A FUNCTIONAL LIMIT THEOREM FOR MOVING AVERAGES WITH WEAKLY DEPENDENT HEAVY-TAILED INNOVATIONS

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Abstract. Recently a functional limit theorem for sums of moving averages with random coefficients and i.i.d. heavy tailed innovations has been obtained under the assumption that all partial sums of the series of coefficients are a.s. bounded between zero and the sum of the series. The convergence takes place in the space $D[0,1]$ of càdlàg functions with the Skorohod $M_2$ topology. In this article we extend this result to the case when the innovations are weakly dependent in the sense of strong mixing and local dependence condition $D'$. 

1. Introduction

Let $(Z_i)_{i \in \mathbb{Z}}$ be a strictly stationary sequence of regularly varying random variables with index of regular variation $\alpha \in (0,2)$. This means that

$$P(|Z_i| > x) = x^{-\alpha} L(x), \quad x > 0,$$

where $L$ is a slowly varying function at $\infty$. Let $(a_n)$ be a sequence of positive real numbers such that

$$n P(|Z_1| > a_n) \to 1,$$

as $n \to \infty$. Then regular variation of $Z_i$ can be expressed in terms of vague convergence of measures on $E = \mathbb{R} \setminus \{0\}$:

$$n P(a_n^{-1} Z_i \in \cdot) \xrightarrow{v} \mu(\cdot) \quad \text{as } n \to \infty,$$

where $\mu$ is a measure on $E$ given by

$$\mu(dx) = (p 1_{(0,\infty)}(x) + r 1_{(-\infty,0)}(x)) \alpha x^{-\alpha-1} dx,$$

with

$$p = \lim_{x \to \infty} \frac{P(Z_i > x)}{P(|Z_i| > x)} \quad \text{and} \quad r = \lim_{x \to \infty} \frac{P(Z_i \leq -x)}{P(|Z_i| > x)}.$$ (1.5)

We study the moving average process with random coefficients, defined by

$$X_i = \sum_{j=0}^{\infty} C_j Z_{i-j}, \quad i \in \mathbb{Z},$$

where $(C_i)_{i \geq 0}$ is a sequence of random variables independent of $(Z_i)$ such that the above series is a.s. convergent. One sufficient condition for that, which is commonly used in the literature is

$$\sum_{j=0}^{\infty} \mathbb{E}|C_j|^\delta < \infty \quad \text{for some } \delta < \alpha, \ 0 < \delta \leq 1.$$ (1.7)

2010 Mathematics Subject Classification. Primary 60F17; Secondary 60G51.

Key words and phrases. Functional limit theorem, Regular variation, $M_2$ topology, Moving average process.
The moment condition (1.7), stationarity of the sequence \((Z_i)\) and \(E|Z_1|^\beta < \infty\) for every \(\beta \in (0, \alpha)\) (which follows from the regular variation property and Karamata's theorem) imply the a.s. convergence of the series in (1.6), since

\[
E|X_i|^\delta \leq \sum_{j=0}^{\infty} E|C_j|^\delta E|Z_i-j|^\delta = E|Z_1|^\delta \sum_{j=0}^{\infty} E|C_j|^\delta < \infty.
\]

Another condition that assures the a.s. convergence of the series in the definition of linear processes with \(E(Z_1) = 0\), if \(\alpha \in (1, 2)\), \(Z_1\) is symmetric, if \(\alpha = 1\), and a.s. bounded coefficients can be deduced from the results in Astrauskas [1] for linear processes with deterministic coefficients:

\[
\sum_{j=0}^{\infty} c_j^\alpha L(c_j^{-1}) < \infty,
\]

where \((c_j)\) is a sequence of positive real numbers such that \(|C_j| \leq c_j\) a.s. for all \(j\) (c.f. Balan et al. [3]).

In the case when the \(Z_i\)'s are independent, under some usual regularity conditions and the assumption that all partial sums of the series \(C = \sum_{i=0}^\infty C_i\) are a.s. bounded between zero and the sum of the series, i.e.

\[
0 \leq \sum_{i=0}^{s} C_i \leq \sum_{i=0}^{\infty} C_i \leq 1 \quad \text{a.s. for every } s = 0, 1, 2, \ldots,
\]

a functional limit theorem for the corresponding partial sum stochastic process

\[
V_n(t) = \frac{1}{a_n} \sum_{i=1}^{\lfloor nt \rfloor} X_i, \quad t \in [0, 1],
\]

in the space \(D[0, 1]\) with the Skorohod \(M_2\) topology was recently derived in Krizmanić [17]. More precisely,

\[
V_n(\cdot) \overset{d}{\rightarrow} CV(\cdot) \quad \text{as } n \rightarrow \infty,
\]

in \(D[0, 1]\) endowed with the \(M_2\) topology, where \(V\) is an \(\alpha\)-stable Lévy process with characteristic triple \((0, \mu, b)\), with \(\mu\) as in (1.4),

\[
b = \begin{cases}
0, & \alpha = 1, \\
(p-r) \frac{\alpha}{1-\alpha}, & \alpha \in (0, 1) \cup (1, 2),
\end{cases}
\]

\(\tilde{C}\) is a random variable, independent of \(V\), such that \(\tilde{C} \overset{d}{=} C\), and \(D[0, 1]\) is the space of real–valued right continuous functions on \([0, 1]\) with left limits. When the sequence of coefficients \((C_j)\) is deterministic, relation (1.10) reduces to

\[
V_n(\cdot) \overset{d}{\rightarrow} CV(\cdot) \quad \text{as } n \rightarrow \infty
\]

(see Proposition 3.2 in Krizmanić [17]). This functional convergence, as shown by Avram and Taqqu [2], can not be strengthened to the Skorohod \(J_1\) convergence on \(D[0, 1]\), but if all coefficients are nonnegative, then it holds in the \(M_1\) topology.

More precisely, let \(X_j = \sum_{j \in \mathbb{Z}} C_j Z_{i-j}\) be a linear process with independent, regularly varying innovations \(Z_i\) with index of regular variation \(\alpha \in (0, 2)\), and
deterministic coefficients that are summable: $\sum_{j \in \mathbb{Z}} |c_j| < \infty$. Assume also $E(Z_1) = 0$ if $\alpha \in (1, 2)$, and $Z_1$ is symmetric if $\alpha = 1$. Then it is known that

$$V_n(\cdot) \xrightarrow{\text{fd}i} \left( \sum_{j \in \mathbb{Z}} c_j \right) V(\cdot) \quad \text{as } n \to \infty, \quad (1.11)$$

where $V$ is an $\alpha$–stable Lévy process and “$\xrightarrow{\text{fd}i}$” denotes convergence of finite-dimensional distributions (see Astrauskas [1], Theorem 1i, and Balan et al. [3], Theorem 2.1). Avram and Taqqu [2] in their Theorem 1 showed that in the case of finite-order moving average with at least two non-zero coefficients the convergence in (1.11) does not hold in the Skorohod $J_1$ topology (when only one coefficient is non-zero, the $J_1$ convergence holds by Basrak and Krizmanic [5], Theorem 2), but the results by Louhichi and Rio [22] show that it can be dropped. Theorem 2, Proposition 1 and (1.11) does not hold in the Skorohod $J_1$ topology. Recently, Balan et al. [3] obtained functional convergence in a weaker topology: if $c_j = 0$ for $j < 0$, $c_0, c_1, \ldots \in \mathbb{R}$ and for every $s = 0, 1, 2, \ldots$

$$0 \leq \sum_{j=0}^{s} c_j / \sum_{j=0}^{\infty} c_j \leq 1$$

(i.e. all partial sums of the series of coefficients are bounded between zero and the sum of the series), then (1.11) holds in the $M_2$ topology, the weakest of the four Skorohod topologies. Recently, Balan et al. [3] obtained functional convergence in the $S$ topology under every of the the following three sets of conditions: (i) $\alpha \in (1, 2)$ and $\sum_{j \in \mathbb{Z}} |c_j| < \infty$; (ii) $\alpha \leq 1$, $\sum_{j \in \mathbb{Z}} |c_j|^\alpha < \infty$ and the function $L$ from (1.11) satisfies $L(\lambda x)/L(x) \leq M$ for $\lambda > 1$ and $x \geq x_0$ (for some constants $M$, $x_0$); (iii) $\alpha < 1$, $\sum_{j \in \mathbb{Z}} |c_j|^\alpha < \infty$ and there exists a constant $0 < \gamma < \alpha$ such that

$$\frac{\max_{j+1 \leq k \leq j+n} |c_k|^{\frac{1-\alpha(\gamma-\alpha)}{1-\gamma}}}{\sum_{k=j+1}^{j+n} |c_k|^\alpha} \leq K_+ < \infty, \quad j \geq 0,$$

$$\frac{\max_{j-n \leq k \leq j-1} |c_k|^{\frac{1-\alpha(\gamma-\alpha)}{1-\gamma}}}{\sum_{k=j-n}^{j-1} |c_k|^\alpha} \leq K_- < \infty, \quad j \leq 0$$

(with the convention that $0/0 \equiv 1$). The $S$ topology, introduced in Jakubowski [15], is a sequential and non-metric topology, weaker than the $M_1$ topology.
In this paper we aim to extend the functional convergence in (1.10) to the case when the innovations $Z_i$ are weakly dependent, i.e. $(Z_i)$ is a strongly mixing sequence which satisfies the local dependence condition $D'$ as is given in Davis [8]:

$$\lim_{k \to \infty} \limsup_{n \to \infty} n \sum_{i=1}^{[n/k]} P\left( \frac{|Z_0|}{a_n} > x, \frac{|Z_i|}{a_n} > x \right) = 0 \quad \text{for all } x > 0. \quad (1.12)$$

For instance, a process which is an instantaneous function of a stationary Gaussian process with covariance function $r_n$ behaving like $r_n \log n \to 0$ as $n \to \infty$ satisfies Condition (1.11), see Davis [8]. Other examples of time series that satisfy Condition (1.11), related to stochastic volatility models and ARMAX processes, can be found in Davis and Mikosch [9] and Ferreira and Canto e Castro [12]. This condition, together with the strong mixing property, assures that, as in the i.i.d. case, the extremes of the sequence $(Z_i)$ are isolated. This corresponds to the situation when the extremal index $\theta$ of the sequence $(Z_i)$, which can be interpreted as the reciprocal mean cluster size of large exceedances (c.f. Hsing et al. [13]), is equal to 1. When $\theta < 1$ clustering of extreme values occurs, and in general condition (1.12) and the convergence in (1.10) fail to hold, see Example 2.1 below for an illustration.

We also impose the following standard regularity conditions on $Z_1$:

$$EZ_1 = 0, \quad \text{if } \alpha \in (1, 2), \quad (1.13)$$

$$Z_1 \text{ is symmetric, } \quad \text{if } \alpha = 1. \quad (1.14)$$

Beside condition (1.7) we will require some other moment conditions, which will be specified in Section 3. Further, in the case $\alpha \in [1, 2)$ we will need to assume the following condition:

$$\lim_{u \downarrow 0} \limsup_{n \to \infty} P\left[ \max_{1 \leq k \leq n} \left( \sum_{i=1}^{k} \left( \frac{Z_i}{a_n 1\{|Z_i|/a_n \leq u\}} - \frac{Z_i}{a_n 1\{|Z_i|/a_n \leq u\}} \right) \right) > \epsilon \right] = 0 \quad (1.15)$$

for all $\epsilon > 0$. This condition holds if the sequence $(Z_i)$ is $\rho$-mixing at a certain rate (see Lemma 4.8 in Tyran-Kamińska [26]). In case $\alpha \in (0, 1)$ it is a simple consequence of regular variation and Karamata’s theorem. Similar conditions are used often in the related literature on the limit theory for partial sums, see [2, 4, 10, 26].
The Skorohod $M_2$ topology on $D[0,1]$ is defined using completed graphs and their parametric representations (see Section 12.11 in Whitt [28] for details). We will use the following characterization of the $M_2$ topology with the Hausdorff metric on the spaces of graphs: for $x_1, x_2 \in D[0,1]$, the $M_2$ distance between $x_1$ and $x_2$ is given by

$$d_{M_2}(x_1, x_2) = \left( \sup_{a \in \Gamma_{x_1}} \inf_{b \in \Gamma_{x_2}} d(a, b) \right) \vee \left( \sup_{a \in \Gamma_{x_2}} \inf_{b \in \Gamma_{x_1}} d(a, b) \right),$$

where $\Gamma_x$ is the completed graph of $x \in D[0,1]$ defined by

$$\Gamma_x = \{(t, z) \in [0,1] \times \mathbb{R} : z = \lambda x(t-) + (1 - \lambda)x(t) \text{ for some } \lambda \in [0,1] \},$$

where $x(t-)$ is the left limit of $x$ at $t$, $d$ is the metric on $\mathbb{R}^2$ defined by $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| \vee |y_1 - y_2|$ for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, $i = 1, 2$, and $a \vee b = \max\{a, b\}$. The metric $d_{M_2}$ induces the $M_2$ topology, which is weaker than the more frequently used $M_1$ and $J_1$ topologies.

The paper is organized as follows. In Section 2 we derive functional convergence of partial sum stochastic processes for finite order moving averages, and then in Section 3 we extend this result to infinite order moving average processes.

2. Finite order MA processes

Fix $q \in \mathbb{N}$ and let $C_0, C_1, \ldots, C_q$ be random variables satisfying

$$0 \leq \sum_{i=0}^{s} C_i \leq \sum_{i=0}^{q} C_i \leq 1 \text{ a.s. for every } s = 0, 1, \ldots, q. \tag{2.1}$$

Condition (2.1) implies that $C = \sum_{i=0}^{q} C_i$, $\sum_{i=0}^{s} C_i$ and $\sum_{i=s}^{q} C_i$ are a.s. of the same sign for every $s = 0, 1, \ldots, q$. Note that condition (2.1) is satisfied if the $C_j$’s are all nonnegative or all nonpositive.

Let $(X_t)$ be a moving average process defined by

$$X_t = \sum_{i=0}^{q} C_i Z_{t-i}, \quad t \in \mathbb{Z},$$

and let the corresponding partial sum process be

$$V_n(t) = \frac{1}{a_n} \sum_{i=1}^{\lfloor nt \rfloor} X_i, \quad t \in [0,1], \tag{2.2}$$

where the normalizing sequence $(a_n)$ satisfies (1.2).

**Theorem 2.1.** Let $(Z_i)_{i \in \mathbb{Z}}$ be a strictly stationary and strongly mixing random sequence of regularly varying random variables with index $\alpha \in (0,2)$, such that conditions (1.12), (1.13), and (1.14) hold. If $\alpha \in [1,2)$, also suppose that condition (1.15) holds. Assume $C_0, C_1, \ldots, C_q$ are random variables, independent of $(Z_i)$, that satisfy (2.1). Then

$$V_n(\cdot) \overset{d}{\to} \widetilde{C}V(\cdot) \text{ as } n \to \infty,$$

in $D[0,1]$ endowed with the $M_2$ topology, where $V$ is an $\alpha$–stable Lévy process with characteristic triple $(0, \mu, b)$, with $\mu$ as in (1.4),

$$b = \begin{cases} 0, & \alpha = 1, \\ (p-r)^{\frac{\alpha}{1-\alpha}}, & \alpha \in (0,1) \cup (1,2). \end{cases}$$
and $\tilde{C}$ is a random variable, independent of $V$, such that $\tilde{C} \sim C$.

In the proof of the above theorem we will use the following lemma (which can be proven as in Basrak and Krizmanić [4]), functional convergence for regularly varying time series with isolated extremes and appropriate modifications of Theorem 2.1 in Krizmanić [17].

Lemma 2.2. With the notation $C_i = 0$ for $i < 0$, it holds that:

(i) For $k < q$
\[ \sum_{i=1}^{k} \frac{C_i}{a_n} \sum_{i=1}^{k} \frac{X_i}{a_n} = \sum_{u=0}^{k-1} \frac{Z_{k-u}}{a_n} \sum_{s=u+1}^{q} C_s - \sum_{u=0}^{q-1} \frac{Z_{u}}{a_n} \sum_{s=u+1}^{q} C_s \]

(ii) For $k \geq q$
\[ \sum_{i=1}^{k} \frac{C_i}{a_n} \sum_{i=1}^{k} \frac{X_i}{a_n} = \sum_{u=0}^{q-1} \frac{Z_{k-u}}{a_n} \sum_{s=u+1}^{q} C_s - \sum_{u=0}^{q-1} \frac{Z_{u}}{a_n} \sum_{s=u+1}^{q} C_s \]

(iii) For $q \leq k \leq n - q$
\[ \sum_{i=1}^{k} \frac{C_i}{a_n} \sum_{i=1}^{k+q} \frac{X_i}{a_n} = -\sum_{u=0}^{q-1} \frac{Z_{u}}{a_n} \sum_{s=u+1}^{q} C_s - \sum_{u=0}^{q-1} \frac{Z_{k+u}}{a_n} \sum_{s=0}^{q-u} C_s \]

Proof. (Theorem 2.1) Condition (1.12) and the strong mixing property imply that the extremes of the sequence $(Z_i)$ are isolated, i.e. $\theta = 1$ (see Leadbetter and Rootzén [20], page 439, and Leadbetter et al. [19], Theorem 3.4.1) and the tail process $(Y_i)$ of the sequence $(Z_i)$ defined by $P(|Y_0| > y) = y^{-\alpha}$ for $y \geq 1$ and

\[ \left( x^{-1}Z_i \right)_{i \in \mathbb{Z}} \rightarrow d \left( Y_i \right)_{i \in \mathbb{Z}} \text{ as } x \rightarrow \infty, \]

is the same as for an i.i.d. sequence, that is, $Y_i = 0$ for $i \neq 0$, and $Y_0$ is as described above, see Basrak et al. [4]. As a special case of their main theorem on functional $M_1$ convergence of partial sum processes of stationary, regularly varying sequences for which all extremes within each cluster of large values have the same sign (i.e. the corresponding tail process almost surely has no two values of the opposite sign), Basrak et al. [4] obtained $M_1$ convergence for processes with isolated extremes. More precisely, they showed that for strictly stationary and strongly mixing sequences of regularly varying random variables that satisfy the dependence condition (1.12) and the vanishing small values condition (1.15), the properly centered partial sum process converges in distribution to an $\alpha$–stable Lévy process in $D[0,1]$ with the $M_1$ topology (see Basrak et al. [4], Theorem 3.4 and Example 4.1). We apply this result directly to our case to conclude that, as $n \rightarrow \infty$,

\[ \sum_{i=1}^{[nt]} \frac{Z_i}{a_n} - [nt]E\left( \frac{Z_1}{a_n} \mathbb{1}_{\{|Z_1| \leq a_n\}} \right), \quad t \in [0,1], \]
converges in distribution in $D[0, 1]$ with the $M_1$ topology to an $\alpha$–stable Lévy process with characteristic triple $(0, \mu, 0)$ if $\alpha = 1$ and $(0, \mu, (p - r)\alpha/(1 - \alpha))$ if $\alpha \in (0, 1) \cup (1, 2)$.

Since the space $D[0, 1]$ equipped with the $M_1$ topology is a Polish space (see Section 14 in Billingsley [6] and Section 12.8 in Whitt [28]), by Corollary 5.18 in Kallenberg [16] we can find a random variable $\tilde{C}$, independent of $V$, such that $\tilde{C} \overset{d}{=} C$. This and the fact that $C$ is independent of $V_n^Z$, by an application of Theorem 3.29 in Kallenberg [16], imply

$$(B(\cdot), V_n^Z(\cdot)) \overset{d}{\rightarrow} (\tilde{B}(\cdot), V(\cdot)), \quad \text{as } n \to \infty, \quad (2.3)$$

in $D([0, 1], \mathbb{R}^2)$ with the product $M_1$ topology, where $B(t) = C$ and $\tilde{B}(t) = \tilde{C}$ for $t \in [0, 1]$. Applying the continuous mapping theorem to relation (2.3) we obtain $B(\cdot)V_n^Z(\cdot) \overset{d}{\rightarrow} \tilde{B}(\cdot)V(\cdot)$ as $n \to \infty$, i.e. $CV_n^Z(\cdot) \overset{d}{\rightarrow} \tilde{C}V(\cdot)$ in $D[0, 1]$ with the $M_1$ topology. Since $M_1$ convergence implies $M_2$ convergence, we have

$$(CV_n^Z(\cdot)) \overset{d}{\rightarrow} \tilde{C}V(\cdot), \quad \text{as } n \to \infty, \quad (2.4)$$

in $(D[0, 1], d_{M_2})$ as well. It remains to show that for every $\epsilon > 0$

$$\lim_{n \to \infty} P[d_{M_2}(CV_n^Z, V_n) > \epsilon] = 0,$$

since then by an application of Slutsky’s theorem (see for instance Theorem 3.4 in Resnick [23]) it will follow that $V_n(\cdot) \overset{d}{\rightarrow} \tilde{C}V(\cdot)$ in $(D[0, 1], d_{M_2})$.

Fix $\epsilon > 0$ and let $n \in \mathbb{N}$ be large enough, i.e. $n > \max\{2q, 2q/\epsilon\}$. By the definition of the metric $d_{M_2}$ we have

$$d_{M_2}(CV_n^Z, V_n) = \left( \sup_{a \in \Gamma_{CV_n^Z}} \inf_{b \in \Gamma_{V_n}} d(a, b) \right) \lor \left( \sup_{a \in \Gamma_{V_n}} \inf_{b \in \Gamma_{CV_n^Z}} d(a, b) \right)$$

$$=: Y_n \lor T_n,$$

and therefore

$$P[d_{M_2}(CV_n^Z, V_n) > \epsilon] \leq P(Y_n > \epsilon) + P(T_n > \epsilon). \quad (2.5)$$
By the same arguments as in the proof of Theorem 2.1 in Krizmanić [17] (see also Basrak and Krizmanić [5]) for the first term on the right hand side of (2.5) we have

\[ \{ Y_n > \epsilon \} \subseteq \{ \exists a \in \Gamma_{CV_n} \text{ such that } d(a, b) > \epsilon \text{ for every } b \in \Gamma_{V_n} \} \]

\[ \subseteq \{ \exists k \in \{1, \ldots, q-1\} \text{ such that } |CV_n^Z(k/n) - V_n(k/n)| > \epsilon \} \]

\[ \cup \{ \exists k \in \{q, \ldots, n-q\} \text{ such that } |CV_n^Z(k/n) - V_n(k/n)| > \epsilon \}

\[ \text{and } |CV_n^Z((k+q)/n) - V_n((k+q)/n)| > \epsilon \}

\[ \cup \{ \exists k \in \{n-q+1, \ldots, n\} \text{ such that } |CV_n^Z(k/n) - V_n(k/n)| > \epsilon \} \]

\[ =: A_n^Y \cup B_n^Y \cup C_n^Y. \quad (2.6) \]

By Lemma 2.2 (i) we obtain

\[ P(A_n^Y) \leq \sum_{k=1}^{q-1} P\left( \left| \sum_{i=1}^{k} CZ_i - \sum_{i=1}^{k} X_i \right| > \epsilon \right) \]

\[ \leq \sum_{k=1}^{q-1} \left[ P\left( \sum_{u=0}^{k-1} \frac{|Z_{k-u}|}{a_n} \sum_{s=u+1}^{q} |C_s| > \frac{\epsilon}{3} \right) + P\left( \sum_{u=k-q}^{q-1} \frac{|Z_{u-k}|}{a_n} \sum_{s=u+1}^{q} |C_s| > \frac{\epsilon}{3} \right) \]

\[ + P\left( \sum_{u=0}^{q-k-1} \frac{|Z_{u-k}|}{a_n} \sum_{s=u+1}^{q} |C_s| > \frac{\epsilon}{3} \right) \]

\[ \leq 3(q-1)(2q-1) P\left( \frac{|Z_0|}{a_n} C_* > \frac{\epsilon}{3(2q-1)} \right), \quad (2.7) \]

where \( C_* = \sum_{s=0}^{q} |C_s| \). Take now \( M > 0 \) arbitrary and note

\[ P\left( \frac{|Z_0|}{a_n} C_* > \frac{\epsilon}{3(2q-1)} \right) \]

\[ = P\left( \frac{|Z_0|}{a_n} C_* > \frac{\epsilon}{3(2q-1)}, C_* > M \right) + P\left( \frac{|Z_0|}{a_n} C_* > \frac{\epsilon}{3(2q-1)}, C_* \leq M \right) \]

\[ \leq P\left( C_* > M \right) + P\left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{3(2q-1)M} \right). \]

Since by the regular variation property it holds that

\[ \lim_{n \to \infty} P\left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{3(2q-1)M} \right) = 0, \]

from (2.7) we get

\[ \limsup_{n \to \infty} P(A_n^Y) \leq P\left( C_* > M \right). \]

Letting \( M \to \infty \) we conclude

\[ \lim_{n \to \infty} P(A_n^Y) = 0. \quad (2.8) \]
Similarly

\[ \lim_{n \to \infty} P(C_n^V) = 0. \]  

(2.9)

Next, using Lemma 2.2 (ii) and (iii), for an arbitrary \( M > 0 \) we obtain

\[
\begin{align*}
P(B_n^V \cap \{ C_s \leq M \}) &= P \left( \exists k \in \{ q, \ldots, n-q \} \text{ such that } |H_n^k - G_n| > \epsilon \\
&\quad \text{and } | - G_n - T_n^k | > \epsilon, C_s \leq M \right) \\
&\leq P \left( |G_n| > \frac{\epsilon}{2}, C_s \leq M \right) + \sum_{k=q}^{n-q} P \left( \left| H_n^k \right| > \frac{\epsilon}{2}, \left| T_n^k \right| > \frac{\epsilon}{2}, C_s \leq M \right).
\end{align*}
\]  

(2.10)

Note that

\[
P \left( |G_n| > \frac{\epsilon}{2}, C_s \leq M \right) \leq P \left( C_s \sum_{u=0}^{q-1} \left| \frac{Z_{-u}}{a_n} \right| > \frac{\epsilon}{2}, C_s \leq M \right)
\]

\[
\leq P \left( \sum_{u=0}^{q-1} \left| \frac{Z_{-u}}{a_n} \right| > \frac{\epsilon}{2M} \right)
\]

\[
\leq q P \left( \left| \frac{Z_0}{a_n} \right| > \frac{\epsilon}{2qM} \right),
\]

and an application of the regular variation property yields

\[
\lim_{n \to \infty} P \left( |G_n| > \frac{\epsilon}{2}, C_s \leq M \right) = 0.
\]  

(2.11)

Further, since

\[
H_n^k = \sum_{u=0}^{q-1} \frac{Z_{k-u}}{a_n} \sum_{s=u+1}^{q} C_s \quad \text{and} \quad T_n^k = \sum_{u=1}^{q} \frac{Z_{k+u}}{a_n} \sum_{s=0}^{q-u} C_s,
\]

for a fixed \( k \in \{ q, \ldots, n-q \} \), on the event \( \{ |H_n^k| > \epsilon/2 \text{ and } |T_n^k| > \epsilon/2, C_s \leq M \} \) there exist \( i \in \{ k-(q-1), \ldots, k \} \) and \( j \in \{ k+1, \ldots, k+q \} \) such that

\[
\frac{|Z_i|}{a_n} > \frac{\epsilon}{2qM} \text{ and } \frac{|Z_j|}{a_n} > \frac{\epsilon}{2qM}.
\]

Therefore, using the stationarity of the sequence \((Z_i)\) we obtain

\[
P \left( |H_n^k| > \frac{\epsilon}{2} \text{ and } |T_n^k| > \frac{\epsilon}{2}, C_s \leq M \right)
\]

\[
\leq \sum_{i=k-(q-1), \ldots, k} \sum_{j=k+1, \ldots, k+q} P \left( \frac{|Z_i|}{a_n} > \frac{\epsilon}{2qM}, \frac{|Z_j|}{a_n} > \frac{\epsilon}{2qM} \right)
\]

\[
\leq q \sum_{j=1}^{2q-1} P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{2qM}, \frac{|Z_j|}{a_n} > \frac{\epsilon}{2qM} \right),
\]
and hence for all positive integers $s \leq n/(2q - 1)$ it holds
\[
\sum_{k=q}^{n-q} P \left( |H_n^k| > \frac{\epsilon}{2} \text{ and } |T_n^k| > \frac{\epsilon}{2}, C_s \leq M \right) \leq nq \sum_{j=1}^{\lfloor n/s \rfloor} P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{2qM}, |Z_j|/a_n > \frac{\epsilon}{2qM} \right).
\]

From this, taking into account condition (1.12), we conclude that
\[
\lim_{n \to \infty} \sum_{k=q}^{n-q} P \left( |H_n^k| > \frac{\epsilon}{2} \text{ and } |T_n^k| > \frac{\epsilon}{2}, C_s \leq M \right) = 0.
\]
Together with relations (2.10) and (2.11) this implies
\[
\lim_{n \to \infty} P \left( B_n^Y \cap \{ C_s \leq M \} \right) = 0.
\]
Thus
\[
\limsup_{n \to \infty} P \left( B_n^Y \right) \leq \limsup_{n \to \infty} P \left( B_n^Y \cap \{ C_s > M \} \right) \leq P(C_s > M),
\]
and letting again $M \to \infty$ we conclude
\[
\lim_{n \to \infty} P(B_n^Y) = 0. \tag{2.12}
\]
From relations (2.6), (2.8), (2.9) and (2.12) we obtain
\[
\lim_{n \to \infty} P(Y_n > \epsilon) = 0. \tag{2.13}
\]

In order to estimate the second term on the right hand side of (2.7) define for each $k \geq q$ the numbers $V_k^{Z,\min} = \min\{ CV_n^{Z,((k-q)/n)}, CV_n^{Z,k/n} \}$ and $V_k^{Z,\max} = \max\{ CV_n^{Z,((k-q)/n)}, CV_n^{Z,k/n} \}$. Following the arguments in the proof of Theorem 2.1 in Krizmanić [17] we obtain
\[
\{ T_n > \epsilon \} \subseteq \{ \exists a \in \Gamma_{V_n} \text{ such that } d(a, b) > \epsilon \text{ for every } b \in \Gamma_{CV_n^{Z}} \}
\]
\[
\subseteq \{ \exists k \in \{1, \ldots, 2q - 1\} \text{ such that } |V_n(k/n) - CV_n^{Z}(k/n)| > \epsilon \}
\]
\[
\cup \left\{ \exists k \in \{2q, \ldots, n\} \text{ such that } \tilde{d}(V_n(k/n), [V_k^{Z,\min}, V_k^{Z,\max}]) > \epsilon \right\}
\]
\[
=: A_n^{T} \cup B_n^{T}, \tag{2.14}
\]
where $\tilde{d}$ is the Euclidean metric on $\mathbb{R}$. Using Lemma 2.2 (i) and (ii), one could similarly as before for the set $A_n^{T}$ obtain
\[
\lim_{n \to \infty} P(A_n^{T}) = 0. \tag{2.15}
\]
Note that $P(B_n^T)$ is bounded above by
\[
P \left( \exists k \in \{2q, \ldots, n\} \text{ such that } \sum_{i=1}^{k} \frac{X_i}{a_n} > V_k^{Z,\max} + \epsilon \right) + P \left( \exists k \in \{2q, \ldots, n\} \text{ such that } \sum_{i=1}^{k} \frac{X_i}{a_n} < V_k^{Z,\min} - \epsilon \right).
\]
We consider only the first of these two probabilities, since the other one can be handled in a similar manner. Using Lemma 2.2 the first probability can be bounded by
\[
P \left( \exists k \in \{2q, \ldots, n\} \text{ such that } G_n - H_n^k > \epsilon \text{ and } G_n + T_n^{k-q} > \epsilon \right)
\leq \ P \left( G_n > \frac{\epsilon}{2} \right) + P \left( \exists k \in \{2q, \ldots, n\} \text{ such that } H_n^k < -\frac{\epsilon}{2} \text{ and } T_n^{k-q} > \frac{\epsilon}{2} \right).
\]
From the calculations yielding (2.12) we conclude that \( P(G_n > \epsilon/2) \to 0 \) as \( n \to \infty \).

The second term is bounded by
\[
P(C_* > M) + \sum_{k=2q}^{n} P \left( H_n^k < -\frac{\epsilon}{2} \text{ and } T_n^{k-q} > \frac{\epsilon}{2}, C_* \leq M \right) \quad (2.16)
\]
for an arbitrary \( M > 0 \). Note that
\[
H_n^k = \sum_{u=0}^{q-1} \frac{Z_{k-u}}{a_n} \sum_{s=0}^{q} C_s \quad \text{and} \quad T_n^{k-q} = \sum_{u=0}^{q-1} \frac{Z_{k-u}}{a_n} \sum_{s=0}^{n} C_s.
\]
Hence for a fixed \( k \in \{2q, \ldots, n\} \), on the event \( \{H_n^k < -\epsilon/2 \text{ and } T_n^{k-q} > \epsilon/2, C_* \leq M\} \) there exist \( i, j \in \{0, \ldots, q-1\} \) such that
\[
\frac{Z_{k-i}}{a_n} \sum_{s=0}^{q} C_s < -\frac{\epsilon}{2q} \quad \text{and} \quad \frac{Z_{k-j}}{a_n} \sum_{s=0}^{j} C_s > \frac{\epsilon}{2q}.
\]
Condition (2.11) implies the sums \( \sum_{s=i+1}^{q} C_s \) and \( \sum_{s=i+1}^{q} C_s \) are a.s. of the same sign, and since their absolute values are bounded by \( C_* \), we obtain \( |Z_{k-i}|M/a_n > \epsilon/(2q) \) and \( |Z_{k-j}|M/a_n > \epsilon/(2q) \). The case \( i = j \) is not possible since then we would have \( Z_{k-i} < 0 \) and \( Z_{k-i} > 0 \). From this, using the stationarity of the sequence \( (Z_t) \), we conclude that the expression in (2.16) is bounded by
\[
P(C_* > M) + nP \left( \exists i, j \in \{0, \ldots, q-1\}, i \neq j \text{ s.t. } M \left| \frac{Z_{i-1}}{a_n} \right| > \frac{\epsilon}{2q} \text{ and } M \left| \frac{Z_{j-1}}{a_n} \right| > \frac{\epsilon}{2q} \right)
\leq P(C_* > M) + 2n(q-1) \sum_{j=1}^{q-1} P \left( \left| \frac{Z_j}{a_n} \right| > \frac{\epsilon}{2qM} \text{ and } M \left| \frac{Z_{j-1}}{a_n} \right| > \frac{\epsilon}{2qM} \right),
\]
which, similarly as before when considering the set \( D_n^Y \), tends to 0 if we first let \( n \to \infty \) and then \( M \to \infty \). Together with relations (2.14) and (2.15) this implies
\[
\lim_{n \to \infty} P(T_n > \epsilon) = 0. \quad (2.17)
\]
Now from (2.15), (2.13) and (2.17) we obtain
\[
\lim_{n \to \infty} P[d_{M_1}(CV_{Z_n}^{\cdot}, V_n) > \epsilon] = 0, \quad (2.18)
\]
and finally we conclude that \( V_n(\cdot) \overset{d}{\to} CV(\cdot) \), as \( n \to \infty \), in \( (D[0,1], d_{M_1}) \). This concludes the proof.

\textit{Example 2.1.} Condition (1.12) prohibits clustering of extreme values in the sequence \( (Z_t) \), which means that it has extremal index \( \theta = 1 \). Here we give an example when clustering of extreme values occurs, and all conditions in Theorem 2.1 hold except
condition (1.12), but the convergence of the partial sum stochastic process \( V_n \), as defined in (2.2), in \( D[0,1] \) with the \( M_2 \) topology fails to hold.

Let \( (\xi_i) \) be a sequence of i.i.d. regularly varying random variables with index of regular variation \( \alpha \in (0,2) \). Define

\[
Z_i = \xi_i + \xi_{i-2}, \quad i \in \mathbb{Z}
\]

and assume conditions (1.13) and (1.14) hold. The sequence \((Z_i)\) is strictly stationary and consists of regularly varying random variables (see Proposition 7.4 in Resnick [23] and Theorem 1.28 in Lindskog [21]). Consider the finite order moving average process

\[
X_t = Z_t - Z_{t-1} + Z_{t-2}, \quad t \in \mathbb{Z}.
\]

Hence \( q = 2 \), \( C_0 = 1 \), \( C_1 = -1 \), \( C_2 = 1 \), and condition (2.1) clearly holds. Let \((a_n)\) be a sequence of positive real numbers for which (1.2) holds. By Lemma 1.2 in Cline [7]

\[
\lim_{x \to \infty} \frac{P(|Z_0| > x)}{P(|\xi_0| > x)} = 3,
\]

which yields

\[
\lim_{n \to \infty} n P(|\xi_0| > a_n) = 1/3. \tag{2.19}
\]

This together with the regular variation property of \( \xi_i \) and the following inequality

\[
n P \left( \frac{|Z_0|}{a_n} > x, \frac{|Z_2|}{a_n} > x \right) \geq n P \left( \frac{|\xi_0|}{a_n} > 2x, \frac{|\xi_2|}{a_n} \leq x, \frac{|\xi_2| - 2}{a_n} \leq x \right)
\]

implies

\[
\liminf_{n \to \infty} n P \left( \frac{|Z_0|}{a_n} > x, \frac{|Z_2|}{a_n} > x \right) \geq \frac{1}{3} (2x)^{-\alpha} > 0,
\]

for \( x > 0 \), and therefore we conclude that condition (1.12) does not hold. The sequence \((Z_i)\) is 2–dependent, and hence strongly mixing, with extremal index \( \theta = 1/2 \) (see Embrechts et al. [11], page 415).

Next we show that \( V_n \) does not converge in distribution under the \( M_2 \) topology on \( D[0,1] \). For this, according to Skorohod [24] (cf. Proposition 2 in Avram and Taqqu [2]), it suffices to show that

\[
\lim_{\delta \to 0} \limsup_{n \to \infty} P[\Delta_{M_2}(\delta, V_n) > \epsilon] > 0 \tag{2.20}
\]

for some \( \epsilon > 0 \), where

\[
\Delta_{M_2}(\delta, x) = \sup_{0 \leq t \leq 1} M(x(t_1), x(t), x(t_2))
\]

\[
= \begin{cases} 0, & t_5 \leq t_1 \leq t_5 + \delta/2 \\
\min\{|x_2 - x_1|, |x_3 - x_2|\}, & \text{otherwise}
\end{cases}
\]

\((x \in D[0,1], \delta > 0), t_5 = \max\{0, t - \delta\}, t_5^* = \min\{1, t + \delta\}\), and

\[
M(x_1, x_2, x_3) = \begin{cases} 0, & \text{if } x_2 \in [x_1, x_3], \\
\min\{|x_2 - x_1|, |x_3 - x_2|\}, & \text{otherwise}
\end{cases}
\]

Note that \( M(x_1, x_2, x_3) \) is the distance from \( x_2 \) to \( [x_1, x_3] \), and \( \Delta_{M_2}(\delta, x) \) is the \( M_2 \) oscillation of \( x \). To show (2.20) we use, with appropriate modifications, the procedure of Avram and Taqqu [2] in the proof of their Theorem 1.
Let \( i' = i'(n) \) be the index at which \( \max_{1 \leq i \leq n-2} |\xi_i| \) is obtained. Fix \( \epsilon > 0 \) and introduce the events

\[
A_{n,\epsilon} = \{ \max_{1 \leq i \leq n-2} |\xi_i| > \epsilon a_n \}
\]

and

\[
B_{n,\epsilon} = \{ |\xi_i| > \epsilon a_n \text{ and } \exists l \neq 0, -i' - 3 \leq l \leq 1, \text{ such that } |\xi_{i'+l}| > \epsilon a_n/8. \}
\]

Using the facts that \( (\xi_i) \) is an i.i.d. sequence and \( n P(|\xi_0| > \lambda a_n) \to \lambda^{-\alpha}/2 \) as \( n \to \infty \), for \( \lambda > 0 \) (which follows from the regular variation property of \( \xi_0 \) and (2.19)) we get

\[
P(A_{n,\epsilon}) = 1 - \left[ 1 - \frac{n P(|\xi_0| > \epsilon a_n)}{n} \right]^{n-2} \to 1 - e^{-\epsilon^{\alpha}/2},
\]

as \( n \to \infty \), and

\[
\limsup_{n \to \infty} P(B_{n,\epsilon}) \leq \limsup_{n \to \infty} P \left( \bigcup_{i=1}^{n-2} \left\{ |\xi_i| > \epsilon a_n, |\xi_{i+l}| > \epsilon a_n/8 \right\} \right)
\]

\[
\leq \limsup_{n \to \infty} (n-2)(n+2) P(|\xi_0| > \epsilon a_n) P(|\xi_0| > \epsilon a_n/8)
\]

\[
= \frac{e^{-2\alpha}}{4 \cdot 8^{-\alpha}}.
\]

On the event \( A_{n,\epsilon} \setminus B_{n,\epsilon} \) one has \( |\xi_{i'}| > \epsilon a_n \) and \( |\xi_{i'+l}| \leq \epsilon a_n/8 \) for every \( l \in \{-i' - 3, \ldots 1\} \setminus \{0\} \), and hence since

\[
V_n(t) = \sum_{i=1}^{[nt]} X_i = a_n \left[ \xi_{[nt]} + 2\xi_{[nt]-2} + \xi_{[nt]-3} + 2(\xi_{[nt]-4} + \ldots + \xi_1) + \xi_0 + 2\xi_{-1} + \xi_{-3} \right],
\]

we have

\[
\left| V_n \left( \frac{i'}{n} \right) - V_n \left( \frac{i'-1}{n} \right) \right| = \frac{|\xi_{i'} - \xi_{i'-1} + 2\xi_{i'-2} - \xi_{i'-3} + \xi_{i'-4}|}{a_n} > \epsilon - \frac{5\epsilon}{8} = \frac{3\epsilon}{8}
\]

and

\[
\left| V_n \left( \frac{i'+1}{n} \right) - V_n \left( \frac{i'}{n} \right) \right| = \frac{|\xi_{i'+1} - \xi_{i'} + 2\xi_{i'-1} - \xi_{i'-2} + \xi_{i'-3}|}{a_n} > \epsilon - \frac{5\epsilon}{8} = \frac{3\epsilon}{8}.
\]

Further, on the event \( A_{n,\epsilon} \setminus B_{n,\epsilon} \) it also holds that

\[
V_n \left( \frac{i'}{n} \right) \notin \left[ V_n \left( \frac{i'-1}{n} \right), V_n \left( \frac{i'+1}{n} \right) \right],
\]

since

\[
\max \left\{ V_n \left( \frac{i'-1}{n} \right), V_n \left( \frac{i'+1}{n} \right) \right\} < V_n \left( \frac{i'}{n} \right) \text{ if } \xi_{i'} > 0,
\]

and

\[
\min \left\{ V_n \left( \frac{i'-1}{n} \right), V_n \left( \frac{i'+1}{n} \right) \right\} > V_n \left( \frac{i'}{n} \right) \text{ if } \xi_{i'} < 0.
\]
Since that stochastic process may fail to hold. Here the fact that average process in that index $\theta M$ condition (1.12) and functional $Z$ where

\[ x^\delta \geq M \left( V_n \left( \frac{i^\prime - 1}{n} \right), V_n \left( \frac{i^\prime}{n} \right), V_n \left( \frac{i^\prime + 1}{n} \right) \right) \]

Taking into account (2.23) and (2.24) we obtain

\[ \Delta_{M_2} \left( 1/n, V_n \right) = \sup_{0 \leq t \leq 1} M(V_n(t_1), V_n(t), V_n(t_2)) \]

\[ \geq M \left( V_n \left( \frac{i^\prime - 1}{n} \right), V_n \left( \frac{i^\prime}{n} \right), V_n \left( \frac{i^\prime + 1}{n} \right) \right) > \frac{3\epsilon}{8} \]

on the event $A_{n,\epsilon} \setminus B_{n,\epsilon}$. Therefore, since $\Delta_{M_2}(\delta, V_n)$ is nondecreasing in $\delta$, it holds that

\[ \lim_{n \to \infty} \inf P(A_{n,\epsilon} \setminus B_{n,\epsilon}) \leq \lim_{n \to \infty} \inf P(\Delta_{M_2}(1/n, V_n) > 3\epsilon/8) \]

\[ \leq \lim_{\delta \to 0} \limsup_{n \to \infty} P(\Delta_{M_2}(\delta, V_n) > 3\epsilon/8). \quad (2.26) \]

Since $x^{2\alpha} (1 - e^{-x^{\alpha}/2})$ tends to infinity as $x \to \infty$, we can find $\epsilon > 0$ such that $e^{2\alpha} (1 - e^{-\epsilon^{\alpha}/2}) > 8^\alpha/4$, i.e.

\[ 1 - e^{-\epsilon^{\alpha}/2} > \frac{e^{-2\alpha}}{4 \cdot 8^{\alpha}}. \]

For this $\epsilon$, by relations (2.21) and (2.22), it holds that

\[ \lim_{n \to \infty} P(A_{n,\epsilon}) > \limsup_{n \to \infty} P(B_{n,\epsilon}), \]

i.e.

\[ \lim_{n \to \infty} \inf P(A_{n,\epsilon} \setminus B_{n,\epsilon}) \geq \lim_{n \to \infty} P(A_{n,\epsilon}) - \limsup_{n \to \infty} P(B_{n,\epsilon}) > 0. \]

Therefore by (2.20) we obtain

\[ \lim_{\delta \to 0} \limsup_{n \to \infty} P(\Delta_{M_2}(\delta, V_n) > 3\epsilon/8) > 0 \]

and relation (2.20) holds, which means that $V_n$ does not converge in distribution in $D[0,1]$ endowed with the $M_2$ topology.

Using similar arguments, one can obtain the same conclusion for the moving average process

\[ X_i^t = Z_i^t + Z_i^t, \quad t \in \mathbb{Z}, \]

where $Z_i^t = \xi_i - \xi_{i-1}$, $i \in \mathbb{Z}$, but in this case the sequence $(Z_i^t)$ has extremal index $\theta = 1$. Therefore, even when clustering of extreme values do not occur, condition (1.12) and functional $M_2$ convergence of the corresponding partial sum stochastic process may fail to hold. Here the fact that $V_n$ does not converge in $D[0,1]$ with the $M_2$ topology can be seen also by the following reasoning. Observe that $X_i^t = \xi_i - \xi_{i-2}$, and hence, as $n \to \infty$,

\[ V_n(\cdot) = \frac{1}{a_n} \sum_{i=1}^{[n]} X_i^t = \frac{\xi_{[n]} + \xi_{[n] - 1} - \xi_0 - \xi_{-1}}{a_n} \to 0. \quad (2.27) \]
Since, as is known, \( \sup_{t \in [0,1]} \xi_{[nt]}/a_n \) converges in distribution to a non-zero limit (cf. Proposition 7.2 in Resnick [23]) and the functional \( \sup_{t \in [0,1]} \) is continuous in the \( M_2 \) topology (see Skorohod [24]), the “fidi” convergence in (2.27) can not be replaced by convergence in distribution under the \( M_2 \) topology (neither under the other Skorohod’s topologies).

3. Infinite order MA processes

For infinite order moving averages the idea is to approximate them by a sequence of finite order moving averages, for which Theorem 2.1 holds, and to show that the error of approximation is negligible in the limit. In the case \( \alpha \in (0,1) \) one can follow directly the lines in the proof of Theorem 3.1 in Krizmanić [17] to obtain the functional convergence of the corresponding partial sum stochastic processes. In the case \( \alpha \in [1,2) \) the arguments from the proof of Theorem 3.1 in Krizmanić [17] can not be applied to our setting, since due to the dependence in the sequence \( (Z_i) \), certain sequences constructed from \( (C_i) \) and \( (Z_i) \) are no longer martingales and martingale-difference sequences, which was crucial in obtaining functional convergence for infinite order moving averages with i.i.d. innovations. The idea in this case is to use the arguments from Lemma 2 in Tyran-Kamińska [27] to show that functional convergence of the partial sum stochastic processes still holds. More precisely, we have the following result.

**Theorem 3.1.** Let \( (X_i) \) be a moving average process defined by

\[
X_i = \sum_{j=0}^{\infty} C_j Z_{i-j}, \quad i \in \mathbb{Z},
\]

where \( (Z_i)_{i \in \mathbb{Z}} \) is a strictly stationary and strongly mixing sequence of regularly varying random variables with index \( \alpha \in (0,2) \), such that conditions (1.12), (1.13) and (1.14) hold, and \( (C_i)_{i \geq 0} \) is a sequence of random variables, independent of \( (Z_i) \), satisfying conditions (1.17) and (1.18). If \( \alpha \in (0,1) \) suppose further

\[
\sum_{i=0}^{\infty} \mathbb{E}|C_i|^\gamma < \infty \quad \text{for some } \gamma \in (\alpha,1),
\]  
(3.1)

while if \( \alpha \in [1,2) \) suppose condition (1.15) holds,

\[
\lim_{n \to \infty} \sup_{j \geq 0} \mathbb{E} \left[ \max_{1 \leq l \leq n} \left| \frac{1}{a_n} \sum_{i=1}^{l} Z_{i-j} 1\{|Z_{i-j}| \leq a_n\} \right|^r \right] < \infty \quad \text{for some } r \geq 1,
\]  
(3.2)

and

\[
\sum_{j=0}^{\infty} \mathbb{E}|C_j| < \infty.
\]  
(3.3)

Then

\[
V_n(\cdot) \xrightarrow{d} \tilde{C} V(\cdot), \quad n \to \infty,
\]

in \( D[0,1] \) endowed with the \( M_2 \) topology, where \( V \) is an \( \alpha \)-stable Lévy process with characteristic triple \( (0,\mu,b) \), with \( \mu \) as in (1.4) and

\[
b = \left\{ \begin{array}{ll}
0, & \alpha = 1, \\
(p-r)\frac{\alpha}{1-\alpha}, & \alpha \in (0,1) \cup (1,2),
\end{array} \right.
\]

and \( \tilde{C} \) is a random variable, independent of \( V \), such that \( \tilde{C} \xrightarrow{d} \sum_{i=0}^{\infty} C_i \).
Proof. Take \( q \in \mathbb{N} \) and define
\[
X_i^q = \sum_{j=0}^{q-1} C_j Z_{i-j} + C_q' Z_{i-q}, \quad i \in \mathbb{Z},
\]
where \( C_q' = \sum_{i=q}^{\infty} C_i \), and
\[
V_{n,q}(t) = \sum_{i=1}^{[nt]} \frac{X_i^q}{a_n}, \quad t \in [0,1].
\]
The coefficients \( C_0, \ldots, C_{q-1}, C_q' \) satisfy condition (2.1), and hence an application of Theorem 3.5 in Resnick [23] to a finite order moving average process \((X_i^q)\) yields that, as \( n \to \infty \),
\[
V_{n,q}(\cdot) \overset{d}{\to} \widetilde{CV}(\cdot)
\]
in \((D[0,1], d_{M_2})\), where \( V \) is an \( \alpha \)-stable Lévy process with characteristic triple as in Theorem 2.1 and \( \widetilde{C} \) is a random variable, independent of \( V \), such that \( \widetilde{C} \overset{d}{\to} \sum_{i=0}^{\infty} C_i \).

In the case \( \alpha \in (0,1) \) by repeating the arguments from the proof of Theorem 3.1 in Krizmanič [17] we obtain
\[
\lim_{q \to \infty} \limsup_{n \to \infty} \mathbb{P}[d_{M_2}(V_{n,q}, V_n) > \epsilon] = 0, \quad (3.4)
\]
for every \( \epsilon > 0 \). Therefore, by a generalization of Slutsky’s theorem (see for instance Theorem 3.5 in Resnick [23]) it follows that \( V_n(\cdot) \overset{d}{\to} \widetilde{CV}(\cdot) \), as \( n \to \infty \), in \((D[0,1], d_{M_2})\).

Assume now \( \alpha \in [1,2) \). We will use the arguments from the proof of Lemma 2 in Tyran-Kamińska [27] adapted to linear processes with random coefficients instead of deterministic. Define \( Z_{n,j} = a_n^{-1} Z_j 1_{\{|Z_j| \leq a_n\}} \) for \( j \in \mathbb{Z} \) and \( n \in \mathbb{N} \),
\[
\widetilde{C}_j = \begin{cases} 
C_j, & \text{if } j > q, \\
C_q - C_q', & \text{if } j = q,
\end{cases}
\]
and note that
\[
V_n(t) - V_{n,q}(t) = \sum_{i=1}^{[nt]} \frac{1}{a_n} \left( \sum_{j=q}^{\infty} C_j Z_{i-j} - C_q' Z_{i-q} \right)
\]
\[
= \sum_{i=1}^{[nt]} \sum_{j=q}^{\infty} \frac{\widetilde{C}_j Z_{n,i-j}}{a_n} + \sum_{i=1}^{[nt]} \sum_{j=q}^{\infty} \frac{\widetilde{C}_j Z_{i-j}}{a_n} 1_{\{|Z_{i-j}| > a_n\}}.
\]
Since the Skorohod \( M_2 \) metric on \([0,1]\) is bounded above by the uniform metric on \([0,1]\), we have
\[
\mathbb{P}[d_{M_2}(V_{n,q}, V_n) > \epsilon] \leq \mathbb{P} \left( \sup_{0 \leq t \leq 1} |V_n(t) - V_{n,q}(t)| > \epsilon \right)
\]
\[
\leq \mathbb{P} \left( \max_{1 \leq i \leq n} \sum_{i=1}^{[nt]} \sum_{j=q}^{\infty} \frac{\widetilde{C}_j Z_{i,j}}{a_n} > \epsilon \right) + \mathbb{P} \left( \max_{1 \leq i \leq n} \sum_{i=1}^{[nt]} \sum_{j=q}^{\infty} \frac{\widetilde{C}_j Z_{i,j}}{a_n} 1_{\{|Z_{i,j}| > a_n\}} > \frac{\epsilon}{2} \right)
\]
\[
= I_1 + I_2. \quad (3.5)
\]
By Hölder’s inequality we have
\[
\left( \sum_{j=q}^{\infty} |\bar{C}_j| \cdot \left| \sum_{i=1}^{l} Z_{n,i-j} \right| \right)^r \leq \left( \sum_{j=q}^{\infty} |\bar{C}_j| \right)^{r-1} \sum_{j=q}^{\infty} |\bar{C}_j| \cdot \left| \sum_{i=1}^{l} Z_{n,i-j} \right|^r,
\]
with \( r \) as in (3.6), and therefore using Markov’s inequality we obtain
\[
I_1 \leq P \left( \sum_{j=q}^{\infty} |\bar{C}_j| > 1 \right) + P \left( \max_{1 \leq l \leq n} \left| \sum_{i=1}^{l} \sum_{j=q}^{\infty} \bar{C}_j Z_{n,i-j} \right|^r > \left( \frac{\epsilon}{2} \right)^r, \sum_{j=q}^{\infty} |\bar{C}_j| \leq 1 \right)
\]
\[
\leq E \left( \sum_{j=q}^{\infty} |\bar{C}_j| \right) + P \left( \max_{1 \leq l \leq n} \left| \sum_{i=1}^{l} \sum_{j=q}^{\infty} \bar{C}_j Z_{n,i-j} \right|^r > \left( \frac{\epsilon}{2} \right)^r, \sum_{j=q}^{\infty} |\bar{C}_j| \leq 1 \right)
\]
\[
\leq E \left( \sum_{j=q}^{\infty} |\bar{C}_j| \right) + P \left( \max_{1 \leq l \leq n} \left| \sum_{i=1}^{l} \sum_{j=q}^{\infty} \bar{C}_j \cdot \left| \sum_{i=1}^{l} Z_{n,i-j} \right|^r \right) > \left( \frac{\epsilon}{2} \right)^r, \sum_{j=q}^{\infty} |\bar{C}_j| \leq 1 \right)
\]
\[
\leq E \left( \sum_{j=q}^{\infty} |\bar{C}_j| \right) + P \left( \max_{1 \leq l \leq n} \sum_{j=q}^{\infty} |\bar{C}_j| \cdot \left| \sum_{i=1}^{l} Z_{n,i-j} \right|^r \right) > \left( \frac{\epsilon}{2} \right)^r, \sum_{j=q}^{\infty} |\bar{C}_j| \leq 1 \right).
\]

Now, using again Markov’s inequality and the fact that the sequence \((C_i)_{i \geq 0}\) is independent of \((Z_i)\) we obtain
\[
I_1 \leq E \left( \sum_{j=q}^{\infty} |\bar{C}_j| \right) + \frac{\epsilon^r}{\epsilon} E \left[ \sum_{j=q}^{\infty} |\bar{C}_j| \cdot \max_{1 \leq l \leq n} \left| \sum_{i=1}^{l} Z_{n,i-j} \right|^r \right] \]
\[
\leq E \left( \sum_{j=q}^{\infty} |\bar{C}_j| \right) + \frac{\epsilon^r}{\epsilon} \sum_{j=q}^{\infty} E|\bar{C}_j| \cdot E \left( \max_{1 \leq l \leq n} \left| \sum_{i=1}^{l} Z_{n,i-j} \right|^r \right)
\]
\[
\leq E \left( \sum_{j=q}^{\infty} |\bar{C}_j| \right) + \frac{\epsilon^r}{\epsilon} \sum_{j=q}^{\infty} E|\bar{C}_j| \cdot \sup_{k \geq q} E \left( \max_{1 \leq l \leq n} \left| \sum_{i=1}^{l} Z_{n,i-k} \right|^r \right).
\]

Noting that \( \sum_{j=q}^{\infty} |\bar{C}_j| \leq 2 \sum_{j=q}^{\infty} |C_j| \), from condition (3.2) we now conclude that there exists a positive constant \( D_1 \) such that for all \( q \in \mathbb{N} \) it holds that
\[
\limsup_{n \to \infty} I_1 \leq D_1 \sum_{j=q}^{\infty} E|C_j|. \tag{3.6}
\]

In order to estimate \( I_2 \) we consider separately the cases \( \alpha \in (1, 2) \) and \( \alpha = 1 \). Assume first \( \alpha \in (1, 2) \). Applying Markov’s inequality, the fact that the sequence \((C_i)_{i \geq 0}\) is independent of \((Z_i)\) and the stationarity of the sequence \((Z_i)\) we obtain
\[
I_2 \leq P \left( \sum_{i=1}^{n} \sum_{j=q}^{\infty} \frac{\bar{C}_j Z_{i-j}}{a_n} 1_{\{|Z_{i-j}| > a_n\}} \right) > \frac{\epsilon}{2}
\]
\[
\leq \frac{2}{\epsilon a_n} E \left( \sum_{i=1}^{n} \sum_{j=q}^{\infty} \bar{C}_j Z_{i-j} 1_{\{|Z_{i-j}| > a_n\}} \right)
\]
\[
\leq \frac{2n}{\epsilon a_n} \sum_{j=q}^{\infty} E|\bar{C}_j| \cdot E \left( |Z_1| 1_{\{|Z_1| > a_n\}} \right). \tag{3.7}
\]
By Karamata’s theorem, as \( n \to \infty \),
\[
\frac{n}{a_n} \mathbb{E}\left( \left| Z_1 1_{\{|Z_1| > a_n\}} \right| \right) \to \frac{\alpha}{\alpha - 1},
\]
and hence from (3.7) we conclude that there exists a positive constant \( D_2 \) such that
\[
\limsup_{n \to \infty} I_2 \leq D_2 \sum_{j=q}^{\infty} \mathbb{E}|C_j|.
\]
(3.8)

Now assume \( \alpha = 1 \). Markov’s inequality implies
\[
I_2 \leq \frac{2\delta}{e^\delta a_n^\delta} \sum_{i=1}^{n} \sum_{j=q}^{\infty} \mathbb{E}\left( \left| \tilde{C}_j Z_{i-j} 1_{\{|Z_{i-j}| > a_n\}} \right| \right)^\delta
\]
with \( \delta \) as in relation (1.7). Since \( \delta < 1 \), a double application of the triangle inequality \( \sum_{i=1}^{\infty} a_i^s \leq \sum_{i=1}^{\infty} |a_i|^s \) with \( s \in (0,1] \) yields
\[
I_2 \leq \frac{2\delta}{e^\delta a_n^\delta} \sum_{i=1}^{n} \sum_{j=q}^{\infty} \mathbb{E}\left( \left| \tilde{C}_j Z_{i-j} 1_{\{|Z_{i-j}| > a_n\}} \right| \right)^\delta
\]
\[
\leq \frac{2\delta}{e^\delta a_n^\delta} \sum_{i=1}^{n} \sum_{j=q}^{\infty} \mathbb{E}\left( \tilde{C}_j Z_{i-j} 1_{\{|Z_{i-j}| > a_n\}} \right)^\delta.
\]

Using again the fact that \( (C_i) \) is independent of \( (Z_i) \) and the stationarity of \( (Z_i) \)
we obtain
\[
I_2 \leq \frac{2\delta n}{e^\delta a_n^\delta} \mathbb{E}\left( |Z_1|^\delta 1_{\{|Z_1| > a_n\}} \right) \sum_{j=q}^{\infty} \mathbb{E}|\tilde{C}_j|^\delta
\]
From this, since by Karamata’s theorem
\[
\lim_{n \to \infty} \frac{n}{a_n^\alpha} \mathbb{E}\left( |Z_1|^\delta 1_{\{|Z_1| > a_n\}} \right) = \frac{1}{1 - \delta},
\]
it follows that there exists a positive constant \( D_3 \) such that
\[
\limsup_{n \to \infty} I_2 \leq D_3 \sum_{j=q}^{\infty} \mathbb{E}|C_j|^\delta.
\]
This together with (3.5), (3.6) and (3.3) shows that
\[
\limsup_{n \to \infty} \mathbb{P}[d_{M_2}(V_{n,q}, V_n) > \epsilon] \leq D_1 \sum_{j=q}^{\infty} \mathbb{E}|C_j| + (D_2 + D_3) \sum_{j=q}^{\infty} \mathbb{E}|C_j|^s,
\]
where
\[
s = \left\{ \begin{array}{ll}
\delta, & \text{if } \alpha = 1, \\
1, & \text{if } \alpha \in (1,2).
\end{array} \right.
\]

Now, the dominated convergence theorem and conditions (1.7) and (3.3) yield (3.4). Therefore we again obtain \( V_n(\cdot) \overset{d}{\to} \tilde{C}V(\cdot) \) in \( (D[0,1], d_{M_2}) \).

□

Remark 3.1. If the sequence \( (Z_i) \) is an i.i.d. or \( \rho \)-mixing sequence with \( \sum_{i=1}^{\infty} \rho(2^i) < \infty \), where
\[
\rho(n) = \sup \{|\text{corr}(f,g)| : f \in L^2(\mathcal{F}_1^n), g \in L^2(\mathcal{F}_k^{\infty+n}), k = 1,2,\ldots\},
\]
then it is known that condition (5.2) holds with \( r = 2 \), see Tyran-Kamińska [27].
In the case when the sequence \((C_j)\) is deterministic, conditions (3.1) and (3.3) can be dropped since they are implied by (1.7). To see this note that by condition (1.7) it holds that \(|C_j|^\delta < 1\) for large \(j\). Now since \(|C_j|^{\delta x}\) is decreasing in \(x\), it follows that for large \(j\)

\[|C_j|^\gamma = (|C_j|^\delta)^{\gamma/\delta} \leq |C_j|^\delta,\]

and similarly \(|C_j| \leq |C_j|^\delta\). This suffice to conclude that (3.1) and (3.3) hold. In general this does not hold when the coefficients are random (see for an example Krizmanić [17]).

**Acknowledgment**

This work has been supported in part by University of Rijeka research grants uniri-prirod-18-9 and uniri-pr-prirod-19-16 and by Croatian Science Foundation under the project IP-2019-04-1239.

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