Blended smoothing splines on Riemannian manifolds

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Abstract – We present a method to compute a fitting curve B to a set of data points \(d_0, \ldots, d_m\) lying on a manifold \(M\). That curve is obtained by blending together Euclidean Bézier curves obtained on different tangent spaces. The method guarantees several properties among which B is \(C^1\) and is the natural cubic smoothing spline when \(M\) is the Euclidean space. We show examples on the sphere \(S^2\) as a proof of concept.

1 Introduction

We address the problem of curve fitting on a Riemannian manifold \(M\). From a set of data points \(d_0, \ldots, d_m\) associated with times \(t_0, \ldots, t_m\) on a given time-interval \([0, n]\), we seek a \(C^1\) curve \(B : [0, n] \to M\) that is “sufficiently straight”, while approximating “sufficiently well” the data points at the given times.

Curve fitting on manifold appears in several applications where denoising or resampling time-dependent data is required. For instance, in Arnould et al. [2], the evolution of an organ is observed by interpolating several contours of a tumoral tissue on a shape manifold. Regression is also of interest in problems where 3D rigid rotations of objects are involved, as in motion planning of rigid bodies or in computer graphics [9]. In that case, the manifold would be the special orthogonal group \(SO(3)\).

A widely used strategy to address the fitting problem in general is to encapsulate the fitting and straightness constraints in a single optimization problem

\[
\min_{\gamma \in \Gamma} E_\lambda(\gamma) := \int_{t_0}^{t_m} \left\| D^2 \gamma(t) / dt^2 \right\|^2 \gamma(t) \, dt + \lambda \sum_{i=0}^{m} d^2(\gamma(t_i), d_i),
\]

where \(\Gamma\) is an admissible set of curves \(\gamma\) on \(M\), \(\| \cdot \|_\gamma(t)\) is the (Levi-Civita) second covariant derivative, \(\| \cdot \|_{\gamma(t)}\) is the Riemannian metric at \(\gamma(t)\), and \(d(\cdot, \cdot)\) is the Riemannian distance. The parameter \(\lambda\) permits to strike the balance between the regularizer \(\int_{t_0}^{t_m} \left\| D^2 \gamma(t) / dt^2 \right\|^2 \gamma(t) \, dt\) and the fitting term \(\sum_{i=0}^{m} d^2(\gamma(t_i), d_i)\).

This problem has been tackled in different ways in the past few years. We cite for instance Samit et al. [10] that approached the solution of problem (1) on a manifold-valued steepest-descent method on an infinite dimensional Sobolev space equipped with the Palais-metric. In Boumal et al. [3], the search space is reduced to the product manifold \(M^M\), as the curve \(B\) is discretized in \(M\) points, and the covariant derivative from (1) is approached with finite differences on manifolds. A technique for regression based on unwrapping and unrolling has been recently proposed by Kim et al. [7]. Finally, we mention Lin et al. [8], who proposed a polynomial regression technique based on projections on tangent spaces.

The limit case when \(\lambda \to \infty\) concerns interpolation. We cite here several works that solve this problem by means of Bézier curves [2, 1]. In those works, the search space \(\Gamma\) is reduced to composite cubic Bézier splines \(B\) and the optimality of (1) is guaranteed only when \(M = \mathbb{R}^r\). However, the main advantages of these methods are twofold: (i) the search space is drastically reduced to the so-called control points of \(B\) (see, e.g., [5] for an overview on Bézier curves); (ii) they are very simple to implement on any Riemannian manifold, as only two objects are required: the Riemannian exponential and the Riemannian logarithm, while most of the other techniques require a gradient or heavy computations of parallel transportation.

Our method aims to extend these works to fitting, and is extensively described in [6] for the case where \(m = n\). We build several polynomial pieces by solving the problem (1) on carefully chosen tangent spaces, and then blend together these curves in such a way that \(B\) (i) is differentiable, (ii) is the natural cubic smoothing spline when \(M = \mathbb{R}^r\), (iii) interpolates the data points if \(m = n\) when \(\lambda \to \infty\). Furthermore, we assess that the method is easy-to-use, as (iv) it only requires the knowledge of the Riemannian exponential and the Riemannian logarithm on \(M\); (v) the curve can be stored with only \(O(n)\) tangent vectors; and, finally, (vi) given this representation, computing \(\gamma(t)\) at \(t \in [0, n]\) only requires \(O(1)\) exponential and logarithm evaluations.

We present here the above-mentioned method and give results for fitting on the sphere \(S^2\). We refer to [6] for more details and for the proof of the six properties.

2 Method

Framework. Consider a Riemannian manifold \(M\) and a set of \(m + 1\) data points \(d_0, \ldots, d_m\) associated with parameters \(t_0, \ldots, t_m\) over an interval \([0, n]\). Our method relies on computations on tangent spaces. For this, we define the points \(d(i), i = 0, \ldots, n\), where \(d(i) = d_k\) is the data point whose associated parameter \(t_k\) is the closest to \(t = i\). We denote \(T_{d(i)} M\) its associated tangent space. Consider finally the search space \(\Gamma\) from (1) reduced to the space of \(C^1\) composite
curves

\[ \mathbf{B} : [0, n] \to \mathcal{M} : f_i(t - i), \quad i = [t], \]

where the functions \( f_i : [i, i + 1] \to \mathcal{M} \) are called blended functions. They are given by

\[ f_i(t - i) = \text{av}([L_i(t), R_i(t)], (1 - w(t), w(t))], \]

for \( i = 0, \ldots, n \) and where \( \text{av}([x, y], (1 - a, a)] \) is a Riemannian weighted mean. The fitting technique we present here consists in computing the functions \( L_i(t) \) and \( R_i(t) \) and choosing the weight function \( w(t) \) such that the six above-mentioned properties are met.

**Optimal curves.** The functions \( L_i(t) \) and \( R_i(t) \) are obtained as follows. We note \( \tilde{x} = \log_{d(i)}(x) \) and \( \tilde{y} = \log_{d(i+1)}(x) \), the representation of the point \( x \in \mathcal{M} \) in the tangent spaces at \( d(i) \) and \( d(i + 1) \) respectively. We define \( L_i(t) = \exp_{d(i)}\left( \mathbf{B}(t) \right) \) and \( R_i(t) = \exp_{d(i+1)}\left( \mathbf{B}(t) \right) \), where \( \mathbf{B}(t) \) is the natural cubic spline fitting the data points \( \tilde{d}_0, \ldots, \tilde{d}_m \) on \( T_d(i) \mathcal{M} \) and accordingly for \( \mathbf{B}(t) \). Note that \( \mathbf{B}(t) \) (resp. \( \mathbf{B}(t) \)) are therefore solutions of (1) on the corresponding tangent space.

**Riemannian averaging.** Finally, the choice of the weight function \( w(t) \) is of high importance in order to meet the differentiability property. The weight function must thus be chosen such that \( L_i(0) = f_i(0), R_i(1) = f_i(1), L_i(0) = f_i(0) \) and \( R_i(1) = f_i(1) \). This is obtained for \( w(1) = 1 \), and \( w(0) = w'(0) = w'(1) = 0 \). Among all the possible weight functions, we choose \( w(t) = 3t^2 - 2t^3 \).

The blending method is represented in Figure 1.

### 3 Results

We show two examples on \( S^2 \). Figure 2a presents a smoothing curve fitting 100 noisy points at times \( t_i \in [0, 4] \) with \( \lambda = 100 \). Figure 2b shows the fitting curve obtained for 10 data points at times \( t_i = i, i = 0, \ldots, 9 \), for \( \lambda = 10^6 \). We observe in both cases that the curve is \( C^1 \) (property (ii)) and that the data points are interpolated (property (iii)) when \( \lambda \to \infty \). Property (i) is obtained by construction. Properties (iv-vi) are shown and proved in [6]. Additional examples on the special orthogonal group \( SO(3) \) or on the manifold of positive semidefinite matrices of size \( p \) and rank \( q, S_+(p, q) \), are also provided in [6].

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