On solving the constraints by integrating a strongly hyperbolic system

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Abstract

It was shown recently that the constraints on the initial data for Einstein’s equations may be posed as an evolutionary problem \cite{1}. In one of the proposed two methods the constraints can be replaced by a first order symmetrizable hyperbolic system and a subsidiary algebraic relation. Here, by assuming that the initial data surface is smoothly foliated by a one-parameter family of topological two-spheres, the basic variables are recast in terms of spin-weighted fields. This allows one to replace all the angular derivatives in the evolutionary system by the Newman-Penrose $\bar{\nabla}$ and $\nabla$ operators which, in turn, opens up a new avenue to solve the constraints by integrating the resulting system using suitable numerical schemes. In particular, by replacing the $\bar{\nabla}$ and $\nabla$ operators either by a finite difference or by a pseudo-spectral representation or by applying a spectral decomposition in terms of spin-weighted spherical harmonics, the evolutionary equations may be put into the form of a coupled system of non-linear ordinary differential equations.

1 Introduction

This paper is intended to be a technical report providing a firm analytic background to support the numerical integration of the Einstein constraint equations, when cast into the form of a first order symmetrizable hyperbolic system and a subsidiary algebraic condition. The main steps in deriving this form of the constraint equations is outlined in this section.

Consider first the initial data specification in general relativity comprised by a Riemannian metric $h_{ij}$ and a symmetric tensor field $K_{ij}$ on a three-dimensional manifold $\Sigma$. The pair $(h_{ij}, K_{ij})$ is said to satisfy the vacuum constraints (see e.g. Refs. \cite{14, 15})
if the relations
\begin{align}
(3) R + (K^{ij}_j)^2 - K_{ij} K^{ij} &= 0, \\
D_i K^{ij}_i - D_i K^{ij}_j &= 0
\end{align}
(1.1)
(1.2)
hold on $\Sigma$, where $(3) R$ and $D_i$ denote the scalar curvature and the covariant derivative operator associated with $h_{ij}$, respectively.

$\Sigma$ is assumed to be smoothly foliated by a one-parameter family of topological two-spheres $\mathcal{S}_\rho$ which may also be considered as the level surfaces $\rho = \text{const}$ of a smooth function $\rho : \Sigma \to \mathbb{R}$.

Applying a vector field $\rho^i$ on $\Sigma$, satisfying the relation $\rho^i \partial_i \rho = 1$, the unit normal $\hat{n}^i$ to the level surfaces $\mathcal{S}_\rho$ decomposes as
\begin{equation}
\hat{n}^i = \hat{N}^{-1} [\rho^i - \hat{N}^i],
\end{equation}
where the ‘lapse’ $\hat{N}$ and ‘shift’ $\hat{N}^i$ of the vector field $\rho^i$ are determined by $\hat{n}_i = \hat{N} \partial_i \rho$ and $\hat{N}^i = \hat{\gamma}^i_j \rho^j$, where $\hat{\gamma}^i_j = \delta^i_j - \hat{n}^i \hat{n}_j$.

The Riemannian metric $h_{ij}$ on $\Sigma$ can then be decomposed as
\begin{equation}
h_{ij} = \hat{\gamma}_{ij} + \hat{n}_i \hat{n}_j,
\end{equation}
where $\hat{\gamma}_{ij}$ is the metric induced on the surfaces $\mathcal{S}_\rho$, while the extrinsic curvature $\hat{K}_{ij}$ of $\mathcal{S}_\rho$ is given by
\begin{equation}
\hat{K}_{ij} = \hat{\gamma}^l_i D_l \hat{n}_j = \frac{1}{2} \hat{\gamma}_n \hat{\gamma}_{ij}.
\end{equation}

The other part of the initial data represented by the symmetric tensor field $K_{ij}$ has decomposition
\begin{equation}
K_{ij} = \kappa \hat{n}_i \hat{n}_j + [\hat{n}_i k_j + \hat{n}_j k_i] + K_{ij},
\end{equation}
where $\kappa = \hat{n}^k \hat{n}^l K_{kl}$, $k_i = \hat{\gamma}^k_i \hat{n}^l K_{kl}$ and $K_{ij} = \hat{\gamma}^k \hat{\gamma}^l_j K_{kl}$. Note that all boldfaced symbols stand for tensor fields which are well-defined on the individual leaves $\mathcal{S}_\rho$. In recasting the Hamiltonian and momentum constraints (1.1) and (1.2) the traces
\begin{equation}
\hat{K}^l_l = \hat{\gamma}^{kl} \hat{K}_{kl} \quad \text{and} \quad K^l_l = \hat{\gamma}^{kl} K_{kl}
\end{equation}
(1.7)
and the trace free part of $K_{ij}$, defined as
\begin{equation}
\hat{K}_{ij} = K_{ij} - \frac{1}{2} \hat{\gamma}_{ij} K^l_l,
\end{equation}
(1.8)
will also be involved.

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By making use of the above variables, the pair \((h_{ij}, K_{ij})\) may be replaced by the fields \(\hat{N}, \hat{N}^i, \hat{\gamma}_{ij}, \hat{K}_{ij}, \kappa, k_i, \text{ and } K^l_i\), and in turn, the Hamiltonian and momentum constraints (1.1) and (1.2) can be re-expressed as [9] (see also [10, 11, 12])

\[
\mathcal{L}_n(K^l_i) - \hat{D}^l_k k_l + 2 \hat{n}^l k_l - [\kappa - \frac{1}{2} (K^l_i)^2] (\hat{K}^l_i) + \hat{K}_{kl} \hat{K}^{kl} = 0, \tag{1.9}
\]

\[
\mathcal{L}_n(k_i) + (K^l_i)^{-1} [\kappa \hat{D}_i(K^l_i) - 2 k^l \hat{D}_l k_i] + (2 K^l_i)^{-1} \hat{D}_i \kappa_0 + (\hat{K}^l_i) k_i + [\kappa - \frac{1}{2} (K^l_i)^2] \hat{n}_i - \hat{n}^l \hat{K}_{li} + \hat{D}^l \hat{K}_{li} = 0, \tag{1.10}
\]

where \(\kappa\) and \(\kappa_0\) are given by the algebraic expressions

\[
\kappa = (2 K^l_i)^{-1} [2 k^l k_i - \frac{1}{2} (K^l_i)^2 - \kappa_0], \tag{1.11}
\]

\[
\kappa_0 = (\hat{\gamma}^l_i) k_i, \tag{1.12}
\]

and where \(\hat{D}_i\) and \(\hat{R}\) denote the covariant derivative operator and scalar curvature associated with \(\hat{\gamma}_{ij}\), respectively, and \(\hat{n}_k = \hat{n}^l D_k \hat{n}_l = -\hat{D}_k (\ln \hat{N})\).

Note that (1.11) replaces the Hamiltonian constraint (1.1) which acquires, thereby, an algebraic form (for more details see [9]). Note also that in virtue of (1.9)-(1.12) the four basic variables \(\kappa, k_i, K^l_i\) are subject to the constraints whereas the remaining eight variables, represented by the fields \(\hat{N}, \hat{N}^i, \hat{\gamma}_{ij}, \hat{K}_{ij}\), are freely specifiable throughout \(\Sigma\).

### 2 The Newman-Penrose \(\bar{\delta}\) and \(\bar{\partial}\) operators

Equations (1.9)-(1.11) are intended to be solved by decomposing the involved basic variables in terms of spin-weighted spherical fields. In doing so we shall replace all angular derivatives by the Newman-Penrose \(\bar{\delta}\) and \(\bar{\partial}\) operators [8, 5], using the notation introduced in [6, 14] throughout this paper.

Consider first the unit sphere metric \(q_{ab}\), given in standard \((\theta, \phi)\) coordinates by

\[
ds^2 = q_{ab} dx^a dx^b = d\theta^2 + \sin^2 \theta d\phi^2. \tag{2.1}
\]

In terms of the complex stereographic coordinate\(^1\)

\[
z = e^{-i \phi} \cot \frac{\theta}{2} = z_1 + i z_2, \tag{2.2}
\]

on the unit sphere \(S^2\), the line element (2.1) can also be written as

\[
ds^2 = 4 \left(1 + z \bar{z}\right)^{-2} \left[ (dz_1)^2 + (dz_2)^2 \right]. \tag{2.3}
\]

\(^1\)Expressions relevant for the south hemisphere will only be given explicitly. From these the ones which apply to the north hemisphere can be deduced by using the replacement \(z_N = 1/\bar{z}_S[6, 14]\).
Choose now the complex dyad on $S^2$
\[ q^a = 2^{-1} P \left[ (\partial z_1)^a + i (\partial z_2)^a \right] = P (\partial \varphi)^a, \] (2.4)
where
\[ P = 1 + z \overline{z}. \] (2.5)

We also have
\[ q_a = q_{ab} q^b = 2 P^{-1} \left[ (dz_1)_a + i (dz_2)_a \right] = 2 P^{-1} (dz)_a. \] (2.6)

Note that the complex dyad $q^a$ has normalization
\[ q^a \overline{q}_a = 2, \quad q^a q_a = 0, \] (2.7)
and that the unit sphere metric $q_{ab}$ satisfies
\[ q_{ab} = q(a \overline{b}), \quad q^{ab} = q^a q^b, \quad q^{ae} q_{eb} = \delta^a_b. \] (2.8)

Note also that the metric (2.3) is conformally flat,
\[ q_{ab} = \Omega^2 \delta_{ab}, \] (2.9)
with conformal factor
\[ \Omega = 2 (1 + z \overline{z})^{-1} = 2 P^{-1}. \] (2.10)

The Newman-Penrose $\overline{\mathcal{D}}$ and $\mathcal{D}$ operators are then given by (see, e.g. (A4) in [6])
\[ \overline{\mathcal{D}} \mathcal{L} = P^{1-s} \partial_\varphi (P^s \mathcal{L}) \] (2.11)
\[ \mathcal{D} \mathcal{L} = P^{1+s} \partial_\varphi (P^{-s} \mathcal{L}), \] (2.12)
where the spin-weight $s$ function $\mathcal{L}$ on the unit two-sphere is defined by the contraction
\[ \mathcal{L} = q^{a_1} \ldots q^{a_s} \mathcal{L}_{(a_1 \ldots a_s)} \] (2.13)
for some totally symmetric traceless tensor field $\mathcal{L}_{a_1 \ldots a_s}$ on $S^2$.

As pointed out in [6, 14], this choice of $\overline{\mathcal{D}}$ and $\mathcal{D}$ corresponds to the standard conventions in [6, 5, 14]. Therefore, the action of $\overline{\mathcal{D}}$ and $\mathcal{D}$ on spin-weighted spherical harmonics $s \mathcal{Y}_{l,m}$ is given by (see e.g. (2.6)-(2.8) in [6])
\[ s \mathcal{Y}_{l,m} = (-1)^{m+s-s} s \mathcal{Y}_{l,m} \] (2.14)
\[ \overline{\mathcal{D}} s \mathcal{Y}_{l,m} = \sqrt{(l-s)(l+s+1)} s+1 \mathcal{Y}_{l,m} \] (2.15)
\[ \mathcal{D} s \mathcal{Y}_{l,m} = - \sqrt{(l+s)(l-s+1)} s-1 \mathcal{Y}_{l,m} \] (2.16)
\[ \overline{\mathcal{D}} \mathcal{D} s \mathcal{Y}_{l,m} = - (l-s)(l+s+1) s \mathcal{Y}_{l,m}. \] (2.17)
Also, the $\partial$ and $\bar{\partial}$ operators are related to the torsion free covariant derivative operator $D_a$ determined by $q_{ab}$ by (see the Appendix of this paper for verification)

$$\partial L = q^b q^{a_1} \ldots q^{a_s} D_b L(a_1 \ldots a_s), \quad (2.18)$$

$$\bar{\partial} L = q^b q^{a_1} \ldots q^{a_s} D_b L(a_1 \ldots a_s). \quad (2.19)$$

The applied conventions are such that the volume element $\epsilon_{AB}$ on $S^2$ is $\epsilon_{AB} = i q_A q_B$ and for a spin-weight $s$ field $f$ the relation $[\partial, \bar{\partial}] f = 2 s f$ holds on $S^2$.

3 Reduction to spin-weighted fields

Consider now one of the level surfaces $\mathcal{S}_{\rho_0}$ of the foliation $\mathcal{S}_{\rho}$. As $\mathcal{S}_{\rho_0}$ is diffeomorphic to the unite sphere $S^2$ we may assume that standard spherical coordinates $(\theta, \phi)$ as in (2.1) are chosen on $\mathcal{S}_{\rho_0}$. By using the vector field $\rho^i = (\partial_{\rho})^i$ these coordinates can be Lie dragged onto all the other leaves of the foliation $\mathcal{S}_{\rho}$ by keeping their values constant along the integral curves of $\rho^i$.

Note that in terms of the coordinates $(\theta, \phi)$, by making use of the line element (2.1), the metric $q_{ab}$ can immediately be defined on each of the level surfaces $\mathcal{S}_{\rho}$. Similarly, the complex dyad vector $q^a$ can be defined on the individual $\mathcal{S}_{\rho}$ surfaces.

It is then an important consequence of the above construction that not only the coordinates $\theta$ and $\phi$ but the complex dyad $q^a$, along with $q_a$ and the unit sphere metric $q_{ab}$, will be Lie dragged from $\mathcal{S}_{\rho_0}$ onto the other level surfaces of the foliation $\mathcal{S}_{\rho}$, i.e.

$$\mathcal{L}_{\rho} q^a = 0, \quad \mathcal{L}_{\rho} q_a = 0 \quad \text{and} \quad \mathcal{L}_{\rho} q_{ab} = 0. \quad (3.1)$$

3.1 The decomposition of the metric $\widehat{\gamma}_{ab}$

The metric $\widehat{\gamma}_{ab}$ induced on the $\mathcal{S}_{\rho}$ level surfaces can then be decomposed as

$$\widehat{\gamma}_{ab} = \mathfrak{a} q_{ab} + \breve{\gamma}_{ab}, \quad (3.2)$$

where

$$\mathfrak{a} = \frac{1}{2} \widehat{\gamma}_{ab} q^a q^b \quad (3.3)$$

is a positive spin-weight zero function on the $\mathcal{S}_{\rho}$ level surfaces and $\breve{\gamma}_{ab}$ is the trace-free part of $\widehat{\gamma}_{ab}$ with respect to the unit sphere metric $q_{ab}$, i.e.

$$\breve{\gamma}_{ab} = \left[ \delta^e_a \delta^f_b - \frac{1}{2} q_{ab} q^{ef} \right] \widehat{\gamma}_{ef} = \widehat{\gamma}_{ab} - \mathfrak{a} q_{ab}. \quad (3.4)$$

As $\breve{\gamma}_{ab}$ is a symmetric trace-free tensor it is given by

$$\breve{\gamma}_{ab} = \frac{1}{2} \left[ \mathfrak{b} q_a q_b + \overline{\mathfrak{b}} q_b q_a \right], \quad (3.5)$$
where the spin-weight 2 function $b$ is given by the contraction

$$b = \frac{1}{2} \tilde{\gamma}_{ab} q^a q^b = \frac{1}{2} \tilde{q}_{ab} q^a q^b.$$  

(3.6)

It may also be verified that the inverse $\tilde{\gamma}^{ab}$ metric can be given by

$$\tilde{\gamma}^{ab} = d^{-1} \left\{ a q^{ab} - \frac{1}{2} \left[ b \bar{q}^a \bar{q}^b + \bar{b} q^a q^b \right] \right\},$$

(3.7)

where

$$d = a^2 - b \bar{b}$$

(3.8)

stands for the ratio $\text{det}(\tilde{\gamma}_{ab})/\text{det}(q_{ab})$ of the determinants of $\tilde{\gamma}_{ab}$ and $q_{ab}$.

As an immediate application, using (2.8), (3.2) and (3.4), along with the notation

$$k = q' k_l, \quad \bar{k} = q' \bar{k}_l,$$

(3.9)

$k' k_l$ can be expressed as

$$k' k_l = \tilde{\gamma}^{kl} k_k k_l = \frac{1}{2} d^{-1} \left\{ a \left[ q^k \bar{q}^l + q^l \bar{q}^k \right] \right\} k_k k_l$$

$$= \frac{1}{2} d^{-1} \left\{ 2 a k k_l - b \bar{b} k_l - \bar{b} k_l^2 \right\}. \quad (3.10)$$

### 3.2 Terms involving the covariant derivative $\hat{D}_a$

The covariant derivative operators $\hat{D}_a$ and $D_a$ can be related by the (1,2) type tensor field (see e.g. (3.1.28) and (D.3) in [15])

$$C_{e}^{ab} = \frac{1}{2} \tilde{\gamma}^{ef} \left\{ D_a \tilde{\gamma}_{fb} + D_b \tilde{\gamma}_{af} - D_f \tilde{\gamma}_{ab} \right\}. \quad (3.11)$$

In particular,

$$\hat{D}_a k_b = D_a k_b - C_{e}^{ab} k_e,$$

(3.12)

and thereby

$$\hat{D}^l k_l = \tilde{\gamma}^{kl} \hat{D}_k k_l = \frac{1}{2} d^{-1} \left\{ a \left[ q^k \bar{q}^l + q^l \bar{q}^k \right] \right\} \hat{D}_k k_l$$

$$= \frac{1}{2} d^{-1} \left\{ 2 a \left( \bar{\partial} \bar{k} - \partial \bar{k} \right) - b \left( 2 \bar{\partial} k - \partial k - \bar{k} \bar{k} \right) \right\} + “CC” \right\}, \quad (3.13)$$

where

$$A = q^a q^b C_{e}^{ab} \bar{q}_e = d^{-1} \left\{ a \left[ 2 \bar{\partial} a - \bar{\partial} b \right] - \bar{b} \bar{\partial} b \right\}$$

$$B = \bar{q}^i q^b C_{e}^{ab} q_e = d^{-1} \left\{ a \bar{\partial} b - b \bar{\partial} b \right\}$$

$$C = q^a q^b C_{e}^{ab} q_e = d^{-1} \left\{ a \bar{\partial} b - b \left[ 2 \bar{\partial} a - \bar{\partial} b \right] \right\}. \quad (3.14)$$

Hereafter “CC” stands for the complex conjugate of the terms at the pertinent level of the hierarchy.
3.3 The scalar curvature \( (3) R \)

In expressing the scalar curvature \( (3) R \) in terms of spin-weighted fields we can use the relation

\[
(3) R = \tilde{R} - [2 \mathcal{L}\hat{\gamma}(\hat{K}^l_i) + (\hat{K}^l_i)^2 + \hat{K}_{kl}\hat{K}^{kl} + 2 \hat{N}^{-1}\hat{D}^i\hat{D}_i\hat{N}],
\]

where the scalar curvature \( \tilde{R} \) of the metric \( \hat{\gamma}_{ab} \) is given by

\[
\tilde{R} = \frac{1}{2} d^{-1} \left\{ 2 a - \partial \hat{\sigma} a + \hat{\sigma}^2 b + \frac{1}{2} d^{-1} \left[ 2 (\partial a) (a \hat{\sigma} a - a \hat{\sigma} b - b \hat{\sigma} b) \right] \right.
\]
\[
+ (\partial b) \left( b \hat{\sigma} b + \frac{1}{2} a \hat{\sigma} b \right) + (\hat{\sigma} b) \left( b \hat{\sigma} b - \frac{1}{2} a \hat{\sigma} b \right) \left\} + \text{"CC"} \right.,
\]

or, by using (3.14) to replace first order derivatives, \( \tilde{R} \) can also be given in the shorter form

\[
\tilde{R} = \hat{R} = b^{-1} \left\{ \hat{\sigma} C - \hat{\sigma} B + \frac{1}{2} \left[ C \hat{A} + A B - B^2 - \hat{B} C \right] \right\}.
\]

3.4 Terms involving the lapse \( \hat{N} \)

Using the notation

\[
\hat{N} = \tilde{N}
\]

we obtain

\[
\hat{D}^i\hat{D}_i\hat{N} = \hat{\gamma}^{kl} [\hat{D}_k\hat{D}_l\hat{N}] = \hat{\gamma}^{kl} [\hat{D}_k\hat{D}_l \tilde{N} - C^f_{kl}\hat{D}_f\tilde{N}] \]
\[
= d^{-1} \left\{ a q^{kl} - \frac{1}{2} [b q^l q^l + \hat{\sigma} q^l q^l] \right\} [\hat{D}_k\hat{D}_l \tilde{N} - \frac{1}{2} C^f_{kl} [q_f q^l + q^l q^f] \hat{D}_f\tilde{N}] \]
\[
= \frac{1}{2} d^{-1} \left\{ a \left\{ (\partial\hat{\sigma}\hat{N}) - \hat{B} (\hat{\sigma}\hat{N}) \right\} - b \left\{ (\hat{\sigma}^2\hat{N}) - \frac{1}{2} \hat{A} (\hat{\sigma}\hat{N}) - \frac{1}{2} \hat{C} (\hat{\sigma}\hat{N}) \right\} + \text{"CC"} \right].
\]

In virtue of the relation \( \hat{n}_k = \hat{n}^l D_l \hat{n}_k = -\hat{D}_k(\ln \hat{N}) \) we also have

\[
q^i \hat{n}_i = -\hat{N}^{-1}\hat{\sigma}\hat{N}
\]

and

\[
\hat{k}^i \hat{n}_i = - (2 d \hat{N})^{-1} \left\{ (\hat{\sigma}\hat{N}) \left[ a \hat{k} - \hat{B} k \right] + \text{"CC"} \right\}.
\]
3.5 Terms involving the shift $\hat{N}^i$ and $K^l_l$

By making use of the relations

$$\hat{N} = q_i \hat{N}^i = q_i \hat{\gamma}^{ij} \hat{N}_j = d^{-1} (a q^i - b \bar{q}^i) \hat{N}_j = d^{-1} (a N - b \bar{N})$$

(3.23)

or alternatively

$$N = q^i \hat{N}_i = q^i \hat{\gamma}^{lk} \hat{N}_k = (a q_k + b \bar{q}_k) \hat{N}_k = a \hat{N} + b \bar{N},$$

(3.24)

the Lie derivative $\mathcal{L}_\hat{n} (K^l_l)$ appearing in (1.9) can be expressed as

$$\mathcal{L}_\hat{n} (K^l_l) = \hat{n}^i D_i K^l_l = \hat{N}^{-1} [(\partial_{\rho})^i - \hat{N}^i] D_i K^l_l = \hat{N}^{-1} [\partial_{\rho} K^l_l - \hat{N}^i \mathcal{D}_i K^l_l]$$

$$= \mathcal{L}_\hat{n} K = \hat{N}^{-1} [\partial_{\rho} K] - \frac{1}{2} \hat{N} (\nabla K) - \frac{1}{2} \hat{N} (\overline{\nabla} K),$$

(3.25)

where

$$K = K^l_l = \hat{\gamma}^{kl} K_{kl}$$

(3.26)

and we have used $\hat{N}^i D_i K^l_l = \hat{N}^i \mathcal{D}_i K^l_l = \frac{1}{2} \hat{N}^i (q_i q^j + \bar{q}_i \bar{q}^j) \mathcal{D}_j K^l_l$.

3.6 Terms involving the trace-free part of $K_{kl}$

By setting

$$\hat{K} = q^k q^l \hat{K}_{kl}$$

(3.27)

and

$$\hat{\gamma} = q^k \bar{q}^l \hat{K}_{kl},$$

(3.28)

in virtue of (1.8), we obtain

$$\hat{K}_{ij} = \frac{1}{2} q_{ij} \hat{K} + \frac{1}{4} [q_i q_j \hat{K} + \bar{q}_i \bar{q}_j \hat{K}] .$$

(3.29)

Note that, since $\hat{K}_{kl}$ is trace free, $\hat{\gamma}$ and $\hat{K}$ are not functionally independent. Indeed, the trace-free condition $\hat{\gamma}^{kl} \hat{K}_{kl} = 0$ implies

$$\hat{K} = (2 a)^{-1} [b \hat{K} + \overline{b} \hat{K}].$$

(3.30)

For both $a^{-1}$ and $\hat{\gamma}$, to be well-defined $a$ cannot vanish. This is, however, guaranteed because $\hat{\gamma}_{ij}$ is a positive definite Riemannian metric so that $d = a^2 - b \overline{b}$ is positive.

We then have

$$q^i \hat{n}^k \hat{K}_{ki} = -\frac{1}{2} (\hat{N} d)^{-1} \left[ a (\overline{\partial} \hat{N}) \hat{K} + a (\partial \hat{N}) \hat{K} - b (\overline{\partial} \hat{N}) \hat{K} - \overline{b} (\partial \hat{N}) \hat{K} \right]$$

(3.31)
\[ q^i \hat{D}^k \hat{K}_{ki} = \frac{1}{2} d^{-1} \left\{ a \overline{\partial} \hat{K} + a \overline{\partial} \hat{K} - b \overline{\partial} \hat{K} - b \overline{\partial} \hat{K} \right\} \]
\[ - \frac{a}{4d} \left\{ 3 \overline{\partial} \hat{K} + 3 \overline{\partial} \hat{K} + A \hat{K} + \overline{\partial} \hat{K} \right\} \]
\[ + \frac{b}{4d} \left\{ \overline{\partial} \hat{K} + A \hat{K} + B \hat{K} + B \overline{\partial} \hat{K} \right\} + \frac{\overline{\partial}}{2d} \left\{ A \hat{K} + C \hat{K} \right\} \]
\[ (3.32) \]

\[ \hat{K}_{ij} \hat{K}^{ij} = \frac{1}{2} d^{-2} \left\{ \overline{\partial} \left( a^2 \hat{K} + b^2 \overline{\partial} \hat{K} \right) + "CC" \right\} + 2 (a^2 + b b) \hat{K}^2 \]
\[ (3.33) \]

### 3.7 The determination of \( \mathcal{L}_{\hat{n}} k_i \) and \( \mathcal{L}_\rho k_i \)

The Lie derivative \( \mathcal{L}_{\hat{n}} k_i \) appearing in (1.10), can be re-expressed as follows.

Note first that
\[ (\mathcal{L}_{\hat{n}} k_i) \hat{n}^i = \mathcal{L}_{\hat{n}} (k_i \hat{n}^i) = 0 \]
\[ (3.34) \]
which implies
\[ \mathcal{L}_{\hat{n}} k_i = \hat{\gamma}_i^i \mathcal{L}_{\hat{n}} k_i. \]
\[ (3.35) \]

Then, it is straightforward to verify that
\[ \mathcal{L}_{\hat{n}} k_i = \hat{\gamma}_i^i \mathcal{L}_{\hat{n}} k_i = \hat{N}^{-1} \hat{\gamma}_i^i \left[ \mathcal{L}_\rho k_i - \mathcal{L}_{\hat{N}} k_i \right] \]
\[ = \hat{N}^{-1} \left[ \hat{\gamma}_i^j (\mathcal{L}_{\rho} k_i) - \hat{N}^j \hat{D}_j k_i - k_f \hat{D}_l \hat{N}^j \right] \]
\[ = \hat{N}^{-1} \left[ \hat{\gamma}_i^j (\mathcal{L}_{\rho} k_i) - \hat{N}^j \hat{D}_j k_i - k_f \hat{D}_l \hat{N}^j \right], \]
\[ (3.36) \]

where in the second line we have used the freedom in choosing a torsion free connection when evaluating \( \mathcal{L}_{\hat{N}} k_i \).

In determining \( q^i \mathcal{L}_{\hat{n}} k_i \) we use
\[ q^i \hat{\gamma}_i^j (\mathcal{L}_{\rho} k_i) = q^i q_j^i (\mathcal{L}_{\rho} k_i) = (\partial_\rho k) \]
\[ (3.37) \]
and
\[ q^i \left[ \hat{N}^j \hat{D}_j k_i + k_f \hat{D}_l \hat{N}^j \right] = \frac{1}{2} \left[ \hat{N} \overline{\partial} k + \overline{\partial} \hat{N} k \right] + \frac{1}{2} \left[ k \overline{\partial} \hat{N} + \overline{\partial} \overline{\partial} \hat{N} \right]. \]
\[ (3.38) \]

Then
\[ q^i \mathcal{L}_{\hat{n}} k_i = \hat{N}^{-1} \left( \partial_\rho k - \frac{1}{2} \left[ \hat{N} \overline{\partial} k + \overline{\partial} \hat{N} k + k \overline{\partial} \hat{N} + \overline{\partial} \overline{\partial} \hat{N} \right] \right). \]
\[ (3.39) \]
3.8 The decomposition of $\hat{D}_k \hat{N}_l$

We also need to evaluate the auxiliary expressions $q^k q^l (\hat{D}_k \hat{N}_l)$ and $q^k q^l (\hat{D}_k \hat{N}_l)$. To do so notice first that

$$\hat{D}_k \hat{N}_l = \mathbb{D}_k \hat{N}_l - C^f_{kl} \hat{N}_l$$  \hspace{1cm} (3.40)

from which one gets

$$q^k q^l (\hat{D}_k \hat{N}_l) = q^k q^l (\mathbb{D}_k \hat{N}_l) - q^k q^l C^f_{kl} [\frac{1}{2} (q_f \overline{\varphi} + \overline{q}_f q^e)] \hat{N}_e$$

$$= \overline{\mathbb{D}} \mathbb{N} - \frac{1}{2} \overline{C} \mathbb{N} - \frac{1}{2} A \mathbb{N} ,$$  \hspace{1cm} (3.41)

$$q^k q^l (\hat{D}_k \hat{N}_l) = q^k q^l (\mathbb{D}_k \hat{N}_l) - q^k q^l C^f_{kl} [\frac{1}{2} (q_f \overline{\varphi} + \overline{q}_f q^e)] \hat{N}_e$$

$$= \overline{\mathbb{D}} \mathbb{N} - \frac{1}{2} \overline{B} \mathbb{N} - \frac{1}{2} \overline{F} \mathbb{N} .$$  \hspace{1cm} (3.42)

3.9 Terms involving $\hat{K}_{ij}$

Before determining $q^l [\hat{\gamma}^i j k, \hat{K}_{jl}]$ we need also to evaluate the extrinsic curvature $\hat{K}_{ij}$ of $\mathcal{I}_p$ as given by (3.4),

$$\hat{K}_{ij} = \frac{1}{2} \mathcal{L}_a \hat{\gamma}_{ij} = \frac{1}{2} \hat{N}^{-1} [\mathcal{L}_a \hat{\gamma}_{ij} - (\hat{D}_i \hat{N}_j + \hat{D}_j \hat{N}_i)]$$

$$= \frac{1}{2} \hat{N}^{-1} [((\partial_p a) q_{ij} + \frac{1}{2} [(\partial_p b) \overline{q} q_j + (\partial_p \overline{b}) q_i q_j] - (\hat{D}_i \hat{N}_j + \hat{D}_j \hat{N}_i)] ,$$  \hspace{1cm} (3.43)

where in the last step (3.41) was applied. As a result,

$$\hat{K} = \hat{K}^l_{ij} = \hat{\gamma}^i j \hat{K}_{ij} = \mathbb{d}^{-1} \{ a q^j - \frac{1}{2} [b \overline{q} \overline{q} + \overline{b} q q^j] \} \hat{K}_{ij} = \frac{1}{2} (\hat{N} \mathbb{d})^{-1} \times$$

$$\times [a ((\partial_p a) - q^j q^i) (\hat{D}_i \hat{N}_j + \hat{D}_j \hat{N}_i)] - b ((\partial_p \overline{b}) - \overline{q} \overline{q} (\hat{D}_i \hat{N}_j)) + "CC"]$$

$$= \frac{1}{2} (\hat{N} \mathbb{d})^{-1} \{ a ((\partial_p a) - (\overline{\mathbb{D}} \mathbb{N} + \overline{B} \mathbb{N})$$

$$- b [(\partial_p \overline{b}) - (\overline{\mathbb{D}} \mathbb{N} + \frac{1}{2} \overline{C} \mathbb{N} + \frac{1}{2} \overline{A} \mathbb{N})] + "CC" .$$  \hspace{1cm} (3.44)

Set now

$$\hat{K} = q^i q^j \hat{K}_{ij} = \frac{1}{2} \hat{N}^{-1} \{ 2 \partial_p a - 2 \overline{\mathbb{D}} \mathbb{N} + \overline{C} \mathbb{N} + \overline{A} \mathbb{N} \} ,$$  \hspace{1cm} (3.45)

$$\hat{K} = q^i q^j \hat{K}_{ij} = \frac{1}{2} \hat{N}^{-1} \{ 2 \partial_p a - \overline{\mathbb{D}} \mathbb{N} - \overline{\mathbb{D}} \mathbb{N} + \overline{B} \mathbb{N} + \overline{F} \mathbb{N} \} .$$  \hspace{1cm} (3.46)

Then, because the symmetric 2-tensor $\hat{K}^l_{ij}$ is determined by three real functions, it follows that $\hat{K}$, $\hat{K}$ and $\hat{K}$ are functionally dependent. In determining their algebraic relation it is advantageous to introduce the auxiliary variables

$$\star \hat{K} = q^i q^j [\hat{K}_{ij} - \frac{1}{2} \hat{\gamma}_{ij} \hat{K}^l_{ij}] = \hat{K} - a \hat{K}$$  \hspace{1cm} (3.47)

$$\star \hat{K} = q^i q^j [\hat{K}_{ij} - \frac{1}{2} \hat{\gamma}_{ij} \hat{K}^l_{ij}] = \hat{K} - b \hat{K} .$$  \hspace{1cm} (3.48)
The analog of the trace relation (3.30) then gives

\[ \overset{\star}{\mathbf{K}} = (2 \ a)^{-1} [ \overset{\star}{\mathbf{b}} \overset{\star}{\mathbf{K}} + \overset{\star}{\mathbf{b}} \overset{\star}{\mathbf{K}} \mathbf{]} , \quad (3.49) \]

from which it follows, in virtue of (3.47) and (3.48),

\[ \mathbf{K} = a^{-1} \{ \mathbf{a} \cdot \mathbf{K} + \frac{1}{2} [ \mathbf{b} \mathbf{K} + \mathbf{b} \mathbf{K} \mathbf{]} \} . \quad (3.50) \]

Then, by making use of all the above \( \mathbf{K}^{ij} \) related variables, we obtain

\[ q_i q_j \hat{\mathbf{K}}^{ij} = (2^2 \mathbf{C} \mathbf{P}^{-1} - 1) \{ \mathbf{a} \cdot \mathbf{K} + \mathbf{b} \mathbf{K} \mathbf{]} + (2^2 \mathbf{C} \mathbf{Q}) \hat{\mathbf{K}}^{ij} \]

and

\[ q_i \overline{q}_j \hat{\mathbf{K}}^{ij} = (2^2 \mathbf{C} \mathbf{P}^{-1} - 1) \{ (\mathbf{a} + \mathbf{b}) \mathbf{K} \mathbf{]} - \mathbf{a} \mathbf{b} \mathbf{K} \mathbf{]} \]. \quad (3.52)

These relations, along with (3.29), imply

\[ q_i q_j \hat{\mathbf{K}}^{ij} = (2^2 \mathbf{C} \mathbf{P}^{-1} - 1) \{ \mathbf{a} \cdot \mathbf{K} + \mathbf{b} \mathbf{K} \mathbf{]} + (2^2 \mathbf{C} \mathbf{Q}) \hat{\mathbf{K}}^{ij} \]

and

\[ q_i \overline{q}_j \hat{\mathbf{K}}^{ij} = (2^2 \mathbf{C} \mathbf{P}^{-1} - 1) \{ (\mathbf{a} + \mathbf{b}) \mathbf{K} \mathbf{]} - \mathbf{a} \mathbf{b} \mathbf{K} \mathbf{]} \]. \quad (3.52)

Finally, by making use of all the above \( \hat{\mathbf{K}}^{ij} \) related variables, we obtain

\[ \hat{\mathbf{K}}_{ij} \hat{\mathbf{K}}^{ij} = \frac{1}{4} \mathbf{q} \mathbf{q} \left[ 2 \mathbf{K} \left( [ \mathbf{a} \cdot \mathbf{K} + \mathbf{b} \mathbf{K} ] \right) - a \left( \mathbf{b} \mathbf{K} + \mathbf{b} \mathbf{K} \mathbf{]} \right) \right] + \left( \mathbf{K} \left[ \mathbf{a} \cdot \mathbf{K} + \mathbf{b} \mathbf{K} \mathbf{]} - 2 \mathbf{a} \mathbf{b} \mathbf{K} \mathbf{] \} \right) + \frac{1}{2} \mathbf{d} \left( \mathbf{a} \cdot \mathbf{b} \mathbf{K} \mathbf{]} \mathbf{K} \mathbf{]} \right. \]

and

\[ \hat{\mathbf{K}}_{ij} \hat{\mathbf{K}}^{ij} = \frac{1}{4} \mathbf{q} \mathbf{q} \left( \mathbf{K} \left[ \mathbf{a} \cdot \mathbf{K} + \mathbf{b} \mathbf{K} \mathbf{]} - 4 \mathbf{a} \mathbf{b} \mathbf{K} \mathbf{] \} \right) + \frac{1}{2} \mathbf{d} \left( \mathbf{a} \cdot \mathbf{b} \mathbf{K} \mathbf{]} \mathbf{K} \mathbf{]} \right. \]

Finally, the analogue of (3.25) is

\[ \mathfrak{L}_n (\hat{\mathbf{K}}_{il}) = \mathfrak{L}_n \hat{\mathbf{K}} = \hat{\mathbf{N}}^{-1} \left[ (\mathbf{\partial}_l \hat{\mathbf{K}}) - \frac{1}{2} \hat{\mathbf{N}} (\overline{\mathbf{\partial}} \hat{\mathbf{K}}) - \frac{1}{2} \hat{\mathbf{N}} (\overline{\mathbf{\partial}} \hat{\mathbf{K}}) \right] . \quad (3.54) \]

4 The constraints in terms of spin-weighted variables

This section presents the explicit form of the constraints in terms of the spin-weighted fields introduced in Section 3. To provide a clear outline of the analytic setup, these spin-weighted fields are collected in Table 1.

By applying these fields and their relations, the constraint system comprised of (1.9)–(1.11) takes the form

\[ \mathbf{D}_n \hat{\mathbf{K}} = 0 \]

\[ \mathbf{D}_n \hat{\mathbf{K}} = 0 \]

\[ \mathbf{D}_n \hat{\mathbf{K}} = 0 \]

\[ \mathbf{D}_n \hat{\mathbf{K}} = 0 \]

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\[ \mathbf{D}_n \hat{\mathbf{K}} = 0 \]
| notation | definition | spin-weight |
|----------|------------|------------|
| $a$      | $\frac{1}{2} q^i \bar{q}^j \hat{\gamma}_{ij}$ | 0          |
| $b$      | $\frac{1}{2} q^i q^j \hat{\gamma}_{ij}$ | 2          |
| $c$      | $a^2 - b \bar{b}$ | 0          |
| $k$      | $q^i k_i$ | 1          |
| $A$      | $q^a q^b C_{ab} \bar{q}_c = d^{-1} \{ a [2 \bar{\partial} a - \bar{\partial} b] - \bar{b} \bar{\partial} b \} | 1 |
| $B$      | $\bar{q}^i q^b C_{ab} \bar{q}_c = d^{-1} \{ a \bar{\partial} b - \bar{b} \partial b \} | 1 |
| $C$      | $q^a q^b C_{ab} \bar{q}_c = d^{-1} \{ a \bar{\partial} b - \bar{b} [2 \bar{\partial} a - \bar{\partial} b] \} | 3 |
| $\hat{R}$ | $\hat{R} = b^{-1} \{ 3 C - \bar{\partial} B + \frac{1}{2} [C \bar{A} + A B - B^2 - E C] \} | 0 |
| $\hat{N}$ | $\hat{N}$ | 0          |
| $N$      | $q^i \hat{N}_i = q^i \bar{\gamma}_{ij} \hat{N}_j$ | 1          |
| $\bar{N}$ | $q_i \bar{N}_i = q_i \bar{\gamma}^{ij} \bar{N}_j = d^{-1} (a N - b \bar{N})$ | 1          |
| $K$      | $K^i_j = \gamma^{kl} K_{kl}$ | 0          |
| $\dot{K}$ | $q^k q^l \dot{K}_{kl}$ | 2          |
| $\ddot{K}$ | $q^k \bar{q}^l \ddot{K}_{kl} = (2 a)^{-1} [\bar{b} \dot{K} + \bar{b} \ddot{K}]$ | 0          |
| $\dot{\bar{K}}$ | $\dot{\bar{K}}^i_j = \bar{\gamma}^{ij} \dot{K}_{ij}$ | 0          |
| $\ddot{\bar{K}}$ | $q^k q^l \ddot{K}_{kl} = \frac{1}{2} \bar{N}^{-1} \{ 2 \partial_{ab} - 2 \bar{\partial} N + C \bar{N} + A N \} | 2 |
| $\dddot{\bar{K}}$ | $q^k \bar{q}^l \dddot{K}_{kl} = a^{-1} \{ d \cdot \ddot{K} + \frac{1}{2} [b \dddot{K} + b \dddot{K}] \} | 0 |

Table 1: The spin-weighted fields as they appear in various terms of (4.1)–(4.2).

where, in virtue of (1.12), $\kappa_0$ can be evaluated by means of (3.16), (3.18), (3.33), (3.44), (3.53) and (3.54).

In (4.1)–(4.2), the lower order forcing terms $F_K$ and $\mathfrak{f}_K$ are spin-weight 0 and 1 fields, respectively, on the level surfaces of the $\mathcal{S}_\rho$ foliation. They are both smooth undifferentiated functions of the constrained variables $\kappa$, $\bar{K}$, $k$; and they are also smooth functions of the freely specifiable variables $a$, $b$, $\bar{N}$, $\bar{K}$ and their various $\partial$, $\bar{\partial}$ and $\rho$.  


derivatives. The explicit form of the forcing terms is

\[ F_K = \frac{1}{4} \hat{N} d^{-1} \left\{ 2 a \mathbb{B} \mathbb{K} - b (\mathbb{C} \mathbb{K} + \mathbb{A} \mathbb{K}) + "CC" \right\} \]  

\[ \hat{f}_K = - \frac{1}{2} \left[ k \hat{\partial} \hat{N} + \hat{K} \hat{\partial} \hat{N} \right] + \frac{1}{2} \hat{N} (dK)^{-1} \left[ (a \mathbb{K} - b \mathbb{K}) (\mathbb{B} \mathbb{K} + \mathbb{B} \mathbb{K}) + (a \mathbb{K} - b \mathbb{K}) (C \mathbb{K} + A \mathbb{K}) \right] \]

\[ - [\kappa - \frac{1}{2} \mathbb{K}] \hat{\partial} \hat{N} + \hat{N} \left[ \frac{1}{2} K^{-1} \hat{\partial} \kappa_0 + \hat{K} \hat{\partial} + q' \hat{N} \hat{K}_l + q' \hat{D} \hat{K}_l \right] \]

where the terms with parenthetical sub-indices are obtained by referring to the designated equations.

5 Final remarks

A chief motivation for these rather heavy calculations is to yield evolution equations which can be integrated numerically in the radial \( \rho \)-direction as a coupled system of ordinary differential equations (ODEs), e.g. by applying the method of lines to a finite difference or a pseudo-spectral representation of the \( \hat{\partial} \) and \( \hat{D} \) operators, as described in [6, 14, 7]. Note, however, that other numerical methods can be applied to integrate (4.1)–(4.2). For instance, the following spectral method may be preferable in various circumstances.

This method is based upon the spectral expansion of the spin-weighted fields \( a, b, \hat{N}, \mathbb{N}; \kappa, \mathbb{K}, \mathbb{K} \) on the \( \mathcal{S}_\rho \) level surfaces, which can be expressed in the general pattern

\[ x = \sum_{l,m} x^{l,m}(\rho) \cdot s \mathbb{V}_{l,m}, \]  

(5.6)

where \( x \) has spin-weight \( s \) and \( s \mathbb{V}_{l,m} \) stands for the corresponding spin-weighted spherical harmonics. In particular, \( \mathbb{K} \) and \( \mathbb{K} \) have the decompositions

\[ \mathbb{K} = \sum_{l,m} K^{l,m}(\rho) \cdot s \mathbb{V}_{l,m} \quad \text{and} \quad \mathbb{K} = \sum_{l,m} k^{l,m}(\rho) \cdot 1 \mathbb{V}_{l,m}. \]  

(5.7)

Accordingly, after using (4.3) to substitute for \( \kappa \), (4.1)–(4.2) become a system of coupled ODEs for the expansion coefficients \( K^{l,m}(\rho) \) and \( k^{l,m}(\rho) \). This system can be then be implemented numerically and integrated, e.g. by means of a suitable adaptation of the numerical package GridRipper [4, 3] or by that of the method described in [2].
Given the resulting solution for $K$ and $\mathbb{K}$, their substitution back into (4.3) yields $\kappa$ on $\Sigma$ and thereby the full set of constrained variables. It then follows from Theorem 4.3 of [9] that the complete initial data $h_{ij}$ and $K_{ij}$ determined from the constrained variables and the freely specifiable part of the initial data is guaranteed to satisfy the constraints (1.1)-(1.2) in the “domain of dependence” of $\mathcal{S}_{\rho_0}$ in $\Sigma$.

Recall, in this procedure, that the initial data for $K$ and $\mathbb{K}$ must also be freely specified on the level surface $\rho = \rho_\circ$. The particular choice

$$K^l,m = \begin{cases} -\frac{8\sqrt{\pi} M}{\rho_\circ^2 \sqrt{1+2M\rho_\circ}}, & \text{if } l = m = 0; \\ 0, & \text{otherwise} \end{cases}, \quad k^l,m = 0 \quad \forall \ l, m \quad (5.8)$$

yields the Schwarzschild initial data, provided that all the freely specifiable functions take their associated Schwarzschild values, as specified in [13].

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Appendix

This appendix verifies the basic relations between the operators $\tilde{\partial}$, $\bar{\partial}$ and the covariant derivative operator $\nabla_a$ on $\mathbb{S}^2$.

To do so denote by $\nabla_a$ and $\partial_a$ the torsion free covariant derivative operators with respect to the metrics $q_{ab}$ and $\delta_{ab}$ in (2.9), respectively. They are related to the Christoffel symbols by

$$\Gamma^e_{ab} = \frac{1}{2} q^{ef} \left\{ \partial_a q_{fb} + \partial_b q_{af} - \partial_f q_{ab} \right\} = \Omega^{-1} \left\{ 2 \delta^e_{(a} \partial_{b)} \Omega - \delta_{ab} \delta^{ef} \partial_f \Omega \right\} . \quad (A.1)$$

Using (2.10) it is straightforward to verify that

$$q^a q^b \Gamma^e_{ab} = P \left\{ 2 P^2 \delta^e_z \left( \partial_z P^{-1} \right) \right\} = -2 q^e (\partial_z P) \quad (A.2)$$

$$\bar{\partial} q^b \Gamma^e_{ab} = P \left\{ q^e \left( \partial_z P^{-1} \right) + \bar{\partial} \left( \partial_z P^{-1} \right) \right\} - \left( \bar{\partial} q^b q_{ab} \right) q^{ef} \left( \partial_f P^{-1} \right) = 0 , \quad (A.3)$$

where in the last step of (A.3) we used the relation

$$\left( \bar{\partial} q^b q_{ab} \right) q^{ef} \left( \partial_f P^{-1} \right) = 2 \left[ \frac{1}{2} \left( q^e \bar{\partial}^f + q^f \bar{\partial}^e \right) \right] \left( \partial_f P^{-1} \right) = P \left[ q^e \left( \partial_z P^{-1} \right) \right] \left( \partial_f P^{-1} \right) + \bar{\partial} \left( \partial_z P^{-1} \right) \quad (A.4)$$

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along with
\[ q^a \partial_a q^b = P \delta^b_\pi (\partial_\pi P) = q^b (\partial_\pi P) \] (A.5)
and
\[ \overline{q}^a \partial_a q^b = P \delta^b_\pi (\partial_\pi P) = q^b (\partial_\pi P) \] , (A.6)
whereas in verifying (A.4) we used the normalization condition (2.7), along with the definition (2.4).

**Lemma A.1** As a consequence of (2.11)-(2.13)
\[ \overline{d} L = q^b q^{a_1} \ldots q^{a_s} \mathbb{D}_b L_{(a_1 \ldots a_s)} , \] (A.7)
\[ \overline{\delta} L = \overline{q}^b q^{a_1} \ldots q^{a_s} \mathbb{D}_b L_{(a_1 \ldots a_s)} . \] (A.8)

**Proof:** As
\[ \mathbb{D}_b L_{(a_1 \ldots a_s)} = \partial_b L_{(a_1 \ldots a_s)} - \sum_{i=1}^s \Gamma_{ba_i}^{c} L_{(a_1 \ldots \hat{e} \ldots a_s)} , \] (A.9)
we have
\[ q^b q^{a_1} \ldots q^{a_s} \mathbb{D}_b L_{(a_1 \ldots a_s)} = \{ q^b \partial_b L - \sum_{i=1}^s (q^b \partial_b q^{a_i}) q^{a_1} \ldots \hat{i} \ldots q^{a_s} L_{(a_1 \ldots a_s)} \} \\
- \sum_{i=1}^s (\Gamma_{ba_i}^{c} q^b q^{a_i}) L_{(a_1 \ldots \hat{e} \ldots a_s)} q^{a_1} \ldots \hat{i} \ldots q^{a_s} , \] (A.10)
where \( \hat{i} \) indicates that the omission of the \( i \)th dyad element.

This, in virtue of (2.4), (A.2) and (A.5), implies that
\[ q^b q^{a_1} \ldots q^{a_s} \mathbb{D}_b L_{(a_1 \ldots a_s)} = \{ P (\partial_\pi L) - s (\partial_\pi P) L \} + 2s (\partial_\pi P) L \\
= P (\partial_\pi L) + s (\partial_\pi P) L = P^{1-s} \partial_\pi (P^s L) = \overline{d} L . \] (A.11)

By replacing \( q^b \) by \( \overline{q}^b \), and \( \overline{\delta} \) by \( \overline{\delta} \) in the above argument, the application of (2.4), (A.3) and (A.6) yields the analogous relation
\[ \overline{q}^b q^{a_1} \ldots q^{a_s} \mathbb{D}_b L_{(a_1 \ldots a_s)} = P (\partial_\pi L) - s (\partial_\pi P) L = P^{1+s} \partial_\pi (P^{-s} L) = \overline{\delta} L , \] (A.12)
as intended to be shown. \( \square \)
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