ON PROBLEMS OF U. SIMON
CONCERNING MINIMAL SUBMANIFOLDS
OF THE NEARLY KAHLER 6-SPHERE
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ABSTRACT. We classify the complete 3-dimensional totally real submanifolds with sectional curvature \( K > \frac{1}{16} \) in the nearly Kaehler 6-sphere \( S^6(1) \), and, as a corollary, we solve a problem for compact 3-dimensional totally real submanifolds of \( S^6(1) \) related to U. Simon's conjecture for compact minimal surfaces in spheres.

1. The nearly Kaehler 6-sphere. It is well known that a 6-dimensional sphere \( S^6 \) does not admit any Kaehler structure, and whether \( S^6 \) does or does not admit a complex structure, as far as we know, is still an open question. However, using the Cayley algebra \( \mathbb{O} \), a natural almost complex structure \( J \) can be defined on \( S^6 \) considered as a hypersurface in \( \mathbb{R}^7 \), which itself is viewed as the set \( \mathbb{O}_+ \) of the purely imaginary Cayley numbers (see, for instance, E. Calabi [1]). Together with the standard metric \( g \) on \( S^6 \), \( J \) determines a nearly Kaehler structure in the sense of A. Gray [9], i.e. one has \( \forall X \in \mathbb{O}(S^6): (\tilde{\nabla}_X J)(X) = 0 \), where \( \tilde{\nabla} \) is the Levi Civita connection of \( g \). For reasons of normalization only, in the following we will always work with this nearly Kaehler structure on the sphere \( S^6(1) \), of radius 1 and constant sectional curvature 1. The compact simple Lie group \( G_2 \) is the group of automorphisms of \( \mathbb{O} \) and acts transitively on \( S^6(1) \). Moreover, \( G_2 \) preserves both \( J \) and \( g \).

2. Special submanifolds of \( (S^6(1), g, J) \). With respect to \( J \), two natural particular types of submanifolds \( M \) of \( S^6(1) \) can be investigated: those which are almost complex (i.e. for which the tangent space of \( M \) at each point is invariant under the action of \( J \)) and those which are totally real (i.e. for
which the tangent space of $M$ at each point is mapped into the normal space by $J$). There only exist 2-dimensional almost complex submanifolds in $S^6(1)$, and these are always minimal [10]. Curvature properties for such surfaces were first obtained by K. Sekigawa [15]. Totally real submanifolds of $S^6(1)$ have either dimension 2 or 3. N. Ejiri [7] showed that every 3-dimensional totally real submanifold of $S^6(1)$ is orientable and minimal, and he first obtained curvature properties for such submanifolds. The 3-dimensional totally real submanifolds of $S^6(1)$ were also considered by J. B. Lawson Jr. and R. Harvey [11] in their study of calibrated geometries and by K. Mashimo [13] in his classification of such compact submanifolds which are orbits of closed subgroups of $G_2$. In our study of submanifolds of the nearly Kaehler 6-sphere, we concentrated on the following problems.

**Problem A.** Which real numbers can be realised as the constant sectional curvatures of almost complex or minimal totally real submanifolds $M$ of $S^6(1)$?

**Problem B.** Let $K_1$ and $K_2$ be two consecutive numbers as in Problem A. Then, do there exist compact submanifolds $M$ of $S^6(1)$ whose sectional curvatures $K$ satisfy $K_1 \leq K \leq K_2$, other than those for which $K = K_1$ or $K = K_2$?

3. On minimal submanifolds of arbitrary spheres. For minimal surfaces in a unit sphere $S^n(1)$ of arbitrary dimension $n$, one has a complete answer to Problem A (given by O. Boruvka, E. Calabi and N. Wallach for the case of positive Gauss curvature, the solutions being $K = 2/m(m+1)$, $m \in \mathbb{N}_0$, and by R. Bryant, proving the nonexistence of minimal surfaces of constant negative Gauss curvature in any sphere). Concerning Problem B, U. Simon conjectured the following.

**Conjecture of U. Simon [12].** Let $M$ be a compact surface whose Gauss curvature $K$ satisfies $2/m(m+1) \leq K \leq 2/m(m-1)$, for some $m \in \mathbb{N}\{0,1\}$, which is minimally immersed in $S^n(1)$. Then $K = 2/m(m+1)$ or $K = 2/m(m-1)$ (and hence $M$ is a Boruvka sphere).

For $m = 2$ and $m = 3$, this conjecture is known to be true, as was shown by H. B. Lawson Jr., U. Simon, M. Kothe, K.-D. Semmler, K. Benko and M. Kozlowski. Recently, quite a number of people have been working on this problem; in particular, T. Ogata, S. Montiel, T. Itoh, G. Jensen, M. Rigoli, J. Bolton, L. Woodward and U. Simon, A. Schwenk, B. Opozda together with the present authors. As far as we know however, in general this conjecture is still open for $m \geq 3$. In view of U. Simon's conjecture, we would like to call problems of type A and B, as stated above for almost complex and totally real submanifolds of $S^6(1)$, "problems of U. Simon".

4. Solutions of problems A and B.

**Theorem 1 [15].** If an almost complex surface $M$ in $S^6(1)$ has constant Gauss curvature $K$, then either $K = 1$ (and $M$ is totally geodesic) or $K = 0$.

**Theorem 2 [4, 2].** Let $M$ be a compact almost complex surface in $S^6(1)$ with Gauss curvature $K$. 
(i) Let $\frac{1}{6} \leq K \leq 1$; then either $K \equiv \frac{1}{6}$ or $K \equiv 1$.

(ii) If $0 \leq K \leq \frac{1}{6}$, then either $K \equiv 0$ or $K \equiv \frac{1}{6}$.

**THEOREM 3** [6]. If a minimal totally real surface $M$ in $S^6(1)$ has constant Gauss curvature $K$, then either $K = 1$ (and $M$ is totally geodesic) or $K = 0$.

**THEOREM 4** [6]. For a compact minimal totally real surface $M$ in $S^6(1)$ with nonnegative Gauss curvature $K$ (or equivalently $0 \leq K \leq 1$), either $K \equiv 0$ or $K \equiv 1$.

In 1981, making use of a special choice of local orthonormal frames, N. Ejiri solved Problem A in the remaining case as follows.

**THEOREM 5** [7]. If a 3-dimensional totally real submanifold $M$ of $S^6(1)$ has constant sectional curvature $K$, then either $K = 1$ (and $M$ is totally geodesic) or $K = \frac{1}{16}$.

Totally real 3-dimensional totally geodesic submanifolds in $S^6(1)$ are not hard to construct. On the other hand, N. Ejiri [8] proved that $S^3(\frac{1}{16})$ can be immersed totally real and isometrically in $S^6(1)$. K. Mashimo [13] found an orbit of a closed subgroup of $G_2$ with constant curvature $\frac{1}{16}$. Later we will explicitly describe these immersions, obtaining for instance as extra information that they are in fact 56-fold coverings of $S^3(\frac{1}{16})$. Compared to the solutions given in Theorems 2 and 4, the solution of Problem B seems more involved in the present case. In our approach, the solution represented by Theorem 6 is an immediate consequence of the Main Theorem.

**THEOREM 6.** A compact 3-dimensional totally real submanifold of $S^6(1)$ whose sectional curvature $K$ satisfies $\frac{1}{16} \leq K \leq 1$ has constant sectional curvature $K = \frac{1}{16}$ or $K = 1$.

**MAIN THEOREM.** Let $x: M^3 \rightarrow S^6(1)$ be a totally real isometric immersion of a complete 3-dimensional Riemannian manifold $M^3$ into the nearly Kaehler 6-sphere $S^6(1)$. If the sectional curvatures $K$ of $M^3$ satisfy $K \geq \frac{1}{16}$, then either $M^3$ is simply connected and $x$ is $G_2$-congruent to $x_1: M_1 \rightarrow S^6(1)$ (in which case $K$ attains all values in the closed interval $[\frac{1}{16}, \frac{24}{16}]$) or to $x_2: M_2 \rightarrow S^6(1)$ (in which case $K \equiv 1$), or else $\tilde{x}$, the composition of the universal covering map of $M^3$ with $x$, is $G_2$-congruent to $x_3: M_3 \rightarrow S^6(1)$ (in which case $K \equiv \frac{1}{16}$).

**SKETCH OF PROOF** (details will appear elsewhere [3]). As in our partial solution of Problem B [5], a crucial role is played by some integral formulas of A. Ros, of which we’ll state one next. We do believe that these formulas provide a powerful tool to study problems in global Riemannian geometry.

**LEMMA OF A. ROS** [14]. Let $M$ be a compact Riemannian manifold, $UM$ its unit tangent bundle and $UM_p$ the fibre of $UM$ over a point $p$ of $M$. Let $dp$, $du$ and $du_p$ denote the canonical measures on $M$, $UM$ and $UM_p$, respectively. For any continuous function $f: UM \rightarrow \mathbb{R}$, one has

$$\int_{UM} f \, du = \int_M \left\{ \int_{UM_p} f \, du_p \right\} \, dp.$$
Now, let $T$ be any $k$-covariant tensor field on $M$. Then one has the integral formula

$$\int_M (\nabla T)(u, u, \ldots, u) \, du = 0,$$

where $\nabla$ is the Levi Civita connection on $M$.

We apply this lemma for some particular tensors $T$ constructed in terms of the second fundamental form $h$ of the immersion $\pi$. Then, under the assumption $K \geq \frac{1}{16}$, amongst others, we obtain that

$$R(v, A_j v; A_j v, v) = \frac{1}{16} \{ \|A_j v\|^2 - \langle A_j v, v \rangle \}$$

for all $p \in M^3$ and all $v \in UM_p$, where $R$ is the Riemann-Christoffel curvature tensor of $M^3$ and $A$ is the Weingarten map with respect to a normal section $\xi$. From this, working with special frames, using the Gauss equation and with the help of computer manipulation of formulas, we can prove that at each point $p$ the second fundamental form $h_p$ has either one of three possible forms, leading respectively to the possibilities $K(p) \equiv 1$, $K(p) \equiv \frac{1}{16}$, and $K(p) \in [\frac{1}{16}, \frac{21}{16}]$, where $K(p)$ is the sectional curvature function of $M^3$ at $p$. In the following, we will give comments concerning only $x_1$ ($x_2$ is the totally geodesic case, and for $x_3$ we will confine ourselves to give precise formulas for the immersion). The existence of $x_1$ is guaranteed by the following result taken from a preprint by N. Ejiri.

Theorem of N. Ejiri [8]. Let $M$ be a 3-dimensional simply connected Riemannian manifold with metric $(\cdot, \cdot)$. Suppose there exist a $(1, 2)$-symmetric tensor field $T$ on $M$ such that

(i) $\text{Tr} T = 0$,  $(T(X, Y), Z) = \langle T(X, Z), Y \rangle,$

(ii) $\langle R(X, Y)W, Z \rangle = (X, Z)\langle Y, W \rangle - (X, W)\langle Y, Z \rangle + \langle T(X, Z), T(Y, W) \rangle - \langle T(X, W), T(Y, Z) \rangle$,

(iii) $(\nabla_X T)(Y, Z) - (\nabla_Y T)(X, Z) + T(Z, X \wedge Y) = 0$, where $\wedge$ is the vector product determined by some orientation on $M$.

Then, up to a transformation of $G_2$, there exists a unique isometric immersion $x$ of $M$ into $S^6$ as a totally real submanifold with second fundamental form $J(x^* T)$ and with normal connection $D$ defined by $D_x J(x^* Y) = J(x, (\nabla_X Y + X \wedge Y))$.

Namely, on the unit sphere $S^3(1) = \{ y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 | \sum y_i^2 = 1 \}$ we can define a metrix $(\cdot, \cdot)$, vector product $\wedge$ and tensor field $T$ satisfying the conditions of this theorem and for which $K$ attains all values in $[\frac{1}{16}, \frac{21}{16}]$. This leads to the immersion $x_1: S^3(1) \subset \mathbb{R}^4 \to S^6(1) \subset \mathbb{R}^7: y \to z = (z_1, \ldots, z_7)$, where

$z_1(y) = \frac{1}{9} (5y_1^2 + 5y_2^2 - 5y_3^2 - 5y_4^2 + 4y_1),$

$z_2(y) = -\frac{2}{3} y_2, \quad z_3(y) = \frac{2\sqrt{5}}{9} (y_1^2 + y_2^2 - y_3^2 - y_4^2 - y_1),$

$z_4(y) = \frac{\sqrt{3}}{9\sqrt{2}} (-10y_1y_3 - 10y_2y_4), \quad z_5(y) = \frac{\sqrt{15}}{9\sqrt{2}} (2y_1y_4 - 2y_4 - 2y_2y_3),$

$z_6(y) = \frac{\sqrt{15}}{9\sqrt{2}} (2y_1y_3 - 2y_3 + 2y_2y_4), \quad z_7(y) = -\frac{\sqrt{3}}{9\sqrt{2}} (10y_1y_4 + 2y_4 - 10y_2y_3).$
In practice, $x_1$ was found solving the system of differential equations (1) on p. 67 of M. Spivak’s volume IV [16]; the rigidity of course follows from the fundamental theorem of submanifolds.

Finally, we mention the formulas of $X_3: S^3(\frac{1}{16}) = \{y \in \mathbb{R}^4 | \sum y_j^2 = 16\} \subset \mathbb{R}^4 \to S^6(1) \subset \mathbb{R}^7$: $y \mapsto z(y)$; we have

$$z_1(y) = \sqrt{15} \cdot 2^{-10} \cdot (y_1 y_3 + y_2 y_4) \cdot (y_1 y_4 - y_2 y_3)(y_1^2 + y_2^2 - y_3^2 - y_4^2),$$

$$z_2(y) = 2^{-12} \left[ -\sum_j y_j^6 + 5 \sum_{i<j} y_i^2 y_j^2 (y_i^2 + y_j^2) - 30 \sum_{i<j<k} y_i^2 y_j^2 y_k^2 \right],$$

$$z_3(y) = 2^{-10}[y_3 y_4 (y_3^2 - y_4^2)(y_3^2 + y_4^2 - 5y_1^2 - 5y_2^2)
+ y_1 y_2 (y_1^2 - y_2^2)(y_1^2 + y_2^2 - 5y_3^2 - 5y_4^2)],$$

$$z_4(y) = 2^{-12}\{y_2 y_4 (y_2^2 + 3y_3^2 - y_4^2 - 3y_1^2) + y_1 y_3 (y_3^2 + 3y_4^2 - y_1^2 - 3y_4^2)
+ 2(y_1 y_3 - y_2 y_4)(y_2^2 + 4y_3^2 - y_3^2(y_2^2 + 4y_3^2))\},$$

$$z_5(y_1, y_2, y_3, y_4) = z_4(y_2, -y_1, y_3, y_4),$$

$$z_6(y) = \sqrt{6} \cdot 2^{-12} \cdot [y_1 y_3 (y_1^4 + 5y_2^2 - y_3^4 - 5y_4^2) - y_2 y_4 (y_2^4 + 5y_1^2 - y_4^4 - 5y_3^2)
+ 10(y_1 y_3 - y_2 y_4)(y_3^2 y_4^2 - y_1^2 y_2^2)],$$

$$z_7(y_1, y_2, y_3, y_4) = z_6(y_2, -y_1, y_3, y_4).$$

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