A \( q \)-deformation of enriched \( P \)-partitions

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Abstract. We introduce a \( q \)-deformation that generalises in a single framework previous works on classical and enriched \( P \)-partitions. In particular, we build a new family of power series with a parameter \( q \) that interpolates between Gessel’s fundamental \((q = 0)\) and Stembridge’s peak quasisymmetric functions \((q = 1)\) and show that it is a basis of \( \text{QSym} \) when \( q \notin \{-1, 1\} \). Furthermore we build their corresponding monomial bases parametrised with \( q \) that cover our previous work on enriched monomials and the essential quasisymmetric functions of Hoffman.

Résumé. Nous introduisons une \( q \)-déformation qui généralise dans un cadre unique les travaux antérieurs sur les \( P \)-partitions classiques et enrichies. En particulier, nous construisons une famille de séries formelles avec un paramètre \( q \) qui interpole entre les fonctions quasisymétriques fondamentales de Gessel \((q = 0)\) et les fonctions de pic de Stembridge \((q = 1)\) et montrons qu’il s’agit d’une base de \( \text{QSym} \) quand \( q \notin \{-1, 1\} \). De plus, nous construisons leur bases de monômes associées paramétrées par \( q \) qui généralisent nos travaux sur les monômes enrichis et les fonctions essentielles de Hoffman.

Keywords: Quasisymmetric functions, enriched \( P \)-partitions, peak functions.

1 Introduction

Introduced by Stanley in [6], \( P \)-partitions are order preserving maps from a partially ordered set \( P \) to the set of positive integers with many significant applications in algebraic combinatorics. In particular, they are the building block of Gessel’s ring of quasisymmetric functions (\( \text{QSym} \)) in [1]. Replacing positive integers by signed ones, Stembridge introduces in [8] an enriched version of \( P \)-partitions to build the algebra of peaks, a subalgebra of \( \text{QSym} \). The generating functions of classical (enriched) \( P \)-partitions on labelled chains are the fundamental (peak) quasisymmetric functions, an important basis of \( \text{QSym} \) (the algebra of peaks) related to the descent (peak) statistic on permutations. More recently, in [3], we redefine these generating functions on weighted posets to extend their nice properties to the monomial and enriched monomial bases of...
QSym. However the classical and enriched frameworks remained so far separated. We merge them into one via a new $q$-deformation of the generating function for enriched $P$-partitions that interpolates between Gessel’s and Stembridge’s works.

1.1 Posets and enriched $P$-partitions

We recall the main definitions regarding posets and (enriched) $P$-partitions. The reader is referred to [7, 1, 8] for further details.

**Definition 1** (Labelled posets). Let $[n] = \{1, 2, \ldots, n\}$. A labelled poset $P = ([n], <_P)$ is an arbitrary partial order $<_P$ on the set $[n]$.

**Definition 2** ($P$-partition). Let $P = \{1, 2, 3, \ldots\}$ and let $P = ([n], <_P)$ be a labelled poset. A $P$-partition is a map $f : [n] \rightarrow P$ that satisfies the two following conditions:

(i) If $i <_P j$, then $f(i) \leq f(j)$.

(ii) If $i <_P j$ and $i > j$, then $f(i) < f(j)$.

The relations $< and > stand for the classical order on $P$. Let $L_P(P)$ denote the set of $P$-partitions.

**Definition 3** (Enriched $P$-partition). Let $\mathbb{P}^\pm$ be the set of positive and negative integers totally ordered by $-1 < 1 < -2 < 2 < -3 < 3 < \ldots$. We embed $P$ into $\mathbb{P}^\pm$ and let $-P \subseteq \mathbb{P}^\pm$ be the set of all $-n$ for $n \in P$. Given a labelled poset $P = ([n], <_P)$, an enriched $P$-partition is a map $f : [n] \rightarrow \mathbb{P}^\pm$ that satisfies the two following conditions:

(i) If $i <_P j$ and $i < j$, then $f(i) < f(j)$ or $f(i) = f(j) \in P$.

(ii) If $i <_P j$ and $i > j$, then $f(i) < f(j)$ or $f(i) = f(j) \in -P$.

Further, let $L_{P^\pm}(P)$ be the set of enriched $P$-partitions.

Finally recall the weighted variants of posets introduced in [3].

**Definition 4** ([3]). A labelled weighted poset is a triple $P = ([n], <_P, \epsilon)$ where $([n], <_P)$ is a labelled poset and $\epsilon : [n] \rightarrow P$ is a map (called the weight function).

Each node of a labelled weighted poset is marked with its label and weight (Figure 1).

1.2 Quasisymmetric functions

Consider the set of indeterminates $X = \{x_1, x_2, x_3, \ldots\}$, the ring $k[[X]]$ of formal power series on $X$ where $k$ is a commutative ring, and let $Z \in \{\mathbb{P}, \mathbb{P}^\pm\}$. Given a labelled weighted poset $([n], <_P, \epsilon)$, define its generating function $\Gamma_Z(([n], <_P, \epsilon)) \in k[[X]]$ by

$$\Gamma_Z(([n], <_P, \epsilon)) = \sum_{f \in L_Z([n], <_P)} \prod_{1 \leq i \leq n} x^{\epsilon(f(i))}$$  \hspace{2cm} (1.1)
where \( |f(i)| = -f(i) \) (resp. \( f(i) \)) for \( f(i) \in -\mathbb{P} \) (resp. \( \mathbb{P} \)). Let \( S_n \) be the symmetric group on \([n] \). Given a composition, i.e. a sequence of positive integers \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) with \( n \) entries, and a permutation \( \pi = \pi_1 \ldots \pi_n \) of \( S_n \), we let \( P_{\pi, \alpha} = ([n], \alpha, \pi) \) be the labelled weighted poset on the set \([n] \), where the order relation \( \pi_i < \pi_j \) is such that \( \pi_i < \pi \pi_j \) if and only if \( i < j \) and \( \alpha \) is the weight function sending the vertex labelled \( \pi_i \) to \( \alpha_i \) (see Figure 2). For \( Z \in \{ \mathbb{P}, \mathbb{P}^\pm \} \), its generating function \( U^Z_{\pi, \alpha} = \Gamma_Z([n], \alpha, \pi) \) is called the universal quasisymmetric function ([3]) indexed by \( \pi \) and \( \alpha \).

\[
\pi_1, \alpha_1 \rightarrow \pi_2, \alpha_2 \rightarrow \cdots \cdots \rightarrow \pi_n, \alpha_n
\]

**Figure 2:** The labelled weighted poset \( P_{\pi, \alpha} \).

**Definition 5.** Let \([1^n] \) denote the composition with \( n \) entries equal to 1. For each \( \pi \in S_n \), let \( L_\pi = U^\mathbb{P}_{\pi,[1^n]} \) and \( K_\pi = U^\mathbb{P}_{\pi,[1^n]} \). The power series \( L_\pi \) (resp. \( K_\pi \)) are Gessel’s fundamental (resp. Stembridge’s peak) quasisymmetric functions indexed by the permutation \( \pi \).

The power series \( L_\pi \) and \( K_\pi \) belong to the subalgebra of \( k[[X]] \) called the ring of quasisymmetric functions (QSym), i.e. for any strictly increasing sequence of indices \( i_1 < i_2 < \cdots < i_p \) the coefficient of \( x_1^{k_1} x_2^{k_2} \cdots x_p^{k_p} \) is equal to the coefficient of \( x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_p}^{k_p} \).

Furthermore they are related to two major statistics on permutations. Given \( \pi \in S_n \), define its descent set \( \text{Des(\pi)} = \{1 \leq i \leq n - 1 | \pi(i) > \pi(i + 1) \} \) and its peak set \( \text{Peak(\pi)} = \{2 \leq i \leq n - 1 | \pi(i - 1) < \pi(i) > \pi(i + 1) \} \). The peak set of a permutation is peak-lacunar, i.e. it neither contains 1 nor contains two consecutive integers.

**Proposition 1** ([1, 8]). For any permutation \( \pi \in S_n \), the fundamental quasisymmetric function \( L_\pi \) and the peak quasisymmetric function \( K_\pi \) satisfy

\[
L_\pi = \sum_{i_1 \leq \cdots \leq i_n;\ j \in \text{Des(\pi)} \Rightarrow i_j < i_{j+1}} x_{i_1} x_{i_2} \cdots x_{i_n},
\]

\[
K_\pi = \sum_{i_1 \leq \cdots \leq i_n;\ j \in \text{Peak(\pi)} \Rightarrow i_{j-1} < i_{j+1}} 2^{[i_1,i_2,\ldots,i_n]} x_{i_1} x_{i_2} \cdots x_{i_n}.
\]
As a result $L_\pi (K_\pi)$ depends only on $n$ and $\text{Des}(\pi)$ ($\text{Peak}(\pi)$) and we may use set indices and write $L_{n,\text{Des}(\pi)} (K_{n,\text{Peak}(\pi)})$ instead of $L_\pi (K_\pi)$. Furthermore $(L_{n,1})_{n\geq 0,I\subseteq [n-1]}$ is a basis of $\text{QSym}$ (we assume $[-1] = [0] = \emptyset$), and $(K_{n,1})_{n\geq 0,1}$ is a basis of a subalgebra of $\text{QSym}$ called the algebra of peaks when $I$ runs over all peak-lacunar subsets of $[n-1]$ for all integers $n$.

**Definition 6.** Let $\text{id}_n$ and $\text{\overline{id}}_n$ denote the permutations in $S_n$ given by $\text{id}_n = 1 2 3 \ldots n$ and $\text{\overline{id}}_n = n n-1 \ldots 1$. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $n$ entries, define the monomial $M_\alpha$ ([4]), essential $E_\alpha$ ([4]) and enriched monomial $\eta_\alpha$ ([5, 3]) quasisymmetric functions

$$M_\alpha = U_{\text{id}_n,\alpha}^P = \sum_{i_1 < \cdots < i_n} x_{i_1}^{\alpha_1} \cdots x_{i_n}^{\alpha_n}, \quad E_\alpha = U_{\text{id}_n,\alpha}^{p,\pm} = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1}^{\alpha_1} \cdots x_{i_n}^{\alpha_n},$$

$$\eta_\alpha = U_{\text{\overline{id}}_n,\alpha}^{\pm} = \sum_{i_1 \leq \cdots \leq i_n} 2^{|\{i_1, \ldots, i_n\}|} x_{i_1}^{\alpha_1} \cdots x_{i_n}^{\alpha_n}.$$

Compositions $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $\alpha_1 + \cdots + \alpha_n = s$ are in bijection with subsets of $[s-1]$. For $I \subseteq [s-1]$, we also use the following alternative indexing for monomial, essential and enriched monomials. References to $s$ in indices are removed for clarity.

$$M_I = \sum_{j \in I \Rightarrow i_j = i_{j+1}} x_{i_1} \cdots x_{i_s}, \quad E_I = \sum_{j \in I \Rightarrow i_j = i_{j+1}} x_{i_1} \cdots x_{i_s}, \quad \eta_I = \sum_{j \in I \Rightarrow i_j = i_{j+1}} 2^{|\{i_1, \ldots, i_s\}|} x_{i_1} \cdots x_{i_s}.$$

**Proposition 2.** Let $s \geq 0$. Let $I$ and $J$ be a subset and a peak-lacunar subset of $[s-1]$. Then,

$$L_I = \sum_{U \subseteq I} (-1)^{|U|} E_U, \quad K_J = \sum_{V \subseteq J} (-1)^{|V|} \eta_{(V-1) \cup V}, \quad (1.2)$$

where for $V$ peak-lacunar, we set $V - 1 = \{v - 1 \mid v \in V\}$.

## 2 A $q$-deformed generating function for $P$-partitions

Equation (1.1) and Propositions 1 and 2 exhibit the strong similarities between enriched and classical $P$-partitions. As we will see, both are special cases of a more general theory. Looking at Equation (1.1), one may notice that the generating function does not depend on the sign of $f(i)$. Let $\omega$ be the map that sends the element $i$ of a labelled weighted poset $([n], <, \epsilon)$ and an enriched $P$-partition $f$ to the contributing monomial in $\Gamma$. That is, $\omega(i, f) = x_{f(i)}^{e(i)}$. As proposed by Stembridge, the value of $\omega$ does not depend on the sign of $f$. We break this assumption and write for an additional parameter $q$:

$$\omega(i, f, q) = x_{f(i)}^{e(i)} \text{ if } f(i) \in \mathbb{P}, \quad \omega(i, f, q) = qx_{-f(i)}^{e(i)} \text{ if } f(i) \in -\mathbb{P}.$$
Definition 7. Let \( q \in k \) (the base ring of the power series). The \( q \)-generating function for enriched \( P \)-partitions on the weighted poset \( ([n], <_P, \epsilon) \) is

\[
\Gamma_q([n], <_P, \epsilon) = \sum_{f \in \mathcal{L}_{P^\pm}( [n], <_P)} \prod_{1 \leq i \leq n} \omega(i, f, q) = \sum_{f \in \mathcal{L}_{P^\pm}( [n], <_P)} \prod_{1 \leq i \leq n} q^{[f(i) < 0]} x^{e(i)}_{|f(i)|},
\]

(2.1)

where \( [f(i) < 0] = 1 \) if \( f(i) < 0 \) and 0 otherwise.

This definition covers the case of Gessel \( (q = 0) \) with no negative numbers allowed and the one of Stembridge \( (q = 1) \) where the sign of \( f \) is ignored in the generating function. Define also the \( q \)-universal quasisymmetric function

\[
U^q_{\pi, \alpha} = \Gamma_q([n], <_\pi, \alpha).
\]

Proposition 3. Let \( q \in k \), \( \pi \in S_n \) and \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) be a composition with \( n \) entries. Then,

\[
U^q_{\pi, \alpha} = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n, \\ j < \text{Peak}(\pi) \Rightarrow i_j < i_{j+1}}} q^{\{j \in \text{Des}(\pi) | i_j = i_{j+1}\}} (q + 1)^{|\{i_1, i_2, \ldots, i_n\}|} x^{\alpha_1, \alpha_2, \ldots, \alpha_n}_{i_1 \cdot i_2 \cdot \ldots \cdot i_n}.
\]

(2.3)

Proof. Let \( ([n], <_\pi, \alpha) \) be the weighted chain poset associated to \( \pi \in S_n \) and to the composition \( \alpha \) with \( n \) entries. Consider an enriched \( P \)-partition \( f \in \mathcal{L}_{P^\pm}( [n], <_\pi) \) and an \( a \in \mathbb{P} \). All the \( i \in [n] \) satisfying \( |f(\pi_i)| = a \) form an interval \( [j, k] = \{j, j+1, \ldots, k\} \) for some positive integers \( j \) and \( k \). By Definition 3, we have \([j, k] \cap \text{Peak}(\pi) = \emptyset\). As a result, there exists \( l \) such that \( \pi_j > \cdots > \pi_l < \cdots < \pi_k \). We have \( f(\pi_j) = \cdots = f(\pi_{l-1}) = -a \), \( f(\pi_{l+1}) = \cdots = f(\pi_k) = a \) and \( f(\pi_l) \in \{-a, a\} \). The two contributions in \( x_a \) are

\[
x^{\alpha_1, \alpha_2, \ldots, \alpha_k}_{i_1 \cdot i_2 \cdot \ldots \cdot i_n} [q^{l-j} + q^{l-j+1}] = (q + 1)q^{l-j}x^{\alpha_1, \alpha_2, \ldots, \alpha_k}_{i_1 \cdot i_2 \cdot \ldots \cdot i_n}.
\]

Note that \( l - j = \{i \in \text{Des}(\pi) | |f(\pi_i)| = a\} \) to complete the proof. \( \square \)

The nice formula for the product of two generating functions of chain posets extends naturally to this \( q \)-deformation. Recall the definition of coshuffle from [3]:

Definition 8. Let \( \pi \in S_n \) and \( \sigma \in S_m \) be two permutations. Let \( \alpha \) and \( \beta \) be two compositions with \( n \) and \( m \) entries, respectively. The coshuffle of \((\pi, \alpha)\) and \((\sigma, \beta)\), denoted \((\pi, \alpha) \circledast (\sigma, \beta)\), is the set of pairs \((\tau, \gamma)\) where

- \( \tau \in S_{n+m} \) is a shuffle of \( \pi \) and \( n + \sigma = (n + \sigma_1, n + \sigma_2, \ldots, n + \sigma_m) \), and

- \( \gamma \) is a composition with \( n + m \) entries, obtained by shuffling the entries of \( \alpha \) and \( \beta \) using the same shuffle used to build \( \tau \) from the letters of \( \pi \) and \( n + \sigma \).

Example 1. \((132, (2, 1, 2))\) is a coshuffle of \((12, (2, 2))\) and \((1, (1))\).
Proposition 4. Let \( q \in k \), let \( \pi \) and \( \sigma \) be two permutations in \( S_n \) and \( S_m \), and let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_m) \) be two compositions with \( n \) and \( m \) entries. The product of two \( q \)-universal quasisymmetric functions is given by
\[
U^q_{\pi, \alpha} U^q_{\sigma, \beta} = \sum_{(\tau, \gamma) \in (\pi, \alpha) \circ (\sigma, \beta)} U^q_{\tau, \gamma}. \tag{2.4}
\]

Proof. The proof is similar to [3, Thm. 3]. \( \square \)

3 Enriched \( q \)-monomials

3.1 Definition, relation to \( q \)-universal quasisymmetric functions and product formula

We introduce a new basis of QSym that generalises the essential and enriched monomial quasisymmetric functions in Definition 6.

Definition 9 (Enriched \( q \)-monomials). Let \( q \in k \) and \( \alpha \) be a composition with \( n \) entries. The enriched \( q \)-monomial indexed by \( \alpha \) is defined as
\[
\eta^{(q)}_\alpha = U^q_{id_n, \alpha}. \tag{3.1}
\]

As an immediate consequence of Definition 9, one has \( \eta^{(0)}_\alpha = E_\alpha \) and \( \eta^{(1)}_\alpha = \eta_\alpha \).

Proposition 5. Let \( q \in k \), and let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) be a composition with \( n \) entries. Then,
\[
\eta^{(q)}_\alpha = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} (q + 1)^{|\{i_1, i_2, \ldots, i_n\}|} x^{\alpha_1}_{i_1} x^{\alpha_2}_{i_2} \cdots x^{\alpha_n}_{i_n}. \tag{3.2}
\]

Proof. This is a direct consequence of Proposition 3. \( \square \)

Interestingly, one may express general \( q \)-universal quasisymmetric functions in terms of the \( \eta^{(q)}_\alpha \). To state this result we need the following definition.

Definition 10. Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be a composition with \( n \) entries. For any integer \( 1 \leq i \leq n - 1 \), we let \( \alpha^{\downarrow i} \) denote the following composition with \( n - 1 \) entries:
\[
\alpha^{\downarrow i} = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n).
\]

Furthermore, for any subset \( I \subseteq [n - 1] \), we set
\[
\alpha^{\downarrow I} = \left( \cdots (\alpha^{\downarrow i_k}) \cdots \right)^{i_2}_{i_1},
\]
where \( i_1, i_2, \ldots, i_k \) are the elements of \( I \) in increasing order. Finally, if \( I \) and \( J \) are two subsets of \( [n - 1] \), with \( J \) being peak-lacunar, then we set \( \alpha^{\downarrow \downarrow I} = \alpha^{\downarrow K} \), where \( K = I \cup J \cup (J - 1) \).
Thus, the above computation becomes

\[ U_{\pi,\alpha}^q = \sum_{K \subseteq \text{Des}(\pi)} \sum_{J \subseteq \text{Peak}(\pi)} q^{\left| K \right|} (q + 1)^{|I|} x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots x_{i_n}^{a_n} \]

**Proof.** For any subset \( V \subseteq [n - 1] \), set \( \overline{V} = [n - 1] \setminus V \). Then, (2.3) becomes

\[ U_{\pi,\alpha}^q = \sum_{K \subseteq \text{Des}(\pi)} \sum_{J \subseteq \text{Peak}(\pi)} q^{\left| K \right|} (q + 1)^{|I|} x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots x_{i_n}^{a_n} \]

If we set \( I = U \cup K \setminus J \) and \( U' = I \setminus U = K \setminus J \), then \( |U'| = |K| - |J| \) and \( I \subseteq \text{Des}(\pi) \setminus J \). Thus, the above computation becomes

\[ U_{\pi,\alpha}^q = \sum_{U' \subseteq I} \sum_{J \subseteq \text{Peak}(\pi)} q^{\left| U' \right|} (q + 1)^{|I'|} x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots x_{i_n}^{a_n} \]

Summing over \( U' \) yields formula (3.3). \( \square \)

**Corollary 1.** Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_m) \) be two compositions. Let \( \alpha \cup \beta \) be the multiset of compositions obtained by shuffling \( \alpha \) and \( \beta \). As in [3], given \( \gamma \in \alpha \cup \beta \), let \( S_\beta(\gamma) \) be the set of the positions of the entries of \( \beta \) in \( \gamma \). Set furthermore \( S_\beta(\gamma) - 1 = \{ i - 1 \mid i \in S_\beta(\gamma) \} \) and \( T_\beta(\gamma) = S_\beta(\gamma) \setminus (S_\beta(\gamma) - 1) \). Then,

\[ \eta_{\alpha}^{(q)} \eta_{\beta}^{(q)} = \sum_{\gamma \in \alpha \cup \beta; I \subseteq T_\beta(\gamma) \setminus \{n + m\}; J \subseteq T_\beta(\gamma) \setminus \{1, n + m\}; I \cap J = \emptyset} (q - 1)^{|I|} (q - 1)^{|J|} \eta_{\gamma}^{(q)} \eta_{\gamma}^{(q)} \eta_{\gamma}^{(q)} \eta_{\gamma}^{(q)} \eta_{\gamma}^{(q)}. \] (3.4)

**Proof.** Corollary 1 is a consequence of Theorem 1, Equation (3.1) and Proposition 4. \( \square \)

\(^1\)We understand \( i_{j-1} \) to be 0 whenever \( j = 1 \).
3.2 Relation to the monomial and fundamental bases

We consider the alternative indexing with sets proposed at the end of Section 1.2. Given a set of positive integers \( I \subseteq [s-1] \), the enriched \( q \)-monomial may be written as

\[
\eta^{(q)}_I = \sum_{i_1 \leq \cdots \leq i_s} (q + 1)^{|I|} x_{i_1} \cdots x_{i_s}.
\]

**Proposition 6.** Let \( I \subseteq [s-1] \) be a set of positive integers. One has

\[
\eta^{(q)}_I = \sum_{J \subseteq I} (q + 1)^{|J|} M_J.
\]  

(3.5)

**Theorem 2.** Let \( q \in k \) be such that \( q + 1 \) is invertible. The family of enriched \( q \)-monomial quasisymmetric functions \((\eta^{(q)}_{s,I})_{s \geq 0, I \subseteq [s-1]}\) is a basis of \( \text{QSym} \). Furthermore

\[
(q + 1)^{s-|J|} M_J = \sum_{J \subseteq I} (-1)^{|I \setminus J|} \eta^{(q)}_I.
\]  

(3.6)

**Proof.** Follows from Equation (3.5) by Möbius inversion. \( \square \)

We develop further the properties of the enriched \( q \)-monomial basis of \( \text{QSym} \).

**Proposition 7.** Let \( s \) be a positive integer and \( I \subseteq [s-1] \). One may expand the enriched \( q \)-monomials in the fundamental basis as

\[
\eta^{(q)}_I = (q + 1) \sum_{J \subseteq [s-1]} (-1)^{|J|} (-q)^{|I \setminus J|} L_J.
\]  

(3.7)

**Proof.** The expression above is a consequence of Equation (3.5) and the expansion of monomial quasisymmetric functions in the fundamental basis (see e.g. [1]). \( \square \)

**Proposition 8.** Let \( s \) be a positive integer, \( J \subseteq [s-1] \) and let \( q \in k \). Then,

\[
(q + 1)^s L_J = \sum_{I \subseteq [s-1]} (-1)^{|I|} (-q)^{|J \setminus I|} \eta^{(q)}_I.
\]  

(3.8)

Equations (3.7) and (3.8) expand the fundamental and enriched \( q \)-monomial bases in terms of one another, and thus suggest a duality relation between the two. Let \( \text{QSym}_s \) be the vector subspace of \( \text{QSym} \) containing the homogeneous quasisymmetric functions of degree \( s \). Define \( f : \text{QSym}_s \to \text{QSym}_s \) as the \( k \)-linear map that sends each \( L_I \) to \( \eta^{(q)}_I \) for \( I \subseteq [s-1] \). Then \( f^2 \) is a scaling by \((q + 1)^{s+1}\) (that is, \( f^2 = (q + 1)^{s+1} \text{id} \)). Moreover,

\[
f(M_I) = (q + 1)^{|I|+1} M_{[s-1] \setminus I}
\]

for any \( I \subseteq [s-1] \).
3.3 Antipode

For an integer $s$ and a subset $I \subseteq [s-1]$, we set $s-I = \{s-i| i \in I\}$. The antipode of QSym (see [2, Chapter 5]) can be defined as the unique $k$-linear map $S : \text{QSym} \to \text{QSym}$ that satisfies

$$S(M_I) = (-1)^{s-|I|} \sum_{(s-I) \subseteq J} M_J.$$ 

**Proposition 9.** Assume that $q$ is invertible in $k$, and let $p = \frac{1}{q}$. Then, for $I \subseteq [s-1]$,

$$S\left(\eta_I^{(q)}\right) = (-q)^{s-|I|} \eta_{s-I}^{(p)}.$$ \hspace{1cm} (3.9)

**Proof.** This can be derived from Equation (3.7). \hfill \square

4 A $q$-interpolation between Gessel and Stembridge quasisymmetric functions

4.1 $q$-fundamental quasisymmetric functions

We introduce a new family of quasisymmetric functions that interpolate between Gessel’s fundamental and Stembridge peak quasisymmetric functions and show that it is a basis of QSym in all but the Stembridge case.

**Definition 11** ($q$-fundamental quasisymmetric functions). Let $\pi$ be a permutation in $S_n$ and $q \in k$. Define the $q$-fundamental quasisymmetric function indexed by $\text{Des}(\pi)$ as

$$L^{(q)}_{n,\text{Des}(\pi)} = U^{q}_{\pi,[1^n]}.$$ \hspace{1cm} (4.1)

Let $I$ be a subset of $[n-1]$. Set $I+1 = \{i+1| i \in I\}$, and let $\text{Peak}(I) = I \setminus (I+1) \setminus \{1\}$ the peak-lacunar subset obtained from $I$ (so $\text{Peak}(I) = \text{Peak}(\pi)$ for every $\pi \in S_n$ satisfying Des($\pi$) = $I$). One recovers immediately that for $q = 0$, $L^{(0)}_{n,I} = L_{n,I}$ is the Gessel fundamental quasisymmetric function indexed by the set $I$. For $q = 1$, $L^{(1)}_{n,I} = K_{n,\text{Peak}(I)}$ is the Stembridge peak function indexed by the relevant peak-lacunar set. In the sequel we remove the reference to $n$ in indices when it is clear from context. **Proposition 2** admits a nice generalisation to this $q$-deformation.

**Theorem 3.** Let $I \subseteq [n-1]$ and $q \in k$. The $q$-fundamental quasisymmetric functions may be expressed in the enriched $q$-monomial basis as

$$L^{(q)}_I = \sum_{K \subseteq \text{Peak}(I)} (-q)^{|K|} (q-1)^{|I|} \eta_I^{(q)}_{I \cup (K-1) \cup K}.$$ \hspace{1cm} (4.2)
Proof. This a consequence of Equation (3.3). \hfill \Box

Proposition 10. Recall the antipode $S$ of Section 3.3. Let $q \in k$ be invertible, and set $p = \frac{1}{q}$. Let $I \subseteq [n - 1]$, and set $n - I = \{n - i \mid i \in I\}$. Then,

$$S(L_I^{(q)}) = (-q)^n L_{n-I}^{(p)}. \quad (4.3)$$

Proof. This a consequence of Equations (4.2) and (3.9). \hfill \Box

To know whether \( (L_n^{(q)})_{n \geq 0, I \subseteq [n-1]} \) is a basis of QSym for some value of $q$ appears as a natural question. For example, for $n = 3$, we can invert Equation (4.2) as follows:

\begin{itemize}
  \item $\eta^{(q)}_{\emptyset} = L^{(q)}_{\emptyset};$
  \item $(q - 1)\eta^{(q)}_{\{1\}} = L^{(q)}_{\{1\}} - L^{(q)}_{\emptyset};$
  \item $(q - 1)\eta^{(q)}_{\{2\}} = \frac{(q-1)^2}{(q-1)^2+q}(L^{(q)}_{\{2\}} - L^{(q)}_{\emptyset}) + \frac{q}{(q-1)^2+q}(L^{(q)}_{\{1,2\}} - L^{(q)}_{\{1\}});$
  \item $\eta^{(q)}_{\{1,2\}} = \frac{1}{(q-1)^2+q} \left( L^{(q)}_{\{1,2\}} - L^{(q)}_{\{2\}} - L^{(q)}_{\{1\}} + L^{(q)}_{\emptyset} \right).$
\end{itemize}

We see that except for the case of Stembridge $q = 1$ (and the degenerate case $q = -1$), \( (L_{2,I}^{(q)})_{I \subseteq [2]} \) seems to be a basis of QSym. We state one of our main theorems:

**Theorem 4.** Let $k$ be the set $\mathbb{R}$ of real numbers. The family of $q$-fundamental quasisymmetric functions \( (L_n^{(q)})_{n \geq 0, I \subseteq [n-1]} \) is a basis of QSym for $q \notin \{-1, 1\}$.

**Remark 1.** We set $k = \mathbb{R}$ for the sake of simplicity. For a more general field, \( (L_{n,I}^{(q)})_{n,I \subseteq [n-1]} \) is a basis if and only if $q \notin \{\rho^i \mid \rho^k = 1 \text{ for some integer } k > 0\}$.

### 4.2 Proof of Theorem 4

To prove Theorem 4 we characterise the transition matrix between the $q$-fundamental and enriched $q$-monomial quasisymmetric functions and show it is invertible for $q \neq -1, 1$.

**Definition 12.** Let $B_n$ be the transition matrix between \( (L_I^{(q)})_{I \subseteq [n-1]} \) and \( (\eta_J^{(q)})_{J \subseteq [n-1]} \) with coefficients given by Equation (4.2). Columns and rows are indexed by subsets $I$ of $[n - 1]$ sorted in reverse lexicographic order. A subset $I$ is before subset $J$ iff the word obtained by writing the elements of $I$ in decreasing order is before the word obtained from $J$ for the lexicographic order.
Example 2. For \( n = 4 \), let us show the transition matrix \( B_4 \) between \( (L^{(q)}_I)_{I \subseteq [3]} \) and \( (\eta^{(q)}_I)_{I \subseteq [3]} \). The entry at row index \( I \) and column index \( J \) is the coefficient in \( \eta_I^{(q)} \) of \( L_I^{(q)} \) in Equation (4.2).

\[
B_4 = \begin{pmatrix}
\emptyset & \{1\} & \{2\} & \{2, 1\} & \{3\} & \{3, 1\} & \{3, 2\} & \{3, 2, 1\} \\
\emptyset & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\{1\} & 1 & q - 1 & 0 & 0 & 0 & 0 & 0 \\
\{2\} & 1 & 0 & q - 1 & -q & 0 & 0 & 0 \\
\{2, 1\} & 1 & q - 1 & q - 1 & (q - 1)^2 & 0 & 0 & 0 \\
\{3\} & 1 & 0 & 0 & 0 & q - 1 & 0 & -q \\
\{3, 1\} & 1 & q - 1 & 0 & 0 & q - 1 & (q - 1)^2 & -q \\
\{3, 2\} & 1 & 0 & q - 1 & -q & q - 1 & 0 & (q - 1)^2 \\
\{3, 2, 1\} & 1 & q - 1 & q - 1 & (q - 1)^2 & q - 1 & (q - 1)^2 & (q - 1)^3
\end{pmatrix}
\]

Using Definition 12 and Equation (4.2), one can deduce the following lemmas.

Lemma 1. The matrix \( B_n \) is block triangular. To be more specific:

For each \( k \in [n] \), let \( A_k \) denote the transition matrix from \( (L^{(q)}_I)_{I \subseteq [n-1]} \) to \( (\eta^{(q)}_I)_{I \subseteq [n-1]} \) (where \( \max(\emptyset) := 0 \)); this actually does not depend on \( n \). Note that \( A_k \) is a \( 2^{k-2} \times 2^{k-2} \)-matrix if \( k \geq 2 \), whereas \( A_1 \) is a \( 1 \times 1 \)-matrix. We have

\[
B_n = \begin{pmatrix}
A_1 & 0 & 0 & \ldots & 0 \\
* & A_2 & 0 & \ldots & 0 \\
* & * & A_3 & \ldots & 0 \\
* & * & * & \ldots & 0 \\
* & * & * & * & A_n
\end{pmatrix}.
\]

Lemma 2. The matrices \( (B_n)_n \) and \( (A_n)_n \) satisfy the following recurrence relations (for \( n \geq 1 \) and \( n \geq 2 \), respectively):

\[
B_n = \begin{pmatrix}
B_{n-1} & 0 \\
B_{n-1} & A_n
\end{pmatrix}, \quad A_n = \begin{pmatrix}
(q - 1)B_{n-2} & -qB_{n-2} \\
(q - 1)B_{n-2} & (q - 1)A_{n-1}
\end{pmatrix}.
\]

Thanks to Lemmas 1 and 2, we are ready to state and show the main proposition of this section and prove Theorem 4.

Proposition 11. The matrix \( B_n \) is invertible for \( q \neq 1 \).

Proof. For any square matrix \( M \), let \( |M| \) denote its determinant. We want to show that for all \( n \), \( |B_n| \neq 0 \) or equivalently that \( |A_n| \neq 0 \). To this end we compute for any rational functions in \( q, \alpha \) and \( \beta \):

\[
|\alpha A_n + \beta B_{n-1}| = ((q - 1)\alpha + \beta)|B_{n-2}||((q - 1)\alpha + \beta)A_{n-1} + q\alpha B_{n-2}|. \quad (4.4)
\]
Equation (4.4) exhibits a recurrence relation on the determinants that we solve by defining the sequence of coefficients:

\[
\begin{pmatrix}
\alpha_0 \\
\beta_0
\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix}
\alpha_i \\
\beta_i
\end{pmatrix} = \begin{pmatrix} q - 1 & 1 \\ q & 0 \end{pmatrix} \begin{pmatrix}
\alpha_{i-1} \\
\beta_{i-1}
\end{pmatrix} = \begin{pmatrix} q - 1 & 1 \\ q & 0 \end{pmatrix}^i \begin{pmatrix}
\alpha_0 \\
\beta_0
\end{pmatrix}.
\]

We have:

\[
|A_n| = \prod_{i=0}^{n-3} |B_{n-2-i}| (q - 1 \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}) \begin{pmatrix} A_2 & B_1 \end{pmatrix} \begin{pmatrix}
\alpha_{n-2} \\
\beta_{n-2}
\end{pmatrix}.
\]

But \( A_2 = (q - 1), B_1 = (1) \) and one may compute that (left to the reader):

\[
(q - 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = \frac{1}{q+1} \left( q^{i+2} - (-1)^{i+2} \right) = (-1)^{(i+1)}[i+2]_{-q},
\]

where for integer \( p \), \([p]_q\) is the \( q\)-\textit{number}, \([p]_q = 1 + q + q^2 + \cdots + q^{p-1}\). Define the \( q\)-\textit{factorial} \([p]_q! = [1]_q \cdot [2]_q \cdots [p]_q\). We find

\[
|A_n| = (-1)^{n(n-1)/2} [n]_{-q}! \prod_{i=1}^{n-2} |B_i|.
\]

Then, notice that \([n]_{-q}!\) is 0 if and only if \( q = 1 \) and \( n > 1 \) (when \( q \) runs over real numbers). Finish the proof with a simple recurrence argument on \(|B_i|\). 

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