Quantifying coherence with respect to general measurements via quantum Fisher information

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The standard quantum coherence theory is defined with respect to an orthonormal basis of a Hilbert space. Recently, Bischof, Kampermann and Bruß generalized the notion of coherence into the case of general measurements, and also, they established a rigorous resource theory of coherence with respect to general measurements. In this paper, we propose such a coherence measure with respect to general measurements via quantum Fisher information, and provide an application of this measure.

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I. INTRODUCTION

Coherence is a fundamental ingredient in quantum physics. Since Baumgratz, Cramer and Plenio [1] established a rigorous framework for quantifying coherence (we call this framework BCP framework), fruitful results about coherence have been achieved both in theories and experiments, for reviews see [2][3]. For a quantum system associating with a \( d \)-dimensional Hilbert space \( H \), the BCP framework considers the coherence defined with respect to an orthonormal basis \( \{|j\}_{j=1}^{d} \), we call such coherence standard coherence. An orthonormal basis \( \{|j\}_{j=1}^{d} \) corresponds to a rank-1 projective measurement \( \{|j\rangle\langle j|\}_{j=1}^{d} \), hence we may ask whether or not we can extend the standard coherence to the case of general measurements. A general measurement, or called a POVM (Positive Operator-Valued Measurement) [4], is described by a set of positive-definite operators \( E = \{E_j\}_{j=1}^n \) satisfying \( \sum_{j=1}^n E_j = I_d \) with \( I_d \) the identity operator on \( H \). Recently, Bischof, Kampermann and Bruß generalized the notion of coherence into the case of general measurements [5], and established a framework of quantifying coherence with respect to general measurements (we call this framework BKB framework) [6]. BKB framework includes BCP framework as a special case when the POVM \( E \) is a rank-1 projective measurement. We call the coherence under BKB framework as POVM coherence.

Many standard coherence measures have been found and some of them possess physical interpretations and applications [2][3]. So we expect that some of these standard coherence measures can be generalized to be the corresponding POVM coherence measures. In fact, several standard coherence measures have been properly generalized to be POVM coherence measures, such as the relative entropy of POVM coherence [1, 5, 7], robustness of POVM coherence [6, 8, 9], \( l_1 \) norm of POVM coherence [1, 6, 10], and the POVM coherence based on the Tsallis entropy [10–14]. Also, there is a physical interpretation for the relative entropy of POVM coherence [6]. Recently, the relationship between POVM coherence and entanglement was investigated [15].

In this work, we propose a POVM coherence measure via the quantum Fisher information (QFI) and provide an application of it. Quantum Fisher information is a core concept in quantum metrology [16], we hope the POVM coherence measure via the quantum Fisher information proposed in this work has potential applications in quantum metrology. This paper is organized as follows. In section II, we review the framework of block coherence and the BKB framework for POVM coherence. In section III, we propose the POVM coherence measure via QFI and prove that this is a valid POVM coherence measure. In section IV, we provide a physical application for the POVM coherence measure via QFI. Section V is a brief summary.

II. BLOCK COHERENCE AND POVM COHERENCE

In this section we review the standard coherence, block coherence and POVM coherence, they can be viewed as special cases of quantum resource theory [17]. We first review the BCP framework about standard coherence. BCP framework is defined with respect to a fixed orthonormal basis \( \{|j\}_{j=1}^{d} \) of Hilbert space \( H \). In BCP framework, a state \( \rho \) is defined as an incoherent state if

\[
\rho = \sum_{j=1}^{d} p_j |j\rangle \langle j|,
\]

where \( \{p_j\}_{j=1}^{d} \) is a probability distribution. Equivalently, an incoherent state is a diagonal state in the fixed orthonormal basis \( \{|j\}_{j=1}^{d} \), or we say, a state \( \rho \) is incoherent if and only if

\[
(j|\rho|k) = 0, j \neq k.
\]

In BCP framework, a channel [4] \( \phi \) is called incoherent if \( \phi \) allows a Kraus operator decomposition \( \phi = \{K_l\}_l \) such that \( K_l \rho K_l^\dagger \) is diagonal in the fixed orthonormal basis \( \{|j\}_{j=1}^{d} \) for any \( l \) and any incoherent state \( \rho \). We

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call such Kraus operator decomposition \( \phi = \{ K_i \}_i \) an incoherent decomposition.

Block coherence is a generalization of standard coherence to the case of projective measurements [6, 18]. For a projective measurement \( P = \{ P_j \}_j \), that is, \( P_j P_k = \delta_{jk} P_j \), \( \sum_j P_j = I_d \), a state \( \rho \) is called block incoherent (with respect to \( P \)) if and only if
\[
P_j \rho P_k = 0, \forall j \neq k. \tag{3}
\]
A channel \( \phi \) is called block incoherent if \( \phi \) allows a Kraus operator decomposition \( \phi = \{ K_i \}_i \) such that for any block incoherent state \( \rho \),
\[
P_j K_i \rho K_j^\dagger P_k = 0, \forall l, \forall j \neq k. \tag{4}
\]
We call such decomposition \( \phi = \{ K_i \}_i \) an block incoherent decomposition.

In Ref. [6], the authors established a rigorous framework for quantifying the block coherence that a real-valued functional \( C(\rho, P) \) with respect to \( P \) is a block coherence measure if it satisfies (B1) to (B4) below.

(B1). Nonnegativity: \( C(\rho, P) \geq 0 \), and \( C(\rho, P) = 0 \) if and only if \( \rho \) is block incoherent.

(B2). Monotonicity: \( C(\phi(\rho), P) \leq C(\rho, P) \) for any state \( \rho \) and any block incoherent channel \( \phi \).

(B3). Strong monotonicity: \( \sum_i \text{tr}(K_i \rho K_i^\dagger)C(\frac{K_i \rho K_i^\dagger}{\text{tr}(K_i \rho K_i^\dagger)}) P_i \leq C(\rho, P) \) for any state \( \rho \) and any block incoherent channel \( \phi \) with \( \phi = \{ K_i \}_i \) a block incoherent decomposition.

(B4). Convexity: \( C(\sum_i p_i \rho_i, P) \leq \sum_i p_i C(\rho_i, P) \) for any states \( \{ \rho_i \}_i \) and any probability distribution \( \{ p_i \}_i \).

One can check that (B3) and (B4) together imply (B2), and also (B1) to (B4) returns to the case of standard coherence when \( P = \{ P_j \}_j \) is rank-1 [1], i.e., rank\( P_j = 1 \) for all \( j \).

In Ref. [10], the authors proposed (B5) below.

(B5). Block additivity:
\[
C(\rho_1 \oplus (1 - p) \rho_2, P) = p C(\rho_1, P) + (1 - p) C(\rho_2, P), \tag{5}
\]
for \( p \in [0, 1] \), states \( \rho_1 \) and \( \rho_2 \) satisfy that there exists a partition \( P = \{ P_{l_1} \}_i \cup \{ P_{l_2} \}_j \) such that
\[
\{ P_{l_1} \}_i \cap \{ P_{l_2} \}_j = \emptyset, \rho_1 P_{l_2} = \rho_2 P_{l_1} = 0 \text{ for any } l_1 \text{ and } l_2.
\]
It has been proved that (B2)+(B5) is equivalent to (B3)+(B4) [10] hence (B5) provide an alternative way for verifying a block coherence measure. For many cases verifying (B2)+(B5) is easier than verifying (B3)+(B4) [10].

We can check that the definitions of block incoherent state and block incoherent channel, and also the conditions (B1) to (B4) all return to the corresponding ones of standard coherence [1, 19].

Now we turn to review the BKB framework of POVM coherence. For a given POVM \( E = \{ E_j \}_j \), a state \( \rho \) is specified as a POVM incoherent state (with respect to POVM \( E \)) if [5, 6]
\[
E_j \rho E_k = 0, j \neq k. \tag{6}
\]
The definition of POVM incoherent channel is related to the canonical Naimark extension. For the POVM \( E = \{ E_j \}_j \) on the \( d \)-dimensional Hilbert space \( H \), introduce an \( n \)-dimensional Hilbert space \( H_R \) with \( \{ |j\rangle \}_j \) an orthonormal basis of \( H_R \). Write \( E = \{ E_j = A_j^\dagger A_j \}_j \). Notice that for any unitaries \( \{ U_j \}_j \) acting on \( \{ A_j \}_j \) respectively, it holds that \( E_j = (U_j A_j)^\dagger(U_j A_j) \). Since POVM coherence is defined with respect to \( E = \{ E_j \}_j \), then when we express POVM coherence in terms of \( \{ A_j \}_j \), the POVM coherence should be invariant under the unitary transformations \( \{ A_j \}_j \rightarrow \{ U_j A_j \}_j \). We define the unitary operator \( V \) on \( H_{\epsilon} = H \otimes H_R \)
\[
V = \sum_{j=1}^n V_{\epsilon j} \otimes |j\rangle\langle k|, \tag{7}
\]
\[
V_{j1} = A_j, \forall j. \tag{8}
\]
For the projective measurement \( P = \{ P_j \}_j \) as
\[
\tilde{P}_j = I_d \otimes |j\rangle\langle j|, \tag{9}
\]
define the projective measurement \( \tilde{P} = \{ \tilde{P}_j \}_j \) as
\[
\tilde{P}_j = V^\dagger \tilde{P}_j V. \tag{10}
\]
The projective measurement \( \tilde{P} = \{ \tilde{P}_j \}_j \) is a canonical Naimark extension of the POVM \( E = \{ E_j \}_j \) which satisfies \( \text{tr}(E_j \rho) = \text{tr}(\tilde{P}_j \rho \otimes |1\rangle\langle 1|) \) [5, 6]. A channel \( \phi \) is defined as POVM incoherent if \( \phi \) allows a Kraus operator decomposition \( \phi = \{ K_i \}_i \) and there exists a block incoherent channel \( \phi' \) with its Block incoherent decomposition \( \phi' = \{ K_i \}_i \) with respect to the projective measurement \( P = \{ P_j \}_j \) on \( H_{\epsilon} = H \otimes H_R \) such that
\[
K_i \rho K_i^\dagger \otimes |1\rangle\langle 1| = K_i \rho \otimes |1\rangle\langle 1| K_i^\dagger, \forall \rho. \tag{11}
\]
We call such decomposition \( \phi = \{ K_i \}_i \) a POVM incoherent decomposition.

With the definitions of POVM incoherent state and POVM incoherent channel, the authors in Ref. [6] established the following conditions that any POVM coherence measure \( C(\rho, E) \) should satisfy.

(P1). Nonnegativity: \( C(\rho, E) \geq 0 \), and \( C(\rho, E) = 0 \) if and only if \( \rho \) is POVM incoherent.

(P2). Monotonicity: \( C(\phi(\rho), E) \leq C(\rho, E) \) for any state \( \rho \) and any POVM incoherent channel \( \phi \).

(P3). Strong monotonicity: \( \sum_i \text{tr}(K_i \rho K_i^\dagger)C(\frac{K_i \rho K_i^\dagger}{\text{tr}(K_i \rho K_i^\dagger)}) P_i \leq C(\rho, E) \) for any state \( \rho \) and any POVM incoherent channel \( \phi \) with \( \phi = \{ K_i \}_i \) a POVM incoherent decomposition.

(P4). Convexity: \( C(\sum_i p_i \rho_i, E) \leq \sum_i p_i C(\rho_i, E) \) for any states \( \{ \rho_i \}_i \) and any probability distribution \( \{ p_i \}_i \).

One can check that when the POVM \( E \) is a projective measurement, the definitions of POVM incoherent state and incoherent channel return to the definitions of block incoherent state and block incoherent channel, and also
is unitarily invariant. Together with Eqs. (7,8,9,10) we have
\[ F\left(\sum_{k,k'=1}^{n} A_{k} \rho A_{k'}^{\dagger} \otimes |k\rangle\langle k'|, \overline{\mathcal{P}}\right) = F\left(Vh_{\rho}U^\dagger, U\overline{\mathcal{P}}U^\dagger\right) = F\left(\rho, E_j\right), \]
where \( \rho = \sum_{i=1}^{d} \lambda_i |\varphi_i\rangle \langle \varphi_i| \) is the eigendecomposition of \( \rho\). Then we get \( C_{F}(\rho_{E}, \overline{\mathcal{P}}) = C_{F}(\rho, E) \).

Applying Lemma 1, we now prove that \( C_{F}(\rho_{E}, \overline{\mathcal{P}}) \) is a valid block coherence measure. We see, (F2) implies that \( C_{F}(\rho_{E}, \overline{\mathcal{P}}) \) satisfies (B2), (F3) implies that \( C_{F}(\rho_{E}, \overline{\mathcal{P}}) \) satisfies (B4). To prove \( C_{F}(\rho_{E}, \overline{\mathcal{P}}) \) satisfies (B5), suppose \( \rho_{E} = p\rho_{E1} + \rho_{E2} \) as described in (B5). Let \( \rho_{E1} = \sum_{i=1}^{d} \lambda_{1,i} |\varphi_{1,i}\rangle \langle \varphi_{1,i}| \) be the eigendecomposition of \( \rho_{E1} \), \( \rho_{E2} = \sum_{i=1}^{d} \lambda_{2,i} |\varphi_{2,i}\rangle \langle \varphi_{2,i}| \) be the eigendecomposition of \( \rho_{E2} \). From Eq. (13) we have
\[ F(p\rho_{E1} \otimes \rho_{E2}, \overline{\mathcal{P}}) = p_{1} F(\rho_{E1}, \overline{\mathcal{P}}) + p_{2} F(\rho_{E2}, \overline{\mathcal{P}}), \]

Summing over \( j \), we get that \( C_{F}(\rho_{E}, \overline{\mathcal{P}}) \) satisfies (B5).

We then only need to prove that \( C_{F}(\rho_{E}, \overline{\mathcal{P}}) \) satisfies (B1). To this end, we prove Lemma 2 below.

Lemma 2. For the projective measurement \( P = \{P_{j}\}_{j=1}^{n} \) on the \( d \)-dimensional Hilbert space \( H \),
\[ C_{F}(\rho, P) := \sum_{j=1}^{n} F(\rho, P_{j}) = 0 \quad \text{if and only if} \quad P_{j}\rho P_{k} = 0, \forall j \neq k. \]

Proof of Lemma 2. Suppose \( C_{F}(\rho, P) = 0 \). Since \( F(\rho, P_{j}) \geq 0 \) for any \( j \), then \( C_{F}(\rho, P) = 0 \) if and only if \( F(\rho, P_{j}) = 0 \) for any \( j \). Notice that \( P_{j}\rho P_{k} = 0 \) for any \( j \neq k \) is equivalent to say that \( \rho \) is of the form \( \rho = \sum_{i=1}^{d} p_{i} \rho_{i} \) with \( \{p_{i}\}_{i=1}^{d} \) a probability distribution and \( \rho_{j} = P_{j}\rho_{j} P_{j} \) for any \( j \).
Let $P_j = \sum_{a=1}^{d_j} |\phi_{ja}\rangle\langle\phi_{ja}|$ be an eigendecomposition of $P_j$, then $\{|\phi_{ja}\rangle\rangle_j$ constitute an orthonormal basis of $H$ and $\sum_{a=1}^{d_j} d_j = d$. We relabel the orthonormal basis $\{|\phi_{ja}\rangle\rangle_j$ as $\{|\phi_{ja}\rangle\rangle_j = \{|\phi_{11}\rangle, |\phi_{12}\rangle, \ldots, |\phi_{1d_1}\rangle, |\phi_{21}\rangle, |\phi_{22}\rangle, \ldots, |\phi_{2d_2}\rangle, \ldots\rangle\rangle_j = \{|1\rangle, |2\rangle, \ldots, |d_1\rangle, |d_1 + 1\rangle, |d_1 + 2\rangle, \ldots, |d_1 + d_2\rangle, \ldots\rangle\rangle_j = \{|m\rangle\rangle_{m=1}.$ Let

$$\rho = \sum_{l=1}^{d} \lambda_l |\varphi_l\rangle\langle\varphi_l|$$

(18)

be the eigendecomposition of $\rho$, with $\lambda_l \geq 0$ for any $l$ and $\sum_{l=1}^{d} \lambda_l = 1$. Without loss of generality, we always assume that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$. We express $\{|\varphi_l\rangle\rangle_l$ in $\{|m\rangle\rangle_{m=1}$ as

$$|\varphi_l\rangle = \sum_{m=1}^{d} \varphi_{lm} |m\rangle.$$ 

(19)

Introduce a $d \times d$ matrix

$$\Phi = (\varphi_{lm})_{1 \leq l \leq d, 1 \leq m \leq d}$$

$$= \begin{pmatrix}
\varphi_{11} & \varphi_{12} & \ldots & \varphi_{1d} \\
\varphi_{21} & \varphi_{22} & \ldots & \varphi_{2d} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_{d1} & \varphi_{d2} & \ldots & \varphi_{dd}
\end{pmatrix}$$

(20)

We set three steps to consider different situations of $\{|\lambda_j\rangle\rangle_j$.

1. Situation 1: $\lambda_1 > \lambda_2 > \ldots > \lambda_d$. In this situation, $\{|\varphi_l\rangle\rangle_l$ forms an orthonormal basis of $H$, $\Phi$ is a unitary matrix, from Eq. (13), $F(\rho, P_1) = 0$ reads

$$\sum_{a=1}^{d_1} \langle\varphi_l|\phi_{1a}\rangle\langle\phi_{1a}|\varphi_{l'}\rangle = 0, \forall l \neq l',$$

(21)

that is

$$\sum_{m=1}^{d_1} \varphi^*_{lm} \varphi_{l'm} = 0, \forall l \neq l'.$$ 

(22)

Consider the $d \times d_1$ matrix $(\varphi_{lm})_{1 \leq l \leq d, 1 \leq m \leq d_1}$, since $d_1$ column vectors $(\varphi_{lm})_{1 \leq l \leq d_1}$ are orthogonal to each other, then the column rank

$$\text{rank}((\varphi_{lm})_{1 \leq l \leq d_1})_{m=1}^{d_1} = d_1.$$ 

(23)

We know that for any matrix, the rank of column vectors equals the rank of row vectors, thus the row rank

$$\text{rank}((\varphi_{lm})_{1 \leq m \leq d_1})_{l=1}^{d_1} = d_1.$$ 

(24)

Eq. (22) says $d$ row vectors $(\varphi_{lm})_{1 \leq l \leq d_1}$ are orthogonal to each other, then there must be $d_1$ row vectors being nonzero (not all elements are zero), and other $d - d_1$ row vectors being zero (all elements are zero). Let $(\varphi_{im})_{1 \leq m \leq d_1}$ denotes the $d_1$ nonzero rows in $(\varphi_{lm})_{1 \leq m \leq d_1}$, that is, the set $\{|l_1\rangle\rangle = \{|l_1\rangle\rangle_1$, we write the number of $\{|l_1\rangle\rangle$ as $\{|l_1\rangle\rangle = \{|l_1\rangle\rangle_1$. Similarly, $F(\rho, P_j)$ yields

$$\{|l_1\rangle\rangle = \{|l_1\rangle\rangle_1.$$ 

(25)

Since $\Phi = (\varphi_{lm})_{1 \leq l \leq d_1, 1 \leq m \leq d}$ is a unitary matrix, then rank $\Phi = d$, here rank $\Phi$ is the rank of column vectors $(\varphi_{lm})_{1 \leq l \leq d_1}$ and also the row vectors $(\varphi_{lm})_{1 \leq m \leq d_1}$. This fact implies that

$$\{|l_1\rangle\rangle \subset \text{Range}(P_j),$$

(27)

then $\rho$ must be of the form $\rho = \sum_{j=1}^{n} \rho_j p_j$ with $\{|p_j\rangle\rangle_j$ a probability distribution and $\rho_j = P_j$, $P_j$ for any $j$.

2. Situation 2: $\lambda_1 = \lambda_2 = \ldots = \lambda_{D_1} > \lambda_{D_1+1} = \lambda_{D_1+2} = \ldots = \lambda_{D_1+D_2} > \lambda_{D_1+D_2+1} = \ldots = \lambda_{D_1+D_2+\ldots+D_N} \geq 0$, with $d = D_1 + D_2 + \ldots + D_N$. We also let $D_0 = 0$. For this situation, we rewrite Eq. (18) as

$$\rho = \lambda_1 \sum_{l=1}^{D_1} |\varphi_l\rangle\langle\varphi_l| + \lambda_{D_1+1} \sum_{l=1}^{D_1+D_2} |\varphi_l\rangle\langle\varphi_l| + \ldots;$$

(28)

and rewrite Eq. (20) as a block matrix

$$\Phi = (\varphi_{lm})_{1 \leq l \leq d_1, 1 \leq m \leq d}$$

$$= \begin{pmatrix}
\Phi_{11} & \Phi_{12} & \ldots & \Phi_{1n} \\
\Phi_{21} & \Phi_{22} & \ldots & \Phi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\Phi_{N1} & \Phi_{N2} & \ldots & \Phi_{Nn}
\end{pmatrix}$$

(29)

with the block elements for example

$$\Phi_{11} = (\varphi_{lm})_{1 \leq l \leq D_1, 1 \leq m \leq d_1}$$

$$= \begin{pmatrix}
\varphi_{11} & \varphi_{12} & \ldots & \varphi_{1d_1} \\
\varphi_{21} & \varphi_{22} & \ldots & \varphi_{2d_1} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_{d_1} & \varphi_{d_2} & \ldots & \varphi_{d_1d_1}
\end{pmatrix}.$$ 

(30)

From Eq. (13), we see that $F(\rho, P_1) = 0$ if and only if

$$\Phi_1 \Phi_1^\dagger = 0, \forall l \neq l'.$$ 

(31)

Notice that if $P_j = \sum_{a=1}^{d_j} |\phi_{ja}\rangle\langle\phi_{ja}|$ is an eigendecomposition of $P_j$, then so is

$$P_j = \sum_{a=1}^{d_j} U_j^* |\phi_{ja}\rangle\langle\phi_{ja}| U_j^\dagger$$

(32)
for any $d_1 \times d_1$ unitary $U_j$, where $U_j^*$ stands for the conjugate of $U_j$, $U_j^+$ stands for the transpose of $U_j$, and $U_j^t = U_j^*$.

Similarly, Eq. (18) can be rewritten as

$$\rho = \lambda_1 \sum_{l=1}^{D_1} W_1^t |\varphi_l\rangle\langle \varphi_l| W_1 + \lambda_{D_1+D_2} \sum_{l=1}^{D_1+D_2} W_2^t |\varphi_l\rangle\langle \varphi_l| W_2 + \ldots$$

(33)

for any $\{W_k\}_{k=1}^N$ with each $W_k$ a $D_k \times D_k$ unitary matrix. Under the definition of Eq. (20), Eqs. (32, 33) correspond to

$$\Phi' = (\oplus_{k=1}^N W_k) \Phi(\oplus_{j=1}^n U_j)$$

$$= \left( \begin{array}{ccc} W_{11} U_1 & W_{12} U_2 & \cdots & W_{1n} U_n \\ W_{21} U_1 & W_{22} U_2 & \cdots & W_{2n} U_n \\ \vdots & \vdots & \ddots & \vdots \\ W_{n1} U_1 & W_{n2} U_2 & \cdots & W_{nn} U_n \end{array} \right)$$

(34)

which is a unitary matrix, we denote it by $\Phi' = (\varphi'_{lm})_{1 \leq l \leq d: 1 \leq m \leq d} + (\varphi'_{kj})_{1 \leq k \leq N: 1 \leq j \leq n}$. Below we will use the notation $\Phi^{(i)} = (\varphi^{(i)}_{lm})_{1 \leq l \leq d: 1 \leq m \leq d} + (\varphi^{(i)}_{kj})_{1 \leq k \leq N: 1 \leq j \leq n}$ similarly defined. Notice that rank$\Phi^{(i)} = \text{rank}(W_k U_k)$. Suppose $\{\text{rank} \Phi^{(i)} = r_k\}_{k=1}^N$, since the row rank equals the column rank, then the total rank equals the dimensions. If rank$\Phi^{(i)} = r_1 > 1$, using the singular value decomposition, there exist unitary $\{W_1^{(i)}, U_1^{(i)}\}$ such that

$$W_1^{(i)} \Phi^{(i)} U_1^{(i)} = \left( \begin{array}{cccc} c_1 & 0 & \ldots & 0 \\ 0 & c_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & c_r \end{array} \right).$$

(35)

For such $\{W_1^{(i)}, U_1^{(i)}\}$, and other unitary matrices $\{W_k^{(i)}\}_{k=2}^N; \{U_j^{(i)}\}_{j=2}^n$ being identity matrices with corresponding dimensions, now consider the matrix $\Phi^{(i)} = (\oplus_{k=1}^N W_k^{(i)}) \Phi(\oplus_{j=1}^n U_j^{(i)}).$ From $\Phi^{(i)}(\Phi^{(i)})^* = 0, \forall i \neq i'$, we get $\{\varphi^{(i)}_{lm}\}_{1 \leq l \leq d: 1 \leq m \leq d}$. Since $\Phi^{(i)}$ is unitary, each row or each column has just one of them. In unitary $\Phi^{(i)}$, for any $c_j \in \{c_j\}_{j=1}^{d_1}$, the row of containing $c_j$ has just one nonzero element $c_j$. We denote the set of $d_1$ rows in $\Phi^{(i)}$ each just has one of $\{c_j\}_{j=1}^{d_1}$ as $\Psi_1 = \{\psi^{(i)}_{lm}\}_{1 \leq m \leq d: D_1+1 \leq l \leq D_1+d_1, k = 0, 1, \ldots, N-1\}$, clearly,

$$\Psi_1 \subset \text{Range}\{P_1\},$$

(41)

$$|\Psi_1| = d_1.$$

(42)

where $|\Psi_1|$ denote the number of vectors in $\Psi_1$. In unitary matrix $\Phi^{(i)}$, delete the rows and columns containing the elements $\{c_j\}_{j=1}^{d_1}$ above, the remained matrix is a $(d-d_1) \times (d-d_1)$ unitary matrix with all row vectors belonging to $\text{Range}\{I_d - P_1\}$. This remained matrix corresponds to the remained state

$$\rho_1 = \rho - \lambda_1 \sum_{l=1}^{r_1} W_1^t |\varphi_l\rangle\langle \varphi_l| W_1^* - \lambda_{D_1+1} \sum_{l=D_1+r_1}^{D_1+r_2} W_2^t |\varphi_l\rangle\langle \varphi_l| W_2^* - \ldots,$$

(43)

then we taken out the part of $\rho$ projecting into the subspace of $\text{Range}\{P_1\}$. Now repeat the procedure for $P_2, P_3, \ldots, P_n$, we will get that there exists an eigen decomposition $\rho = \sum_{l=1}^d \lambda_l |\varphi_l\rangle\langle \varphi_l|$ such that each $|\varphi_l\rangle$ just belongs to one $\text{Range}\{P_j\}$. 


Combine situations 1-2 above, we then end the proof of lemma 2. Having lemma 2, we immediately get that $C_F(\rho, \{j\})$ satisfies (B1), and then end the proof of Theorem 1.

As a special case of Theorem 1, when POVM $E$ is a rank-1 projective measurement, Theorem 1 becomes the Corollary 1 below.

**Corollary 1.** Suppose $\{|j\rangle\}_{j=1}^d$ is an orthonormal basis of $d$-dimensional Hilbert space, then

$$C_F(\rho, \{|j\rangle\}_{j=1}^d) = \sum_{j=1}^d F(\rho, |j\rangle\langle j|) \quad (44)$$

is a valid standard coherence measure under BCP framework with respect to $\{|j\rangle\}_{j=1}^d$.

Remark that in Ref. [21], the authors claimed that $C_F(\rho, \{|j\rangle\}_{j=1}^d)$ is a valid standard coherence measure (which coincides with Corollary 1 above, up to a factor 1/4), and proved this assertion by proving $C_F(\rho, \{|j\rangle\}_{j=1}^d) = C_F^C(\rho, \{|j\rangle\}_{j=1}^d)$ in Proposition 4 with $C_F^C(\rho, \{|j\rangle\}_{j=1}^d)$ defined in definition 3. In appendix A we will show that that proof is not justified, i.e., $C_F(\rho, \{|j\rangle\}_{j=1}^d) = C_F^C(\rho, \{|j\rangle\}_{j=1}^d)$ is not true in general.

**IV. AN APPLICATION OF POVM COHERENCE VIA QFI**

We provide an application of $C_F(\rho, E)$ in quantum metrology. QFI gives the ultimate precision bound on the estimation of parameters encoded in a quantum state. Assuming that we start from an initial state $\rho$ and $A$ is a Hermitian operator, $\rho$ evolves to $\rho(\theta)$ under the unitary dynamics $\rho(\theta) = U\rho U^\dagger$ with $U = \exp(-iA\theta)$. QFI $F(\rho, A)$ constrains the achievable precision in estimation of the parameter $\theta$ via the quantum Cramér-Rao bound as

$$(\Delta\theta)^2 \geq \frac{1}{NF(\rho, A)}, \quad (45)$$

where $(\Delta\theta)^2$ is the variance of $\theta$, $N$ is the number of independent repetitions.

Let $U = \exp(-iE_j\theta_j)$, Eqs. (45,14) yield

$$(\Delta\theta_j)^2 \geq \frac{1}{NF(\rho, E_j)}, \quad (46)$$

$$\frac{1}{(\Delta\theta_j)^2} \leq NF(\rho, E_j), \quad (47)$$

$$\sum_{j=1}^n \frac{1}{(\Delta\theta_j)^2} \leq NC_F(\rho, E). \quad (48)$$

Eq. (48) sets an uncertainty relation on $\{((\Delta\theta_j)^2)_{j=1}^n\}$ by the upper bound $NC_F(\rho, E)$.

Remark that when $E$ is a rank-1 projective measurement, Eq. (48) recovers the corresponding results in [21].

**V. SUMMARY**

We proposed a POVM coherence measure via QFI under the BKB framework and provide an application in estimation theory. QFI is a core concept in quantum metrology, we hope that this POVM coherence measure will provide new insights in quantum metrology and in some quantum information processings.

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**Appendix: Disproof of $C_F(\rho, \{|j\rangle\}_{j=1}^d) = C_F^C(\rho, \{|j\rangle\}_{j=1}^d)$**

In this section, we disprove the result $C_F(\rho, \{|j\rangle\}_{j=1}^d) = C_F^C(\rho, \{|j\rangle\}_{j=1}^d)$ claimed in Proposition 4 of Ref. [21].

Suppose $\{|j\rangle\}_{j=1}^d$ is an orthonormal basis of a $d$-dimensional Hilbert space $H$. For state $\rho$, $C_F(\rho, \{|j\rangle\}_{j=1}^d)$ and $C_F^C(\rho, \{|j\rangle\}_{j=1}^d)$ are defined as [21] (in [21] the authors defined $C_F(\rho, \{|j\rangle\}_{j=1}^d) = \frac{1}{4}C_F(\rho, \{|j\rangle\}_{j=1}^d)$, we omit the factor $\frac{1}{4}$)

$$C_F(\rho, \{|j\rangle\}_{j=1}^d) = \sum_{j=1}^d F(\rho, |j\rangle\langle j|), \quad (A1)$$

$$C_F^C(\rho, \{|j\rangle\}_{j=1}^d) = \min_{\{p_k, \{|\psi_k\rangle\}_k\}} \sum_k p_k C_F(|\psi_k\rangle, \{|j\rangle\}_{j=1}^d), \quad (A2)$$

where min runs over all pure state ensembles $\{p_k, \{|\psi_k\rangle\}_k\}$ of $\rho$, i.e., $\rho = \sum_k p_k |\psi_k\rangle\langle \psi_k|$. With $\{p_k, \{|\psi_k\rangle\}_k\}$ a probability distribution and $\{|\psi_k\rangle\}_k$ normalized pure states. Since $F(\rho, A)$ is convex then

$$F(\rho, |j\rangle) \leq \sum_k p_k F(|\psi_k\rangle, |j\rangle), \forall j, \quad (A3)$$

$$C_F(\rho, \{|j\rangle\}_{j=1}^d) \leq C_F^C(\rho, \{|j\rangle\}_{j=1}^d), \quad (A4)$$

and the question remained is whether the equality can be achieved.

We would like to consider the more general case of $C_F(\rho, E)$ and $C_F^C(\rho, E)$ with respect to the POVM $E = \{E_j\}_{j=1}^n$ as

$$C_F(\rho, E) = \sum_{j=1}^n F(\rho, E_j), \quad (A5)$$

$$C_F^C(\rho, E) = \min_{\{p_k, \{|\psi_k\rangle\}_k\}} \sum_k p_k C_F(|\psi_k\rangle, E). \quad (A6)$$
Since $F(\rho, A)$ is convex then

$$F(\rho, E_j) \leq \sum_k p_k F(|\psi_k\rangle, E_j), \quad \forall j, \quad (A7)$$

$$C_F(\rho, E) \leq C_F^C(\rho, E), \quad (A8)$$

and $C_F(\rho, E) = C_F^C(\rho, E)$ if and only if there exists a pure state ensemble $\{p_k, |\psi_k\rangle\}_k$ such that $F(\rho, E_j) = \sum_k p_k F(|\psi_k\rangle, E_j)$ for any $j$. When $\rho$ is a pure state $|\psi\rangle$, $C_F(|\psi\rangle, E) = C_F^C(|\psi\rangle, E)$ evidently holds, then in the following we mainly discuss the case of mixed state $\rho$.

We now discuss which pure state ensemble $\{p_k, |\psi_k\rangle\}$ can achieve the equality above.

Any pure state ensemble $\{p_k, |\psi_k\rangle\}_{k=1}^{d'}$ can be generated by the eigendecomposition $\rho = \sum_{l=1}^d \lambda_l |\varphi_l\rangle \langle \varphi_l|$ and a $d' \times d'$ unitary matrix $U = (U_{jk})_{jk}$ as $[4]$

$$\sqrt{p_k} |\psi_k\rangle = \sum_{l=1}^d U_{kl} \sqrt{\lambda_l} |\varphi_l\rangle. \quad (A15)$$

We can always suppose $d' \geq d$ since $p_k = 0$ is permitted. We can write $\rho = \sum_{l=1}^{d'} \lambda_l |\varphi_l\rangle \langle \varphi_l|$ by adding

$$\lambda_{d+1} = \lambda_{d+2} = \ldots = \lambda_{d'} = 0,$$

and $\{|\varphi_l\rangle\}_{l=1}^{d'} \cup \{|\varphi_l\rangle\}_{l=d+1}^{d'}$ an orthonormal basis of a $d'$-dimensional Hilbert space $H'$. Consequently,

$$p_k = \sum_{l=1}^d \lambda_l |U_{kl}|^2, \quad (A16)$$

$$\langle \psi_k | A | \psi_k \rangle = \frac{1}{p_k} \sum_{l,l'=1}^d U_{kl} U_{kl'}^* \langle \varphi_l | A | \varphi_{l'} \rangle \quad (A17)$$

with

$$\Gamma_k = \frac{d'}{2} \sum_{l,l'=1} \frac{\lambda_l + \lambda_{l'}}{\sqrt{\lambda_l} \sqrt{\lambda_{l'}}} U_{kl} U_{kl'}^* |\varphi_l\rangle \langle \varphi_{l'}| \quad (A18)$$

As a result

$$\sum_{k=1}^{d'} p_k \langle \psi_k | A | \psi_k \rangle^2 \leq \sum_k \left( \text{tr} (Z_A \Gamma_k) \right)^2 / \sqrt{p_k} \quad (A19)$$

and Eq. (A14) becomes

$$\sum_{k=1}^{d'} \left( \text{tr} (Z_A \Gamma_k) \right)^2 \leq \text{tr} (Z_A^2). \quad (A20)$$

We know that all Hermitian matrices on $d'$-dimensional Hilbert space $H'$ form a $d'^2$-dimensional real Hilbert space $H'_d$ with the inner product $\langle A | A' \rangle = \text{tr}(AA')$ for any Hermitian matrices $A, A'$ on $H'_d$. We can check that

$$\text{tr} \left( \frac{\Gamma_{l_1}}{\sqrt{p_{l_1}}} \frac{\Gamma_{l_2}}{\sqrt{p_{l_2}}} \right) = \sum_{l_1,l_2=1}^{d'} \frac{\lambda_{l_1} + \lambda_{l_2}}{2} U_{l_1 j_1}^* U_{j_1 l_2} U_{l_2 k_1} U_{k_1 l_2}^* \delta_{j_1 k_1} \quad (A21)$$

this says $\{\Gamma_k\}_{k=1}^{d'}$ are normalized and orthogonal to each other in $H'_d$, but not a complete orthonormal basis for $H'_d$ since $H'_d$ is of dimension $d'^2$. Hence equality in Eq. (A20) holds if and only if there exist real number $\{r_k\}_{k=1}^{d'}$ such that

$$Z_A = \sum_{k=1}^{d'} r_k \Gamma_k. \quad (A22)$$

There is a subtlety to be aware of that when for example $p_1 = 0$ in Eq. (A16) then $p_1 |\psi_1\rangle \langle |\psi_1| \rangle = 0$ does not appear in the eigendecomposition $\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k|$ and Eqs. (A19,A20,A21,A22) all exclude $k = 1$.

Taking Eqs. (A11,A18) into Eq. (A22), we get that for $\{|l\rangle_{l=1}^{d'}$ and $\{|l'\rangle_{l=1}^{d'}$,

$$2 \sqrt{\frac{\lambda_l \lambda_{l'}}{\lambda_l + \lambda_{l'}} \langle \varphi_l | A | \varphi_{l'} \rangle} = \sum_{k=1}^{d'} r_k U_{kl} U_{kl'}^*. \quad (A23)$$
\[ UY_A U^\dagger = \sum_{k=1}^{d'} r_k |\varphi_k\rangle \langle \varphi_k|, \]  
(A24)

where

\[ Y_A = 2 \sum_{l,t'=1}^d \sqrt{\lambda_l \lambda_{t'}} \frac{|\varphi_l\rangle \langle \varphi_{t'}| + |\varphi_{t'}\rangle \langle \varphi_l|}{\lambda_l + \lambda_{t'}}. \]  
(A25)

We then conclude that, any \( d' \times d' \) (\( d' \geq d \)) unitary matrix \( U \) which diagonalizes \( Y_A \) yields a pure state ensemble which achieves the equality in Eq. (A13). Apply this result to Eqs. (A7,A8), we get that \( C_F(\rho, E) = C_{F}^C(\rho, E) \) if and only if they are commute, i.e., \( A \times A' = A' \times A \). Hence, \( C_F(\rho, E) = C_{F}^C(\rho, E) \) if and only if \( \{Y_{E_j}\}_{j=1}^n \) commute to each other.  

For \( d = 2 \) and \( \rho = \sum_{l=1}^2 \lambda_l |\varphi_l\rangle \langle \varphi_l| \) the eigendecomposition of \( \rho \), from Eq. (A25) we have

\[ Y_{1}\langle 1| = \left( \begin{array}{cc} \frac{|\langle \varphi_1| \langle \varphi_1|\rangle^2}{2\sqrt{\lambda_1 \lambda_2}} & 2\sqrt{\lambda_1 \lambda_2} |\varphi_1\rangle \langle \varphi_2| \\ 2\sqrt{\lambda_1 \lambda_2} |\varphi_2\rangle \langle \varphi_1| & \frac{|\langle \varphi_2| \langle \varphi_2|\rangle^2}{2\sqrt{\lambda_1 \lambda_2}} \end{array} \right), \]

\[ Y_{2}\langle 2| = \left( \begin{array}{cc} \frac{|\langle \varphi_2| \langle \varphi_2|\rangle^2}{2\sqrt{\lambda_1 \lambda_2}} & 2\sqrt{\lambda_1 \lambda_2} |\varphi_2\rangle \langle \varphi_1| \\ 2\sqrt{\lambda_1 \lambda_2} |\varphi_1\rangle \langle \varphi_2| & \frac{|\langle \varphi_1| \langle \varphi_1|\rangle^2}{2\sqrt{\lambda_1 \lambda_2}} \end{array} \right). \]

Notice that \( |\langle \varphi_1| \rangle^2 \rangle = |\langle \varphi_2| \rangle^2 \rangle \)  

\[ |\langle \varphi_1| \rangle^2 \rangle = -|\langle \varphi_2| \rangle^2 \rangle = |\langle \varphi_1| \rangle^2 \rangle = |\langle \varphi_2| \rangle^2 \rangle, \]

we can directly check that \( \{Y_{|j\rangle \langle j|}\}_{j=1}^d \) commute, then \( C_F(\rho, \{\langle j| \rangle^2 \}_{j=1}^d) = C_{F}^C(\rho, \{\langle j| \rangle^2 \}_{j=1}^d). \)

For \( d = 3 \), consider the state

\[ \rho = \frac{1}{2} |\varphi_1\rangle \langle \varphi_1| + \frac{1}{2} |\varphi_2\rangle \langle \varphi_2|, \]  
(A26)

\[ |\varphi_1\rangle = \frac{1}{\sqrt{3}} (|1\rangle + |2\rangle + |3\rangle), \]  
(A27)

\[ |\varphi_2\rangle = \frac{1}{\sqrt{3}} (|1\rangle + e^{i\frac{2\pi}{3}} |2\rangle + e^{-i\frac{2\pi}{3}} |3\rangle). \]  
(A28)

From Eq. (A25) we have

\[ Y_{1}\langle 1| = \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array} \right), \]

\[ Y_{2}\langle 2| = \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} e^{i\frac{2\pi}{3}} & 0 \\ \frac{1}{2} e^{-i\frac{2\pi}{3}} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array} \right), \]

\[ Y_{3}\langle 3| = \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} e^{i\frac{2\pi}{3}} & 0 \\ \frac{1}{2} e^{-i\frac{2\pi}{3}} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array} \right). \]

We can directly check that any two of \( \{Y_{|j\rangle \langle j|}\}_{j=1}^d \) do not commute, then \( C_F(\rho, \{\langle j| \rangle^2 \}_{j=1}^d) < C_{F}^C(\rho, \{\langle j| \rangle^2 \}_{j=1}^d). \) This example disproves the assertion \( C_F(\rho, \{\langle j| \rangle^2 \}_{j=1}^d) = C_{F}^C(\rho, \{\langle j| \rangle^2 \}_{j=1}^d). \)