Optimum Distance Flag Codes from Spreads in Network Coding

Clementa Alonso-González, Miguel Ángel Navarro-Pérez, Xaro Soler-Escrivà

Abstract

We study multishot codes in network coding given by families of flags on a vector space $\mathbb{F}_q^n$, being $q$ a prime power and $\mathbb{F}_q$ the finite field of $q$ elements. In particular, we characterize the type vector of flag codes that attain the maximum distance (optimum distance flag codes) having a spread as the subspace code sent at some shot and we also provide a construction of these codes. A maximum distance constant dimension code with the best possible size, whenever the dimension divides $n$, must be a spread. In this paper we show that optimum distance flag codes attaining the best possible size, given an admissible type vector, must have a spread as the subspace code used at the corresponding shot.

Keywords: Network coding, subspace codes, spreads, flag codes.

1 Introduction

Network coding is a part of coding theory awaking a lot of interest during the last years. Subspace codes were introduced for the first time in [11] as adequate error-correction codes in random network coding, that is, when the network is non-coherent. These codes consist of families of subspaces of a given $n$-dimensional vector space over a finite field $\mathbb{F}_q$ endowed with a specific distance. If the dimension of all the subspaces is fixed, we say that the code is a constant dimension code. Due to their good properties, spread codes or just spreads have become one of the most important families of constant dimension codes ([15, 16]). Spreads are objects widely studied in classical finite geometry without paying attention to their application to coding theory.

1Dpt. de Matemàtiques, Universitat d’Alacant, Sant Vicent del Raspeig, Ap. Correus 99, E – 03080 Alacant.
E-mail adresses: clementa.alonso@ua.es, miguelangel.np@ua.es, xaro.soler@ua.es.
In [11], the authors proposed a suitable single-source multicast network channel that is used only once, so that subspace codes can be considered as one-shot codes. The use of the channel more than once was suggested originally in [17] as an option when the field size $q$ and the packet size $n$ could not be increased. This gives rise to the so-called multishot codes or multishot constant dimension codes if the dimension at each shot is constant. In a multishot constant dimension code $C$, codewords consist of sequences of $r$ subspaces of $\mathbb{F}_q^n$ with respective fixed dimensions $t_1, \ldots, t_r$, sent in $r$ successive shots. In particular, if $0 < t_1 < \ldots < t_r < n$ and these subspaces are nested, we have flags on $\mathbb{F}_q^n$. In that case, we say that $C$ is a flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$. A channel model for flag codes in the setting of network coding was also described in [14].

In this paper we focus on the construction of flag codes of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ with the maximum possible distance (optimum distance flag codes) such that the subspace code sent at some shot is a $k$-spread, being $k$ a divisor of $n$. We show that this construction is not possible in general and we conclude that an admissible type vector $(t_1, \ldots, t_r)$ (for $k$) has to satisfy the following: $k \in \{t_1, \ldots, t_r\} \subseteq \{1, \ldots, k, n-k, \ldots, n-1\}$, that is, the dimension $k$ must appear in the type vector whereas no dimension between $k+1$ and $n-k-1$ is allowed. In [2], it was proved that optimum distance full flag codes (type $(1, \ldots, n-1)$) such that the subspace code sent at some shot is a $k$-spread can be constructed just if $n = 2k$ or $n = 3$ and $k = 1$. In these two situations any full type vector is admissible. For $n = 2k$, in [2] it is also described a concrete construction of optimum distance full flag codes from planar spreads using the fact that $k = n-k$. However, this equality does not hold for a general admissible type vector and we need to overcome the gap between dimensions $k$ and $n-k$. We propose a gradual construction starting with the type $(1, n-1)$ (this includes the type vector $(1, 2)$ for the case $n = 3, k = 1$), following with the type $(k, n-k)$, to finish with the full admissible type $(1, \ldots, k, n-k, \ldots, n-1)$. Our construction provides codes with the best possible size and the maximum distance for the given admissible type vector. Furthermore, we prove that optimum distance flag codes attain the maximum possible size for a fixed admissible type vector if, and only if, they have a spread as the subspace code used at the corresponding shot.

The paper is organized as follows: In Section 2 we recall some background on finite fields, constant dimension codes, flag codes and graphs. In Section 3 we characterize first the admissible type vectors to have a flag code with the maximum possible distance and a spread as the subspace code used at some shot. Then, we undertake the construction of our codes in several stages: we consider first the type $(1, n-1)$ and construct optimum distance flag codes from the spread of lines. Then, using the field reduction map, we translate the previous construction to the type $(k, n-k)$. Finally, by taking advantage of some properties satisfied by the mentioned map, we finish with the full admissible type $(1, \ldots, k, n-k, \ldots, n-1)$ and any other admissible type vector. We show that our codes have the best size for the given admissible type vector and the associated
maximum distance. Furthermore, we prove that optimum distance flag codes attain the maximum possible size for a fixed admissible type vector if, and only if, the subspace code used at the corresponding shot is a spread. We complete this section with an example of our construction to obtain an optimum distance flag code of type \((2, 4)\) on \(\mathbb{F}_2^6\) having a 2-spread as the subspace code used at the first shot.

\section{Preliminaries}

We devote this section to recall some background we will need along this paper. This background involves finite fields, subspace and flag codes and graph theory.

\subsection{Results on finite fields}

Most of the following definitions and results about finite fields as well as the corresponding proofs can be found in [13].

Let \(q\) be a prime power and \(\mathbb{F}_q\) the finite field with \(q\) elements. Consider \(f(x) \in \mathbb{F}_q[x]\) a monic irreducible polynomial of degree \(k\) and \(\alpha \in \mathbb{F}_{q^k}\) a root of \(f(x)\). Then we have that \(\mathbb{F}_{q^k} \cong \mathbb{F}_q[\alpha]\), which allows us to realize the field \(\mathbb{F}_{q^k}\) as \(\mathbb{F}_q[\alpha]\). If \(f(x) = x^k + \sum_{i=0}^{k-1} a_i x^i \in \mathbb{F}_q[x]\), the following square matrix

\[
P = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_0 & -a_1 & -a_2 & \cdots & -a_{k-1}
\end{pmatrix}
\]

is called the companion matrix of \(f(x)\) and satisfies that \(\mathbb{F}_q[\alpha] \cong \mathbb{F}_q[P]\). Then, \(\mathbb{F}_q[P]\) is a field with \(q^k\) elements. We also have the natural field isomorphism

\[
\phi : \mathbb{F}_{q^k} \to \mathbb{F}_q[P], \quad \sum_{i=0}^{k-1} v_i \alpha^i \mapsto \sum_{i=0}^{k-1} v_i P^i.
\] (1)

For any positive integer \(n\), we denote by \(\mathcal{P}_q(n)\) the projective geometry of \(\mathbb{F}_q^n\), that is, the set of all vector subspaces of \(\mathbb{F}_q^n\). The Grassmann variety \(\mathcal{G}_q(k, n)\) is the set of all \(k\)-dimensional subspaces of \(\mathbb{F}_q^n\). Any subspace \(U \in \mathcal{G}_q(k, n)\) can be generated by the rows of some full-rank matrix \(U \in \mathbb{F}_q^{k \times n}\). In that case, we write \(U = \text{rowsp}(U)\) and say that \(U\) is a generator matrix of \(U\). By taking the generator matrix in reduced row echelon form (RREF) we get uniqueness in the matrix representation of the subspace \(U\).

Let us take \(n = ks\) with \(k > 1\). The field isomorphism \(\phi\) provided by (1), in turn, naturally induces a map \(\varphi\) between \(\mathcal{P}_{q^k}(s)\) and \(\mathcal{P}_q(ks)\) given by:
This map is known as field reduction since it maps subspaces over \( \mathbb{F}_{q^k} \) into subspaces over the subfield \( \mathbb{F}_q \) (see \([6, 12, 15, 16]\)). Let us recall some useful properties of the map \( \varphi \) pointed out in \([12]\) that we will use in Section 3.2.2.

**Proposition 2.1.** The map \( \varphi \) defined in (2) satisfies the following:

1. \( \varphi \) is injective.
2. For any pair of subspaces \( U, V \) of \( \mathbb{F}_{q^k} \), we have \( \varphi(U \cap V) = \varphi(U) \cap \varphi(V) \).
3. Given \( U, V \) subspaces of \( \mathbb{F}_{q^k} \) with \( U \subseteq V \), then \( \varphi(U) \subseteq \varphi(V) \).
4. For any \( m \in \{1, \ldots, s - 1\} \), it holds that \( \varphi(\mathcal{G}_{q^k}(m, s)) \subseteq \mathcal{G}_q(mk, sk) \).

### 2.2 Constant dimension codes

The Grassmannian \( \mathcal{G}_q(k, n) \) can be considered as a metric space with the subspace distance defined as

\[
d_S(U, V) = \dim(U + V) - \dim(U \cap V) = 2(k - \dim(U \cap V)),
\]

for all \( U, V \in \mathcal{G}_q(k, n) \) (see \([11]\)).

A constant dimension (subspace) code of dimension \( k \) and length \( n \) is any non-empty subset \( C \subseteq \mathcal{G}_q(k, n) \). The minimum subspace distance of the code \( C \) is defined as

\[
d_S(C) = \min\{d_S(U, V) \mid U, V \in C, U \neq V\}
\]

(see \([19]\) and references therein, for instance). It follows that the minimum distance of a constant dimension code \( C \) is upper bounded by

\[
d_S(C) \leq \begin{cases} 2k & \text{if } 2k \leq n, \\ 2(n-k) & \text{if } 2k > n. \end{cases}
\]

Constant dimension codes \( C \subseteq \mathcal{G}_q(k, n) \) in which the distance between any pair of different codewords is \( d_S(C) \) are said to be equidistant. For such codes, there exists some value \( c < k \) such that, given two different subspaces \( U, V \in C \), it holds that \( \dim(U \cap V) = c \). Hence, the minimum distance of the code is precisely \( d_S(C) = 2(k - c) \), and \( C \) is also called an equidistant \( c \)-intersecting constant
dimension code. In case the value $c$ is the minimum possible dimension of the intersection between $k$-dimensional subspaces of $\mathbb{F}_q^n$, that is,

$$c = \begin{cases} 
0 & \text{if } 2k \leq n, \\
2k - n & \text{if } 2k > n,
\end{cases}$$

equidistant $c$-intersecting codes attain the bound given in (4) and are called \textit{maximum distance constant dimension codes}. In particular, for dimensions $k \leq \left\lfloor \frac{n}{2} \right\rfloor$, we have that these codes are 0-intersecting codes known as \textit{partial spreads}. In [7], it was proved that the cardinality of any partial spread $C$ in $\mathcal{G}_q(k, n)$ is upper bounded by

$$|C| \leq \left\lfloor \frac{q^n - 1}{q^k - 1} \right\rfloor.$$  \hfill (5)

Whenever $k$ divides $n$, this bound is attained by the so-called \textit{spread codes} (or \textit{k-spreads}) of $\mathbb{F}_q^n$. Notice that a $k$-spread $\mathcal{S}$ is a subset of $\mathcal{G}_q(k, n)$ whose elements give a partition of $\mathbb{F}_q^n$. For further information related to spread codes, we refer the reader to [6, 15, 16, 19].

The following construction of spreads is given in [15] and will be widely used later on. Denote by $GL_k(q)$ the general linear group of degree $k$ over the field $\mathbb{F}_q$. Let $P \in GL_k(q)$ be the companion matrix of a monic irreducible polynomial in $\mathbb{F}_q[x]$. We will write $I_k$ and $0_k$ to denote the identity matrix and the zero matrix of size $k \times k$, respectively. Take $s \in \mathbb{N}$ such that $n = sk$. Then, the following family of $k$-dimensional subspaces is a spread code:

$$\mathcal{S}(s, k, P) = \{\text{rowsp}(S) \mid S \in \Sigma \} \subseteq \mathcal{G}_q(k, n),$$  \hfill (6)

where $\Sigma$ is the set of $k \times ks$ matrices

$$\Sigma = \{(A_1|A_2|\ldots|A_s) \mid A_i \in \mathbb{F}_q[P]\}$$  \hfill (7)

with the first non-zero block from the left equal to $I_k$.

**Remark 2.2.** Notice that the matrices in $\Sigma$ are in reduced row echelon form and it is clear that the field reduction map $\varphi$ defined in (2) gives a bijection between the Grassmannian of lines $\mathcal{G}_q^k(1, s)$ and the spread code $\mathcal{S}(s, k, P)$

$$\varphi\big|_{\mathcal{G}_q^k(1, s)} : \mathcal{G}_q^k(1, s) \longrightarrow \mathcal{S}(s, k, P)$$

$$\text{rowsp} \left( x_{11}, \ldots, x_{1s} \right) \overset{\text{rowsp} \left( \phi(x_{11})|\ldots|\phi(x_{1s}) \right)}{\longrightarrow} \text{rowsp} \left( \phi(x_{11})|\ldots|\phi(x_{1s}) \right).$$  \hfill (8)

We will come back to this fact in Section 3.2.2.

Given a constant dimension code $\mathcal{C} \subseteq \mathcal{G}_q(k, n)$, the \textit{dual code} of $\mathcal{C}$ is the subset of $\mathcal{G}_q(n - k, n)$ given by

$$\mathcal{C}^\perp = \{U^\perp \mid U \in \mathcal{C}\},$$
where $\mathcal{U}^\perp$ is the orthogonal of $\mathcal{U}$ with respect to the usual inner product in $\mathbb{F}_q^n$. In [11], it was proved that $\mathcal{C}$ and $\mathcal{C}^\perp$ have both the same cardinality and minimum distance. Notice that the dual of a partial spread of dimension $k \leq \lfloor \frac{n}{2} \rfloor$ is an equidistant $(n - 2k)$-intersecting code of dimension $n - k$ and conversely.

2.3 Flag codes

Subspace codes were introduced for the first time in [11] as error-correction codes in random network coding. In that paper, the authors propose a suitable network channel with a single transmitter and several receivers that is used just once, so that subspace codes can be considered as one-shot codes. The use of the channel more than once was suggested originally in [17] and gives rise to the so-called multi-shot codes as a generalization of subspace codes. We call multishot code of length $r \geq 1$, or $r$-shot code, to any non-empty subset $\mathcal{C} \subseteq \mathcal{P}_q(n)^r$. In particular, if codewords in $\mathcal{C}$ are sequences of nested subspaces, we say that $\mathcal{C}$ is a flag code.

Flag codes were also studied as orbits of group actions in [14]. Let us recall some concepts in the setting of flag codes.

A flag of type $(t_1, \ldots, t_r)$, with $0 < t_1 < \cdots < t_r < n$, on the vector space $\mathbb{F}_q^n$ is a sequence of subspaces $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ in $\mathcal{G}_q(t_1, n) \times \cdots \times \mathcal{G}_q(t_r, n) \subseteq \mathcal{P}_q(n)^r$ such that

$$\{0\} \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_r \subsetneq \mathbb{F}_q^n.$$  

With this notation, $\mathcal{F}_i$ is said to be the $i$-th subspace of $\mathcal{F}$. In case the type vector is $(1, 2, \ldots, n - 1)$, we say that $\mathcal{F}$ is a full flag.

The space of flags of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is denoted by $\mathcal{F}_q((t_1, \ldots, t_r), n)$ and can be endowed with the flag distance $d_f$ that naturally extends the subspace distance defined in (3): given two flags $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r)$ and $\mathcal{F}' = (\mathcal{F}'_1, \ldots, \mathcal{F}'_r)$ in $\mathcal{F}_q((t_1, \ldots, t_r), n)$, the flag distance between them is

$$d_f(\mathcal{F}, \mathcal{F}') = \sum_{i=1}^{r} d_S(\mathcal{F}_i, \mathcal{F}'_i).$$

A flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is defined as any non-empty subset $\mathcal{C} \subseteq \mathcal{F}_q((t_1, \ldots, t_r), n)$. The minimum distance of a flag code $\mathcal{C}$ of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ is given by

$$d_f(\mathcal{C}) = \min\{d_f(\mathcal{F}, \mathcal{F}') \mid \mathcal{F}, \mathcal{F}' \in \mathcal{C}, \mathcal{F} \neq \mathcal{F}'\}.$$  

Given a type vector $(t_1, \ldots, t_r)$, for every $i = 1, \ldots, r$, we define the $i$-projection to be the map

$$p_i : \mathcal{F}_q((t_1, \ldots, t_r), n) \rightarrow \mathcal{G}_q(t_i, n)$$

$$\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_r) \mapsto p_i(\mathcal{F}) = \mathcal{F}_i.$$  

(9)
The *i-projected code* of \( C \) is the set \( C_i = \{ p_i(F) \mid F \in C \} \). By definition, this code \( C_i \) is a constant dimension code in the Grassmannian \( G_q(t_i, n) \) and its cardinality satisfies \( |C_i| \leq |C| \). We say that \( C \) is a *disjoint* flag code if \( |C_1| = \cdots = |C_r| = |C| \), that is, the \( i \)-projection \( p_i \) is injective for any \( i \in \{1, \ldots, r\} \).

The distance of a flag code \( C \) of type \( (t_1, \ldots, t_r) \) is upper bounded by

\[
d_f(C) \leq 2 \left( \sum_{t_i \leq \lfloor \frac{n}{2} \rfloor} t_i + \sum_{t_i > \lfloor \frac{n}{2} \rfloor} (n - t_i) \right).
\]

In particular, if \( C \) is a full flag code, we have that (10) becomes

\[
d_f(C) \leq \begin{cases} 
\frac{n^2}{2}, & \text{for } n \text{ even}, \\
\frac{n^2 - 1}{2}, & \text{for } n \text{ odd}.
\end{cases}
\]

### 2.4 Matchings in graphs

Now we introduce some basic concepts and results on graphs in order to use them in the construction of a specific family of flag codes with the maximum distance in Section 3. All these definitions and results together with their proofs can be found in [3].

A graph \( G = (V, E) \) consists of a *vertex set* \( V \) and an *edge set* \( E \subset V \times V \) where an edge is an unordered pair of vertices. Two vertices \( v, v' \in V \) are *adjacent* if \( (v, v') \in E \). Also, we say that \( (v, v') \) is an *incident* edge with \( v \) and \( v' \). Two edges are *adjacent* if they have a common vertex. Given a vertex \( v \in V \) we call the *degree* of \( v \) to the number of incident edges with \( v \). A graph \( G \) is said to be *k-regular*, if each vertex in \( G \) has degree \( k \).

On the other hand, a set of vertices (or edges) is *independent* if it does not contain adjacent elements. A set \( M \subseteq E \) of independent edges of a graph \( G = (V, E) \) is called a *matching*. A matching \( M \) matches \( S \subseteq V \) if every vertex in \( S \) is incident with an edge in \( M \) and \( M \) is *perfect* if it matches \( V \).

A graph \( G \) is *bipartite* if the vertex set can be partitioned into two sets \( V = A \cup B \) such that there is no pair of adjacent vertices neither in \( A \) nor in \( B \). For this class of graphs, perfect matchings are just bijections between \( A \) and \( B \) given by a subset of edges of the graph connecting each vertex in \( A \) with another vertex in \( B \). The following classic result whose proof can be found in [3] (pages 37 – 38) states the existence of perfect matchings in a family of graphs:

**Theorem 2.3.** Any \( k \)-regular bipartite graph admits a perfect matching.

This theorem will be used through Section 3 to give perfect matchings of a particular regular bipartite graph of our interest. Such matchings will allow us to construct disjoint flag codes of a specific type as we will show later.
3 Optimum distance flag codes from spreads

Flag codes attaining the bound in (10) are called optimum distance flag codes and can be characterized in terms of their projected codes in the following way:

**Theorem 3.1.** (see [2]) Let $C$ be a flag code of type $(t_1, \ldots, t_r)$. The following statements are equivalent:

(i) $C$ is an optimum distance flag code.

(ii) $C$ is disjoint and every projected code $C_i$ attains the maximum possible subspace distance.

As a consequence, the $i$-projected codes of an optimum distance flag code have to be partial spreads if $t_i \leq \lfloor \frac{n}{2} \rfloor$ and equidistant $(2t_i - n)$-intersecting subspace codes for dimensions $t_i > \lfloor \frac{n}{2} \rfloor$.

As mentioned in Section 2.2, whenever $k$ divides $n$, $k$-spread codes are partial spread codes (maximum distance constant dimension codes) with the best size. This good property of spreads naturally gives rise to the question of finding optimum distance flag codes having a spread as their $i$-projected code when the dimension $t_i$ is a divisor of $n$. Note that, due to the disjointness property, we could have at most one spread among the projected codes. In [2] it was proved that, if the type vector is $(1, \ldots, n - 1)$, it is possible to find optimum distance full flag codes having a spread as a $k$-projected code only when $n = 2k$ or $n = 3$ and $k = 1$. However, out of the family of full flag codes, not all the type vectors are allowed. Let us describe the admissible ones and provide a construction of optimum distance flag codes for them.

3.1 Admissible type vectors

This paper is devoted to explore the existence of optimum distance flag codes of a general type vector $(t_1, \ldots, t_r)$, not necessarily the full type, having a spread as their $i$-projected code when $t_i$ is a divisor of $n$. Next result states the necessary conditions that the type vector $(t_1, \ldots, t_r)$ must satisfy.

**Theorem 3.2.** Let $C$ be an optimum distance flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$. Assume that some dimension $t_i = k$ divides $n$ and the associated projected code $C_i$ is a $k$-spread. Then, for each $j \in \{1, \ldots, r\}$, either $t_j \leq k$ or $t_j \geq n - k$.

**Proof.** Notice that in case $i = r$, clearly $t_j \leq t_r = k$, for every $j = 1, \ldots, r$. Suppose that $i < r$. Let us show that $t_{i+1} \geq n - k$.

Since $t_i = k$ divides $n$, we can write $n = sk$, for some $s \geq 2$. If $s = 2$, we have that $n - k = k$ and the result trivially holds. In case $s > 2$, then $s < 2(s - 1)$ and we have that $2k < n < 2(s - 1)k = 2(n - k)$. We deduce that $k \leq \lfloor \frac{n}{2} \rfloor < n - k$. Now, by contradiction, assume that $t_{i+1} < n-k$. We distinguish two possibilities:
(1) If \( k < t_{i+1} \leq \lfloor \frac{n}{2} \rfloor \), since \( C \) is an optimum distance flag code, by Theorem 3.1, its projected code \( C_{i+1} \) must be a partial spread of dimension \( t_{i+1} \) and cardinality \( |C_{i+1}| = |C_i| = \frac{q^n - 1}{q^k - 1} \). Contradiction with (5).

(2) If \( \lfloor \frac{n}{2} \rfloor < t_{i+1} < n - k \), the projected code \( C_{i+1} \) has to be an equidistant \((2t_{i+1} - n)\)-intersecting constant dimension code. In other words, the subspace distance of \( C_{i+1} \) is \( 2(n - t_{i+1}) \). Hence, its dual code \( C_{i+1}^\perp \) is a partial spread of dimension \( n - t_{i+1} > n - k > k \) and cardinality \( |C_{i+1}| = |C_i| = \frac{q^n - 1}{q^k - 1} \), which again contradicts (5).

We conclude that \( t_{i+1} \geq n - k \).

**Remark 3.3.** This result provides a necessary condition over the type vector of any optimum distance flag code on \( \mathbb{F}_q^n \) if we require the condition of having a \( k \)-spread as a projected code: clearly the dimension \( k \) must appear on the type vector but no dimension between \( k+1 \) and \( n-k+1 \) can be part of it. Notice that in case \( n = 2k \), any type vector containing the dimension \( k \) is admissible since \( k = n - k \). Moreover, in that case, it was proved in [2] that optimum distance flag codes of any type vector containing the dimension \( k \) can be constructed from a \( k \)-spread (planar spread).

### 3.2 Our codes construction

This part is devoted to describe a specific construction of optimum distance flag codes on \( \mathbb{F}_q^n \) from a \( k \)-spread of a given admissible type vector \((t_1, \ldots, t_r)\). By means of Theorem 3.2, if such codes exist, their type vector must satisfy \( k \in \{t_1, \ldots, t_r\} \subseteq \{1, \ldots, k, n-k, \ldots, n-1\} \). For the sake of simplicity, we undertake this construction in several phases: we consider first the admissible type vector \((1, n-1)\), that is, the construction of optimum distance flag codes from the spread of lines. Secondly, by using the field reduction map defined in Section 2.1, we properly translate the construction in the first step to get optimum distance flag codes of type vector \((k, n-k)\) having the \( k \)-spread \( S \) introduced in (6) as its first projected code. Then, taking advantage of certain properties of the \( k \)-spread \( S \), we extend the construction in the second step to obtain optimum distance flag codes of the **full admissible type**, that is, \( \{1, \ldots, k, n-k, \ldots, n-1\} \). Finally, this last construction gives optimum distance flag codes of any admissible type vector after a suitable **puncturing process**. Let us explain in detail all these stages.

#### 3.2.1 The type vector \((1, n-1)\): starting from the spread of lines

Take \( n \geq 3 \). In this section we provide a construction of optimum distance flag codes on \( \mathbb{F}_q^n \) from the spread of lines, that is, having the Grassmannian \( G_q(1, n) \) as a projected code. By Theorem 3.2, the only admissible type vector in this
case is $(1, n-1)$. In other words, to give an optimum distance flag code from the spread of lines of $\mathbb{F}^n_q$, we have to provide a family of $|G_q(1, n)|$ pairwise disjoint flags of length two, all of them consisting of a line contained in a hyperplane. To do so, we translate this problem into a problem of matching in graphs and make use of the results given in Section 2.4. Let us precise this.

Consider the graph $G = (V, E)$, with set of vertices $V = G_q(1, n) \cup G_q(n-1, n)$ and set of edges $E$ defined by

$$E = \{(l, H) \in G_q(1, n) \times G_q(n-1, n) \mid l \subseteq H\}.$$  

Notice that the set of vertices in $G$ consists of the lines and hyperplanes of $\mathbb{F}^n_q$. An edge $(l, H)$ of $G$ exists if, and only if, the line $l$ is contained in the hyperplane $H$. With this notation, next result holds.

**Proposition 3.4.** The graph $G = (V, E)$ is bipartite and $\frac{q^{(n-1)}-1}{q-1}$-regular.

**Proof.** It is clear that $G$ is a bipartite graph by definition. Moreover, the number of hyperplanes containing a fixed line coincides with the number of lines lying on a given hyperplane. This number is precisely $\frac{q^{(n-1)}-1}{q-1}$. Then, the degree of any vertex in $G$ coincides with this value and then $G$ is $\frac{q^{(n-1)}-1}{q-1}$-regular.

Note that the problem of giving a family of flags with the desired conditions can be seen as the problem of giving a perfect matching in $G$. Since $G$ is a regular bipartite graph, we can use Theorem 2.3 to conclude that there exist perfect matchings in $G$. More precisely, there exists a subset $M \subseteq E$ that matches $V$, that is, each edge in $M$ has an extremity in $G_q(1, n)$ and the other one in $G_q(n-1, n)$. In particular, the set $M$ has a number of edges equal to $|G_q(1, n)|$. This matching $M$ induces naturally a bijection, also denoted by $M$, between the set of lines and the set of hyperplanes in $\mathbb{F}^n_q$. Moreover, by the definition of $E$, we have that the map $M : G_q(1, n) \rightarrow G_q(n-1, n)$ satisfies that $l \subseteq M(l)$ for any $l \in G_q(1, n)$. This fact allows us to construct a family of flags of type $(1, n-1)$ on $\mathbb{F}^n_q$ in the following way:

$$\tilde{C} = \tilde{C}_M = \{(l, M(l)) \mid l \in G_q(1, n)\}. \quad (11)$$

Let us see that the family $\tilde{C}$ is a flag code with projected codes $\tilde{C}_1 = G_q(1, n)$ and $\tilde{C}_2 = G_q(n-1, n)$ satisfying the desired conditions.

**Theorem 3.5.** Given $n \geq 3$, the code $\tilde{C}$ defined in (11) is an optimum distance flag code of type $(1, n-1)$ on $\mathbb{F}^n_q$ with the spread of lines as a projected code.

**Proof.** Since the map $M$ defined above is bijective, the code $\tilde{C}$ must be a disjoint flag code with projected codes $\tilde{C}_1 = G_q(1, n)$ and $\tilde{C}_2 = G_q(n-1, n)$. In particular, as $d_S(\tilde{C}_1) = d_S(\tilde{C}_2) = 2$ is the maximum possible distance for constant dimension codes of dimension 1 and $n-1$ in $\mathbb{F}^n_q$. By Theorem 3.1, we have that $\tilde{C}$ is an optimum distance flag code with $G_q(1, n)$ as a projected code.
Remark 3.6. Observe that, since an optimum distance flag code of type \((1, n - 1)\) on \(\mathbb{F}_q^n\) is disjoint, its cardinality is at most \(|\mathcal{G}_q(1, n)| = q^{n-1} + q^{n-2} + \ldots + q + 1\).

Hence, our code \(\tilde{C}\) defined as above, attains the maximum possible cardinality for flag codes of type \((1, n - 1)\) and distance 4. In the case \(n = 3\), the previous construction gives optimum distance full flag codes from a non-planar spread. Apart from this particular case, optimum distance full flag codes with a \(k\)-spread as a projected code can just be constructed on \(\mathbb{F}_q^{2k}\), as it was proved in [2].

Note that, despite the fact that Theorem 2.3 guarantees the existence of perfect matchings in regular bipartite graphs, in order to provide an example of optimum distance flag codes of type \((1, n - 1)\) on \(\mathbb{F}_q^n\), we need to exhibit a precise matching in \(G\). The reader can find an algorithm to construct perfect matchings in the general setting of regular bipartite graphs in [3] and [5]. In Section 3.3 we give an example of a flag code constructed from a perfect matching generated by an algorithm programmed in GAP and adapting the procedure described in [3].

3.2.2 The type vector \((k, n - k)\)

Take \(n = ks\) a natural number with \(k \geq 2\) and \(s \geq 3\). In order to construct optimum distance flag codes of type \((k, n - k)\) on \(\mathbb{F}_q^n\), we will use the construction of optimum distance flag codes of type \((1, s - 1)\) on \(\mathbb{F}_q^s\) given in Section 3.2.1 together with the field reduction map defined in Section 2.1. Let us explain this construction.

Let \(M : \mathcal{G}_q^k(1, s) \rightarrow \mathcal{G}_q^k(s-1, s)\) be a bijection such that \(l \subset M(l)\) for any \(l \in \mathcal{G}_q^k(1, s)\). By Theorem 3.5 we know that the code

\[
\hat{C} = \tilde{C}_M = \{(l, M(l)) \mid l \in \mathcal{G}_q^k(1, s)\}
\]

is an optimum distance flag of type \((1, s - 1)\) on \(\mathbb{F}_q^s\). In particular, the code \(\hat{C}\) is disjoint. On the other hand, given \(P \in GL_k(q)\) the companion matrix of a monic irreducible polynomial of degree \(k\) in \(\mathbb{F}_q[x]\), the associated field isomorphism \(\phi : \mathbb{F}_q^k \rightarrow \mathbb{F}_q^k[P]\) induces the field reduction \(\varphi : \mathcal{P}_q^k(s) \rightarrow \mathcal{P}_q(ks)\) as in (2).

Notice that, by Proposition 2.1, we have that for any \(m \in \{1, \ldots, s - 1\}\), it holds that \(\varphi(\mathcal{G}_q^k(m, s)) \subseteq \mathcal{G}_q(mk, sk)\). Moreover, given \(U, V\) subspaces of \(\mathbb{F}_q^s\) with \(U \subseteq V\), then \(\varphi(U) \subseteq \varphi(V)\). As a consequence, if \((l, M(l)) \in \hat{C}\), then \((\varphi(l), \varphi(M(l)))\) is a flag of type \((k, n - k)\) on \(\mathbb{F}_q^n\). This fact allows us to define a family of flags over \(\mathbb{F}_q^n\) as follows

\[
\hat{C} = \{(\varphi(l), \varphi(M(l))) \mid l \in \mathcal{G}_q^k(1, s)\}. \tag{12}
\]

By Remark 2.2 we know that \(\varphi\) gives a bijection between \(\mathcal{G}_q^k(1, s)\) and the \(k\)-spread \(S = S(s, k, P)\) defined in (6). Hence, the family \(\hat{C}\) is a flag code with projected codes

\[
\hat{C}_1 = \varphi(\mathcal{G}_q^k(1, s)) = S, \quad \hat{C}_2 = \varphi(\mathcal{G}_q^k(s - 1, s))
\]
and the following result holds:

**Theorem 3.7.** The code \( \hat{C} \) defined in (12) is an optimum distance flag code of type \((k, n-k)\) on \( \mathbb{F}_q^n \) having the spread \( S \) as a projected code.

**Proof.** Since \( \hat{C}_1 = \varphi(G_q^k(1,s)) = S \), we have \( |\hat{C}| = |\hat{C}_1| = |\varphi(G_q^k(1,s))| = |G_q^k(1,s)| \). Furthermore, by the injectivity of \( \varphi \) (see Proposition 2.1), we also have that \( |\hat{C}_2| = |\varphi(G_q^k(s-1,s))| = |G_q^k(s-1,s)| \). As \( |G_q^k(1,s)| = |G_q^k(s-1,s)| \) we conclude that \(|\hat{C}| = |\hat{C}_1| = |\hat{C}_2| \) and \( \hat{C} \) is disjoint. Let us now prove that the projected codes of \( \hat{C} \) are constant dimension codes with the maximum possible distance. Since the projected code \( \hat{C}_1 \) is a spread, it is enough to check this property for \( \hat{C}_2 \). Given any two different subspaces \( \varphi(H), \varphi(H') \in \hat{C}_2 = \varphi(G_q^k(s-1,s)) \), by means of Proposition 2.1, we have that \( H, H' \) are different hyperplanes. Moreover, since the intersection of any two hyperplanes in \( \mathbb{F}_q^{s_k} \) is a \((s-2)\)-dimensional subspace of \( \mathbb{F}_q^k \) we have that

\[
\dim(\varphi(H) \cap \varphi(H')) = \dim(\varphi(H \cap H')) = k(s-2) = n-2k.
\]

Notice that \( n-2k = 2(n-k) - n \) is the minimum among the possible dimensions of the intersection of subspaces in \( G_q(n-k,n) \). Hence \( \hat{C}_2 \) is an equidistant \((n-2k)\)-intersecting constant dimension code and, by applying Theorem 3.1, we are done.

\[\square\]

### 3.2.3 The full admissible type vector

In this subsection we finally tackle the construction of optimum distance flag codes of the full admissible type, that is, of type \((1, \ldots, k, n-k, \ldots, n-1)\) on \( \mathbb{F}_q^n \) having the \( k \)-spread \( S \) defined in (6) as a projected code. To do this, we start from the optimum distance flag code \( \hat{C} \) of type \((k, n-k)\) defined in (12). Recall that the construction of this code depends on the choice of a bijection \( M : G_q^k(1,s) \to G_q^k(s-1,s) \) such that \( l \subset M(l) \) for any \( l \in G_q^k(1,s) \).

Let us fix an order in the set of lines of \( \mathbb{F}_q^k \) and write \( G_q^k(1,s) = \{l_1, l_2, \ldots, l_L\} \), where \( L = |G_q^k(1,s)| \). This order in \( G_q^k(1,s) \) naturally induces respective orders in the sets \( G_q^k(s-1,s) \), \( S \) and \( H = \varphi(G_q^k(s-1,s)) \) as follows:

\[
H_i = M(l_i), \quad S_i = \varphi(l_i), \quad H_i = \varphi(H_i)
\]

for \( i = 1, \ldots, L \). Denote also by \( S_i \) the RREF generator matrix of \( S_i \). Notice that \( S_i \in \Sigma \) where \( \Sigma \) is the set defined in (7), and \( S_i = (\phi(x_{i1})| \ldots | \phi(x_{is})) \), where \( (x_{i1}, \ldots, x_{is}) \in \mathbb{F}_q^k \) is the RREF matrix generating the line \( l_i \).

Now, given a hyperplane \( H_i = M(l_i) \) of \( \mathbb{F}_q^k \), we can write

\[
H_i = l_i \oplus l_{i_2} \oplus \ldots \oplus l_{i_{s-1}},
\]
for $l_{i_2}, \ldots, l_{i_s-1}$ some lines of $\mathbb{F}_{q^k}^s$. By the properties of the field reduction $\varphi$ described in Proposition 2.1, we have that

$$H_i = \varphi(H_i) = S_{i_1} \oplus S_{i_2} \oplus \ldots \oplus S_{i_s-1}.$$  

So, any subspace $H_i \in \mathcal{H}$ can be decomposed as a direct sum of subspaces in $S$. This representation is not unique since $H_i$ can be written as direct sum of different collections of lines. Moreover, given that for any line $l_{i_s}$ in $\mathbb{F}_{q^k}^s \setminus H_i$, it holds that $H_i \oplus l_{i_s} = \mathbb{F}_{q^k}^s$, by using Proposition 2.1 again, we conclude that $H_i \oplus S_{i_s} = \mathbb{F}_q^n$. As a consequence, the rows of the matrix

$$W_i = \begin{pmatrix} S_{i_1} \\ S_{i_2} \\ \vdots \\ S_{i_s-1} \\ S_{i_s} \end{pmatrix}$$  

form a basis of $\mathbb{F}_q^n$. Moreover, any collection of $j \leq n$ rows of $W_i$ generates a $j$-dimensional subspace of $\mathbb{F}_q^n$.

Denote by $W_i^{(j)}$ the submatrix of $W_i$ given by its first $j$ rows. We also denote by $W_i^{(j)} = \text{rowsp}(W_i^{(j)})$. With this notation, it is clear that $W_i^{(k)} = S_i$ and $W_i^{(n-k)} = H_i$. In addition, for any $1 \leq j_1 < j_2 \leq n$, it holds that $W_i^{(j_1)} \subsetneq W_i^{(j_2)}$. This fact allows us to define $\mathcal{F}_{W_i}$ the flag of type $(1, \ldots, k, n-k, \ldots, n-1)$ associated to $W_i$ in the following way:

$$\mathcal{F}_{W_i} = (W_i^{(1)}, \ldots, W_i^{(k-1)}, S_i, H_i, W_i^{(n-k+1)}, \ldots, W_i^{(n-1)}).$$  

Finally, given the family of matrices $\{W_i\}_{i=1}^L$, we define the family of associated flags of type $(1, \ldots, k, n-k, \ldots, n-1)$:

$$\mathcal{C} = \{\mathcal{F}_{W_i} \mid i = 1, \ldots, L\}.$$  

Let us see that $\mathcal{C}$ is an optimum distance flag code. To do so, we analyze the structure of its projected codes:

$$\mathcal{C}_j = \{W_i^{(j)} \mid i = 1, \ldots, L\}$$  

and

$$\mathcal{C}_{k+j} = \{W_i^{(n-k+j-1)} \mid i = 1, \ldots, L\},$$  

for all $j = 1, \ldots, k$.

**Proposition 3.8.** Given the flag code $\mathcal{C}$ defined as above, for each $j = 1, \ldots, k$ the following is satisfied:

1. The code $\mathcal{C}_j$ is a partial spread in $\mathcal{G}_q(j, n)$ with cardinality $L = \frac{q^{n-1}}{q^j-1}$. 

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(2) The code $C_{k+j}$ is an equidistant $(n - 2k + 2(j - 1))$-intersecting constant dimension code in $G_q(n - k + j - 1, n)$ with cardinality $L = \frac{q^n}{q^k-1}$. As a consequence, $C_{k+j}$ is a maximum distance constant dimension code.

In particular, we have that $C_k = S$ and $C_{k+1} = H$.

Proof. By construction it is clear that $C_k = S$ and $C_{k+1} = H$. Now, for any $1 \leq j \leq k$, given two different indices $i_1, i_2 \in \{1, \ldots, L\}$, we have that

$$\mathcal{W}_{i_1}^{(j)} \cap \mathcal{W}_{i_2}^{(j)} \subset S_{i_1} \cap S_{i_2} = \{0\}.$$ 

Hence, $C_j$ is a partial spread in the Grassmannian $G_q(j, n)$ with $|C_j| = L$.

To prove (2), consider subspaces $\mathcal{W}_{i_1}^{(n-k+j-1)}, \mathcal{W}_{i_2}^{(n-k+j-1)} \in C_{k+j}$. We know that $\dim(\mathcal{H}_{i_1} \cap \mathcal{H}_{i_2}) = (s - 2)k = n - 2k$, then the sum subspace $\mathcal{H}_{i_1} + \mathcal{H}_{i_2}$ is the whole space $F_q^n$. As a consequence,

$$n = \dim(\mathcal{H}_{i_1} + \mathcal{H}_{i_2}) \leq \dim(\mathcal{W}_{i_1}^{(n-k+j-1)} + \mathcal{W}_{i_2}^{(n-k+j-1)}) \leq n$$

and then it follows that

$$n = \dim(\mathcal{W}_{i_1}^{(n-k+j-1)} + \mathcal{W}_{i_2}^{(n-k+j-1)}) = 2(n - k + j - 1) - \dim(\mathcal{W}_{i_1}^{(n-k+j-1)} \cap \mathcal{W}_{i_2}^{(n-k+j-1)}).$$

Hence, we obtain

$$\dim(\mathcal{W}_{i_1}^{(n-k+j-1)} \cap \mathcal{W}_{i_2}^{(n-k+j-1)}) = 2(n - k + j - 1) - n = n - 2k + 2(j - 1),$$

which is the minimum possible dimension of the intersection between subspaces of dimension $n - k + j - 1$ of $F_q^n$. Thus, we conclude that $C_{k+j}$ is an equidistant $(n - 2k + 2(j - 1))$-intersecting constant dimension code with exactly $L$ elements. In particular, we have that $d_S(C_{k+j}) = 2(k - (j - 1))$ and $C_{k+j}$ is a constant dimension code with the maximum distance. \[\blacksquare\]

**Theorem 3.9.** The flag code $C$ defined in (15) is an optimum distance flag code of type $(1, \ldots, k, n-k, \ldots, n-1)$ on $F_q^n$ with the $k$-spread $S$ as a $k$-projected code. This code has cardinality $|C| = L = \frac{q^n}{q^k-1}$ and distance $d_f(C) = 2k(k+1)$.

Proof. By means of Proposition 3.8 we conclude that $C$ is a disjoint flag code of cardinality $L$ with maximum distance subspace codes as projected codes. Then, by Theorem 3.1, $C$ is an optimum distance flag code, that is, $d_f(C) = 2k(k+1)$. \[\blacksquare\]

**Remark 3.10.** The code $C$ defined in (15) attains the maximum possible distance for flag codes of type $(1, \ldots, k, n-k, \ldots, n-1)$ on $F_q^n$. Furthermore, we will see that $C$ has also the best possible size among the optimum distance flag codes with this type vector. In other words, optimum distance flag codes of type $(1, \ldots, k, n-k, \ldots, n-1)$ on $F_q^n$ with the maximum cardinality must have a $k$-spread as a projected code.
Theorem 3.11. Let $C$ be an optimum distance flag code of type $(1, \ldots, k, n-k, \ldots, n-1)$ on $\mathbb{F}_q^n$. Then, $|C| \leq q^{a-1} = |S|$. The equality holds if, and only if, its $k$-projected code $C_k$ is a $k$-spread of $\mathbb{F}_q^n$.

Proof. Given that $C$ is an optimum distance flag code, in particular $C$ is disjoint and $|C| = |C_k|$. Besides, the projected code $C_k$ must be a partial spread. Since $k|n$, we have that $|C_k| \leq |S| = q^{a-1}$. This equality holds, if and only if, $C_k$ is a $k$-spread.

3.2.4 The general case

Finally, in order to get an optimum distance flag code of any admissible type vector with a $k$-spread as a projected code, we apply a puncturing process to the code $C$ defined in (15). This process was already used in [2] to get optimum distance flag codes having a planar spread as a projected code. Let us recall it. Fix a type vector $(t_1, \ldots, t_r)$, where $k \in \{t_1, \ldots, t_r\} \subseteq \{1, \ldots, k, n-k, \ldots, n-1\}$. Consider $F_{W_i} \in C$ the flag associated to the matrix $W_i$, for some $i \in \{1, \ldots, L\}$, defined as in (14). We define its punctured flag of type $(t_1, \ldots, t_r)$ as

$$F_{W_i}^{(t_1, \ldots, t_r)} = (W_i^{(t_1)}, \ldots, W_i^{(t_r)}).$$

The $(t_1, \ldots, t_r)$-punctured flag code of $C$ is the code given by

$$C^{(t_1, \ldots, t_r)} = \{F_{W_i}^{(t_1, \ldots, t_r)} \mid i = 1, \ldots, L\}.$$  

With this notation we have several results, whose proofs follow straightforwardly.

Theorem 3.12. The code $C^{(t_1, \ldots, t_r)}$ defined as above is an optimum distance flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ with a $k$-spread as a projected code.

Furthermore, optimum distance flag codes constructed from spreads attain the maximum possible cardinality for the corresponding admissible type.

Theorem 3.13. Let $n \geq 2$ be an integer number and $k$ a divisor of $n$. Consider $(t_1, \ldots, t_r)$ a type vector with $k \in \{t_1, \ldots, t_r\} \subseteq \{1, \ldots, k, n-k, \ldots, n-1\}$. Then any optimum distance flag code of type $(t_1, \ldots, t_r)$ on $\mathbb{F}_q^n$ has, at most, $L = q^{a-1}$ elements. The equality holds if, and only if, it has a $k$-spread as a projected code.

Corollary 3.14. Given a vector type $(t_1, \ldots, t_r)$ such that $k \in \{t_1, \ldots, t_r\} \subseteq \{1, \ldots, k, n-k, \ldots, n-1\}$, the $(t_1, \ldots, t_r)$-punctured flag code of the optimum distance flag code $C$ defined in (15) is also an optimum distance flag code with the maximum possible cardinality, that is, $L = q^{a-1}$. 

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3.3 Example

We construct an example of optimum distance flag code of type $(2, 4)$ on $\mathbb{F}_4^3$ from a 2-spread. To do this, we follow the steps given in Section 3.2.

Consider the bipartite graph $G = (V, E)$ where $V = \mathcal{G}_4(1, 3) \cup \mathcal{G}_4(2, 3)$ and $E$ is the set of pairs $(l, H) \in \mathcal{G}_4(1, 3) \times \mathcal{G}_4(2, 3)$ with $l \subset H$. Take $\alpha \in \mathbb{F}_4$ with $\alpha \neq 0, 1$. Then, we have that $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$. By using the package GRAPe of GAP, and following the process described in [3] to get perfect matchings, we have designed an algorithm that provides a perfect matching of $V$. The induced bijection $M : \mathcal{G}_4(1, 3) \to \mathcal{G}_4(2, 3)$ is explicitly given by:

\[
\begin{align*}
M((0, 0, 1)) &= \operatorname{rowsp} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad & M((1, 1, \alpha)) &= \operatorname{rowsp} \begin{pmatrix} 1 & 1 & \alpha \\ 0 & 1 & \alpha \end{pmatrix} \\
M((0, 1, 0)) &= \operatorname{rowsp} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \alpha^2 \end{pmatrix} \quad & M((1, 1, \alpha^2)) &= \operatorname{rowsp} \begin{pmatrix} 1 & 1 & \alpha^2 \\ 0 & 0 & 1 \end{pmatrix} \\
M((0, 1, 1)) &= \operatorname{rowsp} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad & M((1, 1, 0)) &= \operatorname{rowsp} \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
M((0, 1, \alpha)) &= \operatorname{rowsp} \begin{pmatrix} 0 & 1 & \alpha \\ 1 & 0 & \alpha \end{pmatrix} \quad & M((1, 1, 1)) &= \operatorname{rowsp} \begin{pmatrix} 1 & \alpha & 1 \\ 0 & 1 & 1 \end{pmatrix} \\
M((0, 1, \alpha^2)) &= \operatorname{rowsp} \begin{pmatrix} 0 & 1 & \alpha^2 \\ 1 & 0 & 1 \end{pmatrix} \quad & M((1, 1, \alpha)) &= \operatorname{rowsp} \begin{pmatrix} 1 & \alpha & \alpha \\ 0 & 1 & 0 \end{pmatrix} \\
M((1, 0, 0)) &= \operatorname{rowsp} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha^2 \end{pmatrix} \quad & M((1, 1, \alpha^2)) &= \operatorname{rowsp} \begin{pmatrix} 1 & \alpha & \alpha^2 \\ 0 & 0 & 1 \end{pmatrix} \\
M((1, 0, 1)) &= \operatorname{rowsp} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad & M((1, 2, 0)) &= \operatorname{rowsp} \begin{pmatrix} 1 & \alpha^2 & 0 \\ 0 & 1 & \alpha \end{pmatrix} \\
M((1, 0, \alpha)) &= \operatorname{rowsp} \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 1 \end{pmatrix} \quad & M((1, 2, 1)) &= \operatorname{rowsp} \begin{pmatrix} 1 & \alpha^2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
M((1, 0, \alpha^2)) &= \operatorname{rowsp} \begin{pmatrix} 1 & 0 & \alpha^2 \\ 0 & 1 & \alpha \end{pmatrix} \quad & M((1, 2, \alpha)) &= \operatorname{rowsp} \begin{pmatrix} 1 & \alpha^2 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \\
M((1, 1, 0)) &= \operatorname{rowsp} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & \alpha^2 \end{pmatrix} \quad & M((1, 2, \alpha^2)) &= \operatorname{rowsp} \begin{pmatrix} 1 & \alpha^2 & \alpha^2 \\ 0 & 1 & 1 \end{pmatrix} \\
M((1, 1, 1)) &= \operatorname{rowsp} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \alpha^2 \end{pmatrix} \\
\end{align*}
\]

Observe that every line $l \in \mathcal{G}_4(1, 3)$ is a subspace of the (hyper)plane $M(l)$. Even more, we have expressed every subspace $M(l)$ as the rowspace of a $2 \times 3$ matrix whose first row is precisely a generator of the line $l$. In this way, we obtain the optimum distance flag code of type $(1, 2)$ on $\mathbb{F}_4^3$

\[
\tilde{C} = \{ (l, M(l)) \mid l \in \mathcal{G}_4(1, 3) \}.
\]

Now, let $f(x) \in \mathbb{F}_2[x]$ be the minimal polynomial of $\alpha$, which has degree 2, and $P \in GL_2(2)$ its companion matrix. If $\phi$ is the field isomorphism in (1), we have that $\phi(0) = 0_2$, $\phi(1) = I_2$ and $\phi(\alpha) = P$. Taking the previous matching
$M$ and the field reduction $\varphi$ induced by $\phi$ (2), we define the following optimum distance flag code of type $(2, 4)$ on $\mathbb{F}_2^6$

$$\tilde{C} = \{ (\varphi(l), \varphi(M(l))) \mid l \in \mathcal{G}_4(1, 3) \}.$$ 

If we take $l = \langle (0, 1, \alpha) \rangle$, for instance, the corresponding element on $\tilde{C}$ is the flag

$$\mathcal{F} = \left( \text{rowsp} \left( \begin{array}{cc} 0_2 & I_2 \\ I_2 & P \end{array} \right), \text{rowsp} \left( \begin{array}{ccc} 0_2 & I_2 & P \end{array} \right) \right).$$

Note that $\tilde{C}_1 = \mathcal{S}(3, 2, P) = \mathcal{S}$. Also, for every $\varphi(l) \in \mathcal{S}$ with $l \in \mathcal{G}_4(1, 3)$, we have that $\varphi(M(l))$ is a 4-dimensional subspace over $\mathbb{F}_2$ that contains $\varphi(l)$. Moreover, $\varphi(M(l))$ is the vector space generated by the rows of a $4 \times 6$ full-rank matrix, whose two first rows span $\varphi(l)$.

4 Conclusions and future work

In this paper we have addressed the problem of obtaining flag codes of general type $(t_1, \ldots, t_r)$ on a space $\mathbb{F}_q^n$ with the maximum possible distance and the property of having a $k$-spread as a projected code whenever $k$ divides $n$. Firstly, we have showed that the existence of such codes might be not possible for an arbitrary type vector and have characterized the admissible ones. They have to satisfy the condition: $k \in \{ t_1, \ldots, t_r \} \subseteq \{ 1, \ldots, k, n-k, \ldots, n-1 \}$.

Given an admissible type vector, we have proved the existence of optimum distance flag codes of such a type with a spread as a projected code by describing a gradual construction starting from type $(1, n-1)$, following with type $(k, n-k)$, to finish with the full admissible type $\{ 1, \ldots, k, n-k, \ldots, n-1 \}$. This construction is mainly based on two ideas: on one side, we exploit the existence of perfect matchings on the bipartite graph with set of vertices given by the lines and the hyperplanes of $\mathbb{F}_q^n$ and edges given by the containment relation. On the other hand, we use the properties of the field reduction map that allow us to translate the spread of lines to a $k$-spread and to build our code from it. We also have proved that our construction provides codes with the best possible size. Furthermore, we have proved that optimum distance flag codes attain the maximum possible cardinality for the corresponding type if, and only if, they have a spread as a projected code.

In current work we investigate the algebraic structure and features of this family of codes and explore other possible constructions. We also study the family of flag codes from spreads not necessarily having the maximum distance as well as the existence and performance of decoding algorithms for them.
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