On the speed of convergence of Picard iterations of backward stochastic differential equations

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Abstract
It is a well-established fact in the scientific literature that Picard iterations of backward stochastic differential equations with globally Lipschitz continuous nonlinearity converge at least exponentially fast to the solution. In this paper we prove that this convergence is in fact at least square-root factorially fast. We show for one example that no higher convergence speed is possible in general. Moreover, if the nonlinearity is z-independent, then the convergence is even factorially fast. Thus we reveal a phase transition in the speed of convergence of Picard iterations of backward stochastic differential equations.

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1 Introduction
Since their introduction by Pardoux & Peng in [17] backward stochastic differential equations (BSDEs) have been extensively studied in the scientific literature and have found numerous applications. For example, BSDEs provide a solution approach for stochastic optimal control

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problems, BSDEs appear in the pricing and hedging of options in mathematical finance, and BSDEs provide stochastic representations of semilinear parabolic partial differential equations (PDEs).

A standard approach for proving existence results for BSDEs is to construct a contraction mapping whose fixed point is the solution \( (Y, Z) \) of the BSDE. The associated fixed point iterations, the so-called Picard iterations, are a key component of several numerical approximation methods for BSDEs. We refer, e.g., to [2, 3] for numerical approximation methods for BSDEs based on Picard iterations and the so-called Monte Carlo method, we refer, e.g., to [10, 15] for numerical approximation methods for BSDEs based on Picard iterations and adaptive control variates, we refer, e.g., to [4, 9] for numerical approximation methods for BSDEs based on Picard iterations and Wiener chaos expansions, and we refer, e.g., to [6, 13, 7, 14, 11, 1, 12] for numerical approximation methods for BSDEs based on Picard iterations and a multilevel technique. Precise estimates on the speed of convergence of the Picard iterations \( Y, Z \) converge at least square-root factorially fast. Somewhat surprisingly this speed of convergence is due to the nonlinearities that the Picard iterations converge in fact factorially fast. Moreover, we show that BSDEs provide stochastic representations of semilinear parabolic partial differential equations (PDEs).

Theorem 1.1. Let \( T \in (0, \infty) \), \( d, m \in \mathbb{N} \), \( L_0, L_j \in [0, \infty) \), \( b \in \mathbb{R}^m \), let \( \|\cdot\| : \cup_{n \in \mathbb{N}} \mathbb{R}^n \to [0, \infty) \) satisfy for all \( n \in \mathbb{N} \) that \( \|\cdot\|_{\mathbb{R}^n} \) is the standard norm on \( \mathbb{R}^n \), let \( \|\cdot\|_F : \mathbb{R}^{d \times \infty} \to [0, \infty) \) denote the Frobenius norm on \( \mathbb{R}^{d \times \infty} \), let \( (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]}) \) be a filtered probability space which satisfies the usual conditions\(^1\), let \( f : [0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times m} \to \mathbb{R}^d \) be measurable, assume for all \( t \in [0, T], y, \tilde{y} \in \mathbb{R}^d, z, \tilde{z} \in \mathbb{R}^{d \times m} \) it holds a.s. that

\[
\|f(t, y, z) - f(t, \tilde{y}, \tilde{z})\| \leq L_0\|y - \tilde{y}\| + L_j\|z - \tilde{z}\|_F,
\]

let \( W : [0, T] \times \Omega \to \mathbb{R}^m \) be a standard \((\mathcal{F}_t)_{t \in [0,T]}\)-Brownian motion with continuous sample paths, let \( \xi : \Omega \to \mathbb{R}^d \) be \( \mathcal{F}_T \)-measurable, let \( Y^k : [0, T] \times \Omega \to \mathbb{R}^d \), \( k \in \mathbb{N}_0 \cup \{\infty\} \), be adapted with continuous sample paths, let \( Z^k : [0, T] \times \Omega \to \mathbb{R}^{d \times m} \), \( k \in \mathbb{N}_0 \cup \{\infty\} \), be progressively measurable, assume that for all \( s \in [0, T], k \in \mathbb{N}_0 \cup \{\infty\} \) it holds a.s. that

\[
\begin{align*}
\int_0^T \mathbb{E}[\|\xi\|^2 + \|f(t, 0, 0)\|^2 + \|Y_t^\infty\|^2 + \|Z_t^k\|^2] \, dt < \infty, \\
Y_s^0 = 0, Z_s^0 = 0, \\
Y_s^{k+1} = \xi + \int_s^T f(t, Y_t^k, Z_t^k) \, dt - \int_s^T Z_t^{k+1} \, dW_t,
\end{align*}
\]

\(^{1}\)Let \( T \in (0, \infty) \) and let \( \Omega = (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]}) \) be a filtered probability space. Then we say that \( \Omega \) satisfies the usual conditions if and only if it holds for all \( t \in [0, T) \) that \( \{A \in \mathcal{F} : \mathbb{P}(A) = 0\} \subseteq \mathcal{F}_t = \cap_{s \in (t, T]} \mathcal{F}_s \).
and let \( e_k \in [0, \infty], k \in \mathbb{N} \), satisfy for all \( k \in \mathbb{N} \) that
\[
e_k = \left( E \left[ \sup_{t \in [0,T]} \left( \| Y_t^k - Y_t^\infty \|^2 \right) + \int_0^T \| Z_t^k - Z_t^\infty \|^2 dt \right] \right)^{1/2}.
\] (3)

Then

(i) there exists \( c \in (0, \infty) \) such that for all \( k \in \mathbb{N} \) it holds that \( e_k \leq \frac{c^k}{\sqrt{k!}} \).

(ii) if, in addition to the above assumptions, it holds that \( L_3 = 0 \), then there exists \( c \in [0, \infty) \) such that for all \( k \in \mathbb{N} \) it holds that \( e_k \leq \frac{c^k}{k!} \), and

(iii) if, in addition to the above assumptions, \( d = T = 1, \xi = 2^{m/2} e^{-\frac{1}{2} W_1^2} \), and for all \( t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^{1 \times m} \) it holds a.s. that \( f(t, y, z) = z \cdot b \), then there exists \( c \in [0, \infty) \) such that for all \( k \in \mathbb{N} \cap [\| b \|^2 - 1, \infty) \) it holds that \( \frac{1}{2} \left( \frac{\| b \|^2}{4} \right)^{\frac{k+1}{2}} \leq e_k \leq \frac{c^k}{\sqrt{k!}} \).

Item (i) of Theorem 1.1 is a direct consequence of Proposition 4.1 and Remark 4.2. Item (ii) of Theorem 1.1 follows from Proposition 4.1 and Remark 4.3. Item (i) of Theorem 1.1 and Corollary 2.2 prove Item (iii) of Theorem 1.1. The proof of Item (i) of Theorem 1.1 is based on Lemma 3.1 which shows for all \( t \in [0, T], k \in \mathbb{N}, \lambda \in (0, \infty) \) that
\[
E \left[ e^{\lambda (\| Y_t^k - Y_t^\infty \|^2 + \int_t^T e^{\lambda s} | Z_s^k - Z_s^\infty |^2 ds) \right]
\leq \frac{1}{\lambda} E \left[ \int_t^T e^{\lambda s} | f_s(Y_s^{k-1}, Z_s^{k-1}) - f_s(Y_s^\infty, Z_s^\infty) |^2 ds \right],
\] (4)

based on the Lipschitz continuity of \( f \), iterating (4) \( k \in \mathbb{N} \) times and then setting \( \lambda = k \) to get an upper bound of the form \( c^k/\sqrt{k^k} \) for \( e_k \).

We finally discuss some possible consequences of Item (iii) of Theorem 1.1 on the performance of numerical approximation methods for BSDEs based on Picard iterations in high-dimensional situations. To this end we consider a sequence of BSDEs indexed by the dimension \( m \in \mathbb{N} \) of the driving Brownian motion \( W \) whose associated Lipschitz constants \( L_{3,m} \in [0, \infty) \), \( m \in \mathbb{N}, \) grow for some \( \alpha \in (0, \infty) \) like \( m^\alpha \) as \( m \to \infty \). Item (iii) of Theorem 1.1 shows that it is possible in such a situation that the approximation errors \( e_{k,m}, k, m \in \mathbb{N} \), grow faster in the dimension \( m \in \mathbb{N} \) than any polynomial in the sense that for all \( p \in [0, \infty) \) there exists \( N \in \mathbb{N} \) such that for all \( k \in \mathbb{N} \cap [N, \infty) \) it holds that \( \liminf_{m \to \infty} \frac{e_{k,m}}{m^p} = \infty \).

The remainder of this article is organized as follows. In Section 2 we provide lower bounds for the convergence speed of Picard iterations. In Section 2.2 we establish in Corollary 2.2 lower bounds for the convergence speed of Picard iterations for a linear example BSDE. In our proof of Corollary 2.2 we employ lower bounds for the convergence speed of Picard iterations for a linear example PDE which we prove in Lemma 2.1 in Section 2.1. In Lemma 3.1 in Section 3 we establish explicit a priori estimates for certain backward Itô processes in appropriate \( L^2 \)-norms. In Section 4 we provide upper bounds for the convergence speed of Picard iterations of BSDEs. Proposition 4.1 establishes an explicit bound for the \( L^2 \)-distance between the Picard iterations and the solution of a BSDE with a globally Lipschitz continuous nonlinearity. In Remark 4.2 we employ the estimate of Proposition 4.1 to obtain the square root-factorial speed of convergence of Picard iterations. In Remark 4.3 we employ the estimate of Proposition 4.1 to obtain the factorial speed of convergence of Picard iterations in the \( z \)-independent case.
2 Lower bounds for the convergence speed of Picard iterations

In this section we provide lower bounds for the convergence speed of Picard iterations of BSDEs. In Lemma 2.1 in Section 2.1 we establish lower bounds for the convergence speed of Picard iterations for a linear example PDE. We employ Lemma 2.1 in our proof of Corollary 2.2 in Section 2.2 to provide lower bounds for the convergence speed of Picard iterations for a linear example BSDE. Corollary 2.2 shows that square-root factorial convergence speed cannot be improved up to exponential factors in the case of $z$-dependent drivers. Item (ii) of Theorem 1.1 shows in the case of $z$-independent drivers that factorial speed of convergence is possible. Lemma 2.3 observes that factorial speed of convergence cannot be improved up to exponential factors in the case of $y$-dependent drivers.

2.1 Lower bounds for the convergence speed of Picard iterations for an example PDE

**Lemma 2.1.** Let $d \in \mathbb{N}$, $b = (b_1, b_2, \ldots, b_d) \in \mathbb{R}^d$, let $\langle \cdot, \cdot \rangle: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ denote the standard scalar product on $\mathbb{R}^d$, let $\| \cdot \|: \mathbb{R}^d \to [0, \infty)$ denote the standard norm on $\mathbb{R}^d$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W = (W^1, W^2, \ldots, W^d): [0, 1] \times \Omega \to \mathbb{R}^d$ be a standard Brownian motion, let $v^n: [0, 1] \times \mathbb{R}^d \to \mathbb{R}$, $n \in \mathbb{N}_0 \cup \{\infty\}$, satisfy for all $t \in [0, 1]$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ that $v^0(t, x) = 0$,

$$v^n(t, x) = \mathbb{E}\left[ 2^{d/2} \exp\left(-\frac{\|x+W_1-W_t\|^2}{2}\right) \right] + \sum_{k=1}^{n-1} \sum_{\mu_1, \mu_2, \ldots, \mu_k=1}^{d} \left[ \frac{(1-\gamma)^k}{k!} b_{\mu_1} b_{\mu_2} \cdots b_{\mu_k} \mathbb{E}\left[ 2^{d/2} \frac{\partial^k}{\partial x_{\mu_1} \partial x_{\mu_2} \cdots \partial x_{\mu_k}} \exp\left(-\frac{\|x+W_1-W_t\|^2}{2}\right) \right] \right],$$

and

$$v^\infty(t, x) = \mathbb{E}\left[ 2^{d/2} \exp\left(-\frac{\|x+b(1-t)+W_1-W_t\|^2}{2}\right) \right].$$

Then

(i) it holds for all $t \in [0, 1]$, $x \in \mathbb{R}^d$ that $v^\infty \in C^\infty([0, 1] \times \mathbb{R}^d, \mathbb{R})$, $v^\infty(1, x) = 2^{d/2} e^{-\frac{1}{2}\|x\|^2}$, and

$$\frac{\partial v^\infty}{\partial t}(t, x) + \frac{1}{2}(\Delta_x v^\infty)(t, x) + \langle b, (\nabla_x v^\infty)(t, x) \rangle = 0,$$

(ii) it holds for all $n \in \mathbb{N}_0$, $t \in [0, 1]$, $x \in \mathbb{R}^d$ that $v^n \in C^\infty([0, 1] \times \mathbb{R}^d, \mathbb{R})$, $v^n(1, x) = 2^{d/2} e^{-\frac{1}{2}\|x\|^2}$,

$$v^{n+1}(t, x) = \mathbb{E}\left[ 2^{d/2} \exp\left(-\frac{\|x+W_1-W_t\|^2}{2}\right) \right] + \int_t^1 \mathbb{E}\left[ \langle b, (\nabla_x v^n)(s, x+W_s-W_t) \rangle \right] ds,$$

and

$$\nabla_x v^{n+1}(t, x) = \mathbb{E}\left[ 2^{d/2} \exp\left(-\frac{\|x+W_1-W_t\|^2}{2}\right) \frac{W_1-W_t}{1-t} \right] + \int_t^1 \mathbb{E}\left[ \langle b, (\nabla_x v^n)(s, x+W_s-W_t) \rangle \frac{W_s-W_t}{s-t} \right] ds.$$
(iii) it holds for all \( n \in \mathbb{N} \) that
\[
v^n(0, 0) = 1 + \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{(-1)^i \|b\|^{2i}}{4^i i!},
\]
(10)

(iv) it holds that
\[
v^\infty(0, 0) = \exp\left(-\frac{\|b\|^2}{4}\right),
\]
and
\[
(\|b\|^2/4)^{\lfloor n/2 \rfloor} \frac{1 - \epsilon}{\lfloor n/2 \rfloor!} \leq |v^\infty(0, 0) - v^n(0, 0)| \leq \left(\frac{\|b\|^2}{4}\right)^{\lfloor n/2 \rfloor} \frac{1}{\lfloor n/2 \rfloor!} 1 - \epsilon.
\]
(12)

Proof of Lemma 2.1. First note that the Feynman-Kac formula (cf., e.g., [16, Theorem 8.2.1]) proves (i).

Next observe that (5) proves for all \( \ell \in \{1, 2, \ldots, d\}, s \in [0, 1), x \in \mathbb{R}^d, n \in \mathbb{N} \) that \( v^n \in C^\infty([0, 1] \times \mathbb{R}^d, \mathbb{R}), v^n(1, x) = q^{d/2}e^{-\frac{\|x\|^2}{4}}, \) and
\[
\begin{align*}
\frac{\partial v^n}{\partial x}(s, x) - \mathbb{E}\left[2^{d/2} \frac{\partial}{\partial x} \exp\left(-\frac{\|x + W_{1/2} - W_{1/2}\|^2}{2}\right)\right] &= \frac{\partial v^n}{\partial x}(s, x) - \frac{\partial}{\partial x} \mathbb{E}\left[2^{d/2} \exp\left(-\frac{\|x + W_{1/2} - W_{1/2}\|^2}{2}\right)\right] \\
&= \sum_{k=1}^{n-1} \sum_{\mu_1, \mu_2, \ldots, \mu_k=1}^{d} \left[ \frac{(1-s)^k}{k!} b_{\mu_1} b_{\mu_2} \ldots b_{\mu_k} \frac{\partial}{\partial x} \mathbb{E}\left[2^{d/2} \frac{\partial}{\partial x_{\mu_1}} \frac{\partial}{\partial x_{\mu_2}} \ldots \frac{\partial}{\partial x_{\mu_k}} \exp\left(-\frac{\|x + W_{1/2} - W_{1/2}\|^2}{2}\right)\right] \right]
\end{align*}
\]
(13)

This, the disintegration theorem, and independence of Brownian increments show for all \( t \in [0, 1), s \in (t, 1), x \in \mathbb{R}^d, \ell \in \{1, 2, \ldots, d\}, n \in \mathbb{N} \) that
\[
\begin{align*}
\mathbb{E}\left[\frac{\partial v^n}{\partial x}(s, x + W_s - W_t)\right] - \mathbb{E}\left[2^{d/2} \frac{\partial}{\partial x} \exp\left(-\frac{\|x + W_{1/2} - W_{1/2}\|^2}{2}\right)\right] &= \mathbb{E}\left[\frac{\partial v^n}{\partial x}(s, x + W_s - W_t)\right] - \mathbb{E}\left[2^{d/2} \frac{\partial}{\partial x} \exp\left(-\frac{\|x + W_{1/2} - W_{1/2}\|^2}{2}\right)\right]_{|z=x+W_s-W_t} \\
&= \sum_{k=1}^{n-1} \sum_{\mu_1, \mu_2, \ldots, \mu_k=1}^{d} \left[ \frac{(1-s)^k}{k!} b_{\mu_1} b_{\mu_2} \ldots b_{\mu_k} \right.
\end{align*}
\]
(14)

This, the fact that \( \forall k \in \mathbb{N}_0, t \in [0, 1]: \int_t^1 \frac{(1-s)^k}{k!} ds = \frac{(1-t)^{k+1}}{(k+1)!}, \) and (5) show for all \( t \in [0, 1), x \in \mathbb{R}^d, n \in \mathbb{N} \) that
\[
\int_t^1 \mathbb{E}\left[\langle b, (\nabla_x v^n)(s, x + W_s - W_t)\rangle\right] ds = \sum_{\mu_{k+1}=1}^{d} \int_t^1 \mathbb{E}\left[b_{\mu_{k+1}} \frac{\partial v^n}{\partial x_{\mu_{k+1}}}(s, x + W_s - W_t)\right] ds
\]
\[
= \sum_{k=0}^{n-1} \sum_{\mu_1, \mu_2, \ldots, \mu_k = 1}^{d} \int_0^1 \frac{(1-s)^k}{k!} b_{\mu_1} b_{\mu_2} \cdots b_{\mu_k+1} \mathbb{E} \left[ 2^{d/2} \frac{\partial^{k+1}}{\partial x_{\mu_1} \partial x_{\mu_2} \cdots \partial x_{\mu_k+1}} \exp \left( -\frac{\|x+\hat{W}_t-W_t\|^2}{2} \right) \right] ds
\]
\[
= \sum_{k=0}^{n-1} \sum_{\mu_1, \mu_2, \ldots, \mu_k = 1}^{d} \left[ \frac{(1-s)^k}{(k+1)!} b_{\mu_1} b_{\mu_2} \cdots b_{\mu_k} \mathbb{E} \left[ 2^{d/2} \frac{\partial^{k+1}}{\partial x_{\mu_1} \partial x_{\mu_2} \cdots \partial x_{\mu_k}} \exp \left( -\frac{\|x+\hat{W}_t-W_t\|^2}{2} \right) \right] \right]
\]
\[
= \sum_{k=1}^{n} \sum_{\mu_1, \mu_2, \ldots, \mu_k = 1}^{d} \left[ \frac{(1-s)^k}{k!} b_{\mu_1} b_{\mu_2} \cdots b_{\mu_k} \mathbb{E} \left[ 2^{d/2} \frac{\partial^{k}}{\partial x_{\mu_1} \partial x_{\mu_2} \cdots \partial x_{\mu_k}} \exp \left( -\frac{\|x+\hat{W}_t-W_t\|^2}{2} \right) \right] \right]
\]
\[
v^{n+1}(t, x) - \mathbb{E} \left[ 2^{d/2} \exp \left( -\frac{\|x+\hat{W}_t-W_t\|^2}{2} \right) \right].
\] (15)

This, the fact that \( \forall t \in [0, 1], x \in \mathbb{R}^d: v^0(t, x) = 0 \), and (5) show for all \( t \in [0, 1], x \in \mathbb{R}^d, n \in \mathbb{N}_0 \) that
\[
v^{n+1}(t, x) = \mathbb{E} \left[ 2^{d/2} \exp \left( -\frac{\|x+\hat{W}_t-W_t\|^2}{2} \right) \right] + \int_0^t \mathbb{E} \left[ \left( b, (\nabla_x v^n)(s, x + W_s - W_t) \right) \right] ds.
\] (16)

Next note that Stein’s lemma proves for all \( \ell \in \{1, 2, \ldots, d\}, x \in \mathbb{R}^d, s \in (0, 1], t \in [0, s), h \in C^1(\mathbb{R}^d, \mathbb{R}) \) with \( \sup_{y \in \mathbb{R}^d} (|h(y)| + |\frac{\partial h}{\partial y}(y)|) < \infty \) that
\[
\frac{\partial}{\partial x_\ell} \mathbb{E}[h(x + W_s - W_t)] = \mathbb{E} \left[ \frac{\partial h}{\partial x_\ell}(x + W_s - W_t) \right] = \mathbb{E} \left[ h(x + W_s - W_t) \frac{W_\ell^s - W_\ell^t}{s - t} \right].
\] (17)

This, (16), differentiation under integrals, the fact that \( \forall \ell \in \{1, 2, \ldots, d\}, n \in \mathbb{N}_0, s \in (0, 1]: \sup_{x \in \mathbb{R}^d} \left[ \exp \left( -\frac{\|x\|^2}{2} \right) + \frac{\partial}{\partial x_\ell} \exp \left( -\frac{\|x\|^2}{2} \right) \right] + \left| \left( b, (\nabla_x v^n)(s, x) \right) \right| + \left| \frac{\partial}{\partial x_\ell} \left( b, (\nabla_x v^n)(s, x) \right) \right| < \infty \), and the fact that \( \forall t \in [0, 1], x \in \mathbb{R}^d: v^0(t, x) = 0 \) show for all \( \ell \in \{1, 2, \ldots, d\}, n \in \mathbb{N}_0, t \in [0, 1], x \in \mathbb{R}^d \) that
\[
\frac{\partial v^{n+1}}{\partial x_\ell}(t, x) = \frac{\partial}{\partial x_\ell} \mathbb{E} \left[ 2^{d/2} \exp \left( -\frac{\|x+\hat{W}_t-W_t\|^2}{2} \right) \right] + \int_0^1 \frac{\partial}{\partial x_\ell} \mathbb{E} \left[ \left( b, (\nabla_x v^n)(s, x + W_s - W_t) \right) \right] ds
\]
\[
= \mathbb{E} \left[ 2^{d/2} \exp \left( -\frac{\|x+\hat{W}_t-W_t\|^2}{2} \right) \right] \frac{W_\ell^s - W_\ell^t}{s - t}
\]
\[
+ \int_0^1 \mathbb{E} \left[ \left( b, (\nabla_x v^n)(s, x + W_s - W_t) \right) \right] \frac{W_\ell^s - W_\ell^t}{s - t} ds.
\] (18)

This and (16) show (ii).

For the next step let \( 0^0 = 1 \), let \( H_k: \mathbb{R} \to \mathbb{R}, k \in \mathbb{N}_0 \), satisfy for all \( k \in \mathbb{N}_0, x \in \mathbb{R} \) that
\[
H_k(x) = \sum_{\ell = 0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!(-1)^\ell x^{k-2\ell}}{\ell!(k-2\ell)!} \left( \frac{x}{2} \right)^{k-2\ell}
\] (19)

and for every \( n \in \mathbb{N}_0 \cup \{-1\} \) let \( n!! \in \mathbb{N} \) satisfy that \( n!! = \prod_{k=0}^{[n/2]-1} (n - 2k) \). A well-known fact on Hermite polynomials shows for all \( x \in \mathbb{R}, k \in \mathbb{N}_0 \) that \( \frac{\partial^k}{\partial x^k} (e^{-x^2}) = e^{-x^2} H_k(x) \). Furthermore, a well-known fact on moments of normally distributed random variables shows for all \( k \in \mathbb{N}_0, \ell \in [0, k] \cap \mathbb{N}_0 \) that
\[
\frac{1}{\sqrt{\pi}} \int_\mathbb{R} z^{2k+1-2\ell} e^{-z^2} dz = \left[ \frac{1}{\sqrt{2\pi\sigma^2}} \int_\mathbb{R} z^{2k-2\ell} e^{-\frac{z^2}{2\sigma^2}} dz \right]_{\sigma^2=\frac{1}{2}} \left( \frac{2k-2\ell-1}{2k-\ell} \right)!
\] (20)
This, (19), and the binomial theorem imply for all $k \in \mathbb{N}_0$ that

$$
\mathbb{E} \left[ \sqrt{2} \exp \left( -\frac{|W_1|^2}{2} \right) H_{2k}(W_1) \right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{2} e^{-\frac{z^2}{2}} \left[ \sum_{\ell=0}^{k} \frac{(2k)!(-1)^{\ell}}{\ell!(2\ell-2k)!} \frac{2k^{2\ell}}{2^{\ell}} \right] e^{-\frac{z^2}{2}} \, dz
$$

$$
= \sum_{\ell=0}^{k} \frac{(2k)!(-1)^{\ell}}{\ell!(2\ell-2k)!} \left( \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} z^{2\ell-2k} e^{-z^2} \, dz \right) = \sum_{\ell=0}^{k} \frac{(2k)!(-1)^{\ell}}{\ell!(2\ell-2k)!} \frac{(2k-2\ell-1)!}{2^{k-\ell}}
$$

$$
= \sum_{\ell=0}^{k} \frac{(2k)!(-1)^{\ell}}{\ell!(k-\ell)!} \frac{(-2)^{\ell}}{2^{k-\ell}} = \frac{(2k)!(-1)^{\ell}}{4^{k-\ell} k!} \sum_{i=0}^{k} \frac{(-2)^i}{(k-i)!} = \frac{(2k)!(-1)^{k}}{4^{k-1} k!} \tag{21}
$$

and

$$
\mathbb{E} \left[ \sqrt{2} \exp \left( -\frac{|W_1|^2}{2} \right) H_{2k+1}(W_1) \right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{2} e^{-\frac{z^2}{2}} \left[ \sum_{\ell=0}^{(2k+1)/2} \frac{(2k+1)!(-1)^{\ell}}{\ell!(2k+1-2\ell)!} \frac{2^{k-\ell}}{2^{k-\ell}} \right] e^{-\frac{z^2}{2}} \, dz = 0. \tag{22}
$$

Thus, it holds for all $k \in \mathbb{N}_0$ that

$$
\mathbb{E} \left[ \sqrt{2} \exp \left( -\frac{|W_1|^2}{2} \right) H_k(W_1) \right] = 1_{2\mathbb{N}_0}(k) \frac{k!(-1)^{|k/2|}}{4^{k/2}|k/2|!}. \tag{23}
$$

Furthermore, the combinatorial interpretation of multinomial coefficients yields for all $k \in \mathbb{N}$, $\alpha \in (\mathbb{N}_0)^k$ with $|\alpha| = k$ that

$$
\# \bigg\{ \bigcap_{i=1}^{d} \left\{ (\mu_1, \mu_2, \ldots, \mu_k) \in \{1, 2, \ldots, d\}^{k}: \# \{\ell \in \{1, 2, \ldots, k\}: \mu_\ell = i\} = \alpha_i \right\} \bigg\} = \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_d!}
$$

$$
= \# \left\{ \left( \bigcap_{i=1}^{d} \bigcup_{\alpha \in (\mathbb{N}_0)^k: |\alpha| = k} \bigg\{ (\mu_1, \mu_2, \ldots, \mu_k) \in \{1, 2, \ldots, d\}^{k}: \# \{\ell \in \{1, 2, \ldots, k\}: \mu_\ell = i\} = \alpha_i \right\} \right\} \bigg\}.
$$

This, the fact that $W^1, W^2, \ldots, W^d$ are independent, (5), and the fact that $\forall x \in \mathbb{R}, k \in \mathbb{N}_0: \frac{d^k}{dx^k}(e^{-\frac{x^2}{2}}) = e^{-\frac{x^2}{2}} H_k(x)$ show for all $n \in \mathbb{N}$ that

$$
v^n(0, 0) - \prod_{\ell=1}^{d} \mathbb{E} \left[ \sqrt{2} \exp \left( -\frac{(W_\ell^1)^2}{2} \right) \right] = e^n(0, 0) - \mathbb{E} \left[ \frac{2^{d/2}}{\sqrt{\pi}} \exp \left( -\frac{|W_1|^2}{2} \right) \right]
$$

$$
= \sum_{k=1}^{n-1} \sum_{\mu_1, \mu_2, \ldots, \mu_k=1}^{d} \left[ \frac{1}{k!} b_{\mu_1} b_{\mu_2} \cdots b_{\mu_k} \mathbb{E} \left[ \frac{\partial^k}{\partial x_{\mu_1} \partial x_{\mu_2} \cdots \partial x_{\mu_k}} \exp \left( -\frac{|x + W_1|^2}{2} \right) \right] \right]_{x=0}
$$

$$
= \sum_{\alpha \in (\mathbb{N}_0)^k: |\alpha| = k} \prod_{\ell=1}^{d} \mathbb{E} \left[ \frac{b_{\ell}}{\alpha_\ell!} \frac{\partial^{\alpha_\ell}}{\partial x_{\ell}^{\alpha_\ell}} \exp \left( -\frac{(x_{\ell} + W_{\ell}^1)^2}{2} \right) \right]_{x_{\ell}=0}
$$

$$
= \sum_{\alpha \in (\mathbb{N}_0)^k: |\alpha| = k} \prod_{\ell=1}^{d} \mathbb{E} \left[ \frac{b_{\ell}^{\alpha_\ell}}{\alpha_\ell!} \frac{\partial^{\alpha_\ell}}{\partial x_{\ell}^{\alpha_\ell}} \exp \left( -\frac{(W_{\ell}^1)^2}{2} \right) H_{\alpha_\ell}(W_{\ell}^1) \right]. \tag{25}
$$
This, \( (23) \), and the multinomial theorem show for all \( n \in \mathbb{N} \) that
\[
\begin{align*}
v^n(0, 0) &= \sum_{k=0}^{n-1} \left( \sum_{\alpha \in (\mathbb{N}_0)^d : |\alpha| = k} \prod_{\ell=1}^d \frac{b_{\ell}^{\alpha_\ell}}{\alpha_\ell!} 1_{2N_0}(\alpha_\ell) \frac{\alpha_\ell!(-1)^{|\alpha_\ell|/2}}{4^{\alpha_\ell/2}[\alpha_\ell/2]!} \right) \\
&= \sum_{i=0}^{n-1} \left( \prod_{\beta \in (\mathbb{N}_0)^d : |\beta| = i} \frac{b_{\ell}^{2\beta_\ell}}{4^\ell \beta_\ell!} \right) \sum_{i=0}^{n-1} \left( \prod_{\beta \in (\mathbb{N}_0)^d : |\beta| = i} \frac{(-1)^i}{4^i} \right) \\
&= \sum_{i=0}^{n-1} \left( \prod_{\beta \in (\mathbb{N}_0)^d : |\beta| = i} \frac{(-1)^i}{4^i \beta_\ell!} \right) \sum_{i=0}^{n-1} \left( \prod_{\beta \in (\mathbb{N}_0)^d : |\beta| = i} \frac{b_{\ell}^{2\beta_\ell}}{4^\ell \beta_\ell!} \right). \quad (26)
\end{align*}
\]
This establishes (iii).

Next observe that for all \( a \in \mathbb{R}, \ell \in \{1, 2, \ldots, d\} \) it holds that
\[
\mathbb{E} \left[ \exp \left( -\frac{(a + W_1^\ell)^2}{2} \right) \right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{(a + x)^2}{2}} e^{-z^2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{a^2}{2}} e^{-z^2} e^{\frac{a^2}{2}} dx = e^{-\frac{a^2}{2}}. \quad (27)
\]
This, \( (6) \), and the fact that \( W_1^1, W_1^2, \ldots, W_1^d \) are independent prove that
\[
\begin{align*}
v^\infty(0, 0) &= \mathbb{E} \left[ 2^{d/2} \exp \left( -\frac{\|b + W_1^\ell\|^2}{2} \right) \right] = 2^{d/2} \prod_{\ell=1}^d \mathbb{E} \left[ \exp \left( -\frac{(b_\ell + W_1^\ell)^2}{2} \right) \right] = e^{-\frac{\|b\|^2}{2}}. \quad (28)
\end{align*}
\]
This shows (iv).

Next note that (iii), (iv), and the fact that \( \forall x \in \mathbb{R}: e^{-\frac{x^2}{2}} = 1 + \sum_{i=1}^{\infty} \frac{(-1)^i x^{2i}}{4^i i!} \) show for all \( n \in \mathbb{N} \) that
\[
\begin{align*}
v^\infty(0, 0) - v^n(0, 0) &= \sum_{i=1}^{\infty} \frac{(-1)^i \|b\|^{2i}}{i!4^i} = \sum_{i=\lfloor \frac{n+1}{2} \rfloor}^{\infty} \frac{(-1)^i \|b\|^{2i}}{i!4^i}. \quad (29)
\end{align*}
\]
Therefore, for all \( \epsilon \in (0, 1), n \in \mathbb{N} \) with \( j = \left\lfloor \frac{n+1}{2} \right\rfloor \) it holds that
\[
\frac{1}{2} \|b\|^2 \leq n + 1 = 2 \frac{n+1}{2} \leq 2\left( \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \right) = 2(j+1), \quad \frac{\|b\|^2}{(j+1)!} \leq \epsilon,
\]
\[
\begin{align*}
&(v^\infty(0, 0) - v^n(0, 0))(-1)^j = \left( \frac{\|b\|^{2j}}{j!4^j} - \frac{\|b\|^{2(j+1)}}{(j+1)!4^{j+1}} \right) + \left( \frac{\|b\|^{2(j+2)}}{(j+2)!4^{j+2}} - \frac{\|b\|^{2(j+3)}}{(j+3)!4^{j+3}} \right) + \ldots \\
&= \frac{\|b\|^{2j}}{j!4^j} \left( 1 - \frac{\|b\|^2}{4(j+1)} \right) + \frac{\|b\|^{2(j+2)}}{(j+2)!4^{j+2}} \left( 1 - \frac{\|b\|^2}{4(j+3)} \right) + \ldots \\
&\geq \frac{\|b\|^{2j}}{j!4^j} \left( 1 - \frac{\|b\|^2}{4(j+1)} \right) \geq \left( \frac{\|b\|^2}{4} \right)^j \frac{1}{j!}(1 - \epsilon), \quad (30)
\end{align*}
\]
and
\[
\left| v^\infty(0, 0) - v^n(0, 0) \right| \leq \sum_{i=1}^{\infty} \frac{\|b\|^{2i}}{i!4^i} \left( 1 + \sum_{i=1}^{\infty} \frac{\|b\|^2}{4(i+1)} \right)^{i-j} \leq \frac{\|b\|^{2j}}{j!4^j} \sum_{i=0}^{\infty} \epsilon^i = \left( \frac{\|b\|^2}{4} \right)^j \frac{1}{j!} \frac{1}{1 - \epsilon}. \quad (31)
\]
This establishes (v). The proof of Lemma 2.1 is thus completed.
2.2 Lower bounds for the convergence speed of Picard iterations for an example BSDE

**Corollary 2.2.** Let \( d \in \mathbb{N}, b \in \mathbb{R}^d \), let \( \langle , \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) denote the standard scalar product on \( \mathbb{R}^d \), let \( \| \cdot \| : \mathbb{R}^d \to [0, \infty) \) denote the standard norm on \( \mathbb{R}^d \), let \( (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0,1]}) \) be a filtered probability space which satisfies the usual conditions, let \( W : [0,1] \times \Omega \to \mathbb{R}^d \) be a standard \((\mathbb{F}_t)_{t \in [0,1]}\)-Brownian motion with continuous sample paths, let \( Y^n : [0,1] \times \Omega \to \mathbb{R} \), \( n \in \mathbb{N}_0 \cup \{\infty\} \), be adapted with continuous sample paths, let \( Z^n : [0,1] \times \Omega \to \mathbb{R}^d, n \in \mathbb{N}_0 \cup \{\infty\} \), be progressively measurable, and assume for all \( s \in [0,1], n \in \mathbb{N} \cup \{\infty\} \) that a.s. it holds that \( Y^n_0 = 0, Z^n_0 = 0 \) for all \( n \in \mathbb{N} \cap [\|b\|^2 - 1, \infty) \) it holds a.s. that \( |Y^n_0 - Y^n_0| \geq \frac{1}{2} \left( \frac{\|b\|^2}{4} \right)^{\frac{n+1}{2}} \frac{1}{\sqrt{n}} \).

**Proof of Corollary 2.2.** Throughout this proof let \( v^n : [0,1] \times \mathbb{R}^d \to \mathbb{R}, n \in \mathbb{N}_0 \cup \{\infty\} \), satisfy for all \( t \in [0,1], x \in \mathbb{R}^d, n \in \mathbb{N} \) that \( v^n(t, x) = 0 \),

\[
v^n(s, x) = \mathbb{E}\left[2^{d/2} \exp\left(-\frac{\|x + b(1 - s) + W_1 - W_s\|^2}{2}\right)\right],
\]

and

\[
v^n(s, x) = \mathbb{E}\left[2^{d/2} \exp\left(-\frac{\|x + W_1 - W_s\|^2}{2}\right)\right]
+ \sum_{k=1}^{n-1} \sum_{\mu_1, \mu_2, \ldots, \mu_k=1}^d \left[ \frac{(1 - s)^k}{k!} b_{\mu_1} b_{\mu_2} \cdots b_{\mu_k} \mathbb{E}\left[2^{d/2} \partial^{\mu_1} \partial^{\mu_2} \cdots \partial^{\mu_k} \exp\left(-\frac{\|x + W_1 - W_s\|^2}{2}\right)\right] \right].
\]

Then Lemma 2.1 proves

a) for all \( t \in [0,1], x \in \mathbb{R}^d \) that \( v^n \in C^\infty([0,1] \times \mathbb{R}^d, \mathbb{R}) \) and

\[
\frac{\partial v^n}{\partial t}(t, x) + \frac{1}{2} (\Delta_x v^n)(t, x) + \langle b, (\nabla_x v^n)(t, x) \rangle = 0,
\]

b) for all \( s \in [0,1], x \in \mathbb{R}^d, n \in \mathbb{N}_0 \) that \( v^n \in C^\infty([0,1] \times \mathbb{R}^d, \mathbb{R}) \) and

\[
v^{n+1}(s, x) = \mathbb{E}\left[2^{d/2} \exp\left(-\frac{\|x + W_1 - W_s\|^2}{2}\right)\right] + \int_s^1 \mathbb{E}\left[\langle b, (\nabla_x v^n)(t, x + W_1 - W_s) \rangle\right] dt,
\]

and

c) for all \( n \in \mathbb{N} \cap [\|b\|^2 - 1, \infty) \) that

\[
|v^n(0, 0) - v^n(0, 0)| \geq \frac{1}{2} \left( \frac{\|b\|^2}{4} \right)^{\frac{n+1}{2}} \frac{1}{\sqrt{n}}.
\]
This and Itô’s formula prove that for all $s \in [0, 1]$ it holds a.s. that
\[
2^{d/2}e^{-\frac{|W_1|^2}{2}} - v^\infty(s, W_s) = v^\infty(1, W_1) - v^\infty(s, W_s)
\]
\[
= \int_s^1 \left( \frac{\partial v^\infty}{\partial t} + \frac{1}{2} \triangle_x v^\infty \right) (t, W_t) \, dt + \int_s^1 \langle (\nabla_x v^\infty)(t, W_t), dW_t \rangle
\]
\[
= -\int_s^1 \langle b, (\nabla_x v^\infty)(t, W_t) \rangle \, dt + \int_s^1 \langle (\nabla_x v^\infty)(t, W_t), dW_t \rangle.
\] (38)

This, (32), and a standard result on uniqueness of backward stochastic differential equations (cf., e.g., [21, Theorem 4.3.1]) prove for all $s \in [0, 1]$ that $P(Y_s^\infty = v^\infty(s, W_s)) = 1$ and $P(Z_s^\infty = (\nabla_x v^\infty)(s, W_s)) = 1$.

Next, we prove by induction on $n \in \mathbb{N}_0$ that for all $n \in \mathbb{N}_0$, $s \in [0, 1]$ it holds that $P(Y_s^n = v^n(s, W_s)) = 1$ and $P(Z_s^n = (\nabla_x v^n)(s, W_s)) = 1$. First, the fact that $\forall t \in [0, 1], x \in \mathbb{R}^d$: $v^0(t, x) = 0$ and the fact that $\forall s \in [0, 1]: P(Y_s^0, Z_s^0) = (0, 0) = 1$ establish the base case $n = 0$. For the induction step $\mathbb{N}_0 \ni n \mapsto n + 1 \in \mathbb{N}$ let $n \in \mathbb{N}_0$ satisfy for all $s \in [0, 1]$ that $P(Y_s^n = v^n(s, W_s)) = 1$ and $P(Z_s^n = (\nabla_x v^n)(s, W_s)) = 1$. This, (36), the Markov property of $W$, the fact that for all $s \in [0, 1]$ it holds a.s. that $E[\int_s^1 (Z_t^{n+1}, dW_t)|F_s] = 0$, (32), and adaptedness of $Y^{n+1}$ imply that for all $s \in [0, 1]$ it holds a.s. that
\[
v^{n+1}(s, W_s) = E\left[2^{d/2}e^{-\frac{|W_1|^2}{2}}\right|F_s] + \int_s^1 E\left[\langle b, (\nabla_x v^n)(t, W_t) \rangle\right|F_s] \, dt
\]
\[
= E\left[2^{d/2}e^{-\frac{|W_1|^2}{2}}\right|F_s] + \int_s^1 E\left[\langle b, Z_t^n \rangle\right|F_s] \, dt
\]
\[
= E\left[2^{d/2}e^{-\frac{|W_1|^2}{2}}\right] + \int_s^1 \langle b, Z_t^n \rangle \, dt + \int_s^1 \langle Z_t^{n+1}, dW_t \rangle \right|F_s] = E[Y_s^{n+1}|F_s] = Y_s^{n+1}.
\] (39)

This, Itô’s formula, the fact that $v^{n+1} \in C^2([0, 1] \times \mathbb{R}^d, \mathbb{R})$, (34), and (32) show that for all $s \in [0, 1]$ it holds a.s. that
\[
0 = v^{n+1}(s, W_s) - Y_s^{n+1} = -\int_s^1 \left( \frac{\partial v^{n+1}}{\partial t} + \frac{1}{2} \triangle_x v^{n+1} \right)(t, W_t) - \langle b, Z_t^n \rangle \, dt
\]
\[
-\int_s^1 \langle (\nabla_x v^n)(t, W_t) - Z_t^{n+1}, dW_t \rangle.
\] (40)

This and the uniqueness of the decomposition of continuous semimartingales show for all $s \in [0, 1]$ that $P(Y_s^{n+1} = v^{n+1}(s, W_s)) = 1$ and $P(Z_s^{n+1} = (\nabla_x v^{n+1})(s, W_s)) = 1$. This completes the induction step. Induction thus shows for all $n \in \mathbb{N}_0$, $s \in [0, 1]$ that $P(Y_s^n = v^n(s, W_s)) = 1$ and $P(Z_s^n = (\nabla_x v^n)(s, W_s)) = 1$. This and the fact that $P(Y_0^\infty = v^\infty(0, W_0)) = 1$ imply that for all $n \in \mathbb{N}$ it holds a.s. that $Y_0^n - Y_0^\infty = v^n(0, 0) - v^\infty(0, 0)$. This, the fact that for all $k, n \in \mathbb{N}_0$ with $n = 2k$ it holds that
\[
\left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{2k+1}{2} \right\rfloor = k.
\] (41)

the fact that for all $k, n \in \mathbb{N}_0$ with $n = 2k + 1$ it holds that
\[
\left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{2k+1+1}{2} \right\rfloor = (k+1)!
\] (42)
and (37) imply that for all $n \in \mathbb{N} \cap [\|b\|^2 - 1, \infty)$ it holds a.s. that
\[
|Y_0^n - Y_0^\infty| = |v^n(0, 0) - v^\infty(0, 0)| \geq \frac{1}{2} \left( \frac{\|b\|^2}{4} \right)^{\frac{1}{2n}} \frac{1}{\sqrt{n}}.
\] (43)

This completes the proof of Corollary 2.2. \qed

The following Lemma 2.3 gives an example where the BSDE solution is an exponential function (see Item (i)) and the Picard approximations are just partial sums of the exponential series (see Item (ii)). Thereby, Lemma 2.3 shows that factorial speed of convergence cannot be improved up to exponential factors in the case of $y$-dependent drivers.

**Lemma 2.3.** Let $T \in (0, \infty)$ and let $Y^k \in C([0, T], \mathbb{R})$, $k \in \mathbb{N}_0 \cup \{\infty\}$, satisfy for all $k \in \mathbb{N}_0$, $s \in [0, T]$ that $Y_s^0 = 1$, $Y_s^{k+1} = 1 + \int_s^T Y_r^k \, dr$, and $Y_\infty^s = 1 + \int_s^T Y_r^\infty \, dr$. Then
\begin{itemize}
  \item[i)] it holds for all $s \in [0, T]$ that $Y_\infty^s = e^{T-s},$
  \item[ii)] it holds for all $s \in [0, T]$, $n \in \mathbb{N}_0$ that $Y_s^n = 1 + \sum_{k=1}^{n} \frac{(T-s)^k}{k!}$, and
  \item[iii)] it holds that $\sup_{s \in [0,T]} |Y_\infty^s - Y_s^n| = \sum_{k=n+1}^{\infty} \frac{T^k}{k!(n+1)!} \geq \frac{T^{n+1}}{(n+1)!}.$
\end{itemize}

**Proof of Lemma 2.3.** The fact that $\forall s \in [0,T]: Y_\infty^s = 1 + \int_s^T Y_r^\infty \, dr$ and the substitution rule show for all $s \in [0, T]$ that $Y_\infty^s = 1 + \int_s^T Y_r^\infty \, dr = 1 + \int_0^s Y_s^\infty \, dr$. This, the fact that $\forall s \in [0,T]: e^{T-s} = 1 + \int_0^s e^{T-r} \, dr$, and the Picard–Lindelöf theorem show (i). Next, we prove (ii) by induction on $n \in \mathbb{N}_0$. The fact that $\forall s \in [0,T]: Y_s^0 = 1$ shows the base case $n = 0$. For the induction step $\mathbb{N}_0 \ni n \mapsto n+1 \in \mathbb{N}$ let $n \in \mathbb{N}_0$ and assume for all $s \in [0,T]$ that $Y_s^n = 1 + \sum_{k=1}^{n} \frac{(T-s)^k}{k!}$. The assumptions of Lemma 2.3 then show for all $s \in [0, T]$ that
\[
Y_s^{n+1} = 1 + \int_s^T Y_r^n \, dr = 1 + \int_s^T \left( 1 + \sum_{k=1}^{n} \frac{(T-r)^k}{k!} \right) \, dr
\] (44)
\[
= 1 + \sum_{k=1}^{n+1} \frac{-(T-r)^{k+1}}{(k+1)!} \bigg|_{r=s} = 1 + \sum_{k=1}^{n+1} \frac{(T-s)^k}{k!}.
\]
This completes the induction step. Induction hence shows (ii). Combining (i), (ii), and the fact that $\forall x \in \mathbb{R}: e^x = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$ yields (iii). The proof of Lemma 2.3 is thus completed. \qed

### 3 A priori estimates for backward Itô processes

In this section we establish a priori estimates for certain backward Itô processes. Results of this form are well-known in the scientific literature on BSDEs (see, e.g., [17, Proof of Theorem 3.1], [8, Proposition 2.1], [21, Theorem 4.2.1], [18, Proposition 5.2]). Lemma 3.1 below establishes estimates for an Itô process and its diffusion process in terms of the drift process and in terms of the terminal value of the Itô process. The contribution of Lemma 3.1 is to provide explicit universal constants. Moreover, the Itô process in Lemma 3.1 and its drift process are not assumed to be square-integrable and, in particular, the right-hand sides of (46), (47), and (48) are allowed to be infinite (with positive probability). We note that square-integrability of the diffusion process $Z$ in Lemma 3.1, however, is in general required; e.g., choose $A \equiv 0$ and $Z$ such that the Itô isometry does not hold for the Itô integral $Y_T$. 

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Lemma 3.1. Let \( T \in (0, \infty), \, d, m \in \mathbb{N} \), let \( \langle \cdot, \cdot \rangle: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) denote the standard scalar product on \( \mathbb{R}^d \), let \( \| \cdot \|: \mathbb{R}^d \to [0, \infty) \) denote the standard norm on \( \mathbb{R}^d \), let \( \| \cdot \|_F: \mathbb{R}^{d \times m} \to [0, \infty) \) denote the Frobenius norm on \( \mathbb{R}^{d \times m} \), let \( (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]}) \) be a filtered probability space which satisfies the usual conditions, let \( W: [0, T] \times \Omega \to \mathbb{R}^m \) be a standard \((\mathbb{F}_t)_{t \in [0, T]}\)-Brownian motion with continuous sample paths, let \( A: [0, T] \times \Omega \to \mathbb{R}^d \) be adapted with continuous sample paths, let \( Y: [0, T] \times \Omega \to \mathbb{R}^d \) be measurable, let \( Z: [0, T] \times \Omega \to \mathbb{R}^{d \times m} \) be progressively measurable, and assume that for all \( s \in [0, T] \) it holds a.s. that

\[
\int_0^T \left( \| A_t \| + \mathbb{E}[\| Z_t \|_F^2] \right) dt < \infty \quad \text{and} \quad Y_s = Y_T + \int_s^T A_t dt - \int_s^T Z_t dW_t. \tag{45}
\]

Then

(i) for all \( s \in [0, T], \lambda \in (0, \infty) \) it holds a.s. that

\[
\mathbb{E}\left[ e^{\lambda s} \| Y_s \|^2 + \int_s^T e^{\lambda t} \| Z_t \|_F^2 dt \right] \leq \mathbb{E}\left[ e^{\lambda T} \| Y_T \|^2 + \int_s^T \frac{e^{\lambda t}}{\lambda} \| A_t \|^2 dt \right], \tag{46}
\]

(ii) for all \( s \in [0, T], \lambda \in (0, \infty) \) it holds a.s. that

\[
\mathbb{E}\left[ \sup_{t \in [s, T]} \left( e^{\lambda t} \| Y_t \|^2 + \int_t^T e^{\lambda u} \| Z_u \|_F^2 du \right) \right] \leq 34 \mathbb{E}\left[ e^{\lambda T} \| Y_T \|^2 + \int_s^T \frac{e^{\lambda t}}{\lambda} \| A_t \|^2 dt \right], \tag{47}
\]

and

(iii) it holds for all \( \alpha, \lambda \in (0, \infty) \) that

\[
\int_0^T \left[ \frac{t^{\alpha-1} e^{\lambda t} \mathbb{E}[\| Y_t \|^2]}{\Gamma(\alpha)} + \frac{t^\alpha e^{\lambda t} \mathbb{E}[\| Z_t \|_F^2]}{\Gamma(\alpha + 1)} \right] dt \leq \frac{e^{\lambda T} T^\alpha \mathbb{E}[\| Y_T \|^2]}{\Gamma(\alpha + 1)} + \frac{1}{\lambda} \int_0^T e^{\lambda t} T^\alpha \mathbb{E}[\| A_t \|^2] dt. \tag{48}
\]

Proof of Lemma 3.1. Throughout this proof for every \( s \in [0, T] \) let \( B_s \in \mathbb{F}_s \) satisfy that a.s. on \( B_s \) it holds that \( \mathbb{E}[\| Y_T \|^2 + \int_s^T \| A_t \|^2 dt | \mathbb{F}_s] < \infty \) and a.s. on \( \Omega \setminus B_s \) it holds that \( \mathbb{E}[\| Y_T \|^2 + \int_s^T \| A_t \|^2 dt | \mathbb{F}_s] = \infty \), let \( \{e_1, e_2, \ldots, e_m\} \subseteq \mathbb{R}^m \) be an orthonormal basis of \( \mathbb{R}^m \), and let \( \alpha, \lambda \in (0, \infty) \). First note that (45), Jensen’s inequality, and the Burkholder-Davis-Gundy inequality (see, e.g., [5, Lemma 7.2]) yield that for all \( s \in [0, T] \) it holds a.s. on \( B_s \) that

\[
\mathbb{E}\left[ \sup_{t \in [s, T]} \| Y_t \|^2 \right] \leq 3 \left( \| Y_s \|^2 + \mathbb{E}\left[ \left( \int_s^T \| A_t \| dt \right)^2 + \sup_{u \in [s, T]} \left\| \int_s^u Z_t dW_t \right\|_F^2 \right] \right) \tag{49}
\]

\[
\leq 12 \left( \| Y_s \|^2 + \mathbb{E}\left[ T \int_s^T \| A_t \|^2 dt \right] \right) + \mathbb{E}\left[ \left( \int_s^T \| Z_t \|_F^2 dt \right) \right] < \infty.
\]

This, the \( L^1 \)-Burkholder-Davis-Gundy inequality (e.g., [20, Theorem 1]), the Cauchy-Schwarz
inequality, and Hölder’s inequality imply that for all \( s \in [0, T] \) it holds a.s. on \( B_s \) that

\[
\mathbb{E} \left[ \sup_{u \in [s, T]} \left| \int_s^u e^{\lambda t} \langle Y_t, Z_t dW_t \rangle \right| \mathbb{F}_s \right] \leq \sqrt{8} \mathbb{E} \left[ \left( \int_s^T \mathbb{E} \left[ \sup_{t \in [s, T]} e^{\lambda t} \| Y_t \|^2 \right] \mathbb{F}_s \right)^{1/2} \left( \int_s^T \mathbb{E} \left[ \sup_{t \in [s, T]} e^{\lambda t} \| Y_t \|^2 \right] \mathbb{F}_s \right)^{1/2} \right]^{1/2} dt \leq \sqrt{8} \mathbb{E} \left[ \int_s^T \mathbb{E} \left[ \sup_{t \in [s, T]} e^{\lambda t} \| Y_t \|^2 \right] \mathbb{F}_s \right]^{1/2} \left( \int_s^T \mathbb{E} \left[ \sup_{t \in [s, T]} e^{\lambda t} \| Y_t \|^2 \right] \mathbb{F}_s \right)^{1/2} \right]^{1/2}.
\]

This, (45), and (49) yield that for all \( s \in [0, T] \) it holds a.s. that \( (1_B, \int_s^u e^{\lambda t} \langle Y_t, Z_t dW_t \rangle)_{u \in [s, T]} \) is a martingale with respect to \( \mathbb{P} (\cdot | \mathbb{F}_s) \) and

\[
\mathbb{E} \left[ \sup_{u \in [s, T]} \left| \int_s^u e^{\lambda t} \langle Y_t, Z_t dW_t \rangle \right| \mathbb{F}_s \right] = 0. \tag{51}
\]

Next note that (45) and Itô’s formula show that for all \( s \in [0, T] \) it holds a.s. that

\[
e^{\lambda T} \| Y_T \|^2 - e^{\lambda s} \| Y_s \|^2 = \int_s^T d(\mathbb{E} e^{\lambda t} \| Y_t \|^2) = \int_s^T \lambda e^{\lambda t} \| Y_t \|^2 dt + \int_s^T e^{\lambda t} d(\| Y_t \|^2) = \int_s^T \lambda e^{\lambda t} \| Y_t \|^2 dt + \int_s^T e^{\lambda t} (\| Z_t \|^2 - 2 \langle Y_t, A_t \rangle) dt + \int_s^T 2 e^{\lambda t} \langle Y_t, Z_t dW_t \rangle = \int_s^T e^{\lambda t} \| Z_t \|^2 dt + \int_s^T \frac{e^{\lambda t}}{\lambda} [\lambda \langle Y_t, A_t \rangle - \| A_t \|^2] dt + \int_s^T 2 e^{\lambda t} \langle Y_t, Z_t dW_t \rangle. \tag{52}
\]

This shows that for all \( s \in [0, T] \) it holds a.s. that

\[
e^{\lambda t} \| Y_s \|^2 + \int_s^T e^{\lambda t} \| Z_t \|^2 dt + \int_s^T \frac{e^{\lambda t}}{\lambda} [\lambda \langle Y_t, A_t \rangle - \| A_t \|^2] dt = e^{\lambda T} \| Y_T \|^2 + \int_s^T \frac{e^{\lambda t}}{\lambda} \| A_t \|^2 dt - \int_s^T 2 e^{\lambda t} \langle Y_t, Z_t dW_t \rangle. \tag{53}
\]

This and (51) show that for all \( s \in [0, T] \) it holds a.s. on \( B_s \) that

\[
\mathbb{E} \left[ e^{\lambda s} \| Y_s \|^2 + \int_s^T e^{\lambda t} \| Z_t \|^2 dt \big| \mathbb{F}_s \right] \leq \mathbb{E} \left[ e^{\lambda T} \| Y_T \|^2 + \int_s^T \frac{e^{\lambda t}}{\lambda} \| A_t \|^2 dt \big| \mathbb{F}_s \right]. \tag{54}
\]

This and the definition of \( B_s \), \( s \in [0, T] \), prove (i).

Next observe that (50), (51) and (i) yield that for all \( s \in [0, T] \) it holds a.s. on \( B_s \) that

\[
\mathbb{E} \left[ \sup_{t \in [s, T]} \left( -2 \int_t^T e^{\lambda u} \langle Y_u, Z_u dW_u \rangle \right) \big| \mathbb{F}_s \right] = 2 \mathbb{E} \left[ \sup_{t \in [s, T]} \left( \int_s^T e^{\lambda u} \langle Y_u, Z_u dW_u \rangle \right) \big| \mathbb{F}_s \right] - 2 \mathbb{E} \left[ \int_s^T e^{\lambda u} \langle Y_u, Z_u dW_u \rangle \big| \mathbb{F}_s \right] \leq 2 \sqrt{8} \left( \mathbb{E} \left[ \int_s^T e^{\lambda t} \| Z_t \|^2 dt \big| \mathbb{F}_s \right] \mathbb{E} \left[ \sup_{t \in [s, T]} e^{\lambda t} \| Y_t \|^2 \big| \mathbb{F}_s \right] \right)^{1/2}.
\]

\[
\leq 2 \sqrt{8} \left( \mathbb{E} \left[ e^{\lambda T} \| Y_T \|^2 + \int_s^T \frac{e^{\lambda t}}{\lambda} \| A_t \|^2 dt \big| \mathbb{F}_s \right] \mathbb{E} \left[ \sup_{t \in [s, T]} e^{\lambda t} \| Y_t \|^2 \big| \mathbb{F}_s \right] \right)^{1/2}
\]

\[
13
\]
This, (53), (45), and (49) yield that for all \( s \in [0, T] \) it holds a.s. on \( B_s \) that

\[
\mathbb{E} \left[ \sup_{t \in [s,T]} \left( e^{\lambda t} ||Y_t||^2 + \int_t^T e^{\lambda u} ||Z_u||^2 du \right) \right] \leq \mathbb{E} \left[ e^{\lambda T} ||Y_T||^2 + \int_0^T e^{\lambda u} ||A_u||^2 du \right] + \mathbb{E} \left[ \sup_{t \in [s,T]} \left( -2 \int_t^T e^{\lambda u} \langle Y_u, Z_u dW_u \rangle \right) \right] \\
\leq \mathbb{E} \left[ e^{\lambda T} ||Y_T||^2 + \int_0^T e^{\lambda u} ||A_u||^2 du \right] + 2\sqrt{8} \left( \mathbb{E} \left[ e^{\lambda T} ||Y_T||^2 + \int_0^T e^{\lambda u} ||A_u||^2 du \right] \right)^{\frac{1}{2}} \leq 2\sqrt{8} \left( \mathbb{E} \left[ e^{\lambda T} ||Y_T||^2 + \int_0^T e^{\lambda u} ||A_u||^2 du \right] \right)^{\frac{1}{2}} < \infty.
\]

This and the fact that \( \forall t \in [0, \infty) : (|x| \leq c + 2\sqrt{8\sqrt{c}} \Rightarrow |x| \leq (\sqrt{8} + \sqrt{1}) c \leq 34c) \) imply for all \( s \in [0, T] \) that a.s. on \( B_s \) it holds that

\[
\mathbb{E} \left[ \sup_{t \in [s,T]} \left( e^{\lambda t} ||Y_t||^2 + \int_t^T e^{\lambda u} ||Z_u||^2 du \right) \right] \leq 34 \mathbb{E} \left[ e^{\lambda T} ||Y_T||^2 + \int_0^T e^{\lambda u} ||A_u||^2 du \right].
\]

This and the definition of \( B_s, \ s \in [0, T], \) prove (ii). Next, the fact that \( \forall t \in [0, T] : \frac{t}{\Gamma(\alpha + 1)} = \int_0^t s^{\alpha-1} ds, \) Tonelli’s theorem, the tower property, and (i) show that

\[
\int_0^T \left[ \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda t} \mathbb{E} [||Y_t||^2] + \frac{t^\alpha e^{\lambda t}}{\Gamma(\alpha + 1)} \mathbb{E} [||Z_t||^2] \right] dt \\
= \int_0^T \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda t} \mathbb{E} [||Y_t||^2] dt + \int_0^T \int_0^t \frac{s^{\alpha-1} e^{\lambda s}}{\Gamma(\alpha)} \mathbb{E} [||Z_s||^2] ds dt \\
= \int_0^T \frac{s^{\alpha-1}}{\Gamma(\alpha)} e^{\lambda s} \mathbb{E} [||Y_s||^2] ds + \int_0^T \int_s^T \frac{s^{\alpha-1} e^{\lambda s}}{\Gamma(\alpha)} \mathbb{E} [||Z_s||^2] dt ds \\
= \int_0^T \frac{s^{\alpha-1}}{\Gamma(\alpha)} \mathbb{E} \left[ \mathbb{E} [e^{\lambda s} ||Y_s||^2 + \int_s^T e^{\lambda t} ||Z_t||^2 dt] \right] ds \\
\leq \int_0^T \frac{s^{\alpha-1}}{\Gamma(\alpha)} \mathbb{E} \left[ \mathbb{E} [e^{\lambda t} ||Y_t||^2 + \int_0^t e^{\lambda s} \mathbb{E} [||Z_s||^2] ds] \right] dt \\
= e^{\lambda T} \mathbb{E} [||Y_T||^2] \left( \int_0^T \frac{s^{\alpha-1} ds}{\Gamma(\alpha)} \right) + \int_0^T \int_0^t \frac{s^{\alpha-1} e^{\lambda s}}{\Gamma(\alpha)} \mathbb{E} [||A_t||^2] ds dt \\
= \frac{e^{\lambda T} T^\alpha \mathbb{E} [||Y_T||^2]}{\Gamma(\alpha + 1)} + \frac{1}{\lambda} \int_0^T \frac{e^{\lambda t} \mathbb{E} [||A_t||^2]}{\Gamma(\alpha + 1)} dt. \tag{58}
\]

This shows (iii). The proof of Lemma 3.1 is thus completed.

\[\square\]

4 Upper bounds for the convergence speed of Picard iterations

In this section we provide upper bounds for the convergence speed of Picard iterations of BSDEs. Proposition 4.1 establishes an explicit bound for the \( L^2 \)-distance between the Picard
Proposition 4.1. Let $T \in (0, \infty)$, $d, m \in \mathbb{N}$, $L_0, L_\lambda \in [0, \infty)$, let $t^0 = 1$, let $(\cdot, \cdot): \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ denote the standard scalar product on $\mathbb{R}^d$, let $\| \cdot \|: \mathbb{R}^d \to [0, \infty)$ denote the standard norm on $\mathbb{R}^d$, let $\| \cdot \|: \mathbb{R}^{d \times m} \to [0, \infty)$ denote the Frobenius norm on $\mathbb{R}^{d \times m}$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]})$ be a filtered probability space which satisfies the usual conditions, let $f: [0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times m} \to \mathbb{R}^d$ be measurable, assume that for all $t \in [0, T]$, $y, \tilde{y} \in \mathbb{R}^d$, $z, \tilde{z} \in \mathbb{R}^{d \times m}$ it holds a.s. that

$$
\| f(t, y, z) - f(t, \tilde{y}, \tilde{z}) \| \leq L_\alpha \| y - \tilde{y} \| + L_\beta \| z - \tilde{z} \|, \tag{59}
$$

let $W: [0, T] \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t \in [0, T]}$-Brownian motion with continuous sample paths, let $\xi: \Omega \to \mathbb{R}^d$ be $\mathcal{F}_T$-measurable, let $Y: [0, T] \times \Omega \to \mathbb{R}^d$, $k \in \mathbb{N}_0 \cup \{\infty\}$, be adapted with continuous sample paths, let $Z^k: [0, T] \times \Omega \to \mathbb{R}^{d \times m}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, be progressively measurable, and assume that for all $s \in [0, T]$, $k \in \mathbb{N}_0 \cup \{\infty\}$ it holds a.s. that $f_s^T \mathbb{E}[\| \xi \|^2 + \| f(t, 0, 0) \|^2 + \| Y_s^k \|^2 + \| Z^k \|^2] dt < \infty$, $Y_s^0 = 0$, $Z_s^0 = 0$, and

$$
Y_{s+1}^k = \xi + \int_s^T f(t, Y_t^k, Z_t^k) dt - \int_s^T Z_{t+1}^k dW_t. \tag{60}
$$

Then it holds for all $k \in \mathbb{N}$ that

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} \left( \| Y_t^k - Y_t^\infty \|^2 \right) + \int_0^T \| Z_t^k - Z_t^\infty \|^2 dt \right] \leq 35 \left( \frac{T e}{k} \right)^k \left( \sum_{\ell=0}^k \frac{k! L_\alpha T^{\ell/2}}{\ell!(k-\ell)! \sqrt{\ell!}} \right)^2 \left( \mathbb{E}[\| \xi \|^2] + \frac{T}{k} \int_0^T \mathbb{E}[\| f(t, Y_t^\infty, Z_t^\infty) \|^2] dt \right) < \infty. \tag{61}
$$

Proof of Proposition 4.1. First note that (60) proves that for all $s \in [0, T]$, $k \in \mathbb{N}_0$ it holds a.s. that

$$
Y_s^k - Y_s^\infty = -1_{\{0\}}(k) \xi + \int_s^T 1_{\mathbb{N}}(k) f(s, Y_s^{k-1}, Z_s^{k-1}) - f(s, Y_s^\infty, Z_s^\infty) dt - \int_s^T [Z_t^k - Z_t^\infty] dW_t. \tag{62}
$$

This, the tower property, Tonelli’s theorem, and Lemma 3.1 (applied for every $k \in \mathbb{N}_0$ with $Y \leftarrow Y^k - Y^\infty$, $A \leftarrow 1_{\mathbb{N}}(k) f(\cdot, Y_s^{k-1}, Z_s^{k-1}) - f(\cdot, Y^\infty, Z^\infty)$, $Z \leftarrow Z^k - Z^\infty$ in the notation of Lemma 3.1) prove

(i) that for all $k \in \mathbb{N}_0$, $\lambda \in (0, \infty)$ it holds that

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} \left( e^{\lambda t} \| Y_t^k - Y_t^\infty \|^2 \right) + \int_0^T e^{\lambda t} \| Z_t^k - Z_t^\infty \|^2 dt \right] \leq 35 \lambda \mathbb{E} \left[ \lambda e^{\lambda T} \| \xi \|^2 1_{\{0\}}(k) + \int_0^T e^{\lambda t} \left( 1_{\mathbb{N}}(k) f(t, Y_t^{k-1}, Z_t^{k-1}) - f(t, Y_t^\infty, Z_t^\infty) \right)^2 dt \right], \tag{63}
$$

and
(ii) that for all $k \in \mathbb{N}_0$, $\alpha, \lambda \in (0, \infty)$ it holds that

$$\int_0^T t^{\alpha-1} e^{\lambda t} \mathbb{E} \left[ \left| Y^k_t - Y^\infty_t \right|^2 \right] \frac{dt}{\Gamma(\alpha)} + \frac{t^{\alpha} e^{\lambda t} \mathbb{E} \left[ \left| Z^k_t - Z^\infty_t \right|^2 \right]}{\Gamma(\alpha + 1)} dt$$

$$\leq e^{\lambda T} T^\alpha \mathbb{E} \left[ \left| \xi \right|^2 \right] \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\lambda} \int_0^T e^{\lambda t} t^\alpha \mathbb{E} \left[ \left| 1_{N(t)} f(t, Y^k_{t-1}, Z^k_{t-1}) - f(t, Y^\infty_t, Z^\infty_t) \right|^2 \right] \frac{dt}{\Gamma(\alpha + 1)}.$$

This, (59), and the fact that $\forall \alpha \in \mathbb{N}_0: \Gamma(\alpha + 1) = \alpha!$ show for all $k \in \mathbb{N}_0$, $\lambda \in (0, \infty)$ that

$$\left( \int_0^T t^\alpha e^{\lambda t} \frac{1}{\alpha!} \mathbb{E} \left[ \left| f(t, Y^k_t, Z^k_t) - f(t, Y^\infty_t, Z^\infty_t) \right|^2 \right] dt \right)^{1/2}$$

$$\leq L_0 \left( \int_0^T \frac{t^\alpha}{\alpha!} e^{\lambda t} \mathbb{E} \left[ \left| Y^k_t - Y^\infty_t \right|^2 \right] dt \right)^{1/2} + L_3 \left( \int_0^T \frac{t^\alpha}{\alpha!} e^{\lambda t} \mathbb{E} \left[ \left| Z^k_t - Z^\infty_t \right|^2 \right] dt \right)^{1/2}$$

$$\leq \frac{1}{\nu_0} \left[ \frac{L_0 L_1}{\sqrt{\lambda}} \left( \frac{T^{\alpha+\nu}}{(\alpha+\nu)!} \right) \right] \lambda e^{\lambda T} \mathbb{E} \left[ \left| \xi \right|^2 \right] \frac{1}{\Gamma(\alpha + 1)} (k)$$

$$+ \int_0^T \frac{t^\alpha}{\alpha!} e^{\lambda t} \mathbb{E} \left[ \left| 1_{N(t)} f(t, Y^k_{t-1}, Z^k_{t-1}) - f(t, Y^\infty_t, Z^\infty_t) \right|^2 \right] \frac{dt}{\Gamma(\alpha + 1)}.$$

This and induction prove for all $k \in \mathbb{N} \cap [2, \infty)$, $\lambda \in (0, \infty)$ that

$$\left( \int_0^T t^0 e^{\lambda t} \mathbb{E} \left[ \left| f(t, Y^k_{t-1}, Z^k_{t-1}) - f(t, Y^\infty_t, Z^\infty_t) \right|^2 \right] dt \right)^{1/2}$$

$$\leq \sum_{\nu_1, \nu_2, \ldots, \nu_{k-1} = 0}^1 \left[ \int_0^T t^{\nu_1} \frac{\lambda e^{\lambda t} \mathbb{E} \left[ \left| f(t, Y^0_t, Z^0_t) - f(t, Y^\infty_t, Z^\infty_t) \right|^2 \right]}{(\nu_1)!} \frac{dt}{\Gamma(k)} \right]^{1/2}$$

$$\leq \sum_{\nu_1, \nu_2, \ldots, \nu_{k-1} = 0}^1 \left[ \int_0^T t^{\nu_1} \frac{\lambda e^{\lambda t} \mathbb{E} \left[ \left| f(t, Y^0_t, Z^0_t) - f(t, Y^\infty_t, Z^\infty_t) \right|^2 \right]}{(\nu_1)!} \frac{dt}{\Gamma(k)} \right]^{1/2}$$

$$= \sum_{\nu_1, \nu_2, \ldots, \nu_{k-1} = 0}^1 \left[ \int_0^T t^{\nu_1} \frac{\lambda e^{\lambda t} \mathbb{E} \left[ \left| f(t, Y^0_t, Z^0_t) - f(t, Y^\infty_t, Z^\infty_t) \right|^2 \right]}{(\nu_1)!} \frac{dt}{\Gamma(k)} \right]^{1/2}.$$
This and (65) show for all $k \in \mathbb{N}$, $\lambda \in (0, \infty)$ that
\[
\left( \int_0^T e^{\lambda t} \mathbb{E} \left[ \| f(t, Y_t^{k-1}, Z_t^{k-1}) - f(t, Y_t^\infty, Z_t^\infty) \|_F^2 \right] dt \right)^{1/2} 
\leq \sum_{\ell=0}^k \frac{k!}{\ell!(k-\ell)!} \frac{L_0^\ell L_3^k}{\lambda^{k/2}} \left( \frac{T^\ell}{\ell!} \lambda e^{\lambda T} \mathbb{E} \left[ \| \xi \|_2^2 \right] + \int_0^T \frac{t^\ell}{\ell!} e^{\lambda t} \mathbb{E} \left[ \| f(t, Y_t^\infty, Z_t^\infty) \|_F^2 \right] dt \right)^{1/2} 
\leq \sum_{\ell=0}^k \frac{k!}{\ell!(k-\ell)!} \frac{L_0^\ell L_3^k}{\lambda^{k/2}} \left( \frac{T^\ell}{\ell!} e^{\lambda T} \mathbb{E} \left[ \| \xi \|_2^2 \right] + \int_0^T \mathbb{E} \left[ \| f(t, Y_t^\infty, Z_t^\infty) \|_F^2 \right] dt \right)^{1/2}.
\] (67)

This and (63) prove for all $k \in \mathbb{N}$, $\lambda \in (0, \infty)$ that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left( e^{\lambda t} \| Y_t^k - Y_t^\infty \|_2^2 \right) + \int_0^T e^{\lambda t} \| Z_t^k - Z_t^\infty \|_F^2 dt \right] 
\leq \frac{35}{\lambda} \mathbb{E} \left[ \int_0^T e^{\lambda t} \| f(t, Y_t^{k-1}, Z_t^{k-1}) - f(t, Y_t^\infty, Z_t^\infty) \|_F^2 dt \right] 
\leq \frac{35}{\lambda} \left( \sum_{\ell=0}^k \frac{k!}{\ell!(k-\ell)!} \frac{L_0^\ell L_3^k}{\lambda^{k/2}} \left( \frac{T^\ell}{\ell!} e^{\lambda T} \mathbb{E} \left[ \| \xi \|_2^2 \right] + \int_0^T \mathbb{E} \left[ \| f(t, Y_t^\infty, Z_t^\infty) \|_F^2 \right] dt \right)^{1/2} \right)^2.
\] (68)

Furthermore, observe for all $k \in \mathbb{N}$ that
\[
\left( \sum_{\ell=0}^k \frac{k!}{\ell!(k-\ell)!} \frac{L_0^\ell L_3^k}{\lambda^{k/2}} \left( \frac{T^\ell}{\ell!} e^{\lambda T} \mathbb{E} \left[ \| \xi \|_2^2 \right] + \int_0^T \mathbb{E} \left[ \| f(t, Y_t^\infty, Z_t^\infty) \|_F^2 \right] dt \right)^{1/2} \right)^2 = \left( \frac{T e}{k} \right)^k \left( \sum_{\ell=0}^k \frac{k!}{\ell!(k-\ell)!} \frac{L_0^\ell L_3^k}{\lambda^{k/2}} \left( \frac{T^\ell}{\ell!} e^{\lambda T} \mathbb{E} \left[ \| \xi \|_2^2 \right] + \int_0^T \mathbb{E} \left[ \| f(t, Y_t^\infty, Z_t^\infty) \|_F^2 \right] dt \right)^{1/2} \right)^2.
\] (69)

This and (68) yield for all $k \in \mathbb{N}$ that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left( \| Y_t^k - Y_t^\infty \|_2^2 \right) + \int_0^T \| Z_t^k - Z_t^\infty \|_F^2 dt \right] 
\leq \frac{35T}{k} \left( \frac{T e}{k} \right)^k \left( \sum_{\ell=0}^k \frac{k!}{\ell!(k-\ell)!} \frac{L_0^\ell L_3^k}{\lambda^{k/2}} \left( \frac{T^\ell}{\ell!} e^{\lambda T} \mathbb{E} \left[ \| \xi \|_2^2 \right] + \int_0^T \mathbb{E} \left[ \| f(t, Y_t^\infty, Z_t^\infty) \|_F^2 \right] dt \right)^{1/2} \right)^2.
\] (70)

Next note that (59) ensures that
\[
\left( \int_0^T \mathbb{E} \left[ \| f(t, Y_t^\infty, Z_t^\infty) \|_F^2 \right] dt \right)^{1/2} 
\leq \left( \int_0^T \mathbb{E} \left[ \| f(t, 0, 0) \|_2^2 \right] dt \right)^{1/2} + L_3 \left( \int_0^T \mathbb{E} \left[ \| Y_t^\infty \|_2^2 \right] dt \right)^{1/2} + L_3 \left( \int_0^T \mathbb{E} \left[ \| Z_t^\infty \|_F^2 \right] dt \right)^{1/2} < \infty.
\] (71)

This, the fact that $\mathbb{E} \left[ \| \xi \|_2^2 \right] < \infty$, and (70) complete the proof of Proposition 4.1. \qed

**Remark 4.2.** Assume the setting of Proposition 4.1. Then it holds for all $k \in \mathbb{N}$ that
\[
35 \left( \frac{T e}{k} \right)^k \left( \sum_{\ell=0}^k \frac{k!}{\ell!(k-\ell)!} \frac{L_0^\ell L_3^k}{\lambda^{k/2}} \left( \frac{T^\ell}{\ell!} \right)^{1/2} e^{\lambda T} \mathbb{E} \left[ \| \xi \|_2^2 \right] + \int_0^T \mathbb{E} \left[ \| f(t, Y_t^\infty, Z_t^\infty) \|_F^2 \right] dt \right)^{1/2} \leq 35 \left( \frac{4 \max \{ T^2, 1 \} e \max \{ L_0^2, L_3^2 \} }{k} \right)^k \left( \sum_{\ell=0}^k \frac{k!}{\ell!(k-\ell)!} \right)^2 \left( \frac{T e}{k} \right)^k \left( \sum_{\ell=0}^k \frac{k!}{\ell!(k-\ell)!} \frac{L_0^\ell L_3^k}{\lambda^{k/2}} \left( \frac{T^\ell}{\ell!} \right)^{1/2} e^{\lambda T} \mathbb{E} \left[ \| \xi \|_2^2 \right] + \int_0^T \mathbb{E} \left[ \| f(t, Y_t^\infty, Z_t^\infty) \|_F^2 \right] dt \right)^{1/2}.
\] (72)
Remark 4.3. Assume the setting of Proposition 4.1 and assume that $L_z = 0$. Then it holds for all $k \in \mathbb{N}$ that

$$
35 \left( \frac{T e_k}{k} \right)^2 \left[ \sum_{\ell=0}^{k} \frac{k! L_{\ell} L_{k-\ell} T^{\ell/2}}{\ell! (k-\ell)! \sqrt{\ell!}} \right]^2 = 35 \left( \frac{T e_k}{k} \right)^2 \left[ \frac{L_{k} T^{k/2}}{\sqrt{k!}} \right]^2 \leq 35 \left( \frac{T e L_{2}^2}{(k!)^2} \right)^2.
$$

(73)

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References

[1] Beck, C., Hornung, F., Hutzenthaler, M., Jentzen, A., and Kruse, T. Overcoming the curse of dimensionality in the numerical approximation of Allen–Cahn partial differential equations via truncated full-history recursive multilevel Picard approximations. Journal of Numerical Mathematics 28, 4 (2020), 197–222.

[2] Bender, C., and Denk, R. A forward scheme for backward SDEs. Stochastic Process. Appl. 117, 12 (2007), 1793–1812.

[3] Bender, C., and Zhang, J. Time discretization and Markovian iteration for coupled FBSDEs. The Annals of Applied Probability 18, 1 (2008), 143–177.

[4] Briand, P., and Labart, C. Simulation of BSDEs by Wiener chaos expansion. Ann. Appl. Probab. 24, 3 (2014), 1129–1171.

[5] Da Prato, G., and Zabczyk, J. Stochastic equations in infinite dimensions, vol. 44 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1992.

[6] E, W., Hutzenthaler, M., Jentzen, A., and Kruse, T. Multilevel Picard iterations for solving smooth semilinear parabolic heat equations. arXiv:1607.03295 (2016). Accepted in Partial Differential Equations and Applications.

[7] E, W., Hutzenthaler, M., Jentzen, A., and Kruse, T. On multilevel Picard numerical approximations for high-dimensional nonlinear parabolic partial differential equations and high-dimensional nonlinear backward stochastic differential equations. Journal of Scientific Computing 79, 3 (2019), 1534–1571.

[8] El Karoui, N., Peng, S., and Quenez, M. C. Backward stochastic differential equations in finance. Math. Finance 7, 1 (1997), 1–71.

[9] Geiss, C., and Labart, C. Simulation of BSDEs with jumps by Wiener chaos expansion. Stochastic Process. Appl. 126, 7 (2016), 2123–2162.

[10] Gobet, E., and Labart, C. Solving BSDE with adaptive control variate. SIAM J. Numer. Anal. 48, 1 (2010), 257–277.
[11] Hutzenthaler, M., Jentzen, A., and Kruse, T. Overcoming the curse of dimensionality in the numerical approximation of parabolic partial differential equations with gradient-dependent nonlinearities. arXiv:1912.02571 (2019). Accepted in Foundations of Computational Mathematics.

[12] Hutzenthaler, M., Jentzen, A., Kruse, T., and Nguyen, T. A. Multilevel Picard approximations for high-dimensional semilinear second-order PDEs with Lipschitz nonlinearities. arXiv:2009.02484 (2020).

[13] Hutzenthaler, M., Jentzen, A., Kruse, T., Nguyen, T. A., and von Wurstemberger, P. Overcoming the curse of dimensionality in the numerical approximation of semilinear parabolic partial differential equations. Proceedings of the Royal Society A 476, 2244 (2020), 20190630.

[14] Hutzenthaler, M., and Kruse, T. Multilevel Picard approximations of high-dimensional semilinear parabolic differential equations with gradient-dependent nonlinearities. SIAM Journal on Numerical Analysis 58, 2 (2020), 929–961.

[15] Labart, C., and Lelong, J. A parallel algorithm for solving BSDEs. Monte Carlo Methods Appl. 19, 1 (2013), 11–39.

[16] Øksendal, B. Stochastic differential equations. Universitext. Springer-Verlag, Berlin, 1985. An introduction with applications.

[17] Pardoux, É., and Peng, S. Adapted solution of a backward stochastic differential equation. Systems Control Lett. 14, 1 (1990), 55–61.

[18] Pardoux, E., and Răşcanu, A. Stochastic differential equations, Backward SDEs, Partial differential equations. Springer, 2016.

[19] Pham, H. Continuous-time stochastic control and optimization with financial applications, vol. 61. Springer Science & Business Media, 2009.

[20] Ren, Y.-F. On the Burkholder–Davis–Gundy inequalities for continuous martingales. Statistics & probability letters 78, 17 (2008), 3034–3039.

[21] Zhang, J. Backward Stochastic Differential Equations. From Linear to Fully Nonlinear Theory. Springer, 2017.