On parsimonious edge-colouring of graphs with maximum degree three

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Abstract

In a graph $G$ of maximum degree $\Delta$ let $\gamma$ denote the largest fraction of edges that can be $\Delta$ edge-coloured. Albertson and Haas showed that $\gamma \geq \frac{13}{15}$ when $G$ is cubic [1]. We show here that this result can be extended to graphs with maximum degree 3 with the exception of a graph on 5 vertices. Moreover, there are exactly two graphs with maximum degree 3 (one being obviously the Petersen graph) for which $\gamma = \frac{13}{15}$. This extends a result given in [14]. These results are obtained by giving structural properties of the so called $\delta$–minimum edge colourings for graphs with maximum degree 3.

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1 Introduction

Throughout this paper, we shall be concerned with connected graphs with maximum degree 3. We know by Vizing’s theorem [15] that these graphs can be edge-coloured with 4 colours. Let $\phi : E(G) \to \{\alpha, \beta, \gamma, \delta\}$ be a proper edge-colouring of $G$. It is often of interest to try to use one colour (say $\delta$) as few as possible. When an edge colouring is optimal, following this constraint, we shall say that $\phi$ is $\delta$–minimum. In [3] we gave without proof (in French, see [5] for a translation) results on $\delta$ – minimum edge-colourings of cubic graphs. Some of them have been obtained later and independently by Steffen [13, 14]. Some other results which were not stated formally in [4] are needed here. The purpose of Section 2 is to give those results as structural properties of $\delta$–minimum edge-colourings as well as others which will be useful in Section 3.

An edge colouring of $G$ using colours $\alpha, \beta, \gamma, \delta$ is said to be $\delta$–improper provided that adjacent edges having the same colours (if any) are coloured with $\delta$. It is clear that a proper edge colouring (and hence a $\delta$–minimum edge-colouring) of $G$ is a particular $\delta$–improper edge colouring. For a proper or $\delta$–improper edge colouring $\phi$ of $G$, it will be convenient to denote $E_\phi(x) (x \in \{\alpha, \beta, \gamma, \delta\})$ the set of edges coloured with $x$ by $\phi$. For $x, y \in \{\alpha, \beta, \gamma, \delta\}, x \neq y$, $\phi(x, y)$ is the partial subgraph of $G$ spanned by these two colours, that is $E_\phi(x) \cup E_\phi(y)$ (this subgraph being a union of paths and even cycles where the colours $x$ and $y$ alternate). Since any two $\delta$–minimum edge-colourings of $G$ have the same number of edges coloured $\delta$ we shall denote by $s(G)$ this number (the colour number as defined in [13]).
As usual, for any undirected graph $G$, we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges and we suppose that $|V(G)| = n$ and $|E(G)| = m$. A strong matching $C$ in a graph $G$ is a matching $C$ such that there is no edge of $E(G)$ connecting any two edges of $C$, or, equivalently, such that $C$ is the edge-set of the subgraph of $G$ induced on the vertex-set $V(C)$.

2 On $\delta$–minimum edge-colouring

The graph $G$ considered in the following series of Lemmas will have maximum degree 3.

Lemma 1 [3, 4, 5] Any 2-factor of $G$ contains at least $s(G)$ disjoint odd cycles.

Lemma 2 [3, 4, 5] Let $\phi$ be a $\delta$–minimum edge-colouring of $G$. Any edge in $E_\phi(\delta)$ is incident to $\alpha, \beta$ and $\gamma$. Moreover each such edge has one end of degree 2 and the other of degree 3 or the two ends of degree 3.

Lemma 3 Let $\phi$ be a $\delta$–improper colouring of $G$ then there exists a proper colouring of $G \phi'$ such that $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$

Proof Let $\phi$ be a $\delta$-improper edge colouring of $G$. If $\phi$ is a proper colouring, we are done. Hence, assume that $uv$ and $uw$ are coloured $\delta$. If $d(u) = 3$ we can change the colour of $uv$ to $\alpha, \beta$ or $\gamma$ since $v$ is incident to at most two colours in this set.

If $d(u) = 3$ assume that the third edge $uz$ incident to $u$ is also coloured $\delta$, then we can change the colour of $uw$ for the same reason as above.

If $uz$ is coloured with $\alpha, \beta$ or $\gamma$, then $v$ and $w$ are incident to the two remaining colours of the set $\{\alpha, \beta, \gamma\}$ otherwise one of the edges $uw, uv$ can be recoloured with the missing colour. W.l.o.g., consider that $uz$ is coloured $\alpha$ then $v$ and $w$ are incident to $\beta$ and $\gamma$. Since $u$ has degree 1 in $\phi(\alpha, \beta)$ let $P$ be the path of $\phi(\alpha, \beta)$ which ends on $u$. We can assume that $v$ or $w$ (say $v$) is not the other end vertex of $P$. Exchanging $\alpha$ and $\beta$ along $P$ does not change the colours incident to $v$. But now $uz$ is coloured $\alpha$ and we can change the colour of $uw$ with $\beta$.

In each case, we get hence a new $\delta$-improper edge colouring $\phi_1$ with $E_{\phi_1}(\delta) \subseteq E_\phi(\delta)$. Repeating this process leads us to construct a proper edge colouring of $G$ with $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$ as claimed. $\square$

Proposition 4 Let $v_1, v_2, \ldots, v_k \in V(G)$ such that $G - \{v_1, v_2, \ldots, v_k\}$ is 3-edge colourable. Then $s(G) \leq k$.

Proof Let us consider a 3-edge colouring of $G - \{v_1, v_2, \ldots, v_k\}$ with $\alpha, \beta$ and $\gamma$ and let us colour the edges incident to $v_1, v_2, \ldots, v_k$ with $\delta$. We get a $\delta$-improper edge colouring $\phi$ of $G$. Lemma 3 gives a proper colouring of $G \phi'$ such that $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$. Hence $\phi'$ has at most $k$ edges coloured with $\delta$ and $s(G) \leq k$ $\square$

Proposition 4 above has been obtained by Steffen [13] for cubic graphs.
Lemma 5 Let $\phi$ be a $\delta$--improper colouring of $G$ then $|E_\phi(\delta)| \geq s(G)$

Proof Applying Lemma 3, let $\phi'$ be a proper edge colouring of $G$ such that $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$. We clearly have $|E_\phi(\delta)| \geq |E_{\phi'}(\delta)| \geq s(G)$. \qed

Lemma 6 [3, 4, 5] Let $\phi$ be a $\delta$--minimum edge-colouring of $G$. For any edge $e = uv \in E_\phi(\delta)$ there are two colours $x$ and $y$ in $\{\alpha, \beta, \gamma\}$ such that the connected component of $\phi(x, y)$ containing the two ends of $e$ is an even path joining these two ends.

Remark 7 An edge of $E_\phi(\delta)$ is in $A_\phi$ when its ends can be connected by a path of $\phi(\alpha, \beta)$, $B_\phi$ by a path of $\phi(\beta, \gamma)$ and $C_\phi$ by a path of $\phi(\alpha, \gamma)$. It is clear that $A_\phi$, $B_\phi$ and $C_\phi$ are not necessarily pairwise disjoint since an edge of $E_\phi(\delta)$ with one end of degree 2 is contained in 2 such sets. Assume indeed that $e = uv \in E_\phi(\delta)$ with $d(u) = 3$ and $d(v) = 2$ then, if $u$ is incident to $\alpha$ and $\beta$ and $v$ is incident to $\gamma$, we have an alternating path whose ends are $u$ and $v$ in $\phi(\alpha, \gamma)$ as well as in $\phi(\beta, \gamma)$. Hence $e$ is in $A_\phi \cap B_\phi$. When $e \in A_\phi$ we can associate to $e$ the odd cycle $C_{A_\phi}(e)$ obtained by considering the path of $\phi(\alpha, \beta)$ together with $e$. We define in the same way $C_{B_\phi}(e)$ and $C_{C_\phi}(e)$ when $e$ is in $B_\phi$ or $C_\phi$. In the following lemma we consider an edge in $A_\phi$, an analogous result holds true whenever we consider edges in $B_\phi$ or $C_\phi$ as well.

Lemma 8 [3, 4, 5] Let $\phi$ be a $\delta$--minimum edge-colouring of $G$ and let $e$ be an edge in $A_\phi$ then for any edge $e' \in C_{A_\phi}(e)$ there is a $\delta$--minimum edge-colouring $\phi'$ such that $E_{\phi'}(\delta) = E_\phi(\delta) - \{e\} \cup \{e'\}$, $e' \in A_\phi$ and $C_{A_\phi}(e) = C_{A_\phi}(e')$. Moreover, each edge outside $C_{A_\phi}(e)$ but incident with this cycle is coloured $\gamma$, $\phi$ and $\phi'$ only differ on the edges of $C_{A_\phi}(e)$.

For each edge $e \in E_\phi(\delta)$ (where $\phi$ is a $\delta$--minimum edge-colouring of $G$) we can associate one or two odd cycles following the fact that $e$ is in one or two sets among $A_\phi$, $B_\phi$ or $C_\phi$. Let $C$ be the set of odd cycles associated to edges in $E_\phi(\delta)$.

Lemma 9 [3, 4, 5] For each cycle $C \in C$, there are no two consecutive vertices with degree two.

Lemma 10 [3, 4, 5] Let $e_1, e_2 \in E_\phi(\delta)$ and let $C_1, C_2 \in C$ be such that $C_1 \neq C_2$, $e_1 \in E(C_1)$ and $e_2 \in E(C_2)$ then $C_1$ and $C_2$ are (vertex) disjoint.

By Lemma 8 any two cycles in $C$ corresponding to edges in distinct sets $A_\phi$, $B_\phi$ or $C_\phi$ are at distance at least 2. Assume that $C_1 = C_{A_\phi}(e_1)$ and $C_2 = C_{A_\phi}(e_2)$ for some edges $e_1$ and $e_2$ in $A_\phi$. Can we say something about the subgraph of $G$ whose vertex set is $V(C_1) \cup V(C_2)$? In general, we have no answer to this problem. However, when $G$ is cubic and any vertex of $G$ lies on some cycle of $C$ (we shall say that $C$ is spanning), we have a property which will be useful later. Let us remark first that whenever $C$ is spanning, we can consider that $G$ is edge-coloured in such a way that the edges of the cycles of $C$ are alternatively coloured with $\alpha$ and $\beta$ (except one edge coloured $\delta$) and the remaining perfect matching is coloured with $\gamma$. For this $\delta$--minimum edge-colouring of $G$ we have $B_\phi = \emptyset$ as well as $C_\phi = \emptyset$. 

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Lemma 11 Assume that \( G \) is cubic and \( C \) is spanning. Let \( e_1, e_2 \in A_\phi \) and let \( C_1, C_2 \in C \) such that \( C_1 = C_{A_\phi}(e_1) \) and \( C_2 = C_{A_\phi}(e_2) \). Then at least one of the followings is true

(i) \( C_1 \) and \( C_2 \) are at distance at least 2

(ii) \( C_1 \) and \( C_2 \) are joined by at least 3 edges

(iii) \( C_1 \) and \( C_2 \) have at least two chords each

Proof Since \( e_1, e_2 \in A_\phi \) and \( C \) is spanning we have \( B_\phi = C_\phi = \emptyset \). Let \( C_1 = v_0v_1 \ldots v_{2k_1} \) and \( C_2 = w_0w_1 \ldots w_{2k_2} \). Assume that \( C_1 \) and \( C_2 \) are joined by the edge \( v_0w_0 \). By Lemma 6, we can consider a \( \delta \)–minimum edge-colouring \( \phi \) such that \( \phi(v_0v_1) = \phi(w_0w_1) = \delta, \phi(v_1v_2) = \phi(w_1w_2) = \beta \) and \( \phi(v_0v_{2k_1}) = \phi(w_0w_{2k_2}) = \alpha \). Moreover each edge of \( G \) (in particular \( v_0w_0 \)) incident with these cycles is coloured \( \gamma \). We can change the colour of \( v_0w_0 \) in \( \beta \). We obtain thus a new \( \delta \)–minimum edge-colouring \( \phi' \). Performing that exchange of colours on \( v_0w_0 \) transforms the edges coloured \( \delta \) \( v_0v_1 \) and \( w_0w_1 \) in two edges of \( C_{\phi'} \) lying on odd cycles \( C'_1 \) and \( C'_2 \) respectively. We get hence a new set \( C' = C - \{C_1, C_2\} \cup \{C'_1, C'_2\} \) of odd cycles associated to \( \delta \)–coloured edges in \( \phi' \).

From Lemma 8 \( C'_1 \) (\( C'_2 \) respectively) is at distance at least 2 from any cycle in \( C - \{C_1, C_2\} \). Hence \( V(C'_1) \cup V(C'_2) \subseteq V(C_1) \cup \{C_2\} \). It is an easy task to check now that (ii) or (iii) above must be verified. \( \square \)

Lemma 12 [3, 4, 5] Let \( e_1 = u_1v_1 \) be an edge of \( E_\phi(\delta) \) such that \( v_1 \) has degree 2 in \( G \). Then \( v_1 \) is the only vertex in \( N(u) \) of degree 2 and for any edge \( e_2 = u_2v_2 \in E_\phi(\delta), \{e_1, e_2\} \) induces a \( 2K_2 \).

Lemma 13 [3, 4, 5] Let \( e_1 \) and \( e_2 \) be two edges of \( E_\phi(\delta) \). If \( e_1 \) and \( e_2 \) are contained in two distinct sets of \( A_\phi, B_\phi \) or \( C_\phi \) then \( \{e_1, e_2\} \) induces a \( 2K_2 \) otherwise \( e_1, e_2 \) are joined by at most one edge.

Lemma 14 [3, 4, 5] Let \( e_1, e_2 \) and \( e_3 \) be three distinct edges of \( E_\phi(\delta) \) contained in the same set \( A_\phi, B_\phi \) or \( C_\phi \). Then \( \{e_1, e_2, e_3\} \) induces a subgraph with at most four edges.

3 Applications and problems

3.1 On a result by Payan

In [10] Payan showed that it is always possible to edge-colour a graph of maximum degree 3 with three maximal matchings (with respect to the inclusion) and introduced henceforth a notion of strong-edge colouring where a strong edge-colouring means that one colour is a strong matching while the remaining colours are usual matchings. Payan conjectured that any \( d \)–regular graph has \( d \) pairwise disjoint maximal matchings and showed that this conjecture holds true for graphs with maximum degree 3.

The following result has been obtained first by Payan [10], but his technique does not exhibit explicitly the odd cycles associated to the edges of the strong matching and their properties.
Theorem 15 Let $G$ be a graph with maximum degree at most 3. Then $G$ has a $\delta$–minimum edge-colouring $\phi$ where $E_\phi(\delta)$ is a strong matching and, moreover, any edge in $E_\phi(\delta)$ has its two ends of degree 3 in $G$.

Proof Let $\phi$ be a $\delta$–minimum edge-colouring of $G$. From Lemma 13, any two edges of $E_\phi(\delta)$ belonging to distinct sets from among $A_\phi$, $B_\phi$ and $C_\phi$ induce a strong matching. Hence, we have to find a $\delta$–minimum edge-colouring where each set $A_\phi$, $B_\phi$ or $C_\phi$ induces a strong matching (with the supplementary property that the end vertices of these edges have degree 3). That means that we can work on each set $A_\phi$, $B_\phi$ and $C_\phi$ independently. Without loss of generality, we only consider $A_\phi$ here.

Assume that $A_\phi = \{e_1, e_2, \ldots, e_k\}$ and $A'_\phi = \{e_1, \ldots, e_i\}$ ($1 \leq i \leq k - 1$) is a strong matching and each edge of $A'_\phi$ has its two ends with degree 3 in $G$. Consider the edge $e_{i+1}$ and let $C = C_{e_{i+1}}(\phi) = (u_0, u_1 \ldots u_2p)$ be the odd cycle associated to this edge (Lemma 6).

Let us mark any vertex $v$ of degree 3 on $C$ with a $+$ whenever the edge of colour $\gamma$ incident to this vertex has its other end which is a vertex incident to an edge of $A'_\phi$ and let us mark $v$ with $-$ otherwise. By Lemma 9 a vertex of degree 2 on $C$ has its two neighbours of degree 3 and by Lemma 12 these two vertices are marked with $-$. By Lemma 14 we cannot have two consecutive vertices marked with $+$. Hence, $C$ must have two consecutive vertices of degree 3 marked with $-$ whatever is the number of vertices of degree 2 on $C$.

Let $u_{j}$ and $u_{j+1}$ be two vertices of $C$ of degree 3 marked with $-$ ($j$ being taken modulo $2p + 1$). We can transform the edge colouring $\phi$ by exchanging colours on $C$ uniquely, in such a way that the edge of colour $\delta$ of this cycle is $u_{j}u_{j+1}$. In the resulting edge colouring $\phi_1$ we have $A_{\phi_1} = A_\phi - \{e_{i+1}\} \cup \{u_j, u_{j+1}\}$ and $A'_{\phi_1} = A'_\phi \cup \{u_j, u_{j+1}\}$ is a strong matching where each edge has its two ends with degree 3. Repeating this process we are left with a new $\delta$–minimum colouring $\phi'$ where $A_{\phi'}$ is a strong matching. 

Corollary 16 Let $G$ be a graph with maximum degree 3 then there are $s(G)$ vertices of degree 3 pairwise non-adjacent $v_1 \ldots v_{s(G)}$ such that $G - \{v_1 \ldots v_{s(G)}\}$ is 3-colourable.

Proof Pick a vertex on each edge coloured $\delta$ in a $\delta$–minimum colouring $\phi$ of $G$ where $E_\phi(\delta)$ is a strong matching (Theorem 15). We get a subset $S$ of vertices satisfying our corollary.

Steffen [13] obtained Corollary 16 for bridgeless cubic graphs.

3.2 Parsimonious edge colouring

Let $\chi'(G)$ be the classical chromatic index of $G$. For convenience let

$$e(G) = \max\{|E(H)| : H \subseteq G, \chi'(H) = 3\}$$

$$\gamma(G) = \frac{e(G)}{|E(G)|}$$

Staton [12] (and independently Locke [9]) showed that whenever $G$ is a cubic graph distinct from $K_4$ then $G$ contains a bipartite subgraph (and hence a 3-edge colourable graph, by König’s theorem [8]) with at least $\frac{7}{8}$ of the edges of
G. Bondy and Locke [2] obtained \( \frac{1}{3} \) when considering graphs with maximum degree at most 3.

In [1] Albertson and Haas showed that whenever \( G \) is a cubic graph, we have \( \gamma(G) \geq \frac{13}{15} \) while for graphs with maximum degree 3 they obtained \( \gamma(G) \geq \frac{26}{31} \).

Our purpose here is to show that \( \frac{13}{15} \) is a lower bound for \( \gamma(G) \) when \( G \) has maximum degree 3, with the exception of the graph \( G_5 \) depicted in Figure 1 below.

\[ \text{Figure 1: } G_5 \]

**Lemma 17** Let \( G \) be a graph with maximum degree 3 then \( \gamma(G) = 1 - \frac{s(G)}{m} \).

**Proof** Let \( \phi \) be a \( \delta \)-minimum edge-colouring of \( G \). The restriction of \( \phi \) to \( E(G) - E_\phi(\delta) \) is a proper 3-edge-colouring, hence \( c(G) \geq m - s(G) \) and \( \gamma(G) \geq 1 - \frac{s(G)}{m} \).

If \( H \) is a subgraph of \( G \) with \( \chi(H) = 3 \), consider a proper 3-edge-colouring \( \phi : E(H) \to \{\alpha, \beta, \gamma\} \) and let \( \psi : E(G) \to \{\alpha, \beta, \gamma, \delta\} \) be the continuation of \( \phi \) with \( \psi(e) = \delta \) for \( e \in E(G) - E(H) \). By Lemma 3 there is a proper edge-colouring \( \psi' \) of \( G \) with \( E_\psi'(\delta) \subseteq E_\psi(\delta) \) so that \( |E(H)| = |E(G) - E_\psi(\delta)| \leq |E(G) - E_\psi'(\delta)| \leq m - s(G) \), \( c(G) \leq m - s(G) \) and \( \gamma(G) \leq 1 - \frac{s(G)}{m} \). \( \square \)

In [11], Rizzi shows that for triangle free graphs of maximum degree 3, \( \gamma(G) \geq 1 - \frac{2}{s_{odd}(G)} \) (where the odd girth of a graph \( G \), denoted by \( s_{odd}(G) \), is the minimum length of an odd cycle).

**Theorem 18** Let \( G \) be a graph with maximum degree 3 then \( \gamma(G) \geq 1 - \frac{2}{s_{odd}(G)} \).

**Proof** Let \( \phi \) be a \( \delta \)-minimum edge-colouring of \( G \) and \( E_\phi(\delta) = \{e_1, e_2 \ldots e_{s(G)}\} \). \( \mathcal{C} \) being the set of odd cycles associated to edges in \( E_\phi(\delta) \), we write \( \mathcal{C} = \{C_1, C_2 \ldots C_{s(G)}\} \) and suppose that for \( i = 1, 2 \ldots s(G) \), \( e_i \) is an edge of \( C_i \). We know by Lemma 10 that the cycles of \( \mathcal{C} \) are vertex-disjoint.

Let us write \( \mathcal{C} = \mathcal{C}_2 \cup \mathcal{C}_3 \), where \( \mathcal{C}_2 \) denotes the set of odd cycles of \( \mathcal{C} \) which have a vertex of degree 2, while \( \mathcal{C}_3 \) is for the set of cycles in \( \mathcal{C} \) whose all vertices have degree 3. Let \( k = |\mathcal{C}_2| \), obviously we have \( 0 \leq k \leq s(G) \) and \( \mathcal{C}_2 \cap \mathcal{C}_3 = \emptyset \).

If \( C_1 \in \mathcal{C}_2 \), we may assume that \( e_1 \) has a vertex of degree 2 (see Lemma 8) and we can associate to \( e_1 \) another odd cycle say \( C'_1 \) (Remark 7) whose edges distinct from \( e_1 \) form an even path using at least \( \frac{s_{odd}(G)}{2} \) edges which are not
edges of $C_i$. Hence, $C_i \cup C'_i$ contains at least $\frac{3}{2}g_{\text{odd}}(G)$ edges. Consequently there are at least $\frac{3}{2} \times k \times g_{\text{odd}}(G)$ additionnal edges which are incident to a vertex of $\bigcup_{C_i \in C_3} C_i$. 

When $C_i \in C_3$, $C_i$ contains at least $g_{\text{odd}}(G)$ edges, moreover, each vertex of $C_i$ being of degree 3, there are $\frac{s(G)-k}{2} \times g_{\text{odd}}(G)$ additionnal edges which are incident to a vertex of $\bigcup_{C_i \in C_3} C_i$. 

Since $C_i \cap C_j = \emptyset$ and $C_i \cap C_j = \emptyset$ (1 $\leq i, j \leq s(G)$, $i \neq j$), we have 

$$m \geq \frac{3}{2}g_{\text{odd}}(G) \times k \times (s(G)-k) \times g_{\text{odd}}(G) + \frac{s(G)-k}{2} \times g_{\text{odd}}(G) = \frac{3}{2} \times s(G) \times g_{\text{odd}}(G).$$ 

Consequently $\gamma(G) = 1 - \frac{s(G)}{m} \geq 1 - \frac{2}{3g_{\text{odd}}(G)}$. 

\[\square\]

**Lemma 19** [1] Let $G$ be a graph with maximum degree 3. Assume that $v \in V(G)$ is such that $d(v) = 1$ then $\gamma(G) > \gamma(G-v)$. 

A triangle $T = \{a, b, c\}$ is said to be reducible whenever its neighbours are distinct. When $T$ is a reducible triangle in $G$ ($G$ having maximum degree 3) we can obtain a new graph $G'$ with maximum degree 3 by shrinking this triangle into a single vertex and joining this new vertex to the neighbours of $T$ in $G$.

**Lemma 20** [1] Let $G$ be a graph with maximum degree 3. Assume that $T = \{a, b, c\}$ is a reducible triangle and let $G'$ be the graph obtained by reduction of this triangle. Then $\gamma(G) > \gamma(G')$.

**Theorem 21** Let $G$ be a graph with maximum degree 3, $V_2$ be the set of vertices with degree 2 in $G$ and $V_3$ those of degree 3. If $G \neq G_5$ then $\gamma(G) \geq 1 - \frac{3}{4 + \frac{9}{|V_3|}}$.

**Proof** From Lemma 19 and Lemma 20 we can consider that $G$ has only vertices of degree 2 or 3 and that $G$ contains no reducible triangle.

Assume that we can associate a set $P_e$ of at least 5 distinct vertices of $V_3$ for each edge $e \in E_\phi(\delta)$ in a $\phi$–minimum edge-colouring $\phi$ of $G$. Assume moreover that 

$$\forall e, e' \in E_\phi(\delta) \quad P_e \cap P_{e'} = \emptyset$$ 

(1)

Then 

$$\gamma(G) = 1 - \frac{s(G)}{m} = 1 - \frac{s(G)}{2|V_3| + |V_2|} \geq 1 - \frac{|V_2|}{2|V_3| + |V_2|}$$

Hence 

$$\gamma(G) \geq 1 - \frac{3}{4 + \frac{9}{|V_3|}}$$

It remains to see how to construct the sets $P_e$ satisfying Property (1). Let $C$ be the set of odd cycles associated to edges in $E_\phi(\delta)$ (see Lemma 10). Let $e \in E_\phi(\delta)$, assume that $e$ is contained in a cycle $C \in C$ of length 3. By Lemma 10, the edges incident to that triangle have the same colour in $\{\alpha, \beta, \gamma\}$. This triangle is hence reducible, impossible. We can thus consider that each cycle of
C has length at least 5. By Lemma 2 and Lemma 12, we know that whenever such a cycle contains vertices of $V_2$, their distance on this cycle is at least 3. Which means that every cycle $C \in \mathcal{C}$ contains at least 5 vertices in $V_2$ as soon as $C$ has length at least 7 or $C$ has length 5 but does not contain a vertex of $V_2$. For each edge $e \in E_C(\delta)$ contained in such a cycle we associate $P_e$ as any set of 5 vertices of $V_3$ contained in the cycle.

There may exist edges in $E_C(\delta)$ contained in a 5-cycle of $\mathcal{C}$ having exactly one vertex in $V_2$. Let $C = a_1a_2a_3a_4a_5$ be such a cycle and assume that $a_1 \in V_2$. By Lemma 2 and Lemma 14, a is the only vertex of degree 2 and by exchanging colours along this cycle, we can suppose that $a_1a_2 \in E_C(\delta)$. Since $a_1 \in V_2$, $e = a_1a_2$ is contained in a second cycle $C'$ of $\mathcal{C}$ (see Remark 7). If $C'$ contains a vertex $x \in V_3$ distinct from $a_2, a_3, a_4$ and $a_5$ then we set $P_e = \{a_2, a_3, a_4, a_5, x\}$. Otherwise $C' = a_1a_2a_3a_4a_5$ and $G$ is isomorphic to $G_5$, impossible.

The sets $\{P_e | e \in E_C(\delta)\}$ are pairwise disjoint since any two cycles of $\mathcal{C}$ associated to distinct edges in $E_C(\delta)$ are disjoint. Hence property 1 holds and the proof is complete. \[\square\]

Albertson and Haas [1] proved that $\gamma(G) \geq \frac{2n}{3}$ when $G$ is a graph with maximum degree 3 and Rizzi [11] obtained $\gamma(G) \geq \frac{5}{6}$. From Theorem 21 we get immediately $\gamma(G) \geq \frac{13}{16}$, a better bound. Let us remark that we get also $\gamma(G) \geq \frac{13}{15}$ by Theorem 18 as soon as $g_{odd} \geq 5$.

**Lemma 22** Let $G$ be a cubic graph which can be factored into $s(G)$ cycles of length 5 and without reducible triangle. Then every 2-factor of $G$ contains $s(G)$ cycles of length 5.

**Proof** Since $G$ has no reducible triangle, all cycles in a 2-factor have length at least 4. Let $\mathcal{C}$ be any 2-factor of $G$. Let us denote $n_4$ the number of cycles of length 4, $n_5$ the number of cycles of length 5 and $n_{6+}$ the number of cycles on at least 6 vertices in $\mathcal{C}$. We have $5n_5 + 6n_{6+} \leq 5s(G) - 4n_4$.

When $n_4 + n_{6+} > 0$, if $n_{6+} > 0$ then $n_5 + n_{6+} < n_5 + \frac{6n_{6+}}{5} \leq \frac{5s(G) - 4n_4}{5} \leq s(G)$ and if $n_{6+} = 0$, we have $n_4 > 0$ and $n_5 \leq \frac{5s(G) - 4n_4}{5} < s(G)$. A contradiction in both cases with Lemma 1. \[\square\]

**Corollary 23** Let $G$ be a graph with maximum degree 3 such that $\gamma(G) = \frac{13}{16}$. Then $G$ is a cubic graph which can be factored into $s(G)$ cycles of length 5. Moreover every 2-factor of $G$ has this property.

**Proof** The optimum for $\gamma(G)$ in Theorem 21 is obtained whenever $s(G) = \frac{|V_1|}{15}$ and $|V_2| = 0$. That is, $G$ is a cubic graph admitting a 2-factor of $s(G)$ cycles of length 5. Moreover by Lemma 20 $G$ has no reducible triangle, the result comes from Lemma 22. \[\square\]

As pointed out by Albertson and Haas [1], the Petersen graph with $\gamma(G) = \frac{13}{16}$ supplies an extremal example for cubic graphs. Steffen [14] proved that the only cubic bridgeless graph with $\gamma(G) = \frac{13}{16}$ is the Petersen graph. In fact, we can extend this result to graphs with maximum degree 3 where bridges are allowed (excluding the graph $G_5$). Let $P'$ be the cubic graph on 10 vertices
obtained from two copies of $G_5$ (Figure 1) by joining by an edge the two vertices of degree 2.

**Theorem 24** Let $G$ be a connected graph with maximum degree 3 such that $\gamma(G) = \frac{13}{15}$. Then $G$ is isomorphic to the Petersen graph or to $P'$.

**Proof** Let $G$ be a graph with maximum degree 3 such that $\gamma(G) = \frac{13}{15}$.

From Corollary 23, we can consider that $G$ is cubic and $G$ has a 2-factor of cycles of length 5.

Let $C = \{C_1 \ldots C_{s(G)}\}$ be such a 2-factor ($C$ is spanning).

Let $\phi$ be a $\delta-$minimum edge-colouring of $G$ induced by this 2-factor.

Without loss of generality consider two cycles in $C$, namely $C_1$ and $C_2$, and let us denote $C_1 = v_1v_2v_3v_4v_5$ while $C_2 = u_1u_2u_3u_4u_5$ and assume that $v_1u_1 \in G$. From Lemma 11, $C_1$ and $C_2$ are joined by at least 3 edges or each of them has two chords. If $s(G) > 2$ there is a cycle $C_3 \in C$. Without loss of generality, $G$ being connected, we can suppose that $C_3$ is joined to $C_1$ by an edge. Applying once more time Lemma 11, $C_1$ and $C_3$ have two chords or are joined by at least 3 edges, contradiction with the constraints imposed by $C_1$ and $C_2$. Hence $s(G) = 2$ and $G$ has 10 vertices and no 4-cycle, which leads to a graph isomorphic to $P'$ or the Petersen graph as claimed. \(\square\)

We can construct cubic graphs with chromatic index 4 (snarks in the literature) which are cyclically 4-edge connected and having a 2-factor of $C_5$’s.

Indeed, let $G$ be a cubic cyclically 4-edge connected graph of order $n$ and $M$ be a perfect matching of $G$, $M = \{x_iy_i|i = 1 \ldots \frac{n}{2}\}$. Let $P_1 \ldots P_{\frac{n}{2}}$ be copies of the Petersen graph. For each $P_i$ ($i = 1 \ldots \frac{n}{2}$) we consider two edges at distance 1 apart $e_1^i$ and $e_2^i$. Let us observe that $P_i - \{e_1^i, e_2^i\}$ contains a 2-factor of two $C_5$’s ($C_1^i$ and $C_2^i$).

We construct then a new cyclically 4-edge connected cubic graph $H$ with chromatic index 4 by applying the well known operation dot-product (see Figure 2 for a description and [7] for a formal definition) on $\{e_1^i, e_2^i\}$ and the edge $x_iy_i$ ($i = 1 \ldots \frac{n}{2}$). We remark that the vertices of $G$ vanish in the operation and the resulting graph $H$ has a 2 factor of $C_5$, namely $\{C_1^1, C_2^1, \ldots C_1^n, C_2^n\}$.

We do not know example an of a cyclically 5-edge connected snark (except the Petersen graph) with a 2-factor of induced cycles of length 5.

**Problem 25** Is there any cyclically 5-edge connected snark distinct from the Petersen graph with a 2-factor of $C_5$’s ?

As a first step towards the resolution of this Problem we propose the following Theorem. Recall that a permutation graph is a cubic graph having some 2-factor with precisely 2 odd cycles.

**Theorem 26** Let $G$ be a cubic graph which can be factored into $s(G)$ induced odd cycles of length at least 5, then $G$ is a permutation graph. Moreover, if $G$ has girth 5 then $G$ is the Petersen graph.

**Proof** Let $\mathcal{F}$ be a 2-factor of $s(G)$ cycles of length at least 5 in $G$, every cycle of $\mathcal{F}$ being an induced odd cycle of $G$. We consider the $\delta-$minimum edge-colouring $\phi$ such that the edges of all cycles of $\mathcal{F}$ are alternatively coloured $\alpha$ and $\beta$ except for exactly one edge per cycle which is coloured with $\delta$, all the
Figure 2: The dot product operation on graphs $G_1, G_2$. 
remaining edges of $G$ being coloured $\gamma$. By construction we have $B_\phi = C_\phi = \emptyset$ and $A_\phi = F$.

Let $xy$ be an edge connecting two distinct cycles of $\mathcal{F}$, say $C_1$ and $C_2$ ($x \in C_1$, $y \in C_2$). By Lemma 8 we may assume that there is an edge in $C_1$, say $e_1$, adjacent to $x$ and coloured with $\delta$, similarly there is on $C_2$ an edge $e_2$ adjacent to $y$ and coloured with $\delta$. Let $z$ be the neighbour of $y$ on $C_2$ such that $e_2 = yz$ and let $t$ be the neighbour of $z$ such that $zt$ is coloured with $\gamma$. If $t \notin C_1$, there must be $C_3 \neq C_1$ such that $t \in C_3$, by Lemma 8 again there is an edge $e_3$ of $C_3$, adjacent to $t$ and coloured with $\delta$. But now $\{e_1, e_2, e_3\}$ induces a subgraph with at least 5 edges, a contradiction with Lemma 14.

It follows that $C$ contains exactly two induced cycles of equal $\gamma$. Consequently $G$ is a permutation graph. When this cycles have length 5, $G$ is obviously the Petersen graph. □

**Comments:** The index $s(G)$ used here is certainly not greater than $o(G)$ the oddness of $G$ used by Huck and Kochol [6]. The oddness $o(G)$ is the minimum number of odd cycles in any 2-factor of a cubic graph (assuming that we consider graphs with that property). Obviously $o(G)$ is an even number and it is an easy task to construct a cubic graph $G$ with $s(G)$ odd which satisfies $0 < s(G) < o(G)$. We can even construct cyclically-5-edge-connected cubic graphs with that property with $s(G) = k$ for any integer $k \neq 1$ (see [4] and [14]). It can be pointed out that, using a parity argument (see [13]) in a graph of oddness at least 2 the colour of minimum frequency is certainly used at least twice. In other words, $o(G) = 2 \Leftrightarrow s(G) = 2$.

When $G$ is a cubic bridgeless planar graph, we know from the Four Colour Theorem that $G$ is 3–edge colourable and hence $\gamma(G) = 1$. Albertson and Haas [1] gave $\gamma(G) \geq \frac{3}{2} - \frac{\sqrt{5}}{2}$, when $G$ is a planar bridgeless graph with maximum degree 3. Our Theorem 21 improves this lower bound (allowing moreover bridges). On the other hand, they exhibit a family of planar graphs with maximum degree 3 (bridges are allowed) for which $\gamma(G) = \frac{3}{2} - \frac{\sqrt{5}}{2}$.

As in [14] we denote $g(\mathcal{F}) = \min\{|V(C)| : C \text{ is an odd cycle of } \mathcal{F}\}$ and $g^+(G) = \max\{g(\mathcal{F}) : \mathcal{F} \text{ is a 2–factor of } G\}$. We suppose that $g^+(G)$ is defined, that is $G$ has at least one 2-factor (when $G$ is a cubic bridgeless graphs this condition is obviously fulfilled).

When $G$ is cubic bridgeless, Steffen [14] showed that we have :

$$\gamma(G) \geq \max\{1 - \frac{2}{3g^+(G)} \} \frac{11}{12}$$

The difficult part being to show that $\gamma(G) \geq \frac{11}{12}$.

**Theorem 27** Let $G$ be a graph with maximum degree 3. Let $V_i$ ($i = 1..3$) be the set of vertices of degree $i$.

Then $\gamma(G) \geq 1 - \frac{2n}{(3n - |V_2|)|V_3|} (2)$.

**Proof** By Lemma 19, we may assume $V_1 = \emptyset$. Hence, $m = \frac{1}{2} (2|V_2| + 3|V_3|)$, moreover $n = |V_2| + |V_3|$, henceforth $m = \frac{3n - |V_2|}{2}$. We have $\gamma(G) = 1 - \frac{s(G)}{m}$ obviously, $s(G) \leq \frac{n}{g^+(G)}$. The result follows.
Theorem 28  Let $G$ be a graph with maximum degree 3 having at least one 2-factor. Let $V_i$ (i = 1..3) be the set of vertices of degree $i$. Assume that $|V_2| \leq \frac{n}{3}$ and $g^+(G) \geq 11$ then $\gamma(G) \geq \max\{1 - \frac{3}{4g^+(G)} \frac{11}{12}\}$.

Proof  By assumption we have $V_1 = \emptyset$. From Theorem 27 we have just to prove that $\gamma(G) \geq \frac{11}{12}$. Following the proof of Theorem 21, we try to associate a set $P_e$ of at least 8 distinct vertices of $V_3$ for each edge $e \in E(\delta)$ in a $\delta$-minimum edge-colouring $\phi$ of $G$ such that

$$\forall e, e' \in E(\delta) \quad P_e \cap P_{e'} = \emptyset$$

Indeed, let $F$ be a 2-factor of $G$ where each odd cycle has length at least 11 and let $C_1, C_2 \ldots C_{2k}$ be its set of odd cycles. We have, obviously $s(G) \leq 2k$. Let $V'_3$ and $V'_2$ be the sets of vertices of degree 3 and 2 respectively contained in these odd cycles. As soon as $|V'_3| \geq 8s(G)$ we have

$$\gamma(G) = 1 - \frac{s(G)}{m} = 1 - \frac{s(G)}{\frac{2}{3}|V_3| + |V_2|} \geq 1 - \frac{|V'_3|}{\frac{2}{3}|V_3| + |V_2|}$$

which leads to

$$\gamma(G) \geq 1 - \frac{2|V'_3|}{1 + \frac{2}{3}|V_2|}$$

Since $|V_3| \geq |V'_3|$, we have

$$\gamma(G) \geq 1 - \frac{2}{1 + \frac{2}{3}|V_2|}$$

and

$$\gamma(G) \geq \frac{11}{12}$$

as claimed.

It remains the case where $|V'_3| < 8s(G)$. Since each odd cycle has at least 11 vertices we have $|V'_2| > 11 \times 2k - |V'_3| > 3s(G)$.

$$\gamma(G) = \frac{m - s(G)}{m} \geq \frac{m - |V'_2|}{m}$$

We have

$$\frac{m - |V'_2|}{m} \geq \frac{11}{12}$$

when

$$m \geq 4|V'_2|$$

(4)

Since $|V_2| \leq \frac{n}{3}$ we have $|V_3| \geq \frac{2n}{3}$ and

$$m = 3\frac{|V_3|}{2} + |V_2| = 3\frac{n - |V_2|}{2} + |V_2| = \frac{3n}{2} - \frac{|V_2|}{2} \geq 4\frac{n}{3} \geq 4|V'_2|$$

(5)

and the result holds. □
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