THE NUMBER OF POSITIVE SOLUTIONS TO THE BREZIS-NIRENBERG PROBLEM

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Abstract. In this paper we are concerned with the well-known Brezis-Nirenberg problem
\[
\begin{aligned}
-\Delta u &= u^{\frac{N+2}{N-2}} + \varepsilon u, & & \text{in } \Omega, \\
u > 0, & & \text{in } \Omega, \\
u = 0, & & \text{on } \partial \Omega.
\end{aligned}
\]

The existence of multi-peak solutions to the above problem for small \(\varepsilon > 0\) was obtained in [23]. However, the uniqueness or the exact number of positive solutions to the above problem is still unknown. Here we focus on the local uniqueness of multi-peak solutions and the exact number of positive solutions to the above problem for small \(\varepsilon > 0\).

By using various local Pohozaev identities and blow-up analysis, we first detect the relationship between the profile of the blow-up solutions and Green’s function of the domain \(\Omega\) and then obtain a type of local uniqueness results of blow-up solutions. Lastly we give a description of the number of positive solutions for small positive \(\varepsilon\), which depends also on Green’s function.

Keywords: Critical Sobolev exponent, Local Pohozaev identity, Existence of solutions, Exact number of solutions, Green’s function

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1. Introduction and main results

In this paper, we consider the following Brezis-Nirenberg problem
\[
\begin{aligned}
-\Delta u &= u^{\frac{N+2}{N-2}} + \varepsilon u, & & \text{in } \Omega, \\
u > 0, & & \text{in } \Omega, \\
u = 0, & & \text{on } \partial \Omega,
\end{aligned}
\]

where \(N \geq 3\), \(\varepsilon > 0\) is a small parameter, \(\Omega\) is a smooth and bounded domain in \(\mathbb{R}^N\).

In 1983, Brezis and Nirenberg proved in their celebrated paper [5] that if \(N \geq 4\), problem (1.1) has a solution for \(\varepsilon \in (0, \lambda_1)\), where \(\lambda_1\) denotes the first eigenvalue of \(-\Delta\) with 0-Dirichlet boundary condition on \(\partial \Omega\). Also it is well known in [24] that problem (1.1) admits no solutions when \(\Omega\) is star-shaped and \(\varepsilon = 0\). On the other hand, Bahri and Coron [1] gave an existence result of a positive solution to problem (1.1) for \(\Omega\) with a nontrivial topology and \(\varepsilon = 0\). Since then a lot of attention has been paid to the limiting behavior of the solutions \(u_\varepsilon\) of (1.1) as \(\varepsilon \to 0\). To state such type of results, we introduce some facts on Green’s function.

The Green’s function \(G(x, \cdot)\) is the solution of
\[
\begin{aligned}
-\Delta G(x, \cdot) &= \delta_x, & & \text{in } \Omega, \\
G(x, \cdot) &= 0, & & \text{on } \partial \Omega,
\end{aligned}
\]

where \(\delta_x\) is the Dirac function. For \(G(x, y)\), we have the following form
\[
G(x, y) = S(x, y) - H(x, y), \quad (x, y) \in \Omega \times \Omega,
\]
where $S(x, y) = \frac{1}{(N-2)\omega_N |y - x|^{N-2}}$ is the singular part and $H(x, y)$ is the regular part of $G(x, y)$, $\omega_N$ is a measure of the unit sphere of $\mathbb{R}^N$. For any $x \in \Omega$, we denote $R(x) := H(x, x)$, which is called the Robin function.

Rey [25] proved that if a solution $u_\varepsilon$ of (1.1) satisfies

$$|\nabla u_\varepsilon|^2 \to S^{N/2} \delta_{x_0}, \text{ as } \varepsilon \to 0,$$

with $S$ the best Sobolev constant defined by

$$S = \inf \left\{ \int_\Omega |\nabla u|^2 \mid u \in H^1_0(\Omega), \int_\Omega |u|^2 = 1 \right\},$$

then $x_0$ is a critical point of $R(x)$. Conversely if $x_0$ is a nondegenerate critical point of $R(x)$ and $N \geq 5$, then (1.1) has a solution $u_\varepsilon$ satisfying (1.2). Similar results are also proved in [20]. Later, Glangetas [16] proved that the solution $u_\varepsilon$ of (1.1) satisfying (1.2) is unique for $\varepsilon$ small enough under some additional conditions.

A natural question is whether (1.1) has a solution $u_\varepsilon$ concentrated at multi-points. In this aspect, Musso and Pistoia [23] gave an affirmative answer. To state their results, we need to introduce some notations. As is well-known, the equation $-\Delta u = u^{N/2}$ in $\mathbb{R}^N$ has a family of solutions

$$U_{x, \lambda}(y) = C_N \frac{\lambda^{(N-2)/2}}{(1 + \lambda^2 |y - x|^2)^{(N-2)/2}},$$

where $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^+$ and $C_N = \left( N(N-2) \right)^{(N-2)/4}$. Set

$$A = \int_{\mathbb{R}^N} U_{0,1}^N, \quad B = \int_{\mathbb{R}^N} U_{0,1}^2.$$

(1.3)

Let $\Psi_k : \Omega^k \times (\mathbb{R}^+)^k \to \mathbb{R}$ be defined by

$$\Psi_k(x, \lambda) = A^2 \left( M_k(x) \lambda^{(N-2)/2}, \lambda^{(N-2)/2} \right) - B \sum_{j=1}^k \lambda_j^2,$$

where $\lambda^{(N-2)/2} = \left( \lambda_1^{(N-2)/2}, \ldots, \lambda_k^{(N-2)/2} \right)^T$, the matrix $M_k(x) = (m_{ij}(x))_{1 \leq i, j \leq k}$ is defined by

$$m_{ii}(x) = R(x_i), \quad m_{ij}(x) = -G(x_i, x_j), \text{ if } i \neq j.$$

Musso and Pistoia [23] proved that there exists a family of solutions $u_\varepsilon$ to (1.1) satisfying

$$|\nabla u_\varepsilon|^2 \to S^{N/2} \sum_{i=1}^k \delta_{a_i}, \text{ as } \varepsilon \to 0,$$

(1.4)

if $N \geq 5$ and $(a^k, \Lambda^k)$ is a nondegenerate critical point of $\Psi_k$ with $a^k = (a_1, \ldots, a_k)$ and some $\Lambda^k = (\lambda_1, \ldots, \lambda_k)$.

On the other hand, for any given $f \in H^1(\Omega)$, let $P$ denote the projection from $H^1(\Omega)$ onto $H^1_0(\Omega)$, i.e., $u = Pf$ is the solution of

$$\begin{cases}
\Delta u = -\Delta f, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}$$

Now for any $x \in \Omega$ and $\lambda \in \mathbb{R}^+$, we define

$$E_{x, \lambda} = \left\{ v \in H^1_0(\Omega) \mid \left\langle \frac{\partial PU_{x, \lambda}}{\partial \lambda}, v \right\rangle = \left\langle \frac{\partial PU_{x, \lambda}}{\partial x_i}, v \right\rangle = 0, \text{ for } i = 1, \ldots, N \right\},$$

where $\|\cdot\|$ denotes the basic norm in the Sobolev space $H^1_0(\Omega)$ and $\langle \cdot, \cdot \rangle$ means the corresponding inner product. Then our first result is on the structure of the blow-up solutions of (1.1).
Theorem 1.1. Let $N \geq 5$ and suppose that $u_\varepsilon(x)$ is a solution of (1.1) with (1.4). Then $M_k(a^k)$ is a non-negative matrix with $a^k = (a_1, \cdots, a_k)$ and $u_\varepsilon(x)$ can be written as

$$u_\varepsilon = \sum_{j=1}^{k} P U_{x_{j, \varepsilon}, \lambda_{j, \varepsilon}} + w_\varepsilon,$$

satisfying, for $j = 1, \cdots, k$, $\lambda_{j, \varepsilon} = (u_\varepsilon(x_{j, \varepsilon}))^{\frac{1}{4N}}$,

$$x_{j, \varepsilon} \to a_j, \quad \lambda_{j, \varepsilon} \to +\infty, \quad \|w_\varepsilon\| = o(1) \text{ and } w_\varepsilon \in \bigcap_{j=1}^{k} E_{x_{j, \varepsilon}, \lambda_{j, \varepsilon}}.$$

Moreover if $M_k(a^k)$ is a positive matrix, then there exist two constants $C_1, C_2$ such that

$$0 < C_1 \leq \varepsilon \frac{1}{N+1} \lambda_{j, \varepsilon} \leq C_2 < +\infty.$$

Furthermore if we denote (by choosing subsequence)

$$\lambda_j := \lim_{\varepsilon \to 0} (\varepsilon \frac{1}{N+1} \lambda_{j, \varepsilon})^{-1}, \text{ for } j = 1, \cdots, k,$$

then $(a^k, \Lambda^k)$ is a critical point of $\Psi_k$ with $a^k = (a_1, \cdots, a_k)$ and $\Lambda^k = (\lambda_1, \cdots, \lambda_k)$.

When $\Omega$ is a convex domain, it is known from [18] that $\Psi_k(x, \lambda)$ has no critical points in $\Omega^k \times (\mathbb{R}^+)^k$ for $k \geq 2$. Hence combining Theorem 1.1, we conclude that (1.1) has no solutions blowing-up at multiple points on convex domains. On the other hand, from [6, 11, 17], we know that Robin function $R(x)$ has a unique critical point on convex domains, which is also non-degenerate under some conditions. Therefore, considering the uniqueness result of Glangetas [16], we see that problem (1.1) has a unique solution for $\varepsilon$ small enough and a convex domain $\Omega$.

Next, to study the number of concentrated solutions, for any given $a^k = (a_1, \cdots, a_k)$ satisfying $\nabla_x \Psi_k(a^k, \Lambda^k) = 0$ for some $\Lambda^k \in (\mathbb{R}^+)^k$, we define

$$S_k = \left\{ \Lambda^k = (\lambda_1, \cdots, \lambda_k), \nabla_x \Psi_k(a^k, \Lambda^k) = 0, \nabla_\lambda \Psi_k(a^k, \Lambda^k) = 0 \right\}.$$

Now we can count the number of solutions to (1.1) satisfying (1.4), which can be stated as follows.

Theorem 1.2. For $N \geq 6$ and any given $a^k = (a_1, \cdots, a_k)$, suppose that $M_k(a^k)$ is a positive matrix and $(a^k, \Lambda^k)$ is a nondegenerate critical point of $\Psi_k$ for any $\Lambda^k \in S_k$. Then for $\varepsilon > 0$ sufficiently small,

$$\text{the number of solutions to (1.1) satisfying (1.4) = } z S_k,$$

where $z S_k$ is the number of the elements in the set $S_k$.

In Theorem 1.2, the existence and non-degeneracy of critical points to $\Psi_k$ play a crucial role. In fact, the existence of critical points to $\Psi_k$ and their non-degeneracy are very important topics. Musso and Pistoia [23] constructed a class of $\Omega_\delta$ for small $\delta$ and proved the existence of stable critical points of $\Psi_k$ on $(\Omega_\delta)^k \times (\mathbb{R}^+)^k$ for some domain $\Omega_\delta$. Specially, let $\Omega_0 = \bigcup_{i=1}^{k} \Omega_i$, where $\Omega_1, \cdots, \Omega_k$ are $k$ smooth bounded domains such that $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. Then the function $\Psi_k$ has a strict minimum point in the connected component $\Omega_1 \times \cdots \times \Omega_k \times (\mathbb{R}^+)^k$ of the set $(\Omega_0)^k \times (\mathbb{R}^+)^k$. Moreover, assume that

$$\Omega_i \subset \{ (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} | a_i \leq x_1 \leq b_i \}, \quad \text{with } b_i < a_{i+1}, \quad i = 1, \cdots, k.$$

For any $\delta > 0$, let $C_\delta = \{ (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} | x_1 \in (a_1, b_k), \|x'\| \leq \delta \}$ and $\Omega_\delta$ be a smooth connected domain such that $\Omega_0 \subset \Omega_\delta \subset \Omega_0 \cup C_\delta$. Then if $\delta$ is small enough, the function $\Psi_k$ has a strict minimum point on $(\Omega_\delta)^k \times (\mathbb{R}^+)^k$, which is stable.

Very recently, Bartsch, Micheletti and Pistoia [3] proved that all critical points of $\Psi_k$ are non-degenerate for most domains. Specially, for a bounded domain $\Omega \subset \mathbb{R}^N$ of class $C^{m+2, \alpha}, m \geq 0, 0 < \alpha < 1, \psi \in C^{m+2, \alpha}$, the set

$$\Omega_\psi := (id + \psi)(\Omega) = \{ x + \psi(x) : x \in \Omega \}$$
is again a bounded domain of class $C^{m+2,\alpha}$ provided $\|\psi\|_{C^{1,1}} < \rho(\Omega)$ is small. Setting

$$B^{m+2,\alpha}(\Omega) := \{ \psi \in C^{m+2,\alpha}(\Omega, \mathbb{R}^N) : \|\psi\|_{C^{1,1}} < \rho(\Omega) \},$$

then the set

$$\mathcal{M}^{m+2,\alpha}(\Omega) := \{ \psi \in B^{m+2,\alpha}(\Omega) : \text{all critical points of } \Psi_k \text{ are non-degenerate on } (\Omega, \mathbb{R}^k) \}$$

is a dense subset of $B^{m+2,\alpha}(\Omega)$.

The above results give us that there exist some domains $\Omega$ such that $\Psi_k$ possesses some critical points and all these critical points are non-degenerate on $(\Omega)^k \times (\mathbb{R}^+)^k$. We can also refer to [4, 22] and the references therein.

Furthermore, to obtain the exact number of solutions to (1.1), we need to impose some assumption on the domain $\Omega$. The following one will be used later.

**Assumption A:** The problem

$$\begin{cases}
-\Delta u = u^{\frac{N+2}{N-2}}, & u > 0, \quad \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases} \quad (1.5)$$

has no solutions.

It follows from [13, 21] and Theorem 1.1 that all blow-up points of (1.1) are simple and isolated. Also from the well-known results in [2], we find that the number of the blow-up points to (1.1) are finite. Now we denote the largest number of blow-up points by $k_0$ and define

$$T_k = \left\{ (a^k, \Lambda^k) = (a_1, \cdots, a_k, \lambda_1, \cdots, \lambda_k), \nabla_2 \Psi_k(a^k, \Lambda^k) = 0, \nabla_\lambda \Psi_k(a^k, \Lambda^k) = 0 \right\}.$$ 

Then the following result confirms the number of solutions to problem (1.1).

**Theorem 1.3.** Let $N \geq 6$. For any integer $k \in [1, k_0]$, suppose that $M_k(a^k)$ is a positive matrix, $(a^k, \Lambda^k)$ is a nondegenerate critical point of $\Psi_k$ for any $(a^k, \Lambda^k) \in T_k$ and the domain $\Omega$ satisfies **Assumption A**. Then for $\varepsilon > 0$ sufficiently small,

$$\text{the number of solutions to (1.1) } = \sum_{k=1}^{k_0} \sharp T_k,$$

where $\sharp T_k$ is the number of the elements in the set $T_k$.

**Remark 1.4.** From the above statements after Theorem 1.2(see also [3, 23]), we know that there are some non-convex domains such that $\Psi_k$ admits some critical points and all critical points of $\Psi_k$ are non-degenerate. A special example on which the function $\Psi_k$ has at least two critical points is as follows. Let $\Omega = \bigcup_{i=1}^{k+1} \Omega_i$, where $\Omega_1, \cdots, \Omega_{k+1}$ are $k + 1$ smooth bounded domains such that $dist\{\Omega_i, \Omega_j\}$ is large if $i \neq j$, then the function $\Psi_k$ has two strict minimum points in $\Omega_1 \times \cdots \times \Omega_k \times (\mathbb{R}^+)^k$ and $\Omega_1 \times \cdots \times \Omega_{k-1} \times \Omega_{k+1} \times (\mathbb{R}^+)^k$ correspondingly. Moreover, assume that

$$\Omega_i \subset \left\{ (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} | a_i \leq x_1 \leq b_i \right\}, \text{ with } b_i < a_{i+1}, \quad i = 1, \cdots, k + 1.$$ 

For any $\delta > 0$, let $C_\delta = \left\{ (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} | x_1 \in (a_i, b_{k+1}), |x'| \leq \delta \right\}$ and $\Omega_\delta$ be a smooth connected domain such that $\Omega_0 \subset \Omega_\delta \subset \Omega_0 \bigcup C_\delta$. Then if $\delta$ is small enough, the function $\Psi_k$ has at least two strict minimum points on $(\Omega_\delta)^k \times (\mathbb{R}^+)^k$, which are non-degenerate. Hence problem (1.1) admits at least two solutions concentrated at $k$ points for above domain $\Omega_\delta$. And Theorem 1.2 gives us the description on the exact number of solutions concentrated at $k$ points for above domain $\Omega_\delta$.

On the other hand, it is known in [24] that **Assumption A** is satisfied for a star-shaped domain. And from [12], we can also find some non-star-shaped domains on which **Assumption A** holds. However whether there exists a non-convex domain such that **Assumption A** holds and the function $\Psi_k$ admits non-degenerate critical points simultaneously seems to be interesting and difficult. Since the function $\Psi_k$ will depend on Green’s function on $\Omega$ and we know little information on Green’s function. The properties of the critical points of $\Psi_k$ will be a substantive and important project. A known example
concerning that $\Psi_k$ admits some non-degenerate critical points is above domain constructed in [23]. But it seems to be not easy to determine whether Assumption A holds in this case.

Here we would like to point out that without Assumption A, we can relax the result in Theorem 1.3 into “the number of concentrated solutions to (1.1) = $\sum_{k=1}^{k_0} F_k$”.

Whether or not theorems 1.2 and 1.3 are true for $N = 5$ are not clear due to our methods, which can be found in Remark 4.5 below for more details. To prove our main results, the crucial step is to prove a local uniqueness result of blow-up solutions. To this end, a widely used method is to reduce into finite dimensional problems and count the local degree. We refer to [8, 16] for examples. However, for the multi-peak solution of (1.1), it is extremely complicated to calculate the corresponding degree. Here inspired by [14, 19], our proofs mainly depend on the local Pohozaev type identities:

$$- \int_{\partial \Omega} \frac{\partial u_\varepsilon}{\partial \nu} \frac{\partial u_\varepsilon}{\partial x_i} + \frac{1}{2} \int_{\partial \Omega} |\nabla u_\varepsilon|^2 \nu_i = \frac{N-2}{2N} \int_{\partial \Omega} \frac{\partial \nu}{\partial x_i} u_\varepsilon^2 \nu_i + \frac{\varepsilon}{2} \int_{\partial \Omega} u_\varepsilon^2 \nu_i,$$

(1.6)

and

$$- \int_{\partial \Omega} \frac{\partial u_\varepsilon}{\partial \nu} \langle x-x_{j,\varepsilon}, \nabla u_\varepsilon \rangle + \frac{1}{2} \int_{\partial \Omega} |\nabla u_\varepsilon|^2 \langle x-x_{j,\varepsilon}, \nu \rangle + \frac{2-N}{2} \int_{\partial \Omega} \frac{\partial u_\varepsilon}{\partial \nu} u_\varepsilon = \frac{N-2}{2N} \int_{\partial \Omega} u_\varepsilon^2 \langle x-x_{j,\varepsilon}, \nu \rangle + \frac{\varepsilon}{2} \int_{\partial \Omega} u_\varepsilon^2 \langle x-x_{j,\varepsilon}, \nu \rangle - \varepsilon \int_{\partial \Omega} u_\varepsilon^2,$$

(1.7)

where $\Omega' \subset \Omega$ is a smooth domain and $\nu(x) = (\nu_1(x), \cdots, \nu_N(x))$ is the outward unit normal of $\partial \Omega'$. The local Pohozaev identities (1.6) and (1.7) can be deduced by multiplying $\frac{\partial u_\varepsilon}{\partial x_i}$ and $\langle x-x_{j,\varepsilon}, \nabla u_\varepsilon \rangle$ on both sides of (1.1) and integrating on $\Omega'$ respectively. With the absence of potential function in (1.1), only surface integrals appear in the local Pohozaev identities (1.6) and (1.7). So we need to study carefully each surface integral to determine which one dominates all the others. The concentrated points of (1.1) depend on Green’s function of $\Omega$, which causes new difficulties in the estimates of each term in local Pohozaev identities. Here inspired by [7], we establish some new entire estimates to overcome these difficulties caused by Green’s function. Last but not least, since any solution of (1.1) with (1.4) decays algebraically, we need to estimate the order of each terms in the local Pohozaev identities precisely. Here we also point out that the interaction between the bumps must be taken into careful consideration.

This paper is organized as follows. In Section 2, we establish some basic estimates of the solutions with concentration and give the proof of Theorem 1.1. In Section 3, we estimate the regularization of difference between two solutions. Then combining these calculations and the local Pohozaev identities, we prove Theorem 1.2 and Theorem 1.3 in Section 4. In Section 5, we give the proofs of some crucial estimates involving the Green’s function. In order that we can give a clear line of our framework, we list some basic estimates and calculations in Appendix A.

Throughout our paper, we use the same $C$ to denote various generic positive constants independent of $\varepsilon$. We will use $\partial$ or $\nabla$ to denote the partial derivative for any function $h(y, x)$ with respect to $y$, while we will use $D$ to denote the partial derivative for any function $h(y, x)$ with respect to $x$.

2. SOME ESTIMATES ON BLOW-UP SOLUTIONS AND PROOF OF THEOREM 1.1

In this section, we obtain some basic estimates for solutions of (1.1) satisfying (1.4). These estimates are crucial for discussions in next sections. We start with the following decomposition result concerning with solutions of (1.1).

**Proposition 2.1.** Let $N \geq 5$. Suppose that $u_\varepsilon(x)$ is a solution of (1.1) satisfying (1.4). Then $u_\varepsilon$ can be written as

$$u_\varepsilon = \sum_{j=1}^{k} PU_{x_j, \xi_j} + w_\varepsilon,$$

(2.1)
satisfying, for \( j = 1, \cdots, k \), \( \lambda_{j, \varepsilon} = \left( u_{\varepsilon}(x_{j, \varepsilon}) \right)^{\frac{2}{N-2}} \),

\[
x_{j, \varepsilon} \to a_j, \quad \lambda_{j, \varepsilon} \to +\infty, \quad \| w_{\varepsilon} \| = o(1) \quad \text{and} \quad w_{\varepsilon} \in \bigcup_{j=1}^{k} E_{x_{j, \varepsilon}, \lambda_{j, \varepsilon}}.
\] (2.2)

**Proof.** Since \( u_{\varepsilon}(x) \) is a solution of (1.1) satisfying (1.4), we find that \( u_{\varepsilon}(x) \) blows up at \( a_1, \cdots, a_k \). Then there exist \( x_{j, \varepsilon} \in \Omega \) for \( j = 1, \cdots, k \) satisfying

\[
x_{j, \varepsilon} \to a_j \quad \text{and} \quad u_{\varepsilon}(x_{j, \varepsilon}) \to +\infty.
\]

Let \( v_{1, \varepsilon} = \lambda_{1, \varepsilon}^{-(N-2)/2} u_{\varepsilon} \left( \frac{x}{\lambda_{1, \varepsilon}} + x_{1, \varepsilon} \right) \), then

\[
-\Delta v_{1, \varepsilon} = v_{1, \varepsilon}^{2\ast -1} + \frac{\varepsilon}{\lambda_{1, \varepsilon}^2} v_{1, \varepsilon}, \quad \text{in} \quad \mathbb{R}^N.
\]

For any fixed small \( d \), \( \max_{B_{d \lambda_{j, \varepsilon}(0)}} v_{1, \varepsilon} = 1 \), which means that

\[
u_{\varepsilon} = PU_{x_{1, \varepsilon}, \lambda_{1, \varepsilon}} + u_{1, \varepsilon}, \quad \text{with} \quad \int_{B_{d}(x_{1, \varepsilon})} \left( |\nabla u_{1, \varepsilon}|^2 + u_{1, \varepsilon}^2 \right) = o(1).
\]

Repeating the above process and setting \( w_{\varepsilon}(x) := u_{\varepsilon} - \sum_{j=1}^{k} PU_{x_{j, \varepsilon}, \lambda_{j, \varepsilon}} \), we get

\[
\int_{\bigcup_{j=1}^{k} B_{d}(x_{j, \varepsilon})} \left( |\nabla w_{\varepsilon}|^2 + w_{\varepsilon}^2 \right) = o(1).
\]

This and (1.4) imply \( \| w_{\varepsilon} \| = o(1) \). Then we find

\[
\left\langle \frac{\partial PU_{x_{j, \varepsilon}, \lambda_{j, \varepsilon}}}{\partial \lambda}, w_{\varepsilon} \right\rangle = \left\langle \frac{\partial PU_{x_{j, \varepsilon}, \lambda_{j, \varepsilon}}}{\partial x_i}, w_{\varepsilon} \right\rangle = o(1).
\]

Now we can move \( x_{j, \varepsilon} \) a bit (still denoted by \( x_{j, \varepsilon} \)), so that the error term \( w_{\varepsilon} \in \bigcup_{j=1}^{k} E_{x_{j, \varepsilon}, \lambda_{j, \varepsilon}} \). \( \square \)

**Proposition 2.2.** Let \( u_{\varepsilon} \) be a solution of (1.1) with (1.4), then for any small fixed \( d > 0 \), it holds

\[
u_{\varepsilon}(x) = A \left( \sum_{j=1}^{k} \frac{G(x_{j, \varepsilon}, x)}{(\lambda_{j, \varepsilon}|(N-2)/2)} \right) + O \left( \frac{1}{\lambda_{\varepsilon}^2} \left( N-2)/2 + \frac{\varepsilon}{\lambda_{j, \varepsilon}(N-2)/2} \right) + o \left( \frac{\varepsilon}{\lambda_{\varepsilon}^2} \right) \quad \text{in} \quad C^1 \left( \Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x_{j, \varepsilon}) \right),
\] (2.3)

where \( A \) is the constant in (1.3) and \( \lambda_{\varepsilon} := \min \{ \lambda_{1, \varepsilon}, \cdots, \lambda_{k, \varepsilon} \} \).

**Proof.** First for \( x \in \Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x_{j, \varepsilon}) \), we have

\[
u_{\varepsilon}(x) = \int_{\Omega} G(y, x) \left( u_{\varepsilon}^{\frac{N+2}{N-2}}(y) + \varepsilon u_{\varepsilon}(y) \right) dy
\]

\[
\begin{align*}
= & \sum_{j=1}^{k} \int_{B_{d}(x_{j, \varepsilon})} G(y, x) u_{\varepsilon}^{\frac{N+2}{N-2}}(y) dy + \int_{\Omega \setminus \bigcup_{j=1}^{k} B_{d}(x_{j, \varepsilon})} G(y, x) u_{\varepsilon}^{\frac{N+2}{N-2}}(y) dy \\
& + \varepsilon \sum_{j=1}^{k} \int_{B_{d}(x_{j, \varepsilon})} G(y, x) u_{\varepsilon}(y) dy + \varepsilon \int_{\Omega \setminus \bigcup_{j=1}^{k} B_{d}(x_{j, \varepsilon})} G(y, x) u_{\varepsilon}(y) dy.
\end{align*}
\] (2.4)
And by Taylor’s expansion, we know
\[
\int_{B_d(x_j,\varepsilon)} G(y, x) u_{\varepsilon}^{N+2} (y) dy
\]
\[
= G(x_j,\varepsilon, x) \int_{B_d(x_j,\varepsilon)} u_{\varepsilon}^{N+2} dy + \sum_{i=1}^{N} D_{x_i} G(x_j,\varepsilon, x) \int_{B_d(x_j,\varepsilon)} (y_i - x_{j,\varepsilon,i}) u_{\varepsilon}^{N+2} (y) dy
\]
\[
+ \sum_{i=1}^{N} \sum_{m=1}^{N} D_{x_i,x_m}^2 G(x_j,\varepsilon, x) \int_{B_d(x_j,\varepsilon)} (y_i - x_{j,\varepsilon,i})(y_m - x_{j,\varepsilon,m}) u_{\varepsilon}^{N+2} (y) dy
\]
\[
+ O \left( \int_{B_d(x_j,\varepsilon)} |y - x_{j,\varepsilon}|^3 u_{\varepsilon}^{N+2} (y) dy \right).
\] (2.5)

Also from the symmetry and the fact that
\[
\sum_{i=1}^{N} D_{x_i,x_m}^2 G(x_j,\varepsilon, x) = 0 \quad \text{for} \quad x \in \Omega \setminus B_d(x_j,\varepsilon),
\]
we get
\[
\sum_{i=1}^{N} \sum_{m=1}^{N} D_{x_i,x_m}^2 G(x_j,\varepsilon, x) \int_{B_d(x_j,\varepsilon)} (y_i - x_{j,\varepsilon,i})(y_m - x_{j,\varepsilon,m}) u_{\varepsilon,x_j,\lambda_j,\varepsilon}^{N+2} = 0.
\] (2.6)

Next for \( x \in \Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x_j,\varepsilon) \), from (A.1)–(A.3), it holds
\[
\varepsilon \int_{B_d(x_j,\varepsilon)} G(y, x) u_{\varepsilon}(y) dy = O \left( \varepsilon \int_{B_d(x_j,\varepsilon)} u_{\varepsilon}(y) dy \right) = O \left( \frac{\varepsilon}{\lambda_{\varepsilon}^{(N-2)/2}} + o \left( \frac{\varepsilon}{\lambda_{\varepsilon}^2} \right) \right).
\] (2.7)

Then (2.4)–(2.7) and (A.4)–(A.8) imply
\[
u_{\varepsilon}(x) = A \left( \sum_{j=1}^{k} \frac{G(x_j,\varepsilon,x)}{(\lambda_{j,\varepsilon})^{(N-2)/2}} \right) + O \left( \frac{1}{\lambda_{\varepsilon}^{(N+2)/2}} \right) + o \left( \frac{\varepsilon}{\lambda_{\varepsilon}^2} \right), \quad \text{in} \quad \Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x_j,\varepsilon).
\]

On the other hand, from (A.1), for \( x \in \Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x_j,\varepsilon) \), we have
\[
\frac{\partial u_{\varepsilon}(x)}{\partial x_i} = \int_{\Omega} D_{x_i} G(y, x) \left( u_{\varepsilon}^{N+2} (y) + \varepsilon u_{\varepsilon}(y) \right) dy
\]
\[
= \sum_{j=1}^{k} \int_{B_d(x_j,\varepsilon)} D_{x_i} G(y, x) \left( u_{\varepsilon}^{N+2} (y) + \varepsilon u_{\varepsilon}(y) \right) dy + O \left( \frac{1}{\lambda_{\varepsilon}^{(N+2)/2}} \right).
\] (2.8)

Similar to the above estimates, for \( x \in \Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x_j,\varepsilon) \) and \( j = 1, \ldots, k \), we can prove
\[
\int_{B_d(x_j,\varepsilon)} D_{x_i} G(y, x) \left( u_{\varepsilon}^{N+2} (y) + \varepsilon u_{\varepsilon}(y) \right) dy = \frac{A}{\left( \lambda_{j,\varepsilon} \right)^{(N-2)/2}} D_{x_i} G(x_{j,\varepsilon}, x) + O \left( \frac{1}{\lambda_{\varepsilon}^{(N+2)/2}} \right).
\] (2.9)

Then (2.8) and (2.9) imply
\[
\frac{\partial u_{\varepsilon}(x)}{\partial x_i} = A \left( \sum_{j=1}^{k} \frac{D_{x_i} G(x_{j,\varepsilon},x)}{(\lambda_{j,\varepsilon})^{(N-2)/2}} \right) + O \left( \frac{1}{\lambda_{\varepsilon}^{(N+2)/2}} + \frac{\varepsilon}{\lambda_{\varepsilon}^{(N-2)/2}} \right) + o \left( \frac{\varepsilon}{\lambda_{\varepsilon}^2} \right), \quad \text{in} \quad \Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x_j,\varepsilon).
\]
Proposition 2.3. Let $u_\varepsilon$ be a solution of (1.1) with (1.4), then $M_k(a^k)$ is a non-negative matrix. Moreover if $M_k(a^k)$ is a positive matrix, it holds

$$0 < C_1 \leq \varepsilon \frac{1}{N-2} \lambda_{j,\varepsilon} \leq C_2 < +\infty, \text{ for } j = 1, \cdots, k,$$

and

$$\nabla \Psi_k(a^k, \Lambda^k) = 0, \text{ with } a^k = (a_1, \cdots, a_k) \text{ and } \Lambda^k = (\lambda_1, \cdots, \lambda_k).$$

Here we denote (by subsequence) $\lambda_j := \lim_{\varepsilon \to 0} \left( \varepsilon \frac{1}{N-2} \lambda_{j,\varepsilon} \right)^{-1}$ for $j = 1, \cdots, k$.

Proof. We define the following quadratic form

$$P(u, v) = -\frac{\theta}{2} \int_{\partial B_\varepsilon(x_j, \epsilon)} \langle \nabla u, \nabla v \rangle + \frac{\theta}{2} \int_{\partial B_\varepsilon(x_j, \epsilon)} \langle \nabla u, \nabla v \rangle + \frac{2 - N}{4} \int_{\partial B_\varepsilon(x_j, \epsilon)} \langle \nabla u, \nabla v \rangle + \frac{2 - N}{4} \int_{\partial B_\varepsilon(x_j, \epsilon)} \langle \nabla v, \nabla v \rangle u.$$

Note that if $u$ and $v$ are harmonic in $B_\varepsilon(x_j, \epsilon)$, then $P(u, v)$ is independent of $\theta > 0$. Let $\Omega^\varepsilon = B_\varepsilon(x_j, \epsilon)$ in (1.7), then from (2.3) and (A.3), we have

$$\sum_{l=1}^{k} \sum_{m=1}^{k} \frac{P(G(x_{m,\varepsilon}, x), G(x_{l,\varepsilon}, x))}{\lambda_{m,\varepsilon}^{(N-2)/2} \lambda_{l,\varepsilon}^{(N-2)/2}} = -\frac{B\varepsilon}{A^2 \lambda_{j,\varepsilon}^2} + O\left( \frac{1}{\lambda_{j,\varepsilon}^N} + \varepsilon \frac{1}{\lambda_{j,\varepsilon}^{N-2}} \right) + o\left( \frac{\varepsilon}{\lambda_{j,\varepsilon}^{N+2}/2} + \varepsilon^2 \lambda_{j,\varepsilon}^2 \right),$$

where $A, B$ are the constants in (1.3).

Next we have the following estimate for which the proof is left in Section 5:

$$P\left(G(x_{m,\varepsilon}, x), G(x_{l,\varepsilon}, x)\right) = \begin{cases} -\frac{(N-2)R(x_j, \epsilon)}{2}, & \text{for } l, m = j, \text{ and } \text{if } \epsilon \frac{1}{N-2} \lambda_{j,\varepsilon} \leq \lambda_{l,\varepsilon} \leq \lambda_{m,\varepsilon}, \\ \frac{(N-2)G(x_{j,\varepsilon}, x_{l,\varepsilon})}{4}, & \text{for } m = j, l \neq j, \\ \frac{(N-2)G(x_{j,\varepsilon}, x_{m,\varepsilon})}{4}, & \text{for } m \neq j, l = j, \\ 0, & \text{for } l, m \neq j. \end{cases}$$

Then (2.12) and (2.13) imply

$$\frac{R(x_{j,\varepsilon})}{\lambda_{j,\varepsilon}^{N-2}} - \sum_{l \neq j}^{k} \frac{G(x_{j,\varepsilon}, x_{l,\varepsilon})}{\lambda_{j,\varepsilon}^{(N-2)/2} \lambda_{l,\varepsilon}^{(N-2)/2}} = \frac{2B\varepsilon}{A^2(N-2) \lambda_{j,\varepsilon}^2} + O\left( \frac{1}{\lambda_{j,\varepsilon}^N} + \varepsilon \frac{1}{\lambda_{j,\varepsilon}^{N-2}} \right) + o\left( \frac{\varepsilon}{\lambda_{j,\varepsilon}^{N+2}/2} + \varepsilon^2 \lambda_{j,\varepsilon}^2 \right).$$

Let $\Lambda_{l,\varepsilon} := \left( \epsilon \frac{1}{N-2} \lambda_{j,\varepsilon} \right)^{-1}$. From Corollary 3.7 in [13], we find

$$\Lambda_{l,\varepsilon} \geq C > 0, \text{ for } j = 1, \cdots, k.$$

Now we define $\Lambda_{j,\varepsilon}^{N-2} = \max \{ \Lambda_{j,\varepsilon}, j = 1, \cdots, k \}$. Then

$$\Lambda_{j,\varepsilon}^{N-2} R(x_{j,\varepsilon}) - \sum_{l \neq j}^{k} \frac{G(x_{j,\varepsilon}, x_{l,\varepsilon})}{\lambda_{j,\varepsilon}^{(N-2)/2} \lambda_{l,\varepsilon}^{(N-2)/2}} = \frac{2B}{A^2(N-2)} \Lambda_{j,\varepsilon}^2 + o\left( \Lambda_{j,\varepsilon}^{N-2} \right).$$

Since \( \frac{1}{C} \sum_{l=1}^{k} \Lambda_{l,\varepsilon}^{N-2} \leq (\Lambda_{j,\varepsilon}^{N-2})^{N-2} \leq \sum_{l=1}^{k} \Lambda_{l,\varepsilon}^{N-2} \), (2.15) gives us

$$\left( M_k(x_{\varepsilon}) + o(1) \right) \hat{\mu}_{k,\varepsilon} = \frac{2B}{A^2(N-2)} (\Lambda_{j,\varepsilon}^{N-2}, \cdots, \Lambda_{k,\varepsilon}^{N-2})^T,$$

where $\hat{\mu}_{k,\varepsilon} = (\Lambda_{1,\varepsilon}^{(N-2)/2}, \cdots, \Lambda_{k,\varepsilon}^{(N-2)/2})$ and $x_{\varepsilon} = (x_{1,\varepsilon}, \cdots, x_{k,\varepsilon})$. Now we recall that the first eigenvector of a symmetric matrix may be chosen with all its components strictly positive (see also Appendix
A in [2]). So if \( \rho(a^k) \) is the first eigenvalue of \( M_k(a^k) \), then there exists a first eigenvector \( \vec{\chi}(a^k) \) of \( M_k(a^k) \) such that all its components are strictly positive. Then (2.16) gives us that
\[
\vec{\chi}(a^k) \left( M_k(x_\varepsilon) + o(1) \right) \vec{\mu}_k^T = \left( \frac{2B}{A^2(N-2)} \right) \vec{\chi}(a^k) \left( \Lambda_\varepsilon^N, \ldots, \Lambda_{k,\varepsilon}^N \right)^T > 0.
\]

Also we know
\[
\rho(a^k) \left( \vec{\chi}(a^k) \vec{\mu}_k^T \right) = \vec{\chi}(a^k) M_k(a^k) \vec{\mu}_k^T \text{ and } \vec{\chi}(a^k) \vec{\mu}_k^T > 0.
\]

Then these mean that \( \rho(a^k) \geq 0 \) and \( M_k(a^k) \) is a non-negative matrix. Moreover, if \( M_k(a^k) \) is a positive matrix, we find \( \Lambda_{j,\varepsilon} \) is bounded for \( j = 1, \ldots, k \). And then these imply (2.10). Moreover letting \( \varepsilon \to 0 \) in (2.15), we find (2.11).

**Proposition 2.4.** Under the conditions in Proposition 2.3, it holds
\[
u'_\varepsilon(x) = A \left( \sum_{j=1}^k \frac{G(x_j, x)}{(\lambda_j)^{(N-2)/2}} \right) + \begin{cases} O \left( \frac{1}{\lambda_j^N} \right), & \text{if } N = 5, \\
O \left( \frac{1}{\lambda_j^{N-2}} \right), & \text{if } N \geq 6,
\end{cases} \quad \text{in } C^1(\Omega \setminus \bigcup_{j=1}^k B_{2d}(x_j)) \quad (2.17)
\]

**Proof.** The estimate (2.17) can be deduced by (2.3) and (2.10).

**Proposition 2.5.** Let \( u_\varepsilon \) be a solution of (1.1) with (1.4) and \( M_k(a^k) \) be a positive matrix. Then
\[
\nabla_x \Psi_k(a^k, \Lambda^k) = 0, \quad \text{with } a^k = (a_1, \ldots, a_k) \text{ and } \Lambda^k = (\lambda_1, \ldots, \lambda_k),
\]

where \( \lambda_j := \lim_{\varepsilon \to 0} \left( \varepsilon \frac{\Lambda_j}{\lambda_j} \right)^{-1} \) for \( j = 1, \ldots, k \). Moreover if \( (a^k, \Lambda^k) \) is a nondegenerate critical point of \( \Psi_k \), then for \( j = 1, \ldots, k \), it follows
\[
|x_{j,\varepsilon} - a_j| = \begin{cases} O \left( \frac{1}{\lambda_j^N} \right), & \text{if } N = 5, \\
O \left( \frac{1}{\lambda_j^{N-2}} \right), & \text{if } N \geq 6,
\end{cases} \quad \text{and } |\lambda_j - \left( \varepsilon \frac{\Lambda_j}{\lambda_j} \right)^{-1}| = \begin{cases} O \left( \frac{1}{\lambda_j^N} \right), & \text{if } N = 5, \\
O \left( \frac{1}{\lambda_j^{N-2}} \right), & \text{if } N \geq 6. \quad (2.19)
\end{cases}
\]

**Proof.** First, we define the following quadratic form
\[
Q(u, v) = -\int_{\partial B_\theta(x_{j,\varepsilon})} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} - \int_{\partial B_\theta(x_{j,\varepsilon})} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + \int_{\partial B_\theta(x_{j,\varepsilon})} \langle \nabla u, \nabla v \rangle \nu_i.
\]

Note that if \( u \) and \( v \) are harmonic in \( B_d(x_j, \varepsilon) \setminus \{ x_{j,\varepsilon} \} \), then \( Q(u, v) \) is independent of \( \theta \in (0, d) \). Letting \( \Omega' = B_\theta(x_{j,\varepsilon}) \) in (1.6) and using (2.17), we have
\[
\sum_{k=1}^k \sum_{m=1}^k \frac{Q(G(x_m, x), G(x_l, x))}{\lambda_j^{(N-2)/2}} = \begin{cases} O \left( \frac{1}{\lambda_j^N} \right), & \text{if } N = 5, \\
O \left( \frac{1}{\lambda_j^{N-2}} \right), & \text{if } N \geq 6. \quad (2.20)
\end{cases}
\]

Next we have the following estimate for which the proof is left in Section 5:
\[
Q(G(x_m, x), G(x_l, x)) = \begin{cases} \frac{-\partial R(x_{j,\varepsilon})}{\partial x_i}, & \text{for } l, m = j, \\
D_{x_l} G(x, x_{j,\varepsilon}), & \text{for } m \neq j, l = j, \\
D_{x_j} G(x, x_{j,\varepsilon}), & \text{for } m = j, l \neq j, \\
0, & \text{for } l, m \neq j. \quad (2.21)
\end{cases}
\]

Then (2.20) and (2.21) imply
\[
\frac{1}{2 \lambda_j^{N-2}} \frac{\partial R(x_{j,\varepsilon})}{\partial x_i} - \sum_{l=1, l 
eq j}^k \frac{1}{\lambda_j^{(N-2)/2}} \frac{\partial G(x_{j,\varepsilon}, x_{l,\varepsilon})}{\partial x_i} = \begin{cases} O \left( \frac{1}{\lambda_j^N} \right), & \text{if } N = 5, \\
O \left( \frac{1}{\lambda_j^{N-2}} \right), & \text{if } N \geq 6. \quad (2.22)
\end{cases}
\]
Let $\Lambda_{j,\varepsilon} := \left(\varepsilon \frac{\partial}{\partial \lambda_{j,\varepsilon}}\right)^{-1}$, we can rewrite (2.22) as follows:

$$\frac{\Lambda_{j,\varepsilon}^{N-2}}{2} \partial R(x_{j,\varepsilon}) - \sum_{i=1, i \neq j}^{k} \Lambda_{j,\varepsilon}^{(N-2)/2} \partial G(x_{j,\varepsilon}, x_{l,\varepsilon}) \frac{\partial}{\partial x_{i}} = \begin{cases} O\left(\frac{1}{\varepsilon}\right), & \text{if } N = 5, \\ O\left(\frac{1}{\lambda_{j,\varepsilon}}\right), & \text{if } N \geq 6. \end{cases}$$  \hspace{1cm} (2.23)

Then taking $\varepsilon \to 0$ in (2.23), we find (2.18). Moreover by the assumption that $(a^{k}, \Lambda^{k})$ is a nondegenerate critical point of $\Psi_{k}$, we get (2.19) from (2.10), (2.14) and (2.23).

**Proof of Theorem 1.1.** Theorem 1.1 can be deduced by (2.1), (2.2), (2.10), (2.11) and (2.18). \hfill \square

### 3. Regularization and blow-up analysis

To estimate the number of concentrated solutions to (1.1), we need first to obtain local uniqueness of such type of solutions. To this end, we need to estimate the difference between two solutions concentrating at the same points.

Let $u^{(1)}_{\varepsilon}(x)$, $u^{(2)}_{\varepsilon}(x)$ be two different solutions of (1.1) satisfying (1.4). Under the assumption that $M_{k}(a^{k})$ is a positive matrix, we find from Theorem 1.1 that $u^{(l)}_{\varepsilon}(x)$ can be written as

$$u^{(l)}_{\varepsilon}(x) = \sum_{j=1}^{k} PU_{x_{j,\varepsilon}, \lambda^{(l)}_{j,\varepsilon}} + w^{(l)}_{\varepsilon}(x),$$

satisfying, for $j = 1, \cdots, k$, $l = 1, 2$, $\lambda^{(l)}_{j,\varepsilon} = (u_{\varepsilon}(x^{(l)}_{j,\varepsilon}))^{-1}$,

$$x^{(l)}_{j,\varepsilon} \to a_{j}, \left(\varepsilon \frac{\partial}{\partial \lambda_{j,\varepsilon}}\right)^{-1} \to \lambda_{j}, \|w^{(l)}_{\varepsilon}\| = o(1) \text{ and } w^{(l)}_{\varepsilon} \in \bigcap_{j=1}^{k} E_{x_{j,\varepsilon}, \lambda^{(1)}_{j,\varepsilon}}.$$

Let $Q_{\varepsilon}$ be a quadratic form on $H_{0}^{1}(\Omega)$ given by

$$\langle Q_{\varepsilon}u, v \rangle = \langle u, v \rangle - \int_{\mathbb{R}^{N}} \left(2^{*-1} - 1\right) \left(\sum_{j=1}^{k} PU_{x_{j,\varepsilon}, \lambda^{(1)}_{j,\varepsilon}}\right)^{2^{*-2}} \| \varepsilon \right) uv, \forall u, v \in H_{0}^{1}(\Omega).$$

**Proposition 3.1.** For any $\varepsilon > 0$ sufficiently small, there exists a constant $\rho > 0$ such that

$$\langle Q_{\varepsilon}v, v \rangle \geq \rho \|v\|^{2}, \forall v \in \bigcap_{j=1}^{k} E_{x_{j,\varepsilon}, \lambda^{(1)}_{j,\varepsilon}}.$$

**Proof.** This is standard and can be found in Lemma 1.7 of [23]. Also one can refer to Proposition B.1 in [9] and Proposition 2.4.3 in [10]. \hfill \square

Now we define $\bar{\lambda}_{\varepsilon} := \min \left\{ \lambda^{(1)}_{1,\varepsilon}, \cdots, \lambda^{(1)}_{k,\varepsilon}, \lambda^{(2)}_{1,\varepsilon}, \cdots, \lambda^{(2)}_{k,\varepsilon} \right\}$.

**Proposition 3.2.** For $N \geq 6$, it holds

$$\|w^{(1)}_{\varepsilon} - w^{(2)}_{\varepsilon}\| = o\left(\frac{1}{\bar{\lambda}_{\varepsilon}^{(N+2)/2}}\right).$$  \hspace{1cm} (3.1)

**Proof.** First we define $\bar{w}_{\varepsilon} := w^{(1)}_{\varepsilon} - w^{(2)}_{\varepsilon}$, then

$$Q_{\varepsilon} \bar{w}_{\varepsilon} = R^{(1)}_{\varepsilon}(w^{(1)}_{\varepsilon}) - R^{(2)}_{\varepsilon}(w^{(2)}_{\varepsilon}) + l_{\varepsilon},$$

where

$$R^{(l)}_{\varepsilon}(w^{(l)}_{\varepsilon}) = \left(\sum_{i=1}^{k} PU_{y_{i,\varepsilon}, \lambda^{(l)}_{i,\varepsilon}} + w^{(l)}_{\varepsilon}\right)^{\frac{N+2}{N+4}} - \left(\sum_{i=1}^{k} PU_{y_{i,\varepsilon}, \lambda^{(l)}_{i,\varepsilon}}\right)^{\frac{N+2}{N+4}} - \frac{N+2}{N-2} \left(\sum_{i=1}^{k} PU_{y_{i,\varepsilon}, \lambda^{(l)}_{i,\varepsilon}}\right)^{\frac{N+2}{N-2}} w^{(l)}_{\varepsilon}. $$
and
\[ l_\varepsilon = (2^s - 1) \left[ \left( \sum_{j=1}^{k} PU_{x_j,\varepsilon,\lambda_j} \right)^{2^s - 2} - \left( \sum_{j=1}^{k} PU_{x_j,\varepsilon,\lambda_{j,\varepsilon}} \right)^{2^s - 2} \right] w_\varepsilon^{(2)}. \]

Now we write \( \varpi_\varepsilon = \varpi_{\varepsilon,1} + \varpi_{\varepsilon,2} \) with \( \varpi_{\varepsilon,1} \in \bigcap_{j=1}^{k} E_{x_j,\varepsilon,\lambda_j}^{(1)} \) and \( \varpi_{\varepsilon,2} \perp \bigcap_{j=1}^{k} E_{x_j,\varepsilon,\lambda_j}^{(1)} \). Then
\[ \varpi_{\varepsilon,1}(x) = \varpi_\varepsilon(x) - \sum_{i=1}^{k} \left( \alpha_{\varepsilon,i,0} \frac{\partial PU_{x_i,\varepsilon,\lambda_i}^{(1)}}{\partial \lambda} + \sum_{j=1}^{N} \alpha_{\varepsilon,i,j} \frac{\partial PU_{x_i,\varepsilon,\lambda_i}^{(1)}}{\partial x_j} \right), \]
for some constants \( \alpha_{\varepsilon,i,j} \) with \( i = 1, \ldots, k \) and \( j = 0, \ldots, N \). Then
\[
\alpha_{\varepsilon,i,0} \left\| \frac{\partial PU_{x_i,\varepsilon,\lambda_i}^{(1)}}{\partial \lambda} \right\|^2 = \left\langle \varpi_\varepsilon(x), \frac{\partial PU_{x_i,\varepsilon,\lambda_i}^{(1)}}{\partial \lambda} \right\rangle - \sum_{l=1, l \neq i}^{k} \alpha_{\varepsilon,l,0} \left\langle \frac{\partial PU_{x_i,\varepsilon,\lambda_i}^{(1)}}{\partial \lambda}, \frac{\partial PU_{x_j,\varepsilon,\lambda_j}^{(1)}}{\partial \lambda} \right\rangle \\
+ \sum_{l=1}^{k} \sum_{j=1}^{N} \alpha_{\varepsilon,i,j} \left\langle \frac{\partial PU_{x_i,\varepsilon,\lambda_i}^{(1)}}{\partial \lambda}, \frac{\partial PU_{x_j,\varepsilon,\lambda_j}^{(1)}}{\partial x_j} \right\rangle \\
= - \left\langle w_\varepsilon^{(2)}, \frac{\partial PU_{x_i,\varepsilon,\lambda_i}^{(1)}}{\partial \lambda} \right\rangle - \frac{\lambda_{\varepsilon,i}^{(1)}}{\lambda_{\varepsilon}^2} \left\| w_\varepsilon^{(2)} \right\| + o\left( \sum_{l=1, l \neq i}^{k} \frac{\alpha_{\varepsilon,l,0}}{\lambda_{\varepsilon}^2} + \sum_{l=1}^{k} \sum_{j=1}^{N} \alpha_{\varepsilon,l,j} \right) \\
= O\left( \frac{\left| x_{\varepsilon,i}^{(1)} - x_{\varepsilon,i}^{(2)} \right|}{\lambda_{\varepsilon}^2} \right) + \frac{\lambda_{\varepsilon,i}^{(1)} - \lambda_{\varepsilon,i}^{(2)}}{\lambda_{\varepsilon}^2} \left\| w_\varepsilon^{(2)} \right\| + o\left( \sum_{l=1, l \neq i}^{k} \frac{\alpha_{\varepsilon,l,0}}{\lambda_{\varepsilon}^2} + \sum_{l=1}^{k} \sum_{j=1}^{N} \alpha_{\varepsilon,l,j} \right). \]

Also from (2.19), we know \( \left| x_{\varepsilon,i}^{(1)} - x_{\varepsilon,i}^{(2)} \right| = O\left( \frac{1}{\lambda_{\varepsilon}^2} \right) \) and \( \left| \lambda_{\varepsilon,i}^{(1)} - \lambda_{\varepsilon,i}^{(2)} \right| = O(1) \) for \( N \geq 6 \). Then
\[
\alpha_{\varepsilon,i,0} = O\left( \left\| w_\varepsilon^{(2)} \right\| \right) + o\left( \sum_{l=1, l \neq i}^{k} \frac{\alpha_{\varepsilon,i,0}}{\lambda_{\varepsilon}^2} + \sum_{l=1}^{k} \sum_{j=1}^{N} \lambda_{\varepsilon}^2 |\alpha_{\varepsilon,l,j}| \right). \tag{3.2} \]

Similarly, we find
\[
\lambda_{\varepsilon}^2 |\alpha_{\varepsilon,i,j}| = O\left( \left\| w_\varepsilon^{(2)} \right\| \right) + o\left( \sum_{l=1}^{k} \alpha_{\varepsilon,i,0} + \sum_{l=1}^{k} \sum_{m=1, m \neq j}^{N} \lambda_{\varepsilon}^2 |\alpha_{\varepsilon,i,m}| \right). \tag{3.3} \]

Hence it follows from (3.2) and (3.3) that
\[
\sum_{i=1}^{k} \alpha_{\varepsilon,i,0} + \sum_{i=1}^{k} \sum_{j=1}^{N} \lambda_{\varepsilon}^2 |\alpha_{\varepsilon,i,j}| = O\left( \left\| w_\varepsilon^{(2)} \right\| \right). \tag{3.4} \]
Then by the definition of $Q_\varepsilon$ and the fact $\bar{w}_{\varepsilon,1} \in \bigcap_{j=1}^{k} E_{x_j,\lambda_j,\varepsilon}$, we have

$$
\langle Q_\varepsilon \left( \alpha_{\varepsilon,0} \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial \lambda} + \sum_{j=1}^{N} \alpha_{\varepsilon,i,j} \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial x_j} \right), \bar{w}_{\varepsilon,1} \rangle
$$

$$= O \left( \int_{\Omega} \left| W(x) \right| \cdot \left| \alpha_{\varepsilon,0} \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial \lambda} + \sum_{j=1}^{N} \alpha_{\varepsilon,i,j} \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial x_j} \right| \cdot \left| \bar{w}_{\varepsilon,1} \right| \right)
$$

$$+ O \left( \varepsilon \int_{\Omega} \left| \alpha_{\varepsilon,i,0} \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial \lambda} + \sum_{j=1}^{N} \alpha_{\varepsilon,i,j} \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial x_j} \right| \cdot \left| \bar{w}_{\varepsilon,1} \right| \right),
$$

(3.5)

where $W(x) := \left( \sum_{i=1}^{k} PU_{x_{i,e},\lambda_{i,e},\varepsilon} \right)^{2^{*}-2} - \left( \sum_{i=1}^{k} PU_{x_{i,e},\lambda_{i,e},\varepsilon}^{(2)(2)} \right)^{2^{*}-2}$. Also we compute

$$
\int_{\Omega} \left| W(x) \right| \cdot \left| \alpha_{\varepsilon,0} \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial \lambda} \right| \cdot \left| \bar{w}_{\varepsilon,1} \right|
$$

$$= \int_{\Omega} \sum_{i=1}^{k} \left( PU_{x_{i,e},\lambda_{i,e},\varepsilon}^{(1)} - PU_{x_{i,e},\lambda_{i,e},\varepsilon}^{(2)} \right)^{2^{*}-2} \cdot \left| \alpha_{\varepsilon,0} \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial \lambda} \right| \cdot \left| \bar{w}_{\varepsilon,1} \right|
$$

$$= O \left( \sum_{i=1}^{k} \left| \alpha_{\varepsilon,0} \right| \cdot \left( \lambda_{i,e}^{(1)} - \lambda_{i,e}^{(2)} \right)^{2^{*}-2} \cdot \int_{\Omega} \left| \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial \lambda} \right| \cdot \left| \bar{w}_{\varepsilon,1} \right| \right)
$$

$$+ O \left( \sum_{i=1}^{k} \left| \alpha_{\varepsilon,0} \right| \cdot \left( \lambda_{i,e}^{(1)} - \lambda_{i,e}^{(2)} \right)^{2^{*}-2} \cdot \int_{\Omega} \left| \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial \lambda} \right| \cdot \left| \bar{w}_{\varepsilon,1} \right| \right).
$$

(3.6)

By Hölder’s inequality, we have

$$
\int_{\Omega} \left| \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial \lambda} \right|^{2^{*-1}} \cdot \left| \bar{w}_{\varepsilon,1} \right| = O \left( \frac{1}{\lambda_{i,e}^{(1)}} \cdot \left\| PU_{x_j,\lambda_j,\varepsilon}^{(1)} \right\|_{2^{*}-1} \cdot \left\| \bar{w}_{\varepsilon,1} \right\| \right) = O \left( \frac{1}{\lambda_{i,e}^{(1)}} \cdot \left\| \bar{w}_{\varepsilon,1} \right\| \right),
$$

(3.7)

and

$$
\int_{\Omega} \left| \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial \lambda} \right| \cdot \left| \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial x_j} \right|^{2^{*-2}} \cdot \left| \bar{w}_{\varepsilon,1} \right| = O \left( \frac{1}{\lambda_{i,e}^{(1)}} \cdot \left\| PU_{x_j,\lambda_j,\varepsilon}^{(1)} \right\|_{2^{*-1}} \cdot \left\| \bar{w}_{\varepsilon,1} \right\| \right) = O \left( \frac{1}{\lambda_{i,e}^{(1)}} \cdot \left\| \bar{w}_{\varepsilon,1} \right\| \right).
$$

(3.8)

Hence using (2.19), (3.4) and (3.6)–(3.8), we can deduce

$$
\int_{\Omega} \left| W(x) \right| \cdot \left| \alpha_{\varepsilon,0} \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial \lambda} \right| \cdot \left| \bar{w}_{\varepsilon,1} \right| = O \left( \frac{1}{\lambda_{i,e}^{(1)}} \cdot \left\| u_{\varepsilon}^{(2)} \right\| \cdot \left\| \bar{w}_{\varepsilon,1} \right\| \right).
$$

(3.9)

And similar to the estimate of (3.9), we can also find

$$
\int_{\Omega} \left| W(x) \right| \cdot \left| \sum_{j=1}^{N} \alpha_{\varepsilon,i,j} \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial x_j} \right| \cdot \left| \bar{w}_{\varepsilon,1} \right| = O \left( \frac{1}{\lambda_{i,e}^{(1)}} \cdot \left\| u_{\varepsilon}^{(2)} \right\| \cdot \left\| \bar{w}_{\varepsilon,1} \right\| \right).
$$

(3.10)

Next using (3.4) and Hölder’s inequality, we have

$$
\varepsilon \int_{\Omega} \left| \alpha_{\varepsilon,0} \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial \lambda} + \sum_{j=1}^{N} \alpha_{\varepsilon,i,j} \frac{\partial PU_{x_j,\lambda_j,\varepsilon}}{\partial x_j} \right| \cdot \left| \bar{w}_{\varepsilon,1} \right| = O \left( \varepsilon \cdot \left\| u_{\varepsilon}^{(2)} \right\| \cdot \left\| \bar{w}_{\varepsilon,1} \right\| \right).
$$

(3.11)
Hence from (3.5) and (3.9)–(3.11), we find
\[
\left\langle Q_\varepsilon \left( \alpha_{\varepsilon,0} \frac{\partial PU_{x_i,\lambda_{x_i}^{(1)}}}{\partial \lambda} + \sum_{j=1}^N \alpha_{\varepsilon,i,j} \frac{\partial PU_{x_i,\lambda_{x_i}^{(1)}}}{\partial x_j} \right), \overline{w}_{\varepsilon,1} \right\rangle = O \left( \frac{1}{\lambda_{x_i}^{(1)^2}} \| w_{\varepsilon,1}^{(2)} \| \right). \tag{3.12}
\]
From \( \overline{w}_{\varepsilon,1} \in \bigcap_{j=1}^k E_{\varepsilon,j,\lambda_{\varepsilon,j}^{(1)}} \) and (3.12), we obtain
\[
\left\langle Q_\varepsilon \overline{w}_{\varepsilon,1}, \overline{w}_{\varepsilon,1} \right\rangle = \left\langle Q_\varepsilon \overline{w}_{\varepsilon,1}, \overline{w}_{\varepsilon,1} \right\rangle + \left\langle Q_\varepsilon \left( \alpha_{\varepsilon,0} \frac{\partial PU_{x_i,\lambda_{x_i}^{(1)}}}{\partial \lambda} + \sum_{j=1}^N \alpha_{\varepsilon,i,j} \frac{\partial PU_{x_i,\lambda_{x_i}^{(1)}}}{\partial x_j} \right), \overline{w}_{\varepsilon,1} \right\rangle
\geq \rho \| \overline{w}_{\varepsilon,1} \|^2 - \frac{C}{\lambda_{x_i}^{(1)^2}} \| w_{\varepsilon,1}^{(2)} \| \cdot \| \overline{w}_{\varepsilon,1} \|. \tag{3.13}
\]
Also, we have
\[
\left\langle R_\varepsilon^{(1)}(w_{\varepsilon}^{(1)}) - R_\varepsilon^{(2)}(w_{\varepsilon}^{(2)}), \overline{w}_{\varepsilon,1} \right\rangle = O \left( \left\| \sum_{l=1}^2 R_\varepsilon^{(l)}(w_{\varepsilon}^{(l)}) \right\| \cdot \| \overline{w}_{\varepsilon,1} \| \right) = O \left( \sum_{l=1}^2 \| w_{\varepsilon}^{(l)} \|^{\frac{n+2}{2}} \cdot \| \overline{w}_{\varepsilon,1} \| \right). \tag{3.14}
\]
And
\[
\left\langle l_\varepsilon, \overline{w}_{\varepsilon,1} \right\rangle = O \left( \lambda_{\varepsilon} \sum_{i=1}^k \| x_i^{(1)} - x_i^{(2)} \| + \sum_{i=1}^k \| \lambda_{i,\varepsilon}^{(1)} - \lambda_{i,\varepsilon}^{(2)} \| \right) \| w_{\varepsilon}^{(2)} \| \cdot \| \overline{w}_{\varepsilon,1} \|. \tag{3.15}
\]
Combining (2.19) and (3.13)–(3.15), we obtain
\[
\| \overline{w}_{\varepsilon,1} \| = O \left( \sum_{l=1}^2 \| w_{\varepsilon}^{(l)} \|^{\frac{n+2}{2}} \right) + O \left( \frac{\| w_{\varepsilon}^{(2)} \|}{\lambda_{\varepsilon}} \right). \tag{3.16}
\]
Then from (3.4) and (3.16), we see
\[
\| \overline{w}_{\varepsilon} \| = O \left( \| \overline{w}_{\varepsilon,1} \| + \sum_{i=1}^k \left( \alpha_{\varepsilon,0} \frac{\partial PU_{x_i,\lambda_{x_i}^{(1)}}}{\partial \lambda} + \sum_{j=1}^N \alpha_{\varepsilon,i,j} \frac{\partial PU_{x_i,\lambda_{x_i}^{(1)}}}{\partial x_j} \right) \right)
\leq O \left( \sum_{l=1}^2 \| w_{\varepsilon}^{(l)} \|^{\frac{n+2}{2}} \right) + O \left( \frac{1}{\lambda_{\varepsilon}} \left( \sum_{i=1}^k \| \alpha_{\varepsilon,i,0} \| + \sum_{i=1}^k \sum_{j=1}^N \| \lambda_{\varepsilon,i,j} \| \right) \right)
\leq O \left( \sum_{l=1}^2 \| w_{\varepsilon}^{(l)} \|^{\frac{n+2}{2}} \right) + O \left( \frac{\| w_{\varepsilon}^{(2)} \|}{\lambda_{\varepsilon}} \right),
\]
which and (A.3) give (3.1). \qed

**Proposition 3.3.** For \( N \geq 6 \) and \( j = 1, \cdots, k \), it holds
\[
|x_j^{(1)} - x_j^{(2)}| = o \left( \frac{1}{\lambda_{x_i}^{(1)^2}} \right) \quad \text{and} \quad |\lambda_{x_i}^{(1)} - \lambda_{x_i}^{(2)}| = o \left( \frac{1}{\lambda_{x_i}^{(1)^2}} \right). \tag{3.17}
\]

**Proof.** First for \( N \geq 6 \), from (2.19), we know \( |x_j^{(1)} - x_j^{(2)}| = O \left( \frac{1}{\lambda_{x_i}^{(1)^2}} \right) \). Also by direct calculations, we find
\[
PU_{x_i,\lambda_{x_i}^{(1)}}(y) = G(x_j^{(2)}, y) + O \left( \frac{1}{\lambda_{x_i}^{(1)^2}} \right), \quad \text{in} \quad \Omega \setminus B_\theta(x_j^{(1)}).
\]
Now we define the following quadratic form
\[
Q_1(u, v) = -\int_{\partial B_\theta(x_j^{(1)})} \frac{\partial v}{\partial \nu^{(1)}} \frac{\partial u}{\partial \nu^{(1)}} + \int_{\partial B_\theta(x_j^{(1)})} \frac{\partial u}{\partial \nu^{(1)}} \frac{\partial v}{\partial \nu^{(1)}} + \int_{\partial B_\theta(x_j^{(1)})} \langle \nabla u, \nabla v \rangle u^{(1)}.\]
where \( \nu^{(1)}(x) = (\nu^{(1)}_1(x), \ldots, \nu^{(1)}_N(x)) \) is the outward unit normal of \( \partial B_d(x_j^{(1)}) \). Note that if \( u \) and \( v \) are harmonic in \( B_d(x_j^{(1)}) \setminus \{x_j^{(1)}\} \), then \( Q_1(u, v) \) is independent of \( \theta \in (0, d] \). Let \( u_{\epsilon} = u_{\epsilon}^{(q)} \) with \( q = 1, 2 \) and \( \Omega' = \partial B_d(x_j^{(1)}) \) in (1.6). Then from (2.3), we have

\[
\sum_{l=1}^{k} \sum_{m=1}^{k} \frac{1}{(\lambda_{m,\epsilon}^{(l)})^{(N-2)/2}(\lambda^{(1)}_{l,\epsilon})^{(N-2)/2}} \left[ (\sum_{j=1}^{k} PU_{x_j^{(1)}, \lambda^{(1)}_{l,\epsilon}}) - (\sum_{j=1}^{k} PU_{x_j^{(1)}, \lambda^{(1)}_{l,\epsilon}}) \right] + o\left( \frac{1}{\lambda^{(1)}_{l,\epsilon}} \right) = o\left( \frac{1}{\lambda^{(1)}_{l,\epsilon}} \right)
\]

Let \( \Lambda^{(q)}_{j,\epsilon} := (\frac{\lambda^{(q)}_{j,\epsilon}}{\lambda^{(q)}_{j,\epsilon}^{(N-2)/2}})^{-1} \) with \( q = 1, 2 \). Then similar to (2.23), we find

\[
\nabla_x \Psi_k(x, \lambda) \big|_{(x, \lambda) = (x_j^{(1)}, \Lambda^{(1)}_{j,\epsilon})} - \nabla_x \Psi_k(x, \lambda) \big|_{(x, \lambda) = (x_j^{(1)}, \Lambda^{(2)}_{j,\epsilon})} = o\left( \frac{1}{\lambda^{(2)}_{j,\epsilon}} \right), \quad (3.18)
\]

where \( x_j^{(1)} = (x_j^{(1)}, \ldots, x_j^{(N)}) \) and \( \Lambda^{(2)}_{j,\epsilon} = (\Lambda^{(1)}_{1,\epsilon}, \ldots, \Lambda^{(1)}_{k,\epsilon}) \). And similar to the estimate of (3.18), we conclude

\[
\nabla_x \Psi_k(x, \lambda) \big|_{(x, \lambda) = (x_j^{(1)}, \Lambda^{(1)}_{j,\epsilon})} - \nabla_x \Psi_k(x, \lambda) \big|_{(x, \lambda) = (x_j^{(1)}, \Lambda^{(2)}_{j,\epsilon})} = o\left( \frac{1}{\lambda^{(2)}_{j,\epsilon}} \right), \quad (3.19)
\]

Then (3.17) follows by (3.18) and (3.19).

Now we set

\[
\xi_{\epsilon}(x) = \frac{u_{\epsilon}^{(1)}(x) - u_{\epsilon}^{(2)}(x)}{\|u_{\epsilon}^{(1)} - u_{\epsilon}^{(2)}\|_{L^\infty(\Omega)}}
\]

then \( \xi_{\epsilon}(x) \) satisfies \( \|\xi_{\epsilon}\|_{L^\infty(\Omega)} = 1 \) and

\[
-\Delta \xi_{\epsilon}(x) = C_{\epsilon}(x) \xi_{\epsilon}(x) + \epsilon \xi_{\epsilon}(x), \quad (3.21)
\]

where

\[
C_{\epsilon}(x) = \left( \frac{N + 2}{N - 2} \right) \int_0^1 \left( tu_{\epsilon}^{(1)}(x) + (1 - t)u_{\epsilon}^{(2)}(x) \right) \frac{\lambda}{t} \, dt.
\]

**Lemma 3.4.** For any constant \( 0 < \sigma \leq N - 2 \), there is a constant \( C > 0 \), such that

\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \frac{1}{(1 + |z|)^{2+\sigma}} \, dz \leq \begin{cases} C(1 + |y|)^{-\sigma}, & \sigma < N - 2, \\ C \ln |y|(1 + |y|)^{-\sigma}, & \sigma = N - 2. \end{cases} \quad (3.22)
\]

**Proof.** See Lemma B.2 in [27].

**Proposition 3.5.** For \( \xi_{\epsilon}(x) \) defined by (3.20), we have

\[
\int_{\Omega} \xi_{\epsilon}(x) \, dx = O\left( \frac{\ln \lambda_{\epsilon}}{\lambda_{\epsilon}^{N-2}} \right) \quad \text{and} \quad \xi_{\epsilon}(x) = O\left( \frac{\ln \lambda_{\epsilon}}{\lambda_{\epsilon}^{N-2}} \right), \quad \text{in} \ \Omega \setminus \bigcup_{j=1}^{k} B_d(x_j^{(1)}).
\]

where \( d > 0 \) is any small fixed constant.
Proof. By the potential theory, (3.21) and (3.22), we have
\[
\xi_\varepsilon(x) = \int_\Omega G(y, x) (C_\varepsilon(y) + \varepsilon) \xi_\varepsilon(y) dy
\]
\[
= O\left( \sum_{j=1}^{k} \sum_{l=1}^{2} \int_\Omega \frac{1}{|x-y|^{N-2}} U_{x_{j,\varepsilon}^{(l)}}^{N-2}(y) dy \right) + O(\varepsilon)
\]
\[
= O\left( \sum_{j=1}^{k} \sum_{l=1}^{2} \frac{1}{(1 + \lambda_{j,\varepsilon}(l)|x - x_{j,\varepsilon}(l)|^2) + O(\varepsilon).}
\]

Next repeating the above process, we know
\[
\xi_\varepsilon(x) = O\left( \int_\Omega \frac{1}{|x-y|^{N-2}} (C_\varepsilon(y) + \varepsilon) \left( \sum_{j=1}^{k} \sum_{l=1}^{2} \frac{1}{(1 + \lambda_{j,\varepsilon}(l)|x - x_{j,\varepsilon}(l)|^2) + O(\varepsilon) \right) dy \right)
\]
\[
= O\left( \sum_{j=1}^{k} \sum_{l=1}^{2} \frac{1}{(1 + \lambda_{j,\varepsilon}(l)|x - x_{j,\varepsilon}(l)|^2) + O(\varepsilon^2).}
\]

Then we can proceed as in the above argument for finite number of times to prove
\[
\xi_\varepsilon(x) = O\left( \sum_{j=1}^{k} \sum_{l=1}^{2} \frac{\ln \lambda_{j}}{(1 + \lambda_{j,\varepsilon}(l)|x - x_{j,\varepsilon}(l)|^{N-2}) + O(\varepsilon^2).}
\]

Hence (3.23) can be deduced by (3.25).

Proposition 3.6. For \( N \geq 6 \) and \( j = 1, \cdots, k \), let \( \xi_{\varepsilon,j}(x) = \xi_{\varepsilon}(\overrightarrow{x_{j,\varepsilon}} + x_{j,\varepsilon}^{(1)}) \). Then by taking a subsequence if necessary, we have
\[
\left| \xi_{\varepsilon,j}(x) - \sum_{i=0}^{N} c_{j,i}\psi_i(x) \right| = o\left( \frac{1}{\lambda_{j}} \right), \text{ uniformly in } C^1(B_R(0)) \text{ for any } R > 0,
\]
where \( c_{j,i}, i = 0, 1, \cdots, N \) are some constants and
\[
\psi_0(x) = \frac{\partial U_{0,\lambda}(x)}{\partial \lambda} |_{\lambda = 1}, \quad \psi_i(x) = \frac{\partial U_{0,1}(x)}{\partial x_i}, \quad i = 1, \cdots, N.
\]

Proof. Since \( \xi_{\varepsilon,j}(x) \) is bounded, by the regularity theory in [15], we find
\[
\xi_{\varepsilon,j}(x) \in C^{1,\alpha}(B_r(0)) \text{ and } \|\xi_{\varepsilon,j}\|_{C^{1,\alpha}(B_r(0))} \leq C,
\]
for any fixed large \( r \) and \( \alpha \in (0,1) \) if \( \varepsilon \) is small, where the constants \( r \) and \( C \) are independent of \( \varepsilon \) and \( j \). So we may assume that \( \xi_{\varepsilon,j}(x) \rightarrow \xi_j(x) \) in \( C(B_r(0)) \). By direct calculations, we know
\[
-\Delta \xi_{\varepsilon,j}(x) = -\frac{1}{(\lambda_{j,\varepsilon}^{(1)})^2} \Delta \xi_{\varepsilon}(\frac{x}{\lambda_{j,\varepsilon}} + x_{j,\varepsilon}^{(1)}) = \frac{1}{(\lambda_{j,\varepsilon}^{(1)})^2} C_{\varepsilon} \frac{x}{\lambda_{j,\varepsilon}} + \frac{\varepsilon}{\lambda_{j,\varepsilon}^{(1)}} \xi_{\varepsilon,j}(x).
\]

Now, we estimate \( C_{\varepsilon}(x + x_{j,\varepsilon}^{(1)}) \). By (2.19), we have
\[
U_{x_{j,\varepsilon}^{(1)}}(x) - U_{x_{j,\varepsilon}^{(2)}}(x) = O\left( |x_{j,\varepsilon}^{(1)} - x_{j,\varepsilon}^{(2)}| \cdot (\nabla_y U_{y,\lambda_{j,\varepsilon}^{(1)}(x)}{y = x_{j,\varepsilon}^{(1)}}) + |\lambda_{j,\varepsilon}^{(1)} - \lambda_{j,\varepsilon}^{(2)}| \cdot (\nabla_\lambda U_{x_{j,\varepsilon}^{(1)}}{\lambda = \lambda_{j,\varepsilon}^{(1)}}) \right)
\]
\[
= O\left( \left( |x_{j,\varepsilon}^{(1)} - x_{j,\varepsilon}^{(2)}| + (\lambda_{j,\varepsilon}^{(1)})^{-1} \lambda_{j,\varepsilon}^{(2)} \right) \right) U_{x_{j,\varepsilon}^{(1)}}(x) = o\left( \frac{1}{\lambda_{j,\varepsilon}} \right) U_{x_{j,\varepsilon}^{(1)}}(x),
\]
which means
\[ u^{(1)}(x) - u^{(2)}(x) = o\left(\frac{1}{\lambda \varepsilon}\right) \left( \sum_{j=1}^{k} U_{x_{j, \varepsilon}, \lambda_{j, \varepsilon}}(x) \right) + O\left( \frac{2}{|x|} \right). \] (3.28)

Then for a small fixed $d$ and $x \in B_d(x_{j, \varepsilon}^{(1)})$, by (A.1) and (A.2), we find
\[ C_{\varepsilon}(x) = \left( \frac{N + 2}{N - 2} \right) \int_{\mathbb{R}^N} \Phi(x) - \frac{l}{2} \int_{\mathbb{R}^N} \Phi(x) dx. \] (3.29)

Next, for any given $\Phi(x) \in C_0^\infty(\mathbb{R}^N)$ with $supp \Phi(x) \subset B_{\lambda_{j, \varepsilon}}(x^{(1)}_{j, \varepsilon})$, we have
\[ \frac{1}{(\lambda_{j, \varepsilon})^2} \int_{B_{\lambda_{j, \varepsilon}}(x^{(1)}_{j, \varepsilon})} C_{\varepsilon}(x) \frac{x - x^{(1)}_{j, \varepsilon}}{\lambda_{j, \varepsilon}} \Phi(x) dx = \left( \frac{N + 2}{N - 2} \right) \int_{\mathbb{R}^N} \Phi(x) dx. \] (3.30)

Also from the fact that $\|\xi_\varepsilon\|_{L^\infty(\Omega)} = 1$, we know
\[ \frac{\varepsilon}{(\lambda_{j, \varepsilon})^2} \int_{B_{\lambda_{j, \varepsilon}}(x^{(1)}_{j, \varepsilon})} \xi_{\varepsilon, j}(x) \Phi(x) dx = o\left( \frac{1}{\lambda \varepsilon} \right). \] (3.31)

Then (3.27), (3.30) and (3.31) imply
\[ - \int_{B_{\lambda_{j, \varepsilon}}(x^{(1)}_{j, \varepsilon})} \Delta \xi_{\varepsilon, j}(x) \Phi(x) dx = \left( \frac{N + 2}{N - 2} \right) \int_{\mathbb{R}^N} \Phi(x) dx. \] (3.32)

Letting $\varepsilon \to 0$ in (3.32) and using the elliptic regularity theory, we find that $\xi_j(x)$ satisfies
\[ - \Delta \xi_j(x) = \left( \frac{N + 2}{N - 2} \right) U_{0,1}(x) \xi_j(x), \] in $\mathbb{R}^N$, (3.33)

which gives $\xi_j(x) = \sum_{i=0}^{N} c_{j,i} \psi_i(x)$. Moreover combining (3.32) and (3.33), we find (3.26).

\[ \square \]

**Proposition 3.7.** Let $\xi_\varepsilon(x)$ be defined as in (3.20). Then it holds
\[ \xi_\varepsilon(x) = \sum_{j=1}^{k} A_{\varepsilon, j} G(x_{j, \varepsilon}^{(1)}, x) + \sum_{j=1}^{k} \sum_{i=1}^{N} B_{\varepsilon, j, i} \partial_i G(x_{j, \varepsilon}^{(1)}, x) \]
\[ + \begin{cases} O\left(\frac{\lambda}{\lambda \varepsilon}^2\right), & N = 5, \\ O\left(\frac{\lambda}{\lambda \varepsilon}^2\right), & N \geq 6, \end{cases} \] in $C^1(\Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x_{j, \varepsilon}))$. (3.34)

where $d > 0$ is any small fixed constant, and $\partial_i G(y, x) = \frac{\partial G(y, x)}{\partial y_i}$,
\[ A_{\varepsilon, j} = \int_{B_{d}(x_{j, \varepsilon}^{(1)})} C_{\varepsilon}(x) \xi_\varepsilon(x) dx \quad \text{and} \quad B_{\varepsilon, j, i} = \int_{B_{d}(x_{j, \varepsilon}^{(1)})} (x_i - x_{j, \varepsilon}^{(1)}) C_{\varepsilon}(x) \xi_\varepsilon(x) dx. \] (3.35)

**Proof.** By the potential theory and (3.21), we have
\[ \xi_\varepsilon(x) = \int_{\Omega} G(y, x) \xi_\varepsilon(y) dy + \varepsilon \int_{\Omega} G(y, x) \xi_\varepsilon(y) dy. \] (3.36)
Next for $x \in \Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x_{j,\varepsilon}^{(1)})$, by (3.23) we find

$$
\int_{\Omega} G(y, x) \xi_{e}(y)dy = \sum_{j=1}^{k} \int_{B_{d}(x_{j,\varepsilon}^{(1)})} G(y, x) \xi_{e}(y)dy + \int_{\Omega \setminus \bigcup_{j=1}^{k} B_{d}(x_{j,\varepsilon}^{(1)})} G(y, x) \xi_{e}(y)dy \\
= \sum_{j=1}^{k} \int_{B_{d}(x_{j,\varepsilon}^{(1)})} \xi_{e}(y)dy + O\left(\frac{\ln \lambda_{e}}{\lambda_{e}}\right) \int_{\Omega} G(y, x)dy = O\left(\frac{\ln \lambda_{e}}{\lambda_{e}^{N-2}}\right).
$$

(3.37)

Also using (3.23) and (3.26), we can get

$$
\int_{\Omega} G(y, x) C_{e}(y) \xi_{e}(y)dy \\
= \sum_{j=1}^{k} \int_{B_{d}(x_{j,\varepsilon}^{(1)})} G(y, x) C_{e}(y) \xi_{e}(y)dy + \int_{\Omega \setminus \bigcup_{j=1}^{k} B_{d}(x_{j,\varepsilon}^{(1)})} G(y, x) C_{e}(y) \xi_{e}(y)dy \\
= \sum_{j=1}^{k} A_{e,j} G(x_{j,\varepsilon}^{(1)}, x) + \sum_{j=1}^{k} \sum_{i=1}^{N} B_{e,j,i} \partial_{i} G(x_{j,\varepsilon}^{(1)}, x) \\
+ O\left(\sum_{j=1}^{k} \int_{B_{d}(x_{j,\varepsilon}^{(1)})} |y-x_{j,\varepsilon}^{(1)}|^{2} C_{e}(y) \xi_{e}(y)dy\right) + O\left(\frac{\ln \lambda_{e}}{\lambda_{e}^{N}}\right) \int_{\Omega \setminus \bigcup_{j=1}^{k} B_{d}(x_{j,\varepsilon}^{(1)})} G(y, x)dy \\
= \sum_{j=1}^{k} A_{e,j} G(x_{j,\varepsilon}^{(1)}, x) + \sum_{j=1}^{k} \sum_{i=1}^{N} B_{e,j,i} \partial_{i} G(x_{j,\varepsilon}^{(1)}, x) + O\left(\frac{\ln \lambda_{e}}{\lambda_{e}^{N}}\right),
$$

(3.38)

where $A_{e,j}$ and $B_{e,j,i}$ are defined in (3.35). Hence (2.10), (3.36), (3.37) and (3.38) imply

$$
\xi_{e}(x) = \sum_{j=1}^{k} A_{e,j} G(x_{j,\varepsilon}^{(1)}, x) + \sum_{j=1}^{k} \sum_{i=1}^{N} B_{e,j,i} \partial_{i} G(x_{j,\varepsilon}^{(1)}, x) + O\left(\frac{\ln \lambda_{e}}{\lambda_{e}^{N}} + \frac{\varepsilon \ln \lambda_{e}}{\lambda_{e}^{N-2}}\right) \\
= \sum_{j=1}^{k} A_{e,j} G(x_{j,\varepsilon}^{(1)}, x) + \sum_{j=1}^{k} \sum_{i=1}^{N} B_{e,j,i} \partial_{i} G(x_{j,\varepsilon}^{(1)}, x) \\
+ \begin{cases} 
O\left(\frac{\ln \lambda_{e}}{\lambda_{e}^{N}}\right), & N = 5, \\
O\left(\frac{\ln \lambda_{e}}{\lambda_{e}^{N}}\right), & N \geq 6,
\end{cases} \quad \text{for } x \in \Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x_{j,\varepsilon}^{(1)}).
$$

On the other hand, from (3.36), we know

$$
\frac{\partial \xi_{e}(x)}{\partial x_{i}} = \int_{\Omega} D_{x_{i}} G(y, x) C_{e}(y) \xi_{e}(y)dy + \varepsilon \int_{\Omega} D_{x_{i}} G(y, x) \xi_{e}(y)dy, \quad \text{for } i = 1, \ldots, N.
$$

Then similar to the above estimates of $\xi_{e}(x)$, we can complete the proof of (3.34). \hfill \Box

4. PROOFS OF THEOREM 1.2 AND THEOREM 1.3

In this section we prove Theorem 1.2 and Theorem 1.3. Theorem 1.2 will be verified by using an indirect argument to show $\xi_{e}(x) \equiv 0$ for $\varepsilon$ small. To do this we first show that $\xi_{e}(x)$ is small both near and away from the points at which $u_{e}^{(l)}(x)$ ($l = 1, 2$) concentrates. Therefore we need to obtain quantitative behaviors for $u_{e}^{(1)}(x)$, $u_{e}^{(2)}(x)$ and $\xi_{e}(x)$. 


Proposition 4.1. Let \( u^{(l)}_\varepsilon(x) \) with \( l = 1, 2 \) be the solutions of (1.1) satisfying (1.4). Then for small fixed \( d > 0 \), it holds

\[
u^{(l)}_\varepsilon(x) = A \left( \sum_{j=1}^{k} \frac{G(x^{(1)}_{j,\varepsilon}, x)}{(\lambda^{(1)}_{j,\varepsilon})^{(N-2)/2}} \right) + \begin{cases} O \left( \frac{1}{\lambda^{(1)}_{j,\varepsilon}} \right), & N = 5, \\ O \left( \frac{1}{\lambda^{(1)}_{j,\varepsilon} + \theta} \right), & N \geq 6, \end{cases} \Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x^{(1)}_{j,\varepsilon}), \quad (4.1)
\]

where \( A \) is the constant in (1.3).

Proof. First, (2.3) implies that (4.1) holds for \( l = 1 \) and

\[
u^{(2)}_\varepsilon(x) = A \left( \sum_{j=1}^{k} \frac{G(x^{(2)}_{j,\varepsilon}, x)}{(\lambda^{(2)}_{j,\varepsilon})^{(N-2)/2}} \right) + \begin{cases} O \left( \frac{1}{\lambda^{(1)}_{j,\varepsilon}} \right), & N = 5, \\ O \left( \frac{1}{\lambda^{(1)}_{j,\varepsilon} + \theta} \right), & N \geq 6, \end{cases} \Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x^{(2)}_{j,\varepsilon}), \quad (4.2)
\]

Also we calculate

\[
\frac{G(x^{(2)}_{j,\varepsilon}, x)}{(\lambda^{(2)}_{j,\varepsilon})^{(N-2)/2}} = \frac{G(x^{(1)}_{j,\varepsilon}, x)}{(\lambda^{(1)}_{j,\varepsilon})^{(N-2)/2}} + O \left( \frac{|x^{(1)}_{j,\varepsilon} - x_{j,\varepsilon}|}{\lambda^{(1)}_{j,\varepsilon}} \right) + O \left( \frac{|\lambda^{(1)}_{j,\varepsilon} - \lambda^{(2)}_{j,\varepsilon}|}{\lambda^{(1)}_{j,\varepsilon}} \right).
\]

Since \( B_d(x^{(1)}_{j,\varepsilon}) \subset B_{2d}(x^{(2)}_{j,\varepsilon}) \) for small \( \varepsilon \), we get (4.1) for \( l = 2 \) from (2.19) and (4.2).

Proposition 4.2. For \( \xi \) defined by (3.20), we have the following local Pohozaev identities:

\[
- \int_{\partial\Omega'} \frac{\partial \xi}{\partial \nu} \frac{\partial u^{(1)}_\varepsilon}{\partial x_i} - \int_{\partial\Omega'} \frac{\partial u^{(2)}_\varepsilon}{\partial \nu} \frac{\partial \xi}{\partial x_i} + \frac{1}{2} \int_{\partial\Omega'} \langle \nabla(u^{(1)}_\varepsilon + u^{(2)}_\varepsilon), \nabla\xi \rangle \nu_i = \frac{N - 2}{2N} \int_{\partial\Omega'} D_\varepsilon(x) \xi \nu_i \nu_i + \varepsilon \int_{\partial\Omega'} (u^{(1)}_\varepsilon + u^{(2)}_\varepsilon) \xi \nu_i \nu_i,
\]

and

\[
\frac{1}{2} \int_{\partial\Omega'} \langle \nabla(u^{(1)}_\varepsilon + u^{(2)}_\varepsilon), \nabla\xi \rangle \langle x - x^{(1)}_{j,\varepsilon}, \nu \rangle - \int_{\partial\Omega'} \frac{\partial \xi}{\partial \nu} \langle x - x^{(1)}_{j,\varepsilon}, \nabla u^{(1)}_\varepsilon \rangle - \int_{\partial\Omega'} \frac{\partial u^{(2)}_\varepsilon}{\partial \nu} \frac{\partial \xi}{\partial x_i} + \frac{2 - N}{2} \int_{\partial\Omega'} \frac{\partial u^{(2)}_\varepsilon}{\partial \nu} \frac{\partial \xi}{\partial x_i} \nabla \xi = \int_{\partial\Omega'} D_\varepsilon(x) \xi \langle x - x^{(1)}_{j,\varepsilon}, \nu \rangle + \varepsilon \int_{\partial\Omega'} (u^{(1)}_\varepsilon + u^{(2)}_\varepsilon) \xi \langle x - x^{(1)}_{j,\varepsilon}, \nu \rangle - \varepsilon \int_{\partial\Omega'} (u^{(1)}_\varepsilon + u^{(2)}_\varepsilon) \xi \langle x - x^{(1)}_{j,\varepsilon}, \nu \rangle - \varepsilon \int_{\partial\Omega'} (u^{(1)}_\varepsilon + u^{(2)}_\varepsilon) \xi \langle x - x^{(1)}_{j,\varepsilon}, \nu \rangle,
\]

where \( \Omega' \subset \Omega \) is a smooth domain, \( \nu(x) = (\nu_1(x), \cdots, \nu_N(x)) \) is the outward unit normal of \( \partial\Omega' \) and

\[
D_\varepsilon(x) = \int_0^1 \left( tu^{(1)}_\varepsilon(x) + (1 - t)u^{(2)}_\varepsilon(x) \right) \frac{dt}{\varepsilon}.
\]

Proof. Taking \( u_\varepsilon = u^{(l)}_\varepsilon \) with \( l = 1, 2 \) in (1.6) and (1.7), and then making a difference between those respectively, we can obtain (4.3) and (4.4).

Proposition 4.3. For \( N \geq 6 \), it holds

\[
c_{j,0} = 0, \quad \text{for} \quad j = 1, \cdots, k,
\]

where \( c_{j,0} \) are the constants in Proposition 3.6.

Proof. First, we define the following quadric form

\[
P_1(u, v) = -\theta \int_{\partial B_d(x^{(1)}_{j,\varepsilon})} \langle \nabla u, \nu \rangle \langle \nabla v, \nu \rangle + \frac{\theta}{2} \int_{\partial B_d(x^{(1)}_{j,\varepsilon})} \langle \nabla u, \nabla v \rangle + \frac{2 - N}{4} \int_{\partial B_d(x^{(1)}_{j,\varepsilon})} \langle \nabla u, v \rangle + \frac{2 - N}{4} \int_{\partial B_d(x^{(1)}_{j,\varepsilon})} \langle \nabla v, u \rangle.
\]
Note that if $u$ and $v$ are harmonic in $B_d(x_j^{(1)} \setminus \{x_j^{(1)}\})$, then $P_1(u, v)$ is independent of $\theta \in (0, d)$. So using (3.34) and (4.1), we get by taking $\Omega' = B_\theta(x_j^{(1)} \setminus \{x_j^{(1)}\})$ in (4.4), for $N \geq 6$,

$$
\text{LHS of (4.4)} = \sum_{l=1}^{k} \sum_{m=1}^{k} \sum_{h=1}^{N} \frac{2AA_{l,m}}{(\lambda_{l,m})^{(N-2)/2}} P_1(G(x_{m,l}, x), G(x_{l,m}, x)) + \sum_{l=1}^{k} \sum_{m=1}^{k} \frac{2AB_{l,m}}{(\lambda_{l,m})^{(N-2)/2}} P_1(G(x_{m,l}, x), \partial_h G(x_{l,m}, x)) + O\left(\frac{1}{\lambda_l^{(N-2)/2}}\right) \tag{4.6}
$$

Similar to (2.13), we have

$$
P_1(G(x_{m,l}, x), G(x_{l,m}, x)) = \begin{cases} 
- \frac{(N-2)R(x_{j,l})}{2}, & \text{for } l, m = j. \\
- \frac{(N-2)G(x_{j,l})x_{j,l}^2}{4}, & \text{for } m = j, l \neq j. \\
- \frac{(N-2)G(x_{j,l})x_{j,l}^2}{4}, & \text{for } m \neq j, l = j. \\
0, & \text{for } l, m \neq j.
\end{cases} \tag{4.7}
$$

Also we have the following estimates for which the proof is postponed until Section 5:

$$
P_1(G(x_{m,l}, x), \partial_h G(x_{l,m}, x)) = \begin{cases} 
\frac{(N-2) + \frac{1}{2(N-2)}d_{j,k}}{2}, & \text{for } l, m = j. \\
\frac{(N-2)}{4} \partial_h G(x_{j,l}, x_{j,l}), & \text{for } m = j, l \neq j. \\
\frac{(N-2)}{4} + \frac{1}{N} \partial_h G(x_{j,l}, x_{j,l}), & \text{for } m \neq j, l = j. \\
0, & \text{for } l, m \neq j.
\end{cases} \tag{4.8}
$$

On the other hand, from (3.23), (3.26), (3.29), (3.35) and (A.3), we deduce

$$
A_{l,m} = \frac{N + 2}{N - 2} \int_{B_\theta(x_{j,l})} \frac{\hat{u}^{(1)} - \hat{v}^{(1)}}{x_{j,l}^2 - x_{j,l}} \, dx + o\left(\frac{1}{\lambda_l^{(N-2)/2}}\right) = - \frac{(N-2)A_{l,m}}{2(\lambda_{l,m})^{(N-2)/2}} + o\left(\frac{1}{\lambda_l^{(N-2)/2}}\right). \tag{4.9}
$$

Now let $d_{j,k} = \frac{(N-2)A_{l,m}}{8(\lambda_{l,m})^{(N-2)/2}}$, for $j = 1, \ldots, k$. Then (4.6)–(4.9) imply

$$
\text{LHS of (4.4)} = 2(N-2)d_{j,k}\left(\frac{R(x_{j,k})}{(\lambda_{j,k})^{(N-2)/2}} - \sum_{l \neq k} \frac{G(x_{j,l}, x_{l,k})}{(\lambda_{j,l})^{(N-2)/2}} \right) + 2(N-2)\left(\frac{d_{j,k}G(x_{j,k})}{(\lambda_{j,k})^{(N-2)/2}} - \sum_{l \neq k} \frac{d_{l,k}G(x_{j,l}, x_{l,k})}{(\lambda_{j,l})^{(N-2)/2}} \right) + \frac{N-2}{2} \sum_{h=1}^{N} B_{l,m,h} \left(\frac{\partial_h R(x_{j,k})}{(\lambda_{j,k})^{(N-2)/2}} - \sum_{l \neq k} \frac{\partial_h G(x_{j,l}, x_{l,k})}{(\lambda_{j,l})^{(N-2)/2}} \right) - \frac{N-2}{2} \sum_{h=1}^{N} \sum_{l \neq k} B_{l,m,h} \left(\frac{\partial_h R(x_{j,k})}{(\lambda_{j,k})^{(N-2)/2}} - \sum_{l \neq k} \frac{\partial_h G(x_{j,l}, x_{l,k})}{(\lambda_{j,l})^{(N-2)/2}} \right) + \frac{(N-1)}{N} \sum_{h=1}^{N} B_{l,m,h} \left(\frac{\partial_h R(x_{j,k})}{(\lambda_{j,k})^{(N-2)/2}} - \sum_{l \neq k} \frac{\partial_h G(x_{j,l}, x_{l,k})}{(\lambda_{j,l})^{(N-2)/2}} \right). \tag{4.10}
$$
Also, from (2.14), (3.23), (3.28) and (A.3), we know

\[
\text{RHS of (4.4)} = -2\varepsilon \int_{B_\varepsilon} U(x^{(1)}_{j,\varepsilon})(x^{(1)}_{j,\varepsilon}) \xi_e + o\left(\frac{1}{\lambda_e^{(3N-4)/2}}\right)
\]
\[
= \frac{2B_{\varepsilon}\epsilon}{(\lambda_e^{(1)}(N+2)/2)} + o\left(\frac{1}{\lambda_e^{(3N-4)/2}}\right)
\]
\[
= 8d_{j,\varepsilon} \left(\frac{R(x^{(1)}_{j,\varepsilon})}{(\lambda_e^{(1)}(N-2)/2)} - \sum_{l \neq j}^k \frac{G(x^{(1)}_{j,\varepsilon}, x^{(1)}_{l,\varepsilon})}{(\lambda_e^{(1)}(N-2)/2)(\lambda_e^{(1)}(N-2)/2)}\right) + o\left(\frac{1}{\lambda_e^{(3N-4)/2}}\right).
\]

It follows from (2.22) that

\[
\sum_{h=1}^k B_{\varepsilon,j,h} \left(\frac{\partial h R(x^{(1)}_{j,\varepsilon})}{(\lambda_e^{(1)}(N-2)/2)} - \sum_{l \neq j}^k \frac{\partial h G(x^{(1)}_{j,\varepsilon}, x^{(1)}_{l,\varepsilon})}{(\lambda_e^{(1)}(N-2)/2)(\lambda_e^{(1)}(N-2)/2)}\right)
\]
\[
= O\left(\sum_{h=1}^k B_{\varepsilon,j,h} \left(\partial h \Psi_k(a^k, \lambda^k) + o(1)\right)\right) = o\left(\frac{1}{\lambda_e^{(3N-2)/2}}\right).
\]

Hence (4.6) and (4.10)–(4.12) imply, for \(j = 1, \ldots, k\),

\[
(N - 6)d_{j,\varepsilon} \left(\frac{R(x^{(1)}_{j,\varepsilon})}{(\lambda_e^{(1)}(N-2)/2)} - \sum_{l \neq j}^k \frac{G(x^{(1)}_{j,\varepsilon}, x^{(1)}_{l,\varepsilon})}{(\lambda_e^{(1)}(N-2)/2)(\lambda_e^{(1)}(N-2)/2)}\right)
\]
\[
+ (N - 2) \left(d_{j,\varepsilon} R(x^{(1)}_{j,\varepsilon}) - \sum_{l \neq j}^k \frac{d_{l,\varepsilon} G(x^{(1)}_{j,\varepsilon}, x^{(1)}_{l,\varepsilon})}{(\lambda_e^{(1)}(N-2)/2)(\lambda_e^{(1)}(N-2)/2)}\right)
\]
\[
+ \frac{N - 2}{4} \sum_{h=1}^k \sum_{l \neq j}^k B_{\varepsilon,j,h} \left(\frac{\partial h R(x^{(1)}_{j,\varepsilon})}{(\lambda_e^{(1)}(N-2)/2)} - \sum_{l \neq j}^k \frac{\partial h G(x^{(1)}_{j,\varepsilon}, x^{(1)}_{l,\varepsilon})}{(\lambda_e^{(1)}(N-2)/2)}\right)
\]
\[
- \frac{N - 2}{4} \sum_{h=1}^k \sum_{l \neq j}^k B_{\varepsilon,j,h} \left(\frac{\partial h \Psi_k(a^k, \lambda^k)}{(\lambda_e^{(1)}(N-2)/2)} - \sum_{l \neq j}^k \frac{\partial h G(x^{(1)}_{j,\varepsilon}, x^{(1)}_{l,\varepsilon})}{(\lambda_e^{(1)}(N-2)/2)}\right) = o\left(\frac{1}{\lambda_e^{(3N-2)/2}}\right).
\]

Let \(a^k = (a_1, \ldots, a_k) \in \Omega^k\) and \(a_j = (y_{(j-1)N+1}, y_{(j-1)N+2}, \ldots, y_{jN}) \in \Omega\). Since for any \(i \in \{1, \ldots, kN\}\), there exists some \(j \in \{1, \ldots, k\}\) satisfying \(i \in [(j - 1)N + 1, jN] \cap \mathbb{N}^+\). Then by direct calculation, we have

\[
\frac{\partial^2 \Psi_k(x, \lambda)}{\partial y_i \partial \lambda_m} = (N - 2)\left(\lambda_j^{s-3} \frac{\partial R(x)}{\partial y_i} - \sum_{l \neq j}^k \lambda_j^{\frac{s-2}{2}} \frac{\partial G(x, x_l)}{\partial y_i}\right), \text{ if } m \in [(j - 1)N + 1, jN] \cap \mathbb{N}^+,
\]

and

\[
\frac{\partial^2 \Psi_k(x, \lambda)}{\partial y_i \partial \lambda_m} = -(N - 2)\lambda_j^{s-3} \frac{\partial G(x, x_s)}{\partial y_i}, \text{ if } m \in [(s - 1)N + 1, sN] \cap \mathbb{N}^+ \text{ and } s \neq j.
\]

Now we rewrite (4.13) as follows:

\[
\overline{M}_{k,\varepsilon} \overline{D}_k + \frac{2}{\lambda_0^2} \overline{D}_h \left(\overline{D}_{\lambda,\varepsilon} \Psi_k(x, \lambda)\right)_{(x,\lambda) = (a^k, \lambda^k)} \overline{B}_{\varepsilon,k} = \left(a\left(\frac{1}{\lambda_0^{3N-1}}, \ldots, \frac{1}{\lambda_0^{3N-1}}\right)\right)^T,
\]

with the vector \(\overline{D}_k\), the matrix \(\overline{M}_{k,\varepsilon} = (a_{i,j,\varepsilon})_{1 \leq i, j \leq k}\) defined by

\[
\overline{D}_k = (c_{1,0}, \ldots, c_{k,0})^T, \quad \overline{D}_h = diag(\lambda_1^{1/2}, \ldots, \lambda_k^{1/2}), \text{ with } \lambda_j := \lim_{\varepsilon \to 0} (\varepsilon^{3N-4}/\lambda_{j,\varepsilon})^{-1},
\]

and

\[
\overline{B}_{\varepsilon,k} = (b_{1,0}^{1/2}, \ldots, b_{k,0}^{1/2}), \text{ with } b_{j,0} := \lim_{\varepsilon \to 0} (\varepsilon^{3N-4}/\lambda_{j,\varepsilon})^{-1}.
\]
Proposition 4.4. For $N \geq 6$, it holds

$$c_{j,k} = 0, \text{ for } j = 1, \ldots, k \text{ and } i = 1, \ldots, N,$$

where $c_{j,k}$ are the constants in Proposition 3.6.

Proof. First taking $\Omega' = B_{e}(x_{j,e})$ in (4.3), from (3.34) and (4.1), we have

\[
\text{LHS of (4.3)} = \sum_{l=1}^{k} \sum_{m=1}^{k} A \frac{A_{e,l} Q_{1}(G(x_{m,e}, x), G(x_{l,e}, x))}{\lambda_{m,e}^{(1)}(N-2)/2} + \sum_{l=1}^{k} \sum_{h=1}^{N} \sum_{m=1}^{k} A B_{e,l,h} Q_{1}(G(x_{m,e}, x), \partial_{h} G(x_{l,e}, x)) \frac{1}{\lambda_{m,e}^{(1)}(N-2)/2} + O\left(\frac{\ln \lambda_{e}}{\lambda_{e}^{(3N-2)/2}}\right),
\]

and

\[
\text{RHS of (4.3)} = O\left(\frac{1}{\lambda_{e}^{(3N-2)/2}}\right).
\]

Then from (4.16), (4.19) and (4.20), we find

\[
\sum_{l=1}^{k} \sum_{h=1}^{N} \sum_{m=1}^{k} B_{e,l,h} Q_{1}(G(x_{m,e}, x), \partial_{h} G(x_{l,e}, x)) \frac{1}{\lambda_{m,e}^{(1)}(N-2)/2} = O\left(\frac{1}{\lambda_{e}^{(3N-4)/2}}\right).
\]

Also we have the following estimates for which the proof is postponed until Section 5:

$$Q_{1}(G(x_{m,e}, x), \partial_{h} G(x_{l,e}, x)) = \begin{cases} -\partial_{x_{l},x_{h}}^{2} R(x_{l,e}^{(1)}), & \text{for } l, m = j, \\
\partial_{x_{l},x_{h}}^{2} G(x_{m,e}^{(1)}, x_{j,e}^{(1)}), & \text{for } m \neq j, l = j, \\
\partial_{x_{l},x_{h}}^{2} G(x_{l,e}^{(1)}, x_{j,e}^{(1)}), & \text{for } m = j, l \neq j, \\
0, & \text{for } l, m \neq j. \end{cases}$$

Then (4.21) and (4.22) imply

\[
\sum_{h=1}^{N} B_{e,l,h} \frac{\partial_{x_{l},x_{h}}^{2} R(x_{j,e}^{(1)})}{\lambda_{j,e}^{(1)}(N-2)/2} - \sum_{i \neq j} \partial_{x_{l},x_{h}}^{2} G(x_{j,e}^{(1)}, x_{i,e}^{(1)}) \frac{1}{\lambda_{j,e}^{(1)}(N-2)/2} - \sum_{h=1}^{N} B_{e,m,h} \partial_{x_{l}} G(x_{j,e}^{(1)}, x_{m,e}^{(1)}) \frac{1}{\lambda_{j,e}^{(1)}(N-2)/2} = O\left(\frac{1}{\lambda_{e}^{(3N-4)/2}}\right).
\]
Since for any \(i = \{1, \cdots, kN\}\), there exists some \(j \in \{1, \cdots, k\}\) satisfying \(i \in [(j-1)N+1, jN] \cap \mathbb{N}^+\), by direct calculations, we have
\[
\frac{\partial^2 \Psi_k(x, \lambda)}{\partial y_i \partial y_m} = \lambda_j^{N-2} \frac{\partial^2 R(x_j)}{\partial y_i \partial y_m} - \sum_{l \neq j}^{k} \lambda_l^{N-2} \lambda_j^{N-2} \frac{\partial^2 G(x_j, x_l)}{\partial y_i \partial y_m}, \text{ if } m \in [(j-1)N+1, jN] \cap \mathbb{N}^+,
\]
and
\[
\frac{\partial^2 \Psi_k(x, \lambda)}{\partial y_i \partial y_m} = - \lambda_j^{N-2} \lambda_s^{N-2} \frac{\partial^2 G(x_j, x_s)}{\partial y_i \partial y_m}, \text{ if } m \in [(s-1)N+1, sN] \cap \mathbb{N}^+ \text{ and } s \neq j.
\]
So from (4.23), we can obtain
\[
(D^2_{xx} \Psi_k(x, \lambda))_{(x, \lambda)=(\epsilon, \Lambda^k)} \bar{B}_{\epsilon, k} = \left( o\left( \frac{1}{\Lambda^N} \right), \cdots, o\left( \frac{1}{\Lambda^N} \right) \right)^T,
\] (4.24)
where \(\bar{B}_{\epsilon, k}\) is the vector in (4.15). Noting that \((\epsilon, \Lambda^k)\) is a nondegenerate critical point of \(\Psi_k\), we see
\[
\text{Rank} \left( D^2_{xx} \Psi_k(x, \lambda) \right)_{(x, \lambda)=(\epsilon, \Lambda^k)} = Nk.
\] (4.25)
Hence (4.17), (4.24) and (4.25) imply that
\[
B_{\epsilon, j, h} = o\left( \frac{1}{\Lambda^N} \right), \quad \text{for } j = 1, \cdots, k \text{ and } h = 1, \cdots, N.
\] (4.26)
On the other hand, from (3.26), (3.29) and (3.35), we find
\[
B_{\epsilon, j, h} = \int_{B_{\lambda_j(1), \epsilon}(0)} x_k C_r \left( \frac{x}{\lambda_j^{(1)}} \right) \xi_{\epsilon, j}(x) dx
\]
\[
= \frac{1}{\lambda_j^{(1)} N-1} \int_{\mathbb{R}^N} x_k U_{0, 1} \left( \sum_{l=1}^{N} c_{j, l} \psi_l(x) \right) dx + o\left( \frac{1}{\Lambda^N} \right)
\] (4.27)
\[
= - \frac{N-2}{2N} \int_{\mathbb{R}^N} \frac{|x|^2}{(1+|x|^2) \lambda_j^{(1)} N-1} + o\left( \frac{1}{\Lambda^N} \right).
\]
Then (4.26) and (4.27) imply (4.18). \(\square\)

We are now ready to show Theorem 1.2.

**Proof of Theorem 1.2.** For any given \((\epsilon, \Lambda^k)\), since \(M_k(\epsilon)\) is a positive matrix and \((\epsilon, \Lambda^k)\) is a nondegenerate critical point of \(\Psi_k\), then from [23], we find a solution of (1.1) with (1.4). Next, we prove the local uniqueness of solutions to (1.1) with (1.4).

From (3.24), it holds
\[
|\xi_{\epsilon}(x)| = O\left( \frac{1}{R^2} \right) + O(\epsilon), \quad \text{for } x \in \Omega \setminus \bigcup_{j=1}^{k} B_{R(\lambda_j^{(1)})^{-1}}(x_{j, \epsilon}^{(1)}),
\]
which implies that for any fixed \(\gamma \in (0, 1)\) and small \(\epsilon\), there exists \(R_1 > 0\),
\[
|\xi_{\epsilon}(x)| \leq \gamma, \quad x \in \Omega \setminus \bigcup_{j=1}^{k} B_{R_1(\lambda_j^{(1)})^{-1}}(x_{j, \epsilon}^{(1)}).
\] (4.28)

Also for the above fixed \(R_1\), from (4.5) and (4.18), we have
\[
\xi_{\epsilon, j}(x) = o(1) \text{ in } B_{R_1}(0), \quad j = 1, \cdots, k.
\]
We know \( \xi_{\varepsilon,j}(x) = \xi_{\varepsilon}(\frac{x}{\lambda_{j,\varepsilon}}, x_{j,\varepsilon}^{(1)}) \), so

\[
\xi_{\varepsilon}(x) = o(1), \quad x \in \bigcup_{j=1}^{k} B_{R_{\varepsilon}(\lambda_{j,\varepsilon})^{-1}}(x_{j,\varepsilon}^{(1)}).
\] (4.29)

Hence for any fixed \( \gamma \in (0,1) \) and small \( \varepsilon \), (4.28) and (4.29) imply \( |\xi_{\varepsilon}(x)| \leq \gamma \) for all \( x \in \Omega \), which is in contradiction with \( \|\xi_{\varepsilon}\|_{L^\infty(\Omega)} = 1 \). As a result, \( u^{(1)}_{\varepsilon}(x) \equiv u^{(2)}_{\varepsilon}(x) \) for small \( \varepsilon \). \( \square \)

**Remark 4.5.** Here we point out the reasons why our methods are unsuitable for \( N = 5 \). In fact, a first problem is that we can only get \( |x_{j,\varepsilon}^{(1)} - x_{j,\varepsilon}^{(2)}| = O\left(\frac{1}{\sqrt{\lambda_{j,\varepsilon}}}\right) \) for \( N = 5 \). However, to obtain

\[
\xi_{\varepsilon,j}(x) \to \sum_{i=0}^{N} c_{j,i}v_{i}(x), \quad \text{uniformly in } C^{1}\left(B_{R}(0)\right) \text{ for any } R > 0
\]
as in Proposition 3.6, a necessary estimate is \( |x_{j,\varepsilon}^{(1)} - x_{j,\varepsilon}^{(2)}| = o\left(\frac{1}{\lambda^{2}_{j,\varepsilon}}\right) \). On the other hand, even if we would obtain more precise estimate \( |x_{j,\varepsilon}^{(1)} - x_{j,\varepsilon}^{(2)}| = o\left(\frac{1}{\lambda_{j,\varepsilon}}\right) \) for \( N = 5 \), then similar to the estimate of (4.16), we can find \( A_{j} = O\left(\frac{\ln \lambda_{j,\varepsilon}}{\lambda_{j,\varepsilon}}\right) \), for \( j = 1, \cdots, k \), which and (4.9) imply \( c_{j,0} = 0 \), for \( j = 1, \cdots, k \).

But similar to the estimate of (4.26), we can only find

\[
B_{j} = O\left(\frac{\ln \lambda_{j,\varepsilon}}{\lambda_{j,\varepsilon}}\right), \quad \text{for } j = 1, \cdots, k \text{ and } i = 1, \cdots, 5.
\] (4.30)

Then from (3.35) and (4.30), we can only get \( c_{j,i} = O\left(\ln \lambda_{j,\varepsilon}\right) \), for \( j = 1, \cdots, k \) and \( i = 1, \cdots, 5 \). Why above phenomena occur is that the error estimate \( u_{\varepsilon} \) is not enough for us in Proposition A.2 when \( N = 5 \). However the error estimate \( u_{\varepsilon} \) in Proposition A.2 is basic and can’t be improved.

Now we are in the position to show Theorem 1.3.

**Proof of Theorem 1.3.** First, **Assumption A** implies that the solution of (1.1) must blow up. In fact, if \( u_{\varepsilon}(x) \) is a solution of (1.1) which does not blow up, then letting \( \varepsilon \to 0 \), we can find a nontrivial solution of (1.5), which is a contradiction with **Assumption A**. Also [13] gives us that all blow-up points are isolated and it implies the uniform bound \( \int_{\Omega} |\nabla u_{\varepsilon}|^{2}dx \leq C \), for some positive constant \( C \). Now by the global compactness result in [26], \( u_{\varepsilon}(x) \) can be written as

\[
u_{\varepsilon} = u_{0} + \sum_{j=1}^{k} P_{U_{x_{j,\varepsilon},\lambda_{j,\varepsilon}}} + w_{\varepsilon}(x),
\]
where

\[
x_{j,\varepsilon} \to a_{j}, \quad \varepsilon \to 0, \quad \lambda_{j,\varepsilon} \to \infty, \quad \|w_{\varepsilon}\|_{\varepsilon} = o(1),
\]
and \( u_{0} \) is a nonnegative solution of \(-\Delta u = u^{\frac{N+2}{4}} \) in \( \Omega \). By maximum principle and **Assumption A**, we get \( u_{0} = 0 \). Then using Theorem 1.2, the number of solutions to (1.1) with \( k \) blow-up points is \( 2^{T_{k}} \).
Since the number of the blow-up points to (1.1) are finite, we complete the proof of Theorem 1.3. \( \square \)

5. **Key estimates on Green’s function**

In this section, we give proofs of (2.13), (2.21), (4.8) and (4.22) involving Green’s function, which have been used in sections 3 and 4.

**Proof of (2.13).** By the bilinearity of \( P(u, v) \), we have

\[
P\left(G(x_{j,\varepsilon}, x), G(x_{j,\varepsilon}, x)\right) = P\left(S(x_{j,\varepsilon}, x), S(x_{j,\varepsilon}, x)\right) - 2P\left(S(x_{j,\varepsilon}, x), H(x_{j,\varepsilon}, x)\right)
\]
\[
+ P\left(H(x_{j,\varepsilon}, x), H(x_{j,\varepsilon}, x)\right).
\] (5.1)
After direct calculations, we know
\[ D_x, S(x_{j, \varepsilon}, x) = -\frac{x_i - x_{j, \varepsilon}}{\omega_N|x_{j, \varepsilon} - x|} \], and \( \nu_i = \frac{x_i - x_{j, \varepsilon}}{|x_{j, \varepsilon} - x|} \). \hfill (5.2)

Putting (5.2) in the term \( P\left(S(x_{j, \varepsilon}, x), S(x_{j, \varepsilon}, x)\right) \), we get
\[ P\left(S(x_{j, \varepsilon}, x), S(x_{j, \varepsilon}, x)\right) = 0. \hfill (5.3) \]

Now we calculate \( P\left(S(x_{j, \varepsilon}, x), H(x_{j, \varepsilon}, x)\right) \). Since \( D_v H(x_{j, \varepsilon}, x) \) is bounded in \( B_d(x_{j, \varepsilon}) \), we know
\[ \theta \int_{\partial B_d(x_{j, \varepsilon})} \langle DS(x_{j, \varepsilon}, x), \nu \rangle \langle DH(x_{j, \varepsilon}, x), \nu \rangle = O\left(\theta \int_{\partial B_d(x_{j, \varepsilon})} |x - x_{j, \varepsilon}|^{-(N-1)} \right) = O(\theta), \hfill (5.4) \]
\[ \theta \int_{\partial B_d(x_{j, \varepsilon})} \langle DS(x_{j, \varepsilon}, x), DH(x_{j, \varepsilon}, x) \rangle = O\left(\theta \int_{\partial B_d(x_{j, \varepsilon})} |x - x_{j, \varepsilon}|^{-(N-1)} \right) = O(\theta), \hfill (5.5) \]
and
\[ \int_{\partial B_d(x_{j, \varepsilon})} \langle DH(x_{j, \varepsilon}, x), \nu \rangle S(x_{j, \varepsilon}, x) = O\left(\int_{\partial B_d(x_{j, \varepsilon})} |x - x_{j, \varepsilon}|^{-(N-2)} \right) = O(\theta). \hfill (5.6) \]

Next, we obtain
\[ \int_{\partial B_d(x_{j, \varepsilon})} \langle DS(x_{j, \varepsilon}, x), \nu \rangle H(x_{j, \varepsilon}, x) = H(x_{j, \varepsilon}, x) \int_{\partial B_d(x_{j, \varepsilon})} \langle DS(x_{j, \varepsilon}, x), \nu \rangle = O(\theta) = - R(x_{j, \varepsilon}) + O(\theta). \hfill (5.7) \]

Then from (5.4)–(5.7), we get
\[ P\left(S(x_{j, \varepsilon}, x), H(x_{j, \varepsilon}, x)\right) = \frac{N - 2}{4} R(x_{j, \varepsilon}) + O(\theta). \hfill (5.8) \]

Also since \( H(x_{j, \varepsilon}, x) \) and \( D_v H(x_{j, \varepsilon}, x) \) are bounded in \( B_d(x_{j, \varepsilon}) \), it holds that
\[ P\left(H(x_{j, \varepsilon}, x), H(x_{j, \varepsilon}, x)\right) = O(\theta^{N-1}). \hfill (5.9) \]

Letting \( \theta \to 0 \), from (5.1), (5.3), (5.8) and (5.9), we get
\[ P\left(G(x_{j, \varepsilon}, x), G(x_{j, \varepsilon}, x)\right) = - \frac{N - 2}{2} R(x_{j, \varepsilon}). \hfill (5.10) \]

Next, for \( m \neq j \),
\[ P\left(G(x_{j, \varepsilon}, x), G(x_{m, \varepsilon}, x)\right) = P\left(S(x_{j, \varepsilon}, x), G(x_{m, \varepsilon}, x)\right) - P\left(H(x_{j, \varepsilon}, x), G(x_{m, \varepsilon}, x)\right). \hfill (5.10) \]

Since \( D_v G(x_{m, \varepsilon}, x) \) is bounded in \( B_d(x_{j, \varepsilon}) \), we know
\[ \theta \int_{\partial B_d(x_{j, \varepsilon})} \langle DS(x_{j, \varepsilon}, x), \nu \rangle \langle DG(x_{m, \varepsilon}, x), \nu \rangle = O\left(\theta \int_{\partial B_d(x_{j, \varepsilon})} |x - x_{j, \varepsilon}|^{-(N-1)} \right) = O(\theta), \hfill (5.11) \]
\[ \theta \int_{\partial B_d(x_{j, \varepsilon})} \langle DS(x_{j, \varepsilon}, x), DG(x_{m, \varepsilon}, x) \rangle = O\left(\theta \int_{\partial B_d(x_{j, \varepsilon})} |x - x_{j, \varepsilon}|^{-(N-1)} \right) = O(\theta), \hfill (5.12) \]
\[ \int_{\partial B_d(x_{j, \varepsilon})} \langle DG(x_{m, \varepsilon}, x), \nu \rangle S(x_{j, \varepsilon}, x) = O\left(\int_{\partial B_d(x_{j, \varepsilon})} |x - x_{j, \varepsilon}|^{-(N-2)} \right) = O(\theta). \hfill (5.13) \]
and
\[
\int_{\partial B_0(x_j,\varepsilon)} \langle DS(x_j,\varepsilon, x), \nu \rangle G(x_m,\varepsilon, x)
= G(x_m,\varepsilon, x_j) \int_{\partial B_0(x_j,\varepsilon)} D_\nu S(x_j,\varepsilon, x) + O(\theta) \int_{\partial B_0(x_j,\varepsilon)} |D_\nu S(x_j,\varepsilon, x)|
= -G(x_m,\varepsilon, x_j) + O(\theta). \tag{5.14}
\]

The combination of (5.11)–(5.14) gives
\[
P \left( S(x_j,\varepsilon, x), G(x_m,\varepsilon, x) \right) = \frac{N-2}{4} G(x_m,\varepsilon, x_j) + O(\theta). \tag{5.15}
\]

Also since \(H(x_j,\varepsilon, x), D_\nu H(x_j,\varepsilon, x), G(x_m,\varepsilon, x)\) and \(D_\nu G(x_m,\varepsilon, x)\) are bounded in \(B_d(x_j,\varepsilon)\), it holds
\[
P \left( H(x_j,\varepsilon, x), G(x_m,\varepsilon, x) \right) = O(\theta^{N-1}). \tag{5.16}
\]

Letting \(\theta \to 0\), from (5.10), (5.15) and (5.16), we know
\[
P \left( G(x_j,\varepsilon, x), G(x_m,\varepsilon, x) \right) = \frac{N-2}{4} G(x_m,\varepsilon, x_j), \text{ for } m \neq j. \tag{5.17}
\]

By the symmetry of \(P(u,v)\), (5.17) implies
\[
P \left( G(x_m,\varepsilon, x), G(x_j,\varepsilon, x) \right) = \frac{N-2}{4} G(x_m,\varepsilon, x_j), \text{ for } m \neq j.
\]

Finally, for \(l, m \neq j\), the boundedness of \(G(x_m,\varepsilon, x), D_\nu G(x_m,\varepsilon, x), G(x_l,\varepsilon, x)\) and \(D_\nu G(x_l,\varepsilon, x)\) are bounded in \(B_d(x_j,\varepsilon)\) yields that \(P \left( G(x_l,\varepsilon, x), G(x_m,\varepsilon, x) \right) = O(\theta^{N-1})\). Letting \(\theta \to 0\), we know
\[
P \left( G(x_l,\varepsilon, x), G(x_m,\varepsilon, x) \right) = 0, \text{ for } l, m \neq j.
\]

\[\square\]

**Proof of (2.21).** First, by the bilinearity of \(Q(u,v)\), we have
\[
Q \left( G(x_j,\varepsilon, x), G(x_j,\varepsilon, x) \right) = Q \left( S(x_j,\varepsilon, x), S(x_j,\varepsilon, x) \right) - 2Q \left( S(x_j,\varepsilon, x), H(x_j,\varepsilon, x) \right)
+ Q \left( H(x_j,\varepsilon, x), H(x_j,\varepsilon, x) \right). \tag{5.18}
\]

Then for \(Q \left( S(x_j,\varepsilon, x), S(x_j,\varepsilon, x) \right)\), the oddness of integrands means
\[
Q \left( S(x_j,\varepsilon, x), S(x_j,\varepsilon, x) \right) = 0. \tag{5.19}
\]

Now we calculate the term \(Q \left( S(x_j,\varepsilon, x), H(x_j,\varepsilon, x) \right)\). First, we know
\[
\int_{\partial B_0(x_j,\varepsilon)} D_\nu S(x_j,\varepsilon, x) D_\xi H(x_j,\varepsilon, x)
= D_\xi H(x_j,\varepsilon, x) \int_{\partial B_0(x_j,\varepsilon)} D_\nu S(x_j,\varepsilon, x) + O(\theta) \int_{\partial B_0(x_j,\varepsilon)} |D_\nu S(x_j,\varepsilon, x)|
= -\frac{1}{2} \frac{\partial R(x_j,\varepsilon)}{\partial x_i} + O(\theta), \tag{5.20}
\]
$$\int_{\partial B_\theta(x_j,\epsilon)} D_v H(x_j,\epsilon, x) D_x S(x_j,\epsilon, x)$$

$$= \sum_{l=1}^N D_{x_l} H(x_j,\epsilon, x_j, \epsilon) \int_{\partial B_\theta(x_j,\epsilon)} D_{x_l} S(x_j,\epsilon, x) \nu_l + O(\theta) \int_{\partial B_\theta(x_j,\epsilon)} |D_{x_l} S(x_j,\epsilon, x)|$$

$$= D_{x_j} H(x_j,\epsilon, x_j, \epsilon) \int_{\partial B_\theta(x_j,\epsilon)} D_{x_j} S(x_j,\epsilon, x) \nu_l + O(\theta),$$

and

$$\int_{\partial B_\theta(x_j,\epsilon)} \langle DS(x_j,\epsilon, x), DH(x_j,\epsilon, x) \rangle \nu_i$$

$$= \sum_{l=1}^N D_{x_l} H(x_j,\epsilon, x_j, \epsilon) \int_{\partial B_\theta(x_j,\epsilon)} D_{x_l} S(x_j,\epsilon, x) \nu_l + O(\theta) \int_{\partial B_\theta(x_j,\epsilon)} |DS(x_j,\epsilon, x)|$$

$$= D_{x_j} H(x_j,\epsilon, x_j, \epsilon) \int_{\partial B_\theta(x_j,\epsilon)} D_{x_j} S(x_j,\epsilon, x) \nu_l + O(\theta),$$

which together imply

$$Q \left( S(x_j,\epsilon, x), H(x_j,\epsilon, x) \right) = \frac{1}{2} \frac{\partial R(x_j,\epsilon)}{\partial x_i} + O(\theta). \tag{5.23}$$

Also since $D_v H(x_j,\epsilon, x)$ is bounded in $B_d(x_j,\epsilon)$, it holds

$$Q \left( H(x_j,\epsilon, x), H(x_j,\epsilon, x) \right) = O(\theta^{N-1}). \tag{5.24}$$

Letting $\theta \to 0$, from (5.18), (5.19), (5.23) and (5.24), we get

$$Q \left( G(x_j,\epsilon, x), G(x_j,\epsilon, x) \right) = - \frac{\partial R(x_j,\epsilon)}{\partial x_i}. \tag{5.25}$$

Next, for $m \neq j$,

$$Q \left( G(x_j,\epsilon, x), G(x_m,\epsilon, x) \right) = Q \left( S(x_j,\epsilon, x), G(x_m,\epsilon, x) \right) - Q \left( H(x_j,\epsilon, x), G(x_m,\epsilon, x) \right). \tag{5.26}$$

Similar to (5.20)–(5.22), we know

$$\int_{\partial B_\theta(x_j,\epsilon)} D_v S(x_j,\epsilon, x) D_x G(x_m,\epsilon, x)$$

$$= D_{x_j} G(x_m,\epsilon, x_j, \epsilon) \int_{\partial B_\theta(x_j,\epsilon)} D_v S(x_j,\epsilon, x) + O(\theta) \int_{\partial B_\theta(x_j,\epsilon)} |D_v S(x_j,\epsilon, x)|$$

$$= - D_{x_j} G(x_m,\epsilon, x_j, \epsilon) + O(\theta),$$

$$\int_{\partial B_\theta(x_j,\epsilon)} D_v G(x_m,\epsilon, x) D_x S(x_j,\epsilon, x)$$

$$= \sum_{l=1}^N D_{x_l} G(x_m,\epsilon, x_j, \epsilon) \int_{\partial B_\theta(x_j,\epsilon)} D_{x_l} S(x_j,\epsilon, x) \nu_l + O(\theta) \int_{\partial B_\theta(x_j,\epsilon)} |D_{x_l} S(x_j,\epsilon, x)|$$

$$= D_{x_j} G(x_m,\epsilon, x_j, \epsilon) \int_{\partial B_\theta(x_j,\epsilon)} D_{x_j} S(x_j,\epsilon, x) \nu_l + O(\theta),$$
and
\[ \int_{\partial B_\theta(x,j,\varepsilon)} \langle \nabla S(x_j, \varepsilon, x), \nabla H(x_j, \varepsilon, x) \rangle \nu_i = \sum_{l=1}^{N} D_{x_l} G(x_m, x, x_j, \varepsilon) \int_{\partial B_\theta(x,j,\varepsilon)} D_{x_l} S(x_j, \varepsilon, x) \nu_i + O(\theta \int_{\partial B_\theta(x,j,\varepsilon)} \left| D_{\varepsilon} S(x_j, \varepsilon, x) \right|) \]
\[ = D_{x_j} G(x_m, x, x_j, \varepsilon) \int_{\partial B_\theta(x,j,\varepsilon)} D_{x_j} S(x_j, \varepsilon, x) + O(\theta), \]
which together imply
\[ Q \left( S(x_j, \varepsilon, x), G(x_m, \varepsilon, x) \right) = D_{x_j} G(x_m, \varepsilon, x_j, \varepsilon) + O(\theta). \]  
(5.26)
Since \( D_{\varepsilon} H(x_j, \varepsilon, x) \) and \( D_{\varepsilon} G(x_m, x) \) are bounded in \( B_\delta(x_j, \varepsilon) \), it holds
\[ Q \left( H(x_j, \varepsilon, x), G(x_m, x) \right) = O(\theta^{-1}). \]  
(5.27)
Letting \( \theta \to 0 \), from (5.25)–(5.27), we know
\[ Q \left( G(x_j, \varepsilon, x), G(x_m, \varepsilon, x) \right) = D_{x_j} G(x_m, \varepsilon, x_j, \varepsilon), \quad \text{for } m \neq j. \]  
(5.28)
By the symmetry of \( Q(u, v) \), (5.28) imply
\[ Q \left( G(x_m, \varepsilon, x), G(x_j, \varepsilon, x) \right) = D_{x_j} G(x_m, \varepsilon, x_j, \varepsilon), \quad \text{for } m \neq j. \]
Finally, since \( D_{\varepsilon} G(x_l, \varepsilon, x) \) and \( D_{\varepsilon} G(x_m, x) \) are bounded in \( B_\delta(x_j, \varepsilon) \) for \( l, m \neq j \), it holds that
\[ Q \left( G(x_l, \varepsilon, x), G(x_m, \varepsilon, x) \right) = O(\theta^{-1}). \]  
So letting \( \theta \to 0 \), we know
\[ Q \left( G(x_l, \varepsilon, x), G(x_m, \varepsilon, x) \right) = 0, \quad \text{for } l, m \neq j. \]

\[ \square \]

**Proof of** (4.8). By the bilinearity of \( P(u, v) \), we have
\[ P_1 \left( G(x_j^{(1)}, x), \partial_h G(x_j^{(1)}, x) \right) = P_1 \left( S(x_j^{(1)}, x), \partial_h S(x_j^{(1)}, x) \right) - P_1 \left( S(x_j^{(1)}, x), \partial_h H(x_j^{(1)}, x) \right) \]
\[ - P_1 \left( H(x_j^{(1)}, x), \partial_h S(x_j^{(1)}, x) \right) + P_1 \left( H(x_j^{(1)}, x), \partial_h H(x_j^{(1)}, x) \right). \]  
(5.29)
For \( P_1 \left( S(x_j^{(1)}, x), \partial_h S(x_j^{(1)}, x) \right) \), the oddness of the integrands yields
\[ P_1 \left( S(x_j^{(1)}, x), \partial_h S(x_j^{(1)}, x) \right) = 0. \]  
(5.30)
Now we calculate \( P_1 \left( S(x_j^{(1)}, x), \partial_h H(x_j^{(1)}, x) \right) \). Since \( \partial_h D_{\varepsilon} H(x_j^{(1)}, x) \) is bounded in \( B_\delta(x_j^{(1)}, x) \), we know
\[ \theta \int_{\partial B_{\delta}(x_j^{(1)}, x)} \langle DS(x_j^{(1)}, x), \nu \rangle \langle \partial_h DH(x_j^{(1)}, x), \nu \rangle = O \left( \theta \int_{\partial B_{\delta}(x_j^{(1)}, x)} |x - x_j^{(1)}|^{-(N-1)} \right) = O(\theta), \]  
(5.31)
and
\[ \theta \int_{\partial B_{\delta}(x_j^{(1)}, x)} \langle DS(x_j^{(1)}, x), \partial_h DH(x_j^{(1)}, x) \rangle = O \left( \theta \int_{\partial B_{\delta}(x_j^{(1)}, x)} |x - x_j^{(1)}|^{-(N-1)} \right) = O(\theta), \]  
(5.32)
and
\[ \int_{\partial B_{\delta}(x_j^{(1)}, x)} \langle \partial_h DH(x_j^{(1)}, x), \nu \rangle S(x_j^{(1)}, x) = O \left( \int_{\partial B_{\delta}(x_j^{(1)}, x)} |x - x_j^{(1)}|^{-(N-2)} \right) = O(\theta). \]  
(5.33)
Next, we obtain
\[
\int_{\partial B_h(x_j^{(1)})} \langle DS(x_j^{(1)}, x), \nu \rangle \partial_h H(x_j^{(1)}, x)
= \partial_h H(x_j^{(1)}, x) \int_{\partial B_h(x_j^{(1)})} \langle DS(x_j^{(1)}, x), \nu \rangle + O(\theta) = -\frac{1}{2} \partial_h R(x_j^{(1)}) + O(\theta).
\] (5.34)

Then from (5.31)–(5.34), we get
\[
P_1 \left( S(x_j^{(1)}, x), H(x_j^{(1)}, x) \right) = \frac{N - 2}{8} \partial_h R(x_j^{(1)}) + O(\theta).
\] (5.35)

Next, we calculate \( P_1 \left( H(x_j^{(1)}, x), \partial_h S(x_j^{(1)}, x) \right) \). First, let \( y = x - x_j^{(1)} \), then we get
\[
\partial_h S(x_j^{(1)}, x) = -\frac{y_h}{\omega_N|y|^N}, \quad \langle D \partial_h S(x_j^{(1)}, x), \nu \rangle = \frac{(1 - N)y_h}{\omega_N|y|^{N+1}},
\] (5.36)
and
\[
D_{x_i} \partial_h S(x_j^{(1)}, x) = \frac{\delta_h}{\omega_N|y|^N} - \frac{N y_h y_i}{\omega_N|y|^{N+2}}.
\] (5.37)

Then we know
\[
\theta \int_{\partial B_h(x_j^{(1)})} \langle \partial_h DS(x_j^{(1)}, x), \nu \rangle \langle DH(x_j^{(1)}, x), \nu \rangle = \theta D_h H(x_j^{(1)}, x) \int_{\partial B_h(0)} \frac{(1 - N)y_h^2}{\omega_N|y|^{N+2}} + O(\theta) = \frac{1 - N}{2N} \partial_h R(x_j^{(1)}) + O(\theta),
\]
\[
\theta \int_{\partial B_h(x_j^{(1)})} \frac{\langle \partial_h DS(x_j^{(1)}, x), DH(x_j^{(1)}, x) \rangle}{\omega_N|y|^N} = \theta D_h H(x_j^{(1)}, x) \int_{\partial B_h(0)} \frac{1}{\omega_N|y|^N} - \frac{N y_h^2}{\omega_N|y|^{N+2}} + O(\theta) = O(\theta),
\]
\[
\int_{\partial B_h(x_j^{(1)})} \langle DH(x_j^{(1)}, x), \nu \rangle \partial_h S(x_j^{(1)}, x) = -\partial_h H(x_j^{(1)}, x) \int_{\partial B_h(0)} \frac{y_h^2}{\omega_N|y|^{N+1}} + O(\theta) = -\frac{1}{2N} \partial_h R(x_j^{(1)}) + O(\theta),
\]
and
\[
\int_{\partial B_h(x_j^{(1)})} \langle \partial_h DS(x_j^{(1)}, x), \nu \rangle H(x_j^{(1)}, x) = \partial_h H(x_j^{(1)}, x) \int_{\partial B_h(x_j^{(1)})} \frac{(1 - N)y_h^2}{\omega_N|y|^{N+1}} + O(\theta) = \frac{1 - N}{2N} \partial_h R(x_j^{(1)}) + O(\theta),
\]
which together imply
\[
P_1 \left( H(x_j^{(1)}, x), \partial_h S(x_j^{(1)}, x) \right) = \left( \frac{N - 2}{8} + \frac{1 - N}{2N} \right) \partial_h R(x_j^{(1)}) + O(\theta).
\] (5.38)

Also since \( H(x_j^{(1)}, x) \) and \( D_{x_i} H(x_j^{(1)}, x) \) are bounded in \( B_d(x_j^{(1)}) \), it holds that
\[
P_1 \left( H(x_j^{(1)}, x), \partial_h H(x_j^{(1)}, x) \right) = O(\theta^{N-1}).
\] (5.39)

Letting \( \theta \to 0 \), from (5.29), (5.30), (5.35), (5.38) and (5.39), we get
\[
P_1 \left( G(x_j^{(1)}, x), \partial_h G(x_j^{(1)}, x) \right) = \left( \frac{N - 2}{4} + \frac{1 - N}{2N} \right) \partial_h R(x_j^{(1)}).
\]
Next, for $m \neq j$,
\[
P_l \left( G(x^{(1)}_{j,\varepsilon}, x), \partial_h G(x^{(1)}_{m,\varepsilon}, x) \right) = P_l \left( S(x^{(1)}_{j,\varepsilon}, x), \partial_h G(x^{(1)}_{m,\varepsilon}, x) \right) - P_l \left( H(x^{(1)}_{j,\varepsilon}, x), \partial_h G(x^{(1)}_{m,\varepsilon}, x) \right).
\]  
(5.40)

Since $\partial_h D_v G(x^{(1)}_{m,\varepsilon}, x)$ is bounded in $B_d(x^{(1)}_{j,\varepsilon})$, we know
\[
\theta \int_{\partial B_d(x^{(1)}_{j,\varepsilon})} \langle DS(x^{(1)}_{j,\varepsilon}, x), \nu \rangle \langle \partial_h DG(x^{(1)}_{m,\varepsilon}, x), \nu \rangle = O \left( \theta \int_{\partial B_d(0)} \frac{1}{|y|^{N-1}} \right) = O(\theta),
\]  
(5.41)
\[
\theta \int_{\partial B_d(x^{(1)}_{j,\varepsilon})} \langle DS(x^{(1)}_{j,\varepsilon}, x), \partial_h DG(x^{(1)}_{m,\varepsilon}, x) \rangle = O \left( \theta \int_{\partial B_d(0)} \frac{1}{|y|^{N-1}} \right) = O(\theta),
\]  
(5.42)
\[
\int_{\partial B_d(x^{(1)}_{j,\varepsilon})} \langle \partial_h DG(x^{(1)}_{m,\varepsilon}, x), \nu \rangle S(x^{(1)}_{j,\varepsilon}, x) = O \left( \int_{\partial B_d(0)} \frac{1}{|y|^{N-2}} \right) = O(\theta),
\]  
(5.43)
and
\[
\int_{\partial B_d(x^{(1)}_{j,\varepsilon})} \langle DS(x^{(1)}_{j,\varepsilon}, x), \nu \rangle \partial_h G(x^{(1)}_{m,\varepsilon}, x)
\]
\[= \partial_h G(x^{(1)}_{m,\varepsilon}, x^{(1)}_{j,\varepsilon}) \int_{\partial B_d(x^{(1)}_{j,\varepsilon})} D_v S(x^{(1)}_{j,\varepsilon}, x) + O(\theta) = -\partial_h G(x^{(1)}_{m,\varepsilon}, x^{(1)}_{j,\varepsilon}) + O(\theta).
\]  
(5.44)

From (5.41)–(5.44), we get
\[
P_l \left( S(x^{(1)}_{j,\varepsilon}, x), G(x^{(1)}_{m,\varepsilon}, x) \right) = \frac{N - 2}{4} \partial_h G(x^{(1)}_{m,\varepsilon}, x^{(1)}_{j,\varepsilon}) + O(\theta).
\]  
(5.45)

Also since $H(x^{(1)}_{j,\varepsilon}, x)$, $D_v H(x^{(1)}_{j,\varepsilon}, x)$, $\partial_h G(x^{(1)}_{m,\varepsilon}, x)$ and $\partial_h D_v G(x^{(1)}_{m,\varepsilon}, x)$ are bounded in $B_d(x^{(1)}_{j,\varepsilon})$, it holds that
\[
P_l \left( H(x^{(1)}_{j,\varepsilon}, x), G(x^{(1)}_{m,\varepsilon}, x) \right) = O(\theta^{N-1}).
\]  
(5.46)

Letting $\theta \to 0$, from (5.40), (5.45) and (5.46), we know
\[
P_l \left( G(x^{(1)}_{j,\varepsilon}, x), \partial_h G(x^{(1)}_{m,\varepsilon}, x) \right) = \frac{N - 2}{4} \partial_h G(x^{(1)}_{m,\varepsilon}, x^{(1)}_{j,\varepsilon}), \text{ for } m \neq j.
\]

Next, we calculate the term $P_l \left( G(x^{(1)}_{m,\varepsilon}, x), \partial_h G(x^{(1)}_{j,\varepsilon}, x) \right)$. Similar to the estimate of (5.38), we find
\[
P_l \left( G(x^{(1)}_{m,\varepsilon}, x), \partial_h G(x^{(1)}_{j,\varepsilon}, x) \right) = \left( \frac{N - 2}{4} + \frac{1 - N}{N} \right) \partial_h G(x^{(1)}_{j,\varepsilon}, x^{(1)}_{m,\varepsilon}), \text{ for } m \neq j.
\]

Finally, for $l, m \neq j$, since $G(x^{(1)}_{l,\varepsilon}, x)$, $D_v G(x^{(1)}_{m,\varepsilon}, x)$, $\partial_h G(x^{(1)}_{l,\varepsilon}, x)$ and $\partial_h D_v G(x^{(1)}_{l,\varepsilon}, x)$ are bounded in $B_d(x^{(1)}_{j,\varepsilon})$, we conclude $P_l \left( G(x^{(1)}_{l,\varepsilon}, x), \partial_h G(x^{(1)}_{m,\varepsilon}, x) \right) = O(\theta^{N-1})$. So letting $\theta \to 0$, we know
\[
P_l \left( G(x^{(1)}_{l,\varepsilon}, x), \partial_h G(x^{(1)}_{m,\varepsilon}, x) \right) = 0, \text{ for } l, m \neq j.
\]

\[\Box\]

**Proof of (4.22).** By the bilinearity of $Q_1(u, v)$, we have
\[
Q_1 \left( G(x^{(1)}_{j,\varepsilon}, x), \partial_h G(x^{(1)}_{j,\varepsilon}, x) \right) = Q_1 \left( S(x^{(1)}_{j,\varepsilon}, x), \partial_h S(x^{(1)}_{j,\varepsilon}, x) \right) - Q_1 \left( H(x^{(1)}_{j,\varepsilon}, x), \partial_h S(x^{(1)}_{j,\varepsilon}, x) \right)
\]
\[\quad - Q_1 \left( S(x^{(1)}_{j,\varepsilon}, x), \partial_h H(x^{(1)}_{j,\varepsilon}, x) \right) + Q_1 \left( H(x^{(1)}_{j,\varepsilon}, x), \partial_h H(x^{(1)}_{j,\varepsilon}, x) \right).
\]  
(5.47)

Then for $Q_1 \left( S(x^{(1)}_{j,\varepsilon}, x), \partial_h G(x^{(1)}_{j,\varepsilon}, x) \right)$, the integrand is odd which means
\[
Q_1 \left( S(x^{(1)}_{j,\varepsilon}, x), \partial_h S(x^{(1)}_{j,\varepsilon}, x) \right) = 0.
\]  
(5.48)
Now we calculate the term $Q_1 \left( S(x_{j,\varepsilon}^{(1)}, x), \partial_{\theta} H(x_{j,\varepsilon}^{(1)}, x) \right)$. First, we know

$$
\int_{\partial B_0(x_{j,\varepsilon}^{(1)})} D_y S(x_{j,\varepsilon}^{(1)}, x) D_{x,\nu} \partial_{\theta} H(x_{j,\varepsilon}^{(1)}, x)
= D_z \partial_{\theta} H(x_{j,\varepsilon}^{(1)}, x) \int_{\partial B_0(x_{j,\varepsilon}^{(1)})} D_y S(x_{j,\varepsilon}^{(1)}, x) + O(\theta) \int_{\partial B_0(x_{j,\varepsilon}^{(1)})} |D_y S(x_{j,\varepsilon}^{(1)}, x)|
= - D_z \partial_{\theta} H(x_{j,\varepsilon}^{(1)}, x) + O(\theta),
$$

and

$$
\int_{\partial B_0(x_{j,\varepsilon}^{(1)})} \langle D_y S(x_{j,\varepsilon}^{(1)}, x), D_{\theta} H(x_{j,\varepsilon}^{(1)}, x) \rangle \nu_i
= \sum_{i=1}^{N} D_z \partial_{\theta} H(x_{j,\varepsilon}^{(1)}, x) \int_{\partial B_0(x_{j,\varepsilon}^{(1)})} D_y S(x_{j,\varepsilon}^{(1)}, x) \nu_i + O(\theta) \int_{\partial B_0(x_{j,\varepsilon}^{(1)})} |D_y S(x_{j,\varepsilon}^{(1)}, x)|
= D_z \partial_{\theta} H(x_{j,\varepsilon}^{(1)}, x) \int_{\partial B_0(x_{j,\varepsilon}^{(1)})} D_y S(x_{j,\varepsilon}^{(1)}, x) \nu_i + O(\theta),
$$

which together imply

$$
Q_1 \left( S(x_{j,\varepsilon}^{(1)}, x), \partial_{\theta} H(x_{j,\varepsilon}^{(1)}, x) \right) = D_z \partial_{\theta} H(x_{j,\varepsilon}^{(1)}, x) + O(\theta). \tag{5.49}
$$

Next we calculate the term $Q_1 \left( \partial_{\theta} S(x_{j,\varepsilon}^{(1)}, x), H(x_{j,\varepsilon}^{(1)}, x) \right)$. Using (5.36) and (5.37), we find

$$
\int_{\partial B_0(x_{j,\varepsilon}^{(1)})} D_y \partial_{\theta} S(x_{j,\varepsilon}^{(1)}, x) D_{x,\nu} H(x_{j,\varepsilon}^{(1)}, x)
= \int_{\partial B_0(x_{j,\varepsilon}^{(1)})} D_y \partial_{\theta} S(x_{j,\varepsilon}^{(1)}, x) \langle D D_{x, \nu} H(x_{j,\varepsilon}^{(1)}, x), x - x_{j,\varepsilon}^{(1)} \rangle + O(\theta)
= \sum_{i=1}^{N} (1 - N) \omega^{-1} D_{x, \nu} H(x_{j,\varepsilon}^{(1)}, x) \int_{|y|=\theta} \frac{y_i y_i}{|y|^{N+1}} + O(\theta)
= \frac{1 - N}{N} D_{x, \nu}^2 H(x_{j,\varepsilon}^{(1)}, x_{j,\varepsilon}^{(1)}) + O(\theta), \tag{5.50}
$$
\[
\int_{\partial B_\theta(x_j^{(i)}, \epsilon)} D_\nu H(x_j^{(i)}, x) D_x, \partial_\nu S(x_j^{(i)}, x) \\
= \int_{\partial B_\theta(x_j^{(i)}, \epsilon)} D_x, \partial_\nu S(x_j^{(i)}, x) (D^2 H(x_j^{(i)}, x_j^{(i)})(x - x_j^{(i)}), \nu) + O(\theta) \\
= \sum_{i=1}^{N} \sum_{i=1}^{N} \omega_N^{-1} D_{x, x_j^{(i)}} H(x_j^{(i)}, x_j^{(i)}) \int_{|y| = \theta} y_i y_i \left( \frac{\delta_{j i}}{|y|^N} - N \frac{y_i y_i}{|y|^{N+2}} \right) + O(\theta) \\
= \left\{ \begin{array}{ll}
- \frac{2}{N} D_{x, x_j^{(i)}} H(x_j^{(i)}, x_j^{(i)}) + O(\theta), & \text{for } i \neq h, \\
\frac{1}{N} D_{x, x_j^{(i)}} H(x_j^{(i)}, x_j^{(i)}) + O(\theta), & \text{for } i = h.
\end{array} \right.
\]

and
\[
\int_{\partial B_\theta(x_j^{(i)}, \epsilon)} (D \partial_\nu S(x_j^{(i)}, x), D H(x_j^{(i)}, x)) \nu_i \\
= \int_{\partial B_\theta(x_j^{(i)}, \epsilon)} (D^2 H(x_j^{(i)}, x_j^{(i)})(x - x_j^{(i)}), D \partial_\nu S(x_j^{(i)}, x)) \nu_i + O(\theta) \\
= \sum_{i=1}^{N} \sum_{i=1}^{N} \omega_N^{-1} D_{x, x_j^{(i)}} H(x_j^{(i)}, x_j^{(i)}) \int_{|y| = \theta} y_i \left( \frac{\delta_{j i}}{|y|^N} - N \frac{y_i y_i}{|y|^{N+2}} \right) + O(\theta) \\
= \left\{ \begin{array}{ll}
- \frac{1}{N} D_{x, x_j^{(i)}} H(x_j^{(i)}, x_j^{(i)}) + O(\theta), & \text{for } i \neq h, \\
\frac{1}{N} D_{x, x_j^{(i)}} H(x_j^{(i)}, x_j^{(i)}) + O(\theta), & \text{for } i = h.
\end{array} \right.
\]

From (5.50)–(5.52), we get
\[
Q_1 \left( S(x_j^{(i)}, x), \partial_\nu H(x_j^{(i)}, x) \right) = D_{x, x_j^{(i)}} H(x_j^{(i)}, x_j^{(i)}) + O(\theta).
\]

Here the last two equalities hold by the fact \( \Delta G(x, x_j^{(i)}) = 0 \) for \( x \in \Omega \setminus B_\theta(x_j^{(i)}) \). Also since \( H(x_j^{(i)}, x) \) and \( D_\nu H(x_j^{(i)}, x) \) are bounded in \( B_\theta(x_j^{(i)}) \), it holds that
\[
Q_1 \left( H(x_j^{(i)}, x), \partial_\nu H(x_j^{(i)}, x) \right) = O(\theta^{N-1}).
\]

Letting \( \theta \to 0 \), from (5.47)–(5.49), (5.53) and (5.54), we get
\[
Q_1 \left( G(x_j^{(i)}, x), \partial_\nu G(x_j^{(i)}, x) \right) = -D_x, \partial_\nu H(x_j^{(i)}, x_j^{(i)}) - D_{x, x_j^{(i)}} H(x_j^{(i)}, x_j^{(i)}) = -\frac{\partial^2 R(x_j^{(i)})}{\partial x_i \partial x_j}.
\]

Next, for \( m \neq j \),
\[
Q_1 \left( G(x_j^{(i)}, x), \partial_\nu G(x_j^{(i)}, x) \right) = Q_1 \left( S(x_j^{(i)}, x), \partial_\nu G(x_j^{(i)}, x) \right) - Q_1 \left( H(x_j^{(i)}, x), \partial_\nu G(x_j^{(i)}, x) \right).
\]

Similar to the estimates of (5.49), we know
\[
Q_1 \left( S(x_j^{(i)}, x), \partial_\nu G(x_j^{(i)}, x) \right) = D_x, \partial_\nu G(x_j^{(i)}, x_j^{(i)}) + O(\theta).
\]

Obviously,
\[
Q_1 \left( H(x_j^{(i)}, x), \partial_\nu G(x_j^{(i)}, x) \right) = O(\theta^{N-1}).
\]

Letting \( \theta \to 0 \), from (5.55)–(5.57), we obtain
\[
Q_1 \left( G(x_j^{(i)}, x), \partial_\nu G(x_j^{(i)}, x) \right) = D_x, \partial_\nu G(x_j^{(i)}, x), \text{ for } m \neq j.
\]
Also, for \( m \neq j \),
\[
Q_1\left(G(x_m^{(1)} \epsilon), \partial_h G(x_j^{(1)} \epsilon)\right) = Q_1\left(S(x_m^{(1)} \epsilon), \partial_h G(x_j^{(1)} \epsilon)\right) - Q_1\left(H(x_m^{(1)} \epsilon), \partial_h G(x_j^{(1)} \epsilon)\right).
\] (5.58)

Similar to the estimates of (5.53), we know
\[
Q_1\left(S(x_m^{(1)} \epsilon), \partial_h G(x_j^{(1)} \epsilon)\right) = D_2^2 x_m \epsilon \partial_h G(x_m^{(1)} \epsilon) + O(\theta).
\] (5.59)

Obviously,
\[
Q_1\left(H(x_m^{(1)} \epsilon), \partial_h G(x_j^{(1)} \epsilon)\right) = O(\theta^{N-1}).
\] (5.60)

Letting \( \theta \to 0 \), from (5.58)–(5.60), we obtain
\[
Q_1\left(G(x_m^{(1)} \epsilon), \partial_h G(x_j^{(1)} \epsilon)\right) = D_2^2 x_m \epsilon \partial_h G(x_m^{(1)} \epsilon), \text{ for } m \neq j.
\]

Finally, since \( G(x_m^{(1)} \epsilon) \) and \( D_\nu H(x_j^{(1)} \epsilon) \) are bounded in \( B_d(x_j^{(1)} \epsilon) \) for \( l \neq j \), it holds that
\[
Q_1\left(G(x_m^{(1)} \epsilon), \partial_h G(x_l^{(1)} \epsilon)\right) = 0, \text{ for } m, l \neq j.
\]

\[ \square \]

A. Some basic estimates

In this appendix, we give estimates that have been used in the previous sections.

**Lemma A.1.** For any small fixed \( d > 0 \) and \( j = 1, 2, \ldots, k \), it holds
\[
PU_{x_j, \lambda_j, \epsilon}(x) = O\left(\frac{1}{\lambda_j^{(N-2)/2}}\right), \text{ in } C^1\left(\Omega \setminus B_d(x_j, \epsilon)\right).
\] (A.1)

Moreover, if we define \( \varphi_{x_j, \lambda_j, \epsilon}(x) = U_{x_j, \lambda_j, \epsilon}(x) - PU_{x_j, \lambda_j, \epsilon}(x) \), then it holds
\[
\varphi_{x_j, \lambda_j, \epsilon}(x) = O\left(\frac{1}{\lambda_j^{(N-2)/2}}\right), \text{ in } C^1(\Omega).
\] (A.2)

**Proof.** See [25, Proposition 1].

**Proposition A.2.** Let \( N \geq 5 \) and \( u_\epsilon(x) \) be a solution of (1.1) with (1.4). Then
\[
\|w_\epsilon\| = \begin{cases} O\left(\frac{1}{\lambda_\epsilon^{(N-2)/2}}\right), & \text{if } N = 5, \\
O\left(\frac{(\log \lambda_\epsilon)^{2/3}}{\lambda_\epsilon^{2/3}} + \epsilon (\log \lambda_\epsilon)^{2/3}\right), & \text{if } N = 6, \\
O\left(\frac{1}{\lambda_\epsilon^{(N-2)/2}} + \frac{1}{\lambda_\epsilon^2}\right), & \text{if } N > 6,
\end{cases}
\] (A.3)

where \( \lambda_\epsilon := \min \{\lambda_{1, \epsilon}, \ldots, \lambda_{k, \epsilon}\} \).

**Proof.** See Proposition 4 in [25].

**Proposition A.3.** Let \( N \geq 5 \) and \( u_\epsilon(x) \) be a solution of (1.1) with (1.4). Then
\[
u_\epsilon(x) = O\left(\frac{1}{\lambda_\epsilon^{(N-2)/2}}\right), \text{ in } C^1\left(\Omega \setminus \bigcup_{j=1}^k B_{2d}(x_j, \epsilon)\right).
\] (A.4)
Proof. First, by Theorem 8.17 in [15], for any \( y \in \Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x_j, \varepsilon) \), we find
\[
\sup_{B_{d/2}(y)} u_\varepsilon(x) \leq C \left( \|u_\varepsilon\|_{L^2(B_d(y))} + \|f_\varepsilon\|_{L^{n/2}(B_d(y))} \right),
\]
where \( f_\varepsilon = \frac{u_\varepsilon^{N+2}}{\varepsilon} + \varepsilon u_\varepsilon \) and some \( q > N \).

Since \( u_\varepsilon(x) \) is a solution of (1.1) with (1.4), then \( u_\varepsilon(x) \) is uniformly bounded on \( \Omega \setminus \bigcup_{j=1}^{k} B_{2d}(x_j, \varepsilon) \).

From this, we can get
\[
\|f_\varepsilon\|_{L^{n/2}(B_d(y))} = o(1) \sup_{B_d(y)} u_\varepsilon(x). \tag{A.4}
\]

Next, by Moser iteration, we find (A.4).

\[\square\]

Lemma A.4. It holds
\[
\int_{B_d(x, \varepsilon)} u_\varepsilon^{\frac{N+2}{N-2}}(y)dy = \frac{A}{(\lambda_{j, \varepsilon})^{(N-2)/2}} + O \left( \frac{1}{\lambda_{\varepsilon}^{(N+2)/2}} \right) + o \left( \frac{\varepsilon}{\lambda_{\varepsilon}^2} \right), \tag{A.5}
\]
where \( A \) is the constant in (1.3).

Proof. By Hölder’s inequality, we calculate
\[
\int_{B_d(x, \varepsilon)} PU_{x, \lambda_{j, \varepsilon}}^{\frac{N}{N-2}} w_\varepsilon(x) = o \left( \frac{\varepsilon}{\lambda_{\varepsilon}^2} \right), \text{ and } \int_{B_d(x, \varepsilon)} PU_{x, \lambda_{j, \varepsilon}}^{\frac{N}{N-2}} w_\varepsilon^2(x) = o \left( \frac{\varepsilon}{\lambda_{\varepsilon}^2} \right).
\]

Also, we know
\[
\int_{B_d(x, \varepsilon)} \left( \sum_{l=1,l \neq j}^{k} PU_{x, \lambda_{j, \varepsilon}}^{\frac{N}{N-2}} + w_\varepsilon \right)^{\frac{N+2}{N-2}} = O \left( \frac{1}{\lambda_{\varepsilon}^{(N+2)/2}} \right) + o \left( \frac{\varepsilon}{\lambda_{\varepsilon}^2} \right).
\]

On the other hand, for any \( a, b \in \mathbb{R}^+ \) and \( p > 1 \), we have the following inequality:
\[
(a + b)^p - a^p - pa^{p-1}b = O(b^p + a^{p-1}b^p), \text{ with } p^* = \min\{2, p\}.
\]

Combining the above estimates, we get (A.5).

\[\square\]

Similar to the proof of (A.5), we can find following estimates.

Lemma A.5. For any \( j = 1, \ldots, N \), it holds
\[
\int_{B_d(x, \varepsilon)} |y - x, \varepsilon| u_\varepsilon^{\frac{N+2}{N-2}}(y)dy = O \left( \frac{1}{\lambda_{\varepsilon}^{(N+2)/2}} \right) + o \left( \frac{\varepsilon}{\lambda_{\varepsilon}^2} \right), \tag{A.6}
\]
and
\[
\int_{B_d(x, \varepsilon)} |y - x, \varepsilon|^3 u_\varepsilon^{\frac{N+2}{N-2}}(y)dy = O \left( \frac{1}{\lambda_{\varepsilon}^{(N+2)/2}} \right) + o \left( \frac{\varepsilon}{\lambda_{\varepsilon}^2} \right). \tag{A.7}
\]

Also for any \( i, j, m = 1, \ldots, N \), it holds
\[
\int_{B_d(x, \varepsilon)} (y_i - x, \varepsilon, \varepsilon)(y_m - x, \varepsilon, \varepsilon) u_\varepsilon^{\frac{N+2}{N-2}}(y)dy = \frac{\delta_{im} \log \lambda_{i, \varepsilon}}{\lambda_{i, \varepsilon}^{(N+2)/2}} + O \left( \frac{1}{\lambda_{\varepsilon}^{(N+2)/2}} \right) + o \left( \frac{\varepsilon}{\lambda_{\varepsilon}^2} \right), \tag{A.8}
\]
where \( \delta_{im} = 1 \) if \( i = m \) and \( \delta_{im} = 0 \) if \( i \neq m \).

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