du, \quad W_q(n,p) := \sum_{1 \leq j \leq q-1} \binom{q-j}{j} N_j,q(n,p). \]

The inequality

\[ M_q(n_{k+1}) \leq 2M_q(n_k) + W_q(n_k, P_{k+1})1_{\{\omega(n) > k+1\}} \]

is an equality if \( \omega(n) \geq k + 1 \) and holds trivially otherwise. Since \( M_2(n_{k+1}) \geq 2M_2(n_k) \) and \( L(n_{k+1}) \leq 2L(n_k) \), it follows that

\[ F_{k+1,q}(\sigma) \leq F_{k,q}(\sigma) + \sum_{m \in M_k} \sum_{\omega(m) = k}^* \sum_{P^-(m) > p \geq 2k} W_q(m,p)L(m)^{(q-1)/2} \sum_{n_{k+1} = mp}^* \frac{g(n)}{n^\sigma}, \quad (2.9) \]

where \( M_k := \{ m \geq 1 : \exists n \in A_x : nk = m \} \).

Let \( \mathcal{H}_k := \{ h \geq 1 : \mu(h)^2 = 1, \omega(h) \geq j \Rightarrow \log_2 p_j(h) \geq (j + k + 1)/5 \} \). The inner sum in (2.9) does not exceed

\[ \frac{\mu(mp)^2 g(mp)}{(mp)^\sigma} \sum_{h \in \mathcal{H}_k} \frac{g(h)}{h^\sigma}, \]

hence

\[ F_{k+1,q}(\sigma) - F_{k,q}(\sigma) \leq \sum_{m \in M_k} \frac{\mu(m)^2 g(m)L(m)^{(q-1)/2}}{2kM_2(m)^{(q-1)/2}m^\sigma} \sum_{P^-(h) > p^+(m)}^* \frac{g(h)}{h^\sigma} \sum_{P^+(m)} g(p)W_q(m,p). \quad (2.10) \]
Now, observe that, for all \( z > 1 \) and \( 1 \leq j < q \),
\[
\log z \sum_{p > z} \frac{g(p)N_{j,q}(m,p)}{p} \leq \sum_{p > z} \frac{g(p)N_{j,q}(m,p) \log p}{p} \\
= \int_{\mathbb{R}} \Delta(m,u)^j \sum_{d_1, \ldots, d_{q-1} | m} \sum_{p > z} \frac{g(p) \log p}{p} du.
\]
The inner \( p \)-sum does not exceed
\[
\left\{ 1 - \log \left( \frac{\max_n d_n}{\min_n d_n} \right) + O(e^{-c(\log z)^q}) \right\} \mathbf{1}_{\{ \max_n d_n / \min_n d_n < e \}},
\]
and so
\[
\log z \sum_{p > z} \frac{g(p)N_{j,q}(m,p)}{p} \leq \sum_{p > z} \frac{g(p)N_{j,q}(m,p) \log p}{p} \\
\leq AM_j(m)M_{q-j}(m) + O(e^{-c(\log z)^q}M_j(m)M_{q-j}^*(m)),
\]
where
\[
M_j^*(n) := \sum_{d_n | m} 1 \leq 2^\ell M_j(n) \quad (\ell \geq 1, \, n \geq 1),
\]
by [8, (6)].

At this stage we note that Hölder’s inequality furnishes for \( 2 \leq \ell \leq q - 2 \) and \( m \geq 1 \),
\[
M_j(m) \leq M_2(m)^{(q-\ell-2)/(q-4)}M_{q-2}(m)^{(\ell-2)/(q-4)}.
\]
Applying twice for \( \ell = j \) and \( \ell = q - j \), we get for any \( 2 \leq j < q - 2 \),
\[
M_j(m)M_{q-j}(m) \leq M_2(m)M_{q-2}(m)
\]
so that for any \( 2 \leq j < q - 2 \),
\[
\sum_{p > z} \frac{g(p)N_{j,q}(m,p) \log p}{p} \leq AM_2(m)M_{q-2}(m)\{ 1 + R_{j,q}(m,z) \},
\]
where
\[
R_{j,q}(m,z) \ll 2^{q-j}e^{-c(\log z)^q}.
\]
Carrying back into (2.10) and taking into account the fact that the integer \( mh \) belongs to \( A_x \) when \( m \in M_k, \, h \in \mathcal{H}_k, \, P^-(h) > P^+(m) \) and \( \mu(mh)^2 = 1 \), we obtain
\[
F_{k+1,q}(\sigma) - F_{k,q}(\sigma) \ll q(1 + 2^k \varepsilon_k)H_{k,q}(\sigma) + (2^k + 3^k \varepsilon_k)J_{k,q}(\sigma),
\]
with
\[
\varepsilon_k := e^{-c\exp(qk/5)},
\]
\[
H_{k,q}(\sigma) := \sum_{\omega(n) \geq k} \frac{g(n_k)M_{q-1}(n_k)L(n_k)(q-1)/2}{M_2(n_k)^{(q-1)/2}n^\sigma \log p_k(n)},
\]
\[
J_{k,q}(\sigma) := \sum_{\omega(n) \geq k} \frac{g(n_k)M_{q-2}(n_k)L(n_k)(q-1)/2}{2^kM_2(n_k)^{(q-3)/2}n^\sigma \log p_k(n)}.
\]
From the inequalities
\[
\frac{L(n_k)(q-1)/2}{\log p_k(n)} \leq kL(n_k)(q-3)/2, \quad \frac{1}{L(n_k)} \leq \frac{M_2(n_k)}{4^k} \quad (\omega(n) \geq k),
\]
where
we deduce that

\[
H_{k,q}(\sigma) \leq \sum_{\omega(n) \geq k}^{\ast} \frac{kg(n_k)M_{q-1}(n_k)L(n_k)(q-2)/2}{2^kM_2(n_k)^{(q-2)/2}n^{\sigma}} = kF_{k,q-1}(\sigma),
\]

\[
J_{k,q}(\sigma) \leq \sum_{\omega(n) \geq k}^{\ast} \frac{kg(n_k)M_{q-2}(n_k)L(n_k)(q-3)/2}{2^kM_2(n_k)^{(q-3)/2}n^{\sigma}} = kF_{k,q-2}(\sigma),
\]

whence

\[
F_{k+1,q}(\sigma) - F_{k,q}(\sigma) \leq kq(1 + 2^q\varepsilon_k)F_{k,q-1}(\sigma) + k(2^q + 3^q\varepsilon_k)F_{k,q-2}(\sigma).
\]

Let \( k_0(q) := B\log q \), where \( B \) is sufficiently large to ensure \( \varepsilon_k2^q \leq 1 \) whenever \( q \geq 2 \) and \( k \geq k_0(q) \). We thus have, for a suitable constant \( C > 0 \),

\[
F_{k+1,q}(\sigma) - F_{k,q}(\sigma) \leq CkqF_{k,q-1}(\sigma) + Ck2^qF_{k,q-2}(\sigma) \quad (k \geq k_0(q), \quad \sigma > 1).
\]

We now show by induction on \( k \) that this implies, for a suitable constant \( D \),

\[
F_{k,q}(\sigma) \leq \frac{Dq^{Bq}k^{3q}2^{q^2/4}}{\sigma - 1} \prod_{1 \leq j \leq k} \left( 1 + \frac{4C}{j^2} \right) \quad (q \geq 2, \quad k \geq 1, \quad 1 < \sigma \leq 2).
\]

(2.12)

For \( k \leq k_0(q) \), this follows from the trivial bound \((\sigma - 1)F_{k,q}(\sigma) \leq 2^{k(q-1)}\), in view of the inequalities \(1/M_2(n) \leq L(n)/\tau(n)^2\), \( L(n) \leq \tau(n)\) and \( M_q(n) \leq \tau(n)^q\). Assuming that (2.12) holds for \( k \geq k_0(q) \), we deduce from (2.11) that

\[
F_{k+1,q}(\sigma) \leq \frac{Dq^{Bq}(k + 1)^{3q}2^{q^2/4}}{\sigma - 1} \prod_{1 \leq j \leq k+1} \left( 1 + \frac{4C}{j^2} \right),
\]

since \( q \leq 2^{(q+1)/2} \) and

\[
k^{3q}2^{q^2/4} + Ck^{3q-2}\tau(n)^{1/2}(q-1)^{2/4} + Ck^{3q-2}\tau(n)^{1/2}(q-2)^{2/4} \leq (k + 1)^{3q}2^{q^2/4} \left\{ 1 + \frac{4C}{(k + 1)^2} \right\}.
\]

Therefore, we may state that

\[
F_{k,q}(\sigma) \leq \frac{q^{Bq}k^{3q}2^{q^2/4}}{\sigma - 1} \quad (q \geq 2, \quad k \geq 1, \quad 1 < \sigma \leq 2).
\]

(2.13)

Now invoking once more the inequality \( 4^k = \tau(n_k)^2 \leq L(n_k)M_2(n_k) \), we get via Hölder’s inequality that

\[
G_{k,q}(\sigma) \leq 2^{k/q}F_{k,q}(\sigma)^{1/q} \left\{ \sum_{\omega(n) \geq k} \frac{\mu(n)^2g(n)\sqrt{M_2(n_k)}}{n^{\sigma}\sqrt{L(n_k)}} \right\}^{1-1/q} \leq \frac{q^{Bq}k^{3q}2^{q^2/4}}{(\sigma - 1)^{1/q}} \left\{ \sum_{\omega(n) \geq k} \frac{\mu(n)^2g(n)M_2(n_k)}{2^k\log x} \right\}^{1-1/q} \leq \frac{q^{Bq}k^{3q}2^{q^2/4}}{\sigma - 1},
\]

(2.14)

by (2.6).

Applying this for all \( k \leq K_x \), with \( q = q(k) = \lceil 2\sqrt{k} \rceil \) and \( \sigma = 1 + 1/\log x \), we yield (2.8) and thus finish the proof.

3 Normal order: Proof of Theorem 1.3

This is a reappraisal of [9, Theorem 1.3]. By (2.7), it is sufficient to bound \( M_q(n) \).
Let \( \lambda \in [1,2] \) and let \( \gamma \) and \( \delta \) be real numbers satisfying
\[
\delta (\log 2)/\lambda < \gamma < 1, \quad 1 < \delta < \lambda (\gamma - 1) + 1/\log 2.
\] (3.1)

We shall show by induction on \( k \) that
\[
M_q(n_k) \leq 2^\delta k (q!)^\gamma \quad (1 \leq q \leq \lambda k)
\] pp. (3.2)

Here, as in [5], the mention pp indicates that a formula holds for all integers \( n \leq x \) but at most \( o(x) \) as \( x \to \infty \).

Given an integer-valued function \( \xi = \xi(x) \) tending to infinity arbitrarily slowly, we put
\[
K = K(n,x) := \max\{k : 1 \leq k \leq \omega(n), \log_2 p_k(n) < \log_2 x - \xi(x)\}
\]
and redefine
\[
n_k := \begin{cases} \prod_{\xi<j\leq k} P_j(n) & \text{if } k \leq K, \\ n_K & \text{if } k > K. \end{cases}
\]

Assuming (3.2) holds for \( \xi < k \leq K \), we aim at showing that this bound persists at rank \( k + 1 \).

Let \( e_1 < e \). By [9, (3.2)], for \( \xi < k \leq K \) and \( q \geq 1 \), we have
\[
M_q(n_{k+1}) \leq 2M_q(n_k) + e_1^{-k} \sum_{1 \leq j \leq q - 1} \binom{q}{j} M_j(n_k) M_{q-j}(n_k) \quad \text{pp. (3.3)}
\]

Note that Hölder’s inequality implies
\[
\sum_{1 \leq j \leq q - 1} \binom{q}{j}^{1-\gamma} \leq 2^{(1-\gamma)q(q-1)^\gamma}.
\]

When \( q \leq q_k := \lfloor \lambda k \rfloor \), we appeal to the induction bound (3.2) to majorize the right-hand side of (3.3).

This yields
\[
M_q(n_{k+1}) \leq 2^\delta (k+1) (q!)^\gamma \left\{ 2^{1-\delta} + (2^\delta/e_1)^k \sum_{1 \leq j \leq q - 1} \binom{q}{j}^{1-\gamma} \right\}
\leq 2^\delta (k+1) (q!)^\gamma \left\{ 2^{1-\delta} + (2^\delta/e_1)^k 2^{(1-\gamma)q} \right\}
\leq 2^\delta (k+1) (q!)^\gamma \left\{ 2^{1-\delta} + (2^{(1-\gamma)\lambda/e_1})^k 2^{\gamma} \right\} \leq 2^\delta (k+1) (q!)^\gamma \] (3.4)
for sufficiently large \( \xi \), by the second condition (3.1).

When \( q_k < q \leq \lfloor \lambda (k + 1) \rfloor \), we apply [9, (3.3)]: given \( \alpha > 0 \) and \( r > 1/\alpha \) we have
\[
\Delta(n_k) \leq r + e^{\alpha k/q} M_q(n_k)^{1/q} \quad (\xi < k \leq K, q \geq 1) \quad \text{pp.}
\]

Since \( \lambda < 2 \), we have \( q = q_k + 1 \) or \( q = q_k + 2 \). If \( r \) is sufficiently large and \( \alpha > 1/r \), our induction hypothesis (3.2) furnishes
\[
M_{q_k+1}(n_k) \leq \Delta(n_k) M_{q_k}(n_k) \leq \left\{ e^{\alpha/\lambda} M_{q_k}(n_k)^{1/q_k} + r \right\} M_{q_k}(n_k)
\leq 2^\delta \left\{ (q_k + 1)! \right\}^\gamma \left\{ 2^{\delta \lambda/e^{\alpha/\lambda}} (q_k^{1/q_k})^\gamma / (q_k + 1)^\gamma \right\} + r / (q_k + 1)^\gamma
\leq 2^\delta \left\{ (q_k + 1)! \right\}^\gamma \left\{ 2^{\delta \lambda/e^{\alpha/\lambda-\gamma}} + o(1) \right\}.
\] (3.5)

Carrying back into (3.3) and writing
\[
b := 2^{1-\delta+\delta/\lambda} e^{\alpha/\lambda-\gamma} < 1,
\]
we get, taking (3.4) into account,

\[ M_{q_k+1}(n_{k+1}) \leq 2M_{q_k+1}(n_k) + e_{k-1}^{-k} \sum_{1 \leq j \leq q_k} \frac{(q_k + 1)}{j} M_j(n_k)M_{q_k+1-j}(n_k) \]

\[ \leq 2\delta(k+1)\{(q_k + 1)!\} \gamma \left\{ b + o(1) + \left( \frac{2\delta}{e_k} \right)^k \sum_{1 \leq j \leq q_k} \frac{(q_k + 1)}{j} \right\}^{1-\gamma} \]

\[ \leq 2^\delta(k+1)\{(q_k + 1)!\} \gamma \left\{ b + o(1) + \left( \frac{2\delta}{e_k} \right)^k 2^{1-\gamma}(\lambda_k+1)(q_k + 1) \right\} \]

\[ \leq 2^\delta(k+1)\{(q_k + 1)!\} \gamma \{ b + o(1) \} \leq 2^\delta(k+1)\{(q_k + 1)!\} \gamma, \quad (3.6) \]

by (3.1).

If \( q_k + 2 \leq \lambda(k+1) \), we also need to bound \( M_{q_k+2}(n_{k+1}) \). Put \( g := 2^\delta/\lambda e^{\alpha/\lambda - \gamma} < 1 \). By (3.2), [9, (3.3)] and (3.6), we have, for large \( \xi \),

\[ M_{q_k+2}(n_{k+1}) \leq \Delta(n_{k+1})M_{q_k+1}(n_{k+1}) \]

\[ \leq \left\{ r + e^{\alpha/\lambda}M_{q_k+1}(n_{k+1})^{1/(q_k+1)} \right\} M_{q_k+1}(n_{k+1}) \]

\[ \leq 2^\delta(k+1)\{(q_k + 2)!\} \gamma \left\{ b + o(1) \right\} \left( \frac{r + e^{\alpha/\lambda}2^\delta/\lambda^{\delta/\lambda}2^{1-\gamma}}{(q_k + 1)!} \right)^{(q_k + 1)!/(q_k + 1)} \]

\[ \leq 2^\delta(k+1)\{(q_k + 2)!\} \gamma \left\{ b + o(1) \right\} \{ g + o(1) \} \leq 2^\delta(k+1)\{(q_k + 2)!\} \gamma, \]

still by (3.1).

Selecting \( \alpha \) sufficiently small, we see that the induction hypothesis is still valid at rank \( k + 1 \). We may take \( \delta \) arbitrarily close to 1, and so \( \gamma \) is arbitrarily close to \( \gamma_3 \). This yields

\[ \Delta(n_K) \leq 2^{\delta K/q_k} (q_k!)^{\gamma/q_k} \ll K^\gamma. \]

Since we have classically \( K(n, x) \sim \log_x x \) pp as \( x \to \infty \), we may conclude as in [9] by invoking the bound

\[ \Delta(n) \leq \Delta(n_K)2^{O(n/n_K)} \ll \Delta(n_K)4^k \quad \text{ppx}. \]

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