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RELATIVE GROWTH OF ENTIRE DIRICHLET SERIES WITH DIFFERENT GENERALIZED ORDERS

For entire functions $F$ and $G$ defined by Dirichlet series with exponents increasing to $+\infty$ formulas are found for the finding the generalized order $\varrho_{\alpha, \beta}[F]_G = \lim_{\sigma \to +\infty} \frac{\alpha(M^{-1}_G(M_F(\sigma)))}{\beta(\sigma)}$
and the generalized lower order $\lambda_{\alpha, \beta}[F]_G = \lim_{\sigma \to +\infty} \frac{\alpha(M^{-1}_G(M_F(\sigma)))}{\beta(\sigma)}$ of $F$ with respect to $G$, where $M_F(\sigma) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ and $\alpha$ and $\beta$ are positive increasing to $+\infty$ functions.

Key words and phrases: Dirichlet series, generalized order, relative growth.

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INTRODUCTION

Let $f$ and $g$ be entire transcendental functions and $M_F(r) = \max\{|f(z)| : |z| = r\}$. For the study of relative growth of the functions $f$ and $g$ Ch. Roy [1] used the order $\varrho_g[f] = \lim_{r \to +\infty} \ln M_F^{-1}(M_F(r))/\ln r$ and the lower order $\lambda_g[f] = \lim_{r \to +\infty} \ln M_F^{-1}(M_F(r))/\ln r$ of the function $f$ with respect to the function $g$. Researches of relative growth of entire functions was continued by S.K. Data, T. Biswas and other mathematicians (see, for example, [2, 3, 4, 5]) in terms of maximal terms, Nevanlinna characteristic function and $k$-logarithmic orders. In [6] it is considered a relative growth of entire functions of two complex variables and in [7] the relative growth of entire Dirichlet series is studied in terms of $R$-orders.

Suppose that $\Lambda = (\lambda_n)$ is an increasing to $+\infty$ sequence of non-negative numbers, and by $S(\Lambda)$ we denote a class of entire Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f_n \exp\{s\lambda_n\}, \quad s = \sigma + it.$$ (1)

For $\sigma < +\infty$ we put $M_F(\sigma) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$. We remark that the function $M_F(\sigma)$ is continuous and increasing to $+\infty$ on $(-\infty, +\infty)$ and, therefore, there exists the function $M^{-1}_F(x)$ inverse to $M_F(\sigma)$, which increase to $+\infty$ on $(x_0, +\infty)$.

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By $L$ we denote a class of continuous non-negative on $(-\infty, +\infty)$ functions $\alpha$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 < x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i. e. $\alpha$ is a slowly increasing function. Clearly, $L_{si} \subset L^0$.

If $\alpha \in L$, $\beta \in L$ and $F \in S(\Lambda, +\infty)$ then the quantities

$$\varrho_{\alpha, \beta}[F] = \lim_{\sigma \to +\infty} \alpha(\ln M_F(\sigma)) / \beta(\sigma), \quad \lambda_{\alpha, \beta}[F] = \lim_{\sigma \to +\infty} \alpha(\ln M_F(\sigma)) / \beta(\sigma)$$

are called the generalized $(\alpha, \beta)$-order and the generalized lower $(\alpha, \beta)$-order of $F$ accordingly. We say that $F$ has the generalized regular $(\alpha, \beta)$-growth, if $0 < \lambda_{\alpha, \beta}[F] = \varrho_{\alpha, \beta}[F] < +\infty$.

We define the generalized $(\alpha, \beta)$-order $\varrho_{\alpha, \beta}[F]_G$ and the generalized lower $(\alpha, \beta)$-order $\lambda_{\alpha, \beta}[F]_G$ of the function $F \in S(\Lambda)$ with respect to a function $G \in S(\Lambda)$, given by Dirichlet series $G(s) = \sum_{n=1}^{\infty} g_n \exp\{s \lambda_n\}$, as follows

$$\varrho_{\alpha, \beta}[F]_G = \lim_{\sigma \to +\infty} \alpha(M^{-1}_G(\ln M_F(\sigma))) / \beta(\sigma), \quad \lambda_{\alpha, \beta}[F]_G = \lim_{\sigma \to +\infty} \alpha(M^{-1}_G(\ln M_F(\sigma))) / \beta(\sigma).$$

The following theorems are proved in [9].

**Theorem (A).** Let $\beta \in L$ and $\gamma \in L$. Except for the cases, when $\varrho_{\gamma, \beta}[F] = \varrho_{\gamma, \beta}[G] = 0$ or $\varrho_{\gamma, \beta}[F] = \varrho_{\gamma, \beta}[G] = +\infty$, the inequality $\varrho_{\beta, \beta}[F]_G \geq \varrho_{\gamma, \beta}[F]_G / \varrho_{\gamma, \beta}[G]$ is true and subject to the condition of the generalized regular $(\gamma, \beta)$-growth of $G$ this inequality converts into an equality.

Except for the cases, when $\lambda_{\gamma, \beta}[F] = \lambda_{\gamma, \beta}[G] = 0$ or $\lambda_{\gamma, \beta}[F] = \lambda_{\gamma, \beta}[G] = +\infty$, the inequality $\lambda_{\beta, \beta}[F]_G \leq \lambda_{\gamma, \beta}[F]_G / \lambda_{\gamma, \beta}[G]$ is true and subject to the condition of the generalized regular $(\gamma, \beta)$-growth of $G$ this inequality converts into an equality.

**Theorem (B).** Let $0 < p < +\infty$ and one of conditions is executed:

a) $\gamma \in L^0$, $\beta(\ln x) \in L^0$, $\frac{d\beta^{-1}(c\gamma(x))}{d\ln x} \to \frac{1}{p} \ (x \to +\infty)$ for each $c \in (0, +\infty)$ and $\ln n = o(\lambda_n) \ (n \to \infty)$;

b) $\gamma \in L_{si}$, $\beta \in L^0$, $\varrho_{\gamma, \beta}[F] < +\infty$, $\frac{d\beta^{-1}(c\gamma(x))}{d\ln x} = O(1) \ (x \to +\infty)$ and $\ln n = o(\lambda_n \beta^{-1}(c\gamma(\lambda_n))) \ (n \to \infty)$ for each $c \in (0, +\infty)$.

Suppose that $\gamma(\lambda_{n+1}/p) = (1 + o(1))\gamma(\lambda_n/p)$ as $n \to \infty$.

If the function $G$ has generalized regular $(\gamma, \beta)$-growth and $\kappa_n[G] := \frac{\ln |g_n| - \ln |g_{n+1}|}{\lambda_{n+1} - \lambda_n} \to +\infty$ as $n_0 \leq n \to \infty$ then

$$\varrho_{\beta, \beta}[F]_G = \lim_{n \to \infty} \beta \left( \frac{1}{p} + \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) / \beta \left( \frac{1}{p} + \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right)$$

except for the cases, when $\varrho_{\gamma, \beta}[F] = \varrho_{\gamma, \beta}[G] = 0$ or $\varrho_{\gamma, \beta}[F] = \varrho_{\gamma, \beta}[G] = +\infty$.

If, moreover, $\kappa_n[F] \not\to +\infty$ as $n_0 \leq n \to \infty$ then

$$\lambda_{\beta, \beta}[F]_G = \lim_{n \to \infty} \beta \left( \frac{1}{p} + \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) / \beta \left( \frac{1}{p} + \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right)$$
except for the cases, when \( \lambda_{\gamma,\beta}[F] = \lambda_{\gamma,\beta}[G] = 0 \) or \( \lambda_{\gamma,\beta}[F] = \lambda_{\gamma,\beta}[G] = +\infty \).

Similar results in terms of \( R \)-types are obtained in [10].

Here we consider the general case when \( \alpha \neq \beta \).

1 Analogues of Theorem (A)

We begin from the following general theorem.

**Theorem 1.** If \( \alpha \in L \) and \( \beta \in L \) then:

1) the inequalities

\[
\frac{\varrho_{\gamma,\beta}[F]}{\varrho_{\gamma,\alpha}[F]} \leq \varrho_{\alpha,\beta}[F] \leq \frac{\varrho_{\gamma,\beta}[F]}{\lambda_{\gamma,\alpha}[F]},
\]

(2)

are true for each function \( \gamma \in L \) except for the cases \( \varrho_{\gamma,\beta}[F] = \varrho_{\gamma,\alpha}[G] = 0 \), \( \varrho_{\gamma,\beta}[F] = \varrho_{\gamma,\alpha}[G] = +\infty \), \( \varrho_{\gamma,\beta}[F] = \lambda_{\gamma,\alpha}[G] = +\infty \);  

2) the inequalities

\[
\frac{\lambda_{\gamma,\beta}[F]}{\varrho_{\gamma,\alpha}[G]} \leq \lambda_{\alpha,\beta}[F] \leq \frac{\lambda_{\gamma,\beta}[F]}{\lambda_{\gamma,\alpha}[G]},
\]

(3)

are true for each function \( \gamma \in L \) except for the cases \( \lambda_{\gamma,\beta}[F] = \lambda_{\gamma,\alpha}[G] = 0 \), \( \lambda_{\gamma,\beta}[F] = \varrho_{\gamma,\alpha}[G] = +\infty \), \( \lambda_{\gamma,\beta}[F] = \varrho_{\gamma,\alpha}[G] = +\infty \).

**Proof.** Indeed,

\[
\varrho_{\alpha,\beta}[F] = \lim_{x \to +\infty} \frac{\alpha(M^{-1}_G(x))}{\beta(M^{-1}_F(x))} = \lim_{x \to +\infty} \frac{\gamma(ln x)}{\beta(M^{-1}_F(x))} \cdot \frac{\alpha(M^{-1}_G(x))}{\gamma(ln x)} 
\]

\[
\geq \lim_{x \to +\infty} \frac{\gamma(ln x)}{\beta(M^{-1}_F(x))} \cdot \lim_{x \to +\infty} \frac{\alpha(M^{-1}_G(x))}{\gamma(ln x)} = \lim_{x \to +\infty} \frac{\gamma(ln M_F(\sigma))}{\beta(\sigma)} \cdot \lim_{x \to +\infty} \frac{\alpha(\sigma)}{\gamma(ln M_G(\sigma))} = \frac{\varrho_{\gamma,\beta}[F]}{\varrho_{\gamma,\alpha}[G]},
\]

and

\[
\varrho_{\alpha,\beta}[F] = \lim_{x \to +\infty} \frac{\gamma(ln x)}{\beta(M^{-1}_F(x))} \cdot \lim_{x \to +\infty} \frac{\alpha(M^{-1}_G(x))}{\gamma(ln x)} = \lim_{x \to +\infty} \frac{\gamma(ln M_F(\sigma))}{\beta(\sigma)} \cdot \lim_{x \to +\infty} \frac{\alpha(\sigma)}{\gamma(ln M_G(\sigma))} = \frac{\varrho_{\gamma,\beta}[F]}{\lambda_{\gamma,\alpha}[G]},
\]

i.e., inequalities (2) are proved.

The proof of (3) is similar. Indeed,

\[
\lambda_{\alpha,\beta}[F] = \lim_{x \to +\infty} \frac{\gamma(ln x)}{\beta(M^{-1}_F(x))} \cdot \frac{\alpha(M^{-1}_G(x))}{\gamma(ln x)} \leq \lim_{x \to +\infty} \frac{\gamma(ln x)}{\beta(M^{-1}_F(x))} \cdot \lim_{x \to +\infty} \frac{\alpha(M^{-1}_G(x))}{\gamma(ln x)} = \frac{\lambda_{\gamma,\beta}[F]}{\lambda_{\gamma,\alpha}[G]},
\]

and

\[
\lambda_{\alpha,\beta}[F] \geq \lim_{x \to +\infty} \frac{\gamma(ln x)}{\beta(M^{-1}_F(x))} \cdot \lim_{x \to +\infty} \frac{\alpha(M^{-1}_G(x))}{\gamma(ln x)} = \frac{\lambda_{\gamma,\beta}[F]}{\varrho_{\gamma,\alpha}[G]},
\]

whence (3) follows. Theorem 1 is proved. \( \Box \)
Remark 1. In the statements 1) and 2) of Theorem 1 the conditions for the function $\gamma$ hold if $0 < \gamma_{\alpha}[G] \leq \gamma_{\beta}[G] < +\infty$. From (2) and (3) it follows that if $G$ has the generalized regular $(\gamma, \alpha)$-growth then $\varphi_{\alpha, \beta}[F]_G = \varphi_{\gamma, \alpha}[G]$. If we choose $\alpha(x) = \ln x$ and $\beta(x) = x$ for $x \geq 3$ then from the definition of $\varphi_{\gamma, \beta}[F]$ and $\lambda_{\gamma, \beta}[F]$ we obtain the definition of the $R$-order $\varphi_R[G]$ and the lower $R$-order $\lambda_R[G]$ introduced by J. Ritt [11], and if we choose $\alpha(x) = \beta(x) = \ln x$ for $x \geq 3$ then we obtain the definition of the logarithmic order $\varphi_l[G]$ and the lower logarithmic order $\lambda_l[G]$.

For the characteristic of the relative growth of the function $F$ with respect to a function $G$ in Ritt’s scale we use

$$\varphi_{R, R}[F]_G = \lim_{\sigma \to +\infty} M^{-1}_G(M_F(\sigma))/\sigma, \quad \lambda_{R, R}[F]_G = \lim_{\sigma \to +\infty} M^{-1}_G(M_F(\sigma))/\sigma,$$

in the logarithmic scale we use

$$\varphi_{l, l}[F]_G = \lim_{\sigma \to +\infty} \ln M^{-1}_G(M_F(\sigma))/\ln \sigma, \quad \lambda_{l, l}[F]_G = \lim_{\sigma \to +\infty} \ln M^{-1}_G(M_F(\sigma))/\ln \sigma$$

and in the mixed scale we use

$$\varphi_{R, l}[F]_G = \lim_{\sigma \to +\infty} \ln M^{-1}_G(M_F(\sigma))/\sigma, \quad \lambda_{R, l}[F]_G = \lim_{\sigma \to +\infty} \ln M^{-1}_G(M_F(\sigma))/\sigma.$$

Then Theorem 1 implies the following statement.

Corollary 1. If $0 < \lambda_R[G] \leq \varphi_R[G] < +\infty$ then

$$\frac{\varphi_R[F]}{\varphi_R[G]} \leq \varphi_{R, R}[F]_G \leq \frac{\varphi_R[F]}{\lambda_R[G]} \quad \text{and} \quad \frac{\lambda_R[F]}{\varphi_R[G]} \leq \lambda_{R, R}[F]_G \leq \frac{\lambda_R[F]}{\lambda_R[G]}.$$

If $0 < \lambda_l[G] \leq \varphi_l[G] < +\infty$ then

$$\frac{\varphi_l[F]}{\varphi_l[G]} \leq \varphi_{l, l}[F]_G \leq \frac{\varphi_l[F]}{\lambda_l[G]} \quad \text{and} \quad \frac{\lambda_l[F]}{\varphi_l[G]} \leq \lambda_{l, l}[F]_G \leq \frac{\lambda_l[F]}{\lambda_l[G]}.$$

If $0 < \lambda_l[G] \leq \varphi_l[G] < +\infty$ then

$$\frac{\varphi_R[F]}{\varphi_l[G]} \leq \varphi_{R, l}[F]_G \leq \frac{\varphi_R[F]}{\lambda_l[G]} \quad \text{and} \quad \frac{\lambda_R[F]}{\varphi_l[G]} \leq \lambda_{R, l}[F]_G \leq \frac{\lambda_R[F]}{\lambda_l[G]}.$$

For a more detailed description of the growth of Dirichlet series of finite nonzero order use the type. If $0 < \varphi_R[F] < +\infty$ then the quantities

$$T_R[F] = \lim_{\sigma \to +\infty} \ln M_F(\sigma)/\exp\{\sigma \varphi_R[F]\} \quad \text{and} \quad t_R[F] = \lim_{\sigma \to +\infty} \ln M_F(\sigma)/\exp\{\varphi_R[F]\}$$

are called the $R$-type and the lower $R$-type of function $F$. Similarly, the quantities

$$T_l[F] = \lim_{\sigma \to +\infty} \ln M_F(\sigma)/\sigma \varphi_l[F] \quad \text{and} \quad t_l[F] = \lim_{\sigma \to +\infty} \ln M_F(\sigma)/\sigma \varphi_l[F]$$
are called the logarithmic type and the lower logarithmic type of function $F$. Therefore, by analogy, if $0 < \varrho_{\alpha, \beta}[F] < +\infty$ then we define the generalized $(\alpha, \beta)$-type and the lower generalized $(\alpha, \beta)$-type of $F$ as follows

$$T_{\alpha, \beta}[F] = \lim_{\sigma \to +\infty} \frac{\exp\{\alpha(\ln M_F(\sigma))\}}{\exp\{\beta(\varrho_{\alpha, \beta}[F])\}}, \quad t_{\alpha, \beta}[F] = \lim_{\sigma \to +\infty} \frac{\exp\{\beta(\varrho_{\alpha, \beta}[F])\}}{\exp\{\beta(\varrho_{\alpha, \beta}[F])\}}.$$

Similarly, if $0 < \varrho_{\alpha, \beta}[F]_G < +\infty$ then we define the generalized $(\alpha, \beta)$-type and the lower generalized $(\alpha, \beta)$-type of the function $F$ with respect to the function $G$ as follows

$$T_{\alpha, \beta}[F]_G = \lim_{\sigma \to +\infty} \frac{\exp\{\alpha(M^{-1}_G(\sigma))\}}{\exp\{\beta(\varrho_{\alpha, \beta}[F])_G\}}, \quad t_{\alpha, \beta}[F]_G = \lim_{\sigma \to +\infty} \frac{\exp\{\beta(\varrho_{\alpha, \beta}[F])_G\}}{\exp\{\beta(\varrho_{\alpha, \beta}[F])_G\}}.$$

**Theorem 2.** Let $\alpha \in L$, $\beta \in L$ and $\gamma \in L$. If the function $G$ has the regular generalized $(\gamma, \alpha)$-growth and $0 < t_{\gamma, \alpha}[G] \leq T_{\gamma, \alpha}[G] < +\infty$ then

$$\frac{T_{\gamma, \beta}[F]}{T_{\gamma, \alpha}[G]} \leq \frac{(T_{\alpha, \beta}[F]_G)^{t_{\gamma, \alpha}[G]}}{(T_{\alpha, \beta}[F]_G)^{t_{\gamma, \alpha}[G]}} \leq \frac{T_{\gamma, \beta}[F]}{t_{\gamma, \alpha}[G]} \quad (4)$$

and

$$\frac{T_{\gamma, \beta}[F]}{T_{\gamma, \alpha}[G]} \leq \frac{(T_{\alpha, \beta}[F]_G)^{t_{\gamma, \alpha}[G]}}{(T_{\alpha, \beta}[F]_G)^{t_{\gamma, \alpha}[G]}} \leq \frac{T_{\gamma, \beta}[F]}{t_{\gamma, \alpha}[G]} \quad (5).$$

**Proof.** Since $G$ has the regular generalized $(\gamma, \alpha)$-growth, by Theorem 1 (see Remark 1) we have $\varrho_{\alpha, \beta}[F]_G = \frac{\varrho_{\gamma, \beta}[F]}{\varrho_{\gamma, \alpha}[G]}$. Therefore,

$$(T_{\alpha, \beta}[F]_G)^{t_{\gamma, \alpha}[G]} = \lim_{\sigma \to +\infty} \frac{\exp\{\varrho_{\gamma, \alpha}[G]_\alpha(M^{-1}_G(\sigma))\}}{\exp\{\beta(\varrho_{\gamma, \beta}[F]_G)\}} = \lim_{x \to +\infty} \frac{\exp\{\varrho_{\gamma, \alpha}[G]_\alpha(M^{-1}_G(x))\}}{\exp\{\varrho_{\gamma, \beta}[F]_G\beta(M^{-1}_G(x))\}} = \frac{T_{\gamma, \beta}[F]}{T_{\gamma, \alpha}[G]}\gamma(\ln x).$$

Estimates (4) are proved. The proof of (5) is similar and we will omit it. $\square$

Theorem 2 implies the following statement.
Corollary 2. If the function $G$ has the regular growth and $0 < t_R[G] \leq T_R[G] < +\infty$ then
\[
\frac{T_R[F]}{T_R[G]} \leq \frac{(T_R,F[G])^{\varrho[G]}}{t_R[F]} \leq \frac{T_R[F]}{t_R[G]}, \quad \frac{T_R[F]}{T_R[G]} \leq \frac{(T_R,F[G])^{\varrho[G]}}{t_R[F]} \leq \frac{T_R[F]}{t_R[G]},
\]
where
\[
T_{R,R}[F]_G = \lim_{\sigma \to +\infty} \frac{\exp\{M_1(G,F)\}}{\exp\{\varrho_\sigma^{R,R}[F]_G\}}, \quad t_{R,R}[F]_G = \lim_{\sigma \to +\infty} \frac{\exp\{M_1(G,F)\}}{\exp\{\varrho_\sigma^{R,R}[F]_G\}}.
\]

If the function $G$ has the regular logarithmic growth and $0 < t_l[G] \leq T_l[G] < +\infty$ then
\[
\frac{T_l[F]}{T_l[G]} \leq \frac{(T_l,F[G])^{\varrho[G]}}{t_l[F]} \leq \frac{T_l[F]}{t_l[G]}, \quad \frac{T_l[F]}{T_l[G]} \leq \frac{(T_l,F[G])^{\varrho[G]}}{t_l[F]} \leq \frac{T_l[F]}{t_l[G]},
\]
where
\[
T_{l,l}[F]_G = \lim_{\sigma \to +\infty} \frac{-\varrho_{\sigma,l}[F]_G M^{-1}_1(G,F)}{\varrho_{\sigma,l}[F]_G}, \quad t_{l,l}[F]_G = \lim_{\sigma \to +\infty} \frac{-\varrho_{\sigma,l}[F]_G M^{-1}_1(G,F)}{\varrho_{\sigma,l}[F]_G}.
\]

If the function $G$ has the regular logarithmic growth and $0 < t_l[G] \leq T_l[G] < +\infty$ then
\[
\frac{T_R[F]}{T_l[G]} \leq \frac{(T_R,F[G])^{\varrho[G]}}{t_l[F]} \leq \frac{T_R[F]}{t_l[G]}, \quad \frac{T_R[F]}{T_l[G]} \leq \frac{(T_R,F[G])^{\varrho[G]}}{t_l[F]} \leq \frac{T_R[F]}{t_l[G]},
\]
where
\[
T_{R,l}[F]_G := \lim_{\sigma \to +\infty} \frac{M^{-1}_1(G,F)}{\exp\{\varrho_{\sigma,R,l}[F]_G\}}, \quad t_{R,l}[F]_G := \lim_{\sigma \to +\infty} \frac{M^{-1}_1(G,F)}{\exp\{\varrho_{\sigma,R,l}[F]_G\}}.
\]

2 Analogues of Theorem (B)

We need the following lemma.

Lemma 1 ([8]). Let $\alpha \in L_{si}$, $\beta \in L^0$ and \( \frac{d\beta^{-1}(\alpha(x))}{d\ln x} = O(1) \) as $x \to +\infty$ for each $c \in (0, +\infty)$. If $\ln n = o(\lambda_n^{-1}(\alpha(\lambda_n)))$ as $n \to \infty$ for each $c \in (0, +\infty)$ and $G \in S(\Lambda)$ then
\[
\varrho_{\alpha,\beta}[G] = \lim_{n \to \infty} \alpha(\lambda_n)/\beta \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right).
\]

If, moreover, $\alpha(\lambda_{n+1}) \sim \alpha(\lambda_n)$ and $\kappa_n[G] \to +\infty$ as $n_0 \leq n \to \infty$ then
\[
\lambda_{\alpha,\beta}[G] = \lim_{n \to \infty} \alpha(\lambda_n)/\beta \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right).
\]

Now we prove the following theorem.

Theorem 3. Let $\alpha \in L^0$, $\beta \in L^0$, $\gamma \in L_{si}$, \( \frac{d\alpha^{-1}(\gamma(x))}{d\ln x} = O(1) \) and \( \frac{d\beta^{-1}(\gamma(x))}{d\ln x} = O(1) \) as $x \to +\infty$ for each $c \in (0, +\infty)$. Suppose that $\ln n = o(\lambda_n^{-1}(\gamma(\lambda_n)))$ and $\ln n = o(\lambda_n^{-1}(\gamma(\lambda_n)))$ as $n \to \infty$ for each $c \in (0, +\infty)$.
If the function $G$ has generalized regular $(\gamma, \alpha)$-growth, $\gamma(\lambda_{n+1}) \sim \gamma(\lambda_n)$ and $\kappa_n(G) \nearrow +\infty$ as $n_0 \leq n \to \infty$ then

$$\varrho_{\alpha, \beta}[F]_G = P_{\alpha, \beta} := \lim_{n \to \infty} \alpha \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) / \beta \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right). \tag{8}$$

If, moreover, $\kappa_n[F] \nearrow +\infty$ as $n_0 \leq n \to \infty$ then

$$\lambda_{\alpha, \beta}[F]_G = p_{\alpha, \beta} := \lim_{n \to \infty} \alpha \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) / \beta \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right). \tag{9}$$

Proof. Since $G$ has generalized regular $(\gamma, \alpha)$-growth, by Theorem 1 $\varrho_{\alpha, \beta}[F]_G = \varrho_{\gamma, \beta}[F] / \varrho_{\gamma, \alpha}[G]$, $\lambda_{\alpha, \beta}[F]_G = \frac{\lambda_{\gamma, \beta}[F]}{\lambda_{\gamma, \alpha}[G]}$ and by Lemma 1

$$\varrho_{\gamma, \beta}[F] = \lim_{n \to \infty} \frac{\gamma(\lambda_n)}{\beta \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right)}, \quad \varrho_{\gamma, \alpha}[G] = \lim_{n \to \infty} \frac{\gamma(\lambda_n)}{\alpha \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right)}.$$

Therefore,

$$\varrho_{\alpha, \beta}[F]_G = \lim_{n \to \infty} \frac{\gamma(\lambda_n)}{\beta \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right)} \lim_{n \to \infty} \alpha \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) / \gamma(\lambda_n) \leq$$

$$\leq \lim_{n \to \infty} \left( \frac{\gamma(\lambda_n)}{\beta \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right)} \right) \left( \alpha \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) / \gamma(\lambda_n) \right) = P_{\alpha, \beta}.$$

On the other hand, let $P_{\alpha, \beta} > 0$. Then for every $\varepsilon \in (0, P_{\alpha, \beta})$ there exists an increasing to $+\infty$ sequence $(n_k)$ of integers such that

$$\alpha \left( \frac{1}{\lambda_{n_k}} \ln \frac{1}{|g_{n_k}|} \right) > (P_{\alpha, \beta} - \varepsilon) \beta \left( \frac{1}{\lambda_{n_k}} \ln \frac{1}{|f_{n_k}|} \right)$$

i. e.

$$\gamma(\lambda_{n_k}) / \beta \left( \frac{1}{\lambda_{n_k}} \ln \frac{1}{|f_{n_k}|} \right) > (P_{\alpha, \beta} - \varepsilon) \gamma(\lambda_{n_k}) / \alpha \left( \frac{1}{\lambda_{n_k}} \ln \frac{1}{|g_{n_k}|} \right)$$

and, thus,

$$\varrho_{\gamma, \beta}[F] = \lim_{n \to \infty} \frac{\gamma(\lambda_n)}{\beta \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right)} \geq (P_{\alpha, \beta} - \varepsilon) \lim_{n \to \infty} \frac{\gamma(\lambda_n)}{\alpha \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right)} =$$

$$= (P_{\alpha, \beta} - \varepsilon) \lambda_{\gamma, \alpha}[G] = (P_{\alpha, \beta} - \varepsilon) \varrho_{\gamma, \alpha}[G],$$

whence in view of the arbitrariness of $\varepsilon$ we get $\varrho_{\alpha, \beta}[F]_G \geq P_{\alpha, \beta}$. For $P_{\alpha, \beta} = 0$ the last inequality is obvious. Equality (8) is proved.

For the proof of (9) we remark that since $G$ has generalized regular $(\alpha, \beta)$-growth, by Theorem 1 and Lemma 1

$$\lambda_{\alpha, \beta}[F]_G = \lim_{n \to \infty} \frac{\gamma(\lambda_n)}{\beta \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right)} \lim_{n \to \infty} \alpha \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) / \gamma(\lambda_n) \geq$$
inequality is obvious. Equality (9) is proved, and the proof of Theorem 3 is complete.

On the other hand, let \( p_{\alpha,\beta} < +\infty \). Then for every \( \varepsilon > 0 \) there exists an increasing to \( +\infty \) sequence \((n_k)\) of integers such that

\[
\alpha \left( \frac{1}{\lambda_{n_k}} \ln \frac{1}{|f_{n_k}|} \right) < (p_{\alpha,\beta} + \varepsilon) \beta \left( \frac{1}{\lambda_{n_k}} \ln \frac{1}{|f_{n_k}|} \right)
\]

and, as above,

\[
\lambda_{\gamma,\beta} [F] = \lim_{n \to \infty} \gamma(\lambda_n) / \beta \left( \frac{1}{\lambda_{n_k}} \ln \frac{1}{|f_{n_k}|} \right) \leq (p_{\alpha,\beta} + \varepsilon) \lim_{n \to \infty} \gamma(\lambda_n) / \alpha \left( \frac{1}{\lambda_{n_k}} \ln \frac{1}{|f_{n_k}|} \right) = (p_{\alpha,\beta} + \varepsilon) \varrho_{\gamma,\alpha} [G] = (p_{\alpha,\beta} + \varepsilon) \lambda_{\gamma,\alpha} [G],
\]

whence in view of the arbitrariness of \( \varepsilon \) we get \( \lambda_{\alpha,\beta} [F] G \leq p_{\alpha,\beta} \). For \( p_{\alpha,\beta} = +\infty \) the last inequality is obvious. Equality (9) is proved, and the proof of Theorem 3 is complete. \( \square \)

For the study of the relative growth in classical scales we need the following lemmas.

**Lemma 2** ([11], [12], [13], [14]). If \( \ln n = o(\lambda_n \ln \lambda_n) \) as \( n \to \infty \) then

\[
\varrho_R [F] = \lim_{n \to \infty} \lambda_n \ln \lambda_n / (\ln |f_n|)
\]

and if, moreover, \( \ln \lambda_{n+1} \sim \ln \lambda_n \) and \( \kappa_n [G] \nearrow +\infty \) as \( n_0 \leq n \to \infty \) then

\[
\lambda_R [F] = \lim_{n \to \infty} \lambda_n \ln \lambda_n / (\ln |f_n|).
\]

If \( \ln n = o(\lambda_n) \) as \( n \to \infty \) then

\[
T_R [F] = (1/(e \varrho_R [F])) \lim_{n \to \infty} \lambda_n |f_n|^{|\varrho_R [F]/\lambda_n|}
\]

and if, moreover, \( \lambda_{n+1} \sim \lambda_n \) and \( \kappa_n [G] \nearrow +\infty \) as \( n_0 \leq n \to \infty \) then

\[
t_R [F] = (1/(e \varrho_R [F])) \lim_{n \to \infty} \lambda_n |f_n|^{|\varrho_R [F]/\lambda_n|}.
\]

**Lemma 3** ([8]). If \( \lim_{n \to \infty} \ln \ln n / \ln \lambda_n \leq 1 \) then

\[
\varrho [F] = 1 + \lim_{n \to \infty} \ln \ln \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right)
\]

and if, moreover, \( \ln \lambda_{n+1} \sim \ln \lambda_n \) and \( \kappa_n [G] \nearrow +\infty \) as \( n_0 \leq n \to \infty \) then

\[
\lambda [F] = 1 + \lim_{n \to \infty} \ln \ln \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right).
\]

If \( 1 \leq \varrho [F] < +\infty \) and \( \ln n = o(\lambda_n^{\varrho [F]/(\varrho[F] - 1)}) \) as \( n \to \infty \) then

\[
T_i [F] = A(\varrho [F]) \lim_{n \to \infty} \lambda_n^{\varrho [F]} \left( \ln \frac{1}{|f_n|} \right)^{1-\varrho [F]}, \quad A(\varrho) = (\varrho - 1)^{\varrho - 1} \varrho^\varrho,
\]

and if, moreover, \( \lambda_{n+1} \sim \lambda_n \) and \( \kappa_n [G] \nearrow +\infty \) as \( n_0 \leq n \to \infty \) then

\[
t_i [F] = A(\varrho [F]) \lim_{n \to \infty} \lambda_n^{\varrho [F]} \left( \ln \frac{1}{|f_n|} \right)^{1-\varrho [F]},
\]
Choosing $\alpha(x) = \beta(x) = x$ and $\gamma(x) = \ln x$, from Theorem 3 we obtain the following statement.

**Proposition 1.** If the function $G$ has regular growth, \( \kappa_n[G] \nearrow +\infty \), $\ln n = o(\lambda_n \ln \lambda_n)$ and $\ln \lambda_{n+1} \sim \ln \lambda_n$ as $n_0 \leq n \to \infty$ then $q_{R,R}[F]_G = \lim_{n \to \infty} \ln |g_n| / \ln |f_n|$.

If, moreover, $\kappa_n[F] \nearrow +\infty$ then $\lambda_{R,R}[F]_G = \lim_{n \to \infty} \ln |g_n| / \ln |f_n|$.

This result can be directly obtained using Lemma 2. It is easy to see also that the functions $\alpha(x) = \beta(x) = \gamma(x) = \ln x$ do not satisfy the conditions of Theorem 3. However, the following statement is correct.

**Proposition 2.** If the function $G$ has regular logarithmic growth, $\overline{\lim}_{n \to \infty} \ln n / \ln \lambda_n \leq 1$, $\ln \lambda_{n+1} \sim \ln \lambda_n$ and $\kappa_n[G] \nearrow +\infty$ as $n_0 \leq n \to \infty$ then

\[
q_{l,l}[F]_G = p_l := \lim_{n \to \infty} \frac{\ln \ln \frac{1}{|f_n|} \ln \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right)}{\ln \ln \frac{1}{|g_n|} \ln \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right)}.
\tag{10}
\]

If, moreover, $\kappa_n[F] \nearrow +\infty$ as $n_0 \leq n \to \infty$ then

\[
\lambda_{l,l}[F]_G = p_l := \lim_{n \to \infty} \frac{\ln \ln \frac{1}{|f_n|} \ln \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right)}{\ln \ln \frac{1}{|g_n|} \ln \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right)}.
\tag{11}
\]

**Proof.** Since $G$ has generalized regular logarithmic growth, by Corollary 2 $q_{l,l}[F]_G = \frac{q_l[F]}{q_l[G]}$, $\lambda_{l,l}[F]_G = \frac{\lambda_l[F]}{\lambda_l[G]}$ and by Lemma 3

\[
q_{l,l}[F]_G = \overline{\lim}_{n \to \infty} \ln \frac{1}{|f_n|} / \ln \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) \lim_{n \to \infty} \ln \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) / \ln \frac{1}{|g_n|} \leq p_l.
\]

On the other hand, if $p_l > 0$ then for every $\varepsilon \in (0, p_l)$ there exists an increasing to $+\infty$ sequence $(n_k)$ of integers such that

\[
\ln \frac{1}{|f_{n_k}|} / \ln \left( \frac{1}{\lambda_{n_k}} \ln \frac{1}{|f_{n_k}|} \right) \geq (p_l - \varepsilon) \ln \frac{1}{|g_{n_k}|} / \ln \left( \frac{1}{\lambda_{n_k}} \ln \frac{1}{|g_{n_k}|} \right),
\]

i. e. by Lemma 3

\[
q_l[F] = \overline{\lim}_{n \to \infty} \ln \frac{1}{|f_n|} / \ln \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right) \geq (p_l - \varepsilon) \lim_{n \to \infty} \ln \frac{1}{|g_n|} / \ln \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) = (p_l - \varepsilon) \lambda_l[G] = (p_l - \varepsilon) q_l[G],
\]

whence in view of the arbitrariness of $\varepsilon$ we get $q_{l,l}[F]_G \geq p_l$. For $p_l = 0$ the last inequality is obvious. Equality (10) is proved.

The proof of (11) is similar. \( \square \)
The condition \( \ln n = o(\lambda_n \ln \lambda_n) \) as \( n \to \infty \) implies the condition \( \lim_{n \to \infty} \ln n / \ln \lambda_n \leq 1 \). Therefore, using Lemmas 2 and 3 it is easy to prove the following statement.

**Proposition 3.** If the function \( G \) has regular logarithmic growth, \( \ln n = o(\lambda_n \ln \lambda_n) \), \( \ln \lambda_{n+1} \sim \ln \lambda_n \) and \( \kappa_n[G] \nearrow +\infty \) as \( n_0 \leq n \to \infty \) then

\[
\varrho_{RL}[F]_G = \lim_{n \to \infty} \frac{\lambda_n \ln \lambda_n (1/|g_n|)}{(1/f_n) \ln (1/|f_n|)}.
\]

If, moreover, \( \kappa_n[F] \nearrow +\infty \) as \( n_0 \leq n \to \infty \) then

\[
\lambda_{RL}[F]_G = \lim_{n \to \infty} \frac{\lambda_n \ln \lambda_n (1/|g_n|)}{(1/f_n) \ln (1/|f_n|)}.
\]

Let us turn to the results about the relative growth of functions in terms of their types. For classic growth scales, we can use Lemmas 2 and 3, and for generalized orders we need such lemma.

**Lemma 4.** Let \( \alpha \in L, \beta \in L, x\alpha'(x) = o(1), x\beta'(x) = O(1) \) and \( \frac{d\beta^{-1}(c_1 + c_2\alpha(x))}{d\ln x} = O(1) \) as \( x \to +\infty \) for each \( 0 < c_1, c_2 < +\infty \). If \( \ln n = o(\lambda_n \beta^{-1}(c\alpha(\lambda_n))) \) as \( n \to \infty \) for each \( c \in (0, +\infty) \) and \( G \in S(\Lambda) \) then

\[
\lim_{n \to \infty} \exp \left\{ \frac{\alpha(\lambda_n) - \varrho_{a,\beta}[G]\beta left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} right)}{\lambda_n \ln |g_n|} \right\} = T_{a,\beta}[G].
\]

If, moreover, \( \alpha(\lambda_{n+1}) - \alpha(\lambda_n) \to 0 \) and \( \kappa_n[G] \nearrow +\infty \) as \( n_0 \leq n \to \infty \) then

\[
\lim_{n \to \infty} \exp \left\{ \frac{\alpha(\lambda_n) - \varrho_{a,\beta}[G]\beta left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} right)}{\lambda_n \ln |g_n|} \right\} = t_{a,\beta}[G].
\]

**Proof.** Put \( \alpha_1(x) = \exp\{\alpha(x)\} \) and \( \beta_1(x) = \exp\{\varrho_{a,\beta}[G]\beta(x)\} \). Then

\[
T_{a,\beta}[G] = \lim_{\sigma \to +\infty} \frac{\alpha_1(\ln M_G(\sigma))}{\beta_1(\sigma)} = \varrho_{a_1,\beta_1}[G].
\]

If, for example, \( c > 1 \) then

\[
\alpha(cx) - \alpha(x) = o(x(x-1)x \leq (c-1)\xi a'(\xi)
\]

for some \( \xi \in [x, cx] \) and, since \( x\alpha'(x) = o(1) \) as \( x \to +\infty \), we have \( \alpha(cx) - \alpha(x) \to 0 \) as \( x \to +\infty \), i.e. \( \alpha_1 \in L_{si} \). Similarly, in view of condition \( x\beta'(x) = O(1) \) as \( x \to +\infty \), we have \( \beta((1 + O(1))x) - \beta(x) = \beta'(\xi) o(x) \to 0 \) as \( x \to +\infty \), whence it follows that \( \beta_1 \in L^0 \).

Since \( \beta_1^{-1}(x) = \beta^{-1}(\ln x/\varrho_{a,\beta}[G]) \), we have

\[
\frac{d\beta_1^{-1}(c\alpha_1(x))}{d\ln x} = \frac{d\beta^{-1}(\ln c + \alpha(x)/\varrho_{a,\beta}[G])}{\ln x} = \frac{d\beta^{-1}(c_1 + c_2\alpha(x))}{\ln x} = O(1), \quad x \to +\infty.
\]

Finally, the condition \( \ln n = o(\lambda_n \beta^{-1}(c\alpha(\lambda_n))) \) as \( n \to \infty \) holds if \( \ln n = = o(\lambda_n \beta^{-1}(\ln c + \alpha(\lambda_n)/\varrho_{a,\beta}[G])) \) as \( n \to \infty \). But \( \ln(\ln c + \alpha(\lambda_n))/\varrho_{a,\beta}[G] \geq \alpha(\lambda_n)/(2\varrho_{a,\beta}[G]) \) for \( n \geq n_0 \). Therefore, the last condition holds if \( \ln n = o(\lambda_n \beta^{-1}(c\alpha(\lambda_n))) \) as \( n \to \infty \) for \( c = 1/(2\varrho_{a,\beta}[G]) \). Thus, the functions \( \alpha_1 \) and \( \beta_1 \) satisfy the conditions of Lemma 1 and in view of (8) formula (12) is proved. Also \( \frac{\alpha_1(\lambda_{n+1})}{\alpha_1(\lambda_n)} = \exp\{\alpha(\lambda_{n+1}) - \alpha(\lambda_n)\} \to 1 \) as \( n \to \infty \) and, therefore, by Lemma 1 formulas (13) is correct. \( \square \)
Using Lemma 4, we prove the following theorem.

**Theorem 4.** Let \( \alpha \in L, \beta \in L, \gamma \in L, x\alpha'(x) = O(1), x\beta'(x) = O(1), x\gamma'(x) = o(1), \frac{d\alpha^{-1}(c_1 + c_2 \gamma(x))}{d \ln x} = O(1) \) and \( \frac{d\beta^{-1}(c_1 + c_2 \gamma(x))}{d \ln x} = O(1) \) as \( x \to +\infty \) for each \( 0 < c_1, c_2 < +\infty \). Suppose that \( \ln n = o(\lambda_n \alpha^{-1}(c\gamma(\lambda_n))) \) and \( \ln n = o(\lambda_n \beta^{-1}(c\gamma(\lambda_n))) \) as \( n \to \infty \) for each \( c \in (0, +\infty) \).

If the function \( G \) has strongly regular generalized \((\gamma, \alpha)\)-growth (i. e. \( 0 < t_{\gamma,\alpha}[G] = T_{\gamma,\alpha}[G] < +\infty \)), \( \gamma(\lambda_{n+1}) - \gamma(\lambda_n) \to 0 \) and \( \kappa_n[G] \nearrow +\infty \) as \( n_0 \leq n \to \infty \) then

\[
(T_{\alpha,\beta}[F])^{e_{\gamma,\alpha}[G]} = Q := \exp\{ \lim_{n \to \infty} Q_n(F, G) \},
\]

where

\[
Q_n(F, G) = \varrho_{\gamma,\alpha}[G] \alpha \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) - \varrho_{\gamma,\beta}[F] \beta \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right).
\]

If, moreover, \( \kappa_n[F] \nearrow \infty \) as \( n_0 \leq n \to \infty \) then

\[
(t_{\alpha,\beta}[F])^{e_{\gamma,\alpha}[G]} = q := \exp\{ \lim_{n \to \infty} Q_n(F, G) \}.
\]

**Proof.** Since the function \( G \) has strongly regular generalized \((\gamma, \alpha)\)-growth, by Theorem 2 \((T_{\alpha,\beta}[F])^{e_{\gamma,\alpha}[G]} = T_{\gamma,\beta}[F]/T_{\gamma,\alpha}[G] \) and \((t_{\alpha,\beta}[F])^{e_{\gamma,\alpha}[G]} = t_{\gamma,\beta}[F]/t_{\gamma,\alpha}[G]\). Therefore, by Lemma 4

\[
(T_{\alpha,\beta}[F])^{e_{\gamma,\alpha}[G]} = \lim_{n \to \infty} \exp \left\{ \gamma(\lambda_n) - \varrho_{\gamma,\beta}[F] \beta \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right) \right\} \lim_{n \to \infty} \exp \left\{ \varrho_{\gamma,\alpha}[G] \alpha \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) - \gamma(\lambda_n) \right\} \leq \lim_{n \to \infty} \exp \{ Q_n(F, G) \} = Q
\]

and

\[
(t_{\alpha,\beta}[F])^{e_{\gamma,\alpha}[G]} = \lim_{n \to \infty} \exp \left\{ \gamma(\lambda_n) - \varrho_{\gamma,\beta}[F] \beta \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right) \right\} \lim_{n \to \infty} \exp \left\{ \varrho_{\gamma,\alpha}[G] \alpha \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) - \gamma(\lambda_n) \right\} \geq \lim_{n \to \infty} \exp \{ Q_n(F, G) \} = q.
\]

On the other hand, let \( Q > 0 \). Then for every \( Q_1 \in (0, Q) \) there exists an increasing to \( \infty \) sequence \((n_k)\) of integers such that \( \exp \{ Q_{n_k}(F, G) \} \geq Q_1 \), i. e.

\[
\exp \left\{ \gamma(\lambda_{n_k}) - \varrho_{\gamma,\beta}[F] \beta \left( \frac{1}{\lambda_{n_k}} \ln \frac{1}{|f_{n_k}|} \right) \right\} > Q_1 \exp \left\{ \gamma(\lambda_{n_k}) - \varrho_{\gamma,\alpha}[F] \alpha \left( \frac{1}{\lambda_{n_k}} \ln \frac{1}{|g_{n_k}|} \right) \right\}
\]

and, thus,

\[
T_{\gamma,\beta}[F] = \lim_{n \to \infty} \exp \left\{ \gamma(\lambda_n) - \varrho_{\gamma,\beta}[F] \beta \left( \frac{1}{\lambda_n} \ln \frac{1}{|f_n|} \right) \right\} \geq Q_1 \lim_{n \to \infty} \exp \left\{ \gamma(\lambda_n) - \varrho_{\gamma,\alpha}[F] \alpha \left( \frac{1}{\lambda_n} \ln \frac{1}{|g_n|} \right) \right\} Q_1 t_{\gamma,\alpha}[G] = Q_1 T_{\gamma,\alpha}[G],
\]

whence in view of the arbitrariness of \( Q_1 \) we get \( T_{\alpha,\beta}[F] \geq Q \). For \( Q = 0 \) the last inequality is obvious. The equality \((T_{\alpha,\beta}[F])^{e_{\gamma,\alpha}[G]} = Q\) is proved.

Similar we prove the inequality \( t_{\alpha,\beta}[F] \leq q \), i. e. we get the equality \((t_{\alpha,\beta}[F])^{e_{\gamma,\alpha}[G]} = q\).

The proof of Theorem 4 is complete. \(\square\)
Next three statements are proved in general a way and we will drop their proofs. Using Corollary 2 and Lemma 2, we get the following statement.

**Proposition 4.** If the function $G$ has the strongly regular growth (i.e. $0 < T_R[G] = \ln n = o(\lambda_n)$, $\lambda_{n+1} \sim \lambda_n$ and $\kappa_n[G] \nearrow +\infty$ as $n_0 \leq n \to \infty$ then

$$(T_{R,R}[F]_G)^{\varrho_R[G]} = \lim_{n \to +\infty} \frac{\varrho_R[G]}{\varrho_R[F]}|f_n|^{|\varrho_R[F]|/\lambda_n}|g_n|^{-\varrho_R[G]/\lambda_n}.$$

If, moreover, $\kappa_n[F] \nearrow +\infty$ as $n_0 \leq n \to \infty$ then

$$(t_{R,R}[F]_G)^{\varrho_R[G]} = \lim_{n \to +\infty} \frac{\varrho_R[G]}{\varrho_R[F]}|f_n|^{|\varrho_R[F]|/\lambda_n}|g_n|^{-\varrho_R[G]/\lambda_n}.$$

Since the condition $\ln \ln n = o(\ln \lambda_n)$ as $n \to \infty$ implies the condition $\ln n = o(\lambda_n^{\varrho/(\varrho - 1)})$ as $n \to \infty$ for every $\varrho > 1$, using Corollary 2 and Lemma 3, we get the next statement.

**Proposition 5.** Let $1 < \varrho[F], \varrho[G] < +\infty$. If the function $G$ has the strongly regular logarithmic growth (i.e. $0 < t_I[G] = T_I[G] < +\infty$), $\ln \ln n = o(\ln \lambda_n)$, $\lambda_{n+1} \sim \lambda_n$ and $\kappa_n[G] \nearrow +\infty$ as $n_0 \leq n \to \infty$ then

$$\frac{A(\varrho[I]_G)}{A(\varrho[I]_F)}(T_{I,I}[F]_G)^{\varrho_R[G]} = \lim_{n \to +\infty} \lambda_n^{\varrho[I]_F - \varrho[I]_G}(\ln \frac{1}{|f_n|})^{1 - \varrho[I]_F}(\ln \frac{1}{|g_n|})^{\varrho[I]_G - 1}.$$

If, moreover, $\kappa_n[F] \nearrow +\infty$ as $n_0 \leq n \to \infty$ then

$$\frac{A(\varrho[I]_G)}{A(\varrho[I]_F)}(t_{R,R}[F]_G)^{\varrho_R[G]} = \lim_{n \to +\infty} \lambda_n^{\varrho[I]_F - \varrho[I]_G}(\ln \frac{1}{|f_n|})^{1 - \varrho[I]_F}(\ln \frac{1}{|g_n|})^{\varrho[I]_G - 1}.$$

Finally, since the condition $\ln \ln n = o(\ln \lambda_n)$ as $n \to \infty$ implies the condition $\ln n = o(\lambda_n)$ as $n \to \infty$, using Corollary 2 and Lemmas 2 and 3, we get the next statement.

**Proposition 6.** Let $1 < \varrho[I] < +\infty$. If the function $G$ has the strongly regular logarithmic growth, $\ln \ln n = o(\ln \lambda_n)$, $\lambda_{n+1} \sim \lambda_n$ and $\kappa_n[G] \nearrow +\infty$ as $n_0 \leq n \to \infty$ then

$$e^{\varrho_R[F]}A(\varrho[I]_G)(T_{R,I}[F]_G)^{\varrho_R[G]} = \lim_{n \to +\infty} \lambda_n^{1 - \varrho[I]_G}|f_n|^{|\varrho[I]_F|/\lambda_n}(\ln \frac{1}{|g_n|})^{\varrho[I]_G - 1}.$$

If, moreover, $\kappa_n[F] \nearrow +\infty$ as $n_0 \leq n \to \infty$ then

$$e^{\varrho_R[F]}A(\varrho[I]_G)(t_{R,R}[F]_G)^{\varrho_R[G]} = \lim_{n \to +\infty} \lambda_n^{1 - \varrho[I]_G}|f_n|^{|\varrho[I]_F|/\lambda_n}(\ln \frac{1}{|g_n|})^{\varrho[I]_G - 1}.$$
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