Abstract. We analyze the weak solution concept for the Fornberg-Whitham equation in case of traveling waves with a piecewise smooth profile function. The existence of discontinuous weak traveling wave solutions is shown by means of analysis of a corresponding planar dynamical system and appropriate patching of disconnected orbits.

1. Basic concepts

1.1. Introduction. The Fornberg-Whitham equation has been introduced as one of the simplest shallow water wave models which are still capable of incorporating wave breaking (cf. [4], [6], [7], [10], [12], [14]). The wave height is described by a function of space and time $u : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$, $(x, t) \mapsto u(x, t)$, we will occasionally write $u(t)$ to denote the function $x \mapsto u(x, t)$. Suppose that an initial wave profile $u_0$ is given as a real-valued function on $\mathbb{R}$. The Cauchy problem for the Fornberg-Whitham equation is

$$u_t + uu_x + K * u_x = 0,$$

(1)

$$u(x, 0) = u_0(x),$$

(2)

where the convolution is in the $x$-variable only and

$$K(x) = \frac{1}{2}e^{-|x|},$$

(3)

which satisfies $(1 - \partial^2_x)K = \delta$.

We note that formally applying $1 - \partial^2_x$ to (1) produces a third order partial differential equation

$$u_t - uu_{xx} - 3u_xu_{xx} - uu_{xxx} + uu_x + u_x = 0,$$

but we will stay with the above non-local equation which corresponds to the original model and is also more suitable for the weak solution concept.

Remark 1.1. Note that we follow here in (1) the sign convention for the convolution term as used in [4, Equation (4)] (or also in [14, Section 13.14]), but used a rescaling of the solution by $3/2$ to get rid of an additional constant factor in the nonlinear term.

Well-posedness results on short time intervals for (1-2) with spatial regularity according to Sobolev or Besov scales have been obtained in [8], [9]. For example, in terms of Sobolev spaces these read as follows: If $s > 3/2$ and $u_0 \in H^s(\mathbb{R})$, then there exists $T_0 > 0$ such that (1) possesses a unique solution $u \in C([0, T_0], H^s(\mathbb{R})) \cap C^1([0, T_0], H^{s-1}(\mathbb{R}))$; moreover, the map $u_0 \mapsto u$ is continuous $H^s(\mathbb{R}) \to C([0, T_0], H^s(\mathbb{R}))$ and $\sup_{t \in [0, T_0]} \|u(t)\|_{H^s(\mathbb{R})} < \infty$.

1.2. Weak solution concept. Equation (1) can formally be rewritten in the form

$$\partial_t u + \partial_x \left( \frac{u^2}{2} \right) + K' * u = 0,$$

(4)

which suggests to define weak solutions in the context of locally bounded measurable functions in the following way.

Date: October 8, 2018.
Definition 1.2. A function \( u \in L^\infty_{loc}(\mathbb{R} \times [0, \infty]) \) is called a weak solution of the Cauchy problem (1–4) with initial value \( u_0 \in L^\infty_{loc}(\mathbb{R}) \), if
\[
\int_0^\infty \int_{-\infty}^\infty \left( -u(x,t)\partial_t \phi(x,t) - \frac{u^2(x,t)}{2} \partial_x \phi(x,t) + (K' * u(., t))(x)\phi(x,t) \right) \, dx \, dt = \int_{-\infty}^\infty u_0(x)\phi(x,0) \, dx
\]
holds for every test function \( \phi \in \mathcal{D}(\mathbb{R}^2) \).

Remark 1.3. In the current paper we will not discuss uniqueness or well-posedness of general weak solutions, which might also require to introduce the concept of an entropy solution \( u \in L^\infty_{loc}(\mathbb{R} \times [0, \infty]) \) with initial value \( u_0 \in L^\infty_{loc}(\mathbb{R}) \) in the sense that
\[
0 \leq \int_0^\infty \int_{-\infty}^\infty \left( |u(x,t) - \lambda| \partial_t \phi(x,t) + \text{sgn}(u(x,t) - \lambda) \frac{u^2(x,t) - \lambda^2}{2} \partial_x \phi(x,t) \right) \, dx \, dt
\]
\[
- \int_{-\infty}^\infty \int_0^\infty \left( \text{sgn}(u(x,t) - \lambda)(K' * (u(., t) - \lambda))(x)\phi(x,t) \right) \, dx \, dt + \int_{-\infty}^\infty |u_0(x) - \lambda| \phi(x,0) \, dx
\]
holds for every nonnegative test function \( \phi \in \mathcal{D}(\mathbb{R}^2) \) and every \( \lambda \in \mathbb{R} \). We note that this entropy condition implies (5), since for any given \( \phi \) we may choose \( \lambda = -r \) and \( \lambda = r \), where \( r > 0 \) is sufficiently large such that \( |u| < r \) holds on \( \text{supp}(\phi) \). Thus, every entropy solution is a weak solution of the Cauchy problem (1–4).

Example 1.4 (Peakon as weak solution). We consider the well-known peakon-type traveling wave solutions (2–6) for the Fornberg-Whitham equation, namely
\[
p(x,t) = \frac{4}{3} \exp\left(-\frac{1}{2} |x - \frac{4}{3} t| \right) = U(x - \frac{4}{3} t),
\]
where \( U(y) := \frac{4}{3} \exp\left(-\frac{1}{2} |y| \right) \) is the profile function. It can be described as the solitary wave of greatest height (see (14, Section 13.14), (6, Section 6) or the more detailed discussions of traveling solitary-wave solutions of the governing equations for two-dimensional water waves propagating in irrotational flow over a flat bed in (3 and 1)). We observe that
\[
p(x,0) = U \in H^s(\mathbb{R}) \iff s < \frac{3}{2},
\]
which is easily seen from the fact that \( \tilde{U}(\xi) \) is proportional to \( 1/(1 + \xi^2) \) (the precise constants depending on the convention of the Fourier transform). Therefore, \( p \) has less spatial regularity than required for the strong solution concept and for the well-posedness result mentioned in the introductory subsection.

It is easy to see that from the peakon given in (6) we obtain a weak solution with initial value \( u_0 = U \): We may calculate directly (e.g., as in (8, Appendix)) \((1 - \partial_x^2)p = 3p/4 + 4\delta(x - 4t/3)/3\) and \((1 - \partial_x^2)(p^2) = 32\delta(x - 4t/3)/9\), which implies that \((1 - \partial_x^2)(\partial_x p + \partial_x (p^2)/2) = \partial_x((1 - \partial_x^2)p) + \partial_x(1 - \partial_x^2)(p^2)/2 = -\partial_x p\) holds in the sense of distributions on \( \mathbb{R}^2 \) and therefore, upon applying \((1 - \partial_x^2)^{-1}\) in the form of spatial convolution with \( K \), that (4) holds for \( p \) on all of \( \mathbb{R}^2 \); now putting \( u(x,t) := p(x,t)H(t) \) (where \( H \) denotes the Heaviside function) and observing that \( \partial_t (p(x,t)H(t)) = p(x,0)\delta(t) + \partial_x p(x,t)H(t) \) (by checking the action on a test function) we arrive at \( \partial_t u + \partial_x (u^2/2) + K' * u = p(.,0)\delta(t) + (\partial_t p + \partial_x (p^2/2) + K' * p)H(t) = p(.,0) \otimes \delta + 0 = u_0 \otimes \delta \), which means exactly (5) when applied to a test function (and noting that \( t \geq 0 \) in \( \text{supp}(u) \) by construction).

The following section is devoted to the construction of traveling wave solutions which are bounded and discontinuous.
2. Bounded traveling waves with discontinuity

The possible continuous traveling waves for the Fornberg-Whitham equation have been obtained and classified successfully by means of studying the properties of corresponding ordinary differential equations for the profile function, e.g., in [2, 15–17]. In our current attempt to construct a discontinuous bounded traveling wave, we will make use of a similar basic strategy and draw on many ideas from these references. In particular, we have to make a somewhat refined analysis of several steps along the way to a corresponding first-order system of ordinary differential equations for the profile function and its derivative. Finally, we will have to find a correct way for patching up a profile function from two disconnected orbits in the topological dynamics. The inspiration for the whole construction stems from a discussion of traveling waves with shocks for a model of radiating gas, the so-called Rosenau model, given in [11].

The typical Ansatz for a traveling wave solution is \( u(x, t) = W(x - ct) \) with a profile function \( W : \mathbb{R} \to \mathbb{R} \) and \( c \in \mathbb{R} \). We suppose that \( W \) is piecewise \( C^2 \) in the following sense

\[
W \text{ is a } C^2 \text{ function off 0 and } W, W', W'' \text{ possess one-sided limits at 0}
\]

and, in addition, we require that there exist \( A, B \in \mathbb{R} \) such that

\[
\lim_{\xi \to -\infty} W(\xi) = A \quad \text{and} \quad \lim_{\xi \to +\infty} W(\xi) = B.
\]

In particular, \( W \) belongs to \( L^\infty(\mathbb{R}) \) and \( u(x, 0) = W(x) \).

2.1. Traveling waves as weak solutions. A traveling wave \( u \) with piecewise smooth profile function \( W \) is a weak solution, if and only if for every test function \( \phi \) on \( \mathbb{R}^2 \) we have (upon a change of variables \( \xi = x - ct \) in the integrals on the left-hand side of (5) and with explicit convolution integral)

\[
\int_0^\infty \int_{-\infty}^\infty \left( -W(\xi)\partial_2 \phi(\xi + ct, t) - \frac{W^2(\xi)}{2} \partial_1 \phi(\xi + ct, t) \right) d\xi dt
\]

\[+ \int_0^\infty \int_{-\infty}^\infty K'(z)W(\xi - z)dz \phi(\xi + ct, t) d\xi dt = \int_{-\infty}^\infty W(x)\phi(x, 0) dx.
\]

In the first integral we make use of the relation \( \partial_2 \phi(\xi + ct, t) = \frac{d}{dt}(\phi(\xi + ct, t)) - c\partial_1 \phi(\xi + ct, t) \), split the \( \xi \)-integral into two parts according to \( \xi < 0 \) and \( \xi > 0 \), and apply integration by parts. Thus, we obtain

\[
\int_0^\infty \phi(ct, t)\left( \frac{W^2(0+) - W^2(0-)}{2} + c(W(0-) - W(0+)) \right) dt
\]

\[+ \int_0^\infty \int_{-\infty}^0 \phi(\xi + ct, t) \left( W(\xi)W'(\xi) - cW'(\xi) + (K' * W)(\xi) \right) d\xi dt
\]

\[+ \int_0^\infty \int_0^\infty \phi(\xi + ct, t) \left( W(\xi)W'(\xi) - cW'(\xi) + (K' * W)(\xi) \right) d\xi dt = 0.
\]

Observe that due to the properties of \( W \), the \( \xi \)-integrals could be re-combined into one integration over \( \mathbb{R} \), but this could cause a misunderstanding about the exact meaning of the differential equation we want to extract from the above condition. First we note the intermediate result.

**Proposition 2.1.** A traveling wave \( u \) with piecewise smooth profile function \( W \) (in the sense of (7)) is a weak solution, if and only if (9) holds for every \( \phi \in \mathcal{D}(\mathbb{R}^2) \).
We note that the linear span of test functions of the form \( \phi(x, t) = \varphi_1(x - ct)\varphi_2(t) \) with \( \varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}) \) is a dense subspace of \( \mathcal{D}(\mathbb{R}^2) \), hence \( \phi \) may be reduced to

\[
\varphi_1(0) \int_0^\infty \varphi_2(t) \left( \frac{W^2(0+) - W^2(0-)}{2} + c(W(0-) - W(0+)) \right) dt
\]

\[
+ \int_0^\infty \varphi_2(t) dt \left( \int_{-\infty}^0 \varphi_1(\xi) \left( W(\xi)W'(\xi) - cW'(\xi) + (K' \ast W)(\xi) \right) d\xi dt
\]

\[
+ \int_0^\infty \varphi_1(\xi) \left( W(\xi)W'(\xi) - cW'(\xi) + (K' \ast W)(\xi) \right) d\xi \right) = 0.
\]

Choosing \( 0 \leq \varphi_1 \leq 1 \) with support arbitrarily close to 0 and \( \varphi_1(0) = 1 \) while letting \( \varphi_2 \) vary in \( \mathcal{D}(\mathbb{R}) \) we deduce \( (W^2(0+) - W^2(0-))/2 = c(W(0+) - W(0-)) \), which yields the Rankine-Hugoniot condition

\[
W(0+) + W(0-) = 2c,
\]

if \( W(0-) \neq W(0+) \). Having observed this we may now choose \( \varphi_2 \) such that \( \int_0^\infty \varphi_2(t) dt = 1 \) and \( \varphi_1 \) with support in \( \xi < 0 \) or in \( \xi > 0 \), but otherwise arbitrary, and deduce

\[
(W(\xi) - c)W'(\xi) + (K' \ast W)(\xi) = 0 \quad \forall \xi \neq 0.
\]

On the other hand, we see that \( \text{(10)} \) and \( \text{(11)} \) together imply \( \text{(9)} \), which along with the above proposition proves the following statement.

**Theorem 2.2.** A traveling wave \( u \) with piecewise smooth, but discontinuous, profile function \( W \) is a weak solution of the Cauchy problem \( \text{(4)} \) with initial value \( u_0 = W \), if and only if \( W \) satisfies the Rankine-Hugoniot condition \( \text{(10)} \) and the integro-differential equation \( \text{(11)} \).

**Remark 2.3.** Note that constant functions \( u \) are obviously strong solutions to \( \text{(4)} \), whereas a piecewise constant, discontinuous, profile function \( W \) cannot produce a weak traveling wave solution \( u \): If \( W(\xi) = AH(-\xi) + BH(\xi) \) with \( A \neq B \), then \( W' = (B - A)\delta \) and this leads to a contradiction in \( \text{(11)} \) due to the convolution term producing \( (B - A)K(\xi) \) in this case.

Now suppose that we have a discontinuous traveling wave solution according to the theorem. We may take the limits \( \xi \to 0- \) and \( \xi \to 0+ \) in Equation \( \text{(11)} \), take the difference of the equations thus obtained, and note that \( K' \ast W \in L^1 \ast L^\infty \) is (uniformly) continuous on \( \mathbb{R} \) \( 14.10.6(ii) \]) to deduce

\[
(W(0+) - c)W'(0+) = (W(0+) - c)W'(0-).
\]

The Rankine-Hugoniot condition \( \text{(10)} \) means \( W(0+) - c = c - W(0-) \), which by discontinuity of \( W \) requires \( W(0-) \neq c \) and \( W(0+) \neq c \), so that we obtain a relation for the one-sided derivatives

\[
W'(0+) + W'(0-) = 0
\]

as a further necessary condition.

### 2.2. Analysis of the integro-differential equation for the traveling wave profile.

We observe that Equation \( \text{(11)} \) can be written as

\[
\left( \frac{(W - c)^2}{2} \right)'(\xi) + (K \ast W)'(\xi) = 0 \quad \forall \xi \neq 0
\]

and we will argue that it may be understood as an equation of distributions gobally on \( \mathbb{R} \), if \( W \) is supposed to satisfy \( \text{(10)} \).

For a piecewise continuous function \( f \) on \( \mathbb{R} \) denote by \( [f] \) the measurable function with \( [f](\xi) := f(\xi) \) for every \( \xi \neq 0 \) and \( [f](0) := 0 \) and recall that for a piecewise \( C^1 \) function \( g \) on \( \mathbb{R} \) we have
for its distributional derivative $g' = [g'] + (g(0+) - g(0-)) \cdot \delta$ (where $[g']$ uses the value of the pointwise classical derivative $g'(\xi)$ for $\xi \neq 0$). Employing this notation, we obtain thanks to (10)

\[
\frac{(W - c)^2}{2}'' = \left[ \frac{(W - c)^2}{2}' \right]' = \left[ \frac{(W - c)^2}{2} \right]' + \frac{(W(0+) - c)^2 - (W(0-) - c)^2}{2} \cdot \delta,
\]

\[
= \left[ \frac{(W - c)^2}{2} \right]' + \frac{(W(0+) - W(0-))(W(0+) + W(0-) + 2c)}{2} \cdot \delta.
\]

Therefore, we have in the sense of distributions on $\mathbb{R}$

\[
0 = \left( \frac{(W - c)^2}{2} \right)' + (K * W)' = \left( \frac{(W - c)^2}{2} + K * W \right)'
\]

which implies that there is a constant $\alpha \in \mathbb{R}$ such that

\[
\frac{(W - c)^2}{2} + K * W = \alpha.
\]

Similarly to the reasoning above, but now in addition employing Equation (12) (which was a consequence of (10) and (11)), we obtain

\[
\frac{(W - c)^2}{2}'' = \left[ (W - c)W' \right]'
\]

\[
= \left[ (W - c)W' \right]' = \left[ \frac{(W - c)W'}{2} + \frac{W'(0+) + W'(0-)}{2} \cdot \delta \right] = \left[ (W - c)W' \right] = \left[ (W')^2 + (W - c)W'' \right],
\]

which allows us to conclude upon differentiating in Equation (14) that

\[
\left[ (W')^2 + (W - c)W'' \right] + K'' * W = 0.
\]

Taking now the difference of the Equations (15) and (17) and recalling that $K - K'' = \delta$ we have the following equation

\[
\frac{(W - c)^2}{2} - \left[ (W')^2 + (W - c)W'' \right] + W = \alpha,
\]

which gives a classical second-order differential equation on $\mathbb{R} \setminus \{0\}$. So far, we have shown the first part of the following

**Proposition 2.4.** For every discontinuous profile function $W$ the Rankine-Hugoniot condition (10) and the integro-differential equation (11) imply Equation (18). On the other hand, in combination with conditions (10) and (12) implies (11) and thus, according to Theorem 2.2, defines a weak traveling wave solution to the Cauchy problem (13).

**Proof.** It remains to prove the second part of the statement. We recall from the details of the above reasoning that conditions (10) and (12) imply the equality (10) (while (10) implies (13)). Applying this to (18) and using again the fact that $(1 - \partial_x^2)K = K - K'' = \delta$ we get

\[
\alpha = \frac{(W - c)^2}{2} - \left( \frac{(W - c)^2}{2} \right)'' + W = \frac{(W - c)^2}{2} (1 - \partial_x^2) (W - c)^2 + (K - K'') * W.
\]

Noting that $K * W = \alpha$ we deduce $\alpha = \frac{(W - c)^2}{2} + K * W$ and differentiate once to obtain (11). \qed

We now determine the constant $\alpha$ that appeared for the first time in (15) with the help of the boundary conditions (8). All the distributional equations above have a pointwise classical meaning in $\mathbb{R} \setminus \{0\}$, hence we may evaluate (15) at any $\xi < 0$ or $\xi > 0$. Moreover, supposing (8) we know that the term $(W(\xi) - c)^2$ possesses a limit when $\xi \to -\infty$ or $\xi \to \infty$. A brief inspection of the
integral defining the convolution $K \ast W$ and appealing to the theorem of dominated convergence shows that this term also has a limit as $\xi \to -\infty$ or $\xi \to \infty$, namely $A$ or $B$, respectively. Therefore, we derive from (15) the relations
\begin{equation}
\frac{(A - c)^2}{2} + A = \frac{(B - c)^2}{2} + B,
\end{equation}
in particular,
\begin{equation}
(A - B)\left(1 + \frac{A + B}{2} - c\right) = 0, \quad \text{hence} \quad A = B \quad \text{or} \quad c = 1 + \frac{A + B}{2}.
\end{equation}

**Remark 2.5.** In case $A = B$ there are plenty of continuous solutions for the profile function $W$. In fact, the constant $A$ clearly is one, but also the peakon $p(x, t) + A$ with $A = (3c - 4)/3$ and $p$ as in [6], and many more solitary wave solutions are given in [2].

As a further observation, to be made use of later, we note that from the boundary condition (8) and the boundedness of $W$ we may deduce that
\begin{equation}
\lim_{\xi \to \pm \infty} W'(\xi) = 0,
\end{equation}
since the rule of de l’Hospital implies
\begin{equation}
\lim_{\xi \to \pm \infty} W(\xi) = \lim_{\xi \to \pm \infty} \frac{\xi W(\xi)}{\xi} = \lim_{\xi \to \pm \infty} (W(\xi) - W'(\xi)).
\end{equation}

2.3. **Transformation to a first-order system of differential equations.** According to Proposition 2.4 we may make use of (18) to construct a weak traveling wave solution by patching together two pieces of solutions, say, $W_1$ defined on $\xi < 0$ and $W_2$ defined on $\xi > 0$, such that the jump conditions (10) and (12) at $\xi = 0$ are satisfied. Therefore, we may extract from (18) the second-order differential equation
\begin{equation}
\frac{(W - c)^2}{2} - (W')^2 - (W - c)W'' = \alpha
\end{equation}
and consider pieces of solutions that are defined (at least) on closed half lines $]-\infty, 0]$ or $[0, \infty[$.

Since a shift $\xi \mapsto \xi - \xi_0$ in the independent variable does not alter the structure of the various equations for $W$ considered so far (once the notation $[f]$ is adapted to jumps at $\xi_0$ and (11) is required for $\xi \neq \xi_0$), hence we may always apply a shift to any appropriate smooth solution piece defined on some half line in order to produce a part of the prospective traveling wave profile to be patched at $\xi = 0$.

Let $W : I \to \mathbb{R}$ denote a solution to (22), where $I = ]-\infty, 0]$ or $I = [0, \infty[$. We will see below how to remove the factor $W(\xi) - c$ in front of the second order derivative in (22), if we require
\begin{equation}
(\forall \xi \in I : W(\xi) < c) \quad \text{or} \quad (\forall \xi \in I : W(\xi) > c).
\end{equation}
Before doing so, we briefly discuss the situations where these assumptions are not met in the following

**Remark 2.6.** Apart from the trivial case with $W$ being the constant solution $c$ (implying thus also $c = A = B$) we have the following instances where $W$ takes on the value $c$ at some point:

(i) If $W(0^-) = c$ or $W(0^+) = c$, then $W$ has to be continuous, for otherwise we obtain a contradiction in the Rankine-Hugoniot condition (10) stating $W(0^+) + W(0^-) = 2c$.

(ii) If $W(\xi) = c$ for some $\xi \neq 0$, then evaluation of (18) at $\xi$ implies $\alpha = c - W'(\xi)^2$, hence
\begin{equation}
\alpha \leq c.
\end{equation}

We obtain then from (19) also that $|A - B| \leq 2$, since $2c = 2A + B$ (in case $A \neq B$) yields
\begin{equation}
0 \leq 2c - 2\alpha = 2 + A + B - ((A - c)^2 + 2A) = 2 + A + B - \left(\frac{(A - B)^2}{4} + 1 + A + B\right) = 1 - \frac{(A - B)^2}{4}.
\end{equation}

In each of the two cases in (23) we attempt the change of coordinate $\xi = h(\xi)$, where $h : J \to I$ with $J = ]-\infty, 0]$ or $J = [0, \infty[$ is the $C^3$ function determined from of the initial value problem
\begin{equation}
h'(\xi) = W(h(\xi)) - c,
\end{equation}
\begin{equation}
h(0) = 0.
\end{equation}
Lemma 2.7. Suppose \( c \neq A, c \neq B, \) and consider \( h \) given by \((24)-(25)\), then the following hold:

(a) In the first case of \((23)\), \( W < c \), we may put \( J = -I \) and obtain that \( h \) is strictly decreasing and bijective.

(b) In the second case of \((23)\), \( W > c \), we may put \( J = I \) and obtain that \( h \) is strictly increasing and bijective.

Proof. We discuss the details in case (b) and for the subcase \( I = [0, \infty) \) only, since the proof for the other configurations is analogous except for obvious sign changes.

From \( c \neq B \) and from the boundary condition \( \lim_{\xi \to \infty} W(\xi) = B \) we deduce for the \( C^2 \) function \( W \) that \( G(\xi) := \int_0^\xi dy/(W(y) - c) \) is finite for every \( \xi > 0 \) while \( \int_0^\infty dy/(W(y) - c) = \infty \), i.e., \( \lim_{\xi \to \infty} G(\xi) = \infty \). Thus, the function \( G: [0, \infty[ \to [0, \infty] \) is \( C^3 \), strictly increasing, and bijective with \( G(0) = 0 \). From \((24)-(25)\) we obtain upon division, integration, and a change of variables that \( h(z) = G^{-1}(z) \) for every \( z \geq 0 \). \( \square \)

We put \( U(z) := W(h(z)) \) and \( V(z) := W'(h(z)) \), then we have

\[
U'(z) = W'(h(z))h'(z) = W''(h(z))(W(h(z)) - c) = V(z)(U(z) - c)
\]

and, with a view on \((22)\), also

\[
V'(z) = W''(h(z))h'(z) = W''(h(z))(W(h(z)) - c) = (W(h(z)) - c)^2 - W'(h(z))^2 + W(h(z)) - \alpha = \frac{(U(z) - c)^2}{2} - V(z)^2 + U(z) - \alpha = -V(z)^2 + \frac{U(z)^2}{2} + (1 - c)U(z) + \frac{c^2}{2} - \alpha.
\]

Thus, we have obtained the following first-order system of ordinary differential equations

\[
\begin{pmatrix} U' \\ V' \end{pmatrix} = \begin{pmatrix} -V^2 + U - cV \\ \frac{U^2}{2} + (1 - c)U + \frac{c^2}{2} - \alpha \end{pmatrix} =: F(U, V),
\]

which is equivalent to \((22)\) under the conditions \( c \neq A \) and \( c \neq B \) as in Lemma 2.7 for solutions restricted to any of the half planes \( U < c \) or \( U > c \), because \( W \) can be recovered from \( U \) via

\[
U(z) = \int_0^z (U(r) - c)dr \quad \text{and} \quad W(\xi) := U(h^{-1}(\xi)).
\]

We have to keep in mind that, due to the strictly decreasing transformation \( \xi = h(z) \) in the case \( W < c \) above, the trajectories of \((22)\) in the half plane \( U < c \) “run backward” in relation to the original part \( \xi \mapsto W_1(\xi) \) of the solution in \( \xi < 0 \).

Remark 2.8. (i) Regarding boundary conditions at infinity, that is, if \( U \) and \( V \) are defined on some interval unbounded above or below, we obtain from \((8)\) and the properties of \( h \) that

\[
\lim_{z \to \mp \infty} U(z) = \lim_{\xi \to \pm \infty} W(\xi) \quad \text{in case } U < c, \quad \text{and} \quad \lim_{z \to \mp \infty} U(z) = \lim_{\xi \to \pm \infty} W(\xi) \quad \text{in case } U > c.
\]

And for the \( V \)-component we deduce

\[
\lim_{z \to \pm \infty} V(z) = 0
\]

directly from \((21)\).

(ii) The function \( H: \mathbb{R}^2 \to \mathbb{R} \), given by (and adapted from \([15]\))

\[
H(U, V) = (U - c)^2 \left(V^2 - \frac{U^2}{4} + \frac{3c - 4}{6}U + \alpha - \frac{c^2}{4} - \frac{e}{3}\right),
\]

is constant along the solutions of \((26)\), i.e., the orbits are subsets of the level sets of \( H \), as can be verified by direct computation.
Below we are going to study a little of the qualitative properties of (26) which will enable us to construct a discontinuous wave profile \( W \) as indicated in the beginning of the current subsection. To outline the construction in more detail, suppose for example that \( B < c < A \) holds—this is a situation to be considered later on, see (34) and the discussion preceding it,—then we have \( W(\xi) > c \) for \( \xi \) “near \(-\infty\)” and \( W(\xi) < c \) for \( \xi \) “near \(\infty\)” from the boundary conditions (5).

Patching up the solution \( W \) then requires: 1. Searching for two solutions to (26) in the form \( P = (U_1, V_1) : ] - \infty, b_1] \to \mathbb{R}^2 \) and \( Q = (U_2, V_2) : ] - \infty, b_2] \to \mathbb{R}^2 \) which satisfy

\[
\begin{align*}
(30) \quad & \lim_{z \to -\infty} U_1(z) = A, \quad U_1(b_1) + U_2(b_2) = 2c, \quad \lim_{z \to -\infty} U_2(z) = B, \\
(31) \quad & \lim_{z \to -\infty} V_1(z) = 0, \quad V_1(b_1) + V_2(b_2) = 0, \quad \lim_{z \to -\infty} V_2(z) = 0
\end{align*}
\]

(recall that the conditions on \( U_2 \) and \( V_2 \) account for the “backward running” in the region \( U < c \).

2. A backtransformation via \( z = h^{-1}(\xi) \) as in (27) of appropriately shifted versions of \( z \to U_1(z) \) and \( z \to U_2(z) \) as patches for \( \xi \to W(\xi) \). In this process, the conditions in (30) imply that the original boundary conditions (8) as well as the Rankine-Hugoniot condition (10) are satisfied by \( W \), while (31) guarantee that also (12) and (21) hold for \( W' \).

The relevant equilibrium points of the vector field \( F \) in (26) are determined from

\[
(U - c)V = 0 \quad \text{and} \quad V^2 = \frac{U^2}{2} + (1 - c)U + \frac{c^2}{2} - \alpha
\]

with the additional restriction to \( U \neq c \) in case of our intended construction of wave profile functions. (Note that \( U = c \) implies \( V^2 = c - \alpha \), which gives again the necessary condition \( \alpha \leq c \) found already in Remark 2.6(ii).) We obtain then the two solutions \( S_- := (U_0^-, 0) \) and \( S_+ := (U_0^+, 0) \) with

\[
U_0^\pm = c - 1 \pm \sqrt{1 + 2(\alpha - c)},
\]

provided that \( 1 \pm 2(\alpha - c) \geq 0 \).

Recall from (19) that in case \( A \neq B \) we have \( 2c = 2 + A + B \), hence \( 2\alpha = A + B + 1 + \frac{(A-B)^2}{4} \) and therefore

\[
U_0^\pm = \frac{A + B}{2} \pm \frac{|A - B|}{2},
\]

which means that \( U_0^- = \min(A, B) \) while \( U_0^+ = \max(A, B) \).

(32) In all further analysis we focus on the case \( A > B \), hence \( U_0^- = B \) and \( U_0^+ = A \).

The Jacobian of the vector field \( F \) is

\[
DF(U, V) = \begin{pmatrix} -cV & U - c \\ U + 1 - c & -2V \end{pmatrix},
\]

which defines linearizations of the system at the equilibrium points \( S_- = (B, 0) \) and \( S_+ = (A, 0) \) with the respective constant coefficient matrices

\[
L_- = \begin{pmatrix} 0 & -A-B \\ -A-B & 0 \end{pmatrix} \quad \text{and} \quad L_+ = \begin{pmatrix} 0 & A+B \\ A+B & 0 \end{pmatrix}.
\]

The eigenvalues of \( L_- \) are

\[
\lambda_1 := -\frac{1}{2}\sqrt{(A-B)(2+A-B)} < 0 \quad \text{and} \quad \lambda_2 := \frac{1}{2}\sqrt{(A-B)(2+A-B)} =: \lambda_2,
\]

hence we have a saddle at \( S_- = (B, 0) \). We have the eigenvectors

\[
r_1 := \left( \begin{array}{c} \sqrt{2+A-B} \\ \sqrt{A-B} \end{array} \right), \quad r_2 := \left( \begin{array}{c} \sqrt{2+A-B} \\ -\sqrt{A-B} \end{array} \right)
\]

for \( \lambda_1, \lambda_2 \), respectively.

The eigenvalues \( \mu \) of \( L_+ \) are determined from \( \mu^2 = (A - B)(A - B - 2)/4 \). If we strengthen (32) to the requirement

\[
(33) \quad A > B + 2,
\]
then \( S_+ = (A, 0) \) is a saddle point as well, since

\[
\mu_1 := -\frac{1}{2} \sqrt{(A - B)(A - B - 2)} < 0 < \frac{1}{2} \sqrt{(A - B)(A - B - 2)} =: \mu_2.
\]

In this case, there are the eigenvectors

\[
s_1 := \left( \frac{\sqrt{A - B - 2}}{\sqrt{A - B}}, -\frac{\sqrt{A - B}}{\sqrt{A - B}} \right), \quad s_2 := \left( \frac{\sqrt{A - B - 2}}{\sqrt{A - B}}, \frac{\sqrt{A - B}}{\sqrt{A - B}} \right)
\]

for the eigenvalues \( \mu_1, \mu_2 \), respectively.

**Remark 2.9.** Recall from Remark 2.6(ii) that the inequality (33) also ensures that we cannot have \( W(\xi) = c \), thus supporting the separation into the open half planes \( U < c \) and \( U > c \).

We observe that (33) in combination with (20), i.e., \( c = 1 + \frac{A + B}{2} \), gives the refined condition

(34) \hspace{1cm} B + 2 < c < A.

In particular, we see that \( S_- = (B, 0) \) lies in the left half plane \( U < c \), while the saddle point \( S_+ = (A, 0) \) belongs to the region with \( U > c \). A prospective discontinuous wave profile function thus has to be constructed from a trajectory \( P \) in the right half plane “emerging at \( z = -\infty \)” from \((A, 0)\) with a jump to a trajectory \( Q \) in the left half plane connecting to \((B, 0)\) asymptotically, where the points of “departure” from \( P \) and of “arrival” on \( Q \) have to be chosen such that the middle parts in the conditions (30-31) are satisfied.

2.4. **Existence of a discontinuous traveling wave as weak solution.** We suppose that (34) holds, i.e., \( B + 2 < c < A \), such that we have saddle points at \( S_- = (B, 0) \) and \( S_+ = (A, 0) \) for the dynamics according to (26) as discussed in the previous subsection. From the eigenvectors \( r_2 \) and \( s_2 \) corresponding to the positive eigenvalues in each of the saddle points, we see that there is a unique trajectory \( P \) with \( \lim_{z \to -\infty} P(z) = (A, 0) \) leaving at \( (A, 0) \) in the direction up (growing \( V \)) and to the right (growing \( U \)) and a unique trajectory \( Q \) with \( \lim_{z \to -\infty} Q(z) = (B, 0) \) leaving at \( (B, 0) \) in the direction down (decreasing \( V \)) and to the right (growing \( U \)).
Lemma 2.10. Suppose that \( (34) \) holds and let \( P = (U_1, V_1) \) and \( Q = (U_2, V_2) \) be the trajectories defined above. There are unique parameter values \( b_1, b_2 \in \mathbb{R} \), such that
\[
U_1(b_1) + U_2(b_2) = 2c \quad \text{and} \quad V_1(b_1) + V_2(b_2) = 0
\]
hold.

Proof. We proceed in several steps, proving first separate claims for \( P \) and \( Q \).

Claim 1: \( U_1 \) and \( V_1 \) are both strictly increasing.

The definition of \( P = (U_1, V_1) \) implies that
\[
U_1(z) > A \quad \text{and} \quad V_1(z) > 0
\]
holds for \( z \) at least in some interval of the form \( ] - \infty, z_0[ \). For every \( z \in \mathbb{R} \) satisfying \( (36) \), the first line in \( (26) \) reads

\[
U_1(z) = (U_1(z) - c)V_1(z) > (A - c)V_1(z) > 0,
\]
hence \( U_1 \) is strictly increasing and the first condition in \( (36) \) stays valid.

The second line of system \( (26) \) and \( (19) \) give
\[
V_1(z) = -V_1(z)^2 + \frac{U_1(z)^2}{2} + (1 - c)U_1(z) + \frac{c^2}{2} - \alpha = -V_1(z)^2 + \frac{U_1(z)^2}{2} + (1 - c)U_1(z) - \frac{A^2}{2} + (c - 1)A.
\]
From Remark 2.8 we also know that \( P(z) \) lies on the level set of the function \( H \) for the value \( H(A, 0) \), which implies that the relation
\[
(U_1(z) - c)^2 \left( V_1(z)^2 - \frac{U_1(z)^2}{4} + \frac{3c - 4}{6}U_1(z) + \alpha - \frac{c^2}{4} - \frac{c}{3} \right) = \left( A - c \right)^2 \left( \frac{A^2}{4} + \left( \frac{1}{3} - \frac{c}{2} \right)A + \frac{c^2}{4} - \frac{c}{3} \right)
\]
holds. By \( (36) \) we have \( U_1(z) - c > A - c \), hence the reverse inequality holds for the second factors in the above equation, i.e.,
\[
V_1(z)^2 - \frac{U_1(z)^2}{4} + \frac{3c - 4}{6}U_1(z) + \alpha - \frac{c^2}{4} - \frac{c}{3} < \frac{A^2}{4} + \left( \frac{1}{3} - \frac{c}{2} \right)A + \frac{c^2}{4} - \frac{c}{3},
\]
which, upon calling again on \( (19) \), we may rewrite in the form
\[
V_1(z)^2 < \frac{U_1(z)^2}{4} - \frac{3c - 4}{6}U_1(z) - \frac{A^2}{4} + \left( \frac{c}{2} - \frac{2}{3} \right)A.
\]
Thus, we have a lower bound for \( -V_1(z)^2 \) that we insert in the equation for \( V_1'(z) \) and find (using the condition \( (36) \) towards the end of the following estimate)
\[
V_1'(z) > -\frac{U_1(z)^2}{4} + \frac{3c - 4}{6}U_1(z) + \frac{A^2}{4} - \left( \frac{c}{2} - \frac{2}{3} \right)A + \frac{U_1(z)^2}{2} + (1 - c)U_1(z) - \frac{A^2}{2} + (c - 1)A
\]
\[
= \frac{U_1(z)^2}{4} + \left( \frac{1}{3} - \frac{c}{2} \right)U - \frac{A^2}{4} - \left( \frac{1}{3} - \frac{c}{2} \right)A = \frac{U_1(z)^2 - A^2}{4} + \left( \frac{1}{3} - \frac{c}{2} \right)(U - A)
\]
\[
= (U - A)(\frac{U_1(z)^2 + \frac{1}{3} - \frac{c}{2}}{4} > (U - A)(\frac{A + A}{4} + \frac{1}{3} - \frac{c}{2}) = (U - A)(\frac{A - c}{2} + \frac{1}{3}) > 0.
\]
Hence we see that \( V_1 \) is strictly increasing and the conditions in \( (38) \) remain valid throughout.

Claim 2: Let \( ] - \infty, p[ \) be the maximal interval of existence for the solution \( P \) (regardless whether \( p \) is finite or \( p = \infty \), though we conjecture the latter), then \( U_1(z) \to \infty \) and \( V_1(z) \to \infty \) as \( z \to p \).

We recall that \( P \) stays entirely in the region \( (36) \), where in particular \( U_1 > A > c \). Hence the norm \( \| P(z) \| \) has to become unbounded as \( z \to p \), since the solution cannot reach the boundary points of the domain along \( U = c \). Thus, at least one of the components \( U_1(z) \) or \( V_1(z) \) is unbounded as \( z \to p \). But observing \( U_1(z) - c > A - c > 0 \) we may deduce from \( (37) \) that either both component functions, \( U_1(z) \) and \( V_1(z) \), stay bounded or both are unbounded as \( z \to p \). We conclude that both have to be unbounded.
Claim 3: The function $U_2$ is strictly increasing, while $V_2$ is strictly decreasing. Moreover, the maximal interval of existence for the solution curve $Q$ is of the form $]-\infty, q[ \text{ with some } q \in \mathbb{R}$ and, as $z \to q$, we have $V_2(z) \to -\infty$ and $U_2(z) \to c$.

By definition of $Q = (U_2, V_2)$, we have

\[(38) \quad B < U_2(z) < c \quad \text{and} \quad V_2(z) < 0\]

for $z$ at least in some interval bounded only on the right (note that by connectedness, $U < c$ has to hold for every solution starting somewhere in the left half plane). For every $z \in \mathbb{R}$ satisfying (38), we immediately deduce from the first line in (26) that

\[U_2'(z) = (U_2(z) - c) V_2(z) > 0,\]

hence $U_2$ is strictly increasing and the first condition in (38) will continue to be valid.

From the second line in (26) we obtain

\[V_2'(z) = -V_2(z)^2 + \frac{U_2(z)^2}{2} + (1-c)U_2(z) + \frac{c^2}{2} - \alpha = -V_2(z)^2 + \frac{(U_2(z) + 1-c)^2 - 1}{2} + c - \alpha.\]

We note that (38) and (20) imply $1 = c + 1 - c > U_2(z) + 1 - c > B + 1 - c = B - A$ and by (34) we then have

\[|U_2(z) + 1 - c| \leq \max(1, \frac{A - B}{2}) = \frac{A - B}{2}.\]

Moreover, (19) gives $c - \alpha = \frac{1}{2} - \frac{(A-B)^2}{8}$ and therefore,

\[V_2'(z) \leq -V_2(z)^2 + \frac{(A-B)^2}{4} - 1 + \frac{1}{2} - \frac{(A-B)^2}{8} = -V_2(z)^2 < 0,\]

which implies that $V_2$ is strictly decreasing and the second condition in (38) will hold throughout. In particular, we obtained the inequality $-V_2'/V_2^2 \geq 1$, which we may integrate over an interval $[z_0, z]$ to obtain the chain of inequalities

\[0 > \frac{1}{V_2(z)} \geq \frac{1}{V_2(z_0)} + z - z_0,\]

which implies $z < z_0 - 1/V_2(z_0)$ and that $V_2(z) \to -\infty$ when $z$ approaches the upper bound. Therefore, the maximal interval of existence of $Q$ is of the form $]-\infty, q[ \text{ with finite } q \in \mathbb{R}$.

By boundedness and monotonicity of $U_2$, there exists $c_0 := \lim_{z \to q} U_2(z)$ satisfying $B < c_0 \leq c$. We may argue as in (21) to see that $\lim_{z \to q} U_2'(z) = 0$. On the other hand, $U_2'(z) = (U_2(z) - c)V_2(z)$ would necessarily tend to $+\infty$ (as $z \to q$), unless $c_0 = c$.

Combining now the information from the claims proved above, we complete the proof by the following observation: If $\bar{P}$ denotes the pointwise reflection of $P$ at $(c,0)$, i.e., $\bar{P}(z) = 2(c,0) - P(z)$, then $Q$ and $\bar{P}$ have a unique intersection point (in the region $B < U < c, V < 0$), which corresponds to a unique parameter value $b_2$ along $Q$ and to a unique parameter value $b_1$ along $P$. The condition of reflection at $(c,0)$ reproduces precisely the relation (35). \hfill \Box

The above lemma shows that we can indeed find solutions to (26) satisfying (30-31), which can therefore be used as solution patches for a discontinuous wave profile $W$.

**Theorem 2.11.** If $B + 2 < c < A$, then there exists a discontinuous wave profile $W$ defining a weak traveling wave solution $u$ to the Cauchy problem (12) with initial value $u_0 = W$.

**Proof.** Let $U_1$ and $U_2$ be the first components of the solution curves $P$ and $Q$ introduced above and let $b_1, b_2$ be as in Lemma 2.10. Note that the hypothesis $B + 2 < c < A$ implies that Lemma 2.7 and the subsequent transformations between the second-order equation for $W$ and the first-order system for $(U, V)$ are applicable and preserve equivalence. Construct $h$ from $U$ as in (27) and put $W(\xi) := U_1(h^{-1}(\xi) + b_1)$, if $\xi < 0$, and $W(\xi) := U_2(h^{-1}(\xi) + b_2)$, if $\xi > 0$. From (30-31) and the observations in the discussion of these conditions, we see that the proof is complete by appealing to the second statement in Proposition 2.4. \hfill \Box
These specific discontinuous traveling wave solutions serve here more as a mathematical test case for the weak solution concept and will not be useful as models of water wave profiles. One might see them as a reminiscence of shock wave solutions for the Burgers equation in its nonlocal perturbation described by [4], or rather in its weak form by [5].

References

[1] C. J. Amick, L. E. Fraenkel, and J. F. Toland. On the Stokes conjecture for the wave of extreme form. Acta Math. 148:193–214, 1982.
[2] A. Chen, J. Li, and W. Huang. Single peak solitary wave solutions for the Fornberg-Whitham equation. Appl. Anal. 91 (3):587–600, 2012.
[3] A. Constantin and J. Escher. Particle trajectories in solitary water waves. Bull. Amer. Math. Soc. (N.S.) 44 (3):423–431, 2007.
[4] A. Constantin and J. Escher. Wave breaking for nonlinear nonlocal shallow water equations. Acta Math. 181 (2):229–243, 1998.
[5] J. Dieudonné. Treatise on analysis. Vol. II. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976. Enlarged and corrected printing, Pure and Applied Mathematics, 10-II.
[6] B. Fornberg and G. B. Whitham. A numerical and theoretical study of certain nonlinear wave phenomena. Philos. Trans. Roy. Soc. London Ser. A 289 (1361):373–404, 1978.
[7] S. V. Haziot. Wave breaking for the Fornberg-Whitham equation. J. Differential Equations 263 (12):8178–8185, 2017.
[8] J. Holmes. Well-posedness of the Fornberg-Whitham equation on the circle. J. Differential Equations 260 (12):8530–8549, 2016.
[9] J. Holmes and R. C. Thompson. Well-posedness and continuity properties of the Fornberg–Whitham equation in Besov spaces. J. Differential Equations 263 (7):4355–4381, 2017.
[10] G. Hörmann. Wave breaking of periodic solutions to the Fornberg-Whitham equation. Discr. Cont. Dyn. Syst. Ser. A 38 (3):1605–1613, 2018.
[11] S. Kawashima and S. Nishibata. Shock waves for a model system of the radiating gas. SIAM J. Math. Anal. 30 (1):95–117, 1999.
[12] P. I. Naumkin and I. A. Shishmarëv. Nonlinear nonlocal equations in the theory of waves. Translations of Mathematical Monographs, vol. 133. American Mathematical Society, Providence, RI, 1994. Translated from the Russian manuscript by Boris Gommerstadt.
[13] R. L. Seliger. A note on the breaking of waves. Proc. Roy. Soc. A 303:493–496, 1968.
[14] G. B. Whitham. Linear and nonlinear waves. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974. Pure and Applied Mathematics.
[15] J. Zhou and L. Tian. A type of bounded traveling wave solutions for the Fornberg-Whitham equation. J. Math. Anal. Appl. 346 (1):255–261, 2008.
[16] J. Zhou and L. Tian. Periodic and solitary wave solutions to the Fornberg-Whitham equation. Math. Probl. Eng.:Art. ID 507815, 10, 2009.
[17] J. Zhou and L. Tian. Solitons, peakons and periodic cusp wave solutions for the Fornberg-Whitham equation. Nonlinear Anal. Real World Appl. 11 (1):356–363, 2010.

Fakultät für Mathematik, Universität Wien, Austria
E-mail address: guenther.hoermann@univie.ac.at

12