INJECTIVE ENVELOPES AND LOCAL MULTIPLIER ALGEBRAS OF C*-ALGEBRAS

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Abstract. The local multiplier C*-algebra $\mathcal{M}_{\text{loc}}(A)$ of any C*-algebra $A$ can be $\ast$-isomorphically embedded into the injective envelope $I(A)$ of $A$ in such a way that the canonical embeddings of $A$ into both these C*-algebras are identified. If $A$ is commutative then $\mathcal{M}_{\text{loc}}(A) \equiv I(A)$. The injective envelopes of $A$ and $\mathcal{M}_{\text{loc}}(A)$ always coincide, and every higher order local multiplier C*-algebra of $A$ is contained in the regular monotone completion $\overline{A} \subseteq I(A)$ of $A$. For C*-algebras $A$ with a center $Z(A)$ such that $Z(A) \circ A$ is norm-dense in $A$ the center of the local multiplier C*-algebra of $A$ is the local multiplier C*-algebra of the center of $A$, and both they are $\ast$-isomorphic to the injective envelope of the center of $A$. A Wittstock type extension theorem for completely bounded bimodule maps on operator bimodules taking values in $\mathcal{M}_{\text{loc}}(A)$ is proven to hold if and only if $\mathcal{M}_{\text{loc}}(A) \equiv I(A)$. In general, a solution of the problem for which C*-algebras $\mathcal{M}_{\text{loc}}(A)$ is injective is shown to be equivalent to the solution of I. Kaplan- sky’s 1951 problem whether all AW*-algebras are monotone complete.

The injective envelope of a C*-algebra in the category of C*-algebras and completely positive linear maps is defined by an extrinsic algebraic characterization. M. Hamana showed its general existence and uniqueness up to $\ast$-isomorphism, cf. [15]. The main problem is to determine the injective envelope $I(A)$ of a given C*-algebra $A$ from the structure of $A$, i.e. intrinsically. For commutative C*-algebras an intrinsic characterization was given by H. Gonshor in [3, 4]. He relied on I. M. Gel’fand’s theorem for commutative C*-algebras and on the topology of the locally compact Hausdorff space $X$ corresponding to a commutative C*-algebra $A = C_0(X)$. Unfortunately, there seems to be no obvious way to extend his results to the non-commutative case. The injective envelope of non-commutative C*-algebras has been described only for some examples and special classes of C*-algebras yet.

One of the goals of this short note is an intrinsic algebraic characterization of the injective envelopes of commutative C*-algebras identifying them with their local multiplier algebras. In the present paper we emphasize the topological approach which relies on strict topologies. For the more algebraic point of

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In the general non-commutative case we show the existence of a canonical injective \(*\)-homomorphism mapping the local multiplier algebra into the corresponding injective envelope (Th. 1). The question whether the image of this map coincides with the entire injective envelope has a negative answer, in general, and is connected with I. Kaplansky’s still unsolved problem on the monotone completeness of AW*-algebras in the case of AW*-algebras. So this class of C*-algebras cannot be characterized in full at present. We indicate some examples (Cor. 7). The center of the local multiplier C*-algebra of \(A\) always coincides with the local multiplier C*-algebra of the center \(Z(A)\) of \(A\), and both they can be identified with the injective envelope of the center of \(A\), in case \(Z(A) \circ A\) is dense in \(A\), i.e. \(Z(M_{loc}(A)) \equiv M_{loc}(Z(A)) \equiv I(Z(A))\) in that situation (Th. 2). Moreover, the injective envelopes of \(A\) and \(M_{loc}(A)\) are always \(*\)-isomorphic, and every higher order local multiplier C*-algebra of \(A\) is contained in the regular monotone completion \(\overline{A} \subseteq I(A)\) of \(A\) (Th. 5).

Additionally, we establish extension properties of completely bounded maps from operator spaces or operator (bi)modules into the (local) multiplier C*-algebra of a certain C*-algebra (Prop. 4).

1. Preliminaries

A C*-algebra \(B\) is said to be injective if given any self-adjoint linear subspace \(S\) of a unital C*-algebra \(C\) with \(1 \in S\) any completely positive linear map of \(S\) into \(B\) extends to a completely positive linear map of \(C\) into \(B\), cf. \([15, \text{Def. 2.1}]\). Equivalently, \(B\) is injective if and only if it is an injective object of the category consisting of C*-algebras as objects and completely positive linear maps as morphisms. For any C*-algebra \(A\) we can define an injective envelope in the following way (cf. \([15]\)): let \(B \supseteq A\) be an injective C*-algebra such that \(A\) is embedded into \(B\) as a C*-subalgebra. If the identity map on \(A\) possesses a unique extension to a completely positive linear map on \(B\) then \(B\) is said to be an injective envelope \(I(A)\) of \(A\). The injective envelope \(I(A)\) of a C*-algebra \(A\) is uniquely determined up to \(*\)-isomorphism, \([15, \text{Th. 4.1}]\). Injective C*-algebras are monotone complete, and so is \(I(A)\), i.e. any bounded increasingly directed net of self-adjoint elements of \(I(A)\) admits a least upper bound in \(I(A)\) (cf. \([4]\)). In particular, \(I(A)\) is unital. Note that the injective envelope of a non-unital C*-algebra equals the injective envelope of its unitization. A C*-algebra \(B\) containing \(A\) as a C*-subalgebra equals \(I(A)\) if and only if \(B\) is injective and the restriction of any completely isometric linear map \(\phi : B \to I(A)\) to \(A \subseteq B\) is completely isometric again, cf. \([15, \text{Prop. 4.7}]\). For more information on injective envelopes of C*-algebras in various categories we refer to \([15, 6, 12]\).

Injective commutative C*-algebras \(A = C(X)\) can be characterized easily: \(A = C(X)\) is injective if and only if \(X\) is stonean (\([14, \text{Th. 7.3}]\)). This follows also from \([25, \text{Th. 25.5.1}]\) and the fact that bounded linear maps between C*-algebras are positive whenever their norm equals the norm of their evaluation at the identity of the C*-algebra. What is more, by \([15, \text{Th. 6.3}]\) the equality

\[ \text{view see } [12]. \]
$x = \sup\{a \in A_{sa} : a \leq x\}$ holds for every self-adjoint element $x$ of the injective envelope $I(A)$ of a commutative $C^*$-algebra $A$. Since for $x \in I(A)_{sa}$ and $A$ commutative the inequality $a \leq x$ for some $a \in A_{sa}$ implies $0 \leq (a \vee 0) \leq x$ the set $\{a \in A_{sa}^+ : a \leq x\}$ is non-trivial for any positive $x \in I(A)$, and the supremum of this set equals exactly $x$. By \cite{17, Lemma 1.7} the latter condition is equivalent to the assertion that every non-zero projection of the injective envelope $I(A)$ of a commutative $C^*$-algebra $A$ majorizes some non-zero positive element of $A$. Analyzing this inner order structure of an injective envelope $I(A)$ of a unital commutative $C^*$-algebra $A = C(X)$ H. Gonshor identified $I(A)$ with the $C^*$-algebra $B(X)$ of all bounded Borel functions on $X$ modulo the ideal of all functions supported on meager subsets of $X$, cf. \cite{14, Th. 1}. The background of this coincidence is that every remainder class of $B(X)$ contains a lower semi-continuous on $X$ function (\cite{10, Lemma 7.5}).

The second main construction in the present paper is also essentially an algebraic one - the local multiplier algebra $M_{loc}(A)$ of a given $C^*$-algebra $A$. A norm-closed one-(resp., two-)sided ideal $I$ of $A$ is said to be essential if its intersection with any other one-(resp., two-)sided ideal of $A$ is non-trivial. For commutative $C^*$-algebras $A = C_0(X)$ an ideal $I$ is essential if and only if there exists a closed meager subset $S_I \subseteq X$ such that $I = \{f \in C_0(X) : f$ is continuous on $X \setminus S_I\}$, where $X$ is the locally compact Hausdorff space corresponding to $A$ by I. M. Gelfand’s theorem. If for an arbitrary $C^*$-algebra $A$ and any pair of essential two-sided ideals $J \subseteq I \subseteq A$ their multiplier $C^*$-algebras $M(J) \supseteq M(I) \supseteq M(A)$ are considered then the induced partial order of canonical embeddings of those multiplier algebras $\{M(I) : I$ essential two-sided in $A\}$ gives rise to the direct limit of this partially ordered set:

$$Q_b(A) = \text{alg lim} \{M(I) : I$ essential two-sided ideal in $A\}$$

Performing the construction in the set $B(H)$ of all bounded linear operators on a Hilbert space $H$ wherein $A$ is faithfully represented, the norm-closure $M_{loc}(A)$ of $Q_b(A)$ becomes a unital $C^*$-algebra. It is said to be the local multiplier algebra of $A$, cf. \cite{11, 20, 1}. Note that the local multiplier algebra of a non-unital $C^*$-algebra equals the local multiplier algebra of its unitization. If $A = C(X)$ is commutative then $M_{loc}(C(X))$ can be identified with the set of all continuous functions over the inverse limit $\lim \leftarrow \{\beta U : U$ open dense in $X\}$ which is a compact Hausdorff space, \cite{11, Th. 1}. Moreover, it is a commutative AW*-algebra by \cite{11, Th. 1}, and so it is monotone complete by \cite{11, Prop. 2.2}. For more information on local multiplier algebras we refer to \cite{11, 2, 24, 25}.

2. Results

The connection between the local multiplier algebra and the injective envelope of $C^*$-algebras is described by the following theorem:
Theorem 1: Let $A$ be a (unital) C*-algebra, $I(A)$ be its injective envelope and $M_{loc}(A)$ be its local multiplier algebra. Then $M_{loc}(A)$ can be canonically identified with a C*-subalgebra of $I(A)$, i.e. the canonical embedding $A \hookrightarrow I(A)$ extends to an isometric algebraic monomorphism $M_{loc}(A) \hookrightarrow I(A)$. If $A$ is commutative then $M_{loc}(A)$ coincides with $I(A)$.

Proof: Without loss of generality we assume $A$ to be unital. The C*-algebra $M_{loc}(A)$ is an operator $A$-$A$ bimodule by its construction, and $A$ is a C*-subalgebra of $M_{loc}(A)$. Consider the canonical embedding of $A$ into $I(A)$. By the injectivity of $I(A)$ and by G. Wittstock’s extension theorems for completely bounded $A$-$A$ bimodule maps ([26]) this canonical embedding $A \subseteq I(A)$ extends to a completely contractive $A$-$A$ bimodule map $\psi : M_{loc}(A) \rightarrow I(A)$ with the same complete boundedness norm one, see [26, Th. 3.1], [3]. Let us show that this extension is actually an isometric algebraic embedding of $M_{loc}(A)$ into $I(A)$.

Consider an essential two-sided ideal $I \subseteq A$ and its multiplier algebra $M(I) \subseteq M_{loc}(A)$. Every element $x \in M(I)$ is representable as the strict limit of a net of elements $\{x_\alpha\}$ of $I$, i.e.

$$\lim_\alpha \|xy - x_\alpha y\| = \lim_\alpha \|yx - yx_\alpha\| = 0$$

for every $y \in I$. Therefore, inside $I(A)$ we obtain the equalities

$$\psi(x)y = \psi(xy) = \lim_\alpha \psi(x_\alpha y) = \lim_\alpha \psi(1_A)x_\alpha y = xy$$

$$yx_\alpha = \psi(yx) = \lim_\alpha \psi(yx_\alpha) = \lim_\alpha \psi(1_A)y x_\alpha = yx$$

for every $y \in I$, where all limits are limits in norm. So $\psi(x)$ is the strict limit of the net $\{x_\alpha\} \subset I \subseteq I(A)$ with respect to $I$ since $I(A)$ is an AW*-algebra and this strict limit is realized in a unique way as an element of $I(A)$ by [21, Th.]. Consequently, every multiplier algebra $M(I)$ of an essential two-sided ideal $I$ of $A$ is isometrically algebraically embedded by the existing map $\psi$ into $I(A)$, and the selected embedding preserves canonical containment relations between essential two-sided ideals of $A$, the C*-algebra $A$ itself and their multiplier algebras. So the map $\psi$ is actually an injective *-homomorphism of C*-algebras.

If $A$ is commutative then $M_{loc}(A)$ is a commutative AW*-algebra by [4, Th. 1]. Hence, $M_{loc}(A)$ is an injective C*-algebra, and by W. B. Arveson’s extension theorem ([3]) for completely positive maps applied to the identity map on $A$ and by the definition of injective envelopes and their uniqueness, the identity $I(A) \equiv M_{loc}(A)$ holds. □

By [17, Cor. 1.6] there are C*-algebras $A$ such that the injective envelope $I(Z(A))$ of the center $Z(A)$ of $A$ is different from the center $Z(I(A))$ of the injective envelope $I(A)$ of $A$. We identify $I(Z(A))$ with the center $Z(M_{loc}(A))$ of $M_{loc}(A) \subseteq I(A)$ that belongs to the center of $I(A)$ whenever $Z(I) \circ I$ is dense in $I$ for any essential two-sided ideal $I \subseteq A$. 

Theorem 2: Let $A$ be a $C^*$-algebra with a center $Z(A)$ such that $Z(A) \circ A$ is norm-dense in $A$. Then the center of the multiplier $C^*$-algebra of $A$ coincides with the multiplier $C^*$-algebra of the center of $A$, i.e. $Z(M(A)) \equiv M(Z(A))$. Moreover, $Z(I) \circ I$ is norm-dense in $I$ for any essential two-sided ideal $I$ of $A$, then the center of the local multiplier $C^*$-algebra of $A$ coincides with the local multiplier $C^*$-algebra of the center of $A$, i.e. $Z(M_{loc}(A)) \equiv M_{loc}(Z(A))$. Moreover, $M_{loc}(Z(A)) \equiv I(Z(A))$.

Proof: At the beginning we show that the center of the multiplier $C^*$-algebra of any $C^*$-algebra $B$ coincides with the multiplier $C^*$-algebra of its center, i.e. $Z(M(B)) \equiv M(Z(B))$, as soon as $Z(B) \circ B$ is norm-dense in $B$. The inclusion $M(Z(B)) \subseteq Z(M(B))$ is immediate. Indeed, $M(Z(B))$ consists of all strict limits of strictly converging norm-bounded nets of $Z(B)$. The multiplier $C^*$-algebra $M(B)$ is a Banach $Z(M(B))\cdot Z(M(B))$ bimodule such that both $Z(M(B)) \circ M(B)$ and $M(B) \circ Z(M(B))$ are norm-dense in $M(B)$. Hence, [22, Th. 4.1] applies saying that every element $x \in M(B)$ can be represented as $x = z_1 y_1$ and as $x = y_2 z_2$ for certain positive $z_1, z_2 \in Z(M(B))$ with $\|z\| \leq 1$ and certain $y_1, y_2 \in M(B)$. Let $\{z_\alpha\}$ be a norm-bounded strictly converging net of $Z(A)$. Then for any $x \in M(B)$ both the nets $\{z_\alpha x\}$ and $\{xz_\alpha\}$ converge in norm inside the Banach $Z(M(B))\cdot Z(M(B))$-module $M(B)$ since $z_\alpha x = (z_\alpha z_1) y_1$, $xz_\alpha = (z_\alpha z_2) y_2$ and the coefficients at the right sides are converging in norm. Therefore, every strictly converging norm-bounded net of $Z(B)$ preserves strict convergence if considered as a norm-bounded net of $M(B)$. So its strict limit belongs to $M(B)$ and, moreover, to $Z(M(B))$ since strict convergence preserves commutation relations in the limits.

Conversely, fix $t \in Z(M(B))$ and an approximate identity $\{s_\alpha\}$ of $Z(B)$. We have $(ts_\alpha)x = ts_\alpha x = x(s_\alpha t)$ for any $x \in B$. So $ts_\alpha = s_\alpha t \in Z(B)$ for any index $\alpha$. What is more, $(ts_\alpha)u = t(s_\alpha u)$ for any $u \in Z(B)$. The right side converges in norm, so does the left side showing the strict convergence of the net $\{ts_\alpha\} \in Z(B)$. Since $Z(B) \circ B$ is norm-dense in $B$ the net $\{ts_\alpha\} \in Z(B)$ converges strictly to the element $t \in M(Z(B))$.

The $C^*$-algebra $B$ can be replaced by any essential two-sided norm-closed ideal $I$ of the given $C^*$-algebra $A$ by assumption. The additional condition claimed above ensures that the idea of the first part of the present proof can be applied again. If two essential ideals $I, J$ of $A$ fulfill $I \subseteq J$ then their centers satisfy $Z(I) \subseteq Z(J)$. This can be seen switching to the respective multiplier algebras $M(J) \subseteq M(I)$. We derive

$$Z(\text{alg lim } M(I)) = \text{alg lim } Z(M(I)) = \text{alg lim } M(Z(I))$$

for the algebraic direct limit of the partially ordered net of multiplier $C^*$-algebras of these essential ideals $I \subseteq A$. If $J$ is an essential norm-closed ideal of $Z(A)$ then the norm-closure of $AJA$ is an essential two-sided norm-closed ideal of $A$ with center $J$. So at the right side the multiplier $C^*$-algebra of any
essential two-sided norm-closed ideal \( J \) of \( Z(A) \) appears, and we obtain

\[
Z(\text{alg lim} \to M(I)) = \text{alg lim} \to M(J)
\]

for \( I \subseteq A \) and \( J \subseteq Z(A) \) as described. Forming the norm-closure of both the sides of the latter equality we get \( M_{\text{loc}}(Z(A)) \) at the right side and \( Z(M_{\text{loc}}(A)) \) at the left by \([1\text{, Th. 1, Lemma 1, Cor. 1}]\). This shows the first set identity claimed. The other identity of C*-algebras is a simple consequence of Theorem 1 applied to the center \( Z(A) \) of the C*-algebra \( A \).

Theorem 2 is surely not true for \( A = K(H) \) and \( M_{\text{loc}}(A) = M(A) = B(H) \), where \( H = l_2 \) is the standard separable Hilbert space. Similar problems appear if one of the essential two-sided ideals \( I \) of a certain C*-algebra \( A \) has a center \( Z(I) \) for which \( Z(I) \circ I \) does not cover one or more block-diagonal components of \( I \). In connection with this phenomenon it would be interesting to know whether the additional conditions on \( A \) can be weakened for Theorem 2 to hold, or not. Unfortunately, we are not able to formulate any conjecture on the diversity of the appearing situations at present. Nevertheless, Theorem 2 shows the way to a new short proof of \([1\text{, Cor. 2}]\) in a particular situation:

**Corollary 3:** Let \( A \) be a C*-algebra with the property that \( Z(I) \circ I \) is norm-dense in \( I \) for any essential two-sided ideal \( I \) of \( A \). Then we have the set identity \( Z(M_{\text{loc}}(M_{\text{loc}}(A))) \equiv Z(M_{\text{loc}}(A)) \) of the respective centers, i.e. taking higher order local multiplier C*-algebras does not change the center any more.

**Proof:** By the Theorems 1 and 2 and by the local multiplier C*-algebra properties of injective C*-algebras we conclude

\[
Z(M_{\text{loc}}(M_{\text{loc}}(A))) \equiv M_{\text{loc}}(Z(M_{\text{loc}}(A))) \equiv M_{\text{loc}}(I(Z(A))) \\
\equiv I(Z(A)) \equiv Z(M_{\text{loc}}(A)),
\]

cf. \([21\text{, Th.}]\).

The observation formulated at Theorem 1 shows the way to a criterion on the coincidence of the (local) multiplier algebra of a given C*-algebra \( A \) with the injective envelope of \( A \) in terms of a Wittstock type extension theorem. Recall the definition of operator \( B\text{-}C \) bimodules \( B_\mathcal{M}C \) over unital C*-algebras \( B \) and \( C \), \([22, 3]\). Operator spaces \( \mathcal{M} \) can be characterized as norm-closed subspaces of C*-algebras. Then operator \( B\text{-}C \) bimodules are operator spaces \( \mathcal{M} \) which are \( B\text{-}C \) bimodules such that the trilinear module pairing \( B \times \mathcal{M} \times C \to \mathcal{M}, \ (b,x,c) \to bxc \) is completely contractive in the sense of E. Christensen and A. Sinclair \([8]\), i.e. it extends to be a completely contractive linear map on the Haagerup tensor product \( B \otimes_h \mathcal{M} \otimes_h C \). We may assume that all module actions are unital.

**Proposition 4:** Let \( A \) be a C*-algebra, \( B, C \) be unital C*-subalgebras of \( M(A) \) (resp., of \( M_{\text{loc}}(A) \)). The following two conditions are equivalent:
For any operator $B$-$C$ bimodule $M$, any operator $B$-$C$ subbimodule $N$ and any completely bounded $B$-$C$ bilinear map $\phi : N \to M(A)$ (resp., $\phi : N \to M_{\text{loc}}(A)$) there exists a completely bounded $B$-$C$ bilinear map

$\psi : M \to M(A)$ (resp., $\psi : M \to M_{\text{loc}}(A)$) such that $\psi|A = \phi$ and $\|\psi\|_{\text{c.b.}} = \|\phi\|_{\text{c.b.}}$.

(ii) The $C^*$-algebras $M(A)$ and $I(A)$ (resp., $M_{\text{loc}}(A)$ and $I(A)$) are $*$-isomorphic.

**Proof:** Condition (ii) implies condition (i) by G. Wittstock’s extension theorem, see [26, 19]. Conversely, consider the canonical $*$-monomorphism of $M(A)$ (resp., $M_{\text{loc}}(A)$) into $I(A)$ existing by Theorem 1. Both $M(A)$ (resp., $M_{\text{loc}}(A)$) and $I(A)$ are operator $B$-$C$ bimodules if considered in a faithful representation of $I(A)$. By [12, Th. 2.1] the identity map of $A$ onto its canonical copy inside $I(A)$ admits a unique completely bounded extension to $I(A)$ with the same complete boundedness norm in case we require one-sided $A$-linearity: the identity map on $I(A)$. If condition (i) is supposed to hold then the canonical copy of $M(A)$ (resp., $M_{\text{loc}}(A)$) inside $I(A)$ has to coincide with $I(A)$. This implies (ii). •

We do not know whether the process of taking higher order local multiplier algebras of a $C^*$-algebra $A$ in general ever stabilizes (or at least converges), or not. However, it takes place in a fixed monotone complete $C^*$-algebra, the injective envelope $I(A)$ of $A$, and more precisely in the monotone closure $\overline{A}$ of $A$ in $I(A)$ that is a uniquely determined monotone complete $C^*$-algebra by [16]. $\overline{A}$ is said to be the regular monotone completion of $A$. We remark that $C^*$-subalgebras $A$ of injective $C^*$-algebras $B$ might not admit an embedding of their injective envelopes $I(A)$ as a $C^*$-subalgebra of $B$ that extends the given embedding of $A$ into $B$. An example was given by M. Hamana, [16, Rem. 3.9].

**Theorem 5:** Let $A$ be a $C^*$-algebra, $M(A)$ be its multiplier algebra and $M_{\text{loc}}(A)$ be its local multiplier $C^*$-algebra. Then the injective envelopes $I(A)$, $I(M(A))$ and $I(M_{\text{loc}}(A))$ of these three $C^*$-algebras, respectively, coincide. Every higher order local multiplier algebra of $A$ is a $C^*$-subalgebra of the regular monotone completion $\overline{A}$ of $A$ inside $I(A)$.

**Proof:** Since $M(A)$ and $M_{\text{loc}}(A)$ are $*$-isomorphically embedded into $I(A)$ extending the canonical embedding of $A$ into $I(A)$, the $C^*$-algebra $I(A)$ serves as an injective extension of both $M(A)$ and $M_{\text{loc}}(A)$. However, the identity map on $M(A)$ or $M_{\text{loc}}(A)$, respectively, admits a unique extension to a completely positive map on $I(A)$ of the same complete boundedness norm one because $A \subseteq M_{\text{loc}}(A) \subseteq I(A)$ by construction and $I(A)$ is the injective envelope of $A$ by definition, cf. [12, Th. 2.1]. So $I(A)$ serves as the injective envelope of $M(A)$ and $M_{\text{loc}}(A)$, too.

By [21, Th.] the multiplier $C^*$-algebra of any $C^*$-subalgebra $B$ of the regular monotone completion $\overline{A}$ of $A$ inside $I(A)$ can be realized as a $C^*$-subalgebra
of \( \overline{A} \). So the process of taking the higher order local multiplier C*-algebras of \( A \) can be always carried out inside \( \overline{A} \). 

D. W. B. Somerset pointed out to us some non-standard examples of C*-algebras \( A \) beyond the set of commutative C*-algebras for which \( \text{M}_{\text{loc}}(A) \equiv \text{M}_{\text{loc}}(\text{M}_{\text{loc}}(A)) \), \[25\]. However, the class of C*-algebras that coincide with their own local multiplier C*-algebra is very heterogeneous from the point of view of existing classifications. So it also contains all AW*-algebras (like von Neumann algebras) together with all unital simple C*-algebras (like \( C^*_r(F_2) \), irrational \( A_\theta \), \( O_n \)). For example, while in AW*-algebras the linear span of the set of projections is norm-dense the C*-algebra \( C^*_r(F_2) \) does not contain any non-trivial projection. Nevertheless, every C*-algebra of this class possesses a commutative AW*-algebra as its center, cf. \[1, 21, 1, 2\].

**Problem 6:** Whether any C*-algebra \( A \) possess the property \( \text{M}_{\text{loc}}(A) \equiv \text{M}_{\text{loc}}(\text{M}_{\text{loc}}(A)) \), and if not, does the process of consecutively taking local multiplier C*-algebras stabilize? Are there intrinsic characterizations of C*-algebras \( A \) for which \( A \equiv \text{M}_{\text{loc}}(A) \)?

For non-commutative C*-algebras \( A \) with non-injective multiplier C*-algebra \( M(A) \) the injective envelope may be strictly larger than \( \text{M}_{\text{loc}}(A) \) since some of the essential one-sided ideals may have larger algebras of left multipliers enjoying an isometric algebraic embedding into \( I(A) \) by G. Wittstock’s extension theorem for completely bounded one-sided module maps \[24\, \text{Th. 4.1}\], the properties of the left-strict (resp., right-strict) topology on the algebra of left multipliers \( LM(A) \) (resp., right multipliers \( RM(A) \)) and by the monotone completeness of \( I(A) \).

However, even if \( M(I) \equiv LM(I) \) for any two-sided ideal \( I \) of a certain C*-algebra \( A \) (as valid for any AW*-algebra - see \[21\, \text{Th.}\]) the coincidence of \( \text{M}_{\text{loc}}(A) \) and \( I(A) \) may fail. Any non-injective von Neumann algebra \( A \) serves as an example.

**Corollary 7:** Let \( A \) be an AW*-algebra and \( I \subseteq A \) be an essential two-sided norm-closed ideal of \( A \). For any C*-algebra \( B \) with \( I \subseteq B \subseteq A \) we obtain \( \text{M}_{\text{loc}}(B) \equiv A \). The C*-algebra \( A \) is injective if and only if \( \text{M}_{\text{loc}}(B) \equiv I(B) \equiv A \).

**Proof:** Since \( I \subseteq B \) and \( I \) is an essential two-sided norm-closed ideal of \( A \) the C*-algebra \( I \) is an essential two-sided norm-closed ideal of \( B \) and \( M(I) \equiv A \). So \( A \subseteq \text{M}_{\text{loc}}(B) \). At the other side the multipliers of any C*-subalgebra \( J \) of an AW*-algebra \( A \) can be found inside \( A \) by \[21\, \text{Th.}\]. In other words, \( A \) always contains a faithful \( * \)-representation of \( M(J) \) that extends the given embedding of \( J \) into \( A \). Hence, \( \text{M}_{\text{loc}}(B) \subseteq A \). The coincidence of both these C*-algebras follows. \( \text{M}_{\text{loc}}(B) = A \) is injective if and only if \( A \) is injective, otherwise \( I(B) \equiv I(\text{M}_{\text{loc}}(B)) \equiv I(A) \supset A \) by Theorem 5. 

To indicate some concrete examples, we can consider the pair of $C^*$-algebras $I = K(H)$ and $A = B(H)$ of all compact linear operators on an infinite-dimensional Hilbert space $H$ and of all bounded linear operators on $H$, the pair of an injective II$_1$ von Neumann factor $A$ and its unique non-trivial two-sided norm-closed ideal $I \subset A$, or the pair of the $C^*$-algebra $I = C_0((0, 1])$ of all continuous functions on the unit interval that vanish at zero and the Dixmier algebra $A = D([0, 1])$ which is defined to be the set of all bounded Borel functions on $[0, 1]$ modulo the ideal of all functions supported on meager subsets of $[0, 1]$.

Finally, we arrive at the following open questions which are surely difficult to answer:

**Problem 8:** Characterize the $C^*$-algebras $A$ for which the local multiplier $C^*$-algebra $M_{loc}(A)$ of $A$ coincides with the injective envelope $I(A)$ of $A$, or at least with the regular monotone completion $\overline{A}$ of $A$ in $I(A)$.

Combining Theorem 2 and Corollary 3 with [17, Cor. 1.6] we may obtain necessary conditions to the center of certain $C^*$-algebras $A$ to satisfy $M_{loc}(A) \equiv \overline{A}$ and $M_{loc}(A) \equiv I(A)$. Investigating AW*-algebras $A$, for example, the local multiplier algebra $M_{loc}(A)$ of $A$ coincides with $A$ itself by [21, Th.]. However, $A$ coincides with its regular monotone completion $\overline{A}$ if and only if $A$ is monotone complete. So we arrive at a long standing open problem of $C^*$-theory dating back to the work of I. Kaplansky in 1951 ([18]): Are all AW*-algebras monotone complete, or do there exist counterexamples?

Consider the universal $\ast$-representation of $A$ in $B(H)$. By the Arveson-Wittstock theorem [3, 21] the canonical embedding of $A$ into its bidual von Neumann algebra $A^{**}$ realized as the bicommutant $A''$ of $A$ in $B(H)$ extends to a completely isometric linear map $\psi$ of $I(A)$ into $B(H)$. The image $\psi(I(A)) \subseteq B(H)$ is an operator system. The domain of $\psi$ contains $M_{loc}(A)$ as a $C^*$-algebra on which $\psi$ acts as an algebraic $\ast$-monomorphism by Theorem 1, canonical identifications provided.

**Problem 9:** Is $M_{loc}(A)$ the largest $C^*$-subalgebra of $I(A)$ on which $\psi$ acts as an algebraic $\ast$-monomorphism?

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