Quantization of Thermodynamics, Supersecondary Quantization, and a New Variational Principle

V. P. Maslov

Abstract

In solving the problem of finding a temperature distribution which, at zero temperature, corresponds to superfluidity, i.e., to nonzero energy, the author tried to quantize free energy. This was done on the basis of supersecondary quantization whose special case is the usual secondary quantization for bosons and with the help of which new representations of the Schrödinger equation were obtained. The supersecondary quantization allowed the author to construct a variational method whose zero approximation are the Hartree–Fock and Bogolyubov–BCSCh variational principles. This method works especially well in the case of not a large number of particles. The new quantization and the variational method are of general character and can be used in the quantum field theory.

1 Introduction

1.1 Different pictures of $N$ bodies and their representations

There are two aspects in the many-body problem. The many bodies can be located in our three-dimensional space and move in some way. They can collide or interact. All this takes place in our three-dimensional space. However, we can consider these bodies in a different way. Namely, we can treat these $N$ bodies as one point (here the term “$N$ bodies” means “$N$ material points”) in the three-dimensional space. In this three-dimensional space this single point moves in some way, and we can observe its motion. In some cases one of these two approaches (two aspects) is efficient. In other cases the other approach can be useful. So, from the mathematical viewpoint, it is simpler to deal with a single point. From the physical viewpoint, since we live in the three-dimensional space, we would like to understand how material points behave in our three-dimensional space. This is a more natural “picture.” For example, in the three-dimensional space we can study a particle located in the field of all other particles. In particular, this approach underlies Vlasov’s theory of self-consistent field. In this theory it is assumed that all particles are approximately the same and have the same distribution. We can assume that one particle behaves as if it interacts with another particle, and this other particle is the same particle, i.e., it has the same distribution as the first particle.

*This research was supported by the Russian Foundation for Basic Research under grant No. 99-01-01198.
It turns out that this approach can be realized not only approximately, as it was realized by Vlasov, but exactly by using the secondary quantization, i.e., in other words, in the three-dimensional space we consider the behavior of operators rather than the behavior of functions. Namely, we study some creation and annihilation operators. For the first time, this was realized by Dirac in quantum mechanics. However, it is not necessary to consider just the quantum mechanics. So, Schoenberg studied the classical mechanics (approximately in 1953–1956) and applied this secondary quantization method to classical objects and to classical statistical physics.

Indeed, this is a very general thing. For example, I together with my pupil used this method in the case of $N$ fields. We acted as if it was the third quantization, which, in fact, is a version of the secondary quantization [1]. Here we have a natural generalization. If we consider $N$ particles as a single particle in the three-dimensional space, why cannot we consider, for example, any two particles of these $N$ particles? I can treat this pair as one particle in the six-dimensional space and then study these pairs of six-dimensional particles. Each of them is similar to one particle in the $3N/2$-dimensional space. So I can simultaneously consider a pair of particles and a single particle. In other words, there exist some six-dimensional particles (pairs) and some three-dimensional particles and all this is embedded in the corresponding space. If there are $k$ pairs and $m$ “single” particles, then we have $2k + m = 3N$, where $N$ is the general number of particles.

In fact, in addition to the coordinate of the particle (the location of the point), there can be some other degrees of freedom. For example, in the classical mechanics there can be momenta. In the quantum theory, there can be spins and some other degrees of freedom. In the theory developed here, I introduce one more degree of freedom, the minimal degree which is just a number. In addition to the location of a particle in some part of the space, this particle is equipped with a number. This viewpoint had a good interpretation in the early quantum mechanics. In the Bohr theory we deal with electrons. We can interchange their places, but nothing changes. This is the identity principle. However, besides these electrons, there are orbit numbers. They can also be treated as numbers assigned to electrons.

For example, I have $N$ particles and I consider some pairs of particles and single particles, which I have numbered. Perhaps, it is convenient also to number the pairs. Whether I have numbered then or not, this does not change anything, since this numbering is not related to the fact that I fix these particles and they cannot interchange their places. Thus all this is the same picture. These are simply different representations of the same phenomenon. The secondary quantization allows us to write equations in the three-dimensional space but with operators, i.e., an operator equation. This operator equation is equivalent to one equation in the $3n$-dimensional space (which is simply an equation) not for operators but for functions. The same problem is obtained in terms of representations, which I propose and call supersecondary quantized.

### 1.2 Series and serials

Probabilities of transition from level to level play an essential role. In particular, as is known, the energy quantum issued by an atom correspond to transitions from level to level. If the probability of transition is zero or very small, i.e., if this transition is practically forbidden, then the corresponding levels are treated as if they belong to different series. In fact, the...
partition into series means a partition into systems that are isolated from each other. If a given series of energy levels contains the minimal level, which is the principal state of the series, then this is a metastable state. If the system is in this state and transitions to the lower levels of other series are almost forbidden, then the system stays in this state sufficiently long.

The existence of a superfluid liquid or a superconducting current corresponds to such metastable series. The current can flow without friction during 100000 years, which means that the series is almost stable. In a finite large volume, where we consider a system with a large number of particles, the velocities and currents are discrete, and each value determines its own metastable series. A set of series that differ in velocities we call a serial.

To each picture discussed above, there corresponds its own serial in the asymptotic approximation. Although the physicists do not say this directly, in reality, they try to find a serial such that the lower level of its lower series (i.e., the level at the zero velocity) would coincide with the principal state of the entire original system.

In the above-mentioned paper by N. N. Bogolyubov (1947), such a serial was found. It corresponds to the following picture: $N$ particles are in the three-dimensional space and all their numbers coincide.

However, it turns out and this will be proved in the present work that the serial corresponding to the picture in which pairs of particles are not numbered but the particles are numbered has a level which is lower than that of the Bogolyubov serial. In this sense, apparently, the serial corresponds to the true phenomenon of superfluidity and, to a larger extent, corresponds to the experiment. Although, if the nature “set” a system to another serial, the system will stay in this state for a long time.

However, it is possible that resonances may occur between different series. In this case the series is destroyed. An example is provided by the one-dimensional Schrödinger equation with two minima. If the maximum between the minima is very large and if we have a semiclassical approximation, then to each hollow there corresponds its own series and if there are no resonances, then transitions from one hollow to the other are almost forbidden (the “tunnel” effects). If some eigenvalues of two series coincide, then the resonance is possible and in this case the position of a particle in one of the hollows at this eigenvalue is destroyed. For example, in the potential symmetric with respect to the maximum, all eigenvalues “almost” coincide, all are in resonance, and the system does not split into series.

### 1.3 Quantization of free energy

We assume that the eigenfunctions of the Schrödinger equation (the Hamilton operator) remain the same. From each eigenvalue we subtract the temperature multiplied by entropy corresponding to this eigenvalue. In the simplest case of a discrete spectrum, as is known [2], this is the logarithm of the multiplicity of the eigenvalue.

As is known, the Hamilton operator can be represented as the sum of its eigenvalues multiplied by the projectors on the corresponding subspaces of eigenfunctions. The free energy operator, as was already pointed out, is the sum of the same eigenvalues from which the temperature multiplied by entropy is subtracted and then they are multiplied by the same projectors. If we consider the principal state of this quantum free energy, then we obtain all the results of the former thermodynamics. But we can consider not only the lower
level of this serial but also the series corresponding to the velocities (currents). Then we obtain answers to the question of how the metastable states vary with temperature.

We shall show that there exist serials that contain superfluid and superconducting series at any temperature. However, these series necessarily do not contain the principal state, i.e., there exist serials whose lower level is lower. By idea, the nature must choose the latter. On the other hand, by the argumentation of the great Dirac, “it were astonishing if the nature did not use this possibility.” I add “somewhere”, in nuclei, in stars, etc. in one word, somewhere.

Is it possible to create such serials “by hand”? We have already created by hand metastable series, higher than the principal state, by letting currents flow. Is it possible, by using resonance, also to “pump” high-temperature serials?

1.4 Necessary conditions for metastable states

If a large parameter is given as some concrete number, it is sometimes not clear whether this parameter is “large” or “not very large.” Say, if the number of particles is 100, this number must be a sufficiently large parameter, but the logarithm of this large parameter is already not too large. In this case, first of all, we would like to know how to determine a metastable state. I introduce, at least, necessary conditions for a metastable state.

To this end, I first consider a very simple example, the one-dimensional Schrödinger equation whose potential well has two hollows, i.e., there are two minima: one minimum is global, i.e., it corresponds to the principal state, and the other minimum is local. In the case of the classical asymptotics, it determines a metastable state. What does it mean?

If the system is at this lowest level corresponding to the local minimum or to the smaller hollow, i.e., in other words, if the eigenfunction is concentrated near the local minimum and rapidly decreases everywhere else, then the following question arises: what happens if the system is perturbed? In other words, what is the probability of the transition from this state to the principal state, i.e., to the state that corresponds to the global minimum and whose eigenfunction is concentrated near this minimum and decreases sufficiently rapidly everywhere else.

If we have a semiclassical approximation, then the tails of these eigenfunctions decrease very rapidly. Hence if we consider the matrix element obtained as a result of perturbation by some operator of multiplication by an external potential, then we see that this matrix element is very small, since the product of functions is very small. In other words, I take the following matrix element: $\psi_0$ is the eigenfunction corresponding to the principal state and $\psi_1$ is the first eigenfunction corresponding to the other hollow. $V'$ is the potential by which we perturb the system. If the matrix element $(\psi_0, V'\psi_1^*)$ is small, then this means that the transition probability is small. Hence this state is metastable, i.e., it is very difficult to “fall” to the lower level, while transitions to the higher levels whose supports are also near the minimum of this smaller hollow are quite possible.

However, since we say that this metastable state is, to some extent, a model of superfluidity and superconductivity, it is very important for us that the energy did not become less. If the energy increases, then the larger the better. (If the velocity increases due to this perturbation, this is also very good.) So, we perturbed the problem as if by friction, as Landau would say, and nevertheless, the energy did not decrease, since the transition to
lower levels for such an asymptotic problem is practically forbidden.

However, if we take an operator not from the class I described but from a more complicated class, e.g., from the class where the translation operator is also involved, then the matrix element is not at all small. Hence the first problem is to specify the operators by which we perturb our problem.

The operators by which we perturb the problem are operators that preserve the support of a function, i.e., in other words, if a function \( \psi \) is sufficiently smooth and equal to zero outside some interval (thus is interval is called just the support of this function) and we apply the multiplication operator or the differentiation operator of any order, then these operators do not take the function outside its support. We usually perturb our problem by such operators. What does it mean?

This means that, prior saying that we have a metastable state, we must define the class of operators with respect to which this state is metastable and for which the transition operators are small. Natural operators of this type are just the potential forces. Further, the question arises: perhaps, this is not a semiclassical approximation, the parameter \( \hbar \) is not small, perhaps, the barrier is not very high, what happens then? Can a metastable state exist in this situation and how long? In this section we shall try to answer this question.

The first question: is such a state possible at all or not? The second question: how small are the matrix elements of the transition? Thus the first question is reduced to the following one: does the minimum exist, or not, and of what is this minimum?

We spoke about the minimum of a potential well. However, about what minimum must we speak in the quantum problem itself, without a small parameter?

In the quantum problem without a small parameter we can reason in the following way: we consider a set of perturbing operators with support on the interval \([a, b]\), in other words, we take the operators of multiplication by smooth finite functions that differ from zero on the interval \([a, b]\). Then we act by the set of such operators on the entire space \(L_2\). We obtain the set of values of the result of this action. We study whether the original Hamiltonian attains its local minimum on the functions from this set. Namely, this minimum is understood as the true local minimum with positive second-order derivatives, i.e., the minimum does not lie on the “edge” of the interval but lies somewhere in the middle. If the interval \([a, b]\) includes the minimum point of our potential, i.e., the smaller hollow, then the local minimum necessarily exists for a sufficiently small \(\hbar\). Such a minimum can also exist if \(\hbar\) is not very small, but just the existence of such a minimum answers our question of whether there exist a metastable state.

If there is no minimum, we cannot speak about a metastable state. If there is no second hollow, then, apparently, the Hamiltonian does not have a minimum, at least for sufficiently small \(\hbar\).

At all events, we, first of all, must answer the question of whether the quantum problem with classical Hamiltonian, which is not related to the classics, has a local minimum. If such a local minimum exists, then there exists an eigenvalue, which is the nearest to this minimum, and the corresponding eigenfunction determines some metastable state of the original operator. In this case a small perturbation of the length of the interval does not lead to the transition to another value.

The value of the dimensionless variable \(V_0 a^2 m / \hbar^2\) at which the local (not principal) minimum disappears will be called the critical value. Here \(m\) is the mass, \(\hbar\) is the Planck constant,
and the potential has the form \( V_0 v(x/a) \), where \( V_0 \) is a constant, \( a \) is the characteristic length, and \( v(y) \) is the potential well with two hollows mentioned above.

The question of how long this metastable state lives is still open, hence I only say that this condition is necessary for the existence of a metastable state.

1.5 Necessary conditions for the existence of superfluidity and superconductivity

Obviously, a metastable state does not depend on the representation. However, we relate the problem of finding a minimum to some asymptotic problem, where the representation in which we seek the minimum (namely, the \( k \)-representation) is chosen in some natural way. Similarly, we act in the many-body problem, in the situation in which the number of particles is not very large. We first determine the representation for which these metastable states are natural in our asymptotic problem.

Hence in the many-body problem we choose the representation in which, as \( N \to \infty \) we already have series with metastable lower levels. The most convenient way is to deal with the representation of occupation numbers, which is introduced in the supersecondary quantization, and hence, in the variational method, which will be considered later.

These occupation numbers \( n_j \) attain integer nonnegative values for \( j = 0, 1, 2, \ldots \).

Let us consider the set of functions of occupation numbers \( A(n_0, n_1, n_2, \ldots) \) such that

\[
\sum_{\{n\}} |A(n_0, n_1, n_2, \ldots)|^2 < \infty
\]

and \( A(n_0, n_1, n_2, \ldots) = 0 \) if \( N \neq n_0 + n_1 + n_2 + \ldots \). The subset of functions of occupation numbers such that \( A(n_0, n_1, n_2, \ldots) = 0 \) for \( \alpha N < n_0 + n_1 + n_2 + \ldots \) will be the characteristic class of functions corresponding to the number \( 0 < \alpha < 1 \).

To determine a metastable state in the problem of \( N \) bodies, we seek the minimum of the Hamiltonian in the representation of occupation numbers on the set of functions from the characteristic class. By analogy with the preceding section, we can find the critical value \( N_{cr} \) for which the metastable state disappears. Superfluidity and superconductivity in the system are possible for \( N \) larger than the critical values. Thus, \( N_{cr} \) is the lower bound for \( N \) for which superfluidity and superconductivity are possible at least for a short time.

1.6 Variational methods and variational principles

There exist many variational methods, for example, the variational Riesz method, the variational Galerkin method, the variational Tamm–Dankov method. The method of finding the solution is, so to say, successive, i.e., there \emph{a priori} exists a sequence that converges to the solution. In the Tamm–Dankov method this convergence is not proved, but each successive approximation must be, in principle, better than the preceding approximation.

The situation with variational methods is somewhat different. The variational methods are formulated so that they directly lead to some equation and are postulates that follow from the equations. There also exist variational principles that are derived from the original variational principle which is considered on a narrower function class. They lead to some
new equations. The relation between these new equations and the original “true” equations is determined experimentally.

Suppose that some variational method, say, the Hartree–Fock method is used. The results obtained are compared with experimental results, and the coincidence is good. In the variational principles such as the Hartree–Fock principle, the variational Bogolyubov principle, the variational Schriffer principle, we try to guess the form of the solution by substituting this solution and obtaining some new equation. So, these principles lead to some new equations and, instead of the original equations, we solve some other equations obtained from the variational principle. We could determine the relation between the variational principle and the equations if we had a sequence (as in the Riesz method, in the Galerkin method, and in some other methods) of test functions with the help of which we could step-by-step approach the solution, but not a single test function with the help of which we need to verify the variational principle.

Just in this case, the supersecondary quantization, which I propose, naturally leads to a variational method, i.e., to the method of successive approximations. To some extent this ideology is close to that of the Tamm–Dankov method. The variational Bogolyubov and Hartree–Fock principles constitute only the first approximation. For some problems this yields exact asymptotic methods, as was shown by Bogolyubov for the Cooper–Bardeen model. He showed that this is an exact asymptotics as $N \to \infty$. It is of interest that the variational method proposed here leads to better results if the number of particles is not very large, just where the asymptotic theory cannot be applied.

If the number of particles is $10^4$, as in traps, then this case cannot be an asymptotics as $N \to \infty$, since the logarithm, which tends to infinity as $N \to \infty$, is here equal to 4. If we have $10^{23}$, just as in the weakly nonideal Bose gas, then the logarithm is already a sufficiently large number, and this case can be an asymptotics for large $N$. In this case it is not so convenient to use the variational method.

## 2 Supersecondary quantization in the boson case

Following [3], we introduce the averaging of operators in the more general case of a large number of two types of clusters. Let $k \geq 1$ and $N_1, \ldots, N_k \geq 0$ be integers. Consider the Hilbert space of functions of the form

$$
\Psi(x_1^1, j_1^1; \ldots; x_{N_1}^1, j_{N_1}^1; x_1^2, j_1^2; \ldots; x_{2N_2-1}^2, x_{2N_2}^2, j_{N_2}^2; \ldots \ x_k^k, j_1^k; \ldots; x_{kN_k-k+1}^k, \ldots; x_{kN_k}^k, j_{N_k}^k),
$$

where $x_{l}^l \in T^3$, $T^3$ is the three-dimensional torus with side length $L$, $j_q^l = 1, \ldots, \infty$ for all $l = 1, \ldots, k$ and $p_l = 1, \ldots, lN_l$, $q_l = 1, \ldots, N_l$. Moreover, the functions (4) are symmetric with respect to the permutations of any pairs $(x_{pt-l+1}^l, \ldots, x_{pt}^l, j_p^l)$ and $(x_{ql-l+1}^l, \ldots, x_{ql}^l, j_q^l)$ of sets of variables, where $l = 1, \ldots, k$ and $p, q = 1, \ldots, N_l$.

The inner product has the form

$$
(\Psi, \Phi) \overset{\text{def}}{=} \sum_{j_1^1=1}^{\infty} \ldots \sum_{j_{N_1}^1=1}^{\infty} \ldots \sum_{j_1^k=1}^{\infty} \ldots \sum_{j_{N_k}^k=1}^{\infty} \int \ldots \int dx_1^1 \ldots dx_{N_1}^1 \ldots dx_1^k \ldots dx_{kN_k}^k \times
$$

7
respect to the permutations of the variables \( x \). Assign the operators \( \hat{A} \) and acting as the identity operators with respect to the discrete variables.

This space is the infinite direct sum in the subspaces of \( L \) and is a special case of the Fock spaces [4]. Let \( \delta \) and let \( \hat{\epsilon} \) be the operator of embedding of \( F \nabla \times \cdots \times F \), be the projection on the corresponding component of the direct sum, and let \( \hat{i} \) be the operator of embedding of \( F \nabla \times \cdots \times F \) in \( F \):

\[
\hat{i}_{N_1, \ldots, N_k} : F \nabla \times \cdots \times F \rightarrow F, \quad \hat{i}_{N_1, \ldots, N_k} \varphi = \{ \delta_{N_1 M_1} \cdot \cdots \cdot \delta_{N_k M_k} \varphi \},
\]

where \( \delta_{N M} \) is the Kronecker delta.

We introduce the notion of an averaged operator. Let \( \{ \hat{A}_N \} \) be a sequence of operators in the subspaces of \( L^2(T^3 \times \cdots \times T^3) \) formed by elements \( \psi(x_1, \ldots, x_N) \) symmetric with \( N \) times respect to the permutations of the variables \( x_j \) and \( x_k, x_j \in T^3 \). To these operators, we assign the operators \( \hat{A}_{N_1, \ldots, N_k} : N_1 + 2N_2 + \cdots + kN_k = N \) defined on the subspaces \( F_{N_1, \ldots, N_k} \) and acting as the identity operators with respect to the discrete variables \( j_p^l \).

**Definition.** The averaged operator \( \overline{A} \) is the operator

\[
\overline{A} = \sum_{N_1=0}^{\infty} \cdots \sum_{N_k=0}^{\infty} (N_1 + \cdots + kN_k) \hat{i}_{N_1, \ldots, N_k} \hat{A}_{N_1, \ldots, N_k} \hat{i}_{N_1, \cdots, N_k} \hat{P}_{N_1, \ldots, N_k} \hat{P}_{N_1, \ldots, N_k}
\]
in the Fock space $\mathcal{F}$.

**Remark.** The operator $\tilde{\gamma}_{N_1,...,N_k} \tilde{A}_{N_1+...,kN_k} \tilde{P}_{N_1,...,N_k} \tilde{P}_{N_1,...,N_k}$ represented as

$$
\frac{1}{N_1!...N_k!} \sum_{j_1^1=1}^{\infty} \ldots \sum_{j_{N_k}^k=1}^{\infty} \int \ldots \int dx_1 \ldots dx_{N_1} \ldots dx_k \ldots dx_{kN_k} \\
\times b_1^+(x_{1}^{j_1^1}) \ldots b_1^+(x_{N_1}^{j_{N_1}^1}) \ldots b_k^+(x_k^{j_k^1}) \ldots b_k^+(x_{kN_k}^{j_{kN_k}^k}) \tilde{\gamma}_{N_1+...,kN_k} \tilde{A}_{N_1+...,kN_k} \tilde{P}_{N_1,...,N_k} \tilde{P}_{N_1,...,N_k} \\
\times \text{Symm}_{x_1^1...x_{N_1}^1...x_k^1...x_{kN_k}^k} \left\{ b_1^+(x_{1}^{j_1^1}) \ldots b_1^+(x_{N_1}^{j_{N_1}^1}) \ldots b_k^+(x_k^{j_k^1}) \ldots b_k^+(x_{kN_k}^{j_{kN_k}^k}) \right\} \\
\times \exp \left( -\sum_{j=1}^{\infty} \int dz b_1^+(z,j) b_1^-(z,j) - \ldots - \right.

$$

$$
-\sum_{j=1}^{\infty} \int dz_1 \ldots dz_k b_1^+(z_1,j) b_1^-(z_1,j) - \ldots - \\
$$

where $\text{Symm}_{x_1...x_N}$ is the operator of symmetrization with respect to the variables $x_1, \ldots, x_N$ and numbers over operators indicate the order of action of these operators [5]. The expression (2) was obtained in [6] for the special case $k = 2$ and can be proved in a similar way for arbitrary $k$.

According to this remark, the following functional will be called the *symbol* of the averaged operator $\overline{A}$:

$$
A(b_1^+(x_{1}^{j_1^1})b_1(x_{1}^{j_1^1}), \ldots, b_k^+(x_k^{j_k^1})b_k(x_k^{j_k^1})) = \\
= \sum_{N_1=0}^{\infty} \ldots \sum_{N_k=0}^{\infty} \frac{(N_1 + 2N_2 + \ldots + kN_k)!}{N_1!N_2!\ldots N_k!} \times \\
\times \sum_{j_1^1=1}^{\infty} \ldots \sum_{j_{N_k}^k=1}^{\infty} \int \ldots \int dx_1 \ldots dx_{N_1} \ldots dx_k \ldots dx_{kN_k} \\
\times b_1^+(x_{1}^{j_1^1}) \ldots b_k^+(x_{kN_k}^{j_{kN_k}^k}) \times \\
\times \overline{A}_{N_1+...,kN_k} \text{Symm}_{x_1^1...x_{N_1}^1...x_k^1...x_{kN_k}^k} \left\{ b_1(x_{1}^{j_1^1}) \ldots b_1(x_{N_1}^{j_{N_1}^1}) \ldots b_k(x_k^{j_k^1}) \ldots b_k(x_{kN_k}^{j_{kN_k}^k}) \right\} \\
\times \exp \left( -\sum_{j=1}^{\infty} \int dz b_1^+(z,j) b_1(z,j) - \ldots - \right.

$$

$$
-\sum_{j=1}^{\infty} \int dz_1 \ldots dz_k b_1^+(z_1,j) b_1(z_1,j) - \ldots - \\
$$

(3)
where \( b_l(\cdot, \ldots, j_l) \in L^2(T^3) \), \( l = 1, \ldots, k \), for all \( j_l = 0, 1, \ldots \) and the inequality
\[
\sum_{j=0}^{\infty} \int \ldots \int dx_1 \ldots dx_l b^*_l(x_1, \ldots, x_l, j) b_l(x_1, \ldots, x_l, j) < \infty
\]
is valid. To the sequence of operators \( \{ \hat{A}_N \} \), there corresponds a secondary quantized operator \( \hat{A} \) in the Fock space \( \mathcal{H}_B \).

The following assertion holds.

**Lemma.** The symbol of the averaged operator \( \overline{\hat{A}} \) can be expressed as follows:
\[
\begin{align*}
A(b^*_1(x^1_1, j^1), b_1(x^1_1, j^1), \ldots, b^*_k(x^k_1, \ldots, x^k_k, j^k), b_k(x^k_1, \ldots, x^k_k, j^k)) &= \\
&= \text{Sp} (\hat{A}\hat{\rho}) \exp \left( -\sum_{j=1}^{\infty} \int dz b^*_1(z, j) b_1(z, j) - \right. \\
&\left. - \ldots - \sum_{j=1}^{\infty} \int \ldots \int dz_1 \ldots dz_k b^*_k(z_1, \ldots, z_k, j) b_k(z_1, \ldots, z_k, j) \right),
\end{align*}
\]
where \( \hat{\rho} \) is an operator in \( \mathcal{H}_B \) and has the form
\[
\begin{align*}
\hat{\rho} &= \sum_{N_1=0}^{\infty} \ldots \sum_{N_k=0}^{\infty} \frac{1}{N_1! \ldots N_k!} \prod_{l=1}^{k} \left( \sum_{j_l=0}^{\infty} \int \ldots \int dx^l_1 dy^l_1 \ldots dx^l_l dy^l_l \times \\
&\times b_l(x^l_1, \ldots, x^l_l, j^l) \frac{2}{\psi^+ (x^l_1) \ldots \psi^+ (x^l_l)} \times \\
&\times b^*_l(y^l_1, \ldots, y^l_l, j^l) \frac{1}{\psi^- (y^l_1) \ldots \psi^- (y^l_l)} \right)^{N_l} \exp \left( -\int dz \frac{2}{\psi^+ (z) \psi^- (z)} \right).
\end{align*}
\]

This lemma follows from a lemma proved in [3].

As was shown in the Introduction, it is impossible to find how metastable states depend on the temperature, if the quantization of free energy is not introduced.

In the simplest case of microcanonical distribution, we have the following expression for free energy \( F \):
\[
F = \lambda - \theta \ln n_\lambda \tag{5}
\]
where \( \theta \) is the temperature, \( \lambda \) is the energy level, \( n_\lambda \) is its multiplicity, and \( S = \ln n_\lambda \) is entropy by definition. Let \( dE_\lambda \) be the spectral family of a self-adjoint energy operator \( \hat{H} \) with discrete spectrum. Then, as is known, we have
\[
\hat{H} = \int \lambda dE_\lambda.
\]
It is natural to define the quantum free energy as
\[
\hat{F} = \int (\lambda - \theta \ln n_\lambda) dE_\lambda. \tag{6}
\]
As a rule, in statistical physics we deal with the canonical distribution. The passage from (5) to the canonical distribution is not trivial and here can be carried out only in the simplest
cases. Similarly, the method for obtaining the values of quantum entropy, which we propose later, can be derived from (6) not in the general case in which we shall use it. This separate problem lies beyond the framework of the present work.

Next, let \( \hat{H} \) and \( \hat{E} \) be the Hamiltonian and the identity operator, respectively, averaged by this method. By the lemma, the symbols of these operators, up to a factor, are equal to \( \text{Sp} \hat{\rho} \) and \( \text{Sp}(\hat{H}\hat{\rho}) \), respectively. If \( \hat{\rho} \) is treated as a statistical operator (a density matrix), then \( \text{Sp}\hat{\rho} \) is the norm of this operator and \( \text{Sp}(\hat{H}\hat{\rho}) \) is the mean value of the Hamiltonian multiplied by \( \text{Sp}\hat{\rho} \). Furthermore, entropy corresponding to \( \hat{\rho} \) is given by

\[
S = \text{Sp}\left( \hat{\rho}\ln\left( \frac{\hat{\rho}}{\text{Sp}\hat{\rho}} \right) \right) (\text{Sp}\hat{\rho})^{-1}.
\]

By analogy with statistical physics, we refer to the functional

\[
S(b_1(x_1^1,j_1^1), b_1(x_1^1,j_1^1), \ldots, b_k^*(x_1^k,x_k^k), b_k(x_1^k,x_k^k)) = \\
\text{Sp}\left( \hat{\rho}\ln\left( \frac{\hat{\rho}}{\text{Sp}\hat{\rho}} \right) \right) \exp \left( -\sum_{j=1}^{\infty} \int dz b_1^*(z,j)b_1(z,j) - \ldots - \sum_{j=1}^{\infty} \int \ldots \int dz_1 \ldots dz_k b_k^*(z_1,\ldots,z_k,j)b_k(z_1,\ldots,z_k,j) \right)
\]

(7)

as the symbol of averaged entropy. To this symbol, we assign the following operator in \( \mathcal{F} \):

\[
\overline{S} = \text{Reg}S(b_1^+(x_1^1,j_1^1), b_1^-(x_1^1,j_1^1), \ldots, b_k^+(x_1^k,x_k^k), b_k^-(x_1^k,x_k^k)),
\]

(8)

where Reg stands for the regularization if necessary.

**Definition.** Numbers \( \lambda \) such that

\[
(\hat{H} + \theta\overline{S})\Phi = \lambda \hat{E}\Phi, \quad \Phi \in \mathcal{F}, \quad \Phi \not\equiv 0,
\]

(9)

are called quantum eigenvalues of free energy for a system of bosons with Hamiltonian \( \hat{H} \) at temperature \( \theta \) for \( k \) cluster types.

### 3 Supersecondary quantization for fermions

Now we introduce the averaging of operators and the quantum free energy for fermions. We assume that the fermions may have a spin variable \( s \) ranging in a discrete finite set \( \Sigma \). Consider the Hilbert space \( \mathcal{F}_{N_1,\ldots,N_k} \) of functions (1), where \( x_{pl}^l \in T^3 \times \Sigma \). Throughout this section, we use the notation

\[
\int dx = \sum_{s \in \Sigma} \int d\xi,
\]

(10)

where \( x = (\xi, s), \xi \in T^3, s \in \Sigma \). Let us define the fermion space \( \mathcal{F}_{N_1,\ldots,N_k}^F \).

**Definition.** The space \( \mathcal{F}_{N_1,\ldots,N_k}^F \) is the subspace of \( \mathcal{F}_{N_1,\ldots,N_k} \) formed by elements \( \mathcal{F}_{N_1,\ldots,N_k} \) that are antisymmetric with respect to the permutations of arbitrary \( x_{pl}^l \) and \( x_{qm}^m \), \( l, m =
1, \ldots, k, p = 1, \ldots, lN_l, q = 1, \ldots, mN_m. The projection on this subspace will be denoted by \( \tilde{P}_F^{E_{N_1, \ldots, N_k}} \):
\[
\tilde{P}_F^{E_{N_1, \ldots, N_k}} : \mathcal{F}_{N_1, \ldots, N_k} \to \mathcal{F}_{E_{N_1, \ldots, N_k}}.
\]

Let us introduce the notion of an averaged operator in the fermion case. Let \( \{ \tilde{A}_N \} \) be a sequence of operators in the subspaces \( L_2(T^3 \times \Sigma \times \ldots \times T^3 \times \Sigma) \) formed by elements \( \psi(x_1, \ldots, x_N) \) antisymmetric with respect to the permutations of the variables \( x_j \) and \( x_k \), \( x_j \in T^3 \times \Sigma \). To these operators, we assign the operators \( \tilde{A}_{N_1, \ldots, N_k}, N_1 + 2N_2 + \ldots + kN_k = N \), defined on the subspaces \( \mathcal{F}_{E_{N_1, \ldots, N_k}} \) and acting as the identity operators with respect to the discrete variables \( j^p \).

**Definition.** The averaged operator \( \overline{A} \) is the operator
\[
\overline{A} = \sum_{N_1=0}^{\infty} \ldots \sum_{N_k=0}^{\infty} (N_1 + \ldots + kN_k) \tilde{A}_{N_1, \ldots, N_k} \tilde{A}_{N_1, \ldots, +kN_k} \tilde{P}_F^{E_{N_1, \ldots, N_k}} \tilde{P}_N \ldots N_k
\]
in the Fock space \( \mathcal{F} \).

**Remark.** The operator \( \tilde{A}_{N_1, \ldots, N_k} \tilde{A}_{N_1, \ldots, +kN_k} \tilde{P}_F^{E_{N_1, \ldots, N_k}} \tilde{P}_N \ldots N_k \) can be written as
\[
\frac{1}{N_1! \ldots N_k!} \sum_{j^1_1=1}^{2} \ldots \sum_{j^1_{N_1}=1}^{2} \ldots \sum_{j^k_1=1}^{2} \ldots \sum_{j^k_{N_k}=1}^{2} \int \ldots \int dx_1^{k_1} \ldots dx_{N_1}^{k_1} \ldots dx_{k_{N_k}}^{k_k}
\]
\[
\times \left\{ b_1^{-1} (x_1, j^1_1) \ldots b_1^{-1} (x_{N_1}, j^1_{N_1}) \ldots b_k^{-1} (x_1, \ldots, x_{k_{N_k}})ight\}
\]
\[
\times \text{Asymm}_{x_1^{k_1} \ldots x_{N_1}^{k_1} \ldots x_{k_{N_k}}^{k_k}} \left\{ b_1^+ (x_1, j^1_1) \ldots b_1^+ (x_{N_1}, j^1_{N_1}) \ldots b_k^+ (x_1, \ldots, x_{k_{N_k}})ight\}
\]
\[
\times \exp \left( -\sum_{j=1}^{\infty} \int dz \frac{2}{b^+_1} (z, j) \frac{1}{b^-_1} (z, j) - \ldots - \sum_{j=1}^{\infty} \int \ldots \int dz_1 \ldots dz_k \frac{2}{b^+_k} (z_1, \ldots, z_k) \frac{1}{b^-_k} (z_1, \ldots, z_k, j) \right),
\]
where Asymm\(_{x_1 \ldots x_N}\) is the operator of antisymmetrization with respect to the variables \( x_1, \ldots, x_N \). This expression is the fermion counterpart of (2).

The symbol of the averaged operator \( \overline{A} \) in the fermion case is defined as the functional
\[
A(b_1^*(x_1, j^1_1), b_1(x_1, j^1_1), \ldots, b_k^*(x_1, \ldots, x_{k_{N_k}}), b_k(x_1, \ldots, x_{k_{N_k}})) =
\]
Lemma. The symbol of the averaged operator \( \tilde{A} \) is given by the formula

\[
A(b_1^*(x_1, j^i), b_1(x_1, j^i), \ldots, b_k^*(x_1^k, j^k), b_k(x_1^k, \ldots, x_k^k)) = \text{Sp} (\tilde{A}\rho) \exp \left( -\sum_{j=1}^{\infty} \int dz b_1^*(z, j)b_1(z, j) - \ldots - \right)
\]

\[
-\sum_{j=1}^{\infty} \int \ldots \int dz_1 \ldots dz_k b_k^*(z_1, \ldots, z_k, j)b_k(z_1, \ldots, z_k, j) \right), \tag{13}
\]

where \( b_l(x_1, \ldots, x^{j^l}) \in L^2(\mathbf{T}^N), \ l = 1, \ldots, k, \ j^l = 0, 1, \ldots, \) and the inequality

\[
\sum_{j=0}^{\infty} \int \ldots \int dx_1 \ldots dx_i b_i^*(x_1, \ldots, x_i, j)b_l(x_1, \ldots, x_i, j) < \infty
\]

is valid. To the sequence \( \{\tilde{A}_N\} \) of operators, there corresponds a secondary quantized operator \( \tilde{A} \) in the fermion Fock space \( \mathcal{H}_F \), which consists of sequences of antisymmetric functions and is an analog of the space \( \mathcal{H}_B \) (see [4] for details).

The following assertion holds for the fermion case.

\[
\text{Lemma. The symbol of the averaged operator } \tilde{A} \text{ is given by the formula}
\]

\[
A(b_1^*(x_1^k, j^k), b_1(x_1^k, j^k), \ldots, b_k^*(x_1^k, \ldots, x_k^k), b_k(x_1^k, \ldots, x_k^k)) = \text{Sp} (\tilde{A}\rho) \exp \left( -\sum_{j=1}^{\infty} \int dz b_1^*(z, j)b_1(z, j) - \ldots - \right)
\]

\[
-\sum_{j=1}^{\infty} \int \ldots \int dz_1 \ldots dz_k b_k^*(z_1, \ldots, z_k, j)b_k(z_1, \ldots, z_k, j) \right), \tag{14}
\]

where \( \rho \) is the operator in \( \mathcal{H}_F \) given by

\[
\tilde{\rho} = \sum_{N_1=0}^{\infty} \sum_{N_k=0}^{\infty} \frac{1}{N_1! \ldots N_k!} \sum_{j_1=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_k=0}^{\infty} \sum_{j_k=0}^{\infty} \times
\]

\[
\times \int \ldots \int dx_1^k dy_1^k \ldots dx_{N_1}^k dy_{N_1}^k \ldots dx_{N_k}^k dy_{N_k}^k \tilde{\psi}^+(x_1^k) \ldots \tilde{\psi}^+(x_{N_1}^k) \times
\]

\[
\times \tilde{\psi}^+(x_{N_1}^k) \ldots \tilde{\psi}^+(x_{N_k}^k) \tilde{P}_0 \times
\]

\[
\times b_1(x_1^k, j_1^k) \ldots b_1(x_{N_1}^k, j_{N_1}^k) b_k(x_1^k, \ldots, x_k^k, j_1^k) \ldots b_k(x_{N_k}^k, \ldots, x_k^k, j_{N_k}^k) \times
\]

\[
\times b_1^*(y_1^k, j_1^k) \ldots b_1^*(y_{N_1}^k, j_{N_1}^k) b_k^*(y_1^k, \ldots, y_k^k, j_1^k) \ldots b_k^*(y_{N_k}^k, \ldots, y_k^k, j_{N_k}^k) \times
\]

\[
\times \tilde{\psi}^-(y_{N_1}^k) \ldots \tilde{\psi}^-(y_{N_k}^k) \tilde{\psi}^-(y_1^k) \ldots \tilde{\psi}^-(x_1^k); \tag{15}
\]
here \( \hat{P}_0 \) is the projection on the vacuum vector in \( \mathcal{H}_F \).

In the fermion case, we define the symbol of averaged entropy as the functional

\[
S(b_1^*(x_1^1, j^1), b_1(x_1^1, j^1), \ldots, b_k^*(x_k^k, j^k), b_k(x_k^k, j^k)) =
\]

\[
= Sp \left( \hat{\rho} \ln \left( \frac{\hat{\rho}}{Sp \hat{\rho}} \right) \right) \exp \left( -\sum_{j=1}^{\infty} \int dz b_1^*(z, j)b_1(z, j) - \ldots - \right.
\]

\[
- \sum_{j=1}^{\infty} \int \ldots \int dz_1 \ldots dz_k b_k^*(z_1, \ldots, z_k, j)b_k(z_1, \ldots, z_k, j) \right)
\]

To this symbol, we assign the operator

\[
\overline{S} = \text{Reg} S(b_1^+(x_1^1, j^1), b_1^-(x_1^1, j^1), \ldots, b_k^+(x_k^k, j^k), b_k^-(x_k^k, j^k))
\]

in the space \( \mathcal{F} \).

**Definition** The numbers \( \lambda \) such that

\[
(\overline{\mathcal{H}} + \theta \overline{S}) \Phi = \lambda \overline{E} \Phi, \quad \Phi \in \mathcal{F}, \quad \Phi \neq 0,
\]

are called quantum eigenvalues of free energy for a system of fermions with Hamiltonian \( \hat{H} \) at temperature \( \theta \) for \( k \) cluster types.

### 4 Asymptotic series as \( N \to \infty \) for a fermion system

Further, we assume that the secondary quantized fermion Hamiltonian in the space \( \mathcal{H}_F \) has the form

\[
\hat{H} = \iint dx dy \bar{\psi}^+(x)T(x, y)\bar{\psi}^-(y)
+ \frac{1}{2} \iiint dx dx' dy dy' \bar{\psi}^+(x)\psi^+(x')V(x, x', y, y')\bar{\psi}^-(y')\bar{\psi}^-(y),
\]

where, just as everywhere, \( x, x', y, y' = (\zeta, s), \zeta \in \mathbb{T}^3, s \in \Sigma, \) and the functions \( T(x, y), V(x, x', y, y') \) have the properties

\[
T(x, y) = T^*(y, x), \quad V(x, x', y, y') = V^*(y, y', x, x').
\]

Let us consider the supersecondary quantized equations for free energy of fermions in another representation. Suppose that \( k \) is the number of clusters of the first type, \( M \) is the number of clusters of the second type, and there are no clusters of any other type. We choose some functions \( \Psi(x, y) \in L_2((\mathbb{T}^3 \times \Sigma) \times (\mathbb{T}^3 \times \Sigma)), \phi(x, j) \in L_2(\mathbb{T}^3 \times \Sigma) \times l_2 \) such that

\[
\Psi(x, y) = -\Psi(y, x),
\]

\[
k = Sp \left( \hat{G}^t - 1 + (1 + (1 - \hat{G}^t)^{-1}\hat{R}(1 - \hat{G}^t)^{-1}\hat{R}^+) \right)^{-1},
\]

\[
M = \frac{1}{2} Sp \left( (1 + (1 - \hat{G}^t)^{-1}\hat{R}(1 - \hat{G})^{-1}\hat{R}^+) \right)^{-1}(1 - \hat{G}^t)^{-1}\hat{R}(1 - \hat{G})^{-1}\hat{R}^+ \right),
\]

(20)
where \( \hat{G}, \hat{R} \) are operators in \( L_2(\mathbb{T}^3 \times \Sigma) \) of the form

\[
\hat{G} = (1 + \hat{A}^t)^{-1} \left( \hat{A}^t + \hat{B}^+(1 + \hat{A})^{-1} \hat{B}(1 + \hat{A}^t)^{-1} (1 + \hat{B}^+(1 + \hat{A})^{-1} \hat{B}(1 + \hat{A}^t)^{-1}) \right)^{-1}
\]

\[
\hat{R} = -(1 + \hat{A})^{-1} \hat{B}(1 + \hat{A}^t)^{-1} (1 + \hat{B}^+(1 + \hat{A})^{-1} \hat{B}(1 + \hat{A}^t)^{-1})^{-1},
\]

and \( \hat{A}, \hat{B} \) are the following operators in \( L_2(\mathbb{T}^3 \times \Sigma) \):

\[
(\hat{A}u)(x) = \sum_{j=1}^{\infty} \int dy \varphi(x,j) \varphi^*(y,j)u(y), \quad (\hat{B}u)(x) = \int dy \Psi(x,y)u(y),
\]

where the superscript + stands for Hermitian conjugation and \( t \) for transposition. We choose

an arbitrary complete orthonormal system of functions \( \Psi_a(x,y), a = 1, 2, \ldots \), in a subspace of the space \( L_2((\mathbb{T}^3 \times \Sigma) \times (\mathbb{T}^3 \times \Sigma)) \) whose elements are functions orthogonal to \( \Psi(x,y) \), and

an arbitrary complete orthonormal system of functions \( \varphi_c(x,j), c = 1, 2, \ldots \) in the subspace of the space \( L_2(\mathbb{T}^3 \times \Sigma) \times l_2 \) consisting of functions orthogonal to \( \varphi(x,j) \). The following vector system corresponding to these systems is complete and orthonormal in the subspace \( \mathcal{F}_{k,M} \) of the space \( \mathcal{F} \)

\[
\Phi_{\{n\},\{m\}} = \prod_{a=1}^{\infty} \frac{1}{\sqrt{n_a}} \left( \int dx_1 dx_2 \Psi_a(x_1,x_2) \hat{b}^+_2(x_1,x_2) \right)^{n_a}
\]

\[
\times \prod_{c=1}^{\infty} \frac{1}{\sqrt{m_c}} \left( \sum_{j=1}^{\infty} \int dy \varphi_c(y,j) \hat{b}^+_1(y,j) \right)^{m_c}
\]

\[
\times \frac{1}{C^{\tilde{N}_2} \sqrt{\tilde{N}_2}!} \left( \int dx dx' \Psi(x,x') \hat{b}^+_2(x,x') \right)^{\tilde{N}_2}
\]

\[
\times \frac{1}{D^{\tilde{N}_1} \sqrt{\tilde{N}_1}!} \left( \sum_{j'=1}^{\infty} \int dy' \varphi(y',j') \hat{b}^+_1(y',j') \right)^{\tilde{N}_1} \Phi_0,
\]

where \( \Phi_0 \) is the vacuum vector of the space \( \mathcal{F} \); \( \{n\}, \{m\} \) are sets of integers \( n_a \geq 0 \), \( a = 1, 2, \ldots \), and \( m_c \geq 0 \), \( c = 1, 2, \ldots \), such that

\[
\sum_{a=1}^{\infty} n_a \leq M, \quad \sum_{c=1}^{\infty} m_c \leq k,
\]

and the following notation is used:

\[
\tilde{N}_2 = M - \sum_{a=1}^{\infty} n_a, \quad \tilde{N}_1 = k - \sum_{c=1}^{\infty} m_c,
\]

and

\[
C = \sqrt{\int \int dx dx' |\Psi(x,x')|^2}, \quad D = \sqrt{\sum_{j=1}^{\infty} \int dx |\varphi(x,j)|^2}.
\]
We introduce the Fock space $\mathcal{M}$ generated by the vacuum vector $Y_0$ and the following boson creation and annihilation operators of two kinds: $\hat{\beta}_a^\pm, a = 1, 2, \ldots, \hat{d}_c^\pm, c = 1, 2, \ldots$. To each vector (22) we assign the following vector of the space $\mathcal{M}$:

$$Y_{\{n\},\{m\}} = \prod_{a=1}^{\infty} \frac{1}{\sqrt{n_a!}} (\hat{\beta}_a^+)^{n_a} \prod_{c=1}^{\infty} \frac{1}{\sqrt{m_c!}} (\hat{d}_c^+)^{m_c} Y_0.$$  \hfill (23)

Taking into account the relation between (22) and (23), we can write the equation for quantum values of free energy in the space $\mathcal{M}$ as

$$\hat{F}_\theta Y = \lambda \hat{E} Y, \quad Y \in \mathcal{M}, \quad Y \neq 0,$$  \hfill (24)

where $\hat{F}_\theta, \hat{E}$ are the following operators in $\mathcal{M}$:

$$\hat{F}_\theta = \exp \left( -\sum_{l=1}^{\infty} \frac{2^{l-1}}{\hat{\beta}_l^+ \hat{\beta}_l^-} - \sum_{p=1}^{\infty} \frac{2^{p-1}}{\hat{d}_p^+ \hat{d}_p^-} \right) \sum_{\{n\},\{m\}} \sum_{\{n'\},\{m'\}} \left( \Phi_{\{n\},\{m\}}, (\hat{H} + \theta \hat{S}) \Phi_{\{n'\},\{m'\}} \right) \times$$

$$\times \prod_{a=1}^{\infty} \frac{2^{n_a}(\hat{\beta}_a^-)^{n'_a}}{n_a! n'_a!} \prod_{c=1}^{\infty} \frac{2^{m_c}(\hat{d}_c^-)^{m'_c}}{m_c! m'_c!},$$

$$\hat{E} = \exp \left( -\sum_{l=1}^{\infty} \frac{2^{l-1}}{\hat{\beta}_l^+ \hat{\beta}_l^-} - \sum_{p=1}^{\infty} \frac{2^{p-1}}{\hat{d}_p^+ \hat{d}_p^-} \right) \sum_{\{n\},\{m\}} \sum_{\{n'\},\{m'\}} \left( \Phi_{\{n\},\{m\}}, \hat{E} \Phi_{\{n'\},\{m'\}} \right) \times$$

$$\times \prod_{a=1}^{\infty} \frac{2^{n_a}(\hat{\beta}_a^-)^{n'_a}}{n_a! n'_a!} \prod_{c=1}^{\infty} \frac{2^{m_c}(\hat{d}_c^-)^{m'_c}}{m_c! m'_c!}.$$  

We assume that equation (24) has the solutions

$$Y = \sum_{\{n\},\{m\}} \gamma_{\{n\},\{m\}} Y_{\{n\},\{m\}},$$  \hfill (25)

such that in the limit as $N \to \infty$, $L \to \infty$, $N/L^3 \to \text{const}$ we have

$$\frac{\sum_{\{n\},\{m\}} \left( \sum_{a=1}^{\infty} n_a \right) |\gamma_{\{n\},\{m\}}|^2}{\sum_{\{n\},\{m\}} |\gamma_{\{n\},\{m\}}|^2} = O(1), \quad \frac{\sum_{\{n\},\{m\}} \left( \sum_{c=1}^{\infty} m_c \right) |\gamma_{\{n\},\{m\}}|^2}{\sum_{\{n\},\{m\}} |\gamma_{\{n\},\{m\}}|^2} = O(1).$$

To such solutions of equation (24) there correspond the quantum value of free energy

$$\lambda(\Psi(x,y), \varphi(x,j)) = \int \int dx dy T(x,y)G(x,y) +$$

$$+ \frac{1}{2} \int \int \int \int dx dx' dy dy' V(x, x', y, y') \left( G(x,y)G(x',y') - G(x,y')G(x',y) + R^*(x',x)R(y',y) \right) +$$

$$+ \theta \text{Sp} f \left( \frac{\hat{G}}{\hat{R}} - \frac{1}{2} \frac{\hat{R}^+}{\hat{G}^t + \frac{1}{2}} \right) + O(1),$$  \hfill (26)
where \( f(\xi), \xi \in \mathbb{R} \) has the form

\[
f(\xi) = \frac{1}{2}(\frac{1}{2} + \xi) \ln\left(\frac{1}{2} + \xi\right) + \frac{1}{2}(\frac{1}{2} - \xi) \ln\left(\frac{1}{2} - \xi\right),
\]

and the functions \( G(x, y), R(x, y) \) are the kernels of the operators (21):

\[
(\hat{G}u)(x) = \int dy \, G(x, y)u(y), \quad (\hat{R}u)(x) = \int dy \, R(x, y)u(y).
\]

The vector \( \Phi \in \mathcal{F} \) corresponding to (25) we denote by \( \Phi(\Psi(x, y), \varphi(x, j)) \). For two distinct pairs of functions \( \Psi(x, y), \varphi(x, j) \) and \( \Psi'(x, y), \varphi'(x, j) \) satisfying conditions (20), the following matrix element of an arbitrary supersecondary quantized operator \( \hat{A} \), which is bounded in \( N \):

\[
\begin{vmatrix}
\Phi(\Psi'(x, y), \varphi'(x, j)) & \mathbf{A}\Phi(\Psi(x, y), \varphi(x, j)) \\
\Phi(\Psi'(x, y), \varphi'(x, j)) & \Phi(\Psi(x, y), \varphi(x, j))
\end{vmatrix}
\]

is exponentially small in \( N \).

Therefore, in the case in which the values \( \lambda(\Psi(x, y), \varphi(x, j)) \) and \( \lambda(\Psi'(x, y), \varphi'(x, j)) \) corresponding to these function pairs do not coincide, the vectors \( \Phi(\Psi(x, y), \varphi(x, j)) \) and \( \Phi(\Psi'(x, y), \varphi'(x, j)) \) belong to different series. If the resonance occurs, i.e., the value (26) is the same for \( \Psi(x, y), \varphi(x, j) \) and \( \Psi'(x, y), \varphi'(x, j) \), then the series is formed by linear combinations of the vectors \( \Phi(\Psi(x, y), \varphi(x, j)) \) and \( \Phi(\Psi'(x, y), \varphi'(x, j)) \). From all series of solutions of the equation for quantum values of free energy corresponding to some chosen \( k \) and \( M \), we choose the series with the least value (26). The functions \( \Psi(x, y), \varphi(x, j) \) corresponding to this series are determined from the equation of self-consistent type:

\[
\begin{pmatrix}
\hat{G} - \frac{1}{2} & \hat{R}^+ \\
\hat{R} & -\hat{G}^t + \frac{1}{2}
\end{pmatrix} = -\frac{1}{2^\theta} \begin{pmatrix}
\hat{T}^t & \hat{V}^t \\
\hat{V} & -\hat{T}
\end{pmatrix}
\]

(27)

where the operators \( \hat{T}, \hat{V} \) are expressed in terms of \( \hat{G}, \hat{R} \) as follows:

\[
\hat{T} = \hat{X} - \mu + (2\mu - \omega)(1 - \hat{G}^t)^{-1}\hat{R}(1 - \hat{G})^{-1}\hat{R}^+(1 - \hat{G}^t)^{-1}
\]

\[
\times \left(1 + \hat{R}(1 - \hat{G})^{-1}\hat{R}^+(1 - \hat{G}^t)^{-1}\right)^{-2},
\]

\[
\hat{V} = \hat{Z} + (2\mu - \omega)(1 - \hat{G}^t)^{-1}\hat{R}(1 - \hat{G})^{-1}(1 + \hat{R}^+(1 - \hat{G}^t)^{-1}\hat{R}(1 - \hat{G})^{-1})^{-2},
\]

(28)

and the operators \( \hat{X}, \hat{Z} \) in the space \( L_2(T^3 \times l_2) \) have the form

\[
(\hat{X}u)(x) = \int dy \, T(x, y)u(y) + \int dx \int dy \, \{V(x, x', y, y') - V(x, x', y', y')\}G(x', y')u(y),
\]

\[
(\hat{Z}u)(x) = \int dx \int dy \, V(x', x, y, y')R(y', y)u(x').
\]

(29)

The numbers \( \mu, \omega \) in (28) are determined by conditions (20). The minimal quantum value of free energy (20) for some chosen \( k \) and \( M \) we denote by \( \lambda_{k,M} \). Let us consider the minimum of \( \lambda_{k,M} \) with respect to \( k \) and \( M \) provided \( k + 2M = N = \text{const} \). We denote this minimum
by \( \lambda_N \). The functions \( \Psi(x, y), \varphi(x, j) \) corresponding to this minimum can be found from the equation

\[
\left( \frac{\hat{G} - \frac{1}{2}}{\hat{R}} \begin{pmatrix} \hat{R}^+ - \hat{G}^t + \frac{1}{2} \\ -\hat{G}^t + \frac{1}{2} \end{pmatrix} \right) = -\frac{1}{2} \frac{1}{2\theta} \left( \begin{pmatrix} \bar{X}^t - \mu \\ \bar{Z}^t \end{pmatrix} - \begin{pmatrix} \bar{Z}^t \\ -\bar{X}^t + \mu \end{pmatrix} \right)
\]

(30)

where the operators \( \bar{X}, \bar{Z} \) are the same as (24) and the number \( \mu \) is determined by the condition \( k + 2M = N \), where \( k \) and \( M \) have the form (20). Equation (34) is well known. It coincides with the Bardeen–Cooper–Schriffer–Bogolyubov equation in the theory of superconductivity. The solutions of this equation with different \( \theta \) determine the functions \( k(\theta), M(\theta) \) by formulas (20) and the function \( \lambda_N(\theta) \) by formula (29). The temperature \( \theta_c \) at which \( M(\theta_c) = 0 \) is called critical. At this temperature the heat capacity

\[
C = -\theta \lambda''_N(\theta)
\]

(31)

has a discontinuity. Since resonances can occur for \( \lambda_{k,M} = \lambda_{k',M'} \), with \( k \neq k' \) and \( M \neq M' \), we can replace the functions \( k(\theta), M(\theta) \) by some other \( k_1(\theta), M_1(\theta) \) such that \( M_1(\theta_1) = 0 \) for \( \theta_1 > \theta_c \) and thus to increase the critical temperature. To calculate the heat capacity corresponding to a temperature series distinct from the Bardeen–Cooper–Schriffer–Bogolyubov-series, we need to replace \( \lambda_N(\theta) \) by \( \lambda_{k(\theta),M(\theta)} \) in formula (29).

## 5 Asymptotic series as \( N \to \infty \) for a boson series

In a similar way, we can find series of quantum values of free energy for bosons in the case in which the number of cluster types is 2.

Here we present an asymptotics for the case in which the condensate can exist.

The principal state of the series is expressed in the same way as for fermions via the vectors of some system of the form (21). For the sets \( \{n\}, \{m\} \) the same condition as that for fermions holds in the thermodynamical limit. Let us study the series such that

\[
\Psi(x, y) = \frac{1}{L^3} \sum_p \phi(p)e^{ip(x-y)}, \quad \phi(p) = \phi(-p), \quad \varphi(x, j) = \sqrt{\frac{a(p_j)}{L^3}} e^{ip_jx},
\]

(32)

where \( p \) are three-dimensional vectors of the form \( 2\pi(l_1, l_2, l_3)/L, l_1, l_2, l_3 \in \mathbb{Z} \), \( \sum_p \) stands for the summation over all such vectors, and \( p_j \) is a single-valued mapping of the set \( j = 1, 2, \ldots \) on the set of such vectors. Moreover, \( \phi(q), a(q) \) are continuous functions of the variable \( q \in \mathbb{R}^3 \) such that \( \phi(q) = \phi(-q), a(q) = a^*(q), 0 \leq a(q) \leq 1 \). If the condensate consists of particles with zero momentum, then the functions \( \phi(q), a(q) \) satisfy the inequality

\[
(1 - a(q))(1 - a(-q)) = \phi^*(q)\phi(q) \geq 0.
\]

(33)

Note that we have the equality only for \( q = 0 \). In this case the asymptotics of the ratio of the quantum value of free energy to the volume in the thermodynamical limit has the form

\[
\frac{\lambda}{L^3} = \frac{1}{(2\pi)^3} \int dq \frac{h^2q^2}{2m} G(q) + \frac{U_0a^3n_0\bar{v}(0)}{2} + \frac{U_0a^3n_0}{(2\pi)^3} \int dq (\bar{v}(0) + \bar{v}(aq)) G(q) +
\]

18
\[ + \frac{U_0 a^3 n_0}{2(2\pi)^3} \int dq \tilde{v}(aq)(R(q) + R^*(q)) + \frac{U_0 a^3}{2(2\pi)^6} \int dqdl (\tilde{v}(0) + \tilde{v}(a(q - l)))G(q)G(l) + \]
\[ + \frac{U_0 a^3}{2(2\pi)^6} \int dqdl \tilde{v}(a(q - l))R^*(q)R(l) + \]
\[ + \theta \frac{1}{(2\pi)^3} \int dq \left( n(q) \ln n(q) - (1 + n(q)) \ln(1 + n(q)) \right), \quad (34) \]

where \( n_0 \) is the condensate density,
\[ n_0 = \frac{N}{L^3} - \frac{1}{(2\pi)^3} \int dq G(q), \]
\( N \) is the number of particles, and just as for fermions we have \( N = k + 2M \). The functions \( G(q), R(q), n(q) \) in formula (34) are expressed via \( a(q), \phi(q) \) as follows
\[ G(q) = \frac{a(q)(1 - a(-q)) + \phi^*(q)\phi(q)}{(1 - a(q))(1 - a(-q)) - \phi^*(q)\phi(q)}, \]
\[ R(q) = \frac{\phi(q)}{(1 - a(q))(1 - a(-q)) - \phi^*(q)\phi(q)}, \]
\[ n(q) = \frac{1}{2} \sqrt{(1 + G(q) + G(-q))^2 - 4R^*(q)R(q)} - \frac{1}{2}(1 - G(q) + G(-q)). \quad (35) \]
Expression (34) corresponds to the quantum free energy of the principal state of some series if the functions \( \phi(q), a(q) \) satisfy the relations
\[ k = L^3 n_0 a(0) + \frac{L^3}{2(2\pi)^3} \int dq \frac{a(q) + a(-q) - 2a(q)a(-q)}{(1 - a(q))(1 - a(-q)) - \phi^*(q)\phi(q)}, \]
\[ M = \frac{L^3 n_0}{2} (1 - a(0)) + \frac{L^3}{2(2\pi)^3} \int dq \frac{\phi^*(q)\phi(q)}{(1 - a(q))(1 - a(-q)) - \phi^*(q)\phi(q)} \quad (36) \]
and (34) attains the minimum on these functions provided that conditions (36) are satisfied.

One can easily see that if \( n_0 \neq 0 \) then
\[ \frac{\delta}{\delta \phi^*(q) \frac{\lambda}{L^3}} \bigg|_{\phi=0} = \frac{U_0 a^3 n_0}{2(2\pi)^3} \frac{\tilde{v}(aq)}{(1 - a(q))(1 - a(-q))} \neq 0. \]
This implies that for any function \( a(p) \) there always exists a function \( \phi(p) \) such that the value of free energy (34) determined by the functions \( a(p), \phi(p) \) is less than the value of free energy for \( a(p), \phi(p) = 0 \). Although the nature can “set” the system on any metastable state, the physicists always assume that the state with the least energy is realized. Hence here we do not consider boson systems with clusters only of the first type and we always assume that there exists at least two types of clusters.

### 6 Variational method for the supersecondary quantized problem

We consider a variational method which allows us to construct successive approximates to the solutions of the supersecondary quantized equations for the quantum free energy both in the boson and fermion cases. We study the fermion case in detail.
Suppose that the number of clusters of the first type is $k$, the number of clusters of the second type is $M$, and there are no clusters of any other type. We study some complete system of functions $\Psi_a(x,y)$, $a = 0, 1, \ldots,$ in the space $L_2((T^3 \times \Sigma) \times (T^3 \times \Sigma))$ and some complete system of functions $\varphi_c(x,j)$, $c = 0, 1, \ldots,$ in the space $L_2(T^3 \times \Sigma) \times l_2$. In this section we choose function systems no in the same way as in Sect. 3. Below we explain how we choose these function systems.

To these functions, by formula (22) we assign a vector system that is complete in the subspace $F_{k,M}$ of the space $F$. We assume that the numbers $k$ and $M$ are sufficiently large so that it is possible to regulate the entropy operator $\hat{S}$. Let us consider the sequence

$$\lambda_b = \frac{\left(\Phi_b, \left(\hat{H} + \theta \hat{S}\right) \Phi_b\right)}{\left(\Phi_b, \hat{E} \Phi_b\right)}, \quad b = 0, 1, \ldots,$$  \hspace{1cm} (37)

where $\Phi_b \in F$ is a vector of the form (22) corresponding to the set of numbers $n_a = M\delta_{ab}$, $m_c = k\delta_{cb}$, and $\delta_{ab}$ is the Kronecker delta. We choose function systems $\Psi_a(x,y)$, $a = 0, 1, \ldots$, and $\varphi_c(x,j)$, $c = 0, 1, \ldots$, so that the following conditions hold:

$$\frac{\delta \lambda_b}{\delta \Psi_b(x,y)} = 0, \quad \frac{\delta \lambda_b}{\delta \varphi_b(x,j)} = 0$$  \hspace{1cm} (38)

and, in addition,

$$\min_b \lambda_b = \lambda_0.$$

Conditions (38) are Hartree-Fock–Bogolyubov–Bardeen–Cooper–Schriffer type equations for the functions $\Psi_b(x,y)$, $\varphi_b(x,j) = 0$.

For example, for $\Psi(x,y) = 0$ at zero temperature, we can write the equation for the function $\varphi_b(x,j)$ as a system of Hartree–Fock type equations for the functions $u_j(x) = \varphi_b(x,j)$.

In the equation for the quantum free energy we expand the vector $\Phi$ with respect to the vector system (22):

$$\Phi = \sum_{\{n\},\{m\}} A_{\{n\},\{m\}} \Phi_{\{n\},\{m\}},$$  \hspace{1cm} (39)

where

$$\sum_{\{n\},\{m\}}$$

stands for the summation over all possible sets of $\{n\}$, $\{m\}$. Taking into account (39), we can rewrite the equation for free energy as the linear system

$$\sum_{\{n\},\{m\}} F_{\{n'\},\{m'\},\{n\},\{m\}} A_{\{n\},\{m\}} \Phi_{\{n\},\{m\}} = \lambda \sum_{\{n'\},\{m'\},\{n\},\{m\}} E_{\{n'\},\{m'\},\{n\},\{m\}} A_{\{n\},\{m\}} \Phi_{\{n\},\{m\}},$$  \hspace{1cm} (40)

where the coefficients $F_{\{n'\},\{m'\},\{n\},\{m\}}$, $E_{\{n'\},\{m'\},\{n\},\{m\}}$ are expressed via the supersecondary quantized operators $\hat{H}$, $\hat{S}$, $\hat{E}$ and the vectors (22). Equation (40) is always the exact representation for the equation of quantum free energy. Approximate solutions of the equation
for free energy can be obtained from the linear system

\[
\sum_{\{n\},\{m\}} F_{\{n'\},\{n\},\{m\} \{n\},\{m\} A_{\{n\} \{m\}} = \lambda \sum_{\{n\},\{m\}} E_{\{n'\},\{n\},\{m\} \{n\},\{m\} A_{\{n\} \{m\}},
\]

(41)

where \(0 \leq \kappa \leq k\), \(0 \leq \tilde{M} \leq M\) are integers and

\[
\sum_{\{n\},\{m\}} F_{\{n'\},\{n\},\{m\} \{n\},\{m\} A_{\{n\} \{m\}} = \lambda \sum_{\{n\},\{m\}} E_{\{n'\},\{n\},\{m\} \{n\},\{m\} A_{\{n\} \{m\}}.
\]

(42)

For \(k = 0\), \(\tilde{M} = 0\), relation (41) implies the upper bound for free energy, which is equal to \(\lambda_0\) (37). For \(k > 0\), \(\tilde{M} > 0\), this estimate can be improved and, for \(k \to k\), \(\tilde{M} \to M\), coincides with the exact value of the quantum free energy.

## 7 Metastable states and the case of many clusters

The same procedure can be used for any number of clusters both in the fermion and boson cases. Namely, we construct the supersecondary quantization for this set of clusters and set the creation and annihilation operators equal to functions (i.e., we assume that they commute, and hence we take the symbols of these operators; the physicists say: “we assume that the creation and annihilation operators are \(^\prime\)-numbers).

Next, according to this nonlinear problem, we construct a set of independent functions complete in the corresponding space, which, in turn, determines the passage to the occupation numbers. Then we employ the ideology of the Tamm–Dankov method.

The set of such functions is constructed just as above, namely, the vector system (22) is naturally generalized to the case of an arbitrary number of clusters and the ratios of matrix elements similar to (37) are studied for this system.

Expression (37) is the upper bound for the least quantum value of free energy. Hence it is natural to choose the function system so that this expression for each function attain its extremum value. Previously, this was performed in such a way that equation (38) is an extremum equation. For an arbitrary number of clusters, the approximate solutions of the equation for free energy can be constructed by using a variational method similar to (41).

Hence, in some sense, this method is similar to the Tamm–Dankov method, since inequality (42) is used. This variational method is especially effective for systems of third quantized fields and automatically can be used for such systems.

The supersecondary quantization method can be used in quantum electrodynamics and quantum chromodynamics, when the “charge” is preserved rather than the number of particles. Such an example is studied earlier. However, in this case we encounter the renormalization problem and must construct our variational method following the method proposed by Dyson [8], i.e., following the “new Tamm–Dankov” method.
To find metastable states corresponding to superfluidity and superconductivity, we perform constructions similar to those considered in the Introduction. Let $0 < \alpha < 1$. We consider the following class of vectors in the space $\mathcal{F}$:

$$\tilde{\Phi} = \sum_{\{n\},\{m\}} B_{\{n\}\{m\}} \Phi_{\{n\},\{m\}}, \tag{43}$$

where $B_{\{n\}\{m\}}$ depend on the sets of numbers $\{n\}, \{m\}$ so that

$$B_{\{n\}\{m\}} = 0$$

if at least one of the conditions

$$\sum_{a=1}^{\infty} n_a \leq \alpha M, \quad \sum_{c=1}^{\infty} m_c \leq \alpha k \tag{44}$$

is not satisfied. The metastable state corresponding to our function system exists if the functional

$$\frac{(\tilde{\Phi}, (\tilde{\mathcal{H}} + \theta \tilde{\mathcal{S}}) \tilde{\Phi})}{(\tilde{\Phi}, \tilde{E} \tilde{\Phi})} \tag{45}$$

has a minimum on the class of vectors (43) for some value of $\alpha$. In the number of cluster types is $\geq 2$, the class of vectors is determined similar to (43) by using inequalities similar to (47). And the existence of a metastable state for a larger number of clusters is determined by the existence of a minimum of the corresponding functional of type (45) on the corresponding vector class.

8

We proposed a quantization of entropy and of free energy for many particles.

The results can readily be carried over to the case of many fields.

In particular, we can define quantum thermodynamics in the case of many fields by generalizing our operation of quantization over clusters to the case of third quantization [9] (this operation was performed for the second quantization).

Let us perform the quantization of thermodynamics in the theory of many scalar fields. In [9], a third quantization of the model of $N$ scalar fields is presented for the Hamiltonian of the model of the form

$$\tilde{\mathcal{H}}_N = \frac{1}{2} \sum_{a=1}^{N} \int dx \left( \pi_a(x) \pi_a(x) + \nabla \varphi_a(x) \nabla \varphi_a(x) + m^2 \varphi_a(x) \varphi_a(x) \right) + \varepsilon \sum_{a,b=1}^{N} \int dx \varphi_a^2(x) \varphi_b^2(x), \tag{46}$$

where $x \in \mathbb{R}^3$, $\varepsilon$, and $m$ are given quantities and $\pi_a(x)$ and $\varphi_a(x)$ are operators in the space $\mathcal{L}_N$ of the symmetric functionals, $\Psi_N[\varphi_1(\cdot), \ldots, \varphi_N(\cdot)]$, of $N$ functions $\varphi_1(\cdot), \ldots, \varphi_N(\cdot)$,

$$\varphi_a(\cdot) \Psi_N[\varphi_1(\cdot), \ldots, \varphi_N(\cdot)] = \varphi_a(\cdot) \Psi_N[\varphi_1(\cdot), \ldots, \varphi_N(\cdot)],$$

$$\pi_a(\cdot) \Psi_N[\varphi_1(\cdot), \ldots, \varphi_N(\cdot)] = -i \frac{\delta}{\delta \varphi_a(\cdot)} \Psi_N[\varphi_1(\cdot), \ldots, \varphi_N(\cdot)].$$
Let us consider the Fock space \( \mathcal{L} = \oplus_{N=0}^{\infty} \mathcal{L}_N \) whose elements are sequences of symmetric functionals \( \Psi_N[\varphi_1(\cdot), \ldots, \varphi_N(\cdot)] \), \( N = 0, 1, \ldots \), and introduce operators of generalized functionals \( \hat{A}^\pm[\varphi(\cdot)] \) on \( \mathcal{L} \) as follows:

\[
\left( \int D\varphi \hat{A}^+ [\varphi(\cdot)] X[\varphi(\cdot)] \Psi \right)_k [\varphi_1(\cdot), \ldots, \varphi_k(\cdot)] =
\frac{1}{\sqrt{k}} \sum_{a=1}^k X[\varphi_a(\cdot)] \Psi_{k-1}[\varphi_1(\cdot), \ldots, \varphi_{a-1}(\cdot), \varphi_{a+1}(\cdot), \ldots, \varphi_k(\cdot)] , \quad \text{(47)}
\]

\[
\left( \int D\varphi \hat{A}^- [\varphi(\cdot)] X^*[\varphi(\cdot)] \Psi \right)_{k-1} [\varphi_1(\cdot), \ldots, \varphi_{k-1}(\cdot)] =
\sqrt{k} \int D\varphi X^*[\varphi(\cdot)] \Psi_k[\varphi(\cdot), \varphi_1(\cdot), \ldots, \varphi_{k-1}(\cdot)].
\]

To the Hamiltonian of \( N \) fields of the form (46), we assign the following operator \( \hat{H} \) on the Fock space \( \mathcal{L} \):

\[
(\hat{H} \Psi)_N[\varphi_1(\cdot), \ldots, \varphi_N(\cdot)] = \hat{H}_N \Psi_N[\varphi_1(\cdot), \ldots, \varphi_N(\cdot)],
\]

which can be represented as

\[
\hat{H} = \frac{1}{2} \int D\varphi \hat{A}^+ [\varphi(\cdot)] \int dx \left( -\frac{\delta^2}{\delta\varphi(x)\delta\varphi(x)} + \nabla\varphi(x)\nabla\varphi(x) + m^2\varphi(x)\varphi(x) \right) \hat{A}^- [\varphi(\cdot)] +
+ \varepsilon \int D\varphi D\phi \int dx \varphi^2(x) \varphi(x) \hat{A}^+ [\varphi(\cdot)] \hat{A}^- [\phi(\cdot)] \hat{A}^+ [\phi(\cdot)] \hat{A}^- [\phi(\cdot)]. \quad \text{(48)}
\]

The expression (48) is the triply quantized representation of the Hamiltonian (46).

Now we introduce the \textit{cluster} spaces \( \mathcal{M}_{N_1, \ldots, N_k} \), where \( k \geq 1, N_1, \ldots, N_k \geq 0 \) are integers. The elements of these spaces are functionals of the form

\[
\Psi_{N_1, \ldots, N_k}[\varphi^1_1(\cdot), j^1_1, \ldots, \varphi^1_{N_1}(\cdot), j^1_{N_1}; \varphi^2_1(\cdot), j^2_1; \ldots, \varphi^2_{N_2-1}(\cdot), j^2_{N_2-1}; \varphi^2_{N_2}(\cdot), j^2_{N_2}; \ldots; \varphi^k_1(\cdot), j^k_1; \ldots; \varphi^k_{N_k}(\cdot), j^k_{N_k}],
\]

that are symmetric with respect to the permutations of

\[
\varphi^l_{pl+1}(\cdot), \ldots, \varphi^l_{pl+1}(\cdot), j^l_{pl+1}(\cdot), \varphi^l_{ql+1}(\cdot), \ldots, \varphi^l_{ql+1}(\cdot), j^l_{ql+1}(\cdot)
\]

for all \( l = 1, \ldots, k \) and \( p, q = 0, \ldots, N_l - 1 \), where \( j^l_s, s = 1, \ldots, N_l \), take values in the set \( 0, 1, \ldots, I_l \). We also introduce the \textit{cluster} Fock space \( \mathcal{M} = \oplus_{N_1, \ldots, N_k=0}^{\infty} \mathcal{M}_{N_1, \ldots, N_k} \), which consists of sequences of functionals (49). Similarly to (47), we can introduce operator generalized functionals \( \hat{B}_l^\pm[\varphi^l_1(\cdot), \ldots, \varphi^l_l(\cdot)] \).

We introduce an \textit{operator measure} \( \hat{\rho} \), which is an operator on \( \mathcal{L} \) and depends on the functionals \( B_l[\varphi^l_1(\cdot), \ldots, \varphi^l_l(\cdot), j^l_j] \):

\[
\hat{\rho} = \exp \left( -\int D\varphi \hat{A}^2 [\varphi(\cdot)] \hat{A}^1 [\varphi(\cdot)] \right) \times
\times \exp \left( \sum_{l=1}^k \sum_{I_l=0}^{I_1} \int \ldots \int D\varphi^l_1 D\phi^l_1 \ldots D\varphi^l_l D\phi^l_l \right.
\times B^l_1[\varphi^l_1(\cdot), \ldots, \varphi^l_l(\cdot), j^l_j] B_l[\varphi^l_1(\cdot), \ldots, \varphi^l_l(\cdot), j^l_j] \times
\times \hat{A}^2 [\varphi^l_1(\cdot)] \hat{A}^1 [\varphi^l_1(\cdot)] \hat{A}^2 [\varphi^l_1(\cdot)] \hat{A}^1 [\varphi^l_1(\cdot)] \right), \quad \text{(50)}
\]
where the numbers over the operators stand for the order of their execution [5]. We omit
the arguments $\hat{\rho}$ in what follows. Furthermore, let
\[
\mathcal{H} (B_1^1[\cdot], B_1^2[\cdot], \ldots B_k^k[\cdot], B_k^1[\cdot]) =
\exp \left( \sum_{l=1}^{k} \sum_{j=0}^{l} \int \cdots \int D\varphi_1^l \cdots D\varphi_l^1 \varphi_1^l(\cdot), \ldots, \varphi_l^1(\cdot) B_l[\varphi_1^l(\cdot), \ldots, \varphi_l^1(\cdot)] \right) \times
\times \text{Sp} \left( \hat{\rho} \hat{H} \right).
\] (51)

By the \textit{cluster representation} of the triply quantized Hamiltonian (48) we mean the
following operator on $\mathcal{M}$:
\[
\hat{\mathcal{H}} = \mathcal{H} \left( \hat{B}_1^+ [\cdot], \hat{B}_1^- [\cdot], \ldots, \hat{B}_k^+ [\cdot], \hat{B}_k^- [\cdot] \right),
\] (52)

We introduce the operator of \textit{entropy} of many fields on $\mathcal{M}$ by the relation
\[
\hat{S} = \mathcal{S}_{\text{reg}} \left( \hat{B}_1^+ [\cdot], \hat{B}_1^- [\cdot], \ldots, \hat{B}_k^+ [\cdot], \hat{B}_k^- [\cdot] \right),
\] (53)
where $\mathcal{S}_{\text{reg}}$ is the regularized functional
\[
\mathcal{S} = \text{Sp} \left( \hat{\rho} \ln \left( \frac{\hat{\rho}}{\text{Sp} \hat{\rho}} \right) \right).
\] (54)

\textbf{Definition}. The numbers $\lambda$ such that
\[
\left( \hat{\mathcal{H}} + \theta \hat{S} \right) \Phi = \lambda \Phi, \quad \Phi \in \mathcal{M}, \, \Phi \neq 0,
\] (55)
are called \textit{quantum values of free energy} of many fields under the temperature $\theta$.

In the general case of a system of many fields, the quantization of entropy and free energy
can be carried out just as was done in the present note.

9 Supersecondary quantization and quantization of en-
tropy with preserved charge

Earlier the supersecondary quantization and quantization of thermodynamics was introduced
in the case in which the secondary quantized Hamiltonian of the system commutes with
the operator of the number of particles. In this section the secondary quantization and
quantization of thermodynamics with number of cluster types equal to 2 is generalized to
the case of charge preservation.

Let $n_1, n_2, m \geq 0$ be integers. Consider the functions
\[
\Phi_{n,n_1,m_2}(x_1, j_1, \ldots, x_{n_1}, j_{n_1}, y_1, i_1, \ldots, y_{n_2}, i_{n_2}, z_1, w_1, \ldots, z_m, w_m)
\] (56)
that are symmetric with respect to permutations of the pairs \(x_p, j_p\) and \(x_q, j_q\), with respect to permutations of the pairs \(y_r, i_r\) and \(y_s, i_s\), and with respect to permutations of the pairs \(z_t, w_t\) and \(z_t, w_t\), where the variables \(x_p, y_r, z_t, w_t\) take values on the three-dimensional torus \(\mathbb{T}^3\), and the variables \(j_p\) and \(i_r\) attain values \(1, 2, \ldots\) and are called numbers. The Hilbert space of functions (56) with the norm

\[
\|\Phi_{n_1, n_2, m}\|^2 = \\
= \sum_{j_1=1}^{\infty} \cdots \sum_{j_{n_1}=1}^{\infty} \cdots \sum_{i_{n_2}=1}^{\infty} \|\Phi_{n_1, n_2, m}(x_1, j_1, \ldots, x_{n_1}, j_{n_1}, y_1, i_1, \ldots, y_{n_2}, i_{n_2}, z_1, w_1, \ldots, z_m, w_m)\|^2
\]

we denote by \(\mathcal{F}_{n_1, n_2, m}\). Now we introduce the cluster space \(\mathcal{F}\):

\[
\mathcal{F} = \bigoplus_{n_1, n_2, m=0}^{\infty} \mathcal{F}_{n_1, n_2, m}
\]

whose elements are sets of functions (56):

\[
\Phi = \{\Phi_{n_1, n_2, m}\}, \quad n_1, n_2, m = 0, 1, \ldots,
\]

\[
\|\Phi\|^2 = \sum_{n_1, n_2, m=0}^{\infty} \|\Phi_{n_1, n_2, m}\|^2.
\]

The space \(\mathcal{F}\) is a special case of the Fock space in which we introduce the creation and annihilation operators \(\hat{b}^+_1(x, j), \hat{b}^+_2(y, i), \hat{B}^\pm(z, w)\):

\[
(\hat{b}^+_1(x, j)\Phi)_{n_1, n_2, m}(x_1, j_1, \ldots, x_{n_1}, j_{n_1}, y_1, i_1, \ldots, y_{n_2}, i_{n_2}, z_1, w_1, \ldots, z_m, w_m) = \\
= \frac{1}{\sqrt{n_1}} \sum_{j=1}^{n_1} \delta_{ij} \delta(x - x_j) \Phi_{n_1-1, n_2, m}(x_1, j_1, \ldots, x_{j-1}, j_{j-1}, x_{j+1}, j_{j+1}, \ldots, x_{n_1}, j_{n_1}, y_1, i_1, \ldots, y_{n_2}, i_{n_2}, z_1, w_1, \ldots, z_m, w_m),
\]

\[
(\hat{b}^+_2(y, i)\Phi)_{n_1, n_2, m}(x_1, j_1, \ldots, x_{n_1}, j_{n_1}, y_1, i_1, \ldots, y_{n_2}, i_{n_2}, z_1, w_1, \ldots, z_m, w_m) = \\
= \sqrt{n_2} \hat{\Phi}_{n_1+1, n_2, m}(x_1, j_1, \ldots, x_{n_1}, j_{n_1}, y_1, i_1, \ldots, y_{n_2}, i_{n_2}, z_1, w_1, \ldots, z_m, w_m),
\]

\[
(\hat{B}^+(z, w)\Phi)_{n_1, n_2, m}(x_1, j_1, \ldots, x_{n_1}, j_{n_1}, y_1, i_1, \ldots, y_{n_2}, i_{n_2}, z_1, w_1, \ldots, z_m, w_m) = \\
= \frac{1}{\sqrt{m}} \sum_{l=1}^{m} \delta(z - z_l) \delta(w - w_l) \Phi_{n_1, n_2, m-1}(x_1, j_1, \ldots, x_{n_1}, j_{n_1}, y_1, i_1, \ldots, y_{n_2}, i_{n_2}, z_1, w_1, \ldots, z_m, w_m),
\]

\[
(\hat{B}^-(z, w)\Phi)_{n_1, n_2, m}(x_1, j_1, \ldots, x_{n_1}, j_{n_1}, y_1, i_1, \ldots, y_{n_2}, i_{n_2}, z_1, w_1, \ldots, z_m, w_m) = \\
= \sqrt{m} \hat{\Phi}_{n_1, n_2, m-1}(x_1, j_1, \ldots, x_{n_1}, j_{n_1}, y_1, i_1, \ldots, y_{n_2}, i_{n_2}, z_1, w_1, \ldots, z_m, w_m),
\]

25
where $\delta_{jj'}$ is the Kronecker delta. The operators $\hat{b}_1^+(x, j)$ and $\hat{b}_1^-(x, j)$ will be called the \textit{creation and annihilation operators} for charge 1, the operators $\hat{b}_2^+(x, j)$ and $\hat{b}_2^-(x, j)$ will be called the \textit{creation and annihilation operators} for charge $-1$, and the operators $\hat{b}_1^+(x, j)$ and $\hat{b}_1^-(x, j)$ will be called the \textit{creation and annihilation operators} for charge 0. We consider the subspace of the space $\mathcal{F}_{n_1, n_2, m}$, which consists of functions (50) symmetric with respect to permutations of the variables $x_l$ and $x_p$, symmetric with respect to permutations of the variables $x_l$ and $z_r$, symmetric with respect to permutations of the variables $y_q$ and $y_s$, and symmetric with respect to permutations of the variables $y_k$ and $y_r$. In this space the creation and annihilation operators $\hat{\psi}_1^+(x)$ for bosons with charge 1 and the creation and annihilation operators $\hat{\psi}_1^-(y)$ for bosons with charge $-1$ are defined as

\[
(\hat{\psi}_1^+(x)\phi)_{q_1, q_2}(x_1, \ldots, x_{q_1}, y_1, \ldots, y_{q_2}) = \frac{1}{\sqrt{q_1}} \sum_{l=1}^{q_1} \delta(x - x_l)\phi_{q_1-1, q_2}(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{q_1}, y_1, \ldots, y_{q_2}),
\]

\[
(\hat{\psi}_1^-(x)\phi)_{q_1, q_2}(x_1, \ldots, x_{q_1}, y_1, \ldots, y_{q_2}) = \sqrt{q_1} \phi_{q_1+1, q_2}(x, x_1, \ldots, x_{q_1}, y_1, \ldots, y_{q_2}),
\]

\[
(\hat{\psi}_2^+(y)\phi)_{q_1, q_2}(x_1, \ldots, x_{q_1}, y_1, \ldots, y_{q_2}) = \frac{1}{\sqrt{q_2}} \sum_{l=1}^{q_2} \delta(y - y_l)\phi_{q_1, q_2-1}(x_1, \ldots, x_{q_1}, y_1, \ldots, y_{l-1}, y_{l+1}, \ldots, y_{q_2}),
\]

\[
(\hat{\psi}_2^-(y)\phi)_{q_1, q_2}(x_1, \ldots, x_{q_1}, y_1, \ldots, y_{q_2}) = \sqrt{q_2} \phi_{q_1, q_2+1}(x_1, \ldots, x_{q_1}, y, y_1, \ldots, y_{q_2}).
\]

An operator in $\mathcal{H}$ of the form

\[
\hat{Q} = \int dx \hat{\psi}_1^+(x)\hat{\psi}_1^-(x) - \int dx \hat{\psi}_2^+(x)\hat{\psi}_2^-(x)
\]

is called the \textit{charge operator} $\hat{Q}$. Let $\hat{A}$ be an operator in the space $\mathcal{H}$ commuting with the charge operator $\hat{Q}$. The \textit{supersecondary quantized operator} $\overrightarrow{A}$ is the following operator in $\mathcal{F}$:

\[
\overrightarrow{A} = A[\hat{b}_1^+(x, j), \hat{b}_1^-(x, j), \hat{b}_2^+(y, i), \hat{b}_2^-(y, i), \hat{B}_1^+(z, w), \hat{B}_2^-(z, w)],
\]

where the numbers over the operators stand for order of their action and the functional $A[b_1^+(\cdot), b_1^-(\cdot), b_2^+(\cdot), b_2^-(\cdot), B^+(\cdot), B^-(\cdot)]$ has the form

\[
A[b_1^+(\cdot), b_1^-(\cdot), b_2^+(\cdot), b_2^-(\cdot), B^+(\cdot), B^-(\cdot)] = \text{Sp} \left( \hat{A}\hat{\rho}[b_1^+(\cdot), b_1^-(\cdot), b_2^+(\cdot), b_2^-(\cdot), B^+(\cdot), B^-(\cdot)] \right) \times
\]

26
\[
\times \exp \left( -\sum_{j=1}^{\infty} \int dx b_1^*(x, j) b_1(x, j) - \sum_{i=1}^{\infty} \int dy b_2^*(y, i) b_2(y, i) \right),
\]

where \( \hat{\rho}[b_1^*(\cdot), b_1(\cdot), b_2^*(\cdot), b_2(\cdot), B^*(\cdot), B(\cdot)] \) is an operator valued functional ranging in the set of operators in the space \( \mathcal{H} \) and having the form

\[
\hat{\rho}[b_1^*(\cdot), b_1(\cdot), b_2^*(\cdot), b_2(\cdot), B^*(\cdot), B(\cdot)] = \\
\exp \left( \frac{1}{2} \int \int dz_1 dw_1 B(z_1, w_1) \frac{2}{\psi_1^+ (z_1) \psi_2^+ (w_1)} \right) \times \\
\exp \left( \frac{1}{2} \int \int dz_2 dw_2 B^*(z_2, w_2) \frac{1}{\psi_1^- (z_2) \psi_2^- (w_2)} \right) \times \\
\exp \left( \sum_{j=1}^{\infty} \int \int dx_1 dx_2 b_1(x_1, j) b_1^*(x_2, j) \frac{2}{\psi_1^+(x_1) \psi_2^- (x_2)} \right) \times \\
\exp \left( \sum_{i=1}^{\infty} \int \int dy_1 dy_2 b_2(y_1, i) b_2^*(y_2, i) \frac{2}{\psi_2^+(y_1) \psi_2^- (y_2)} \right) \times \\
\exp \left( -\int dx \frac{2}{\psi_1^- (x)} \frac{1}{\psi_1^+(x)} - \int dy \frac{2}{\psi_2^- (y)} \frac{1}{\psi_2^+(y)} \right).
\]

To each element \( \Phi \) of the boson subspace \( \mathcal{F}^B \) of the space \( \mathcal{F} \), we bijectively assign the set \( \{\phi(j_1, \ldots, j_{q_1}, i_1, \ldots, i_{q_2})\} \), \( q_1, q_2 = 0, 1, \ldots \), with values in the space \( \mathcal{H} \) of the functions of the numbers

\[
(\phi(j_1, \ldots, j_{q_1}, i_1, \ldots, i_{q_2}))_{n_1,n_2}(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}) = \\
= \delta_{q_1-n_1, q_2-n_2} \theta(n_1 - q_1) \Phi_{j_1, j_2, n_1-q_1}(x_1, j_1, \ldots, x_{q_1}, j_{q_1}, y_1, i_1, \ldots, y_{q_2}, i_{q_2}, \ldots),
\]

where \( \theta(\cdot) \) is the Heaviside theta function. Hence, to the secondary quantized operator \( \hat{A} \), there naturally corresponds an operator \( \tilde{A} \) in the space \( \mathcal{F}^B \) such that

\[
(\tilde{A}\Phi)_{q_1,q_2,m}(x_1, j_1, \ldots, x_{q_1}, j_{q_1}, y_1, i_1, \ldots, y_{q_2}, i_{q_2}, x_{q_1+1}, y_{q_2+1}, \ldots, x_{n_1}, y_{n_2}) = \\
= (A\phi(j_1, \ldots, j_{q_1}, i_1, \ldots, i_{q_2}))_{q_1+m,q_2+m}(x_1, \ldots, x_{q_1+m}, y_1, \ldots, y_{q_2+m}).
\]

We have the following theorem.

**Theorem.** The projection of the supersecondary quantized operator \( \tilde{A} \) on the boson subspace is equal to \( \tilde{A} \).

We introduce an operator of supersecondary quantized entropy \( \tilde{S} \) for systems with preserved charge

\[
\tilde{S} = S_{reg}[b_1^2 (x, j), b_1^1 (x, j), b_2^2 (y, i), b_2^1 (y, i), B^+ (z, w), B^- (z, w)],
\]

(66)
where the functional $S_{reg}[b_1^*(\cdot), b_1(\cdot), b_2^*(\cdot), b_2(\cdot), B^*(\cdot), B(\cdot)]$ is equal to the regularized functional

$$S[b_1^*(\cdot), b_1(\cdot), b_2^*(\cdot), b_2(\cdot), B^*(\cdot), B(\cdot)] =$$

$$= \text{Sp} \left( \hat{\rho}[b_1^*(\cdot), b_1(\cdot), b_2^*(\cdot), b_2(\cdot), B^*(\cdot), B(\cdot)] \frac{\hat{\rho}[b_1^*(\cdot), b_1(\cdot), b_2^*(\cdot), b_2(\cdot), B^*(\cdot), B(\cdot)]}{\text{Sp} \hat{\rho}[b_1^*(\cdot), b_1(\cdot), b_2^*(\cdot), b_2(\cdot), B^*(\cdot), B(\cdot)]} \right) \times$$

$$\times \exp \left( -\sum_{j=1}^{\infty} \int dx \, b_1^*(x, j)b_1(x, j) - \sum_{i=1}^{\infty} \int dy \, b_2^*(y, i)b_2(y, i) \right).$$

(67)

References

[1] V.P. Maslov, O.Yu. Shvedov. // Phys. Rev. D V. 60. N105012 1999.

[2] I.A. Kvasnikov. Thermodynamics and Statistical Physics. Theory of Equilibrium Systems. Izd. Moskov. Univ. 1991. (in Russian).

[3] V.P. Maslov. // Teoret. Mat. Fiz. v. 125. N. 2. 297–314. 2000.

[4] F.A. Berezin. Secondary Quantization Method. Nauka. Moscow. 1986. (in Russian).

[5] V.P. Maslov. Operator Methods. Nauka. Moscow. 1973. (in Russian).

[6] V.P. Maslov. // Funktsional. Anal. i Prilozhen. v. 33. N 4. p. 50. 1999.

[7] V.P. Maslov. // Dokl. Ross. Akad. Nauk. 2000

[8] F. Dyson. // Phys. Rev., v. 90. p. 994. 1953.

[9] V.P. Maslov, O.Yu. Shvedov. Complex Germ Method. Moscow. Editorial URSS. 2000. (in Russian).