FERMIONS ON LATTICE AND CHIRAL INVARIANCE

H. Banerjee* and Asit K. De

a S.N. Bose National Centre for Basic Sciences
Block-JD, Sector-III, Salt Lake, Calcutta-700091, India
e-mail: banerjee@bose.ernet.in

b Saha Institute of Nuclear Physics
AF-Block, Sector I, Salt Lake Calcutta - 700064, India
e-mail: de@tnp.saha.ernet.in

Abstract

A model for lattice fermion is proposed which is, (i) free from doublers, (ii) hermitian, and (iii) chirally invariant. The price paid is the loss of hypercubic and reflection symmetries in the lattice action. Thanks to the $\epsilon$-prescription, correlation functions are free from the ill effects due to the loss of these symmetries. In weak coupling approximation, the U(1) vector current of a gauge theory of lattice fermion in this model is conserved in the continuum limit. As for the U(1) axial vector current, one obtains the ABJ anomaly if the continuum limit is implemented before the chiral limit $m = 0$. The anomaly disappears, as in the Wilson model, if the order of the two limits is reversed.

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1 Introduction

Lattice formulation provides a framework for quantum field theory where one can, in principle, address questions which are non-perturbative. Interacting gauge fields are easily incorporated through the link variables and the lattice spacing ‘$a$’ provides built-in gauge-invariant regularisation. In practical applications, however, a serious impediment is the problem of fermion doubling\(^1\). The ‘naive’ lattice action leads\(^1\) to fifteen ‘doublers’ all with the same mass as the physical fermion and survive in the spectrum in the continuum limit.

A ‘no-go’ theorem\(^2\) states that the unwanted doublers can be evaded, but only at the price of some basic, desirable properties\(^3\) of lattice fermion action. A variety of models have appeared in literature which get rid of the doublers, either partially or completely, by abandoning one or more of these basic properties. The most popular among these, the Wilson model, abandons chiral symmetry\(^1,4\). The Wilson term in the lattice action of this model vanishes formally in the continuum limit, i.e. is an irrelevant term, and breaks chiral symmetry explicitly. It does the job of removing all the doublers by giving each of them a mass of $0(1/a)$, i.e., of the order of the cut-off. For a vector-like gauge theory, e.g., QCD, the Wilson model is adequate. In weak coupling perturbation theory (wcpt) it reproduces in the continuum limit the correct Ward identities, and, most importantly, the ABJ anomaly in the $U(1)$ axial current\(^1,5\). The explicit breaking of chiral symmetry, however, renders the Wilson model unsuitable for chiral gauge theories like the Standard Model. The difficulty with chiral gauge theory is not specific for the Wilson model, but an outstanding problem of lattice formulation and quantum field theory in general.

It is clear that in order to be compatible with chiral gauge theories the lattice action for fermion should have exact chiral symmetry. Even the irrelevant terms, like the Wilson term, needed to remove the doublers from the spectrum, must leave chiral symmetry unscathed. In view of the no-go theorem, it is also clear that the chirally symmetric irrelevant term must pay some price. What, however, is not clear is whether the model will still be physically acceptable. A decisive criterion for this would be the Ward identities in wcpt. We report here the results of a search for a chirally symmetric lattice fermion action which satisfies this criterion.

In sect.2 we introduce various options for finite differences on lattice, all of which are candidates for derivative in continuum. The specific choice of finite difference for our lattice action (sect.3) is motivated by the three principles, (i) absence of doublers, (ii) strict chiral symmetry, and (iii) hermiticity. The price paid\(^3\) is the loss of hypercubic and reflection symmetries\(^6,7\). We, however, give a prescription to remedy the lack of these
symmetries in correlation functions. In sect. 4 we derive the Ward identities of the U(1)
currents in \textit{wcpt} and show that the vector current is conserved while the axial current
has the ABJ anomaly in the continuum limit. Important steps in the derivation of these
identities are given in the Appendix. We conclude with some remarks in sect. 5.

2 Finite Differences and Fermion Doublers

There are several options for finite differences on lattice, all of which coincide with derivative in the continuum limit:

(a) forward difference
\[
\delta^f_\mu \equiv \frac{e^{-ip_\mu a} - 1}{a} \equiv -iP_\mu
\]  
(2.1)

(b) backward difference
\[
\delta^b_\mu \equiv \frac{1 - e^{ip_\mu a}}{a} \equiv -iP^+_\mu
\]  
(2.2)

(c) symmetric difference
\[
\delta^s_\mu = \frac{e^{-ip_\mu a} - e^{ip_\mu a}}{2a} \equiv -i\frac{P_\mu + P^+_\mu}{2}
\]  
(2.3)

where \(p_\mu\)'s are generators of translation. The forward and the backward differences may
be expressed in terms of the symmetric difference
\[
\delta^f_\mu = \delta^s_\mu + \Delta_\mu^a, \quad \delta^b_\mu = \delta^s_\mu - \Delta_\mu^a
\]  
(2.4)

where the antisymmetric combination \(\Delta_\mu^a\).
\[
\Delta_\mu^a \equiv \frac{e^{-ip_\mu a} + e^{ip_\mu a} - 2}{2a}
\]  
(2.5)

vanishes in the continuum limit.

The ‘naive’ lattice action is the result of substituting the symmetric difference \(\delta^s_\mu\) for
the derivative \(\partial_\mu\) in the continuum euclidean action

\[
S^{naive} = -\sum_\mu \bar{\psi} \gamma_\mu \left( \frac{e^{-ip_\mu a} - e^{ip_\mu a}}{2a} \right) \psi
\]  
(2.6)

\[
= -\sum_{x,\mu} \frac{1}{2a} \left[ \bar{\psi}(x + \mu) \gamma_\mu \psi(x) - \bar{\psi}(x) \gamma_\mu \psi(x + \mu) \right]
\]
where \( \mu = a e_\mu \) with \( e_\mu \) a unit vector in the \( \mu \)-direction. Within the Brillouin zone \(-\pi/a \leq k_\mu \leq \pi/a \) the inverse propagator

\[
G_F^{\text{naive}}(k)^{-1} = i \gamma_\mu \frac{\sin(k_\mu a)}{a}
\]

(2.7)
derived from the naive action vanishes at \( k_\mu = 0 \) and \( k_\mu = \pi/a \). The entire Brillouin zone, therefore, splits up into 16 distinct sectors corresponding to the domains

\[
0 \leq |k_\mu| \leq \frac{\pi}{2a}, \quad \frac{\pi}{2a} \leq |k_\mu| \leq \frac{\pi}{a}
\]

(2.8)
of each component of the four momenta, in each of which the fermion propagator assumes the standard continuum form

\[
G_F^{\text{naive}}(k)^{-1} = i \gamma^A k_\mu
\]

with \( \gamma^A (A = 1, 2, ..., 16) \) unitarily equivalent to \( \gamma_\mu \). The physical fermion resides in the first hypercube \( 0 \leq |k_\mu| \leq \pi/2a \), all \( \mu \), and corresponds to \( \gamma^A = \gamma_\mu \), while the 15 doublers live in the remaining sectors\(^{1,4}\).

Appearance of the doublers is not peculiar to the naive action, but, according to the no-go theorem\(^2\), is generic. A simple version of the theorem states\(^3\) that the 15 doublers are unavoidable if the lattice inverse propagator has (i) continuity, (ii) hypercubic symmetry, (iii) reflection symmetry i.e., \( \gamma_4 G_F^{-1}(k) \gamma_4 = G_F^{-1}(k) \gamma_4 \), and (iv) chiral symmetry \( \gamma_5 G_F^{-1}(k) \gamma_5 = -G_F^{-1}(k) \). Models have been proposed which eliminate the doublers, either completely or partially, at the expense of one or more of the above properties.

In Wilson model chiral symmetry is abandoned to achieve complete elimination of the doublers in the continuum spectrum

\[
S^W = - \sum_\mu \langle \bar{\psi}(x) \gamma_\mu \left( \frac{e^{-i p_\mu a} - e^{i p_\mu a}}{2a} \right) \psi(x) \rangle + r \left( \frac{e^{-i p_\mu a} + e^{i p_\mu a} - 2}{2a} \right) \psi(x) \rangle
\]

(2.9)

The Wilson term makes a mass-like contribution of \( 0 (r/a) \) in the inverse propagator

\[
G_F^W(k)^{-1} = \sum_\mu \left[ i \gamma_\mu \frac{\sin(k_\mu a)}{a} + \frac{r}{a} (1 - \cos(k_\mu a)) \right]
\]

(2.10)
in the neighbourhood of \( k_\mu \approx \pi/a \) where the doublers live. This lifts the degeneracy between the physical fermion and the doublers, and removes the latter from the spectrum in the limit \( a = 0^{1,4} \).

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3 Chirally Invariant Model for Lattice Fermion

The species doublers in the naive lattice action (2.6) owe their origin to the substitution of the symmetric difference for continuum derivative. A recipe for cure, which follows naturally\textsuperscript{7}, is to use instead forward (2.1) or backward (2.2) difference. A serious fallout of this recipe is the loss of hermiticity of the action. The integrand in the partition function ceases to have the interpretation of a probability measure, and Feynman rules would yield terms which do not have correct reality properties in the continuum limit\textsuperscript{4}.

Intimately related to the hermiticity of action is the chiral structure and anomaly of a gauge theory. Consider the continuum Dirac operator of a non-abelian chiral gauge theory

\[ i\mathcal{D}_\pm(A) \equiv -\left( i\mathcal{D} - ig \frac{1 \pm \gamma_5}{2} A \right) \]  \hspace{1cm} (3.1)

with \( A_\mu = A^\dagger_\mu \). The effective actions \( \Gamma_+(A) \) and \( \Gamma_-(A) \) defined by

\[ \Gamma_\pm(A) = -\ln \det (i\mathcal{D}_\pm(A)) \]  \hspace{1cm} (3.2)

are related to each other through hermitian conjugation

\[ \Gamma_+(A) = \Gamma_-(A)^\dagger \]  \hspace{1cm} (3.3)

The relation (3.3) is of essence for the chiral properties of the theory\textsuperscript{8} and implies that chiral anomaly is related to imaginary part of the effective actions \( \Gamma_\pm(A) \). Eq.(3.3) is a direct consequence of the hermiticity of the Dirac operator \( \mathcal{D}(A) \).

\[ \mathcal{D}(A) \equiv (i\mathcal{D} + gA) \]
\[ = \begin{pmatrix} 0 & D(A) \\ D^\dagger(A) & 0 \end{pmatrix} \]  \hspace{1cm} (3.4)

The Weyl components \( D(A), D^\dagger(A) \)

\[ D(A) \equiv \sigma_\mu (i\partial_\mu + gA_\mu) \]  \hspace{1cm} (3.5)

with \( \sigma_\mu = (i, \sigma_k) \), are hermitian conjugates of each other and eq.(3.3) follows from

\[ \Gamma_+(A) = -\ln \det [D(A)D^\dagger(0)] \]
\[ \Gamma_-(A) = -\ln \det [D(0)D^\dagger(A)] \]  \hspace{1cm} (3.6).
The naive lattice action (2.6) with gauge field interactions through link variables $U_\mu$

$$S^{\text{naive}} = - \sum_\mu < \bar{\psi} | \gamma_\mu \left( \frac{e^{-ip_\mu a} U_\mu^\dagger - U_\mu e^{ip_\mu a}}{2a} \right) | \psi >$$

(3.7)

obeys the hermiticity condition even for finite lattice spacing. The fact that it fails to reproduce the ABJ anomaly is well understood in terms of cancellation of the contribution from the physical fermion by those from the doublers.

The key elements for a chirally invariant formulation of lattice Dirac fermion, therefore, are (i) absence of doublers (ii) chiral symmetry, and (iii) hermiticity. Note that there is no requirement of hermiticity for the Weyl components of a Dirac operator. All these elements are realised if in the continuum Dirac operator we substitute forward (backward) and backward (forward) finite differences for the derivatives in the right-handed and the left-handed Weyl components respectively

$$\gamma_\mu \partial_\mu \rightarrow \begin{pmatrix} 0 & \sigma_\mu \left( \frac{e^{-ip_\mu a} - 1}{a} \right) \\ \sigma_\mu^\dagger \left( \frac{1 - e^{ip_\mu a}}{a} \right) & 0 \end{pmatrix}$$

(3.8)

In Dirac basis the wave operator (3.8) assumes the form

$$\gamma_\mu \left( \frac{e^{-ip_\mu a} - e^{ip_\mu a}}{2a} \right) - \gamma_\mu \gamma_5 \left( \frac{e^{-ip_\mu a} + e^{ip_\mu a} - 2}{2a} \right) = \gamma_\mu \delta_\mu^s - \gamma_\mu \gamma_5 \Delta_\mu^a$$

(3.9)

The ‘irrelevant term’ $\gamma_\mu \gamma_5 \Delta_\mu^a$ coincides with the Wilson term in (2.9) if in the latter the parameter $r$ is replaced by $\gamma_\mu \gamma_5$. The representation (3.9) has the correct hermiticity properties, is chirally invariant, and, as is easily verified, free from doublers.

The price paid for eliminating the doublers is the loss of hypercubic and reflection symmetries. Hypercubic symmetry is the remnant of $SO(4)$ symmetry on lattice and is the invariance under rotation through $\pi/2$. This is lost because the antisymmetric combination $\Delta_\mu^a$ does not transform as a vector. As for reflection symmetry, $\gamma_4 \gamma_5 \Delta_4^a$ is compatible but the remaining pieces in the irrelevant term are not. Note that the sum of squares $\sum_\mu (\gamma_\mu \gamma_5 \Delta_\mu^a)^2$, does not suffer from these incompatibilities. This suggests as remedy to use instead the wave operator

$$\gamma_\mu \delta_\mu^s - \gamma_\mu \epsilon_\mu \gamma_5 \Delta_\mu^a \equiv \gamma_\mu \delta_\mu^s - \gamma_\mu \gamma_5 \Delta_\mu^a$$

(3.10)

where $\epsilon_\mu = \pm 1$, with the prescription that all correlation functions are to be obtained after averaging over $\epsilon_\mu$. What $\epsilon_\mu$ achieves is the decoupling of the finite difference, forward or
backward, used for the \( \mu \)-th component of the derivative from those used for the other components. The (anti) correlation of finite differences used for the right-handed and the left-handed Weyl components is, however, maintained by the wave operator (3.10), so that the chiral structure (3.4) is preserved. The wave operator (3.10) leads to the ‘free’ fermion propagator

\[
G^0_F(k) = \frac{i \gamma_\mu}{a} \frac{\sin(k_\mu a)}{a} + \gamma_\mu \gamma_5 \frac{1 - \cos(k_\mu a)}{a}
\] (3.11)

Interactions with gauge fields are introduced, as usual, through the link variables \( U_\mu \), and the lattice action corresponding to the wave operator (3.10) is given by

\[
S_F = - \sum_\mu \frac{1}{2a} \langle \bar{\psi}(x + \mu) \gamma_\mu U_\mu^\dagger(x) \psi(x) - \bar{\psi}(x) \gamma_\mu U_\mu(x) \psi(x + \mu) \rangle
\]

\[
+ \frac{1}{2} \bar{\psi}(x) \gamma_\mu \gamma_5 \psi(x) - \bar{\psi}(x + \mu) \gamma_\mu \gamma_5 U_\mu^\dagger(x) \psi(x)
\]

\[
- \bar{\psi}(x) \gamma_\mu \gamma_5 U_\mu(x) \psi(x + \mu) - m \bar{\psi}(x) \psi(x)
\] (3.12)

We have introduced a mass ‘\( m \)’ for the fermion which vanishes in the chiral limit.

Ward identities for the \( U(1) \) vector and axial vector currents are derived by requiring invariance of the partition function under the local transformations

\[
\psi(x) \to e^{i\alpha(x)} \psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x) e^{-i\alpha(x)}
\]

and

\[
\psi(x) \to e^{i\beta(x) \gamma_5} \psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x) e^{i\beta(x) \gamma_5}
\] (3.13)

respectively. The Jacobians of fermion measure in the lattice fermion action are trivial\(^9\) for both the transformations (3.13). Anomalies, if any, can arise only from lattice artifacts, the ‘irrelevant terms’. The Ward identities on lattice, therefore, are

\[
\frac{1}{a} < J^+_\mu(x) - J^+_\mu(x - \mu) > = \frac{1}{a} < J^-_{\mu 5}(x) - J^-_{\mu 5}(x - \mu) >,
\] (3.14)

\[
\frac{1}{a} < J^+_{\mu 5}(x) - J^+_{\mu 5}(x - \mu) > = \frac{1}{a} < J^-_{\mu 5}(x) - J^-_{\mu 5}(x - \mu) >,
\] (3.15)

where

\[
J^+_{\mu}(x) \equiv \frac{1}{2} \left[ \bar{\psi}(x) \gamma_\mu U_\mu(x) \psi(x + \mu) + \bar{\psi}(x + \mu) \gamma_\mu U_\mu^\dagger(x) \psi(x) \right],
\]
\[ J_{\mu 5}^+(x) \equiv \frac{1}{2} \left[ \bar{\psi}(x)\gamma_\mu \gamma_5 U_\mu(x)\psi(x + \mu) + \bar{\psi}(x + \mu)\gamma_\mu \gamma_5 U_\mu^\dagger(x)\psi(x) \right], \]
\[ J_{\mu 5}^-(x) \equiv \frac{1}{2} \left[ \bar{\psi}(x)\gamma_\mu \gamma_5 U_\mu(x)\psi(x + \mu) - \bar{\psi}(x + \mu)\gamma_\mu \gamma_5 U_\mu^\dagger(x)\psi(x) \right], \]
\[ J_{\mu 5}^-(x) \equiv \frac{1}{2} \left[ \bar{\psi}\gamma_\mu \gamma_5 U_\mu(x)\psi(x + \mu) + \bar{\psi}(x + \mu)\gamma_\mu \gamma_5 U_\mu^\dagger(x)\psi(x) \right]. \tag{3.16} \]

The right hand sides of eqs. (3.14) and (3.15), if non-zero, are to be identified as anomalies in \( U(1) \) vector and axial vector currents respectively

\[ < \partial_\mu J_\mu(x) > = \lim_{a \to 0} \frac{1}{a} < J_{\mu 5}^-(x) - J_{\mu 5}^-(x - \mu) > \tag{3.17} \]
\[ < \partial_\mu J_{\mu 5}(x) > = \lim_{a \to 0} \frac{1}{a} < J_{\mu 5}^- - J_{\mu 5}^- - J_{\mu 5}^-(x - \mu) > \tag{3.18} \]

## 4 \textit{U(1) Ward Identities in WCPT}

To evaluate \( < X > \) and \( < Y > \) it is convenient to start from their operator representations

\[ < Y > = Tr \gamma_5 < x | (G_F \mathcal{R}^\epsilon + \mathcal{R}^\epsilon G_F) | x > \]
\[ < X > = Tr < x | (G_F \mathcal{R}^\epsilon - \mathcal{R}^\epsilon G_F) | x > \tag{4.1} \]

where \( \mathcal{R}^\epsilon \equiv \gamma_\lambda \epsilon_\lambda R_\lambda \) with

\[ R_\lambda \equiv \frac{2 - U_\lambda e^{ip_\lambda a} - e^{-ip_\lambda a} U_\lambda^\dagger}{2a} \tag{4.2} \]

The fermion propagator is given by

\[ G_F^{-1} = i (\not{p} - i \mathcal{R}^\epsilon \gamma_5 - i m) \tag{4.3} \]

where

\[ D_\lambda = \frac{U_\lambda e^{ip_\lambda a} - e^{-ip_\lambda a} U_\lambda^\dagger}{2ia} \tag{4.4} \]

In (4.1) ‘Tr’ stands for trace over \( \gamma \)-matrices, and, in non-abelian gauge theory, also over the symmetry matrices.
In \( wcpt \) the link variable \( U_\lambda \approx 1 + i a g A_\lambda \), so that for small lattice spacing

\[
\begin{align*}
D_\lambda &= (s_\lambda + gA_\lambda \cos(p_\lambda a)) + 0(a) \\
R_\lambda &= c_\lambda + 0(a)
\end{align*}
\] (4.5)

where

\[
\begin{align*}
s_\lambda &= \sin \left( \frac{p_\lambda a}{a} \right), \\
c_\lambda &= \frac{1 - \cos(p_\lambda a)}{a}
\end{align*}
\] (4.6)

The commutators in \( wcpt \) are given by

\[
\begin{align*}
[R_\lambda, R_\rho] &= [D_\lambda, R_\rho] = 0 \\
[D_\lambda, D_\rho] &= F_{\lambda\rho} \cos(p_\lambda a) \cos(p_\rho a) \equiv f_{\lambda\rho}
\end{align*}
\] (4.7)

The quadratic form of the propagator \( G \), defined by

\[
G^{-1} \equiv \left( G_F G_F^\dagger \right)^{-1} = (\slashed{D} + i \slashed{R} \gamma_5 - i m) (\slashed{D} + i \slashed{R} \gamma_5 + i m)
\]

\[
= \sum_\lambda \left( D^2_\lambda + R^2_\lambda \right) + \frac{i}{2} \sigma_{\lambda\rho} f_{\lambda\rho} - 2 \gamma_5 \sigma_{\lambda\rho} \epsilon_\rho R_\rho D_\lambda + m^2
\] (4.8)

commutes with \( \gamma_5 \) and its trace with odd powers of \( \gamma \)-matrices vanish. These properties lead to a simpler structures for \( < Y > \) and \( < X > \)

\[
< Y > = Tr \gamma_5 < x | (\slashed{D} G \slashed{R} + \slashed{R} \slashed{G} \slashed{D}) | x > \\
< X > = Tr < x | (\slashed{D} G \slashed{R} - \slashed{R} \slashed{G} \slashed{D}) | x > - 2 i Tr \gamma_5 < x | \slashed{R} \slashed{G} \slashed{R} | x >
\] (4.9)

In \( wcpt \) where both the lattice spacing ‘\( a \)’ and the gauge coupling ‘\( g \)’ are regarded as small, one can define a ‘potential’ \( V \)

\[
G^{-1} = G_o^{-1} + V
\]

and develop a perturbative series\(^5\)

\[
G = G_o - G_o V G_o + G_o V G_o V G_o + .......
\] (4.10)

where

\[
G_o^{-1} = \sum_\lambda \left( s^2_\lambda + c^2_\lambda \right) + m^2
\] (4.11)

and

\[
V = \sum_\lambda \left( 2 g A_\lambda s_\lambda \cos (p_\lambda a) + g^2 A^2_\lambda \right) + \frac{i}{2} \sum_{\lambda, \rho} \sigma_{\lambda\rho} f_{\lambda\rho} - 2 \gamma_5 \sum_{\lambda, \rho} \sigma_{\lambda\rho} \epsilon_\rho R_\rho D_\lambda
\] (4.12).
Two factors play key role in the calculation of $<X>$ and $<Y>$: (i) terms odd in $R_\lambda$ drop out because of $\epsilon$-averaging, and (ii) in the ‘physical’ sector of the loop momentum $0 \leq |k_\lambda| \leq \pi/2a$, $R_\lambda$ is of $0(a)$ whereas in the ‘doubler’ sector $\pi/2a \leq |k_\lambda| \leq \pi/a$ it is of $0(1/a)$ and behaves like the mass of a Pauli-Villars regulator field\textsuperscript{10}. One obtains (see Appendix)

$$\lim_{a \to 0} <Y> = 0$$

so that the U(1) vector current is conserved

$$<\partial_\mu J_\mu(x)> = 0$$

In the expression for $<X>$, however, only the first term in (4.9) vanishes so that

$$<X> = -2iTr\gamma_5 <x|R^e G F^e|x> \sum_\lambda c_\mu^2 \Pi_\alpha (\cos(k_\alpha a))$$

$$= 2iTr \left(F_{\lambda \rho}\bar{F}_{\lambda \rho}\right) \int_{\pi/2a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \left[\sum_\lambda (s^2_\lambda + c^2_\lambda) + m^2\right]^{3/2} \sum_{\nu=1}^4 (-1)^{\nu^2} C_{\nu-1} I_\nu$$

where $\bar{F}_{\lambda \rho}$ is the dual field tensor, ‘tr’ stands for trace of symmetry matrices, and

$$I_\nu = \int_{-\pi/2a}^{\pi/a} \frac{d^4k}{2a} \left[\sum_{\lambda=1}^4 s^2_\lambda + \sum_{\lambda=1}^4 \left(1 + \frac{\cos(k_\lambda a)}{a}\right)^2 + m^2\right]^{3/2}$$

(4.16)

In the continuum limit $I_\nu = \pi^2/2\nu$. Thus the model reproduces the ABJ anomaly

$$<\partial_\mu J_\mu^5(x)> = \lim_{a \to 0} <X> = -(i/16\pi^2)tr \left(F_{\lambda \rho}\bar{F}_{\lambda \rho}\right)$$

(4.17)

Note that in (4.9) the subscript $\nu$ of $I_\nu$ has the meaning of the number of components of the loop momentum with support in the doubler sector $\pi/2a \leq |k_\mu| \leq \pi/a$. For finite ‘$m$’ the contribution from $\nu = 0$ vanishes in the continuum limit. If, however, $m$ is zero the latter would exactly cancel the right hand side of (4.15). For a nonvanishing anomaly it is, therefore, essential that the continuum limit is taken first and chiral limit $m = 0$, if necessary, afterwards. This is true also for the calculation of anomaly in the Wilson model\textsuperscript{5}. 

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Our search for a model of lattice fermion which is, (i) free from doublers, (ii) hermitian, and (iii) chirally invariant, has paid rich dividends. Not only is the model free from doublers, it reproduces in the continuum limit the correct Ward identities, and, most importantly, the ABJ anomaly in the U(1) axial vector current. In this respect the model measures up to the Wilson model. In a very crucial way, however, the two models differ. Whereas chiral symmetry is broken explicitly by the Wilson term, the irrelevant term in the proposed model is chirally invariant. This makes the latter a viable candidate for chiral lattice gauge theories.

An important lesson of the present exercise is that the ABJ anomaly can be realised even with a chirally invariant term. What is important is that the physical fermion has, to start with, a small but non-zero mass and the chiral limit \( m = 0 \) is taken only after the continuum limit. If the order of the two limits is reversed the ABJ anomaly disappears. The same observation, interestingly enough, is true also in the chiral symmetry breaking Wilson model. In the absence of the mass term, a chirally invariant action for lattice gauge theory breaks up into two completely decoupled Weyl components. In Weyl basis, vector and axial vector currents lose their individual identity and both the currents are conserved. By coupling the two Weyl components, the mass term brings about an asymmetry between the vector and the axial vector currents. This asymmetry is apparently the genesis of the ABJ anomaly. The ABJ anomaly, once it materialises, persists even when the parameter \( m \), which triggered it, vanishes. There is a parallelism here with the phenomenon of ferromagnetism. A magnetic field, however weak, is needed to trigger the magnetised state of ferromagnet, and, once realised, the state persists even in the symmetry limit corresponding to zero field. For ferromagnetism, we have to take the thermodynamic limit first and the symmetry limit afterwards just as for ABJ anomaly the continuum limit \( a = 0 \) must precede the chiral limit.
Appendix

We show in the following how the first term of (4.9), let us call it $<X_o>$, and $<Y>$ vanish in the continuum limit. The crucial ingredients are Dirac trace, $\epsilon_{\mu}$-averaging and locality of the matrix elements.

To illustrate the locality property, concentrate on a typical term

$$<x|R_\alpha G_o f_{\lambda \rho} G_o R_\lambda D_\mu G_o D_\beta|x>$$

$$= \int \int <x|c_\alpha G_o |k> \frac{d^4k}{(2\pi)^4} <k|f_{\lambda \rho}|y> d^4y <y|G_o R_\lambda D_\mu G_o D_\beta|x>$$

$$= \int \int \frac{d^4k}{(2\pi)^4} \left[ \sum_{\lambda} \frac{e^{ik(x-y)}c_\alpha}{(c_\lambda^2 + s_\lambda^2) + m^2} \right] d^4y f_{\lambda \rho}(y) <y|G_o R_\lambda D_\mu G_o D_\beta|x>$$

In the physical region $0 \leq |k_\mu| \leq \pi/2a, c_\alpha \sim 0(a)$ and the integral vanishes in the continuum limit, whereas in the nonphysical or doubler sector $\pi/2a \leq |k_\mu| \leq \pi/a$, the denominator in the square bracket is of $O(1/a^2)$ but the numerator is at most of order $O(1/a)$ unless $x = y$. The same argument can be continued and extended for the remaining factors and hence proves the locality property. Note the parallelism here of the role of $c_\lambda$ in doubler sector with that of the mass of the Pauli-Villars regulator field in the calculation of chiral anomalies.

We now illustrate (4.13). Using locality and Dirac trace

$$<Y> = tr\gamma_5 <x|\bar{\psi}G\gamma_5 \epsilon + \gamma_5 \epsilon \gamma D|x>$$

$$= -2tr\gamma_5 <x|R_\alpha^2 D_\alpha G_o V G_o V G_o|x>$$

where $V = V_5$ or $i/2\sigma_{\lambda \rho} f_{\lambda \rho}$. Averaging of $\epsilon_\mu$ and locality now gives

$$<Y> = -itr (\sigma_{\lambda \rho} \sigma_{\lambda \rho}) <x|R_\alpha^2 D_\alpha G_o (D_\beta G_o f_{\lambda \rho} + f_{\lambda \rho} G_o D_\beta) G_o|x>$$

$$= -itr (\sigma_{\lambda \rho} \sigma_{\alpha \beta}) <x|\frac{1}{(R_\alpha^2 + R_\beta^2)} (D_\beta D_\beta + D_\beta D_\alpha) f_{\lambda \rho} G_o G_o|x>$$

$$= 0$$

Next we turn our attention to $<X_o>$.

$$<X_o> = <\tilde{X}_o> - <\tilde{X}_o^\dagger>$$

where

$$<\tilde{X}_o> = Tr <x|\bar{\psi} G^\dagger \gamma D|x>$$
now, 
\[ \langle \tilde{X}_o \rangle = \langle \tilde{X}_{o1} \rangle + \langle \tilde{X}_{o2} \rangle, \]
with, 
\[ \langle \tilde{X}_{o1} \rangle = tr < x | R^\epsilon \mathcal{G}_o V_5 \mathcal{G}_o \frac{i}{2} \sigma_{\lambda \rho} f_{\lambda \rho} \mathcal{G}_o \mathcal{D} | x > \]
and 
\[ \langle \tilde{X}_{o2} \rangle = tr < x | R^\epsilon \mathcal{G}_o \frac{i}{2} \sigma_{2 \rho} f_{2 \rho} \mathcal{G}_o V_5 \mathcal{G}_o \mathcal{D} | x >, \]
after plugging in the perturbative expansion for \( \mathcal{G} \) in terms of \( \mathcal{G}_o \) and \( V \).

After \( \epsilon_\mu \)-averaging, putting in \( R^2/4 \) in place of \( R^2 = \sum_\lambda R^2_\lambda \), and using locality, we finally obtain,
\[ \langle \tilde{X}_{o1} \rangle = \frac{i}{4} \sum_{\beta, \lambda, \nu, \rho} Tr (\gamma_5 \sigma_{\beta \nu} \sigma_{\lambda \rho}) < x | R^2 D_\beta \mathcal{G}_o D_\nu \mathcal{G}_o f_{\lambda \rho} \mathcal{G}_o | x > \]
Similarly, we find, \( \langle \tilde{X}_{o2} \rangle = - \langle \tilde{X}_{o1} \rangle \), so that \( \langle \tilde{X}_o \rangle = 0 \), implying \( \langle X_o \rangle = 0 \) in the continuum limit.
References

[1] L.H. Karsten and J. Smit, Nucl. Phys. B 183, 103 (1981).

[2] H.B. Nielsen and M. Ninomiya, Nucl. Phys. B 185, 20 (1981).

[3] A. Pilessetto, Ann. Phys. (N.Y.) 182, 177 (1988); M. Pernici, Phys. Lett. B 346, 99 (1995).

[4] J. Smit, Acta Phys. Polonica B 17, 531 (1986).

[5] W. Kerler, Phys. Rev. D 23, 2384 (1981); E. Seiler and I.O. Stamatescu, Phys. Rev. D 25, 2177 (1982).

[6] F. Wileczek, Phys. Rev. Lett. 59, 2397 (1987); L.H. Karsten, Phys. Lett. B 104, 315 (1981).

[7] J.L. Alonso and J.L. Cortes, Phys. Lett. B 187, 146 (1987); I.O. Stamatescu and T.T. Wu, Nucl. Phys. (Proc. Suppl.) B 42, 838 (1995).

[8] M. Gockeler and G. Schierholz, Nucl. Phys. (Proc. Suppl.), B 29, 114 (1992); L. Alvarez-Gaume, S. Della Pietra and V. Della Pietra, Phys. Lett. B 166, 177 (1986).

[9] K. Fujikawa, Z. Phys. C 25, 179 (1984).

[10] H. Banerjee and R. Banerjee, Phys. Lett. B 174, 313 (1986); H. Banerjee, R. Banerjee and P. Mitra, Z. Phys. C 32, 445 (1986).