Graviton in a Curved Space-Time Background
and Gauge Symmetry

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Pauli-Fierz approach to description of a massless spin-2 particle is investigated in the framework of 30-component first order relativistic wave equation theory on a curved space-time background. It is shown that additional gauge symmetry of massless equations established by Pauli-Fierz can be extended only to curved space-time regions where Ricci tensor vanishes. In all such space-time models the generally covariant S=2 massless wave equation exhibits gauge symmetry property, otherwise it is not so.

1 Introduction

The theory of the massive spin-2 field has received much attention over the years since the initial construction of Lagrangian formulation by Fierz and Pauli [1-2]. The original Fierz-Pauli theory for spin was second order in derivatives $\partial_\alpha$ (and involved scalar and tensor auxiliary fields). It is highly satisfactory as long as we restrict ourselves to a free particle case. However this approach turned out not to be so good at considering spin-2 theory in presence of an external electromagnetic field. Federbush [3] showed that to avoid a loss of constrains problem the minimal coupling had to be supplemented by a direct non-minimal to the electromagnetic field strength. There followed a number of works on modification or generalizations of the Fierz-Pauli theory (Rivers [4], Nath [5], Bhargava and Watanabe [6], Tait [7], Reilly [8]). At the same time interest in general high-spin fields was generated by the discovery of the now well-known inconsistency problems of Johnson and Sudarshan [9] and Velo and Zwanzinger [10]. In the course of investigating their acausality problems for other then 3/2, Velo-Zwanzinger rediscovered the spin-2 loss of constrains problem, but were not at first aware of the non-minimal term solution of it. Velo [11] later made a thorough analysis of the external field problem for the 'correct' non-minimally coupled spin-2 theory, showing that it is acausal too.

All the work mentioned above dealt with a second-order formalism for the spin-2 theory. Much of the confusion which arose over this theory could be traced to the so-called "derivative ordering ambiguity (Naglal [12]). This problem can be avoided by working from the start with

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a first-order formalism (for example see Gel’fand et al [13]) and for which the minimal coupling procedure is unambiguous.

The work by Fedorov [14] was likely to be the first one where consistent investigation of the spin-2 theory in the framework of first-order theory was carried out in detail. The 30-component wave equation [14] referred to the so-called canonical basis, transition from which to the more familiar tensor formulation is possible but laborious task and it was not done in [14]. Subsequently the same 30-component theory was rediscovered and fundamentally elaborated in tensor-based approach by a number of authors (Regge [15], Schwinger [16], Chang [17], Hagen [18], Mathews et al [19], Cox [20]). Also a matrix formalism for the spin-2 theory was developed (Fedorod, Bogush, Krylov, Kisel [21-25]).

Concurrently else one theory for spin-2 particle was advanced that requires 50 field components (Adler [26], Deser et al [27], Fedorov and Krylov [28, 23], Cox [20]). It appears to be more complicated, however some evident correlation between the corresponding massless theory and the non-linear gravitational equation is revealed (Fedorov [28]).

Possible connections between two variants of spin-2 theories have been investigated. Seemingly, the most clarity was achieved by Bogush and Kisel [25], who showed that 50-component equation in presence of an external electromagnetic field can be reduced to 30-component equation with additional interaction that must be interpreted as anomalous magnetic momentum term.

In the present work the 30-component first order theory is investigated in the case of vanishing mass of the particle and external curved space-time background.

2 Particle in the flat space-time

A system of first order wave equations describing a massless spin-2 particle in a flat space-time has the form

\[ \partial^2 \Phi_a = 0 , \]  
\[ \frac{1}{2} \partial_a \Phi - \frac{1}{3} \partial^k \Phi_{ab} = \Phi_a , \]  
\[ \frac{1}{2} (\partial^k \Phi_{kab} + \partial^k \Phi_{kba} - \frac{1}{2} g_{ab} \partial^k \Phi_{kn} \phi_n ) + \partial_a \Phi_b + \partial_b \Phi_a - \frac{1}{2} g_{ab} \partial^k \Phi_k = 0 , \]  
\[ \partial_a \Phi_{bc} - \partial_b \Phi_{ac} + \frac{1}{3} (g_{bc} \partial^k \Phi_{ak} - g_{ac} \partial^k \Phi_{bk} ) = \Phi_{abc} . \]  

A 30-component wave function consists of a scalar \( \Phi \), vector \( \Phi_a \), symmetric 2-rank tensor \( \Phi_{ab} \), and 3-rank tensor \( \Phi_{abc} \) antisymmetric in two first indices. From (4) it follows four conditions that are satisfied by the 3-index field:

\[ \Phi_{abc} + \Phi_{cba} + \Phi_{cab} = 0 \text{ or } \epsilon^{kabc} \Phi_{abc} = 0. \]  

Simplifying Eq. (4) in indices \( b \) and \( c \), one produces

\[ \partial_a \Phi^b_a = \Phi^c_{ac} . \]  

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Thus, a total number of independent components entering the theory equals 31 (instead of 30 in massive case):

\[ \Phi(x) \to 1, \quad \Phi_a \to 4, \quad \Phi_{ab} \to 10, \]
\[ \Phi_{abc} \to 6 \times 4 - 4 - 4 = 16. \]

After excluding fields \( \Phi_a \) and \( \Phi_{kab} \) from (1-4) one gets to a pair of second order equations on fields \( \Phi(x) \) and \( \Phi_{ab}(x) \):

\[
\frac{1}{2} \nabla^2 \Phi - \frac{1}{3} \partial^k \partial^j \Phi_{kl} = 0, \quad (7)
\]
\[
(\partial_a \partial_b - \frac{1}{4} g_{ab} \nabla^2) \Phi - \frac{1}{4} g_{ab} \nabla^2 \Phi_c + \nabla^2 \Phi_{ab} - \partial_a \partial^j \Phi_{bl} - \partial_b \partial^j \Phi_{al} + \frac{1}{2} g_{ab} \partial^k \partial^j \Phi_{kl} = 0. \quad (8)
\]

Allowing for (7), Eq. (8) can be rewritten as

\[
(\partial_a \partial_b + \frac{1}{2} g_{ab} \nabla^2) \Phi - \frac{1}{4} g_{ab} \nabla^2 \Phi_c + \nabla^2 \Phi_{ab} - \partial_a \partial^j \Phi_{bl} - \partial_b \partial^j \Phi_{al} = 0. \quad (9)
\]

The fact of prime significance in the theory under consideration is that these equations permit specific gauge principle\(^2\). That means the following: the above second order system (9) is satisfied by a substitution (class of trivial or gradient-like solution)

\[
\Phi^{(0)} = \partial^j \Lambda_t, \quad \Phi_{ab}^{(0)} = \partial_a \Lambda_b + \partial_b \Lambda_a - \frac{1}{2} g_{ab} \partial^j \Lambda_t, \quad (10)
\]

at any 4-vector function \( \Lambda_a(x) \). Indeed,

\[
- \frac{1}{3} \partial^a \partial^b \Phi_{ab}^{(0)} = - \frac{1}{2} \nabla^2 \partial^j \Lambda_t = - \frac{1}{2} \nabla^2 \Phi^{(0)}, \quad (11)
\]

and therefore the set (10) turns Eq. (7) into identity. Further, taking into account

\[
\frac{1}{2} (\partial^k \Phi_{kab} + \partial^k \Phi_{kba} - g_{ab} \partial^k \Phi_{kn}) = + \frac{1}{3} \partial^j (\partial_b \Lambda_a + \partial_a \Lambda_b) - \frac{2}{3} \partial_a \partial_b \partial^j \Lambda_t,
\]
\[
\partial_a \Phi_b + \partial_b \Phi_a - \frac{1}{2} g_{ab} \partial^k \Phi_k = - \frac{1}{3} \partial^j (\partial_b \Lambda_a + \partial_a \Lambda_b) + \frac{2}{3} \partial_a \partial_b \partial^j \Lambda_t,
\]

one can verify that the set (10) satisfies Eq. (8) as well.

So, a massless spin-2 field in Minkowski space-time can be described by the first order system, or by the second order system (Pauli-Fierz [1-2]). At this their solutions are not determined uniquely; in general, to any chosen one we may add an arbitrary \( \Lambda_a \) -dependent term.

### 3 Particle in curved space-time

With the use of principle of minimal coupling to a curved space-time background (external gravitational field), expected generally covariant equations for a spin-2 particle are to be taken

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1. The notation \( \nabla^2 = \partial^a \partial_a \) is used.
2. The fact was firstly established by Pauli and Fierz [1-2].
in the form

\[ \nabla^\alpha \Phi_\alpha = 0, \tag{12} \]

\[ \frac{1}{2} \nabla_\alpha \Phi - \frac{1}{3} \nabla^\beta \Phi_{\alpha \beta} = \Phi_\alpha, \tag{13} \]

\[ \frac{1}{2} \left( \nabla^\rho \Phi_{\rho \alpha \beta} + \nabla^\rho \Phi_{\rho \beta \alpha} - \frac{1}{2} g_{\alpha \beta}(x) \nabla^\rho \Phi_{\rho \sigma} \right) + \left( \nabla_\alpha \Phi_\beta + \nabla_\beta \Phi_\alpha - \frac{1}{2} g_{\alpha \beta}(x) \nabla^\rho \Phi_\rho \right) = 0, \tag{14} \]

\[ \nabla_\alpha \Phi_{\beta \sigma} - \nabla_\beta \Phi_{\alpha \sigma} + \frac{1}{3} (g_{\beta \sigma}(x) \nabla^\rho \Phi_{\alpha \rho} - g_{\alpha \sigma}(x) \nabla^\rho \Phi_{\beta \rho}) = \Phi_{\alpha \beta \sigma}. \tag{15} \]

Here \( \nabla_\alpha \) designates a generally covariant derivative. As in the flat space-time, the system exhibits the property

\[ \nabla_\alpha \Phi_\beta = \Phi_{\alpha \beta}. \tag{16} \]

Now we are to investigate the question of possible gauge symmetry of the system. To this end we will try to satisfy these equations by a substitution

\[ \Phi^{(0)} = \nabla^\beta \Lambda_\beta, \]

\[ \Phi^{(0)}_{\alpha \beta} = \nabla_\alpha \Lambda_\beta + \nabla_\beta \Lambda_\alpha - \frac{1}{2} g_{\alpha \beta}(x) \nabla^\sigma \Lambda_\sigma, \tag{17} \]

where \( \Lambda(x) \) is an arbitrary 4-vector function. With the use of Eq. \( 13 \), a vector field corresponding to the set \( 17 \) takes the form

\[ \Phi^{(0)}_{\alpha} = \frac{2}{3} \nabla_\alpha \nabla^\beta \Lambda_\beta - \frac{1}{3} \nabla^\beta \nabla_\alpha \Lambda_\beta - \frac{1}{3} (\nabla^\beta \nabla_\beta) \Lambda_\alpha. \tag{18} \]

After substitution it into Eq. \( 12 \) one produces

\[ 0 = \frac{2}{3} (\nabla^\alpha \nabla_\alpha) \nabla^\beta \Lambda_\beta - \frac{1}{3} \nabla^\alpha \nabla^\beta \nabla_\alpha \Lambda_\beta - \frac{1}{3} \nabla^\beta (\nabla^\alpha \nabla_\alpha) \Lambda_\beta. \tag{19} \]

Employing conventionally the Riemann and Ricci tensors

\[ (\nabla_\beta \nabla_\alpha - \nabla_\alpha \nabla_\beta) \Lambda_\rho = R_{\beta \alpha \rho \sigma} \Lambda_\sigma, \quad R^{\beta}_{\alpha \rho \sigma} = R_{\alpha \sigma}, \]

the second term in \( 19 \) can be rewritten as

\[ -\frac{1}{3} \nabla^\alpha \nabla_\beta \nabla_\alpha \Lambda_\beta = -\frac{1}{3} \nabla^\alpha (\nabla_\alpha \nabla_\beta \Lambda_\beta + R_{\alpha \beta} \Lambda^\beta), \]

with the use of which Eq. \( 19 \) will take the form

\[ 0 = \frac{1}{3} (\nabla^\alpha \nabla_\alpha, \nabla^\beta) \Lambda_\beta - \frac{1}{3} \nabla^\alpha (R_{\alpha \beta} \Lambda^\beta). \tag{20} \]

The latter, with the commutator

\[ [\nabla^\alpha \nabla_\alpha, \nabla^\beta] \Lambda_\beta = -\nabla^\alpha (R_{\alpha \beta} \Lambda^\beta), \tag{21} \]

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will read as

\[ 0 = -\frac{2}{3} \nabla^\alpha (R_{\alpha\beta} \Lambda^\beta). \]  

(22)

This equation means: if \( R_{\alpha\beta} \neq 0 \), the present spin-2 particle equations do not have any trivial \( \lambda_a \)-based solution. In other terms, a gauge principle in accordance with Einstein gravitational equations the equality \( R_{\alpha\beta} \neq 0 \) speaks that at those \( x^\alpha \)-points any material fields vanish. However, in \( (R_{\alpha\beta} = 0) \) -region the wave equation under consideration includes such \( \lambda_a \)-based solutions and corresponds a gauge principle. Now, analogously, we should consider Eq. (14):

what will we have had on substituting \( \Lambda_{\alpha} \)-set into it. We must exclude all auxiliary fields from Eq. (14):

\[
\begin{align*}
\frac{1}{2} (\nabla^\rho \Phi^{(0)}_{\rho\alpha\beta} + \nabla^\rho \Phi^{(0)}_{\rho\beta\alpha}) - \frac{1}{2} g_{\alpha\beta}(x) \nabla^\rho \Phi^{(0)}_{\rho\sigma} \sigma \\
+ \nabla_\alpha \Phi^{(0)}_{\beta} + \nabla_\beta \Phi^{(0)}_{\alpha} - \frac{1}{2} g_{\alpha\beta}(x) \nabla^\rho \Phi^{(0)}_{\rho} = 0,
\end{align*}
\]

(23)

Let us step by step calculate all terms entering Eq. (23). For first (1) term we have that

\[
\begin{align*}
(1) \overset{\text{def}}{=} & \frac{1}{2} \nabla^\rho \Phi^{(0)}_{\rho\alpha\beta} = \frac{1}{2} (\nabla^\rho \nabla_\rho) (\nabla_\alpha \Lambda_\beta) \\
& + \frac{1}{2} (\nabla^\rho \nabla_\rho) (\nabla_\beta \Lambda_\alpha) - \frac{1}{4} g_{\alpha\beta} (\nabla^\rho \nabla_\rho) (\nabla^\gamma \Lambda_\gamma) \\
& - \frac{1}{2} (\nabla^\rho \nabla_\rho) \nabla_\alpha \Lambda_\beta - \frac{1}{2} \nabla^\rho (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) - \nabla_\beta \nabla_\beta (\nabla^\rho \Lambda_\rho) \\
& + \frac{1}{4} (\nabla_\beta \nabla_\alpha) (\nabla^\gamma \Lambda_\gamma) + \frac{1}{6} g_{\alpha\beta} (\nabla^\rho \nabla_\rho) (\nabla^\sigma \Lambda_\sigma) \\
& + \frac{1}{6} g_{\alpha\beta} [\nabla^\rho, \nabla_\rho] - \Lambda_\sigma + \frac{1}{6} g_{\alpha\beta} (\nabla^\sigma \nabla_\sigma) (\nabla^\rho \Lambda_\rho) \\
& + \frac{1}{6} g_{\alpha\beta} [\nabla^\rho, \nabla^\sigma \nabla_\sigma] - \Lambda_\rho - \frac{1}{12} g_{\alpha\beta} (\nabla^\rho \nabla_\rho) (\nabla^\gamma \Lambda_\gamma) \\
& - \frac{1}{6} \nabla_\beta \nabla_\alpha (\nabla^\rho \Lambda_\rho) - \frac{1}{6} \nabla_\beta [\nabla^\sigma, \nabla_\sigma] - \Lambda_\sigma + \frac{1}{12} \nabla_\beta \nabla_\alpha (\nabla^\gamma \Lambda_\gamma).
\end{align*}
\]

Second term in Eq. (23) can be produced on straightforward symmetry considerations from Eq. (23). Third term in Eq. (23) turns out to vanish

\[
\begin{align*}
(3) \overset{\text{def}}{=} & -\frac{1}{4} g_{\alpha\beta} \nabla^\rho \Phi^{(0)}_{\rho\alpha\beta} = -\frac{1}{2} g_{\alpha\beta} \nabla^\rho \nabla_\rho \Phi^{(0)}_{\rho\beta} \\
& = -\frac{1}{4} g_{\alpha\beta} \nabla^\rho \nabla_\rho (\nabla_\beta \Lambda_\beta + \nabla_\beta \Lambda_\beta - \frac{1}{2} \delta_\beta^\gamma \nabla^\gamma \Lambda_\gamma) = 0.
\end{align*}
\]

For fourth and fifth terms we will have
\[
\begin{align*}
(4) \text{ def } \nabla_{\alpha} \Phi_{\beta}^{(0)} &= \frac{1}{2} \nabla_{\alpha} \nabla_{\beta} \nabla^{\gamma} \Lambda_{\gamma} - \frac{1}{3} \nabla_{\alpha} \nabla_{\beta} (\nabla^{\rho} \Lambda_{\rho}) \\
&- \frac{1}{3} \nabla_{\alpha} [\nabla^{\rho}, \nabla_{\beta}] \Lambda_{\rho} - \frac{1}{3} (\nabla^{\rho} \nabla_{\rho}) \nabla_{\alpha} \Lambda_{\beta} \\
&- \frac{1}{3} \nabla_{\alpha} [\nabla^{\rho} \nabla_{\beta}] - \Lambda_{\beta} + \frac{1}{6} \nabla_{\alpha} \nabla_{\beta} \nabla^{\gamma} \Lambda_{\gamma},
\end{align*}
\]

\[
(5) \text{ def } \nabla_{\beta} \Phi_{\alpha}^{(0)} = \frac{1}{2} \nabla_{\beta} \nabla_{\alpha} \nabla^{\gamma} \Lambda_{\gamma}
\]

\[
\begin{align*}
- \frac{1}{3} \nabla_{\beta} \nabla_{\alpha} (\nabla^{\rho} \Lambda_{\rho}) - \frac{1}{3} \nabla_{\beta} [\nabla^{\rho}, \nabla_{\alpha}] \Lambda_{\rho} - \frac{1}{3} (\nabla^{\rho} \nabla_{\rho}) \nabla_{\beta} \Lambda_{\alpha} \\
&- \frac{1}{3} [\nabla_{\beta}, \nabla^{\rho} \nabla_{\rho}] - \Lambda_{\alpha} + \frac{1}{6} \nabla_{\beta} \nabla_{\alpha} \nabla^{\gamma} \Lambda_{\gamma};
\end{align*}
\]

and term (6) is

\[
(6) \text{ def } = - \frac{1}{2} g_{\alpha \beta} \nabla^{\rho} \Phi_{\rho}^{(0)} - \frac{1}{2} g_{\alpha \beta} \nabla^{\rho} \Phi_{\rho}^{(0)}
\]

\[
\begin{align*}
= & - \frac{1}{4} g_{\alpha \beta} (\nabla^{\rho} \nabla_{\rho}) (\nabla^{\gamma} \Lambda_{\gamma}) + \frac{1}{6} g_{\alpha \beta} (\nabla^{\rho} \nabla_{\rho}) (\nabla^{\sigma} \Lambda_{\sigma}) \\
&+ \frac{1}{6} g_{\alpha \beta} \nabla^{\rho} [\nabla^{\sigma}, \nabla_{\rho}] - \Lambda_{\sigma} + \frac{1}{6} g_{\alpha \beta} (\nabla^{\sigma} \nabla_{\sigma}) (\nabla^{\rho} \Lambda_{\rho}) \\
&+ \frac{1}{12} g_{\alpha \beta} [\nabla^{\rho}, \nabla^{\sigma} \nabla_{\sigma}] - \Lambda_{\rho} - \frac{1}{12} g_{\alpha \beta} (\nabla^{\rho} \nabla_{\rho}) (\nabla^{\gamma} \Lambda_{\gamma}).
\end{align*}
\]

Summing up all six expressions and taking into account similar terms (factors at all terms without commutators turn out to be equal zero as should be expected):

\[
0 = (\nabla^{\rho} \nabla_{\rho}) (\nabla_{\alpha} \Lambda_{\beta}) \left[ \left( \frac{1}{2} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{6} \right) - \frac{1}{3} \right] \\
+ (\nabla^{\rho} \nabla_{\rho}) (\nabla_{\beta} \Lambda_{\alpha}) \left[ \left( \frac{1}{2} - \frac{1}{6} \right) + \left( \frac{1}{2} - \frac{1}{2} \right) - \frac{1}{3} \right] \\
+ g_{\alpha \beta} (\nabla^{\rho} \nabla_{\rho}) (\nabla^{\sigma} \Lambda_{\gamma}) \left[ \left( -\frac{1}{4} + \frac{1}{6} + \frac{1}{6} - \frac{1}{12} \right) + \left( -\frac{1}{4} + \frac{1}{6} + \frac{1}{6} - \frac{1}{12} \right) \right] \\
+ \nabla_{\alpha} \nabla_{\beta} (\nabla^{\rho} \Lambda_{\rho}) \left[ \left( -\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{12} \right) + \left( -\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{12} \right) \right. \\
\left. + \left( \frac{1}{2} - \frac{1}{3} + \frac{1}{6} \right) + \left( \frac{1}{2} - \frac{1}{3} + \frac{1}{6} \right) \right] \\
+ \left\{ - \frac{1}{2} \nabla^{\rho} [\nabla_{\alpha}, \nabla_{\rho}] - \Lambda_{\beta} - \frac{1}{2} [\nabla^{\rho}, \nabla_{\alpha} \nabla_{\beta}] - \Lambda_{\rho} \right\}
\]
With Eq. (26), Eq. (24) takes the form

\[ 0 = g_{\alpha \beta} \nabla_\rho (R^\rho{}_{\alpha \beta \sigma} \Lambda_\sigma) + \Lambda^\sigma \left[ \nabla_\rho R^\rho{}_{\alpha \beta \sigma} + \nabla_\rho R^\rho{}_{\beta \alpha \sigma} \right] + (\nabla_\rho \Lambda_\sigma) \left[ R^\rho{}_{\alpha \beta} \sigma + R^\rho{}_{\beta \alpha} \sigma \right] - \Lambda^\rho \left[ \nabla_\alpha R^\rho{}_{\beta \rho} + \nabla_\beta R^\rho{}_{\alpha \rho} \right] - \frac{3}{2} \left[ R^\rho{}_{\beta} (\nabla_\alpha \Lambda_\rho) + R^\rho{}_{\alpha} (\nabla_\beta \Lambda_\rho) \right] + \frac{1}{2} \left[ R^\rho{}_{\beta} (\nabla_\rho \Lambda_\alpha) + R^\rho{}_{\alpha} (\nabla_\rho \Lambda_\beta) \right] \]. \tag{24}

Calculating in series all commutators, after simple calculation we will produce

\[ 0 = g_{\alpha \beta} \nabla_\rho (R^\rho{}_{\alpha \beta \sigma} \Lambda_\sigma) + \Lambda^\sigma \left[ \nabla_\rho R^\rho{}_{\alpha \beta \sigma} + \nabla_\rho R^\rho{}_{\beta \alpha \sigma} \right] + (\nabla_\rho \Lambda_\sigma) \left[ R^\rho{}_{\alpha \beta} \sigma + R^\rho{}_{\beta \alpha} \sigma \right] - \Lambda^\rho \left[ \nabla_\alpha R^\rho{}_{\beta \rho} + \nabla_\beta R^\rho{}_{\alpha \rho} \right] - \frac{3}{2} \left[ R^\rho{}_{\beta} (\nabla_\alpha \Lambda_\rho) + R^\rho{}_{\alpha} (\nabla_\beta \Lambda_\rho) \right] + \frac{1}{2} \left[ R^\rho{}_{\beta} (\nabla_\rho \Lambda_\alpha) + R^\rho{}_{\alpha} (\nabla_\rho \Lambda_\beta) \right] + \frac{1}{6} g_{\alpha \beta} \left\{ \frac{1}{3} \nabla_\lambda \left[ \nabla_\rho (\nabla_\rho \nabla_\lambda - \Lambda_\lambda) + (\nabla_\rho \nabla_\sigma - \Lambda_\sigma) \right] \right\}. \tag{27}

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\[ 0 = g_{\alpha \beta} \nabla_\rho (R^\rho{}_{\alpha \beta \sigma} \Lambda_\sigma) + \Lambda^\sigma \left[ \nabla_\rho R^\rho{}_{\alpha \beta \sigma} + \nabla_\rho R^\rho{}_{\beta \alpha \sigma} \right] + (\nabla_\rho \Lambda_\sigma) \left[ R^\rho{}_{\alpha \beta} \sigma + R^\rho{}_{\beta \alpha} \sigma \right] - \Lambda^\rho \left[ \nabla_\alpha R^\rho{}_{\beta \rho} + \nabla_\beta R^\rho{}_{\alpha \rho} \right] - \frac{3}{2} \left[ R^\rho{}_{\beta} (\nabla_\alpha \Lambda_\rho) + R^\rho{}_{\alpha} (\nabla_\beta \Lambda_\rho) \right] + \frac{1}{2} \left[ R^\rho{}_{\beta} (\nabla_\rho \Lambda_\alpha) + R^\rho{}_{\alpha} (\nabla_\rho \Lambda_\beta) \right] + \frac{1}{6} g_{\alpha \beta} \left\{ \frac{1}{3} \nabla_\lambda \left[ \nabla_\rho (\nabla_\rho \nabla_\lambda - \Lambda_\lambda) + (\nabla_\rho \nabla_\sigma - \Lambda_\sigma) \right] \right\}. \tag{27}

It must be noticed that contrary to the expectations the equation obtained contains explicitly the curvature Riemann tensor. It enters into Eq. (24) in two combinations:

\[ \Lambda^\sigma \left[ \nabla_\rho R^\rho{}_{\alpha \beta \sigma} + \nabla_\rho R^\rho{}_{\beta \alpha \sigma} \right] + (\nabla_\rho \Lambda_\sigma) \left[ R^\rho{}_{\alpha \beta} \sigma + R^\rho{}_{\beta \alpha} \sigma \right] \tag{25} \]

The curvature tensor in combination (25) can be readily escaped. To this end, it suffices for the Bianchi identity

\[ \nabla_\gamma R^\rho{}_{\alpha \beta \sigma} + \nabla_\sigma R^\rho{}_{\alpha \gamma \beta} + \nabla_\beta R^\rho{}_{\alpha \sigma \gamma} = 0, \quad \nabla_\rho R^\rho{}_{\alpha \beta \sigma} + \nabla_\sigma R^\rho{}_{\beta \alpha \sigma} - \nabla_\beta R^\rho{}_{\alpha \sigma} = 0. \tag{26} \]

Thus,

\[ \nabla_\rho R^\rho{}_{\alpha \beta \sigma} + \nabla_\rho R^\rho{}_{\beta \alpha \sigma} = (\nabla_\alpha R^\rho{}_{\beta \sigma} + \nabla_\beta R^\rho{}_{\alpha \sigma}) - 2 \nabla_\sigma R^\rho{}_{\beta \alpha} \]. \tag{26}

With Eq. (26), Eq. (24) takes the form

\[ 0 = g_{\alpha \beta} \nabla_\rho (R^\rho{}_{\alpha \beta \sigma} \Lambda_\sigma) - 2 \Lambda^\sigma \nabla_\sigma R^\rho{}_{\alpha \beta} + (\nabla_\rho \Lambda_\sigma) \left[ R^\rho{}_{\alpha \beta} \sigma + R^\rho{}_{\beta \alpha} \sigma \right] - \frac{3}{2} \left[ R^\rho{}_{\beta} (\nabla_\alpha \Lambda_\rho) + R^\rho{}_{\alpha} (\nabla_\beta \Lambda_\rho) \right] + \frac{1}{2} \left[ R^\rho{}_{\beta} (\nabla_\rho \Lambda_\alpha) + R^\rho{}_{\alpha} (\nabla_\rho \Lambda_\beta) \right] \]. \tag{27}
However, the curvature tensor still remains to enter Eq. (27). And this means that in regions involving curvature the above massless spin-2 equation does not allow any gauge principle.

Now we will show that in order to overcome such a difficulty the above starting equations should be slightly altered. To this end, let us add special term (a not minimal gravitational interaction term) into Eq. (14):

\[
\begin{align*}
& \frac{1}{2} \left( \nabla^\rho \Phi_{\rho \alpha \beta} + \nabla^\rho \Phi_{\rho \beta \alpha} - \frac{1}{2} g_{\alpha \beta}(x) \nabla^\rho \Phi_{\rho \sigma} \right) \\
& + \left( \nabla_\alpha \Phi_\beta + \nabla_\beta \Phi_\alpha - \frac{1}{2} g_{\alpha \beta}(x) \nabla^\rho \Phi_\rho \right) = A[R^\rho_{\alpha \beta} \sigma + R^\rho_{\beta \alpha} \sigma] \Phi_{\rho \sigma}.
\end{align*}
\]

(28)

Let us show that at special parameter \( A \) the theory of massless spin-2 particle can be done satisfactory in the sense of the above gauge principle. Indeed,

\[
AR^\rho_{\alpha \beta} \sigma \Phi^{(0)}_{\rho \sigma} = AR^\rho_{\alpha \beta} \sigma \left( \nabla_\rho \Lambda_\sigma + \nabla_\sigma \Lambda_\rho - \frac{1}{2} g_{\rho \sigma} \nabla^\gamma \Lambda_\gamma \right)
\]

\[
= A \left[ R^\rho_{\alpha \beta} \sigma \nabla_\rho \Lambda_\sigma + R^\rho_{\beta \alpha} \sigma \nabla_\rho \Lambda_\sigma + \frac{1}{2} R_{\alpha \beta} \nabla^\gamma \Lambda_\gamma \right]
\]

and therefore a contribution of that additional term into (27) is equal to

\[
A[R^\rho_{\alpha \beta} \sigma + R^\rho_{\beta \alpha} \sigma] \Phi^{(0)}_{\rho \sigma} = 2A[R^\rho_{\alpha \beta} \sigma + R^\rho_{\beta \alpha} \sigma](\nabla_\rho \Lambda_\sigma) + AR_{\alpha \beta} (\nabla^\gamma \Lambda_\gamma).
\]

(29)

So, instead of Eq. (27) we have

\[
2A(\nabla_\rho \Lambda_\sigma)[R^\rho_{\alpha \beta} \sigma + R^\rho_{\beta \alpha} \sigma] + AR_{\alpha \beta} (\nabla^\gamma \Lambda_\gamma)
\]

\[
= g_{\alpha \beta} \nabla_\rho (R^\rho_{\alpha \beta} \sigma) - 2\Lambda^\sigma \nabla_\sigma R_{\alpha \beta} + (\nabla_\rho \Lambda_\sigma)[R^\rho_{\alpha \beta} \sigma + R^\rho_{\beta \alpha} \sigma]
\]

\[
- \frac{3}{2} [R^\rho_{\beta \sigma} (\nabla_\alpha \Lambda_\rho) + R^\rho_{\alpha \sigma} (\nabla_\beta \Lambda_\rho)] + \frac{1}{2} [R^\rho_{\beta \sigma} (\nabla_\rho \Lambda_\alpha) + R^\rho_{\alpha \sigma} (\nabla_\rho \Lambda_\beta)].
\]

(30)

Setting \( A = \frac{1}{2} \), both terms with curvature tensor will be cancelled by each other:

\[
\frac{1}{2} R_{\alpha \beta} (\nabla^\gamma \Lambda_\gamma) = g_{\alpha \beta} \nabla_\rho (R^\rho_{\alpha \beta} \sigma) - 2\Lambda^\sigma \nabla_\sigma R_{\alpha \beta}
\]

\[
- \frac{3}{2} [R^\rho_{\beta \sigma} (\nabla_\alpha \Lambda_\rho) + R^\rho_{\alpha \sigma} (\nabla_\beta \Lambda_\rho)] + \frac{1}{2} [R^\rho_{\beta \sigma} (\nabla_\rho \Lambda_\alpha) + R^\rho_{\alpha \sigma} (\nabla_\rho \Lambda_\beta)].
\]

(31)

Finally the obtained relationship does not contain the curvature tensor and will turn into identity at \( R_{\alpha \beta}(x) = 0 \) which was required. So, the required system is which one changes Eq. (14) by

\[
\frac{1}{2} \left( \nabla^\rho \Phi_{\rho \alpha \beta} + \nabla^\rho \Phi_{\rho \beta \alpha} - \frac{1}{2} g_{\alpha \beta}(x) \nabla^\rho \Phi_{\rho \sigma} \right) \\
+ \left( \nabla_\alpha \Phi_\beta + \nabla_\beta \Phi_\alpha - \frac{1}{2} g_{\alpha \beta}(x) \nabla^\rho \Phi_\rho \right) = \frac{1}{2} (R^\rho_{\alpha \beta} \sigma + R^\rho_{\beta \alpha} \sigma) \Phi_{\rho \sigma}.
\]

(32)
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