QUANTUM-ELECTRODYNAMICS WITHOUT RENORMALIZATION

V. BOSONIC CONTRIBUTIONS TO THE PHOTON MASS

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The bosonic contributions to the photon mass are shown to be of the same form as the fermionic contributions, but of opposite sign. A mass sum rule for bosons and fermions follows from gauge invariance. Assuming completeness of the standard model the top quark mass can be predicted. Effectively only the W-bosons contribute, giving a lowest order prediction of $85.1 \text{ GeV}$ for the mass of top quark, on the low side of currently accepted estimates.

1. Introduction

In the preceding parts (I–IV) of this series it was shown that the symmetrical theory of generalised functions can be applied to computations in quantum field theory. This paper can be read without consulting the preceding parts of the series. The needed results are summarized briefly.

The symmetrical theory of generalised functions never has infinity in any result, and the available simple model is strong enough to handle all special functions which occur in quantum field theory, so divergencies do not occur when this theory is applied to quantum field theory. All results are automatically finite and renormalization is never needed to remove infinities. Infinity of integrals is replaced by the less restricted concept of determinacy, which is determined by the scale transformation properties.

In part III of this series the photon mass was found to be determinate, and therefore physically relevant. It is (of course) finite and it is found to be non-zero, in disagreement with gauge invariance and therefore with all experience.

An attempt (in part III) to obtain cancellation between orders, and thereby an equation fixing the fine-structure constant $\alpha$, did not lead to useful results. In this paper the contribution of charged bosons to the photon mass is evaluated and it is shown that bosonic contributions may cancel the fermionic contributions, provided that a mass sum rule holds. If this idea is realized in Nature it is possible to predict the mass of the top quark. The result is $m_t = 85.1 \pm 0.3 \text{ GeV}$ to leading order.

2. Fermions

In the previous parts of this series the convention $g_{\mu \nu} = (-,+,+,+)$ was used, in agreement with most of the original QED literature. In this paper the computations are transcribed to the nowadays more popular convention $g_{\mu \nu} = (+,-,-,-)$. 

The fermionic contribution to the photon mass $m_\gamma$ is found from

\[ F \text{eynman diagram to be added} \]  

with photon momentum $k = 0$. Substitution of the Feynman rules for Dirac fermions from Ap. A (including the fermion loop minus sign) gives

\[ i m_\gamma^2 g_{\mu\nu} = \Pi_{\mu\nu}(0) = -4\pi\alpha q^2 \text{Tr} \int \frac{d^4p}{(2\pi)^4} \gamma_\mu \frac{\phi + m}{(p^2 - m^2)} \gamma_\nu \frac{\phi + m}{(p^2 - m^2)}, \]  

in which $\alpha$ is the fine-structure constant, and $q^2$ is the squared relative charge of the fermion. It is convenient to contract with $g_{\mu\nu}$ to obtain

\[ 4i m_\gamma^2 = \Pi^{\mu}(0) = 2\frac{\alpha q^2}{\pi^3} m^2 \int d^4p \frac{p^2 - 2m^2}{m^2(p^2 - m^2)^2} = 2\alpha q^2 m^2 I_0/\pi^3, \]  

where $I_0$ is a convenient abbreviation for the remaining dimensionless integral.

\[ I_0 := \int d^4p \frac{p^2 - 2m^2}{m^2(p^2 - m^2)^2} = -i \pi^2, \]  

which was shown to be finite, non-zero, and determinate in part III, despite its quadratic divergence. Correcting for the new normalization the integral equals $+i\pi^2$ by (I-E.5,11), so the squared photon mass correction produced by fermions of mass $m$ is

\[ m_\gamma^2 = \frac{\alpha q^2}{2\pi^3} m^2, \]  

as derived (III.7) before. This result was first obtained and printed (as far as I am aware) by Keller\(^4\). It seems likely that this result has been found before without being published, since (as commonly understood) it is in complete disagreement with all experience.

3. Scalar bosons

The Lagrangian and the Feynman rules are summarized in appendix A. Instead of the simple diagram (1) for fermions there are now two contributions to the photon self-energy, the analogue of the fermion loop, and the bubble diagram

Feynman diagram to be added

Putting $k = 0$ and contracting with $g^{\mu\nu}$ gives the integral

\[ \Pi_{\mu}(0) = \frac{\alpha q^2}{\pi^3} \int d^4p \frac{p^2}{(p^2 - m^2)^2}. \]
for the loop diagram and

$$\Pi_\mu^\mu(0) = -2 \frac{\alpha q^2}{\pi^3} \int d^4p \frac{1}{(p^2 - m^2)},$$  \hfill (8)

for the bubble diagram. Combining these gives the total contribution

$$\Pi_\mu^\mu(0) = - \frac{\alpha q^2}{\pi^3} m^2 \int d^4p \frac{p^2 - 2m^2}{m^2(p^2 - m^2)^2} = - \alpha q^2 m^2 I_0/\pi^3. \hfill (9)$$

Comparing this result with the fermion contribution (3) it is seen that the contribution of charged scalar bosons to the photon mass equals minus half the contribution from a spin \( \frac{1}{2} \) fermion of the same mass.

The minus sign is due to the fact that fermion loops carry an additional minus sign, which boson loops do not have. The scalar boson contribution is determinate and finite, as it is for fermions, since it is proportional to the same integral.

4. **Vector bosons**

Fundamental charged scalar bosons are not known, but charged intermediate vector bosons are seen as carriers of the weak interaction. For vector bosons there are again two Feynman diagrams contributing to the photon self energy

Feynman diagram to be added \hfill (10)

obtained from the vertices (A12,13). It is not clear in advance that the vector boson contribution will be proportional to the same integral, since the first diagram contains terms with (in power counting terminology) logarithmic, quadratic, quartic, and even sextic degree of divergence, while the bubble diagram has quartic, quadratic and logarithmic terms. However, in the first diagram the sextic terms involving \( p^4 p_\mu p_\nu \) cancel, leaving upon contraction with \( g_{\mu\nu} \) the quartic and quadratic terms

$$\Pi_\mu^\mu(0) = - 3 \frac{\alpha q^2}{\pi^3} \int d^4p \frac{p^4 - 3p^2 m^2}{2m^2(p^2 - m^2)^2},$$  \hfill (11)

with the W-subscript on the mass omitted for clarity. The seagull vertex (A13) gives the bubble diagram, which yields upon substitution

$$\Pi_\mu^\mu(0) = 3 \frac{\alpha q^2}{\pi^3} \int d^4p \frac{p^2 - 4m^2}{2m^2(p^2 - m^2)}. \hfill (12)$$

The quartic terms cancel in the sum of the two diagrams, so the total vector boson contribution to the photon mass is

$$\Pi_\mu^\mu(0) = - 3 \frac{\alpha q^2}{\pi^3} m^2 \int d^4p \frac{p^2 - 2m^2}{m^2(p^2 - m^2)^2} = - 3\alpha q^2 m^2 I_0/\pi^3, \hfill (13)$$
which is again proportional to the same determinate quadratic integral (3) found in the fermionic case. It may be noted that the freedom from subtractions and the full linearity of the symmetrical theory of generalised functions makes it unnecessary to invent special tricks to avoid the canceling terms in (11) and (12).

5. The mass sum rule

Since the boson and fermion contributions to the photon mass have opposite sign they will cancel if the mass sum rule

\[
\sum_{\text{fermions}} g_f q_f^2 c_j m_f^2 = \sum_{\text{bosons}} g_b q_b^2 c_j m_b^2 ,
\]

(14)
is satisfied. The sums are over all fundamental fermions and bosons. The factors \( g_f \) and \( g_b \) are the multiplicity of the particles, and the \( q^2 \)'s are the charges measured in units of the squared electron charge. The factors \( c_j \) are the relative coefficients of the integrals obtained from the vacuum polarization diagrams, which were calculated (3,13) in the previous sections.

If the leptons, quarks and the W-bosons of the standard model are assumed to be the only fundamental charged particles, all masses in (14) are known with the exception of the top quark mass. The top mass can therefore be calculated from the sum rule.

To presently available accuracy all other quarks and leptons are effectively (squared) massless, leaving only the top- and the W-contribution. Taking the \( W^{\pm} \) bosons to be a unit charge doublet we must nevertheless take \( g_W=1 \). The required factor 2 has already been accounted for in the Feynman rules (A12,13). Assuming threefold colour degeneracy and charge 2/3 for the top quark substitution of the integral coefficients \( c_{1/2}=2, \ c_1=3 \) yields

\[
m_t^2 = \frac{9}{8} m_W^2 ,
\]

(15)
or

\[
m_t = \frac{3}{4} \sqrt{2} m_W .
\]

(16)
Substitution the value\(^5\) \( m_W = 80.22 \pm 0.26 \) GeV for the W-mass yields a top mass of

\[
m_t = 85.1 \pm 0.3 \text{ GeV} \quad + \quad \cdots .
\]

(17)
The given error reflects the experimental W-mass uncertainty only. In addition there will be higher order electro-weak and strong corrections which have not yet been estimated, which will probably be larger than the experimental W-mass uncertainty.

The result is on the low side of currently accepted estimates\(^5\) of the top mass, but these estimates are based on difficult experiments, which in addition depend on difficult theoretical computations for their interpretation, so a judgement on the correctness of the prediction (17) must await experiments which can observe the top quark unambiguously by obtaining a measurement of its mass. This is expected to be possible in the near future.
6. Discussion

The photon mass has been a source of difficulties in QED from the very beginning. It is obvious upon computation that the photon mass is not zero automatically, as required by charge conservation and gauge invariance, so it needs special treatment.

Special renormalization methods have been devised to get rid of it, as discussed in part IV of this series of papers. The best known methods, in order of appearance, are brute force subtraction, Pauli-Villars regularization, or dimensional regularization. These methods all lack a natural mathematical and physical interpretation.

For the argument presented in this paper the method of regularization is irrelevant, since all contributions to the photon mass are proportional to the same integral (4). For any non-zero interpretation of this integral the mass sum rule (14) leads to a vanishing photon mass, and conversely the sum rule makes a forced evaluation to zero unnecessary.

The determinacy of the integral (4) in the generalised function framework makes it impossible to put it equal to zero, so the validity of the sum rule (14) is the only way left to obtain gauge invariance. A proof of determinacy to all orders and of the sufficiency of (14) in higher orders remains to be given, but the result can be made plausible.

Sum rules such as (14) cannot be falsified until we possess a complete theory of everything. Conversely experimental verification of the sum rule would imply that the standard model is probably complete. Additional bosons and fermions would again have to satisfy the sum rule by themselves, which seems unlikely.

If the top quark is not found at the predicted mass, heavier charged bosons are necessary to balance the sum rule. Charged Higgs bosons are a possibility which can be accommodated in the standard model. If this happens to be realized it will take much longer until the relevant experiments can be performed.

The derivation given in this paper is clearly inadequate, since the validity of a fundamental result like the mass sum rule (14) should be clear beforehand, instead of appearing afterwards in perturbation theory. This will have to await the development of a more fundamental theory from which the mass spectrum of the quarks, leptons, and vector bosons can be derived. Nevertheless, the electromagnetic coupling of the charges should be given correctly even by the present incomplete theory, so the mass sum rule (14) may be expected to hold also in future, more complete theories. (Application of minimal coupling to the generic Proca Lagrangian yields the same electromagnetic interaction).

It is interesting to see that meaningful cancellation of bosonic and fermionic contributions to undesirable results may occur without invoking supersymmetry. There is a difference; with supersymmetry one hopes for cancellation of undesirable infinities, in the symmetrical theory of generalised functions applied to quantum field theory there are no infinities. Instead one has cancellation of unobserved physical predictions.

It remains to be seen if the value of the top quark mass of 85.1 GeV, predicted from gauge invariance and the assumed completeness of the standard model, will be confirmed by experiment.

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Appendix A  Feynman rules

For ease of reference the Feynman rules for charged fermions and bosons with the modern choice $g_{\mu\nu} = (1, -1, -1, -1)$ for the metric tensor.

For charged scalar bosons with Lagrangian

$$L = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi,$$  \hspace{1cm} (A1)

the assumption of minimal coupling leads to the electromagnetic interaction Lagrangian

$$L_{\text{int}} = ieA^\mu (\partial_\mu \varphi^\dagger \varphi - \varphi^\dagger \partial_\mu \varphi) + e^2 A^2 \varphi^\dagger \varphi.$$  \hspace{1cm} (A2)

The standard derivation of the Feynman rules gives the wave equation and propagator

$$(\partial_\mu \partial^\mu + m^2) \varphi = 0, \quad \text{and} \quad \frac{i}{(p^2 - m^2)},$$  \hspace{1cm} (A3)

the three wave vertex factor

$$3\text{-Vertex} = i(p_1 - p_2) \frac{1}{(2\pi)^4} \delta^4(k + p_1 + p_2),$$  \hspace{1cm} (A4)

and the four wave (or seagull) vertex factor

$$4\text{-Vertex} = 2ie^2 g_{\mu\nu} \frac{1}{(2\pi)^4} \delta^4(k_1 + k_2 + p_1 + p_2),$$  \hspace{1cm} (A5)

with all momenta taken as incoming.

For charged Dirac fermions with free Lagrangian and minimal electromagnetic coupling

$$L = i\bar{\psi} \gamma_\mu \partial_\mu \psi - m \bar{\psi} \psi - e\bar{\psi} \gamma_\mu \gamma_5 A_\mu,$$  \hspace{1cm} (A6)

the standard derivation of the Feynman rules\textsuperscript{6} gives the Dirac equation and propagator

$$i\gamma_\mu \partial_\mu \psi - m \psi = 0, \quad \text{and} \quad i\frac{\not{p} + m}{(p^2 - m^2)},$$  \hspace{1cm} (A7)

and the electromagnetic fermion vertex factor

$$3\text{-Vertex} = i e \gamma_\mu \frac{1}{(2\pi)^4} \delta^4(k + p_1 + p_2).$$  \hspace{1cm} (A8)
There is no four wave coupling for fermions.
For intermediate vector bosons the relevant part of the Lagrangian\(^7\) is
\[
\mathcal{L} = \frac{1}{2} \text{Tr}(W_{\lambda\rho} W^{\lambda\rho}) + m_W^2 W^+ W^-.
\] (A9)

The field strength tensor is given by
\[
W_\lambda = W^a_\lambda \tau_a/2, \quad W^a_{\lambda\rho} = \partial_\lambda W^a_\rho - \partial_\rho W^a_\lambda - g \epsilon_{abc} W^b_\lambda W^c_\rho,
\] (A10)

with \(g\) the weak coupling constant, \(\tau_a\) the Pauli matrices, and \(\epsilon_{abc}\) the completely antisymmetric SU(2) structure constants. Working out the product and computing the Feynman rules (in the unitary gauge) gives the propagator
\[
3\text{-Vertex} = \frac{-i}{m_W^2} \frac{m_W^2 g_{\mu\nu} - p_\mu p_\nu}{p^2 - m_W^2},
\] (A11)

and with \(W^\pm_\lambda = \left(W^1_\lambda \mp iW^2_\lambda \right)/\sqrt{2}, W^3_\lambda = A_\lambda \sin \theta_W, g \sin \theta_W = e\), the three boson vertex coupling the \(W^\pm\) to the photon becomes
\[
3\text{-Vertex} = i e \left( (k - p_1)_\sigma g_{\mu\nu} + (p_1 - p_2)_\mu g_{\rho\sigma} + (p_2 - k)_\rho g_{\mu\sigma} \right)
\] (A12)

with the momentum conserving \(\delta\)-function understood. The quadratic term in \(g\) gives the four boson vertex coupling the \(W^\pm\) to two photons
\[
4\text{-Vertex} = -i e^2 \left( 2 g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right),
\] (A13)

which are analogous to the corresponding terms in the scalar case. The boson vertex factors depend on the momenta at the vertex, giving rise to stronger divergence.

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If the top quark is heavier than 85.1 GeV it must be much heavier, since light charged bosons are known not to exist. Assuming charged scalar bosons the missing mass is

$$m_H^2 = \frac{8}{3} m_t^2 - 3 m_W^2,$$

which yields $m_H = 205$ GeV for 170 GeV top quarks. Charged Higgs bosons may be a possibility.