SIGN-CHANGING BLOWING-UP SOLUTIONS FOR SUPERCRITICAL
BAHRI-CORON’S PROBLEM

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Abstract: Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n \geq 3 \) with smooth boundary \( \partial \Omega \) and a small hole. We give the first example of sign-changing bubbling solutions to the nonlinear elliptic problem

\[
-\Delta u = |u|^{\frac{n+2}{n-2} + \varepsilon - 1} u \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega,
\]

where \( \varepsilon \) is a small positive parameter. The basic cell in the construction is the sign-changing nodal solution to the critical Yamabe problem

\[
-\Delta w = |w|^{\frac{4}{n-2}} w, \quad w \in D^{1,2}(\mathbb{R}^n)
\]

which has large number \( (3n) \) of kernels.

1. Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n \geq 3 \) with smooth boundary \( \partial \Omega \). In this paper we establish existence of a new type of bubbling solutions to the nonlinear elliptic problem

\[
-\Delta u = |u|^{\frac{n+2}{n-2} + \varepsilon - 1} u \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega,
\]

where \( \varepsilon \) is a small positive parameter.

It is known that solvability for Problem

\[
-\Delta u = |u|^{q-1} u \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega,\tag{2}
\]

is an elementary fact when \( 1 < q < \frac{n+2}{n-2} \). This is no longer the case for \( q \geq \frac{n+2}{n-2} \) due to the loss of compactness of Sobolev embeddings. Our aim is to analyze solutions exhibiting bubbling behavior to the above problem when one lets the exponent \( q \) approach \( \frac{n+2}{n-2} \) from above.

Pohozaev [27] showed that if \( \Omega \) is strictly star-shaped then no solution of (2) exists if \( q \geq \frac{n+2}{n-2} \). In contrast Kazdan and Warner [19] showed that, if \( \Omega \) is a radially symmetric annulus, \( \Omega = \{ a < |x| < b \} \), there exists a radial positive solution to Problem (2) for any exponent \( q > 1 \). Without symmetry the question is harder. This issue was first considered by Coron [9] who found that (2) has a positive solution when \( q = \frac{n+2}{n-2} \) in any domain exhibiting a small hole. Also a second solution exists in Coron’s setting, as shown in [7], see also the results in [6, 8] and reference therein. The most general result concerning existence of positive solutions to (2) for \( q = \frac{n+2}{n-2} \) is obtained by Bahri and Coron [2]: if some homology group of \( \Omega \) with coefficients in \( \mathbb{Z}_2 \) is not trivial, then (2) has at least one positive solution, in particular in any three-dimensional domain which is not contractible to a point. Examples showing that this condition is actually not necessary for solvability were found by Dancer [10], Ding [12] and Passaseo [23], for \( q = \frac{n+2}{n-2} \) and also for very super critical powers \( q \geq \frac{n+1}{n-3} > \frac{n+2}{n-2} \), see [24]. The question of existence for super-critical powers close to critical has been addressed in [15, 14, 15, 25, 4], where existence of positive solutions to (2) is established. These solutions become unbounded as the exponent \( q \downarrow \frac{n+2}{n-2} \) and they develop a blowing-up profile.

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By a blowing-up solution for (2) near the critical exponent we mean an unbounded sequence of solutions $u_j$ of (2) for $q = q_j \to \frac{n+2}{n-2}$. Setting

$$M_j = \max_{\Omega} |u_j| = |u_j(x_j)| \to +\infty$$

we see then that the scaled function

$$v_j(y) = M_j u_j(x_j + M_j^{(q_j-1)/2} y),$$

satisfies

$$\Delta v_j + |v_j|^{q_j-1}v_j = 0$$

in the expanding domain $\Omega_j = M_j^{(q_j-1)/2}(\Omega - x_j)$. Assuming for instance that $x_j$ stays away from the boundary of $\Omega$, elliptic regularity implies that locally over compacts around the origin, $v_j$ converges up to subsequences to a solution of

$$(3) \quad \Delta w + |w|^{q_j-2}w = 0 \quad \text{in} \quad \mathbb{R}^n.$$  
Back to the original variable, “near $x_j$” the behavior of $u_j(y)$ can be approximated as

$$u_j(x) \sim \frac{1}{M_j} w \left( \frac{x - x_j}{M_j^{1/2}} \right)$$

If the solution $u_j$ develops a positive bubbling around $x_j$, then the limit profile (3) is necessarily positive. It is known, see [5] [22], that for the convenient choice $\alpha_n = (n(n-2))^{\frac{n-2}{4}}$, this solution is explicitly given by

$$\bar{w}(z) = \alpha_n \left( \frac{1}{1 + |z|^2} \right)^{\frac{n-2}{2}}$$

which corresponds precisely to an extremal of $\sigma_n$, the best constant in the critical Sobolev embedding,

$$\sigma_n = \inf_{u \in C_0^1(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2}{\left( \int_{\mathbb{R}^n} |u|^{q_j - 2} \right)^{\frac{q_j - 2}{2}}},$$

see [1] [30]. Thus, a solution blowing-up positively near $x_j$ looks at main order as

$$u_j(x) \sim \alpha_n \left( \frac{1}{1 + M_j^{\frac{n-2}{2}} |x - x_j|^2} \right)^{\frac{n-2}{2}} M_j.$$  
In [13] [14] this issue has been addressed for a class of domains which includes that considered by Coron in [9]. It is established that a positive solution to (2) exists for $q = \frac{n+2}{n-2} + \varepsilon$ with any small $\varepsilon > 0$, or equivalently to (1), if for instance $\Omega$ is a smooth domain exhibiting a sufficiently small hole: considering $\varepsilon$ as a small parameter, the solution exhibits single-bubbling around exactly two points and ceases to exist when $\varepsilon = 0$. More precisely, let $\mathcal{D}$ be a bounded, smooth domain in $\mathbb{R}^n$, $n \geq 3$, and $P$ a point of $\mathcal{D}$. Let us consider the domain

$$\Omega = \mathcal{D} \setminus \overline{B}(P, \delta)$$

where $\delta > 0$ is a small number. Then there exists a $\delta_0 > 0$, which depends on $\mathcal{D}$ and the point $P$ such that if $0 < \delta < \delta_0$ is fixed and $\Omega$ is the domain given by (8), then the following holds: There exists $\varepsilon_0 > 0$ and a solution $u_\varepsilon$, $0 < \varepsilon < \varepsilon_0$ of (1) of the form

$$u_\varepsilon(x) = \sum_{j=1}^{2} \alpha_n \left( \frac{1}{1 + \varepsilon^{-\frac{n-2}{2}} \Lambda_j^{-2} |x - \xi_j|^2} \right)^{\frac{n-4}{2}} \Lambda_j^{\frac{n-2}{2}} \varepsilon^{\frac{1}{2}} (1 + o(1)).$$
where \(o(1) \to 0\) uniformly as \(\varepsilon \to 0\). The numbers \(\lambda_j\) and the points \(\xi_j\) converge (up to subsequences) to a critical point of certain function built upon the Green’s function of \(\Omega\).

Another kind of construction for positive solutions to (11) has been recently proposed by Vaira [31]: if \(\Omega\) is such that Problem (11) at \(\varepsilon = 0\) admits a positive non-degenerate solution \(u_0\), then Problem (11) has a solution that at main order looks like the sum of \(u_0\) and a blowing-up profile as the one described in (7). Consequently, if we define the energy by

\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} dx,
\]

we have

\[
E(Q_k) = \begin{cases} 
(k + 1) S_n \left( 1 + O(k^{2-n}) \right) & \text{if } n \geq 4, \\
(k + 1) S_3 \left( 1 + O(k^{-1} \log k)^{-1} \right) & \text{if } n = 3
\end{cases}
\]

as \(k \to \infty\), where \(S_n\) is a positive constant, depending on \(n\). The solution \(Q = Q_k\) decays at infinity like the fundamental solution, namely

\[
\lim_{|x| \to \infty} |x|^{n-2} Q_k(x) = \left[ \frac{4}{n(n-2)} \right]^{\frac{n-2}{n-1}} 2^{\frac{n-2}{2}} (1 + c_k)
\]

where

\[
c_k = \begin{cases} 
O(k^{-1}) & \text{if } n \geq 4, \\
O(k^{-1} \log k)^2 & \text{if } n = 3
\end{cases}
\]

as \(k \to \infty\). Furthermore, the solution \(Q = Q_k\) has a positive global non degenerate maximum at \(x = 0\). To be more precisely we have

\[
Q(x) = [n(n-2)]^{\frac{2}{n-2}} \left( 1 - \frac{n-2}{2} |x|^2 + O(|x|^3) \right) \quad \text{as } |x| \to 0,
\]

and also there exists \(\eta > 0\), depending on \(k_0\), but independent of \(k\), so that

\[
\eta \leq Q(x) \leq Q(0) \quad \text{for all } |x| \leq \frac{1}{2},
\]

for any \(k\). Another property for the solution \(Q = Q_k\) is that it is invariant under rotation of angle \(\frac{2\pi}{k}\) in the \(x_1, x_2\) plane, namely

\[
Q(e^{\frac{2\pi}{k}} x, x') = Q(x, x'), \quad x = (x_1, x_3), \quad x' = (x_3, \ldots, x_n).
\]
It is even in the $x_j$-coordinates, for any $j = 2, \ldots, n$
\begin{equation}
Q(x_1, \ldots, x_j, \ldots, x_n) = Q(x_1, \ldots, -x_j, \ldots, x_n), \quad j = 2, \ldots, n.
\end{equation}

It respects invariance under Kelvin’s transform:
\begin{equation}
Q(x) = |x|^{2-n}Q(|x|^{-2}x).
\end{equation}
A detailed description of these solutions is given in Appendix 8. These solutions are non-degenerate, as proved in \cite{21}, in the sense precised in Section 2. More precisely the dimensional of the kernels of the linearized operator at $Q$
\begin{equation}
-\Delta \phi = p|Q|^{p-1}\phi
\end{equation}
is shown to be $3n$.

In this paper we show that, if the domain $\Omega$ has a small hole, like in \cite{13}, then a large number of sign-changing solutions to (1) exist: they blow-up with a profile $Q$ near two points of the domain, and they converges to 0, as $\varepsilon \to 0$, far from these two points.

We have the validity of the following result

**Theorem 1.1.** Let $D$ be a smooth bounded connected domain in $\mathbb{R}^n$ containing the origin 0. There exists $\delta_0$ such that, if $\delta \in (0,\delta_0)$ is fixed and $\Omega$ is the set defined by $\Omega = D \setminus \omega$, for any smooth domain $\omega \subset \overline{B}(0,\delta) \subset D$, then there exists a sequence of $\varepsilon_k$ such that, for any $\varepsilon \in (0,\varepsilon_k)$ there exists a sign changing solution $u_\varepsilon$ to (1), with
\begin{equation}
\|u_\varepsilon\|_{H^2_0(\Omega)} \leq \varepsilon^{-1} (1 + o(1))
\end{equation}

where $o(1) \to 0$ uniformly as $\varepsilon \to 0$. Up to subsequence,
\begin{equation}
\lim_{\varepsilon \to 0} \Lambda_{j_\varepsilon} = \Lambda_j \in (0,\infty), \quad \lim_{\varepsilon \to 0} \xi_{j_\varepsilon} = \xi_j \in \Omega, \quad \text{with} \quad \xi_1 \neq \xi_2.
\end{equation}

Furthermore
\begin{equation}
\frac{1}{2} \int_{\Omega} \|\nabla u_\varepsilon\|^2 dx - \frac{n-2}{2n} \int_{\Omega} |u_\varepsilon|^{\frac{2n}{n-2}+\varepsilon} = 2(k+1)S_n + O(\varepsilon \ln \varepsilon).
\end{equation}

Theorem 1.1 exhibits new concentration phenomena in which the basic cell of bubbling is not the positive solution. In the positive bubbling case, the kernel at the basic cell always contains $n+1$ dimensional kernels which corresponds to precisely the translation and scaling. For sign-changing bubbling solution, which is non-radial, the kernel at the basic cell contains not only the translations and scaling but also effects of rotation and Kelvin transform. The main difficulty is to find enough parameters to adjust. This is the main contributions of this paper. As we discover, the dominating role played is still the translation and scaling.

We mention related results on sign-changing bubbling solutions. For scalar curvature type equations, examples of sign-changing blowing-up solutions are constructed by Robert-Vetois in \cite{28}-\cite{29}. There negative bubbles are added to a positive solution to the Yamabe problem. The basic cell is still the single positive solution. Sign-changing bubbling solutions are constructed in the slightly subcritical problems ($\epsilon < 0$) by Pistoia and Vetois \cite{26}. The basic cell is a combination of positive and negative solution.

In Section 2 we precise the notion of non-degeneracy for $Q$. In Section 3 we describe the projection of the function $Q$ into $H^1_0(\Omega)$ and we give the expansion of the energy associated to the sum of two projected copies of $Q$. Section 4 is devoted to explain the construction of our solution and the scheme of the proof.
2. About the non-degeneracy of the basic cell

In [21], we proved that these solutions are non degenerate. To explain this, let us fix one solution $Q = Q_k$ of the family and define the linearized equation around $Q$ for Problem (10) as follows

$$L(\phi) = \Delta \phi + p|Q|^{p-1} \phi.$$ 

The invariances (15), (16), (17), together with the natural invariance of any solution to (10) under translation (if $u$ solves (10) then also $u(x + \xi)$ solves (10) for any $\xi \in \mathbb{R}^n$) and under dilation (if $u$ solves (10) then $\lambda^{-\frac{n-2}{2}} u(\lambda^{-1} x)$ solves (10) for any $\lambda > 0$) produce some natural functions $\varphi$ in the kernel of $L$, namely

$$L(\varphi) = 0.$$ 

These are the $3n$ linearly independent functions we introduce next:

$$z_0(x) = \frac{n-2}{2} Q(x) + \nabla Q(x) \cdot x,$$

$$z_\alpha(x) = \frac{\partial}{\partial x_\alpha} Q(x), \quad \text{for } \alpha = 1, \ldots, n,$$

and

$$z_{n+1}(x) = -x_2 \frac{\partial}{\partial x_1} Q(x) + x_1 \frac{\partial}{\partial x_2} Q(x),$$

$$z_{n+2}(x) = -2x_1 z_0(x) + |x|^2 z_1(x), \quad z_{n+3}(x) = -2x_2 z_0(x) + |x|^2 z_2(x)$$

and, for $l = 3, \ldots, n$

$$z_{n+l+1}(x) = -x_l z_1(x) + x_1 z_l(x), \quad z_{2n+l-1}(x) = -x_l z_2(x) + x_2 z_l(x).$$

Indeed, a direct computation gives that

$$L(z_\alpha) = 0, \quad \text{for all } \alpha = 0, 1, \ldots, 3n - 1.$$ 

The function $z_0$ defined in (19) is related to the invariance of Problem (10) with respect to dilation $\lambda^{-\frac{n-2}{2}} Q(\lambda^{-1} x)$. The functions $z_i$, $i = 1, \ldots, n$, defined in (20) are related to the invariance of Problem (10) with respect to translation $Q(x + \xi)$. The function $z_{n+1}$ defined in (21) is related to the invariance of $Q$ under rotation in the $(x_1, x_2)$ plane. The two functions $z_{n+2}$ and $z_{n+3}$ defined in (22) are related to the invariance of Problem (10) under Kelvin transformation (17). The functions defined in (23) are related to the invariance under rotation in the $(x_1, x_2)$ plane and in the $(x_2, x_1)$ plane respectively.

Let us be more precise. Denote by $O(n)$ the orthogonal group of $n \times n$ matrices $M$ with real coefficients, so that $M^T M = I$, and by $SO(n) \subset O(n)$ the special orthogonal group of all matrices in $O(n)$ with $\det M = 1$. $SO(n)$ is the group of all rotations in $\mathbb{R}^n$, it is a compact group, which can be identified with a compact set in $\mathbb{R}^{n(n-1)/2}$. Consider the subgroup $\hat{S}$ of $SO(n)$ generated by rotations in the $(x_1, x_2)$-plane, in the $(x_j, x_\alpha)$-plane, for any $j = 1, 2$ and $\alpha = 3, \ldots, n$. We have that $\hat{S}$ is compact and can be identified with a compact manifold of dimension $2n - 3$, with no boundary. In other words, there exists a smooth injective map $\chi : \hat{S} \to \mathbb{R}^{n(n-1)/2}$ so that $\chi(\hat{S})$ is a compact manifold of dimension $2n - 3$ with no boundary and $\chi^{-1} : \chi(\hat{S}) \to \hat{S}$ is a smooth parametrization of $\hat{S}$ in a neighborhood of the Identity. Thus we write

$$\theta \in K = \chi(\hat{S}), \quad R_{\theta} = \chi^{-1}(\theta)$$

where $K$ a compact manifold of dimension $2n - 3$ with no boundary and $R_{\theta}$ denotes a rotation in $\hat{S}$. 
Let $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$, and define
\begin{equation}
\Theta_A(x) = \lambda^{-\frac{n-2}{2}} |\eta_{\lambda, \xi, a}(x)|^{2-n} Q \left( \frac{R_\theta \left( \frac{x-\xi}{\lambda} - a \frac{x-\xi}{|x-\xi|} \right)}{|\eta_{\lambda, \xi, a}(x)|^2} \right),
\end{equation}
where
\begin{equation}
\eta_{\lambda, \xi, a}(x) = \frac{x - \xi}{|x - \xi|} - a \frac{|x - \xi|}{\lambda}
\end{equation}
and $Q$ is our fixed non degenerate solution to Problem (10) described above. In [18] it is proven that for any choice of $A$, the function $\Theta_A$ is still a solution of (10), namely
\[ \Delta \Theta_A + |\Theta_A|^{p-1} \Theta_A = 0, \quad \text{in} \quad \mathbb{R}^n. \]

For any set of parameters $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$, we introduce the function
\begin{equation}
Q_A(x) = \Theta_A(R_\theta^{-1} x)
\end{equation}
\begin{equation}
= \lambda^{-\frac{n-2}{2}} |\eta_{\lambda, \xi, a}(R_\theta^{-1} x)|^{2-n} Q \left( \frac{R_\theta \left( \frac{R_\theta^{-1} x-\xi}{\lambda} - a \frac{R_\theta^{-1} x-\xi}{|R_\theta^{-1} x-\xi|} \right)}{|\eta_{\lambda, \xi, a}(R_\theta^{-1} x)|^2} \right)
\end{equation}
More explicitly
\begin{equation}
Q_A(x) = \lambda^{-\frac{n-2}{2}} \frac{x - R_\theta \xi}{|x - R_\theta \xi|} - R_\theta a \frac{|x - R_\theta \xi|}{\lambda} \right)^{2-n} \times
\end{equation}
\begin{equation}
\times Q \left( \frac{x-R_\theta \xi}{\lambda} - R_\theta a \frac{|x-R_\theta \xi|}{\lambda} \right) \left( \frac{1}{1 - 2 R_\theta a \cdot (\frac{2}{R_\theta} x-R_\theta \xi) + |a|^2 |\frac{x-R_\theta \xi}{\lambda}|^2} \right).
\end{equation}

Easy but long computations give the following natural relations between $z_\alpha$ and differentiation of $Q_A$ with respect to each component of $A$. More precisely, one has
\begin{equation}
z_0(y) = - \frac{\partial}{\partial \lambda} [ Q_A(x) ]|_{\lambda=1, \xi=0, a=0, \theta=0}
\end{equation}
\begin{equation}
z_\alpha(y) = - \frac{\partial}{\partial \xi_\alpha} [ Q_A(x) ]|_{\lambda=1, \xi=0, a=0, \theta=0}, \quad \alpha = 1, \ldots, n,
\end{equation}
\begin{equation}
z_{n+2}(y) = \frac{\partial}{\partial a_1} [ Q_A(x) ]|_{\lambda=1, \xi=0, a=0, \theta=0},
\end{equation}
\begin{equation}
z_{n+3}(y) = \frac{\partial}{\partial a_2} [ Q_A(x) ]|_{\lambda=1, \xi=0, a=0, \theta=0}.
\end{equation}
Now, let $\theta = (\theta_{12}, \theta_{13}, \ldots, \theta_{1n}, \theta_{23}, \ldots, \theta_{2n})$, where $\theta_{ij}$ represents the rotation in the $(i, j)$-plane. Then
\begin{equation}
z_{n+1}(y) = \frac{\partial}{\partial \theta_{12}} [ Q_A(x) ]|_{\lambda=1, \xi=0, a=0, \theta=0}
\end{equation}
and, for any $l = 3, \ldots, n,$
\begin{equation}
z_{n+l+1}(y) = \frac{\partial}{\partial \theta_{1l}} [ Q_A(x) ]|_{\lambda=1, \xi=0, a=0, \theta=0}
\end{equation}
\begin{equation}
z_{2n+l-1}(y) = \frac{\partial}{\partial \theta_{2l}} [ Q_A(x) ]|_{\lambda=1, \xi=0, a=0, \theta=0}.
\end{equation}
Following [15], a solution $Q$ is said to be non degenerate if
\begin{equation}
\text{Kernel}(L) = \text{Span}\{z_\alpha : \alpha = 0, 1, 2, \ldots, 3n - 1\},
\end{equation}
or equivalently, any bounded (or any solution in $\mathcal{D}^{1,2}$) of $L(\varphi) = 0$ is a linear combination of the functions $z_\alpha$, $\alpha = 0, \ldots, 3n - 1$.

In [21] we proved that, under certain condition on the dimension $n$, the solution $Q$ is non-degenerate. Indeed, in [21] we showed that in all dimensions $n \leq 48$, any solution $Q = Q_k$ is non-degenerate in the sense defined above. If dimension $n \geq 49$, our result [21] guarantees the existence of a subsequence of solutions $Q_k$, each one of which is non-degenerate.

3. First approximation and expansion of the energy

The existence of a non-degenerate solution in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ of (10) is the basic element for our construction. In fact, we can perform our construction starting from any one of the infinitely many solutions $Q = Q_k$ with the property of being non-degenerate. Thus, from now on, we fix a function $Q = Q_k$, and for simplicity of notation we drop the index $k$.

The function $Q_A$ defined in (26) will be the building block of our construction. We first correct it so that it satisfies zero boundary condition on $\partial \Omega$. This is done defining $PQ_A$ to be the projection of $Q_A$ onto $H^1_0(\Omega)$, namely the unique solution to

$$\Delta u = \Delta Q_A \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega.$$  

In other words, $PQ_A = Q_A - \varphi_A$ where $\varphi_A$ solves

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u = Q_A \quad \text{on} \quad \partial \Omega.$$  

Next Lemma provides a precise description of $PQ_A$ and $\varphi_A$, when $\lambda \to 0$. To state the result we need to recall the following. Let us denote by $G(x,y)$ the Green’s function of the domain, namely $G$ satisfies

$$\Delta_x G(x,y) = \delta(x-y), \quad x \in \Omega, \quad G(x,y) = 0, \quad x \in \partial \Omega,$$

where $\delta(x)$ denotes the Dirac mass at the origin. We denote by $H(x,y)$ its regular part, namely

$$H(x,y) = \Gamma(x-y) - G(x,y)$$

where $\Gamma$ denotes the fundamental solution of the Laplacian,

$$\Gamma(x) = b_n|x|^{2-n},$$

so that $H$ satisfies

$$\Delta_x H(x,y) = 0, \quad x \in \Omega, \quad H(x,y) = \Gamma(x-y), \quad x \in \partial \Omega.$$  

Its diagonal $H(x,x)$ is usually called Robin function.

Lemma 3.1. Assume that $a \in \mathbb{R}^2$, $\xi \in \mathbb{R}^n$, and $\theta \in K$ are fixed so that $R_\theta \xi \in \Omega$. We have the validity of the following estimates:

$$\varphi_A(x) = b_n^{-1} \lambda \frac{x-R_{\theta}a}{|x-R_{\theta}a|} Q(R_{\theta}a) H(x,R_{\theta} \xi) + O(\lambda^\frac{4}{2})$$

uniformly for $x \in \Omega$, as $\lambda \to 0$. Furthermore,

$$PQ_A(x) = b_n^{-1} \lambda \frac{x-R_{\theta}a}{|x-R_{\theta}a|} Q(R_{\theta}a) G(x,R_{\theta} \xi) + O(\lambda^\frac{4}{2})$$

uniformly for $x$ in compact sets of $\Omega \setminus \{R_{\theta} \xi\}$, as $\lambda \to 0$. In (40) and (41), $H$ and $G$ are the functions defined respectively in (38) and (37), $b_n$ is a positive constant defined in (33), and $O(1)$ denotes a smooth function of $x$ which is uniformly bounded as $\lambda \to 0$.

Proof. Let $x \in \partial \Omega$ and let $\delta > 0$ so that dist $(x,R_{\theta}\xi) > \delta$. Define $X = \frac{x-R_{\theta} \xi}{\lambda}$. The argument of the function $Q$ in (26) gets re-written as

$$\frac{x-R_{\theta} \xi - R_{\theta}a|X|^2|X|}{1 - 2R_{\theta}a \cdot (\frac{x-R_{\theta} \xi}{\lambda}) + |a|^2|\frac{x-R_{\theta} \xi}{\lambda}|^2} = \frac{X - R_{\theta}a|X|^2}{1 - 2R_{\theta}a \cdot X + |a|^2|X|^2}.$$
Observe that
\[ X = \frac{x - R_\theta \xi}{\lambda} \rightarrow \infty \text{ uniformly on } x \in \partial \Omega, \text{ as } \lambda \rightarrow 0. \]
Thus we have
\[
\frac{X - R_\theta a |X|^2}{1 - 2 R_\theta a \cdot (X) + |a|^2 |X|^2} = \frac{X - R_\theta a |X|^2}{|a|^2 |X|^2 \left[ 1 - 2 \frac{R_\theta a}{|a|^2} \cdot \frac{X}{|X|^2} + \frac{1}{|a|^2 |X|^2} \right]^2}
\]
\[
= - \frac{R_\theta a}{|a|^2} \left[ 1 - 2 \frac{R_\theta a}{|a|^2} \cdot \frac{X}{|X|^2} + \frac{1}{|a|^2 |X|^2} \right]^{-2}
+ \frac{X}{|a|^2 |X|^2} \left[ 1 - 2 \frac{R_\theta a}{|a|^2} \cdot \frac{X}{|X|^2} + \frac{1}{|a|^2 |X|^2} \right]^{-2}
\]
(42)

A Taylor expansion gives that, uniformly for \( x \in \partial \Omega \), one has
\[
\left[ 1 - 2 \frac{R_\theta a}{|a|^2} \cdot \frac{X}{|X|^2} + \frac{1}{|a|^2 |X|^2} \right]^{-2} = 1 - 2 \left( -2 \frac{R_\theta a}{|a|^2} \cdot \frac{X}{|X|^2} + \frac{1}{|a|^2 |X|^2} \right)^2 + O(\lambda^2)
\]
\[
+ 3 \left( -2 \frac{R_\theta a}{|a|^2} \cdot \frac{X}{|X|^2} + \frac{1}{|a|^2 |X|^2} \right)^2 + O(\lambda^3)
\]
\[
= 1 + 4 \frac{R_\theta a}{|a|^2} \cdot \frac{X}{|X|^2} - \left( \frac{2}{|a|^2 |X|^2} + 12 \frac{R_\theta a}{|a|^2} \cdot \frac{X}{|X|^2} \right)^2 + O(\lambda^3)
\]
\[
= 1 + 4 \frac{R_\theta a}{|a|^2} \cdot \frac{X}{|X|^2} + O(\lambda^3)
\]

as \( \lambda \rightarrow 0 \). Inserting this information in (42), we get that
\[
\frac{X - R_\theta a |X|^2}{1 - 2 R_\theta a \cdot (X) + |a|^2 |X|^2} = - \frac{R_\theta a}{|a|^2} + \frac{X}{|a|^2 |X|^2} + O(\lambda^2)
\]
(43)

\[
R_\theta a \frac{X}{|a|^2 |X|^2} + O(\lambda^2)
\]
\[
R_\theta a \frac{x - R_\theta \xi}{|a|^2 |x - R_\theta \xi|^2} \lambda
\]
\[
- 4 \frac{R_\theta a}{|a|^2} \frac{x - R_\theta \xi}{|x - R_\theta \xi|^2} \lambda + O(\lambda^2).
\]

Thanks to (43), we get that
\[
Q \left( \frac{X - R_\theta a |X|^2}{1 - 2 a \cdot (X) + |a|^2 |X|^2} \right) = Q(- \frac{R_\theta a}{|a|^2})
\]
\[
+ \nabla Q(- \frac{R_\theta a}{|a|^2}) \cdot \frac{x - R_\theta \xi}{|a|^2 |x - R_\theta \xi|^2} - 4 \frac{R_\theta a}{|a|^2 |x - R_\theta \xi|^2} \lambda + O(\lambda^2)
\]
uniformly for \( x \in \partial \Omega \), as \( \lambda \rightarrow 0 \).

On the other hand, in the same region \( x \in \partial \Omega \), we have that
\[
\left| \frac{x - R_\theta \xi}{|x - R_\theta \xi|} - \frac{x - R - \theta \xi}{\lambda} \right|^{2-n} = \left| \frac{\lambda^{n-2}}{|a|^{n-2} |x - R_\theta \xi|^n} \right|^2 \lambda^{-n+2}
\]
\[
+ \left| \frac{n-2}{|a|^{n-2} |x - R_\theta \xi|^n} \frac{R_\theta a}{|a|^2 |x - R_\theta \xi|^2} \lambda^{n-1} \right| + O(\lambda^n).
\]
(45)
Lemma 3.2. Given \( \delta > 0 \) we have the validity of the expansion

\[
J_0(PQ_1 + PQ_2) = 2\gamma_n + \alpha_n H(\tilde{\xi}_1, \tilde{\xi}_2) Q(\hat{a}_1)^2 \lambda_1^{n-2} + \alpha_n H(\tilde{\xi}_2, \tilde{\xi}_2) Q(\hat{a}_2)^2 \lambda_2^{n-2} - \alpha_n 2G(\tilde{\xi}_1, \tilde{\xi}_2) Q(\hat{a}_1)Q(\hat{a}_2) \lambda_1^{n-2} \lambda_2^{n-2} + o(\max\{\lambda_1, \lambda_2\}^{n-2})
\]

as \( \lambda_1, \lambda_2 \to 0 \), uniformly in the set satisfying constraints (49). In (51), \( \alpha_n \) denotes a fixed positive constant, independent of \( \varepsilon \).
Proof. The full expansion (51) is consequence of the following formulas:

\[ \int_\Omega |\nabla PQ_i|^2 = \int_{\mathbb{R}^n} |\nabla Q|^2 - 2\alpha_n H(\hat{\xi}_i, \hat{\xi}_i) Q^2(\hat{a}_i) \lambda_i^{n-2} 
+ o(\lambda_i^{n-2}), \]

(52)

\[ \int_\Omega \nabla PQ_i \nabla PQ_2 = 2\alpha_n G(\hat{\xi}_1, \hat{\xi}_2) Q(\hat{a}_1)Q(\hat{a}_2) \lambda_1^{n-2} \lambda_2^{n-2} + o(\lambda_1^{n-2}), \lambda_2^{n-2} \]

(53)

\[ \frac{1}{p+1} \int_\Omega |PQ_1 + PQ_2|^{p+1} - |PQ_1|^{p+1} - |PQ_2|^{p+1} \]

\[ = 4\alpha_n G(\hat{\xi}_1, \hat{\xi}_2) Q(\hat{a}_1)Q(\hat{a}_2) \lambda_1^{n-2} \lambda_2^{n-2} + o(\max\{\lambda_1, \lambda_2\}^{n-2}) \]

(54)

and

\[ \frac{1}{p+1} \int_\Omega |PQ_i|^{p+1} = \frac{1}{p+1} \int_{\mathbb{R}^n} |Q|^{p+1} - 2\alpha_n H(\hat{\xi}_i, \hat{\xi}_i) Q^2(\hat{a}_i) \lambda_i^{n-2} + o(\lambda_i^{n-2}). \]

(55)

Indeed, we decompose

\[ J_0(PQ_1 + PQ_2) = \sum_{i=1,2} \frac{1}{2} \left[ \int_\Omega |\nabla PQ_i|^2 - \frac{1}{p+1} \int_\Omega |PQ_i|^{p+1} \right] \]

\[ + \int_\Omega \nabla PQ_1 \nabla PQ_2 \]

\[ - \frac{1}{p+1} \int_\Omega [|PQ_1 + PQ_2|^{p+1} - |PQ_1|^{p+1} - |PQ_2|^{p+1}]. \]

Thus substituting estimates (52), (53), (54) and (55) in this relation we obtain the thesis.

In what is left of this proof, we shall show the validity of (52), (53), (54) and (55).

Proof of (52). For simplicity, we write \( \varphi_i(x) = \varphi_A_i(x) \). An integration by parts gives that

\[ \int_\Omega |\nabla PQ_i|^2 dx = \int_\Omega |Q_i|^{p+1} dx - \int_\Omega |Q_i|^{p-1} Q_i \varphi_i dx \]

(56)

taking into account that \( -\Delta PQ_i = |Q_i|^{p-1} Q_i \). Let us first compute \( \int_\Omega |Q_i|^{p+1} dx \), writing

\[ \int_\Omega |Q_i|^{p+1} dx = \int_{|x-\hat{\xi}_i|<\delta} |Q_i|^{p+1} dx + \int_{|x-\hat{\xi}_i|>\delta} |Q_i|^{p+1} dx \]
for some $\delta > 0$ fixed and small. In the ball $|x - \xi_i| < \delta$ we introduce the change of variables $y = \frac{x - \xi_i}{\lambda_i}$, so that

$$
\int_{|x - \xi_i| < \delta} |Q_i|^{p+1} \, dx = \int_{|y| < \frac{\delta}{\lambda_i}} \left[ \frac{y}{|y|} - \frac{\hat{a}_i}{|\hat{a}_i|} \right] |Q|^{p+1} \left( \frac{\hat{\nu}}{|\hat{\nu}|^2} - \frac{\lambda_i}{|\lambda_i|^2} \right) \, dy
$$

$$
= \int_{|y| < \frac{\delta}{\lambda_i}} \left[ \frac{y}{|y|^2} - \frac{\hat{a}_i}{|\hat{a}_i|} \right] |Q|^{p+1} \left( \frac{\hat{\nu}}{|\hat{\nu}|^2} - \frac{\lambda_i}{|\lambda_i|^2} \right) \, dy
$$

since $|w|^{2-n}Q(\frac{w}{|w|^2}) = Q(w)$

$$
= \int_{|y| < \frac{\delta}{\lambda_i}} |y|^{2-n} |Q| \left( \frac{y}{|y|^2} - \frac{\hat{a}_i}{|\hat{a}_i|} \right) \, dy
$$

$$
= \int_{|y| < \frac{\delta}{\lambda_i}} |y|^{2n} \left[ |Q| \left( \frac{y}{|y|^2} - \frac{\hat{a}_i}{|\hat{a}_i|} \right) \right] \, dy
$$

using the change of variables $z = \frac{y}{|y|^2}$

$$
= \int_{|x| > \frac{\lambda_i}{\delta}} |Q(z - \hat{a}_i)|^{p+1} \, dz = \int_{\mathbb{R}^n} |Q|^{p+1} \, dz + O(\lambda_i^n),
$$

where $O(1)$ denotes a generic smooth function of the parameters that is uniformly bounded as $\lambda_i \to 0$. Observe that the last expansion is consequence of (13), (14) and (19). On the other hand, in the set $|x - \xi_i| > \delta$ we have the validity of the expansion (47), so that we conclude that

$$
\left| \int_{|x - \xi_i| > \delta} |Q_i|^{p+1} \, dx \right| \leq C \lambda_i^{\frac{n+2}{2}} |Q(\hat{a}_i)| \leq C \lambda_i^{\frac{n+2}{2}}
$$

where again we use the assumption in (19) that $|a_i| \leq \frac{1}{2}$, and also (13)-(14).

We thus conclude that

$$
\int_{\Omega} |PQ_i|^{p+1} \, dx = \int_{\mathbb{R}^n} |Q|^{p+1} \, dz + O(\lambda_i^{\frac{n+2}{2}}).
$$

We turn now to $\int_{\Omega} |Q_i|^{p-1}Q_i\varphi_i \, dx$. We claim that

$$
\int_{\Omega} |Q_i|^{p-1}Q_i\varphi_i \, dx = b_i^{-2} \lambda_i^{-2} Q^2(\hat{a}_i) H(\xi_i, \xi_i) + O(\lambda_i^{-1}),
$$

where $O(1)$ is uniformly bounded, as $\lambda_i \to 0$, in the set of parameters satisfying (19). We decompose

$$
\int_{\Omega} |Q_i|^{p-1}Q_i\varphi_i \, dx = \int_{|x - \xi_i| < \delta} |Q_i|^{p-1}Q_i\varphi_i \, dx + \int_{|x - \xi_i| > \delta} |Q_i|^{p-1}Q_i\varphi_i \, dx
$$

Recalling the validity of the expansion (40) for $\varphi_i$, we get that

$$
\int_{|x - \xi_i| < \delta} |Q_i|^{p-1}Q_i\varphi_i \, dx = b_i^{-2} \lambda_i^{\frac{n+2}{2}} Q(\hat{a}_i) H(\xi_i, \xi_i) \left( \int_{|x - \xi_i| < \delta} |Q_i|^{p-1}Q_i \, dx \right)
$$

$$
+ \left( \int_{|x - \xi_i| < \delta} |Q_i|^{p-1}Q_i \, dx \right) O(\lambda_i^{\frac{n+2}{2}})
$$

$$
+ \left( \int_{|x - \xi_i| < \delta} |Q_i|^{p-1}Q_i |x - \xi_i| \, dx \right) O(\lambda_i^{\frac{n+2}{2}}).$$
In the ball $|x - \hat{\xi}_i| < \delta$ we introduce the change of variables $y = \frac{x - \hat{\xi}_i}{\lambda_i}$, and using the invariance of $Q$ under Kelvin transform,
\[
\int_{|x - \hat{\xi}_i| < \delta} |Q_1|^{p-1} Q_1 \, dx = \lambda_i^{n-2} \int_{|y| < \frac{\delta}{\lambda_i}} \left[ |y|^{2-n} |Q_1| \left( \frac{y}{|y|^2} - \hat{a}_i \right) \right]^p \, dy
\]
\[
= \lambda_i^{n-2} \int_{|y| < \frac{\delta}{\lambda_i}} |y|^{n+2} \left[ |Q_1| \left( \frac{y}{|y|^2} - \hat{a}_i \right) \right]^p \, dy
\]
using the change of variables $z = \frac{y}{|y|^2}$
\[
= \lambda_i^{n-2} \int_{|z| > \frac{\delta}{\lambda_i}} \frac{1}{|z|^{n-2}} |Q_1(z - \hat{a}_i)|^p \, dz
\]
\[
= \lambda_i^{n-2} \int_{\mathbb{R}^n} \frac{1}{|z + \hat{a}_i|^{n-2}} |Q_1|^p \, dz + o(\lambda_i^{n-2}),
\]
as $\lambda_i \to 0$. Recall now that
\[
Q(\hat{a}_i) = b_n \int_{\mathbb{R}^n} \frac{1}{|z + \hat{a}_i|^{n-2}} |Q|^p \, dz.
\]
Thus we get
\[
(59) \quad \int_{|x - \hat{\xi}_i| < \delta} |Q_1|^{p-1} Q_1 \, dx = b_n^{-1} \lambda_i^{n-2} Q(\hat{a}_i) + o(\lambda_i^{n-2}).
\]
On the other hand, using again the change of variables $y = \frac{x - \hat{\xi}_i}{\lambda_i}$ one finds directly that
\[
\int_{|x - \hat{\xi}_i| < \delta} |Q_1|^{p-1} Q_1 |x - \hat{\xi}_i| \, dx = O(\lambda_i^{n-2}).
\]
On the other hand, we have
\[
\int_{|x - \hat{\xi}_i| > \delta} |Q_1|^{p-1} Q_1 \varphi_i \, dx = O(\lambda_i^{n-1})
\]
as direct consequence of (40) and (47). Collecting the above estimates we get the validity of (58). Expansion (52) follows directly from (57) and (58).

Proof of (73). Arguing as in the proof of (52), it holds
\[
\int_\Omega \nabla P Q_1 \nabla P Q_2 = \int_\Omega |Q_1|^{p-1} Q_1 P Q_2 = \left( \int_{|x - \hat{\xi}_2| < \delta} |Q_1|^{p-1} Q_1 P Q_2 \, dx \right) (1 + o(1)),
\]
as $\lambda_i \to 0$, $i = 1, 2$. Now, using (11), we get
\[
\int_{|x - \hat{\xi}_1| < \delta} |Q_1|^{p-1} Q_1 P Q_2 \, dx = b_n^{-1} \lambda_2^{n-2} Q(\hat{a}_2) G(\hat{\xi}_1, \hat{\xi}_2) \times
\]
\[
\left( \int_{|x - \hat{\xi}_1| < \delta} |Q_1|^{p-1} Q_1 \, dx \right) (1 + O(\lambda_2))
\]
Thanks to (57), we conclude that
\[
\int_{|x - \hat{\xi}_1| < \delta} |Q_1|^{p-1} Q_1 P Q_2 \, dx = b_n^{-2} \lambda_1^{n-2} \lambda_2^{n-2} Q(\hat{a}_1) Q(\hat{a}_2) G(\hat{\xi}_1, \hat{\xi}_2)
\]
\[
+ o(\max\{\lambda_1, \lambda_2\}^{n-2}).
\]
This gives the validity of (53).
Proof of (54). We write
\[
\int_\Omega \left[ |PQ_1 + PQ_2|^{p+1} - |PQ_1|^{p+1} - |PQ_2|^{p+1} \right] = \int_{|x-\hat{\xi}_1|<\delta} |PQ_1|^{p-1}PQ_1PQ_2 \\
+ \int_{|x-\hat{\xi}_2|<\delta} |PQ_2|^{p-1}PQ_2PQ_1 + O((\lambda_1\lambda_2)^{\frac{n+2}{2}}) \\
= \int_{|x-\hat{\xi}_1|<\delta} |Q_1|^{p-1}Q_1PQ_2 \\
+ \int_{|x-\hat{\xi}_2|<\delta} |Q_2|^{p-1}Q_2PQ_1 + O((\lambda_1\lambda_2)^{\frac{n+2}{2}})
\]
At this point, (54) follows directly from (60).

Proof of (55). Using the estimate (47), we see that
\[
\int_\Omega |PQ_i|^{p+1} dx = \int_{|x-\hat{\xi}_i|<\delta} |PQ_i|^{p+1} dx + O(\lambda_i^{n+2}).
\]
On the other hand, a Taylor expansion gives
\[
\int_{|x-\hat{\xi}_i|<\delta} |PQ_i|^{p+1} dx = \frac{1}{p+1} \int_{|x-\hat{\xi}_i|<\delta} |PQ_i|^{p+1} dx \\
+ \left( \int_{|x-\hat{\xi}_i|<\delta} |PQ_i|^{p-1}PQ_i\varphi_i dx \right) (1 + o(1))
\]
\[
= \frac{1}{p+1} \int_{|x-\hat{\xi}_i|<\delta} |PQ_i|^{p+1} dx \\
+ \left( \int_{|x-\hat{\xi}_i|<\delta} |Q_i|^{p-1}Q_i\varphi_i dx \right) (1 + o(1))
\]
We now apply (57) and (58) to get (55). \(\square\)

We shall now choose the numbers \(\lambda_i\) in terms of \(\epsilon\): we will assume
\[
\lambda_i^{n-2} = \int_{\mathbb{R}^n} |Q_i|^{p+1} \frac{1}{n\alpha_n} \Lambda_2^2 \epsilon.
\]
Let us consider the energy functional, associated to problem (61),
\[
J_\epsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1+\epsilon} \int_\Omega u^{p+1+\epsilon}.
\]
Arguing like in [13], we show that
\[
J_\epsilon(PQ_1 + PQ_2) = J_0(PQ_1 + PQ_2) \\
+ 2\epsilon \left[ \frac{1}{(p+1)^2} \int_{\mathbb{R}^n} |Q|^{p+1} - \frac{1}{p+1} \int_{\mathbb{R}^n} |Q|^{p+1} \log |U| \right] \\
+ \frac{(n-2)^2}{4n} \epsilon \log(\lambda_1\lambda_2) \int_{\mathbb{R}^n} |Q|^{p+1} + o(\epsilon).
\]
Combining this estimate with the previous lemma, and our choice (61) for \(\lambda_i\), we get the following result.
Lemma 3.3. Let $\delta > 0$ and assume that
\[
(\hat{\xi}_1, \hat{\xi}_2) \in \Omega \times \Omega : \text{dist}(\hat{\xi}_i, \partial \Omega) > \delta, |\hat{\xi}_1 - \hat{\xi}_2| > \delta
\]
\[
\delta < \Lambda_i < \delta^{-1}, \quad \theta_i \in K, \quad |a_i| \leq \frac{1}{2}, \quad i = 1, 2,
\]
where
\[
\chi_i^{n-2} = \int_{\mathbb{R}^n} |Q|^{p+1} - \frac{1}{n} \Lambda_i^2 \varepsilon, \quad \hat{\xi}_i = R_{\theta_i} \xi_i, \quad \hat{a}_i = R_{\theta_i} a_i, \quad i = 1, 2.
\]
Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ we have
\[
J_\varepsilon(U_1 + U_2) = 2\gamma_n + \chi_n \varepsilon - \frac{1}{p+1} \int_{\mathbb{R}^n} |Q|^{p+1} \Psi(A_1, A_2) + o(\varepsilon),
\]
uniformly with respect to sets of parameters $A_1$ and $A_2$ satisfying [71]. Here
\[
\Psi(A_1, A_2) = \frac{1}{2} H(\hat{\xi}_1, \hat{\xi}_2) Q^2(\hat{a}_1) \Lambda_1^2 + \frac{1}{2} H(\hat{\xi}_2, \hat{\xi}_2) Q^2(\hat{a}_2) \Lambda_2^2
\]
\[
- G(\hat{\xi}_1, \hat{\xi}_2) Q(\hat{a}_1) Q(\hat{a}_2) \Lambda_1 \Lambda_2 + \log \Lambda_1 \Lambda_2,
\]
while $\eta_n$ and $\chi_n$ are the constants defined by
\[
\eta_n = 2(\frac{1}{p+1}) \int_{\mathbb{R}^n} |Q|^{p+1} - \frac{1}{p+1} \int_{\mathbb{R}^n} |Q|^{p+1} \Psi(A_1, A_2)
\]
\[
+ \frac{1}{p+1} \chi_n \log \left( \frac{\int_{\mathbb{R}^n} |Q|^{p+1}}{n \alpha_n} \right)
\]
and $\chi_n = \frac{1}{p+1} \int_{\mathbb{R}^n} |Q|^{p+1}$.

4. Scheme of the proof

The solution predicted by Theorem [14] will have the form
\[
u(x) = (1 + \zeta) \left( u_0(x) + \hat{\phi}(x) \right),
\]
where $\zeta$ is the number defined by
\[
\zeta = \varepsilon \frac{1}{\alpha(p+1)} - 1
\]
and $u_0$ is the function defined by
\[
u_0(x) = PQ_1(x) + PQ_2(x)
\]
where for $j = 1, 2$, $PQ_j$ is the $H^1_0(\Omega)$-projection of the function $Q_{A_j}$ defined in [23], for some parameter $A_j = (\lambda_j, \xi_j, a_j, \theta_j)$. In [33], the function $\hat{\phi}$ has to be determined to have that $u$ is a solution to [1].

It is useful to rephrase Problem [1] in some expanded domain $\Omega_\varepsilon = \varepsilon^{-\frac{1}{p-1}} \Omega$. After the change of variables
\[
v(y) = \varepsilon^{\frac{1}{2}} u(\varepsilon^{\frac{1}{n-2}} y), \quad y = \varepsilon^{-\frac{1}{n-2}} x \in \Omega_\varepsilon,
\]
Problem [1] gets re-written as
\[
\Delta v + v^{p+\varepsilon} = 0, \quad v > 0, \quad \text{in } \Omega_\varepsilon \quad v = 0, \quad \text{on } \partial \Omega_\varepsilon,
\]
In the expanded variables, the solution in [33] gets into the form
\[
v(y) = v_0(y) + \phi(y), \quad \text{where } v_0(y) = \varepsilon^{\frac{1}{2}} u_0(\varepsilon^{\frac{1}{n-2}} y).
\]
In order to determine the unknown function $\phi$, we proceed in two steps. In the first step, we fix the parameters $A_1$ and $A_2$, and we find $\phi$ as solution of a proper non linear projected problem. With abuse of notation, we denote

$$A_j = (\Lambda_j, \xi_j, a_j, \theta_j) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^2 \times K$$

and we assume the following constraints on $A_j$, $j = 1, 2$:

$$\langle \xi_1, \xi_2 \rangle \in \Omega \times \Omega : \text{dist}(\xi_i, \partial \Omega) > \delta, \ |\xi_1 - \xi_2| > \delta$$

(71)

$$\delta < \Lambda_i < \delta^{-1}, \quad \theta_i \in \chi(\hat{S}), \quad |a_i| \leq \frac{1}{2}, \quad i = 1, 2,$$

for some fixed $\delta > 0$, where

$$\lambda_i^{n-2} = \beta_n \Lambda_i^n \varepsilon, \quad \hat{\xi}_i = R_{\theta_i} \xi_i, \quad \hat{a}_i = R_{\theta_i} a_i, \quad i = 1, 2,$$

with $\beta_n$ the positive number defined by $\beta_n = \frac{\int_0^\alpha |Q|^{p-1}}{n \alpha}$. Observe that a direct computation gives that, as $\varepsilon \to 0$,

$$v_0(y) = \sum_{j=1}^{2} (\Lambda_j)^{-\frac{n-2}{2}} \left| \eta_{\Lambda_j, R_{\theta_j} \xi_j, a_j}(y) \right|^2 - n \int_{\Omega} \left( \frac{y - R_{\theta_j} \xi_j}{\Lambda_j} - \frac{y - R_{\theta_j} \xi_j}{\Lambda_j} \right) \eta_{\Lambda_j, R_{\theta_j} \xi_j, a_j}(y)$$

(72)

$$+ o(1)$$

where the function $\eta$ is defined in (26), $\Lambda_i^n = (\beta_n \Lambda_i^n)^{\frac{1}{n-1}}$, see (11), and $\xi_j = \varepsilon^{-\frac{n-2}{2}} \xi_j.

Consider the following functions, for any $\alpha = 0, 1, \ldots, 3n - 1$, and $j = 1, 2$

$$Z_{\alpha j} = (\Lambda_j)^{-\frac{n-2}{2}} \left| \eta_{\Lambda_j, R_{\theta_j} \xi_j, a_j}(y) \right|^2 - n \int_{\Omega} \left( \frac{y - R_{\theta_j} \xi_j}{\Lambda_j} - \frac{y - R_{\theta_j} \xi_j}{\Lambda_j} \right) \eta_{\Lambda_j, R_{\theta_j} \xi_j, a_j}(y)$$

where the functions $z_\alpha$ are defined in (19), (20), (21), (22), (23), and the function $\eta$ is defined in (25). Consider furthermore their $H_0^1(\Omega_\varepsilon)$-projections $Z_{\alpha j}$, namely the unique solutions of

$$\Delta Z_{\alpha j} = \Delta Z_{\alpha j} \quad \text{in} \ \Omega_\varepsilon, \quad Z_{\alpha j} = 0 \quad \text{on} \ \partial \Omega_\varepsilon.$$

The nonlinear projected problem that we first solve consists in finding a function $\phi$ such that the following equation holds

$$\begin{cases}
\Delta (v_0 + \phi) + (v_0 + \phi)^{p+\varepsilon} = \sum_{\alpha, j} c_{\alpha j} v_{j}^{p-1} Z_{\alpha j} & \text{in} \ \Omega_\varepsilon, \\
\phi = 0 & \text{on} \ \partial \Omega_\varepsilon, \\
\int_{\Omega_\varepsilon} \phi v_{j}^{p-1} Z_{\alpha j} = 0 & \text{for all} \ \alpha, j,
\end{cases}$$

(73)

for some constants $c_{\alpha j}$, where

$$v_i(y) = \varepsilon^\frac{n-2}{2} PQ_i(\varepsilon^\frac{n-2}{2} - 1), \quad i = 1, 2.$$

A fundamental tool to solve properly Problem (73) consists in developing an invertibility theory for the following linear problem. Given $h \in C^\alpha(\Omega_\varepsilon)$, find a function $\phi$ such that for certain constants $c_{\alpha j}$, $j = 1, 2$, $\alpha = 0, \ldots, 3n - 1$ one has

$$\begin{cases}
\Delta \phi + (p + \varepsilon) v_0^{p+\varepsilon-1} \phi = h + \sum_{\alpha, j} c_{\alpha j} v_{j}^{p-1} Z_{\alpha j} & \text{in} \ \Omega_\varepsilon, \\
\phi = 0 & \text{on} \ \partial \Omega_\varepsilon, \\
\int_{\Omega_\varepsilon} v_{i}^{p-1} Z_{\alpha j} \phi \, dy = 0 & \text{for all} \ i, j.
\end{cases}$$

(74)
We do solve (74) in proper weighted $L^\infty$-norms: for a function $\psi$ defined on $\Omega_\varepsilon$, we define
\[
\|\psi\|_* = \sup_{x \in \Omega_\varepsilon} \left( \sum_{j=1}^2 \left( 1 + |x - \xi_j|^2 \right)^{-\frac{n-2}{2}} \right)^{-\frac{1}{2}} \psi(x),
\]
\[
+ \sup_{x \in \Omega_\varepsilon} \left( \sum_{j=1}^2 \left( 1 + |x - \xi_j|^2 \right)^{-\frac{n-2}{2}} \right)^{-\frac{1}{2}} D\psi(x),
\]
where $\beta = 1$ if $n = 3$ and $\beta = \frac{2}{n-2}$ otherwise, and
\[
\|\psi\|_{**} = \sup_{x \in \Omega_\varepsilon} \left( \sum_{j=1}^2 \left( 1 + |x - \xi_j|^2 \right)^{-\frac{n-2}{2}} \right)^{-\frac{1}{2}} \psi(x).
\]
In Section 5 we shall establish

**Proposition 4.1.** Assume constraints (71) hold on the parameter sets $A_1$ and $A_2$. Then there are numbers $\varepsilon_0 > 0$, $C > 0$, such that for all $0 < \varepsilon < \varepsilon_0$ and all $h \in C^0(\Omega_\varepsilon)$, problem (74) admits a unique solution $\phi \equiv L_\varepsilon(h)$. Besides,
\[
\|L_\varepsilon(h)\|_* \leq C\|h\|_{**}, \quad |c_{\alpha j}| \leq C\|h\|_{**}
\]
and
\[
\|\nabla_{\Lambda, \xi', a, \theta} \phi\|_* \leq C\|h\|_{**}.
\]
A fixed point argument using contraction mapping Theorem gives as a direct byproduct of the previous Proposition the following result that states unique solvability of the nonlinear Problem (63), for any given sets of parameters $A_1$ and $A_2$.

**Proposition 4.2.** Assume constraints (71) hold on the parameter sets $A_1$ and $A_2$. Then there is a constant $C > 0$, such that for all small $\varepsilon$ there exists a unique solution $\phi = \phi(\xi', \Lambda)$ to Problem (75) with
\[
\|\phi\|_* \leq C\varepsilon, \quad \|\nabla_{\Lambda, \xi', a, \theta} \phi\|_* \leq C\varepsilon.
\]
The proof of Proposition 4.2 is postponed to Section 6.

Looking back at Problem (73), it is immediate to observe that $v_0 + \phi$ is a solution to the scaled Problem (63) if the constants $c_{\alpha j}$ appearing in (63) are all zero. The second step in our argument consists in showing that the constants $c_{\alpha j}$ can be made all equal to zero provided the parameter sets $A_1$ and $A_2$ are properly chosen. Let us explain this second part of our argument.

Consider the function of $A_1$ and $A_2$ defined by
\[
I(A_1, A_2) \equiv J_\varepsilon((1 + \zeta) \left( u_0(x) + \tilde{\phi}(x) \right)),
\]
where $J_\varepsilon$ is defined in (62), $u_0$ is given by (67) and $\tilde{\phi}(x) = \varepsilon^{-\frac{1}{2}} \phi(\varepsilon^{-\frac{1}{2}} x)$, where $\phi$ is the unique solution to Problem (63) as predicted by Proposition 1.2. Recall that
\[
A_j = (\Lambda_j, \xi_j, a_j, \theta_j) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^2 \times K, \quad j = 1, 2.
\]
A consequence of Proposition 1.2 and estimate (77), is that the function $I$ depends in a $C^1$ sense on the parameters $A_1$ and $A_2$. Furthermore, we see that
\[
I(A_1, A_2) = (1 + \zeta)^2 F_\varepsilon(v_0 + \phi),
\]
where $\zeta$ is defined in (66) and
\[
F_\varepsilon(v) = \frac{1}{2} \int_{\Omega_\varepsilon} |Dv|^2 - \frac{1}{p + \varepsilon + 1} \int_{\Omega_\varepsilon} v^{p+\varepsilon+1}.
\]
The key observation of our argument is the following

**Lemma 4.3.** $u = (1 + \zeta)(u_0 + \tilde{\phi})$ is a solution of problem (1) if and only if $(A_1, A_2)$ is a critical point of $I$.

Before giving the proof of Lemma 4.3, an observation is in order. Let us recall that, for $A = (\lambda, \xi, a, \theta) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{2n-3}$, the function $Q_A$ is defined as

$$Q_A(x) = \lambda^{-\frac{n-2}{2}} \left| \frac{x - \acute{\xi}}{|x - \acute{\xi}|} - \frac{\bar{a}|x - \acute{\xi}|^2}{\lambda} \right|^{2-n} Q \left( \frac{x - \acute{\xi} - \bar{a}|x - \acute{\xi}|^2}{1 - 2\bar{a} \cdot \frac{x - \acute{\xi}}{|x - \acute{\xi}|^2}} \right),$$

with

$$\acute{\xi} = R_\theta \xi, \quad \bar{a} = R_\theta a.$$

Recall the functions $z_\alpha$, $\alpha = 0, \ldots, 3n - 1$, defined in (19), (20), (21), (22), (23). These are the only elements in the kernel of the linear operator $L(\varphi) = \Delta \varphi + p|Q|^{p-2}Q\varphi$ (see (21)). We are in a position to prove

**Proof of Lemma 4.3.** Consider, for example, the derivative of $I$ with respect to $a_{11}$, the first component of $a_1 = (a_{11}, a_{12}) \in \mathbb{R}^2$. We start observing that that $\frac{\partial}{\partial a_{11}} I = 0$ is equivalent to say that $\frac{\partial}{\partial a_{11}} F_\varepsilon(v_0 + \phi) = 0$. Next we compute $\frac{\partial}{\partial a_{11}} F_\varepsilon(v_0 + \phi) = DF_\varepsilon(v_0 + \phi)[\frac{\partial}{\partial a_{11}} v_0 + \frac{\partial}{\partial a_{11}} \phi]$. On the other hand, using (22), one sees that

$$\frac{\partial}{\partial a_{11}} v_0 = \bar{Z}_{n+2,1} = Z_{n+2,1} + o(1)$$

where

$$\bar{Z}_{\alpha j} = (\Lambda_j^* - \frac{n-2}{2}) \eta_{\alpha j} R_\theta \xi_j a_j(y) \left| \frac{\eta_{\alpha j} R_\theta \xi_j a_j(y)}{y - R_{\alpha j} \xi_j} - \frac{\eta_{\alpha j} R_{\alpha j}}{\Lambda_j^*} \right|^2$$

and $Z_{\alpha j}$ is the $H_0^1(\Omega_\varepsilon)$-projection of $\bar{Z}_{\alpha j}$. Taking into account that $\|\frac{\partial}{\partial a_{11}} \phi\| = o(1)$, as $\varepsilon \to 0$, we get that $\frac{\partial}{\partial a_{11}} I = 0$ is equivalent to say $DF_\varepsilon(v_0 + \phi)[\bar{Z}_{n+2,1} + o(1)] = 0$. Based on this argument, we can say that the conditions $\nabla I(A_1, A_2) = 0$ are equivalent to say that

$$DF_\varepsilon(v_0 + \phi)[\bar{Z}_{\alpha j} + o(1)] = 0$$

for all $\alpha$, for all $j$. Digging further in the above set of equalities, and using the fact that, by definition, $DF_\varepsilon(v_0 + \phi)[g] = 0$ for all functions such that $\int_{\Omega_\varepsilon} |g|^{p-1} Z_{\beta k} g = 0$, we can be more precise in (79): indeed, one has that

$$DF_\varepsilon(v_0 + \phi)[\bar{Z}_{\alpha j} + o(1) \Theta] = 0 \quad \text{for all } \alpha, j$$

where $\Theta$ is a uniformly bounded function, that belongs to the vector space generated by the functions $Z_{\beta k}$. From the above relation we thus conclude that the $6n$ conditions $\nabla I(A_1, A_2) = 0$ are equivalent to the $6n$ conditions

$$DF_\varepsilon(v_0 + \phi)[Z_{\alpha j}] = 0$$

for all $\alpha, j$. By definition of the $c_{\alpha j}$, it is easily seen that this is indeed equivalent to $c_{\alpha j} = 0$ for all $\alpha, j$. This concludes the proof of the Lemma.

The result of Lemma 4.3 says that the function defined in (65)

$$u(x) = (1 + \zeta)(u_0(x) + \tilde{\phi}(x)),$$

where $\tilde{\phi}(x) = \varepsilon^{-\frac{1}{2}} \phi(\varepsilon^{-\frac{1}{2}} x)$ and $\phi$ is the unique solution to Problem (73) as predicted by Proposition 4.2 is a solution to (1) if $(A_1, A_2)$ is a critical point for $I$ defined in (78).
Our purpose is thus to establish the existence of a critical point for $I(A_1, A_2)$. To this purpose, we first give an asymptotic estimate for the function $I(A_1, A_2)$. We prove

**Proposition 4.4.** Let $\zeta$ be given by (60). Then we have the expansion,

$$
\varepsilon^{2\zeta - 1} I(A_1, A_2) = 2\gamma_n + \chi_n \varepsilon \log \varepsilon + \eta_n \varepsilon 
$$

(80)

uniformly with respect to $(A_1, A_2)$ satisfying constraint (77), where $\theta$ and its derivatived $D\theta$ are smooth functions that are uniformly bounded, independently of $\varepsilon$. Here, we recall

$$
\Psi(A_1, A_2) = \frac{1}{2} H(\hat{\xi}_1, \hat{\xi}_2) Q^2(\hat{\alpha}_1) \Lambda_1^2 + \frac{1}{2} H(\hat{\xi}_2, \hat{\xi}_2) Q^2(\hat{\alpha}_2) \Lambda_2^2 
$$

$$
- G(\hat{\xi}_1, \hat{\xi}_2) Q(\hat{\alpha}_1) Q(\hat{\alpha}_2) A_1 A_2 + \log A_1 A_2, 
$$

and the constants in (80) are those in Lemma 3.2.

The proof of this result is postponed to the end of Section 6.

The final argument to get our Theorem 1.1 is to show that the function $\Psi$ in (80) defined also in (64) has a critical point, in fact a robust critical point of min max type, that persists under small $C^1$ perturbation. This is where we need that our domain $\Omega$ has the shape of a smooth bounded connected domain with a small removed hole. We show the existence of a min max structure for $\Psi$ in Section 7. This completes the proof of Theorem 1.1.

The rest of the paper is devoted to give detailed proofs of all our previous statements.

5. The linear problem: proof of Proposition 4.1

**Proof of Proposition 4.1.** The proof of this result is divided into two steps: we first assume the existence of a solution, and we prove the estimates (75), then we show existence of $\phi$. To prove (75), assume that there exists sequence $\varepsilon = \varepsilon_n \to 0$ such that there are functions $\phi_{\varepsilon}$ and $h_{\varepsilon}$ with $\|h_{\varepsilon}\|_{**} = o(1)$ such that

$$
\Delta \phi_{\varepsilon} + (p + \varepsilon) v_0^{p-1+\varepsilon} \phi_{\varepsilon} = h_{\varepsilon} + \sum_{\alpha,j} c_{\alpha,j} v_j^{p-1} Z_{\alpha,j} \quad \text{in } \Omega_{\varepsilon}
$$

$$
\phi_{\varepsilon} = 0 \quad \text{on } \partial \Omega_{\varepsilon},
$$

$$
\int_{\Omega_{\varepsilon}} v_j^{p-1} Z_{\alpha,j} \phi_{\varepsilon} \, dx = 0 \quad \text{for all } \alpha, j,
$$

for certain constants $c_{\alpha,j}$, depending on $\varepsilon$. We shall show that $\|\phi_{\varepsilon}\|_{*} \to 0$.

We first establish that

$$
\|\phi_{\varepsilon}\|_{\rho} = \sup_{x \in \Omega_{\varepsilon}} \left| \sum_{j=1}^{2} (1 + |x - \xi_j'|^2)^{-\frac{n-2}{2}} \phi_{\varepsilon}(x) \right|^{\beta - \rho} + \sup_{x \in \Omega_{\varepsilon}} \left| \sum_{j=1}^{2} (1 + |x - \xi_j'|^2)^{-\frac{n-2}{2}} D\phi_{\varepsilon}(x) \right| \to 0
$$

with $\rho > 0$ a small fixed number. To do this, we assume the opposite, so that with no loss of generality we may take $\|\phi_{\varepsilon}\|_{\rho} = 1$. Testing the above equation against $Z_{\beta,k}$, integrating by parts twice we get that

$$
\sum_{\alpha,j} c_{\alpha,j} \int_{\Omega_{\varepsilon}} v_j^{p-1} Z_{\alpha,j} Z_{\beta,k} = \int_{\Omega_{\varepsilon}} [\Delta Z_{\beta,k} + (p + \varepsilon) v_0^{p-1+\varepsilon} Z_{\beta,k} \phi_{\varepsilon}] - \int_{\Omega_{\varepsilon}} h_{\varepsilon} Z_{\beta,k}.
$$

(81)
Formula (81) defines a linear system in the 6n variables \( c_{\alpha,j} \), which is uniquely solvable, with bounded inverse. This is due to the following facts: fix \( \alpha \) and \( j \). If \( k \neq j \), then
\[
\int_{\Omega_\varepsilon} v_j^{p-1}Z_{\alpha j}Z_{\beta k} = O(\varepsilon^{\frac{n}{p-1}}), \quad \text{for any } \beta.
\]
If \( k = j \), then we have
\[
\int_{\Omega_\varepsilon} v_j^{p-1}Z_{\alpha j}Z_{\beta j} = \begin{cases} 
\int_{\mathbb{R}^n} |Q|^{p-1}z_0^2 + O(\varepsilon^{\frac{n}{p-1}}) & \text{if } \alpha = \beta, \\
\int_{\mathbb{R}^n} |Q|^{p-1}z_1 z_{n+2} + O(\varepsilon^{\frac{n}{p-1}}) & \text{if } \alpha = 1, \beta = n + 2, \\
\int_{\mathbb{R}^n} |Q|^{p-1}z_2 z_{n+3} + O(\varepsilon^{\frac{n}{p-1}}) & \text{if } \alpha = 2, \beta = n + 3,
\end{cases}
\]
otherwise.

Moreover, the numbers \( \int_{\mathbb{R}^n} |Q|^{p-1}z_0^2, \alpha = 0, 1, \ldots, 3n-1 \), and \( \int_{\mathbb{R}^n} |Q|^{p-1}z_1 z_{n+2}, \int_{\mathbb{R}^n} |Q|^{p-1}z_2 z_{n+3} \) are fixed numbers, different from zero, that are independent of \( \varepsilon \). The above computations tell us several facts: first of all, the linear system (81) of 6n equations in the 6n variables \( c_{\alpha j} \) at main order decouples in two systems, the first a 3n \( \times \) 3n system in the variables \( c_{\alpha 1} \) and the second in a 3n \( \times \) 3n system in the variables \( c_{\alpha 2} \). Second, if we analyze for instant the system in the \( c_{\alpha 1} \), we see that the coefficients \( c_{01}, \ldots, c_{(3n-1)1} \) are coupled, at main order, but the coupling is very clear, in fact only two coupling occurs: the variable \( c_{11} \) with the variable \( c_{n+2,1} \) and the variable \( c_{21} \) with the variable \( c_{n+3,1} \). Except for these coupling, the system in the \( c_{\alpha 1} \) decouples at main order, as \( \varepsilon \to 0 \). We finally observe that the matrices
\[
\frac{\varepsilon}{\varepsilon}
\]
are invertible. Thus we conclude that (81) defines a linear system in the 6n variables \( c_{\alpha,j} \), which is uniquely solvable, with bounded inverse.

On the other hand, it is easy to see that we have, for \( l = 1, 2 \),
\[
\int_{\Omega_\varepsilon} \Delta Z_{\alpha k} + (p + \varepsilon) v_0^{p+\varepsilon-1} Z_{\alpha k} \phi = o(1)\|\phi\|_{\rho}, \quad \text{and} \quad \int_{\Omega_\varepsilon} h_\varepsilon, Z_{\beta k} \leq C\|h_\varepsilon\|_{**}.
\]
Thus, we conclude that
\[
|c_{\alpha j}| \leq C\|h_\varepsilon\|_{**} + o(1)\|\phi_\varepsilon\|_{\rho}
\]
so that \( \phi_{\varepsilon} = o(1) \). Let \( G_\varepsilon \) denotes the Green’s function of \( \Omega_\varepsilon \). We have for \( x \in \Omega_\varepsilon \)
\[
\phi_\varepsilon(x) = (p + \varepsilon) \int_{\Omega_\varepsilon} G_\varepsilon(x, y)v_0^{p+\varepsilon-1} \phi_\varepsilon dy
\]
\[
- \int_{\Omega_\varepsilon} G_\varepsilon(x, y)h_\varepsilon dy - \sum c_{\alpha j} \int_{\Omega_\varepsilon} v_j^{p-1} Z_{\alpha j} G_\varepsilon(x, y) dy
\]
Furthermore, the function \( \phi_\varepsilon \) is of class \( C^1 \) and
\[
\partial_\varepsilon \phi_\varepsilon(x) = p \int_{\Omega_\varepsilon} \partial_\varepsilon G_\varepsilon(x, y)v_0^{p+\varepsilon-1} \phi_\varepsilon dy =
\]
Direct estimates give

\[ - \int_{\Omega_\varepsilon} \partial_x \varphi \Omega \varphi \|e\| dy - \sum c_{\alpha j} \int_{\Omega_\varepsilon} v_0^{p-1} Z_{\alpha j} \partial_x \varphi \Omega \varphi \|e\| dy \quad x \in \Omega_\varepsilon. \]

Equation (84) and the above estimates imply that

\[
\int_{\Omega_\varepsilon} \varphi \Omega \varphi \|e\| dy \leq \|h_\varepsilon\|_{**} C \int_{\mathbb{R}^n} \Gamma(x-y) \sum_{j=1}^2 \left( \frac{1}{1 + |y - \xi_j'|^2} \right)^{1 - \frac{\alpha_j}{2}} dy
\]

\[
\leq C \|h_\varepsilon\|_{**} \sum_{j=1}^2 \left( \frac{1}{1 + |x - \xi_j'|^2} \right)^{\frac{\alpha_j}{2}}
\]

and

\[
|\int_{\Omega_\varepsilon} v_0^{p-1} Z_{\alpha j} \partial_x \varphi \Omega \varphi \|e\| dy| \leq C \int_{\mathbb{R}^n} \Gamma(x-y) \sum_{j=1}^2 \left( \frac{1}{1 + |y - \xi_j'|^2} \right)^{\frac{\alpha_j}{2}} dy
\]

\[
\leq C \sum_{j=1}^2 \left( \frac{1}{1 + |x - \xi_j'|^2} \right)^{\frac{\alpha_j}{2}}
\]

Analogously we get

\[
|\partial_x \varphi \Omega \varphi \|e\| dy \leq \|h_\varepsilon\|_{**} C \sum_{j=1}^2 \int_{\mathbb{R}^n} \varphi \Omega \varphi \|e\| \left( \frac{1}{|x-y|^{n-1}} (1 + |y - \xi_j'|^2)^{-2} \right) dy
\]

\[
\leq C \|h_\varepsilon\|_{**} \sum_{j=1}^2 \left( \frac{1}{|x-y|^{n-1}} (1 + |y - \xi_j'|^2)^{-\frac{n+1}{2}} \right)
\]

and

\[
|\int_{\Omega_\varepsilon} v_0^{p-1} Z_{\alpha j} \partial_x \varphi \Omega \varphi \|e\| dy| \leq C (\|\varphi \|_{\rho} + \|h_\varepsilon\|_{**}) \sum_{j=1}^2 \left( \frac{1}{|x-y|^{n-1}} (1 + |y - \xi_j'|^2)^{-\frac{n+1}{2}} \right)
\]

\[
\leq C (\|\varphi \|_{\rho} + \|h_\varepsilon\|_{**}) \sum_{j=1}^2 \left( \frac{1}{|x-y|^{n-1}} (1 + |y - \xi_j'|^2)^{-\frac{n+1}{2}} \right)^{\beta + \frac{1}{n-2}}
\]

Equation (84) and the above estimates imply that

\[
|\varphi \|_{\rho} \leq C (\|\varphi \|_{\rho} + \|h_\varepsilon\|_{**}) \left( \sum_{j=1}^2 \left( \frac{1}{1 + |x - \xi_j'|^2} \right)^{\frac{\alpha_j}{2}} \right)^{\beta}
\]

and

\[
|D\varphi \|_{\rho} \leq C (\|\varphi \|_{\rho} + \|h_\varepsilon\|_{**}) \left( \sum_{j=1}^2 \left( \frac{1}{1 + |x - \xi_j'|^2} \right)^{\frac{\alpha_j}{2}} \right)^{\beta + \frac{1}{n-2}}.
\]

In particular

\[
\left( \sum_{j=1}^2 \left( \frac{1}{1 + |x - \xi_j'|^2} \right)^{\frac{\alpha_j}{2}} \right)^{-(\beta - \rho)} |\varphi \|_{\rho} \leq C \left( \sum_{j=1}^2 \left( \frac{1}{1 + |x - \xi_j'|^2} \right)^{\frac{\alpha_j}{2}} \right)^{\rho}.
\]
Since \( \|\phi_\varepsilon\|_\rho = 1 \), we assume that \( \|\phi_\varepsilon\|_{L^\infty(B_R(x_1))} = \gamma \) for certain \( R > 0 \) and \( \gamma > 0 \) independent of \( \varepsilon \), for either \( i = 1 \) or \( i = 2 \). Then local elliptic estimates and the bounds above yield that, up to a subsequence, \( \tilde{\phi}_\varepsilon(x) = \phi_\varepsilon(x - \xi_1^\varepsilon) \) converges uniformly over compacts of \( \mathbb{R}^N \) to a nontrivial solution \( \tilde{\phi} \) of

\begin{equation}
\Delta \tilde{\phi} + p|Q|^{p-1}\tilde{\phi} = 0,
\end{equation}

which besides satisfies

\begin{equation}
|\tilde{\phi}(x)| \leq C|x|^{(2-n)\beta}.
\end{equation}

In dimension \( n = 3 \) this means \( |\tilde{\phi}(x)| \leq C|x|^{2-n} \). In higher dimension, a bootstrap argument of \( \tilde{\phi} \) solution of \( (83) \), using estimate \( (87) \), gives \( |\tilde{\phi}(x)| \leq C|x|^{2-n} \). Thanks to non degenerate result in \( [21] \), this implies that \( \tilde{\phi} \) is a linear combination of the functions \( z_\alpha \), defined in \( [10], [20], [21], [22] \) and \( [23] \). On the other hand, dominated convergence Theorem gives that the orthogonality conditions \( \int_{\Omega_\varepsilon} \phi_\varepsilon v_j^{p-1}Z_{\alpha j} = 0 \) pass to the limit, thus getting

\[ \int_{\mathbb{R}^n} |Q|^{p-1}z_\alpha \tilde{\phi} = 0 \quad \text{for all} \quad \alpha = 0, \ldots, 3n - 1. \]

Hence the only possibility is that \( \tilde{\phi} \equiv 0 \), which is a contradiction which yields the proof of \( \|\phi_\varepsilon\|_\rho \to 0 \). Moreover, we observe that

\[ \|\phi_\varepsilon\|_* \leq C(\|h\|_{**} + \|\phi_\varepsilon\|_\rho), \]

hence \( \|\phi_\varepsilon\|_* \to 0 \).

Now we are in a position to prove the existence of \( \phi \) solution to \( (74) \). To do this, let us consider the space

\[ H = \{ \phi \in H^1_0(\Omega_\varepsilon) \mid \int_{\Omega_\varepsilon} v_j^{p-1}Z_{\alpha j}, \phi >= 0 \forall \alpha, j \} \]

destroyed with the usual inner product \( [\phi, \psi] = \int_{\Omega_\varepsilon} \nabla\phi\nabla\psi \). Problem \( (74) \) expressed in weak form is equivalent to that of finding a \( \phi \in H \) such that

\[ [\phi, \psi] = \int_{\Omega_\varepsilon} \left( (p + \varepsilon)v_0^{p+\varepsilon-1}\phi - h \right) \psi \quad \forall \psi \in H. \]

With the aid of Riesz’s representation theorem, this equation gets rewritten in \( H \) in the operational form

\begin{equation}
\phi = T_\varepsilon(\phi) + \tilde{h}
\end{equation}

with certain \( \tilde{h} \in H \) which depends linearly in \( h \) and where \( T_\varepsilon \) is a compact operator in \( H \). Fredholm’s alternative guarantees unique solvability of this problem for any \( h \) provided that the homogeneous equation \( \phi = T_\varepsilon(\phi) \) has only the zero solution in \( H \). Assume it has a nontrivial solution \( \phi = \phi_\varepsilon \), which with no loss of generality may be taken so that \( \|\phi_\varepsilon\|_* = 1 \). But for what we proved before, necessarily \( \|\phi_\varepsilon\|_* \to 0 \). This is certainly a contradiction that proves that this equation only has the trivial solution in \( H \). We conclude then that for each \( h \), problem \( (74) \) admits a unique solution. Standard arguments give then the validity of \( (76) \).

We go now to the issue of the dependence of the solution \( \phi \) to \( (74) \) on the parameters \( A_1' = (\lambda_1, \xi_1', a_1, \theta_1) \) and \( A_2' = (\lambda_2, \xi_2', a_2, \theta_2) \). Let us fix \( j = 1 \) and define \( A_1' = (A_{11}, A_{12}, \ldots, A_{13n}) \) the components of the vector \( A_1 \). Let us differential \( \phi \) with respect to \( A_{1l} \), for some \( l = 1, \ldots, 3n \). We set formally \( Z = \frac{\partial}{\partial A_{1l}} \phi \).
We define the number $b_{\alpha j}$ so that
\[
\int_{\Omega_\varepsilon} v_i^{p-1} Z_{\beta i} [Z - \sum_{\alpha j} b_{\alpha j} Z_{\alpha j}] = 0, \quad \text{for all } \beta, i.
\]
This amounts to solving a linear system in the constants $b_{\alpha j}$.

\[
\sum_{\beta j} b_{\beta j} \int_{\Omega_\varepsilon} v_i^{p-1} Z_{\alpha i} Z_{\beta j} = \int_{\Omega_\varepsilon} \frac{\partial}{\partial A_{1l}} (v_i^{p-1} Z_{\alpha i}) \phi,
\]
as a direct differentiation with respect to $A_{1l}$ of the orthogonal conditions $\int_{\Omega_\varepsilon} v_i^{p-1} Z_{\alpha i} \phi = 0$ directly shows. Arguing as in (81), we see that (89) is uniquely solvable and that
\[
b_{\beta i} = O(\|\phi\|_*)
\]
uniformly for parameters $A'_1$ and $A'_2$ in the considered region. Thus $\eta \in H^1_0(\Omega_\varepsilon)$ and
\[
\int_{\Omega_\varepsilon} v_i^{p-1} Z_{\alpha i} \eta = 0 \quad \text{for all } \alpha, i.
\]
On the other hand, a direct but long computation shows that
\[
\Delta \eta + (p + \varepsilon) v_0^{p-1+\varepsilon} \eta = f + \sum_{\alpha j} d_{\alpha j} v_j^{p-1} Z_{\alpha j} \quad \text{in } \Omega_\varepsilon,
\]
where $d_{\alpha j} = \frac{\partial}{\partial A_{1l}} c_{\alpha j}$ and
\[
f = \sum_{\alpha j} b_{\alpha j} (- (\Delta + (p + \varepsilon) v_0^{p-1+\varepsilon}) Z_{\alpha j} + c_{\alpha j} \partial A_{1l} (v_j^{p-1} Z_{\alpha j})) - (p + \varepsilon) \partial A_{1l} (v_0^{p-1+\varepsilon} \phi),
\]
Thus we have that $\eta = L_\varepsilon (f)$. Moreover, we easily see that
\[
\| \phi \partial A_{1l} (v_0^{p-1+\varepsilon}) \|_* \leq C \| \phi \|_*.\]
On the other hand
\[
| \partial A_{1l} (v_i^{p-1} Z_{\alpha i} (x)) | \leq C |x - \xi_i^l|^{-n+4},
\]
hence
\[
\| c_{\alpha i} \partial A_{1l} v_i^{p-1} Z_{\alpha i} \|_* \leq C \| h \|_*
\]
since we have that $c_{\alpha i} = O(\| h \|_*)$. We conclude that
\[
\| f \|_* \leq C \| h \|_*.
\]
Reciprocally, if we define
\[
Z = L_\varepsilon (f) + \sum_{\alpha j} b_{\alpha j} v_j^{p-1} Z_{\alpha j},
\]
with $b_{\alpha j}$ given by relations (89) and $f$ by (92), we check that indeed $Z = \partial A_{1l} \phi$. In fact $Z$ depends continuously on the parameters $A'_1$, $A'_2$ and $h$ for the norm $\| \|_*$, and $\| Z \|_* \leq C \| h \|_*$, for parameters in the considered region. The corresponding result for differentiation with respect to the $A'_2$ follow similarly.

In other words, we proved that $(A'_1, A'_2) \rightarrow L_\varepsilon$ is of class $C^1$ in $L(L_{0*}, L_{0*}^\infty)$ and, for instance,
\[
(D A_{1l} L_\varepsilon) (h) = L_\varepsilon (f) + \sum_{\alpha j} b_{\alpha j} Z_{\alpha j},
\]
where $f$ is given by (92) and $b_{\alpha j}$ by (89). This concludes the proof. \qed
6. The non-linear Problem: proof of Proposition 4.2

Proof of Proposition 4.2: We write the equation in (73) as
\[ \Delta \phi + (p + \varepsilon) v_0^{p+\varepsilon - 1} \phi = E - N_\varepsilon(\phi) + \sum_{\alpha, j} c_{\alpha j} v_j^{p-1} Z_{\alpha j} \quad \text{in} \quad \Omega_\varepsilon \]
where
\[ N_\varepsilon(\phi) = (v_0 + \phi)^{p+\varepsilon} - v_0^{p+\varepsilon} - (p + \varepsilon) v_0^{p+\varepsilon - 1} \phi, \quad E = v_0^{p+\varepsilon} - Q_1^p - Q_2^p. \]

Observe that
\[ |E| \leq C \left( |v_0^{p+\varepsilon} - v_0^p| + |v_0^{p} - Q_1^p - Q_2^p| \right) \]
\[ \leq C \varepsilon \left( |Q_i|^{\alpha} \log |Q_i|(x) + \frac{1}{1 + |x - \xi|^2} \right) \]
in the regions where \( |x - \xi| \leq \bar{\delta} \varepsilon^{-\frac{1}{p-2}} \), for small \( \bar{\delta} > 0 \). Taking into account that
\[ |E| \leq C \varepsilon \frac{\bar{\delta}}{\varepsilon^{-\frac{1}{p-2}}} \] in the complement of these two regions, we get
\[ \|E\|_{**} \leq C \varepsilon. \]

To estimate \( N_\varepsilon(\phi) \), it is convenient, and sufficient for our purposes, to assume \( \|\phi\|_* < 1 \). Note that, if \( n \leq 6 \), then \( p \geq 2 \) and we can estimate
\[ |N_\varepsilon(\phi)| \leq C |v_1 + v_2|^{p-2} |\phi|^2 \]
and hence
\[ \|N_\varepsilon(\phi)\|_* \leq C \|\phi\|_*^2. \]

Assume now that \( n > 6 \). If \( |\phi| \geq \frac{1}{2} \), we have
\[ |N_\varepsilon(\phi)| \leq C |\phi|^p \]
\[ \|N_\varepsilon(\phi)\|_* \leq C \varepsilon^{-\frac{n-6}{2}} \|\phi\|_*^2. \]

Let us consider now the case \( |\phi| \leq \frac{1}{2} v_0 \). In the region where \( \text{dist}(y, \partial \Omega_\varepsilon) \geq \delta \varepsilon^{-\frac{1}{p-2}} \), for some \( \delta > 0 \), then \( v_0(y) \geq \alpha_\delta U(y) \) for some \( \alpha_\delta > 0 \); hence in this region, we have
\[ |N(\phi)| \leq C \varepsilon^{2} \|\phi\|_*^2 \leq C \varepsilon^{2} \|\phi\|_*^2. \]

On the other hand, when \( \text{dist}(y, \partial \Omega_\varepsilon) \leq \delta \varepsilon^{-\frac{1}{p-2}} \), the following facts occur: \( U(y) \), \( v_0(y) = O(\varepsilon) \) and, as \( y \to \partial \Omega_\varepsilon \), \( v_0(y) = C \varepsilon^{-\frac{1}{p-2}} \text{dist}(y, \partial \Omega_\varepsilon)(1 + o(1)) \). This second assertion is a consequence of the fact that the Green function of the domain \( \Omega \) vanishes linearly with respect to \( \text{dist}(x, \partial \Omega) \) as \( x \to \partial \Omega \). These two facts imply that, if \( \text{dist}(y, \partial \Omega_\varepsilon) \leq \delta \varepsilon^{-\frac{1}{p-2}} \) and \( \phi(y) \neq 0 \) (otherwise \( N(\phi)(y) = 0 \)), then
\[ \|N(\phi)\|_* \leq U^{-\frac{1}{p-2}} v_0^{p-2} |\phi|^2 \]
\[ \leq C \varepsilon^{-\frac{1}{p-2} + 2} \left( \frac{n-2}{p-2} \text{dist}(y, \partial \Omega_\varepsilon) \right)^{p-2} \text{dist}(y, \partial \Omega_\varepsilon)^2 |D\phi(y)|^2 \]
\[ \leq C \varepsilon^{-\frac{n-6}{2}} \|\phi\|_*^2 \leq C \varepsilon^{-\frac{n-6}{2}} \|\phi\|_*^2. \]

Combining these relations we get
\[ \|N(\phi)\|_* \leq \begin{cases} C \|\phi\|_*^2 & \text{if } n \leq 6 \\ C \varepsilon^{-\frac{n-6}{2}} \|\phi\|_*^2 & \text{if } n > 6. \end{cases} \]

Now, we are in position to prove that problem (73) has a unique solution \( \phi = \tilde{\phi} + \tilde{\psi} \), with
\[ \tilde{\psi} := -T_\varepsilon(E), \]
with the required properties. Here $T_{\varepsilon}$ denotes the linear operator defined by Proposition 4.1. Namely, $T_{\varepsilon}(h) = \phi$ is $L_{\varepsilon}\phi = h$. We see that problem (73) is equivalent to solving a fixed point problem. Indeed $\tilde{\phi} = \tilde{\phi} + \tilde{\psi}$ is a solution of (73) if and only if

$$\tilde{\phi} = -T_{\varepsilon}(N(\tilde{\phi} + \tilde{\psi})) \equiv A_{\varepsilon}(\tilde{\phi}).$$

We proceed to prove that the operator $A_{\varepsilon}$ defined above is a contraction inside a properly chosen region. Since $\|E\|_\varepsilon \leq C\varepsilon$, the result of Proposition 4.1 gives that

$$\|\tilde{\psi}\|_\varepsilon \leq C\varepsilon$$

and

$$\|N(\tilde{\psi} + \eta)\|_\varepsilon \leq \left\{ \begin{array}{ll}
C(\varepsilon^2 + \varepsilon\|\eta\|_\varepsilon + \|\eta\|^2) & \text{if } n \leq 6 \\
C(\varepsilon^{1+4n} + \varepsilon^{4n/2}\|\eta\|_\varepsilon + \varepsilon^{n/2}\|\eta\|^2) & \text{if } n > 6.
\end{array} \right.$$ (97)

Call

$$F = \{\eta \in H_0^1_\varepsilon : \|\eta\|_\varepsilon \leq R\varepsilon\}.$$ (97)

From Proposition 4.1 and (97) we conclude that, for $\varepsilon$ sufficiently small and any $\eta \in lF$ we have

$$\|A_{\varepsilon}(\eta)\|_\varepsilon \leq C\varepsilon.$$ (97)

If we choose $R$ big enough in the definition of $F$, we get then that $A_{\varepsilon}$ maps $F$ in itself. Now we will show that the map $A_{\varepsilon}$ is a contraction, for any $\varepsilon$ small enough. That will imply that $A_{\varepsilon}$ has a unique fixed point in $F$ and hence problem (73) has a unique solution. For any $\eta_1, \eta_2$ in $F$ we have

$$\|A_{\varepsilon}(\eta_1) - A_{\varepsilon}(\eta_2)\|_\varepsilon \leq C\|N_{\varepsilon}(\tilde{\psi} + \eta_1) - N_{\varepsilon}(\tilde{\psi} + \eta_2)\|_\varepsilon,$$

hence we just need to check that $N$ is a contraction in its corresponding norms. By definition of $N$

$$D_{\tilde{\eta}}N_{\varepsilon}(\tilde{\eta}) = (p + \varepsilon)[(v_0 + \tilde{\eta})_+^{p+\varepsilon-1} - v_0^{p+\varepsilon-1}].$$ (97)

Hence we get

$$|N_{\varepsilon}(\tilde{\psi} + \eta_1) - N_{\varepsilon}(\tilde{\psi} + \eta_2)| \leq C\varepsilon^{p-2}\|\tilde{\eta}\|_\varepsilon|\eta_1 - \eta_2|,$$

for some $\tilde{\eta}$ in the segment joining $\tilde{\psi} + \eta_1$ and $\tilde{\psi} + \eta_2$. Hence, we get for small enough $\|\tilde{\eta}\|_\varepsilon$,

$$\|N(\tilde{\psi} + \eta_1) - N(\tilde{\psi} + \eta_2)\|_\varepsilon \leq C\varepsilon^{p-2+2\beta}\|\tilde{\eta}\|_\varepsilon|\eta_1 - \eta_2|.$$ (97)

We conclude that there exists $c \in (0, 1)$ such that

$$\|N(\tilde{\psi} + \eta_1) - N(\tilde{\psi} + \eta_2)\|_\varepsilon \leq c\|\eta_1 - \eta_2\|_\varepsilon.$$ (97)

This concludes the proof of existence of $\phi$ solution to (73), and the first estimate in (77). We devote the rest to prove the second estimate in (77).

We recall that $\phi$ is defined through the relation

$$B(A_1, A_2, \phi) \equiv \phi + L_\varepsilon(N_\varepsilon(\phi + \psi)) = 0.$$ (77)

We have that

$$D_{\phi}B(A_1, A_2, \phi)[\theta] = \theta + L_\varepsilon(\theta D_{\phi}N(\phi + \psi)) \equiv \theta + M(\theta)$$

where

$$D_{\phi}N(A_1, A_2, \tilde{\phi}) = (p + \varepsilon)[(v_0 + \tilde{\phi})_+^{p+\varepsilon-1} - v_0^{p+\varepsilon-1}].$$ (97)

Now,

$$\|M(\theta)\|_\varepsilon \leq C\|\theta D_{\phi}N(\phi + \psi)\|_\varepsilon \leq C\|v_0^{n/2-\varepsilon}D_{\phi}N(\phi + \psi)\|_\varepsilon \|\theta\|_\varepsilon,$$

and

$$v_0^{n/2-\varepsilon}|D_{\phi}N(\phi + \psi)| \leq v_0^{2\beta-1}\|\phi + \psi\|_\varepsilon \leq C\varepsilon^\min(2\beta, 1).$$ (97)

It follows that for small $\varepsilon$, the linear operator $D_{\phi}B(A_1, A_2, \phi)$ is invertible in $L_\varepsilon^\infty$, with uniformly bounded inverse. It also depends continuously on its parameters. Define
again $A'_1 = (A_1, \xi'_1, a_1, \theta_1)$ and $A'_2 = (A_2, \xi'_2, a_2, \theta_2)$. Let us fix $j = 1$ and define $A'_1 = (A_{11}, A_{12}, \ldots, A_{13n})$ the components of the vector $A'_1$. Let us differential $\phi$ with respect to $A_{ll}$, for some $l = 1, \ldots, 3n$. We have

$$D_{A_{ll}} N(A_1, A_2, \tilde{\phi}) = (p + \varepsilon)[(v_0 + \tilde{\phi})_{\varepsilon}^{p+\varepsilon-1} - v_0^{p+\varepsilon-1} - (p + \varepsilon - 1)v_0^{p+\varepsilon-2}\varepsilon]D_{A_{ll}} v_0,$$

and

$$D_{A_{ll}} B(A_1, A_2, \phi) = (D_{A_{ll}} L_\varepsilon)(N(\phi + \psi)) + [L_\varepsilon((D_{A_{ll}})N(A_1, A_2, \phi + \psi)) + L_\varepsilon((D_{\tilde{\phi}} N)(A_1, A_2, \phi + \psi)D_{A_{ll}} \psi)].$$

Here $D_{A_{ll}} L_\varepsilon$ is the operator defined by the expression (93) and the second quantity by (95). Observe also that

$$D_{A_{ll}} \psi = (D_{A_{ll}} L_\varepsilon)(E) + L_\varepsilon(D_{A_{ll}} E).$$

see (95). Also,

$$D_{A_{ll}} E = (p + \varepsilon)v_0^{p+\varepsilon-1}D_{A_{ll}} v_1 - pv_1^{p-1}D_{A_{ll}} v_1.$$ 

These expressions also depend continuously on their parameters.

The implicit function theorem then applies to yield that $\phi(A_1, A_2)$ indeed defines a $C^1$ function into $L^\infty$. Moreover, we have for instance

$$D_{A_{ll}} \phi = -(D_{\tilde{\phi}} B(A_1, A_2, \phi))^{-1}[(D_{A_{ll}} L_\varepsilon)(N(\phi + \psi)) + [L_\varepsilon((D_{A_{ll}})N(A_1, A_2, \phi + \psi)) + L_\varepsilon((D_{\tilde{\phi}} N)(A_1, A_2, \phi + \psi)D_{A_{ll}} \psi)].$$

Hence,

$$\|D_{A_{ll}} \phi\|_{**} \leq C(\|N(\phi + \psi)\|_{**} + \|D_{\tilde{\phi}} N(A_1, A_2, \phi + \psi)D_{A_{ll}} \psi\|_{**}),$$

thanks to (93). On the other hand, we get

$$\|N_\varepsilon(\phi + \psi)\|_{**} \leq \begin{cases} C\varepsilon^2 & \text{if } n \leq 6 \\ C\varepsilon^{p+\beta+1} & \text{if } n > 6. \end{cases}$$

Thus, from (95) we have

$$\|(D_{A_{ll}})N(A_1, A_2, \tilde{\phi})\| \leq C\varepsilon^{-\frac{1}{2}}[(v_0 + \tilde{\phi})_{\varepsilon}^{p+\varepsilon-1} - v_0^{p+\varepsilon-1} - (p + \varepsilon - 1)v_0^{p+\varepsilon-2}\varepsilon] \leq C\varepsilon^{-\frac{1}{2}}(p+\varepsilon+\beta)\|\tilde{\phi}\|_{**},$$

hence

$$\|(D_{A_{ll}})N(A_1, A_2, \phi + \psi)\|_{**} \leq C\|\phi + \psi\|_{**} \leq C\varepsilon.$$ 

In similar way we get that

$$\|D_{\tilde{\phi}} N(A_1, A_2, \phi + \psi)D_{A_{ll}} \psi\|_{**} \leq C\varepsilon.$$ 

Hence, we finally get

$$\|D_{A_{ll}} \phi\|_{**} \leq C\varepsilon,$$

as desired. A similar estimate holds for differentiation with respect to the other variables. This concludes the proof.
Proof of Proposition 4.4. We write
\[ I(A_1, A_2) = J_\varepsilon((1 + \zeta)(u_0 + \tilde{\phi})) - J_\varepsilon((u_0 + \tilde{\phi})) + J_\varepsilon((u_0 + \tilde{\phi})) - J_\varepsilon(PQ_1 + PQ_2). \]

Since \( \tilde{\phi}(x) = \varepsilon^{-\frac{1}{2}}\phi(\varepsilon^{-\frac{1}{2}}x) \), and \( \|\phi\|_* \leq C\varepsilon \), we have that
\[ J_\varepsilon((1 + \zeta)(u_0 + \tilde{\phi})) - J_\varepsilon((u_0 + \tilde{\phi})) = J_\varepsilon((1 + \zeta)(u_0)) - J_\varepsilon((u_0)) + o(\varepsilon) \]

At this point, arguing like in the proof of Lemma 3.3 we are able to show that
\[ J_\varepsilon((1 + \zeta)(u_0)) - J_\varepsilon((u_0)) = 2\tilde{\gamma}_n \log \varepsilon + O(\varepsilon^2 |\log \varepsilon|), \]

where \( \tilde{\gamma}_n \) is a fixed constant, independent of \( \varepsilon \). Observe also that
\[ \nabla A_1 A_2 [J_\varepsilon((1 + \zeta)(u_0)) - J_\varepsilon((u_0))] = O(\varepsilon^2 |\log \varepsilon|) \]

uniformly for parameters \( A_1 \) and \( A_2 \) in the considered region.

Given the result of Lemma 3.3 we need to show that
\[ I(A_1, A_2) - J_\varepsilon(PQ_1 + PQ_2) = o(\varepsilon) \]

and
\[ \nabla A_1 A_2 [I(A_1, A_2) - J_\varepsilon(PQ_1 + PQ_2)] = o(\varepsilon). \]

Recall now that \( u_0 = PQ_1 + PQ_2 \). Let us apply a Taylor expansion
\[ J_\varepsilon((u_0 + \tilde{\phi})) - J_\varepsilon(PQ_1 + PQ_2) = \int_0^1 t dt D^2 J_\varepsilon(u_0 + t\tilde{\phi})[\tilde{\phi}, \tilde{\phi}], \]

since \( 0 = DF_\varepsilon(v_0 + \phi)[\phi] = (1 + \zeta)^2 DJ_\varepsilon(u_0\tilde{\phi})[\tilde{\phi}] \). Now, from the definition of \( \phi \), we see that
\[ \int_0^1 t dt D^2 J_\varepsilon(u_0 + t\tilde{\phi})[\tilde{\phi}, \tilde{\phi}] = (1 + \zeta)^2 \int_0^1 t dt D^2 F_\varepsilon(v_0 + t\phi)[\phi, \phi] = \]

\[ (1 + \zeta)^2 \int_0^1 t dt \int_{\Omega_\varepsilon} |\nabla \phi|^2 - (p + \varepsilon)(v_0 + t\phi)^{p+\varepsilon-1}\phi^2. \]

(105)

Since, we recall \( \|\phi\|_* + \|\tilde{\psi}\|_* = O(\varepsilon) \), the above relation together with (101) yield in particular,
\[ I(A_1, A_2) - J_\varepsilon(u_0 + \tilde{\phi}) = \begin{cases} \frac{O(\varepsilon^2)}{\varepsilon^2 |\log \varepsilon|} & \text{if } n < 6 \\ \frac{O(\varepsilon^2 |\log \varepsilon|)}{\varepsilon^2 |\log \varepsilon|} & \text{if } n = 6 \\ \frac{O(\varepsilon^{1+\frac{1}{n-2}})}{\varepsilon^{1+\frac{1}{n-2}}} & \text{if } n \geq 7, \end{cases} \]

uniformly on \( A_1, A_2 \) in the considered region. Let us estimate now difference in derivatives. Define again \( A'_1 = (A_1, \xi, a_1, \theta_1) \) and \( A'_2 = (A_2, \xi_2, a_2, \theta_2) \). Let us fix \( j = 1 \) and define \( A'_1 = (A_{11}, A_{12}, \ldots, A_{13n}) \) the components of the vector \( A'_1 \). Let us differential \( \phi \) with respect to \( A_{1l} \), for some \( l = 1, \ldots, 3n \). Differentiating with respect to \( A_{1l} \) variables we get form (105) that
\[ D_{A_{1l}}[I(A_1, A_2) - J_\varepsilon(u_0 + \tilde{\phi})] = (1 - \zeta)^2 \int_0^1 t dt \int_{\Omega_\varepsilon} D_{A_{1l}}[(N(\phi + \psi))\phi] + (p + \varepsilon) \int_{\Omega_\varepsilon} \nabla_{A_{1l}}[(v_0 + \psi + t\phi)^{p+\varepsilon-1} - v_0^{p+\varepsilon-1}\phi^2]. \]

Using the computations in the proof of Proposition 4.3 we get that
\[ D_{A_{1l}}[I(A_1, A_2) - J_\varepsilon(u_0 + \tilde{\psi})] = o(\varepsilon). \]
Now,
\[ J_\varepsilon(u_0 + \hat{\psi}) - J_\varepsilon(u_0) = (1 - \zeta)^2 [F_\varepsilon(v_0 + \psi) - F_\varepsilon(v_0)] = \]

(107) \[ (1 - \zeta)^2 \left\{ \int_0^1 (1 - t) dt \left[ (p + \varepsilon) \int_{\Omega_\varepsilon} \left( (v_0 + t\psi)^{p_\varepsilon - 1} - v_0^{p_\varepsilon - 1} \right) \psi^2 \right] - 2 \int_{\Omega_\varepsilon} E\psi \right\} \]

where we have used that
\[ DF_\varepsilon(v_0)[\psi] = -\int_{\Omega_\varepsilon} E\psi. \]

Arguing as before and taking into account that (106) holds, we get (102). On the other hand, using (107), we see that
\[ DA_{11}[J_\varepsilon(u_0 + \hat{\psi}) - J_\varepsilon(u_0)] = (1 - \zeta)^2 DA_{11} \left\{ \int_0^1 (1 - t) dt \times \left[ (p + \varepsilon) \int_{\Omega_\varepsilon} \left( (v_0 + t\psi)^{p_\varepsilon - 1} - v_0^{p_\varepsilon - 1} \right) \psi^2 \right] - 2 \int_{\Omega_\varepsilon} E\psi \right\} = o(\varepsilon) - 2DA_{11} \left( \int_{\Omega_\varepsilon} E\psi \right). \]

On the other hand, we have that
\[ DA_{11} (\int_{\Omega_\varepsilon} R^x \psi) = \begin{cases} O(\varepsilon^{2-\frac{n-2}{n-1}}) & \text{if } n \leq 5 \\ O(\varepsilon^{\frac{7}{4}} |\log \varepsilon|) & \text{if } n = 6 \\ O(\varepsilon^{1+\frac{4}{n-2}-\frac{1}{n-2}}) & \text{if } n \geq 7. \end{cases} \]

This concludes the proof of the Proposition.

\[ \square \]

7. The min-max

In this section we set up a min-max scheme to find a critical point of the function $\Psi$, defined in (64).

We write
\[ \Psi(\Lambda, \xi, a, \theta) = \Psi(\Lambda_1, \Lambda_2, \xi_1, \xi_2, a_1, a_2, \theta_1, \theta_2), \]

where
\[ (\Lambda, \xi, a, \theta) \in \mathbb{R}^2 \times (\Omega \times \Omega \setminus \{\xi_1 = \xi_2\}) \times B^2 \times K^2 \]

where
\[ B = \{(x_1, x_2) : \sqrt{x_1^2 + x_2^2} \leq \frac{1}{2}\}, \]

and $K$ is a compact manifold of dimension $2n - 3$, without boundary. Recall that $\Omega = D \setminus \omega$, where $\omega \subset \bar{B}(0, \delta) \subset D$. Define
\[ \varphi(\xi_1, \xi_2) = H(\xi_1, \xi_1)^{\frac{1}{2}} H(\xi_2, \xi_2)^{\frac{1}{2}} - G(\xi_1, \xi_2). \]

The following result holds true (see Corollary 2.1, [13])

**Corollary 7.1.** For any (fixed) sufficiently small $\sigma > 0$ there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and for any smooth domain $\omega \subset B(0, \delta)$ it holds
\[ \varphi(\xi_1, \xi_2) < 0 \quad \forall (\xi_1, \xi_2) \in S, \]

where the manifold $S$ is defined by
\[ S = \{ (\xi_1, \xi_2) \in \Omega^2 \mid |x_1| = |x_2| = \sigma \}. \]
For any \( \xi = (\xi_1, \xi_2) \in S \) we let \( d(\xi) = (d_1(\xi), d_2(\xi)) \in \mathbb{R}^2_+ \) be the negative direction of the quadratic form defining \( \Psi \). We easily see that there is a constant \( c > 0 \) so that \( c < d_1(\xi)d_2(\xi) < c^{-1} \) for all \( \xi \in S \).

In the following we will construct a critical point of “min-max” type of the function \( \Psi \). This construction has similarities with the ones developed in [13] and [20]. We start with the observation that the functions

\[
a \to \Psi, \quad \theta \to \Psi
\]

have a maximum respectively in \( B^2 \) and \( K^2 \). Let us now introduce for \( l > 0 \) and \( \rho > 0 \) the following manifold

\[
W^l_\rho = \{ \xi \in \Omega^2 \mid \varphi(\xi) < -l \} \cap V_\rho,
\]

where

\[
V_\rho = \{ (\xi_1, \xi_2) \in \Omega^2 \mid \text{dist}(\xi_1, \partial \Omega) > \rho, \text{dist}(\xi_2, \partial \Omega) > \rho, |\xi_1 - \xi_2| > \rho \}.
\]

If we take \( \lambda_0 = -\max_{\xi \in S} \varphi(\xi) \) and \( \rho_0 = \text{dist}(S, \partial \Omega) \), then for any \( \rho \in (0, \rho_0) \) and \( l \in (0, l_0) \) we have that \( S \subset W^l_\rho \). Moreover, for any \( R > 1 \)

\[
\max_{\xi \in S, r \leq R, a \in B^2, \theta \in K^2} \Psi(rd(\xi), \xi, a, \theta) > \max_{\xi \in S, r \leq R, a \in B^2, \theta \in K^2} \Psi(rd(\xi), \xi, a, \theta),
\]

where \( d(\xi) = (d_1(\xi), d_2(\xi)) \in \mathbb{R}^2_+ \) is the negative direction of the quadratic form defining \( \Psi \). This is a direct consequence of Corollary 7.2.

Now let \( A \) and \( B \) be fixed numbers defined as follows

\[
B = \max_{\xi \in S, r \leq R, a \in B^2, \theta \in K^2} \Psi(rd(\xi), \xi, a, \theta) > A
\]

(109)

\[
\max_{\xi \in S, r \leq R, a \in B^2, \theta \in K^2} \Psi(rd(\xi), \xi, a, \theta) > \max_{\xi \in S, r \leq R, a \in B^2, \theta \in K^2} \Psi(rd(\xi), \xi, a, \theta).
\]

There exists \( R > 0 \) large such that for any \( l \in (0, l_0) \) it holds

\[
B = \max_{\xi \in S, r \leq R, a \in B^2, \theta \in K^2} \Psi(rd(\xi), \xi, a, \theta) \geq \max_{\xi \in S, r \leq R, a \in B^2, \theta \in K^2} \Psi(\Lambda, \xi, a, \theta)
\]

(110)

\[
\max_{\xi \in S, r \leq R, a \in B^2, \theta \in K^2} \Psi(rd(\xi), \xi, a, \theta) > A \max_{\xi \in S, r \leq R, a \in B^2, \theta \in K^2} \Psi(rd(\xi), \xi, a, \theta),
\]

where \( I \) is the hyperbola in \( \mathbb{R}^2_+ \) defined by \( I = \{ \Lambda \in \mathbb{R}^2_+ \mid \Lambda_1\Lambda_2 = 1 \} \). Indeed, for any \( \Lambda \in I \), we have

\[
\Psi(\Lambda, \xi, a, \theta) \geq -G(\xi_1, \xi_2) \geq -\frac{1}{\rho^{n-2}} > A,
\]

provided that \( R \) is chosen properly.

**Lemma 7.2.** There exist \( l_0 > 0 \) and \( \rho_0 > 0 \) such that for any \( l \in (0, l_0) \) and \( \rho \in (0, \rho_0) \) the function \( \Psi \) satisfies the following property:

for any sequence \( (\Lambda_n, \xi_n, a_n, \theta_n) \) in \( [R^{-1}, R]^2 \times W^l_\rho \times B^2_2 \times K^2 \) such that

\[
\lim_n(\Lambda_n, \xi_n, a_n, \theta_n) = (\Lambda, \xi, a, \theta) \in \partial([R^{-1}, R]^2 \times W^l_\rho \times B^2 \times K^2)
\]

and \( \psi(\Lambda_n, \xi_n, a_n, \theta_n) \in [A, B] \) there exists a vector \( T \) tangent to \( \partial([R^{-1}, R]^2 \times W^l_\rho \times B^2 \times K^2) \) at \( (\xi, \Lambda, a, \theta) \), such that

\[
\nabla \Psi(\xi, \Lambda, a, \theta) \cdot T \neq 0.
\]
Let us call \( A \) such that for some \( n \) there exists a continuous deformation \( \nabla K \) can define an appropriate negative gradient flow that will remain in \( \Psi \) has a maximum in \( B \) below by a positive constant. First we claim that the function \( \Psi \) constrained to \( \Lambda \) satisfies the Palais-Smale condition in \( [A, B] \). Indeed, let \( \Lambda_n, \xi_n, a_n, \theta_n \) in \( \mathbb{R}^2_t \times W^l \times B^2_\rho \times K^2 \) be such that \( \lim \nabla \Psi(\Lambda_n, \xi_n, a_n, \theta_n) \in [A, B] \) and \( \lim \nabla \Psi(\Lambda_n, \xi_n, a_n, \theta_n) = 0 \). Arguing as in the proof of Lemma 7.2 it can be shown that \( \Lambda_n \) remains bounded component-wise from above and below by a positive constant.

Assume now by contradiction that there are no critical levels in the interval \( [A, B] \). We can define an appropriate negative gradient flow that will remain in \( [R^{-1}, R]^2 \times W^l \times B^2_\rho \times K^2 \) at any level \( c \in [A, B] \). Moreover the Palais-Smale condition holds in \( [A, B] \). Hence there exists a continuous deformation

\[
\eta : [0, 1] \times \Psi^B \to \Psi^B
\]

such that for some \( A' \in (0, A) \)

\[
\eta(0, u) = u \quad \forall \; u \in \Psi^B \\
\eta(t, u) = u \quad \forall \; u \in \Psi^{A'} \\
\eta(1, u) \in \Psi^{A'}. 
\]

Let us call

\[
A = \{(\Lambda, \xi, a, \theta) \in \mathbb{R}_t^2 \times W^l \times B^2 \times K^2 \mid \xi \in S, \; \Lambda = rd(\xi), \; R^{-1} \leq r \leq R \},
\]

\[
\partial A = \{(\Lambda, \xi, a, \theta) \in \mathbb{R}_t^2 \times W^l \times B^2 \times K^2 \mid \xi \in S, \; \Lambda = R^{-1} \text{ or } \Lambda = Rd(\xi) \},
\]

\[
C = I_r \times W^l. 
\]
From (110) we deduce that $A \subset \Psi B$, $\partial A \subset \Psi A'$ and $\Psi A' \cap C = \emptyset$. Therefore

$$\eta(0, u) = u \quad \forall \ u \in A,$$
$$\eta(t, u) = u \quad \forall \ u \in \partial A,$$

(113)
$$\eta(1, A) \cap C = \emptyset.$$

For any $(\Lambda, \xi, a, \theta) \in A$ and for any $t \in [0, 1]$ we denote

$$\eta(t, (\Lambda, \xi, a, \theta)) = (\tilde{\Lambda}, \tilde{\xi}, \tilde{a}, \tilde{\theta}) \in \mathbb{R}_+^2 \times W_\rho^l \times B^2 \times K^2.$$

We define the set

$$B = \{(\Lambda, x) \in A \mid \tilde{\Lambda} \in I\}.$$

Since $\eta(1, A) \cap C = \emptyset$ it holds $B = \emptyset$. Now let $\mathcal{U}$ be a neighborhood of $B$ in $\mathbb{R}_+^2 \times W_\rho^l \times B^2 \times K^2$ such that $H^*(\mathcal{U}) = H^*(B)$. If $\pi : \mathcal{U} \to S$ denotes the projection, arguing like in Lemma 7.1 of [10] we can show that

$$\pi^* : H^*(S) \to H^*(\mathcal{U})$$

is a monomorphism.

This condition provides a contradiction, since $H^*(\mathcal{U}) = \{0\}$ and $H^*(S) \neq \{0\}$. \qed
8. Appendix

To give a first description of these solutions, let us introduce some notations. Fix an integer \( k \). For any integer \( l = 1, \ldots, k \), we define angles \( \theta_l \) and vectors \( n_l, t_l \) by

\[
\theta_l = \frac{2\pi}{k} (l-1), \quad n_l = (\cos \theta_l, \sin \theta_l, 0), \quad t_l = (-\sin \theta_l, \cos \theta_l, 0).
\]

Here \( 0 \) stands for the zero vector in \( \mathbb{R}^{n-2} \). Notice that \( \theta_1 = 0, n_1 = (1, 0, 0) \), and \( t_1 = (0, 1, 0) \).

In [16] it was proved that there exists \( k_0 \) such that for all integer \( k > k_0 \) there exists a solution \( Q = Q_k \) to (10) that can be described as follows

\[
Q_k(x) = U_*(x) + \tilde{\phi}(x).
\]

where

\[
U_*(x) = U(x) - \sum_{j=1}^{k} U_j(x),
\]

while \( \tilde{\phi} \) is smaller than \( U_* \). The functions \( U \) and \( U_j \) are positive solutions to [10], respectively defined as

\[
U(x) = \gamma \left( \frac{2}{1 + |x|^2} \right)^{\frac{n-2}{2}}, \quad U_j(x) = \mu_k^{-\frac{n-2}{2}} U(\mu_k^{-1}(x - \xi_j)),
\]

where \( \gamma = \left[ \frac{n(n-2)}{4} \right]^{\frac{n-2}{2}} \). For any integer \( k \) large, the parameters \( \mu_k > 0 \) and the \( k \) points \( \xi_l, l = 1, \ldots, k \) are given by

\[
\mu_k = \sum_{l=1}^{k} \frac{1}{(1 - \cos \theta_l)^{\frac{n-2}{2}}} \cdot \mu_k^{-\frac{n-2}{2}} = 1 + O\left( \frac{1}{k} \right), \quad \text{for} \quad k \to \infty
\]

in particular \( \mu_k \sim k^{-2} \) if \( n \geq 4 \), and \( \mu_k \sim k^{-2} \log k \) if \( n = 3 \), as \( k \to \infty \), and

\[
\xi_l = \sqrt{1 - \mu_k^2} (n_l, 0).
\]

The function \( \tilde{\phi} \) in (115) can be further decomposed. Let us introduce some cut-off functions \( \xi_j \) to be defined as follows. Let \( \zeta(s) \) be a smooth function such that \( \zeta(s) = 1 \) for \( s < 1 \) and \( \zeta(s) = 0 \) for \( s > 2 \). We also let \( \zeta^-(s) = \zeta(2s) \). Then we set

\[
\xi_j(y) = \begin{cases} 
\zeta(k\eta^{-1}|y|^2 - |y - \xi_j|^2) & \text{if } |y| > 1, \\
\zeta(k\eta^{-1}|y - \xi_j|) & \text{if } |y| \leq 1,
\end{cases}
\]

in such a way that

\[
\zeta_j(y) = \zeta_j(y/|y|^2).
\]

The function \( \tilde{\phi} \) has the form

\[
\tilde{\phi} = \sum_{j=1}^{k} \phi_j + \psi.
\]

In the decomposition (120) the functions \( \phi_j \), for \( j > 1 \), are defined in terms of \( \phi_1 \)

\[
\tilde{\phi}_j(y, y') = \phi_1(e^{2\pi i j y}, y'), \quad j = 1, \ldots, k - 1.
\]

We have that

\[
\|\psi\|_{n-2} \leq C k^{1-\frac{2}{n}} \quad \text{if} \quad n \geq 4, \quad \|\psi\|_{n-2} \leq \frac{C}{\log k} \quad \text{if} \quad n = 3,
\]
where

\begin{equation}
\|\phi\|_{n-2} := \| (1 + |y|^{n-2})\phi \|_{L^\infty(\mathbb{R}^n)}.
\end{equation}

On the other hand, if we rescale and translate the function \(\tilde{\phi}_1\)

\begin{equation}
\phi_1(y) = \mu^{n-2} \tilde{\phi}_1(\xi_1 + \mu y)
\end{equation}

we have the validity of the following estimate for \(\phi_1\)

\begin{equation}
\|\phi_1\|_{n-2} \leq Ck^{-\frac{n}{q}} \quad \text{if} \quad n \geq 4, \quad \|\phi_1\|_{n-2} \leq \frac{C}{k \log k} \quad \text{if} \quad n = 3.
\end{equation}

In terms of the function \(\tilde{\phi}\) in the decomposition (115), equation (10) gets re-written as

\begin{equation}
\Delta \tilde{\phi} + p\gamma |U_*|^{p-1} \tilde{\phi} + E + \gamma N(\tilde{\phi}) = 0
\end{equation}

where \(E\) is defined by

\begin{equation}
E = \Delta U_* + f(U_*)
\end{equation}

and

\[N(\phi) = |U_* + \phi|^{p-1}(U_* + \phi) - |U_*|^{p-1}U_* - p|U_*|^{p-1}\phi.\]

One has a precise control of the size of the function \(E\) when measured for instance in the following norm. Let us fix a number \(q\), with \(\frac{n}{2} < q < n\), and consider the weighted \(L^q\) norm

\begin{equation}
\|h\|_{*s} = \| (1 + |y|)^{n+2-\frac{2n}{q}} h \|_{L^q(\mathbb{R}^n)}.
\end{equation}

In [16] it is proved that there exist an integer \(k_0\) and a positive constant \(C\) such that for all \(k \geq k_0\) the following estimates hold true

\begin{equation}
\|E\|_{*s} \leq Ck^{1-\frac{n}{q}} \quad \text{if} \quad n \geq 4, \quad \|E\|_{*s} \leq \frac{C}{\log k} \quad \text{if} \quad n = 3.
\end{equation}

To be more precise, we have estimates for the \(\| \cdot \|_{*s}\)-norm of the error term \(E\) in the exterior region \(\bigcap_{j=1}^{k} \{ |y - \xi_j| > \frac{\eta}{k} \}\), and also in the interior regions \(\{ |y - \xi_j| < \frac{\eta}{k} \}\), for any \(j = 1, \ldots, k\). Here \(\eta > 0\) is a positive and small constant, independent of \(k\).

In the exterior region. We have

\[\| (1 + |y|)^{n+2-\frac{2n}{q}} E(y) \|_{L^q(\bigcap_{j=1}^{k} \{ |y - \xi_j| > \frac{\eta}{k} \})} \leq Ck^{1-\frac{n}{q}}\]

if \(n \geq 4\), while

\[\| (1 + |y|)^{n+2-\frac{2n}{q}} E(y) \|_{L^q(\bigcap_{j=1}^{k} \{ |y - \xi_j| > \frac{\eta}{k} \})} \leq \frac{C}{\log k}\]

if \(n = 3\).

In the interior regions. Now, let \( |y - \xi_j| < \frac{\eta}{k} \) for some \(j \in \{ 1, \ldots, k \}\) fixed. It is convenient to measure the error after a change of scale. Define

\[\tilde{E}_j(y) := \mu^{\frac{n+2}{q}} E(y), \quad |y| < \frac{\eta}{\mu k}\]

We have

\[\| (1 + |y|)^{n+2-\frac{2n}{q}} \tilde{E}_j(y) \|_{L^q(|y - \xi_j| < \frac{\eta}{\mu k})} \leq Ck^{-\frac{n}{q}} \quad \text{if} \quad n \geq 4\]

and

\[\| (1 + |y|)^{n+2-\frac{2n}{q}} \tilde{E}_j(y) \|_{L^q(|y - \xi_j| < \frac{\eta}{\mu k})} \leq \frac{C}{k \log k} \quad \text{if} \quad n = 3\]

We refer the readers to [16].
Let us now define the following functions
\begin{equation}
\pi_{\alpha}(y) = \frac{\partial}{\partial y_{\alpha}} \tilde{\phi}(y), \quad \text{for} \quad \alpha = 1, \ldots, n; 
\end{equation}
\begin{equation}
\pi_0(y) = \frac{n-2}{2} \tilde{\phi}(y) + \nabla \tilde{\phi}(y) \cdot y.
\end{equation}
In the above formula \( \tilde{\phi} \) is the function defined in (115) and described in (120). Observe that the function \( \pi_0 \) is even in each of its variables, namely
\[ \pi_0(y_1, \ldots, y_j, \ldots, y_n) = \pi_0(y_1, \ldots, -y_j, \ldots, y_n) \quad \forall j = 1, \ldots, n, \]
while \( \pi_{\alpha} \), for \( \alpha = 1, \ldots, n \) is odd in the \( y_{\alpha} \) variable, while it is even in all the other variables. Furthermore, all functions \( \pi_{\alpha} \) are invariant under rotation of \( \frac{2\pi}{k} \) in the first two coordinates, namely they satisfy \( \text{(15)} \). The functions \( \pi_{\alpha} \) can be further described, as follows.

The functions \( \pi_{\alpha} \) can be decomposed into
\begin{equation}
\pi_{\alpha}(y) = \sum_{j=1}^{k} \tilde{\pi}_{\alpha,j}(y) + \hat{\pi}_{\alpha}(y)
\end{equation}
where
\[ \tilde{\pi}_{\alpha,j}(y) = \tilde{\pi}_{\alpha,1}(e^{\frac{2\pi}{k} j i} \bar{y}, y'). \]
Furthermore, there exists a positive constant \( C \) so that
\[ \| \hat{\pi}_0 \|_{n-2} \leq C k^{1-rac{n}{2}}, \quad \| \hat{\pi}_j \|_{n-1} \leq C k^{1-rac{n}{4}}, \quad j = 1, \ldots, k, \]
if \( n \geq 4 \), and
\[ \| \hat{\pi}_0 \|_{n-2} \leq \frac{C}{\log k}, \quad \| \hat{\pi}_j \|_{n-1} \leq \frac{C}{\log k}, \quad j = 1, \ldots, k, \]
if \( n = 3 \). Furthermore, if we denote \( \pi_{\alpha,1}(y) = \mu \frac{\partial}{\partial y_{\alpha}} \tilde{\pi}_{\alpha,1}(\xi_1 + \mu y) \), then
\[ \| \pi_{0,1} \|_{n-2} \leq C k^{1-rac{n}{2}}, \quad \| \pi_{\alpha,1} \|_{n-1} \leq C k^{1-rac{n}{4}}, \quad \alpha = 1, \ldots, n \]
if \( n \geq 4 \), and
\[ \| \pi_{0,1} \|_{n-2} \leq \frac{C}{k \log k}, \quad \| \pi_{\alpha,1} \|_{n-1} \leq \frac{C}{k \log k}, \quad \alpha = 1, \ldots, 3 \]
if \( n = 3 \).

For further details we refer the interested reader to [16].

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