One-Dimensional Consolidation Analysis Considering Non-Darcian Flow and Self-Weight Based on Continuous Drainage Boundary

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Abstract. The one-dimensional consolidation problem of soft soil with self-weight and non-Darcian flow is studied by introducing continuous drainage boundary. A numerical solution is derived by using finite difference method and its correctness is proved by comparing with existing analytical and numerical solutions. Based on the present solution, the consolidation behavior of soil is detailedly analyzed by adjusting the different parameters. The results show that, the soil consolidation rate increases with the increase of interface parameters or external loading rate. The difference between the average consolidation degree obtained by the solution with continuous drainage boundary and that with traditional boundary is mainly in the early stage of consolidation, but in the late stage of consolidation, the difference between these two solutions becomes smaller. Compared with the soil consolidation solution considers non-Darcian flow, the soil consolidation rate with Darcy flow is faster.

1. Introduction
The consolidation law of soft soils in practical engineering is very complicated owing to the complexity of soil properties, flow patterns, drainage boundary conditions and external loads acting on soil. The conventional consolidation theory usually assumes that the flow in soil is Darcy flow. However, a large number of laboratory tests show that the flow in soft clay sometimes deviates from Darcy’s law. The non-Darcian flow model commonly used in consolidation theory is the Hansbo flow model proposed by Hansbo [1-2]. Hansbo [1-2] analyzed the consolidation by vertical drains and the one-dimensional consolidation of saturated soil with non-Darcian flow. Following the non-Darcian flow proposed by Hansbo, Teh et al. [3] conducted consolidation analysis of the vertical drains by taking the nonlinear compression characteristics of soft soils into consideration. Subsequently, Liu et al. [4] utilized the finite difference method to solve the one-dimensional consolidation problem of saturated clay under transient loading. Li et al. [5] applied the finite difference method to solve the one-dimensional consolidation problem considering variable loading and non-Darcian flow.

The traditional drainage boundary cannot reflect the process of pore pressure dissipation over time at the boundary. Therefore, Mei et al. [6] proposed a continuous drainage boundary that can reflect the dissipation of pore pressure at the boundary over time. Subsequently, many scholars have studied the one-dimensional consolidation problem of soil with continuous drainage boundary by considering self-weight [7], unsaturated property, [8], time-dependent loading [9] and nonlinear characteristics [10]. However, the aforementioned consolidation theory based on continuous drainage boundary
assumes that the soil obeys Darcy’s law, but some dense clays with low permeability often have non-Darcian flow.

In this work, the one-dimensional consolidation problem of soft soil with self-weight and non-Darcian flow under time-dependent loading is investigated by introducing continuous drainage boundary. Then, a parametric study is conducted to study the consolidation behavior of soft soil based on the present method.

2. Mathematical Modeling

Figure 1 shows the soft soil foundation model. In Figure 1, $H$, $q(t)$ and $z$ denote the thickness, variable load applied on the surface and vertical direction of the soft soil, respectively.

![Soft soil foundation model](image)

**Figure 1.** Soft soil foundation model.

The relation between loading and time is shown in Figure 2 and expressed as:

$$q(t) = \begin{cases} 
q_0 + \frac{q_u - q_0}{t_0} t & t \leq t_0 \\
q_u & t > t_0 
\end{cases}$$

(1)

where, $q_0$ and $q_u$ are the initial and final loads, respectively. $t$ and $t_0$ denote the loading time and the moment when the loading stage is completed.

The expression of the non-Darcy flow model proposed by Hansbo is as follows:

$$v = \begin{cases} 
\kappa \iota^m, & i \leq i_1 \\
k_v (i - i_0), & i \geq i_1 
\end{cases}$$

(2)

where, $v$ is the percolation velocity. $\kappa$ and $k_v$ denote the permeability coefficients for the exponential segment and the linear segment, respectively. $i$ and $i_0$ represent the hydraulic gradient and the threshold hydraulic gradient for the linear relationship. $m$ is a constant determined by experiment.

The expressions of $i_0$ and $\kappa$ are written as:

$$i_0 = \left\{ \frac{i_1 (m-1)}{m} \right\}$$

(3)

$$\kappa = \left\{ \frac{k_v}{m i_1^{m-1}} \right\}$$

According to the continuous conditions of flow, the one-dimensional consolidation equation of the soft soil with non-Darcian flow can be expressed as equation [4]:

$$\frac{\partial u}{\partial t} = \frac{1}{H} \frac{\partial}{\partial z} (v \frac{\partial u}{\partial z})$$

where $u$ is the excess settlement.
\[
\begin{align*}
\frac{c_v}{\gamma_w^{m-1}} \left( 1 - \frac{\partial u}{\partial z} \right)^{m-1} \left( \frac{\partial^2 u}{\partial z^2} \right) &= \frac{\partial u}{\partial t} - \frac{dq}{dt}, \quad i \leq I_1 \\
\frac{\partial^2 u}{\partial z^2} &= \frac{\partial u}{\partial t} - \frac{dq}{dt}, \quad i \geq I_1
\end{align*}
\]

where, \( u \) and \( \gamma_w \) denote the excess pore water pressure and the gravity of water. \( c_v \) is the consolidation coefficient and satisfy \( c_v = k_v/\gamma_w m_v \), in which \( m_v \) is the compression coefficient.

According to the continuous drainage boundary proposed by Mei [6], the initial and boundary conditions for the consolidation problem can be written as:

\[
\begin{align*}
&u(z,0) = q_0 + \gamma' z \\
&u(0,t) = e^{-\gamma z} \left( q_0 + \int_0^t e^{\gamma z} q'(\xi) d\xi \right) \\
&\left. \frac{\partial u}{\partial z} \right|_{z=0} = 0
\end{align*}
\]

where, \( \gamma' \) is the effective weight of the soil. \( \alpha \) is the interface parameter reflecting the drainage capacity at the top surface of the soil. The interface parameter can be obtained by back-calculating the variations of excess pore water pressures at the interface through experimental or field measurements.

3. Numerical Solution of Consolidation Equation

For convenience, the following dimensionless parameters are defined:

\[
\begin{align*}
U &= \frac{u}{q_u}, Z &= \frac{z}{H}, T_v &= \frac{c_v t}{H^2}, I &= \frac{i \gamma H}{q_u} \\
I_1 &= \frac{i \gamma H}{q_u}, \eta &= \frac{\gamma H}{q_u}, Q &= \frac{q(t)}{q_u}, Q_0 &= \frac{q_0}{q_u}
\end{align*}
\]

Substituting the dimensionless parameters into equations (4)-(7), the corresponding consolidation equations, initial and boundary conditions can be written as:

\[
\begin{align*}
\frac{1}{I_1^{m-1}} \left( \frac{\partial U}{\partial Z} \right)^{m-1} \left( \frac{\partial^2 U}{\partial Z^2} \right) &= \frac{\partial U}{\partial T_v} - \frac{dQ}{dT_v}, \quad I \leq I_1 \\
\frac{\partial^2 U}{\partial Z^2} &= \frac{\partial U}{\partial T_v} - \frac{dQ}{dT_v}, \quad I \geq I_1
\end{align*}
\]

\[
U(Z,0) = Q_0 + \eta Z
\]

\[
U(0,T_v) = \begin{cases} 
Q_0 e^{-\alpha T_v} + \frac{1 - Q_0}{\alpha T_{v0}} (1 - e^{-\alpha T_v}) T_v & \leq T_{v0} \\
Q_0 e^{-\alpha T_v} + \frac{1 - Q_0}{\alpha T_{v0}} (e^{-\alpha(T_v - T_{v0})} - e^{-\alpha T_v}) T_v & \leq T_{v0}
\end{cases}
\]

\[
\left. \frac{\partial U}{\partial Z} \right|_{Z=0} = 0
\]
The consolidation equation (9) is a complex nonlinear partial differential equation, and finite difference method is utilized to solve this problem. The Crank-Nicolson difference method format is utilized to solve the consolidation equation for it has high accuracy and is completely stable. Equation (9) can be expressed as:

$$\alpha^l_{j} \lambda U^{l+1}_{j+1} - 2(\alpha^l_{j} \lambda + 1) U^{l}_{j} + \alpha^l_{j} \lambda U^{l+1}_{j} = 2(\alpha^l_{j} \lambda - 1) U^{l-1}_{j} - \alpha^l_{j+1} \lambda U^{l+1}_{j+1} - 2(Q^{l} - Q^{l-1})$$

(13)

where \( l \) and \( j \) represent space and time nodes, respectively, and \( l = 0,1,2,\ldots,n \). \( j = 0,1,2,\ldots,N \). \( \tau \) and \( h \) are time step and space step, respectively.

$$\alpha^l_{j} = \begin{cases} \frac{1}{I^{l+1}_{j} - I^{l}_{j}} \left( \frac{U^{l}_{j+1} - U^{l}_{j-1}}{2h} \right)^{m-1}, & I^{l}_{j} \leq I^{l}_{j} \\ 1, & I^{l}_{j} \geq I^{l}_{j} \end{cases}$$

(14)

$$U^{l}_{j} = \frac{\left| U^{l}_{j+1} - U^{l}_{j-1} \right|}{2h}$$

(15)

Under the condition of single-sided drainage, the flow velocity at the bottom surface of the soil is always 0, which makes it difficult to solve the mathematical equations by difference method. When the hydraulic gradient equals to 0, the non-Darcian flow can be degenerated to the Darcy flow. Therefore, the flow form at the bottom surface of the soil is calculated according to the Darcy flow.

The corresponding initial and boundary conditions can be expressed as:

$$U^{0}_{j} = Q_{o} + \eta \|h|l = 0,1,2,L,n;$$

(16)

$$U^{l}_{0} = \begin{cases} Q_{o} e^{-\alpha T_{v0}} + \frac{1 - Q_{o}}{\alpha T_{v0}} (1 - e^{-\alpha T_{v0}}), & T_{v} \leq T_{v0} \\ Q_{o} e^{-\alpha T_{v0}} + \frac{1 - Q_{o}}{\alpha T_{v0}} (e^{-\alpha (T_{v} - T_{v0})} - e^{-\alpha T_{v0}}), & T_{v} > T_{v0} \end{cases}$$

(17)

$$U^{l}_{n+1} = U^{l}_{n-1}$$

(18)

\( \alpha^l_{j} \) contains the \( j \)-node excess pore water pressure, therefore, equation (13) is a series of nonlinear equations of unknown variables \( U^{l}_{j} \). Using iterative method to calculate \( \alpha^l_{j} \), \( U^{l-1}_{j} \) can be adopted as the initial iterative value to obtain the values of \( \alpha^l_{j} \). Then using the chasing method to solve \( U^{l}_{j} \), and equation (13) can be expressed as the following matrix:

$$A^{l-1} U^{l} = B^{l-1}$$

(19)

where, \( A \) is an unknown tridiagonal matrix. The elements in the matrix are expressed as:

$$A_{ll} = -2(\alpha^l_{j} \lambda + 1), \ l = 1,2,3,L,n-1;$$

(20)

$$A_{l(l+1)} = A_{l(l+1)} = \alpha^l_{j} \lambda$$

(21)

$$A_{ll} = -2(1 + \lambda)$$

(22)

$$A_{ll} = 2 \lambda$$

(23)

$$B_{l} = 2 \left( \alpha^{l-1}_{j} \lambda - 1 \right) U^{l-1}_{j} - \alpha^{l-1}_{j} \lambda U^{l-1}_{j} + \alpha^{l-1}_{j} \lambda U^{l-1}_{j+1} - \alpha^{l-1}_{j} \lambda U^{l-1}_{j-1} - \alpha^{l-1}_{j} \lambda U^{l-1}_{j} + \frac{\lambda}{2} \left( Q - Q \right)$$

(24)
\[ B_i = 2(\alpha_i^{j+1} \lambda - 1)U_i^{j+1} - \alpha_i^{j+1} \lambda U_i^{j+1} - \alpha_i^{j+1} \bar{\lambda} U_i^{j+1} - 2(Q_i^{j+1} - Q_i^{j+1}), \quad i = 2, 3, \ldots, n - 1; \]  
\[ B_n = 2(\lambda - 1)U_n^{j+1} - 2\lambda U_n^{j+1} - 2(Q_n^{j+1} - Q_n^{j+1}) \]  

The average degree of consolidation \( U_s \) defined by soil settlement can be expressed as:

\[
U_s = \frac{m_n}{m_1} \int_0^1 (Q + \eta Z - U) dZ - \frac{2}{2 + \eta} \left( 1 - Q + \frac{1}{n} \sum_{i=1}^n U_i + U_{i+1} \right)
\]  

4. Evaluation of the Proposed Solution

\( Q_0 = 1 \) means that the variable loading is degraded to constant loading, and \( m = 1 \) means that non-Darcian flow is degraded to Darcy flow. In the case of \( Q_0 = 1 \) and \( m = 1 \), figure 3 compares the proposed numerical solution with the analytical solution of Mei [6]. It can be seen from figure 3 that the average consolidation degree obtained by the present method is consistent with that obtained by the analytical solution, which demonstrates the correctness of the present solution. In addition, the consolidation rate of the soil increases with the increase of the interface parameter \( \alpha \). This is because the drainage rate at the interface becomes faster as the \( \alpha \) value increases.

![Figure 3. Consolidation curve for non-Darcian flow degradation.](image)

![Figure 4. Influence of \( m \) value on pore pressure curves.](image)

Figure 4 compares the excess pore water pressure obtained by the present method when \( \alpha = 1000 \) with that obtained by the solution of Li [5] based on the traditional boundary. As shown in figure 4, the solution of pore pressure obtained by the proposed method is consistent with that obtained by Li’s method, indicating that the continuous drainage boundary can be degenerated to traditional boundary when \( \alpha \) value is relatively large. At the same time, the correctness of the present solution is demonstrated again. According to the flow velocity expressed by equation (2), the flow velocity decreases with the increase of the flow parameter \( m \) value, which leads to a slower rate of pore water dissipation in the soil. Therefore, the pore pressure in figure 4 increases with the increase of \( m \) value.

5. Parametric Analysis

Figure 5 shows the effect of the interface parameter \( \alpha \) on the overall average degree of consolidation \( U_s \) of soil with constant loading. As shown in figure 5, with the increase of \( \alpha \) value, the average degree of consolidation obtained by the present method gradually increases, and the \( U_s \) curve obtained by the present method gradually approaches that obtained by the analytical solution proposed by Liu with Terzaghi boundary [5]. In addition, with the increase of \( T_s \) value, the difference between the \( U_s \) obtained by the present method and that obtained by Liu’s solution gradually decreases. This indicates that the consolidation rate obtained by Liu's method is faster in the early stage of consolidation, but the
The consolidation rate obtained by the present solution is faster in the later stage of consolidation. In practice, the required consolidation drainage rate can be designed by adjusting the $\alpha$ value according to the solution with continuous drainage boundary.

Figure 5. Influence of $\alpha$ value on $U_s$ curves.

Figure 6. Influence of $m$ value on $U_s$ curves.

Figure 6 depicts the effect of $m$ value on $U_s$ under the present method with continuous drainage boundary and Li's solution [5] with Terzaghi boundary. It can be found that the $U_s$ decreases as the value of $m$ increases. This also illustrates that the consolidation rate of soil without considering the influence of non-Darcian flow will be faster. In addition, figure 6 shows that the difference between the average consolidation degree obtained by the present method and that obtained by Li's method is mainly in the early stage of consolidation. As the value of $m$ increases, the difference between these two solutions in the entire consolidation phase is gradually reduced.

Figure 7 illustrates the effect of $\eta$ value on the average consolidation degree of soil. $\eta$ is the coefficient of self-weight of soil determined by the ratio of the self-weight pressure to the final external load. In figure 7, the average consolidation degree of soil increases with the increase of the $\eta$ value, indicating that the self-weight pressure of soil can accelerate the consolidation rate of soil. On the contrary, the consolidation rate obtained without considering the influence of self-weight will be small.

Figure 7. Influence of $\eta$ value on the average consolidation degree curves.

Figure 8. Influence of $T_{v0}$ value on the average consolidation degree curves.

Figure 8 shows the influence of $T_{v0}$ on soil consolidation rate when other parameters are fixed. When $Q_0$ value remains unchanged, the loading rate of external load increases with the decrease of $T_{v0}$ value. As shown in figure 8, the $U_s$ increases with the decrease of $T_{v0}$, indicating that increasing the
loading rate of external load is conducive to soil consolidation. In addition, with the increase of $T_{v0}$ value, the difference between the average consolidation degree obtained by the present method with continuous drainage boundary and that obtained by Li’s method with traditional boundary gradually decreases. This is because when the loading rate is slow, the excess pore water pressure in the soil increases slowly, and the continuous drainage boundary has enough ability to dissipate the pore water pressure. However, the drainage capacity provided by the Terzaghi boundary is too large and is wasted. This also indicates that when the external load increases slowly, if the consolidated drainage boundary is designed according to the continuous drainage boundary, not only the material can be saved, but also the drainage capacity of the designed drainage boundary can achieve the result of complete drainage boundary.

6. Conclusions
Based on continuous drainage boundary, this paper studies the one-dimensional consolidation characteristics of soil considering non-Darcy flow, and main results as follow:

(1) The difference between the average consolidation degree obtained by the solution with continuous drainage boundary and that with Terzaghi boundary is mainly in the early stage of consolidation, but the difference between these two solutions becomes smaller in the late stage of consolidation.

(2) The larger the value of $m$, the more difficult it is to dissipate the pore pressure and the slower the soil consolidation rate, and the greater the difference between the average degree of consolidation with non-Darcian flow and the average degree of consolidation obtained with Darcy flow.

(3) The self-weight pressure can improve the consolidation rate of soil.

(4) The consolidation rate of soil increases with the increase of external loading rate. The difference between the method with continuous drainage boundary and that with traditional boundary decreases with the slow loading rate of external load.

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References
[1] Hansbo S 1997 Aspects of vertical drain design: Darcian or non-Darcian flow Geotechnique 47(5) 983-992.
[2] Hansbo S 2003 Deviation from Darcy’s law observed in one-dimensional consolidation Geotechnique 53(6) 601-605.
[3] Teh C I and Nie X Y 2002 Coupled consolidation theory with non-Darcian flow Comput. Geotech. 29(3) 169-209.
[4] Liu Z Y, Sun L Y, Yue J C and Ma C W 2009 One-dimensional consolidation theory of saturated clay based on non-Darcy flow Chinese Journal of Rock Mechanics and Engineering 28(5) 973-979 (in Chinese).
[5] Li C X, Xie K H and Wang K 2010 Analysis of 1D consolidation with non-Darcian flow described by exponent and threshold gradient J. Zhejiang Univ., Sci., A 11(9) 656-667.
[6] Mei G X and Chen Q M 2013 Solution of Terzaghi one-dimensional consolidation equation with general boundary conditions J. Cent. South Univ. (Engl. Ed.) 22(8) 2239-44.
[7] Feng J X, Ni P P and Mei G X 2019 One-dimensional self-weight consolidation with continuous drainage boundary conditions: Solution and application to clay-drain reclamation Int. J. Numer. Anal. Methods Geomech. DOI: 10.1002/nag.2928.
[8] Wang L, Sun D A and Qin A F 2018 Semi-analytical solution to one-dimensional consolidation for unsaturated soils with exponentially time-growing drainage boundary conditions Int. J. Geomech. 18(2) 04017144.
[9] Sun M, Zong M F, Ma S J, Wu W B and Liang R Z 2018 Analytical solution for one-dimensional consolidation of soil with exponentially time-growing drainage boundary under a Ramp Load *Math. Probl. Eng.* doi.org/10.1155/2018/9385615.

[10] Zong M F, Wu W B, Mei G X, Liang R Z and Tian Y 2018 An analytical solution for one-dimensional nonlinear consolidation of soils with continuous drainage boundary *Chinese Journal of Rock Mechanics and Engineering* 37(12) 2829-38 (in Chinese).