Black chain of pearls in 5D de Sitter spacetime

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Black holes are a fascinating class of solutions appearing in Einstein theory of gravity as well as many modified theories of gravity. In 4 spacetime dimensions there exist a number of uniqueness theorems which guarantee the uniqueness of black hole solution under a given set of conditions. In higher dimensions black hole uniqueness does not hold and a number of exotic black hole solutions, known as black rings, were found [1–3] in the last few years. These black rings have horizons of topological 3-spheres and are mostly living in 5 dimensional asymptotically flat spacetime, usually carrying spin [2] and/or electromagnetic charges [3] in order to avoid conical singularities. One astonishing property of the black ring solutions is that they can be more stable than the corresponding spherical black holes with the same mass, charge and spin. Most recently it was found [4, 5] that a particular black hole configuration called black Saturn which consists of a spherical black hole surrounded by a thin black ring is even more stable than black rings. The existence of all these unusual black solutions in 5 dimensions makes the phase diagram of 5D gravity an active field of study.

In a recent paper by Chu and Dai [6], a 5-dimensional (5D) “black ring” solution with positive cosmological constant was constructed. The construction is based on the warped decomposition of the 5D geometry in terms of the 4-dimensional charged de Sitter C-metric, i.e.

\[
ds_5^2 = dz^2 + \cos^2(kz) dS_4^2,
\]

\[
ds_4^2 = \frac{1}{A^2(x-y)^2} \left[ G(y)dt^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \tilde{G}(x) d\varphi^2 \right],
\]

where

\[
G(\xi) = q^2 A^2 \xi^4 + a_3 \xi^3 + a_5 \xi^2 + a_1 \xi + a_0,
\]

\[
\tilde{G}(\xi) = G(\xi) - k^2 / A^2,
\]

and \(ds_4^2\) must be accompanied by a Maxwell potential

\[
A_\varphi = qx + c_0
\]

in order to constitute a solution to the Einstein-Maxwell equation in the presence of a cosmological constant. The argument which makes the solution (1) a black ring is as follows. First, the metric \(ds_4^2\), as a de Sitter generalization of the well known C-metric, corresponds to two black holes accelerating apart in 4 dimensions. Then, since the warped factor \(\cos^2(kz)\) is periodic in \(z\), the topology becomes \(S^1 \times S^2\) in 5 dimensions. So the standard method for analyzing black rings in asymptotically flat spacetime can be systematically applied to the de Sitter black ring which constitutes the major part of [6].

In this article we first point out that actually the horizon topology of warped geometries like (1) is not \(S^1 \times S^2\), because at \(kz = s \mod \pi\), the \(S^2\) factor in the horizon shrinks to zero size and at those points the local geometry of the horizons look like two cones concatenating each other at the tops. Moreover, since the metric and the horizon is not translationally invariant along the \(z\) axis, it is not necessary to require the coordinate \(z\) to extend only over a single period: allowing \(z\) to extend over several periods of \(\cos(kz)\) would correspond to very different horizon topologies. Despite the above remarks, we would like to adopt the construction used in [6] to illustrate some simpler axial symmetric de Sitter solutions in 5-dimensions.

Our constructions will be based on the following formula which also appeared in [6]: a generic \(D\)-dimensional metric with positive cosmological constant can be embedded into a \((D+1)\)-dimensional geometry also with positive cosmological constant via

\[
ds_{(D+1)}^2 = dz^2 + \cos^2(kz) ds_D^2, \quad (2)
\]

\[
ds_5^2 = dz^2 + \cos^2(kz) dS_4^2,
\]

\[
ds_4^2 = \frac{1}{A^2(x-y)^2} \left[ G(y)dt^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \tilde{G}(x) d\varphi^2 \right],
\]

\[
G(\xi) = q^2 A^2 \xi^4 + a_3 \xi^3 + a_5 \xi^2 + a_1 \xi + a_0,
\]

\[
\tilde{G}(\xi) = G(\xi) - k^2 / A^2,
\]

and \(ds_4^2\) must be accompanied by a Maxwell potential

\[
A_\varphi = qx + c_0
\]
and the corresponding Ricci tensors are related by

\[ R_{\mu\nu} - Dk^2 g_{\mu\nu} = R_{\mu\nu} - (D - 1)k^2 g_{\mu\nu}, \]
\[ R_{zz} = Dk^2. \]

Now at \( D = 4 \), instead of de Sitter C-metric, we insert the usual 4-dimensional Schwarzschild-de Sitter metric

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (3) \]
\[ f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}, \quad \Lambda = 3k^2 \quad (4) \]

into (2). The resulting 5D metric \( ds_5^2 \) will then be a solution to the vacuum Einstein equation with positive cosmological constant \( 4k^2 \). As mentioned earlier, since the metric is periodic but not translationally invariant along \( z \), we have no reason to restrict \( z \) to take values in only a single period. Instead, we allow \( z \) to take values in the range of several integral multiples of the period of the metric functions. This will not affect the local geometry but generically will change the topology of the horizon. To make it more explicit, one can calculate the curvature invariants of the metric described by (2), (3) and (4). One of these turns out to be

\[ R^\mu_{\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 8 \left( 5k^4 + \frac{6M^2}{r^6 \cos^4(kz)} \right). \]

The singularity at \( r = 0 \) is surrounded by roots \( r_\pm \) of \( f(r) \), i.e. the horizons, while the singularities at \( \cos(kz) = 0 \) are naked. The metric at the horizons can be written as

\[ ds_{H_\pm}^2 = dz^2 + \cos^2(kz)r_\pm^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \]

To illustrate the horizon topology, we depict the outer horizon of the metric corresponding to allowing \( z \) to run over 4 periods. Of course the number of periods 4 is picked for random here, any integer number of periods as allowed. This horizon is 3-dimensional and while depicting it we have omitted the angular coordinate \( \varphi \) appearing in the 4-dimensional Schwarzschild-de Sitter part. Such solutions will be referred to as black chain of pearls due to the particular shape of the horizons. The 2-dimensional slices of the horizon at constant \( z \) are almost all 2-spheres corresponding to the Schwarzschild-de Sitter horizons except at the naked singularities located at \( kz = n + \pi/2 \). These naked singularities are referred to as nodes on the chain, and the segments of the chain between two nodes are called a pearl. Note that the topology of each pearl is a 3-sphere and the black chain horizon as a whole is topologically equivalent to several 3-spheres concatenating at isolated nodes. We postulate that the appearance of these nodes signifies that such solutions might correspond to the “critical stage” for the gravitational phase transition from an array of 5-dimensional black holes or a uniform 5D black string into a final stable configuration, presumably something similar to a 5D black ring, because there have been numerous arguments stating that the large black ring phase is thermodynamically more favorable than an array of black holes, and the uniform black string is unstable due to Gregory-Laflamme instability [7].

Besides different horizon topologies, there is another fundamental difference between the black chain of pearls and black strings/rings. As shown in Figure 1, the black chain spacetimes are not translationally invariant in the fifth dimension (i.e. the \( z \)-direction). Instead, there is only a discrete symmetry in that direction which is \( Z_p \), with \( p \) the number of pearls on the chain. Gravitational solutions with discrete symmetries are always of great interests in the literature.

It is tempting to study the properties of the multi-period solution in detail. First comes the area of the horizon per unit period in \( z \). We have

\[ A = 4\pi r_+^2 \int_0^{\pi/k} \cos^2(kz) dz = \frac{2\pi^2 r_+^2}{k} = 2\pi^2 r_+^2 \ell, \]

where \( r_+ \) is the radius of the outer horizon of the 4-dimensional Schwarzschild-de Sitter spacetime, and

\[ \ell = \frac{1}{k} = \sqrt{\frac{3}{\Lambda}} \]

is just the de Sitter radius. The Bekenstein-Hawking entropy of the outer horizon then reads

\[ S = \frac{pA}{4} = \frac{p\pi^2 r_+^2 \ell}{2}. \]

To determine the temperature of the horizons, however, we need to take care of a subtlety which generally appear in any black hole solutions admitting more than one horizon. It is believed that such spacetimes are not in
thermal equilibrium and one can define Hawking temperatures separately if the separation between the horizons is large enough and the shell of spacetime in between the horizons can be thought of as adiabatic. Under such assumptions it is an easy practice to derive the Hawking temperature for the outer horizon, which turn out to be equal to the Hawking temperature on the outer horizon of 4D Schwarzschild-de Sitter black hole under the same assumptions [8–10],

\[ T = \frac{1}{2\pi \sqrt{1-(2y)^{1/3}}} \left| \frac{M}{r^2} - \frac{r_+}{\ell^2} \right|, \]

where \( y = M^2/\ell^2 \).

The black chain of pearls described in this article is of the simplest form, i.e. they are static and neutral. We can check that they are unstable against the gravitational perturbations to the first order. This can be done following the standard perturbation process usually pursued in the test of Gregory-Laflamme instability of various black strings in 5-dimensions. To do this, we first change the metric into another gauge in which it looks as follows:

\[ ds^2 = e^{B(\zeta)} (ds^2_4 + d\zeta^2). \]

It is easy to see that the desired coordinate transformation is

\[ z \rightarrow z(\zeta) = \frac{2}{k} \tan^{-1} \left( \tanh \left( \frac{k\zeta}{2} \right) \right), \]

and the function \( B(\zeta) \) reads

\[ B(\zeta) = \log(\text{sech}^2(\zeta)). \]

Then we can repeat the procedure of perturbation as we did recently in [11] to get the following set of perturbation equations:

\[
\begin{align*}
\left[ \Box \gamma \right] h_{\mu\nu}(x) + 2R_{\mu\rho\nu\lambda}(\gamma) h^{\rho\lambda}(x) &= m^2 h_{\mu\nu}(x), \quad (5) \\
-\partial_\zeta^2 + \frac{3k^2}{4} (3 - 5 \text{sech}^2(k\zeta)) \xi(\zeta) &= m^2 \xi(\zeta), \quad (6)
\end{align*}
\]

where the 5D metric \( ds^2_5 \) is perturbed as

\[
g_{MN} \rightarrow g_{MN} + \delta g_{MN}, \quad ds^2 \rightarrow ds^2 = e^{B(\zeta)}
\times \left( \left( \gamma_{\mu\nu} + e^{-B(\zeta)} h_{\mu\nu}(x, \zeta) \right) dx^\mu dx^\nu + d\zeta^2 \right), \quad (7)
\]

\( \gamma_{\mu\nu} \) represent components of the metric \( ds^2_4 \), and \( m \) is a constant coming from the separation of variables which represents the energy scale of the perturbation. In the above, the metric perturbation \( h_{\mu\nu}(x, \zeta) \) is related to \( h_{\mu\nu}(x) \) and \( \xi(\zeta) \) via

\[ h_{\mu\nu}(x, \zeta) = e^{B(\zeta)/4} h_{\mu\nu}(x) \xi(\zeta), \]

and in order to cast the perturbation equations into the simple form (6) the following transverse traceless and Lorentzian gauge condition is adopted,

\[ h \equiv \gamma^{\mu\nu} h_{\mu\nu} = 0, \quad \nabla_\mu h^{\mu\nu} = 0. \]

The first of the perturbation equations (eq. (5)) is the well known Lichnerowicz equation in 4-dimensional Schwarzschild-de Sitter background, which was shown to be always unstable by Hirayama and Kang in [12, 13]. The second of the perturbation equations (eq.(6)) is a Schrödinger-like equation in the fifth dimension with a potential approaching a finite constant value \( 9k^2/4 \) at large \( \zeta \) (see Figure 2). Thus the normalizable solutions thereof contain a spectrum with continuous eigenvalue \( m^2 \) and hence any unstable modes with strength bigger than \( 9k^2/4 \) will travel along \( \zeta \) direction without any barrier. Thus both parts of the perturbation equation signify instability of the solution. In fact, such instability should already be forecasted when we first encountered the naked singularities at the nodes.

![FIG. 2: Plot of the potential in (6) at \( k = 1 \)](image)

Notice, however, that the discussion made above using the \( \zeta \) coordinate is a little bit flawed. This is because the \( \zeta \) coordinate actually cannot represent the full range of \( z \) values allowed in the solution. In fact, \( \zeta \) going from \( -\infty \) to \( \infty \) only corresponds to \( k z \) going from \( -\pi/2 \) to \( \pi/2 \), i.e. only one period of \( z \). Thus the ability for the instability modes to travel from \( \zeta = -\infty \) to \( \zeta = \infty \) does not mean that they can travel all along the chain. Indeed, changing back to the \( z \) coordinate, the second perturbation equation becomes

\[ [-\partial_z^2 + 2k^2 (\sec^2(kz) - 2)] \eta(z) = m^2 \sec^2(kz) \eta(z), \]

where \( \eta(z) = \cos^{1/2}(kz) \xi(z) \). A similar equation also arises in [14], in which stability of AdS black string is analysed. This equation does not look like a Schrödinger equation due to the appearence of the factor \( \sec^2(kz) \) on the right hand side. Nevertheless, the potential term on the left hand side is a confining function, thus the
propagating modes along z axis cannot travel through the nodes located at $kz = n + \pi/2$. In other words, the nodes cannot be blown up into $S^2$ with nonvanishing radius by the perturbation modes. Therefore the number $p$ of pearls on the chain (which equals the number of nodes plus one) is an invariant property of the solution even in the presence of perturbation, so it might be regarded as yet another hair of the black chain solution.

It will be interesting to make further understanding of the instability described above in the framework of Gregory-Laflamme instabilities. As pointed out by Horowitz and Maeda in [15], black string (of which black ring is a particular branch) horizons cannot pinch off. So whatever the nature and strength the perturbation is, a black string cannot segregate into black holes. Therefore, the black chain solution described in this article should not be broken into smaller segments after the perturbation.

Combining the arguments made in the last two paragraphs, the black chains can neither be blown up into black strings without singular nodes, nor can they be broken into single pearls which are equivalent to short black strings with compact horizons. It remains to answer what is the final configuration of the black chains after the perturbation.

It is tempting to find other black chain solutions with more complex properties, e.g. those carrying nontrivial electro-magnetic charges and/or rotation parameters, because in general the inclusion of these parameters will improve the stability of the solution. To this end it should be mentioned that the solution found by Chu and Dai in [6] is actually a black di-chain carrying a pure magnetic charge if we allow $z$ to extend over several periods.

It is also tempting to find the closed analogues of the black chains (black “necklaces”) which has horizons of the shape as skematically depicted in Figure 3. If ever such a configuration exist, it would arguably be more stable than the black chains described here, just as black rings are more stable than uniform black strings. However we need more direct evidence to justify the last statement.

Changing the 5D de Sitter bulk to spacetimes with different asymptotics will be another direction of further investigation. Last but not the least, it is also of great interests to compare the thermodynamic properties of black strings/rings and black chains/necklaces in detail. We hope to answer these questions in later works.

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