NORMAL ART GALLERIES: WALL IN - ALL IN

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Abstract. We introduce the notion of a normal gallery, a gallery in which any configuration of guards that visually covers the walls necessarily covers the entire gallery. We show that any star gallery is normal and any gallery with at most two reflex corners is normal. A polynomial time algorithm is provided deciding if, for a given gallery and a finite set of positions within the gallery, there exists a configuration of guards in some of these positions that visually covers the walls, but not the entire gallery.

1. Introduction and main results

An art gallery is a simple polygon (the boundary is a simple closed curve consisting of a finite number of line segments) and a guard is a designated point in the gallery. A guard $G$ in an art gallery $\Gamma$ visually covers every point $A$ in the gallery for which the segment $GA$ is entirely within $\Gamma$ (including the possibility that $GA$ intersects the boundary). A configuration $F$ of guards that visually covers the gallery is a set of points in the gallery such that every point in the gallery is visually covered by at least one of the guards in $F$. It is known that a gallery with $n$ corners can always be visually covered by a configuration of $\lfloor n/3 \rfloor$ guards [7]. The subject of visual coverage of art galleries has developed quite substantially since that paper, for instance by restricting or extending the types of polygons considered (orthogonal polygons [12, 15, 19] or polygons with holes [17, 9, 10]), considering guards that can move [10], restricting the positions that guards can occupy [4], considering combinations of existing variations, exploring higher dimensions, and so on. A survey on the various achievements and directions of study can be found in [20].

We consider the relation between the visual coverage of the walls and the rest of the gallery.

Example 1. It is known that there exists an art gallery $\Gamma$ and a configuration of guards in $\Gamma$ that visually covers the walls of $\Gamma$, but does not visually cover the entire gallery. One such gallery is exhibited in [17, page 4] and we reproduce (a version of) it, denoted by $\Gamma_6$, in Figure 1. The guards $A, B,$ and $C$ visually cover the walls, but none of them covers $D$ (the dashed lines in the figure represent the sight lines of $A, B$ and $C$).

We are interested in conditions under which any configuration of guards that visually covers the walls of a gallery necessarily covers the entire gallery.

Definition 1. An art gallery $\Gamma$ is called normal if any configuration of guards in $\Gamma$ that visually covers the walls of the gallery necessarily covers the entire gallery.

Key words and phrases. art galleries, guards, visibility in polygons.

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Figure 1. The guards $A$, $B$ and $C$ visually cover the walls, but not the entire gallery.

Note that, from a practical point of view, checking if a configuration of guards covers the walls of a gallery may be an easier task than checking if it covers the entire gallery. For instance, there might be situations in which it is easy to implement sensor control (or some other type of control) along the walls, while the access to the interior is restricted in some way. If we know that the gallery is normal and we have a way of monitoring that the walls are visually covered at all times, then, as long as the walls are covered, we are sure that the guards are covering the entire gallery. In such a situation, for as long as the walls are covered, we may even allow relatively free movement of the guards within the gallery without the need to constantly guide them, communicate with them, or even have an information about their location.

The terminology for normal galleries is modeled, albeit superficially, on the terminology used for normal field extensions. Namely, the defining property of normal field extensions is the “one in - all in” property of the roots, while the defining property of the normal galleries is “wall in - all in”.

We provide two (independent) sufficient conditions for a gallery to be normal. Recall that a reflex corner in a gallery is a corner at which the interior angle is greater than 180 degrees.

**Theorem 1.** Every gallery with no more than two reflex corners is normal.

A star gallery is a gallery that can be visually covered by a single guard (more formally, there exists a point $P$ within the gallery such that, for every point $X$ in the gallery, the segment $PX$ is entirely within the gallery).

**Theorem 2.** Every star gallery is normal.

The proof of Theorem 1 is based on the concept of support lines, which is also helpful in establishing the following result.

**Proposition 3.** If the walls of a gallery are visually covered by one or two guards, then these guards cover the entire gallery (even if the gallery is not normal).

Deciding if a given gallery is normal seems to be a nontrivial task. However, the task simplifies if the positions where the guards could be placed are restricted.
**Definition 2.** An art gallery $\Gamma$ is called normal with respect to a set $A \subseteq \Gamma$ if any configuration of guards in $A$ that visually covers the walls of the gallery necessarily covers the entire gallery.

In particular, a gallery $\Gamma$ is normal if it is normal with respect to $\Gamma$. Another special case of interest is when the positions of the guards are restricted to the corners (or any particular finite set of positions within the gallery). In this case there is an algorithm that decides, given as input a gallery, if the given gallery is normal with respect to its set of corners. More generally (and more precisely), the following holds.

**Theorem 4.** There exists an algorithm that decides, given as input a gallery $\Gamma$ with $n$ corners and a set $A$ of $m$ points in $\Gamma$, if $\Gamma$ is normal with respect to $A$. The algorithm runs in $O(m^2 n(m + n) \log m)$ time.

The provided algorithm is based on the decomposition into visibility regions with respect to a finite set of points, which is the preprocessing step in the work of Bose, Lubiw, and Munro [5] on visibility queries. In particular, the complexity estimate is based on the tight bound on the number of certain special regions in the visibility decomposition (called sinks) provided in [5].

### 2. Normal galleries

#### 2.1. Support lines and galleries with no more than 2 reflex corners.

Before we are ready for the proof of Theorem 1 we need to develop the concept of a support line.

For the duration of the present subsection we keep the following setting and notation. Let $\Gamma$ be a gallery that is not normal and let $F$ be a configuration of guards that visually covers the walls of the gallery without covering the entire gallery. Let $R$ be the region in the gallery that is not visually covered by any guard (the hidden region for the configuration $F$). Since the walls of the gallery are visually covered by the guards, the region $R$ is (an open set) in the interior of the gallery and the entire boundary $\partial R$ of $R$, which consists of one or more polygonal lines, is visually covered.

Call the lines that support the boundary segments of the closure $\overline{R}$ the support lines of $R$. Since each boundary segment of $\overline{R}$ is visually covered by the guards, while $R$ is not, there must be a guard on each support line.

Each connected component of $R$ is the interior of a polygon and must have at least three boundary segments.

**Example 2.** Note that the hidden region $R$ does not have to be connected. For instance, the gallery on the left in Figure 2 is not normal (guards are placed at 1,2,3,4,5) and the hidden region has two components.

The same example shows that the boundary of the hidden region $R$ may in parts coincide with the boundary of the gallery, that some support lines may have more than one guard (2 and 3 are on the same support line, as well as 3 and 4), and that some guards may be on more than one support line (guard 3).

**Lemma 5.** The closure of every connected component of the hidden region is a convex polygon.

**Proof.** Assume otherwise. Let $C$ be a reflex corner of the closure at the intersection of the boundary segments $AC$ and $BC$ of the connected component $R_0$ of the hidden
region, and let \( G_1 \) be a guard on the support line of the side \( AC \) visually covering the side \( AC \) (see Figure 3). The sight line from \( G_1 \) towards \( C \) cannot be interrupted at \( C \), since the immediate neighborhood of \( C \) on the line \( G_1C \) and inside the region \( R_0 \) is inside the polygon. This would imply that this sight line extends inside \( R_0 \), a contradiction.

Even though different support lines may share the same guard, this may not happen if the support lines come from the same connected component of the hidden region.

**Lemma 6.** No guard can be on two different support lines bounding the same connected component of the hidden region.

**Proof.** Assume otherwise. Let \( R_0 \) be a connected component of the hidden region \( R \) and let \( G \) be a guard on the lines supporting the boundary segments \( AB \) and \( CD \) of \( R_0 \) (see Figure 4). Since \( R_0 \) is convex the segment \( AD \) is entirely within \( R_0 \), except for the endpoints. No part of the walls of \( \Gamma \) can be in the interior of the triangle \( GAD \) (since each of the three sides is within the polygon), which shows that \( G \) has an unobstructed view of some part of the region \( R_0 \), a contradiction. 

The following is an immediate corollary (note that Proposition 3 is just a re-statement of this corollary).

**Corollary 7.** Let \( \Gamma \) be a gallery that is not normal. Any configuration of guards in \( \Gamma \) that covers the walls, but not the entire gallery, must have at least three members.
Proof. Let $\Gamma$ be a gallery that is not normal and assume that a configuration of guards that visually covers the walls but not the entire gallery is given.

Since the closure $\overline{R}_0$ of any connected component $R_0$ of the hidden region $R$ has at least 3 sides, each support line contains a guard, and no guard is on two support lines corresponding to the same connected component, there are at least three guards in the given configuration. □

Proof of Theorem 1. Let $\Gamma$ be a gallery that is not normal and $F$ a configuration of guards in $\Gamma$ that visually covers the walls but not the entire gallery. Consider a connected component $R_0$ of the hidden region $R$.

For each support line $\ell$ supporting a boundary segment $s$ of $R_0$ let $G_\ell$ be a guard on $\ell$ that visually covers $s$ (if there is more than one such guard select any of them). Recall that there are at least three different support lines and that the guards chosen on different support lines must be different.

The reason the guard $G_\ell$ cannot see the hidden region $R_0$ must be a presence of a reflex corner $C$ between $G_\ell$ and the boundary segment $s$ of $R_0$ that is visually covered by $G_\ell$ such that the region $R_0$, and the two neighboring corners $A$ and $B$ of $C$ are all on the same side of $\ell$ (including the possibility that one of $A$ and $B$ is on $\ell$) as indicated in Figure 5.

Figure 4. No guard is on two support lines

Figure 5. Reflex corner obstructs the guard’s view
Since $R_0$ is convex, there can be only one more support line of $R_0$ passing through $C$, call it $\ell'$, but the view of the guard $G_{\ell'}$ on $R_0$ cannot be obstructed by the corner at $C$ since $R_0$ is not on the same side of $\ell'$ as $A$ and $B$.

Therefore, a different reflex corner obstructs guards on different support lines, and since there are at least three such lines (and guards), there are at least three reflex corners in $\Gamma$. □

2.2. Star galleries and covers by convex polygons.

Proof of Theorem 2. Let $\Gamma$ be a star gallery and $S$ be a point from which the entire gallery is visible. Assume that a configuration of guards that visually covers the walls of $\Gamma$ is given.

We will show that for every point $W$ on the walls, there exists at least one guard that visually covers the entire segment $SW$, which will show that the guards visually cover the entire gallery.

Given a point $W$ on the walls, let $G$ be a guard that visually covers it. Since both $S$ and $W$ are visible from $G$ and the entire segment $SW$ is within $P$, the entire segment $SW$ is visible from $G$. □

It is tempting to conjecture that the converse of Theorem 2 is true, but the next example shows that it is not.

Example 3. Consider the gallery on the right in Figure 2. It is not a star gallery, but it is normal. Indeed, any configuration of guards that visually covers the points 1, 2, 3, and 4 covers the entire gallery.

The last example can be placed in a larger context. For a point $A$ in a gallery, call the set of points visible from $A$ the view of $A$.

Proposition 8. If $\Gamma$ is a gallery in which there exist a set $S$ of points on the walls such that the points in $S$ have convex views and the union of these views covers the entire gallery, then $\Gamma$ is normal.

Proof. This is clear since any guard that visually covers a point on the wall with convex view belongs to this view and visually covers it. □

Remark 1. We point out that there is a polynomial time algorithm that, given a gallery $\Gamma$ with $n$ corners, decides if there exist a set $S$ of points on the walls such that the points in $S$ have convex views and the union of these views covers the entire gallery. The main point to observe is that, for a given non-convex gallery, it is sufficient to check the convexity of the views of the points in the set $S'$ consisting of the non-reflex corners on edges adjacent to reflex-corners and the midpoints of edges that are adjacent to two reflex corners, and then consider the union of the views of the points in $S'$ that have convex views. Details will be provided in a subsequent work.

By using Proposition 8 one may easily show that there is no upper bound on the number of guards needed to visually cover normal galleries nor on the number of reflex corners for such galleries. Indeed, there are spiral, right-angled galleries (in the spirit of the one shown on the right in Figure 2 but with more and more “turns”) that require an arbitrarily large number of guards and have an arbitrarily large number of reflex corners, while still being normal.
The minimal number of guards that visually cover $\Gamma_6$ is 2. Since all star galleries are normal, $\Gamma_6$ is, with respect to the number of guards that can visually cover it, minimal among the galleries that are not normal. The next example shows that $\Gamma_6$ is not minimal in a different sense.

**Example 4.** The gallery $\Gamma_8$ in Figure 6 is not normal, since guards at corners 4, 5 and 8 visually cover the walls of $\Gamma_8$, but do not cover the entire gallery. Indeed,

![Figure 6](image)

none of them covers the points in the (open) triangular region $H_{4,5,8}$ (the region is bounded by the dashed lines). Note that $\Gamma_8$ can be decomposed into three convex polygons. On the other hand, $\Gamma_6$ can be decomposed into 4, but not fewer than 4, convex polygons (none of the points A, B, C, D are visible from each other, showing that no two of them belong to a convex subset of $\Gamma_6$). Since it is clear that galleries that can be decomposed into two convex polygons are normal (in fact, they are star galleries; any point on the boundary between the two convex polygons in the decomposition visually covers the entire gallery), we see that $\Gamma_8$ is, with respect to the number of convex polygons needed for its decomposition, minimal among the galleries that are not normal.

Note also that $\Gamma_6$ and $\Gamma_8$ are minimal among galleries that are not normal both with respect to the number of reflex corners (three) and the number of guards required to visually cover the walls but not the entire gallery (three).

3. **Galleries normal with respect to a set**

We begin by providing an example that shows that a gallery may be normal with respect to a set without being normal.

**Example 5.** The gallery $\Gamma_9$ in Figure 7 is normal with respect to its corners, but it is not normal.
Observe that any corner guard that can see corner 9 (any of 1, 2, 8 or 9) can see the entire trapezoid 1289, any corner guard that can see the points near corner 8 on the wall between 8 and 7 (any of 2, 3, 7 or 8) can see the entire trapezoid 2378, and any corner guard that can see corner 4 (any of 3, 4, 5, 6, or 7) can see the entire trapezoid 4567. Since the trapezoids 1289, 2378, and 4567 cover the entire gallery, this gallery is normal with respect to its corners.

On the other hand, guards at G, 6, and 9 visually cover the walls without covering the entire gallery.

Before we provide the polynomial time algorithm announced in Theorem 3 let us point out a rather simple approach that leads to an exponential time solution. Namely, for a given point P in a finite set A of m points in a gallery with n corners, there exists an algorithm of time complexity O(n) that determines the view of P (the first such algorithm for finding the visibility polygon of a point was given by ElGindy and Avis [9]; see also the work of Lee [14], and Joe and Simpson [11]). The same algorithm can be easily modified to determine, in the same time, the portion of the walls that can be seen from P. Thus, in time O(mn) we may determine both the view and the wall view of every point in A. At this moment, a straightforward approach would be to check (potentially) all $2^m - 1$ nonempty subsets of A in search of a subset that visually covers the walls without covering the entire gallery, but this is certainly not an approach we want to follow. The search time for the right subset of A can be reduced considerably by utilizing the fact that the gallery is simple and planar, which reduces the number of possible candidates to polynomially many, and can be reduced even further by considering only a certain class of candidates (corresponding to minimal visibility regions).

3.1. Decomposition into visibility regions. Let $\Gamma$ be a gallery with n corners and A a set of m points within the gallery. The results and notions we are using are based on the work of Bose, Lubiw, and Munro [5].

A pair of points $(P, C)$ is called feasible if $P \neq C$, P is a point in A, C is a reflex corner that is visible from P, and the two walls meeting at C are in the same half-plane with respect to PC (by definition, each of the two half-planes includes the line PC). For each feasible pair $(P, C)$, draw the segment $CW_{P,C}$ on the line
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$PC$, where $C$ is between $P$ and $WP_C$ and $WP_C$ is the furthest point on the walls of $\Gamma$ visible from $P$. Following [3], the segment $CW_{P,C}$ is called the window of $P$ with base $C$.

**Example 6.** Consider the gallery $\Gamma_8$ in Figure 6 and let $A$ be the set of corners. The point $W_{3,7}$ is clearly indicated in the figure. Further, $W_{3,7} = W_{4,7} = 1$, $W_{5,6} = 2$, and so on.

As another example, consider the gallery $\Gamma_9$ in Figure 7. In this gallery $W_{8,3} = G$, the pair $(4,8)$ is not feasible because 8 is not visible from 4, and the pair $(6,3)$ is not feasible because 2 and 4 are on different sides of the line 6-3.

The collection of all windows together with the gallery walls, decomposes the gallery $\Gamma$ into a finite number of regions called visibility regions with respect to $A$. The decomposition is called the visibility decomposition of $\Gamma$ with respect to $A$. Since there can be no more than $mn$ feasible pairs the number of regions into which $\Gamma$ is decomposed is $O((mn)^2)$. However, using the planarity of the structure and a more careful analysis, Bose, Lubiw, and Munro give a better estimate (they also provide examples showing that their estimate is sharp).

**Theorem 9** (Bose, Lubiw, Munro [5], Theorem 7). The number of regions in the visibility decomposition of a gallery $\Gamma$ with $n$ corners with respect to a set $A$ of $m$ points is $O(m^2n)$.

The significance of the regions in the visibility decomposition for our purposes is derived from the following observation.

**Lemma 10** (Bose, Lubiw, Munro [5], Lemma 19). Any two points in the interior of any visibility region in the visibility decomposition of $\Gamma$ with respect to $A$ can be seen by the same subset of points in $A$.

**Example 7.** The visibility regions with respect to the corners of $\Gamma_8$ are indicated in Figure 6. The boundaries of the regions are represented by dashed lines. The notation $H_{4,5,8}$ is used as an indicator that this is the region that is hidden from the corners 4, 5, and 8 (and visible from any other corner).

3.2. **Sinks.** We define some special regions, called sinks, in the visibility decomposition. These regions have minimal visibility with respect to $A$ and play a crucial role in the work of Bose, Lubiw, and Munro. As in [5], we assume that no three distinct points chosen among the corners of the gallery and the set $A$ are collinear (in fact, it is sufficient to assume that no two windows are collinear).

For any region $R$ in the visibility decomposition, let $V(R)$ be the subset of points in $A$ that visually cover the interior of the region $R$ (note that, by Lemma 10, a point in $A$ either visually controls the entire interior of the region $R$ or it does not control any point in it).

**Definition 3.** Let $\Gamma$ be a gallery with $n$ corners and $A$ a set of $m$ points within $\Gamma$. A region $R$ in the visibility decomposition of $\Gamma$ with respect to $A$ is called a sink if, for every region $R'$ that shares a common boundary edge with $R$, the set $V(R')$ contains the set $V(R)$.

The dual graph to the visibility decomposition is defined as follows. Every region in the decomposition is represented by a vertex and a directed edge from the vertex representing $R$ to the vertex representing $R'$ is placed whenever $R$ and $R'$ share a
common boundary edge and $V(R)$ contains $V(R')$. The non-collinearity condition ensures that the interiors of the regions $R$ and $R'$ can be seen by the same subset of points in $A$ except for a single point (the set $V(R)$ has one point more than $V(R')$ and this is the point $P$ defining the window $PW_{P,C}$ supporting the common boundary edge of $R$ and $R'$). Thus, any two vertices representing adjacent regions in the decomposition are also adjacent in the dual graph, the graph is acyclic, the sinks are precisely the regions corresponding to graph theoretic sinks in the graph (vertices without outgoing edges), and no two sink regions share a boundary edge [5]. The dual graph leads, in the work of Bose, Lubiw, and Munro, to a structure that can be used to recover the sets $V(R)$ for any visibility region $R$ without keeping the full information for every region in the memory ($V(R)$ is only memorized for the sinks). A particularly useful fact about sinks leading to good time complexity estimates in their work, as well as in ours, is their low count (compared to the number of all regions).

**Theorem 11** (Bose, Lubiw, Munro [5], Theorem 8). The number of sinks in the visibility decomposition of a gallery $\Gamma$ with $n$ corners with respect to a set $A$ of $m$ points is $O(m(m + n))$.

### 3.3. The algorithm.

The idea of the algorithm is simply to construct the visibility decomposition and then, for each sink region $R$ in the decomposition, consider the set $\overline{V}(R)$, the complement of $V(R)$ in $A$, consisting of the vertices in $A$ for which $R$ is hidden, and check if guards placed at all points in $\overline{V}(R)$ visually cover the walls.

We first prove that it is indeed enough to check the sets $\overline{V}(R)$ only for sink regions $R$.

**Lemma 12.** A gallery $\Gamma$ is not normal with respect to a finite set $A$ if and only if there exists a sink region $R$ in the visibility decomposition of $\Gamma$ with respect to $A$ such that guards placed at all points in $\overline{V}(R)$ visually cover the walls of the gallery.

**Proof.** If such a sink region $R$ exists, the guards placed at all points in $\overline{V}(R)$ visually control the walls and none of them controls any point inside $R$, which, by definition, implies that $\Gamma$ is not normal with respect to $A$.

Assume that $\Gamma$ is not normal with respect to $A$. Then there exists a configuration of guards at some subset $F$ of $A$ that visually controls the walls, but not the entire gallery. This means that there exists an open set within $\Gamma$ that is hidden to all points in $F$. This open set must contain an interior point of some visibility region $R$ in the visibility decomposition of $\Gamma$ with respect to $A$. Since none of the points in $F$ visually controls $R$, $F$ is a subset of $\overline{V}(R)$. If we enlarge $F$ to $\overline{V}(R)$, guards placed in all points in $\overline{V}(R)$ still have no visual control of the region $R$ and they still control the walls (since the guards in $F$ already do). If $R$ is a sink we are done. Otherwise, there is an adjacent region $R'$ such that $V(R') \subset V(R)$ and, consequently, $\overline{V}(R) \subset \overline{V}(R')$. We may then enlarge the configuration of guards to the set of all points in $\overline{V}(R')$. This enlarged configuration of guards has no visual control of the region $R'$, but still controls the walls (since the guards in $\overline{V}(R)$ already do). If $R'$ is a sink we are done. Otherwise we continue the procedure by moving to an adjacent region $R''$ with an even larger set $\overline{V}(R'')$. The finiteness of the visibility decomposition (the finiteness of $A$) and the fact that we are adding guards at every step ensures that this procedure must eventually end in a sink region.

$\square$
Proof of Theorem 4. The input of the algorithm is $\Gamma$, a gallery with $n$ walls, and $A$, a set of $m$ points within the gallery. The corners of $\Gamma$ and the points in $A$ are given by their coordinates in the plane. The corners are given in a sequence that indicates their order along the boundary of the gallery (say counterclockwise). The output is NORMAL, if the gallery $\Gamma$ is normal with respect to $A$, and NOT NORMAL, otherwise. The algorithm proceeds through three steps (phases) described below, along with time estimates. The steps are sequential, so the time complexity of the entire algorithm is equal to the maximal time complexity of the individual steps.

**Step 1:** Construct the visibility decomposition, identify the sink regions, and calculate $V(R)$ for each sink region $R$

This step is, essentially, the preprocessing step described in Section 4 in [5]. It uses the visibility polygon algorithm (ElGindy and Avis [9], Lee [14], Joe and Simpson [11]) and the Bentley-Ottmann Algorithm for segment intersections [2] (see [8, Chapter 2] for a more current treatment). As indicated in [5, Theorem 9], this step takes $O(m^2(m + n) \log n)$ time.

**Step 2:** Determine the portion of the wall covered by each point in $A$

For every point $P$ in $A$ determine the portion $W(P)$ of the gallery walls that is visually covered by $P$. The output $W(P)$ should be given as follows. Fix a corner of the gallery, and identify the boundary of the gallery with the interval $[0, L]$, where $L$ is the total length of the gallery walls, $0$ represents the chosen corner, and the point $x$ in the interval $[0, L]$ represents the point at distance $x$ along the walls as they are traversed in the counterclockwise direction. The portion of the walls $W(P)$ visible to $P$ is represented as the union of $O(n)$ subintervals of the interval $[0, L]$. Moreover, the visibility polygon algorithm provides this union in a sorted manner, i.e., $W(P)$ consists of $O(n)$ subintervals of $[0, L]$ that are given as a sorted list of endpoints. Since there are $m$ points in $A$ and the visibility polygon algorithm requires $O(n)$ time for each point in $A$, this step can be completed in $O(mn)$ time.

**Step 3:** For each sink region $R$, check if $V(R)$ controls the walls

Consider a fixed sink region $R$. The set $V(R)$ can be determined in $O(m)$ time (since $A$ has $m$ elements and $V(R)$ is already known from Step 1). Consider the union $\bigcup_{P \in V(R)} W(P)$. We need to check if this union covers the entire interval $[0, L]$. Each of the $O(m)$ wall portions $W(P)$, for $P$ in $V(R)$, comes as a sorted list of $O(n)$ subintervals of $[0, L]$. All these lists can be merged in time $O(mn \log m)$ into a single sorted list of $O(mn)$ endpoints (for each endpoint one only needs to keep its value in the interval $[0, L]$ and the information if it is a left or a right endpoint; all endpoints are sorted by their value on $[0, L]$, and for endpoints that have the same value the left endpoints are considered smaller than the right endpoints). At this point, we may use Klee’s algorithm [13] (see also [18, Chapter 8]) that determines the measure of the union of intervals in linear time with respect to the number of intervals, as long as the endpoints are already presorted (this is precisely why we first perform the merge sort indicated above). Thus, after the presorting is completed, it may be checked in $O(mn)$ time if $\bigcup_{P \in V(R)} W(P)$ is equal to $[0, L]$, i.e., if guards placed at all points in $V(R)$
cover the walls of the gallery. Since there are $O(m(m + n))$ sink regions, the merging of the intervals takes $O(mn \log m)$ time and Klee's algorithm takes $O(mn)$ time, this whole step takes $O(m(m + n))O((mn \log m) + mn) = O(m^2 n(m + n) \log m)$ time.

If, for some sink region $R$, the union $\bigcup_{P \in \mathcal{V}(R)} W(P)$ turns out to be the whole interval $[0, L]$ stop and report NOT NORMAL. Otherwise, after all sink regions are checked, stop and report NORMAL.

\[\square\]

**Remark 2.** Since the time complexity of the algorithm is carried by Step 3, there is no need to try to improve the bounds in Step 1 by replacing the Bentley-Ottmann Algorithm by any of the faster algorithms for segment intersections and map overlays such as Chazelle-Edelsbrunner \[6\] or Balaban \[1\]. The time complexity of Step 3 is only affected by the number of sink regions and the complexity of the merge sort of several presorted lists. Since the estimate $O(m(m + n))$ for the number of sink regions is shown to be sharp in \[5\], it seems that there is not much room for improvements (unless an entirely different approach is taken).

4. **Concluding Remarks**

In this work, we defined the notion of a normal gallery, a gallery in which every configuration of guards that visually controls the walls necessarily controls the interior. We established several sufficient conditions for a gallery to be normal and provided an algorithm, running in polynomial time, that checks if a given gallery is normal with respect to a given finite set of positions within the gallery.

We mention some natural follow up problems/questions.

Given that a full characterization of normal galleries seems to be a difficult task, there are two ways to proceed. One is to replace the attempt to characterize by an attempt to find good sufficient and/or necessary conditions and the other is to limit the domain on which a characterization is sought. We briefly discuss both approaches.

4.1. **Sufficient conditions.** As mentioned in the introduction, there are situations in which checking if a configuration of guards visually covers the entire gallery may be impractical, while checking if they cover the walls may be relatively easy to implement. If we know ahead of time that the gallery is normal, we know that it is actually sufficient to check and ensure only the visual coverage of the walls. This motivates a search for a wide range of sufficient conditions for normality that are simple to verify (three such simple sufficient conditions are already given here in Theorem \[1\] Theorem \[2\] and Proposition \[8\]).

4.2. **Normality in classes of galleries.** Since all galleries with at most two reflex corners are normal, and some galleries with three reflex corners are not normal, a natural place to start is to try to characterize the class of normal galleries with exactly three reflex corners.

In case a sufficiently simple and useful characterization is not possible, one may try to find an algorithm that decides, given a gallery with three reflex corners as input, if the given gallery is normal.
Similarly, since all galleries that can be decomposed into two convex parts are normal, one may try to characterize the normal galleries that can be decomposed into three convex parts.

Clearly, such questions (characterization and/or algorithmic decidability of normality) may be posed within any other well defined class of galleries.

4.3. Size and location of witness sets. Assume a gallery $\Gamma$ with $n$ corners is not normal. Call the smallest size of a configuration of guards that visually covers the walls, but not the entire gallery, the witness set size of $\Gamma$, and denote it by $wss(\Gamma)$. Corollary 4 states that, for every non-normal gallery $\Gamma$, $wss(\Gamma) \geq 3$. It would be interesting to establish upper bounds on the maximum witness set size

$$wss(n) = \max\{wss(\Gamma) \mid \Gamma \text{ a non-normal gallery with } n \text{ corners}\}.$$ 

in non-normal galleries in terms of the number of corners $n$.

It is desirable to limit the search for witness sets to particular regions or even particular locations, if and when possible. For instance, is it true that every non-normal gallery has at least one witness set within the gallery walls? Note that Example 5 shows that the search for a witness set cannot be limited to the corners of the gallery. On the other hand, it would be interesting to effectively characterize the non-normal galleries that have witness sets within the set of corners.

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