Rips Complexes as Nerves and a Functorial Dowker-Nerve Diagram

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Abstract. Using ideas related to Dowker duality, we prove that the Rips complex at scale $r$ is homotopy equivalent to the nerve of a cover consisting of sets of prescribed diameter. We then develop a functorial version of the Nerve theorem coupled with Dowker duality, which is presented as a Functorial Dowker-Nerve Diagram. These results are incorporated into a systematic theory of filtrations arising from covers. As a result, we provide a general framework for reconstruction of spaces by Rips complexes, a short proof of the reconstruction result of Hausmann, and completely classify reconstruction scales for metric graphs. Furthermore, we introduce a new extraction method for homology of a space based on nested Rips complexes at a single scale, which requires no conditions on neighboring scales nor the Euclidean structure of the ambient space.

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1. Introduction

One of the most common approaches in mathematics is approximation: to study an object we can often represent or approximate it by simpler objects, of which we have better understanding. In the context of topology, the approximations of spaces are most frequently performed using Čech or Rips simplicial complexes. Each such complex represents a snapshot of a space at a chosen scale. Taking into account all scales, the individual complexes can be bound together to form a filtration of a space. These filtrations have been initially used to study local properties of spaces: they are used to define the Čech (co)homology and form a basis of shape theory [27] (see [14] for more on

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The Rips complexes, originally discovered by Vietoris [25, p. 271], [33], were rediscovered by Rips in the context of geometry of groups and popularized by Gromov [20, 2.2]. In the asymptotic setting, both mentioned filtrations have been used to provide approximations of coarse spaces [7,31]. Our focus, however, will be on the perspective of computational topology: filtrations are used to reconstruct spaces at small scales and to compute persistent homology at all scales, which reflects the size of holes in the space.

The choice of a filtration, based on Čech or Rips complexes, obviously matters. While both filtrations are interleaved (meaning that the limiting objects as scale goes to 0 of $\infty$ do not depend on the choice), the complexes at each scale differ, and precise connection between is known only in a few settings [1,34]. Čech complexes are nerves of covers; hence, their homotopy type can be usually interpreted in a convenient way as neighborhoods of a space using the Nerve theorem. Consequently, the corresponding theory is rich and contains strong reconstruction results [29]. Rips complexes on the other hand are easier to define and compute, making them the favorite choice for applications. However, there is no clear interpretation of their homotopy type, and the reconstruction results [2,22,24] require significantly more structure and hold only for small scales.

In this paper, we provide a systematic treatment of nerve and Vietoris complexes arising from covers, of which the Čech and Rips complexes are a special case. The main results are the following:

- Rips complexes as nerves (Theorem 3.9): we can express the homotopy type of Rips complex as a nerve of a cover by sets of prescribed diameter.
- Functorial Dowker-nerve diagram (Theorem 5.4): we present a functorial version of the Nerve theorem and Dowker duality, jointly incorporated into a diagram.

The ability to treat the homotopy type of Rips complexes as nerves (in a functorial way) allows us to gain an insight into reconstruction results:

- we provide a short proof of Hausmann’s reconstruction for Riemannian manifolds [22], and extend it to persistent setting and closed Rips complexes;
- we completely classify reconstruction scales for metric graphs;
- we present a general way of extracting homology of a space using nested Rips complexes at a single scale arising from a single good cover of a space.

The structure of the paper is the following. Section 2 provides preliminaries on topologies on infinite complexes and partitions of unity. Section 3 contains a systematic study of complexes arising from covers, Dowker duality, and contains Nerve theorem for Rips complexes. In Sect. 4, we use results of the previous section to develop reconstruction results using Rips complexes, in particular for Riemannian manifolds and metric graphs. Section 5 contains functorial Dowker-Nerve diagram and applications to reconstruction. In Sect. 6, we describe how to extract homology of a space from nested Rips complexes of finite subsets and provide a corresponding example. Section 7
(Appendix) demonstrates how to use the Nerve theorem to reconstruct the homotopy type of Rips complexes and the restrictions connected to it.

2. Preliminaries on Complexes and Partitions of Unity

In this section, we present background on infinite simplicial complexes and partitions of unity. Our interest in the later stems from the fact that appropriate partitions of unity represent convenient maps to simplicial complexes. We provide several technical results that will be used in the future sections. The two interpretations of a simplicial complex (abstract and geometric simplicial complex) will be used interchangeably when no confusion may occur.

2.1. Infinite Simplicial Complexes

Given \( n \geq 0 \), the standard (geometric) \( n \)-simplex is the convex hull of the collection of \( n + 1 \) points of the form \((0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}\). Suppose \( K \) is a (geometric) simplicial complex. We usually equip \( K \) with a topology, for which the restriction onto each simplex coincides with the topology on the standard simplex. If \( K \) is locally finite, meaning that each of its vertices is contained in only finitely many simplices, then there is a unique such topology on \( K \). However, if \( K \) is not locally finite, there are more topologies to choose from. We recall two of them that appear standardly in textbooks:

1. **The weak topology.** It is obtained by gluing simplices together via quotient maps along their boundaries, and hence, the resulting topology is the weak topology with respect to the identification of the boundaries. This is the standard topology used for infinite simplicial or CW (also called cell) complexes. Unless stated otherwise, each simplicial complex will be equipped with this topology. A convenient aspect of this topology is related to constructions of maps defined on it [27, Theorem 2, p. 290]: if \( K \) is a simplicial complex and \( f : K \to X \) is a map, then \( f \) is continuous iff the restriction \( f|_\sigma \) is continuous for all simplices \( \sigma \) of \( K \).

2. **The metric (or \( \ell_1 \) or strong) topology,** see for example [13, Remark 7.14, p.116], [15, Section 5], [27] or [32]. To define it, we first have to introduce the barycentric coordinates. Let \( K^{(0)} = \{v_i\}_{i \in J} \) be the set of vertices of \( K \). Note that the points in the standard simplex are presented as convex combinations of the vertices of that simplex. Similarly, each simplex \( \sigma = [v_0, v_1, \ldots, v_n] \) in a simplicial complex \( K \) can be parameterized by the barycentric coordinates \((t_0, t_1, \ldots, t_n) \mapsto \sum_i t_i v_i \) with \( \sum_i t_i = 1 \) and \( t_i \in [0, 1] \), \( \forall i \). For example, vertex \( v_0 \) has only one non-zero barycentric coordinate: \( t_0 = 1 \). The midpoint on the edge between \( v_0 \) and \( v_1 \) has two non-zero barycentric coordinates: \( t_0 = t_1 = 1/2 \). For each \( i \in J \), let \( \lambda_i : K \to [0, 1] \) denote the \( i \)th barycentric coordinate, i.e., \( \lambda_i(\sum_j t_j v_j) = t_i \). The strong topology on \( K \) is the coarsest topology, under which all barycentric coordinates are continuous. More conveniently, we may define a metric on \( K \) using the barycentric coordinates:

\[
d_{\ell_1} \left( \sum_i t_i v_i, \sum_i t_i' v_i \right) = \sum_i |t_i - t_i'|.
\]
Metric $d_{\ell_1}$ is actually obtained by embedding $K$ into the metric space $\ell_1(K_0)$, so that the restriction to each simplex is a linear map. The resulting metric space will be denoted by $K_m$. A convenient aspect of this topology is related to constructions of maps to it [27, Theorem 8, p. 301]: a map $f : X \to K_m$ is continuous iff $\lambda_i \circ f$ is continuous for all $i \in J$.

Conveniently enough, the identity map $K \to K_m$ is a homotopy equivalence ([13, Corollary 2.9, p. 354] or [32, Theorem 4.9.6]). Maps $f, g : X \to K$ are contiguous, if, for each $x \in X$, there exists a simplex in $K$ containing both $f(x)$ and $g(x)$. Contiguous maps are homotopic ([13, Remark 2.22, p. 359] or [32, Theorem 4.9.7]), although the obvious straight-line homotopy is only continuous in $K_m$.

2.2. Partitions of Unity

A partition of unity on a topological space $X$ is a collection of continuous functions $\{f_i : X \to [0, 1]\}_{i \in I}$, such that for each $x \in X$ we have $\sum_{i \in I} f_i(x) = 1$. Such a partition is locally finite, if for each $x \in X$, there exist a neighborhood $U_x$ of $x$ and a finite $I_x \subset I$, such that $f_i(y) = 0, \forall y \in U_x, \forall i \in I \setminus I_x$. A partition of unity is point-wise finite, if for each $x \in X$, only finitely many $f_i(x)$ are non-zero.

Interpretations of partitions of unity as barycentric coordinates and vice versa provide a correspondence between appropriate partitions of unity and maps to simplicial complexes. Point-wise finite partitions of unity are in bijective correspondence with continuous maps to metric simplicial complexes. A locally finite partition of unity on a space represents a continuous map to a simplicial complex in the weak topology. See [15, Theorem 6.5] for more details of such correspondences.

We will be interested in partitions of unity arising from a cover. For a continuous function $f : X \to [0, 1]$, define support $\text{supp}(f)$ as the closure of the subset $f^{-1}(0, 1]$ in $X$. Given a cover $U = \{U_j\}_{j \in A}$ of $X$, a partition of unity $\{f_i : X \to [0, 1]\}_{i \in I}$ on $X$ is subordinate to $U$, if for each $i \in I$, there exists $j_i \in A$ so that $\text{supp}(f_i) \subset U_{j_i}$. We will be particularly interested in locally finite partitions of unity on $X$ subordinated to a cover $U$ on $X$, as these encode maps to the nerve of $U$, i.e., maps $X \to \mathcal{N}(U)$. (For a definition of the nerve, see Definition 3.4 provided in Sect. 3 within a broader context of constructions of complexes.)

- A cover $U$ of $X$ is numerable if it admits a locally finite partition of unity on $X$ subordinate to $U$.
- Space $X$ is paracompact, if each open cover of $X$ is numerable. Consequently, a closed cover of a paracompact space $X$ is numerable if the interiors of the elements of $U$ cover $X$. The class of paracompact spaces includes all metric spaces.

Suppose $U$ is a numerable cover of $X$. We can define a map $X \to \mathcal{N}(U)$ by choosing a partition of unity subordinate to $U$ and declaring it represents barycentric coordinates. It is easy to verify (see, for example, [32, Corollary 4.9.2, p. 198]) that a different choice of a partition of unity induces a different
but contiguous (hence homotopic) map. Consequently, we will denote by $i_U: X \to \mathcal{N}(U)$ any such map and keep in mind that its homotopy type does not depend on the choice of the generating partition of unity.

3. Dowker Duality and Complexes Arising from Covers

In this section, we provide a systematic introduction of complexes arising from covers, and conclude with a statement describing the homotopy type of a Rips complex as a nerve of a specific cover. Our main tool will be the notion of Dowker duality (Theorem 3.2) introduced in [14].

Suppose $R \subset Y \times Z$ is a subset of a product. It can be thought of as a relation [14]. For our setting, though it will be convenient to think of the corresponding (usually infinite) binary matrix $M_R = (m_{y,z})_{y \in Y, z \in Z}$ defined by the following rule:

- $m_{y,z} = 1$ if $(y, z) \in R$;
- $m_{y,z} = 0$ else.

For any $z_0 \in Z$, let $C_{z_0} = (m_{y,z_0})_{y \in Y}$ denote the column corresponding to $z_0$. Similarly, for any $y_0 \in Y$, let $R_{y_0} = (m_{y_0,z})_{z \in Z}$ denote the row corresponding to $y_0$.

Following [14], we introduce two complexes associated with a binary matrix $M_R$.

**Definition 3.1.** Suppose $R \subset Y \times Z$, and $M_R$ is the corresponding binary matrix.

The **column complex** of $M_R$ is a simplicial complex $C(M_R)$ defined by the following conditions:

1. The set of vertices of $C(M_R)$ consists of all elements of $z \in Z$, for which the column $C_z$ has a non-zero entry;
2. A finite subset $\sigma \subset Z$ is a simplex of $C(M_R)$ iff there exists $y \in Y$, for which $m_{y,z} = 1, \forall z \in \sigma$.

Alternatively, we could define $C(M_R)$ by the following rule: a finite subset $\sigma \subset Z$ is a simplex in $C(M_R)$ iff there exists $y \in Y$, so that $\sigma$ appears is a finite subset of $R \cap (\{y\} \times Z)$.

In an analogous manner, we can define the **row complex** $\mathcal{R}(M_R)$, although a quicker definition would be using a transposed matrix: $\mathcal{R}(M_R) = C(M_R^T)$.

**Theorem 3.2** (Dowker duality [14]). Suppose $R \subset Y \times Z$ and $M_R$ is the corresponding binary matrix. Then, $C(M_R)$ and $\mathcal{R}(M_R)$ are homotopy equivalent.

Dowker duality can be proved easily using the version of the Nerve Lemma stated in this paper as Theorem 4.3, by considering the following steps:

- Columns of $M_R$ represent a good cover of $\mathcal{R}(M_R)$ in the sense that for each $z \in Z$, the set $U_z = \{y \in Y : (y, z) \in R\}$ spans a full contractible complex $K_z$ (i.e., each finite subset of $U_z$ is a simplex in $K_z$) in $\mathcal{R}(M_R)$.
- While the collection of complexes $\{K_z\}_{z \in Z}$ is a closed cover of $\mathcal{R}(M_R)$,
we can thicken each of them slightly to an open neighborhood by preserving the intersection properties (i.e., the nerve) of \( \{ U_z \}_{z \in Z} \).

- By definition, \( C(M_R) \) is the nerve of \( \{ U_z \}_{z \in Z} \).

Our aim is to apply Theorem 3.2 to the case of complexes arising from covers. Following the terminology of [14], we introduce two complexes associated with a collection of subsets of \( X \). Before we do that, we would like to bring attention to a technical detail.

Remark 3.3. We will often be talking about collection of subsets, or covers, the later being a collection of subsets, whose union is the whole space. Such collections may contain multiple copies of the same set, even of the empty set. For example, a collection \( \{ \{ 1, 2 \}, \emptyset, \{ 2 \}, \{ 2 \} \} \) is a collection of subsets of \( \{ 1, 2 \} \). We will use the same notation for collections and sets, and the context should make sure that there is no confusion. For example, if \( U = \{ U_\alpha \}_{\alpha \in A} \) is a cover of \( X \), then \( U_\alpha \) are subsets of \( X \), but there may exist \( \alpha_1, \alpha_2, \alpha_3 \in A \) with \( U_\alpha_1 = U_\alpha_2 \) or \( U_\alpha_3 = \emptyset \).

Definition 3.4. Suppose \( U \) is a collection of subsets of \( X \).

The **nerve** of \( U \) is the simplicial complex \( \mathcal{N}(U) \) defined by the following conditions:

1. the set of vertices of \( \mathcal{N}(U) \) consists of all non-trivial elements of \( U \);
2. a finite subset \( \sigma \subset U \) is a simplex of \( \mathcal{N}(U) \) iff \( \cap_{U \in \sigma} U \neq \emptyset \).

Strictly speaking, (2) implies (1).

The **Vietoris complex** of \( U \) is the simplicial complex \( \mathcal{V}(U) \) defined by the following conditions:

1. the set of vertices of \( \mathcal{V}(U) \) consists of all points of \( X \), which are contained in some element of \( U \);
2. a finite subset \( \sigma \subset X \) is a simplex of \( \mathcal{V}(U) \) iff there exists \( U \in \mathcal{U} \), so that \( \sigma \subset U \).

Remark 3.5. It is easy to see that any simplicial complex \( K \) can be expressed as a nerve (using open stars of vertices as a cover of \( K \)) and as a Vietoris complex (using the cover consisting of all \( n \)-tuples of vertices that span a simplex in \( K \) as a cover of the collection of all vertices of \( K \)).

To connect the nerve and the Vietoris construction to Dowker duality, we assign a binary matrix to a cover. This connection is a classical idea appearing in [14,37].

Definition 3.6. Suppose \( U = \{ U_i \}_{i \in A} \) is a cover of \( X \). The associated binary matrix \( M_U = (a_{x,U})_{x \in X, U \in U} \) is defined by:

- \( a_{x,U} = 1 \) if \( x \in U \);
- \( a_{x,U} = 0 \) else.

Matrix \( M_U \) can also be regarded as a subset of \( X \times A \) encoding a binary relation of containment between the points of \( X \) and the elements of the cover \( U \).

Note that each column \( C_U = \{ a_{x,U} \}_{x \in X} \) of \( M_U \) encodes an element \( U \in \mathcal{U} \): the non-zero entries correspond precisely to the elements of \( U \). Similarly,
each row $R_x = (a_{x, U})_{U \in \mathcal{U}}$ of $M_U$ locates an element $x \in X$: the non-zero entries correspond precisely to the sets from $\mathcal{U}$ containing $x$.

Proposition 3.7 follows from Theorem 3.2.

Proposition 3.7. Suppose $\mathcal{U} = \{U_i\}_{i \in A}$ is a cover of $X$. Then, $\mathcal{N}(\mathcal{U}) = \mathcal{C}(M_\mathcal{U})$ and $\mathcal{V}(\mathcal{U}) = \mathcal{R}(M_\mathcal{U})$. In particular, $\mathcal{N}(\mathcal{U}) \simeq \mathcal{V}(\mathcal{U})$.

In the setting of metric spaces, two specific constructions of complexes feature prominently: the Čech complex and the Rips complex [16,18,30]

Definition 3.8. Suppose $X$ is a metric space, $A \subset X$ and $r > 0$. For $x \in X$, let $B(x, r)$ and $\overline{B}(x, r)$ denote the open and closed balls respectively.

The (open) Čech complex $\text{Čech}_X(A, r)$ is defined by the following rules:

- its vertices are elements of $A$;
- a finite subset $\sigma \subset A$ is a simplex in $\text{Čech}_X(A, r)$ iff $\cap_{a \in \sigma} B(a, r) \cap X \neq \emptyset$.

The closed Čech complex $\overline{\text{Čech}}_X(A, r)$ is defined analogously using closed balls.

The (open) Rips complex $\text{Rips}_X(A, r)$ is defined by the following rules:

- its vertices are elements of $A$;
- a finite subset $\sigma \subset A$ is a simplex in $\text{Rips}_X(A, r)$ iff $\text{Diam}(\sigma) < r$.

The closed Rips complex $\overline{\text{Rips}}_X(A, r)$ is defined analogously using the non-strict inequality.

The Čech complex can also be defined as the nerve of the collection of open balls, i.e., for $A \subset X$ metric, $r > 0$, and $\mathcal{U} = \{B(a, r)\}_{a \in A}$, $\text{Čech}_X(A, r) = \mathcal{N}(\mathcal{U})$. Keeping in mind the identification of vertices $a \in A$ (in the Čech complex) by $B(a, r)$ (in the column and the nerve complex) for the rest of this paragraph, we observe that the two definitions of the Čech complex agree, and that $\text{Čech}_X(A, r) = \mathcal{C}(M_\mathcal{U})$. In case when $A = X$, we can use this fact to deduce two interesting statements from Dowker duality:

1. $\text{Čech}_X(X, r) = \mathcal{N}(\mathcal{U}) = \mathcal{C}(M_\mathcal{U}) \simeq \mathcal{R}(M_\mathcal{U}) = \mathcal{V}(\mathcal{U})$. Actually, basic geometry implies $\mathcal{N}(\mathcal{U}) = \mathcal{V}(\mathcal{U})$ (via the above-mentioned identification of vertices), since a collection of balls of radius $r$ intersects iff its centers themselves are contained in a ball of radius $r$ (based at any point of the aforementioned intersection). The coincidence $\mathcal{N}(\mathcal{U}) = \mathcal{V}(\mathcal{U})$ via a natural identification of vertices of both simplices seems to be specific to the cover by open balls.

2. Suppose $\mathcal{U}'$ is a cover of $X$ with the following property for each finite subset $\sigma \subset X$: $\sigma$ is contained in some element of $\mathcal{U}'$ iff $\cap_{a \in \sigma} B(a, r) \cap X \neq \emptyset$. Then, $\mathcal{R}(M_\mathcal{U}) = \mathcal{R}(M_{\mathcal{U}'})$ by definition, and thus, by Dowker duality, $\text{Čech}_X(X, r) = \mathcal{N}(\mathcal{U}) \simeq \mathcal{N}(\mathcal{U}')$. This trick allows us to change a cover and keep the homotopy type of the nerve. It will come handy in the context of Rips complexes.

Analogous conclusions hold for closed Čech complexes, as well.

The Rips complex has, to the best of the author’s knowledge, never been expressed as a nerve of a covering. However, using the setup of this section, we can do just that.
Theorem 3.9 (Rips complex as a nerve). Suppose $X$ is a metric space and $r > 0$. Let $\mathcal{U}$ be a cover of $X$ with the following property for each finite subset $\sigma \subset X$: $\sigma$ is contained in some element of $\mathcal{U}$ iff $\text{Diam}(\sigma) < r$. Then, $\text{Rips}(X, r) \simeq \mathcal{N}(\mathcal{U})$.

Let $\mathcal{U}'$ be a cover of $X$ with the following property for each finite subset $\sigma \subset X$: $\sigma$ is contained in some element of $\mathcal{U}$ iff $\text{Diam}(\sigma) \leq r$. Then, $\text{Rips}(X, r) \simeq \mathcal{N}(\mathcal{U}')$.

Proof. Equality $\text{Rips}(X, r) = \mathcal{R}(M_\mathcal{U})$ follows by definition, and by Dowker duality, we have $\mathcal{R}(M_\mathcal{U}) \simeq \mathcal{C}(M_\mathcal{U}) = \mathcal{N}(\mathcal{U})$. The same argument works for closed Rips complexes. □

Remark 3.10. Here, we provide some examples of coverings $\mathcal{U}$ satisfying the conditions of Theorem 3.9.

1. Suppose $\mathcal{U}$ is a cover by all finite subsets of $X$ of diameter less than $r$. Then, $\text{Rips}(X, r) \simeq \mathcal{N}(\mathcal{U})$.
2. Suppose $\mathcal{U}$ is a cover by all subsets of $X$ of diameter less than $r$. Then, $\text{Rips}(X, r) \simeq \mathcal{N}(\mathcal{U})$.
3. Suppose $\mathcal{U}$ is a cover by all finite subsets of $X$ of diameter at most $r$. Then, $\text{Rips}(X, r) \simeq \mathcal{N}(\mathcal{U})$.
4. Suppose $\mathcal{U}$ is a cover by all subsets of $X$ of diameter at most $r$. Then, $\text{Rips}(X, r) \simeq \mathcal{N}(\mathcal{U})$.
5. Suppose $X$ is a flat (locally Euclidean, see [5, Definition 3.3.4]) compact manifold of dimension $n$, i.e., assume there exists $q > 0$, so that for each $x \in X$ the ball $B(x, q)$ is isometric to the Euclidean $n$-ball of radius $q$. Assume now that $r < q/4$. For each finite subset $\sigma \subset X$ of diameter at most $r$, we can well define the convex hull of $\sigma$ via the mentioned local isomorphism with the Euclidean ball (see Definition 4.7 for details). Now, suppose $\mathcal{U}$ consists of the convex hulls of all finite subsets $\sigma \subset X$ of diameter less than $r$ (it essentially consists of edges, triangles, polygons, tetrahedra, etc...). Then, $\text{Rips}(X, r) \simeq \mathcal{N}(\mathcal{U})$.

More examples will be provided in the next section in the context of the reconstruction results.

4. Reconstruction Results

In this section, we combine the setting of Sect. 3 with the Nerve lemma and provide a number of reconstruction results. To be more precise, we are interested in reconstructing the homotopy type of a space via Rips or Čech complexes. The idea of reconstructing spaces via Čech complexes is rather old and well understood. The reconstruction is usually based on the Nerve lemma, the first versions of which were apparently proved by Borsuk [4] and Leray [26]. More results aimed at the computational setting were proved in this millennium [8,29]. An absence of formulation of the Rips complex in terms of a nerve made similar reconstruction results for Rips complexes much more rare and not as general as the results for the Čech complexes [2,22,24]. With Theorem 3.9, we aim to bridge this gap of understanding.
Definition 4.1. A cover $\mathcal{U}$ of a space $X$ is good if the intersection of each finite subcollection of $\mathcal{U}$ is either contractible or empty.

The version of the Nerve Lemma suitable for this paper will be derived from the following result.

Theorem 4.2 (Theorem 1 of [12] with $B$ being a singleton). Let $f : X \to Y$ be a map. Suppose $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ are numerable covers of $X$ and $Y$, respectively, so that $f(U_\alpha) \subset V_\alpha$ for each $\alpha \in A$. Assume that for each finite $\sigma \subset A$, the restriction $f|_{\bigcap_{\alpha \in \sigma} U_\alpha} : \bigcap_{\alpha \in \sigma} U_\alpha \to \bigcap_{\alpha \in \sigma} V_\alpha$ is a homotopy equivalence. Then, $f$ is a homotopy equivalence.

For the following theorem, recall the definition of map $i_\mathcal{U} : X \to \mathcal{N}(\mathcal{U})$ and properties of numerable covers from the last part of Sect. 2.

Theorem 4.3 (Nerve Lemma). If $\mathcal{U}$ is a good numerable cover of a space $X$, then $i_\mathcal{U} : X \to \mathcal{N}(\mathcal{U})$ is a homotopy equivalence.

Proof. Apply Theorem 4.2 with $f$ being $i_\mathcal{U}$ and $\mathcal{V}$ being the cover of $\mathcal{N}(\mathcal{U})$ by the collection of open stars of vertices. \hfill \Box

A standard application of the Nerve lemma is the following reconstruction result via the Čech complex: if, for some $r > 0$, a cover by open $r$-balls of a paracompact space $X$ is a good cover, then the corresponding Čech complex is homotopy equivalent to $X$ [16, 29]. Using Theorem 3.9, we can state an equivalent result for the Rips complexes.

Theorem 4.4 (Reconstruction Theorem for Rips complexes). Suppose $r > 0$ and $\mathcal{U}$ is a good numerable cover of a space $X$ with the following property for each finite subset $\sigma \subset X$: $\sigma$ is contained in some element of $\mathcal{U}$ iff $\text{Diam}(\sigma) < r$ (or $\text{Diam}(\sigma) \leq r$, respectively). Then, $\text{Rips}(X, r) \simeq X$ (or $\overline{\text{Rips}}(X, r) \simeq X$, respectively).

Proof. Follows from Theorems 3.9 and 4.3. \hfill \Box

Example 4.5. If $X$ is a compact locally flat manifold, then cover $\mathcal{U}$ of Remark 3.10 (5) is a good cover and hence $X \simeq \text{Rips}(X, r)$.

4.1. Reconstruction of Riemannian Manifolds and More General Geodesic Spaces

A reconstruction theorem for Riemannian manifolds via Rips complexes was first proved in [22]. Using our framework, we can provide a much simpler proof. Recall that a metric space is geodesic if every pair of points $x, y$ is connected by a path of length $d(x, y)$ called a geodesic. Note that our scope of a geodesic is a bit more restrictive than a more general notion of a geodesic in differential geometry (and also from the notion of a geodesic in [5, Definition 2.5.27]), which is determined by the local curvature: in our setting, a geodesic is the trace of an isometric embedding of a closed line segment in a general geodesic space, not necessarily in a manifold. A geodesic, as used throughout this paper, is a shortest path [5, Definition 2.5.15] realizing the distance between its endpoints.
Definition 4.6. [22] Suppose \( X \) is a geodesic space. Define \( r = r(X) \geq 0 \) as the least upper bound of the set of real numbers \( r \) satisfying the following two conditions:

1. For all \( x, y \in X \) with \( d(x, y) < 2r \), there exists a unique geodesic from \( x \) to \( y \) of length \( d(x, y) \).
2. Let \( x, y, z, u \in X \) with \( d(x, y) < r, d(u, x) < r, d(u, y) < r \), and let \( z \) be a point on the geodesic joining \( x \) and \( y \). Then, \( d(u, z) \leq \max\{d(u, x), d(u, y)\} \).
3. If \( \gamma \) and \( \gamma' \) are arc-length parameterized geodesics, such that \( \gamma(0) = \gamma'(0) \), and if \( 0 \leq s, s' < r \) and \( 0 \leq t < 1 \), then \( d(\gamma(ts), \gamma'(ts')) \leq d(\gamma(s), \gamma'(s')) \).

As was stated in [22, Remark on p. 179], \( r(X) > 0 \) if \( X \) is a Riemannian manifold with a uniform upper bound on sectional curvature and a positive lower bound on the injectivity radius. In particular, each compact Riemannian manifold has \( r(X) > 0 \).

Definition 4.7. Suppose \( X \) is a geodesic space and \( A \subset X \). The subset \( A \) is geodesically convex if each geodesic connecting two points in \( A \) lies entirely in \( A \). The geodesic hull of \( A' \subset X \) is the smallest geodesically convex set containing \( A' \).

Remark 4.8. It is easy to observe that intersections of geodesically convex sets are geodesically convex.

Proposition 4.9. Suppose \( X \) is a geodesic space with \( r(X) > 0 \), \( 0 < q < r(X) \), and \( A \subset X \) is of diameter \( \text{Diam}(A) \leq q - 2\varepsilon \). Then, there exists an open geodesically convex subset \( A_\infty \) containing \( A \) of diameter at most \( \text{Diam}(A_\infty) \leq q - \varepsilon \).

Proof. We will construct \( A_\infty \) inductively. Define \( A_0 = A \) and let \( A'_0 \) be the union of traces of all unique geodesics from Definition 4.6(1), whose endpoints lie in \( A_0 \). It follows from Definition 4.6(2) that \( \text{Diam}(A_0) = \text{Diam}(A'_0) \).

We now present an inductive step. For each \( i \in \mathbb{N} \):

- Define \( A_i \) to be the open \( 2^{-i-1}\varepsilon \) neighborhood of \( A'_{i-1} \). Note that \( \text{Diam}(A_i) \leq \text{Diam}(A'_{i-1}) + 2^{-i}\varepsilon \leq \text{Diam}(A_0) + (1 - 2^{-i})\varepsilon \leq q - \varepsilon - 2^{-i}\varepsilon \).
- Define \( A'_i \) as the union of traces of all unique geodesics from Definition 4.6(1), whose endpoints lie in \( A_i \). Since \( \text{Diam}(A_i) \) is less than \( r(X) \), we can use Definition 4.6(2) to conclude \( \text{Diam}(A_i) = \text{Diam}(A'_i) \).

Then, \( A_\infty = \bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i \in \mathbb{N}} A'_i \) satisfies the conclusions of the proposition. \( \square \)

Theorem 4.10 (Hausmann’s Theorem [22]). Suppose \( X \) is a geodesic space with \( r(X) > 0 \) (for example, a compact Riemannian manifold). Then, \( X \cong \text{Rips}(X, q) \), for each positive \( q \leq r(X) \).

An example of a geodesic space with \( r(X) = 0 \) is Hawaiian earring space with a geodesic metric.
Proof. For each finite subset $\sigma \subset X$ of diameter less than $q$ choose a geodesically convex open subset $\sigma_\infty$ of diameter less than $q$ by Proposition 4.9. Define an open cover consisting of such sets:

$$U = \{ \sigma_\infty \mid \sigma \subset X, |\sigma| < \infty, \text{Diam}(\sigma) < q \}.$$ 

By Theorem 3.9, $\mathcal{N}(U) \simeq \text{Rips}(X, q)$. The cover $U$ is a good cover because of the following two reasons:

- $U$ consists of geodesically convex sets (see Remark 4.8);
- geodesically convex sets of diameter less than $r(X)$ are contractible by Definition 4.7.

Hence, $\mathcal{N}(U) \simeq X$ by Theorem 4.3. □

An alternative proof of this fact was obtained by Henry Adams and Florian Frick (Personal communication).

In a similar way, we can prove a version Hausmann’s Theorem for closed Rips complexes.

**Theorem 4.11** (A version of the Hausmann’s Theorem for closed Rips complexes). Suppose $X$ is a geodesic space with $r(X) > 0$ (for example, a compact Riemannian manifold). Then, $X \simeq \text{Rips}(X, q)$, for each positive $q < r(X)$.

Proof. The proof goes along the same lines as the proof of Theorem 4.10, with $U$ of that proof being modified by adding geodesic hulls of the collection \{ $\sigma \subset X : |\sigma| < \infty, \text{Diam}(\sigma) = q$ \}. After the modification, the obtained cover is still good (as we only added geodesically convex sets of diameter less than $r(x)$), numerable (as it contains a numerable cover), and the diameters of the added sets are precisely $q$ by Definition 4.6 (2). □

### 4.2. Reconstruction of Metric Graphs

A (finite) metric graph $X$ is a geodesic metric space homeomorphic to a finite connected 1-dimensional simplicial complex. In particular, each edge is assigned a length and the geodesic distance is then naturally generated by shortest paths. To avoid unnecessary technicalities and also to keep with the established notation in the setting of applied topology [17], we will not consider metric graphs with infinitely many edges or vertices. For details on the definition of metric graphs in general, see [28].

The length of the shortest non-contractible loop in a metric graph will be denoted by $\ell(X)$. If $X$ is a tree (i.e., an acyclic metric graph), then $\ell(X) = \infty$.

In this subsection, we use the Reconstruction Theorem for Rips complexes to completely classify scales $r$, for which the Rips complex of a metric graph is homotopy equivalent to $X$.

**Remark 4.12.** While the bounds on reconstruction results in the previous subsection were expressed in terms of $r(X)$ in line with the established terminology of [22], the bounds in this subsection will be expressed in terms of $\ell(X)$ in line with the usual treatment of metric graphs (see, for example, [17]). The two notions, however, are connected. The results and proofs of the previous subsection also hold for $r^*(X)$, which is obtained by modifying condition (1) of Definition 4.6 to:
(1*) For all \( x, y \in X \) with \( d(x, y) < r \), there exists a unique geodesic from \( x \) to \( y \) of length \( d(x, y) \).

With this definition, it is not hard to see that for a metric graph \( X \) Proposition 4.13 implies \( \ell(X)/3 = r^*(X) \), which means that Theorems 4.10 and 4.11 provide the same upper reconstruction bound as Theorems 4.14 and 4.15.

**Proposition 4.13.** Suppose \( X \) is a metric graph.

1. If \( d(x, y) < \ell(X)/2 \) for some \( x, y \in X \), then there exist precisely one simple (non-selfintersecting) path \( \gamma \) from \( x \) to \( y \) of length less than \( \ell(X)/2 \). In particular, a shortest geodesic between \( x \) and \( y \) is unique and equals \( \gamma \).
2. If \( A \subset X \) is geodesically convex and \( \text{Diam}(A) < \ell(X)/2 \), then \( A \) is a tree.
3. Let \( A \subset X \) be compact, \( \text{Diam}(A) < \ell(X)/3 \), \( x, y \in A \), and let \( \gamma \) be the (trace of the) unique shortest geodesic between \( x \) and \( y \). Then, \( \text{Diam}(A \cup \gamma) = \text{Diam}(A) \).
4. Let \( A \subset X \) be finite, \( \text{Diam}(A) < \ell(X)/3 \), and for \( x, y \in A \), let \( \gamma_{x,y} \) be the (trace of the) unique shortest geodesic between \( x \) and \( y \). Define \( A' = \bigcup_{x, y \in A} \gamma_{x,y} \). Then, \( A' \) is compact, geodesically convex, and \( \text{Diam}(A') = \text{Diam}(A) \). In particular, by (2) \( A' \) is a tree.

**Proof.** (1) If this was not so, the two different paths between them would form a non-contractible loop of length less than \( \ell(X) \), a contradiction.

(2) It suffices to show that \( A \) contains no non-contractible loop. Assume \( \gamma \) is the shortest non-contractible loop in \( A \) and suppose \( x, y \in \gamma \subset A \) are diametrically opposite points on \( \gamma \), i.e., both segments \( \gamma_1 \) and \( \gamma_2 \) of \( \gamma \) between \( x \) and \( y \) are of the same length at least \( \ell(X)/2 \). The shortest geodesic \( \nu \) between \( x \) and \( y \) is of length less than \( \ell(X)/2 \) and cannot be homotopic to both \( \gamma_1 \) and \( \gamma_2 \) (rel endpoints). Without the loss of generality, we assume that \( \nu \) is not homotopic to \( \gamma_1 \). Concatenating \( \nu \) and \( \gamma_1 \), we obtain a non-contractible loop in \( A \) of length at most \( \text{Diam}(A) + \text{length}(\gamma)/2 < \text{length}(\gamma) \), a contradiction to the minimality of \( \gamma \).

(3) Choose \( z \in A \). Let \( \gamma_x \) and \( \gamma_y \) denote the geodesics from \( z \) to \( x \) and \( y \), respectively. Both \( \gamma_x \) and \( \gamma_y \) follow the same trace to some point on \( \gamma \), and then, each of them runs along \( \gamma \) towards its respective point: if this was not the case, \( \gamma, \gamma_x \) and \( \gamma_y \) would form a non-contractible loop of length less than \( \ell(X) \), a contradiction. Geodesics \( \gamma_x \) and \( \gamma_y \) jointly visit all the points of \( \gamma \), and hence, for each \( w \in \gamma \), we have \( d(z, w) \leq \text{Diam}(A) \).

(4) We proceed by induction. Let \( A = \{x_1, x_2, \ldots, x_k\} \) and \( A_i = \{x_1, x_2, \ldots, x_i\} \). Now, \( A'_2 \) is geodesically convex of diameter \( d(x_1, x_2) = \text{Diam}(A_2) \) as it is just a geodesic.

Assume now that \( A'_{j} \) is geodesically convex of diameter \( \text{Diam}(A_j) \). By inductive use of (3), we conclude \( \text{Diam}(A'_{j+1}) = \text{Diam}(A_{j+1}) \). Furthermore, the argument of (3) shows that the traces of geodesics \( \{\gamma_{j+1,i}\}_{i \leq j} \) cover \( A'_j \); hence, by (1), \( A'_j \) is geodesically convex. \( \square \)

**Theorem 4.14.** Suppose \( X \) is a metric graph and \( r > 0 \). Then, \( \text{Rips}(X, r) \simeq X \) iff \( r \leq \ell(X)/3 \).
Proof. Using the notation of Proposition 4.13(4) define a closed cover of $X$ in the following way:

$$U = \{ U' \subset X \mid U \subset X, U \text{ finite, Diam}(U) < r \}.$$ 

By Theorem 4.4, it suffices to show that $U$ is a good numerable cover of $X$. Cover $U$ is numerable as its interiors cover $X$. Furthermore, each element of $U$ is geodesically convex of diameter less than $\ell(X)/3$, hence so are all finite intersections of elements of $U$. Proposition 4.13(2) then implies $U$ is a good cover.

For $r > \ell(X)/3$, the first homology groups of $\text{Rips}(X, t)$ and $X$ are known to differ, see the main results of [34] or [17] for details. □

Again, the same argument can be constructed for closed Rips complexes to prove the following result.

**Theorem 4.15.** Suppose $X$ is a metric graph and $r > 0$. Then, $\overline{\text{Rips}}(X, r) \simeq X$ iff $r < \ell(X)/3$.

**Remark 4.16.** Suppose $X$ is a metric graph and $r > 0$. Analogous arguments to those in the proof of Theorem 4.14 could be made to prove that $\text{Cech}(X, r) \simeq X$ iff $r \leq \ell(X)/4$. Similarly, $\text{Cech}(X, r) \simeq X$ iff $r < \ell(X)/4$.

### 5. Functorial Dowker-Nerve Diagram

The results of the previous sections relate to complexes obtained at a fixed scale. In the context of filtrations, however, we are interested in preserving or relating these results through various scales. With this aim, we present in this section a functorial version of the Nerve Lemma coupled with functorial Dowker duality spanning various scales. We call the result the Functorial Dowker-Nerve diagram, as it combines functorial versions of Dowker Theorem and of the Nerve lemma.

Functorial versions of these results have been considered before. A functorial version of the Nerve Lemma appears in [9], [30, Lemma 4.12] and later in [10] for pairs of finite good open covers of a paracompact space. Approximate homological versions were obtained in [19] and [6]. On the other hand, a functorial version of Dowker duality was proved in [10]. The functorial versions of the Nerve Lemma (Lemma 5.1) and of Dowker duality (Theorem 5.2) presented in this paper are more general than the previously known versions:

- The first one is proved for nested numerable covers connected by a continuous map, while previous known versions dealt with bijectively nested finite open covers connected by an inclusion.
- The second one is also proved for nested numerable covers connected by a continuous map, while previously known version was only stated for bijectively nested covers connected by an inclusion and phrased as a subset relation [10].

We first introduce notation of our functorial setting which is applicable for the rest of this section. Let $f: X \to Y$ be a map. Suppose $U = \{ U_\alpha \}_{\alpha \in A}$
and \( W = \{ W_\beta \}_{\beta \in B} \) are numerable covers of \( X \) and \( Y \), respectively, so that \( f(\mathcal{U}) \) is a refinement of \( W \), i.e., \( \forall \alpha \in A \), \( \exists \beta_\alpha \in B : f(U_\alpha) \subset W_{\beta_\alpha} \).

Define the following maps:

- \( \varphi_N : \mathcal{N}(\mathcal{U}) \rightarrow \mathcal{N}(W) \) is the simplicial map mapping \( U_\alpha \mapsto W_{\beta_\alpha} \). While \( \varphi_N \) depends on the choice of pairing \( \alpha \mapsto \beta_\alpha \) (each \( f(U_\alpha) \) can typically be contained in more than one element of \( W \)), its homotopy type does not. To see this, note that a different choice of pairing induces a contiguous (hence homotopic) map.
- \( \varphi_V : \mathcal{V}(\mathcal{U}) \rightarrow \mathcal{V}(W) \) is the simplicial map mapping \( x \mapsto f(x) \).
- maps \( i_\mathcal{U} : X \rightarrow \mathcal{N}(\mathcal{U}) \) and \( i_\mathcal{W} : Y \rightarrow \mathcal{N}(\mathcal{W}) \) arise via locally finite partitions of unity subordinate to \( \mathcal{U} \) and \( \mathcal{W} \) as explained in Sect. 2.

**Lemma 5.1** (Functorial Nerve Lemma). The following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
Y & \overset{i_\mathcal{W}}{\longrightarrow} & \mathcal{N}(W) \\
\downarrow f & & \downarrow \varphi_N \\
X & \overset{i_\mathcal{U}}{\longrightarrow} & \mathcal{N}(\mathcal{U})
\end{array}
\]

Furthermore, if \( \mathcal{U} \) and \( \mathcal{W} \) are good covers, then \( i_\mathcal{W} \) and \( i_\mathcal{U} \) are homotopy equivalences, and hence, \( f \) is a homotopy equivalence iff \( \varphi_N \) is one.

**Proof.** Maps \( i_\mathcal{W} \circ f \) and \( \varphi_N \circ i_\mathcal{U} \) are contiguous hence homotopic. For the second part, use Theorem 4.3. \( \square \)

We will also require a version of Dowker Theorem, which is functorial in our setting described above.

**Theorem 5.2** (The Functorial Dowker Theorem for covers). There exist homotopy equivalences \( \tilde{\gamma}_\mathcal{U} \) and \( \tilde{\gamma}_\mathcal{W} \) for which the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
\mathcal{N}(\mathcal{W}) & \overset{\tilde{\gamma}_\mathcal{W}}{\leftarrow} & \mathcal{V}(\mathcal{W}) \\
\uparrow \varphi_N & & \uparrow \varphi_V \\
\mathcal{N}(\mathcal{U}) & \overset{\tilde{\gamma}_\mathcal{U}}{\leftarrow} & \mathcal{V}(\mathcal{U})
\end{array}
\]

The statement is true even if \( \mathcal{U} \) and \( \mathcal{W} \) are not numerable.

**Proof.** For any \( C \subset X \), let \( \Delta_C \) denote the full simplicial complex on \( C \), i.e., each finite subset of \( C \) spans a simplex in \( \Delta_C \). Recall that full simplicial complexes are contractible and note that if \( C \subset U \) for some \( U \in \mathcal{U} \), then \( \Delta_C \subset \mathcal{V}(\mathcal{U}) \).

We see that \( \{ \Delta_{U_\alpha} \}_{\alpha \in A} \) is a good cover of \( \mathcal{V}(\mathcal{U}) \) as for each \( A' \subset A \), the intersection

\[
\bigcap_{\alpha \in A'} \Delta_{U_\alpha} = \Delta_{(\cap_{\alpha \in A'} U_\alpha)}
\]
is a full simplicial complex. However, it may not be numerable, as its interiors may not cover \( \mathcal{V}(\mathcal{U}) \). To remedy this shortcoming, we modify it by slightly thickening the elements of the cover. For a moment, let us assume that each simplex of \( \mathcal{V}(\mathcal{U}) \) is isometric to the standard simplex. We enlarge each \( \Delta_{U\alpha} \) to \( \tilde{\Delta}_{U\alpha} \), so that for each simplex \( \sigma \in \mathcal{V}(\mathcal{U}) \):

\[
\tilde{\Delta}_{U\alpha} \cap \sigma = N(\Delta_{U\alpha}, 0.1) \cap \sigma,
\]

i.e., we thicken the sets by 0.1 in each adjacent simplex. Note that for each \( A' \subset A \), the intersection \( \cap_{\alpha \in A'} \tilde{\Delta}_{U\alpha} \) deformation contracts to \( \cap_{\alpha \in A'} \Delta_{U\alpha} \).

In particular, \( \cap_{\alpha \in A'} \tilde{\Delta}_{U\alpha} = \emptyset \) iff \( \cap_{\alpha \in A'} \Delta_{U\alpha} = \emptyset \) and \( \tilde{\mathcal{U}} = \{ \tilde{\Delta}_{U\alpha} \}_{\alpha \in A} \) is a good numerable cover of \( \mathcal{V}(\mathcal{U}) \). Furthermore, \( \mathcal{N}(\mathcal{U}) \) is isomorphic to \( \mathcal{N}(\tilde{\mathcal{U}}) \) by identifying vertices \( U \in \mathcal{N}(\mathcal{U}) \) with \( \tilde{\Delta}_U \in \mathcal{N}(\tilde{\mathcal{U}}) \). Using this identification and Lemma 5.1, we define a homotopy equivalence \( \tilde{\gamma}_\mathcal{U} = i_\tilde{\mathcal{U}} : \mathcal{V}(\mathcal{U}) \to \mathcal{N}(\mathcal{U}) \).

As \( \varphi_{\mathcal{N}}(\tilde{\mathcal{U}}) \) is a refinement of \( \tilde{\mathcal{W}} \), the diagram commutes up to homotopy by Lemma 5.1.

Throughout the proof, we never required \( \mathcal{U} \) or \( \mathcal{W} \) to be numerable. \( \square \)

**Remark 5.3.** Note that Theorem 5.2 is stronger than the Functorial Dowker Theorem of [10]. The setting of [10] requires nested relations on a fixed product \( Y \times Z \) which, by the theory of Sect. 3, translates to covers \( \mathcal{U} = \{ U_\alpha \}_{\alpha \in A} \) and \( \mathcal{W} = \{ W_\beta \}_{\beta \in B} \) of \( X \) with \( U_\alpha \subset W_\beta \). In particular, in that setting, the covers have to be indexed by the same set, nesting has to precisely respect the indexing, and covers have to be of the same space (hence, function \( f \) is the identity).

We may now combine the results of Lemma 5.1 and Theorem 5.2 into a single diagram. For the sake of completeness, the statement of Theorem 5.4 also contains the assumptions which we made throughout this section.

**Theorem 5.4** (Functorial Dowker-Nerve diagram). Let \( f : X \to Y \) be a map. Suppose \( \mathcal{U} = \{ U_\alpha \}_{\alpha \in A} \) and \( \mathcal{W} = \{ W_\beta \}_{\beta \in B} \) are numerable covers of \( X \) and \( Y \), respectively, so that \( \forall \alpha \in A, \exists \beta_\alpha \in B : f(U_\alpha) \subset W_\beta \).

Define the following maps:

- \( \varphi_{\mathcal{N}} : \mathcal{N}(\mathcal{U}) \to \mathcal{N}(\mathcal{W}) \) is the simplicial map mapping \( U_\alpha \mapsto W_\beta_\alpha \);
- \( \varphi_{\mathcal{V}} : \mathcal{V}(\mathcal{U}) \to \mathcal{V}(\mathcal{W}) \) is the simplicial map mapping \( x \mapsto f(x) \);
- maps \( i_{\mathcal{U}} : X \to \mathcal{N}(\mathcal{U}) \) and \( i_{\mathcal{W}} : Y \to \mathcal{N}(\mathcal{W}) \) arise via locally finite partitions of unity subordinate to \( \mathcal{U} \) and \( \mathcal{W} \);
- homotopy equivalences \( \gamma_{\mathcal{U}} : \mathcal{N}(\mathcal{U}) \to \mathcal{V}(\mathcal{U}) \) and \( \gamma_{\mathcal{W}} : \mathcal{N}(\mathcal{W}) \to \mathcal{V}(\mathcal{W}) \) arise from Theorem 5.2 as homotopy inverses of \( \tilde{\gamma}_{\mathcal{U}} \) and \( \tilde{\gamma}_{\mathcal{W}} \).

Then, the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
Y & \xrightarrow{i_\mathcal{W}} & \mathcal{N}(\mathcal{W}) \xrightarrow{\gamma_{\mathcal{W}}} \mathcal{V}(\mathcal{W}) \\
| \varphi_{\mathcal{N}} | & & | \varphi_{\mathcal{V}} | \\
\downarrow f & & \downarrow \gamma_{\mathcal{U}} & \downarrow \gamma_{\mathcal{W}} \\
X & \xrightarrow{i_{\mathcal{U}}} & \mathcal{N}(\mathcal{U}) \xrightarrow{\gamma_{\mathcal{U}}} \mathcal{V}(\mathcal{U})
\end{array}
\]
Furthermore, if $U$ and $W$ are good covers, then $i_U$ and $i_W$ are homotopy equivalences. In this case, the following holds: if any of the maps \( \{f, \varphi_N, \varphi_V\} \) is a homotopy equivalence, then all three are.

**Remark 5.5.** A consequence of Theorem 5.4 is that in the case of good numerable covers $U$ and $W$, $f$ is a homotopy equivalence iff $\varphi_N$ or $\varphi_V$ is. In effect, this means that we can decide whether $f$ is a homotopy equivalence by considering (testing) only the induced maps on the nerve or Vietoris complex.

By Theorem 5.4, the existence of good covers at small scale implies all sorts of homotopy equivalences between the induced complexes. Here, we provide some of them in two specific cases.

**Theorem 5.6** (Local Rips-Čech coincidence for metric graphs). Suppose $X$ is a metric graph and $0 < r \leq \ell(X)/4$. Then, the inclusions

\[
\text{Cech}(X, r/2) \hookrightarrow \text{Rips}(X, r) \hookrightarrow \text{Cech}(X, r)
\]

are homotopy equivalences.

The same statement holds for closed complexes if $0 < r < \ell(X)/3$.

**Theorem 5.7** (Local Rips-Čech coincidence for geodesic spaces). Suppose $X$ is a geodesic space with $r(X) > 0$ and $0 < r \leq r(X)/2$. Then, the inclusions

\[
\text{Cech}(X, r/2) \hookrightarrow \text{Rips}(X, r) \hookrightarrow \text{Cech}(X, r)
\]

are homotopy equivalences.

The same statement holds for closed complexes if $0 < r < r(X)/2$.

**Proof.** The following argument is a proof for Theorems 5.6 and 5.7.

Let $U^r$ and $U^{r/2}$ be the covers of $X$ by all open balls of radius $r$ and $r/2$, respectively. Note that $\text{Cech}(X, r/2) = \mathcal{N}(U^{r/2}) = \mathcal{V}(U^{r/2})$ and $\text{Cech}(X, r) = \mathcal{N}(U^r) = \mathcal{V}(U^r)$. Let $W$ be a good open cover of $X$ from Theorems 4.10 or 4.14, respectively, so that $\text{Rips}(X, r) = \mathcal{V}(W)$. The conclusion holds by Theorem 5.4, since maps $\varphi_V$ are inclusions and the covers are appropriately nested. \(\Box\)

**Remark 5.8.** The passages of parameter from $r/2$ to $r$ to $r$ in $\text{Cech}(X, r/2) \hookrightarrow \text{Rips}(X, r) \hookrightarrow \text{Cech}(X, r)$ are essentially due to two facts:

- balls of radius $r/2$ are of diameter $r$;
- subsets of diameter $r$ are contained in a ball of radius $r$.

The second parameter passage can be tightened in specific cases (for example in Euclidean spaces) by suitable versions of Jung’s Theorem [11,23].

**Theorem 5.9** (Initial persistence invariance for Rips complexes). Suppose $X$ is:

1. a metric graph and $0 < r_1 \leq r_2 < r_3 = \ell(X)/3$, or
2. a geodesic space with $r(X) > 0$ and $0 < r_1 \leq r_2 < r_3 = r(X)/2$. 


Then, all the maps in the following (up to homotopy commutative) diagram are homotopy equivalences between spaces:

\[
\begin{array}{ccc}
\text{Rips}(X, r_3) & \xrightarrow{\gamma_W \circ i_W} & \text{Rips}(X, r_2) \\
\downarrow & & \downarrow \\
\text{Rips}(X, r_2) & \xrightarrow{\gamma_U \circ i_U} & \text{Rips}(X, r_1) \\
\end{array}
\]

with maps \( \gamma_U \circ i_U, \gamma_W \circ i_W, \) and \( \gamma'_W \circ i'_W \) arising from Theorem 5.4 for appropriate covers described in Sect. 4.

Proof. Follows from Theorem 5.4 for appropriate covers described in Sect. 4.

\[ \square \]

Remark 5.10. The same theorem holds for Čech complexes as well with \( X \) being:

1. a metric graph and \( 0 < r_1 \leq r_2 < r_3 = \ell(X)/4 \), or
2. a geodesic space with \( r(X) > 0 \) and \( 0 < r_1 \leq r_2 < r_3 = r(X)/4 \).

Remark 5.11. While the initial persistence invariance results do not seem to differ much in the case of open filtrations as compared to the closed filtrations, the understanding of the underlying theoretical difference could have important consequence on computational setting, i.e., the case where such filtrations are approximated by finite subcomplexes. The results of [1] suggest that the open and closed filtrations of a geodesic circle differ by ephemeral summands, i.e., they differ only at a discrete set of points. These summands, however, seem to grow in size when approximated by finite sample [1]. A similar effect was observed in [36]. We plan to delve deeper in that direction in the future research.

6. Reconstruction Results by Subsets

Many of the results of the previous sections were aimed at reconstructing the homotopy type of a space \( X \) from its Rips complex. In this section, we focus on the reconstruction of \( X \) or its homology using Rips complexes of subspaces. In the ideal case, we would want to carry out reconstruction using only finite subspaces.

Given a good open cover \( U \) of a compact space \( X \), we can always find a finite subcover and thus obtain a finite nerve, which is homotopy equivalent to the initial space. Passing to finite covers when using nerves is thus easy. On the other hand, there is no known construction which would be able to produce a finite collection of points \( S \subset X \) from a good open cover \( U \) of a space \( X \), with the property that the Vietoris complex of \( U \) restricted to \( S \) is homotopy equivalent to \( X \). Passing to finite subsets in the context of Rips complexes is hard.
The idea presented in this section is to use the Functorial Dowker-Nerve diagram to combine the simplicity of passing to a finite cover in the context of nerves with the targeted construction of Rips complexes, resulting in the diagram in the proof of Theorem 6.1.

Theorem 6.1 states that the homotopy type of a sufficiently nice compact space can be reconstructed as the Rips complex of a countable subset $C$. It is an open question whether in the context of Theorem 6.1, the subset $C$ can be taken to be a finite subset of $X$. Such a result is known to hold for sufficiently nice Riemannian manifolds [24] and certain subsets of the Euclidean space [2]. However, the currently known proofs of both of these results rely strongly on the underlying structure of the space (either a manifold structure or that of certain Euclidean neighborhoods). To the best of the author’s knowledge, it is currently unknown how these conditions compare with the assumptions of Theorem 6.1.

The conclusion of Theorem 6.1 could be proved for separable metric spaces (instead of compact spaces) using the same argument. However, both of these statements actually follow from [22, Proposition 2.4], which states that for each dense subset $Y$ of a pseudo-metric space $X$ and for each $r > 0$, we have $\text{Rips}(Y, r) \simeq \text{Rips}(X, r)$. The added value of Theorem 6.1 lies in its proof method:

- Unlike [22, Proposition 2.4], which actually does not hold for closed complexes, the setting and the proof of Theorem 6.1 can be easily adjusted to accommodate closed complexes; see Theorem 6.2.
- The finiteness of the spaces in the diagram in the proof of Theorem 6.1 can be used to extract the homology of a space from finite subspaces; see Theorem 6.4.
- The proof of Theorem 6.1 suggests a potential direction to answer the open question from the previous paragraph. The remaining step would be to prove, possibly under stronger conditions, that the inductive construction in the proof of Theorem 6.1 stabilizes after finitely many steps.

Let us introduce some new notation: if $\mathcal{U}$ is a cover of $X$ and $A \subset X$, then $\mathcal{U}|_A = \{U \cap A \mid U \in \mathcal{U}\}$ is a cover of $A$.

**Theorem 6.1.** Suppose $X$ is a compact metric space, $r > 0$ and $\mathcal{U}$ is a good open cover of $X$ satisfying the following condition for each finite subset $\sigma \subset X$: $\sigma$ is contained in some element of $\mathcal{U}$ iff $\text{Diam}(\sigma) < r$. Then, there exists a countable $C \subset X$ satisfying $\text{Rips}(C, r) \simeq X$.

**Proof.** Note that $\text{Rips}(X, r) \simeq X$ by Theorem 4.4.

Let $\mathcal{U}_1 \subset \mathcal{U}$ be a finite subcover of $\mathcal{U}$ and choose a finite subset $X_1 \subset X$, so that each non-trivial intersection of elements of $\mathcal{U}_1$ contains a point of $X_1$; i.e., $\mathcal{N}(\mathcal{U}_1|_{X_1}) = \mathcal{N}(\mathcal{U}_1)$.

We proceed by inductive definition of covers $\mathcal{U}_n$ and finite subsets $X_n$ for $n > 1$:
• Let $\mathcal{U}_n$ be a finite subcover of $\mathcal{U}$ containing $\mathcal{U}_{n-1}$, so that for each finite $\sigma \subset X_{n-1}$ of diameter less than $r$, there exists $U_\sigma \in \mathcal{U}_n$ containing $\sigma$. Consequently, $\mathcal{V}(\mathcal{U}_n|X_{n-1}) = \text{Rips}(X_{n-1}, r)$.

• Let $X_n \subset X$ be a finite subset containing $X_{n-1}$, so that each non-trivial intersection of elements of $\mathcal{U}_n$ contains a point of $X_n$, i.e., $\mathcal{N}(\mathcal{U}_n|X_n) = \mathcal{N}(\mathcal{U}_n)$.

Define $C = \bigcup_n X_n$ and note that $\text{Rips}(C, r) = \bigcup_n \text{Rips}(X_n, r)$. We can now construct the following diagram:

$$
\begin{array}{ccc}
X & \longrightarrow & \mathcal{N}(\mathcal{U}_1) = \mathcal{N}(\mathcal{U}_1|X_1) \\
\downarrow & & \mathcal{V}(\mathcal{U}_1|X_1) \\
\mathcal{N}(\mathcal{U}_2|X_1) & \longrightarrow & \mathcal{V}(\mathcal{U}_2|X_1) = \text{Rips}(X_1, r) \\
\downarrow & & \text{Rips}(C, r) \\
X & \longrightarrow & \mathcal{N}(\mathcal{U}_2) = \mathcal{N}(\mathcal{U}_2|X_2) \\
\downarrow & & \mathcal{V}(\mathcal{U}_2|X_2) \\
\mathcal{N}(\mathcal{U}_3|X_2) & \longrightarrow & \mathcal{V}(\mathcal{U}_3|X_2) = \text{Rips}(X_2, r) \\
\downarrow & & \text{Rips}(C, r) \\
\end{array}
$$

By Theorem 5.4, the diagram commutes up to homotopy, all horizontal maps are homotopy equivalences, and all inclusions $\mathcal{V}(\mathcal{U}_{n+1}|X_n) \rightarrow \mathcal{V}(\mathcal{U}_{n+2}|X_{n+1})$ are homotopy equivalences. Consequently, $\bigcup_n \mathcal{V}(\mathcal{U}_{n+1}|X_n) \simeq X$, which proves the theorem as $\bigcup_n \mathcal{V}(\mathcal{U}_{n+1}|X_n) = \bigcup_n \text{Rips}(X_n, r) = \text{Rips}(C, r)$. □

**Theorem 6.2** (A version of Theorem 6.1 for closed complexes). Suppose $X$ is a compact metric space, $r > 0$ and $\mathcal{U}$ is a good closed numerable cover of $X$ satisfying the following condition for each finite subset $\sigma \subset X$: $\sigma$ is contained in some element of $\mathcal{U}$ iff $\text{Diam}(\sigma) \leq r$. Then, there exists a countable $C \subset X$ satisfying $\text{Rips}(C, r) \simeq X$.

**Proof.** The proof is analogous to that of Theorem 6.1. We give it here for completeness. We emphasise differences through the use of italics. Note that $\text{Rips}(X, r) \simeq X$ by Theorem 4.4.

Let $\mathcal{U}_1 \subset \mathcal{U}$ be a finite subcover of $\mathcal{U}$ whose interiors cover $X$, and choose a finite subset $X_1 \subset X$, so that each non-trivial intersection of elements of $\mathcal{U}_1$ contains a point of $X_1$, i.e., $\mathcal{N}(\mathcal{U}_1|X_1) = \mathcal{N}(\mathcal{U}_1)$.

We proceed by giving the inductive definitions of both the covers $\mathcal{U}_n$ and the finite subsets $X_n$ for $n > 1$:

• Let $\mathcal{U}_n$ be a finite subcover of $\mathcal{U}$ containing $\mathcal{U}_{n-1}$, so that for each finite $\sigma \subset X_{n-1}$ of diameter less or equal to $r$, there exists $U_\sigma \in \mathcal{U}_n$ containing $\sigma$. Consequently, $\mathcal{V}(\mathcal{U}_n|X_{n-1}) = \text{Rips}(X_{n-1}, r)$. 

Let \( X_n \subset X \) be a finite subset containing \( X_{n-1} \), so that each non-trivial intersection of elements of \( U_n \) contains a point of \( X_n \), i.e., \( \mathcal{N}(U_n|X_n) = \mathcal{N}(U_n) \).

Define \( C = \bigcup_n X_n \) and note that \( \overline{\mathrm{Rips}}(C,r) = \bigcup_n \overline{\mathrm{Rips}}(X_n,r) \). We can now construct the same diagram as in the proof of Theorem 6.1, with the only difference being that all Rips complexes appearing there are closed Rips complexes.

By Theorem 5.4, the diagram commutes up to homotopy, all horizontal maps are homotopy equivalences, and all inclusions \( V(U_{n+1}|X_n) \to V(U_{n+2}|X_{n+1}) \) are homotopy equivalences. Consequently, \( \bigcup_n V(U_{n+1}|X_n) = \bigcup_n \overline{\mathrm{Rips}}(X_n,r) = \overline{\mathrm{Rips}}(C,r) \).

While it is unknown whether the set \( C \) of Theorem 6.1 can be taken to be finite (i.e., whether we can construct the homotopy type of \( X \) as the Rips complex of a finite subset), we can make use of a trick from [9] to reconstruct the homology of \( X \) through nested finite Rips complexes in Theorem 6.4.

**Lemma 6.3** (Excerpt from Lemma 3.2 of [9]). If \( A \to B \to C \to E \to F \) is a sequence of homomorphisms, such that \( \text{rank}(A \to F) = \dim C \), then \( \text{rank}(B \to E) = \dim C \).

**Theorem 6.4.** Suppose \( X \) is a compact metric space, \( r > 0 \) and \( \mathcal{U} \) is a good open cover of \( X \) satisfying the following condition for each finite subset \( \sigma \subset X \): \( \sigma \) is contained in some element of \( \mathcal{U} \) iff \( \text{Diam}(\sigma) < r \). Then, there exist finite subsets \( X_1 \subset X_2 \subset X \), so that for each \( k \in \mathbb{N} \) and each field \( \mathbb{F} \):

\[
H_k(X,\mathbb{F}) \cong \text{Im} j^*,
\]

where \( j^*: H_k(\overline{\mathrm{Rips}}(X_1,r),\mathbb{F}) \to H_k(\overline{\mathrm{Rips}}(X_2,r),\mathbb{F}) \) is the inclusion-induced homomorphism.

**Proof.** Apply Lemma 6.3 to the right vertical thread of the diagram from the proof of Theorem 6.1.

**Remark** 6.5. Note that the nesting of Theorem 6.4 is horizontal; that is, it only considers one scale \( r \) and increases the subspace (sample). This is in contrast to the use in [9] where nesting is vertical (i.e., includes multiple scales at fixed sample), confined to small \( r \), and constrained by the weak feature size of [9] (and, consequently, the structure of the Čech complex in Euclidean space). Therefore Theorem 6.4 can be used at any individual scale without extra requirements on the ambient space (provided the conditions of Theorem 6.4 hold) and irrespectively of other scales. In particular, we may get appropriate reconstruction even for scales above the weak feature size. For a demonstration, see Example 6.6.

**Example 6.6.** Let \( \varepsilon < \frac{1}{10} \). Space \( X \subset \mathbb{R}^2 \) is presented by Fig. 1: a smaller circle of diameter 1 has an \( \varepsilon \)-gap on the left and is connected on the right by a line of length 10 to a circle of diameter \( D > 3 \). It is easy to verify that by our results, \( \overline{\mathrm{Rips}}(X,r) \cong X \) for any \( r < \varepsilon \). A similar reconstruction can
be obtained by [2]: as $\varepsilon/2 > 0$ is the weak feature size of $X$, we can choose a finite $A \subset X$ and a small $r > 0$, so that Rips$(A, r) \simeq X$. Furthermore, the homology of $X$ with coefficients in a field can be extracted by [9] using a pair of Rips complexes of a finite subset at different scales or using Theorem 6.4. However, it is not hard to see that also for $R \in (1, D/3)$, there exists a cover of $X$ satisfying the conditions of Theorem 4.4, hence Rips$(X, R) \simeq X$ for scale $R$, as well. Consequently, the homotopy type of $X$ can be extracted using Theorem 6.1 at scale $R$ by a countable subset. Similarly, its homology with coefficients in a field may be extracted using Theorem 6.4 at scale $R$ using two nested finite subsets. Since the interval $(1, D/3)$ between two critical values of the distance function $x \mapsto d(x, X)$ can be arbitrarily small, results of [2, 9] do not apply in this case for sufficiently small $D > 3$.

7. Appendix: Homotopy Types of Rips Complexes

Suppose $X \subset \mathbb{R}^n$ is a finite subset and $r > 0$. A convenient implication of the nerve theorem for Čech complexes is that Čech$_{\mathbb{R}^n}(X, r)$ is homotopy equivalent to a subspace of the ambient space $\mathbb{R}^n$ (in particular to the $r$-neighborhood of $X$). It is easy to see, however, that the same does not always hold for Rips complexes. In this section, we explain why this is the case despite the presented version of the nerve theorem for Rips complexes, and how the difficulties surrounding this issues can sometimes be circumvented. We will do that by analyzing the case of a regular planar hexagon using Theorem 3.9. A general strategy is the following: for $r > 0$, first use Theorem 3.9 to construct a cover $\mathcal{U}$ of $X$ with Rips$(X, r) = \mathcal{V}(\mathcal{U})$, and then replace each element of $\mathcal{U}$ by a convex set in $\mathbb{R}^n$ to obtain a good cover $\tilde{\mathcal{U}}$, for which Rips$(X, r) \simeq \bigcup_{\tilde{U} \in \tilde{\mathcal{U}}} \tilde{U}$ potentially holds.

Let $X_1 = \{x_1, x_2, \ldots, x_6\} \subset \mathbb{R}^2$ denote the vertices of a regular hexagon in the plane with the side length 1, i.e., $d(x_i, x_{i+1}) = 1 = d(x_1, x_6)$. Define $r_2 = d(x_1, x_3)$ and $r_3 = d(x_1, x_4)$. The open Rips complexes attain four different homotopy types.

(a) $r \leq 1$ Rips$(X_1, r) = \mathcal{V}(\{\{x_1\}, \{x_2\}, \ldots, \{x_6\}\}) = X_1$ is a discrete space.

(b) $1 < r \leq r_2$ Rips$(X_1, r) = \mathcal{V}(\mathcal{U}_1)$ for:

$$\mathcal{U}_1 = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_5, x_6\}, \{x_6, x_1\}\}.$$
Replacing each element of $U_1$ by the open $(r - 1)/4$-neighborhood of its convex hull, we get a good planar cover $\tilde{U}_1$ of the boundary of the corresponding regular hexagon in the plane. Since every nonempty intersection of sets of $\tilde{U}_1$ contains a point of $X_1$, we have $\text{Rips}(X_1, r) \simeq \mathcal{N}(U_1) = \mathcal{N}(\tilde{U}_1) \simeq S^1$.

In this case, $\text{Rips}(X_1, r)$ is actually homeomorphic to $S^1$. To demonstrate a less trivial example, let $X_1$ be obtained by replacing each $x_i$ by a subset $A_i$ of $n$ points in $B(x_i, (r - 1)/4)$. In this case, $\text{Rips}(X_2, r)$ is a $(2n - 1)$-dimensional complex, but the above argument (with the open cover consisting of six sets, each of which is the open $(r - 1)/4$-neighborhood of the convex hull of consecutive sets $A_i$) still holds hence: $\text{Rips}(X_2, r) \simeq S^1$.

(c) $r_2 < r \leq r_3$ $\text{Rips}(X_1, r) \simeq \mathcal{N}(U_2)$ for

$$U_2 = \{\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \ldots, \{x_5, x_6, x_1\}, \{x_6, x_1, x_2\}\}$$

$$\cup\{\{x_1, x_3, x_5\}, \{x_2, x_4, x_6\}\}.$$ 

Replacing each element of $U_1$ by the open $(r - r_2)/4$-neighborhood of its convex hull, we get a good planar cover $\tilde{U}_2$. However, $\mathcal{N}(U_2) \neq \mathcal{N}(\tilde{U}_2)$, since the sets of $\tilde{U}_2$ corresponding to element $\{x_1, x_3, x_5\}, \{x_2, x_4, x_6\} \in U_1$ intersect in $\mathbb{R}^2$ but not in $X_1$. In fact, $\mathcal{N}(U_2)$ is contractible, while $\mathcal{N}(\tilde{U}_2) \simeq S^2$. To see the later note that if the vertices of $X_1$ are appropriately arranged as the vertices of an octahedron in $\mathbb{R}^3$, then small neighborhoods of the faces of the octahedron form a good cover of the boundary of the octahedron, so that:

- the combinatorial structure of the cover matches that of $U_2$;
- each nonempty intersection contains a point of $X_2$.

Hence, $\text{Rips}(X_1, r) \simeq \mathcal{N}(U_2) \simeq S^2$. As before, the same holds if we replace each vertex of $X_1$ by a large collection of points in its small neighborhood.

(d) $r_3 < r$ $\text{Rips}(X_1, r)$ is the full simplex on $X_1$ and hence contractible.

8. Conclusion

For a nice representation of the homotopy type of $\text{Rips}(X, r)$, one needs to:

- choose a small cover $U$ satisfying Theorem 3.9 (note that the cover of $X_2$ in (b) above consists of only six sets despite $|X_2| = 6n$), and
- extend $U$ to a good cover $\tilde{U}$, so that the nonempty intersections of $\tilde{U}$ contain an element of $X$.

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