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SOME CARDINAL CHARACTERISTICS RELATED TO THE COVERING NUMBER AND THE UNIFORMITY OF THE MEAGRE IDEAL

A b s t r a c t. We extend the concepts of splitting, reaping, and independent families to families of functions and permutations on \( \omega \) and define associated cardinal characteristics \( s_f, s_p, r_f, r_p, i_f, \) and \( i_p \). We study relationships among \( \text{cov}(\mathcal{M}), \text{non}(\mathcal{M}) \), and these cardinals. In this paper, we show that \( s_f = \text{non}(\mathcal{M}) = s_p, r_f = \text{cov}(\mathcal{M}) \leq r_p \), and \( \text{cov}(\mathcal{M}) \leq i_f, i_p \).

1. Introduction

The covering number of the meagre ideal \( \mathcal{M}, \text{cov}(\mathcal{M}) \), is the smallest size of a family of meagre subsets of \( ^\omega \omega \) whose union is \( ^\omega \omega \) and the uniformity

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of $\mathcal{M}$, non($\mathcal{M}$), is the smallest size of a non-meagre subset of $\omega$ (see [3] or [7, Chapter III] for more details). It is well-known that $\aleph_1 \leq \mathfrak{p} \leq \text{cov}(\mathcal{M}) \leq \mathfrak{r} \leq \mathfrak{i} \leq \mathfrak{c}$ and $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{s} \leq \text{non}(\mathcal{M}) \leq \mathfrak{c}$, where $\mathfrak{p}, \mathfrak{s}, \mathfrak{r},$ and $\mathfrak{i}$ are the pseudo-intersection, the splitting, the reaping, and the independence numbers respectively (for more details about these numbers see [3] or [6, Chapter 9]).

The almost disjoint number $\mathfrak{a}$ is the smallest size of a maximal almost disjoint family of infinite subsets of $\omega$. It has been shown that both $\mathfrak{a}$ and non($\mathcal{M}$) lie between the bounding number $\mathfrak{b}$ and $\mathfrak{c}$ (see [3] and [6]). Almost disjoint families of functions and permutations on $\omega$ and associated cardinal characteristics, denoted by $\mathfrak{a}_f$ and $\mathfrak{a}_p$, respectively, were studied by Zhang in [9]. Brendle, Spinas, and Zhang showed in [4] that non($\mathcal{M}$) is a lower bound of both $\mathfrak{a}_f$ and $\mathfrak{a}_p$ (cf. [4, Theorem 2.2 and Proposition 4.6]).

Independent families of functions and permutations on $\omega$ and associated cardinal characteristics $\mathfrak{i}_f$ and $\mathfrak{i}_p$ were studied by us in [8]. We have shown that $\mathfrak{p} \leq \mathfrak{i}_f, \mathfrak{i}_p \leq \mathfrak{i}$ and also mentioned that $\text{cov}(\mathcal{M})$ is a lower bound of both $\mathfrak{i}_f$ and $\mathfrak{i}_p$. In this paper, we give a full direct proof of this fact.

We also extend the concepts of splitting and reaping families to families of functions and permutations on $\omega$ and define associated cardinal characteristics $\mathfrak{s}_f, \mathfrak{s}_p, \mathfrak{r}_f, \text{and} \mathfrak{r}_p$. We study relationships among $\text{cov}(\mathcal{M}), \text{non}(\mathcal{M})$, and these cardinals. As mentioned above, $\mathfrak{s} \leq \text{non}(\mathcal{M})$ and $\text{cov}(\mathcal{M}) \leq \mathfrak{r}$. In this paper, we show that $\mathfrak{s}_f = \text{non}(\mathcal{M}) = \mathfrak{s}_p$ and $\mathfrak{r}_f = \text{cov}(\mathcal{M}) \leq \mathfrak{r}_p$.

2. Splitting and reaping families

A set $A \subseteq \omega$ splits an infinite set $B \subseteq \omega$ if both $B \cap A$ and $B \setminus A$ are infinite. A splitting family $\mathcal{S}$ is a family of infinite subsets of $\omega$ such that each infinite set $B \subseteq \omega$ is split by at least one $A \in \mathcal{S}$. A reaping family $\mathcal{R}$ is a family of infinite subsets of $\omega$ such that there is no infinite subset of $\omega$ which splits every member of $\mathcal{R}$. The splitting number $\mathfrak{s}$ is the smallest cardinality of any splitting family and the reaping number $\mathfrak{r}$ is the smallest cardinality of any reaping family.

We write $\omega^\omega$ and $\text{Sym}(\omega)$ for the set of functions and the set of permutations, respectively, on $\omega$. We extend the concepts of splitting and reaping families to families of functions and permutations on $\omega$. To be precise, we say $f \in \omega^\omega$ splits $g \in \omega^\omega$ if both $g \cap f$ and $g \setminus f$ are infinite. A splitting family
S of functions (permutations) is a family of functions (permutations) on \(\omega\) such that each \(g \in {}^\omega \omega\) \((g \in \text{Sym}(\omega))\) is split by an \(f \in S\). A reaping family \(R\) of functions (permutations) is a family of functions (permutations) on \(\omega\) such that there is no function (permutation) on \(\omega\) which splits every member of \(R\). We define corresponding cardinal characteristics \(s_f\), \(s_p\), \(r_f\), and \(r_p\) as follows.

\[
s_f = \min\{|S| : S \subseteq {}^\omega \omega \text{ is a splitting family}\},\]
\[
s_p = \min\{|S| : S \subseteq \text{Sym}(\omega) \text{ is a splitting family}\},\]
\[
r_f = \min\{|R| : R \subseteq {}^\omega \omega \text{ is a reaping family}\},\]
\[
r_p = \min\{|R| : R \subseteq \text{Sym}(\omega) \text{ is a reaping family}\}.
\]

It is easy to see that the above definitions are well-defined since \(\omega\) and \(\text{Sym}(\omega)\) are splitting and reaping families of functions and permutations respectively.

First, we shall show that \(s_f = \text{non}(\mathcal{M})\) and \(r_f = \text{cov}(\mathcal{M})\). The following is Theorem 5.9 in [3]. The first statement is also from [1, Corollary 1.8].

**Theorem 2.1.**

\[
\text{cov}(\mathcal{M}) = \min\{|C| : C \subseteq {}^\omega \omega \land \neg \exists f \in {}^\omega \omega \forall g \in C \ (f \cap g \text{ is infinite})\}, \text{ and}
\]
\[
\text{non}(\mathcal{M}) = \min\{|C| : C \subseteq {}^\omega \omega \land \forall f \in {}^\omega \omega \exists g \in C \ (f \cap g \text{ is infinite})\}.
\]

**Theorem 2.2.** \(s_f = \text{non}(\mathcal{M})\) and \(r_f = \text{cov}(\mathcal{M})\).

**Proof.** It follows immediately from the above theorem that \(r_f \leq \text{cov}(\mathcal{M})\) and \(\text{non}(\mathcal{M}) \leq s_f\). To show that \(s_f \leq \text{non}(\mathcal{M})\), let \(C \subseteq {}^\omega \omega\) be an infinite family such that for all \(f \in {}^\omega \omega\), there exists a \(g \in C\) such that \(f \cap g\) is infinite.

For each \(g \in C\), define \(\tilde{g} \in {}^\omega \omega\) by

\[
\tilde{g}(n) = \begin{cases} 
g(n) & \text{if } n \text{ is even}, \\
g(n) + 1 & \text{if } n \text{ is odd}. \end{cases}
\]

Let \(D = C \cup \{\tilde{g} : g \in C\}\). To show that \(D\) is a splitting family, let \(f \in {}^\omega \omega\). By the property of \(C\), there is a \(g \in C\) such that \(f \cap g\) is infinite. If \(f \setminus g\) is finite, then there is an \(n_0 < \omega\) such that \(f(n) = g(n)\) for all \(n \geq n_0\), and hence \(\tilde{g}\) splits \(f\). Otherwise, \(g\) splits \(f\). Thus \(s_f \leq |D| = |C|\). Since \(C\) is arbitrary, \(s_f \leq \text{non}(\mathcal{M})\).
To show that $\text{cov}(\mathcal{M}) \leq \tau_f$, let $\mathcal{C} \subseteq \omega^\omega$ be an infinite family such that $|\mathcal{C}| < \text{cov}(\mathcal{M})$. We shall show that $\mathcal{C}$ is not a reaping family.

For each $g \in \mathcal{C}$, let $g \oplus 1 \in \omega^\omega$ be defined by $(g \oplus 1)(n) = g(n) + 1$. Let $\mathcal{D} = \mathcal{C} \cup \{g \oplus 1 : g \in \mathcal{C}\}$. Then $|\mathcal{D}| = |\mathcal{C}| < \text{cov}(\mathcal{M})$. By the above theorem, there is an $f \in \omega^\omega$ such that $f \cap h$ is infinite for any $h \in \mathcal{D}$. Consider a $g \in \mathcal{C}$. Since $f \cap (g \oplus 1)$ is infinite, there are infinitely many $k \in \omega$ such that $f(k) \neq g(k)$. Hence $g \setminus f$ is infinite. Since $f \cap g$ is infinite, $f$ splits $g$. Therefore, $\mathcal{C}$ is not a reaping family. \[\square\]

Next, we shall show that $\text{cov}(\mathcal{M}) \leq \tau_p$. The proofs make use of Martin’s Axiom. We start with some relevant definitions and known facts.

**Definition 2.3.** $\text{MA}_\kappa(P)$ is the statement that whenever $D$ is a family of dense subsets of a poset $P$ with $|D| \leq \kappa$, there exists a filter $G$ on $P$ such that $G \cap D \neq \emptyset$ for all $D \in D$.

By the Generic Filter Existence Lemma [7, Lemma III.3.14], we obtain the following theorem.

**Theorem 2.4.** $\text{MA}_\kappa(P)$ holds for any poset $P$ and $\kappa \leq \aleph_0$.

**Definition 2.5.** A subset $C$ of a poset $P$ is centered if, for any $n \in \omega$ and any $p_1, p_2, \ldots, p_n \in C$ there is a $q \in P$ such that $q \leq p_i$ for all $i$. $P$ is $\sigma$-centered if $P$ is a countable union of centered subsets of $P$.

**Definition 2.6.** $m_\sigma$ is the least $\kappa$ such that there is a $\sigma$-centered poset $P$ for which $\text{MA}_\kappa(P)$ fails, and $m_{\text{ctbl}}$ is the least $\kappa$ such that there is a countable poset $P$ for which $\text{MA}_\kappa(P)$ fails.

We have shown, in Theorem 2.2, that $\tau_f = \text{cov}(\mathcal{M})$. Now, we show that $\text{cov}(\mathcal{M}) \leq \tau_p$ by using the following theorem which is Proposition (d) in [5].

**Theorem 2.7.** $m_{\text{ctbl}} = \text{cov}(\mathcal{M})$.

**Theorem 2.8.** $\text{cov}(\mathcal{M}) \leq \tau_p$.

**Proof.** It suffices to show that $m_{\text{ctbl}} \leq \tau_p$. To show this, let $\mathcal{C} \subseteq \text{Sym}(\omega)$ be such that $\aleph_0 \leq |\mathcal{C}| < m_{\text{ctbl}}$. Consider the poset $P = \text{Fn}_{1-1}(\omega, \omega)$, i.e. $\{s \subseteq \omega \times \omega : s$ is a finite injection$\}$. For each $n \in \omega$ and $f \in \mathcal{C}$, let

$$A_n = \{p \in P : n \in \text{dom}(p) \cap \text{ran}(p)\},$$

$$B_{n,f} = \{p \in P : \exists k \geq n \exists \ell \geq n (p(k) = f(k) \land p(\ell) \neq f(\ell))\}.$$
Then $A_n$ and $B_{n,f}$ are dense in $\mathbb{P}$ for any $n \in \omega$ and $f \in \mathcal{C}$. Let
$$\mathcal{D} = \{A_n : n \in \omega\} \cup \{B_{n,f} : n \in \omega, f \in \mathcal{C}\}.$$ Since $\mathcal{D}$ is of size $< \mathfrak{m}_{ctbl}$, there is a filter $G$ on $\mathbb{P}$ such that $G \cap A_n \neq \emptyset \neq G \cap B_{n,f}$ for any $n \in \omega$ and $f \in \mathcal{C}$. Let $g = \bigcup G$. Then $g \in \text{Sym}(\omega)$ and for any $n \in \omega$ and any $f \in \mathcal{C}$, we have that $g(k) = f(k)$ and $g(\ell) \neq f(\ell)$ for some $k, \ell \geq n$. Hence for any $f \in \mathcal{C}$, $f \cap g$ and $f \setminus g$ are infinite, so $g$ splits $f$. Thus $\mathcal{C}$ is not a reaping family. □

It is well-known that $\mathfrak{p} \leq \mathfrak{s}$ (cf. [6, Chapter 9]). Now, we shall use the fact below to show that $\mathfrak{p}$ is also a lower bound of $\mathfrak{s}_p$. The following theorem is from Bell ([2]), and is also Theorem III.3.61 in [7].

**Theorem 2.9.** $m_\sigma = \mathfrak{p}$.

**Theorem 2.10.** $\mathfrak{p} \leq \mathfrak{s}_p$.

**Proof.** It suffices to show that $m_\sigma \leq \mathfrak{s}_p$. To show this, let $\mathcal{C} \subseteq \text{Sym}(\omega)$ be such that $\aleph_0 \leq |\mathcal{C}| < m_\sigma$. Define the poset $\mathbb{P} = \text{Fn}_{1-1}(\omega, \omega) \times [\mathcal{C}]^{< \omega}$, where $(s, E) \leq (t, F)$ if and only if
$$s \supseteq t, E \supseteq F \text{ and } \forall n \in \text{dom}(s) \setminus \text{dom}(t) \forall f \in F (s(n) \neq f(n)).$$

Clearly this poset is $\sigma$-centered, as the set $\{(s, E) \in \mathbb{P} : E \in [\mathcal{C}]^{< \omega}\}$ is centered for any fixed $s$ and $\text{Fn}_{1-1}(\omega, \omega)$ is countable. For each $n \in \omega$ and $f \in \mathcal{C}$, let
$$A_n = \{(s, E) \in \mathbb{P} : n \in \text{dom}(s) \cap \text{ran}(s)\},$$
$$B_f = \{(s, E) \in \mathbb{P} : f \in E\}.$$ It is easy to see that $B_f$ is dense in $\mathbb{P}$ for all $f \in \mathcal{C}$. To show that $A_n$ is dense in $\mathbb{P}$ for any $n \in \omega$, let $n \in \omega$ and $(s, E) \in \mathbb{P}$. Since $s$ is a finite function and $E$ is a finite set of injections, we can pick $k \in \omega \setminus \text{dom}(s)$ and $\ell \in \omega \setminus \text{ran}(s)$ so that $(k, n), (n, \ell) \notin \bigcup E$. We choose
$$t = \begin{cases} s & \text{if } n \in \text{dom}(s) \cap \text{ran}(s), \\ s \cup \{(k, n)\} & \text{if } n \in \text{dom}(s) \setminus \text{ran}(s), \\ s \cup \{(n, \ell)\} & \text{if } n \in \text{ran}(s) \setminus \text{dom}(s), \\ s \cup \{(k, n), (n, \ell)\} & \text{if } n \notin \text{dom}(s) \cup \text{ran}(s). \end{cases}$$

Then $(t, E) \leq (s, E)$ where $(t, E) \in A_n$. So $A_n$ is dense in $\mathbb{P}$. Let
$$\mathcal{D} = \{A_n : n \in \omega\} \cup \{B_f : f \in \mathcal{C}\}.$$
Since $D$ is of size $|C| < \mathfrak{m}_\sigma$, there is a filter $G$ on $\mathbb{P}$ such that $G \cap A_n \neq \emptyset \neq G \cap B_f$ for any $n \in \omega$ and $f \in C$. Let $g = \bigcup \text{dom}(G)$. Then $g \in \text{Sym}(\omega)$.

To show that $g \cap f$ is finite for any $f \in C$, let $f \in C$. Since $G \cap B_f \neq \emptyset$, there is a $(s, E) \in G$ such that $f \in E$. Let $m \in \text{dom}(g) \setminus \text{dom}(s)$. We shall show that $g(m) \neq f(m)$. Since $(m, g(m)) \in g = \bigcup \text{dom}(G)$, there is a $(t, F) \in G$ such that $(m, g(m)) \in t$. Since $G$ is a filter, there is a $(s', E') \in G$ such that $(s', E') \leq (s, E)$ and $(s', E') \leq (t, F)$. Then $m \in \text{dom}(s') \setminus \text{dom}(s)$ and hence $g(m) = t(m) = s'(m) \neq f(m)$. Therefore, $g(m) \neq f(m)$ for any $m \in \text{dom}(g) \setminus \text{dom}(s)$. So $\{m : g(m) = f(m)\} \subseteq \text{dom}(s)$, which implies that $g \cap f$ is finite. Therefore, $C$ is not a splitting family. \qed

The above proof shows the relationship between $p$ and $s_p$ by using the fact that $m_\sigma = p$. However, since $p \leq s \leq \text{non}(\mathcal{M})$, a stronger result can be obtained as shown in the following theorem. The notation $\exists^\infty n$ means “there are infinitely many” and $\forall^\infty n$ means “for all but finitely many”.

**Theorem 2.11.** $\text{non}(\mathcal{M}) = s_p$.

**Proof.** We first show that $s_p \leq \text{non}(\mathcal{M})$. Note that $\text{Sym}(\omega)$ is homeomorphic to $^{\omega}\omega$, so the notion of “the smallest size of a meagre set” in both (topological) spaces are the same. Let $S \subseteq \text{Sym}(\omega)$ be such that $|S| < s_p$. We shall show that $S$ is meagre in $\text{Sym}(\omega)$. By the definition of $s_p$, there is a $g \in \text{Sym}(\omega)$ such that, for each $f \in S$, $\forall^\infty n[g(n) \neq f(n)]$ or $\forall^\infty n[g(n) = f(n)]$. Let $S_0 = \{f \in S : \forall^\infty n[g(n) \neq f(n)]\}$. We claim that $S_0$ is meagre in $\text{Sym}(\omega)$. For each $n < \omega$, let

$$C_n = \{f \in \text{Sym}(\omega) : \forall m > n[g(m) \neq f(m)]\}.$$ 

It is straightforward to show that $C_n$ is closed nowhere dense and $S_0 \subseteq \bigcup_{n<\omega} C_n$, and hence $S_0$ is meagre. Since $S \setminus S_0 = \{f \in S : \forall^\infty n[g(n) = f(n)]\}$ is countable (and hence is meagre), $S = S_0 \cup (S \setminus S_0)$ is meagre.

We next show that $\text{non}(\mathcal{M}) \leq s_p$. Let $S \subseteq \text{Sym}(\omega)$ be such that $|S| < \text{non}(\mathcal{M})$, and we shall show that $S$ is not a splitting family.

Claim. There exists an injection $f \in {^\omega}\omega$ such that $f(n) > n$ for all $n < \omega$ and for all $g \in S$, $\forall^\infty n[f(n) \neq g(n)]$ and $\forall^\infty n[f(n) \neq q^{-1}(n)]$.

**Proof.** Let $\pi : {^\omega}\omega \to \omega$ be a one-to-one map such that $\pi(n, m) > n$ for all $n, m < \omega$. For any $q \in \text{Sym}(\omega)$, we define $q^+ \in {^\omega}\omega$ by

$$q^+(n) = \begin{cases} m & \text{if } q(n) = \pi(n, m), \\ 0 & \text{otherwise}. \end{cases}$$

Note that $q^+(n) = m$ if and only if $q(n) = \pi(n, m)$. Therefore, $q^+ \in {^\omega}\omega$ and $q^+ \neq q$ for any $q \in S$. Hence, $\forall^\infty n[q^+(n) \neq q(n)]$ and $\forall^\infty n[q^+(n) \neq q^{-1}(n)]$. Thus, $S$ is not a splitting family.

\qed
Put $S^{-1} = \{ p^{-1} : p \in S \}$ and $S^+ = S \cup S^{-1} \cup \{ q^+ : q \in S \cup S^{-1} \}$.

Since $|S| < \operatorname{non}(M)$, by Theorem 2.1, there exists an $\hat{f} \in {}^\omega \omega$ such that for all $g \in S^+$, $\forall \in n [\hat{f}(n) \neq g(n)]$. In particular, for each $q \in S \cup S^{-1}$, $\forall \in n [\hat{f}(n) \neq q^+(n)]$. Define $f \in {}^\omega \omega$ by $f(n) = \pi(n, \hat{f}(n))$. Clearly $f$ is one-to-one and $f(n) > n$ for all $n < \omega$. Notice that $\forall q \in S \cup S^{-1} \forall n < \omega [f(n) = q(n) \rightarrow \hat{f}(n) = q^+(n)]$.

Hence, for each $q \in S \cup S^{-1}$, $\forall \in n [f(n) \neq q(n)]$, and the proof of the claim is complete.

Let $f(k) = n_k$, and note that $n_k > k$ for all $k$ and $n_k$’s are distinct. Define $p \in \operatorname{Sym}(\omega)$ recursively as follows. Suppose we have already defined $p(k)$. If there exists an $i < k$ such that $p(i) = k$ then put $p(k) = i$; otherwise, put $p(k) = n_k$. Note that, after the construction is done, if $p(x) = y$ then $(x, y) = (k, n_k)$ or $(x, y) = (n_k, k)$ for some $k$. So $p(p(x)) = x$ for all $x < \omega$, and hence $p$ is bijective.

We finally show that $\forall \in k [p(k) \neq q(k)]$ for all $q \in S$. Suppose to the contrary that there is a $q \in S$ such that $\exists \in k [p(k) = q(k)]$. Let $X = \{ k : p(k) = n_k \}$. Note that $\omega \setminus X = \{ n_k : p(n_k) = k \}$. Then either $\exists \in k \in X [p(k) = q(k)]$ or $\exists \in i \in \omega \setminus X [p(i) = q(i)]$.

In the former case, we have $\exists \in k [\hat{f}(k) = n_k = p(k) = q(k)]$. In the latter case, we have $\exists \in k [k = p(n_k) = q(n_k)]$, so $\exists \in k [q^{-1}(k) = n_k = f(k)]$. Both cases contradict the above claim. Therefore $S$ is not a splitting family. □

3. Independent families

An infinite set $I \subseteq \mathcal{P}(\omega)$ is said to be an independent family (or shortly i.f.) if, for any disjoint finite sets $A, B \subseteq I$, $\bigcap A \setminus \bigcup B$ is infinite. We interpret $\bigcap \emptyset = \omega$. The cardinal $i$ is defined as the least cardinality of a maximal independent family. We extend the concept of independent families to families of functions and permutations on $\omega$ and define corresponding cardinal characteristics $\inf$ and $\inp$ as follows.

\[
\inf = \min\{|I| : I \subseteq {}^\omega \omega \text{ is a maximal independent family}\}
\]

\[
\inp = \min\{|I| : I \subseteq \operatorname{Sym}(\omega) \text{ is a maximal independent family}\}.
\]

Since there is an i.f. of permutations of cardinality $\epsilon$ (see Proposition 2.1 in [8]), $\inf$ and $\inp$ are well-defined.
We shall show that \( \text{cov}(\mathcal{M}) \) is a lower bound of \( i_p \) (and also \( i_f \)). First, we need the following fact.

**Fact.** \( \text{cov}(\mathcal{M}) \) is the least cardinality of a family of open dense subsets of \( \omega^\omega \) whose intersection is empty.

This fact follows from Proposition (a) in [5] by viewing a Polish space \( X \) as the Baire space \( \omega^\omega \) (with the basic open sets of the form \( [p] = \{ f \in \omega^\omega : f \supseteq p \} \) for \( p \in \omega^\omega \)) together with the fact from topology that “a subset \( O \) of a topological space \( X \) is open dense if and only if \( X \setminus O \) is closed nowhere-dense”. Note the fact that \( D \subseteq \mathbb{P} \) is dense in the Cohen poset \( \mathbb{P} = \omega^\omega \) if and only if \( [D] = \{ f \in \omega^\omega : f \supseteq p \text{ for some } p \in D \} \) is open dense in the Baire space \( \omega^\omega \).

For an infinite family \( \mathcal{C} \subseteq \omega^\omega \), let \( \text{bc}(\mathcal{C}) = \{ \bigcap A \setminus \bigcup B : A, B \in \text{fin}(\mathcal{C}), A \cap B = \emptyset \text{ and } A \neq \emptyset \} \). Then each member of \( \text{bc}(\mathcal{C}) \) is a function and is an injection if \( \mathcal{C} \) is a family of permutations. Notice that \( \mathcal{C} \) is an independent family if and only if every member of \( \text{bc}(\mathcal{C}) \) is infinite.

**Theorem 3.1.** \( \text{cov}(\mathcal{M}) \leq i_p \).

**Proof.** Let \( \mathcal{C} \subseteq \text{Sym}(\omega) \) be an i.f. of permutations such that \( \aleph_0 \leq |\mathcal{C}| < \text{cov}(\mathcal{M}) \). We shall show that \( \mathcal{C} \) is not maximal.

For any \( p \in \omega^\omega \cup \omega^\omega \), let \( \hat{p} \) be the one-to-one sequence obtained from \( p \) by removing all repetitions of each occurrence of \( p(i) \) except its first one. Let \( \mathbb{P} = \omega^\omega \) and for each \( x \in \text{bc}(\mathcal{C}) \) and \( n < \omega \), let

\[
D_{x,n} = \{ p \in \mathbb{P} : \exists k, \ell \geq n (k, \ell \in \text{dom}(\hat{p}) \cap \text{dom}(x) \land \hat{p}(k) = x(k) \\
\land \hat{p}(\ell) \neq x(\ell)) \}, \\
A_n = \{ p \in \mathbb{P} : n \in \text{ran}(p) \}.
\]

It is easy to see that each \( A_n \) is dense in \( \mathbb{P} \). To show that each \( D_{x,n} \) is dense in \( \mathbb{P} \), let \( x \in \text{bc}(\mathcal{C}), n < \omega \), and \( p \in \mathbb{P} \). Pick distinct \( k, \ell \geq \max\{ n, \text{dom}(p) \} \) such that \( k, \ell \in \text{dom}(x) \) and \( k < \ell \) where \( x(k) \) and \( x(\ell) \) are not in \( \text{ran}(p) \). Choose a \( q \in \mathbb{P} \) such that \( q \supseteq p \) and the \( k \)-th and the \( \ell \)-th unrepeated elements are equal to \( x(k) \) and not equal to \( x(\ell) \), respectively. Rigorously, let \( s = \text{dom}(p), t = |\text{ran}(p)| \), pick distinct \( a_0, a_1, \ldots, a_{k-t-1}, b_0, b_1, \ldots, b_{\ell-k-1} \in \omega \setminus (\text{ran}(p) \cup \{ x(k), x(\ell) \}) \), and define \( q = p \cup \{ (s+i, a_i) : i < k-t \} \cup \{ (s-t+k, x(k)) \} \cup \{ (s-t+k+1+j, b_j) : j < \ell-k \} \). Thus \( \hat{q}(k) = x(k) \) and \( \hat{q}(\ell) \neq x(\ell) \), so \( q \in D_{x,n} \). Let
\[ \mathcal{D} = \{ [D_{x,n}] : x \in \text{bc}(\mathcal{C}) \text{ and } n < \omega \} \cup \{ [A_n] : n < \omega \} \]

Then \( \mathcal{D} \) is a family of open dense subsets of the Baire space \( \omega^\omega \) where \(|\mathcal{D}| \leq |\mathcal{C}| < \text{cov}(\mathcal{M}) \). By the above fact, \( \bigcap \mathcal{D} \neq \emptyset \), and we can pick an element \( f \in \bigcap \mathcal{D} \). Thus \( x \cap \hat{f} \) and \( x \setminus \hat{f} \) are infinite where \( \hat{f} \in \text{Sym}(\omega) \). So \( \mathcal{C} \cup \{ \hat{f} \} \) is an i.f. of permutations. \( \square \)

Another way to prove the above theorem is, by using the fact that \( \text{mctbl} = \text{cov}(\mathcal{M}) \) and showing that \( \text{mctbl} \leq i_f \) instead. This can be done by consider the countable poset \( F_{n_1-1}(\omega, \omega) \). We leave the details for the reader.

By simplifying the proof of Theorem 3.1, we can show that \( \text{cov}(\mathcal{M}) \leq i_f \).

However, the following theorem gives a better lower bound of \( i_f \). Recall that the cardinal \( \mathfrak{d} \) is the dominating number, the smallest size of a dominating family of functions on \( \omega \).

**Theorem 3.2.** \( \mathfrak{d} \leq i_f \).

**Proof.** Suppose \( \mathcal{I} \subseteq \omega^\omega \) is an independent family with \( \aleph_1 \leq |\mathcal{I}| < \mathfrak{d} \). We shall show that \( \mathcal{I} \) is not maximal.

Take a model \( M \) of sufficiently large finite fragment of ZFC with \( \mathcal{I} \in M \) and \( |M| = |\mathcal{I}| \).

Claim. There is a strictly increasing sequence \( \{ n_k : k < \omega \} \subseteq \omega \) with \( n_0 = 0 \) so that for any \( g \in M \cap \omega^\omega \), there are infinitely many \( k \) such that \( g(n_k) < n_{k+1} \).

**Proof.** Since \( |M| < \mathfrak{d} \), \( \omega^\omega \cap M \) is not a dominating family. Hence there is a strictly increasing function \( f \in \omega^\omega \) such that \( \exists^\infty n [g(n) < f(n)] \) for all \( g \in \omega^\omega \cap M \). Define \( n_0 = 0 \) and \( n_{k+1} = f(n_k) \) for each \( k < \omega \).

Let \( g \in \omega^\omega \cap M \). We shall show that \( \exists^\infty k [g(n_k) < n_{k+1}] \). In \( M \), define \( G \in \omega^\omega \cap M \) by \( G(0) = 1 \) and

\[ G(n+1) = \max \{ \{ g(i) : i \leq G(n) \} \cup \{ G(n) \} \} + 1. \]

If there is an \( \ell < \omega \) such that \( |\text{ran}(G) \cap [n_k, n_{k+1})| \leq 1 \) for all \( k \geq \ell \), then \( G(k) \geq n_{k+1} = f(n_k) \geq f(k) \) for all \( k \geq n_\ell + 1 \), which is impossible by the property of \( f \). So there are infinitely many \( k \) such that \( |\text{ran}(G) \cap [n_k, n_{k+1})| \geq 2 \). For such a \( k \), there is an \( a_k \) such that \( n_k \leq G(a_k) < G(a_k + 1) < n_{k+1} \) and hence, by the definition of \( G \), \( g(n_k) \leq G(a_k + 1) < n_{k+1} \), and the proof of the claim is done.
Let \( \{f_k : k < \omega\} \subseteq \mathcal{I} \) be a sequence in \( M \) without repetitions. Define

\[
h = \bigcup_{k<\omega} f_k[n_k, n_{k+1}).
\]

We shall show that \( \mathcal{I} \cup \{h\} \) is an independent family and \( h \notin \mathcal{I} \). To show this, let \( \mathcal{A}, \mathcal{B} \subseteq \mathcal{I} \) be disjoint finite sets. Note that \( h \) and \( f_k \) agree on \([n_k, n_{k+1})\) for each \( k \). It suffices to show that

\[
\exists \infty k < \omega \exists a \in [n_k, n_{k+1}) \left[ \forall f \in \mathcal{A}[f(a) = f_k(a)] \land \forall g \in \mathcal{B}[g(a) \neq f_k(a)] \right].
\]

Choose an \( \ell < \omega \) so that \( f_k \notin \mathcal{A} \cup \mathcal{B} \) for all \( k > \ell \). Since, for any \( n \) with \( \ell < n < \omega \), three sets \( \mathcal{A}, \mathcal{B} \) and \( \{f_k : \ell < k \leq n\} \) are disjoint subset of an independent family \( \mathcal{I} \), working in \( M \), we can construct a \( d \in \omega \cap M \) such that for any \( n, k \) with \( \ell < k \leq n \),

\[
\exists a \in [n, d(n)) \left[ \forall f \in \mathcal{A}[f(a) = f_k(a)] \land \forall g \in \mathcal{B}[g(a) \neq f_k(a)] \right].
\]

Since there are infinitely many \( k \) such that \( d(n_k) < n_{k+1} \) (by the above claim), we are done. \( \square \)

4. Summary and open problems

We summarise relationships among the cardinals studied in this paper and other well-known ones in the following diagram. A line connecting two cardinals indicates that the lower cardinal is less than or equal to the upper one.
By Cohen forcing, we have that $\aleph_1 = \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = \mathfrak{c}$ is relatively consistent with ZFC (cf. [3, Section 11.3, pages 472–473]). Therefore, the following statement is consistent with ZFC:

$$\aleph_1 = p = s = s_f = sp = \text{non}(\mathcal{M}) < \text{cov}(\mathcal{M}) = r = r_f = r_p = i_f = i_p = i = \mathfrak{c}.$$ 

By Random forcing, we have that $\aleph_1 = s = \text{cov}(\mathcal{M}) < \text{non}(\mathcal{M}) = r = \mathfrak{c}$ is relatively consistent with ZFC (cf. [3, Section 11.4, pages 473–474]). Thus it is relatively consistent with ZFC that

$$\aleph_1 = p = r_f = s = \text{cov}(\mathcal{M}) < \text{non}(\mathcal{M}) = r = s_f = sp = ae = ap = i = \mathfrak{c}.$$ 

Since there are many models of ZFC in which $\text{cov}(\mathcal{M}) = \aleph_1$ and $\mathfrak{d} = \aleph_2$, e.g., Laver, Mathias, or Miller forcing (cf. [3, Sections 11.7-11.9, pages 478–479]), by Theorem 3.2, $\text{cov}(\mathcal{M}) < i_f$ in these models.

From the above results, there are some interesting open problems below.

1. Is $r_p = \text{cov}(\mathcal{M})$?
2. Is $\mathfrak{d}$ a lower bound of $i_p$?
3. Is there any model of ZFC in which $\text{cov}(\mathcal{M}) < i_p$?

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References

[1] T. Bartoszyński, Combinatorial aspects of measure and category, *Fundamenta Mathematicae*, 127 (1987), 225–239.

[2] M. G. Bell, On the combinatorial principle $P(c)$, *Fundamenta Mathematicae*, 114 (1981), 149–157.

[3] A. Blass, Combinatorial cardinal characteristics of the continuum, In: *Handbook of Set Theory*, vol. 1, M. Foreman, A. Kanamori (eds.), (Springer, Berlin, 2010), 395–490.
[4] J. Brendle, O. Spinas, Y. Zhang, Uniformity of the meager ideal and maximal cofinitary groups, *Journal of Algebra, 232* (2000), 209–225.

[5] D. H. Fremlin, S. Shelah, On partitions of the real line, *Israel Journal of Mathematics, 32* (1979), 299–304.

[6] L. J. Halbeisen, Combinatorial set theory: With a gentle introduction to forcing, 2 ed., Springer International Publishing (2017).

[7] K. Kunen, Set theory, vol. 34 of Studies in logic, College Publications (2011).

[8] N. Sonpanow, P. Vejjajiva, Independent families of functions and permutations, *Mathematical Logic Quarterly, 66* (2020), 311–315.

[9] Y. Zhang, On a class of m.a.d. families, *Journal of Symbolic Logic, 64* (1999), 737–746.

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