On the existence of solutions for a nonlinear stochastic partial differential equation arising as a model of phytoplankton aggregation

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Abstract

In this paper, we are interested in the analytical study of a nonlinear Stochastic Partial Differential Equation (SPDE) arising as a model of phytoplankton aggregation. This SPDE consists in a diffusion equation with a chemotaxis term responsible of self-attraction of phytoplankton cells and a multiplicative branching noise. Existence of mild solutions is established through weak and tightness arguments.

Keywords: Phytoplankton aggregation, Nonlinear stochastic partial differential equation, Semigroups, Chemotaxis, Gaussian space-time white noise, Weak convergence, Tightness, Skorohod representation theorem.

Introduction

In this paper, we are interested in the following nonlinear stochastic partial differential equation:

\[
\frac{\partial}{\partial t} u(t, x) = D \frac{\partial^2}{\partial x^2} u(t, x) - \frac{\partial}{\partial x} \left( u(t, x) \left[ G * u^0(t.,.) \right](x) \right) + \sqrt{\lambda u(t, x)} W(t, x),
\]

in \([0, T] \times \Omega\), where \(\Omega = [0, L]\) is a bounded domain with boundary \(\partial \Omega\) in \(\mathbb{R}\), \(x\) is a one dimensional coordinate, \(t\) is time. The motivation in studying this SPDE is that equation (1) arises as a model of phytoplankton aggregation ([9],[10],[11],[12]). In fact, the authors in ([9],[10],[11],[12]) have investigated an individual-based model (IBM) of a population of phytoplankton that takes into account small scales biological mechanisms for phytoplankton cells. These processes are: random dispersal of phytoplankton cells due to turbulence, spatial interactions between phytoplankton cells caused by chemical signals and random division and death of phytoplankton cells. The aim of such modelling was

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to capture the main features of the individual dynamics at small scales that are responsible at a larger scale for the formation of aggregating patterns. An Eulerian version of the IBM has been derived in \([9],[11]\) and the SPDE \((1)\) is obtained as a continuum limit of this IBM. \(u(t,x)\) represents the spatio-temporal distribution density of phytoplankton on the vertical water column. We denote by \(u_+(t,x)\) the positive part of \(u(t,x)\), that is \(u_+(t,x) = \max(u(t,x),0)\). The diffusion term in \((1)\) takes into account the random spread of the phytoplankton cells with the coefficient of diffusion \(D\); while the transport term, which is a chemotaxis term, describes the interaction mechanisms between the phytoplankton cells via the velocity \(G^*u_0\). The latter has the form of convolution as in \([23]\), i.e.,

\[
\left[G * u^0(t,.)\right](x) = \int_{\mathbb{R}} G(x - x') u^0(t,x')dx',
\]

where \(G\) is the attractive force given by:

\[
G(x) = \left[([|x| - r_1]([|x| - r_0])1_{[-r_1,-r_0]∪[r_0,r_1]}(x)\right).
\]

and

\[
u^0(x) = \begin{cases} u(x) & 0 < x < L \\ 0 & x \leq 0 \text{ or } x \geq L. \end{cases}
\]

The biological explanation of the interactions between phytoplankton cells is based on the following works \((14),(15),(18),(21),(22),(24),(25),(26)\) which report that species of phytoplankton such as dinoflagellates and motile algae have chemosensory abilities i.e., they can sense the chemical field generated by the presence of other dinoflagellates and motile algae that are present at a certain distance. More precisely, dinoflagellates and motile algae leak organic matter such as sugar and amino acids, forming regions around them having concentrations higher than average \([3]\). Experiments studies on the chemosensory abilities in dinoflagellates and motile algae \((26),(15)\) show that the released products attract other dinoflagellates and motile algae that are present in a suitable neighborhood. It has also been observed that high concentrations of these products inhibit the chemosensory behavior in dinoflagellates and motile algae \((15)\).

So, if a phytoplankton cell is located at a position \(x\), the extracellular products released by this cell form a concentration around \(x\), which is highest in the closest vicinity of \(x\) (for instance, on a radius of length \(r_0\) \((r_0\) small)) and then decreases progressively. Via their chemosensory abilities, all dinoflagellates at positions \(x'\) such that \(r_0 \leq |x - x'| \leq r_1\) \((r_1 > r_0\) and \(r_0\) too small relative to \(r_1\)) detect the differences of concentration in water and hence are attracted to the cell in \(x\). Beyond \(r_1\), cells cannot perceive the difference in concentration because they are sensory limited \((4),(19),(20)\). Hence, they are not attracted.

\((G * u^0)(x)\) describes the velocity induced at the site \(x\) by the net effects of all phytoplankton cells at various sites \(x'\). The kernel \(G(x-x')\) associates a strength of interaction per unit density as a function of the distance \(x-x'\) between any two phytoplankton cells in sites \(x\) and \(x'\). \(G\) behaves as a gradient, that is, a cell is attracted to the region of high density.

\(W(t,x)\) is a white noise in space and time defined on some probability space
\( (A, \mathcal{F}, P) \) \[27\] and \( \lambda \) is the branching rate. The term \( \sqrt{\lambda u(t, x)}W(t, x) \) describes the stochastic fluctuations of the number of phytoplankton cells as a result of the random birth and death events.

Here, \( \mathbb{R} \) represents the vertical axis oriented downward from the surface to the seabed. The point 0 is at the surface of water and \( L \) is the limit of the "euphotic zone" i.e. the upper layers in the vertical water column. The restriction of model (1) to this domain is due to the fact that phytoplankton cells can survive and multiply only in the "euphotic zone". Therefore, von-Neumann boundary conditions are imposed at the surface 0 and at \( L \):

\[
\frac{\partial}{\partial x} u(t, x) = 0, \text{ on } [0, T] \times \partial \Omega. \tag{2}
\]

and the initial density is

\[
u(0, x) = u_0(x) \geq 0, \text{ in } \Omega. \tag{3}
\]

To our knowledge, SPDE (1) is new and unknown in both biological and mathematical literature. From the mathematical point of view, it is a complication. The major difficulties come from the two nonlinear terms: the chemotaxis term with a convolution and the stochastic branching term with the nonlinear multiplicative noise. We point out that previous works (\[2\], \[13\]) have concerned the study of the deterministic part of the SPDE (1) (i.e. equation (1) without the noise term) with the goal of analyzing the influence of the spatial interactions between phytoplankton cells on the aggregation process. Here in this paper, we are interested by the whole equation (1) which includes the effect of the stochastic branching term. The volatility \( \sqrt{\lambda u(t, x)} \) of the space-time white noise is non-Lipschitzian; hence, we are not able to expect the existence of strong or mild solutions for (1). However, we might expect the existence of a mild solution to (1) in the weak sense; that is we shall construct a probability basis \( (\Lambda, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P) \), on which there exists a Gaussian space-time white noise \( W(t, x) \) and a mild solution \( \tilde{u}(t, x) \) to (1). This will be done by using tightness arguments which are useful in obtaining weak convergence. The rest of the paper is organized as follows. In the next section, we present the abstract formulation of the problem and in section 3, we present our results on existence of mild solutions at the weak sense. Several steps are arranged for proving the main theorem. Some estimates on the semi-group generated by the operator \( \frac{\partial^2}{\partial x^2} \) are listed in the Appendix.

1 Abstract formulation

We reformulate problem (1) with conditions (2) and (3) as a stochastic version of an abstract Cauchy problem which can be treated by using the theory of semigroups of operators:

\[
\begin{cases}
du(t) = (Au(t) - B[u(t)g_G(u(t))])dt + C(u(t))dW(t) \\
u(0) = u_0
\end{cases} \tag{4}
\]
The operator $A : \mathcal{D}(A) \subset X = L^2(\Omega) \rightarrow X$ is defined by

$$A w = D \frac{d^2 w}{dx^2},$$

$$\mathcal{D}(A) = \left\{ w \in H^2(\Omega) : w'_{\partial \Omega} = 0 \right\},$$

and the operator $B : \mathcal{D}(B) \subset X \rightarrow X$ by

$$B w = \frac{d}{dx} w,$$

$$\mathcal{D}(B) = H^1(\Omega).$$

$H^1(\Omega)$ and $H^2(\Omega)$ denote usual Sobolev functions spaces. We will denote by $\langle , \rangle$ and $\| \|$, respectively, the scalar product and the norm in $X$. The operator $A$ commutes with $B$ and they are related by the following formula

$$\langle B u, D B u \rangle = - \langle u, A u \rangle, \forall u \in \mathcal{D}(A).$$

We endow $D(B)$ with the graph norm $|x|_{\mathcal{D}(B)} = \| B x \| + \| x \|$ for $x \in \mathcal{D}(B)$. The operator $g_G$ is defined as follows:

$$g_G(\varphi)(x) = [G * \varphi](x) = \int_{\mathbb{R}} G(x - y) \varphi(y) \, dy.$$  

By straightforward consequence of standard calculations, we can establish that $g_G : \mathcal{D}(B) \rightarrow \mathcal{D}(B)$, continuously so there exists a constant $\delta$, so that

$$|g_G(\varphi)|_{\mathcal{D}(B)} \leq \delta |\varphi|_{\mathcal{D}(B)}, \forall \varphi \in \mathcal{D}(B).$$

Note also that $G * u^0$ is uniformly bounded. Hence, $g_G(u)$ is uniformly bounded. As a result of Hölder’s inequality, we get

$$|g_G(u)|_{\infty} \leq \sqrt{L} |G|_{\infty} \| u \|, \forall u \in \mathcal{D}(B).$$

On the other hand, we have

$$|B g_G(u)|_{\infty} \leq \sqrt{L} |G|_{\infty} |u|_{\mathcal{D}(B)}, \forall u \in \mathcal{D}(B).$$

$C$ is the non linear operator,

$$C : X \longrightarrow L_{HS}(\mathcal{D}(B), X)$$

where $C(u)$ is the linear operator of multiplication by the function $\sqrt{\lambda u_+}$, that is:

$$\forall u \in X, \ w \in \mathcal{D}(B), \ (C(u) w)(x) = \sqrt{\lambda u_+}(x) w(x), \quad x \in \Omega$$
\(L_{HS}(\mathcal{D}(B), X)\) denotes the space of Hilbert-Schmidt operators from \(\mathcal{D}(B)\) to \(X\) equipped with the norm

\[\|T\|_2 = \left( \sum_{k=1}^{\infty} \|Te_k\|^2 \right)^{1/2} < \infty,\]

where \(\{e_k\}\) is a complete orthonormal basis in \(X\).

Suppose that the compact embedding:

\[J : \mathcal{D}(B) \hookrightarrow X\]

is Hilbert-Schmidt, (12) then \((W(t))_{t \in [0, T]}\) can be considered as a cylindrical Wiener process on \(\mathcal{D}(B)\) with values in \(X\), defined on the probability space \((\Lambda, \mathcal{F}, P)\) with a filtration \((\mathcal{F}_t)_{t \in [0, T]}\).

**Proposition 1** The operator \(A\) defined by (5) is the generator of an analytic semigroup of contractions in \(X\), \((T(t))_{t \geq 0}\), compact for \(t > 0\). The restrictions \(T(t)/\mathcal{D}(B)\) send \(\mathcal{D}(B)\) into itself and are uniformly bounded in \(\mathcal{D}(B)\) (that is, there exists \(C_1 \geq 0\), such that, \(|T(t)/\mathcal{D}(B)|_{\mathcal{D}(B)} \leq C_1\), for \(t \geq 0\)).

**Proof.** The proof is similar to that one in [2].

We propose to solve, on time interval \([0, T]\), equation (4) in integrated form by using the stochastic generalization of the classical variation of constants formula

\[u(t) = T(t)u_0 - \int_0^t T(t-s)B[u(s)g_G(u(s))]\,ds + \int_0^t T(t-s)C(u(s))\,dW(s).\]  

(13)

We remind that a predictable \(X\)-valued stochastic process \((u(t))_{t \in [0, T]}\) defined on the probability space \((\Lambda, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)\) is called a mild solution of the differential equation (4) if \(u(t)\) is a solution of (13). On another hand, the stochastic process \((u(t))_{t \in [0, T]}\) is called a weakened solution of (4) if \(u(t)\) is a solution of

\[u(t) = u_0 + A\int_0^t u(s)\,ds - \int_0^t B[u(s)g_G(u(s))]\,ds + \int_0^t C(u(s))\,dW(s).\]  

(14)

It is well known that solutions of (13) and (14) are equivalent ([6], [7]). Note that the stochastic integral \(\int_0^t T(t-s)C(u(s))\,dW(s)\) is well defined since

\[E\left(\int_0^t \|T(t-s)C(u(s))\|_2^2\,ds\right) < \infty, \forall t \in [0, T].\]

This is due to the fact that \(C(u(s))\) is Hilbert-Schmidt operator and \(T(t-s)\) is linear bounded operator then, basing on the theory of Hilbert-Schmidt operators (e.g. [16], Chapter I), \(T(t-s)C(u(s))\) is Hilbert-Schmidt operator too.
2 Existence of solutions

This section is concerned with the existence of mild solutions for (4). For this purpose, we need first to establish several propositions and lemmas. We start by giving some useful estimates.

Lemma 2 1) There exists a constant $M$, such that, for all $u, v \in D(B)$, we have

$$\|B[uG(u)] - B[vG(v)]\| \leq M \max(|u|_{D(B)}, |v|_{D(B)}) |u - v|_{D(B)}.$$  

2) There exists a positive constant $Q$, such that, for all $u \in D(B)$, it holds that

$$\|B[uG(u)]\| \leq Q |u|_{D(B)} \|u\|.$$  

3) There exists a positive constant $C$, such that, for all $u \in X$, it holds that

$$\|BT(t)u\| \leq \frac{C}{\sqrt{t}} \|u\|, \forall t > 0.$$  

(15)

Proof. For the claim, the proof is the same as the one given in (Lemma 2.1, [1]). ■

By using Lemma 2, we can prove

Proposition 3 Consider the following abstract problem

$$\begin{aligned}
\begin{cases}
\text{du}(t) &= (Au(t) - B[u(t)g_G(u(t))]) \, dt + a(u(t))dW(t) \\
\text{u}(0) &= u_0.
\end{cases}
\end{aligned}$$

(16)

Suppose that the operator $a$ satisfies

$$a : X \to L_{HS}(D(B), X)$$

and there exists a constant $K$ such that for all $x, y \in X,

$$\|a(x) - a(y)\|_2 \leq K \|x - y\|.$$  

Then, for each $u_0 \in D(B)$, problem (16) has a unique mild solution $u(t)$, $t \in [0, T]$ such that $\sup_{t \in [0, T]} E(\|u(t)\|^2_{D(B)}) < \infty$.  

Proof. We use the method of successive approximations. Let $u_0 \in D(B)$ and define the sequence $(u_n)_{n \geq 1}$ by

$$u_{n+1}(t) = T(t)u_0(t) - \int_0^t T(t-s)B[u_n(s)g_G(u_n(s))]ds + \int_0^t T(t-s)a(u_n(s))dW(s).$$

Let us denote by $Y$ the Banach space of all nonanticipating $D(B)$-valued continuous stochastic processes $\{U(t)\}_{t \in [0, T]}$ endowed with the norm

$$|||U||| = \left\{ \sup_{0 \leq t \leq T} E(\|U(t)\|^2_{D(B)}) \right\}^{1/2} < \infty.$$
Assume that \((u_n(t))_{n \geq 1}\) is bounded in \(D(B)\), that is, \(\exists R > 0\) such that \(|u_n(t)|_{D(B)} \leq R\), \(\forall t \in [0, T], \forall n \geq 1\). Hence \((u_n(t), t \in [0, T])_{n \geq 1}\) is also bounded in \(Y\), since
\[
|||u_n||| = \left\{ \sup_{0 \leq t \leq T} E |u_n(t)|_{D(B)}^2 \right\}^{1/2} \leq R.
\]
Let
\[
h_{n+1}(t) = E(|u_{n+1}(t) - u_n(t)|_{D(B)}^2), \quad n \geq 0
\]
We have
\[
h_{n+1}(t) \leq 2E \left| \int_0^t T(t-s) [a(u_n(s)) - a(u_{n-1}(s))] dW(s) \right|_{D(B)}^2 + 2E \left| \int_0^t T(t-s) (B[u_n(s)g_G(u_n(s))] - B[u_{n-1}(s)g_G(u_{n-1}(s))]) ds \right|_{D(B)}^2
\]
\[
\leq 2E \left( \int_0^t \|T(t-s) [a(u_n(s)) - a(u_{n-1}(s))] \|^2 ds \right) + 2E \left( \int_0^t \|T(t-s) (B[u_n(s)g_G(u_n(s))] - B[u_{n-1}(s)g_G(u_{n-1}(s))]) \|^2 ds \right)
\]
Since \(\exists C_1 \geq 0\) such that \(|T(t)|_{D(B)} \leq C_1\), for \(t \geq 0\), hence
\[
I \leq 2C_1^2 E \left( \int_0^t \|a(u_n(s)) - a(u_{n-1}(s)) \|^2 ds \right) \leq 2C_1^2 K^2 E \left( \int_0^t \|u_n(s) - u_{n-1}(s) \|^2 ds \right) \leq 2C_1^2 K^2 T E(|u_n(s) - u_{n-1}(s)|_{D(B)}^2),
\]
Therefore
\[
\begin{align*}
&\int_0^t T(t-s) \left( B[u_n(s)g_G(u_n(s))] - B[u_{n-1}(s)g_G(u_{n-1}(s))] \right) \, ds
= \int_0^t T(t-s) \left( B[u_n(s)g_G(u_n(s))] - B[u_{n-1}(s)g_G(u_{n-1}(s))] \right) \, ds
+ B \int_0^t T(t-s) \left( B[u_n(s)g_G(u_n(s))] - B[u_{n-1}(s)g_G(u_{n-1}(s))] \right) \, ds \\
&\leq \int M \max\{ |u_n(s)|_{D(B)} , |u_{n-1}(s)|_{D(B)} \} |u_n(s) - u_{n-1}(s)|_{D(B)} \, ds \\
&+ \int \frac{C}{\sqrt{t-s}} \|B[u_n(s)g_G(u_n(s))] - B[u_{n-1}(s)g_G(u_{n-1}(s))]\| \, ds \\
&\leq MR(T + 2\sqrt{T}) \sup_{0 \leq t \leq T} |u_n(t) - u_{n-1}(t)|_{D(B)}.
\end{align*}
\]

To calculate II, we have
\[
II \leq 2M^2R^2(T + 2\sqrt{T})^2E(\sup_{0 \leq t \leq T} |u_n(t) - u_{n-1}(t)|^2_{D(B)}).
\]

Hence, we obtain
\[
h_{n+1}(t) \leq 2[M^2R^2(T + 2\sqrt{T})^2 + C_1^2K^2T]E(\sup_{0 \leq t \leq T} |u_n(t) - u_{n-1}(t)|^2_{D(B)}).
\]

By choosing \(T > 0\) small enough so that
\[
2[M^2R^2(T + 2\sqrt{T})^2 + C_1^2K^2T] < \frac{1}{2}, \quad (17)
\]
we obtain
\[
h_{n+1}(t) \leq \frac{1}{2^n}E(\sup_{0 \leq t \leq T} |u_1(t) - u_0(t)|^2_{D(B)}).
\]

\(u_{n+1}(t) - u_n(t)\) is the general term of an absolutely convergent series in the Banach space \(Y\). Hence, \(\{u_n(.)\}\) is a convergent sequence in \(Y\). The limit in \(Y\), \(u_\infty(t) = \lim_{n \to \infty} u_n(t)\) is a solution of (16) for fixed \(t\). To complete the proof of the proposition, we have to show that the sequence \(u_n(t)\) remains in \(Y\), i.e. \(E(\sup_{0 \leq t \leq T} |u_{n+1}(t)|^2_{D(B)}) < \infty\).

\[
\begin{align*}
E(|u_{n+1}(t)|^2_{D(B)}) &\leq 2C_1^2 |u_0(t)|^2_{D(B)} + 2M^2R^2(T + 2\sqrt{T})^2 \\
&\quad \cdot E(\sup_{0 \leq t \leq T} |u_n(t)|^2_{D(B)}) + 2C_1^2K^2T E(\sup_{0 \leq s \leq T} |u_n(s)|^2_{D(B)}) \\
&\leq 2C_1^2R^2 + 2R^2 \left[ M^2R^2(T + 2\sqrt{T})^2 + C_1^2K^2T \right], \forall t \in [0, T].
\end{align*}
\]
Using $T > 0$ such that (17) holds, we have

$$\sup_{0 \leq t \leq T} E(|u_{n+1}(t)|^2_{D(B)}) \leq 2C_1^2R^2 + \frac{1}{2}R^2 < \infty.$$ 

To prove the uniqueness of the solution (up to equivalence), consider $u$ and $v$ two mild solutions of (16) on $[0,T]$. We might show that

$$\sup_{0 \leq t \leq T} E|u(t) - v(t)|^2_{D(B)} = 0.$$

We have

$$E|u(t) - v(t)|^2_{D(B)} \leq 2E \left[ \int_0^t (T-t)(B[u(s)g_G(u(s))] - B[v(s)g_G(v(s))])ds \right]^2_{D(B)}$$

$$+ 2E \left[ \int_0^t |a(u(s)) - a(v(s))|dW(s) \right]^2_{D(B)}$$

(1)

We obtain

$$E|u(t) - v(t)|^2_{D(B)} \leq 2C^2K^2TE(\sup_{0 \leq s \leq T} |u(s) - v(s)|^2_{D(B)})$$

(2)

To calculate (1), we have

$$\left| \int_0^t (T-t)(B[u(s)g_G(u(s))] - B[v(s)g_G(v(s))])ds \right|^2_{D(B)}$$

$$\leq MR(T + 2C\sqrt{T}) \sup_{0 \leq t \leq T} |u(t) - v(t)|_{D(B)}$$

Then

$$E|u(t) - v(t)|^2_{D(B)} \leq 2M^2\gamma^2R^2(T + 2C\sqrt{T})^2E(\sup_{0 \leq t \leq T} |u(t) - v(t)|^2_{D(B)})$$

Finally, we obtain

$$E|u(t) - v(t)|^2_{D(B)} \leq \left[ 2M^2R^2(T + 2C\sqrt{T})^2 + 2C_1^2K^2T \right] E(\sup_{0 \leq t \leq T} |u(t) - v(t)|^2_{D(B)}) \quad \forall t \in [0,T]$$

Since $T$ is chosen such that $2M^2R^2(T + 2C\sqrt{T})^2 + 2C_1^2K^2T < \frac{1}{2}$, hence we have

$$\sup_{0 \leq t \leq T} E(|u(t) - v(t)|^2_{D(B)}) = 0$$
which leads to
\[ u(t) = v(t), \quad \forall t \in [0, T], \quad P - a.s. \]

We return to problem (4). We propose as in [5] to construct a sequence of Lipschitz continuous mappings \((a_n(u))_{n \in \mathbb{N}}\) to approximate the non-Lipschitzian function \(\sqrt{\lambda u^+}\) on \(\mathbb{R}\). For \(n \in \mathbb{N}\), define \(a_n(\cdot)\) by

\[
a_n(u) = \begin{cases} 
0 & u < 0 \\
\sqrt{n}u & 0 \leq u < \frac{1}{n} \\
\sqrt{\lambda u} & u \geq \frac{1}{n}
\end{cases}
\]

It is clear that the sequence \((a_n(u))_n\) converges to \(\sqrt{\lambda u^+}\) uniformly on \(u \in \mathbb{R}\) as \(n \to +\infty\).

**Lemma 4** Suppose that the sequence \((a_n(u))_n\) defined by (18) satisfies
\[
a_n : X \to L_{HS}(\mathcal{D}(B), X)
\]
so that
\[
\forall u \in X, \ w \in \mathcal{D}(B), \quad a_n(u) w = a_n(u) w.
\]
Then, there exists a constant \(K'\) such that for all \(u, v \in X\),
\[
\|a_n(u) - a_n(v)\|_2 \leq K' \|u - v\|.
\]

**Proof.** Let \(u, v \in X\) and \(\{e_k\}\) a complete orthonormal basis in \(\mathcal{D}(B)\). Hence
\[
\|a_n(u) - a_n(v)\|_2^2 = \sum_{k=1}^{+\infty} \|e_k\|_2^2 \sum_{k=1}^{+\infty} f(x) e_k(x) \int_0^1 |a_n(u(x)) - a_n(v(x))|^2 |e_k(x)|^2 \, dx
\]
As \(a_n(u)\) is lipschitz in \(u\) on \(\mathbb{R}\), then there exists \(K > 0\) such that
\[
\|a_n(u) - a_n(v)\|_2^2 \leq K^2 \sum_{k=1}^{+\infty} \int_0^1 |u(x) - v(x)|^2 |e_k(x)|^2 \, dx
\]
\[
\leq K^2 \sum_{k=1}^{+\infty} \|u - v\|_2^2 \sum_{k=1}^{+\infty} \|e_k\|^2 \|D(B)\|
\]
Using the hypothesis (12), we have
\[
\sum_{k=1}^{+\infty} \|e_k\|^2 \|D(B)\| = \sum_{k=1}^{+\infty} \|Je_k\|_2^2 = \|J\|_2^2 < \infty,
\]

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which leads to the fact that there exists $M_1 > 0$ such that
\[ \|a_n(u) - a_n(v)\|_2^2 \leq M_1 K^2 \|u - v\|^2 \]

Let us now consider the following approximating SPDE’s:

\[ u_n(t) = I_0(t) - I_n(t) + J_n(t), \quad n \in \mathbb{N} \quad (22) \]

with

\[ I_0(t) = T(t)u_0(t) \]
\[ I_n(t) = \int_0^T T(t - s)B[u_n(s)g_G(u_n(s))]ds \]
\[ J_n(t) = \int_0^T T(t - s)a_n(u_n(s))dW(s) \]

where the sequence $(a_n(\cdot))_n$ is defined by $[18]$ and satisfies $[19]$.

We aim to prove weak convergence of the sequence $\{u_n(t), t \in [0,T]\}_{n \in \mathbb{N}}$ by proving the tightness of $\{I_n(t), t \in [0,T]\}_{n \in \mathbb{N}}$ and $\{J_n(t), t \in [0,T]\}_{n \in \mathbb{N}}$ since $I_0(t)$ is deterministic.

**Proposition 5** Let the initial condition $u_0 \in D(B)$ be a continuous deterministic mapping on $\Omega$. The sequences $\{I_n(t), t \in [0,T]\}_{n \in \mathbb{N}}$ and $\{J_n(t), t \in [0,T]\}_{n \in \mathbb{N}}$ are tight in $C([0,T], D(B))$.

**Proof.** By Lemma 4, it follows that for each $n \in \mathbb{N}$, the sequence $(a_n(\cdot))_n$ satisfies the conditions of Proposition $3$. Hence, there exists a unique mild solution $u_n(t), t \in [0,T]$ for problem $(22)$ such that $\sup_{t \in [0,T]} E(|u_n(t)|_{D(B)}^2) < \infty$.

For given $0 \leq t' \leq t \leq T$ and $\eta \in [0, \frac{1}{2}]$, it follows from Lemma $7$ (in Appendix A) that

\[
E |J_n(t) - J_n(t')|^2_{D(B)} \leq 2E\left(\int_0^t \| (T(t - s) - T(t' - s))a_n(u_n(s))\|_2^2 \, ds \right) + 2E\left(\int_t^T \| (T(t' - s)a_n(u_n(s))\|_2^2 \, ds \right) \\
\leq 2C^4 K^2 E\left(\sup_{0 \leq s \leq T} |u_n(s)|_{D(B)}^2 \right) |t - t'|^{\eta} + 2E\left(\int_t^T |B(T(t' - s))|_{D(B)}^2 |u_n(s)g_G(u_n(s))|_{D(B)}^2 \, ds \right) \\
\leq C_1 |t - t'|^{\eta}. 
\]
On another hand, by using Lemma 8 (in Appendix A), we have for given $0 \leq t' \leq t \leq T$ and $\eta \in [0, \frac{1}{2}]$

\[
E |I_n(t) - I_n(t')|^2_{D(B)} \leq 2E( \int_0^t |(T(t-s) - T(t'-s))B[u_n(s)g_G(u_n(s))]|^2_{D(B)} ds
\]
\[
+ 2E( \int_{t'}^t |B(T(t-s) - T(t'-s))B[u_n(s)g_G(u_n(s))]|^2_{D(B)} ds
\]
\[
\leq 2E( \int_0^{t'} |B(T(t-s) - T(t'-s))B[u_n(s)g_G(u_n(s))]|^2_{D(B)} ds
\]
\[
+ 2E( \int_t^T |B(T(t'-s)B[u_n(s)g_G(u_n(s))]|^2_{D(B)} ds
\]

using (9), we obtain

\[
E |I_n(t) - I_n(t')|^2_{D(B)} \leq 2C'' \delta^2 E( \sup_{0 \leq s \leq T} |u_n(s)|^4_{D(B)} |t - t'|^\eta
\]
\[
+ 2C' \delta^2 E( \sup_{0 \leq s \leq T} |u_n(s)|^4_{D(B)} |t - t'|^\eta
\]
\[
\leq C_2 |t - t'|^{\eta/2}.
\]

It follows that

\[
E \int_0^T \int_0^T \left( \frac{|J_n(t) - J_n(t')|_{D(B)}}{|t - t'|^\gamma} \right)^2 dt'dt' \leq \int_0^T \int_0^T C_1 |t - t'|^{\eta - 2\gamma} dt'dt'
\]
\[
< \infty \quad \text{for all } 0 < \gamma < \frac{\eta + 1}{2}
\]

For $R > 0$, we define $A_R \subset \Lambda$

\[
A_R = \left\{ \omega \in \Lambda / \int_0^T \int_0^T \left( \frac{|J_n(t, \omega) - J_n(t', \omega)|_{D(B)}}{|t - t'|^\gamma} \right)^2 dt'dt' \leq R \right\}
\]

If we let in the Garsia-Rodemich-Rumsey Lemma (Lemma 9, in Appendix A):

$f = J_n(., w), x = t, y = t', p(x - y) = |x - y|^\gamma$ ie $p(x) = |x|^\gamma$ and $\psi(x) = x^2$, then

\[
\|J_n(t, \omega) - J_n(t', \omega\|_{D(B)} \leq 8 \int_0^{|t-t'|} \psi^{-1}(\frac{4R}{u^2}) dp(u).
\]
Since $\psi^{-1}(\frac{4R}{\omega^2}) = \sup_{v^2 \leq \frac{4R}{\omega^2}} \frac{\sqrt{R}}{u}$, thus

$$|J_n(t, \omega) - J_n(t', \omega)|_{D(B)} \leq \left| t - t' \right| \leq 8 \int_0^\gamma 2 \frac{\sqrt{R}}{u} \gamma u^{-1} du \leq 16 \frac{\gamma}{\gamma - 1} R^{\frac{1}{2}} |t - t'|^{\gamma - 1}.$$  

Finally

$$|J_n(t, \omega) - J_n(t', \omega)|_{D(B)}^2 \leq M' R |t - t'|^{2\gamma - 2}.$$  

It follows that for any $0 < \delta < \gamma$

$$\sup_{t, t' \in [0, T], t \neq t'} \frac{|J_n(t, \omega) - J_n(t', \omega)|_{D(B)}}{|t - t'|^\delta} \leq \frac{C}{\delta} \left| t - t' \right|^{\gamma - 1 - \frac{\delta}{\gamma}} R^{1/2} \leq C' R^{1/2}.$$  

Hence, for $\omega \in A_R$, we can define

$$\|J_n(\cdot, \omega)\|_{C[0, T]} = \sup_{t \in [0, T]} |J_n(t, \omega)|_{D(B)} + \sup_{t, t' \in [0, T], t \neq t'} \frac{|J_n(t, \omega) - J_n(t', \omega)|_{D(B)}}{|t - t'|^\delta} \leq C'' R^{1/2}.$$  

We have then for every $R > 0$

$$P(A_R) = P \left\{ \omega \in A / \int_0^T \int_0^T \left( \frac{|J_n(t) - J_n(t')|_{D(B)}}{|t - t'|^\gamma} \right)^2 dt dt' \leq R \right\} \leq \left\{ \|J_n(\cdot, \omega)\|_{C[0, T]} \leq C'' R^{1/2} \right\}.$$  

Let us now show that $\{J_n(t, t \in [0, T])\}_{n \in \mathbb{N}}$ is tight on $C([0, T], D(B))$. For $R > 0$, we define

$$B_R = \left\{ f \in C([0, T], D(B)) / \|f\|_{C[0, T]} \leq R \right\}.$$  

$B_R$ is a compact set of $C([0, T], D(B))$ (see the Proof B1 in Appendix B) and we have the following:

$$\sup_n P[J_n \in B_R] = \sup_n \|J_n(\cdot, \omega)\|_{C[0, T]} \leq \left( \frac{A_{R^2}}{C'' R^{1/2}} \right) \leq \sup_n \left\{ \int_0^T \int_0^T \left( \frac{|J_n(t) - J_n(t')|_{D(B)}}{|t - t'|^\gamma} \right)^2 dt dt' > \frac{R^2}{C'' R^{1/2}} \right\}$$  

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Using Markov inequality, we obtain
\[ \sup_n P[J_n \in \overline{B_R}] \leq \frac{C''}{R^2} \sup_n \left\{ \int_0^T \int_0^T \left( \frac{|J_n(t) - J_n(t')|_{D(B)}}{|t-t'|^\gamma} \right)^2 dt dt' \right\}. \tag{23} \]

Since \( E \left\{ \int_0^T \int_0^T \left( \frac{|J_n(t) - J_n(t')|_{D(B)}}{|t-t'|^\gamma} \right)^2 dt dt' \right\} < +\infty \), (23) leads to
\[ \sup_n P[J_n \in \overline{B_R}] \leq \frac{M''}{R^2} < \epsilon \]
for \( R \) large enough. This proves the tightness of \( \{J_n(t), t \in [0,T]\}_{n \in \mathbb{N}} \). A similar argument implies that \( \{I_n(t), t \in [0,T]\}_{n \in \mathbb{N}} \) is also tight on the same space. \( \square \)

Now, we are able to state our main result

**Theorem 6** Let the initial condition \( u_0 \in D(B) \) be a continuous deterministic mapping on \( \Omega \). Then, there exists a probability basis \((\mathcal{X}, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathcal{P})\) on which there is a cylindrical Wiener process \((\mathcal{W}(t))_{t \in [0,T]}\) and a mild solution \((u(t))_{t \in [0,T]}\) of (13) in \( C([0,T], D(B)) \).

**Proof.** The tightness of \( \{I_n(t), t \in [0,T]\}_{n \in \mathbb{N}} \) and \( \{J_n(t), t \in [0,T]\}_{n \in \mathbb{N}} \) implies that \( \{u_n(t), t \in [0,T]\}_{n \in \mathbb{N}} \) converges weakly on \( C([0,T], D(B)) \), this means that there exists a subsequence \( \{N_k\}_{k \in \mathbb{N}} \subset \mathbb{N} \) and a \( C([0,T], D(B)) \)-valued random variable \( v \) such that
\[ u_{N_k} \rightarrow v \quad \text{on} \quad C([0,T], D(B)), \quad \text{as} \quad k \rightarrow +\infty. \]

By Skorohod Representation Theorem, there exists a probability space \((\overline{\mathcal{X}}, \overline{\mathcal{F}}, \overline{\mathcal{P}})\) with a filtration \((\mathcal{F}_t)_{t \in [0,T]}\) and \( C([0,T], D(B)) \)-valued random variables \((\overline{u}_N)_{N \in \mathbb{N}}\) and \( u \) such that as \( N \rightarrow +\infty \)
\[ \overline{u}_N \rightarrow u \quad \text{in law} \quad \text{on} \quad C([0,T], D(B)) \]
and
\[ \overline{u}_N \overset{in law}{=} u_N \quad \text{and} \quad \overline{u}_N \overset{in law}{=} v. \]

The mild solution \( u_N(\cdot) \) of (22) is also a weakened solution of (22) (see for instance [6]), that is \( u_N(\cdot) \) satisfies the following integral equation
\[ u_N(t) = u_0 + A \left( \int_0^t u_N(s) ds \right) - B \left[ u_N(s) g_G (u_N(s)) \right] ds + \int_0^t a_N(u_N(s)) dW(s). \]

Hence, for each \( N \),
\[ M_N(t) = \int_0^t a_N(u_N(s)) dW(s) \]
\[ = u_N(t) - u_0 - A \left( \int_0^t u_N(s) ds \right) + B \left[ u_N(s) g_G (u_N(s)) \right] ds \]

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is a square integrable martingale with respect to $\mathcal{F}_t^N = \sigma\{u_N(s), s \leq t\}$ and has the following quadratic variation

$$\langle M_N(t), M_N(t) \rangle = \int_0^t a_N(u_N(s))a_N^*(u_N(s))ds.$$  \hspace{1cm} (24)

Since $\mathcal{L}(\tilde{u}_N) = \mathcal{L}(u_N)$ ($\mathcal{L}$ denotes the probability distribution),

$$\tilde{M}_N(t) = \tilde{u}_N(t) - u_0 - A\int_0^t \tilde{u}_N(s)ds + \int_0^t B[\tilde{u}_N(s)g_G(\tilde{u}_N(s))]ds$$

has the same distribution as $M_N(t)$ and hence

$$\mathbb{E}\left|\tilde{M}_N(t)\right|_{D(B)}^2 = E |M_N(t)|_{D(B)}^2 < +\infty$$ \hspace{1cm} (25)

(we denote by $E$ the expectation with respect to the probability measure $\mathbb{P}$).

On another hand, since $\{M_N(t)\}_t$ is a martingale, that is

$$E(M_N(t) - M_N(s) / \mathcal{F}_s^N) = 0, \hspace{.5cm} \forall \hspace{.2cm} 0 \leq s < t \leq T$$

and using the fact that

$$M_N(t) - M_N(s) = u_N(t) - u_N(s) - A\int_s^t u_N(s)ds + \int_s^t B[u_N(s)g_G(u_N(s))]ds$$

has the same law as

$$\tilde{M}_N(t) - \tilde{M}_N(s) = \tilde{u}_N(t) - \tilde{u}_N(s) - A\int_s^t \tilde{u}_N(s)ds + \int_s^t B[\tilde{u}_N(s)g_G(\tilde{u}_N(s))]ds,$$

it holds that

$$\mathbb{E}(\tilde{M}_N(t) - \tilde{M}_N(s) / \mathcal{F}_s^N) = 0, \hspace{.5cm} \forall \hspace{.2cm} 0 \leq s < t \leq T$$

which implies that $\{\tilde{M}_N(t)\}_t$ is a square integrable martingale on $(\mathbb{X}, \mathcal{F}, \mathbb{P})$ with respect to $\mathcal{F}_t^N = \sigma\{\tilde{u}_N(s), s \leq t\}$.

On another hand, since $M_N(t)$ has the quadratic variation (24), this means that for $x, y \in D(B),$

$$\langle M_N(t), x \rangle \langle M_N(t), y \rangle - \int_0^t \langle a_N(u_N(z)), x \rangle \langle a_N(u_N(z)), y \rangle dz$$

is an $\mathcal{F}_t^N$-martingale.
Then
\[
E(\langle MN(t), x \rangle \langle MN(t), y \rangle - \langle MN(s), x \rangle \langle MN(s), y \rangle)
- \int_0^t \langle a_N(u_N(z)), x \rangle \langle a_N(u_N(z)), y \rangle dz + \int_0^s \langle a_N(u_N(z)), x \rangle \langle a_N(u_N(z)), y \rangle dz)
/ \mathcal{F}_s = 0.
\]

It follows that
\[
E(\langle \tilde{M}_N(t), x \rangle \langle \tilde{M}_N(t), y \rangle - \langle \tilde{M}_N(s), x \rangle \langle \tilde{M}_N(s), y \rangle)
- \int_0^t \langle a_N(\tilde{u}_N(z)), x \rangle \langle a_N(\tilde{u}_N(z)), y \rangle dz + \int_0^s \langle a_N(\tilde{u}_N(z)), x \rangle \langle a_N(\tilde{u}_N(z)), y \rangle dz) / \mathcal{F}_s = 0.
\]

Hence
\[
E(\langle \tilde{M}_N(t), x \rangle \langle \tilde{M}_N(t), y \rangle - \int_0^t \langle a_N(\tilde{u}_N(z)), x \rangle \langle a_N(\tilde{u}_N(z)), y \rangle dz) / \mathcal{F}_s = 0.
\]

and as a consequence, \( \tilde{M}_N(t) \) has the unique quadratic variation process
\[
\langle \tilde{M}_N(t), \tilde{M}_N(t) \rangle = \int_0^t a_N(\tilde{u}_N(s))a_N^*(\tilde{u}_N(s)) ds.
\]

From [25], since
\[
\forall N \in \mathbb{N}, \sup_{0 \leq t \leq T} E \left| \tilde{M}_N(t) \right|^2_{D(B)} = \sup_{0 \leq t \leq T} E |M_N(t)|^2_{D(B)} < +\infty,
\]
this means that \( (\tilde{M}_N)_{N \in \mathbb{N}} \) is a sequence of uniformly integrable martingales. Therefore, there exists on \( (\Lambda, \mathcal{F}, \mathbb{P}) \) a square integrable martingale \( \tilde{M} \) such that for \( N \to +\infty, \quad \tilde{M}_N \to \tilde{M} \) and
\[
M(t) = u(t) - u_0 - A(\int_0^t u(s) ds) + \int_0^t B[u(s)g_G(u(s))] ds
\]
with
\[
\langle M(t), M(t) \rangle = \lim_{N \to +\infty} \int_0^t a_N(\tilde{u}_N(s))a_N^*(\tilde{u}_N(s)) ds = \int_0^t \lambda u^+(s) ds.
\]

By the representation theorem for square integrable martingales [3], there exists a cylindrical Wiener process \( \tilde{W} \) defined on \( (\Lambda, \mathcal{F}, \mathbb{P}) \) such that
\[
M(t) = \int_0^t \sqrt{\lambda u^+(s)} d\tilde{W}(s), \quad t \in [0, T].
\]

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Then \( u \) satisfies that
\[
  u(t) = u_0 + A \int_0^t u(s) ds - \int_0^t B [u(s)g_G(u(s))] ds + \int_0^t \sqrt{\lambda u^+} d\tilde{W}(s), \quad t \in [0, T]
\]
or equivalently
\[
  u(t) = T(t)u_0 - \int_0^t T(t-s)B [u(s)g_G(u(s))] ds + \int_0^t T(t-s)C(u(s)) d\tilde{W}(s).
\]

\[\blacksquare\]

Appendix A

Lemma 7 For \( \eta \in [0, \frac{1}{2}] \), there exists constants \( C', C'_1 > 0 \), such that
\[
  \int_0^s |T(t-u) - T(s-u)|^2_{D(B)} du \leq C' |t-s|^\eta \tag{26}
\]
and
\[
  \int_s^t |T(t-u)|^2_{D(B)} du \leq C'_1 |t-s|^\eta \tag{27}
\]
with \( 0 \leq s < t \leq T \).

**Proof.** We represent \( T(t) \) in terms of the eigenvalues of \( A = D \frac{d^2}{dx^2} \) denoted \( -w_j^2 \) to which correspond the eigenvectors \( \varphi_j \):
\[
  T(t) = \sum_{j=1}^\infty \exp\{-w_j^2 t\} \langle \cdot, \varphi_j \rangle \varphi_j
\]
Then, we have

\[
\int_0^s |T(t - u) - T(s - u)|_{D(B)}^2 du \\
\leq \int_0^s \sum_{j=1}^{\infty} \left| \exp\{-w_j^2(t - u)\} - \exp\{-w_j^2(s - u)\} \right|_{D(B)}^2 \| \langle \cdot, \varphi_j \rangle \|_{D(B)}^2 \| \varphi_j \|_{D(B)}^2 du \\
\leq \int_0^s \sum_{j=1}^{\infty} \left| \exp\{-w_j^2(s - u)\} \right|^2 \| 1 - \exp\{-w_j^2(t - s)\} \|_{D(B)}^2 \| \langle \cdot, \varphi_j \rangle \|_{D(B)}^2 \| \varphi_j \|_{D(B)}^2 du \\
\leq \sum_{j=1}^{\infty} \| \langle \cdot, \varphi_j \rangle \|_{D(B)}^2 \| \varphi_j \|_{D(B)}^2 |w_j^2(t - s)|^{2\lambda} \int_0^s \exp\{-2w_j^2(s - u)\} du \\
\leq \sup_j \| \langle \cdot, \varphi_j \rangle \|_{D(B)}^2 \sup_j \| \varphi_j \|_{D(B)}^2 |t - s|^{2\lambda} \sum_{j=1}^{\infty} w_j^{-2(2-4\lambda)} \\
\leq C_0 |t - s|^{2\lambda} \sum_{j=1}^{\infty} w_j^{-2(2-4\lambda)}
\]

where we have used the fact that for all \( u \in [0, 1] \),

\[ 1 - \exp\{-x\} \leq x^u \quad (x > 0). \]

For \( \lambda \in [0, \frac{1}{2}] \), we have \( \sum_{j=1}^{\infty} w_j^{-2(2-4\lambda)} < \infty \). If we set \( \eta = 2\lambda \quad (\eta \in [0, \frac{1}{2}]) \), hence

\[
\int_0^s |T(t - u) - T(s - u)|_{D(B)}^2 du \leq C'|t - s|^{\eta}
\]

Similarly, for the same \( \eta \in [0, \frac{1}{2}] \) as above, we have

\[
\int_s^t |T(t - u)|_{D(B)}^2 du \leq \sum_{j=1}^{\infty} \| \langle \cdot, \varphi_j \rangle \|_{D(B)}^2 \| \varphi_j \|_{D(B)}^2 \int_s^t \exp\{-2w_j^2(t - u)\} du \\
\leq \frac{1}{2} \sup_j \| \langle \cdot, \varphi_j \rangle \|_{D(B)}^2 \sup_j |\varphi_j|_{D(B)}^2 \sum_{j=1}^{\infty} w_j^{-2} (1 - \exp\{-2w_j^2(t - s)\}) \\
\leq C''_0 |t - s|^{\eta} \sum_{j=1}^{\infty} w_j^{-(2-2\eta)} \\
\leq C''_1 |t - s|^{\eta}
\]

Lemma 8 For \( \eta \in [0, \frac{1}{2}] \), there exists constants \( C''_1, C''_1 > 0 \), such that
\[
\int_0^s |B[T(t-u)-T(s-u)]|_{D(B)}^2 \, du \leq C'' |t-s|^{\eta} 
\]
with \(0 \leq s < t \leq T\).

**Proof.** From the representation
\[
T(t) = \sum_{j=1}^{\infty} \exp\{-w_j^2 t\} \langle \cdot, \varphi_j \rangle \varphi_j.
\]
we obtain
\[
BT(t) = \sum_{j=1}^{\infty} \exp\{-w_j^2 t\} \langle \cdot, \varphi_j \rangle B\varphi_j.
\]
Hence
\[
\int_0^s |B[T(t-u)-T(s-u)]|_{D(B)}^2 \, du
\leq \int_0^s \sum_{j=1}^{\infty} \left| \exp\{-w_j^2 (t-u)\} - \exp\{-w_j^2 (s-u)\} \right|^2 |\langle \cdot, \varphi_j \rangle |_{D(B)}^2 |B\varphi_j|_{D(B)}^2 \, du
\leq \int_0^s \sum_{j=1}^{\infty} \left| \exp\{-w_j^2 (s-u)\} \right|^2 \left| 1 - \exp\{-w_j^2 (t-s)\} \right|^2 |\langle \cdot, \varphi_j \rangle |_{D(B)}^2 |B\varphi_j|_{D(B)}^2 \, du
\leq \sum_{j=1}^{\infty} |\langle \cdot, \varphi_j \rangle |_{D(B)}^2 |B\varphi_j|_{D(B)}^2 |w_j^2 (t-s)|^{2\lambda} \int_0^s \exp\{-2w_j^2 (s-u)\} \, du
\leq \sup_j |\langle \cdot, \varphi_j \rangle |_{D(B)}^2 \sup_j |B\varphi_j|_{D(B)}^2 |t-s|^{2\lambda} \sum_{j=1}^{\infty} w_j^{-(2-4\lambda)}
\leq C_2 |t-s|^{2\lambda} \sum_{j=1}^{\infty} w_j^{-(2-4\lambda)}
\]
where we have used the fact that for all \(u \in [0, 1]\),
\[
1 - \exp\{-x\} \leq x^u \quad (x > 0).
\]
For \(\lambda \in [0, \frac{1}{4}]\), we have \(\sum_{j=1}^{\infty} w_j^{-(2-4\lambda)} < \infty\). If we set \(\eta = 2\lambda (\eta \in [0, \frac{1}{2}]\)), hence
\[
\int_0^s |B[T(t-u)-T(s-u)]|_{D(B)}^2 \, du \leq C_1 |t-s|^{\eta}.
\]
Similarly, for the same \( \eta \in [0, \frac{1}{2}] \) as above, we have

\[
\int_{s}^{t} |BT(t-u)|_{D(B)}^2 \, du \leq \sum_{j=1}^{\infty} |\langle \cdot, \varphi_j \rangle|_{D(B)}^2 |B\varphi_j|_{D(B)}^2 \int_{s}^{t} \exp\{-2w_j^2(t-u)\} \, du
\]

\[
\leq \frac{1}{2} \sup_j |\langle \cdot, \varphi_j \rangle|_{D(B)}^2 \sup_j |B\varphi_j|_{D(B)}^2 \sum_{j=1}^{\infty} w_j^{-2}(1 - \exp\{-2w_j^2(t-s)\})
\]

\[
\leq C \cdot |t-s|^{\eta} \sum_{j=1}^{\infty} w_j^{-(2-2\eta)}
\]

\[
\leq C' \cdot |t-s|^{\eta}
\]

Lemma 9 (Garsia-Rodemich-Rumsey) Let the function \( \psi : [0, \infty[ \rightarrow [0, \infty[ \) non-decreasing with \( \lim_{u \rightarrow +\infty} \psi(u) = +\infty \). Let the function \( p : [0, 1] \rightarrow [0, 1] \) be continuous and non-decreasing with \( p(0) = 0 \).

Set \( \psi^{-1}(u) = \sup_{v < u} \psi(v) \) if \( \psi(0) \leq u < \infty \). Let \( f \) be a continuous function on \([0, 1]\) and suppose that

\[
\int_{0}^{1} \int_{0}^{1} \psi\left(\frac{|f(x) - f(y)|}{p(x-y)}\right) \, dx \, dy \leq B < \infty
\]

Then, for all \( x, y \in [0, 1] \), we have

\[
|f(x) - f(y)| \leq 8 \int_{0}^{\frac{|y-x|}{A}} \psi^{-1}\left(\frac{4B}{u^2}\right) \, dp(u).
\]

Appendix B

Proof B1

Since

\[
\|f\|_{C([0,T],D(B))} = \sup_{t \in [0,T]} |f(t)|_{D(B)} + \sup_{t, t' \in [0,T], t \neq t'} \left\| \frac{|f(t) - f(t')|_{D(B)}}{|t - t'|^{\eta}} \right\|
\]

this means that \( \forall t, t' \in [0, T], t \neq t' \)

\[
|f(t) - f(t')|_{D(B)} \leq \|f\|_{C([0,T],D(B))} |t - t'|^{\eta}
\]

As \( f \in B_R \), we have

\[
|f(t) - f(t')|_{D(B)} \leq R |t - t'|^{\eta}
\]

This means that \( B_R \) is uniformly equicontinuous. By the theorem of Ascoli-Arzela, \( B_R \) is a compact of \( C([0,T], D(B)) \).
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