SEARCH FOR THE QCD GROUND STATE

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Abstract

Within the Euclidean effective action approach we propose criteria for the ground state of QCD. Despite a nonvanishing field strength the ground state should be invariant with respect to modified Poincaré transformations consisting of a combination of translations and rotations with suitable gauge transformations. We have found candidate states for QCD with four or more colours. The formation of gluon condensates shows similarities with the Higgs phenomenon.
1. It has been known for a long time that the perturbative vacuum cannot be the true ground state of QCD \[1\]. If one considers the Euclidean effective action \(\Gamma\) as a function of the colour-magnetic and colour-electric fields \(B^z_i, E^z_i\), one finds states with \(B, E\) different from zero for which the value of \(\Gamma\) is lower than for \(B = E = 0\). This observation constitutes the basis for many models of condensates of composite operators as \(F^{\mu\nu}_z F^z_{\mu\nu}\) \[2\]. Since the existence of states with lower energy than the perturbative vacuum is clearly visible in the effective action for the gluon field \(A_\mu\), one may wonder what is the true QCD ground state in this language.\[3\] The Euclidean effective action for a pure Yang-Mills theory should be bounded below and we know that the state \(A_\mu = 0\) (or a gauge-equivalent state) is not the state of lowest action. There must therefore exist an absolute minimum of \(\Gamma\) with \(F^z_{\mu\nu} \neq 0\). An immediate worry is then the apparent breakdown of Euclidean rotation symmetry (corresponding to Lorentz-symmetry) for any nonvanishing value of the antisymmetric tensor field \(F_{\mu\nu}\). This seems to be in contradiction with the observed Poincaré symmetry (global \(d\)-dimensional rotations and translations in the Euclidean language) of our world which requires a ground state invariant with respect to these symmetries. Furthermore, no parity \(P\), time reversal \(T\) or charge conjugation \(C\) violation is observed in strong interactions and the QCD ground state must respect these discrete symmetries as well.

The situation is less dramatic if one realizes that the standard implementation of rotations and translations is not the only way to realize the Poincaré symmetry. One may define modified Poincaré transformations by combining the standard space transformations with appropriate gauge transformations. An example for the \(SO(3)\) rotation group is well known for instantons \[3\] where space rotations are combined with \(SU(2)\) gauge rotations to form a new rotation group. With respect to the combined symmetry transformations the instanton is invariant. From the observational

\[1\] In the present context “ground state” means the classical field configuration which constitutes the absolute minimum of the Euclidean effective action. It should not be confused with the quantum mechanical vacuum state.
point of view there is no way of distinction between the new “combined” rotations and the standard rotations. Furthermore, the combined rotations act as standard rotations on any gauge-invariant state. Another example for a ground state with a modified translation group are spin waves [4].

The aim of the present letter is to search for a ground state for QCD with $F_{\mu\nu}^z \neq 0$ which nevertheless preserves a new version of Poincaré symmetry as well as $P, T$ and $C$ symmetry.

The four-dimensional rotation symmetry $SO(4)$ is locally equivalent to the direct product $SU(2)_L \times SU(2)_R$ with generator $\vec{\tau}_L, \vec{\tau}_R$ obeying

$$
\begin{align*}
[\tau^i_L, \tau^j_L] &= i\epsilon^{ijk} \tau^k_L \\
[\tau^i_R, \tau^j_R] &= i\epsilon^{ijk} \tau^k_R \\
[\tau^i_L, \tau^j_R] &= 0
\end{align*}
$$

It is obvious that this group structure remains unchanged if we combine $SU(2)_L$ or $SU(2)_R$ or both with appropriate $SU(2)$ subgroups of the gauge group $SU(N)_C$. (We discuss here general $N$ and will later discuss special properties for $N = 3$.)

Denoting by $\vec{\tau}_1, \vec{\tau}_2$ the generators of two groups $SU(2)_1, SU(2)_2$ commuting with $SU(2)_L, SU(2)_R$ one obtains new rotation symmetries

$$
\begin{align*}
SU(2)'_L &= \text{diag}(SU(2)_L, SU(2)_1) \\
SU(2)'_R &= \text{diag}(SU(2)_R, SU(2)_2)
\end{align*}
$$

with generators

$$
\begin{align*}
\vec{\tau}'_L &= \vec{\tau}_L + \epsilon_L \vec{\tau}_1 \\
\vec{\tau}'_R &= \vec{\tau}_R + \epsilon_R \vec{\tau}_2
\end{align*}
$$

We will also consider the possibility that only $SU(2)_L$ (or $SU(2)_R$) is modified and therefore admit $\epsilon_L, \epsilon_R = 0$ or $1$. The new generators $\vec{\tau}'_L$ and $\vec{\tau}'_R$ fulfil the same

[2] Also in the classic example of the “magnetic translation group” [?] the generators are similarly modified.
commutation relations (1) as $\vec{\tau}_L$ and $\vec{\tau}_R$ provided the subgroups $SU(2)_1$ and $SU(2)_2$ commute

$$[\vec{\tau}_1^i, \vec{\tau}_2^j] = 0$$ (4)

2. Let us first investigate the possibility that $SU(2)_{1,2}$ are subgroups of global $SU(N)_C$ transformations. Then the group structure of $SU(N)$ implies that we can modify both $SU(2)_L$ and $SU(2)_R$ ($\epsilon_L = \epsilon_R = 1$) only for $N \geq 4$. For $N = 2, 3$ two commuting $SU(2)$ subgroups do not exist within the global $SU(N)$ transformations and either $\epsilon_L$ or $\epsilon_R$ must vanish. (We will choose $\epsilon_L = 1, \epsilon_R = 0$, but the opposite choice is equivalent.) The ground-state structure may therefore be different for $N \geq 4$ and for $N = 2, 3$.

We begin with the case $N = 4$ where $SU(4)_C$ has a $SO(4)_C$ subgroup. (This discussion can be generalized to all gauge groups containing an $SO(4)$ subgroup.) The 15-dimensional adjoint representation of $SU(4)_C$ transforms with respect to $SO(4)_C = SU(2)_{C1} \times SU(2)_{C2}$ as

$$15 \rightarrow (3, 1) + (1, 3) + (3, 3) \quad (I)$$

or

$$15 \rightarrow (1, 1) + (3, 1) + (1, 3) + (2, 2) + (2, 2) \quad (II)$$ (5)

(There are two inequivalent embeddings I, II of $SO(4)$ in $SU(4)$.) For the embedding II one may put all gauge fields $A_\mu^\alpha$ to zero except for those in one $(2,2)$ representation which we denote in the standard $SO(4)$ vector notation by $A_\mu^\alpha, \alpha = 1\ldots4$. The state

$$\langle A_\mu^\alpha \rangle = a\delta_\mu^\alpha$$ (6)

with constant nonvanishing $a$ is manifestly invariant under standard translations as well as under the combined $SO(4)$ rotation group (2) (with $\epsilon_L = \epsilon_R = 1$). An arbitrary space rotation acting on the index $\mu$ of $A_\mu^\alpha$ can be compensated by an appropriate $SO(4)_C$ gauge rotation acting on $\alpha$. (In a more group-theoretical language $A_\mu^\alpha$
belongs to the representation \( (2,2,2,2) \) with respect to \( SU(2)_L \times SU(2)_R \times SU(2)_{C1} \times SU(2)_{C2} \) and this representation contains a singlet (proportional to \( \mathbf{1} \)) with respect to the subgroup \( SU(2)'_L \times SU(2)'_R, SU(2)'_L = \text{diag}(SU(2)_L, SU(2)_{C1}), SU(2)'_R = \text{diag}(SU(2)_R, SU(2)_{C2}) \). It is easy to verify that the commutation relations between modified rotations \( (\vec{\tau}'_L, \vec{\tau}'_R) \) and translations \( P_\mu \) are the same as for the usual rotations since the gauge transformations of \( SO(4)_C \) are coordinate independent \( ([\vec{\tau}_{1,2}, P_\mu] = 0) \). The state \( (\mathbf{6}) \) is therefore invariant under a modified version of Poincaré symmetry. (The discrete symmetries \( P, T, C \) will be discussed later.) Hence it is a possible candidate for the ground state of a pure \( SU(4)_C \) gauge theory. The field strength for the gauge field \( (\mathbf{6}) \) is easily computed with \( T_\alpha \) the generators of \( SO(4)_C \) and

\[
A_\mu = A_\mu^\alpha T_\alpha
\]  

(7)

One finds a nonvanishing value

\[
F_{\mu\nu} = -ig[A_\mu, A_\nu] = -ig\alpha^2 \delta_\mu^\alpha \delta_\nu^\beta [T_\alpha, T_\beta] = g\alpha^2 f_{\mu\nu\gamma} T_\gamma
\]  

(8)

with \( g \) the gauge coupling and \( f_{\alpha\beta\gamma} \) the structure constants of \( SO(4) \). We emphasize, nevertheless, that \( (\mathbf{6}) \) is not the only possible ground state candidate. Ground state candidates corresponding to the embedding (I) can also be found. They can be described in a way similar to the discussion for \( N = 2, 3 \) to which we will turn next.

Let us now address the realistic case of \( SU(3)_C \). There are two inequivalent embeddings of \( SU(2)_C \) according to which the gluon octet transforms as

\[
\begin{align*}
8 & \to 3 + 5 & \text{(I)} \\
8 & \to 1 + 2 + 2 + 3 & \text{(II)}
\end{align*}
\]

(9)

For the first embedding the fundamental three-dimensional representation (the quarks) transform as a triplet with respect to the \( SO(3) \) subgroup of \( SU(3) \). The second
embedding corresponds to the $SU(2)$-“isospin” symmetry and the results based on this embedding can be applied to the case $N = 2$ as well. In four dimensions there is obviously no possible choice of a constant gauge field $A^\mu_\alpha$ as in (8) since the representation $(2,2)$ with respect to $SO(4)$ space rotations cannot be matched with any of the representations (9). A constant field $A_\mu$ does not contain a singlet with respect to $SU(2)_L \times SU(2)_R$. There is a mismatch between a large number of dimensions ($d = 4$ in the present case) and a small number of colours ($N_C = 2, 3$).

The possibility of a constant gauge field exists, nevertheless, for the three-dimensional theory, which is relevant as an effective theory at high temperature. Here the rotation symmetry is reduced to $SO(3)$ and it is now easy to find a rotation and translation invariant ground state

$$A^\alpha_i = a \delta^\alpha_i$$
$$A_0 = s$$

with $a, s$ constant. Here $\alpha$ denotes the vector index of an $SU(2)$-triplet contained in the decomposition (8) (or the vector of an $SU(2)$ gauge group) and $A_0$ is in the singlet direction. (We denote by $0$ the index corresponding to Euclidean time and $i$ runs from one to three.) We observe that for $a \neq 0$ one can have a non-vanishing $s$ only for the embedding II. (For the pure three dimensional Yang-Mills theory there is no field $A_0$.)

In four dimensions the state (10) is not an acceptable ground state for the zero temperature theory since it is not invariant with respect to “boosts” (or the full $SO(4)$ rotations). It is, however, a candidate for the ground state at high temperatures where the $SO(4)$ symmetry is not respected for “Euclidean time” compactified on a torus. We observe that the configuration (10) gives rise to non-vanishing constant colour magnetic fields $\sim a^2$ whereas the colour electric fields vanish for $s = 0$. If there is a transition from a zero temperature ground state with a symmetry between electric and magnetic fields to the “asymmetric state” (II) at high temperature the difference between the electric and magnetic condensate could be an
interesting signal.

3. For the search of a Poincaré invariant state for $SU(3)_C$ in four dimensions we have to abandon $A_\mu = \text{const}$. We have to consider space dependent $A_\mu(x)$ for which the general discussion becomes more complicated. Let us first ask under what conditions we can have at least a constant field strength

$$F^z_{\mu\nu} = \partial_\mu A^z_\nu - \partial_\nu A^z_\mu + g f^z_{wy} A^w_\mu A^y_\nu$$

which is invariant under a suitable combination of Lorentz rotations and global gauge transformations. As we will see the requirement of an invariant constant field strength is weaker than the corresponding one for the gauge field. Invariance of the ground state field strength under a suitably modified Poincaré transformation is necessary for obtaining Poincaré-covariant Green functions. If such a field strength is found, one may, in a second step, attempt the construction of a ground state gauge field which guarantees the covariance of the Green functions.

With respect to $SU(2)_L \times SU(2)_R$ the field strength $F^z_{\mu\nu}$ decomposes into two irreducible representations $G^z_{\mu\nu}$ and $H^z_{\mu\nu}$ transforming as $(3,1)$ and $(1,3)$:

$$G^z_{\mu\nu} = \frac{1}{2} (F^z_{\mu\nu} + \tilde{F}^z_{\mu\nu})$$

$$H^z_{\mu\nu} = \frac{1}{2} (F^z_{\mu\nu} - \tilde{F}^z_{\mu\nu})$$

$$\tilde{F}^z_{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F^z_{\rho\sigma}$$

with $\varepsilon_{\mu\nu\rho\sigma}$ the totally antisymmetric tensor $\varepsilon_{1234} \equiv \varepsilon_{1230} = 1$.

In terms of colour-magnetic and colour-electric fields

$$F^z_{oi} = E^z_i \quad , \quad \tilde{F}^z_{oi} = B^z_i$$

$$F^z_{jk} = \varepsilon^i_{jk} B^z_i \quad , \quad \tilde{F}^z_{jk} = \varepsilon^i_{jk} E^z_i$$

$^3$We use often the index 0 instead of 4 to be close to a Minkowski notation. Greek indices always run from 1 to 4 (or 0) whereas latin indices run from 1 to 3.
the self-dual and anti-self-dual fields can be written in the form

\[
G^z_i = G^z_{\alpha i} = \frac{1}{2} \varepsilon^{jk} G^z_{jk} = \frac{1}{2} (E^z_i + B^z_i)
\]

\[
H^z_i = H^z_{\alpha i} = -\frac{1}{2} \varepsilon^{jk} H^z_{jk} = \frac{1}{2} (E^z_i - B^z_i).
\] (14)

Now \(G\) is invariant under \(SU(2)_R\) and transforms as a triplet with respect to \(SU(2)_L\). We may therefore form a singlet with respect to \(SU(2)'_L\) if \(G^z_{\mu\nu}\) belongs to a triplet representation \(G^z_{\alpha\mu\nu}(\alpha = 1...3)\) in the decomposition (9). Similar to (3) the singlet with respect to \(SU(2)'_L \times SU(2)_R\) reads now

\[
B^\alpha_i = E^\alpha_i = b \delta^\alpha_i
\] (15)

With constant \(b\) this configuration is also translation invariant. Again, this state with constant magnetic and electric colour fields is invariant under a modified version of Poincaré symmetry. Since the state (14) is self-dual \((F^z_{\mu\nu} = \tilde{F}^z_{\mu\nu})\) it follows immediately that it also obeys the Yang-Mills equation \(F^\mu_{\mu\nu} = 0\). Written more explicitly in terms of the usual \(SU(3)_C\) generators (Gell-Mann matrices \(\lambda_z\)) the state (14) reads for the embeddings (I), (II) with \(F^\mu_{\mu\nu} = F^z_{\mu\nu}\lambda_z\)

\[
F^z_{\mu\nu} = -ib \begin{pmatrix} 0 & s_3 & -s_2 \\ -s_3 & 0 & s_1 \\ s_2 & -s_1 & 0 \end{pmatrix} \] (I)

\[
F^z_{\mu\nu} = b \begin{pmatrix} s_3 & s_1 - is_2 & 0 \\ s_1 + is_2 & -s_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \] (II) (16)

\[
s_1 = \delta_{0\mu}\delta_{i\nu} - \delta_{i\mu}\delta_{0\nu}
\]

\[
+\varepsilon_{ijk}(\delta^j_{i\mu}\delta^k_{\nu} - \delta^k_{i\mu}\delta^j_{\nu})
\] (17)

From the transformation property of the gluon field strength as \((2,2,8)\) with respect to \(SU(2)_L \times SU(2)_R \times SU(3)_C\) we obtain the representations of the modified
“Lorentz” group $SU(2)'_L \times SU(2)_R$ for the two embeddings (I)

\[
\begin{align*}
(3, 1, 8) & \rightarrow (1, 1) + (3, 1) + (3, 1) + (5, 1) + (7, 1) \\
(1, 3, 8) & \rightarrow (3, 3) + (5, 3) \\
\end{align*}
\] (I)

\[
\begin{align*}
(3, 1, 8) & \rightarrow (1, 1) + (2, 1) + (2, 1) + (3, 1) + (4, 1) + (4, 1) + (5, 1) \\
(1, 3, 8) & \rightarrow (1, 3) + (2, 3) + (2, 3) + (3, 3) \\
\end{align*}
\] (II)

The first striking observation is the appearance of spin 1/2 and spin 3/2 states with respect to the modified three dimensional rotation group $diag \ (SU(2)'_L \times SU(2)_R)$ for the embedding (II). If we identify the representations (18) with (parts of) the glueball spectrum the embedding (II) would predict not only bosonic but also fermionic glue ball states! This partial boson-fermion transmutation would also occur for the embedding (II) in (I). In contrast, the embedding (I) predicts integer spin for all states in (18) (and similar for the embedding (I) in (I)). In addition, the $SU(3)_C$ group is here completely broken without any residual $U(1)_C$ symmetry commuting with $SU(2)'_L \times SU(2)_R$. With respect to the three dimensional rotation group $diag \ (SU(2)'_L \times SU(2)_R)$ the fields in $G_{\mu\nu}$ and $H_{\mu\nu}$ have the same spectrum, i.e. singlets, vectors, spin 2 and spin 3 tensors for the embedding I and additional half integer spin states for the embedding II. With respect to the modified boosts, however, the fields in $G_{\mu\nu}$ and $H_{\mu\nu}$ transform quite differently. This is not a worry of principle since colour neutral bound states will always have standard Lorentz-transformation properties. It points, nevertheless, to a strong violation of left-right symmetry and one wonders how such a spectrum can be consistent with parity.

4. In addition to the Poincaré symmetry we will now also require parity conservation for the ground state. Again, we envisage the possibility that parity is realized as a modified transformation in combination with a suitable discrete gauge transformation. The standard parity transformation $(x \rightarrow x')$

\[
\begin{align*}
x_i & \rightarrow -x_i, \quad \partial_i & \rightarrow -\partial_i \\
x_0 & \rightarrow x_0, \quad \partial_0 & \rightarrow \partial_0
\end{align*}
\] (19)
reverses the sign of the electric fields

\[ P(A_i(x)) = -A_i(x') \]
\[ P(A_0(x)) = A_0(x') \]
\[ P(E_i(x)) = -E_i(x') \]
\[ P(B_i(x)) = B_i(x') \] (20)

It therefore maps

\[ G^z_i \rightarrow -H^z_i \]
\[ H^z_i \rightarrow -G^z_i \] (21)

and is obviously violated by the state (15). Let us ask if there could be a modified parity transformation with the property (19) and leaving the state (15) invariant. (Such a transformation would belong to the class \( \tilde{P} \) of generalized parity transformations discussed in [5].) The existence of such a transformation requires a discrete symmetry of the action which does not act on coordinates but nevertheless implies a mapping \( G \rightarrow H \). The modified parity transformation would then be a combination of this symmetry with the standard parity reflection \( P \). Any such mapping \( G \rightarrow H \) must act as an automorphism of the group \( SO(4) \) exchanging the role of \( SU(2)_L \) and \( SU(2)_R \). It cannot be a subgroup of \( SO(4) \) since those transformations cannot “switch” from one representation to another. The same holds for global gauge transformations (the latter commute with \( SO(4) \).)

In the case of QCD the only symmetry transformation exchanging \( SU(2)_L \) and \( SU(2)_R \) representations also acts as a reflection of an odd number of components of any \( SO(4) \) vector. This holds in particular for the coordinate vector \( x_\mu \) in contradiction to what we are looking for. We conclude that the state (15) spontaneously breaks all possible generalized parity symmetries \( \tilde{P} \). It could correspond to the ground state of a parity-violating theory (like a Yang-Mills theory with a chiral fermion content or explicit parity violating \( F_{\mu\nu}\tilde{F}^{\mu\nu} \) interactions). It is, however, not
a realistic candidate for QCD which is known to conserve parity. In consequence, a realistic QCD ground state cannot have a constant field strength $F_{\mu\nu}$ either!

We should mention at this place that the parity problem is absent for the ground state candidate (6) for $SU(4)_C$ (or other gauge groups containing $O(4)$). In fact, the configuration (6) violates the standard parity transformation $P$. We may nevertheless combine $P$ with an automorphism of the $SO(4)$ gauge group which reverses the sign of three components of $A^\alpha_\mu$, i.e.

$$A^\alpha_\mu \rightarrow -A^\alpha_\mu$$

for $\alpha = 1...3$. The state (6) is left invariant by the combined reflection. The global $SU(4)_C$ gauge transformations contain the required automorphism. For $SU(4)_C$ one may also construct a ground state candidate based on the embedding (I) by having a nonvanishing $G_{\mu\nu}$ for the $(3,1)$ representation according to (13) and similarly nonvanishing $H_{\mu\nu}$ for the $(1,3)$ representation. The modified parity reflection is again combined from the standard parity $P$ and a suitable discrete gauge transformation. We remember that such a state would have constant $F_{\mu\nu}$, but not constant $A_\mu$.

5. We may summarize the preceding observations by the statement that neither the gauge field nor the field strength can be constant for a realistic ground state of four-dimensional QCD with three colours. This implies that also translation symmetry cannot be realized in the standard way. Standard translations have to be combined with suitable gauge transformations. We require that any infinitesimal translation of the ground state gauge field $A^z_\mu(x)$ can be compensated by a corresponding infinitesimal gauge transformation with gauge parameter $\theta^z_\mu(x)$

$$- \partial_\mu A^z_\mu(x) = i \theta^w_\mu(x)(T_w)^z_y A^y_\mu(x) + \frac{1}{g} \partial_\mu \theta^z(x)$$  \hspace{1cm} (22)

The combined transformations form the modified translation group with generators

$$P_\mu = -i \partial_\mu \theta^w_\mu(x)T_w$$  \hspace{1cm} (23)

\(^4\)An example of such a combination of standard translations with global abelian gauge transformations is given by the symmetry leaving the spin waves of ref. [4] invariant.
(The modified “momentum operators” act in a standard way on tensors, whereas for
gauge fields the inhomogeneous part of the transformation \(^{[22]}\) has to be included.)

Invariance of the ground state gauge field \(A^z_\mu(x)\) under generalized Lorentz rota-
tions requires similarly

\[
-x_\mu \partial_\nu A^z_\rho + x_\nu \partial_\mu A^z_\rho + \delta_{\mu\rho} A^z_\nu - \delta_{\nu\rho} A^z_\mu \\
= i \eta^w_{\mu\nu}(x) (T^z)(x)^{\nu}_{\rho} A^w_\rho + \frac{1}{g} \partial_\rho \eta^z_{\mu\nu}(x)
\]

The corresponding generalized (Lorentz) rotation operators are

\[
M_{\mu\nu} = S_{\mu\nu} + L_{\mu\nu} + G_{\mu\nu} \\
L_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu) \\
G_{\mu\nu} = \eta^w_{\mu\nu}(x) T^w
\]

Here the “spin operators” \(S_{\mu\nu}\) correspond to rotations acting on the vector index of
\(A_\mu\) and commute with angular momentum \(L_{\mu\nu}\) and the “gauge part” \(G_{\mu\nu}\)

\[
[S_{\mu\nu}, L_{\rho\sigma}] = 0, \quad [S_{\mu\nu}, G_{\rho\sigma}] = 0
\]

Both \(S_{\mu\nu}\) and \(L_{\mu\nu}\) obey separately the \(SO(4)\) commutation relations

\[
[L_{\mu\nu}, L_{\rho\sigma}] = i(\delta_{\mu\rho} L_{\nu\sigma} - \delta_{\mu\sigma} L_{\nu\rho} - \delta_{\nu\rho} L_{\mu\sigma} + \delta_{\nu\sigma} L_{\mu\rho})
\]

(The spin generators \(S_{\mu\nu}\) are linear combinations of \(\tau_L, \tau_R\) in eq. \([1]\).)

In consequence, we associate to every element \(l_i\) of the standard Poincaré group
an element \(g_i\) of the gauge group \(G\) such that the combination \(g_i l_i\) leaves \(A_\mu\) invariant.
This defines a map

\[
f : l_i \longrightarrow g_i l_i
\]

With respect to the group multiplication the invariance of \(A_\mu\) implies for this map

\[
l_1 l_2 \longrightarrow h_{12} g_1 l_1 g_2 l_2
\]

and it is easy to see that \(h_{12}\) must be a gauge transformation which leaves \(A_\mu\)
invariant. (We denote by \(H\) the subgroup of gauge transformations leaving \(A_\mu\)
invariant \((h_{12} \in H)\). Let us discuss the case that \(H\) is trivial (only the identity element) or that the \(g_i\) can be chosen from a subgroup \(\tilde{G} \subset G\) commuting with \(H\). Then we can put \(h_{12} = 1\) in eq. (29) and the mapping (28) is a group homomorphism. The elements \(g_i\) are uniquely determined in this case by the invariance condition \(g_i t_i(A_\mu) = A_\mu\). The image of \(f\) is then isomorphic to the Poincaré group \(P_4\).

By virtue of this isomorphism the generalized translation generators \(P_\mu\) (23) in the different directions must commute

\[
[P_\mu, P_\nu] = -i(\partial_\mu \theta^z_\nu - \partial_\nu \theta^z_\mu) T_w z = 0
\]

\[
\partial_\mu \theta^z_\nu - \partial_\nu \theta^z_\mu = 0
\]  

(30)

Similarly, the modified “angular momenta” \(M_{\mu\nu}\) must obey the same \(SO(4)\) commutation relations as \(L_{\mu\nu}\). In general \(L_{\mu\nu}\) and \(G_{\mu\nu}\) do not commute for \(\eta\) depending on \(x\), and we obtain similar to (30) the consistency condition

\[
\delta_{\mu\rho} \eta_{\nu\sigma} - \delta_{\mu\sigma} \eta_{\nu\rho} - \delta_{\nu\rho} \eta_{\mu\sigma} + \delta_{\nu\sigma} \eta_{\mu\rho} - i \eta_{\mu\nu} \eta_{\rho\sigma} f_{xy} z
+x \mu \partial_\nu \eta_{\rho\sigma} - x \nu \partial_\mu \eta_{\rho\sigma} - x \rho \partial_\sigma \eta_{\mu\nu} + x \sigma \partial_\mu \eta_{\mu\nu} = 0
\]

(31)

The operators \(M_{\mu\nu}\) and \(P_\rho\) must generate the generalized Poincaré group. From

\[
[M_{\mu\nu}, P_\rho] = i \delta_{\mu\rho} P_\nu - i \delta_{\nu\rho} P_\mu
\]  

(32)

we get the additional consistency relation

\[
\delta_{\rho} \eta_{\mu\nu} - x \mu \partial_\nu \theta^z_\rho + x \nu \partial_\mu \theta^z_\rho - \delta_{\mu\rho} \theta^z_\nu + \delta_{\nu\rho} \theta^z_\mu + \eta_{\mu\nu} \theta^y_\rho f_{xy} z = 0
\]

(33)

The problem of finding simultaneous solutions to the conditions (30), (31), and (33) is equivalent to the problem of finding embeddings of the Poincaré group \(P_4\) into the infinite-dimensional group generated by local gauge transformations and standard translations and (Lorentz) rotations. We have presented before a few simple solutions for \(N \geq 4\), but the general problem is quite difficult to solve. The problem of finding simultaneous solutions to eqs. (22) and (24) amounts to the problem of finding gauge fields invariant under the modified \(P_4\) transformation.
Any solution of eqs. (22), (24) should automatically fulfil eqs. (30), (31), and (33) if $h_{12} = 1$ in eq. (29) (see above).

It is amazing to see how the ground state problem for QCD in the framework of the Euclidean effective action turns into an interesting but difficult group-theoretical problem. It is not yet clear to us what is the best way for its solution. The observation may be helpful that all gauge singlets contracted from the ground state field $A_\mu$ must be invariant under the standard Poincaré and parity transformations, i.e.

$$F^z_{\mu\nu}F^\rho\sigma_{z} = \text{const}(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\mu)$$

or

$$F^z_{\mu\nu;\rho}F^\mu\nu_{z} = 0$$

6. Even though we have not yet been able to solve the ground state problem of four-dimensional QCD based on the gauge group $SU(3)_C$, several lessons can be learned from our preliminary investigations. We have demonstrated for the gauge group $SU(4)_C$ or larger groups that there exist indeed translation, rotation, and parity-invariant states with nonvanishing field strength. If such a state can be identified as the ground state and the form of the effective action for field configurations in the vicinity of this state is known, one can derive the spectrum of excitations from the second functional variation of the effective action evaluated at the ground state. This gives the masses of (some of) the glueballs. In this context it is interesting to observe that the excitations of the gauge field have indeed different spins. The field $A_\mu$ can describe a rather rich glueball spectrum which is not restricted to spin one states. On the other hand the gauge group may be completely broken. (For certain ground state candidates a residual global gauge symmetry could also persist.) The phenomenon of “gluon condensates” can be associated in this language with the “spontaneous symmetry-breaking of the gauge symmetry” by non-perturbative

One may check that the configuration (16) does not obey (34).
effects! In this respect there are analogies with the Higgs phenomenon: The role of the scalar field is now played by gauge fields proportional to the ground state field $A_\mu$. This field corresponds to a scalar with respect to the modified Poincaré transformations. The ground state field is equivalent to the vacuum expectation value of this “scalar” excitation\[7\]. The (generalized) scalar excitation corresponding to the Higgs boson is always present within the spectrum of glueball states since the gauge field must contain a singlet with respect to the generalized (Lorentz) rotations.

We have seen that the “embedding problem” of finding a generalized $P_4$ subgroup and a corresponding state left invariant by this subgroup changes qualitatively for a small number of colours $N$. If this is connected to an important quantitative change in the ground state properties for $N \geq 4$ and $N < 4$ an expansion in $1/N$ for a large number of colours may sometimes produce misleading results for $SU(3)_C$.

Another observation is the appearance of more than one ground state candidate for $SU(4)_C$. This suggests that the ground state may not be fixed uniquely by the requirement of Poincaré and parity invariance and $F_{\mu\nu} \neq 0$. In order to distinguish between different possible candidates and to select the true ground state one needs details of the effective action. (The ground state corresponds to the absolute minimum of the effective action.) A reliable computation of the effective action is not easy because of the severe infrared problems in perturbative QCD. In this context the concept of the scale-dependent effective average action $\Gamma_k$\[8\] may prove a useful tool. For the effective average action only the quantum fluctuations with momenta larger than an infrared cutoff, $q^2 > k^2$, are included. The effective action obtains then in the limit $k \to 0$. The dependence of $\Gamma_k$ on the scale $k$ is governed by an exact evolution equation \[9\]. It is encouraging to observe in this context that the lowest order invariant

$$
\Gamma_k^{(0)} = \frac{Z_{F,k}}{4} \int d^4 x F^{z}_{\mu\nu} F^z_{\mu\nu}
$$

changes its sign for small $k$ \[10\], i.e.

$$
Z_{F,k} < 0 \quad \text{for} \quad k < \Lambda_{conf}
$$
Here the vanishing of $Z_{F,k}$ is directly related to a diverging renormalized gauge coupling and occurs at the confinement scale $\Lambda_{\text{conf}}$. The negative value of $Z_{F,k}$ for small $k$ is a clear sign of the instability of the perturbative vacuum and the onset of “gluon condensation” or “nonperturbative spontaneous symmetry-breaking of the gauge group”. A calculation of the $k$-dependence of the coefficients of higher invariants (e.g. $\sim (F_{\mu}^z F_{\nu}^{\mu z})^2$) is in progress [11] and should shed some light on the properties of the ground state in terms of the values of various gauge invariants formed from the ground state field $A_{\mu}$. In particular, it would be interesting to know if a covariantly constant field strength

$$F_{\mu\nu;\rho} = 0$$

is favoured for the ground state or not. Needless to say that a determination of the QCD ground state either by group-theoretical methods or by the use of detailed properties of the effective action would offer new insights into various phenomena of the theory of strong interactions.

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