HOFER’S NORM AND DISK TRANSLATIONS IN AN ANNULUS

MICHAEL KHANEVSKY

ABSTRACT. Let $\mathcal{D}$ be a non-displaceable disk in an annulus $\mathbb{A}$. Suppose that $\phi$ is a compactly supported Hamiltonian such that $\phi(\mathcal{D}) = \mathcal{D}$ with translation number $n$. We show that Hofer’s norm $\|\phi\|$ is bounded from below by $\kappa \cdot |n|$ for a certain constant $\kappa$ which depends on $\mathcal{D}$ but not on $n$. We also give example of such Hamiltonian which is more efficient than the obvious rotation of $\mathcal{D}$. This answers question 5 from the list provided by F. Le Roux in [Ro].

1. INTRODUCTION AND RESULTS

Let $\mathbb{A} = S^1 \times (0,1)$ be an annulus equipped with the standard symplectic form $\omega$. We use the convention $S^1 = \mathbb{R}/\mathbb{Z}$, so $\int_{\mathbb{A}} \omega = 1$. Let $\mathcal{D} \subset \mathbb{A}$ be a disk with $A = \text{Area}(\mathcal{D}) = \int_{\mathcal{D}} \omega > 1/2$, $L = \partial \mathcal{D}$. Denote by $S = \{ \phi \in \text{Ham}(\mathbb{A}) | \phi(\mathcal{D}) = \mathcal{D} \}$ the stabilizer of $\mathcal{D}$ in the group $\text{Ham}(\mathbb{A})$ of compactly supported Hamiltonian diffeomorphisms. For a $\phi \in S$ we define the translation number $\tau_{\mathcal{D}}(\phi)$ in the following way. Roughly speaking, $\tau_{\mathcal{D}}$ is the number of loops around the $S^1$ coordinate that $\mathcal{D}$ does under an isotopy from the identity to $\phi$. For the formal definition, let $\tilde{\phi}$ be the lift of $\phi$ to the universal cover $\tilde{\mathbb{A}} \cong \mathbb{R} \times (0,1)$ which restricts to $1$ near the boundary. Pick $p \in \mathcal{D}$ and pick a lift $\tilde{p}$ of $p$ to $\tilde{\mathbb{A}}$. Then

$$\tau_{\mathcal{D}}(\phi) = \lim_{n \to \infty} \frac{\pi_0(\tilde{\phi}^n(\tilde{p}))}{n}.$$ 

It is easy to see that $\tau_{\mathcal{D}}$ does not depend on the choices made. If $p \in \mathcal{D}$ is a fixed point of $\phi$ then $\tau_{\mathcal{D}}(\phi)$ agrees with the usual notion of translation number for $\phi$ and $p$. We denote $S_n = \{ \phi \in S | \tau_{\mathcal{D}}(\phi) = n \}$.

Denote by $\| \cdot \|$ the Hofer norm on the group $\text{Ham}(\mathbb{A})$:

$$\|\phi\| = \inf_{p \in \mathbb{A}} \int_0^1 \max_{p \in \mathbb{A}} H(p, t) - \min_{p \in \mathbb{A}} H(p, t) dt,$$

where the infimum goes over all compactly supported Hamiltonians $H : \mathbb{A} \times [0,1] \to \mathbb{R}$ such that $\phi$ is the time-1 map of the corresponding flow.

We prove the following:

Theorem 1.

$$\frac{2A - 1}{2} \cdot |n| \leq \inf_{\phi \in S_n} \|\phi\| < (2A - 1) \cdot |n| + 1.$$ 

This answers question 5 from the list provided by F. Le Roux in [Ro]. The author provides there the following background for this question. Let $A$ be the annulus $S^1 \times (a,b) \subset \mathbb{A}$ ($0 < a < b < 1$) and $D' \subset \mathbb{A}$ be a displaceable disk. Consider Hamiltonians $\phi_A, \phi_{D'}$ which fix $A$ (resp., $D'$) pointwise with translation number $n$. Then, according to [Ro], the energy-capacity inequality in the universal cover $\tilde{\mathbb{A}}$ of $\mathbb{A}$ implies $\|\phi_A\| \geq |n| \cdot \text{Area}(A)$. On the other hand, $\|\phi_{D'}\|$ could be less than 1 regardless of $n$. The case of a non-displaceable disk $\mathcal{D}$ may be considered as an “intermediate” between $A$ and $D'$. F. Le Roux asked whether the translations of $\mathcal{D}$ behave similarly to those of one of the sets above. Theorem [Ro] shows that Hofer’s norm of translations of $\mathcal{D}$ grows linearly with

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respect to the translation number (in a similar way to the annulus $A$) but the coefficient is strictly less than $\text{Area}(D)$, in contrast to the case of an annulus.

The rest of this paper is organized as follows. In Section 3 we extend the translation number $\tau_D$ to a quasimorphism $\rho : \text{Ham}(\mathbb{A}) \to \mathbb{R}$. $\rho$ is Hofer-Lipschitz, hence gives a lower bound on Hofer’s distance. This implies the left-hand side inequality in Theorem 4. For the right-hand side inequality we describe an explicit Hamiltonian flow whose time-1 map $\psi_n$ belongs to $S_n$ and whose length is bounded by $(2A - 1) \cdot |n| + 1$. The details are carried out in Section 5.

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2. Lower bound

Let $G$ be a group. A function $r : G \to \mathbb{R}$ is called a quasimorphism if there exists a constant $R$ such that $|r(fg) - r(f) - r(g)| \leq R$ for all $f, g \in G$. $R$ is called the defect of $r$. The quasimorphism $r$ is homogeneous if it satisfies $r(g^n) = nr(g)$ for all $g \in G$ and $n \in \mathbb{Z}$. Any homogeneous quasimorphism satisfies $r(fg) = r(f) + r(g)$ for commuting elements $f, g$.

Pick a function $\tilde{H} : \mathbb{A} \to \mathbb{R}$ with compact support such that $\tilde{H}(\theta, h) = h$ away from a small neighborhood of $\partial \mathbb{A}$. Denote by $\tilde{\Phi}$ the time-1 map of the Hamiltonian flow generated by $\tilde{H}$. It is easy to see that $\tilde{\Phi} \in S$ with translation number $\tau_D(\tilde{\Phi}) = 1$. Note that $S = \bigcup_{n \in \mathbb{Z}} S_n = \bigcup_{n \in \mathbb{Z}} \tilde{\Phi}^n S_0$.

In what follows we construct a homogeneous quasimorphism $\rho : \text{Ham}(\mathbb{A}) \to \mathbb{R}$ which is Hofer-Lipschitz ($|\rho(\phi)| \leq k \|\phi\|$ for all $\phi \in \text{Ham}(\mathbb{A})$) and such that $\rho(S_0) = 0$, $\rho(\tilde{\Phi}) = \alpha > 0$. Any $\phi_n \in S_n$ decomposes as $\phi_n = \tilde{\Phi}^n \circ s$ for some $s \in S_0$. Hence

$$|\rho(\phi_n) - \alpha n| = |\rho(\tilde{\Phi}^n \circ s) - n\rho(\tilde{\Phi}) - 0| = |\rho(\tilde{\Phi}^n \circ s) - \rho(\tilde{\Phi}^n) - \rho(s)| < R.$$  

($R$ denotes the defect of $\rho$). It follows that

$$\rho(\phi_n) = \alpha \cdot n + \delta_{\phi_n}$$

where $|\delta_{\phi_n}| \leq R$. Note that $\phi_n^k \in S_{nk}$, hence by homogeneity

$$\rho(\phi_n) = \rho(\phi_n^k) = \frac{\alpha \cdot nk + \delta_{\phi_n^k}}{k}$$

which implies $\rho(\phi_n) = \alpha \cdot n$ in the limit as $k \to \infty$. This way, $\rho$ extends (up to a rescaling by $\alpha$) the translation number $\tau_D$. The Lipschitz property of $\rho$ implies the desired lower bound:

$$\|\phi_n\| \geq \frac{|\rho(\phi_n)|}{k} = \frac{|n| \cdot \alpha}{k}.$$  

In order to build $\rho$ we use the Calabi quasimorphism on $\text{Ham}(S^2)$ which was constructed by M. Entov and L. Polterovich in [EP]. We give a brief recollection of the relevant facts.

Let $D$ be an open disk equipped with a symplectic form $\omega$. Let $F_t : D \to \mathbb{R}$, $t \in [0, 1]$ be a time-dependent smooth function with compact support. We define $\text{Cal}(F_t) = \int_0^1 \left( \int_D F_t \omega \right) dt$. As $\omega$ is exact on $D$, $\text{Cal}$ descends to a homomorphism $\text{Cal}_D : \text{Ham}_c(D) \to \mathbb{R}$ which is called the Calabi homomorphism. Clearly, for $\phi \in \text{Ham}_c(D)$, $|\text{Cal}_D(\phi)| \leq \text{Area}(D) \cdot \|\phi\|$.

Let $S^2$ be a sphere equipped with a symplectic form $\omega$. Let $\text{Area}(S^2) = 2A$. For a smooth function $F : S^2 \to \mathbb{R}$ the Reeb graph $T_F$ is defined as the set of connected
components of level sets of $F$ (for a more detailed definition we refer the reader to [EP]). For a generic Morse function $F$ this set, equipped with the topology induced by the projection $\pi_F : S^2 \to T_F$, is homeomorphic to a tree. We endow $T_F$ with a measure given by $\mu(A) = \int_{\pi_F^{-1}(A)} \omega$ for any $X \subseteq T_F$ with measurable $\pi_F^{-1}(X)$. $x \in T_F$ is the median of $T_F$ if the measure of each connected component of $T_F \setminus \{x\}$ does not exceed $A$. This construction can be extended to functions $F$ such that $F|_{\text{supp}(F)}$ is Morse.

[EP] describes construction of a homogeneous quasimorphism $\text{Cal}_{S^2} : \text{Ham}(S^2) \to \mathbb{R}$. It has the following properties: $\text{Cal}_{S^2}$ is Hofer-Lipschitz ($|\text{Cal}_{S^2}(\phi)| \leq 2A \cdot \|\phi\|$). In the case when $\phi \in \text{Ham}(S^2)$ is supported in a disk $D$ which is displaceable in $S^2$, $\text{Cal}_{S^2}(\phi) = \text{Cal}_D(\phi|_D)$. Moreover, for $\phi \in \text{Ham}(S^2)$ generated by an autonomous function $F : S^2 \to \mathbb{R}$, $\text{Cal}_{S^2}(\phi)$ can be computed in the following way. Let $x$ be the median of $T_F$ and $X = \pi_F^{-1}(x)$ be the corresponding subset of $S^2$. Then

$$\text{Cal}_{S^2}(\phi) = \int_{S^2} F\omega - 2A \cdot F(X).$$

Given a symplectic embedding $j : \mathbb{A} \to S^2$ into a sphere of area $2A$, consider the pullback $\text{Cal}_j = j^*(\text{Cal}_{S^2}) : \text{Ham}(\mathbb{A}) \to \mathbb{R}$. Namely, given $\phi \in \text{Ham}(\mathbb{A})$, extend $j_\ast(\phi)$ to $\tilde{\phi} \in \text{Ham}(S^2)$ by identity on the complement of $j(\mathbb{A})$. Then $\text{Cal}_j(\phi) = \text{Cal}_{S^2}(\tilde{\phi})$. Clearly, $\text{Cal}_j$ is a homogeneous quasimorphism. It has the following properties:

- $\text{Cal}_j(\phi) = \text{Cal}_D(\phi|_D)$ for any $\phi$ supported in a disk $D$ of area $A$. To see that note that the corresponding $\tilde{\phi} \in \text{Ham}(S^2)$ is supported in a displaceable disk $j(D)$ in $S^2$.

- $|\text{Cal}_j(\phi)| = |\text{Cal}_{S^2}(\tilde{\phi})| \leq 2A \cdot \|\tilde{\phi}\|_{S^2} \leq 2A \cdot \|\phi\|_\mathbb{A}$.

- For an autonomous $\phi$ generated by a compactly supported function $H : \mathbb{A} \to \mathbb{R}$,

$$\text{Cal}_j(\phi) = \int_{\mathbb{A}} H\omega - 2A \cdot H(X)$$

where $X \subseteq \mathbb{A}$ is the level set component which is sent by $j$ to the median set of $j_\ast(H)$ in $S^2$.

Consider the embeddings $j_s : \mathbb{A} \to S^2$ ($0 \leq s \leq 2A - 1$) into a sphere of area $2A$ ($A = \text{Area}(D)$) that are given by gluing a disk of area $s$ to $S^1 \times \{0\}$ and a disk of area $(2A - 1 - s)$ to $S^1 \times \{1\}$. This construction ensures that $j_s(L)$ bisects $S^2$ into two displaceable disks.

We pick $0 \leq s_1 < s_2 \leq 2A - 1$ and set

$$\rho = \rho_{s_1,s_2} = \text{Cal}_{j_{s_2}} - \text{Cal}_{j_{s_1}}.$$  

(2)

Obviously, $\rho$ is a homogeneous quasimorphism on $\text{Ham}(\mathbb{A})$ which satisfies the Lipschitz property:

$$|\rho(\phi)| \leq |\text{Cal}_{j_{s_2}}(\phi)| + |\text{Cal}_{j_{s_1}}(\phi)| \leq 4A \cdot \|\phi\|.$$  

Consider the function $\bar{H}$ which was used to define the Hamiltonian $\tilde{\Phi}$ described above. It is easy to see that the “median” level set $X_s$ of $\bar{H}$ which is relevant for the computation of $\text{Cal}_{j_s}(\tilde{\Phi})$ is $S^1 \times \{A - s\}$. Hence

$$\text{Cal}_{j_s}(\tilde{\Phi}) = \int_{\mathbb{A}} \bar{H}\omega - 2A \cdot H(X_s) = \int_{\mathbb{A}} \bar{H}\omega - 2A \cdot (A - s).$$

This implies

$$\rho(\tilde{\Phi}) = \text{Cal}_{j_{s_2}}(\tilde{\Phi}) - \text{Cal}_{j_{s_1}}(\tilde{\Phi}) = 2A \cdot [(A - s_2) + (A - s_1)] = 2A \cdot (s_2 - s_1) > 0.$$  

Substitute $s_1 = 0$, $s_2 = 2A - 1$ into the definition of $\rho$. Using the computation above, $\alpha = \rho(\tilde{\Phi}) = 2A \cdot (s_2 - s_1) = 2A(2A - 1)$. $\rho$ is $4A$-Lipschitz, hence for any $\phi_n \in S_n(\tilde{\Phi})$
implies the lower bound of Theorem \[ \|H\| \geq |n| \cdot \frac{\alpha}{k} = |n| \cdot \frac{2A(2A - 1)}{4A} = |n| \cdot \frac{2A - 1}{2}. \]

It is left to show that \( \rho \) vanishes on \( S_0 \). Let \( S_0' = \{ \phi \in S_0 \mid \text{supp}(\phi) \subset \mathbb{A} \setminus L \} \) be the subgroup which fixes a neighborhood of \( L \) pointwise.

**Lemma 2.** Let \( q \) be a homogeneous quasimorphism which is Hofer-continuous and vanishes on \( S_0' \). Then \( q \) vanishes on \( S_0 \).

**Proof.** Pick an open disk \( D \subset \mathbb{A} \setminus L \). \( \text{Ham}_c(D) \subset S_0' \), therefore \( q \) vanishes on \( \text{Ham}_c(D) \). It follows from the results of [EPF] that \( q \) is continuous in the \( C^0 \)-topology.

Let \( \phi \in S_0 \). Applying an appropriate \( \psi \in S_0 \) with arbitrary small Hofer norm, we may ensure that \( \psi \circ \phi = \mathbb{I} \) on \( L \). Further, we may find a \( C^0 \)-small diffeomorphism \( h \in \text{Ham}(\mathbb{A}) \) such that \( h \circ \psi \circ \phi = \mathbb{I} \) in a neighborhood of \( L \). It follows that \( h \circ \psi \circ \phi \in S_0' \) and \( q(h \circ \psi \circ \phi) = 0 \). Hofer and \( C^0 \)-continuity of \( q \) imply that \( q(\phi) = 0 \).

Let \( \phi \in S_0' \). We show that \( \rho(\phi) = 0 \). \( \phi \) splits to a composition \( \phi = \phi_D \circ \phi_P \) where \( \phi_D \) is supported in \( D \) and \( \phi_P \) is supported in the pair of pants \( P = \mathbb{A} \setminus D \). \( \phi_D, \phi_P \) have disjoint supports, therefore they commute. Hence \( \rho(\phi) = \rho(\phi_D) + \rho(\phi_P) \). Note that \( \phi_D \in \text{Ham}_c(D) \) and \( \text{Area}(D) = A \), so \( \text{Cal}_{j_s}(\phi_D) = \text{Cal}_D(\phi_D) \) for all \( s \). Therefore \( \rho(\phi_P) = 0 \).

\( \phi_P \) is Hamiltonian on \( \mathbb{A} \), but after the restriction to \( P \) we have just \( \phi_P = \phi \big|_P \in \text{Symp}_c(P) \). In the argument below we apply a sequence of deformations to \( \phi_P \) in order to get \( \tilde{\phi}_P \) whose restriction \( \tilde{\phi}_P \big|_P \in \text{Ham}_c(P) \). All deformations involved in the process preserve the value \( \rho(\phi_P) \). Finally, we show that \( \rho(\phi_P) = 0 \) by explicit computation.

The mapping class group \( \pi_0(\text{Symp}_c(P)) \) is isomorphic to \( \mathbb{Z}^2 \) and is generated by Dehn twists near the three boundary components. For the proof of this fact we refer the reader to [FM] where the authors show that \( \pi_0(\text{Diff}_c(P)) \simeq \mathbb{Z}^3 \) and is generated by Dehn twists. Note that \( \phi, \psi \in \text{Symp}_c(P) \) are isotopic in \( \text{Symp}_c \) if and only if they are isotopic in \( \text{Diff}_c \). As Dehn twists belong to \( \text{Symp}_c(P) \), the statement for \( \pi_0(\text{Symp}_c(P)) \) follows.

Denote by \( T_1, T_0 \) Dehn twists near \( S^1 \times \{1\}, S^1 \times \{0\} \) and by \( T_L \) a Dehn twist in \( P \) near \( L = \partial D \). Note that we may find a Hamiltonian \( \psi_L \) in \( S_0' \) with arbitrary small Hofer norm whose restriction to \( P \) realizes the Dehn twist \( T_L \). \( \tilde{\phi}_P \) is isotopic in \( \text{Symp}_c(P) \) to some \( T_1 \circ T_0 \circ T_L \) \( k_0 \in \mathbb{Z} \). If \( k_L \neq 0 \) we replace the original \( \phi \) by \( \psi_L^{k_L} \circ \phi \in S_0' \). As \( \|\psi_L\| \) can be chosen to be arbitrarily small, by continuity of \( \rho \) it is enough to show the desired statement for the Dehn \( \phi \). After the replacement \( k_L \) vanishes, hence the modified \( \phi_P \approx T_1 \circ T_0 \circ T_L \). Note that \( \tilde{\phi}_P \) is induced by a Hamiltonian \( \phi \in S \). The definition of \( \tau_{\phi} \) implies that \( k_1 = \tau_{\phi}(\phi) = -k_0 \). The minus sign appears because the opposite orientation of the boundary components results in the opposite directions of the corresponding Dehn twist. Moreover, as \( \phi \in S_0' \), \( k_1 = \tau_{\phi}(\phi) = 0 \). Therefore the restriction \( \phi_P \) belongs to the identity component of \( \text{Symp}_c(P) \).

Pick \( K : \mathbb{A} \to \mathbb{R} \) supported in a small neighborhood of \( D \) such that \( K = 1 \) in a neighborhood of the closure \( \overline{D} \). Denote by \( \chi^t \) the time-\( t \) map generated by the Hamiltonian flow of \( K \). The median set for \( K \) which is used to compute \( \text{Cal}_{j_s}(\chi^1) \) is \( K^{-1}(1) \). It follows that the value \( \text{Cal}_{j_s}(\chi^1) = \int K \omega - 2A \cdot 1 \) does not depend on \( s \), hence \( \rho(\chi^1) = 0 \). By homogeneity and continuity of \( \rho \), \( \rho(\chi^t) = 0 \) implies that \( \rho(\chi^1) = 0 \) for all \( t \).

Consider the homomorphism \( i_* : H^1_c(P; \mathbb{R}) \to H^1_c(\mathbb{A}; \mathbb{R}) \) induced by inclusion \( i : P \to \mathbb{A} \). Both \( \chi^t, \phi_P \) are Hamiltonian in \( \mathbb{A} \), hence their fluxes are zero in \( H^1_c(\mathbb{A}; \mathbb{R}) \). After the restriction to \( P \), \( \text{flux}(\chi^1|_P), \text{flux}(\phi_P) \) belong to the one-dimensional subspace \( \ker i_* \subset H^1_c(P; \mathbb{R}) \). \( \text{flux}(\chi^1|_P) \neq 0 \), therefore one can find an appropriate \( t_0 \in \mathbb{R} \) such that the restriction \( \tilde{\phi}_P = \chi^{t_0} \big|_P \circ \phi_P = \chi^{t_0} \circ \phi_P \) has zero flux in \( P \). Hence, as it is shown in [Bar], \( \tilde{\phi}_P \in \text{Ham}_c(P) \).
Pick a compactly supported function \( F_1 : P \times [0,1] \to \mathbb{R} \) whose flow generates \( \phi_p' \). Denote by \( U_s \) the complement of the closed disk \( j_s(D) \) in \( S^2 \), it is a displaceable disk. \((j_s)_*(\phi_p') \in Ham(S^2)\) and it is supported in \( U_s \), therefore

\[
\text{Cal}_j_s(\phi_p') = \text{Cal}_{U_s}(j_s)_*(\phi_p') = \int_0^1 \left( \int_{U_s} (j_s)_* F_1 \omega \right) dt = \int_0^1 \left( \int_P F_1 \omega \right) dt
\]
is independent of \( s \). Hence \( \rho(\phi_p') = 0 \). From \( \phi_p' = \chi_{t_0} \circ \phi_p \), we obtain \(|\rho(\phi_p') - \rho(\chi_{t_0}) - \rho(\phi_p)| = |\rho(\phi_p)| < R \). It follows that \( \rho \) is bounded on the subgroup \( S_0 \). As \( \rho \) is homogeneous, it vanishes there.

**Remark 3.** The extension \( \frac{\rho}{\rho(\phi)} \) (where \( \rho \) is the quasimorphism constructed above) of the translation number \( \tau_0 \) is not unique. Choice of different parameters \( s_1, s_2 \) in (\( \mathbb{B} \)) gives rise to different extension quasimorphisms. In particular, their Lipschitz constant varies.

### 3. Upper Bound

Pick \( n \in \mathbb{Z} \). We construct an explicit deformation \( \psi_n \in S_n \) such that

\[ \|\psi_n\| \leq (2A - 1) \cdot |n| + 1. \]

This implies the right-hand side inequality in Theorem \( \mathbb{B} \).

Consider the Hamiltonian flow described in Figure \( \mathbb{B} \) on the left. It is supported in a small neighborhood of two disks and two paths connecting them. The flow is generated by an autonomous function \( H \) such that \( H = 1 \) in the internal rectangle, zero outside and is reasonably smoothed in between. We ask from “reasonable smoothing” that \( H \) is approximately linear on each of the two disks. Clearly, the time-\( t \) map of the corresponding flow \( \phi_t \) transfers area \( t \) from the right disk towards the left one and vice versa. The Hofer length \( l(\phi_t) = t \). We may deform this construction to make the internal rectangle very thin. Then it looks as depicted on the right, namely, the flow is supported in a small neighborhood of the two disks and a single path connecting them.

We apply this construction on \( A \) as drawn in Figure \( \mathbb{B} \) on the left. Namely, we change the coordinate system, if necessary, so that \( D \) is approximated by rectangle \( (\delta, 1 - \delta) \times (\delta, A + \delta') \subseteq S^1 \times (0,1) \). Here \( A = \text{Area}(D) \) and \( \delta, \delta' \ll 1 \). Denote by \( D' \) a disk given by smoothing the rectangle \( (\delta + \epsilon, 1 - \delta) \times (A + \delta' + \delta, 1 - A) \subseteq A \setminus D \). Note that \( \text{Area}(D') = 1 - A - \epsilon \) where \( \epsilon \) is comparable to \( \delta + \delta' \). Connect \( D \) with \( D' \) by a spiral which lies in \( S^1 \times [A + \delta', A + \delta' + \delta] \) and makes \( n \) loops around the \( S^1 \) coordinate. Now apply the flow as in Figure \( \mathbb{B} \) (right) to transfer a portion of area from \( D \) to \( D' \) using a small tubular neighborhood of the spiral. We stop the process when entire \( D' \) is covered by the deformed \( D \) (see Figure \( \mathbb{B} \) on the right). We may arrange this construction so that the deformed \( D \) lies above \( S^1 \times \{1 - A - \delta''\} \) where \( \delta'' \) is comparable to \( \delta + \delta' \). The length of this deformation is \( \text{Area}(D') + \epsilon' = 1 - A - \epsilon + \epsilon' \).

In the next step we rotate horizontal circles \( S^1 \times \{\hbar\} \). The rotation angle is chosen in a way that the annulus \( S^1 \times (1 - A - \delta''', A + \delta') \) (which contains the intersection of the deformed \( D \) with the original \( D \)) is rotated \( n \) times around the \( S^1 \) coordinate, and the angle gradually decreases to zero along the annulus containing the spiral \( (S^1 \times (A + \delta', A + \delta')) \).
\(\delta' + \delta\)). As the result, the spiral is unfolded to a vertical line. The remaining part of \(A\) is remains fixed. Denote by \(H\) an autonomous Hamiltonian which generates this flow, its graph is shown schematically in Figure 3 (left). \(H\) is cut off in the \(\delta\)-neighborhood of \(\partial A\) to make it compactly-supported and is smoothed near the singular points. The cutoff will result in a strong flow near \(S^1 \times \{0, 1\}\), but this has no effect on our construction as this flow stays away from our area of interest. Energy needed for this deformation is

\[
|n| \cdot \text{Area}(S^1 \times (1 - A - \delta'', A + \delta')) + \varepsilon'' = |n| \cdot (2A - 1 + \delta' + \delta''') + \varepsilon''
\]

where \(\varepsilon''\) covers energy consumed by rotation of the spiral area and smoothing costs. \(\varepsilon''\) is bounded by \(|n| \cdot \delta\).

Finally we are in the situation shown in Figure 3 (right) where all points of the deformed \(D\) are already rotated \(n\) times around the \(S^1\) coordinate. We move the deformed disk in an obvious way down in order to fill \(D\). The length of such deformation is bounded by \(\text{Area}(\mathbb{R}) - \text{Area}(D) = 1 - A\).

Summarizing the argument above, we have constructed a Hamiltonian isotopy connecting \(\mathbb{R}\) to \(S_n\) whose Hofer length is bounded by

\[
(1 - A - \varepsilon' + \varepsilon') + |n| \cdot (2A - 1 + \delta' + \delta''') + \varepsilon'' + (1 - A) = |n| \cdot (2A - 1) + (2 - 2A) + \varepsilon < |n| \cdot (2A - 1) + 1
\]

for appropriate choices of parameters \(\delta\) and \(\varepsilon\).
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Michael Khanevsky, School of Mathematics, Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA
E-mail address: khanév@math.ias.edu