Reconstruction of Inclusion from Boundary Measurements

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Abstract

We consider the problem of reconstructing of the boundary of an unknown inclusion together with its conductivity from the localized Dirichlet-to-Neumann map. We give an exact reconstruction procedure and apply the method to an inverse boundary value problem for the system of the equations in the theory of elasticity. AMS: 35R30

1 Introduction and statement of the results

This paper is the sequel to [7]. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n = 2, 3 \) with connected Lipschitz boundary. We denote its conductivity by \( \gamma \). In what follows, unless otherwise stated, we assume (C):

\[
\left\{ \begin{array}{l}
\gamma \text{ is an essentially bounded real-valued function on } \Omega \\
\text{and uniformly positive definite.}
\end{array} \right. \quad (C)
\]

We define the Dirichlet-to-Neumann map

\[
\Lambda_\gamma : H^{1/2}(\partial \Omega) \longrightarrow H^{-1/2}(\partial \Omega)
\]

by the formula

\[
< \Lambda_\gamma f, g > = \int_\Omega \gamma \nabla u \cdot \nabla \varphi dx,
\]

where \( g \in H^{1/2}(\partial \Omega) \), \( \varphi \) is any \( H^1(\Omega) \) function with \( \varphi|_{\partial \Omega} = g \), \( u \) is in \( H^1(\Omega) \) and the weak solution of the Dirichlet problem

\[
\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega,
\]

\[
u|_{\partial \Omega} = f.
\]

Notice that \( \Lambda_\gamma \) is a bounded linear operator. \( \Lambda_\gamma f \) is the electric current on \( \partial \Omega \) corresponding to a voltage potential \( f \) on \( \partial \Omega \).

Let \( \Gamma \) be a given nonempty open subset of \( \partial \Omega \). Set

\[
\mathcal{D}(\Gamma) = \{ f \in H^{1/2}(\partial \Omega) \mid \text{supp } f \subset \Gamma \}.
\]

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We call the map
\[ \mathcal{D}(\Gamma) \ni f \mapsto \Lambda_\gamma f|_\Gamma \]
the localized Dirichlet-to-Neumann map.

We assume that \( \Omega \) contains an unknown inclusion “discontinuously” imbedded in a reference medium with known conductivity \( \gamma_0 \). We denote by \( D \) the shape of the inclusion and by \( \gamma|_D \) its conductivity.

The problem is to find a reconstruction formula (procedure) of \( D \) together with \( \gamma|_D \) by means of the localized Dirichlet-to-Neumann map.

Let us describe it more precisely. We assume that \( \gamma_0 \) satisfies (C) and that \( \gamma_0 \in C^0,1(\Omega) \) if \( n = 3 \);
\( \gamma_0 \in C^0,\theta(\Omega) \) with \( 0 < \theta \leq 1 \) if \( n = 2 \).

The pair \((D, \gamma)\) satisfies
\[ \begin{align*}
(i) & \quad D \text{ is an open subset of } \Omega \text{ with Lipschitz boundary and satisfies } \overline{D} \subset \Omega; \\
(ii) & \quad \Omega \setminus \overline{D} \text{ is connected}; \\
(iii) & \quad \gamma(x) = \gamma_0(x) \text{ for almost all } x \in \Omega \setminus \overline{D}; \\
(iv) & \quad \gamma \text{ satisfies the following jump condition:} \\
& \forall a \in \partial D \exists \delta > 0 \\
& \gamma - \gamma_0 \text{ is uniformly positive definite on } D \cap B(a, \delta) \\
& \text{ or} \\
& \gamma_0 - \gamma \text{ is uniformly positive definite on } D \cap B(a, \delta). 
\end{align*} \]

Our main result is

\textbf{Main Theorem.} Assume that \( \gamma|_D \in W^{2,p}(D) \) with \( 2p > n \). Let \( \Gamma \) be a given nonempty open subset of \( \partial \Omega \). There exists a set \( \mathcal{D}(\gamma_0, \Gamma) \) contained in \( \mathcal{D}(\Gamma) \) such that both \( \partial D \) and \( \gamma|_D \) can be reconstructed from the set
\[ \{ < (\Lambda_\gamma - \Lambda_{\gamma_0})f, f > | f \in \mathcal{D}(\gamma_0, \Gamma) \}. \]

\textbf{Remark.} \( \mathcal{D}(\gamma_0, \Gamma) \) is independent of \( \Lambda_\gamma \) and depends on \( \Omega, \gamma_0 \) and \( \Gamma \).

Isakov [8] has proved the uniqueness of reconstruction in the case when \( n = 3, \gamma_0 \in C^2(\Omega), \partial \Omega \) is \( C^2 \) and \( \gamma|_D \in C^2(\overline{D}) \). However his proof does not provide us how to reconstruct \( D \) and \( \gamma|_D \). Main Theorem does it. It should be noted that Kohn-Vogelius [9] has proved the uniqueness of reconstruction of the piecewise real analytic conductivity. Alessandrini [1] gave further uniqueness results in this direction. Their proofs also do not provide us any reconstruction procedure.

Nachman [13] (see also [12]) proved

\textbf{Theorem A.} Let \( \gamma \) satisfy (C). Assume that \( \gamma \in W^{2,p}(\Omega) \) with \( 2p > n \). \( \gamma \) can be reconstructed from \( \Lambda_\gamma \).

Since \( \gamma \) in Main Theorem has a first kind of discontinuity, one can not immediately get Main Theorem from Theorem A. Furthermore in Main Theorem we need only the
localized Dirichlet-to-Neumann map. It is not clear whether one can replace the Dirichlet-
to-Neumann map in the statement of Theorem A with the localized one. Notice that in
the case when $n = 3$, Sylvester - Uhlmann [17] proved the uniqueness of the reconstruction
of $\gamma \in C^\infty(\Omega)$ from the full knowledge of $\Lambda_\gamma$.

We briefly describe the steps of the proof of Main Theorem. First in Section 2 we
prove

**Theorem B.** Let $\Gamma$ be a given nonempty open subset of $\partial \Omega$. Under (CI), there exists a
set $D(\gamma_0, \Gamma)$ contained in $D(\Gamma)$ such that $\partial D$ can be reconstructed from the set
\[
\{ \langle (\Lambda_\gamma - \Lambda_{\gamma_0})f, f \rangle \mid f \in D(\gamma_0, \Gamma) \}.
\]

**Remark.** We do not require any regularity on $\gamma$ inside the inclusion; if $\gamma_0 \equiv 1$, Theorem
B is included in [7] and the uniqueness of the reconstruction has been proved in [6]; for
the concrete description of $D(\gamma_0, \Gamma)$ see Remark under Proposition 2.5 in Section 2.

From this theorem one gets $D$. So the next problem is to reconstruct $\gamma|_D$. To do it
we calculate the Dirichlet-to-Neumann map inside the inclusion. Let us describe it more
precisely.

**Definition (Dirichlet-to-Neumann Map inside Inclusion).** Consider $D$ and $\gamma
satisfying (i), (ii) of (CI) and (C) respectively. We define the Dirichlet-to-Neumann map
inside inclusion $\Lambda_{\gamma^-} : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$
by the formula
\[
\langle \Lambda_{\gamma^-} f, g \rangle = \int_D \gamma \nabla u \cdot \nabla \varphi \, dx,
\]
where $g \in H^{1/2}(\partial D)$, $\varphi$ is any $H^1(D)$ function with $\varphi|_{\partial D} = g$, $u$ is in $H^1(D)$ and the
weak solution of the Dirichlet problem
\[
\nabla \cdot \gamma \nabla u = 0 \text{ in } D,
\]
\[
u|_{\partial D} = f.
\]

In Section 3 we prove

**Theorem C.** Let $\Gamma$ be a given nonempty open subset of $\partial \Omega$. Assume that (i)~(iii) of (CI)
hold. There exists a set $D(\gamma_0, \Gamma, D)$ contained in $D(\Gamma)$ such that $\Lambda_{\gamma^-}$ can be calculated
from the set
\[
\{ \langle (\Lambda_\gamma - \Lambda_{\gamma_0})f, f \rangle \mid f \in D(\gamma_0, \Gamma, D) \}.
\]

**Remark.** In this theorem, it is assumed that both $\gamma_0$ and $D$ are known; $D(\gamma_0, \Gamma, D)$ is
independent of $\Lambda_\gamma$ and depends on $\Omega, \Gamma$ and $D$; notice that we do not assume any regularity
on $\gamma$ inside the inclusion; for the concrete description of $D(\gamma_0, \Gamma, D)$ see Remark under
Proposition 3.3 in Section 3.

Thus if once we get $D$, from Theorem C we get $\Lambda_{\gamma^-}$. Then from Theorem A
(Nachman’s reconstruction procedure) in the case where $\Omega = D, \gamma = \gamma|_D$ we get $\gamma|_D$. For the
summary of the procedure see Section 4.

Theorem C has an interesting conclusion in the context of heat conduction. Suppose
you have a heat conductive body $D$ with unknown heat conductivity $\gamma^-$. We assume that
there is no heat sources inside $D$. You can measure the temperature distribution $f$ on $\partial D$. But how can you measure the heat flux $\Lambda_{\gamma} f$ on $\partial D$ which produces $f$ on $\partial D$? Since $\gamma$ is unknown, to do it you can not make use of the temperature distribution at the boundary layer of $\partial D$. A way to overcome this difficulty is to insert $D$ into a conductive body $\Omega$ with known heat conductivity $\gamma_0$. Then give any temperature distribution $g$ on $\partial \Omega$ and measure the temperature at the boundary layer of $\partial \Omega$. Since $\gamma_0$ is known, from such data you can calculate the heat flux on $\partial \Omega$ which produces $g$. If you do this procedure infinitely many times, Theorem C says that, in principle, you can know $\Lambda_{\gamma} f$.

As an application of our method we consider an inverse problem for elastic material occupying $\Omega \subset \mathbb{R}^3$. For general information about the linear theory of elasticity we refer the reader to Gurtin’s beautiful survey paper [4]. We consider $\Omega$ as an isotropic elastic body with Lamé parameters $\lambda, \mu$. In what follows, unless otherwise stated, $(\lambda, \mu)$ satisfies

\[
\begin{align*}
\lambda, \mu & \text{ are essentially bounded functions on } \Omega \text{ and satisfy } \\
\exists C > 0 \mu(y) > C \text{ and } 3\lambda(y) + 2\mu(y) > C & \text{ for almost all } y \in \Omega.
\end{align*}
\]

$(E)$

Let $\text{Sym} A$ denote the symmetric part of the matrix $A$. One can easily prove that, for each $f \in \{H^{1/2}(\partial \Omega)\}^3$ there exists a unique $u \in \{H^1(\Omega)\}^3$ such that

\[
\mathcal{L}_{\lambda, \mu} u \equiv \left( \sum_{1 \leq j \leq 3} \frac{\partial}{\partial y_j} \{ \lambda (\nabla \cdot u) \delta_{ij} + 2\mu(\text{Sym} \nabla u)_{ij} \} \right) = 0 \text{ in } \Omega,
\]

\[u = f \text{ on } \partial \Omega.\]

We define the Dirichlet-to-Neumann map, denoted by $\Lambda_{\lambda, \mu}$, by the formula

\[
< \Lambda_{\lambda, \mu} f, g > = \int_{\Omega} \lambda \nabla \cdot u \nabla \cdot v + 2\mu \text{Sym} \nabla u \cdot \text{Sym} \nabla v \, dy
\]

where $v \in \{H^1(\Omega)\}^n$ with $v|_{\partial \Omega} = g \in \{H^{1/2}(\partial \Omega)\}^3$. $\Lambda_{\lambda, \mu} f$ is the traction on $\partial \Omega$ corresponding to a displacement field $f$ on $\partial \Omega$.

Let $\Gamma$ be a given nonempty open subset of $\partial \Omega$. Set

\[\mathcal{D}(\Gamma) = \{ f \in \{H^{1/2}(\partial \Omega)\}^3 | \text{supp } f \subset \Gamma \}.
\]

We call the map

\[\mathcal{D}(\Gamma) \ni f \mapsto \Lambda_{\lambda, \mu} f|_{\Gamma}\]

the localized Dirichlet-to-Neumann map.

We assume that $\Omega$ contains an unknown inclusion "discontinuously" imbedded in a reference medium with known Lamé parameters $\lambda_0, \mu_0$. We denote by $D$ the shape of the inclusion and by $\lambda|_D, \mu|_D$ its Lamé parameters.

The problem is to find a reconstruction procedure of $D$ by means of the localized Dirichlet-to-Neumann map. In this problem we do not assume that $\lambda|_D, \mu|_D$ are known. Let us describe the result. We assume that $\lambda_0, \mu_0$ satisfy (E) and that

\[
\partial \Omega \text{ is } C^3, \\
\lambda_0 \in C^2(\Omega) \text{ and } \mu_0 \in C^2(\Omega).
\]
In the sequel we will assume the following hypotheses and notation to be in force.

\[
\begin{align*}
(i) \ D \text{ is an open subset of } \Omega \text{ with Lipschitz boundary} \\
(ii) \ \Omega \setminus \overline{D} \text{ is connected;} \\
(iii) \ (\lambda(x), \mu(x)) = (\lambda_0(x), \mu_0(x)) \text{ for almost all } x \in \Omega \setminus \overline{D;} \\
(iv) \ \forall a \in \partial D \exists (\alpha, \beta) \forall \epsilon > 0 \exists \delta > 0 \ \left| \lambda(x) - \alpha \right| + |\mu(x) - \beta| < \epsilon \text{ for almost all } x \in B(a, \delta) \cap D.
\end{align*}
\] 

(EI)

Since \( \text{supp } f \subset \Gamma \), we use only the restriction of \( \Lambda_{\lambda, \mu} f \) to \( \Gamma \) for recovering \( D \); we do not require any regularity on \( \lambda, \mu \) inside the inclusion; for the concrete description of \( \mathcal{D}(\lambda_0, \mu_0, \Gamma) \) see Remark under Proposition 5.6.

The condition (v) of (EI) makes the problem difficult. Such type of condition never appeared in inverse conductivity problem. To overcome the difficulty we fully make use of the speciality of isotropic elastic body.

The second problem is to find a reconstruction formula of \( (\lambda|_D, \mu|_D) \) by means of the localized Dirichlet-to-Neumann map provided \( D \) is known. To do it a result similar to Theorem C shall be useful.

**Definition (Dirichlet-to-Neumann Map inside Inclusion).** Consider \( D \) and \( \lambda, \mu \) satisfying (i), (ii) of (EI) and (E) respectively. We define the Dirichlet-to-Neumann map inside inclusion

\[ \Lambda_{\lambda-, \mu-} : \{H^{1/2}(\partial D)\}^3 \to \{\{H^{1/2}(\partial D)\}^3\}^* \]

by the formula

\[ < \Lambda_{\lambda-, \mu-} f, g > = \int_D \lambda \nabla \cdot u \nabla \cdot v + 2\mu \text{Sym} \nabla u \cdot \text{Sym} \nabla v \, dx, \]

where \( g \in \{H^{1/2}(\partial D)\}^3 \), \( v \) is any \( \{H^1(D)\}^3 \) function with \( v|_{\partial D} = g \), \( u \) is in \( \{H^1(D)\}^3 \) and the weak solution of the Dirichlet problem

\[ L_{\lambda, \mu} u = 0 \text{ in } D, \]

\[ u|_{\partial D} = f. \]

The proof of Theorem E stated below proceeds along the same lines with that of Theorem C and we omit it.

**Theorem E.** Let \( \Gamma \) be a given nonempty open subset of \( \partial \Omega \). Assume that (i)–(iii) of (EI) hold. There exists a set \( \mathcal{D}(\lambda_0, \mu_0, \Gamma, D) \) contained in \( \mathcal{D}(\Gamma)^3 \) such that \( \Lambda_{\lambda-, \mu-} \) can be calculated from the set

\[ \{ < \Lambda_{\lambda, \mu} - \Lambda_{\lambda_0, \mu_0} f, f > \mid f \in \mathcal{D}(\lambda_0, \mu_0, \Gamma, D) \}. \]
Remark. In this theorem, it is assumed that \( \lambda_0, \mu_0 \) and \( D \) are known; \( D(\gamma_0, \mu_0, \Gamma, D) \) is independent of \( \Lambda_{\lambda, \mu} \) and depends on \( \Omega, \Gamma \) and \( D \); notice that we do not assume any regularity on \( \lambda, \mu \) inside the inclusion.

By the way, Nakamura-Uhlmann [15] proved

**Theorem F.** Let \( (\lambda_i, \mu_i), i = 1, 2, \) be two pairs of Lamé parameters. Assume that

\[
\lambda_i, \mu_i \text{ are smooth on } \overline{\Omega}, i = 1, 2,
\]

and

\[
\partial \Omega \text{ is smooth.}
\]

If

\[
\Lambda_{\lambda_1, \mu_1} = \Lambda_{\lambda_2, \mu_2},
\]

then

\[
(\lambda_1, \mu_1) = (\lambda_2, \mu_2).
\]

In contrast to what Theorem A tells Theorem F is a uniqueness theorem and does not contain any reconstruction procedure. So far such procedure is not known. This affects Theorem G mentioned below. We refer the reader to their survey paper [16] for more precise information about related results.

We consider two arbitrary triples \((D_1, \lambda_1, \mu_1), (D_2, \lambda_2, \mu_2)\) both satisfying (EI). From Theorems D, E and F we immediately get

**Theorem G.** Assume that for \( i = 1, 2 \)

\[
\partial D_i \text{ is smooth}
\]

and

\[
\lambda_i|_{D_i}, \mu_i|_{D_i} \text{ are smooth on } \overline{D_i}.
\]

There exists a set \( D(\lambda_0, \mu_0, \Gamma) \) contained in \( D(\Gamma)^3 \) such that if

\[
\Lambda_{\lambda_1, \mu_1} f|\Gamma = \Lambda_{\lambda_2, \mu_2} f|\Gamma
\]

for all \( f \in D(\lambda_0, \mu_0, \Gamma) \), then

\[
D_1 = D_2 \text{ and } (\lambda_1|_{D_1}, \mu_1|_{D_1}) = (\lambda_2|_{D_2}, \mu_2|_{D_2}).
\]

This is a uniqueness theorem. Nakamura-Uhlmann’s result cannot cover Theorem G without using Theorem E since their method heavily relies on the regularity of \( \lambda, \mu \).

## 2 The probe method and proof of Theorem B

Let us recall some definitions in [6]. We insert a “needle” into \( \Omega \) defined below.

**Definition (Needle).** We call a continuous map \( c : [0, 1] \rightarrow \overline{\Omega} \) satisfying (i) and (ii) a needle:

\[
(i) \quad c(0), c(1) \in \partial \Omega \\
(ii) \quad c(t) \in \Omega \quad (0 < t < 1).
\]
Definition (Impact parameter). It is easy to verify that if a needle \( c \) touches a point on \( \partial D \), there exists a unique \( t(c; D) \in ]0, 1[ \) such that if \( 0 < t < t(c; D) \), \( c(t(c; D)) \in \partial D \). We set \( t(c; D) = 1 \) if \( c \) does not touch any point on \( \partial D \). We call \( t(c; D) \) the impact parameter of \( c \) with respect to \( D \). Notice that \( t(c; D) \) has the form
\[
G_0^0(\cdot; c(t; D)) = \sup \{ s \in ]0, 1[ | \forall t \in ]0, s[ \},
\]
where \( \nabla \cdot \gamma_0(\cdot) = 0 \) in \( \Omega \), \( \nabla \cdot \gamma_0(\cdot) \) is bounded in \( H^1(\Omega) \), and \( \nabla \cdot \gamma_0(\cdot) + \delta(\cdot - x) = 0 \) in \( \Omega \).

By virtue of (ii) of (CI) we get
\[
\partial D = \{ c(t(c; D)) | c \text{ is a needle and satisfies } t(c; D) < 1 \}.
\]

Our purpose is to construct the analytic version of the impact parameter by means of the localized Dirichlet-to-Neumann map and show the coincidence of both impact parameters. Then from (2.2) we get \( \partial D \).

Let us explain its construction. First we have to prove

Proposition 2.1. For each \( x \in \Omega \) let \( G_0^0(\cdot; x) \) denote the standard fundamental solution for the operator \( \nabla \cdot \gamma_0(x) \nabla \). Then, there exists a family \( \{ G_x^0(\cdot) \} \) in \( \cap_{1 \leq p < n} L^p(\Omega) \) such that
\[
G_x^0(\cdot) - G_0^0(\cdot - x; x) \in H^1(\Omega),
\]
\[
\nabla \cdot \gamma_0(\cdot) G_x^0(\cdot) + \delta(\cdot - x) = 0 \text{ in } \Omega.
\]

For the proof see Section 6. Second we prepare

Proposition 2.2 (Selection of boundary data). Let \( c \) be a needle. Then, for any \( t \in ]0, 1[ \), there exists a sequence \( \{ f_n(\cdot; c(t)) \} \) of \( \mathcal{D}(\Gamma) \) such that the weak solution \( v_n \) of
\[
\nabla \cdot \gamma_0(\cdot) v_n = 0 \text{ in } \Omega,
\]
\[
v_n|_{\partial \Omega} = f_n(\cdot; c(t))
\]
converges to \( G_{c(t)}^0(\cdot) \) in \( H^1_{loc}(\Omega \setminus \{ c(t')|0 < t' \leq t \}) \) as \( n \to \infty \).

Proof. By virtue of the connectedness of \( \partial \Omega \), we can take a sequence \( \{ O_n \} \) of relatively compact open subsets of \( \Omega \) such that
\[
\overline{O_n} \subset O_{n+1},
\]
\[
\Omega \setminus \{ c(t')|0 < t' \leq t \} = \bigcup_n O_n,
\]
\[
\Omega \setminus \overline{O_n} \text{ is connected.}
\]
Then, for each \( O_n \), apply the Runge approximation property in Appendix.

\( \square \)

In [5] the author proved

Proposition 2.3. Let \( \gamma_j, j = 1, 2 \), be two conductivities. Let \( v_j \in H^1(\Omega) \) denote the weak solution of the Dirichlet problem
\[
\nabla \cdot \gamma_j(\cdot) v = 0 \text{ in } \Omega,
\]
\[
v|_{\partial \Omega} = f.
\]
It holds that
\[ \int_{\Omega} \{ \gamma_1^{-1} - \gamma_2^{-1} \} \gamma_1 \nabla v_1 \cdot \gamma_1 \nabla v_1 \, dx \leq (\Lambda_{\gamma_2} - \Lambda_{\gamma_1}) f, f >, \]
\[ < (\Lambda_{\gamma_2} - \Lambda_{\gamma_1}) f, f > \leq \int_{\Omega} (\gamma_2 - \gamma_1) \nabla v_1 \cdot \nabla v_1 \, dx, \quad (2.3) \]
\[ \int_{\Omega} \{ \gamma_2^{-1} - \gamma_1^{-1} \} \gamma_2 \nabla v_2 \cdot \gamma_2 \nabla v_2 \, dx \leq (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f, f >, \]
\[ < (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f, f > \leq \int_{\Omega} (\gamma_1 - \gamma_2) \nabla v_2 \cdot \nabla v_2 \, dx. \]

Remark. These however are not the best possible estimates. In fact, therein the author proved also
\[ \int_{\Omega} \{ \gamma_1^{-1} - \gamma_2^{-1} \} \gamma_1 \nabla v_1 \cdot \gamma_1 \nabla v_1 \, dx \leq \frac{< \Lambda_{\gamma_1} f, f >}{< \Lambda_{\gamma_2} f, f >} < (\Lambda_{\gamma_2} - \Lambda_{\gamma_1}) f, f >, \]
\[ \int_{\Omega} \{ \gamma_2^{-1} - \gamma_1^{-1} \} \gamma_2 \nabla v_2 \cdot \gamma_2 \nabla v_2 \, dx \leq \frac{< \Lambda_{\gamma_2} f, f >}{< \Lambda_{\gamma_1} f, f >} < (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f, f >. \]

The proof is not so trivial. Notice that if $\gamma_1$ and $\gamma_2$ are constant, the inequalities indicated above become the equalities.

These inequalities can be derived just from the definition of the Dirichlet-to-Neumann map like as the Schwartz inequality in Hilbert space theory. For complete details, we refer the reader to [5] or [6].

Definition($T(c)$ and $I(t, c)$). Denote the set of all $s \in ]0, 1[$ such that
\[ I(t, c) \equiv \lim_{n \to \infty} < (\Lambda_\gamma - \Lambda_{\gamma_0}) f_n(\cdot; c(t)) ; f_n(\cdot; c(t)) > \]
exists for $0 < t < s$ and $\sup_{0 < t < s} |I(t, c)| < \infty$ by $T(c)$.

Remark. The definition of $T(c)$ is slightly different from that of [7].

A combination of Proposition 2.2 and (2.3) gives

Proposition 2.4. Let $c$ be a needle. If $0 < t < t(c; D)$, $I(t, c)$ exists and it holds that
\[ \int_D \{ \gamma_0^{-1} - \gamma^{-1} \} \gamma_0 \nabla G_{c(t)}^0 \cdot \gamma_0 \nabla G_{c(t)}^0 \, dx \]
\[ \leq I(t, c) \leq \int_D (\gamma - \gamma_0) \nabla G_{c(t)}^0 \cdot \nabla G_{c(t)}^0 \, dx. \quad (2.4) \]

Proof. Let $u_n \in H^1(\Omega)$ be the weak solution of
\[ \nabla \cdot \gamma \nabla u_n = 0 \text{ in } \Omega, \]
\[ u_n|_{\partial \Omega} = f_n(\cdot; c(t)). \]

Since $w_n = u_n - v_n \in H_0^1(\Omega)$ satisfies
\[ \nabla \cdot \gamma \nabla w_n = -\nabla \cdot \chi_D(\gamma - \gamma_0) \nabla v_n \text{ in } \Omega \]
and
\[ v_n \rightarrow G_{c(t)}^0(\cdot) \text{ in } H^1(D) \text{ for } 0 < t < t(c; D), \]
we know that \( w_n \) converges in \( H^1(\Omega) \) to the weak solution of
\[ \nabla \cdot \gamma \nabla w = -\nabla \cdot \chi_D(\gamma - \gamma_0) \nabla G_{c(t)}^0 \text{ in } \Omega, \]
\[ w|_{\partial \Omega} = 0. \]

Then from Alessandini’s identity we get
\[ I(t, c) = \lim_{n \to \infty} \int_D (\gamma - \gamma_0) \nabla u_n \cdot \nabla v_n \, dx \]
\[ = \int_D (\gamma - \gamma_0) \nabla (G_{c(t)}^0 + w) \cdot \nabla G_{c(t)}^0 \, dx. \quad (2.5) \]

(2.4) is clearly valid.

\[ \square \]

Theorem B is a consequence of Proposition 2.5(Reconstruction of impact parameter).

**Proposition 2.5(Reconstruction of impact parameter).**

*For any needle \( c \) it holds that*
\[ T(c) = [0, t(c; D)] \]

*and thus the formula*
\[ t(c; D) = \sup T(c), \quad (2.6) \]

*is valid. \( \partial D \) has the form*
\[ \partial D = \{ c(\sup T(c)) \mid c \text{ is a needle and satisfies } \sup T(c) < 1 \}. \quad (2.7) \]

**Remark.** \( D(\gamma_0, \Gamma) \) in Theorem B is
\[ \{ f_n(\cdot; c(t)) \mid n = 1, \cdots \mid c \text{ is a needle and } 0 < t < 1 \}. \]

The proof of Proposition 2.5 proceeds along the almost same lines with that of Theorem A in [7] and so we describe its outline. First from Propositions 2.1 and 2.4 we get \([0, t(c; D)] \subset T(c)\). Since \( T(c) \subset [0, 1] \), if \( t(c; D) = 1 \), we know that \( T(c) = [0, 1[ = ]0, t(c; D)] \). So the problem is the case where \( t(c; D) < 1 \). Set \( a = c(t(c; D))(\in \partial D) \). We assume that \([0, t(c; D)] \) does not coincide with \( T(c) \). Then there exists \( s \in T(c) \) such that \( t(c; D) \leq s \). Then \( \sup_{0 < t < t(c; D)} |I(t, c)| < C \) for a constant. But a combination of (iv) of (CI), Propositions 2.1 and 2.4 gives
\[ \lim_{t \uparrow t(c; D)} |I(t, c)| = \infty. \]

This is a contradiction.

Proposition 2.5 tells us that using the localized Dirichlet-to-Neumann map, we can predict whether you encounter a point on the boundary of unknown inclusion when you go down along the given needle from \( \partial \Omega \).
3 Proof of Theorem C

For each \( f \in H^{-1/2}(\partial D) \) define

\[
T_f(\varphi) = \langle f, \varphi|_{\partial D} \rangle, \quad \varphi \in H^1_0(\Omega).
\]

From the trace theorem we know \( T_f \in H^{-1}(\Omega) \equiv (H^1_0(\Omega))^* \) and thus there exists a unique weak solution \( u_f \) in \( H^1_0(\Omega) \) of

\[
\nabla \cdot \gamma \nabla u_f = -T_f \text{ in } \Omega,
\]

\[
u_f|_{\partial \Omega} = 0.
\]

Set

\[
G_f = u_f|_{\partial D}.
\]

It is easy to see that \( G \) is a bounded linear operator from \( H^{-1/2}(\partial D) \) to \( H^{1/2}(\partial D) \).

**Definition (Dirichlet-to-Neumann map outside inclusion).** We define the Dirichlet-to-Neumann map outside inclusion

\[
\Lambda_{\gamma^+} : H^{1/2}(\partial D) \longrightarrow H^{-1/2}(\partial D)
\]

by the formula

\[
\langle \Lambda_{\gamma^+} f, g \rangle = -\int_{\Omega \setminus \overline{D}} \gamma \nabla u \cdot \nabla \varphi dx,
\]

where \( g \in H^{1/2}(\partial D) \), \( \varphi \) is any \( H^1(\Omega \setminus \overline{D}) \) function with \( \varphi|_{\partial D} = g \) and \( \varphi|_{\partial \Omega} = 0 \), \( u \) is in \( H^1(\Omega \setminus \overline{D}) \) and the weak solution of the Dirichlet problem

\[
\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega \setminus \overline{D},
\]

\[
u|_{\partial \Omega} = 0,
\]

\[
u|_{\partial D} = f.
\]

We start with

**Proposition 3.1.**

(i) \( \Lambda_{\gamma^-} - \Lambda_{\gamma^+} \) is injective;

(ii) the formula

\[
(\Lambda_{\gamma^-} - \Lambda_{\gamma^+})Gf = f, \forall f \in H^{-1/2}(\partial D)
\]

is valid.

From (i) and (ii) of Proposition 3.1 we can conclude that \( \Lambda_{\gamma^-} - \Lambda_{\gamma^+} \) is bijective and thus we know \( G = (\Lambda_{\gamma^-} - \Lambda_{\gamma^+})^{-1} \). Therefore \( G \) is bijective, too and hence the formula

\[
\Lambda_{\gamma^-} - \Lambda_{\gamma^+} = G^{-1},
\]

(3.2) is valid. Nachman ((6.15) in [13]) proved the corresponding fact in the case where \( \Omega \subset \mathbb{R}^2 \) and \( \gamma \in W^{2,p}(\Omega) \) with \( p > 1 \). Our proof is quite elementary than that of Nachman and notice that we do not assume any regularity of \( \gamma \) on \( \Omega \). This is because of the weak formulation of \( G \).
Proof of Proposition 3.1. (i) is well known, and is proved here only for the convenience of the reader. Assume that $g \in H^{1/2}(\partial D)$ satisfies $(\Lambda_{\gamma} - \Lambda_{\gamma^+})g = 0$. Let $u^- \in H^1(D)$ be the weak solution of

$$\nabla \cdot \gamma \nabla u^- = 0 \text{ in } D,$$

$$u^-|_{\partial D} = g.$$ 

Let $u^+ \in H^1(\Omega \setminus \overline{D})$ be the weak solution of

$$\nabla \cdot \gamma \nabla u^+ = 0 \text{ in } \Omega \setminus \overline{D},$$

$$u^+|_{\partial D} = g,$$

$$u|_{\partial \Omega} = 0.$$ 

Set

$$u = \begin{cases} 
  u^- & \text{in } D \\
  u^+ & \text{in } \Omega \setminus \overline{D}.
\end{cases}$$

Then $u \in H^1_0(\Omega)$ and from the assumption on $g$ we know that $u$ is the weak solution of

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega,$$

$$u|_{\partial \Omega} = 0.$$ 

Therefore $u = 0$ and thus $g = 0$.

Let us give the proof of (ii). Let $\varphi \in H^1_0(\Omega)$. From (3.1) we get

$$< f, \varphi|_{\partial D} > = T_f(\varphi)$$

$$= \int_{\Omega} \gamma \nabla u_f \cdot \nabla \varphi dx$$

$$= \int_{\Omega \setminus \overline{D}} \gamma \nabla u_f \cdot \nabla \varphi dx + \int_{\overline{D}} \gamma \nabla u_f \cdot \nabla \varphi dx$$

$$= - < \Lambda_{\gamma^+} (u_f|_{\partial D}), \varphi|_{\partial D} > + < \Lambda_{\gamma^-} (u_f|_{\partial D}), \varphi|_{\partial D} >$$

$$= < (\Lambda_{\gamma^-} - \Lambda_{\gamma^+}) G f, \varphi|_{\partial D} >.$$ 

From this identity we get (ii) since the map

$$H^1_0(\Omega) \ni \varphi \mapsto \varphi|_{\partial D} \in H^{1/2}(\partial D)$$

is surjective.

$\square$

In the remainder of this section we make use of (i)∼(iii) of (Cl). Let $F$ be a given element of $H^{-1}(\Omega)$ with

$$\text{supp } F \subset \Omega \setminus \overline{D}.$$
Let \( u_0 \in H^1_0(\Omega) \) be the weak solution of
\[
\nabla \cdot \gamma_0 \nabla u_0 = -F \text{ in } \Omega,
\]
\[
u_0|_{\partial \Omega} = 0
\]
Set
\[
G_0 F = u_0.
\]
Let \( w \in H^1_0(\Omega) \) be the weak solution of
\[
\nabla \cdot \gamma \nabla w = -\nabla \cdot \chi_D(\gamma - \gamma_0) \nabla u_0 \text{ in } \Omega,
\]
\[
w|_{\partial \Omega} = 0.
\]
Set
\[
G F = G_0 F + w.
\]
We prove the one of two crucial identities, needed for calculating \( G \).

**Proposition 3.2.** The formula
\[
F(u_f) = \langle f, G F|_{\partial D} \rangle,
\]
is valid.

**Proof.** A combination of (3.1) and (3.4) yields
\[
\int_D (\gamma - \gamma_0) \nabla u_f \cdot \nabla u_0 dx = -\int_\Omega \gamma \nabla w \cdot \nabla u_f dx
\]
\[
= -T_f(w)
\]
\[
= - \langle f, w|_{\partial D} \rangle.
\]
On the other hand, from (3.1) and (3.3) we get
\[
\langle f, u_0|_{\partial D} \rangle = T_f(u_0)
\]
\[
= \int_\Omega \gamma \nabla u_f \cdot \nabla u_0 dx
\]
\[
= \int_\Omega (\gamma - \gamma_0) \nabla u_f \cdot \nabla u_0 dx + \int_\Omega \gamma_0 \nabla u_f \cdot \nabla u_0 dx
\]
\[
= \int_D (\gamma - \gamma_0) \nabla u_f \cdot \nabla u_0 dx + F(u_f).
\]
Combining this with (3.6), we get (3.5).

\[\square\]

Let \( H \) be a given element of \( H^{-1}(\Omega) \) with
\[
\text{supp } H \subset \Omega \setminus D.
\]
Let \( v_0 \in H_0^1(\Omega) \) be the weak solution of

\[
\nabla \cdot \gamma_0 \nabla v_0 = -H \text{ in } \Omega,
\]

\[
v_0|_{\partial \Omega} = 0.
\]

(3.7)

Notice that both \( u_0 \) and \( v_0 \) satisfies \( \nabla \cdot \gamma_0 \nabla u = 0 \) in an open neighbourhood of \( \overline{D} \). Therefore from the Runge approximation property proved in Appendix we get two sequences \{\( u_n \}\}, \{\( v_n \}\) of \( H^1(\Omega) \) functions with

\[
\text{supp} (u_n|_{\partial \Omega}), \text{supp} (v_n|_{\partial \Omega}) \subset \Gamma
\]

such that

\[
\nabla \cdot \gamma_0 \nabla u_n = 0 \text{ in } \Omega,
\]

\[
u_n \longrightarrow u_0 \text{ in } H^1(D),
\]

\[
\nabla \cdot \gamma_0 \nabla v_n = 0 \text{ in } \Omega,
\]

\[
v_n \longrightarrow v_0 \text{ in } H^1(D).
\]

**Proposition 3.3.** The formula

\[
-H(GF - G_0F) = \frac{1}{4} \lim_{n \to \infty} \{< (\Lambda_\gamma - \Lambda_{\gamma_0})f_n^+, f_n^+ > - < (\Lambda_\gamma - \Lambda_{\gamma_0})f_n^-, f_n^- >\},
\]

(3.8)

is valid where

\[
f_n^+ = (u_n + v_n)|_{\partial \Omega}, f_n^- = (u_n - v_n)|_{\partial \Omega}.
\]

**Remark.** \( \mathcal{D}(\gamma_0, \Gamma, D) \) in Theorem C is

\[
\{f_n^+, f_n^-, n = 1, \ldots \mid F, H \in H^{-1}(\Omega) \text{ with } \text{supp } F, \text{supp } H \subset \Omega \setminus \overline{D}\}.
\]

**Proof of Proposition 3.3.** Let \( a_n \in H^1(\Omega) \) be the weak solution of

\[
\nabla \cdot \gamma \nabla a_n = 0 \text{ in } \Omega,
\]

\[
a_n|_{\partial \Omega} = u_n|_{\partial \Omega}.
\]

Set

\[
w_n = a_n - u_n \in H_0^1(\Omega).
\]

\( w_n \) solves

\[
\nabla \cdot \gamma \nabla w_n = -\nabla \cdot \chi_D(\gamma - \gamma_0) \nabla u_n \text{ in } \Omega.
\]

It is easy to see that \( w_n \longrightarrow w \) in \( H^1(\Omega) \). Thus from Alessandrinis identity and the
symmetry of the Dirichlet-to-Neumann map we get

\[
\frac{1}{4} \{ < (\Lambda_{\gamma} - \Lambda_{\gamma_0}) f_n^+, f_n^+ > - < (\Lambda_{\gamma} - \Lambda_{\gamma_0}) f_n^-, f_n^- > \}
\]

\[= < (\Lambda_{\gamma} - \Lambda_{\gamma_0}) u_n|_{\partial \Omega}, v_n|_{\partial \Omega} >
\]

\[= \int_{\Omega} (\gamma - \gamma_0) \nabla a_n \cdot \nabla v_n dx
\]

\[= \int_{\partial D} (\gamma - \gamma_0) \nabla a_n \cdot \nabla v_n dx
\]

\[= \int_{\partial D} (\gamma - \gamma_0) \nabla u_n \cdot \nabla v_n dx + \int_{\partial D} (\gamma - \gamma_0) \nabla w_n \cdot \nabla v_n dx
\]

\[\rightarrow \int_{\partial D} (\gamma - \gamma_0) \nabla u_0 \cdot \nabla v_0 dx + \int_{\partial D} (\gamma - \gamma_0) \nabla w \cdot \nabla v_0 dx.
\]

From (3.4) and (3.7) we get

\[H(w) = \int_{\Omega} \gamma_0 \nabla v_0 \cdot \nabla w dx
\]

\[= \int_{\Omega} \{ \gamma - \chi_D(\gamma - \gamma_0) \} \nabla v_0 \cdot \nabla w dx
\]

\[= \int_{\Omega} \gamma \nabla v_0 \cdot \nabla w dx - \int_{\partial D} (\gamma - \gamma_0) \nabla v_0 \cdot \nabla w dx
\]

\[= - \int_{\partial D} (\gamma - \gamma_0) \nabla u_0 \cdot \nabla v_0 dx - \int_{\partial D} (\gamma - \gamma_0) \nabla v_0 \cdot \nabla w dx.
\]

Combining this with (3.9), we get (3.8).

\[\square
\]

Proof of Theorem C. First from (3.8) we get \(G F - G_0 F\) in \(\Omega \setminus \overline{D}\) and thus its trace on \(\partial D\). From (3.5) we get \(u_f\) in \(\Omega \setminus \overline{D}\) and thus its trace on \(\partial D\), that is \(G f\). Therefore we get \(G\) and from (3.2) \(\Lambda_{\gamma^-}\), too.

\[\square
\]

4 Summary of reconstruction procedure

Reconstruction of the shape of inclusion.

1. First construct a family \((G_0^x(t))_{x \in \Omega}\) each of which is a special solution of \(\nabla \cdot \gamma_0 \nabla u = 0\) in \(\Omega \setminus \{x\}\).

2. For each needle \(c\) and \(t \in ]0, 1[\), take a sequence \(\{f_n(t; c(t))\}\) of functions on \(\partial \Omega\) with \(\text{supp} \ f_n \subset \Gamma\) in such a way that the solution \(v_n\) of

\[\nabla \cdot \gamma_0 \nabla v = 0 \ \text{in} \ \Omega
\]

\[v|_{\partial \Omega} = f_n
\]
converges to $G^0_{c(t)}$ in $H^1_{loc}(\Omega \setminus \{c(t')|0 < t' \leq t\})$ as $n \to \infty$.

(3) Calculate $T(c)$.

(4) Use formula $t(c; D) = \sup T(c)$ to recover $t(c; D)$ from $T(c)$.

(5) Use formula $\partial D = \{c(t(c; D))|c \text{ is a needle and } t(c; D) < 1\}$ to recover $\partial D$ from $t(c; D)$.

Reconstruction of the Dirichlet-to-Neumann map inside inclusion.

(1) Give $F, H$ being elements of $H^{-1}(\Omega)$ with supp $F, \text{supp } H \subset \Omega \setminus \overline{D}$.

(2) Construct a sequence $\{u_n\}$ of functions in $H^1(\Omega)$ such that
\[
\nabla \cdot \gamma_0 \nabla u_n = 0 \text{ in } \Omega,
\]
\[
\text{supp}(u_n|_{\partial \Omega}) \subset \Gamma,
\]
\[
u_n \to G_0 F \text{ in } H^1(D).
\]

(3) Construct a sequence $\{v_n\}$ of functions in $H^1(\Omega)$ such that
\[
\nabla \cdot \gamma_0 \nabla v_n = 0 \text{ in } \Omega,
\]
\[
\text{supp}(v_n|_{\partial \Omega}) \subset \Gamma,
\]
\[
v_n \to G_0 H \text{ in } H^1(D).
\]

(4) Calculate
\[
f_n^+ = (u_n + v_n)|_{\partial \Omega}, f_n^- = (u_n - v_n)|_{\partial \Omega}.
\]

(5) Use formula
\[
-H(GF - G_0 F) = \frac{1}{4} \lim_{n \to \infty} \{<(\Lambda_\gamma - \Lambda_{\gamma_0}) f_n^+, f_n^+>, -(\Lambda_\gamma - \Lambda_{\gamma_0}) f_n^-, f_n^->\}
\]
to recover $H(GF - G_0 F)$ from the set
\[
\{<(\Lambda_\gamma - \Lambda_{\gamma_0}) f_n^+, f_n^+>, -(\Lambda_\gamma - \Lambda_{\gamma_0}) f_n^-, f_n^->\}.
\]

(6) Calculate $GF - G_0 F$ in $\Omega \setminus \overline{D}$ from the set
\[
\{H(GF - G_0 F) | H \in H^{-1}(\Omega) \text{ with supp } H \subset \Omega \setminus \overline{D}\}.
\]

(7) Calculate $GF|_{\partial D}$.

(8) Give $f \in H^{-1/2}(\partial D)$.

(9) Use formula
\[
F(u_f) = <f, GF|_{\partial D}>
\]
to recover $F(u_f)$ from $GF|_{\partial D}$.

(10) Calculate $u_f$ in $\Omega \setminus \overline{D}$ from the set
\[
\{F(u_f) | F \in H^{-1}(\Omega) \text{ with supp } F \subset \Omega \setminus \overline{D}\}.
\]

(11) Calculate $Gf = u_f|_{\partial D}$. 

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(12) Calculate $G$ from the set
\[ \{ Gf \mid f \in H^{-1/2}(\partial D) \} \].

(13) Use formula
\[ \Lambda_\gamma^- - \Lambda_\gamma^+ = G^{-1} \]
to recover $\Lambda_\gamma^-$ from $G$.

5 The multiprobe method and proof of Theorem D

The starting point is

**Proposition 5.1.** Let $(\lambda_j, \mu_j)$, $j = 1, 2$, be two pairs of Lamé parameters. Let $u_j \in \{H^1(\Omega)^3\}$ denote the weak solution of
\[ \mathcal{L}_{\lambda_j, \mu_j} u_j = 0 \quad \text{in } \Omega \]
\[ u_j|_{\partial \Omega} = f. \]

It holds that
\[ \int_\Omega \frac{3 \lambda_2 + 2 \mu_2}{3(3 \lambda_2 + 2 \mu_2)} \left\{ 3(\lambda_1 - \lambda_2) + 2(\mu_1 - \mu_2) \right\} |\nabla \cdot u_2|^2 \]
\[ + \frac{2(\mu_1 - \mu_2)}{\mu_1} \left| \text{Sym} \nabla u_2 - \frac{\nabla \cdot u_2}{3} I_3 \right|^2 dx \]
\[ \leq < (\Lambda_{\lambda_1, \mu_1} - \Lambda_{\lambda_2, \mu_2}) f, f >, \]
\[ < (\Lambda_{\lambda_1, \mu_1} - \Lambda_{\lambda_2, \mu_2}) f, f > \]
\[ \leq \int_\Omega \frac{3(\lambda_1 - \lambda_2) + 2(\mu_1 - \mu_2)}{3} |\nabla \cdot u_2|^2 \]
\[ + 2(\mu_1 - \mu_2) \left| \text{Sym} \nabla u_2 - \frac{\nabla \cdot u_2}{3} I_3 \right|^2 dx, \]
\[ \int_\Omega \frac{3 \lambda_1 + 2 \mu_1}{3(3 \lambda_2 + 2 \mu_2)} \left\{ 3(\lambda_2 - \lambda_1) + 2(\mu_2 - \mu_1) \right\} |\nabla \cdot u_1|^2 \]
\[ + \frac{2(\mu_2 - \mu_1)}{\mu_2} \left| \text{Sym} \nabla u_1 - \frac{\nabla \cdot u_1}{3} I_3 \right|^2 dx \]
\[ \leq < (\Lambda_{\lambda_2, \mu_2} - \Lambda_{\lambda_1, \mu_1}) f, f >, \]
\[ < (\Lambda_{\lambda_2, \mu_2} - \Lambda_{\lambda_1, \mu_1}) f, f > \]
\[ \leq \int_\Omega \frac{3(\lambda_2 - \lambda_1) + 2(\mu_2 - \mu_1)}{3} |\nabla \cdot u_1|^2 \]
\[ + 2(\mu_2 - \mu_1) \left| \text{Sym} \nabla u_1 - \frac{\nabla \cdot u_1}{3} I_3 \right|^2 dx. \]
Remark. These inequalities are a simple consequence of the system of integral inequalities in [5], the identities (3.8) and (3.9) in [5] and the factorization of symmetric matrix $B$:

$$B = \frac{\text{Trace} B}{3} I_3 + \left( B - \frac{\text{Trace} B}{3} I_3 \right).$$

However, for reader’s convenience, we present a direct proof.

Proof. It is easy to see that

$$< (\Lambda_{\lambda_2, \mu_2} - \Lambda_{\lambda_1, \mu_1}) f, f > = \int_{\Omega} \lambda_1 |\nabla \cdot (u_1 - u_2)|^2 + 2\mu_1 |\text{Sym} \nabla (u_1 - u_2)|^2 + (\lambda_2 - \lambda_1) |\nabla \cdot u_2|^2 + 2(\mu_2 - \mu_1) |\text{Sym} \nabla u_2|^2 dx. \quad (5.5)$$

Set

$$B_j = \text{Sym} \nabla u_j - \frac{\nabla \cdot u_j}{3} I_3.$$

Note that $B_j \cdot I_3 = \text{Trace} B_j = 0$. Since for any $\alpha, \beta$ and $3 \times 3$ matrix $A$

$$\alpha |\text{Trace} A|^2 + 2\beta |\text{Sym} A|^2 = \frac{3\alpha + 2\beta}{3} |\text{Trace} A|^2 + 2\beta \left| \text{Sym} A - \frac{\text{Trace} A}{3} I_3 \right|^2,$$

(5.5) can be rewritten as

$$< (\Lambda_{\lambda_2, \mu_2} - \Lambda_{\lambda_1, \mu_1}) f, f > = \int_{\Omega} \frac{3\lambda_1 + 2\mu_1}{3} |\nabla \cdot (u_1 - u_2)|^2 + 2\mu_1 |B_1 - B_2|^2 + \frac{3(\lambda_2 - \lambda_1) + 2(\mu_2 - \mu_1)}{3} |\nabla \cdot u_2|^2 + 2(\mu_2 - \mu_1) |B_2|^2. \quad (5.6)$$

A combination of (E) and (5.6) gives

$$< (\Lambda_{\lambda_2, \mu_2} - \Lambda_{\lambda_1, \mu_1}) f, f > \geq \int_{\Omega} \frac{3(\lambda_2 - \lambda_1) + 2(\mu_2 - \mu_1)}{3} |\nabla \cdot u_2|^2 + 2(\mu_2 - \mu_1) |B_2|^2 dx.$$
Replacing 2 with 1 and 1 with 2, we get (5.4). On the other hand, from (E) we get

\[
\frac{3\lambda_1 + 2\mu_1}{3}|\nabla \cdot (u_1 - u_2)|^2 + 2\mu_1|B_1 - B_2|^2
\]

\[
+ \frac{3(\lambda_2 - \lambda_1) + 2(\mu_2 - \mu_1)}{3}|\nabla \cdot u_2|^2 + 2(\mu_2 - \mu_1)|B_2|^2
\]

\[
= \frac{3\lambda_2 + 2\mu_2}{3}|\nabla \cdot u_2|^2 - \frac{2(3\lambda_1 + 2\mu_1)}{3}\nabla \cdot u_1 \nabla \cdot u_2
\]

\[
+ 2\mu_2|B_2|^2 - 4\mu_1 B_1 \cdot B_2
\]

\[
+ \frac{3\lambda_1 + 2\mu_1}{3}|\nabla \cdot u_1|^2 + 2\mu_1|B_1|^2
\]

\[
= \sqrt{\frac{3\lambda_2 + 2\mu_2}{3}}|\nabla \cdot u_2| - \sqrt{\frac{3}{3\lambda_2 + 2\mu_2}}\frac{3\lambda_1 + 2\mu_1}{3}|\nabla \cdot u_1|
\]

\[
+ \left| \sqrt{2\mu_2 B_2} - \frac{2\mu_1}{\sqrt{2\mu_2}} B_1 \right|^2
\]

\[
+ \left\{ \frac{3\lambda_1 + 2\mu_1}{3} - \frac{3}{3\lambda_2 + 2\mu_2} \left( \frac{3\lambda_1 + 2\mu_1}{3} \right)^2 \right\}|\nabla \cdot u_1|^2
\]

\[
+ \left\{ 2\mu_1 - \frac{(\mu_1)^2}{\mu_2} \right\}|B_1|^2
\]

\[
\geq \frac{3\lambda_1 + 2\mu_1}{3(3\lambda_2 + 2\mu_2)} \{3(\lambda_2 - \lambda_1) + 2(\mu_2 - \mu_1)\} |\nabla \cdot u_1|^2 + \frac{2\mu_1(\mu_2 - \mu_1)}{\mu_2}|B_1|^2
\]

Combining this with (5.6), we immediately get (5.3). Similarly, we obtain (5.1) and (5.2), too.

We construct for each \( x \in \Omega \) two kinds of singular solutions of the equation \( \mathcal{L}_{\lambda_0, \mu_0} u = 0 \) in \( \Omega \setminus \{x\} \).

Let \( G(\cdot) \) denote the standard fundamental solution for \(-\Delta:\)

\[
G(z) = \frac{1}{4\pi|z|}.
\]

The lemma below is a result of speciality of ”isotropic” and the proof is given in Section 6.

**Proposition 5.2.** Assume that

\[
\partial \Omega \text{ is } C^{2,1}
\]

and that

\[
\lambda_0 \in C^{0,1}(\overline{\Omega}), \ \mu_0 \in C^{2,1}(\overline{\Omega}).
\]
There exists a family \((u_x^0 \in \{H^1_{\text{loc}}(\Omega \setminus \{x\})\})_{x \in \Omega}\) such that

\[
\left( u_x^0 - \nabla G(\cdot - x) + \frac{G(\cdot - x)}{\lambda_0(x) + 2\mu_0(x)} \left( I_3 - \frac{\cdot - x}{|\cdot - x|} \otimes \frac{\cdot - x}{|\cdot - x|} \right) \nabla \mu_0(x) \right)_{x \in \Omega}
\]

is bounded in \(\{H^1(\Omega)\}^3\) and

\[\mathcal{L}_{\lambda_0, \mu_0} u_x^0 = 0 \text{ in } \Omega \setminus \{x\}.\]

Remark. \(\nabla G(\cdot - x)\) satisfies the equation

\[\mathcal{L}_{\lambda_0(x), \mu_0(x)} v + (\lambda_0(x) + 2\mu_0(x)) \nabla \delta(\cdot - x) = 0 \text{ in } \mathbb{R}^3\]

and its divergence vanishes in \(\mathbb{R}^3 \setminus \{x\}\).

Let \(E_0(\cdot; x)\) denote the standard fundamental solution (Kelvin matrix) for the operator \(\mathcal{L}_{\lambda_0(x), \mu_0(x)}\) (see for example [10]):

\[
E_0(z; x) = \frac{1}{8\pi} \left( \frac{1}{\mu_0(x)} + \frac{1}{\lambda_0(x) + 2\mu_0(x)} \right) I_3 + \frac{1}{8\pi} \left( \frac{1}{\mu_0(x)} - \frac{1}{\lambda_0(x) + 2\mu_0(x)} \right) \frac{z \otimes z}{|z|^3}.\]

**Proposition 5.3.** Assume that \(\lambda_0, \mu_0 \in C^{0,1}(\Omega)\). There exists a family \((E^0_x(\cdot))_{x \in \Omega}\) in \(\cap_{1 \leq p < 3} L^p(\Omega, M_3(\mathbb{R}))\) such that \((E^0_x(\cdot) - E_0(\cdot - x; x))_{x \in \Omega}\) is bounded in \(H^1(\Omega, M_3(\mathbb{R}))\) and for each constant vector \(b\)

\[\mathcal{L}_{\lambda_0, \mu_0} (E^0_x(\cdot)b) + \delta(\cdot - x)b = 0 \text{ in } \Omega.\]

The proof of this proposition proceeds along the same lines with that of Proposition 2.1 and we may omit it. We note that \(E_0(\cdot - x; x)\) and \(G(\cdot - x)\) are not independent of each other. We will use the formula

\[
\nabla \cdot \{E_0(\cdot - x; x)b\} = \frac{1}{\lambda_0(x) + 2\mu_0(x)} \nabla G(\cdot - x) \cdot b,
\]

where \(b\) is a constant vector. That can be easily checked by direct computation.

**Proposition 5.4 (Selection of boundary data).** Assume that \(\partial \Omega\) is \(C^3\) and that

\[\lambda_0 \in C^2(\Omega); \mu_0 \in C^3(\Omega)\].

Let \(c\) be a needle.

(i) For each \(t \in [0, 1]\) there exists a sequence \(\{f_n(\cdot; c(t))\}\) of \(D(\Gamma)^3\) such that the weak solution \(u_n\) of

\[\mathcal{L}_{\lambda_0, \mu_0} u = 0 \text{ in } \Omega,\]

\[u|_{\partial \Omega} = f_n(\cdot; c(t))\]

converges to \(u^0_c(\cdot)\) in \(\{H^1_{\text{loc}}(\Omega \setminus \{c(t)|0 < t' \leq t\})\}^3\) as \(n \rightarrow \infty\);
(ii) Let \( \{e_1, e_2, e_3\} \) be the standard orthonormal basis for \( \mathbb{R}^3 \). For each \( t \in ]0, 1[ \) and \( j = 1, 2, 3 \) there exists a sequence \( \{g_{n,j}(\cdot; c(t))\} \) of \( D(\Omega)^3 \) such that the weak solution \( v_n \) of
\[
L_{\lambda_0, \mu_0} v = 0 \quad \text{in } \Omega,
\]
\[
v|_{\partial \Omega} = g_{n,j}(\cdot; c(t))
\]
converges to \( E_c^{0}(\cdot) e_j \) in \( \{H^1_{loc}\}^3 (\Omega \setminus \{c(t')|0 < t' \leq t\}) \) as \( n \to \infty \).

Proof. Apply the Runge approximation property for the equation \( L_{\lambda_0, \mu_0} u = 0 \). It is easily proved by using the unique continuation property established in [3].

\[ \square \]

Definition(\( T^f(c), T^g(c), I^f(t, c) \) and \( I^g(t, c) \)).

(i) Denote by \( T^f(c) \) the set of all \( s \in ]0, 1[ \) such that
\[
I^f(t, c) \equiv \lim_{n \to \infty} <(\Lambda_{\lambda, \mu} - \Lambda_{\lambda_0, \mu_0}) f_n (\cdot; c(t)), f_n (\cdot; c(t)) >
\]
exists for \( 0 < t < s \) and \( \sup_{0 < t < s} |I^f(t, c)| < \infty \).

(ii) Denote by \( T^g(c) \) the set of all \( s \in ]0, 1[ \) such that
\[
I^g(t, c) \equiv \lim_{n \to \infty} \sum_{1 \leq j \leq 3} <(\Lambda_{\lambda, \mu} - \Lambda_{\lambda_0, \mu_0}) g_{n,j}(\cdot; c(t)), g_{n,j}(\cdot; c(t)) >
\]
exists for \( 0 < t < s \) and \( \sup_{0 < t < s} |I^g(t, c)| < \infty \).

A combination of Propositions 5.1 and 5.4 yields

Proposition 5.5. Let \( c \) be a needle. If \( 0 < t < t(c; D) \), \( I^f(t, c) \) and \( I^g(t, c) \) exist and it holds that
\[
\int_{D} \frac{3\lambda_0 + 2\mu_0}{3(3\lambda + 2\mu)} \{3(\lambda - \lambda_0) + 2(\mu - \mu_0)\} |\nabla \cdot u_{c(t)}^0|^2
\]
\[
+ \frac{\mu_0}{\mu} 2(\mu - \mu_0) \left| \text{Sym} \nabla u_{c(t)}^0 - \frac{\nabla \cdot u_{c(t)}^0}{3} I_3 \right|^2 \, dy \leq I^f(t, c),
\]
\[
I^f(t, c) \leq \sum_{1 \leq j \leq 3} \int_{D} \frac{3\lambda_0 + 2\mu_0}{3(3\lambda + 2\mu)} \{3(\lambda - \lambda_0) + 2(\mu - \mu_0)\} |\nabla \cdot (E_{c(t)}^0 e_j)|^2
\]
\[
+ \frac{\mu_0}{\mu} 2(\mu - \mu_0) \left| \text{Sym} \nabla (E_{c(t)}^0 e_j) - \frac{\nabla \cdot (E_{c(t)}^0 e_j)}{3} I_3 \right|^2 \, dy \leq I^g(t, c),
\]
\[
I^g(t, c) \leq \sum_{1 \leq j \leq 3} \int_{D} \frac{3(\lambda - \lambda_0) + 2(\mu - \mu_0)}{3} |\nabla \cdot (E_{c(t)}^0 e_j)|^2
\]
\[
+ 2(\mu - \mu_0) \left| \text{Sym} \nabla (E_{c(t)}^0 e_j) - \frac{\nabla \cdot (E_{c(t)}^0 e_j)}{3} I_3 \right|^2 \, dy.
\]
The proof is same as that of Proposition 2.4 and we may omit it.

Theorem D is a consequence of

**Proposition 5.6 (Reconstruction of impact parameter).**

For any needle \( c \) it holds that

\[
T^f(c) \cap T^g(c) = ]0, t(c; D)[
\]

and thus the formula

\[
t(c; D) = \sup T^f(c) \cap T^g(c),
\]

is valid. \( \partial D \) has the form

\[
\partial D = \{ c(\sup T^f(c) \cap T^g(c)) \mid c \text{ is a needle and satisfies } \sup T^f(c) \cap T^g(c) < 1 \}.
\]

**Remark.** \( \mathcal{D}(\lambda_0, \mu_0, \Gamma) \) in Theorem D is

\[
\{ f_n(\cdot; c(t)), g_{n,j}(\cdot; c(t)) \mid j = 1, 2, 3, n = 1, \cdots \mid c \text{ is a needle and } 0 < t < 1 \}.
\]

**Proof.** First from Propositions 5.2, 5.3 and 5.5 we get \([0, t(c; D)] \subset T^f(c) \cap T^g(c). \) Since \( T^f(c) \cap T^g(c) \subset [0, 1[ \), if \( t(c; D) = 1 \), we know that \( T^f(c) \cap T^g(c) = [0, 1[ = [0, t(c; D)]. \)

So the problem is the case where \( t(c; D) < 1 \). Set \( a = c(t(c; D))(\in \partial D) \). We assume that \([0, t(c; D)] \) does not coincide with \( T^f(c) \cap T^g(c). \) Then there exists \( s \in T^f(c) \cap T^g(c) \) such that \( t(c; D) \leq s. \) Then from the definitions of \( T^f(c) \), \( T^g(c) \) we get

\[
\exists C > 0 \forall t \in ]0, t(c; D)[ \mid |I^f(t, c)| < C \text{ and } |I^g(t, c)| < C.
\]

(5.14)

**Case 1.** \( \mu_D(a) \neq \mu_0(a) \)

We consider first the case when \( \mu_D(a) > \mu_0(a). \) In what follows, \( C_1, C_2, \cdots \) denote positive constants independent of \( x \in \Omega. \) Then we may assume that

\[
\exists r > 0 \exists C_1 > 0 \forall y \in B(a, 2r) \cap D \mu(y) - \mu_0(y) \geq C_1.
\]

(5.15)

take \( t_1 \in ]0, t(c; D)[ \) such that \( c(t) \in B(a, r) \) for \( t \in ]t_1, t(c; D)[. \) Then we know that \( |y - c(t)| \geq r \) for \( y \in \Omega \setminus B(a, 2r). \) From Proposition 5.2 we get

\[
\exists C_2 > 0 \int_{D \setminus B(a, 2r)} |\text{Sym} \nabla u^0_{c(t)}|^2 dy < C_2.
\]

(5.16)

On the other hand, since \( c(t) \in \Omega \setminus \overline{D} \) for \( t_1 < t < t(c; D), \) we deduce that

\[
\nabla \cdot u^0_{c(t)} = \nabla \cdot \{ u^0_{c(t)} - \nabla G(\cdot - c(t)) \} \text{ in } D \cap B(a, 2r).
\]

Thus we get

\[
\int_{D \cap B(a, 2r)} |\nabla \cdot u^0_{c(t)}|^2 dy \leq C_3 \int_{D \cap B(a, 2r)} \frac{1}{|y - c(t)|^4} dy + C_4.
\]

(5.17)

Noting that \( |\nabla \cdot u^0_{c(t)}| \leq \sqrt{3}|\text{Sym} \nabla u^0_{c(t)}|, \) from (5.10), (5.16) and (5.17) we get

\[
\int_{D \cap B(a, 4r)} \frac{\mu}{\mu_0} 2(\mu - \mu_0) \left|\text{Sym} \nabla u^0_{c(t)} - \frac{\nabla \cdot u^0_{c(t)}}{3} I_3 \right|^2 dy
\]

\[
\leq I^f(t, c) + C_5 \int_{D \cap B(a, 2r)} \frac{1}{|y - c(t)|^4} dy + C_6.
\]

(5.18)
Since
\[ \left| \text{Sym} \nabla u_0^\alpha(t) - \frac{\nabla \cdot u_0^\alpha(t)}{3} I_3 \right|^2 \geq \frac{1}{2} \left| \text{Sym} \nabla u_0^\alpha(t) \right|^2 - \frac{\left| \nabla \cdot u_0^\alpha(t) \right|^2}{3} I_3 , \]
it follows from (5.15), (5.17) and (5.18) that
\[ \int_{D \cap B(a,2r)} \left| \text{Sym} \nabla u_0^\alpha(t) \right|^2 dy \leq C_7 I^f(t,c) + C_8 \int_{D \cap B(a,2r)} \frac{1}{|y-c(t)|^4} dy + C_9. \] (5.19)
A combination of the triangle inequality and Proposition 5.2 yields
\[ \int_{D \cap B(a,2r)} \left| \text{Sym} \nabla u_0^\alpha(t) \right|^2 dy \geq C_{10} \int_{D \cap B(a,2r)} \frac{1}{|y-c(t)|^6} dy - C_{11} \int_{D \cap B(a,2r)} \frac{1}{|y-c(t)|^4} dy - C_{12}. \] (5.20)
From (5.19), (5.20) we get
\[ \int_{D \cap B(a,2r)} \frac{1}{|y-c(t)|^6} dy \leq C_7 I^f(t,c) + C_{13} \int_{D \cap B(a,2r)} \frac{1}{|y-c(t)|^4} dy + C_{14}. \] (5.21)
The Young inequality yields
\[ \int_{D \cap B(a,2r)} \frac{1}{|y-c(t)|^4} dy \leq \frac{1}{3c^3} |D \cap B(a,2r)| + \frac{2c^{3/2}}{3} \int_{D \cap B(a,2r)} \frac{1}{|y-c(t)|^6} dy, \forall \epsilon > 0. \] (5.22)
If we take a sufficiently small \( \epsilon \), from (5.21) and (5.22) we get
\[ C_{15} \int_{D \cap B(a,2r)} \frac{1}{|y-c(t)|^6} dy - C_{16} \leq I^f(t,c). \] (5.23)
Since \( \partial D \) is Lipschitz, it is easy to see that
\[ \lim_{t \uparrow (c,D)} \int_{D \cap B(a,2r)} \frac{1}{|y-c(t)|^6} dy = \infty \]
and (5.23) therefore yields \( \lim_{t \uparrow (c,D)} I^f(t,c) = \infty \). This is a contradiction to (5.14).

We now consider the case when \( \mu_D(a) < \mu_0(a) \). Using (5.11), we get \( \lim_{t \uparrow (c,D)} I^f(t,c) = -\infty \) and a contradiction to (5.14). The proof proceeds along the same lines.

**Case 2.** \( \mu_D(a) = \mu_0(a) \)
From (v) of (EI) we know that $\lambda_D(a) \neq \lambda_0(a)$. Let $\epsilon > 0$. Then there exists a positive constant $C_a$ such that if we take a small $r > 0$, we may assume that
\[
|\mu(y) - \mu_0(y)| < \epsilon, \forall y \in D \cap B(a, 2r) \tag{5.24}
\]
and
\[
\lambda(y) - \lambda_0(y) \geq C_a, \forall y \in D \cap B(a, 2r) \tag{5.25}
\]
or
\[
\lambda(y) - \lambda_0(y) \leq -C_a, \forall y \in D \cap B(a, 2r). \tag{5.26}
\]
Let us consider the case when (5.24) and (5.25) are satisfied. Take $t_1 \in [0, t(c; D)]$ such that $c(t) \in B(a, r)$ for $t \in [t_1, t(c; D)]$. Then we know that $|y - c(t)| \geq r$ for $y \in \Omega \setminus B(a, 2r)$.

From Proposition 5.3 we get
\[
\left| \sum_{1 \leq j \leq 3} \int_{D \cap B(a, 2r)} \frac{3\lambda_0 + 2\mu_0}{3(3\lambda + 2\mu)} \left\{ (\lambda - \lambda_0) + 2(\mu - \mu_0) \right\} |\nabla \cdot (E_{c(t)}^0 e_j)|^2
\right. \
\left. + \frac{\mu_0}{\mu} 2(\mu - \mu_0) \left| \text{Sym} \nabla (E_{c(t)}^0 e_j) - \frac{\nabla \cdot (E_{c(t)}^0 e_j)}{3} I_3 \right|^2 \right| dy \leq C(r),
\]
and therefore (5.12) yields
\[
\sum_{1 \leq j \leq 3} \int_{D \cap B(a, 2r)} \frac{3\lambda_0 + 2\mu_0}{3(3\lambda + 2\mu)} \left\{ (\lambda - \lambda_0) + 2(\mu - \mu_0) \right\} |\nabla \cdot (E_{c(t)}^0 e_j)|^2
\right. \
\left. + \frac{\mu_0}{\mu} 2(\mu - \mu_0) \left| \text{Sym} \nabla (E_{c(t)}^0 e_j) - \frac{\nabla \cdot (E_{c(t)}^0 e_j)}{3} I_3 \right|^2 \right| dy \leq C(r)
\]
\[
\leq I^g(t, c). \tag{5.27}
\]
A combination of (5.24), Proposition 5.3 and the triangle inequality yields
\[
\left| \sum_{1 \leq j \leq 3} \int_{D \cap B(a, 2r)} \frac{3\lambda_0 + 2\mu_0}{3(3\lambda + 2\mu)} 2(\mu - \mu_0) |\nabla \cdot (E_{c(t)}^0 e_j)|^2
\right. \
\left. + \frac{\mu_0}{\mu} 2(\mu - \mu_0) \left| \text{Sym} \nabla (E_{c(t)}^0 e_j) - \frac{\nabla \cdot (E_{c(t)}^0 e_j)}{3} I_3 \right|^2 \right| dy \leq C_{17} \epsilon \left( 1 + \int_{D \cap B(a, 2r)} \frac{1}{|y - c(t)|^4} dy \right). \tag{5.28}
\]
From (5.9), (5.25), Proposition 5.3 and the triangle inequality we get
\[
\sum_{1 \leq j \leq 3} \int_{D \cap B(a, 2r)} \frac{3\lambda_0 + 2\mu_0}{3(3\lambda + 2\mu)} 3(\lambda - \lambda_0) |\nabla \cdot (E_{c(t)}^0 e_j)|^2 dy
\]
\[
\geq C_{18} \int_{D \cap B(a, 2r)} |\nabla G(y - c(t))|^2 dy - C_{19}. \tag{5.29}
\]
A combination of (5.27) to (5.29) yields
\[
(C_{18} - C_{17}\epsilon) \int_{D \cap B(a,2r)} \frac{1}{|y - c(t)|^4} dy - C_{17}\epsilon - C_{19} - C(r) \leq I^g(t,c).
\] (5.30)

Notice that $C_{17}$ and $C_{18}$ are independent of $r$ and $\epsilon$. So if we take $\epsilon$ in such a way that $C_{18} - C_{17}\epsilon > 0$ in advance, we get $\lim_{t \to t_0} I^g(t,c) = \infty$. This is a contradiction to (5.14). We can apply a similar argument to (5.13) in the case when (5.24) and (5.26) are satisfied and get $\lim_{t \to t_0} I^g(t,c) = -\infty$. This completes the proof of Proposition 5.6.

\[\square\]

6 Construction of singular solutions

Proof of Proposition 2.1. Set
\[
s = \begin{cases} 
1 & \text{if } n = 3 \\
\theta & \text{if } n = 2.
\end{cases}
\]

We construct $G_x^0(\cdot)$ in the form
\[
G_x^0(\cdot) = G_0(\cdot - x; x) + \epsilon(\cdot; x).
\]

Define a functional $f_x$ by the formula
\[
f_x(\varphi) = \int_{\Omega} (\gamma_0(x) - \gamma_0(y)) \nabla G_0(y - x; x) \cdot \nabla \varphi(y) dy, \varphi \in C^\infty_0(\Omega).
\]

By the assumption, we have
\[
|\gamma_0(x) - \gamma_0(y)| \leq M|x - y|^s.
\]

Since
\[
|\nabla G_0(y - x; x)| \leq \frac{C}{|y - x|^{n-1}},
\]
we get
\[
|f_x(\varphi)| \leq \int_{\Omega} |\gamma_0(x) - \gamma_0(y)||\nabla G_0(y - x; x)||\nabla \varphi(y)| dy
\]
\[
\leq C \left\{ \int_{\Omega} \frac{dy}{|y - x|^{2(n-1-s)}} \right\}^{1/2} |\varphi|_{1,\Omega}.
\]

Therefore $f_x$ is in $(H^1_0(\Omega))^*$ since $2(n-1-s) < n$. Let $\epsilon(\cdot; x) \in H^1_0(\Omega)$ be the weak solution of
\[
\nabla \cdot \gamma_0 \nabla \epsilon(\cdot; x) = -f_x \text{ in } \Omega,
\]
\[
\epsilon(\cdot; x)|_{\partial \Omega} = 0.
\]

$\epsilon(\cdot; x)$ satisfies
\[
|\epsilon(\cdot; x)|_{1,\Omega} \leq C \left\{ \int_{\Omega} \frac{dy}{|y - x|^{2(n-1-s)}} \right\}^{1/2}.
\]
For any \( \varphi \in C_0^\infty(\Omega) \)
\[
\int_\Omega \gamma_0(y) \nabla G_x^0(y) \cdot \nabla \varphi(y) \, dy \\
= \int_\Omega \gamma_0(y) \nabla G_0(y - x; x) \cdot \nabla \varphi(y) + \gamma_0(y) \nabla \epsilon(y; x) \cdot \nabla \varphi(y) \, dy \\
= \int_\Omega \gamma_0(y) \nabla G_0(y - x; x) \cdot \nabla \varphi(y) \, dy \\
= \varphi(x).
\]

\( G_x^0(\cdot) \) satisfies
\[
|G_x^0(\cdot) - G_0(\cdot - x; x)|_{1, \Omega} \leq C \left\{ \int_\Omega \frac{dy}{|y - x|^{2(n-1-s)}} \right\}^{1/2}.
\]

From this inequality, we know that \((G_x^0(\cdot) - G_0(\cdot - x; x))_{x \in \Omega}\) is bounded in \( H^1(\Omega) \).

\( \square \)

**Proof of Proposition 5.2.**

We construct \( u_0^\phi(t, \cdot) \) in the form
\[
u_0^\phi(t, \cdot) = \nabla G(\cdot - x) + \mathcal{E}(\cdot; x).
\]

Since
\[
\mathcal{L}_{\lambda, \mu} \nabla G(\cdot - x) = 0 \text{ in } \Omega \setminus \{x\},
\]
it suffices to construct \( \mathcal{E}(\cdot; x) \) such that
\[
\mathcal{L}_{\lambda, \mu} \mathcal{E}(\cdot; x) = \{ \mathcal{L}_{\lambda, \mu} \nabla G(\cdot - x) \text{ in } \Omega \setminus \{x\} \}
\]

In fact, this is divided into two steps.

First we construct \( \mathcal{E}_1(\cdot; x) \) such that
\[
\mathcal{L}_{\lambda, \mu} \mathcal{E}_1(\cdot; x) = \{ \mathcal{L}_{\lambda, \mu} \nabla G(\cdot - x) \text{ in } \Omega \setminus \{x\} \}
\]

Second we construct \( \mathcal{E}_2(\cdot; x) \) such that
\[
\mathcal{L}_{\lambda, \mu} \mathcal{E}_2(\cdot; x) = \{ \mathcal{L}_{\lambda, \mu} \mathcal{E}_1(\cdot; x) \text{ in } \Omega \}
\]

Then
\[
\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2
\]
satisfies (6.1). Let us start with the explanation of

**Construction of \( \mathcal{E}_1(\cdot; x) \)**

This is based on

**Claim 1.** Assume that \( w \) and \( \mathcal{E}_1^0 \) satisfy
\[
\triangle w = -\nabla G(\cdot - x) \cdot \nabla \mu_0(\cdot) \text{ in } \Omega \setminus \{x\}
\]

(6.4)
and
\[ \mathcal{L}_{\lambda_0(x), \mu_0(x)} \mathcal{E}^0_1 = 2 \{ \nabla \nabla \mu_0(\cdot) \} \nabla G(\cdot - x) \text{ in } \Omega \setminus \{x\}. \] (6.5)

Then
\[ \mathcal{E}_1 = \frac{2 \nabla w}{\lambda_0(x) + 2 \mu_0(x)} + \mathcal{E}^0_1 \] (6.6)
satisfies (6.2).

This claim is easily checked if one knows
\[ \mathcal{L}_{\lambda_0(x), \mu_0(x)} (\nabla w) = \{ \lambda_0(x) + 2 \mu_0(x) \} \nabla \triangle w \]
and
\[ \{ \mathcal{L}_{\lambda_0(x), \mu_0(x)} - \mathcal{L}_{\lambda_0, \mu_0} \} \nabla G(\cdot - x) = -2 \nabla \{ \nabla G(\cdot - x) \cdot \nabla \mu_0(\cdot) \} + 2 \{ \nabla \nabla \mu_0(\cdot) \} \nabla G(\cdot - x). \]

Therefore we have to explain how to construct such \( w \) and \( \mathcal{E}^0_1 \).

**Construction of \( w \).**

We construct \( w \) in the form
\[ w = -\nabla \cdot \xi^0 - \nabla \cdot \xi^1 + \eta \] (6.7)
where
\[ \xi^0 = \xi^0(\cdot; x) = \frac{\cdot - x}{8\pi} \nabla \mu_0(x); \]
\[ \xi^1 = \xi^1(\cdot; x) \in \{ H^1_0(\Omega) \}^3 \]
is the weak solution of
\[ \triangle \xi^1 = G(\cdot - x) \{ \nabla \mu_0(\cdot) - \nabla \mu_0(x) \} \text{ in } \Omega; \] (6.9)
\[ \eta = \eta(\cdot; x) \in H^1_0(\Omega) \]
is the weak solution of
\[ \triangle \eta = G(\cdot - x) \triangle \mu_0(\cdot) \text{ in } \Omega. \] (6.10)

Notice that \( \xi^0 \) satisfies the equation
\[ \triangle \xi^0 = G(\cdot - x) \nabla \mu_0(x) \text{ in } \Omega. \] (6.11)

A combination of (6.9), (6.10) and (6.11) implies that \( w \) given by (6.7) satisfies (6.4).

From (5.7), (5.8) and the regularity theory we know \( (\xi^1(\cdot; x))_{x \in \Omega} \) and \( (\eta(\cdot; x))_{x \in \Omega} \) are bounded in \( \{ H^3(\Omega) \}^3 \) and \( H^2(\Omega) \), respectively. Therefore we can conclude
\[ \left( \nabla w + \frac{1}{8\pi} \{ \nabla \nabla \{ \cdot - x \} \} \nabla \mu_0(x) \right)_{x \in \Omega} \text{ is bounded in } \{ H^1(\Omega) \}^3. \] (6.12)

**Construction of \( \mathcal{E}^0_1 \).** Since
\[ \{ \nabla \nabla \mu_0(\cdot) - \nabla \nabla \mu_0(x) \} \nabla G(\cdot - x) \in \{ L^2(\Omega) \}^3, \]
We can find the weak solution \( \mathcal{E}^{0,1}_1 = \mathcal{E}^{0,1}_1(\cdot; x) \in \{ H^1_0(\Omega) \}^3 \) of
\[ \mathcal{L}_{\lambda_0(x), \mu_0(x)} \mathcal{E}^{0,1}_1 = 2 \{ \nabla \nabla \mu_0(\cdot) - \nabla \nabla \mu_0(x) \} \nabla G(\cdot - x) \text{ in } \Omega. \] (6.14)
We construct $E^0_1$ in the form
\[ E^0_1 = E^{0,0}_1 + E^{0,1}_1, \]
where $E^{0,0}_1 = E^{0,0}_1(\cdot; x)$ solves
\[ \mathcal{L}_{\lambda_0(x), \mu_0(x)} E^{0,0}_1 = 2\{\nabla \nabla \mu_0(x)\} \nabla G(\cdot - x) \text{ in } \Omega. \]
(6.16)
The construction of $E^{0,0}_1$ is based on

**Claim 2.** Assume that $f$ and $u$ satisfy
\[ (\lambda_0(x) + 2\mu_0(x)) \Delta f = -(\lambda_0(x) + \mu_0(x)) \nabla \cdot u. \]
(6.17)
Then the formula
\[ \mathcal{L}_{\lambda_0(x), \mu_0(x)} (u + \nabla f) = \mu_0(x) \Delta u, \]

is valid.

This is directly checked and we may omit the proof. Thus first we solve
\[ \mu_0(x) \Delta u = 2\{\nabla \nabla \mu_0(x)\} \nabla G(\cdot - x) \text{ in } \Omega. \]

Fortunately, we can find the explicit solution given by
\[ u = u(\cdot; x) = \left\{ \frac{\nabla \nabla \mu_0(x)}{4\pi\mu_0(x)} \right\} \frac{(\cdot - x)}{|\cdot - x|}. \]

$(u(\cdot; x))_{x \in \Omega}$ is bounded in $\{H^1(\Omega)\}^3$ and thus we get the unique weak solution $f = f(\cdot; x) \in H^1_0(\Omega)$ of (6.17). From the regularity theory and (5.7) we know $(f(\cdot; x))_{x \in \Omega}$ is bounded in $H^2(\Omega)$. Therefore $E^{0,0}_1$ given by
\[ E^{0,0}_1 = u + \nabla f \]
satisfies (6.16) and hence $E^0_1$ given by (6.15) is in $\{H^1(\Omega)\}^3$. The boundedness of $(E^0_1(\cdot; x))_{x \in \Omega}$ in $\{H^1(\Omega)\}^3$ is clear. This completes the construction of $E^0_1$.

**Construction of $E^1_2(\cdot; x)$.**

From the construction of $E_1$ and (5.8) we get the functional
\[ \{\mathcal{L}_{\lambda_0(x), \mu_0(x)} - \mathcal{L}_{\lambda_0, \mu_0}\} E^0_1(\cdot; x) \]
belongs to the dual space of $\{H^1(\Omega)\}^3$. Thus one can find the weak solution $E^1_2 = E^1_2(\cdot; x) \in \{H^1_0(\Omega)\}^3$ of (6.3). The boundedness of $(E^1_2(\cdot; x))_{x \in \Omega}$ in $\{H^1(\Omega)\}^3$ is clear. This completes the construction of $E^1_2$.

\[ \square \]

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7 Appendix

For reader’s convenience, we give a proof of

The Runge approximation property. Let \( \gamma_0 \in C^{0,1}(\Omega)(n \geq 3) \), \( \in L^\infty(\Omega)(n = 2) \). Let \( \Gamma \) be a given nonempty open subset of \( \partial \Omega \). Let \( U \) be an open subset of \( \Omega \). Assume that \( \overline{U} \subset \Omega \) and that \( \Omega \setminus \overline{U} \) is connected. Then for any \( u \) satisfying \( \nabla \cdot \gamma_0 \nabla u = 0 \) in an open neighbourhood of \( \overline{U} \) there exists a sequence \((u_k)\) of \( H^1(\Omega) \) functions such that

\[
\nabla \cdot \gamma_0 \nabla u_k = 0 \text{ in } \Omega, \\
\text{supp}(u_k|_{\partial \Omega}) \subset \Gamma, \\
\lim_{k \to \infty} u_k = u \text{ in } H^1(U).
\]

Remark. Notice that if \( \Gamma = \partial \Omega \), this is a usual Runge approximation theorem. A relationship between the uniqueness for the Cauchy problem and the Runge approximation property are described in [11] on pp.761-763.

Proof. Extend \( \gamma_0 \) outside \( \Omega \) as 1 if \( n = 2 \), as a uniformly positive Lipschitz function if \( n \geq 3 \). Take an open ball \( B \) with small radius centered at a point in \( \Gamma \) such that \( B \cap \partial \Omega \) is contained in \( \Gamma \) and represented by a graph of a Lipschitz function on \( \mathbb{R}^{n-1} \). Then modifying it, we can construct a domain \( \Omega_0 \) with Lipschitz boundary in such a way that \( \Omega \subset \Omega_0 \), \( \Omega \neq \Omega_0 \), and \( \partial \Omega \setminus (B \cap \partial \Omega) \) is contained in \( \partial \Omega_0 \). Give \( F \in (H^1_0(\Omega_0))^* \) and find \( u \in H^1_0(\Omega_0) \) such that

\[
\nabla \cdot \gamma_0 \nabla u = -F \text{ in } \Omega_0, \\
u|_{\partial \Omega_0} = 0.
\]

Set

\[
GF = u.
\]

We show that:

Let \( X \) denote the set of all \( u|_U \) satisfying \( u \in H^1 \) in an open neighbourhood of \( \overline{U} \) and therein \( \nabla \cdot \gamma_0 \nabla u = 0 \); let \( Y \) denote the set of all \( GF|_U \) with \( \text{supp } F \subset \Omega_0 \setminus \overline{\Omega} \). Then \( Y \) is dense in \( X \) with respect to \( H^1 \)-topology.

By the Hahn-Banach theorem, this is equivalent to the following statement. Let \( f \in (H^1(U))^* \). If

\[
f(GF|_U) = 0
\]

for all \( F \in (H^1_0(\Omega_0))^* \) with \( \text{supp } F \subset \Omega_0 \setminus \overline{\Omega} \) then \( f \) vanishes on \( X \), too.

So we prove it. Give \( \varphi \in H^1_0(\Omega_0) \). Set

\[
\tilde{f}(\varphi) = f(\varphi|_U).
\]

Of course \( \tilde{f} \) is in \( (H^1_0(\Omega_0))^* \). Consider \( G\tilde{f} \). Since \( \text{supp } \tilde{f} \subset \overline{U} \), we know that

\[
\nabla \cdot \gamma_0 \nabla (G\tilde{f}) = 0 \text{ in } \Omega_0 \setminus \overline{U}.
\]
Give $F \in (H^1_0(\Omega_0))^*$ with $\text{supp} \ F \subset \Omega_0 \setminus \overline{\Omega}$. Then we get

$$
0 = f(GF|_U) = \tilde{f}(GF)
$$

$$
= \int_{\Omega_0} \gamma_0 \nabla(G\tilde{f}) \cdot \nabla(GF) \, dx
$$

$$
= \int_{\Omega_0} \gamma_0 \nabla(GF) \cdot \nabla(G\tilde{f}) \, dx
$$

$$
= F(G\tilde{f}).
$$

Since $F$ is arbitrary, we get

$$
G\tilde{f} = 0, \text{ in } \Omega_0 \setminus \overline{\Omega}.
$$

From the unique continuation property (see [2] for $n = 2$) and the connectedness of $\Omega \setminus \overline{U}$ we get

$$
G\tilde{f} = 0 \text{ in } \Omega \setminus \overline{U}.
$$

Now let $u|_U \in X$ where $u \in H^1(V)$ in an open neighbourhood $V$ of $\overline{U}$ and satisfies $\nabla \cdot \gamma_0 \nabla u = 0$ in $V$. By cutting off $u$ outside $\overline{U}$, we know that there exists $\tilde{u} \in H^1_0(\Omega_0)$ such that $\tilde{u}|_U = u|_U$. Then

$$
f(u|_U) = f(\tilde{u}|_U) = \tilde{f}(\tilde{u})
$$

$$
= \int_{\Omega_0} \gamma_0 \nabla(G\tilde{f}) \cdot \nabla \tilde{u} \, dx
$$

$$
= \int_{\Omega_0} \gamma_0 \nabla(G\tilde{f}) \cdot \nabla u \, dx
$$

$$
= \int_{V} \gamma_0 \nabla u \cdot \nabla(G\tilde{f}) \, dx = 0.
$$

This completes the proof. □

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