Bihamiltonian Geometry, Darboux Coverings, and Linearization of the KP Hierarchy

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\textbf{Abstract}

We use ideas of the geometry of bihamiltonian manifolds, developed by Gel’fand and Zakharevich, to study the KP equations. In this approach they have the form of local conservation laws, and can be traded for a system of ordinary differential equations of Riccati type, which we call the Central System. We show that the latter can be linearized by means of a Darboux covering, and we use this procedure as an alternative technique to construct rational solutions of the KP equations.

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1 Introduction

In this paper we study some aspects of the KP theory from the point of view of the bihamiltonian approach to integrable systems. Our purpose is twofold. At first we describe how the KP theory can be defined by means of a suitable application of the method of Poisson pencils, in the light of the so-called Gel’fand–Zakharevich (hereinafter GZ) theorem on the local geometry of a bihamiltonian manifold \[10\]. Secondly we discuss, from this point of view, how one can trade the KP hierarchy of partial differential equations for a system of ordinary differential equations, which we term Central System. We show how this system can be linearized and solved by means of a Darboux transformation.

Our approach is inductive. We start from the KdV theory, which we tackle as the GZ theory of the Poisson pencil

\( P_\lambda = -\frac{1}{2} \partial_x^3 + 2(u + \lambda) \partial_x + u_x, \) \hspace{1cm} (1.1)

defined on the manifold of \( C^\infty \)-functions on \( S^1 \). Following the GZ scheme we study the Casimir function \( H_\lambda \) of this pencil. We show that it can be written in the integral form

\( H(z) = 2z \int_{S^1} h(x, z) dx \) \hspace{1cm} (1.2)

where the local density \( h \) is a Laurent series

\( h = z + \sum_{j \geq 1} h_j z^{-j} \) \hspace{1cm} (1.3)

in \( z = \sqrt{\lambda} \). This density is related to the point \( u \) of the phase space by the Riccati equation \[17, 14\]

\( u + z^2 = h_x + h^2, \) \hspace{1cm} (1.4)

which, as is well known (see, e.g., \[4\]) defines all the coefficients \( h_j, j \geq 1 \), as differential polynomials of \( u \).

The next step is to study the conservation laws associated with the KdV hierarchy. A trivial but far reaching consequence of the involutivity of the KdV Hamiltonians is that the KdV flows imply the local conservation laws \[24, 7, 4\]

\( \frac{\partial}{\partial t_j} h = \partial_x H^{(j)} \) \hspace{1cm} (1.5)

for the density \( h \).
These equations introduce the principal characters of our picture: the currents \( H^{(j)} \). We show that they can be computed in the following way. Among the (finite) linear combinations \( \sum C_l h^{(l)} \) of the Faà di Bruno iterates
\[
h^{(j)} = (\partial_x + h)h^{(j-1)}, \quad h^{(0)} = 1,
\] (1.6)
of \( h \), with coefficients \( C_l \) independent of \( z \), we pick up, for every \( j \geq 0 \), the unique combination having the asymptotic expansion
\[
H^{(j)} = z^j + \sum_{k \geq 1} H^j_k z^{-k}.
\] (1.7)
When we insert these currents into the conservation laws (1.3) we obtain a hierarchy of mutually commuting vector fields on a generic Laurent series of the type (1.3). They are (a possible form of) the celebrated KP equations. When \( h \) is required to be a solution of the Riccati equations (1.4), these equations collapse into the conservation laws associated with the Poisson pencil (1.1).

As a further step we study the time evolution of the currents \( H^{(j)} \). We show that they satisfy a closed system of ordinary differential equations which has the form of a generalized Riccati system:
\[
\frac{\partial H^{(k)}}{\partial t_j} = H^{(j+k)} - H^{(j)}H^{(k)} + \sum_{l=1}^{k} H^j_l H^{(k-l)} + \sum_{l=1}^{j} H^k_l H^{(j-l)}.
\] (1.8)
This we call the Central System (CS). It encompasses and extends the KP hierarchy. In [5] we have discussed how the KP equations can be recovered from CS by a projection on the orbit space of the first vector field \( \frac{\partial}{\partial t_1} \). Different projections allow to obtain those KP systems related to fractional KdV hierarchies [3, 11].

Finally we turn to the problem of solving the Central System. By means of the method of Darboux covering, discussed in [15], we prove that the Miura–like map
\[
H^{(j)} = \left( \sum_{l=0}^{j} W_{j-l}^0 W^{(l)} \right) / W^{(0)}
\] (1.9)
connects the Central System with another system of Riccati equations, defined on the space of sequences of Laurent series \( \{W^{(k)}\}_{k \geq 0} \) of the form
\[
W^{(k)} = z^k + \sum_{l \geq 1} W^k_l z^{-l},
\] (1.10)
which reads
\[
\frac{\partial}{\partial t_j} W^{(k)} + z^j W^{(k)} = W^{(j+k)} + \sum_{l=1}^{j} W^k_l W^{(j-l)}.
\] (1.11)
This system (which we call the *Sato System*, see [23]) can be explicitly linearized using methods well known from the theory of Riccati equations [20].

As a result the Miura map (1.9), which in the present picture is the analog of a dressing transformation, allows to construct explicit families of solutions of the Central System, and hence of the KP hierarchy.

In our opinion, this paper clarifies some issues in the analysis of the different pictures of the KP hierarchy, their mutual relations, and the linearization of the KP flows. More specifically we refer to the following three representations for KP:

- a) The Lax representation in the space of pseudodifferential operators;
- b) The Sato representation as linear flows of a maximal torus in $GL_\infty$ on the Universal Sato Grassmannian $UGr$;
- c) The “bihamiltonian representation” as conservation laws (1.5) satisfied by the Hamiltonian density $h$.

The analysis of the representations a) and b), and of their equivalence, has been deeply expounded in a number of nowadays classical papers and lecture notes (see, e.g., [8, 9, 16, 18, 19, 21, 22, 23]), while the picture of the KP equations as conservation laws, although already introduced in [7, 12, 21, 24], has somehow been left aside from the main stream of the research work on the subject. By fully developing the bihamiltonian approach, this paper aims to show that the KP theory can also be approached on a traditional ground, in the spirit of geometrical methods of classical mechanics [2].

The paper is organized as follows. It starts with a brief résumé of the bihamiltonian theory, devoted only to those aspects of the GZ theorem relevant to the paper. The next three sections provide the description of the path from KdV to KP and the Central System, briefly discussed above. In Section 6 we consider the action of Darboux transformations on the Central System. In the final sections we take advantage of such a point of view to linearize the KP theory and we address the problem of writing explicit solutions, which are Hirota–like polynomial solutions.

2 The method of Poisson pencils (to construct integrable Hamiltonian system)

In the simplest setting of this method one considers a Poisson manifold $\mathcal{M}$ and a vector field $X$. The vector field is used to deform the Poisson bracket $\{\cdot, \cdot\}$ on $\mathcal{M}$. 
We denote by
\[
\{f, g\}' = \{X(f), g\} + \{f, X(g)\} - X(\{f, g\})
\]
and
\[
\{f, g\}'' = \{X(f), g\}' + \{f, X(g)\}' - X(\{f, g\}')
\]
(2.1)
the first two Lie derivatives of this bracket along \(X\). As the unique condition on \(X\), we demand that the second derivative identically vanishes on \(M\):
\[
\{f, g\}'' \equiv 0.
\]
(2.2)
In this case, the pull–back
\[
\{f, g\}_\lambda := \{f \circ \phi_{-\lambda}, g \circ \phi_{-\lambda}\} \circ \phi_\lambda
\]
(2.3)
of the given bracket with respect to the flow \(\phi_\lambda : M \to M\) associated with \(X\) depends linearly on \(\lambda\),
\[
\{f, g\}_\lambda = \{f, g\} - \lambda\{f, g\}',
\]
(2.4)
and, therefore, it defines a linear pencil of Poisson brackets. Under these circumstances, we say that \(M\) is an (exact) bihamiltonian manifold and that \(X\) is its Liouville vector field. The names are chosen to suggest the analogy with the case of exact symplectic manifolds.

The basic idea of the method is to use the Casimir functions of the pencil (2.4) to construct integrable Hamiltonian systems on \(M\). We describe this technique in the case of an odd–dimensional manifold endowed with a Poisson pencil of maximal rank. This entails that \(\{\cdot, \cdot\}_\lambda\) has a unique Casimir function \(H_\lambda\), depending on \(\lambda\). Let us set \(\text{dim } M = 2n + 1\). Gel’fand and Zakharevich \([10]\) have shown that \(H_\lambda\) is a degree \(n\) polynomial in \(\lambda\),
\[
H_\lambda = H_0 \lambda^n + H_1 \lambda^{n-1} + \cdots + H_n,
\]
(2.5)
which starts with the Casimir function \(H_0\) of \(\{\cdot, \cdot\}'\) and ends with the Casimir function \(H_n\) of \(\{\cdot, \cdot\}\). The coefficients \(H_j\) verify the recursion relations
\[
\{\cdot, H_{j+1}\}' = \{\cdot, H_j\},
\]
(2.6)
and therefore are in involution with respect to all the brackets of the pencil:
\[
\{H_j, H_k\}_\lambda = 0.
\]
(2.7)
In the compact case, their level surfaces are \(n\)–dimensional tori defining a Lagrangian foliation of \(M\).
To convert this result in a statement on dynamical systems, we consider the pencil of vector fields
\[ X_\lambda(f) := \{f, H_\lambda\}' \] (2.8)
associated with \( H_\lambda \) through the deformed bracket \( \{\cdot, \cdot\}' \). We make two remarks. First we notice that \( X_\lambda \) is a bihamiltonian vector field since we can write
\[ X_\lambda(f) = \{f, H_\lambda\}' = \{f, H_\lambda'\}_\lambda. \] (2.9)
The derivative
\[ H_\lambda' = X(H_\lambda) \] (2.10)
is the second Hamiltonian function. Then we notice that \( X_\lambda \) is a completely integrable system in the sense of Liouville since
\[ X_\lambda(H_j) = 0. \] (2.11)
We call the polynomial family of vector fields,
\[ X_\lambda = X_0\lambda^n + X_1\lambda^{n-1} + \cdots + X_n, \] (2.12)
the canonical hierarchy defined on the exact bihamiltonian manifold \( \mathcal{M} \).

3 The KdV hierarchy

In this section we define the KdV hierarchy as the canonical hierarchy on a special exact bihamiltonian manifold, and we use this point of view to pave the way to the KP theory.

In this example the manifold \( \mathcal{M} \) is the space of scalar–valued \( C^\infty \)-functions on \( S^1 \), the Liouville vector field is
\[ \dot{u} := X(u) = 1, \] (3.1)
and the Poisson pencil is given in the form of a one–parameter family of skew–symmetric maps from the cotangent to the tangent bundle \([1, 6]\):
\[ \dot{u} = (P_\lambda)_uv = -\frac{1}{2}v_{xxx} + 2(u + \lambda)v_x + u_xv. \] (3.2)
In this formula \( u \) is a point of \( \mathcal{M} \), \( v \) is a covector attached at \( u \), and the value of \( v \) on a generic tangent vector \( \dot{u} \) is given by
\[ \langle v, \dot{u} \rangle = \int_{S^1} v(x)\dot{u}(x) \, dx, \] (3.3)
where \( x \) is the coordinate on \( S^1 \).

The first problem is to compute the Casimir function \( H_\lambda \) and its derivative \( H'_\lambda \) along \( X \). In the present infinite–dimensional context they can be conveniently written as integrals

\[
H = 2z \int_{S^1} h \, dx \quad (3.4)
\]
\[
H' = \int_{S^1} h' \, dx \quad (3.5)
\]
of local densities

\[
h(z) = z + \sum_{j \geq 1} h_j z^{-j} \quad (3.6)
\]
\[
h'(z) = 1 + \sum_{j \geq 1} h'_j z^{-j}, \quad (3.7)
\]

which are Laurent series (rather than polynomials) in \( z = \sqrt{\lambda} \).

**Proposition 3.1** Let \( h \) and \( h' \) be the unique solutions of the Riccati system

\[
h_x + h^2 = u + z^2 \quad (3.8)
\]
\[
-\frac{1}{2} h'_x + hh' = z \quad (3.9)
\]

admitting the asymptotic expansions (3.6) and (3.7). Then their integrals \( H_\lambda \) and \( H'_\lambda \) are respectively the Casimir function of the Poisson pencil (3.2) and its associated second Hamiltonian function.

**Proof.** We use the identity

\[
v \left( \frac{1}{2} v_{xxx} + 2(u + z^2)v_x + u_x v \right) = \frac{d}{dx} \left( \frac{1}{4} v_x^2 - \frac{1}{2} vv_{xx} + (u + z^2)v^2 \right) \quad (3.10)
\]
to prove that the solution of the equation

\[
\frac{1}{4} v_x^2 - \frac{1}{2} vv_{xx} + (u + \lambda)v^2 = \lambda \quad (3.11)
\]
belongs to the kernel of (3.2). Setting \( \lambda = z^2 \) we note that equation (3.11) can be written in the form of a Riccati equation,

\[
\left( \frac{z}{v} + \frac{v_x}{2v} \right)_x + \left( \frac{z}{v} + \frac{v_x}{2v} \right)^2 = u + z^2, \quad (3.12)
\]
on

\[
h(z) := \frac{z}{v} + \frac{v_x}{2v} \quad (3.13)
\]
By deriving this equation along any curve \( u(t) \) in \( \mathcal{M} \), we have \( \dot{u} = \dot{h}_x + 2h\dot{h} \). Therefore,

\[
\langle v, \dot{u} \rangle = \int_{S^1} v(x)(\dot{h}_x + 2h\dot{h}) \, dx = 2 \int_{S^1} (-\frac{1}{2}v_x + hv) \dot{h} \, dx = \frac{d}{dt} \left( 2z \int_{S^1} h \, dx \right).
\]

This formula proves that \( v \) is the differential of the first Hamiltonian \( H_\lambda \), which, consequently, is the Casimir function we were looking for. To compute the second Hamiltonian \( H'_\lambda \), it suffices to notice that

\[
H'_\lambda = X(H_\lambda) = \langle v, 1 \rangle = \int_{S^1} v \, dx.
\]

This suggests to set

\[
h' = v.
\]

Equations (3.8) and (3.9) follow from (3.12) and (3.13) respectively.

The next problem is to study the canonical hierarchy associated with \( H_\lambda \). It admits three different representations, according to the use of equations (2.8), (2.9), or (2.11) of Section 2. In the first representation, based on formula (2.8), the KdV equations are written as Hamiltonian equations

\[
\begin{align*}
\frac{\partial u}{\partial t_{2j}} &= -2\partial_x v_{2j+1} = 0 \\
\frac{\partial u}{\partial t_{2j+1}} &= -2\partial_x v_{2j+2}
\end{align*}
\]

with respect to the derived bracket \( \{\cdot, \cdot\}' \). In the second representation, based on (2.9), they are written as Hamiltonian equations with respect to the pencil \( P_\lambda \). After some straightforward computations \[1, 6\], we get

\[
\begin{align*}
\frac{\partial u}{\partial t_{2j}} &= 0 \\
\frac{\partial u}{\partial t_{2j+1}} &= -\frac{1}{2} \partial_x^3 + 2(u + z^2)\partial_x + u_x \cdot (\lambda^j v(\lambda))_+,
\end{align*}
\]

where

\[
(\lambda^j v(\lambda))_+ = \sum_{i=0}^{j} v_{j-i} \lambda^i.
\]

In the third representation, based on formula (2.11), the attention is focused on the local Hamiltonian density \( h(z) \). It must obey local conservation laws of the form

\[
\frac{\partial h}{\partial t_{2j}} = \partial_x H^{(j)},
\]

\[7\]
where the $H^{(j)}$ are suitable “current densities”, since the Hamiltonian $H_\lambda$ is constant along the flows of the KdV hierarchy. The final problem is to compute these densities.

**Proposition 3.2** The current densities $H^{(j)}$ of the KdV hierarchy are given by the formulas

\[
H^{(2j)} = \lambda^j \quad (3.23)
\]
\[
H^{(2j+1)} = -\frac{1}{2}(\lambda^j v)_+ + h(\lambda^j v)_+. \quad (3.24)
\]

**Proof.** From equation (3.8) we have

\[
\frac{\partial u}{\partial t_j} = (\partial_x + 2h) \left( \frac{\partial h}{\partial t_j} \right), \quad (3.25)
\]

and from the second representation (3.20) of the KdV equations we deduce

\[
\frac{\partial u}{\partial t_{2j+1}} = (\partial_x + 2h) \left( \frac{1}{2} \partial_x (-\partial_x + 2h)(\lambda^j v)_+ \right), \quad (3.26)
\]

by noticing that

\[
-\frac{1}{2} v_{xxx} + 2(u + \lambda) v_x + u_x v = (\partial_x + 2h) \left( \frac{1}{2} \partial_x (-\partial_x + 2h) \cdot v. \right) \quad (3.27)
\]

Therefore

\[
(\partial_x + 2h) \left( \frac{\partial h}{\partial t_{2j+1}} \right) = (\partial_x + 2h) \left( \frac{1}{2} \partial_x \left( -\lambda^j v + 2h(\lambda^j v)_+ \right) \right)
\]
\[
= (\partial_x + 2h) \partial_x \left( -\frac{1}{2} (\lambda^j v)_+ + h(\lambda^j v)_+ \right), \quad (3.28)
\]

proving (3.24). The formula $H^{(2j)} = z^{2j}$ is obvious, since the even times are trivial.

The definition (3.3) of the Poisson pencil, the Riccati system (3.8, 3.9) for the Hamiltonians, and the definition (3.24, 3.23) of the currents are the basic formulas of the Hamiltonian theory of the KdV equations. They introduce a new object, the currents $H^{(j)}$. Their study is the leading theme of the paper.

### 4 The KP hierarchy

The aim of this section is to give a new characterization of the current $H^{(j)}$ in terms of the Hamiltonian density $h(z)$. To this end we write (3.24) in the equivalent forms

\[
H^{(2j+1)} = z^{2j} \left( -\frac{1}{2} v_x + h v \right) + \frac{1}{2} (z^{2j} v)_- x - h(z^{2j} v)_- \quad (4.1)
\]
\[
H^{(2j+1)} = \sum_{l=1}^{j} \left[ -\frac{1}{2} v_{j-l,x} (z^{2l} \cdot 1) + v_{j-l}(z^{2l} \cdot h) \right], \quad (4.2)
\]
where \((z^{2j}v)_-\) denotes the strict negative part of the expansion of \(z^{2j}v\) in powers of \(z\). Each of these representations points out an important property of the currents \(H^{(j)}\). Equation (4.1) allows to control the expansion of \(H^{(2j+1)}\) in powers of \(z\). Indeed, from it we obtain

\[
H^{(j)} = z^j + O(z^{-1}) \quad (4.3)
\]

by noticing that the second Riccati equation (3.9) implies \(z^{2j}(-\frac{1}{2}v_x + hv) = z^{2j+1}\).

The interpretation of (4.2) is more subtle: it provides a different type of expansion of the current \(H^{(2j+1)}\) on a basis attached to the Hamiltonian density \(h\). To display this expansion, we consider the Faà di Bruno iterates of \(h^{(0)} = 1\) at the point \(h\), defined by

\[
h^{(j+1)} = (\partial_x + h) \cdot h^{(j)}. \quad (4.4)
\]

The linear space spanned by them (over \(C^\infty\)-functions) is denoted by \(H_+\). Since

\[
h^{(2)} = h_x + h^2, \quad (4.5)
\]

we can write the Riccati equation (3.8) in the form

\[
z^2 = h^{(2)} - uh^{(0)}, \quad (4.6)
\]

showing that \(z^2 \in H_+\). Applying the operators \((\partial_x + h)^j\) to both sides of this equation, one shows that \(z^2(H_+) \subset H_+\). In particular, \(z^{2j} \cdot 1 \in H_+\) and \(z^{2j} \cdot h \in H_+\) for \(j \geq 0\). Then equation (4.2) means that the currents \(H^{(2j+1)}\) belong to \(H_+\). The same is trivially true for \(H^{(2)}\). Therefore we conclude that all the currents \(H^{(j)}\) are Faà di Bruno polynomials

\[
H^{(j)} = \sum_{l=0}^{j} c^j_l h^{(l)} \quad (4.7)
\]

with coefficients \(c^j_l\) independent of \(z\). For any \(j \geq 0\) there is a unique choice of these coefficients leading to a “degree” \(j\) Faà di Bruno polynomial with the asymptotic expansion (4.3) in power of \(z\): the currents \(H^{(j)}\) are (with alternating signs) the
principal minors of the infinite matrix

\[
\mathbb{H} = \begin{bmatrix}
    h^{(0)} & h^{(1)} & h^{(2)} & h^{(3)} & \ldots \\
    \text{res} \frac{h^{(0)}}{z} & \text{res} \frac{h^{(1)}}{z} & \text{res} \frac{h^{(2)}}{z} & \text{res} \frac{h^{(3)}}{z} & \ldots \\
    0 & \text{res} \frac{h^{(1)}}{z^2} & \text{res} \frac{h^{(2)}}{z^2} & \text{res} \frac{h^{(3)}}{z^2} & \ldots \\
    0 & \text{res} \frac{h^{(2)}}{z^3} & \text{res} \frac{h^{(3)}}{z^3} & \text{res} \frac{h^{(4)}}{z^3} & \ldots \\
    \ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}.
\]  

(4.8)

Summarizing:

Proposition 4.1 The current densities of the KdV theory are the principal minors of the matrix (4.8), i.e., they are the unique Faà di Bruno polynomials (4.7) having the asymptotic expansion (4.3) in powers of \( z \).

The advantage of this definition with respect to the one of Proposition (3.2) is that it does no longer require that the Laurent series

\[
h(z) = z + \sum_{l \geq 1} h_l z^{-l}
\]

be a solution of the Riccati equation (3.8). We have already encoded the Hamiltonian origin of the currents \( H^{(j)} \) into the Faà di Bruno expansion (4.7). We can thus forget the Poisson pencil (3.2) and the associated Riccati system, and retain simply the property stated in Proposition 4.1. Henceforth, we shall regard it as the definition of the currents \( H^{(j)} \) associated with any monic Laurent series (4.9). This allows to extend equations (3.22) to these series.

Definition 4.2 The KP equations are the equations

\[
\frac{\partial}{\partial t_j} h = \partial_x H^{(j)}
\]

on an arbitrary monic Laurent series (4.3), where the currents \( H^{(j)} \) are the Faà di Bruno polynomials considered in Proposition 4.1.

After a suitable change of variables [4], this definition reproduces the standard one, usually written in the language of pseudodifferential operators (see, e.g., [8, 9]). Let us briefly explain this relation. First of all, we consider the negative Faà di Bruno iterates, obtained by solving backwards the recursion relations

\[
h^{(j+1)} = (\partial_x + h)h^{(j)}, \quad j < 0.
\]

(4.11)
The coefficients of the $h^{(j)}$ can be computed recursively, and one can easily show that $h^{(j)} = \pm\bigl(z^j + O(z^{j-1})\bigr)$. Then we develop $z$ on the basis $\{h^{(j)}\}_{j \in \mathbb{Z}}$:

$$z = h - \sum_{j \geq 1} q_j h^{(-j)}.$$  \hfill (4.12)

This gives an invertible relation between the coefficients $h_i$ of $h$ and the $q_i$. For instance, the first relations are

$$q_1 = h_1, \quad q_2 = h_2, \quad q_3 = h_3 + h_1^2,$$

$$q_4 = h_4 + 3h_1h_2 - h_1h_{1x}.$$  \hfill (4.13)

Finally, we introduce the pseudodifferential operator

$$Q = \partial - \sum_{j \geq 1} q_j \partial^{-j}.$$  \hfill (4.14)

One can show \cite{4} that the KP equations (4.10) on $h$ entail the Lax equations on $Q$,

$$\frac{\partial Q}{\partial t_j} = [Q, (Q^j)_{+}],$$  \hfill (4.15)

where $(Q^j)_{+}$ is the purely differential part of the $j$–th power of $Q$.

The transformation (4.12) can be usefully compared with the change of representation in the classical treatment of the equations of motion of the Euler top. If we use the space representation we simply write the conservation law of the angular momentum as

$$\frac{dL}{dt} = 0.$$  \hfill (4.16)

If we use the body representation, we write the Euler equation

$$\frac{dL}{dt} = [L, \Omega].$$  \hfill (4.17)

The same happens in the KP theory. When we pass from the Hamiltonian representation (4.10) to the Lax representation (4.15) we are performing the analog of the passage from the space representation to the body representation of classical mechanics.

We will not discuss the KP equations any longer, but rather consider them as an intermediate step towards the main topic of this paper, i.e., the analysis of the equations on the currents $H^{(j)}$ themselves.
5 The Central System

In this section we shall see the Riccati equations appear again in a disguised form. They arise here from the study of the time evolution of the currents \( H^{(j)} \). We interpret the KP equations (4.10) as the commutativity conditions of the operators \( \partial_x + h \) and \( \partial_{t_j} + H^{(j)} \):

\[
\left[ \partial_x + h, \partial_{t_j} + H^{(j)} \right] = 0. \tag{5.1}
\]

Since \( H^{(j)} \in H_+ \) and \( H_+ \) is invariant with respect to the operator \( \partial_x + h \), we see that \( H_+ \) is invariant also with respect to the operators \( \partial_{t_j} + H^{(j)} \),

\[
\left( \frac{\partial}{\partial t_j} + H^{(j)} \right) (H_+) \subset H_+, \tag{5.2}
\]

as shown by the following simple argument:

\[
\left( \frac{\partial}{\partial t_j} + H^{(j)} \right) h^{(k)} = \left( \frac{\partial}{\partial t_j} + H^{(j)} \right) (\partial_x + h)^k 1 = (\partial_x + h)^k H^{(j)} \in H_+. \tag{5.3}
\]

Let us now remark that the sequence \( \{H^{(j)}\}_{j \geq 0} \) is, in its turn, a basis in \( H_+ \). Then the previous invariance condition implies that there exist coefficients \( \gamma^j_l \) (independent of \( z \)) such that

\[
\frac{\partial H^{(k)}}{\partial t_j} + H^{(j)} H^{(k)} = \sum_{l=0}^{j+k} \gamma^j_l (h) H^{(l)}. \tag{5.4}
\]

They can be easily identified by comparing the expansion of both sides of this equation in powers of \( z \).

**Proposition 5.1** Along the trajectories of the KP hierarchy, the current densities \( H^{(k)} \) obey the equations

\[
\frac{\partial H^{(k)}}{\partial t_j} + H^{(j)} H^{(k)} = H^{(j+k)} + \sum_{l=1}^{k} H^j_l H^{(k-l)} + \sum_{l=1}^{j} H^k_l H^{(j-l)}, \tag{5.5}
\]

which we call the Central System (CS) associated with the KP theory.

In these equations the Hamiltonian density \( h(z) \) plays no special role. Hence we can forget the Faà di Bruno rule to construct the polynomials \( H^{(j)} \), and we can look at them simply as a collection \( \{H^{(j)}\}_{j \in \mathbb{N}} \) of Laurent series,

\[
H^{(j)} = z^j + \sum_{l \geq 1} H^j_l z^{-l}. \tag{5.6}
\]
with independent coefficients \( H_l^j \). From this point of view, the Central System, written in the componentwise form

\[
\frac{\partial H_m^k}{\partial t_j} + H_{m+k}^j + H_{j+m}^k + \sum_{l=1}^{m-1} H_l^j H_{m-l}^{j+k} + \sum_{l=1}^{j-1} H_l^k H_{m-l}^{j-k} + \sum_{l=1}^{k-1} H_l^j H_{m-l}^{k-l},
\]

is manifestly a system of ordinary differential equations of Riccati type for the new variables \( H_l^j \).

Through our two–steps process from the KdV equations to the Central System, we eventually passed from a hierarchy of partial differential equations to a dynamical system. This result, which is originally due to Sato, admits an interesting geometrical interpretation [3], which we briefly recall. The idea is that the KP and KdV equations are “reduction” of the Central System, and the problem is to understand this reduction process. We have to recall that the vector fields of the Central System pairwise commute. Then we are allowed to perform two kind of “reductions”. The first is a restriction to the submanifold of singular points of any vector field of the system. The second is a projection onto the orbit space of any vector field of the system along its trajectories. The two processes commute. In [3] we have shown that KP can be obtained from CS as the projection along the trajectories of the first vector field of CS. It is this process which converts the original family of ordinary differential equations into a hierarchy of partial differential equations. The projection is defined by the Fa`a di Bruno condition

\[
H^{(k)} = \sum_{l=0}^{k} c^k_l h^{(l)}. \tag{5.8}
\]

On the other hand, as it is well known, KdV is a restriction of KP on the manifold of singular points of the second vector field of the hierarchy. The restriction is defined by the Riccati equation

\[
h_x + h^2 = u + z^2. \tag{5.9}
\]

Therefore the passage from CS to KdV is a combined process, involving both a projection and a restriction. This gives the geometrical meaning of the Fa`a di Bruno condition and of the Riccati equation. Of course this is only the simplest example of such a kind of procedure. Other examples are the so–called fractional KdV hierarchies [11], see [3].

**Remark 5.2** We end this section on the Central System with some cursory remarks on its relation with the theory of the \( \tau \)–function and of the Baker–Akhiezer function \( \psi \) (see, e.g., [3, 8, 13, 18, 21, 22]).
The link with the Baker–Akhiezer function $\psi$ and the alternative forms of the KP equations suggested by Sato and Sato in the seminal paper [21] rests on the following argument. A glance at the Central System shows the symmetry conditions

$$\frac{\partial H^{(j)}}{\partial t_k} = \frac{\partial H^{(k)}}{\partial t_j}. \quad (5.10)$$

Therefore, there exists a function $\psi$ such that

$$\frac{\partial \psi}{\partial t_j} = H^{(j)} \psi. \quad (5.11)$$

With a suitable normalization this is a Baker–Akhiezer function. Moreover, by the same conditions, the differential operators

$$D_j := \frac{\partial}{\partial t_j} + H^{(j)} \quad (5.12)$$

commute. By acting recursively with these operators on the lowest order current $H^{(0)} = 1$ one obtains vectors $D_{j_1} \cdots D_{j_k}(H^{(0)})$ belonging to $H_+$ thanks to the invariance condition (5.2). One can see that they satisfy remarkable constraints. For instance, we have, for $a, b \in \mathbb{C}$,

$$(aD_2 + bD_1^2)(1) = aH^{(2)} + b(H^{(2)} + 2H_1^{(1)}H^{(0)}) = (a + b)H^{(2)} + 2bH_1^{(1)}H^{(0)},$$

so that setting $a = -b$ we can force the above linear combination to be a multiple of $H^{(0)}$. Analogously

$$(aD_3 + bD_1D_2 + cD_1^3)(1) = (a + b + c)H^{(3)} + (b + 3c)H_1^{(1)}H^{(1)} + [(b - 3c)H_2^{(1)} + (b + 3c)H_1^{(2)}]H^{(0)}$$

is independent of $z$ for $a = 2c$ and $b = -3c$. In general it can be proved that the vectors $D_{j_1} \cdots D_{j_k}(1)$ verify the constraints

$$p_k(-\frac{1}{l}D_l)(1) = H_{k-1}^l, \quad (5.13)$$

where the Schur polynomials $p_k(t_1, t_2, \ldots)$ are as usual defined via the relation

$$\exp\left(\sum_{i \geq 1} t_i z^i\right) = \sum_{k \geq 0} p_k(t_l) z^k. \quad (5.14)$$

Then the simple identity

$$\psi \cdot p_k(-\frac{1}{l}D_l)(1) = p_k(-\frac{1}{l} \frac{\partial}{\partial t_l})(\psi) \quad (5.15)$$
allows us to recover the Sato constraints

\[ p_k(-\frac{1}{l} \frac{\partial}{\partial t_l})(\psi) = H_{k-1}^1 \cdot \psi \]  

(5.16)
presented in [21].

The link with the \( \tau \)-function is provided by the second Hamiltonian function \( H'_\lambda \) discussed in Section 3. It is preserved along the flows of the KP hierarchy too, and therefore its Hamiltonian density \( h' \) verifies local conservation laws of the form

\[ \frac{\partial h'}{\partial t_j} = \partial_x H'_j, \]  

(5.17)
which we call dual KP equations. Setting

\[ H(l) = z^{l-1} - \sum_{k \geq 1} H_k^j z^{-(k+1)}, \]  

(5.18)
one finds [3] that the dual currents \( H'_j \) are given by

\[ H'_j = \sum_{l=1}^{j} H(l) H^{(j-l)}. \]  

(5.19)
This formula allows us to compute the coefficients \( H'_{jk} \) of the expansion of \( H'_j \) in powers of \( z \),

\[ H'_j = jz^{j-1} - \sum_{l \geq 1} H'_{jl} z^{-(l+1)}, \]  

(5.20)
as quadratic polynomials of the coefficients \( H_k^i \) of the primal currents \( H^i \). By using this representation one can show the symmetry property

\[ H'_{jk} = H'_{kj}. \]  

(5.21)
Furthermore, as functions of the times \((t_1, t_2, \ldots)\), they verify the differential conditions

\[ \frac{\partial H'_{jk}}{\partial t_l} = \frac{\partial H'_{lk}}{\partial t_j}. \]  

(5.22)
Therefore, there exists a function \( \tau(t_1, t_2, \ldots) \) independent of \( z \) such that

\[ H'_{jk} = \frac{\partial^2}{\partial t_j \partial t_k} \log \tau. \]  

(5.23)
This function is the Hirota \( \tau \)-function associated with the Central System. As it is well known, by introducing this function it is possible to set the KP hierarchy in the form of Hirota bilinear equations. However, it is also possible to set directly the equations of the Central System in the form of a linear system by means of a suitable transformation, to be discussed in the next sections.
6 Darboux Coverings and the Central System

To linearize the Central System we shall use the method of Darboux coverings [13]. The hallmark of such a rather “unconventional” approach to the classical subject of Darboux maps and symmetries is quite simple. In a first instance, one replaces the search for a transformation between two vector fields $X$ and $Z$ defined on the manifolds $\mathcal{M}$ and $\mathcal{P}$ respectively, by that of a third vector field $Y$ (defined in general on a bigger manifold $\mathcal{N}$) separately related to $X$ and $Z$ by two maps $\pi : \mathcal{N} \to \mathcal{M}$ and $\sigma : \mathcal{N} \to \mathcal{P}$:

$$X = \pi_*(Y), \quad Z = \sigma_*(Y). \quad (6.1)$$

By definition, integral curves of $Y$ are mapped to integral curves of $X$ by $\pi$, and to integral curves of $Y$ by $\sigma$. We say for short that $Y$ intertwines $X$ with $Z$. Moreover, we say that $Y$ is a Darboux covering of $X$ if $\mathcal{N}$ is a fiber bundle over $\mathcal{M}$ and $\pi$ is the canonical projection. In this case, any section $\rho$ of $\pi$, invariant under $Y$, allows us to define a (Miura) map $\mu : \mathcal{M} \to \mathcal{P}$ relating directly the vector fields $X$ and $Z$. In pictures:

$$\begin{array}{cccc}
Y & \xrightarrow{\sigma_*} & \xrightarrow{\pi_*} & Y \\
\sigma_* & \xleftarrow{\rho_*} & \xleftarrow{\mu_*} & \sigma_* \\
Z & X & Z & X
\end{array}$$

In the present instance, $\mathcal{M}$ is the space of sequences of Laurent series $\{W^{(k)}\}_{k \geq 0}$ of the form

$$W^{(k)} = z^k + \sum_{l \geq 1} W^k_l z^{-l}. \quad (6.2)$$

This space is a natural parameter space for the big cell in the Sato Grassmannian [21, 22]. The manifold $\mathcal{P}$ is a second copy of $\mathcal{M}$, formed by sequences $\{H^{(k)}\}_{k \geq 0}$ of the form

$$H^{(k)} = z^k + \sum_{l \geq 1} H^k_l z^{-l}. \quad (6.3)$$

Since the sequences $W^{(k)}$ and $H^{(k)}$ will play different roles in the sequel, it is convenient to regard the spaces $\mathcal{M}$ and $\mathcal{P}$ as distinct. Finally, the manifold $\mathcal{N}$ is the Cartesian product $\mathcal{M} \times \mathcal{G}$ of $\mathcal{M}$ by the group $\mathcal{G}$ of invertible Laurent series of the form

$$w = 1 + \sum_{l \geq 1} w_l z^{-l}. \quad (6.4)$$
The vector field $Z$ on $\mathcal{P}$ is any vector field of the Central System (5.5). We recall that it is completely characterized by the property

$$\left( \frac{\partial}{\partial t_j} + H^{(j)} \right) H_+ \subset H_+, \quad (6.5)$$

where $H_+$ is the linear span of the Laurent series $\{H^{(k)}\}_{k \geq 0}$. The vector field $X$ on $\mathcal{M}$ is analogously characterized by the property

$$\left( \frac{\partial}{\partial t_j} + z^j \right) W_+ \subset W_+, \quad (6.6)$$

where $W_+$ is the linear span of the Laurent series $\{W^{(k)}\}_{k \geq 0}$. By comparing the expansions of both sides of equation (6.6) in powers of $z$, it is easily seen that the equations defining $X$ are

$$\frac{\partial}{\partial t_j} W^{(k)} + z^j W^{(k)} = W^{(j+k)} + \sum_{l=1}^{j} W^k_l W^{(j-l)}. \quad (6.7)$$

It can be shown [23] that these are precisely the linear flows on the Sato Grassmannian. Finally, to define the vector field $Y$ on $\mathcal{N}$ we impose the further invariance condition

$$\left( \frac{\partial}{\partial t_j} + z^j \right) (w) \in W_+, \quad (6.8)$$

which is tantamount to defining

$$\frac{\partial}{\partial t_j} w + z^j w = \sum_{l=0}^{j} w_l W^{(j-l)}. \quad (6.9)$$

We summarize this discussion in the following

**Definition 6.1** The Central System (CS) is the family of vector fields

$$\frac{\partial H^{(k)}}{\partial t_j} + H^{(j)} H^{(k)} = H^{(j+k)} + \sum_{l=1}^{k} H^j_l H^{(k-l)} + \sum_{l=1}^{j} H^k_l H^{(j-l)}$$

on $\mathcal{P}$ uniquely characterized by the invariance condition (6.5).

The Sato System (S) is the family of vector fields

$$\frac{\partial}{\partial t_j} W^{(k)} + z^j W^{(k)} = W^{(j+k)} + \sum_{l=1}^{j} W^k_l W^{(j-l)}$$

on $\mathcal{M}$ uniquely characterized by the invariance condition (6.6).
The Darboux Sato System (DS) is the family of vector fields

\[
\frac{\partial}{\partial t_j} W^{(k)} + z^j W^{(k)} = W^{(j+k)} + \sum_{l=1}^j W^l W^{(j-l)}
\]
\[
\frac{\partial}{\partial t_j} w + z^j w = \sum_{l=0}^j w_l W^{(j-l)}
\]

on \( \mathcal{N} \) uniquely characterized by the invariance conditions (6.6) and (6.8).

To complete the geometrical scheme of Darboux covering we have yet to define the maps \( \sigma \) and \( \pi \). The map \( \pi \) is of course the canonical projection

\[
\pi(w, \{W^{(k)}\}) = \{W^{(k)}\}. \tag{6.10}
\]

The map \( \sigma \) is defined by imposing the intertwining condition

\[
w \cdot (H_+) = W_+ \tag{6.11}
\]
on the linear spans \( H_+ \) and \( W_+ \). It means that multiplying any element \( H^{(j)} \) by \( w \) we get an element of \( W_+ \). This happens if and only if

\[
w H^{(j)} = \sum_{l=0}^j w_{j-l} W^{(l)} \quad \forall j \geq 0. \tag{6.12}
\]

**Definition 6.2** We say that the sequence \( \{H^{(k)}\}_{k \geq 0} \) is related to the sequence \( \{W^{(k)}\}_{k \geq 0} \) by the Darboux transformation generated by \( w \), and we write \( H_+ = D_w(W_+) \), if \( w \cdot H_+ = W_+ \).

We are now in a position to prove the main property of the DS equations.

**Proposition 6.3** The DS system is a Darboux covering of the Sato System, intertwining it with CS.

**Proof.** The only thing to show is that \( \sigma_*(\text{DS}) = \text{CS} \). Notice that the definitions of \( \sigma \) and DS entail

\[
\frac{\partial w}{\partial t_j} + z^j w = w H^{(j)}. \tag{6.13}
\]

This means that the operators \( \left( \frac{\partial}{\partial t_j} + z^j \right) \) and \( \left( \frac{\partial}{\partial t_j} + H^{(j)} \right) \) are intertwined by the multiplication by \( w \), i.e.,

\[
w \cdot \left( \frac{\partial}{\partial t_j} + H^{(j)} \right) = \left( \frac{\partial}{\partial t_j} + z^j \right) \cdot w. \tag{6.14}
\]
Therefore
\[ w \cdot \left( \frac{\partial}{\partial t_j} + H^{(j)} \right) (H_+) = \left( \frac{\partial}{\partial t_j} + z^j \right) (w(H_+)) \]
\[ = \left( \frac{\partial}{\partial t_j} + z^j \right) W_+ \subset W_+, \tag{6.15} \]
and consequently \( \left( \frac{\partial}{\partial t_j} + H^{(j)} \right) (H_+) \subset H_+ \), so that CS follows. □

We now exploit this result to define a Miura map relating directly S to CS. We consider in \( \mathcal{P} \) the submanifold
\[ H^{(0)} = 1, \tag{6.16} \]
which is clearly invariant under CS, and we construct its inverse image in \( \mathcal{N} \),
\[ w = W^{(0)}, \tag{6.17} \]
with respect to the Darboux map \( \sigma : \mathcal{N} \to \mathcal{P} \). The corresponding section \( \rho : \mathcal{M} \to \mathcal{P} \) is given by \( \rho(\{W^{(k)}\}) = (W^{(0)}, \{W^{(k)}\}) \).

**Proposition 6.4** The submanifold defined by equation (6.17) in \( \mathcal{N} \) is a section of \( \pi : \mathcal{N} \to \mathcal{M} \) which is invariant under DS.

**Proof.** We have to compare the DS equations for the pair \((W^{(0)}, w)\). They are:
\[
\begin{cases}
\frac{\partial}{\partial t_j} w = -z^j w + \sum_{l=0}^{j} w_l W^{(j-l)} \\
\frac{\partial}{\partial t_j} W^{(0)} = -z^j W^{(0)} + \sum_{l=0}^{j} W_l^0 W^{(j-l)},
\end{cases}
\tag{6.18}
\]
since \( W_0^0 = 1 \). Hence
\[
\frac{\partial}{\partial t_j} (w - W^{(0)}) = -z^j (w - W^{(0)}) + \sum_{l=0}^{j} (w_l - W_l^0) W^{(j-l)}, \tag{6.19}
\]
proving the statement. □

Motivated by this result, we give the following

**Definition 6.5** The nonlinear map
\[
\mu = \sigma \circ \rho : \mathcal{M} \to \mathcal{P} \tag{6.20}
\]
given by
\[ H^{(j)} = \left( \sum_{l=0}^{j} W_{j-l}^0 W^{(l)} \right) / W^{(0)}. \] (6.21)

is the Miura map relating S to CS.

It enjoys the property of mapping any solution of the Sato System into a solution of CS satisfying the constraint \( H^{(0)} = 1. \)

7 Linearization of the Sato System and families of solutions

The final step is to show that the Sato System can be explicitly linearized. To this end it is useful to write it in matrix form. Notice that in components it reads
\[ \frac{\partial W^j_m}{\partial t_k} + W^j_{k+m} - W^{j+k}_m = \sum_{l=1}^{k} W^j_l W^{k-l}_m. \] (7.1)

If we consider the infinite shift matrix
\[ \Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ & & \ddots & \ddots \\ & & & \ddots & \ddots \\ & & & & \ddots \end{bmatrix} \] (7.2)

and the convolution matrix of level \( k \)
\[ \Gamma_k = \begin{bmatrix} 0 & \cdots & 1 & 0 & \cdots \\ \vdots & & 1 & 0 & \cdots & \cdots \\ 1 & 0 & \cdots & \cdots \\ \vdots & & & & \ddots \end{bmatrix} \] (7.3)
we can write (7.1) in the matrix form:

\[
\frac{\partial}{\partial t_k} \mathcal{W} + \mathcal{W} \cdot T \Lambda^k - \Lambda^k \cdot \mathcal{W} = \mathcal{W} \Gamma_k \mathcal{W}.
\] (7.4)

This equation belongs to a well-known class of linearizable matrix Riccati equations [20].

**Proposition 7.1** The infinite matrix \( \mathcal{W} \) is a solution of the matrix Riccati equation (7.4) if and only if it has the form \( \mathcal{W} = V \cdot U - U^{-1} \), with \( U \) and \( V \) satisfying the constant coefficients linear system

\[
\begin{cases}
\frac{\partial}{\partial t_k} U &= T \Lambda^k U - \Gamma_k V \\
\frac{\partial}{\partial t_k} V &= \Lambda^k V
\end{cases}
\] (7.5)

**Proof.** We only have to check that *every* solution of equation (7.4) can be obtained from (7.5). Let \( \mathcal{W} \) be such a solution, and let

\[
\begin{cases}
\mathcal{V} &= \exp(\sum_{k \geq 1} t_k \Lambda^k) \\
\mathcal{U} &= \mathcal{W}^{-1} \mathcal{V}
\end{cases}
\] (7.6)

Then \((\mathcal{U}, \mathcal{V})\) is a solution of (7.5).

\[\square\]

Since \( U \) and \( V \) are infinite matrices, in general one cannot explicitly solve the linear system (7.3), and this procedure would imply the discussion of suitable notions of convergence for formal series in infinite variables. Nevertheless, a lot of solutions can be constructed as follows\(^1\). Let us notice that the constraints \( W_{ij} = 0 \forall i > n, j > m \), is compatible with equations (7.4). In other words, the space \( \mathbb{W}_{m,n} \) of matrices \( \mathcal{W} \) which have zero entries outside the first \( m \) rows and the first \( n \) columns is invariant for the Sato System. If we denote with \( \mathcal{M}_{m,n} \) the \( m \times n \) matrix obtained by the infinite matrix \( \mathcal{M} \) by taking its \( m \times n \) upper corner, then for the matrices \( \mathcal{W} \in \mathbb{W}_{m,n} \) equations (7.4) can be written as

\[
\frac{\partial}{\partial t_k} \mathcal{W}_{m,n} + \mathcal{W}_{m,n} \cdot (T \Lambda_{n,n})^k - (\Lambda_{m,m})^k \cdot \mathcal{W}_{m,n} = \mathcal{W}_{m,n} (\Gamma_k)_{n,m} \mathcal{W}_{m,n}.
\] (7.7)

These are matrix Riccati equations for finite matrices. They can be linearized as in Proposition 7.1, and explicitly solved. Their solutions depend only on \( \{t_k\}_{k=1,\ldots,m+n-1} \), and should be compared with the Hirota polynomial solutions of the KP hierarchy.

\(^1\)This is part of a joint work with J.P. Zubelli, which will appear in a forthcoming paper.
8 An explicit example

To make more concrete the discussion about the finite rank solutions, we present some explicit computations for the case of $W_{3,2}$. To avoid clumsy notations, let us redefine the matrix coefficients appearing in (7.7) as follows:

$$W_{3,2} = W, \quad \Lambda_{3,3} = A, \quad ^T \Lambda_{2,2} = B, \quad (\Gamma_k)_{2,3} = C_k, \quad (8.1)$$

for $k = 1, \ldots, 4$. Hence, the only non–vanishing coefficients are:

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^2, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (8.2)$$

This shows that the Sato System on $W_{3,2}$ can be seen as a system of four Riccati–type ordinary differential equations in $C^6$. According to the recipe of Proposition 7.1, we set $W = VU^{-1}$, where $V$ is a $3 \times 2$ matrix, and $U$ is a non–singular matrix of rank 2. We study the Cauchy problems

$$\begin{cases} \frac{\partial}{\partial t_k} V = A^k V \\ V(0) = W(0) \end{cases}, \quad \begin{cases} \frac{\partial}{\partial t_k} U = B^k U - C_k V \\ U(0) = I \end{cases} \quad (8.3)$$

We first consider the equation for $V$. Since $A$ is nilpotent, the solution is the polynomial

$$V(t) = \exp(\sum_{l=1}^2 t_l A^l) W(0). \quad (8.4)$$

Now we recall the definition of the Schur polynomials $p_l(t_1, t_2, \ldots)$ given in Equation (5.14) and of their “adjoint” ones,

$$\exp(-\sum_{i=1}^\infty t_i z^i) = \sum_{l=0}^\infty \tilde{p}_l(t_1, t_2, \ldots) z^l. \quad (8.5)$$
Thus we can rewrite (8.4) as

\[
V(t) = \sum_{l=0}^{2} p_l(t)A^lW(0) = \begin{bmatrix} 1 & p_1 & p_2 \\ 0 & 1 & p_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot W(0),
\]

As far as the second set of equations in (8.3) is concerned, we put

\[
U = \exp(t_1 B)U_0(t),
\]

so that these equations can be rewritten as

\[
\frac{\partial}{\partial t_k} U_0 = -(\sum_{l=0}^{1} \tilde{p}_l B^l) \cdot C_k(\sum_{j=0}^{2} p_j A^j) \cdot W(0),
\]

with \( U_0(0) = I \). The Cauchy problems with initial data \( U_0(0) = I \) can thus be easily solved; we get

\[
U_0(t) = I - \begin{bmatrix} t_1 & t_2 + \frac{1}{2}t_1^2 & t_3 + t_1t_2 + \frac{1}{6}t_1^3 \\ t_2 - \frac{1}{2}t_1^2 & t_3 - \frac{1}{3}t_1^3 & t_4 + \frac{1}{2}t_2^2 - \frac{1}{2}t_1^2t_2 - \frac{1}{8}t_1^4 \end{bmatrix} \cdot W(0).
\]

In particular, for

\[
W(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},
\]

we obtain

\[
W(t) = V(t)U(t)^{-1} = \tau^{-1} \begin{bmatrix} -t_1t_2 - \frac{1}{2}t_1^3 & t_2 + \frac{1}{2}t_1^2 \\ -t_1^2 & t_1 \\ -t_1 & 1 \end{bmatrix},
\]

where \( \tau = \det U(t) = 1 - t_4 - \frac{1}{2}t_2^2 + \frac{1}{2}t_1^2t_2 + \frac{1}{3}t_1^4 \). Therefore the corresponding solution of \( S \) is

\[
W^{(0)} = 1 + \tau^{-1} \left[ -(t_1t_2 + \frac{1}{2}t_1^3)z^{-1} + (t_2 + \frac{1}{2}t_1^2)z^{-2} \right]
\]

\[
W^{(1)} = z + \tau^{-1} \left[ -t_1^2z^{-1} + t_1z^{-2} \right]
\]

\[
W^{(2)} = z^2 + \tau^{-1} \left[ -t_1z^{-1} + z^{-2} \right]
\]

\[
W^{(k)} = z^k \quad \text{for} \quad k \geq 3.
\]
Using the Miura map (6.21) we obtain a solution \( \{H^{(j)}\} \) of the Central System such that \( H^{(k)} = z^k \) for \( k = 0 \) and \( k \geq 5 \). For example, \( H^{(1)} \) is given by

\[
W^{(0)} H^{(1)} = W^{(1)} + W^{(0)} W^{(0)}.
\]  

(8.10)

This means that the coefficients \( H^1_k \) can be computed using the recursion relations

\[
H^1_1 = W^1_1 + (W^0_1)^2 - W^0_2
\]

\[
H^1_2 = W^1_2 + 2W^0_1 W^0_2 - W^0_1 W^1_1 - (W^0_1)^3
\]

\[
H^1_{j+2} = -(H^1_{j+1} W^0_1 + H^1_j W^0_2) \quad \text{for } j \geq 1.
\]

In a more compact form,

\[
H^{(1)} = (W^{(1)} + W^0_1 W^{(0)}) / W^{(0)}
\]

\[
= -\tau^{-1} (t_1 t_2 + \frac{1}{2} t_1^2 z^{-1}) + \frac{z + \tau^{-1} [-t_1^2 z^{-1} + t_1 z^{-2}]}{1 + \tau^{-1} [- (t_1 t_2 + \frac{1}{2} t_1^3) z^{-1} + (t_2 + \frac{1}{2} t_2^2) z^{-2}]}.
\]

As we have seen in Section 3, \( h = H^{(1)} \) is a solution of the KP equations after putting \( t_1 = x \).

9 Summary

In this paper we have tried to give an overview of the bihamiltonian approach to the KP theory, starting from the primitive idea of Poisson pencil to arrive to the polynomial solutions of these equations. The approach consists of two parts, dealing with the equations and with their solutions respectively. In the first part we have traced the way from KdV to CS (through a double process of extension), and backwards from CS to KdV (through a projection and a restriction). In the second part we have shown how to use the technique of Darboux coverings to linearize the equations and, therefore, to construct explicit solutions. We hope that, by providing an alternative view of the theory, the present paper may clarify the logical structure of the Hamiltonian approach to the KP theory.

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