Quantifying Entanglement in Cluster States Built with Error-Prone Interactions

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Measurement-based quantum computing is an alternative paradigm to the circuit-based model. This approach can be advantageous in certain scenarios, such as when read-out is fast and accurate, but two-qubit gates realized via inter-particle interactions are slow and can be parallelized to efficiently create a cluster state. However, understanding how two-qubit errors impact algorithm accuracy and developing experimentally viable approaches to characterize cluster-state fidelity are outstanding challenges. Here, we consider one-dimensional cluster states built from controlled phase, Ising, and XY interactions with slow two-qubit error in the interaction strength, consistent with error models of interactions found in a variety of qubit architectures. We detail an experimentally viable teleportation fidelity that offers a measure of the impact of these errors on the cluster state. Our fidelity calculations show that the error has a distinctly different impact depending on the underlying interaction used for the two-qubit entangling gate. In particular, the Ising and XY interactions can allow perfect teleportation through the cluster state even with large errors, but the controlled phase interaction does not.

I. INTRODUCTION

Cluster states are entangled quantum many-body states of matter that can serve as resource states for measurement-based quantum computing (MBQC) [1, 2]. The MBQC formalism shows that any quantum algorithm can be executed with properly arranged measurements of a cluster state. Cluster states are also symmetry protected topological phases [3–6]. As such, certain Hilbert space sectors of cluster states offer robust routes to store and process information.

MBQC with cluster states can offer advantages over the usual circuit-based approach to quantum information processing. In systems with slow and possibly error-prone two qubit gates, the MBQC formalism front-loads the burden of executing two-qubit gates to the initial phase of the algorithm. Furthermore, all two-qubit gates can be run in parallel (at the same time) to quickly build the cluster state. Single-qubit measurements then execute the algorithm and therefore avoid the later use of two-qubit gates. The MBQC approach works best in systems with a large number of available qubits that can be entangled in parallel into a cluster state while allowing subsequent fast and accurate single-qubit measurements.

Observable properties of cluster states depend on the level of entanglement left over after error-prone gates are used to create them. Characterization of entanglement in an N-qubit many-body state with full quantum state tomography has been done with small cluster states built from entangled photons [7, 8] and with ions [9]. But, in general, full quantum state tomography scales exponentially with N and is well known to be prohibitive.

Teleportation fidelity offers a route to measure entanglement between qubits [10]. Teleportation measures of fidelity typically envision operations on a pair of well-separated qubits. But teleportation through a cluster state is different. The process was originally introduced [11, 12] as a logical identity gate where information encoded in one end of the cluster state is teleported via measurement to the other side. It passes information within a many-body state of matter in its entirety, engaging all qubits in the measurement. The process of measurement-induced teleportation relies on entanglement in the cluster state itself and is therefore an implicit test of cluster state entanglement. As a result, several theoretical studies have examined the interplay of teleportation and errors in cluster states [11, 13] where thresholds [14–17] for teleportation along a quantum channel (as opposed to classical transmission of information) were studied. Recent works have used teleportation to test the ability of symmetry-protected topological order in cluster states to protect against errors (see, e.g., Refs. [5, 6]).

Cluster states are built from two-qubit entangling gates. The most compact gate set used to construct cluster states utilizes the controlled phase gate [1, 2]. This two-qubit gate has been constructed as a composite gate (built from other gates or operations) between ion [18, 19], superconducting [20, 21], and photonic [22] qubits, which show considerable promise in MBQC [22, 23]. But the controlled phase gate also arises from physical interactions between particles used as qubits, e.g., neutral atoms with Rydberg excitations [24–28] or controlled collisions in optical lattices [29, 30], that do not rely on a composite two-qubit gate construction. Other two-qubit entangling interactions, the Ising interaction [1, 2] and the XY (conditional phase flip) interaction [31] can also be used to efficiently create cluster states directly from inter-particle interactions. The Ising interaction characterizes the

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Mølmer-Sørenson gate between ion qubits \cite{19, 32, 33}, the interactions between NMR qubits \cite{36}, and the interaction between superconducting charge qubits \cite{37, 38}. And the XY interaction characterizes the entangling interaction between qubits formed from rotational states of polar molecules \cite{39–41}, quantum dots in cavities \cite{42}, and other types of superconducting qubits \cite{38}.

We consider the intertwined obstacles of growing and diagnosing entanglement in cluster states from the point of view of challenges and goals in the laboratory setting. Moving from a few qubits to several requires a growth and measurement protocol that avoids costly $N$-qubit quantum state tomography along with a roadmap to mitigate errors. Adding to the complexity, we find that the impact of errors on entanglement depends on the physical two-qubit gate used to build the cluster state. To tackle these obstacles, we study cluster states built from three types of error-prone interactions: controlled phase, Ising, and XY, because these interactions are commonly used to characterize the physical (native) interaction between particles in many architectures and they lead directly to well known entangling gates.

While all of the above architectures have a variety of error and noise sources, here we focus on slow errors in the native two-qubit coupling. Two qubit gates tend to have a lower fidelity than the single-qubit gates \cite{43}. We therefore ignore single qubit errors as a first approximation. Furthermore, in many qubit platforms, particularly those formed from atoms and molecules, fluctuations in control fields, e.g., laser power or orientation, perturb interactions thereby causing slow two-qubit gate errors. For example, consider two qubits defined by rotational states of polar molecules trapped in optical tweezers. Slow relative perturbations to the position of the laser focal point will perturb the dipolar interaction and therefore lead to two-qubit gate errors \cite{39–41}. In general, we consider two-qubit interactions of strength $J$, applied for a time $t$, with unknown errors $\epsilon$, such that the gate strength is perturbed by an error: $Jt \rightarrow Jt(1 + \epsilon)$, where $\epsilon$ does not depend on time for a single application of a two-qubit gate (errors are slow on time scales of the interaction energy). We allow for the possibility that $\epsilon$ changes from gate to gate.

We construct protocols for experiments to use teleportation fidelity as a low-cost, $O(N)$, operation to benchmark the impact of errors on entanglement in cluster states \cite{11, 12}. We numerically test teleportation fidelity benchmarking on cluster states to predict what experiments should see. We focus on cluster state chains but our results apply to two and three dimensions without loss of generality because analogues of the logical identity gate apply to higher dimensions \cite{1, 2}. The fidelity measures the ability of measurements performed to teleport information from an input qubit on one end of the cluster state chain to the other. Figure \ref{fig:1} shows a schematic of the process used to measure the fidelity. By using this process as a fidelity, we diagnose the entanglement in the cluster state, in so far as it can be used in teleportation. The fidelity defined here generalizes to higher dimensional cluster states, but we will focus on one dimension because it requires the lowest number of qubit resources.

To define the fidelity we denote the one-dimensional cluster state wavefunction containing $N−1$ qubits by $|\Phi_C⟩$. We then consider a single input qubit, $|\psi_{\hat{r}}⟩$, that is prepared and entangled with $|\Phi_C⟩$ to create a combined $N$-qubit state: $|\psi_{\hat{r}}, \Phi_C⟩$, where $\hat{r}$ is a unit vector pointing to a location on the Bloch sphere. The fidelity measures the ability of measurements performed on the cluster state to move information stored in $\hat{r}$ along the chain.

To move the information along the chain, a series of measurements act on $|\psi_{\hat{r}}, \Phi_C⟩$ to effectively teleport. The initial $N$-qubit wavefunction is measured to create a post-measurement state. Assuming measurement outcomes are recorded, we can write the final outcome of measurements as:

$$P(M_{\hat{r}'}|P(M_{\hat{r}})P(M_{\hat{r}}^{N−1})|\psi_{\hat{r}}, \Phi_C⟩,$$

II. CLUSTER STATE TELEPORTATION, FIDELITY, AND ENTANGLEMENT

We define the cluster state fidelity using the MBQC identity gate \cite{1, 2, 11, 12}. The MBQC identity gate relies on measurements to teleport information from an input qubit on one end of the cluster state chain to the other. Figure \ref{fig:1} shows a schematic of the process used to measure the fidelity. By using this process as a fidelity, we diagnose the entanglement in the cluster state, in so far as it can be used in teleportation. The fidelity defined here generalizes to higher dimensional cluster states, but we will focus on one dimension because it requires the lowest number of qubit resources.

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$$P(M_{\hat{r}'}|P(M_{\hat{r}})P(M_{\hat{r}}^{N−1})|\psi_{\hat{r}}, \Phi_C⟩,$$
where measurements along the qubit-x direction on the first $N-1$ qubits are defined by the projectors: $\mathcal{P}(M_{x}^{N-1}) = \prod_{i=1}^{N-1} (\sigma_{x} + (-1)^{s_{i}} \sigma_{z})/2$, and the measurement along the qubit-$\tilde{r}'$ direction on the final $N^{th}$ qubit is defined by the projector: $\mathcal{P}(M_{\tilde{r}'} = (\sigma_{x} + (-1)^{s_{N}} \sigma_{z})/2$. Here, the integers $s_{i} = 0, 1$ are the eigenvalues that result from the measurement of the $i^{th}$ qubit and $\sigma_{i} = (\sigma_{x}^{i}, \sigma_{y}^{i}, \sigma_{z}^{i})$ is a vector of the usual Pauli matrices for qubit $i$. In what follows, $\sigma_{0}^{i}$ denotes the identity matrix for qubit $i$.

Repeating this process allows quantum state tomography on the final $N^{th}$ qubit to yield the single-qubit output density matrix, $\rho_{O}$. The output qubit orientation should be identical to the chosen input orientation $\tilde{r}$, so that we retrieve $(\sigma_{0} + \tilde{r} \cdot \tilde{\sigma})/2$ for $\rho_{O}$. But there are two caveats. The first caveat is that errors lead to a reorientation of the output qubit. The role of two-qubit errors will be discussed in later sections.

The second caveat is well known in usual teleportation schemes. Even in the absence of errors, the measurement process inherently randomizes the measurement basis needed to interpret output on the last qubit. A classical channel is needed to feedforward and interpret the output measurement. MBQC uses measurement outcomes to construct a byproduct matrix (made from a combination of Pauli operations) that gives the correct basis in which to interpret the measurements on the final qubit. Specifically, we record all measurement outcomes, $s_{i}$ on the first $N-1$ qubits. The $s_{i}$ are then used to construct the appropriate byproduct matrix, $U_{\Sigma}$.

![Diagram](Image)

**FIG. 1.** Schematic of two stages of the fidelity measurement shown for 5 qubits. The first column depicts preparation of qubit 1 defining the input state, $|\psi_{i}^{1}\rangle$, where information is encoded in the qubit orientation, $\tilde{r}$. This qubit is entangled with the remaining qubits (2-5) that define a cluster state $|\Phi_{C}\rangle$. The resulting state $|\psi_{C}\rangle$ is then measured. The second column depicts the measurement stage where measurement along the qubit-x direction for qubits 1 through 5 effectively moves the information encoded in qubit 1 to qubit 5. Quantum state tomography on qubit 5 allows reconstruction of an output state, $|\psi_{O}\rangle$, that is, in the absence of error, identical to the input, $|\psi_{i}^{5}\rangle$, up to a matrix, $U_{\Sigma}$, defined from measurement results of qubits 1 to 5. An error-free cluster state teleports information along a quantum channel from qubit 1 to qubit 5. Sufficiently strong gate errors will destroy entanglement in the cluster state which can be detected in degraded teleportation. The information recorded in the measurements is used offline to construct the fidelity $F_{r}$ of the cluster state, where $F_{r} > 2/3$ guarantees a quantum channel.

The resulting state $|\psi_{O}\rangle$ is then $\rho_{O} = |\langle \psi_{O} | \psi_{i}^{5} \rangle |^{2}$. We then have $F_{r} \rightarrow |\langle \psi_{O} | \psi_{i}^{5} \rangle |^{2}$.

Equation (3) shows that if the output qubit is aligned along $\tilde{r}$, the process used to define $F_{r}$ effectively executes the logical identity gate in MBQC and we have $F_{r} = 1$ regardless of our choice of $\tilde{r}$. In such a case, the cluster state is perfectly entangled and teleportation occurs along a quantum channel. But in the presence of two-qubit error, the output qubit will not be aligned along $\tilde{r}$ and we might have $F_{r} < 1$ for some or all choices of $\tilde{r}$. $F_{r}$ therefore measures the deviation from the logical identity gate induced by error.

The minimum fidelity aids in quantifying the extent to which error will degrade the fidelity. To benchmark over a random sample of input qubit orientations, we repeat the procedure for various input qubit directions $\tilde{r}$. Some initial orientations $\tilde{r}$ have a lower fidelity than others. The entanglement in the cluster state is ensured to contain a quantum channel $\{1-17\}$ if $F_{r} > 2/3$. For $2/3 \geq F_{r} > 1/2$, the channel could be either quantum or classical. For $F_{r} \leq 1/2$, entanglement in the cluster state has degraded to a point where there is only a classical channel. The best measure of entanglement is then the minimum fidelity found for the worst case $\tilde{r}$, defined to be $\min(F)$.

The maximum fidelity, by contrast, helps locate protected channels. It might be possible to find protected routes of teleportation that are, due to the interplay of symmetry in the cluster state $\{5-6\}$ and the error model, less sensitive to errors than other routes. In Sec. $X$ and $Y$ we show that certain input orientations $\tilde{r}$ allow perfect transmission, i.e., unity fidelity, for certain types of two-qubit errors.

Before closing this section, we mention a generalization of the above fidelity that will be needed in the presence of realistic single and two-qubit errors. In general, single and two-qubit errors will lead to mixed input and output states with density matrices, $\rho_{C}$ and $\rho_{O}$, respectively. Here we have $[2]$: 

$$U_{\Sigma} = \prod_{i=1}^{(N-1)/2} (\sigma_{z}^{2i})^{s_{2i}} (\sigma_{z}^{2i-1})^{s_{2i-1}},$$  

(2)

where the product runs over all measurements except the last qubit. Here we see that the measurement outcomes, $s_{i}$, feedforward into interpretation of the quantum state tomography data on the $N^{th}$ qubit.

Once the results of all measurements are recorded offline, we can construct the cluster state fidelity as the MBQC identity gate $[1-2]$. As above, we assume that the input density matrix is a pure state, so that $\rho_{C}^{i} = |\psi_{C}^{i}\rangle|\langle \psi_{C}^{i} |$ and define the fidelity to be:

$$F_{\tilde{r}} \equiv |\langle \psi_{C}^{i} | U_{\Sigma} \rho_{0} U_{\Sigma} | \psi_{C}^{i} \rangle|^{2},$$

(3)

where $U_{\Sigma}$ is defined in the limit of no error, Eq. (2), to rotate our measurement outcome $\rho_{0}$ so that the output qubit orientation perfectly aligns with $\tilde{r}$ if there is no error. $\rho_{0}$ can, in general, describe a mixed state, but in what follows we will focus only on slow two-qubit gate errors. In the presence of slow two-qubit error, the system stays closed. This leaves the input and output states as pure states, i.e., $\rho_{0} \rightarrow |\psi_{O}\rangle|\langle \psi_{O} |$. We then have $F_{\tilde{r}} \rightarrow |\langle \psi_{O} | \psi_{O} \rangle|^{2}$.

The second caveat is well known in usual teleportation schemes. Even in the absence of errors, the measurement process inherently randomizes the measurement basis needed to interpret output on the last qubit. A classical channel is needed to feedforward and interpret the output measurement. MBQC uses measurement outcomes to construct a byproduct matrix (made from a combination of Pauli operations) that gives the correct basis in which to interpret the measurements on the final qubit. Specifically, we record all measurement outcomes, $s_{i}$ on the first $N-1$ qubits. The $s_{i}$ are then used to construct the appropriate byproduct matrix, $U_{\Sigma}$. For odd $N$, we have $[2]$: 

$$U_{\Sigma} = \prod_{i=1}^{(N-1)/2} (\sigma_{z}^{2i})^{s_{2i}} (\sigma_{z}^{2i-1})^{s_{2i-1}},$$

(2)

where the product runs over all measurements except the last qubit. Here we see that the measurement outcomes, $s_{i}$, feedforward into interpretation of the quantum state tomography data on the $N^{th}$ qubit.

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$$F_{\tilde{r}} \equiv |\langle \psi_{C}^{i} | U_{\Sigma} \rho_{0} U_{\Sigma} | \psi_{C}^{i} \rangle|^{2},$$

(3)
\[ \rho_i^t = (\sigma^0 + r \cdot \hat{\sigma})/2 \text{ and } r \text{ is a Bloch sphere vector with } |r| \leq 1. \] A mixed state fidelity is then given by [47–49]:

\[ \text{Tr} \left( \sqrt{\sqrt{\rho_i^t} U \rho_0 U \sqrt{\rho_i^t}} \right)^2. \]

This generalization of fidelity reduces to the case we study in this paper, Eq. (4), in the limit that the input state is pure, i.e., \( \rho_i^t \rightarrow |\psi_i^t\rangle \langle \psi_i^t| \).

The next sections describe how to build cluster states and measure fidelity. Cluster states can be constructed from many different entangling gates. But we use three different two-qubit gates built from three different types of interaction that we consider to be physical interactions: the controlled phase interaction, the Ising interaction, and the XY interaction. We test fidelity measures on small cluster state chains to examine the impact of interaction errors on the fidelity.

### III. CLUSTER STATES FROM THE ERROR-PRONE CONTROLLED PHASE INTERACTION

The controlled phase interaction between two qubits allows construction of cluster states with the fewest number of operations compared to the interactions considered in the following sections. The controlled phase gate has also been realized with ionic [18, 19, 34], superconducting [20, 21], and photonic [22, 23] qubits. Furthermore, certain physical systems have native controlled phase interactions. Rydberg interactions can directly implement the controlled phase interaction [24, 28]. Also, controlled collisions [29] using neutral atoms in hyperfine state-dependent optical lattices have realized parallel implementation of the controlled phase gate [30].

Even though the controlled phase interaction requires the fewest gates to create a cluster state, we will see that it has trade-offs in response to two-qubit error. In this section, we will see that the controlled phase interaction does not allow trade-offs in response to two-qubit error. In this section, we will see that it has trade-offs in response to two-qubit error. In this section, we will see that it has trade-offs in response to two-qubit error. In this section, we will see that it has trade-offs in response to two-qubit error.

The controlled phase interaction can be used to construct the two-qubit entangling interactions, the controlled phase shift gate. We assume that the physical two-qubit interaction can be characterized by:

\[ H_{ij}^{\text{CP}} = J_{ij}^{\text{CP}} (\sigma_i^0 - \sigma_j^x)(\sigma_j^0 - \sigma_i^x), \]

where \( J_{ij}^{\text{CP}} \) is the interaction energy between qubits \( i \) and \( j \) and is tunable in time. \( H_{ij}^{\text{CP}} \) is useful for studying two-qubit error sources approximated by perturbations in \( J_{ij}^{\text{CP}} \).

To build the cluster state, we start with all qubits aligned along the \( x \)-direction:

\[ |\Phi_0^C\rangle = \prod_i |+\rangle_{x,i}, \]

where \(|\pm\rangle_{x,i} = (|0\rangle_i \pm |1\rangle_i)/\sqrt{2}\) are the eigenstates of \( \sigma_i^x \).

Unitaries constructed from \( H_{ij}^{\text{CP}} \) can then be used to entangle qubits into the cluster state. For the one-dimensional cluster state, we have [1, 2]:

\[ |\Phi_C\rangle = \prod_{<i,j>} U_{\text{CP},i,j}^{\pi/4}|\Phi_0^C\rangle, \]

where \( \langle i,j \rangle \) denotes a product over unique nearest neighbor pairs, i.e., \( (1, 2), (2, 3), \ldots \). Here we have used the controlled phase shift gate (the conditional phase gate) between qubits \( i \) and \( j \):

\[ U_{\text{CP},i,j}^{\pi/4} = e^{-i J_{ij}^{\text{CP}} t (\sigma_i^0 - \sigma_i^x)(\sigma_j^0 - \sigma_j^x)}, \]

where \( h \) is set to unity throughout. In matrix form, we see that the interaction appears in only one entry:

\[ U_{\text{CP},i,j}^{\pi/4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-i J_{ij}^{\text{CP}} t} \end{pmatrix}, \]

Here and in the following, we construct matrices with the two-qubit basis of eigenstates of \( \sigma_i^x \) in the following order: \( \{ |+\rangle_z, |-\rangle_z, |+\rangle_x, |-\rangle_x \} \).

If we set \( \theta = J_{ij}^{\text{CP}} t \) correctly, we can use the controlled phase interaction to realize the controlled-Z gate (controlled-phase flip gate). For \( J_{ij}^{\text{CP}} t = \pi/4 \), we have:

\[ U_{\text{CP},i,j}^{\pi/4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]

To study the impact of weak errors that are slow on the time scales of the two-qubit gates, we assume a static perturbation of the interaction strength. We introduce the dimensionless perturbation \( \varepsilon \) such that

\[ J_{ij}^{\text{CP}} t \rightarrow J_{ij}^{\text{CP}} t (1 + \varepsilon). \]
The fidelity for large chains can be derived using cluster state techniques. We take the initial state to have orientation \( \hat{x} \) on the Bloch sphere (blue points). We therefore see that the error in the fidelity to compute Min \( \varepsilon \). We now turn to numerical simulations to estimate the impact of error on the fidelity.

Errors will degrade the entanglement in the cluster state. To study the role of error on the fidelity, we will assume that the single-qubit gates and all measurements are error-free, thus leaving error in just the controlled phase interaction. We start by assuming that the two-qubit error is the same on all bonds. We will then, as a second step, randomize the two-qubit error as we move from qubit to qubit along the chain.

To quantify the procedure depicted in Fig. 2 we first assume a fixed error \( \varepsilon \) that is the same for all two-qubit gates. We then vary the input qubit orientation, \( \hat{r} \), using a uniform random distribution on the Bloch sphere for fixed \( \varepsilon \). We then find the orientation where the error makes the largest impact on the fidelity to compute Min \( \varepsilon \).

Figure 3 shows an example simulation of the fidelity for an \( N = 5 \) cluster state for two different error strengths. Each point plots a fidelity as measured from the origin. The axes correspond to three respective directions in qubit space for the input qubit, \( \hat{r} \). Here the \( x \), \( y \), and \( z \) axes correspond to three respective directions in qubit space for the input qubit, \( \hat{r} \). The distance of the plotted points from the origin corresponds to the output fidelity, Eq. (3). For low error, e.g., \( \varepsilon = 1/(2\pi) \) (red circles), the output qubit is nearly identical to the input qubit leading to an approximate map of the Bloch sphere with nearly unity radius. The blue circles correspond to a larger error, \( \varepsilon = 1/\pi \), where the fidelity is well below unity. The resulting map of the Bloch sphere is a smaller shape. The resulting shape approximates a sphere but with weak anisotropy.

We set a fixed interaction time using \( \theta = \pi/4 \) in Eq. (11), so that the two-qubit interaction returns the desired controlled-Z gate (and therefore a perfect cluster state) in the absence of error. The error is chosen as a static perturbation of the controlled phase interaction strength: \( \pi/4(1+\varepsilon) \), so that \( \varepsilon \) is the fractional change in the interaction strength. The initial state orientation \( \hat{r} \) is chosen from a uniform distribution on the Bloch sphere to display benchmarking over varied initial states. Here the \( x \), \( y \), and \( z \) axes correspond to three respective directions in qubit space for the input qubit, \( \hat{r} \). The distance of the plotted points from the origin corresponds to the output fidelity, Eq. (3). For low error, e.g., \( \varepsilon = 1/(2\pi) \) (red circles), the output qubit is nearly identical to the input qubit leading to an approximate map of the Bloch sphere with nearly unity radius. The blue circles correspond to a larger error, \( \varepsilon = 1/\pi \), where the fidelity is well below unity. The resulting map of the Bloch sphere is a smaller shape. The resulting shape approximates a sphere but with weak anisotropy.

We can derive closed formulae for the fidelity for small cluster states. We take the initial state to have orientations \( \hat{r} \) with the usual spherical coordinate angles \( \theta_0 \) and \( \phi_0 \) on the Bloch sphere, \( |\psi_j\rangle = |\hat{r}(\theta_0,\phi_0)\rangle = \cos(\theta_0/2)|0\rangle + e^{i\phi_0}\sin(\theta_0/2)|1\rangle \). For \( N = 3 \), we find

\[
F_{\hat{r}|N=3} = 1 - (1 - \sin^2\theta_0 \cos^2\phi_0) \sin^2(\pi\varepsilon/2)/2 - \sin^2(\theta_0/2) \sin^2(\pi\varepsilon)/2.
\]

This expression shows that the fidelity is only slightly anisotropic (nearly mapping out the uniform Bloch sphere). So, even though the minimum fidelity only appears for an initial state corresponding to a single point on the Bloch sphere, the minima can be approximately assumed to be at any angle. The fidelity for large chains can be derived using cluster state refreshing for mathematical induction. Appendix A discusses refreshing that allows us to grow the cluster state by concatenation and therefore obtain analytic expressions for the fidelity for larger \( N \). We do not find a simple closed form for the minimum fidelity for larger \( N \) for the controlled phase interaction, but we see that the Ising and XY interactions yield a simple closed form.

Refreshing and concatenation of qubits (Appendix A) can be used to systematically grow cluster states in experiments starting with the fewest number of qubits as possible. One can use as few as three qubits with concatenation and refreshing to effectively grow the number of qubits used in teleportation. However, in this limit, the measurement-based protocol is effectively the same as a circuit-based scheme to do the same (Appendix A). By systematically increasing the minimum number of qubits used to form the cluster state, e.g., as few as 5 qubits at once, one moves to the limit where teleportation along a cluster state chain begins to differ from the circuit-based scheme.

\[
U_{\text{CP}} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & e^{-i4\theta_0\sin^2(1+\varepsilon)}
\end{pmatrix}.
\]
σ for 7 qubits with orientations as well as two-qubit error configurations. allows a complete benchmark using averaging over the input rations, the calculation of fidelity for randomly selected input orienta-

Figure 4 plots the minimum fidelity as a function of the two-
qubit error strength for several different cluster state chain lengths. The horizontal dashed line plots 2/3. The plot shows that the lowest fidelity is still above the threshold guaranteeing quantum teleportation (2/3) with perturbations to the interaction energy as large as 15%. This assumes a cluster state with no more than 9 qubits. This error is particularly large. Our results for errors in the controlled phase interaction are consistent with the previous results [11, 12] that averaged over all input configurations for the controlled phase interaction. The conclusion here is that one-dimensional cluster states allow a quantum channel for teleportation even for relatively large error strengths. In the following sections, we will see that the Ising interaction allows simple analytic formulas to predict the scaling as we increase the chain length.

We now turn to numerical simulations which are more physically realistic in approximating how an experiment can map fidelity. We assume random errors in all two-qubit gates used to build the cluster state such that ε is sampled from a normal distribution with standard deviation σ = 0.1 that disorders the two-qubit gate along the chain (ε₁, ε₂, ..., ε₇ in Fig. 4 are separate random variables). The input qubit orientation is also randomly chosen, but from a uniform distribution of angles on the Bloch sphere. The process depicted in Fig. 4 is repeated for 7 qubits. The histogram is truncated because the distribution of errors is still small enough to allow cases near the maximum fidelity (F = 1) for certain input states (See discussion in Sec. III). The star indicates the minimum half width at half maximum. Main: Minimum half widths at half maximum of fidelities plotted as a function of the number of qubits for various error distribution widths, σ. The red star plots the same point as in the inset. Fidelities above the dashed line are guaranteed to have used a quantum channel for teleportation. We see that rather large errors are needed to pull the minimum fidelities below the dashed line for these small clusters states.

Figure 5 shows that even for a broad distribution of two- qubit errors σ ≤ 0.15, experiments can, on average, detect teleportation along a quantum channel. This is consistent with robustness found for uniform error in Fig. 4. But we also see that randomized error overall lowers the fidelity as one would expect. The distribution tail in the inset of Fig. 5 even shows that some extreme cases of disorder drawn from our random sample of ε strongly suppresses teleportation.
The matrix form for the Ising unitary is given by

\[
H_{ij}^{ZZ} = J_{ij}^{Z} \sigma_i^z \sigma_j^z \tag{13}
\]

where \( J_{ij}^{Z} \) is the interaction energy between qubits \( i \) and \( j \) and is tunable in time. To build the cluster state, we start with all qubits aligned along the \( x \)-direction \( |\Phi_0^x \rangle \). Unitaries constructed from \( H_{ij}^{ZZ} \) can then be used to build the cluster state:

\[
|\Phi_C \rangle = \prod_{(i,j)} R_{Z,i,j}^{-\pi/2} R_{Z,j,i}^{-\pi/2} U_{ZZ,i,j}^{\pi/4} |\Phi_0^x \rangle, \tag{14}
\]

where

\[
U_{ZZ,i,j}^{\pi/4} = e^{-i J_{ij}^{Z} t} \sigma_i^z \sigma_j^z. \tag{15}
\]

The two qubit interaction can be recast in a matrix form. The matrix form for the Ising unitary is given by

\[
U_{ZZ,i,j}^{\pi/4} = \begin{pmatrix}
e^{-i J_{ij}^{Z} t} & 0 & 0 & 0 \\
0 & e^{i J_{ij}^{Z} t} & 0 & 0 \\
0 & 0 & e^{i J_{ij}^{Z} t} & 0 \\
0 & 0 & 0 & e^{-i J_{ij}^{Z} t}
\end{pmatrix}. \tag{16}
\]

If we set \( \theta = J_{ij}^{Z} t \) to parameterize time in units of the Ising interaction, then \( \theta = \pi/4 \) defines the Ising gate needed to build the cluster state:

\[
U_{ZZ,i,j}^{\pi/4} = e^{-i \pi/4} \begin{pmatrix}1 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & i & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \tag{17}
\]

We see from Eq. (14) that additional single qubit rotations are needed to build the cluster state that were not needed for the controlled phase interaction. A single qubit rotation about the \( z \)-axis is

\[
R_{Z,i}^{\phi} = e^{-i \phi \sigma_i^z/2}, \tag{18}
\]

where \( \phi \) parameterizes time evolution in units of the single qubit control field energy.

\[
F_{r=\pm y} = 1, \tag{21}
\]

**IV. CLUSTER STATES FROM THE ERROR-PRONE ISING INTERACTION**

We now turn to cluster state chains built from the Ising interaction. The Ising interaction characterizes NMR-based qubits \[36\] as well as superconducting charge-based qubits \[37, 38\]. The Ising gate also characterizes the Mølmer-Sørensen gate between ions \[19, 32\].

This section will show that the cluster state and measurement process for the Ising interaction are nearly identical to the same procedure as discussed for the controlled phase interaction in Sec. [III].

We therefore might expect that the cluster state fidelity responds in the same way. We find similarities, but also find considerable differences in the fidelity and in re-focusing schemes discussed later in Sec. [IV].

We first revisit the protocol introduced in Refs. [1][2] to construct the cluster state from the Ising interaction. We assume that the physical two-qubit interaction is given by

\[
H_{ij}^{ZZ} = J_{ij}^{Z} \sigma_i^z \sigma_j^z, \tag{13}
\]

where \( J_{ij}^{Z} \) is the interaction energy between qubits \( i \) and \( j \) and is tunable in time. To build the cluster state, we start with all qubits aligned along the \( x \)-direction \( |\Phi_0^x \rangle \). Unitaries constructed from \( H_{ij}^{ZZ} \) can then be used to build the cluster state:

\[
|\Phi_C \rangle = \prod_{(i,j)} R_{Z,i,j}^{-\pi/2} R_{Z,j,i}^{-\pi/2} U_{ZZ,i,j}^{\pi/4} |\Phi_0^x \rangle, \tag{14}
\]

where

\[
U_{ZZ,i,j}^{\pi/4} = e^{-i J_{ij}^{Z} t} \sigma_i^z \sigma_j^z. \tag{15}
\]

The two qubit interaction can be recast in a matrix form. The matrix form for the Ising unitary is given by

\[
U_{ZZ,i,j}^{\pi/4} = \begin{pmatrix}
e^{-i J_{ij}^{Z} t} & 0 & 0 & 0 \\
0 & e^{i J_{ij}^{Z} t} & 0 & 0 \\
0 & 0 & e^{i J_{ij}^{Z} t} & 0 \\
0 & 0 & 0 & e^{-i J_{ij}^{Z} t}
\end{pmatrix}. \tag{16}
\]

If we set \( \theta = J_{ij}^{Z} t \) to parameterize time in units of the Ising interaction, then \( \theta = \pi/4 \) defines the Ising gate needed to build the cluster state:

\[
U_{ZZ,i,j}^{\pi/4} = e^{-i \pi/4} \begin{pmatrix}1 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & i & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \tag{17}
\]

We see from Eq. (14) that additional single qubit rotations are needed to build the cluster state that were not needed for the controlled phase interaction. A single qubit rotation about the \( z \)-axis is

\[
R_{Z,i}^{\phi} = e^{-i \phi \sigma_i^z/2}, \tag{18}
\]

where \( \phi \) parameterizes time evolution in units of the single qubit control field energy.

\[
F_{r=\pm y} = 1, \tag{21}
\]
for arbitrary error. We also find that the minimum fidelity occurs for initial qubits oriented in the x-z plane, i.e., orientations such that $\hat{r} \cdot \hat{y} = 0$. We therefore see that the error in Eq. (20) acts in a manner akin to a dephasing channel, in contrast to the depolarizing channel behavior seen for the controlled phase interaction.

The shape of Fig. 7 shows that the quality of quantum communication channels for the Ising interaction depends on the input state. This is due to the interplay of symmetry in the cluster state and the particular choice of Ising-gate error used in Eq. (20) (See Appendix A for a perturbative argument). We note that perfect transmission with error-free single-qubit gates arises in a trivial entangled limit, for $\varepsilon = -1$ because the single-qubit rotations alone are sufficient to leave $F_{\hat{r}=\pm \hat{y}} = 1$. But it is surprising that, away from the trivial $\varepsilon = -1$ point, we have perfect transmission for $0 < \varepsilon < 1$, even where we have low (but non-zero) entanglement. Future work will explore the role of symmetry in protecting these channels.

Perfect transmission implies that, in experiments, we can use the fidelity to characterize departures from the error model captured by Eq. (19). If the error is only Ising error, then input qubits oriented along $\pm \hat{y}$ will reveal unity fidelity. The presence of other error types, e.g., single-qubit error or measurement error, are therefore the only possible sources for departures from unity fidelity assuming $\hat{r} = \pm \hat{y}$. Specifically, observing $F_{\hat{r}=\pm \hat{y}} < 1$ would measure all other types of error besides Ising errors. $F_{\hat{r}=\pm \hat{y}}$, by contrast, includes all types of errors. By combining measurements to find the following ratio: $F_{\hat{r}=\pm \hat{y}} / F_{\hat{r}=0}$, one measures the relative strength of the non-Ising errors to all errors as they impact cluster state teleportation. We therefore propose $F_{\hat{r}=\pm \hat{y}} / F_{\hat{r}=0}$ as a useful experimental diagnostic of error types.

The Ising interaction also allows the fidelity to be quantified by closed formulas. We find that the minimum fidelity for the Ising interaction has a simple expression for uniform error on all gates defining the $N$-qubit cluster state chain. As for the controlled phase interaction, we start with the $N = 3$ chain fidelity for a qubit initially oriented along $\hat{r} = (\theta_0, \phi_0)$ on the Bloch sphere to find:

$$F_{\hat{r}}|_{N=3} = 1 - (1 - \sin^2 \theta_0 \sin^2 \phi_0) \sin^2(\pi \varepsilon / 2) / 2. \quad (22)$$

Here, we see that $F_{\hat{r}=\pm \hat{y}}|_{N=3} = 1$ for states along $\pm \hat{y}$, i.e., $\theta_0 = \pi / 2$ and $\phi_0 = \pm \pi / 2$. But we also now see explicitly that there are minima for $\phi = 0$ or $\pi$ for any $\theta$, i.e., the minima occurs in the $x$-$z$ plane with $\hat{r} \cdot \hat{y} = 0$. We then find that $\text{Min}(F)|_{N=3} = F_{\hat{r}=\hat{y}}|_{N=3} = [3 + \cos(\pi \varepsilon)] / 4$. The $N = 3$ case can be extended to larger $N$ using refreshing and concatenation (see the Appendix A). For $N$ odd, we find

$$\text{Min}(F) = \frac{1 + \cos^{N-1}(\pi \varepsilon / 2)}{2}. \quad (23)$$

From this expression, we see that the maximum error allowing teleportation [obtained by setting $\text{Min}(F) = 2 / 3$] is

$$\varepsilon_{\max} = \frac{2}{\pi} \arccos \left(3 \frac{1}{1 - 4} \right)$$

$$= \sqrt{\frac{2 \log 3}{N} + O(N^{-3/2})}. \quad (24)$$

This shows that significant two-qubit Ising errors can be tolerated in long cluster state chains since the fidelity scales as
FIG. 9. The same as Fig. 5 but for the Ising interaction protocol described in Fig. 6 and with the star chosen for \( \sigma = 0.2 \). Here the truncation of the Gaussian is due to the perfect transmission of two initial states through the disordered cluster state. See Sec. IV for a discussion of perfect transmission.

1/\( \sqrt{N} \). We note, for comparison, that the wavefunction overlap between the error-free cluster state and the cluster state with Ising error is \( \cos^{N-1}(\pi \varepsilon/4) \), which diminishes rapidly with increasing \( N \).

We now compare the fidelity in the presence of two-qubit Ising error to the results for the controlled-phase error by plotting them in the same manner. We assume a benchmarking protocol identical to the one discussed in Sec. III. To compare with the controlled phase interaction, Fig. 8 plots Eq. (23). By looking at where the solid lines cross the dashed line, we find the error \( \varepsilon_{\text{max}} \) where the teleportation is no longer guaranteed to be along a quantum channel. Here we also see that the minimum fidelity has the same qualitative behavior as shown for the controlled phase interaction in Fig. 4.

Figure 9 plots the same as Fig. 5 but for the Ising interaction instead of the controlled phase interaction. The inset shows a histogram that is truncated due to perfect transmission. The truncation in the fidelity distribution is an observable qualitative difference between dominant error in the Ising interaction and the controlled phase interaction as we benchmark with the fidelity. As we sample the orientations of the input qubits with a uniform distribution, we find that the Gaussian distributed two-qubit errors still allow a significant number of cases of perfect transmission.

The main panel of Fig. 9 appears to be qualitatively the same as Fig. 5 in spite of the cases of perfect transmission. But quantitatively we see that the minimum fidelity is higher for the error-prone Ising interaction. We therefore find that the minimum fidelity distribution reveals a somewhat more robust quantum channel for the error-prone Ising interaction than for the error-prone controlled phase interaction.

V. CLUSTER STATES FROM THE ERROR-PRONE XY INTERACTION

We now turn to cluster states constructed from the error-prone XY interaction. The XY interaction directly implements the iSWAP gate and characterizes interactions between qubits in several different qubit architectures. Examples include the rotational states of a polar molecule placed in an optical tweezer trap [39–41], quantum dots in cavities [42], and certain types of superconducting qubits [38].

We assume that the physical two-qubit interaction is given by

\[
H^{XY}_{ij} = J^{XY}_{ij} (\sigma^x_i \sigma^x_j + \sigma^y_i \sigma^y_j),
\]

where the interaction energy between qubits \( i \) and \( j \), \( J^{XY}_{ij} \), is tunable in time. To build the cluster state we start with all qubits aligned along the \( y \)-direction:

\[
|\Phi_0^\text{\{\}}\rangle = \prod_i |+\rangle_{y,i},
\]

where \( |\pm\rangle_{y,i} \equiv (|0\rangle_i \pm |1\rangle_i)/\sqrt{2} \) are the eigenstates of \( \sigma^y_i \).

Unitaries constructed from \( H^{XY}_{ij} \) can be used to build the cluster state. We have an efficient parallelizable protocol for building the “twisted” cluster state [31]:

\[
|\Phi^\text{\{\}}\rangle = \prod_{i \neq 1} R_{z,i}^{\pi/2} \prod_{i+1,j+1} U^{\pi/4}_{XY,i,i+1,j,j+1} \prod_{i,i+1} U^{\pi/4}_{XY,i,i+1} |\Phi^\text{\{\}}_0\rangle,
\]

FIG. 10. Circuit diagram depicting the construction of a 4-qubit twisted cluster state using the XY interaction followed by an entanglement measure. The twisted cluster state, Eq. (27), is the same as the cluster state but with neighboring qubit labels on every other bond swapped. The initial kets denote four qubits prepared as eigenstates of \( \sigma_y, |+\rangle_y \). These four qubits are entangled with the imperfect XY interaction between two qubits, \( U^{\pi/4}_{XY} \), where \( \varepsilon \) is a dimensionless error parameter, Eq. (31). The \( U^{\pi/4}_{XY} \) gates do not commute on bonds sharing a qubit and the order of execution must be respected, e.g., the XY interaction on all even bonds are executed simultaneously, followed by the simultaneous execution of the XY interaction on all odd bonds. The series of measurements used to teleport the input state \( |\psi_0\rangle \) to the end of the chain with output \( |\psi_0\rangle \) is the same as Fig. 2. The single-qubit rotation gates can also be implemented by changing the measurement angles.
where the notation $\prod'_{i,i+1}$ and $\prod''_{i,i+1}$ indicates a product over non-touching bonds $(i, i + 1), (i + 2, i + 3), ...$ and non-touching bonds $(i + 1, i + 2), (i + 3, i + 4), ...$, respectively. $U_{XY,ij}^{J}$ is given by:

$$U_{XY,ij}^{J} = e^{-iJ_{XY}^{J}(\sigma_{i}^{x}\sigma_{j}^{x} + \sigma_{i}^{y}\sigma_{j}^{y})}.$$

In matrix form, this becomes:

$$U_{XY,ij}^{J} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2J_{XY}^{J}t) & -i\sin(2J_{XY}^{J}t) & 0 \\ 0 & -i\sin(2J_{XY}^{J}t) & \cos(2J_{XY}^{J}t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The twisted cluster state is the same as $|\Phi_{C}\rangle$ but with states at certain qubits swapped according to a twisting protocol.

To build the cluster state with the XY interaction, we relied on the iSWAP gate on qubits $i$ and $j$ with $J_{XY}^{J}t = \pi/4$:

$$U_{XY,ij}^{\pi/4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

But a faulty time evolution operator corresponding to the two-qubit interaction arises from the replacement

$$J_{XY}^{J}t \rightarrow J_{XY}^{J}t(1 + \varepsilon),$$

in Eq. (25) and is given by:

$$U_{XY}^{\theta,\varepsilon} = e^{-i\theta(1+\varepsilon)(\sigma_{i}^{x}\sigma_{j}^{x} + \sigma_{i}^{y}\sigma_{j}^{y})},$$

where we set $\theta = J_{XY}^{J}t$. We will now examine the impact of these two-qubit XY errors on the cluster state fidelity.

Figure 10 depicts the circuit diagram needed to construct the cluster state with the error-prone XY interaction and measure the fidelity. Note that the XY interaction on neighboring bonds does not commute, i.e., $[U_{XY,ij}^{J}, U_{XY,jk}^{J}] \neq 0$, where $i \neq k$. This results in two key differences between the protocol constructed for the Ising interaction and that for the XY interaction. First, care must be taken in the pulse order to create the twisted cluster state as opposed to the Ising interaction. Second, we note that the twisted cluster state can be thought of as the original cluster state with qubits 1, 2, ..., $N-1$ swapped in neighboring pairs on every other bond. We must therefore redefine the byproduct matrix $U_{XY}^{J}$. For the cluster state defined with the XY interaction, the $x$ and $z$ in Eq. (2) are swapped. With these two primary differences accounted for, the procedure for defining the cluster state with the XY interaction follows in the same manner as with the Ising interaction.

We simulated the process of measuring the fidelity of the cluster state formed from the error-prone XY interaction as in the previous sections. We used the XY interaction and found results identical to Fig. 7 obtained using the error-prone Ising interaction. For example, we find perfect transmission for $\hat{r} = \pm \hat{y}$ and the fidelity minima at $\hat{r} \cdot \hat{y} = 0$. We also find the same analytic expressions for the error bound on teleportation along quantum channels, Eqs. (33) and (34).

Figures 11 and 12 quantify the response of the fidelity to errors in the XY interaction. We find that the fidelity degrades in essentially the same manner as for the Ising interaction. These results show that the protocol defined by Fig. 10 puts the cluster state constructed from the XY interaction on the same footing as that constructed from the Ising interaction.

VI. REFOCUSING PULSES TO CORRECT ERRORS IN TWO-QUBIT GATES

Cluster state sizes can be limited experimentally. Refreshing can be used to increase cluster state size in experiments with a limited number of qubits (See the Appendix A).
also saw in previous sections that, for fixed error strength, increasing the length of the cluster state eventually degrades the fidelity below 2/3. Errors therefore limit the length of the cluster state chain that allows teleportation along a quantum channel. If the dominant error sources are slow on time scales of two-qubit gates, we can use refocusing schemes to correct these gate errors [36]. Refocusing is a powerful tool because it does not rely on a specific input state or on knowledge of the exact error strength.

By correcting slow gate errors, we can increase the length of the cluster state chains allowing teleportation along a quantum channel. Refocusing schemes for single-qubit gate errors have been examined extensively [36, 391]. In what follows, we assume no error in single-qubit gates or in measurements. We limit our analysis to refocusing schemes for slow two-qubit gate errors because these can dominate. Such refocusing schemes can be used to correct errors to very high orders in ε [45, 51] but at the expense of gate overhead. We construct the simplest possible two-qubit refocusing pulse sequences to correct the errors studied above (Ising, XY, and controlled phase) to the lowest order.

A. Refocusing Pulses for the Ising Interaction

We start with refocusing the error-prone Ising gate, Eq. (20). Assuming an error-prone Ising interaction, \( U_{ZZ}^{θ,ε} \), we can construct a sequence of refocusing pulses to improve the accuracy of the two-qubit gate in approximating the exact Ising interaction, \( U_{ZZ}^{θ,0} \). We construct sequences that simplify the two-qubit refocusing schemes following the usual BB1-type protocols of Refs. [44–46]. We use fewer pulses to refocus the interaction to only correct the lowest order errors (as opposed to the lowest and next-lowest order errors corrected by the longer BB1-type sequences [44]).

The Ising gate, \( U_{ZZ}^{θ,ε} \), will have errors \( O(ε) \) that can be corrected to \( O(ε^2) \). For any two qubits \( i = 1 \) and \( j = 2 \), we assume, in addition to the Ising interaction, Eq. (13), an error-free single-qubit control Hamiltonian:

\[
H_X(B_x)_{1,2} = B_x σ_i^x σ_j^x,
\]

which can be pulsed for a fixed duration in time leading to a propagator:

\[
U_X^δ = e^{-i(δ/2)σ_i^x σ_j^x},
\]

where \( δ = 2B_x t \).

The two Hamiltonians [Eqs. (13) and (33)] lead to two different time scales \( θ \) and \( δ \) characterizing the interaction and single-qubit magnetic field, respectively. They are related by

\[
θ = \frac{J^2}{2B_x δ}.
\]

For convenience, we assume a time parameterization such that

\[
\frac{J^2}{B_x} = \frac{8π cos(δ)}{δ},
\]

which can always be solved for at least one δ.

The central result of Ref. [44] was to show that known single-qubit refocusing schemes apply to errors in the two-qubit Ising gate as well. Pulse sequences were shown to correct the two lowest orders of errors. But we would like to construct a pulse sequence that is as short as possible and corrects only the lowest order. We find that the improved two-qubit Ising gate that corrects the lowest order of the error-prone Ising gate is

\[
V_{ZZ}^{θ,ε} = U_{X}^{-δ} U_{ZZ}^{-2π,ε}(U_{X}^{-δ})^† U_{ZZ}^{-2π,ε}(U_{X}^{-δ})^† U_{ZZ}^{θ,ε},
\]

with an improved scaling in error: \( V_{ZZ}^{θ,ε} = U_{ZZ}^{θ,0} + ΔU_{ZZ} ε^2 + O(ε^3) \). Here the unwanted error is given by:

\[
ΔU_{ZZ} = \begin{pmatrix}
0 & d_δ & 0 & 0 \\
-d_δ ^* & 0 & 0 & 0 \\
0 & 0 & 0 & -d_δ ^* \\
0 & 0 & d_δ & 0
\end{pmatrix},
\]

with \( d_δ = -4π^2 i e Aπi cos(δ) sin(2δ) \).

The first row in Fig. 13 depicts the circuit diagram used to create the refocused Ising gate, Eq. (37). Replacing the two qubit gates in Fig. 6 with \( V_{ZZ}^{θ,ε} \) from the first row in Fig. 13 will improve the fidelity of the cluster state. To characterize the improved two-qubit gate, we construct a two-qubit gate fidelity:

\[
F_{ZZ}^{θ,ε} = \frac{Tr[V_{ZZ}^{θ,ε}(U_{ZZ}^{θ,0})^†] }{Tr[U_{ZZ}^{θ,0}(U_{ZZ}^{θ,0})^†] }.
\]

The infidelity for the refocused gate, Eq. (37), is \( 1 - F_{ZZ}^{θ,ε} = 8(πε)^2 sin^2(2δ) + O(ε^6) \) which is improved by \( O(ε^2) \) over the infidelity for the original gate, Eq. (20), \( 8(πε)^2 cos^2(δ) + O(ε^4) \). Longer pulse sequences can be constructed to further improve the fidelity [44].
B. Refocusing Pulses for the XY Interaction

The errors in the XY gate, Eq. (32), will lead to $O(\varepsilon)$ errors as we attempt to construct the cluster state with the iSWAP gate. These errors can also be corrected to $O(\varepsilon^2)$, in a procedure similar to that for the Ising gate. The goal of this section is to implement a good approximation to the exact XY time evolution operator, $U_{XY}^{t,0}$, most importantly, the iSWAP gate, $U_{XY}^{t/4,0}$, used in constructing the cluster state from the XY interaction.

We assume an error-free single-qubit control Hamiltonian:

$$H_Z(B_\gamma)_{i,j} = B_\gamma \sigma^0_i \sigma^2_j,$$

where $H_Z$ is applicable for a fixed duration on qubits $i = 1$ and $j = 2$. The time evolution operator inducing single-qubit rotations of the second qubit about the Z-axis [obtained from pulsing Eq. (40)] is

$$U_\gamma^Z = e^{-i(\alpha/2)\sigma^0_i \sigma^2_j},$$

where we use $\alpha$ to parameterize time

$$B_\gamma t = \alpha/2.$$

The two Hamiltonians, Eqs. (25) and (40), define two time scales. The two time scales are related by

$$\theta = \frac{J_{XY}}{2B_\gamma} \alpha.$$

For convenience, we assume a time parameterization such that

$$\frac{J_{XY}}{B_\gamma} = \frac{8\pi \cos(\alpha)}{\alpha},$$

which can always be solved for at least one $\alpha$.

To improve the accuracy of the two-qubit XY gate, we consider the refocused pulse sequence obtained in direct analogy to the Ising gate refocusing (Sec. VI A). It is given by

$$V_{XY}^{\theta,\varepsilon} = U_\gamma^Z U_{XY}^{t-2\pi,\varepsilon} (U_\gamma^Z)^\dagger U_\gamma^Z U_{XY}^{t-2\pi,\varepsilon} (U_\gamma^Z)^\dagger U^{\theta,\varepsilon}. \quad (45)$$

Here $V_{XY}^{\theta,\varepsilon}$ is an improved approximation to the iSWAP gate if we choose $\theta = \pi/4$. The middle row in Fig. 13 depicts the circuit diagram used to create the refocused XY gate. Replacing the two qubit gates in Fig. 10 with $V_{XY}^{\theta,\varepsilon}$ from the middle row in Fig. 13 will improve the fidelity of the cluster state.

To see that $V_{XY}^{\theta,\varepsilon}$ offers an improved approximation to the iSWAP gate, we expand in powers of $\varepsilon$: $V_{XY}^{\theta,\varepsilon} = V_{XY}^{t,0} + \Delta U_{XY} \varepsilon^2 + O(\varepsilon^3)$ where the correction is given by

$$\Delta U_{XY} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_\alpha & -c_\varepsilon & 0 \\ 0 & c_\varepsilon & c_\alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (46)$$

with $c_\alpha = i16\pi^2 \cos[8\pi \cos(\alpha)] \sin(2\alpha)$ and $c_\varepsilon = -16\pi^2 \times \sin[8\pi \cos(\alpha)] \sin(2\alpha)$. We then define the two-qubit gate fidelity:

$$F_{XY}^* = \frac{\text{Tr} \left[ V_{XY}^{\theta,\varepsilon} (U_{XY}^{t,0})^\dagger \right]}{\text{Tr} \left[ (U_{XY}^{t,0})^\dagger \right]}. \quad (47)$$

We find that the infidelity is then $1 - F_{XY}^* = 64(\pi \varepsilon)^4 \times \sin^2(2\alpha) + O(\varepsilon^6)$. This shows that refocusing to implement $V_{XY}$ offers an $O(\varepsilon^2)$ improvement to the faulty iSWAP gate since the infidelity of the un-refocused pulse, Eq. (32), is much larger, specifically, $16(\pi \varepsilon)^2 \cos^2(\alpha)$.

C. Refocusing for the Controlled Phase Interaction

Refocusing of the controlled phase interaction can, in principle, use the method constructed for the Ising interaction above, because the Ising and controlled phase gates are equivalent up to single-qubit rotations. But these added single-qubit rotations imply the need for an additional extraction procedure [45] which adds to the gate overhead.

To construct the refocusing scheme for the controlled phase interaction, we assume that the physical two-qubit interaction is given by $H_{ij}^{CP}$ acting on qubits $i$ and $j$ leading to the propagator $U_{ij}^{\theta,\varepsilon}$ for any time $t$. But we must now also assume two different error-free single-qubit control Hamiltonians. First we assume that

$$H_X(B_\gamma)_{i,j} = B_\gamma \sigma^0_i \sigma^\pi_j,$$

can be pulsed for a fixed duration in time leading to a propagator $U_{ij}^{\theta,\varepsilon}$ where $\gamma = 2B_\gamma t$. For convenience, we set $\gamma = \arccos(\theta/16\pi)$. We also assume the single-qubit control Hamiltonian is given by

$$H_X(B_\gamma)_{i,j} = B_\gamma \sigma^0_i \sigma^2_j,$$

leading to a propagator $U_{ij}^{\gamma'}$ where $\gamma' = 2B_\gamma t$.

To refocus and improve the accuracy of the controlled phase gate, we rely on the pulse sequence for the Ising interaction. But we must extract [44, 45] the Ising term from the controlled phase interaction by noting

$$U_{ij}^{t,0} = e^{-i\theta \sigma^0_i \sigma^\pi_j} U_{ZZ}^{t,0}.$$

The Ising term can be isolated using $e^{i\theta/2}(\sigma^\pi_j \sigma^0_i + \sigma^0_i \sigma^\pi_j) U_{ij}^{t,0}$ since these rotations essentially remove the role of the leading $Z$ rotations.

$$U_{EX}^{\theta,\varepsilon} \equiv e^{-i\pi/2}(\sigma^0_i \sigma^\pi_j + \sigma^\pi_j \sigma^0_i) U_{CP}^{t,0} e^{-i\pi/2}(\sigma^\pi_j \sigma^0_i + \sigma^0_i \sigma^\pi_j) U_{CP}^{t,0} \sigma^0_i \sigma^\pi_j, \quad (50)$$

where the Ising term is extracted from the controlled phase interaction (up to a phase of $e^{i\theta(1+\varepsilon)}$).
We can now use $U_{EX}^{θ,ε}$ in place of the Ising pulse in the refocusing sequence discussed in Sec. VI A. The composite refocusing sequence for the controlled phase interaction then becomes nearly the same as Eq. (37):

$$V_{CP}^{θ,ε} = e^{i(θ + σ_i^0 + σ_i^z)}U_{EX}^{−γ}U_{EX}^{−2π,ε} \times (U_{X}^{−γ})^iU_{X}^{−2π,ε}(U_{X}^0)U_{EX}^{θ/4,ε},$$

(51)

where the primary difference is the leading exponential term which reinserts the prefactors needed for the corrected controlled phase gate.

The bottom row in Fig. 13 shows the circuit diagram for $V_{CP}^{θ,ε}$ written in terms of the composite pulse Eq. (50) for comparison with the Ising and XY sequences. Note that extracted gates, Eq. (50), are themselves composite gates. Figure 14 depicts Eq. (51) in a full circuit diagram written in terms of the physical controlled phase gate rather than Eq. (50).

Replacing $U_{EX}^{θ,ε}$ with $V_{CP}^{θ,ε}$ in Fig. 2 will reduce error in constructing the controlled phase gate. To quantify the improvement, we define the two-qubit gate fidelity:

$$F_{CP}^{2} = \left[ \frac{\text{Tr}[V_{CP}^{θ,ε}(U_{CP}^{θ,0})^d]}{\text{Tr}[U_{CP}^{θ,0}(U_{CP}^{θ,0})^d]} \right].$$

(52)

The corrected gate sequence then has an infidelity $1 - F_{CP}^{2} = \frac{θ^4(θ^2−256π^2ε^4)}{2048} + O(ε^6)$ which improves over the infidelity of $U_{CP}^{θ,ε}$, which is $3ε^2θ^2/32 + O(ε^4)$. The improved two-qubit fidelity for the controlled phase interactions shows that we can replace the two qubit gates in Fig. 2 with the corrected sequence from Fig. 14. While this should, in principle, help improve the overall cluster state fidelity, we see that the overall gate count for the refocused XY and Ising interaction are much smaller than that for the controlled phase interaction. Further compactifications of these gate sequences could shorten them.

VII. SUMMARY

We have systematically analyzed the construction of cluster state chains from faulty interactions along with a cluster state teleportation-based fidelity measure of entanglement. We focused on the controlled phase, Ising, and XY interactions. We find that errors in different interaction strengths lead to different fidelity responses. By running numerical experiments designed to simulate key pieces of experimental benchmarking, we find that errors in the Ising and XY interaction have a preferred direction in qubit space. We find, in particular, a case of perfect transmission in spite of errors in the Ising and XY interaction. The fidelity discussed here can therefore be used as a measure of interaction error in comparison to all other errors. We also find that the $1/√N$ scaling of maximum error for teleportation along a quantum channel leaves room for growing cluster state chains while preserving entanglement.

We have also discussed refocusing schemes for improving the two-qubit gate fidelities in the presence of slow two-qubit gate errors. The refocusing schemes discussed here can be made more compact, extended to correct error to higher orders [44, 45], and can also be paired with single-qubit refocusing [46]. Our work sets the stage for combining randomized benchmarking with refocusing to experimentally grow and test entanglement on cluster states.

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FIG. 15. Schematic depicting qubit refreshing of a 5-qubit cluster state as we teleport $|\psi_i\rangle$ using measurements on at most 3 qubits. The bold boxes contain the information regarding $|\psi_i\rangle$. The greyed-out qubits are not needed. The first column depicts the initial qubit prepared in the state $|\psi_i\rangle$ and entangled with two other qubits in a cluster state. The second column depicts measurements that move $|\psi_i\rangle$ from the first to the third qubit. The third column depicts the entangling of two new qubits into the cluster state after the first two qubits are discarded. The final row depicts the final measurements that moves $|\psi_i\rangle$ to the last qubit. The entire refreshing process depicted here uses at most 3 qubits at the same time while effectively teleporting along a 5-qubit cluster state.

dependent graph. Fig. 15 shows a schematic of the refreshing process for teleportation (MBQC identity gate) along a cluster state chain. The example in the figure shows that only 3 qubits are needed at any one time to effectively teleport along a 5-qubit cluster state chain. More generally, refreshing shows that only 3 qubits are needed for measuring fidelity along a chain of arbitrary length.

We relied on refreshing in the main text. We used it in our derivations where, in Sec. III, we used refreshing to derive Eq. (23). We also relied on refreshing to argue that it can also be used experimentally. If experiments are limited in qubit resources, refreshing can be used to build larger cluster states. As the minimum number of qubits used increases, the process of MBQC teleportation along the chain begins to differ from the circuit-based scheme. In MBQC, more single qubit measurements are done after the application of the two-qubit gates. This is to be compared with the circuit-based scheme where two-qubit gates are applied throughout the algorithm rather than upfront.

**Appendix B: Perfect transmission for weak error**

In this section, we show analytically that, for $\varepsilon \ll 1$, the cluster state exhibits perfect transmission for input qubit aligned along the $y$ direction for the Ising interaction (similarly for the XY interaction). We also show that this is not the case for the controlled phase interaction. For this purpose, it is sufficient to consider just the $N = 3$ cluster state. Using refreshing (Appendix A), one can extend the argument here to larger $N$.

We first focus on the error-prone Ising interaction. We make a perturbation expansion in $\varepsilon$:

$$U_{ZZ,i,j}^{\pi/4,\varepsilon} = U_{ZZ,i,j}^{\pi/4,0} \sum_{n=0}^{\infty} \Gamma_{ZZ,i,j}^{(n)} \varepsilon^n, \quad (B1)$$

where the coefficient matrices are defined by

$$\Gamma_{ZZ,i,j}^{(n)} = \frac{1}{n!} (-i \frac{\pi}{4})^n \left\{ \begin{array}{ll} \sigma_0^0 \sigma_0^0, & n \in \text{even}, \\ \sigma_i^\sigma_i \sigma_j^\sigma_j, & n \in \text{odd}. \end{array} \right. \quad (B2)$$

Accordingly, the cluster state is expanded as:

$$|\psi_I, \Phi_C\rangle_\varepsilon = \prod_{(i,j)} \left( \sum_{n=0}^{\infty} \Gamma_{ZZ,i,j}^{(n)} \varepsilon^n \right) |\psi_I, \Phi_C\rangle_0. \quad (B3)$$

For the density matrix $\rho_{\varepsilon}(|\psi_I, \Phi_C\rangle_\varepsilon \langle \psi_I, \Phi_C|)$, the perturbation expansion reads:

$$\rho_{\varepsilon}(|\psi_I, \Phi_C\rangle_\varepsilon \langle \psi_I, \Phi_C|) = \rho_0(|\psi_I, \Phi_C\rangle \langle \psi_I, \Phi_C|) + \sum_{n=1}^{\infty} \rho_{ZZ}^{(n)} (|\psi_I, \Phi_C\rangle_\varepsilon \langle \psi_I, \Phi_C|) \varepsilon^n. \quad (B4)$$

Here, we are interested in the first two coefficient matrices $(n = 1, 2)$ given by

$$\begin{align*}
\hat{\rho}_{ZZ}^{(1)} (|\psi_I, \Phi_C\rangle_\varepsilon \langle \psi_I, \Phi_C|) &= \hat{\Gamma}_{ZZ}^{(1)} (|\psi_I, \Phi_C\rangle \langle \psi_I, \Phi_C|), \\
\hat{\rho}_{ZZ}^{(2)} (|\psi_I, \Phi_C\rangle_\varepsilon \langle \psi_I, \Phi_C|) &= \hat{\Gamma}_{ZZ}^{(2)} (|\psi_I, \Phi_C\rangle \langle \psi_I, \Phi_C|) + \rho_0 (|\psi_I, \Phi_C\rangle \langle \psi_I, \Phi_C|).
\end{align*} \quad (B5)$$

where, for $N = 3$, we define

$$\begin{align*}
\hat{\Gamma}_{ZZ}^{(1)} &= \Gamma_{ZZ,1,2}^{(1)} \Gamma_{ZZ,2,3}^{(0)} + \Gamma_{ZZ,1,2}^{(0)} \Gamma_{ZZ,2,3}^{(1)} \\
&= -i \frac{\pi}{4} (\sigma_3^\sigma_3 \sigma_3^0 + \sigma_0^\sigma_3 \sigma_3^0) \sigma_2^z, \\
\hat{\Gamma}_{ZZ}^{(2)} &= \Gamma_{ZZ,1,2}^{(2)} \Gamma_{ZZ,2,3}^{(0)} + \Gamma_{ZZ,1,2}^{(0)} \Gamma_{ZZ,2,3}^{(2)} + \Gamma_{ZZ,1,2}^{(1)} \Gamma_{ZZ,2,3}^{(1)} \\
&= -\frac{\pi^2}{16} (\sigma_0^\sigma_0 \sigma_0^3 + \sigma_3^\sigma_3 \sigma_3^0) \sigma_2^z.
\end{align*} \quad (B7)$$

It can be shown that, for a qubit initially oriented along $\hat{r} = (\theta_0, \phi_0)$ on the Bloch sphere, $|\psi_I\rangle = (\theta_0, \phi_0) = \cos (\theta_0/2) |0\rangle + \sin (\theta_0/2) |1\rangle$, the output density matrix and the fidelity have the following closed form (up to second order in $\varepsilon$):

$$\rho_{O\mid N=3} = \hat{\rho}_{I} + \hat{\rho}_{O}^{(2)} \varepsilon^2 + O(\varepsilon^3), \quad (B9)$$

$$\mathcal{F}_{I\mid N=3} = 1 + \hat{\mathcal{F}}_{I}^{(2)} \varepsilon^2 + O(\varepsilon^3), \quad (B10)$$

where the leading correction arises in the second order:

$$\hat{\rho}_{O} = -\frac{\pi^2}{8} \left( \hat{r}_I^{(\theta_0, \phi_0)} - \hat{r}_I^{(\pi - \theta_0, \pi - \phi_0)} \right), \quad (B11)$$

$$\hat{\mathcal{F}}_{I}^{(2)} = -\frac{\pi^2}{8} (1 - \sin^2 \theta_0 \sin^2 \phi_0). \quad (B12)$$

We can use Eqs. (B11) and (B12) to show perfect transmission. The first term of Eq. (B11) does not perturb the direction of the input qubit, giving an angle-independent contribution to Eq. (B12). The second term of Eq. (B11) describes a qubit...
oriented along $\hat{r}' = (\pi - \theta_0, \pi - \phi_0)$ (preserving the sum $\hat{r} + \hat{r}'$), giving rise to the angle dependence in Eq. (B12). For the symmetric choice of $\hat{r} = \hat{r}' = (\pi/2, \pm \pi/2)$, two terms in Eqs. (B11) and (B12) cancel each other, leading to perfect transmission in the second order. The same argument is expected to be applied to the higher orders as already shown in Eq. (22).

For comparison, the perturbation expansion for the controlled phase interaction is also made as follows:

$$U_{\pi/4, e}^{\pi/4} = U_{\pi/4, 0}^{\pi/4} \sum_{n=0}^{\infty} \Gamma_{\pi/4, 0}^{(n)} e^n,$$  \hspace{1cm} (B13)

where the coefficient matrices are defined by

$$\Gamma_{\pi/4, 0}^{(n)} = \frac{(-i\pi)^n}{4n!} (\sigma_i^0 - \sigma_i^z)(\sigma_j^0 - \sigma_j^z).$$  \hspace{1cm} (B14)

As before, the leading correction to the fidelity starts from the second order:

$$\tilde{F}^{(2)} = -\frac{\pi^2}{8} [1 - \sin^2 \theta_0 \cos^2 \phi_0 + 4 \sin^2(\theta_0/2)].$$  \hspace{1cm} (B15)

Here, we see that Eq. (B15) always gives a negative contribution for any $\theta_0$ and $\phi_0$ in contrast to Eq. (B12). The controlled phase interaction therefore does not allow perfect transmission.