SCALAR CURVATURE, MEAN CURVATURE AND HARMONIC MAPS TO THE CIRCLE

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Abstract. We study harmonic maps from a 3-manifold with boundary to $S^1$ and prove a special case of Gromov dihedral rigidity of three dimensional cubes whose dihedral angles are $\pi/2$. Furthermore we give some applications to mapping torus hyperbolic 3-manifolds.

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1. Introduction

Stern [Ste20] showed an interesting formula relating the level set of harmonic functions and the scalar curvature of a 3-manifold. With Bray [BS19], they generalized the result to 3-manifold with boundary where a harmonic 1-form with vanishing normal component along the boundary was studied. Different from [BS19], we send the boundary to a point i.e. we study a Dirichlet boundary condition. We obtain a similar formula, and then we combine the technique with [Ste20, BS19] to study the dihedral rigidity of standard cubes in $\mathbb{R}^3$.

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Theorem 1.1. For a harmonic map $u : N^3 \to S^1$ with a Dirichlet condition $u|_{\partial N} = [0] \in S^1$ and for almost all $\theta \in S^1$ the level set $\Sigma_{\theta} = u^{-1}(\theta)$ being closed, we have the inequality

$$
\int_N \frac{1}{2} \left| \text{Hess} u \right|^2 + R_N |du|^2 + \int_{\partial N} H_{\partial N} |du| \leq 2\pi \int_{S^1} \chi(\Sigma_{\theta}),
$$

where $R_N$ is the scalar curvature of $N$ and $H_{\partial N}$ is the mean curvature of $\partial N$.

Remark 1.2. The first author learned from Pengzi Miao that the theorem appeared in general form [HMT20, Proposition 2.2] for harmonic functions.

This result and the one in [BS19] suggest that it is worthwhile to consider harmonic maps to $S^1$ with mixed boundary conditions. We study the problem on three dimensional cubes. We identify $S^1$ with $\mathbb{R}/\mathbb{Z}$ and use $[r], r \in \mathbb{R}$ to denote an element of $S^1$. Let $(Q^3, g)$ be a Riemannian manifold diffeomorphic to a standard unit cube in $\mathbb{R}^3$, $T$ and $B$ be respectively the top and bottom face, $F$ be the union of side faces, $\nu$ be the outward unit normal to each face. The angle $\gamma$ formed by neighboring two unit normals is called exterior dihedral angle and $\pi - \gamma$ is called the dihedral angle between two neighboring faces. We assume that the dihedral angles are everywhere equal to $\pi/2$. For the general case, we can use the bending construction of Gromov ([Gro18], [Li20b]) to reduce to the case where dihedral angles are everywhere equal to $\pi/2$.

Theorem 1.3 ([Li20a], [Li20b]). Unless $(Q^3, g)$ is isometric (up to constant multiple of the metric) to the standard Euclidean cube, the following three conditions cannot be satisfied at the same time on the cube $(Q^3, g)$:

1. The scalar curvature of $Q^3$ is nonnegative;
2. Every face of $Q^3$ is mean convex;
3. The dihedral angle is equal to $\pi/2$ everywhere along each edge.

We assume that the metric is $C^{2,\alpha}$ for some $\alpha \in (0, 1)$.

This theorem was conjectured as the dihedral rigidity by Gromov [Gro14] and was verified by Li [Li20a], [Li20b] using free boundary minimal surfaces. We solve the Laplace equation with Dirichlet condition on bottom and top faces, and with Neumann boundary condition along side faces, then we apply the level set method of Stern. We address the regularity of the problem in the Appendix.

We mention two interesting questions. For the cone type polyhedron, it seems desirable to find a boundary condition which is analogue to the capillary condition of the minimal surface. The second question is about the spacetime version of the problem. Recently [BKKS19] proved the Riemannian positive mass theorem using harmonic coordinate method, which later was generalized to spacetime by [HKK20].

Recall that a function $u \in C^2(Q)$ is called spacetime harmonic if

$$
\Delta_s u + \text{tr}_g k|\nabla u| = 0 \text{ in } Q,
$$

where $k$ is a prescribed symmetric 2-tensor. The natural boundary condition is $u = 0$ along bottom and top faces of $Q$ and $\frac{\partial u}{\partial n} = 0$ along side faces. A preliminary calculation shows that with these mixed boundary conditions it will give a spacetime version of the [Li20a] if we can control the level set topology and assuming corresponding convexity conditions.

In the last section, we give one application to a hyperbolic 3-manifold, a mapping torus by a pseudo-Anosov map. Usually, people use Kleinian group theory to attack
such a problem, but we use a simple differential geometric approach of harmonic maps.

**Theorem 1.4.** Let \( M_\phi \) be a mapping torus of a closed surface \( S \) of genus \( g \geq 2 \) via a pseudo-Anosov map \( \phi \). Then

\[
g \leq \frac{3}{4C\pi \|\phi\|} \text{vol}(M_\phi) + 1,
\]

where \( \|\phi\| \) is the translation length of a hyperbolic isometry defined by \( \phi \) and \( C \) is a constant depending only on \( S \) and the injectivity radius of \( M_\phi \).

It seems there would be more applications to 3-manifold topology as in [BS19], which should be explored more in a near future.

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2. **Formula for Laplacian of energy of harmonic map to the circle**

In this section we collect some basic formulas for Laplacian of energy of harmonic map to the circle. The reference is [Ste20]. Let \( u : N \to S^1 \) be a harmonic map from a closed Riemannian 3-manifold to the unit circle.

Choose an orthonormal frame \( e_1, e_2, e_3 \) adapted to \( \Sigma = u^{-1}(\theta) \), so that \( e_1, e_2 \) are tangential to \( \Sigma_\theta \), and \( e_3 = \frac{\nabla u}{|\nabla u|} \). Let \( R_{ij} \) denote the sectional curvature of \( N \) for the section \( e_i \wedge e_j \). The symmetric quadratic tensor \( (h_{ij} = \langle D_{e_i} e_3, e_j \rangle) \) is the second fundamental form \( k_{\Sigma_\theta} \) for \( \Sigma_\theta \). Note that \( k_{\Sigma_\theta} = (|\nabla u|^{-1} Dd \sigma |_{\Sigma_\theta}) \).

Then Gauss equation gives

\[
K = R_{12} + h_{11} h_{22} - h_{12}^2,
\]

and the scalar curvature \( R_N \) of \( N \) is

\[
R_N = 2(R_{12} + R_{13} + R_{23})
\]

and

\[
\text{Ric}(e_3, e_3) = R_{13} + R_{23}.
\]

The mean curvature \( H_{\Sigma_\theta} = \text{tr} k_{\Sigma_\theta} \) is \( h_{11} + h_{22} \) and the scalar curvature \( R_{\Sigma_\theta} \) is \( 2K \).

Hence

\[
\text{Ric}(e_3, e_3) = \frac{1}{2}(R_N - R_{\Sigma_\theta} + H_{\Sigma_\theta}^2 - |k_{\Sigma_\theta}|^2).
\]

Using harmonicity of \( u \), one can verify that

\[
|\nabla u|^2 (H_{\Sigma_\theta}^2 - |k_{\Sigma_\theta}|^2) = 2|d|h|^2 - |Dd|^2,
\]

and one can rewrite

\[
\text{Ric}(\nabla u, \nabla u) = |\nabla u|^2 \text{Ric}(e_3, e_3)
\]

\[
-\frac{1}{2} |\nabla u|^2 (R_{\Sigma_\theta} - R_{\Sigma_\theta}) + \frac{1}{2} (2|d|n u|^2 - |Dd u|^2).
\]

Then using the standard Bochner identity for \( du \)

\[
\Delta_g \frac{1}{2} |\nabla u|^2 = |Dd u|^2 + \text{Ric}(du, du),
\]
one can deduce the formula as in \cite{Ste20}

\begin{equation}
2 \int_N \frac{|du|}{2} R_{\Sigma_\theta} = 4\pi \int_\theta \chi(\Sigma_\theta) d\theta \geq \int_N R_N |du|,
\end{equation}

if \(N\) is closed where the first equality follows from the coarea formula and Gauss-Bonnet theorem. Let \(\varphi_\delta = \sqrt{|h|^2 + \delta}\) with \(\delta > 0\), it follows that

\[\Delta \varphi_\delta = \frac{1}{\varphi_\delta} \left[ \frac{1}{2} \Delta |h|^2 - \frac{|h|^2}{\varphi_\delta^2} |d|h|^2 \right] \geq \frac{1}{\varphi_\delta} \left[ |Dh|^2 - |d|h|^2 + \text{Ric}(h, h) \right].\]

Inserting (2.1) and Bochner formula (2.2) into the above, we have that along regular level sets \(\Sigma_\theta\) of \(u\),

\begin{equation}
\Delta_g \varphi_\delta \geq \frac{1}{2\varphi_\delta} \left[ \text{Hess}u^2 + |du|^2 (R_N - R_{\Sigma_\theta}) \right].
\end{equation}

3. Application to rigidity of scalar curvature and mean curvature

Let \(N\) be a 3-manifold with boundary \(\partial N \neq \emptyset\). We require that \(H_2(N; \mathbb{Z})\) is nontrivial. Let \(\alpha\) be a nontrivial element of \(H_2(N; \mathbb{Z})\), then according to Poincaré-Lefschetz duality that \(H^1(N, \partial N; \mathbb{Z})\) is isomorphic to \(H_2(N, \mathbb{Z})\). Let \([\tilde{u}^* (d\theta)]\) be the homotopy classes of maps from \(N\) to \(S^1\) sending the boundary \(\partial N\) to a point. Hence \(\alpha\) determines a nontrivial homotopy class in \([\tilde{u}] \in [N, \partial N : S^1]\). We minimize the energy in this homotopy class, we obtain a harmonic map \(u \in [\tilde{u}]\) satisfying the conditions in Theorem 1.1. The Hodge-Morrey theory \cite[Chapter 5]{GMS98} applied to the relative cohomology class \([\tilde{u}^* (d\theta)] \in H^1(N, \partial N; \mathbb{Z})\) yields an energy minimizing representative \(u : N \to S^1\).

**Proof of Theorem 1.1.** Main computation was done already in \cite{Ste20}. Note that every level set \(u^{-1}(\theta)\) does not intersect the boundary \(\partial N\) except at the level \([0] \in S^1\). Let \(h = u^* (d\theta)\) be gradient 1-form, so \(h\) is harmonic. Let \(\varphi_\delta = \sqrt{|h|^2 + \delta}\) with \(\delta > 0\). From \cite[(14)]{Ste20}, we have that along regular level sets \(\Sigma\) of \(u\),

\[\Delta_g \varphi_\delta \geq \frac{1}{2\varphi_\delta} \left[ \text{Hess}u^2 + |du|^2 (R_N - R_{\Sigma_\theta}) \right].\]

Let \(A \subset S^1\) be an open set containing the critical values of \(u\) and let \(B\) be the complement subset. So \(B\) contains only regular values. We have

\begin{equation}
\int_{\partial N} \frac{\partial \varphi_\delta}{\partial \nu} = \int_{u^{-1}(A)} \Delta_g \varphi_\delta + \int_{u^{-1}(B)} \Delta_g \varphi_\delta.
\end{equation}

Since \(u\) is smooth by elliptic regularity, using local coordinates of \(S^1\), \(u\) is a harmonic function. By a direct calculation and \(u|_{\partial N} = [0]\) is constant along \(\partial N\), so

\[\Delta_{\partial N} u = 0, \quad \Delta_g u = 0 = \Delta_{\partial N} u + H_{\partial N} \langle h, \nu \rangle + \text{Hess}(u)(\nu, \nu).\]
The above equality can be deduced via the following: Let $e_i$, $\nu$ be an orthonormal frame of $N$, then
\[
\Delta u = \sum_{e_i / \partial N} \text{Hess}_N u(e_i, e_i) + \text{Hess}(\nu, \nu)
\]
\[
= \sum_{e_i / \partial N} (e_i e_i u - \nabla_{e_i}^N e_i u) + \text{Hess}(\nu, \nu)
\]
\[
= \sum_{e_i / \partial N} (e_i e_i u - \nabla_{e_i}^N e_i u) - \langle \nabla_{e_i}^N e_i, \nu \rangle \langle \nabla u, \nu \rangle + \text{Hess}(\nu, \nu)
\]
\[
= \Delta_{\partial N} u + H_{\partial N} \langle h, \nu \rangle + \text{Hess}(u)(\nu, \nu).
\]

Also, we have that
\[
\frac{\partial \varphi_\delta}{\partial \nu} = \frac{1}{\varphi_\delta} \text{Hess}(u)(h, \nu)
\]
\[
= \frac{\langle h, \nu \rangle}{\varphi_\delta} \text{Hess}(u)(\nu, \nu)
\]
\[
= \frac{|h|^2}{\varphi_\delta} H_{\partial N}.
\]

We have also globally on $N$ that (see [Ste20])
\[
\Delta g \varphi_\delta \geq -C_N |h|
\]
for some constant $C_N > 0$ depending only on $N$. We see that
\[
- \int_{u^{-1}(A)} \Delta \varphi_\delta \leq C_N \int_{u^{-1}(A)} |h| = C_N \int_A |\Sigma_\theta|,
\]
where we have applied coarea formula. So we have from (2.1), (3.1) and taking limits as $\delta \to 0$,
\[
\int_{u^{-1}(B)} \frac{|du|}{2} \left( \frac{\text{Hess}(u)^2}{|du|^2} + R_N - R_\Sigma \right)
\]
\[
\leq \lim_{\delta \to 0} \left[ \int_{\partial N} \frac{\partial \varphi_\delta}{\partial \nu} - \int_{u^{-1}(A)} \Delta g \varphi_\delta \right]
\]
\[
\leq - \int_{\partial N} H_{\partial N} |du| + C_N \int_A |\Sigma_\theta|.
\]

Rearranging and applying the coarea formula once again, we have that
\[
\int_{u^{-1}(B)} \frac{|du|}{2} \left( \frac{\text{Hess}(u)^2}{|du|^2} + R_N \right) + \int_{\partial N} H_{\partial N} |du|
\]
\[
\leq \frac{4}{\delta} \int_{\theta \in B} \int_{\Sigma_\theta} R_\Sigma + C_N \int_A |\Sigma_\theta|
\]
\[
= 2\pi \int_{\theta \in B} \chi(\Sigma_\theta) + C_N \int_A |\Sigma_\theta|.
\]

In the last line, we have used Gauss-Bonnet theorem. Since $\mathcal{H}^1(A)$, the Hausdorff measure, can be made arbitrarily small using the Sard theorem and $\theta \mapsto |\Sigma_\theta|$ is integrable over $S^1$ by coarea formula, sending $\mathcal{H}^1(A)$ to zero leads to our inequality (1.1).
Now we discuss Theorem 1.3. Let \((x_1, x_2, x_3)\) be the coordinates on \(Q = Q^3\) induced by the diffeomorphism \(\Phi\) from \([0, 1]^3\), and we identify the top face with \(\{x_3 = 1, 0 \leq x_1, x_2 \leq 1\}\) and the bottom face with \(\{x_3 = 0, 0 \leq x_1, x_2 \leq 1\}\). We consider the class of maps which are homotopic to the map \(\tilde{u} : Q \to S^1\) given by

\[\tilde{u}(x_1, x_2, x_3) = [x_3]\]

and takes the value \([0] \in S^1\) at \(x_3 = 0\) and \(x_3 = 1\). By Poincaré-Lefschetz duality,

\[H^1(Q, T \cup B; \mathbb{Z}) \cong H_2(Q, F; \mathbb{Z}).\]

One can associate the homotopy class \([\tilde{u}]\) to an element \(\alpha \in H_2(Q, F; \mathbb{Z})\) where \(\tilde{u}^{-1}(\theta)\) represents \(\alpha\).

We do minimization of the energy in this homotopy class. The Hodge-Morrey theory [GMS98, Chapter 5] slightly modified to mixed boundary conditions applied to the relative cohomological class \([\tilde{u}^*(d\theta)] \in H^1(Q, T \cup B; \mathbb{Z})\) yields an energy minimizing representative \(u : Q \to S^1\) with Sobolev regularity. The existence of a harmonic map is equivalent to the existence of a solution \(u\) to the following mixed boundary value problem

\[
\Delta_g u = 0 \text{ in } Q, \frac{\partial u}{\partial \nu} = 0 \text{ along } F, u = 1 \text{ on } T, u = 0 \text{ on } B.
\]

Indeed, if \(0 \leq u \leq 1\) in \(\tilde{Q}\), after identifying 0 and 1, the solution to the above gives a harmonic map \(u\) from \(Q\) to \(S^1\). In fact, we can show that \(u \in C^{2,\alpha}(\bar{Q}, S^1)\) (See Theorem A.1).

**Proof of Theorem 1.3.** The harmonic map equation reduces to the harmonicity of the pull back 1-form \(h = u^*(d\theta)\). That is, the harmonic map equation gives

\[dh = 0, \quad d^*h = 0 \text{ in } Q.\]

Since \(u\) takes fixed values at top and bottom faces, the 1-form \(h\) has only a normal component along \(T\) and \(B\). The Dirichlet boundary condition gives

\[h \wedge \nu = 0 \text{ on } T \cup B.\]

And \(h\) satisfies the Neumann condition

\[(h, \nu) = 0 \text{ on } F.\]

From Sard’s theorem, the level set \(u^{-1}(\theta)\) is a \(C^1\) submanifold of \(Q\) and hence multiple copies of squares.

Let \(A \subset S^1\) be an open set containing the critical values of \(u\) and let \(B\) be the complement subset. So \(B\) contains only regular values. From integration by parts,

\[
\int_{T \cup B} \frac{\partial \varphi}{\partial \nu} + \int_F \frac{\partial \varphi}{\partial \nu} = \int_{u^{-1}(A)} \Delta_g \varphi \delta + \int_{u^{-1}(B)} \Delta_g \varphi \delta.
\]

Similar to (3.2), we have

\[
\int_{T \cup B} \frac{\partial \varphi}{\partial \nu} = -\int_{T \cup B} H_{\partial N} \frac{|h|^2}{\varphi^2} \to -\int_{T \cup B} H_{\partial N} |d\varphi|.
\]
as $\delta \to 0$. Similar to \[BS19\], with $\frac{\partial \varphi_\delta}{\partial \nu} = \langle d\varphi_\delta, \nu \rangle = -\varphi_\delta^{-1}(h, Du)\nu$ using the Neumann condition on $F$, and $(\kappa_{\partial \Sigma^\delta} - H_{\partial N})|du| = -|h|^{-1}\langle h, Du\rangle \nu$ we have that

$$
\lim_{\delta \to 0} \int_F \frac{\partial \varphi_\delta}{\partial \nu} = \lim_{\delta \to 0} \int_{\Gamma \cap u^{-1}(B)} \frac{\partial \varphi_\delta}{\partial \nu} + \int_{\Gamma \cap u^{-1}(B)} \frac{\partial \varphi_\delta}{\partial \nu} \leq C_Q \int_{\theta \in A} \partial \Sigma^\theta |u| + \int_{\Gamma \cap u^{-1}(B)} \left(\kappa_{\partial \Sigma^\delta} - H_{\partial N}\right)|du|$$

(3.7)

as $\delta \to 0$. From (3.4), (3.3) and (3.5),

$$
\int_{u^{-1}(B)} \left(\frac{|du|}{|\Hess u|^2} + R_Q - R_\Sigma\right) \leq \int_{T \cup B} \frac{\partial \varphi_\delta}{\partial \nu} + \int_F \frac{\partial \varphi_\delta}{\partial \nu} + C_Q \int_A |\Sigma^\theta|.
$$

Inserting $\frac{\partial \varphi_\delta}{\partial \nu} = -|h|H_{\partial Q}$ on $T \cup B$ we have

$$
\int_{u^{-1}(B)} \frac{|du|}{2} \left(\frac{|\Hess u|^2}{|du|^2} + R_Q\right) \leq \int_{F \cap u^{-1}(B)} H_{\partial Q}|du| + \int_{T \cup B} H_{\partial Q}|du|
$$

$$
\leq \int_{u^{-1}(B)} \frac{1}{2} R_\Sigma|du| + \int_{F \cap u^{-1}(B)} \kappa_{\partial \Sigma^\delta}|du| + C_Q \left(\int_{\theta \in A} |\Sigma^\theta| + |\partial \Sigma^\theta|\right)
$$

$$
= \int_{E \cap u^{-1}(B)} \frac{1}{2} R_\Sigma + \int_{E \cap \partial \Sigma} \kappa_{\partial \Sigma^\delta} + C_Q \left(\int_{\theta \in A} |\Sigma^\theta| + |\partial \Sigma^\theta|\right)
$$

$$
= \int_{E \cap u^{-1}(B)} \left[2\pi \chi(\Sigma^\theta) - \sum_j \gamma_j\right] + C_Q \left(\int_{\theta \in A} |\Sigma^\theta| + |\partial \Sigma^\theta|\right).
$$

Here we have used the coarea formula and the Gauss-Bonnet theorem (with turning angle). By definition of Euler characteristic, we have that $\chi(\Sigma^\delta) \leq 1$. Now we analyze the turning angle $\gamma_i$. Note that $\gamma_i$ is $\pi$ minus the interior turning angle. By coarea formula, we get the integrability of $\theta \mapsto |\Sigma^\theta|$

Let $F_1$ and $F_2$ be any pair of neighboring side faces, $E = F \cap F_2$, $\nu_1$ be the normal of the face $F_i$, $\tau$ be the tangent vector of $E$. We pick a point $p \in E$ which is not a vertex. We analyze the gradient vector field $\nabla u$. Since that $\nabla u$ has no component in $\nu_1$ direction according to the boundary condition and the vector $\tau$ is normal to both $\nu_1$ and $\nu_2$, so $\nabla u$ must be parallel to $\tau$ along $E$. Therefore, $E$ intersects the level set $u^{-1}(\theta)$ orthogonally. So $\nu_1$ coincides with the tangent vector of $u^{-1}(\theta) \cap F_j$ along $E$, and the exterior turning angle of $u^{-1}(\theta) \cap F_1$ to $u^{-1}(\theta) \cap F_j$ is the same as the angle forming by $\nu_1$ and $\nu_2$. So we have that

$$
\int_{E \cap u^{-1}(B)} \left[2\pi \chi(\Sigma^\theta) - \sum_j \gamma_j\right] \leq \int_{E \cap u^{-1}(B)} \sum_{j=1}^4 \left(\frac{\pi}{2} - \gamma_j\right) \leq 0
$$

using that the dihedral angle is everywhere equal to $\pi/2$. Here $\gamma_j$ is the four turning angles. By Sard type theorem, we can take $\mathcal{H}^1(A)$ arbitrarily small, we have that

$$
\int_Q \frac{|du|}{2} \left(\frac{|\Hess u|^2}{|du|^2} + R_Q\right) + \int_{\partial Q} H_{\partial Q}|du| \leq 0.
$$

By nonnegativity of $R_Q$ and $H_{\partial Q}$, we have that $\Hess(u) \equiv 0$. We have that $R_Q \equiv 0$, $H_{\partial Q} \equiv 0$ and $\gamma_j \equiv \pi/2$ and every regular level set intersect the vertical edges only
four times. Fixing any component $S$ of the regular level set $u^{-1}(\theta)$. The map

$$\Psi : S \times \mathbb{R} \to Q, \quad \frac{\partial \Psi}{\partial t} = \frac{\text{grad} u}{|\text{grad} u|} \circ \Psi$$

gives a local isometry. Then $\Sigma$ has vanishing curvature and $\partial \Sigma$ has vanishing geodesic curvature. This says that $Q$ is a three dimensional Euclidean cube (up to a constant multiple of the metric). \qed

4. Application to hyperbolic mapping torus

In his geometrization program, Thurston proved that a mapping torus of a surface of genus at least 2 by a pseudo-Anosov map is hyperbolizable. By Mostow rigidity, this mapping torus has a unique hyperbolic structure, and hence it has an associated hyperbolic volume. Then it is a natural question to find the genus bound of the surface in terms of the volume. As an application of a current technique, we give an upper bound for the genus in terms of the volume and the hyperbolic translation length of the pseudo-Anosov map. The main estimate is

**Theorem 4.1.** Let $M_\phi$ be a mapping torus of a closed surface $S$ of genus $g \geq 2$ via a pseudo-Anosov map $\phi$. Then

$$g \leq \frac{3}{4C\pi \|\phi\|} \text{vol}(M_\phi) + 1,$$

where $\|\phi\|$ is the translation length of a hyperbolic isometry defined by $\phi$ and $C$ is a constant depending only on $S$ and the injectivity radius of $M_\phi$.

In the last section, we compare $\|\phi\|$ to the entropy of the pseudo-Anosov map $\phi$. The entropy of a pseudo-Anosov map is $\log \lambda(\phi)$ where $\lambda(\phi)$ is the dilatation of $\phi$. Another interpretation of the entropy is the translation length of the action of $\phi$ on the Teichmüller space with respect to the Teichmüller metric.

Using Minsky’s geometric model, one can show that the entropy $\text{ent}(\phi)$ and $\|\phi\|$ are comparable once $S$ is fixed and the injectivity radius of the mapping torus $M_\phi$ is bounded below. Hence we obtain

**Theorem 4.2.**

$$\frac{1}{3\pi|\chi(S)|} \text{ent}(M_\phi) \leq \text{ent}(\phi) \leq \frac{3}{2\pi|\chi(S)|K} \text{ent}(M_\phi),$$

where $K$ depends only on $S$ and the injectivity radius of $M_\phi$.

In general, if the injectivity radius goes to zero, $K$ also tends to zero. Since there exist families of pseudo-Anosov maps whose entropy tends to infinity while the volume of the mapping torus remains bounded, this is the best that we can hope for except the explicit calculation of $K$. This inequality is obtained in [KKT09] using Brock’s inequality [Bro03]. Our proof relies on the harmonic map technique in [Ste20], and it is simpler.

4.1. Genus bound for mapping torus. Let $M_\phi$ be a hyperbolic mapping torus of $S$ via a pseudo-Anosov map $\phi$ and

$$u : M_\phi = S \times [0,1]/(x,0) \sim (\phi(x),1) \to [0,1]/0 \sim 1$$

the projection. On the infinite cyclic cover $S \times \mathbb{R}$, $\phi$ acts as a translation. Since $\pi_1(M_\phi) = \langle \pi_1(S), \gamma \tau \gamma^{-1} = \phi_\ast(\gamma), \gamma \in \pi_1(S) \rangle$, $\phi$ corresponds to a hyperbolic isometry $t$, and we denote $\|\phi\|$ the translation length of $t$ on $\mathbb{H}^3$. This is the hyperbolic
translation length of \( \phi \) on the infinite cyclic cover \( S \times \mathbb{R} \). Hence \( \| \phi \| \) denotes the width of the fundamental domain of \( M_\phi \) on the cyclic cover \( S \times \mathbb{R} \) where the left and right sides are identified by the action of \( \phi \).

Choose orthonormal basis \( e_1, e_2 \) tangent to \( \Sigma_\theta = u^{-1}(\theta) \), \( e_3 \) such that \( e_3 = \nabla u / \| \nabla u \| \).

Note that for any \( x \in S \times \{0\} \), along the gradient flow starting from \( x \) in \( u^{-1}(0) \) to \( u^{-1}(1) \), \( u \) behaves like the projection from \([0, \text{length of the gradient flow}]\) to \([0, 1]\). Hence 

\[
\| du \|_{L^2} \leq \frac{\sqrt{\text{vol}(M_\phi)}}{(\text{W=width of the fundamental domain})}.
\]

Let \( u' \) be a harmonic map homotopic to \( u \). For each regular \( \theta \in S^1 \), \( u^{-1}(\theta) = S \) and \( u'^{-1}(\theta) = \Sigma_\theta \) are homotopic in \( M_\phi \). Since \( M_\phi \) is a mapping torus of \( S \), the genus of \( \Sigma_\theta \) is bigger than the genus of \( S \). By Sard’s theorem, \( |\chi(\Sigma_\theta)| \geq |\chi(S)| \) for almost all \( \theta \in S^1 \). Hence by equation (2.3), we get

\[
-4\pi \chi(S) \leq \| du' \|_{L^2} - 6 \| u \|_{L^2} \leq 6 \sqrt{\text{vol}(M_\phi)} \| du \|_{L^2} = \frac{6}{W} \text{vol}(M_\phi).
\]

Hence we get the genus bound of \( S \)

\[
g \leq \frac{3}{4\pi W} \text{vol}(M_\phi) + 1.
\]

Once \( S \) is fixed and if there is a lower bound for the injectivity radius of \( M_\phi \), the diameter of \( S \) is bounded in \( M_\phi \). Since \( \phi \) identifies \( S \times \{0\} \) to \( S \times \{1\} \), \( \| \phi \| \) and \( W \) are comparable, i.e., there exists \( C = C(\text{inj}M_\phi, S) \) such that \( W \geq C \| \phi \| \). Hence the above inequality becomes and prove Theorem 4.1

\[
g \leq \frac{3}{4C\pi \| \phi \|} \text{vol}(M_\phi) + 1.
\]
4.2. Interpretation of $\|\phi\|$ and some applications. In this section, we interpret $\|\phi\|$ as a quantity comparable to $\text{ent}(\phi)$ and give an independent proof of Theorem 4.2.

By Minsky [Min01], it is known that the infinite cyclic cover $\tilde{M}_\phi = S \times \mathbb{R}$ has a geometric model, the universal curve over the Teichmüller geodesic $\Gamma$ invariant by $\phi$ parametrized by the arc length. More precisely, the geometric model is built as follows. Fix a hyperbolic surface $X_0 = X$ on $\Gamma$. The universal curve $C_{\Gamma}$ over $\Gamma$ is the collection of $X_t$ where $X_t$ is a hyperbolic surface at time $t$. The fundamental domain of $\phi$ is the subset over $[X, \phi(X)]$. The Teichmüller distance $d_T(X, \phi(X))$ is known to be $\text{ent}(\phi)$. Then there exists a biLipschitz map $\Phi : \tilde{M}_\phi \to C_{\Gamma}$ with biLipschitz constant $K$ depending only on $S$ and the injectivity radius of $M_\phi$. Hence the hyperbolic translation distance $\|\phi\|$ of $\phi$ on $\tilde{M}_\phi$ is comparable to $\text{ent}(\phi)K$, i.e.

$$K(S, \text{inj}(M_\phi)) \text{ent}(\phi) \leq \|\phi\|.\]$$

By equation (4.1), we get

$$\text{ent}(\phi) \leq \frac{3}{2\pi |\chi(S)|} \text{vol}(M_\phi).\]$$

By combining the result of Kojima-McShane [KM18], we obtain

$$\frac{1}{3\pi |\chi(S)|} \text{vol}(M_\phi) \leq \text{ent}(\phi) \leq \frac{3}{2\pi |\chi(S)|} \text{vol}(M_\phi).\]$$

One can compare this inequality with the inequality obtained by Brock [Bro03] by relating the entropy to Weil-Petersson translation length. Indeed, the Weil-Petersson metric $g_{WP}$ and the Teichmüller metric $g_T$ satisfy the inequality $g_{WP} \leq 2\pi |\chi(S)| g_T$ in general, once the injectivity radius of $M_\phi$ has a lower bound, there exists a constant $C$ depending only on the topology of $S$ and the lower bound of the injectivity radius [KKT09] such that

$$C^{-1}\|\phi\|_{WP} \leq \text{ent}(\phi) \leq C\|\phi\|_{WP}.\]$$

APPENDIX A. REGULARITY OF MIXED BOUNDARY VALUE PROBLEMS

In this appendix, our goal is the existence of mixed boundary value problem

$$Lu = f \quad \text{in } Q, \quad u = \varphi \quad \text{on } T \cup B, \quad \frac{\partial u}{\partial n} = \psi \quad \text{on } F,$$

on a three dimensional cube whose dihedral angles are all $\frac{\pi}{2}$. Here $L$ is defined to be $L = g^{ij} \partial_i \partial_j u + b^i \partial_i u + cu$ and $g^{ij}$ is the inverse metric of the cube $Q$. To achieve a solution in $C^{2,\alpha}(\tilde{Q})$, we assume that there exists a function $u' \in C^{2,\alpha}(\tilde{Q})$ such that $u' = \varphi$ on $T \cup B$ and $\frac{\partial u'}{\partial n} = \psi$ on $F$. This is nothing more than an easy way of prescribing the compatibility of boundary conditions.

**Theorem A.1.** There exists a solution $u$ in $C^{2,\alpha}(\tilde{Q})$ to (A.1).

To show this theorem, the appendix is outline as follows: First, we establish a maximum principle for mixed boundary value problems. Second, we establish the $C^{2,\alpha}$ estimates. Using the method of continuity, we solve a similar mixed boundary value problems on two model domains which are respectively half ball and a quarter ball. By a classical technique [LU68] we map a neighborhood of a point in a cube to a neighborhood of the model domains obtaining a $C^{2,\alpha}$ regular solution for mixed boundary value problem in the cube. The reasons we do not attempt to solve the problem on a cube directly are: The Green function on a standard cube is not
Lemma A.2. If $p \in \partial Q$ belong to $k$ faces $F_1, \ldots, F_k$ where $1 \leq k \leq 3$, then there exists a local coordinate system such that $\{z_j = 0\}_{j=1}^3$ where each face is given by constant level set $\{z_j = 0\}$ or $\{z_j = 1\}$ with labeling consistent with the coordinate system $\{x_j\}$ given by the diffeomorphism $\Psi$ to the standard cube, and on each face $F_i$,

$$g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}) = 0, g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}) = 1 \text{ on } F_i$$

for all $j \neq i$.

A.1. Hopf boundary point lemma. We have a Hopf boundary point lemma for $u$.

Lemma A.3. If $L$ is uniformly elliptic, $c = 0$ and $Lu \geq 0$, let $x_0 \in \partial Q$ be a point at an open edge where faces $F_1$ and $F_2$ meet such that $u$ is continuous at $x_0$, $u(x_0) > u(x)$ for all $x \in Q$. If $\frac{\partial u}{\partial \nu_1} \leq 0$ along $F_1$, then the outer normal derivative of $u$ at $x_0$, if exists satisfies the strict inequality

$$\frac{\partial u}{\partial \nu_2}(x_0) > 0.$$

If $c \leq 0$, the same conclusion holds provided $u(x_0) \geq 0$, and if $u(x_0) = 0$, the same conclusion holds irrespective of the sign of $c$.

Proof. Near $x_0$, there exists a local coordinate $\{z_j\}$ in a neighborhood $U$ of $x_0$ from Lemma A.2. Without loss of generality, we assume that $x_0 = 0$, each $F_i$ is a small piece of $\{z_j = 0\}$ and $U$ is the intersection of a small ball $B(0, \delta_1)$ with the wedge $W = \{z_1 > 0, z_2 > 0\}$. In the local coordinate $\{z_j\}$, we can pick a point $y \in F_1 \cap U$ and a ball $B(y, \delta_2)$ such that $x_0 = 0 \in \partial B(y, \delta_2)$ and $B(y, \delta_2) \cap W \subset U$.

Setting these things up, we can follow [GTS3] Lemma 3.4. For $0 < \rho < \delta_2$, for a constant $\alpha > 0$ to be determined, we define the auxiliary function

$$v(z) = e^{-\alpha r^2} - e^{-\alpha \delta_2^2},$$

where $r$ denotes the $z$-distance between $z$ and $y$. Direct calculation shows that

$$Lv(z) = e^{-\alpha r^2}[4\alpha^2 g^{ij}(z_i - y_i)(z_j - y_j) - 2\alpha(g^{ij} + b^i(z_i - y_i))] + cv$$

$$\geq e^{-\alpha \delta_2^2}[4\alpha^2 \lambda(z) r^2 - 2\alpha(g^{ij} + b^i|r| + c)],$$

where $\lambda(z)$ is the smallest eigenvalue of $g^{ij}(z)$. Since the domain we are dealing with is compact and $g^{ij}$ is never degenerate, hence $\alpha$ could be chosen large enough such that $Lv \geq 0$ in the annular region $A = (B(y, \delta_2) \setminus B(y, \rho)) \cap W$.

Since $u - u(x_0) < 0$ on $\partial B(y, \rho)$, there is a constant $\varepsilon > 0$ for which $u - u(x_0) + \varepsilon v \leq 0$ on $\partial B(y, \rho)$. The inequality is also satisfied on $\partial B(y, \delta_2)$ since $v = 0$ along $\partial B(y, \delta_2)$. So we have that

$$L(u - u(x_0) + \varepsilon v) \geq -cu(x_0) \geq 0 \text{ in } A.$$
Figure 2. Illustration of Hopf boundary point lemma

Hopf boundary point lemma, the maximum of \( u - u(x_0) + \varepsilon v \) cannot be attained at \( F_1 \cap \tilde{A} \). So by weak maximum principle, \( u - u(x_0) + \varepsilon v \leq 0 \) in \( A \), so

\[
\frac{\partial u}{\partial \nu_2}(x_0) \geq -\varepsilon \frac{\partial v}{\partial \nu_2} = -\varepsilon v'(\delta_2) > 0
\]
as claimed. \( \square \)

It follows directly that the solution \( u \in C^2(\bar{Q}) \) to (3.4) cannot attain its maximum at open vertical edges. This could be shown via an alternative argument. If the maximum is attained at \( x' \) of an open vertical edge. Then locally we even reflect \( u \) across a neighboring side face using coordinate system at \( x' \) from Lemma A.2. The reflected \( u \) is easily verified to be \( C^2 \) in the interior. Then the reflected \( u \) contradicts the Hopf maximum principle. From the usual Hopf maximum principle, the maximum cannot be attained in the interior of \( Q \) and at open side faces. To conclude, we have \( 0 \leq u \leq 1 \) if \( u \in C^1(\bar{Q}) \cap C^2(\tilde{Q}) \).

A.2. Schauder estimate. Let \( W_k = \{ x \in \mathbb{R}^3 : x_i \geq 0 \text{ for all } 1 \leq i \leq k \} \) when \( 1 \leq k \leq 3 \) and \( W_k = \mathbb{R}^3 \) when \( k = 0 \). Let \( \{ x_i = 0 \} \) be a face, denote by \( \partial_{(i)} \) the partial derivatives on \( \{ x_i = 0 \} \) which is with respect to \( x_j \) for \( j \neq i \). We define the seminorms for \( u \) along \( \{ x_i = 0 \} \) using \( \partial_{(i)} \) i.e.

|partial_{(i)} g|_\alpha \text{ and } |partial_{(i)} partial_{(j)} g|_\alpha

for \( g \in C^{2,\alpha}(\{ x_i = 0 \}) \) and \( g \in C^{2,\alpha}(\{ x_i = 0 \}) \).

**Lemma A.4.** Any \( u \in C^{2,\alpha}(\tilde{W}_k) \) satisfies the estimate

\[
|\partial u|_\alpha \leq C|\Delta u|_\alpha + E_k(u)
\]
where $\Delta$ is the standard Laplace operator and $E_k(u)$ contains seminorms on faces of $W_k$ in either form of $[A.4]$.

Proof. This could be proved following the method of [Sim97]. First, one should establish suitable Liouville theorem for harmonic functions with homogeneous Dirichlet and Neumann boundary condition along faces of $\partial W$. Let $v$ be such a harmonic function with the growth rate

$$\sup_{B_r \cap W_k} |v| \leq C r^{2+\varepsilon}$$

for $\varepsilon \in (0,1)$. One can prove that $v$ is a quadratic polynomial. This is done via reflection. By odd (or Schwarz) reflection across faces where $v$ vanishes, even reflection across faces where $v$ has vanishing normal derivative, one can reduce to the standard Liouville theorem. Then one can use the method of scaling to prove $[A.5]$. See [Sim97, Theorem 4].

We give a local boundary estimate of harmonic functions for some special balls centered at points of $\partial Q$.

We write the equation $\Delta_g u = 0$ in a local coordinate, we have

$$\frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g} \partial_j u) = g^{ij} \partial_i \partial_j u + \frac{1}{\sqrt{g}} \partial_i (g^{ij} \sqrt{g} \partial_j u) =: g^{ij} u_{ij} + b_i u_i = 0.$$

The coefficient of the second order is given by $g^{ij}$. The local coordinate system in Lemma $[A.2]$ then plays the same role as flattening the boundary in theory of Dirichlet boundary value problems of elliptic equations if $P$ lies in the open face.

Moreover, it is important in Neumann boundary value problem as well. We illustrate this by the simplest example $\Delta_g v = f$ in $\mathbb{R}^3$ and $\frac{\partial v}{\partial \nu} = 0$ along $\partial \mathbb{R}^3$. Here we assume $\partial \mathbb{R}^3 = \{x_3 = 0\}$ and temporarily that $g^{ij}$ are defined on all of $\mathbb{R}^3$. In order to get estimates for this variable coefficient equation, we have to get estimates for constant coefficient equation at the boundary $g^{ij}(0) v_{ij} = f'$ and $g^{ij}(0) v_i \nu_j(0) = 0$. This is the standard procedure of freezing coefficients. In order to apply the estimates $[A.5]$, the condition $[A.2]$ with $i = 3$ is exactly what we need. The coefficient $g^{ij}(0)$ may not be $\delta^{ij}$ for $1 \leq i, j \leq 2$, but one can make a linear change of variables.

Let $z_0$ be a point at the boundary $\partial Q$. In a neighborhood of $z_0$, there exists a local coordinate system satisfying properties in Lemma $[A.2]$. We define balls using the $\{z_j\}$ coordinates, that is

$$B_r(z_0) = \{ z : |z - z_0| \leq r \}$$

for $r > 0$.

We write short $B_r$ for $B_r(z_0)$.

**Lemma A.5.** We assume that $B_{2r}$ does not intersect top face $\bar{T}$; and if $z_0$ is on an open face, then $B_{2r}$ does not intersect with any edges; if $z_0$ is on an open edge, then $B_{2r}$ does not contain any vertices. If $u \in C^{2,\alpha}(\bar{Q})$ solves $[A.1]$ in $B_{2r}$, then

$$|u|_{C^{2,\alpha}(B_{2r} \cap \partial Q)} \leq C|u|_{C^{2,\alpha}(B_{2r} \cap \partial Q)} + |Lu|_{C^{\alpha}(B_{2r} \cap \partial Q)} + C|\varphi|_{C^{2,\alpha}(B_{2r} \cap \partial Q)} + C|\psi|_{C^{1,\alpha}(B_{2r} \cap \partial Q)}.$$  

(A.6)

**Proof.** Let $v = \xi u$, where $\xi$ vanishes outside the ball $B_{2r}$. So by earlier estimate $[A.3]$, we have that

$$|\partial \bar{v}|_{\alpha} \leq C|g^{ij}(z_0) v_{ij}|_{\alpha} + C \mathcal{E}(v).$$
We analyze the contribution of $\mathcal{E}(v)$ from top or bottom faces first. The contribution from the top face for example is

$$\mathcal{E}(v) = \|\xi u\|_{C^{2,\alpha}(\partial B_{2r})} = \|\xi \varphi\|_{C^{2,\alpha}(\partial B_{2r})} \leq C\|\varphi\|_{C^{2,\alpha}(\text{spt } \xi \cap \partial Q)}.$$ 

The contribution from the bottom face is of the same form. And along side faces for instance $\{z_1 = 0\}$, $\frac{\partial u}{\partial z_1} = \psi$ and

$$|\partial_1 \partial_1 (\xi u)|_\alpha = |\partial_1 \partial_1 (\xi u) + \partial_1 (\xi \psi)|_\alpha \leq C\|\psi\|_{C^{1,\alpha}(\text{spt } \xi \cap \partial Q)},$$

by choosing a special $\xi$ such that $\partial_1 \xi = 0$. A standard cutoff would suffice for this purpose. So

$$\|\partial \psi v\|_\alpha \leq C|g^{ij}(z_0)v_{ij}|_\alpha + C\|u\|_{C^2(\text{spt } \xi \cap Q)}.$$ 

Freezing the coefficients, we have that

$$g^{ij}(z_0)v_{ij} = Lv - (g^{ij} - g^{ij}(z_0))v_{ij} - b_i v_i - cv.$$ 

Note that the steps above differ very little from standard methods of obtaining Schauder estimates except that we have to keep track of the special property of the local coordinate so that we can make use of (A.6). See Lemma A.2. Using that

$$|fh|_\alpha \leq \|f\|_{L^\infty} |h|_\alpha + |f|_\alpha \|h\|_{L^\infty},$$

and that $g_{ij} \in C^{2,\alpha}(\bar{B}_{2r})$, we have that

$$|(g^{ij} - g^{ij}(z_0))v_{ij}|_\alpha \leq C\tau^\alpha |\partial \psi v|_\alpha + C\|\partial \psi v\|_{L^\infty(\bar{B}_{2r} \cap Q)},$$

and

$$|b^i v_i|_\alpha \leq C\tau^\alpha \|v_i\|_{L^\infty} + |v_i|_\alpha |b_i|_{L^\infty} \leq C\|v\|_{C^{1,\alpha}(\bar{B}_{2r} \cap Q)} \leq C\|v\|_{C^2(\bar{B}_{2r} \cap Q)},$$

and

$$|cv|_\alpha \leq C\|v\|_{L^\infty(\bar{B}_{2r} \cap Q)} + C\|v\|_{C^1(\bar{B}_{2r} \cap Q)}.$$ 

By choosing $\tau$ small, so that we can absorb $C\tau^\alpha |\partial \psi v|_\alpha$ and combining with (A.7) and (A.8), we have that

$$|\partial \psi v|_\alpha \leq C\|Lv\|_\alpha + C\|v\|_{C^2(\bar{B}_{2r} \cap Q)},$$

where $C$ only also depends on $\alpha$ and $\tau$. Then pick $\xi$ to be $\xi = \phi(|x-y|)$ with $\phi = 1$ in $[0, \tau]$, $\phi = 0$ in $[2\tau, \infty)$, $\tau|\phi'| + \tau^2|\phi''| \leq C$, then the above estimate gives (A.6) since $Lu = f$ and

$$Lv = \xi Lu + 2g^{ij} \partial_i \xi \partial_j u + u(g^{ij} \partial_i \xi + b^i \partial_i \xi).$$

□

Observing the proof above does not yet immediately give a real local boundary estimate, because there is a special requirement in Lemma A.5 on the location of the center $x_0$ of the ball. We have maybe missed some part of the boundary. We have to show that all balls satisfying the requirement of Lemma A.5 still covers $\partial Q$, and hence a neighborhood of $\partial Q$.

We can cover the vertices first, then along each edge, we only have to cover a segment (not including any vertices) shorter than the entire edge with such balls. The number of balls to cover all edges we used is finite by compactness, hence there is a lower bound of their radius. Therefore, we only have to cover a smaller piece on each face. After this process, we have covered $\partial Q$. We decrease the radius of
each ball if needed. This covering argument also removes dependence of $C$ on $\tau$ in (A.6). We obtain a global Schauder estimate.

**Proposition A.6.** If $u \in C^{2,\alpha}(\bar{\Omega})$ solves (A.1), then

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq |Lu|_{C^{\alpha}(\bar{\Omega})} + C\|u\|_{L^\infty(\bar{\Omega})} + C\|\varphi\|_{C^{2,\alpha}(\partial\Omega)} + C\|\psi\|_{C^{1,\alpha}(\partial\Omega)}$$

(A.9)

**Proof.** By patching (A.6) up with interior Schauder estimates we have

$$\|u\|_{C^2(\bar{\Omega})} \leq C\|\varphi\|_{C^{2,\alpha}(\partial\Omega)} + C\|\psi\|_{C^{1,\alpha}(\partial\Omega)}$$

Applying the global interpolation inequality

$$\|u\|_{C^2(\bar{\Omega})} \leq \varepsilon \|u\|_{C^{2,\alpha}(\bar{\Omega})} + C\varepsilon\|u\|_{L^\infty(\bar{\Omega})}$$

and choosing $\varepsilon > 0$ sufficiently small, we can eliminate $\|u\|_{C^{2,\alpha}(\bar{\Omega})}$ and get (A.9). □

**Remark A.7.** It is possible to bound $\|u\|_{C^{2,\alpha}(B_2 \cap \Omega)}$ by $sup_{B_2 \cap \Omega} |u|$, $|Lu|_{C^{\alpha}(B_2 \cap \Omega)}$, $\|\varphi\|_{C^{2,\alpha}(B_2 \cap \partial\Omega)}$ and $\|\psi\|_{C^{1,\alpha}(B_2 \cap \partial\Omega)}$.

### A.3. Solvability on model domains.

Let the (open) half ball be $B_+ = \{x : |x| < 1, x_1 > 0\}$. We solve

$$\Delta u = f \text{ in } B_+, \frac{\partial u}{\partial x_1} = f_1 \text{ on } \{|x| \leq 1, x_1 = 0\}, u = f_2 \text{ on } \{|x| = 1, x_1 \geq 1\}.$$  

Without loss of generality, we assume that $f_2 = 0$. Let $G(x, y)$ be the Green function of the three dimensional unit ball, then by reflection method, we can write down explicitly that

$$G(x, y) = \frac{1}{4\pi|x-y|} + \frac{1}{4\pi|y-x|}$$

where $y = \frac{y}{|y|^2}$ is the spherical reflection with respect to the unit sphere and denote

$$\tilde{y} = (-y_1, y_2, y_3), \hat{y} = (y_1, -y_2, y_3).$$

Let $\tilde{G}(x, y) = G(x, y) + G(x, \tilde{y})$, so $\Delta_x \tilde{G}(x, y) = \delta(x-y)$, and $\frac{\partial^2 \tilde{G}}{\partial x_1^2} = 0$ along $B \cap \{x_1 = 0\}$.

**Lemma A.8.** We have that the solution $u$ to (A.10) on half ball can be represented as

$$u(x) = \int_{B^+} \tilde{G}(x, y)f(y)dy + \int_{B \cap \{x_1 = 0\}} \tilde{G}(x, y)f_1(y)dy,$$

(A.11)

and $u \in C^{2,\alpha}(\bar{B}^+)$. \hfill □

**Proof.** First, the solution (A.11) follows easily from Green’s formula. We only have to show the $C^{2,\alpha}$ regularity of $u$. Performing an even reflection of $f$ across the plane $\{x_1 = 0\}$, we get a reflected $\tilde{f} \in C^{\alpha}(B)$. By a change of variables,

$$\int_{B^+} \tilde{G}(x, y)f(y)dy = \int_{B^+_+} G(x, y)f(y)dy + \int_{B^+} G(x, y)f(\tilde{y})dy = \int_B G(x, y)\tilde{f}(y)dy \in C^{2,\alpha}(B^+_+).$$

Since $g$ vanishes on $\{|x| = 1, x_1 = 0\}$, extend $f_1$ to all of $\{x_1 = 0\}$ by setting $\tilde{f}_1(x)$ to be $-|x|^{-1}f_1\left(\frac{x}{|x|}\right)$ if $|x| \geq 1$ (This is the Kelvin transform in dimension
three). We denote the extended version by $\tilde{f}_1$. It is not difficult to verify that $\tilde{f}_1 \in C^{1,\alpha}(\{|x|=0\})$. Indeed, $\tilde{f}_1$ vanishes along $\{|x|=1, x_1 = 0\}$ since $f_1$ has to satisfy the compatibility condition, so we have $\tilde{f}_1$ is continuous. By direct calculation if $|x| \geq 1$,

\begin{equation}
(A.12) \quad x^i \partial_i \tilde{f}_1 = |x|^{-1}f_1(\frac{x}{|x|^2}) + |x|^{-3}x^i \partial_j f_1(\frac{x}{|x|^2})
\end{equation}

We see that $x^i \partial_i \tilde{f}_1 = x^i \partial_i f_1$ on $\{|x|=1, x_1 = 0\}$. And derivatives of $\tilde{f}_1$ along tangential direction of $\{|x|=1, x_1 = 0\}$ vanishes, this implies that $\tilde{f}_1 \in C^1(\{|x|=0\})$. The Hölder continuity of $x^i \partial_i \tilde{f}_1$ readily follows from

$$|x^i \partial_i f_1(x) - z^i \partial_i f_1(z)| \leq C|x-z|^\alpha$$

for each $z \in \{|x|=1, x_1 = 0\}, x \in \{|x| \leq 1, x_1 = 0\}$ and (A.12). Other derivatives are similar proved. We use again a change of variables, and

$$\int_{\{|y| \leq 1, y_1 = 0\}} \tilde{G}(x,y) f_1(y)dy$$

$$= \int_{\{|y| \leq 1, y_1 = 0\}} \psi(x,y) f_1(y)dy$$

$$+ \int_{\{|y| \leq 1, y_1 = 0\}} \left( \frac{1}{4\pi|y||x-y|} + \frac{1}{4\pi|\tilde{y}||x-\tilde{y}|} \right) f_1(y)dy$$

$$= \int_{\{|y| \leq 1, y_1 = 0\}} \psi(x,y) f_1(y)dy$$

$$+ \int_{\{|y| \geq 1, y_1 = 0\}} \left( \frac{1}{4\pi|y||x-y|} + \frac{1}{4\pi|\tilde{y}||x-\tilde{y}|} \right) f_1(y)\left(\frac{1}{|y|^\alpha}\right)dy$$

$$= \int_{\{y_1 = 0\}} \psi(x,y) \tilde{f}_1(y)dy.$$

We conclude from Schauder regularity of Neumann problem for half space that this contribution of $u$ is also in $C^{2,\alpha}(\bar{B}_+)$. Here

$$\psi(x,y) = -\frac{1}{4\pi|x-y|} - \frac{1}{4\pi|x-\tilde{y}|}$$

is the Neumann Green function for the half space, and we have used that Kelvin transform and reflection across $\{y_1 = 0\}$ commutes, and that $|\tilde{y}| = |y|$. $\square$

The problem (A.10) illustrates the reflection principle of how to handle the mixed boundary value problems. Now we turn to a slightly more complicated model. Let $B_{+,*}$ be the (open) quarter ball

$$\{x : |x| < 1, x_1 > 0, x_2 > 0\},$$

and $\Sigma_1 = \{|x| \leq 1, x_1 = 0, x_2 \geq 0\}$, $\Sigma_2 = \{|x| \leq 1, x_1 \geq 0, x_2 = 0\}$. Again, using reflection, we have,

**Lemma A.9.** Assume that there exists a function $v \in C^{2,\alpha}(\bar{B}_{+,*})$ with $\partial v \partial_{x_{1}^{i}} = f_i$ on $\Sigma_1$ and $v = f_3$ on $\{|x| = 1, x_1 \geq 0, x_2 \geq 0\}$. There exists a unique solution $u$ in $C^{2,\alpha}(\bar{B}_{+,*})$.

$$\Delta u = f \text{ in } B_{+,*}, \partial u \partial_{x_{1}^{i}} = f_i \text{ on } \Sigma_1, u = f_3 \text{ on } \{|x| = 1, x_1 \geq 0, x_2 \geq 0\}.$$

**Proof.** The boundary of the quarter ball has two point corners $v_{\pm} = \{x_3 = \pm 1, x_1 = x_2 = 0\}$ and three edge corners which are $E_1 = \{x_2 = 0, x_1 \geq 0, |x| = 1\}$, $E_2 = \{x_1 = 0, x_2 \geq 0, |x| = 1\}$ and $E_3 = \{-1 \leq x_3 \leq 1, x_1 = 0, x_2 = 0\}$.


By subtracting from $u$ a $C^{2,\alpha}(\bar{B}_{+})$ function, we can assume that $f_3 = 0$. The Green function to the problem is
\[ \tilde{G}(x, y) = G(x, y) + \hat{G}(x, y). \]

We can perform even reflection on $f$, two times, we can subtract $\int_{B} G(x, y)f(y)dy$ from $u$, so we are reduce to the case that $f = 0$.

We extend $f_1$ to $\{x_1 = 0, x_2 \geq 0\}$ and $f_2$ to $\{x_1 \geq 0, x_2 = 0\}$ similarly as Lemma [A.8] using the Kelvin transform. It is easily checked that $\partial_z f_1 = \partial_z f_2$ along $\{x_1 = x_2 = 0\}$. Let $v$ be a function such that $\frac{\partial v}{\partial x_1} = f_1$, then we see that $u - v$ satisfies the equation
\[ \Delta (u - w) = -\Delta w \quad \text{in} \quad \{x_1 \geq 0, x_2 \geq 0\}, \]
\[ \frac{\partial (u - w)}{\partial x_1} = 0 \quad \text{on} \quad \{x_1 = 0, x_2 \geq 0\}, \]
\[ \frac{\partial (u - w)}{\partial x_2} = f_2 - \frac{\partial w}{\partial x_2} \quad \text{on} \quad \{x_1 \geq 0, x_2 = 0\}. \]

We solve separately by reflection technique
\[ \Delta u_1 = -\Delta w \quad \text{in} \quad \{x_1 \geq 0, x_2 \geq 0\}, \]
\[ \frac{\partial u_1}{\partial x_1} = 0 \quad \text{on} \quad \{x_1 = 0, x_2 \geq 0\}, \]
\[ \frac{\partial u_1}{\partial x_2} = 0 \quad \text{on} \quad \{x_1 \geq 0, x_2 = 0\}, \]

and
\[ \Delta u_2 = 0 \quad \text{in} \quad \{x_1 \geq 0, x_2 \geq 0\}, \]
\[ \frac{\partial u_2}{\partial x_1} = 0 \quad \text{on} \quad \{x_1 = 0, x_2 \geq 0\}, \]
\[ \frac{\partial u_2}{\partial x_2} = f_2 - \frac{\partial w}{\partial x_2} \quad \text{on} \quad \{x_1 \geq 0, x_2 = 0\}. \]

We see then $u = v + w_1 + w_2$ by linearity and it is easily verified to be in $C^{2,\alpha}(\bar{B}_{+})$.

Now we are ready to establish the following more general existence theorem. We assume that $g$ is a metric on the quarter ball $B_{+}$ satisfying that $g^{11} = 1$, $g^{12} = g^{13} = 0$ on $\Sigma_1$ and $g^{22} = 1$, $g^{21} = g^{23} = 0$ on $\Sigma_2$, and the spherical piece of $\partial B_{+}$ meet other pieces $\Sigma_i$ with a constant contact angle $\frac{\pi}{2}$. This condition is for the sake of compatibility of two Neumann boundary conditions along $\{x | x_1 = 0, x_2 = 0\}$ as we shall see later. This is sufficient for our use, see [Li20b, Lemma 2.2].

**Lemma A.10.** There exists a unique solution $u \in C^{2,\alpha}(B_{+})$ if $c \leq 0$ to the mixed boundary value problem
\[ (A.13) \quad Lu = f \text{ in } B_{+}, \partial_{x_i} u = f_i \text{ on } \Sigma_i, u = f_3 \text{ on } \{x_1 = 1, x_1 \geq 1\}. \]

**Proof.** As before, we can assume that $f_3 = 0$. We use the method of continuity. Denote $g_0$ be the metric $g$ and let $S$ be the set of all $t \in [0, 1]$ such that the problem parametrized by $t$
\[ (A.14) \quad L_t u = f \text{ in } B_{+}, \partial_{x_i} u = f_i \text{ on } \Sigma_i, u = 0 \text{ on } \{x_1 = 1, x_1 \geq 1\} \]

is solvable where the operators $L_t = (1 - t)\Delta + t L$. We define the metric $g_t$ to be the inverse of $tg^{ij} + (1 - t)\delta^{ij}$ and we see that the assumptions on $g$ are preserved by such a path $g_t$. 
We see first that $0 \in S$ by Lemma\ref{lem:compactness}. From Schauder estimate \ref{lem:schauder}, the set $S$ is closed. It remains to show that $S$ is open.

We fix $t_0 \in S$. Then at $t$, for every $z \in C^{2,\alpha}(\bar{B}_{+,+})$ with $z = 0$ on $\{|x| = 1, x_1 \geq 1\}$, we define the following problem

$$L_{t_0}v = f + L_{t_0}z - L_tz \in B_{+,+}, \quad \frac{\partial v}{\partial x_i} = f_i \text{ on } \Sigma_i, \quad v = 0 \text{ on } \{|x| = 1, x_1 \geq 1\}.$$ 

By assumption that $t_0 \in S$, there exists a solution in $C^{2,\alpha}(\bar{B}_{+,+})$ which denote by $v \equiv Az$. We wish to find a fixed point $v = Av$, which gives a solution of \ref{eq:fixed_point} in the metric $g_t$.

We show that $A$ is a contraction operator for all $t$ with $|t - t_0| < \varepsilon$ where $\varepsilon$ will be determined later. Let $v_i = Az_i$ where $z_i \in C^{2,\alpha}(\bar{B}_{+,+})$. The difference $v_i - v_2$ is a solution to the problem

$$L_{t_0}(v_i - v_2) = (L_t - L_{t_0})(z_1 - z_2) \in B_{+,+},$$

$$\frac{\partial v}{\partial x_i} = 0 \text{ on } \Sigma_i, \quad v = 0 \text{ on } \{|x| = 1, x_1 \geq 1\}.$$ 

By Schauder estimates \ref{lem:schauder}, we have that

$$\|v_i - v_2\|_{C^{2,\alpha}(\bar{B}_{+,+})} \leq C\|L_t - L_{t_0}\|(z_1 - z_2)\|_{C^{\alpha}(\bar{B}_{+,+})}.$$ 

Note that

$$\|L_t - L_{t_0}\|(z_1 - z_2)\|_{C^{\alpha}(\bar{B}_{+,+})} \leq C|t - t_0|\|z\|_{C^{2,\alpha}(\bar{B}_{+,+})}.$$ 

We choose $\varepsilon > 0$ so small that we have

$$\|v_i - v_2\|_{C^{2,\alpha}(\bar{B}_{+,+})} \leq \frac{1}{2}\|z_1 - z_2\|_{C^{2,\alpha}(\bar{B}_{+,+})}.$$ 

Applying the contraction mapping principle, there exists a unique $u \in C^{2,\alpha}(\bar{B}_{+,+})$ such that

$$u = Au \in C^{2,\alpha}(\bar{B}_{+,+})$$

obtaining the openness of $S$. \hfill \qed

A.4. **Existence of $C^{2,\alpha}$ solutions on a cube.** We are now ready to show that the solution on a cube is in $C^{2,\alpha}(\bar{Q})$. We use a technique from Chapter 3, Section 1 of \cite{Li20} (see also \cite{GT83} Lemma 6.10).

Consider a point $x_0 \in \partial Q$, $x_0$ lies at an open edge or at the vertex. Then near $x_0$, the problem is either mixed Neumann-Neumann, mixed Dirichlet-Neumann or mixed Dirichlet-Neumann-Neumann type. We assume that $x_0$ is a vertex at the top face $T$ and will show the following theorem. Other cases is similar.

**Theorem A.11.** Near $x_0$ the solution $u$ to \ref{eq:dirichlet} is $C^{2,\alpha}$ up to the vertices.

**Proof.** Note that $u' - u$ satisfies the homogeneous boundary condition. We can perform even reflection on $u' - u$ across side faces two times (see \cite{GT83}), then apply the weak theory of Dirichlet boundary value problems. We see that $u' - u \in C^\mu(\bar{Q}) \cap C^{2,\alpha}(Q)$ for some $\mu \in (0, 1)$ using weak theory. So $u \in C^{\alpha}(\bar{Q}) \cap C^{2,\alpha}(Q)$.

Similar to Lemma \ref{lem:coordinate_system} there exists a coordinate system in a neighborhood $\Omega$ of $x_0$ such that the a piece of the top face near $x_0$ is mapped into a piece of spherical boundary $\{|x| = 1, x_1 \geq 0, x_2 \geq 0\}$ and side faces are mapped into pieces of planar boundaries $\{x_i = 0, |x| \leq 1\} (i \text{ is } 1 \text{ or } 2)$. And the metric satisfies that $g^{11} = 1$, $g^{12} = g^{13} = 0$ on $\Sigma_1$ and $g^{22} = 1$, $g^{21} = g^{23} = 0$ on $\Sigma_2$, and the spherical piece
of \( \partial B_{+,+} \) meet other pieces \( \Sigma_i \) with a constant contact angle \( \frac{\pi}{2} \). This is the same condition as in Lemma A.10.

We show that \( u \in C^{2,\alpha}(\bar{Q}) \) by approximation. We may now regard \( u \) now as a solution to a problem of the form (A.13). We can take a sequence of (compatible) boundary data \( f^{(k)}_i \) such that they converge uniformly to \( f_i \). Let \( u^{(k)}_i \) be the solution to

\[
Lu_k = f \text{ in } B_{+,+}, \quad \frac{\partial u_k}{\partial x_i} = f^{(k)}_i \text{ on } \Sigma_i, \quad u = f^{(k)}_3 \text{ on } \{ |x| = 1, x_1 \geq 1 \}.
\]

Using the maximum principle Lemma A.3, we can show that \( u_k \) converges uniformly to \( u \), and by Schauder estimates (A.9) \( u^{(k)} \in C^{2,\alpha}(\bar{B}_{+,+}) \). Combining with Arzela theorem, we have that \( u \in C^{2,\alpha}(\bar{B}_{+,+}) \). Returning to the cube \( Q \), hence we have shown that near \( x_0 \) the solution \( u \) is \( C^{2,\alpha} \) up to the vertices. \( \Box \)

**Remark A.12.** It might be possible that the assertion \( u \in C^{\mu}(\bar{Q}) \) follows directly from a weak theory for mixed boundary value problems. Then the \( C^{2,\alpha}(\bar{Q}) \) regularity follows directly from uniqueness.

**Remark A.13.** Alternatively, we can further divide the quarter ball by half and prescribe a Dirichlet boundary condition on the extra neighboring face of the origin. Then near the origin, the boundary condition is mixed Dirichlet-Neumann-Neumann type. We can study the problem in a similar fashion as in Lemma A.10. The Green function of the model problem is obtained via one more reflection. This approach is simply longer in presentation and the idea is the same.

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