Abstract. In this paper, we introduce and motivate the studies of Quantum Weyl Gravity (also known as Conformal Gravity). We discuss some appealing features of this theory both on classical and quantum level. The construction of the quantum theory is described in detail to the one-loop level. To facilitate computations we use only physical degrees of freedom, which are singled out through the York decomposition. At the one-loop level we compute the partition function around a general Einstein space. Next, the functional renormalization group of couplings in Quantum Weyl Gravity is investigated. We reproduce completely previous results obtained on maximally symmetric and Ricci-flat backgrounds. Finally, we comment on further directions and on the issue of conformal anomaly.

1. Introduction

Conformal gravity is a field-theoretical system with a very peculiar form of relativistic dynamics. The evolution of Green functions in this model is governed by two principles of the fundamental character. First is the principle of general relativity since this model describes gravitational interactions. Therefore we have the freedom to choose an arbitrary coordinate system to describe the physics and nothing observable depends on this choice. The second principle is the invariance under local conformal transformations since this is a conformal gravity, where the conformal symmetry is realized in the local version. In turn, again we have a freedom to choose an arbitrary system of units and scales to measure lengths, times, energies, etc. In conformal gravity nothing observable depends on this choice of mass scale. When these two principles are realized, the resulting dynamical system is of a very special nature.

Relativistic dynamics is a very broad topic. In general, it describes theories where the fundamental dynamical principles governing the dynamics of the system are Lorentz invariant. When the Lorentz symmetry is made local (gauged) then we end up with theories describing the dynamical structure and evolution of spacetime itself. The requirement of local Lorentz invariance puts very strong dynamical constraints on the possible form of the relativistic theory of gravitation. Similarly, by gauging conformal symmetry we produce a very particular gravitational theory, which is unique in four-dimensional spacetime. The symmetry in the global version describes matter systems and models which are in particular scale-invariant, so they do not possess any characteristic invariant mass scale and their dynamics looks the same at any scale of observation. In conformal gravity we describe the dynamics of the relativistic gravitational field which does not contain any mass scale (like Planck scale for example). This situation is
very special regarding gravitational theories and one can realize that new additional constraints are put on the dynamics of the gravitational field. At the end, the system is very constrained, but not contradictory and instead it leads to unique very restrictive predictions for gravitational observables. This is why it is interesting to investigate this relativistic system closer.

Although conformal symmetry was first used by Weyl in what would be called now a gravitational context, it was rediscovered in models of particle physics in the second half of the last century [3]. It was easier to find conformal high energy models without gravity than with it. Conformal symmetry as well as scale invariance was first better understood in massless and scaleless models of particle physics. In the realm of quantum field theories (QFT) some very special models show conformality also on the quantum level [4]. Since then this symmetry has been realized and investigated both on the classical and quantum level of matter models [5, 6, 7, 8, 9, 10, 11, 12]. However, somehow in parallel, conformal methods found their applications in classical research on gravitational theories [13, 14, 15, 16, 17, 18]. This put back conformal symmetry to its natural gravitational setup, where, however, it was most times analyzed in sufficient details only in the classical domain. As it is well known, the question concerning the definition of a consistent quantum gravitational theory is still open although there are very interesting proposals on the market nowadays. The quantum role of conformal symmetry in the gravitational setup is currently being actively investigated [19, 20, 21, 22, 23, 24, 25, 26]. The constraints mentioned above select a unique theory in $d = 4$ dimensions which bears the name of Weyl gravity or conformal gravity (although most of the preliminary works in this theory were done by Bach in 1920’s [1]).

Quantum Weyl Gravity (QWG) [27] is a proposal for a fundamental theory realizing symmetries of relativity and conformality in the local version both on the classical and quantum level. We will discuss the status of this theoretical model, its overall consistency and its current problems below a bit. Most of them are related to the ultraviolet (UV) regime of energies. One can assume hypothetically that they are solved by an existence of some UV fixed point (FP) realizing also tentative UV-completion of the theory. The situation on the other side of the spectrum, namely in the infrared (IR) part of energies, is also interesting. In this contribution we review the results obtained previously for the IR renormalization group (RG) flows in QWG. We also put them in a perspective relating them with a very special relativistic dynamics happening in QWG.

2. Motivations for QWG
First, we discuss here the motivations for Weyl gravity (WG) on the classical level. Later we add also some special features visible only on the quantum level. Both particle models (without gravity) and gravitational models with conformal symmetry are discussed briefly.

The first motivation for conformal symmetry in our Universe comes from the rough observation of cosmic microwave background radiation (CMB). Its spectrum (giving us some clues about the dynamics of the very early universe) is very close to be scale-invariant [28]. The spectral index of CMB is

$$n_s = 1.00 - \varepsilon \quad \text{with} \quad \varepsilon \ll 1.$$  \hspace{1cm} (1)

Hence as a first approximation we can assume that early Universe is described by a scale-invariant gravitational theory coupled also to scale-invariant matter models (like electromagnetic radiation). As it is well known Maxwell theory of electromagnetic interactions is not only globally scale-invariant, but is invariant under full local conformal transformations (depending on the spacetime location). The same extension can be performed on the level of gravitational dynamics to end up with 4-dimensional Weyl gravity. Moreover, it is also a common wisdom that in a fully quantized theory involving relativistic and dynamical gravity, all global symmetries must be made local. And this procedure when applied to scale-invariance results in full invariance under local conformal transformations. So the early Universe must be described by a theory enjoying
local (gauged) conformal dynamical symmetry. This is one of the first phenomenological or observational motivation for WG.

Conformal symmetry has its deep roots in the differential geometry of two-dimensional curved surfaces. There it was first observed that the following transformations

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = \Omega^2(x)g_{\mu\nu}$$

make any curved 2-dimensional manifold flat. (This is why all 2-manifolds are conformally flat). The transformations in (2) are not induced by any coordinate transformation $x \rightarrow x' = x'(x)$, hence the differential structure of the manifolds is not preserved. Under the conformal transformations the metric is “charged” whereas the coordinates remain untouched, so that the invariant infinitesimal line element in differential geometry is not preserved and scales according to the law

$$ds^2 \rightarrow ds'^2 = \Omega^2(x)ds^2.$$ (3)

From this we derive that distances (lengths, times, energies, etc.) are not invariant. They are relative. And they depend on the choice of the gauge; in this case we speak about a conformal gauge.

We conclude that by performing conformal transformations we abandon the absolute meaning of physical distances and scales [29]. With physical relativity we know that velocities are relative, here we add the conformal gauge-dependence of scales. But one element of the geometrical description remains invariant under the conformal transformations, even when their parameter $\Omega = \Omega(x)$ depends on the location on the manifold. These are angles between vectors.

The angle between two vectors $\vec{u}$ and $\vec{v}$ could be defined as

$$\angle(\vec{u}, \vec{v}) = \arccos \left( \frac{\vec{u} \cdot \vec{v}}{\sqrt{|\vec{u}|^2 |\vec{v}|^2}} \right).$$ (4)

This is an example of conformal invariant since it does not depend on the scales.

In Minkowskian framework (when we include temporal relations to our differential geometry description of curved spacetime), the structure which is left invariant under conformal transformations is a causal structure on the spacetime manifold. The light-like trajectories of massless particles remain such since their defining condition $ds^2 = 0$ is untouched by conformal rescalings [2]. This was one of the reasons for the success of methods of conformal transformation during the golden years of general relativity (GR). In the gravitational framework, conformal methods revealed secrets of black holes and their horizons as well as they were instrumental in flattening of cosmologies. (All Friedmann-Lemaître-Robertson-Walker gravitational spacetimes are conformally flat.) Only the causal or conformal structure of the spacetime manifold is preserved after the process of conformal change of the metric. Moreover, in the gravitational context global scale symmetry (equivalent to absence of mass dimension parameter in the theory, so all the couplings are dimensionless) can be gauged like other global gauge symmetries, which can be made local and dynamical, for example in some models of particle physics. For the gravitational dynamics, the local conformal symmetry (invariance with respect to transformations [2] with the spacetime-dependent parameter $\Omega(x)$) can be an additional gravitational symmetry constraining the dynamics even more strongly, as we will show below (both on the quantum and classical level). The conformal symmetry can be treated like a gauge symmetry of gravity and can be placed on the same level as local Lorentz symmetry (so invariance under local boosts together with local translations) in the full gravitational theory.

Conformal symmetry should be the last dynamical symmetry (making impacts on the dynamics of the system) to be discovered in particle physics. For its verification one needs to go to arbitrarily high energy scales. However, the first time this symmetry was realized on the quantum level was in theoretical models without gravity and on flat Minkowski spacetime backgrounds [4].
In high energy models, one intuitively knows that almost every time one deals with massless particles or excitations, then the conformal symmetry emerges in the description of the system. Even if the system is not intended to possess some conformal properties, the limit where all particles become effectively without mass, leads to a miraculous enhancement of the symmetries of the considered model. And there are some properties that could be only explained by invoking conformal symmetry. Hence conformal symmetry is a quite frequently an emergent approximate symmetry of models of QFT, when the regime of high energy is analyzed. The particles do not have to be strictly with vanishing mass, their mass can be effectively very small compared to other energy scales in the very high energy approximation. For the classical considerations done on flat Minkowski spacetime background, conformal symmetry can be seen just as global scale-invariance. However, full conformal group contains more generators than just dilatation generator from scale-invariance. And also its predictions are more firm and it constrains the dynamical system enjoying invariance under the full conformal group more.

The full constraining power of conformal symmetry is seen on the Green functions of the model enjoying it. It is more than just scale-invariance, which forbids only the presence of dimensionful parameters in Green functions. For global scale-invariance dimensionful momenta $p$, energies $E$, and fields $\phi$ can still be present on the legs of Green functions and can enter into the corresponding expression for these Green functions in arbitrary combinations not constrained by any further symmetry principle. Full conformal symmetry changes this last fact. The dynamics of a Quantum Field Theory (QFT) with conformal symmetry at quantum level is so strongly constrained that it is customary to refer to this special class of theories under the name of conformal field theories (CFT). Quantum Green functions are very strongly constrained in such models. For example, the form of all 2-point and 3-point functions is completely fixed and 4-point functions at arbitrary loop order depends only on one arbitrary function of very specially constructed conformal ratios [30]. Quantum conformality enjoyed by the theories in this class means that all quantum fluctuations look the same at all energy scales. So we do not have standard suppression of quantum fluctuations in IR or in UV. In quantum CFT models they show the same strength in every energy regime. But quantum conformality is more than quantum scale-invariance since these two group of symmetries are not isomorphic to each other. Moreover, as a next piece of motivation, one can notice that the current standard model (SM) of particle physics is a classically almost scale-invariant theory and the breaking of scale-invariance is related to the spontaneous breaking of electroweak symmetry (so it is proportional to the vacuum expectation value of the Higgs field, or in turn to its mass).

Relativistic models with scale-invariance enjoy a symmetry under an 11-parameter group, which includes the generator of dilatations, apart from those from the Poincaré group. Instead, full conformal group in $d = 4$ contains 15 generators because of 4 additional generators of conformal boosts. To require invariance under these additional transformations of all Green functions (also on the quantum level) is a big constraint, which signifies that there is an additional symmetry on the quantum level. This is of course also a symmetry of the full quantum effective action. Conformal symmetry as additional symmetry on the quantum level constrains further (besides relativistic constraints imposed by symmetry under transformations from the Lorentz group) quantum dynamics in such a way that UV-divergences are completely avoided in such models. Since there cannot be any scale or mass generated in quantum dynamics of CFT model, then there is no need to renormalize the quantum theory by introducing an arbitrary renormalization scale $\mu$. And the fact of need for doing infinite UV-renormalization is tightly related to the presence of infinities in the UV regime of the theory. This additional symmetry is essential for avoiding quantum singularities in CFT models. However, the similar arguments can be also applied to classical gravitational theory enjoying conformal symmetry to solve the ubiquitous problems with classical spacetime singularities. This is a next motivation for classical Weyl gravity, which should keep its conformal properties also on the quantum level for this
argument to work.

The CFT matter models considered above are indeed very special QFT’s since in them there are no UV-divergences. This implies that there are no $\beta$-functions both on the perturbative (to all orders) level as well as non-perturbatively due to conformal algebra in which we must have the energy-momentum tensor operator of CFT. If we do not need to use the renormalization scale $\mu$ to define properly the theory on the quantum level, then also the form of the quantum effective action is very special. One can view the quantization procedure $Q$ as a several mapping in theory space $T$:

$$Q : T_{cl} \rightarrow T_q,$$

where both the classical theory $T_{cl}$ and the quantum theory $T_q$ (regularized and renormalized with some physical renormalization conditions) are elements of the theory space $T$. The quantum theory $T_q$ is specified completely by quantum effective action $\Gamma$ (regularized and renormalized), similarly like $T_{cl}$ is defined by classical action functional $S_{cl}$. The conformal symmetry constrains the effective action of the quantum theory $T_q$ so much that $T_q = T_{cl}$ (up to a finite renormalization of couplings). In other words, the quantization procedure $Q$ finds its mathematical fixed point for very special initial elements $T_{cl}$, which leads to conformal symmetry both on the classical level. And what is more important it leads also to conformality on the quantum level which is preserved by quantum corrections. In a mathematical sense this gives a fully idempotent definition of the quantum effective action $\Gamma$ in CFT. Quantization process $Q$ produces quantum corrections to the classical dynamics, but they are scale-invariant, so their dynamics is completely captured by the original classical action, if it is properly conformally invariant. We conclude that quantum effective action $\Gamma$ is the same (up to finite dimensionless coupling rescalings) as the classical conformally invariant action $S_{cl}$.

The conformal symmetry of CFT is explicitly realized on the form of all correlation functions of the theory. But not only there. Due to conformal symmetry the special set of Ward identities is satisfied in CFT relating different $n$-point function of the theory. They are quantum operatorial manifestation of the conformal symmetry present in the quantum effective action $\Gamma$. They encode on the quantum level, for example, the fact that the trace of the effective energy-momentum tensor $T = g^{\mu\nu}T_{\mu\nu}$ of the full theory is off-shell zero even after inclusion of quantum corrections. And this statement has the operatorial meaning, not only in the sense of expectation values. Generally, the set of all conformal Ward identities satisfied by any CFT (including these models when gravity is dynamical) is an expression of the fact that the quantum system is highly constrained and that its relativistic dynamics is not an ordinary one. Moreover, quantum CFT models are finite (no UV divergences), so these are very well behaved quantum models which do not need regularization for their definitions on the quantum level. The conformal symmetry in CFT’s is very constraining such that the form of classical action describing the starting point of the theory $S_{cl}$ is very restrictive, provided that the number of spacetime dimension $d$ is fixed. Another specification of the model is the listing of all fields in it. However, not any composition is allowed. There are also restrictions on this. For example, if we desire too construct a CFT with gauge interactions, this must take a form of a very special theory of $N = 4$ supersymmetric Yang-Mills theory. If we want to add also gravitational interactions, this must be a model of $\mathcal{N} = 4$ conformal supergravity first discovered in 1985 by Fradkin and Tseytlin. Other actions, when conformal symmetry is preserved on the quantum level (so it is not broken) are not possible in $d = 4$.

We see that in CFT there are no $\beta$-functions of (necessarily) dimensionless coupling parameters. So they are really coupling constants, in the old sense, they are not RG running constants. There is no RG flow in models of CFT. The only difference from the classical field theory is that the dimension of operators may receive some finite corrections (though not related to any RG running), so there could be some anomalous dimensions of various operators in CFT’s. The fact that there is no RG flow is equivalent to the statement that conformal field theories
sit at the FP of the RG flows. They describe completely the situation (scaling dimensions, also known as critical exponents) of all operators present in a given CFT. For a different point of view, see [31]. The analysis of FP’s of RG flows is a very important topic for its own interest. However, knowing the general structure of CFT’s in a form of CFT data (conformal algebra, set of primary operators, their scaling dimensions) helps a lot in describing the FP’s of RG flows which are very special points in theory space $T$. As it is well known FP’s of RG are starting (or ending) points of RG flows since the situation at the FP’s could be seen as very extraordinary. But the existence of non-trivial RG flows breaks explicitly conformal symmetry of the CFT model. To do this in a more gentle manner, one can imagine turning smoothly various deformation operators and then the breaking of conformal symmetry could be achieved in a spontaneous way. In this way CFT’s are starting points in the theory space $T$ for various non-conformal theory modifications, in which there is RG flow, but still there is some remnant and memory of the constraints on the dynamics that were imposed at the FP. Therefore this kind of relativistic dynamics enjoys RG flows and only some remnants of full conformal symmetry. The full symmetry is recovered in the UV limit of the theory. In the next sections of this contribution, we will present one quantum computation which is done in the theory, which originates from full CFT with gravitational interactions, but in which conformal symmetry is spontaneously broken and the RG flows towards IR is possible. As we will see even in such partially broken model, the amount of constraints which were left after fully realized conformal symmetry is significant and the ensuing relativistic dynamics is constrained and highly interesting.

3. Weyl gravity

Now we proceed with the construction of Weyl gravity in $d = 4$ spacetime dimensions. The number of dimensions here has to be fixed and even integer. In quantum theory we will not allow playing with the “effective” dimensionality of spacetime similarly to what happens for example in dimensional regularization (DIMREG) scheme). This requirement has to do with the construction of the classical action of gravity which is supposed to be conformally invariant. For such a construction we must use Weyl tensors as building blocks, and if we require locality of the action, then they can appear only in positive integer powers. Hence the dimension $d$ of spacetime has to be even integer, and we choose $d = 4$ to be consistent with our every day experience of relativistic physics.

The main building block for the action is the Weyl tensor (tensor of conformal curvature), which is defined in $d = 4$ as

$$C_{\mu \nu \rho \sigma} = R_{\mu \nu \rho \sigma} - 2R_{[\mu | \rho} g_{\sigma | \nu]} + \frac{1}{3} g_{[\mu | \rho} g_{\sigma | \nu]} R,$$

(6)

where $R_{\mu \nu \rho \sigma}$, $R_{\mu \nu}$ and $R$ denote the Riemann tensor, Ricci tensor and Ricci scalar respectively. Its main useful property is the conformal transformation law

$$C_{\mu \nu \rho \sigma} \rightarrow \Omega^2 C_{\mu \nu \rho \sigma}$$

(7)

or for versions with different positions of indices (such position matters!)

$$C_{\mu \nu \rho \sigma} \rightarrow C_{\mu \nu \rho \sigma}^\sigma, \quad \text{and} \quad C^{\mu \nu \rho \sigma} \rightarrow \Omega^{-6} C^{\mu \nu \rho \sigma}$$

(8)

Of course, Weyl tensor plays the role of the gravitational curvature and for example the vanishing of all its components for some given spacetime implies it is conformally flat. Then such a spacetime can be mapped to a completely flat one by a conformal transformation (2) of the metric tensor. Generally, in conformal gravity different spacetimes differ only by their conformal structures (so how the Weyl tensor $C_{\mu \nu \rho \sigma}$ looks like), and the details about the differential metric structure are irrelevant. Moreover, metrics related by conformal transformations are in the same
conformal equivalence class and hence they describe the same gravitational spacetime from the point of view of conformal symmetry. Not only we have to be independent of the particular choice of the coordinate system, but also of the choice of the conformal gauge, so of the rescaling factor $\Omega^2$ in the definition of conformal transformation for metric (2). This conformal factor $\Omega^2$ (known also as a scale factor in cosmology) is strictly non-negative due to the usage of the square of a completely arbitrary real scalar function $\Omega(x)$ of the point $x$ of the spacetime. To describe the gravitational field in conformal gravity, one must exploit conformal invariants, which are both diffeomorphism invariants (scalars) as well as they do not transform under general conformal transformations. The set of them is quite restricted. The gravitational action must be one of them.

This action \[1\] in $d = 4$ is constructed exclusively with the help of Weyl tensor as

$$S_{\text{conf}} = \int d^4x \sqrt{|g|} \alpha_C C^2,$$

where the square of the Weyl tensor has the following expansion in known quadratic in curvatures invariants

$$C^2 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2,$$

which shows a big resemblance to the formula (6) regarding the numerical coefficients. This is easily explained by the tracelessness of the Weyl tensor in any pair of its indices. To check that the action (9) is conformally invariant in $d = 4$ spacetime dimension, one must also recall a conformal transformation law for the volume element, namely

$$\sqrt{|g|} \rightarrow \sqrt{|g'|} = \Omega^4 \sqrt{|g|},$$

while the coordinate volume differentials collected in $d^4x$ do not transform (that is $d^4x \rightarrow d^4x$) since the coordinates are not changed under the transformation $(x^\mu \rightarrow x'^\mu)$ and all conformal transformations are induced from the way how the metric changes in (2).

One notices that the scalar invariant $C^2$ transforms co-covariantly, that is we have

$$C^2 \rightarrow \Omega^{-4} C^2$$

and only its properly densitized version

$$\sqrt{|g|} C^2 \rightarrow \sqrt{|g'|} \sqrt{|g|} C^2$$

is conformal invariant. However, this last one is not a scalar invariant from the perspective of GR, since it is a density. Therefore in the local framework we are in conflict of finding local conformal invariants which are also scalars from the point of view of diffeomorphisms. This observation has some important consequences for the study of black hole solutions and the question of the fate of singularities there. Probably, one should look at some non-local invariants which are integrated versions of the densitized scalar invariant in (13), i.e. at the values of the action integral (9) over some small region of spacetimes.

The action integral is a scalar and it is conformally invariant, but as a main feature of the action functionals it is obtained via integration, so that is why it must be highly non-local (or strongly non-local). This action in (9) describes completely the classical conformal Weyl gravity in $d = 4$. This theory is both diffeomorphically (relativistic coordinate changes) and conformally invariant (changes of the conformal factors of the metric). This is a unique theory in $d = 4$ since there are no other local conformal invariants (besides $\sqrt{|g|} C^2$ in (13)) possible to be constructed here. Additionally, the theory is described by only one conformal coupling constant $\alpha_C$. In this way this theory is a very similar to Maxwell or Yang-Mills action, which is also quadratic in gauge
field strengths and which also comes with one dimensionless coupling constant. The fact that $\alpha_C$ is dimensionless and unique constrains the choice of the theory even more in the gravitational setting. Compare this situation with Einstein-Hilbert with cosmological constant which already comes with two coupling parameters. Models with higher derivatives and quadratic in curvatures contain even more such coupling parameters. (The general Stelle theory has four such parameters appearing in the Lagrangian $\lambda_{cc} + \kappa N R + \alpha_R R^2 + \alpha_C C^2$). The uniqueness of the theory is a very elegant and economic property to select a correct gravitational theory. For definiteness in Euclidean setup we require that the coupling $\alpha_C$ is given by a real number and strictly positive.

One should examine the advantages of classical Weyl gravity as a theory of relativistic gravitational field. First, it is noticed that the energy-momentum tensor which is a source of gravity must be here traceless to comply with the tracelessness of the tensor of equations of motion (EOM) coming from the action (9) required by conformal invariance of both matter (source) and gravity sides of the coupled EOM. Explicitly, the tensor of EOM for Weyl conformal gravity has the form

$$B^{\mu\nu} = \frac{1}{\alpha_C} \frac{1}{\sqrt{|g|}} \delta g_{\mu\nu}^{\text{conf}} = 2 \nabla_\kappa \nabla_\lambda C^{\mu\kappa\nu\lambda} + C^{\mu\kappa\nu\lambda} R_{\kappa\lambda}$$

and it bears the name of Bach tensor. Generally, in $d = 4$ the Bach tensor is symmetric ($B^{\mu\nu} = B^{\nu\mu}$), automatically traceless ($g_{\mu\nu} B^{\mu\nu} = 0$) and it originates from the conformal action $S_{\text{conf}}$ as the variational derivative of the action integral in (9). The field equations in Weyl gravity read

$$B^{\mu\nu} = \frac{1}{2\alpha_C} T^{\mu\nu}$$

This form of EOM puts a stringent condition on the form of a possible energy-momentum tensor of the matter source because it has to be traceless, so that only massless or scale-invariant matter on their equations of motion can give rise to it. This condition is satisfied in the deep UV (regime of high energies), where all the matter becomes “effectively” massless and conformal symmetry is restored. Since the effects of relativistic gravity are not well investigated and understood in the regime of very high energies and also of high matter densities, then it is more practical to look for tests of CWG in the setup of vacuum solutions, where there is no matter at all ($T^{\mu\nu} = 0$).

Here, for empty space solutions, we can perform and verify three basic tests of gravitational theory which originally gave impetus to the Einstein gravitational theories. Those were the perihelion of Mercury, the light deflection by the Sun and the redshift of spectral lines due to gravitational effects. To this one could also add the time delay of radar echoes and precessions of gyroscopes. All these tests were carried out in empty space and they gave a verification of Einstein’s theory. But they are also equally valid and satisfied in CWG since they are based in particular on static and spherically symmetric exact vacuum solutions in the form of the Schwarzschild metric. This metric is also an exact vacuum solution in CWG. A bolder statement is that all exact vacuum solutions of Einstein gravitational theory (without cosmological constant) are also exact vacuum solutions of classical Weyl gravity. For this, one have to notice that in the expansion in (10) only the Ricci tensor appears, hence the EOM of CWG are quadratic in Ricci tensors or in Ricci scalars (and not of Riemann tensor), and hence they vanish identically on Ricci-flat spacetime which are known to be gravitational vacuum in original Einsteinian theory. We conclude that metrics describing vacuum spacetimes like Schwarzschild, Kerr, etc are also solutions there. This implies that CWG reproduces all precise tests of relativistic gravitation like Einstein theory in the vacuum.

But CWG contains more solutions that just Einsteinian theory. And this is one of its powers. For example, the solutions found by Mannheim and Kazanas for static and spherically symmetric situations in conformal gravitational vacuum [32] are a family of solutions with more parameters than in Einsteinian theory. But they can serve as a solution of “dark matter” or missing matter
problems in galaxies \[16, 17, 18\]. These solutions describe accurately more than 200 galactic rotation curves with just one fittable parameter and mimic the behavior of the gravitational potential produced by the conjectured dark matter halo. However, in CWG dark matter is not necessary at all and the power of classical conformal gravity is to explain flat rotation curves as exact vacuum solutions of the theory. Moreover, as emphasized at the beginning of the previous section conformal Weyl gravity provides also an explanation for “almost” scale-invariant power spectrum of scalar cosmological fluctuations therefore explaining one of the issues in current cosmology. Other problems in cosmology where conformal gravity helps is the problem of cosmological constant, which is not needed in conformal cosmology any more, and the problem of flatness. According to \[33, 34\] conformal gravity also solves the problem related to inflation and dark energy problems (vacuum energy) since all cosmological manifolds are conformally flat. This fact also simplifies the analysis of the cosmological fluctuations both in the scalar as well as in the tensorial sectors.

Classical CWG provides also a natural and very robust resolution of the problem of spacetime singularities, which are probably the most serious inconsistencies of classical relativistic gravitational physics. These singularities signal the breakdown of classical relativistic physics where the theory inherently sets its own limits of validity. One would say that the fate of classical singularities of the gravitational field is to be eventually solved somehow by a hypothetical theory of quantum gravity. However, we think that for the full resolution of them, one has to use the power of additional new local symmetry in the gravitational framework. And this symmetry is a dynamical symmetry related to arbitrary conformal rescalings of the metric of the spacetime \[2\]. By allowing for conformal rescalings one resolves the problem of classical GR singularities (which put in danger exact solutions of the gravitational theory). For example, the Big Bang or Big Crunch singularities in the cosmological framework, or black hole singularities (Schwarzschild, Kerr, and other metrics, which are singular at the origin of the black hole) \[35, 36, 37\]. The idea is to relegate the “observable” general relativistic singularity into a (conformal) gauge-dependent unobservable conformal factor \(\Omega^2\). In this way something which was for example the curvature singularity (like measured by the Kretschmann invariant scalar \(R^2\)) in Einsteinian theory, now is not anymore a good conformal invariant to look at. By performing a change of the conformal gauge according to \[2\], one shows that such an invariant scalar from GR loses its unambiguous meaning and its actual value changes.

When one analyzes the conformal invariants (there are very few of them), then one sees that the singularities can be completely resolved and that the conformal equivalence class of metrics is not singular, similarly like the coordinate “singularity” at the horizon of Schwarzschild metric is not present in the equivalence class of metrics related by coordinate transformations. The point is that there exist metrics (describing physical gravitational spacetimes), which are not singular in these classes, compared to one particular example metric which was singular but the singularity was assessed by looking at “wrong” invariants, that is scalars which are not invariant under the full group of symmetries of the theory in question. Then the choice of non-singular metrics should be a preferred one, and the one of original singular metric should be discarded as driven by a singular conformal gauge-dependent scale factor \(\Omega^2\) of the metric. Since the choice of any metric in the equivalence class is possible (and should be done to perform actual physical computations), then the one not giving rise to singularity is completely equivalent to others, but about the resolution of singularities one decides by choosing a most convenient one from the physical purposes. And that is why one choose Kruskal-Szekeres coordinates for example over original Schwarzschild coordinates to cover the whole black hole horizon region. One concludes that there is not a true singularity at the location of the horizon, it was only a coordinate gauge choice singularity. In complete analogy for the conformal resolution of singularities, one derives that the conformally related metrics which do not develop singularities are better and physically more convenient than the original metrics in Einsteinian theory. One concludes that there is not a true singularity with
a conformally invariant meaning here, it was only an original conformal gauge choice that was singular and led to not “conformally” invariant notion of curvature singularities like in Einsteinian GR. This again shows the interplay and similarities between relativistic and conformal ways of treatment of some things - a theme which is constantly recurring in this original contribution.

We also believe that the true resolution of classical singularities still should be achieved on the classical level of the theory. Even if the quantum effects play an important role for this issue, they all are captured by the quantum effective action. The true quantum-corrected solutions of the theory are understood as classical solutions to the quantum effective action, so the same problem returns to the classical level. Our take on is to assume that additional new symmetry in the gravitational relativistic dynamics comes with the rescue to these annoying problems of ubiquitous and inevitable classical singularities of gravitational spacetimes.

Another virtue of CWG is that this approach to relativistic dynamics of the gravitational field is fully Machian. This realizes completely the old ideas of Mach, which originally inspired Einstein in his construction of the gravitational theory. However, as it is well known in Einsteinian gravity the dynamics is only partially Machian, as this is obvious from Einstein equations in vacuum, which depend on the matter distribution only locally. However, in CWG gravitational field at a particular location is determined completely by the conformal matter distribution everywhere else in the Universe (in Machian sense, although of course the EOM of CWG are local despite that they are with four derivatives on the metric tensor). This particular feature is seen also from the explicit form of Mannheim-Kazanas solutions for which the particular slope of the galactic rotation curve is determined from cosmological reasons, so the matter density in the whole universe influences the gravitational potential inside galaxies. In particular, Mannheim-Kazanas solutions provide one exact solution to the problem of embedding the local galactic gravity solution in a bigger scale picture of cosmological expansion of the Universe as a whole. The fully Machian character of CWG is obviously related to the preservation of only causal structure of the gravitational spacetimes under conformal transformations. And only the last structure of the spacetime manifold retains its meaning in full CWG.

Conformal symmetry is quite often seen on the classical level of the relativistic dynamics. For example, classical electromagnetic and Yang-Mills (YM) (non-Abelian) gauge theories, and the theory of massless Dirac fermions in $d = 4$ spacetime dimensions, are perfectly conformally invariant. They enjoy spacetimes symmetries from the full 15-parameter conformal group containing as its subgroup the 10-element usual Poincaré group of standard relativistic symmetries. Moreover, a scalar field conformally coupled to gravity in $d = 4$ with the non-minimal coupling of the form

$$ S_{\text{int}} = \int d^4 x \sqrt{|g|} \frac{1}{6} R \phi^2 $$  \hspace{1cm} (16)$$

represents another classically conformal model with matter. One can also ask what are the corresponding consequences of this fact on the quantum level. For this one has to discuss the quantum Weyl gravity QWG.

4. Quantum Weyl Gravity

There are various reasons for quantum Weyl gravity. More clear conceptually is to couple first quantum matter fields to fixed but non-trivial background geometry. This setup of quantum fields in curved spacetime is well known and it has very profound consequences also for the construction of resulting quantum gravitational theory, where the gravity is taken as dynamical and the dynamics of the related field of the graviton is fully quantum. One can induce gravity from quantum matter fluctuations $\text{35}$. In this way one can find a correct quantum gravity theory, the same like this was originally done with YM theory when at the beginning only fermions transforming non-trivially under non-Abelian symmetry groups were known.
The idea is the following. It is the embodiment of the original deWitt-Utiyama argument \cite{39}. When the matter fields are put in a non-trivial classical external non-dynamical gravitational field, then as quantum fields described by QFT on non-trivial backgrounds they produce various UV divergences, both in the matter and in the gravitational (geometric sector). The correct quantum gravitational theory should absorb all the divergences of the gravitational character in its counterterms. For this the original action of the gravitational theory should contain the corresponding terms. And here the results of the computation about these UV divergences is as follows. When one uses the conformal models described above as matter models on curved spacetimes, then in $d = 4$ spacetime dimensions the only non-trivial gravitational counterterm which is generated by quantum corrections induced from the matter side is of the conformal character, that is, it is precisely

\[ S_{\text{div}} = \int d^4 x \sqrt{|g|} C^2. \]

(17)

Hence all matter-generated UV-divergences will be completely absorbed in the gravitational counterterm coming from the action of WG. This means that QWG allows for the quantum coupling of conformal matter to gravity in a way preserving conformal symmetry. The main message is that the conformal symmetry of classical matter models is not destroyed by quantum corrections when coupled to conformal gravity. Moreover, gravity enhances this coupling providing the whole uniform treatment of the classical system, which is described in a fully conformally invariant way.

One can introduce QWG completely from the scratch along similar lines of thoughts. For this, one needs to consider the path integral on curved fixed spacetime backgrounds, and to integrate out the quantum fields of conformal matter models (like electromagnetism, YM or SM Dirac fermions before the electroweak symmetry breaking). As proved by 't Hooft and Veltman \cite{40, 41}, this generates the very specific action for quantum gravitational theories. Since the conformal symmetry was originally in the matter model, then the careful quantization procedure cannot break it and it must be inherited by the resulting induced gravitational theory. This must be the theory described by the $C^2$ action. Hence QWG is inevitable, if one uses conformal matter models and if the conformal symmetry is treated with sufficient care. If one did not know that dynamical gravity is out there (like one did not know about the dynamical theory of the gluonic field related to non-Abelian gauge symmetries), then one would discover that theory (dynamical gravity) by integrating out example massless Dirac fermions. And this dynamical quantum gravity must be described by QWG theory. There is also a related line of argumentation on how to rederive Einsteinian theory starting from Weyl gravity in $d = 4$ (due to Maldacena and others, see e.g. \cite{12}), but we will not discuss this direction here, since it goes against the main spirit of our presentation, although we admit that there are striking similarities between the two theories (apparent for example on the level of exact vacuum solutions discussed at length above).

It is also interesting with this line of thought to study processes of black hole evaporation within classical conformal gravity as it was done in \cite{43}.

Having discussed motivation for Weyl gravity as the theory of quantum gravity, now we analyze properties of QWG understood as the QFT of gravitational interactions with conformal symmetry. This last element is additional and requires a special care. This is because the conformal symmetry is in the local version here. It is not a global symmetry as it was for example in highly (super)-symmetric $\mathcal{N} = 4$ matter models. Since it is in the gauged version, then this additional symmetry of local character must be properly taken care of during the formal process of quantization of gravity. The framework of Faddeev-Popov quantization allows for this since it parallels the treatment of diffeomorphism symmetries, which are from their nature local transformations. On the other side, the conformal symmetry is here gauged (localized) and there are some subtle differences compared to the other case of gauge symmetries of relativistic gravitation (understood as the local gauge theory of the Poincaré group). We refer the reader
to a special treatment of local gauge symmetries in this case. Of course, on the quantum level one must secure that these local symmetries are not anomalous. Otherwise, the whole program of quantization would miserably fail. The issue of conformal anomaly is very important and we discuss some comments about it at the end of this contribution.

For covariant quantization the best theoretical framework is that one with functional path integrals. Since in gauge theory we have a local symmetry and also degeneracy of the kinetic operator on the level of path integral, then this symmetry must be constrained to define properly the functional integral. One typically does a gauge fixing. Here this should be done separately for diffeomorphisms and for conformal symmetry of WG. The framework lets all this to be done in a covariant manner. Next, one has to add appropriate Faddeev-Popov quantum ghost fields. We remark that these ghost fields are good and useful and they have nothing in common with the Boulware-Deser ghosts of Weyl gravity (bad and malevolent ghosts). They have to be specific to the diffeomorphism part of the symmetry group, as well as to the conformal group. With such conformal ghosts new Feynman rules need to be added to the perturbative rules of the quantum gravitational theory turning it therefore into QWG theory. Moreover, finally we can unambiguously find the propagator of the QWG graviton field when some gauge choice is made. Of course, this propagator will depend on the gauge fixing parameters used to kill the additional infinite gauge freedom on the level of path integral.

The properties of QWG on the perturbative level of analysis in QFT are very interesting. This quantum system gives rise to a renormalizable model of quantum gravitation. The perturbative UV-divergences are fully under control and they are all absorbable in the original covariant terms of the theory, namely the known and seen before counterterm $\sqrt{|g|}C^2$ of the original action of the theory. QWG has a good control not only over divergences generated from the matter side, it also constrains the divergences generated by quantum graviton loops. The counterterm $R_2$ is not generated (at least on the level of first quantum loop). The renormalizability of this model of quantum conformal gravity is a very remarkable feature, and it persists even after coupling to conformal matter models (which in $d = 4$ are characterized by dimensionless coupling constants). The QWG in its pure gravitational sector has one unique coupling constant $\alpha_C$. Of course, this coupling exhibits RG running due to the fact that the corresponding $\beta$-function as read from the UV-divergences of the theory is non-vanishing. However, asymptotically in the UV the control over this running of $\alpha_C$ is gained since the QWG is asymptotically free in deep UV regime. In this way, this model resembles very much the situation met in QCD, where the theory also contains only one positive unique coupling which in the UV limit is asymptotically free.

One knows that QCD as a YM theory of the non-Abelian group of $SU(3)$ is defined without any mass scale and the QCD coupling parameter is dimensionless although it runs under RG. However, in QCD one sees in the IR region spontaneous creation of mass scales, in the form of generation of $\Lambda_{\text{QCD}}$ or effectively the pion mass $m_\pi$ (this last particle mediates effectively the strong interactions between protons and neutrons at very low energies). This effect is entirely due to the phenomenon known as dimensional transmutation. Similar effects will take lead in QWG, where in the low energy regime we should see a spontaneous generation of mass scales (like the Planck scale), which describe effectively quantum gravitational interactions at sufficiently low energies.

The fact that the QWG is asymptotically free in the ultraviolet regime can be also seen as a particular case of a special situation when the fixed point (FP) of RG is met in the UV. And the theory in the UV reaches some CFT describing conformal gravitational interactions also on the quantum level. The quest for such CFT theory with local conformal symmetry and with gravitational interactions is very restricting. Especially, the constraint that in the UV gravitational CFT there is no conformal anomaly fixes the final UV theory quite uniquely. The theory in the UV has to be an $\mathcal{N} = 4$ conformal supergravity as discovered by Fradkin and Tseytlin with four copies of $\mathcal{N} = 4$ super-Yang-Mills theories [44]. In such a coupled model,
there are no UV-divergences and the quantum model is completely UV-finite. This framework of supergravity is very constrained and the total symmetry algebra has many generators, hence the divergences are constrained to vanish. Moreover, the conformal current, the energy-momentum tensor and all other supersymmetric currents find themselves in the same supersymmetry multiplet and that is why they have to be simultaneously conserved, which implies the absence of any violation of conformal symmetry, the preservation of the quantum structure of CFT, the independence of any mass scale and UV-finiteness. Only with extended supersymmetries it is possible to relate the conservation of the energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$ to the conservation of conformal current $\nabla_\mu j^\nu = 0$ and hence to the situation with unbroken scale and conformal invariance. In this very constrained framework, the relativistic dynamics is very special, and for example there is no dynamics related to RG flows. Still constraints put a big amount of control on possible places from which divergences could pop out requiring regularization. In such a constrained theory it is still possible to perform a very well defined quantum computation, which is the main result of this paper. For this it is enough to use powers of the relativistic symmetries and of conformal symmetries in the QFT theoretical framework.

The model due to Fradkin and Tseytlin realized fully the idea of quantum conformality with gravitational interactions. In this way the conformality of $\mathcal{N} = 4$ SYM theory is extended away from flat spacetime to a general curved background. It is very reassuring that when $\mathcal{N} = 4$ SYM theory is put on curved background then even the conformally invariant gravitational counterterm $\sqrt{|g|}C^2$ is not generated (of course in the gauge sector the $F^2$ counterterm is not generated to the flat spacetime UV-finiteness of this gauge theory model). Only the coupling of the two theories (supergravity and super-Yang-Mills) provides at the end the theory with no conformal (gauged) anomaly on the quantum level. This theory is with quantum scale invariance promoted to quantum conformal invariance. There are no beta functions, no running couplings, no RG flows, only anomalous dimensions of quantum operators. The conformal Ward identities are clearly preserved and they constrain highly the Green functions. And as we discussed this above, this theory sits all the time at the UV FP of RG, however the breaking of conformal symmetry by adding some deformation operator can be achieved. This will result with some non-trivial RG flows. For some very particular situation, one can view this as a spontaneous conformal symmetry breaking (induced and related to supersymmetry breaking as well) and one can hope that when this symmetry is broken in a soft way, then there is a remnant of some information in conformal Ward identities of the theory even away from the UV FP, when the theory was described by a gravitational CFT.

Another virtue of QWG in Euclidean framework is that the Euclidean theory defined by action functional with $C^2$ as Lagrangian density is bounded from below. This implies that the partition function of the corresponding statistical mechanical model is completely positive-definite and well-defined, does not need any regularization, and the functional integrals are mathematically very well-defined. Contrary to Euclidean Einsteinian gravitational theory in QWG there is no conformal instability problem, of course because here we do not have a conformal mode with negative sign. In QWG we have only propagating modes which are invariant under symmetry properties of the theory, so the conformal mode, (trace of the metric tensor fluctuations) does not show up in the perturbative spectrum. One can analyze with all available methods from statistical mechanics and field theory the quantum partition function $Z$ here, which is positive-definite and describes the quantum statistical partition function. It is one of the very first consistent model of field theory which in the context of quantum statistical mechanics gives statistical and stochastic properties of Euclidean field theory. Especially, since in Euclidean QWG this is a field theory with propagating metric degrees of freedom describing differential geometry of 4-dimensional manifolds (surfaces). It could be said as an analogue of Euclidean gravity - quantum statistical mechanical theory of differential manifolds. Since the metric structure does not matter here, only the conformal properties, and angles as mentioned in the introductions, this is really a study of
(scaleless) shape of manifolds. This theory can be viewed as a first consistent model of quantum shape dynamics.

As emphasized a lot previously QWG is a system with a very peculiar relativistic dynamics. Diffeomorphism symmetry is fully realized (symmetry under the change of coordinate system used to describe physics). Moreover, it is also conformal symmetry that it is gauged here (appears in the local version). And this is a symmetry under an arbitrary changes of the conformal factor used to measure distances. This puts a lot of constraints on the ensuing relativistic dynamics of the gravitational field. One could say from the historical perspective, that this is a first modified gravitational theory (introduced by Weyl in 1918), although it was not called like this neither by Weyl, nor by Einstein - his opponent. His theory was clearly different than Einsteinian gravity (introduced in 1918) because it embodies more symmetries and the resulting dynamics is with higher derivatives, particularly in $d = 4$ this is with four derivatives, while Einstein theory gives a dynamics with two derivatives only. One can see this difference also on the level of the spectra of two theories. For example, in QWG as analyzed on the tree-level the physical degrees of freedom around any background are only transverse traceless (TT) gravitons as this is obvious from the York decomposition (to be explained in detail below). And we do not have for example a problematic conformal mode of Einsteinian gravity.

Moreover, QWG is different than any other higher derivative (HD) gravitational theory containing four derivatives in $d = 4$ spacetime dimensions. Such a general Stelle theory is characterized by two coupling parameters and two covariant terms in the gravitational Lagrangian, namely $\alpha R R^{2} + \alpha C C^{2}$. However, QWG is different and not a similar to a general quadratic Stelle theory. The spectra are different since only in QWG we have only TT gravitons present. One cannot also take in a simple continuous way the limit $\alpha R$ in the results of the general Stelle theory. This is another incarnation of the famous Veltman discontinuities. This time it is to be analyzed in HD gravitational theories. It is not like original problem with the mass gap of gauge theories and counting of propagating degrees of freedom there (2 in massless vs 3 in massive gauge theories). Here the discontinuity has to do not only with transition from massive to massless excitations, but also with a drastic change of the character of fluctuations in the two theories. For example, they fill different representations of the Lorentz group which is used to classify particle excitations around flat spacetime. So the particles are in different tensorial representation. For example, in QWG on the tree-level we have 2 massless tensors and a massless vector. This last particle is not seen in any other spectrum of ordinary HD gravitational theories in $d = 4$. This vector could be viewed as a remnant and the signal of the remaining (vectorial) conformal symmetry on the level of the action of the theory. One should see that in general the Veltman discontinuity means that in the discontinuous limiting theory there exists an enhancement of local symmetries. The theory with $\alpha R = 0$ is very special and highly symmetric, higher than any ordinary HD gravitational theories. And this is of course, due to the presence of local conformal symmetry there.

5. Quantum computations

We would like to present here an example of a quantum computation done in the framework of QWG. This computation has to be done in the framework of quantum field theory (QFT) of gravitational (and conformal) interactions. For this we will use the formalism of Faddeev-Popov quantization where special care should be exerted in the treatment of local conformal symmetry on the quantum level. We do not wish that this symmetry is broken by the mere quantization process. This quantum computation (at the one-loop level) will be performed within the background-independent formalism and for this we will use the background field method (BFM). Moreover, as our background we will choose a general Einstein spacetime (ES) background, characterized by the condition for Ricci tensor

$$R_{\mu\nu} = \Lambda g_{\mu\nu},$$

(18)
where $\Lambda$ is an arbitrary value constant parameter, that is we have $\Lambda = \text{const}$ ($\nabla_\mu \Lambda = 0$). The values and the form of the Weyl tensor $C_{\mu\nu\rho\sigma}$ can be arbitrary on general ES and only in this way the generic ES differs from a maximally symmetric spacetimes (MSS) characterized by the parameter $\Lambda$. Selecting a general ES background we get in one stroke the result of the previous computations on two different backgrounds: namely on Ricci-flat background and MSS. They were considered in [15].

We will perform the computation considering only the physical degrees of freedom which are present as dynamical degrees of freedom in QWG. As it is well known due to diffeomorphism symmetry and due to the conformal symmetry they are transverse and traceless (TT) gravitons respectively. Of course, a quantum graviton is described as a fluctuation of the metric tensor of the spacetime, so these fluctuations can be considered as a symmetric rank-2 tensor.

Our computation is an RG-improved one-loop level computation of the RG running of couplings in QWG. In this calculation we take into account the special role of non-perturbative effects that improve over the standard quantum one-loop computation which was achieved in 1980’s. In this sense, we can say that our computation is a 1.5-loop one since we still acknowledge that the effects of two perturbative loops will give a more accurate description of quantum phenomena. We start our computation with the one-loop partition function of QWG on ES and make it more sensitive to quantum effects by working with it in the non-perturbative Wetterich equation describing the exact RG flows. This means that our one-loop beta functions are functionally improved in the sense of using functional Renormalization Group (FRG) approach.

The novelty of this approach is that for the first time we analyze the general situation on ES, and not separately on MSS [16] or Ricci-flat backgrounds. This approach follows [17]. After analyzing the RG flow mainly in the deep IR regime we will search for FP’s of RG (defined by condition that beta functions vanish $\beta = 0$). Finally, we discuss the physical applications and interpretations of these results about the IR behavior of QWG. This conclusion will open a new window for cosmology.

Before we embark on showing details of the computation, we should explain why it is sensible to perform such quantum computation with running couplings and beta function in QWG. We emphasized before that if the conformality is preserved on the quantum level, then the theory must be at a FP of RG, and there are no beta functions and no RG flow. For consistency such a regime of the theory must be also conformal anomaly-free. This implies that this theory has to be (very probably) uniquely the superconformal gravity constructed by Fradkin and Tseytlin [14]. We believe such theory describes the conformal gravitational physics in the deep UV regime, where all the symmetries (including supersymmetry in local version) are restored. Therefore we find more interesting to look at the deep UV behavior of the QWG where we can find some non-trivial FP of RG describing probably new infrared physics. What happens in the IR regime is the soft breaking of conformal symmetry so that only the global part of the full 15-dimensional conformal group is broken. In particular, the generator related to global scale-invariance, i.e. to the dilatation transformations on the physical system, is broken. This breaking in a spontaneous way is a deformation of the original UV conformal theory by giving a vev to some operator, which is globally scale-invariant but not invariant under full conformal group.

This is what may happen in QWG described by pure gravitational action (without local supersymmetry) when analyzed consequently at one-loop and two-loop perturbative levels of the theory. In the UV this theory meets a UV FP of RG, where the scale-invariance is regained. Also conformal anomaly vanishes there (due to participation of other fields in the supermultiplet). When one lowers the energy scale and decouples the superpartners, one sees that the quantum
operator $R^2$ acquires a non-vanishing vev due to contribution at the two-loop order. This is also related to the fact that the beta function $\beta_{R^2}$ of the coupling in front of the $R^2$ term appears on the second perturbative loop order, while it was not present there at the first loop order. In such a theory we start a non-trivial RG running, which when approaching the IR limit, generates mass scales (like the Planck scale) and the symmetries of the theory are changed. Likewise, symmetries in QCD are changed when we start describing strong interactions via interactions with massive pions.) This happens because of the effect of dimensional transmutation occurring in this theory. The quantum theory which describes these phenomena towards IR regime of the spectrum is QWG with non-vanishing beta functions of couplings and running coupling parameters. Hence, it is useful to describe quantum RG effects in such theory described by the quantized version of Weyl gravity, when we still had the conformal symmetry on the classical level, and when the quantization procedure did not break this and was done very delicately. One sees that if sufficient care is exerted then conformal Ward identities are still satisfied like in softly spontaneously broken local gauge theories.

The issue of conformal anomaly (CA) does not prevent sensible computations in such a theory since the computation is done towards the IR regime. One notices that the CA is really an UV problem or in a different disguise a problem with the UV-completion of the theory. In other words, for a good definition of the theory one must find a UV FP with good conformal properties. We already have discussed such a candidate theory, namely the $\mathcal{N} = 4$ conformal supergravity (CSG). Moreover, the CA at one-loop, for example, is completely expressed via the perturbative beta functions of the theory. And it is well known that these beta functions are related to the UV-divergences of the theory, so to the UV limits of the theory. Hence physics in the IR does not matter for this. If we assume that the UV-completion is done by CSG, then the problem with CA is gone since this last one probes the UV physics. We know that in IR the conformal symmetry has to be broken anyway, but this will not influence at all the securement of CA issue in the UV regime. With this understanding in mind, we can still reasonably well and logically correct describe the situation with running in the IR. In order to have the power of the remnants of the conformal symmetry expressed on the quantum level via conformal Ward identities, the conformal symmetry towards the IR regime has to be broken but only in a softly way as described schematically above. Of course, this violation of conformality has to be analyzed better and in more details, and we leave it for future investigations. In this way, we can still sensibly talk about RG flow in QWG in the IR regime, where the softly violated conformal Ward identities still constrain the physics in a sense that the quantized theory which takes into account these phenomena in the IR regime is the quantized version of Weyl gravity, and for example not a generic higher derivative (quadratic in curvatures) gravitational theory. This is the perspective which gives sense to the computation that we present below. This is exactly a quantum computation in the IR regime of the QWG theory (non-supersymmetric).

In order to proceed with the quantum computation we need a few technical details. First in order to get rid of ambiguities related to gauge degrees of freedom and other non-physical degrees of freedom in QWG and also explicit addition of Faddeev-Popov diffeomorphic and conformal ghosts, we use the York decomposition of the metric fluctuations. Working in BFM, we start with writing a general expansion of the full quantum metric $\tilde{g}_{\mu\nu}$

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu},$$

where the fluctuations fields $h_{\mu\nu}$ are general and the background metric $g_{\mu\nu}$ is a general ES metric. We perform the York decomposition of the gravitational fluctuations in two steps. First, we remove the spurious trace degree of freedom by writing that

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{4}g_{\mu\nu}\phi \quad \text{with} \quad \phi = h_{\mu}^{\mu} = g^{\mu\nu} h_{\mu\nu},$$

(20)
and with the tracelessness condition on the new fluctuations

\[ g^{\mu\nu} \tilde{h}_{\mu\nu} = \tilde{h}_\mu = 0. \]  

(21)

In the second step of York decomposition we remove the transverse part from the traceless metric fluctuations \( \tilde{h}_{\mu\nu} \). This is achieved by recalling the following formula

\[ \tilde{h}_{\mu\nu} = \tilde{h}^\perp_{\mu\nu} + \nabla_\mu \eta^\perp_\nu + \nabla_\nu \eta^\perp_\mu + \nabla_\mu \nabla_\nu \sigma - \frac{1}{4} g_{\mu\nu} \Box \sigma, \]  

(22)

where the transversality conditions on new metric fluctuations \( \tilde{h}^\perp_{\mu\nu} \) and on the vector field \( \eta^\perp_\mu \) read

\[ \nabla^\mu \tilde{h}^\perp_{\mu\nu} = 0 \quad \text{and} \quad \nabla^\mu \eta^\perp_\mu = 0. \]  

(23)

Moreover, still the metric fluctuation field \( \tilde{h}^\perp_{\mu\nu} = h^{TT}_{\mu\nu} \) retains its traceless character

\[ g^{\mu\nu} \tilde{h}^\perp_{\mu\nu} = \tilde{h}^\perp_\mu = 0 \]  

(24)

and this implies that for convenience we can also redefine the trace scalar field \( \phi \to \tilde{\phi} = \phi - \Box \sigma \), where \( \sigma \) is another scalar field. We notice that the final York decomposition reads

\[ h_{\mu\nu} = \tilde{h}^\perp_{\mu\nu} + \nabla_\mu \eta^\perp_\nu + \nabla_\nu \eta^\perp_\mu + \nabla_\mu \nabla_\nu \sigma - \frac{1}{4} g_{\mu\nu} \tilde{\phi}. \]  

(25)

The purpose of this decomposition is its main result namely the irreducible part of the fluctuation, which is in the traceless transverse tensor \( \tilde{h}^\perp_{\mu\nu} = h^{TT}_{\mu\nu} \). One can see that in the QWG the physical field which propagates around any background metric \( g_{\mu\nu} \) (it does not even have to be an ES metric) is only a traceless transverse graviton. This is a physical field of QWG, where all redundancies due to the local gauge symmetries of the formalism are removed. This feature is due to diffeomorphism (transversality) and due to conformality (tracelessness). One can see it explicitly when writing the second variation of the Weyl action functional (9) around any on-shell background (a Bach-flat background). In such an on-shell situation the spurious quantum fields \( \eta^\perp_\mu, \sigma \) and \( \tilde{\phi} \) completely decouple and they do not show up in the expression for the second variation of the action integral. On a completely general background there appear terms with fields \( \eta^\perp_\mu, \sigma \) and \( \tilde{\phi} \) but they are all proportional to the Bach tensor \( B^{\mu\nu} \), so they correctly vanish when EOM in (14) are used. But below we consider only on-shell backgrounds (all ES are automatically Bach-flat in \( d = 4 \) spacetime dimensions).

Next, we discuss the most general expansion of terms quadratic in generalized curvatures in \( d = 4 \) dimensions. We consider here only local scalar invariants of dimension four. This implies that in the action they come only with dimensionless coupling coefficients, which is important for the property of soft breaking of conformal symmetry. We also analyze their properties under global and local scale transformations and also their first variations.

The first invariant is known already to us conformal scalar invariant \( C^2 \) whose expansion in other terms in curvature reads

\[ C^2 = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2 = 2R^2_{\mu\nu} - \frac{2}{3}R^2, \]  

(26)

where in the last line we used the integration by parts under spacetime volume integral and the Gauss-Bonnet topological relation. As we discussed at length in the introduction, this invariant when properly densitized is perfectly conformally invariant, which we express as the following statement below. Its infinitesimal local conformal variation vanishes identically, that is we have

\[ \delta_c \left( \sqrt{|g|} C^2 \right) = 0. \]  

(27)
Of course, this is also true for finite conformal transformation which is also a basis for CWG.

The next curvature scalar invariant that we can consider here is a Gauss-Bonnet invariant (also known as topological Euler invariant). Its expansion reads explicitly

\[ \text{GB} = E = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2. \]  

(28)

In four dimensions one can call it topological since it is related by Gauss-Bonnet theorem to the topological (completely metric-independent) characteristics of the spacetime, known as Euler topological invariant or Euler characteristics \( \chi_4 \). This also implies that any its variation under the integral integrates to zero, that is we have

\[ \delta \left( \int d^4 x \sqrt{|g|} \text{GB} \right) = 0. \]

(29)

In particular, this means that the variation is zero also under conformal transformations, both infinitesimal and finite, local and global.

The last invariant we can call as a Starobinsky invariant since first was invented by him in 1980 [48]. It is simply the square of the Ricci curvature scalar \( R^2 \). Besides it being globally scale-invariant (in \( d = 4 \)) related to the fact that the coefficient \( \alpha_R \) is dimensionless, it also shows some more extended form of global conformal invariance. Namely, it is invariant with respect to restricted local conformal invariance. The general local conformal transformations are parametrized by arbitrary scalar field \( \Omega = \Omega(x) \) as in (2). However, the Starobinsky invariant \( R^2 \) does not change under conformal transformations if the parameter \( \Omega \) satisfies the GR-covariant d’Alembertian (wave) equation, namely if

\[ \Box \Omega(x) = 0. \]

(30)

This is less than general local conformal transformations with arbitrary \( \Omega \), but still it is more than just global scale-invariance or no invariance at all. Therefore usage of this operator for the breaking of the conformal symmetry in \( d = 4 \) dimensions in the softest possible way may be a preferable solution to get a soft RG flow in QWG.

Now, we characterize the background ES manifolds. They are more general than MSS and Ricci-flat spacetimes, but they include both these cases as special subsets. To describe a general ES, one needs to specify the value of the \( \Lambda \) parameter in (18) and also the form of the Weyl tensor \( C_{\mu\nu\rho\sigma} \) consistent with all symmetries (so in results 10 algebraically independent components in \( d = 4 \)). As a consequence of defining condition of ES, we derive the following curvature relations

\[ R = 4\Lambda, \quad \text{GB} = R^2_{\mu\nu\rho\sigma} \]

(31)

and since the expression for the Riemann tensor on a general ES is given by

\[ R_{\mu\nu\rho\sigma} = \frac{\Lambda}{3} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) + C_{\mu\nu\rho\sigma}, \]

(32)

which is up to the Weyl tensor the same like on a general MSS with \( \Lambda \) parameter, we derive also relations for various curvature square invariants

\[ C^2 = R^2_{\mu\nu\rho\sigma} - \frac{8}{3} \Lambda^2 \quad \text{and} \quad \text{GB} = C^2 + \frac{8}{3} \Lambda^2. \]

(33)

We remark that in \( d = 4 \) all ES are vacuum solutions to CWG, i.e. they satisfy Bach EOM [14]. This is an important consequence for the evaluation of the partition function on on-shell backgrounds and also for the possibility to use WKB approximation to describe one-loop
quantum physics. It has also implications for the form of the second variation on the considered backgrounds. In what follows we will only consider classical on-shell backgrounds. However, quantum perturbations do not have to satisfy classical EOM of the theory since QFT effects are typically off-shell.

The next step consists of writing the expression for the one-loop partition function in QWG on ES in terms of functional determinants of some differential operators considered on this background. We first start with writing the second variation of the conformal action of classical conformal Weyl gravity. It reads

\[ \delta^2 S_{\text{conf}} = \int d^4x \sqrt{|g|} h^{TT}_{\mu\nu} \left( \hat{\Box} - \frac{2}{3} \Lambda \hat{\Delta} + 2 \hat{C} \right) \left( \hat{\Box} - \frac{4}{3} \Lambda \hat{\Delta} + 2 \hat{C} \right) h^{TT}_{\mu\nu}, \]  

(34)

where we used York decomposition of fluctuations [25] and we commuted and integrated by parts covariant derivatives under the volume spacetime integral. We also exploited the definition of the Weyl tensor as a matrix \( \hat{C} \) whose action on the graviton fluctuations is the following

\[ \left( \hat{C} h \right)_{\mu\nu} = C_{\mu\alpha\nu\beta} h^{\alpha\beta}. \]  

(35)

Finally, we define the identity matrix \( \hat{1} \) in the space of rank-2 symmetric tensor fluctuations by the formula

\[ \left( \hat{1} \right)^{\mu\nu}_{\alpha\beta} = \delta^{\mu\nu}_{\alpha\beta} = \delta^{(\alpha}_{(\mu} \delta^{\nu)}_{\beta)} = \frac{1}{2} \left( \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} + \delta^{\mu}_{\beta} \delta^{\nu}_{\alpha} \right). \]  

(36)

The formula [34] was proven by an explicit computation where we also used various non-trivial identities coming from taking variational derivatives of the GB term in (28). One notices that for ES being on-shell background there is no participation of spurious gauge-redundant fields of transverse vectors \( \eta_{\mu} \) and two scalars \( \sigma \) and \( \bar{\phi} \). Moreover, one sees that in the kernel of the expression in (34) between traceless and transverse fluctuations \( h^{TT}_{\mu\nu} \) there is a matrix-differential operator, which has a direct factorization in a product of two single two-derivative operators (shifted by some constant factors proportional to \( \Lambda \) and also by some matrix of the Weyl tensor \( \hat{C} \)). It is remarkable that there is not a mixing term \( \Lambda \hat{C} \) which could be allowed by dimensional reasons, and moreover such a term separately would not be seen by taking a limit to MSS (\( \hat{C} \to 0 \)) or to Ricci-flat spacetime (\( \Lambda \to 0 \)). This lets the factorization of the kernel to be successful here. One can think additionally of this kernel as the simplest generalization of the kernels that appear separately in the expressions for the second variations on MSS and on Ricci-flat spacetimes. In the last case, the second variation gets simplified even to a quadratic form

\[ \delta^2 S_{\text{conf}} = \int d^4x \sqrt{|g|} h^{TT}_{\mu\nu} \left( \hat{\Box} + 2 \hat{C} \right)^2 h^{TT}_{\mu\nu}. \]  

(37)

One could naively derive the expression for the partition function from the result in (34). The result would be

\[ \tilde{Z}^2_{\text{1-loop}} = \frac{1}{\det_{2TT} \left( \hat{\Box} - \frac{4}{3} \Lambda \hat{\Delta} + 2 \hat{C} \right) \det_{2TT} \left( \hat{\Box} - \frac{4}{3} \Lambda \hat{\Delta} + 2 \hat{C} \right)}. \]  

(38)

But since in the theory we have local symmetries then the true expression for the one-loop partition function is more complicated. We have to take care of the Jacobian of the path integral variable transformation, of fixing all the local gauge symmetries, and also of adding Faddeev-Popov determinant (both for diffeomorphism and conformal part of the local symmetry group here in QWG). At the end, one has to take care of possible zero modes on ES and carefully exclude them from the computation of determinants of the differential operators here.
When one uses only physical and gauge-invariant degrees of freedom (only TT gravitons), then there is no need to additionally care about fixing all the local gauge symmetries, and also about the addition of Faddeev-Popov determinants. This is the advantage of using real degrees of freedom which do not come with any redundancy related to local symmetries. However, one still has to treat delicately the Jacobian of the change of integration variables and the issue of zero modes. The first problem of the functional change of the integration variables under field theory path integral from original unconstrained spin-2 field $h_{\mu\nu}$ to a set of constrained fields $(h_{TT}^{\mu\nu}, \eta_{\mu}^{\perp}, \sigma, \bar{\phi})$ is easily done even on a general spacetime background. For this issue, one can easily understand that the situation on a general ES is the same like on a general MSS with the same $\Lambda$ since the terms in the Jacobian depend only on contractions of the Riemann tensor, and not on the Riemann tensor itself. And one can understand from the formula (32) that as for Ricci tensors and Ricci scalars there is no any difference between MSS and ES. Therefore, following what was done in the analysis of Jacobian on MSS we can write

$$
\tilde{Z}^{2}_{1-\text{loop}} = \frac{\det_{1T} \left( \Box + \Lambda \hat{\Box} \right) \det_{0} \left( \Box + \frac{4}{3} \Lambda \hat{\Box} \right)}{\det_{2TT} \left( \Box - \frac{2}{3} \Lambda \hat{\Box} + 2 \hat{\Box} \right) \det_{2TT} \left( \Box - \frac{4}{3} \Lambda \hat{\Box} + 2 \hat{\Box} \right)}.
$$

(39)

The above expression is closer to the final one for the one-loop partition function on the general ES, but not yet the final one. We observe that the factors in the numerator originate only from the analysis on MSS background. However, they cannot be “corrected” by the addition of the matrix of the Weyl tensor $\hat{\Box}$ since this matrix does not makes sense between spin-1 or spin-0 fluctuations, which are used when we take determinants of the operators in the numerators. These determinants in the subspaces of spin-1 and spin-0 fluctuations respectively originate completely from the treatment of the Jacobian of the field variables transformation (to the TT gravitational fluctuations).

The last problem is related to the presence of zero modes of the differential operators which appear in (39) on a general ES background. In full generality, such a problem is very complicated to be tackled analytically. What we need actually, is a much more modest answer to the problem of rewriting the determinants of operators in the above formula. As it is well known zero modes appear mostly due to constraints on operators that they have to act only in the subspace of transverse fluctuations (both spin-1 and spin-2) since these last constraints are of the differential character. The constraint on the operator to act on traceless rank-2 tensor fluctuations $\hat{h}_{\mu\nu}$ does not create any problem for this since the last one is an algebraic condition. Therefore it is desirable to change the determinants of operators acting in subspaces of transverse fluctuations to determinants of operators acting in extended subspaces, where the condition of transversality is not imposed anymore. In our candidate expression for the one-loop partition function on the ES background, we have to change determinants in the space of spin-2 TT graviton fluctuations and for spin-1 transverse vector fluctuations.

In general, formulas relating determinants in subspace of transverse and extended fluctuations are derivable around any background. For this purpose one must consider the following expressions

$$
\tilde{h}^{\mu\nu} \Box \tilde{h}_{\mu\nu} \quad \text{and} \quad \eta^{\mu} \Box \eta_{\mu}
$$

(40)

under the volume spacetime integrals and expand them using the York decomposition for both traceless spin-2 fluctuations $\tilde{h}_{\mu\nu}$ and for general unconstrained spin-1 fluctuations $\eta_{\mu}$ separately. One has to commute derivatives, integrate them by parts, exploit various tracelessness and transversality conditions as stipulated in formulas (23) and (24). At the end one has to use various curvature relations valid on a general ES from (18) and (32).

One sees that the relation between determinants of operators for transverse and unconstrained spin-1 fluctuations does depend only on the Ricci tensor of the background spacetime. Hence in
this place we again find no difference between MSS and ES case. On MSS background spacetime, the final formula reads
\[
\det_{1T} \left( \Box + X \hat{1} \right) = \frac{\det_{1} \left( \Box + X \hat{1} \right)}{\det_{0} \left( \Box + X \hat{1} - \Lambda \hat{1} \right)}
\]  
(41)

and this formula allows to perform change from the subspace (1T) of transverse spin-1 fluctuations to the space of unconstrained spin-1 fluctuations. The correction term is proportional to the functional determinant of some scalar box operator, which receives also an additional shift by the term proportional to the \( \Lambda \) parameter (which is the same for both ES and MSS backgrounds).

The situation with the sector of spin-2 TT fluctuations is more complicated. The expressions for the expansion of \( \bar{h} \mu \nu \Box \bar{h} \mu \nu \) using the York decomposition on a generic ES background contains terms proportional to the powers of the \( \Lambda \) parameter as well as terms proportional to the matrix of Weyl tensor \( \hat{C} \). However, the terms from the last group come without derivatives, they are only of the endomorphic character without any remnant differential operator. Again, one does not note any mixing terms (proportional for example to \( \Lambda \hat{C} \)), which would be invisible in MSS or in Ricci-flat limits. This signifies that the contribution to the exclusion of zero modes for differential operators is done separately from MSS backgrounds (parametrized by the value of the \( \Lambda \) parameter) and from Ricci-flat backgrounds (parametrized by arbitrary Weyl tensor \( \hat{C} \)).

Since in the last case the resulting terms in \( \bar{h} \mu \nu \Box \bar{h} \mu \nu \) proportional to \( \hat{C} \) are not of the differential characters, then one concludes that there is no contribution that has to be subtracted from the determinants of the operators on Ricci-flat background due to zero modes. The same happens on general ES and one can safely forget about the contributions with Weyl tensor \( \hat{C} \) there, and only concentrate on the contribution that really have to be subtracted, namely those which are proportional to the \( \Lambda \) parameter of ES. For this last aspect the computation is identical to the general case of MSS. One ends up with the following formula
\[
\det_{2TT} \left( \Box + X \hat{1} \right) = \frac{\det_{2} \left( \Box + X \hat{1} \right)}{\det_{1TT} \left( \Box + X \hat{1} - \frac{2}{3} \Lambda \hat{1} \right) \det_{0} \left( \Box + X \hat{1} - \frac{8}{3} \Lambda \hat{1} \right)}
\]  
(42)

and this formula allows to perform change from the subspace (2TT) of traceless transverse spin-2 gravitational fluctuations to the space of traceless spin-2 fluctuations. There are two correction terms. They are proportional to the functional determinant of the differential operator acting on transverse spin-1 fluctuations and to the functional determinants of some scalar box operator, which receives also an additional shift by the term proportional to the \( \Lambda \) parameter on ES. In accordance with the previous remarks, one sees in general that such correction terms could not have a term with the matrix of the Weyl tensor \( \hat{C} \) because such matrix cannot act in the subspace of spin-1 or spin-0 fluctuations due to simple algebraic reasons (there is not a sufficient number of indices to contract with Weyl to produce an endomorphism of the vector or scalar bundle).

Finally, one gets the expressions for the one-loop partition around a general ES in the following form
\[
Z_{1-loop}^{2} = \frac{\det_{1T} \left( \Box + \Lambda \hat{1} \right) \det_{1} \left( \Box + \frac{1}{3} \Lambda \hat{1} \right) \det_{0} \left( \Box + \frac{4}{3} \Lambda \hat{1} \right)}{\det_{2T} \left( \Box - \frac{2}{3} \Lambda \hat{1} + 2 \hat{C} \right) \det_{2T} \left( \Box - \frac{2}{3} \Lambda \hat{1} + 2 \hat{C} \right) \det_{0} \left( \Box + 2 \Lambda \hat{1} \right)}
\]  
(43)

To get this expression we applied (41) and (42) to (39) and finally once again (41) to the intermediate result. In this form, this expression resembles very much the one obtained as the final one on MSS backgrounds. The only difference is the addition of two terms with the matrix of the Weyl tensor \( \hat{C} \) in the denominator, when this algebraically makes sense so only for spin-2
fluctuations. This is the final correct expression for the one-loop partition function of QWG theory on ES background. Its correctness was verified by performing the explicit computation of the $b_0$, $b_2$ and $b_4$ coefficients related to UV divergences of the QWG theory. For example, for the $b_4$ coefficient, they reproduce the famous results by Fradkin and Tseytlin of divergences at one-loop level proportional to $\Lambda^2$ and $C$ in one stroke.

One can study various limiting situations of the one-loop partition function we have just computed in (43). First, one can easily take the limit $\hat{C} \to 0$ to reduce to MSS case. This is only a small cosmetic modification since the terms with matrix of the Weyl tensor $\hat{C}$ will be gone, but the structure of the partition function $Z$ is exactly the same in MSS like in (43). The limit $\Lambda \to 0$ brings us back the situation on general Ricci-flat spacetime. Then one sees that the form gets reduced to the following

$$Z_{1-\text{loop},\text{Ricci-flat}}^2 = \frac{\det \det^3 \left( \Box \right)}{\det^2 \left( \Box + 2\hat{C} \right)}. \quad (44)$$

The last partition function can be extended to contain only fully unconstrained fields of fluctuations (in particular of spin-2 fluctuations) and then it reads

$$Z_{1-\text{loop},\text{Ricci-flat}}^2 = \frac{\det^3 \left( \Box \right) \det^2 \left( \Box \right)}{\det^2 \left( \Box + 2\hat{C} \right)}. \quad (45)$$

The similar operation of extension can be also performed on the level of the one-loop partition function on a general ES background. Namely, we can write

$$Z_{1-\text{loop}}^2 = \frac{\det^2 \left( \Box + \Lambda \hat{1} \right) \det \left( \Box + \frac{1}{3} \Lambda \hat{1} \right) \det \left( \Box - \frac{3}{2} \Lambda \hat{1} \right) \det \left( \Box - \frac{3}{2} \Lambda \hat{1} \right) \det \left( \Box + \frac{4}{3} \Lambda \hat{1} \right)}{\det^2 \left( \Box - \frac{4}{3} \Lambda \hat{1} + 2\hat{C} \right) \det^2 \left( \Box - \frac{4}{3} \Lambda \hat{1} + 2\hat{C} \right) \det^2 \left( \Box + 2\Lambda \hat{1} \right)}. \quad (46)$$

The two formulas in (43) and in (46) are completely equivalent due to the following extension formula

$$\det^2 \left( \Box + X \hat{1} + 2\hat{C} \right) = \frac{\det^2 \left( \Box + X \hat{1} + 2\hat{C} \right)}{\det^2 \left( \Box + X \hat{1} \right)}. \quad (47)$$

This last formula has nothing to do with zero modes and this extension of the operator to fully unconstrained spin-2 representation from the traceless one. In this form this is valid on any spacetime. One notices that due to algebraic reasons, the matrix of the Weyl tensor $\hat{C}$ cannot appear in the denominator of the expression above.

The next step in the computation is the application of the functional RG flow equation due to Wetterich [49, 50]. In a schematic form for a two-derivative theory defined by the following kinetic part of the action in arbitrary fluctuation field $\phi$ in whatever representation

$$S_{\text{kin}} = \int d^4x \sqrt{|g|} \phi \left( \Box + a \hat{1} \right) \phi$$

the Wetterich equation takes the form

$$k \partial_k \Gamma_k = \frac{1}{2} \text{Tr} \left( \frac{k \partial_k R_k \hat{1}}{\Box + R_k \hat{1} + a \hat{1}} \right). \quad (49)$$

where $\Gamma_k$ denotes the scale-dependent effective (average) action for all fields in the theory, $a$ is a general constant shift of the differential operator, and the cutoff kernel $R_k(z)$ is a function of the leading part in derivatives of the differential operator $z = \Box$ and also of the RG energy scale $k$. The IR-cutoff kernel $R_k(z)$ leads to a suppression of the contribution of modes with small eigenvalues of the covariant Laplacian operator $\Box$ (namely these for which we have $-\Box \ll k^2$ in momentum space representation), while the factor $k\partial_k R_k \hat{1}$ in the numerator of the Wetterich equation removes contributions from large eigenvalues $-\Box \gg k^2$. In this way, the loop integrals to be done as functional traces in Wetterich equation are both IR- and UV-finite. The functional traces in (49) are traces both in the spacetime integration sense as well as in the normal traces over all internal indices the fields may carry with themselves. In our case, this will be traces over Lorentz indices on metric fluctuations $h_{\mu\nu}$ (or on traceless fluctuations $\hat{h}_{\mu\nu}$) and on vector field fluctuations. Obviously, for scalars such internal traces are identical to multiplication by unity.

Following the developments of the Wetterich equation for the constrained fields and for the quantum theory read from the expression of its one-loop partition function understood as product of various functional determinants (raised to various powers) of simple two-derivative operators $\Box$ possibly shifted by some constant endomorphic terms $a\hat{1}$, the resulting reduced Wetterich equation looks as follows

$$k\partial_k \Gamma_k = \frac{1}{2} \sum_i \sum_j \pm \text{Tr} \phi_i \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box + R_k \hat{1} + a_{i,j} \hat{1}} \right),$$

(50)

where we have denoted the anomalous dimension (identical for all fluctuation fields) as $\eta$. Additionally, the $\pm$ signs depend on what was in the exponent on the corresponding term in the one-loop partition function (43) or (46). In other words, this decides whether the general factor of the type $\left( \frac{\Box + R_k \hat{1} + a_{i,j} \hat{1}}{\Box + a_{i,j} \hat{1}} \right)$ was originally in the denominator or the numerator of the partition function, respectively and with which power. The functional traces were separately considered in each subspace of fluctuation fields $\phi_i$, and in each subsector of fluctuations we allowed for different factors of the type $\left(\frac{\Box + R_k \hat{1} + a_{i,j} \hat{1}}{\Box + a_{i,j} \hat{1}}\right)$ with different shifts $a_{i,j}$ (the index $j$ here is internal to each subsector of fluctuations, while the index $i$ counts different subspaces with different representation of fields $\phi_i$). The correctness of the above steps that lead to the above FRG flow equation in a reduced form (50) is, for example, verified by an independent check of the coefficients for one-loop perturbative beta functions of dimensionless coupling parameters in QWG.

One should write the Wetterich equation in a reduced form in an explicit way on a general ES background using the two form of the one-loop partition function. Here are the results of these expansions. From formula in (43) one gets

$$k\partial_k \Gamma_k = \frac{1}{2} \text{Tr}_{2T} \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box - \frac{2}{3}\Lambda 1 + 2C} \right) + \frac{1}{2} \text{Tr}_{2T} \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box - \frac{2}{3}\Lambda 1 + 2C} \right) + \frac{1}{2} \text{Tr}_0 \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box + 2\Lambda 1} \right) -$$

$$- \text{Tr}_1 \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box + \Lambda 1} \right) - \frac{1}{2} \text{Tr}_1 \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box + \frac{1}{3}\Lambda 1} \right) - \frac{1}{2} \text{Tr}_0 \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box + \frac{4}{3}\Lambda 1} \right),$$

(51)

while when we use the alternative (but equivalent) expression from (46) one finds instead a slightly more complicated formula which reads

$$k\partial_k \Gamma_k = \frac{1}{2} \text{Tr}_{2T} \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box - \frac{2}{3}\Lambda 1 + 2C} \right) + \frac{1}{2} \text{Tr}_{2T} \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box - \frac{2}{3}\Lambda 1 + 2C} \right) + \frac{1}{2} \text{Tr}_0 \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box + 2\Lambda 1} \right) -$$

$$- \text{Tr}_1 \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box + \Lambda 1} \right) - \frac{1}{2} \text{Tr}_1 \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box + \frac{1}{3}\Lambda 1} \right) - \frac{1}{2} \text{Tr}_0 \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box + \frac{4}{3}\Lambda 1} \right),$$

(52)
\[ -\text{Tr}_1 \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box + \Lambda} \right) - \frac{1}{2} \text{Tr}_1 \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box + \frac{1}{3} \Lambda} \right) - \frac{1}{2} \text{Tr}_0 \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box + \frac{1}{3} \Lambda} \right) - \frac{1}{2} \text{Tr}_0 \left( \frac{(k\partial_k R_k - \eta R_k) \hat{1}}{\Box - \frac{2}{3} \Lambda} \right) \]

(52)

In the next step, we need to write and simplify the form of the beta functional (LHS of the Wetterich equation) on a general ES background. This is of course, related to the choice of the action ansatz for the running effective action \( \Gamma_k \) and to our abilities to read various beta functions from it.

In general, we are interested only in running of dimensionless couplings since their RG flow appears first when the conformal symmetry is broken softly and spontaneously. In our 4-dimensional setup such couplings appear in front of three quadratic invariants in curvatures which we defined above and which are consistent with diffeomorphism symmetry of the quantum theory (since conformal symmetry is already softly violated by the mere presence of RG flow for dimensionless couplings). Therefore the general beta functional of the theory may have the form

\[ k\partial_k \Gamma_k = \int d^4x \sqrt{|g|} \left( \beta_R R^2 + \beta_C C^2 + \beta_{\text{GB}} \text{GB} \right) \]

(53)

In the above formula we defined that the general running effective action had the form

\[ \Gamma_k = \int d^4x \sqrt{|g|} \left( \alpha_R(k)R^2 + \alpha_C(k)C^2 + \alpha_{\text{GB}}(k) \text{GB} \right) \]

(54)

and that the beta functions of the running couplings parameter \( \alpha_i = \alpha_i(k) \) are defined as logarithmic derivative of them with respect to RG scale \( k \) that is the following formula holds

\[ \beta_i = k \frac{d}{dk} \alpha_i(k) \]

(55)

We notice that as it is common in some FRG approaches the \( k \)-dependence for the one-loop RG-improved effective action \( \Gamma_k \) sits only in the running couplings, while the diffeomorphically invariant terms there are not touched at all by the derivative with respect to RG scale \( k \) needed when defining non-perturbatively the beta function\( \beta_i \). Moreover, this is not the end of our choice of the general ansatz on ES. It is well known that perturbatively in QWG the beta function of the Starobinsky \( R^2 \) term is highly suppressed compared to other beta functions \( \beta_C \) and \( \beta_{\text{GB}} \) (that is we have parametrically that \( \beta_R \ll \beta_C, \beta_{\text{GB}} \)). This hierarchy of the system of beta functions is already seen on the level of first loop. There holds that \( \beta_R = 0 \) exactly to one-loop level. If one considers higher loops then the expression for \( \beta_R \) is smaller because it contains higher powers of the perturbative expansion parameter which for us here is assumed to be small. These parameters are the couplings of the original theory. Following this important observation we may neglect in our computation the presence of the \( \beta_R \) function both on the perturbative and also on non-perturbative levels. Therefore our reduced beta functional of the theory describing running effective action \( \Gamma_k \) has the form

\[ k\partial_k \Gamma_k = \int d^4x \sqrt{|g|} \left( \beta_C C^2 + \beta_{\text{GB}} \text{GB} \right) \]

(56)

derived from

\[ \Gamma_k = \int d^4x \sqrt{|g|} \left( \alpha_C(k)C^2 + \alpha_{\text{GB}}(k) \text{GB} \right) \]

(57)
while we unanimously agree that on the RHS of the Wetterich flow equation we have the partition function coming exactly from the QWG action functional described by the truncation ansatz $\alpha_C C^2$ in agreement with (9). On the RHS of the FRG flow equation the full $k$-dependence is brought about by IR cutoff kernels $R_k$ and we decide not to include any additional $k$-dependence in the Weyl coupling $\alpha_C$ on this RHS (a source side which is a reason for all effects of RG running of the LHS of the RG flow equation).

The beta functional (56) evaluated on the generic ES background gives

$$\beta_C C^2 + \beta_{GB} GB\big|_{ES} = \beta_C C^2 + \beta_{GB} \left(C^2 + \frac{8}{3} \Lambda^2\right) = (\beta_C + \beta_{GB}) C^2 + \frac{8}{3} \beta_{GB} \Lambda^2$$

(58)

Therefore on a general ES manifold we can read in principle independently two beta functions $\beta_C$ and $\beta_{GB}$. This is possible due to an extraction of terms proportional to the invariant quantities $C^2$ and $\Lambda^2$ respectively from the RHS of the FRG flow equation in (49). Again, we emphasize that the advantage of using ES as background spaces is that we read these two beta functions from one expression evaluated on one single background, hence there is no any issue related to changing the background and to mutual interrelations and interdependences between them. Of course, such a background should be understood only as a theoretical tool and a trick which is very helpful in extracting the beta functions of the quantum theory. One should not think that our theory has to be considered always on ES background. For consistency of the quantum theory the QWG in the form we present here can be considered on general Bach-flat backgrounds.

In order to proceed with the quantum computation of beta functions in the one-loop RG improved scheme due to non-perturbative FRG phenomena, one has to make a choice for the explicit cutoff kernel function $R_k(z)$. We follow the standard choice given by the Litim cutoff

$$R_k = R_k(z) = (k^2 - z) \theta (k^2 - z),$$

(59)

where $\theta(x)$ denotes standard Heaviside theta step function. For such a choice of the cutoff kernel we find that the functional traces (the spacetime integration part of them) of some general function of the general at most two-derivative operator $\hat{O}$ is given by

$$\text{Tr} f(\hat{O}) = B_4 \left(\hat{O}\right) Q_0[f],$$

(60)

where the $B_4$ coefficients of any differential operator are well known in QFT and in the theory of operators. The $Q_0$ functional of the function $f$ can be simplified in our case of interest, if we recall what is the generic form of the function of the operator $f = f(z)$ that we usually meet in our consideration related to the Wetterich equation. This form reads

$$f(z) = \frac{k \partial_k R_k(z)}{z + R_k(z) + a_{i,j}}$$

(61)

for a generic two-derivative factor of the type $\left(\hat{\Box} + R_k \hat{1} + a_{i,j} \hat{1}\right)$. In our case we also accept the identification that $\hat{O} = \hat{\Box}$ as the leading in the number of derivatives part of the operator $\hat{\Box} + a_{i,j} \hat{1}$. Then in such circumstances the expression for $Q_0$ functional simplifies to

$$Q_0[f] = Q_0 \left[\frac{k \partial_k R_k(z)}{z + R_k(z) + a_{i,j}}\right] = (2 - \eta) \left(1 + \frac{a_{i,j}}{k^2}\right)^{-1}.$$  

(62)

Using such a form of IR-decoupling we are sure that we correctly take care of the threshold effects related to massive modes of the excitations present around a given background. This decoupling is explicitly realized by the second factor in the above formula, while the first factor describes
the corrections due to the inclusion of the anomalous dimension for the quantum fields. Here $\eta$ denotes the uniform anomalous dimension for the graviton field, since all $T T$ graviton fields, vector fields and scalars must have it the same (due to the requirements of original conformal symmetry of the initial theory and due to always unbroken diffeomorphism symmetry).

With the formulas (60) and (62) at hand, one can easily compute the ensuing functional traces in the FRG flow equation. First, we write it for the compact form of the one-loop partition function on a general ES given in (51). Then the flow simplifies to

$$
\frac{2}{2 - \eta} k \partial_k \Gamma_k = \left( 1 - \frac{2 A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_{2T} - \frac{2}{3} \Lambda \hat{1}_{2T} + 2 \hat{C} \right) + \left( 1 - \frac{4 A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_{2T} - \frac{4}{3} \Lambda \hat{1}_{2T} + 2 \hat{C} \right) +
$$

$$
\left( 1 + \frac{2 A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_0 + 2 \Lambda \hat{1}_0 \right) - \left( 1 + \frac{4 A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_0 + \frac{4}{3} \Lambda \hat{1}_0 \right) -
$$

$$
2 \left( 1 + \frac{A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_1 + \Lambda \hat{1}_0 \right) - \left( 1 + \frac{4 A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_1 + \frac{4}{3} \Lambda \hat{1}_0 \right). \quad (63)
$$

In the case of the alternative equation for the one-loop partition function given in (62) we instead find

$$
\frac{2}{2 - \eta} k \partial_k \Gamma_k = \left( 1 - \frac{3 A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_2 - \frac{2}{3} \Lambda \hat{1}_2 + 2 \hat{C} \right) + \left( 1 - \frac{4 A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_2 - \frac{4}{3} \Lambda \hat{1}_2 + 2 \hat{C} \right) +
$$

$$
\left( 1 + \frac{2 A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_0 + 2 \Lambda \hat{1}_0 \right) - \left( 1 + \frac{4 A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_0 + \frac{4}{3} \Lambda \hat{1}_0 \right) -
$$

$$
\left( 1 - \frac{4 A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_0 - \frac{2}{3} \Lambda \hat{1}_0 \right) - \left( 1 - \frac{4 A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_0 - \frac{4}{3} \Lambda \hat{1}_0 \right) -
$$

$$
2 \left( 1 + \frac{4 A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_1 + \Lambda \hat{1}_0 \right) - \left( 1 + \frac{4 A}{k^2} \right)^{-1} B_4 \left( \tilde{\square}_1 + \frac{4}{3} \Lambda \hat{1}_0 \right). \quad (64)
$$

However, already on this level one sees that these two forms of the FRG flows are equivalent because of the following formula

$$
B_4 \left( \tilde{\square}_{2T} - X \hat{1}_{2T} + 2 \hat{C} \right) = B_4 \left( \tilde{\square}_2 - X \hat{1}_2 + 2 \hat{C} \right) - B_4 \left( \tilde{\square}_0 - X \hat{1}_0 \right) \quad (65)
$$

for any constant $X$. Therefore for future considerations, we will consider only the explicit RG flow given by the formula (63). In the formulas above one explicitly sees the presence of massive threshold functions of the general type

$$
\left( 1 + \frac{a_{i,j}}{k^2} \right)^{-1}. \quad (66)
$$

As we will see in a moment they are very important for the consideration of the RG flows towards the deep IR regime in QWG. To eventually simplify the expression for the RG flow and read the non-perturbative expressions for the beta functions with the threshold phenomena included, one needs to evaluate the $B_4$ coefficients related to perturbative UV-divergences of the theory. Precisely they are related to logarithmic dimensionless UV-divergences in front of the terms proportional to $C^2$ and GB term in the quantum theory.
Explicit computations on the general ES give us the results for these $B_4$ coefficients of various operators

$$
B_4 \left( \Box_{2T} - \frac{2}{3} \Lambda 1_{2T} + 2 \hat{C} \right) = \frac{21}{20} C^2 - \frac{7}{5} \Lambda^2
$$

(67)

$$
B_4 \left( \Box_{2T} - \frac{4}{3} \Lambda 1_{2T} + 2 \hat{C} \right) = \frac{21}{20} C^2 + \frac{3}{5} \Lambda^2
$$

(68)

$$
B_4 \left( \Box_1 + \Lambda \hat{1}_0 \right) = -\frac{11}{180} C^2 + \frac{716}{135} \Lambda^2
$$

(69)

$$
B_4 \left( \Box_1 + \frac{1}{3} \Lambda \hat{1}_0 \right) = -\frac{11}{180} C^2 + \frac{236}{135} \Lambda^2
$$

(70)

$$
B_4 \left( \Box_0 + 2 \Lambda \hat{1}_0 \right) = \frac{1}{180} C^2 + \frac{479}{135} \Lambda^2
$$

(71)

$$
B_4 \left( \Box_0 + \frac{4}{3} \Lambda \hat{1}_0 \right) = \frac{1}{180} C^2 + \frac{269}{135} \Lambda^2
$$

(72)

We conveniently above wrote these formulas in terms of $C^2$ and $\Lambda^2$ invariant to be prepared for the extraction of two beta functions $\beta_C$ and $\beta_{GB}$. Once again one notices that in these formulas there are no interference terms between $\Lambda$ and matrix $\hat{C}$ terms. Of course, they are not possible due to algebraic reasons here, because the final expression for the $B_4$ coefficient must be a scalar and the matrix $\hat{C}$ is completely traceless. On the other hand this result signifies that the contributions from the MSS part and from the Ricci-flat part to the general ES are independent of each other and there is no any mixing between them and also that they can be computed separately.

Now, we read the beta functions exploiting together formulas (63), (56), (58) and a set of formulas from (67) to (72). We therefore find that the following equation holds

$$
\frac{2}{2 - \eta} \left( \beta_C + \beta_{GB} \right) C^2 + \frac{8}{3} \beta_{GB} \Lambda^2 = \left( 1 - \frac{2 \Lambda}{k^2} \right)^{-1} \left( \frac{21}{20} C^2 - \frac{7}{5} \Lambda^2 \right) + \left( 1 - \frac{4 \Lambda}{k^2} \right)^{-1} \left( \frac{21}{20} C^2 + \frac{3}{5} \Lambda^2 \right) + \left( 1 + \frac{2 \Lambda}{k^2} \right)^{-1} \left( \frac{1}{180} C^2 + \frac{479}{135} \Lambda^2 \right) - \left( 1 + \frac{4 \Lambda}{k^2} \right)^{-1} \left( \frac{1}{180} C^2 + \frac{269}{135} \Lambda^2 \right) - 2 \left( 1 + \Lambda \right)^{-1} \left( -\frac{11}{180} C^2 + \frac{716}{135} \Lambda^2 \right) - \left( 1 + \frac{2 \Lambda}{k^2} \right)^{-1} \left( -\frac{11}{180} C^2 + \frac{236}{135} \Lambda^2 \right). \tag{73}
$$

By comparing both sides of this equation and treating the two invariants $C^2$ and $\Lambda^2$ as exclusive and mutually independent, we arrive to the two equations

$$
\frac{2}{2 - \eta} \left( \frac{8}{3} \beta_{GB} \Lambda^2 \right) = \left( 1 - \frac{2 \Lambda}{k^2} \right)^{-1} \left( -\frac{7}{5} \Lambda^2 \right) + \left( 1 - \frac{4 \Lambda}{k^2} \right)^{-1} \left( \frac{3}{5} \Lambda^2 \right) + \left( 1 + \frac{2 \Lambda}{k^2} \right)^{-1} \left( \frac{479}{135} \Lambda^2 \right) - \left( 1 + \frac{4 \Lambda}{k^2} \right)^{-1} \left( \frac{269}{135} \Lambda^2 \right) - 2 \left( 1 + \Lambda \right)^{-1} \left( \frac{716}{135} \Lambda^2 \right) - \left( 1 + \frac{2 \Lambda}{k^2} \right)^{-1} \left( \frac{236}{135} \Lambda^2 \right) \tag{74}
$$
\[
\frac{2}{2 - \eta} (\beta_C + \beta_{GB}) C^2 = 2 \times \left( \frac{21}{20} C^2 \right) + \left( \frac{1}{180} C^2 \right) - \left( \frac{1}{180} C^2 \right) - 3 \times \left( \frac{-11}{180} C^2 \right).
\]

(75)

These two equations represent respectively the RG flows as considered separately on MSS and Ricci-flat backgrounds. In order to obtain the equation in (75) we also took a zeroth order series coefficient in expansion in the \(\Lambda\) parameter, since this parameter is non-vanishing only on MSS. In this way we prove that the results as obtained on a general ES completely reproduce the previous results obtained separately on MSS and Ricci-flat backgrounds. This also confirms that the original method with using of two distinct backgrounds is consistent and does not lead to any contradiction. These two backgrounds are mutually exclusive since the MSS can be Ricci-flat only if this is a flat spacetime, however this observation regarding the physical background does not lead to any inconsistency here. We used these two backgrounds as a trick to get two combinations of beta functions but none should understand that we consider a QWG theory put simultaneously on these two backgrounds. The results we get there are consistent with each other. Moreover, they are also consistent with the general considerations that we make on ES, when the background used again as a mathematical trick is a single one. However, we do not find any problem in taking respective limits \(\Lambda \to 0\) or \(\hat{\Lambda} \to 0\) and we recover previous results from Ricci-flat and MSS respectively. Moreover, there are no mixing terms \(\Lambda \hat{C}\) which would invalidate the independent analysis of MSS and Ricci-flat to get beta functions \(\beta_C\) and \(\beta_{GB}\) separately. This verifies a general philosophy that the quantum beta functions of the theory (in a given scheme) are universal and one can use arbitrary backgrounds to evaluate them. Before we used MSS and Ricci-flat because of the reasons of convenience. Now, we have proven that the same consistent results are also obtained on the general ES.

By doing algebraic simplifications in the formula one finally arrives to the system

\[
\frac{2}{2 - \eta} \beta_{GB} \Lambda^2 = \left( 1 - \frac{2\Lambda}{3k^2} \right)^{-1} \left( -\frac{21}{40} \Lambda^2 \right) + \left( 1 - \frac{4\Lambda}{k^2} \right)^{-1} \left( \frac{9}{40} \Lambda^2 \right) + \left( 1 + \frac{2\Lambda}{k^2} \right)^{-1} \left( \frac{479}{360} \Lambda^2 \right)
\]

\[
- \left( 1 + \frac{4\Lambda}{k^2} \right)^{-1} \left( \frac{269}{360} \Lambda^2 \right) - \left( 1 + \frac{\Lambda}{k^2} \right)^{-1} \left( \frac{179}{45} \Lambda^2 \right) - \left( 1 + \frac{1}{3} \Lambda \right)^{-1} \left( \frac{59}{90} \Lambda^2 \right)
\]

(76)

and

\[
\frac{2}{2 - \eta} (\beta_C + \beta_{GB}) C^2 = \frac{137}{60} C^2.
\]

(77)

From this system one solves for the beta functions \(\beta_{GB}\) and \(\beta_C\) in the following way

\[
\beta_{GB} = \frac{1}{2} (2 - \eta) \left[ -\frac{21}{40} \left( 1 - \frac{2\Lambda}{k^2} \right)^{-1} + \frac{9}{40} \left( 1 - \frac{4\Lambda}{k^2} \right)^{-1} + \frac{479}{360} \left( 1 + \frac{2\Lambda}{k^2} \right)^{-1} - \frac{269}{360} \left( 1 + \frac{4\Lambda}{k^2} \right)^{-1} - \frac{179}{45} \left( 1 + \frac{\Lambda}{k^2} \right)^{-1} - \frac{59}{90} \left( 1 + \frac{1}{3} \Lambda \right)^{-1} \right]
\]

(78)

and

\[
\beta_C = \frac{1}{2} (2 - \eta) \left[ \frac{137}{60} + \frac{21}{40} \left( 1 - \frac{2\Lambda}{k^2} \right)^{-1} - \frac{9}{40} \left( 1 - \frac{4\Lambda}{k^2} \right)^{-1} - \frac{479}{360} \left( 1 + \frac{2\Lambda}{k^2} \right)^{-1} + \right.
\]

\[
- \frac{179}{45} \left( 1 + \frac{\Lambda}{k^2} \right)^{-1} - \frac{59}{90} \left( 1 + \frac{1}{3} \Lambda \right)^{-1} \right]
\]

(79)
\begin{align}
\frac{269}{360} \left( 1 + \frac{4\Lambda}{k^2} \right)^{-1} + \frac{179}{45} \left( 1 + \frac{\Lambda}{k^2} \right)^{-1} + \frac{59}{90} \left( 1 + \frac{4\Lambda}{k^2} \right)^{-1} \right]
\end{align}

(79)

Now, everything depends on the expression for the anomalous dimension \( \eta \) of the TT graviton field. Using its expression motivated by perturbative one-loop considerations, we get that

\[ \eta = -\frac{1}{\omega_C} \frac{d}{dk} \omega_C(k) \]

(80)

where the coupling \( \omega_C \) is related to the original Weyl coupling by the inverse relation (i.e. \( \omega_C = \alpha_C^{-1} \)). This extension to include a non-trivial anomalous dimension is here for free since it does not change anything related to the structure of the two beta functions that we have obtained.

To check the consistency of these results for the beta functions \( \beta_C \) and \( \beta_{GB} \) one can obtain their asymptotic values in the UV asymptotic regimes of energies. This means that we should take the limit \( k \to +\infty \) together with \( \eta \to 0 \) since this anomalous dimension is a quantum effect of higher order compared to the perturbative one-loop running that the true running asymptotes to in the deep UV regime. These limits result in the following expressions

\[ \beta_{GB} = -\frac{87}{20} \]

(81)

\[ \beta_C + \beta_{GB} = \frac{137}{60} \]

(82)

which are solved by

\[ \beta_C = \frac{199}{30} \quad \text{and} \quad \beta_{GB} = -\frac{87}{20}. \]

(83)

These two results perfectly agree with the one-loop perturbative results as obtained earlier by Fradkin and Tseytlin.

Having obtained the results for the non-perturbatively improved beta functions with the inclusion of the threshold effects of massive modes and of anomalous dimension of the graviton \( \eta \), we can now discuss the implications for the RG flows and its topology in QWG. The situation in the UV regime is quite trivial and boring since in QWG both couplings reach a trivial UV FP of RG. The coupling \( \beta_{GB} \) is induced by quantum corrections to the action of QWG theory in (9), but as we have seen in (29) it non-zero value does not lead to any violation of conformal symmetry even in the local version. This means that the perturbative couplings that should be defined as \( \left( \sqrt{\omega_C} \right)^{-1} = \sqrt{\alpha_C} \) and \( \left( \sqrt{-\omega_{GB}} \right)^{-1} = \sqrt{-\alpha_{GB}} \) reach an asymptotically free FP in the UV, when the FP values of these couplings both vanish. (The coupling of the Gauss-Bonnet term is negative since its beta function \( \beta_{GB} < 0 \) is also negative and this is why we took the definitions with the change of the sign under the square roots.) We conclude that in the UV the quantum theory of Weyl gravity meets a Gaussian FP with trivial scaling of couplings and operators. This behavior is in agreement with that in the UV regime the QWG theory should meet a conformally invariant FP where this model is probably merged to a broader theory such as CSG in which the UV FP with fully realized quantum conformality is present. In this UV-complete theory there is no running and the value of the Weyl coupling can be arbitrary (even non-perturbatively big), but still with any value of \( \alpha_C \) the conformal symmetry is explicitly present and not violated by quantum effects.

Instead, the situation towards the IR regime of the energy spectrum is more interesting. One looks in that regime for non-trivial new FP of RG flows. They are defined by the simultaneous condition that \( \beta_C = 0 \) and \( \beta_{GB} = 0 \). One can see that for this system of equations the presence of the anomalous dimension \( \eta \) in the expressions for the beta functions (78) and (79) is completely
spurious and this higher order quantum effect can be safely neglected here. The results we have found are really curious since for some finite value of the energy (in the IR regime) we see with a very good precision of 2% a non-trivial point where the RG flow for the two beta functions $\beta_C$ and $\beta_{GB}$ stops completely for a moment (of energies, or a while in RG time $t = \log \frac{k}{k_0}$). This confluence of two FP’s for respectively $\alpha_C$ and $\alpha_{GB}$ couplings is a very stunning feature that is present only in QWG. Other higher derivative quadratic gravities do not enjoy such a behavior in the IR regime. When one uses a rescaled dimensionless energy RG scale variable, here conveniently defined as

$$\kappa = \frac{k}{\sqrt{|\Lambda|}},$$

then one can numerically determine the location of these FP’s for both of the couplings separately. First, in the case of $\Lambda > 0$ we have a very good agreement and the FP for the $\alpha_C$ coupling is found around $\kappa_C \approx 1.17709$ and for the $\alpha_{GB}$ coupling is found around $\kappa_{GB} \approx 1.19163$. For the case of $\Lambda < 0$ we still find a good agreement and the particular values for energy locations of FP’s are $\kappa_C \approx 1.49722$ and $\kappa_{GB} \approx 1.52128$ for the couplings $\alpha_C$ and $\alpha_{GB}$ respectively.

Now, we provide some interpretation to this special point (happening almost at the same energy scales for both couplings) at some finite energy scale. It cannot be a true IR FP of RG flow since this happens at finite energy scale, so the flow may momentarily stop, turn back but generally must continue, when we decrease the value of the RG time $t$ more towards the deeper IR regime. We decided to call such a special point of RG flow as turning point (TP) in distinction to a true IR FP. One can plot the beta functions of the two couplings analyzed here. One sees that the curves for RG beta functions cross zero line almost at the same energy scale. Moreover, one expects that by the inclusion of higher order phenomena (like by going to the two-loop level) one will improve the accuracy of matching such that in a truly non-perturbative theory the two curves cross zero exactly at the same RG time. We can call this position as a TP. The estimates that we have made above suggest that its value is around 1.2 and definitely it is for values bigger than 1. We emphasize once again that this TP is not a true FP of RG that still has to be searched for in the deeper IR regime. Instead, for this turning point there exists a holographic interpretation that we describe in short below.

When one wants to describe the IR regime, one leaves the turning point and continue the RG flow still towards the IR regime. Here we can employ the perturbation calculus in value of the couplings that we met at the TP. We remind that only the beta functions of couplings vanish there, while this is not the case for the coupling themselves. We denote a general coupling there by $\omega_s$ at the location of TP. We therefore later perform perturbations in the small $\omega_s$ parameter treating the TP as a starting point for this perturbation and assuming that $\omega_s \ll 1$. The consequences of this are the following. We found a non-trivial IR FP at $k = 0$ so in the true IR regime. This IR FP of RG is completely stable in a sense that the dimensions of two operators $C^2$ and GB near it are finite negative and bounded. Their actual values are equal to each other and given by $\theta = -\frac{1}{3}$. This implies that they are relevant operators near this IR FP. They describe the RG flow of the two originally dimensionless couplings $\alpha_C$ and $\alpha_{GB}$ of the theory. One sees that thanks to the quantum effects the operator and couplings acquire some anomalous but controllable dimensions and in the IR regime they are not anymore dimensionless (and the corresponding operators $C^2$ and GB are not of energy dimension 4 in $d = 4$). This means that the perturbation by addition of these two operators to the CFT of IR FP of the theory is completely stable and does not destabilize the physics which in the deep IR regime is controlled and dictated by IR FP of the theory. Moreover, one finds the following relation between the couplings $\omega_s$ defined at the TP and the ensuing couplings $\omega_{ss}$ at the true IR FP,

$$\omega_{ss} = \omega_s + \frac{9}{2} \kappa \beta'_{\omega}(\kappa),$$

(85)
where $\kappa$ is the schematic notation for the location of the TP, and also the derivatives of the beta functions of the corresponding couplings are evaluated at the TP location. This formula is valid for both types of couplings $\omega_C$ and $\omega_{GB}$. Additionally, one finds that the couplings $\omega_1$ are not constrained at all and the couplings $\omega_{**}$ are only forced to stay in relation (85), where the couplings $\omega_1$ are completely arbitrary. This means that also resulting true IR FP values of the couplings $\omega_{**}$ can be both in the perturbative as well as in some other non-perturbative regimes.

6. Interpretation and further discussion of the results

Firstly, we would like to discuss the possible interpretation of our findings about the TP of RG flow that occurs at some finite location in energy scales (around $\kappa = 1.2$, so for $k = 1.2\sqrt{\Lambda}$ for $\Lambda > 0$). The whole fact that this is a turning point of RG corresponds to the existence of a multi-branch holographic RG flow. A possibly correct analysis of such an IR behavior is provided by the AdS/CFT correspondence, which describes the geometrized understanding of RG flows in various QFT’s (see [51] and references within). The geometrization comes by promoting RG flows to some gravitational spacetime (characterized by some metric tensor) in some higher dimensional spacetimes which resembles the deformations of exact AdS spacetime. In our case, this extended completely artificial spacetime is 5-dimensional and has nothing to do with real physical gravitational spacetime considered as exact or background solution in CWG. The geometrization of the RG flow is only a useful theoretical tool to visualize various properties of the otherwise complicated RG flows in quantum field theories [52]. As the name suggests only in the case of FP of RG (when we know that they are described by some CFT’s) we are sure that the resulting gravitationally dual 5-dimensional bulk is exactly identical to some patch of the anti-de Sitter spacetime. The departure from the FP is encoded on the dual side of the theory by some radial geometry that differs from the AdS. Therefore in the case of the RG running of couplings we need to speak about the modified AdS spacetime used for geometrization of RG flow. This bulk spacetime can be also understood as a gravitational dual in 5-dimension to the field theory which was considered in $d = 4$ dimensions of real physical spacetime. When the running takes place only as a function of one variable, like in our case, where $k$ is its argument, then in the 5-dimensional bulk we can use a metric of 5-dimensional spacetime which has a very special metric structure. Namely this could be a conformally flat metric (in a 5-dimensional sense) with only a non-trivial scale factor and in this way this analogue of the FLRW spacetime used very frequently in description of physical 4-dimensional cosmology. This is the framework and the metric ansatz with which we will try to describe the geometrization of the interesting RG flow phenomena that we have found towards the IR regime in QWG. In particular, the situation with TP of RG is well described by bounce cosmological solutions.

One recalls that this TP happens at some finite energy scales $\kappa$. They in general, correspond to some finite radial locations in the asymptotically AdS spacetime, where we geometrize the RG flow. Of course, due to the fact that both in the UV regime as well as in the IR we find a FP of RG in QWG, the gravitational dual spacetime must be asymptotically (that is for zero and infinite values of the radial locations) approaching some 5-dimensional AdS types of spacetimes. Precisely, in the dual gravitational spacetime, the IR FP corresponds to infinite radial location $\rho \to +\infty$ and UV FP is in turn located at the origin of the AdS spacetime, that is for $\rho \to 0$. In general, there exists a relation between the energy scale in RG flow $k$ and the radial coordinate $\rho$ of the dual AdS, which reads $\rho \sim k^{-1}$, which is true up to a constant of proportionality. The particular spacelike location of the TP of RG flow is dual to 4-dimensional surface (so co-dimension 1 in 5-dimensional bulk spacetime) embedded in AdS-like 5-dimensional geometry, which is located at particular constant value of the radial coordinate $\rho = \text{const}$ of the AdS-like geometry. From this setup one easily understands that the true IR FP of RG flow must corresponds to the conformal boundary of AdS spacetime which is asymptotically located at infinite values of the radial coordinate $\rho$. Therefore the TP cannot be a true IR FP as we
have also found earlier.

One can consider the gravitational cosmological spacetimes describing the bounce situation. A particular useful example is the case of a FLRW spacetime with a specific form of the scale factor $a(t)$. In cosmology if the scale factor is not an injective function but a continuous function of the cosmological time $t$, then there must exist a bouncing point. This moment of the cosmological time corresponds to the physical bounce like in the motion of a material point in the given gravitational potential, when it bounces from the floor. This means that the scale factor just after the bounce grows in time and just before it must decrease in time. Such a gravitational spacetime is not very common in cosmology since a bounce cannot be caused by the presence of normal matter (satisfying the energy condition). The gravitational source for the bounce must be an exotic matter. Typically if the bounce has to take effect just for a very short period of cosmological time $t$, then this non-physical source of matter must be localized and put on the 4-dimensional brane located at some specific location in radial coordinate $\rho = \text{const}$. In different vein, one can see this gravitational spacetime as solution in Einsteinian gravity, where the energy matter source is solely determined due to the presence of some bulk scalar field with some non-trivial spacetime profile. The fact that the bounce is sourced by exotic matter corresponds to the fact that the scalar field must come with a negative kinetic term (so then this is opposite to the standard scalar field and the scalar field is of the phantom character). In particular, one can take the scalar profile to be only radially dependent, while in other transverse dimensions it can be completely translationally invariant. Then with such a bulk scalar field the bounce situation can be easily realized. Actually by Friedmann equations of the ensuing cosmological model, the profile of the scalar field is related to the beta function in the RG flow. The TP of the RG flow is the situation when the beta function from both sides of the running stops. This is a non-analytic behavior in coupling when it is written as a function of the running energy scale $k$.

Such a behavior happens of course because the TP is at some finite energy scale. The profile of the dual scalar field in the 5-dimensional bulk spacetime must in such circumstances also show some singularity. The detailed analysis shows that in the generic case (when $\beta'(\kappa)$ is non-zero so the zero of the beta function is a single, not multiple zero) the singularity of the scalar field profile must be of the type of a square-root like singularity $\sqrt{x}$. And obviously, such a behavior is non-analytic in the energy scale $\kappa$.

The introduction of the scalar field is one of the way to interpret geometrically the RG flow which exhibits the TP in the IR regime, but for some finite energy scale. The non-analytic behavior in the solution for the profile of the bulk scalar field can be analyzed also from a different point of view. The TP is a critical point of the RG flow and it corresponds to the joining (branching point) of the two real solutions for the profile. This is seen clearly from the type of the singularity which is here $\sqrt{x}$, so it describes two real solutions which are exactly non-analytic at the point of joining. This is interpreted as a very interesting situation for the bulk scalar field, when the profile is not analytic function of the radial coordinate $\rho$ of the AdS-like spacetime. The TP describes then precisely a bifurcation point in the theory of scalar field on this spacetime, where the two real solutions are joined. Each of the two real-valued solutions describes a different RG flow, a different branch of the RG flow. Hence we conclude that using the holographic dictionary we are able to find an interpretation of the TP of RG flow as a bifurcation point of a two-branch holographic RG flow in the 5-dimensional bulk spacetime.

Let us summarize what were our findings about the IR behavior of the QWG. First, we found a TP (turning point) for RG flow at some finite energy scales. These values of energies were moreover dependent on the $\Lambda$ parameter which conveniently describes the curvature scales on a general ES. However, we also found perturbatively a true IR FP happening at $k = 0$, so in the deep IR regime. This last true IR FP fulfils all the requirements for the scale-invariant and terminal point of the evolution of couplings described by RG flows. There formally we have $k = 0$ as the IR energy scale (or the limit $t \to 0$ in the RG time coordinate). It should be described by
a CFT with gravitational interactions in the infrared regime. Hence the evidence for such a FP that we have collected, is quite remarkable and give rise to a new FP with quantum gravitational interactions which also enjoy conformal symmetry.

The TP that we found for some finite energy scales (of the order of the curvature on ES) has origin that clearly can be explained. It is there entirely due to the threshold phenomena related to the decoupling of massive modes present in the theory around a general ES backgrounds. Another non-perturbative phenomena is related to the inclusion of an impact of the anomalous dimension $\eta$ of the quantum graviton field. But as we have seen it does not change anything related to the quest for IR FP’s of the RG flows, even when its supposed truly non-perturbative exact behavior is used in the computation. We find that the TP happens for any initial values of the two couplings $\omega_C$ and $\omega_{GB}$ that could be set at some UV energy scale $t_{UV}$; we did not see any constraint for them and they remain completely arbitrary. But still the location of the TP of RG was determined unambiguously and independently on these initial values for the RG flow. As for what regards the IR FP (true one at $k = 0$) we derived only a relation (85) which still leaves some arbitrariness in the FP values of the couplings $\omega_{C**}$ and $\omega_{GB**}$. We see that there is no any constraint from which we could possibly determine the special values of the coupling constants $\omega_{C*}$ and $\omega_{GB*}$ at the TP and hence also the values of $\omega_{C**}$ and $\omega_{GB**}$ at IR FP. Therefore we conclude that in the IR FP’s of QWG we are on a 2-dimensional surface of FP’s (which is of course a simple generalization of a line of FP). This is indeed considered as a very special situation for the IR behavior of the quantum field theory. But here all the speciality comes because of the special character of quantum gravitational theory with conformal interactions.

One can also analyze the applications of the IR behavior for the actual physical cosmology of 4-dimensional Universe. For this we need the Universe to be classically described by conformal gravitational theory. The quantum effects captured by RG flow phenomena may have some interesting implications and features for inflationary cosmology. First, one can identify the UV FP of QWG as the moment when the characteristic energies of all particles are very high, so this is a moment of the Big Bang for our Universe. However, in this paper we put more emphasis on the IR behavior due to quantum RG effects. One knows that near the IR FP the symmetries of the theory may change and the theory undergoes a phase transition. The same could happen with the Universe. Therefore the IR FP may be an onset for the inflation where the dynamics of the quantum gravitational field changes completely and start a new phase with exponential expansion and with exponential growth of the cosmological scale factor $a(t)$. One could also ask for the interpretation of the scale $\Lambda$ which was crucial for a determination of the location of TP of the RG flow. As we defined this was a characteristic scale of the curvature on a general ES, in particular related to the radius of curvature in the case of MSS. In the 4-dimensional cosmological framework the MSS are realized as physical de Sitter spacetimes characterized by some radius describing actually the speed of the exponential expansion. Hence, when $\Lambda > 0$ one sees that the de Sitter background that we use in our computation may be interpreted as inverse radius (up to some constant of proportionality) of the inflationary phase of the Universe. In this way our theoretical considerations related to the RG effects may have some direct interpretations and implications for physical cosmology of our Universe.

One of the two pending problems of QWG is the issue of conformal anomaly (CA) [53, 54]. Here we will comment on it in more details. The other problem is related to the presence of Boulware-Deser ghosts (or also known as Weyl ghost in Weyl gravity) which quite likely endanger unitarity of the theory when it is quantized in a standard way. Therefore the quantization procedure of classical Weyl gravity probably requires some modifications. We will not dwell on this interesting issue with apparent violation of unitarity in QWG anymore. Instead we discuss some issues with CA.

The trace anomaly has to do with the trace of the energy momentum tensor of the physical system in question. When gravity is coupled to matter then for this issue we have to consider...
the whole system (gravity and matter). For such a system we define the classical trace of the total energy momentum tensor (EMT)

$$T = g_{\mu\nu}T_{\text{tot}}^{\mu\nu} \quad \text{or} \quad T = \langle \hat{g}_{\mu\nu}T_{\text{tot}}^{\mu\nu} \rangle \quad (86)$$

for the quantum version of the theory when we have to deal with quantum operators and their expectation values. The energy momentum tensor of the total system we obtain as a variational derivative of the total action of the system with respect to metric fluctuations, that is we have

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_{\text{tot}}}{\delta h_{\mu\nu}} \quad (87)$$

One finds according to Weinberg definition and understanding of the total EMT \[55\] that on-shell (so using gravitational equation of motion), this tensor for the whole system vanishes. One could interpret this as the fact that the EMT of matter completely balances the one coming from gravitational sector. On the classical level one finds that off-shell the EMT is non-zero of the total system. This also implies that off-shell the trace $T$ in general theories is not vanishing.

Let us consider the off-shell trace $T = T_{\mu}^{\mu}$ of the total EMT of the system read from the total action (could be classical tree-level action or quantum effective action). If in the gravitational sector the action is given by

$$S = \int d^4x \sqrt{|g|} R^2, \quad (88)$$

which is a diffeomorphic invariant and if this is the total system (no matter present), then

$$\frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}} = T_{\text{tot}}^{\mu\nu} \neq 0 \quad (89)$$

and this is off-shell non-zero, so off-shell the trace $T \neq 0$ in general. Only for the Weyl action \(9\) we have that off-shell

$$T = g_{\mu\nu} \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}} = g_{\mu\nu}B^{\mu\nu} = 0 \quad (90)$$

as the result of special geometric properties of the Bach tensor. We look for a similar speciality on the quantum level of gravitational theories with conformal symmetries present not only on the classical level.

The special situation happens in classically conformally invariant theories (both with gravity or just of pure matter). Then one finds that the trace of EMT even off-shell vanishes identically and this is a clear sign of conformal invariance of the total classical theory. Of course, the total trace vanishes on-shell but this is due to the fact that on-shell $T_{\text{tot}}^{\mu\nu} = 0$, so the condition $T = 0$ on-shell does not say nothing about the conformal invariance. By closer inspection one notices that in some models of matter which are specially non-minimally coupled to gravity, the following complication arises. One sees that if the action of the total theory is conformally invariant, then the trace of the EMT of the total system vanishes but only with using of matter sector EOM. But still the gravitational sector EOM are not used to find that $T = 0$, so this is mixed off-shell and on-shell situation, but it is important that for the main degrees of freedom for EMT, that is for metric fluctuations the situation is off-shell. To derive any conclusion about conformal invariance (in the GR sense) we must not use gravitational EOM, but it is possible to use the ones from the matter sector, or it is even necessary. The specific models in question for which such situation happens are for example conformally coupled scalar fields in $d = 4$ with the action

$$S_\phi = \int d^4x \sqrt{|g|} \phi \left( \Box + \frac{R}{6} \right) \phi \quad (91)$$
or also higher derivative conformally coupled Dirac fermions and gauge vectors in higher number of dimensions. In this way we can easily derive about the conformal invariance in the theory on the classical level. And this is indeed verified by explicit computations for all known classical conformal models.

On the quantum level if we want to preserve conformality, then we should again require that $T = 0$ (or even in the sense of expectation values $\langle T \rangle = 0$). This is also the condition for the absence of CA on the quantum level. Known models with fully preserved quantum conformality satisfy this condition. Now, the pertinent question is whether there is or there is not CA in pure QWG theory analyzed by the means of the standard quantization procedure. We should accept that this must be viewed as a truly off-shell problem from the point of view of gravitational EOM. It is interesting also to study in details the situation when matter models are coupled in a conformal way to CWG gravity. Historically it was first that the CA appeared as generated due to matter loops in matter models that were put on curved external gravitational backgrounds. It was the inconsistency between the classical conformal symmetry of the matter models (like for example electrodynamics with massless fermions) and the breaking of it on the quantum level, already for one-loop level for example. Due to a presence of non-vanishing beta functions of classically dimensionless couplings, or in general due to RG flow, the anomaly shows up. Only in some special models like in $\mathcal{N} = 4$ supersymmetric Yang-Mills theories, the CA was cancelled and there were no beta functions, and in the result the theory was UV-finite [4]. There was a hope that in matter models with CA the coupling to gravity will balance contributions to beta functions and the problem of CA would be solved in this way.

However, explicit computations of CA, performed even to the simplest one-loop level, showed us that this is generally impossible. The conformal anomaly shows up in particular in the computation of triangle diagrams that we have at the one-loop level in a general matter theory. One could also notice a very interesting observation that at the one-loop level in $d = 4$ spacetime dimensions the contributions to anomaly from gravitational and matter sector are completely independent one of each other and the total anomaly is the sum of all of them without any interference terms. This is originated from the expressions for the beta functions of gravitational terms $R^2$, $C^2$ and GB, which are linear in the contributions coming respectively from the matter sectors and gravitational sector. Therefore if there is a balance in the total anomaly, then this is achieved without any participation of mixing terms. But this also implies that the CA in pure QWG cannot vanish since matter is needed for the balance and the linear superposition is at work here. Therefore it is very crucial to consider coupled matter plus gravity models, when the balance of these two contributions is theoretically possible. The example of coupled theory is $\mathcal{N} = 4$ CSG with two copies of the $\mathcal{N} = 4$ SYM matter species, where the CA completely vanishes as this theory is called as anomaly-free by its authors (Fradkin and Tseytlin).

The fact that on-shell the trace $T$ of the total energy-momentum tensor of the system vanishes [56] has implications only for anomaly of global conformal symmetry. By adding a spin-2 field (understood as matter not as a dynamical spin-2 geometrical current), coupled to external geometry represented here by the tensor of rank-2 namely the metric of the spacetime, one can cancel the global anomaly (‘t Hooft anomaly) of the conformal invariance. This is evident from the fact that on-shell the energy-momentum tensor of the total system (gravity+matter) vanishes $T_{\mu\nu} = 0$ [55]. For the cancellation of the global anomaly this condition is exactly necessary and sufficient and should hold on-shell. Similarly, a gauge current conservation in some globally gauge anomalous theories (because of the presence of chiral fermions) is due to on-shell condition and after all (after adding some new matter fields) the total current is conserved, so there is not a global anomaly anymore. For the ‘t Hooft global anomaly (and current conservation or the issue of the trace of the EMT) the off-shell situation does not matter at all.

We agree that on-shell we have $T_{\mu\nu} = 0$ for the total system (gravity+matter). But this does not cancel the anomaly of the local gauged conformal symmetry which is supposed to be
a local symmetry in conformal gravity. Analogously, $\partial_\mu j^\mu = 0$ does not guarantee that local
gauge anomaly is cancelled in models with chiral fermions and gauge symmetries. It would only
ensure that local gauge anomaly is not there in the model. But for local gauge anomalies if
they are there the theory is sick and should not be considered on the quantum level. Moreover, the
argument from [56] about vanishing of the total EMT on-shell seems too robust and holds
for any gravitational theory (with diffeomorphism symmetry) and with any conformal matter as
a gravitational source (so this matter must have $T_{\mu\nu} = 0$ on matter EOM to be conformally
coupled). The original problem with CA was originated in matter sectors, and this was global
conformal anomaly. Some people claim that the anomaly from the matter side is cancelled by
coupling to conformal gravity. But pure conformal gravity should solve this problem by its own.

So how is it that the contribution of gravity is enough for gravity alone, but adapts to the matter
in such a way that this reaches a balance and the total contribution to CA is zero off-shell? It
is impossible, if one knows that the beta functions at the one-loop level are independent
and additive (for two sectors: matter and gravity), but here the contributions do not adapt to
whether there is matter added or not to always balance, so the total anomaly does not vanish in
full generality.

Moreover, that argument seems to be too naive, because for any diffeomorphism-invariant
gravitational theory on-shell for the total system $T_{\mu\nu} = 0$. For this to happen, this does not have
to be a conformal gravity. And then automatically $T = 0$ on-shell, so there is not a conformal
anomaly in any gravitational theory, no matter if conformal or not, or whether it is coupled to
any matter sector or not. This seems to be too robust argumentation. But we think that the
conformal symmetry should put some constraints, like it puts in Fradkin Tseytlin CSG. And
then you have to add a very special matter sector to QWG to make it anomaly-free.

As we emphasized in the introduction if we have that in the quantum model $T \neq 0$, then
we generally run into problems in quantum theory. One sees that the classical theory (before
quantization) was with local conformal symmetry, while quantum theory is without it. There is
no enough symmetry on the quantum level to bring the positive features of conformality which
is very constraining as we have seen before. This symmetry is for example needed to constrain
the form of all possible UV-divergences (or the absence of them in completely UV-finite theories
[57]) and also is essential to restrict very much the form of the quantum effective action in such a
theory on every loop level and also non-perturbatively. If one understands that this symmetry is
not present on the quantum level, then the problems are with Green functions which now do not
respect the full symmetries of the classical theory. This means that quantum effects break the
gauged conformal symmetry of the model - the symmetry is not there on the full quantum level
and conformal Ward identities are not satisfied. The theory on the quantum level is not very
much controlled anymore. One basically ends up with quantum version of the theory quadratic
in gravitational curvatures (a general higher derivative Stelle theory) and the traces of conformal
symmetry which was present only on the classical tree-level are gone. The quantum theory is
not special anymore. Of course, such situation is very bad and should be avoided by all means.
So, in the sense CA in the UV regime should be made vanishing.

When one analyzes the pure $C^2$ gravity, so the Weyl conformal gravity in $d = 4$, one finds
that there is a non-trivial RG flow that we exploited in this paper. The beta function of the
Weyl coupling $\beta_C$ does not vanish even at the level of first quantum loop. For higher loop this
case is even more dangerous. One could be misled by this fact since this is the UV-divergence
proportional to the conformally invariant counterterm $\sqrt{|g|}C^2$ in the UV-divergent part of the
effective action. Somehow, this situation knows about the remnants of conformal symmetry
since counterterm is conformally invariant, however just the mere fact of its presence signifi es
that there is a non-vanishing CA and that there are severe problems with the preservation of
the conformal symmetry on the quantum level. Moreover, one also remembers that on the one-
loop level there is a non-vanishing beta function of the Gauss-Bonnet term. However, one can
waive this counterterm since its variation (to get for example quantum EOM or any contribution for Green functions) is a total derivative and it is in particular conformally invariant. The counterterm which would explicitly break the conformal symmetry in a full local version, namely the $R^2$ counterterm is not present there on the level of one-loop. However, it is heralded due to the presence of CA already on the one-loop level and one expects the need for presence of the $R^2$ counterterm starting from the two-loop level on. One can also dismiss these arguments with beta functions since they are not directly related to any observable effects, and the RG running of couplings is only a theoretical effect predicted by theory. In general, it is very difficult to find an observable consequence of the violation of the conformal symmetry on the quantum level explicitly such that everyone would agree with it unanimously. Similarly the trace of the quantum EMT of the total system is difficult to measure directly and one does not have any direct physical phenomena derived from the breaking of conformal symmetry.

In this quite disappointing situation, one has to look at other objects that still have good physical meaning. One of such an object is the quantum effective action, in particular not only its UV-divergences but also finite terms. Therefore it is very useful to analyze the situation with some selected terms of the quantum effective action. We choose some of them from the infinite number of them that they are expected in the effective action (even restricted to the one-loop level) due to the infinite series expansion in number of derivatives and another series expansion in powers of fields present in some terms of it. The analysis presented below is more transparent regarding the issues of conformal symmetry than the analysis with UV-divergences at the one-loop level since as written above we found that all counterterms generated at the one-loop level are conformally invariant. We will see the explicit violation of conformal symmetry from some terms in the effective action.

Due to the RG invariance of the total effective action $\Gamma$ and the existence of the non-vanishing beta function $\beta_C$, one derives that in the finite terms of the effective action at the one-loop level one must have a term

$$\Gamma_{\text{fin}} \supset \beta_C C_{\mu\nu\rho\sigma} \log \left( \frac{\Box}{\mu^2} \right) C^{\mu\nu\rho\sigma},$$

where $\mu$ is an arbitrary renormalization scale needed to absorb the UV-divergences of the theory. One could say that just the fact that we have to introduce this dimensionful parameter $\mu$ to make sense of the theory (i.e. to renormalize it) explicitly breaks the conformal invariance of QWG. But one could wait with this argumentation by saying that $\mu$ is a spurious non-physical parameter that all observable measurable effects should be independent of, so about its existence we derive only theoretically, while experimentally it is completely invisible. None can polemize with this fact. However, our argumentation goes one step further. And we consider the full quantum action (classical tree-level action of the QWG theory and the term with first quantum corrections at the one-loop level). Such an object in $d = 4$ reads explicitly

$$\Gamma_{\text{tot}} = \Gamma_{C^2} + \Gamma_{\text{eff}} \supset \int d^4 x \sqrt{|g|} \left[ \alpha_C C^2 + \beta_C C_{\mu\nu\rho\sigma} \log \left( \frac{\Box}{\mu^2} \right) C^{\mu\nu\rho\sigma} \right].$$

The formula above contains some selected terms in the full effective action to the one-loop level with all quantum corrections taken into account. But one should view this functional as giving the action description of the theory to the one-loop level with full quantum corrections included. So one should treat it as the classical action and one could in principle derive all correlation functions and scattering amplitudes from it to the level of one-loop accuracy. But if this functional should be seen as classical one could also answer the question about the presence of symmetries on the one-loop quantum level, in particular about the presence or absence of the conformal symmetry. This question is equivalent to asking whether the classical theory governed by the action (93) is conformally invariant. One can even simplify his life by neglecting the presence of the scale $\mu$
there and we can be silent about its conformal transformation law. This will be immaterial for what follows next. When one considers the action term

$$\sqrt{|g|} \beta_C C_{\mu\nu\rho\sigma} \log \Box C^{\mu\nu\rho\sigma},$$

then one sees that the only case in which this is conformally invariant in $d = 4$ is when $\beta_C = 0$ (everything is spoiled here by the presence of the box operator which transforms non-trivially and non-homogeneously under conformal transformations, and on top of that by the logarithm function of such an operator which acts on a tensor representation of Weyl tensor naturally appearing with its four indices). This term in the total effective action is not conformally invariant in $d = 4$ unless its front coefficient vanishes. The condition $\beta_C = 0$ is then more direct condition for the vanishing of the CA on the quantum level. The terms in the effective action are more related to observable quantities than counterterms or beta functions, which serve here theoretically. One concludes that in pure QWG there is CA, so there is a problem with preservation of the full conformal symmetry on the quantum level.

Since we have confirmed that on the quantum level the conformal symmetry is not fully realized in QWG, then one can ask for reasons of that behavior. One explanation is due to the fact that on the quantum level quantum polarization effects touch conformal symmetry in a dramatic way and the results are very devastating since the conformal symmetry is hardly broken there, when analyzed in the UV regime. Pure QWG theory is anomalous which puts a big question about the consistency of this theory since this conformal symmetry was gauged in such a model. This is in simple words that we expected the symmetry to be on the quantum level and we quantized a theory putting such a lot of care to treat conformal symmetry delicately, but the result is opposite. The conformal symmetry was there only on the classical, but it is not there anymore on the quantum level. Of course, like we explained above this breaking of conformal symmetry is clearly related to the RG running of couplings and to non-trivial RG flow in general. However, we think that this is a problem with the UV-completion of the theory, similarly like the renormalizability was the problem with quantum Einsteinian gravity before it was properly UV completed by terms with higher derivatives in $d = 4$ dimensions. If the UV-completion or UV-embedding of the QWG theory is done by means of CSG, then the problem in the UV limit is solved. One has to worry only about the problem of the CA in the UV limit so for very high energies (when the notion of energy still make sense). If UV-completion is achieved and there is not CA in the fundamental embedding theory, then the problem of overall consistency of the QWG model at lower energies is not an issue. One can also for still lower energies study the RG flow and analyze the situation with soft breaking of conformal symmetry in the domain of applicability of QWG theory (so in the regime of energies when there is a running of gravitational couplings). This is the direction that we have pursued in this contribution.

Now, one can think about some of the remedies for the situation of CA of QWG on the quantum level. Restricting to the first quantum loop level, one can invent some ingenious solutions with adding proper number and proper representations of matter species, in such a way that in the linear superposition the total beta functions for the dimensionless gravitational couplings like for $\alpha_C$, $\alpha_R$ and $\alpha_{GB}$ are balanced to zero thanks to matter contribution which act against the pure QWG gravity. This means that the special matter content must be added to make the theory anomaly-free. Then the dangerous non-conformally invariant terms like $C \log \Box C$ are not generated in the quantum effective action. This could be viewed as a provisory solution on the level of one-loop action, but these new matter species can be promoted to fully gauge-invariant copies of $N = 4$ SYM theories by subjecting to Noether procedure. In this way quantum conformality will be present on the quantum level of the full coupled theory. This will give back the possibility to constrain the scattering amplitudes and expressions for Green functions and various other correlation functions thanks to the presence of quantum conformality. Finally, if on the full level of quantum effective action the conformal symmetry is present in
a preserved way, only then one can successfully resolve the issues of GR-like singularities of classical gravitational theories using the quantum effects from the quantized gravitational theory and using the fully realized quantum conformal symmetry. If this would be done only on the level of classical action of WG, then one runs into the problem that quantum correction may completely invalidate this conclusion about the resolution of singularities and they may change drastically the quantum solutions of the theory bringing back singularities. Instead, when one has a conformal symmetry on the quantum level one is sure that resolution is final, stable and that various quantum corrections cannot destroy it.

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