Casimir effect for the massless Dirac field in two-dimensional Reissner-Nordström spacetime

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In this paper, the two-dimensional Reissner-Nordström black hole is considered as a system of the Casimir type. In this background the Casimir effect for the massless Dirac field is discussed. The massless Dirac field is confined between two “parallel plates” separated by a distance \( L \) and there is no particle current drilling through the boundaries. The vacuum expectation values of the stress tensor of the massless Dirac field at infinity are calculated separately in the Boulware state, the Hartle-Hawking state and the Unruh state.

Keywords: Casimir effect, massless Dirac field, Reissner-Nordström spacetime

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I. INTRODUCTION

In 1948, Casimir first predicted that two neutral parallel plates in vacuum would attract each other [1], a phenomenon later known as the Casimir effect. This effect is a manifestation of the non-trivial properties of the vacuum state in quantum field theory, and it is a macroscopic quantum effect.

The Casimir effect is an interdisciplinary subject. It has been studied in many fields of physics, such as quantum field theory, atomic and molecular physics, condensed matter physics, gravitation and cosmology [2].

Since the first prediction, people have investigated cases of various geometries, boundaries, fields and backgrounds [3]. With improved experiment techniques, a lot of theoretical predictions have been verified, arousing even more interest.

When one studies the Casimir effect, the zero point energy of the confined system should be calculated, and regularized [4,5]. There are many methods of regularization [6–9] such as Green’s function method, zeta function regularization, dimensional regularization, and point-splitting method, etc. The zero point energy after regularization is divergent, and thus it needs renormalization, which, briefly, aims at shifting the divergent part of the ground state energy from the ground state energy to the classical energy [2], and can be written as

\[
E = E^{\text{class}} + E_0 = (E^{\text{class}} + E^{\text{div}}) + (E_0 - E^{\text{div}}) = \tilde{E}^{\text{class}} + E^{\text{ren}}.
\]

In order to study the Casimir effect in a curved spacetime, the vacuum expectation value of the energy-momentum tensor needs to be calculated. But it is very difficult to calculate the energy-momentum tensor directly using the formula

\[
T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^\mu\nu},
\]

especially in higher dimensions. In Refs [10–12], Wald’s axioms [13–15] for renormalized energy-momentum tensor were used to calculate the Casimir effect indirectly in two-dimensional curved spacetime. The massless scalar field in two-dimensional Schwarzschild black hole background [10], two-dimensional string black hole background [11], and two-dimensional Achucarro-Ortiz black hole background [12] were studied, respectively. On the one hand, as far as we know, the Casimir effect for the Dirac field in curved spacetime is yet to be studied. On the other hand, we know that the energy-momentum tensor in four-dimensional black hole background could be calculated in some cases, for example in Refs. [16,17], but there is no boundary to confine the field in those works. The energy-momentum tensor for a field confined by some boundary is very difficult to calculate in curved spacetime. In this paper we will study the Casimir effect for the massless Dirac field using Wald’s axioms, however the background spacetime is chosen to be two-dimensional Reissner-Nordström (RN) black hole, not four-dimensional background. We want to know if there are some differences in Casimir energy momentum tensor and Casimir force between scalar field and massless Dirac

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field in the same background. We also research the dependent relationship of Casimir force to background and massless field.

Wald’s axioms are as follows: (1) Expectation values of the energy-momentum tensor are covariantly conserved. (2) Causality holds. (3) Standard results in Minkowski spacetime can be obtained. (4) Standard results for the ‘off-diagonal’ elements can be obtained. (5) The energy-momentum tensor is a local functional of the metric; i.e., it depends only on the metric and its derivatives which appear through the Riemann curvature tensor and the metric’s covariant derivatives up to the second order. Here condition (3) means that the normal ordering procedure in the Minkowski spacetime should be valid, and condition (4) is simply the observation that, as \( \langle \Phi | T_{\mu\nu} | \Psi \rangle \) is in any case finite for orthogonal states, \( \langle \Phi | \Psi \rangle = 0 \), this quantity should be of the usual value [15].

Due to the existence of the horizon and the effective potential barrier, the two-dimensional RN black hole is considered as a system of the Casimir type. Applying Wald’s axioms for the energy-momentum tensor, and the anomaly trace of the energy-momentum tensor, the regularized energy-momentum tensor of the massless Dirac field in two-dimensional RN background is calculated and the Casimir effect is discussed in this paper. The organization is as follows. In Sec. II, the background spacetime is given and the most general form of the energy-momentum tensor in the given background is obtained. Then in Sec. III, the renormalized energy-momentum tensor for the massless Dirac field in the two-dimensional Minkowski spacetime is calculated. In Sec. IV, the expectation values of the renormalized energy-momentum tensor for the massless Dirac field in the Boulware vacuum, the Hartle-Hawking vacuum and the Unruh vacuum are calculated respectively, and from that the Casimir force acting on two “parallel plates” in the given background is extracted. Finally, Sec. V contains the summarization and the conclusions.

II. BACKGROUND SPACETIME AND THE GENERAL DESCRIPTION OF THE ENERGY-MOMENTUM TENSOR

The RN spacetime is the background of this paper. The line element of a two-dimensional RN black hole is

\[
\mathrm{d}s^2 = \left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right) \mathrm{d}t^2 - \left( 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \right)^{-1} \mathrm{d}r^2,
\]

where \( m \) and \( Q \) are the mass and the charge of the black hole respectively.

In order to make the best use of the conclusions in the Minkowski spacetime, metric (1) is written in the following form under conformal transformation

\[
\mathrm{d}s^2 = f(r) \left( \mathrm{d}t^2 - \mathrm{d}u^2 \right),
\]

in which

\[
f(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}, \quad \frac{df(r)}{dr} = f(r).
\]

The non-zero Christoffel symbols of metric (2) are

\[
\Gamma^u_{tt} = \Gamma^t_{ut} = \Gamma^u_{uu} = \Gamma^u_{uu} = \frac{1}{2} \frac{df(r)}{dr} = \frac{mr - Q^2}{r^3}.
\]

The scalar curvature of metric (2) is

\[
R = -\frac{2 \left( 2mr - 3Q^2 \right)}{r^4}.
\]

The conservation equation of energy-momentum tensor \( \nabla_a \langle T^a_b \rangle_{\text{ren}} = 0 \) can be extended as [18]

\[
\partial_u \langle T^u_t \rangle_{\text{ren}} + \Gamma^t_{ua} \langle T^a_t \rangle_{\text{ren}} - \Gamma^u_{tt} \langle T^t_t \rangle_{\text{ren}} = 0,
\]

\[
\partial_u \langle T^u_u \rangle_{\text{ren}} + \Gamma^t_{ua} \langle T^a_u \rangle_{\text{ren}} - \Gamma^u_{tt} \langle T^t_u \rangle_{\text{ren}} = 0,
\]

where

\[
\langle T^t_t \rangle_{\text{ren}} = -\langle T^u_u \rangle_{\text{ren}}, \quad \langle T^t_t \rangle_{\text{ren}} = \langle T^a_a \rangle_{\text{ren}} - \langle T^u_u \rangle_{\text{ren}},
\]

\[
\langle T^t_t \rangle_{\text{ren}} = \langle T^u_u \rangle_{\text{ren}}.
\]
in which \( \langle T^\alpha_\alpha \rangle_{\text{ren}} \) is the anomaly trace \([19]\).

References \([20,21]\) tell us that the anomaly trace of the energy-momentum tensor in two dimensions of a massless Dirac field after renormalization is the same as a scalar field

\[
\langle T^\alpha_\alpha \rangle_{\text{ren}} = -\frac{R}{24\pi}.
\]  

(9)

In the RN background, applying Eq. (5), Eq. (9) becomes

\[
\langle T^\alpha_\alpha \rangle_{\text{ren}} = -\frac{1}{24\pi} \frac{2(2mr - 3Q^2)}{r^4}.
\]  

(10)

Equations (4) and (8) being used, Eq. (6) becomes

\[
\frac{\partial}{\partial r} (f(r) \langle T^u_u \rangle_{\text{ren}}) = 0,
\]  

(11)

The solution of Eq. (11) is

\[
\langle T^u_u \rangle_{\text{ren}} = \alpha f^{-1}(r),
\]  

(12)

where \( \alpha \) is a constant of integration. In the same way Eq. (7) becomes

\[
\frac{\partial}{\partial r} (f(r) \langle T^u_u \rangle_{\text{ren}}) = \frac{1}{2} \left( \frac{df(r)}{dr} \right) \langle T^\alpha_\alpha \rangle_{\text{ren}}.
\]  

(13)

The solution of Eq. (13) is

\[
\langle T^u_u \rangle_{\text{ren}} = [H(r) + \beta] f^{-1}(r),
\]  

(14)

in which \( \beta \) is a constant of integration and

\[
H(r) = \frac{1}{2} \int_{r_H}^r \langle T^u_u \rangle_{\text{ren}} \frac{d}{dr'} f(r') dr'.
\]  

(15)

Substituting Eqs. (3) and (10) to Eq. (15), we get

\[
H(r) = -\frac{1}{24\pi} \left( \frac{m^2 - 2mQ^2 + Q^4}{r^4} \right)
+ \frac{1}{24\pi} \left[ \frac{m^2}{(m + \sqrt{m^2 - Q^2})^4} - \frac{2mQ^2}{(m + \sqrt{m^2 - Q^2})^5} + \frac{Q^4}{(m + \sqrt{m^2 - Q^2})^6} \right].
\]  

(16)

Using Eqs. (8), (12) and (14), we get the general expression of the energy-momentum tensor for a massless Dirac field in the given two-dimensional RN background

\[
\langle T^u_u \rangle_{\text{ren}} = \begin{pmatrix}
\langle T^\alpha_\alpha \rangle_{\text{ren}} - f^{-1}(r) H(r) & 0 \\
0 & f^{-1}(r) H(r)
\end{pmatrix}
+ f^{-1}(r) \begin{pmatrix}
-\beta & -\alpha \\
\alpha & -\beta
\end{pmatrix}.
\]  

(17)

In expression (17) the values of the undetermined constants \( \alpha \) and \( \beta \) depend on which vacuum state the massless Dirac field is in. The energy-momentum tensor can have different expression in different vacuum states. In the following we will determine the value of \( \alpha \) and \( \beta \) by imposing Wald’s axioms (3) and (4). In this way, we will work out the energy-momentum tensors for a massless Dirac field in the given two-dimensional RN background in three vacuum states.
III. RENORMALIZED ENERGY-MOMENTUM TENSOR FOR A MASSLESS DIRAC FIELD IN THE MINKOWSKI SPACETIME

Two “parallel plates” are placed at \( r_1 \) and \( r_2 = r_1 + L \). The massless Dirac field is confined in the interval of \( r_1 \leq r < r_1 + L \). The physical boundary condition is that there is no particle current through the “walls”, i.e., \( \hat{n} \cdot \hat{j}(x) = 0 \) at \( r_1 \) and \( r_2 = r_1 + L \), where \( \hat{n} \) is the unit vector normal to the boundary surface. \( \hat{n} \) equals \( \hat{r} \) at \( r_1 \), and \( \hat{n} \) equals \( -\hat{r} \) at \( r_2 = r_1 + L \).

To satisfy this boundary condition, the momentum of a massless Dirac field in the confined direction should be

\[
p = p(n, L) = \left(n + \frac{1}{2}\right) \frac{\pi}{L}.\tag{18}
\]

Thus the zero point energy of the massless Dirac field satisfying the given boundary condition is

\[
E_B = -\sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) \frac{\pi}{L}.\tag{19}
\]

We can easily write down the zero point energy of the massless Dirac field without boundary

\[
E_0 = -\frac{1}{2\pi} \int_{0}^{\infty} kdk,\tag{20}
\]

in which \( k \) is the wave vector.

Thus, the Casimir energy for the massless Dirac field is

\[
E_c = E_B - E_0
= -\sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) \frac{\pi}{L} + \frac{1}{2\pi} \int_{0}^{\infty} kdk
= -\frac{\pi}{L} \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) - \left(\frac{L}{\sqrt{2\pi}}\right)^2 \int_{0}^{\infty} kdk
= -\frac{\pi}{L} \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) - \int_{0}^{\infty} t dt,\tag{21}
\]

in which \( t = \frac{k}{\sqrt{2\pi}} \). We define \( F(t) = t \), and Eq. (21) becomes

\[
E_c = -\frac{\pi}{L} \left[ \sum_{n=0}^{\infty} F\left(n + \frac{1}{2}\right) - \int_{0}^{\infty} F(t) dt \right].\tag{22}
\]

Applying the Abel-Plana formula [23]

\[
\sum_{n=0}^{\infty} F\left(n + \frac{1}{2}\right) - \int_{0}^{\infty} dtF(t) = -i \int_{0}^{\infty} \frac{dt}{e^{2\pi t} + 1} [F(it) - F(-it)],\tag{23}
\]

we get

\[
E_c = -\frac{\pi}{L} (-i) \int_{0}^{\infty} \frac{dt}{e^{2\pi t} + 1} [F(it) - F(-it)]
= -\frac{2\pi}{L} \int_{0}^{\infty} \frac{tdt}{e^{2\pi t} + 1}
= -\frac{\pi}{24L}.\tag{24}
\]

So the Casimir energy density is

\[
\rho = \frac{E_c}{L} = -\frac{\pi}{24L^2}.\tag{25}
\]
Equation (25) is the same as the result of Ref. [24] in which \( D = 2 \). Therefore we obtain the standard Casimir energy-momentum tensor of the massless Dirac field confined by two “parallel plates” in the 1 + 1 dimensional Minkowski spacetime

\[
\langle T^\mu_\nu \rangle_\text{ren} = \begin{pmatrix} \frac{-\pi L}{24L^2} & 0 \\ 0 & \frac{\pi L}{24L^2} \end{pmatrix} = \frac{\pi}{24L^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\]

which is identical to the result of the massless scalar field in the two-dimensional flat spacetime with the same constraint condition [25].

IV. RENORMALIZED ENERGY-MOMENTUM TENSOR FOR A MASSLESS DIRAC FIELD IN THREE VACUUM STATES

We consider the two-dimensional RN black hole as a system of the Casimir type. Because the two-dimensional RN black hole is asymptotically flat at infinity, we require the renormalized energy-momentum tensor (17) in the two-dimensional RN background at infinity be equal to the standard Casimir energy-momentum tensor in the Minkowski spacetime (26). In this way \( \alpha \) and \( \beta \) are determined. Then we get the Casimir energy-momentum tensors for the massless Dirac field under the given boundary condition in the two-dimensional RN background in three vacuum states separately.

A. Boulware vacuum

The Boulware vacuum (denoted by \( |B\rangle \)) [26] assumes that there is no particle at infinity (towards \( J^+ \)). Comparing Eq. (26) with Eq. (17) at \( r \to \infty \), we get

\[
\alpha = 0,
\]

and

\[
\beta = \frac{\pi}{24L^2} - \frac{1}{24\pi} \left[ \frac{m^2}{(m + \sqrt{m^2 - Q^2})} - \frac{2mQ^2}{(m + \sqrt{m^2 - Q^2})^2} + \frac{Q^4}{(m + \sqrt{m^2 - Q^2})^2} \right].
\]

Substituting expressions (27) and (28) into the general expression of the energy-momentum tensor (17), we get

\[
\langle B | T^\mu_\nu | B \rangle_\text{ren} = \left( \frac{\pi}{24L^2} - \frac{1}{24\pi} \right) \begin{pmatrix} f^{-1}(r)H(r) & 0 \\ 0 & f^{-1}(r)H(r) \end{pmatrix} + f^{-1}(r) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
\times \left[ \begin{pmatrix} \frac{m^2}{(m + \sqrt{m^2 - Q^2})} & \frac{2mQ^2}{(m + \sqrt{m^2 - Q^2})^2} & \frac{Q^4}{(m + \sqrt{m^2 - Q^2})^2} \\ \frac{m^2}{(m + \sqrt{m^2 - Q^2})^2} & \frac{2mQ^2}{(m + \sqrt{m^2 - Q^2})^3} & \frac{Q^4}{(m + \sqrt{m^2 - Q^2})^3} \\ \frac{m^2}{(m + \sqrt{m^2 - Q^2})^3} & \frac{2mQ^2}{(m + \sqrt{m^2 - Q^2})^4} & \frac{Q^4}{(m + \sqrt{m^2 - Q^2})^4} \end{pmatrix} \right].
\]

This expression is the renormalized energy-momentum tensor for a confined massless Dirac field in the given two-dimensional RN background in the Boulware vacuum.

B. Hartle-Hawking vacuum

In the Hartle-Hawking vacuum (denoted by \( |H\rangle \)) [27], the black hole is considered to be in a thermal equilibrium state at Hawking temperature \( T \), and there is an infinite heat reservoir of black body radiation around the black hole. In order to calculate the energy-momentum tensor of the gas in a thermal equilibrium state, the system is considered as a canonical system. The number of microscopic states is
The particle population in the energy interval \( \varepsilon \rightarrow \varepsilon + d\varepsilon \) is

\[
\frac{\varepsilon dn}{L} = g_s \frac{1}{\hbar c} e^{\varepsilon/\hbar} + 1.
\]

(31)

From Eqs. (30) and (31), the energy density of black body radiation in one dimensional space is

\[
\rho (\varepsilon, T) d\varepsilon = \frac{\varepsilon dn}{L} = g_s \frac{\hbar \omega}{\sqrt{\hbar \omega}} e^{\omega/\hbar} + 1.
\]

(32)

In natural units, Eq. (32) becomes

\[
\rho (\varepsilon, T) d\varepsilon = g_s \frac{\omega d\omega}{2\pi e^{\omega/\hbar} + 1}.
\]

(33)

Thus in two-dimensional flat spacetime, the energy density of a massless Dirac field in a thermal equilibrium state is

\[
\left< T_{tt} \right> = \int g_s \frac{\omega d\omega}{2\pi e^{\omega/\hbar} + 1} = g_s \frac{\pi}{24} T^2 = \frac{\pi}{24} T^2 \times 2.
\]

(34)

There is no momentum flow in a thermal equilibrium state at temperature \( T \), and thus the energy-momentum tensor of the gas in a thermal equilibrium state can be written as

\[
\left< H \left< T^\mu_\nu \right> H \right> = \frac{\pi}{24} T^2 \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = -\frac{\pi}{12} T^2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(35)

In the Hartle-Hawking vacuum, the standard Casimir energy-momentum tensor (26) is modified by Eq. (35) to

\[
\left< H \left< T^\mu_\nu \right> H \right>_{\text{ren}} = \frac{\pi}{24L^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\pi}{12} T^2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\]

(36)

in which \( T \) is the Hawking temperature of the two-dimensional RN black hole. The temperature \( T \) is the same as the temperature in the four dimensional RN black hole [28]. Therefore

\[
T = \frac{1}{2\pi} \left( \frac{\sqrt{m^2 - Q^2}}{m + \sqrt{m^2 - Q^2}} \right)^2.
\]

(37)

Substituting Eq. (37) into Eq. (36), we get the renormalized energy-momentum tensor for a massless Dirac field in the two dimensional RN background in the Hartle-Hawking vacuum at infinity (towards \( J^+ \))

\[
\left< H \left< T^\mu_\nu \right> H \right>_{\text{ren}} = \frac{\pi}{24L^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\pi}{12} \frac{m^2 - Q^2}{4\pi^2 \left( m + \sqrt{m^2 - Q^2} \right)^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(38)

Expression (38) corresponds to (17) at \( r \rightarrow \infty \), we get the values of \( \alpha \) and \( \beta \)

\[
\alpha = 0,
\]

(39)

\[
\beta = -\frac{\pi}{24L^2} - \frac{1}{24\pi} \left[ \frac{\frac{3}{4} m^2 - Q^2}{\left( m + \sqrt{m^2 - Q^2} \right)^4} - \frac{2mQ^2}{\left( m + \sqrt{m^2 - Q^2} \right)^5} + \frac{Q^4}{\left( m + \sqrt{m^2 - Q^2} \right)^6} \right].
\]

(40)

Substituting expressions (39) and (40) into the general expression of the energy-momentum tensor (17), we get
Expression (44) corresponds to (17) at equilibrium state at Hawking temperature direction is half of that in Eq. (34). Thus the energy-momentum tensor in the limit but also momentum density component. The energy density of the massless black body radiation spreading in the dimensional RN background in the Hartle-Hawking vacuum state.

\[ \langle H | T^\mu_\nu | H \rangle_{ren} = \left( \langle T^\alpha_\alpha \rangle_{ren} - f^{-1}(r)H(r) \right) + f^{-1}(r) \times \left\{ \frac{\pi}{24L^2} \frac{m^2 - \frac{1}{2}Q^2}{(m + \sqrt{m^2 - Q^2})} - \frac{2mQ^2}{(m + \sqrt{m^2 - Q^2})} + \frac{Q^4}{(m + \sqrt{m^2 - Q^2})^5} \right\} \times \left( \begin{array}{ccc} \frac{5}{2}m^2 - \frac{1}{2}Q^2 \\ m + \sqrt{m^2 - Q^2} \end{array} \right) \times \left( \begin{array}{c} -1 \\ 0 \\ 0 \end{array} \right) \].

(41)

This expression is the renormalized energy-momentum tensor for a confined massless Dirac field in the given two-dimensional RN background in the Hartle-Hawking vacuum state.

C. Unruh vacuum

The Unruh vacuum (denoted by \(|U\)) is considered as a vacuum in which the two-dimensional RN black hole is in an equilibrium state at Hawking temperature \(T\) [29], but due to the Hawking radiation, massless particles are detected at infinity (towards \(J^+\)) in this vacuum state. The energy-momentum tensor has not only energy density component but also momentum density component. The energy density of the massless black body radiation spreading in the \(u\) direction is half of that in Eq. (34). Thus the energy-momentum tensor in the limit \(r \to \infty\) and in the absence of the boundary conditions is as follows

\[ \langle U | T^\mu_\nu | U \rangle = \frac{\pi}{24} T^2 \left( \begin{array}{ccc} 1 & 1 \\ -1 & -1 \end{array} \right) \].

(42)

This means that the standard Casimir energy-momentum tensor of a massless Dirac field (26) must be modified by Eq. (42) to

\[ \langle U | T^\mu_\nu | U \rangle_{ren} = \frac{\pi}{24L^2} \left( \begin{array}{ccc} -1 & 0 \\ 0 & 1 \end{array} \right) - \frac{\pi}{24} T^2 \left( \begin{array}{ccc} -1 & -1 \\ 1 & 1 \end{array} \right) \],

(43)

where \(T\) is the Hawking temperature of the two-dimensional RN black hole. Substituting expression (37) into (43) we get the renormalized energy-momentum tensor for a massless Dirac field in the two dimensional RN background in the Unruh vacuum at infinity (towards \(J^+)\)

\[ \langle U | T^\mu_\nu | U \rangle_{ren} = \frac{\pi}{24L^2} \left( \begin{array}{ccc} -1 & 0 \\ 0 & 1 \end{array} \right) - \frac{\pi}{24} T^2 \frac{m^2 - Q^2}{(m + \sqrt{m^2 - Q^2})^2} \left( \begin{array}{ccc} -1 & -1 \\ 1 & 1 \end{array} \right). \]

(44)

Expression (44) corresponds to (17) at \(r \to \infty\), and in this way we determine the values of \(\alpha\) and \(\beta\)

\[ \alpha = \frac{1}{96\pi} \frac{m^2 - Q^2}{(m + \sqrt{m^2 - Q^2})^4}, \]

(45)

\[ \beta = \frac{\pi}{24L^2} - \frac{1}{24\pi} \left[ \frac{5}{2}m^2 - \frac{1}{2}Q^2 \\ (m + \sqrt{m^2 - Q^2}) \right] - \frac{2mQ^2}{(m + \sqrt{m^2 - Q^2})^5} + \frac{Q^4}{(m + \sqrt{m^2 - Q^2})^6} \].

(46)

Substituting expressions (45) and (46) into the general expression of the energy-momentum tensor (17), we get

\[ \langle U | T^\mu_\nu | U \rangle_{ren} = \left( \langle T^\alpha_\alpha \rangle_{ren} - f^{-1}(r)H(r) \right) + f^{-1}(r) \times \left\{ \frac{\pi}{24L^2} \frac{m^2 - \frac{1}{2}Q^2}{(m + \sqrt{m^2 - Q^2})^5} - \frac{2mQ^2}{(m + \sqrt{m^2 - Q^2})^5} + \frac{Q^4}{(m + \sqrt{m^2 - Q^2})^6} \right\} \times \left( \begin{array}{ccc} \frac{5}{2}m^2 - \frac{1}{2}Q^2 \\ m + \sqrt{m^2 - Q^2} \end{array} \right) \times \left( \begin{array}{c} -1 \\ 0 \\ 0 \end{array} \right) \].

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\[
\frac{\pi}{24L^2} - \frac{1}{24\pi} \left[ \frac{\frac{1}{2}m^2 - \frac{1}{2}Q^2}{(m + \sqrt{m^2 - Q^2})^2} - \frac{2mQ^2}{(m + \sqrt{m^2 - Q^2})^6} + \frac{Q^4}{(m + \sqrt{m^2 - Q^2})^8} \right].
\]

This expression is the renormalized energy-momentum tensor for a confined massless Dirac field in the given two-dimensional RN background in the Unruh vacuum state.

**D. Casimir force**

Expressions (29), (41) and (47) are the renormalized energy-momentum tensors for a confined massless Dirac field in the given two-dimensional RN background in the Boulware vacuum, the Hartle-Hawking vacuum and the Unruh vacuum respectively. In expressions (29), (41) and (47),

\[
\frac{\pi}{24L^2} f^{-1}(r) \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right)
\]

is the contribution to vacuum fluctuation due to the presence of boundaries. The energy-momentum tensors in three vacuum states (29), (41) and (47) are separable as follows:

\[
\langle B | T^\mu_\nu | B \rangle_{\text{ren}} = \langle B | T^\mu_\nu(\text{gravitation}) | B \rangle + \langle B | T^\mu_\nu(\text{boundary}) | B \rangle,
\]

\[
\langle H | T^\mu_\nu | H \rangle_{\text{ren}} = \langle H | T^\mu_\nu(\text{gravitation}) | H \rangle + \langle H | T^\mu_\nu(\text{boundary}) | H \rangle + \langle H | T^\mu_\nu(\text{bath}) | H \rangle,
\]

\[
\langle U | T^\mu_\nu | U \rangle_{\text{ren}} = \langle U | T^\mu_\nu(\text{gravitation}) | U \rangle + \langle U | T^\mu_\nu(\text{boundary}) | U \rangle + \langle U | T^\mu_\nu(\text{radiation}) | U \rangle,
\]

where \( \langle T^\mu_\nu(\text{gravitation}) \rangle \) and \( \langle T^\mu_\nu(\text{boundary}) \rangle \) denote the contribution to vacuum zero point energy from the gravitation and boundary constraints, respectively. \( \langle T^\mu_\nu(\text{bath}) \rangle \) denotes the contribution to vacuum zero point energy from the thermal bath at temperature \( T \). \( \langle T^\mu_\nu(\text{radiation}) \rangle_{\text{ren}} \) denotes the contribution to vacuum zero point energy from Hawking radiation at temperature \( T \). Considering (49), (50) and (51), we analyze the energy-momentum tensors for massless Dirac field in three vacuum states (29), (41) and (47) and find that

\[
\langle B | T^\mu_\nu(\text{gravitation}) | B \rangle = \langle H | T^\mu_\nu(\text{gravitation}) | H \rangle = \langle U | T^\mu_\nu(\text{gravitation}) | U \rangle
\]

\[
= \left( \begin{array}{cc} (T^\alpha_\alpha)_{\text{ren}} - f^{-1}(r)H(r) & 0 \\ 0 & f^{-1}(r)H(r) \end{array} \right) + f^{-1}(r) \frac{1}{24\pi}
\]

\[
\times \left[ -\frac{m^2}{(m + \sqrt{m^2 - Q^2})^4} - \frac{2mQ^2}{(m + \sqrt{m^2 - Q^2})^6} + \frac{Q^4}{(m + \sqrt{m^2 - Q^2})^8} \right] \times \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right),
\]

\[
\langle B | T^\mu_\nu(\text{boundary}) | B \rangle = \langle H | T^\mu_\nu(\text{boundary}) | H \rangle = \langle U | T^\mu_\nu(\text{boundary}) | U \rangle
\]

\[
= \frac{\pi}{24L^2} f^{-1}(r) \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right),
\]

\[
\langle H | T^\mu_\nu(\text{bath}) | H \rangle = f^{-1}(r) \frac{1}{24\pi} \left( \begin{array}{cc} -\frac{1}{2}m^2 - \frac{1}{2}Q^2 & -1 \\ 0 & 1 \end{array} \right),
\]

8
Expression (52) shows that the contribution to vacuum zero point energy from gravitation is the same in the three vacuum states. Expression (53) shows that the contribution to vacuum zero point energy from boundary constraint is also the same in three vacuum states. Expressions (54) and (55) denote the contributions to vacuum zero point energy from the thermal bath at temperature $T$ and Hawking radiation at temperature $T$, respectively.

Now we can distinguish the different contributions from the gravitational background (including the trace anomaly), boundaries, thermal bath and the Hawking radiation. Obviously, the presence of boundary, i.e., the two “parallel plates”, leads to a pressure

$$P_b^{(1,2)} = -\left< T_{\mu\nu}^{(\text{boundary})} \right> = -f^{-1}(r_{1,2}) \frac{\pi}{24L^2}. \tag{56}$$

That is the Casimir force acting on two “parallel plates”, appearing as an attraction. It should be noted that the additional pressure created by the other parts in Eqs. (49)-(51) are the same from both sides on the plates.

V. CONCLUSION

There are many ways to group different cases of the Casimir effect: to give some examples, zero temperature or nonzero temperature according to the temperature of the field, static boundaries or moving boundaries according to the motion state of boundaries, flat spacetime or curved spacetime according to the background, scalar field, Dirac field, or gravitation field, ...according to the spin of the field, and so on. To the best of our knowledge, the issue of the Casimir effect for the massless Dirac field in curved spacetime has not been studied before the present work.

From Refs. [15,10–12], we know that, in the lower-dimensional case, if the energy-momentum tensor of a certain field confined by an exterior boundary in the Minkowski spacetime can be obtained, the energy-momentum tensor of the same field confined by the same boundary in curved spacetime can be obtained too. Here the Casimir effect for a massless Dirac field in the two-dimensional RN background has been studied. The massless Dirac field is confined between two “parallel plates” separated by a distance $L$, and there is no particle current through the boundaries. We have derived the energy-momentum tensors, using only the general properties of stress tensor, in the Boulware vacuum, the Hartle-Hawking vacuum and the Unruh vacuum, respectively. We have found that the Casimir energy-momentum tensors in the three vacuum states for a massless Dirac field in the two-dimensional RN background when $Q \to 0$ are different from those for a massless scalar field in the two-dimensional Schwarzschild background. However, we have found that under the given constraints the Casimir force for a massless Dirac field in the two-dimensional RN background is the same as that for a massless scalar field in the two-dimensional Schwarzschild background [10].

It is interesting to compare some of the previous conclusions. In Ref. [24], de Paola et al. studied the Casimir effect for a massless Dirac field confined between two parallel plates in a $D$ dimensional flat spacetime, and the massless Dirac field satisfies the Dirichlet boundary condition. They obtained the Casimir energy in even dimensional spacetime

$$\varepsilon_D(L) = -\frac{f(D)}{L^{D-2}} \Gamma \left( \frac{1-D}{2} \right) (2^{1-D} - 1) \frac{B_D}{D}. \tag{57}$$

In $D = 2$ dimension, the Casimir energy for the massless scalar field

$$\varepsilon_2(L) = -\frac{f(2)}{L^2} \Gamma \left( \frac{1}{2} \right) (2^{-1} - 1) \frac{B_2}{2} = -\frac{\pi}{24L} \tag{58}$$

in which $L$ is the distance between two ‘parallel plates’. In Ref. [25], Milton studied the Casimir effect for the massless scalar field confined between two ‘parallel plates’ in two dimensional flat spacetime, and the confined massless scalar field satisfies the Dirichlet boundary condition on the ‘parallel plates’. The author obtained the Casimir energy for a massless scalar field

$$E = -\frac{\pi}{24a}, \tag{59}$$

in which $a$ is the distance between two ‘parallel plates’. Thus in the two-dimensional case, the Casimir energy for the massless Dirac field confined between two ‘parallel plates’ is the same as the Casimir energy for massless scalar field.
under the same condition. However, in four dimensional flat spacetime, the Casimir energy for a massless Dirac field
confined between two ‘parallel plates’ is [24]

$$\varepsilon_4 (L) = -\frac{7\pi^2}{2880L^3},$$  \hfill (60)

and the Casimir energy for a massless scalar field confined between two ‘parallel plates’ is [25]

$$E_c (a) = -\frac{\pi^2}{1440a^3}. \hfill (61)$$

From the analysis above, it is only in higher-dimensional cases that the massless Dirac field confined between two
‘parallel plates’ produces a different Casimir force on the plates from a scalar field in the same condition. However,
the Casimir effect for a massless Dirac field does coincide with the Casimir effect for a massless scalar field in two-
dimensional spacetime, at least in the cases studied in the present work and the cited references above.

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