PACKING METRIC MEAN DIMENSION OF SETS OF GENERIC POINTS

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Abstract. We obtain some new formulas of packing metric mean dimension of the sets of generic points of ergodic measures without additional conditions compared with our previous work [YCZ22].

1. Introduction

By a pair \((X, T)\) we mean a topological dynamical system (TDS for short), where \(X\) is a compact metric space with metric \(d\) and \(T : X \to X\) is a homeomorphism transformation. By \(M(X)\), \(M(X, T)\), \(E(X, T)\) we respectively denote the sets of all Borel probability measures on \(X\) endowed with the weak\(^*\)-topology, all \(T\)-invariant Borel probability measures on \(X\), all ergodic measures on \(X\).

In 1973, Bowen [Bow73] introduced the notion of Bowen topological entropy \(h_{\text{Btop}}(T, Z)\) for arbitrary non-empty subset \(Z \subset X\) by resembling the definition of Hausdorff dimension. He proved a remarkable result which states that if \(\mu \in E(X, T)\), then the Bowen topological entropy of the set of generic points of \(\mu\) is equal to the measure-theoretic entropy \(h_{\mu}(T)\). Namely, \(h_{\text{Btop}}(T, G_{\mu}) = h_{\mu}(T)\), where \(G_{\mu} := \{x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)} = \mu\}\) denotes the set of the generic points of \(\mu\). Later, Zhao et al. [ZCZ18] generalized this result to weighted Bowen topological entropy. The versions of this result for amenable group and fixed-point free flows can be respectively founded in [ZC18, WCL20]. If \(\mu \in M(X, T)\), Pfister and Sullivan [PS07] showed that \(h_{\text{top}}(T, G_{\mu}) = h_{\mu}(T)\) still holds under the condition of the \(g\)-almost product property, and this work was extended to amenable group by Zhang [Z18] by using quasi-tiling method. Wang [W21b] further showed if \(\mu\) is ergodic, then the packing topological entropy of \(G_{\mu}\) denoted by \(h_{\text{top}}^p(T, G_{\mu})\) also coincides with \(h_{\mu}(T)\), see also [DZZ20] for the amenable version of this result.

In 2000, Lindenstrauss and Weiss [LW00] introduced the notion of metric mean dimension and showed that metric mean dimension is

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an upper bound of mean topological dimension introduced by Gromov [Gro99], which provides a powerful method to bound the mean topological dimension. In the context of metric mean dimension, the left hand side of the formula $h^p_{\text{top}}(T,G_\mu) = h^p_{\text{top}}(T,G_\mu) = h_\mu(T)$ are respectively replaced by Bowen and packing metric mean dimensions, and the right hand of formula $h_\mu(T)$ are replaced by several types of candidates related to measure-theoretic entropy. If $\mu \in E(X,T)$, Wang [W21a] established inequalities between Bowen upper metric mean dimension of $G_\mu$ and rate-distortion dimension coming from information theory. Later, the authors [YCZ22] further verified that these inequalities can be equal under an additional condition. However, in the present paper we can establish some new formulas for packing metric mean dimension of $G_\mu$ without additional conditions compared with our previous work [YCZ22, Theorem 1.5]. To this end, the key proof is along the following two steps:

Step 1. To use Katok’s entropy to characterize the $\text{mdim}^P(T,G_\mu,d)$, we need to investigate the relationships between upper Katok’s entropy and lower Katok’s entropy. This allows us to establish a version of Katok’s entropy formula for ergodic measures in sense of measure-theoretic metric mean dimension, which is an analogue of the classical Katok’s entropy formula [Kat80].

Theorem 1.1. Let $(X,T)$ be a TDS and $\mu \in E(X,T)$. Then for every $\delta \in (0,1)$, we have

$$\limsup_{\varepsilon \to 0} \frac{K_\mu(T,\varepsilon,\delta)}{\log \frac{1}{\varepsilon}} = \limsup_{\varepsilon \to 0} \frac{h^K_\mu(T,\varepsilon,\delta)}{\log \frac{1}{\varepsilon}} = \limsup_{\varepsilon \to 0} \frac{\overline{h}_\mu(T,\varepsilon)}{\log \frac{1}{\varepsilon}} = \limsup_{\varepsilon \to 0} \frac{\underline{h}_\mu(T,\varepsilon)}{\log \frac{1}{\varepsilon}}.$$

The first equality is independent of the choice of $\delta \in (0,1)$, and the above formulas are also valid if we change $\limsup_{\varepsilon \to 0}$ into $\liminf_{\varepsilon \to 0}$.

Step 2. To use Bowen and packing upper measure-theoretic metric mean dimensions of $\mu$, which are defined by Carathéodory-Pesin structures, to characterize $\text{mdim}^P(T,G_\mu,d)$, we need to compare the two quantities with measure-theoretic upper and lower Brin-Katok entropies of $\mu$, this additionally allows us to establish two new variational principles for Bowen and packing upper metric mean dimensions on subsets.
Theorem 1.2. Let $(X, T)$ be a TDS with a metric $d$ and $K$ be a non-empty compact subset of $X$. Then
\[
\overline{\text{mdim}}^B(T, K, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup \{ M_\mu(T, d, \epsilon) : \mu \in M(X), \mu(K) = 1 \},
\]
\[
\overline{\text{mdim}}^P(T, K, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup \{ P_\mu(T, d, \epsilon) : \mu \in M(X), \mu(K) = 1 \},
\]
where $\overline{\text{mdim}}^B(T, K, d)$ and $\overline{\text{mdim}}^P(T, K, d)$ respectively denote Bowen and packing upper metric mean dimensions of $T$ on $K$.

The “new variational principles” means that we find two candidates
\[
M_\mu(T, d, \epsilon), P_\mu(T, d, \epsilon)
\]
given by dimensional characterizations to replace the previous measure-theoretic upper and lower Brin-Katok entropies used in [W21a, YCZ22]. We refer readers to [VV17, LT18, LT19, Shi22, Tsu20, GS21, Li21] for more details about the interplay between metric mean dimension theory and ergodic theory. After finishing the above two steps, we can obtain some new formulas for $\overline{\text{mdim}}^P(T, G_\mu, d)$ as follows.

Theorem 1.3. Let $(X, T)$ be a TDS with a metric $d$ and $\mu \in E(X, T)$. Then for every $\delta \in (0, 1)$, we have
\[
\overline{\text{mdim}}^P(T, G_\mu, d) = \limsup_{\epsilon \to 0} \frac{F(\mu, \epsilon)}{\log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \frac{\inf \{ P(T, d, Z, \epsilon) : \mu(Z) = 1 \}}{\log \frac{1}{\epsilon}},
\]
where the candidate
\[
F(\mu, \epsilon) \in \begin{cases}
\overline{h}_K(\mu(T, \epsilon, \delta)), \overline{\mu}_K(T, \epsilon, \delta), \overline{h}_K(\mu(T, \epsilon)), \overline{\mu}_K(T, \epsilon), \overline{h}^B_{\mu}(T, \epsilon), \\
P_\mu(T, d, \epsilon), PS_\mu(T, \epsilon), R_{\mu, L^\infty}(\epsilon), \inf_{\text{diam}P \leq \epsilon} h_\mu(T, P), \inf_{\text{diam}(U) \leq \epsilon} h_\mu(U)
\end{cases},
\]
and the infimum in the second equality can be attained for some subsets of $X$. $P$ ranges over all finite Borel partitions of $X$ with diameter at most $\epsilon$, and $U$ ranges over all finite open covers of $X$ with diameter at most $\epsilon$. Moreover, it is also valid if we change $\limsup_{\epsilon \to 0}$ into $\liminf_{\epsilon \to 0}$ and $\overline{\text{mdim}}^P(T, G_\mu, d)$ into $\text{mdim}^P(T, G_\mu, d)$.

We gives some remarks for Theorem 1.3 as follows.

(i). The second equality in Theorem 1.3 establishes the relationship between the candidates and the packing metric mean dimension on subsets, which is analogue of the inverse variational principle for measure-theoretic entropy stating that $h_\mu(T) = \inf \{ h_{\text{top}}^B(T, Z) : \mu(Z) = 1 \}$ for any $\mu \in E(X, T)$ obtained in [Bow73, Pes97].
(ii). The formula \( \limsup_{\epsilon \to 0} \frac{h_{BK}(T, \epsilon)}{\log \frac{1}{\epsilon}} = \text{mdim}_{M}(T, G, d) \) can be regarded as the partial Brin-Katok formula \([BK83]\) in sense of measure-theoretic metric mean dimension.

The rest of this paper is organized as follows. In section 2, we recall the definitions of metric mean dimension and collect several types of candidates related to measure-theoretic entropy. In section 3, we prove Theorems 1.1, 1.2, 1.3.

2. Preliminary

2.1. Upper metric mean dimension. In this subsection, we recall the definitions of metric mean dimension on the whole phase space and Bowen and packing metric mean dimensions on subsets.

Let \( n \in \mathbb{N} \), \( x, y \in X \), we define the \( n \)-th Bowen metric \( d_n \) on \( X \) as \( d_n(x, y) := \max_{0 \leq j \leq n-1} d(T^j(x), T^j(y)) \). Let \( \epsilon > 0 \), the Bowen open ball and closed ball of radius \( \epsilon \) and order \( n \) in the metric \( d_n \) around \( x \) are respectively given by

\[
B_n(x, \epsilon) = \{ y \in X : d_n(x, y) < \epsilon \}, \\
\overline{B}_n(x, \epsilon) = \{ y \in X : d_n(x, y) \leq \epsilon \}.
\]

Let \( Z \subset X \) be a non-empty subset. A set \( F \subset Z \) is an \((n, \epsilon)\)-separated set of \( Z \) if \( d_n(x, y) > \epsilon \) for any \( x, y \in F \) with \( x \neq y \). The maximal cardinality of \((n, \epsilon)\)-separated set of \( Z \) is denoted by \( s_n(T, d, \epsilon, Z) \). We define

\[
s(T, X, d, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log s_n(T, d, \epsilon, X).
\]

**Definition 2.1.** Let \((X, T)\) be a TDS with a metric \( d \). The lower and upper metric mean dimensions of \( T \) on \( X \) are respectively given by

\[
\text{mdim}_M(T, X, d) = \liminf_{\epsilon \to 0} \frac{s(T, X, d, \epsilon)}{\log \frac{1}{\epsilon}}, \\
\overline{\text{mdim}}_M(T, X, d) = \limsup_{\epsilon \to 0} \frac{s(T, X, d, \epsilon)}{\log \frac{1}{\epsilon}}.
\]

It is clear that \( \overline{\text{mdim}}_M(T, X, d) \) (or \( \text{mdim}_M(T, X, d) \)) depends on the metric on \( X \) and hence is not a topological invariant. Recall that topological entropy of \( X \) is given by \( h_{\text{top}}(T, X) = \lim_{\epsilon \to 0} s(T, X, d, \epsilon) \). Hence \( s(T, X, d, \epsilon) < \infty \) for all \( \epsilon > 0 \) if the system has finite topological entropy, this implies that \( \text{mdim}_M(T, X, d) = 0 \). Consequently, metric mean dimension is a useful quantity to characterize the systems with infinite topological entropy.

We continue to recall the definitions of Bowen metric mean dimension on subsets \([W21a]\) and packing metric mean dimension on subsets \([YCZ22]\) defined by Carathéodory-Pesin structures.
Definition 2.2. Let $Z \subset X$, $\epsilon > 0$, $N \in \mathbb{N}$, $s \in \mathbb{R}$. Define

$$M(T, d, Z, s, N, \epsilon) = \inf \left\{ \sum_{i \in I} e^{-n_i s} \right\},$$

where the infimum is taken over all finite or countable covers $\{B_{n_i}(x_i, \epsilon)\}_{i \in I}$ of $Z$ with $n_i \geq N, x_i \in X$.

Since $M(T, d, Z, s, N, \epsilon)$ is non-decreasing when $N$ increases, so we define

$$M(T, d, Z, s, \epsilon) = \lim_{N \to \infty} M(T, d, Z, s, N, \epsilon).$$

It is readily to check that the quantity $M(T, d, Z, s, \epsilon)$ has a critical value of parameter $s$ jumping from $\infty$ to $0$. We define such critical value as

$$M(T, d, Z, s, \epsilon) : = \inf \left\{ s : M(T, d, Z, s, \epsilon) = 0 \right\} = \sup \left\{ s : M(T, d, Z, s, \epsilon) = \infty \right\}.$$ 

The Bowen upper metric mean dimension of $T$ on the set $Z$ is given by

$$\overline{\text{mdim}}^B(T, d, Z) = \limsup_{\epsilon \to 0} \frac{M(T, d, Z, \epsilon)}{\log \frac{1}{\epsilon}}.$$ 

Replacing $\limsup_{\epsilon \to 0}$ by $\liminf_{\epsilon \to 0}$, one can similarly define Bowen lower metric mean dimension $\underline{\text{mdim}}^B(T, d, Z)$ of $T$ on the set $Z$.

Definition 2.3. Let $Z \subset X$, $\epsilon > 0$, $N \in \mathbb{N}$, $s \in \mathbb{R}$. Define

$$P(T, d, Z, s, N, \epsilon) = \sup \left\{ \sum_{i \in I} e^{-n_i s} \right\},$$

where the supremum is taken over all finite or countable pairwise disjoint closed families $\{B_{n_i}(x_i, \epsilon)\}_{i \in I}$ of $Z$ with $n_i \geq N, x_i \in Z$.

The quantity $P(T, d, Z, s, N, \epsilon)$ is non-increasing as $N$ increases, so we define

$$P(T, d, Z, s, \epsilon) = \lim_{N \to \infty} P(T, d, Z, s, N, \epsilon).$$

Set

$$\mathcal{P}(T, d, Z, s, \epsilon) = \inf \left\{ \sum_{i=1}^{\infty} P(T, d, Z_i, s, \epsilon) : \cup_{i \geq 1} Z_i \supseteq Z \right\}.$$ 

It is readily to check that the quantity $\mathcal{P}(T, d, Z, s, \epsilon)$ has a critical value of parameter $s$ jumping from $\infty$ to $0$. We define such critical value as

$$\mathcal{P}(T, d, Z, s, \epsilon) : = \inf \left\{ s : \mathcal{P}(T, d, Z, s, \epsilon) = 0 \right\} = \sup \left\{ s : \mathcal{P}(T, d, Z, s, \epsilon) = \infty \right\}.$$
The packing upper metric mean dimension of $T$ on the set $Z$ is given by
\[
\overline{\text{mdim}}^P(T, Z, d) = \limsup_{\epsilon \to 0} \frac{\mathcal{P}(T, d, Z, \epsilon)}{\log \frac{1}{\epsilon}}.
\]
Replacing $\limsup_{\epsilon \to 0}$ by $\liminf_{\epsilon \to 0}$, one can similarly define packing lower metric mean dimension $\underline{\text{mdim}}^P(T, Z, d)$ of $T$ on the set $Z$.

It is straightforward to check the following statements.

**Proposition 2.4.** Let $(X, T)$ be a TDS. Then

1. If $Z_1 \subset Z_2 \subset X$, then $
\overline{\text{mdim}}^P(T, Z_1, d) \leq \overline{\text{mdim}}^P(T, Z_2, d),
\underline{\text{mdim}}^P(T, Z_1, d) \leq \underline{\text{mdim}}^P(T, Z_2, d).
\]

2. If $Z$ is a countable union of $Z_i$, then
$\mathcal{M}(T, d, Z, \epsilon) = \sup_{i \geq 1} \mathcal{M}(T, d, Z_i, \epsilon),$
$\mathcal{P}(T, d, Z, \epsilon) = \sup_{i \geq 1} \mathcal{P}(T, d, Z_i, \epsilon).$

3. If $Z$ is a finite union of $Z_j$, $j = 1, \ldots, N$, then
$\max_{1 \leq j \leq N} \overline{\text{mdim}}^B(T, Z_j), \underline{\text{mdim}}^P(T, Z) = \max_{1 \leq j \leq N} \underline{\text{mdim}}^B(T, Z_j).$

2.2. **Several types of measure-theoretic entropies.** In this subsection, we collect several types of candidates related to measure-theoretic entropy to describe the metric mean dimension of $G_\mu$.

2.2.1. **Shapira’s entropy.** Measure-theoretic entropy defined by fixed open covers.

Let $\mathcal{U}$ be a finite open cover of $X$. By $\text{diam}(\mathcal{U}) = \max_{A \in \mathcal{U}} \text{diam} U$ we denote the diameter of $\mathcal{U}$. By $\text{Leb}(\mathcal{U})$ we denote the Lebesgue number of $\mathcal{U}$. Namely, the largest positive number $\delta > 0$ such that every open ball of radius $\delta$ is contained in an element of $\mathcal{U}$. We write $P \succ \mathcal{U}$ to denote $P$ refining $\mathcal{U}$, that is, every atom $A \in P$ is contained in some members of $\mathcal{U}$.

Let $P, Q$ be two finite partitions of $X$, the join of $P$ and $Q$ is defined by $P \lor Q := \{ A \cap B : A \in P, B \in Q \}$. Set $P^n := \lor_{j=0}^{n-1} T^{-j} P$, and the measure-theoretic entropy of $\mu$ w.r.t. $T$ and $P$ is given by
\[
h_\mu(T, P) = \lim_{n \to \infty} \frac{1}{n} H_\mu(P^n),
\]
where $H_\mu(P)$ is the partition entropy of $P$.

The measure-theoretic entropy of $\mu$ is defined by
\[
h_\mu(T) = \sup_P h_\mu(T, P)
\]
with supremum over all finite Borel partitions of $X$.

Let $\mu \in E(X, T)$ and $\mathcal{U}$ be a finite open cover of $X$. Put
\[
N_\mu(\mathcal{U}, \delta) := \min \{ \# \mathcal{U}' : \mu(\bigcup_{A \in \mathcal{U}'} A) \geq \delta, \mathcal{U}' \subset \mathcal{U} \},
\]
Shapira [Sha07, Theorem 4.2] showed that the limit
\[ h^S_\mu(U) := \lim_{n \to \infty} \frac{\log N_\mu(U^n, \delta)}{n} \]
exists and does not depend on the choice of \( \delta \in (0, 1) \). Besides, Shapira [Sha07, Theorem 4.4] proved that Shapira’s entropy can be given by measure-theoretic entropy of \( \mu \) as follows.
\[ h^S_\mu(U) = \inf_{P \succ U} h_\mu(T, P), \]
where \( P \) ranges over all partitions refining \( U \). In fact, the infimum can only consider all finite partitions of \( X \) refining \( U \).

2.2.2. Brin-Katok local entropy. Measure-theoretic entropy is given from the viewpoint of the local perspective. Let \( \mu \in \mathcal{M}(X) \). We define
\[ h^{BK}_\mu(T, \epsilon) := \int \limsup_{n \to \infty} \frac{-\log \mu(B_n(x, \epsilon))}{n} d\mu \]
\[ h^{BK}_\mu(T, \epsilon) := \int \liminf_{n \to \infty} \frac{-\log \mu(B_n(x, \epsilon))}{n} d\mu, \]

Given \( \mu \in E(X, T) \) and \( \epsilon > 0 \), Brin and Katok [Kat80] showed that \( h^{BK}_\mu(T, x, \epsilon), h^{BK}_\mu(T, x, \epsilon) \) are both constants for \( \mu \)-a.e. \( x \in X \) and are equal to \( h^\mu(T, \epsilon), h^{BK}_\mu(T, \epsilon) \), respectively. Moreover, for each \( \mu \in E(X, T) \),
\[ \lim_{\epsilon \to 0} h^{BK}_\mu(T, \epsilon) = \lim_{\epsilon \to 0} h^{BK}_\mu(T, \epsilon) = h^\mu(T). \]

2.2.3. Katok’s entropy. Measure-theoretic entropy defined by spanning set. Let \( \mu \in \mathcal{M}(X) \), \( \epsilon > 0 \) and \( n \in \mathbb{N} \). Given \( \delta \in (0, 1) \) and put
\[ R^\delta_\mu(T, n, \epsilon) := \min\{\#E : \mu(\cup_{x \in E} B_n(x, \epsilon)) > 1 - \delta\}. \]
Define
\[ h^K_\mu(T, \epsilon, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log R^\delta_\mu(T, n, \epsilon) \]
\[ h^K_\mu(T, \epsilon, \delta) = \liminf_{n \to \infty} \frac{1}{n} \log R^\delta_\mu(T, n, \epsilon). \]
Let \( \mu \in E(X, T) \), Katok [Kat80] showed that for every \( \delta \in (0, 1) \),
\[ \lim_{\epsilon \to 0} h^K_\mu(T, \epsilon, \delta) = \lim_{\epsilon \to 0} h^K_\mu(T, \epsilon, \delta) = h^\mu(T). \]

We define two new quantities related to Katok’s entropy by an alternative way. Let \( \mu \in M(X) \), since the quantities \( h^K_\mu(T, \epsilon, \delta), h^K_\mu(T, \epsilon, \delta) \) are increasing when \( \delta \) decreases, so we define
\[ h^K_\mu(T, \epsilon) := \lim_{\delta \to 0} h^K_\mu(T, \epsilon, \delta), h^K_\mu(T, \epsilon) := \lim_{\delta \to 0} h^K_\mu(T, \epsilon, \delta). \]

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2.2.4. Bowen and packing measure-theoretic metric mean dimensions.
By means of Carathéodory-Pesin structures [PS07], we define two dimensionallike quantities related to Katok’s entropy by covering method and packing method used in fractal geometry.

Let $\epsilon > 0, s \in \mathbb{R}, N \in \mathbb{N}, \mu \in M(X)$ and $\delta \in (0, 1)$. Define

$$M^\delta(T, d, \mu, s, N, \epsilon) = \inf \left\{ \sum_{i \in I} e^{-n_i s} \right\},$$

where the infimum is taken over all finite or countable covers $\{B_{n_i}(x_i, \epsilon)\}_{i \in I}$ satisfying $\mu(\bigcup_{i \in I} B_{n_i}(x_i, \epsilon)) \geq 1 - \delta$ with $n_i \geq N, x_i \in X$.

We define $M^\delta(T, d, \mu, s, \epsilon) = \lim_{N \to \infty} M^\delta(T, d, \mu, s, N, \epsilon)$. It is readily to check that the quantity $M^\delta(T, d, \mu, s, \epsilon)$ has a critical value of parameter $s$ jumping from $\infty$ to 0. We define such critical value as

$$M^\delta(T, d, \mu, s, \epsilon) : = \inf \left\{ s : M^\delta(T, d, \mu, s, \epsilon) = 0 \right\} = \sup \left\{ s : M^\delta(T, d, \mu, s, \epsilon) = \infty \right\}.$$

Let $\mu \in M(X)$ and $\delta \in (0, 1)$. Set

$$P^\delta(T, d, \mu, s, \epsilon) = \inf \left\{ \sum_{i=1}^{\infty} P(T, d, Z_i, s, \epsilon) : \mu(\bigcup_{i \in I} Z_i) \geq 1 - \delta \right\}.$$

It is readily to check that the quantity $P^\delta(T, d, \mu, s, \epsilon)$ has a critical value of parameter $s$ jumping from $\infty$ to 0. We define the critical value as

$$P^\delta(T, d, \mu, s, \epsilon) : = \inf \left\{ s : P^\delta(T, d, \mu, s, \epsilon) = 0 \right\} = \sup \left\{ s : P^\delta(T, d, \mu, s, \epsilon) = \infty \right\}.$$

Let $\mu \in M(X)$ and $\delta \in (0, 1)$. Set

$$P^\delta(T, d, \mu, s, \epsilon) = \inf \left\{ \sum_{i=1}^{\infty} P(T, d, Z_i, s, \epsilon) : \mu(\bigcup_{i \in I} Z_i) \geq 1 - \delta \right\}.$$

It is readily to check that the quantity $P^\delta(T, d, \mu, s, \epsilon)$ has a critical value of parameter $s$ jumping from $\infty$ to 0. We define the critical value as

$$P^\delta(T, d, \mu, s, \epsilon) : = \inf \left\{ s : P^\delta(T, d, \mu, s, \epsilon) = 0 \right\} = \sup \left\{ s : P^\delta(T, d, \mu, s, \epsilon) = \infty \right\}.$$

2.2.5. Pfister and Sullivan’s entropy. Measure-theoretic entropy defined by separated set. Let $\mu \in E(X, T)$ and $\epsilon > 0$. Define

$$PS_\mu(T, d, \epsilon) = \inf \limsup_{n \to \infty} \frac{1}{n} \log s_n(T, d, \epsilon, X_n,F),$$

where the infimum ranges over all neighborhoods of $\mu$ in $M(X)$ and $X_n,F = \{ x \in X : \frac{1}{n} \sum_{j=1}^{n-1} 1_{T^j(x) \in F} \}$. Pfister and Sullivan [PS07] proved that $h_\mu(T) = \lim_{\epsilon \to 0} PS_\mu(T, d, \epsilon)$. 

2.2.6. $L^\infty$-rate distortion dimension. The following notions come from information theory. Recall that upper and lower $L^\infty$ rate distortion dimensions are respectively given by

$$\overline{\text{rdim}}_{L^\infty}(T, X, d, \mu) = \limsup_{\epsilon \to 0} \frac{R_{\mu,L^\infty}(T, \epsilon)}{\log \frac{1}{\epsilon}},$$

$$\underline{\text{rdim}}_{L^\infty}(T, X, d, \mu) = \liminf_{\epsilon \to 0} \frac{R_{\mu,L^\infty}(T, \epsilon)}{\log \frac{1}{\epsilon}}.$$ 

Due to our forthcoming proof does not refer to the definition of $R_{\mu,L^\infty}(T, \epsilon)$, so we omit its definition and refer readers to [CT06, LT18, LT19] for more details.

3. PROOFS OF MAIN RESULTS

In this section, we first prove Theorems 1.1 and 1.2, then we use these auxiliary results obtained in Theorems 1.1 and 1.2 to deduce Theorem 1.3.

We firstly give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\delta \in (0, 1)$. For each $\epsilon > 0$, we can find a finite open cover of $X$ with $\text{diam}(U) \leq \epsilon$ and $\text{Leb}(U) \geq \frac{\epsilon}{4}$ (see [GS21, Lemma 3.4] for its proof). Let $\gamma > 0$. By $h^S_{\mu}(U) = \inf_{P \succ U} h_{\mu}(T, P)$ [Sha07, Theorem 4.4], there exists a finite Borel partition $P$ refining $U$ such that $h_{\mu}(T, P) < h^S_{\mu}(U) + \gamma$.

Let $P^n(x)$ denote the (unique) atom of $P^n = \bigvee_{j=0}^{n-1} T^{-j} P$ containing $x$, then we have $P^n(x) \subset B_n(x, 2\epsilon)$ for any $n \in \mathbb{N}$ and $x \in X$. Applying Shannon-McMillan-Breiman theorem yields that $h_{BK}(T, 2\epsilon) \leq h_{BK}(T, 2\epsilon) \leq h^S_{\mu}(U)$. Letting $\gamma \to 0$ gives us $h_{BK}(T, 2\epsilon) \leq h^S_{\mu}(U)$.

Note that any Bowen ball $B_n(x, \frac{\epsilon}{4})$ is contained in some members of $U^n$, then $N_{\mu}(U^n, 1 - \delta) \leq R^K_{\mu}(T, n, \frac{\epsilon}{4})$. Therefore,

\begin{equation}
(3\cdot1) \quad h^S_{\mu}(U) = \lim_{n \to \infty} \frac{\log N_{\mu}(U^n, 1 - \delta)}{n} \text{ by [Sha07, Theorem 4.2]}
\leq \liminf_{n \to \infty} \frac{1}{n} \log R^K_{\mu}(T, n, \frac{\epsilon}{4})
\leq \limsup_{n \to \infty} \frac{1}{n} \log R^K_{\mu}(T, n, \frac{\epsilon}{4}) \leq \limsup_{\delta \to 0} \frac{1}{n} \log R^K_{\mu}(T, n, \frac{\epsilon}{4}).
\end{equation}

Combining the fact $h^K_{\mu}(T, \frac{\epsilon}{4}) \leq h^K_{\mu}(T, \frac{\epsilon}{8})$ obtained in [YCZ22, Proposition 3.21], we have $h^K_{\mu}(T, 2\epsilon) \leq h^K_{\mu}(T, 2\epsilon, \delta) \leq h^K_{\mu}(T, 2\epsilon, \delta) \leq h^K_{\mu}(T, \frac{\epsilon}{4}) \leq h^K_{\mu}(T, \frac{\epsilon}{8})$. It follows that for every $\delta \in (0, 1)$

$$\limsup_{\epsilon \to 0} \frac{h^K_{\mu}(T, \epsilon, \delta)}{\log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \frac{h^K_{\mu}(T, \epsilon, \delta)}{\log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \frac{h^K_{\mu}(T, \epsilon)}{\log \frac{1}{\epsilon}} = C,$$
where \( C = \limsup_{\epsilon \to 0} \frac{K^B\mu(T, \epsilon)}{\log \frac{1}{\epsilon}} \) is a constant. Together with the fact
\[
h^K_\mu(T, \frac{\epsilon}{4}, \delta) \leq h^K_\mu(T, \epsilon) \leq \bar{h}^K_\mu(T, \epsilon),
\]
this completes the proof. \( \square \)

Similar to Theorem 1.1, we can deduce the following proposition, which will be used in the proof of Theorem 1.3.

**Proposition 3.1.** Let \((X, T)\) be a TDS and \(\mu \in E(X, T)\). Then for every \(\delta \in (0,1)\) and \(\epsilon > 0\), we have
\[
\inf_{\text{diam}(\mathcal{U}) \leq \epsilon} h^S_\mu(\mathcal{U}) \leq \bar{h}^K_\mu(T, \frac{\epsilon}{4}, \delta) \leq \bar{h}^K_\mu(T, \frac{\epsilon}{4}) \leq \bar{h}^{BK}_\mu(T, \frac{\epsilon}{8}).
\]

**Proof.** By (3·1), we have
\[
\inf_{\text{diam}(\mathcal{U}) \leq \epsilon} h^S_\mu(\mathcal{U}) \leq \bar{h}^K_\mu(T, \frac{\epsilon}{4}, \delta) \leq \bar{h}^K_\mu(T, \frac{\epsilon}{4}) \leq \bar{h}^{BK}_\mu(T, \frac{\epsilon}{8}).
\]

Now, let \(\mathcal{U}\) be a finite open cover of \(X\) with diameter at most \(\epsilon\). Let \(\gamma > 0\), then we can find a finite Borel partition \(P\) refining \(\mathcal{U}\) such that
\[
\inf_{\text{diam}(\mathcal{U}) \leq \epsilon} h_\mu(T, P) \leq h_\mu(T, P) < h^S_\mu(\mathcal{U}) + \gamma \text{ by [Sha07, Theorem 4.4].}
\]
This implies that
\[
\inf_{\text{diam}(\mathcal{U}) \leq \epsilon} h_\mu(T, P) \leq \inf_{\text{diam}(\mathcal{U}) \leq \epsilon} h^S_\mu(\mathcal{U}).
\]

Similarly, using Shannon-McMillan-Breiman theorem one can obtain that
\[
\bar{h}^{BK}_\mu(T, 2\epsilon) \leq \inf_{\text{diam}(\mathcal{U}) \leq \epsilon} h_\mu(T, P).
\]

By inequalities (3·2) (3·3),(3·4), we get the desired result. \( \square \)

Next, we give the proof of Theorem 1.2. The following lemma is from [YCZ22, Theorem 1.4].

**Lemma 3.2.** Let \((X, T)\) be a TDS with a metric \(d\) and \(K\) be a non-empty compact subset of \(X\). Then
\[
\overline{\text{mdim}}^B(T, K, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup\{h^{BK}_\mu(T, \epsilon) : \mu \in M(X), \mu(K) = 1\},
\]
\[
\overline{\text{mdim}}^P(T, K, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup\{\overline{h}^{BK}_\mu(T, \epsilon) : \mu \in M(X), \mu(K) = 1\}.
\]

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Proof of Theorem 1.2. By Lemma 3.2, it suffices to show for any \( \epsilon > 0 \) and \( \mu \in M(X) \), we have

\[
(3.5) \quad h^{BK}_\mu(T, \epsilon) \leq M_\mu(T, d, \frac{\epsilon}{2}) \leq \inf \{ M(T, d, Z, \frac{\epsilon}{2}) : \mu(Z) = 1 \},
\]

\[
(3.6) \quad \overline{h}^{BK}_\mu(T, \epsilon) \leq P_\mu(T, d, \frac{\epsilon}{10}) \leq \inf \{ P(T, d, Z, \frac{\epsilon}{10}) : \mu(Z) = 1 \}.
\]

Fix \( \epsilon > 0 \) and \( \mu \in M(X) \), we firstly show inequality

\[
h^{BK}_\mu(T, \epsilon) \leq M_\mu(T, d, \frac{\epsilon}{2}) \leq \inf \{ M(T, d, Z, \frac{\epsilon}{2}) : \mu(Z) = 1 \}.
\]

It is clear that \( M_\mu(T, d, \frac{s}{2}) \leq \inf \{ M(T, d, Z, \frac{s}{2}) : \mu(Z) = 1 \} \). Let \( h^{BK}_\mu(T, \epsilon) > 0 \) and \( 0 < s < h^{BK}_\mu(T, \epsilon) \), then there exists a Borel set \( A \) with \( \mu(A) > 0 \) and \( N_0 \) such that

\[
\mu(B_n(x, \epsilon)) < e^{-ns}
\]

for any \( x \in A \) and \( n \geq N_0 \). Let \( \delta = \frac{1}{2} \mu(A) > 0 \) and \( N \geq N_0 \). Let \( \{B_n, (x_i, \frac{s}{2})\}_{i \in I} \) be a finite or countable family so that \( \mu(\cup_{i \in I} B_n(x_i, \frac{s}{2})) \geq 1 - \delta \), \( x_i \in X \) and \( n_i \geq N \). Then we have \( \mu(A \cap \cup_{i \in I} B_n(x_i, \frac{s}{2})) \geq \frac{1}{2} \mu(A) > 0 \). Put

\[
I_1 = \{ i \in I : A \cap B_n(x_i, \frac{\epsilon}{2}) \neq \emptyset \}.
\]

For each \( i \in I_1 \), we have \( B_n(y_i, \epsilon) \supset A \cap B_n(x_i, \frac{s}{2}) \) by choosing a point \( y_i \in A \cap B_n(x_i, \frac{s}{2}) \). This yields that

\[
\sum_{i \in I_1} e^{-sn_i} \geq \sum_{i \in I_1} e^{-sm_i} \geq \sum_{i \in I_1} \mu(B_n(y_i, \epsilon)) \geq \mu(A \cap \cup_{i \in I} B_n(x_i, \frac{\epsilon}{2})) \geq \frac{1}{2} \mu(A) > 0,
\]

which implies that \( M^d(T, d, \mu, s, \frac{s}{2}) \geq M^d(T, d, \mu, s, N, \frac{s}{2}) > 0 \). It follows that \( M_\mu(T, d, \frac{s}{2}) \geq M_\mu(T, d, \mu, \frac{s}{2}) \geq s \). Letting \( s \to h^{BK}_\mu(T, \epsilon) \), we have \( h^{BK}_\mu(T, \epsilon) \leq M_\mu(T, d, \frac{s}{2}) \).

Next, we show the inequality

\[
\overline{h}^{BK}_\mu(T, \epsilon) \leq P_\mu(T, d, \frac{\epsilon}{10}) \leq \inf \{ P(T, d, Z, \frac{\epsilon}{10}) : \mu(Z) = 1 \}.
\]

The inequality \( P_\mu(T, d, \frac{\epsilon}{10}) \leq \inf \{ P(T, d, Z, \frac{\epsilon}{10}) : \mu(Z) = 1 \} \) is clear. Now, let \( 0 < s < \overline{h}^{BK}_\mu(T, \epsilon) \). We can choose \( \theta > 0 \) and a Borel set \( A \) with \( \mu(A) > 0 \) so that

\[
\limsup_{n \to \infty} \frac{-\log \mu(B_n(x, \epsilon))}{n} > s + \theta
\]

for all \( x \in A \).

Let \( \delta \in (0, \mu(A)) \) and \( \{ Z_i \}_{i \geq 1} \) be a family with \( \mu(\cup_{i \geq 1} Z_i) \geq 1 - \delta \). Then \( \mu(A \cap Z_i) > 0 \) for some \( i \geq 1 \). Fix such \( i \), put

\[
E_n := \{ x \in A \cap Z_i : \mu(B_n(x, \epsilon)) < e^{-(s+\theta)n} \}.
\]
Then we have \( A \cap Z_i = \bigcup_{n \geq N} E_n \) for every \( N \in \mathbb{N} \). For any fixed \( N \), there exists \( n \geq N \) so that
\[
\mu(E_n) \geq \frac{1}{n(n+1)} \mu(A \cap Z_i).
\]

Fix such \( n \), consider a family of closed covers of \( E_n \) formed by the balls \( \{ B_n(x, \epsilon) : x \in E_n \} \). By 5r-covering lemma [Mat95, Theorem 2.1], there exists a finite pairwise disjoint subfamily \( \{ B_n(x_i, \epsilon) : i \in I \} \) so that
\[
\bigcup_{i \in I} B_n(x_i, \epsilon) \supseteq \bigcup_{i \in I} B_n(x_i, \epsilon/2) \supseteq \bigcup_{x \in E_n} B_n(x, \epsilon/10).
\]
Hence,
\[
P(T, d, A \cap Z_i, N, s, \frac{\epsilon}{10}) \geq P(T, d, E_n, N, s, \frac{\epsilon}{10}) \geq e^{n\theta} \sum_{i \in I} e^{-n(s+\theta)} \geq e^{n\theta} \mu(B_n(x_i, \epsilon)) \geq e^{n\theta} \mu(E_n) \geq e^{n\theta} \frac{\mu(A \cap Z_i)}{n(n+1)}.
\]
We obtain that \( P(T, d, A \cap Z_i, s, \frac{\epsilon}{10}) = \infty \) by letting \( N \to \infty \). It follows that \( \sum_{i \geq 1} P(T, d, Z_i, N, s, \frac{\epsilon}{10}) = \infty \) and hence
\[
\mathcal{P}_\mu(T, d, s, \frac{\epsilon}{10}) \geq \mathcal{P}^B(T, d, \mu, \frac{\epsilon}{10}) \geq s.
\]
Letting \( s \to h_{\mu}^{BK}(T, \epsilon) \) gives us the desired result. \( \square \)

By means of Theorem 1.2, we can also obtain the following variational principles for metric mean dimension.

**Corollary 3.3.** Let \((X, T)\) be a TDS with a metric \(d\). Then
\[
\overline{\text{mdim}}_M(T, X, d) = \lim_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{M}(X)} \text{M}_\mu(T, d, \epsilon),
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{M}(X,T)} \mathcal{P}_\mu(T, d, \epsilon)
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{M}(X,T)} \mathcal{P}_\mu(T, d, \epsilon)
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}(X,T)} \mathcal{P}_\mu(T, d, \epsilon).
\]
**Proof.** By [YCZ22, Proposition 3.4.(iii)], we have
\[
\overline{\text{mdim}}_M(T, X, d) = \overline{\text{mdim}}^B(T, X, d) = \overline{\text{mdim}}^P(T, X, d).
\]
Therefore, we have

\[
\dim_M(T, X, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in M(X)} M_\mu(T, d, \epsilon),
\]

\[
= \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in M(X)} P_\mu(T, d, \epsilon)
\]

by Theorem 1.2

\[
\geq \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in M(X, T)} P_\mu(T, d, \epsilon)
\]

\[
\geq \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in E(X, T)} P_{\mu}^B(T, \epsilon)
\]

by inequality (3.6)

\[
= \dim_M(T, X, d)
\]

by [Shi22, Theorem 5.2].

This finishes the proof.

\[\square\]

We remark that whether

\[
\dim_M(T, X, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in E(X, T)} P_{\mu}^B(T, \epsilon)
\]

or not is still an open problem posed by Shi [Shi22, Section 6, Problem 1]. This means that it is unclear whether the variational principle for \(\dim_M(T, X, d)\) in terms of \(P_\mu(T, d, \epsilon)\) can take supremum over all invariant (or ergodic) measures.

**Proposition 3.4.** Let \((X, T)\) be a TDS with a metric \(d\) and \(\mu \in E(X, T)\). Then for any \(\epsilon > 0\),

\[
P(T, d, G_\mu^N, \epsilon) \leq P S_\mu(T, \epsilon).
\]

**Proof.** Let \(F\) be neighborhood of \(\mu\). Then we have \(G_\mu = \cup_{N \geq 1} G_\mu^N, \)
when \(G_\mu^N := \{x \in G_\mu : \frac{1}{n} \sum_{j=1}^{n-1} \delta_{T^j(x)} \in F, \forall n \geq N\}\). Fix \(N_0 \in \mathbb{N}\),
then \(G_\mu^{N_0} \subset X_{n,F}\) for any \(n \geq N_0\). Without loss generality, we assume \(P(T, d, G_\mu^{N_0}, \epsilon) > 0\). Let \(0 < s < t < P(T, d, G_\mu^{N_0}, \epsilon)\). Then we know \(P(T, d, G_\mu^{N_0}, t, \epsilon) \geq P(T, d, G_\mu^{N_0}, t, \epsilon) = \infty\). This implies that for any \(N \geq N_0\),
we can find a finite or countable pairwise disjoint closed family \(\{B_n(x_i, \epsilon)\}_{i \in I}\) of \(G_\mu^{N_0}\) with \(n_i \geq N, x_i \in G_\mu^{N_0}\) satisfying

\[
\sum_{i \in I} e^{-n_i t} > 1.
\]

Put \(I_k := \{i \in I : n_i = k\}, k \geq N\). Then \(\sum_{k \geq N} \sum_{i \in I_k} e^{kt} > 1\), and hence there exists \(k \geq N\) such that \(#I_k > (1 - e^{s-t})e^{sk}\). It follows that \(\{x_i : i \in I_k\}\) is a \((k, \epsilon)\)-separated set of \(X_{k,F}\). This implies that \(s_k(T, d, \epsilon, X_{k,F}) \geq (1 - e^{s-t})e^{sk}\), which yields that

\[
\limsup_{n \to \infty} \frac{\log s_n(T, d, \epsilon, X_{n,F})}{n} \geq s.
\]
Letting \( s \to \mathcal{P}(T, d, G_{\mu,F}^N, \epsilon) \), we get
\[
\limsup_{n \to \infty} \frac{\log s_n(T, d, X_n,F)}{n} \geq \mathcal{P}(T, d, G_{\mu,F}^N, \epsilon).
\]
By Proposition 2.4 and the arbitrariness of \( F \), one has
\[
PS_{\mu}(T, \epsilon) \geq \mathcal{P}(T, d, G_{\mu,F}^N, \epsilon).
\]

Now, we give the proof of Theorem 1.3.

**Proof of Theorem 1.3.** By [W21a, Proposition 4.3], we have
\[
PS_{\mu}(T, \epsilon) \leq R_{\mu,L^\infty}(T, 6\epsilon) \leq h_{BK}\mu(T, 6\epsilon).
\]
Using inequality (3.7) and the fact \( \mu(G_{\mu}) = 1 \), we know that
\[
\overline{h}_{\mu}^K(T, 10\epsilon) \leq P_{\mu}(T, \epsilon) \leq \inf\{P(T, d, Z, \epsilon) : \mu(Z) = 1\} \leq P(T, d, G_{\mu}, \epsilon).
\]
Using [YCZ22, Proposition 3.21] again, we have
\[
\overline{h}_{\mu}^K(T, 20\epsilon) \leq \overline{h}_{\mu}^{BK}(T, 10\epsilon).
\]
Finally, we obtain that
\[
\overline{h}_{\mu}^K(T, 20\epsilon) \leq \overline{h}_{\mu}^{BK}(T, 10\epsilon) \leq P_{\mu}(T, \epsilon) \leq \inf\{P(T, d, Z, \epsilon) : \mu(Z) = 1\} \leq P(T, d, G_{\mu}, \epsilon).
\]
Combing the Theorem 1.1, Proposition 3.1 and the inequality (3.8), this finishes the proof.

**Remark 3.5.** (i). As a consequence of the inequality of (3.8) and the Proposition 3.1, using the well-known Brin-katok formula [BK83] and Kalok’s entropy formula [Kat80], we obtain some (new) entropy formulas for measure-theoretic entropy and variational principles for topological entropy (except that we have known).
\[
h_{\mu}(T) = \lim_{\epsilon \to 0} F(\mu, \epsilon),
\]
\[
h_{\text{top}}(T, X) = \sup_{\mu \in E(X,T)} \lim_{\epsilon \to 0} F(\mu, \epsilon),
\]
where \( F(\mu, \epsilon) \) is chosen from the candidate set given in Theorem 1.3.

(ii). The value \( \limsup_{\epsilon \to 0} \frac{F(\mu, \epsilon)}{\log \frac{1}{\epsilon}} \) (or we prefer to say "speed") can be interpreted as how fast these candidates \( F(\mu, \epsilon) \) approximate the (infinite) measure-theoretic entropy \( h_{\mu}(T) \) as \( \epsilon \to 0 \). It intuitively seems that these candidates \( F(\mu, \epsilon) \) may have different speeds that approximates the measure-theoretic entropy \( h_{\mu}(T) \) as \( \epsilon \to 0 \), but Theorem 1.3 offers somewhat surprising fact that these different candidates have the
same speed and the speed is exactly equal to \( \text{mdim}^P(T, G_\mu, d) \). Hence, when \( \epsilon > 0 \) is sufficiently small, we may approximate \( F(\mu, \epsilon) \) as

\[
F(\mu, \epsilon) \approx \text{mdim}^P(T, G_\mu, d) \log \frac{1}{\epsilon},
\]

where \( F(\mu, \epsilon) \) is taken from the candidate set given in Theorem 1.3.

As we mentioned above, the formula \( h_{\text{top}}^B(T, G_\mu) = h_{\text{top}}^P(T, G_\mu) = h_\mu(T) \) holds for every \( \mu \in E(X, T) \). Unfortunately, we only get such a formula for Bowen and packing metric mean dimensions under an additional condition.

**Corollary 3.6.** Let \( (X, T) \) be a TDS with a metric \( d \in \mathcal{D}(X) \). Suppose that \( \mu \in E(X, T) \) satisfying

\[
\limsup_{\epsilon \to 0} \frac{h^K_{\mu}(T, \epsilon, \delta)}{\log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \frac{h^{BK}_{\mu}(T, \epsilon)}{\log \frac{1}{\epsilon}},
\]

then for every \( \delta \in (0, 1) \)

\[
\text{mdim}^B(T, G_\mu, d) = \text{mdim}^P(T, G_\mu, d) = \limsup_{\epsilon \to 0} \frac{F(\mu, \epsilon)}{\log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \inf \{ M(T, d, Z, \epsilon) : \mu(Z) = 1 \},
\]

where the candidate

\[
F(\mu, \epsilon) \in \left\{ h^K_{\mu}(T, \epsilon, \delta), h^K_{\mu}(T, \epsilon, \delta), h^K_{\mu}(T, \epsilon), h^{BK}_{\mu}(T, \epsilon), h^{BK}_{\mu}(T, \epsilon), M_\mu(T, \epsilon, \delta), P_\mu(T, \epsilon, \delta), P_\mu(T, \epsilon), R_\mu, \inf_{\text{diam}P \leq \epsilon} h_\mu(T, P), \inf_{\text{diam}(U) \leq \epsilon} h^S_\mu(U) \right\},
\]

and the infimum in the third equality can be attained for some subsets of \( X \). \( P \) ranges over all finite Borel partitions of \( X \) with diameter at most \( \epsilon \), and \( U \) ranges over all finite open covers of \( X \) with diameter at most \( \epsilon \). Moreover, it is also valid if we change \( \limsup_{\epsilon \to 0} \) into \( \liminf_{\epsilon \to 0} \) and \( \text{mdim}^P(T, G_\mu, d) \) into \( \text{mdim}^P(T, G_\mu, d) \).

**Proof.** By [YCYZZ22, Proposition 3.21], we have \( h^K_{\mu}(T, 2\epsilon) \leq h^{BK}_{\mu}(T, \epsilon) \). Together with the additional condition, we obtain

\[
(3.9) \quad \limsup_{\epsilon \to 0} \frac{h^K_{\mu}(T, \epsilon)}{\log \frac{1}{\epsilon}} \leq \limsup_{\epsilon \to 0} \frac{h^{BK}_{\mu}(T, \epsilon)}{\log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \frac{h^{BK}_{\mu}(T, \epsilon)}{\log \frac{1}{\epsilon}}.
\]

By (3.5), we have

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\[
\limsup_{\epsilon \to 0} \frac{h_{BK}^{\mu}(T, \epsilon)}{\log \frac{1}{\epsilon}} \leq \limsup_{\epsilon \to 0} \frac{M_{\mu}(T, \frac{2}{\epsilon})}{\log \frac{1}{\epsilon}} \\
\leq \limsup_{\epsilon \to 0} \inf \left\{ M(T, d, Z, \frac{2}{\epsilon}) : \mu(Z) = 1 \right\} \\
\leq \text{mdim}^B(T, G_{\mu}, d) \\
\leq \text{mdim}^P(T, G_{\mu}, d) \text{ by [YCZ22, Proposition 3.4,(iii)]}
\]
\[
(3·10) = \limsup_{\epsilon \to 0} \frac{h_{BK}^{\mu}(T, \epsilon)}{\log \frac{1}{\epsilon}} \text{ by Theorem 1.3.}
\]

By the inequalities (3·9), (3·10) and Theorem 1.3, we get desired result.

\[
\square
\]

Remark 3.7. (i). The partial statements about \(\text{mdim}^B(T, G_{\mu}, d)\) in Corollary 3.6 were given in [YCZ22, Theorem 1.5,(ii)] under the same assumption. An example satisfying the additional condition can be found in [YCZ22, Example 3.25].

(ii) The inverse variational principles for Bowen upper metric mean dimension are given in Corollary 3.6 under the additional condition. Actually, such an inverse variational principle is valid for \(h_{BK}^{\mu}(T, \epsilon)\) without the extra condition. We state it as follows for completeness.

Corollary 3.8. Let \((X, T)\) be a TDS with a metric \(d\) and \(\mu \in E(X, T)\), then

\[
\limsup_{\epsilon \to 0} \frac{h_{BK}^{\mu}(T, \epsilon)}{\log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \frac{\inf \left\{ M(T, d, Z, \epsilon) : \mu(Z) = 1 \right\}}{\log \frac{1}{\epsilon}}.
\]

Proof. One side is clear by inequality (3·5). We show the converse inequality

\[
\inf \left\{ M(T, d, Z, 3\epsilon) : \mu(Z) = 1 \right\} \leq h_{BK}^{\mu}(T, \epsilon)
\]
by modifying the method used in [MW08, Theorem 1].

Let \(0 < \epsilon < 1\) and we set \(E = \left\{ x \in X : h_{BK}^{\mu}(T, x, \epsilon) = h_{BK}^{\mu}(T, \epsilon) \right\}\), where \(h_{BK}^{\mu}(T, x, \epsilon) = \liminf_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon))}{n}\). Then \(\mu(E) = 1\) by \(\mu \in E(X, T)\). Let \(s > h_{BK}^{\mu}(T, \epsilon)\), then for every \(x \in E\) we can choose a strictly increasing sequence \(n_j(x)\) that converges to \(\infty\) so that \(\mu(B_{n_j}(x, \epsilon)) > e^{-n_j(x)s}\) for each \(j \geq 1\). Therefore, for each \(N \geq 1\), the set \(E\) is contained in the family \(\mathcal{F}_N := \left\{ x \in E : \mu(B_{n_j}(x, \epsilon)) > e^{-n_j(x)s}, n_j(x) \geq N \right\}\). By \(3\epsilon\)-covering lemma [MW08, Lemma 1], there exists a pairwise disjoint sub-family \(\left\{ B_{n_i}(x_i, \epsilon) \right\}_{i \in I} \subset \mathcal{F}_N\) such that

\[
E \subset \bigcup_{i \in I} B_{n_i}(x_i, 3\epsilon).
\]
Since \( B_n(x, \epsilon) \) has positive measure for each \( i \in I \), then the cardinality of \( I \) is at most countable by the fact that \( \mu(X) = 1 \). It follows that

\[
M(T, d, E, s, N, 3\epsilon) \leq \sum_{i \in I} e^{-n_is} \leq \sum_{i \in I} \mu(B_n(x, \epsilon)) \leq 1,
\]

which implies that \( M(T, d, E, 3\epsilon) \leq s \). Letting \( s \to h_{\mu}^{BK}(T, \epsilon) \) gives us

\[
\inf \{ M(T, d, Z, 3\epsilon) : \mu(Z) = 1 \} \leq M(T, d, E, 3\epsilon) \leq h_{\mu}^{BK}(T, \epsilon). \quad \square
\]

**Example 3.9.** Let \( \sigma : [0, 1]^Z \to [0, 1]^Z \) be the shift on alphabet \([0, 1]\), where \([0, 1]\) is the unit interval with the standard metric. Equipped \([0, 1]^Z\) with a metric given by

\[
d(x, y) = \sum_{n \in \mathbb{Z}} 2^{-|n|} |x_n - y_n|.
\]

Let \( \mu = L \otimes \mathbb{Z} \), where \( L \) is the Lebesgue measure on \([0, 1]\).

It is shown that \([YCZ22, Example 3.24]\) (or see \([Shi22, Example 6.1]\))

\[
\overline{\text{mdim}}^B(\sigma, G_\mu, d) = \limsup_{\epsilon \to 0} \frac{h_{\mu}^{BK}(\sigma, \epsilon)}{\log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \frac{h_{\mu}^{BK}(\sigma, \epsilon)}{\log \frac{1}{\epsilon}} = 1.
\]

It is well-known that \( \overline{\text{mdim}}_M(\sigma, [0, 1]^Z, d) = 1 \) \([LT18]\). By \([YCZ22, Proposition 3.4,(iii)]\), we have \( \overline{\text{mdim}}_M(\sigma, [0, 1]^Z, d) = \overline{\text{mdim}}^B(\sigma, [0, 1]^Z, d) = \overline{\text{mdim}}^P(\sigma, [0, 1]^Z, d) = 1 \) and \( \overline{\text{mdim}}^B(\sigma, A, d) \leq \overline{\text{mdim}}^P(\sigma, A, d) \) for any non-empty subset \( A \subset [0, 1]^Z \). Hence, we obtain

\[
1 = \overline{\text{mdim}}^B(\sigma, G_\mu, d) \leq \overline{\text{mdim}}^P(\sigma, G_\mu, d) \leq \overline{\text{mdim}}^P(\sigma, [0, 1]^Z, d) = 1.
\]

This shows that

\[
\overline{\text{mdim}}^B(\sigma, G_\mu, d) = \overline{\text{mdim}}^P(\sigma, G_\mu, d)
= \limsup_{\epsilon \to 0} \frac{h_{\mu}^{BK}(\sigma, \epsilon)}{\log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \frac{h_{\mu}^{BK}(\sigma, \epsilon)}{\log \frac{1}{\epsilon}} = 1.
\]

We finally finish this paper with three questions.

**Question 1** Does for every \( \delta \in (0, 1) \),

\[
\overline{\text{mdim}}_M(T, X, d) = \sup_{\mu \in E(X, T)} \limsup_{\epsilon \to 0} \frac{h_{\mu}^{K}(T, \epsilon, \delta)}{\log \frac{1}{\epsilon}}?
\]

Note that for every \( \delta \in (0, 1) \), Shi \([Shi22, Theorem 4.2]\) showed that

\[
\overline{\text{mdim}}_M(T, X, d) = \limsup_{\epsilon \to 0} \sup_{\mu \in E(X, T)} \frac{h_{\mu}^{K}(T, \epsilon, \delta)}{\log \frac{1}{\epsilon}}.
\]

Hence, the question can be reduced to whether we can change the order of sup and limsup in the variational principle.
Question 2 For every TDS $(X, T)$, does the equality
\[
\limsup_{\epsilon \to 0} \frac{h^B_{\mu}(T, \epsilon)}{\log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \frac{h^B_{\mu}(T, \epsilon)}{\log \frac{1}{\epsilon}}
\]
hold for all invariant (or ergodic) measures $\mu$?

We remark that the positive answer to Question 2 allows us to drop the additional condition in Corollary 3.6.

Question 3 If a TDS admits $g$-almost product property and $\mu \in M(X, T)$, can we establish analogous results for Bowen (or packing) metric mean dimension of $G_\mu$?

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