Liouville Vortex And $\varphi^4$ Kink Solutions Of The Seiberg–Witten Equations

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Abstract

The Seiberg–Witten equations, when dimensionally reduced to $\mathbb{R}^2$, naturally yield the Liouville equation, whose solutions are parametrized by an arbitrary analytic function $g(z)$. The magnetic flux $\Phi$ is the integral of a singular Kaehler form involving $g(z)$; for an appropriate choice of $g(z)$, $N$ coaxial or separated vortex configurations with $\Phi = \frac{2\pi N}{e}$ are obtained when the integral is regularized. The regularized connection in the $\mathbb{R}^1$ case coincides with the kink solution of $\varphi^4$ theory.

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The Seiberg–Witten equations \[ \text{[1]} \] do not admit nonsingular solutions unless the curvature of the four dimensional base manifold \( M \) happens to be negative over some regions of \( M \). In particular, if \( M = \mathbb{R}^4 \), the Weitzenbock formula implies that the modulus squared of the spinor field \( \psi \) must either vanish everywhere, or exhibit singularities instead of local maxima \[ \text{[2]} \]. There is also a global restriction on flat-space Seiberg–Witten solutions: Integrating the Weitzenbock formula, Witten \[ \text{[3]} \] showed that all nontrivial flat solutions, including dimensionally reduced ones based on \( \mathbb{R}^3, \mathbb{R}^2 \), or \( \mathbb{R}^1 \), are all necessarily non-\( L^2 \).

Such singular, non-\( L^2 \) solutions, while probably not useful for Donaldson theory, may nevertheless be of physical interest. For example, Freund recently recognized that a singular \( U(1) \) magnetic monopole field and an accompanying spinor, found earlier by Gürsey in a different setting, solve the \( \mathbb{R}^3 \)-reduced Seiberg–Witten equations \[ \text{[4]} \]. The monopole being the characteristic topological object in \( \mathbb{R}^3 \), one may inquire whether there are \( \mathbb{R}^2 \) and \( \mathbb{R}^1 \) Seiberg–Witten solutions corresponding to vortices and kinks, respectively. The chief purpose of the present note is to show that such solutions indeed exist. A novel aspect of the \( \mathbb{R}^n \) (\( n \leq 2 \)) case is that the three coupled Seiberg–Witten equations (we should properly count \( F = dA \) as one of the three) can be reduced to a single nonlinear one. This happens to be the Liouville equation \[ \text{[5]} \], which has recently been related to \( N = 2 \) supersymmetric Seiberg–Witten theory in another context \[ \text{[6]} \].
We follow the conventions of Akbulut [3] in the choice of the Dirac \( \gamma \)-matrices

\[
\begin{align*}
\gamma^1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\gamma^2 &= \begin{pmatrix} 0 & i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}, \\
\gamma^3 &= \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \\
\gamma^4 &= \begin{pmatrix} 0 & -i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix},
\end{align*}
\] (1)

and the self–dual \( \Sigma_{ij} \)

\[
\begin{align*}
\Sigma_{12} &= \frac{1}{4} \{ [\gamma^1, \gamma^2] + [\gamma^3, \gamma^4] \} = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & 0 \end{pmatrix}, \\
\Sigma_{13} &= \frac{1}{4} \{ [\gamma^1, \gamma^3] + [\gamma^4, \gamma^2] \} = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & 0 \end{pmatrix}, \\
\Sigma_{23} &= \frac{1}{4} \{ [\gamma^1, \gamma^4] + [\gamma^2, \gamma^3] \} = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\] (2)

Taking a spinor \( \psi^T = (a, b, 0, 0) \), a connection \( iA_\mu \) and its curvature \( iF_{\mu\nu} = i(\partial_\mu A_\nu - \partial_\nu A_\mu) \), where \( \mu, \nu = 1, 2, 3, 4 \), the first of the Seiberg–Witten pair is nothing but the Dirac equation

\[
\gamma^\mu (\partial_\mu + iA_\mu) \psi = 0 .
\] (3)

In the notation of [3], the second Seiberg–Witten equation becomes

\[
\rho(iF^+_A) = \sigma(\psi) ,
\] (4)

where

\[
\sigma(\psi) \equiv \left( \frac{(|a|^2 - |b|^2)}{2ba} \frac{ab}{(|b|^2 - |a|^2)} \right)
\] (5)

and

\[
\rho(iF^+_A) = \frac{i}{8} (F_{\mu\nu} + \tilde{F}_{\mu\nu}) \cdot \Sigma_{\mu\nu} = \frac{i}{4} F_{\mu\nu} \Sigma_{\mu\nu} .
\] (6)

4
The vortex solutions in $\mathbb{R}^2$ follow from the Ansatz

$$A_\mu = (A_1, A_2, 0, 0) ,$$  \hfill (7)

$$\psi^T = (a, b, 0, 0) ,$$  \hfill (8)

where all quantities are assumed to depend only on $x_1$ and $x_2$. Putting (7) and (8) in (4), one finds two possibilities: either $(a \neq 0, b = 0)$ or $(a = 0, b \neq 0)$. Choosing the first, (4) reduces to

$$-F_{12} = -B_3 = |a|^2$$  \hfill (9)

while (3) yields

$$(-\partial_1 + i\partial_2)a = (iA_1 + A_2)a .$$  \hfill (10)

We now set

$$a = \alpha \exp(\omega x + i\omega y) ,$$  \hfill (11)

where $\alpha$ is a constant with the dimensions of inverse length as required by (4). This unusual dimension for the spinor field of course comes from the vacuum expectation value of the Higgs field in the twisted supersymmetric Yang–Mills theory underlying the Seiberg–Witten approach [7]. Dividing both sides of (10) by $a$, applying $(-\partial_1 - i\partial_2)$ and separating real and imaginary parts, we find

$$(\partial_1^2 + \partial_2^2)\omega_x = -B_3 = \alpha^2 \exp(2\omega_x)$$  \hfill (12)

and

$$(\partial_1^2 + \partial_2^2)\omega_y = -(\partial_1 A_1 + \partial_2 A_2) .$$  \hfill (13)
In (12), we have also used (9). It is convenient to introduce the dimensionless coordinates \( x = \alpha x_1 \), \( y = \alpha x_2 \) with \( \alpha > 0 \) and to define \( z(x) = x + (-)iy \). The equation (12) then becomes

\[
4 \partial_z \partial_{\bar{z}} \omega_x = \exp(2 \omega_x) .
\]  

(14)

This is of course the well–known Liouville equation. Using (10) and (11) we obtain

\[
A_1 = \alpha (\partial_y \omega_x - \partial_x \omega_y)
\]  

(15)

and

\[
A_2 = -\alpha (\partial_x \omega_x + \partial_y \omega_y) ,
\]  

(16)

which show that (13) is automatically satisfied. (14) has the solution

\[
\omega_x = \frac{1}{2} \ln \frac{4(dg/dz)(d\bar{g}/d\bar{z})}{(1 - gg)^2} ,
\]  

(17)

due to Liouville [4]. At this point, \( g(z) \) is an arbitrary analytic function. Comparing (17) with (11), we see that

\[
|a| = 2\alpha \frac{|dg/dz|}{(1 - gg)} ,
\]  

(18)

which naturally suggests

\[
\omega_y = \pm \arg \frac{dg}{dz} = \mp \arg \frac{d\bar{g}}{d\bar{z}} .
\]  

(19)

Note that this also makes \( \omega_y \) harmonic and enforces \( \vec{\nabla} \cdot \vec{A} = 0 \) via (13). We finally take

\[
a = 2\alpha \frac{dg/dz}{(1 - gg)}
\]  

(20)
leading to
\[ B_3 = -\frac{4\alpha^2 |dg/dz|^2}{(1 - g\overline{g})^2} \]  
(21)

The \(U(1)\) curvature is thus seen to be the Kaehler 2–form

\[ F = \frac{1}{e} F_{12} \, dx_1 \wedge dx_2 = \frac{i}{2\epsilon \alpha^2} F_{12} \, dz \wedge \overline{dz} \]

\[ = \frac{-2i \, dg \wedge d\overline{g}}{e \, (1 - g\overline{g})^2}, \]
(22)

where we have brought out the coupling constant \(e\) which was hidden in \(A_\mu\) all along. We can also combine (15) and (16) into the 1–form

\[ A = \frac{i}{e} \left( \frac{g \, dg}{(1 - g\overline{g})} - \frac{\overline{g} \, d\overline{g}}{(1 - g\overline{g})} \right). \]
(23)

A number of remarks are in order:

(i) The singularity dictated by the Weitzenbock formula manifests itself in (20)–(23). The solutions are singular in the \(z\)–plane along a curve defined by \(g\overline{g} = 1\). One can easily trace the minus in \((1 - g\overline{g})\) to the relative plus sign between the two sides of the Liouville equation (14); introducing a relative minus sign in (14) changes all the \((1 - g\overline{g})\) factors to \((1 + g\overline{g})\) without affecting anything else. (ii) Remarkably, (14) and (22) also arise in a twice-dimensionally reduced Ansatz leading to vortexlike solutions of the self–dual Yang–Mills equations [8]. Since the \(R^4\) self–dual Yang–Mills system is conjectured to generate all integrable systems through various dimensional reductions, the appearance of (14) in both the SDYM and the Seiberg-Witten contexts may be regarded as an additional
clue for similar integrability properties of the latter. Furthermore, putting $e^{uw} = u$ in the Liouville equation (14) and performing a further dimensional reduction by demanding $u = u(|z| = r)$ results in the differential equation for the 3rd Painlevé transcendent with $\gamma = 1$, $\alpha = \beta = \delta = 0$, in the notation of Ince [9]. Thus the Seiberg–Witten equations also exhibit a Painlevé property, considered an indication of integrability [10].

(iii) In the SDYM case, passing from the $(++++) \mathbb{R}^4$ to the twistor–based $(+-++) \mathbb{R}^{2,2}$ supplies the change in the relative sign in (14) converting the singular $(1 - g\varphi)^{-1}$ factor into $(1 + g\varphi)^{-1}$; the same obviously holds in our problem as well. (iv) The curvature form (22) remains unchanged under $g \rightarrow 1/g$, just as it would under a gauge transformation. Indeed, this inversion of $g$ precisely gives rise to the $U(1)$ gauge transformations

$$a \rightarrow a' = \frac{g}{\overline{g}} a$$

(24)

and

$$eA \rightarrow eA' = eA + id\ln \frac{g}{\overline{g}}$$

(25)

on the spinor field and the connection, respectively.

We now wish to restrict the choice of $g(z)$ by physical considerations. In Freund’s case, $\int F$ gives the quantized magnetic charge; it would be natural to expect that $\int F \equiv \Phi$ is a quantized magnetic flux in $\mathbb{R}^2$ for an appropriate $g(z)$. This requires that we somehow “regularize” the singular integrand (22); happily, different approaches to making sense out of $\int F$ gives the same result, as we shall
see. Let us start by considering the $\mathbb{R}^{2,2}$ version of (22), which becomes the area of the Riemann sphere stereographically expressed onto the $g$–plane:

$$
\Phi = \int F = \frac{2i}{\pi} \int \frac{dg \wedge d\bar{g}}{(1 + gg)^2}.
$$

(26)

Note the overall sign change due to the change in the RHS of (14).

Trying $g = z^\nu$ for an axisymmetric solution centered at the origin, we find

$$
\Phi = \frac{4\pi \nu}{\pi} \int_0^\infty \frac{(2\nu r^{2\nu-1})}{(1 + r^{2\nu})^2} dr = \frac{4\pi \nu}{\pi} \int_1^\infty \frac{dw}{w^2} = \frac{4\pi \nu}{\pi} ,
$$

(27)

where we have put $w = 1 + r^{2\nu}$. We note that the gauge transformation (24) results in

$$
a' = \frac{z^\nu}{z} - \nu a = e^{-2\nu \theta} a .
$$

(28)

The singlevaluedness of $a'$ at $\theta = 2\pi$ allows the $\nu$ values

$$
\nu = \frac{1}{2}, 1, \frac{3}{2}, \ldots .
$$

(29)

We need not consider $\nu \rightarrow -\nu$ as this only amounts to the gauge transformation $g \rightarrow 1/g$ mentioned earlier. Of the values in (29), it is the $\nu = \frac{1}{2}$ that corresponds to $2\pi/e$, i.e., the Nielsen–Olesen [11] unit of flux. Thus $g = z^{1/2}$ represents the basic single–vertex solution, while $g = z^{n/2}$ corresponds to a single vortex with $n$ units of flux. It is now easy to verify that

$$
g(z) = \prod_{k=1}^n (z - a_k)^{1/2}
$$

(30)
describes $n$ vortices centered at the locations $a_k = (a_{kx} + ia_{ky})$. To do this, we first switch to the compactified version of (23), which becomes

$$A = -\frac{i}{e} \left( \frac{\overline{g} \, dg}{1 + g \overline{g}} - \frac{g \, d\overline{g}}{1 + g \overline{g}} \right).$$

(31)

Next we use the $g(z)$ of (30) in

$$\Phi = \int_{\mathbb{R}^2} F = \int_{\partial \mathbb{R}^2} A,$$

(32)

where $\partial \mathbb{R}^2$ is a clockwise circle whose radius goes to infinity. Since $|g| \to |z^n/2| \gg 1$ on $\partial \mathbb{R}^2$, we obtain

$$\Phi = -\frac{i}{2e} \oint_{\partial \mathbb{R}^2} \left\{ n \frac{dz}{z} - n \frac{d\overline{z}}{\overline{z}} \right\} = \frac{2n\pi}{e}.$$

(33)

The similarity of expression (30) to Weierstrassian functions suggests we might consider a doubly–periodic solution on a two dimensional lattice, with one vortex per unit lattice cell. Let us take $\omega_1$ and $\omega_2$ as the two basic lattice vectors, subject to the usual restriction $\text{Im}(\omega_2/\omega_1) \neq 0$. For a pair of integers $(n_1, n_2)$, $\omega = n_1 \omega_1 + n_2 \omega_2$ is a point in the lattice. We can now choose for $g(z)$ the square root of the Weierstrassian quasi–periodic function, i.e.,

$$g(z) = \sigma^{1/2}(z) = z^{1/2} \prod_{\omega \neq 0} \left( 1 - \frac{z}{\omega} \right)^{1/2} \exp \left( \frac{z^2}{4\omega^2} + \frac{z}{2\omega} \right).$$

(34)

The exponential factor is needed to ensure the convergence of the product.

Another method for defining the integral of the singular expression (22) is as follows. If we attempt to calculate $\Phi$ starting from (22), we obtain

$$\Phi = -\frac{4\pi \nu}{e} \int_0^\infty \frac{(2\nu r^{2\nu-1})}{(r^{2\nu} - 1)^2} \, dr = -\frac{4\pi \nu}{e} \int_{-1}^\infty \frac{dw}{w^2} \quad (w = r^{2\nu} - 1)$$

(35)
instead of (27). Adopting Speer’s analytic regularization [12], we define

\[ I(-1, \infty) = \int_{-1}^{\infty} \frac{dw}{w^2} = \left[ \int_{-1}^{\infty} w^\lambda dw \right]_{\lambda=-2} = -1 . \] (36)

This is equivalent to writing \( I(-1, \infty) = I(-\infty, \infty) - I(-\infty, -1) \) and throwing away the infinite “constant” \( I(-\infty, \infty) \). Thus we get the same answer as in the compactified \( \mathbb{R}^{2,2} \) formulation.

Now let us return to equation (8) and ask what happens if we take \( a = 0 \), \( b \neq 0 \). It is easy to check that one still ends up with the Liouville equation (14); the changes consist of \( B_3 \rightarrow -B_3 \) and

\[ b = 2\alpha \frac{(d\overline{T}/dz)}{(1 - g\overline{T})} . \] (37)

Thus while it is not possible to change the direction of the magnetic field by \( g \rightarrow 1/g \), anti–vortices can be obtained by \( (a(g), 0) \rightarrow (0, b(\overline{g})) \), \( B_3 \rightarrow -B_3 \).

Finally, let us briefly examine the \( n = 1 \) case. A possible Ansatz is

\[ A_\mu = (0, 0, 0, A_4(x_1)) \] (38)

and

\[ \psi^T = (a(x_1), b(x_1), 0, 0) \] (39)

The Seiberg–Witten equation (6) demands either \( \psi^T = (a, a, 0, 0) \) or \( \psi^T = (a, -a, 0, 0) \). Taking \( a = \alpha \exp(\omega_x + i\omega_y) \) as before, these two cases yield

\[ \partial_1(\omega_x + i\omega_y) = \pm A_4 \] (40)
respectively. Thus in order for $A_4$ be real, $\omega_y$ can at most be a constant, which we may take to be zero. Then, using (3) we obtain

$$\partial_1^2\omega_x = \alpha^2 \exp(2\omega_x)$$  \hspace{1cm} (41)

Calling $\alpha x_1 = x$ again, (41) is seen to be a $y$–independent version of the Liouville equation (12). While (41) may be integrated directly by elementary methods, it is simpler to read off the solution from (17) by picking a $g(z)$ such that the variable $y$ disappears in $\omega_x$. This happens only for $g(z) = \exp \kappa (z + x_0)$, $\kappa$ and $x_0$ being constant real numbers (an imaginary constant added to $x_0$ cancels out along with the $iy$ in (17)). We may as well set $x_0 = 0$, which gives

$$\omega_x = \frac{1}{2} \ln \frac{4\kappa^2 e^{2\kappa x}}{(1 - e^{2\kappa x})^2} .$$  \hspace{1cm} (42)

This results in

$$|a| = \frac{\alpha \kappa}{\sinh \kappa x} ,$$  \hspace{1cm} (43)

$$A_4 = \pm \alpha \kappa \coth \kappa x ,$$  \hspace{1cm} (44)

and

$$E_1 = \mp \frac{\alpha^2 \kappa^2}{\sinh^2 \kappa x} .$$  \hspace{1cm} (45)

The expected singularity appears at $x = 0$ in (42)–(45). In contrast, the non-singular version obtained by $x_1 \rightarrow ix_1$ has $(1+e^{2\kappa x})^{-2}$ in (42); in addition, $\cosh \kappa x$ and $\sinh \kappa x$ in (13)–(15) are now switched. Thus $A_4$ changes into $\pm \alpha \kappa \tanh \kappa x$, which is the well–known kink (antikink) solution of $\varphi^4$ theory.
In conclusion, we see that the dimensionally reduced Seiberg–Witten equations in $\mathbb{R}^n$ ($n = 1, 2, 3$) yield singular version of topological solitons characteristic of each $n$, the $n = 3$ case being represented by Freund’s monopole solution. The accompanying spinors are of course the new feature associated with these familiar solitons. The $n = 2$ case indicates connections between integrable systems and the Seiberg–Witten equations. Finally, it should be interesting to look for solutions of the Seiberg–Witten equations reduced to a two–dimensional manifold admitting negative local values for the scalar curvature and see how this affects the singularities.

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References

[1] E. Witten, Math. Research Letters 1 (1994) 769.

[2] P. Kronheimer and T. Mrowka, Math. Research Letters 1 (1994) 797.

[3] P.G.O. Freund, J. Math. Phys. 36 (1995) 2673 ; F. Gürsey, in *Gauge Theories and Modern Field Theory*, edited by R. Arnowitt and P. Nath (MIT, Cambridge, 1976), p. 369

[4] J. Liouville, J. Math. Appl. 18 (1853) 71.
[5] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, “Integrability and Seiberg–Witten Exact Solution”, HEP–TH–9505035.

[6] S. Akbulut, Journal reine angew. Math 447 (1994) 83; “Lectures on Seiberg–Witten Invariants” (unpublished).

[7] E. Witten, J. Math. Phys. 35 (1994) 5101.

[8] C. Saçhoğlu, J. Math. Phys. 25 (1984) 3214.

[9] E.L. Ince, *Ordinary Differential Equations*, (Dover, New York, 1956) p.345

[10] M.J. Ablowitz, A. Ramani and H. Segur, J. Math. Phys. 21 (1980) 715.

[11] H. Nielsen and P. Olesen, Nucl. Phys. B 61 (1973) 45.

[12] E. Speer, Ann. Math. Stud. 62 (1969).