FINITE SIZE CORRECTIONS ON THE BOUNDARY BETWEEN THE
SPIN-GLASS AND THE FERROMAGNETIC PHASES OF DERRIDA’s
MODEL

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Abstract

The boundary line between the ferromagnetic and the spin-glass phases was investigated. Finite size corrections to the free energy and magnetization were calculated.

The situation coincides with the case in the information theory, when the transmission rate equals to the capacity of channel.

1. Introduction.

Derrida’s model [1] is connected with optimal coding, as was suggested in [2] and proved in [3]. Different aspects of this connection were considered in [4,5].

In [6,7] were considered finite size effects to the magnetization for the fully connected and diluted models in the ferromagnetic phase. This is interesting both for physics and information theory applications. It corresponds to decoding error probability. Rich fine structure of the ferromagnetic phase was considered in [8]. In [9] was calculated vanishing magnetization in the spin-glass phase.

It’s also very interesting to calculate finite size effects on the boundary line between the ferromagnetic and the spin-glass phases.
2. Finite size corrections to the free energy.

We consider Derrida’s model with the Hamiltonian:

\[ H = - \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_p \leq N} \left( J_0 \frac{N}{C_N} + J_{i_1 \cdots i_p} \sqrt{\frac{N}{C_N}} \right) \sigma_{i_1} \cdots \sigma_{i_p} \] (1)

where \( J_{i_1 \cdots i_p} \) are quenched random coupling constants with a Gaussian distribution:

\[ < J_{i_1 \cdots i_p}^2 > = \frac{1}{2} \] (2)

The ferromagnetic phase appears (in low temperatures) [8]:

\[ J_0 > \sqrt{\ln 2} \] (3)

We consider just the case:

\[ J_0 = \sqrt{\ln 2} \] (4)

As argued Derrida, with high (exponential) accuracy our system (1) is equivalent to Random Energy Model (REM) of \( 2^N \) energy levels, distributed independently. The first energy level has the probability distribution:

\[ P(E) = \frac{1}{\sqrt{\pi N}} e^{-\frac{(E + J_0 N B)^2}{N}} \] (5)

and another \( 2^N - 1 \) levels:

\[ P(E) = \frac{1}{\sqrt{\pi N}} e^{-\frac{E^2}{N}} \] (6)

We therefore find that the free energy of REM is given by:

\[ < \ln Z >= \ln \sum_{\alpha=1}^{2^N} e^{-BE_\alpha} \] (7)

and the averaging is over the distribution of \( E_\alpha \).

To perform the averaging in (7) we use the trick introduced by Derrida [1]:

\[ < \ln Z > = \Gamma'(1) + \int_\infty^\infty u \, d[e^{-\phi(u)}] \] (8)
where \( u = \ln t; \ e^{-\phi} = <e^{-tZ}> \) and \( \Gamma' \) is derivative of the gamma function.

It is easy to see that:

\[
e^{-\phi(u)} = f(u + J_0 NB) f(u)^{2N-1}
\]  

(9)

where

\[
f(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dy e^{-y^2 - e^{-\lambda y + u}}, \ \lambda = B\sqrt{N}
\]  

(10)

Derrida has given approximate expressions for the function \( f(u) \):

\[
f(u) \approx \Gamma\left(\frac{2u}{\lambda^2}\right) e^{-u^2/\lambda^2}; \quad u > 0
\]  

(11)

\[
f(u) \approx 1 - \frac{\left|\Gamma\left(\frac{2u}{\lambda^2}\right)\right|}{\sqrt{\pi} \lambda} e^{-u^2/\lambda^2}; \quad \frac{\lambda^2}{2} < u < 0
\]  

(12)

\[
f(u) \approx 1 - e^{u + \frac{\lambda^2}{2}}; \quad -\lambda^2 < u < -\frac{\lambda^2}{2}
\]  

(13)

In the appendix we have derived the expression for \( f(u) \) in the case \( 1 << |u| << \lambda \).

To evaluate approximately (8) we write it in a suitable form:

\[
<\ln Z> = u_0 + \int_{-\infty}^{\infty} f(u + u_0) \Psi(u) \, du
\]  

(14)

where \( u_0 = BJ_0 N \) and \( \Psi(u) = 1 - f(u)^{2N-1} \).

Let’s denote:

\[f(u + u_0) = F'(u); \quad F(u) = \int_{0}^{u+u_0} f(x) \, dx\]

For our purposes it’s enough to know the values of \( F(u) \) near to \(-u_0\).

To calculate the integral in (14) it is convenient to integrate by parts as the derivation of the function \( \Psi(u) \) behaves like \( \delta- \) function.

The thing is that both functions \( F'(u) \) and \( \Psi(u) \) are like step-functions at the same point (as we are on the boundary between the phases). Moreover we can replace the integration \( \int_{-\infty}^{\infty} \) by \( \int_{-u_0-\delta}^{-u_0+\delta} \) because of the behavior of our functions \( f(u) \) and \( \Psi(u) \) (where \( \lambda << \delta << \lambda^2 \)), outside the interval \([-u_0 - \delta, -u_0 + \delta]\) \( f(u) \ast \Psi(u) \) vanishes exponentially.
That is:

\[
\int_{-\infty}^{\infty} F'(u) \Psi(u) \, du = F(u) \Psi(u) \bigg|_{-u_0-\delta}^{u_0+\delta} - \int_{-u_0-\delta}^{u_0+\delta} F(u) \Psi'(u) \, du \tag{15}
\]

As has been proved in the appendix:

\[
F(u) \simeq \int_{0}^{x} \frac{1}{\sqrt{\pi}} \int_{\delta}^{\infty} e^{-t^2} \, dt - \frac{C}{\sqrt{\pi}} \exp\left[-u^2/\lambda^2\right] \tag{16}
\]

where \( C \) is Euler constant. As the function \( \Psi(u) \) is very close to a step function: it is very near one for \( u > -u_0 \) and near zero for \( u < -u_0 \), so

\[
F(-u_0 + \delta) \Psi(-u_0 + \delta) \simeq F(-u_0 + \delta) = \simeq \frac{\lambda}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t^2} \, dt - C/2 \tag{17}
\]

\[
F(-u_0 - \delta) \Psi(-u_0 - \delta) \simeq 0 \quad \text{because} \quad \Psi(-u_0 - \delta) \sim e^{-\frac{\delta^2}{\lambda^2}} \to 0
\]

The integral in (15) we denote by:

\[
I = - \int_{-u_0-\delta}^{-u_0+\delta} F(u) \, d\Psi(u) = \int_{-u_0-\delta}^{-u_0+\delta} F(u) df(u)^2N \tag{18}
\]

and after expansion the function \( F(u) \) near the point \( u = -u_0 \)

\[
I \simeq - \int_{-u_0-\delta}^{-u_0+\delta} \left[ F(-u_0) + F'(-u_0)(u + u_0) + F''(-u_0) \frac{(u + u_0)^2}{2} \right] df(u)^M \tag{19}
\]

We’ll need \( u \) expressed in terms of \( \phi \).

Using Derrida’s tricks

\[
f^M = e^{-\phi} = e^{-\frac{A}{\sqrt{\lambda^2 + \frac{u^2}{\lambda^2} + N \ln 2}}}; \quad \text{where} A \sim 1;
\]

\[
\ln \phi = -\frac{u^2}{\lambda^2} + N \ln 2 - \frac{1}{2} \ln N + \ln A;
\]

\[
\frac{u^2}{\lambda^2} = N \ln 2 - \frac{1}{2} \ln N + \ln A - \ln \phi
\]

It is easy to get that:

\[
u \simeq -BN \sqrt{\ln 2} + \frac{B \ln N}{4\sqrt{\ln 2}} + \frac{B \ln \phi}{2\sqrt{\ln 2}} \tag{20}
\]
Using this we obtain:

\[ I = -F(-u_0) + F'(-u_0) \int_{-u_0-\delta}^{-u_0+\delta} (u + u_0) \, df + \frac{F''(-u_0)}{2} \int_{-u_0-\delta}^{-u_0+\delta} (u + u_0)^2 \, df^2 = 0 + f(0)I_1 + f'(0)I_2 \]  

(21)

where by \( I_1, I_2 \) are denoted the integrals, given above.

Using (19) we find that:

\[ I_1 \approx -\frac{B \ln N}{4\sqrt{\ln 2}} \]  

(22)

In the appendix we derived the expressions for \( f(0) \) and \( f'(0) \).

After all this tricks and conserving the major elements we obtain:

\[ <\ln Z> = -u_0 - \frac{1}{4} \frac{B}{B_c} \ln N + \frac{B\sqrt{N}}{\sqrt{\pi}} \int_0^\infty dx \int_0^\infty e^{-t^2} dt - C/2; \]

\[ B_c = 2\sqrt{\ln 2}; \]

\[ u_0 = J_0 BN \]  

(23)

We have got a strange result: the finite size effects on the SG-FM boundary are of the order of \( \sqrt{N} \).
3. Calculation of the magnetization.

The magnetization for particularly values of $E_1 \ldots E_M$ equals:

$$m = \left\langle \frac{e^{-BE_1} - e^{-BE_2}}{\sum_{\alpha=1}^{M} e^{-BE_\alpha}} \right\rangle; \quad (M = 2^N) \quad (24)$$

With the accuracy $\sim 2^{-N}$ we neglect by second term in the nominator and transform (23) to the form:

$$m = \int P(E_1 \ldots E_M) e^{-BE_1} \int_{0}^{\infty} e^{-t \sum_{\alpha} e^{-BE_\alpha}} = - \int_{-\infty}^{\infty} du f'(u + u_0) f^{M-1}(u) =$$

$$= 1 - \int_{-\infty}^{\infty} du f'(u + u_0) [f^{M-1}(u) - 1] = 1 + \int_{-\infty}^{\infty} f(u + u_0) df^{M-1} =$$

$$= 1 + \int_{-\infty}^{\infty} f(u + u_0) de^{-\phi} \quad (25)$$

Let’s expand the function $f(u + u_0)$ near the point $u = -u_0$:

$$f(u + u_0) = f(0) + f'(0)(u + u_0) + f''(0)\frac{(u + u_0)^2}{2} \quad (26)$$

Using formula (19), which gives $u$ expressed by $\phi$, we obtain:

$$m = \frac{1}{2} + \frac{1}{\sqrt{\pi NB}} C - \frac{1}{\sqrt{\pi N 2B_c}} \ln N \quad (27)$$

where $C$ is Euler’s constant (see Appendix A). In the limit $B \to \infty$ the second term disappears.

This formula is the main result of our work.

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Appendix A

In this appendix we calculate the values of $f(0)$ and $f'(0)$, where

$$f(u) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2-e^{\lambda x+u}} \, dx$$  \hspace{1cm} (A.1)

By changing the variables $e^{\lambda x+u} = y$, we obtain

$$f(u) = \frac{1}{\sqrt{\pi \lambda}} e^{-u^2/\lambda^2} \int_{0}^{\infty} e^{-\left(\ln y\right)^2+(\frac{2u}{\lambda}-1)\ln y-y^\lambda} \, dy$$  \hspace{1cm} (A.2)

Then split it like this:

$$f(u) = \frac{1}{\sqrt{\pi \lambda}} e^{-u^2/\lambda^2} \int_{0}^{1} \ldots + \frac{1}{\sqrt{\pi \lambda}} e^{-u^2/\lambda^2} \int_{1}^{\infty} \ldots$$  \hspace{1cm} (A.3)

By changing the variables in the first integral (which we denote by $J_1$) according to $x = y^\lambda$, we get:

$$J_1 = \frac{1}{\sqrt{\pi}} e^{-u^2/\lambda^2} \int_{0}^{1} \ldots$$  \hspace{1cm} (A.4)

After the expansion $e^{-y^\lambda} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} e^{\lambda k \ln y} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} y^{\lambda k}$ and replacement $\ln y = x$, we have

$$J_1 = \frac{1}{\sqrt{\pi}} e^{-u^2/\lambda^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{0}^{1} e^{-x^2-\frac{2u}{k\lambda}x-\lambda k x} \, dx$$

For $f(u)$ we obtain:

$$f(u) = \frac{1}{\sqrt{\pi}} \int_{0}^{1} e^{-z^2} \, dz + \frac{1}{\sqrt{\pi}} e^{-u^2/\lambda^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} e^{\left(\frac{u}{k\lambda}\right)^2} \int_{1}^{\infty} e^{-z^2} \, dz$$

$$+ \frac{1}{\sqrt{\pi \lambda}} e^{-u^2/\lambda^2} \int_{1}^{\infty} e^{-\left(\ln x\right)^2+(\frac{2u}{\lambda^2}-1)\ln x-x} \, dx$$  \hspace{1cm} (A.5)

Now it is not difficult to check that:

$$f(0) \simeq \frac{1}{2} + \frac{1}{\sqrt{\pi \lambda}} \sum_{k=1}^{\infty} \frac{(-1)^k}{k! k} + \frac{1}{\sqrt{\pi \lambda}} \int_{1}^{\infty} \frac{e^{-x}}{x} \, dx + O\left(\frac{1}{\lambda^2}\right); f'(0) \simeq -\frac{1}{\sqrt{\pi \lambda}} + O\left(\frac{1}{\lambda^2}\right)$$

If we note that:

$$C = -\int_{1}^{\infty} \frac{e^{-x}}{x} \, dx - \sum_{k=1}^{\infty} \frac{(-1)^k}{k! k}$$  \hspace{1cm} (A.6)
then

\[ f(0) \simeq \frac{1}{2} - \frac{1}{\sqrt{\pi} \lambda} C + O\left(\frac{1}{\lambda^2}\right) \]  \hspace{1cm} (A.7)

\[ f'(0) \simeq -\frac{1}{\sqrt{\pi} \lambda} + O\left(\frac{1}{\lambda^2}\right) \]  \hspace{1cm} (A.8)

where \( C = 0.577\ldots \) is Euler’s constant.

We also need the integral:

\[ \int_0^\infty f(u) du \simeq \frac{B \sqrt{N}}{\sqrt{\pi}} \int_0^\infty dx \int_x^\infty e^{-t^2} dt - C/2 \]  \hspace{1cm} (A.9)

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