QUADRATIC FORMULA: REVISITING A PROOF THROUGH THE LENS OF TRANSFORMATIONS

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ABSTRACT

In this paper, we derive the quadratic formula as a consequence of constructively proving the existence of standard and factored forms for general form real quadratic functions. Emphasis is put on connections to graphing of corresponding parabolas through function transformation perspectives of scaling and translation.

Keywords Quadratic Formula · Proof · Transformation · Quadratics

1 Introduction

In the routine of teaching college mathematics at an introductory and general education level, frequent reconsideration of how best to present certain topics is a common challenge. Derivations of significant results certainly have their place, but tradition and routine can often create some intellectual stagnancy, and a shift in perspective can be refreshing for instructors and students.

Po-Shen Loh’s expositions [1][2] on deriving the quadratic formula through factorization and symmetry arguments is a less common algebraic approach than a traditional completing-the-square method. In 2019, it caught some attention in mathematics education circles, likely due to its refreshing derivation method. As Loh points out, the approach is hardly new in the sense of the history of algebra, but certainly may have fallen out of favor in modern western mathematics education.

Perhaps the most strikingly satisfying features of the method Loh revisits to solve a quadratic equation is the symmetry of the set-up using linear factorization blended with the connection to an ancient mathematics problem: solving for two unknowns given their known product and sum. Though this approach has its own form of algebraic elegance, there are some that may find the symbolic algebraic arguments only marginally more accessible than a traditional completing-the-square based derivation of the quadratic formula.

In some presentations of college algebra and precalculus level content (see Blitzer’s Precalculus [3]), analysis of quadratic functions and derivation of the quadratic formula follows development of both the topics of function transformation and a basic introduction to the complex plane C and its component-based algebra. For those craving a visually accessible motivation for the linear factorization based derivation, as well as to simultaneously build connections between general, standard, and factored forms of quadratics, we revisit a derivation of the quadratic formula inspired by Loh’s perspective and additionally under an assumption of a familiarity with function transformation.
1.1 A Foundation of Transformations

At the Precalculus level in the development of introductory function theory in $\mathbb{R}$, one encounters the concept of relating basic “parent” functions and their Cartesian graphs to the family of transformed functions and graphs through scaling (colloquially stretching, squishing, and reflecting) and translation (shifting).

Given a parent function $g(t)$ with translation constants $C, D \in \mathbb{R}$ and non-zero scaling constants $A, B \in \mathbb{R}$, a transformed version of function $g$ into a function $f$ may be given by

$$f(x) = A \cdot g(B \cdot x + C) + D.$$ 

Using the domain and range variable correspondence system

$$\begin{cases} 
  t = B \cdot x + C \\
  A \cdot s + D = y 
\end{cases}$$

(1)

to connect parent-graph data to transformed-graph data, a correspondence of points $(t, s)$ on the graph of $s = g(t)$ with points $(x, y)$ on the graph of $y = f(x)$ through the point correspondence of

$$\begin{pmatrix} t \\ s \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t - C \\ B \end{pmatrix}, \quad A \cdot s + D = \begin{pmatrix} t - C \\ B \end{pmatrix}, \quad A \cdot g(t) + D = \begin{pmatrix} x \\ f(x) \end{pmatrix}$$

may be achieved.

Avoiding reuse of traditional $x$ and $y$ variables for both parent-function input and output as well as transformed-function input and output allows for some simplicity in using substitutions to relate data on the corresponding graphs, as well as more readily adapt to tabular functions’ data in the abstraction of transformations.

**Example 1.1.** Suppose it is given that parent wave function $g : \mathbb{R} \to \mathbb{R}$ given by $g(t) = \sin(t)$ includes points $(-\frac{\pi}{2}, -1), (0, 0), (\frac{\pi}{2}, 1)$ on the graph $s = g(t)$ in the $ts$-plane, and as well has a graph that exhibits rotational symmetry about the origin due to $g$ being an odd function. Consider the transformed wave function given by

$$f(x) = 3 \cdot g \left( \pi \cdot x + \frac{\pi}{2} \right) + 2.$$ 

The data correspondence system

$$\begin{cases} 
  t = \pi \cdot x + \frac{\pi}{2} \\
  3 \cdot s + 2 = y 
\end{cases}$$

implies that by indexing the correspondence between data points we can calculate

$$\begin{pmatrix} t_k \\ s_k \end{pmatrix} \to \begin{pmatrix} x_k \\ y_k \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} t_k \pi - \frac{1}{2} \\ 3 \cdot s_k + 2 \end{pmatrix}, \quad \text{hence}$$

$$\begin{pmatrix} t_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} -\frac{\pi}{2} \\ -1 \end{pmatrix} \to \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \text{as} \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -\frac{\pi}{2} \cdot \frac{1}{2} \\ 3 \cdot (-1) + 2 \end{pmatrix},$$

$$\begin{pmatrix} t_2 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1/2 \\ 2 \end{pmatrix} \quad \text{as} \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \pi - \frac{1}{2} \\ 3 \cdot (0) + 2 \end{pmatrix},$$

and $$\begin{pmatrix} t_3 \\ s_3 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 5 \end{pmatrix} \quad \text{as} \quad \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} \pi - \frac{1}{2} \\ 3 \cdot (1) + 2 \end{pmatrix}.$$ 

As an example of a transformed wave function, the graph of $y = 3 \cdot \sin \left( \pi \cdot x + \frac{\pi}{2} \right) + 2$ is readily viewed having its the parent wave-function’s amplitude vertically-scaled by multiplying by 3, its equilibrium vertically-shifted up by 2, its period rescaled to cycle lengths of 2 (by horizontally-scaling through division by $\pi$), and introducing a phase-shift of $-\frac{1}{2}$ (by finally left-shifting the graph of the wave). This sort of analysis also conveniently identifies the transformed extrema points where a new minimum is achieved at $(-1, -1)$ and a new maximum is achieved at $(0, 5)$. Additionally, we identify a new point $\left(\frac{1}{2}, 2\right)$ about which $y = f(x)$ has rotational symmetry.

The correspondence approach summarized in[1] also can help students to understand the qualitative graphical interpretations of the ordered effects of the constants used in the transformation. The intuitive vertical-scaling of output data caused by multiplication by $A$ followed by vertical-shifting (up) by addition of $D$ can be paired with the often counter-intuitive effects of horizontal-scaling by division of input-data by $B$ followed by horizontal-shifting (right) by $-\frac{C}{B}$. Alternatively, there is also the perspective of first horizontally-shifting (right) by $-C$ followed by the horizontal-scaling through division by $B$, in which the addition (shifting) ends up taking order of operation priority over multiplication (scaling) due to the inversion steps needed to connect[1] to[2]. Regardless of choice of the non-commutative order of horizontal transformation effects, substitution can be used to establish point correspondence and subsequently interpret graphical effects.
1.1.1 Transformations of Parabolas

Students initially become familiar with the basic squaring function \( g : \mathbb{R} \to \mathbb{R} \) given by \( g(t) = t^2 \) and the graphical features of its representative parabola \( s = g(t) \) (i.e. \( s = t^2 \)) within the Cartesian \( ts \)–plane, including its graph’s reflective symmetry across the vertical axis due to \( g \) being an even function, as well as the function’s absolute (and relative) minimum \( 0 = \min\{g(t) | t \in \mathbb{R}\} = g(0) \). Following understanding these basics, it is an appropriate extension to consider the general real quadratic functions.

Definition 1.1 (General form). For fixed constants \( b, c \in \mathbb{R} \) and nonzero \( a \in \mathbb{R} \), the function \( f : \mathbb{R} \to \mathbb{R} \) given by

\[
f(x) = a \cdot x^2 + b \cdot x + c
\]

is a (real) quadratic function written in **general form**.

Definition 1.2 (Standard form). For a (real) quadratic function \( f \) given by \( f(x) = a \cdot x^2 + b \cdot x + c \), if constants \( h, k \in \mathbb{R} \) satisfy

\[
f(x) = a \cdot (x - h)^2 + k,
\]

then this form is said to be the **standard form** of \( f \).

What is not immediately obvious in the introduction of quadratic functions is that it is possible to write any real quadratic in a unique standard form. A common verification of this result is to complete-the-square, which includes much of the algebra needed to also derive the quadratic formula. Since we will use a different method, at this point we we treat the existence of the standard form as a proposition.
Advantages of the standard form given in 4 include analysis of the behavior of the standard form quadratic as a transformation of a basic squaring function, as indicated in Figure 2. This results in the identification of horizontally-shifted reflective symmetry in the graph of \( y = f(x) \), about the axis-of-symmetry \( x = h \) where we find the location of vertex \((h, k)\) (i.e. the point corresponding to the location of the minimum of the parent function \( g \)) and the extreme value of the function \( f = f(h) \). Furthermore, the identification of the extremum of a quadratic directly relates to the identification of the vertex of the function’s parabolic graph. Hence, the standard form of a quadratic function is extremely useful in precalculus optimization applications that rely on quadratic form models, and delays the need for development of calculus-based optimization techniques.

1.1.2 Transformations of Parabolas: Visualization of Cases for \( x \)-intercepts

Inspired by the transformed graph example in Figure 2ii, through consideration of variations of vertical-scaling factor \( a \in (-\infty, 0) \cup (0, \infty) \) and vertical-shifting factor \( k \in \mathbb{R} \), and without loss of generality for choice of horizontal-shifting factor \( h \in \mathbb{R} \), we produce results as seen in Figures 3 and 4.

With the reasonable assumption of a continuous dependence on parameters, these cases clearly indicate the possibility of two distinct \( x \)-intercepts (as in Figures 3i and 4iii, and labeled as \( r, s \in \mathbb{R} \)), one intercept, or none. Of course, the expectation is that the study of the distinctions between these cases will lead to development of the discriminant of the quadratic, but at this point we merely use the visualization to motivate the following insight:

If the graph of a quadratic function written in standard form is to have distinct real \( x \)-intercepts \((r, s \in \mathbb{R})\), they must be equidistant from the axis-of-symmetry \((x = h)\). So \( h = \frac{r + s}{2} \) necessarily.

It is this insight into the symmetry inherited through the transformation perspective that we see paralleled in the purely algebraic approach to deriving the quadratic formula. As we see play out in latter argumentation, the convenience of identifying the distinct \( x \)-intercepts of the graph as \( r = h - z \) and \( s = h + z \) for some \( z \in \mathbb{C} \) will play a significant role in generalizing from the case of real quadratic roots to the case of non-real quadratic roots. Additionally, when continuously parametrizing quadratic transformations based on \( z \in \mathbb{R} \cup i\mathbb{R} \), one can observe the transition between cases such as in Figure 3i when \( z \in (0, \infty) \) to those in 3iii when \( z = i\xi \) for \( \xi \in (0, \infty) \), with the critical case of 3ii for \( z = 0 \).

Figure 3: Example cases of a transformed quadratic \( f(x) = a \cdot (x - h)^2 + k \) with \( a > 0 \)

Figure 4: Example cases of a transformed quadratic \( f(x) = a \cdot (x - h)^2 + k \) with \( a < 0 \)
1.2 Towards Complex Linear Factorization of Real Quadratics

Having previously encountered the Zero Product Property on \( \mathbb{R} \) (and likely as well on \( \mathbb{C} \)), one can infer from cases in Figures 3i, 3ii, 4ii, and 4iii that the existence of a standard form for a quadratic (and hence the transformation perspective on the parabolic graph) will necessarily imply the existence of a (real) factored form as well.

**Definition 1.3** (Linear factored form). For a (real) quadratic function \( f \) given by \( f(x) = a \cdot x^2 + b \cdot x + c \), if constants \( r, s \in \mathbb{C} \) satisfy

\[
f(x) = a \cdot (x - r)(x - s),
\]

then this form is said to be a (complex) linear factored form of \( f \).

In some precalculus presentations (see [3]), what becomes clear soon after the development of the theory of real quadratics is the value of complete linear factorization of polynomials for purposes of solving polynomial equations and graphing interpretations. The Fundamental Theorem of Algebra and corollaries support the goal of pursuing linear factorization of polynomials for such purposes, at least in as much as an existence theorem can support.

Similar to our perspective on the standard form in [4] we treat the existence of the factored form as a proposition to establish. In fact, it is constructively proving the existence of such roots \( r, s \in \mathbb{C} \) that is equivalent to deriving the quadratic formula.

Unlike the unique representation of the standard form, the factored form has some ambiguity built in through choice of how to identify a pair of quadratic roots with \( r \) and \( s \), in particular once the natural ordering of \( \mathbb{R} \) is unable to be analogously applied when \( r, s \in \mathbb{C} \). Still, the standard result of complex polynomial factorization being unique up to ordering still applies.

The non-constructive Fundamental Theorem of Algebra requires consideration of potentially non-real roots of a real quadratic polynomial prior to an approach based on appealing to the Factor Theorem to extract a complete linear factorization of general degree polynomials. In deriving a quadratic formula for real quadratic functions, one is forced to consider non-real roots at some point, so it is convenient to have at least introduced \( \mathbb{C} \) prior to computational derivation of the quadratic formula.

1.3 A Classic System of Equations

**Example 1.2.** Let \( \alpha, \beta \in \mathbb{R} \) be given. Solve for \( r, s \) such that

\[
\begin{align*}
\alpha &= r \cdot s \\
\beta &= r + s
\end{align*}
\]

(6)

The statement of such a simple nonlinear system hides within it a deeper philosophical prompting than perhaps is apparent on the surface. Even in the special case of \( \beta = 0 \), when \( s = -r \) and \( r^2 = -\alpha \), there is an imperative to develop irrational solutions \( r \in \mathbb{R} \setminus \mathbb{Q} \) for some choices of \( \alpha \in \mathbb{Q} \) (such as for prime \( p \in \mathbb{N} \) and \( \alpha = -p \in \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \)). Further yet, for real \( \alpha > 0 \), imaginary solutions \( r, s \in \mathbb{C} \setminus \mathbb{R} \) would be needed to satisfy \(-\alpha = r^2\). The apparent ease of solving the system (6) has only been earned through generations of mathematical discovery and invention.

Yet systems like this are accessible enough to have earned their place in the collective memory of human history, having been preserved over the ages in cuneiform and other mediums, but also entering into the collection of math puzzles that are likely to be encountered at some point in the pre-algebra era of primary education.

Eves [4](p. 58) refers to Babylonian mathematics problems and a Louvre tablet dating to 300 B.C. essentially presenting solutions to two-variable systems given a known product and sum. Such systems could be motivated from geometric interpretations of known area and semi-perimeter. Eves (p. 87) continues later in discussing an method attributed to the Pythagoreans in the text section “Geometric Solution of Quadratic Equations” in which applications of areas and appealing to propositions from Euclid’s *Elements* results in a satisfying melding of the geometric and algebraic problems. The long standing history of connecting systems such as (6) and their connection to the study of quadratics and geometric modeling is clear based on extant mathematical sources.

Solving problems equivalent to solving a quadratic equation have inspired varied approaches through the history of algebra, but the nature of products and the connection to areas in basic geometry make it unsurprising that many approaches take a geometric perspective. However, the perspective we take in solving quadratic equations is informed from development of function transformation theory, as we see in [1.1.1] and the subsequent symmetry arguments to which we previously alluded.
2 A Symmetry Motivated Proof of the Quadratic Formula

Theorem 2.1. Let $b, c \in \mathbb{R}$ and nonzero $a \in \mathbb{R}$ yield real quadratic polynomial function

$$f(x) = a \cdot x^2 + b \cdot x + c. \quad (7)$$

Then there exists real $h, k \in \mathbb{R}$ and complex $r, s \in \mathbb{C}$ such that $f$ has standard form

$$f(x) = a \cdot (x - h)^2 + k \quad (8)$$

and a linear factored form

$$f(x) = a \cdot (x - r)(x - s), \quad (9)$$

where constructively

$$h = -\frac{b}{2a}, \quad k = \frac{4 \cdot a \cdot c - b^2}{4a}, \quad r, s = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4 \cdot a \cdot c}}{2a}. \quad (10)$$

Proof. Let $a, b, c \in \mathbb{R}$ with $a \neq 0$, and $f$ be given by $f(x) = a \cdot x^2 + b \cdot x + c$.

Two polynomial functions of equal degree are equivalent precisely when every coefficient of their general form correspond and are numerically equal. Through normalization of (7), (8), and (9), and comparison of equivalent monic quadratic polynomials, it is clear the lead coefficient ($a$) across any equivalent forms must be equal. We proceed under this conclusion.

For any transformed parabolic function graph given by standard form (8), if it is presumed equivalent to a factored form (9) and general form (7), then we have three cases for existence of $x-$intercepts:

i.) The graph of $y = f(x)$ has distinct $x-$intercepts at $(r, 0)$ and $(s, 0)$ that are equidistant from the axis of symmetry $x = h$,

ii.) the graph of $y = f(x)$ has a single $x-$intercept at the vertex with $(r, 0) = (h, k)$, or

iii.) the graph of $y = f(x)$ has no real $x-$intercepts, and $r, s \in \mathbb{C} \setminus \mathbb{R}$.

From our cases above, it is clear for such $h \in \mathbb{R}$ and $r, s \in \mathbb{C}$ to exist, it must hold by symmetry of the transformed parabola that whenever the graph $y = f(x)$ has precisely one or two $x-$intercepts, then necessarily

$$h = \frac{r + s}{2}. \quad (11)$$

When there are no $x-$intercepts and $r, s \notin \mathbb{R}$, then $r, s$ must form a complex conjugate pair with $s = \bar{r}$, as expansion of presumed equivalent forms (8) and (9) into a general form given by (7) requires the coefficient of $x$ in each expansion equates

$$b = -a \cdot 2h = a \cdot (-r - s), \quad (10)$$

and the average of the roots $r, s$ must be equal to the horizontal shifting $h$, as previously concluded in the distinct intercept case.

Based on (10), if a standard form for a general quadratic (7) exists, then it must hold that the axis of symmetry $x = h$ satisfies

$$h = -\frac{b}{2a}. \quad (11)$$

Furthermore, as (complex) roots $r, s$ must have real average $h$, and without loss of generality assuming $s > r$ when they are real and distinct, we may write both roots in a (real or complex) symmetric form

$$r = h - z, \quad s = h + z, \quad (12)$$

where $z \in [0, \infty) \cup i(0, \infty)$, depending on whether we encounter the cases of

i.) distinct $x-$intercepts at $(r, 0)$ and $(s, 0)$, where $z = \frac{s - r}{2} \in (0, \infty),$

ii.) a single $x-$intercept, where $z = 0$, or
At this point, since we have exhausted the extent to which the axis-of-symmetry and linear coefficient analysis can produce results readily, and we begin consideration of the quadratic function’s extremum (k) at the parabola’s vertex. Returning to expansions of each form, with attention given now to the constant term, and by subsequently using (13), we conclude that the following must hold:

\[ c = a \cdot h^2 + k = a \cdot (r \cdot s) \]

\[ = a \cdot (h^2 - z^2). \]

It is also at this point that we see a nonlinear system for \( r, s \) based on parameters \( a, b, c \) arise based on the correspondence of constant terms in (14) and the linear coefficient terms in (10). Respectively isolating the product and sum of proposed roots \( r, s \) in these results yields a classic system known as an example of Vieta’s formulas [5] (as formulaic representations of polynomial coefficients with respect to roots). The resulting system

\[
\begin{cases}
\frac{c}{a} = r \cdot s \\
-\frac{b}{a} = r + s
\end{cases}
\]

is of the form discussed in [1,3]. For the purpose of identifying quadratic roots or \( x \) -intercepts of a parabola, the equivalence of using the quadratic formula to solving this system should be recognized. In fact, it is almost this very system which is under consideration when the algebraic task of quadratic factorization is posed in the typical algebra course. Following normalization to a monic quadratic, the imperative to find two numbers with a given sum and product is familiar, though rarely applied for complex solutions.

That said, our purposes go beyond merely deriving quadratic roots, but also include relating said roots to the parabola’s vertex when written in standard form, so we continue with those goals in mind.

By isolating the \( z^2 \) term in (14), a substitution of the axis-of-symmetry result (11) yields

\[ z^2 = a \cdot h^2 - c = a \cdot \left(\frac{-b}{2a}\right)^2 - c = \frac{b^2 - 4 \cdot a \cdot c}{4a^2}. \]

At this point, since \( a, b, c \in \mathbb{R} \) by assumption, we have \( z^2 \in \mathbb{R} \) necessarily. However, it is precisely the numerator in (16) (that is, traditionally referred to as the discriminant \( \Delta = b^2 - 4 \cdot a \cdot c \)) which distinguishes between our cases of:

i.) distinct \( x \) -intercepts as in Figures 5i, 5iv and distinct real roots \( r, s = h \pm z = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4 \cdot a \cdot c}}{2a} \),

ii.) and a single \( x \) -intercept as in Figures 5ii, 5v and a double real root \( r = s = h \pm z = -\frac{b}{2a} \pm \frac{\sqrt{0}}{2a} = -\frac{b}{2a} \),

iii.) or no \( x \) -intercepts as in Figures 5iii, 5vi and non-real conjugate roots \( r, s = h \pm z = -\frac{b}{2a} \pm i \cdot \frac{\sqrt{4 \cdot a \cdot c - b^2}}{2a} \).

Up to case-based insight into the nature of solutions \( z \) to (16), where purely imaginary solutions arise when discriminant \( b^2 - 4 \cdot a \cdot c < 0 \) and we are required to interpret square roots of a negative number, we see that all cases follow the same formula - the quadratic formula:

\[ r, s = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4 \cdot a \cdot c}}{2a}, \]

where we choose to retain the emphasis of the symmetry about axis of symmetry \( x = h = -\frac{b}{2a} \).

Having formulaically constructed \( h, r, s \) using parameters \( a, b, c \), all that remains to finish our simultaneous constructive proof of both factored and standard forms of quadratics is to construct extremum \( k \). Rather than the typical approach of merely evaluating \( k = f \left( -\frac{b}{2a} \right) \) in the general form of \( f \), we instead again appeal to coefficient correspondence of terms in our expansions. By (14), it is clear that we may solve for

\[ k = c - a \cdot h^2 = -a \cdot z^2 \]

\[ = -a \cdot \left(\frac{b^2 - 4 \cdot a \cdot c}{4a^2}\right) = \frac{4 \cdot a \cdot c - b^2}{4a}, \]

where \( a, b, c \in \mathbb{R} \) imply \( k \in \mathbb{R} \).
Corollary 2.1 (The Quadratic Formula). Let \( b, c \in \mathbb{R} \) and non-zero \( a \in \mathbb{R} \). Then the quadratic equation

\[
0 = a \cdot x^2 + b \cdot x + c
\]

has solutions in \( \mathbb{C} \) of

\[
x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4 \cdot a \cdot c}}{2a}.
\]

\( \Box \)

Furthermore, as coefficients \( a, b, c \in \mathbb{R} \subset \mathbb{C} \), a field, the polynomial ring \( \mathbb{C}[x] \) is a Euclidean Domain (with Euclidean valuation of polynomial degree) which admits a division algorithm, and is therefore a Unique Factorization Domain. (See Judson [6] or [7], Chapter 18.) Hence, the roots \( r, s \in \mathbb{C} \) and associated factors are unique up to reordering.

3 Comments on Quadratic Formula Extensions and Limitations

As much as the historical study of quadratics prompted an expansion of the number system from \( \mathbb{Q} \) to \( \mathbb{R} \) to \( \mathbb{C} \), of interest to many is how quadratic polynomial structures in other mathematical systems (numerical or otherwise) may have constructive proofs of zeros (roots) or appropriate analogs.

An advanced undergraduate development of abstract algebra tends to resolve discussions of quadratic formula extensions within the context of fields, and prior to expansion into the Galois-theory results which prove non-existence of coefficient based radical formula for roots for general degree polynomials (i.e. for quintic or greater).

3.1 Quadratics in polynomial rings over fields

Proposition 3.1. Let \( F \) be a field. Let \( b, c \in F \) and non-zero \( a \in F \). Then the quadratic equation

\[
0 = a \cdot x^2 + b \cdot x + c
\]

has solutions in splitting field \( F(z) \supseteq F \), for some \( z \) satisfying \( z^2 = \frac{b^2 - 4 \cdot a \cdot c}{4a^2} \). Solutions \( x \in F(z) \) are then of the form

\[
x = \frac{-b}{2a} \pm z.
\]
Thomas Judson develops the theory of splitting fields for polynomials of general degree in [6] (and [7]) (Chapter 21), and we leave the proof of formal theory to such works.

In the case of $F = \mathbb{R}$, this just restates that either a real quadratic has real roots (with non-negative discriminant) or a discriminant $\Delta < 0$ implies roots $x \in \mathbb{R} \left( i \cdot \sqrt{\frac{-\Delta}{2a}} \right) = \mathbb{R}(i) = \mathbb{C}$.

This result is more interesting in a field such as $F = \mathbb{Q}^*$ with irreducible quadratics like $x^2 - 2x - 2$ and roots $1 \pm \sqrt{3} \in \mathbb{Q}^*(\sqrt{3}) \subset \mathbb{C}$. Such considerations also remind of the ancient controversy over existence of irrational numbers as a foreshadowing of the controversy of developing imaginary numbers.

Also of interest are the polynomial rings generated by finite fields, such as modular arithmetic fields $F = \mathbb{Z}_p \cong \mathbb{Z}/(p)$ for prime $p$. In such cases, many authors will take additional care to clarify the operation of division in the fraction form of (19) may be better understood as modular multiplication by the multiplicative inverse $\beta \in \mathbb{Z}_p$ to $2a \in \mathbb{Z}_p$ such that $\beta \cdot (2a) \equiv 1 \pmod{p}$. Though in a prime modulus this may seem little more than a semantic distinction, in attempts to extend to quadratic congruences for composite moduli it may be more troublesome to clearly convey results without explicitly adapting notation.

### 3.2 General Quadratic Congruences

**Proposition 3.2.** Let modulus $n \in \mathbb{N}$, modular arithmetic ring $R = \mathbb{Z}_n$, with $b, c \in R$ and (nonzero) unit $a \in R$. Then the quadratic congruence

$$\tag{20} 0 \equiv a \cdot x^2 + b \cdot x + c \pmod{n}$$

has solutions provided $\gcd(2a, n) = 1$, and the discriminant $\Delta \equiv b^2 - 4 \cdot a \cdot c$ is a quadratic residue modulo $n$. The solutions $x \in \mathbb{Z}_n$ to (20) are of the form

$$x \equiv -b \cdot (2a)^{-1} + z \pmod{n}$$

where $z \equiv s \cdot (2a)^{-1}$ for solutions $s \in \mathbb{Z}_n$ to $s^2 \equiv \Delta$.

Karl-Dieter Crisman develops the theory of quadratic residues, congruences, and reciprocity in [8] (Chapters 16-17), which we will not recreate here. We have notationally adapted statements of results there to align with the symmetry approaches in [2].

The validity of modular multiplicative inversion of $2a \in R$ is dependent on both $2, a$ being units in $R$, which is in turn dependent on $n$ being odd and relatively prime to $2a$. Furthermore, the restriction on discriminant $\Delta$ to be a quadratic residue (i.e. a perfect square modulo $n$) is analogous to the non-negative discriminant restriction on real quadratics when seeking real quadratic roots.

For the special case of prime $n$ with cyclic (multiplicative) group of units $U_n = \langle g \rangle$ and primitive root (and generator) $g \in \mathbb{Z}_n$, the valid quadratic residues are $\langle g^2 \rangle = Q_n$, with $\Delta \in Q_n$ being necessary. However, for a general composite modulus $n$, development of Legendre and Jacobi symbol theory is needed for a more thorough assessment of Proposition 3.2.

Of note in the modular congruence setting is the failure of the Zero Product Property in $\mathbb{Z}_n$ when modulus $n$ is composite (and $\mathbb{Z}_n$ is not a field nor integral domain). This introduces uncertainty (non-uniqueness) in factorization, such as in

$$x^2 \equiv (x + 2)^2 \pmod{4},$$

and results in a quadratic formula for a general composite modulus and arbitrary coefficients to be unobtainable, with coefficient compatibility conditions being more restrictive than the traditional discriminant compatibility conditions.

### 3.3 Beyond Fields and Commutative Rings

Without going into details, there is research on additional extensions of quadratic formula to quadratics over noncommutative division rings such as the quaternion ring $\mathbb{H}$, or for the physics-relevant adaptation of the split-quaternions $\mathbb{H}_s$ (see [5]). However, the complexity of the computational results go well beyond what we wish to present here, as accessible connections to a perspective of transformation have yet to be uncovered.
References

[1] Po-Shen Loh. A simple proof of the quadratic formula. *arXiv preprint arXiv:1910.06709*, 2019.

[2] Po-Shen Loh. Quadratic method: Detailed explanation. Retrieved from [https://www.poshenloh.com/quadraticdetail/](https://www.poshenloh.com/quadraticdetail/), 2020/09/15.

[3] Robert Blitzer. *Precalculus*. Pearson, 6th edition, 2018.

[4] Howard Eves. *An Introduction to the History of Mathematics*. Cengage Learning, 6th edition, 1990.

[5] Wensheng Cao. Quadratic formulas for split quaternions. Retrieved from [https://arxiv.org/abs/1905.08153v3](https://arxiv.org/abs/1905.08153v3), 2020.

[6] Thomas Judson. *Abstract Algebra: Theory and Applications*. Orthogonal Publishing L3C, 2020.

[7] Thomas Judson. Abstract algebra: Theory and applications. Retrieved from [http://abstract.ups.edu/](http://abstract.ups.edu/), 2020/10/23. (2020 annual ed. open text).

[8] Karl-Dieter Crisman. Number theory: In context and interactive. Retrieved from [http://math.gordon.edu/ntic/](http://math.gordon.edu/ntic/), 2020/10/23. (2020/1 ed. open text).