Geometry of Cyclic Quotients, I:
Knotted Totally Geodesic Submanifolds in Positively Curved Spheres

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Theorem 0.1. Given a pair of coprime integers $(m, n), |m|, |n| \geq 2$, there exists a metric of positive sectional curvature in a compact three-manifold $M$, with the following properties:

(a) $M$ is diffeomorphic to $S^3$

(b) There exist geodesic embeddings of $S^1$ in $M$, isotopic to torus knots $(m, n)$ and $(m - n)$.

In connection to the Theorem 0.1 above, we would like to propose the following problem.

0.2. Is it true, that a positively curved metric in $S^3$ admits only finitely many different geodesic knot types?

The problem 0.2 seems to be conceptually related to a theorem of Choi-Schoen on compactness of the space of embedded minimal surfaces of a given genus in a Ricci-positive three-manifold (see 3.4.)

1. Topology of cyclic quotients.

1.1. Here we collect the facts we need on the topology of cyclic quotients. Let $N$ be a smooth manifold with a smooth action of a cyclic group $\mathbb{Z}_n$. Assume that

(a) All stationary subgroups of points in $N$ are either trivial or $\mathbb{Z}_n$ itself (this is automatically so, if $n$ is prime)

(b) All components of the fixed point set $\text{Fix} (N)$ are of codimension 2.

Then the quotient $N/\mathbb{Z}_n$ has a canonical structure of a smooth manifold. Indeed, let $Q \subset \text{Fix} (N)$ be a connected component. Fix an invariant Riemannian metric in $N$. Let $\eta$ be the rank two normal bundle to $Q$ and let $\varepsilon D$ be the disc bundle $\eta$ of radius $\varepsilon$. For $\varepsilon$ small
the exponential map establishes a diffeomorphism of $\varepsilon D$ onto a tubular neighbourhood of $Q$. We may assume $\eta$ to be orientable (see below). Consider $\eta$ as a complex line bundle. Then cut $\text{Exp}(\varepsilon D)$ off and glue the unit disc bundle of $\eta^{\otimes n}$ to $(N \setminus \text{Exp}(\varepsilon D))/\mathbb{Z}_n$. Doing this simultaneously for all $Q$, we get a new manifold, homeomorphic to $N/\mathbb{Z}_n$. If $\eta$ is not orientable, consider the double covering $\tilde{Q} \xrightarrow{\pi} Q$ such that $\tilde{\eta} = \pi^*\eta$ is orientable. Denote $\tau$ the involution of $\tilde{Q}$ and $\tilde{\eta}$ such that $\tilde{Q}/\tau = Q$ and $\tilde{\eta}/\tau = \eta$. Observe that $\tau$ is an orthogonal and antilinear automorphism of $\tilde{\eta}$, so it induces an orthogonal and antilinear automorphism of $\tilde{\eta}^{\otimes n}$ which we denote again by $\tau$. Now, glue $(\tilde{\eta}^{\otimes n})/\tau$ to $(N \setminus \text{Exp}(\varepsilon D))/\mathbb{Z}_n$.

1.2 Example: Let $N$ be a smooth quasiprojective variety over $\mathbb{R}$. Let $\tau$ be the canonical involution of $N(\mathbb{C})$, coming from $\text{Gal} (\mathbb{C}/\mathbb{R})$. Then $N(\mathbb{C})/\tau$ is a (real) smooth manifold. In particular, $\mathbb{C}P^2/\tau$ is a four-sphere, [5], [6].

1.3. Cyclic quotients of the three-sphere: Consider $S^3$ as a unit sphere in the Hermitian space $\mathbb{C}^2$ with the coordinates $(z_1, z_2)$. Denote by $K$ and $L$ the geodesic circles $z_2 = 0$ and $z_1 = 0$. For $a, b > 0$ with $a^2 + b^2 = 1$ consider a torus $T_{a,b} : |z_1| = a, |z_2| = b$. The family $T_{a,b}$ form a fibration of $S^3 \setminus (K \cup L)$, “converging” to $K$ and $L$. Now, any Hopf circle, i.e. an intersection of $S^3$ with a complex line, lies in one of $T_{a,b}$, namely, $\{z_2 = \lambda z_1\} \cap S^3$ lies on $T_{a,b}$ with $a = \frac{1}{\sqrt{1+|\lambda|^2}}$ and $b = \frac{|\lambda|}{\sqrt{1+|\lambda|^2}}$. Observe that all tori $T_{a,b}$ are equidistant from $K$ and $L$ in spherical metric. Any such torus is a Heegard surface of the decomposition $S^3 = D^2 \times S^1 \cup S^1 \times D^2$.

Now, consider a $\mathbb{Z}_m$-action $(z_1, z_2) \mapsto (z_1, e^{\frac{2\pi k}{m}}z_2)$. It has $K$ as a fixed point set and acts free in $S^3 \setminus K$. According to 1.1, $S^3/\mathbb{Z}_m$ is a manifold. We claim $S^3/\mathbb{Z}_m \approx S^3$. Indeed, the action in the handle, which contains $L$, is free and the quotient is obviously a handle again. The action in the other handle, which contains $K$ is fiber like, as in 1.1, and since the normal bundle to $K$ is trivial, the quotient is again a handle, which proves the statement.

Let $n$ be an integer, coprime to $m$. Consider action of $\mathbb{Z}_m \times \mathbb{Z}_n$ by $(z_1, z_2) \mapsto (e^{\frac{2\pi k}{m}}z_1, e^{\frac{2\pi l}{n}}z_2)$. Applying the diffeomorphism above twice, we get a following lemmas.

Lemma 1.4. The quotient $S^3/\mathbb{Z}_m \times \mathbb{Z}_n$ is diffeomorphic to $S^3$.

2. Constructing a metric in the cyclic quotient.
The construction below uses the computations of Gronov and Thurston [4]. In that paper, Gronov and Thurston introduced negatively curved metrics on ramified coverings of hyperbolic manifolds. Our situation is “dual” to that considered in [4], in particular, lifting to a ramified covering is replaced by descending to a cyclic quotient.

**Lemma 2.1.** (comp [4], p.4). Given \( n \in \mathbb{N} \) and \( \rho > 0 \), there exists a smooth function \( \sigma(r) \) with the following properties:

(i) \( \sigma(r) = \sin r \) for small \( r \)

(ii) \( \sigma'(r) > 0 \) and \( \sigma''(r) < 0 \)

(iii) \( \sigma(r) = \frac{\sin r}{n} \) for \( r \geq \rho \)

The proof is immediate.

Now, the metric of \( S^3 \setminus L \) can be written as

\[
g = dr^2 + \sin^2 r d\theta^2 + \cos^2 r dt^2
\]

Here \( t \) is the length parameter along \( K \) and \( (r, \theta) \) are polar coordinates in geodesic two-spheres, orthogonal to \( K \).

Consider the cyclic quotient \( S^3/\mathbb{Z}_n \) and equip it with the metric

\[
\tilde{g} = dr^2 + \sigma^2(r)d\theta^2 + \cos^2 r dt^2
\]

where \( \theta \) here is the new angle parameter. Outside the \( \rho \)-neighbourhood of \( K/\mathbb{Z}_n \), this is just a descend of the spherical metric by the (free) action of \( \mathbb{Z}_n \). The crucial fact is the following.

**Lemma 2.2.** The metric \( \tilde{g} \) is a well-defined smooth metric on \( S^3/\mathbb{Z}_n \), of strictly positive curvature, which is invariant under the (descend of) the \( \mathbb{Z}_m \)-action. Out of the small neighbourhood of \( K \), the curvature of \( \tilde{g} \) is constant.

**Proof:** It is elementary to check that \( \tilde{g} \) is smooth with respect to the manifold structure of \( S^3/\mathbb{Z}_n \). The positivity of curvature follows form computations of [4], p. 4-5, with obvious changes (\( \cosh \rightarrow \cos \) etc.). the invariance under the \( \mathbb{Z}_m \)-action is obvious from the construction.

2.3: Taking \( \rho \) small and repeating the construction with respect to the \( \mathbb{Z}_m \)-action, we come to the following lemma.
Lemma 2.3. The quotient $S^3/\mathbb{Z}_m \times \mathbb{Z}_n$ can be equipped with the metric $\tilde{g}$ with following properties:

(a) the curvature of $(S^3/\mathbb{Z}_m \times \mathbb{Z}_n, \tilde{g})$ is strictly positive

(b) outside arbitrary small neighbourhood of $K/\mathbb{Z}_m$ and $L/\mathbb{Z}_n$ the metric $\tilde{g}$ is a descend of the spherical metric of $S^3$.

3. Knotted geodesics.

Lemma 3.1. The image of any Hopf geodesic circle in $S^3$ outside the $\rho$-neighbourhoods of $K, L$ is a torus knot $(m, n)$ in $S^3/\mathbb{Z}_m \times \mathbb{Z}_n \approx S^3$.

Proof: Let $\gamma = (z_2 = \lambda z_1) \cap S^3, \lambda \in \mathbb{C}$, be a Hopf circle. According to 1.3, $\gamma \subset T_{a,b}$ with $a = \frac{1}{\sqrt{1+|\lambda|^2}}$. In angle coordinates $(e^{i\theta}, e^{i\tau})$ on $T_{a,b}$, $\gamma$ may be written in parametric form as $t \mapsto (e^{it}, e^{it})$. The quotient map $T_{a,b} \to T_{a,b}/\mathbb{Z}_m \times \mathbb{Z}_n \approx T^2$ can be written as $(e^{i\theta}, e^{i\tau}) \mapsto (e^{mit}, e^{nit})$. So the image of $\gamma$ is $t \mapsto (e^{mit}, e^{nit})$ which is $(m, n)$ torus knot.

3.2. Geodesics of different knot type: Consider a new complex structure in $\mathbb{C}^2$, defined by the matrix $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$. Observe that $K$ and $L$ are still Hopf circles which respect to this new complex structure. The tori $T_{a,b}$ have the same equation $|w_1| = a, |w_2| = b$ with respect to new coordinates $w_1 = z_1, w_2 = \bar{z}_2$. Hence they contain a Hopf circle $w_2 = \lambda w_1$, which descends to a geodesic in $M$, which is isotopic to the torus knot $(m, -n)$. Since torus knots are invertible, but not amphichiral ([3]), we get two different knot types among geodesics in $M$. This concludes the proof of the Theorem 0.1.

3.3. Questions and Remarks 3.3.1.: May knots other than torus knots be realized as geodesics of positively curved metric in $S^3$?

3.3.2.: Fix a pinching constant $\delta$. Can a three-sphere with a $\delta$-pinched metric have geodesics of arbitrary torus knot type?

3.4.: Suppose $S^g \subset (S^3, \text{can})$ be an immersed minimal surface of genus $g$ whose image do not touch $K \cup L$. Applying the construction above, we come to a minimal surface in $M$ with "a lot of” selfintersections. In particular, one may start with a Clifford torus, close to $T_{1/\sqrt{3},1/\sqrt{2}}$. This situation contrasts the compactness theorem of embedded minimal surface of a given genus [7]. For $g \geq 2$, are may therefore ask a following question:

May a compact minimal surface in $S^3$ of genus $g \geq 2$ avoid a geodesic circle?
3.5. We sketch a different type of examples which lead to Theorem 0.1. in case when both \( m, n \) are odd. Start with a positively curved metric in \( S^2 \). Let \( \tilde{M} = US^2 \), a unit tangent bundle with the Sasaki metric (making \( US^2 \to S^2 \) be a Riemannian submersion). If the curvature of \( S^2 \) is less than \( 1/\sqrt{3} \), then \( US^2 \) is positively curved. This follows from the O’Neil formulas for Riemannian submersions with totally geodesic fibers ([2], chap. 9). Since \( US^2 \cong \mathbb{R}P^2 \), the double cover of \( US^2 \) is the three-sphere.

Now, we may take the metric of \( S^2 \) to be rotationally invariant. Then the computations of ([1]) show that we have the full control on closed geodesics of \( S^2 \). In particular, there are closed trajectories of the geodesic flow in \( US^2 \), whose lift to \( S^3 \) will be a torus knot of a given type \( (m, n) \) if both \( m, n \) are odd. Unfortunately, already the trefoil knot may not be realized in this way.

3.6. Concluding remarks: It looks like that there exists a positively curved metric on \( S^4 \) with a totally geodesic \( \mathbb{R}P^2 \) having the normal Euler number four. Indeed, look at the standard Kähler metric in \( \mathbb{C}P^2 \). The canonical autoholomorphic involution \( \tau : \mathbb{C}P^2 \to \mathbb{C}P^2 \) is an isometry and \( \mathbb{C}P^2/\tau \cong S^4 \). The fixed point set \( \text{Fix}(\tau) \) is a totally geodesic \( \mathbb{R}P^2 \subset \mathbb{C}P^2 \). It is possible to mimic the construction of 2.2. and find a perturbation of the quotient metric in \( \mathbb{C}P^2/\tau \) in directions, orthogonal to \( \mathbb{R}P^2 \) (recall that there exists exactly one totally geodesic surface, also isometric to \( \mathbb{R}P^2 \), meeting \( \text{Fix}(\tau) \) orthogonally at a given point.) The curvature tensor, however, is no more diagonal, and it is a nontrivial problem to check if the curvature is positive. Observe that the resulting metric admit an isometric \( SO(3) \)-action, and the equidistant from \( \mathbb{R}P^2 \) manifolds are lense spaces \( SO(3)/\mathbb{Z}_2 \cong S^3/\mathbb{Z}_4 \) with a homogeneous metric, which is different from Berger’s metrics.

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