A class of finite two-dimensional sigma models and string vacua

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We consider a two-dimensional Minkowski signature sigma model with a $2 + N$-dimensional target space metric having a null Killing vector. It is found that the model is finite to all orders of the loop expansion if the dependence of the “transverse” part of the metric $g_{ij}(u,x)$ on the light cone coordinate $u$ is subject to the standard renormalization group equation of the $N$-dimensional sigma model,

$$\frac{dg_{ij}}{du} = \beta g_{ij} = R_{ij} + \ldots$$

In particular, we discuss the ‘one-coupling’ case when $g_{ij}(u,x)$ is a metric of an $N$-dimensional symmetric space $\gamma_{ij}(x)$ multiplied by a function $f(u)$. The theory is finite if $f(u)$ is equal to the “running” coupling of the symmetric space sigma model (with $u$ playing the role of the RG “time”). For example, the geometry of space-time with $\gamma_{ij}$ being the metric of the $N$-sphere is determined by the form of the $\beta$-function of the $O(N+1)$ model. The “asymptotic freedom” limit of large $u$ corresponds to the weak coupling limit of small $2+N$-dimensional curvature. We show that there exists a dilaton field which together with the $2 + N$-dimensional metric solves the sigma model Weyl invariance conditions. The resulting backgrounds thus represent new tree level string vacua. We remark on possible connections with some $2d$ quantum gravity models.

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1. Only few examples of exact solutions of the string (tree–level) effective equations [1] are explicitly known at present. They include, in particular, group manifolds [2] and plane wave – type spaces [3]. Below we shall present a new time - dependent solution corresponding to a finite 2d sigma model. Like in the case of the spaces considered in [3] the metric will have the Minkowski signature and a null covariantly constant Killing vector. However, the “transverse” part of the metric will not be flat but will represent an arbitrary (for example, symmetric) space. Also, the mechanism by which conformal invariance conditions will be satisfied will be novel. The $\sigma$ - model divergences will not be absent automatically at each order of the loop expansion but will vanish “on - shell”, i.e. one will be able to redefine them away. The corresponding terms in the Weyl anomaly coefficients ($\bar{\beta}$ - functions) will be cancelled by the contributions of the dilaton and “reparametrisation terms” [4].

Our model provides an example of how one can construct a conformal invariant theory by adding two extra (one time - like and one space - like) dimensions to a non - conformal one (in our case – an arbitrary $N$ - dimensional sigma model). This suggests a close connection with a particular model of 2d scalar - tensor quantum gravity\(^1\) coupled to a $\sigma$ - model.\(^2\) In fact, a 2d gravitational system represented in the conformal gauge turns out to be identical to the “1-loop finite” form of our $2 + N$ - dimensional $\sigma$ - model (with $N$ being the dimension of a symmetric space and the conformal factor and an extra scalar playing the role of the light cone coordinates). This correspondence implies that the properly modified ‘symmetric space $\sigma$ - model – 2d gravity’ model is also finite to all orders in the loop expansion.\(^3\)

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1 For recent discussions of scalar–tensor 2d quantum gravity from various points of view see e.g. [5–7].

2 At the classical level similar models (with symmetric target spaces) appear as a result of dimensional reduction of higher dimensional Einstein theory to $D = 2$ (see e.g. [8]).

3 Quantum properties of supersymmetric extensions of such 2d gravity - matter models which can be obtained e.g. by a dimensional reduction from $D = 4$ supergravity (for an early discussion of r-trivial “scalar - tensor” $D = 2$ supergravity models see [9]) were recently studied in [10]. Our
2. Let us consider the following metric

\[ ds^2 = G_{\mu \nu} dx^\mu dx^\nu = -2dudv + g_{ij}(u, x) dx^i dx^j , \]

\( \mu, \nu = 0, 1, ..., N, N + 1 \), \( i, j = 1, ..., N \).

We shall show that if the dependence of \( g_{ij} \) on \( u \) is subject to a certain first order differential equation (nothing but the standard RG equation of the \( N \)-dimensional sigma model with the metric \( g_{ij}(u, x) \) and \( u = \) const playing the role of the RG “time” parameter) then the \( 2 + N \)-dimensional \( \sigma \)-model with the target space metric (1) is UV finite.

As an important example we shall consider the ‘one - coupling’ case when

\[ g_{ij}(u, x) = f(u) \gamma_{ij}(x) , \]

where \( \gamma_{ij}(x) \) will be assumed to be a metric of a symmetric (constant curvature) space (one can of course consider a trivial generalization of (2) corresponding to a direct product of symmetric spaces with separate “couplings” \( f_n(u) \)). Once the metric is chosen in the “factorized” form the restriction that the \( N \)-dimensional space is symmetric is necessary in order to be able to represent the “transverse” part of the divergences in terms of the original metric \( \gamma_{ij} \) (i.e. to have renormalisability in terms of one essential coupling constant \( f \)). The model will be finite under the specific choice of \( f \), i.e. for \( f(u) \) equal to the “running” coupling of the symmetric space \( \sigma \)-model.

While all the basic formulas will be true for a general \( g_{ij} \) or a symmetric space \( f \gamma_{ij} \) we shall derive the explicit perturbative expressions (eqs.(14),(24),(30)–(32), etc) in the case when \( \gamma_{ij} \) corresponds to a maximally symmetric \( N \)-space with the curvature tensor

\[ R_{ijkl} = \frac{k}{(N - 1)} (\gamma_{ik} \gamma_{jl} - \gamma_{il} \gamma_{jk}) , \quad k \equiv \frac{R}{N} = \text{const} . \]

Conclusion about finiteness of the corresponding generalized bosonic models suggests that similar “modified” supersymmetric models may be finite to all loop orders. Let us note also that it should be possible to construct a finite model with explicitly known metric by replacing the “transverse” part of the bosonic action by an \( N = 2 \) supersymmetric sigma model with a homogeneous target space. The presence of the extra two “longitudinal” dimensions will make possible to get rid of the 1-loop divergences preserving at the same time the higher loop finiteness of the supersymmetric “sub - model”.
It is useful first to give a heuristic “proof” of the finiteness of the model (1),(2). The path integral over $v$ produces a delta - function which constrains $u$ to be constant. Then we are left with the standard $\sigma$ - model for $x^i$ (with $f^{-1}$ playing the role of a constant coupling or a Planck constant ). The corresponding divergences are proportional to
\[ \int d^2z \, T_{ij}(x, f) \partial_a x^i \partial^a x^j , \]
where $T_{ij}$ is constructed from the curvature tensor of $\gamma_{ij}$ and its derivatives [11]. If $g_{ij}$ corresponds to a symmetric space $T_{ij}$ is proportional to $\gamma_{ij}$ with a constant coefficient.

The final step is to note that the classical $u$ - equation of motion implies that $\gamma_{ij} \partial_a x^i \partial^a x^j$ is proportional to the total derivative term $\partial^2 v$ ($u = \text{const}$ according to $v$ - equation of motion). That is why all the divergences are absent “on shell”.

This “proof” does not seem to depend on a form of $f(u)$. What it actually proves, however, is not the finiteness of the model but the absence of divergences on a particular solution ($u = \text{const}$) of the classical equations. As it is easy to understand, the latter property does not by itself guarantee that the divergences are proportional to the classical equations of motion and hence can be eliminated by redefinitions of the 2d fields $u(z), \, v(z), \, x^i(z)$.

To establish the finiteness of the model on a flat 2d background one should check that the $\beta$ - function for the “full” $\sigma$ - model with the target space metric $G_{\mu\nu}$ (1) vanishes up to a “reparametrisation” term [12], i.e.
\[ \beta^G_{\mu\nu} + 2D(\mu,M_{\nu}) = 0 . \] (4)

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4 It is not straightforward to generalize the above argument to the case when $g_{ij}$ is arbitrary, i.e. when the condition (2) is relaxed. Given that the 2d sigma model is renormalisable in the generalized sense [12,11] we can still represent the counterterms in terms of the “classical” metric with shifted $u$ (i.e. in terms of renormalized metric). However, the shifted $u$ depends on $x$ and hence is no longer constant. In contrast to the factorized case (2), to achieve finiteness even on the $u = \text{const}$ background we do need to subject $g_{ij}(u,x)$ to a (differential) constraint.

5 This is related to the fact that $\partial^2 u = 0$ is a second order equation, i.e. its general solution is given by $u(z) = u_0 + b_a z^a$. 
As we shall show, (4) is satisfied only for a particular \( g_{ij}(x,u) \) as a function of \( u \) (i.e. for a particular \( f(u) \) in the case of (2)). Moreover, the resulting metric (1) together with an appropriate dilaton background will represent a solution of the Weyl invariance conditions of the \( \sigma \)-model on a curved 2d background (and thus also a solution of the string effective equations).

The Weyl invariance conditions for the \( \sigma \)-model

\[
I = \frac{1}{4\pi\alpha'} \int d^{2}z \sqrt{g} \left[ G_{\mu\nu}(x) \partial_{\mu}x^{\nu} \partial_{\rho}x^{\rho} + \alpha' R^{(2)}(x) \phi(x) \right],
\]

have the following general structure \([1,4]\)

\[
\tilde{\beta}^{G}_{\mu\nu} = \beta^{G}_{\mu\nu} + 2\alpha' D_{\mu}D_{\nu}\phi + D_{(\mu} W_{\nu)} = 0 ,
\]

(5)

\[
\tilde{\beta}^{\phi} = \beta^{\phi} + \alpha'(\partial_{\mu}\phi)^{2} + \frac{1}{2} W^{\mu} \partial_{\mu}\phi = 0 .
\]

(6)

The leading terms in the \( \beta \)-functions and \( W_{\mu} \) are well known (in the dimensional regularisation and minimal subtraction scheme) \([11,12,1,4,13,14]\)

\[
\beta^{G}_{\mu\nu} = \alpha' R_{\mu\nu} + \frac{1}{2} \alpha'^{2} R_{\mu\alpha\beta\gamma} R^{\alpha\beta\gamma}_{\nu} + \frac{1}{16} \alpha'^{3} \left[ 8 R_{\alpha\rho\sigma\beta} R_{\mu}^{\rho\sigma\gamma} R_{\nu}^{\alpha\beta} - 6 R_{\mu\alpha\beta\nu} R^{\alpha\sigma\rho\kappa} R^{\beta} R_{\sigma\rho\kappa} \right]
\]

\[
+ 2 D_{\rho} R_{\mu\alpha\beta\gamma} D^{\rho} R_{\nu}^{\alpha\beta\gamma} - D_{\mu} R_{\alpha\beta\gamma\rho} D_{\nu} R^{\alpha\beta\gamma\rho} + O(\alpha'^{4}) ,
\]

(7)

\[
\beta^{\phi} = \frac{1}{6} (2 + N - 26) - \frac{1}{2} \alpha' D^{2}\phi + \frac{1}{16} \alpha'^{2} R_{\mu\alpha\beta\gamma} R^{\mu\alpha\beta\gamma} + O(\alpha'^{3}) ,
\]

(8)

\[
W_{\mu} = \frac{1}{32} \alpha'^{3} \partial_{\mu}(R_{\alpha\beta\gamma\rho} R^{\alpha\beta\gamma\rho}) + O(\alpha'^{4}) .
\]

(9)

If (5) has a solution, the finiteness condition (4) is satisfied for \( M_{\mu} = \alpha' \partial_{\mu}\phi + \frac{1}{2} W_{\mu} \), i.e. a Weyl invariant \( \sigma \)-model is also UV finite on a flat 2d background (the opposite may not necessarily be true \([15,4]\)).

The non-vanishing components of the connection and curvature of the metric \( G_{\mu\nu} \) (1) are (we shall finally specify all the formulas to the case of the metric (2))

\[
\hat{\Gamma}_{jk}^{i} = \Gamma_{jk}^{i} , \quad \hat{\Gamma}_{ij}^{u} = \frac{1}{2} \hat{g}_{ij} = \frac{1}{2} \hat{f}\gamma_{ij} , \quad \hat{\Gamma}_{j}^{i} = \frac{1}{2} g^{ik} \hat{g}_{kj} = \frac{1}{2} f^{-1} \hat{f}\delta_{j}^{i} , \quad (10)
\]
\[ \dot{g}_{ij} \equiv \frac{dg_{ij}}{du} , \quad \dot{f} \equiv \frac{df}{du} , \]

\[ \dot{R}_{ijkl} = R_{ijkl}^i , \quad \dot{R}_{ij} \equiv H_{ij} , \quad \dot{R}_{ij}^i = g_{ik}H_{kj} , \]

\[ H_{ij} = -\frac{1}{2}(\ddot{g}_{ij} - \frac{1}{2}g_{mn}\dot{g}_{im}\dot{g}_{nj}) = -\frac{1}{2}(\ddot{f} - \frac{1}{2}f^{-1}\dot{f}^2)\gamma_{ij} , \tag{11} \]

\[ \dot{R}_{ijk} = \dot{R}_{uijk} \equiv F_{ij} , \quad F_{ijk} = \frac{1}{2}[\partial_j\dot{g}_{ik} + \dot{g}_{jm}\Gamma_{ik}^m - (j \leftrightarrow k)] = \frac{1}{2}(g_{km}\Gamma_{ij}^m - g_{jm}\Gamma_{ik}^m) , \tag{12} \]

\[ F_{ijk}(g_{ij} = f\gamma_{ij}) = 0 . \]

For a general \( g_{ij} \) the non-zero elements of \( \beta^G_{\mu\nu} \) (7) are \( \beta^G_{ij} \), \( \beta^G_{uu} \) and \( \beta^G_{iu} \) (the latter vanishes in the factorized case (2)). In general they are non-trivial functions of \( u \) and \( x^i \). It is easy to understand that there are no non-vanishing contractions of the curvature components with \( u,v \) indices. \( \beta^G_{ij} \) depends only on \( R_{ijkl}^i \), its covariant derivatives and \( g_{ij} \) and therefore coincides with the \( \beta \)-function of the \( \sigma \)-model with the metric \( g_{ij} \). In the special case of (2),(3) \( f^{-1} \) plays the role of the coupling of the symmetric space \( \sigma \)-model (we may absorb the scale of \( \gamma_{ij} \) into \( f \) making \( k \) in (3) equal, for example, +1 or –1 depending on a sign of the curvature \( R \)). As a result, we find for \( \beta^G_{ij} \) (see (7))

\[ \beta^G_{ij} = \beta^g_{ij} = \beta(f)\gamma_{ij} , \tag{13} \]

\[ \beta^g_{ij} = \alpha' R_{ij} + O(\alpha'^2) , \quad \beta = f \sum_{n=1}^{\infty} c_n (af^{-1})^n , \]

\[ \beta(f) = a + (N - 1)^{-1}a^2f^{-1} + \frac{1}{4}(N - 1)^{-2}(N + 3)a^3f^{-2} + O(a^4f^{-3}) , \tag{14} \]

\[ a \equiv \alpha' k . \]

The leading terms in \( \beta^G_{uu} \) and \( \beta^G_{iu} \) are given by (see (7),(11),(12))\[\]

\[ \beta^G_{uu} = \alpha' R_{uu} + \frac{1}{2}\alpha'^2 R_{uijk} R_{ijkl}^i + O(\alpha'^3) = \alpha' g^{ij}H_{ij} + \frac{1}{2}\alpha'^2 F_{ijk} F_{ijkl} + O(\alpha'^3) , \]

\[ ^6 \text{In the symmetric space case (2) } \beta^G_{uu} \text{ does not receive the 2-loop (and certain higher loop) contributions if the } \beta^G \text{ - function is computed in the 'dimensional regularisation plus minimal subtraction' scheme (i.e. if there are no terms in } \beta^G_{\mu\nu} \text{ with Ricci tensors as factors). However, there are 3- and higher loop contributions to it. For example, there is the 3-loop term in (13) } -\frac{1}{4}\alpha'^3 f^{-4}f^2 R_{ijkl} R_{ijkl} \text{ which originates from the derivative } D_{\mu} R_{\alpha\beta\gamma\rho} D_{\nu} R^{\alpha\beta\gamma\rho} \text{ - term in (7).} \]
\begin{align}
\beta^G_{uu} &= -\frac{1}{2} \alpha' N f^{-1} \dot{f} - \frac{1}{2} \dot{f}^2 + O(\alpha^3) , \quad (15) \\
\beta^G_{iu} &= \alpha' g^{jk} F_{jki} - \frac{1}{2} \alpha'^2 F_{mnk} R_{mik}^{mn} + O(\alpha^3) , \quad \beta^G_{iu}(g_{ij} = f \gamma_{ij}) = 0 .
\end{align}

Let us now try to find such vectors $M_{\mu}$ which may satisfy the finiteness condition (4). Using (13) and that $\beta^G_{iv} = 0$, $\beta^G_{uv} = 0$ we can get the following equations (the covariant derivatives are now defined with respect to $g_{ij}$, see (10))

\begin{align}
\beta^g_{ij} + 2 D(i M_j) - 2 \hat{\Gamma}_{ij}^v M_v &= 0 , \\
\beta^g_{ij} &= \dot{g}_{ij} M_v , \quad \beta^g_{ij} \equiv \beta^g_{ij} + 2 D(i M_j) , \quad (16) \\
\beta^G_{uu} &= -2 \partial_u M_u , \quad (17) \\
\beta^G_{iu} &= -\partial_i M_u - \partial_u M_i + g^{jk} \dot{g}_{ij} M_k , \quad (18) \\
\partial_i M_v + \partial_v M_i &= 0 , \quad \partial_u M_v + \partial_v M_u = 0 . \quad (19)
\end{align}

Since all the components of $\beta^G_{\mu\nu}$ do not depend on $v$, the only $v$ - dependence that is possible in $M_{\mu}$ is a linear term in $M_u$. Then the general solution of (19) is given by

\begin{align}
M_v = m u + p , \quad M_u = -m v + Q(u, x) , \quad M_i = M_i(u, x) , \quad p , \ m = \text{const} . \quad (20)
\end{align}

For a given $g_{ij}(u, x)$ $\beta^G_{uu}$ and $\beta^G_{iu}$ are some given $N + 1$ functions of $u$ and $x$ so that one can always satisfy the equations (17) and (18) by properly chosen $N + 1$ functions $M_u$ and $M_i$ (once we solved (17), we can put (18) in the form $\partial_u M_i + h_i^j(u, x) M_j = E_i(u, x)$ which always has a solution).

Having determined $M_u$ and $M_i$ as functionals of $g_{ij}$ we are left with the final equation (16). It should be interpreted as an equation for $g_{ij}(u, x)$. Using (20) and introducing\footnote{We are considering non-zero $m$ for generality; to get a Weyl invariant model one must actually set $m = 0$ (see below).}

\begin{align}
\tau = m^{-1} \ln(m u + p) , \quad m \neq 0 ;
\end{align}
we can represent (16) in the form
\[ \frac{dg_{ij}}{d\tau} = \beta^g_{ij}. \] (21)

This concludes the proof of the statement that if the metric \( g_{ij} \) depends on \( u \) in such a way that it satisfies the standard RG equation of the \( N \)-dimensional sigma model (with some particular reparametrisation vectors) then the \( 2 + N \)-dimensional model based on (1) is UV finite to all orders of the loop expansion.

The above argument for finiteness is simplified in the particular case of symmetric space (2). Since \( \beta^G_{uu} = 0 \) and scalar functions (e.g. \( \beta^G_{uu} \)) are \( x \)-independent we set \( M_i = 0, \partial_i M_u = 0 \) and thus solve (16),(17),(18) by (see (13),(14))

\[ M_u = mu + p, \quad M_u = -mv + Q(u), \]

\[ \dot{Q} = \frac{1}{4} \alpha' N(f^{-1} \tilde{f} - haf^{-2} \tilde{f}^2) + O(\alpha'^3), \] (22)

\[ M_v \dot{\gamma}_{ij} = \beta(f) \gamma_{ij}, \]

i.e.

\[ \frac{df}{d\tau} = \beta(f). \] (23)

Eq. (23) with \( \beta \) given by (14) has the obvious perturbative solution

\[ f(u) = a[\tau + (N - 1)^{-1} \ln \tau + O(\tau^{-1})]. \] (24)

Having found \( f(u) \) from (19) one determines \( Q \) from (18) by integration. This proves the existence of a finite sigma model based on (1),(2).

3. To solve the Weyl invariance conditions (5),(6) one needs to establish that \( M_\mu \) in (4) can be represented in the form

\[ M_\mu = \alpha' \partial_\mu \phi + \frac{1}{2} W_\mu. \] (25)
Since it is known [1,4] that $W_\mu$ vanishes in the one- and two-loop approximation \(^8\) $m$ in (20) must vanish in order for a solution to exist. In fact, eq.(19) is consistent with $M_\mu$ being a gradient only if $\partial_v M_u = 0$. We find from (17),(18) (in the 2-loop approximation)

$$
\beta_{uu}^G = -2\alpha' \partial_u^2 \phi + O(\alpha'^3) \ ,
$$

$$
\beta_{iu}^G = -2\alpha' \partial_i \partial_u \phi + g^{jk} \dot{g}_{ij} \partial_k \phi + O(\alpha'^3) \ .
$$

Using the expression for $\beta_{\mu\nu}^G$ it is possible to check that the integrability condition for these equations is satisfied, i.e. there exists a solution

$$
\phi = \alpha'^{-1} [pv + F(u,x)] \ , \quad p = \text{const} \ , \ (26)
$$

which together with the metric (1) satisfies (5),(6). At the higher loop level the expression for $M_\mu$ found in the previous section will be “distributed” between the $\phi$- and $W_\mu$-terms in (25).

All is more transparent in the symmetric space case (2) where the differential equations become the ordinary ones and their integrability is obvious. We get

$$
\phi = \alpha'^{-1} [pv + F(u)] \ ,
$$

$$
\dot{F} = Q - \frac{1}{2} W_u \ , \quad W_v = 0 \ , \ (27)
$$

To find $Q$, $W_u$ and $\phi$ from (22), (9) and (27) it is useful to change the integration variable from $u$ to $f$ using the “RG equation” (23)

$$
Q = q + \frac{1}{4} \alpha' N p^{-1} \int_{f}^{\tilde{f}} d\tilde{f} [\tilde{f}^{-1} d\beta \tilde{f}^{-2} \beta(\tilde{f})] + O(\alpha'^3) \ , \quad q = \text{const} \ , \ (28)
$$

$$
W_u = -\frac{1}{8} \alpha' a^2 N (N-1)^{-1} f^{-3} \dot{f} + O(\alpha'^4) = -\frac{1}{8} \alpha' a^3 p^{-1} N (N-1)^{-1} f^{-3} + O(f^{-4}) \ ,
$$

\(^8\) In general, $W_\mu$ is constructed from the curvature and its covariant derivatives and thus depends only on $u$. 

8
\[
\phi = \alpha'^{-1}(qu + pv) + \int_\infty^f d\bar{f} \beta^{-1}(\bar{f}) \left\{ \frac{1}{4} N \int_\infty^\bar{f} d\bar{f} \left[ \frac{d\beta}{df} \right]^{-1} + \frac{1}{2} \bar{f}^{-2} \beta(\bar{f}) + O(\alpha'^3) \right\} - \frac{1}{2} \alpha' p W_u \} .
\]

(29)

Using the expression for \(\beta(f)\) (14) we get the following large \(u\) expansions in powers of the “coupling” \(f^{-1}\)

\[
f(u) = a[p^{-1}u + (N - 1)^{-1} \ln(p^{-1}u) + O(u^{-1} \ln u)] ,
\]

(30)

\[
Q = q + \frac{1}{16} \alpha' N p^{-1} [2af^{-1} + 3a^2(N - 1)^{-1}f^{-2} + O(f^{-3})] \\
= q + \alpha'\left[\frac{1}{8} Nu^{-1} + O(u^{-2} \ln u)\right] , \quad W_u = O(u^{-3}) ,
\]

(31)

\[
\phi = \phi_0 + \alpha'^{-1}(qu + pv) + \frac{1}{8} N [\ln f - a(N - 1)^{-1}f^{-1} + O(f^{-2})] \\
= \phi_0 + \alpha'^{-1}(qu + pv) + \frac{1}{8} N \ln(ap^{-1}u) + O(u^{-1} \ln u) .
\]

(32)

The resulting dilaton contribution to \(\bar{\beta}^\phi\) (eqs.(6),(8)) is given by

\[
\Delta \bar{\beta}^\phi = -\frac{1}{2} \alpha' D^2 \phi + \alpha'(\partial_\mu \phi)^2 + \frac{1}{2} W^\mu \partial_\mu \phi = \frac{1}{4} pN f^{-1} \dot{f} - 2\alpha'^{-1}pQ + \frac{1}{2} p W_u .
\]

(33)

It cancels the contribution of the higher loop \(\phi\) - independent terms in \(\beta^\phi\) (8) so that the central charge of the model \(\bar{\beta}^\phi\) is equal to that of the free \(N + 2\) - dimensional theory plus the contribution of the linear terms in the dilaton. To prove this one is to note that once (5) is satisfied \(\bar{\beta}^\phi\) is constant [16] and hence can be computed at any value of \(u\), e.g. \(u = \infty\). Given that all higher loop contributions should vanish in the weak coupling limit of large \(u\) it is sufficient to compute \(\bar{\beta}^\phi\) in the leading order approximation. Substituting (31),(32) into (33) we get

\[
\Delta \bar{\beta}^\phi = -2\alpha'^{-1}pq - \frac{1}{8} N(N - 1)^{-1}a^2 f^{-2} + O(f^{-3}) .
\]

(34)

The 1 - loop \(O(f^{-1})\) terms in (34) cancelled automatically while the 2 - loop \(O(f^{-2})\) term cancels against the contribution of the \(R^2\) term in (8), etc. The final expression for the central charge coefficient \(\bar{\beta}^\phi\) is

\[
\bar{\beta}^\phi = \frac{1}{6} (N - 24) - 2\alpha'^{-1}pq .
\]

(35)
Thus one can satisfy the zero central charge condition (6) for arbitrary $N$ by a proper choice of the constants $p$ and $q$.\footnote{We are assuming that $u$, $v$, $x^i$ have dimension $+1$ and $G_{\mu\nu}$, $\gamma_{ij}$, $\phi$, $f$, $\beta$ are dimensionless. Then $[\alpha'] = 2$, $[p] = [q] = 1$, $[a] = 0$, $[k] = -2$, $[Q] = 1$, etc. Since $p$ is a free parameter we may set it, for example, equal to $\sqrt{\alpha'}$ (or 1 if $\alpha' = 1$). Then $\tau$ in (24) becomes equal simply to $u$.}

4. We have found the standard (first order) RG equation of an $N$-dimensional theory (21),(23) in the process of solving the (second order) finiteness conditions (4) of a higher $N+2$-dimensional model. This is reminiscent of the "semiclassical" correspondence between the conformal invariance conditions in an $N+1$-dimensional theory and the RG equations in an $N$-dimensional one. The latter relation can be established in the presence of an (asymptotically) linear dilaton background (see e.g. [17] and Appendix C of ref.[18]). In fact, consider a generic equation for a "massless" (marginal perturbation) coupling $\psi$ ( $f$ in (1),(2) is an example of such coupling)

$$\bar{\beta} \psi = \beta - \frac{1}{2} \alpha' D^2 \psi + \alpha' D^\mu \phi \partial_\mu \psi = 0 \ , \ \beta = \beta(\psi) = \frac{\partial V}{\partial \psi} . \quad (36)$$

If the $N+1$-dimensional metric and the dilaton are given by $ds^2 = -b dt^2 + ds_N^2 \ , \ \phi = \phi_0 + bt \ (b = \text{const})$ and $\psi$ depends only on $t$ then in the limit of $b \to \infty$ eq.(36) reduces to $\frac{d \psi}{dt} = \beta$ (we set $\alpha' = 1$). If, on the other hand, we use the $N+2$-dimensional metric (1) and the dilaton (26) which is linear in $v$ and assume that $\psi = \psi(u)$ then (36) reduces to (23), i.e. $p \frac{d \psi}{du} = \beta$.

The "1-loop finite" form of the symmetric case metric (1),(2),(30) is particularly simple (we set $a = p = 1$)

$$ds^2 = -2 du dv + u \gamma_{ij}(x) dx^i dx^j . \quad (37)$$

It is closely related to the following $2d$ gravity model

$$S = - \int d^2 z \sqrt{\hat{g}} \ e^{-2\phi} \left[ \hat{R} + 4 (\partial \phi)^2 - \gamma_{ij}(x) \partial_a x^i \partial^a x^j \right] . \quad (38)$$
In the conformal gauge $\hat{g}_{ab} = e^{2\rho} \delta_{ab}$ one can put (33) into the $2 + N$ - dimensional sigma model form by identifying 

$$u = e^{-2\phi}, \quad v = \rho + \frac{1}{2} e^{-2\phi}.$$ (39)

To get a model which is finite to all loop orders one should modify (38) by replacing the coefficient $u$ of the $\sigma$ - model part of (38) by the full $f(u)$.

The fact that $u$ plays the role of an RG “time” parameter (equal to logarithm of a 2d cutoff) which in the case of a covariant regularisation is coupled to the conformal factor of a 2d metric suggest a connection with a different ‘2d quantum gravity – sigma model’ theory

$$S = \int d^2 z \sqrt{\hat{g}} \left[ - v\hat{R} + g_{ij}(\tau, x) \partial_a x^i \partial^a x^j \right].$$ (40)

Here $v(z)$ is a scalar field and the 2d metric in the conformal gauge is

$$\hat{g}_{ab} = e^{2u} \delta_{ab}, \quad u = u(z)$$ (41)

(note that these identifications are opposite to what was assumed in (38),(39)). The argument $\tau$ of $g_{ij}$ in (40) indicates the dependence of the coupling on an RG parameter on a flat 2d background. It is effectively replaced by $u(z)$ once a covariant cutoff is used (see e.g. [19]). Then the statement that the model (36) is finite if $g_{ij}$ depends on $u$ according to the RG equation is similar to what one expects to find in the 2d quantum gravity context (cf.[20,17,19]).

In conclusion, let us mention some particular cases. If the metric $\gamma_{ij}$ is flat (this includes, for example, the general case of $N = 1$, i.e. $D \equiv 2 + N = 3$) then $\beta$ in (14) is zero (but $\beta^G_{uu}$ in (15) is non-trivial). Eq.(23) implies that we should set $p = 0$ ($\dot{f} = 0$ gives a flat space solution). As a consequence, we get a finite and Weyl invariant model for an arbitrary $f(u)$ (with $Q$ and the dilaton $\phi$ being defined by (22) and (26) and $W_\mu = 0$).  

\footnote{10 Though an interpretation of $v$ remains obscure let us mention that the presence of both $u$ and $v$ may be related to the presence of two string world sheet coordinates.}
Note that this $2 + N$ - dimensional space has a conformally flat metric (1) and a non-vanishing curvature (11).

The model with $D = 4$ (i.e. $N = 2$) is represented by (1),(2) with $f(u)$ being the running coupling of the $O(3)$ sigma model (we assume that $k$ in (3) is positive, i.e. the metric is conformal to that of $R^2 \times S^2$). As a result, the geometry of this $D = 4$ space-time is determined by the behaviour of the $\beta$ - function of the $O(3)$ $\sigma$ - model.

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