THE DIRICHLET PROBLEM FOR MIXED HESSIAN EQUATIONS ON HERMITIAN MANIFOLDS

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Abstract. In this paper we study the Dirichlet problem for a class of Hessian type equation with its structure as a combination of elementary symmetric functions on Hermitian manifolds. Under some conditions with the initial data on manifolds and admissible subsolutions, we derive a priori estimates for this complex mixed Hessian equation and solvability of the corresponding Dirichlet problem.

1. Introduction

Let \((M, \omega)\) be a closed Hermitian manifold of complex dimension \(n \geq 2\) with smooth boundary \(\partial M\) and \(M = M \cup \partial M\), fix a real smooth closed \((1, 1)\)-form \(\chi_0\) on \(M\). For any \(C^2\) function \(u : M \to \mathbb{R}\), we can obtain a new real \((1, 1)\)-form

\[
\chi_u = \chi_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u.
\]

We consider the following Dirichlet problem of Hessian type equation on \((M, \omega)\)

\[
\begin{aligned}
\chi_u^k \wedge \omega^{n-k} &= \sum_{l=0}^{k-1} \alpha_l(z) \chi_u^l \wedge \omega^{n-l}, & \text{in } M, \\
 u &= \varphi & \text{on } \partial M,
\end{aligned}
\]

where \(1 < k \leq n\), \(\alpha_l(z)\) and \(\varphi\) are real smooth functions on \(M\) and \(\partial M\), respectively. Note that this is a class of fully nonlinear equation with its structure as a combination of elementary symmetric functions on Hermitian manifolds. In order to keep the ellipticity of equation (1.1), we require the eigenvalues of \((1, 1)\)-form \(\chi_u\) with respect to \(\omega\) belong to the Gårding’s cone \(\Gamma_{k-1}\). Hence we introduce the following definition.

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Definition 1.1. A function \( u \in C^2(M) \) is called \( k \)-admissible if \( \chi_u \in \Gamma_k(M) \) for any \( z \in M \), where \( \Gamma_k(M) \) is the Gårding cone

\[
\Gamma_k(M) = \{ \chi \in A^{1,1}(M) : \sigma_i(\lambda[\chi]) > 0, \forall 1 \leq i \leq k \}.
\]

The equation in (1.1)

\[
\chi_k^u \wedge \omega^{n-k} = \sum_{l=0}^{k-1} \alpha_l(z) \chi_l^u \wedge \omega^{n-l}
\]

includes some of the most partial differential equations in complex geometry and analysis. When \( k = n \) and \( \alpha_1 = \cdots = \alpha_{n-1} = 0 \), the equation becomes the complex Monge-Ampère equation \( \chi_n^u = \alpha_0 \omega^n \), which was solved by Yau [41] on closed Kähler manifolds in the resolution of the Calabi conjecture. Then Tosatti-Weinkove [40, 39] have solved the analogous problem for the equation on closed Hermitian manifolds. The corresponding Dirichlet problems on manifolds were studied by Cherrier-Hanani [9] and Guan-Li [16, 17, 18].

In fact, the Hessian equation \( \chi_k^u \wedge \omega^{n-k} = \alpha_0 \omega^n \) and the Hessian quotient equation \( \chi_k^u \wedge \omega^{n-k} = \alpha_l(z) \chi_l^u \wedge \omega^{n-l} \) are also the special case of (1.2). For equation \( \chi_k^u \wedge \omega^{n-k} = \alpha_0 \omega^n \), Hou-Ma-Wu [22] established the second order estimates for the equation without boundary on Kähler manifolds, and then Dinew-Kolodziej [12] solved the equation by combining the Liouville theorem and Hou-Ma-Wu’s results. Zhang [42] and Székelyhidi [30] have solved the equation without boundary on Hermitian manifolds. The corresponding Dirichlet problem on manifolds has also attracted the interest of many researchers, such as Gu-Nguyen [15] were able to obtain continuous solutions to the equation with boundary on Hermitian manifolds, and Collins-Picard [11] solved the problem under the existence of a subsolution. For \((k, l)\)-Hessian quotient equation \( \chi_k^u \wedge \omega^{n-k} = \alpha_l(z) \chi_l^u \wedge \omega^{n-l} \), the \((n, n-1)\)-Hessian quotient equation have appeared in a problem proposed by Donaldson in the setting of moment maps and was solved by Song-Weinkove [32]. Then \((n, l)\)-Hessian quotient equation was considered by Fang-Lai-Ma [13] on Kähler manifold, and by Guan-Li [17], Guan-Sun [20] on Hermitian manifolds. The general \((k, l)\)-Hessian quotient equation with \( k < n \) without boundary on Hermitian manifolds was studied by Székelyhidi [30] for constants \( \alpha_l \) and by Sun [34] for functions \( \alpha_l \). The corresponding Dirichlet problem on Hermitian manifolds was studied by Feng-Ge-Zheng [14], since they can only obtain the gradient estimates in some special cases, the existence of solution can be solved in these kinds of special cases.

When \( k = n \) and \( \alpha_l \in \mathbb{R} \), equation (1.2) was raised as a conjecture by Chen [8] in the study of Mabuchi energy. The conjecture was solved by Collins-Székelyhidi [10] for some special constant \( \alpha_l \). Later, Phong-Tô [29] generalized Collins-Székelyhidi’s result for nonnegative constants \( \alpha_l \). Moreover, a generalized equation of Chen’s problem was studied by Sun [35, 36] and Pingali [26, 27, 28].
The initial motivation of our work is the following: As an important example for the applications of the general notion of fully nonlinear elliptic equations, Krylov [24] studied the Dirichlet problem of the equation

$$
\sigma_k(D^2u) = \sum_{l=0}^{k-1} \alpha_l(x)\sigma_l(D^2u)
$$

in a \((k-1)\)-convex domain in \(\mathbb{R}^n\) with \(\alpha_l > 0 (0 \leq l \leq k-1)\). Recently, Guan-Zhang [21] observed that equation (1.3) can be rewritten as the following equation

$$
\frac{\sigma_k(D^2u)}{\sigma_{k-1}(D^2u)} - \sum_{l=0}^{k-2} \alpha_l(x)\frac{\sigma_l(D^2u)}{\sigma_{k-1}(D^2u)} = -\alpha_{k-1}.
$$

Actually, the equation is elliptic and concave in \(\Gamma_{k-1}\). Then they obtained an a priori \(C^2\) estimate of the \((k-1)\)-admissible solution of equation (1.4) without sign requirement for \(\alpha_{k-1}\) and solved the Dirichlet problem for the corresponding equation. Later the corresponding Neumann problem and prescribed curvature equations were also discussed in \([4, 5, 45, 7]\). Recently, Zhang [43] considered the Dirichlet problem for (1.3) on complex domains in \(\mathbb{C}^n\), which can be seen as the extension of Guan-Zhang’s result. Chen [9] and Zhou [44] provide a sufficient and necessary condition for the solvability of the mixed hessian equation on Kähler manifold without boundary. A natural problem is raised whether we can consider the Dirichlet problem for equation (1.3) on complex manifolds. The following is our main result.

**Theorem 1.1.** Let \((M, \omega)\) be a closed Hermitian manifold of complex dimension \(n \geq 2\), \(\chi_0 \in \Gamma_{k-1}(M)\) be a \((1,1)\)-form, \(\alpha_{k-1}(z), \varphi(z), \alpha_l\) be smooth functions and \(\alpha_l(z) > 0\) for \(l = 0, 1, \cdots, k-2\). Suppose there exists an \((k-1)\)-admissible subsolution \(u \in C^2(M)\) such that

$$
\begin{cases}
\chi_k^\omega \land \omega^{n-k} \geq \sum_{l=0}^{k-1} \alpha_l(z)\chi_l^\omega \land \omega^{n-l}, & \text{in } M, \\
u = \varphi & \text{on } \partial M.
\end{cases}
$$

Then there exists a unique \((k-1)\)-admissible solution \(u \in C^\infty(M)\) of Dirichlet problem (1.1). Moreover, we have

$$
\|u\|_{C^2(M)} \leq C,
$$

where the constant \(C\) depends on \(M, \omega, \chi_0, \varphi, \|u\|_{C^2}, \inf_M \alpha_l\) with \(0 \leq l \leq k-2\) and \(\|\alpha_l\|_{C^2}\) with \(0 \leq l \leq k-1\).

Note that, equation (1.1) can be seen a class of fully nonlinear elliptic equations and there is no sign requirement of the coefficient function \(\alpha_{k-1}\). Using some notations on Hermitian manifolds, equation (1.1) can be rewritten as a combination of elementary symmetric functions on Hermitian manifolds, which is similar to equation (1.4). Therefore, we will construct barrier functions to obtain a prior estimates for equation (1.1) and solve the
Dirichlet problem. However, compared with the classical method, we apply a blow-up argument and Liouville-type theorem due to Dinew-Kolodziej [12] to obtain the gradient estimate. For this purpose, we follow a technique of Hou-Ma-Wu [22] (also see [11]) to derive a second derivative bound of the form

\[ \sup_M |\bar{\partial} u| \leq C(1 + \sup_M |\nabla u|^2). \]

The organization of the paper is as follows. In Section 2 we start with some preliminaries. In Section 3 we prove $C^0$ estimates and second order derivative interior estimates. Boundary second order derivative estimates are given in Section 4 and 5. In Section 6, we use a blow-up argument and Liouville-type theorem to obtain gradient estimates and Theorem 1.1 is proved by the standard continuity method.

2. Preliminaries

In this section, we give some basic properties of elementary symmetric functions, which could be found in [25, 33], and establish some key lemmas. Throughout this paper, repeated indices will be summed unless otherwise stated.

2.1. Basic properties of elementary symmetric functions. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, the $k$-th elementary symmetric function is defined by

\[ \sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}. \]

We also set $\sigma_0 = 1$ and denote by $\sigma_k(\lambda | i)$ the $k$-th symmetric function with $\lambda_i = 0$. The generalized Newton-MacLaurin inequality and some well-known result (See [1]) are as follows, which will be used later.

**Proposition 2.1.** For $\lambda \in \Gamma_k := \{ \lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0, \forall 1 \leq i \leq k \}$ and $n \geq k > l \geq 0, r > s \geq 0, k \geq r, l \geq s$, we have

\[ \left[ \frac{\sigma_k(\lambda)/C_n^k}{\sigma_l(\lambda)/C_n^l} \right]^{\frac{k-l}{s-t}} \leq \left[ \frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s} \right]^{\frac{r-s}{s-t}}. \]  

(2.1)

**Proposition 2.2.** Let $A = (a_{ij})$ be a Hermitian matrix, $\lambda(A) = (\lambda_1, \cdots, \lambda_n)$ be the eigenvalues of $A$ and $F = F(A) = f(\lambda(A))$ be a symmetric function of $\lambda(A)$. Then for any Hermitian matrix $B = (b_{ij})$, we have

\[ \frac{\partial^2 F}{\partial a_{ij} \partial a_{st}} b_{ij} b_{st} = \frac{\partial^2 f}{\partial \lambda_p \partial \lambda_q} b_{pq} b_{pp} b_{qq} + 2 \sum_{p < q} \frac{\partial f}{\partial \lambda_p} b_{pq} \left( \frac{\partial f}{\partial \lambda_q} - \frac{\partial f}{\partial \lambda_p} \right) b_{pq}^2. \]

(2.2)

In addition, if $f$ is concave and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then we have

\[ f_1 \geq f_2 \geq \cdots \geq f_n, \]

where $f_i = \frac{\partial f}{\partial \lambda_i}$.
Proposition 2.3. Let $W = W_{ij}$ be an $n \times n$ symmetric matrix and $\lambda(W) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the eigenvalues of the symmetric matrix $W$. Suppose that $W = W_{ij}$ is diagonal and $\lambda_i = W_{ii}$, then we have

\begin{equation}
\frac{\partial \lambda_i}{\partial W_{ii}} = 1, \quad \frac{\partial \lambda_k}{\partial W_{ij}} = 0 \quad \text{otherwise,}
\end{equation}

\begin{equation}
\frac{\partial^2 \lambda_i}{\partial W_{ij} \partial W_{ji}} = \frac{1}{\lambda_i - \lambda_j} \quad \text{for} \ i \neq j \ \text{and} \ \lambda_i \neq \lambda_j.
\end{equation}

\begin{equation}
\frac{\partial^2 \lambda_i}{\partial W_{kl} \partial W_{pq}} = 0 \quad \text{otherwise.}
\end{equation}

Lemma 2.1. Let $\lambda(z) \in C^0(M, \mathbb{R}^n) \cap \Gamma_{k-1}$ satisfy

\begin{equation}
f(\lambda(z), z) := \frac{\sigma_k(\lambda(z))}{\sigma_{k-1}(\lambda(z))} - \sum_{l=0}^{k-1} \beta_l(z) \frac{\sigma_l(\lambda(z))}{\sigma_{k-1}(\lambda(z))} = \beta(z),
\end{equation}

where $\beta_l, \beta$ are smooth functions with $\beta_l \geq 0$, then

\begin{equation}
\sum_{i=1}^n f_i(\lambda) \mu_i \geq f(\mu) + (k - l) \sum_{l=0}^{k-2} \beta_l \frac{\sigma_l(\lambda)}{\sigma_{k-1}(\lambda)}.
\end{equation}

Proof. Since $f$ is concave in $\Gamma_{k-1}$, we have

\[ f(\mu) \leq \sum_{i=1}^n f_i(\lambda)(\mu_i - \lambda_i) + f(\lambda), \]

which implies

\[ \sum_{i=1}^n f_i(\lambda) \mu_i \geq f(\mu) + \sum_{l=0}^{k-2} (k - l) \beta_l \frac{\sigma_l(\lambda)}{\sigma_{k-1}(\lambda)}. \]

So, the proof is completed. \hfill \Box

Lemma 2.2. Let $\lambda(z) \in C^0(M, \mathbb{R}^n) \cap \Gamma_{k-1}$ satisfy (2.4) and $\beta_l, \beta$ be smooth functions with $\beta_l \geq 0$. Assume $\mu \in C^0(M, \mathbb{R}^n) \cap \Gamma_{k-1}$ satisfy

\begin{equation}
\frac{\sigma_{k-1}(\mu^i)}{\sigma_{k-2}(\mu^i)} - \sum_{l=1}^{k-2} \beta_l(z) \frac{\sigma_{l-1}(\mu^i)}{\sigma_{k-2}(\mu^i)} > \beta(z) \quad \forall z \in M.
\end{equation}

Then there exist constants $N, \theta > 0$ depending on $\|\mu\|_{C^0(M)}, \|\beta\|_{C^0(M)}$ and $\|\beta_l\|_{C^0(M)}$ such that if

\[ \lambda_{\max}(z) := \max_{1 \leq i \leq n} \{\lambda_i(z)\} \geq N \]

we have at $z$

\begin{equation}
\sum_i f_i(\lambda)(\mu_i - \lambda_i) \geq \theta + \theta \sum_i f_i(\lambda)
\end{equation}
or
\[(2.8)\]
\[f_{\text{max}} \lambda_{\text{max}} \geq \theta, \]
where \(f_{\text{max}} = \frac{\partial f}{\partial \lambda_{\text{max}}}.\)

**Proof.** The proof can be seen in [6, Lemma 2.7].

Let \(\mathcal{A}^{1,1}(M)\) be the space of real smooth \((1,1)\)-forms on the Hermitian manifold \((M, \omega)\). For any \(\chi \in \mathcal{A}^{1,1}(M)\), we write in a local coordinate chart \((z^1, \ldots, z^n)\)
\[\omega = \frac{\sqrt{-1}}{2} g_{ij} dz^i \wedge d\bar{z}^j\]
and
\[\chi = \frac{\sqrt{-1}}{2} \chi_{ij} dz^i \wedge d\bar{z}^j.\]

In particular, in a local normal coordinate system \(g_{ij} = \delta_{ij}\), the matrix \((\chi_{ij})\) is a Hermitian matrix. We denote \(\lambda(\chi_{ij})\) by the eigenvalues of the matrix \((\chi_{ij})\). We define \(\sigma_k(\chi)\) with respect to \(\omega\) as
\[\sigma_k(\chi) = \sigma_k(\lambda(\chi_{ij})),\]
and the Gårding’s cone on \(M\) is defined by
\[\Gamma_k(M) = \{ \chi \in \mathcal{A}^{1,1}(M) : \sigma_i(\chi) > 0, \ \forall 1 \leq i \leq k \}.\]

In fact, the definition of \(\sigma_k(\chi)\) is independent of the choice of local normal coordinate system, and it can be defined without the use of local normal coordinate by
\[\sigma_k(\chi) = C^k_n \frac{\chi^k \wedge \omega^{n-k}}{\omega^n},\]
where \(C^k_n = \frac{n!}{(n-k)!k!}\).

Using the above notation, we can rewrite equation (1.1) as the following local form:
\[(2.9)\]
\[
\begin{cases}
\sigma_k(\chi_u) - \sum_{l=0}^{k-2} \beta_l(z) \frac{\sigma_l(\chi_u)}{\sigma_{k-1}(\chi_u)} = \beta(z), & \text{in } M, \\
u = \varphi & \text{on } \partial M,
\end{cases}
\]
where \(\beta_l(z) = \frac{C^k_l}{C^k_n} \alpha_l(z)\) for \(0 \leq l \leq k - 2\) and \(\beta(z) = \frac{C^k_k}{C^k_n} \alpha_{k-1}(z)\).

According to an important observation by Guan-Zhang in [21], we know that

**Proposition 2.4.** If \(u \in C^2(M)\) with \(\chi_u \in \Gamma_{k-1}(M)\) and \(\beta_l(z) \geq 0\) for \(0 \leq l \leq k - 2\), then the operator
\[(2.10)\]
\[G(\chi_u) = \frac{\sigma_k(\chi_u)}{\sigma_{k-1}(\chi_u)} - \sum_{l=0}^{k-2} \beta_l(z) \frac{\sigma_l(\chi_u)}{\sigma_{k-1}(\chi_u)}\]
is elliptic and concave.
For the convenience of notations, we will denote
\[ G(\chi) := \frac{\sigma_k(\chi)}{\sigma_{k-1}(\chi)} - \sum_{l=0}^{k-2} \beta_l(z) \frac{\sigma_l(\chi)}{\sigma_{k-1}(\chi)}, \]
\[ G_k(\chi) := \frac{\sigma_k(\chi)}{\sigma_{k-1}(\chi)}, \quad G_l(\chi) := -\frac{\sigma_l(\chi)}{\sigma_{k-1}(\chi)}, \]
and
\[ G_{ij} := \frac{\partial G}{\partial \chi_{ij}}, \quad G_{ij,rs} := \frac{\partial^2 G}{\partial \chi_{ij} \partial \chi_{rs}}, \quad G_{ij} := \frac{\partial G_l}{\partial \chi_{ij}}, \quad G_{ij,rs} := \frac{\partial^2 G_l}{\partial \chi_{ij} \partial \chi_{rs}} \]
for any \( 1 \leq i,j,r,s \leq n \) and \( 0 \leq l \leq k - 2 \).

**Lemma 2.3.** If \( u \in C^2(M) \) with \( \chi_u \in \Gamma_{k-1}(M) \) satisfy
\[ G(\chi_u) = \beta, \quad \text{in } M, \]
where \( \beta_l > 0 \) for \( 0 \leq l \leq k - 2 \). Then
\[ 0 < \frac{\sigma_l}{\sigma_{k-1}} (\lambda(\chi_u)) \leq C(n,k,\inf_M \beta_l, \sup_M |\beta|), \quad 0 \leq l \leq k - 2; \]
\[ -\sup_M |\beta| < \frac{\sigma_k}{\sigma_{k-1}} (\lambda(\chi_u)) \leq C(n,k,\sum_{l=0}^{k-1} \sup_M \beta_l); \]
\[ \frac{n-k+1}{k} \leq \sum_i G_{ii} \leq n - k - 1 + \frac{(n-k+2)\sigma_{k-2}}{\sigma_{k-1}} (\lambda(\chi_u)) \beta; \]
\[ \sum_i G_{ii} \lambda_i(\chi_u) = \beta + \sum_{l=0}^{k-2} (k-l) \beta_l \frac{\sigma_l}{\sigma_{k-1}} (\lambda(\chi_u)). \]

**Proof.** The proofs of Lemma are quite similar to that given for [6, Lemma 2.8, 2.9] and so they are omitted. \( \square \)

3. **C^0 estimates and Second order interior estimates**

3.1. **C^0 estimates.**

**Theorem 3.1.** Let \( u \in C^\infty(\tilde{M}) \) be an \((k-1)\)-admissible solution for equation (1.1). Under the assumptions mentioned in Theorem 1.1, then there exists a positive constant \( C \) depending only on \((M,\omega),\chi_0, \alpha_l, \varphi\) and the subsolution \( u \) such that
\[ \sup_M |u| \leq C. \]

**Proof.** On the one hand, we know that the subsolution \( u \in C^2(\tilde{M}) \) satisfies
\[ G(\chi_u) - G(\chi_u) \geq 0 \]
\[ \text{in } M \text{ and } u - u = 0 \text{ on } \partial M. \]
By the ellipticity of \( G \) and the maximum principle,
\[ u(z) \leq u(z), \quad \forall z \in \tilde{M}. \]
On the other hand, let $v$ be a function satisfying
\[ \omega^k (\chi_0)_{\bar{k}i} + \partial_i \partial_{\bar{k}} v = 0 \quad \text{in } M; \quad v = \varphi \quad \text{on } \partial M. \]

Note that $\chi_u \in \Gamma_k(M) \subset \Gamma_1(M)$. By the comparison principle,
\[ u(z) \leq v(z), \quad \forall z \in \bar{M}. \]

According to the proof of Theorem 3.1, we know that $u - v$ and $u - \bar{u}$ attain their maximums on the boundary. Combining with the Hopf lemma, we obtain
\[ \sup_{z \in \partial M} |\nabla u| \leq C, \]
where $C$ depends on $(M, \omega), \chi_0, \alpha_l$ and $u$.

3.2. Notations and some lemmas. In local complex coordinates $(z^1, ..., z^n)$, the subscripts of a function $u$ always denote the covariant derivatives of $u$ with respect to $\omega$ in the directions of the local frame $\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}$. Namely,
\[ u_i = \nabla \frac{\partial}{\partial z_i} u, \quad u_{\bar{j}} = \nabla \frac{\partial}{\partial z_j} \nabla \frac{\partial}{\partial z_{\bar{j}}} u, \quad u_{\bar{j}k} = \nabla \frac{\partial}{\partial z_k} \nabla \frac{\partial}{\partial z_{\bar{j}}} \nabla \frac{\partial}{\partial z_{\bar{j}}^*} u. \]
But, the covariant derivatives of a $(1,1)$-form $\chi$ with respect to $\omega$ will be denoted by indices with semicolons, e.g.,
\[ \chi_{\bar{j};k} = \frac{\partial}{\partial z_k} \chi \left( \frac{\partial}{\partial z_{\bar{j}}}, \frac{\partial}{\partial z_{\bar{\bar{j}}}} \right), \quad \chi_{\bar{j}k;\bar{l}} = \frac{\partial}{\partial z_k} \frac{\partial}{\partial z_{\bar{l}}} \chi \left( \frac{\partial}{\partial z_{\bar{j}}}, \frac{\partial}{\partial z_{\bar{\bar{j}}}} \right). \]

We recall the following commutation formula on Hermitian manifolds $(M, \omega)$ [22, 16, 17].

**Lemma 3.1.** For $u \in C^4(M)$, we have
\[ u_{\bar{j}k} - u_{k\bar{j}} = T_{ik} u_{\bar{j}} , \]
\[ u_{\bar{i}k} - u_{k\bar{i}} = T_{ij} u_{\bar{j}} , \]
\[ u_{\bar{j}k} = u_{ikj} - R_{iklj} \rho^l u_i , \]
\[ u_{\bar{j}k;\bar{l}} - u_{k\bar{i}\bar{j}} = g^{\bar{m}} (R_{ik\bar{l}\bar{m}u_{\bar{j}l}} - R_{ij\bar{l}\bar{m}u_{\bar{j}l}}) + T_{ik} u_{\bar{j}l} + T_{ij} u_{\bar{i}l} - T_{ik} \overline{T_{ij}} u_{\bar{i}l}, \]
where $R$ is the curvature tensor of $(M, \omega)$.

**Theorem 3.2.** Let $u \in C^\infty(\bar{M})$ be an $(k-1)$-admissible solution for equation (1.1). Under the assumptions mentioned in Theorem 1.1, then there exists a positive constant $C$ depending only on $(M, \omega), \chi_0, \alpha_l, \varphi$ and the subsolution $\underline{u}$ such that
\[ \sup_M |\sqrt{-1} \partial \bar{\partial} u| \leq C \left( K + \sup_{\partial M} |\sqrt{-1} \partial \bar{\partial} u| \right), \]
where $K := 1 + \sup_M |\nabla u|^2$, and $C$ depending on $(M, \omega), \chi_0, \alpha_l, \varphi$ and the subsolution $\underline{u}$.
3.3. The proof of Theorem 3.2

In order to study the second order interior estimate of (2.9) (or (1.1)), it is sufficient to consider the following equation

\[
\begin{align*}
\sigma_k(\tilde{\chi} u) - \sum_{l=0}^{k-2} \beta_l(z) \sigma_l(\tilde{\chi} u) &= \beta(z), \quad \text{in } \bar{M}, \\
u &= 0 \quad \text{on } \partial M,
\end{align*}
\]

where \( \tilde{\chi}_u := \tilde{\chi}_0 + u \) and \( \tilde{\chi}_0 = \chi_0 + \sqrt{-1} \frac{i}{2} \partial \bar{\partial} u \). It is clear that 0 is an admissible subsolution to (3.2). Without causing confusion, we denote \( \tilde{\chi}_u \) by \( \chi \) and \( \tilde{\chi}_0 \) by \( \chi_0 \).

Following the work of Hou-Ma-Wu [22], we define a function \( W \) on \( M \) as

\[
W(z) = \log \lambda_1(z) + \varphi(|\nabla u|^2) + \psi(u),
\]

where \( \lambda_1 : M \to \mathbb{R} \) is the largest eigenvalue of the matrix \( A = \omega^{ij} \chi_{ij} \) at each point with \( \omega \). Since \( \lambda_1 \) is not a smooth function, we will perturb \( \tilde{A} \) slightly as in [31, 30]. There are also other methods to deal with this issue: one is to use a viscosity type argument as in [37], another is to replace \( \lambda_1 \) by a carefully chosen quadratic function of \( \chi_{ij} \) as in [38].

The function \( W \) must achieve its maximum at the interior point \( z_0 \in M \). Around \( z_0 \), we choose a normal chart such that \( \tilde{A} \) is diagonal with eigenvalues

\[
\lambda_1 \geq \ldots \geq \lambda_n.
\]

We perturb \( \tilde{A} \) by a diagonal matrix \( B \) with \( B_{11} = 0 \), small \( 0 < B_{22} < \ldots < B_{nn} \) and \( B_{nn} < 2B_{22} \). Thus, the new matrix \( \tilde{A} = A - B \) has eigenvalues at \( z_0 \)

\[
\tilde{\lambda}_1 = \lambda_1, \quad \tilde{\lambda}_i = \lambda_1 - B_{ii} \quad \text{for } \quad i > 1.
\]

Then, we can calculate the first and second derivatives of \( \tilde{\lambda}_1 \) at \( z_0 \) from Proposition 2.3

\[
\tilde{\lambda}_{1;i} = \tilde{\lambda}_1^{pq}(A_{pq})_i = \chi_{1i} - (B_{11})_i,
\]

\[
\tilde{\lambda}_{1;ii} = \tilde{\lambda}_1^{pq,rs}(A_{pq})_i(A_{rs})_i + \chi_1^{pq}(A_{pq}_{ii})_i
\]

\[
= \chi_{1i}^2 + \sum_{p>1} \frac{|\chi_{1p;i}|^2 + |\chi_{1p;i}|^2}{\lambda_1 - \lambda_p} + (B_{11})_{ii} - 2\text{Re} \sum_{p>1} \frac{\chi_{1p;i}(B_{1p})_i + \chi_{1p;i}(B_{p1})_i}{\lambda_1 - \lambda_p} + \tilde{\lambda}_1^{pq,rs}(B_{pq})_i (B_{rs})_i.
\]
Since $\sum_i \lambda_i > 0$, we can choose $B$ sufficient small such that $\sum_i \bar{\lambda}_i > 0$. Thus, $|\lambda_i| < (n-1)\lambda_1$, which gives
\[
\frac{1}{\lambda_1 - \lambda_i} > \frac{1}{n\lambda_1}.
\]
We can absorb the term $\chi_{p\vec{\tau}_i}(B_{1\vec{p}}\vec{\tau})$ using
\[
|\chi_{p\vec{\tau}_i}(B_{1\vec{p}}\vec{\tau})| \leq \frac{1}{4}|\chi_{p\vec{\tau}_i}|^2 + C.
\]
Moreover, for $p > 1$ there is
\[
\frac{1}{\lambda_1 - \lambda_p} < \frac{1}{B_{22}}.
\]
It follows that
\[
(3.3) \quad \tilde{\lambda}_{1;\vec{i}} \geq \chi_{1\vec{\tau},\vec{\tau}} + \frac{1}{2n\lambda_1} \sum_{p>1} (|u_{p\vec{\tau}_i}|^2 + |u_{1\vec{p}_i}|^2) - C,
\]
where $C$ depends on $|B|_{C^2(M)}$.

From Lemma 3.1, we have
\[
u_{1\vec{\tau}} = \nu_{1\vec{\tau}} - 2\text{Re}(u_{\vec{p}_1 T_{11}}) + \partial \bar{\partial} u * R + \bar{\partial} u * T * T.
\]
where $*$ denotes a contraction. Then, we get
\[
(3.4) \quad \tilde{\lambda}_{1;\vec{i}} \geq \nu_{1\vec{\tau}} + \frac{1}{3n\lambda_1} \sum_{p>1} (|u_{p\vec{\tau}_i}|^2 + |u_{1\vec{p}_i}|^2)
- 2\text{Re}(u_{\vec{p}_1 T_{11}}) - C\lambda_1 - C.
\]
Using $u_{j\vec{p}} = u_{1\vec{p}} + T_{11} u_{p\vec{\tau}}$ and Cauchy-Schwartz inequality, we have
\[
(3.5) \quad 2 \sum_{p>1} |\text{Re}(u_{j\vec{p} T_{11}})| \leq \frac{1}{3n\lambda_1} \sum_{p>1} (|u_{p\vec{\tau}_i}|^2 + |u_{1\vec{p}_i}|^2) + C\lambda_1.
\]
Plugging (3.5) into (3.4), we obtain
\[
(3.6) \quad \tilde{\lambda}_{1;\vec{i}} \geq \nu_{1\vec{\tau}} - 2\text{Re}(u_{1\vec{p} T_{11}}) - C\lambda_1
\]
\[
\geq \chi_{1\vec{\tau},\vec{\tau}} - 2\text{Re}(\chi_{1\vec{\tau},\vec{\tau}} T_{11}) - C\lambda_1,
\]
where we use $\nu_{1\vec{\tau}} = \nu_{1\vec{\tau}} + T_{11} u_{p\vec{\tau}}$ and choose $\lambda_1$ large enough ($\lambda_1 >> 1$) to absorb a constant into $C\lambda_1$.

Differentiating the equation (3.2) twice, we have
\[
(3.7) \quad \nabla_p \beta = G^{ij} \chi_{ij;\vec{p}} + \sum_{l=0}^{k-2} (\nabla_p \beta_{l}) G_{l},
\]
Combining with (3.7), (3.8), (3.9) and (3.10), we have
\[ (3.11) \]
and
\[ (3.9) \]
\[ (3.8) \]
\[ \delta \]
\[ \Gamma \]
\[ k \]
Therefore,
\[ \nabla_p \nabla_p \beta = G^{ij,rs} \chi_{ij,p} \chi_{rs,p} \]
\[ + \sum_{l=0}^{k-2} (\nabla_p \nabla_p \beta_l) G_l . \]
Note that the operator \( \left( \frac{\sigma_k}{\sigma_l} \right) \) and the operator \( \frac{\sigma_k}{\sigma_{k-1}} \) is concave in \( \Gamma_{k-1} \), we have
\[ (3.9) \]
\[ (3.10) \]
Combining with (3.7), (3.8), (3.9) and (3.10), we have
\[ (3.8) \]
which implies that

\[
G^\tilde{\sigma}(\log \tilde{\lambda}_1)_{ii} = G^\tilde{\sigma}\bar{\lambda}_{1;ii} - G^\tilde{\sigma}\frac{|\chi_{1;ii}|^2}{\lambda_1^2} \\
\geq \frac{1}{\lambda_1} G^\tilde{\sigma}\tilde{\lambda}_{1;ii} - \frac{2}{\lambda_1} G^\sigma \Re(\chi_{1;ii} T_{1;ii}^\dagger) - C \sum_i G^\tilde{\sigma} \frac{1}{\lambda_1} G^\tilde{\sigma}|\chi_{1;ii} - B_{1;ii}|^2 \\
\geq G^\tilde{\sigma} \frac{1}{\lambda_1} \nabla_i \nabla_1 \beta - \frac{1 - \delta^2}{\lambda_1} G^\sigma \tilde{\lambda}_{1;ii} \chi_{ij;1} \chi_{rs;1} - \frac{1}{\lambda_1} \sum_{l=0}^{k-2} (\nabla_l \nabla_1 \beta_l) G_l \\
\geq \frac{1 - \delta^2}{\lambda_1} G^\sigma \tilde{\lambda}_{1;ii} \chi_{ij;1} \chi_{rs;1} - (1 + \delta^4) \frac{G^\sigma |\chi_{1;ii}|^2}{\lambda_1^2} - C \sum_i G^\tilde{\sigma} \\
\geq - \frac{1}{\lambda_1} G^\sigma \tilde{\lambda}_{1;ii} \chi_{ij;1} \chi_{rs;1} - (1 + \delta^4) \frac{G^\sigma |\chi_{1;ii}|^2}{\lambda_1^2} - C \sum_i G^\tilde{\sigma} - C.
\]

(3.12)

Now we begin to prove Theorem 3.2. We redefine the auxiliary function

\[
W(z) = \log \tilde{\lambda}_1 + \varphi(|\nabla u|^2) + \psi(u),
\]

where

\[
\varphi(s) = -\frac{1}{2} \log \left(1 - \frac{s}{2K}\right) \quad \text{for} \quad 0 \leq s \leq K - 1
\]

and

\[
\psi(t) = -A \log \left(1 + \frac{t}{2L}\right) \quad \text{for} \quad -L + 1 \leq t \leq 0.
\]

Here, we set

\[
K = \sup_M |\nabla u|^2 + 1, \quad L = \sup_M |u| + 1, \quad A = 2L \Lambda,
\]

and \( \Lambda \) is a large constant that which will be chosen later. Clearly, \( \varphi \) satisfies

\[
\frac{1}{2K} \geq \varphi' \geq \frac{1}{4K}, \quad \varphi'' = 2(\varphi')^2 > 0,
\]

and \( \psi \) satisfies

\[
2\Lambda \geq -\psi' \geq \Lambda, \quad \psi'' \geq \frac{2\varepsilon}{1 - \varepsilon}(\psi')^2 \quad \text{for all} \quad \varepsilon \leq \frac{1}{2A + 1}.
\]

The function \( W \) must achieve its maximum at the interior point \( z \in M \).

Thus, we arrive at

\[
W_i = \frac{\chi_{1T;i}}{\chi_{1T}} + \varphi' \nabla_i (|\nabla u|^2) + \psi' u_i = 0
\]

and

\[
W_{ii} = (\log \tilde{\lambda}_1)_{ii} + \varphi'' \nabla_i (|\nabla u|^2)^2 + \varphi' \nabla_i \nabla_i (|\nabla u|^2)
\]

\[
+ \psi'' |u_i|^2 + \psi' u_{ii} \leq 0.
\]

(3.14)
Multiplying (3.14) by $G^i\bar{\tau}$ and summing it over index $i$, we can know from (3.12) 

$$0 \geq -\frac{1 - \delta^2}{\chi_{11}} G^{ij,rs} \chi_{ij:1} \chi_{rs:1} - (1 + \delta^4) \frac{G^i |\chi_{\tau i}|^2}{\chi_{11}^2} - C \sum_i G^i - C \tag{3.15}$$

$$+ \varphi'' G^i \nabla_i (|\nabla u|^2) + \varphi' G^i \nabla_i (|\nabla u|^2) + \psi'' G^i |u_i|^2 + \psi' G^i u_i \nabla_i G^i.$$

To proceed, we need the following calculation

$$\nabla_i (|\nabla u|^2) = \sum_p (u_p \bar{u}_{pi} + u_{\bar{p}} u_{pi}).$$

and

$$\nabla_i (|\nabla u|^2) = \sum_p (u_p \bar{u}_{pi} + u_{\bar{p}} u_{pi} + u_p u_{\bar{p}} + u_{\bar{p}} u_{pi}).$$

Note that

$$u_{\bar{p}i} = u_{\bar{p}i} + R^p_{\bar{p}q} g^m u_m = \chi_{\bar{p},p} - (\chi_0)_{\bar{p},p} + T^q_{\bar{p}i} \chi_{\bar{i},q} - T^q_{\bar{p}i} (\chi_0)_{\bar{i},q}$$

and

$$u_{pi} = u_{pi} + R^p_{\bar{p}q} g^m u_m = u_{\bar{p}i} + R^p_{\bar{p}q} g^m u_m + T^q_{pi} u_i = \chi_{\bar{i},p} - (\chi_0)_{\bar{i},p} + R^q_{\bar{i}p} g^m u_m + T^q_{pi} \chi_{\bar{i},r} - T^q_{pi} (\chi_0)_{\bar{i},r}.$$

Combining with (3.7), we have

$$G^i u_{\bar{p}i} u_p \geq G^i \chi_{\bar{p},p} u_p + G^i T^q_{\bar{p}i} \chi_{\bar{i},q} u_p - C_i K \sum_i G^i \chi_{\bar{i},q} u_p$$

$$\geq (\beta)_{\bar{p}} u_p - \frac{\epsilon}{2} G^i |\chi_{\bar{i},q}|^2 - C_i K \sum_i G^i \chi_{\bar{i},q} u_p.$$

Similarly,

$$G^i u_{pi} u_{\bar{p}} \geq (\beta)_{p} u_{\bar{p}} - \frac{\epsilon}{2} G^i |\chi_{\bar{i},q}|^2 - C_i K \sum_i G^i \chi_{\bar{i},q} u_p.$$

Thus,

$$G^i \nabla_i (|\nabla u|^2) \geq \sum_p G^i (|u_{\bar{p}i}|^2 + |u_{pi}|^2) - \sum_p \text{Re} \{ (\beta)_{p} u_{\bar{p}} \}$$

$$\geq \frac{\epsilon}{2} G^i |\chi_{\bar{i},q}|^2 - C_i K \sum_i G^i \chi_{\bar{i},q} u_p. \tag{3.16}$$

Since

$$\sum_p G^i |u_{\bar{p}i}|^2 \geq G^i |u_{\bar{p}i}|^2 \geq G^i |\chi_{\bar{i},q}|^2 - C_i K \sum_i G^i \chi_{\bar{i},q} u_p \tag{3.17}$$

$$\geq \frac{1}{2} G^i |\chi_{\bar{i},q}|^2 - C \sum_i G^i,$$
we can use half of the term $\sum_p G^{\tilde{\alpha}} |u_{\tilde{\alpha}i}|^2$ to absorb the negative term $-\epsilon G^{\tilde{\alpha}} |\chi_{i\tilde{a}}|^{2}$ if we chose $\epsilon = \frac{1}{8}$. From now on we can replace $C_{\epsilon}$ with $C$ since $\epsilon$ is fixed. It follows from (3.15)

$$G^{\tilde{\alpha}} \nabla_i \nabla_i (|\nabla u|^2) \geq \frac{1}{8} G^{\tilde{\alpha}} |\chi_{i\tilde{a}}|^2 + \frac{1}{2} \sum_p G^{\tilde{\alpha}} (|u_{pi}|^2 + |u_{\tilde{p}i}|^2)$$

(3.18) 

$$+ 2 \sum_p \text{Re}\{\beta_p u_{\tilde{p}i}\} - CK \sum_i G^{\tilde{\alpha}}.$$ 

Taking the inequality (3.18) into (3.15), it yields

$$0 \geq -\frac{1 - \delta^2}{\chi_{11}} G^{\tilde{\alpha}} |\chi_{1\tilde{a}}|^2 + \psi' G^{\tilde{\alpha}} |\chi_{i\tilde{a}}|^2 + \psi' G^{\tilde{\alpha}} u_{\tilde{a}i} + \varphi'' G^{\tilde{\alpha}} |\nabla_i (|\nabla u|^2)|^2 + \frac{\varphi'}{8} G^{\tilde{\alpha}} |\chi_{i\tilde{a}}|^2$$

(3.19) 

$$+ 2 \varphi' \sum_p \text{Re}\{\beta_p u_{\tilde{p}i}\} - C \sum_i G^{\tilde{\alpha}} - C.$$ 

Now, we divide our proof into two cases separately, depending on whether $\chi_{0\tilde{a}} < -\delta \chi_{1\tilde{a}}$ or not, for a small $\delta$ to be chosen later.

Case 1. $\chi_{n\tilde{a}} < -\delta \chi_{1\tilde{a}}$. In this case, it follows that $\chi_{1\tilde{a}}^2 \leq \frac{1}{\delta^2} \chi_{n\tilde{a}}^2$. So, we only need to bound $\chi_{1\tilde{a}}^2$. Clearly, we can obtain if we throw some positive terms in (3.19)

$$0 \geq -(1 + \delta^4) G\tilde{\alpha} \frac{|\chi_{1\tilde{a}}|^2}{\chi_{11}^2} + \psi' G^{\tilde{\alpha}} u_{\tilde{a}i} + \varphi'' G^{\tilde{\alpha}} |\nabla_i (|\nabla u|^2)|^2 + \frac{\varphi'}{8} G^{\tilde{\alpha}} |\chi_{i\tilde{a}}|^2$$

(3.20) 

$$+ 2 \varphi' \sum_p \text{Re}\{\beta_p u_{\tilde{p}i}\} - C \sum_i G^{\tilde{\alpha}} - C.$$ 

From (2.5), we obtain

$$-\psi' \sum_i G^{\tilde{\alpha}} u_{\tilde{a}i} = -\psi' \sum_i G^{\tilde{\alpha}} \left[ \chi_{\tilde{a}} - ((\chi_0)_{\tilde{a}} - \tau) - \tau \right]$$

$$= -\psi' \left[ \beta + \sum_{l=0}^{k-2} (l - k) \beta_l G_l - \sum_i G^{\tilde{\alpha}} (\chi_{0\tilde{a}} - \tau) - \tau \sum_i G^{\tilde{\alpha}} \right]$$

(3.21) 

$$\leq CA - \tau \Lambda \sum_i G^{\tilde{\alpha}}.$$ 

Plugging (3.21) into (3.20) and choosing $\Lambda$ large enough,

$$\varphi'' G^{\tilde{\alpha}} |\nabla_i (|\nabla u|^2)|^2 + \frac{\varphi'}{8} G^{\tilde{\alpha}} |\chi_{i\tilde{a}}|^2$$

$$\leq (1 + \delta^4) G^{\tilde{\alpha}} \frac{|\chi_{1\tilde{a}}|^2}{\chi_{11}^2} - 2 \varphi' \sum_p \text{Re}\{\beta_p u_{\tilde{p}i}\} + C(1 + \Lambda).$$
Note that we get from (3.13)

\[(1 + \delta^4) \sum_i \frac{G_{|i|}^\tilde{\pi} |\chi_{1i|\tilde{\pi}|}|^2}{\chi_{1|\tilde{\pi}|}^2} = (1 + \delta^4) \sum_i G_{|i|}^\tilde{\pi} |\varphi'| \nabla_i(|\nabla u|^2) + |\psi'| u_i|^2
\]

\[\leq 2(\varphi')^2 \sum_i G_{|i|}^\tilde{\pi} |\nabla_i(|\nabla u|^2)|^2 + \frac{8(1 + \delta^4)}{1 - \delta^4} \Lambda^2 K \sum_i G_{|i|}^\tilde{\pi} \]

\[\leq 2(\varphi')^2 \sum_i G_{|i|}^\tilde{\pi} |\nabla_i(|\nabla u|^2)|^2 + 16\Lambda^2 K \sum_i G_{|i|}^\tilde{\pi}, \]

(3.23)

where we choose \(\delta \leq 3^{-\frac{1}{4}}\). In fact,

\[(3.24) \quad \sum_i G_{|i|}^\tilde{\pi} \chi_{n|}\tilde{\pi}| \geq G_{n|\tilde{\pi}|} \chi_{n|\tilde{\pi}|} \geq \frac{1}{n} \chi_{n|\pi}| \sum_i G_{|i|}^\tilde{\pi}.\]

Substituting (3.23) and (3.24) into (3.22), we get

\[\frac{1}{32nK \chi_{n|\pi}|} \sum_i G_{|i|}^\tilde{\pi} \leq 16\Lambda^2 K \sum_i G_{|i|}^\tilde{\pi} + C(1 + \Lambda).\]

Combining with [2.13], it is easy to derive that

\[\chi_{1|\tilde{\pi}|} \leq CK.\]

**Case 2.** \(\chi_{n|\pi}| \geq -\delta \chi_{1|\tilde{\pi}|}\). Define

\[I = \left\{ i \in \{1, 2, \ldots, n\} : G_{|i|}^\tilde{\pi} > \delta^{-1} G_{1|\tilde{\pi}|} \right\}.\]

It follows from [2.22]

\[-\frac{1}{\chi_{1|\tilde{\pi}|}} G_{|i|}^\tilde{\pi} \chi_{i|\tilde{\pi}|} \chi_{1|\tilde{\pi}|} \geq \frac{1 - \delta}{1 + \delta} \frac{1}{\chi_{1|\tilde{\pi}|}} \sum_{i \in I} G_{|i|}^\tilde{\pi} \chi_{i|\tilde{\pi}|}^2 \]

\[\geq \frac{1 - \delta}{1 + \delta} \frac{1}{\chi_{1|\tilde{\pi}|}} \sum_{i \in I} G_{|i|}^\tilde{\pi} \left( |\chi_{1|\tilde{\pi}|}|^2 + 2Re\{\chi_{1|\tilde{\pi}|, i|\tilde{\pi}|}\} \right), \]

(3.25)

where \(e'_i = T_{1i}^p u_{p|\tilde{\pi}|} + (\chi_0)_{1i|\tilde{\pi}|} - (\chi_0)_{11|\tilde{\pi}|}i\).

Note that \(\varphi'' = 2(\varphi')^2\), using (3.13), we get

\[\varphi'' \sum_{i \in I} G_{|i|}^\tilde{\pi} |\nabla_i(|\nabla u|^2)|^2 \geq 2 \sum_{i \in I} G_{|i|}^\tilde{\pi} \left( \frac{\chi_{1|\tilde{\pi}|, i|\tilde{\pi}|}^2}{\chi_{1|\tilde{\pi}|}^2} - \frac{\delta}{1 - \delta} |\psi'| u_i|^2 \right), \]

(3.26)
Choosing $\delta \leq \min\{\frac{1}{2A+1}, 3^{-\frac{1}{2}}\}$ and hence $\psi'' \geq \frac{2\delta}{1-\delta}(\psi')^2$. Combining (3.25) with (3.26),

$$-(1 + \delta^4) \sum_{i \in I} G_{\vec{x}} |\chi_{i,11}|^2 \chi_{\vec{x}}^2 - \frac{1 - \delta^2}{\chi_{\vec{x}}^2} G_{\vec{x}} \chi_{i,11} \chi_{i,11}^\star$$

$$+ \varphi'' \sum_{i \in I} G_{\vec{x}} |\nabla_i (|\nabla u|^2)|^2 + \psi'' \sum_{i \in I} G_{\vec{x}} |u|^2$$

$$\geq \frac{\delta^2}{2} \sum_{i \in I} G_{\vec{x}} |\chi_{i,11}|^2 \chi_{\vec{x}}^2 + 2(1 - \delta)^2 \frac{1}{\chi_{\vec{x}}^2} \sum_{i \in I} G_{\vec{x}} \chi_{i,11}^2$$

$$\geq \frac{\delta^2}{4} \sum_{i \in I} G_{\vec{x}} |\chi_{i,11}|^2 \chi_{\vec{x}}^2 - C_\delta \frac{1}{\chi_{\vec{x}}^2} \sum_{i \in I} G_{\vec{x}} \chi_{i,11}^2$$

(3.27)

$$\geq \frac{\delta^2}{4} \sum_{i \in I} G_{\vec{x}} |\chi_{i,11}|^2 \chi_{\vec{x}}^2 - C \sum_{i \in I} G_{\vec{x}} \chi_{i,11}^2$$

where we choose $\chi_{i,11}$ large enough to get the last inequality.

For the terms without an index in $I$, by (3.23) and the fact $\frac{1}{2} \in I$, it follows that

(3.28)

$$-(1 + \delta^4) \sum_{i \notin I} G_{\vec{x}} |\chi_{i,11}|^2 \chi_{\vec{x}}^2 + \varphi'' \sum_{i \notin I} G_{\vec{x}} |\nabla_i (|\nabla u|^2)|^2$$

$$\geq -\frac{16n\Lambda^2 K}{\delta} G_{\vec{x}}\chi_{i,11}^2.$$ 

Substituting (3.27) and (3.28) into (3.19),

(3.29)

$$C \sum_i G_{\vec{x}} + C + \frac{16n\Lambda^2 K}{\delta} G_{\vec{x}}\chi_{i,11}^2 \geq \frac{1}{8K} G_{\vec{x}} \chi_{i,11}^2 + \psi' G_{\vec{x}} u_{i,11}^\star.$$ 

Note that $\mu := \lambda(\chi_0 + \frac{\chi_{i,11}}{2} \partial \partial u)$ satisfies the condition (2.6) since (1.5). Without loss of generality we assume that $\chi_{i,11} \geq N$. From Lemma 2.2 we will divide the following argument into two case:

**Case 2.1:**

$$\sum_i G_{\vec{x}} u_{i,11}^\star \leq -\theta - \theta \sum_i G_{\vec{x}}.$$ 

It implies that

$$\psi' \sum_i G_{\vec{x}} u_{i,11}^\star \geq \Lambda \theta (1 + \sum_i G_{\vec{x}}).$$ 

Combining with (3.29), we have

(3.30) $C \sum_i G_{\vec{x}} + C + \frac{16n\Lambda^2 K}{\delta} G_{\vec{x}}\chi_{i,11}^2 \geq \frac{1}{8K} G_{\vec{x}} \chi_{i,11}^2 + \Lambda \theta (1 + \sum_i G_{\vec{x}}).$
We can choose $\Lambda$ large enough, then (3.30) gives
\[
\frac{1}{8K}G^{1\mathbf{T}}\chi_{11}^2 \leq \frac{1}{8K}G^{\bar{i}\bar{i}}\chi_{\bar{i}\bar{i}}^2 \leq \frac{16n\Lambda^2K}{\delta}G^{1\mathbf{T}}.
\]
So
\[
|\chi_{11}| \leq 8\Lambda K \sqrt{\frac{2n}{\delta}}.
\]

**Case 2.2:**
\[
G^{11} \chi_{11} \geq \theta.
\]
From (3.21), we can absorb the last term of (3.29) into $\tau \Lambda \sum_i G^{\bar{i}\bar{i}}$,
\[
C(1 + \Lambda) + \frac{16n\Lambda^2K}{\delta}G^{1\mathbf{T}} \geq \frac{1}{8K}G^{1\mathbf{T}}\chi_{11}^2 + \tau \Lambda \sum_i G^{\bar{i}\bar{i}}.
\]
Choosing $\Lambda$ large enough, it follows that
\[
(3.32) \quad C(1 + \Lambda) + \frac{16n\Lambda^2K}{\delta}G^{1\mathbf{T}} \geq \frac{1}{8K}G^{1\mathbf{T}}\chi_{11}^2.
\]
Using (3.31), we choose $\chi_{1\mathbf{T}}$ large enough such that
\[
\frac{1}{16K}G^{1\mathbf{T}}\chi_{11}^2 \geq \frac{\theta}{16K} \chi_{1\mathbf{T}} \geq C(1 + \Lambda).
\]
Taking the above inequality into (3.32), we arrive
\[
\chi_{1\mathbf{T}} \leq 16\sqrt{\frac{n}{\delta}} \Lambda K.
\]
So, we complete the proof.

4. **Boundary mixed tangential-normal estimates**

The goal of this section is to prove an estimate for the boundary mixed tangential-normal derivatives. For $p \in \partial M$, we choose a coordinate $z = (z^1, \ldots, z^n)$ such that $p$ corresponds to the origin and $g_\mathbf{\bar{g}} = \delta_{ij}$. We denote $z^i = x^i + \sqrt{-1}y^i$ and
\[
t^1 = y^1, t^2 = y^2, \ldots, t^n = y^n, t^{n+1} = x^1, \ldots, t^{2n-1} = x^{n-1}.
\]
Let $\rho$ be the distance function to 0, i.e.,
\[
\rho(z) =: \text{dist}_{g_\mathbf{\bar{g}}}(z, 0), \quad x \in M
\]
and set
\[
M_\delta = \{z \in M : \rho(z) < \delta\} \quad \text{for} \quad \delta > 0.
\]
Let $d$ be the distance function to the boundary $\partial M$ with respect to the background metric $g$
\[
d(z) =: \text{dist}_{g_\mathbf{\bar{g}}}(z, \partial M), \quad z \in M.
\]
Since $\partial M$ is smooth and $|\nabla d| = 1$ on $\partial M$, we can choose $\delta > 0$ sufficiently small so that $d$ is smooth,

$$\frac{1}{2} \leq |\nabla d| \leq 2 \quad \text{in} \quad M_{\delta}$$

and

$$C_2 \leq |\nabla^2 d| \leq C_1 \quad \text{in} \quad M_{\delta},$$

where the constants $C_1$ and $C_2$ are independent of $\delta$.

Suppose near the origin, the boundary $\partial M$ is represented by $\rho(z) = 0$ and $d\rho \neq 0$ on $\partial M$. Then, there exists a function $\zeta(t)$ such that $\rho(t,\zeta(t)) = 0$.

Differentiating the boundary condition $(u - \underline{u})(t,\zeta(t)) = 0$, we derive the relation on $\partial M \cap M_{\delta}$

$$\partial_{t^\alpha} (u - u) = -\partial_{x^\alpha} (u - \underline{u}) \partial_{x^\alpha} \zeta$$

for any $\alpha = 1, \cdots, 2n - 1$. Moreover, differentiating the boundary condition again gives

$$\partial_{t^\alpha} \partial_{t^\gamma} (u - \underline{u})(0) = -\partial_{x^\alpha} (u - \underline{u})(0) \partial_{t^\alpha} \partial_{x^\gamma} \zeta(0),$$

which implies that

$$|\partial_{t^\alpha} \partial_{t^\gamma} u(0)| \leq C$$

for any $\alpha, \gamma = 1, \cdots, 2n - 1$.

To derive the boundary tangential-normal estimates, we use the following locally defined auxiliary function in $M_{\delta}$

$$\Psi = A\sqrt{K}v + B\sqrt{K}|z|^2 - \frac{1}{\sqrt{K}} \sum_{\rho=1}^{n} \left| \partial_{\rho^\alpha} (u - \underline{u}) \right|^2 - \frac{1}{\sqrt{K}} \sum_{a=1}^{n-1} \left| \nabla_a (u - \underline{u}) \right|^2$$

$$+ T_\alpha (u - \underline{u}),$$

where $A, B \gg 1$ are constants to be determined. Here,

$$v = u - \underline{u} + td - \frac{N}{2}d^2,$$

introduced by Guan in [19], and $t$ and $N$ will be determined later.

**Lemma 4.1.** There exist uniform positive constants $t, \delta, \varepsilon$ small enough and $N \gg 1$ such that $v$ satisfies

$$G^\gamma v_{ij} \leq -\frac{\varepsilon}{4} \left( 1 + \sum_{i=1}^{n} G^{\alpha} \right), \quad \text{in} \quad M_{\delta},$$

$$v \geq 0 \quad \text{on} \quad \partial M_{\delta},$$

**Proof.** The proof is quite similar to that given for [43, Lemma 2.4] and so they are omitted. □
4.1. Estimates of tangential derivatives $T_{\alpha}(u-u)$. For $\alpha \in \{1, 2, \ldots, 2n-1\}$, we define the real vector fields

$$T_{\alpha} = \frac{\partial}{\partial t^\alpha} - \frac{\rho_{t^\alpha}}{\rho_{x^n}} \frac{\partial}{\partial x^n},$$

which are clearly tangential vector on $\partial M$. Then, we have

**Lemma 4.2.**

$$|G^i_j \partial_i \partial_j T_{\alpha}(u-u)| \leq \frac{1}{\sqrt{K}} G^i_j \partial_i \partial_j y^n (u-u) + C \sum_{i=1}^{n} (1 + G^i_i |\lambda_i|) + C (1 + \sqrt{K}) \sum_{i=1}^{n} G^{ii}.$$  

(4.7)

Proof. We start with the following computation

$$G^i_j \partial_i \partial_j T_{\alpha}(u-u) = G^i_j \partial_i \partial_j \partial_t \alpha (u-u) - \rho_{t^\alpha} G^i_j \partial_i \partial_x (u-u) - 2 \text{Re} [G^i_j \partial_i \rho_{t^\alpha} \partial_x (u-u)] - G^i_j \partial_i \partial_x (u-u).$$  

(4.8)

Differentiating the equation (2.9) once, we have

$$G^i_j \nabla_{t^\alpha} \nabla_i \nabla_j u + G^i_j \nabla_{t^\alpha} (\chi_0)_{ji} = \nabla_{t^\alpha} \beta - \sum_{l=0}^{k-2} (\nabla_{t^\alpha} \beta_l) G_l.$$  

(4.9)

Note that

$$\frac{\partial}{\partial t^\alpha} = \frac{1}{\sqrt{-1}} \left( \frac{\partial}{\partial z^\alpha} - \frac{\partial}{\partial \bar{z}^\alpha} \right) \quad \text{for} \quad 1 \leq \alpha \leq n,$$

and

$$\frac{\partial}{\partial h^\alpha} = \left( \frac{\partial}{\partial z^{\alpha-n}} + \frac{\partial}{\partial \bar{z}^{\alpha-n}} \right) \quad \text{for} \quad n+1 \leq \alpha \leq 2n-1.$$

Then, we can convert covariant derivatives to partial derivatives for $n+1 \leq \alpha \leq 2n-1$

$$\nabla_{t^\alpha} \nabla_i \nabla_j u = \nabla_{i} \nabla_j \nabla_{t^\alpha} u + \nabla_{\bar{z}} \nabla_i \nabla_{t^\alpha} u$$

(4.10)

$$\quad = \partial_i \partial_j \partial_{t^\alpha} u - \Gamma_{\alpha \beta}^{r} u_{jr} - \Gamma_{\alpha \beta}^{r} u_{ri}.$$  

Substituting (4.10) into (4.9) gives

$$G^i_j \partial_i \partial_j T_{\alpha}(u-u) = -G^i_j \nabla_{t^\alpha} (\chi_0)_{ji} + \nabla_{t^\alpha} \beta - \sum_{l=0}^{k-2} (\nabla_{t^\alpha} \beta_l) G_l$$

(4.11)

$$\quad + G^i_j \Gamma_{\alpha \beta}^{r} u_{jr} + G^i_j \Gamma_{\alpha \beta}^{r} u_{ri}.$$
It follows that for \( n + 1 \leq \alpha \leq 2n - 1 \)

\[
|G^\alpha_i \partial_i \partial_y u| \leq CG^\alpha(1 + |\lambda_i|) + C(|\nabla \beta| + \sum_{l=0}^{k-2} |\nabla \beta_l|).
\]

Similarly, we can get the estimate (4.12) for \( 1 \leq \alpha \leq n \). Thus, the estimate (4.12) holds for all \( 1 \leq \alpha \leq 2n - 1 \). Moreover, we also have a similar estimate

\[
|G^\alpha_i \partial_i \partial_y u| \leq CG^\alpha(1 + |\lambda_i|) + C(|\nabla \beta| + \sum_{l=0}^{k-2} |\nabla \beta_l|).
\]

Plugging (4.12) and (4.13) into (4.8), it yields

\[
G^\alpha_i \partial_i \partial_y T\alpha(u - \underline{u}) \leq -2\text{Re}\left[G^\alpha_i \left( \partial_i \frac{\rho^{\alpha}}{\rho_x} \right) \partial_x (u - \underline{u}) \right] + \sqrt{K} \sum_{i=1}^{n} G^\alpha_i
\]

\[
+ CG^\alpha(1 + |\lambda_i|) + C.
\]

The first term on the right hand side of (4.14) can be written as

\[
G^\alpha_i \left( \partial_i \frac{\rho^{\alpha}}{\rho_x} \right) \partial_x (u - \underline{u}) = 2G^\alpha_i \left( \partial_i \frac{\rho^{\alpha}}{\rho_x} \right) \partial_x (u - \underline{u})
\]

\[
+ \sqrt{-1} G^\alpha_i \left( \partial_i \frac{\rho^{\alpha}}{\rho_x} \right) \partial_x (u - \underline{u})
\]

it follows from the Cauchy-Schwarz inequality that

\[
|G^\alpha_i \left( \partial_i \frac{\rho^{\alpha}}{\rho_x} \right) \partial_x (u - \underline{u})| \leq \frac{1}{2\sqrt{K}} G^\alpha_i \partial_i \partial_y n(u - \underline{u}) \partial_y (u - \underline{u})
\]

\[
+ C \left( \sum_{i=1}^{n} G^\alpha_i |\lambda_i| + 1 \right) + C(1 + \sqrt{K}) \sum_{i=1}^{n} G^\alpha_i.
\]

Combining (4.14) and (4.15) gives (4.7).

\[
\square
\]

4.2. Estimates of \( K^{-\frac{1}{2}} \sum_{p=1}^{n} \left[ \partial_{y^p} (u - \underline{u}) \right]^2 \). Differentiating the boundary condition \( (u - \underline{u})(t, \zeta(t)) = 0 \), we have

\[
\partial_{y^p} (u - \underline{u}) = -\partial_{x^p} (u - \underline{u}) \partial_{y^p} \zeta, \quad \text{on} \quad \partial M \cap \overline{M}\delta.
\]

Note that \( |\partial_{y^p} \zeta| \leq C|t| \) in view of \( \partial_{y^p} \zeta(0) = \partial_{y^p} \rho(0) = 0 \), hence

\[
(\partial_{y^p} (u - \underline{u}))^2 \leq C |z|^2 \quad \text{on} \quad \partial M \cap \overline{M}\delta.
\]
Combining with (4.12), we can obtain

\[ (4.16) \quad \frac{1}{\sqrt{K}} G^{ij} \partial_i \partial_j \left[ \partial_y^p (u - u) \right]^2 \]

\[ = \frac{2}{\sqrt{K}} G^{ij} \partial_i \partial_j \partial_y^p (u - u) + \frac{2}{\sqrt{K}} \partial_y^p (u - u) G^{ij} \partial_i \partial_j \partial_y^p (u - u) \]

\[ \geq \frac{2}{\sqrt{K}} G^{ij} \partial_i \partial_j \partial_y^p (u - u) - C|G^{ij} \partial_i \partial_j u| - C \sum_{i=1}^{n} G^{ii} - C. \]

4.3. Estimates of \( \frac{1}{\sqrt{K}} \sum_{a=1}^{n-1} |\nabla_a (u - u)|^2 \). For each \( a \in \{1, 2, ..., n-1\} \), we define local sections of \( T^{1,0}M \) around the origin

\[ E_a(0) = \frac{\partial}{\partial z^a} - \left[ \frac{\partial_y^a}{\partial \bar{z}^a} \right] \frac{\partial}{\partial z^n}, \quad \text{for} \quad 1 \leq a \leq n-1. \]

Clearly, those are tangential to \( \partial M \). Using the metric \( \omega \), we perform the Gram-Schmidt process to obtain a local orthonormal frame \( \{e_a\}_{a=1}^{n-1} \) of \( T^{1,0}M \). Thus, \( \{e_a\}_{a=1}^{n-1} \) are tangential to \( \partial M \), \( \omega(e_a, e_b) = \delta_{ab} \) and

\[ e_a(0) = \frac{\partial}{\partial z^a}, \quad \text{for} \quad 1 \leq a \leq n-1. \]

Furthermore, let

\[ e_n = \frac{E_n}{|E_n|_\omega}, \quad E_n = \frac{\partial}{\partial z^n} - \sum_{a=1}^{n-1} \omega(\partial_n, e_a) e_a. \]

Thus,

\[ \frac{1}{\sqrt{K}} G^{ij} \partial_i \partial_j |\nabla_a (u - u)|^2 = \frac{1}{\sqrt{K}} G^{ij} \partial_i \partial_j \nabla_a (u - u) \nabla_a (u - u) \]

\[ = \frac{1}{\sqrt{K}} G^{ij} \partial_i \partial_j \left[ e_a^p \bar{e_a}^q \partial_q (u - u) \bar{\partial}_q (u - u) \right] \]

\[ \geq \frac{1}{\sqrt{K}} G^{ij} e_a^p \partial_i \partial_p (u - u) \bar{e_a}^q \bar{\partial}_q \partial_j (u - u) \]

\[ + \frac{1}{\sqrt{K}} G^{ij} \bar{e_a}^p \partial_i \partial_p (u - u) e_a^q \partial_q \bar{\partial}_j (u - u) \]

\[ + \frac{2}{\sqrt{K}} \text{Re} \left( G^{ij} \partial_i (\bar{e_a}^q) \partial_j (u - u) \partial_q (u - u) \right) \]

\[ (4.17) \quad -CG^{ii} (1 + |\lambda_i|) - C(1 + \sqrt{K}) \sum_{i=1}^{n} G^{ii} - C. \]
We deal with the first term on the right side of the inequality (4.17) by

\[ G^J e_a^p \partial_J \partial_p (u - w) = G^J (\chi - \chi_u) \partial_J \partial_p (u - w) \]

(4.18)

\[ \geq \frac{1}{2} G^J \lambda_j a \lambda_m - C \sum_{i=1}^{n} G^i a. \]

Next, we can rewrite the third term on the right side of the inequality (4.17) as

\[ \frac{2}{\sqrt{K}} G^J \partial_J (e_a^p e_a^q) \partial_p \partial_i (u - w) \partial_J (u - w) = \frac{2}{\sqrt{K}} G^J \partial_J (e_a^p e_a^q) \partial_p \partial_i (u - w) \partial_J (u - w) \]

(4.19)

where we used the relation \( \frac{\partial}{\partial \eta^p} = \frac{\partial}{\partial \eta^p} - \sqrt{-1} \frac{\partial}{\partial y^p} \). Note that

\[ \frac{2}{\sqrt{K}} \sum_{p,q} \left| G^J \partial_J (e_a^p e_a^q) \partial_p \partial_i (u - w) \partial_J (u - w) \right| \]

\[ \leq 2 \sum_{p,q} \left[ G^J \partial_i \partial_J (u - w) \partial_J (u - w) \right]^{\frac{1}{2}} \left[ G^J \partial_i (e_a^p e_a^q) \partial_J (e_a^p e_a^q) \right]^{\frac{1}{2}} \]

\[ \leq \frac{1}{\sqrt{K}} \sum_{i} G^J \partial_i \partial_J (u - w) \partial_J (u - w) + C \sqrt{K} \sum_{i=1}^{n} G^i a. \]

Substituting (4.18) and (4.19) into (4.17), dropping the second term on the right side of the inequality (4.17), we can obtain

\[ \frac{1}{\sqrt{K}} G^J \partial_i \partial_J \nabla_a (u - w)^2 \geq \frac{1}{2} \sqrt{K} G^J \lambda_j a \lambda_m - \frac{1}{\sqrt{K}} G^J \partial_i \partial_J \partial_p (u - w) \partial_J \partial_p (u - w) \]

(4.20)

\[ -C \sum_{i=1}^{n} G^i |\lambda_i| - C(1 + \sqrt{K}) \sum_{i=1}^{n} G^i a - C. \]

By the Lemma 2.7 in [14], there exists an index \( r \) such that

\[ G^J \lambda_j a \lambda_m \geq \frac{1}{2} \sum_{i \neq r} G^i a \lambda_i^2. \]

Going back to (4.20), we get

\[ \frac{1}{\sqrt{K}} G^J \partial_i \partial_J \nabla_a (u - w)^2 \geq \frac{1}{4 \sqrt{K}} \sum_{i \neq r} G^i a \lambda_i^2 - \frac{1}{\sqrt{K}} \sum_{p=1}^{n} G^J \partial_i \partial_J \partial_p (u - w) \partial_J \partial_p (u - w) \]

(4.21)

\[ -C \sum_{i} G^i |\lambda_i| - C(1 + \sqrt{K}) \sum_{i=1}^{n} G^i a - C. \]
4.4. Mixed tangential-normal estimates. Combining with (4.6), (4.7), (4.16) and (4.21), we obtain

\[ G^\bar{\mu}\bar{\nu} \partial_i \partial_{\bar{j}} \Psi \leq -\frac{1}{\sqrt{K}} \sum_{p=1}^{n-1} G^\bar{\mu}\bar{\nu} (u - \bar{u}) \partial_j \partial_{\bar{\nu}} (u - \bar{u}) - \frac{1}{4\sqrt{K}} \sum_{i \neq r} G^{\bar{\mu}} \lambda_i^2 \]

\[ -\varepsilon \frac{A}{4} \sqrt{K} \left( 1 + \sum_{i=1}^{n} G^{\bar{\mu}} \right) + B\sqrt{K} \sum_{i=1}^{n} G^{\bar{\mu}} \]

\[ + C(1 + \sum_{i} G^{\bar{\mu}} |\lambda_i|) + C(1 + \sqrt{K}) \sum_{p=1}^{n} G^{\mu\bar{\nu}} \]

\[ \leq -\varepsilon \frac{A}{4} \sqrt{K} \left( 1 + \sum_{i=1}^{n} G^{\bar{\mu}} \right) + B\sqrt{K} \sum_{i=1}^{n} G^{\bar{\mu}} + C(1 + \sqrt{K}) \sum_{i=1}^{n} G^{\bar{\mu}} + C \]

\[ -\frac{1}{4\sqrt{K}} \sum_{i \neq r} G^{\bar{\mu}} \lambda_i^2 + C G^{\bar{\mu}} |\lambda_i| \].

Choosing \( \frac{\varepsilon A}{4} \geq B + 2C + A_0 \) with \( A_0 \) large enough, we have

\[ (4.22) G^\bar{\mu}\bar{\nu} \partial_i \partial_{\bar{j}} \Psi \leq -A_0 \sqrt{K} \sum_{i=1}^{n} G^{\bar{\mu}} - \frac{1}{4\sqrt{K}} \sum_{i \neq r} G^{\bar{\mu}} \lambda_i^2 + C \sum_{i} G^{\bar{\mu}} |\lambda_i| \].

By [19] Corollary 2.8, for any \( \varepsilon_0 > 0 \), we have

\[ (4.23) \sum_{i} G^{\bar{\mu}} |\lambda_i| \leq \varepsilon_0 \sum_{i \neq r} G^{\bar{\mu}} \lambda_i^2 + C_1(1 + \frac{1}{\varepsilon_0}) \sum_{i} G^{\bar{\mu}} \].

Choosing \( \varepsilon_0 = \frac{1}{4C\sqrt{K}} \) and substituting the above inequality into (4.22), choosing \( A_0 \) sufficiently large results in

\[ G^\bar{\mu}\bar{\nu} \partial_i \partial_{\bar{j}} \Psi \leq 0. \]

Lastly, we consider the boundary value for \( \Psi \) which consists of two pieces.

First, on \( \partial M \cap M_\delta \), if we take \( B \) large enough, we have

\[ (4.24) \Psi \geq A\sqrt{K}v + B\sqrt{K}|z|^2 - C|z|^2 \geq 0. \]

Secondly, on \( \partial M_\delta \cap M \), if we take \( B \) large enough, we have

\[ (4.25) \Psi \geq A\sqrt{K}v + B\sqrt{K}\delta^2 - C\sqrt{K} \geq 0, \]

combining (4.24) and (4.25) gives

\[ \Psi \geq 0 \text{ on } \partial M_\delta. \]

Then, applying the maximum principle, we get

\[ \Psi \geq 0 \text{ in } M_\delta. \]

Note that \( \Psi(0) = 0 \), it yields

\[ \partial_{\nu^n} \Psi(0) \geq 0. \]
It follows that
\[
0 \leq A\sqrt{K}\partial_{x^a} v(0) - (\partial_{x^a} \frac{\rho_{x^a}}{\rho_{x^n}})(0)\partial_{x^a} (u - u)(0) + \partial_{x^n} \partial_{t^a} (u - u).
\]
Since \(|\partial_{x^a} v| \leq C\) on \(\partial M\), we conclude
\[
\partial_{x^a} \partial_{t^a} u(0) \geq -C\sqrt{K}.
\]
We can apply the same argument to the function
\[
\tilde{\Psi} : = A\sqrt{K}v + B\sqrt{K}|z|^2 - \frac{1}{\sqrt{K}} \sum_{i=1}^{n} \left[ \partial_{y^i}(u - u) \right]^2 - \frac{1}{\sqrt{K}} \sum_{i=1}^{n-1} |\nabla_{\bar{i}}(u - u)|^2
\]
\[\text{(4.26)} \quad -T_\alpha(u - u).
\]
It follows that
\[
\partial_{x^a} \partial_{t^a} u(0) \leq C\sqrt{K}.
\]
Thus,
\[
|\chi_{\alpha'}(0)| \leq C\sqrt{K} \quad \forall \alpha' \in \{1, 2, \cdots, n - 1\}.
\]

5. Boundary double normal estimate

For \(p \in \partial M\), we choose coordinates \(z = (z^1, \ldots, z^n)\) such that \(z(p) = 0\) and \(g_{i\bar{j}} = \delta_{ij}\). We denote \(z^i = x^i + \sqrt{-1}y^i\) and rotate the coordinates such that \(\frac{\partial}{\partial x^n}\) is the unit inner normal vector at \(p\). Then, we perform an orthogonal change of coordinates in the tangential directions to arrange that
\[
\chi_{ij} = (\chi_0)_{ij} + u_{\bar{i}j} = \lambda'_{ij} \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq n - 1.
\]
Since \(\chi \in \Gamma_{k-1}(M)\), we have
\[
\chi_{n\bar{\pi}} + \sum_{i=1}^{n-1} \lambda'_i \geq 0,
\]
which implies
\[
\chi_{n\bar{\pi}} \geq -C.
\]
Thus, it remains to estimate \(\chi_{n\bar{\pi}}\) from above. We give the following lemma before starting the estimate.

**Lemma 5.1.** Let \(A = (a_{ij})\) be a \(n \times n\) Hermitian matrix and \(A' = (a'_{ij})\) be the \((n-1) \times (n-1)\) Hermitian matrix. Suppose that \(\lambda_1(A) \leq \cdots \leq \lambda_n(A)\) are the eigenvalues of \(A\) and \(\lambda'_1(A') \leq \cdots \leq \lambda'_{n-1}(A')\) are the eigenvalues of \(A'\). Then we have
\[
\text{(5.1)} \quad \lambda_j(A) \leq \lambda'_j(A') \leq \lambda_{j+1}(A), \quad 1 \leq j \leq n - 1,
\]
and
\[
\text{(5.2)} \quad \begin{cases}
\lambda_j(A) = \lambda'_j(A') + o(1), & 1 \leq j \leq n - 1, \\
\lambda_{n-1}(A) \leq \lambda_{n-1}(A') + o(1), & \text{when} \ A = A'.
\end{cases}
\]
as \(|a_{nn}| \to +\infty\).

**Proof.** (5.1) can be obtained by the Cauchy’s interlace inequality (see for example [23]) and (5.2) is follows from [3, Lemma 1.2] \(\Box\)

We set

\[\Gamma_\infty := \{(\lambda_1, \cdots, \lambda_{n-1}) \mid (\lambda_1, \cdots, \lambda_{n-1}, \lambda_n) \in \Gamma_{k-1} \text{ for some } \lambda_n\},\]

and

\[f(\lambda(z)) := \frac{\sigma_k(\lambda(z))}{\sigma_{k-1}(\lambda(z))} - \sum_{l=0}^{k-1} \beta_l(z) \frac{\sigma_l(\lambda(z))}{\sigma_{k-1}(\lambda(z))}\]

for any continuous function \(\lambda\). For any \((n-1) \times (n-1)\) Hermitian matrix \(E\) with \(\lambda'(E) \in \Gamma_\infty\), we define

\[\widetilde{G}(E) = f_\infty(\lambda'(E)) := \lim_{\lambda_n \to \infty} f(\lambda_1'(E), \cdots, \lambda_{n-1}'(E), \lambda_n).\]

**The upper bound estimate of \(\chi_{n\overline{m}}\).** For simplicity, we denote by \(\lambda = \lambda(\chi_u(z))\) and \(\lambda' = \lambda'(\chi''_u(z))\), where \(\chi_u(z) = ((\chi_u)_{ij})_{1 \leq i, j \leq n-1}\). The proof is devided into two claims:

**Claim 1:**

\[(5.3) \quad P_\infty := \min \limits_{z \in \partial M} \left(\widetilde{G}(\chi''_u(z)) - \beta(z)\right) > c_0\]

for some uniform constant \(c_0\).

We assume that \(P_\infty = \widetilde{G}(\chi''_u(z_0)) - \beta(z_0)\) at point \(z_0 \in \partial M\). Since \(\lambda(\chi_u) \in \Gamma_{k-1}\) and \(\sigma_k(\lambda) = \lambda_{n}\sigma_{k-1}(\lambda \mid i) + \sigma_k(\lambda \mid i)\), then

\[\widetilde{G}(\chi''_u(z)) = \frac{\sigma_{k-1}}{\sigma_{k-2}}(\chi''_1, \cdots, \chi''_{n-1}) - \sum_{l=1}^{k-2} \beta_l(z) \frac{\sigma_{l-1}}{\sigma_{k-2}}(\chi''_1, \cdots, \chi''_{n-1}).\]

Denote \(\widetilde{G}^{\overline{ij}}_0 := \frac{\partial \widetilde{G}}{\partial (\chi''_u)_{ij}}(\chi''_u(z_0))\) for \(1 \leq i, j \leq n-1\). By the concavity of \(G\) and \(\widetilde{G}\), we have

\[\widetilde{G}^{\overline{ij}}_0((\chi''_{ij})_{ij}(z) - (\chi''_{ij})_{ij}(z_0))\]

\[= \left(\frac{\sigma_{k-1}}{\sigma_{k-2}}\right)^{\overline{ij}} \mid_{z_0} (\chi''_{ij}(z) - (\chi''_{ij})_{ij}(z_0)) + \sum_{l=1}^{k-2} \beta_l(z_0) \left(\frac{\sigma_{l-1}}{\sigma_{k-2}}\right)^{\overline{ij}} ((\chi''_{ij}(z) - (\chi''_{ij})_{ij}(z_0)))\]

\[\geq \frac{\sigma_{k-1}}{\sigma_{k-2}}(\chi''_u(z)) - \frac{\sigma_{k-1}}{\sigma_{k-2}}(\chi''_u(z_0)) + \sum_{l=1}^{k-2} \beta_l(z_0) \left(\frac{\sigma_{l-1}}{\sigma_{k-2}}(\chi''_u(z_0)) + \frac{\sigma_{l-1}}{\sigma_{k-2}}(\chi''_u(z))\right)\]

\[\geq \widetilde{G}(\chi''_u(z)) - C \sum_{l=1}^{k-2} |\beta_l| |C^1| |z - z_0|,\]
\[
\begin{align*}
\tilde{G}_0^{ij}(\chi'_u)_{ij}(z) - \beta(z) - \tilde{G}_0^{ij}(\chi'_u)_{ij}(z_0) + \beta(z_0) \\
\geq \tilde{G}(\chi'_u(z)) - \beta(z) - c_0 - C \sum_{l=1}^{k-2} |\beta_l|_{C^1} |z - z_0| \\
(5.4) \geq -C \sum_{l=1}^{k-2} |\beta_l|_{C^1} |z - z_0|.
\end{align*}
\]

Note that
\[
(5.5) \quad c_\infty := \min_{z \in \partial M} (\tilde{G}(\chi'_u(z)) - G(\chi_u(z))) > 0.
\]

Using (4.4), we have
\[
(\chi_u)_{ij}(z_0) = (\chi_u)_{ij}(z_0) - \partial_n (u - \bar{u})(z_0) \zeta_{ij}(z_0)
\]
for any \(1 \leq i, j \leq n - 1\). Then
\[
(5.6) \quad \partial_n (u - \bar{u})(z_0) \sum_{i,j=1}^{n-1} \zeta_{ij}(z_0) \tilde{G}_0^{ij}(z_0) = \tilde{G}_0^{ij} ((\chi'_u)_{ij}(z_0) - (\chi'_u)_{ij}(z_0)) \\
\geq \tilde{G}(\chi'_u(z_0)) - \tilde{G}(\chi_u(z_0)) \\
= \tilde{G}(\chi'_u(z_0)) - \beta(z_0) - c_0 \\
\geq \tilde{G}(\chi_u(z_0)) - G(\chi_u(z_0)) - c_0 \\
\geq c_\infty - c_0.
\]

Consequently, if \(\partial_n (u - \bar{u})(z_0) \sum_{i,j=1}^{n-1} \zeta_{ij}(z_0) \tilde{G}_0^{ij}(z_0) \leq \frac{c_\infty}{2}\), then \(c_0 \geq \frac{c_\infty}{2}\) and we are done.

Suppose now that
\[
\partial_n (u - \bar{u})(z_0) \sum_{i,j=1}^{n-1} \zeta_{ij}(z_0) \tilde{G}_0^{ij}(z_0) \geq \frac{c_\infty}{2}.
\]

Let \(\eta(z) = \sum_{i,j=1}^{n-1} \zeta_{ij}(z) \tilde{G}_0^{ij}(z)\). Note that
\[
\eta(z_0) \geq \frac{c_\infty}{2\partial_n (u - \bar{u})(z_0)} \geq 2\epsilon_1 c_\infty
\]
for some uniform \(\epsilon_1 > 0\) since (3.1). We may assume that
\[
\eta \geq \epsilon_1 c_\infty \quad \text{on } \bar{M}_\delta(z_0)
\]
by requiring $\delta$ small, where $M_\delta(z_0) := \{z \in M \mid \text{dist}_{g_0}(z, z_0) < \delta\}$. We consider the function
\[
\Phi(z) = -\partial_{x^n}(u - u_\nu)(z) + \frac{1}{\eta(z)} \sum_{i,j=1}^{n-1} G^i_0 \left((\chi'_{u})_{ij}(z) - (\chi'_{u})_{ij}(z_0)\right) - \frac{\beta(z) - \beta(z_0)}{\eta(z)}
+ \frac{C}{\eta(z)} \sum_{l=1}^{k-2} |\beta_l| |z - z_0|.
\]
We deduce from (5.4) and the fact $(\chi_{u})_{ij}(z) = (\chi'_{u})_{ij}(z) - \partial_{x^n}(u - u_\nu)(z_0)\zeta_{ij}(z)$ on $\partial M \cap \overline{M_\delta(z_0)}$ that
\[
\Phi(z_0) = 0, \quad \Phi(z) \geq 0 \quad \forall z \in \partial M \cap \overline{M_\delta(z_0)},
\]
while by (4.7) and (4.13),
\[
G^{ij} \Phi_{ij} \leq C \sqrt{K}(1 + \sum_i G^{ii}) + C \sum_i G^{ii} |\lambda_i(\chi_u)|^2).
\]
Consider the function $\tilde{\Psi}$ defined in (4.26), it follows from (4.6), (4.7), (4.16), (4.21) and (4.22) that
\[
\begin{cases}
G^{ij}(\Phi + \tilde{\Psi})_{ij} \leq 0, & \text{in } M_\delta(z_0), \\
\Phi + \tilde{\Psi} \geq 0 & \text{on } \partial M_\delta(z_0).
\end{cases}
\]
The maximum principle yields that $\Phi + \tilde{\Psi} \geq 0$ in $\partial M_\delta(z_0)$, and then $\partial_{x^n} \Phi(z_0) = -\phi_{\nu}(z_0) \geq \tilde{\Psi}_{\nu}(z_0)$. This, together with the definition of $\Phi$, yields that
\[
\partial_{x^n} \partial_{x^n} u(z_0) \leq CK.
\]
Since
\[
u_{\bar{a}n} = \frac{1}{4} \left(\frac{\partial}{\partial x^n} + \sqrt{-1} \frac{\partial}{\partial y^n}\right)\left(\frac{\partial}{\partial x^n} - \sqrt{-1} \frac{\partial}{\partial y^n}\right)u,
\]
it follows that $\lambda(\chi_u)(z_0)$ is contained in a compact subset of $\Gamma$. Therefore
\[
P_\infty \geq f'(\lambda'_{\chi_{u}}(z_0), R) - \beta(z_0) > 0
\]
for $R$ sufficiently large since $f_i > 0$, which yields (5.3).

**Claim 2:** There holds
\[
\begin{equation}
(5.7)
\chi_u\pi(z) \leq CK \quad z \in \partial M
\end{equation}
\]
for some constant $C$.

Let $z \in \partial M$, we suppose that $\lambda_1 \leq \cdots \leq \lambda_n$ and $\lambda'_1 \leq \cdots \leq \lambda'_{n-1}$. From the boundary tangential-tangential and tangential-normal estimates, it follows that $\lambda'(\chi'_u)$ lies in a compact set $L \subset \Gamma_\infty$. Combining with (5.3), we know that there exist uniform positive constant $c_0$ and $R_0$ depending on the range of $\lambda'(\chi'_u)$ such that for any $R > R_0$
\[
f(\lambda', R) > \sup_{M} \beta + c_0,
\]
which implies
\[(5.8)\quad f(\lambda, R) > \sup_M \beta + \frac{c_0}{2}\]
for any \(\lambda \in U_L\) and \(R > R_0\), where \(U_L\) is the neighborhood of \(L\).

Assuming that there exists a constant \(R_1 > R_0\) such that \(\chi_{n\bar{n}}(z) \geq R_1\).

According to (5.1) and (5.2), it is easy to see that
\[
\lambda_n \geq \chi_{n\bar{n}}(z) \geq R_0, \quad 1 \leq j \leq n - 1.
\]
Combining with (5.8), we know that \(G(\chi_u(z)) = f(\lambda) > \beta(z) + \frac{c_0}{2}\), This contradicts with the equation (2.9), and hence (5.7) holds. \(\Box\)

Combining this with the second order interior estimate, i.e., Theorem 3.2, we have

**Theorem 5.1.** Let \(u \in C^\infty(\bar{M})\) be an \((k - 1)\)-admissible solution for equation (1.1). Under the assumptions mentioned in Theorem 1.1, then there exists a positive constant \(C\) depending only on \((M, \omega), \chi_0, \alpha, \varphi\) and the subsolution \(u\) such that
\[
\sup_M |\sqrt{-1} \partial \bar{\partial} u| \leq C K,
\]
where \(K := 1 + \sup_M |\nabla u|^2\).

6. **Gradient estimates**

In this section, we combine the second derivative estimate with a blow-up argument and Liouville type theorem due to Dinew-Kolodziej [12] to obtain gradient estimates.

**Theorem 6.1.** Let \(u \in C^\infty(\bar{M})\) be an \((k - 1)\)-admissible solution for equation (1.1). Under the assumptions mentioned in Theorem 1.1, then there exists a positive constant \(C\) depending only on \((M, \omega), \chi_0, \alpha, \varphi\) and the subsolution \(u\) such that
\[
\sup_M |\nabla u| \leq C.
\]

**Proof.** Suppose that there exists a sequence of function \(u_m \in C^4(\bar{M})\) to the equation (1.1) such that
\[
N_m = \sup_M |\nabla u_m| \to +\infty \quad \text{as} \quad m \to +\infty.
\]

Hence
\[(6.1)\quad \chi_{u_m}^k \wedge \omega^{n-k} = \sum_{l=0}^{k-1} \alpha_l(z) \chi_{\bar{u}_m}^l \wedge \omega^{n-l}, \quad u = \varphi \text{ on } \partial M.
\]

For any \(m\), we assume that \(|\nabla u_m|\) attains its maximum value at \(z_m \in M\).

Then, after passing a subsequence we may assume that \(z_m \to z'\) for some point \(z' \in M\). By Theorem 5.1 we have
\[(6.2)\quad \sup_M |\sqrt{-1} \partial \bar{\partial} u_m| \leq C(1 + N_m^2),
\]
where the constant independs of \( m \). we divide our proof into two cases separately.

**Case 1:** \( z' \in \overset{\circ}{M} \).

Choosing a small coordinate ball centered at \( z' \), which we identify with an open set in \( \mathbb{C}^n \) with coordinates \( (z^1, \cdots, z^n) \), and such that \( \omega(0) = \omega_0 = \sqrt{-1} \delta_{ij} dz^i \wedge d\bar{z}^j \). We can assume that all \( z_m \) are within this coordinate ball. Let \( R > 0 \), we define

\[
v_m(z) := u_m\left(\frac{z}{N_m} + z_m\right), \quad \forall z \in B_R(0),
\]

which is well-defined for any \( m \) such that \( N_m \) is large enough. Clearly,

\[
|\nabla v_m(0)| = 1, \quad |v_m|_{C^2(B_R(0))} \leq C.
\]

Let \( R \to +\infty \) and taking a diagonal subsequence again, we can assume that \( v_m \to v \) in \( C^1_{\text{loc}}(\mathbb{C}^n) \) with \( \nabla v(0) = 1 \). Then we have from (6.1)

\[
\left[ \chi_0\left(\frac{z}{N_m} + z_m\right) + N_m^2 \frac{\sqrt{-1}}{2} \partial \bar{\partial} v_m \right]^k \wedge \left[ \omega\left(\frac{z}{N_m} + z_m\right) \right]^{n-k} = \sum_{l=0}^{k-1} \alpha_l \left( \frac{z}{N_m} + z_m \right)^l \left[ \chi_0\left(\frac{z}{N_m} + z_m\right) + N_m^2 \frac{\sqrt{-1}}{2} \partial \bar{\partial} v_m \right]^l \wedge \left[ \omega\left(\frac{z}{N_m} + z_m\right) \right]^{n-l}.
\]

We can take a limit of the equation and obtain

\[
(\sqrt{-1} \partial \bar{\partial} v)^k \wedge \omega_0^{n-k} = 0,
\]

which is in the pluripotential sense. Moreover, we have for any \( 1 \leq l \leq k - 1 \) by a similar reasoning

\[
(\sqrt{-1} \partial \bar{\partial} v)^l \wedge \omega_0^{n-l} \geq 0.
\]

Combining with the result of Blocki [2], we know that \( v \) is a maximal \( k \)-subharmonic function in \( \mathbb{C}^n \). Then the Liouville theorem in [12] implies that \( v \) is a constant, which contradicts the fact \( \nabla v(0) = 1 \).

**Case 2:** \( z' \in \partial M \).

Let \( \Omega \subset M \) be a coordinate chart centered at \( z' \). Then there exists a smooth function \( \rho: B_{2s} \to \mathbb{R} \) such that

\[
\partial M \cap \Omega = \{ \rho = 0 \}, \quad M \cap \Omega \subset \{ \rho \leq 0 \}.
\]

Where \( B_{2s} \subset \mathbb{C}^{2n} \) is the ball of radius \( 2s \) at \( 0 = z' \). Without loss of generality, we may assume \( z_m \in \Omega \) with \( |z_m| < s \) in local coordinates and

\[
r_m := \text{dist}(z_m, \partial M \cap \Omega) = |z_m - y_m|
\]

for a unique point \( y_m \in \partial M \cap \Omega \). Clearly, \( y_m \to z \) as \( m \to +\infty \). Set

\[
v_m(z) := u_m\left(\frac{z}{M_m} + z_m\right), \quad z \in \Omega_m,
\]

where \( \Omega_m := \{ z \in \Omega \mid \frac{z}{M_m} + z_m \in B_{2s} \cap \{ \rho \leq 0 \} \} \). Therefore

\[
|\nabla v_m(0)| = 1, \quad |v_m|_{C^2(\Omega_m)} \leq C.
\]
Let $\rho_m := \rho\left(\frac{z_m}{N_m} + z_m\right)$, then $B_{sN_m} \cap \{\rho_m \leq 0\} \subset \Omega_i$ since $|z_m| < s$. By the standard elliptic theory, we know that

(6.3) \[ |v_m|_{C^{2,\gamma}(\Omega_i)} \leq C. \]

Then we divide the proof into the following two sub-cases

Case 2.1

\[ \liminf_{m \to \infty} N_m r_m = +\infty. \]

After passing to a subsequence $v_m$ converges in $C^{2,\gamma}$ on compact sets to $v \in C^{2,\gamma}(\mathbb{C}^n)$ since $N_m r_m \to \infty$ as $m \to \infty$. Similar to Case 1, we know that $v$ is a constant, which contradicts the fact $\nabla v(0) = 0$.

Case 2.2

\[ \liminf_{m \to \infty} N_m r_m = L \in [0, \infty). \]

The proof for Case 2.2 is quite similar to that given for Case 2b in [11, Proposition 6.1], and so they are omitted.

Last we apply the standard continuity method to solve the Dirichlet problem.

**Proof of Theorem 1.1.** For any $t \in [0, 1]$, we consider the equation

(6.4) \[
\begin{cases}
G(\chi_{u_t}) = t\beta + (1-t)G(\chi_u), & \text{in } M, \\
\quad u_t = \varphi & \text{on } \partial M,
\end{cases}
\]

where

\[
G(\chi_{u_t}) = \frac{\sigma_k(\chi_{u_t})}{\sigma_{k-1}(\chi_{u_t})} - \sum_{l=0}^{k-2} \beta_l(z) \frac{\sigma_l(\chi_{u_t})}{\sigma_{k-1}(\chi_{u_t})}.
\]

Set

\[ S = \{t \in [0, 1] \mid \text{there exists } u_t \in C^{4,\alpha}(\bar{M}) \text{ with } \lambda(\chi_{u_t}) \in \Gamma_{k-1} \text{ solving (6.4)}\}. \]

Clearly the set $S$ is non-empty since $u$ solve the equation (6.4) for $t = 0$. On the one hand, by the generalized Newton-MacLaurin inequality, we have

\[ \sigma_{k-1}(\chi_{u_t}) \geq C, \]

which implies that the equation (6.4) is uniformly elliptic. Therefore $S$ is an open set by the implicit function theorem.

On the other hand, let $u_{t_i}$ be the solution of (6.4) for $t_i \in [0, 1]$ with $t_i \to t_0$. From 3.1, theorem 5.1, 5.1 and 6.1, we know that

\[ |u_{t_i}|_{C^{2}(M)} \leq C \]

for some uniformly constants. Then, higher-order estimates follow from the Evan-Krylov theorem and the Schauder estimate, i.e.,

\[ |u_{t_i}|_{C^{4,\alpha}(M)} \leq C. \]
We can take convergent subsequence to a limiting function $\bar{u} \in C^{4,\alpha}$ which solves the equation (6.4) for $t_0$. Hence $S$ is also closed and we complete the proof of Theorem 1.1.

References

[1] B. Andrews, Contraction of convex hypersurfaces in Euclidean space, Calc. Var. Partial Differ. Equ., 2 (1994), 151-171.
[2] Z. Blocki, Weak solutions to the complex Hessian equation, Ann. Inst. Fourier (Grenoble), 55 (2005), no. 5, 1735-1756.
[3] L. Caffarelli, L. Nirenberg, J. Spruck, Dirichlet problem for nonlinear second order elliptic equations III, Functions of the eigenvalues of the Hessian, Acta Math., 155 (1985), 261-301.
[4] C. Q. Chen, L. Chen, X. Q. Mei, N. Xiang, The Classical Neumann Problem for a class of mixed Hessian equations, To appear in Studies in Applied Mathematics, 2021.
[5] C. Q. Chen, L. Chen, X. Q. Mei, N. Xiang, The Neumann problem for a class of mixed complex Hessian equations, Preprint, 2019.
[6] L. Chen, Hessian equations of Krylov type on Kähler manifolds, Preprint, arXiv:2107.12035, 2021.
[7] L. Chen, A. G. Shang, Q. Tu, A class of prescribed Weingarten curvature equations in Euclidean space, Comm. Partial Differential Equations, 46 (2021), no. 7, 1326-1343.
[8] X. X. Chen, On the lower bound of the Mabuchi energy and its application, Int. Math. Res. Not., 12 (2000), 607-623.
[9] P. Cherrier, A. Hanani, Le probleme de Dirichlet pour des equations de Monge-Ampere en metrique hermitienne, Bull. Sci. Math., 123 (1999), 577-597.
[10] T. Collins, G. Székelyhidi, Convergence of the J-flow on toric manifolds, J. Differential Geom., 107 (2017), no. 1, 47-81.
[11] T. Collins, S. Picard, The Dirichlet Problem for the k-Hessian Equation on a complex manifold, Preprint, arXiv:1909.00447, 2019.
[12] S. Dinew, S. Kolodziej, Liouville and Calabi-Yau type theorems for complex Hessian equations, Amer. J. Math., 139(2017), 403-415.
[13] H. Fang, M. J. Lai, X. N. Ma, On a class of fully nonlinear flows in Kähler geometry, J. Reine Angew. Math., 653 (2011), 189-220.
[14] K. Feng, H. B. Ge, T. Zheng, The Dirichlet problem of fully nonlinear equations on Hermitian manifolds, Preprint, arXiv:1905.02412v4, 2020.
[15] D. Gu, N.C. Nguyen, The Dirichlet problem for a complex Hessian equation on compact Hermitian manifolds with boundary, Annali della Scuola Normale Superiore di Pisa Classe di scienze, 18 (2018), 1189-1248.
[16] B. Guan, Q. Li, Complex Monge-Ampère equations and totally real submanifolds, Adv. Math., 225 (2010), no. 3, 1185-1223.
[17] B. Guan, Q. Li, A Monge-Ampère type fully nonlinear equation on Hermitian manifolds, Discrete Contin. Dyn. Syst., 17 (2012), 1991-1999.
[18] B. Guan, Q. Li, The Dirichlet problem for a complex Monge–Ampère type equation on Hermitian manifolds, Adv. Math., 246 (2013), 351-367.
[19] B. Guan, Second order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds, Duke Math. J., 163 (2014), 1491–1524.
[20] B. Guan, W. Sun, On a class of fully nonlinear elliptic equations on Hermitian manifolds, Calc. Var. Partial Differential Equations, 54 (2015), no. 1, 901-916.
[21] P. F. Guan, X. W. Zhang, A class of curvature type equations. Pure and Applied Math Quarterly, 17 (2021), No. 3, 865-907.
[22] Z. Hou, X.N. Ma, D. M. Wu, A second order estimate for complex Hessian equations on a compact Kähler manifold, Math. Res. Lett., 17 (2010), no. 3, 547-561.
[23] S. Hwang, Cauchy’s interlace theorem for eigenvalues of Hermitian matrices, American Mathematical Monthly, 111 (2004), 157–159.
[24] N. V. Krylov, On the general notion of fully nonlinear second order elliptic equation, Trans. Amer. Math. Soc., 347 (1995), 857-895.
[25] G. Lieberman, Second order parabolic differential equations, World Scientific, 1996.
[26] Vamsi P. Pingali, A fully nonlinear generalized Monge-Ampère PDE on a torus, Electron. J. Differential Equations, 211 (2014), pp.
[27] Vamsi P. Pingali, A generalised Monge-Ampère equation, J. Partial Differ. Equ., 27 (2014), no. 4, 333-346.
[28] Vamsi P. Pingali, A priori estimates for a generalized Monge-Ampère PDE on some compact Kähler manifolds, Complex Var. Elliptic Equ., 61 (2016), no. 8, 1037-1051.
[29] D. H. Phong, D. T. Tô, Fully non-linear parabolic equations on compact Hermitian manifolds, Ann. Scient. Ec. Norm. Sup., 54 (2021), no.3, 793-829.
[30] G. Székelyhidi, Fully non-linear elliptic equations on compact Hermitian manifolds, J. Differ. Geom., 109 (2018), 337-378.
[31] G. Székelyhidi, V. Tosatti, B. Weinkove, Gauduchon metrics with prescribed volume form, Acta Math., 219 (2017), no. 1, 181-211.
[32] J. Song, B. Weinkove, On the convergence and singularities of the J-flow with applications to the Mabuchi energy, Comm. Pure Appl. Math., 61 (2008), 210-229.
[33] J. Spruck, Geometric aspects of the theory of fully nonlinear elliptic equations, Clay Mathematics Proceedings, 2 (2005), 283-309.
[34] W. Sun, On a class of fully nonlinear elliptic equations on closed Hermitian manifolds II: $L^\infty$ estimate, Commun. Pure Appl. Math., 70 (2017), 172-199.
[35] W. Sun, Generalized complex Monge-Ampère type equations on closed Hermitian manifolds, Preprint, arXiv:1412.8192.
[36] W. Sun, Parabolic Flow for Generalized complex Monge-Ampère type equations, Preprint, arXiv:1501.04255.
[37] V. Tosatti, B. Weinkove, The complex Monge-Ampère equation with a gradient term, Pure and Applied Mathematics Quarterly, 17 (2021), 1005-1024.
[38] V. Tosatti, B. Weinkove, Hermitian metrics, $(n-1, n-1)$ forms and Monge-Ampère equations, J. Reine Angew. Math., 755 (2019), 67-101.
[39] V. Tosatti, B. Weinkove, Estimates for the complex Monge-Ampère equation on Hermitian and balanced manifolds, Asian J. Math., 14 (2010), 19-40.
[40] V. Tosatti, B. Weinkove, The complex Monge-Ampère equation on compact hermitian manifolds, J. Amer. Math. Soc., 23 (2010), 1187-1195.
[41] S.T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I, Comm. Pure Appl. Math., 31 (1978), 339-411.
[42] D. K. Zhang, Hessian equations on closed Hermitian manifolds, Pac. J. Math., 291 (2017), 485-510.
[43] Q. Zhang, Regularity of the Dirichlet Problem for the Non-degenerate Complex Quotient Equations, Int. Math. Res. Not., 23 (2021), 17673-17694.
[44] J. D. Zhou, A class fo the non-degenerate complex quotient equations on compact Kähler manifolds, Comm. Pure Appl. Anal., 20 (2021), 2361-2377.
[45] J. D. Zhou, The interior gradient estimate for a class of mixed Hessian curvature equations, J. Korean Math. Soc. 59 (2022), 53-69.