Billiard Arrays and finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules

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Abstract

We introduce the notion of a Billiard Array. This is an equilateral triangular array of one-dimensional subspaces of a vector space $V$, subject to several conditions that specify which sums are direct. We show that the Billiard Arrays on $V$ are in bijection with the 3-tuples of totally opposite flags on $V$. We classify the Billiard Arrays up to isomorphism. We use Billiard Arrays to describe the finite-dimensional irreducible modules for the quantum algebra $U_q(\mathfrak{sl}_2)$ and the Lie algebra $\mathfrak{sl}_2$.

Keywords. Quantum group, quantum enveloping algebra, Lie algebra, flag.

2010 Mathematics Subject Classification. Primary: 17B37. Secondary: 15A21.

1 Introduction

Our topic is informally described as follows. As in the game of Billiards, we start with an array of billiard balls arranged to form an equilateral triangle. We assume that there are $N + 1$ balls along each boundary, with $N \geq 0$. For $N = 3$ the balls are centered at the following locations:

For us, each ball in the array represents a one-dimensional subspace of an $(N+1)$-dimensional vector space $V$ over a field $\mathbb{F}$.

We impose two conditions on the array, that specify which sums are direct. The first condition is that, for each set of balls on a line parallel to a boundary, their sum is direct. The second condition is described as follows. Three mutually adjacent balls in the array are said to form a 3-clique. There are two kinds of 3-cliques: $\Delta$ (black) and $\nabla$ (white). The second condition is that, for any three balls in the array that form a black 3-clique, their sum is not direct.

Whenever the above two conditions are met, our array is called a Billiard Array on $V$. We say that the Billiard Array is over $\mathbb{F}$, and call $N$ the diameter.
We have some remarks about notation. For $1 \leq i \leq N + 1$ let $\mathcal{P}_i(V)$ denote the set of subspaces of $V$ that have dimension $i$. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $\Delta_N$ denote the set consisting of the 3-tuples of natural numbers whose sum is $N$. Thus

$$\Delta_N = \{(r, s, t) \mid r, s, t \in \mathbb{N}, \ r + s + t = N\}.$$ 

We arrange the elements of $\Delta_N$ in a triangular array. For $N = 3$, the array looks as follows after deleting all punctuation:

\[
\begin{array}{cccc}
0 & 3 & 0 & \\
120 & 021 & \\
210 & 111 & 012 & \\
300 & 201 & 102 & 003 & \\
\end{array}
\]

An element in $\Delta_N$ is called a location. We view our Billiard Array on $V$ as a function $B : \Delta_N \to \mathcal{P}_1(V), \lambda \mapsto B_\lambda$. For $\lambda \in \Delta_N$, $B_\lambda$ is the billiard ball/subspace at location $\lambda$.

In this paper we obtain three main results, which are summarized as follows: (i) we show that the Billiard Arrays on $V$ are in bijection with the 3-tuples of totally opposite flags on $V$; (ii) we classify the Billiard Arrays up to isomorphism; (iii) we use Billiard Arrays to describe the finite-dimensional irreducible modules for the quantum algebra $U_q(\mathfrak{sl}_2)$ and the Lie algebra $\mathfrak{sl}_2$.

We now describe our results in more detail. By a flag on $V$ we mean a sequence $\{U_i\}_{i=0}^N$ such that $U_i \in \mathcal{P}_{i+1}(V)$ for $0 \leq i \leq N$ and $U_{i-1} \subseteq U_i$ for $1 \leq i \leq N$. Suppose we are given three flags on $V$, denoted $\{U_i\}_{i=0}^N$, $\{U_i'\}_{i=0}^N$, $\{U_i''\}_{i=0}^N$. These flags are called totally opposite whenever $U_{N-r} \cap U_{N-s} \cap U_{N-t} = 0$ for all $r, s, t (0 \leq r, s, t \leq N)$ such that $r + s + t > N$. Assume that the flags in line (1) are totally opposite. Using these flags we now construct a Billiard Array on $V$. For each location $\lambda = (r, s, t)$ in $\Delta_N$ define

$$B_\lambda = U_{N-r} \cap U_{N-s} \cap U_{N-t}.$$ 

We will show that $B_\lambda$ has dimension one, and the map $B : \Delta_N \to \mathcal{P}_1(V), \lambda \mapsto B_\lambda$ is a Billiard Array on $V$. We just went from flags to Billiard Arrays; we now reverse the direction. Let $B$ denote a Billiard Array on $V$. Using $B$ we now construct a 3-tuple of totally opposite flags on $V$. By the 1-corner of $\Delta_N$ we mean the location $(N, 0, 0)$. The 2-corner and 3-corner of $\Delta_N$ are similarly defined. For $0 \leq i \leq N$ let $U_i$ (resp. $U_i'$) (resp. $U_i''$) denote the sum of the balls in $B$ that are at most $i$ balls over from the 1-corner (resp. 2-corner) (resp. 3-corner). We will show that $\{U_i\}_{i=0}^N$, $\{U_i'\}_{i=0}^N$, $\{U_i''\}_{i=0}^N$ are totally opposite flags on $V$. Consider the following two sets:

(i) the Billiard Arrays on $V$;
(ii) the 3-tuples of totally opposite flags on $V$.

We just described a function from (i) to (ii) and a function from (ii) to (i). We will show that these functions are inverses, and hence bijections.

We now describe our classification of Billiard Arrays up to isomorphism. Let $B$ denote a Billiard Array on the above vector space $V$. Let $V'$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$, and let $B'$ denote a Billiard Array on $V'$. The Billiard Arrays $B, B'$ are called isomorphic whenever there exists an $\mathbb{F}$-linear bijection $V \to V'$ that sends $B_{\lambda} \mapsto B'_{\lambda}$ for all $\lambda \in \Delta_N$. By a value function on $\Delta_N$ we mean a function $\Delta_N \to \mathbb{F}\{0\}$. For $N \leq 1$, up to isomorphism there exists a unique Billiard Array over $\mathbb{F}$ that has diameter $N$. For $N \geq 2$ we will obtain a bijection between the following two sets:

(i) the isomorphism classes of Billiard Arrays over $\mathbb{F}$ that have diameter $N$;

(ii) the value functions on $\Delta_{N-2}$.

We now describe the bijection. Let $V$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$, and let $B$ denote a Billiard Array on $V$. Given adjacent locations $\lambda, \mu$ in $\Delta_N$, we define an $\mathbb{F}$-linear map $\tilde{B}_{\lambda,\mu} : B_{\lambda} \to B_{\mu}$ as follows. There exists a unique location $\nu \in \Delta_N$ such that $\lambda, \mu, \nu$ form a black 3-clique. Pick $0 \neq u \in B_{\lambda}$. There exists a unique $v \in B_{\mu}$ such that $u + v \in B_{\nu}$. The map $\tilde{B}_{\lambda,\mu}$ sends $u \mapsto v$. By construction the maps $\tilde{B}_{\lambda,\mu} : B_{\lambda} \to B_{\mu}$ and $\tilde{B}_{\mu,\lambda} : B_{\mu} \to B_{\lambda}$ are inverses. Let $\lambda, \mu, \nu$ denote locations in $\Delta_N$ that form a 3-clique, either black or white. Consider the composition of maps around the clique:

$$
\begin{array}{ccc}
B_{\lambda} & \longrightarrow & B_{\mu} \\
\tilde{B}_{\lambda,\mu} & & \tilde{B}_{\mu,\nu} \\
& \longrightarrow & \quad B_{\nu} \\
& & \quad \quad \tilde{B}_{\nu,\lambda} \\
& \longrightarrow & B_{\lambda}
\end{array}
$$

This composition is a nonzero scalar multiple of the identity map on $B_{\lambda}$. First assume that the 3-clique is black. Then the scalar is 1. Next assume that the 3-clique is white. Then the scalar is called the clockwise $B$-value (resp. counterclockwise $B$-value) of the 3-clique whenever the sequence $\lambda, \mu, \nu$ runs clockwise (resp. counterclockwise) around the clique. The clockwise $B$-value and counterclockwise $B$-value are reciprocal. By the $B$-value of the 3-clique, we mean its clockwise $B$-value. For $N \leq 1$ the set $\Delta_N$ has no white 3-clique. For $N \geq 2$ we now give a bijection from $\Delta_{N-2}$ to the set of white 3-cliques in $\Delta_N$. The bijection sends each element $(r, s, t)$ in $\Delta_{N-2}$ to the white 3-clique in $\Delta_N$ consisting of the locations

$$
(r, s + 1, t + 1), \quad (r + 1, s, t + 1), \quad (r + 1, s + 1, t).
$$

Using $B$ we define a function $\tilde{B} : \Delta_{N-2} \to \mathbb{F}$ as follows: $\tilde{B}$ sends each element $(r, s, t)$ in $\Delta_{N-2}$ to the $B$-value of the corresponding white 3-clique in $\Delta_N$. By construction $\tilde{B}$ is a value function on $\Delta_{N-2}$. The map $B \mapsto \tilde{B}$ induces our bijection, from the set of isomorphism classes of Billiard Arrays over $\mathbb{F}$ that have diameter $N$, to the set of value functions on $\Delta_{N-2}$.

We now use Billiard Arrays to describe the finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-modules. We recall $U_q(\mathfrak{sl}_2)$. We will use the equitable presentation, which was introduced in [11]. See...
also [13, 15, 7, 10, 14, 17]. Fix a nonzero $q \in \mathbb{F}$ such that $q^2 \neq 1$. By [11, Theorem 2.1] the equitable presentation of $U_q(\mathfrak{sl}_2)$ has generators $x, y, z$ and relations $yy^{-1} = 1$, $y^{-1}y = 1$,

$$
\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \quad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \quad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.
$$

Following [15, Definition 3.1] define

$$
\nu_x = q(1 - yz), \quad \nu_y = q(1 - zx), \quad \nu_z = q(1 - xy).
$$

By [15, Lemma 3.5],

$$
x\nu_y = q^2\nu_y x, \quad x\nu_z = q^{-2}\nu_z x, \\
y\nu_z = q^2\nu_z y, \quad y\nu_x = q^{-2}\nu_x y, \\
z\nu_x = q^2\nu_x z, \quad z\nu_y = q^{-2}\nu_y z.
$$

For the moment, assume that $q$ is not a root of unity. Pick $N \in \mathbb{N}$ and let $V$ denote an irreducible $U_q(\mathfrak{sl}_2)$-module of dimension $N + 1$. By [16, Lemma 8.2] each of $\nu_x^{N+1}, \nu_y^{N+1}, \nu_z^{N+1}$ is zero on $V$. Moreover by [16, Lemma 8.3], each of the following three sequences is a flag on $V$:

$$
\{\nu_x^{N-i}V\}_{i=0}^N, \quad \{\nu_y^{N-i}V\}_{i=0}^N, \quad \{\nu_z^{N-i}V\}_{i=0}^N.
$$

We will show that these three flags are totally opposite. We will further show that for the corresponding Billiard Array on $V$, the value of each white 3-clique is $q^{-2}$.

We just obtained a Billiard Array from each finite-dimensional irreducible $U_q(\mathfrak{sl}_2)$-module, under the assumption that $q$ is not a root of unity. Similarly, for the Lie algebra $\mathfrak{sl}_2$ over $\mathbb{F}$, we will obtain a Billiard Array from each finite-dimensional irreducible $\mathfrak{sl}_2$-module, under the assumption that $\mathbb{F}$ has characteristic 0. We will show that for these Billiard Arrays, the value of each white 3-clique is 1.

We have been discussing Billiard Arrays. Later in this paper we will introduce the concept of a Concrete Billiard Array and an edge-labelling of $\Delta_N$. These concepts help to clarify the Billiard Array theory, and will be used in our proofs. They may be of independent interest.

This paper is organized as follows. Section 2 contains some preliminaries. In Sections 3–5 we consider the set $\Delta_N$ from various points of view. Section 6 is about flags. Section 7 contains the definition and basic facts about Billiard Arrays. Section 8 contains a similar treatment for Concrete Billiard Arrays. Sections 9–12 are devoted to the correspondence between Billiard Arrays and 3-tuples of totally opposite flags. Sections 13–19 are devoted to our classification of Billiard Arrays up to isomorphism. Section 20 contains some examples of Concrete Billiard Arrays. In Section 21 we use Billiard Arrays to describe the finite-dimensional irreducible modules for $U_q(\mathfrak{sl}_2)$ and $\mathfrak{sl}_2$.

## 2 Preliminaries

We now begin our formal argument. Let $\mathbb{R}$ denote the field of real numbers. We will be discussing the vector space $\mathbb{R}^3$ (row vectors). We will refer to the basis

$$
e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1).
$$
Define a subset \( \Phi \subseteq \mathbb{R}^3 \) by
\[
\Phi = \{ e_i - e_j \mid 1 \leq i, j \leq 3, \ i \neq j \}.
\]
The set \( \Phi \) is often called the root system \( A_2 \). For notational convenience define
\[
\alpha = e_1 - e_2, \quad \beta = e_2 - e_3, \quad \gamma = e_3 - e_1. \tag{2}
\]
Note that
\[
\Phi = \{ \pm \alpha, \pm \beta, \pm \gamma \}, \quad \alpha + \beta + \gamma = 0.
\]
An element \((r, s, t) \in \mathbb{R}^3\) will be called nonnegative whenever each of \(r, s, t\) is nonnegative. Define a partial order \(\leq\) on \(\mathbb{R}^3\) such that for \(\lambda, \mu \in \mathbb{R}^3\), \(\mu \leq \lambda\) if and only if \(\lambda - \mu\) is nonnegative.

Let \(\{u_i\}_{i=0}^n\) denote a finite sequence. We call \(u_i\) the i-component or i-coordinate of the sequence. By the inversion of the sequence \(\{u_i\}_{i=0}^n\) we mean the sequence \(\{u_{n-i}\}_{i=0}^n\).

### 3 The set \(\Delta_N\)

Throughout this section fix \(N \in \mathbb{N}\).

**Definition 3.1.** Let \(\Delta_N\) denote the subset of \(\mathbb{R}^3\) consisting of the three-tuples of natural numbers whose sum is \(N\). Thus
\[
\Delta_N = \{(r, s, t) \mid r, s, t \in \mathbb{N}, \ r + s + t = N \}. \tag{3}
\]
We arrange the elements of \(\Delta_N\) in a triangular array, as discussed in Section 1. An element in \(\Delta_N\) is called a location. For notational convenience define \(\Delta_{-1} = \emptyset\).

**Definition 3.2.** For \(\eta \in \{1, 2, 3\}\) the \(\eta\)-corner of \(\Delta_N\) is the location in \(\Delta_N\) that has \(\eta\)-coordinate \(N\) and all other coordinates 0. By a corner of \(\Delta_N\) we mean the 1-corner or 2-corner or 3-corner. The corners in \(\Delta_N\) are listed below.
\[
Ne_1 = (N, 0, 0), \quad Ne_2 = (0, N, 0), \quad Ne_3 = (0, 0, N).
\]

**Definition 3.3.** For \(\eta \in \{1, 2, 3\}\) the \(\eta\)-boundary of \(\Delta_N\) is the set of locations in \(\Delta_N\) that have \(\eta\)-coordinate 0.

**Example 3.4.** The 1-boundary of \(\Delta_N\) consists of the locations
\[
(0, N - i, i) \quad i = 0, 1, \ldots, N.
\]

**Definition 3.5.** The boundary of \(\Delta_N\) is the union of its 1-boundary, 2-boundary, and 3-boundary. By the interior of \(\Delta_N\) we mean the set of locations in \(\Delta_N\) that are not on the boundary.
Definition 3.6. For $\eta \in \{1, 2, 3\}$ we define a binary relation on $\Delta_N$ called $\eta$-collinearity. By definition, locations $\lambda, \lambda'$ in $\Delta_N$ are $\eta$-collinear whenever the $\eta$-coordinate of $\lambda - \lambda'$ is 0. Note that $\eta$-collinearity is an equivalence relation. Each equivalence class will be called an $\eta$-line.

Example 3.7. Pick $\eta \in \{1, 2, 3\}$. For $0 \leq i \leq N$ there exists a unique $\eta$-line of $\Delta_N$ that has cardinality $i + 1$. For $i = 0$ (resp. $i = N$) this $\eta$-line is the $\eta$-corner (resp. $\eta$-boundary) of $\Delta_N$.

Example 3.8. For $0 \leq i \leq N$ the following locations make up the unique 1-line of $\Delta_N$ that has cardinality $i + 1$:

$$\{(N - i, i - j, j) | j = 0, 1, \ldots, i\}.$$ (4)

Definition 3.9. Locations $\lambda, \lambda'$ in $\Delta_N$ are called collinear whenever they are 1-collinear or 2-collinear or 3-collinear. By a line in $\Delta_N$ we mean a 1-line or 2-line or 3-line.

4 $\Delta_N$ as a graph

Throughout this section fix $N \in \mathbb{N}$. In this section we describe $\Delta_N$ using notions from graph theory. For each result that we mention, the proof is routine and omitted.

Definition 4.1. Locations $\lambda, \mu$ in $\Delta_N$ are called adjacent whenever $\lambda - \mu \in \Phi$.

Example 4.2. Assume $N \geq 1$, and pick a location $\lambda \in \Delta_N$.

(i) Assume that $\lambda$ is a corner. Then $\lambda$ is adjacent to exactly 2 locations in $\Delta_N$.

(ii) Assume that $\lambda$ is on the boundary, but not a corner. Then $\lambda$ is adjacent to exactly 4 locations in $\Delta_N$.

(iii) Assume that $\lambda$ is in the interior. Then $\lambda$ is adjacent to exactly six locations in $\Delta_N$.

Definition 4.3. By an edge in $\Delta_N$ we mean a set of two adjacent locations.

Definition 4.4. For $n \in \mathbb{N}$, by a walk of length $n$ in $\Delta_N$ we mean a sequence of locations $\{\lambda_i\}_{i=0}^n$ in $\Delta_N$ such that $\lambda_{i-1}, \lambda_i$ are adjacent for $1 \leq i \leq n$. This walk is said to be from $\lambda_0$ to $\lambda_n$. By a path of length $n$ in $\Delta_N$ we mean a walk $\{\lambda_i\}_{i=0}^n$ in $\Delta_N$ such that $\lambda_{i-1} \neq \lambda_{i+1}$ for $1 \leq i \leq n - 1$. By a cycle in $\Delta_N$ we mean a path $\{\lambda_i\}_{i=0}^n$ in $\Delta_N$ of length $n \geq 3$ such that $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are mutually distinct and $\lambda_0 = \lambda_n$.

Definition 4.5. For locations $\lambda, \lambda' \in \Delta_N$ let $\partial(\lambda, \lambda')$ denote the length of a shortest path from $\lambda$ to $\lambda'$. We call $\partial(\lambda, \lambda')$ the distance between $\lambda$ and $\lambda'$.

Example 4.6. A location $\lambda = (r, s, t)$ in $\Delta_N$ is at distance $N - r$ (resp. $N - s$) (resp. $N - t$) from the 1-corner (resp. 2-corner) (resp. 3-corner) of $\Delta_N$.

Definition 4.7. For a location $\lambda \in \Delta_N$ and a nonempty subset $S \subseteq \Delta_N$, define

$$\partial(\lambda, S) = \min\{\partial(\lambda, \lambda') | \lambda' \in S\}.$$ We call $\partial(\lambda, S)$ the distance between $\lambda$ and $S$. 

6
Example 4.8. A location \( \lambda = (r, s, t) \) in \( \Delta_N \) is at distance \( r \) (resp. \( s \)) (resp. \( t \)) from the 1-boundary (resp. 2-boundary) (resp. 3-boundary) of \( \Delta_N \).

**Lemma 4.9.** For \( \eta \in \{1, 2, 3\} \) and \( 0 \leq n \leq N \) the following sets coincide:

(i) the locations in \( \Delta_N \) that have \( \eta \)-coordinate \( n \);

(ii) the locations in \( \Delta_N \) at distance \( n \) from the \( \eta \)-boundary;

(iii) the locations in \( \Delta_N \) at distance \( N - n \) from the \( \eta \)-corner;

(iv) the \( \eta \)-line of cardinality \( N - n + 1 \).

We now consider the distance function \( \partial \) in more detail.

**Lemma 4.10.** Let \( \lambda \) and \( \lambda' \) denote locations in \( \Delta_N \), written \( \lambda = (r, s, t) \) and \( \lambda' = (r', s', t') \). Then \( \partial(\lambda, \lambda') \) is equal to the maximum of the absolute values

\[
|r - r'|, \quad |s - s'|, \quad |t - t'|.
\]

**Lemma 4.11.** Given locations \( (r, s, t) \) and \( (r', s', t') \) in \( \Delta_N \), consider the three quantities in line (5). Let \( d_1, d_2, d_3 \) denote an ordering of these quantities such that \( d_1 \leq d_2 \leq d_3 \). Then \( d_1 + d_2 = d_3 \).

**Lemma 4.12.** For locations \( \lambda, \lambda' \) in \( \Delta_N \) we have \( \partial(\lambda, \lambda') \leq N \). Moreover the following are equivalent:

(i) \( \partial(\lambda, \lambda') = N \);

(ii) there exists \( \eta \in \{1, 2, 3\} \) such that one of \( \lambda, \lambda' \) is the \( \eta \)-corner of \( \Delta_N \), and the other one is on the \( \eta \)-boundary of \( \Delta_N \).

We mention several types of paths in \( \Delta_N \).

**Definition 4.13.** Pick \( \eta \in \{1, 2, 3\} \). A path \( \{\lambda_i\}_{i=0}^n \) in \( \Delta_N \) is called an \( \eta \)-boundary path whenever \( \lambda_i \) is on the \( \eta \)-boundary for \( 0 \leq i \leq n \). The path \( \{\lambda_i\}_{i=0}^n \) is called \( \eta \)-linear whenever there exists an \( \eta \)-line that contains \( \lambda_i \) for \( 0 \leq i \leq n \).

**Definition 4.14.** Pick \( \eta \in \{1, 2, 3\} \). A subset \( S \) of \( \Delta_N \) is called \( \eta \)-geodesic whenever distinct elements in \( S \) do not have the same \( \eta \)-coordinate.

**Lemma 4.15.** Pick \( \eta \in \{1, 2, 3\} \). For a path \( \{\lambda_i\}_{i=0}^n \) in \( \Delta_N \) the following are equivalent:

(i) the path \( \{\lambda_i\}_{i=0}^n \) is \( \eta \)-geodesic;

(ii) there exists \( \varepsilon \in \{1, -1\} \) such that for \( 1 \leq i \leq n \), the \( \eta \)-coordinate of \( \lambda_i \) is equal to \( \varepsilon \) plus the \( \eta \)-coordinate of \( \lambda_{i-1} \).

**Lemma 4.16.** Pick \( \eta \in \{1, 2, 3\} \). For a path \( \{\lambda_i\}_{i=0}^n \) in \( \Delta_N \) the following are equivalent:

(i) the path \( \{\lambda_i\}_{i=0}^n \) is \( \eta \)-linear;
(ii) the path \( \{\lambda_i\}_{i=0}^n \) is \( \xi \)-geodesic for each \( \xi \in \{1, 2, 3\} \) other than \( \eta \).

**Definition 4.17.** A path \( \{\lambda_i\}_{i=0}^n \) in \( \Delta_N \) is said to be geodesic whenever \( \partial(\lambda_0, \lambda_n) = n \).

**Lemma 4.18.** For a path \( \{\lambda_i\}_{i=0}^n \) in \( \Delta_N \) the following are equivalent:

(i) the path \( \{\lambda_i\}_{i=0}^n \) is geodesic;

(ii) there exists \( \eta \in \{1, 2, 3\} \) such that the path \( \{\lambda_i\}_{i=0}^n \) is \( \eta \)-geodesic.

**Lemma 4.19.** Let \( \lambda \) and \( \lambda' \) denote locations in \( \Delta_N \), written \( \lambda = (r, s, t) \) and \( \lambda' = (r', s', t') \). Then the number of geodesic paths in \( \Delta_N \) from \( \lambda \) to \( \lambda' \) is equal to the binomial coefficient \( \binom{d_1 + d_2}{d_1} \) with \( d_1, d_2 \) from Lemma 4.11.

**Lemma 4.20.** For locations \( \lambda, \lambda' \) in \( \Delta_N \) the following are equivalent:

(i) the locations \( \lambda, \lambda' \) are collinear;

(ii) there exists a unique geodesic path in \( \Delta_N \) from \( \lambda \) to \( \lambda' \).

Assume that (i), (ii) hold. Define \( \eta \in \{1, 2, 3\} \) such that \( \lambda, \lambda' \) are \( \eta \)-collinear. Then the geodesic path mentioned in (ii) is \( \eta \)-linear.

To motivate the next definition we make some comments. Let \( \lambda, \lambda' \) denote distinct corner locations in \( \Delta_N \). Then \( \lambda, \lambda' \) are collinear, and \( \partial(\lambda, \lambda') = N \). There exists a unique geodesic path in \( \Delta_N \) from \( \lambda \) to \( \lambda' \). This is a boundary path.

**Definition 4.21.** For distinct \( \eta, \xi \in \{1, 2, 3\} \) let \( [\eta, \xi] \) denote the unique geodesic path in \( \Delta_N \) from the \( \eta \)-corner of \( \Delta_N \) to the \( \xi \)-corner of \( \Delta_N \).

**Lemma 4.22.** For distinct \( \eta, \xi \in \{1, 2, 3\} \) the path \( [\eta, \xi] \) is the inversion of the path \( [\xi, \eta] \).

**Lemma 4.23.** Pick distinct \( \eta, \xi \in \{1, 2, 3\} \) and let \( \{\lambda_i\}_{i=0}^N \) denote the path \( [\eta, \xi] \). For \( 0 \leq i \leq N \) the location \( \lambda_i \) is described in the table below:

| \([\eta, \xi]\) | \( \lambda_i \) |
|-----------------|----------------|
| [1, 2]          | \((N - i, i, 0)\) |
| [2, 1]          | \((i, N - i, 0)\) |
| [2, 3]          | \((0, N - i, i)\) |
| [3, 2]          | \((0, i, N - i)\) |
| [3, 1]          | \((i, 0, N - i)\) |
| [1, 3]          | \((N - i, 0, i)\) |

**Definition 4.24.** A subset \( S \) of \( \Delta_N \) is said to be geodesically closed whenever for each geodesic path \( \{\lambda_i\}_{i=0}^n \) in \( \Delta_N \), if \( S \) contains \( \lambda_0, \lambda_n \) then \( S \) contains \( \lambda_i \) for \( 0 \leq i \leq n \).

**Example 4.25.** For \( \eta \in \{1, 2, 3\} \) and \( 0 \leq n \leq N \), the set of locations in \( \Delta_N \) that have \( \eta \)-coordinate at least \( n \) is geodesically closed.
Lemma 4.26. Let $S$ and $S'$ denote geodesically closed subsets of $\Delta_N$. Then $S \cap S'$ is geodesically closed.

**Definition 4.27.** By a spanning tree of $\Delta_N$ we mean a set $T$ of edges for $\Delta_N$ that has the following property: for any two distinct locations $\lambda, \lambda'$ in $\Delta_N$ there exists a unique path in $\Delta_N$ from $\lambda$ to $\lambda'$ that involves only edges in $T$.

**Note 4.28.** Let $T$ denote a spanning tree of $\Delta_N$. Then the cardinality of $T$ is one less than the cardinality of $\Delta_N$.

**Definition 4.29.** By a 3-clique in $\Delta_N$ we mean a set of three mutually adjacent locations in $\Delta_N$. There are two kinds of 3-cliques: $\Delta$ (black) and $\nabla$ (white).

**Example 4.30.** Assume $N = 0$. Then $\Delta_N$ does not contain a 3-clique. Assume $N = 1$. Then $\Delta_N$ contains a unique black 3-clique and no white 3-clique. Assume $N = 2$. Then $\Delta_N$ contains three black 3-cliques and a unique white 3-clique.

**Lemma 4.31.** Assume $N \geq 1$. We describe a bijection from $\Delta_{N-1}$ to the set of black 3-cliques in $\Delta_N$. The bijection sends each $(r, s, t) \in \Delta_{N-1}$ to the black 3-clique consisting of the locations

$$(r + 1, s, t), \quad (r, s + 1, t), \quad (r, s, t + 1).$$

**Lemma 4.32.** Assume $N \geq 2$. We describe a bijection from $\Delta_{N-2}$ to the set of white 3-cliques in $\Delta_N$. The bijection sends each $(r, s, t) \in \Delta_{N-2}$ to the white 3-clique consisting of the locations

$$(r, s + 1, t + 1), \quad (r + 1, s, t + 1), \quad (r + 1, s + 1, t).$$

**Lemma 4.33.** For $\Delta_N$, each edge is contained in a unique black 3-clique and at most one white 3-clique.

## 5 The poset $\Delta_{\leq N}$

Throughout this section fix $N \in \mathbb{N}$. We now describe $\Delta_N$ using notions from the theory of posets. We will follow the notational conventions from [13].

**Definition 5.1.** Let $\Delta_{\leq N}$ denote the poset consisting of the set $\bigcup_{n=0}^N \Delta_n$ together with the partial order $\leq$ from Section 2. An element $\lambda$ in the poset is said to have rank $n$ whenever $\lambda \in \Delta_n$.

We mention some facts about the poset $\Delta_{\leq N}$. Pick elements $\lambda, \mu$ in the poset. Then $\lambda$ covers $\mu$ if and only if $\lambda - \mu$ is one of $e_1, e_2, e_3$. In this case $\text{rank}(\lambda) = 1 + \text{rank}(\mu)$. For $0 \leq n \leq N - 1$, each element in $\Delta_n$ is covered by precisely three elements in the poset. An element in $\Delta_N$ is not covered by any element in the poset. For $1 \leq n \leq N$, each element in $\Delta_n$ covers at least one element in the poset. The element in $\Delta_0$ does not cover any element in the poset.

Given elements $\lambda, \mu$ in $\Delta_{\leq N}$ we define an element $\lambda \wedge \mu$ in $\Delta_{\leq N}$ as follows: for $\eta \in \{1, 2, 3\}$ the $\eta$-coordinate of $\lambda \wedge \mu$ is the minimum of the $\eta$-coordinates for $\lambda$ and $\mu$. Note that $\lambda \wedge \mu \leq \lambda$ and $\lambda \wedge \mu \leq \mu$. Moreover $\nu \leq \lambda \wedge \mu$ for all elements $\nu$ in $\Delta_{\leq N}$ such that $\nu \leq \lambda$ and $\nu \leq \mu$. 

Given elements $\lambda, \mu$ in $\Delta_{\leq N}$ we define an element $\lambda \vee \mu$ in $\mathbb{R}^3$ as follows: for $\eta \in \{1, 2, 3\}$ the $\eta$-coordinate of $\lambda \vee \mu$ is the maximum of the $\eta$-coordinates for $\lambda$ and $\mu$. Note that $\lambda \vee \mu$ is contained in $\Delta_{\leq N}$ if and only if the sum of its coordinates is at most $N$. In this case we say that $\lambda \vee \mu$ exists in $\Delta_{\leq N}$. Assume that $\lambda \vee \mu$ exists in $\Delta_{\leq N}$. Then $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$. Moreover $\lambda \vee \mu \leq \nu$ for all elements $\nu$ in $\Delta_{\leq N}$ such that $\lambda \leq \nu$ and $\mu \leq \nu$. Now assume that $\Delta_{\leq N}$ does not contain an element $\nu$ such that $\lambda \leq \nu$ and $\mu \leq \nu$.

**Lemma 5.2.** For locations $\lambda, \mu$ in $\Delta_N$ the rank of $\lambda \wedge \mu$ is equal to $N - \partial(\lambda, \mu)$.

**Proof.** Write $\lambda = (r, s, t)$ and $\mu = (r', s', t')$. By construction $r + s + t = N$ and $r' + s' + t' = N$. The sum of $r - r'$, $s - s'$, $t - t'$ is zero. Permuting the coordinates of $\mathbb{R}^3$ and interchanging $\lambda, \mu$ if necessary, we may assume without loss that each of $r - r'$, $s' - s$, $t' - t$ is nonnegative. By Lemma 4.10 $\partial(\lambda, \mu) = r - r'$. By construction $\lambda \wedge \mu = (r', s, t)$ has rank $r' + s + t$. The result follows. $\square$

For $\mu \in \Delta_{\leq N}$ we now describe the set

$$\{ \lambda \in \Delta_N \mid \mu \leq \lambda \}.$$

(6)

**Lemma 5.3.** For $\mu = (r, s, t) \in \Delta_{\leq N}$ and $\lambda \in \Delta_N$, the following are equivalent:

(i) $\mu \leq \lambda$;

(ii) the 1-coordinate (resp. 2-coordinate) (resp. 3-coordinate) of $\lambda$ is at least $r$ (resp. $s$) (resp. $t$);

(iii) $\lambda$ is at distance at least $r$ (resp. $s$) (resp. $t$) from the 1-boundary (resp. 2-boundary) (resp. 3-boundary) of $\Delta_N$;

(iv) $\lambda$ is at distance at most $N - r$ (resp. $N - s$) (resp. $N - t$) from the 1-corner (resp. 2-corner) (resp. 3-corner) of $\Delta_N$;

(v) the 1-line (resp. 2-line) (resp. 3-line) of $\Delta_N$ that contains $\lambda$ has cardinality at most $N - r + 1$ (resp. $N - s + 1$) (resp. $N - t + 1$).

**Proof.** (i) $\Leftrightarrow$ (ii) By the definition of the partial order $\leq$ in Section 2.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) By Lemma 4.9 $\square$

**Lemma 5.4.** For $0 \leq n \leq N$ and $\mu \in \Delta_{N-n}$ the set (6) is equal to $\Delta_n + \mu$.

**Proof.** By construction. $\square$

**Lemma 5.5.** For $\mu \in \Delta_{\leq N}$ the set (6) is geodesically closed.

**Proof.** By Example 4.25 and Lemma 4.26 along with Lemma 5.3(i),(ii). $\square$

We have been discussing the set (6). We now consider a special case.
Lemma 5.7. Referring to Lemma 5.3, we now consider the case in which \( \mu \) is a corner of \( \Delta_{\leq N} \).

Definition 5.6. Pick \( \eta \in \{1,2,3\} \). Note by Definition 3.2 that for \( 0 \leq n \leq N \) the element \( ne_\eta \) is the \( \eta \)-corner of \( \Delta_n \). By an \( \eta \)-corner of \( \Delta_{\leq N} \) we mean one of \( \{ne_\eta\}_{n=0}^N \). By a corner of \( \Delta_{\leq N} \) we mean a 1-corner or 2-corner or 3-corner of \( \Delta_{\leq N} \).

Lemma 5.7. For \( \eta \in \{1,2,3\} \) and \( 0 \leq n \leq N \) and \( \lambda \in \Delta_N \), the following are equivalent:

(i) \( \mu \leq \lambda \), where \( \mu \) is the \( \eta \)-corner of \( \Delta_{N-n} \);

(ii) the \( \eta \)-coordinate of \( \lambda \) is at least \( N-n \);

(iii) \( \lambda \) is at distance at least \( N-n \) from the \( \eta \)-boundary of \( \Delta_N \);

(iv) \( \lambda \) is at distance at most \( n \) from the \( \eta \)-corner of \( \Delta_N \);

(v) the \( \eta \)-line containing \( \lambda \) has cardinality at most \( n+1 \).

Proof. For the \( \eta \)-corner \( \mu \) of \( \Delta_{N-n} \), the \( \eta \)-coordinate is \( N-n \) and the other two coordinates are 0. The result follows in view of Lemma 5.3.

Definition 5.8. By a flag in \( \Delta_{\leq N} \) we mean a sequence \( \{\lambda_n\}_{n=0}^N \) such that \( \lambda_n \in \Delta_n \) for \( 0 \leq n \leq N \) and \( \lambda_{n-1} \leq \lambda_n \) for \( 1 \leq n \leq N \).

Definition 5.9. Pick \( \eta \in \{1,2,3\} \). Observe that the sequence of \( \eta \)-corners \( \{ne_\eta\}_{n=0}^N \) is a flag in \( \Delta_{\leq N} \). We denote this flag by \([\eta]\).

6 The vector space \( V \) and the poset \( \mathcal{P}(V) \)

From now on, let \( F \) denote a field. Fix \( N \in \mathbb{N} \), and let \( V \) denote a vector space over \( F \) with dimension \( N+1 \). Let \( \text{End}(V) \) denote the \( F \)-algebra consisting of the \( F \)-linear maps from \( V \) to \( V \). Let \( I \in \text{End}(V) \) denote the identity map on \( V \). By a decomposition of \( V \) we mean a sequence \( \{V_i\}_{i=0}^N \) consisting of one-dimensional subspaces of \( V \) such that \( V = \sum_{i=0}^N V_i \) (direct sum). For a decomposition of \( V \), its inversion is a decomposition of \( V \). Let \( \{v_i\}_{i=0}^N \) denote a basis for \( V \) and let \( \{V_i\}_{i=0}^N \) denote a decomposition of \( V \). We say that \( \{v_i\}_{i=0}^N \) induces \( \{V_i\}_{i=0}^N \) whenever \( v_i \in V_i \) for \( 0 \leq i \leq N \).

Let \( \mathcal{P}(V) \) denote the set of nonzero subspaces of \( V \). The containment relation \( \subseteq \) is a partial order on \( \mathcal{P}(V) \). For \( 1 \leq i \leq N+1 \) let \( \mathcal{P}_i(V) \) denote the set of \( i \)-dimensional subspaces of \( V \). This yields a partition \( \mathcal{P}(V) = \bigcup_{i=1}^{N+1} \mathcal{P}_i(V) \).

By a flag in \( \mathcal{P}(V) \) we mean a sequence \( \{U_i\}_{i=0}^N \) such that \( U_i \in \mathcal{P}_{i+1}(V) \) for \( 0 \leq i \leq N \) and \( U_{i-1} \subseteq U_i \) for \( 1 \leq i \leq N \). For notational convenience, a flag in \( \mathcal{P}(V) \) is said to be on \( V \). The following construction yields a flag on \( V \). Let \( \{V_i\}_{i=0}^N \) denote a decomposition of \( V \). For \( 0 \leq i \leq N \) define \( U_i = V_0 + \cdots + V_i \). Then the sequence \( \{U_i\}_{i=0}^N \) is a flag on \( V \). This flag is said to be induced by the decomposition \( \{V_i\}_{i=0}^N \). Suppose we are given two flags on \( V \), denoted \( \{U_i\}_{i=0}^N \) and \( \{U'_i\}_{i=0}^N \). Then the following are equivalent: (i) \( U_i \cap U'_j = 0 \) if \( i+j < N \) \((0 \leq i,j \leq N)\); (ii) there exists a decomposition \( \{V_i\}_{i=0}^N \) of \( V \) that induces \( \{U_i\}_{i=0}^N \) and whose inversion induces \( \{U'_i\}_{i=0}^N \). The flags are called opposite whenever (i), (ii) hold. In this case \( V_i = U_i \cap U'_{N-i} \) for \( 0 \leq i \leq N \).

We mention a result for later use.
Lemma 6.1. Let \( \{U_i\}_{i=0}^N \) and \( \{U'_i\}_{i=0}^N \) denote opposite flags on \( V \). Pick integers \( r, s \) \((0 \leq r, s \leq N)\) such that \( r + s \geq N \). Then \( U_r \cap U'_s \) has dimension \( r + s - N + 1 \).

Proof. There exists a decomposition \( \{V_i\}_{i=0}^N \) of \( V \) that induces \( \{U_i\}_{i=0}^N \) and whose inversion induces \( \{U'_i\}_{i=0}^N \). Observe that \( U_r \cap U'_s = \sum_{i=N-s}^r V_i \). The sum is direct, so it has dimension \( r + s - N + 1 \). \( \square \)

7 Billiard Arrays

Throughout this section the following notation is in effect. Fix \( N \in \mathbb{N} \). Let \( V \) denote a vector space over \( \mathbb{F} \) with dimension \( N + 1 \).

Definition 7.1. By a Billiard Array on \( V \) we mean a function \( B : \Delta_N \to \mathcal{P}_1(V), \lambda \mapsto B_\lambda \) that satisfies the following conditions:

(i) for each line \( L \) in \( \Delta_N \) the sum \( \sum_{\lambda \in L} B_\lambda \) is direct;
(ii) for each black 3-clique \( \mathcal{C} \) in \( \Delta_N \) the sum \( \sum_{\lambda \in \mathcal{C}} B_\lambda \) is not direct.

We say that \( B \) is over \( \mathbb{F} \). We call \( V \) the underlying vector space. We call \( N \) the diameter of \( B \).

Note 7.2. We will show that the Billiard Array \( B \) in Definition 7.1 is injective.

Let \( B \) denote a Billiard Array on \( V \), as in Definition 7.1. We view \( B \) as an arrangement of one-dimensional subspaces of \( V \) into a triangular array, with \( B_\lambda \) at location \( \lambda \) for all \( \lambda \in \Delta_N \). For \( U \in \mathcal{P}_1(V) \) and \( \lambda \in \Delta_N \), we say that \( U \) is included in \( B \) at location \( \lambda \) whenever \( U = B_\lambda \).

Lemma 7.3. Let \( B \) denote a Billiard Array on \( V \). Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a black 3-clique. Then the subspace \( B_\lambda + B_\mu + B_\nu \) is equal to each of

\[
B_\lambda + B_\mu, \quad B_\mu + B_\nu, \quad B_\nu + B_\lambda.
\]

Moreover, in line (7) each sum is direct.

Proof. Any two of \( \lambda, \mu, \nu \) are collinear, so in line (7) each sum is direct by Definition 7.1(i). The sum \( B_\lambda + B_\mu + B_\nu \) is not direct by Definition 7.1(ii). The result follows. \( \square \)

Corollary 7.4. Let \( B \) denote a Billiard Array on \( V \). Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a black 3-clique. Then each of \( B_\lambda, B_\mu, B_\nu \) is contained in the sum of the other two.

Lemma 7.5. Let \( B \) denote a Billiard Array on \( V \). Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a white 3-clique. Then the sum \( B_\lambda + B_\mu + B_\nu \) is direct.

Proof. Permuting \( \lambda, \mu, \nu \) if necessary, we may assume that

\[
\mu - \lambda = \alpha, \quad \nu - \mu = \beta, \quad \lambda - \nu = \gamma,
\]

where \( \alpha, \beta, \gamma \) are from (2). Consider the locations \( \mu + \gamma, \mu, \mu - \gamma \) in \( \Delta_N \). These locations are collinear, so by Definition 7.1(i) the sum \( B_{\mu+\gamma} + B_\mu + B_{\mu-\gamma} \) is direct. The locations \( \lambda, \mu, \mu + \gamma \) form a black 3-clique, so \( B_{\mu+\gamma} \) is contained in \( B_\lambda + B_\mu \) by Corollary 7.4. The locations \( \mu, \nu, \mu - \gamma \) form a black 3-clique, so \( B_{\mu-\gamma} \) is contained in \( B_\mu + B_\nu \) by Corollary 7.4. By these comments \( B_{\mu+\gamma} + B_\mu + B_{\mu-\gamma} \) is contained in \( B_\lambda + B_\mu + B_\nu \). The result follows. \( \square \)
Lemma 7.6. Let $B$ denote a Billiard Array on $V$. Pick $\eta \in \{1, 2, 3\}$ and let $L$ denote the $\eta$-boundary of $\Delta_N$. Then the sum $V = \sum_{\lambda \in L} B_\lambda$ is direct.

Proof. The sum $\sum_{\lambda \in L} B_\lambda$ is direct by Definition 7.1(i) and since $L$ is a line. The sum is equal to $V$, since $L$ has cardinality $N + 1$ and $V$ has dimension $N + 1$. \qed

Corollary 7.7. Let $B$ denote a Billiard Array on $V$. Then $V$ is spanned by $\{B_\lambda\}_{\lambda \in \Delta_N}$.

Proof. By Lemma 7.6. \qed

We now describe the Billiard Arrays of diameter 0 and 1.

Example 7.8. Let $B : \Delta_N \to \mathcal{P}_1(V)$ denote any function.

(i) Assume $N = 0$, so that $\Delta_N$ and $\mathcal{P}_1(V)$ each have cardinality 1. Then $B$ is a Billiard Array on $V$.

(ii) Assume $N = 1$. Then $B$ is a Billiard Array on $V$ if and only if $B$ is injective.

Definition 7.9. Let $V'$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$. Let $B$ (resp. $B'$) denote a Billiard Array on $V$ (resp. $V'$). By an isomorphism of Billiard Arrays from $B$ to $B'$ we mean an $\mathbb{F}$-linear bijection $\sigma : V \to V'$ that sends $B_\lambda \mapsto B'_\lambda$ for all $\lambda \in \Delta_N$. The Billiard Arrays $B$ and $B'$ are called isomorphic whenever there exists an isomorphism of Billiard Arrays from $B$ to $B'$.

Example 7.10. For $N = 0$ or $N = 1$, any two Billiard Arrays over $\mathbb{F}$ of diameter $N$ are isomorphic.

8 Concrete Billiard Arrays

Throughout this section the following notation is in effect. Fix $N \in \mathbb{N}$. Let $V$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$.

Definition 8.1. By a Concrete Billiard Array on $V$ we mean a function $\mathcal{B} : \Delta_N \to V$, $\lambda \mapsto \mathcal{B}_\lambda$ that satisfies the following conditions:

(i) for each line $L$ in $\Delta_N$ the vectors $\{\mathcal{B}_\lambda\}_{\lambda \in L}$ are linearly independent;

(ii) for each black 3-clique $C$ in $\Delta_N$ the vectors $\{\mathcal{B}_\lambda\}_{\lambda \in C}$ are linearly dependent.

We say that $\mathcal{B}$ is over $\mathbb{F}$. We call $V$ the underlying vector space. We call $N$ the diameter of $\mathcal{B}$.

Lemma 8.2. Let $\mathcal{B}$ denote a Concrete Billiard Array on $V$. Then $\mathcal{B}_\lambda \neq 0$ for all $\lambda \in \Delta_N$.

Proof. There exists a line $L$ in $\Delta_N$ that contains $\lambda$. The result follows in view of Definition 8.1(i). \qed
Billiard Arrays and Concrete Billiard Arrays are related as follows. Let $B$ denote a Billiard Array on $V$. For $\lambda \in \Delta_N$ let $B_\lambda$ denote a nonzero vector in $B_\lambda$. Then the function $B : \Delta_N \to V$, $\lambda \mapsto B_\lambda$ is a Concrete Billiard Array on $V$. Conversely, let $B$ denote a Concrete Billiard Array on $V$. For $\lambda \in \Delta_N$ let $B_\lambda$ denote the subspace of $V$ spanned by $B_\lambda$. Note that $B_\lambda \in P_1(V)$ by Lemma 8.2. The function $B : \Delta_N \to P_1(V)$, $\lambda \mapsto B_\lambda$ is a Billiard Array on $V$.

**Definition 8.3.** Let $B$ denote a Billiard Array on $V$, and let $B$ denote a Concrete Billiard Array on $V$. We say that $B$, $B$ correspond whenever $B_\lambda$ spans $B_\lambda$ for all $\lambda \in \Delta_N$.

Let $B$ denote a Concrete Billiard Array on $V$. For $v \in V$ and $\lambda \in \Delta_N$, we say that $v$ is included in $B$ at location $\lambda$ whenever $v = B_\lambda$.

**Lemma 8.4.** Let $B$ denote a Concrete Billiard Array on $V$. Let $\lambda, \mu, \nu$ denote locations in $\Delta_N$ that form a black 3-clique. Then $B_\lambda, B_\mu, B_\nu$ span a 2-dimensional subspace of $V$. Any two of $B_\lambda, B_\mu, B_\nu$ form a basis for this subspace.

**Proof.** Use Lemma 7.3. \qed

**Lemma 8.5.** Let $B$ denote a Concrete Billiard Array on $V$. Let $\lambda, \mu, \nu$ denote locations in $\Delta_N$ that form a white 3-clique. Then the vectors $B_\lambda, B_\mu, B_\nu$ are linearly independent.

**Proof.** Use Lemma 7.5. \qed

**Lemma 8.6.** Let $B$ denote a Concrete Billiard Array on $V$. Pick $\eta \in \{1, 2, 3\}$ and let $L$ denote the $\eta$-boundary of $\Delta_N$. Then the vectors $\{B_\lambda\}_{\lambda \in L}$ form a basis for $V$.

**Proof.** Use Lemma 7.6. \qed

**Corollary 8.7.** Let $B$ denote a Concrete Billiard Array on $V$. Then $V$ is spanned by the vectors $\{B_\lambda\}_{\lambda \in \Delta_N}$.

**Proof.** By Corollary 7.7 or Lemma 8.6. \qed

We now describe the Concrete Billiard Arrays of diameter 0 and 1.

**Example 8.8.** Let $B : \Delta_N \to V$ denote any function.

(i) Assume $N = 0$, so that $\Delta_N$ contains a unique element $\lambda$. Then $B$ is a Concrete Billiard Array on $V$ if and only if $B_\lambda \neq 0$.

(ii) Assume $N = 1$. Then $B$ is a Concrete Billiard Array on $V$ if and only if any two of $\{B_\lambda\}_{\lambda \in \Delta_N}$ are linearly independent.

**Lemma 8.9.** Let $B$ denote a Concrete Billiard Array on $V$. For all $\lambda \in \Delta_N$ let $\kappa_\lambda$ denote a nonzero scalar in $\mathbb{F}$. Then the function $B' : \Delta_N \to V$, $\lambda \mapsto \kappa_\lambda B_\lambda$ is a Concrete Billiard Array on $V$.

**Definition 8.10.** Let $B, B'$ denote Concrete Billiard Arrays on $V$. We say that $B, B'$ are associates whenever there exist nonzero scalars $\{\kappa_\lambda\}_{\lambda \in \Delta_N}$ in $\mathbb{F}$ such that $B'_{\lambda} = \kappa_\lambda B_\lambda$ for all $\lambda \in \Delta_N$. The relation of being associates is an equivalence relation.
Lemma 8.11. Let $\mathcal{B}, \mathcal{B}'$ denote Concrete Billiard Arrays on $V$. Then $\mathcal{B}, \mathcal{B}'$ are associates if and only if they correspond to the same Billiard Array.

**Proof.** By Definitions 8.3, 8.10. \qed

**Example 8.12.** Referring to Definition 8.10 if $N = 0$ then $\mathcal{B}, \mathcal{B}'$ are associates.

**Definition 8.13.** Let $\mathcal{B}, \mathcal{B}'$ denote Concrete Billiard Arrays on $V$. Then $\mathcal{B}, \mathcal{B}'$ are called *relatives* whenever there exists $0 \neq \kappa \in \mathbb{F}$ such that $\mathcal{B}'_\lambda = \kappa \mathcal{B}_\lambda$ for all $\lambda \in \Delta_N$. The relation of being relatives is an equivalence relation.

**Example 8.14.** Referring to Definition 8.13 if $N = 0$ then $\mathcal{B}, \mathcal{B}'$ are relatives.

**Definition 8.15.** Let $V'$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$. Let $\mathcal{B}$ (resp. $\mathcal{B}'$) denote a Concrete Billiard Array on $V$ (resp. $V'$). By an *isomorphism of Concrete Billiard Arrays from $\mathcal{B}$ to $\mathcal{B}'$* we mean an $\mathbb{F}$-linear bijection $\sigma : V \to V'$ that sends $\mathcal{B}_\lambda \mapsto \mathcal{B}'_\lambda$ for all $\lambda \in \Delta_N$. We say that $\mathcal{B}, \mathcal{B}'$ are isomorphic whenever there exists an isomorphism of Concrete Billiard Arrays from $\mathcal{B}$ to $\mathcal{B}'$. In this case the isomorphism is unique. Isomorphism is an equivalence relation.

**Example 8.16.** Referring to Definition 8.15 if $N = 0$ then $\mathcal{B}, \mathcal{B}'$ are isomorphic.

**Definition 8.17.** Let $V'$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$. Let $\mathcal{B}$ (resp. $\mathcal{B}'$) denote a Concrete Billiard Array on $V$ (resp. $V'$). Then $\mathcal{B}, \mathcal{B}'$ are called *similar* whenever their corresponding Billiard Arrays are isomorphic in the sense of Definition 7.9. Similarity is an equivalence relation.

**Example 8.18.** Referring to Definition 8.17 assume $N = 0$ or $N = 1$. Then $\mathcal{B}, \mathcal{B}'$ are similar.

In each of the last four definitions we gave an equivalence relation for Concrete Billiard Arrays.

**Lemma 8.19.** The above equivalence relations obey the following logical implications:

- relative $\longrightarrow$ isomorphic
- associates $\longrightarrow$ similar

**Proof.** This is routinely verified. \qed

We have a few comments.

**Proposition 8.20.** Let $V'$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$. Let $\mathcal{B}$ (resp. $\mathcal{B}'$) denote a Concrete Billiard Array on $V$ (resp. $V'$). Then the following are equivalent:

(i) $\mathcal{B}$ and $\mathcal{B}'$ are similar;

(ii) $\mathcal{B}$ is isomorphic to a Concrete Billiard Array associated with $\mathcal{B}'$;
(iii) \( \mathcal{B} \) is associated with a Concrete Billiard Array isomorphic to \( \mathcal{B}' \).

**Proof.** Let \( B \) (resp. \( B' \)) denote the Billiard Array on \( V \) (resp. \( V' \)) that corresponds to \( \mathcal{B} \) (resp. \( \mathcal{B}' \)).

(i) \( \Rightarrow \) (ii) By Definition 8.17 \( B, B' \) are isomorphic. Let \( \sigma : V \to V' \) denote an isomorphism of Billiard Arrays from \( B \) to \( B' \). Define a function \( B'' : \Delta_N \to V', \lambda \mapsto \sigma(B_\lambda) \). By construction, \( \mathcal{B}'' \) is a Concrete Billiard Array that is isomorphic to \( \mathcal{B} \) and corresponds to \( B' \). Note that \( \mathcal{B}', \mathcal{B}'' \) are associates since they both correspond to \( \mathcal{B} \).

(ii) \( \Rightarrow \) (i) Let \( \mathcal{B}'' \) denote a Concrete Billiard Array that is isomorphic to \( \mathcal{B} \) and associated with \( \mathcal{B}' \). Since \( \mathcal{B}', \mathcal{B}'' \) are associates, \( \mathcal{B}'' \) must correspond to \( \mathcal{B}' \). Let \( \sigma \) denote an isomorphism of Concrete Billiard Arrays from \( \mathcal{B} \) to \( \mathcal{B}'' \). Then \( \sigma \) is an isomorphism of Billiard Arrays from \( B \) to \( B' \). Therefore \( B, B' \) are isomorphic so \( \mathcal{B}, \mathcal{B}' \) are similar.

(i) \( \Leftrightarrow \) (iii) Interchange the roles of \( \mathcal{B}, \mathcal{B}' \) in the proof of (i) \( \Leftrightarrow \) (ii). \( \square \)

**Lemma 8.21.** Consider the map which takes a Concrete Billiard Array to the corresponding Billiard Array. This map induces a bijection between the following two sets:

(i) the similarity classes of Concrete Billiard Arrays over \( \mathbb{F} \) that have diameter \( N \);

(ii) the isomorphism classes of Billiard Arrays over \( \mathbb{F} \) that have diameter \( N \).

**Proof.** By Definition 8.17 the given map induces an injective function from set (i) to set (ii). The function is surjective by the comments above Definition 8.13. Therefore the function is a bijection. \( \square \)

## 9 Billiard Arrays and the poset \( \Delta_{\leq N} \)

Throughout this section the following notation is in effect. Fix \( N \in \mathbb{N} \). Let \( V \) denote a vector space over \( \mathbb{F} \) with dimension \( N + 1 \). Let \( B \) denote a Billiard Array on \( V \).

By definition, \( B \) is a function \( \Delta_N \to \mathcal{P}(V) \). We now extend \( B \) to a function \( B : \Delta_{\leq N} \to \mathcal{P}(V) \).

**Definition 9.1.** For \( \mu \in \Delta_{\leq N} \) define

\[
B_{\mu} = \sum_{\lambda \in \Delta_N, \mu \leq \lambda} B_{\lambda}.
\]  

(8)

Thus \( B_{\mu} \in \mathcal{P}(V) \).

Pick \( \mu \in \Delta_{\leq N} \). Let \( n = N - \text{rank}(\mu) \), so that \( \mu \in \Delta_{N-n} \). Evaluating (8) using Lemma 5.4 we obtain

\[
B_{\mu} = \sum_{\nu \in \Delta_n} B_{\mu+\nu}.
\]  

(9)

Our next goal is to show that the function \( \Delta_n \to \mathcal{P}_1(\mu), \nu \mapsto B_{\mu+\nu} \) is a Billiard Array on \( B_{\mu} \). To this end, we show that \( B_{\mu} \) has dimension \( n + 1 \).
Lemma 9.2. With the above notation, pick $\xi \in \{1, 2, 3\}$ and let $K$ denote the $\xi$-boundary of $\Delta_n$. Then the sum

$$B_\mu = \sum_{\nu \in K} B_{\mu + \nu} \quad (10)$$

is direct. Moreover $B_\mu$ has dimension $n + 1$.

Proof. Define $B'_\mu = \sum_{\nu \in K} B_{\mu + \nu}$. The preceding sum is direct, by Definition 7.1(i) and since $\{\mu + \nu\}_{\nu \in K}$ are collinear locations in $\Delta_N$. We show $B_\mu = B'_\mu$. We have $B'_\mu \subseteq B_\mu$ by (9). To obtain $B_\mu \subseteq B'_\mu$, for $\nu \in \Delta_n$ use Corollary 7.4 and induction on the $\xi$-coordinate of $\nu$ to obtain $B_{\mu + \nu} \subseteq B'_\mu$. This and (9) yield $B_\mu \subseteq B'_\mu$, and therefore $B_\mu = B'_\mu$. We have shown that the sum (10) is direct. The set $K$ has cardinality $n + 1$, so $B_\mu$ has dimension $n + 1$. \[\square\]

We emphasize one aspect of Lemma 9.2.

Lemma 9.3. For $\mu \in \Delta_{\leq N}$ we have $B_\mu \in \mathcal{P}_{n+1}(V)$, where $n = N - \text{rank}(\mu)$.

Proof. We verify that the given function satisfies the conditions of Definition 7.1. Let $L$ denote a line in $\Delta_n$. Then the elements $\{\mu + \nu\}_{\nu \in L}$ are collinear locations in $\Delta_N$. Consequently the given function satisfies Definition 7.1(i). Let $C$ denote a black 3-clique in $\Delta_n$. Then the locations $\{\mu + \nu\}_{\nu \in C}$ form a black 3-clique in $\Delta_N$. Therefore the given function satisfies Definition 7.1(ii). The result follows. \[\square\]

As we work with the elements $B_\mu$ from Definition 9.1 we often encounter the case in which $\mu$ is a corner of $\Delta_{\leq N}$. We now consider this case.

Lemma 9.5. Pick $\eta \in \{1, 2, 3\}$ and $0 \leq n \leq N$. Let $\mu$ denote the $\eta$-corner of $\Delta_{N-n}$. Then $B_\mu$ is described as follows.

(i) We have $B_\mu = \sum_\lambda B_\lambda$, where the sum is over the locations $\lambda$ in $\Delta_N$ at distance $\leq n$ from the $\eta$-corner of $\Delta_N$.

(ii) The sum $B_\mu = \sum_\lambda B_\lambda$ is direct, where the sum is over the locations $\lambda$ in $\Delta_N$ at distance $n$ from the $\eta$-corner of $\Delta_N$.

(iii) Pick $\xi \in \{1, 2, 3\}$ other than $\eta$. Then the sum $B_\mu = \sum_\lambda B_\lambda$ is direct, where the sum is over the locations $\lambda$ on the $\xi$-boundary of $\Delta_N$ at distance $\leq n$ from the $\eta$-corner of $\Delta_N$.

Proof. (i) By Lemma 5.7(i),(iv) and Definition 9.1

(ii), (iii) In Lemma 9.2, take $\mu$ to be the $\eta$-corner of $\Delta_{N-n}$. \[\square\]

At the end of Section 5 we discussed the flags in the poset $\Delta_{\leq N}$. In Section 6 we discussed the flags in the poset $\mathcal{P}(V)$. We now consider how the flags in these two posets are related.

Lemma 9.6. Let $\mu, \nu$ denote elements in $\Delta_{\leq N}$ such that $\mu \leq \nu$. Then $B_\nu \subseteq B_\mu$.

Proof. Use Definition 9.1. \[\square\]
Lemma 9.7. Let $\{\lambda_n\}_{n=0}^N$ denote a flag in $\Delta_{\leq N}$. Define $U_i = B_{\lambda_{N-i}}$ for $0 \leq i \leq N$. Then the sequence $\{U_i\}_{i=0}^N$ is a flag on $V$.

Proof. By Lemma 9.6 $U_{i-1} \subseteq U_i$ for $1 \leq i \leq N$. By Lemma 9.3 the subspace $U_i$ has dimension $i+1$ for $0 \leq i \leq N$. The result follows.

Definition 9.8. Referring to Lemma 9.7 the flag $\{U_i\}_{i=0}^N$ is called the $B$-flag induced by $\{\lambda_n\}_{n=0}^N$.

Definition 9.9. Pick $\eta \in \{1, 2, 3\}$. Recall from Definition 5.9 the flag $[\eta]$ in $\Delta_{\leq N}$. The $B$-flag induced by $[\eta]$ will be called the $B$-flag $[\eta]$. This is a flag on $V$.

The next result is meant to clarify Definition 9.9.

Lemma 9.10. Pick $\eta \in \{1, 2, 3\}$. Let $\{U_n\}_{n=0}^N$ denote the $B$-flag $[\eta]$. Then for $0 \leq n \leq N$, $U_n$ is equal to the space $B_{\mu}$ from Lemma 9.3.

Proof. By Lemma 9.7 and Definitions 9.8, 9.9.

Lemma 9.11. Pick $\eta \in \{1, 2, 3\}$. Let $\{U_n\}_{n=0}^N$ denote the $B$-flag $[\eta]$. Then for $1 \leq n \leq N$ the sum $U_n = U_{n-1} + B_{\lambda}$ is direct, where $\lambda$ is any location in $\Delta_N$ at distance $n$ from the $\eta$-corner.

Proof. We first show that $U_{n-1} + B_{\lambda}$ is independent of the choice of $\lambda$. Let $L$ denote the set of locations in $\Delta_N$ that are at distance $n$ from the $\eta$-corner. By Lemma 4.3 $L$ is the $\eta$-line with cardinality $n+1$. By construction $\lambda \in L$. Let $\mu \in L$ be adjacent to $\lambda$. By Lemma 4.33 there exists a unique $\nu \in \Delta_N$ such that $\lambda, \mu, \nu$ form a black 3-clique. The location $\nu$ is at distance $n-1$ from the $\eta$-corner of $\Delta_N$, so $B_{\nu} \subseteq U_{n-1}$ in view of Lemmas 9.5(i), 9.10. By Lemma 7.3 we have $B_{\lambda} + B_{\nu} = B_{\mu} + B_{\nu}$. By these comments $U_{n-1} + B_{\lambda} = U_{n-1} + B_{\mu}$. It follows that $U_{n-1} + B_{\lambda}$ is independent of the choice of $\lambda$. So we may assume that $\lambda$ is on the boundary of $\Delta_N$. Now there exists $\xi \in \{1, 2, 3\}$ such that $\xi \neq \eta$ and $\lambda$ is on the $\xi$-boundary of $\Delta_N$. By Lemmas 9.3(iii), 9.10 we have a direct sum $U_{n-1} + \sum_\xi B_\xi$, where the sum is over the locations $\xi$ on the $\xi$-boundary of $\Delta_N$ at distance $\leq n-1$ from the $\eta$-corner. Similarly we have a direct sum $U_n = \sum_\xi B_\xi$, where the sum is over the locations $\xi$ on the $\xi$-boundary of $\Delta_N$ at distance $\leq n$ from the $\eta$-corner. By assumption $\lambda$ is on the $\xi$-boundary of $\Delta_N$ at distance $n$ from the $\eta$-corner. By these comments the sum $U_n = U_{n-1} + B_{\lambda}$ is direct.

Proposition 9.12. Pick $\eta \in \{1, 2, 3\}$. Let $S$ denote a subset of $\Delta_N$ that is $\eta$-geodesic in the sense of Definition 4.14. Then the sum $\sum_{\lambda \in S} B_{\lambda}$ is direct.

Proof. We assume that the assertion is false, and get a contradiction. By assumption there exists a counterexample $S$. Without loss, we may assume that among all the counterexamples, the cardinality of $S$ is minimal. This cardinality is at least 2. Let $\mu$ denote the location in $S$ that has minimal $\eta$-coordinate, and abbreviate $R = S \setminus \{\mu\}$. By construction the sum $\sum_{\lambda \in S} B_{\lambda}$ is not direct, and the sum $\sum_{\lambda \in R} B_{\lambda}$ is direct. By Definition 7.1 $B_{\mu}$ has dimension 1. By these comments $B_{\mu} \subseteq \sum_{\lambda \in R} B_{\lambda}$. Denote the $\eta$-coordinate of $\mu$ by $N-n$, and observe $n \geq 1$. By construction $\mu$ is at distance $n$ from the $\eta$-corner of $\Delta_N$, and each element of $R$ is at distance $\leq n-1$ from the $\eta$-corner of $\Delta_N$. Let $\{U_i\}_{i=0}^N$ denote the $B$-flag $[\eta]$. By Lemma 9.11 the sum $U_n = U_{n-1} + B_{\mu}$ is direct. Using Lemmas 9.3(i), 9.10 we obtain $B_{\lambda} \subseteq U_{n-1}$ for all $\lambda \in R$. Therefore $B_{\mu} \subseteq U_{n-1}$, for a contradiction. The result follows.
Corollary 9.13. The function $B$ is injective.

Proof. Let $\lambda, \mu$ denote locations in $\Delta_N$ such that $B_\lambda = B_\mu$. By Proposition 9.12, the locations $\lambda, \mu$ have the same $\eta$-coordinate for all $\eta \in \{1, 2, 3\}$. Therefore $\lambda = \mu$. \hfill \Box

The following definition is for notational convenience.

Definition 9.14. Let $B$ denote a Billiard Array on $V$. Define $B_\lambda = 0$ for all $\lambda \in \mathbb{R}^3$ such that $\lambda \notin \Delta_{\leq N}$.

10 Bases, decompositions, and flags

Throughout this section the following notation is in effect. Fix $N \in \mathbb{N}$. Let $V$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$. Let $B$ denote a Concrete Billiard Array on $V$, and let $B$ denote the corresponding Billiard Array on $V$.

Lemma 10.1. Let $\{\lambda_i\}_{i=0}^n$ denote a geodesic path in $\Delta_N$, and define $\mu = \lambda_0 \land \lambda_n$. Then the following (i)–(iii) hold:

(i) $\mu \leq \lambda_i$ for $0 \leq i \leq n$;
(ii) the sequence $\{B_{\lambda_i}\}_{i=0}^n$ is a decomposition of $B_\mu$;
(iii) the sequence $\{B_{\lambda_i}\}_{i=0}^n$ is a basis for $B_\mu$.

Proof. (i) The set $\{\lambda \in \Delta_N \mid \mu \leq \lambda\}$ is geodesically closed by Lemma 5.3. By construction $\mu \leq \lambda_0$ and $\mu \leq \lambda_n$. Now by Definition 4.24, $\mu \leq \lambda_i$ for $0 \leq i \leq n$.
(ii) By Definition 9.1 and (i) above, $B_{\lambda_i} \subseteq B_{\mu}$ for $0 \leq i \leq n$. Observe that $n = \partial(\lambda_0, \lambda_n)$ by Definition 4.17, so $\mu \in \Delta_{N-n}$ by Lemma 5.2. Now $B_{\mu}$ has dimension $n + 1$ by Lemma 9.3.
(iii) By (ii) and since $B_{\lambda}$ spans $B_{\lambda}$ for all $\lambda \in \Delta_N$. \hfill \Box

Definition 10.2. Referring to Lemma 10.1, the decomposition $\{B_{\lambda_i}\}_{i=0}^n$ (resp. basis $\{B_{\lambda_i}\}_{i=0}^n$) is said to be $B$-induced (resp. $B$-induced) by the path $\{\lambda_i\}_{i=0}^n$.

Definition 10.3. For distinct $\eta, \xi \in \{1, 2, 3\}$ we define a decomposition of $V$ called the $B$-decomposition $[\eta, \xi]$. This decomposition is $B$-induced by the path $[\eta, \xi]$ in $\Delta_N$ from Definition 4.21. Similarly, we define a basis of $V$ called the $B$-basis $[\eta, \xi]$. This basis is $B$-induced by the path $[\eta, \xi]$ in $\Delta_N$. By construction, the $B$-basis $[\eta, \xi]$ of $V$ induces the $B$-decomposition $[\eta, \xi]$ of $V$, in the sense of the first paragraph of Section 6.

Lemma 10.4. For distinct $\eta, \xi \in \{1, 2, 3\}$ consider the $B$-decomposition (resp. $B$-basis) $[\eta, \xi]$ of $V$. For $0 \leq i \leq N$ the $i$-component of this decomposition (resp. basis) is included in $B$ (resp. $B$) at the location $\lambda_i$ given in Lemma 4.23.

Proof. By Lemma 4.23 and Definitions 10.2, 10.3. \hfill \Box
Lemma 10.5. For distinct $\eta, \xi \in \{1, 2, 3\}$ the $B$-decomposition (resp. $B$-basis) $[\eta, \xi]$ is the inversion of the $B$-decomposition (resp. $B$-basis) $[\xi, \eta]$.

Proof. By Lemma 4.22 or Lemma 10.4. $\square$

Lemma 10.6. For distinct $\eta, \xi \in \{1, 2, 3\}$ the $B$-decomposition $[\eta, \xi]$ of $V$ induces the $B$-flag $[\eta]$ on $V$.

Proof. By Lemmas 9.5(iii), 9.10. $\square$

Lemma 10.7. The $B$-flags $[1]$, $[2]$, $[3]$ on $V$ are mutually opposite.

Proof. Pick distinct $\eta, \xi \in \{1, 2, 3\}$. By Lemma 10.6, the $B$-decomposition $[\eta, \xi]$ induces the $B$-flag $[\eta]$, and the inversion of this decomposition induces the $B$-flag $[\xi]$. Now the $B$-flags $[\eta]$, $[\xi]$ are opposite in view of Lemma 10.5. $\square$

Lemma 10.8. For distinct $\eta, \xi \in \{1, 2, 3\}$ and $0 \leq i \leq N$, the $i$-component of the $B$-decomposition $[\eta, \xi]$ is equal to the intersection of the following two subspaces of $V$:

(i) component $i$ of the $B$-flag $[\eta]$;

(ii) component $N - i$ of the $B$-flag $[\xi]$.

Proof. The $B$-decomposition $[\eta, \xi]$ induces the $B$-flag $[\eta]$, and the inversion of this decomposition induces the $B$-flag $[\xi]$. $\square$

11 More on flags

Throughout this section the following notation is in effect. Fix $N \in \mathbb{N}$. Let $V$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$. Let $B$ denote a Billiard Array on $V$.

Proposition 11.1. Pick integers $r, s, t$ ($0 \leq r, s, t \leq N$) and consider the intersection of the following three sets:

(i) component $N - r$ of the $B$-flag $[1]$;

(ii) component $N - s$ of the $B$-flag $[2]$;

(iii) component $N - t$ of the $B$-flag $[3]$.

If $r + s + t > N$ then the intersection is zero. If $r + s + t \leq N$ then the intersection is $B_\mu$, where $\mu = (r, s, t)$.

Proof. Denote the sets in (i), (ii), (iii) by $S_1, S_2, S_3$ respectively. Let $L$ denote the set of locations in $\Delta_N$ at distance $N - s$ from the 2-corner of $\Delta_N$. Thus $L$ is the 2-line with cardinality $N - s + 1$. The set $L$ consists of the locations $\{\lambda_i\}_{i=0}^{N-s}$ where $\lambda_i = (N - s - i, s, i)$ for $0 \leq i \leq N - s$. By Lemmas 9.5(ii), 9.10 the sequence $\{B_{\lambda_i}\}_{i=0}^{N-s}$ is a decomposition of $S_2$. We now describe $S_1 \cap S_2$. By Lemma 10.7 the $B$-flags $[1]$, $[2]$ are opposite. For the moment assume $r + s > N$. Then $S_1 \cap S_2 = 0$, so $S_1 \cap S_2 \cap S_3 = 0$ and we are done. Next assume $r + s \leq N$. Then $S_1 \cap S_2$ has dimension $N - r - s + 1$ by Lemma 6.1. For $0 \leq i \leq N - s$
the location \( \lambda_i \) is at distance \( s + i \) from the 1-corner of \( \Delta_N \). For \( 0 \leq i \leq N - r - s \) this distance is at most \( N - r \), so \( B_{\lambda_i} \subseteq S_1 \) by Lemmas 9.5(i), 9.10 so \( B_{\lambda_i} \subseteq S_1 \cap S_2 \). Therefore \( \sum_{i=0}^{N-r-s} B_{\lambda_i} \subseteq S_1 \cap S_2 \). In this inclusion each side has dimension \( N - r - s + 1 \), so equality holds. In other words

\[
S_1 \cap S_2 = \sum_{i=0}^{N-r-s} B_{\lambda_i}.
\] (11)

We now describe \( S_2 \cap S_3 \). By Lemma 10.7 the B-flags [2], [3] are opposite. For the moment assume \( s + t > N \). Then \( S_2 \cap S_3 = 0 \), so \( S_1 \cap S_2 \cap S_3 = 0 \) and we are done. Next assume \( s + t \leq N \). Then \( S_2 \cap S_3 \) has dimension \( N - s - t + 1 \) by Lemma 6.1. For \( 0 \leq i \leq N - s \) the location \( \lambda_i \) is at distance \( N - i \) from the 3-corner of \( \Delta_N \). For \( t \leq i \leq N - s \) this distance is at most \( N - t \), so \( B_{\lambda_i} \subseteq S_3 \) by Lemmas 9.5(i), 9.10 so \( B_{\lambda_i} \subseteq S_2 \cap S_3 \). Therefore \( \sum_{i=t}^{N-s} B_{\lambda_i} \subseteq S_2 \cap S_3 \). In this inclusion each side has dimension \( N - s - t + 1 \), so equality holds. In other words

\[
S_2 \cap S_3 = \sum_{i=t}^{N-s} B_{\lambda_i}.
\] (12)

Combining (11), (12) we obtain

\[
S_1 \cap S_2 \cap S_3 = \sum_{i=t}^{N-r-s} B_{\lambda_i}.
\] (13)

Consider the sum on the right in (13). First assume \( r + s + t > N \). Then the sum is empty, so \( S_1 \cap S_2 \cap S_3 = 0 \). Next assume \( r + s + t \leq N \), and define \( n = N - r - s - t \). Then \( 0 \leq n \leq N \) and \( \mu = (r, s, t) \) is contained in \( \Delta_{N-n} \). Let \( K \) denote the 2-boundary of \( \Delta_n \), and consider the set \( K + \mu \). This set consists of the locations \( \{\lambda_i\}_{i=r}^{N-r-s} \). Now by Lemma 9.2 we obtain \( B_{\mu} = \sum_{i=r}^{N-r-s} B_{\lambda_i} \). By this and (13) we obtain \( S_1 \cap S_2 \cap S_3 = B_{\mu} \).

**Corollary 11.2.** Pick a location \( \lambda = (r, s, t) \) in \( \Delta_N \). Then \( B_{\lambda} \) is equal to the intersection of the following three sets:

(i) component \( N - r \) of the B-flag [1];
(ii) component \( N - s \) of the B-flag [2];
(iii) component \( N - t \) of the B-flag [3].

**Proof.** In Proposition 11.1 assume \( r + s + t = N \). By Corollary 11.2 the Billiard Array \( B \) is uniquely determined by the B-flags [1], [2], [3]. This idea will be pursued further in Section 12. In the meantime we have a few comments.

**Proposition 11.3.** Let \( \mu, \nu \) denote elements in \( \Delta_{\leq N} \).

(i) Assume that \( \mu \vee \nu \) exists in \( \Delta_{\leq N} \). Then \( B_\mu \cap B_\nu = B_{\mu \vee \nu} \).
(ii) Assume that \( \mu \vee \nu \) does not exist in \( \Delta_{\leq N} \). Then \( B_\mu \cap B_\nu = 0 \).

**Proof.** This is a routine application of Proposition 11.1.
Proposition 11.4. Let $\mu, \nu$ denote elements in $\Delta_{\leq N}$. Then $\mu \leq \nu$ if and only if $B_\nu \subseteq B_\mu$.

Proof. First assume that $\mu \leq \nu$. Then $B_\nu \subseteq B_\mu$ by Lemma 9.6. Next assume that $B_\nu \not\subseteq B_\mu$. Then $B_\nu = B_\mu \cap B_\nu$, so $B_\mu \cap B_\nu \neq 0$. Now by Proposition 11.3 we see that $\mu \lor \nu$ exists in $\Delta_{\leq N}$, and $B_\mu \cap B_\nu = B_\mu \lor B_\nu$. Therefore $B_\nu = B_\mu \lor B_\nu$. By this and Lemma 9.3 $\nu$ and $\mu \lor \nu$ have the same rank. By construction $\nu \leq \mu \lor \nu$. By these comments $\nu = \mu \lor \nu$, so $\mu \leq \nu$. \qed

12 A characterization of Billiard Arrays using totally opposite flags

Throughout this section the following notation is in effect. Fix $N \in \mathbb{N}$. Let $V$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$.

In Lemma 10.7 we encountered three flags on $V$ that are mutually opposite. We now introduce a stronger condition involving three flags.

Definition 12.1. Suppose we are given three flags on $V$, denoted $\{U_i\}_{i=0}^N$, $\{U_i\}'_{i=0}^N$, $\{U_i\}''_{i=0}^N$. These flags are said to be totally opposite whenever $U_{N-r} \cap U_{N-s} \cap U_{N-t}' = 0$ for all $r, s, t$ $(0 \leq r, s, t \leq N)$ such that $r + s + t > N$.

Lemma 12.2. Referring to Definition 12.1 assume that $\{U_i\}_{i=0}^N$, $\{U_i\}'_{i=0}^N$, $\{U_i\}''_{i=0}^N$ are totally opposite. Then they are mutually opposite.

Proof. Setting $r = 0$ (resp. $s = 0$) (resp. $t = 0$) in Definition 12.1, we find that the flags $\{U_i\}'^N_{i=0}$ and $\{U_i\}''^N_{i=0}$ (resp. $\{U_i\}'^N_{i=0}$ and $\{U_i\}''^N_{i=0}$) (resp. $\{U_i\}'^N_{i=0}$ and $\{U_i\}''^N_{i=0}$) are opposite. \qed

Lemma 12.3. Referring to Definition 12.1 the following are equivalent:

(i) the flags $\{U_i\}_{i=0}^N$, $\{U_i\}'_{i=0}^N$, $\{U_i\}''_{i=0}^N$ are totally opposite;

(ii) for $0 \leq n \leq N$ the sequences

\[
\{U_i\}_{i=0}^{N-n}, \quad \{U_{N-n} \cap U'_{n+i}\}_{i=0}^{N-n}, \quad \{U_{N-n} \cap U''_{n+i}\}_{i=0}^{N-n}
\]

are mutually opposite flags on $U_{N-n}$;

(iii) for $0 \leq n \leq N$ the sequences

\[
\{U_i\}'_{i=0}^{N-n}, \quad \{U_{N-n} \cap U''_{n+i}\}_{i=0}^{N-n}, \quad \{U'_{N-n} \cap U_{n+i}\}_{i=0}^{N-n}
\]

are mutually opposite flags on $U'_{N-n}$;

(iv) for $0 \leq n \leq N$ the sequences

\[
\{U_i\}''_{i=0}^{N-n}, \quad \{U''_{N-n} \cap U_{n+i}\}_{i=0}^{N-n}, \quad \{U''_{N-n} \cap U'_{n+i}\}_{i=0}^{N-n}
\]

are mutually opposite flags on $U''_{N-n}$.
Proof. (i) ⇒ (ii) We first show that each sequence in (14) is a flag on $U_{N-n}$. For the first sequence this is clear, so consider the second sequence. By construction, the subspaces $\{U_{N-n} \cap U_{n+i}^{'}\}_{i=0}^{N-n}$ are nested and contained in $U_{N-n}$. For $0 \leq i \leq N-n$ the subspace $U_{N-n} \cap U_{n+i}^{'}$ has dimension $i+1$ by Lemmas 12.1, 12.2. Therefore the sequence $\{U_{N-n} \cap U_{n+i}^{'}\}_{i=0}^{N-n}$ is a flag on $U_{N-n}$. Similarly the sequence $\{U_{N-n} \cap U_{n+i}^{''}\}_{i=0}^{N-n}$ is a flag on $U_{N-n}$. We have shown that each sequence in (14) is a flag on $U_{N-n}$. We check that these three flags are mutually opposite. We show that the flags $\{U_{i}^{N-n}\}_{i=0}^{N-n}$ and $\{U_{N-n} \cap U_{n+i}^{'}\}_{i=0}^{N-n}$ are opposite. For $0 \leq i, j \leq N-n$ such that $i+j < N-n$, the intersection of $U_i$ and $U_{N-n} \cap U_{n+i}^{'}$ is equal to $U_i \cap U_{n+i}^{'}$, which is zero by Lemma 12.2. Therefore the flags $\{U_{i}^{N-n}\}_{i=0}^{N-n}$ and $\{U_{N-n} \cap U_{n+i}^{'}\}_{i=0}^{N-n}$ are opposite. Similarly the flags $\{U_{i}^{N-n}\}_{i=0}^{N-n}$ and $\{U_{N-n} \cap U_{n+i}^{''}\}_{i=0}^{N-n}$ are opposite. We now show that the flags $\{U_{N-n} \cap U_{n+i}^{'}\}_{i=0}^{N-n}$ and $\{U_{N-n} \cap U_{n+i}^{''}\}_{i=0}^{N-n}$ are opposite. For $0 \leq i, j \leq N-n$ such that $i+j < N-n$, the intersection of $U_{N-n} \cap U_{n+i}^{'}$ and $U_{N-n} \cap U_{n+i}^{''}$ is equal to $U_{N-n} \cap U_{n+i}^{'} \cap U_{n+i}^{''}$, which is zero by Definition 12.1. Therefore the flags $\{U_{N-n} \cap U_{n+i}^{'}\}_{i=0}^{N-n}$ and $\{U_{N-n} \cap U_{n+i}^{''}\}_{i=0}^{N-n}$ are opposite. By these comments the three flags in (14) are mutually opposite.

(ii) ⇒ (i) Setting $n = 0$ in (14) we see that the flags $\{U_{i}^{N}\}_{i=0}^{N}$, $\{U_{i}^{N}\}_{i=0}^{N}$, $\{U_{i}^{N}\}_{i=0}^{N}$ are mutually opposite. Pick integers $r, s, t$ (0 ≤ $r, s, t$ ≤ $N$) such that $r+s+t > N$. We show that $U_{N-r} \cap U_{N-s} \cap U_{N-t} = 0$. We may assume that $r + s \leq N$ (resp. $s + t \leq N$) (resp. $t + r \leq N$); otherwise $U_{N-r} \cap U_{N-s} = 0$ (resp. $U_{N-s} \cap U_{N-t} = 0$) (resp. $U_{N-r} \cap U_{N-t} = 0$) and we are done. Define $i = N-r-s$ and $j = N-r-t$. By construction 0 ≤ $i, j \leq N-r$ and $i+j < N-r$. Observe that $U_{r+i}^{'} = U_{r+i}^{''}$ and $U_{N-t} = U_{N-t}^{''}$. Therefore $U_{N-r} \cap U_{N-s} \cap U_{N-t}$ is the intersection of $U_{N-r} \cap U_{r+i}^{'}$ and $U_{N-r} \cap U_{r+j}^{''}$, which is zero by assumption.

The implications (i) ⇒ (ii) and (ii) ⇒ (iv) are similarly obtained. □

Theorem 12.4. Let $B$ denote a Billiard Array on $V$, and recall the B-flags [1], [2], [3] on $V$ from Definition 12.4. These flags are totally opposite.

Proof. By Proposition 11.1 and Definition 12.4 □

Lemma 12.5. Suppose we are given three totally opposite flags on $V$, denoted $\{U_{i}^{N}\}_{i=0}^{N}$, $\{U_{i}^{N}\}_{i=0}^{N}$, $\{U_{i}^{N}\}_{i=0}^{N}$. Then $U_{N-r} \cap U_{N-s} \cap U_{N-t}$ has dimension $N-r-s-t+1$ for all $(r, s, t) \in \Delta_{N}$.

Proof. The given flags satisfy Lemma 12.3(i), so they satisfy Lemma 12.3(ii). Therefore the sequences $\{U_{N-r} \cap U_{r+i}^{'}\}_{i=0}^{N-r}$ and $\{U_{N-r} \cap U_{r+i}^{''}\}_{i=0}^{N-r}$ are opposite flags on $U_{N-r}$. Define $i = N-r-s$ and $j = N-r-t$. By construction 0 ≤ $i, j \leq N-r$ and $i+j \geq N-r$. Observe that $U_{N-s} = U_{r+i}^{'}$ and $U_{N-t} = U_{r+j}^{''}$. Therefore $U_{N-r} \cap U_{N-s} \cap U_{N-t}$ is the intersection of $U_{N-r} \cap U_{r+i}^{'}$ and $U_{N-r} \cap U_{r+j}^{''}$, which is equal to $N-r-s-t+1$, which is equal to $N-r-s-t+1$. □

Corollary 12.6. Suppose we are given three totally opposite flags on $V$, denoted $\{U_{i}^{N}\}_{i=0}^{N}$, $\{U_{i}^{N}\}_{i=0}^{N}$, $\{U_{i}^{N}\}_{i=0}^{N}$. Then $U_{N-r} \cap U_{N-s} \cap U_{N-t}$ has dimension one for all $(r, s, t) \in \Delta_{N}$.

Proof. In Lemma 12.5 assume $r+s+t = N$. □

Theorem 12.7. Suppose we are given three totally opposite flags on $V$, denoted $\{U_{i}^{N}\}_{i=0}^{N}$, $\{U_{i}^{N}\}_{i=0}^{N}$, $\{U_{i}^{N}\}_{i=0}^{N}$. For each location $\lambda = (r, s, t)$ in $\Delta_{N}$ define

$$B_{\lambda} = U_{N-r} \cap U_{N-s} \cap U_{N-t}.$$ (15)
Then the map \( B : \Delta_N \to P_1(V) \), \( \lambda \mapsto B_\lambda \) is a Billiard Array on \( V \).

**Proof.** The map \( B \) is well defined by Corollary 12.6. We check that \( B \) satisfies the conditions of Definition 7.1. We show that \( B \) satisfies Definition 7.1(i). Let \( L \) denote a line in \( \Delta_N \). We show that the sum \( \sum_{\lambda \in L} B_\lambda \) is direct. For notational convenience, assume that \( L \) is a 1-line. Denote the cardinality of \( L \) by \( N - n + 1 \). The sum \( \sum_{\lambda \in L} B_\lambda \) is direct, since the summands make up the decomposition of \( U_{N-n} \) associated with the last two flags in \( (14) \). We have shown that \( B \) satisfies Definition 7.1(i). Next we show that \( B \) satisfies Definition 7.1(ii). Assume \( N \geq 1 \); otherwise we are done. Pick \((r,s,t) \in \Delta_N - 1\) and let \( C \) denote the corresponding black 3-clique in \( \Delta_N \) from Lemma 4.31. By construction, for all \( \lambda \in C \) the subspace \( B_\lambda \) is contained in \( U_{N-r} \cap U'_{N-s} \cap U''_{N-t} \). This space has dimension 2 by Lemma 12.5. Therefore the sum \( \sum_{\lambda \in C} B_\lambda \) is not direct. We have shown that \( B \) satisfies Definition 7.1(ii). By the above comments \( B \) is a Billiard Array on \( V \).

**Proposition 12.8.** Referring to Theorem 12.7, the \( B \)-flags \([1],[2],[3]\) coincide with \( \{U_i\}_{i=0}^N \), \( \{U'_i\}_{i=0}^N \), \( \{U''_i\}_{i=0}^N \) respectively.

**Proof.** We show that the \( B \)-flag \([1]\) coincides with \( \{U_i\}_{i=0}^N \). Denote the \( B \)-flag \([1]\) by \( \{U_i\}_{i=0}^N \). We show that \( U_i = U_i \) for \( 0 \leq i \leq N \). Let \( i \) be given. By Lemmas 9.5(i), 9.10 the subspace \( U_i \) is spanned by the subspaces \( B_\lambda \) such that \( \lambda \) is a location in \( \Delta_N \) at distance \( \leq i \) from the 1-corner of \( \Delta_N \). Each such \( B_\lambda \) is contained in \( U_i \) by \( (15) \), so \( U_i \subseteq U_i \). Each of \( U_i,U_i \) has dimension \( i + 1 \) by the definition of a flag. Therefore \( U_i = U_i \). We have shown that the \( B \)-flag \([1]\) coincides with \( \{U_i\}_{i=0}^N \). The remaining assertions are similarly shown.

Consider the following two sets:

(i) the Billiard Arrays on \( V \);

(ii) the 3-tuples of totally opposite flags on \( V \).

Theorem 12.4 describes a function from (i) to (ii). Theorem 12.7 describes a function from (ii) to (i).

**Theorem 12.9.** The above functions are inverses. Moreover they are bijections.

**Proof.** The functions are inverses by Corollary 11.2 and Proposition 12.8. It follows that they are bijections.

### 13 The braces for a Billiard Array

Our next general goal is to classify the Billiard Arrays up to isomorphism. This goal will be completed in Secton 19. Throughout the present section the following notation is in effect. Fix \( N \in \mathbb{N} \). Let \( V \) denote a vector space over \( \mathbb{F} \) with dimension \( N + 1 \). Let \( B \) denote a Billiard Array on \( V \).

**Definition 13.1.** Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a black 3-clique. By an *affine brace* (or *abrace*) for this clique, we mean a set of vectors

\[
  u \in B_\lambda, \quad v \in B_\mu, \quad w \in B_\nu
\]

that are not all zero, and \( u + v + w = 0 \).
Example 13.2. Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a black 3-clique. Pick any nonzero vectors
\[
u \in B_\lambda, \quad v \in B_\mu, \quad w \in B_\nu.
\]
The vectors \( u, v, w \) are linearly dependent by Definition 7.1(ii). So there exist scalars \( a, b, c \) in \( \mathbb{F} \) that are not all zero and \( au + bv + cw = 0 \). The vectors \( au, bv, cw \) form an abrace for the clique.

We have a few comments about abraces.

Lemma 13.3. Referring to Definition 13.1, assume that \( u, v, w \) form an abrace. Then each of \( u, v, w \) is nonzero. Moreover any two of \( u, v, w \) are linearly independent.

Proof. If at least one of \( u, v, w \) is zero then the other two are opposite and nonzero, contradicting the last assertion of Lemma 7.3. Therefore each of \( u, v, w \) is nonzero. By this and the last assertion of Lemma 7.3 we find that any two of \( u, v, w \) are linearly independent. \( \square \)

Lemma 13.4. Referring to Definition 13.1, assume that \( u, v, w \) form an abrace. Let \( a, b, c \) denote scalars in \( \mathbb{F} \) such that \( au + bv + cw = 0 \). Then \( a = b = c \).

Proof. Using \( u + v + w = 0 \) we obtain \( (b - a)v + (c - a)w = 0 \). The vectors \( v, w \) are linearly independent so \( b - a \) and \( c - a \) are zero. \( \square \)

Lemma 13.5. Referring to Definition 13.1, assume that \( u, v, w \) form an abrace. Then for \( 0 \neq \delta \in \mathbb{F} \) the vectors \( \delta u, \delta v, \delta w \) form an abrace.

Proof. By Definition 13.1 \( \square \)

Lemma 13.6. Referring to Definition 13.1, assume that \( u, v, w \) form an abrace. Then for any abrace
\[
u' \in B_\lambda, \quad v' \in B_\mu, \quad w' \in B_\nu
\]
there exists \( 0 \neq \delta \in \mathbb{F} \) such that
\[
u' = \delta u, \quad v' = \delta v, \quad w' = \delta w.
\]

Proof. By (16) and since \( u, v, w \) are nonzero, we see that \( u, v, w \) are bases for \( B_\lambda, B_\mu, B_\nu \) respectively. Therefore there exist scalars \( a, b, c \) in \( \mathbb{F} \) such that
\[
u' = au, \quad v' = bv, \quad w' = cw.
\]
Now \( au + bv + cw = u' + v' + w' = 0 \) so \( a = b = c \) by Lemma 13.4. Let \( \delta \) denote this common value and observe that \( \delta \) satisfies the requirements of the lemma. \( \square \)

Lemma 13.7. Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a black 3-clique. Then each nonzero \( u \in B_\lambda \) is contained in a unique abrace for this clique.

Proof. Concerning existence, pick nonzero vectors \( v \in B_\mu \) and \( w \in B_\nu \). By Example 13.2 there exist scalars \( a, b, c \) in \( \mathbb{F} \) such that \( au, bv, cw \) is an abrace. The scalars \( a, b, c \) are nonzero by Lemma 13.3. Now by construction the vectors \( u, ba^{-1}v, ca^{-1}w \) form an abrace that contains \( u \). The abrace containing \( u \) is unique by Lemma 13.6. \( \square \)
**Definition 13.8.** Let $\lambda, \mu$ denote locations in $\Delta_N$ that form an edge. By Lemma 4.33 there exists a unique location $\nu \in \Delta_N$ such that $\lambda, \mu, \nu$ form a black 3-clique. We call $\nu$ the completion of the edge.

**Definition 13.9.** Let $\lambda, \mu$ denote locations in $\Delta_N$ that form an edge. By a brace for this edge, we mean a set of nonzero vectors $u \in B_\lambda, \quad v \in B_\mu$ such that $u + v \in B_\nu$. Here $\nu$ denotes the completion of the edge.

**Lemma 13.10.** Referring to Definition 13.1, assume that $u, v, w$ form an abrace. Then $u, v$ form a brace for the edge $\lambda, \mu$.

**Proof.** The vectors $u, v$ are nonzero by Lemma 13.3. Moreover $u + v = -w \in B_\nu$. The result follows in view of Definition 13.9.

**Lemma 13.11.** Referring to Definition 13.4, assume that $u, v$ form a brace, and define $w = -u - v$. Then $u, v, w$ form an abrace for the black 3-clique $\lambda, \mu, \nu$.

**Proof.** By Definitions 13.1 and 13.9.

**Lemma 13.12.** Let $\lambda, \mu$ denote locations in $\Delta_N$ that form an edge. Each nonzero $u \in B_\lambda$ is contained in a unique brace for this edge.

**Proof.** By Lemmas 13.10, 13.11 together with Lemma 13.7.

# 14 The maps $\tilde{B}_{\lambda,\mu}$ and the value function $\hat{B}$

Throughout this section the following notation is in effect. Fix $N \in \mathbb{N}$. Let $V$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$. Let $B$ denote a Billiard Array on $V$.

**Definition 14.1.** Let $\lambda, \mu$ denote adjacent locations in $\Delta_N$. We define an $\mathbb{F}$-linear map $\tilde{B}_{\lambda,\mu} : B_\lambda \to V$ as follows. For each brace $u \in B_\lambda, \quad v \in B_\mu$ the map $\tilde{B}_{\lambda,\mu}$ sends $u \mapsto v$. The map $\tilde{B}_{\lambda,\mu}$ is well defined by Lemma 13.12.

**Note 14.2.** Let $\lambda, \mu$ denote adjacent locations in $\Delta_N$. By Definition 14.1 $B_\mu$ is the image of $B_\lambda$ under $\tilde{B}_{\lambda,\mu}$. Consequently we sometimes view $\tilde{B}_{\lambda,\mu}$ as a bijection $\tilde{B}_{\lambda,\mu} : B_\lambda \to B_\mu$.

The following definition is for notational convenience.

**Definition 14.3.** Define $\tilde{B}_{\lambda,\mu} = 0$ for all $\lambda, \mu \in \mathbb{R}^3$ such that $\lambda \notin \Delta_N$ or $\mu \notin \Delta_N$.

**Lemma 14.4.** Let $\lambda, \mu$ denote adjacent locations in $\Delta_N$. For vectors $u \in B_\lambda$ and $v \in B_\mu$ the following are equivalent:

(i) the vectors $u, v$ form a brace;
(ii) the vector $u \neq 0$ and the map $\tilde{B}_{\lambda,\mu}$ sends $u \mapsto v$;

(iii) the vector $v \neq 0$ and the map $\tilde{B}_{\mu,\lambda}$ sends $v \mapsto u$.

Proof. By Definition 14.4.

Lemma 14.5. Let $\lambda, \mu$ denote adjacent locations in $\Delta_N$. Then the maps $\tilde{B}_{\lambda,\mu} : B_\lambda \to B_\mu$ and $\tilde{B}_{\mu,\lambda} : B_\mu \to B_\lambda$ are inverses.

Proof. Compare parts (ii), (iii) of Lemma 14.4.

Lemma 14.6. Let $\lambda, \mu, \nu$ denote locations in $\Delta_N$ that form a black 3-clique. Pick an abrace $u \in B_\lambda$, $v \in B_\mu$, $w \in B_\nu$.

Then

| the map      | $\tilde{B}_{\lambda,\mu}$ | $\tilde{B}_{\mu,\lambda}$ | $\tilde{B}_{\mu,\nu}$ | $\tilde{B}_{\nu,\mu}$ | $\tilde{B}_{\nu,\lambda}$ | $\tilde{B}_{\lambda,\nu}$ |
|--------------|----------------------------|---------------------------|----------------------|----------------------|---------------------------|---------------------------|
| sends        | $u \mapsto v$              | $v \mapsto u$            | $v \mapsto w$       | $w \mapsto v$       | $w \mapsto u$            | $u \mapsto w$            |

Proof. By Lemma 13.10 and Lemma 14.4.

Lemma 14.7. Let $\lambda, \mu, \nu$ denote locations in $\Delta_N$ that form a black 3-clique. Then on $B_\lambda$,

$$I + \tilde{B}_{\lambda,\mu} + \tilde{B}_{\lambda,\nu} = 0. \quad (17)$$

Proof. Use Definition 13.1 and Lemma 14.6.

Lemma 14.8. Let $\lambda, \mu, \nu$ denote locations in $\Delta_N$ that form a black 3-clique. Then the composition

$$B_\lambda \xrightarrow{\tilde{B}_{\lambda,\mu}} B_\mu \xrightarrow{\tilde{B}_{\mu,\nu}} B_\nu \xrightarrow{\tilde{B}_{\nu,\lambda}} B_\lambda$$

is equal to the identity map on $B_\lambda$.

Proof. Use Lemma 14.6.

Definition 14.9. Let $\lambda, \mu, \nu$ denote locations in $\Delta_N$ that form a white 3-clique. Then the composition

$$B_\lambda \xrightarrow{\tilde{B}_{\lambda,\mu}} B_\mu \xrightarrow{\tilde{B}_{\mu,\nu}} B_\nu \xrightarrow{\tilde{B}_{\nu,\lambda}} B_\lambda$$

is a nonzero scalar multiple of the identity map on $B_\lambda$. The scalar is called the clockwise $B$-value (resp. counterclockwise $B$-value) of the clique whenever the sequence $\lambda, \mu, \nu$ runs clockwise (resp. counterclockwise) around the clique.

Lemma 14.10. For each white 3-clique in $\Delta_N$, its clockwise $B$-value and counterclockwise $B$-value are reciprocal.

Proof. By Lemma 14.5 and Definition 14.9.
Definition 14.11. For each white 3-clique in $\Delta_N$, by its $B$-value we mean its clockwise $B$-value.

Definition 14.12. By a value function on $\Delta_N$ we mean a function $\psi : \Delta_N \to \mathbb{F}\{0\}$.

Definition 14.13. Assume $N \geq 2$. We define a function $\hat{B} : \Delta_{N-2} \to \mathbb{F}$ as follows. Pick $(r, s, t) \in \Delta_{N-2}$. To describe the image of $(r, s, t)$ under $\hat{B}$, consider the corresponding white 3-clique in $\Delta_N$ from Lemma 4.32. The $B$-value of this 3-clique is the image of $(r, s, t)$ under $\hat{B}$. Observe that $\hat{B}$ is a value function on $\Delta_{N-2}$, in the sense of Definition 14.12. We call $\hat{B}$ the value function for $B$.

We are going to show that for $N \geq 2$ the map $B \mapsto \hat{B}$ induces a bijection from the set of isomorphism classes of Billiard Arrays over $\mathbb{F}$ that have diameter $N$, to the set of value functions on $\Delta_{N-2}$. The proof of this result will be completed in Section 19.

15 The scalars $\tilde{B}_{\lambda, \mu}$

Throughout this section the following notation is in effect. Fix $N \in \mathbb{N}$. Let $V$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$. Let $B$ denote a Concrete Billiard Array on $V$. Let $B$ denote the corresponding Billiard Array on $V$, from Definition 8.3.

Definition 15.1. Let $\lambda, \mu$ denote adjacent locations in $\Delta_N$. Recall the bijection $\tilde{B}_{\lambda, \mu} : B_\lambda \to B_\mu$. Recall that $B_\lambda$ is a basis for $B_\lambda$ and $B_\mu$ is a basis for $B_\mu$. Define a scalar $\tilde{B}_{\lambda, \mu} \in \mathbb{F}$ such that $\tilde{B}_{\lambda, \mu}$ sends $B_\lambda \mapsto \tilde{B}_{\lambda, \mu}B_\mu$. Note that $\tilde{B}_{\lambda, \mu} \neq 0$.

Lemma 15.2. For all $\lambda \in \Delta_N$ let $\kappa_\lambda$ denote a nonzero scalar in $\mathbb{F}$. Consider the Concrete Billiard Array $B' : \Delta_N \to V, \lambda \mapsto \kappa_\lambda B_\lambda$. Then for all adjacent $\lambda, \mu$ in $\Delta_N$,

$$\tilde{B}_{\lambda, \mu}\kappa_\lambda = \tilde{B}_{\lambda, \mu}'\kappa_\mu.$$

Proof. By Definition 15.1 the map $\tilde{B}_{\lambda, \mu}$ sends $B_\lambda \mapsto \tilde{B}_{\lambda, \mu}B_\mu$ and $B'_\lambda \mapsto \tilde{B}'_{\lambda, \mu}B'_\mu$. Compare these using $B'_\lambda = \kappa_\lambda B_\lambda$ and $B'_\mu = \kappa_\mu B_\mu$. \hfill $\square$

We mention a special case of Lemma 15.2.

Lemma 15.3. Pick $0 \neq \kappa \in \mathbb{F}$ and consider the Concrete Billiard Array $B' : \Delta_N \to V, \lambda \mapsto \kappa B_\lambda$. Then $\tilde{B}_{\lambda, \mu} = \tilde{B}'_{\lambda, \mu}$ for all adjacent $\lambda, \mu$ in $\Delta_N$.

Lemma 15.4. Let $\lambda, \mu$ denote adjacent locations in $\Delta_N$. Then the scalars $\tilde{B}_{\lambda, \mu}$ and $\tilde{B}_{\mu, \lambda}$ are reciprocal.

Proof. By Lemma 14.5 and Definition 15.1. \hfill $\square$

Lemma 15.5. Let $\lambda, \mu$ denote adjacent locations in $\Delta_N$. Then the following are equivalent:

(i) the vectors $B_\lambda, B_\mu$ form a brace for $B$;

(ii) $\tilde{B}_{\lambda, \mu} = 1$;
(iii) \( \bar{B}_{\mu,\lambda} = 1. \)

**Proof.** By Lemma 14.4 and Definition 15.1

**Lemma 15.6.** Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a black 3-clique. Then

\[
B_{\lambda} + \bar{B}_{\lambda,\mu} B_{\mu} + \bar{B}_{\lambda,\nu} B_{\nu} = 0. 
\]

**Proof.** Use Lemma 14.7 and Definition 15.1

We now consider the linear dependency (18) from a slightly more general point of view. Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a black 3-clique. By Definition 8.1(ii) the vectors \( B_{\lambda}, B_{\mu}, B_{\nu} \) are linearly dependent. Therefore there exist scalars \( a, b, c \) in \( \mathbb{F} \) that are not all zero, and

\[
a B_{\lambda} + b B_{\mu} + c B_{\nu} = 0. 
\]

By Definition 13.1 the vectors \( a B_{\lambda}, b B_{\mu}, c B_{\nu} \) form an abrace for the given black 3-clique. Each of \( a, b, c \) is nonzero by Lemma 13.3.

**Lemma 15.7.** With the above notation,

\[
\bar{B}_{\lambda,\mu} = \frac{b}{a}, \quad \bar{B}_{\mu,\nu} = \frac{c}{b}, \quad \bar{B}_{\nu,\lambda} = \frac{a}{c}, \\
\bar{B}_{\mu,\lambda} = \frac{a}{b}, \quad \bar{B}_{\nu,\mu} = \frac{b}{c}, \quad \bar{B}_{\lambda,\nu} = \frac{c}{a}. 
\]

**Proof.** To obtain the first and last equation, compare (18), (19) in light of Lemma 8.4. The remaining equations are obtained from these by cyclically permuting the roles of \( \lambda, \mu, \nu \).

**Lemma 15.8.** Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a black 3-clique. Then

\[
\bar{B}_{\lambda,\mu} \bar{B}_{\mu,\nu} \bar{B}_{\nu,\lambda} = 1. 
\]

**Proof.** Use Lemma 14.8 and Definition 15.1, or use Lemma 15.7

**Lemma 15.9.** Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a white 3-clique. Then the clockwise (resp. counterclockwise) \( B \)-value of the clique is equal to

\[
\bar{B}_{\lambda,\mu} \bar{B}_{\mu,\nu} \bar{B}_{\nu,\lambda} 
\]

whenever the sequence \( \lambda, \mu, \nu \) runs clockwise (resp. counterclockwise) around the clique.

**Proof.** Use Definitions 14.9, 15.1

### 16 Edge-labellings of \( \Delta_N \)

Throughout this section fix \( N \in \mathbb{N} \).

**Definition 16.1.** By an *edge-labelling* of \( \Delta_N \) we mean a function \( \beta \) that assigns to each ordered pair \( \lambda, \mu \) of adjacent locations in \( \Delta_N \) a scalar \( \beta_{\lambda,\mu} \in \mathbb{F} \) such that:
(i) for adjacent locations \( \lambda, \mu \) in \( \Delta_N \),

\[ \beta_{\lambda,\mu} \beta_{\mu,\lambda} = 1; \]

(ii) for locations \( \lambda, \mu, \nu \) in \( \Delta_N \) that form a black 3-clique,

\[ \beta_{\lambda,\mu} \beta_{\mu,\nu} \beta_{\nu,\lambda} = 1. \]

**Lemma 16.2.** Let \( B \) denote a Concrete Billiard Array over \( \mathbb{F} \) that has diameter \( N \). Define a function \( \tilde{B} \) that assigns to each ordered pair \( \lambda, \mu \) of adjacent locations in \( \Delta_N \) the scalar \( \tilde{B}_{\lambda,\mu} \) from Definition 15.7. Then \( \tilde{B} \) is an edge-labelling of \( \Delta_N \).

**Proof.** The function \( \tilde{B} \) satisfies the conditions (i), (ii) of Definition 16.1 by Lemmas 15.4, 15.8 respectively.

For the rest of this section the following notation is in effect. Let \( \beta \) denote an edge-labelling of \( \Delta_N \). Let \( V \) denote a vector space over \( \mathbb{F} \) that has dimension \( N + 1 \).

Our next goal is to describe the Concrete Billiard Arrays \( B \) on \( V \) such that \( \beta = \tilde{B} \).

**Lemma 16.3.** For adjacent locations \( \lambda, \mu \) in \( \Delta_N \) we have \( \beta_{\lambda,\mu} \neq 0 \).

**Proof.** By Definition 16.1 (i).

**Definition 16.4.** A function \( \beta : \Delta_N \rightarrow V \), \( \lambda \mapsto B_\lambda \) is said to be \( \beta \)-dependent whenever

\[ B_\lambda + \beta_{\lambda,\mu} B_\mu + \beta_{\lambda,\nu} B_\nu = 0 \]

(20)

for all locations \( \lambda, \mu, \nu \) in \( \Delta_N \) that form a black 3-clique.

**Lemma 16.5.** Let \( B \) denote a Concrete Billiard Array on \( V \). Then \( B \) is \( \beta \)-dependent if and only if \( \beta = \tilde{B} \).

**Proof.** Compare (18), (20) in light of Lemma 8.4.

Let \( \beta : \Delta_N \rightarrow V \), \( \lambda \mapsto B_\lambda \) denote a \( \beta \)-dependent function. Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a black 3-clique. Then the vectors \( B_\lambda, B_\mu, B_\nu \) satisfy (20). Cyclically permuting the roles of \( \lambda, \mu, \nu \) we obtain

\[ B_\lambda + \beta_{\lambda,\mu} B_\mu + \beta_{\lambda,\nu} B_\nu = 0, \]

(21)

\[ \beta_{\mu,\lambda} B_\lambda + B_\mu + \beta_{\mu,\nu} B_\nu = 0, \]

(22)

\[ \beta_{\nu,\lambda} B_\lambda + \beta_{\nu,\mu} B_\mu + B_\nu = 0. \]

(23)

The linear dependencies (21)–(23) are essentially the same. To see this, consider the coefficient matrix

\[
\begin{pmatrix}
1 & \beta_{\lambda,\mu} & \beta_{\lambda,\nu} \\
\beta_{\mu,\lambda} & 1 & \beta_{\mu,\nu} \\
\beta_{\nu,\lambda} & \beta_{\nu,\mu} & 1
\end{pmatrix}
\]

(24)

Using Definition 16.1 one checks that this matrix has rank 1.

30
Lemma 16.6. Assume $N \geq 1$. For a function $B : \Delta_N \to V$, $\lambda \mapsto B_\lambda$ the following (i), (ii) are equivalent:

(i) $B$ is $\beta$-dependent;

(ii) for all $(r, s, t) \in \Delta_{N-1}$ we have

$$B_\lambda + \beta_{\lambda, \mu} B_\mu + \beta_{\lambda, \nu} B_\nu = 0,$$  \hspace{1cm} (25)

where

$$\lambda = (r + 1, s, t), \quad \mu = (r, s + 1, t), \quad \nu = (r, s, t + 1).$$

Proof. By Lemma 4.31 and Definition 16.4 along with the discussion around (24).

Given a $\beta$-dependent function $B : \Delta_N \to V, \lambda \mapsto B_\lambda$, using the dependencies (25) we can solve for $\{B_\lambda\}_{\lambda \in \Delta_N}$ in terms of $\{B_\lambda\}_{\lambda \in L}$, where $L$ denotes the 1-boundary of $\Delta_N$. We give the details after a few definitions.

Definition 16.7. For each walk $\omega = \{\lambda_i\}_{i=0}^n$ in $\Delta_N$ we define the scalar $\beta_\omega = \prod_{i=1}^n \beta_{\lambda_{i-1}, \lambda_i}$ called the $\beta$-value of $\omega$. Note that $\beta_\omega \neq 0$ by Lemma 16.3. We interpret $\beta_\omega = 1$ if $n = 0$.

Lemma 16.8. For any walk $\omega$ in $\Delta_N$ the following scalars are reciprocal:

(i) the $\beta$-value for $\omega$;

(ii) the $\beta$-value for the inversion of $\omega$.

Proof. By Definition 16.1(i) and Definition 16.7.

Definition 16.9. For $\lambda, \mu \in \Delta_N$ we define the scalar

$$\beta_{\lambda, \mu} = \sum_\omega \beta_\omega,$$  \hspace{1cm} (26)

where the sum is over all geodesic paths $\omega$ in $\Delta_N$ from $\lambda$ to $\mu$.

Note 16.10. Let $\lambda, \mu$ denote collinear locations in $\Delta_N$. By Lemma 4.20 there exists a unique geodesic path $\omega$ in $\Delta_N$ from $\lambda$ to $\mu$. Now $\beta_{\lambda, \mu} = \beta_\omega$ in view of Definition 16.9. In this case $\beta_{\lambda, \mu} \neq 0$.

Recall the path $[2, 3]$ in $\Delta_N$, from Definition 4.21 and Lemma 4.23. This path runs along the 1-boundary of $\Delta_N$, from the 2-corner to the 3-corner.

Proposition 16.11. Let $\{\lambda_i\}_{i=0}^N$ denote the path $[2, 3]$ in $\Delta_N$. Let $\{v_i\}_{i=0}^N$ denote arbitrary vectors in $V$. Then there exists a unique $\beta$-dependent function $B : \Delta_N \to V, \lambda \mapsto B_\lambda$ that sends $\lambda_i \mapsto v_i$ for $0 \leq i \leq N$. For $\lambda = (r, s, t)$ in $\Delta_N$ we have

$$B_\lambda = (-1)^r \sum_{i=t}^{r+t} \beta_{\lambda, \lambda_i} v_i,$$  \hspace{1cm} (27)
Proof. We first show that $\mathcal{B}$ exists. We define the vectors $\{\mathcal{B}_\lambda\}_{\lambda \in \Delta_N}$ in the following recursive way. For $\xi = 0, 1, \ldots, N$ we define $\mathcal{B}_\lambda$ for those locations $\lambda \in \Delta_N$ that have 1-coordinate $\xi$. Let $\xi$ and $\lambda$ be given. First assume $\xi = 0$, so that $\lambda$ is on the 1-boundary of $\Delta_N$. Define $\mathcal{B}_\lambda = v_i$ where $\lambda = \lambda_i$. Next assume $1 \leq \xi \leq N$. For notational convenience define $r = \xi - 1$ and write $\lambda = (r + 1, s, t)$. Consider the locations $\mu = (r, s + 1, t)$ and $\nu = (r, s, t + 1)$ in $\Delta_N$. The locations $\lambda, \mu, \nu$ form a black 3-clique. Each of $\mu, \nu$ has 1-coordinate $r$, so the vectors $\mathcal{B}_\mu$ and $\mathcal{B}_\nu$ were defined earlier in the recursion. Define $\mathcal{B}_\lambda$ so that (25) holds. The vectors $\{\mathcal{B}_\lambda\}_{\lambda \in \Delta_N}$ are now defined. Consider the function $\mathcal{B} : \Delta_N \to V, \lambda \mapsto \mathcal{B}_\lambda$. By construction $\mathcal{B}$ satisfies condition (ii) of Lemma [16.6] so by that lemma $\mathcal{B}$ is $\beta$-dependent. By construction $\mathcal{B}$ sends $\lambda_i \mapsto v_i$ for $0 \leq i \leq N$. We have shown that $\mathcal{B}$ exists. Now let $\mathcal{B} : \Delta_N \to V, \lambda \mapsto \mathcal{B}_\lambda$ denote any $\beta$-dependent function that sends $\lambda_i \mapsto v_i$ for $0 \leq i \leq N$. Then $\mathcal{B}$ satisfies the equivalent conditions (i), (ii) of Lemma [16.6]. In condition (ii) there is a parameter $r$; using condition (ii) and induction on $r$ one routinely obtains (27). It follows from (27) that $\mathcal{B}$ is unique. \hfill $\blacksquare$

**Proposition 16.12.** With reference to Proposition [16.11], the following are equivalent:

(i) $\mathcal{B}$ is a Concrete Billiard Array;

(ii) the vectors $\{v_i\}_{i = 0}^N$ are linearly independent.

Proof. (i) $\Rightarrow$ (ii) By Definition [8.1] (i) and since $\{\lambda_i\}_{i = 0}^N$ form a line in $\Delta_N$.

(ii) $\Rightarrow$ (i) We show that $\mathcal{B}$ satisfies the conditions of Definition [8.1]. Concerning Definition [8.1] (i), let $L$ denote a line in $\Delta_N$. Pick a location $\lambda = (r, s, t)$ in $L$ and consider the sum in line (27). In that sum, by Note [16.10] the coefficient of $v_i$ is nonzero for $i = t$ and $i = r + t$. Therefore the vectors $\{\mathcal{B}_\lambda\}_{\lambda \in L}$ are linearly independent. We have shown that $\mathcal{B}$ satisfies Definition [8.1] (i). The function $\mathcal{B}$ satisfies Definition [8.1] (ii), since for each black 3-clique $C$ in $\Delta_N$ the vectors $\{\mathcal{B}_C\}_{\lambda \in C}$ are linearly dependent by (20). We have shown that $\mathcal{B}$ satisfies the conditions of Definition [8.1], so $\mathcal{B}$ is a Concrete Billiard Array. \hfill $\blacksquare$

We give two variations on Propositions [16.11, 16.12]

**Proposition 16.13.** Let $\{\lambda_i\}_{i = 0}^N$ denote the path $[2, 3]$ in $\Delta_N$. Then for any function $\mathcal{B} : \Delta_N \to V, \lambda \mapsto \mathcal{B}_\lambda$ the following (i), (ii) are equivalent:

(i) $\mathcal{B}$ is a Concrete Billiard Array and $\beta = \bar{\beta}$;

(ii) $\mathcal{B}$ is $\beta$-dependent and $\{\mathcal{B}_\lambda\}_{i = 0}^N$ are linearly independent.

Proof. (i) $\Rightarrow$ (ii) The function $\mathcal{B}$ is $\beta$-dependent by Lemma [16.5]. The vectors $\{\mathcal{B}_\lambda\}_{i = 0}^N$ are linearly independent by Definition [8.1] (i).

(ii) $\Rightarrow$ (i) Define $v_i = \mathcal{B}_\lambda$ for $0 \leq i \leq N$. By assumption, the function $\mathcal{B}$ is $\beta$-dependent and sends $\lambda_i \mapsto v_i$ for $0 \leq i \leq N$. Therefore $\mathcal{B}$ meets the requirements of Proposition [16.11]. By assumption, the vectors $\{v_i\}_{i = 0}^N$ are linearly independent. Now $\mathcal{B}$ is a Concrete Billiard Array in view of Proposition [16.12]. We have $\beta = \bar{\beta}$ by Lemma [16.5]. \hfill $\blacksquare$

**Proposition 16.14.** Let $\{v_i\}_{i = 0}^N$ denote a basis for $V$. Then there exists a unique Concrete Billiard Array $\mathcal{B}$ on $\mathcal{V}$ such that $\beta = \bar{\beta}$ and $\{v_i\}_{i = 0}^N$ is the $\mathcal{B}$-basis $[2, 3]$. 

32
Proof. Let \( \{\lambda_i\}_{i=0}^{N} \) denote the path \([2, 3]\) in \(\Delta_N\). We first show that \(B\) exists. By Proposition \([16.11]\) there exists a \(\beta\)-dependent function \(B : \Delta_N \to V, \lambda \mapsto B_\lambda\) that sends \(\lambda_i \mapsto v_i\) for \(0 \leq i \leq N\). By construction \(B\) satisfies condition (ii) of Proposition \([16.13]\). By that proposition \(B\) is a Concrete Billiard Array and \(\beta = B\). By construction \(\{v_i\}_{i=0}^{N}\) is the \(B\)-basis \([2, 3]\). We have shown that \(B\) exists. Concerning uniqueness, let \(B\) denote any Concrete Billiard Array on \(V\) such that \(\beta = B\) and \(\{v_i\}_{i=0}^{N}\) is the \(B\)-basis \([2, 3]\). Then \(B\) is \(\beta\)-dependent by Proposition \([16.13]\). By construction the function \(B\) sends \(\lambda_i \mapsto v_i\) for \(0 \leq i \leq N\). Therefore \(B\) meets the requirements of Proposition \([16.11]\). By that proposition \(B\) is unique. \(\square\)

Corollary 16.15. Let \(B : \Delta_N \to V, \lambda \mapsto B_\lambda\) denote a \(\beta\)-dependent function. Then \(B\) is a Concrete Billiard Array if and only if the vectors \(\{B_\lambda\}_{\lambda \in \Delta_N}\) span \(V\).

Proof. First assume that \(B\) is a Concrete Billiard Array. Then the vectors \(\{B_\lambda\}_{\lambda \in \Delta_N}\) span \(V\) by Corollary \([8.7]\). Conversely, assume that the vectors \(\{B_\lambda\}_{\lambda \in \Delta_N}\) span \(V\). Let \(\{\lambda_i\}_{i=0}^{N}\) denote the path \([2, 3]\) in \(\Delta_N\) and define \(v_i = B_\lambda\) for \(0 \leq i \leq N\). By Proposition \([16.11]\) the vectors \(\{B_\lambda\}_{\lambda \in \Delta_N}\) are contained in the span of \(\{v_i\}_{i=0}^{N}\). Therefore the vectors \(\{v_i\}_{i=0}^{N}\) span \(V\). The dimension of \(V\) is \(N + 1\) so \(\{v_i\}_{i=0}^{N}\) are linearly independent. Now \(B\) is a Concrete Billiard Array by Proposition \([16.12]\). \(\square\)

17 Concrete Billiard Arrays and edge-labellings

Throughout this section fix \(N \in \mathbb{N}\).

Let \(B\) denote a Concrete Billiard Array over \(F\) that has diameter \(N\). Recall from Lemma \([16.2]\) the edge-labelling \(\tilde{B}\) of \(\Delta_N\).

Proposition 17.1. With the above notation, the map \(B \mapsto \tilde{B}\) induces a bijection between the following two sets:

(i) the isomorphism classes of Concrete Billiard Arrays over \(F\) that have diameter \(N\);

(ii) the edge-labellings of \(\Delta_N\).

Proof. The map \(B \mapsto \tilde{B}\) induces a function from set (i) to set (ii), and this function is surjective by Proposition \([16.14]\). We now show that the function is injective. Let \(B, B'\) denote Concrete Billiard Arrays over \(F\) that have diameter \(N\) and \(\tilde{B} = B'\). We show that \(B, B'\) are isomorphic. Let \(V\) (resp. \(V'\)) denote the vector space underlying \(B\) (resp. \(B'\)). Let \(\{v_i\}_{i=0}^{N}\) denote the \(B\)-basis \([2, 3]\) for \(V\), and let \(\{v'_i\}_{i=0}^{N}\) denote the \(B'\)-basis \([2, 3]\) for \(V'\). Consider the \(F\)-linear bijection \(\sigma : V \to V'\) that sends \(v_i \mapsto v'_i\) for \(0 \leq i \leq N\). For \(\lambda = (r, s, t)\) in \(\Delta_N\) the vector \(B_\lambda\) satisfies \((27)\), where \(\beta = B = B'\). Applying \(\sigma\) to each side of \((27)\) we see that \(\sigma\) sends \(B_\lambda \mapsto B'_\lambda\). So by Definition \([8.15]\) \(\sigma\) is an isomorphism of Concrete Billiard Arrays from \(B\) to \(B'\). Consequently \(B, B'\) are isomorphic, as desired. The result follows. \(\square\)

Lemma 17.2. Let \(\beta\) denote an edge-labelling of \(\Delta_N\). Let \(\{\kappa_\lambda\}_{\lambda \in \Delta_N}\) denote nonzero scalars in \(F\). Then there exists an edge-labelling \(\beta'\) of \(\Delta_N\) such that \(\beta_{\lambda,\mu} = \beta'_{\lambda,\mu} \kappa_\lambda\) for all adjacent \(\lambda, \mu\) in \(\Delta_N\).
Proof. For all adjacent $\lambda, \mu \in \Delta_N$ define $\beta'_{\lambda, \mu} = \beta_{\lambda, \mu} \kappa_\lambda / \kappa_\mu$. One checks that $\beta'$ is an edge-labelling of $\Delta_N$ with the required features. \hfill \Box

**Definition 17.3.** Edge-labellings $\beta, \beta'$ of $\Delta_N$ are called *similar* whenever there exist nonzero scalars $\{\kappa_\lambda\}_{\lambda \in \Delta_N}$ in $F$ such that $\beta_{\lambda, \mu} \kappa_\lambda = \beta'_{\lambda, \mu} \kappa_\mu$ for all adjacent $\lambda, \mu$ in $\Delta_N$. This similarity relation is an equivalence relation.

**Example 17.4.** Assume $N = 1$. Then any two edge-labellings of $\Delta_N$ are similar.

**Lemma 17.5.** Let $B$ and $B'$ denote Concrete Billiard Arrays over $F$ that have diameter $N$. Then the following are equivalent:

(i) $B$ and $B'$ are similar in the sense of Definition 17.3.

(ii) the edge-labellings $\tilde{B}$ and $\tilde{B}'$ are similar in the sense of Definition 17.3.

**Proof.** (i) $\Rightarrow$ (ii) By Proposition 17.1 there exists a Concrete Billiard Array $B''$ over $F$ that is associated with $B$ and isomorphic to $B'$. By Definition 8.10 there exist nonzero scalars $\{\kappa_\lambda\}_{\lambda \in \Delta_N}$ in $F$ such that $B''_{\lambda, \mu} = B_{\lambda, \mu} \kappa_\lambda$ for all $\lambda \in \Delta_N$. For all adjacent $\lambda, \mu \in \Delta_N$ we have $B_{\lambda, \mu} \kappa_\lambda = B''_{\lambda, \mu} \kappa_\mu$ by Lemma 15.2 and $B''_{\lambda, \mu} = B'_{\lambda, \mu}$ by Proposition 17.1, so $B_{\lambda, \mu} \kappa_\lambda = B'_{\lambda, \mu} \kappa_\mu$. Now $\tilde{B}$ and $\tilde{B}'$ are similar in the sense of Definition 17.3.

(ii) $\Rightarrow$ (i) By Definition 17.3 there exist nonzero scalars $\{\kappa_\lambda\}_{\lambda \in \Delta_N}$ in $F$ such that $\tilde{B}_{\lambda, \mu} \kappa_\lambda = B'_{\lambda, \mu} \kappa_\mu$ for all adjacent $\lambda, \mu$ in $\Delta_N$. Let $B''$ denote the Concrete Billiard Array over $F$ that sends $\lambda \mapsto \kappa_\lambda B_\lambda$ for all $\lambda \in \Delta_N$. Then $B''$ is associated with $B$. For all adjacent $\lambda, \mu \in \Delta_N$ we have $\tilde{B}_{\lambda, \mu} \kappa_\lambda = B''_{\lambda, \mu} \kappa_\mu$ by Lemma 15.2 so $\tilde{B}_{\lambda, \mu} = B''_{\lambda, \mu}$. Therefore $B', B''$ are isomorphic by Proposition 17.1. The Concrete Billiard Array $B''$ is associated with $B$ and isomorphic to $B'$. Now by Proposition 8.20, $B$ and $B'$ are similar in the sense of Definition 8.17. \hfill \Box

The following result is a variation on Proposition 17.1.

**Proposition 17.6.** The map $B \mapsto \tilde{B}$ induces a bijection between the following two sets:

(i) the similarity classes of Concrete Billiard Arrays over $F$ that have diameter $N$;

(ii) the similarity classes of edge-labellings for $\Delta_N$.

**Proof.** By Proposition 17.1 and Lemma 17.5. \hfill \Box

## 18 Edge-labellings and value functions

Throughout this section fix $N \in \mathbb{N}$.

For this section our goal is to describe the similarity classes of edge-labellings for $\Delta_N$. Note that if $N = 0$ then $\Delta_N$ has no edges. If $N = 1$ then by Example 17.4 any two edge-labellings of $\Delta_N$ are similar. For $N \geq 2$ we will display a bijection between the following two sets: (i) the similarity classes of edge-labellings for $\Delta_N$; (ii) the value functions on $\Delta_{N-2}$.
Definition 18.1. Let $\beta$ denote an edge-labelling of $\Delta_N$. Let $\lambda, \mu, \nu$ denote locations in $\Delta_N$ that form a white 3-clique. Then the scalar

$$\beta_{\lambda,\mu}\beta_{\mu,\nu}\beta_{\nu,\lambda}$$

is called the clockwise $\beta$-value (resp. counterclockwise $\beta$-value) of the clique whenever the sequence $\lambda, \mu, \nu$ runs clockwise (resp. counterclockwise) around the clique.

Lemma 18.2. Let $\beta$ denote an edge-labelling of $\Delta_N$. For each white 3-clique in $\Delta_N$, its clockwise $\beta$-value and counterclockwise $\beta$-value are reciprocal.

Proof. By Definition 16.1(i) and Definition 18.1.

Definition 18.3. Let $\beta$ denote an edge-labelling of $\Delta_N$. For each white 3-clique in $\Delta_N$, by its $\beta$-value we mean its clockwise $\beta$-value.

Definition 18.4. Assume $N \geq 2$, and let $\beta$ denote an edge-labelling of $\Delta_N$. We define a function $\hat{\beta} : \Delta_{N-2} \to \mathbb{F}$ as follows. Pick $(r, s, t) \in \Delta_{N-2}$. To describe the image of $(r, s, t)$ under $\hat{\beta}$, consider the corresponding white 3-clique in $\Delta_N$ from Lemma 4.32. The $\beta$-value of this 3-clique is the image of $(r, s, t)$ under $\hat{\beta}$. Note that $\hat{\beta}$ is a value function on $\Delta_{N-2}$, in the sense of Definition 14.12. We call $\hat{\beta}$ the value function for $\beta$.

Lemma 18.5. Let $B$ denote a Concrete Billiard Array over $\mathbb{F}$ that has diameter $N$. Let $\hat{B}$ denote the corresponding Billiard Array. Then for each white 3-clique in $\Delta_N$ the $B$-value is equal to the $\hat{B}$-value. In other words, $B$ and $\hat{B}$ have the same value function.

Proof. Compare Lemma 15.9 and Definition 18.1.

Proposition 18.6. Let $\beta$ denote an edge-labelling of $\Delta_N$. Given a cycle in $\Delta_N$ that runs clockwise (resp. counterclockwise), consider the white 3-cliques that are surrounded by the cycle. Then the following scalars coincide:

(i) the $\beta$-value of the cycle, in the sense of Definition 16.7;

(ii) the product of the clockwise (resp. counterclockwise) $\beta$-values for these white 3-cliques.

Proof. Use Definitions 16.1, 16.7, 18.1.

Let $\beta$ denote an edge-labelling of $\Delta_N$. Let $T$ denote a spanning tree of $\Delta_N$, as in Definition 4.27. Consider the scalars

$$\beta_{\lambda,\mu} \quad \lambda, \mu \in \Delta_N, \quad \lambda, \mu \text{ form an edge in } T.$$  \hfill (28)

For adjacent $\lambda, \mu \in \Delta_N$ we now compute $\beta_{\lambda,\mu}$ in terms of (28) and the value function $\hat{\beta}$. First assume that the edge formed by $\lambda, \mu$ is in $T$. Then $\beta_{\lambda,\mu}$ is included in (28), so we are done. Next assume that the edge formed by $\lambda, \mu$ is not in $T$. There exists a unique path in $\Delta_N$ from $\mu$ to $\lambda$ that involves only edges in $T$. Denote this path by $\{\lambda_i\}_{i=1}^n$. By construction $\lambda_1 = \mu$ and $\lambda_n = \lambda$. Define $\lambda_0 = \lambda$ and note that $\{\lambda_i\}_{i=0}^n$ is a cycle in $\Delta_N$. Apply Proposition 18.6 to this cycle. For this cycle, the scalar in Proposition 18.6(i) is equal to $\prod_{i=1}^n \beta_{\lambda_{i-1},\lambda_i}$, and...
the scalar in Proposition 18.6(ii) is determined by \( \hat{\beta} \). Therefore \( \prod_{i=1}^{n} \beta_{\lambda_{i-1}, \lambda_{i}} \) is determined by \( \hat{\beta} \). In the product \( \prod_{i=1}^{n} \beta_{\lambda_{i-1}, \lambda_{i}} \) consider the \( i \)-factor for \( 1 \leq i \leq n \). This \( i \)-factor is \( \beta_{\lambda, \mu} \) for \( i = 1 \), and is included in (28) for \( 2 \leq i \leq n \). Using these comments we routinely solve for \( \beta_{\lambda, \mu} \) in terms of (28) and \( \hat{\beta} \).

The scalars (28) are “free” in the following sense.

**Proposition 18.7.** Assume \( N \geq 2 \), and let \( \psi \) denote a value function on \( \Delta_{N-2} \). Let \( T \) denote a spanning tree of \( \Delta_{N} \). Consider a collection of scalars

\[
b_{\lambda, \mu} \in \mathbb{F}, \quad \lambda, \mu \in \Delta_{N}, \quad \lambda, \mu \text{ form an edge in } T
\]  

such that \( b_{\lambda, \mu} = 1 \) for all \( \lambda, \mu \in \Delta_{N} \) that form an edge in \( T \). Then there exists a unique edge-labelling \( \beta \) of \( \Delta_{N} \) that has value function \( \psi \) and \( \beta_{\lambda, \mu} = b_{\lambda, \mu} \) for all \( \lambda, \mu \in \Delta_{N} \) that form an edge in \( T \).

**Proof.** We first show that \( \beta \) exists. To do this we mimic the argument from below (28). For adjacent \( \lambda, \mu \in \Delta_{N} \) we define a scalar \( \beta_{\lambda, \mu} \) as follows. First assume that the edge formed by \( \lambda, \mu \) is in \( T \). Define \( \beta_{\lambda, \mu} = b_{\lambda, \mu} \). Next assume that the edge formed by \( \lambda, \mu \) is not in \( T \). There exists a unique path in \( \Delta_{N} \) from \( \mu \) to \( \lambda \) that involves only edges in \( T \). Denote this path by \( \{ \lambda_{i} \}_{i=1}^{n} \). By construction \( \lambda_{1} = \mu \) and \( \lambda_{n} = \lambda \). Define \( \lambda_{0} = \lambda \) and note that \( \{ \lambda_{i} \}_{i=0}^{n} \) is a cycle in \( \Delta_{N} \). Define \( \varepsilon = 1 \) (resp. \( \varepsilon = -1 \)) if this cycle is clockwise (resp. counterclockwise). Define \( \beta_{\lambda, \mu} = c^{i}/d \) where \( d = \prod_{i=2}^{n} b_{\lambda_{i-1}, \lambda_{i}} \) and \( c \) is the product of the \( \psi \)-values of the white 3-cliques surrounded by the cycle. We have defined the scalar \( \beta_{\lambda, \mu} \) for all adjacent \( \lambda, \mu \in \Delta_{N} \). One checks that these scalars satisfy the conditions of Definition 16.1. This gives an edge-labelling \( \beta \) of \( \Delta_{N} \). By construction \( \beta \) has value function \( \psi \) and \( \beta_{\lambda, \mu} = b_{\lambda, \mu} \) for all \( \lambda, \mu \in \Delta_{N} \) that form an edge in \( T \). We have shown that \( \beta \) exists. The edge-labelling \( \beta \) is unique by the discussion below (28). \( \square \)

**Corollary 18.8.** Assume \( N \geq 2 \), and let \( \psi \) denote a value function on \( \Delta_{N-2} \). Let \( T \) denote a spanning tree of \( \Delta_{N} \). Then there exists a unique edge-labelling \( \beta \) of \( \Delta_{N} \) that has value function \( \psi \) and \( \beta_{\lambda, \mu} = 1 \) for all \( \lambda, \mu \in \Delta_{N} \) that form an edge in \( T \).

**Proof.** Apply Proposition 18.7 with \( b_{\lambda, \mu} = 1 \) for all \( \lambda, \mu \in \Delta_{N} \) that form an edge in \( T \). \( \square \)

Assume \( N \geq 2 \). By Definition 18.4 we get a map \( \beta \mapsto \hat{\beta} \) from the set of edge-labellings of \( \Delta_{N} \), to the set of value functions on \( \Delta_{N-2} \). This map is surjective by Corollary 18.8. We now consider the issue of injectivity.

**Lemma 18.9.** Assume \( N \geq 2 \). Let \( \beta \) and \( \beta' \) denote edge-labellings of \( \Delta_{N} \). Then the following are equivalent:

(i) \( \beta \) and \( \beta' \) are similar;

(ii) \( \beta \) and \( \beta' \) have the same value function.

**Proof.** (i) \( \Rightarrow \) (ii) Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_{N} \) that form a white 3-clique. We show that

\[
\beta_{\lambda, \mu} \beta_{\mu, \nu} \beta_{\nu, \lambda} = \beta'_{\lambda, \mu} \beta'_{\mu, \nu} \beta'_{\nu, \lambda}.
\]  

(30)
By Definition \[17.3\] there exist nonzero \(\kappa_\lambda, \kappa_\mu, \kappa_\nu \in \mathbb{F}\) such that
\[
\beta_{\lambda,\mu} \kappa_\lambda = \beta'_{\lambda,\mu} \kappa_\mu, \quad \beta_{\mu,\nu} \kappa_\mu = \beta'_{\mu,\nu} \kappa_\nu, \quad \beta_{\nu,\lambda} \kappa_\nu = \beta'_{\nu,\lambda} \kappa_\lambda.
\] (31)

Line (30) is routinely verified using (31).

(ii) \(\Rightarrow\) (i) Let \(T\) denote a spanning tree of \(\Delta_N\). Fix a location \(\nu \in \Delta_N\). For \(\lambda \in \Delta_N\) (and with reference to Definition \[16.7\]) define \(\kappa_\lambda = \beta'_\omega / \beta_\omega\) where \(\omega\) denotes the unique path in \(\Delta_N\) from \(\lambda\) to \(\nu\) that involves only edges in \(T\). Note that \(\kappa_\lambda \neq 0\). Using Proposition \[18.6\] one checks that \(\beta_{\lambda,\mu} \kappa_\lambda = \beta'_{\lambda,\mu} \kappa_\mu\) for all adjacent \(\lambda, \mu\) in \(\Delta_N\). Therefore \(\beta, \beta'\) are similar in view of Definition \[17.3\] \(\square\)

Assume \(N \geq 2\), and let \(\beta\) denote an edge-labelling of \(\Delta_N\). Recall from Definition \[18.3\] the value function \(\hat{\beta}\) on \(\Delta_{N-2}\).

**Proposition 18.10.** With the above notation, the map \(\beta \mapsto \hat{\beta}\) induces a bijection between the following two sets:

(i) the similarity classes of edge-labellings for \(\Delta_N\);

(ii) the value functions on \(\Delta_{N-2}\).

**Proof.** By Lemma \[18.9\] the map \(\beta \mapsto \hat{\beta}\) induces an injective function from set (i) to set (ii). The function is surjective by Corollary \[18.8\] Therefore the function is a bijection. \(\square\)

## 19 Billiard Arrays and value functions

From now until Proposition \[19.3\] fix an integer \(N \geq 2\).

To motivate the next result, we review a few points. Let \(BA_N(\mathbb{F})\) denote the set of isomorphism classes of Billiard Arrays over \(\mathbb{F}\) that have diameter \(N\). Let \(CBA_N(\mathbb{F})\) denote the set of similarity classes of Concrete Billiard Arrays over \(\mathbb{F}\) that have diameter \(N\). Let \(EL_N(\mathbb{F})\) denote the set of similarity classes of edge-labellings on \(\Delta_N\). Let \(VF_N(\mathbb{F})\) denote the set of value functions on \(\Delta_N\). The map \(B \mapsto \hat{B}\) induces a function \(BA_N(\mathbb{F}) \rightarrow VF_{N-2}(\mathbb{F})\) which we will denote by \(\theta\). In Lemma \[8.21\] we displayed a function \(CBA_N(\mathbb{F}) \rightarrow VF_{N-2}(\mathbb{F})\) which we will denote by \(g\). In Proposition \[17.6\] we displayed a bijection \(CBA_N(\mathbb{F}) \rightarrow EL_N(\mathbb{F})\) which we will denote by \(f\). In Proposition \[18.10\] we displayed a bijection \(EL_N(\mathbb{F}) \rightarrow VF_{N-2}(\mathbb{F})\) which we will denote by \(h\).

**Lemma 19.1.** With the above notation, the following diagram commutes:

\[
\begin{array}{ccc}
CBA_N(\mathbb{F}) & \xrightarrow{f} & BA_N(\mathbb{F}) \\
\downarrow{g} & & \downarrow{\theta} \\
EL_N(\mathbb{F}) & \xrightarrow{h} & VF_{N-2}(\mathbb{F})
\end{array}
\]

Moreover \(\theta\) is a bijection.

**Proof.** The diagram commutes by Lemma \[18.5\]. It follows that \(\theta\) is a bijection. \(\square\)

37
We emphasize one aspect of Lemma 19.1.

**Corollary 19.2.** The map \( B \mapsto \hat{B} \) induces a bijection between the following two sets:

(i) the isomorphism classes of Billiard Arrays over \( \mathbb{F} \) that have diameter \( N \);

(ii) the value functions on \( \Delta_{N-2} \).

**Proof.** The map \( \theta \) in Lemma 19.1 is a bijection. \( \Box \)

Summarizing the above discussion, we obtain a bijection between any two of the following sets:

- the isomorphism classes of Billiard Arrays over \( \mathbb{F} \) that have diameter \( N \);
- the similarity classes of Concrete Billiard Arrays over \( \mathbb{F} \) that have diameter \( N \);
- the similarity classes of edge-labellings for \( \Delta_N \);
- the value functions on \( \Delta_{N-2} \).

We have some remarks. For the rest of this section fix \( N \in \mathbb{N} \). Let \( V \) denote a vector space over \( \mathbb{F} \) with dimension \( N + 1 \).

**Proposition 19.3.** Let \( B \) and \( B' \) denote Concrete Billiard Arrays on \( V \). Then the following are equivalent:

(i) \( B \) and \( B' \) are relatives;

(ii) \( B \) and \( B' \) are associates and isomorphic.

**Proof.** (i) \( \Rightarrow \) (ii) By Lemma 8.19.

(ii) \( \Rightarrow \) (i) By Definition 8.10 and since \( B, B' \) are associates, there exist nonzero scalars \( \{ \kappa_{\lambda} \}_{\lambda \in \Delta_N} \) in \( \mathbb{F} \) such that \( B'_\lambda = \kappa_{\lambda}B_\lambda \) for all \( \lambda \in \Delta_N \). For all adjacent \( \lambda, \mu \in \Delta_N \) we have \( B'_{\lambda, \mu} = B'_{\lambda, \mu} \), since \( B, B' \) are isomorphic, so \( \kappa_{\lambda} = \kappa_{\mu} \). Consequently there exists \( 0 \neq \kappa \in \mathbb{F} \) such that \( \kappa_{\lambda} = \kappa \) for all \( \lambda \in \Delta_N \). Now \( B \) and \( B' \) are relatives by Definition 8.13. \( \Box \)

**Proposition 19.4.** Let \( B \) denote a Billiard Array on \( V \). For an \( \mathbb{F} \)-linear map \( \sigma : V \rightarrow V \) the following are equivalent:

(i) the map \( \sigma \) is an isomorphism of Billiard Arrays from \( B \) to \( B' \);

(ii) there exists \( 0 \neq \kappa \in \mathbb{F} \) such that \( \sigma = \kappa I \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( B \) denote a Concrete Billiard Array on \( V \) that corresponds to \( B \). Consider the Concrete Billiard Array \( B' : \Delta_N \rightarrow V, \lambda \mapsto \sigma(B_\lambda) \). By construction \( \sigma \) is an isomorphism of Concrete Billiard Arrays from \( B \) to \( B' \); therefore \( B \) and \( B' \) are isomorphic. By construction \( B' \) corresponds to \( B \); therefore \( B \) and \( B' \) are associates. Now \( B \) and \( B' \) are relatives by Proposition 19.3. By Definition 8.13 there exists \( 0 \neq \kappa \in \mathbb{F} \) such that \( B'_{\lambda} = \kappa B_\lambda \) for all \( \lambda \in \Delta_N \). Now \( \kappa I \) is an isomorphism of Concrete Billiard Arrays from \( B \) to \( B' \). By Definition 8.15 there does not exist another isomorphism of Concrete Billiard Arrays from \( B \) to \( B' \). Therefore \( \sigma = \kappa I \).

(ii) \( \Rightarrow \) (i) Clear. \( \Box \)
We make a definition for later use.

**Definition 19.5.** Let \( B \) denote a Concrete Billiard Array on \( V \), and let \( \mathcal{B} \) denote the corresponding Billiard Array on \( V \). Let \( C \) denote a white 3-clique in \( \Delta_N \). By the \( B \)-value of \( C \) we mean the \( B \)-value of \( C \), which is the same as the \( \mathcal{B} \)-value of \( C \) by Lemma 18.5.

## 20 Examples of Concrete Billiard Arrays

Throughout this section the following notation is in effect. Let \( x, y, z \) denote mutually commuting indeterminates. Let \( P = \mathbb{F}[x, y, z] \) denote the \( \mathbb{F} \)-algebra consisting of the polynomials in \( x, y, z \) that have all coefficients in \( \mathbb{F} \). For \( n \in \mathbb{N} \) let \( P_n \) denote the subspace of \( P \) consisting of the homogeneous polynomials that have total degree \( n \). The sum \( P = \sum_{n \in \mathbb{N}} P_n \) is direct.

Fix an integer \( N \geq 1 \). In this section we give some examples of Concrete Billiard Arrays of diameter \( N \). For these examples, the underlying vector space will be a subspace of \( P_N \).

We now describe our first example.

**Definition 20.1.** For each location \( \lambda = (r, s, t) \) in \( \Delta_N \) define
\[
B_\lambda = (x - y)^r(y - z)^s(z - x)^t.
\]

We will show that the function \( B \) from Definition 20.1 is a Concrete Billiard Array.

**Lemma 20.2.** Let \( \lambda, \mu, \nu \) denote locations in \( \Delta_N \) that form a black 3-clique. Then
\[
B_\lambda + B_\mu + B_\nu = 0. \tag{32}
\]

**Proof.** By Lemma 4.31 we may take
\[
\lambda = (r + 1, s, t), \quad \mu = (r, s + 1, t), \quad \nu = (r, s, t + 1)
\]
with \( (r, s, t) \in \Delta_{N-1} \). By Definition 20.1
\[
B_\lambda = (x - y)^{r+1}(y - z)^s(z - x)^t, \\
B_\mu = (x - y)^r(y - z)^{s+1}(z - x)^t, \\
B_\nu = (x - y)^r(y - z)^s(z - x)^{t+1}.
\]

From this we routinely obtain (32). \( \square \)

**Lemma 20.3.** The function \( B \) from Definition 20.1 is a Concrete Billiard Array.

**Proof.** We show that \( B \) satisfies the two conditions in Definition 8.1. Concerning Definition 8.1(i), let \( L \) denote a line in \( \Delta_N \) and consider the locations in \( L \). We show that their images under \( B \) are linearly independent. Without loss we may assume that \( L \) is a 1-line. Denote the cardinality of \( L \) by \( i + 1 \). The locations in \( L \) are listed in (4). For these locations their images under \( B \) are
\[
(x - y)^{N-i}(y - z)^{i-j}(z - x)^j \quad 0 \leq j \leq i. \tag{33}
\]

39
We show that the vectors \((33)\) are linearly independent. The vectors
\[(y - z)^{i-j}(z - x)^j \quad 0 \leq j \leq i\] (34)
are linearly independent; to see this set \(z = 0\) in (33) and note that \(x, y\) are algebraically independent. It follows that the vectors \((33)\) are linearly independent. We have shown that \(B\) satisfies Definition 8.1(i). The function \(B\) satisfies Definition 8.1(ii) by Lemma 20.2.

For the Concrete Billiard Array \(B\) in Definition 20.1 we now compute the corresponding edge labelling and value function.

**Lemma 20.4.** Let \(B\) denote the Concrete Billiard Array from Definition 20.1. For adjacent locations \(\lambda, \mu\) in \(\Delta_N\) we have \(\tilde{B}_{\lambda,\mu} = 1\).

**Proof.** By Lemma 4.33 there exists a location \(\nu \in \Delta_N\) such that \(\lambda, \mu, \nu\) form a black 3-clique. Now \(B_\lambda + B_\mu + B_\nu = 0\) by Lemma 20.2. Comparing this with (18) we find that in (18) the terms \(B_\mu\) and \(B_\nu\) have coefficient 1. Therefore \(\tilde{B}_{\lambda,\mu} = 1\).

**Proposition 20.5.** Let \(B\) denote the Concrete Billiard Array from Definition 20.1. Then each white 3-clique in \(\Delta_N\) has \(B\)-value 1.

**Proof.** By Lemmas 15.9, 20.4 along with Definition 19.5.

We are done with our first example. We now describe our second example.

We will use the following notation. Fix \(0 \neq q \in \mathbb{F}\) such that \(q \neq 1\). For elements \(a, b\) in any \(\mathbb{F}\)-algebra, define
\[
(a, b; q)_n = (a - b)(a - bq)(a - bq^2) \cdots (a - bq^{n-1}) \quad n = 0, 1, 2, \ldots
\]
We interpret \((a, b; q)_0 = 1\).

**Definition 20.6.** Pick three scalars \(x, y, z\) in \(\mathbb{F}\) such that \(x, y, z\) are algebraically independent. For each location \(\lambda = (r, s, t)\) in \(\Delta_N\) define
\[
B_\lambda = (x, y, z; q)_r (y, z, x; q)_s (z, x, y; q)_t.
\]
We are going to show that the function \(B\) from Definition 20.6 is a Concrete Billiard Array. Pick a location \((r, s, t) \in \Delta_{N-1}\) and consider the corresponding black 3-clique in \(\Delta_N\) from Lemma 4.31.

\[
\lambda = (r + 1, s, t), \quad \mu = (r, s + 1, t), \quad \nu = (r, s, t + 1).
\]

**Lemma 20.7.** With the above notation,
\[
aB_\lambda + bB_\mu + cB_\nu = 0
\] (35)
where each of \(a, b, c\) is nonzero and
\[
b/a = q^r/x,\quad c/b = q^s/y,\quad a/c = q^t/z.\] (36)
Proof. By Definition 20.6 and the construction,

\[ B_\lambda = (x, y; q)_{r+1}(y, y; z; q)_s(z, x; q)_t, \]
\[ B_\mu = (x, y; q)_r(y, y; z; q)_{s+1}(z, x; q)_t, \]
\[ B_\nu = (x, y; q)_r(y, y; z; q)_s(z, x; q)_{t+1}. \]

Define

\[ F = (x, y; q)_r(y, y; z; q)_s(z, x; q)_t, \]

so

\[ B_\lambda = F(x - yq^r), \quad B_\mu = F(y - zq^s), \quad B_\nu = F(z - xq^t). \]  \hspace{1cm} (37)

Using (36) along with \( r + s + t = N - 1 \) and \( x, y, z = q^{N-1} \), we obtain

\[ a(x - yq^r) + b(y - zq^s) + c(z - xq^t) = 0. \]  \hspace{1cm} (38)

Equation (35) follows from (37) and (38).

Lemma 20.8. The function \( B \) from Definition 20.6 is a Concrete Billiard Array.

Proof. We show that \( B \) satisfies the two conditions in Definition 8.1. Concerning Definition 8.1(i), let \( L \) denote a line in \( \Delta_N \) and consider the locations in \( L \). We show that their images under \( B \) are linearly independent. Without loss we may assume that \( B \) is a 1-line. Denote the cardinality of \( L \) by \( i + 1 \). The locations in \( L \) are listed in (4). For these locations their images under \( B \) are

\[ (x, y; q)_{N-i}(y, y; z; q)_{i-j}(z, x; q)_j \hspace{1cm} 0 \leq j \leq i. \]  \hspace{1cm} (39)

We show that the vectors (39) are linearly independent. The vectors

\[ (y, y; z; q)_{i-j}(z, x; q)_j \hspace{1cm} 0 \leq j \leq i \]  \hspace{1cm} (40)

are linearly independent; to see this set \( z = 0 \) in (40) and note that \( x, y \) are algebraically independent. It follows that the vectors (39) are linearly independent. We have shown that \( B \) satisfies Definition 8.1(i). The function \( B \) satisfies Definition 8.1(ii) by Lemma 20.7.

For the Concrete Billiard Array \( B \) in Definition 20.6 we now compute the corresponding edge labelling and value function.

Lemma 20.9. With the notation from above Lemma 20.7,

\[ \tilde{B}_{\lambda, \mu} = q^r / y, \quad \tilde{B}_{\mu, \nu} = q^s / z, \quad \tilde{B}_{\nu, \lambda} = q^t / x, \]

\[ \tilde{B}_{\mu, \lambda} = \bar{y} / q^r, \quad \tilde{B}_{\nu, \mu} = \bar{z} / q^s, \quad \tilde{B}_{\lambda, \nu} = \bar{x} / q^t. \]

Proof. Compare Lemma 15.7 and Lemma 20.7.

Proposition 20.10. Let \( B \) denote the Concrete Billiard Array from Definition 20.6. Then each white 3-clique in \( \Delta_N \) has \( B \)-value \( q \).
Proof. Assume $N \geq 2$; otherwise $\Delta_N$ has no white 3-clique. Let $(r,s,t) \in \Delta_{N-2}$ and consider the corresponding white 3-clique in $\Delta_N$ from Lemma 4.32. This 3-clique consists of the locations

\[
\lambda = (r, s+1, t+1), \quad \mu = (r+1, s, t+1), \quad \nu = (r+1, s+1, t).
\]

Using Lemma 20.9

\[
\tilde{B}_{\lambda,\mu} = \overline{y}/q^{r}, \quad \tilde{B}_{\mu,\nu} = \overline{z}/q^{s}, \quad \tilde{B}_{\nu,\lambda} = \overline{x}/q^{t}.
\]

By Lemma 15.9 and Definition 19.5 the $B$-value of the above white 3-clique is $\tilde{B}_{\lambda,\mu}\tilde{B}_{\mu,\nu}\tilde{B}_{\nu,\lambda}$ which is equal to

\[
q^{N-1}/q^{N-2} = q.
\]

Corollary 20.11. For the Concrete Billiard Array $B$ from Definition 20.6, the similarity class is independent of the choice of $\overline{x}$, $\overline{y}$, $\overline{z}$ and depends only on $q$, $N$.

Proof. By Lemma 19.1 and Proposition 20.10.

Corollary 20.12. For the Concrete Billiard Array $B$ from Definition 20.6, the isomorphism class of the corresponding Billiard Array is independent of the choice of $\overline{x}$, $\overline{y}$, $\overline{z}$ and depends only on $q$, $N$.

Proof. By Corollary 19.2 and Proposition 20.10.

21 The Lie algebra $\mathfrak{sl}_2$ and the quantum algebra $U_q(\mathfrak{sl}_2)$

In this section we use Billiard Arrays to describe the finite-dimensional irreducible modules for the Lie algebra $\mathfrak{sl}_2$ and the quantum algebra $U_q(\mathfrak{sl}_2)$. We now recall $\mathfrak{sl}_2$. We will use the equitable basis, which was introduced in [4] and comprehensively described in [2]. Until the end of Corollary 21.15 assume that the characteristic of $\mathbb{F}$ is not 2.

Definition 21.1. [4, Lemma 3.2] Let $\mathfrak{sl}_2$ denote the Lie algebra over $\mathbb{F}$ with basis $x, y, z$ and Lie bracket

\[
[x, y] = 2x + 2y, \quad [y, z] = 2y + 2z, \quad [z, x] = 2z + 2x.
\]

(41)

Until the end of Corollary 21.15 the following notation is in effect. Fix $N \in \mathbb{N}$. Let $V$ denote a vector space over $\mathbb{F}$ with dimension $N+1$. Let $B$ denote a Billiard Array on $V$. Assume that each white 3-clique in $\Delta_N$ has $B$-value 1. Using $B$ we will turn $V$ into a $\mathfrak{sl}_2$-module. We acknowledge that our construction is essentially the same as the one in [2, Proposition 8.17]; the details given here are meant to illuminate the role played by the maps $\tilde{B}_{\lambda,\mu}$. Recall the $B$-decompositions $[\eta, \xi]$ of $V$ from Definition 10.3.

Definition 21.2. Define $X, Y, Z$ in $\text{End}(V)$ such that for $0 \leq i \leq N$, $X - (2i - N)I$ (resp. $Y - (2i - N)I$) (resp. $Z - (2i - N)I$) vanishes on component $i$ of the $B$-decomposition $[2, 3]$ (resp. $[3, 1]$) (resp. $[1, 2]$).
The next result is meant to clarify Definition 21.2.

**Lemma 21.3.** Pick a location \( \lambda = (r, s, t) \) on the boundary of \( \Delta_N \). Then on \( B_\lambda \),

\[
X = (2t - N)I = (N - 2s)I \quad \text{if } r = 0; \\
Y = (2r - N)I = (N - 2t)I \quad \text{if } s = 0; \\
Z = (2s - N)I = (N - 2r)I \quad \text{if } t = 0.
\]

**Proof.** By Lemma 10.4 and Definition 21.2.

Pick \((r, s, t) \in \Delta_{N-1}\) and consider the corresponding black 3-clique in \( \Delta_N \) from Lemma 4.31:

\[
\lambda = (r + 1, s, t), \quad \mu = (r, s + 1, t), \quad \nu = (r, s, t + 1).
\]

**Proposition 21.4.** With the above notation, for each abrace

\[
u \in B_\lambda, \quad v \in B_\mu, \quad w \in B_\nu
\]

we have

\[
Xu = (N - 2t)v + (2s - N)w, \quad (42) \\
Yv = (N - 2r)w + (2t - N)u, \\
Zw = (N - 2s)u + (2r - N)v.
\]

**Proof.** We verify (42). Our proof is by induction on \( r \). First assume \( r = 0 \), so that \( s + t = N - 1 \). By construction \( \mu = (0, N - t, t) \) and \( \nu = (0, s, N - s) \). So \( Xu = (2t - N)v \) and \( Xw = (N - 2s)w \) in view of Lemma 21.3. To obtain (42), in the equation \( u + v + w = 0 \) apply \( X \) to each term and evaluate the result using the above comments. We have verified (42) for \( r = 0 \). Next assume \( r \geq 1 \). For the location \((r - 1, s, t + 1) \in \Delta_{N-1}\) the corresponding black 3-clique in \( \Delta_N \) is

\[
\lambda' = (r, s, t + 1), \quad \mu' = (r - 1, s + 1, t + 1), \quad \nu' = (r - 1, s, t + 2).
\]

Observe \( \lambda' = \nu \). By Lemma 13.7 and since \( 0 \neq w \in B_\nu = B_{\lambda'} \) there exists a unique abrace

\[
u' \in B_{\lambda'}, \quad v' \in B_{\mu'}, \quad w' \in B_{\nu'}
\]

such that \( u' = w \). By Definition 13.1 \( u' + v' + w' = 0 \). By induction

\[
Xu' = (N - 2t - 2)v' + (2s - N)w'. \quad (43)
\]

For the location \((r - 1, s + 1, t) \in \Delta_{N-1}\) the corresponding black 3-clique in \( \Delta_N \) is

\[
\lambda'' = (r, s + 1, t), \quad \mu'' = (r - 1, s + 2, t), \quad \nu'' = (r - 1, s + 1, t + 1).
\]

Observe \( \lambda'' = \mu \). By Lemma 13.7 and since \( 0 \neq v \in B_\mu = B_{\lambda''} \) there exists a unique abrace

\[
u'' \in B_{\lambda''}, \quad v'' \in B_{\mu''}, \quad w'' \in B_{\nu''}
\]
such that \( u'' = v \). By Definition 13.1 \( u'' + v'' + w'' = 0 \). By induction
\[
Xu'' = (N - 2t)v'' + (2s + 2 - N)w''. \tag{44}
\]
We claim that \( w'' = v' \). We now prove the claim. Observe \( \mu' = \nu'' \). The three locations
\[
\mu' = \nu'', \quad \lambda' = \nu, \quad \lambda'' = \mu
\]
race clockwise around a white 3-clique in \( \Delta_N \), which we recall has \( B \)-value 1. By construction \( v', u' \) is a brace for the edge \( \mu', \lambda \) so \( \tilde{B}_{\mu', \lambda} \) sends \( v' \mapsto u' \). Similarly \( w, v \) is a brace for the edge \( \nu, \mu \) so \( \tilde{B}_{\nu, \mu} \) sends \( w \mapsto v \). Similarly \( u'', w'' \) is a brace for the edge \( \lambda'', \nu \) so \( \tilde{B}_{\lambda'', \nu} \) sends \( u'' \mapsto w'' \). Consider the composition
\[
\begin{array}{c}
B_{\mu'} \xrightarrow{\tilde{B}_{\mu', \lambda'}} B_{\lambda'} = B_{\nu} \xrightarrow{\tilde{B}_{\nu, \mu}} B_{\mu} = B_{X''} \xrightarrow{\tilde{B}_{X'', \nu}} B_{\nu} = B_{\mu'}.
\end{array}
\]
On one hand, this composition sends
\[
v' \mapsto u' = w \mapsto v = u'' \mapsto w''.
\]
On the other hand, by Definition 14.9 this composition is equal to the identity map on \( B_{\mu'} \). Therefore \( w'' = v' \) and the claim is proved. Now to obtain (42), in the equation \( u + v + w = 0 \) apply \( X \) to each term and evaluate the result using (43), (44) and nearby comments. We have verified equation (42). The remaining two equations are similarly verified.

We now reformulate Proposition 21.4

**Corollary 21.5.** Referring to the notation above Proposition 21.4, the following (i)–(iii) hold.

(i) On \( B_\lambda \),
\[
X = (N - 2t)\tilde{B}_{\lambda, \mu} + (2s - N)\tilde{B}_{\lambda, \nu}.
\]

(ii) On \( B_\mu \),
\[
Y = (N - 2r)\tilde{B}_{\mu, \nu} + (2t - N)\tilde{B}_{\mu, \lambda}.
\]

(iii) On \( B_\nu \),
\[
Z = (N - 2s)\tilde{B}_{\nu, \lambda} + (2r - N)\tilde{B}_{\nu, \mu}.
\]

**Proof.** Use Lemma 14.6 and Proposition 21.4.

Recall the vectors \( \alpha, \beta, \gamma \) from line (2).

**Proposition 21.6.** For each location \( \lambda = (r, s, t) \) in \( \Delta_N \) the following hold on \( B_\lambda \):

\[
\begin{align*}
X - (N - 2s)I &= 2r\tilde{B}_{\lambda, \alpha}, & X - (2t - N)I &= -2r\tilde{B}_{\lambda, \gamma}, \\
Y - (N - 2t)I &= 2s\tilde{B}_{\lambda, \beta}, & Y - (2r - N)I &= -2s\tilde{B}_{\lambda, \alpha}, \\
Z - (N - 2r)I &= 2t\tilde{B}_{\lambda, \gamma}, & Z - (2s - N)I &= -2t\tilde{B}_{\lambda, \beta}.
\end{align*}
\]
Proof. We verify the equations in the top row. First assume \( r = 0 \), so that \( s + t = N \). In each equation the left-hand side is zero by Lemma 21.3. In each equation the right-hand side is zero since \( r = 0 \). Next assume \( r \geq 1 \). Define
\[
\mu = (r - 1, s + 1, t) = \lambda - \alpha, \quad \nu = (r - 1, s, t + 1) = \lambda + \gamma.
\]
The locations \( \lambda, \mu, \nu \) form a black 3-clique in \( \Delta_N \) that corresponds to the location \((r - 1, s, t)\) in \( \Delta_{N-1} \) under the bijection of Lemma 14.31. Apply Corollary 21.5(ii) to this 3-clique and in the resulting equation (45) eliminate \( \tilde{B}_{\lambda,\mu} \) or \( \tilde{B}_{\lambda,\nu} \) using (17). We have verified the equations in the top row. The other equations are similarly verified. \( \square \)

**Corollary 21.7.** For each location \( \lambda = (r, s, t) \) in \( \Delta_N \),
\[
(X - (N - 2s)I)B_\lambda \subseteq B_{\lambda-\alpha}, \quad (X - (2t - N)I)B_\lambda \subseteq B_{\lambda+\gamma},
\]
\[
(Y - (N - 2t)I)B_\lambda \subseteq B_{\lambda-\beta}, \quad (Y - (2r - N)I)B_\lambda \subseteq B_{\lambda+\alpha},
\]
\[
(Z - (N - 2r)I)B_\lambda \subseteq B_{\lambda-\gamma}, \quad (Z - (2s - N)I)B_\lambda \subseteq B_{\lambda+\beta}.
\]
Moreover, equality holds in each inclusion provided that the characteristic of \( \mathbb{F} \) is 0 or greater than \( N \).

**Proof.** By Note 14.2 and Proposition 21.6. \( \square \)

Our next goal is to show that the elements \( X, Y, Z \) from Definition 21.2 satisfy the defining relations for \( sI_2 \) given in (41).

**Lemma 21.8.** For each location \( \lambda = (r, s, t) \) in \( \Delta_N \) the following hold on \( B_\lambda \):
\[
XY - 2Y = X(2r - N) + Y(N - 2s) + (N + 2)(2t - N)I = XY + 2X,
\]
\[
YZ - 2Z = Y(2s - N) + Z(N - 2t) + (N + 2)(2r - N)I = ZY + 2Y,
\]
\[
ZX - 2X = Z(2t - N) + X(N - 2r) + (N + 2)(2s - N)I = XZ + 2Z.
\]

**Proof.** We verify the equation on the left in the bottom row. For this equation let \( M \) denote the left-hand side minus the right-hand side. We show \( M = 0 \) on \( B_\lambda \). First assume \( r = 0 \), so that \( s + t = N \). By Lemma 21.3 \( X = (2t - N)I \) on \( B_\lambda \). By these comments we routinely obtain \( M = 0 \) on \( B_\lambda \). Next assume \( r \geq 1 \). Observe \( \lambda + \gamma = (r - 1, s, t + 1) \in \Delta_N \). By Lemma 14.5 the maps \( \tilde{B}_{\lambda,\lambda+\gamma} : B_\lambda \to B_{\lambda+\gamma} \) and \( \tilde{B}_{\lambda+\gamma,\lambda} : B_{\lambda+\gamma} \to B_\lambda \) are inverses. Using Proposition 21.3 we find that on \( B_\lambda \),
\[
X - (2t - N)I = -2r\tilde{B}_{\lambda,\lambda+\gamma},
\]
and on \( B_{\lambda+\gamma} \),
\[
Z - (N - 2r + 2)I = 2(t + 1)\tilde{B}_{\lambda+\gamma,\lambda}.
\]
Therefore on \( B_\lambda \),
\[
(Z - (N - 2r + 2)I)(X - (2t - N)I) = -4r(t + 1)I. \tag{46}
\]
Evaluating (46) using \( r + s + t = N \) we find that \( M = 0 \) on \( B_\lambda \). We have verified the equation on the left in the bottom row. The remaining equations are similarly verified. \( \square \)
Proposition 21.9. The elements $X, Y, Z$ from Definition 21.2 satisfy
\[ XY - YX = 2X + 2Y, \quad YZ - ZY = 2Y + 2Z, \quad ZX - XZ = 2Z + 2X. \]

Proof. By Lemma 21.8 these equations hold on $B_\lambda$ for all $\lambda \in \Delta_N$. The result holds in view of Corollary 7.7. \qed

Theorem 21.10. Let $V$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$. Let $B$ denote a Billiard Array on $V$. Assume that each white 3-clique in $\Delta_N$ has $B$-value 1. Then there exists a unique $\mathfrak{sl}_2$-module structure on $V$ such that for $0 \leq i \leq N$, $x - (2i - N)I$ (resp. $y - (2i - N)I$) (resp. $z - (2i - N)I$) vanishes on component $i$ of the $B$-decomposition $[2, 3]$ (resp. $[3, 1]$) (resp. $[1, 2]$). The $\mathfrak{sl}_2$-module $V$ is irreducible, provided that the characteristic of $\mathbb{F}$ is 0 or greater than $N$.

Proof. The $\mathfrak{sl}_2$-module structure exists by Proposition 21.9. It is unique by construction. The last assertion of the theorem is readily checked. \qed

Definition 21.11. [2] Section 2] Define $\nu_x, \nu_y, \nu_z$ in $\mathfrak{sl}_2$ by
\[ -2\nu_x = y + z, \quad -2\nu_y = z + x, \quad -2\nu_z = x + y. \]

Our next goal is to describe the actions of $\nu_x, \nu_y, \nu_z$ on $\{B_\lambda\}_{\lambda \in \Delta_N}$.

Proposition 21.12. For each location $\lambda = (r, s, t)$ in $\Delta_N$ the following hold on $B_\lambda$:
\[ \nu_x = s \tilde{B}_{\lambda, \alpha} - t \tilde{B}_{\lambda, -\gamma}, \quad \nu_y = t \tilde{B}_{\lambda, \beta} - r \tilde{B}_{\lambda, -\alpha}, \quad \nu_z = r \tilde{B}_{\lambda, \gamma} - s \tilde{B}_{\lambda, -\beta}. \]

Proof. Evaluate the equations in Definition 21.11 using Proposition 21.6. \qed

Corollary 21.13. For each location $\lambda \in \Delta_N$,
\[ \nu_x B_\lambda \subseteq B_{\lambda + \alpha} + B_{\lambda - \gamma}, \quad \nu_y B_\lambda \subseteq B_{\lambda + \beta} + B_{\lambda - \alpha}, \quad \nu_z B_\lambda \subseteq B_{\lambda + \gamma} + B_{\lambda - \beta}. \]

Proof. By Note 14.2 and Proposition 21.12. \qed

Recall the $B$-flags $[1, 2, 3]$ from Definition 9.9.

Proposition 21.14. Assume that the characteristic of $\mathbb{F}$ is 0 or greater than $N$. Then the $B$-flags $[1, 2, 3]$ are, respectively,
\[ \{\nu_x^{N-i}V\}_{i=0}^N, \quad \{\nu_y^{N-i}V\}_{i=0}^N, \quad \{\nu_z^{N-i}V\}_{i=0}^N. \]

Proof. Consider the $B$-flag $[1]$. Denote this by $\{U_i\}_{i=0}^N$. Denote the $B$-decomposition $[1, 2]$ by $\{V_i\}_{i=0}^N$. Recall from Lemma 10.4 that for $0 \leq i \leq N$, $V_i$ is included in $B$ at location $(N - i, i, 0)$. Recall from Lemma 10.6 that $U_i = V_0 + \cdots + V_i$ for $0 \leq i \leq N$. By Corollary 21.13 we find $\nu_x V_i \subseteq V_{i-1}$ for $1 \leq i \leq N$ and $\nu_x V_0 = 0$. Going back to Proposition 21.12 and invoking our assumption about the characteristic of $\mathbb{F}$, we see that in fact $\nu_x V_i = V_{i-1}$ for $1 \leq i \leq N$. By the above comments $\nu_x U_i = U_{i-1}$ for $1 \leq i \leq N$. Now since $U_N = V$ we obtain $U_i = \nu_x^{N-i}V$ for $0 \leq i \leq N$. We have shown that the $B$-flag $[1]$ is equal to $\{\nu_x^{N-i}V\}_{i=0}^N$. The remaining assertions are similarly shown. \qed
Corollary 21.15. Assume that $\mathbb{F}$ has characteristic 0. Let $V$ denote an irreducible $\mathfrak{sl}_2$-module of dimension $N + 1$. Then

(i) the following are totally opposite flags on $V$:
\[
\{\nu_x^{N-i}V\}_{i=0}^N, \quad \{\nu_y^{N-i}V\}_{i=0}^N, \quad \{\nu_z^{N-i}V\}_{i=0}^N.
\]

(ii) for the corresponding Billiard Array on $V$, the value of each white 3-clique is 1.

Proof. By [6, Theorem 7.2], up to isomorphism there exists a unique irreducible $\mathfrak{sl}_2$-module with dimension $N + 1$. The $\mathfrak{sl}_2$-module from Theorem 21.10 is irreducible with dimension $N + 1$. Therefore the $\mathfrak{sl}_2$-module $V$ is isomorphic to the $\mathfrak{sl}_2$-module from Theorem 21.10. Via the isomorphism we identify these $\mathfrak{sl}_2$-modules. Consider the Billiard Array $B$ on $V$ from Theorem 21.10. By Theorem 12.4 and Proposition 21.14, the three sequences in line (47) are totally opposite flags on $V$, and $B$ is the corresponding Billiard Array. By the assumption of Theorem 21.10, the $B$-value of each white 3-clique is 1. We are done discussing $\mathfrak{sl}_2$. For the rest of this section assume the field $\mathbb{F}$ is arbitrary. Fix a nonzero $q \in \mathbb{F}$ such that $q^2 \neq 1$. We recall the quantum algebra $U_q(\mathfrak{sl}_2)$. We will use the equitable presentation, which was introduced in [11].

Definition 21.16. [11, Theorem 2.1] Let $U_q(\mathfrak{sl}_2)$ denote the associative $\mathbb{F}$-algebra with generators $x, y^\pm 1, z$ and relations
\[
qxy - q^{-1}yx = 1, \quad qyz - q^{-1}zy = 1, \quad qzx - q^{-1}xz = 1.
\]

(48)

For the rest of this section the following notation is in effect. Fix $N \in \mathbb{N}$. Let $V$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$. Let $B$ denote a Billiard Array on $V$. Assume that each white 3-clique in $\Delta_N$ has $B$-value $q^{-2}$. Using $B$ we will turn $V$ into a $U_q(\mathfrak{sl}_2)$-module. Recall the $B$-decompositions $[\eta, \xi]$ of $V$ from Definition 10.3

Definition 21.17. Define $X, Y, Z$ in $\text{End}(V)$ such that for $0 \leq i \leq N$, $X - q^{N-2i}I$ (resp. $Y - q^{N-2i}I$) (resp. $Z - q^{N-2i}I$) vanishes on component $i$ of the $B$-decomposition $[2, 3]$ (resp. $[3, 1]$) (resp. $[1, 2]$). Note that each of $X, Y, Z$ is invertible.

The next result is meant to clarify Definition 21.17

Lemma 21.18. Pick a location $\lambda = (r, s, t)$ on the boundary of $\Delta_N$. Then on $B_\lambda$,
\[
X = q^{N-2t}I = q^{2s-N}I \quad \text{if } r = 0; \\
Y = q^{N-2r}I = q^{2s-N}I \quad \text{if } s = 0; \\
Z = q^{N-2s}I = q^{2r-N}I \quad \text{if } t = 0.
\]

Proof. By Lemma 10.3 and Definition 21.17.

Pick $(r, s, t) \in \Delta_{N-1}$ and consider the corresponding black 3-clique in $\Delta_N$ from Lemma 4.31
\[
\lambda = (r+1, s, t), \quad \mu = (r, s+1, t), \quad \nu = (r, s, t+1).
\]
Proposition 21.19. With the above notation, for each abrace
\[ u \in B_\lambda, \quad v \in B_\mu, \quad w \in B_\nu \]
we have
\[ Xu = -q^{N-2t}v - q^{2s-N}w, \quad Yv = -q^{N-2r}w - q^{2t-N}u, \quad Zw = -q^{N-2s}u - q^{2r-N}v. \]  

Proof. Our proof is similar to the proof of Proposition 21.14, we give the details for the sake of clarity. We verify (49). Our proof is by induction on \( r \). First assume \( r = 0 \), so that \( s + t = N - 1 \). By construction \( \mu = (0, N - t, t) \) and \( \nu = (0, s, N - s) \). So \( Xu = q^{N-2t}v \) and \( Xw = q^{2s-N}w \) in view of Lemma 21.18. To obtain (49), in the equation \( u + v + w = 0 \) apply \( X \) to each term and evaluate the result using the above comments. We have verified (49) for \( r = 0 \). Next assume \( r \geq 1 \). For the location \((r - 1, s, t + 1) \in \Delta_{N-1}\) the corresponding black 3-clique in \( \Delta_N \) is
\[ \lambda' = (r, s, t + 1), \quad \mu' = (r - 1, s + 1, t + 1), \quad \nu' = (r - 1, s, t + 2). \]
Observe \( \lambda' = \nu \). By Lemma 13.7 and since \( 0 \neq w \in B_\nu = B_\lambda \) there exists a unique abrace
\[ u' \in B_\lambda, \quad v' \in B_\mu, \quad w' \in B_\nu \]
such that \( u' = w \). By Definition 13.1 \( u' + v' + w' = 0 \). By induction
\[ Xu' = -q^{N-2t-2}v' - q^{2s-N}w'. \]  

For the location \((r - 1, s + 1, t) \in \Delta_{N-1}\) the corresponding black 3-clique in \( \Delta_N \) is
\[ \lambda'' = (r, s + 1, t), \quad \mu'' = (r - 1, s + 2, t), \quad \nu'' = (r - 1, s + 1, t + 1). \]
Observe \( \lambda'' = \mu \). By Lemma 13.7 and since \( 0 \neq v \in B_\mu = B_\lambda \) there exists a unique abrace
\[ u'' \in B_\lambda, \quad v'' \in B_\mu, \quad w'' \in B_\nu \]
such that \( u'' = v \). By Definition 13.1 \( u'' + v'' + w'' = 0 \). By induction
\[ Xu'' = -q^{N-2t}v'' - q^{2s+2-N}w''. \]  

We claim that \( w'' = q^{-2}v' \). We now prove the claim. Observe \( \mu' = \nu'' \). The three locations
\[ \mu' = \nu'', \quad \lambda' = \nu, \quad \lambda'' = \mu \]
rung clockwise around a white 3-clique in \( \Delta_N \), which we recall has \( B \)-value \( q^{-2} \). By construction \( v', u' \) is a brace for the edge \( \mu', \lambda \) so \( B_{\mu', \lambda} \) sends \( v' \mapsto u' \). Similarly \( w, v \) is a brace for the edge \( \nu, \mu \) so \( B_{\nu, \mu} \) sends \( w \mapsto v \). Similarly \( u'', w'' \) is a brace for the edge \( \lambda'', \nu'' \) so \( B_{\lambda'', \nu''} \) sends \( u'' \mapsto w'' \). Consider the composition
\[ B_{\mu'} \xrightarrow{B_{\mu', \lambda'}} B_{\lambda'} = B_{\nu} \xrightarrow{B_{\nu, \mu}} B_{\mu} = B_{\lambda''} \xrightarrow{B_{\lambda'', \nu''}} B_{\nu''} = B_{\mu'} . \]
On one hand, this composition sends
\[ u' \mapsto u' = w \mapsto v = u'' \mapsto w''. \]

On the other hand, by Definition \[14.9\] this composition is equal to \( q^{-2} \) times the identity map on \( B'_\mu \). Therefore \( w'' = q^{-2}v' \) and the claim is proved. Now to obtain \([49]\), in the equation \( u + v + w = 0 \) apply \( X \) to each term and evaluate the result using \([50], [51]\) and nearby comments. We have verified equation \([49]\). The remaining two equations are similarly verified.

We now reformulate Proposition \[21.19\].

**Corollary 21.20.** Referring to the notation above Proposition \[21.19\], the following (i)–(iii) hold.

(i) On \( B_\lambda \),
\[ X = -q^{N-2t} B_{\lambda,\mu} - q^{2s-N} B_{\lambda,\nu}. \]

(ii) On \( B_\mu \),
\[ Y = -q^{N-2r} B_{\mu,\nu} - q^{2t-N} B_{\mu,\lambda}. \]

(iii) On \( B_\nu \),
\[ Z = -q^{N-2s} B_{\nu,\lambda} - q^{2r-N} B_{\nu,\mu}. \]

**Proof.** Use Lemma \[14.6\] and Proposition \[21.19\].

**Proposition 21.21.** For each location \( \lambda = (r, s, t) \) in \( \Delta_N \) the following hold on \( B_\lambda \):

\[
\begin{align*}
X - q^{2s-N} I & = (q^{2s-N} - q^{-N-2t}) B_{\lambda,\alpha}, & X - q^{N-2t} I & = (q^{N-2t} - q^{2s-N}) B_{\lambda,\gamma}, \\
Y - q^{2t-N} I & = (q^{2t-N} - q^{-N-2r}) B_{\lambda,\beta}, & Y - q^{N-2r} I & = (q^{N-2r} - q^{2t-N}) B_{\lambda,\alpha}, \\
Z - q^{2r-N} I & = (q^{2r-N} - q^{-N-2s}) B_{\lambda,\gamma}, & Z - q^{N-2s} I & = (q^{N-2s} - q^{2r-N}) B_{\lambda,\beta}.
\end{align*}
\]

**Proof.** We verify the equations in the top row. First assume \( r = 0 \), so that \( s + t = N \). In each equation the left-hand side is zero by Lemma \[21.18\]. In each equation the right-hand side is zero since \( 2s - N = N - 2t \). Next assume \( r \geq 1 \). Define
\[ \mu = (r - 1, s + 1, t) = \lambda - \alpha, \quad \nu = (r - 1, s, t + 1) = \lambda + \gamma. \]

The locations \( \lambda, \mu, \nu \) form a black 3-clique in \( \Delta_N \) that corresponds to the location \( (r - 1, s, t) \) in \( \Delta_{N-1} \) under the bijection of Lemma \[4.31\]. Apply Corollary \[21.20\](i) to this 3-clique and in the resulting equation \([52]\) eliminate \( B_{\lambda,\mu} \) or \( B_{\lambda,\nu} \) using \([17]\). We have verified the equations in the top row. The other equations are similarly verified.

**Corollary 21.22.** For each location \( \lambda = (r, s, t) \) in \( \Delta_N \),
\[
\begin{align*}
(X - q^{2s-N} I)B_\lambda & \subseteq B_{\lambda-\alpha}, & (X - q^{N-2t} I)B_\lambda & \subseteq B_{\lambda+\gamma}, \\
(Y - q^{2t-N} I)B_\lambda & \subseteq B_{\lambda-\beta}, & (Y - q^{N-2r} I)B_\lambda & \subseteq B_{\lambda+\alpha}, \\
(Z - q^{2r-N} I)B_\lambda & \subseteq B_{\lambda-\gamma}, & (Z - q^{N-2s} I)B_\lambda & \subseteq B_{\lambda+\beta}.
\end{align*}
\]

Moreover, equality holds in each inclusion provided that \( q^{2i} \neq 1 \) for \( 1 \leq i \leq N \).
Proof. By Note 14.2 and Proposition 21.21.

Our next goal is to show that the elements $X, Y, Z$ from Definition 21.17 satisfy the defining relations for $U_q(\mathfrak{sl}_2)$ given in (48).

Lemma 21.23. For each location $\lambda = (r, s, t)$ in $\Delta_N$ the following hold on $B_\lambda$:

\[
q(I - XY) = -q^{N-2r+1}(X - q^{N-2}I) - q^{2s-N-1}(Y - q^{2t-N}I) = q^{-1}(I - YX),
\]
\[
q(I - YZ) = -q^{N-2s+1}(Y - q^{N-2}I) - q^{2t-N-1}(Z - q^{2r-N}I) = q^{-1}(I - ZY),
\]
\[
q(I - ZX) = -q^{N-2t+1}(Z - q^{N-2s}I) - q^{2r-N-1}(X - q^{2s-N}I) = q^{-1}(I - XZ).
\]

Proof. We verify the equation on the left in the bottom row. For this equation let $M$ denote the left-hand side minus the right-hand side. We show $M$ vanishes on the right in the bottom row. For this equation let $r \geq 1$. Observe $\lambda + \gamma = (r - 1, s, t + 1) \in \Delta_N$. By Lemma 21.23 the maps $\tilde{B}_{\lambda,\lambda+\gamma} : B_\lambda \to B_{\lambda+\gamma}$ and $\tilde{B}_{\lambda+\gamma,\lambda} : B_{\lambda+\gamma} \to B_\lambda$ are inverses. Using Proposition 21.21 we find that on $B_\lambda$,

\[
X - q^{N-2t}I = (q^{N-2t} - q^{2s-N})\tilde{B}_{\lambda,\lambda+\gamma},
\]

and on $B_{\lambda+\gamma}$,

\[
Z - q^{2r-2-N}I = (q^{2r-2-N} - q^{N-2s})\tilde{B}_{\lambda+\gamma,\lambda}.
\]

Therefore on $B_\lambda$,

\[
(Z - q^{2r-2-N}I)(X - q^{N-2t}I) = (q^{2r-2-N} - q^{N-2s})(q^{N-2t} - q^{2s-N})I.
\]

(53)

Evaluating (53) using $r + s + t = N$ we find that $M = 0$ on $B_\lambda$. We have verified the equation on the left in the bottom row. The remaining equations are similarly verified.

Lemma 21.24. The elements $X, Y, Z$ from Definition 21.17 satisfy

\[
q(I - XY) = q^{-1}(I - YX),
\]
\[
q(I - YZ) = q^{-1}(I - ZY),
\]
\[
q(I - ZX) = q^{-1}(I - XZ).
\]

Proof. By Lemma 21.23 these equations hold on $B_\lambda$ for all $\lambda \in \Delta_N$. The result holds in view of Corollary 7.27.

Proposition 21.25. The elements $X, Y, Z$ from Definition 21.17 satisfy

\[
\frac{qXY - q^{-1}YX}{q - q^{-1}} = I, \quad \frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I, \quad \frac{qZX - q^{-1}XZ}{q - q^{-1}} = I.
\]

Proof. These equations are a reformulation of the equations in Lemma 21.24.

Theorem 21.26. Let $V$ denote a vector space over $\mathbb{F}$ with dimension $N + 1$. Let $B$ denote a Billiard Array on $V$. Assume that each white 3-clique in $\Delta_N$ has $B$-value $q^{-2}$. Then there exists a unique $U_q(\mathfrak{sl}_2)$-module structure on $V$ such that for $0 \leq i \leq N$, $x - q^{N-2i}I$ (resp. $y - q^{N-2i}I$) (resp. $z - q^{N-2i}I$) vanishes on component $i$ of the $B$-decomposition $[2, 3]$ (resp. $[3, 1]$) (resp. $[1, 2]$). The $U_q(\mathfrak{sl}_2)$-module $V$ is irreducible, provided that $q^{2i} \neq 1$ for $1 \leq i \leq N$. 50
Proof. The $U_q(\mathfrak{sl}_2)$-module structure exists by Proposition 21.25. It is unique by construction. The last assertion of the theorem is readily checked.

**Definition 21.27.** [13] Definition 3.1 Let $\nu_x, \nu_y, \nu_z$ denote the following elements in $U_q(\mathfrak{sl}_2)$:

\[
\begin{align*}
\nu_x &= q(1 - yz) = q^{-1}(1 - zy), \\
\nu_y &= q(1 - zx) = q^{-1}(1 - xz), \\
\nu_z &= q(1 - xy) = q^{-1}(1 - yx).
\end{align*}
\]

Our next goal is to describe the actions of $\nu_x, \nu_y, \nu_z$ on $\{B_\lambda\}_{\lambda \in \Delta_N}$.

**Proposition 21.28.** For each location $\lambda = (r, s, t)$ in $\Delta_N$ the following hold on $B_\lambda$:

\[
\begin{align*}
\nu_x &= q^{1 - 2s - 1}(q^s - q^{-t}) \bar{B}_{\lambda, \lambda - \gamma} - q^{2t - s + 1}(q^s - q^{-s}) \bar{B}_{\lambda, \lambda + \alpha}, \\
\nu_y &= q^{r - 2t - 1}(q^r - q^{-r}) \bar{B}_{\lambda, \lambda - \alpha} - q^{2r - t + 1}(q^r - q^{-t}) \bar{B}_{\lambda, \lambda + \beta}, \\
\nu_z &= q^{s - 2r - 1}(q^s - q^{-r}) \bar{B}_{\lambda, \lambda - \beta} - q^{2s - r + 1}(q^s - q^{-r}) \bar{B}_{\lambda, \lambda + \gamma}.
\end{align*}
\]

**Proof.** Evaluate the equations in Lemma 21.23 using Proposition 21.21 and Definition 21.27.

**Corollary 21.29.** For each location $\lambda \in \Delta_N$,

\[
\begin{align*}
\nu_x B_\lambda &\subseteq B_{\lambda + \alpha} + B_{\lambda - \gamma}, & \nu_y B_\lambda &\subseteq B_{\lambda + \beta} + B_{\lambda - \alpha}, & \nu_z B_\lambda &\subseteq B_{\lambda + \gamma} + B_{\lambda - \beta}.
\end{align*}
\]

**Proof.** By Note 14.2 and Proposition 21.28.

Recall the $B$-flags [1], [2], [3] from Definition 9.9.

**Proposition 21.30.** Assume that $q^{2i} \neq 1$ for $1 \leq i \leq N$. Then the $B$-flags [1], [2], [3] are, respectively,

\[
\{ \nu_x^{N-i} V \}_{i=0}^N, \quad \{ \nu_y^{N-i} V \}_{i=0}^N, \quad \{ \nu_z^{N-i} V \}_{i=0}^N.
\]

**Proof.** Similar to the proof of Proposition 21.14.

**Corollary 21.31.** Assume that $q$ is not a root of unity. Let $V$ denote an irreducible $U_q(\mathfrak{sl}_2)$-module with dimension $N + 1$. Then

(i) the following are totally opposite flags on $V$:

\[
\{ \nu_x^{N-i} V \}_{i=0}^N, \quad \{ \nu_y^{N-i} V \}_{i=0}^N, \quad \{ \nu_z^{N-i} V \}_{i=0}^N.
\]

(ii) for the corresponding Billiard Array on $V$, the value of each white 3-clique is $q^{-2}$.

**Proof.** By [11] Theorem 2.1 and [12] Theorem 2.6, each of $x, y, z$ is diagonalizable on $V$. Moreover by [11] Theorem 2.1 and [12] Theorem 2.6, there exists $\varepsilon \in \{1, -1\}$ such that for each of $x, y, z$ the eigenvalues on $V$ are $\{\varepsilon q^{N-2i}\}_{i=0}^N$. The $U_q(\mathfrak{sl}_2)$-module $V$ has type $\varepsilon$ in the sense of [12] Section 5.2. Replacing $x, y, z$ by $\varepsilon x, \varepsilon y, \varepsilon z$ respectively, the type becomes 1 and $\nu_x, \nu_y, \nu_z$ are unchanged. By [12] Theorem 2.6, up to isomorphism there exists a unique irreducible $U_q(\mathfrak{sl}_2)$-module with type 1 and dimension $N + 1$. The $U_q(\mathfrak{sl}_2)$-module from Theorem 21.26 is irreducible, with type 1 and dimension $N + 1$. Therefore the $U_q(\mathfrak{sl}_2)$-module $V$ is isomorphic to the $U_q(\mathfrak{sl}_2)$-module from Theorem 21.26. Via the isomorphism we identify these $U_q(\mathfrak{sl}_2)$-modules. Consider the Billiard Array $B$ on $V$ from Theorem 21.26. By Theorem 21.4 and Proposition 21.30 the three sequences in line (54) are totally opposite flags on $V$, and $B$ is the corresponding Billiard Array. By the assumption of Theorem 21.26 the $B$-value of each white 3-clique is $q^{-2}$.

51
22 Acknowledgments

The author thanks Jae-ho Lee and Kazumasa Nomura for giving this paper a close reading and offering valuable suggestions.

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