Twist number and order properties of periodic orbits

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Abstract

A less studied numerical characteristic of periodic orbits of area preserving twist maps of the annulus is the twist or torsion number, called initially the amount of rotation \cite{Mather}. It measures the average rotation of tangent vectors under the action of the derivative of the map along that orbit, and characterizes the degree of complexity of the dynamics.

The aim of this paper is to give new insights into the definition and properties of the twist number, and to relate its range to the order properties of periodic orbits. We derive an algorithm to deduce the exact value or a demi–unit interval containing the exact value of the twist number.

We prove that at a period–doubling bifurcation threshold of a mini-maximizing periodic orbit, the new born doubly periodic orbit has the absolute twist number larger than the absolute twist of the original orbit after bifurcation. We give examples of periodic orbits having large absolute twist number, that are badly ordered, and illustrate how characterization of these orbits only by their residue can lead to incorrect results.

In connection to the study of the twist number of periodic orbits of standard–like maps we introduce a new tool, called 1–cone function. We prove that the location of minima of this function with respect to the vertical symmetry lines of a standard–like map encodes a valuable information on the symmetric periodic orbits and their twist number.

1 Introduction

The study of the dynamics of area preserving positive twist maps of the annulus is mainly concerned with the characterization of its invariant sets, and the dynamical behaviour of a map restricted to such sets. Among invariant sets, rotational periodic orbits have been thoroughly studied and classified according to their linear stability, extremal type, order properties (see \cite{BC} for a survey). A numerical characteristic associated to a periodic orbit is the rotation number, which measures the average rotation of the orbit around the annulus. In \cite{Mather} Mather defined also the amount of rotation, which is called twist number in \cite{G} or torsion number in \cite{T}. The twist number of a periodic orbit characterizes the average rotation of tangent vectors under the action of the tangent map along the map’s orbit. Mather related the twist number with the Morse index of the corresponding
critical sequence (configuration), and Angenent \cite{2} proved that in the space of \((p, q)\)-sequences a critical point of the \(W_{pq}\)-action, corresponding to a periodic orbit of twist number greater than 0 is connected by the negative gradient flow of the action, through a heteroclinic connection, with a sequence corresponding to an orbit of zero twist number. Using topological arguments, Crovisier \cite{10} proved the existence of orbits of zero–torsion (twist) number for any real number in the rotation number set of a twist map of the annulus (not necessarily area preserving).

In order to detect new classes of dynamical systems exhibiting periodic orbits of non–zero twist (torsion), Béguin and Boubaker gave \cite{5} conditions ensuring that some area preserving diffeomorphisms of the disc \(D^2\), and particular diffeomorphisms of the torus \(T^2\) exhibit such orbits.

In Mather’s and Angenent’s definitions of the twist number this a positive quantity. However its natural definition leads to a signed number \cite{10}. The twist number of periodic orbits of positive twist maps is zero or negative, while for those of negative twist maps it is zero or positive. In our approach we keep its natural sign and call absolute twist, the absolute value of the twist number.

During almost 30 years since the definition of the twist number was given, no periodic orbit of absolute twist number greater than \(1/2\) was detected in the dynamics of classical twist maps (standard map, Fermi–Ulam map, etc). That is why we wonder under what conditions can such orbits appear.

From Aubry–Mather theory \cite{3}, \cite{22} it is known that non–degenerate minimizing and associated mini–maximizing \((p, q)\)-type periodic orbits are well ordered and the minimizing orbit has zero twist number. Thus it is natural to investigate whether \((p, q)\)-periodic orbits that are not given by this theory can be well–ordered or not and how order properties are related to the value of their twist number.

Our starting point was not however the study of the twist number. In an attempt to characterize the dynamical behaviour of twist maps after the breakdown of the last KAM invariant torus (the case of standard map) or in a half annulus where no invariant circle exist (the case of Fermi–Ulam map \cite{15}, or tokamap \cite{4}), we identified in the phase space of such maps, special regions where no minimizing periodic orbits can land. Instead we noticed that any periodic orbit having at least two points in such a region is badly ordered and typically has absolute twist greater than \(1/2\).

Thus we were led to a deeper study of the twist number of periodic orbits, and their properties. In this paper we complement the results on twist number reported in \cite{23}, \cite{2}, and show that periodic orbits of non–zero twist numbers are typically born through a period doubling bifurcation. We give examples of periodic orbits that have large absolute twist number. As far as we know no such orbits were identified before in the study of the most known twist maps, standard–like maps. Some of these orbits are ordered and other are unordered. In order to explain why some twist maps can exhibit a sequence of bifurcations of a periodic orbit leading to increasing the absolute twist number up to its maximal value we introduce two subclasses of standard–like maps, USF maps and TSF maps.

The paper is organized as follows. In Section 2 we give the basic properties of twist
area preserving diffeomorphisms of the annulus, relevant to our study. In order to derive
sufficient conditions that favor the existence of ordered periodic orbits of
large absolute twist number, we set up in Section 3 a framework of study, defining the
class of twist maps exhibiting the so called strong folding property, and a function, called
1–cone function.

The 1–cone function is defined on the phase–space of a twist map, and takes negative
values within the region where the map exhibits strong folding property. We prove that
the restriction of this function to a periodic orbit gives information on the eigenvalues of
the Hessian matrix associated to that orbit (Lemma 3.1). Analyzing the 1–cone function
associated to a standard–like map, we show that such maps can have either a connected
strong folding region including one of the vertical symmetry lines or a two–component
strong folding region including both vertical symmetry lines. These results will be ex-
plained in Section 5 to explain the variation of the twist number.

In Section 4 we revisit the definition and properties of the twist number of a periodic
orbit based on the structure of the universal covering group of the group $SL(2, \mathbb{R})$ (a
system of coordinates on this group allowing to decipher its topology is presented in [2]).
The twist number is defined as the translation number of a circle map induced by the
monodromy matrix associated to the periodic orbit. We review in [1] the properties of the
translation number of an orientation preserving homeomorphism of the unit circle and
point out the particularities of the translation number of circle homeomorphisms induced
by matrices in $SL(2, \mathbb{R})$.

Theorem 4.2 gives the relationship between the twist number value of a $(p,q)$–periodic
orbit, and the position of the real number 0 with respect to the sequence of interlaced
eigenvalues of the Hessian matrix $H_q$, associated to the corresponding $(p,q)$–sequence,
and of a symmetric matrix derived from $H_q$. This theorem complements results from [23]
and [2].

Based on this theorem we derive an algorithm to deduce the exact value of the twist
number or a demi–unit interval that contains the twist number.

The main result in Section 4 is the Proposition 4.1, which shows that periodic orbits
of large absolute twist number are born through a period doubling bifurcation. More
precisely it states that if at some threshold, a period doubling bifurcation of a mini–
maximizing $(p,q)$–periodic orbit occurs, with transition from elliptic to inverse hyperbolic
orbit, then the new born $2q$–periodic orbit is elliptic, having the twist number within the
interval $(-1, -1/2)$.

In Section 5 we give examples of periodic orbits that have large absolute twist number
and are unordered. A natural question is whether a positive twist map can also exhibit
ordered $(p,q)$–periodic orbits $(p,q$ relative prime integers), of twist number less than
$-1/2$ (we note that periodic orbits of large absolute twist number, born through a period
doubling bifurcation are of type $(2p,2q)$). We give a positive answer to this question,
giving an example of three–harmonic standard map having an ordered $(1,2)$–periodic
orbit, of twist number $\tau = -1$.

In order to illustrate the different range of the twist number of a periodic orbit in the
three-harmonic standard map in comparison to that of a periodic orbit of the same type
of the standard map, we deduce (Proposition 5.1) the bifurcations that a periodic orbit of a family of standard–like maps, with a uni-component strong folding region, undergo necessarily.

The concrete examples of periodic orbits of large absolute twist number, given in this section, illustrate that the classical method used for more than 30 years in classification of periodic orbits can lead to an erroneous conclusion.

2 Background on twist maps

We recall basic properties of twist maps, relevant to our approach. For a detailed presentation of classical results concerning dynamics of this type of maps, the reader is referred to [15], [24].

Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the unit circle, $\mathbb{A} = S^1 \times \mathbb{R}$, the infinite annulus, and $\pi : \mathbb{R}^2 \to \mathbb{A}$, the covering projection, $\pi(x, y) = (x \mod 1, y)$.

We consider a $C^1$–diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (x', y')$, satisfying the following properties:

i) $F$ is exact area preserving map, isotopic to the identity;

ii) $F(x + 1, y) = F(x, y) + (1, 0)$, for all $(x, y) \in \mathbb{R}^2$;

iii) $F$ has uniform positive twist property, i.e. $\frac{\partial F_1}{\partial y} \geq c > 0$.

The map $F$ defines a $C^1$–diffeomorphism, $f : \mathbb{A} \to \mathbb{A}$, such that $\pi \circ F = f \circ \pi$. Both maps $f$ and $F$ are called area preserving positive twist maps or simply twist maps. $F$ is a lift of $f$.

In the sequel we will switch from $f$ to $F$, and conversely, without comment.

The motion of a point around the annulus is characterized by its rotation number. The orbit of a point $z \in \mathbb{A}$ has a rotation number if there exists the limit:

$$\rho = \lim_{n \to \infty} \frac{x_n - x}{n},$$

where $(x, y) \in \mathbb{R}^2$ is a lift of $z$, and $(x_n, y_n) = F^n(x, y)$. The rotation number does not depend on the chosen point $z$ on the orbit or the lift $(x, y)$. For different lifts of the map $f$, the corresponding rotation numbers differ by an integer.

Let $p, q$ be two relative prime integers, $q > 0$. A $(p, q)$–type orbit of the twist map, $F$, is an orbit $(x_n, y_n) = F^n(x_0, y_0)$, $n \in \mathbb{Z}$, such that $x_{n+q} = x_n + p$, for every $n \in \mathbb{Z}$. The $\pi$–projection of such an orbit onto the annulus is called $(p, q)$– periodic orbit or simply $q$–periodic orbit of the map $f$, and its rotation number is $p/q$.

The linear stability of a $(p, q)$–periodic orbit, $O(z)$, is characterized by the multipliers of the derivative, $D_z f^q \in SL(2, \mathbb{R})$, which in turn are determined by the trace, $\text{tr}(D_z f^q)$. Instead of trace one can also use the residue [14] to classify the $q$–periodic orbits. Namely, the residue is defined by:

$$R = \frac{2 - \text{trace}(D_z f^q)}{4},$$

where $q$ is the least period of the orbit.
The dynamics of a twist map, $F$, has a variational formulation, that is the $F$–orbits are associated to critical points of an action, and vice–versa. We discuss only the case of $(p, q)$–type orbits.

More precisely, $F$ admits a $C^2$–generating function $h : \mathbb{R}^2 \to \mathbb{R}$ (unique up to additive constants), such that $h_{12}(x, x') = \partial h/\partial x \partial x'< 0$ on $\mathbb{R}^2$, and $F(x, y) = (x', y')$ iff $y = -h_1(x, x'), y' = h_2(x, x')$ ($h_1 = \partial h/\partial x, h_2 = \partial h/\partial x'$).

The generating function defines an action $W_{pq}$, on the space $X_{pq}$, of $(p, q)$–type sequences of real numbers, $x = (x_n)_{n \in \mathbb{Z}}$, i.e. sequences satisfying the property, $x_{n+q} = x_n + p$, for all $n \in \mathbb{Z}$. The space $X_{pq}$ is identified with the affine subspace of $\mathbb{R}^{q+1} = \{(x_0, x_1, \ldots, x_q)\}$, of equation $x_q = x_0 + p$, and the action is defined by:

$$W_{pq}(x) = \sum_{k=0}^{q-1} h(x_k, x_{k+1})$$

There is a one to one correspondence between critical points, $x = (x_n)$, of the $W_{pq}$–action, and $(p, q)$–type orbits, $(x_n, y_n)$ of the positive twist map, $F$:

$$(x_n, y_n) = (x_n, -h_1(x_n, x_{n+1})) \quad \forall \ n \in \mathbb{N} \tag{3}$$

By Aubry–Mather theory [3], [22], for each pair of relative prime integers $(p, q)$, $q > 0$, a twist area preserving map has at least two periodic orbits of type $(p, q)$. One corresponds to a non–degenerate minimizing $(p, q)$–sequence of the action, and by abuse of language it is also called in the following, minimizing $(p, q)$–type orbit. Associated to such an orbit there is a second one, corresponding to a mini–maximizing sequence, and called mini–maximizing $(p, q)$–type periodic orbit.

Let $H_q$ be the Hessian matrix of the action, $W_{pq}$, at a critical point $x = (x_n)$. The signature (the number of negative and positive eigenvalues) of the Hessian $H_q$, at a non–degenerate minimizing sequence is $(0, q)$, while at the corresponding mini–maximizing sequence it is $(1, q – 1)$. The number of negative eigenvalues is called the Morse index of the critical sequence.

A salient property of a non–degenerate minimizing $(p, q)$–periodic orbit of a positive twist map, $f$, is that it is well-ordered.

An invariant set $M$ of a positive twist map, $f$, is well ordered if for every $(x, y), (x', y') \in \pi^{-1}(M)$, we have $x < x'$ iff $F_1(x, y) < F_1(x', y')$, where $F_1$ is the first component of the lift $F$.

A well–ordered $(p, q)$–orbit is also called Birkhoff orbit, while a badly–ordered orbit is called non–Birkhoff.

A $(p, q)$–periodic orbit of the twist map $f$ is well ordered iff the corresponding $(p, q)$–sequence $x$ is cyclically ordered, i.e.

$$x \leq \tau_{ij}x \quad \text{or} \quad \tau_{ij}x \leq x, \quad \forall \ i, j \in \mathbb{Z}, \tag{4}$$

where $\tau_{ij}$ is the translation defined by:

$$(\tau_{ij}x)_k = x_{k+i} + j, \quad \forall \ x = (x_k) \in \mathbb{R}^\mathbb{Z}, i, j, k \in \mathbb{Z}$$
By Aubry–Mather theory $W_{pq}$–minimizing sequences are cyclically ordered. Moreover, the mini–maximizing sequences are also ordered with respect to minimizing sequences \[22\].

The space $X_{pq}$ of $(p,q)$–sequences is partially ordered with respect to the order inherited from $\mathbb{R}^\mathbb{Z}$. Namely if $x = (x_k), y = (y_k) \in \mathbb{R}^\mathbb{Z}$, then:

\[
x \leq y \Leftrightarrow x_k \leq y_k, \forall k \in \mathbb{Z}
\]

One also defines the following relations:

\[
x < y \Leftrightarrow x \leq y, \text{ but } x \neq y
\]

\[
x \prec y \Leftrightarrow x_k < y_k, \forall k \in \mathbb{Z}
\]

To a $(p,q)$–sequence $x = (x_k)$ one associates the Aubry diagram, i.e the graph of the piecewise affine function $A : \mathbb{R} \to \mathbb{R}$, that interpolates linearly the points $(k, x_k), k \in \mathbb{Z}$. The Aubry function associated to a minimizing $(p,q)$–sequence is increasing. If the Aubry diagram of a sequence $y = (y_k)$ do not intersect the Aubry diagram of the minimizing sequence $x = (x_k)$, then the two sequences are comparable, i.e. either $x \prec y$ or $y \prec x$. If the two diagrams intersect transversally, then the sequence $y = (y_k)$ corresponds to an unordered $(p,q)$–periodic orbit.

The most studied twist maps, both theoretically and numerically, are the standard–like maps. A standard–like map is a twist map, $F_\epsilon$, defined by a Lagrangian generating function of the form $h(x, x') = \frac{1}{2}(x - x')^2 - \epsilon V(x)$, where $V$ is a fixed 1–periodic even function of class $C^r$, $r \geq 3$, and $\epsilon \in \mathbb{R}$ is a parameter:

\[
x' = x + y - \epsilon V'(x)
y' = y - \epsilon V'(x),
\]

Classical standard map \[28\] corresponds to the potential $V(x) = -\frac{1}{(2\pi)^2} \cos(2\pi x)$.

The twist map, $F_\epsilon$, is reversible, i.e. it factorizes as $F_\epsilon = I \circ R$, where $R$ and $I$ are the involutions, $R(x, y) = (-x, y - \epsilon V'(x)), I(x, y) = (-x + y, y)$. The symmetry lines, that are the fixed point sets $\text{Fix}(R), \text{Fix}(I)$, have both two components: $\Gamma_0 : x = 0, \Gamma'_0 = 0.5$, respectively $\Gamma_1 : x = y/2, \Gamma'_1 : x = (y + 1)/2$. The $R$–invariant orbits are called symmetric orbits.

As we shall show in Section 4, the twist number of a $(p,q)$–periodic orbit one deduces analyzing the relative position with respect to 0 of the eigenvalues of the associated Hessian matrix and of a companion symmetric matrix. In the following section we prove that diagonal entries of these matrices encode information on the position of the smallest eigenvalue. In order to extract such an information we introduce a new tool of study of a twist map, the 1–cone function, whose restriction to $(p,q)$–periodic orbit gives the diagonal entries in the two symmetric matrices. Depending on the location of the minima of the 1–cone function, with respect to the vertical symmetry–lines of a standard–like map we define two subclasses of this set of twist maps: the class of USF maps, respectively
TSF maps. Then in Section 5 we prove that for a family of USF maps, respectively TSF maps, one can predict the sequence of bifurcations that their symmetric periodic orbits can undergo, as well as the variation of their twist number, only from the analysis of the diagonal entries of the Hessian and the companion matrix.

3 1–cone function and strong folding property of a twist map

Let $f$ be a positive or negative twist APM defined on the annulus $A$. The derivative of the map at an arbitrary point has the entries denoted as follows:

$$D_z f = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}$$

The twist APM, $f$, is called $(+d)$–map if $d(z) > 0$, for all $z \in A$, respectively $(+a)$–map, if $a(z) > 0$, for all $z \in A$. The two conditions are not equivalent.

We study only twist maps $f$ that are $(+d)$–maps, because their inverses are $(+a)$–maps.

The $(+d)$–condition ensures that the lines of the vertical foliation of the annulus are mapped to graphs of functions over the action, while the sign of $a(z)$ illustrates how the map acts on the horizontal foliation of the annulus. If for a $(+d)$–twist map the function $a(z)$ is negative on a segment of an horizontal circle, $S^1 \times \{y\}$, then the $f$–image of that circle is not a graph over $x$. We say that the circle is folded by the map.

**Definition 3.1** A $(+d)$–twist map is said to satisfy the folding property if there exist regions in the phase space where $a \leq 0$. The subset $\mathcal{F} = \{z \in A | a(z) \leq 0\}$ is called folding region.

The standard map is a $(+d)$–twist APM which exhibits folding property for $\epsilon \geq 1$. The vertical strip:

$$\mathcal{F}_\epsilon = \{(x,y) | -\frac{1}{2\pi} \arccos(1/\epsilon) \leq x \leq \frac{1}{2\pi} \arccos(1/\epsilon)\}, \quad \epsilon \geq 1$$

is its folding region.

In the sequel we refer only to $(+d)$–positive twist maps, and all results can be modified accordingly for $(+d)$–negative twist ones. In order to reveal the properties of twist maps exhibiting a folding region, we use geometrical arguments, that are more intuitive and allow visualizations in the phase space. The same properties could be presented in configuration space in a more formal setting.

We associate to each point of the phase space a pair of vectors:

$$v^+_1(z) = D_{f^{-1}(z)} f(e_2) = (b(f^{-1}(z)), d(f^{-1}(z)))^T,$$
$$v^-_1(z) = D_{f(z)} f^{-1}(-e_2) = (b(z), -a(z))^T,$$
where $e_2 = (0,1)^T$ is the vertical tangent vector at $f^-(z)$, respectively at $f(z)$. The directions to which the vectors $v^+_{1}(z)$, $v^-_{1}(z)$ are pointing illustrate how the linear maps $D_{f^{-1}}f$, $D_{f(z)}f^{-1}$, deviate the the upward vertical through $f^{-1}(z)$, respectively the downward vertical through $f(z)$. If at a point the slopes of the two vectors, $s^+_1(z) = \frac{d}{b}(f^{-1}(z))$, $s^-_1(z) = \frac{a}{b}(z)$, are related by $s^-_1(z) \geq s^+_1(z)$, one says that 1–cone crossing occurs at $z$. It is known that no minimizing orbit can pass through such a point [20].

In a folding region, since $a(z) \leq 0$, the two slopes are both positive, and 1–cone crossing is possible. It is easy to check that 1–cone crossing can occur in the phase space of a $(+d)$–twist map only if the map has folding property, and the set

$$
\text{Cone}_1 = \{ z \in \mathbb{A} \mid s^-_1(z) \geq s^+_1(z) \},
$$

of the points where 1–cone crossing occurs is necessarily included in the folding region.

A $(+d)$–twist map with the property that the 1–cone crossing occurs in a subset of a folding region is said to exhibit strong folding property, and the set $\text{Cone}_1$ is called strong folding region.

The standard map exhibits a strong folding region for each $\epsilon \geq 2$:

$$
\text{Cone}_1(\epsilon) = \{(x,y) \in \mathcal{F} \mid -\frac{1}{2\pi} \arccos(2/\epsilon) \leq x \leq \frac{1}{2\pi} \arccos(2/\epsilon) \}
$$

In order to show how the strong folding property influences the dynamical behaviour of a twist map, we define a real function $\delta$, which associates to each point $z = (x,y)$ the difference between the slopes $s^+_1(z)$, $s^-_1(z)$:

$$
\delta(x,y) = \frac{d(x-1,y-1)}{b(x-1,y-1)} + \frac{a(x,y)}{b(x,y)}, \quad (x-1,y-1) = f^{-1}(x,y)
$$

We call this function, 1–cone function associated to the twist map. $\delta$ takes negative values at the points of a strong folding region of a $(+d)$–twist map.

In Fig[1] we illustrate the graph of $\delta$ over the phase space $\mathbb{A}$ identified to $[-0.5,0.5] \times \mathbb{R}$ (considered as a subset of the plane $xOy$), for different values of the perturbation parameter $\epsilon$ of the standard map, while in Fig[2] is illustrated the graph of the same function associated to the standard–like maps $F_{\epsilon}$, defined by a three-harmonic potential,

$$
V(x) = \frac{\eta_1}{2\pi} \cos(2\pi x) + \frac{\eta_2}{4\pi} \cos(4\pi x) + \frac{\eta_3}{6\pi} \cos(6\pi x),
$$

where the parameters $\eta_i$, $i = 1,3$, are respectively 0.18, −0.42, −0.11.

We note that in variational setting, the function $\delta(x,y(x,x'))$ is the Hessian of the orbital action $W(x,x') = h(x,x')$, corresponding to the 1–segment of trajectory $(x,y)$.

Moreover, the restriction of the function $\delta$ to the points of a $(p,q)$–sequence gives the diagonal entries of the Hessian at that sequence. Actually, if $(x_i,y_i)$, $i = 0,q-1$, are points on a $(p,q)$–type orbit, and

$$
D_{(x_i,y_i)}F = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}
$$
then the Hessian expressed in \((x, y)\)–coordinates is:

\[
H_q = \begin{bmatrix}
\hat{\alpha}_0 & \hat{\beta}_0 & 0 & \ldots & \hat{\beta}_{q-1} \\
\hat{\beta}_0 & \hat{\alpha}_1 & \hat{\beta}_1 & \ldots & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & \hat{\beta}_{q-3} & \hat{\alpha}_{q-2} & \hat{\beta}_{q-2} \\
\hat{\beta}_{q-1} & 0 & \ldots & \hat{\beta}_{q-2} & \hat{\alpha}_{q-1}
\end{bmatrix},
\]

(11)

where \(\hat{\alpha}_i = d_{i-1} b_{i-1}^{-1} + a_i b_i^{-1}, \hat{\beta}_i = -1/b_i, i = 0, q - 1\).

**Lemma 3.1** Let \((x_i, y_i)\) be a \((p, q)\)–type orbit, and \(H_q\) its Hessian matrix (11). If \(\lambda_{\text{min}}, \lambda_{\text{max}}\) are the smallest, respectively the largest eigenvalues of \(H_q\) then the values of the 1–cone function, \(\delta(x_i, y_i)\), at each point of the orbit obeys the inequalities:

\[
\lambda_{\text{min}} \leq \delta(x_i, y_i) \leq \lambda_{\text{max}}
\]

(12)

Proof: The smallest and the largest eigenvalues of \(H_q\) are as follows:

\[
\lambda_{\text{min}} = \min_{v \neq 0} R(v), \quad \lambda_{\text{max}} = \max_{v \neq 0} R(v),
\]

(13)

where \(R(v) = \frac{\langle v, H_q v \rangle}{\langle v, v \rangle}, v \neq 0\), is the Rayleigh quotient associated to the symmetric matrix \(H_q\). Because \(\lambda_{\text{min}} \leq R(v) \leq \lambda_{\text{max}}, \forall v \neq 0\), and \(R(e_i) = \delta(x_i, y_i), \forall i = 1, q\), where \(e_i\) is a vector of the standard basis in \(\mathbb{R}^q\), it follows the sequence of inequalities.

By the Gershgorin Theorem [26] each eigenvalue, \(\lambda\), of a symmetric matrix \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\), belongs at least to one of the intervals

\[
[a_{ii} - r_i(A), a_{ii} + r_i(A)], \quad r_i(A) = \sum_{j=1, j \neq i}^{n} |a_{ij}|, \quad i = 1, n
\]

It follows that in the case of the matrix \(H_q\), associated to a periodic orbit, the smallest eigenvalue belongs to the interval:

\[
\lambda_{\text{min}} \in [\delta_k - r_k(A), \delta_k], \quad \delta_k = \min_{i=0, q-1} \delta(x_i, y_i)
\]

Since the smallest value of the function \(\delta\) restricted to a periodic orbit is an upper bound for the smallest eigenvalue of \(H_q\), whose sign is an indicator for the Aubry–Mather periodic orbits, the location of the absolute minimum of the function can give valuable information on extremal type of these orbits.

Let us study the influence of location of minima of the function \(\delta\), on the dynamical behaviour of the standard–like maps.

We restrict our study to standard–like maps corresponding to \(\epsilon \geq 0\). Standard–like maps are \((+-d)\)–twist APMs, that exhibit folding regions beyond a threshold \(\epsilon'\), which is
the smallest parameter \( \epsilon \), for which the set \( \mathcal{F} = \{(x, y) \mid a(x, y) \leq 0\} \) is non-empty or equivalently the set \( \{x \in [0, 1] \mid V''(x) \geq 1/\epsilon\} \neq \emptyset \).

Standard–like maps also exhibit strong folding property, beyond a parameter \( \epsilon^* > \epsilon' \) which is the smallest parameter for which the 1–cone crossing occurs within the folding region: \( \delta(x, y) = 2 - \epsilon V''(x) \leq 0 \). Both folding and strong folding regions, are union of vertical strips, i.e. subsets of the form \( \{(x, y) \mid A \leq x \leq B, y \in \mathbb{R}\} \).

Depending on the potential \( V \), the corresponding standard–like map can exhibit one or more strong folding regions. Since \( V \) is an even periodic function, \( V'' \) is also even, and its critical point set contains the points \( x = 0, x = 0.5 \). If at least one of these two points is a maximum point for \( V'' \), then beyond a perturbation parameter, the corresponding symmetry line (i.e., \( \Gamma_0 \) or/and \( \Gamma'_0 \)) is included within the strong folding region. Such a line is a line of global or local minima for the function \( \delta(x, y) \), because \( \delta \) does not depend on \( y \) (see Figs. 1, 2).

**Remark 3.1** If one of the symmetry lines \( \Gamma_0, \Gamma'_0 \), of equation \( x = x_0, x_0 \in \{-0.5, 0\} \), is a line of minima for the function \( \delta \), then the restriction of the function \( \delta \) to that symmetry line, \( \delta(x_0, y; \epsilon) \), is a decreasing function with respect to \( \epsilon \).

We call USF map (uni–component strong folding region map) a standard–like map exhibiting only one connected strong folding region, and TSF (two–component strong folding region map), a map exhibiting a strong folding region with two connected components (one including the symmetry line \( \Gamma_0 \), and another, \( \Gamma'_0 \)).

Standard–like maps defined by a potential \( V \), whose second derivative has only two critical points is an USF map. Classical standard map (28) obeys this condition, as well as some maps having an analytic potential with infinitely many harmonics, such as the map:

\[
\begin{align*}
x' &= x + y' \\
y' &= \frac{\epsilon \sin 2\pi x}{2\pi 1 - a \cos 2\pi x}, |a| < 1
\end{align*}
\]

(14)

For \( a = -0.3 \), for example, it is an USF map.

The map corresponding to the three harmonic potential (10) is a TSF map. In Fig 2 one can see that the function \( \delta \) associated to this map gets negative beyond a parameter, within two vertical strips.

Obviously in the case of a potential given by a trigonometric polynomial the coefficients of the polynomial can be tuned such that the map be a USF map.

In Section 5 we show how the number of strong folding components influences the dynamical behaviour of a standard–like map, respectively the order properties of periodic orbits of large absolute twist number.

In the sequel we reconsider the definition of the twist number and derive new properties. We measure the twist number of an orbit with respect to the canonical trivialization of the tangent bundle of the annulus. The general framework of its study is based on the structure of \( \widehat{SL}(2, \mathbb{R}) \), and properties of translation number of circles maps given in the A and B.
Figure 1: The graph of the function $\delta$ with respect to the plane $xOy$, associated to the standard map corresponding respectively to $\epsilon = 0.5, 1.2, 2.3$.

Figure 2: The graph of the function $\delta$ with respect to the plane $xOy$, associated to the standard–like map corresponding to a fixed three harmonic potential (see the text) and respectively to perturbation parameters $\epsilon = 0.05, 0.35, 0.48$.

4 The twist number revisited

Let $f$ be a twist area preserving (APM) of the annulus $\mathbb{A}$, and $z_0, z_1, \ldots, z_{q-1}$ a $q$–periodic orbit. We identify the tangent spaces at $z_i$, $T_{z_i}\mathbb{A}$, $i = 0, q - 1$, with $\mathbb{R}^2$. If $v = v_0$ is a tangent vector at a point $z$ of the orbit (let us say that $v \in T_{z_0}\mathbb{A}$), then the corresponding (forward) tangent orbit is:

$$(z_k, v_k), \quad \text{where } v_{k+1} = D_{z_k}f(v_k), k \in \mathbb{N}$$

The amount of rotation about a periodic orbit is the average rotation of tangent vectors under the action of the tangent map along that orbit. In order to characterize the amount of rotation we interpret the derivative matrices, $D_{z}f \in SL(2, \mathbb{R})$, as circle maps (see $\mathbb{A}$), and a particular lift $\widetilde{D_{z}}f$ allows then to measure how much is rotated a vector $v \in T_{z}\mathbb{A}$, by $D_{z}f$.

To be more precise, we consider $\mathbb{S}^1$ as being either the unit circle, $C$, in $\mathbb{R}^2$ or as the quotient group $\mathbb{R}/\mathbb{Z}$.

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R})$ be the matrix $D_{z}f$, with $b$ positive or negative depending on whether $f$ has a positive twist or a negative twist. $D_{z}f$ maps the vector $(0, 1)^T$ to $(b, d)^T$, i.e the twist map $f$ tilts the vertical line through $z \in \mathbb{A}$ to the right ($b > 0$) or to the left ($b < 0$). Regarding $D_{z}f$ as a circle map this means that the point $1/4 + \mathbb{Z} \in \mathbb{S}^1$ (the
north pole in $\mathbb{S}^1 \equiv C$) is displaced clockwise/anticlockwise less than $1/2$.

This property of $D_z f$, transferred to a particular lift, amounts to set a condition on its displacement map value at $x = \frac{1}{4}$ ($x$ is a lift to $\mathbb{R}$ of $\frac{1}{4} + z \in \mathbb{S}^1$):

**Definition 4.1** A lift $\widetilde{D}_z f$ of the derivative matrix, $D_z f$, of a positive (negative) twist map, such that its displacement map satisfies, $\Psi(1/4) \in (-1/2, 0)$ ($\Psi(1/4) \in (0, 1/2)$) is called basic lift.

In Fig.3 we illustrate the action of $D_z f$ on the unit vector $v = (0, 1)^T$ or equivalently on $p = 1/4 + z \in \mathbb{S}^1$, when $b > 0$, as well as how the basic lift $\widetilde{D}_z f$ displaces the point $x = 1/4$, $\pi(x) = p$, along $\mathbb{R}$ ($\pi : \mathbb{R} \to \mathbb{S}^1$, $\pi(x) = x + Z$).

![Figure 3: Illustration of the action of $D_z f$ on a unit vector $v \equiv p \in \mathbb{S}^1$, and of the corresponding basic lift, on a point $x \in \pi^{-1}(p)$. $g$ denotes the circle map defined by the matrix $D_z f$.](image)

Let $\widetilde{D}_{z_k} f$ be the basic lifts associated to matrices $D_{z_k} f \in SL(2, \mathbb{R})$, $k = 0, q - 1$, where $z_0, z_1, \ldots, z_{q-1} \in \Lambda$ are points of a $(p, q)$–periodic orbit, of least period $q$. Because the covering projection $\Pi : SL(2, \mathbb{R}) \to SL(2, \mathbb{R})$ is a group homomorphism, each lift $G_k = D_{f_k(z)} f \circ \cdots \circ D_{f(z)} f \circ D_z f$ is a lift of $D_z f^k$, $k = 1, q$. We denote by $\Psi_k$ the corresponding displacement map, and call $G_k$ the the $k$–basic lift associated to the $q$–periodic orbit, $k = 1, q$.

**Definition 4.2** The twist number of a $(p, q)$–periodic orbit, $O(z)$, is the translation number of the associated $q$–basic lift, $G_q$:

$$\tau(z) = \lim_{n \to \infty} \frac{(D_{f^{q-1}(z)} f \circ \cdots \circ D_{f(z)} f \circ D_z f)^n(x) - x}{n}$$  \hspace{1cm} (15)
If \( x \in \mathbb{R} \) then its displacement:
\[
\Psi_q(x) = D_{f^{q-1}(z)} f \circ \cdots \circ D_{f(z)} f \circ D_z f(x) - x
\]
measures the amount of rotation in a period, of a vector, \( v \in T_z \mathbb{A} \equiv \mathbb{R}^2 \), colinear with \( u = (\cos 2\pi x, \sin 2\pi x)^T, v = \mu u, \mu > 0 \).

The twist number defined as a translation number, is a \( \tilde{SL}(2, \mathbb{R}) \)–conjugacy invariant, and thus it does not depend on the point \( z \) of the orbit, to which one associates the \( q \)–basic lift.

In the sequel we discuss the twist number of periodic orbits of positive twist maps. In this case the basic lift \( \tilde{D}_z f \), at an arbitrary point \( z \in \mathbb{A} \), belongs to one of the subsets: \( \mathcal{H}_0, \mathcal{E}_{-1}, \mathcal{H}'_{-1} \subset \tilde{SL}(2, \mathbb{R}) \) or to the common boundary of two of them (see Fig.13). Thus the basic lifts \( \tilde{D}_{f^k(z)} \), \( k = 0, q-1 \), associated to a \( q \)–periodic orbit have the translation number less or equal to zero.

The positive twist property of the map leads also to:

**Lemma 4.1** If \( z \) belongs to a \( q \)–periodic orbit, and for some \( 1 < k < q \), the principal lift \( G_k \) associated to \( z \) has its displacement at \( 1/4 \), less than \( j/2 \), i.e. \( \Psi_k(1/4) < j/2 \), for some negative integer, \( j \), then no one of the lifts \( G_{k+1}, \ldots, G_q \) can have the displacement at \( 1/4 \), greater than \( j/2 \).

In view of this Lemma and taking into account that the twist number of a \( q \)–periodic orbit belongs to the range of the displacement map, \( \Psi_q, \text{ i.e. } \tau \in [m(G_q), M(G_q)] \), we can conclude that the twist number of a periodic orbit of a positive twist map can be zero or negative.

The twist number defined as above is minus the twist number defined by Angenent [2]. The negative sign illustrates the clockwise sense of rotation about the periodic orbit. Its natural sign allows a better understanding and explanation of its properties in connection with the displacement map of the lift involved in its definition.

In [23] the amount of rotation about a periodic orbit of a positive twist map is defined in a similar way, but in order to get a positive twist number, it was chosen for each matrix \( D_{f^k(z)} f \), \( k = 0, q-1 \), a lift \( G \), with the translation number \( \tau(G) \in [0, 1/2) \). Let us argue that such a choice of a lift is not always possible.

First we recall shortly the method used in [23] to establish a relationship between the twist number of a periodic \((p,q)\)–type orbit, and the Morse index of the corresponding critical sequence of the \( W_{pq} \)–action.

By a non-symplectic change of coordinates:
\[
(x, x') \mapsto (x, y(x, x'))
\]
the derivative \( D_x F^q = D_{F^{q-1}(z)} F \cdots D_{F(z)} F \cdot D_x F \in SL(2, \mathbb{R}) \), at a point \( z = (x, y) \) of a \((p,q)\)–orbit, is expressed as:
\[
M = \begin{bmatrix}
0 & 1 \\
-\beta_{q-1} & \alpha_0 \\
\beta_0 & \beta_0
\end{bmatrix} \cdots \begin{bmatrix}
0 & 1 \\
-\beta_1 & \alpha_2 \\
\beta_2 & \beta_2
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
-\beta_0 & \alpha_1 \\
\beta_1 & \beta_1
\end{bmatrix}
\]
The entries $\alpha_i, \beta_i$ are defined by:

$$
\beta_i = h_{12}(x_i, x_{i+1}), \quad \alpha_i = h_{11}(x_i, x_{i+1}) + h_{22}(x_{i-1}, x_i), \quad i \in \mathbb{Z} \pmod{q},
$$

where $h$ is the Lagrangian generating function of $F$, and $h_{kt}, \ k, \ell \in \{1, 2\}$, are second order partial derivatives of $h$.

Thus $D_z F^\theta$ is $GL^+(2, \mathbb{R})$–conjugated to the matrix $M$ ($GL^+(2, \mathbb{R})$ is the group of real $2 \times 2$–matrices of positive determinant), and the twist number is defined as the translation number of some lift of $M \in GL^+(2, \mathbb{R})$.

The lifts $G_B$ of circle maps, $g_B$, defined by matrices $B \in GL^+(2, \mathbb{R})$ were considered in [23] as having the properties of general circle maps. The lack of information that the difference between the maximum and minimum displacement of a lift $G_B$ is less than $1/2$ led to a non concordance.

Let us point out that the lifts $G_B$, $B \in GL^+(2, \mathbb{R})$, have the same particularities as the lifts in $SL(2, \mathbb{R})$. Indeed, to each matrix $B \in GL^+(2, \mathbb{R})$ one associates the matrix $A = \frac{1}{\sqrt{\det(B)}} B \in SL(2, \mathbb{R})$, and as a consequence $GL^+(2, \mathbb{R})$ can be identified to the direct product of multiplicative groups, $SL(2, \mathbb{R}) \times (0, \infty)$:

$$
B \in GL^+(2, \mathbb{R}) \mapsto \left( \frac{1}{\sqrt{\det(B)}} B, \sqrt{\det(B)} \right) \in SL(2, \mathbb{R}) \times (0, \infty)
$$

Thus a lift of the circle map defined by a matrix in $B \in GL^+(2, \mathbb{R})$ is a pair $(G, \zeta) \in \tilde{SL}(2, \mathbb{R}) \times (0, \infty)$, such that $B = \zeta \cdot \Pi(G)$, where $\Pi : SL(2, \mathbb{R}) \to SL(2, \mathbb{R})$ is the covering projection. Therefore all properties of lifts in $\tilde{SL}(2, \mathbb{R})$ and their translation numbers are shared with the lifts $G_B, B \in GL^+(2, \mathbb{R})$.

Now we are able to show that if $(x_i, y_i)_{i \in \mathbb{Z}}$ is a $(p, q)$–type orbit of a positive twist map, $F$, then some matrices $D_{(x_i, y_i)} F \in SL(2, \mathbb{R})$, and hence their $GL^+(2, \mathbb{R})$–conjugate matrices that represent the derivative in $(x, x')$–coordinate, cannot have a lift of translation number in $(0, 1/2)$, as it was supposed in [23]. Actually, if for some $j$, $D_{(x_j, y_j)} F$ has the trace in $(-2, 2)$, its basic lift, $G$, belongs to the subset $E_{-1} \subset SL(2, \mathbb{R})$, i.e. $\tau(G) \in (-1/2, 0)$, and any other lift $G + j, j \in \mathbb{Z}$, has the translation number in the interval $(-1/2 + j, j)$. Thus no lift $G + j$ can have $\tau \in (0, 1/2)$.

Modifying correspondingly the Mather proof in [23], that is taking as lift for each factor in (17) the basic lift, one gets the following result:

**Theorem 4.1** If $(x_n, y_n), n \in \mathbb{Z}$ is a $(p, q)$–type orbit of the positive twist map $F$, $H_q$ is the Hessian of the action $W_{pq}$ at $x = (x_n)$, $I$ the Morse index of $(x_n)$, and $\tau$ is the twist number of the corresponding $f$–orbit, then $I$ and $\tau$ are related as follows:

i) If $I$ is even, then $\tau = -I/2$;

ii) If $I$ is odd, then $\lceil -I/2 \rceil - 1 \leq \tau < \lceil -I/2 \rceil$, with equality if and only if $x$ is a degenerate critical point. $\lceil a \rceil$ denotes the least integer number greater or equal to $a \in \mathbb{R}$.

From Theorem 4.1 and the general properties of lifts in $\tilde{SL}(2, \mathbb{R})$ we have the following:
Corollary 4.1  i) The twist number of a minimizing orbit is zero.

ii) Along an elliptic mini-maximizing $q$–periodic orbit, the maps $f^k$, $k = 1, q$ have positive twist property.

Proof: ii) Actually, if $f^k$, $k = 2, q$ violated the twist property, then the $k$– basic lift, $G_k$, associated to the orbit would have a displacement, such that $\Psi_k(1/4) \leq -1/2$, thus contradicting, in view of Lemma 4.1, that the twist number is $\tau \in (-1/2, 0)$.

Naturally, one raises the following questions: under what conditions a periodic orbit of twist number less than $-1/2$ can exist? Do such orbits so rarely appear that no reference to their existence in the dynamics of most studied twist maps can be found in literature?

The Proposition below points out that such orbits typically appear at a period doubling bifurcation (details on this type of bifurcation can be found in [21], [25]):

Proposition 4.1 Let $f_\epsilon$, $\epsilon \geq 0$, be a positive twist map, which is a perturbation of an integrable map, $f_0$. If at some threshold $\epsilon'$, a period doubling bifurcation of a mini–maximizing $(p, q)$–periodic orbit occurs, with transition from elliptic to inverse hyperbolic orbit, then the new born $2q$–periodic orbit is elliptic, having the twist number within the interval $(-1, 1/2)$. If an inverse period doubling bifurcation occurs, i.e. with transition from an inverse hyperbolic to an elliptic $q$–periodic orbit, then at the threshold of bifurcation a new regular hyperbolic $2q$–periodic orbit emerges, and its twist number is $-1$.

Proof: We prove only the first part, the proof of the second one being similar. Let $\epsilon \mapsto z_\epsilon$, $\epsilon \in (\epsilon_1, \epsilon_2)$, be the continuous path in the annulus described by the $(p, q)$–periodic orbit that at $\epsilon' \in (\epsilon_1, \epsilon_2)$ undergoes a period doubling bifurcation. One associates to this path the continuous path, $\epsilon \mapsto G_q(\epsilon)$, in the covering space, $\widetilde{SL}(2, \mathbb{R})$, described by the corresponding $q$–basic lift. The latter one starts in the region $\mathcal{E}_{-1}$ (Fig. 4), intersects transversally the parabolic cone $\mathcal{P}_{2k-1}$, and enters $\mathcal{H}'_{-1}$.

Figure 4: Illustration of the paths in the universal covering space $\widetilde{SL}(2, \mathbb{R})$, related to the period doubling bifurcation of a $q$–periodic orbit.
On the other hand, each $z_\varepsilon$ is also a fixed point of $f^q_{\varepsilon}$, and thus the associated $2q$–basic lift is $G_q^2(\varepsilon)$. The twist number along the path $\varepsilon \mapsto G_q^2(\varepsilon)$ is double the twist number along the path $\varepsilon \mapsto G_q(\varepsilon)$, because $\tau(G_q^2(\varepsilon)) = \tau(G_q(\varepsilon))$, i.e. $\tau(G_q^2(\varepsilon)) \in (-1, 0)$, for $\varepsilon \in (\varepsilon_1, \varepsilon')$, it gets $-1$ at $\varepsilon'$, and remains $-1$ within the interval $(\varepsilon', \varepsilon_2)$. Thus $G_q^2(\varepsilon)$ describes a path from $\mathcal{E}_{-2}$ to $\mathcal{H}_{-2}$, that intersects transversally the parabolic cone, $\mathcal{P}_{2k}^-$. From the point of intersection to $\mathcal{P}_{2k}$, a new path emerges, namely the path described by the $2q$–basic lift $K_{2q}(\varepsilon)$, associated to the new born $2q$–periodic orbit, $p_\varepsilon$, $\varepsilon \in (\varepsilon', \varepsilon_2)$. Because of the continuity of the maps $\varepsilon \in [\varepsilon', \varepsilon_2) \rightarrow p_\varepsilon \rightarrow K_{2q}(\varepsilon)$, and $\tau : SL(2, \mathbb{R}) \rightarrow \mathbb{R}$, the new path enters the region $\mathcal{E}_2$. Thus the new born elliptic $2q$–periodic orbit has the twist number within the interval $(-1, -1/2)$.

Because the $2q$–basic lift associated to the new born elliptic $2q$–periodic orbit has the range of its displacement map, $[m(K_{2q}), M(K_{2q})] \subset (-1, -1/2)$, each tangent vector at a point of that orbit is rotated clockwise in one period, with an angle between $\pi$ and $2\pi$.

By Theorem 4.1 the twist number of a periodic orbit is well defined by an even Morse index of the corresponding critical sequence, but there is an ambiguity in deciding the demi–unit interval (an interval of length $1/2$) that contains the twist number, when the Morse index is odd. For instance if the Morse index is $1$, the twist number can belong either to $[-1/2, 0)$ or to $(-1, -1/2)$.

Because the translation number is not a group homomorphism (see 4), we cannot deduce the twist number of a $(p, q)$–periodic orbit, from the translation numbers of the basic lifts, $D_z f, \ldots, D_{f^{q-1}(z)} f$, associated to that orbit.

If the Morse index of a $(p, q)$–critical sequence is an odd number, and the corresponding orbit is elliptic, we can use a naive method to get the demi–unit interval (i.e. an interval of the form $(- (k+1)/2, -k/2)$, $k \in \mathbb{N}$) containing its twist number. Such a method is based on the property that if $z, z'$ are any two points on the same (elliptic in our case) $(p, q)$–periodic orbit, then $D_z f^q$ and $D_{z'} f^q$, are $SL(2, \mathbb{R})$–conjugated. Hence the corresponding $q$–basic lifts, $G_q, H_q$, are $SL(2, \mathbb{R})$–conjugated and by Proposition 3.1 their displacement maps have the range included in the same demi–unit interval $(- (k+1)/2, -k/2)$, for some integer $k \geq 0$.

In order to get the actual value of $k$, we can choose an arbitrary point $z$ on the elliptic $q$–periodic orbit, and compute a $(q+1)$–length segment of tangent trajectory, $(z_i, w_i)$, $w_{i+1} = D_{z_i} f(w_i), i = 0, q$, with $z_0 = z$ and $w_0 = (1, 0)^T \in T_z \mathbb{A}$. Let $c_i$ be the $x$–coordinates of the vectors $w_i$, $i = 0, q$. If $k$ changes of sign in the sequence $(c_i), i = 0, q$, are recorded, then the twist number belongs to the demi–unit interval, $(- (k+1)/2, -k/2)$.

This method works well for low values of the period $q$. In this way the reader can check easily the result in Proposition 4.1 in the case of standard map (Eq.28) corresponding to $\varepsilon = 2$, respectively $\varepsilon = 4$, for which period doubling bifurcation of the $2$–periodic orbit $(0, 0.5)$, respectively of the fixed point $(0, 0)$, occurs.

The naive method is not suitable for hyperbolic periodic orbits, because in this case $m(G_q)$ and $M(G_q)$ belong to distinct demi–unit intervals.

In the sequel, we present a general method to deduce the smallest interval $I$ of ends in $\mathbb{Z}/2$ that contains the interval $[m(G_q), M(G_q)]$ of the twist number of a $(p, q)$–periodic
orbit. The method is based on the spectral properties of the Hessian matrix, $H_q$, and of an associated symmetric matrix.

The Hessian matrix $H_q$ of the $W_{pq}$–action associated to a $(p,q)$–periodic orbit, $q \geq 3$, is a Jacobi periodic matrix (i.e. a symmetric tridiagonal matrix with non-null entries in the upper right, and left lower corners, and the next–to–diagonal entries have the same sign). For $q = 2$, $H_2$ is simply a symmetric matrix.

$$
H_q = \begin{bmatrix}
\alpha_0 & \beta_0 & 0 & \cdots & \beta_{q-1} \\
\beta_0 & \alpha_1 & \beta_1 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \beta_{q-3} & \alpha_{q-2} & \beta_{q-2} \\
\beta_{q-1} & 0 & \cdots & \beta_{q-2} & \alpha_{q-1}
\end{bmatrix}, \quad q \geq 3, \quad H_2 = \begin{bmatrix}
\alpha_0 & \beta_0 + \beta_1 \\
\beta_0 + \beta_1 & \alpha_1
\end{bmatrix}, \quad (19)
$$

where $\alpha_i \in \mathbb{R}, \beta_i < 0, i = 0,q-1$, are defined in (18). Such a Jacobi matrix is called periodic because the sequences $(\alpha_i), (\beta_i), i \in \mathbb{Z}$, are $q$–periodic.

The spectral properties of periodic Jacobi matrices were studied extensively in the late seventies. In his approach to the twist number, Angenent [2] exploited a result by van Moerbeke [27]. The partial characterization of the spectrum of periodic Jacobi matrices given by van Moerbeke is complemented in [12], [7], and a thoroughly presentation of its properties, in connection to discrete Hill equation, can be found in [28].

Exploiting the properties of periodic Jacobi matrices we are able to visualize the results in Theorem 4.1, and moreover to enhance it. We give a method to deduce the smallest interval $I$ of ends in $\mathbb{Z}/2$, containing the twist number of a $(p,q)$–periodic orbit, from the position of the real number zero, with respect to an interval whose ends are eigenvalues of the Hessian $H_q$, and/or eigenvalues of the symmetric matrix, $H_q^-$, which is a slight modification of $H_q$:

$$
H_q^- = \begin{bmatrix}
\alpha_0 & \beta_0 & 0 & \cdots & -\beta_{q-1} \\
\beta_0 & \alpha_1 & \beta_1 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \beta_{q-3} & \alpha_{q-2} & \beta_{q-2} \\
-\beta_{q-1} & 0 & \cdots & \beta_{q-2} & \alpha_{q-1}
\end{bmatrix}, \quad q \geq 3, \quad H_2^- = \begin{bmatrix}
\alpha_0 & \beta_0 - \beta_1 \\
\beta_0 - \beta_1 & \alpha_1
\end{bmatrix}, \quad (20)
$$

The Theorem 4.1 can be enhanced to:

**Theorem 4.2** If $(x_n, y_n), n \in \mathbb{Z}$ is a $(p,q)$–type orbit of the positive twist map $F$, $H_q$ is the Hessian of the action $W_{pq}$ at $x = (x_n)$, $I$, the Morse index of $(x_n)$, $\tau$, the twist number of the corresponding $f$–orbit, and $I'$ is the number of negative eigenvalues of the symmetric matrix $H_q^-$, associated to $H_q$, then $I, I'$ and $\tau$ are related as follows:

i) If $I$ is even, then $\tau = -I/2$;

ii) If $I$ is odd and $H_q$ has at least one zero eigenvalue, then $\tau = -(I + 1)/2$;

iii) If $I$ is odd and $H_q$ has no zero eigenvalue, then:
a) If \( I' = I - 1 \), then \( \tau \in (-I/2, -(I - 1)/2) \);

b) If \( I' = I \), then \( \tau = -I/2 \);

c) If \( I' = I + 1 \), then \( \tau \in (-(I + 1)/2, -I/2) \);

Proof:

In order to relate the spectrum of the Hessian \( H_q \) to the trace of the derivative \( D_z F^q \), at a point \( z = (x,y) \) of a \((p,q)\)–type orbit, one associates to the Jacobi periodic matrix, \( J = H_q \), the discrete Hill equation [28], \( J\xi = \lambda \xi, \lambda \in \mathbb{R} \), where \( \xi = (\xi_i) \), is a \(q\)–periodic sequence in \( \mathbb{R} \). The coordinate-wise version of this equation is:

\[
\beta_{i-1}\xi_{i-1} + (\alpha_i - \lambda)\xi_i + \beta_i\xi_{i+1} = 0, \quad \lambda \in \mathbb{R}
\]  

Among the solutions \( \xi(\lambda) = (\xi_i(\lambda)) \) of this second-order difference equation, of particular interest is the Floquet solution, i.e. the solution , that satisfies the condition \( \xi_{i+q}(\lambda) = \eta(\lambda)\xi_i(\lambda) \), with \( \xi_0(\lambda), \xi_1(\lambda) \) given, and \( \eta(\lambda) \) an eigenvalue of the monodromy matrix:

\[
M(\lambda) = \begin{bmatrix}
0 & 1 & & \\
-\beta_{q-1} & \beta_q & \alpha_q & \\
\vdots & \ddots & \ddots & \ddots \\
0 & \beta_0 & \lambda - \alpha_1 & \beta_1
\end{bmatrix}
\]  

The trace of the matrix \( M(\lambda) \) is a polynomial of order \( q \), with real coefficients,

\[
\text{tr}(M(\lambda)) = (\beta_0\beta_1 \cdots \beta_{q-1})^{-1} \lambda^q + \cdots
\]  

called the Hill discriminant [28].

The characteristic polynomial of a periodic Jacobi matrix, \( J \), defining the discrete Hill equation is related to the Hill discriminant by the so called Hill formula [12], [28]:

\[
\det(J - \lambda I) = (-1)^q \beta_0\beta_1 \cdots \beta_{q-1}(\text{tr}(M(\lambda)) - 2)
\]

The history of Hill formula and generalization to multidimensional discrete and continuous Lagrangian systems can be found in [6].

From (24) it follows that the eigenvalues, \( \lambda_0, \lambda_1, \ldots, \lambda_{q-1} \), of the Hessian, \( H_q \), are the roots of the equation \( \text{tr}(M(\lambda)) = 2 \). The eigenvalues can be ordered in the following way [28]:

\[
\lambda_{2i} < \lambda_{2i+1} \leq \lambda_{2i+2}, \forall \ i \geq 0
\]  

On the other hand the roots of the equation \( \text{tr}(M(\lambda)) = -2 \) can be ordered [28] as:

\[
\lambda'_{2i} \leq \lambda'_{2i+1} < \lambda'_{2i+2}, \forall \ i \geq 0,
\]  

and the two series are interlaced:

\[
\lambda_0 < \lambda'_0 \leq \lambda'_1 < \lambda_1 \leq \lambda_2 < \lambda'_2 \leq \cdots
\]
Figure 5: The graph of the trace of monodromy matrix, $M(\lambda)$, the eigenvalues of a Hessian matrix $H_6$ and its associated matrix $H_6^-$.  

Being roots of the equations $\text{tr}(M(\lambda)) = 2$, $\text{tr}(M(\lambda)) = -2$, with $M(\lambda) \in SL(2, \mathbb{R})$, $\lambda_{2i+1}$, respectively $\lambda'_{2i}$, $i \geq 1$, can have at most double multiplicity. Thus each root occurs in $\mathbb{R}$ as many times as its multiplicity.

For practical purposes it is important to note that the roots $\lambda'_{2i}$, $i = 0, q - 1$, are the eigenvalues of the symmetric matrix $H^-_{2q}$, defined in (20), associated to the Hessian $H_q$. Indeed, by [7] the characteristic polynomial of $H^-_{2q}$ is for any $q \geq 2$:

$$\det(H^-_{2q} - \lambda I_q) = \det(H_q - \lambda I_q) + 4(-1)^{q} \beta_0 \beta_1 \cdots \beta_{q-1}$$

Comparing to the expression of the Hill discriminant (23) one gets, that $\text{tr}(M(\lambda)) + 2 = 0$ iff $\det(H^-_{2q} - \lambda I_q) = 0$.

For the reader convenience we illustrate in Fig.5 the graph of a Hill discriminant corresponding to an even integer, $q$.

Because the leading coefficient of the polynomial $\text{tr}(M(\lambda))$ is $(\beta_0 \beta_1 \cdots \beta_{q-1})^{-1}$, it follows that in both cases, $q$ even and odd, $\lim_{\lambda \to -\infty} \text{tr}(M(\lambda)) = +\infty$. Hence the dynamical type of the monodromy matrix (Fig.5) is as follows:

a) If $\lambda < \lambda_0$ or $\lambda \in (\lambda_{2i+1}, \lambda_{2i+2})$, $i \geq 0$ with $\lambda_{2i+1} \neq \lambda_{2i+2}$, then $M(\lambda)$ is regular hyperbolic.

b) For $\lambda \in (\lambda_{2i}, \lambda'_{2i})$ or $\lambda \in (\lambda'_{2i+1}, \lambda_{2i+1})$, $i \geq 0$, $M(\lambda)$ is elliptic.

c) For $\lambda \in (\lambda'_{2i}, \lambda_{2i+1})$, $\lambda'_{2i} \neq \lambda_{2i+1}$, $i \geq 0$, $M(\lambda)$ is inverse hyperbolic.

Now the location of the real number zero with respect to the interlaced series of eigenvalues (27) gives information on the Morse index of $H_q$, the signature of the matrix $H^-_{2q}$, and the dynamical type (regular hyperbolic, elliptic or inverse hyperbolic) of the $(p,q)$-periodic orbit we are studying.

If the Morse index $I$ of the Hessian $H_q$ is odd, and $H_q$ has no zero eigenvalue, i.e. $0 \in (\lambda_{2i}, \lambda_{2i+1})$, for some $i \geq 0$, then by Theorem 4.1 the twist number belongs to an interval of length 1, $[-I/2] - 1 < \tau < [-I/2]$. It follows that the $q$-basic lift $G_q$.
associated to our orbit can belong to $\mathcal{E}_{-2I+1}$, $\mathcal{H}_{-2I+1}$, or to $\mathcal{E}_{-2I-1} \subset SL(2, \mathbb{R})$, and the twist number can belong to different subintervals of the above 1-length interval. In order to deduce the right demi–unit interval to which the twist number belongs we analyze the position of 0 with respect to the eigenvalues $\lambda_{2i}$, $\lambda'_{2i+1}$, and deduce the index $I'$ of the symmetric matrix $H_q^-$. Due to the continuity of the translation number $\tau : SL(2, \mathbb{R}) \to \mathbb{R}$, the case iii) from the Theorem follows directly.

Similarly one can prove the case ii).

**Corollary 4.2** The smallest twist number a $(p, q)$–periodic orbit of a positive twist APM can have is $-q/2$.

This property was also stated in [2] but no arguments were given.

In numerical investigations of the twist number, if the Lagrangian generating function of the twist map is defined only locally (as in the case of tokamap [4]), then one can reduce the computations and improve accuracy, evaluating the Hessian $H_q$ associated to a periodic orbit, not in $(x, x')$–coordinates, but in the original symplectic $(x, y)$–coordinates. In this case the entries of $H_q$ are as in (11).

Thus, a simple algorithm that in the first step calculates the eigenvalues of the Hessian (11), order them increasingly, and then tests the cases described in Theorem 4.2 (computing also the eigenvalues of $H_q^-$, if necessary, and ordering them increasingly), returns the value of the twist number or the demi–unit interval containing the twist number.

We note that the $2q$–periodic orbits born through a period–doubling bifurcation are ordered orbits (Birkhoff orbits). In the next section we give examples of badly ordered (non–Birkhoff) $(p, q)$–periodic orbits that are born through a saddle–center bifurcation and have the absolute twist number greater than $1/2$. We also show that the so called two–component strong folding region maps can exhibit ordered periodic orbits of large absolute twist number.

## 5 Twist number and order properties of periodic orbits

Computer experiments reveal that plenty of pairs of periodic orbits of absolute twist number, $|\tau| > 1/2$, can be found in the folding region of the standard map corresponding to the perturbation parameter $\varepsilon > 1$.

The standard map we consider here is defined by the lift $F : (x, y) \mapsto (x', y')$:

$$
\begin{align*}
    x' &= x + y', \\
    y' &= y - \frac{\varepsilon}{2\pi} \sin(2\pi x), \quad \varepsilon \in \mathbb{R},
\end{align*}
$$

(28)

**Example 1.** The standard map corresponding to the perturbation parameter $\varepsilon = 1.4578$ has the minimizing $(8, 3)$–periodic orbit represented by $(0.5, 0.693821664066481)$, and the mini–maximizing one by $(0, 0.586319735666097)$ (Fig. 6 left panel).
The point \( z = (0.5, 0.7237177352891645) \) is also a \((8, 13)\)–periodic point. Applying the ubiquitous method used in the study of twist area preserving maps, namely, computing only the residue, \( R = 0.44794945246558 \), we can mistakenly conclude that it is a mini–maximizing periodic orbit. Computing further the eigenvalues of the Hessian \( H_{13} \) we get the Morse index \( I = 1 \), and analogously we are tempted to conclude that it is mini-maximizing (the index equal to one is only a necessary condition to be mini-maximizing, not a sufficient one). But the orbit is unordered and thus it cannot be mini-maximizing. The eigenvalues of the associated matrix \( H_{-13} \) reveal that its twist number belongs to \( (-1, -1/2) \). Hence this orbit is an elliptic periodic orbit with the property that each vector \( v \in T_zA \) is rotated clockwise about the orbit of \( z \), more than \( \pi \) radians in one period.

The orbit of \( z \) was born through a saddle–center bifurcation at a slightly lower value of \( \varepsilon \), and its pair is the regular hyperbolic orbit of the point \( z' = (0.5, 0.72337247461496) \). \( z' \) has the twist number \( \tau = -1 \), and its orbit is also unordered. They are illustrated in Fig. 6 right panel.

One can observe that projecting each of the four orbits onto the first factor, \( S^1 \), identified with \([-0.5, 0.5]\), and denoting by \( x_L, x_R \) the abscissas of the minimizing–orbit, nearest to 0 (\( x_L \) at left and \( x_R \) at right of 0), then within the interval \((x_L, x_R)\) each projected non–Birkhoff orbit has two points.

Example 2. The point \( z = (-0.387429398213243, 0.225141203573513) \) of the standard map corresponding to \( \varepsilon = 3 \) has an inverse hyperbolic \((1, 7)\)–periodic orbit of translation number \( \tau = -1.5 \), and its pair \( z' = (0.389952663038891, 0.536877082952284) \) has a regular hyperbolic \((1, 7)\)–periodic orbit of twist number \( \tau = -1 \).

For the same parameter the Aubry–Mather orbits of type \((1, 7)\) are the following: \((0, 0.390375216334041475)\), mini–maximizing, and \((-0.5, 0.008258333079127612)\), minimizing.

Considering again the interval \((x_L, x_R)\) as in the previous example, we note that within this interval each orbit of large absolute twist number has three points.

In Fig. 7 are illustrated all four orbits of type \((1, 7)\).

In an attempt to asses numerically the efficiency of the Boyland converse KAM criterion [8], Leage and MacKay [19] looked for badly ordered periodic orbits of the standard map. They detected such orbits for large perturbations parameters.
Figure 7: Four (1, 7)–periodic orbits of the standard map corresponding to $\varepsilon = 3$. At left is illustrated the pair of orbits given by Aubry–Mather theory, and at right a pair of orbits having the twist number $-1$, respectively $-1.5$.

We note that the badly ordered orbits reported in [19] have also non–zero twist number and $|\tau| > 1/2$. They are orbits born through a saddle-center bifurcation, at a parameter value greater than the threshold of the period doubling bifurcation of the mini–maximizing periodic orbit of the same type $(p,q)$.

Example 3. The point $z = (0.181826060531142, 0.6818260605309654)$ found in [19] for $\varepsilon = 7.221365$ is an elliptic point of type $(1, 3)$, twist number $\tau \in (-1, -1/2)$, and its pair $z' = (0.181803685309653, 0.681803685309654)$ has a regular hyperbolic $(1, 3)$–orbit, and $\tau = -1$.

The above examples can suggest that $(p,q)$–periodic orbits of large absolute twist number are unordered. A natural question is whether a positive twist map can also exhibit ordered $(p,q)$–periodic orbit $(p,q$ relative prime integers) of twist number less than $-1/2$. The following example shows that the answer is positive.

Example 4. We consider a multi–harmonic standard map defined by the following lift:

$$
x' = x + y + P(x),
\quad y' = y + P(x),
$$

(29)

where $P$ is the trigonometric polynomial, $P(x) = \eta_1 \sin(2\pi x) + \eta_2 \sin(4\pi x) + \eta_3 \sin(6\pi x)$, corresponding to the point $\eta = (0.18, -0.42, -0.11)$ in the parameter space. This map is reversible and has four $(1, 2)$–periodic orbits (Fig.X third line, right):

a) $(0.22048151875563, 0.440963103751125)$ is a symmetric minimizing orbit.

b) $(0, 0.5)$ is a regular hyperbolic point having the twist number equal to $-1$ (the associated Hessian has negative eigenvalues $\lambda_1 = -5.48881762834532$, $\lambda_2 = -1.06693368771638$, and thus the corresponding $(1, 2)$–sequence is a sequence of maximum action).

c) \begin{align*}
z_1 &= (0.099868507451075, 0.699123941758688), \\
z_2 &= (0.900131492548929, 0.300876058241312)
\end{align*}

are two inverse hyperbolic points having distinct orbits, of twist number $-1/2$.

All four $(1, 2)$–periodic orbits are cyclically ordered.

We are wondering why in this case an orbit of large absolute twist is ordered, while in the case of standard map such orbits are badly ordered. In order to give an answer we
connect this map to an integrable one, i.e. we consider the maps $F_\epsilon$, $\epsilon \in [0,1]$ defined by:
\[
x' = x + y + \epsilon P(x) \\
y' = y + \epsilon P(x),
\] (30)
and we analyze the dynamical type and twist number of $(1,2)$–periodic orbits, as $\epsilon$ increases from 0 to 1.

The map $F_\epsilon$, close to the integrable twist map, has two $(1,2)$–periodic orbits: a minimizing one that persists down to $\epsilon = 1$, and the orbit of the point $(0,0.5)$ which starts as a mini–maximizing orbit of elliptic type (Fig.8 first line, left). The latter one passes through a sequence of bifurcations (Fig.8) that lead to the following change of its twist number (see Fig.9):

elliptic mini–maximizing $\rightarrow$ inverse hyperbolic of $\tau = -1/2$ $\rightarrow$ elliptic of twist number in $(-1,-1/2)$ $\rightarrow$ regular hyperbolic of $\tau = -1$.

The first two bifurcations are period doubling, while the last one is a Rimmer–type bifurcation that leads to the creation of the two distinct asymmetric periodic orbits, $O(z_1)$, $O(z_2)$.

Hence the maximizing orbit $(0,0.5)$ of the map $F_1$ was born as an ordered mini–maximizing orbit, and through a sequence of bifurcations it changed the twist number as it is illustrated in Fig.9 but not the order property, because the period doubling, and Rimmer bifurcation of ordered orbits leads to ordered orbits.

We also note that while the $(1,2)$–minimizing orbit does not undergo any bifurcation as the perturbation parameter increases, the $(1,2)$–mini-maximizing orbit bifurcates, changing also the twist number value after each period–doubling bifurcation. Moreover at the Rimmer bifurcation threshold, the symmetric mini–maximizing orbit is continued by a maximizing one, and two asymmetric mini–maximizing orbits are born. Thus at $\epsilon = 1$ there exist 3 orbits given by Aubry–Mather theory: a symmetric minimizing orbit, and two asymmetric mini–maximizing orbits.

**Remark 5.1** From the analysis of the map (30) it follows that the mini–maximizing $(p,q)$–periodic orbit of a twist map can have the twist number $\tau$ either in the interval $[-1/2,0)$ or in the interval $(-1,-1/2)$, while the twist number of the corresponding minimizing orbit is constant and equal to 0.

In the sequel we show that the unexpected behaviour of the map (30) is due to the fact that it is a TSF map.

First we show that if a twist map is either USF or TSF we can predict the bifurcations that the orbits intersecting $\Gamma_0^*$ can undergo, as the perturbation parameter varies.

**Proposition 5.1** Let $F_\epsilon$ be an USF map for $\epsilon \geq \epsilon^*$, and $\Gamma^*$ the symmetry line that get included in the strong folding region for $\epsilon > \epsilon^*$. If after a slight perturbation of the integrable map $F_0$, a $(p,q)$–periodic orbit, $q > 1$, has a point $(x_0(\epsilon),y_0(\epsilon))$ on $\Gamma^*$, then the orbit can be either mini–maximizing or minimizing. Increasing $\epsilon$ from 0, the orbit can undergo the following bifurcations:
Figure 8: The phase portrait of the map $F_\epsilon$, Eq. (30), illustrating the sequence of bifurcations that the orbit of the point $(0, 0.5)$ undergoes as $\epsilon$ increases from 0 to 1. First line: at left, $\epsilon = 0.028$, $(0, 0.5)$ has an elliptic mini–maximizing orbit; at right, $\epsilon = 0.4$, slightly above a period doubling bifurcation. Second line: at left, $\epsilon = 0.46$, slightly above the second period doubling bifurcation (from $\tau = -1/2$ to $\tau \in (-1, -1/2)$); at right, $\epsilon = 0.65$, $(0, 0.5)$ has an elliptic orbit of twist number $\tau \in (-1, -1/2)$. Third line: at left $\epsilon = 0.8$, after a Rimmer bifurcation (from $\tau \in (-1, -1/2)$ to $\tau = -1$); at right, $\epsilon = 1$, $(0, 0.5)$ has regular hyperbolic orbit of twist number $\tau = -1$, and the periodic orbits born through Rimmer bifurcation (the orbits marked with +, respectively x, are inverse hyperbolic of $\tau = -1/2$).

Figure 9: The twist number of the $(1, 2)$–periodic orbits map (Eq. 30), as a function of $\epsilon$. The dashed line is the twist number of the minimizing orbit, the continuous line is the twist number of the point $(0, 0.5)$, and the dotted one is the twist number for the two asymmetric periodic orbits, born through a Rimmer bifurcation.
a) If \((x_0(\epsilon), y_0(\epsilon))\) is mini–maximizing of elliptic type it undergoes necessarily a period doubling bifurcation at some \(0 < \epsilon_d \leq \epsilon^*\).

b) If \((x_0(\epsilon), y_0(\epsilon))\) is minimizing, then at a threshold \(\epsilon_r\) it gets mini–maximizing through a Rimmer–type bifurcation, and at some \(\epsilon_d > \epsilon_r\) it undergoes a period doubling bifurcation, and gets inverse hyperbolic of twist number \(\tau = -1/2\).

Proof.

At \(\epsilon = 0\) \(\delta(x, y) = 2\), for every point \((x, y)\). Let \(H_q\) be the Hessian matrix associated to the \((p, q)\)-orbit, having a point on \(\Gamma^*\), \(H_q^{-}\) the symmetric matrix \((20)\), and \(\lambda_0\), respectively \(\lambda'_0\), the smallest eigenvalues of these matrices. From Remark 3.1 it follows that the function \(\delta(x_0, y_0; \epsilon)\) decreases from 2, as \(\epsilon\) increases. By Lemma 3.1 we have that \(\lambda_0, \lambda'_0 \leq \delta(x_0, y_0) < 2\), and by Gershgorin Theorem \(\lambda_0\) belongs necessarily to the interval \([\delta(x_0, y_0; \epsilon) - 2, \delta(x_0, y_0; \epsilon)]\). Hence for \(\epsilon\) very close to zero, \(\lambda_0\) can be either negative or positive, i.e. after a slight perturbation of the integrable map the considered periodic orbit can be mini–maximizing of elliptic type or minimizing.

a) If the orbit is mini–maximizing, then since the function \(\delta(x_0, y_0; \epsilon)\) is decreasing, the eigenvalue \(\lambda'_0(\epsilon)\) is pushed to the left on the real axis, and at a threshold \(\epsilon_d \leq \epsilon^*\) it crosses 0, i.e. the twist number crosses \(-1/2\) and a period doubling bifurcation occurs.

b) If the orbit is minimizing, using similar arguments it follows that necessarily the stated sequence of bifurcations must occur.

**Corollary 5.1** An USF map exhibits a total or a partial dominant symmetry line, that is either for each perturbation parameter or for \(\epsilon\) greater than a threshold, \(\epsilon^*\), any periodic orbit having a point on \(\Gamma^*_0\) has the twist number less than zero.

It is conjectured that the standard map has \(\Gamma_0\) as a total dominant symmetry line (as far as we know, no proof was given for this numerical observation). The map (Eq.15) corresponding to \(a = -3\) has \(\Gamma_0\) as a partial dominant symmetry line.

In the case of a TSF map a similar behaviour have the orbits intersecting \(\Gamma_0\), but moreover a second period doubling bifurcation from \(\tau = -1/2\) to \(\tau \in (-1, -1/2)\) is also possible.

Let us analyze the \((1, 2)\)-periodic orbit of the twist map \((30)\), intersecting \(\Gamma_0\).

After a slight perturbation of the integrable map the mini–maximizing \((1, 2)\)-periodic orbit has its points \((x_0, y_0), (x_1, y_1)\) on the symmetry lines \(\Gamma_0, \Gamma'_0\). The restriction of the \(1\)-cone function \(\delta\) to each of these symmetry lines is a decreasing function with respect to \(\epsilon\), namely it decreases from the value 2 corresponding to \(\epsilon = 0\), to negative values corresponding to \(\epsilon\) for which \(\text{Cone}_1(\epsilon)\) has two connected components. It follows that as \(\epsilon\) increases, the smallest eigenvalue, \(\lambda_0 \leq \min\{\delta(x_0, y_0), \delta(x_1, y_1)\}\), of \(H_2\) is pushed to the left on the real axis, as well as the smallest eigenvalue \(\lambda'_0 = \min\{\delta(x_0, y_0), \delta(x_1, y_1)\}\) of the matrix \(H_2^{-}\) (see \([20]\)). Hence a sequence of bifurcations of the ordered \((1, 2)\)-periodic orbit occurs leading to a continuous decrease of the twist number. Beyond an \(\epsilon\) both points of the orbit are within the strong folding region. In this case the Hessian \(H_2\) and its associated matrix \(H_2^{-}\) have negative diagonal entries, and as a consequence \(H_2^{-}\) has
both eigenvalues negative. Thus the (1,2)-periodic orbits intersecting Fix(R) is in this case an ordered orbit, of twist number $\tau < -1/2$.

Let us show that in the case of an USF map the second period doubling bifurcation of a mini–maximizing (1,2)–periodic orbit is not possible (i.e. transition from $\tau = -1/2$ to $\tau \in (-1, -1/2)$).

By Proposition 5.1 The points of the mini–maximizing (1,2)–orbit of an USF map are forced (eventually after a threshold) to belong to the symmetry lines $\Gamma_0, \Gamma'_0$. The points, $(x'_0, y'_0), (x'_1, y'_1)$, of the minimizing periodic orbit belong to $\Gamma_1$, respectively $\Gamma'_1$, and $R(x'_0, y'_0) = (x'_1, y'_1)$. Taking into account that for standard–like maps the 1–cone function is $R$–invariant, i.e.

$$\delta(R(x, y)) = \delta(x, y),$$

for any point $(x, y)$ on a symmetric periodic orbit, we have the following Hessian matrices (19) $H_2, H'_2$, associated to the two orbits, and the corresponding symmetric matrices (20) $H_2^-, H'_2^-$:

$$H_2 = \begin{bmatrix} \delta(x_0, y_0) & -2 \\ -2 & \delta(x_1, y_1) \end{bmatrix}, \quad H_2^- = \begin{bmatrix} \delta(x_0, y_0) & 0 \\ 0 & \delta(x_1, y_1) \end{bmatrix}$$

$$H'_2 = \begin{bmatrix} \delta(x'_0, y'_0) & -2 \\ -2 & \delta(x'_1, y'_1) \end{bmatrix}, \quad H'_2^- = \begin{bmatrix} \delta(x'_0, y'_0) & 0 \\ 0 & \delta(x'_1, y'_1) \end{bmatrix}$$

In Fig.10 is illustrated the profile of the $\delta$ function for the standard map, as an USF map, and for the TSF map (30), as well as the values of $\delta$ along the mini–maximizing, respectively minimizing (1,2)–periodic points of these maps.

A simple algebraic computation reveals that the different relative position of diagonal entries of $H'_2^-$ with respect to those of $H_2^-$, in the two cases, leads to distinct relative position of the corresponding eigenvalues (Fig.11). In the case of the USF map (Fig.11, left) the period doubling bifurcation from $\tau = -1/2$ to $\tau \in (-1, -1/2)$ is obstructed because the double eigenvalue of $H'_2^-$ is positive, and less than the second eigenvalue, $\lambda_1(H^-_2)$, of $H^-_2$. 

Figure 10: The profile of $\delta$ function associated to the standard map (left), respectively a TSF map (right) and the value of $\delta$ at mini–maximizing points of the (1,2)–orbit (marked by $\circ$), and minimizing points (marked by $\ast$).
Figure 11: The graphs of the Hill discriminants associated to (1, 2)–periodic orbits of the standard map (as a USF map), corresponding to $\epsilon = 1.3$ (left) respectively of the TSF map (20), corresponding to $\epsilon = 0.8$ (right). These maps have the profile of $\delta$ function illustrated in Fig. 10.

In the case of the TSF map (Fig 11, right) as the perturbation parameter increases, the double eigenvalues of $H'^2_2$ increases, while $\lambda_0(H^2_2)$, $\lambda_1(H^2_2)$ decrease and successively they get zero, without any obstruction.

We conjecture that in the case of an USF map no mini–maximizing $(p, q)$–type orbit of twist number $-1/2$ can bifurcate to an orbit of twist number $\tau \in (-1, -1/2)$, because of a similar obstruction. In this case the only periodic orbits of large absolute twist number are born through a saddle–center bifurcation.

After the discussion concerning the residue and the index of the Hessian matrix associated to periodic orbits given in Example 1 we are tempted to state that among residue, the Morse index, and the twist number of a $(p, q)$–periodic orbit, the twist number is the only numerical quantity that leads to a correct decision whether or not that orbit is one given by Aubry-Mather theory. However this is not the case, because as the following example illustrates, there can exist badly ordered periodic orbits of twist number zero.

**Example 5.** The point $z = (0.5, 0.32302066964897189)$ of the standard map corresponding to $\epsilon = 2.87$ is a $(3, 5)$–regular hyperbolic point, because its Hessian $H_5$ has the eigenvalues:

$$0.04121176201302, \quad 0.53521014626564, \quad 3.88911846604733,$$
$$5.36539881978110, \quad 5.89051890157580$$

Its twist number is zero, but it is not minimizing because it is badly ordered.

The Hessian associated to its pair, $z' = (0.4941763525132936290, 0.3344329777751245200)$, has the eigenvalues:

$$-0.07386797172616, \quad 0.56782907421433, \quad 3.88372973742859,$$
$$5.35939921395639, \quad 5.88621274711297$$

The orbit of $z'$ is also badly ordered. The minimizing and mini–maximizing periodic orbits of the type $(3, 5)$ are respectively:

$$(0.5, 0.792447770847267734), \quad (0, 0.567936349180897305)$$
The action along the orbits of these four periodic points of type \((3, 5)\), in the order of their mention, is respectively:

\[ 0.891078196747568, \quad 0.891083047652180, \quad 0.804181852816563, \quad 0.831286899496951 \]

### 6 Conclusions

Deciphering the structure of the universal covering space of the group \(SL(2, \mathbb{R})\), and exploiting the properties of the translation number of circle maps, we developed a general framework for the study of the twist number of periodic orbits of twist maps.

We developed an algorithm to compute the value of the twist number or an interval containing this value. Applying this algorithm we illustrated that the classical method used in dynamical systems theory, to classify periodic orbits of a twist map only by their residue or the Morse index of the associated Hessian matrix can lead to errors.

We proved that ordered periodic orbits of large absolute twist number are born through a period doubling bifurcation. We also gave examples of periodic orbits born through a saddle–center bifurcation that have large absolute twist number and are unordered.

In order to explain the variation of the twist number in a subclass of standard–like maps we introduced a new tool of study of the dynamical behaviour of these maps, namely the so called 1–cone function. The location of minima of this function, as well as its values along a periodic orbit appears to give valuable information on that orbit.

The examples given in this paper illustrate that a correct characterization of a periodic orbit of a twist map needs the analysis of the residue, the twist number (i.e. the sign of eigenvalues of the Hessian matrix, \(H\), and of its companion matrix \(H^\sim\)), as well as the order properties of that orbit.

The twist number allows a better characterization of the periodic orbits given by Aubry–Mather theory: a minimizing orbit is ordered and has the twist number equal to zero, while a mini-maximizing orbit is also ordered, but its twist number can belong either to the interval \([-1/2, 0]\) or to the interval \((-1, -1/2)\).

The study initiated in this paper allows to investigate the rotational component of chaos, that along with the Lyapunov component can give a deeper insight into the dynamics of twist area preserving maps.

The present study is not exhaustive and implications of the existence of orbits of large twist numbers are to be studied further. We mention here only few problems that deserve to be investigated:

1. What particular properties of a twist area preserving map lead to the birth of badly ordered periodic orbits of large absolute twist number.
2. How invariant manifolds of hyperbolic periodic orbits of absolute twist number greater or equal to 1 intersect with the invariant manifolds of minimizing periodic orbits and how such intersections influence the transport in the phase space or the dynamics in a region of instability.
A. Translation number of circle homeomorphisms induced by matrices in $SL(2, \mathbb{R})$

A.1 Translation number of an orientation preserving circle homeomorphism

Our approach of the twist number of a $q$-periodic orbit of a twist APM exploits some properties of orientation preserving homeomorphisms (OPHs) of the unit circle, and of the translation number of their orbits. For details concerning dynamics of OPHs of the circle we refer to [16, 17, 13].

In the sequel we consider $\mathbb{S}^1$ either as the unit circle, $C$, in $\mathbb{R}^2$ or as the quotient group $\mathbb{R}/\mathbb{Z}$. The two definitions are equivalent through the identification:

$$\theta + \mathbb{Z} \in \mathbb{R}/\mathbb{Z} \leftrightarrow u_\theta = (\cos(2\pi\theta), \sin(2\pi\theta)) \in C \subset \mathbb{R}^2$$  \hspace{1cm} (33)

We denote by $\pi : \mathbb{R} \to \mathbb{S}^1$, the covering projection, $\pi(x) = x \pmod{1}$. A lift of an OPH $g : \mathbb{S}^1 \to \mathbb{S}^1$ is a homeomorphism, $G : \mathbb{R} \to \mathbb{R}$, such that $\pi \circ G = g \circ \pi$. The orientation property of $g$ ensures that $G(x + 1) = G(x) + 1, \forall x \in \mathbb{R}$. Thus the function $\Psi = G - \text{id}_\mathbb{R}$ is 1-periodic and can be written as $\Psi = \psi \circ \pi$, where $\psi : \mathbb{S}^1 \to \mathbb{R}$, $\psi$ is called the displacement map of $g$ associated to $G$. The value of $\psi$ at a point of $\mathbb{S}^1$ measures the amount that point is displaced by $g$ around the circle.

In the sequel we especially use the function $\Psi = G - \text{id}_\mathbb{R}$, and call it the displacement map of $G$.

We denote by $m(G) = \min_{x \in \mathbb{R}} \Psi(x)$, $M(G) = \max_{x \in \mathbb{R}} \Psi(x)$.

Two lifts $G, H$ of $g$ differ by an integer, $H = G + k$, as well as their displacement maps, $\Psi_H = \Psi_G + k, k \in \mathbb{Z}$.

The group $\text{Homeo}^+_+(\mathbb{S}^1)$ of OPHs of the circle has the universal covering group [29]:

$$\tilde{\text{Homeo}}^+_+(\mathbb{S}^1) = \{G : \mathbb{R} \to \mathbb{R} \mid G \text{ strictly increasing continuous map, } G(x+1) = G(x)+1\},$$

i.e. the elements of this groups are lifts of OPHs of $\mathbb{S}^1$. We denote by $\Pi$ the covering projection:

$$\Pi : \tilde{\text{Homeo}}^+_+(\mathbb{S}^1) \to \text{Homeo}^+_+(\mathbb{S}^1), \quad \Pi(G) = g,$$

where $g$ is the unique homeomorphism in $\text{Homeo}^+_+(\mathbb{S}^1)$ such that $g\pi = \pi G$. $\Pi$ is a group homomorphism.

An important role in the qualitative description of the dynamics of OPHs of the circle has the Poincaré translation number.
The translation number of a homeomorphism $G \in \widetilde{\text{Homeo}}^+(S^1)$ is defined by:

$$\tau(G) = \lim_{n \to \infty} \frac{G^n(x) - x}{n}$$

This number exists and is independent on $x$.

For a self-contained presentation we list some properties of the homeomorphisms in $\widetilde{\text{Homeo}}^+(S^1)$ and their translation numbers:

**Proposition A.1** Let $g : S^1 \to S^1$ be an OPH, and $G$ a lift of $g$. Then the following properties hold:

1. The translation number is invariant under the conjugacy in $\widetilde{\text{Homeo}}^+(S^1)$, that is $\tau(H^{-1} \circ G \circ H^{-1}) = \tau(G)$, for every $G, H \in \widetilde{\text{Homeo}}^+(S^1)$.

2. The translation number $\tau : \text{Homeo}^+(S^1) \to \mathbb{R}$ is not a homomorphism of groups, that is, in general

$$\tau(G_2 \circ G_1) \neq \tau(G_2) + \tau(G_1),$$

and its deviation from being a homomorphism is bounded, existing a positive number $D$ such that:

$$|\tau(G_2 \circ G_1) - \tau(G_2) - \tau(G_1)| \leq D, \ \forall \ G_1, G_2 \in \widetilde{\text{Homeo}}^+(S^1).$$

The least constant $D$ with this property is $1$.

3. $\tau(G^n) = n\tau(G)$, for any positive integer $n$.

4. The translation number of the inverse of a homeomorphism $G \in \widetilde{\text{Homeo}}^+(S^1)$ is $\tau(G^{-1}) = -\tau(G)$.

5. The difference between the max and min displacement of a lift $G$ is less than 1, $M(G) - m(G) < 1$.

6. The translation number of a homeomorphism, $G$, belongs to the range of the displacement map, $\tau \in [m(G), M(G)]$.

A.2 The translation number of the circle homeomorphisms defined by matrices in $SL(2, R)$

$SL(2, \mathbb{R})$ is the group of $2 \times 2$ real matrices of determinant equal to 1. It acts on the vector space $\mathbb{R}^2$. This action induces a natural action of the group $SL(2, \mathbb{R})$ on the unit circle, $S^1$, i.e. there is an injective homomorphism $\phi : SL(2, \mathbb{R}) \to \text{Homeo}^+(S^1)$ between $SL(2, \mathbb{R})$ and the group of orientation preserving homeomorphisms of the circle $C = \{u_\theta = (\cos(2\pi \theta), \sin(2\pi \theta)) \in \mathbb{R}^2, \theta \in \mathbb{R}\}$:

$$A \in SL(2, \mathbb{R}) \mapsto g_A \in \text{Homeo}^+(S^1),$$

where $g_A(u_\theta) = \frac{Au_\theta}{||Au_\theta||}$, and $|| \cdot ||$ is the Euclidean norm in $\mathbb{R}^2$. 

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Proof: We denote by $T_{\mathsf{Homeo}^+(S^1)}$, and its universal covering group $\tilde{SL}(2, \mathbb{R})$ with a subgroup of $\mathsf{Homeo}^+(S^1)$. We denote by $\Pi : \tilde{SL}(2, \mathbb{R}) \to SL(2, \mathbb{R})$ the covering projection.

Our aim is to characterize the translation number of the lifts $G \in \tilde{SL}(2, \mathbb{R})$. Its particular properties in comparison to the translation number of a general circle homeomorphism derive from the following:

**Proposition A.2** Any lift $G_A$ of an OPH, $g_A, A \in SL(2, \mathbb{R})$, has the property

$$G_A(x + 1/2) = G_A(x) + 1/2, \forall \ x \in \mathbb{R}$$

(36)

Proof: We denote by $T_\omega, \omega \in \mathbb{R}$, the shift homeomorphism of $\mathbb{R}$, $T_\omega(x) = x + \omega$. The center of the group $SL(2, \mathbb{R})$ is $\{ \pm I_2 \}$. Because each matrix $A$ commutes with $-I_2$, the circle map, $g_A$, commutes with the circle homeomorphism, $h$, defined by:

$$h : \theta + Z \in \mathbb{R}/Z \mapsto \theta + 1/2 + Z \in \mathbb{R}/Z$$

(37)

The lifts of this map are the shifts $T_{(2k+1)/2}, k \in \mathbb{Z}$. Thus a lift $G_A$ commutes with $T_{1/2}$, and this amounts to $G(x + 1/2) = G(x) + 1/2, \forall \ x \in \mathbb{R}$.

Let $\mathsf{Homeo}_h^+(S^1)$ be the subgroup of orientation preserving homeomorphisms of the circle, that commute with the map $h$ defined in (37). Because $SL(2, \mathbb{R}) \subset \mathsf{Homeo}_h^+(S^1)$, we deduce the particularities of the translation number of the homeomorphisms in $\mathsf{Homeo}_h^+(S^1)$ and those properties will be inherited by the translation numbers of lifts in $\tilde{SL}(2, \mathbb{R})$. The key point in deriving these particularities is that $\mathsf{Homeo}_h^+(S^1)$ is a rescaled version of $\mathsf{Homeo}^+(S^1)$. Indeed, let $\varpi : \mathsf{Homeo}^+(S^1) \to \mathsf{Homeo}_h^+(S^1)$ be a map defined by:

$$\varpi(H)(x) = \frac{1}{2} H(2x), x \in \mathbb{R}$$

(38)

$\varpi$ is a monomorphism of groups, and its image is $\mathsf{Homeo}_h^+(S^1)$. Hence each $G \in \mathsf{Homeo}_h^+(S^1)$ is represented by a general homeomorphism $H \in \mathsf{Homeo}^+(S^1), H = \varpi^{-1}(G), H(x) = 2G(x/2)$.

**Proposition A.3** Let $G$ be a homeomorphism in $\mathsf{Homeo}_h^+(S^1)$. Then its translation number is half the translation number of the corresponding rescaled version $H = \varpi^{-1}(G)$: $\tau(G) = \tau(H)/2$.

Proof. From $G(x) = \frac{1}{2} H(2x)$, we get $G^n(x) = \frac{1}{2} H^n(2x)$, and the translation number is:

$$\tau(G) = \frac{1}{2} \lim_{n \to \infty} \frac{H^n(2x) - 2x}{n} = \frac{\tau(H)}{2}$$

From the definition of $\varpi$, the Propositions [A.2, A.3] and the properties of general circle homeomorphisms, we have the following properties of the displacement maps of lifts of circle maps defined by matrices in $SL(2, \mathbb{R})$, relevant to our approach to twist numbers:
Corollary A.1  

a) The displacement map of a lift $G_A \in \widetilde{SL}(2, \mathbb{R})$ is a periodic function of period $1/2$ (Fig. A.2, left).

b) If $\Psi = G_A - \text{id}_\mathbb{R}$ is the displacement map of a lift $G_A \in \widetilde{SL}(2, \mathbb{R})$, and $\Phi$ is the displacement map of the lift $H_A = \varpi^{-1}(G_A)$, then $\Psi(x) = \frac{1}{2} \Phi(2x), \forall x \in \mathbb{R}$ (Fig. A.2).

c) The difference between the max and min displacement of a homeomorphism $G_A \in \widetilde{SL}(2, \mathbb{R})$ is:

$$0 \leq M(G_A) - m(G_A) \leq 1/2$$  \hspace{1cm} (39)

d) The restriction of the translation number $\tau$ to the universal covering space of the group $SL(2, \mathbb{R})$ has the property:

$$|\tau(G_A \circ G_B) - \tau(G_A) - \tau(G_B)| \leq 1/2, \quad \forall G_A, G_B \in \widetilde{SL}(2, \mathbb{R})$$  \hspace{1cm} (40)

B  The structure of the universal covering space of $SL(2, \mathbb{R})$

We define a system of coordinates on the universal cover group $\widetilde{SL}(2, \mathbb{R})$, that allows us to understand and visualize the topology of this group.

We recall that the matrices in $SL(2, \mathbb{R})$ are classified according to their trace. Namely, a matrix $A \in SL(2, \mathbb{R}) \setminus \{\pm I_2\}$ is called regular hyperbolic, inverse hyperbolic, elliptic, positive parabolic, respectively negative parabolic if $\text{tr}(A) > 2$, $\text{tr}(A) < -2$, $|\text{tr}(A)| < 2$, $\text{tr}(A) = 2$, respectively $\text{tr}(A) = -2$.  

Figure 12: The graph of displacement maps associated to two homeomorphisms $G_A, H_A, H_A = \varpi^{-1}(G_A)$.
Each matrix $A \in SL(2, \mathbb{R})$ is $SL(2, \mathbb{R})$–conjugated with one of the normal forms:

$$R_{2\pi \theta} = \begin{pmatrix} \cos 2\pi \theta & -\sin 2\pi \theta \\ \sin 2\pi \theta & \cos 2\pi \theta \end{pmatrix}, \quad \theta \neq 0,$$

$$H^r = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \quad \lambda \in (0, \infty) \setminus \{1\},$$

$$H^i = -H^r$$

$$P^{1,1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad P^{1,-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$P^{-1,1} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad P^{-1,-1} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \quad (41)$$

It is well known that two conjugated matrices have the same trace. The converse is not always true. The matrices within the pairs, $(R_{2\pi \theta}, R_{-2\pi \theta}), (P^{1,1}, P^{1,-1}), (P^{-1,1}, P^{-1,-1})$ for example, have the same trace, but they are not $SL(2, \mathbb{R})$–conjugated.

In the sequel we point out under what conditions a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R})$ is $SL(2, \mathbb{R})$–conjugated to one of the above normal forms.

a) An elliptic matrix, $A \in SL(2, \mathbb{R})$, is $SL(2, \mathbb{R})$–conjugated with a unique matrix $R_{2\pi \theta}$. More precisely according to whether $b - c > 0$ or $b - c < 0$ (conditions that are equivalent respectively to $b > 0$ or $b < 0$), it is conjugated to $R_{2\pi \theta}, \theta \in (-1/2, 0)$ or to $R_{2\pi \theta}, \theta \in (0, 1/2)$ (this property can be checked by a straightforward matricial computation). A matrix with $b = 0$ cannot be elliptic.

Thus $SL(2, \mathbb{R})$ contains two subsets of elliptic type matrices, one consisting in matrices that are $SL(2, \mathbb{R})$–conjugated to a clockwise rotation (or equivalently, they have the entry $b > 0$), and another is formed by matrices conjugated to an anticlockwise rotation (their $b$-entry is negative).

b) If $A \in SL(2, \mathbb{R})$ is regular hyperbolic, respectively inverse hyperbolic, then $A$ is conjugated to an $H^r$, respectively an $H^i$–type matrix.

c) If $A \in SL(2, \mathbb{R})$ is a positive (negative) parabolic matrix, then it is conjugated with the first or the second matrix of the pairs $(P^{1,1}, P^{1,-1}), (P^{-1,1}, P^{-1,-1})$, according to whether $b - c > 0$ or $b - c < 0$.

We can conclude that:

Two hyperbolic matrices are $SL(2, \mathbb{R})$–conjugated iff they have the same trace. Two elliptic, matrices, are $SL(2, \mathbb{R})$–conjugated if and only if they have the same trace and the same sign of the entries $b$, and two parabolic matrices are $SL(2, \mathbb{R})$–conjugated iff they have the same trace and the same sign of the differences $b - c$, associated to each matrix.

Topologically, $SL(2, \mathbb{R})$ is the interior of a three dimensional solid torus $[11]$, pag. 14 i.e. it is homeomorphic to $D \times S^1$, where $D$ is an open disc centered to the origin in $\mathbb{R}^2$. Hence its universal covering group, $SL(2, \mathbb{R})$, is homeomorphic(as well as diffeomorphic) to $D \times \mathbb{R}$. Obviously, $SL(2, \mathbb{R})$ is also diffeomorphic to $\mathbb{R}^3$, and a few such diffeomorphisms are constructed in $[19]$. We define a diffeomorphism that fits best to our purpose of studying
the twist number of orbits in area preserving maps. More precisely we choose a system of coordinates, such that the visual representation of \( \widetilde{SL}(2, \mathbb{R}) \) in this system looks like the Penrose diagram of \( \widetilde{SL}(2, \mathbb{R}) \), regarded as an anti-de-Sitter space \([1]\). Any matrix \( A \in SL(2, \mathbb{R}) \) decomposes uniquely as \( A = RS \) (polar decomposition), where \( R = R_{2\pi \theta} \) is a rotation matrix, and \( S \) is a symmetric positive definite matrix. Let \( D \) be the open disc of radius \( 1/2 \), centered at the origin in \( \mathbb{R}^2 \), and \((\rho, \alpha)\) polar coordinates within this disc. We define \( \chi: D \times \mathbb{R} \to \widetilde{SL}(2, \mathbb{R}) \), such that \( \chi(0, 0, 0) = id \in \widetilde{SL}(2, \mathbb{R}) \), and the polar decomposition of the matrix \( A = \Pi(\chi(\theta, \rho, \alpha)) \) be:

\[
A = R_{2\pi \theta}R_{2\pi \alpha} \begin{bmatrix}
\sqrt{1 - \sin(\pi \rho)} & 0 \\
1 + \sin(\pi \rho) & 0 \\
0 & \sqrt{1 + \sin(\pi \rho)}
\end{bmatrix} R_{-2\pi \alpha}
\]

(42)

One can prove that \( \chi \) is a diffeomorphism. Since the trace of the polar decomposition, \( RS \), is \( \text{trace}(R)\text{trace}(S)/2 \), we get:

\[
\text{trace}(\Pi(\chi(\theta, \rho, \alpha))) = \frac{2 \cos(2\pi \theta)}{\cos(\pi \rho)}
\]

(43)

In the sequel the trace of a homomorphism \( G \in \widetilde{SL}(2, \mathbb{R}) \) is meant as the trace of the corresponding projection, \( \Pi(G) \), onto \( SL(2, \mathbb{R}) \). According to their traces we also call the lifts, elliptic, hyperbolic or parabolic lifts. Denoting by \( \text{tr}: \widetilde{SL}(2, \mathbb{R}) \to \mathbb{R} \) the function trace, the connected components of the level sets \( L_t = \{ \text{tr}(G) = t \}, t \in \mathbb{R} \), are 2-dimensional surfaces and each such a component represents a conjugacy class in \( \widetilde{SL}(2, \mathbb{R}) \) \([9]\).

From (43) it follows that \( \text{tr} \) does not depend on \( \alpha \), and thus each connected component of a level set, \( L_t \), is a surface of revolution about the \( \theta \) axis. This property implies that the relative position of different conjugacy classes can be visualized in any planar section of the infinite solid cylinder, through a plane containing its symmetry axis (Fig.13).

In order to analyze a section in the solid cylinder, we locate the particular lifts, that project to normal forms in \( SL(2, \mathbb{R}) \). We denote by \( T_\omega \in SL(2, \mathbb{R}) \), the shift homeomorphism, defined by \( T_\omega(x) = x + \omega \). \( \Pi(T_\omega) = R_{2\pi \omega} \).

The center of \( SL(2, \mathbb{R}) \), i.e. the set, \( Z(SL(2, \mathbb{R})) \), of the lifts that commutes with any other lift, consists in the set of shifts \( T_{k/2}, k \in \mathbb{Z} \). \( \Pi(T_{k/2}) = \pm I_2 \), according to as \( k \) is even or odd. \( \Pi(\chi(\rho, \alpha, k)) \), \( k \in \mathbb{Z} \) is a regular hyperbolic matrix, and \( \Pi(\chi(\rho, \alpha, (2k + 1)/2)) \), \( k \in \mathbb{Z} \), is an inverse hyperbolic matrix.

The components of the level set \( L_2 \), and \( L_{-2} \) i.e. the set of positive, respectively negative parabolic lifts are cones in \( D \times \mathbb{R} \) with the vertex removed.

Analyzing the regions inside the infinite open solid cylinder, defined by the two families of cones, and the equations of the conjugacy classes surfaces, \( \frac{2 \cos(2\pi \theta)}{\cos(\pi \rho)} = t \), corresponding to \( t \in (-2, 2) \), \( t > 2 \) and \( t < -2 \) we can conclude that:
1. The parabolic cones have the vertices at a shift $T_{k/2}$, $k \in \mathbb{Z}$. More precisely, if $k$ is even (odd), then $T_{k/2}$ is the vertex of a positive (negative) parabolic cone. Each set \{T_{k/2}\} is a conjugacy class.

2. Inside a region bounded by two patches of nearby cones with vertices at $T_{k/2}$, $T_{(k+1)/2}$, $k \in \mathbb{Z}$, lie elliptic lifts. These regions are denoted by $\mathcal{E}_k$. For $k$ even integer, the matrices $\Pi(\mathcal{E}_k)$ have $b < 0$, while for odd $k$ they have $b > 0$.

3. Outside the two sheets of the same cone lie hyperbolic lifts. For $\theta \in \left(\frac{2k - 1}{4}, \frac{2k + 1}{4}\right)$, $k$ an even integer, the points $(\rho, \alpha, \theta) \in D \times \mathbb{R}$, represent regular hyperbolic lifts, while for $k$ odd integer they represent inverse hyperbolic lifts. We denoted by $\mathcal{H}_{2j}$, respectively $\mathcal{H}'_{2j+1}$, $j \in \mathbb{Z}$ the subsets of regular, respectively inverse hyperbolic lifts.

Denoting by $\partial(\mathcal{E}_i, \mathcal{H}_j)$ the boundary between two nearby regions we note that:

4. The cones $\mathcal{P}_{2k}^{1,1} = \partial(\mathcal{E}_{2k}, \mathcal{H}_{2k}) \setminus \{T_k\}$, $k \in \mathbb{Z}$, contain lifts of matrices $SL(2, \mathbb{R})$–conjugated to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, i.e. matrices with $b - c > 0$. The boundary $\mathcal{P}_{2k}^{1,-1} = \partial(\mathcal{E}_{2k}, \mathcal{H}_{2k}) \setminus \{T_k\}$, $k \in \mathbb{Z}$, consists in lifts $G$, whose projections are $SL(2, \mathbb{R})$–conjugated to $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, i.e. matrices with $b - c < 0$.

5. The cones $\mathcal{P}_{2k-1}^{-1,1} = \partial(\mathcal{E}_{2k-1}, \mathcal{H}'_{2k-1}) \setminus \{T_{(2k-1)/2}\}$, $k \in \mathbb{Z}$, contain lifts of matrices $SL(2, \mathbb{R})$–conjugated to $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$, i.e. matrices with $b - c > 0$, while $\mathcal{P}_{2k-1}^{1,-1} =$
\[ \partial(\mathcal{E}_{2k}, \mathcal{H}_{2k+1}^\prime) \setminus \{T_{(2k-1)/2}\}, \ k \in \mathbb{Z}, \] contains lifts of matrices $SL(2, \mathbb{R})$–conjugated to
\[ \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}, \] i.e. matrices with $b - c < 0$.

In the study of twist numbers we are interested in the range, $[m(G), M(G)]$, of the displacement map $\Psi$ of a lift $G$. In order to deduce the position of such intervals for different regions in $\widetilde{SL(2, \mathbb{R})}$, we show how are related the intervals $[m(G), M(G)]$, $[m(H^{-1}GH), M(H^{-1}GH)]$, associated to $SL(2, \mathbb{R})$–conjugated lifts.

**Proposition B.1** The minimum, respectively the maximum displacements, of two $\widetilde{SL(2, \mathbb{R})}$–conjugated homeomorphisms, belong to the same demi-unit interval, i.e. an interval of the form $(k/2, (k + 1)/2)$, for some $k \in \mathbb{Z}$:

Proof:

We give the proof only for the minimum displacement of two $\widetilde{SL(2, \mathbb{R})}$–conjugated homeomorphisms. More precisely we show that:

\[
\max\{\frac{n}{2} \in \mathbb{Z}/2 \mid \frac{n}{2} \leq m(H^{-1}GH)\} = \max\{\frac{n}{2} \in \mathbb{Z}/2 \mid \frac{n}{2} \leq m(G)\}
\]

(44)

and

\[
\min\{\frac{n}{2} \in \mathbb{Z}/2 \mid \frac{n}{2} \geq m(H^{-1}GH)\} = \min\{\frac{n}{2} \in \mathbb{Z}/2 \mid \frac{n}{2} \geq m(G)\}
\]

In order to get the first relation in (44) we prove that for any $G \in \widetilde{SL(2, \mathbb{R})}$ we have:

\[
\max\{\frac{n}{2} \in \mathbb{Z}/2 \mid \frac{n}{2} \leq x' - x\} = \max\{\frac{n}{2} \in \mathbb{Z}/2 \mid \frac{n}{2} \leq G(x') - G(x)\}
\]

(45)

Let $x, x' \in \mathbb{R}$ be such that $n/2 \leq x' - x < (n + 1)/2$, and distinguish two cases: i) $x' = x + n/2$, and ii) $x' = x + \delta, n/2 < \delta < (n + 1)/2, n \in \mathbb{Z}$. In the former case we have $G(x + n/2) = G(x) + n/2$, i.e. $G(x') - G(x) = n/2$, while in the latter one, taking into account that $G$ is increasing, we get:

\[ G(x + n/2) < G(x + \delta) < G(x + (n + 1)/2), \text{i.e. } n/2 < G(x') - G(x) < (n + 1)/2 \]

In relation (45) for the lift $H^{-1}$:

\[
\max\{\frac{n}{2} \in \mathbb{Z}/2 \mid \frac{n}{2} \leq x' - x\} = \max\{\frac{n}{2} \in \mathbb{Z}/2 \mid \frac{n}{2} \leq H^{-1}(x') - H^{-1}(x)\},
\]

we take $y = H^{-1}(x)$, and $x' = G(x) = GH(y)$, and get:

\[
\max\{\frac{n}{2} \in \mathbb{Z}/2 \mid \frac{n}{2} \leq G(H(y)) - H(y)\} = \max\{\frac{n}{2} \in \mathbb{Z}/2 \mid \frac{n}{2} \leq H^{-1}GH(y) - y\},
\]

\[ \forall y \in \mathbb{R}. \] Taking the minimum for $y \in \mathbb{R}$ we get the first relation in (44).

Similarly one proves the second one.

Because each region $\mathcal{E}_k, \mathcal{H}_{2k}, \mathcal{H}_{2k-1}', k \in \mathbb{Z}$ is foliated by conjugacy classes, we have:
Proposition B.2 The displacement intervals for maps $G$ in different regions of the $\tilde{SL}(2,\mathbb{R})$ are located as follows:

a) If $G \in \mathcal{E}_k$, $[m(G), M(G)] \subset \left(\frac{k}{2}, \frac{k+1}{2}\right)$;

b) If $G \in \mathcal{H}_{2k}$, then $k \in (m(G), M(G))$;

c) If $G \in \mathcal{H}_{2k+1}'$, then $\frac{2k+1}{2} \in (m(G), M(G))$;

Proof. a) A conjugacy class in $\mathcal{E}_k$ is represented by the shift $T_\omega$, with $\omega \in \left(\frac{k}{2}, \frac{k+1}{2}\right)$. The displacement map of $T_\omega$ is the constant function $\Psi(x) = \omega$, and thus $m(T_\omega) = M(T_\omega) \in \left(\frac{k}{2}, \frac{k+1}{2}\right)$. From Proposition B.1 we get a).

b) A conjugacy class in $\mathcal{H}_{2k}$ is represented by a lift, $G$, of the normal form, $H^r$ (41), with $m(G) = k - \delta$, $M(G) = k + \delta$, for some $0 < \delta < 1/4$.

c) A conjugacy class in $\mathcal{H}_{2k-1}'$, $k \in \mathbb{Z}$ is represented by a lift $G$ of the normal $H^i$, and $m(G) = \left(2k - 1\right)/2 - \delta$, $M(G) = \left(2k - 1\right)/2 + \delta$, for some $0 < \delta < 1/4$.

Taking into account the dynamical properties of circle maps [17], the properties of the translation numbers stated in A, and that the translation number of a shift $T_\omega$ is $\omega$ we get that, if $G \in \tilde{SL}(2,\mathbb{R})$ is a lift of a circle homeomorphism $g_A$, then the translation number of $G$ is (see also [23], [10]):

1. an integer number, if $G$ is a regular hyperbolic or a positive parabolic lift;

2. a demi-integer, i.e. a number in $(2\mathbb{Z}+1)/2$, if $G$ is an inverse hyperbolic or a negative parabolic lift;

3. a real number different from integers and demi-integers, if $G$ is an elliptic lift. More precisely $\tau(G) = \theta + k$, $k \in \mathbb{Z}$, where $\theta \in (-1/2, 0)$ or $(0, 1/2)$, depending on the sign of $b$ in $\Pi(G)$.

In Fig.13 a) we illustrate the values of translation number function, in each region in the space $SL(2,\mathbb{R})$.

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