Long-range multi-body interactions and three-body anti-blockade in a trapped Rydberg ion chain

Filippo M. Gambetta,1,2 Chi Zhang,3 Markus Hennrich,3 Igor Lesanovsky,1,2,4 and Weibin Li1,2

1School of Physics and Astronomy, University of Nottingham, Nottingham, NG7 2RD, United Kingdom
2Centre for the Mathematics and Theoretical Physics of Quantum Non-equilibrium Systems, University of Nottingham, Nottingham NG7 2RD, United Kingdom
3Department of Physics, Stockholm University, 10691 Stockholm, Sweden
4Institut für Theoretische Physik, University of Tübingen, 72076 Tübingen, Germany

(Dated: July 10, 2020)

Trapped Rydberg ions represent a flexible platform for quantum simulation and information processing which combines a high degree of control over electronic and vibrational degrees of freedom. The possibility to individually excite ions to high-lying Rydberg levels provides a system where strong interactions between pairs of excited ions can be engineered and tuned via external laser fields. We show that the coupling between Rydberg pair interactions and collective motional modes gives rise to effective long-range and multi-body interactions, consisting of two, three, and four-body terms. Their shape, strength, and range can be controlled via the ion trap parameters and strongly depends on both the equilibrium configuration and vibrational modes of the ion crystal. By focusing on an experimentally feasible quasi one-dimensional setup of 88Sr+ Rydberg ions, we demonstrate that multi-body interactions are enhanced by the emergence of soft modes associated, e.g., with a structural phase transition. This has a striking impact on many-body electronic states and results, for example, in a three-body anti-blockade effect which can be employed as a sensitive probe to detect structural phase transitions in Rydberg ion chains. Our study unveils the possibilities offered by trapped Rydberg ions for studying exotic phases of matter and quantum dynamics driven by enhanced multi-body interactions.

Introduction.— The coupling between internal atomic states and collective vibrational modes is the hallmark of trapped ion setups. The possibility to engineer phonon-mediated two-body interactions, which can be tuned via laser fields and trapping parameters, combined with single-ion control and high fidelity state preparation, makes them a powerful platform for quantum simulation and information processing [1–14]. A further enhancement of this setup can be achieved in trapped Rydberg ions, where each ion can be individually excited to a high-lying Rydberg level [15–23]. The strong dipole-dipole interactions and the interplay between electronic and vibrational degrees of freedom characterizing this system can be exploited to simulate equilibrium and non-equilibrium quantum many-body spin models [24–26], to devise non-classical motional states [27], and for quantum information processing beyond the scalability limitations of conventional ion settings [28–30].

In this work we demonstrate that the unique intertwining between intrinsically collective vibrational motion and dipole-dipole interactions characterizing trapped Rydberg ions provides a mechanism to engineer long-range and multi-body interactions in state-of-the-art setups. With respect to their neutral counterparts, Rydberg ions offer important experimental advantages. They can be conveniently trapped via state-independent electric potentials and, thus, do not require magic trapping conditions [31–33], while the enhanced control over both electronic and vibrational degrees of freedom allows to manipulate their state with an unprecedented degree of fidelity. We investigate the emergence of multi-body interactions by focusing on a chain of 88Sr+ Rydberg ions confined by harmonic potentials [see Fig. 1(a)], which has been recently experimentally realized [30]. We demonstrate that their strength is significantly enhanced in the presence of soft vibrational modes. These occur, e.g., at the linear-to-zigzag transition in a chain of few ions [34–40] and also in long linear chains. Here, interactions induced by spin-phonon coupling give rise to non-trivial many-body phenomena, such as a three-body anti-blockade effect. The novel capabilities we unveil in our work show that trapped Rydberg ions are a powerful platform for quantum simulation, allowing for the study of exotic kinetically constrained dynamics [41, 42], long-lived quantum information storage [43] and correlated quantum states of matter [44–51].

Spin-phonon coupling induced multi-body interactions.— We consider a quasi one-dimensional chain of N two-level Rydberg ions confined by a harmonic potential, as sketched in

![Fig. 1. Setup and phonon eigenfrequencies of a three-ion crystal.](image-url)
H = H_{\text{ions}} + H_L + H_{\text{int}}, \quad (1)

with $H_{\text{ions}} = \sum_{l,h} \hbar \Omega_{l,h} (a_{l,h}^\dagger a_{l,h} + 1/2)$ describing the vibrational dynamics of an ion crystal confined in a three-dimensional harmonic potential $V_{R}(r_{n}) = M \omega_n^2 r_{n}^2/2$. Here, $\omega_n$ is the $n$-th ion position, $M$ its mass, and $\Omega_l$ the trapping frequency along direction $\mu = \{x,y,z\}$ [40, 52, 53]. Bosonic operators $a_{l,h}^{(t)}$ are associated with the phonon mode $(l,\lambda)$ with eigenfrequency $\Omega_{l,\lambda}$, where $\lambda$ labels the phonon branches [54]. We assume $\omega_{x,y} \ll \omega_z$, so that the motion of the ions is effectively confined to the $x$-$y$ plane.

In Eq. (1), $H_L = \sum_{n,m} \left[ \Omega_{0} r_n^2 + \Delta n_0 \right]$ describes the laser excitation of ions to the Rydberg state. Here, $n_0 = (1 + \sigma_n^z)/2$ and $\sigma_n^\mu$ are the Rydberg number operator and Pauli matrices acting on the $n$-th ion, respectively. The electrostatic dipole-dipole interaction between pairs of Rydberg excited ions is modeled by $H_{\text{int}} = \sum_{m,n} V_R(r_{m},r_{n}) n_m n_n$, where $V_R(r_{m},r_{n}) = V_R[(r_{m} - r_{n})]$ and the prime denotes $m \neq n$. Under typical experimental conditions, the displacements of ions from their equilibrium positions $r_{0}^n$ are much smaller than inter-ion distances. Hence, we expand the two-ion potential $V_R(r_{m},r_{n})$ to first order around $r_0^n [33, 54]$. By substituting the expansion into Eq. (1) and performing a polaron transformation to approximately decouple electronic and vibrational degrees of freedom [5, 6, 33, 54], the full Hamiltonian becomes

$$H' = H_{\text{ions}} + H_{\text{spin}} + H_{\text{res}}. \quad (2)$$

The spin Hamiltonian $H_{\text{spin}} = H_L + H_{\text{int}}^{0} + H_{\text{eff}}^{\text{int}}$ contains, in addition to the bare dipole-dipole-interaction term $H_{\text{int}}^{0} = \sum_{m,n} V_R(r_{0}^m,r_{0}^n) n_m n_n$, also an effective Rydberg interaction contribution

$$H_{\text{eff}}^{\text{int}} = - \sum_{m,n} \sum_{i,j} V_{mnij} n_m n_n n_i n_j. \quad (3)$$

generated as a consequence of the polaron transformation. The latter consists of long-range and multi-body interactions coupling two, three, and four spins. Their strength is encoded in the interaction coefficients

$$V_{mnij} = \frac{G_{mn} G_{ij}}{M} \sum_{\mu, \nu} F_{\mu \nu}^{m} \bar{R}_{mn,\mu} \bar{R}_{ij,\nu}, \quad (4)$$

where we defined $V_{\mu,\nu} = V_R(r_{m},r_{0}^\mu) r_{n}^\nu$, with coefficients $G_{mn}$ describing the magnitude of the gradient of the Rydberg potential and factors $\bar{R}_{mn} = (r_{0}^m - r_{0}^n)/|r_{0}^m - r_{0}^n|$ encoding the geometry of the system. Effective interactions explicitly depend on the trapping regime via the coupling parameters $F_{\mu \nu}^{m} = \sum_{l,h} \bar{\Omega}_{l,h}^{\mu \nu} M_{l,h}^{\mu} M_{l,h}^{\nu}$, where the normal mode matrices $M_{l,h}^{\mu}$ relate local ion displacements to chain normal modes [54]. Hence, the coefficients $V_{mnij}$ can be controlled by both the Rydberg interaction potential, through its gradient coefficients $G_{mn}$, and the vibrational structure of the chain. Finally, in Eq. (2), $H_{\text{res}}$ contains a residual spin-phonon interaction [5, 6, 54]. In the strong interaction regime, which will be the focus of next sections, its contribution to the spin dynamics is negligible when $\Omega \ll \Omega' \sim \text{min}(\Omega_{l,\lambda}^{1/2}) [54]$. In this case, the electronic and vibrational degrees of freedom decouple.

Three-ion chain.— We first investigate the onset of effective multi-body interactions in a minimal setting of three ions. In this case $H_{\text{eff}}^{\text{int}}$ can be explicitly written as

$$H_{\text{eff}}^{\text{int}} = -C_{2b/n}^{2b} (n_1 n_2 + n_2 n_3) - C_{3b/n}^{2b} n_1 n_3 - C_{3b}^{2b} n_1 n_2 n_3. \quad (5)$$

Here, $C_{2b/n}^{2b}$ and $C_{3b/n}^{2b}$ parameterize effective two-body interactions between nearest neighbors (NNs) and next-to-nearest neighbors (NNNs), respectively, while $C_{3b}^{2b}$ describes the three-body contribution. Their behavior in the various trapping regimes can be inferred from Eq. (4). A three-ion chain features a second-order phase transition to a zigzag configuration at the critical value $\alpha^* = \sqrt{12}/5$ of the trap aspect ratio $\alpha = \omega_z/\omega_x$ [40, 54]. The transition is signaled by the emergence of a soft mode with eigenfrequency $\Omega_{l,2} \rightarrow 0$ [see Fig. 1(b)] which, depending on the configuration of the ions, may strongly affect the effective interactions. In the linear regime ($\alpha > \alpha^*$) the longitudinal and transverse modes of the chain are not coupled and the normal mode matrices $M_{l}^{m\lambda}$ vanish when $\mu \neq \lambda$ [54]. In this case, the soft mode is purely transverse and, since $\bar{R}_{mnl} \approx r_{0}^m - r_{0}^n = 0$, it does not contribute to $V_{mnij}$. This traces back to the fact that transverse displacements hardly affect inter-ion distances and only generate higher-order terms in the expansion of the Rydberg interaction potential. Thus, in a linear chain only longitudinal modes contribute to $V_{mnij}$ via the coupling parameter $F_{\mu \nu}^{xx}$. As show in Fig. 1(b), the latter are constant as a function of $\alpha$ and only depends on $\omega_x$ as $F_{\mu \nu}^{xx} \sim \omega_x^{-2}$. We note that, for $\alpha > \alpha^*$, $\omega_x$ uniquely determines the distances between ions, which scale as $\omega_x^{-2/3}$ [54]. Since for dipole-dipole interaction potentials larger distances result in smaller gradients, in a three-ion chain it is not possible to arbitrarily “soften” the longitudinal modes and, thus, the magnitude of the effective interactions that can be achieved is strongly limited. In the zigzag configuration ($\alpha < \alpha^*$), on the contrary, all normal modes possess both longitudinal and transverse components, i.e. $M_{l}^{m\lambda} \neq 0$, $\forall \mu, \lambda$ [54]. As shown in Fig. 2(a), the collective and intertwined nature of phonon modes results in non-vanishing coupling parameters $F_{\mu \nu}^{xx}$, $\forall \mu, \nu$. Moreover, $\bar{R}_{mn\nu} \neq 0$ and hence all the couplings $F_{\mu \nu}^{xx}$ contribute to Eq. (4). Crucially, due to the presence of the soft mode (3, 2), for $\alpha \rightarrow (\alpha^*)^{-1}$ one finds $F_{\mu \nu}^{xx} \sim \Omega_{l,2} \chi^2, \forall \mu, \nu$. We therefore expect that the presence of such a soft mode results in an increase of the effective interaction strength.

Looking at Eq. (4), one notices that large Rydberg potential gradients $G_{mn}$ are essential to maximize effects of the effective multi-body interactions. Unfortunately, in trapped Rydberg ions van der Waals interactions are generally weaker than their neutral counterparts and do not give access to large gradients. To overcome this issue, we exploit the interplay between dipole-dipole interactions of microwave (MW) dressed...
Rydberg interactions between NNs and NNNs contained in interactions \([55–59]\). By denoting with \(V\) facilitation mechanism in the presence of two-body Rydberg anti-blockade regime, a generalization of the well-studied pulsive three-body effective interactions \[\text{see Fig. 2(d)}\]. implying a competition between attractive two-body and re-

If ions are prepared in state \(|↓↓↓⟩\) at time \(t = 0\), an enhancement in the projector on state \(|↑↑↑⟩\) at subsequent times, \(P_{↑↑↑}(t)\), is expected for values of \(\Delta\) satisfying Eq. (6). The behavior of the time-integrated expectation value of \(P_{↑↑↑}(t)\), denoted by \(\langle P_{↑↑↑}\rangle\), is shown in Fig. 3(b, c). Panel (b) shows the case with bare Rydberg interactions only (i.e., with \(C_{\text{NNN}}^2 = C_{\text{NN}}^2 = C_{\text{NNNN}}^3 = 0\), while effects of multi-body interactions are displayed in panel (c). For \(\alpha \rightarrow (\alpha^+)\), effective interactions modify significantly the value of \(\Delta\) satisfying Eq. (6). This results in a shift of the peak of \(\langle P_{↑↑↑}\rangle\). As can be seen in Fig. 3(d), showing the difference \(\delta\langle P_{↑↑↑}\rangle\) between the time-integrated expectation values of \(P_{↑↑↑}(t)\) with and without the contributions of \(H_{\text{eff}}\), the presence of phonon-mediated multi-body interactions leads to a clear spectroscopic signature.

The latter provides a sensitive tool to locate the critical point of the linear-to-zigzag transition, allowing for a significant improvement over state-of-the-art methods \([38, 60, 61]\). Indeed, the typical resolution of current direct camera images of the ions is \(\approx 0.5\ \mu\text{m}\) and, being the transition a second order one, they can hardly reveal the small ion displacements along the traversal direction for \(\alpha \approx \alpha^*\). In contrast, from Fig. 3(c) we see that close to the critical point a traverse displacement of \(\approx 0.1\ \mu\text{m}\) corresponds to a shift of \(\approx 0.2\ \text{MHz}\) in the peak of \(\langle P_{↑↑↑}\rangle\), which can be easily detected via spectroscopic measurements \([62]\).

**Finite linear chain.**— In the previous section we showed that in a three-ion chain spin-phonon coupling induced interactions are strongly amplified by the emergence of a soft mode. We now inspect the behavior of \(H_{\text{eff}}\) in longer chains,
where system properties explicitly depend on the number of ions, \( N \) [36, 37, 63]. In particular, by increasing \( N \), one can decrease the inter-ion distance in the central region of the chain even in the presence of a weak longitudinal confinement. This fact can be exploited to engineer soft modes even in the linear configuration and, hence, it allows to overcome the restrictions on the strength of effective interactions we discussed for a linear three-ion chain. As we will show, the increased flexibility provided by a denser vibrational spectrum also provides a convenient handle to control the range of the effective interactions.

To gain insights into the phenomenology of this case, we consider the infinite chain limit \( N \to \infty \), which provides a good description of the central region of long yet finite chains [39, 64]. In the linear regime [40], the equilibrium positions of the ions are \( r_n^0 = (n\ell, 0) \), with \( \ell \) being the fixed inter-ion distance and \( n \in \{0, \pm 1, \ldots, \pm \infty\} \). To mimic the effect of a longitudinal confinement, we replace the harmonic trapping potential along the \( x \) axis with a periodic one commensurate with the lattice spacing, \( v_x(r_{n,x}) = -M\omega_x(d/2\pi)^2 \cos(2\pi r_{n,x}/d) \). Expanding ions’ coordinates in Fourier modes and generalizing the steps leading to Eq. (2), we obtain a Hamiltonian \( H' \) with the same form as Eq. (2) [54].

In analogy with the three-ion case, in the linear configuration longitudinal and transverse modes are not coupled (i.e., \( F_{mm'}^{\mu \nu} = 0 \) for \( \mu \neq \nu \) and \( F_{mm}^{\mu \mu} = 0 \). Thus, only longitudinal modes contribute to \( \tilde{V}_{mnj} \) via \( F_{mj}^x \equiv F(j) = (2\pi)^{-1} \int_{-\pi}^{\pi} dk \left[ \Omega_x(k) \right]^{-2} e^{-ikx} \), with \( s = m - i, -\pi \leq k < \pi \) defining the wavevector of the first Brillouin zone, and \( \Omega_x(k) \) the eigenfrequency of phonon mode \((k, x)\) [54].

The dense vibrational spectrum leads to two different regimes for the ion dynamics which can be controlled via \( \eta_x \) genuinely collective excitations. In the stiff limit, and four-body terms. On the other hand, for \( \eta_x \leq 1 \) (soft limit), ions behave as independent harmonic oscillators, while for \( \eta_x \approx 0.1 \) (stiff regime), \( \eta_x = 1 \) (intermediate regime), \( \eta_x = 10 \) (soft regime).

Three-body spectroscopy of a long chain.— The previous discussion allows to gain an understanding of the three-body spectroscopy of a long yet finite chain, which can be experimentally investigated in trapped Rydberg ion simulators. Indeed, in a long enough chain, long wavelength soft modes, which give the largest contribution to \( H_{\text{eff}}^3 \), coincide with good approximation with the ones of the corresponding infinite chain limit [39, 64]. Moreover, due to the finite lifetime of Rydberg excitations \( \tau = \gamma^{-1} \), with \( \gamma \) the spontaneous decay rate [see Fig. 1(a)], the vibrational spectrum of the chain can be considered as continuous when energy gaps between phonon modes are smaller than \( \gamma \). For \(^{88}\text{Sr}^+\) ions (with \( \tau \approx 30 \mu s \)) in a trap with \( \omega_x = 2\pi \times 0.2 \text{ MHz} \), the above conditions are both met for chains with \( N \geq 20 \), which are within the reach of current state-of-the-art setups [30]. The possibility to control inter-ion distances in the central region of a long chain by tuning \( N \) allows one to employ a weaker longitudinal confinement, which results in the emergence of soft modes \( \Omega_x(k) \sim \omega_x \) near \( k \approx 0 \). Thus, an enhancement of the effective interactions can be obtained even in the linear configuration. To quantify this effect, in Fig. 4(b) we plot the ratio between effective two- and three-body coefficients for an infinite chain and the corresponding ones for the three-ion setup shown in Fig. 2(c). To make the comparison meaningful, we fix the inter-ion distance in the infinite chain as \( d = d_{\text{NN}} \), with \( d_{\text{NN}} \) given in Fig. 2(c). When the infinite chain trapping frequency, \( \omega_x \), is varied, \( d \) can be kept fixed by adjusting \( N \). In principle, by adding more ions to the chain while keeping \( \omega_x \) constant, smaller values of \( d \) can be accessed, leading to stronger effective interactions. Looking at Fig. 4(b), we also note that in an infinite chain \( C_{\text{NN}}^{2b} \) and \( C_{\text{NN}}^{3b} \) have opposite sign. This fact, due to the different vibrational structure, allows to investigate the competition between attractive and repulsive effective interactions. We therefore expect that trapped Rydberg ions will give access to different interaction regimes, paving the way to the study of quantum magnetism and frustration phenomena in the presence of exotic multi-body effects.

Conclusions.— We investigated the emergence of long-range multi-body interactions in a chain of trapped Rydberg ions induced by the coupling between phonon modes and ion-ion Rydberg interactions. We showed that these interactions are extremely sensitive to the chain equilibrium configuration and vibrational regimes, such as the emergence of soft modes. By employing realistic parameters from a state-of-the-art setup of \(^{88}\text{Sr}^+\) Rydberg ions, we demonstrated that they result in a neat signature of the linear-to-zigzag transition in ion Ryd-
ber state spectroscopic signal. The intertwining between chain configuration, vibrational structure, and effective interactions illustrated in this work provides a versatile mechanism to investigate quantum dynamics in the presence of non-trivial multi-body interactions and exotic constraints.

ACKNOWLEDGMENTS

The research leading to these results has received funding from the EPSRC Grant No. EP/M014266/1, the EPSRC Grant No. EP/R043402X/1 via the QuantERA project “ERyQSenS”, and the Deutsche Forschungsgemeinschaft (DFG) within the SPP 1929 Giant interactions in Rydberg Systems (GiRyd). CZ and MH acknowledge support from the Swedish Research Council (TRIQS), the QuantERA ERA-NET Cofund in Quantum Technologies (ERyQSenS), and the Knut & Alice Wallenberg Foundation (WACQT). CZ acknowledges the hospitality of the University of Nottingham. IL acknowledges funding from the “Wissenschaftler Rückkehrprogramm GSO/CZS” of the Carl-Zeiss-Stiftung and the German Scholars Organization e.V., as well as through The Leverhulme Trust [Grant No. RPG-2018-181]. WL acknowledges funding from the UKIERI-UGC Thematic Partnership No. IN/CONT/G/16-177/3, and the Royal Society through the International Exchanges Cost Share award No. IEC/NSFC/181078.
6

[46] K. P. Schmidt, J. Dorier, and A. M. Läuchli, Phys. Rev. Lett. 101, 150405 (2008).
[47] J. K. Pachos and M. B. Plenio, Phys. Rev. Lett. 93, 056402 (2004).
[48] B. Capogrosso-Sansone, S. Wessel, H. P. Büchler, P. Zoller, and G. Pupillo, Phys. Rev. B 79, 020503(R) (2009).
[49] H. P. Büchler, A. Micheli, and P. Zoller, Nat. Phys. 3, 726 (2007).
[50] L. Bonnes, H. Büchler, and S. Wessel, New J. Phys. 12, 053027 (2010).
[51] S. Will, T. Best, U. Schneider, L. Hackermüller, D.-S. Lühmann, and I. Bloch, Nature 465, 197 (2010).
[52] D. F. V. James, Appl. Phys. B 66, 181 (1998).
[53] D. Kielpinski, B. E. King, C. J. Myatt, C. A. Sackett, Q. A. Turchette, W. M. Itano, C. Monroe, D. J. Wineland, and W. H. Zurek, Phys. Rev. A 61, 032310 (2000).
[54] See Supplemental Material, which contains Ref. [55].
[55] C. Ates, T. Pohl, T. Pattard, and J. M. Rost, Phys. Rev. Lett. 98, 023002 (2007).
[56] I. Lesanovsky and J. P. Garrahan, Phys. Rev. A 90, 011603(R) (2014).
[57] M. M. Valado, C. Simonelli, M. D. Hoogerland, I. Lesanovsky, J. P. Garrahan, E. Arimondo, D. Ciampini, and O. Morsch, Phys. Rev. A 93, 040701(R) (2016).
[58] M. Marcuzzi, J. Minár, D. Barredo, S. de Léséleuc, H. Labuhn, T. Lahaye, A. Browaeys, E. Levi, and I. Lesanovsky, Phys. Rev. Lett. 118, 063606 (2017).
[59] M. Ostmann, M. Marcuzzi, J. P. Garrahan, and I. Lesanovsky, Phys. Rev. A 99, 060101(R) (2019).
[60] Z.-X. Gong, G.-D. Lin, and L.-M. Duan, Phys. Rev. Lett. 105, 265703 (2010).
[61] J. Zhang, G. Pagano, P. W. Hess, A. Kyprianidis, P. Becker, H. Kaplan, A. V. Gorshkov, Z. X. Gong, and C. Monroe, Nature 551, 601 (2017).
[62] The divergence of the shift for $\alpha \rightarrow (\alpha^*)^-$ is an artifact caused by the divergence of the characteristic length associated with the soft mode, $r_s$, which leads to the breakdown of the small displacement approximation [see Eq. (S14) of Ref. [54]]. For $r_{1,y} \approx 0.1 \mu m$ one finds $\ell_{\text{soft}}/d_{\text{NN}} \approx 0.01$, which is well inside the validity region of the expansion.
[63] L. L. Yan, W. Wan, L. Chen, F. Zhou, S. J. Gong, X. Tong, and M. Feng, Sci. Rep. 6, 21547 (2016).
[64] G. Morigni and S. Fishman, Phys. Rev. Lett. 93, 170602 (2004).
[65] J. Sólyom, Fundamentals of the Physics of Solids, Vol. 1 (Springer, 2007).