Möbius quantum walk

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Received 17 June 2017, revised 25 October 2017
Accepted for publication 2 November 2017
Published 22 November 2017

Abstract
By adding an extra Hilbert space to the Hadamard quantum walk on cycle (QWC), we present a new type of QWC, the Möbius quantum walk (MQW). The new space configuration enables the particle to rotate around the axis of movement. We define the factor $\alpha$ as the Möbius factor, which is the number of rotations per cycle. So, by $\alpha = 0$ we have a normal QWC, while $\alpha \neq 0$ defines a new type of QWC (namely the MQW). In particular, $\alpha = \frac{1}{2}$ defines a structure similar to the Möbius strip. We analytically investigate this new type of QWC and found that by tuning the parameter $\alpha$ we can reach uniform distribution for any number of nodes, while it is impossible for a normal QWC. The effects of $\alpha$ on the limiting distribution are investigated and an explicit formula for non-uniform cases is also derived.

Keywords: quantum walk, limiting distribution, Möbius, mixing time

(Some figures may appear in colour only in the online journal)

1. Introduction

Introduced by Aharonov et al [1], a quantum walk (QW) on graphs is a quantum counterpart of a classical random walk (CRW) on graphs. While in a CRW the particle moves with a certain probability, thanks to superposition in quantum mechanics, in a QW a particle can move in all directions simultaneously [2].

In fact a CRW is a dissipative model in which its dispersion (variance of probability distribution) goes as $t$ (time) [3], while a QW can be considered as a tight-binding model [4], so its dispersion goes as $t^2$. The quadratic behavior of QW variance is a direct consequence of the unitary evolution of coherent QWs, but, in practice, the isolation of a QW system from its environment is impossible. It has been shown that environmental effects and noises make the evolution non-unitary and decoherency occurs [4–6], So the variance of a QW ($t^2$) may transit to the classical one ($t$) [7, 8].

Generally there are two types of QW: a continuous-time QW [9] and a discrete-time QW [10].

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Different types of QWs have been studied, such as a 1D quantum walk [2], a two-dimensional quantum walk [11], a quantum walk on graphs [1] and hypercubes [12], and a variety of parameters have been studied, such as hitting time [13], mixing time [2], entanglement [14], decoherency [15], etc.

Aharonov et al [1] designed a new model of a QW known as the quantum walk on cycle (QWC), in which the discrete nodes can be supposed to be distributed on the circumference of a circle. They studied the mixing time and limiting distribution (LD) of the QWC and proved that for an odd number of nodes the LD is uniform, while for an even number of nodes it is not uniform. An explicit formula for the non-uniform LD of the QWC with an even number of nodes was driven by [16, 17]. The study of mixing time as an important aspect of a QW is still interesting and new parameters such as transient temperature and its connection with mixing time have been introduced [18].

We modified the QWC and added rotation ability to the quantum walker and defined the Möbius quantum walk in such a way that while the particle walks along the cycle it can rotate around the movement direction. We defined the parameter $\alpha$ to define the number of rotations per cycle and investigated its effects on the parameters of the QW. The explicit formula for LD was derived.

This paper is organized as follows. Section 2 provides a short introduction to the QWC and the LD of the QWC. We also highlight two situations in the QWC causing uniform and non-uniform distributions. In section 3 we introduce our model of the QW, namely the Möbius quantum walk (MQW) and define its evolution operator and solve its eigenproblem. Finally, in section 4, we find an explicit formula for the LD in degenerate (thus non-uniform) cases and discuss the role of the Möbius factor in suppressing degeneracy and making the LD uniform.

2. Background

QWs and CRWs are very similar, since there is a coin flip followed by a shift in position space [19], but quantum properties like superposition and interference [20] cause the quantum walk to behave completely different. In a one-dimensional discrete-time QW there are two Hilbert spaces known as coin $\mathcal{H}_C$ spanned by vectors $|s\rangle, s = 0, 1$ and position space $\mathcal{H}_P$ spanned by vectors $|j\rangle$ [2], so the state of walker is given by

$$|\psi_t\rangle = \sum_{s=0}^{1} \sum_{j=0}^{\infty} \Psi_{s,j,t} |s,j\rangle. \quad (1)$$

One step of the walk, $U = S(I \otimes U_C)$, consists of two operators: a coin operator $U_C$, which is a unitary $2 \times 2$ operator and makes a superposition in coin space, and a shift operator $S$, which moves the walker according to the state of the coin,

$$S = \sum_{s=0}^{1} \sum_{j=-\infty}^{+\infty} |s,j + (-1)^s \rangle \langle s,j|. \quad (2)$$

Different types of $U_C$ and $S$ introduce different types of QW, which have completely different properties. For example Ambainis et al [2] used a Hadamard coin operator

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (3)$$

and introduced a one-dimensional Hadamard walk and showed that it spreads linearly with the number of steps, while for a CRW it is $O(\sqrt{T})$ [2]. On the other hand, changing the shift
operator can also introduce new types of QW. For example defining $S$ on a circle with a finite number of nodes defines the QWC

$$
S = \sum_{x=0}^{N-1} \sum_{j=0}^{N-1} |s, (j + (-1)^x) \mod N \rangle \langle s, j|.
$$

(4)

In which $N$ is the number of nodes.

Although it is shown that the probability distribution

$$
p_t(v) = \sum_{r=0}^{N-1} |\langle s, v|\psi_t \rangle|^2
$$

(5)

for the QWC does not converge, but the LD $\pi(v)$ does [1], where

$$
\pi(v) = \lim_{T \to \infty} p_T(v)
$$

(6)

in which

$$
p_T(v) = \frac{1}{T} \sum_{t=1}^{T} p_t(v).
$$

(7)

The LD for an odd number of nodes is uniform, despite the form of the initial state and equals $\frac{1}{N}$, while for an even number of nodes the LD is non-uniform and its explicit formula is [17]

$$
\pi(v) = \frac{1}{N} + \left(\frac{-1}{2N^2}\right)^{\frac{N-1}{2}} \sum_{k=0}^{N-1} \frac{\cos(\frac{4\pi k}{N} v) - \cos(\frac{4\pi k}{N} (v + 1))}{1 + \cos^2(\frac{2\pi k}{N})}.
$$

(8)

An example of the LD of non-uniform cases has been plotted in figure 1.

3. Mōbius quantum walk

In this section we modify the QWC in such a way that the walker is able to rotate around the axis of movement while walking on the cycle. Seeking this goal, we added an extra Hilbert space $\mathcal{H}_r$ spanned by $\{|r\rangle | r = 0, 1\}$ to the Hilbert space of the QWC, so the state of the walker can be written as

$$
|\psi_t\rangle = \sum_{x=0}^{1} \sum_{r=0}^{N-1} \sum_{j=0}^{N-1} \Psi_{x,r,j} |s, r, j\rangle.
$$

(9)

We defined the conditional rotation operator as

$$
R_\theta |s\rangle |r\rangle = e^{-\frac{i}{\hbar} (\sigma_z \otimes \sigma_x)} |s\rangle |r\rangle,
$$

(10)

in which $\theta$ is the amount of rotation per step, $S_x = \frac{i}{2} \sigma_x$ is the spin operator along the axis $x$ and $\sigma_z$ and $\sigma_x$ are Pauli matrices. We can rewrite (10) as

$$
R_\theta |s\rangle |r\rangle = |s\rangle R(s) |r\rangle,
$$

(11)

with

$$
R(s) = \cos \left(\frac{\theta}{2}\right) I_R - i(-1)^y \sin \left(\frac{\theta}{2}\right) \sigma_y.
$$

(12)
in which $I_R$ is the identity operator in rotation space.

The conditional rotation operator $R_\theta$ rotates $|r\rangle$ along the axis $x$ according to the coin state $|s\rangle$. In other words, when the walker moves forward, $|r\rangle$ rotates counter-clockwise, and when it moves backward, $|r\rangle$ rotates clockwise. So this structure can be interpreted as a twisted path like a Möbius strip. That is why we call this model a MQW.

In fact, an example of a Möbius strip can be considered as a paper strip, giving it a half-twist, and then joining the ends of the strip to form a loop. If a walker walks along the length of this strip, it can visit both sides of the strip without crossing the edges. In fact, a Möbius strip is a surface (defined by a normal vector to the surface) with only one side and only one boundary [21]. In our model, $|r\rangle$ plays the role of the normal vector to the surface of a strip, so any node on the circumference of a circle not only has a specific index for its location but also a normal vector for the surface on which it is located.

By these modification, one step of a MQW can be defined as

$$U = S \left( H \otimes I_R \otimes I_F \right),$$

where the action of $S$ is

$$S|s, r, j\rangle = |s\rangle R(s)|r\rangle (j + (-1)^s) \mod N$$

and $H$ is a Hadamard operator. By using Fourier transform

$$|\kappa_k\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{\frac{2\pi i n k}{N}} |n\rangle$$

we can change the basis of the representation and simplify our calculations. Therefore matrix elements of $U$ in k-space are

$$\langle s, r, \kappa_k | U | s', r', \kappa_{k'} \rangle = \left( e^{\frac{2\pi i s}{N}} H_0, R(0) + e^{\frac{2\pi i s}{N}} H_1, R(1) \right) \delta_{k,k'}. \quad (16)$$

In fact $M$ is the evolution operator of the system in k-space. Therefore after $t$ steps of walking we need to know $M^t$. We can use spectral decomposition and write $M$ as
After some calculations, one can find the eigenvalues as
\[
\begin{align*}
\lambda_{0,0,k} & = -e^{i\delta}, \\
\lambda_{0,1,k} & = e^{-i\delta}, \\
\lambda_{1,0,k} & = -e^{i\sigma}, \\
\lambda_{1,1,k} & = e^{-i\sigma},
\end{align*}
\] (18)
and the corresponding eigenstates (see appendix).

4. Möbius quantum walk limiting distribution

If we assume the initial state of the walker in k-space as
\[
|\tilde{\psi}_0\rangle = \sum_{s=0}^{1} \sum_{r=0}^{1} \sum_{k=0}^{N-1} C_{s,r,k} |\lambda_{s,r,k}\rangle,
\] (19)
after \(t\) steps we will have
\[
|\tilde{\psi}_t\rangle = M^t|\tilde{\psi}_0\rangle = \sum_{s=0}^{1} \sum_{r=0}^{1} \sum_{k=0}^{N-1} C_{s,r,k} e^{i\lambda_{s,r,t}} |\lambda_{s,r,k}\rangle.
\] (20)

The probability distribution on QWCs does not converge, but the LD (7) as the asymptotic time average of the probability distribution in each node does [1]. By using (5), (7) and (20)
\[
\rho_T(v) = \sum_{b,s',r',s''=0}^{1} \sum_{\rho,r'=0}^{1} \sum_{k'}^{N-1} C_{s,r,k} C^*_{s',r',k'} \langle b, \rho, v | \lambda_{s,r,k} \rangle \langle \lambda_{s',r',k'} | b, \rho, v \rangle \times \frac{1}{T} \sum_{t=0}^{T-1} e^{i(\lambda_{s,r,k} - \lambda_{s',r',k'})t}.
\] (21)
In this equation only the last term is time-dependent. So for large \(T\), by some algebra one can show that [1]
\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} e^{i(\lambda_{s,r,k} - \lambda_{s',r',k'})t} = \begin{cases} 
0 & (\lambda_{s,r,k} \neq \lambda_{s',r',k'}) \\
1 & (\lambda_{s,r,k} = \lambda_{s',r',k'})
\end{cases}.
\] (22)

By putting (21) in (6) for \(T \to \infty\)
\[
\pi(v) = \sum_{s,s'=0}^{1} \sum_{r,r'=0}^{1} \sum_{k,r,k'}^{N-1} C_{s,r,k} C^*_{s',r',k'} \times \sum_{b=0}^{1} \sum_{\rho=0}^{1} \langle b, \rho, v | \lambda_{s,r,k} \rangle \langle \lambda_{s',r',k'} | b, \rho, v \rangle.
\] (23)
Eigenstates of \(|\lambda_{s,r,k}\rangle\) are in k-space so they are in the form of \(|\chi_{s,r,k}\rangle |\kappa\rangle\), so
\[
M = \sum_{s=0}^{1} \sum_{r=0}^{1} \sum_{k=0}^{N-1} e^{i\lambda_{s,r,t}} |\lambda_{s,r,k}\rangle \langle \lambda_{s,r,k}|.
\] (17)
\[ \pi(v) = \sum_{b,s,r,t=0}^{N-1} \sum_{\rho,\rho',r',t'=0}^{N-1} C_{s,r,k} C_{s',r',k'}^* \langle b, \rho | \chi_{s,r,k} \rangle \langle \chi_{s',r',k'} | b, \rho \rangle \langle v | \kappa \rangle \langle \kappa | v \rangle \]

\[ = \sum_{x,r=0}^{N-1} \sum_{x',r'=0}^{N-1} \sum_{\lambda_{x,r,k}, \lambda_{x',r',k'}}^{N-1} C_{s,r,k} C_{s',r',k'}^* \langle \chi_{x',r',k'} | \chi_{x,r,k} \rangle \langle \kappa | v \rangle \langle v | \kappa \rangle \]

(24)

All the eigenvalues \( \lambda_{x,k} \) are in the form of \( e^{i\phi} \), which can be represented by a point on the unit circle in a complex plane. So the degeneracy condition \( \lambda_{x,k} = \lambda_{x',k'} \) in (24) reduces to finding two different points on the same position in a complex plane. From (18) it is clear that \( \lambda_{0,1,k} \) and \( \lambda_{1,1,k} \) belong to quadrant I and IV (right zone) and \( \lambda_{0,0,k} \) and \( \lambda_{1,0,k} \) to quadrant II and III (left zone). Since only points from the same zone can occupy the same position, therefore the degeneracy conditions reduces to:

\[ \lambda_{0,1,k} = \lambda_{0,1,k'} \]
\[ \lambda_{1,1,k} = \lambda_{1,1,k'} \]
\[ \lambda_{0,0,k} = \lambda_{1,0,k} \]

(25)
for the right zone and
\[ \lambda_{0,0,k} = \lambda_{0,0,k'} \]
\[ \lambda_{1,0,k} = \lambda_{1,0,k'} \]
\[ \lambda_{0,0,k} = \lambda_{1,0,k'} \] (26)
for the left zone. We know \( \langle \lambda_{r,s,k} | \lambda_{r',s',k'} \rangle = \delta_{rr'} \delta_{ss'} \), so terms with \( r \neq r' \) and \( s \neq s' \) do not contribute in the summation (24), therefore the corresponding degeneracy conditions, i.e. \( \lambda_{0,0,k} = \lambda_{1,0,k'} \) and \( \lambda_{0,1,k} = \lambda_{1,1,k'} \) are not necessary to be considered. So by some calculation the whole degeneracy cases can be written as
\[ k = k' \]
\[ k = \frac{nN}{2} - k' \pm \alpha \quad n = 1, 3, \ldots \] (27)
Note that we use \( \alpha = \frac{N\theta}{2\pi} \) (called the Möbius factor) as the number of complete rotations per cycle.

The interesting result we would like to emphasize here is that the degeneracy condition can be controlled by the parameter \( \alpha \). So for \( \alpha \neq \frac{m}{2} \), \( m = 1, 2, \ldots \) the term \( k = \frac{nN}{2} - k' \pm \alpha \) never satisfies and we do not have degeneracy. So, all eigenvalues are distinct. Therefore, as discussed in [1], we have a uniform distribution \( \pi(v) = \frac{1}{N} \).

Furthermore, the parameter \( \alpha \) can be used to control the rate of convergence to the LD, which defines the mixing time. The mixing time, \( M_\varepsilon \), measures the number of time steps required for the average distribution to be \( \varepsilon \)-close to the LD [1],
\[ M_\varepsilon = \min \{ T \mid \forall t \geq T : \| \pi(v) - \bar{p}_t(v) \| \leq \varepsilon \} , \] (28)
which is one of the important differences between QWs and CRWs. In particular, Aharonov et al showed that odd-sided N-cycles (which have a uniform LD) converge to the LD in time \( O(n \log_2 n) \), almost quadratically faster than the classical walk. They proved that, for a general QW specified by the unitary matrix \( U \), and any initial state \( |\beta_0\rangle = \sum_i a_i |\phi_i\rangle \),
\[ \| \pi(v) - \bar{p}_t(v) \| \leq 2 \sum_{i,j,\lambda_i \neq \lambda_j} |a_i|^2 \frac{1}{T|\lambda_i - \lambda_j|} . \] (29)
Where \( |\phi_i\rangle \) and \( \lambda_i \) are the eigenvectors and corresponding eigenvalues of \( U \), respectively. As we shall see, the distances \( |\lambda_i - \lambda_j| \) are of crucial importance, and they need to be large for the convergence time to be small. An interesting point in our model is that the distances \( |\lambda_i - \lambda_j| \) can be controlled by \( \theta \) or equivalently \( \alpha \) (see (18)). This means that we are always able to tune \( \alpha \) to have uniform LD with optimized mixing time.

In the case \( \alpha = \frac{m}{2} \), \( m = 0, 1, 2, \ldots \) the term \( k = \frac{nN}{2} - k' \pm \alpha \) can be satisfied in two different situations and degeneracy happens. For odd \( N \), \( \alpha \) must be a half-integer, and for even \( N \), it must be an integer. These two situations lead to non-uniform LDs for which we derive an explicit formula in the next section.

5. Möbius quantum walk non-uniform limiting distribution

Applying degeneracy conditions (27) into (24) and using (15) leads to
\[ \pi(v) = \frac{1}{N} + \frac{1}{N} \text{Re} \left( \sum_{k=0}^{N-1} C_{0,0,k} C^{*}_{0,0,k} \langle \chi_{0,0,\frac{\pi}{2}+\alpha-k} | \chi_{0,0,k} \rangle e^{\frac{2\pi}{N} (2k-\alpha)} (-1)^v \right) \]

\[ \text{for } k - \frac{\alpha}{2} \neq \frac{N}{4}, \frac{3N}{4} \]

\[ + \text{Re} \left( \sum_{k=0}^{N-1} C_{0,1,k} C^{*}_{0,1,k} \langle \chi_{0,1,\frac{\pi}{2}+\alpha-k} | \chi_{0,1,k} \rangle e^{\frac{2\pi}{N} (2k-\alpha)} (-1)^v \right) \]

\[ \text{for } k + \frac{\alpha}{2} \neq \frac{N}{4}, \frac{3N}{4} \]

\[ + \text{Re} \left( \sum_{k=0}^{N-1} C_{1,0,k} C^{*}_{1,0,k} \langle \chi_{1,0,\frac{\pi}{2}-\alpha-k} | \chi_{1,0,k} \rangle e^{\frac{2\pi}{N} (2k-\alpha)} (-1)^v \right) \]

\[ \text{for } k + \frac{\alpha}{2} \neq \frac{N}{4}, \frac{3N}{4} \]

\[ + \text{Re} \left( \sum_{k=0}^{N-1} C_{1,1,k} C^{*}_{1,1,k} \langle \chi_{1,1,\frac{\pi}{2}-\alpha-k} | \chi_{1,1,k} \rangle e^{\frac{2\pi}{N} (2k-\alpha)} (-1)^v \right) \]

\[ \text{for } k + \frac{\alpha}{2} \neq \frac{N}{4}, \frac{3N}{4} \]

(30)

We should note that inserting \( k = k' \) leads to \( \frac{1}{N} \) and since swapping \( k \) and \( k' \) makes complex conjugate terms in the summation, we omit the summation over \( k' \) and use the real part instead. For \( N \) divisible by 4, the eigenvalues for \( k \pm \frac{\alpha}{2} = \frac{N}{4} \) and \( \frac{3N}{4} \) are non-degenerate and unique (see (18)), so we have omitted the corresponding terms in the summation.

By using the explicit form of \( \langle \chi_{x,y,k} | \chi_{x,y,k} \rangle \) (see appendix), calculations of \( C_{x,y,k} C^{*}_{x,y,\frac{\pi}{2}+\alpha-k} \) and \( \langle \chi_{x,y,\frac{\pi}{2}+\alpha-k} | \chi_{x,y,k} \rangle \) are straightforward. We should note that \( C_{x,y,k} = \langle \chi_{x,y,k} | \psi_0 \rangle \) depends on the initial state. For example, if we use \( |\psi_0\rangle = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \) as the initial state, the compact form of the LD (30) can be written as

\[ \pi(v) = \frac{1}{N} + \frac{1}{N} \text{Re} \left( \sum_{k=0}^{N-1} \cos \left( \frac{2\pi}{N} (2k-\alpha) \right) + \sum_{k=0}^{N-1} \cos \left( \frac{2\pi}{N} (2k+\alpha) \right) \right) \]

(31)

As we expected, for \( \alpha = 0 \) (31) reduces to (8) for the QWC. As Bednarska et al [16] showed and (8) represents, for a QWC with even \( N \) divisible by 4, we have two hills at \( p_0 \) and \( p_0 + \frac{N}{4} \) (figure 1). However, factor \( \alpha \) in (31) can change the scenario. For example for an odd \( N \) and \( \alpha = \frac{1}{2} \), we have a single hill at \( p_0 \) and for an even \( N \) and \( \alpha = 1, 2, \ldots \), depending on \( \alpha \) and division of \( N \) by 4, we have different forms of the LD (figures 2 and 3).
6. Conclusion

We introduced a new type of QWC, namely the MQW, in which we added an extra rotation space and defined $\alpha$, which can parameterize the number of complete rotations per cycle. We showed that by parameter $\alpha$ we are always able to break the degeneracy of the evolution operator, which leads to a uniform LD. In other words, it is always possible to reach uniform LD despite the number of nodes, which was impossible for an even number of nodes in QWCs.

We also showed that the rate of convergence to the LD can be controlled by the Möbius factor $\alpha$, which means that we are always able to tune $\alpha$ to have uniform LD with optimized mixing time. Our analysis shows that only for specific values of $\alpha$ the LD is non-uniform, for which the explicit form has been driven, showing that we have two hills at $p_0$ and $p_0 + \frac{N}{2}$ when both conditions $N \mod 4, \alpha = 2m$ are satisfied, or neither of them are satisfied. We have a hill at $p_0$ and a valley at $p_0 + \frac{N}{2}$ when only one of these conditions is true.

Appendix

By solving eigenproblem for the $M$ introduced in (16), we have

$$
\chi_{0,0,k} = \frac{1}{N_0} \begin{bmatrix} a_0 \\ -a_0 \\ b_0 \\ -b_0 \end{bmatrix}, \quad \chi_{0,1,k} = \frac{1}{N_1} \begin{bmatrix} a_1 \\ -a_1 \\ b_1 \\ -b_1 \end{bmatrix}, \\
\chi_{1,0,k} = \frac{1}{N_2} \begin{bmatrix} a_2 \\ b_2 \\ b_2 \\ a_2 \end{bmatrix}, \quad \chi_{1,1,k} = \frac{1}{N_3} \begin{bmatrix} a_3 \\ b_3 \\ b_3 \\ a_3 \end{bmatrix},
$$

(A.1)

where $N_i$ is the normalization factor and

$$
a_0 = \sqrt{2} \cos (\omega_-) - \frac{1}{\sqrt{2}} \sqrt{1 + \cos^2 (\omega_-)} + \frac{1}{\sqrt{2}} \sin (\omega_-), \\
a_1 = -\sqrt{2} \cos (\omega_-) - \frac{1}{\sqrt{2}} \sqrt{1 + \cos^2 (\omega_-)} - \frac{1}{\sqrt{2}} \sin (\omega_-), \\
a_2 = \sqrt{2} \cos (\omega_+) + \frac{1}{\sqrt{2}} \sqrt{1 + \cos^2 (\omega_+)} - \frac{1}{\sqrt{2}} \sin (\omega_+), \\
a_3 = \sqrt{2} \cos (\omega_+) - \frac{1}{\sqrt{2}} \sqrt{1 + \cos^2 (\omega_+)} - \frac{1}{\sqrt{2}} \sin (\omega_+), \\
b_0 = -\frac{1}{\sqrt{2}} \sqrt{1 + \cos^2 (\omega_-)} + \frac{1}{\sqrt{2}} \sin (\omega_-), \\
b_1 = \frac{1}{\sqrt{2}} \sqrt{1 + \cos^2 (\omega_-)} - \frac{1}{\sqrt{2}} \sin (\omega_-), \\
b_2 = \frac{1}{\sqrt{2}} \sqrt{1 + \cos^2 (\omega_+)} + \frac{1}{\sqrt{2}} \sin (\omega_+), \\
b_3 = -\frac{1}{\sqrt{2}} \sqrt{1 + \cos^2 (\omega_+)} + \frac{1}{\sqrt{2}} \sin (\omega_+),
$$

(A.2)

with $\omega_{\pm} = \frac{\pi a}{N} \pm \frac{2\pi k}{N}$.

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References

[1] Aharonov D, Ambainis A, Kempe J and Vazirani U 2001 Quantum walk on graphs STOC ’01 Proc. of the 33rd annual ACM Symp. on Theory of Computing pp 50–9
[2] Ambainis A, Bach E, Nayak A, Vishwanath A and Watrous J 2001 One-dimensional quantum walk Proc. STOC ’01 Proc. of the 33rd annual ACM Symp. on Theory of Computing pp 37–49
[3] Feller W 1968 An Introduction to Probability Theory and its Applications (New York: Wiley)
[4] Nizama M and Cáceres M O 2012 Non-equilibrium transition from dissipative quantum walk to classical random walk J. Phys. A: Math. Theor. 45 335303
[5] Kendon V and Tregenna B 2003 Decoherence can be useful in quantum walks Phys. Rev. A 67 042315
[6] Esposito M and Gaspard P 2005 Emergence of diffusion in finite quantum systems Phys. Rev. B 71 214302
[7] Brun T, Carteret H and Ambainis A 2003 Quantum to classical transition for random walks Phys. Rev. Lett. 91 130602
[8] Annabestani M, Akhtarshenas S J and Abolhassani M R 2016 Incoherent tunneling effects in a one-dimensional quantum walk J. Phys. A: Math. Theor. 49 115301
[9] Farhi E and Gutmann S 1998 Quantum computation and decision trees Phys. Rev. A 58 915
[10] Watrous J 2001 Quantum simulations of classical random walks and undirected graph connectivity J. Comput. Syst. Sci. 62 376–91
[11] Mackay T D, Bartlett S D, Stephenson L T and Sanders B C 2002 Quantum walks in higher dimensions J. Phys. A: Math. Gen. 35 2745
[12] Moore C and Russell A 2002 Quantum Walks on the Hypercube vol 841, ed J D P Rolim and S Vadhan (Berlin: Springer) (https://doi.org/10.1007/3-540-45726-7_14)
[13] Kempe J 2005 Discrete quantum walks hit exponentially faster Probab. Theory Relat. Fields 133 215–35
[14] Abal G, Siri R, Romanelli A and Donangelo R 2002 Quantum walk on the line: entanglement and non-local initial conditions Phys. Rev. A 73 042302
[15] Annabestani M, Akhtarshenas S J and Abolhassani M R 2010 Decoherence in a 1D quantum walk Phys. Rev. A 81 032321
[16] Bednarska M, Grudka A, Kurzynski P, Luczak T and Wojcik A 2003 Quantum walks on cycles Phys. Lett. A 317 21
[17] Portugal R 2013 Quantum Walks and Search Algorithms (Berlin: Springer) (https://doi.org/10.1007/978-1-4614-6336-8_3)
[18] Díaz N, Donangelo R, Portugal R and Romanelli A 2016 Transient temperature and mixing times of quantum walks on cycles Phys. Rev. A 94 012305
[19] Chandrashekar C M, Srikanth R and Laflamme R 2008 Optimizing the discrete time quantum walk using a su(2) coin Phys. Rev. 77 032326
[20] Chandrashekar C M 2009 Discrete-time quantum walks: dynamics and applications PhD Thesis University of Waterloo
[21] Pickover C A 2005 The Möbius Strip: Dr. August Mőbius’s Marvelous Band in Mathematics, Games, Literature, Art, Technology, and Cosmology (New York: Thunder’s Mouth Press)