Methods for Research of Nonconservative Systems States

O A Burtseva¹, N R Abuladze¹, S A Chipko¹

¹ Platov South-Russian State Polytechnic University (Novocherkassk Polytechnic Institute), Prosveshcheniya str., 132, Novocheskarsk 346428, Russia

E-mail: kuzunaolga@yandex.ru

Abstract. Mechanic equations of viscoelastic solid body are used for formulation of research problem of deformable constructions. These equations are considered together with mixed boundary conditions. Nonconservative loadings are determined at one part of body surface, and at the other part of the body surface displacements are determined. Partial differential equations are transformed with variation methods into systems of ordinary differential equations that may have variable coefficients. Research of stability is performed on the base of joint solving of the main state equations and spectrum problem for linearized equations of disturbed motion. Lyapunov-Shmidt method and method of equivalent linearization are used for research of periodical regimes that branch themselves off the main states. Problem of oscillations of high-rise constructions under wind influence are solved.

1. Introduction

Common feature of self-oscillatory systems is nonlinearity and not conservatism. Therefore, researches of stability of their equilibrium states and, especially, movements can't be executed by Euler-Lagrange's methods or initial imperfections. Stability of self-oscillatory systems has to be investigated on the basis of «dynamic» approach - studying of their «indignant» movements and applications of the general criteria of A. M. Lyapunov [1, 2].

We qualify ways of the solution of tasks on self-oscillations and stability of nonconservative deformable systems on the following signs:
- to methods of approximation of real bodies various models with final or infinite number of degrees of freedom;
- to application of "exact" methods of the solution of the received equations (a way of "pointed transformation [3] of surfaces", methods of reduction of the nonlinear algebraic equations to the linear differential equations [4,5]);
- use of confidants (qualitative, asymptotic) methods of the solution of the nonlinear differential equations (methods: small parameter [6], Van-der-Polya [6, 7], averaging [8], equivalent linearization [9-11], Lyapunov-Schmidt [2, 12].

2. The general equations of motion of the viscoelastic bodies

For setting and solving the problem of investigating the auto-oscillations in the systems being deformed, we use the equation of the mechanics of viscoelastic solid body [13].

Let the body occupy the area $V$ of three-dimensional Euclidean space, limited by the surface $S$. We assign the location (configuration) of the deformed body in the space by radius-vectors of its dots at any moment of time. Three configurations of the body are considered as follows: reading $V$, for
which we select the configuration of the underformed body, basic \( V^0 \), corresponding to the process of deformations, agitated \( \vec{V} \). We designate radius-vectors of dots in each of them respectively as \( \mathbf{r}, \mathbf{r}^0(t), \mathbf{\bar{r}}(t) \), moreover
\[
\mathbf{r}^0(t) = f(r, t), \quad \mathbf{\bar{r}}(t) = \bar{f}(r, t), \quad \mathbf{r} \in V.
\] (1)

We call mappings (1) of configuration \( V \) in \( V^0 \) or \( \vec{V} \) as deformations. The Jacobians of these mappings at every moment of time are considered to be not degenerated:
\[
0 < j \equiv \det(\partial_r f) < \infty, \quad 0 < \bar{j} \equiv \det(\partial_{\mathbf{\bar{r}}} \bar{f}) < \infty.
\]

Displacement field it is described by vector \( \mathbf{W}(r, t) \). Gradient of deformation with respect to configuration \( V^0 \) is
\[
\mathbf{G} = \mathbf{I} + (\nabla \mathbf{W})^\tau,
\] (2)
where it is marked that \( \mathbf{I} = \mathbf{r}_i \mathbf{r}_j - \) is a single tensor, \( \mathbf{V} \) - the Hamiltonian operator, \( \tau \) - the sign of transportation.

We select Cauchy- Green tensor [14] as the measure of ultimate deformation.
\[
2 \mathbf{\varepsilon} = \mathbf{G}^\tau \mathbf{G} - \mathbf{I} = \nabla \mathbf{W} + (\nabla \mathbf{W})^\tau + \nabla \mathbf{W} \cdot (\nabla \mathbf{W})^\tau.
\] (3)

We assume the volume forces, which act on the body, to be stationary:
\[
\mathbf{F} = \mathbf{F}(r, \mathbf{W}, \dot{\mathbf{W}}, \nabla \mathbf{W}).
\] (4)

We consider that on the \( S_1 \) part of \( S \) surface from the \( V \) configuration the displacements \( \mathbf{W}_{|_{S_1}} = 0 \) is assigned; the \( S_2 \) part – has nonconservative forces
\[
\mathbf{E}_{\mathbf{P}} = q(r, \mathbf{W}, \dot{\mathbf{W}}, \mathbf{u}),
\] (5)
where the matrix is operator
\[
\mathbf{E}_{\mathbf{P}} = \text{diag} \left[ D + \lambda_1^{-1} |D + \lambda_2^{-1} |D + \lambda_3^{-1} \right].
\]

\( D \) - the derivative of time, \( \lambda \) - the constants, which have the dimensionality of time, \( \mathbf{u} \) - vector of the load parameters.

The connection between the stresses and the deformations should be presented in the following form:
\[
\mathbf{\Sigma} = \mathbf{E} \cdot \mathbf{\varepsilon} + \mathbf{E}_1 \cdot \mathbf{\varepsilon} \cdot \mathbf{\varepsilon} + \ldots,
\] (6)
where the tensor of generalized stresses with respect to the configuration \( V \) is
\[
\mathbf{\Sigma} = J \mathbf{G}^{-1} \cdot \mathbf{T} \cdot (\mathbf{G}^{-1})^\tau.
\] (7)

\( \mathbf{T} \) - is the tensor of actual stresses in of \( V^0 \) configuration.

In (6) and (7) formulas, designations «\( \cdot \)» and «\( -1 \)» correspond to operations convolution and the inversion of tensors.

We determine tensors of the fourth \( \mathbf{E}_1 \) and the sixth \( \mathbf{E}_1 \) rank by the expressions
\[
\mathbf{E}_1 = \mathbf{E} - \frac{\partial}{\partial t} \mathbf{I} \mathbf{\Theta}(t, \theta) \mathbf{d} \theta; \quad \mathbf{E}_1 = \mathbf{E}_1 - \frac{\partial}{\partial t} \mathbf{H}_1(t, \theta) \mathbf{d} \theta.
\]

Here \( \mathbf{E} \) and \( \mathbf{E}_1 \) – are tensors of the elastic characteristics of the material; \( \mathbf{\Theta} \) and \( \mathbf{H}_1 \) - are tensor nuclei of relaxation.

The motion of body in the reading \( V \) configuration is described by the nonlinear equations
\[
\rho \ddot{\mathbf{W}} = (\mathbf{G} \cdot \mathbf{\Sigma}) \cdot \mathbf{V} + \mathbf{F}; \quad \mathbf{\Sigma} = \mathbf{\Sigma} \cdot \mathbf{P} = (\mathbf{G} \cdot \mathbf{\Sigma}) \cdot \mathbf{n}.
\] (8)
Equations (8) together with the boundary conditions
\[
\mathbf{W}_{|_{S_1}} = 0, \quad (\mathbf{G} \cdot \mathbf{\Sigma}) \cdot \mathbf{n} = \mathbf{E}_1^{-1} \cdot q(r, \mathbf{W}, \dot{\mathbf{W}}, \mathbf{u}) = \mathbf{P}(r, \mathbf{W}, \dot{\mathbf{W}}, \mathbf{u}).
\] (9)

Compose the nonlinear mixed boundary problem of moving the viscoelastic body in relation to the reading configuration \( V \).

3. Formulation of the problem about the auto-oscillations of viscoelastic bodies
Let us formulate the problem of the body basic motion, caused by the development of viscous deformations.

2
Let \( W^0(t, r), \Sigma^0(t, r) \) - be the field of displacement and the field of the generalized stresses in the basic motion. Aggregate of \( W^0 \) and \( \Sigma^0 \) describes the process of the slowly developing deformations of body. In this case inertia terms can be disregarded in the equations of dynamics.

The equation of motion, disturbance in relation to the basic state, takes the form accordingly [13]
\[
\rho \delta W = (\bar{\mathbf{F}}' \cdot (\nabla \delta W)^\tau + (\nabla \delta W)^\tau \cdot \Sigma^0) \cdot \mathbf{V} + F_1 \cdot \delta W + F_2 \cdot \delta W + Q_1 \cdot \nabla \delta W +
\]
\[
+ N_1(\delta W, \delta W') + N_2(\delta W) \cdot \nabla.
\]
(10)

The boundary conditions correspond to the equation (10)
\[
\delta W|_{S_1} = 0, \quad (\bar{\mathbf{F}}' \cdot (\nabla \delta W)^\tau + (\nabla \delta W)^\tau \cdot \Sigma^0)n|_{S_2} = P_1(W^0, W^0, u)\delta W|_{S_2} +
\]
\[
+ P_2(W^0, W^0, u)\delta W|_{S_2} + N_3(\delta W, \delta W, u)|_{S_2}.
\]
(11)

In equations (10) and (11) the following designations are introduced:
\[
\mathbf{E}' = E' + \int_0^t \mathbf{E}'(t, r, \tau) d\tau, \quad E' = (I + (\nabla W^0)^\tau) \cdot E \cdot (I + \nabla W^0);
\]
\[
\mathbf{E}' = (I + (\nabla W^0)^\tau(t)) \cdot \mathbf{E} \cdot (I + \nabla W^0(t));
\]
\[
F_1 = \frac{\partial F}{\partial W}|_{W^0, w^0}, \quad F_2 = \frac{\partial F}{\partial W}|_{W^0, w^0}, \quad Q_1 = \frac{\partial Q}{\partial W}|_{W^0, w^0}, \quad P_1 = \frac{\partial P}{\partial W}|_{W^0, w^0}, \quad P_2 = \frac{\partial P}{\partial W}|_{W^0, w^0},
\]

where \( N_1(\delta W, \delta W') \) and \( N_2(\delta W, \delta W, u) \) - are nonlinear parts of the expansions in the series of vector function of \( F \) and \( P \) in the neighbour \( W^0 \) and \( W^0 \),
\[
N_2(\delta W) = \frac{1}{2} (I + \nabla W^0) \cdot \mathbf{E} \cdot \nabla \delta W \cdot (\nabla \delta W)^\tau + (\delta W)^\tau \cdot (\mathbf{E} \cdot (I + \nabla W^0) \cdot \nabla \delta W +
\]
\[
+ \frac{1}{2} \mathbf{E} \cdot \nabla \delta W \cdot (\nabla \delta W)^\tau).
\]

4. Determination of the critical parameters of viscoelastic systems

The boundary value problem (10), (11) is equivalent to the operator equation
\[
\mathbf{M} \delta \mathbf{W} - \mathbf{\Gamma} \delta \mathbf{W} + \mathbf{H} \delta \mathbf{W} - \int_0^t D(t, r) \delta \mathbf{W} d\tau = \mathbf{F}(\delta \mathbf{W}, \delta \mathbf{W}, u).
\]
(12)

Here
\[
\mathbf{M}, \quad \mathbf{\Gamma} = \Gamma_1 + \Gamma_2, \quad \mathbf{H} = \mathbf{I} - \mathbf{A} - \mathbf{B}, \quad \int_0^t D(t, r, \tau) d\tau, \quad \mathbf{F}(\delta \mathbf{W}, \delta \mathbf{W}, u)
\]
respectively, the operators of inertial, dissipative, elastic, viscous and external non-conservative nonlinear forces.

The basic state of system is obtained during the solution of boundary problem (10), (11) at \( \mathbf{F}(\delta \mathbf{W}, \delta \mathbf{W}, u) = \mathbf{0} \). For the analysis of stability of this state we examine the linearized equation of the disturbance motion
\[
\mathbf{M} \delta \mathbf{W} - \mathbf{\Gamma} \delta \mathbf{W} + \mathbf{H} \delta \mathbf{W} - \int_0^t D(t, r) \delta \mathbf{W} d\tau = 0.
\]
(13)

With large \( t \), linearized problem (13) of integrodifferential operator by time can be replaced with the differential equation
\[
\mathbf{M} \delta \mathbf{W} - \mathbf{\Gamma} \delta \mathbf{W} + \mathbf{H}_{\omega} \delta \mathbf{W} = 0,
\]
(14)
in which the complex operator is introduced
\[
\mathbf{H}_{\omega} = \mathbf{H} - \Pi_{\omega},
\]
where \( \Pi_{\omega} \) - «limit viscosity operator».

We investigate differential equation (14) by a standard method. Assuming that
\[
\delta \mathbf{W}(r, t) = e^{\lambda t} \delta \mathbf{W}(r),
\]
we obtain the spectral task
\[
(\lambda^2 \mathbf{M} - \lambda \mathbf{\Gamma} + \mathbf{H}_{\omega}) \delta \mathbf{W}(r) = 0.
\]
(15)
The pair of roots corresponds to the critical assembly of the system parameters \( \lambda = \pm i\omega \); the rest have \( \text{Re} \, \lambda < 0 \).

For finding the critical parameters, problems (10), (11) and (15) should be solved, for example, with

\[
\upsilon = m + \alpha n
\]

where \( m \) and \( n \) - are vectors with the assigned coefficients. A certain straight line in the multidimensional parameter spaces \( \upsilon_e \) (\( e = 1, \ldots, k \)) corresponds to the equation (16). The values of \( \alpha_e \) determine distances from the dot with the coordinates \( m_j \) to the boundary surfaces, which divide the stability and instability areas. Gradually increasing the parameter of \( \alpha \), from equations (10), (11) we find the appropriate displacement of \( W^0 \) and the roots of spectral task (15). Using a method of sequential approximations, we calculate \( \alpha_{kr} \) with the necessary accuracy.

5. Analysis of the auto-oscillations stability

For the analysis of auto-oscillations stability \( \delta W \) we examine operator equation (12), varied near the state \( \delta W \):

\[
M\delta \dot{w} - \tilde{T}\delta w + H_0\delta w = \frac{\partial F}{\partial \delta w} \delta \dot{w} + \frac{\partial F}{\partial \delta w} \delta w,
\]

where the derivatives \( \frac{\partial F}{\partial \delta w} \), \( \frac{\partial F}{\partial \delta w} \) are found in the meaning of Frechet [17]. Assuming that

\[
\alpha = \alpha_{kr} + \upsilon^2, \quad \delta w = e^{\text{wt}} \delta w, \quad \tau = \omega t;
\]

\[
\bar{T}(W^0, \alpha_{kr} + \upsilon^2) = \bar{T}(W^0, \alpha_{kr}) + \sum \upsilon^k \bar{T}_k (W^0, \alpha_{kr});
\]

\[
H_0(W^0, \alpha_{kr} + \upsilon^2) = H_0(W^0, \alpha_{kr}) + \sum \upsilon^k H_k (W^0, \alpha_{kr});
\]

we obtain:

\[
\omega^2 M\delta \dot{w} - \omega T\delta w + H_0 \delta w = -2\omega \sigma M\delta \dot{w} - \sigma^2 M\delta w + \sigma T\delta w +
\]

\[
+ \frac{\partial F}{\partial \delta w} \delta \omega + \frac{\partial F}{\partial \delta w} (\omega \delta w + \sigma \delta w) + \delta w \sum \upsilon^k (\sigma \bar{T}_k - H_k + \frac{\partial F}{\partial \delta w} + \sigma \frac{\partial F}{\partial \delta w}) +
\]

\[
+ \omega \sigma \delta w \sum \upsilon^k (\bar{T}_k + \frac{\partial F}{\partial \delta w})
\]

(17)

We search the solution of the equation (17) in the form

\[
\delta \dot{w} = \sum_{k=1}^{\infty} \upsilon^k \omega_k.
\]

(18)

assuming that

\[
\sigma = \sum_{k=1}^{\infty} \upsilon^k \alpha_k,
\]

(19)

where \( \sigma_k \) - are the unknown numbers.

Substituting series (18) and (19) into operator equation (17) and equalizing terms with the identical degrees \( \upsilon \), we obtain the sequence of the equations:

\[
\omega^2 \sum_{k=1}^{\infty} \upsilon^k \omega_k
\]

(17)

where \( f_k \) - are the vectors, which depend on the form of the operators

\[
\Gamma_k, \quad H_k, \quad \frac{\partial F}{\partial \delta w}, \quad \frac{\partial F}{\partial \delta w}.
\]

Using conditions of solvability of operator equations

\[
\int_0^{\tau} < v, f_k > \ dt = 0 \quad (k = 1, 2, \ldots),
\]

we consequently find values \( \sigma_1, \sigma_2, \ldots \). If \( \text{Re} \, \sigma_1 > 0 \), then the auto-oscillations of system are unstable, with \( \text{Re} \, \sigma_1 < 0 \) - they are stable. In the case of \( \text{Re} \, \sigma_1 = 0 \) we pass to a study of sign \( \text{Re} \, \sigma_2 \) and the like.
6. Research of high-rise buildings auto-oscillations flowed round by air current

As a result of interaction of the high-rise buildings modelled in a form of a bar, with an air current flowing round it, the aerodynamic forces (frontal, tangential, elevating) and the moment proportional to speed of an oncoming current appear. At detached flow of round section the periodical (Karman force) force appears, that is orthogonal to a vector of relative speed with the frequency depending on Strouhal number. The projections of the frontal force on the axis of the fixed coordinate system are determined by the relations in [18, 19], Fig. 1:

The equations of bending oscillations are as follows:

\[ \frac{\partial^4 u_1}{\partial \tau^4} + \beta \frac{\partial^2 u_1}{\partial \tau^2} \left(1 - \varepsilon^2\right) \frac{\partial u_1}{\partial \tau} + \frac{\partial^2 u_1}{\partial \tau^2} + q_{n0}\beta_0 (1 + \cos^2 \alpha) \frac{\partial u_1}{\partial \tau} + q_{n0}\beta_0 \sin \alpha \cos \alpha \frac{\partial u_1}{\partial \tau} + q_{k0}\beta_0 \sin \alpha \sin \omega_0 \tau \frac{\partial u_1}{\partial \tau} + q_{k0}\beta_0 (1 + \sin^2 \alpha) \sin \omega_0 \tau \frac{\partial u_1}{\partial \tau} = q_{n0}\cos \alpha - q_{k0} \sin \omega_0 \tau + N_1; \]

\[ \frac{\partial^4 u_3}{\partial \tau^4} + \beta \frac{\partial^2 u_3}{\partial \tau^2} \left(1 - \varepsilon^2\right) \frac{\partial u_3}{\partial \tau} + \frac{\partial^2 u_3}{\partial \tau^2} + q_{n0}\beta_0 (1 + \cos^2 \alpha) \frac{\partial u_3}{\partial \tau} + q_{n0}\beta_0 \sin \alpha \cos \alpha \frac{\partial u_3}{\partial \tau} + q_{k0}\beta_0 \sin \alpha \sin \omega_0 \tau \frac{\partial u_3}{\partial \tau} + q_{k0}\beta_0 (1 + \sin^2 \alpha) \sin \omega_0 \tau \frac{\partial u_3}{\partial \tau} = q_{n0}\sin \alpha + q_{k0} \cos \sin \omega_0 \tau + N_2. \]

In formulas (20) dimensionless values are introduced:

\[ s = \varepsilon \lambda, \quad u_1 = u_1/l, \quad u_3 = u_3/l, \quad \beta = \frac{pg^2}{EJ}, \quad p_0 = \frac{\rho l^2}{EJ}, \quad \tau = \omega_0 t, \quad \omega_0 = \omega/p_0; \]

\[ \beta = \frac{pg^2}{V_0}, \quad q_{n0} = \frac{C_n ydV_0^3 l^n}{2EJ}, \quad q_{k0} = \frac{C_k ydV_0^3 l^n}{2EJ}, \]

where \( \varepsilon \) - is the angle between the velocity of the oncoming stream \( V_0 \) and the axis; \( \rho \) - is a specific density of a bar; \( EJ \) – bending rigidity; \( l \) – lengths of a bar; \( d \) – bar diameter; \( g \) - gravitational constant; \( \gamma \) – specific air density; \( C_n \) and \( C_k \) - dimensionless constants determined by experiments. In the formulas (20) the summands \( N_1 \) and \( N_2 \) containing nonlinear terms are introduced.

Assuming that \( u_j(\varepsilon, \tau) = \frac{1}{2} f_j(\tau) (1 - \cos \omega_0 \tau), \quad j = 1, 3 \) and using the Bubnov-Galyerkin method we reduce homogeneous equations in partial differentiations (20) to system nonlinear of ordinary second-order differential equations which are transformed below into a system of first order differential equations

\[ \dot{x} = A(u, t)x + N(u, x, t), \]

where \( x = [f_1, f_2, f_3, f_4] \) – is the state vector; \( A(u, t) \) – is a periodic matrix \( 2n \times 2n \), depending on a vector of parameters \( u = [C_n, C_k, \beta_0, \alpha, \alpha] \); \( N(u, x, t) \) – is a nonlinear relatively to \( x \) vector-function and \( 2\pi / \omega_0 \) – a periodic relatively to \( \tau \). On the basis of a linear system of equations

\[ \dot{x} = (A_0 + \Lambda_0 e^{i\omega_0 \tau} + \Lambda_0 e^{-i\omega_0 \tau})x \]

the problem of detection of excitation conditions of parametric vibrations is solved.

The solution of the equation (21) is searched in the form of:

\[ x = \sum_{k=0}^{\infty} \left( x_k e^{i\omega_0 k\tau} + \bar{x}_k e^{-i\omega_0 k\tau} \right) \]

As a result, of substitution (22) into (21) and equating the members at identical \( e^{i\omega_0 k\tau} \) for the sequences \( k = 0, 2, 4, \ldots \) and \( k = 1, 3, 5, \ldots \) the systems of algebraic equations result. Equating zero
determinants of the received equations, we receive formulas for finding the critical parameter $\nu$ and oscillation frequency $\omega_0$.

**Figure 2.** Dependence of arrangement of instability area on parameters values $\beta, C_k, C_n$:

a) $\beta = 5, C_k = 0.3, C_n = 0.005$ (black diagram), $\beta = 0.5, C_k = 0.3, C_n = 0.005$ (blue diagram);

b) $\beta = 2, C_k = 1.5, C_n = 0.005$ (black diagram), $\beta = 2, C_k = 0.55, C_n = 0.005$ (blue diagram);

c) $\beta = 5, C_k = 0.3, C_n = 0.005$ (black diagram), $\beta = 5, C_k = 0.3, C_n = 0.05$ (blue diagram)

7. Conclusions

We consider the parameter $\alpha_1$ to be changing in a given range and $\omega_0$ to be variable (relative to the latter the equation is solved). As a result of numerical experiments, the areas of instability depending on the parameters $C_n, C_k, \beta, \alpha_1, \alpha$ are drawn up. Characteristic boundary curves dividing the areas of stability and instability are presented in Fig. 2. The computation results give us the possibility to make the following conclusions:

- at small parameters $C_n, C_k$ the proportional dependence of the frequency $\omega_0$ of parametric oscillations on critical values of the parameter $\alpha_1$ that characterizes the eddy currents velocity is observed that is coordinated with the known results;
- at increase $C_n, C_k$ the instability areas expand and the proportional dependence $\omega_0(\alpha_1)$ is disturbed;
- at increase of the bar rigidity the increase of the parametric oscillation frequency is observed.

References

[1] Lyapunov A M 1950 *Obshchaja zadacha ob ustojchivosti dvizhenija* [The general task about stability of the movement] (Moscow-Leningrad: Gostekhizdat Publ) 472 p.

[2] Lyapunov A M 1959 *Sbornik sochineniy* [A collection of essays] (Moscow, Prod. Academy of Sciences of the USSR Vol.4) 645 p.

[3] Andronov A A, Witt A A, Khaykin S Z 1981 *Teoriya kolebaniy* [Theory of fluctuations] (Moscow, Fizmatgiz Publ.) 568 p.

[4] Vorovich I I, Zipalova V F 1965 *K resheniyu kraevyh zadach teorii uprugosti metodom perekhoda k zadache Koshi* [To the solution of regional tasks of the theory of elasticity by method of transition to Cauchy's task] Vol. 29(5), pp. 894-901.

[5] Davidenko D F 1953 Ob odnom novom metode chislennogo resheniya sistem nelineynykh uravneniy [About one new method of the numerical decision of systems of the nonlinear equations] (Academy of Sciences of the USSR Vol. 38(2)) pp. 601-602.

[6] Mandelstam L I, Papaleksi N D 1934 *Ob obosnovanii odnogo metoda pribilzhenogo resheniya differentsial'nykh uravneniy* [About justification of one method of the approximate solution of the differential equations] Vol. 34(2)) pp. 117-121.
[7] Wang-der-Paul B 1935 Nelineynaya teoriya elektricheskikh kholebanii [Nonlinear theory of electric fluctuations] (Moscow, Sovyzfirst Publ.) 167 p.

[8] Krylov N M, Bogolyubov N N 1937 Vvedenie v nelineynuyu mehaniku [Introduction to nonlinear mechanics] (Kiev: Academy of Sciences of the USSR) 363 p.

[9] Savinov G V, Avtokolebaniya v sushchestvenno nelineynykh kvazikonservativnykh sistemakh [Self-oscillations in significantly nonlinear quasiconservative systems] (Academy of Sciences of the USSR, 1953, Vol. 89(6)) pp. 995-1011.

[10] Savinov G V 1953 Avtokolebaniya sistem s sil’no vyrazhennoj nelinejnost’yu [Self-oscillations of systems with strongly expressed the nonlinearity] (Vestnik MGU. Ser. fiz., estestv. Nauk Publ., 1953, Vol. (6)) pp. 77-81.

[11] Chezari L 1964 Asimptoticheskoe povedenie i ustoichivost resheniy obyknovenykh differentsialnykh uravneniy [Asymptotic behavior and stability of solutions of the ordinary differential equations] (Moscow, Mir Publ., 1964) 477 p.

[12] Weinberg M M, Trenogin V A 1969 Teoriya vverleniya resheniy nelineynykh uravneniy [Theory of branching of solutions of the nonlinear equations] (Moscow: Nauka Publ.) 527 p.

[13] Gromov V G 1976 Pervyy metod Lyapunova v dinamicheskoy ustoichivosti gibkikh termovyzkoupurugikh tel [The first method of Lyapunov in dynamic stability of flexible viscoelastic bodies] (Academy of Sciences of the USSR, 1976, Vol. 223(4)) pp. 819-822.

[14] Lurye A I, Teoriya uprugosti [Theory of elasticity], Moscow, Nauka Publ., 1970, 940 p.

[15] Thunders V G 1975 Dinamicheskii kriteriy ustoichivosti i zakriticheskoe povedenie gibkikh vyzkoupurugikh tel pri termosilovom zagruzhenii [Dynamic criterion of stability and zakriticheskoy behavior of flexible viscoelastic bodies at a thermopower uploading] (Academy of Sciences of the USSR, 1975, Vol. 220(4)) pp. 805-808.

[16] Neymark Yu I 1948 Struktura D – razbieniya prostranstva polinomov i diagrammy Vyshegradskogo i Naykviasta [Structure of D – splittings space of polynoms and the chart Above-town and Nyquist] (Academy of Sciences of the USSR, 1948, Vol. 59(5)) pp. 859-856.

[17] Kolmogorov A N, Fomin S V 1976 Elementy teorii funktsiy i funktsional’nogo analiz [Elements of the theory of functions and functional analysis] (Moscow, Nauka Publ.) 544 p.

[18] Svetlit’sky V A 1987 Mekhanika sterzhney: Ucheb. dlya vtuzov. V 2–kh ch. CH.1 [Mechanics of cores: Studies. for technical colleges. In 2 parts. Part 1. Statics] (Moscow, Vysshaya shkola Publ.) 320 p.

[19] Svetlit’sky V A 1987 Mekhanika sterzhney: Ucheb. dlya vtuzov. V 2–kh ch. CH.2. Dinamika [Mechanics of cores: Studies. for technical colleges. In 2 parts. Part 2. Dynamics] (Moscow: Vysshaya shkola Publ.) 304 p.