ON THE NAVIER-STOKES EQUATIONS WITH ROTATING EFFECT AND PRESCRIBED OUTFLOW VELOCITY

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Abstract. We consider the equations of Navier-Stokes modeling viscous fluid flow past a moving or rotating obstacle in $\mathbb{R}^d$ subject to a prescribed velocity condition at infinity. In contrast to previously known results, where the prescribed velocity vector is assumed to be parallel to the axis of rotation, in this paper we are interested in a general outflow velocity. In order to use $L^p$-techniques we introduce a new coordinate system, in which we obtain a non-autonomous partial differential equation with an unbounded drift term. We prove that the linearized problem in $\mathbb{R}^d$ is solved by an evolution system on $L^p_\sigma(\mathbb{R}^d)$ for $1 < p < \infty$. For this we use results about time-dependent Ornstein-Uhlenbeck operators. Finally, we prove, for $p \geq d$ and initial data $u_0 \in L^p_\sigma(\mathbb{R}^d)$, the existence of a unique mild solution to the full Navier-Stokes system.

1. Introduction

The mathematical analysis of the Navier-Stokes flow past a rotating or moving obstacle has attracted quite some attention in recent years. It all started with the work of Borchers [Bor92] in the framework of suitable weak solutions. Later Hishida [His99] constructed local mild solutions to the Navier-Stokes problem in the exterior of a rotating obstacle in the context of $L^2$ by using semigroup techniques (see also [His01]). This existence result was extended to the general $L^p$-theory by Geissert, Heck, Hieber [GHH06] and Hishida, Shibata [HS09] showed that this solution is even a global one, provided the data are small enough. However, there are only a few partial results for the case when the fluid flow is subject to an additional outflow condition at infinity (hereby we mean a prescribed velocity of fluid at infinity). In fact, this situation was studied rather recently by Farwig [Par06] and Shibata [Shi08] only for the special case when the outflow direction of the fluid is parallel to the axis of rotation of the obstacle. This assumption ensures – after rewriting the problem on a fixed domain – that the resulting equations are autonomous and thus can be treated e.g. by applying semigroup techniques. The purpose of this paper is to extend the existing results and to combine the rotating effect with a general outflow condition. For this purpose it is necessary to study the Navier-Stokes system perturbed by time-dependent and unbounded lower order terms, which is done here for the whole space case.
To describe the situation more precisely, let $\mathcal{O} \subset \mathbb{R}^d$ be a compact obstacle with smooth boundary and let $\Omega := \mathbb{R}^d \setminus \mathcal{O}$ be the exterior of the obstacle. We are interested in the case where the obstacle undergoes a prescribed motion, particularly a rotation. So we let $M : [0, \infty) \to \mathbb{R}^{d \times d}$ be a continuous matrix-valued function, such that $M(t)$ is skew-symmetric for all $t > 0$, i.e. $M(t) = -M^*(t)$, and $M(t), M(s)$ commute for all $t, s > 0$. The exterior of the rotated obstacle at time $t > 0$ is represented by $\Omega(t) := U(t, 0) \Omega$ where

$$U(t, s) := \exp \left( \int_s^t M(\tau) d\tau \right), \quad t, s \geq 0.$$  

Since $M(t)$ is skew-symmetric for all $t > 0$, the matrices $U(t, s)$ are orthogonal. With a given velocity vector $v_\infty \in \mathbb{R}^d \neq 0$, representing the outflow velocity of the fluid, the Navier-Stokes equations on the time-dependent domain $\Omega(t)$ with the usual no-slip boundary condition now take the form

$$v_t - \Delta v + v \cdot \nabla v + \nabla q = 0 \quad \text{in } \Omega(t) \times (0, \infty),$$

$$\text{div } v = 0 \quad \text{in } \Omega(t) \times (0, \infty),$$

$$v(t, y) = M(t)y \quad \text{on } \partial \Omega(t) \times (0, \infty),$$

$$\lim_{|y| \to \infty} v(t, y) = v_\infty \neq 0 \quad \text{for } t \in (0, \infty),$$

$$v(0, y) = u_0(y) \quad \text{in } \Omega,$$

where $v$ and $q$ are the unknown velocity field and the pressure of the fluid, respectively. The disadvantage of this description is the variability of the domain $\Omega(t)$, and the fact that the equations do not fit into the $L^p$-setting, due the velocity condition at infinity. By setting

$$x = U^*(t, 0)y, \quad u(t, x) = U^*(t, 0)(v(t, y) - v_\infty), \quad p(t, x) = q(t, y),$$  

the above equations can be transformed back to the reference domain $\Omega$ and the new velocity field $u$ vanishes at infinity. We obtain the following system of equations:

$$u_t - \Delta u - M(t)x \cdot \nabla u + M(t)u + U^*(t, 0)v_\infty \cdot \nabla u + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \Omega \times (0, \infty),$$

$$\text{div } u = 0 \quad \text{in } \Omega \times (0, \infty),$$

$$u(t, x) = M(t)x - U^*(t, 0)v_\infty \quad \text{on } \partial \Omega \times (0, \infty),$$

$$\lim_{|x| \to \infty} u(t, x) = 0 \quad \text{for } t \in (0, \infty),$$

$$u(0, x) = u_0(x) \quad \text{in } \Omega.$$  

The prize to pay for this transformation is that we obtain a non-autonomous partial differential equation with an unbounded drift term. Even if we assume that $M(t) \equiv M$ is independent of time, equation (1.4) is still non-autonomous due to the time-dependent first order term $U^*(t, 0)v_\infty \cdot \nabla$. Only in the special situation where the velocity vector $v_\infty$

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This condition can physically be interpreted by the fact that the axis of rotation is fixed.
is parallel to the axis of rotation – in this case $v_\infty$ is a fixed point under the transformation $U^*(t,0)$ – the transformed equations remain autonomous. This shows that if one allows a general outflow condition, it is necessary to study a non-autonomous problem.

In the special case, where $M(t)x = \omega(t) \times x$ and $\omega : [0, \infty) \to \mathbb{R}^3$ is the angular velocity of the obstacle, Borchers [Bor92] constructed weak non-stationary solutions for the equations (1.4). Later, Farwig [Far06] studied the linearized stationary problem with $\Omega = \mathbb{R}^d$ and he proved $\mathcal{L}$-estimates for the second derivative of the velocity field $u$ and for the first derivative of the pressure $p$. However, he only considered the case, where $M(t)x = \omega \times x$ with $\omega \in \mathbb{R}^3$ parallel to $v_\infty$. Recently, Shibata [Shi08] proved, also for $M(t)x = \omega \times x$ with $\omega \in \mathbb{R}^3$ parallel to $v_\infty$, that the solution of the linearized problem is governed by a strongly continuous semigroup on $L^p_\sigma(\Omega)$, $1 < p < \infty$, which is not analytic. His main result is actually the boundedness of the semigroup (see also [HS09] for the case $v_\infty = 0$). By using Kato’s iteration scheme ([Kat84, Gig86]) this allows to prove the existence of a global solution to the full nonlinear problem for small initial data. A time-dependent fundamental solution (Green’s function) to problem (1.4) was derived by Thomann, Guenther in [TG06] for the special case $M(t)x = \omega \times x$ with $\omega \in \mathbb{R}^3$ parallel to $v_\infty$.

Our approach to the non-autonomous equations (1.4) is based on a linearization and on the family of modified time-dependent Stokes operators

$$A(t)u := \mathbb{P} (\Delta u + (M(t)x - U^*(t,0)v_\infty) \cdot \nabla u - M(t)u), \quad t > 0,$$

where $\mathbb{P}$ denotes the Helmholtz-Leray projection from $L^p(\Omega)$ into $L^p_\sigma(\Omega)$, the space of all solenoidal vector fields in $L^p(\Omega)$ (see e.g. [Gal94, Chapter III]). The main difficulty for treating operators of the above kind lies in the fact that the coefficients of the drift term are unbounded and thus the first order term cannot be considered as a “small” perturbation of the classical Stokes operator in unbounded domains. However, it has been shown by Hieber, Sawada [HS05] for $\Omega = \mathbb{R}^d$ and by Geissert, Heck, Hieber [GHH06] for exterior domains $\Omega$, that in the autonomous case, i.e. for fixed $t$, and for $v_\infty = 0$, the operator $A(t)$ with an appropriate domain generates a strongly continuous semigroup on $L^p_\sigma(\Omega)$, $1 < p < \infty$, which is, however, not analytic. The fact that the semigroup is not analytic prevents us from employing standard generation results for evolution systems of parabolic type mainly due to Tanabe [Tan59, Tan60a, Tan60b] or Acquistapace, Terreni [Acq84, AT86, AT87] (see also [Paz83, Chapter 5] or [Tan97, Chapter 6] for more information on this matter). Here lies one of the main difficulties. A first step in the study of the problem is to consider the whole space case rather than the physically more realistic situation of exterior domains. A solution to the whole space problem is not only interesting in its own right but also needed for using a cut-off technique to solve the exterior domain problem in the next step. Therefore, for the rest of this paper we study – in a more general form – the non-autonomous equations

$$u_t - \Delta u - (M(t)x + f(t)) \cdot \nabla u + M(t)u + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty),$$
$$\text{div } u = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty),$$
$$u(0) = u_0 \quad \text{in } \mathbb{R}^d,$$
where $M : [0, \infty) \to \mathbb{R}^{d \times d}$, $f : [0, \infty) \to \mathbb{R}^d$ are continuous functions and where we assume in addition that $M(t)M(s) = M(s)M(t)$ holds for all $t, s > 0$. Here as usual, $u : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}^d$ and $p : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ denote the unknown velocity field and the pressure of the fluid respectively. By setting $f(t) = -U^*(t, 0)v_\infty$ we are in the special situation of equation (1.4).

This paper is organized as follows. In Section 2 we review and prove results on time-dependent Ornstein-Uhlenbeck operators, studied recently by Da Prato, Lunardi [DPL07] and Geissert, Lunardi [GL08]. By using these results in Section 3 we prove that the solution to the linearized problem is given by a strongly continuous evolution system on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, and we derive an explicit formula for the evolution operators, similar to the representation formula known in the case of time-dependent Ornstein-Uhlenbeck operators. Moreover, we prove $L^p-L^q$ as well as gradient estimates for the evolution system. In Section 4 we return to the full Navier-Stokes problem (1.5) and prove the existence of a mild solution by adjusting Kato’s iteration scheme to our situation.

2. **Time-dependent Ornstein-Uhlenbeck Operators**

In this section we assume that $M : \mathbb{R} \to \mathbb{R}^{d \times d}$ and $f : \mathbb{R} \to \mathbb{R}^d$ are continuous functions. Moreover, we define $\tilde{M}(t) := M(-t)$ for $t \in \mathbb{R}$ and denote by $U(t, s)$ and $\tilde{U}(t, s)$ the solutions of the problems

\[
\begin{aligned}
\frac{d}{dt}U(t, s) &= M(t)U(t, s), \quad t, s \in \mathbb{R}, \\
U(s, s) &= I,
\end{aligned}
\tag{2.1}
\]

and

\[
\begin{aligned}
\frac{d}{dt}\tilde{U}(t, s) &= \tilde{M}(t)\tilde{U}(t, s), \quad t, s \in \mathbb{R}, \\
\tilde{U}(s, s) &= I,
\end{aligned}
\tag{2.2}
\]

respectively.

Now we consider time-dependent Ornstein-Uhlenbeck operators $L(t)$, formally defined on smooth functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ by

\[
(L(t)\varphi)(x) = \Delta \varphi(x) + (M(t)x + f(t)) \cdot \nabla \varphi(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d,
\tag{2.3}
\]

and the associated non-autonomous forward Cauchy problem

\[
\begin{aligned}
u_t(t, x) &= L(t)u(t, x), \quad s < t, \quad x \in \mathbb{R}^d, \\
u(s, x) &= \varphi(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\tag{2.4}
\]

where $s \in \mathbb{R}$ is fixed. A straightforward change of variables allows to transform problem (2.4) into an equivalent backward problem. More precisely, the function $(t, x) \mapsto u(t, x)$ is a classical solution to problem (2.4) if and only if the function $(t, x) \mapsto v(t, x) := u(-t, x)$ is a classical solution to the backward problem.

\footnote{The physically reasonable condition that $M(t)$ is skew-symmetric for all $t > 0$ is not needed for our main results and therefore not explicitly assumed for the rest of the paper unless otherwise stated.}
\[ \begin{cases} v_t(t, x) + \tilde{\mathcal{L}}(t)v(t, x) = 0, & t < -s, \ x \in \mathbb{R}^d, \\ v(-s, x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.5} \]

where \( \tilde{\mathcal{L}}(t) := \mathcal{L}(-t) \). Such a backward problem was considered by Da Prato, Lunardi [DPL07] and Geissert, Lunardi [GL08], since their main motivation came from stochastics. In our case, with the application to problem (1.5) in mind, it is more convenient to work with the forward problem. The following proposition follows, via the transformation mentioned above, directly from the analogous result for the backward equation (2.5) proved in [DPL07] Proposition 2.1.

**Proposition 2.1.** Let \( \varphi \in C_c^\infty(\mathbb{R}^d) \) and fix \( s \in \mathbb{R} \). Then problem (2.4) has a unique bounded classical solution \( u \in \mathcal{C}^{1,2}([s, \infty) \times \mathbb{R}^d) \), given by the formula

\[ u(t, x) = \frac{1}{(4\pi)^d (\det Q_{t,s})^{\frac{1}{2}}} \int_{\mathbb{R}^d} \varphi(\tilde{U}(-s, -t)x + g(t, s) - y) e^{-\frac{1}{4} (Q_{t,s}^{-1} y - y)^2} \, dy, \tag{2.6} \]

where \( g(t, s) \) and \( Q_{t,s} \) are defined by

\[ g(t, s) = \int_{-s}^{-t} \tilde{U}(-s, r)f(-r) \, dr \quad \text{and} \quad Q_{t,s} = \int_{-t}^{-s} \tilde{U}(-s, r)\tilde{U}^*(s, r) \, dr \tag{2.7} \]

respectively.

Note that the right hand side of (2.6) is well defined for each \( L^p(\mathbb{R}^d) \)-function \( \varphi \). Thus, in the following this explicit formula serves as a starting point to define an *evolution system* on \( L^p(\mathbb{R}^d) \), \( 1 < p < \infty \), associated with problem (2.4). Before, we have to give equation (2.4) a meaning in the \( L^p \)-setting, i.e., we have to define the \( L^p \)-realizations of the formally defined operators \( \mathcal{L}(t) \). For this purpose we set

\[ \mathcal{D}(L(t)) := \{ \varphi \in W^{2,p}(\mathbb{R}^d) : M(t)x \cdot \nabla \varphi(x) \in L^p(\mathbb{R}^d) \}, \]

\[ L(t)\varphi := \tilde{\mathcal{L}}(t)\varphi. \tag{2.8} \]

Here the domain of \( L(t) \) is depending on \( t \), but \( C_c^\infty(\mathbb{R}^d) \) is a subset of \( \mathcal{D}(L(t)) \) for every \( t \in \mathbb{R} \). It has been shown by Metafune [Met01] and Metafune, Prüss, Rhandi, Schnaubelt [MPRS02] that in the autonomous case, i.e., for fixed \( t \), and for \( f(t) = 0 \), the operator \( L(t) \) with domain \( \mathcal{D}(L(t)) \) generates a strongly continuous semigroup on \( L^p(\mathbb{R}^d) \). However, due to the fact that the coefficients of the drift term are unbounded this semigroup is not analytic on \( L^p(\mathbb{R}^d) \) in general. Thus, the existence of an evolution system with nice regularity properties does not follow from the general theory of parabolic evolution equations. However, formula (2.5) allows to define a family of operators as follows. For \( \varphi \in L^p(\mathbb{R}^d) \) we put \( G(s, t)\varphi = \varphi \) and for \( t > s \) we define the operator \( G(t, s) \) by

\[ G(t, s)\varphi(x) := \frac{1}{(4\pi)^{\frac{d}{2}} (\det Q_{t,s})^{\frac{1}{2}}} \int_{\mathbb{R}^d} \varphi(\tilde{U}(-s, -t)x + g(t, s) - y) e^{-\frac{1}{4} (Q_{t,s}^{-1} y - y)^2} \, dy, \tag{2.9} \]

where \( g(t, s) \) and \( Q_{t,s} \) are defined as in (2.7).

**Lemma 2.2.** For \( t \geq s \) fixed, the linear operator \( G(t, s) \), defined in (2.9), is bounded on \( L^p(\mathbb{R}^d) \), \( 1 < p < \infty \). Moreover, \( G(t, s)\varphi \in \mathcal{D}(L(t)) \) holds for any \( \varphi \in C_c^\infty(\mathbb{R}^d) \) and \( t \geq s \).
Proof. First let us note, that for \( \varphi \in L^p(\mathbb{R}^d) \) and \( t > s \) we can write
\[
G(t, s) \varphi(x) = (\varphi * k_{t,s})(U(-s, -t)x + g(t, s)), \quad x \in \mathbb{R}^d,
\]
where the kernel \( k_{t,s} \) is defined by
\[
k_{t,s}(x) := \frac{1}{(4\pi)^{\frac{d}{2}}(\det Q_{t,s})^\frac{d}{2}} e^{-\frac{1}{4}(Q_{t,s}^{-1}x,x)}, \quad x \in \mathbb{R}^d.
\]
By a change of variable and Young’s inequality we obtain
\[
\|G(t, s) \varphi\|_{L^p(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |(\varphi * k_{t,s})(U(-s, -t)x + g(t, s))|^p \, dx \right)^\frac{1}{p}.
\]
where the kernel \( k_{t,s} \) is defined by
\[
k_{t,s}(x) := \frac{1}{(4\pi)^{\frac{d}{2}}(\det Q_{t,s})^\frac{d}{2}} e^{-\frac{1}{4}(Q_{t,s}^{-1}x,x)}, \quad x \in \mathbb{R}^d.
\]
By a change of variable and Young’s inequality we obtain
\[
\|G(t, s) \varphi\|_{L^p(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |(\varphi * k_{t,s})(U(-s, -t)x + g(t, s))|^p \, dx \right)^\frac{1}{p}.
\]
where the kernel \( k_{t,s} \) is defined by
\[
k_{t,s}(x) := \frac{1}{(4\pi)^{\frac{d}{2}}(\det Q_{t,s})^\frac{d}{2}} e^{-\frac{1}{4}(Q_{t,s}^{-1}x,x)}, \quad x \in \mathbb{R}^d.
\]
for some constant \( C > 0 \). This proves the first assertion.

To prove the second assertion it suffices to show \( M(t)x \cdot \nabla(\varphi * k_{t,s})(x) \in L^p(\mathbb{R}^d) \), since \( U(-s, -t) \) is an invertible matrix. At first we note that
\[
\nabla(\varphi * k_{t,s})(x) = \frac{1}{(4\pi)^{\frac{d}{2}}(\det Q_{t,s})^\frac{d}{2}} \int_{\mathbb{R}^d} \nabla \varphi(y)e^{-\frac{1}{4}(Q_{t,s}^{-1/2}(x-y),y)} \, dy
\]
holds. Now for a function \( h \in L^q(\mathbb{R}^d) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) we obtain
\[
\int_{\mathbb{R}^d} |(M(t)x \cdot \nabla(\varphi * k_{t,s})(x))| h(x) |dx|
\]
\[
\leq C \int_{\mathbb{R}^d} |\nabla \varphi(y)| \int_{\mathbb{R}^d} |M(t)x \exp\left(-\frac{1}{4}Q_{t,s}^{-\frac{1}{2}}(x-y)\right)| h(x) |dx| dy
\]
\[
\leq C \int_{\mathbb{R}^d} |\nabla \varphi(y)| \int_{\mathbb{R}^d} |M(t)x \exp\left(-\frac{1}{4}Q_{t,s}^{-\frac{1}{2}}2 Q_{t,s}^{-\frac{1}{2}}(x,y)\right)| h(x) |dx| dy
\]
\[
\leq C \int_{\text{supp} \varphi} |\nabla \varphi(y)\exp\left(-\frac{1}{4}Q_{t,s}^{-\frac{1}{2}}2 Q_{t,s}^{-\frac{1}{2}}(x,y)\right)| dy \cdot \int_{\mathbb{R}^d} |M(t)x \exp\left(-\frac{1}{4}((Q_{t,s}^{-\frac{1}{2}}x)^2 - 2K|x|))\right)| h(x) |dx|
\]
for constants \( C, K > 0 \). Here we essentially used the fact that \( \text{supp} \varphi \) is compact. Thus, we can conclude that
\[
\int_{\mathbb{R}^d} |(M(t)x \cdot \nabla(\varphi * k_{t,s})(x))| h(x) |dx| < \infty
\]
holds for every \( h \in L^q(\mathbb{R}^d) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). This yields the assertion. \( \square \)

We are now in position to state the main result of this section.

**Proposition 2.3.** Let \( 1 < p < \infty \). The two parameter family of bounded linear operators \( \{ G(t,s) : s \leq t \} \) defines an evolution system on \( L^p(\mathbb{R}^d) \), i.e.,
(i) $G(s, s) = Id$ and $G(t, s) = G(t, r)G(r, s)$ for $-\infty < s \leq r \leq t < \infty$,
(ii) for each $\varphi \in L^p(\mathbb{R}^d)$, $(t, s) \mapsto G(t, s)\varphi$ is continuous on $-\infty < s \leq t < \infty$.

Moreover, for any initial value $\varphi \in C_c^\infty(\mathbb{R}^d)$, the abstract non-autonomous Cauchy problem
\begin{equation}
\begin{aligned}
u'(t) &= L(t)u(t), \quad s < t, \\
u(s) &= \varphi,
\end{aligned}
\end{equation}

admits a classical solution $u$ given by $u(t) = G(t, s)\varphi$.

Proof. In [GL08, Proposition 2.4] it was shown that the law of evolution
\[ G(t, s)G(s, r)\varphi = G(t, r)\varphi, \quad r \leq s \leq t, \]
holds for every $\varphi \in C_c^\infty(\mathbb{R}^d)$. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, property (i) follows.

In order to prove property (ii), we apply the change of the variable $y = Q_{t,s}^{1/2}z$, to see that
\[ G(t, s)\varphi(x) = \frac{(\det Q_{t,s})^{\frac{1}{2}}}{(4\pi)^{\frac{d}{2}}(\det Q_{t,s})^{\frac{1}{2}}} \int_{\mathbb{R}^d} \varphi(\tilde{U}(-s, -t)x + g(t, s) - Q_{t,s}^{\frac{1}{2}}z) e^{-\frac{|z|^2}{4}} dz \]
holds. For $t > s$ fixed, we pick two sequences $(t_n)_{n \in \mathbb{N}}$ and $(s_n)_{n \in \mathbb{N}}$ such that $t_n \geq s_n$ holds for every $n \in \mathbb{N}$ and $(t_n, s_n) \rightarrow (t, s)$ as $n \rightarrow \infty$. For every $\varphi \in C_c^\infty(\mathbb{R}^d)$ and every $x \in \mathbb{R}^d$ we now obtain
\[ \varphi(\tilde{U}(-s_n, -t_n)x + g(t_n, s_n) - Q_{t_n,s_n}^{\frac{1}{2}}z) \rightarrow \varphi(\tilde{U}(-s, -t)x + g(t, s) - Q_{t,s}^{\frac{1}{2}}z) \]
as $n \rightarrow \infty$. Lebegue’s theorem now yields $G(t_n, s_n)\varphi \rightarrow G(t, s)\varphi$ as $n \rightarrow \infty$ for every $\varphi \in C_c^\infty(\mathbb{R}^d)$. The density of $C_c^\infty(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$ yields (ii).

The last assertion follows directly from Proposition 2.4 and Lemma 2.2.

In order to prove $L^p$-$L^q$ and gradient estimates in the following section we need the following estimates for the matrices $Q_{t,s}$.

Lemma 2.4. For $0 < T < \infty$ there exists a constant $C := C(T) > 0$ such that

(i) $\|Q_{t,s}^{\frac{1}{2}}\| \leq C(t - s)^{-\frac{1}{2}}, \quad 0 < s < t < T$,
(ii) $(\det Q_{t,s})^{\frac{1}{2}} \geq C(t - s)^{\frac{d}{2}}, \quad 0 < s < t < T$.

Assertion (i) has been proved by Geissert and Lunardi [GL08 Lemma 3.2]. However, to make the paper as self-contained as possible we provide a proof here.

Proof. Let $T > 0$ and $x \in \mathbb{R}^d$. From (2.7) we obtain
\[ \langle Q_{t,s}x, x \rangle = \int_{-T}^{-s} \langle \tilde{U}(-s, r)\tilde{U}^*(-s, r)x, x \rangle dr = \int_{-T}^{-s} \|\tilde{U}^*(-s, r)x\|^2 dr. \]
The continuity of the map $(-s, -t) \mapsto \tilde{U}(-s, -t)$ yields that there exists a $\delta > 0$ such that $\|\tilde{U}^*(-s, -t)x - x\| \leq \frac{1}{2}\|x\|$ for $t - s \leq \delta$. Thus
\[ \langle Q_{t,s}x, x \rangle \geq \frac{1}{4}(t - s)\|x\|^2 \quad (2.11) \]
We define the family of linear operators \( B \) solution to problem (2.1) for \( M \) are continuous and can easily been seen, since in this case \( U \) for constants \( C \) the evolution operator \( G \) is given by

\[
U(t, s) = \exp \left( \int_s^t M(r) \, dr \right).
\]

In the case that \( \sigma \) change of variables the representation formula (2.9) can be rewritten to the following form.

Thus by applying (i) we obtain

\[
\det Q_{t,s} = (\det Q_{t,s}^{-1})^{-1} \geq C_1 \left( \|Q_{t,s}\| \right)^{-1} \geq C_2(t - s)^d,
\]

for constants \( C_1, C_2 > 0 \) and assertion (ii) directly follows.

In the case that \( M(t), M(s) \) commute for all \( t, s \in \mathbb{R} \), we have \( \tilde{U}(-s, -t) = U(t, s) \). This can easily been seen, since in this case \( U(t, s) \) has the explicit form (1.1). By a simple change of variables the representation formula (2.9) can be rewritten to the following form.

**Corollary 2.5.** Let \( M(t), M(s) \) commute for all \( s, t \in \mathbb{R} \). Then for \( \varphi \in L^p(\mathbb{R}^d) \) and \( t > s \) the evolution operator \( G(t, s) \) associated with the non-autonomous Cauchy problem (2.10) is given by

\[
G(t, s)\varphi(x) := \frac{1}{(4\pi)^{d/2}(\det Q_{t,s})^{1/2}} \int_{\mathbb{R}^d} \varphi(U(t, s)x + g(t, s) - y)e^{-\frac{1}{2}(Q_{t,s}^{-1}y, y)}dy,
\]

where \( g(t, s) \) and \( Q_{t,s} \) are defined by

\[
g(t, s) = \int_s^t U(r, s)f(r)dr \quad \text{and} \quad Q_{t,s} = \int_s^t U(r, s)U^*(r, s)dr,
\]

respectively.

3. **The Linearized Problem: The Evolution System on** \( L^p(\mathbb{R}^d) \)

From now on our standing assumption is that \( M : [0, \infty) \to \mathbb{R}^{d \times d}, f : [0, \infty) \to \mathbb{R}^d \) are continuous and \( M(t), M(s) \) commute for all \( t, s \geq 0 \). We recall that in this case the solution to problem (2.1) for \( t, s \geq 0 \) is given by

\[
U(t, s) = \exp \left( \int_s^t M(r) \, dr \right).
\]

We define the family of linear operators \( B(t), t > 0, \) in \( L^p(\mathbb{R}^d)^d, 1 < p < \infty \), by

\[
\mathcal{D}(B(t)) := \mathcal{D}(L(t))^d, \quad B(t)u := D_{L(t)}u - M(t)u,
\]

(3.2)
where \( u = (u_1, \ldots, u_d) \in L^p(\mathbb{R}^d)^d \). Here \( D_{L(t)} \) is the \( d \times d \) diagonal matrix operator with entries \( L(t) \), defined as in (2.8). For \( u \in L^p(\mathbb{R}^d)^d \) we put \( W(s, s)u = u \) and for \( 0 \leq s < t \) we define

\[
W(t, s)u(x) = \frac{1}{(4\pi)^{\frac{d}{2}}(\det Q_{t,s})^\frac{d}{2}} U(s, t) \cdot \int_{\mathbb{R}^d} u(U(t, s)x + g(t, s) - y) \\
\times e^{-\frac{1}{2}(Q_{t,s}^{-1}y,y)} dy, \quad x \in \mathbb{R}^d,
\]

where \( g(t, s) \) and \( Q_{t,s} \) are defined as in (2.14). Analogously to Lemma 2.2 it follows that, for \( 0 \leq s \leq t \), the operator \( W(t, s) \) is well defined and bounded on \( L^p(\mathbb{R}^d)^d \). Based on Proposition 2.3 and Corollary 2.3 we now obtain the following result.

**Proposition 3.1.** Let \( 1 < p < \infty \). The two parameter family of bounded linear operators \( \{W(t, s) : 0 \leq s \leq t\} \) defines an evolution system on \( L^p(\mathbb{R}^d)^d \), i.e.,

(i) \( W(s, s) = I_d \) and \( W(t, s) = W(t, r)W(r, s) \) for \( 0 \leq s \leq r \leq t < \infty \),

(ii) for each \( u \in L^p(\mathbb{R}^d)^d \), \( (t, s) \mapsto W(t, s)u \) is continuous on \( 0 \leq s \leq t < \infty \).

Moreover, for any initial value \( \varphi \in C^\infty_c(\mathbb{R}^d)^d \), the abstract non-autonomous Cauchy problem

\[
\begin{cases}
  u'(t) = B(t)u(t), & 0 \leq s < t, \\
  u(s) = \varphi,
\end{cases}
\]

admits a classical solution \( u \) given by \( u(t) = W(t, s)\varphi \).

**Proof.** For \( u \in L^p(\mathbb{R}^d)^d, t > s \), and \( x \in \mathbb{R}^d \) we define the operator \( \tilde{G}(t, s) \) by

\[
\tilde{G}(t, s)u(x) := \frac{1}{(4\pi)^{\frac{d}{2}}(\det Q_{t,s})^\frac{d}{2}} \int_{\mathbb{R}^d} u(U(t, s)x + g(t, s) - y)e^{-\frac{1}{2}(Q_{t,s}^{-1}y,y)} dy.
\]

This is just the Ornstein-Uhlenbeck evolution system from Proposition 2.3 applied in each component of the function \( u = (u_1, \ldots, u_d) \). Thus, \( \{\tilde{G}(t, s) : 0 \leq s \leq t\} \) is an evolution system on \( L^p(\mathbb{R}^d)^d \) such that

\[
\frac{\partial}{\partial t}\tilde{G}(t, s)\varphi = D_{L(t)}\tilde{G}(t, s)\varphi
\]

holds for every \( \varphi \in C^\infty_c(\mathbb{R}^d)^d \). Note that for \( u \in L^p(\mathbb{R}^d)^d \) and \( t \geq s \) we can write\(^3\)

\( W(t, s)u = U(s, t)\tilde{G}(t, s)u \). By applying the product rule we obtain

\[
\frac{\partial}{\partial t} W(t, s)u = \frac{\partial}{\partial t} U(s, t)\tilde{G}(t, s)u
= U(s, t)D_{L(t)}\tilde{G}(t, s)u - U(s, t)M(t)\tilde{G}(t, s)u
= B(t)W(t, s)u,
\]

for every \( u \in C^\infty_c(\mathbb{R}^d)^d \). We have used that \( D_{L(t)} - M(t) \) commutes with the multiplication by \( U(s, t) \), which can be easily seen, as \( U(t, s) \) is given by (3.1). Thus for every \( u \in C^\infty_c(\mathbb{R}^d) \) the solution to equation (3.8) is indeed given by \( W(t, s)u \).

---

\(^3\)To be precise, \( U(s, t) \) has to be interpreted here as a multiplication operator.
The law of evolution follows from a similar calculation. For $0 \leq s \leq r \leq t$ we have
\[
W(t,r)W(r,s)u = U(r,t)G(t,r)\left(U(s,r)\dot{G}(r,s)u\right) = U(r,t)U(s,r)\dot{G}(t,r)\dot{G}(r,s)u = W(t,s)u.
\]
Here we have used $U(r,t)U(s,r) = U(s,t)$, which also can be seen from (3.3).

The strong continuity of $(t,s) \mapsto W(t,s)$ follows directly from the strong continuity of $(t,s) \mapsto U(s,t)$ and $(t,s) \mapsto \dot{G}(t,s)$. This completes the proof.

By the Proposition 3.1, \{W(t,s) : 0 \leq s \leq t\} is an evolution system on $L^p(\mathbb{R}^d)$. However, later in Section 4 we shall not work on $L^p(\mathbb{R}^d)$ but, as usual in the theory of the Navier-Stokes equations, on $L^p_\sigma(\mathbb{R}^d)$, the space of all solenoidal vector fields in $L^p(\mathbb{R}^d)$. Therefore we also consider the operators $A(t)$, $t > 0$, in $L^p_\sigma(\mathbb{R}^d)$ defined by
\[
A(t) := B(t)|_{L^p_\sigma(\mathbb{R}^d)},
\]
\[
\mathcal{D}(A(t)) := \mathcal{D}(B(t)) \cap L^p_\sigma(\mathbb{R}^d),
\]
(i.e., $A(t)$ is the restriction of $B(t)$ to $L^p_\sigma(\mathbb{R}^d)$). To ensure that this definition really makes sense we have to show that the operators $B(t)$, $t > 0$, leave $L^p_\sigma(\mathbb{R}^d)$ invariant. An easy calculation shows that
\[
\text{div}\left\{M(t)x \cdot \nabla u + f(t) \cdot \nabla u - M(t)u\right\} = 0
\]
holds for all $u \in C^\infty_{c,\sigma}(\mathbb{R}^d)$. Thus, $(t,s) \mapsto B(t)$, $t > 0$, is indeed a linear operator acting on $L^p_\sigma(\mathbb{R}^d)$.

Similarly, we can show that
\[
\text{div}\left\{(U(s,t) \cdot u(U(t,s)x + g(t,s)) = (\text{div} u) (U(t,s)x + g(t,s)) = 0
\]
holds for all $u \in C^\infty_{c,\sigma}(\mathbb{R}^d)$. It now easily follows from (3.3) that also the evolution system \{W(t,s) : 0 \leq s \leq t\} leaves $L^p_\sigma(\mathbb{R}^d)$ invariant. Thus we can define a family of operators on $L^p_\sigma(\mathbb{R}^d)$ by setting
\[
V(t,s) = W(t,s)|_{L^p_\sigma(\mathbb{R}^d)}, \quad 0 \leq s \leq t,
\]
i.e., $V(t,s)$ is just the restriction of $W(t,s)$ to $L^p_\sigma(\mathbb{R}^d)$. The next result now follows directly from Proposition 3.1.

**Proposition 3.2.** Let $1 < p < \infty$. The two parameter family of bounded linear operators \{V(t,s) : 0 \leq s \leq t\} defines an evolution system on $L^p_\sigma(\mathbb{R}^d)$. Moreover, for any initial value $\varphi \in C^\infty_{c,\sigma}(\mathbb{R}^d)$, the abstract non-autonomous Cauchy problem
\[
\begin{cases}
  u'(t) = A(t)u(t), & 0 \leq s < t, \\
  u(s) = \varphi,
\end{cases}
\]
admits a classical solution $u$ given by $u(t) = V(t,s)\varphi$.

This shows that the Stokes problem corresponding to equation (1.5) is solved by the evolution system \{V(t,s) : 0 \leq s \leq t\} on $L^p_\sigma(\mathbb{R}^d)$. Next we prove $L^p$-$L^q$ and gradient estimates for this evolution system. Since the evolution system is not of parabolic type in the sense of Tanabe or Acquistapace, Terreni, gradient estimates do not follow from the
general theory. However, the explicit formula for $V(t, s)$ allows us to obtain the following result.

**Proposition 3.3.** Let $1 < p < \infty$ and $p \leq q \leq \infty$.

(a) For $T > 0$ there exists a constant $C > 0$ such that for $u \in L^p_0(\mathbb{R}^d)$

$$
\| V(t, s)u \|_{L^p_0(\mathbb{R}^d)} \leq C(t-s)^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}\| u \|_{L^q_0(\mathbb{R}^d)}, \quad \text{for } 0 \leq s < t \leq T.
$$

(b) Assume in addition that $M(t)$ is skew-symmetric for all $t > 0$. Then there exists a constant $C > 0$ such that for $u \in L^p_0(\mathbb{R}^d)$

$$
\| V(t, s)u \|_{L^p_0(\mathbb{R}^d)} \leq C(t-s)^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}\| u \|_{L^q_0(\mathbb{R}^d)}, \quad \text{for } 0 \leq s < t,
$$

$$
\| \nabla V(t, s)u \|_{L^p(\mathbb{R}^d)} \leq C(t-s)^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}\| u \|_{L^q(\mathbb{R}^d)}, \quad \text{for } 0 \leq s < t.
$$

**Proof.** We start by showing (3.9). Let $\| u \|_{L^p_0(\mathbb{R}^d)} < \infty$. We obtain

$$
\| V(t, s)u \|_{L^p_0(\mathbb{R}^d)} \leq \frac{\| U(s, t) \|_{L^p_0(\mathbb{R}^d)}}{(4\pi)^{\frac{d}{2}}(\det Q_{t,s})^{\frac{1}{2}}} | \det U(t, s) | \left( \int_{\mathbb{R}^d} | e^{-\frac{1}{2}(Q_{t,s}^{-1}y, y)} \right|^{\frac{1}{q}} dy \right)^{\frac{1}{2}}\| u \|_{L^q_0(\mathbb{R}^d)},
$$

where $1 < r < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Further, by the change of variable $y = Q_{t,s}^{1/2}z$ we obtain

$$
\left( \int_{\mathbb{R}^d} | e^{-\frac{1}{2}(Q_{t,s}^{-1}y, y)} \right|^{\frac{1}{q}} dy \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}^d} e^{-\frac{|z|^2}{4}}(\det Q_{t,s})^{1/2}dz \right)^{\frac{1}{2}} \leq C(\det Q_{t,s})^{\frac{1}{2}},
$$

for some constant $C > 0$. Now Lemma 2.4 (ii) yields the assertion.

To prove the gradient estimate (3.10), we first observe that

$$
\nabla V(t, s)u(x) = \frac{U(s, t)}{(4\pi)^{\frac{d}{2}}(\det Q_{t,s})^{\frac{1}{2}}} \int_{\mathbb{R}^d} u(U(t, s)x + g(t, s) - y)\nabla e^{-\frac{1}{2}(Q_{t,s}^{-1}y, y)}U(t, s)dy
$$

holds. Similarly as above we now obtain the desired estimate

$$
\| \nabla V(t, s)u \|_{L^p(\mathbb{R}^d)} \leq \frac{\| U(s, t) \|_{L^p(\mathbb{R}^d)}}{(4\pi)^{\frac{d}{2}}(\det Q_{t,s})^{\frac{1}{2}}} | \det U(t, s) | \left( \int_{\mathbb{R}^d} | \nabla e^{-\frac{1}{2}(Q_{t,s}^{-1}y, y)} \right|^{\frac{1}{q}} dy \right)^{\frac{1}{2}}\| u \|_{L^q(\mathbb{R}^d)}
$$

for some constant $C > 0$. Here we used Lemma 2.4 (i) and (ii).

In order to prove (3.11) and (3.12) we first note, that the fact that $M(t)$ is skew-symmetric for all $t > 0$ implies that the evolution operator $U(t, s)$ is orthogonal for all
t, s > 0. Thus \( ||U(t, s)|| = 1 \) and \( |\det U(t, s)| = 1 \) holds for all \( t, s > 0 \). Moreover, we have \( Q_{t,s} = (t-s)I \) for all \( 0 < s < t \) and therefore it is trivial that the estimates in Lemma 2.3 hold for all \( 0 < s < t \). The estimates \( 3.11 \) and \( 3.12 \) now follow from the calculations above.

\[ \text{Proposition 3.4.} \text{ For } 1 < p < q < \infty \text{ and } u \in L^p_{\sigma}(\mathbb{R}^d) \]

\[
(t - s)^{\frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} ||V(t, s)u||_{L^q_{\sigma}(\mathbb{R}^d)} \to 0 \text{ as } t \to s \quad \text{and} \\
(t - s)^{\frac{d}{2}} ||\nabla V(t, s)u||_{L^p(\mathbb{R}^d)} \to 0 \text{ as } t \to s. \tag{3.13}
\]

\[
\text{Proof.} \text{ Let } t - s \leq 1 \text{ and } u_n \in C^\infty_{c,\sigma}(\mathbb{R}^d) \subset L^p_{\sigma}(\mathbb{R}^d) \cap L^q_{\sigma}(\mathbb{R}^d), \ n \in \mathbb{N}, \text{ such that } u_n \to u \text{ in } L^p(\mathbb{R}^d) \text{ as } n \to \infty. \text{ The triangle inequality together with the } L^p-L^q \text{ estimates 3.9 imply that there exist constants } C_1, C_2 > 0 \text{ such that}
\]

\[
(t - s)^{\frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} ||V(t, s)u||_{L^q_{\sigma}(\mathbb{R}^d)} \leq (t - s)^{\frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} ||V(t, s)u - V(t, s)u_n||_{L^q_{\sigma}(\mathbb{R}^d)} + (t - s)^{\frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} ||V(t, s)u_n||_{L^q_{\sigma}(\mathbb{R}^d)} \leq C_1 ||u - u_n||_{L^q_{\sigma}(\mathbb{R}^d)} + C_2(t - s)^{\frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} ||u_n||_{L^q_{\sigma}(\mathbb{R}^d)} \to 0,
\]

by letting first \( t \to s \) and then \( n \to \infty \).

Similarly, by using \( 3.9 \) and \( 3.10 \) we obtain

\[
(t - s)^{\frac{d}{2}} ||\nabla V(t, s)u||_{L^p(\mathbb{R}^d)} \leq (t - s)^{\frac{d}{2}} ||\nabla V(t, s)u - \nabla V(t, s)u_n||_{L^p(\mathbb{R}^d)} + (t - s)^{\frac{d}{2}} ||\nabla V(t, s)u_n||_{L^p(\mathbb{R}^d)} \leq C_1 ||u - u_n||_{L^p_{\sigma}(\mathbb{R}^d)} + (t - s)^{\frac{d}{2}} ||\nabla V(t, s)u_n||_{L^p(\mathbb{R}^d)}. \tag{3.15}
\]

Since \( u_n \in C^\infty_{c,\sigma}(\mathbb{R}^d) \), we observe that

\[
\nabla V(t, s)u_n(x) = \frac{U(s, t)}{(4\pi)^{\frac{d}{2}} (\det Q_{t,s})^{\frac{1}{2}}} \int_{\mathbb{R}^d} \nabla u_n(U(t, s)x + g(t, s) - y) e^{-\frac{d}{2} (Q^{-1}_{t,s})^{-1} y y} U(t, s) dy
\]

holds. Thus, as in the proof of estimate \( 3.9 \) we now obtain

\[ ||\nabla V(t, s)u_n||_{L^p(\mathbb{R}^d)} \leq C_2 ||\nabla u_n||_{L^p(\mathbb{R}^d)} \]

for some constant \( C_2 \). Now the assertion follows from \( 3.15 \) by letting \( t \to s \) and \( n \to \infty \).

4. The Navier-Stokes Flow

By applying the Helmholtz-Leray projection \( P \) to \( 1.3 \) the pressure \( p \) can be eliminated and we may rewrite the equations as a non-autonomous Cauchy problem

\[
\begin{align*}
u'(t) - A(t)u(t) + P((u(t) \cdot \nabla)u(t)) &= 0, \quad \text{for } t > 0, \\
u(0) &= u_0. \tag{4.1}
\end{align*}
\]
with initial value \( u_0 \in L_{\sigma}^p(\mathbb{R}^d) \). By the Duhamel principle this problem is reduced to the integral equation
\[
  u(t) = V(t, 0)u_0 - \int_0^t V(t, s)\mathbb{P}((u(s) \cdot \nabla)u(s))ds, \quad t > 0,
\]
in \( L_{\sigma}^p(\mathbb{R}^d) \). In the following, given \( 0 < T_0 \leq \infty \), we call \( u \in C([0, T_0); L_{\sigma}^p(\mathbb{R}^d)) \) a mild solution of (4.1) if \( u \) satisfies the integral equation (4.2) on \([0, T_0)\). By adjusting Kato’s iteration scheme \((\text{Kat84, Gig86})\) to our situation we now prove the existence of a unique (local) mild solution.

**Proposition 4.1.** Let \( 2 \leq d \leq p \leq q < \infty \) such that \( d \neq q \) and \( u_0 \in L_{\sigma}^p(\mathbb{R}^d) \). Then there exists \( T_0 > 0 \) and a unique mild solution \( u \in C([0, T_0); L_{\sigma}^p(\mathbb{R}^d)) \) of (4.1), which has the properties
\[
  t^{\frac{\sigma}{p} - \frac{1}{q}} u(t) \in C([0, T_0); L_{\sigma}^q(\mathbb{R}^d)),
\]
if \( p < q \), then
\[
  t^{\frac{\sigma}{p} - \frac{1}{q}} \|u(t)\|_{L^q(\mathbb{R}^d)} + t^{\frac{p}{q}} \|\nabla u(t)\|_{L^p(\mathbb{R}^d)} \to 0 \quad \text{as } t \to 0.
\]

**Remark 4.2.** In the case \( p > d \), property (4.5) is not necessary to guarantee the uniqueness of the mild solution \( u \).

**Proof of Proposition 4.1.** Let \( q > p \geq d \) or \( q \geq p > d \) and take \( u_0 \in L_{\sigma}^p(\mathbb{R}^d) \) and \( T > 0 \). We set \( u_1(t) = V(t, 0)u_0 \) and for \( j \geq 1 \) and \( t > 0 \) we define a recursion by
\[
  u_{j+1}(t) = V(t, 0)u_0 - \int_0^t V(t, s)\mathbb{P}((u_j(s) \cdot \nabla)u_j(s))ds.
\]
Our aim is to show that for some \( 0 < T_0 \leq T \), this sequence converges in \( C([0, T_0); L_{\sigma}^p(\mathbb{R}^d)) \) to a solution \( u \) of (4.2).

We set \( \gamma = \frac{\sigma}{p} - \frac{1}{q} \) and for \( j \geq 1 \) we define constants
\[
  K_j := K_j(T_0) := \sup_{0 < t \leq T_0} t^{\gamma} \|u_j\|_{L^q(\mathbb{R}^d)},
\]
\[
  K'_j := K'_j(T_0) := \sup_{0 < t \leq T_0} t^{\frac{p}{q}} \|\nabla u_j\|_{L^p(\mathbb{R}^d)}
\]
and
\[
  L_j := L_j(T_0) := \sup_{0 < t \leq T_0} t^{\gamma} \|u_{j+1}(t) - u_j(t)\|_{L^q(\mathbb{R}^d)},
\]
\[
  L'_j := L'_j(T_0) := \sup_{0 < t \leq T_0} t^{\frac{p}{q}} \|\nabla u_{j+1}(t) - \nabla u_j(t)\|_{L^p(\mathbb{R}^d)}.
\]
Moreover, we set \( R_j := R_j(T_0) := \max\{K_j, K'_j\} \). Note that the \( L^p-L^q \) estimates (3.9) and the gradient estimates (3.10) yield \( R_1 \leq \tilde{C} \|u_0\|_{L_{\sigma}^p(\mathbb{R}^d)} \) for some constant \( \tilde{C} > 0 \).
From (4.6), the \( L^r - L^q \) estimates (3.9) and the boundedness of \( \mathbb{P} \) from \( L^r(\mathbb{R}^d) \) into \( L^q(\mathbb{R}^d) \) it follows that
\[
\|u_{j+1}(t)\|_{L^q(\mathbb{R}^d)} 
\leq \|V(t, 0)u_0\|_{L^q(\mathbb{R}^d)} + \int_0^t \|V(t, s)\mathbb{P}((u_j(s) \cdot \nabla)u_j(s))\|_{L^q(\mathbb{R}^d)} ds 
\leq t^{-\gamma}K_1 + C\int_0^t (t-s)^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{1}{2}}\|u_j(s) \cdot \nabla\|_{L^r(\mathbb{R}^d)} ds,
\] (4.7)
holds, where \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \). Similarly, with the gradient estimate (3.10) we obtain
\[
\|\nabla u_{j+1}(t)\|_{L^p(\mathbb{R}^d)} 
\leq \|\nabla V(t, 0)u_0\|_{L^p(\mathbb{R}^d)} + \int_0^t \|\nabla V(t, s)\mathbb{P}((u_j(s) \cdot \nabla)u_j(s))\|_{L^p(\mathbb{R}^d)} ds 
\leq t^{-\frac{1}{2}}K'_1 + C\int_0^t (t-s)^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{1}{2}}\|u_j(s) \cdot \nabla\|_{L^r(\mathbb{R}^d)} ds.
\] (4.8)

In order to estimate the terms on the right hand side of the inequalities (4.7) and (4.8), we apply Hölder’s inequality to conclude
\[
\|(u_j(s) \cdot \nabla)u_j(s)\|_{L^r(\mathbb{R}^d)} \leq \|u_j(s)\|_{L^q(\mathbb{R}^d)}\|\nabla u_j(s)\|_{L^p(\mathbb{R}^d)} \leq K_j'\gamma^{-\frac{1}{2}}.
\] (4.9)

This implies
\[
\|u_{j+1}(t)\|_{L^q(\mathbb{R}^d)} \leq t^{-\gamma}K_1 + CK_jK'_j\int_0^t (t-s)^{-\frac{d}{2}\gamma} s^{-\frac{1}{2}} ds,
\] (4.10)

and
\[
\|\nabla u_{j+1}(t)\|_{L^p(\mathbb{R}^d)} \leq t^{-\frac{1}{2}}K'_1 + CK_jK'_j\int_0^t (t-s)^{-\frac{d}{2}\gamma} s^{-\frac{1}{2}} ds,
\] (4.11)

respectively. By multiplying inequality (4.10) with \( t^\gamma \) and inequality (4.11) with \( t^{\frac{1}{2}} \) and then by taking \( \sup_{0 < t \leq T_0} \) we obtain
\[
K_{j+1} \leq K_1 + C_1K_jK'_j \quad \text{and} \quad K'_{j+1} \leq K'_1 + C_2K_jK'_j
\] (4.12)
for some positive constants \( C_1, C_2 \) independent of \( j \), but depending on \( T \). Here we have used the estimate
\[
\int_0^t (t-s)^{-\alpha} s^{-\beta} ds = \int_{t/2}^t (t-s)^{-\alpha} s^{-\beta} ds + \int_0^{t/2} (t-s)^{-\alpha} s^{-\beta} ds 
\leq \left( \frac{t}{2} \right)^{-\beta} \int_{t/2}^t (t-s)^{-\alpha} ds + \left( \frac{t}{2} \right)^{-\alpha} \int_0^{t/2} s^{-\beta} ds 
\leq \left( \frac{t}{2} \right)^{1-\beta-\alpha} \left( \frac{1}{1-\alpha} + \frac{1}{1-\beta} \right),
\]
for exponents \( 0 < \alpha, \beta < 1 \).
From (4.12) it now follows that \( R_{j+1} \leq R_1 + \delta R_j^2 \) holds, for some positive constant \( \delta \geq 1 \). If we assume \( R_1 \leq \frac{1}{68} \), then inductively we obtain \( R_j \leq 2R_1 \). From Proposition 3.3 it follows that for any \( \lambda > 0 \), there exists \( T_0 > 0 \) such that \( R_1 < \lambda \). Thus we obtain a bound for \( R_j \) uniformly in \( j \), provided \( T_0 \) is small enough. Using this uniform bound for \( R_j \), it follows that the sequences 

\[
(t \mapsto t^\gamma u_j(t))_{j \geq 1} \quad \text{and} \quad (t \mapsto t^{\gamma+\frac{4}{7}} \nabla u_j(t))_{j \geq 1}
\]

are uniformly bounded in \( L^q(\mathbb{R}^d) \) and \( L^q(\mathbb{R}^d)^{d \times d} \) respectively for \( t \in [0, T_0] \) and all \( j \in \mathbb{N} \). Moreover, from (3.13) and (3.14) we can conclude that the maps \( t \mapsto t^\gamma u_j(t) \) and \( t \mapsto t^{\gamma+\frac{4}{7}} \nabla u_j(t) \) are continuous at \( t = 0 \). The continuity of \( t \mapsto t^\gamma u_j(t) \) and \( t \mapsto t^{\gamma+\frac{4}{7}} \nabla u_j(t) \) for \( j \geq 1 \) now follows by similar arguments as above.

We now derive estimates for the difference \( u_{j+1} - u_j \). First we note that 

\[
(u_j \cdot \nabla)u_j - (u_{j-1} \cdot \nabla)u_{j-1} = (u_j \cdot \nabla)(u_j - u_{j-1}) + ((u_j - u_{j-1}) \cdot \nabla)u_{j-1}
\]

holds. Similarly as above we obtain

\[
\|u_{j+1}(t) - u_j(t)\|_{L^q(\mathbb{R}^d)} \leq \int_0^t \|V(t, s)\| \|u_j(s) \cdot \nabla)u_j(s) - (u_{j-1}(s) \cdot \nabla)u_{j-1}(s)\|_{L^q(\mathbb{R}^d)} ds 
\]

\[
\leq C \int_0^t (t-s)^{-\frac{3}{q}+\frac{4}{7}-(d-1)} \|u_j(s) \cdot \nabla)u_j(s) - (u_{j-1}(s) \cdot \nabla)u_{j-1}(s)\|_{L^q(\mathbb{R}^d)} ds 
\]

\[
\leq C \int_0^t (t-s)^{-\frac{3}{q}+\frac{4}{7}} \left( \|u_j(s)\|_{L^q(\mathbb{R}^d)} \|\nabla(u_j(s) - u_{j-1}(s))\|_{L^p(\mathbb{R}^d)} + \|u_j(s) - u_{j-1}(s)\|_{L^q(\mathbb{R}^d)} \|\nabla u_{j-1}(s)\|_{L^p(\mathbb{R}^d)} \right) ds,
\]

and

\[
\| \nabla u_{j+1}(t) - \nabla u_j(t)\|_{L^p(\mathbb{R}^d)} \leq \int_0^t \|\nabla V(t, s)\| \|u_j(s) \cdot \nabla)u_j(s) - (u_{j-1}(s) \cdot \nabla)u_{j-1}(s)\|_{L^p(\mathbb{R}^d)} ds 
\]

\[
\leq C \int_0^t (t-s)^{-\frac{3}{q}+\frac{4}{7}-(d-1)} \|u_j(s) \cdot \nabla)u_j(s) - (u_{j-1}(s) \cdot \nabla)u_{j-1}(s)\|_{L^p(\mathbb{R}^d)} ds 
\]

\[
\leq C \int_0^t (t-s)^{-\frac{3}{q}+\frac{4}{7}} \left( \|u_j(s)\|_{L^q(\mathbb{R}^d)} \|\nabla(u_j(s) - u_{j-1}(s))\|_{L^p(\mathbb{R}^d)} + \|u_j(s) - u_{j-1}(s)\|_{L^q(\mathbb{R}^d)} \|\nabla u_{j-1}(s)\|_{L^p(\mathbb{R}^d)} \right) ds.
\]

Thus we can conclude

\[
L_j \leq C_3(L_{j-1}'K_j + L_{j-1}K_{j-1}') \leq 2C_3R_1(L_{j-1}' + L_{j-1}) \tag{4.13}
\]

and

\[
L_{j}' \leq C_4(L_{j-1}'K_j + L_{j-1}K_{j-1}') \leq 2C_4R_1(L_{j-1}' + L_{j-1}) \tag{4.14}
\]
Moreover, by (3.9) and (4.9) we obtain

\[ u(t) \text{ to some } \] 

and the connectedness of \( p > d \) respectively. As it was previously mentioned, \( R_1 \) can be made sufficiently small if \( \| u_0 \|_{L^p(\mathbb{R}^d)} \) is small enough or if we choose \( T_0 \) sufficiently small. As a consequence \( t \mapsto t^\gamma u_j(t) \) converges to some \( t^\gamma u(t) \in C([0, T_0), L^q(\mathbb{R}^d)) \) and \( t \mapsto t^{\gamma + \frac{1}{2}} \nabla u_j(t) \) converges to some \( t^{\gamma + \frac{1}{2}} v(t) \in C([0, T_0), L^q(\mathbb{R}^d)^{d \times d}) \). It follows directly from the construction that \( v(t) = \nabla u(t) \) and that \( u \) satisfies (4.2) on \([0, T_0)\). The property (4.5) follows from the construction and Proposition 3.2. Moreover, by (3.9) and (4.9) we obtain

\[ \| u(t) \|_{L^p(\mathbb{R}^d)} \leq \| V(t, 0)u_0 \|_{L^p(\mathbb{R}^d)} + C \int_0^t (t - s)^{-\frac{d}{2}} s^{-\gamma - \frac{1}{2}} ds, \]

for some constant \( C > 0 \) and thus \( \sup_0 \leq T_0 \| u(t) \|_{L^p(\mathbb{R}^d)} < \infty \) holds. The continuity at 0 can be seen similarly, so \( u \in C([0, T_0); L^p(\mathbb{R}^d)) \).

It remains to prove the uniqueness of a mild solution \( u \) with the mentioned properties. To do this let \( u, v \) be two mild solutions of (1.1) satisfying (4.3), (4.4) and (4.5). Moreover, let \( 0 < T \leq T_0 \) and define the constant \( K \) as

\[ K := K(T) := \max \left\{ \sup_{0 \leq t \leq T} t^\gamma \| u(t) \|_{L^p(\mathbb{R}^d)}, \sup_{0 \leq t \leq T} t^{\frac{1}{2}} \| \nabla v(t) \|_{L^p(\mathbb{R}^d)} \right\}. \]

Since \( u \) and \( v \) both solve the integral equation (4.2), we obtain similarly as above

\[ \| u(t) - v(t) \|_{L^p(\mathbb{R}^d)} \leq K C \left( \int_0^t (t - s)^{-\frac{d}{2}} s^{-\gamma - \frac{1}{2}} ds \right) + \sup_{0 < \tau \leq T} \left( t^\gamma \| u(\tau) - v(\tau) \|_{L^p(\mathbb{R}^d)} + t^{\frac{1}{2}} \| \nabla (u(\tau) - v(\tau)) \|_{L^p(\mathbb{R}^d)} \right), \]

and

\[ \| \nabla u(t) - \nabla v(t) \|_{L^p(\mathbb{R}^d)} \leq K C \left( \int_0^t (t - s)^{-\frac{d}{2}} s^{-\gamma - \frac{1}{2}} ds \right) + \sup_{0 < \tau \leq T} \left( t^\gamma \| u(\tau) - v(\tau) \|_{L^p(\mathbb{R}^d)} + t^{\frac{1}{2}} \| \nabla (u(\tau) - v(\tau)) \|_{L^p(\mathbb{R}^d)} \right), \]

for \( 0 < t \leq T \). Thus, for \( 0 < t \leq T \) we have

\[ t^\gamma \| u(t) - v(t) \|_{L^p(\mathbb{R}^d)} + t^{\frac{1}{2}} \| \nabla (u(t) - v(t)) \|_{L^p(\mathbb{R}^d)} \leq 2 K C T^{1 - \frac{d}{2} - \frac{1}{2}} \sup_{0 < \tau \leq T} \left( t^\gamma \| u(\tau) - v(\tau) \|_{L^p(\mathbb{R}^d)} + t^{\frac{1}{2}} \| \nabla (u(\tau) - v(\tau)) \|_{L^p(\mathbb{R}^d)} \right). \]

(4.15) In the case \( p > d \) we can choose \( T \) small, so that \( 2 K C T^{1 - \frac{d}{2} - \frac{1}{2}} < 1 \). This implies \( u = v \) on \([0, T]\). Since \( u, v \in C((\varepsilon, T_0); L^q_\varepsilon(\mathbb{R}^d)) \) for every \( \varepsilon > 0 \), the above argument with initial data \( u(\varepsilon) = v(\varepsilon) \) yields that the set \( \{ t \in (0, T_0) : u(t) = v(t) \} \) is open. The continuity of \( u, v \) and the connectedness of \((0, T_0)\) imply that \( u = v \) on \([0, T_0]\).
Now, it remains to prove the uniqueness in the case $p = d$. Instead of (4.15) we consider
\[
t^\gamma \|u(t) - v(t)\|_{L^q(\mathbb{R}^d)} + t^{\frac{\gamma}{2}} \left\| \nabla (u(t) - v(t)) \right\|_{L^p(\mathbb{R}^d)}
\leq 2KC \sup_{0 < \tau \leq T} \left( \left\| \nabla (u(\tau) - v(\tau)) \right\|_{L^p(\mathbb{R}^d)} + \|u(\tau) - v(\tau)\|_{L^q(\mathbb{R}^d)} \right)
\]
for $0 < t \leq \bar{T}$. By (4.15) the constant $K := K(\bar{T})$ tends to zero as $\bar{T} \to 0$. Thus, we can choose $\bar{T}$ small, so that $2KC < 1$. This shows $u = v$ on $[0, \bar{T})$. Since $u, v \in C([\bar{T}/2, T_0); L^q_\sigma(\mathbb{R}^d))$ for $q > d$ with $u(\bar{T}/2) = v(\bar{T}/2)$, the uniqueness in the case $p > d$ implies $u = v$ on $[\bar{T}/2, T_0)$. The proof is hence complete.

If we assume in addition that $M(t)$ is skew-symmetric for all $t > 0$, then we can even expect to obtain a global solution, provided that $u_0 \in L^d_\sigma(\mathbb{R}^d)$ and that $\|u_0\|_{L^d(\mathbb{R}^d)}$ is sufficiently small.

**Proposition 4.3.** Let $d \geq 2$ and $u_0 \in L^d_\sigma(\mathbb{R}^d)$. Moreover assume that $M(t)$ is skew-symmetric for all $t > 0$. Then there exists $\lambda > 0$, such that if $\|u_0\|_{L^d(\mathbb{R}^d)} < \lambda$, then the mild solution $u \in C([0, T_0); L^d_\sigma(\mathbb{R}^d))$ obtained in Proposition 4.1 is global, i.e. we may take $T_0 = +\infty$.

For the proof one can use the estimates (3.9) and (3.10) and the same argumentation as above.

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