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SOLUTION OF SELF-DUALITY EQUATION IN QUANTUM-GROUP GAUGE THEORY AND QUANTUM HARMONICS

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Abstract

We discuss the gauge theory for quantum group $SU_q(2) \times U(1)$ on the quantum Euclidean space. This theory contains three physical gauge fields and one $U(1)$—gauge field with a zero field strength. We construct the quantum-group self-duality equation (QGSDE) in terms of differential forms and with the help of the field-strength decomposition. A deformed analog of the BPST-instanton solution is obtained. We consider a harmonic (twistor) interpretation of QGSDE in terms of $SU_q(2)/U(1)$ quantum harmonics. The quantum harmonic gauge equations are formulated in the framework of a left-covariant 3D differential calculus on the quantum group $SU_q(2)$.

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An attractive idea of quantum deformations for the gauge theories has been considered in the framework of different approaches [1 - 6]. Formally one can discuss independent deformations of basic spaces and gauge groups and possible correlations between these deformations. We shall here consider a gauge theory with identical one-parameter deformations of the 4-dimensional Euclidean space and the gauge group $SU(2)$. A consistent formulation of the gauge theory for the semisimple quantum group $SU_q(N)$ is unknown to us, so we shall deal with the quantum group $U_q(2) = SU_q(2) \times U(1)$. It will be shown that the $U(1)$-gauge field can be treated as a field with a zero field strength.

Consider the standard relations between elements $T^i_k$ of the quantum $U_q(2)$-matrix [8]

$$RT_1 T_2 = T_1 T_2 R \ , \quad R^2 = I + (q - q^{-1})R$$

where $I$ is a unity matrix, $R$ is a constant symmetric matrix with components $R_{lm}^{ik}(q)$ ($i, k, l, m = 1, 2$) and $q$ is a real deformation parameter.

It is convenient to use the following covariant representation for a deformed antisymmetric symbol

$$\varepsilon_{ik}(q) = \sqrt{q(ik)} \varepsilon_{ik} = -q(ik)\varepsilon_{ki}(q)$$

$$q(12) = [q(21)]^{-1} = q, \quad q(11) = q(22) = 1$$

$$\varepsilon_{ik}(q)\varepsilon^{kl}(q) = \delta^l_i$$

where $\varepsilon_{ik}$ is an ordinary antisymmetric symbol ($\varepsilon_{ik} = \varepsilon^{ki}$).

The $R$-matrix can be written in terms of projection operators $P^{(+)}$ and $P^{(-)}$ [8]

$$R = qP^{(+)} - q^{-1}P^{(-)} = qI - (q + q^{-1})P^{(-)}$$

$$P^{(+)} + P^{(-)} = I \ ,$$

$$(P^{(\pm)})^2 = P^{(\pm)} , \quad P^{(+)}P^{(-)} = 0$$

where matrix $P^{(-)}$ has the following components

$$[P^{(-)}]_{lm}^{ik} = -\frac{q}{1 + q^2} \varepsilon_{kl}(q)\varepsilon_{ml}(q)$$

We shall use also covariant representation for the $SU_q(2)$-metric

$$D^i_k(q) = -\varepsilon_{mk}(q)\varepsilon^{mi}(q)$$

The basic RTT-relations imply the simple equation

$$\varepsilon_{ml}(q)T^l_k T^m_i = \varepsilon_{ki}(q) \ D(T)$$

where $D(T)$ is the quantum determinant [8].

A covariant expression for the inverse quantum matrix $S(T) = T^{-1}$ contains inverse determinant

$$S^l_k = \varepsilon_{kl}(q) \ T_j^l \varepsilon^{ji}(q) \ D^{-1}(T)$$
We shall use the well-known equations for multiplication of the transposed matrices

\[ T_i^i D^m_i(q) S^k_m = D^k_i(q) \]

The unitarity condition for the matrix \( T \) can be formulated with the help of involution

\[ T_k^i \rightarrow \overline{T_k^i} = S^k_i \]

Let us consider the bicovariant differential calculus on the \( U_q(2) \) group [9 - 12]

\[ T_1 dT_2 = R dT_1 T_2 R \]

\[ D(T) dT = q^2 dT D(T) \]

\[ [d, T] = dT, \quad \{d, dT\} = 0 \]

Note that the condition \( D(T) = 1 \) is inconsistent in the framework of this calculus.

Consider the relations for the right-invariant differential forms \( \omega = dTS \)

\[ \omega R \omega + R \omega R \omega R = 0 \]

\[ T_1 \omega_2 = R \omega_1 R T_1 \]

The quantum trace \( \xi \) of the form \( \omega \) plays an important role in this calculus

\[ \xi(T) = D^k_i(q) \omega^j_k(T) \neq 0, \quad \xi^2 = 0, \quad d\xi = 0 \]

\[ d\omega = \omega^2 = -(q^2 \lambda)^{-1} \{\xi, \omega\} \]

The bicovariant calculus makes the basis for consistent formulation of quantum-group gauge theory in the framework of noncommutative algebra of differential complexes [5-7]. Consider formally the quantum group gauge matrix \( T^a_b(x) \) defined on some basic space. Suppose that Eqs(12-15) satisfy locally for each ”point” \( x \). Then one can try to construct the \( U_q(2) \)-connection 1-form \( A^a_b(x) \) which obeys the simplest commutation relation

\[ A R A + R A R A R = 0 \]

Note that the general relation for \( A \) contains a nontrivial right-hand side [7].

Coaction of the gauge quantum group \( U_q(2) \) has the following standard form:

\[ A \rightarrow T(x) A S(T) + dT(x) S(T) = T A S + \omega(T) \]

\[ \alpha = \text{Tr}_q A \rightarrow \alpha + \xi(T) \]

The restriction \( \alpha = 0 \) is inconsistent with [16], but we can use the gauge-covariant relations

\[ \alpha^2 = 0, \quad \text{Tr}_q A^2 = 0 \]

It should be stressed that we can choose the zero field-strength condition \( d\alpha = 0 \) for the \( U(1) \)-gauge field [1]. This constraint is gauge invariant and consistent with (16). The

\[ ^1\text{This condition is consistent also for the case of } GL_q(N) \text{ group} \]
deformed pure gauge field $\alpha$ can be decoupled from the set of physical fields in the limit $q = 1$. We shall further consider the $U_q(2)$-gauge theory with three "physical” gauge fields and one "zero-mode” $U(1)$ field.

The curvature 2-form is $q$-traceless for this model

$$ F = dA - A^2, \quad \text{Tr}_q F = 0 \quad (19) $$

Quantum deformations of Minkowski and Euclidean 4-dimensional spaces have been considered in Refs [13-15]. We shall treat the coordinates $x^\alpha_i$ of $q$-deformed Euclidean space $E_q(4)$ as generators of a noncommutative algebra covariant under the coaction of the quantum group $G_q(4) = SU_q^L(2) \times SU_q^R(2)$

$$ R^m_{lm} x^\alpha_i x^\beta_j = x^\gamma_k x^\delta_p R^\gamma_{\alpha\beta} \quad (20) $$

where we use two identical copies of R-matrices (4) for left and right $SU_q(2)$-indices.

Coactions of the commuting left and right $SU_q(2)$ groups conserve

$$ x^i_\alpha \rightarrow l^i_k x^k_\beta r^\beta_\alpha \quad (21) $$

The $q$-deformed central Euclidean interval $\tau$ can be constructed by analogy with the quantum determinant

$$ \tau = |x|^2 = -\frac{q}{1 + q^2} \varepsilon^{\beta\alpha}(q)\varepsilon_{ki}(q)x^i_\alpha x^k_\beta \quad (22) $$

We do not consider the quantum group structure on $E_q(4)$. It is convenient to use the following $E_q(4)$ involution

$$ \overline{x^i_\alpha} = \varepsilon_{ik}(q)x^k_\beta \varepsilon^{\beta\alpha}(q) = \tau S^\alpha_i(x) \quad (23) $$

$$ \overline{\tau} = \tau, \quad \overline{x^i_\alpha} = x^i_\alpha $$

We shall use an analog of the bicovariant $U_q(2)$-calculus (12-15) for studying differential complexes on $E_q(4)$. Consider the right-invariant 1-forms

$$ \omega(x)_k^i = dx^i_\alpha S^\alpha_k(x) \quad (24) $$

Basic 2-forms on $E_q(4)$ can be decomposed with the help of $P(\pm)$ operators (5)

$$ dx^i_\alpha dx^k_\beta = \frac{q}{1 + q^2}[\varepsilon^{\beta\alpha}(q)d^2 x^i_\alpha + \varepsilon^{\alpha\beta}(q)d^2 x^k_\alpha] \quad (25) $$

By analogy with the classical case we can treat these two parts as self-dual and anti-self-dual 2-forms under the action of a duality operator $\star$. It is convenient to rewrite this decomposition in terms of the right-invariant self-dual and anti-self-dual forms

$$ P(-) dx_1 dx_2 P(+) = qP(-)(q^3 \omega^2 + \omega \xi)P(+)x_1 x_2 P(+) \quad (26) $$

$$ P(+) dx_1 dx_2 P(-) = -(1/q)P(+)\omega R\omega P(-)\tau = (q^{-1} \omega \xi - \omega^2) P(-) \quad (27) $$

Let us introduce the simple ansatz for quantum $U_q(2)$ anti-self-dual gauge fields

$$ A^a_b = dx^i_\alpha A^{a\alpha}_b(x) = \omega^a_b(x)f(\tau), \quad (28) $$

$$ A^{\alpha a}_b(x) = \delta^\alpha_i S^a_b(x)f(\tau), \quad (29) $$
where \( f(\tau) \) is a function of the \( q \)-interval \(^{[22]}\). Note that this ansatz is a partial case of more general construction of differential complex on \( GL_q(2) \) \(^{[5,7]}\). Addition of the term \( \xi(x)g(\tau) \) results in a relation for the connection \( A \) more complicated than \(^{[16]}\).

Consider the \( q \)-traceless curvature form for the connection \(^{29}\)

\[
F = \omega^2 f(\tau)[1 - f(q^2 \tau)] + (q^2 \lambda)^{-1} \omega \xi[f(\tau) - f(q^2 \tau)]
\]

The anti-self-duality equation \( \ast F = -F \) for our ansatz is equivalent to the nonlinear finite-difference equation

\[
f(\tau) - f(q^2 \tau) = (1 - q^2)f(\tau)[1 - f(q^2 \tau)]
\]

This equation has a simple solution analogous to the classical BPST-solution

\[
f(\tau) = \frac{\tau}{c + \tau},
\]

where \( c \) is an arbitrary constant. Note that our solution for connection \( A \) contains parameter \( q \) only through definitions of \( \omega(x) \) and \( \tau \), however, the corresponding curvature has a more explicit \( q \)-dependence.

The curvature form can be written in terms of the field strength

\[
F = dx^i dx^k F^\beta_\alpha_{ki}(x) = d^2 x^{i\beta} F^{\beta\alpha} + d^2 x^{ik} F_{ki},
\]

where Eq\(^{(23)}\) is used.

The QGSD-equation for the field strength has the following form

\[
F_{ki}^{\beta\alpha} = \varepsilon_{ki}(q) F^{\beta\alpha}
\]

It is interesting to discuss the \( q \)-deformation of the harmonic (or twistor) formalism for QGSDE. The \( q \)-deformed harmonics can be considered as elements of \( SU_q(2) \) matrix \( u^i_a \) \( (i = 1, 2, \ a = +, -) \). We shall treat these matrix elements as coordinates of the noncommutative coset space \( SU_q(2)/U(1) \) by analogy with the classical harmonic formalism for a self-duality equation \(^{[16]}\).

Consider \( SU_q(2) \times U(1) \) cotransformations of \( q \)-harmonics

\[
u^i_+ \rightarrow l^i_k u^k_+ \exp(\pm i\alpha),
\]

where \( \alpha \) is \( U(1) \) parameter and \( l \) is a matrix of left \( SU_q(2) \) acting on \( E_q(4) \).

The \( q \)-harmonics satisfy the following relations:

\[
Ru_1 u_2 = u_1 u_2 R, \quad qu_1 x_2 = Rx_1 u_2, \quad \varepsilon_{ki}(q) u^i_- u^k_+ = \sqrt{q},
\]

It is convenient to use the 3-dimensional left-invariant differential calculus on \( SU_q(2) \) \(^{[9, 17]}\) for the harmonic formalism. Consider the \( q \)-traceless left-invariant 1-forms \( \theta = S(u)du \) and introduce the notations:

\[
\theta_0 = \theta^+_+ = -q^{-2} \theta^-_, \quad \theta_{(+2)} = \theta^+_-, \quad \theta_{(-2)} = \theta^+_+
\]
We shall below write the left-invariant relations between $\theta$ and $u$ which allow us to define the operator of harmonic external derivative on $SU_q(2)$
\[ du = \delta_0 + \delta + \bar{\delta} = \theta_0 D_0 + \theta_{(-2)} D_{(+2)} + \theta_{(+2)} D_{(-2)} , \] (36)
where $\delta_0$, $\delta$ and $\bar{\delta}$ are invariant operators satisfying the ordinary Leibniz rules and the following relations
\[ \delta_0^2 = \delta^2 = \bar{\delta}^2 = 0 \]
(37)
\[ \{\delta_0, \delta\} + \{\delta_0, \bar{\delta}\} + \{\delta, \bar{\delta}\} = 0 \]

The left-invariant differential operators $D_0$, $D_{(+2)}$ are the basis of a $q$-deformed Lie algebra equivalent to the universal enveloping algebra $U_q[SU(2)]$ [17].

Let us define an invariant decomposition of the Maurer-Cartan equations for $SU_q(2)/U(1)$
\[ du \theta_0 = 2 \delta \theta_0 = 2 \bar{\delta} \theta_0 = -\theta_{(-2)} \theta_{(+2)} \]
(38)
\[ du \theta_{(+2)} = \delta \theta_{(+2)} = q^2 (1 + q^2) \theta_0 \theta_{(+2)} \]
\[ du \theta_{(-2)} = \delta \theta_{(-2)} = q^2 (1 + q^2) \theta_{(-2)} \theta_0 \]

Define also $\delta_0$, $\delta$ and $\bar{\delta}$ operators on the quantum harmonics
\[ \delta_0 u^i_+ = u^i_+ \theta_0 = q^2 \theta_0 u^i_+ , \quad \delta u^i_+ = 0 \]
(39)
\[ \bar{\delta} u^i_+ = u^i_+ \theta_{(+2)} = q^{-1} \theta_{(+2)} u^i_- \]
\[ \delta_0 u^i_- = -q^2 u^i_- \theta_0 = -\theta_0 u^i_- , \quad \bar{\delta} u^i_- = 0 \]
(40)
\[ \delta u^i_- = u^i_- \theta_{(-2)} = q \theta_{(-2)} u^i_+ \]

Global functions on a quantum 2-sphere can be defined via the invariant condition
\[ \delta_0 f(u) = \theta_0 D_0 f(u) = 0 \]

Consider the harmonic decomposition of the Euclidean coordinates and derivatives
\[ x_{(b)\alpha} = -q \varepsilon_{ik} (q) u^k_b x^i_\alpha , \quad \partial^a_\alpha = u^i_a \partial^a_i \]
(41)
\[ \partial^a_\alpha x_{(b)\beta} = \delta^a_{\beta} \varepsilon_{\alpha \beta} (q) \]
where $a, b = +, -$.

One can use the asymmetric decomposition of the operator $d_x$ on $E_q(4)$
\[ d_x = dx^i_\alpha \partial^a_\alpha = \bar{d} + (d_x - \bar{d}) \]
(42)
\[ \bar{d} \sim dx_{(-)\alpha} \partial^a_\alpha , \quad \bar{d}^2 = 0 \]
\[ \{\bar{d}, \delta\} = 0 \]

It should be remarked that the use of the symmetric decomposition results in a modification of analitcity condition for corresponding invariant operators.

An analyticity condition for the deformed harmonic space has the following form
\[ \partial^a_\alpha \Lambda(x_{(+)}, u) = 0 \iff \bar{d} \Lambda = 0 \]
Multiplying QGSDE (32) by $u^i_+ u^k_+$ one can obtain the $q$-deformed integrability conditions in central basis (CB) CB corresponds to $u$-independent gauge-group matrices $T(x)$ which are analogous to the classical self-dual integrability conditions [16, 18].

Consider the decomposition of the $U_q(2)$-connection in CB corresponding to (42) and let $\bar{a} \sim dx(-)_{\alpha}A_{\alpha}^\alpha(x)$ be a connection for $\bar{d}$. The quantum-group self-duality equation (32) is equivalent to the following zero-curvature equation

$$\bar{d}\bar{a} - \bar{a}^2 = 0$$ (43)

This equation has the following harmonic solution

$$\bar{a} = \bar{d}hS(h) = \omega(h, \bar{d}h)$$ (44)

where $h(x, u)$ is a ”bridge” $U_q(2)$-matrix function. The matrix elements of $h$ and $\bar{d}h$ satisfy the relations analogous to Eqs(12-15). Additional harmonic relations are

$$\delta \bar{a} = 0, \quad \delta_0 h = 0$$ (45)

The bridge solution possesses a nontrivial gauge freedom

$$h \to T(x)h\Lambda(x(+), u), \quad \delta_0 \Lambda = 0$$

where $\Lambda$ is an analytical $U_q(2)$ gauge matrix.

The matrix $h$ can be treated as a transition matrix from the central basis to the analytical basis (AB) where $\bar{d}$ has no connection. The characteristic feature of AB is a nontrivial harmonic connection $V$ that is an invariant component of a new AB-basis $A_{AB}$ in the algebra of $U_q(2)$ differential complexes

$$A_{AB} = S(h)Ah - S(h)dh = \bar{A} - S(h)d_\bar{a}h = \bar{A} + V$$ (46)

$$V = -S(h)\delta h - S(h)\bar{\delta}h = v + \bar{v}$$

where the analytical connection $v = \theta_{(-2)}V_{(+2)}$ contains the analytical prepotential $V_{(+2)}$.

By analogy with the classical harmonic formalism [16] the prepotential $V_{(+2)}$ generates a general solution of QGSDE that can be obtained as a solution of the basic harmonic gauge equation

$$\delta h + hv = 0$$ (47)

One can obtain explicit or perturbative solutions of this equation by using the noncommutative generalizations of classical harmonic expansions and harmonic Green functions [16, 19]. It seems very interesting to study reductions of QGSDE to lower dimensions and to search a more general deformation scheme for the self-duality equation.

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