Abstract

We quantify conditions that ensure that a signed measure on a Riemannian manifold has a well defined centre of mass. We then use this result to quantify the extent of a neighbourhood on which the Riemannian barycentric coordinates of a set of $n + 1$ points on an $n$-manifold provide a true coordinate chart, i.e., the barycentric coordinates provide a diffeomorphism between a neighbourhood of a Euclidean simplex, and a neighbourhood containing the points on the manifold.

Keywords. Riemannian centre of mass, Karcher means, barycentric coordinates, Riemannian manifold, Riemannian simplices

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The focus of this work is on identifying conditions under which barycentric coordinates will parameterise a neighbourhood of a simplex in a Riemannian manifold, and specifically on quantifying the extent of the permissible neighbourhood. Recall that if \( \sigma = \{p_0, \ldots, p_n\} \subset \mathbb{E}^n \) defines a nondegenerate simplex in Euclidean space, i.e., \( \text{aff}(\sigma) = \mathbb{E}^n \), where \( \text{aff}(\sigma) \) is the affine hull of \( \sigma \), then \( \mathbb{E}^n \) can be parameterised by the barycentric coordinate functions; any point \( u \in \mathbb{E}^n \) is uniquely defined by a set of \( (n + 1) \) barycentric coordinates, \( (\lambda_i) \), that satisfy \( \sum \lambda_i = 1 \). The point \( u \) corresponding to \( (\lambda_i) \) can be identified as the point that minimises the function

\[
x \mapsto \frac{1}{2} \sum_{i=0}^{n} \lambda_i d_{\mathbb{E}^n}(x, p_i)^2,
\]

where \( d_{\mathbb{E}^n}(x, y) \) is the Euclidean distance function. In this way the barycentric coordinates define the familiar affine functions that satisfy \( \lambda_i(p_j) = \delta_{ij} \).

We can view a given set of barycentric coordinates \( \lambda = (\lambda_0, \ldots, \lambda_n) \) as a point in \( \mathbb{R}^{n+1} \). The set \( \Delta^n \) of all points in \( \mathbb{R}^{n+1} \) with non-negative coefficients that sum to 1 is called the standard Euclidean \( n \)-simplex. Thus the minimisation of the function \( 1 \) defines a barycentric coordinate map from the the plane \( \{ \lambda \in \mathbb{R}^{n+1} \mid \sum \lambda_i = 1 \} \) to \( \mathbb{E}^n \) that brings the standard Euclidean simplex to the Euclidean manifold \( \sigma_{\mathbb{E}^n} = \text{conv}(\sigma) \subset \mathbb{E}^n \).

This same technique can be used to define barycentric coordinate neighbourhoods in a Riemannian manifold \( M \). If \( \sigma = \{p_i\} \subset M \), we consider the function \( 1 \) except that the Euclidean distance \( d_{\mathbb{E}^n} \) is replaced by \( d_M \), the intrinsic metric of the manifold. The first problem is to identify conditions under which this function admits a unique minimum. This is a particular case of a more general problem of identifying conditions under which the Riemannian centre of mass of a measure is well defined. There are several demonstrations of the existence and uniqueness of Riemannian centres of mass.

The standard reference for the subject is the excellent exposition by Karcher [Kar77] (see also the recent note outlining the earlier history of the subject [Kar14]). He uses Jacobi field estimates to demonstrate that if the support of a measure \( \mu \) is contained in a sufficiently small convex ball, then the energy function \( x \mapsto \frac{1}{2} \int d_M(x, y)^2 \ d\mu(y) \) is convex, and since the gradient is pointing outward on the boundary of the ball, there is a unique minimum in the ball’s interior.

Kendall [Ken90] gave a different proof based on the concept of “convex geometry” which arose in earlier work on Dirichlet problems. Kendal was able to relax Karcher’s bounds on the size of the ball that must contain the support of the measure.

Groisser [Gro04] gives a proof based on a contraction mapping argument, using a map defined from the gradient of the above mentioned energy function. This yields a constructive algorithm for finding the centre of mass, but the constraint on the extent of the admissible neighbourhood is more restrictive than Karcher’s.

These works all assume an unsigned measure, so they do not apply to situations where negative weights are involved. The existence and uniqueness of the Riemannian centre of mass with non-negative weights at least allows us to define a “filled in” geometric Riemannian simplex associated to a finite set of points with small diameter. If \( \sigma \subset M \) is a set of \( n + 1 \) points in a Riemannian \( n \)-manifold, and \( \sigma \) is contained in a sufficiently small convex ball \( B \), then we can define the Riemannian simplex \( \sigma_M \) as the image of the standard simplex \( \Delta^n \) under the barycentric coordinate map.

In order to make the concept of Riemannian simplices useful for triangulating manifolds, we need to ensure that the barycentric coordinate map is not only well-defined, but also an embedding. When this is the case, we say that the Riemannian simplex is nondegenerate. We recently

\footnote{We consider \( \mathbb{E}^n \) to have an affine structure, and Euclidean distance function, but no canonical origin or coordinate system.}
established conditions, based on the local curvatures of the manifold and the quality of the point set $\sigma$, that ensure that the barycentric coordinate map is an embedding $[DVW15]$. Although we used the result on nondegenerate Riemannian simplices to establish criteria for ensuring that a simplicial complex triangulates a Riemannian manifold, we did not establish some simple properties that we expect from geometric simplices. Specifically, if $\sigma_M$ and $\tau_M$ are two Riemannian $n$-simplices in an $n$-manifold, and they have $n$ vertices in common, and agree with a local orientation of the manifold, then is $\tau_M \cap \sigma_M$ exactly the Riemannian simplex that is the common facet? Of course it is impossible to ensure a triangulation by Riemannian simplices without ensuring this property, but in $[DVW15]$ we did not show this property for two isolated simplices.

This is one of the motivations for establishing the barycentric coordinate neighbourhoods that are the focus of the current work. If the barycentric neighbourhood defined by $\sigma_M$ encompasses $\tau_M$ and vice versa, then the above problem is easily resolved, as discussed in Section 3.3.

In order to establish such barycentric coordinate neighbourhoods we must first ensure that the barycentric coordinate map is well defined when the weights may be negative. To this end we give an alternate proof of the existence and uniqueness of Riemannian centres of mass. The result, Theorem 3, can accommodate signed measures, i.e., mass distributions that include negative weights. This is the main technical contribution of this paper.

A demonstration of the existence and uniqueness of Riemannian centres of mass for signed measures has already been published by Sander $[San16$, Theorem 3.19]. He was motivated by a desire to describe higher order interpolation on Riemannian manifolds, which needs to accommodate negative interpolation weights. Sander’s proof is modelled on Groisser’s proof for unsigned measures $[Gro04]$. The result is stated in terms of several parameters governing the neighbourhood of validity, and they are bounded in an intricate manner. We are specifically interested in obtaining an explicit bound on the extent of the admissible neighbourhood in terms of primitive properties of the measure and bounds on the sectional curvatures. Such an explicit bound is not easily obtained from Sander’s result.

We provide an elementary proof of the existence and uniqueness of Riemannian centres of mass for signed measures that is in the spirit of Karcher’s demonstration for the unsigned case. However, instead of using Jacobi field estimates to demonstrate the convexity of the energy function, we rely on a series expansion of the cosine rule in constant curvature spaces. At the expense of this brute force calculation, we obtain a concise and elementary demonstration. Furthermore, our bounds on the admissible neighbourhood are explicit, and they can be reduced to Karcher’s bounds in the case where the measure is unsigned. This is especially evident in the formulation of the result expressed in Corollary 9.

In Section 1 we briefly review the tools we will use for our exposition, including a recap of the main ideas in defining Riemannian centres of mass and barycentric coordinates. We demonstrate our main theorem on Riemannian centres of mass of signed measures in Section 2 and Section 3 is devoted to exploiting this for the development of barycentric coordinate neighbourhoods that motivates this work.

In order to obtain true coordinate neighbourhoods we need to ensure not only that the barycentric coordinate map is well defined, but also that it is an embedding. We observe in Section 3 that we can use the arguments in $[DVW15]$ verbatim since they did not rely on the sign of the weights, only on the convexity of the energy function, but this is established as part of the demonstration of Theorem 8. We obtain Theorem 10 and Proposition 12 which describe the extent of admissible barycentric coordinate neighbourhoods. Finally, in Section 3.3 we make the observation that suitably constrained compatibly oriented Riemannian $n$-simplices that share $n$-vertices intersect only on their shared facet.
1 Background and notation

In this work $M$ refers to a $C^\infty$ Riemannian manifold (without boundary) of dimension $n$. The adjective smooth is synonymous with $C^\infty$. A function defined on a closed set $A \subset M$ is smooth if it can be extended to a smooth function on an open neighbourhood of $A$. The distance between $x, y \in M$ is denoted $d_M(x, y)$, and $B_M(c, r) = \{ x \in M \mid d_M(c, x) < r \}$ is the open geodesic ball of radius $r$ centred at $c \in M$. The topological closure of a set $B \subset M$ is denoted $\bar{B}$. A geodesic is minimizing if it is the shortest path between any two of its points (all geodesics are locally minimizing). A (geodesic) segment is the trace of a minimizing geodesic. We sometimes abuse notation and terminology by identifying a geodesic with its trace.

The injectivity radius $\iota_M$ is the supremum of the distances $r$ such that any two points $x, y \in M$ with $d_M(x, y) < r$ have a unique minimizing geodesic connecting them. This number is positive if $M$ is compact. For any $x \in M$, the exponential map $\exp_x : T_x M \to M$ maps a vector $v\in T_x M$ to the point $\gamma(|v|) \in M$, where $\gamma$ is the unique unit-speed geodesic emanating from $x$ with $\gamma'(0) = v/|v|$, and $|v| = \langle v, v \rangle^{1/2}$ is the norm defined by the Riemannian metric. If $r < \iota_M$, then $\exp_x$ is a diffeomorphism onto its image when restricted to $B_{\iota_M}(x, r)$. Identifying $T_x M$ with $\mathbb{R}^n$, this yields a local coordinate system called Riemann normal coordinates.

For any $x \in M$, and any 2-dimensional subspace $H \subset T_x M$ we associate a sectional curvature, which is the Gaussian curvature at $x$ of the surface $\exp_x(H)$. We denote by $K$ the sectional curvature function, but we are concerned only with bounds on $K$, never individual values.

1.1 Convexity

A set $B \subset M$ is convex if for all $x, y \in B$ there is a unique minimizing geodesic in $M$ connecting $x$ to $y$, and this geodesic is contained in $B$. If $\Lambda_u$ is an upper bound on the sectional curvatures of $M$, then for any point $c \in M$ the ball $B(c, r)$ is convex if $r < \min\left\{ \frac{|v|}{2}, \frac{|v|}{2\sqrt{\Lambda_u}} \right\}$, where we set $\frac{|v|}{2\sqrt{\Lambda_u}} = \infty$ if $\Lambda_u \leq 0$ (see [Cha06, Thm. IX.6.1]).

A function $f : A \to \mathbb{R}$, with $A \subset M$ is convex if for any $p \in A$ and geodesic $\gamma$ with $\gamma(0) = p$, we have $\frac{d}{dt}f(\gamma(t))|_{t=0} \geq 0$. We say $f$ is strictly convex if this inequality is strict. If $A$ is compact and convex with nonempty interior, and $f$ is strictly convex with grad $f$ pointing outwards on $\partial A$ (i.e., for any $x \in \partial A, y \in \text{int}(A)$, and $\gamma$ a minimizing geodesic from $x$ to $y$, we have $\langle \gamma'(0), \text{grad} f \rangle < 0$), then $f$ has a unique minimum in the interior of $A$. (The gradient assumption precludes a minimum on the boundary, and the existence of multiple minima contradicts strict convexity via the same argument as in the Euclidean case.)

1.2 Riemannian centre of mass

Let $\mu$ be an unsigned measure whose support is contained within a convex geodesic ball $B_\rho \subset M$ with radius $\rho$. Consider the energy function $\mathcal{E}_\mu : \bar{B}_\rho \to \mathbb{R}$ defined by

$$\mathcal{E}_\mu(x) = \frac{1}{2} \int d_M(x,y)^2 \, d\mu(y),$$

where $d\mu(y)$ indicates that the integration is with respect to the variable $y$, and the domain of integration is understood to be $M$, or equivalently $B_\rho$, since it contains the support of $\mu$. Karcher [Kar77, Theorem 1.2] showed that $\mathcal{E}_\mu$ has a unique minimum on $B_\rho$, provided that, if there are positive sectional curvatures in $B_\rho$, the radius satisfies $\rho \leq \frac{\pi}{4\sqrt{\kappa}}$, where $\kappa$ is an upper bound on the sectional curvature in $B_\rho$. This unique minimum is called the Riemannian centre of mass of $\mu$.

\footnote{In general $\exp_x$ may not be defined on all of $T_x M$, unless $M$ is complete, but this detail is not a concern for us.}
Since we will be working within these same constraints, it is convenient to define
\[
\rho_0 = \min \left\{ \ell_M, \frac{\pi}{4\sqrt{\Lambda_u}} \right\},
\]  
where \( \Lambda_u \) is an upper bound on the sectional curvatures of \( M \), and we understand \( \frac{1}{\sqrt{\Lambda_u}} = \infty \) if \( \Lambda_u \leq 0 \). Then \( \rho < \rho_0 \) is sufficient to enforce the convexity condition and the curvature sensitive bound on the radius.

In the case of a discrete measure concentrated on a finite set of points \( \sigma = \{p_0, \ldots, p_k\} \subset B_\rho \), then instead of \( \mu \) we use \( \lambda \), with \( \lambda_i = \lambda(p_i) \), and when \( \sum \lambda_i = 1 \) this defines the barycentric coordinates of the centre of mass. In other words, the unique point \( x \in B_\rho \) that has barycentric coordinates \( \lambda \) with respect to \( \sigma \) is the point that minimises
\[
\mathcal{E}_\lambda(x) = \frac{1}{2} \sum_i \lambda_i d_M(x, p_i)^2.
\]  
The “filled-in” Riemannian simplex \( \sigma_M \) defined by \( \sigma \) is defined to be the set of points in \( B_\rho \) defined by all nonnegative barycentric coordinates.

### 1.3 Comparison theorems

We denote by \( \mathbb{H}_\kappa = \mathbb{H}_\kappa^n \) the simply connected space of dimension \( n \) whose sectional curvatures are all equal to the constant \( \kappa \). A comparison theorem provides inequalities relating geometry on a manifold \( M \) to that on an appropriate \( \mathbb{H}_\kappa \). The comparison theorems we state here are equivalent corollaries of the Rauch comparison theorem [BK81, 6.4.1] and can be found in [Kar89, BK81, 6.4.3, 6.4.4], and many other standard references.

The first comparison theorem captures the notion that geodesics diverge more quickly in low curvature than in high curvature. A hinge in \( M \) consists of two geodesic segments of lengths \( \ell_1 \) and \( \ell_2 \) meeting at a common endpoint at an angle \( \alpha \). A comparison hinge is a hinge in \( \mathbb{H}_\kappa \) with the same corresponding side lengths and angle. Three points \( w, x, y \) in a convex ball in \( M \) define a hinge with the angle \( \alpha \) at \( x \) being the angle between the unique geodesic segments connecting \( w \) to \( x \) and \( x \) to \( y \).

**Lemma 1 (Hinge comparison)** Assume the sectional curvature \( K \) in \( M \) is bounded by \( \Lambda_\ell \leq K \leq \Lambda_u \), and let \( B_\rho \subseteq M \) be a convex ball with radius \( \rho \leq \rho_0 \) (as defined in [3]). Suppose \( w, x, y \in B_\rho \) define a hinge with angle \( \alpha \) at \( x \), and let \( \bar{w}, \bar{x}, \bar{y} \in \mathbb{H}_{\Lambda_\ell} \) and \( \bar{w}, \bar{x}, \bar{y} \in \mathbb{H}_{\Lambda_u} \) define comparison hinges.

Then
\[
d_{\mathbb{H}_{\Lambda_u}}(\bar{w}, \bar{y}) \leq d_M(w, y) \leq d_{\mathbb{H}_{\Lambda_\ell}}(\bar{w}, \bar{y}).
\]

A geodesic triangle \( T \) is a collection of three points (vertices) in \( M \), together with geodesic segments between them. A comparison triangle \( T_\kappa \) is a geodesic triangle in \( \mathbb{H}_\kappa \) with the same edge lengths. Thus a vertex, edge, or angle in \( T \) has a corresponding vertex, edge, or angle in \( T_\kappa \).

The second comparison theorem says that triangles are fatter in spaces with higher curvature.

**Lemma 2 (Angle comparison)** Assume the sectional curvature \( K \) in \( M \) is bounded by \( \Lambda_\ell \leq K \leq \Lambda_u \), and let \( T \) be a geodesic triangle contained in a convex ball \( B_\rho \subseteq M \), with radius \( \rho \leq \rho_0 \) (as defined in [3]).

Then comparison triangles \( T_{\Lambda_\ell} \) and \( T_{\Lambda_u} \) exist in \( \mathbb{H}_{\Lambda_\ell} \) and \( \mathbb{H}_{\Lambda_u} \). For any angle \( \alpha \) in \( T \), the corresponding angles in \( T_{\Lambda_\ell} \) and \( T_{\Lambda_u} \) satisfy
\[
\alpha_{\Lambda_\ell} \leq \alpha \leq \alpha_{\Lambda_u}.
\]
1.4 Signed measures

In this work we are only concerned with finite (i.e., bounded) Borel measures. We will be considering signed measures $\mu$ on $M$. The Jordan decomposition theorem [Hal74 §29], states that there are unique unsigned measures $\mu_+$ and $\mu_-$ such that $\mu = \mu_+ - \mu_-$, and for our purposes we can take this as a definition of a signed measure.

If $\nu$ is an unsigned measure on $M$, then the support of $\nu$ is the closed set

$$\text{supp}(\nu) = \{p \in M \mid \nu(U) > 0\} \text{ for all open } U \subseteq M \text{ with } p \in U.$$

The support of a signed measure $\mu$ is defined by $\text{supp}(\mu) = \text{supp}(\mu_+) \cup \text{supp}(\mu_-)$.

2 Center of mass of signed measures

We consider a signed measure $\mu$ with support contained in a convex ball $B_\rho \subset M$. In this section we establish constraints on $\rho$ in terms of $\mu$ and bounds on the sectional curvatures that guarantee two properties which together are sufficient to guarantee that $E_\mu$ has a unique minimum in $B_\rho$. In Section 2.1 we show when $E_\mu$ is guaranteed to be strictly convex, and in Section 2.2 we find conditions that ensure that the gradient of $E_\mu$ is pointing outwards on $\partial B_\rho$. This results in the main theorem of this section:

**Theorem 3 (Centre of mass of signed measures)** Let $M$ be a manifold whose sectional curvature $K$ is bounded by $\Lambda_\ell \leq K \leq \Lambda_u$, and let $\mu$ be a signed measure on $M$, whose support is contained in a geodesic ball $B_M(c, r)$. Suppose $B_\rho = B_M(c, \rho)$, with $r < \rho < \rho_0$ as defined in (3).

Then $E_\mu : \overline{B}_\rho \to \mathbb{R}$ (Equation (2)) has a unique minimum in $B_\rho$ if

$$(\rho - r)\mu_+(M) - (\rho + r)\mu_-(M) > 0,$$

and

$$\frac{\partial u}{\tan \partial u} \frac{\mu_+(M) - \mu_-(M)}{\partial u} > 0 \quad \text{if } 0 \leq \Lambda_\ell \leq \Lambda_u,$$

$$\frac{\partial u}{\tan \partial u} \frac{\mu_+(M) - \mu_-(M)}{\partial u} > 0 \quad \text{if } \Lambda_\ell \leq 0 \leq \Lambda_u,$$

$$\mu_+(M) - \frac{\partial u}{\tanh \partial u} \mu_-(M) > 0 \quad \text{if } \Lambda_\ell \leq \Lambda_u \leq 0,$$

where $\partial u = 2\rho \sqrt{|\Lambda_|}$, and $\partial u = 2\rho \sqrt{|\Lambda_u|}$.

2.1 Ensuring the convexity of $E_\mu$

Our demonstration of the convexity of $E_\mu$ relies on an asymptotic expansion of the squared distance function in constant curvature spaces. We consider points $x, y \in \mathbb{H}_\kappa$, and for any point $w$ close to $x$ we express the distance $d_{\mathbb{H}_\kappa}(w, y)$ in terms of $d_{\mathbb{H}_\kappa}(x, y)$ and terms involving powers of $\delta = d_{\mathbb{H}_\kappa}(x, w)$.

**Lemma 4** Suppose $B_\rho \subset \mathbb{H}_\kappa$ is a geodesic ball of radius $\rho$, and $w, x, y \in B_\rho$ define a hinge with angle $\alpha$ at $x$, and $d_{\mathbb{H}_\kappa}(x, w) = \delta$. If $\kappa > 0$, assume $\rho < \frac{\pi}{4\sqrt{\kappa}}$.

Then

$$d_{\mathbb{H}_\kappa}(w, y)^2 = \left|\exp_x^{-1}(w) - \exp_x^{-1}(y)\right|^2 + (f_\kappa(\alpha, x, y) - 1)\delta^2 + O(\delta^3),$$

where $f_\kappa(\alpha, x, y)$ is the geodesic function for the angle $\alpha$.
where
\[
f_\kappa(\alpha, x, y) = \begin{cases} 
\cos^2 \alpha + \frac{\vartheta}{\tan \alpha} \sin^2 \alpha & \text{if } \kappa \geq 0, \\
\cos^2 \alpha + \frac{\vartheta}{\tanh \alpha} \sin^2 \alpha & \text{if } \kappa \leq 0,
\end{cases}
\]
and \( \vartheta = d_{\mathbb{H}_\kappa}(x, y) \sqrt{|\kappa|} \).

**Proof** The result is obtained directly from a series expansion of the cosine rule for a space of constant curvature. Letting \( a = d_{\mathbb{H}_\kappa}(w, y) \), and \( c = d_{\mathbb{H}_\kappa}(x, y) = |\exp_x^{-1}(y)| \), the cosine rules are (see e.g., [Ber87], §§18.6.8, 19.3.1):
\[
\begin{align*}
\cos(a\sqrt{|\kappa|}) &= \cos(c\sqrt{|\kappa|}) \cos(\delta\sqrt{|\kappa|}) + \sin(c\sqrt{|\kappa|}) \sin(\delta\sqrt{|\kappa|}) \cos \alpha \quad &\text{if } \kappa > 0, \\
a^2 &= c^2 + \delta^2 - 2c\delta \cos \alpha \quad &\text{if } \kappa = 0, \\
\cosh(a\sqrt{|\kappa|}) &= \cosh(c\sqrt{|\kappa|}) \cosh(\delta\sqrt{|\kappa|}) - \sinh(c\sqrt{|\kappa|}) \sinh(\delta\sqrt{|\kappa|}) \cos \alpha \quad &\text{if } \kappa < 0.
\end{align*}
\]

With the aid of a computer algebra system, we compute the series expansion with respect to \( \delta \) and find
\[
\left( \kappa^{-1/2} \arccos \left( \cos(c\sqrt{|\kappa|}) \cos(\delta\sqrt{|\kappa|}) + \sin(c\sqrt{|\kappa|}) \sin(\delta\sqrt{|\kappa|}) \cos \alpha \right) \right)^2
= c^2 - 2c\delta \cos \alpha + \left( \cos^2 \alpha + \frac{c\sqrt{|\kappa|}}{\tan(c\sqrt{|\kappa|})} \sin^2 \alpha \right) \delta^2 + O(\delta^3),
\]
when \( \kappa > 0 \) and \( c\sqrt{|\kappa|} \) and \( \delta\sqrt{|\kappa|} \) are less than \( \frac{\pi}{2} \), and
\[
\left( (-\kappa)^{-1/2} \text{arccosh} \left( \cosh(c\sqrt{|\kappa|}) \cosh(\delta\sqrt{|\kappa|}) + \sinh(c\sqrt{|\kappa|}) \sinh(\delta\sqrt{|\kappa|}) \cos \alpha \right) \right)^2
= c^2 - 2c\delta \cos \alpha + \left( \cos^2 \alpha + \frac{c\sqrt{|\kappa|}}{\tanh(c\sqrt{|\kappa|})} \sin^2 \alpha \right) \delta^2 + O(\delta^3),
\]
when \( \kappa < 0 \).

Since the Euclidean cosine rule in this notation yields
\[
|\exp_x^{-1}(w) - \exp_x^{-1}(y)|^2 = c^2 - 2c \delta \cos \alpha + \delta^2,
\]
the result follows. \( \square \)

Now let \( M \) be an arbitrary Riemannian manifold with sectional curvature \( K \) bounded by \( \Lambda_f \leq K \leq \Lambda_u \), and let \( B_\rho \) be a geodesic ball of radius \( \rho < \rho_0 \), as defined in (4). We consider a hinge defined by \( w, x, y \in B_\rho \), and let \( \bar{w}, \bar{x}, \bar{y} \in \mathbb{H}_\Lambda_f \) and \( \bar{w}, \bar{x}, \bar{y} \in \mathbb{H}_\Lambda_u \) define comparison hinges. Then the hinge comparison theorem (Lemma [H]) says
\[
d_{\mathbb{H}_\Lambda_u}(\bar{w}, \bar{y})^2 \leq d_M(w, y)^2 \leq d_{\mathbb{H}_\Lambda_f}(\bar{w}, \bar{y})^2. \tag{5}
\]

Consider now that \( w = w(t) \) is a point on a unit-speed geodesic with \( w(0) = x \), so that \( d_M(x, w(t)) = |\exp_x^{-1}(w(t))| = \pm t \), and associate the angle \( \alpha \) with the positive values of \( t \), i.e., \( \langle w'(0), \exp_x^{-1}(y) \rangle = |\exp_x^{-1}(y)| \cos \alpha \). Employing Lemma [H] on the bounds (5), and expanding the squared norm using the inner product, we find
\[
d_M(w(t), y)^2 = |\exp_x^{-1}(y)|^2 - 2t |\exp_x^{-1}(y)| \cos \alpha + O(t^2).
\]
From this we conclude that
\[
\frac{d}{dt} E_\mu(w(t))|_{t=0} = - \int \langle w'(0), \exp^{-1}_x(y) \rangle \, d\mu(y),
\]
and since the geodesic \(w\) through \(x\) was arbitrary, and
\[
\frac{d}{dt} E_\mu(w(t))|_{t=0} = \langle w'(0), \text{grad} \, E_\mu(x) \rangle,
\]
by definition of the gradient, we see that
\[
\text{grad} \, E_\mu(x) = - \int \exp^{-1}_x(y) \, d\mu(y). \tag{6}
\]

We use the same technique to get a lower bound on the second derivative. If \(\mu\) were an unsigned measure, then the lower bound in (5), would yield
\[
\frac{d^2}{dt^2} E_\mu(w(t))|_{t=0} \geq \int f_{E\mu_\alpha}(\alpha, x, y) \, d\mu(y). \tag{7}
\]

Our bound \(d_{E\mu_\alpha}(\tilde{x}, \tilde{y}) = d_M(x, y) < 2\rho_0 \leq \frac{\pi}{2}\mu_{\Lambda_u}\) ensures that \(f_{E\mu_\alpha} > 0\) when \(x\) is distinct from \(y\). It follows that if \(\mu\) is an unsigned measure, then \(E_\mu\) is strictly convex in \(B_{\rho}\).

However, for the case of a signed measure that interests us here, more work is required. Using the Jordan decomposition (Section 1.4), our lower bound now becomes
\[
\frac{d^2}{dt^2} E_\mu(w(t))|_{t=0} \geq \int f_{E\mu_\alpha}(\alpha, x, y) \, d\mu_+ + \int f_{E\mu_\alpha}(\alpha, x, y) \, d\mu_- (y), \tag{8}
\]
where we have used both of the inequalities in (5). We wish to ensure that this bound is strictly positive.

Observe that for \(\vartheta \in [0, \frac{\pi}{2})\), the function \(\vartheta / \tan \vartheta\) is monotonically decreasing, and \(\vartheta / \tanh \vartheta\) is monotonically increasing for \(\vartheta \geq 0\). Let \(z = d_M(x, y)\). Writing \(\vartheta_\kappa(z) = z\sqrt{|\kappa|}\) for the variable that appears in the definition of \(f_\kappa\), we have
\[
\frac{\vartheta_\kappa(2\rho)}{\tan \vartheta_\kappa(2\rho)} < \frac{\vartheta_\kappa(z)}{\tan \vartheta_\kappa(z)} \leq 1 \quad \text{if} \ \kappa \geq 0,
\]
\[
1 \leq \frac{\vartheta_\kappa(z)}{\tanh \vartheta_\kappa(z)} < \frac{\vartheta_\kappa(2\rho)}{\tanh \vartheta_\kappa(2\rho)} \quad \text{if} \ \kappa \leq 0.
\]

Finally, by defining \(\vartheta_\ell = 2\rho \sqrt{|\Lambda_\ell|}\) and \(\vartheta_u = 2\rho \sqrt{|\Lambda_u|}\), and expanding \(f_{E\mu_\alpha}\) and \(f_{E\mu_\ell}\) in (8) using these inequalities, we obtain the convexity property of \(E_\mu\) desired for Theorem 3.

**Lemma 5** Let \(M\) be a Riemannian manifold with sectional curvature \(K\) bounded by \(\Lambda_\ell \leq K \leq \Lambda_u\). Let \(B_\rho \subseteq M\) be a geodesic ball of radius \(\rho < \rho_0\), and \(\mu\) a signed measure with support in \(B_\rho\).

The function \(E_\mu_\ell : B_\rho \to \mathbb{R}\) of Equation (2) is strictly convex provided
\[
\frac{\vartheta_u}{\tan \vartheta_u} \mu_+(M) - \mu_-(M) > 0 \quad \text{if} \ 0 \leq \Lambda_\ell \leq \Lambda_u,
\]
\[
\frac{\vartheta_u}{\tan \vartheta_u} \mu_+(M) - \frac{\vartheta_\ell}{\tanh \vartheta_\ell} \mu_-(M) > 0 \quad \text{if} \ \Lambda_\ell \leq 0 \leq \Lambda_u,
\]
\[
\mu_+(M) - \frac{\vartheta_\ell}{\tanh \vartheta_\ell} \mu_-(M) > 0 \quad \text{if} \ 0 \leq \Lambda_\ell \leq \Lambda_u,
\]
where \(\vartheta_\ell = 2\rho \sqrt{|\Lambda_\ell|}\), and \(\vartheta_u = 2\rho \sqrt{|\Lambda_u|}\).
Remark Equation (7) encompasses the bound on the second derivative announced by Karcher [Kar77] (1.2.2 and (1.2.3)) in the following sense. If $\mu$ is an unsigned measure with $\mu(M) = 1$, then if $\Lambda_u < 0$, (7) may be expanded as
\[
\frac{d^2}{dt^2} \mathcal{E}_\mu(w(t))|_{t=0} \geq \int \left( \cos^2 \alpha + \frac{\theta}{\tanh \theta} \sin^2 \alpha \right) d\mu(y) \geq \cos^2 \alpha + \frac{2\rho\sqrt{\Lambda}}{\tan(2\rho\sqrt{\Lambda})} \sin^2 \alpha \geq 1,
\]
and if $\Lambda_u > 0$, then
\[
\frac{d^2}{dt^2} \mathcal{E}_\mu(w(t))|_{t=0} \geq \int \left( \cos^2 \alpha + \frac{\theta}{\tan \theta} \sin^2 \alpha \right) d\mu(y) \geq \cos^2 \alpha + \frac{2\rho\sqrt{\Lambda}}{\tan(2\rho\sqrt{\Lambda})} \sin^2 \alpha \geq \frac{2\rho\sqrt{\Lambda}}{\tan(2\rho\sqrt{\Lambda})}.
\]
The right hand side in each case corresponds to Karcher’s bound.

2.2 The gradient of $\mathcal{E}_\mu$ on the boundary of a ball

We have a signed measure $\mu$ with support in $\overline{B}_M(c, r)$, and as usual the sectional curvatures on $M$ satisfy the upper bound $\Lambda_u$. We consider a convex ball $B_\rho = B_M(c, \rho) \supset \overline{B}_M(c, r)$ and we wish to find conditions on $\mu$ and $\rho$ such that $\mathcal{E}_\mu : \overline{B}_\rho \to \mathbb{R}$ has gradient pointing outward on $\partial B_\rho$. To be specific, let $N$ be an outward pointing unit normal vector field on $\partial B_\rho$. Then grad $\mathcal{E}_\mu$ is pointing outward on $\partial B_\rho$ if $\langle N(x), \text{grad} \mathcal{E}_\mu(x) \rangle > 0$ for all $x \in \partial B_\rho$.

Using the Jordan decomposition in (6), and defining
\[
\begin{align*}
X(x) &= -\int \exp^{-1}_x(y) d\mu_+(y), & Z(x) &= -\int \exp^{-1}_x(y) d\mu_-(y),
\end{align*}
\]
we have grad $\mathcal{E}_\mu = X - Z$, with $\langle N, X \rangle > 0$ and $\langle N, Z \rangle > 0$.

It is convenient to introduce $R = \rho - r > 0$. For the “inward pointing” part of the gradient, we have $\langle N, Z \rangle \leq (2r + R)\mu_-(M)$, because any given point in $\overline{B}_M(c, r)$ lies at most $2r + R$ from a point on $\partial B_\rho$.

The “outward pointing” part of the gradient satisfies the bound $\langle N, X \rangle \geq R\mu_+(M)$. This bound is not as easy to prove as the bound on the inward pointing part of the gradient. We wish to show that if $x \in \partial B_\rho$ and $y \in \partial B_M(c, r)$, then
\[
\langle -\exp^{-1}_x(y), N(x) \rangle = |\exp^{-1}_x(y)| \cos \varphi \geq R, \tag{9}
\]
where $\varphi$ is the angle between $-N(x)$ and $\exp^{-1}_x(y)$. In other words, we want to show that the image of $\overline{B}_M(c, r)$ under $\exp^{-1}_x$ is contained in the half-space $\{v \in T_x M \mid \langle v, -N(x) \rangle \geq R\}$.

Let $b = |\exp^{-1}_x(y)|$ be the length of the geodesic segment between $x$ and $y$. We show that, if $b \leq R/\cos \varphi$, the only geodesic triangle with edge lengths $r, r + R, b$, vertices $c, x, y$, and angle $\varphi$ at $x$, as in Figure 11, is the trivial triangle with $\varphi = 0$.

Observe that $\sin(as) \leq a \sin s$ for $a \geq 1$ and $0 \leq as \leq \pi/2$, and this inequality is strict if $s > 0$ and $a > 1$. This can be seen by differentiating with respect to $s$ and observing that $\cos as \leq \cos s$ under the same conditions.

Returning to the geodesic triangle, the angle comparison theorem (Lemma 2) implies that $\cos \varphi$ is bounded from below by $\cos \varphi_{\Lambda_u}$, where $\varphi_{\Lambda_u}$ denotes the angle corresponding to $\varphi$ for the comparison triangle in the space of curvature $\Lambda_u$. Without loss of generality we can assume that $\Lambda_u$ is positive (for our purposes here, nothing is gained by requiring $\Lambda_u$ to be a least upper bound on the sectional curvatures).
We have $b \geq R$, by the triangle inequality. Using our comparison triangle, and letting $\tilde{r} = r\sqrt{\Lambda_u}$, $\tilde{R} = R\sqrt{\Lambda_u}$, and $\tilde{b} = b\sqrt{\Lambda_u}$, the cosine rule for spaces of positive curvature now yields

$$\cos \tilde{r} = \cos(\tilde{r} + \tilde{R}) \cos(\tilde{b}) + \sin(\tilde{r} + \tilde{R}) \sin(\tilde{b}) \cos \varphi_{\Lambda_u}$$

$$\leq \cos(\tilde{r} + \tilde{R}) \cos(\tilde{b}) + \sin(\tilde{r} + \tilde{R}) \sin(\tilde{R} / \cos \varphi_{\Lambda_u}) \cos \varphi_{\Lambda_u}$$

$$\leq \cos(\tilde{r} + \tilde{R}) \cos(\tilde{b}) + \sin(\tilde{r} + \tilde{R}) \sin(\tilde{R})$$

$$= \cos \tilde{r},$$

and since the inequalities are strict unless $\varphi_{\Lambda_u} = 0$, we must have $\varphi = 0$. This confirms Equation (9), and we obtain our desired result:

**Lemma 7** Let $\mu$ be a signed measure on $M$ with support in $\overline{B}_M(c,r)$, which is contained in a concentric convex geodesic ball $B_\rho = B_M(c,\rho)$. The gradient of $E_\mu$ is pointing outward on $\partial B_\rho$ if

$$(\rho - r)\mu_+(M) - (\rho + r)\mu_-(M) > 0.$$
in Section 3.2 we use these simplified bounds to quantify the extent of well-defined barycentric coordinate neighbourhoods. The size of these neighbourhoods depends on a measure of the quality of \( \sigma \), as discussed in Section 3.2.

Our main result, Theorem 10, quantifies the size of a ball in the parameter domain, \( B \subset \mathbb{R}^n \), on which the barycentric coordinate map \( b : B \rightarrow B_\rho \subseteq M \) is well defined. By earlier work [DVW15] we are ensured that the barycentric coordinate map is not only well defined, but is in fact an embedding on this domain. We also estimate the size of a geodesic ball that is contained in the barycentric coordinate neighbourhood \( b(B) \) (Proposition 12). In Section 3.3 we exploit barycentric coordinate neighbourhoods to show that Riemannian simplices behave like their Euclidean counterparts: simplices with the same orientation that share a facet intersect only in that facet.

### 3.1 Simplifying the setting

Our goal now is to simplify the constraints imposed in Theorem 3, albeit at the expense of weakening the statement. We will assume \( \mu(M) = 1 \); this normalisation was not required for Theorem 3, although we did assume \( \mu^-(M), \mu^+(M) < \infty \). We will also replace \( \Lambda_\ell \) and \( \Lambda_u \) with a single bound \( \Lambda \) on the absolute value of the sectional curvatures. Before exploiting these assumptions, we first replace the bounds of Lemma 5 with constraints linear in \( \rho \).

**Lemma 8** We have

\[
1 - \frac{2\vartheta}{\pi} \leq \frac{\vartheta}{\tan \vartheta} \quad \text{for } \vartheta \in [0, \frac{\pi}{2}),
\]

\[
1 + \vartheta \geq \frac{\vartheta}{\tanh \vartheta} \quad \text{for } \vartheta \geq 0.
\]

**Proof** By taking the second derivatives, and observing that \( \frac{\vartheta}{\tan \vartheta} \geq 1 \) and \( \frac{\vartheta}{\tanh \vartheta} \leq 1 \) in the stated ranges, we find that \( \frac{\vartheta}{\tan \vartheta} \) is concave and \( \frac{\vartheta}{\tanh \vartheta} \) is convex in their ranges. The inequalities now follow from Jensen’s inequality, i.e., by comparing the function to a secant line.

Revisiting Lemma 5 with the assumption that \( \Lambda_\ell = -\Lambda \) and \( \Lambda_u = \Lambda > 0 \), the convexity of \( \mathcal{E}_\mu \) is ensured if the middle inequality is satisfied. Using Lemma 8 and then expanding \( \vartheta_\ell = \vartheta_u = 2\rho\sqrt{\Lambda} \) we get that \( \mathcal{E}_\mu \) is convex if

\[
\left( 1 - \frac{4\rho\sqrt{\Lambda}}{\pi} \right) \mu^+(M) - \left( 1 + 2\rho\sqrt{\Lambda} \right) \mu^-(M) > 0.
\]

Now using the assumption that \( \mu(M) = \mu^+(M) - \mu^-(M) = 1 \), we obtain a convenient simplified version of Theorem 3.

**Corollary 9** (Centre of mass of signed measures) Suppose the sectional curvature \( K \) of \( M \) is bounded by \( |K| \leq \Lambda \), and \( \mu \) is a signed measure on \( M \) with \( \mu(M) = 1 \) and support contained in a geodesic ball \( B_M(c, r) \). If \( B_\rho = B_M(c, \rho) \) with \( r < \rho < \rho_0 \) as defined in (3), then \( \mathcal{E}_\mu : B_\rho \rightarrow \mathbb{R} \) (Equation (2)) has a unique minimum in \( B_\rho \) if

\[
\rho > (1 + 2\mu^-(M))r, \quad (10)
\]

and

\[
\rho < \frac{\pi}{4\sqrt{\Lambda}} (1 + C\mu^-(M))^{-1}, \quad (11)
\]

where \( C = \left( 1 + \frac{\pi}{2} \right) \).
If $\mu$ is an unsigned measure, the conditions of Corollary 9 reduce to the same conditions required in Karcher’s theorem \cite[Thm. 1.2]{Kar77}.

### 3.2 The extent of barycentric coordinate neighbourhoods

We consider a set $\sigma$ of $n+1$ points contained in a geodesic ball $B_\rho \subseteq M$ of radius $\rho < \rho_0$ ($\rho$ will be further constrained below). We will require that there exists a Euclidean $n$-simplex $\tilde{\sigma} \subset \mathbb{E}^n$ whose vertices are in correspondence with the points of $\sigma$ and whose edge lengths are equal to the geodesic distances between the corresponding points in $\sigma$. This in itself is a constraint on $\sigma$ (see \cite[$\S$5.1]{DVW15}).

Let $(\lambda_i)$ be the barycentric coordinate functions associated with $\tilde{\sigma}$. We define $\lambda^+ : \mathbb{E}^n \to \mathbb{R}$ to be the function that gives the sum of the positive barycentric coordinates, and $\lambda^-$ to be the sum of the negative coordinates, so $\lambda^+ + \lambda^- = 1$ is constant. Given any fixed $u \in \mathbb{E}^n$, the barycentric coordinates $(\lambda_i(u))$ define a discrete measure on $M$ (a weight for each point of $\sigma$) via the correspondence between $\sigma$ and the vertices of $\tilde{\sigma}$. In this context, $|\lambda_- (u)|$ plays the role of $\mu_-(M)$ in Corollary 9.

We would like to find a domain $B \subset \mathbb{E}^n$, with $\tilde{\sigma} \subset B$, such that the constraints of Corollary 9 are satisfied for any $u \in B$. So we are interested in finding a bound on $|\lambda_-|$. We choose an arbitrary vertex $\tilde{v} \in \tilde{\sigma}$, and we will find a radius $\tilde{r}$ such that for all points $u$ in $B = B_{\mathbb{E}^n}(\tilde{v}, \tilde{r})$, this goal is attained.

Let $a$ be a lower bound on the altitudes of $\tilde{\sigma}$. The magnitude of the gradient of each barycentric coordinate function $\lambda_i$ is bounded by $a^{-1}$. Therefore $-\lambda_i < \tilde{r}/a$ on $B$. Since at most $n$ coordinate functions can be negative at any point in $B$, we find

$$|\lambda_-| < \frac{n\tilde{r}}{a} = \frac{\tilde{r}}{Lt}, \quad (12)$$

where $L$ is a strict upper bound on the edge lengths, and $t = a/nL$ is the thickness of $\tilde{\sigma}$.

We choose $r = L$ as the radius of a ball containing $\sigma \subset M$. Observing that $|\lambda_-| = \lambda^+ - 1$ and exploiting (12), we find that (10) is satisfied if

$$\rho > L + \frac{2\tilde{r}}{t}, \quad (13)$$

Similarly, (11) may be replaced with

$$\rho < \frac{\pi}{4\sqrt{\Lambda}} \left( 1 + C \frac{\tilde{r}}{Lt} \right)^{-1}, \quad (14)$$

where $C = 1 + \pi/2$.

In order for there to exist a $\rho$ satisfying both of these constraints, we require

$$\left( L + \frac{2\tilde{r}}{t} \right) \left( 1 + C \frac{\tilde{r}}{Lt} \right) < \frac{\pi}{4\sqrt{\Lambda}},$$

or

$$2C \frac{\tilde{r}^2}{Lt^2} + \frac{2 + C}{t} \tilde{r} + \left( L - \frac{\pi}{4\sqrt{\Lambda}} \right) < 0.$$

We require $\tilde{r}$ to be strictly smaller than the largest root:

$$\tilde{r} < \frac{Lt(2 + C)}{4C} \left( -1 + \left( 1 + \frac{8C}{(2 + C)L} \left( \frac{\pi}{4\sqrt{\Lambda}} - L \right) \right)^{1/2} \right). \quad (15)$$
Since $\frac{4}{9} < \frac{2sC}{6L} < \frac{1}{2}$, Equation (15) is satisfied when

$$\tilde{r} \leq \frac{Lt}{3} \left( -1 + \sqrt{-3 + \frac{\pi}{4L\sqrt{\Lambda}}} \right). \quad (16)$$

Observe that this bound is positive provided $L \leq \frac{1}{6\sqrt{\Lambda}} < \frac{\pi}{16\sqrt{\Lambda}}$. Also, the bound (16) is maximised when $L = \frac{\pi}{48\sqrt{\Lambda}} \sim \frac{1}{15\sqrt{\Lambda}}$.

We want to ensure that the entire simplex $\tilde{\sigma}$ is contained in the domain of the barycentric coordinates. Therefore, using a scale parameter $s \geq 1$, we set $\tilde{r} = sL$ in (16) and find that $L$ must satisfy

$$L \leq \frac{\pi}{4\sqrt{\Lambda}} \left( \frac{9s^2}{t^2} + \frac{6s}{t} + 4 \right)^{-1}.$$

Since $t < 1$, and $s \geq 1$ this is satisfied when

$$L \leq \frac{t^2}{25s^2\sqrt{\Lambda}}. \quad (17)$$

This is the criterion bounding $\tilde{r}$ that we sought.

Thus when (17) is satisfied, the barycentric coordinate map $b: B_{E^n}(\tilde{v}, sL) \to B_\rho$ is well defined, provided $\rho$ satisfies (13) and (14), with $\tilde{r} = sL$; the existence of such a $\rho$ is assured. But we can say more. It has been shown [DVW15, Prop. 29] that the differential of the barycentric coordinate map is nondegenerate provided $L \leq t/(3\sqrt{\Lambda})$, which is obviously satisfied when (17) is. Although the argument was made in the context of positive weights, this fact was not used. What is required is that the energy functional (4) that defines the barycentric coordinate map be convex, but we still have that in our current context. Thus we obtain our main result:

**Theorem 10 (Barycentric coordinate neighbourhoods)** Let $M$ be an $n$-dimensional Riemannian manifold with sectional curvature $K$ bounded by $|K| \leq \Lambda$, and let $\sigma \subset M$ be a set of $n + 1$ points such that $d_M(p, q) < L$ for any $p, q \in \sigma$.

Given a scale parameter $s \geq 1$, if $\sigma$ defines a Euclidean simplex $\tilde{\sigma} \subset \mathbb{E}^n$ with the same edge lengths, and with thickness $t$ satisfying

$$t^2 \geq 25s^2L\sqrt{\Lambda},$$

then, the barycentric coordinate map $b: B_{E^n}(\tilde{v}, sL) \to B_\rho$, where $\tilde{v}$ is any vertex of $\tilde{\sigma}$, is well defined, and is in fact an embedding. The ball $B_\rho$ is centred on the vertex of $\sigma$ corresponding to $\tilde{v}$, and its radius $\rho$ must lie in the nonempty interval given by

$$\left( 1 + \frac{2s}{t} \right) L < \rho < \frac{\pi}{4\sqrt{\Lambda}} \left( 1 + \frac{C^s}{t} \right)^{-1},$$

where $C = 1 + \pi/2$.

Although the image of $B_{E^n}(\tilde{v}, sL)$ under the barycentric coordinate map $b$ is guaranteed to be contained in $B_\rho$ for any sufficiently large $\rho$, it may be also useful to know the size of a geodesic ball that is contained in $b(B_{E^n}(\tilde{v}, sL))$. The metric distortion of the barycentric coordinate map has already been calculated [DVW15, §5.2], and although this argument was made in the context of non-negative weights, this fact was not exploited, and the argument carries verbatim to the case of interest here. The argument does require that $\rho$ be bounded by $\rho \leq \frac{1}{6\sqrt{\Lambda}}$ but this is satisfied if the conditions of Theorem 10 are satisfied.
Lemma 11 (DVW15, §5.2) Given the assumptions of Theorem 10, the barycentric coordinate map \( b: B_{\mathbb{E}^n}(\tilde{v}, sL) \to B_p \) satisfies
\[
|d_M(b(u), b(w)) - |u - w|| \leq \frac{50\Lambda \rho^2}{t^2} |u - w|,
\]
for any \( u, w \in B_{\mathbb{E}^n}(\tilde{v}, sL) \).

There is no reason to choose a \( \rho \) larger than necessary; with the assumptions of Theorem 10, we can choose \( \rho^2 = (1 + \frac{2}{7})^2 L^2 + \varepsilon^2 \leq \frac{3^2 t^2}{25 s^2 \Lambda} \), where \( \varepsilon \) is an arbitrarily small distance. Employing this bound in Lemma 11 and choosing a point \( w \in \partial B_{\mathbb{E}^n}(\tilde{v}, sL) \), we find
\[
d_M(b(\tilde{v}), b(w)) \geq \left( 1 - \frac{18}{25s^2} \right) sL.
\]

We desire that the image of \( \tilde{\sigma} \) be contained in this ball. Demanding \( (s - \frac{18}{25s}) L \geq L \), we find this will be satisfied provided \( s \geq \frac{3}{2} \), and we have the following addendum to Theorem 10.

Proposition 12 Assume that the conditions of Theorem 10 are satisfied, and that \( v \in \sigma \) is the vertex corresponding to \( \tilde{v} \in \tilde{\sigma} \). Then the image of \( B_{\mathbb{E}^n}(\tilde{v}, sL) \) under the barycentric coordinate map contains the geodesic ball \( B_M(v, \tilde{r}) \), where
\[
\tilde{r} = \left( s - \frac{18}{25s} \right) L.
\]
If \( s \geq \frac{3}{2} \), then \( \tilde{r} \geq L \), and so \( B_M(v, \tilde{r}) \) contains \( \tilde{\sigma} \).

3.3 Riemannian simplices that share a facet

Suppose \( \sigma_M \subset M \) is a Riemannian \( n \)-simplex defined by \( \sigma = \{p, p_0, \ldots, p_{n-1}\} \). The facet of \( \sigma_M \) opposite \( p \) is the Riemannian \((n - 1)\)-simplex \( \eta_M \) defined by \( \eta = \{p_0, \ldots, p_{n-1}\} \), i.e., the set of points in \( \sigma_M \) where the barycentric coordinate associated with \( p \) is 0. Suppose now that \( \tau_M \subset M \) is another Riemannian simplex defined by \( \tau = \{p_0, q, p_1, \ldots, p_{n-1}\} \), so that \( \tau_M \) shares the facet \( \eta_M \) with \( \sigma_M \).

Suppose that both \( \sigma_M \) and \( \tau_M \) satisfy the conditions of Theorem 10 with scale factor \( s = 3/2 \), and with \( L \) being a common strict upper bound on their edge lengths. Then they are non-degenerate, and we can define their orientation in \( B_p \) (see DVW15 §3.3). If they have the same orientation, then they can be represented by Euclidean simplices \( \tilde{\sigma} = \{\tilde{p}, \tilde{p}_0, \ldots, \tilde{p}_{n-1}\} \) and \( \tilde{\tau} = \{\tilde{p}_0, \tilde{q}, \tilde{p}_1, \ldots, \tilde{p}_{n-1}\} \) that share a common facet, \( \tilde{\eta} = \tilde{\sigma} \cap \tilde{\tau} \), such that \( \tilde{p} \) and \( \tilde{q} \) lie on opposite sides of the hyperplane \( \text{aff}(\tilde{\eta}) \). Thus \( \tilde{\tau}_{\mathbb{E}^n} = \text{conv}(\tilde{\tau}) \) lies in the region where the barycentric coordinate associated with \( p \in \tilde{\sigma} \) is nonpositive, and similarly, \( \tilde{\sigma}_{\mathbb{E}^n} = \text{conv}(\tilde{\sigma}) \) lies in the region where the barycentric coordinate associated with \( \tilde{q} \in \tilde{\tau} \) is nonpositive, and \( \text{aff}(\tilde{\eta}) \) is the common 0 level set of both of these barycentric coordinate functions.

The barycentric coordinate map for \( \sigma \) is an embedding of \( B_{\mathbb{E}^n}(\tilde{p}_0, sL) \) that maps \( \tilde{\sigma}_{\mathbb{E}^n} \) to \( \sigma_M \), preserving the associated barycentric coordinates, and similarly the barycentric coordinate map for \( \tau \) maps \( \tilde{\tau}_{\mathbb{E}^n} \) to \( \tau_M \). These maps agree on \( \text{aff}(\tilde{\eta}) \), and by our choice of scale factor \( s \), Proposition 12 ensures that \( B_M(p_0, L) \) is contained in the image of both maps. Since the \((n - 1)\)-submanifold that is the image of \( \text{aff}(\tilde{\eta}) \) separates this ball into two components according to the sign of the barycentric coordinate of \( p \) (and likewise of \( q \)), it follows that \( \sigma_M \cap \tau_M = \eta_M \).
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