On Belief Propagation Guided Decimation for Random $k$-SAT

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Abstract. Let $\Phi$ be a uniformly distributed random $k$-SAT formula with $n$ variables and $m$ clauses. Non-constructive arguments show that $\Phi$ is satisfiable for clause/variable ratios $m/n \leq r_{k\text{-SAT}} \sim 2^k \ln 2$ with high probability. Yet no efficient algorithm is known to find a satisfying assignment beyond $m/n \sim 2^{k} \ln (k)/k$ with a non-vanishing probability. On the basis of deep but non-rigorous statistical mechanics ideas, a message passing algorithm called Belief Propagation Guided Decimation has been put forward (Mézard, Parisi, Zecchina: Science 2002; Braunstein, Mézard, Zecchina: Random Struct. Alg. 2005). Experiments suggested that the algorithm might succeed for densities very close to $r_{k\text{-SAT}}$ for $k = 3, 4, 5$ (Kroc, Sabharwal, Selman: SAC 2009). Furnishing the first rigorous analysis of this algorithm on a non-trivial input distribution, in the present paper we show that Belief Propagation Guided Decimation fails to solve random $k$-SAT formulas already for $m/n = O(2^k/k)$, almost a factor of $k$ below the satisfiability threshold $r_{k\text{-SAT}}$. Indeed, the proof refutes a key hypothesis on which Belief Propagation Guided Decimation hinges for such $m/n$.

1 Introduction and results

Let $k \geq 3$ and $n > 1$ be integers, let $r > 0$ be a fixed real number (independent of $n$), and set $m = \lceil rn \rceil$. Let $\Phi = \Phi_k(n,m)$ be a propositional formula obtained by choosing a set of $m$ clauses of length $k$ over the variables $x_1, \ldots, x_n$ uniformly at random such that no variable occurs in the same clause more than once (either positively or negatively). For $k, r$ fixed we say that $\Phi$ has some property $\mathcal{P}$ with high probability (‘w.h.p.’) if $\lim_{n \to \infty} P[\Phi \in \mathcal{P}] = 1$.

1.1 Background and motivation

Since the 1990s the random formula $\Phi$ has gained a reputation as an extremely challenging benchmark for SAT solving. More precisely, early computer experiments led to two key hypotheses [14,28,32]. First, that there is a sharp threshold for satisfiability. That is, for any clause length $k$ there is a threshold value $r_{k\text{-SAT}} > 0$ such that the random formula $\Phi$ is satisfiable w.h.p. if $r < r_{k\text{-SAT}}$, while $\Phi$ is unsatisfiable w.h.p. if $r > r_{k\text{-SAT}}$. Second, that standard SAT-solvers such as DPLL-based algorithms require an exponential time to find a satisfying assignment for densities $r$ ‘close’ to $r_{k\text{-SAT}}$. Thus, while these algorithms are highly efficient on “real-world” SAT instances, the simplest conceivable model of random formulas eludes them. These two hypotheses have inspired a considerable amount of research over the years, both experimental and theoretical [1]. Moreover, similar phenomena have been hypothesised in many other random problems [5].

While the precise values (and even the existence) of the $k$-SAT threshold remain unknown for any $k \geq 3$, asymptotically tight upper and lower bounds have been established. Indeed, non-constructive arguments show that $\Phi$ is satisfiable w.h.p. if $r < 2^k \ln 2 - \frac{1}{2} (1 + \ln 2) - \varepsilon_k$, while $\Phi$ is unsatisfiable w.h.p. if $r > 2^k \ln 2 - \frac{1}{2} (1 + \ln 2) + \varepsilon_k$, with $\varepsilon_k \to 0$ for large $k$ [4,19,27]. Thus, the transition from satisfiable to unsatisfiable takes place at about $r_{k\text{-SAT}} \sim 2^k \ln 2$.

With respect to the computational problem, in spite of two decades of extensive research in the CS community no algorithm seemed capable of finding a satisfying assignment for densities $r$ anywhere close to $r_{k\text{-SAT}}$ in polynomial time with a non-vanishing probability. More precisely, the best rigorously analyzed polynomial time algorithm, designed specifically to “beat” random formulas, is known to succeed

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for $r < (1 - \varepsilon_k)2^k \ln(k)/k$ w.h.p., and seems to fail beyond $16$. Furthermore, a plethora of algorithms are known to fail for asymptotically even smaller densities $r = \rho \cdot 2^k/k$ with $\rho > 0$ an absolute constant (independent of $k$). Examples include simple linear-time algorithms such as Unit Clause ($\rho = e/2$) $13$ or Shortest Clause ($\rho = 1.817$) $21$, as well as a wide range of DPLL-type algorithms ($\rho = 11/4$) $2$. In summary, there remained a factor of about $k/\ln k$ between the satisfiability threshold and the density where algorithms are known to find satisfying assignments efficiently.

Against this gloomy background, it came as a considerable surprise when experiments indicated that certain highly efficient message passing algorithms come within a whisker of the conjectured satisfiability threshold $11,24,30,35$. These algorithms, called Belief Propagation Guided Decimation and Survey Propagation Guided Decimation, were put forward on the basis of the “cavity method”, a very insightful but non-rigorous technique from statistical mechanics $11,31$. Conceptually, Belief/Survey Propagation Guided Decimation are more sophisticated than the previously studied algorithms by an order of magnitude; we will give a detailed account in Section $1.3$. As a consequence, the techniques that were developed to analyze previous algorithms fail dramatically for Belief/Survey Propagation.

The performance of the new message passing algorithms can be exemplified nicely in the case $k = 4$. The conjectured threshold for the existence of satisfying assignments is $r_{4-SAT} \approx 9.93$ $34$. According to experiments from $30$, Survey Propagation guided decimation finds satisfying assignments efficiently for densities up to $r = 9.73$. Experiments from $42$ suggest that the “vanilla” version of Belief Propagation Guided Decimation succeeds up to $r = 9.05$. With a certain tweak (the “most biased variable” decimation rule) Belief Propagation Guided Decimation succeeds up to $r = 9.24$ $30$. By comparison, the best “classical” algorithm SSB from $21$ finds satisfying assignments in polynomial time merely up to $r = 5.54$, while zChaff, an industrial SAT solver, is effective up to $r = 5.35$ $30$.

1.2 Unsatisfied with physics

Ever since these stunning experimental results were reported, coming up with a rigorous analysis of the new message passing algorithms has been one of the key challenges in the area of random constraint satisfaction problems (cf. $5$). The present paper contributes the first such analysis. More specifically, we study the “vanilla” version of Belief Propagation Guided Decimation (“BPdec”), the simplest but arguably most natural version. We establish a negative result: BPdec fails to find a satisfying assignment w.h.p. for densities $r > \rho \cdot 2^k/k$ for a certain absolute constant $\rho > 0$. In other words, we prove that, perhaps surprisingly, BPdec does not outperform simpler combinatorial algorithms such as the one from $16$ asymptotically.

Stating the result precisely requires a little care, because it involves two levels of randomness: the choice of the random formula $\Phi$, and the ‘coin tosses’ of the randomized algorithm BPdec. For a (fixed, non-random) $k$-CNF $\Phi$ let $\text{success}(\Phi)$ denote the probability that BPdec($\Phi$) outputs a satisfying assignment. Here, of course, ‘probability’ refers to the coin tosses of the algorithm only. Then, if we apply BPdec to the random $k$-CNF $\Phi$, the success probability $\text{success}(\Phi)$ becomes a random variable. Recall that $\Phi$ is unsatisfiable for $r > 2^k \ln 2$ w.h.p.

Theorem 1. There is a constant $\rho_0 > 0$ such that for any $k, r$ satisfying

$$\rho_0 \cdot 2^k/k \leq r \leq 2^k \ln 2 \quad (1)$$

we have $\text{success}(\Phi) \leq \exp(-\Omega(n))$ w.h.p.

Theorem $1$ contrasts with the very promising experimental results. The explanation for this is that the experiments were conducted for ‘small’ $k = 3, 4, 5$ $30,42$. Indeed, already for $k = 10$ large-scale experiments are difficult to carry out, because the relevant density $r$ scales exponentially with $k$. Thus, the good experimental performance can be attributed to the value of the constant $\rho_0$ in Theorem $1$. Because the analysis is intricate as is, no attempt has been made to compute (or optimize) $\rho_0$.

$1$ The message passing procedure upon which Belief Propagation Guided Decimation is based has been rediscovered several times in the context of different applications, see Section $1.5$ for details. In the physics literature it was originally known under the name “Bethe-Peierls approximation”. By contrast, the message passing technique that underpins Survey Propagation seems to be new.
Since Belief/Survey Propagation guided decimation were suggested [35], there have been various stabs at explaining the performance of Belief/Survey Propagation Guided Decimation by means of non-rigorous physics arguments [11,31,42]. We will review this work in more detail in Section 1.4 below, but roughly speaking the predictions were as follows. In chronological order,

- the authors of [11] opined that Belief Propagation Guided Decimation fails for \( r > (1+\varepsilon)2^k \ln(k)/k \).
- More optimistically, it was predicted in [31] that Belief Propagation Guided Decimation will find satisfying assignments efficiently up to \( r \sim 2^k \ln 2 \).
- Finally and most pessimistically, according to [42] Belief Propagation Guided Decimation ought to fail for \( r > \rho \cdot 2^k/k \) for an absolute constant \( \rho > 0 \).

All of these predictions derived from fairly sophisticated statistical mechanics reasoning, and both [31,42] quote experimental evidence, thereby (unintentionally) highlighting the need for a rigorous analysis. Theorem 1 confirms the scenario put forward in [42], but does not sit well with the predictions from [31]. Furthermore, the present analysis shows that the reasoning from [11], where the demise of Belief Propagation Guided Decimation was attributed to a certain change in the geometry of the set of satisfying assignments, is off the mark.

A potential objection to a negative result like Theorem 1 is that it might hinge on a small detail of the algorithm that could easily be fixed. However, in the sequel we will see that in the regime (1) our analysis refutes a key hypothesis upon which \( \text{BPdec} \) depends. In other words, we show that \( \text{BPdec} \) falls victim to a conceptual issue, not a technicality. Furthermore, some of the arguments used to prove Theorem 1 may be of independent interest as they can be expected to extend to applications of BP beyond random \( k \)-SAT. For instance, we develop a technique for tracing BP on certain quasi-random problem instances.

Finally, we point out that Theorem 1 has no immediate bearing on the potentially more powerful Survey Propagation algorithm. We will comment on Survey Propagation in Section 1.4 below.

### 1.3 The \( \text{BPdec} \) algorithm

Fix a satisfiable \( k \)-CNF \( \Phi \) on the variables \( V = \{x_1, \ldots, x_n\} \). We generally represent truth assignments as maps \( \sigma : V \to \{-1, 1\} \), with \(-1\) representing ‘false’ and \(1\) representing ‘true’. (It turns out that using \( \pm 1 \) instead of the more common \( 0, 1 \) simplifies the description of BP quite a bit.) Let \( S(\Phi) \) denote the set of all satisfying assignments of \( \Phi \). The algorithm \( \text{BPdec} \) is an attempt at implementing the following thought experiment.
Experiment 2. Input: A satisfiable $k$-CNF $\Phi$. Result: An assignment $\sigma : V \to \{-1, 1\}$.

0. Let $\phi_0 = \phi$.
1. For $t = 0, \ldots, n - 1$ do
2. Compute the fraction
   \[ M_{x_{t+1}}(\phi_t) = \frac{|\{\sigma \in S(\phi_t) : \sigma(x_{t+1}) = 1\}|}{|S(\phi_t)|} \]
   of satisfying assignments of $\phi_t$ in which the variable $x_{t+1}$ takes the value 1.
3. Assign $\sigma(x_{t+1}) = \begin{cases} 1 & \text{with probability } M_{x_{t+1}}(\phi_t), \\ -1 & \text{with probability } 1 - M_{x_{t+1}}(\phi_t). \end{cases}$
4. Obtain the formula $\phi_{t+1}$ from $\phi_t$ by substituting the value $\sigma(x_{t+1})$ for $x_{t+1}$ and simplifying, i.e.,
   \begin{itemize}
   \item remove all clauses that got satisfied by setting $x_{t+1}$ to $\sigma(x_{t+1})$,
   \item omit $x_t$ from all the other clauses.
   \end{itemize}
5. Return the assignment $\sigma$.

A moment’s reflection reveals that the above experiment not only produces a satisfying assignment, but that its (random) outcome is in fact uniformly distributed over the set $S(\phi)$. We observe that in the formulas $\phi_t$ obtained at intermediate steps some clauses can (and typically will) have length less than $k$.

Referring to the successive assignments of variables and the corresponding shrinking of the formula, we call the above experiment the \textit{decimation process}. The obvious obstacle to implementing it is the computation of the marginal probabilities $M_{x_{t+1}}(\phi_t)$. Indeed, this task is \#P-hard on worst-case inputs.

Yet, under what conditions could we hope to compute (or approximate) the marginals $M_x(\phi_t)$? Clearly, the marginals are influenced by ‘local’ effects. For instance, if $x$ occurs in a unit clause of $\phi_t$, i.e., a clause whose other $k - 1$ variables have been assigned already without satisfying $a$, then $x$ must be assigned so as to satisfy $a$. Hence, if $x$ appears in a positively, then $M_x(\phi_t) = 1$, and otherwise $M_x(\phi_t) = 0$. Similarly, if $x$ occurs \textit{only} positively in $\phi_t$, then $M_x(\phi_t) \geq 1/2$. Furthermore, these local effects propagate: if $x$ appears in a clause $a$ whose other variables $y$ are subject to influences from other clauses $b_y \neq a$, then the local effects operating on the variables $y$ may impact $x$ via $a$. In the most extreme case, think of a variable $x$ that occurs in a clause $a$ whose other variables are all constrained by unit clauses to take values that fail to satisfy $a$. Then $a$ effectively turns into a unit clause for $x$.

The key hypothesis underlying $\text{BPdec}$ is that in random formulas such local effects determine the marginals $M_x(\phi_t)$ asymptotically. To define ‘local’ precisely, we need a metric on the variables/clauses. This metric is the shortest path distance on the \textit{factor graph} $G = G(\phi_t)$ of $\phi_t$, which is a bipartite graph whose vertices are the variables $V_t = \{x_{t+1}, \ldots, x_n\}$ and the clauses of $\phi_t$. Each clause is adjacent to the variables that occur in it. For an integer $\omega \geq 1$ let $N[\omega](x)$ signify the set of all vertices of $G$ that have distance at most $2\omega$ from $x$. Then the induced subgraph $G[N[\omega](x)]$ corresponds to the sub-formula of $\phi_t^{[\omega]}$ obtained by removing all clauses and variables at distance more than $2\omega$ from $x$. Note that all vertices at distance precisely $2\omega$ are variables. Hence, any satisfying assignment of $\phi$ induces a satisfying assignment of the sub-formula. Let us denote by
\[ M_x(\phi_t) = M_x(\phi_t^{[\omega]}) = \frac{|\{\sigma \in S(\phi_t^{[\omega]}) : \sigma(x) = 1\}|}{|S(\phi_t^{[\omega]})|} \]
the marginal probability that $x_t$ takes the value 1 in a random satisfying assignment of this sub-formula.

Of course, in the worst case the ‘local’ marginals $M_x(\phi_t)$ are just as difficult to compute as the $M_x(\phi_t)$ themselves. But $\text{BPdec}$ employs an efficient heuristic called \textit{Belief Propagation} (‘BP’), which yields certain values $\mu_{x_t}(\phi_t) \in [0, 1]$; we will state this heuristic below. If $G[N[\omega](x_t)]$ is a tree, then clearly $\mu_{x_t}(\phi_t) = M_{x_t}(\phi_t)$. Moreover, standard arguments show that in a random formula $\phi$ actually $G[N[\omega](x_t)]$ is a tree w.h.p. so long as $\omega = o(\ln n)$. More generally, in order to obtain an efficient algorithm it would be sufficient for the BP outcomes $\mu_{x_t}(\phi_t)$ to approximate the true overall marginals.
Furthermore, we define the \( \mu \) well for some (say, polynomially computable, polynomially bounded) function \( \omega = \omega(n) \geq 1 \). This leads to the following hypothesis underpinning \( \text{BPdec} \) (cf. [31]).

**Hypothesis 3.** With probability \( 1 - o(1) \) over the choice of \( \Phi \) and the random decisions in Experiment 2, the following holds for all \( 0 \leq t < n \).

1. For any \( \varepsilon > 0 \) there is \( \omega = \omega(\varepsilon, k, r) \) such that \( |M_{t+1}^a(\Phi_t) - M_{t+1}^{[\omega]}(\Phi_t)| \leq \varepsilon \).
2. For any \( \varepsilon > 0 \) there is \( \omega = \omega(\varepsilon, k, r) \) such that \( |M_{t+1}^a(\Phi_t) - M_{t+1}^{[\omega]}(\Phi_t)| \leq \varepsilon \).

Hypothesis 3 motivates the following algorithm [39], which is called **Belief Propagation Guided Decimation** because it combines BP (Step 2) with a decimation step (Steps 3–4).

**Algorithm 4.** \( \text{BPdec}(\Phi) \)

**Input:** A \( k \)-CNF \( \Phi \) on \( V = \{x_1, \ldots, x_n\} \). **Output:** An assignment \( \sigma : V \to \{-1, 1\} \).

0. Let \( \phi_0 = \Phi \).
1. For \( t = 0, \ldots, n - 1 \) do
2. Use BP to compute \( \mu_{t+1}^{[\omega]}(\Phi_t) \).
3. Assign

\[
\sigma(x_{t+1}) = \begin{cases} 1 & \text{with probability } \mu_{t+1}^{[\omega]}(\Phi_t), \\ -1 & \text{with probability } 1 - \mu_{t+1}^{[\omega]}(\Phi_t). \end{cases}
\]
4. Obtain the formula \( \Phi_{t+1} \) from \( \Phi_t \) by substituting the value \( \sigma(x_{t+1}) \) for \( x_{t+1} \) and simplifying.
5. Return the assignment \( \sigma \).

**Remark 5.** The function \( \omega = \omega(k, r, n) \) is “hard-wired” into the above algorithm, and our analysis does not depend on any assumptions on \( \omega \). In particular, the statement of Theorem 4 is understood to hold for all integer-valued functions \( \omega = \omega(n) \geq 0 \).

Although, strictly speaking, Hypothesis 3 provides neither a necessary nor a sufficient condition for \( \text{BPdec} \) to succeed on random \( k \)-CNFs w.h.p., the hypothesis inspired the algorithm (we will get back to this in Section 1.4). Combining parts of the present analysis of the dynamics of the BP computation (more precisely, Theorem 25 below) with techniques for analyzing the geometry of the space of satisfying assignments, we proved the following in [18].

**Corollary 6.** Both statements of Hypothesis 3 are false for \( k, r \) satisfying [7].

To complete the presentation of the algorithm, we need to define Belief Propagation for \( k \)-SAT; for a detailed derivation we point the reader to [11, 36, 41]. Ultimately, we need to define the value \( \mu_{t+1}^{[\omega]}(\Phi_t) \) in Step 2 of \( \text{BPdec} \).

Let \( N(v) \) denote the neighborhood of a vertex \( v \) of the factor graph \( G(\Phi_t) \). For a variable \( x \in V_t \) and a clause \( a \in N(x) \) we will denote the ordered pair \((x, a)\) by \( x \to a \). Similarly, \( a \to x \) stands for the pair \((a, x)\). Furthermore, we let \( \text{sign}(x, a) = 1 \) if \( x \) occurs in \( a \) positively, and \( \text{sign}(x, a) = -1 \) otherwise.

The **message space** \( \mathcal{M}(\Phi_t) \) is the set of all tuples

\[
(\mu_{x \rightarrow a}(\zeta))_{x \in V_t, a \in N(x), \zeta \in \{-1, 1\}}
\]

such that \( \mu_{x \rightarrow a}(\pm 1) \in [0, 1] \) and \( \mu_{x \rightarrow a}(-1) + \mu_{x \rightarrow a}(1) = 1 \) for all \( x, a, \zeta \). For \( \mu \in \mathcal{M}(\Phi_t) \) we define

\[
\mu_{a \rightarrow x}(\text{sign}(x, a)) = 1 - \prod_{y \in N(a) \setminus \{x\}} \mu_{y \rightarrow a}(\text{sign}(y, a)).
\]

Furthermore, we define the **Belief Propagation operator** BP as follows: for any \( \mu \in \mathcal{M}(\Phi_t) \) we define \( \text{BP}(\mu) \in \mathcal{M}(\Phi_t) \) by letting

\[
(\text{BP}(\mu))_{x \rightarrow a}(\zeta) = \frac{\prod_{b \in N(x) \setminus \{a\}} \mu_{b \rightarrow x}(\zeta)}{\prod_{b \in N(x) \setminus \{a\}} \mu_{b \rightarrow x}(-1) + \prod_{b \in N(x) \setminus \{a\}} \mu_{b \rightarrow x}(1)}.
\]
are computed under the hypothesis that all other clauses $\mu$ is satisfied if $G$ is satisfied from variables to clauses. More precisely, for each vector with all entries equal to $0$, $1 \in M(\Phi_t)$ be the vector with all entries equal to $\frac{1}{2}$. Moreover, define inductively $\mu^{[\ell]} = \text{BP}(\mu^{[\ell]})$ for $0 \leq \ell < \omega$. Then

$$
\mu^{[\omega]}(\Phi_t) = \frac{\prod_{b \in N(x)} \mu_b^{[\omega]}(1)}{\prod_{b \in N(x)} \mu_b^{[\omega]}(-1) + \prod_{b \in N(x)} \mu_b^{[\omega]}(1)}
$$

for any $x \in V_t$, unless the denominator is zero, in which case we set $\mu^{[\omega]}(\Phi_t) = \frac{1}{2}$.

The intuition here is that the $\mu_{x \rightarrow a}(\zeta)$ are ‘messages’ from a variable $x$ to the clauses $a$ in which $x$ occurs, indicating how likely $x$ were to take the value $\zeta$ if clause $a$ were removed from the formula. Based on these, (3) yields messages $\mu_{a \rightarrow x}(\zeta)$ from clauses $a$ to variables $x$, indicating the probability that $a$ is satisfied if $x$ takes the value $\zeta$ and all other variables $y \in N(a) \setminus \{x\}$ are assigned independently with probability $\mu_{b \rightarrow a}(\pm 1)$. The BP operator (3) then uses the messages $\mu_{a \rightarrow x}$ in order to ‘update’ the messages from variables to clauses. More precisely, for each $x$ and $a \in N(x)$ the new messages (BP($\mu$))$_{x \rightarrow a}$ are computed under the hypothesis that all other clauses $b \in N(x) \setminus \{a\}$ are satisfied with probabilities $\mu_{b \rightarrow x}(\zeta)$ independently if $x$ takes the value $\zeta$. Finally, the difference between (3) and (4) is that the latter product runs over all clauses $b \in N(x)$. An inductive proof shows that, if for a variable $x$ the subgraph $G[N^{[\omega]}(x)]$ of the factor graph is acyclic, then in fact $\mu^{[\omega]}(\Phi_t) = M^{[\omega]}(\Phi_t)$ (11).

Variations of the algorithm. BPdec could be called the “vanilla” version of Belief Propagation Guided Decimation. It is the simplest but arguably the most natural variant. Nonetheless, several other installments have been suggested and experimented with. They differ in how the number $\omega$ of iterations is chosen and how exactly the result of the Belief Propagation calculation is used to decimate.

In the “vanilla” variant we used an a priori number $\omega$ of iterations. An alternative idea is to iterate the Belief Propagation operator until it reaches a fixed point. More precisely, to accommodate numerical inaccuracies one could stop after $\omega$ iterations, with $\omega \geq 1$ the least integer such that for some small $\varepsilon > 0$ we have

$$
\max_{x \rightarrow a} |\mu^{[\omega]}_{x \rightarrow a}(1) - \mu^{[\omega-1]}_{x \rightarrow a}(1)| < \varepsilon,
$$

(4)

where the maximum is taken over all edges of the factor graph (e.g., (11,36)). Unfortunately, it is not generally assured that the convergence criterion (4) will ever be met. Hence, one would need to specify how to proceed otherwise. For instance, one could specify an a priori maximum number of iterations. Our analysis can be adapted easily to accommodate these modifications (details omitted).

More importantly, one could come up with a more sophisticated decimation strategy, i.e., a different way of using the BP result $\mu^{[\omega]}(x)$ to choose the variable to be assigned next and its value. In BPdec we went for the “vanilla rule”: the variables are assigned in the natural order, and each time the assignment is performed randomly based on the BP estimate of the marginal.

But in experiments a more common decimation strategy is the “most biased variable” rule: at each time choose a variable $x \in V_t$ that maximizes the “bias” $|\mu^{[\omega]}_{x \rightarrow a}(\Phi_t) - \frac{1}{2}|$, and assign it randomly based on the BP estimate. Experimentally the most biased variable rule allows for slightly better results than the vanilla rule. For instance, in random 4-SAT, experiments indicate that the former succeeds up to $m/n = 9.24$, and the latter up to $m/n = 9.05$ (10,22).

The statistical mechanics ideas that underpin Belief Propagation guided decimation do not endorse a preference for the “most biased variable” rule over the “vanilla” strategy. But a heuristic argument in favor of “most biased variable” is that it might reduce the effect of numerical errors building up (39). The present analysis does not seem to extend to “most biased variable” in a straightforward manner. Thus, analyzing it remains an interesting open problem.

Comparison with combinatorial algorithms. The difference between the previously studied combinatorial algorithms for random $k$-SAT and Belief Propagation can be explained nicely in terms of the factor
graph. Indeed, in order to decide upon the value of a variable the previous algorithms only took the clauses and variables at distance two \( \mathcal{O}(2) \) or four \( \mathcal{O}(4) \) into consideration. Based on this information, the variable is assigned following some simple combinatorial rule.

\( \mathbb{B} \mathbb{P} \mathbb{d} \mathbb{c} \) can be viewed as a systematic way of making a “less shortsighted” decision. The algorithm takes into account clauses/variables at distance up to \( 2 \omega \), where \( \omega \) may be a function that grows with \( n \). Indeed, the idea of determining the marginal \( M_{x}^{[\omega]}(\Phi, \tau) \) yields a meaningful way of incorporating the data from all these clauses/variables. In particular, \( \mathbb{B} \mathbb{P} \mathbb{d} \mathbb{c} \) implicitly implements many of the rules that are used in the combinatorial algorithms (e.g., the “Unit Clause” rule). In this sense, \( \mathbb{B} \mathbb{P} \mathbb{d} \mathbb{c} \) can be seen as a clever generalization of many of these combinatorial algorithms. However, this also means that the techniques used in the previous analyses of combinatorial algorithms are insufficient to tackle \( \mathbb{B} \mathbb{P} \mathbb{d} \mathbb{c} \).

1.4 The statistical physics perspective

Clustering and correlation decay. Closely following the non-rigorous paper \cite{31}, we discuss in this section the statistical mechanics motivation for \( \mathbb{B} \mathbb{P} \mathbb{d} \mathbb{c} \). This will provide the basis for the discussion of the non-rigorous predictions as to the algorithm’s performance.

According to the physicists’ “cavity method”, the random formula \( \Phi \) undergoes several further phase transitions prior to the satisfiability threshold. These phase transitions affect the correlations between the truth values that can be assigned to different variables. Thus, fix a variable \( x \) and let \( \omega = \omega(n) = o(\ln n) \) be a function that tends to infinity slowly, say \( \omega = [\ln \ln n] \). Furthermore, let \( B \) be the set of all variables at distance exactly \( 2 \omega \) from \( x \) in the factor graph. How do the values assigned to variables on the “far away boundary” \( B \) affect the truth value of \( x \)?

The strongest possible decay of correlations occurs when the boundary \( B \) has no impact on \( x \) at all. To formalize this, let \( \tau \mapsto \{ -1, 1 \} \) be a satisfying assignment of \( \Phi \) and let \( M_{x}^{[\omega]}(\Phi, \tau) \) be the fraction of all satisfying assignments of \( \Phi \) that set \( x \) to true and that coincide with \( \tau \) on \( B \). In symbols,

\[
M_{x}^{[\omega]}(\Phi, \tau) = \frac{|\{ \sigma \in \mathcal{S}(\Phi) : \sigma(x) = 1, \sigma(y) = \tau(y) \text{ for all } y \in B\}|}{|\{ \sigma \in \mathcal{S}(\Phi) : \sigma(y) = \tau(y) \text{ for all } y \in B\}|}.
\]

Also recall that \( M_{x}(\Phi) \) denotes the marginal probability that \( x \) takes the value “true” in a random satisfying assignment of \( \Phi \) (without any boundary condition). The Gibbs uniqueness condition requires that

\[
\max_{\tau \in \mathcal{S}(\Phi)} \left| M_{x}^{[\omega]}(\Phi, \tau) - M_{x}(\Phi) \right| = o(1). \tag{5}
\]

In words, fixing the “far away” variables does not make it noticeably more or less like for \( x \) to take the value “true”. Hence, the marginal \( M_{x}(\Phi) \) is governed entirely by the effects of variables at distance less than \( 2 \omega \) from \( x \), i.e., by the local structure of the formula.

Consequently, it seems reasonable to expect that Belief Propagation yields the correct marginals as long as \( \mathbb{E} \) holds. It is known rigorously that w.h.p. \( \mathbb{E} \) holds up to \( r \sim r_{u} = 2 \ln k/k \) (a function that tends to zero for large \( k \)), and that Belief Propagation does indeed yield the correct marginals for such densities \( \mathcal{O}(40) \). That is, for \( r < r_{u} \) w.h.p.

\[
|\mu_{x}^{[\omega]}(\Phi) - M_{x}(\Phi)| = o(1) \quad \text{for any } x \in V. \tag{6}
\]

To define the second correlation decay property, let us denote by \( \tau \) a uniformly random element of \( \mathcal{S}(\Phi) \). Then the non-reconstruction condition is that

\[
\mathbb{E}_{\tau} \left| M_{x}^{[\omega]}(\Phi, \tau) - M_{x}(\Phi) \right| = o(1). \tag{7}
\]

Hence, fixing the far away boundary to a “typical” satisfying assignment has no discernible effect on \( x \). Neglecting a \( o(1) \)-fraction of “atypical” cases \( \tau \), one might still expect \( \mathbb{E} \) to hold so long as \( \mathbb{C} \) is satisfied. However, this conjecture awaits a rigorous proof. According to the cavity method, \( \mathbb{C} \) holds up to \( r \sim r_{d} = 2^{k} \ln(k)/k \). Moreover, the best rigorously analyzed algorithm (which is based on local search) succeeds in finding a satisfying assignment in polynomial time right up to \( r \sim r_{d} \) w.h.p. \( \mathcal{O}(16) \).
To state the third property, let us denote the joint distribution of the truth values of the variables $B$ under a random satisfying assignment by $M_B$. Thus, $M_B$ is a probability distribution over $\{-1, 1\}^B$. Then the *replica symmetry* condition requires that the truth values of the variables in $B$ are asymptotically independent. Formally,

$$
M_B(\tau) - \prod_{x \in B: \tau(x) = 1} M_x(\Phi) \cdot \prod_{x \in B: \tau(x) = -1} (1 - M_x(\Phi)) = o(1) \quad \text{for any } \tau \in \{-1, 1\}^B.
$$

(8)

It has duly been conjectured in [31] that (8) suffices to obtain (9), i.e., to ensure that Belief Propagation yields the correct marginals on $\Phi$. The cavity method predicts that (8) holds for

$$
r \leq r_c = 2^k \ln 2 - 3 \ln 2 + O(2^{-k}),$$

while the conjectured satisfiability threshold [19] is

$$
r_{k-SAT} = 2^k \ln 2 - \frac{1 + \ln 2}{2} + O(2^{-k}) \approx r_c + 0.19.
$$

(9)

The best current rigorous lower bound on $r_{k-SAT}$ matches $r_c$ [19].

The densities $r_d, r_c$ are also conjectured to mark a change in the geometry of the set $S(\Phi)$ of satisfying assignments. Let us turn $S(\Phi)$ into a graph by considering $\sigma, \tau \in S(\Phi)$ adjacent if their Hamming distance is equal to one. While for densities $r < r_d$ the graph $S(\Phi)$ is conjectured to be (essentially) connected, for $r_d < r < r_c$ it shatters into an exponential number of tiny connected components w.h.p. More precisely, $S(\Phi)$ admits a decomposition

$$
S(\Phi) = \bigcup_{i=1}^N C_i.
$$

(10)

into “clusters” $C_i$ such that $|C_i| \leq \exp(-\Omega(n))|S(\Phi)|$ for all $i$ and such that any two satisfying assignments in different clusters have Hamming distance $\Omega(n)$. This decomposition was established rigorously in [3].

Intuitively, the cluster decomposition explains why (2) fails to hold for $r > r_d$: the conditional marginal $M_x^{\omega}(\Phi, \tau)$ corresponds to the marginal of $x$ within the cluster of $\tau$, in contrast to the marginal $M_x(\Phi)$ over the entire set of satisfying assignments.

Further, for $r_c < r < r_{k-SAT}$, the set of satisfying assignments still decomposes into exponentially many well-separated clusters w.h.p. But now a bounded number of clusters are conjectured to dominate. That is, if we order the clusters by size $|C_1| \geq \cdots \geq |C_N|$, then for a bounded number $\gamma = O(1)$ we have

$$
|S(\Phi)| \sim |C_1 \cup \cdots \cup C_\gamma|.
$$

(11)

This structure goes by the name of *condensation* in physics. The values that different variables take within each cluster $C_1, \ldots, C_\gamma$ are conjectured to be heavily dependent. Furthermore, (11) implies that $M_B$ is but a convex combination of a small (bounded) number of such intra-cluster distributions. Hence, the “condensed” geometry [11] appears to be irreconcilable with the factorization property (8).

**Belief Propagation.** Based on this “static” picture, three different hypotheses have been put forward as to the likely performance of Belief Propagation guided decimation. Most optimistically, the authors of [31] argue that Belief Propagation guided decimation ought to find satisfying assignments efficiently for densities right up to $r_c$. Their prediction derives from the opinion that (8) should be sufficient to obtain (9), and that (6) is the key to the success of Belief Propagation Guided Decimation. Specifically, [31] refers to the the “most biased variable” variant. However, the precise decimation strategy is irrelevant to their considerations, which are in effect at odds with Theorem [1].

A second prediction is that Belief Propagation Guided Decimation should fail to find satisfying assignments for $r > r_d$ [11]. This conjecture is based on the hunch that the decomposition of $S(\Phi)$ into “clusters” and the ensuing demise of (2) cause (6) to fail. Agreeing with [31], the authors appear to view (6) as the key to the performance of BPdec.
According to the third prediction [42], $\text{BP}_{\text{dec}}$ fails for densities $r > \rho_0 \cdot 2^k/k$, with $\rho_0 > 0$ an absolute constant (independent of $k$). This prediction is based on a non-rigorous analysis of the decimation process, i.e., the idealized thought experiment that $\text{BP}_{\text{dec}}$ strives to implement (Experiment 2). Crucially, the authors of [42] realize that (6) does not guarantee the success of $\text{BP}_{\text{dec}}$. Instead, their analysis indicates that as the decimation process proceeds to assign variables, the remaining unassigned variables are bound by clauses that become shorter and shorter. In effect, the clauses become more and more difficult to satisfy, and thus the remaining set of satisfying assignments shrinks rapidly. In other words, successive decimation of variables has a similar effect as increasing the density of the formula. Consequently, after a number $t$ of decimations $\text{BP}_{\text{dec}}$ may wind up with a formula $\Phi$, that violates (8) and thus (6), even though the initial formula $\Phi$ may well have satisfied those conditions. The contribution [42] supersedes an earlier attempt at studying the effect of decimation [39]. Theorem 1 is in agreement with the prediction from [42]. But an important advantage of the present work over the (non-rigorous) contribution [42] is that here we manage to analyze the actual algorithm $\text{BP}_{\text{dec}}$. By contrast, [42] only deals with the decimation process (i.e., Experiment 2, the idealized experiment that assumes knowledge of the precise marginals). That is, going significantly beyond the ambition of [42], here we develop a technique for explicitly analyzing the dynamics of the message passing procedure.

In summary, the predictions in [11,31] as to the performance of Belief Propagation are inaccurate because they ignore the effect of decimation. By contrast, as conjectured in [42] and proved here, in actuality $\text{BP}_{\text{dec}}$ gets itself into trouble by assigning and decimating one variable after the other. Thus, computing the correct marginals in the original formula $\Phi$ is one thing, but continuing to do so as decimation proceeds is quite another.²

**Survey Propagation.** Let us briefly comment on Survey Propagation guided decimation, the physicists’ flagship algorithm [11,35]. It is based on the idea of working with a different probability distribution. Namely, instead of the uniform probability over satisfying assignments, Survey Propagation aims for the uniform distribution over the clusters $C_i$ in the decomposition [10]. These clusters can be encoded as generalized assignments $\tau : V \rightarrow \{-1, 0, 1\}$, with $\tau(x) = \pm 1$ indicating that variable $x$ takes the value $\pm 1$ in all the assignments in $C_i$, and $\tau(x) = 0$ indicating that $x$ can take either value [33,36]. Survey Propagation guided decimation combines a message passing algorithm for approximating the marginals of these generalized assignments with a decimation procedure (see [11] for details).

According to the cavity method, the Survey Propagation distribution enjoys a factorization property akin to (8) for densities $r$ right up to $r_{k-SAT}$. In fact, the physicists’ computation of the conjectured $r_{k-SAT}$ depends on this assumption [34,36]. Furthermore, the (conjectured) factorization property nurtured hopes that Survey Propagation may perform well for densities “close to” $r_{k-SAT}$ [11,31,35].

Given the above discussion of Belief Propagation, the obvious problem with this forecast is that it ignores the effect of decimation. More precisely, it might well be that in the undecimated random formula $\Phi$ the Survey Propagation distribution factorizes for densities right up to $r_{k-SAT}$. But this might not be the case in a formula $\Phi_t$ where some variables have been decimated. Since the arguments of [42] do not seem to extend to Survey Propagation, there is currently not even a non-rigorous study of the effect of decimation in this case. Thus, while experiments consistently indicate that Survey Propagation guided decimation outperforms $\text{BP}_{\text{dec}}$, it is unclear how much so. Guided by the present analysis of Belief Propagation, I venture to pose

Conjecture 7. There is an absolute constant $\rho_1 > 0$ such that the Survey Propagation Guided Decimation algorithm as stated in [11] fails to find a satisfying assignment of $\Phi$ w.h.p. for $r > \rho_1 \cdot 2^k/k$.

If true, Conjecture 7 would imply that Survey Propagation Guided Decimation is inferior to conceptually much simpler local-search algorithms (such as [15]), at least for large clause lengths $k$.

² Let us mention, as a cautionary tale, that both [31,42] quote experimental evidence to support their claims. This illustrates the difficulty of producing reliable experimental results on large random CSPs, and thus the need for rigorous results.
1.5 Further related work

In full generality, Belief Propagation is a generic technique for computing the marginals of a probability distribution described by an “acyclic graphical model” [41]. But special instantiations of Belief Propagation have been (re)discovered several times for several applications. Examples include statistical inference [41], coding theory [22] and statistical mechanics [10], where the method is also referred to as “Bethe-Peierls approximation”. For a coherent discussion see [36] and the references therein.

In spite of BP’s practical success (and popularity), rigorous analyses of the algorithm are scarce. A few exist in the context of LDPC decoding (e.g., [17,43]). We also analyzed BP for graph 3-coloring on a certain class of expander graphs. A further related result deals with the conceptually much simpler Warning Propagation algorithm on certain random 3-CNFs (“planted model”) [20]. In the random $k$-XORSAT problem (random linear equations mod 2), Belief Propagation reduces to Warning Propagation due to the algebraic nature of the problem and can thus be analyzed easily [36]. Furthermore, there has been some recent progress on analyzing certain variants of BP (such as the “max-product algorithm”) for certain optimization problems that are polynomial-time solvable in the worst case (e.g., [9,23]).

A distinctive feature of Belief Propagation Guided Decimation in comparison to earlier algorithmic applications of Belief Propagation is the decimation step (that the algorithm assigns one variable at a time and reruns Belief Propagation on the reduced formula). In terms of analyzing the algorithm, decimation poses a substantial challenge. The present paper furnishes the first analysis of this kind of algorithm on a non-trivial type of instances. By contrast, previous analyses deal with algorithms that use BP in a ‘one shot’ fashion, i.e., the supposed marginals obtained via BP are used directly to assign all variables at once [17,43]. Roughly speaking, this approach seems to work best if the problem instances are somewhat over-constrained so that there is (essentially) a unique solution. By contrast, as we saw in Section 1.4 for $r < r_{k\text{-SAT}}$ the random formula $\Phi$ w.h.p. has exponentially many satisfying assignments, whose typical pairwise distance is close to $\frac{n}{2}$ w.h.p. Furthermore, as we saw in Section 1.4 it is the decimation step that precipitates the demise of Belief Propagation Guided Decimation.

In the context of random constraint satisfaction problems, Belief Propagation works out to be a special (the “replica symmetric”) case of a larger statistical mechanics framework called the cavity method [36]. The cavity method provides a toolbox for deriving highly non-trivial exact conjectures on various phase transitions in random CSPs. The conjectured value (9) of the $r_{k\text{-SAT}}$ is an example, but the method is quite general and has been applied to a host of further CSPs as well.

The study of the BP marginals on the undecimated random formula $\Phi$ is somewhat related to the so-called reconstruction problem. This problem has been studied on ‘symmetric’ random CSPs, which include problems such as (hyper)graph coloring [38], but not $k$-SAT. The proofs in [38] are based on indirect arguments (related to the second moment method), which do not seem to extend to an analysis of BPdec.

1.6 Preliminaries and notation

In this section we collect a few well-known results and introduce a bit of notation. First of all, we note for later reference a well-known estimate of the expected number of satisfying assignments (see, e.g., [6] for a derivation).

Lemma 8. We have $E|S(\Phi)| = \Theta(2^n(1-2^{-k})^m) \leq 2^n \exp(-rn/2^k)$.

Furthermore, we are going to need the following Chernoff bound on the tails of a binomially distributed random variable or, more generally, a sum of independent Bernoulli trials [26, p. 21].

Lemma 9. Let $X$ be a sum of independent Bernoulli variables with mean $\mu > 0$. Let

$$\varphi(x) = (1+x) \ln(1+x) - x.$$ 

Then for any $t > 0$,

$$P \{ X > \mu + t \} \leq \exp(-\mu \cdot \varphi(t/\mu)),$$

$$P \{ X < \mu - t \} \leq \exp(-\mu \cdot \varphi(-t/\mu)).$$

In particular, for any $t > 1$ we have $P \{ X > t\mu \} \leq \exp \left[ -t\mu \ln(t/e) \right]$. 

For a real $b \times a$ matrix $A$ let
\[ \|A\|_\square = \max_{C \in \mathbb{R}^a \setminus \{0\}} \|A C\|_1 / \|C\|_\infty. \]
Thus, $\|A\|_\square$ is the norm of $A$ viewed as an operator from $\mathbb{R}^a$ equipped with the $L^\infty$-norm to $\mathbb{R}^b$ endowed with the $L^1$-norm. For a set $A \subseteq [a] = \{1, \ldots, a\}$ we let $1_A \in \{0, 1\}^a$ denote the indicator vector of $A$. The following well-known fact about the norm $\|\|_\square$ of matrices with diagonal entries equal to zero is going to come in handy.

**Fact 10.** For a real $b \times a$ matrix $A$ with zeros on the diagonal we have
\[ \|A\|_\square \leq 24 \max_{A \subseteq [a], B \subseteq [b], A \cap B = \emptyset} |\langle A 1_A, 1_B \rangle|. \]

Finally, throughout the paper we let $S_n$ denote the set of permutations of $[n]$.

## 2 The probabilistic framework for analyzing $\text{BPdec}$

### 2.1 Outline

The single most important technique for analyzing algorithms on the random input $\Phi$ is the “method of deferred decisions”. Where it applies, the dynamics of the algorithm can typically be traced tightly via differential equations, martingales, or Markov chains. Virtually all of the previous analyses of algorithms for random $k$-SAT are based on this approach [2][1][3][4][5][6][7][2][2]. Unfortunately, the ‘deferred decisions’ technique is limited to very simple, ‘shortsighted’ algorithms that decide upon the value of a variable $x$ on the basis of the clauses/variables at distance, say, one or two from $x$ in the factor graph [7]. By contrast, in order to assign some variable $x_t$, $\text{BPdec}$ explores clauses at distance up to $2\omega$ from $x_t$, where (potentially) $\omega = \omega(n) \rightarrow \infty$. This renders a ‘deferred decisions’ approach hopeless.

Therefore, to prove Theorem 1 we need a fundamentally different strategy. In the present section we set up the probabilistic framework for the analysis. We will basically reduce the analysis of $\text{BPdec}$ to the problem of analyzing the BP operator on the formula that is obtained from $\Phi$ by substituting ‘true’ for the first $t$ variables $x_1, \ldots, x_t$ and simplifying (Theorem 5 blow). In the next section we will show that this decimated formula enjoys a few simple quasirandomness properties with probability extremely close to one. Finally, we will show that these properties suffice to trace the BP computation.

Applied to a fix, non-random formula $\Phi$ on $V = \{x_1, \ldots, x_n\}$, $\text{BPdec}$ yields an assignment $\sigma : V \rightarrow \{-1, 1\}$ (that may or may not be satisfying). This assignment is random, because $\text{BPdec}$ itself is randomized. Hence, for any fixed $\Phi$ running $\text{BPdec}(\Phi)$ induces a probability distribution $\beta_\Phi$ on $\{-1, 1\}^V$. With $S(\Phi)$ the set of all satisfying assignments of $\Phi$, the ‘success probability’ of $\text{BPdec}$ on $\Phi$ is just
\[ \text{success}(\Phi) = \beta_\Phi(S(\Phi)). \]

Thus, to establish Theorem 1 we need to show that in the random formula,
\[ \text{success}(\Phi) = \beta_\Phi(S(\Phi)) = \exp(-\Omega(n)) \]
is exponentially small w.h.p. To this end, we are going to prove that the measure $\beta_\Phi$ is ‘rather close’ to the uniform distribution on $\{-1, 1\}^V$ w.h.p., of which $S(\Phi)$ constitutes only an exponentially small fraction.

To facilitate the analysis, we are going to work with a slightly modified version of $\text{BPdec}$. While the original $\text{BPdec}$ assigns the variables in the natural order $x_1, \ldots, x_n$, the modified version $\text{PermBPdec}$ chooses a permutation $\pi$ of $[n]$ uniformly at random and assigns the variables in the order $x_{\pi(1)}, \ldots, x_{\pi(n)}$. Let $\beta_{\Phi}$ denote the probability distribution induced on $\{-1, 1\}^V$ by $\text{PermBPdec}(\Phi)$. Because the uniform distribution over $k$-CNFs is invariant under permutations of the variables, we obtain

**Fact 11.** If $\beta_{\Phi}(S(\Phi)) \leq \exp(-\Omega(n))$ w.h.p., then $\text{success}(\Phi) = \beta_\Phi(S(\Phi)) \leq \exp(-\Omega(n))$ w.h.p.

Let $\Phi$ be a $k$-CNF and let $\delta > 0$. Given a permutation $\pi$ and a partial assignment $\sigma : \{x_{\pi(s)} : s \leq t\} \rightarrow \{-1, 1\}$, we let $\Phi_{t, \pi, \sigma}$ denote the formula obtained from $\Phi$ by substituting the values $\sigma(x_{\pi(s)})$ for the variables $x_{\pi(s)}$ for $1 \leq s \leq t$ and simplifying. Formally, $\Phi_{t, \pi, \sigma}$ is obtained from $\Phi$ as follows:
• remove all clauses \( a \) of \( \Phi \) that contain a variable \( x_{\pi(s)} \) with \( 1 \leq s \leq t \) such that \( \sigma(x_{\pi(s)}) = \text{sign}(x_{\pi(s)}, a) \).
• for all clauses \( a \) that contain a \( x_{\pi(s)} \) with \( 1 \leq s \leq t \) such that \( \sigma(x_{\pi(s)}) \neq \text{sign}(x_{\pi(s)}, a) \), remove \( x_{\pi(s)} \) from \( a \).
• remove any empty clauses (resulting from clauses of \( \Phi \) that become unsatisfied if we set \( x_{\pi(s)} \) to \( \sigma(x_{\pi(s)}) \) for \( 1 \leq s \leq t \)) from the formula.

For a number \( \delta > 0 \) and an index \( l > t \) we say that \( x_{\pi(l)} \) is \((\delta, t)\)-biased if
\[
\left| \mu_{x_{\pi(l)}}(\Phi_{t, \pi, \sigma}) - 1/2 \right| > \delta.
\]
Moreover, the triple \((\Phi, \pi, \sigma)\) is \((\delta, t)\)-balanced if no more than \( \delta(n-t) \) variables are \((\delta, t)\)-biased.

Let \( \pi \) be the permutation chosen by \( \text{PermBPdec}() \), and let \( \sigma \) be the partial assignment constructed in the first \( t \) steps. The variable \( x_{\pi(t+1)} \) is uniformly distributed over the set \( V \setminus \{ x_{\pi(s)} : s \leq t \} \) of currently unassigned variables. Hence, if \((\Phi, \pi, \sigma)\) is \((\delta, t)\)-balanced, then the probability that \( x_{\pi(t+1)} \) is \((\delta, t)\)-biased is bounded by \( \delta \). (This conclusion was the purpose of decimating the variables in a random order.) Furthermore, given that \( x_{\pi(t+1)} \) is not \((\delta, t)\)-biased, the probability that \( \text{PermBPdec}() \) will assign it to ‘true’ lies in the interval \([\frac{1}{2} - \delta, \frac{1}{2} + \delta]\). Consequently,
\[
\left| \frac{1}{2} - P[\sigma(x_{\pi(t+1)}) = 1|\Phi, \pi, \sigma] \right| \leq 2\delta.
\]
(12)
Thus, the smaller \( \delta \), the closer \( \sigma(x_{\pi(t+1)}) \) comes to being uniformly distributed. Hence, if \((\delta, t)\)-balancedness holds for all \( t \) with a ‘small’ \( \delta \), then \( \beta_\Phi \) will be close to the uniform distribution on \([-1, 1]^V\).

To put this observation to work, we define
\[
\delta_t = \exp(-c(1-t/n)k) \quad \text{and} \quad \hat{t} = \left(1 - \frac{\ln(kr/2^k)}{\epsilon^2 k} \right)n,
\]
(13)
where \( c > 0 \) is a small enough absolute constant\(^3\). In addition, we let
\[
\Delta_t = \sum_{s=1}^{t} \delta_t,
\]
(14)
Lemma 12. For any \( 0 \leq t \leq \hat{t} \) we have
\[
\Delta_t \leq (1 + o(1)) \frac{n}{ck} \exp(ck(1-t/n)).
\]
Furthermore, \( \Delta_t \sim \frac{n}{ck} \left[(kr/2^k)^{-\epsilon} - \exp(-ck)\right] \).

Proof. We have
\[
\Delta_t = \sum_{s=1}^{t} \delta_s = \exp(-ck) \sum_{s=1}^{t} \exp(cs/k/n) = \exp(-ck) \left[ \frac{\exp(ck(t+1)/n) - 1}{\exp(ck/n) - 1} - 1 \right].
\]
(15)
Since \( \exp(ck/n) = 1 + ck/n + O(n^{-2}) \) and \( \hat{t} = \Omega(n) \), we obtain from (15)
\[
\Delta_t \sim \frac{n}{ck} \left[ \exp \left( \frac{ck}{n} \left( \frac{\hat{t}}{n} - 1 \right) \right) - \exp(-ck) \right] = \frac{n}{ck} \left[ (kr/2^k)^{-\epsilon} - \exp(-ck) \right].
\]
Furthermore, for \( 1 \leq t \leq \hat{t} \) equation (15) yields the upper bound
\[
\Delta_t \leq \exp(-ck) \cdot \frac{\exp(ck(t+1)/n)}{\exp(ck/n) - 1} = \frac{\exp(ck(t-n)/n)}{1 - \exp(-ck)/n}
\sim \frac{n}{ck} \exp(-ck(1-t/n)),
\]
as \( \exp(-ck/n) = 1 - ck/n + O(n^{-2}) \).

\(^3\) Setting \( c = 10^{-10^{10}} \) will evidently suffice, but no attempt at finding the optimal \( c \) has been made.
Lemma 14. The following is a well-known fact.

\[ \left| \{ (\pi, \sigma) \in S_n \times \{-1, 1\}^V : (\Phi, \pi, \sigma) \text{ is not } (\delta_t, t)\text{-balanced} \} \right| \leq 2^n n! \cdot \exp \left[ -10 (\xi n + \Delta_t) \right]. \]

Proceeding by induction on \( t \), we are going to use (12) to relate the distribution \( \beta_{\Phi} \) to the uniform distribution on \( \{-1, 1\}^V \) for \((t, \xi)\)-uniform formulas. More precisely, in Section 2.2 we are going to prove

Proposition 13. Suppose that \( \Phi \) is \((t, \xi)\)-uniform for all \( 0 \leq t \leq \hat{t} \). Then

\[ \beta_{\Phi}(E) \leq \frac{|E|}{2^t} \cdot \exp \left[ 4\Delta_t \right] + \exp(\xi n/2) \quad \text{ for any } E \subset \{-1, 1\}^V. \quad (16) \]

Proposition 13 reduces the proof of Theorem 11 to showing that \( \Phi \) is \((t, \xi)\)-uniform with some appropriate probability.

To prove this, we need two simple definitions. We call a clause \( a \) of a formula \( \Phi \) redundant if \( \Phi \) has another clause \( b \) such that \( a, b \) have at least two variables in common. Furthermore, we call the formula \( \Phi \) tame if

i. \( \Phi \) has no more than \( \ln n \) redundant clauses, and
ii. no more than \( \ln n \) variables occur in more than \( \ln n \) clauses of \( \Phi \).

The following is a well-known fact.

Lemma 14. The random formula \( \Phi \) is tame w.h.p.

Now, the following result provides the key estimate for proving that \( \Phi \) is \((t, \xi)\)-uniform with a very high probability.

Theorem 15. There is a constant \( \rho_0 > 0 \) such that for any \( k, r \) satisfying \( 2^k \rho_0/k \leq r \leq 2^k \ln 2 \) there is \( \xi = \xi(k, r) > 0 \) so that for \( n \) large enough the following holds. Fix any permutation \( \pi \) of \( [n] \) and any assignment \( \sigma \in \{-1, 1\}^n \). Then for any \( 0 \leq t \leq \hat{t} \) we have

\[ P \left[ (\Phi, \pi, \sigma) \text{ is } (\delta_t, t)\text{-balanced} | \Phi \text{ is tame} \right] \geq 1 - \exp \left[ -3\xi n - 10 \Delta_t \right]. \quad (17) \]

We defer the proof of Theorem 15 to Section 3.

Corollary 16. In the notation of Theorem 15

\[ P \left[ \forall t \leq \hat{t} : (\Phi, \pi, \sigma) \text{ is tame} \right] \geq 1 - \exp(-\xi n). \]

Proof. For \( 1 \leq t \leq \hat{t} \) and a \( k \)-CNF \( \Phi \) we let \( X_t(\Phi) \) signify the number of pairs \((\pi, \sigma) \in S_n \times \{-1, 1\}^V \) such that \((\Phi, \pi, \sigma)\) fails to be \((\delta_t, t)\)-balanced. Then Theorem 15 yields

\[ E[X_t(\Phi)|\Phi \text{ is tame}] \leq 2^n n! \cdot \exp(-3\xi n - 10 \Delta_t). \]

Hence, by Markov’s inequality and the union bound

\[ P \left[ \exists t \leq \hat{t} : X_t(\Phi) > 2^n n! \cdot \exp(-\xi - 10 \Delta_t) | \Phi \text{ is tame} \right] \leq n \exp(-2\xi n) \leq \exp(-\xi n). \quad (18) \]

Since \( \Phi \) is \((t, \xi)\)-uniform if \( X_t(\Phi) \leq 2^n n! \cdot \exp(-\xi n - 10 \Delta_t) \), the assertion follows from (18). \( \Box \)

Proof of Theorem 7 Let us keep the notation of Theorem 15. By Lemma 14 we may condition on \( \Phi \) being tame. Let \( \mathcal{U} \) be the event that \( \Phi \) is \((t, \xi)\)-uniform for all \( 1 \leq t \leq \hat{t} \). Let \( \mathcal{S} \) be the event that \(|\mathcal{S}(\Phi)| \leq n \cdot E[\mathcal{S}(\Phi)]\). By Corollary 16 and Markov’s inequality, we have \( \Phi \in \mathcal{U} \cap \mathcal{S} \) w.h.p. If \( \Phi \in \mathcal{U} \cap \mathcal{S} \), then by Proposition 13

\[ \beta_{\Phi}(\mathcal{S}(\Phi)) \leq \frac{|\mathcal{S}(\Phi)|}{2^t} \cdot \exp \left[ 4\Delta_t \right] + \exp(\xi n/2) \]

\[ \leq n \cdot E[\mathcal{S}(\Phi)] \cdot 2^{-t} \exp \left[ 4\Delta_t \right] + \exp(\xi n/2). \quad (19) \]
By Lemmas 8 and 12 we have \( E(S(\Phi)) \leq 2^n \exp(-rn/2^k) \) and \( \Delta_i \leq \frac{n}{\sqrt{k}} (kr/2^k)^{-c} \). Plugging these estimates and the definition (13) of \( \hat{\ell} \) into (19), we find that given \( \Phi \in \mathcal{U} \cap \mathcal{S} \),

\[
\hat{\beta}_{\Phi}(S(\Phi)) \leq n \exp \left[ n \left( -\frac{r}{2^k} + \frac{\ln(kr/2^k)}{c^2} + \frac{4}{c^2 k} (kr/2^k)^{-c} \right) \right] + \exp(-\xi n/2).
\]

Recalling that \( \rho = kr/2^k \), we thus obtain

\[
\hat{\beta}_{\Phi}(S(\Phi)) \leq n \exp \left[ -\frac{n}{c^k} \left( \rho - \frac{\ln 2 \ln \rho}{c^2} - \frac{4}{c^2} \right) \right] + \exp(-\xi n/2).
\]

Hence, if \( \rho \geq \rho_0 \) for a sufficiently large constant \( \rho_0 > 0 \), then (20) yields \( \hat{\beta}_{\Phi}(S(\Phi)) = \exp(-\Omega(n)) \).

Finally, Theorem 1 follows from Fact 11. \( \square \)

### 2.2 Proof of Proposition 13

We consider an additional variant of \( \text{BPdec} \) that receives the order \( \pi \) in which variables are to be decimated as an input parameter.

#### Algorithm 17. \( \text{BPdec}(\Phi, \pi) \)

**Input:** A \( k \)-SAT formula \( \Phi \) on \( V = \{x_1, \ldots, x_n\} \) and a permutation \( \pi \in S_n \).

**Output:** An assignment \( \tau : V \rightarrow \{-1, 1\} \).

1. Let \( \Phi_0 = \Phi \).
2. For \( t = 0, \ldots, n - 1 \) do
   1. Compute the BP results \( \mu_x^{[\tau]}(\Phi_t) \).
   2. Let
      \[
      \sigma(x_{n(t+1)}) = \begin{cases} 
      1 & \text{with probability } \mu_x^{[\tau]}(\Phi_t), \\
      -1 & \text{with probability } 1 - \mu_x^{[\tau]}(\Phi_t). 
      \end{cases}
      \]
3. Obtain \( \Phi_{t+1} \) from \( \Phi_t \) by substituting the value \( \sigma(x_{n(t+1)}) \) for \( x_{n(t+1)} \) and simplifying.
4. Return the assignment \( \sigma \).

Fix a \( k \)-CNF \( \Phi \) that is \((t, \xi)\)-uniform for all \( 1 \leq t \leq \hat{\ell} \). Let \( S_n \) be the set of all permutations on \([n]\). Let \( \lambda_{\Phi} \) be the probability distribution on pairs \((\pi, \sigma) \in S_n \times \{-1, 1\}^V \) induced by choosing a permutation \( \pi \in S_n \) uniformly at random and letting \( \sigma = \text{BPdec}(\Phi, \pi) \). Then \( \hat{\beta}_{\Phi} \) is the \( \sigma \)-marginal of \( \lambda_{\Phi} \), i.e.,

\[
\hat{\beta}_{\Phi}(\mathcal{E}) = \lambda_{\Phi}(S_n \times \mathcal{E}) \quad \text{for all } \mathcal{E} \subset \{-1, 1\}^V.
\]

In order to study \( \lambda_{\Phi} \), we consider another distribution \( \lambda_{\Phi}' \) on pairs \((\pi, \sigma') \in S_n \times \{0, 1\}^V \) that is easier to analyze and that will turn out to be ‘close’ to \( \lambda_{\Phi} \). To define \( \lambda_{\Phi}' \), let \( B_t \) be the set of all pairs \((\pi, \sigma)\) such that \((\Phi, \pi, \sigma)\) is not \((\delta_t, t)\)-balanced. Moreover, let \( B = \bigcup_{t=0}^{\hat{\ell}} B_t \). The distribution \( \lambda_{\Phi}' \) is induced by choosing a permutation \( \pi \) uniformly at random and running the following algorithm on \( \Phi, \pi \).

#### Algorithm 18. \( \text{BPdec}'(\Phi, \pi) \)

**Input:** A \( k \)-SAT formula \( \Phi \) on \( V = \{x_1, \ldots, x_n\} \) and a permutation \( \pi \in S_n \).

**Output:** An assignment \( \sigma' : V \rightarrow \{-1, 1\} \).

1. Let \( \Phi_0 = \Phi \).
2. For \( t = 0, \ldots, n - 1 \) do
   1. Compute the BP results \( \mu_x^{[\tau]}(\Phi_t) \).
   2. If \((\Phi, \pi, \sigma')\) is \((\delta_t, t)\)-balanced, then let
      \[
      \sigma'(x_{n(t+1)}) = \begin{cases} 
      1 & \text{with probability } \mu_x^{[\tau]}(\Phi_t), \\
      -1 & \text{with probability } 1 - \mu_x^{[\tau]}(\Phi_t). 
      \end{cases}
      \]
Then for any

follows.

Hence, Bayes’ rule yields that for any pair

as desired.

In particular,

Case 2: \( (\pi, \sigma) \notin B_t \) and set \( t \). To relate

Claim that

\[ \lambda'_{\Phi} = 1 \] when \( t \) is

The probability of the event

occurs with probability

Let \( L' \) be the event that

Then for any \( 1 \leq t \leq i \) we can bound the conditional probability

as follows.

Case 1: \( (\Phi, \pi, \sigma) \in B_t \). In this case \( (\Phi, \pi, \sigma) \) is not \((\delta_t, t)\)-balanced. Therefore, step 3 of \( B_{\Phi \pi \sigma} \) chooses the value \( \sigma'(x_{\pi(t)}) \) uniformly. Hence, the event \( \sigma'(x_{\pi(t)}) = \sigma(x_{\pi(t)}) \) occurs with probability \( \frac{1}{2} \).

Case 2: \( (\pi, \sigma) \notin B_t \) and \( A_t(\pi, \sigma) = 0 \). Since \( (\Phi, \pi, \sigma) \) is \((\delta_t, t)\)-balanced, step 3 of \( B_{\Phi \pi \sigma} \) uses the BP marginals \( \mu_{x_{\pi(t)}}(\zeta) \) in order to assign \( x_{\pi(t)} \). Because \( A_t(\pi, \sigma) = 0 \), the variable \( x_{\pi(t)} \) is not \((\delta_t, t)\)-biased, whence \( \mu_{x_{\pi(t)}}(\zeta) \leq \frac{1}{2} + \delta_t \) for both \( \zeta = -1 \) and \( \zeta = 1 \). Hence, the probability that \( \sigma'(x_{\pi(t)}) = \sigma(x_{\pi(t)}) \) is bounded by \( \frac{1}{4} + \delta_t \).

Case 3: \( A_t(\pi, \sigma) = 1 \). In this case we just use the trivial fact that the probability of the event \( \sigma'(x_{\pi(t)}) = \sigma(x_{\pi(t)}) \) is bounded by \( 1 \leq 2 \frac{1}{4} + \delta_t \).
In any case, we obtain the bound \( \lambda'_\Phi [L_t | \pi(t) = \pi(t) \land \bigwedge_{s < t} L_s] \leq 2^{A_t(\pi, \sigma)} (\frac{1}{2} + \delta_t). \) Consequently, as \( \lambda'' \) is the uniform distribution, we get

\[
\frac{\lambda'_\Phi [L_t | \pi(t) = \pi(t) \land \bigwedge_{s < t} L_s]}{\lambda'' [L_t | \pi(t) = \pi(t) \land \bigwedge_{s < t} L_s]} \leq 2^{A_t(\pi, \sigma)} (1 + 2\delta_t). \tag{23}
\]

Multiplying (23) up for \( t \leq i \) yields the assertion. \( \square \)

To put Lemma 20 to work, we need to estimate \( A(\pi, \sigma') \).

**Lemma 21.** We have \( \lambda'_\Phi [A(\pi, \sigma') > 4(\Delta_i + \xi n)] \leq \exp(-\xi n). \)

**Proof.** We are going to bound the probability that \( A_t(\pi, \sigma') = 1 \) given the values \( \pi(s), \sigma'(x_{\pi(s)}) \) for \( 1 \leq s < t \).

**Case 1:** the event \( B_i \) occurs. Then \( A_i = 0 \) by definition.

**Case 2:** the event \( B_i \) does not occur. In this case \( (\Phi, \pi, \sigma) \) is \((\delta_i, t)\)-balanced, which means that no more than \( \delta_i (n - t) \) variables are biased. Since the permutation \( \pi \) is chosen uniformly at random, the probability that \( x_{\pi(t)} \) is \((\delta_i, t)\)-biased is bounded by \( \delta_i \).

Thus, in either case the conditional probability of the event \( A_t = 1 \) is bounded by \( \delta_i \). This implies that the random variable \( A(\pi, \sigma') = \sum_{t \in \hat{t}} A_t(\pi, \sigma') \) is stochastically dominated by a sum of mutually independent Bernoulli variables with means \( \delta_1, \ldots, \delta_i \). Therefore, the assertion follows from Lemma 9 (the Chernoff bound). \( \square \)

**Proof of Proposition 13.** Combining Lemmas 20 and 21 we see that

\[
\lambda'_\Phi [F_i] \leq \lambda'_\Phi [A_i(\pi, \sigma') > 4(\Delta_i + \xi n)] + \lambda'_\Phi [F_i \land A_i(\pi, \sigma') \leq 4(\Delta_i + \xi n)] \\
\leq \exp(-\xi n) + \lambda'' [F_i] \cdot 2^{4(\Delta_i + \xi n)} \prod_{t \leq i} 1 + 2\delta_t \\
\leq \lambda'' [F_i] \cdot \exp(6\Delta_i + 4\xi n) + \exp(-\xi n) \quad \text{for any } F \subset S_n \times \{-1, 1\}^V. \tag{24}
\]

Our assumption that \( \Phi \) is \((t, \xi)\)-uniform ensures that \( \lambda'' [B_i] \leq \exp(-10(\xi n + \Delta_i)) \) for any \( t \leq \hat{t} \). Together with (24), this implies that

\[
\lambda'_\Phi [B_i] \leq \lambda'' [B_i] \exp(6\Delta_i + 4\xi n) + \exp(-\xi n) \leq 2 \exp(-\xi n) \quad \text{for any } t \leq \hat{t}.
\]

Therefore, by the union bound

\[
\lambda'_\Phi [B] \leq 2\hat{t} \exp(-\xi n) \leq \exp(-0.9\xi n). \tag{25}
\]

Finally, consider any \( \mathcal{E} \subset \{-1, 1\}^V \). Let \( \mathcal{F} = S_n \times \mathcal{E} \). Then

\[
\beta_\Phi(\mathcal{E}) = \lambda_\Phi [\mathcal{F}] \\
\leq \lambda'_\Phi [\mathcal{F}] + \lambda'_\Phi [B] \quad \text{[due to (21)]} \\
\leq \lambda'_\Phi [\mathcal{F}] + \exp(-0.9\xi n) \quad \text{[by Lemma 19]} \\
\leq \lambda'' [\mathcal{F}] \exp(6(\Delta_i + \xi n)) + \exp(-\xi n/2) \quad \text{[by (23)]} \\
= \frac{|\mathcal{F}|}{n^{2n}} \cdot \exp(6(\Delta_i + \xi n)) + \exp(-\xi n/2) \quad \text{[as \( \lambda'' \) is uniform]} \\
= \frac{|\mathcal{E}|}{2\hat{t}} \cdot \exp(6(\Delta_i + \xi n)) + \exp(-\xi n/2) \quad \text{[by the definition of \( \mathcal{F}_i \)]},
\]

as desired. \( \square \)
3 Tracing the Belief Propagation Operator

3.1 Overview

The goal in this section (and the rest of the paper) is to establish Theorem 15, which states that for any fixed permutation \( \pi \) and any fixed assignment \( \sigma \) the triple \((\Phi, \pi, \sigma)\) is \((\delta_t, t)\)-balanced with probability very close to one. The basic symmetry properties of the random formula \( \Phi \) allow us to assume without loss of generality that \( \pi = \text{id} \) is the identity and that \( \sigma = 1 \) is the all-true assignment. More precisely, we observe the following.

**Fact 22.** Fix any permutation \( \pi \) of \([n]\) and any assignment \( \sigma \in \{0, 1\}^V \). Then for any \( 0 \leq t \leq \hat{t} \) we have

\[
P[(\Phi, \pi, \sigma) \text{ is } (\delta_t, t)\text{-balanced}] = P[(\Phi, \text{id}, 1) \text{ is } (\delta_t, t)\text{-balanced}].
\]

**Proof.** For a \( k\)-CNF \( \Phi \) let \( \Phi^{\pi, \sigma} \) be the formula obtained by replacing

- each occurrence of the literal \( x_i \) in \( \Phi \) by \( x_{\pi(i)} \) if \( \sigma(x_{\pi(i)}) = 1 \), and by \( \neg x_{\pi(i)} \) if \( \sigma(x_{\pi(i)}) = -1 \), and
- each occurrence of the literal \( \neg x_i \) in \( \Phi \) by \( \neg x_{\pi(i)} \) if \( \sigma(x_{\pi(i)}) = 1 \), and by \( x_{\pi(i)} \) if \( \sigma(x_{\pi(i)}) = -1 \).

Then \((\Phi, \text{id}, 1)\) is \((\delta_t, t)\)-balanced iff \((\Phi^{\pi, \sigma}, \pi, \sigma)\) is. Furthermore, the map \( \Phi \mapsto \Phi^{\pi, \sigma} \) is a bijection. Consequently, for the uniformly random formula \( \Phi \) the resulting formula \( \Phi^{\pi, \sigma} \) is uniformly random as well, for any \( \pi, \sigma \)

Thus, we assume from now on that \( \pi = \text{id} \) and \( \sigma = 1 \). Then the decimated formula \( \Phi_{t, \pi, \sigma} \) is simply obtained from \( \Phi \) by substituting the value ‘true’ for \( x_1, \ldots, x_t \) and simplifying. To unclutter the notation, we are going to denote \( \Phi_{t, \pi, \sigma} \) by \( \Phi^t \) from now on. Let \( G \) be the factor graph of \( \Phi^t \).

Our task is to study the BP operator defined in (4) and (3) on \( \Phi^t \). That is, starting from the initial set of messages \( \mu_{x \rightarrow a}^{[0]}(\pm 1) = \frac{1}{2} \) for all \( x \in V_t, a \in N(x) \), we define inductively for \( \ell \geq 0 \)

\[
\mu_{a \rightarrow x}^{[\ell]}(\zeta) = \begin{cases} 
1 & \text{if } \zeta = \text{sign}(x, a), \\
1 - \prod_{y \in N(a) \setminus \{x\}} \mu_{y \rightarrow a}^{[\ell]}(-\text{sign}(y, a)) & \text{if } \zeta = -\text{sign}(x, a)
\end{cases}
\]

and

\[
\mu_{x \rightarrow a}^{[\ell+1]}(\zeta) = \text{BP}(\mu^{[\ell]}) = \frac{\prod_{b \in N(x) \setminus \{a\}} \mu_{b \rightarrow x}^{[\ell]}(\zeta)}{\prod_{b \in N(x) \setminus \{a\}} \mu_{b \rightarrow x}^{[\ell]}(1) - 1 + \prod_{b \in N(x) \setminus \{a\}} \mu_{b \rightarrow x}^{[\ell]}(1)},
\]

unless the denominator equals zero, in which case \( \mu_{x \rightarrow a}^{[\ell+1]}(\zeta) = \frac{1}{2} \).

A non-rigorous sketch of a rigorous analysis. Before launching into the details of the (long and technical) proof, we are going to give a brief sketch based on heuristic considerations. The aim of this is to develop some intuition. Roughly speaking, Theorem 15 asserts that with probability very close to one, most of the messages \( \mu_{x \rightarrow a}^{[\ell]}(\pm 1) \) are close to 1/2. Hence, letting

\[
\Delta^{[\ell]}_{x \rightarrow a} = \mu_{x \rightarrow a}^{[\ell]}(1) - \frac{1}{2},
\]

we aim to show that \( |\Delta^{[\ell]}_{x \rightarrow a}| \) is small for most \( x, a \). The proof of this is by induction on \( \ell \). That is, given the \( \Delta^{[\ell]}_{x \rightarrow a} \) we need to prove that the biases \( \Delta^{[\ell+1]}_{x \rightarrow a} \) do not “blow up”. More precisely, let us denote by

\[
\theta = 1 - t/n
\]
the fraction of unassigned variables. Then our induction hypothesis is that for all but \( \delta n \) variables we have
\[
\max_{a \in N(a)} |\Delta_{x \rightarrow a}^{[\ell]}| \leq \delta \ell = \exp(-c\theta k),
\]
and the goal is to show that the same holds true for \( \ell + 1 \). To establish this, we need to investigate one iteration of the update rules (26)–(27).

Rewriting (26) in terms of the biases \( \Delta_{y \rightarrow a}^{[\ell]} \), we obtain
\[
\mu_{a \rightarrow x}^{[\ell]}(-\text{sign}(x, a)) = 1 - \prod_{y \in N(a) \setminus \{x\}} \frac{1}{2} - \text{sign}(y, a)\Delta_{y \rightarrow a}^{[\ell]} = 1 - 2^{1 - |N(a)|} \prod_{y \in N(a) \setminus \{x\}} 1 - 2\text{sign}(y, a)\Delta_{y \rightarrow a}^{[\ell]}.
\]  
(29)

How many factors do we expect the product in (29) to have? In the undecimated formula \( \Phi \), each clause has length \( k \). But in \( \Phi^t \), only a \( \theta \) fraction of variables remain unassigned. Hence, the average length of a clause of \( \Phi^t \) should be \( \theta k \). If indeed \( |N(a)| \leq 10\theta k \), say, and if \( |\Delta_{y \rightarrow a}^{[\ell]}| \leq \delta \ell = \exp(-c\theta k) \) for all \( y \in N(a) \setminus \{x\} \), then we can approximate (29) by
\[
\mu_{a \rightarrow x}^{[\ell]}(-\text{sign}(x, a)) \approx 1 - 2^{1 - |N(a)|} \exp\left(-2 \sum_{y \in N(a) \setminus \{x\}} \text{sign}(y, a)\Delta_{y \rightarrow a}^{[\ell]}\right) \approx 1 - 2^{1 - |N(a)|} \left(1 - 2 \sum_{y \in N(a) \setminus \{x\}} \text{sign}(y, a)\Delta_{y \rightarrow a}^{[\ell]}\right).
\]  
(30)

Assume, furthermore, that \( a \) is “not too short” — say, \( |N(a)| \geq 0.1\theta k \). Then \( 2^{1 - |N(a)|} \leq 2^{1 - 0.1\theta k} \) is small, and thus the expression in (30) is close to 1. Hence, we can approximate it by
\[
\mu_{a \rightarrow x}^{[\ell]}(-\text{sign}(x, a)) \approx \exp\left[-2^{1 - |N(a)|} \left(1 - 2 \sum_{y \in N(a) \setminus \{x\}} \text{sign}(y, a)\Delta_{y \rightarrow a}^{[\ell]}\right)\right].
\]  
(31)

To proceed, we are going to plug (31) into (27) to estimate \( \Delta_{x \rightarrow a}^{[\ell + 1]} \). While it is easy enough to multiply the exponentials from (31) together to approximate the numerator of (27), the denominator seems a bit unwieldy. To sidestep this issue, we simply estimate the ratio \( \mu_{x \rightarrow a}^{[\ell + 1]}(1)/\mu_{x \rightarrow a}^{[\ell + 1]}(-1) \) (assuming that \( \mu_{x \rightarrow a}^{[\ell + 1]}(-1) > 0 \)). The denominator cancels. Since \( \mu_{x \rightarrow a}^{[\ell + 1]}(1) + \mu_{x \rightarrow a}^{[\ell + 1]}(-1) = 1 \) by construction, we see that
\[
\frac{\mu_{x \rightarrow a}^{[\ell + 1]}(1)}{\mu_{x \rightarrow a}^{[\ell + 1]}(-1)} = 1 + \frac{2\Delta_{x \rightarrow a}^{[\ell + 1]}}{|\mu_{x \rightarrow a}^{[\ell + 1]}(1)| - |\mu_{x \rightarrow a}^{[\ell + 1]}(-1)|}.
\]

Hence, to show that \( \Delta_{x \rightarrow a}^{[\ell + 1]} \) is close to zero it suffices to prove that \( \mu_{x \rightarrow a}^{[\ell + 1]}(1)/\mu_{x \rightarrow a}^{[\ell + 1]}(-1) \) is close to one. To this end we invoke (31), obtaining
\[
\frac{\mu_{x \rightarrow a}^{[\ell + 1]}(1)}{\mu_{x \rightarrow a}^{[\ell + 1]}(-1)} = \prod_{b \in N(x) \setminus \{a\}} \frac{\mu_{b \rightarrow a}^{[\ell]}(1)}{\mu_{b \rightarrow a}^{[\ell]}(-1)} \approx \exp\left[\sum_{b \in N(x) \setminus \{a\}} 2^{1 - |N(b)|} \left(\text{sign}(x, b) - 2 \sum_{y \in N(b) \setminus \{x\}} \text{sign}(y, b)\Delta_{y \rightarrow b}^{[\ell]}\right)\right].
\]  
(32)

Thus, we need to show that for all but \( \delta n \) variables \( x \) the exponent is close to zero.
To deal with the $\sum_{b \in N(x) \backslash \{a\}} 2^{1-|N(b)|} \text{sign}(x, b)$ bit, we need to estimate in how many clauses of a given length $x$ is likely to appear. Letting

$$\rho = kr/2^k,$$

we find that the expected number of clauses of length $j$ where $x \in V_t$ appears is asymptotically equal to

$$\frac{km}{n} \left( \frac{k-1}{j-1} \right)^{\theta - 1} \left( \frac{1 - \theta}{2} \right)^{k-j} = \rho 2^j \cdot P \left[ \text{Bin}(k-1, \theta) = j-1 \right] \leq \rho 2^j. \quad (34)$$

Indeed, the expected number of clauses of $\Phi$ that $x$ appears in equals $km/n = kr = 2^k \rho$. Furthermore, each of these gives rise to a clause of length $j$ in $\Phi^t$ iff exactly $j - 1$ among the other $k - 1$ variables in the clause are from $V_t$, while the $k - j$ remaining variables are in $V \setminus V_t$ and occur with negative signs. (If one of them had a positive sign, the clause would have been satisfied by setting the corresponding variable to true. It would thus not be present in $\Phi^t$ anymore.) Since $x$ appears with a random sign in each of these clauses, the sum

$$\sum_{b \in N(x) \backslash \{a\}} \text{sign}(x, b)$$

can be viewed as a random walk with an expected length of $\rho 2^j$. Thus, we expect an outcome of $O(\sqrt{2^j \rho})$. In this case, we find that

$$\sum_{b \in N(x) \backslash \{a\}} 2^{1-|N(b)|} \text{sign}(x, b) = 2^{1-j} \cdot O(\sqrt{2^j \rho}) = O(\sqrt{2^{j/2}}).$$

Together with the Chernoff bound, (34) shows that $x$ is unlikely to occur in clauses of lengths less than $0.1\theta k$ or more than $10\theta k$. Furthermore, our assumption that $\theta k \geq \ln(\rho)/e^2$ implies that $\sqrt{2^{-j/2}} \leq \exp(-0.01\theta k)$ for all $j \geq 0.1\theta k$. Hence, we expect that for all but, say, $\delta k \theta n/2$ variables $x \in V_t$

$$\max_{a \in N(x)} \left| \sum_{b \in N(x) \backslash \{a\}} 2^{1-|N(b)|} \text{sign}(x, b) \right| \leq O(\theta k \exp(-0.01\theta k)) \leq \delta/4. \quad (35)$$

The second contribution

$$\sum_{b \in N(x) \backslash \{a\}} \sum_{y \in N(b) \backslash \{x\}} 2^{2-|N(b)|} \text{sign}(x, b) \text{sign}(y, b) \Delta^{[\ell]}_{y \rightarrow b}$$

is a linear function of the bias vector $\Delta^{[\ell]}$ from the previous round. Indeed, this operator can be represented by a matrix

$$\Lambda^* = (\Lambda^*_{x \rightarrow a, y \rightarrow b})_{x \rightarrow a, y \rightarrow b}$$

with entries

$$\Lambda^*_{x \rightarrow a, y \rightarrow b} = \begin{cases} 2^{2-|N(b)|} \text{sign}(x, b) \text{sign}(y, b) & \text{if } a \neq b, x \neq y, \text{ and } b \in N(x), \\ 0 & \text{otherwise}. \end{cases}$$

with $x \rightarrow a, y \rightarrow b$ ranging over all edges of the factor graph of $\Phi^t$.

Since $\Lambda^*$ is based on $\Phi^t$, it is a random matrix. One could therefore try to use standard arguments to bound it in some norm (say, $\|\Lambda^*\|_1$). The problem with this approach is that $\Lambda^*$ is very high-dimensional: it operates on a space whose dimension is equal to the number of edges of the factor graph. In effect, standard random matrix arguments do not apply.

To resolve this problem, consider a “projection” of $\Lambda^*$ onto a space of dimension merely $|V_t| = \theta n$, namely

$$A : R^{V_t} \rightarrow R^{V_t}, \quad \Gamma = (\Gamma_y)_{y \in V_t} \mapsto \left\{ \sum_{b \in N(x)} \sum_{y \in N(b) \backslash \{x\}} 2^{2-|N(b)|} \text{sign}(x, b) \text{sign}(y, b) \Gamma_y \right\}_{x \in V_t}$$
One can think of $A$ as a signed and weighted adjacency matrix of $\Phi'$. Standard arguments easily show that $||A||_\infty \leq \delta t/2$ is small with a very high probability. In effect, we expect that for all but, say, $\delta t/2$ variables $x \in V_t$ we have

$$\max_{a \in N(x)} \left| \sum_{b \in N(x) \setminus \{a\}} \sum_{y \in N(b) \setminus \{x\}} 2^{2-|N(b)|} \text{sign}(x, b) \text{sign}(y, b) \Delta^{[t]}_{y \rightarrow a} \right| \leq \delta t/4.$$ (36)

Combining (35) and (36), we thus expect that for all but $\delta t/2$ variables $x$ the expression (32) is sufficiently close to one to conclude that $\max_{a \in N(x)} |\Delta^{[t]}_{x \rightarrow a}| \leq \delta t$, thereby completing the induction.

Rigorizing the sketch. While the above outlines a strategy for tracing the BP operator, we clearly glossed over numerous issues. The rest of the paper is devoted to rectifying them. To provide a bit of orientation, let us briefly highlight the most important items, and indicate how they are going to be fixed.

The first issue is that Theorem 15 claims a rather strong bound on the probability that $\Phi'$ is $(\delta_t, t)$-balanced. To obtain this bound, we are going to proceed in two steps: in Section 3.2 we will exhibit a property and state the deterministic result about BP on quasirandom formulas (Theorem 25). Then, from property holds on $(\delta_t, t)$. Finally, in Section 3 we establish that the quasirandomness property holds on $\Phi'$ with the required probability.

There are going to be various cases depending on the length of the clause. If, say, $0.1\theta k \leq |N(a)| \leq 10\theta k$ and $|N(a) \cap T^\ell \setminus \{x\}| \leq 1$ (i.e., the second product contains at most one factor), then the above heuristic computation essentially goes through. This case is going to be represented by the set $N_{\leq 1}(x, T^\ell)$ below.

More generally, if $0.1\theta k \leq |N(a)| \leq 10\theta k$, say, then the product (37) is quite close to one, regardless of $|N(a) \cap T^\ell \setminus \{x\}|$. Thus, a single “exposed” clause $a$, or even a small number, are not going to affect the ratio (32) much. To exploit this, we will establish as part of the quasirandomness property that for any possible set $T^\ell$ only very few variables $x$ are “heavily exposed”, meaning that they appear in many clauses that contain several variables from $T^\ell$ (cf. Q2 and Q3 below). Furthermore, we will generally show that there are only very few variables that occur in a clause $a$ such that $|N(a)| \not\in [0.1\theta k, 10\theta k]$ (cf. Q1).

A third issue is the dimension reduction in the linear operator, i.e., that we work with $A$ instead of $A^*$. To vindicate this point, we need to show that for most variables $x$ the bias $\Delta_{x \rightarrow a}^{[t]}$ is essentially independent of $a$. Furthermore, we need to modify the operator $A$ to “cut out” the exceptional set $T^\ell$ where the BP operator has a highly non-linear behavior. This is going to be mirrored in condition Q4 below.

Let us now turn this sketch into an actual proof. In Section 3.2 we introduce the quasirandomness property and state the deterministic result about BP on quasirandom formulas (Theorem 25). Then, from Section 3 onwards, we prove Theorem 25. Finally, in Section 4 we establish that the quasirandomness property holds on $\Phi'$ with the required probability.
3.2 The quasirandomness property

In this section we will exhibit a few simple quasirandomness properties that $\Phi^t$ is very likely to possess. From Section 3.1 we will show that these properties suffice to trace the BP operator.

To state the quasirandomness properties, fix a $k$-$CNF$ $\Phi$. Let $\Phi^t = \Phi_t, id, 1$ denote the CNF obtained from $\Phi$ by substituting ‘true’ for $x_1, \ldots, x_t$ and simplifying (1 $\leq t \leq n$). Let $V_t = \{x_{t+1}, \ldots, x_n\}$ be the set of variables of $\Phi^t$. As before, we will denote the factor graph of $\Phi^t$ by $G = G(\Phi^t)$, and the neighborhood of a vertex $v$ by $N(v)$. We continue to let $\theta$ and $\rho$ be defined as in (28) and (33).

For a variable $x \in V_t$ and a set $T \subset V_t$ let

$$N_{\leq 1}(x, T) = \{b \in N(x) : |N(b) \cap T \setminus \{x\}| \leq 1 \wedge 0.1 \theta k \leq |N(b)| \leq 100 \theta k\}.$$ (38)

Thus, $N_{\leq 1}(x, T)$ is the set of all clauses that contain $x$ (which may or may not be in $T$) and at most one other variable from $T$. In addition, there is a condition on the length $|N(b)|$ of the clause $b$ in the decimated formula $\Phi_t$. Recall from Section 3.1 that having assigned the first $t$ variables, we should ‘expect’ the average clause length to be $\theta k$.

With $c > 0$ as in (13) we let

$$k_1 = \sqrt{c \theta k}.$$  

Moreover, for a variable $x \in V_t$ and a set $T \subset V_t$ let

$$N_{\leq 1}(x, T) = \{b \in N(x) : |N(b) \cap T \setminus \{x\}| \leq \sqrt{c \theta k} \wedge 0.1 \theta k \leq |N(b)| \leq 100 \theta k\}.$$ (38)

$$N_{> 1}(x, T) = \{b \in N(x) : |N(b) \cap T \setminus \{x\}| > 1 \wedge 0.1 \theta k \leq |N(b)| \leq 100 \theta k\}.$$ (38)

**Definition 23.** Let $\delta > 0$. We say that $\Phi$ is $(\delta, t)$-quasirandom if $Q0$–$Q4$ in Figure 4 are satisfied.

Condition Q0 simply bounds the number of redundant clauses and the number of variables of very high degree; it well-known to hold for random $k$-CNFs w.h.p. Apart from a bound on the number of very short/very long clauses, Q1 provides a bound on the ‘weight’ of clauses in which variables $x \in V_t$ typically occur, where the weight of a clause $b$ is $2^{-|N(b)|}$. Moreover, Q2 provides that there is no small set $T$ for which the total weight of the clauses touching that set is very big. In addition, Q2 (essentially) requires that for most variables $x$ the weights of the clauses where $x$ occurs positively/negatively should approximately cancel. Further, Q3 provides a bound on the lengths of clauses that contain many variables from a small set $T$. Finally, the most important condition is Q4, providing a bound on the cut norm of a signed, weighted matrix representation of $\Phi^t$.

**Proposition 24.** There exists a constant $\rho_0 > 0$ such that for any $k, r$ satisfying $\rho_0 \cdot 2^{k/k} \leq r \leq 2^k \ln 2$ there is $\xi = \xi(k, r) > 0$ so that for $n$ large and $\delta_t, t$ as in (23) for any $1 \leq t \leq \ell$ we have

$$P[\Phi \text{ is } (\delta_t, t)-\text{quasirandom} \mid \Phi \text{ is tame}] \geq 1 - \exp[-10(\xi n + \Delta_t)].$$

The proof of Proposition 24 is a necessary evil: it is long, complicated and based on standard arguments. We defer it to Section 4. Together with the following theorem, which we will establish in Section 3.3, Proposition 24 yields Theorem 25.

**Theorem 25.** There is $\rho_0 > 0$ such that for any $k, r$ satisfying $\rho_0 \cdot 2^{k/k} \leq r \leq 2^k \ln 2$ and $n$ sufficiently large the following is true. Let $\Phi$ be a $k$-$CNF$ with $n$ variables and $m$ clauses that is $(\delta_t, t)$-quasirandom for some $1 \leq t \leq \ell$. Then $(\Phi, id, 1)$ is $(\delta_t, t)$-balanced.

The rest of this section deals with the proof of Theorem 25.

*For the rest of Section 3 we keep the notation from Section 3.2 and the assumptions of Theorem 25. To unclutter the notation, we let $\delta = \delta_t.*
Implementing the strategy outlined in Section 3.1, we are going to trace the BP operator when iterated from

**Belief Propagation on quasirandom formulas: proof of Theorem 25**

We say that

Clearly, no variable is

Let

For any

We let

has norm

**Fig. 1.** The conditions for Definition 23.

### 3.3 Belief Propagation on quasirandom formulas: proof of Theorem 25

Implementing the strategy outlined in Section 3.1, we are going to trace the BP operator when iterated from the initial point

\[ \rho x \mapsto a = 2^{-|N(a)|} \]

Let \( \mu^\ell = \text{BP}^\ell(\mu^0) \in M(\Phi) \) be the result of the first \( \ell \) iterations of BP. Let

\[ \Delta^\ell x \mapsto a = \mu^\ell x \mapsto a (1) - \frac{1}{2} \]

We say that \( x \in V_\ell \) is \( \ell \)-biased if

\[ \max_{a \in N(x)} |\Delta^\ell x \mapsto a| \geq 0.1\delta. \]

Clearly, no variable is 0-biased. Let \( B [\ell] \) be the set of all \( \ell \)-biased variables. To prove Theorem 25 the core task will be to bound \( |B [\ell]| \).

To this end, we are going to construct a sequence of sets \( T [\ell] \) whose sizes are easier to estimate and that will turn out to be supersets of the \( B [\ell] \). Actually we will construct sets of variables \( T_1 [\ell] \), \( T_2 [\ell] \) and sets of clauses \( T_3 [\ell] \) inductively and let \( T [\ell] = T_1 [\ell] \cup T_2 [\ell] \cup N(T_3 [\ell]) \).

For \( \ell = 0 \) we let \( T_1 [0] = T_3 [0] = 0 \). Moreover, let \( T_2 [0] \) be the set of all variables \( x \) such that there is a clause \( b \in N(x) \) that is either redundant, or \( |N(b)| < 0.1\delta k \), or \( |N(b)| > 10\delta k \), or that satisfy \( \delta(\theta k)^3 \sum_{b \in N(x)} 2^{-|N(b)|} > 1 \).

To define \( T [\ell + 1] \) inductively for \( \ell \geq 0 \), we need a bit of notation: for \( x \in V \) and \( a \in N(x) \) we let

\[ N_{\leq 1}^\ell (\ell + 1) : a \mapsto x \equiv \left\{ b \in N_{\leq 1} (x, T [\ell]) \setminus \{ a \} : \mu^\ell b \mapsto x (-1) > 0 \right\}. \]
Furthermore, set
\[ P_{\leq 1}^{(\ell+1)}(x \rightarrow a) = \prod_{b \in N(x) \setminus \{a\}} \frac{\mu_{b \rightarrow x}^{(\ell)}(1)}{\mu_{b \rightarrow x}^{(\ell)}(-1)}, \]

In addition, let
\[ N_{\geq 1}^{(\ell+1)}(x \rightarrow a) = \left\{ b \in N(x) \setminus \{a\} : \mu_{b \rightarrow x}^{(\ell)}(1) > 0 \right\}, \]
\[ P_{\geq 1}^{(\ell+1)}(x \rightarrow a) = \prod_{b \in N(x) \setminus \{a\}} \frac{\mu_{b \rightarrow x}^{(\ell)}(1)}{\mu_{b \rightarrow x}^{(\ell)}(-1)}. \]

The motivation behind these definitions is the following. Assume for a moment that \( \mu_{b \rightarrow x}^{(\ell)}(-1) \neq 0 \) for all \( b \in N(x) \). As we saw in Section 3.1, to show that \( \Delta_{(\ell+1)}^{x \rightarrow a} = \mu_{b \rightarrow x}^{(\ell)}(1) - \frac{1}{2} \) is close to zero it suffices to verify that the ratio
\[ \frac{\mu_{x \rightarrow a}^{(\ell+1)}(1)}{\mu_{x \rightarrow a}^{(\ell+1)}(-1)} = \prod_{b \in N(x) \setminus \{a\}} \frac{\mu_{b \rightarrow x}^{(\ell)}(1)}{\mu_{b \rightarrow x}^{(\ell)}(-1)} = P_{\leq 1}^{(\ell+1)}(x \rightarrow a) \cdot P_{\geq 1}^{(\ell+1)}(x \rightarrow a) \] (40)

is close to one, because \( \mu_{x \rightarrow a}^{(\ell+1)}(-1) + \mu_{x \rightarrow a}^{(\ell+1)}(1) = 1 \) by construction. Moreover, (40) is close to one if both factors on the r.h.s. are.

Now, we let \( T_1 \{ \ell + 1 \} \) contain all variables for which \( P_{\leq 1}^{(\ell+1)}(x \rightarrow a) \) fails to be close enough to one:
\[ T_1 \{ \ell + 1 \} = \left\{ x \in V : \max_{a \in N(x)} \left| \frac{P_{\leq 1}^{(\ell+1)}(x \rightarrow a)}{P_{\geq 1}^{(\ell+1)}(x \rightarrow a) - 1} - 0.01 \right| \right\}. \]

To also deal with the second product \( P_{\geq 1}^{(\ell+1)}(x \rightarrow a) \), we define additional sets \( T_2 \{ \ell + 1 \}, T_3 \{ \ell + 1 \} \). To define \( T_2 \{ \ell + 1 \} \), let us say that a variable \( x \) is \((\ell+1)\)-harmless if it enjoys the following four properties.

**H1.** We have \( \delta(\theta k)^0 \sum_{b \in N(x)} 2^{-\|N(b)\|} \leq 1 \), and \( 0.1\theta k \leq \|N(b)\| \leq 100\theta k \) for all \( b \in N(x) \).

**H2.** \( \sum_{b \in N(x) \setminus \{a\}} 2^{-\|N(b)\|} \leq \rho(\theta k)^0\delta \) and \( \sum_{b \in N(x) \setminus \{a\}} 2^{\|N(b)\|-\|T(a)\|-\|N(b)\|} \leq \|N(b)\| \leq \|T(a)\| \).

**H3.** There is at most one clause \( b \in N(x) \) such that \( \|N(b)\| \leq \|T(a)\| \leq k_1 \).

**H4.** \( \sum_{b \in N(x)} \text{sign}(x, b) \cdot 2^{-\|N(b)\|} \geq 0.01\delta. \)

Let \( H \{ \ell + 1 \} \) signify the set of all \((\ell+1)\)-harmless variables. Further, let \( T_2 \{ \ell + 1 \} \) be the set of all variables \( x \) that have at least one of the following properties.

**T2a.** There is a clause \( b \in N(x) \) that is either redundant, or \( \|N(b)\| < 0.1\theta k \), or \( \|N(b)\| > 100\theta k \).

**T2b.** \( \delta(\theta k)^1 \sum_{b \in N(x)} 2^{-\|N(b)\|} > 1 \).

**T2c.** Either
\[ \sum_{b \in N(x) \setminus \{a\}} 2^{-\|N(b)\|} > \rho(\theta k)^0\delta, \text{ or } \sum_{b \in N(x) \setminus \{a\}} 2^{\|N(b)\|-\|T(a)\|-\|N(b)\|} > \delta/(\theta k). \]

**T2d.** \( x \) occurs in more than 100 clauses from \( T_3 \{\ell\} \).

**T2e.** \( x \) occurs in a clause \( b \) that contains fewer than \( 3\|N(b)\|/4 \) variables from \( H \{\ell\} \).

Items Q0 and Q1 from Definition 23 ensure that there are only a very few variables that satisfy H1, T2a, or T2b. We always include these few into the set \( T_2 \{ \ell + 1 \} \) of ‘exceptional’ variables. Moreover, intuitively H2 and T2c–T2e capture variables \( x \) that are highly exposed to the ‘exceptional’ set \( T \{\ell\} \) from the previous round. Furthermore, we let
\[ T_3 \{ \ell + 1 \} = \{ a \in \Phi : \|N(a)\| \geq 100k_1 \land \|N(a)\| \leq k_1 \} \setminus T_3 \{\ell\} \] (41)
Corollary 29. The second assertion follows from the elementary inequality $T[\ell + 1] = T_1[\ell + 1] \cup T_2[\ell + 1] \cup N(T_3[\ell + 1])$.

In Section 3.4 we will verify that $T[\ell]$ does indeed contain the set $B[\ell]$ of biased variables.

Proposition 26. We have $B[\ell] \subset T[\ell]$ for all $\ell \geq 0$.

Furthermore, in Section 3.5 we will establish the following bound on the size of $T[\ell]$.

Proposition 27. We have $|T[\ell]| < \delta n$ for all $\ell \geq 0$.

Finally, in Section 3.8 we will derive Theorem 25 from Proposition 26 and Proposition 27.

3.4 Proof of Proposition 26

The proof will be by induction on $\ell$. We begin with an elementary estimate of the messages $\mu_{b \rightarrow x}$ from clauses to variables.

Lemma 28. Let $x$ be a variable and let $b \in N(x)$ be a clause. Let $t_b = |N(b) \cap B[\ell] \setminus \{x\}|$. Then

$$1 - \mu_{b \rightarrow x}^{[\ell]}(\zeta) \leq 2^{2^{-|N(b)|+t_b} \exp(\delta|N(b)|)} \quad \text{for } \zeta = \pm 1.$$  

Furthermore, if $2^{2^{-|N(b)|+t_b} \exp(\delta|N(b)|)} \leq 1/2$, then

$$\exp\left[-2^{2^{-|N(b)|+t_b} \exp(\delta|N(b)|)}\right] \leq \mu_{b \rightarrow x}^{[\ell]}(\zeta) \leq 1 \quad \text{for } \zeta = \pm 1.$$  

Proof. Since for any $y \in N(b) \setminus \{x\}$ we have $\mu_{y \rightarrow b}^{[\ell]}(1) = \frac{1}{2} + \Delta_{y \rightarrow b}^{[\ell]}$ and $\mu_{y \rightarrow b}^{[\ell]}(-1) + \mu_{y \rightarrow b}^{[\ell]}(1) = 1$, we see that

$$\mu_{y \rightarrow b}^{[\ell]}(-\sign(y, b)) = \frac{1}{2} - \sign(y, b) \Delta_{y \rightarrow b}^{[\ell]}.$$  

Therefore, by the definition (26) of $\mu_{b \rightarrow x}^{[\ell]}(\pm 1)$, we have

$$0 \leq 1 - \mu_{b \rightarrow x}^{[\ell]}(-\sign(x, b)) = \prod_{y \in N(b) \setminus \{x\}} \frac{1}{2} - \sign(y, b) \Delta_{y \rightarrow b}^{[\ell]}$$

$$= 2^{1^{-|N(b)|}} \prod_{y \in N(b) \setminus \{x\}} \left(1 - 2\sign(y, b) \Delta_{y \rightarrow b}^{[\ell]}\right)$$

$$\leq 2^{1^{-|N(b)|}} \cdot 2^{t_b} \cdot \prod_{y \in N(b) \setminus \{x\}} \left(1 + 2|\Delta_{y \rightarrow b}^{[\ell]}|\right) \quad \text{[as } \Delta_{y \rightarrow b}^{[\ell]} \in [-1/2, 1/2] \text{ for all } y]\right]$$

$$\leq 2^{1^{-|N(b)|}+t_b} \cdot \exp\left[2 \sum_{y \in N(b) \setminus \{x\} \cup B[\ell]} |\Delta_{y \rightarrow b}^{[\ell]}|\right]$$

$$\leq 2^{1^{-|N(b)|}+t_b} \cdot \exp(|N(b)|\delta) \quad \text{[as } |\Delta_{y \rightarrow b}^{[\ell]}| \leq 0.1\delta \text{ for all } y \notin B[\ell]).$$

The second assertion follows from the elementary inequality $1 - z \geq \exp(-2z)$ for $0 \leq z \leq 1/2$. 

Corollary 29. Let $x$ be a variable and let $T \subset N(x)$ be a set of clauses. For each $b \in T$ let $t_b = |N(b) \cap B[\ell] \setminus \{x\}|$. Assume that $t_b < |N(b)| - 2$ and $|N(b)| \leq 100k$ for all $b \in T$. Then $\mu_{b \rightarrow x}^{[\ell]}(\pm 1) > 0$ for all $b \in T$ and

$$\left|\ln \prod_{b \in T} \mu_{b \rightarrow x}^{[\ell]}(1)\right| \leq \sum_{b \in T} 2^{4^{-|N(b)|+t_b}}.$$
Proof. For each \( b \in \mathcal{T} \) there is \( y \in N(b) \setminus \{x\} \) such that \( y \not\in B[\ell] \), because \( t_b < |N(b)| - 2 \). Therefore, \( \mu_{b \to y}(\pm 1) > 0 \). Since by definition \( \mu_{b \to x}(\zeta) \leq 1 \) for \( \zeta = \pm 1 \), Lemma 29 implies that for any \( b \in \mathcal{T} \) and we have

\[
\frac{\mu_{b \to x}^{[\ell]}(\zeta)}{\mu_{b \to x}^{[\ell]}(-\zeta)} \geq 1 - 2^{2-|N(b)|+t_b} \exp(\delta|N(b)|). \tag{42}
\]

Our assumptions \( t_b < |N(b)| - 2 \) and \( |N(b)| \leq 10\theta k \) ensure that

\[
2^{2-|N(b)|+t_b} \leq 1/2 \quad \text{and} \quad \exp(\delta|N(b)|) \leq 1.1,
\]

whence \( 2^{2-|N(b)|+t_b} \exp(\delta|N(b)|) \leq 0.6 \). Due to the elementary inequality \( 1 - z \geq \exp(-2z) \) for \( z \in [0, 0.6] \), \( 42 \) thus yields

\[
\frac{\mu_{b \to x}^{[\ell]}(\zeta)}{\mu_{b \to x}^{[\ell]}(-\zeta)} \geq \exp \left[-2^{2-|N(b)|+t_b} \exp(\delta|N(b)|)\right] \geq \exp \left[-2^{2-|N(b)|+t_b}\right]. \tag{43}
\]

Multiplying \( 43 \) up over \( b \in \mathcal{T} \) and taking logarithms yields

\[
\ln \prod_{b \in \mathcal{T}} \frac{\mu_{b \to x}^{[\ell]}(\zeta)}{\mu_{b \to x}^{[\ell]}(-\zeta)} \geq - \sum_{b \in \mathcal{T}} 2^{2-|N(b)|+t_b} \exp(\delta|N(b)|). \tag{44}
\]

Since \( 44 \) holds for both \( \zeta = -1 \) and \( \zeta = 1 \), the assertion follows. \( \square \)

**Corollary 30.** Suppose that \( x \in H[\ell] \) and that \( a \in N(x) \) is a clause such that \( |N(a) \setminus T[\ell - 1]| \leq k_1 \). Moreover, assume that \( B[\ell - 1] \subset T[\ell - 1] \). Then \( |\Delta_{x \to a}^{[\ell]}| \leq 0.01 \).

**Proof.** For each \( b \in N(x) \setminus \{a\} \) let \( t_b = |N(b) \cap B[\ell - 1] \setminus \{x\}| \). Then our assumption that \( B[\ell - 1] \subset T[\ell - 1] \) and condition \( \text{H3} \) ensure that for any \( b \in N(x) \setminus \{a\} \),

\[
t_b \leq |N(b) \cap T[\ell - 1]| \leq |N(b)| - k_1 \leq |N(b)| - 2.
\]

Furthermore, by \( \text{H1} \) we have \( 0.1\theta k \leq |N(b)| \leq 10\theta k \) for all \( b \in N(x) \setminus \{a\} \). Therefore, Corollary 29 applies to the set \( \mathcal{T} = N_{\leq 1}(x, T[\ell]) \setminus \{a\} \). Since Corollary 29 yields \( \mu_{b \to x}^{[\ell-1]}(0) > 0 \) for all \( b \in \mathcal{T} \), we have \( \mathcal{T} = N_{\leq 1}^{[\ell]}(x \to a) \), and thus

\[
|\ln P_{\leq 1}^{[\ell]}(x \to a)| = \left| \ln \prod_{b \in \mathcal{T}} \frac{\mu_{b \to x}^{[\ell]}(1)}{\mu_{b \to x}^{[\ell]}(-1)} \right| \leq \sum_{b \in \mathcal{T}} 2^{2t_b - |N(b)|} \tag{45}
\]

Moreover, \( \text{H2} \) ensures that \( \sum_{b \in \mathcal{T}} 2^{2t_b - |N(b)|} \leq \delta \), whence \( 45 \) entails

\[
|P_{\leq 1}^{[\ell]}(x \to a) - 1| \leq 10^{-4}. \tag{46}
\]

Furthermore, by \( \text{H1} \) all clauses \( b \in N(x) \) have lengths \( 0.1\theta k \leq |N(b)| \leq 10\theta k \). Moreover, for all \( b \in N(x) \setminus \{a\} \) we have \( |N(b) \setminus T[\ell - 1]| \geq k_1 \) by \( \text{H3} \), and thus \( N_{\leq 1}(x, T[\ell - 1]) \subset N_{\leq 1}(x, T[\ell - 1]) \). Further, since \( |N(a) \cap T[\ell - 1]| > 1 \) by assumption, we have

\[
N_{\leq 1}(x \to a)[\ell] = N_{\leq 1}(x, T[\ell - 1]).
\]

Hence, letting \( \mathcal{N} = N_{\leq 1}(x, T[\ell - 1]) \setminus N_{\leq 1}(x, T[\ell - 1]) \), we have

\[
|P_{\leq 1}^{[\ell]}(x \to a) - 1| = \prod_{b \in \mathcal{N}} \frac{\mu_{b \to x}^{[\ell-1]}(1)}{\mu_{b \to x}^{[\ell-1]}(-1)} \prod_{b \in N_{\leq 1}(x, T[\ell - 1])} \mu_{b \to x}^{[\ell-1]}(1) - 1. \tag{47}
\]
Summing these bounds up for \( b \in \mathcal{N} \), we find
\[
\begin{align*}
\ln \prod_{b \in N_1(x,T[\ell-1])} \frac{\mu_{b \to x}^{[\ell-1]}(1)}{\mu_{b \to x}^{[\ell-1]}(-1)} & \leq \sum_{b \in N_1(x,T[\ell-1])} 2^{1-|N(b)|} \\
& \stackrel{\text{H2}}{\leq} 32\rho(\theta k)^5\delta \leq 10^{-6} \quad \text{[as } \delta = \exp(-c\theta k) \text{ with } \theta k \geq \ln(\rho)/c^2\text{].}
\end{align*}
\] (48)

Furthermore, for any \( b \in \mathcal{N} \) we have
\[
\begin{align*}
\mu_{b \to x}^{[\ell-1]}(-\text{sign}(x)) &= 1 - \prod_{y \in N(b) \setminus \{x\}} \frac{1}{2} - \text{sign}(y,b) \Delta_{y \to b}^{[\ell-1]} \\
& = 1 - 2^{1-|N(b)|} \prod_{y \in N(b) \setminus \{x\}} 1 - 2\text{sign}(y,b) \Delta_{y \to b}^{[\ell-1]}.
\end{align*}
\] (49)

Since \( b \in \mathcal{N} \), for all \( y \in N(b) \setminus \{x\} \) we have \( y \notin B[\ell-1] \subset T[\ell-1] \), and thus \( |\Delta_{y \to b}^{[\ell-1]}| \leq 0.1\delta \). Moreover, \( |N(b)| \leq 10k\theta \) by H1. Thus, letting
\[
\alpha_b = 1 - \prod_{y \in N(b) \setminus \{x\}} 1 - 2\text{sign}(y,b) \Delta_{y \to b}^{[\ell-1]},
\]
we find
\[
0 \leq \alpha_b \leq 1 - (1 - 0.2\delta)|N(b)| \leq 8\delta k\theta.
\] (50)

Since \( |N(b)| \geq 0.1k\theta \) by H1, (49) thus yields
\[
\mu_{b \to x}^{[\ell-1]}(-\text{sign}(x)) \geq 1 - 2^{1-|N(b)|}(1 - \delta k\theta) \geq 0.99.
\] (51)

Using the elementary inequality \( -z - z^2 \leq \ln(1 - z) \leq -z \) for \( 0 \leq z \leq 0.5 \), we obtain from (49), (50) and (51)
\[
\begin{align*}
\ln \mu_{b \to x}^{[\ell-1]}(-\text{sign}(x)) & \leq -2^{1-|N(b)|}(1 - \alpha_b) \leq -2^{1-|N(b)|}(1 - 8k\theta\delta), \\
\ln \mu_{b \to x}^{[\ell-1]}(-\text{sign}(x)) & \geq -2^{1-|N(b)|}(1 - \alpha_b) - 2^{2(1-|N(b)|)}(1 - \alpha_b)^2 \\
& \geq -2^{1-|N(b)|}(1 + 8k\theta\delta) \quad \text{[as } |N(b)| \geq 0.1k\theta \text{ by H1].}
\end{align*}
\]

Summing these bounds up for \( b \in \mathcal{N} \), we obtain
\[
\begin{align*}
\ln \prod_{b \in \mathcal{N}} \frac{\mu_{b \to x}^{[\ell-1]}(1)}{\mu_{b \to x}^{[\ell-1]}(-1)} & \leq \sum_{b \in \mathcal{N}} \text{sign}(x,b)2^{1-|N(b)|} + 8k\delta \sum_{b \in \mathcal{N}} 2^{1-|N(b)|} \\
& \leq 2 \sum_{b \in \mathcal{N}} \text{sign}(x,b)2^{-|N(b)|} + 2(\theta k)^{-3} \quad \text{[by H1]} \\
& \leq 2 \sum_{b \in N_1(x,T[\ell-1])} \text{sign}(x,b)2^{-|N(b)|} + 2(\theta k)^{-3} + \sum_{x \in N_1(x,T[\ell-1])} 2^{1-|N(b)|} \\
& \leq 0.02\delta + 2(\theta k)^{-3} + \rho(\theta k)^5\delta \quad \text{[by H2, H4]} \\
& \leq 10^{-6} \quad \text{[because } \delta = \exp(-c\theta k) \text{ and } \theta k \geq \ln(\rho)/c^2\text{].}
\end{align*}
\] (52)

Plugging (43) and (52) into (47), we see that \( |P_{\leq 1}(x \to a) - 1| \leq 10^{-5} \), while \( |P_{\leq 1}(x \to a) - 1| \leq 10^{-4} \) by (46). Therefore, (40) yields
\[
\left| 1 - \frac{1 + 2\Delta_{x \to a}^{[\ell]}}{1 - 2\Delta_{x \to a}^{[\ell]}} \right| = \left| 1 - \frac{\mu_{x \to a}^{[\ell]}(1)}{\mu_{x \to a}^{[\ell]}(-1)} \right| \leq 3 \cdot 10^{-4},
\]
whence \( \Delta_{x \to a}^{[\ell]} \leq 0.01 \), as desired. \( \square \)
Corollary 31. Let $b$ be a clause such that $N(b) \not\subseteq T[\ell]$. Let $x \in N(b)$. Assume that $B[\ell - 1] \subseteq T[\ell - 1]$. Then

$$
\mu_{b \rightarrow x}^{[\ell - 1]}(-1) > 0 \text{ and } \left| \frac{\mu_{b \rightarrow x}^{[\ell - 1]}(1)}{\mu_{b \rightarrow x}^{[\ell - 1]}(-1)} - 1 \right| \leq \exp(-k_1/2).
$$

Proof. We consider two cases.

Case 1: $|N(b) \setminus T[\ell - 1]| > k_1$. Since $N(b) \not\subseteq T[\ell]$, we have $|N(b)| \leq 10k\theta$ (by T2a). Therefore, Lemma 28 yields

$$
\exp(-\exp(-0.6k_1)) \leq \exp(-2^{3-k_1}\exp(\delta|N(b)|)) \leq \mu_{b \rightarrow x}^{[\ell]}(\zeta) \leq 1 \text{ for } \zeta = \pm 1,
$$

whence the assertion follows.

Case 2: $|N(b) \setminus T[\ell - 1]| \leq k_1$. Since $N(b) \not\subseteq T[\ell]$, condition T2a ensures that $0.1\theta k \leq |N(b)| \leq 10\theta k$. The assumption $N(b) \not\subseteq T[\ell]$ implies that $b \not\in T_3[\ell]$. But since $|N(b) \setminus T[\ell - 1]| \leq k_1$, and as $|N(b)| \geq 0.1\theta k \geq 100k_1$, the only possible reason why $b \not\in T_3[\ell]$ is that $b \in T_3[\ell - 1]$ (cf. the definition of $T_3[\ell]$). As $N(b) \not\subseteq T_2[\ell]$, T2e implies

$$
|N(b) \cap H[\ell - 1]| \geq 3|N(b)|/4. \tag{53}
$$

Let $J = N(b) \cap H[\ell - 1]$. Since $b \in T_3[\ell - 1]$, we have $\ell \geq 2$ and $|N(b) \setminus T[\ell - 2]| \leq k_1$. Therefore, Corollary 31 implies that $|\Delta_{y \rightarrow b}| \leq 0.01$ for all $y \in J$. Thus, for all $x \in N(b)$ we have

$$
\mu_{b \rightarrow x}^{[\ell - 1]}(-\text{sign}(x, b)) = 1 - \prod_{y \in N(b) \setminus \{x\}} \mu_{y \rightarrow b}^{[\ell - 1]}(-\text{sign}(y, b)) 
\geq 1 - (0.501)^{|J|-1} \geq 1 - (0.501)^{|N(b)|/4 - 1} \geq 1 - (0.501)^{0.07\theta k},
$$

Consequently, $\mu_{b \rightarrow x}^{[\ell - 1]}(-1) > 0$ and

$$
\left| \frac{\mu_{b \rightarrow x}^{[\ell - 1]}(1)}{\mu_{b \rightarrow x}^{[\ell - 1]}(-1)} - 1 \right| \leq 2 \cdot (0.501)^{0.07\theta k} \leq \exp(-\theta k/100) \leq \exp(-k_1).
$$

Thus, we have established the assertion in either case. \qed

Proof of Proposition 26. \textit{We proceed by induction on $\ell$. Since $B[0] = \emptyset$ the assertion is trivial for $\ell = 0$. Thus, assume that $\ell \geq 0$ and that $B[\ell] \subset T[\ell]$. Let $x \in V_4 \setminus T[\ell + 1]$. We will prove that $x \not\in B[\ell + 1]$.}

Corollary 31 implies that

$$
\mu_{a \rightarrow x}^{[\ell]}(-1) > 0 \text{ and } \left| \frac{\mu_{a \rightarrow x}^{[\ell]}(1)}{\mu_{a \rightarrow x}^{[\ell]}(-1)} - 1 \right| \leq \exp(-k_1/2) \text{ for all } x \not\in T[\ell + 1], a \in N(x). \tag{54}
$$

We claim

$$
|P_{b}^{[\ell + 1]}(x \rightarrow a) - 1| \leq \delta/100 \text{ for all } x \not\in T[\ell + 1], a \in N(x). \tag{55}
$$

To establish (55), we consider two cases.

Case 1: $x \not\in N(T_3[\ell])$. Let $T = N_{b \geq 1}^{[\ell + 1]}(x \rightarrow a)$ be the set of all clauses $b$ that contribute to the product $P_{b \geq 1}^{[\ell + 1]}(x \rightarrow a)$. Since $x \not\in N(T_3[\ell] \cup T_3[\ell + 1])$, none of the clauses $b \in T$ features more than $|N(b)| - k_1$ variables from $T[\ell]$ (just by the definition of $T_3[\ell + 1]$). Furthermore, because $x \not\in T_2[\ell + 1]$, T2c is not satisfied and thus we obtain the bound

$$
\sum_{b \in T} 2^{(N(b) \cap T[\ell](x) \setminus \{x\}) - |N(b)|} \leq \sum_{b \in N_{b \geq 1}(x, T[\ell])} 2^{(N(b) \cap T[\ell](x) \setminus \{x\}) - |N(b)|} \leq \delta/(\theta k) \leq \delta/10^4. \tag{56}
$$

Since $x \not\in T[\ell + 1]$, T2a ensures that $|N(b)| \leq 10\theta k$ for all $b \in T$. Therefore, (55) follows from (56) and Corollary 29.
Case 2: $x \in N(T_3 [\ell])$. Let $\mathcal{T} = N_{\geq 1}^{[\ell + 1]} (x \rightarrow a) \setminus T_3 [\ell]$ be the set of all clauses $b$ that occur in the product $P_{\geq 1}^{[\ell + 1]} (x \rightarrow a)$, apart from the ones in $T_3 [\ell]$. Since $x \not\in T_2 [\ell + 1] \cup N(T_3 [\ell + 1])$, this set $\mathcal{T}$ also satisfies (56). Thus, Corollary 29 yields

$$\left| \ln \prod_{b \in \mathcal{T}} \mu_{b \rightarrow x}^{[\ell + 1]}(1) \right| \leq \delta/10^3. \quad (57)$$

Let $\mathcal{T}' = N_{> 1}^{[\ell + 1]} (x \rightarrow a) \cap T_3 [\ell]$. As condition T2d ensures that $|\mathcal{T}'| \leq |N(x) \cap T_3 [\ell]| \leq 100$, (54) implies

$$\left| \ln \prod_{b \in \mathcal{T}'} \mu_{b \rightarrow x}^{[\ell + 1]}(1) \right| \leq 2 |\mathcal{T}'| \exp(-k_1/2) \leq \delta/1000. \quad (58)$$

Since $N_{> 1}^{[\ell + 1]} (x \rightarrow a) = \mathcal{T} \cup \mathcal{T}'$, (57) and (58) yield $1 - P_{\geq 1}^{[\ell + 1]} (x \rightarrow a) \leq \delta/100$.

Thus, we have established (55) in either case.

If $x \not\in T_1 [\ell + 1] \subset T [\ell + 1]$ and $a \in N(x)$, then $P_{\leq 1}^{[\ell + 1]} (x \rightarrow a) \leq 1/100$. Hence, (54) implies that for all $x \not\in T [\ell + 1]$ and all $a \in N(x)$ we have $\mu_{x \rightarrow a}^{[\ell + 1]}(1) > 0$. Thus,

$$1 - \frac{P_{\leq 1}^{[\ell + 1]} (x \rightarrow a)}{\mu_{x \rightarrow a}^{[\ell + 1]}(1)} = 1 - P_{\leq 1}^{[\ell + 1]} (x \rightarrow a) \cdot P_{\geq 1}^{[\ell + 1]} (x \rightarrow a) \leq \delta/99 \quad \text{[by (55)].}$$

Consequently, $|\Delta_{x \rightarrow a}^{[\ell + 1]}| < 0.1\delta$, and thus $x \not\in B [\ell + 1]$.

\[ \square \]

3.5 Proof of Proposition 27

We are going to proceed by induction on $\ell$. We begin by bounding the sizes of the sets $T_2, T_3$.

**Lemma 32.** Assume that $|T_1 [\ell] \cup T_2 [\ell]| \leq \delta \theta n/3$ and $|N(T_3 [\ell])| \leq \delta \theta n/2$. Then $|N(T_3 [\ell + 1])| \leq \delta \theta n/2$.

**Proof.** By construction we have $T_3 [\ell] \cap T_3 [\ell + 1] = \emptyset$ (cf. 41). Furthermore, also by construction $N(T_3 [\ell]) \subseteq T [\ell]$, and each clause in $T_3 [\ell + 1]$ has at least a 0.99-fraction of its variables in $T [\ell]$. Thus,

$$|N(b) \cap T [\ell]| \geq 0.99 |N(b)| \text{ for all } b \in T_3 [\ell] \cup T_3 [\ell + 1].$$

Hence, Q3 yields

$$|N(T_3 [\ell + 1])| \leq \sum_{b \in T_3 [\ell] \cup T_3 [\ell + 1]} |N(b)| \leq \frac{1.01}{0.99} |T [\ell]| \leq 1.03 \left( |T_1 [\ell]| + |T_2 [\ell]| + |N(T_3 [\ell])| \right).$$

Hence, $|N(T_3 [\ell + 1])| \leq 1.03 \left( |T_1 [\ell]| + |T_2 [\ell]| \right) + 0.03 |N(T_3 [\ell])| \leq \delta \theta n/2$. \[ \square \]

**Lemma 33.** Assume that $|T_1 [\ell] \cup T_2 [\ell]| \leq \delta \theta n/3$ and $|N(T_3 [\ell])| \leq \delta \theta n/2$. Moreover, suppose that $|T [\ell] - 1| \leq \delta \theta n$. Then $|T_2 [\ell + 1]| \leq \delta \theta n/6$.

**Proof.** Conditions Q0 and Q1 readily imply that the number of variables that satisfy either T2a or T2b is at most $0.001 \delta \theta n$. Moreover, we apply Q2 to the set $T [\ell]$ of size

$$|T [\ell]| \leq |T_1 [\ell]| + |T_2 [\ell]| + |N(T_3 [\ell])| \leq 0.9 \delta \theta n \quad (59)$$

to conclude that the number of variables satisfying T2c is at most $0.001 \delta \theta n$ as well.
To bound the number of variables that satisfy $T_{2d}$, consider the subgraph of the factor graph induced on $T_3[\ell] \cup N(T_3[\ell])$. For each $x \in N(T_3[\ell])$ let $D_x$ be the number of neighbors of $x$ in $T_3[\ell]$. Let $\nu$ be the set of all $x \in V_1$ so that $D_x \geq 100$. Then $Q_3$ yields

$$100\nu \leq \sum_{x \in N(T_3[\ell])} D_x = \sum_{a \in T_3[\ell]} |N(a)| \leq 1.01|T[\ell]| \leq \theta\delta n \quad \text{[as $N(b) \subset T[\ell]$ for all $b \in T_3[\ell]$]}. $$

Hence, there are at most $\nu \leq 0.01\theta\delta n$ variables that satisfy $T_{2d}$. In summary, we have shown that

$$|\{x \in V : x \text{ satisfies one of } T_{2a} - T_{2d}\}| \leq 0.015\theta\delta n. \quad (60)$$

To deal with $T_{2e}$, observe that if a clause $a$ has at least $|N(a)|/4$ variables that are not harmless, then one of the following statements is true.

i. $a$ contains at least $|N(a)|/20$ variables $x$ that violate either $H_1$, $H_2$, or $H_4$.

ii. $a$ contains at least $|N(a)|/5$ variables $x$ that violate condition $H_3$.

Let $C_1$ be the set of clauses $a$ for which i. holds, and let $C_2$ be the set of clauses satisfying ii., so that the number of variables satisfying $T_{2e}$ is bounded by $\sum_{a \in C_1 \cup C_2} |N(a)|$.

To bound $\sum_{a \in C_1} |N(a)|$, let $Q$ be the set of all variables $x$ that violate either $H_1$, $H_2$, or $H_4$ at time $\ell$. Then conditions $Q_1$ and $Q_2$ entail that $|Q| \leq 3 \cdot 10^{-4}\theta\delta n$ (because we are assuming $|T(\ell - 1)| \leq \theta\delta n$). Therefore, condition $Q_3$ implies that

$$\sum_{a \in C_1} |N(a)| \leq 21|Q| + 10^{-4}\delta\theta n \leq 0.0064\delta\theta n. \quad (61)$$

To deal with $C_2$ let $B'$ be the set of all clauses $b$ such that $|N(b)| \geq 100k_1$ but $|N(b) \setminus T[\ell]| \leq k_1$. Since we know from $\ref{eq59}$ that $|T[\ell]| \leq \delta\theta n$, condition $Q_3$ applied to $T[\ell]$ implies

$$|N(B')| \leq \sum_{b \in B'} |N(b)| \leq 1.03|T[\ell]| + 10^{-4}\delta\theta n \leq 1.0301\delta\theta n. \quad (62)$$

In addition, let $B''$ be the set of all clauses of length less than $100k_1$. Since $100k_1 = 100\sqrt{\theta k} \leq 0.1\theta k$ by our choice of $c$, $Q_1$ implies that $|N(B'')| \leq 10^{-4}\delta\theta n$. Hence, $\ref{eq62}$ shows that $B = B' \cup B''$ satisfies

$$|N(B)| \leq 1.0302\delta\theta n. \quad (63)$$

Furthermore, let $\mathcal{U}$ be the set of all clauses $a$ such that $N(a) \subset N(B)$. Let $U$ be the set of variables $x \in N(B)$ that occur in at least two clauses from $\mathcal{U}$. Then by $Q_3$

$$|U| + |N(B)| \leq \sum_{a \in \mathcal{U}} |N(a)| \leq 1.01|N(B)| + 10^{-4}\delta\theta n,$$

whence $|U| \leq 0.01|N(B)| + 10^{-4}\delta\theta n \leq 0.02\delta\theta n$ due to $\ref{eq63}$. Since $B \subset \mathcal{U}$, the set $U$ contains all variables that occur in at least two clauses from $B$, i.e., all variables that violate condition $H_3$. Therefore, any $a \in C_2$ contains at least $|N(a)|/5$ variables from $U$. Applying $Q_3$ once more, we obtain

$$\sum_{a \in C_2} |N(a)| \leq 5.05 \cdot 0.02\delta\theta n + 10^{-4}\delta\theta n = 0.1201\delta\theta n. $$

Combining this estimate with the bound $\ref{eq61}$ on $C_1$, we conclude that the number of variables satisfying $T_{2e}$ is bounded by $\sum_{a \in C_1 \cup C_2} |N(a)| \leq 0.127\delta\theta n$. Together with $\ref{eq63}$ this yields the assertion. \qed

In Section 3.6 we will derive the following bound on $|T_1[\ell+1]|$.

**Proposition 34.** If $|T[\ell]| \leq \delta\theta n$, then $|T_1[\ell+1] \setminus T_2[\ell+1]| \leq \delta\theta n/6$. 

Proof (Proposition 27). We are going to show that
\[ |T_1[\ell] \cup T_2[\ell]| \leq 3\delta n/3 \text{ and } |N(T_3[\ell])| \leq 3\delta n/2 \] (64)
for all \( \ell \geq 0 \). This implies that \( |T[\ell]| \leq 3\delta n \) for all \( \ell \geq 0 \), as desired.

In order to prove (64) we proceed by induction on \( \ell \). The bounds for \( \ell = 0 \) are immediate from Q0 and Q1. Now assume that (64) holds for all \( \ell \leq \ell \). Then Lemma 32 shows that \( |N(T_3[\ell])| \leq 3\delta n/2 \). Moreover, Lemma 33 applies (with the convention that \( T[-1] = T[0] \)), giving \( |T_2[\ell + 1]| \leq 3\delta n/6 \). Finally, Proposition 34 shows \( |T_1[\ell + 1] \setminus T_2[\ell + 1]| \leq 3\delta n/6 \), whence \( |T_1[\ell + 1] \cup T_2[\ell + 1]| \leq 3\delta n/3 \).

\[ \square \]

3.6 Proof of Proposition 34

Throughout this section we assume that \( |T[\ell]| \leq 3\delta n \).

For a variable \( x \in V \) and \( a \in N(x) \) we let
\[ \sigma^{[\ell + 1]}_{x \to a} = \sum_{b \in N^{[\ell + 1]}(x \to a)} 2^{1 - |N(b)|} \text{sign}(x, b), \]
\[ \xi^{[\ell + 1]}_{x \to a} = \sum_{b \in N^{[\ell + 1]}(x \to a)} \sum_{y \in N(b) \setminus \{x\}} 2^{1 - |N(b)|} \text{sign}(x, b) \text{sign}(y, b) \delta^{[\ell]}_{y \to b}, \]
and
\[ f^{[\ell + 1]}_{x \to a} = \sigma^{[\ell + 1]}_{x \to a} + \xi^{[\ell + 1]}_{x \to a}. \]

In Section 3.7 we are going to establish the following.

**Proposition 35.** For any variable \( x \notin T_2[\ell + 1] \) and any clause \( a \in N(x) \) we have
\[ \left| f^{[\ell + 1]}_{x \to a} + P^{[\ell + 1]}_{\leq 1} \right| \leq 10^{-3} \delta. \]

**Lemma 36.** For all but at most \( 10^{-4} \delta n \) variables \( x \in V \setminus T_2[\ell + 1] \) we have
\[ \max_{a \in N(x)} \left| \sigma^{[\ell + 1]}_{x \to a} \right| \leq 0.003\delta. \]

**Proof.** Applying Q2 to \( Q = T[\ell] \), we find that for all but at most \( 10^{-4} \delta n \) variables \( x \in V \), we have
\[ \left| \sum_{b \in N_{\leq 1}(x, T[\ell])} 2^{1 - |N(b)|} \text{sign}(x, b) \right| \leq 2 \cdot 10^{-3} \delta. \] (65)

Assume that \( x \) satisfies (65) and that \( x \notin T_2[\ell + 1] \). Let \( a \in N(x) \). Since \( N^{[\ell + 1]}_{\leq 1}(x \to a) = N_{\leq 1}(x, T[\ell]) \setminus \{a\} \), we obtain
\[ \left| \sigma^{[\ell + 1]}_{x \to a} \right| \leq \left| \sum_{b \in N_{\leq 1}(x, T[\ell])} 2^{1 - |N(b)|} \text{sign}(x, b) \right| + 2^{1 - |N(a)|} \leq 2 \cdot 10^{-3} \delta + 2^{1 - |N(a)|} \]
\[ \leq 2 \cdot 10^{-3} \delta + \exp(-0.1\delta k) \leq 0.003\delta \quad \text{[as } |N(a)| \geq 0.1\delta k \text{ due to T2a]}, \]
as desired. \[ \square \]

**Lemma 37.** Let \( x \) be a variable and let \( b_1, b_2 \in N(x) \) be such that \( |N(b_i) \cap T[\ell]| \leq 2 \) and \( |N(b_i)| \geq 0.1\delta k \) for \( i = 1, 2 \). Then
\[ \Delta^{[\ell]}_{x \to b_1} - \Delta^{[\ell]}_{x \to b_2} \leq \delta^3. \]
Proof. By Proposition 26 we have $B \mid \ell - 1 \subset T \mid \ell - 1$. Furthermore, our assumptions ensure that $N(b_1) \setminus T \mid \ell \neq \emptyset$. Hence, Corollary 31 yields

$$\mu_{b_1 \to x}^{(\ell-1)}(1) > 0 \text{ and } \left| \frac{\mu_{b_1 \to x}^{(\ell-1)}(1)}{\mu_{b_1 \to x}(-1)} - 1 \right| \leq \exp(-k_1/2) \leq \delta^6 \quad (66)$$

for $i = 1, 2$. There are three cases.

Case 1: there is $c \in N(b) \setminus \{b_1, b_2\}$ such that $\mu_c^{(\ell-1)}(1) = 0$. Then (27) shows that

$$\mu_{x \to b_1}^{(\ell)}(1) = \mu_{x \to b_2}^{(\ell)}(1) = 0.$$

Thus, $\Delta_{x \to b_1}^{[\ell]} = \Delta_{x \to b_2}^{[\ell]} = -1/2.$

Case 2: there is $c \in N(b) \setminus \{b_1, b_2\}$ such that $\mu_c^{(\ell-1)}(-1) = 0$. Similarly as in Case 1, (27) implies $\mu_{x \to b_i}^{(\ell)}(-1) = 0$ for $i = 1, 2$. Since $\mu_{x \to b_i}^{(\ell)}(-1) + \mu_{x \to b_i}^{(\ell)}(1) = 1$, we thus obtain $\Delta_{x \to b_1}^{[\ell]} = \Delta_{x \to b_2}^{[\ell]} = 1/2.$

Case 3: for all $c \in N(b) \setminus \{b_1, b_2\}$ we have $0 < \mu_c^{(\ell-1)}(1) < 1$. Then (27) yields $0 < \mu_{x \to b_i}^{(\ell)}(-1) < 1$ for $i = 1, 2$. Therefore, we can define

$$q_i = \frac{\mu_{x \to b_1}^{(\ell)}(1)}{\mu_{x \to b_1}^{(\ell)}(-1)} = \frac{\prod_{b \in N(x) \setminus \{b_1\}} \mu_{b \to x}^{(\ell-1)}(1)}{\prod_{b \in N(x) \setminus \{b_1\}} \mu_{b \to x}^{(\ell-1)}(-1)} > 0. \quad (67)$$

Unravelling (27), we see that

$$\mu_{x \to b_i}^{(\ell)} = \frac{\prod_{b \in N(x) \setminus \{b_i\}} \mu_{b \to x}^{(\ell-1)}(1)}{\prod_{b \in N(x) \setminus \{b_i\}} \mu_{b \to x}^{(\ell-1)}(1) + \prod_{b \in N(x) \setminus \{b_i\}} \mu_{b \to x}^{(\ell-1)}(-1)} \quad (68)$$

Hence,

$$\left| \Delta_{x \to b_1}^{[\ell]} - \Delta_{x \to b_2}^{[\ell]} \right| = \left| \mu_{x \to b_1}^{(\ell)}(1) - \mu_{x \to b_2}^{(\ell)}(1) \right|$$

$$= \left| \frac{q_1 - q_2}{(1 + q_1)(1 + q_2)} \right| \quad \text{[by (68)]}$$

$$= \left| \frac{1 - q_2/q_1}{(1 + 1/q_1)(1/q_1 + q_2/q_1)} \right| \leq \frac{q_1}{q_2} \left| 1 - \frac{q_2}{q_1} \right| \quad \text{[as } q_1, q_2 > 0]. \quad (69)$$

Furthermore, by the definition (67) of $q_1, q_2$, we have

$$\frac{q_2}{q_1} = \frac{\mu_{b_2 \to x}^{(\ell-1)}(1)}{\mu_{b_2 \to x}^{(\ell-1)}(-1)} \frac{\mu_{b_2 \to x}^{(\ell-1)}(1)}{\mu_{b_2 \to x}^{(\ell-1)}(-1)} \quad (69)$$

Hence, (66) yields $\left| 1 - \frac{q_2}{q_1} \right| \leq \delta^5$, and thus the desired bound on $\left| \Delta_{x \to b_1}^{[\ell]} - \Delta_{x \to b_2}^{[\ell]} \right|$ follows from (69). Hence, we have established the desired bound in all cases. \( \square \)

**Lemma 38.** For all but at most $0.1 \delta n$ variables $x \not\in T_2 \mid \ell + 1$ we have

$$\max_{a \in N(x)} |\xi_{x \to a}^{(\ell+1)}| \leq 0.001 \delta.$$
Hence, \( \text{Lemma 36 and 38 imply} \) 

\[
\begin{align*}
\text{If } & \quad \text{Proof of Proposition 34.} \\
\therefore & \quad \text{Therefore, the assertion follows from (71).}
\end{align*}
\]

\[\square\]

\[
\text{Because } |T|^{\ell} \leq \delta n, \text{ condition Q4 ensures that } \|T_\ell\|_\infty \leq \delta^{4} n. \text{ Consequently,}
\]

\[
\|\Xi\|_1 = \|T_\ell\| \leq \|A_T_\ell\| \leq \delta^{4} n.
\]

Since \( \|\Xi\|_1 = \sum_{x \in V} |\xi_x| \), (70) implies that

\[
|\{ x \in V : |\xi_x| > \delta^2 \}| \leq \delta^2 n. \tag{71}
\]

To infer the lemma from (71), we need to establish a relation between \( \xi_x \) and \( \xi_{x \rightarrow a} \) for \( x \not\in T_2 [\ell + 1] \) and \( a \in N(x) \). Since \( N(y) \subset N_{\leq 1}(x, T^{\ell}[\ell]) \) for any \( y \in V_t \), we see that \( |N(b) \cap T^{\ell}[\ell]| \leq 2 \) for all \( b \in N(y) \). Furthermore, as \( x \not\in T_2 [\ell + 1] \), we have \( |N(b)| \geq 0.1 \theta k \) for all \( b \in N(y) \) (by T2a). Consequently, Lemma 7 applies to \( b \in N(y) \), whence \( |\Delta_{y \rightarrow b}^{\ell} - \Delta_{y \rightarrow b'}^{\ell}| \leq \delta^3 \) for all \( y \in V_t, b, b' \in N(y) \). Hence,

\[
|\Delta_{y \rightarrow b}^{\ell} - \Delta_{y}^{\ell}| \leq \delta^3 \quad \text{for all } y \in V_t, b \in N(y). \tag{72}
\]

Consequently, we obtain for \( x \not\in T_2 [\ell + 1] \)

\[
\max_{a \in N(x)} \left| 2\xi_x - \xi_{x \rightarrow a}^{\ell+1} \right| = \max_{a \in N(x)} \left| 1_{a \in N_{\leq 2}(x, T^{\ell}[\ell])} \cdot \sum_{y \in N(a) \setminus \{x\}} 2^{1-|N(b)|} \text{sign}(x, b) \text{ sign}(y, b) \Delta_{y}^{\ell} \right| + \sum_{b \in N_{\leq 2}(x, a)} \sum_{y \in N(b) \setminus \{x\}} 2^{1-|N(b)|} \left| \Delta_{y \rightarrow a}^{\ell} - \Delta_{y}^{\ell} \right| \leq |N(a)| \left| 2 \right|^{-|N(a)|} \left| \Delta_{y}^{\ell} \right| + \sum_{b \in N(x)} \sum_{y \in N(b) \setminus \{x\}} 2^{1-|N(b)|} \left| \Delta_{y \rightarrow a}^{\ell} - \Delta_{y}^{\ell} \right| \leq 10k \theta^{-0.1k \theta} + 10 \delta^3 k \theta \sum_{b \in N(x)} 2^{1-|N(b)|} \quad \text{[as } 0.1k \theta \leq |N(b)| \leq 10k \theta \text{ by T2a]} \]

\[
\leq 10^{-4} \delta \quad \text{[by T2b].} \tag{73}
\]

If \( x \not\in T_2 [\ell + 1] \) is such that \( |\xi_x| \leq \delta^2 \), then (73) implies that \( |\xi_{x \rightarrow a}| \leq 2 \cdot 10^{-4} \delta \) for any \( a \in N(x) \). Therefore, the assertion follows from (71). \( \square \)

**Proof of Proposition 34** Let \( S \) be the set of all variables \( x \not\in T_2 [\ell + 1] \) such that \( \max_{a \in N(x)} |\sigma_{x \rightarrow a}^{[r+1]}| \leq 0.003 \delta \) and \( \max_{a \in N(i)} |\xi_{x \rightarrow a}^{[r+1]}| \leq 0.001 \delta \). Then Proposition 35 entails that for any \( x \in S \) and \( a \in N(x) \)

\[
\left| \ln P_{x \rightarrow a}^{[r+1]} \right| = \left| L_{x \rightarrow a}^{[r+1]} + 10^{-3} \delta \right| \leq |\sigma_{x \rightarrow a}^{[r+1]}| + |\xi_{x \rightarrow a}^{[r+1]}| + 10^{-3} \delta \leq 0.005 \delta.
\]

Hence, \( P_{x \rightarrow a}^{[r+1]} = 10^{-4} \delta \) for all \( a \in S \), and therefore

\[
T_r [\ell + 1] \backslash T_2 [\ell + 1] \subset V_t \backslash (S \cup T_2 [\ell + 1]).
\]

Finally, Lemmas 36 and 38 imply \( |T_1 [\ell + 1] \backslash T_2 [\ell + 1]| \leq |V_t \backslash (S \cup T_2 [\ell + 1])| \leq \delta \theta n/6. \) \( \square \)
3.7 Proof of Proposition 35

We begin by approximating \( \ln |\mu_\rho^{[\theta k\delta]}(1)/\mu_\rho^{[\theta k\delta]}(-1)| \) by a linear function. In this section we let \( O_\rho() \) denote an asymptotic bound that holds in the limit of large \( \rho \). That is, \( f(\rho) = O(g(\rho)) \) if there exist \( C > 0 \), \( \rho_* > 0 \) such that \( |f(\rho)| \leq C|g(\rho)| \) for \( \rho > \rho_* \).

Lemma 39. Let \( x \in V_t \), \( a \in N(x) \), and \( b \in N^{[\ell + 1]}_{\leq 1}(x \to a) \). Then \( \mu_{b \to x}^{[\theta k\delta]}(-1) > 0 \) and

\[
\ln \left( \frac{\mu_{b \to x}^{[\theta k\delta]}(1)}{\mu_{b \to x}^{[\theta k\delta]}(-1)} \right) = 2^{1-|N(b)|} \left[ \sum_{y \in N(b) \setminus \{x\}} \text{sign}(x, b) \text{sign}(y, b) \Delta_{y \to b}^{[\theta k\delta]} + \frac{(\theta k\delta + |N(b) \cap T[\ell] \setminus \{x\})}{2^{|N(b)|}} \cdot O_\rho(k\theta\delta) \right].
\]

Proof. The definition of the set \( N^{[\ell + 1]} \) ensures that for all \( b \in N^{[\ell + 1]}_{\leq 1}(x \to a) \) we have \( |N(b) \cap T[\ell]| \leq 2 \), while \( |N(b)| \geq 0.1\theta k \). Therefore, Lemma 28 shows that \( |\frac{1}{2} - \mu_{b \to x}^{[\theta k\delta]}(-1)| \leq \frac{1}{2} \) (recall from Proposition 26 that \( B[\ell] \subseteq T[\ell] \)). Furthermore, \( b \) is not redundant, and thus not a tautology, because otherwise \( N(b) \subseteq T_2[\ell] \) due to T2a.

Let \( s = \text{sign}(x, b) \). Then \( \mu_{b \to x}^{[\theta k\delta]}(s) = 1 \) and thus the definition 2 of the messages \( \mu_{b \to x}^{[\theta k\delta]}(\pm 1) \) yields

\[
\frac{\mu_{b \to x}^{[\theta k\delta]}(-s)}{\mu_{b \to x}^{[\theta k\delta]}(s)} = \mu_{b \to x}^{[\theta k\delta]}(-s) = 1 - \prod_{y \in N(b) \setminus \{x\}} \mu_{y \to b}(-\text{sign}(y, b))
\]

\[
= 1 - \prod_{y \in N(b) \setminus \{x\}} \frac{1}{2} - \text{sign}(y, b) \Delta_{y \to b}^{[\theta k\delta]}
\]

\[
= 1 - 2^{1-|N(b)|} \prod_{y \in N(b) \setminus \{x\}} 1 - 2\text{sign}(y, b) \Delta_{y \to b}^{[\theta k\delta]}. \tag{75}
\]

Let \( \Gamma = N(b) \setminus (T[\ell] \cup \{x\}) \). As Proposition 26 shows that \( T[\ell] \supset B[\ell] \) contains all biased variables, we have \( |\Delta_{y \to b}^{[\theta k\delta]}| \leq \delta \) for all \( y \in \Gamma \). Therefore, we can use the approximation \( |\ln(1 - z) + z| \leq z^2 \) for \( |z| \leq \frac{1}{2} \) to obtain

\[
\left| \ln \prod_{y \in \Gamma} 1 - 2\text{sign}(y, b) \Delta_{y \to b}^{[\theta k\delta]} \right| + \sum_{y \in \Gamma} 2\text{sign}(y, b) \Delta_{y \to b}^{[\theta k\delta]}
\]

\[
= \left| \sum_{y \in \Gamma} \ln \left( 1 - 2\text{sign}(y, b) \Delta_{y \to b}^{[\theta k\delta]} \right) + 2\text{sign}(y, b) \Delta_{y \to b}^{[\theta k\delta]} \right|
\]

\[
\leq 4 \sum_{y \in \Gamma} \Delta_{y \to b}^{[\theta k\delta]} \leq 40\theta k\delta^2; \tag{76}
\]

in the last step, we used that \( |N(b)| \leq 10\theta k \) for all \( b \in N^{[\ell + 1]}_{\leq 1}(x \to a) \). Furthermore, \( |\Gamma| \leq |N(b)| \leq 10\theta k \) and \( |\Delta_{y \to b}^{[\theta k\delta]}| \leq \delta \) for all \( y \in \Gamma \). Hence,

\[
\sum_{y \in \Gamma} 2\text{sign}(y, b) \Delta_{y \to b}^{[\theta k\delta]} \leq 20\delta k\theta. \tag{77}
\]

Therefore, taking exponentials in (76), we obtain

\[
\prod_{y \in \Gamma} 1 - 2\text{sign}(y, b) \Delta_{y \to b}^{[\theta k\delta]} = \exp \left[ O_\rho(\theta k\delta)^2 - \sum_{y \in \Gamma} 2\text{sign}(y, b) \Delta_{y \to b}^{[\theta k\delta]} \right]
\]

\[
= 1 - \sum_{y \in \Gamma} 2\text{sign}(y, b) \Delta_{y \to b}^{[\theta k\delta]} + O_\rho(\theta k\delta)^2. \tag{78}
\]
Finally, the assertion follows by plugging (80) and (81) into (79).

If there is \( y_0 \in N(b) \cap T[\ell] \setminus \{x\} \), then (77) and (78) yield

\[
\prod_{y \in N(b) \setminus \{x\}} 1 - 2\text{sign}(y, b) \Delta^{|\ell|}_{y \to b} = (1 - 2\text{sign}(y_0, b) \Delta^{|\ell|}_{y_0 \to b}) \prod_{y \in T} 1 - 2\text{sign}(y, b) \Delta^{|\ell|}_{y \to b}
\]

\[
= 1 - 2 \left[ \sum_{y \in N(b) \setminus \{x\}} \text{sign}(y, b) \Delta^{|\ell|}_{y \to b} \right] + O_\rho(\theta k\delta).
\]

Hence, in any case we have

\[
\prod_{y \in N(b) \setminus \{x\}} 1 - 2\text{sign}(y, b) \Delta^{|\ell|}_{y \to b} = 1 - 2 \left[ \sum_{y \in N(b) \setminus \{x\}} \text{sign}(y, b) \Delta^{|\ell|}_{y \to b} \right] + (\theta k\delta + |N(b) \cap T[\ell] \setminus \{x\}|) \cdot O_\rho(\theta k\delta).
\]

Combining this with (75) and using the approximation \(|\ln(1 - z) + z| \leq z^2\) for \(|z| \leq 1/2\), we see that

\[
\ln \left( \frac{\mu^{|\ell|}_{b \to x}(s)}{\mu^{|\ell|}_{b \to x}(-s)} \right) = -2^{1-|N(b)|} \left[ 1 - 2 \sum_{y \in N(b) \setminus \{x\}} \text{sign}(y, b) \Delta^{|\ell|}_{y \to b} \right] + 2^{1-|N(b)|} (\theta k\delta + |N(b) \cap T[\ell] \setminus \{x\}|) \cdot O_\rho(\theta k\delta),
\]

whence the assertion follows. \(\Box\)

**Proof (Proposition 35).** Suppose \( x \notin T_2[\ell + 1]\). By the definition of \( P^{|\ell+1|}_{\leq 1}(x \to a) \) we have

\[
\ln P^{|\ell+1|}_{\leq 1}(x \to a) = \sum_{b \in N^{|\ell+1|}_{\leq 1}(x \to a)} \ln \left( \frac{\mu^{|\ell|}_{b \to x}(1)}{\mu^{|\ell|}_{b \to x}(-1)} \right).
\]

Hence, Lemma 39 yields

\[
\ln P^{|\ell+1|}_{\leq 1}(x \to a) = L^{|\ell+1|}_{x \to a} + \sum_{b \in N^{|\ell+1|}_{\leq 1}(x \to a)} 2^{-|N(b)|} (\theta k\delta + |N(b) \cap T[\ell] \setminus \{x\}|) O_\rho(\theta k\delta).
\]

To complete the proof, we need to estimate the second summand. Condition T2b implies

\[
O_\rho(\delta \theta k)^2 \sum_{b \in N^{|\ell+1|}_{\leq 1}(x \to a)} 2^{-|N(b)|} \leq O_\rho(\delta \theta k)^2 \sum_{b \in N(x)} 2^{-|N(b)|} \leq \frac{O_\rho(\delta \theta k)^2}{\delta \theta k} \leq \frac{O_\rho(\delta \theta k)^2}{(\theta k)^3} \leq 10^{-4} \delta.
\]

Furthermore, T2c yields

\[
O_\rho(\theta k\delta) \sum_{b \in N^{|\ell+1|}_{\leq 1}(x \to a)} 2^{-|N(b)|} |N(b) \cap T[\ell] \setminus \{x\}| \leq O_\rho(\theta k\delta) \sum_{b \in N_1(x, T[\ell])} 2^{-|N(b)|} \leq O_\rho(\theta k\delta) \cdot \rho(\theta k)^5 \delta \leq 10^{-4} \delta.
\]

Finally, the assertion follows by plugging (80) and (81) into (79). \(\Box\)
3.8 Completing the proof of Theorem 25

We are going to show that \(|\mu_x(\Phi_t, \omega) - \frac{1}{2}\) \leq \delta = \delta_t\) for all \(x \in V_t \setminus T[\omega + 1]\). This will imply Theorem 25 because \(|T[\omega + 1]| \leq \delta_t(n - t)\) by Proposition 27.

Thus, let \(x \in V_t \setminus T[\omega + 1]\). Corollary 31 shows that \(\mu_{b^{-}x}(\omega) > 0\) for \(\omega = \pm 1\). Hence,

\[P(\omega) = \prod_{b \in N(x)} \mu_{b^{-}x}(\omega) > 0 \text{ for } \omega = \pm 1.\]

Recall from (4) that

\[\mu_x(\Phi_t, \omega) = \frac{P(1)}{P(-1) + P(1)}.\]

If \(N(x) = \emptyset\), then trivially \(P(-1) = P(1) = 1\) and thus \(\mu_x(\Phi_{t-1}, \omega) = \frac{1}{2}\). Thus, assume that \(N(x) \neq \emptyset\) and pick an arbitrary \(a \in N(x)\). Then

\[P(\omega) = \mu_{a^{-}x}(\omega) \cdot \mu_{x^{-}a}(\omega) \text{ for } \omega = \pm 1.\]

Since \(x \not\in T[\omega + 1] \supset B[\omega + 1]\) by Proposition 20, we have

\[|\mu_{a^{-}x}(\omega) - \frac{1}{2}| = |\Delta_{a^{-}x}\omega| \leq 0.1\delta \text{ for } \omega = \pm 1.\]

Therefore,

\[
\ln \frac{\mu_{a^{-}x}(\omega) - 1}{\mu_{a^{-}x}(\omega) + 1} \leq \ln \frac{1 + 0.2\delta}{1 - 0.2\delta} \leq 0.5\delta, \text{ and analogously}
\]

\[
\ln \frac{\mu_{x^{-}a}(\omega) - 1}{\mu_{x^{-}a}(\omega) + 1} \geq -0.5\delta.
\]

Furthermore, since \(x \not\in T[\omega + 1]\) Corollary 31 yields

\[
\left| \ln \frac{\mu_{a^{-}x}(\omega) - 1}{\mu_{a^{-}x}(\omega) + 1} \right| \leq 2\exp(-k_1/2) \leq \delta^2.
\]

Hence,

\[
\left| \ln \frac{P(-1)}{P(1)} \right| \leq \left| \ln \frac{\mu_{a^{-}x}(\omega) - 1}{\mu_{a^{-}x}(\omega) + 1} \right| + \left| \ln \frac{\mu_{x^{-}a}(\omega) - 1}{\mu_{x^{-}a}(\omega) + 1} \right| \leq 0.5\delta + \delta^2 \leq 0.51\delta.
\]

Therefore, letting \(z = \ln \frac{P(-1)}{P(1)}\), we obtain

\[
\left| \frac{1}{2} - \mu_x(\Phi_{t-1}) \right| = \left| \frac{1}{2} - \frac{P(1)}{P(-1) + P(1)} \right| = \left| \frac{1}{2} - \frac{1}{1 + \exp(z)} \right| \leq \left| \frac{1 - \exp(z)}{2} \right| < \delta,
\]
as desired.

4 Proof of Proposition 24

Recall from (13) that \(\delta_x = \exp(-c(1 - s/n))k\) and that \(\tilde{t} = (1 - \frac{\ln p}{\ln 2})\). Suppose that \(1 \leq t \leq \tilde{t}\). Then \(\theta = 1 - t/n\) satisfies \(\theta k \geq \ln(p)/\epsilon^2\). We assume throughout that \(p = kr/2^k \geq \rho_0\) for some large enough number \(\rho_0\); in particular, we assume that \(\rho_0 \geq \exp(1/c)\). Set

\[\delta = \delta_t = \exp(-ck\theta)\]
for brevity. Then Lemma 42 yields
\[ \delta \theta n > 10^{15} \Delta, \] (82)

To prove Proposition 24, we will study two slightly different models of random $k$-CNFs. In the first “binomial” model $\Phi_{\text{bin}}$, we obtain a $k$-CNF by including each of the $(2n)^k$ possible clauses over $V = \{x_1, \ldots, x_n\}$ with probability $p = m/(2n)^k$ independently, where each clause is an ordered $k$-tuple of not necessarily distinct literals. Thus, $\Phi_{\text{bin}}$ is a random set of clauses, and $E[|\Phi_{\text{bin}}|] = m$.

In the second model, we choose a sequence $\Phi_{\text{seq}}$ of $m$ independent $k$-clauses $\Phi_{\text{seq}}(1), \ldots, \Phi_{\text{seq}}(m)$, each of which consists of $k$ independently chosen literals. Thus, the probability of each individual sequence is $(2n)^{-km}$. The sequence $\Phi_{\text{seq}}$ corresponds to the $k$-CNF $\{\Phi_{\text{seq}}(1), \ldots, \Phi_{\text{seq}}(m)\}$ with at most $m$ clauses. The following well-known fact relates $\Phi$ to $\Phi_{\text{bin}}, \Phi_{\text{seq}}$.

**Fact 40.** For any event $E$ we have
\[ P[\Phi \in E] \leq O(\sqrt{m}) \cdot P[\Phi_{\text{bin}} \in E], \]
\[ P[\Phi \in E] \leq O(1) \cdot P[\Phi_{\text{seq}} \in E]. \]

Due to Fact 40 and (82), it suffices to prove that the statements Q1–Q4 hold for either of $\Phi, \Phi_{\text{bin}}, \Phi_{\text{seq}}$ with probability at least $1 - \exp(-10^{-15} \delta \theta n)$.

**Establishing Q1.** We are going to deal with the number of variables that appear in “short” clauses first.

**Lemma 41.** With probability at least $1 - \exp(-10^{-6} \delta \theta n)$ in $\Phi^i$ there are no more than $\theta n \cdot 10^{-5} \frac{\delta}{\theta k}$ clauses of length less than $0.1 \theta k$.

**Proof.** We are going to work with $\Phi_{\text{bin}}$. Let $L_j$ be the number of clauses of length $j$ in $\Phi_{\text{bin}}$. Then for any $j \in [k]$ we have
\[ \lambda_j = E[L_j] = m \cdot 2^{j-k} \binom{k}{j} \theta^j (1 - \theta)^{k-j} = 2^j \rho \theta n \binom{k}{j} \theta^j (1 - \theta)^{k-j}. \]

Indeed, a clause has length $j$ in $\Phi_{\text{bin}}$ iff it contains $j$ variables from the set $V_i$ of size $\theta n$ and $k-j$ variables from $V \setminus V_i$ and none of the $k-j$ variables from $V \setminus V_i$ occurs positively. The total number of possible clauses with these properties is $2^j \binom{k}{j} (\theta n)^j (1-\theta n)^{k-j}$, and each of them is present in $\Phi_{\text{bin}}$ with probability $p = m/(2n)^k$ independently.

Let’s start by bounding the total number $L_* = \sum_{j< \theta k/10} L_j$ of “short” clauses. Its expectation is bounded by
\[ E[L_*] = \sum_{j< \theta k/10} \lambda_j \leq 2^{0.1 \theta k} \rho \theta n \cdot P[\text{Bin}(k-1, \theta) < \theta k/10] \]
\[ \leq 2^{0.1 \theta k} \rho \theta n \cdot \exp(-\theta k/3) \quad \text{[by Lemma 8 (Chernoff)]} \]
\[ \leq \theta \exp(-\theta k/4) n \quad \text{[as } \theta k \geq \ln(\rho)/c^2]. \]

Furthermore, $L_*$ is binomially distributed, because clauses appear independently in $\Phi_{\text{bin}}$. Hence, again by Lemma 8, we have
\[ P[L_* > \theta n \cdot 10^{-5} \delta/(\theta k)] \leq \exp \left[ -\frac{10^{-5} \delta}{\theta k} \cdot \ln \left( \frac{10^{-5} \delta/(\theta k)}{\exp(1-\theta k/4)} \right) \cdot \theta n \right] \]
\[ \leq \exp \left[ -\frac{\delta}{5 \cdot 10^6 \theta k} \cdot \theta k \cdot \theta n \right] \leq \exp \left( -10^{-6} \delta \theta n \right). \] (83)

Hence, the assertion follows from (83) and Fact 40. □

**Corollary 42.** With probability at least $1 - \exp(-10^{-6} \delta \theta n)$ in $\Phi^i$ no more than $10^{-6} \delta \theta n$ variables appear in clauses of length less than $0.1 \theta k$. 
Lemma 45. Let \( x \in V_t \). The expected number of clauses of length \( j \) in \( \Phi^t_{\text{bin}} \), where \( x \) is the underlying variable of the \( i \)-th literal is

\[
\mu_j = \rho \cdot \frac{2^j}{j} \binom{k-1}{j-1} (1 - \theta)^{k-j+1} (\theta)^{j-1} \leq \frac{2^j \rho}{j}.
\]

Proof. This is immediate from Lemma 44. \( \square \)

As a next step, we are going to bound the number of variables that appear in clauses of length \( \geq 10\theta k \).

**Lemma 43.** With probability at least \( 1 - \exp(-10^{-11} \delta \theta n) \) we have

\[
\sum_{b : |N(b)| > 10\delta \theta} |N(b)| \leq 10^{-6} \delta \theta n.
\]

Proof. For a given \( \mu > 0 \) let \( L_\mu \) be the event that \( \Phi^t_{\text{seq}} \) has \( \mu \) clauses so that the sum of the lengths of these clauses is at least \( \lambda = 10\theta k\mu \). Then

\[
P[L_\mu] \leq \binom{m}{\mu} \left( \frac{k\mu}{\lambda} \right)^{\lambda} \left( \frac{1}{2} + \theta \right)^{\frac{1}{2} \lambda} \mu^{\lambda - \mu}.
\]

Indeed, there are \( \binom{m}{\mu} \) ways to choose \( \mu \) places for these \( \mu \) clauses in \( \Phi^t_{\text{seq}} \). Once these have been specified, there are \( k\mu \) literals that constitute the \( \mu \) clauses, and we choose \( \lambda \) whose underlying variables are supposed to be in \( V_t \); the probability that this is indeed the case for all of these \( \lambda \) literals is \( \theta^\lambda \). Moreover, in order for each of the clauses to remain in \( \Phi^t_{\text{seq}} \), the remaining \( k\mu - \lambda \) literals must either be negative or have underlying variables from \( V_t \), leading to the \( (\theta + 1/2)^{\mu - \lambda} \) factor. Thus,

\[
P[L_\mu] \leq \binom{m}{\mu} \left[ \left( \frac{1}{2} + \theta \right) \left( \frac{e}{5} \right)^{10\delta \theta} \right]^{k\mu} \leq \left( \frac{e}{5} \right)^{10\delta \theta} \leq \left( \frac{e}{5} \right)^{10\delta \theta}.
\]

Hence, if \( \lambda \geq 10^{-6} \delta \theta n \) we get

\[
P[L_\mu] \leq \left[ \frac{10^7 e \rho}{\delta} \right]^{1/10\delta \theta} \left( \frac{e}{4} \right)^\lambda \leq \left( \frac{e}{3} \right)^\lambda \leq \exp(-c\delta\theta n).
\]

Thus, we see that in \( \Phi^t_{\text{seq}} \) with probability at least \( 1 - \exp(-10^{-10} \delta \theta n) \) we have

\[
\sum_{b : |N(b)| > 10\delta \theta} |N(b)| \leq 10^{-6} \delta \theta n. \tag{84}
\]

Hence, Fact 40 implies that (84) holds in \( \Phi^t \) with probability at least \( 1 - \exp(-10^{-11} \delta \theta n) \). \( \square \)

**Corollary 44.** With probability at least \( 1 - \exp(-10^{-11} \delta \theta n) \) no more than \( 10^{-6} \delta \theta n \) variables appear in clauses of length greater than \( 10\delta \theta k \).

Proof. The number of such variables is bounded by \( \sum_{b : |N(b)| > 10\theta k} \sum_{N(b)} \). Therefore, the assertion follows from Lemma 43. \( \square \)

We now come to the second part of Q1. We start with the following simple observation.

**Lemma 45.** Let \( x \in V_t \). The expected number of clauses of length \( j \) in \( \Phi^t_{\text{bin}} \), where \( x \) is the underlying variable of the \( i \)-th literal is

\[
\mu_j = \rho \cdot \frac{2^j}{j} \binom{k-1}{j-1} (1 - \theta)^{k-j+1} (\theta)^{j-1} \leq \frac{2^j \rho}{j}.
\]
Proof. There are $2^j(k_j)(\theta n)^{-j}((1 - \theta)n)^{k-j+1}$ possible clauses that have exactly $j$ literals whose underlying variable is in $V_l$ such that the underlying variable of the $j$th such literal is $x$. Each such clause is present in $\Phi_{\text{bin}}$ with probability $p = m/(2n)^k = \frac{1}{2}n^{1-k}$ independently. \qed

Lemma 46. With probability at least $1 - \exp(-10^{-12} \delta n)$ no more than $10^4 \delta n$ variables $x \in V_l$ are such that $2 \delta^{-1}(\theta k)^{-5}/j > 100 \mu_j$. Hence, Lemma 47 (the Chernoff bound) yields

$$P \left[ X_{jl}(x) > 10(\mu_j + 2^j \delta^{-1}(\theta k)^{-5}/j) \right] \leq e^{-\delta \mu_j}.$$

Proof. For $x \in V_l$ let $X_{jl}(x)$ be the number of clauses of length $j$ in $\Phi_{\text{bin}}$ that contain $x$, and let $X_{jl}(x)$ be the number of such clauses where $x$ is the underlying variable of the $j$th literal of that clause ($1 \leq l \leq j$). Then $E[X_{jl}(x)] = \mu_j$, with $\mu_j$ as in (85). Since $1/\delta = \exp(ck\theta)$ and $\theta \geq \ln(\rho)/c^2$, we see that $2^j \delta^{-1}(\theta k)^{-5}/j > 100 \mu_j$. Hence, Lemma 47 (the Chernoff bound) yields

$$P \left[ X_{jl}(x) > 10(\mu_j + 2^j \delta^{-1}(\theta k)^{-5}/j) \right] \leq \exp(\ln(\rho))/2,$$

whence

$$P \left[ X_{jl}(x) > \frac{\delta}{(\theta k)^{\theta}} \cdot \theta n \right] \leq \exp \left[ - \frac{\delta \theta n}{2(\theta k)^{\theta}} \cdot \ln \left( \frac{\delta}{(\theta k)^{\theta}} \right) \right] \leq \exp(-\delta \theta n).$$

Furthermore, if $x \notin V_{jl}$ for all $1 \leq j \leq 100k$ and all $1 \leq l \leq j$, then

$$\sum_{b \in N(x); |N(b)| \leq 100k} 2^{-1} \sum_{j \leq 100k} 2^{-j}(j \mu_j + 2^j \delta^{-1}(\theta k)^{-5}) \leq 100 \delta^{-1}(\theta k)^{-4} + 10 \sum_{j \leq 100k} j^2 \delta^j \mu_j \leq 100 \delta^{-1}(\theta k)^{-4} + 10 \rho < \delta^{-1}(\theta k)^{-3},$$

where we used that $\delta \theta k \geq \ln(\rho)/c^2$, so that $1/\delta \geq (\theta k)^5 \rho$. Hence, the assertion follows from (86). Fact 44 and the bound on the number of variables in clauses of length $> 100k$ provided by Lemma 43. \qed

Establishing Q2. Let $T \subset V_l$ be a set of size $|T| \leq \delta n$. For a variable $x$ we let $Q(x, i, j, l, T)$ be the number of clauses $b$ of $\Phi_{\text{bin}}$ such that the $i$th literal is either $x$ or $\neg x$, $|N(b)| = j$, and $|N(b) \cap Q \setminus \{x\}| = l$.

Lemma 47. Suppose that $l \geq 1$, $j - l > k_1$, and $0.10k \leq j \leq 100k$. Let

$$\gamma_{j,l} = \begin{cases} 10j2^j \delta \rho & \text{if } l = 1, \\ 102^{j-l} \delta^{1.9} & \text{if } l > 1. \end{cases}$$

Then for any $i, x, T$ we have $P \left[ Q(x, i, j, l, T) > \gamma_{j,l} \right] \leq \exp(-\exp(\gamma_{j,l}^{2/3} \theta k))$.

Proof. The random variable $Q(x, i, j, l, T)$ has a binomial distribution, because clauses appear independently in $\Phi_{\text{bin}}$. With $\mu_j$ from (85) we have for $l > 1$

$$E \left[ Q(x, i, j, l, T) \right] \leq \binom{j}{l} \delta^l \mu_j \leq \rho \binom{j}{l} \delta^l 2^j \leq 2^{j-1} \delta^{1.9};$$
in the last step we used that $\delta^{0.05} \leq 1/\rho$, which follows from our assumption that $\theta k \geq \ln(\rho)/\epsilon^2$, and that $2^{\binom{2}{2}} \leq (2j)^2 \leq (20k\theta)^2 \leq \delta^{0.05}$. Hence, by Lemma 49 (the Chernoff bound) in the case $j - l > k = \sqrt{\theta k}, l > 1$ we get

$$
P \left[ Q(x, i, j, l, T) > 10 \cdot 2^{j-l} \delta^{1.9} \right] \leq \exp(-2^{j-l} \delta^{1.9}) \leq \exp(-2^{k} \delta^{1.9}) \leq \exp(-\exp(c^2/3\theta k)),
$$
as $\delta = \exp(-ck\theta)$.

By a similar token, in the case $l = 1$ we have $E[Q(x, i, j, l, T)] \leq j^d \mu_j \leq \rho \delta^2 l$. Hence, once more by the Chernoff bound

$$
P \left[ Q(x, i, j, l, T) > 10 \cdot 2^j \delta \right] \leq \exp(-2^j \delta) \leq \exp(-\exp(c^2/3\theta k)),
$$
as claimed.

\[\square\]

Let $Z(i, j, l, T)$ be the number of variables $x \in V_i$ for which $Q(x, i, j, l, T) > \gamma_{j,l}$. \[\]

**Lemma 48.** Suppose that $l \geq 1, j - l > k$ and $0.1 \theta k \leq j \leq 10 \theta k$. Then for any $i, T$ we have

$$
P \left[ Z(i, j, l, T) > \delta \theta n/(\theta k)^4 \right] \leq \exp \left[ -\frac{\delta \theta n}{2(\theta k)^4} \cdot \exp(c^2/3\theta k) \right].
$$

**Proof.** Whether a variable $x \in V_i$ contributes to $Z(i, j, l, T)$ depends only on those clauses of $\Phi^i_{\text{bin}}$, whose $i$th literal reads either $x$ or $\neg x$. Since these sets of clauses are disjoint for distinct variables and as clauses appear independently in $\Phi^i_{\text{bin}}$, $Z(i, j, l, T)$ is a binomial random variable. By Lemma 49

$$
E[Z(i, j, l, T)] \leq \theta n \cdot \exp(-\exp(c^2/3\theta k)).
$$

Hence, Lemma 49 (the Chernoff bound) yields

$$
P \left[ Z(i, j, l, T) > \delta \theta n/(\theta k)^4 \right] \leq \exp \left[ -\frac{\delta \theta n}{(\theta k)^4} \ln \left( \frac{\delta}{(\theta k)^4 \exp(1 - \exp(c^2/3\theta k))} \right) \right]
$$

\[\leq \exp \left[ -\frac{\delta \theta n}{2(\theta k)^4} \cdot \exp(c^2/3\theta k) \right],
$$
as desired. \[\square\]

**Corollary 49.** With probability $1 - \exp(-\delta \theta n)$ the random formula $\Phi^i_{\text{bin}}$ has the following property.

For all $i, j, l, T$ such that $l \geq 1, j - l > k$, $0.1 \theta k \leq j \leq 10 \theta k$ and $|T| \leq \delta \theta n$ we have $Z(i, j, l, T) \leq \delta \theta n/(\theta k)^4$. \[\]

**Proof.** We apply the union bound. There are at most $n \binom{n}{\delta \theta n}$ ways to choose the set $T$, and no more than $n$ ways to choose $i, j, l$. Hence, by Lemma 48 the probability that there exist $i, j, l, T$ such that $Z(i, j, l, T) > \theta n \cdot \exp(-\exp(c^2/3\theta k))$ is bounded by

$$
n^2 \binom{n}{\delta \theta n} \exp \left[ -\frac{\delta \theta n}{(\theta k)^4} \cdot \exp(c^2/3\theta k) \right] \leq \exp \left[ o(n) + \delta \theta n(1 - \ln(\delta\theta)) - \frac{\delta \theta n}{(\theta k)^4} \cdot \exp(c^2/3\theta k) \right]
$$

\[\leq \exp \left[ \delta \theta n \left( o(1) - 2 \ln \delta - \exp(c^3/3\theta k) \right) \right] \leq \exp [-\delta \theta n],
$$
as claimed. \[\square\]

**Corollary 50.** With probability $1 - \exp(-10^{-12} \delta \theta n)$ the random formula $\Phi^i$ has the following property.

If $T \subset V_i$ has size $|T| \leq \delta \theta n$, then for all but $10^{-4} \delta \theta n$ variables $x$ we have

$$
\sum_{b \in \mathcal{N}_i(x, T)} |\mathcal{N}(b) \cap T \setminus \{x\}| \cdot |\mathcal{N}(b)| \leq \frac{\delta n}{\theta k} \text{ and } \sum_{b \in \mathcal{N}_i(x, T)} 2^{-|\mathcal{N}(b)|} < \rho(\theta k)^5 \delta.
$$
Proof. Given \( T \subset V_t \) of size \( |T| \leq \delta \theta n \), let \( \mathcal{V}_T \) be the set of all variables \( x \) with the following two properties.

i. For all \( b \in N(x) \) we have \( 0.1 \theta k \leq |N(b)| \leq 10 \theta k \).

ii. For all \( 1 \leq i \leq j, 1 \leq l \leq j - k_1 \), and \( 0.1 \theta k \leq j \leq 10 \theta k \) we have \( Q(x, i, j, l, T) \leq \gamma_{j, l} \).

Then for all \( x \in \mathcal{V}_T \) we have

\[
\sum_{b \in N(x, T)} 2^{|N(b) \cap T| - |N(b)|} = \sum_{0.1 \theta k \leq j \leq 10 \theta k} \sum_{i=1}^{j} \sum_{l=2}^{j-k_1} Q(x, i, j, l, T) 2^{l-j} \quad [\text{due to i.}]
\]

\[
\leq 10 k \theta \sum_{0.1 \theta k \leq j \leq 10 \theta k} \sum_{l=2}^{j-k_1} \gamma_{j, l} 2^{l-j} \quad [\text{due to ii.}]
\]

\[
\leq 1000(k \theta)^3 \delta \theta n < \delta/(k \theta) \quad [\text{as } \delta = \exp(-ck \theta)].
\]

Similarly,

\[
\sum_{b \in N(x, T)} 2^{-|N(b)|} \leq \sum_{0.1 \theta k \leq j \leq 10 \theta k} \sum_{i=1}^{j} Q(x, i, j, 1, T) 2^{-j} \quad [\text{due to i.}]
\]

\[
\leq 10 k \theta \sum_{0.1 \theta k \leq j \leq 10 \theta k} 2^{-j} \gamma_{j, 1} \quad [\text{due to ii.}]
\]

\[
\leq 1000(k \theta)^3 \delta \theta n < \rho(k \theta)^2 \delta [\text{due to i.}]
\]

Thus, to complete the proof we need to show that with sufficiently high probability \( \mathcal{V}_T \) is sufficiently big for all \( T \). By Lemmas 41 and 43 with probability \( 1 - 2 \exp(-10^{-11} \delta \theta n) \) the number of variables \( x \) that fail to satisfy i. is less than \( 2 \cdot 10^{-6} \delta \theta n \). Furthermore, by Corollary 42 and Fact 40 with probability \( \geq 1 - \exp(-\delta \theta n/2) \) the random formula \( \Phi' \) satisfies \( (87) \). In this case, for all \( T \) the number of variables that fail to satisfy ii. is bounded by \( \delta \theta n/(k \theta)^3 < 10^{-5} \delta \theta n \). Thus, with probability \( \geq 1 - \exp(-10^{-12} \delta \theta n) \) we have \( |\mathcal{V}_T| > \delta n(1 - 10^{-4} \delta) \) for all \( T \), as desired.

For a set \( T \subset V_t \) and numbers \( i \leq j \) we let \( N_+(x, i, j, T) \) be the number of clauses \( b \in N(x) \) in \( \Phi'_\bin \) such that \( |N(b)| = j \), the 2\textsuperscript{ith} literal of \( b \) is \( x \) and \( |N(b) \cap T \setminus x| \leq 1 \). Similarly, we let \( N_-(x, i, j, T) \) be the number of \( b \in N(x) \) such that \( |N(b)| = j \), the \( i \text{th} \) literal of \( b \) is \( \neg x \) and \( |N(b) \cap T \setminus x| \leq 1 \). Let \( B(i, j, T) \) be the set of variables \( x \in V_t \) such that

\[
|N_+(x, i, j, l) - N_-(x, i, j, l)| > 2^j \delta (\theta k)^{-3}.
\]

**Lemma 51.** Let \( T \subset V_t \) be a set of size \( |T| \leq \delta \theta n \). Let \( i, j \) be such that \( i \leq j \) and \( 0.1 \theta k \leq j \leq 10 \theta k \).

Then in \( \Phi'_\bin \), we have \( P[B(i, j, T) > \delta \theta n/(k \theta)^3] \leq \exp[-\delta \theta n \exp(\theta k/22)] \).

**Proof.** Let \( x \in V_t \). In the random formula \( \Phi'_\bin \) we have

\[
E[N_+(x, i, j, T) + N_-(x, i, j, T)] \leq \mu_j \leq 2^i/\rho \quad [\text{with } \mu_j \text{ as in } (63)].
\]

Furthermore, \( N_+(x, i, j, T) \), \( N_-(x, i, j, T) \) are binomially distributed with identical means, because in \( \Phi'_\bin \) each literal is positive/negative with probability \( \frac{1}{2} \). Hence, for \( j \geq 0.1 \theta k \) Lemma 9 (the Chernoff bound) yields

\[
P[N_+(x, i, j, l) - N_-(x, i, j, l)] > 2^j \delta (\theta k)^{-3}] \leq \exp\left[-\frac{(2^j \delta (\theta k)^{-3})^2}{3(2^j \delta (\theta k)^{-3} + 2/\rho)}\right]
\]

\[
\leq \exp\left[-\frac{2^j \delta^2}{4(\theta k)^6 \rho}\right]
\]

\[
\leq \exp(-\exp(\theta k/20)) \quad [\text{as } \delta = \exp(-ck \theta), j \geq 0.1 \theta k].
\]
Then for all \( x \in V_i \) the random variables \( N_+(x, i, j) - N_-(x, i, j) \) are independent (because we fix the position \( i \) where \( x \) occurs). Hence, \( B(i, j, T) \) is a binomial random variable, and (88) yields
\[
\mathbb{E}[B(i, j, T)] \leq \theta n \exp(-\exp(\theta k/20)).
\]
Consequently, Lemma 9 (the Chernoff bound) gives
\[
P[B(i, j, T) > \theta \delta n/(\theta k)^3] \leq \exp \left[ \frac{-\theta n}{(\theta k)^3} \ln \left( \frac{\delta \theta n/(\theta k)^3}{\exp(1 - \exp(\theta k/20))\theta n} \right) \right] 
\]
provided that \( \rho \geq \rho_0 \) is sufficiently large.

\[ \square \]

**Corollary 52.** With probability \( 1 - \exp(-\delta \theta n) \) the random formula \( \Phi^i_{\text{bin}} \) has the following property.

For all \( T \subset V_i \) of size \( |T| \leq \delta \theta n \) and all \( i, j \) such that \( i \leq j \), \( 0.1\theta k \leq j \leq 10\theta k \) we have \( B(i, j, T) \leq \delta \theta n/(\theta k)^3 \).

\[ (89) \]

**Proof.** Let \( i, j \) be such that \( i \leq j, 0.1\theta k \leq j \leq 10\theta k \). By Lemma 51 and the union bound, the probability that there is a set \( T \) such that \( B(i, j, T) > \delta \theta n/(\theta k)^3 \) is bounded by
\[
n \left( \frac{\theta n}{\delta \theta n} \right) \exp [-\delta \theta n \exp(\theta k/22)] \leq \exp [o(n) + \delta \theta n (1 - \ln(\delta) - \exp(\theta k/22))] 
\]
\[ \leq \exp [-2\delta \theta n] \quad \text{[as } \delta = \exp(-ck\theta)]. \]

Since there are no more than \( (10k\theta)^2 \) ways to choose \( i, j \), the assertion follows.

\[ \square \]

**Corollary 53.** With probability \( 1 - \exp(-10^{-12} \delta \theta n) \) the random formula \( \Phi^i \) has the following property.

If \( T \subset V_i \) has size \( |T| \leq \delta \theta n \), then there are no more than \( 10^{-5} \delta \theta n \) variables \( x \in V_i \) such that
\[
\left| \sum_{b \in N_\leq_1(x, T)} \frac{\text{sign}(x, b)}{2^{1|N(b)|}} \right| > \frac{\delta}{1000}. \]

\[ (90) \]

**Proof.** Given \( T \subset V_i \), let \( V_T \) be the set of all \( x \in V_i \) with the following two properties.

i. For all \( b \in N(x) \) we have \( 0.1\theta k \leq |N(b)| \leq 10\theta k \).

ii. For all \( 1 \leq i \leq j, 0.1\theta k \leq j \leq 10\theta k \) we have \( B(i, j, T) \leq \delta \theta n/(\theta k)^3 \).

Then for all \( x \in V_T \) we have
\[
\left| \sum_{b \in N_\leq_1(x)} \frac{\text{sign}(x, b)}{2^{1|N(b)|}} \right| = \left| \sum_{0.1\theta k \leq j \leq 10\theta k} \sum_{i=1}^j 2^{-j}(N_+(x, i, j, T) - N_-(x, i, j, T)) \right| 
\]
\[ \leq \left| \sum_{0.1\theta k \leq j \leq 10\theta k} \sum_{i=1}^j 2^{-j} |N_+(x, i, j, T) - N_-(x, i, j, T)| \right| 
\]
\[ \leq \sum_{0.1\theta k \leq j \leq 10\theta k} 2^{-j} \cdot 2^j \delta(\theta k)^{-3} \leq 100\delta/(\theta k) < 10^{-3} \delta. \]

Furthermore, by Lemmas 41 and 43 with probability \( \geq 1 - 2\exp(-10^{-11} \delta \theta n) \) the number of variables \( x \) that fail to satisfy i. is less than \( 2 \cdot 10^{-6} \delta \theta n \). In addition, by Corollary 52 and Fact 40 with probability \( \geq 1 - \exp(-\delta \theta n/2) \) the number of variables \( x \) that satisfy ii. in \( \Phi^i \) is bounded by \( 10^{-5} \delta \theta n \). Thus, with probability \( \geq 1 - \exp(-10^{-12} \delta \theta n) \) we have \( V_T \geq 10^{-4} \delta \theta n \) for all \( T \), as claimed.

\[ \square \]
Establishing Q3. We carry the proof out in the model $\Phi_{seq}$. Let $0.01 \leq z \leq 1$ and let $T$ be a set of size $|T| = q\theta n$ with $0.01\delta \leq q \leq 100\delta$.

Lemma 54. Let $S, Z > 0$ be integers and let $\mathcal{E}_z(T, S, Z)$ be the event that $\Phi_{seq}$ contains a set $Z$ of $Z$ clauses with the following properties.

i. $S = \sum_{b \in Z} |N(b)| > 1.009|T|/z$.

ii. For all $b \in Z$ we have $0.1\theta k \leq |N(b)| \leq 100k$.

iii. All $b \in Z$ satisfy $|N(b) \cap T| \geq z|N(b)|$.

Then $P[\mathcal{E}_z(T, S, Z)] \leq q^{0.99999zS}$.

Proof. We claim that in $\Phi_{seq}$,

$$P[\mathcal{E}_z(T, S, Z)] \leq \left(\frac{m}{Z}\right)^{kZ} \left(\frac{S}{zS}\right)^{2^{S-kZ}q^{S}(1-\theta)kZ-S} q^{zS}.$$  

Indeed, $\Phi_{seq}$ is based on the random sequence $\Phi_{seq}$ of $m$ independent clauses. Out of these $m$ clauses we choose a subset $S$ of size $Z$, inducing a $\binom{m}{S}$ factor. Then, out of the $kZ$ literal occurrences of the clauses in $S$ we choose $S$ (leading to the $\left(\frac{kZ}{S}\right)$ factor) of whose underlying variables lie in $V_i$, which occurs with probability $\theta = |V_i|/n$ independently for each literal (inducing a $\theta^S$ factor). Furthermore, all $kZ - S$ literals whose variables are in $V \setminus V_i$ must be negative, because otherwise the corresponding clauses would have been eliminated from $\Phi_{seq}$, and not in $V_i$; this explains the $2^{S-kZ}(1-\theta)kZ-S$ factor. Finally, out of the $S$ literal occurrences in $V_i$ a total of at most $zS$ has an underlying variable from $T$ (a factor of $\left(\frac{S}{zS}\right)$), which occurs with probability $q = |T|/(\theta n)$ independently (hence the $q^{zS}$ factor).

Hence, we obtain

$$P[\mathcal{E}_z(T, S, Z)] \leq \left(\frac{m}{Z}\right)^{2^{-kZ}} \left[\frac{2^{1/z} \cdot q}{z \cdot q}\right]^{zS} \left(\frac{kZ}{S}\right)^{\theta^S(1-\theta)kZ-S} \left(\frac{m}{Z}\right)^{2^{-kZ}(Cq)^zS}$$

for a certain absolute constant $C > 0$, because $z \geq 0.01$. Since all clause lengths are required to be between $0.1\theta k$ and $100\theta k$, we obtain $0.1S/(\theta k) \leq Z \leq 10S/(\theta k)$. Therefore,

$$\left(\frac{m}{Z}\right)^{2^{-kZ}} \leq \left(\frac{cm}{2kZ}\right)^Z \leq \left(\frac{cm}{kZ}\right)^Z \quad [\text{as } m = 2k \rho m/k]$$

$$\leq \left(\frac{100q\theta n}{S}\right)^Z \leq \left(\frac{100q\theta n}{1000q}\right)^Z \quad [\text{as } S \geq 1.009q\theta n/z \geq 1.009q\theta n \text{ by i.}].$$

Since $q \leq 100\delta = 100\exp(-c\theta k)$ and $\theta k \geq \ln(\rho)/c^2$, we have $1/q \geq 100\rho$ for $\rho \geq \rho_0$ sufficiently large. Hence, (92) yields

$$\left(\frac{m}{Z}\right)^{2^{-kZ}} \leq q^{-2Z} \leq q^{-20S/(\theta k)}.$$  

Plugging (93) into (91), we obtain for $\theta k \geq \rho_0$ large enough and $S \geq 1.009|T|/z$

$$P[\mathcal{E}_z(T, S, Z)] \leq q^{-20S/(\theta k)} \cdot (Cq)^zS \leq q^{0.99999zS},$$

as claimed. □

Corollary 55. Let $\mathcal{E}$ be the event that there exist a number $z \in [0.01, 1]$, a set $T \subset V_i$ of size $|T| \leq 100\theta \delta n$ and $S \geq \frac{1}{1001}|T| + 10^{-6}\delta n$, $Z > 0$ such that $\mathcal{E}_z(T, S, Z)$ occurs. Then $\mathcal{E}$ occurs in $\Phi^t$ with probability $\leq \exp(-10^{-6}\delta n)$.
Proof. Let $z \in [0.01, 1]$, let $0 < q \leq 100\delta$, and let $S, Z > 0$ be integers such that $S \geq \frac{100\delta}{q} + 10^{-6}\delta n$. Let $E_z(q, S, Z)$ denote the event that there is a set $T \subset V_i$ of size $|T| = q\theta n$ such that $E_z(T, S, Z)$ occurs. Then by Lemma 55 and the union bound, in $\Phi_{\text{seq}}$ we have

$$P[E(q, S, Z)] \leq \left(\frac{\theta n}{q\theta n}\right)^{0.99999z} \leq \exp\left[q\theta n(1 - \ln q + 1.008\ln q) + 0.9 \cdot 10^{-6}\delta n \ln q\right]$$

$$\leq \exp\left(-0.9 \cdot 10^{-6}\delta n\right) \quad [\text{as } q \leq 100\delta < 1/c] \quad (94)$$

Since there are only $O(n^4)$ possible choices of $S, Z, z$ and $q$, (94) and Fact 40 imply the assertion. \hfill \Box

Corollary 56. With probability at least $1 - \exp(-10^{-12}\delta n)$, $\Phi^i$ has the following property.

Let $0.01 \leq z \leq 1$ and let $T \subset V_i$ have size $0.01\delta n \leq |T| \leq 100\delta n$. Then

$$\sum_{b : |N(b) \cap T| \geq z|N(b)|} |N(b)| \leq \frac{1.01}{2} |T| + 2 \cdot 10^{-5}\delta n.$$ 

Proof. Lemmas 41 and 43 and Corollary 55 imply that with probability at least $1 - 3\exp(-10^{-11}\delta n)$, $\Phi^i$ has the following properties.

i. $E$ does not occur.

ii. $\sum_{b : |N(b) \cap [0.01k, 100k]|} |N(b)| \leq 10^{-5}\delta n$.

Assume that i. and ii. hold and let $T \subset V_i$ be a set of size $|T| \leq 100\delta n$. Let $0.01 \leq z \leq 1$. Let $N_T$ be the set of all clauses $b$ of $\Phi^i$ such that $|N(b) \cap T| \geq z|N(b)|$ and $0.1\theta k \leq |N(b)| \leq 100\theta k$. Then i. implies that

$$\sum_{b \in N_T} |N(b)| \leq \frac{1.009}{z} |T| + 10^{-5}\delta n.$$ 

Furthermore, ii. yields

$$\sum_{b : |N(b) \cap T| \geq z|N(b)|} |N(b)| \leq \sum_{b : |N(b) \cap [0.01k, 100k]|} |N(b)| + \sum_{b \in N_T} |N(b)|$$

$$\leq 1.009 |T|/z + 2 \cdot 10^{-5}\delta n,$$

as desired. \hfill \Box

Establishing Q4. We are going to work with the probability distribution $\Phi_{\text{seq}}$ (sequence of $m$ independent clauses). Let $M$ be the set of all indices $l \in [m]$ such that the $l$th clause $\Phi_{\text{seq}}(l)$ does not contain any of the variables $x_1, \ldots, x_t$ positively. In this case, $\Phi_{\text{seq}}(l)$ is still present in the decimated formula $\Phi_{\text{seq}}^i$ (with all occurrences of $\neg x_1, \ldots, \neg x_t$ eliminated, of course). For each $l \in M$ let $L(l)$ be the number of literals in $\Phi_{\text{seq}}(l)$ whose underlying variable is in $V_i$. We may assume without loss of generality that for any $l \in M$ the $L(l)$ ‘leftmost’ literals $\Phi_{\text{seq}}(l, i), 1 \leq i \leq L(l)$, are the ones with an underlying variable from $V_i$. Let $T \subset V_i$. Analyzing the operator $A_T$ directly is a little awkward. Therefore, we will decompose $A_T$ into a sum of several operators that are easier to investigate. For any $0.1\theta k \leq L \leq 100\theta k, 1 \leq i < j \leq L, l \in M$, and any distinct $x, y \in V_i$ we define

$$m_{xy}(i, j, l, L) = \begin{cases} 1 & \text{if } L(l) = L \land \left(\left[\Phi_{\text{seq}}(l, i) = x \land \Phi_{\text{seq}}(l, j) = y\right) \lor \left(\Phi_{\text{seq}}(l, i) = \neg x \land \Phi_{\text{seq}}(l, j) = \neg y\right)\right), \\
-1 & \text{if } L(l) = L \land \left(\left[\Phi_{\text{seq}}(l, i) = x \land \Phi_{\text{seq}}(l, j) = \neg y\right) \lor \left(\Phi_{\text{seq}}(l, i) = \neg x \land \Phi_{\text{seq}}(l, j) = y\right)\right), \\
0 & \text{otherwise,} \end{cases}$$

while we let $m_{xx}(i, j, l, L) = 0$. Moreover, for $x, y \in V_i$ we let

$$m_{xy}(i, j, l, L) = \sum_{l \in M} m_{xy}(i, j, l, L).$$
For a variable \(x \in V_t\) we let \(\mathcal{N}(x, T)\) be the set of all \(l \in \mathcal{M}\) such that \(0.1\theta k \leq \mathcal{L}(l) \leq 10\theta k\) and the clause \(\Phi_{\text{seq}}(l)\) contains at most one literal whose underlying variable is in \(T \setminus x\). Moreover, for \(l \in \mathcal{M}\) let \(\mathcal{N}(x, l)\) be the set of all variables \(y \in V_t \setminus \{x\}\) that occur in clause \(\Phi_{\text{seq}}(l)\) (either positively or negatively).

We are going to analyze the operators

\[
A_{ij}^{vl} : \mathbb{R}^{|V_t|} \to \mathbb{R}^{|V_t|}, \quad \Gamma = (\Gamma_y)_{y \in V_t} \mapsto \left\{ \sum_{l \in \mathcal{N}(x, T)} \sum_{y \in \mathcal{N}(x, l)} 2^{-L}m_{xy}(i, j, L) \Gamma_y \right\}_{x \in V_t}.
\]

**Lemma 57.** For any \(0.1\theta k \leq L \leq 10\theta k\), \(1 \leq i < j \leq L\) and for any set \(T \subset V_t\) we have

\[
P \left[ \left\|A_{ij}^{vl} \right\|_2 \leq \delta^2 \theta n \right] \geq 1 - \exp(-\theta n).
\]

**Proof.** The proof is based on Fact 10. Fix two sets \(A, B \subset V_t\). For each \(l \in \mathcal{M}\) and any \(x, y \in V_t\) the two 0/1 random variables

\[
\sum \max \{m_{xy}(i, j, l, 0)\}, \quad \sum \max \{-m_{xy}(i, j, l, 0)\}
\]

are identically distributed, because the clause \(\Phi_{\text{seq}}(l)\) is chosen uniformly at random. In effect, the two random variables

\[
\mu(A, B) = \sum_{l \in \mathcal{M}} \sum_{(x, y) \in A \times B} 1_{l \in \mathcal{N}(x, T)} \max \{m_{xy}(i, j, l, 0)\},
\]

\[
\nu(A, B) = \sum_{l \in \mathcal{M}} \sum_{(x, y) \in A \times B} 1_{l \in \mathcal{N}(x, T)} \max \{-m_{xy}(i, j, l, 0)\}
\]

are identically distributed. Furthermore, both \(\mu(A, B)\) and \(\nu(A, B)\) are sums of independent Bernoulli variables, because the clauses \((\Phi_{\text{seq}}(l))_{l \in [m]}\) are mutually independent.

We need to estimate the mean \(E(\mu(A, B)) = E(\nu(A, B))\). As each of the clauses \(\Phi_{\text{seq}}(l)\) is chosen uniformly, for each \(l \in [m]\) we have

\[
P \left[ l \in \mathcal{M} \land \mathcal{L}(l) = L \right] = \binom{k}{L} \theta^L (1 - \theta)^{k - L} 2^{L - k}.
\]

Therefore,

\[
E(\mu(A, B) + \nu(A, B)) = m \binom{k}{L} \theta^L (1 - \theta)^{k - L} 2^{L - k} = \frac{2^L \rho \theta n}{L} \binom{k - 1}{L - 1} \theta^{L - 1} (1 - \theta)^{k - L} \quad \text{[as } m = 2^k \rho/k]\]

\[
\leq \frac{2^L \rho \theta n}{L}.
\]

Hence, Lemma 2 (the Chernoff bound) yields

\[
P \left[ \left| \mu(A, B) - E(\mu(A, B)) \right| > 10 \sqrt{2^L \rho / L \cdot \theta n} \right] \
= P \left[ \left| \nu(A, B) - E(\nu(A, B)) \right| > 10 \sqrt{2^L \rho / L \cdot \theta n} \right]
\leq 16^{-\theta n}.
\]

Hence, by the union bound

\[
P \left[ \exists A, B \subset V_t : \max \left| \mu(A, B) - E(\mu(A, B)) \right|, \left| \nu(A, B) - E(\nu(A, B)) \right| > 10 \sqrt{2^L \rho / L \cdot \theta n} \right] \leq 2 \cdot 4^{\theta n} \cdot 16^{-\theta n} \leq \exp(-\theta n).
\]
Thus, with probability $\geq 1 - \exp(-\theta n)$ we have
\[
\left\langle A_{ij}^{LL} \mathbf{1}_B, \mathbf{1}_A \right\rangle = 2^{-L} (\mu(A, B) - \nu(A, B)) \\
\leq 2^{-L} (|\mu(A, B) - E[\mu(A, B)]| + |\nu(A, B) - E[\nu(A, B)]|) \\
\leq \theta n \cdot 20 \sqrt{\frac{\theta}{L^2L}} \leq 0.01\delta^5\theta n \quad \text{[as $L \geq 0.1k\theta$, $\theta k \geq \ln(\rho)/c^2$, and $\delta = \exp(-ck\theta)$].}
\]

Finally, the assertion follows from Fact 10. \hfill $\Box$

**Corollary 58.** With probability at least $1 - \exp(-0.1\theta n)$ the random formula $\Phi_t^{\text{seq}}$ has the following property.

Let $T \subset V_i$ and let

\[
A_T' = \sum_{0.1\theta k \leq L \leq 100k} \sum_{1 \leq i < j} A_{ij}^{LL}.
\]

Then $\|A_T'\| \leq \delta^{4.9}\theta n$.

**Proof.** By Lemma 57 and the union bound, we have

\[
P\left[\exists T, i, j, L : \left\| A_{ij}^{LL} \right\| > \delta^5\theta n\right] \leq (10\theta k)^3 2^{\theta n} \cdot \exp(-\theta n) \leq \exp(-0.1\theta n).
\]

Furthermore, if $\left\| A_{ij}^{LL} \right\| \leq \delta^5\theta n$ for all $i, j, L$, then by the triangle inequality

\[
\|A_T\| \leq (10\theta k)^3 \delta^5\theta n \leq \delta^{4.9}\theta n \quad \text{[as $\delta = \exp(-ck\theta)$]},
\]

as claimed. \hfill $\Box$

To complete the proof of Q4, we observe that for $(x, y) \in V_i \times V_i$ the $(x, y)$ entries of the matrices $A_T$ and $A_T'$ differ only if either $x$ or $y$ occurs in a redundant clause. Consequently, Q0 ensures that $\|A_T' - A_T\| = o(n)$. Therefore, Fact 50 and Corollary 58 imply $\Phi_t$ satisfies Q4 with probability at least $1 - \exp(-11\Lambda c)$.

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