Lower estimates for random walks on a class of amenable $p$-adic groups

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Abstract

We give central lower estimates for the transition kernels corresponding to symmetric random walks on certain amenable $p$-adic groups.

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1 Introduction

Let $G$ be a locally compact amenable group. We shall denote by $d^r g$ (resp. $d^l g$) the right (resp. left) Haar measure on $G$ and by $\delta(g) = d^r g / d^l g$ the modular function on $G$ normalized by $\delta(e) = 1$ where $e$ denotes the identity of $G$. Let $d \mu(g) = \varphi(g) d^r g \in P(G)$ be a probability measure on $G$ where $\varphi(g) \in L^\infty(G)$ is assumed to have a compact support or a fast decay at infinity. Let us assume that $\mu$ is symmetric (i.e. the involution $g \mapsto g^{-1}$ stabilizes $\mu$) and consider the random walk on $G$ induced by $\mu$, i.e. the $G$-valued process that evolves as follows: if $X_n = g$ is the position at time $n$ then $X_{n+1} = gh$ where $h$ is chosen according to $\mu$. We shall denote by

$$d \mu^n(g) = \varphi_n(g) d^r g$$

the $n^{th}$ convolution power of $\mu$ and examine the behaviour of the decay of $\varphi_n(e)$ as $n \to \infty$.

If we restrict ourselves to unimodular real Lie groups then the answer lies in the behaviour of the volume growth of $G$. Let us recall that if $G$ is a locally compact group that is generated by some symmetric compact neighbourhood $\Omega \subset G$ of the identity in $G$, the volume growth function is (cf. [9])

$$\gamma(t) = \text{Vol}(B_t(e)), \quad t = 1, 2, ...$$

where the volume is taken with respect to $d^r g$ (or $d^l g$) and where $B_t(e)$ is the ball of radius $t$ centred on $e$ defined by

$$B_t(e) = \Omega \cdots \Omega, \quad t \text{ times.}$$

For $x \in G$ the distance from $e$ is defined by $|x| = \inf\{t, \ x \in B_t(e)\}$ and a left invariant distance can be defined on $G$ by setting $d(x, y) = |y^{-1}x|$, $x, y \in G$. If $\Omega_1, \Omega_2$ are two neighbourhoods of $e$ as above it is not difficult to check that there exists $C > 0$ such that $C^{-1} \leq |.| / |.|_1 \leq C$ and that the corresponding growth functions satisfy the obvious equivalence $\gamma_1 \approx \gamma_2$, i.e.

$$\gamma_1(t) \leq C \gamma_2(Ct) + C \leq C' \gamma_1(C't) + C', \quad t \geq 1.$$

For real Lie groups we have the following dichotomy (cf. [9], [13]): either

$$\gamma(t) \approx t^D$$

where $D = D(G) = 1, 2, \ldots$, or

$$\gamma(t) \approx e^t.$$

In the first case we say that $G$ is of polynomial growth and in the second case we say that $G$ is of exponential growth and the answer to our problem in the case of unimodular amenable real Lie groups was given by Varopoulos and is the following

$$\varphi_n(e) \approx n^{-D/2} \iff \gamma(t) \approx t^D$$

$$\varphi_n(e) \approx e^{-n^{1/3}} \iff \gamma(t) \approx e^t$$

(cf. [1], [6], [7], [10], [28]).

Varopoulos showed that the $e^{-n^{1/3}}$ versus polynomial behaviour extends to the non-unimodular amenable case depending on wether the Lie group $G$ is $(C)$ or $(NC)$. This classification introduced
in [24] can be expressed in terms of the roots of the ad-action of the radical of the Lie algebra of the group on its nilradical.

The discret case is more complicated (cf. [1], [11], [16], [17], [22], [23]). First there is no dichotomy in the volume growth (cf. [8]). On the other hand if we suppose that the group G is of exponential growth then one can claim only the upper bound (cf. [11])

\[ \varphi_n(e) \leq C \exp(-cn^{1/3}), \quad n \geq 1. \]

In general, the matching lower bound fails. Ch. Pittet and L. Saloff-Coste showed (cf. [16]) that there are soluble groups with exponential volume growth for which the heat kernel decays as \( \exp(-cn^\alpha) \) with \( \alpha \in (0, 1) \) which can be taken arbitrarily close to 1. This, as mentioned above, can not happen in the case of real Lie groups.

In a recent paper (cf. [17]) Ch. Pittet and L. Saloff-Coste established the lower bound \( \varphi_{2n}(e) \geq c \exp(-Cn^{1/3}) \), \( n = 1, 2, \ldots \) for the large times asymptotic behaviours of the probabilities of return to the origin at even times \( 2n \), for random walks associated with finite symmetric generating sets of solvable groups of finite Prüfer rank. They asked in this paper (cf. [17], §8) if a similiar lower bound is available in the case of analytic \( p \)-adic groups. An answer to this problem was given in [15] (cf. also [14]). The aim of this paper is to show that the \( e^{-n^{1/3}} \) lower bound obtained in [14] can be substantially improved for a large class of amenable \( p \)-adic groups.

## 2 Amenable \( p \)-adic groups

In this section \( G \) will denote an algebraic connected amenable group over \( \mathbb{Q}_p \) the field of \( p \)-adic numbers; \( U \subset Q \subset G \) will denote the radical and the unipotent radical (cf. [3], [5]). Amenability of \( G \) is equivalent to the fact that the semi-simple group \( G/Q \) is compact (cf. [19]). Let \( S \) denote a fixed levi subgroup \( S \) of \( G \) (cf. [3]). The group \( G \) can then be written as a semi-direct product:

\[
G = Q \ltimes S = (U \ltimes A) \ltimes S \cong U \ltimes (A \ltimes S)
\]

where \( A \) is abelian and can be identified to the direct product of a finite group and a \( d \)-torus \( T \cong (\mathbb{Q}_p^*)^d \) (cf. [3], [5]). Here \( \mathbb{Q}_p^* \) denotes the multiplicative group of the field \( \mathbb{Q}_p \). Since \( (\mathbb{Q}_p^*)^d \cong \mathbb{Z}^d \times K \) (where \( K \) is compact, cf. [2], [4]), by considering the projection

\[
\pi : A \longrightarrow \mathbb{Z}^d,
\]

we can fix \( \pi_1, \ldots, \pi_d \in A \) so that each \( z \in A \) admits a unique decomposition

\[
z = \pi_1^{n_1} \cdots \pi_d^{n_d} \tau, \quad n_1, \ldots, n_d \in \mathbb{Z}, \quad \tau \in \bar{K},
\]

where \( \bar{K} \) denotes a compact subgroup of \( A \). Let \( \mathcal{U} = \text{Lie}(U) \) denote the Lie algebra of \( U \). Let \( \overline{\mathbb{Q}_p} \) denotes a finite extension of \( \mathbb{Q}_p \) which contains all the eigenvalues defined by

\[
det \left( \text{Ad}(\pi_j) - \lambda I \right) = 0, \quad j = 1, \ldots, d.
\]

The \( Ad \)-action of \( A \) on \( \mathcal{U} \) extends to \( \mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} \) and it follows from the proof of the Zassenhaus lemma (cf. [12]) that we have a decomposition of \( \mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} \) into a direct sum

\[
\mathcal{U} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p} = W_1 \oplus \ldots \oplus W_r
\]
where the subspaces $W_j$ ($1 \leq j \leq r$) are invariant by $\text{Ad}(\pi_i)$, $i = 1, \ldots, d$, and such that the restriction of $\text{Ad}(\pi_i)$ to $W_j$ is the sum of the scalar $\lambda_j(\pi_i)$ and a nilpotent endomorphism.

Let $\chi_j: A \to \overline{\mathbb{Q}}_p^*$ defined by

$$
\chi_j(z) = \lambda_j(\pi_1)^{n_1} \cdots \lambda_j(\pi_d)^{n_d}, \quad z = \pi_1^{n_1} \cdots \pi_d^{n_d} \tau, \quad j = 1, \ldots, r.
$$

Let $|.|_p$ denote the standard $p$-adic norm. We shall denote by $|.|'_p$ its extension to $\overline{\mathbb{Q}}_p$. We have $|x|'_p \in \{p^n, \ n \in \mathbb{Z}\} \cup \{0\}$, $x \in \overline{\mathbb{Q}}_p$, where $p$ denotes a rational power of the prime $p$ (cf. [2]). Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ denote the different norms of the $\chi_j$'s, i.e. the different homomorphisms obtained by considering $A \to \overline{p}^Z$, $z \to |\chi_j(z)|'_p$, $j = 1, \ldots, r$. Let $\mathcal{L} = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$. We shall assume that $\mathcal{L}$ is nonempty and that

$$
1 \not\in \mathcal{L}.
$$

Here $1$ denotes the homomorphism $A \to \overline{p}^Z$ identically equal to $1$. This assumption guarantees the fact that the group $G$ is compactly generated (cf. [3]). Observe that the group $G$ is then of exponential volume growth (cf. [18]).

Let $\gamma_{j,1}, \ldots, \gamma_{j,d} \in \mathbb{Z}$ the integers defined by

$$
\alpha_j(z) = p^{\gamma_{j,1}n_1+\gamma_{j,2}n_2+\ldots+\gamma_{j,d}n_d}, \quad z = \pi_1^{n_1} \cdots \pi_d^{n_d} \tau, \quad j = 1, \ldots, s.
$$

We shall denote by $L_j$, $j = 1, \ldots, s$, the linear forms on $\mathbb{Z}^d$ defined by

$$
L_j(n_1, \ldots, n_d) = \gamma_{j,1}n_1 + \gamma_{j,2}n_2 + \ldots + \gamma_{j,d}n_d,
$$

$(n_1, \ldots, n_d) \in \mathbb{Z}^d$ and by $\tilde{L}_j$ the linear forms on $\mathbb{R}^d$ induced by the $L_j$'s.

Let now $d\mu(g) = \varphi(g)d\tau g \in \mathbb{P}(G)$ denote a symmetric probability measure on $G$. The density $\varphi(g)$ is assumed to be a continuous compactly supported function on $G$. To avoid unnecessary complications we shall assume that there exists $e \in \Omega = \Omega^{-1} \subseteq G$ such that

$$
\inf \{\varphi(g), \ g \in \Omega\} > 0, \quad G = \bigcup_{n \geq 0} \Omega^n;
$$

the last condition in (7) implies that $\text{supp}(\mu)$ generates the group $G$.

Let

$$
p : G \to G/U \to A
$$

denote the projection that we obtain from the identifications (1) and let

$$
\tilde{\mu} = (\pi \circ p)(\mu) \in \mathbb{P}(\mathbb{Z}^d) \subseteq \mathbb{P}(\mathbb{R}^d),
$$

where $\pi$ denotes the canonical projection (2). It follows from (7) that there exists a choice of coordinates on $\mathbb{R}^d$ for which the covariance matrix of $\tilde{\mu}$ satisfies

$$
\int_{\mathbb{R}^d} x_i x_j d\tilde{\mu}(x) = \delta_{i,j}, \quad 1 \leq i, j \leq d.
$$

We shall assume that $\mathbb{R}^d$ is equipped with the Euclidean structure associated to these coordinates.
From now on we shall assume that the $L_j$’s induce a "Weyl chamber". More precisely we suppose that

$$\Pi_{\mathcal{L}} = \{x \in \mathbb{R}^d, \quad \tilde{L}_j(x) > 0, \quad j = 1, 2, ..., s\} \subset \mathbb{R}^d$$

define a nonempty convex cone in $\mathbb{R}^d$. This condition is the analogue of the $(NC)$-condition introduced by Varopoulos in [24] in the setting of real amenable Lie groups. We shall prove that, under this condition, we have a lower bound of the form $\varphi_n(e) \geq cn^{-\nu}$. The argument follows the approach introduced by Varopoulos in [24].

The exact value of $\nu$ is defined as in the real case and is expressed in terms a parameter $\lambda = \lambda(d, \mathcal{L})$ that is defined as follows. In the the rank one case (i.e. $d = 1$) we shall set

$$(10^a) \quad \lambda = 0.$$  

In the case $d \geq 2$ let us denote by $\Sigma = \{x \in \mathbb{R}^d, \ |x| = 1\}$ the unit sphere in $\mathbb{R}^d$. Let $\Delta_\Sigma$ be the corresponding spherical Laplacian. Let $\Pi_\Sigma = \Sigma \cap \Pi_{\mathcal{L}} \subset \Sigma$. We then set

$$(10^b) \quad \lambda = \inf \{-(\Delta_\Sigma f, f), \ |f| = 1, \ f \in C^\infty_0(\Pi_\Sigma)\};$$
i.e. $\lambda$ is the first Dirichlet eigenvalue of the region $\Pi_\Sigma$. The scalar product and the $L^2$-norm in $(10^b)$ are taken with respect to the Euclidean volume element on $\Sigma$.

**Theorem 1.** Let $G$ and $d\mu^n(g) = \varphi_n(g) d^rg$, $n = 1, 2, ...$ be as above. Then there exists $C > 0$ such that

$$\varphi_n(e) \geq \frac{1}{Cn^{1+\sqrt{(d-2)^2+4}}} \quad n = 1, 2, ...$$

where $\lambda$ is defined by $(10)$.  

The following comments may be helpful in placing the above theorem in its proper perspective.  

(i) It is enough to to prove the estimate $(11)$ when $n$ is even since $\varphi$ satisfies $(7)$.  

(ii) Observe that the group $G$ is automatically non-unimodular, for otherwise we have (cf. [11]):

$$\varphi_n(e) \leq Ce^{-cn^{1/3}}, \quad n = 1, 2, ...$$

(iii) The upper estimate

$$\varphi_n(e) \leq \frac{C}{n^{3/2}}, \quad n = 1, 2, ...$$

is known to hold for general non-unimodular locally compact groups (cf. [26]). This shows that the index $3/2$ cannot be improved in the rank one case.  

(iv) We will show (cf. §4 below) that in the case of metabelian $p$-adic groups, the lower bound $(11)$ can be complemented with a similar upper bound.  

Throughout the remainder of the paper $C$ denotes a positive constant which is not always the same, even in a given line.
3 Proof of Theorem 1

Let $G$, $d\mu(g) = \varphi(g)d^r g \in P(G)$ and $d\mu^n(g) = \varphi_n(g)d^r g$ be as in Theorem 1. Let $\xi_1, \xi_2, \ldots \in G$ be a sequence of independent equidistributed random variables of law $d\mu(g)$ and let $X_n = \xi_1\xi_2\ldots\xi_n$, $n = 1, 2, \ldots$ denote the corresponding random walk starting at $X_0 = e$. Let $B \subset G$ a borel subset, we have

\begin{equation}
\mathbb{P}_e [X_n \in B] = \int_B \varphi_n(g)d^r g, \quad n = 1, 2, \ldots
\end{equation}

The symmetry of $d\mu(g)$ implies that

\begin{equation}
d\mu(g) = d\mu(g^{-1}) = \varphi(g)d^r g = \varphi(g^{-1})d^r g = \varphi(g^{-1})\delta(g)^{-1}d^r g,
\end{equation}

hence

\begin{equation}
\varphi(g^{-1}) = \varphi(g)\delta(g), \quad g \in G.
\end{equation}

We have also

\begin{equation}
\varphi_n(g^{-1}) = \varphi_n(g)\delta(g), \quad g \in G, \quad n = 1, 2, \ldots
\end{equation}

On the other hand we have

\begin{equation}
\varphi_{2n}(e) = \int_G \varphi_n(g^{-1})\varphi_n(g)dg = \int_G \varphi_n(g)\delta(g)\varphi_n(g)dg = \int_G \varphi_n(g)^2d^r g.
\end{equation}

Schwarz inequality applied to (12) gives then

\begin{equation}
\varphi_{2n}(e) \geq \left( \frac{\mathbb{P}_e [X_n \in B]}{|B|} \right)^2, \quad B \subset G, \quad n = 1, 2, \ldots
\end{equation}

where $|B|$ denotes the right Haar measure of $B$.

Let us further observe that the group $G$ (resp. $G/U$) decomposes as a semi-direct (resp. direct) product

\begin{align*}
G &= Q \ltimes S \cong U \ltimes (\mathbb{Z}^d \times \tilde{S}) \\
G/U &\cong A \times S \cong \mathbb{Z}^d \times \tilde{S}
\end{align*}

where $\tilde{S}$ is compact. This follows from (1).

Let us write

\begin{equation}
X_n = \xi_1\xi_2\ldots\xi_n = u_1z_1u_2z_2\ldots u_nz_n, \quad n = 1, 2, \ldots
\end{equation}

where $\xi_j = u_jz_j$ with $u_j \in U$ and $z_j \in A \times S$, $j = 1, 2, \ldots$. Using the interior automorphisms $x^y = yxy^{-1}$, $x, y \in G$, we rewrite (14)

\begin{equation}
X_n = u_1u_2^{z_1}u_3^{z_2}\ldots u_n^{z_1\ldots z_{n-1}}z_2\ldots z_n = \Gamma_n z_n,
\end{equation}

$\Gamma_n \in U$, $Z_n \in A \times S$, $n = 1, 2, \ldots$
We shall use the exponential map and identify $U$ to its Lie algebra $\mathcal{U}$ (cf. [5], [20]) and write each $u_j$ in the above expression

$$u_i = \exp(v_i), \quad v_i \in \mathcal{U}, \quad i = 1, 2, ...$$

We have therefore

$$u_j^{z_1z_2...z_{j-1}} = \exp\left(Ad(z_1...z_{j-1})v_j\right), \quad j \geq 2.$$ 

Let us fix $e_1, e_2, ..., e_m$ a basis of $\mathcal{U} \otimes \mathbb{Q}_p \otimes \mathbb{Q}_p$ adapted to the decomposition (4). For $x = x_1 e_1 + x_2 e_2 + ... + x_m e_m \in \mathcal{U} \otimes \mathbb{Q}_p \otimes \mathbb{Q}_p$, we set

$$\|x\| = \max_{1 \leq i \leq m} |x_i|.$$ 

Since $\mu$ is compactly supported we can suppose that the $v_i$'s in (16) satisfy

$$\|v_j\| \leq C, \quad j = 1, 2, ...$$

where $C > 0$ is an appropriate positive constant.

Let us equip $\text{End}_{\mathcal{U}_p} \left( \mathcal{U} \otimes \mathbb{Q}_p \otimes \mathbb{Q}_p \right)$ with the norm

$$\|T\| = \sup_{\|v\| \leq 1} \|Tv\|, \quad T \in \text{End}_{\mathcal{U}_p} \left( \mathcal{U} \otimes \mathbb{Q}_p \otimes \mathbb{Q}_p \right).$$

It is clear that $||.||$ satisfies the ultrametric property

$$\|T + T'\| \leq \max \left( \|T\|, \|T'\| \right).$$

Let $1 \leq l \leq r$. For

$$z = \pi_1^{n_1} ... \pi_d^{n_d}, \quad \tilde{z} = \tilde{e} \pi \in A \times S \cong \mathbb{Z}^d \times \tilde{S},$$

we have

$$\text{Ad}(z)|_{W_l} = \text{Ad}(\tilde{e})|_{W_l} \circ \left( \tilde{\chi}_l(z)I + \mathcal{T}(z) \right),$$

where $\mathcal{T}(z)$ denotes an upper triangular matrix and where

$$\tilde{\chi}_l(z) = \chi_l(\pi_1^{n_1} ... \pi_d^{n_d}) = \chi_l(\pi_1)_{n_1} ... \chi_l(\pi_d)_{n_d}.$$

On the other hand it is clear that

$$|\tilde{\chi}_l(z)|_p, \quad |\tilde{\chi}_l(z)^{-1}|_p \leq p^{C|\tilde{e}|_{\mathbb{Z}^d}}, \quad z \in A \times S,$$

where $|.|_{\mathbb{Z}^d}$ denotes the Euclidean norm on $\mathbb{Z}^d$. We have also

$$\|\text{Ad}(z)\| \leq C\|\text{Ad}(\pi_1)\|^{n_1} ... \|\text{Ad}(\pi_d)\|^{n_d} \leq C p^{C|\tilde{e}|_{\mathbb{Z}^d}}, \quad z \in A \times S.$$ 

Combining (20), (21), (22) we deduce that the triangular matrix $\mathcal{T}(z)$ that appears in (20) satisfies

$$\|\mathcal{T}(z)\| \leq C \|\mathcal{T}(z)\|, \quad z \in A \times S.$$
Let now $z_1, z_2, ..., z_k \in A \times S$ and $1 \leq l \leq r$. We have

$$ Ad(z_1 z_2 ... z_k)|_{W_l} = \overline{Ad(\tau_1 \tau_2 ... \tau_k)|_{W_l}} \circ \prod_{j=1}^{k} (\tilde{\chi}_l(z_j)I + \mathcal{T}(z_j)) $$

$$ = \tilde{\chi}_l(z_1 z_2 ... z_k) \overline{Ad(\tau_1 \tau_2 ... \tau_k)|_{W_l}} \circ \sum_{\alpha, \beta} (\tilde{\chi}_l(z_{\alpha})...\tilde{\chi}_l(z_{\beta}))^{-1} \mathcal{T}(z_{\alpha})...\mathcal{T}(z_{\beta}). $$

Using the fact that in the last sum all the terms corresponding to indexes $\alpha > n$ vanish and combining this with (21), (23) and the ultrametric property (19) we deduce that

$$ ||Ad(z_1 z_2 ... z_k)|_{W_l}|| \leq C \max_{1 \leq l \leq k} ||l||_{Z_{\alpha}} p^\max_{1 \leq l \leq k} ||l||_{Z_{\beta}}. $$

If we use the linear forms $L_i$ defined by (5) and (6) we then deduce

$$ ||Ad(z_1 z_2 ... z_k)|_{W_l}|| \leq C p^\max_{1 \leq l \leq k} L_i(\zeta_1 + ... + \zeta_k) + C \max_{1 \leq l \leq k} |||l||_{Z_{\alpha}}. $$

By the above considerations we have finally proved that

$$ ||Ad(z_1 z_2 ... z_k)|| \leq C p^\max_{1 \leq l \leq k} L_i(\zeta_1 + ... + \zeta_k) + C \max_{1 \leq l \leq k} ||\zeta||_{Z_{\alpha}}. $$

$$ \zeta_j = p(z_j), \quad z_j \in A \times S, \quad j = 1, ..., k, \quad k = 1, 2, ... $$

If we apply this estimate to $u_j^{z_1 z_2 ... z_{j-1}} = \exp(Ad(z_1 ... z_{j-1})v_j)$ where the $u_j$’s, $v_j$’s and $z_j$’s are as in (16), (17), (18) we then deduce

$$ ||Ad(z_1 ... z_{j-1})v_j|| \leq C p^\max_{1 \leq l \leq k} L_i(\zeta_1 + ... + \zeta_j) + C \max_{1 \leq l \leq k} ||\zeta||_{Z_{\alpha}}. $$

$$ \zeta_j = p(z_j), \quad j \geq 2. $$

Observe that the measure that controls the random walk $(S_j)_{j \in N}$, defined by $S_j = \zeta_0 + \zeta_1 + ... + \zeta_j \quad (\zeta_0 = 0)$ is the symmetric measure $\hat{\mu}$ defined by (8). The random variables $\zeta_j$ are in particular compactly supported and we have

$$ ||Ad(z_1 ... z_{j-1})v_j|| \leq C p^\max_{1 \leq l \leq k} L_i(\zeta_1 + ... + \zeta_j), \quad \zeta_j = p(z_j), \quad j \geq 2. $$

Let us consider, for $n = 1, 2, ...$, the event $E_n$ defined by

$$ E_n = \left( L_i(\zeta_1 + ... + \zeta_j) \leq C, \quad j = 1, ..., n, \quad l = 1, ..., s; \right. $$

$$ \left| \zeta_1 + ... + \zeta_n \right|_{Z_{\alpha}} \leq C \sqrt{n} \right), $$

where $C > 0$ denotes an appropriate large constant. Using (15), Campbell-Hausdorff, (24) and the ultrametric property (19) we see that the event $E_n$ verifies

$$ E_n \subset \left[ X_n \in B_n \right] $$

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where $B_n$ is defined by

$$B_n = \exp (\{u \in \mathcal{U}, \ |u| \leq C \}) \cdot \{z \in A \times S, \ |p(z)|_d \leq An^{1/2} \}.$$  

It is easy to see that $d^*g$ shows that

$$|B_n| \leq C n^d, \ n = 1, 2, ...$$  

It remains to estimate the probability of the event (25). Let $\Pi$ denote the polyhedral region in $\mathbb{Z}^d$ defined by

$$\Pi = \{z \in \mathbb{Z}^d, \ L_j(z) \leq C, \ j = 1, ..., s\},$$

where $C$ denotes the same constant as in (25). Let $h_n(x, y)$, $n = 1, 2, ..., x, y \in \Pi$, denote the transition kernel corresponding to the random walk $(S_j)_{j \in \mathbb{N}}$ with killing outside of $\Pi$. Precise lower estimates for the kernel $h_n(x, y)$ can be obtained thanks to the results of [26] and [27]. To write down these estimates we need the following notations. Let us assume that $d \geq 2$ and let $0 < u_0(\sigma)$, $\sigma \in \Pi_{\Sigma}$, denote the eigenfunction corresponding to the first Dirichlet dirichlet eigenvalue of the region $\Pi_{\Sigma}$ defined by (10). Let $u(x)$ be the function defined on $\Pi_{\Sigma}$ by

$$u(x) = u(r, \sigma) = r^d u_0(\sigma), \ x = (r, \sigma) \in \mathbb{R}^*_+ \times \Pi_{\Sigma}$$

where

$$a = \frac{\sqrt{(d-2)^2 + 4\lambda} - (d-2)}{2},$$

and where $(r, \sigma)$ denote the polar coordinates on $\mathbb{R}^*_+ \times \Sigma$. The function $u$ defined in this way is a positive function harmonic in $\Pi_{\Sigma}$ which vanishes on $\partial \Pi_{\Sigma}$ (cf. [24]). In the case where $d = 1$, the function $u$ is defined by $u(x) \equiv x$, $x \in \mathbb{R}^*_+$.

It follows easily from [27] (cf. estimate (2) p. 359) and [26] (cf. Theorem 1 and estimate (0.3.4)) that there exists $C > 0$ such that

$$h_n(0, y) \geq \frac{u(-y)}{C n^{a+d/2}}, \ |y| \leq C \sqrt{n}, \ n \geq C.$$  

We have therefore

$$\mathbb{P}(E_n) \geq \frac{1}{C n^{a+d/2}} \sum_{y \in \Pi, |y| \leq C \sqrt{n}} u(-y).$$

Using the homogeneity of $u$ we deduce then that

$$\mathbb{P}(E_n) \geq \frac{1}{C n^{a+d/2}} \int_{0}^{C \sqrt{n}} r^{a+d-1} dr = \frac{1}{C n^{d/2}}.$$  

The lower estimate (11) is an immediate consequence of (13), (26), (27) and (28). This completes the proof of Theorem 1.
4 Metabelian $p$-adic groups

Our aim in this section is to show that in the case of metabelian groups the lower estimate (11) can be complemented with a similar upper bound. We keep the notation of §2. We shall denote by $\mathbb{Z}_p^* = \{ x \in \mathbb{Q}_p^*, \ |x|_p = 1 \}$ and by $\mathbb{Z}_p = \{ x \in \mathbb{Q}_p, \ |x|_p \leq 1 \}$. Let $dx$ denote the Haar measure on $\mathbb{Q}_p$ normalized by $dx(\mathbb{Z}_p) = 1$ and let $d^*x$ denote the Haar measure on $\mathbb{Q}_p^*$ normalized by $d^*x(\mathbb{Z}_p^*) = 1$. Let us fix $k, l \geq 2$ and consider

$$G = \mathbb{Q}_p^k \rtimes \sigma\left(\mathbb{Q}_p^l\right)^l$$

the semi-direct product of $\left(\mathbb{Q}_p^l\right)^l$ with the vector space $\mathbb{Q}_p^k$ where $\left(\mathbb{Q}_p^l\right)^l$ acts on $\mathbb{Q}_p^k$ by

$$x = (x_1, ..., x_k) \rightarrow \sigma(y)x = (\chi_1(y)x_1, ..., \chi_k(y)x_k), \ y \in \left(\mathbb{Q}_p^l\right)^l,$$

where $\chi_1, ..., \chi_k$ denote $k$ morphisms

$$\chi_1, ..., \chi_k : \left(\mathbb{Q}_p^l\right)^l \rightarrow \mathbb{Q}_p.$$

More precisely, we assume that the multiplication in $G$ is given by

$$g \cdot g' = (x; y)(x'; y') = (x + \sigma(y)x'; y, y')$$

$$= (x_1 + \chi_1(y)x_1', x_2 + \chi_2(y)x_2', ..., x_k + \chi_k(y)x_k'; y_1, y_1', y_2, y_2', ..., y_l, y_l').$$

$$g = (x, y), \ g' = (x', y') \in G; \ x = (x_1, ..., x_k), \ x' = (x_1', ..., x_k') \in \mathbb{Q}_p^k;$$

$$y = (y_1, ..., y_l), \ y' = (y_1', ..., y_l') \in \left(\mathbb{Q}_p^l\right)^l.$$

We shall denote by

$$d^r g = dxd^*y = dx_1 ... dx_k d^*y_1 ... d^*y_l; \ d^l g = dg = \delta(g)^{-1}d^r g$$

the right and the left invariant Haar measure on $G$. The modular function $\delta(g)$ is given by

$$\delta(g) = \delta(y) = |\chi_1(y)|...|\chi_k(y)|, \ g = (x, y) \in G.$$

Let $d\mu(g) = \varphi(g)d^r(g)$ denote the symmetric, compactly supported, probability measure on $G$ defined by

$$\varphi(g) = a\delta^{-1/2}(g)I_{\mathbb{Z}_p}(x_1) ... I_{\mathbb{Z}_p}(x_k)I_{\mathbb{Z}_p}(\chi_1(y)^{-1}x_1) ... I_{\mathbb{Z}_p}(\chi_k(y)^{-1}x_k)$$

$$\times I_{\left(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup \mathbb{Z}_p^*\right)}(y_1) ... I_{\left(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup \mathbb{Z}_p^*\right)}(y_l),$$

$$g = (x_1, ..., x_k; y_1, ..., y_l) \in G.$$

The notation $I_A$ is used here to denote the indicator of a subset $A$ and $\alpha > 0$ denotes an appropriate positive constant.
We shall assume that the $\chi_j$'s in (30) verify the condition

$$|\chi_j| \neq 1, \quad j = 1, \ldots, k.$$  

This guarantees (cf. §2) that the group $G$ defined by (29) is compactly generated and it is easy to see that $supp(\mu)$ is a generating set of the group $G$. We shall denote, as in §2, by $L_j$, $j = 1, \ldots, k$, the $k$ linear forms on $\mathbb{Z}^l$ induced by the $\chi_j$'s and we shall assume that these linear forms induce a "Weyl chamber" $\Pi = \{x \in \mathbb{R}^l, \quad L_j(x) > 0, \quad j = 1, 2, \ldots, k\}$ as in §2. We shall denote by $\Sigma = \{x \in \mathbb{R}^l, \quad |x| = 1\}$ the unit sphere in $\mathbb{R}^l$, by $\Pi_\Sigma = \Sigma \cap \Pi$ and by $\lambda$ be the first Dirichlet eigenvalue of the region $\Pi_\Sigma$. We have then the following:

**Theorem 2.** Let $G = \mathbb{Q}_p^k \times_\sigma \left( \mathbb{Q}_p^* \right)^l$ and let $\mu \in \mathcal{P}(G)$ be as above. Let $d\mu^n(g) = d (\mu \ast \ast \mu)^{(g)} = \varphi_n(g) d^r g$ denote the $n^{th}$ convolution power of $\mu$. Then there exists $C > 0$ such that

$$\varphi_n(e) \lesssim \frac{C}{n^{1 + \frac{\sqrt{n^2 + 4\lambda}}{2}}}, \quad n = 1, 2, \ldots$$

The first step to prove Theorem 2 consists in establishing an explicit formula for $\varphi_n(g)$, $g \in G$. In what follows we shall use the notation $x_i = (x_{i,1}, \ldots, x_{i,k})$ (resp. $y_i = (y_{i,1}, \ldots, y_{i,l})$) for $x_i \in \mathbb{Q}_p^k$, $i = 1, 2, \ldots$ (resp. $y_i \in \left( \mathbb{Q}_p^* \right)^l$, $i = 1, 2, \ldots$). Let us fix $g = (\xi, \zeta) = (\xi_1, \ldots, \xi_k; \zeta_1, \ldots, \zeta_l) \in G$ and $n = 1, 2, \ldots$. By definition of convolution product we have:

$$\varphi_{n+1}(g) = \int_G \varphi_n(h) \varphi_1(h^{-1}g)d h$$

$$= \int_G \varphi_1(h^{-1}g) \delta(h^{-1}) \varphi_n(h) d h$$

$$= \int_G \varphi_1(h^{-1}g) \delta(h^{-1}) d\mu^n(h)$$

$$= \int_G \ldots \int_G \varphi_1 \left( (g_1 \ldots g_n)^{-1} g \right) \delta(g_1 \ldots g_n)^{-1} d\mu(g_1) \ldots d\mu(g_n)$$

$$= \int_G \ldots \int_G \varphi_1 \left( - (\sigma(y_1 \ldots y_n)^{-1} x_1 + \sigma(y_2 \ldots y_n)^{-1} x_2 + \ldots + \sigma(y_n)^{-1} x_n) + \sigma(y_1 \ldots y_n)^{-1} \xi, (y_1 \ldots y_n)^{-1} \zeta \right) \varphi_1(x_1, y_1) \ldots \varphi(x_n, y_n)$$

$$\delta(y_1 \ldots y_n)^{-1} dx_1 d^* y_1 \ldots dx_n d^* y_n.$$
An obvious change of variables combined with Fubini gives

\[
\varphi_{n+1}(g) = \int_{(\mathbb{Q}_p^n)^n} \cdots \int_{(\mathbb{Q}_p^n)} \left[ \int_{\mathbb{Q}_p^n} \cdots \int_{\mathbb{Q}_p^n} \varphi_1 \left( -(x_1 + x_2 + \ldots + x_n) 
\right.ight.
\]
\[
+ \sigma(y_1 \ldots y_n)^{-1} \xi_1, (y_1 \ldots y_n)^{-1} \xi_n \right) \varphi_1(\sigma(y_1 \ldots y_n)x_1, y_1) 
\]
\[
\varphi_1(\sigma(y_2 \ldots y_n)x_2, y_2) \cdots \varphi_1(\sigma(y_n)x_n, y_n) \right] dx_1 \cdots dx_n 
\]
\[
\prod_{i=2}^n |\chi_1(y_i \ldots y_n)| \cdots |\chi_k(y_i \ldots y_n)| d^*y_1 \ldots d^*y_n,
\]

where we used (31). Let us consider the product

\[
\prod_{i=1}^n \varphi_1(\sigma(y_i \ldots y_n)x_i, y_i)
\]

that appears in equation (33). By (32), this product is equal to

\[
\alpha^n \prod_{i=1}^n \delta^{-1/2}(y_i) \prod_{i=1}^n \left[ I_{\mathbb{Z}_p}(\chi_1(y_i \ldots y_n)x_i, 1) \cdots I_{\mathbb{Z}_p}(\chi_k(y_i \ldots y_n)x_i, 1) \right] 
\]
\[
\times \prod_{i=1}^{n-1} \left[ I_{\mathbb{Z}_p}(\chi_1(y_i+1 \ldots y_n)x_i, 1) \cdots I_{\mathbb{Z}_p}(\chi_k(y_i+1 \ldots y_n)x_i, 1) \right] 
\]
\[
\times I_{\mathbb{Z}_p}(x_{n,1}) \cdots I_{\mathbb{Z}_p}(x_{n,k}) \prod_{i=1}^n I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup \mathbb{Z}_p^*)}(y_{i,1}) \cdots I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup \mathbb{Z}_p^*)}(y_{i,1}) 
\]
\[
= \alpha^n \prod_{i=1}^n \delta^{-1/2}(y_i) \prod_{i=1}^n \left[ I_{\mathbb{Z}_p}(\chi_1(y_i+1 \ldots y_n)x_i, 1) \right] 
\]
\[
\cdots I_{\mathbb{Z}_p}(\chi_k(y_i+1 \ldots y_n)x_i, 1) \right] 
\]
\[
I_{\max(1,|\chi_1(y_i)|)}(x_{n,1}) \cdots I_{\max(1,|\chi_k(y_i)|)}(x_{n,k}) 
\]
\[
\prod_{i=1}^n I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup \mathbb{Z}_p^*)}(y_{i,1}) \cdots I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup \mathbb{Z}_p^*)}(y_{i,1}).
\]

We also have

\[
\varphi_1 \left( -(x_1 + x_2 + \ldots + x_n) + \sigma(y_1 \ldots y_n)^{-1} \xi_1, (y_1 \ldots y_n)^{-1} \xi_n \right) 
\]
\[
= \alpha \delta^{-1/2}(y_i \ldots y_n)^{-1} \xi_i \times I_{\max(1,|\chi_1(y_i)|)} \left( x_{1,1} + x_{2,1} + \ldots + x_{n,1} - \chi_1(y_1 \ldots y_n)^{-1} \xi_1 \right) 
\]
\[
\cdots I_{\max(1,|\chi_k(y_i)|)} \left( x_{1,k} + x_{2,k} + \ldots + x_{n,k} - \chi_k(y_1 \ldots y_n)^{-1} \xi_k \right) 
\]
\[
\times I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup \mathbb{Z}_p^*)}(y_{1,1} \ldots y_{n,1})^{-1} \xi_1 \cdots I_{(p^{-1}\mathbb{Z}_p^* \cup \mathbb{Z}_p^* \cup \mathbb{Z}_p^*)}(y_{1,1} \ldots y_{n,1})^{-1} \xi_l).
\]
On the other hand we shall set
\[
(34) \quad m(y) = a \delta^{-1/2}(y) \prod_{j=1}^{k} \min \left(1, |x_j(y)| \right) I_{(p^{-1}z^*[p]_{1,\ldots,p}^2)}(y_1) \cdots I_{(p^{-1}z^*[p]_{1,\ldots,p}^2)}(y_l),
\]
\[y = (y_1, \ldots, y_l) \in \left(\mathcal{Q}_p^* \right)^l.
\]
What motivates this notation is the fact that
\[
(35) \quad \int_{\mathcal{Q}_p^*} \varphi(x, y) dx = m(y), \quad y \in \left(\mathcal{Q}_p^* \right)^l.
\]
Setting
\[
(36) \quad A(y_1, y_2, \ldots, y_n) = m\left( (y_1 \ldots y_n)^{-1} \xi \right) \prod_{j=1}^{k} \max \left(1, |x_j(y_1 \ldots y_n)\xi^{-1}| \right)
\]
\[\times \prod_{i=1}^{n} \max \left(1, |x_i(y_i)|^{-1} \right) \cdots \max \left(1, |x_k(y_i)|^{-1} \right) m(y_i),
\]
\[
(37) \quad B_j(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) = B_j(x_1, x_2, \ldots, x_n)
\]
\[= I_{\max(1, |x_j(y_1 \ldots y_n)\xi^{-1}|)}(x_{1,j} + x_{2,j} + \ldots
\]
\[+ x_{n,j} - x_j(y_1 \ldots y_n)^{-1} \xi_j) I_{\max(1, |x_j(y_i)|)}(x_{n,j})
\]
\[\times \prod_{i=1}^{n-1} I_{|x_j(y_{i+1} \ldots y_n)| \max(1, |x_j(y_i)|)}(x_{i,j}),
\]
we rewrite (33) as
\[
(38) \quad \varphi_{n+1}(g) = \int_{(\mathcal{Q}_p^*)^l} \ldots \int_{(\mathcal{Q}_p^*)^l} \left[ \int_{\mathcal{Q}_p^k} \ldots \int_{\mathcal{Q}_p^k} B_j(x_1, x_2, \ldots, x_n) dx_1 \ldots dx_n \right]
\]
\[A \left(y_1, y_2, \ldots, y_n \right) \prod_{i=2}^{n} |x_1(y_i \ldots y_n)| \ldots |x_k(y_i \ldots y_n)|
\]
\[d^* y_1 \ldots d^* y_n.
\]
The next step is to calculate
\[
\int_{\mathcal{Q}_p^k} B_j(x_1, x_2, \ldots, x_n) dx_{1,j} dx_{2,j} \ldots dx_{n,j}, \quad j = 1, \ldots, k.
\]
Towards this we observe that:

\[
\int_{Q_p} \cdots \int_{Q_p} I_{\max(1, |\lambda_j(y_{i+1} \ldots y_n)|)}(x_{1,j} + \ldots + x_{n,j} - \lambda_j(y_1 \ldots y_n)^{-1} \xi_j)
\]

\[
I_{\max(1, |\lambda_j(y_n)|)}(x_{i,j}) d x_{1,j} d x_{2,j} \ldots d x_{n,j}
\]

\[
= \int_{Q_p} \left( I_{\max(1, |\lambda_j(y_{i+1} \ldots y_n)|)}(x) * \cdots * I_{\max(1, |\lambda_j(y_{i-1})|)}(x) \right) \left( I_{\max(1, |\lambda_j(y_{i+1})|)}(x) \right) \left( I_{\max(1, |\lambda_j(y_n)|)}(x) \right) dx
\]

where \(*\) denotes the usual convolution in \(Q_p\) (cf. [21]). Taking Fourier transform (cf. [21]) we obtain

\[
\int_{Q_p} \cdots \int_{Q_p} B_j(x_1, x_2, \ldots, x_n) d x_{1,j} d x_{2,j} \ldots d x_{n,j}
\]

\[
= \text{Min} \left[ \min_{1 \leq i \leq n-1} \left( \left| \lambda_j(y_{i+1} \ldots y_n) \right| \max(1, |\lambda_j(y_i)|) \right) \right] \max(1, |\lambda_j(y_n)|)
\]

\[
\min(1, |\lambda_j(y_n)|) \prod_{i=1}^{n-1} |\lambda_j(y_{i+1} \ldots y_n)|^{-1} \min \left( \left| \lambda_j(y_i) \right|^{-1} \right)
\]

\[
\times \left( \text{Min} \left[ \min_{1 \leq i \leq n-1} \left( \left| \lambda_j(y_{i+1} \ldots y_n) \right| \max(1, |\lambda_j(y_i)|) \right) \right] \right) \left( I_{\max(1, |\lambda_j(y_{i+1})|)}(x) \right) \left( I_{\max(1, |\lambda_j(y_n)|)}(x) \right).
\]

Let us denote by \(S_j^\xi(y_1, \ldots, y_n) \subset Q_p, y_1, \ldots, y_n \in Q_p^n\) the subset defined by

\[
S_j^\xi(y_1, \ldots, y_n) = \text{Min} \left[ \min_{1 \leq i \leq n-1} \left( \left| \lambda_j(y_{i+1} \ldots y_n) \right| \max(1, |\lambda_j(y_i)|) \right) \right] \max(1, |\lambda_j(y_1 \ldots y_n)^{-1}|) Z_p.
\]

With this notation we have

\[
(39) \quad \int_{Q_p} \cdots \int_{Q_p} B_j(x_1, x_2, \ldots, x_n) d x_{1,j} d x_{2,j} \ldots d x_{n,j}
\]

\[
= \min \left( 1; \text{Min} \left[ \min_{1 \leq i \leq n-1} \left( \left| \lambda_j(y_{i+1} \ldots y_n) \right| \max(1, |\lambda_j(y_i)|) \right) \right] \right) \max(1, |\lambda_j(y_n)|)
\]

\[
\min \left( 1, \left| \lambda_j(y_1 \ldots y_n)^{-1} \right| \right) \min \left( \left| \lambda_j(y_1 \ldots y_n)^{-1} \right| \right) \min \left( \left| \lambda_j(y_1 \ldots y_n)^{-1} \right| \right) \prod_{i=1}^{n-1} |\lambda_j(y_{i+1} \ldots y_n)|^{-1} \min \left( \left| \lambda_j(y_i) \right|^{-1} \right) I_{S_j^\xi(y_1, \ldots, y_n)}(\lambda_j(y_1 \ldots y_n)^{-1} \xi_j),
\]

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Putting together (33), (36), (37), (38) and (39) and taking into account the fact that

\[
\prod_{j=1}^{k} \min\left(1, |\chi_j(y_n)|^{-1}\right) \prod_{i=1}^{n-1} \left( \prod_{j=1}^{k} \min\left(1, |\chi_j(y_i)|^{-1}\right) \right) \\
\times \prod_{i=1}^{n} \max\left(1, |\chi_1(y_i)|^{-1}\right) \cdots \max\left(1, |\chi_k(y_i)|^{-1}\right) \\
= \prod_{j=1}^{k} |\chi_j(y_1 \cdots y_n)|^{-1} = \delta^{-1}(y_1 \cdots y_n),
\]

and that

\[
\delta^{-1}(y_1 \cdots y_n) \prod_{j=1}^{k} \max\left(1, |\chi_j(y_1 \cdots y_n)\zeta^{-1}|\right) m\left((y_1 \cdots y_n)^{-1}\zeta\right) \\
= \alpha \delta^{-1/2}(y_1 \cdots y_n) \delta^{-1/2}(\zeta) I_{\left(p^{-1}z^*_{p_1} \cup z^*_{p_2} \cup \ldots \cup z^*_{p_k}\right)}((y_1,1 \cdots y_n,1)^{-1}\zeta_1) \\
\cdots I_{\left(p^{-1}z^*_{p_1} \cup z^*_{p_2} \cup \ldots \cup z^*_{p_k}\right)}((y_1,l \cdots y_n,l)^{-1}\zeta_l),
\]

we finally deduce the formula

\[
(40) \quad \varphi_{n+1}(g) = \alpha \delta^{-1/2}(\zeta) \int_{\left(Q_p^*\right)^l} \cdots \int_{\left(Q_p^*\right)^l} \prod_{j=1}^{k} \min\left(1, \min_{1 \leq i \leq n-1} \left( |\chi_j(y_{i+1} \cdots y_n)| \max(1, |\chi_j(y_i)|) \right); \right) \\
\max(1, |\chi_j(y_n)|) \min(1, |\chi_j(y_1 \cdots y_n)|^{-1}\zeta) \right) \\
\delta^{-1/2}(y_1 \cdots y_n) \prod_{j=1}^{k} I_{\left(Q_p^*\right)^l}(\chi_j(y_1 \cdots y_n)^{-1}\zeta_j) \\
\prod_{j=1}^{l} I_{\left(p^{-1}z^*_{p_1} \cup z^*_{p_2} \cup \ldots \cup z^*_{p_k}\right)}((y_1,i \cdots y_n,i)^{-1}\zeta_i) \\
m(y_1) \cdots m(y_n) d^*y_1 \cdots d^*y_n.
\]

The next step is to give a probabilistic interpretation of (40). Towards that let us consider a sequence of independent identically \(\left(Q_p^*\right)^l\)-valued random variables \(Y_1, Y_2, \ldots\) with distribution on \(\left(Q_p^*\right)^l\) given by

\[
(41) \quad \mathbb{P}[Y_j \in d\gamma] = m(\gamma)d^*\gamma
\]
where \( m(y) \) is as in (34). Observe that by (35) we have \( \int m(y) d^* y = 1 \). The formula (40) can be rewritten

\[
\varphi_{n+1}(g) = a \delta^{-1/2}(\zeta) E \left[ \prod_{j=1}^{k} \min \left( 1; \min \left( \min_{1 \leq i \leq n} (|\chi_j(Y_{i+1} \ldots Y_n)| \max(1, |\chi_j(Y_i)|) \right) \right) \right.
\]

\[
\max(1, |\chi_j(Y_n)|) \min(1, |\chi_j(Y_1 \ldots Y_n)^{-1} \zeta_j|) \]

\[
\delta^{-1/2}(Y_1 \ldots Y_n) \prod_{j=1}^{k} I_{\delta_j(Y_1, \ldots, Y_n)}(\chi_j(Y_1 \ldots Y_n)^{-1} \zeta_j) \]

\[
\prod_{j=1}^{l} I_{(p^{-1}Z_p^i \cup Z_p^i \cup pZ_p^i)}((Y_{1,j} \ldots Y_{n,j})^{-1} \zeta_j) \].
\]

In the case \( g = e \) the above formula simplifies considerably and we obtain

\[
\varphi_{n+1}(e) \leq CE \left( \prod_{j=1}^{k} \min \left( 1, \min_{1 \leq i \leq n} |\chi_j(Y_i \ldots Y_n)| \right) \right)
\]

\[
\times \prod_{j=1}^{l} I_{(p^{-1}Z_p^i \cup Z_p^i \cup pZ_p^i)}((Y_{1,j} \ldots Y_{n,j})^{-1}).
\]

Now we project the random walk \( Y_1 \ldots Y_n \) controlled by (41) on \( \mathbb{Z}^l \subset \mathbb{R}^l \) and we assume that \( \mathbb{R}^l \) is equipped with the Euclidean structure ensuring (9). We shall denote by \( S_n = X_1 + X_2 + \ldots + X_n \) \((n \geq 1)\) the random walk obtained via this projection. We have

\[
\varphi_{n+1}(e) \leq CE \left( \prod_{j=1}^{k} \min_{1 \leq i \leq n} p^{L_j(S_i)} \right) \left( S_n \in K \right), \quad n = 1, 2, \ldots
\]

where \( K \) denotes a neighbourhood of the origin in \( \mathbb{Z}^l \). It follows then that

\[
\varphi_{n+1}(e) \leq CE \left( \prod_{j=1}^{k} p^{-\max_{1 \leq i \leq n} L_j(S_i)} \right) \left( S_n \in K \right), \quad n = 1, 2, \ldots
\]

and therefore

\[
\varphi_{n+1}(e) \leq C \sum_{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0} p^{-\sum_{j=1}^{k} \lambda_j \mathbb{P}} \left( \max_{0 \leq i \leq n} L_1(S_i) = \lambda_1, \ldots, \right.
\]

\[
\left. \max_{0 \leq i \leq n} L_k(S_i) = \lambda_k; \quad S_n \in K \right), \quad n = 1, 2, \ldots
\]

Let us now fix \( x_0 \in \Pi \) such that

\[
L_j(x_0) > 0, \quad j = 1, \ldots, k.
\]

Let

\[
\lambda_0 = \min_{1 \leq j \leq k} L_j(x_0).
\]

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We have

\[ L_j(u_0) = L_j(x_0/\lambda_0) \geq 1. \]

It follows then from (42) that

\[
\varphi_{n+1}(e) \leq C \sum_{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0} p^{-\sum_{i=1}^k \lambda_i} \mathbb{P}
\left( L_1(S_i) \leq \lambda_1 L_1(u_0), \ldots, L_k(S_i) \leq \lambda_k L_k(u_0), \right.

\[
0 \leq i \leq n; \quad S_n \in K \bigg) \leq C \sum_{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0} p^{-\sum_{i=1}^k \lambda_i} \mathbb{P}
\left( L_1(S_i) - (\max_{1 \leq i \leq k} \lambda_i + C) u_0 \leq 0, \ldots, \right.

\[
L_k(S_i) - (\max_{1 \leq i \leq k} \lambda_i + C) u_0 \leq 0, \quad 0 \leq i \leq n; \quad S_n \in K \bigg). \]

Choosing \( C > 0 \) large enough, we can apply Theorem 4 of [26] and we deduce

\[
\varphi_{n+1}(e) \leq \frac{C}{n^{1+\frac{\sqrt{(l-2)^2+4k}}{2}}} \sum_{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0} \left( \max_{1 \leq i \leq k} \lambda_i + 1 \right)^C p^{-\sum_{i=1}^k \lambda_i} \leq \frac{C}{n^{1+\frac{\sqrt{(l-2)^2+4k}}{2}}} \sum_{\lambda_1 \geq 0, \ldots, \lambda_k \geq 0} (\sum_{i=1}^k \lambda_i + 1)^C p^{-\sum_{i=1}^k \lambda_i} \leq \frac{C}{n^{1+\frac{\sqrt{(l-2)^2+4k}}{2}}}
\]

This completes the proof of Theorem 2.

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