Abstract

In this paper the whole geometrical set-up giving a conformally invariant holographic projection of a diffeomorphism invariant bulk theory is clarified. By studying the renormalization group flow along null geodesic congruences a holographic version of Zamolodchikov’s c-theorem is proven.
Introduction

Holography, as originally suggested by ’t Hooft \[1\] postulates that \textit{given a closed surface, we can represent all that happens inside it by degrees of freedom on this surface itself}. This postulate implies that quantum gravity is an special type of quantum field theory in which all physical degrees of freedom can be holographically projected onto the boundary. The fundamental aim of holography was to reconcile gravitational collapse and unitarity of quantum mechanics at the Planckian scale.

Taking string theory (or M-theory) as the most natural candidate for a quantum theory of gravity, holography suggests looking for a new description in terms of holographic degrees of freedom living at the boundary of space time. There are until now two independent attempts in this direction, namely the M(atrix) formulation of M theory and the Anti de Sitter/Conformal Field Theory (AdS/CFT) pairs based on Maldacena’s conjecture \[2\]. It is in this second case that a concrete form of holographic projection can be rigorously established. In the simplest AdS/CFT framework (\[3\]) the theories that are holographically related are Type IIB string theory on AdS and N=4 pure Super Yang Mills theory at the boundary.

The discovery of these (gravity/gauge theory) holographic pairs opens the door to a different use of holographic ideas, namely in the direction of finding the gravitational description of pure gauge dynamics and reciprocally. In this approach the first problem to be addressed is that of finding, given a gauge theory on a spacetime manifold $M$, a suitable holographic manifold $\tilde{M}$ with boundary $M$. Presented the holographic problem in this way the appearance of CFT’s at the boundary is not accidental but very much dependent on the underlying geometry. In fact, as pointed out many years ago by Penrose, the very definition of (conformal) spacetime boundary is done in general only up to conformal transformations. More precisely the bulk geometry only fixes a conformal class of geometries at the boundary, forcing us to use theories at the boundary invariant under conformal transformations.
Thus the first step in the construction of holographic pairs, consists in fixing a bulk geometry in terms of a given conformal class at the boundary. This mathematical problem has been worked out in reference [1], leading to the conclusion that bulk geometry is uniquely determined by conformal data at the boundary if the bulk space solves vacuum Einstein’s equations with non vanishing and negative cosmological constant i.e AdS space time. From the physical point of view a necessary condition to achieve uniqueness is to impose vanishing gravitational Bondi news through the boundary. This requirement implies in general the existence of confining gravitational fields preventing all massive degrees of freedom to reach the boundary. This is a very important dynamical ingredient in the construction of holographic pairs where some kind of Kaluza Klein dimensional reduction is involved. In fact decoupling of Kaluza Klein modes is very much dependent on the fact that these modes are locked by the confining gravitational field [5].

In all approaches to holography a physical question arises naturally, namely the meaning of the extra holographic space time dimension. Based on the original Maldacena’s description of near horizon geometry for D-branes, it seems natural to identify this extra holographic dimension as parametrizing somehow the renormalization group scale of the theory. From this perspective the Einstein equations governing the bulk geometry, are promoted to renormalization group equations; a situation that strongly resembles what take place in string theory. In order to get some progress in this direction we must certainly go a trifle beyond the AdS/CFT type of holographic pairs.

Our approach to the problem will be as follows. We will consider as holographic bulk manifold Einstein spaces which are asymptotically AdS and with no Bondi news at the conformal infinity.

Secondly we will use as holographic equations describing the variation of the metric in terms of the holographic coordinate- the transport of a \((d - 2)\)-dimensional surface along null geodesic congruences ending at the conformal infinity.

Finally we will interpret the area of these surfaces, in an appropriated normalization, as
the holographic definition of Zamolodchikov’s c-function with the affine parameter – along
the null congruence entering into the bulk– as renormalization group variable.

This definition of the c-function is based on the IR/UV interpretation of the AdS/CFT
holographic pairs [14], and coincides, in the conformal case, with the central extension. In
this set up we can relate the predicted (by the c-theorem) behaviour of the c-function to
very general gravitational properties of null congruences. More precisely the c-theorem
appears, in this context, as a quite direct consequence of Raychaudhuri’s equation. The
physical meaning of this result goes into the direction of translating general properties of
quantum field theories into gravitational language.

An optimistic interpretation of the results presented in this paper would indicate a deep
interplay between the irreversibility of the renormalization group flow and the famous
singularity theorems in general relativity [18], where now the singularity theorems are
worked out for the holographic bulk space.

Next let us summarize the outline of the paper. In section 1 we review the geometrical
set up of holography, stressing the two different approaches based on Lorentz or Poincaré
metrics. We also review some generalities of Penrose’s construction of conformal infinity
and discuss a possible extension of geometric holography to non local loop variables. In
section 2 we address the question of Bondi news in terms of vanishing Bach tensor at the
boundary. In sections 3 and 4 we discuss the holographic definition of Zamolodchikov c-
function and finally in section 5 we give a purely gravitational proof of the c-theorem. We
end the paper with some general comments and speculations.

1 Geometric Holography

The aim of this section is to explore the geometric setting of the holographic Anti de
Sitter/Conformal Field Theory (AdS/CFT) relationship ([13][2][3]). Heavy use will be
made for this purpose of the mathematical results contained in the paper of Fefferman and
In strict mathematical terms the problem associated with the holographic projection is that of finding the conformal invariants of a given manifold $M$, with generic signature $(p, q) \equiv ((-1)^p, 1^q)$ and dimension $n = p + q$, in terms of the Riemannian invariants of some other manifold $\tilde{M}$ in which $M$ is contained in some precise sense. In order to employ a more physical terminology, we will refer to $M$ as the space-time and to $\tilde{M}$ as the bulk or ambient space.

Following [4] we will work out this geometrical problem from two different points of view, depending on the dimension and signature of the bulk space. In the so-called Lorentzian approach the ambient space will have signature $(p+1, q+1)$ while in the second approach, based on Penrose’s definition of conformal infinity the bulk space will have either $(p, q+1)$ or $(p+1, q)$ signature, leading to two different kinematical types of geometric holography.

### 1.1 Lorentz Holography

Given a Riemannian manifold $M$, a conformal class $[g]$ on $M$ is defined as the equivalence class of metrics: $g_{\mu\nu} \sim g'_{\mu\nu}$ iff $g'_{\mu\nu} = \Omega^2 g_{\mu\nu}$, $\Omega$ being a smooth function on $M$.

Conformal invariant tensors $P(g)$ are defined by the transformation law:

$$P(\lambda g) = \lambda^{-\Delta} P(g)$$

where $\Delta$ is the conformal weight; they are thus associated with a given conformal class, $[g]$.

In order to describe a conformal class $[g]$ on $M$, let us introduce the space $G$ defined in terms of some reference metric $ds^2_{\text{ref}}$ as:

$$G \equiv \{(x, t^2 g^{(\text{ref})}_{\mu\nu}(x)); t > 0; x \in M\}$$

In fact any section of $G$ (i.e., any map from $M$ to $G$ specified by a function $t(x)$) defines a particular representative in the conformal class $[g]$. Conversely, any metric $g_{\mu\nu} \in [g]$ on $M$.

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2A ray sub-bundle $G$ of the bundle of symmetric 2-tensors on $M$. 

Graham [4].
M defines an imbedding of $M$ into $G$. Any conformal transformation on the metric can in this way be interpreted as moving from one section to another. To be specific, we define a dilatation as:

$$\delta_s(x, \bar{g}_{\mu\nu}) \rightarrow (x, s^2 \bar{g}_{\mu\nu}); (x, \bar{g}_{\mu\nu}) \in G \quad (1.3)$$

The big space $G$ is naturally endowed with a natural (degenerate) metric, here denoted by $g_0$, inherited from the one defined at $M$ \footnote{In mathematical terms, this is equivalent to: $g_0(\bar{x}, \bar{y}) = \bar{g}(\pi_* \bar{x}, \pi_* \bar{y})$ for $\bar{x}, \bar{y} \in T_{(x,\bar{g})}(G)$, tangent vectors to $G$ at the point $(x,\bar{g})$, and $\pi_*$ denoting the pull-back of the natural projection to the base, $\pi(x, \bar{g}) \equiv x \in M$.}. In terms of the local coordinates $(x^\mu, t)$

$$ds_0^2(x, t) \equiv t^2 ds_{(ref)}^2(x) \quad (1.4)$$

In this generalized set-up, conformal fields (and conformal weights) on $M$ will be defined by their transformation law with respect to $\delta_s$.

If we write $\delta_s \equiv e^{s\bar{T}}$ ($\bar{T}$ thereby denoting the corresponding infinitesimal generator), then it is plain that $\bar{T}$ is a null vector and, in fact, orthogonal to any other tangent vector, $\bar{X} \in T_G$

$$g_0(\bar{T}, \bar{T}) = g_0(\bar{T}, \bar{X}) = 0 \quad (1.5)$$

(because it acts trivially on the points of the base, $x \in M$).

In order to get a geometrical image of the space $G$, let us consider a simple example of the previous construction. Let $M$ be the one-dimensional sphere $S^1$ of unit radius, with the metric induced by the imbedding in $\mathbb{R}^3$, called $ds^2$. Conformal transformations, $ds^2 \rightarrow t^2 ds^2$ are now equivalent to changes in the radius of the 1-sphere. The big space $G$ defined above is thus in this case a cone $C$, as shown in the Figure \footnote{With each transversal section being identified with a copy of $S^1$ with a particular metric in the conformal class.} with each transversal section being identified with a copy of $S^1$ with a particular metric in the conformal class. The vector $\bar{T}$ generating dilatations goes along the generatrix of the cone $C$ connecting the same point in the one-sphere $S^1$ in different transversal sections.
Once we have this picture in mind, let us proceed to define the ambient space as the product of our big space $G$ with a closed segment:

$$\tilde{G} \equiv G \times [-1, 1]$$

and we formally identify our previous $G$ with a subset of the new construct\footnote{Which is the common boundary of the two complementary regions $\tilde{G}^+ \equiv G \times [0, 1]$ and $\tilde{G}^- \equiv G \times [-1, 0]$, which can be visualized in the figure as the interior and exterior regions to the cone $C$. Incidentally, this defines the natural inclusion $\iota : G \to \tilde{G}$.

with a subset of the new construct $\tilde{G}$:

$$G \equiv G \times \{0\}.$$ 

It remains to define a natural metric $d\tilde{s}^2$ in the bulk space $\tilde{G}$. We would like to define it in such a way that any Riemannian invariant $P(\tilde{g})$ on $\tilde{G}$, induce, when reduced to the
original manifold $M$, conformally invariant tensors on $M$. The way to reduce tensors in
the bulk to tensors in the manifold is simplicity itself: First of all, to reduce from $\tilde{G}$ to $G$,
we use the trivial imbedding, so that in a system of local coordinates $x^M \equiv (x^\mu, t, \rho)$, with
$\rho \in \tilde{G}/G \sim [-1, 1]$, we just put $\rho = 0$

$$\tilde{P}(\tilde{g})|_M \equiv \tilde{P}(\tilde{g}(\rho = 0))$$ (1.7)

Next, to go from $G$ to $M$, we use the imbedding given by the metric $g_{\mu\nu}$ itself

$$g_{\mu\nu}(x) \to t(x)$$ (1.8)

defining in this way a tensor in $M$, which we shall call $\tilde{P}$ Remakably enough, Fefferman
and Graham were able to prove that $\tilde{P}(g)$ will be conformally invariant provided that the
following three conditions are fulfilled:

i) $d\tilde{s}^2(\rho = 0) = ds_0^2$

ii) $d\tilde{s}^2(x, \lambda t, \rho) = \lambda^2 d\tilde{s}^2(x, t, \rho)$

iii) $R_{MN}(\tilde{g}) = 0$ (1.10)

The first condition simply means that $\tilde{g}$ reduces on $G$ to the metric $g_0$ defined by the
conformal class $[g]$. Actually, this first condition $i$ alone (plus the fact that $\tilde{T}^2 = 0$)
forces the signature of $\tilde{G}$ to be $(p + 1, q + 1)$ (i.e., of the two extra dimensions that $\tilde{G}$
gets over those of $M$, one must necessarily be spacelike, and the other one timelike ).

5 In mathematical language, it is just the pull-back under the inclusion: $\tilde{P}(g) \equiv g^* (\tilde{P}(\tilde{g})|_G$. That is:

$\tilde{P} \equiv P(\tilde{g}(x, t(x), \rho = 0))$

6 In (more precise) mathematical language, the three conditions above read:

i) $\iota^* \tilde{g} = g_0$

ii) $\delta^* \tilde{g} = s^2 \tilde{g}$

iii) $\text{Ricci}(\tilde{g}) = 0$ (1.9)
Let us come back to our simple example in order to visualize this result. In this picture the ambient space is defined in terms of the (projective) coordinates $\xi^0, \xi^1, \xi^2$ (so that the sphere $S^1 \equiv (x^1)^2 + (x^2)^2 = 1$ is mapped onto $\xi^2_0 - \xi^2_1 - \xi^2_2 = 0$ by means of $x^i \equiv \xi^i/\xi^0 (i = 1, 2)$). The dilatation generator on the subspace $G$ (i.e., on the cone $C$) is forced to be a null vector with respect to the metric $\tilde{g}$ (restricted to $G$). All this is obviously equivalent to identify the cone $C$ with the light cone of the three-dimensional ambient space with signature $(1, 2)$ (where the coordinate $\xi_0$ is a time). Generically, one goes from signature $(p, q)$ in $M$, to signature $(p + 1, q + 1)$ for the Lorentzian ambient space.

The second condition ii) above, in turn, is crucial in order to get conformal invariants from Riemannian ones. In fact, this condition tells how the metric $\tilde{g}$ transforms with respect to those changes of coordinates in $\tilde{G}$ that induce in $M$ conformal transformations of the metric. In particular, this implies that Riemannian tensors on $\tilde{G}$ are homogeneous in $s$ with respect to dilatations $\delta_s$. This implies that $\tilde{P}(g)$ will be a conformal tensor provided that the dependence of $\tilde{g}$ on $g_0$ is analytic (that is, if $\tilde{g}(g_0)$ admits a power series expansion).

Precisely in order to control this last condition condition iii) above has been introduced. In fact, the Ricci flat condition on $\tilde{G}$ can be thought of as defining a Cauchy problem on the space of metrics with initial condition given by the conformal class $[g]$. By introducing a formal parameter as above, $\rho \in [-1, 1]$, what we get by solving the Ricci-flatness condition is a family of metrics $g_{\mu\nu}(\bar{x}, \rho)$, with $\bar{x}$ denoting coordinates in $M$, and such that $g(\bar{x}, \rho = 0) \in [g]$. It is the variable $\rho$ the one that could be most properly interpreted as a holographic coordinate, and the preceding equations are then describing the holographic flow of the spacetime metric. Once we get a solution of the above problem, we have a systematic way of relating quantities defined on $M$ (the CFT side) and ambient quantities on $\tilde{G}$ (the supergravity side), but we shall refrain from entering into this until we have reviewed Penrose’s conformal infinity approach to holography and its precise relationship with the Lorentzian approach discussed in this section.
1.2 Penrose’s Conformal Infinity

Penrose’s construction of conformal infinity (cf. [13]) was originally designed as a general procedure to address the physics of the asymptotic structure of the space-time in General Relativity. The main idea used to bring infinity back to a finite distance is to Weyl rescale the physical metric, looking for a new manifold \( \tilde{M} \) with boundary, such that the interior of \( \tilde{M} \) coincides with our original spacetime \( M \), endowed with the metric \( \hat{g}_{\mu\nu} \) to be defined in a moment. In the physics literature it is customary to denote the part of \( \partial M \) corresponding to end-points of null geodesics by \( J \).

Based on the previous construction we will say that a spacetime \((M, g_{\mu\nu})\) is asymptotically simple (actually we will be a little bit more general, and allow for asymptotically Einstein behavior) if there exists a smooth manifold \( \tilde{M} \), with metric \( \hat{g}_{\mu\nu} \), and a smooth scalar field \( \Omega(x) \) defined in \( M \) such that:

i) \( M \) is the interior of \( \tilde{M} \).

ii) \( \hat{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} \)

iii) \( \Omega(x) = 0 \) if \( x \in J \); but \( N_{\mu} \equiv -\nabla_{\mu}\Omega \) is nonsingular in \( J \).

iv) Every null geodesic in \( M \) has two endpoints in \( J \).

and, finally, the field equation:

v) \( R_{\mu\nu} \sim \lambda g_{\mu\nu} \) (cf. [15] [11] [18]).

(where we have followed Penrose’s notation \( a \sim b \) to indicate two things that are equal on \( J \) only).

For example, for \( AdS_4 \) [10] one can choose coordinates in such a way that

\[
\begin{align*}
ds^2 &= \sec^2 \rho (d\tau^2 - d\rho^2 - \sin^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2)) \\
&= d\tau^2 - d\rho^2 - \sin^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2)
\end{align*}
\]

where

\[
\begin{align*}
ds^2 &= d\tau^2 - d\rho^2 - \sin^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2)
\end{align*}
\]

is the metric of Einstein’s static universe (ESU). We see that in this example

\[
\Omega \equiv \cos \rho
\]
and the *infinite* $\mathcal{J}$ is located in these coordinates at the finite distance $\rho = \pi/2$.

One of the simplest ways of characterizing the behavior of the conformal factor $\Omega$ is to study null geodesics in its vicinity (cf. [15]). We shall choose an affine parameter $\hat{u}$ on them, (that is, $\hat{l}^\mu \hat{\nabla}_\mu \hat{u} \sim 1$; where $\hat{l}^\mu$ is the tangent (null) vector to the geodesic) and we further fix the origin of the affine parameter by $\hat{u} \sim 0$.

There is a corresponding parameter $u$ associated to the metric $g_{\mu \nu}$, defined by

$$\frac{d\hat{u}}{du} \equiv l^\mu \nabla_\mu \hat{u} \equiv \Omega^2 \hat{l}^\mu \hat{\nabla}_\mu \hat{u} = \Omega^2$$

(1.14)

The conformal factor will have, by analyticity, an expansion of the type

$$\Omega(\hat{u}) = -\sum_{n=1}^{\infty} A_n \hat{u}^n$$

(1.15)

If $\mathcal{M}$ is an Einstein space with scalar curvature given by

$$R = \frac{2n\lambda}{n - 2}$$

(1.16)

it is not difficult to show that the vector $N_\mu \equiv -\nabla_\mu \Omega$ obeys

$$\hat{N}^2 \sim \frac{2\lambda}{(n - 1)(n - 2)}$$

(1.17)

conveying the fact that the sign of the cosmological constant is related to the spacetime properties of the $\mathcal{J}$ boundary (and, in particular, in the ordinary $(1,3)$ case, we see that $\mathcal{J}$ will be timelike when the cosmological constant is negative only).

It is also possible to show that

$$\hat{\nabla}_\nu \hat{\nabla}_\mu \Omega = \frac{1}{n} \hat{\Delta} \Omega \hat{g}_{\mu \nu}$$

(1.18)

which in turn, (using the fact that $l^2 = 0$), enforce $A_2 = 0$ in the preceding expansion (2.13)([15]).

In order to make contact with the previous section, let us remark that we are here associating to a metric in $\mathcal{M}$, a whole conformal class in $\mathcal{J}$.(That is, all the construction above is invariant under $\Omega \to t \Omega$, with $t \in \mathbb{R}^+$).
Let us now relate Penrose’s conformal infinity construction with Lorentz holography as described in the previous subsection. In order to fix the ideas let us come back to the example of Figure 1. We have indeed two complementary pieces $\tilde{G}^\pm$ of $\tilde{G}$ with enjoy a common boundary, which we identify precisely with $G$. The region $\tilde{G}^+$ will be identified with the interior of the light cone. We can use now the holographic coordinate $\rho \in [-1, 1]$ to parametrize the interior cones $C_\rho$, as shown in the Figure 2. Let us denote by $\vec{T}_\rho$ the generator of dilatations at points located in the interior cone $C_\rho$. It is clearly a vector along the cone $C_\rho$. By identifying $C$ with the light cone in $\tilde{G}$, we get

$$\tilde{g}(\vec{T}_\rho, \vec{T}_\rho) < 0$$

(1.19)

(for $\rho \in (0, 1]$; that is, on $\tilde{G}^+$). This means that $\vec{T}_\rho$ is timelike in $\tilde{G}$. In order to make contact with Penrose’s construction let us mode out $\tilde{G}^+$ by the $\mathbb{R}^+$ action of the dilatations.
This can be simply done by imposing

$$\tilde{g}(\tilde{T}_\mu, \tilde{T}_\rho) = -1$$  \hspace{1cm} (1.20)

(again, when $\rho \in (0, 1]$).

In coordinates, it reads

$$\xi_0^2 - \xi_1^2 - \xi_2^2 = 1$$  \hspace{1cm} (1.21)

This defines a surface in $\tilde{G}^+$, which will be denoted by $S_M$. This is a manifold whose boundary in Penrose’s sense is identical to the manifold $M$. In the particular case $M \equiv S^1$, the manifold $S_M$ corresponds to the dashed region in the Figure 2, a region that brings to mind at once the AdS example.

The metric induced in $S_M$ by its imbedding in $\tilde{G}$ can be easily computed by a slight generalization of the usual horospheric coordinates. Actually, for arbitrary ± signs, denoted by $\epsilon_i = \pm 1$, the metric induced on the surface

$$\sum_{i=1}^{n} \epsilon_i x_i^2 = 1$$  \hspace{1cm} (1.22)

by the imbedding on the flat space with metric

$$ds^2 = \sum_{i=1}^{n} \epsilon_i dx_i^2$$  \hspace{1cm} (1.23)

can easily be reduced to a generalization of Poincaré’s metric for the half-plane by introducing the coordinates

$$z \equiv x^-$$

$$y^\mu \equiv z \, x^\mu$$  \hspace{1cm} (1.24)

where we have chosen the two last coordinates, $x^{n-1}$ and $x^n$ in such a way that their contribution to the metric is $dx_{n-1}^2 - dx_n^2$ (this is always possible if we have at least one timelike coordinate); and we define $x^- \equiv x^n - x^{n-1}$. $\mu \in (1, \ldots, n-2)$. The generalization of the Poincaré metric is:

$$ds^2 = \frac{\sum \epsilon_\mu dy_\mu^2 - dz^2}{z^2}$$  \hspace{1cm} (1.25)
It should be clear by the reasoning above that when $M$ is of signature $(p, q)$, $S_M$ enjoys signature $(p, q + 1)$. In local coordinates $(x, t, \rho)$, the metric $\tilde{g}$ on $\tilde{G}^+$ can be written as

$$ds^2 = t^2 ds^2(x, \rho) - dt^2$$

implying at once (whenever $n > 1$)

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + (n - 1)g_{\mu\nu}$$

This means that when $\tilde{G}^+$ is Ricci flat, $S_M \sim \tilde{G}^+/\mathbb{R}^+$ is an Einstein space with negative cosmological constant.

We could, mutatis mutandis repeat the previous construction for $\tilde{G}^-$. We would impose the condition

$$\tilde{g}(\vec{T}_\rho, \vec{T}_\rho) = 1$$

($\rho \in [-1, 0]$) The Figure 2 clearly shows that the main difference with the previous construction is that now we are forced to assume that $M \equiv S^1$ has signature $(1, 0)$ (instead of $(0, 1)$ as we had implicitly assumed before). Furthermore, the line element would contain $+dt^2$ (instead of $-dt^2$). All this implies that $S_M$ is now an Einstein space with positive cosmological constant.

In general this would mean that the holographic coordinate is in this case timelike, and therefore in order to get a boundary with the desired signature $(p, q)$ we should consider a bulk space of signature $(p + 1, q)$ (instead of $(p, q + 1)$).

The preceding remark is potentially interesting for a boundary physical spacetime of Minkowskian signature $(1, 3)$, since in this case we could try to perform a holographic projection with positive cosmological constant, but on a bulk spacetime with signature $(2, 3)$.

Let us now summarize the rules of the geometric holography.

- Given the manifold $M$ of signature $(p, q)$, we first build a Lorentzian ambient space $\tilde{G}$ of signature $(p + 1, q + 1)$
• In the ambient space $\tilde{G}$ look for a light cone defined relative to the extra time in $\tilde{G}$ and with transversal sections isomorphic to the manifold $M$ we started with. This light cone is naturally associated with a conformal class in $M$.

• The bulk space-time is now a manifold $S_M$ of signature $(p, q + 1)$ inside the light cone and with boundary a transversal section which should be identified with $M$. This bulk space is simply defined as the quotient of the space interior to the light cone by dilatations on the ambient spacetime $\tilde{G}$.

Actually, the previous set of holographic rules do allow in principle a further generalization of the holographic map to non-local variables.

### 1.3 Non-Local Holography: Wilson Loops

A first step in order to extend holography to non-local loop variables on $M$ would consist in associating to a closed loop $C$ located in $M$, a two-dimensional manifold, $S_C$, with boundary $C$ itself, and such that $S_C$ is contained in the region $S_M$ precisely defined in the previous paragraph.

A natural definition of $S_C$ would be the locus of points previously defined in eq. (1.20), but now reducing to vectors $\vec{T}_\rho$ associated with dilatations at points $(x, \rho)$ such that $x \in C$. It is plain that the manifold $S_C$ would be precisely of the type depicted in the Figure 2 in the simplest example $M \equiv S^1$. It is this holographic manifold $S_C$ that from a physical point of view, we would like to associated to a string world sheet.

Actually, the preceding construction already conveys the conformal invariance of the holographic representation of the Wilson loop. If we define the vacuum expectation value of the Wilson loop the area of the surface $S_C$ measured in the $\tilde{g}$ metric, it is clear that it will only depend on the conformal class of the metric on $M$ and therefore it would be, by construction, independent of conformal transformations of the loop $C$. (This is actually
well known for Wilson loops in the AdS/CFT case\cite{20}.

From this point of view, the fundamental zig-zag invariance \cite{21} (which physically means that the vacuum expectation value of the loop is not sensitive to foldings of the loop), that is, invariance under orientation-reversal diffeomorphisms of the loop itself (located at $M$), is perhaps related to some bigger invariance present already in the bulk. We hope to come back to this issue in the future.

2 Bondi News and the Holographic Map

The geometric holography just described strongly depends on the existence of a unique solution for a Cauchy problem defined in terms of Einstein’s equations on the bulk and with initial conditions fixed by the conformal class of the physical spacetime metric at the boundary. A necessary condition for the uniqueness of the solution is the vanishing of (gravitational) Bondi news through $J$. The physical meaning of this is that the spacetime boundary $J$ is \emph{opaque} to gravitational radiation; in the four-dimensional case, with topology $S^3 \times \mathbb{R}$ absence of Bondi news requires that the Bach tensor vanishes on $J$. Recall that this construct is defined (in any dimension) using the tensor $A_{\alpha\beta} \equiv \frac{1}{n-2}(R_{\alpha\beta} - \frac{R}{2(n-1)}g_{\alpha\beta})$ as:

$$B_{\mu\nu} \equiv \nabla_{\rho}C_{\rho\mu\nu} + A_{\alpha\beta}W_{\alpha\beta\mu\nu}$$

(2.29)

where $W_{\alpha\beta\mu\nu}$ is the Weyl tensor, defined in terms of the Riemann tensor as: $W_{\alpha\beta\mu\nu} \equiv R_{\alpha\beta\mu\nu} - (A_{\beta\mu}g_{\alpha\nu} + A_{\alpha\nu}g_{\beta\mu} - A_{\beta\nu}g_{\alpha\mu} - A_{\alpha\mu}g_{\beta\nu})$ (so that it vanishes when $n=2$ or $n=3$).

and $C_{\mu\nu\rho}$ is the Cotton tensor, defined by:

$$C_{\alpha\beta\gamma} \equiv \nabla_{\alpha}A_{\beta\gamma} - \nabla_{\beta}A_{\alpha\gamma}$$

(2.30)

In $n=3$ the spacetime is conformally flat iff the Cotton tensor vanishes. It is to be stressed that the Bach tensor is conformally invariant when $n=4$ \emph{only}.

When $n=3$ (with topology $S^2 \times \mathbb{R}$), the Bach tensor vanishes, conveying the fact that
there are no gravitational Bondi news in this case ([7]). This is the simplest instance of a general theorem proved in [4] staying that for a spacetime \( M \) of even dimension there is no obstruction for the existence of a formal power series solution to the Cauchy problem.

In the case of a \( n=3 \) boundary spacetime \( S^2 \times \mathbb{R} \), however, Bondi news exist in general for matter fields if the Cotton tensor does not vanish. This means that for four dimensional bulk spaces there is the possibility of having a well defined Cauchy problem in the Fefferman-Graham sense, and yet, Bondi news for fields with spin different from 2. It is obvious that this is problematic from the holographic point of view (except in the case of pure gravity).

Geometrically, the vanishing of the Cotton tensor in the three-dimensional case is the necessary and sufficient condition for the existence of conformal Killing spinors. (In the four dimensional case, the equivalent condition (implying that the space is conformally Einstein) is the vanishing of the Bach tensor (cf.[8])). Only in this case the conformal symmetry is realized asymptotically in such a way that one can define asymptotically conserved charges associated to the \( O(3,2) \) conformal group. (\( O(4,2) \) in the four-dimensional case).

This reduction of the asymptotic symmetry group to AdS is similar to the reduction from the asymptotic Bondi-Metzner-Sachs [13] group in the asymptotically flat case, towards Poincaré, as has been pointed out in [7]. In this case, however, the condition \( B_{\mu\nu} \sim 0 \) is too strong, and, in particular, it is not stable against gravitational perturbations. There is then a curious discontinuity in the limit \( \lambda \to 0 \).

A curious fact is that it would seem that the vanishing of the conformal anomaly in the three dimensional case in the holographic setting [4] does not need the vanishing of the Cotton tensor.

Another fact worth stressing is that gravitational Bondi news are generically non-vanishing when \( J \) is spacelike, or even null. It is plain that the interplay between holography and Bondi news is related to the existence of a Cauchy surface for asymptotically anti de Sitter spacetimes (i.e. \( J \) timelike). The simplest example is, obviously, AdS it-
self, where in order to define a Cauchy surface one is forced to impose reflective boundary conditions on $\mathcal{J}$,[10] enforcing the desired absence of Bondi news for matter fields..

Remarkably enough, Hawking[6] has proved that the physics of this set of boundary conditions is equivalent to assuming that the gravitational fields tend to AdS at infinity fast enough. Physically, absence of Bondi news on $\mathcal{J}$ is necessary in order that a CFT living on $\mathcal{J}$ could propagate holographically to the bulk in a unique way.

3 Renormalization Group Flow and Einstein Equations

One of the deepest results of the world sheet sigma model approach to string theory is the interpretation of the background spacetime metric as a coupling constant and Einstein’s equations in spacetime as the corresponding renormalization group beta functions (19). The extension to this approach to the physical spacetime metric for ordinary four dimensional quantum field theory is maybe possible provided a holographic projection works.

In a holographic framework one considers generically an ambient bulk spacetime where the real physical spacetime is somehow embedded as a subspace of codimension 1. We shall denote generically by $\rho$ this holographic coordinate.

Let us now consider the one-dimensional family of induced four-dimensional metrics, $g_{\mu\nu}(\rho)$ which can be considered as well as coupling constants. This leads to an interpretation of the $\rho$-dependence as a flow in the Renormalization Group sense.

From the geometric point of view described in previous sections, holography reduces essentially to determine this $\rho$ dependence of the four-dimensional metric in terms of the Einstein equations for the bulk geometry. From a physical point of view, one of the most fascinating aspects of the whole set up of holography is the re-interpretation of the renormalization group equations of the quantum field theory living at the boundary in
terms of Einstein’s equations in the bulk. Following Maldanena’s work on near-horizon geometry of D-branes the holographic coordinate $\rho$ can be related to the ultraviolet scale of the theory.

We would now like to be more specific about the physical meaning of the $\rho$ dependence of the four-dimensional metric. It will actually be proposed to use an appropriate function of $g_{\mu\nu}(\rho)$ as a way of defining Zamolodchikov’s c-function of the theory at the scale $\rho$ (more precisely, as a way to measure the relevant massless degrees of freedom at the scale $\rho$). Based on this interpretation, generic geometric properties of the bulk geometry and the holographic equations suffice to ensure the validity of the c-theorem, to wit, that the number of massless degrees of freedom decreases through the renormalization group flow from the ultraviolet (UV) towards the infrared (IR). An important point will be to understand under what conditions holography is consistent with departing from the CFT fixed point.

In order to address this last question it is important to keep in mind the two different variables used in the Lorentzian (namely, $\tilde{G}$ as bulk space) version of the holographic geometry. One variable $t \in \mathbb{R}^+$ characterizes conformal transformations of the spacetime metric, while the variable $\rho \in [-1,1]$ (that is, the one we can properly interpret as the holographic coordinate) parametrizes the change of the metric in the bulk direction.

The interplay between these two variables can be easily understood in Penrose’s framework. In fact we have already seen that close to $\mathcal{J}$ we have

$$\Omega(\hat{u}) = - \sum_{n=1}^{\infty} A_n \hat{u}^n$$

so that if all $A_n = 0$ for $n > 1$, rescalings of the holographic variable $\hat{u} \rightarrow \lambda \hat{u}$ are equivalent to conformal transformations of the metric. This shall be the typical situation of what we can call conformal holography, or in other words holography describing CFT fixed points.

In the general case, when no coefficient vanish in the above expansion, this simple equivalence does not hold anymore and one can have a nontrivial renormalization group
flow. Very likely the difference between the conformal and the non-conformal case is related to the presence of an energy momentum tensor in the second member of the bulk Einstein’s equations.

After these preliminary remarks on the holographic interpretation of the renormalization group flow, let us proceed to discuss in detail the holographic definition of the c-function.

4 A Proposal for a Holographic c-Function

In [16] Zamolodchikov proved that for local and unitary two-dimensional field theories there exists a function of the couplings, hereinafter called the c-function, \( c(g_i) \), such that

\[
- \beta_i \partial^i c \leq 0 \tag{4.32}
\]

along the renormalization group flow. For fixed points of the flow, the c-function reduces to the central extension of the Virasoro algebra. Some generalizations of the c-function to realistic four-dimensional theories have been suggested; let us mention in particular Cardy’s proposal on \( S^4 \):

\[
c \equiv \int_{S^4} \sqrt{g} < T > \tag{4.33}
\]

where \( T \) is the trace of the energy-momentum tensor of the theory. There is no complete agreement as to whether a convincing proof of the theorem exists for dimension higher than two (cf. [24] for a recent attempt).

It is not difficult to invent a holographic definition of the central extension for N=4 super Yang-Mills theories using the AdS/CFT map [13][25], especially in the light of the IR/UV connection pointed out by Susskind and Witten in [14]. In order to understand this construction properly, let us recall the original ’t Hooft’s presentation of the holographic principle [1], stemming from the Bekenstein-Hawking entropy formula for black holes [22]. Given a bounded region of instantaneous \( d - 1 \) space \( V \) of volume \( \text{vol}_{d-1}(V) \), holography
states that all the physical information on processes in \( V_{d-1} \) can be codified in terms of surface variables, living on the boundary of \( V, \partial V \). More precisely, the number of holographic degrees of freedom is given by:

\[
N_{d.o.f.} \sim \frac{\text{vol}_{(d-2)}(\partial V)}{G^{(d)}}
\]  

(4.34)

Physically this means that we have precisely one degree of freedom in each area cell of size given by the Plack length. In spite of the fact that the Bekenstein bounds would suggest the radical approach that any physics in \( V \) can be mapped into holographic degrees of freedom in \( \partial V \), the list of theories suspected to admit holographic projection is still small, and always involves gravity. (Susskind indeed suggested from the beginning that string theory should be holographic).

In the previous sections of the present paper we have just examined the geometrical set up in which to a given geometrical (diff-invariant) theory defined in the bulk one can associate a CFT living on the boundary. Under these conditions, it is not difficult to generalize ‘t Hooft’s formula in a precise manner.

Let us consider a four dimensional CFT defined on a spacetime with topology \( S^3 \times \mathbb{R} \) and with the natural metric in \( S^3 \). Let us also introduce an ultraviolet cutoff \( \delta \), and let us correspondingly divide the sphere \( S^3 \) into small cells of size \( \delta^3 \). The number of cells is clearly of order \( \frac{1}{\delta^3} \). We would like now to define the number of degrees of freedom in terms of the central extension as

\[
N_{dof} = \frac{c}{\delta^3}
\]  

(4.35)

Notice that here the parameter \( c \) plays the rôle of the number of degrees of freedom in each cell. If the theory we are considering is the holographic projection of some supergravity in the bulk, it is natural to rewrite this in terms of (5.23), but with the \( \text{vol}(\partial V) \) now replaced by a section of the bulk at \( \delta = \text{constant} \), with \( \delta \) being now identified with the holographic parameter. We are thus led to the identification

\[
c \equiv \lim_{\delta \to 0} \frac{\delta^3 \text{Vol}(\partial V_5)}{G_5}
\]  

(4.36)
In the particular example of AdS this yields $c = \frac{R^3}{G_5}$, with $R^4 = \alpha'^2 N g^2 \Phi$.

5 Renormalization Group Flow along Null Geodesics

We shall in this section study the renormalization group evolution of the postulated c-function; that is, its dependence on the holographic variable, $\rho$. In order to do that, the first point is to identify exactly what we understand by area (that is, $vol(\partial V)$). Our definition clearly involves the quotient between an area defined close to the horizon and an inertial area, so that:

$$c(\delta) \equiv \frac{vol_{(d-2)}(\mathcal{J}_\delta) \times vol_{(d-2)}(\text{inertial})}{G_d}$$

The meaning of the preceding formula is as follows (cf. [14]). The first term, $vol_{(d-2)}(\mathcal{J}_\delta)$ is the volume computed on $\mathcal{J}$ regularized with an UV cutoff $\delta$ ($N^2/\delta^3$ in the familiar example of $AdS_5$). The other factor, $vol_{(d-2)}(\text{inertial})$ stands for the equivalent volume measured by an inertial observer which does not feel the gravitational field (that is, $\delta^3$ in $AdS_5$). The whole thing is then divided by the d-dimensional Newton’s constant.

Our c-function will obey a renormalization group equation of the type:

$$\delta \frac{\partial c}{\partial \delta} + \sum_i \beta_i(g) \frac{\partial c}{\partial g_i} = 0$$

which means that to prove the c-theorem we just have to show that:

$$\delta \frac{\partial c}{\partial \delta} \leq 0$$

Now, to study the evolution in the bulk of the c-function, we need to know how this regularized definition of area evolves as we penetrate into the bulk. It is now only natural to identify the UV cutoff $\delta$ with the affine parameter $\hat{u}$ introduced in the previous section such that $\hat{u} \sim 0$.

We would like also to argue that it is quite convenient to study the evolution of the area along null geodesics entering the bulk. First of all, the whole set-up is conformally
invariant (which is not the case for timelike geodesics). In addition, there is a very natural definition of tranverse space there. In a Newman-Penrose ortho normal tetrad which is a sort of complexified light cone, because in terms of a real orthonormal tetrad, \( e^a \),

\[
\begin{align*}
  l^\mu \partial_\mu &\equiv e^+ \equiv \frac{1}{\sqrt{2}}(e^0 + e^3) \\
  n^\mu \partial_\mu &\equiv e^- \equiv \frac{1}{\sqrt{2}}(e^0 - e^3) \\
  m^\mu \partial_\mu &\equiv e_T \equiv \frac{1}{\sqrt{2}}(e^1 - ie^2) \\
  \bar{m}^\mu \partial_\mu &\equiv \bar{e}_T \equiv \frac{1}{\sqrt{2}}(e^1 + ie^2),
\end{align*}
\]

one can easily find the optical scalars \([26]\) of the geodesic congruence. One has, in particular, that

\[
\rho \equiv -\nabla_\mu l_\nu m^\nu \bar{m}^\mu = -(\theta + i\omega) \quad (5.41)
\]

where the expansion, \( \theta \), is defined by \( \theta \equiv \frac{1}{2} \nabla_\alpha l^\alpha \) and the rotation, \( \omega \), is a scalar which measures the antisymmetric part of the covariant derivative of the tangent field:

\[
\omega^2 \equiv \frac{1}{2} \omega_{\alpha\beta} \omega^{\alpha\beta}, \quad \text{with} \quad \omega_{\alpha\beta} \equiv \nabla_{[\alpha} l_{\beta]}.
\]

Let us now consider a congruence of null geodesics. This means that we have a family \( x^\mu(u, v) \), such that \( v \) tells in which geodesic we are, and \( u \) is an affine parameter of the type previously considered. The connecting vector (geodesic deviation) \( Z^\mu \equiv x^\mu(u, v) - x^\mu(u, v + \delta v) \) connects points on neighboring geodesics, and by construction satisfies

\[
\mathcal{L}(l) Z^\mu = 0 \quad (5.42)
\]

that is, \( l^\mu \nabla_\mu Z^\alpha = Z^\mu \nabla_\mu l^\alpha \). Although the molulus of the vector \( Z \) is itself not conserved, it is not difficult to show that its projection on \( l^\mu \) is a constant of motion. Penrose and Rindler call abreast the congruences for which this projection vanishes. In this case one can show that \( h = 0 \), where \( h \) is defined from the projection of the geodesic deviation vector.

\[\text{We choose to present the formulas in the four dimensional case by simplicity, but it should be clear that no essential aspect depends on this.}\]
on the Newman-Penrose tetrad:

\[ Z^\alpha = g^\alpha_l + \zeta^\alpha m^\alpha + \bar{\zeta}^\alpha n^\alpha \]  

(5.43)

Under the preceding circumstances, the triangle \((0, \zeta_1, \zeta_2)\) is contained in \(\Pi\), the 2-plane spanned by the real and imaginary parts of \(m^\alpha\).\(^8\) Now it can be proven \([15]\) that, calling \(A_2\) the area of this elementary triangle,

\[ l^\alpha \nabla_\alpha A_2 = -(\rho + \bar{\rho})A_2 = 2 \theta A_2 \]  

(5.44)

This fact relates in a natural way areas with null geodesic congruences.

Using this information we can write at once:

\[ \dot{u} \frac{dc(\dot{u})}{d\dot{u}} = (\theta \dot{u} + d - 2)c(\dot{u}) \]  

(5.45)

It is worth noting at this point that \(\dot{\theta}\) is finite (it corresponds to Einstein’s static universe in the standard AdS example). The divergence in \(\theta\) stems from the conformal transformation necessary to go from \(\hat{g}_{\alpha\beta}\) to \(g_{\alpha\beta}\), to wit:

\[ \theta = \dot{\theta} + \frac{d - 2 N.Z}{2} \frac{\Omega}{\Omega} \]  

(5.46)

In order that inertial and \(J\) units be the same, it is natural to measure inertial areas in units of \(\delta \equiv \frac{\dot{u}}{(N.Z)J}\), where \((N.Z)J\) represents the scalar product of the vector \(N^\mu \equiv -\nabla^\mu \Omega\) and \(Z^\mu\) computed at \(\dot{u} = 0\). Doing that one gets that the first derivative vanishes to first order:

\[ \dot{u} \frac{dc(\dot{u})}{d\dot{u}} = 0 \]  

(5.47)

But we can now invoke a well known theorem by Raychadhuri \([11]\) \([12]\)

\[ l^\mu \nabla_\mu \theta = \omega^2 - \frac{1}{2} R_{\mu\nu} l^\mu l^\nu - \sigma \dot{\sigma} - \theta^2 \]  

(5.48)

(where the shear \(\sigma \dot{\sigma} \equiv \frac{1}{2} \nabla_{[\beta l^\alpha]} \nabla^{[\beta}l^{\alpha]} - \frac{1}{2} (\nabla_{\alpha} l^\alpha)^2\))
The Ricci term in the above equation vanishes for Einstein spaces, and the rotation must necessarily be zero if we want the flow lines to be orthogonal to the surfaces of transitivity; that is, that there exists a family of hypersurfaces $\Sigma$, such that $l_\mu = \nabla_\mu \Sigma$.

This shows that under these conditions

$$\hat{u} \frac{d\theta}{d\hat{u}} < 0$$

which is enough to prove the c-theorem in the holographic case.

6 Final Comments

We have considered only the null infinity $\mathcal{J}$ in this paper. All particles ending there are massless. It would be exceedingly interesting to extend the geometrical holographic analysis to the massive case. This would presumably mean, in Penrose’s language, to consider the *timelike infinity*, $i^+$. This would answer, in particular, a natural question the reader might like to ask, and that is why we do not see the sphere $S^5$ in our approach (in the simplest case of $AdS_5$). We think that we would see the sphere in the sense of seeing the higher modes of the allowed supergravity Kaluza-Klein spectrum on AdS through boundary conditions at timelike infinity.

Zig-zag symmetry, believed to be an exact symmetry of Wilson loops, does not seem to be naturally implemented in the geometrical approach. It appears that some generalization of it is needed, which is in some sense purely topological at the boundary.

We have mentioned in several places that in some cases bulk spaces could exist with positive cosmological constant. This would mean, in particular, that the Minkowskian holography would correspond to a $(2,3)$ signature five-dimensional bulk manifold. An explicit construction would be most welcome.

Work is in progress in these, and related, matters.
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References

[1] G. ’t Hooft, *Dimensional Reduction in Quantum Gravity* [gr-qc/9310026](http://arxiv.org/abs/gr-qc/9310026)  
L. Susskind, *The world as a Hologram*, [hep-th/9409089](http://arxiv.org/abs/hep-th/9409089)

[2] J. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, [hep-th/9711200](http://arxiv.org/abs/hep-th/9711200)

[3] E. Witten, *Anti de Sitter Space and Holography*, [hep-th/9802150](http://arxiv.org/abs/hep-th/9802150)

[4] C. Fefferman and C. Graham, *Conformal Invariants* Astérisque, hors série, 1985, p.95.

[5] J.L.F. Barbón and E. Rabinovici, *Extensivity versus Holography in Anti de Sitter Spaces*, [hep-th/9805143](http://arxiv.org/abs/hep-th/9805143)

[6] S.W. Hawking, *The Boundary Condition for Gauged Supergravity*. Phys. Lett 126B, (1983) 175.
[7] A. Ashtekar and A. Magnon, *Asymptotically Anti de Sitter space-times*. Class. Quantum Grav. 1(1984)L39

[8] C. Kozameh, E.T.Newman, and K. P. Tod, *Conformal Einstein Spaces* Gen. Rel. Grav.17(1985)343

[9] M. Henningson and K. Skenderis, *The Holographic Weyl Anomaly* hep-th/9806087

[10] S.J. Avis, C.J. Isham and D. Storey, *Quantum Field Theory in anti- de Sitter Space-time*. Phys. Rev. D18 (1978) 3565.

[11] A. Raychaudhuri, Phys. Rev. 98 (1955) 11 23

[12] S.W. Hawking and GFR Ellis, *Singularities in homogeneous world models* Phys. Lett. 17 (1965) 246.

[13] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge Theory Correlators from Non-critical String Theory*, hep-th/9802109

[14] L. Susskind and E. Witten, *The Holographic Bound in AdS-Spaces*, hep-th/9805114

[15] R. Penrose, in *Group Theory in Non-Linear Problems*, A.O. Barut ed., Reidel, Dordrecht, 1974.

S.A. Huggett and P.K. Tod, *An Introduction to Twistor Theory*, Cambridge University Press, 1985.

R. Penrose and W. Rindler, *Spinors and Space-Time*, Cambridge University Press, 1986.

[16] A.B. Zamolodchikov, J.E.T.P Lett. 43 (1986) 730.

[17] J.L. Synge and A. Schild, *Tensor Calculus*, Dover, New York, 1970.

[18] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime*, Cambridge University Press, 1973.
[19] A.M. Polyakov, *Gauge Fields and Strings* (Harwood Academic)

[20] S. J. Rey and Jungtay Yee, *Macroscopic strings as heavy quarks of large N gauge theory and AdS supergravity*, hep-th/9803001

J. Maldacena, *Wilson loops in large N field theories*, hep-th/9803002

E. Witten, *Anti-de Sitter space, thermal phase transitions and confinement in gauge theories*, hep-th/9803131

[21] A.M. Polyakov, *Confining Strings*, hep-th 9607049

A.M. Polyakov, *String Theory and Quark Confinement*, hep-th/9711002

E. Álvarez, C. Gómez and T. Ortín, *String representation of Wilson loops*, hep-th/9806075

[22] J.Bekenstein, Phys. Rev. D49 (1994) 1912

[23] J. Cardy, Phys. Lett B215 (1988) 749.

[24] S. Forte and J.I. Latorre, *A proof of the irreversibility of renormalization group flows in four dimensions*, hep-th/9805015

H. Osborn and D.Z. Friedan, hep-th/9804101

A. Capelli, D. Friedan and J.I. Latorre, Nucl. Phys. B352 (1991) 616.

[25] S. Gubser, A. Hashimoto, I. Klebanov and M Krasnitz, *Scalar absorption and the breaking of the world volume conformal invariance*, hep-th/9803023

[26] D. Kramer, H. Stephani, E. Herlt and M.A.H. MacCallum, *Exact solutions of Einstein’s field equations* (Cambridge, 1980)