NEGATIVITY CONJECTURE FOR THE FIRST HILBERT COEFFICIENT

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Abstract. This gives an alternate proof of the result of [2, Theorem 2.1]: The first Hilbert coefficient of parameter ideals in an unmixed Noetherian local ring is always negative unless the ring is Cohen–Macaulay.

1. Introduction

Let $R$ be a Noetherian local ring with the maximal ideal $m$ of dimension $d > 0$. Let $I$ be an $m$–primary $R$–ideal. For sufficiently large $n$, the length $\lambda(R/I^{n+1})$ is of polynomial type:

$$P_I(n) = \sum_{i=0}^{d} (-1)^i e_i(I) \binom{n + d - i}{d - i}.$$  

The integers $e_i(I)$’s are called the Hilbert coefficients of $I$. The first Hilbert coefficient $e_1(Q)$ of a parameter ideal $Q$ codes structural information about the ring $R$ itself. In response to a question in [7], the following was settled in [2].

Theorem 1.1. ([2, Theorem 2.1]) An unmixed Noetherian local ring $R$ is not Cohen-Macaulay if and only if $e_1(Q) < 0$ for a parameter ideal $Q$.

Meanwhile, for any parameter ideal $Q$ of $R$, it was proved that $e_1(Q) \leq 0$ ([2, Corollary 2.5], [6]). Hence the above theorem can be rephrased as follows:

Corollary 1.2. An unmixed Noetherian local ring $R$ is Cohen–Macaulay if and only if $e_1(Q) = 0$ for some parameter ideal $Q$ of $R$.

In the following section, we give an alternate proof.

2. The Proof

Proof of Theorem 1.1 We use a setup developed in [3]. It is enough to show that if $R$ is not Cohen–Macaulay, then $e_1(Q) < 0$. We may assume that the residue field is infinite.

By passing to $m$–adic completion $\hat{R}$, we may also assume that $R$ is complete. Then there exists a Gorenstein local ring $(S, n)$ of dimension $d = \dim(R)$ such that $R$ is a homomorphic image of...
This means that there exists a canonical module $\omega_R = \text{Hom}_S(R, S)$. Consider the natural homomorphism
$$\varphi : R \to \text{Hom}_S(\omega_R, \omega_R) \simeq \text{Hom}_S(\omega_R, S).$$
Because $R$ is unmixed, this map $\varphi$ is injective (\cite{1}, 1.11.1). Moreover, let $A = \text{Hom}_S(\omega_R, \omega_R)$. Then by applying local cohomology to $0 \to R \to A \to D \to 0$, we obtain $H^1_n(R) \simeq H^0_n(D)$ since depth$(A) \geq 2$ (\cite{4}, \cite{5}).

By dualizing $S^n \to \omega_R \to 0$ into $S$, we obtain another injective map
$$0 \to \text{Hom}_S(\omega_R, S) \to S^n.$$
Composing these two maps, we obtain an embedding $R \hookrightarrow S^n$.

Let $Q$ be a parameter ideal of $R$. Then there exists a parameter ideal $q$ of $S$ such that $qR = Q$ (\cite{3}, Lemma 3.1). Therefore the associated graded ring of $Q$ is isomorphic to the associated graded module of $q$ with respect to the $S$–module $R$:
$$\text{gr}_Q(R) \simeq \text{gr}_q(R),$$
which implies that
$$e_1(Q) = e_1(q, R),$$
where $e_1(q, R)$ denotes the first Hilbert coefficient of $q$ with respect to $S$–module $R$.

Consider the exact sequence of $S$–modules:
$$0 \to R \to S^n \to C \to 0.$$ Let $y$ be a superficial element for $q$ with respect to $R$ such that $y$ is a part of minimal generating set of $q$ and that $y \notin \text{Ass}_S(C) \setminus \{n\}$. By tensoring the exact sequence of $S$–modules with $S/(y)$, we get
$$0 \to T = \text{Tor}_1^S(S/(y)S, C) \to R/(y)R \to S^n/(y)S^n \to C/(y)C \to 0.$$ Let $R' = R/(y)R$ and $S' = \text{Im}(\zeta)$ and consider the short exact sequence :
$$0 \to T \to R' \to S' \to 0.$$ Then either $T = 0$ or $T$ has finite length $\lambda(T) < \infty$.

Now we use induction on $d = \dim(R)$ to show that if $R$ is not Cohen–Macaulay, then $e_1(q, S) < 0$.

Let $d = 2$ and $q = (y, z)$. Then $T \neq 0$ so that $\lambda(T) < \infty$. Applying the Snake Lemma to
$$0 \to T \cap z^nR' \to z^nR' \to z^nS' \to 0$$
we get, for sufficiently large $n$,
$$\lambda(R'/z^nR') = \lambda(T) + \lambda(S'/z^nS').$$
Computing the Hilbert polynomials, we have
$$e_1(q/y, R/(y)R) = -\lambda(T) < 0$$
so that
\[ e_1(q, R) = e_1(q/y, R/yR) - \lambda(0 : R y) = -\lambda(T) - \lambda(0 : R y) < 0. \]

Now suppose that \( d \geq 3 \). From the exact sequence
\[ 0 \rightarrow T \rightarrow R' = R/yR \rightarrow S' \rightarrow 0, \]
we have
\[ e_1(q, R) = e_1(q/(y), R/yR) = e_1(q/(y), S'). \]
By an induction argument, it is enough to show that \( S' \) is not Cohen–Macaulay since \( \dim(S/yS) = d - 1 \).

Suppose that \( S' \) is Cohen–Macaulay. Let \( n \) be the maximal ideal of \( S/yS \). From the exact sequence
\[ 0 \rightarrow T \rightarrow R' = R/yR \rightarrow S' \rightarrow 0, \]
we obtain the long exact sequence:
\[ 0 \rightarrow H^0_n(T) \rightarrow H^0_n(R') \rightarrow H^0_n(S') \rightarrow H^1_n(T) \rightarrow H^1_n(R') \rightarrow H^1_n(S'). \]
By the assumption that \( S' \) is Cohen–Macaulay of dimension \( d - 1 \geq 2 \) and the fact that \( T \) is a torsion module, we get
\[ 0 \rightarrow T \rightarrow H^0_n(R') \rightarrow 0 \rightarrow 0 \rightarrow H^1_n(R') \rightarrow 0. \]
We may assume that \( y \) is a nonzerodivisor on \( R \). From the exact sequence
\[ 0 \rightarrow R \rightarrow y \rightarrow R/yR \rightarrow 0, \]
we obtain the following exact sequence:
\[ 0 \rightarrow T \xrightarrow{y} H^0_n(R') \rightarrow H^1_n(R') \rightarrow 0. \]
Since \( H^1_n(R') \) is finitely generated and \( H^1_n(R) = yH^1_n(R) \), we have \( H^1_n(R) = 0 \). This means that \( T = 0 \). Therefore
\[ 0 \rightarrow T = 0 \rightarrow R/yR \rightarrow S' \rightarrow 0. \]
Since \( S' \) is Cohen–Macaulay, \( R' = R/yR \) is Cohen–Macaulay. Since \( y \) is regular on \( R \), \( R \) is Cohen–Macaulay, which is a contradiction. \( \square \)

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