NEW RIEMANNIAN MANIFOLDS WITH $L^p$-UNBOUNDED RIESZ TRANSFORM FOR $p > 2$

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Abstract. We construct a large class of Riemannian manifolds of arbitrary dimension with Riesz transform unbounded on $L^p(M)$ for all $p > 2$. This extends recent results for Vicsek manifolds, and in particular shows that fractal structure is not necessary for this property.

1. Introduction

Consider a Riemannian manifold $M$ with gradient $\nabla$ and Laplace–Beltrami operator $\Delta$. The Riesz transform $\nabla \Delta^{-1/2}$, with $\Delta^{-1/2}$ defined via the spectral theorem, maps $L^2(M)$ boundedly to the space of square integrable vector fields $L^2(M; TM)$. Much attention has been given to the question of whether this operator extends to a bounded map from $L^p(M)$ to $L^p(M; TM)$ for $p \neq 2$, or equivalently, whether the estimate

$$(R_p) : \| |\nabla f| \|_p \lesssim \| \Delta^{1/2} f \|_p \quad \text{for all } f \in C^\infty_c(M)$$

holds. It is conjectured that for $p \in (1, 2)$ the estimate $(R_p)$ holds whenever $M$ is complete, with implicit constant depending only on $p$; in [1] it is shown that the failure of this uniformity among manifolds of a fixed dimension would imply the existence of a manifold for which $(R_p)$ fails.

One is naturally led to consider also the ‘reverse’ estimate

$$(RR_p) : \| \Delta^{1/2} f \|_p \lesssim \| |\nabla f| \|_p \quad \text{for all } f \in C^\infty_c(M).$$

A duality argument shows that for $p \in (1, \infty)$, $(R_p)$ implies $(RR_{p'})$, where $p' = p/(p - 1)$ is the Hölder conjugate of $p$. If $(R_p)$ and $(RR_p)$ both hold, then we have a norm equivalence

$$\| |\nabla f| \|_p \simeq \| \Delta^{1/2} f \|_p,$$

which says that the homogeneous Sobolev space $\dot{W}^{p}_1(M)$ may be defined either via the gradient or via the square root of the Laplace–Beltrami operator.

Generally $(R_p)$ holds only for some interval of $p \in (1, \infty)$ including 2, and proving $(R_p)$ presents different difficulties depending on whether $p < 2$ or $p > 2$. When $1 < p < 2$, $(R_p)$ is known to follow from the volume doubling property and Gaussian or sub-Gaussian heat kernel upper estimates [10, 9] (see also [15] for examples which do not satisfy such kernel estimates). The volume doubling property and an appropriately scaled $L^2$-Poincaré inequality imply $(R_p)$ for some $p > 2$ [2]. In [3] this is linked with gradient estimates for the heat kernel, and in [5] the $L^2$-Poincaré inequality is replaced by a relative Faber–Krahn inequality and a reverse Hölder inequality.

Some manifolds for which $(R_p)$ fails for some $p > 2$ are known. If $M$ is the connected sum of two copies of $\mathbb{R}^n \setminus B(0, 1)$ with $n \geq 3$ —or more generally, an $n$-dimensional manifold with at least two (and finitely many) Euclidean ends—$(R_p)$
holds if and only if $p \in (1, n)$. Similar results are known for conical manifolds [14] and for 2-hyperbolic, $p$-parabolic manifolds with at least two ends [6].

The most relevant examples to this article are Vicsek manifolds, which are ‘thickenings’ of Vicsek graphs (pictured in the 2-dimensional case in Figure 2). The Vicsek graph, being a graphical realisation of a Vicsek fractal, is a ‘fractal at infinity’. Locally a Vicsek manifold behaves like Euclidean space (it is, of course, a manifold), but at large scale it behaves like a fractal. In [9] it is shown that for a Vicsek manifold of any dimension, $(R_p)$ holds if and only if $p \in (1, 2]$. The result for $p < 2$ is a consequence of volume doubling and sub-Gaussian heat kernel estimates. The proof that $(R_p)$ fails for $p > 2$ directly uses the definition of the Vicsek graph [9, Theorem 5.1].

In this article we construct a class of manifolds of arbitrary dimension for which $(R_p)$ fails for all $p > 2$. These manifolds are thickenings of what we call spinal graphs, satisfying generalised dimension conditions defined in terms of the spinal structure along with a polynomial volume lower bound. The Vicsek graphs satisfy these conditions, but the proof of this exploits their fractal nature. We construct a large class of non-fractal spinal graphs with the desired dimension conditions and volume lower bounds, thus yielding manifolds of arbitrary dimension with no fractal structure that fail $(R_p)$ for all $p > 2$.

**Notation**

The graphs we consider are non-directed, with at most one edge per pair of vertices, and with no edges from a vertex to itself. The set of vertices of a graph $G$ is denoted by $V(G)$, and if two vertices $x, y \in V(G)$ are neighbours we write $x \sim y$. The set of edges of $G$ is denoted by $E(G)$. For a connected graph $G$ we let $d_G(x, y)$ denote the combinatorial distance between $x$ and $y$, given by the minimum length of a path from $x$ to $y$, and for $x \in V(G)$, $r > 0$ let

$$B_G(x, r) := \{y \in V(G) : d_G(x, y) \leq r\}.$$  

2. **Spinal graphs**

**Definition 2.1.** Let $G$ be a connected graph, $\Sigma \subset V(G)$, and let $\pi : V(G) \to \Sigma$ be a function such that

- $\pi(x) = x$ for all $x \in \Sigma$,
- $\pi^{-1}(x)$ is finite for all $x \in \Sigma$,
- if $a, b \in V(G)$ and $\pi(a) \neq \pi(b)$, then every path from $a$ to $b$ contains a subpath from $\pi(a)$ to $\pi(b)$.

We refer to $(G, \Sigma, \pi)$ as a spinal graph, and the set of vertices $\Sigma$ is called the spine.

**Remark 2.2.** One could formulate this definition without the finiteness condition, but it will be convenient for us to keep it.

An example of a spinal graph $(G, \Sigma, \pi)$ is pictured in Figure 1. There the vertices of the spine $\Sigma$ are shaded black, while the other vertices are unshaded; for each vertex $x$, the point $\pi(x)$ is the (uniquely determined) point on $\Sigma$ of minimal distance to $x$. The dotted lines are *not* edges of $G$; if they were to be added to $G$, then the resulting graph would not be a spinal graph.

\[1\text{In fact, we prove the stronger result that } (RR_q) \text{ fails for all } q \in (1, 2).\]
To help the reader familiarise themselves with the definition of a spinal graph we prove the following lemma (which will be useful later).

**Lemma 2.3.** Let \((G, \Sigma, \pi)\) be a spinal graph, and suppose \(a, b \in V(G)\) with \(\pi(a) = \pi(b) =: x\). Then every minimal path from \(a\) to \(b\) is contained entirely in \(\pi^{-1}(x)\).

**Proof.** Suppose this is false. Then there exist \(a, b \in V(G)\) with \(\pi(a) = \pi(b) =: x\) and a path \(\gamma\) from \(a\) to \(b\) of minimal length which passes through a vertex \(c\) with \(\pi(c) \neq x\). Since \(\pi(a) \neq \pi(c)\), by the third condition in the definition of a spinal graph, there exists a subpath \(\heartsuit\) of \(\gamma\) from \(\pi(a) = x\) to \(\pi(\gamma) = \pi(c)\), so we can write \(\gamma\) as a concatenation of paths \(\gamma = \alpha \ast \heartsuit \ast \beta\), where \(\alpha\) is a path from \(a\) to \(x\) and \(\beta\) is a path from \(\pi(c)\) to \(b\); viewed as a commutative diagram,

\[
\begin{array}{ccc}
    a & \xrightarrow{\gamma} & b \\
     \downarrow{\alpha} & & \downarrow{\beta} \\
    x & \xrightarrow{\heartsuit} & \pi(c).
\end{array}
\]

Similarly, there is a subpath \(\heartsuit'\) of \(\beta\) from \(\pi(c)\) to \(\pi(b) = x\), and we can write \(\beta = \delta \ast \heartsuit' \ast \delta'\) as a concatenation of paths, summarised by the commutative diagram

\[
\begin{array}{ccc}
    b & \xrightarrow{\delta'} & x \\
     \downarrow{\beta} & & \downarrow{\heartsuit'} \\
    \pi(c) & \xrightarrow{\delta} & \pi(c).
\end{array}
\]

Putting these commutative diagrams together we get

\[
\begin{array}{ccc}
    a & \xrightarrow{\gamma} & b \xrightarrow{\delta'} x \\
     \downarrow{\alpha} & & \downarrow{\beta} \\
    x & \xrightarrow{\heartsuit} & \pi(c) \xrightarrow{\delta} \pi(c),
\end{array}
\]

from which we can read that

\[\ell(\gamma) = \ell(\alpha \ast \heartsuit \ast \delta \ast \heartsuit' \ast \delta') = \ell(\alpha) + \ell(\heartsuit) + \ell(\delta) + \ell(\heartsuit') + \ell(\delta')\]

where \(\ell(\cdot)\) denotes the length of a path. Since \(x \neq \pi(c)\), the paths \(\heartsuit\) and \(\heartsuit'\) have positive length, so we find that

\[\ell(\alpha \ast \delta') = \ell(\alpha) + \ell(\delta') < \ell(\gamma).\]

Since \(\alpha \ast \delta'\) is a path from \(a\) to \(b\), this contradicts the assumption that \(\gamma\) has minimal length. \(\square\)

Spinal graphs may be constructed by gluing a collection of finite graphs along another graph; this is made precise in the following example. In fact, this construction yields all spinal graphs (up to isomorphism, in the usual graph-theoretical sense), as will be shown in Proposition 2.5.

**Example 2.4.** Let \(\Gamma\) be a connected graph and let \((G_x)_{x \in V(\Gamma)}\) be a collection of finite connected graphs indexed by the vertices of \(\Gamma\). Suppose that for each \(x \in V(\Gamma)\) a
Suppose Proposition 2.5. an isomorphism between \( G \) corresponds to \( \pi \). Every vertex \( x \in V(G) \) and \( \pi \) pass through \( \pi \) and two vertices \((x, z)\). Therefore \((\Gamma), \pi)\) to \( (\Gamma)\) in \( G \). We set \( \Sigma := V(\Gamma) \) in this embedding; in the disjoint union representation \((1)\) we have

\[
\Sigma := \{(x, z) : x \in V(\Gamma)\}.
\]

Every vertex \( z \in V(G) \) belongs to precisely one of the embedded graphs \( G_x \) with \( x \in V(\Gamma) \), and we define \( \pi : V(G) \to \Sigma \) by the relation \( z \in G_{\pi(z)} \); in the disjoint union representation \((1)\) we have \( \pi(x, z) = (x, z_x) \).

It is immediate that \( \pi(x) = x \) for all \( x \in \Sigma \), and that each \( \pi^{-1}(x) \) is finite. Now suppose \( a, b \in V(G) \) with \( \pi(a) \neq \pi(b) \). By construction, every path including \( a \) that does not pass through \( \pi(a) \) must be entirely contained in \( G_{\pi(a)} \). Since \( b \notin G_{\pi(a)} \), every path from \( a \) to \( b \) must pass through \( \pi(a) \), and by symmetry such a path must also pass through \( \pi(b) \). That is, every path from \( a \) to \( b \) contains a subpath from \( \pi(a) \) to \( \pi(b) \). Therefore \((G, \Sigma, \pi)\) is indeed a spinal graph.

**Proposition 2.5.** Suppose \((G, \Sigma, \pi)\) is a spinal graph. Then there exists a connected graph \( \Gamma \) and a collection \( (G_x)_{x \in V(\Gamma)} \) of finite connected graphs such that there exists an isomorphism between \( G \) and the graph constructed in Example 2.4, under which \( \Sigma \) corresponds to \( V(\Gamma) \) and \( \pi \) corresponds to the map \( \pi' \) given by the relation \( z \in G_{\pi'(z)} \).

**Proof.** Let \((G, \Sigma, \pi)\) be a spinal graph. Let \( \Gamma \) be the full subgraph determined by \( \Sigma \), and for every \( x \in \Sigma = V(\Gamma) \) let \( G_x \) be the full subgraph determined by \( \pi^{-1}(x) \). Then \( \Gamma \) is connected, each \( G_x \) is finite and connected (by Lemma 2.3), and we have a bijection

\[
\varphi : V(G) \to \bigcup_{x \in V(\Gamma)} V(G_x), \quad a \mapsto (\pi(a), a).
\]

By the construction in Example 2.4, it suffices to show that \( a, b \in V(G) \) are neighbours if and only if either \( \pi(a) = \pi(b) \) and \( a \sim b \) in \( G_{\pi(a)} \), or \( a = \pi(a) \) and \( b = \pi(b) \) and \( \pi(a) \sim \pi(b) \) in \( \Gamma \).
By Lemma 2.3, if $\pi(a) = \pi(b)$ then every shortest path from $a$ to $b$ is entirely contained in $G_{\pi(a)}$, so in this case $a$ and $b$ are neighbours in $G$ if and only if $a \sim b$ in $G_{\pi(a)}$. On the other hand, if $\pi(a) \neq \pi(b)$, then every path from $a$ to $b$ contains a subpath from $\pi(a)$ to $\pi(b)$, and thus $a$ and $b$ are neighbours if and only if either $a = \pi(a)$ and $b = \pi(b)$ or $a = \pi(b)$ and $b = \pi(a)$. In the first case, since $\Gamma$ is the full subgraph determined by $\Sigma = \pi(V(G))$, we have $\pi(a) \sim \pi(b)$ in $\Gamma$, and we are done. The second case never occurs: since $\pi(b) \in \Sigma$, we would have $\pi(a) = \pi(\pi(b)) = \pi(b)$, which is a contradiction. 

3. Dimensions of a spinal graph

For a spinal graph $(G, \Sigma, \pi)$ we write $d_\Sigma$ and $B_\Sigma$ for the combinatorial distance and balls in the full subgraph determined by $\Sigma$.

**Definition 3.1.** Let $(G, \Sigma, \pi)$ be a spinal graph. For all $x, y \in V(G)$ define the spinal distance $[x, y]$ by

$$[x, y] := d_\Sigma(\pi(x), \pi(y)),$$

and for $r > 0$ we define associated spinal sets by

$$D(x, r) := \{y \in G : [x, y] \leq r\} = \pi^{-1}(B_\Sigma(\pi(x), r)).$$

The spinal distance is a pseudometric on $V(G)$, and the quotient metric space is isometric to $(\Sigma, d_\Sigma)$, but we will not use this fact in what follows.

**Definition 3.2.** Let $\delta_\Sigma, \delta_G \geq 1$. We say that a spinal graph $(G, \Sigma, \pi)$ has dimensions $(\delta_\Sigma, \delta_G)$ if there exists a point $x_0 \in \Sigma$ and an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of natural numbers such that for all $k \in \mathbb{N}$,

$$\begin{align*}
(2) & \quad |D(x_0, 2n_k)| \lesssim |D(x_0, n_k)| , \\
(3) & \quad |B_\Sigma(x_0, 2n_k)| \lesssim n_k^{\delta_\Sigma} , \\
(4) & \quad |D(x_0, n_k)| \simeq n_k^{\delta_G} .
\end{align*}$$

Note that the dimensions of a spinal graph need not be uniquely determined, and may vary for different choices of $x_0$ and $(n_k)_{k \in \mathbb{N}}$.

**Example 3.3.** Let $n \in \mathbb{N}$ and consider the Vicsek graph $\mathcal{V}^n$ in $\mathbb{R}^n$, the construction of which is given in [4, Proof of Theorem 4.1], [8, Chapter 5], and [9, §5]. One can consider $\mathcal{V}^n$ as a graph with $V(\mathcal{V}^n) \subset \mathbb{Z}^n$, defined as an increasing union of subgraphs $\cup_{m=0}^{\infty} \mathcal{V}^n_m$. The subgraph $\mathcal{V}^n_0$ consists of $2^n + 1$ vertices: one at each corner of the unit $n$-cube, and a central vertex at the origin. Each corner vertex is connected to the central vertex. For $m \geq 1$, $\mathcal{V}^n_m$ is constructed inductively by connecting a copy of $\mathcal{V}^n_{m-1}$ to each ‘corner’ of $\mathcal{V}^n_{m-1}$. It follows that $|V(\mathcal{V}^n_m)| \simeq (2^n + 1)^m$ (see [4, equation (4.10)]).

Let $\Sigma \in V(\mathcal{V}^n)$ be the set of vertices along the $2^n$ diagonals: with $V(\mathcal{V}^n) \subset \mathbb{Z}^n$, we have

$$\Sigma = \{(\varepsilon_1 m, \varepsilon_2 m, \ldots, \varepsilon_n m) \in \mathbb{Z}^n : m \in \mathbb{N}, \varepsilon_j \in \{1, -1\}\}.$$ 

For every vertex $x \in V(\mathcal{V}^n)$ there is a unique $y \in \Sigma$ such that $x$ and $y$ are connected by a path which intersects $\Sigma$ only at $y$. Setting $\pi(x) := y$ makes $(\mathcal{V}^n, \Sigma)$ a spinal graph. Pictured in Figure 2 are the first few steps of the construction of $\mathcal{V}^2$, with the spine $\Sigma$ emphasised.
Let $o \in V(V)$ be the ‘center vertex’ of $V^n_0$, and for $k \in \mathbb{N}$ let $n_k := 3^k$. Then $D(o, n_k) = V^n_k$, and so
\[ |D(o, n_k)| = (2^n + 1)^k = n_k^{\log_3(2^n + 1)}. \]
We also have
\[ |D(o, 2n_k)| \leq |D(o, n_{k+1})| = n_{k+1}^{\log_3(2^n + 1)} \approx n_k^{\log_3(2^n + 1)} = |D(o, n_k)| \]
and
\[ |B_{\Sigma}(o, 2n_k)| = 2^n(2n_k) - 1 < 2^{n+1}3^k \approx n_k, \]
which tells us that the spinal graph $(V^n, \Sigma)$ has dimensions $(1, \log_3(2^n + 1))$. In addition, $V^n$ has polynomial volume growth of dimension $\log_3(2^n + 1)$, that is
\[ |B_{V^n}(x, r)| \approx r^{\log_3(2^n + 1)} \]
for all $x \in V(V^n)$ and $r \in \mathbb{N}$. (see [4, page 632]). The lower estimate will allow us to apply Corollary 5.5 to $V^n$.

In Section 6 we construct spinal graphs with global volume lower bounds and dimensions $(1, D)$ with $D > 1$ that do not arise from fractals.

4. Nash-type inequalities and spinal dimensional consequences

Now we assume that $G$ is locally finite. For each vertex $x \in V(G)$ let $m(x) < \infty$ denote the number of neighbours of $x$, and for each $f : V(G) \to \mathbb{C}$ define the length of the gradient $|\nabla f(x)|$ by
\[ |\nabla f(x)| := \left( \frac{1}{2} \sum_{y \sim x} \frac{1}{m(x)} |f(y) - f(x)|^2 \right)^{1/2}. \]
For $1 < p \leq \infty$ and $\beta > 0$, we consider the Nash-type inequality
\[ S(p, \beta) : \|f\|_p^{1 + \frac{\beta}{p}} \lesssim \|f\|_1^{1/2} \||\nabla f||_p \quad (f : V(G) \to \mathbb{C} \text{ finitely supported}), \]
which $G$ may or may not satisfy.

Figure 2. The first three steps of the construction of the Vicsek graph $V^2$, with spine.
In the presence of a spinal structure, the inequality $S(p, \beta)$ gives quantitative information connecting the ‘spinal volume growth’ of $G$ with the volume growth of $\Sigma$. This is shown by constructing test functions, defined in terms of the spinal distance, which are constant on the fibres $\pi^{-1}(x)$. The gradients of these test functions are supported on the spine $\Sigma$, while the functions themselves are supported on spinal sets.

**Lemma 4.1.** Let $(G, \Sigma, \pi)$ be a spinal graph. Fix $p \in (1, \infty)$ and suppose that $G$ satisfies $S(p, \beta)$. Then for every $x_0 \in \Sigma$ and $n \in \mathbb{N}$ we have

$$
|D(x_0, n)|^{\frac{1}{p}(1 + \frac{\beta^p}{p})} \lesssim n^{-1}|D(x_0, 2n)|^{\frac{\beta^p}{p}} |B_\Sigma(x_0, 2n)|^{\frac{1}{p}}.
$$

**Proof.** For each $x_0 \in \Sigma$ and $n \in \mathbb{N}$ define $g_n : V(G) \to [0, 1]$ by

$$
g_n(x) := \frac{\max(0, n - |x, x_0|)}{n}.
$$

Note that $g_n(x) = 0$ if and only if $|x, x_0| \geq n$, so that $\text{supp } g_n = D(x_0, n - 1)$. Furthermore note that $g_n$ is constant on each $\pi^{-1}(x)$.

Since $|g_{2n}| \leq 1$ and $\text{supp } g_{2n} \subset D(x_0, 2n)$ we have

$$
\|g_{2n}\|_1 \lesssim |D(x_0, 2n)|.
$$

Next, since $g_{2n}(x) \geq 1/2$ for $x \in D(x_0, n)$, we have

$$
\|g_{2n}\|_p \geq \left( \sum_{x \in D(x_0, n)} 2^{-p} \right)^{1/p} \simeq |D(x_0, n)|^{1/p}.
$$

Finally, note that $|\nabla g_{2n}(x)| = 0$ whenever $x \in G \setminus \Sigma$ (since $g_{2n}$ is constant on each connected component of $G \setminus \Sigma$) or $x \in G \setminus D(x_0, 2n)$ (since $\text{supp } g_{2n} = D(x_0, 2n - 1)$). When $x \in \Sigma \cap D(x_0, 2n)$ and $y \sim x$, we have

$$
g_{2n}(x) - g_{2n}(y) = \begin{cases} 
(2n)^{-1} & \text{if } y \in \Sigma \cap D(x_0, 2n) \\
0 & \text{otherwise.}
\end{cases}
$$

Therefore

$$
|||\nabla g_{2n}|||_p = \left( \sum_{x \in \Sigma \cap D(x_0, 2n)} \left( \frac{1}{2} \sum_{y \sim x, y \in \Sigma \cap D(x_0, 2n)} \frac{1}{m(x)} (2n)^{-2} \right)^{p/2} \right)^{1/p} \lesssim n^{-1}|\Sigma \cap D(x_0, 2n)|^{1/p} = n^{-1}|B_\Sigma(x_0, 2n)|^{1/p}
$$

Therefore, applying $S(p, \beta)$ to $g_{2n}$, we get (5).

The previous lemma can be used to show that the Nash-type inequalities $S(p, \beta)$ restrict the possible dimensions of a spinal graph.

**Lemma 4.2.** Let $(G, \Sigma, \pi)$ be a spinal graph with dimensions $(\delta_\Sigma, \delta_G)$. Fix $p > 1$ and $\beta > 0$, and suppose $G$ satisfies $S(p, \beta)$. Then

$$
\frac{\delta_G - \delta_\Sigma}{p} - \frac{\delta_G}{\beta} \leq -1.
$$
Proof. Fix \( x_0 \in \Sigma \) and a sequence \((n_k)_{k \in \mathbb{N}}\) as in Definition 3.2. From Lemma 4.1 and assumptions (3) and (2), for all \( k \in \mathbb{N} \) we have
\[
|D(x_0, n_k)|^{\frac{1}{p}} \lesssim n_k^{-1} |D(x_0, 2n_k)|^{\frac{1}{p}} |B_\Sigma(x_0, 2n_k)|^{\frac{1}{p}} 
\lesssim n_k^{-1+\frac{\delta_G}{p}} |D(x_0, n_k)|^{\frac{1}{p}}.
\]
Rearranging yields
\[
|D(x_0, n_k)|^{\frac{1}{p} - \frac{\delta_G}{p}} \lesssim n_k^{-1+\frac{\delta_G}{p}},
\]
and then
\[
\delta_G \left( \frac{1}{p} - \frac{1}{\beta} \right) \lesssim -1 + \frac{\delta_G}{p},
\]
follows by (4). Since \( n_k \) is increasing, taking the limit as \( k \to \infty \) tells us that
\[
\delta_G \left( \frac{1}{p} - \frac{1}{\beta} \right) \leq -1 + \frac{\delta_G}{p},
\]
which is equivalent to (8). \( \square \)

Corollary 4.3. Suppose the conditions of Lemma 4.2 are satisfied, with \( \delta_G > \delta_\Sigma \). Then \( \delta_G > \beta \) and
\[
p \geq \beta \frac{\delta_G - \delta_\Sigma}{\delta_G - \beta}.
\]

Proof. Rearranging (8) gives
\[
\delta_G \geq \beta \left( 1 + \frac{\delta_G - \delta_\Sigma}{p} \right).
\]
Since \( \delta_G - \delta_\Sigma > 0 \), we get \( \delta_G > \beta \). We can then rearrange further to get (9). \( \square \)

5. Riesz transform unboundedness for thickened spinal graphs

Definition 5.1. Let \( G \) be a uniformly locally finite graph (i.e. \( \sup_{x \in V(G)} m(x) < \infty \)) and \( n \in \mathbb{N} \). Then an \( n \)-dimensional thickening of \( G \) is a smooth Riemannian manifold \( M \) constructed by replacing each vertex \( x \in V(G) \) by an \( n \)-sphere, each edge \( e \in E(G) \) by an \( n \)-cylinder, and welding the cylinders smoothly to the balls according to the graph structure of \( G \), in such a way that \( M \) has bounded geometry (i.e. \( M \) has positive injectivity radius, and Ricci curvature bounded from below).

More precisely: define
\[
\tilde{M} := \bigsqcup_{x \in V(G)} B_x \sqcup \bigsqcup_{e \in E(G)} C_e,
\]
where \( B_x \) is isometric to a round \( n \)-sphere \( S^n \) with \( m(x) \) disjoint open balls of fixed small radius \( \varepsilon \) removed, and where each \( C_e \) is isometric to a cylinder \( S^{n-1}_\varepsilon \times [0, 1] \), with \( S^{n-1}_\varepsilon = \partial B(0, \varepsilon) \subset \mathbb{R}^n \). A \( C^0 \) Riemannian manifold \( M' \) is constructed as a quotient of \( \tilde{M} \) by gluing a cylinder \( C_e \) to two spheres \( B_x \) and \( B_y \) if and only if \( x \sim y \) in \( G \) (in such a way that every ‘hole’ in \( B_x \) has a cylinder attached to it).

A thickening \( M \) with bounded geometry may then be defined by smoothing the interface between spheres and the cylinders in \( M' \) arbitrarily (but uniformly among all the interfaces).
Remark 5.2. In what follows, we may replace a thickening of $G$ (in the sense above) by any Riemannian manifold $M$ of bounded geometry that is isometric to $G$ at infinity in the sense of Coulhon–Saloff-Coste [12] (following Kanai [13]); our discretisation/thickening procedures only depend on results in [12].

The following proposition can be proven by directly following the proof of [11, Proposition 6.2] (see also the first part of [9, Theorem 5.1]). The proof involves the discretisation results of [12, §6].

**Proposition 5.3.** Let $G$ be a locally uniformly finite graph, and let $M$ be a thickening of $G$ of any dimension. Fix $p \in (1, \infty)$ and suppose that $M$ satisfies (RR$p$). Furthermore, suppose that the heat kernel $h$ of $M$ satisfies

$$h_t(x, x) \lesssim t^{-\alpha/2} \quad \text{for all } t > 1, x \in M.$$  

Then $G$ satisfies $S(p, \alpha)$.

Since $S(p, \alpha)$ restricts the possible dimensions of a spinal graph, we may argue by contraposition to show that dimension and volume information on a spinal graph implies unboundedness of the Riesz transform on $L^p(M)$ for sufficiently large $p > 2$ (in fact, we prove that (RR$p$) does not hold for sufficiently large $p > 2$, which is strictly stronger).

**Theorem 5.4.** Let $(G, \Sigma, \pi)$ be a locally uniformly finite spinal graph with dimensions $(\delta_G, \delta_G)$, with $\delta_G > \delta_\Sigma$. Furthermore, suppose that $B_G(x, r) \gtrsim r^\nu$ for all $r \geq 1$. Let $M$ be a thickening of $G$ of any dimension. Then for all $p > 2\frac{\delta_G - \delta_\Sigma}{2\delta_G - 2\delta_\Sigma + 2} =: p_c(\delta_\Sigma, \delta_G, \nu)$,

$M$ does not satisfy $(R_p)$.

**Proof.** The volume assumption on $G$ implies a corresponding large-scale volume estimate

$$V(x, r) \gtrsim r^\nu \quad (\text{for all } r > 1, x \in M)$$

on $M$ (this may be derived from the results of [12, §6]). Since $M$ has bounded geometry, [4, Theorem 1.1] implies the heat kernel estimate

$$h_t(x, x) \lesssim t^{-\nu/(\nu + 1)} \quad \text{for all } t > 1$$

on $M$. Fix $q > 1$ and suppose that $M$ satisfies $(R_q)$, hence also (RR$q$). Proposition 5.3 then implies that $G$ satisfies $S(q', 2\nu/(\nu + 1))$, and Corollary 4.3 then yields

$$q' \geq \frac{2\nu}{\nu + 1} \left( \frac{\delta_G - \delta_\Sigma}{2\nu - \nu + 1} \right) = p'_c$$

or equivalently that $q \leq p_c$. Therefore $M$ does not satisfy $(R_p)$ for any $p > p_c$. \hfill $\Box$

Taking $\delta_\Sigma = 1$ and $\nu = \delta_G$ gives the following corollary.

**Corollary 5.5.** Let $(G, \Sigma, \pi)$ be a locally uniformly finite spinal graph with dimensions $(1, \delta_G)$, with $\delta_G > 1$, and suppose that $B_G(x, r) \gtrsim r^{\delta_G}$ for all $r \in \mathbb{N}$. Let $M$ be a thickening of $G$ of any dimension. Then the Riesz transform bound $(R_p)$ for $M$ fails for all $p > 2$.

As remarked in Example 3.3, these assumptions are satisfied by the Vicsek graphs $V^n$, reproving [9, Theorem 5.1].
6. NON-FRACTAL SPINAL GRAPHS WITH VOLUME LOWER BOUNDS

Fix $D > 1$. In this section we show how to construct locally uniformly finite spinal
graphs $(G, \Sigma, \pi)$ with dimensions $(1, D)$ and the volume lower bound

\[ |B_G(x, r)| \gtrsim r^D \quad (x \in V(G), r \in \mathbb{N}), \]

but which need not possess any ‘fractal’ structure (in contrast with the Vicsek graph
example). Corollary 5.5 applies to such spinal graphs, thus yielding many manifolds
$M$ for which $(R_p)$ fails for all $p > 2$.

First we need a technical lemma on volumes of intersections of balls in doubling
graphs. We defer the proof to Section 7. Recall that a graph $G$ is doubling if there
exist constants $C_d, \nu > 0$ such that for all $0 < r < R < \infty$ and $x \in V(G),$

\[ |B_G(x, R)| \leq C_d (R/r)^\nu |B_G(x, r)|. \]

Taking $R = 1$ and $r = 1 - \varepsilon$ for $\varepsilon$ arbitrarily small shows that a doubling graph is
locally uniformly finite, with $m(x) \leq C_d$ for all $x \in V(G)$.

Lemma 6.1. Let $G$ be a doubling graph. Then there exists $C > 0$, depending only
on the doubling constants of $G$, such that for all $y \in V(G)$ and $R > 0$, and for all
$x \in B(y, R)$ and $r \leq 2R$, we have

\[ |B_G(x, r) \cap B_G(y, R)| \geq C |B_G(x, r)|. \]

Now we move on to our construction. This is inspired by the ‘plate’ construction
in [4, Theorem 5.1].

Example 6.2. Fix $\delta > D$, and let $(P_n)_{n \in \mathbb{N}}$ be a family of graphs satisfying

\[ |B_{P_n}(x, r)| \sim r^\delta \quad (x \in V(P_n), r \in \mathbb{N}) \]

with implicit constants independent of $n$. For simplicity one can take each $P_n$ to
be equal to a fixed graph $P$; one could even take $\delta \in \mathbb{N}$ and $P = \mathbb{Z}^d$. We allow for
arbitrary choices to emphasise that self-similarity is not necessary. Let $\alpha = (D-1)/\delta$
(so that $\alpha \delta + 1 = D$ and $\alpha < 1$) and for each $n \in \mathbb{N}$ choose an arbitrary vertex
$o_n \in V(P_n)$. Construct a spinal graph $(G, \Sigma, \pi)$ with $\Sigma = \mathbb{N}$ as in Example 2.4 by
taking $G_n$ to be the full subgraph of $P_n$ determined by $B_{P_n}(o_n, n^\alpha)$, and choosing as
distinguished points $z_n = o_n$.

To show that this spinal graph has dimensions $(1, D)$, take the sequence $n_k = k$
and observe that

\[ |D(1, k)| = \sum_{n=1}^{k} |B_{P_n}(o_n, n^\alpha)| \sim \sum_{n=1}^{k} n^{\alpha \delta} \sim k^{\alpha \delta + 1} = k^D \]

(the second sum may be estimated by comparing with integrals of the function$t \mapsto t^{\alpha \delta}$). In particular we have $|D(1, 2k)| \sim (2k)^D \sim |D(1, k)|$, and furthermore it is clear that $|B_{\mathbb{N}}(x, r)| \sim r$. Therefore the spinal graph has dimensions $(1, D)$.

It is more difficult to show the global volume lower bound (10), but luckily the
proof of [4, Theorem 5.1] already does this for a similar problem. Note that it suffices
to assume $r \geq 2$. 

First we show that $|B_G(n, r)| \gtrsim r^D$ for all $n \in \mathbb{N}$. To see this, write

$$|B_G(n, r)| \geq \sum_{k=0}^{\lfloor r/2 \rfloor} |B_{P_n+k}(o_{n+k}, \min(r-k, (n+k)^\alpha))|$$

$$\geq \sum_{k=0}^{\lfloor r/2 \rfloor} |B_{P_n+k}(o_{n+k}, \min(r-k, k^\alpha))|$$

$$\simeq \sum_{k=0}^{\lfloor r/2 \rfloor} k^{\alpha \delta} \simeq \left\lfloor \frac{r}{2} \right\rfloor^{\alpha \delta+1} \simeq r^D,$$

using that $k^\alpha < r-k$ for $k \leq r/2$ in the third line.

Now suppose $x \in V(G)$ with $\pi(x) = n$. After identifying $x$ with the appropriate vertex $z \in B_{P_n}(o_n, n^\alpha)$ (which, recall, is identified with $\pi(x)$), we get an identification of $B_G(x, r) \cap \pi^{-1}(n)$ with $B_{P_n}(z, r) \cap B_{P_n}(o_n, n^\alpha)$. Thus for $r \leq 2n^\alpha$ we have

$$|B_G(x, r)| \geq |B_G(x, r) \cap \pi^{-1}(n)|$$

$$= |B_{P_n}(z, r) \cap B_{P_n}(o_n, n^\alpha)|$$

$$\gtrsim |B_{P_n}(z, r)| \simeq r^\delta > r^D$$

using Lemma 6.1 in the third line. On the other hand, if $r > 2n^\alpha$, then $B_G(x, r)$ contains both $\pi^{-1}(n)$ and $B_G(n+1, r-1-n^\alpha)$, so

$$|B_G(x, r)| \geq \max(|\pi^{-1}(n)|, |B_G(n+1, r-1-n^\alpha)|)$$

$$\gtrsim |\pi^{-1}(n)| + |B_G(n+1, r-1-n^\alpha)|$$

$$\gtrsim n^\alpha D + (r-1-n^\alpha)^D$$

$$\simeq (n^\alpha + r-1-n^\alpha)^D$$

$$\simeq r^D.$$

This completes the proof of (10).

The following corollary is then an immediate consequence of Corollary 5.5.

**Corollary 6.3.** Suppose $M$ is a thickening of a spinal graph $(G, \Sigma, \pi)$ as constructed as in Example 6.2. Then the Riesz transform bound $(R_p)$ for $M$ fails for all $p > 2$.

**Remark 6.4.** It is probably possible to construct spinal graphs with dimensions $(\delta_\Sigma, \delta_G)$ with $1 < \delta_\Sigma < \delta_G$ and with a polynomial volume lower bound of dimension $\delta_G$, thus yielding manifolds $M$ for which $(R_p)$ fails for all $p > p_c > 2$. Since our construction exploits taking $\Sigma = \mathbb{N}$, this is beyond the scope of this article. It may even be possible to show that $(R_p)$ holds on such manifolds for $p \in (2, p_c)$, but this is very much beyond the scope of this article.

**7. Proof of Lemma 6.1**

Here we prove the technical lemma needed in the construction of the previous section. We write $B(x, r) := B_G(x, r)$ and $d(x, y) := d_G(x, y)$. 
Proof. First note that if the result is true for \( r \leq R \), then it holds for \( r \leq 2R \), because in this case for \( r > R \) we have

\[
|B(x, r) \cap B(y, R)| \geq |B(x, R) \cap B(y, R)| \geq C|B(x, R)|
\]

\[
\geq CC_d^{-1}(R/r)^\nu|B(x, r)|
\]

\[
\geq CC_d^{-1}2^{-\nu}|B(x, r)|
\]

using the doubling condition and \( R \geq r/2 \) in the last step. Thus we assume that \( r \leq R \), and split the proof into two cases.

**Case 1:** \( r > 2d(x, y) \). By definition of the combinatorial distance, there exists a vertex \( z \) such that \( d(y, z) + d(z, x) = d(y, x) \) and \( d(y, z) = \lceil d(y, x)/2 \rceil \). Suppose that \( z' \in B(z, r - d(y, z)) \). Then

\[
d(z', x) \leq r - d(y, z) + d(z, x)
\]

\[
= r - 2\lceil d(y, x)/2 \rceil + d(y, x)
\]

\[
\leq r - 2d(y, x) \leq r
\]

and

\[
d(z', y) \leq r - d(y, z) + d(z, y) = r \leq R,
\]

so \( B(z, r - d(y, z)) \subset B(x, r) \cap B(y, R) \). Therefore

\[
|B(x, r) \cap B(y, R)| \geq |B(z, r - d(y, z))|
\]

\[
\geq \left( \frac{r - d(y, z)}{r + d(x, z)} \right)^\nu |B(z, r + d(x, z))|
\]

\[
\geq \left( \frac{r - d(y, z)}{r + d(x, z)} \right)^\nu |B(x, r)|
\]

for some \( \nu > 0 \) determined by the doubling constant of \( G \). To see that the bracketed expression above is uniformly bounded below, estimate its reciprocal from above by

\[
\frac{r + d(x, z)}{r - d(y, z)} = \frac{r - d(y, z) + d(x, y)}{r - d(y, z)}
\]

\[
= 1 + \frac{d(x, y)}{r - d(y, z)}
\]

\[
< 1 + \frac{d(x, y)}{2d(x, y) - d(y, z)}
\]

\[
= 1 + \frac{d(x, y)}{d(x, y) + d(z, x)}
\]

\[
\leq 2.
\]

using \( r > 2d(x, y) \) in the third line and \( d(z, x) = d(x, y) - d(y, z) \) in the fourth line.

**Case 2:** \( r \leq 2d(x, y) \). Note that the estimate for \( r < 2 \) follows from the fact that \( G \) is locally uniformly finite, so it suffices to consider \( r \geq 2 \). As in the previous case, there exists a vertex \( z \) such that \( d(y, z) + d(z, x) = d(y, x) \) and \( d(x, z) = \lfloor r/2 \rfloor \) (here we use that \( r/2 \leq d(x, y) \)). If \( z' \in B(z, \lfloor r/2 \rfloor) \), then

\[
d(z', x) < 2 \left\lfloor \frac{r}{2} \right\rfloor \leq r
\]

and

\[
d(z', y) < \left\lfloor \frac{r}{2} \right\rfloor + d(z, y) = \left\lfloor \frac{r}{2} \right\rfloor + d(x, y) - \left\lfloor \frac{r}{2} \right\rfloor = d(x, y) \leq R,
\]
using that \( x \in B(y, R) \) by assumption, and so \( B(z, [r/2]) \subset B(x, r) \cap B(y, R) \).

Therefore

\[
|B(x, r) \cap B(y, R)| \geq |B(z, [r/2])| \\
\geq |B(z, r/4)| \\
\geq (8^nC_d)^{-1}|B(z, 2r)| \\
\geq (8^nC_d)^{-1}|B(z, r + d(z, x))| \\
\geq (8^nC_d)^{-1}|B(x, r)|
\]

since \( [r/2] \geq r/4 \) holds whenever \( r \geq 2 \), and using the doubling condition. \( \Box \)

ACKNOWLEDGEMENTS

I thank Li Chen and Thierry Coulhon for many interesting discussions on this material. I also thank the anonymous referee for their careful reading and suggestions. Portions of this work were carried out while the author was a postdoc at Université Grenoble–Alpes and at TU Delft (supported by the VIDI subsidy 639.032.427 of the Netherlands Organisation for Scientific Research (NWO)), and also while the author held an Australian Mathematical Society Lift-off Fellowship.

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