The 2-Ranking Numbers of Graphs

Jordan Almeter, Samet Demircan‡, Andrew Kallmeyer‡, Kevin G. Milans§, Robert Winslow¶
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Abstract

In a graph whose vertices are assigned integer ranks, a path is well-ranked if the endpoints have distinct ranks or some interior point has a higher rank than the endpoints. A ranking is an assignment of ranks such that all nontrivial paths are well-ranked. A k-ranking is a relaxation in which all nontrivial paths of length at most k are well-ranked. The k-ranking number of a graph G, denoted by χk(G), is the minimum t such that there is a k-ranking of G using ranks in {1, . . . , t}.

For the n-dimensional cube Qn, we prove that χ2(Qn) = n + 1. As a corollary, we improve the bounds on the star chromatic number of products of cycles when each cycle has length divisible by 4. We show that Ω(n log m) ≤ χ2(Km □ Kn) ≤ O(nm log(3)−1) when m ≤ n and obtain χ2(Km □ Kn) asymptotically in n when m is constant. We prove that χ2(G) ≤ 7 when G is subcubic, and we also prove the existence of a graph G with maximum degree k and χ2(G) ≥ Ω(k2/ log(k))

1 Introduction

A path consisting of a single vertex is trivial; paths with positive length are nontrivial. In a graph whose vertices are assigned integer ranks, a path is well-ranked if its endpoints have distinct ranks or some interior vertex has a higher rank than the endpoints. A ranking of a graph G is an assignment of ranks to V(G) such that every nontrivial path is well-ranked. Graph rankings have arisen in mathematics and computer science; see the section on rankings in Gallian’s dynamic survey [5] for a summary of results and background. A k-ranking is a relaxation under which each nontrivial path of length at most k is well-ranked. The k-ranking number of G, denoted χk(G), is the minimum number of ranks in a k-ranking of G. The ranking number of G, denoted χ∞(G), is the minimum t such that G has a ranking using ranks in {1, . . . , t}. We note that a 1-ranking requires the endpoints of each edge to be assigned distinct colors, and so a 1-ranking is a proper coloring. It follows that χ1(G) = χ(G), where χ(G) denotes the chromatic number of G.

The notion of a 2-ranking was introduced by Karpas, Neiman, and Smorodinsky [6], who used the term unique-superior coloring. We use the term 2-ranking to emphasize the connection with graph ranking. In our terminology, Karpas, Neiman, and Smorodinsky proved that the maximum,
over all $n$-vertex trees $T$, of $\chi_2(T)$ is $\Theta(\frac{\log n}{\log \log n})$. Trees are $K_3$-minor-free; it turns out that the $k$-ranking number of a graph grows at most logarithmically when some minor is excluded. Specifically, Karpas, Neiman, and Smorodinsky show that for each graph $H$, there is a constant $s$ such that each $n$-vertex $H$-minor-free graph $G$ satisfies $\chi_k(G) \leq s(k + 1) \log n$. A graph $G$ is $d$-degenerate if each subgraph of $G$ has a vertex of degree at most $d$. They also prove that each $n$-vertex $d$-degenerate graph $G$ satisfies $\chi_2(G) \leq d(\sqrt{n} + 1)$ and construct $n$-vertex $2$-degenerate graphs $G$ with $\chi_2(G) > n^{1/3}$.

Our problems are also motivated by star colorings, in which vertices are properly colored and every pair of color classes induces a star forest. The star chromatic number of $G$, denoted $\chi_s(G)$, is the minimum number of colors in a star coloring of $G$. In a 2-ranking of $G$, every pair of ranks induces a star forest in which the centers are all of the higher rank. It follows that $\chi_s(G) \leq \chi_2(G)$ for each graph $G$.

One useful strategy to construct $k$-rankings is to assign distinct ranks to vertices that are close together. The $k$-distance power of $G$, denoted $G^k$, is the graph with vertex set $V(G)$ where $uv \in E(G^k)$ if and only if the distance between $u$ and $v$ in $G$ is at most $k$. It is clear that $\chi_k(G) \leq \chi(G^k)$. Summarizing the relationship between our notions of graph colorings, we have

$$\chi(G) \leq \chi_s(G) \leq \chi_2(G) \leq \chi(G^2)$$

for each graph $G$. In the above chain, it is not possible to bound any of the parameters as a function of its predecessor. Indeed, the star shows that $\chi(G^2)$ can be arbitrarily large even when $\chi_2(G) = 2$. Every tree $T$ satisfies $\chi_s(T) \leq 3$, but Karpas, Neiman, and Smorodinsky [6] show that the maximum, over all $n$-vertex trees, of $\chi_2(T)$ is $\Theta(\frac{\log n}{\log \log n})$. For each integer $n$, we define $[n] = \{1, \ldots, n\}$.

## 2 The hypercube

The $d$-dimensional cube, denoted $Q_d$, is the graph with vertex set $\{0, 1\}^d$ where $u$ and $v$ are adjacent if $u$ and $v$ differ in exactly one coordinate. We prove that $\chi_2(Q_d) = d + 1$. The lower bound follows from a useful proposition. A graph is $k$-degenerate if every subgraph contains a vertex of degree at most $k$. The degeneracy of a graph $G$ is the minimum integer $k$ such that $G$ is $k$-degenerate.

**Proposition 1.** If $G$ is a graph with degeneracy $k$, then $\chi_2(G) \geq k + 1$.

**Proof.** Since $G$ is not $(k - 1)$-degenerate, $G$ contains a subgraph $H$ with minimum degree at least $k$. Consider a $2$-ranking of $G$, and let $v$ be a vertex of minimum rank in $H$. The ranks of the neighbors of $v$ in $H$ are distinct, and the rank of $v$ differs from all of these. It follows that $\chi_2(G) \geq k + 1$. \hfill $\square$

Since $Q_d$ is $d$-regular, it follows that $\chi_2(Q_d) \geq d + 1$. Wan [9] proved that $\chi(Q_d^2) = d + 1$ when $d = 2^k - 1$ for some integer $k$, and it follows that $d + 1 \leq \chi_2(Q_d) \leq \chi(Q_d^2) = d + 1$ in this case. Each color class in a proper coloring of $Q_d^2$ has size at most $\lfloor 2^d/(d + 1) \rfloor$, and it follows that $\chi(Q_d^2) \geq 2^d/\lfloor 2^d/(d + 1) \rfloor$. Therefore $\chi(Q_d^2) > d + 1$ when $d$ does not have the form $2^k - 1$. Nonetheless, we show that $\chi_2(Q_d) = d + 1$ for all $d$. Although determining the exact value of $\chi(Q_d^2)$ remains open, Östergård [7] proved that $\chi(Q_d^2) = (1 + o(1))d$.

We view the vertex set of $Q_d$ as $\mathbb{F}_2^d$, the $d$-dimensional vector space over the finite field $\mathbb{F}_2$ with $2$ elements. For $u \in \mathbb{F}_2^d$, we define the support of $u$ to be the set of coordinates in $[d]$ where $u$ has value $1$. The weight of $u$, denoted $w(u)$, is the size of the support of $u$. Note that for all vertices
Case 4. The support of it follows that \(A\) ranked. If both \(w \neq u\) then the endpoints \(w\) and note that one of the rank in \([0, t]\) consists of the first \(t\) coordinates of \(u\) and we let \(u^-\) be the vector in \(F_2^n\) consisting of the first \(t\) coordinates of \(u\) and \(v\). We have that \(P\) is well-ranked. Therefore \(u\) \((t+2k)\) is assigned rank \(1\), it follows that \(u\) and \(v\) are colored inductively and so \(P\) is well-ranked by induction. Otherwise both \(w(u^-)\) and \(w(v^-)\) are odd, and so \(u\) is assigned rank \(\phi(Av)\) and \(v\) is assigned rank \(\phi(Au)\). Since \(w(u-v) = 1\), it follows that \(A(u-v)\) is the sum of one or two of the first \(t\) columns of \(A\). Since these columns are nonzero and distinct, we have that \(A(u-v) \neq 0\) and it follows that \(u\) and \(v\) are assigned different ranks. Therefore \(P\) is well-ranked.

Case 2. The support of \(u-v\) is contained in the last \(2k\) coordinates and \(\text{dist}(u,v) = 1\).

Since \(w(u-v) = \text{dist}(u,v) = 1\), it follows that \(w(u^+)\) and \(w(v^+)\) have opposite parity, implying that one of \(\{u,v\}\) is assigned a high rank and the other is assigned a low rank.

Case 3. The support of \(u-v\) is contained in the last \(2k\) coordinates and \(\text{dist}(u,v) = 2\).

We have that \(w(u^+)\) and \(w(v^+)\) have the same parity. Let \(x\) be the internal vertex on \(P\), and note that \(w(x^+)\) has opposite parity. If both \(w(u^+)\) and \(w(v^+)\) are even and \(w(x^+)\) is odd, then the endpoints \(u\) and \(v\) are assigned low rank while \(x\) is assigned high rank, and so \(P\) is well-ranked. If both \(w(u^+)\) and \(w(v^+)\) are odd, then \(u\) has rank \(\phi(Av)\) and \(v\) has rank \(\phi(Au)\). Since \(w(u-v) = \text{dist}(u,v) = 2\) and the support of \(u-v\) is contained in the last \(2k\) coordinates, it follows that \(A(u-v)\) is the sum of two columns from the last \(2k\) columns in \(A\). Since these are distinct, it follows that \(A(u-v) \neq 0\). Therefore \(Au \neq Av\), and so \(P\) is well-ranked.

Case 4. The support of \(u-v\) intersects both the first \(t\) coordinates and the last \(2k\) coordinates.

We have that \(w(u^+)\) and \(w(v^+)\) have opposite parity. Therefore one of \(\{u,v\}\) has high rank and the other has low rank.

In all cases, \(P\) is well-ranked.

The 2-ranking given in Theorem 2 assigns the same low rank to \(u\) and \(v\) whenever \(u^- = v^-\) and both \(w(u^+)\) and \(w(v^+)\) are even. Consequently, when \(d \geq 3\), many pairs of vertices at distance 2 share a common rank. When \(d\) is one less than a power of two, a proper coloring of \(Q^2_d\) is a
2-ranking of $Q_d$ in which pairs of vertices at distance 2 receive distinct ranks. It follows that when $d \geq 3$ and $d$ has the form $2^k - 1$, there are non-isomorphic optimal 2-rankings of $Q_d$. The situation when $d$ has the form $2^k$ may be different. For $d \in \{1, 2\}$, there is only one optimal 2-ranking of $Q_d$ up to isomorphism. We suspect that $Q_4$ has only one optimal 2-ranking up to isomorphism. Is it true that $Q_d$ has one optimal 2-ranking up to isomorphism when $d$ is a power of two?

The cartesian product of $G$ and $H$, denoted $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ where $(u, v)$ is adjacent to $(u', v')$ if and only if $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $v = v'$.

**Corollary 3.** If $G$ is the cartesian product of $d$ cycles, each of which has length divisible by 4, then $\chi_s(G) \leq \chi_2(G) = 2d + 1$.

**Proof.** Since $G$ has degeneracy $2d$, Proposition [1] implies that $\chi_2(G) \geq 2d + 1$. Note that $Q_{2d}$ is the cartesian product of $d$ copies of $C_4$. Viewing $V(Q_{2d})$ as $\mathbb{Z}_d^4$, let $f: \mathbb{Z}_d^4 \to [2d + 1]$ be a 2-ranking of $Q_{2d}$. We use $f$ to color $G$. Let $m_1, \ldots, m_d$ be the cycle lengths of the factors of $G$, and view $V(G)$ as $\{(x_1, \ldots, x_d): x_i \in \mathbb{Z}_{m_i}\}$. For $x \in V(G)$, let $x'$ be the vertex in $Q_{2d}$ obtained from $x$ by reducing each coordinate of $x$ modulo 4. We assign $x \in V(G)$ the rank $f(x')$. Since each path $P$ in $G$ of length at most 3 maps to a path in $Q_{2d}$ of the same length whose vertices are assigned the same ranks as in $G$, it follows that $G$ inherits the 2-ranking of $Q_{2d}$. \qed

Let $G$ be the cartesian product of $d$ cycles. Fertin, Raspaud, and Reed [4] proved that $d + 2 \leq \chi_s(G) \leq 2d^2 + d + 1$, and improved the upper bound to $2d + 1$ in the case that $2d + 1$ divides the length of each factor cycle. Pór and Wood [8] proved that $G$ admits a proper $(6d + O(\log d))$-coloring in which each pair of color classes induces a matching and isolated vertices; their result directly implies that $\chi_s(G) \leq 6d + O(\log d)$. Corollary [3] extends the divisibility conditions under which it is known that $\chi_s(G) \leq 2d + 1$.

### 3 Cartesian products of complete graphs

Determining the 2-ranking number of $K_m \Box K_n$ is an interesting problem. For each fixed $m$, we obtain $\chi_2(K_m \Box K_n)$ asymptotically. When $m = n$, our bounds are far apart. A 2-ranking of $K_m \Box K_n$ can be viewed as an $(m \times n)$-matrix $A$ such that $A(i, j)$ is the rank of $(a_i, b_j) \in V(K_m \Box K_n)$. The condition that paths of length 1 are well-ranked is equivalent to the rows and columns of $A$ having distinct entries. The condition that paths of length 2 are well-ranked is equivalent to the property that $A(i, j) = A(i', j')$ implies that the opposite corners $A(i, j')$ and $A(i', j)$ are larger than $A(i, j)$ and $A(i', j')$.

For positive integers $a, b, c, d$, our first result obtains a 2-ranking of $K_{ac} \Box K_{bd}$ from 2-rankings of $K_a \Box K_b$ and $K_c \Box K_d$.

**Proposition 4.** $\chi_2(K_{ac} \Box K_{bd}) \leq \chi_2(K_a \Box K_b) \cdot \chi_2(K_c \Box K_d)$.

**Proof.** Let $k = \chi_2(K_a \Box K_b)$ and $\ell = \chi_2(K_c \Box K_d)$. Let $A$ be an $(a \times b)$-matrix with entries in $\{0, \ldots, k - 1\}$ encoding an optimal 2-ranking of $K_a \Box K_b$, and let $B$ be an $(c \times d)$-matrix with entries in $\{0, \ldots, \ell - 1\}$ encoding an optimal 2-ranking of $K_c \Box K_d$. We use block operations to construct a 2-ranking of $K_{ac} \Box K_{bd}$.

Let $C$ be the $(ac \times bd)$-matrix obtained from $A$ and $B$ by replacing each entry $A(i, j)$ in $A$ with the $(c \times d)$-matrix $\ell A(i, j) + B$. It is easy to see that $C$ encodes a 2-ranking of $K_{ac} \Box K_{bd}$. Since the entries in $C$ belong to $\{0, \ldots, k\ell - 1\}$, we have that $\chi_2(K_{ac} \Box K_{bd}) \leq k\ell$. \qed
Proposition 4 may be iterated to obtain upper bounds on $\chi_2(K_n \square K_n)$.

**Corollary 5.** If $m$ and $n$ are powers of 2 with $m \leq n$, then $\chi_2(K_m \square K_n) \leq nm^{\log_2(3) - 1} \approx nm^{0.585}$.

**Proof.** Observe that $\begin{bmatrix}1 & 0 \\ 0 & 2\end{bmatrix}$ is a 2-ranking witnessing that $\chi_2(K_2 \square K_2) \leq 3$. If $m = 1$, then $\chi_2(K_2 \square K_n) = n$, and so the bound holds. Otherwise, by Proposition 4 and induction, we have that $\chi_2(K_m \square K_n) \leq \chi_2(K_{m/2} \square K_{n/2}) \cdot \chi_2(K_2 \square K_2) \leq \frac{n}{2} (\frac{m}{2})^{\log_2(3) - 1} \cdot 3 = nm^{\log_2(3) - 1}$. □

When $m$ and $n$ are not powers of two, we may apply Corollary 5 to $K_m \square K_n$ for $m'$ and $n'$ are the least powers of two larger than $m$ and $n$, respectively. Since $m' < 2m$ and $n' < 2n$, this gives $\chi_2(K_m \square K_n) < 3nm^{\log_2(3) - 1}$ for general $m$ and $n$. To prove a lower bound on $\chi_2(K_m \square K_n)$, we restrict the number of times that certain ranks can appear.

**Lemma 6.** In a 2-ranking of $K_m \square K_n$, each column of height $m$ contains $k$ ranks which are assigned to at most $k$ vertices for $1 \leq k \leq m$.

**Proof.** Let $A$ be an $(m \times n)$-matrix encoding an optimal 2-ranking of $K_m \square K_n$, and let $x$ be the $j$th column in $A$. Let $R$ be the set of rows containing the $k$ highest ranks in $x$, and let $S = \{A(i,j): i \in R\}$. We claim that each rank in $S$ appears only in rows in $R$. Since each rank appears at most once in each row, it then follows that each of the $k$ ranks in $S$ is assigned to at most $k$ vertices.

Suppose that $A(i,j) = A(i',j')$ where $i \in R$. Since $A$ is a 2-ranking, it must be that $A(i',j) > A(i,j)$, which implies that $A(i',j)$ is among the $k$ highest ranks in $x$. Therefore $i' \in R$ also. It follows that each rank in $S$ appears only in rows in $R$. □

Lemma 6 forces a nontrivial number of ranks in a 2-ranking of $K_m \square K_n$.

**Theorem 7.** We have $\chi_2(K_m \square K_n) \geq n H_m$, where $H_m$ is the Harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$.

**Proof.** Let $A$ be an $(m \times n)$-matrix encoding an optimal 2-ranking of $K_m \square K_n$, and let $a_k$ be the number of ranks that $A$ assigns to exactly $k$ vertices. Note that $\chi_2(K_m \square K_n) = \sum_{k=1}^{n} a_k$. We claim that for $1 \leq k \leq m$, we have that $\sum_{i=1}^{k} ia_i \geq kn$. Indeed, $\sum_{i=1}^{k} ia_i$ counts the number of vertices in $K_m \square K_n$ whose ranks appear at most $k$ times in $A$. By Lemma 6, for $1 \leq k \leq m$, each of the $n$ columns in $A$ is associated with $k$ such vertices. Therefore $\sum_{i=1}^{k} ia_i \geq kn$ as claimed.

Let $a_1, \ldots, a_m$ minimize $\sum_{i=1}^{m} a_i$ subject to the conditions $\sum_{i=1}^{k} ia_i \geq kn$ for $k \in [m]$. We claim that in each constraint, equality holds. Indeed, if $k$ is the least integer such that $\sum_{i=1}^{k} ia_i > kn$, then we may reduce $a_k$ by a positive $\epsilon$ while still satisfying the constraints $\sum_{i=1}^{\ell} ia_i \geq \ell n$ for $1 \leq \ell \leq k$. If we also increase $a_{k+1}$ by $\frac{k}{k+1} \epsilon$, then all constraints are satisfied, but we have reduced $\sum_{i=1}^{m} a_i$ by $\frac{1}{k+1} \epsilon$, contradicting the minimality of $\sum_{i=1}^{m} a_i$.

Since equality holds in all constraints, we conclude $a_k = n/k$ for each $k$ and $\sum_{i=1}^{m} a_i = nH_m$. □

When $n \geq m!$, Theorem 7 gives the correct order of growth of $\chi_2(K_m \square K_n)$. In fact, equality holds when $m! | n$.

**Theorem 8.** If $m! | n$, then $\chi_2(K_m \square K_n) = n H_m$.

**Proof.** Theorem 7 gives the lower bound. We claim that it suffices to prove $\chi_2(K_m \square K_m) \leq (m!) H_m$. Indeed, with $n = tm!$, the general case would then follow from Proposition 4, since $\chi_2(K_m \square K_n) \leq \chi_2(K_m \square K_m) \cdot \chi_2(K_1 \square K_t) = (m!) H_m \cdot t = n H_m$.
We prove that \( \chi_2(K_m \square K_m) = (m!)H_m \) by induction on \( m \). For \( m = 1 \), the statement is trivial. Suppose that \( m \geq 2 \) and let \( A' \) be an \( ((m-1) \times (m-1))! \)-matrix encoding an optimal 2-ranking of \( K_{m-1} \square K_{(m-1)} \). By shifting the ranks appropriately, let \( A_1', \ldots, A_m' \) be copies of \( A' \) that use disjoint intervals of ranks, starting with rank \((m-1)! + 1\). The ranks appearing in \( A_1', \ldots, A_m' \) are high, and the ranks in \([(m-1)!] \) are low.

We construct an \( (m \times ml)! \)-matrix \( A \) encoding a 2-ranking of \( K_m \square K_m \) as follows. Let \( M_i \) be an \( (m \times (m-1)!) \)-matrix such that deleting the \( i \)th row of \( M_i \) gives \( A_i' \) and whose \( i \)th row contains each low rank. Let \( A = [M_1 \cdots M_m] \). The rows and columns of \( A \) have distinct entries. Suppose that \( A(i,j) = A(i',j') \). If \( A(i,j) \) and \( A(i',j') \) are both low ranks, then columns \( j \) and \( j' \) belong to distinct blocks of \( A \) and so \( A(i',j) \) and \( A(i,j') \) are both high ranks. If \( A(i,j) \) and \( A(i',j') \) are both high ranks, then columns \( j \) and \( j' \) belong to the same block of \( A \) and so the opposite corners have higher rank by induction. It follows that \( A \) is a 2-ranking. Since \( A \) uses \((m-1)! \) low ranks and \( m \cdot [(m-1)!]H_{m-1} \) high ranks, we have that \( \chi_2(K_m \square K_m) \leq (m-1)! + m!H_{m-1} = m!(1/m + H_{m-1}) = m!H_m \).

Using that \( H_m = (1 + o(1)) \ln m \), we obtain an asymptotic formula for \( \chi_2(K_m \square K_m) \) when \( m \) is constant.

**Corollary 9.** For each positive integer \( m \), we have that \( \chi_2(K_m \square K_m) = (1+o(1))n \ln m \) as \( n \to \infty \).

**Proof.** The lower bound follows immediately from Theorem 7. For the upper bound, let \( n' \) be the least multiple of \( m! \) that is at least \( n \). By Theorem \( 8 \), we have \( \chi_2(K_m \square K_m) \leq \chi_2(K_{m'} \square K_{n'}) = n'H_m = (1 + (m!)/n) \cdot nH_m = (1 + o(1))n \ln m \).

In the diagonal case, our bounds are far apart. Combining Theorem 7 and Corollary 5 gives \( \Omega(n \log n) \leq \chi_2(K_n \square K_n) \leq O(n \log^3 n) \). What is the order of growth of \( \chi_2(K_n \square K_n) \)?

### 4 The 2-ranking number of graphs with maximum degree \( k \)

Let \( G \) be a graph with \( \Delta(G) = k \), where \( \Delta(G) \) is the maximum degree of \( G \). Since \( \chi_s(G) \leq \chi_2(G) \leq \chi(G^2) \leq \Delta(G^2) + 1 \leq k^2 + 1 \), it is interesting to ask for the maximum of \( \chi_s(G) \) and \( \chi_2(G) \) over all graphs \( G \) with maximum degree at most \( k \). Fertin, Raspaul, and Reed \( 4 \) proved that the maximum of \( \chi_s(G) \) over all graphs with maximum degree at most \( k \) is at least \( \Omega\left(\frac{k^{3/2}}{(\log k)^{1/2}}\right) \) and is at most \( O(k^{3/2}) \). We make slight modifications to their probabilistic construction to show that the maximum of \( \chi_2(G) \) over all graphs with maximum degree \( k \) is at least \( \Omega\left(k^2/\log k\right) \).

**Theorem 10.** For each \( k \), there exists a graph \( G \) with \( \Delta(G) \leq k \) and \( \chi_2(G) \geq \Omega\left(k^2/\log k\right) \).

**Proof.** Choose \( n \) so that \( n \) is even and \( 2np \leq k \), where \( p = c(\log n/n)^{1/2} \) for some constant \( c \) to be chosen later. Since we may assume that \( k \) is sufficiently large, we may assume that \( n \) is also sufficiently large. Let \( G \) be a random graph chosen from \( G(n,p) \). Each vertex in \( G \) has expected degree \( (n-1)p \), and it is well known (see, for example, \( 2 \)) that \( \Pr(\Delta(G) \leq 2np) \rightarrow 1 \) as \( n \to \infty \).

For each function \( f: V(G) \to [n/2] \), let \( A_f \) be the bad event that \( f \) is a 2-ranking of \( G \). Applying the union bound, we have that \( \Pr(\chi_2(G) \leq \frac{n}{2}) = \Pr(\bigcup_f A_f) \leq \sum_f \Pr(A_f) \).

Fix a function \( f: V(G) \to [n/2] \). Discarding one vertex from each rank class with an odd number of vertices, we may partition the remaining vertices into pairs \( S_1, \ldots, S_{\ell} \) such that both vertices on \( S_i \) have the same rank under \( f \). Since at most \( n/2 \) vertices are discarded, we have
\[ \ell \geq (1/2)(n - n/2) = n/4. \]

Index the pairs so that \( i \leq j \) implies that \( f(u) \leq f(v) \) when \( u \in S_i \) and \( v \in S_j \). For each pair \( \{S_i, S_j\} \) with \( i < j \), the probability that \( G \) contains some path \( uvw \) such that \( u, v \in S_j \) and \( w \in S_i \) is at least \( p^2 \). If this happens, then \( f \) is not a 2-ranking since either \( f(u) = f(v) = f(w) \) or \( f(u) = f(v) > f(w) \).

Since the paths \( uvw \) form an edge-disjoint family as we range over the pairs \( \{S_i, S_j\} \), it follows that the pairs \( \{S_i, S_j\} \) give independent chances for \( A_f \) to fail. It follows that

\[
\mathbb{P}(\chi_2(G) \leq \frac{n}{2}) \leq \sum_f \mathbb{P}(A_f) \leq (n/2)^n (1 - p^2)^{(n/2)} \leq (n/2)^n e^{-\frac{c^2n^2}{33}} = \left( \frac{n}{2n^{1/2}} \right)^n.
\]

With \( c = 6 \), we have that \( \mathbb{P}(\chi_2(G) \leq \frac{n}{2}) \rightarrow 0 \) as \( n \rightarrow \infty \). It follows that with probability tending to 1, we have that \( \Delta(G) \leq k \) and \( \chi_2(G) > n/2 \geq c' \frac{k^2}{\ln k} \) for some positive constant \( c' \).

## 5 The 2-ranking number of subcubic graphs

A graph \( G \) is subcubic if \( \Delta(G) \leq 3 \). The star list chromatic number of \( G \), denoted \( \chi_s^\ell(G) \), is the minimum integer \( t \) such that if each vertex \( v \) in \( G \) is assigned a list \( L(v) \) of \( t \) colors, there is a star coloring of \( G \) in which each vertex \( v \) receives a color from its list \( L(v) \). Albertson, Chappell, Kierstead, Kündgen, and Ramamurthi \[1\] gave an elegant proof that every subcubic graph \( G \) satisfies \( \chi_s^\ell(G) \leq 7 \). It follows that \( \chi_s(G) \leq \chi_s^\ell(G) \leq 7 \) when \( G \) is subcubic. Chen, Raspaud, and Wang \[3\] proved that every subcubic graph \( G \) satisfies \( \chi_s(G) \leq 6 \).

Let \( G \) be the 3-regular graph obtained from \( C_8 \) by joining vertices at distance 4. Fertin, Raspaud, and Reed \[4\] proved that \( \chi_s(G) = 6 \), and it follows that the result of Chen, Raspaud, and Wang is best possible.

Here, we show that \( \chi_2(G) \leq 7 \) when \( G \) is subcubic. Since \( \chi_2(G) \geq \chi_s(G) \) always, the example of Fertin, Raspaud, and Reed shows that our bound cannot be reduced by more than 1 in the general case. Nonetheless, we believe the bound can be improved by 2 aside from a single exception; see Conjecture \[2\].

An independent set in \( G \) is a set of vertices that are pairwise nonadjacent. We use \( N_G(u) \) for the set of neighbors of \( u \) in \( G \) and, when \( S \subseteq V(G) \), we use \( G[S] \) for the subgraph of \( G \) induced by \( S \). Vertices \( u \) and \( v \) in a graph \( G \) are antipodal if \( \text{dist}(u, v) = \text{diam}(G) \), where \( \text{diam}(G) \) is the maximum distance between a pair of vertices in \( G \).

**Theorem 11.** If \( G \) is subcubic, then \( \chi_2(G) \leq 7 \).

**Proof.** Let \( G \) be a subcubic graph. We may assume that \( G \) is connected. Let \( S \) be a maximal independent subset of \( V(G) \), and let \( \overline{S} = V(G) - S \). Since \( S \) is maximal, every vertex in \( G \) is in \( S \) or has a neighbor in \( S \). It follows that \( |N_G(u) \cap \overline{S}| \leq 6 \). Indeed, if \( u \) is a neighbor of \( v \), then \( u \) has at most 1 other neighbor in \( \overline{S} \), or else \( u \) would have 3 neighbors in \( \overline{S} \), a contradiction.
It follows that $\Delta(G^2[S]) \leq 6$. If $G^2[S]$ does not contain a copy of $K_7$, then by Brooks’s theorem, $\chi(G^2[S]) \leq 6$. Using a proper coloring of $G^2[S]$ with colors in $[6]$ and assigning rank $0$ to all vertices in $S$ gives a 2-ranking of $G$. Indeed, paths of length 1 are well-ranked and $G$ contains no paths of length 2 joining vertices with nonzero ranks.

Since $G$ is connected, it follows that $G^2[S]$ is connected. Since $G^2[S]$ is connected and has maximum degree at most 6, if $G^2[S]$ contains a copy of $K_7$, then $G^2[S] = K_7$. It follows that $\chi_2(G) \leq 7$ unless $G^2[S] = K_7$.

Suppose that $G^2[S] = K_7$. This has several implications for the structure of $G[S]$. First, we claim that every vertex in $G[S]$ has degree 0 or degree 2. Since each vertex $u \in S$ has a neighbor in $S$, it follows that $u$ has at most 2 neighbors in $G[S]$. If the only neighbor of $u$ in $G[S]$ is $v$, then $v$ has at most 5 neighbors in $G^2[S]$; at most 2 from each of the neighbors of $v$ in $G$ besides $u$, and $u$ itself. This contradicts that $G^2[S] = K_7$.

It follows that $G[S]$ is a 7-vertex graph whose components are isolated vertices and cycles. We claim that each cycle in $G[S]$ has length at least 5. Indeed, suppose that $v$ is in a cycle $C$ in $G^2[S]$ of length at most 4, and let $u_1$ and $u_2$ be the neighbors of $v$ along $C$. Each neighbor of $v$ in $G$ contributes at most 2 neighbors of $v$ in $G^2[S]$. Since $C$ has length at most 4, the contributions of $u_1$ and $u_2$ have nonempty intersection. It follows that $v$ has fewer than 6 neighbors in $G^2[S]$, contradicting that $G^2[S] = K_7$.

Suppose that $G[S]$ contains a 5-cycle $C$, and let $x$ and $y$ be the vertices in $G[S]$ that are not in $C$. Let $u$ be a vertex in $C$. Since $u$ is adjacent to $x$ and $y$ in $G^2[S]$, it must be that $u$ is adjacent in $G$ to a vertex $s_u \in S$ whose other two neighbors in $G$ are $x$ and $y$. The vertices $\{s_u : u \in V(C)\}$ have distinct neighborhoods of size 3 and are therefore distinct. This is not possible, since $x$ and $y$ would have 5 neighbors in $G$. Therefore $G[S]$ does not contain a 5-cycle.

Suppose that $G[S]$ contains a 7-cycle $C$, and let $u$ be a vertex in $C$. Since $u$ is adjacent in $G^2[S]$ to the two vertices $x$ and $y$ at distance 3 from $u$ in $C$, it must be that $u$ is adjacent in $G$ to a vertex $s_u$ such that $N_G(s_u) = \{u, x, y\}$. Again, the vertices $\{s_u : u \in V(C)\}$ have distinct neighborhoods of size 3, and are therefore distinct. This is impossible, since $x$ is adjacent in $G$ to $s_u, s_x$, and its two neighbors on $C$. Therefore, $G[S]$ cannot contain a 7-cycle.

It follows that either $G[S] = C_6 + K_1$ or $G[S] = 7K_1$. Suppose that $G[S]$ contains a 6-cycle $C$ and let $x$ be the isolated vertex. If $u$ is a vertex on $C$, then $u$ is adjacent in $G$ to a vertex $s_u$ whose neighbors are $u, x$, and the vertex on $C$ antipodal to $u$. It follows that $G$ is the Petersen graph. In the diagram below, vertices in $S$ are white and vertices in $\overline{S}$ are black.

Suppose that $G[S] = 7K_1$. It follows that each vertex $u$ in $\overline{S}$ is adjacent in $G$ to 3 neighbors $v_1, v_2, v_3$ in $S$. Moreover, we have $\bigcup_{i=1}^3 N_G(v_i) = S$ and $N_G(v_i) \cap N_G(v_j) = \{u\}$ for $i \neq j$. Since $G$ is connected, we have that $G$ is a 3-regular $(S, \overline{S})$-bigraph, and so $|S| = |\overline{S}| = 7$. It follows also that $G$ does not contain a copy of $C_4$, or else some vertex $u \in \overline{S}$ would have neighbors $v_1$ and $v_2$ with
\(|NG(v_1) \cap NG(v_2)| \geq 2\). Since \(G\) is a bipartite 3-regular graph on 14 vertices with girth at least 6, it follows that \(G\) is the Heawood graph.

As we have seen, if \(G\) is subcubic, then \(\chi_2(G) \leq 7\), or \(G\) is the Petersen graph, or \(G\) is the Heawood graph. If \(G\) is the Petersen graph, then \(G\) contains a maximal independent set \(S\) of size 4. We may repeat the argument with \(G[S]\) having 6 vertices. If \(G\) is the Heawood graph, then a vertex \(u\) and the four vertices antipodal to \(u\) form a maximal independent set \(S\) of size 5. We may repeat the argument with \(G[S]\) having 9 vertices.

Besides the example of Fertin, Raspaud, and Reed, we are not aware of another subcubic graph that requires 6 ranks for a 2-ranking. Plausible candidates such as the Petersen graph and the Heawood graph admit 2-rankings with only 5 ranks.

**Conjecture 12.** If \(G\) is subcubic, then \(\chi_2(G) \leq 6\) and equality holds if and only if \(G\) is the cubic graph obtained from \(C_8\) by joining vertices at distance 4.

### 6 The product of a triangle and a cycle

Applied to the product of a pair of cycles, Corollary \(3\) states that \(\chi_2(C_m \Box C_n) = 5\) when \(m\) and \(n\) are divisible by 4. In this section, we show that the 2-ranking number of cycle products may depend on the parity of the lengths of the factors. In particular, we show that for sufficiently large \(n\), the 2-ranking number of \(C_3 \Box C_n\) is 5 when \(n\) is even and 6 when \(n\) is odd. We represent a 2-ranking of \(C_3 \Box C_n\) with a \((3 \times n)\)-array \(A\) such that \(A(i, j)\) is the rank of \((u_i, v_j) \in V(C_3 \Box C_n)\).

**Lemma 13.** If \(n \geq 24\), then \(\chi_2(C_3 \Box C_n) \leq 6\).

**Proof.** Let \(n = 4q + r\) for integers \(q\) and \(r\) with \(r \in \{0, 1, 2, 3\}\). Since \(q \geq 6\), we have that \(n = 4(q - 2r) + 9r\), and so \(n\) is a nonnegative integer combination of 4 and 9. We give 2-rankings of \(C_3 \Box C_9\) and \(C_3 \Box C_4\) that can be appended together to give a 2-ranking of \(C_3 \Box C_n\).

\[
\begin{array}{cccccc}
2 & 4 & 0 & 3 & 1 & 0 \\
0 & 5 & 1 & 0 & 5 & 2 \\
1 & 3 & 2 & 4 & 0 & 3 \\
\end{array}
\begin{array}{cccc}
0 & 5 & 1 & 3 \\
0 & 5 & 1 & 3 \\
1 & 3 & 2 & 4 \\
\end{array}
\]

To see that these are 2-rankings, observe that the vertices assigned rank 0 form an independent set, and for each positive rank \(t\), the vertices assigned rank \(t\) are independent in \((C_3 \Box C_n)^2\). Because both 2-rankings agree on the first two columns and the last two columns, appending the arrays gives a 2-ranking.

Our upper bound improves for even \(n\).

**Lemma 14.** If \(n\) is even and \(n \geq 4\), then \(\chi_2(C_3 \Box C_n) \leq 5\).

**Proof.** Let \(n = 4q + 6r\) for integers \(q\) and \(r\) with \(r \in \{0, 1\}\). We give 2-rankings of \(C_3 \Box C_4\) and \(C_3 \Box C_6\) which can be appended to give a 2-ranking of \(C_3 \Box C_n\).

\[
\begin{array}{cccc}
0 & 1 & 0 & 2 \\
3 & 2 & 4 & 1 \\
4 & 0 & 3 & 0 \\
\end{array}
\begin{array}{cccc}
0 & 1 & 3 & 0 \\
3 & 0 & 4 & 2 \\
4 & 2 & 0 & 1 \\
\end{array}
\]

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Regardless of how these arrays are appended, vertices assigned rank 0 form an independent set, and for each positive rank \( t \), the vertices assigned rank \( t \) form an independent set in \((C_3 \square C_n)^2\) \( \square \).

Since \( C_3 \square C_n \) has degeneracy 4, it follows that \( \chi_2(C_3 \square C_n) \geq 5 \) always. When \( n \) is odd, our lower bound improves.

**Lemma 15.** If \( n \) is odd, then \( \chi_2(C_3 \square C_n) > 5 \).

**Proof.** Suppose for a contradiction that \( C_3 \square C_n \) has a 2-ranking \( A \) using ranks in \([5]\). Ranks 4 and 5 are \textit{high}; the other ranks are \textit{low}. Note that each high rank appears at most once in every pair of adjacent columns of \( A \). It follows that at most \( k \) vertices are assigned to each high rank. A column containing all of the low ranks is \textit{low}, and a column containing all of the high ranks is \textit{high}. Since \( A \) has \( 2k + 1 \) columns and at most \( 2k \) vertices have high rank, it follows that some column of \( A \) is \textit{low}.

It is easy to check that \( \chi_2(C_3 \square P_2) \geq 5 \). It follows that a column adjacent to a low column must be high. Since high ranks cannot appear in adjacent columns, a column adjacent to a high column must be low. Therefore the columns of \( A \) alternate high and low cyclically, contradicting that \( n \) is odd. \( \square \)

Collecting the lemmas, we obtain the following theorem.

**Theorem 16.** If \( n \) is odd and \( n \geq 25 \), then \( \chi_2(C_3 \square C_n) = 6 \). If \( n \) is even and \( n \geq 6 \), then \( \chi_2(C_3 \square C_n) = 5 \).

It would be interesting to find the 2-ranking number of \( C_m \square C_n \) for general \( m \) and \( n \).

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