Quantum filtering for multiple diffusive and Poissonian measurements

Muhammad F Emzir, Matthew J Woolley and Ian R Petersen

School of Engineering and Information Technology, University of New South Wales, ADFA, Canberra, ACT 2600, Australia

E-mail: m.emzir@student.adfa.edu.au

Received 17 March 2015, revised 21 July 2015
Accepted for publication 6 August 2015
Published 1 September 2015

Abstract
We provide a rigorous derivation of a quantum filter for the case of multiple measurements being made on a quantum system. We consider a class of measurement processes which are functions of bosonic field operators, including combinations of diffusive and Poissonian processes. This covers the standard cases from quantum optics, where homodyne detection may be described as a diffusive process and photon counting may be described as a Poissonian process. We obtain a necessary and sufficient condition for any pair of such measurements taken at different output channels to satisfy a commutation relationship. Then, we derive a general, multiple-measurement quantum filter as an extension of a single-measurement quantum filter. As an application we explicitly obtain the quantum filter corresponding to homodyne detection and photon counting at the output ports of a beam splitter.

Keywords: quantum filtering, stochastic master equation, homodyne detection and photon counting, Poissonian and diffusive processes

(Some figures may appear in colour only in the online journal)

1. Introduction
An optimal filter provides the best estimate of unknown variables through a set of observations of a system. To construct the filter, we need three key ingredients. The first is the probability law corresponding to an observation event. The second is the conditional expectation, which relates an observation result to the unknown variables. Finally, we need to construct the stochastic differential equations, which describe the estimation result.
The quantum filtering problem was considered in the early 1980s in a series of articles by Belavkin [1–3]. In quantum mechanics, any two random variables (represented by operators) do not always commute. This fact requires an extension of Kolmogorov’s classical probability theory to the non-commutative probability theory used in quantum filtering. In the theoretical physics community, the quantum filtering problem is known under the names of the stochastic master equation and quantum trajectory theory [4, 5].

Quantum filters are typically derived for the case of single measurements. The quantum filtering problem with multiple output fields has been developed using quantum trajectory theory in [5, 6] with application to multiple-input multiple-output quantum feedback [7]. In [8], the multiple-output measurements of the generalized ‘dyne’ type were also considered for the case of a zero-mean jointly Gaussian state. It is desirable to extend these results to cover a wider class of measurements, e.g., to include both homodyne detection and photon counting in a quantum optics setting. However, in the case of multiple measurements, it is not clear whether every possible combination of measurements will satisfy the commutation relations required for a joint probability density of the multiple measurements to exist.

A jump-diffusive quantum trajectory has also been derived in [9] using a classical Markov chain approximation of the environmental field, where the system is assumed to interact with the environment over a small time interval. The infinitesimal generator of the Markov process is obtained as the limiting case when the interaction time goes to zero. Recently [10] proved that for a general class of stochastic master equations (SME) driven simultaneously by Wiener and Poissonian process, the quantum filter possesses sub-martingale properties for the fidelity between the estimated state and the actual state. Our quantum filter’s SME descriptions will also fall within a class of the SMEs considered in [10]. However, we derive the filter using quantum stochastic calculus for a class of measurement processes which are functions of bosonic field operators, including combinations of diffusive and Poissonian processes.

Within the experimental quantum optics community, simultaneous measurement of quantum systems are frequently performed [11–14]. Previously, non-classical states of light have been reconstructed via post-processing of homodyne detection measurement data [15, 16]. The thousands of homodyne detection records triggered by photon counts were sampled to construct a Wigner function using a heuristic time window approach. One could replace this procedure with the more systematic quantum filtering approach for multiple measurements that we have derived. In addition to this, the quantum filtering approach using both homodyne and photon counting detections could possibly be used as a solution to the number-resolved photon counting problem [17].

The purpose of this article is to derive using quantum stochastic calculus, the quantum filter corresponding to multiple measurements made on a quantum system. To achieve this, we first investigate the commutativity of multiple measurement processes [3]. We use the definitions of concatenation and series product [18] to describe quantum systems composed of multiple interacting open quantum systems, each of them described by \((S, L, H)\) parameters [19]. We then formulate a general quantum filter for a quantum system with a finite number of commutative measurements.

We will show that the quantum filter for a quantum system with multiple measurement outputs can be described by a stochastic master equation for the conditional density operator as follows
This equation includes an a priori part which is the original unconditional quantum master equation, and a stochastic part which is contained in the innovation term. The innovation term relates the measurement records to the evolution of the conditional density operator. In this equation, $dW$ is a vector of the ‘error’ between the actual measurement and the expected value. $S^{\Sigma^{-T}}$ is the weighting function which associates the contribution of each measurement to the total increment of the conditional density operator. Generally, the evolution of the conditional density operator above will contain both diffusion and jump processes, which allows for simultaneous photon counting and homodyne detection.

We apply our filter to the simple case of photon counting and homodyne measurement at the outputs of a single beam splitter [20]. Figure 1 shows a typical arrangement of photon counting and homodyne detection at the output ports of a beam splitter in a quantum optics experiment. We compare our quantum filter with the un-normalized stochastic Schrödinger equation (SSE) derived in [21] for the similar application.

We refer the readers to [22] for background material, such as an introduction to quantum stochastic calculus, quantum probability, and quantum non-demolition measurements. Furthermore, without loss of generality the reduced Planck constant $\hbar$ is set to one. We will assume that the quantum stochastic differential equation (QSDE) parameters $S, L, H$ are bounded to ensure that the corresponding solution is unique and unitary, as well as reducing the technical difficulties that arise.

2. Preliminary

2.1. Notation

Von Neumann algebras and $\sigma$-algebras are written in calligraphic symbols (e.g. $\mathbb{B}$ for the Borel $\sigma$-algebras on $\mathbb{R}$). As usual, classical probability spaces are written as $(\Omega, \mathcal{F}, \mu)$. Plain capital letters (e.g. $P$) will be used to denote elements of von Neumann algebras. Bold letters (e.g. $X$) will be used to denote a matrix whose elements are Hilbert space operators. Serif symbols (e.g. $H$) are used for Hilbert spaces. Hilbert space adjoints, are indicated by $^*$, while the complex conjugate transpose will be denoted by $^\dagger$, i.e. $(X^*)^T = X^\dagger$. For single-element...
operators we will use * and † interchangeably. The Hilbert space inner product of $X$ and $Y$ is given by $\langle X, Y \rangle$. The commutator of $X$ and $Y$ is given by $[X, Y] = XY - YX$.

2.2. Multiple output and input channels open quantum system

The dynamics of an open quantum system with multiple bosonic field input and output channels can be described via annihilation, creation and conservation processes. First, let $V$ be a countable dimensional Hilbert space, and let $z_k$, $k \geq 1$ be a complete orthonormal basis. A single particle Hilbert space $h$ is defined as $V \otimes L^2(\mathbb{R}_+)$. The symmetric Fock space over $h$ is given by

$$G = \bigotimes_{n=0}^{\infty} h \otimes h^*,$$

where $h \otimes h^*$ is the $n$-fold symmetric tensor product of $h$. The exponential vector $e(u) \in G$ is defined as

$$e(u) = \bigotimes_{n=0}^{\infty} \frac{1}{\sqrt{n!}} u^n.$$

using similar notation to [19, section 19]. The vacuum vector $\Phi$ corresponds to the exponential vector with $u = 0$, while other coherent vectors $e(u)$ are the normalized exponential vectors $e(u)$, $u \neq 0$. The Weyl operator $W(u, U)$, $u \in h$, $U \in B(h)$ is a unique unitary transformation operating on $e(u)$ defined by [19, section 20]

$$W(u, U)e(v) = \left\{ \exp \left( -\frac{1}{2} \| u \|^2 - \langle u, Uv \rangle \right) \right\} e(Uv + u).$$

For any $f \in h$, let us define $f_k(t) \equiv \langle z_k | f(t) \rangle$. The annihilation $A_k(t)$, creation $A_k^\dagger(t)$ and conservation $L_{kl}(t)$ processes related to the orthonormal basis $z_k$ are given by [23],

$$A_k(t) \equiv a \left( z_k \otimes I_{[0,1]} \right),$$

$$A_k^\dagger(t) \equiv a^\dagger \left( z_k \otimes I_{[0,1]} \right),$$

$$L_{kl}(t) \equiv \lambda \left( z_l \otimes z_k \otimes I_{[0,1]} \right),$$

here the operators $a$, $a^\dagger$ and $\lambda$ is the Stone generator of the corresponding Weyl operators, defined for $H = |z_k \rangle \langle z_l| \otimes I_{[0,1]}$ and $u = z_k \otimes I_{[0,1]}$.

$$W(0, \exp(itH)) = \exp(-it\lambda(H)), \quad W(u, I) = \exp(-ip(u)), \quad q = p(u), \quad a(u) = \frac{1}{2}(q(u) + ip(u)), \quad a^\dagger(u) = \frac{1}{2}(q(u) - ip(u)).$$

In the coherent vector domain, the following commutation relations hold

$$[A_k(t), A_l^\dagger(s)] = \left[ A_k^\dagger(t), A_l^\dagger(s) \right] = 0, \quad [A_k(t), A_l^\dagger(s)] = \delta_{kl} \min(t, s).$$

In addition, other commutation relations can be obtained using the general Itô multiplications as below [19, 23],

$$dt \equiv d\Lambda_{00}, \quad da_k \equiv d\Lambda_{0k}, \quad da_k^\dagger \equiv d\Lambda_{kk},$$

$$d\Lambda_k \equiv d\Lambda_{kk}, \quad da_{t}(t)da_{s}(t) = \hat{\delta}_{ts} d\Lambda_{kl},$$

where $\hat{\delta}_{rs} = 0$, $\forall r = 0 \cup s = 0$ and $\hat{\delta}_{rs} = \delta_{rs}$ otherwise. The evolution of a system observable in the Heisenberg picture is given by
Let $\mathcal{G}$ be an open quantum system with parameters $(S, L, H)$, and $SS' = S'S = I$. For any system observable $X$, the following QSDE in the Heisenberg picture is obtained

$$dX_t = (-i [X_t, H_t] + L_t(X_t)) dt + dA_t^r S_i [X_t, L_t] + \left[L_t^r, X_t\right] S_i dA_t^r$$

$$+ \text{tr} \left[\left(S_i X_t S_i - X_t\right) dA_t^r\right],$$

where $L_t(X_t) = \frac{1}{2} L_t^r [X_t, L_t] + \frac{1}{2} [L_t^r, X_t] L_t$, is the Lindbladian super operator, and all operators evolve according to (8), i.e. $L_t \equiv U_t^r (L \otimes I) U_t$. In the Schrödinger picture, the corresponding unitary operator evolution is

$$dU_t = \left[\text{tr} \left((S - I) dA_t^T\right) + dA_t^r L - L/S dA_t - \left(\frac{1}{2} L/L + iI\right)\right] dt U_t, \quad U_0 = I$$

The evolution of the output fields is given by [18]

$$d\hat{A}_t = S_i dA_t^r + L_i dt,$$  \hspace{1cm} (11a)

$$d\hat{A}_t^* = S_i^* dA_t^T + S_i^* dA_t^r L - L_i^* dA_t^T + L_i^* L^T dt.$$  \hspace{1cm} (11b)

3. Main results

3.1. Commutativity of open quantum system output channels

Definition 3.1 (Commutator of two vectors with non-commutative elements). Let $a, b \in B(H)^{n \times 1}$ be vectors whose elements are non-commutative. The commutator of a pair $a, b$ is given by

$$[a, b] = ab^T - (ba)^T.$$  \hspace{1cm} (12)

Notice that in definition 3.1, in general, $(ba)^T \neq ab^T$ due to the non-commutativity of the elements in the pair $a, b$. We further define the self-commutator of a vector $a$ whose elements are non-commutative as $[a, a]$. It is important to see that the self-commutator is not always equal to zero, as the following simple example shows.

Example 3.1. Let $a = [a \ a'^{\dagger}]^T$, where $a$ is an annihilation operator defined for a Fock space $\Gamma(h)$. Then by definition 3.1, the self-commutator of $a$ is given by

$$[a, a] = a a'^{\dagger} - (a a'^{\dagger})^T = \begin{bmatrix} a a & a a'^{\dagger} \\ a' a & a'^{\dagger} \end{bmatrix} - \begin{bmatrix} a a & a a'^{\dagger} \\ a'^{\dagger} & a'^{\dagger} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

As the definition and example clearly show, for self-commutator, commutativity is implied by the symmetry properties of $aa^T$. Now consider a general measurement equation, which is a function of the field output annihilation, creation, and conservation processes.
can have operator valued elements, but to satisfy the quantum stochastic integration, they have to be adapted processes, see [19, section 24]. We require also that every element of $F$, $G$, adapted to a common commutative von Neumann algebra. Furthermore, to have a real measurement value, we require that the elements of $G$ are self-adjoint operators. The measurement class considered in (13) is quite general, in the sense that it includes both diffusive and Poissonian measurement processes. Nevertheless, it does not include the measurement processes with off diagonal element of the conservation process $d\tilde{\lambda}_t$.

Substituting (11) into (13), we can write the general measurement equation as

$$dY_t = F_t^a d\tilde{\lambda}_t^a + F_t d\tilde{\lambda}_t + G_t d\tilde{\lambda}_t,$$  \hspace{1cm} (13)

$F_t$, $G_t$ can have operator valued elements, but to satisfy the quantum stochastic integration, they have to be adapted processes, see [19, section 24]. We require also that every element of $F_t$, $G_t$ adapted to a common commutative von Neumann algebra. Furthermore, to have a real measurement value, we require that the elements of $G_t$ are self-adjoint operators. The measurement class considered in (13) is quite general, in the sense that it includes both diffusive and Poissonian measurement processes. Nevertheless, it does not include the measurement processes with off diagonal element of the conservation process $d\tilde{\lambda}_t$.

Substituting (11) into (13), we can write the general measurement equation as

$$dY_t = F_t^a da_1 + F_t da_2 + da_3 + G_t \{ db_1 + db_2 + db_3 + db_4 \},$$  \hspace{1cm} (14a)

where

$$da_{1,i} = \sum_{k=1}^n S^+_k \tilde{d} \Lambda_{k0}, \text{ } da_{2,i} = \sum_{k=1}^n S^0_k \tilde{d} \Lambda_{ik}, \text{ } da_{3,i} = \sum_{k=1}^n \left[ (FL)^+_{k1} + (FL)_{k1} \right] \tilde{d} r,$$

$$db_{1,i} = \sum_{k,k'=1}^n S^+_k \tilde{d} \Lambda_{kk'} S^+_k \tilde{d} r, \text{ } db_{2,i} = \sum_{k=1}^n S^0_k \tilde{d} \Lambda_{kk} L^+_{kk} \tilde{d} r, \text{ } db_{3,i} = \sum_{k=1}^n L^+_{ij} \tilde{d} \Lambda_{ik} S^+_k \tilde{d} r,$$

$$db_{4,i} = \sum_{k=1}^n L^+_{ii} \tilde{d} r.$$  \hspace{1cm} (14b)

Based on (7), most of the multiplication products between the $dY_t$ elements are zero, while the remaining terms are listed in Table 1.

**Remark 3.1.** A set of measurements $Y_t$ made at the output of a quantum system is self-commutative if and only if $dY_t dY^T_t$ is symmetric.

This fact follows directly from definition 3.1. Now we state the following lemma to prove our main result on the commutation relation for a finite number of outputs of an open quantum system.

**Lemma 3.1.** The off diagonal elements in the Itô table 1 for multiplication between the $dY_t$ elements are zero.
Proof. For every entry in Table 1, we have
\[ \sum_{i=1}^{n} \mathcal{S}_{ii}^* \mathcal{S}_{ji} = \sum_{i=1}^{n} \mathcal{S}_{ii}^* \mathcal{S}_{ij} = (\mathcal{S}_i \mathcal{S}^*)_i = (I)_{i,j} = 0, \quad \forall \; i \neq j, \]
which shows that the non-diagonal elements of the multiplication results are zero.

Theorem 3.1. A set of n general measurements \( \mathcal{Y}_i \) is self-commutative for any multiple-output quantum system with n channels if and only if

\[
\begin{bmatrix}
 F_i & F_i^T \\
 0 & I
\end{bmatrix}
\begin{bmatrix}
 F_i^T \\
 0 & I
\end{bmatrix} = 0. 
\tag{15a}
\]
\[
\begin{bmatrix}
 G_i & F_i^T \\
 0 & I
\end{bmatrix}
\begin{bmatrix}
 G_i^T \\
 0 & I
\end{bmatrix} = 0, 
\tag{15b}
\]
\[
\begin{bmatrix}
 G_i & F_i \\
 0 & I
\end{bmatrix}
\begin{bmatrix}
 G_i^T \\
 0 & I
\end{bmatrix} = 0. 
\tag{15c}
\]

Furthermore, (15a) is equivalent to \( \Re(\mathcal{F}_i) \Im(\mathcal{F}_i^T) \) being symmetric, (15b) is equivalent to both \( \Re(\mathcal{G}_i) \Re(\mathcal{F}_i^T), \Im(\mathcal{G}_i) \Im(\mathcal{F}_i^T) \) being symmetric while (15c) is equivalent to both \( \Re(\mathcal{G}_i) \Im(\mathcal{F}_i^T), \Im(\mathcal{G}_i) \Re(\mathcal{F}_i^T) \) being symmetric.

Proof. Let \( \mathcal{Y} \) be a generalized measurement whose evolution is given by (13). Then to prove the theorem, remark 3.1 shows that it is sufficient to show that \( d\mathcal{Y}_i d\mathcal{Y}_j^T \) is symmetric in order to show that all measurement outputs commute with each other. Simplifying Table 1 to 2 and evaluating the \( d\mathcal{Y}_i d\mathcal{Y}_j^T \), from Table 2, one obtains

\[
\begin{align*}
\left( d\mathcal{Y}_i d\mathcal{Y}_j^T \right)_{ij} &= \left( \mathcal{G}_i \left[ \mathcal{D}_{2} d\mathcal{Y}_i^T + \mathcal{D}_{3} d\mathcal{Y}_i^T + \mathcal{D}_{4} d\mathcal{Y}_i^T + \mathcal{D}_{4} d\mathcal{Y}_i^T \right] \mathcal{G}_j^T \right)_{ij} \\
&\quad + \left( \mathcal{G}_i \left[ \mathcal{D}_{2} d\mathcal{Y}_i^T + \mathcal{D}_{3} d\mathcal{Y}_i^T \right] \mathcal{F}_j^T \right)_{ij} \\
&\quad + \left( \mathcal{F}_i \left[ \mathcal{D}_{2} d\mathcal{Y}_i^T + \mathcal{D}_{3} d\mathcal{Y}_i^T \right] \mathcal{G}_j^T \right)_{ij} \\
&\quad + \left( \mathcal{F}_i \mathcal{D}_{2} d\mathcal{Y}_i^T \mathcal{F}_j^T \right)_{ij} \\
&= \left( \mathcal{G}_i \mathcal{O}_1 \mathcal{G}_j^T + \mathcal{G}_i \mathcal{O}_2 \mathcal{F}_j^T + \mathcal{F}_i \mathcal{O}_3 \mathcal{G}_j^T + \mathcal{F}_i \mathcal{F}_j d\mathcal{Y}_j^T \right)_{ij}.
\end{align*}
\tag{16}
\]

By lemma 3.1, we have \( \mathcal{O}_1, \mathcal{O}_2 \) and \( \mathcal{O}_3 \) are diagonal matrices. Since every diagonal element of \( \mathcal{O}_i, i = 1, 2, 3 \) has different creation, annihilation and conservation processes, by Table 2,
requiring \( dY_t dY_t^\dagger \) to be symmetric is equivalent to the symmetry of \( G_t O_t G_t^\dagger \), \( G_t O_2 F_t^\dagger \), \( F_2 O_t G_t^\dagger \), and \( F_t F_t^\dagger \). We also notice that \( F_t \) and \( G_t \) belong to a common commutative von Neumann algebra, and in addition to that they also commutes to \( O_t \). For the first term, we have \( G_t O_t G_t^\dagger = \sqrt{G_t O_t G_t^\dagger} \), which is satisfied for all \( G_t \). Furthermore, for \( F_t F_t^\dagger \), we have the symmetry condition

\[
F_t F_t^\dagger = F_t^\dagger F_t,
\]

which is equivalent to \( \Re(F_t) \Re(F_t^\dagger) - \Re(F_t^\dagger) \Im(F_t) = 0 \), and in turn equivalent to condition (15a). For \( G_t O_2 F_t^\dagger \), we have the symmetry condition

\[
(\sqrt{G_t} O_2 \sqrt{F_t})_{ij} = (\sqrt{G_t} O_2 \sqrt{F_t})_{ji} = \sum_{k=1}^n G_{t,ik} O_{2,ik} F_{1,ki}^\dagger.
\]

Since every diagonal element of \( O_2 \) will have a different creation process at each \( k \), this condition is equivalent to the equality being satisfied for every \( k \), which is equivalent to the condition \( G_t F_t^\dagger = F_t G_t^\dagger \). This equality is equivalent to the symmetry of \( \Re(G_t) \Re(F_t^\dagger) \) and \( \Im(G_t) \Im(F_t^\dagger) \). Using a similar argument, the third line of (16) is also equivalent to the condition \( G_t F_t^\dagger = F_t G_t^\dagger \), but this equality is equivalent to the symmetry of \( \Re(G_t) \Re(F_t^\dagger) \) and \( \Im(G_t) \Im(F_t^\dagger) \), which completes the proof.

Theorem 3.1 shows that the multiple measurements described in (13) generate a common joint probability density due to their commutativity. Furthermore, we will show that the measurement class described in (13) is well defined, in the sense that they construct a commutative von Neumann algebra. This measurement class also satisfies self-nondemolition and non-demolition properties. Self-nondemolition implies that the measurements in this class are classical stochastic processes while non-demolition ensures that the conditional expectation is defined in quantum probability settings.

**Proposition 3.1.** A set of \( n \) general measurements \( Y_t \) satisfying theorem 3.1 is self non demolition, i.e. \( [Y_s, Y_t] = 0, \forall s < t \). Furthermore, \( Y_t \) is also non-demolition, i.e. \( [X_s, Y_t] = 0, \forall s < t \), for all system observables \( X \).

**Proof.** Let \( dZ_t = F_t^s dA_t^s + F_t d\Lambda_t + G_t d\lambda_t \), where \( Y_t = U_t^r Z_t U_t \). Furthermore, let \( Z_t = vN(Z_t; 0 \leq s \leq t) \). As in [24], we will show that \( Z_t \) is a commutative algebra, for all \( 0 \leq t \leq T \). Let \( \chi_{[t, t']} \) be an indicator function at time interval \([t, t']\). Then we can write,

\[
Z_t = Z_t^f \left( \chi_{[0, t]} \right) G_t \chi_{[0, t]}) \right) \].

Therefore, by quantum Itô rules, we obtain

\[
\left[ Z_t, Z_{t'} \right]_{ij} = \int_0^T \sum_{k=1}^n \left( F_{s,ik}^* \chi_{[t, t']} F_{s,jk} \chi_{[0, t']} - F_{s,ik} \chi_{[0, t']} F_{s,jk}^* \chi_{[0, t']} \right) d\tau
+ \int_0^T \sum_{k=1}^n \left( G_{s,ik}^* \chi_{[0, t']} G_{s,jk} \chi_{[0, t']} - G_{s,ik} \chi_{[0, t']} G_{s,jk}^* \chi_{[0, t']} \right) d\Lambda_t^k
+ \int_0^T \sum_{k=1}^n \left( F_{s,ik}^* \chi_{[0, t']} G_{s,jk} \chi_{[0, t']} - G_{s,ik} \chi_{[0, t']} F_{s,jk}^* \chi_{[0, t']} \right) dA_t^k
+ \int_0^T \sum_{k=1}^n \left( G_{s,ik}^* \chi_{[0, t']} F_{s,jk} \chi_{[0, t']} - G_{s,ik} \chi_{[0, t']} F_{s,jk}^* \chi_{[0, t']} \right) dA_t^k.
\]


Since for any two quantum adapted process $A_s$ and $B_s$, for any $t$, and $t' \in [0, T]$, we have, 
$$A_s \chi_{[0,t')} B_s \chi_{[0,t')} = A_s B_s \chi_{[0,t' \wedge t]}$$ 
therefore, we obtain 
$$Z_s = \{F_{s,ik} G_{s,jk} - F_{s,ik}^* G_{s,jk}^*\} \chi_{[0,t \wedge t']} \mathrm{d}A_{kk}$$
Furthermore, by theorem 3.1, and since elements of $F_t$ and $G_t$ belong to a common commutative von Neumann algebra, we have $[Z_t, Z_{t'}]_{i,j} = 0$, $\forall i, j \leq n$, $\forall t, t' \in [0, T]$, and therefore, $Z_t$ generates a commutative algebra. Self-nondemolition and nondemolition properties then follow from proposition 2.1 of [24].

To clarify theorem 3.1, we provide a few examples. In the case of a quantum system with two output channels, both subject to homodyne detection, $F_t$ and $G_t$ are given by 
$$F_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad G_t = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 
In these cases the self-commutativity condition of theorem 3.1 can be easily verified. However, taking 
$$F_t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad G_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$ 
means that the first measurement is homodyne detection on the first output channel, and the second is the photon counting measurement on the same channel. Now, $F_t G_t^*$ is not symmetric, thus by theorem 3.1 the measurement vector is not self-commutative. In the next subsection, we will present our second result, which gives a general derivation of a quantum filter for a set of commutative measurements.

### 3.2. General quantum filter for multiple compatible measurements

To derive a quantum filter for multiple measurements, we follow the characteristic function method described in [25, 26]. We will use the following notation to denote the conditional expectation 
$$\pi_t(X) = \tilde{X}_t = \mathbb{E}[X | \mathcal{Y}_t],$$
$\mathcal{Y}_t$ is a commutative von Neumann algebra generated by measurements $X_t$ and $Y_t \in \mathcal{Y}_t$, where $\mathcal{Y}_t'$ is the commutant of $\mathcal{Y}_t$. The conditional expectation $\mathbb{E}[X | \mathcal{Y}_t]$ is the orthogonal projection of $X_t$ onto $\mathcal{Y}_t$, with respect to the inner product $\langle X, Y \rangle_\mathbb{P} = \mathbb{P}(X^* Y)$, where $\mathbb{P}$ is a positive normalized linear functional, called a state on $\mathcal{Y}_t$, see [22].

**Theorem 3.2.** Let $(Y_t, i = 1, \ldots, n)$ be a set of $n$ compatible measurement outputs for a quantum system $G$ given in (13). With the field initially at the vacuum state with density
operator \( \omega \), the corresponding joint measurement quantum filter is given by

\[
d\hat{X} = \pi \left[ -i [X_t, H_t] + \mathcal{L}_E(X_t) \right] dt + \sum_{i=1}^{N} \beta_{i,t} dW_{i,t},
\]

(20)

where \( W_{i,t} = Y_{i,t} - \pi_i(Y_{i,t}) \) is a martingale process for each measurement output and \( \beta_{i,t} \) is the corresponding gain given by

\[
\beta = \Sigma^{-1} \zeta, \quad (12a)
\]

\[
\zeta^t = \pi(X_t dY^T_t) - \pi(X_t) \pi(dY^T_t) + \pi(d X_t dY^T_t), \quad (12b)
\]

\[
\Sigma = \pi(dY_t dY^T_t), \quad (12c)
\]

where \( \Sigma \) is assumed to be non-singular.

**Proof.** First, define a \( \mathcal{F}_t \)-measurable Itô exponential \( c_{\hat{f}} \) with respect to arbitrary functions \( \{ f_{i,t} \} \) whose derivative is given by

\[
dc_{\hat{f}} = \hat{f}_{\hat{f}} \left[ \sum_{i=1}^{N} dY_{i,t} \right] = c_{\hat{f}} dY^\top_t f_t.
\]

(22)

Now the dynamics of the conditional expectation are assumed to be in the form of the following equation

\[
d\hat{X} = \alpha_{t} dt + \beta_{t}^T dY_t,
\]

(23)

where \( \alpha_t \) and \( \beta_{t,t} \) are to be determined from the conditional expectation relation

\[
E[X_t c_{\hat{f}} | Y_t] = E[X_t | Y_t] c_{\hat{f}} \quad (24)
\]

\[
\pi_t(X_t c_{\hat{f}}) = \pi_t(\hat{X}_t c_{\hat{f}}) \quad (25)
\]

From the QSDE of a system observable in (9), (22), and the definition of conditional expectation (25), we have

\[
d\pi_t(X_t c_{\hat{f}}) = \pi_t(dX_t c_{\hat{f}} + X_t dc_{\hat{f}} + dX_t dc_{\hat{f}}), \quad (26a)
\]

\[
\pi_t(dX_t c_{\hat{f}}) = \pi_t(\left[ -i [X_t, H_t] + \mathcal{L}_E(X_t) \right] c_{\hat{f}} dt), \quad (26b)
\]

\[
\pi_t(X_t dc_{\hat{f}}) = c_{\hat{f}} \pi_t(X_t dY^T_t) f_t, \quad (26c)
\]

\[
\pi_t(dX_t dc_{\hat{f}}) = c_{\hat{f}} \pi_t(d X_t dY^T_t) f_t, \quad (26d)
\]

while \( d\pi_t(\hat{X}_t c_{\hat{f}}) \) given by

\[
d\pi_t(\hat{X}_t c_{\hat{f}}) = \pi_t(d\hat{X}_t c_{\hat{f}} + \hat{X}_t dc_{\hat{f}} + d\hat{X}_t dc_{\hat{f}}), \quad (27a)
\]

\[
\pi_t(d\hat{X}_t c_{\hat{f}}) = c_{\hat{f}} \left[ \alpha_t dt + \beta_{t}^T \pi_t(dY_t) \right], \quad (27b)
\]

\[
\pi_t(\hat{X}_t dc_{\hat{f}}) = \pi_t(X_t) c_{\hat{f}} \pi_t(dY^T_t) f_t, \quad (27c)
\]
\[ \pi t \left[ d\hat{X}_t, dc_{\beta_t} \right] = c_{\beta_t} \pi t \left( dY_t, dY_t^T \right) f_t. \]  

(27d)

Equating (26b) and (27b), solving for \( \alpha_t \) and then substituting the result into (23), we obtain,

\[ \begin{align*}
\frac{d\hat{X}}{dt} &= \pi t \left[ -i [X_t, H_t] + \mathcal{L}_L(X_t) \right] dt + \beta_t^T \left[ dY_t - \pi_t \left( dY_t \right) \right] \\
&= \pi t \left[ -i [X_t, H_t] + \mathcal{L}_L(X_t) \right] dt + \beta_t^T dW_t.
\end{align*} \]

(28)

Furthermore, using the fact that the function \( f_t \) is arbitrary, we can equate the right-hand side of (26) and (27), which recovers \( \beta_{h,t} \), \( \zeta_i, \Sigma \) are an operator valued row vector and an operator valued matrix, respectively given by

\[ \begin{align*}
\zeta &= \pi i \left( X_t, dY_t^T \right) - \pi_i (X) \pi_i \left( dY_t^T \right) + \pi_i \left( dX_t, dY_t^T \right), \\
\Sigma &= \pi \left( dY_t, dY_t^T \right).
\end{align*} \]

(30a)

(30b)

Furthermore, martingale property of \( W_{k,t} \) can be directly shown by,

\[ \mathbb{P}(W_t - W_0) = \mathbb{P} \left( \int_0^t \left( dY_r - \mathbb{E}[dY_r | \mathcal{F}_r] \right) \right) = \mathbb{P} \left( \int_0^t \left( \mathbb{E}[dY_r - dY_r | \mathcal{F}_r] \right) \right) = 0, \]

which completes the proof. \( \square \)

A restricted form of theorem 3.2 for a class of generalized homodyne detection measurement has been independently proven in [8] using the ‘reference probability’ approach but for rather general zero mean field states settings. This result applied to a class of generalized homodyne detection measurements, i.e. \( G = 0 \).

The dynamics of the quantum filter can also be expressed using the following equation,

\[ \frac{d\hat{X}}{dt} = \pi t \left[ -i [X_t, H_t] + \mathcal{L}_L(X_t) \right] dt + \zeta^T \Sigma^{-1} dW. \]

(31)

From a classical filtering point of view, (31) possesses some similarities to the Kalman filter, where \( \pi t \left[ -i [X_t, H_t] + \mathcal{L}_L(X_t) \right] dt \) is the \textit{a priori} estimate and \( \zeta^T \Sigma^{-1} \) is analogous to the Kalman gain which multiplies the innovation process \( dW \).

\textbf{Remark 3.2.} Theorem 3.2 requires the existence of an invertible differential measurement correlation matrix \( \Sigma \), which is a sufficient condition for the joint measurements to be obtainable from a single quantum filter equation. This condition, however, is not a necessary condition, as we will encounter in section 4.1, where in a case of zero reflectivity, the quantum filter equation exists although \( \Sigma \) is not invertible.

In most cases of nonlinear estimation, (31) is merely a representation for the estimator and cannot be interpreted as an explicit solution to the filtering problem [27]. As in the classical filtering problem, explicit solutions to the general nonlinear filtering problem can be obtained using a variety of approximation methods [28, 29]. However, in the quantum filter, rather than approximating the explicit solution of (31), one can convert the filtering problem in (31), which is given in the Heisenberg picture, into the Schrödinger picture. Then one deals with the evolution of the system’s conditional density operator at time \( t \), \( \rho_t \). For \textit{trace class} operators [19, 30], we have \( \mathbb{P}(\pi_t (X_t)) = \text{tr} (\rho_t X) \equiv \{X_t\} \). Therefore, one can construct from (31),
where $\zeta_t$ in the above equation is now only a function of the conditional density operator, $L$ and $H$, but not of the particular system observable $X$.

Finally, for numerical simulation efficiency, after truncating the Hilbert space dimension to a finite number $n$, instead of solving for the $n \times n$ conditional density operator in (32), one can ‘unravel’ this equation, and solve instead for the state vector $|\psi\rangle$, which is an $n \times 1$ vector. This unravelled equation is of the form

$$d|\psi\rangle = \left[-i\left[H_t, \rho_t\right] + LL^\dagger L^\dagger - \frac{1}{2} L^\dagger L \rho_t - \frac{1}{2} \rho_t L^\dagger L\right]dt + \text{innovation term}$$

where $\zeta_t$ in the above equation is now only a function of the conditional density operator, $L$ and $H$, but not of the particular system observable $X$.

Finally, for numerical simulation efficiency, after truncating the Hilbert space dimension to a finite number $n$, instead of solving for the $n \times n$ conditional density operator in (32), one can ‘unravel’ this equation, and solve instead for the state vector $|\psi\rangle$, which is an $n \times 1$ vector. This unravelled equation is of the form

$$d|\psi\rangle = \left[-i\left[H_t, \rho_t\right] + LL^\dagger L^\dagger - \frac{1}{2} L^\dagger L \rho_t - \frac{1}{2} \rho_t L^\dagger L\right]dt + \text{innovation term}$$

where $\zeta_t$ in the above equation is now only a function of the conditional density operator, $L$ and $H$, but not of the particular system observable $X$.

Finally, for numerical simulation efficiency, after truncating the Hilbert space dimension to a finite number $n$, instead of solving for the $n \times n$ conditional density operator in (32), one can ‘unravel’ this equation, and solve instead for the state vector $|\psi\rangle$, which is an $n \times 1$ vector. This unravelled equation is of the form

$$d|\psi\rangle = \left[-i\left[H_t, \rho_t\right] + LL^\dagger L^\dagger - \frac{1}{2} L^\dagger L \rho_t - \frac{1}{2} \rho_t L^\dagger L\right]dt + \text{innovation term}$$

where $\zeta_t$ in the above equation is now only a function of the conditional density operator, $L$ and $H$, but not of the particular system observable $X$.
and the concatenation product [18], giving \( \mathcal{G} = (G_1 \otimes G_2) \triangleright G_3 \) with parameters 
\[ S, S \left( \frac{L}{0} \right), H. \]

The output field of the system \( G_1, A_{s,t} = U_t^r (I \otimes A_{s,t}) U_t \), is an operator on \( H \otimes \Gamma_{1,t} (h) \), while the vacuum field \( A_{s,t} \) is an operator on \( \Gamma_{2,t} (h) \). We denote the total Hilbert Space as \( H = h \otimes \Gamma_t (h) \otimes \Gamma_t (h) \). The beam splitter equation is given by
\[ S = \begin{pmatrix} \sqrt{1 - r^2} e^{i\theta} & re^{i(\theta + \frac{\pi}{2})} \\ re^{i(\theta + \frac{\pi}{2})} & \sqrt{1 - r^2} e^{i\theta} \end{pmatrix}, \quad r \geq 0. \] (35)

For homodyne measurement in the first output channel and photon counting measurement in the second output channel, we have
\[ dY_i = F_i^* d\hat{A}_{i,t}^* + F_i^* d\hat{A}_{i,t} + G_i^* d\hat{\lambda}_i, \]
\[ F_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad G_i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

By theorem 3.1, the measurement set \( dY_i \) is self-commutative. Substituting the general beam splitter (35) and output field evolution (11), the measurements QSDEs are given by
\[ dY_{1,t} = \sqrt{1 - r^2} \left( \pi \left( e^{i\theta} L_t^+ + e^{-i\theta} L_t^+ \right) \right) dt + \pi \left( e^{i\theta} dA_{i,t} + e^{-i\theta} dA_{i,t}^\dagger \right), \]
\[ + ir \left( e^{i\theta} d\hat{A}_{i,t} - e^{-i\theta} d\hat{A}_{i,t} \right), \] (36a)
\[ dY_{2,t} = r^2 \left[ dA_{i,t} + L_t^+ dA_{i,t}^\dagger + L_t^+ dA_{i,t} + L_t^+ L_t \right] + (1 - r^2) d\hat{A}_{i,t}, \]
\[ + i \left( r\sqrt{1 - r^2} \right) \left[ d\hat{\lambda}_{i,t} - dA_{i,t} + L_t^+ dA_{i,t}^\dagger - L_t^+ L_t^+ \right]. \] (36b)

Next, we can compute the expectation and the correlation of the measurement time derivative as
\[ \pi_i (dY_{1,t}) = \sqrt{1 - r^2} \pi_i \left( e^{i\theta} L_t^+ + e^{-i\theta} L_t^+ \right) dt, \] (37a)
\[ \pi_i (dY_{2,t}, dY_{2,t}) = \pi_i (dY_{2,t}) = r^2 \pi_i \left( L_t^+ L_t \right) dt, \] (37b)
\[ \pi_i (dY_{2,t}, dY_{1,t}) = dt, \] (37c)
\[ \pi_i (dY_{2,t}, dY_{1,t}) = \pi_i (dY_{2,t}, dY_{2,t}) = 0. \] (37d)

Using these values, \( \beta \) is then given by
\[ \beta_1 = \sqrt{1 - r^2} \left( \pi_i \left( X_t e^{i\theta} L_t^+ + e^{-i\theta} L_t^+ X_t \right) - \pi_i (X) \pi_i \left( e^{i\theta} L_t^+ + e^{-i\theta} L_t^+ \right) \right), \] (38)
\[ \beta_2 = \frac{\pi_i \left( L_t^+ X_t L_t \right)}{\pi_i \left( L_t^+ L_t \right)} - \pi_i (X). \] (39)

In the case that \( r \to 1 \), the estimation problem reduces to an estimation problem with a single photon counting process. The opposite case is more interesting. When \( r \to 0 \), the gain \( \beta_2 \) has a non zero value, while the Poisson process has zero arrival rate, and hence the estimation problem reduces to an estimation problem with a single homodyne detection. This is unsurprising since zero reflection ensures all photons pass through to the homodyne detector.
We can unravel the stochastic master equation into the form (33). By using the Itô equivalence
\[ d \rho_t = d | \psi_t \rangle \langle \psi_t | + d | \psi_t \rangle d \{ \psi_t \} + d \{ \psi_t \} d | \psi_t \rangle, \]
one recovers the unraveled stochastic Schrödinger equation for the quantum filter,
\[ d | \psi_t \rangle = -i \left( H - i \frac{1}{2} L^d L \right) | \psi_t \rangle dt + \sigma \left( | \psi_t \rangle \right) dt + \delta_1 \left( | \psi_t \rangle \right) dW + \delta_2 \left( | \psi_t \rangle \right) dN, \]
\[ \sigma \left( | \psi_t \rangle \right) = \left( 1 - \frac{r^2}{2} \pi \left\{ e^{-i \theta L^+} + e^{i \theta L} \right\} \right) L, \]
\[ \delta_1 \left( | \psi_t \rangle \right) = \sqrt{1 - r^2} \left( L - \frac{1}{2} \pi \left\{ e^{-i \theta L^+} + e^{i \theta L} \right\} \right) | \psi_t \rangle, \]
\[ \delta_2 \left( | \psi_t \rangle \right) = \left( \frac{L}{\sqrt{\pi} \langle L^+ L \rangle} - 1 \right) | \psi_t \rangle. \]
Here, dW is equal to dW1, and dN is equal to the Poisson process of the second measurement. The unravelled version of quantum filter given in (41) is normalized. For the case \( r = 0 \) and \( r = 1 \), (41) is equivalent to SSE for homodyne detection and photon counting respectively, given in [31, sections 6.1, 6.4], [32, sections 11.3, 11.4].

4.2. Comparison with results in [21]

In this subsection, we give a comparison of our quantum filter with the results of [21]. Here, the un-normalized SSE for photon counting and homodyne detection was formulated heuristically by the addition of two measurement operations, where every operation determines the infinitesimal evolution of the un-normalized state. The SSE for photon counting and homodyne was given in [21] as
\[ | \tilde{\psi}_{t+dr} \rangle = \left( 1 + Adr + (B - 1) dN + CdW \right) | \tilde{\psi}_t \rangle, \]
\[ A = -iH - \frac{\gamma_1 + \gamma_2}{2} a^d a + \gamma_2 a (a^d + a), \]
\[ B = \sqrt{\gamma_1} dL = r \sqrt{\gamma_1} dL, \]
\[ C = \sqrt{\gamma_2} a = \sqrt{1 - r^2} L, \]
where \( \gamma > 0, i = 1, 2 \) denote the decay rates into the photon counting and homodyne detection channels, respectively. Total cavity coupling operator is given by \( L = \sqrt{\gamma_1 + \gamma_2} a \). Accordingly, the beam splitter reflectivity is given by \( r = \sqrt{\gamma_1 / (\gamma_1 + \gamma_2)} \). In these equations, we set the local oscillator angle to \( \theta = 0 \). To give a comparison of (42) with our result in (41), one can consider the normalization of (42) as detailed in [32, section 11.4]. In general, the
infinitesimal evolution given in (42), can be normalized to the following normalized SSE
\[
d \psi_i = \left[ (A + \overline{A} + CC)d\tau + (BB - 1)dN + (C + \overline{C})dW \right] \psi_i,
\]
(43a)
\[
\overline{A} = \frac{3}{8}(C + C^\dagger)^2 - \frac{1}{2}(A + A^\dagger) - \frac{1}{2}\langle C^\dagger C \rangle,
\]
(43b)
\[
\overline{B} = \frac{1}{\sqrt{\langle B^\dagger B \rangle}},
\]
(43c)
\[
\overline{C} = -\frac{1}{2}(C + C^\dagger).
\]
(43d)
Substituting these values into (42), one can get an SSE in the form of (41a), with
\[
\sigma(\psi_i) = \left[ \left( \frac{1 - r^2}{2} \langle L + L^\dagger \rangle \right) L + \left[ \frac{r^2 \langle L^\dagger L \rangle}{8} - \frac{(1 - r^2)}{8} \langle L + L^\dagger \rangle^2 \right] \right] \psi_i,
\]
(44a)
\[
\delta_1(\psi_i) = \sqrt{1 - r^2} \left( L - \frac{1}{2} \langle L + L^\dagger \rangle \right) \psi_i,
\]
(44b)
\[
\delta_2(\psi_i) = \left( \frac{L}{\sqrt{\langle L^\dagger L \rangle}} - 1 \right) \psi_i.
\]
(44c)
As (44) shows, the un-normalized SSE formulated in [21] is consistent with quantum filter for joint homodyne detection and photon counting given in (41).

5. Simulation results

This section will show a simulation of the proposed quantum filter for a cavity mode with a number state as the initial condition. In this condition, the analytical probability distribution of the number state is given by [31],
\[
\mathbf{P}_N(t) = \binom{n}{N} \mu(t)^N \left( 1 - \mu(t) \right)^{N - n},
\]
(45a)
\[
\mu(t) = 1 - e^{-\gamma t}.
\]
(45b)
Simulation results for different reflectivities are shown in figure 3. The parameters are scaled by the cavity damping rate, that is \( \gamma \) is set to one. Figures 3(a) and (b) show single trajectory simulations of the expected number operator for the case of pure photon counting measurement and homodyne detection. Figure 4 shows the case of a balanced beam splitter \( r^2 = 0.5 \). In this case, the quantum filter average of the expected number operators converges to the analytical result (45) when the trial number is increased.

6. Conclusions

We have derived a sufficient and necessary condition for a class of quantum measurement output channels to satisfy a commutativity relation. Furthermore, this commutativity condition enables us to derive a quantum filter corresponding to multiple measurement outputs. Our
result is more general than previous results on multiple output quantum filtering in terms of
the class of measurements considered, since it covers not only homodyne type measurements,
but also photon counting type measurements. We also provide examples of the quantum
filter for homodyne and photon counting detection. The quantum filter results were shown to be
consistent with the homodyne and photon counting quantum filters for both extreme cases,

Figure 3. Single trajectory Monte-Carlo realizations of the quantum filter with initial
number state and dissipation, with beam splitter reflectivity such that (a) $r^2 = 1$ and (b)
$r^2 = 0$.

Figure 4. Expected number operator as a function of time with number state initial
condition. The average of 100 Monte-Carlo trials is shown by blue line along with the
range of one standard deviation from the mean (dotted blue). This figure shows the case
of half reflective beam splitter $r^2 = 0.5$. The quantum filters expected number operator
converges to the analytical prediction of (45) (dashed black).
where the reflectivity of the beam splitter is zero and one. In addition, the quantum filter results also consistent to the un-normalized SSE given in [21].

Acknowledgments

We acknowledge discussions with Dr Katanya Kuntz of UNSW Canberra and Dr Hendra Nurdin of UNSW Sydney. We would like also to thank two anonymous referees for their helpful comments.

References

[1] Belavkin V P 1992 Commun. Math. Phys. 146 611–35
[2] Belavkin V P 1995 Quantum filtering of Markov signals with white quantum noise Elektronika 25 1445–53
[3] Belavkin V P 1989 Nondemolition measurements, nonlinear filtering and dynamic programming of quantum stochastic processes Modeling and Control of Systems vol 121 ed A Blaquiere (Berlin: Springer) pp 245–65
[4] Carmichael H 1993 An Open Systems Approach to Quantum Optics: Lectures Presented At The Universite Libre de Bruxelles vol 18 (Berlin: Springer) p 1991
[5] Wiseman H M and Milburn G J 2010 Quantum Measurement and Control (Cambridge: Cambridge University Press) ISBN 9780521804424
[6] Wiseman H M and Diósi L 2001 Chem. Phys. 268 91–104
[7] Chia A and Wiseman H M 2011 Phys. Rev. A 84 012120
[8] Nurdin H I 2014 Russ. J. Math. Phys. 21 386–98
[9] Pellegrini C 2010 Markov chains approximation of jump-diffusion stochastic master equations Ann. Inst. Henri Poincare A 46 924–48
[10] Amini H, Pellegrini C and Rouchon P 2014 Russ. J. Math. Phys. 21 297–315
[11] Wieczorek W, Hofer S G, Hoelscher-Obermaier J, Riedinger R, Hammerer K and Aspelmeyer M 2015 Phys. Rev. Lett. 114 223601
[12] Broome M A, Fedrizzi A, Rahimi-Keshari S, Dove J, Aaronson S, Ralph T C and White A G 2013 Science 339 794–8
[13] Spring J B et al 2013 Science 339 798–801
[14] Lang C, Eichler C, Steffen L, Fink J M, Woolley M, Blais A and Wallraff A 2013 Nat. Phys. 9 345–8
[15] Neergaard-Nielsen J S 2008 Generation of single photons and Schrödinger kitten states of light PhD Thesis Danish National Research Foundation Center for Quantum Optics—Quantop Niels Bohr Institute
[16] Kuntz K B, Song H, Webb J G, Wheatley T A, Furusawa A, Ralph T C and Huntington E H 2014 Heralded, frequency-multiplexed, quantum and non-Gaussian states for telecommunications personal communication
[17] Chen Y F, Hover D, Sendelbach S, Maurer L, Merkel S, Pritchett E, Wilhelm F and McDermott R 2011 Phys. Rev. Lett. 107 217401
[18] Gough J and James M R 2009 IEEE Trans. Autom. Control 54 2530–44
[19] Parthasarathy K R 2012 An Introduction to Quantum Stochastic Calculus (Modern Birkhäuser Classics) (Berlin: Springer)
[20] Carmichael H, Castro-Beltran H, Foster G and Orozco L 2000 Phys. Rev. Lett. 85 1855
[21] Kuramochi Y, Watanabe Y and Ueda M 2013 J. Phys. A: Math. Theor. 46 425303
[22] Bouten L, Handel R V and James M R 2007 SIAM J. Control Optim. 46 2199–241
[23] Barchielli A 2006 Continual measurements in quantum mechanics and quantum stochastic calculus Open Quantum Systems III (Berlin: Springer) pp 207–92
[24] Barchielli A and van Handel R 2008 On the separation principle of quantum control Quantum Stochastics and Information: Statistics, Filtering and Control ed V P Belavkin and M Guţă (Singapore: World Scientific) pp 206–38
[25] Handel R V, Stockton J K and Mabuchi H 2005 IEEE Trans. Autom. Control 50 768–80
[26] Gough J E, Guta M, James M R and Nurdin H I 2011 Commun. Inf. Syst. 11 237–68
[27] Segall A, Davis M and Kailath T 1975 IEEE Trans. Inf. Theory 21 143–9
[28] Lototsky S 2006 Appl. Math. Optim. 54 265–91
[29] Crisan D and Rozovskii B 2011 The Oxford Handbook of Nonlinear Filtering (Oxford: Oxford University Press)
[30] Davies E B 1976 Quantum Theory of Open Systems (New York: Academic)
[31] Breuer H and Petruccione F 2007 The Theory of Open Quantum Systems (Oxford: Oxford University Press) ISBN 9780199213900
[32] Gardiner C and Zoller P 2004 Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics (Springer Series in Synergetics) (Berlin: Springer) ISBN 9783540223016