On obtaining the convex hull of quadratic inequalities via aggregations

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Context
Quadratically Constrained Quadratic Program

**QCQP**

Quadratic objective, quadratic constraints:

\[
\begin{align*}
\min & \quad x^T Q_0 x + b_0^T x \\
\text{s.t.} & \quad x^T Q_i x + b_i^T x \leq d_i \quad \forall i \in [m]
\end{align*}
\]
Quadratically Constrained Quadratic Program

**QCQP**
May be equivalently written as:

\[
\begin{align*}
& \text{min } c^T x \\
& \text{s.t. } x^T Q_i x + b_i^T x \leq d_i \; \forall i \in [m]
\end{align*}
\]
QCQP
May be equivalently written as:

\[
\begin{align*}
\text{min} & \quad c^\top x \\
\text{s.t.} & \quad x^\top Q_i x + b_i^\top x \leq d_i \quad \forall i \in [m]
\end{align*}
\]

• Thus, we care about

\[
\text{conv} \{ x \mid x^\top Q_i x + b_i^\top x \leq d_i \quad \forall i \in [m] \}
\]
Quadratically Constrained Quadratic Program

**QCQP**
May be equivalently written as:

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\begin{align*}
\min & \quad c^\top x \\
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\end{align*}
\]

- Thus, we care about
  \[
  \text{conv} \left\{ x \mid x^\top Q_i x + b_i^\top x \leq d_i \quad \forall i \in [m] \right\}
  \]
- Challenging to compute! So we can consider “partial” convexifications
Two-row relaxations

- Single rows are not really useful to convexify.
Two-row relaxations

- Single rows are **not really useful to convexify**.
- We can select **two rows** and try to find the convex hull of their intersection:

\[ C_2 = \{ x \in \mathbb{R}^n \mid x^T Q_i x + b_i^T x \leq d_i \quad \forall i \in [2] \} \]
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\[ C_2 = \{ x \in \mathbb{R}^n \mid x^\top Q_i x + b_i^\top x \leq d_i \ \forall i \in [2] \} \]

- For some technical reasons, we consider the “open version” of the above set:

\[ O_2 = \{ x \in \mathbb{R}^n \mid x^\top Q_i x + b_i^\top x < d_i \ \forall i \in [2] \} \]
Two-row relaxations

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• For some technical reasons, we consider the “open version” of the above set:

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\mathcal{O}_2 = \{ x \in \mathbb{R}^n \mid x^\top Q_i x + b_i^\top x < d_i \ \forall i \in [2] \}
\]

• It turns out the convex hull of \( \mathcal{O}_2 \) is well understood!
Let’s first talk about aggregations

Given \( \lambda \in \mathbb{R}^m \) and

\[
S := \{ x \mid x^T Q_i x + b_i^T x < d_i \ \forall i \in [m] \},
\]

is a relaxation of \( S \). We are multiplying \( i \)-th constraint by \( \lambda_i \) and then adding them together.
Let’s first talk about aggregations

Given $\lambda \in \mathbb{R}_+^m$ and

$$S := \{ x \mid x^\top Q_i x + b_i^\top x < d_i \; \forall i \in [m] \},$$

$$S^\lambda := \left\{ x \mid x^\top \left( \sum_{i=1}^{m} \lambda_i Q_i \right) x + \left( \sum_{i=1}^{m} \lambda_i b_i \right)_x < \left( \sum_{i=1}^{m} \lambda_i d_i \right) \; \forall i \in [m] \right\}$$

is a relaxation of $S$.

We are multiplying $i^{th}$ constraint by $\lambda_i$ and then adding them together.
Convex hull of $O_2$

$$O_2 = \left\{ x \in \mathbb{R}^n \middle| x^T Q_i x + b_i^T x < d_i \, \forall i \in [2] \right\}$$
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$$O_2 = \left\{ x \in \mathbb{R}^n \mid x^T Q_i x + b_i^T x < d_i \; \forall i \in [2] \right\}$$

**Theorem (Yildiran (2009))**

*Given a set $O_2$, such that $\text{conv} \left( O_2 \right) \neq \mathbb{R}^n$, there exists $\lambda_1, \lambda_2 \in \mathbb{R}_+^2$ such that:*

$$\text{conv} \left( O_2 \right) = (O_2)_{\lambda_1} \cap (O_2)_{\lambda_2}.$$
Convex hull of $\mathcal{O}_2$

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\mathcal{O}_2 = \left\{ x \in \mathbb{R}^n \mid x^T Q_i x + b_i^T x < d_i \, \forall i \in [2] \right\}
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**Theorem (Yildiran (2009))**

Given a set $\mathcal{O}_2$, such that $\text{conv}(\mathcal{O}_2) \neq \mathbb{R}^n$, there exists $\lambda_1, \lambda_2 \in \mathbb{R}_+^2$ such that:

\[
\text{conv}(\mathcal{O}_2) = (\mathcal{O}_2)^{\lambda_1} \cap (\mathcal{O}_2)^{\lambda_2}.
\]

- Yildiran (2009) also gives an algorithm to compute $\lambda_1$ and $\lambda_2$.
- The quadratic constraints in $(\mathcal{O}_2)^{\lambda_i} \, i \in \{1, 2\}$ have very nice properties:
  - $\sum_{j=1}^{2} \lambda_j^i Q_j$ has at most one negative eigenvalue for both $i \in \{1, 2\}$.
Convex hull of $\mathcal{O}_2$

$$\mathcal{O}_2 = \left\{ x \in \mathbb{R}^n \left| x^T Q_i x + b_i^T x < d_i \quad \forall i \in [2] \right. \right\}$$

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*Given a set $\mathcal{O}_2$, such that $\text{conv}(\mathcal{O}_2) \neq \mathbb{R}^n$, there exists $\lambda^1, \lambda^2 \in \mathbb{R}^2_+$ such that:*

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  - Basically, the sets $(\mathcal{O}_2)^{\lambda^i}$ $i \in \{1, 2\}$ are either ellipsoids or hyperboloids (union of two convex sets).
Convex hull of $O_2$

$$O_2 = \left\{ x \in \mathbb{R}^n \mid x^\top Q_i x + b_i^\top x < d_i \; \forall i \in [2] \right\}$$

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- The quadratic constraints in $\left( O_2 \right)^{\lambda_i} \; i \in \{1, 2\}$ have very nice properties:
  - $\sum_{j=1}^{2} \lambda_j^i Q_j$ has at most one negative eigenvalue for both $i \in \{1, 2\}$
  - Basically, the sets $\left( O_2 \right)^{\lambda_i} \; i \in \{1, 2\}$ are either *ellipsoids* or *hyperboloids* (union of two convex sets).
  - Henceforth, we call a quadratic constraint with the “quadratic part” having at most one negative eigenvalue a *good constraint.*
Example

\[ S := \left\{ x, y \ \bigg| \begin{array}{c} -xy < -1 \\ x^2 + y^2 < 9 \end{array} \right\} \]
Example

conv(S) := \{ x, y \mid (x - y)^2 < 7, x^2 + y^2 < 9 \}
Example

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With the blue quadratic coming from \(\lambda^1 = (2, 1)\)

\[
\begin{align*}
-xy &< -1 \cdot 2 \\
+ x^2 + y^2 &< 9 \cdot 1
\end{align*}
\]
Example

\[ S := \left\{ x, y \mid \begin{array}{c} -xy < -1 \\ x^2 + y^2 < 9 \end{array} \right\} \]

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With the blue quadratic coming from \( \lambda^1 = (2, 1) \)

\[ -xy < -1 \cdot 2 + x^2 + y^2 < 9 \cdot 1 \]

\[ x^2 - 2xy + y^2 < 7 \equiv (x - y)^2 < 7 \]
Literature survey

Related results:

- [Yildiran (2009)]
- [Burer and Kılınc-Karzan (2017)] (second order cone intersected with a nonconvex quadratic)
- [Modaresi and Vielma (2017)] (closed version of results)
Related results:

- [Yildiran (2009)]
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Other related papers:

- [Tawarmalani, Richard, Chung (2010)] (covering bilinear knapsack)
- [Santana and Dey (2020)] (polytope and one quadratic constraint)
- [Ye and Zhang (2003)], [Burer and Anstreicher (2013)], [Bienstock (2014)]
  [Burer (2015)], [Burer and Yang (2015)], [Anstreicher (2017)] (extended trust-region problem)
- [Burer and Ye (2019)], [Wang and Kılınç-Karzan (2020, 2021)], [Argue, Kılınç-Karzan, and Wang (2020)] (general conditions for the SDP relaxation being tight)
- [Bienstock, Chen, and Muñoz (2020)], [Muñoz and Serrano (2020)] (cuts for QCQP using intersection cuts approach)
- ...
The question we consider...

We want to understand the power of aggregations for $m \geq 3$
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Main contribution
Under some technical conditions, intersection of aggregations yield the convex hull for three quadratic constraints.
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The above result represents the limit of aggregations.
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Under some technical conditions, intersection of aggregations yield the convex hull for three quadratic constraints.

**Additional contribution**
The above result represents the limit of aggregations. Basically, aggregations \( \not\to \) convex hull if the technical sufficient condition does not hold for \( m = 3 \) or when \( m \geq 4 \).
Main results
Theorem

Let $n \geq 3$ and

$$
O_3 = \left\{ x \in \mathbb{R}^n \mid [x \quad 1] \begin{bmatrix} A_i & b_i & c_i \\ b_i^T & c_i & 1 \end{bmatrix} [x] < 0, \ i \in [3] \right\}.
$$
Three rows: main result

Theorem

Let \( n \geq 3 \) and

\[
O_3 = \left\{ x \in \mathbb{R}^n \mid [x \ 1] \begin{bmatrix} A_i & b_i \\ b_i^\top & c_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < 0, \ i \in [3] \right\}.
\]

Assume

- (PDLC) There exists \( \theta \in \mathbb{R}^3 \) such that \( \sum_{i=1}^{3} \theta_i \begin{bmatrix} A_i & b_i \\ b_i^\top & c_i \end{bmatrix} \succ 0. \)

- (Non-trivial convex hull) \( \text{conv}(O_3) \neq \mathbb{R}^n. \)
Theorem

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- *(Non-trivial convex hull)* $\text{conv}(\mathcal{O}_3) \neq \mathbb{R}^n$.

Let $\Omega := \left\{ \lambda \in \mathbb{R}_+^3 \mid \mathcal{O}_3^\lambda \supset \text{conv}(\mathcal{O}_3) \text{ and } (\mathcal{O}_3)^\lambda \text{ is good} \right\}$,
Theorem

Let $n \geq 3$ and

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Let $\Omega := \left\{ \lambda \in \mathbb{R}_+^3 \mid (\mathcal{O}_3)^\lambda \supseteq \text{conv}(\mathcal{O}_3) \text{ and } (\mathcal{O}_3)^\lambda \text{ is good} \right\}$, then

$$\text{conv}(\mathcal{O}_3) = \bigcap_{\lambda \in \Omega} (\mathcal{O}_3)^\lambda.$$
Example

\[ S := \left\{ (x, y, z) \mid \begin{array}{c} x^2 + y^2 < 2 \\ -x^2 - y^2 < -1 \\ -x^2 + y^2 + z^2 + 6x < 0 \end{array} \right\} \]
Example

\[ S := \left\{ (x, y, z) \mid \begin{array}{l} x^2 + y^2 < 2 \\ -x^2 - y^2 < -1 \\ -x^2 + y^2 + z^2 + 6x < 0 \end{array} \right\} \]

\[ \text{conv}(S) := \left\{ (x, y, z) \mid \begin{array}{l} x^2 + y^2 < 2 \\ -2x^2 + z^2 + 6x < -1 \\ -x^2 + y^2 + z^2 + 6x < 0 \end{array} \right\} \]
## Comparsion of results

| Two quadratic constraints | Three quadratic constraints |
|---------------------------|-----------------------------|
| Yildiran (2009)           | This talk                   |
## Comparsion of results

|                      | Two quadratic constraints | Three quadratic constraints |
|----------------------|---------------------------|-----------------------------|
| Yildiran (2009)      | conv($S$) $\neq \mathbb{R}^n$ | PDLC condition, conv($S$) $\neq \mathbb{R}^n$ |
| This talk            |                           |                             |
| When does it hold?   | conv($S$) $\neq \mathbb{R}^n$ |                             |
## Comparison of results

|                        | Two quadratic constraints                                                                 | Three quadratic constraints                                                                 |
|------------------------|------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------|
| When does it hold?     | $\text{conv}(S) \neq \mathbb{R}^n$                                                      | PDLC condition, $\text{conv}(S) \neq \mathbb{R}^n$                                          |
| How many aggregated inequalities needed? | 2                                                                                       | $\infty$ (Conjecture!)                                                                    |

- **Yildiran (2009)**
- **This talk**

*Note: PDLC stands for Pseudo-Dual Linear Condition.*
### Comparison of results

|                  | Two quadratic constraints | Three quadratic constraints |
|------------------|---------------------------|-----------------------------|
| Author           | Yildiran (2009)           | This talk                   |
| When does it hold? | $\text{conv}(S) \neq \mathbb{R}^n$ | $\text{PDLC condition, conv}(S) \neq \mathbb{R}^n$ |
| How many aggregated inequalities needed? | 2                           | $\infty$ (Conjecture!)      |
| Structure of aggregated inequalities | Polynomial-time algorithm exists to find them | Even checking if $\lambda \in \Omega$ is not clear. |
Theorem

Let $n \geq 3$ and let

$$C_3 = \left\{ x \in \mathbb{R}^n \ \mid \begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} A_i & b_i \nosep 1 
\end{bmatrix} \begin{bmatrix} x 
\n1 \end{bmatrix} \leq 0, \ i \in [3] \right\}.$$
The closed case

**Theorem**

Let $n \geq 3$ and let

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Assume

- *(PDLC)* There exists $\theta \in \mathbb{R}^3$ such that $\sum_{i=1}^{3} \theta_i \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \succ 0.$

- *(Non-trivial convex hull)* $\text{conv}(C_3) \neq \mathbb{R}^n.$
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Let $n \geq 3$ and let

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Assume

- **(PDLC)** There exists $\theta \in \mathbb{R}^3$ such that $\sum_{i=1}^{3} \theta_i \begin{bmatrix} A_i \\ b_i^T \\ c_i \end{bmatrix} \succ 0$.

- **(Non-trivial convex hull)** $\text{conv}(C_3) \neq \mathbb{R}^n$.

- **(No low-dimensional components)** $C_3 \subseteq \text{int}(C_3)$.

Let $\Omega := \left\{ \lambda \in \mathbb{R}^3_+ \mid (C_3)^\lambda \supseteq \text{conv}(C_3) \text{ and } (C_3)^\lambda \text{ is good} \right\}$,
The closed case

**Theorem**

Let $n \geq 3$ and let

$$C_3 = \left\{ x \in \mathbb{R}^n \left| \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0, \ i \in [3] \right\}.$$

Assume

- (PDLC) There exists $\theta \in \mathbb{R}^3$ such that $\sum_{i=1}^{3} \theta_i \begin{bmatrix} A_i \\ b_i^T \\ c_i \end{bmatrix} \succ 0$.

- (Non-trivial convex hull) $\text{conv}(C_3) \neq \mathbb{R}^n$.

- (No low-dimensional components) $C_3 \subseteq \text{int}(C_3)$.

Let $\Omega := \left\{ \lambda \in \mathbb{R}_+^3 \mid (C_3)^\lambda \supseteq \text{conv}(C_3) \text{ and } (C_3)^\lambda \text{ is good} \right\}$, then

$$\overline{\text{conv}(C_3)} = \bigcap_{\lambda \in \Omega} (C_3)^\lambda.$$
Counterexamples
$m = 3$ but not satisfying PDLC condition

\[ S := \left\{ (x, y, z) \mid \begin{array}{c} x^2 < 1 \\ y^2 < 1 \\ -xy + z^2 < 0 \end{array} \right\} \]

- PDLC condition does not hold, 
  \( \text{conv}(S) \neq \mathbb{R}^3 \)
$m = 3$ but not satisfying PDLC condition

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  \[ \text{conv}(S) \neq \mathbb{R}^3 \]

\[ \text{conv}(S) \neq \bigcap_{\lambda \in \Omega} S^\lambda \]
$m = 4$ and satisfying PDLC

$S := \left\{ (x, y, z) \right\} \middle| \begin{array}{l}
x^2 + y^2 + z^2 + 2.2(xy + yz + xz) < 1 \\
-2.1x^2 + y^2 + z^2 < 0 \\
x^2 - 2.1y^2 + z^2 < 0 \\
x^2 + y^2 - 2.1z^2 < 0
\end{array}$

- PDLC condition holds, $\text{conv}(S) \neq \mathbb{R}^3$
\( m = 4 \) and satisfying PDLC

\[
S := \left\{ (x, y, z) \mid \begin{array}{l}
  x^2 + y^2 + z^2 + 2.2(xy + yz + xz) < 1 \\
  -2.1x^2 + y^2 + z^2 < 0 \\
  x^2 - 2.1y^2 + z^2 < 0 \\
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\end{array} \right\}
\]

- PDLC condition holds, \( \text{conv}(S) \neq \mathbb{R}^3 \)

\[
\text{conv}(S) \neq \bigcap_{\lambda \in \Omega} S^\lambda
\]
Do we need a finite number of aggregations?

A non-counterexample:

\[ S := \{ x, y \mid x^2 \leq 1, \ y^2 \leq 1, \ (x - 1)^2 + (y - 1)^2 \geq 1 \}, \]

\[ \Omega^+ := \{ \lambda \in \mathbb{R}^3_+ \mid S^\lambda \supseteq \text{conv}(S) \} \]
Do we need a finite number of aggregations?

A non-counterexample:

\[ S := \{ x, y \mid x^2 \leq 1, \ y^2 \leq 1, \ (x - 1)^2 + (y - 1)^2 \geq 1 \} , \]

- Let \( \Omega^+ := \{ \lambda \in \mathbb{R}^3_+ \mid S^\lambda \supseteq \text{conv}(S) \} \)
- \( \text{conv}(S) = \bigcap_{\lambda \in \Omega^+} S^\lambda \).
- \( \text{conv}(S) \not\subseteq \bigcap_{\lambda \in \tilde{\Omega}^+} S^\lambda \) for any \( \tilde{\Omega}^+ \subseteq \Omega^+ \) which is finite.
A non-counterexample:

\[ S := \{ x, y \mid x^2 \leq 1, \ y^2 \leq 1, \ (x - 1)^2 + (y - 1)^2 \geq 1 \} , \]

- Let \( \Omega^+ := \{ \lambda \in \mathbb{R}_+^3 \mid S^\lambda \supseteq \text{conv}(S) \} \)
- \( \text{conv}(S) = \bigcap_{\lambda \in \Omega^+} S^\lambda . \)
- \( \text{conv}(S) \subsetneq \bigcap_{\lambda \in \check{\Omega}^+} S^\lambda \) for any \( \check{\Omega}^+ \subseteq \Omega^+ \) which is finite.

But PDLC does not hold!
Main proof outline
A new S-Lemma for 3 quadratic constraints

Lemma

Let \( n \geq 3 \) and let \( g_1, g_2, g_3 : \mathbb{R}^n \to \mathbb{R} \) be homogeneous quadratic functions:

\[
g_i(x) = x^\top Q_i x.
\]

Assuming there is a linear combination of \( Q_1, Q_2, Q_3 \) that is positive definite, the following equivalence holds

\[
\{ x \in \mathbb{R}^n : g_i(x) < 0, \ i \in [3] \} = \emptyset \iff \exists \lambda \in \mathbb{R}_+^3 \setminus \{0\}, \ \sum_{i=1}^{3} \lambda_i Q_i \succeq 0.
\]
$\text{conv}(S) = \bigcap_{\lambda \in \Omega} S^\lambda$ proof idea

$\text{conv}(S) \subseteq \bigcap_{\lambda \in \Omega} S^\lambda$ is straight-forward
conv(S) = \bigcap_{\lambda \in \Omega} S^\lambda \quad \text{proof idea}

conv(S) \subseteq \bigcap_{\lambda \in \Omega} S^\lambda \quad \text{is straight-forward}

conv(S) \supseteq \bigcap_{\lambda \in \Omega} S^\lambda:
\( \text{conv}(S) = \bigcap_{\lambda \in \Omega} S^\lambda \) proof idea

\( \text{conv}(S) \subseteq \bigcap_{\lambda \in \Omega} S^\lambda \) is straight-forward

\( \text{conv}(S) \supseteq \bigcap_{\lambda \in \Omega} S^\lambda \):

- Pick \( x^* \in \mathbb{R}^n \) such that \( x^* \notin \text{conv}(S) \). We want to show that is lies outside some aggregation
Conv(S) = \bigcap_{\lambda \in \Omega} S^\lambda \text{ proof idea}

Conv(S) \subseteq \bigcap_{\lambda \in \Omega} S^\lambda \text{ is straight-forward}

Conv(S) \supseteq \bigcap_{\lambda \in \Omega} S^\lambda:

- Pick \( x^* \in \mathbb{R}^n \) such that \( x^* \notin \text{conv}(S) \). We want to show that is lies outside some aggregation
- Separation theorem \( \Rightarrow \) there exists \( \alpha^T x < \beta \) valid for conv(S) that separates \( x^* \).
$\text{conv}(S) = \bigcap_{\lambda \in \Omega} S^\lambda$ proof idea

- **(Homogenization)** The above can be shown to imply: $\{x | \alpha^T x = \beta x_{n+1}\}$ (call it $H$) does not intersect homogenization of $S$:

$$H \cap \left\{ (x, x_{n+1}) | \begin{bmatrix} \alpha^T & x_{n+1} \end{bmatrix} \begin{bmatrix} A_i & b_i \\ b_i^T & c_i \end{bmatrix} \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} < 0, \ i \in [3] \right\} = \emptyset.$$
\[ \text{conv}(S) = \bigcap_{\lambda \in \Omega} S^\lambda \]

**Proof Idea**

- **Applying S-lemma** we obtain \( \lambda \in \Omega \) such that

\[
H \cap \left\{ (x, x_{n+1}) \mid [x \ x_{n+1}] \left( \sum_{i=1}^{3} \lambda_i \begin{bmatrix} A_i & b_i & c_i \end{bmatrix} \right) \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} < 0, \right\} = \emptyset.
\]
conv\( (S) = \bigcap_{\lambda \in \Omega} S^\lambda \) proof idea

- **Dehomogenizing**, we obtain \( S^\lambda \supseteq \text{conv}(S) \) that excludes \( x^* \)
• We have shown that, under technical assumptions, aggregations are enough to describe the convex hull of 3 quadratics
• We have also shown that the result is not true if some conditions are relaxed.
Summary and open questions

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