A Beam Network Model Approach to Strength Optimization of Disordered Fibrous Materials

Seyyed Ahmad Hosseini,* Paolo Moretti, and Michael Zaiser

1. Introduction

Network models play an important role for investigating the mechanical properties, fracture, and failure of heterogeneous and disordered materials. Beam network models (BNM) serve as a paradigmatic modeling approach which despite its simple structure captures key features of fracture as a multi-scale process: the existence of local failure thresholds which reflect properties of the material microstructure, the existence of an internal length above which the material can be described as a continuum, and the long-range coupling of different material elements by long-range stress fields that emerge in response to local failure. By abstracting from the bewildering detail of real material microstructures, such models achieve a degree of simplicity that makes them amenable to large-scale simulations and systematic studies over a wide range of system sizes, using ensembles of a size that allows for meaningful statistical predictions. At the same time, BNM, as opposed to even more simplified models such as random fuse models, or random spring models, explicitly preserve fundamental features of continuum mechanics such as the tensorial nature of stress and strain, and the conservation of linear and angular momentum. This allows to “tune” them to reproduce, in principle, macroscopic elastic properties of any type of material. In turn, metamaterials with designed elastic properties often assume the physical form of beam networks. For instance, additively manufactured auxetic metamaterials may often be considered fairly direct physical realizations of beam network structures. Similarly, beam networks may be additively manufactured to act as process-specific catalyst supports or scaffolds for tissue engineering.

In design and optimization of such structures, it is often assumed that materials properties of the matrix are completely known. In contrast, it is well known that the quality of additively manufactured metal parts is subject to huge variations and this feature poses serious problems for application of manufactured parts in terms of reliability and predictability. The problem is particularly relevant when considering mechanical failure, as failure loads may be strongly influenced by disorder. Moreover, this influence is not trivial: While structural disorder at first glance is likely to promote crack nucleation and failure, it has also been reported that artificially introducing “flaws” in the form of geometric defects into periodic metamaterials may actually enhance damage tolerance. For materials that fail by the formation of localized shear bands, it has been demonstrated that enhanced disorder may delay failure, and for cellular materials, disorder has been shown to homogenize the deformation response and prevent formation of localized crushing bands.

To ensure strength and reliability of materials that can be modeled as beam networks, design rules and paradigms need to be adapted to account for the statistical variability of materials properties. In the present investigation, we show this for an extremely simple design problem, namely the optimal configurations of a uniaxially strained bundle of load-carrying (LC) brittle fibers with variable amount of cross links. In the absence of structural disorder, this problem has a trivial solution which, however, turns out to be the worst possible solution if the system is large or the material is strongly disordered.

2. The Method: BNM

The basic structure of our BNM is a 2D square lattice that consists of interconnected beams that are clamped together at their intersections as shown in Figure 1. The points where beams are mutually connected are referred to as nodes; a BNM of size $L$ has $L(L + 1)$ nodes. At the top and bottom boundaries of the BNM, all degrees of freedom (DOFs) are fixed through two rigid bars which are used to apply an axial displacement along one of the two cubic axes of the lattice structure. Periodic boundary conditions.
conditions are imposed in the load perpendicular direction. Beams oriented along the loading axis are denoted as LC beams, their number is \(N_{CL}\); beams oriented in perpendicular direction are denoted as cross-link (CL) beams, their number is \(N_{CL}\). For a fully connected network of size \(L\), \(N_{CL} = L(L - 1)\) and \(N_{CL} = L^2\).

### 2.1. Governing Equations

The constituents of the BNM are assumed to be straight, identical beams of unit length, unit modulus of elasticity and square cross section, which are capable of resisting axial and shear forces as well as bending moments. There are three DOF at each node including two translational DOF (node displacements \(u\) and \(v\) along global axes \(x\) and \(y\)) and one rotational DOF (rotation angle \(\theta\) about global \(z\) axis).

The beams are assumed to deform in a linear-elastic manner and break once a stress-based criterion is met, which we will discuss later. Stresses and strains are evaluated using Timoshenko beam theory. This leads to a matrix equation which relates forces and displacements in the beam’s local 2D coordinate system (\(xy\) plane), which can be written as \(K_i u_i = F_i\), where the stiffness matrix \(K_i\) generalized displacement vector \(u_i\) and generalized force vector \(F_i\) are given by

\[
\begin{bmatrix}
\frac{EA}{I} & 12EI_{x} & \frac{6EI_{y}}{I} & \frac{EI_{z}}{I} \\
0 & \frac{6EI_{x}}{I} & \frac{6EI_{y}}{I} & \frac{EI_{z}}{I} \\
0 & \frac{-6EI_{x}}{I} & \frac{6EI_{y}}{I} & \frac{6EI_{z}}{I} \\
0 & \frac{-6EI_{x}}{I} & \frac{6EI_{y}}{I} & \frac{6EI_{z}}{I}
\end{bmatrix} \begin{bmatrix}
\frac{EI_{x}}{I} & \frac{6EI_{y}}{I} & \frac{EI_{z}}{I} \\
0 & \frac{6EI_{x}}{I} & \frac{EI_{z}}{I} \\
0 & \frac{-6EI_{x}}{I} & \frac{6EI_{y}}{I} \\
0 & \frac{-6EI_{x}}{I} & \frac{6EI_{y}}{I}
\end{bmatrix}
\begin{bmatrix}
u_i \\
\theta_j
\end{bmatrix}
= \begin{bmatrix}
F_i \\
Q_i \\
M_i \\
F_j \\
Q_j \\
M_j
\end{bmatrix}
\]

(1)

Subscripts \(i\) and \(j\) refer to the two end nodes of the beam. In the force vector \(F_i\), the letters \(F\), \(Q\), and \(M\) denote axial and shear forces and bending moment, respectively. In the stiffness matrix \(K_i\), \(E\) is the modulus of elasticity, \(A\) is the beam cross-section area, \(l\) is the beam length, \(I\) is the moment of inertia along the \(z\) axis, \(\Phi_\gamma = 12EI_{z}/(GAL^2)\) is the shear correction factor, and \(G\) is the shear modulus. It is worth mentioning that if \(\Phi_\gamma = 0\), shear strain is being neglected and Equation (1) reduces to Euler–Bernoulli beam theory. However, such a simplified description is not a suitable choice for a BNM because it presumes that the beam is so slender that each cross-section remains perpendicular to the neutral axis during deformation which is not in accordance with our assumptions. Based on our preliminary simulations of BNMs of size \(L = 1024\), using Euler–Bernoulli instead of Timoshenko beam elements will result in significant underestimation of failure stress and fracture energy. To obtain the global equilibrium equation \(Ku = F\) of the BNM, it is required to transform \(K, u, F\) from local to global coordinate system, and to assemble the moments and forces at each node to obtain the relevant component of the global equilibrium equation.

### 2.2. Failure Criteria

The beams are assumed to behave in an elastic-brittle manner. Beam failure occurs depending on a stress-based criterion. We use the Maximum Distortion Energy Theory of Failure (von Mises) criterion. For a 2D geometry with plane stress loading conditions, this criterion can be expressed as

\[
\sigma = \sqrt{\sigma_1^2 + 3\sigma_{12}^2} = \frac{\sigma}{t}
\]

(2)

where \(\sigma_1\) and \(\sigma_{12}\) are axial and shear stress of the beam, respectively, and \(t\) is the equivalent stress at failure (beam failure threshold). Axial stress consists of force and bending components

\[
\sigma_1 = \frac{F}{A} + \frac{M_y}{I}
\]

(3)

Since the loads on each beam are only applied through the end nodes, axial and shear forces are constant. Due to the same reason, the bending moment varies linearly along the beam and therefore is maximum at one of the end nodes. Given that the bending moment is highest in the outer beam layer (\(y_{max}\)) and the material is assumed to be weaker in tension, axial and shear stresses follow the relation

\[
\max(\sigma_i) = \frac{|F_i|}{A} + \max(|M_i|, |M_j|) \frac{y_{max}}{I}
\]

(4)

\[
\sigma_{12} = \frac{|Q|}{A}
\]

(5)

By substituting Equation (4) and (5) into Equation (2), the failure criterion is given by

\[
\frac{\sigma}{T} = 1, \sigma = \sqrt{\frac{F_i}{A} + \max(|M_i|, |M_j|) \frac{y_{max}}{I} + 3 \frac{|Q|}{A}^2}
\]

(6)

A simpler criterion often used in the literature\(^{13,14}\) is obtained by neglecting the shear force \(Q\) in this expression, however, we see no good physical reason for doing so.

Mimicking material heterogeneity, failure thresholds \(t\) of beams are randomly assigned based on a Weibull probability distribution function with probability density

\[
p(t) = \frac{t}{\beta} \left(\frac{t}{\eta}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right]
\]

(7)

With \(p(t) \geq 0\), \(t \geq 0\), \(\beta > 0\), \(\eta > 0\), where \(\beta\) and \(\eta\) are the shape and scale parameters of the distribution, respectively. With such distribution, the mean value of thresholds becomes
\[ I = \eta \Gamma \left( \frac{1}{\beta} + 1 \right) \]  

(8)

where \( \Gamma \) is the gamma function. From Equation (8), the scale parameter can be written as

\[ \eta = \frac{I}{\Gamma \left( \frac{1}{\beta} + 1 \right)} \]  

(9)

The variance of values, in contrast, is a function of the shape and scale parameters

\[ \text{var}(t) = \eta^2 \left[ \Gamma \left( \frac{2}{\beta} + 1 \right) - \Gamma^2 \left( \frac{1}{\beta} + 1 \right) \right] \]  

(10)

Combining Equation (9) and (10) gives

\[ \text{var}(t) = \left[ \frac{I}{\Gamma \left( \frac{1}{\beta} + 1 \right)} \right]^2 \left[ \Gamma \left( \frac{2}{\beta} + 1 \right) - \Gamma^2 \left( \frac{1}{\beta} + 1 \right) \right] \]  

(11)

Equation (11) shows that by assuming a fixed mean (average) for thresholds, their variance will be a function of the shape parameter \( \beta \) only. In this article, we choose a fixed mean value of 0.1 for thresholds. This ensures that strains are of typical order less than 0.1, warranting a small-strain approximation to the elastic problem. The ensuing geometrical linearity ensures, in turn, that the behavior of materials with smaller mean threshold can be recovered by simple multiplication with the threshold ratio.

We consider systems of various sizes and vary the shape parameter \( \beta \) in the range \( 1.5 \leq \beta \leq 20 \), covering the entire spectrum from highly disordered to highly reliable materials. Fixing lower nodes of the network, a unit vertical displacement is imposed on the top-end nodes, whereas periodic boundary conditions are imposed in the horizontal direction. Displacements of all BNM nodes are calculated using the global equilibrium equation \( \mathbf{K} \cdot \mathbf{u} = \mathbf{F} \) and then the local loads of each beam are calculated using Equation (1). The weakest element of the structure is identified as the beam with the highest \( \sigma/t \) ratio \( (\sigma/t)_{\text{max}} \). Using the geometrical linearity of the problem, the imposed displacement is then multiplied with \( 1/((\sigma/t)_{\text{max}}) \), such that the failure criterion Equation (6) is satisfied for the weakest beam which is then removed. The process is repeated iteratively, such that one beam is removed in every iteration, until loss of global connectivity indicates failure of the beam network. Typical damage and failure patterns for a fully cross-linked network are shown in Figure 2 and 3.

3. The Optimization Problem

When an axial load is applied to a fully connected network, it is evident that initially the CL beams do not carry any load. This raises the question to which extent the CL beams can be dispensed with. The basic question we are asking is thus: Assuming that CL beams are associated with a cost function, what is the optimal degree of cross linking for a system of given size \( L \) and given Weibull exponent \( \beta \) of the strength distribution of the elementary beams (i.e., degree of material disorder). We consider this problem in two variants: 1) The cost of a CL beam is associated with its weight. As CL and LC beams are assumed
identical, the optimization problem is then tantamount to achieve, for fixed $\beta$ and $L$, the maximum strength-to-weight ratio (failure load divided by total number of beams) or, equivalently, the highest failure load for a given number of beams. 2) The cost of a CL beam is related to deterioration in strength of the connecting LC beams. A physical example of this type of cost consists of carbon nanotubes (CNTs) connected by chemical cross links, as such cross links are defects in the ideal CNT atomic arrangement (“functionalization defects”), which have been shown to reduce CNT strength in molecular dynamics simulations.$^{[15,16]}$ In this case, the optimization problem consists of achieving the maximum absolute strength for fixed $\beta$ and $L$.

To begin with, we observe that both variants of the optimization problem have the same trivial solution in the limit $\beta \to \infty$ when all beams have identical failure thresholds. In this limit, the solution of the optimization problem is simply to have no cross links, such that the network degenerates into a bundle of equally loaded unconnected fibers. In variant (1), such cross links would add weight without any benefit in failure strength, as all beams fail at the same moment and therefore, the lateral cross links never transmit any load. In variant (2), the cross links simply reduce the strength of the individual beams and therefore the strength of the bundle, again without any benefit.

However, this simplistic answer is incorrect when the beam failure thresholds are statistically distributed. In the limit of large $L$ and low $\beta$ (strong disorder), a bundle of unconnected fibers of length $L$ may, in fact, represent the worst possible choice. In such a bundle, each fiber fails when its weakest beam fails, and from that moment on, all beams in the fiber cease to carry any load. As a consequence, the bundle suffers from a statistical size effect which makes the overall decrease in strength with increasing fiber length as $L^{-1/\nu}$. In the limit of low $\beta$ and large $L$, it is therefore advisable to cross link the fibers to allow for lateral load redistribution. In this case, failure of the weakest beam will not render the entire fiber useless since, through lateral load transfer via the cross links, the remaining intact segments of a fiber will still contribute to carrying the global load imposed on the structure.

We first study which degree of cross linking optimizes the overall strength-to-weight ratio of a cross-linked bundle. To this end, we start with a fully networked structure, which we progressively dilute by random removal of CL beams, thus reducing the CL ratio $n_{CL} = N_{CL}/(L(L-1))$. Results are shown in Figure 4. We first consider the situations where the load-carrying fibers are fully intact ($N_C = L^2$), blue data points in Figure 4. In this case, for high disorder ($\beta = 1.5$), fully cross-linked structures provide an optimal strength-to-load ratio, whereas for low disorder ($\beta = 20$), the optimal structures are bundles of disconnected fibers. For intermediate disorder ($\beta = 4$), the behavior depends on system size: For small systems ($L = 32$), the optimal structure consists of disconnected fibers, whereas for large systems, a finite CL ratio emerges as the optimal structure, though the maximum is rather flat (Figure 4h). From a theoretical point of view, this size dependence results from the different size dependence of strength in fiber bundles and random network structures. In a bundle of disconnected fibers, the bundle strength is proportional to, but always less than, the average strength of the individual fibers.$^{[15]}$ This strength in turn decreases with fiber length according to a statistical size effect: For a fiber consisting of $L$ elements with Weibull-distributed failure thresholds, the average fiber strength (and therefore the bundle strength) decreases as $L^{-1/\nu}$. For a fully cross-linked network, in contrast, the average strength decreases in inverse proportion with the logarithm of system size.$^{[17]}$ Thus, for very large system sizes, the strength of a disconnected bundle (which decreases in inverse proportion with a power of $L$) will always be less than that of a connected network (which decreases in inverse proportion with the logarithm of $L$). We note, however, that for large $\beta$ (low disorder) the cross-over may only occur at extremely large system sizes that are unlikely to be relevant in real-world systems.

Damage of the load-carrying fibers can be introduced by randomly removing a small fraction $f$ of LC beams (orange and green data points in Figure 4). For beams with large strength fluctuations, such damage has only a minor impact on strength (first column of Figure 4). For less-disordered materials, however, the knock-down effect on strength is significant, notably if the systems are large and the cross-link ratio is low (second and third columns in Figure 4). In the limit of large systems with low disorder (Figure 4i), the result can be a reversal of the optimum solution: Whereas in the absence of damage, unconnected fibers represent the optimum strength-to-weight ratio, even small amounts of damage reverse the situation and instead a fully cross-linked network emerges as the optimal structure.

Next, we consider a situation where the introduction of cross links weakens the load-carrying beams as the “welds” connecting the beams weaken the beam structure (an example are chemically cross-linked CNTs where the chemical cross links represent imperfections in the otherwise regular sp² bonding network$^{[16]}$).

In our simple model, we introduce such weakening by taking one of the four LC beams that connect to a newly introduced CL beam, and multiplying their strength by a factor $f_c \leq 1$. This introduces a trade-off between strength reduction of the LC beams, and a potential gain in strength due to cross linking. The resulting overall strength is, for different values of $f_c$ and $\beta$, shown in Figure 5 versus the respective CL ratio.

In the limit of low disorder (Figure 5, right), it is evident from the figure that the introduction of cross links is unfavorable as even low cross-link ratios reduce the overall strength by a factor close to $f_c$. Remarkably, a small knock-down effect on strength is manifested even for $f_c = 1$, indicating that a cross-linked bundle of fibers is here weaker than its unconnected counterpart. This can be understood from the different failure modes of unconnected fiber bundles and of connected beam networks: For a connected network, lateral load redistribution leads to localized damage clusters, which form as weaker-than-average beams fail in a correlated manner and that extend in lateral direction. In studies of random fuse networks, such damage clusters were shown to control system strength in a manner very similar to small cracks that become critical at the failure stress.$^{[17,18]}$ Thus, we observe that in a material with low disorder, cross linking may reduce strength even if it does not introduce damage, because cross links here facilitate the formation and lateral propagation of a critical crack which is impossible as long as the system consists of unconnected fibers. The effect is, of course, exacerbated if in addition the cross links are associated with damage (reduction in strength) of the load-carrying fibers.

In the high disorder limit, in contrast, introduction of cross links leads to a significant strength enhancement even if the
associated damage to the LC beams is significant (Figure 5, left). In this limit, fully networked structures made of intact beams \((f_t = 1)\) are strongest with a strength that exceeds the strength of a bundle of isolated fibers by more than one order of magnitude. Structures where cross linking introduces damage \((f_t < 1)\) are weaker by a factor close to \(f_t\), but still much stronger than the corresponding fiber bundles. For intermediate disorder \((\beta = 4)\), finally, we find an intermediate behavior where the benefit of cross linking depends on the amount of damage introduced (Figure 5, center): With \(f_t = 1\) and \(f_t = 0.8\), the fully cross-linked lattice structure is strongest, whereas for \(f_t = 0.6\), strength is almost independent on cross-linking degree, and for \(f_t < 0.6\), the strongest structure is represented by an unconnected fiber bundle.

4. Discussion and Conclusions

Our investigation shows that structural disorder of materials as reflected in a significant scatter of failure thresholds can have serious consequences for design considerations. Using an apparently trivial example, namely a bundle of fibers carrying an axial load, we demonstrated that the optimal design for a perfect material may perform worst when built of components (beams) made from a strongly disordered and thus unreliable material. Disorder necessitates structural redundancy in the form of creation of alternative load transmission paths through cross linking, even if such cross links appear superfluous from elasticity calculations, but incur a cost in terms of added weight or even in terms of reduced strength of the load-carrying fibers.
Structurally redundant cross links can partly offset statistical size effects in large systems built of unreliable materials. Even if the material disorder is low, such cross links enhance flaw or damage tolerance. The effect does not depend much on the way how, in detail, the cross links are arranged. This is shown in Figure 6a, which compares stress–strain curves obtained for a CL ratio of $\eta_{\text{CL}} = 0.5$ with a regular pattern of system-spanning beams that connect every second row of nodes, and with a random arrangement of the cross links. The regular network is stronger on average, but the effect is less than 10% of the overall strength. Even more sophisticated arrangements of cross links yield similar behavior as shown in Figure 6b, which compares, for a CL ratio of $\eta_{\text{CL}} = 2/3$, a random, a regular hierarchical, and a randomized hierarchical arrangement of cross links (see ref. [19], for method of construction of the hierarchical structures). The random arrangement has slightly higher failure stress but slightly lower work of failure (area under the stress–strain curve).

The introduction of cross links not only introduces structural redundancy but also modifies the mode of failure. In a system of unconnected fibers, failure occurs by sequential failure of individual fibers. The resulting weakest-link effect may, for long fibers consisting of unreliable elements, lead to a low value of absolute strength. If this is mitigated by cross-linking, then a new failure mode emerges, as a local flaw of sufficient size (a critical microcrack) may propagate in a supercritical manner, leading to system failure. Since the size of the largest microcrack, arising from an accidental failure of adjacent weak sites, depends logarithmically on system size, this gives rise to a logarithmic size dependency of strength. The net outcome is that, for large disorder and/or large system sizes, the system strength of the cross-linked structure is higher than that of the unconnected fiber bundle. An interesting question is whether one can combine both effects: Is it possible to reduce statistical size effects by some degree of cross-linking while mitigating against flaw propagation by keeping the cross-link ratio low, thus impeding lateral stress transmission? Our results indicate that, for randomly cross-linked structures, such intermediate solutions are in most cases not favorable, as either full cross-linking or zero

Figure 5. Effect of cross-linking on mechanical strength of axially loaded fiber bundles, when cross linking introduces damage into the load-carrying beams. Strength versus CL ratio, number of realizations $N = 20$ for each data point. After introducing a cross link, 1 of the 4 LC beams connecting to that link is weakened by reducing its strength by a factor $f$. Original beam strengths are Weibull-distributed with shape parameters $\beta = 1.5$, $\beta = 4.0$, and $\beta = 20.0$; system size is $L = 256$.

Figure 6. a) Stress–strain curves of networks with random (RBN) and with regular (ABN) cross links, $\eta_{\text{CL}} = 0.5$; b) Stress–strain curves for networks with $\eta_{\text{CL}} = 0.66$ and CLs distributed randomly (RBN), or according to regular (D-HBN) or randomized (S-HBN) hierarchical patterns.
CLs produce an optimum result. It remains to be seen whether more sophisticated cross linking schemes, such as hierarchical cross linking,\cite{19} are capable of mitigating simultaneously against flaw propagation and statistical size effects.

In conclusion, our exercise has demonstrated that, for materials whose failure strength may be subjected to a large statistical scatter, even very simple design questions may have answers that depend critically on parameters characterizing the statistical scatter of material properties. Thus, the question, “Is it beneficial to introduce statically redundant cross links into a network of parallel LC fibers?” has opposite answers when we are dealing with a very reliable material (low statistical scatter: cross links are superfluous or even detrimental to performance) and with a very unreliable material (cross links significantly improve structural performance). Of course, the optimization problem considered here is ridiculously simple. Nevertheless, it shows that answers to even the most simple design problem that are obtained under the assumption that material properties are perfectly known, may not be robust when material heterogeneity and disorder are considered. In addition, manufactured materials, where the manufacturing methods allow for an almost unprecedented degree of geometrical freedom, may often be comparatively unreliable due to flaws and imperfections introduced by the same manufacturing methods. This calls for new approaches in optimization of such structures to meet simultaneous requirements of functionality and structural resilience.

Acknowledgements

The authors acknowledge support by DFG under grant No 1Za 171/9-1 and under 377472739/GRK 2423/1-2019 FRASCAL. Support by the European Commission under H2020-MSCA-RISE project No. 734485 FRAMED is also gratefully acknowledged.

Conflict of Interest

The authors declare no conflict of interest.

Keywords

beams, cross-links, fibers, networks, optimization

Received: August 22, 2019
Revised: September 22, 2019
Published online: November 4, 2019

\[1\] M. J. Alava, P. K. V. V. Nukala, S. Zapperi, Adv. Phys. 2006, 55, 349.
\[2\] C. Manzato, A. Shekhawat, P. K. V. V. Nukala, M. J. Alava, J. P. Sethna, S. Zapperi, Phys. Rev. Lett. 2012, 108, 065504.
\[3\] F. Warmuth, F. Osmalic, L. Adler, M. A. Lodes, C. Körner, Smart Mater. Struct. 2017, 26, 025013.
\[4\] M. Hanifpour, C. F. Petersen, M. J. Alava, S. Zapperi, Eur. Phys. J. B 2018, 91, 27.
\[5\] T. Knorr, P. Heinl, J. Schwerdtfeger, C. Körner, R. F. Singer, B. J. Etzold, Chem. Eng. J. 2012, 181, 725.
\[6\] P. Heinl, L. Müller, C. Körner, R. F. Singer, F. A. Müller, Acta Biomater. 2008, 4, 1536.
\[7\] J. Schwerdtfeger, F. Wein, G. Leugering, R. F. Singer, C. Körner, M. Stingl, F. Schury, Adv. Mater. 2011, 23, 2650.
\[8\] Z. Hu, S. Mahadevan, Int. J. Adv. Manufact. Technol. 2017, 93, 2855.
\[9\] M. S. Pham, C. Liu, I. Todd, J. Lerthhanasarn, Nature 2019, 565, 305.
\[10\] D. Tüzes, P. D. Ispanovity, M. Zaiser, Int. J. Fract. 2017, 205, 139.
\[11\] M. Zaiser, F. Mill, A. Konstantinidis, K. E. Aifantis, Mater. Sci. Eng. A 2013, 567, 38.
\[12\] J. S. Przemieniecki, Theory of Matrix Structural Analysis, Dover, New York 1985, p. 80.
\[13\] H. J. Herrmann, A. Hansen, S. Roux, Phys. Rev. B 1989, 39, 637.
\[14\] P. K. V. V. Nukala, S. Zapperi, M. J. Alava, S. Simunovic, Phys. Rev. E 2008, 78, 046105.
\[15\] M. Yang, V. Koutsos, M. Zaiser, Nanotechnology 2007, 18, 155708.
\[16\] R. M. Moghadam, S. A. Hosseini, M. Salehi, Phys. E 2014, 62, 80.
\[17\] S. Lennartz-Sassinek, M. Zaiser, I. Main, C. Manzato, S. Zapperi, Phys. Rev. E 2018, 87, 12090.
\[18\] M. Zaiser, S. Lennartz-Sassinek, P. Moretti, J. Stat. Mech. 2015, P08029.
\[19\] P. Moretti, B. Dietemann, N. Esfandiary, M. Zaiser, Sci. Rep. 2018, 8, 12090.