TRUNCATED SOLUTIONS OF PAINLEVÉ EQUATION P_5

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Abstract. We obtain convergent representations (as Borel summed transseries) for the five one-parameter families of truncated solutions of the fifth Painlevé equation with nonzero parameters, valid in half planes, for large independent variable. We also find the position of the first array of poles, bordering the region of analyticity. For a special value of this parameter they represent tri-truncated solutions, analytic in almost the full complex plane, for large independent variable. A brief historical note, and references on truncated solutions of the other Painlevé equations are also included.

1. Introduction

1.1. Historical notes. By the middle of the 19th century it became apparent that solutions of many linear differential equations should be considered new, ”special” functions. The natural question then arose: can nonlinear differential equations have solutions that could be thought of being special functions? Fuchs’s intuition was that the answer is affirmative if ”solutions have only fixed branch points none of which depend on the initial conditions” [1]. This is now called the Painlevé property, the fact that solutions of an equation are meromorphic on a common Riemann surface. Fuchs studied first order equations having this property, concluding in 1884 that all such equations can be solved in terms of previously known functions; his results were later extended by Poincaré [1].

The idea that absence of movable branch points would mean integrability was then used by Sofie Kowalevski in the study of the rotation of a solid about a fixed point, for which she discovered a third integrable case (previous two cases being discovered by Euler and Lagrange), a discovery for which she was awarded the Prix Bordin of the French Academy of Science in 1888.

Around the turn of the 20th century, Painlevé, Picard and Gambier studied nonlinear second order differential equations, rational in \( y \) and \( y' \), discovering that those possessing the Painlevé property can be brought to fifty canonical forms. Of these fifty equations, all but six could be solved in terms of earlier known functions. Painlevé went on to show that generic solutions of the remaining six equations cannot be expressed in terms of earlier known functions, or in terms of each other [3]. These equations are now known as the Painlevé equations, denoted \( P_I \) up to \( P_{VI} \), and their solutions as the Painlevé transcendents.

Having been discovered as a result of a purely theoretical inquiry, the Painlevé transcendents have appeared later in modern geometry, integrable systems [4], statistical mechanics [5], [6], [7], and recently in quantum field theory. Their practical importance makes it necessary to study their properties in detail and to develop good numerical methods for their calculation [8].

1.2. Truncated solutions of Painlevé equations. It is known that generic Painlevé transcendents have poles in any sector towards infinity. But there are special solutions, called tronquées (or truncated) which are free of poles in some sectors, at least for large values of the independent variable. It was later conjectured that truncated solutions have no poles whatsoever in these sectors, the Dubrovin-Novokshenov conjecture [15].
For P_I, Boutroux showed that there are five special sectors in the complex plane, each of opening $2\pi/5$, where solutions may lack poles. There are truncated solutions free of poles in two adjacent such sectors, for large $x$. Among these, there are solutions free of poles in four sectors - tri-truncated solutions [9]. The form of the exponentially small terms as well as the location of the first array of poles of truncated solutions are found in [11]. The Stokes multiplier was found using isomonodromy method [10], and later through a direct method (continuation of a tri-truncated solution through the sector with poles) [16]. The Dubrovin-Novokshenov conjecture was proved for P_I [12].

For P_{II} Boutroux showed that there are six special sectors, with truncated solutions free of poles in two adjacent sectors, and tri-truncated solutions free of poles in four sectors. Existence of tri-truncated solutions of the P_{II} hierarchy was shown in [13]. The Dubrovin-Novokshenov conjecture was established for the Hastings-McLeod solution (tri-truncated in pairs of non-adjacent sectors) [14].

Existence of truncated solution for P_{III} and P_{IV} was established in [17] following methods in [18], and using a different method in [19], where the location of the first array of poles is also found.

An overview of P_{VI} is contained in [21]; see also [22].

The truncated solutions of the fifth Painlevé equation are the subject of the present article.

1.3. Truncated solutions of Painlevé equation P_{V}. The fifth Painlevé equation:

$$P_V(\alpha, \beta, \gamma, \delta):$$

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)w'^2 - \frac{w'}{x} + \frac{(w-1)^2}{x^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{x} + \frac{\delta w(w+1)}{w-1} \quad (1)$$

is known to be reducible to P_{III} if $\delta = 0$ [27].

We assume $\alpha\beta\delta \neq 0$. The following algebraic behavior towards infinity are then possible for solutions of (1) [20]:

I. $w = \pm \sqrt{\frac{\beta}{\delta}} x^{-1} + O(x^{-2}) \quad (x \to \infty)$

II. $w = \pm \sqrt{-\frac{\beta}{\alpha}} x + O(1) \quad (x \to \infty)$

III. $w = -1 + O(x^{-1}) \quad (x \to \infty)$

(2)

It is known that these five families represent asymptotic behaviors of truncated solutions, analytic in (almost) a half plane for large $|x|$; the position of the half plane is determined by the exponentially small terms [23], [25].

In the present paper we express the exponentially small terms using a full formal solution (transseries), Borel summable to actual solutions. This yields one-parameter families of truncated solutions as series which converge in appropriate half-planes and large $|x|$. For special values of the parameter we obtain tri-truncated transcendentals.

Moreover, we find the location of the first array of poles beyond the sector of analyticity. We use techniques introduced in [11]. For completeness, the statements of the theorems used here are included in the Appendix.

1.4. Relations between different truncated solutions.

Remark 1. [27] (i) If $w(x)$ satisfies $P_V(\alpha, \beta, \gamma, \delta)$, then $1/w(x)$ satisfies $P_V(-\beta, -\alpha, -\gamma, \delta)$.

(ii) If $w(x)$ solves $P_V(\alpha, \beta, \gamma, \delta)$ then $w(x/\lambda)$ solves $P_V(\alpha, \beta, \gamma \lambda, \delta \lambda^2)$, for any $\lambda \neq 0$. 
By Remark\[1\](i), truncated solutions in the family II. are obtained as reciprocals of truncated of the family I.

By Remark\[1\](ii), if \(w(x)\) solves \(P_V(\alpha, \beta, \gamma, \delta)\) with \(w(x) \sim \sqrt{2 + x^{-1}} (x \to \infty)\) then \(w(-x)\) solves \(P_V(\alpha, \beta, -\gamma, \delta)\) and satisfies \(w(x) \sim -\sqrt{2 + x^{-1}} (x \to \infty)\).

We can assume any nonzero value for \(\delta\), by rescaling \(x\) and using Remark\[1\](ii).

It is interesting to note that there are other Bäcklund transformations as explained in \[27\] (see Theorem 39.2 there). These transform truncated solutions of the first two families into truncated solutions of the first two families, and truncated solutions in the family III into solutions in the same family.

Due to these relations, it suffices to obtain results for the truncated Painlevé transcendents satisfying

\[
\begin{align*}
I_0 & \quad w = \sqrt{\beta/\delta} x^{-1} + O(x^{-2}) \quad (x \to \infty) \quad \text{for } \delta = -1/2 \quad (3) \\
\text{and for} & \\
III_0 & \quad w = -1 + O(x^{-1}) \quad (x \to \infty) \quad \text{for } \delta = 2 \\
\end{align*}
\]

2. TRUNCATED SOLUTIONS IN THE FAMILY \(I_0\).

In this section we state the main results; their proofs are found in \[4\]

**Theorem 2.** Assume \(\alpha \beta \gamma \neq 0\). Let \(w(x)\) be a solution of \(P_V(\alpha, \beta, \gamma, -\frac{1}{2})\) so that

\[ w(x) = \sqrt{-2\beta} x^{-1} (1 + o(1)) \quad (x \to \infty, \ Re\ x > 0) \]

(i) Then \(w(x)\) is asymptotic to a unique power series solution:

\[ w(x) \sim \tilde{w}_0 = \sum_{n=1}^{\infty} w_{0,n} x^{-n} \quad (x \to \infty, \ Re\ x > 0), \] \[ \text{where } w_{0,1} = \sqrt{-2\beta} \]

(ii) The complete formal solution (transseries) along any half-line in the right half-plane has the form

\[ \tilde{w}(x) = \tilde{w}_0(x) + \sum_{k=1}^{\infty} C^k e^{-kx} x^{-q} \tilde{w}_k(x) \quad \text{where } \tilde{w}_k(x) = \sum_{n=0}^{\infty} w_{n,k} x^{-n}, \]

\(C\) is an arbitrary constant and

\[ q = \gamma + 2\sqrt{-2\beta} \]

(iii) There are unique constants \(C_{\pm}\) so that

\[ w(x) \sim \sum_{n=1}^{\text{Re} q} w_{0,n} x^{-n} + C_{\pm} x^{-q} e^{-x} \quad \text{as } x \to +i\infty \]

and

\[ w(x) \sim \sum_{n=1}^{\text{Re} q} w_{0,n} x^{-n} + C_{\pm} x^{-q} e^{-x} \quad \text{as } x \to -i\infty \]

and \(C_{\mp} - C_{\pm}\) does not depend on the particular solution (but only on the Stokes constant of the equation).

(iv) \(w(x)\) is analytic in the right half plane for \(|x| > R\) (for some \(R > 0\)) and has the Borel summed transseries representation

\[
\begin{align*}
w(x) = \begin{cases} 
w_0(x) + \sum_{k=1}^{\infty} C^k e^{-kx} x^{-q} \tilde{w}_{+,k}(x) & \text{for } \arg x \in \left(0, \frac{\pi}{2}\right) \\
w_0(x) + \sum_{k=1}^{\infty} C^k e^{-kx} x^{-q} \tilde{w}_{-,k}(x) & \text{for } \arg x \in \left(-\frac{\pi}{2}, 0\right) 
\end{cases} \quad (8)
\end{align*}
\]
where \( w_0(x), \ w_{\pm;k}(x) \) are Laplace transforms on half-lines \( e^{i\phi}\mathbb{R}_+ \) (where \( \phi = -\arg x \)) of the Borel transforms of the series \( \tilde{w}_0(x), \ \tilde{w}_k(x) \). If \( \Re q < 0 \) then the series in (8) converge also for \( \arg x = \pm \frac{\pi}{2} \). The series converge uniformly for \( |x| > R \) in the right half plane.

For \( \arg x = 0 \) the transseries is generalized Borel summable (it is obtained by special averages of Laplace transforms in the upper and the lower half plane [28]).

**Consequence 3.** (Existence of tri-truncated solutions) Consider the unique truncated solution \( w(x) \) as in Theorem [3] with \( C_+ = 0 \). Then \( w(x) \) is analytic for large \( |x| \) in the left-half plane, for \( \arg x \in (-\frac{\pi}{2}, \frac{3\pi}{2} - \delta) \), \( |x| > R \).

Similarly, the unique truncated solution with \( C_- = 0 \) is analytic for large \( |x| \) with \( \arg x \in (-\frac{3\pi}{2} + \delta, \frac{\pi}{2}) \).

**Remark.** Truncated solutions in the left half-plane, satisfying

\[
\tilde{w} = \frac{1}{\sqrt{-2\beta} x^{-1}} (1 + o(1)) \quad (x \to \infty, \ \Re x < 0)
\]

form a one-parameter family, which, by Remark [4] are obtained by replacing \((x, \gamma)\) with \((-x, -\gamma)\) in the representation given by Theorem [2].

Truncated solutions \( w(x) \) as in Theorem [2] with \( C_+ \neq 0 \) develop arrays of poles. The following result shows the position of the array of poles closest to \( i\mathbb{R}^+ \).

**Theorem 4.** Assume \( \alpha \beta \gamma \neq 0 \) and

\[
2\alpha \neq (\sqrt{-2\beta} - q - 1)^2
\]

Let \( w(x) \) be as in Theorem [2] with nonzero constant \( C_+ \) in (8). Then \( w(x) \) has two arrays of poles located at

\[
x_{n,1,2} = 2n\pi i + \left( \gamma + 2\sqrt{-2\beta} + 2 \right) \ln(2n\pi i) + \ln C_+ - \ln \zeta_{1,2} + o(1) \quad (n \to +\infty)
\]

where

\[
\zeta_{1,2} = \frac{2}{\sqrt{-2\beta}} \frac{1}{\sqrt{-2\beta} - q - 1 \pm \sqrt{2\alpha}}
\]

3. **Truncated Solutions in the Family III.**

In this section we state the main results; their proofs are found in [5].

**Theorem 5.** Assume \( \alpha \beta \gamma \neq 0 \). Let \( w(x) \) be a solution of \( \mathcal{P}_V(\alpha, \beta, \gamma, 2) \) so that

\[
w(x) = -1 + o(1) \quad (x \to \infty, \ \Re x > 0)
\]

(i) Then \( w(x) \) is asymptotic to a unique power series solution:

\[
w(x) \sim -1 + \tilde{w}_0 = -1 + \sum_{n=1}^{\infty} w_{0;n} x^{-n} \quad (x \to \infty, \ \Re x > 0)
\]

where \( \tilde{w}_{0;1} = \gamma \).

(ii) The complete formal series solution (transseries) along any half-line in the right half-plane has the form

\[
\tilde{w}(x) = \tilde{w}_0(x) + \sum_{k=1}^{\infty} C^k e^{-kx} x^{-k/2} \tilde{w}_k(x)
\]

where \( C \) is an arbitrary constant, \( \tilde{w}_k(x) \) are series in \( x^{-n}, \ n \in \mathbb{N} \).

(iii) There are unique constants \( C_\pm \) so that

\[
w(x) \sim -1 + C_+ x^{-1/2} e^{-x} \quad \text{as} \ x \to +i\infty
\]
and

$$w(x) \sim -1 + C_- x^{-1/2} e^{-x} \quad \text{as } x \to -i\infty$$

(14)

and $C_+ - C_-$ is a constant which does not depend on the particular solution (it only depends on the Stokes constant of the equation).

(iv) $w(x)$ is analytic in the right half plane for $|x| > R$ (for some $R > 0$) and has the Borel summed transseries representation for large $|x|

$$w(x) = \begin{cases} w_0(x) + \sum_{k=1}^{\infty} C^k e^{-kx} x^{-k/2} w_{+,k}(x) & \text{for } \arg x \in (0, \frac{\pi}{2}] \\
-w_0(x) + \sum_{k=1}^{\infty} C^k e^{-kx} x^{-k/2} w_{-,k}(x) & \text{for } \arg x \in [-\frac{\pi}{2}, 0) \end{cases}$$

(15)

where $w_0(x)$, $w_{\pm,k}(x)$ are Laplace transforms on the half-lines $e^{i\phi} \mathbb{R}_+$ (where $\phi = -\arg x$) of the Borel transforms of the series $\tilde{w}_0(x)$, $\tilde{w}_k(x)$.

**Consequence 6.** (Existence of tri-truncated solutions) Consider the unique truncated solution $w(x)$ as in Theorem 2 with $C_+ = 0$. Then $w(x)$ is analytic for large $|x|$ also in the left half-plane, for $\arg x \in (-\frac{\pi}{2}, \frac{3\pi}{2} - \delta)$.

Similarly, the unique truncated solution with $C_- = 0$ is analytic for large $|x|$ with $\arg x \in (-\frac{3\pi}{2} + \delta, \frac{\pi}{2})$.

**Remark.** Truncated solutions in the left half-plane, satisfying

$$w(x) = -1 + o(1) \quad (x \to \infty, \ Re x < 0)$$

form a one-parameter family, which, by Remark 1, are obtained by replacing $(x, \gamma)$ with $(-x, -\gamma)$ in the representation given by Theorem 2.

Truncated solutions $w(x)$ as in Theorem 2 with $C_+ \neq 0$ have arrays of poles near $i\mathbb{R}_+$. The following result shows the position of the closest array of poles.

**Theorem 7.** Assume $\alpha\beta\gamma \neq 0$. Let $w(x)$ be as in Theorem 2 with nonzero constant $C_+$ in (13) and (5). Then $w(x)$ has an array of poles located at

$$x_n = 2n\pi i - \frac{1}{2} \ln(2n\pi i) - \frac{i\pi}{2} - \ln(-C/2) + o(1) \quad (n \to \infty)$$

(16)

We conjecture that there are, in fact, two poles near each $x_n$.

4. Proofs for Family $I_0$

4.1. **Proof of Theorem 2.** Denote

$$m = \sqrt{-2\beta}, \quad q = \gamma + 2m$$

(17)

Then $w(x)$ satisfies

$$w'' = \frac{(3w - 1) w^2}{2w(w - 1)} - \frac{w'}{x} + \frac{\alpha w(w - 1)^2}{x^2} - \frac{m^2 (w - 1)^2}{2x^2w} + \frac{(q - 2m)w}{x} - \frac{1}{2} \frac{w(w + 1)}{w - 1}$$

(18)

for which we study solutions satisfying $w = mx^{-1} + O(x^{-2})$ as $x \to \infty$.

To normalize the equation, substitute

$$w(x) = \frac{m}{x} \left(1 - \frac{q}{x} + u(x)\right)$$

(19)

which transforms (18) into

$$u'' = \left(1 + \frac{2q}{x}\right) u + f(x^{-1}, u, u')$$

(20)
where \( f \) is analytic at \((0,0,0)\) and \( f = O(x^{-2}) + O(u^2) + O(u'^2) + O(uu') \). To show that this a normal form for second order equations, we turn it into a first order system by substituting

\[
\begin{align*}
u(x) &= y_1(x) + y_2(x), \quad \nu'(x) = \left(-1 - \frac{q}{x}\right)y_1(x) + \left(1 + \frac{q}{x}\right)y_2(x) \tag{21}
\end{align*}
\]

upon which (20) is turned into a first order system in normal form (31):

\[
\frac{d}{dx} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 - \frac{q}{x} & 0 \\ 1 + \frac{q}{x} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + g(x^{-1}, y_1, y_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{22}
\]

where

\[
g(x^{-1}, y_1, y_2) = \frac{x}{2x + q} \left[ -f \left(x^{-1}, y_1 + y_2, \left(1 + \frac{q}{x}\right)(y_2 - y_1)\right) + \frac{q(q + 1)}{x^2} y_1 + \frac{q(q - 1)}{x^2} y_2 \right] \tag{23}
\]

There are general theorems that can be applied for differential equations in normal form; these are presented, for convenience, in the Appendix, [6] applying Theorem[9] to the system [22], [23] and then reverting the substitutions [21], [19], Theorem[2] follows.

4.2. Proof of Theorem[4] Searching for asymptotic expansions of the form [39] setting \( \xi = Ce^{-x}x^{-q} \), plugging in an asymptotic series \( u(x) \sim F_0(\xi) + \frac{1}{x} F_1(\xi) + \frac{1}{x^2} F_2(\xi) \ldots \) in (20) and expanding under the assumption that \( \xi \ll x^{-k} \) for all \( k \) we obtain that all \( F_n \) are polynomials:

\[
F_0(\xi) = \xi, \quad F_1(\xi) = c_1 \xi, \quad F_2(\xi) = m(m - q - 1) \xi^2 + c_2 \xi - \frac{1}{2} m^2 + \frac{3}{2} q^2 - a + \frac{1}{2} \ldots
\]

where \( c_1, c_2 \) are uniquely determined in terms of the parameters. By Theorem[10] we have

\[
u(x) \sim \sum_{m=0}^{\infty} x^{-m} F_m(\xi(x)) \quad \text{for} \quad x \to \infty \quad \text{with} \quad \arg x \in \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right], \quad |\xi(x)| < \delta_1 \tag{24}
\]

for some \( \delta_1 > 0 \).

On the other hand, \( F_0 \) has no singularities, so Theorem[11] yields no additional information on the position of the first array of poles, beyond the right half plane.

A direct calculation of the first few terms of the transseries [5] unveils a non-generic structure, namely that \( w_{n,k} = 0 \) for all \( k = 0, 1, \ldots, n - 1 \) (at least for \( n = 1, \ldots, 4 \)). This peculiar structure suggests (by formal re-arrangement of the transseries) to use instead the second scale

\[
\zeta = Ce^{-x}x^{-q - 2}
\]

and look for an expansion of the form

\[
u(x) \sim x^2 \Phi(\zeta) + \sum_{n=-1}^{\infty} \frac{1}{x^n} \Phi_n(\zeta) \quad (\zeta(x) \ll x^{-k} \text{ for all } k) \tag{25}
\]

where \( \Phi(\zeta) = \zeta + O(\zeta^2) \) when \( \zeta \to 0 \) and \( \Phi_n \) are analytic at \( \zeta = 0 \).

Introducing the formal expansion (25) in (20) and expanding in powers of \( x^{-1} \), we obtain that \( \Phi(\zeta) \) must satisfy

\[
\zeta^2 \Phi'' + \zeta \Phi' + \frac{1}{2} \Phi = \alpha m^2 \Phi^3 + \frac{3 \zeta^2 \Phi'^2}{\Phi} \tag{26}
\]

having the general solution

\[
\Phi(\zeta) = \frac{2\zeta/C_1}{8\alpha m^2/C_1^2 - (\zeta - C_2)^2} \tag{27}
\]
where $C_{1,2}$ are determined from the condition that $\Phi(\zeta) = \zeta + O(\zeta^2)$ and that $\Phi_{-1}(\zeta)$ be analytic at $\zeta = 0$, yielding
\[
C_1 = -2 \beta \left[ 2 \alpha - (m - q - 1)^2 \right], \quad C_2 = -\frac{2 m - q - 1}{2 m \alpha - (m - q - 1)^2}
\]  
(28)

With the notation (11) (recall that $m = \sqrt{-2\beta}$) and using (27), then (28) becomes
\[
\Phi(\zeta) = \frac{\zeta \zeta_1 \zeta_2}{(\zeta - \zeta_1)(\zeta - \zeta_2)}
\]
and the next term in the expansion is
\[
\Phi_{-1}(\zeta) = \frac{\zeta (2 \zeta - \zeta_1 - \zeta_2) C_3}{(\zeta - \zeta_2)^2 (\zeta - \zeta_1)^2} + \frac{\zeta (\zeta_2 - \zeta_1 \zeta_2) C_4}{(\zeta - \zeta_2)^2 (\zeta - \zeta_1)^2} - \frac{1}{2a} \frac{\zeta (\zeta_1 \zeta_2 \sqrt{1} (\zeta_1 + \zeta_2) a^{3/2} + (\zeta_1 - \zeta_2) \left( -\frac{1}{\sqrt{a}} (\zeta_1 - \zeta_2) \sqrt{a} + a \zeta_1 \right) \right) (\zeta_1 - \zeta_2)}{(\zeta - \zeta_2)^2 (\zeta - \zeta_1)^2}
\]  
(29)

where $C_{3,4}$ are determined from the condition that the next term, $\Phi_0$, be analytic at $\zeta = 0$.

To justify that the actual truncated solution $w(x)$ is also singular near $\zeta_1$ and near $\zeta_2$, we first note that, since $\zeta = \xi x^{-2}$ and $u(x) \sim x^2 \Phi$ then
\[
x^2 \Phi(\zeta) = \frac{\xi}{(1 - x^{-2} \xi \zeta_1^{-1})(1 - x^{-2} \xi \zeta_2^{-1})} = \xi + \frac{1}{x^2} \xi (\zeta_1^{-1} + \zeta_2^{-1}) + \ldots
\]
which is a convergent series in powers of $x^{-2}$. Also $x \Phi_{-1}(\zeta)$ has a similar convergent expansion.

Therefore $x^2 \Phi(\zeta) = F_0(\xi) + O(x^{-2})$ and, in view of (29), $x \Phi_{-1}(\zeta) = O(x^{-1})$ therefore (21) implies that
\[
u(x) \sim x^2 \Phi(x) + O(x^{-1})
\]
for $x \to \infty$ in the same region where (21) holds (and $|x| > 1$). The same argument as in (29) (in Section 4.6) implies that $u(x)$ has singularities within $o(1)$ distance of the singularities of $\Phi(x)$.

5. Proofs for the family III

The existence of a unique power series formal solution $w_0(x) = -1 + o(1)$ is established by standard techniques.

It is also relatively algorithmic to obtain the form of the transseries. Since the procedure may not be well known some details are provided here, also illustrating why $\delta = 2$ is a natural choice.

Plugging in a formal solution of the type $\tilde{w}(x) = \tilde{w}_0(x) + \epsilon g(x) + O(\epsilon^2)$ in the equation (1) and expanding in power series in $\epsilon$, the coefficient of $\epsilon^0$ vanishes, since $\tilde{w}_0(x)$ is already a formal solution. Next, the coefficient of $\epsilon$ is a linear second order differential equation for $g(x)$, with coefficients given in terms of $\tilde{w}_0(x)$ and its derivatives. It has the form
\[
g''(x) + \frac{1}{x} g'(x) - \frac{\delta}{2} g(x) = \frac{1}{x^2} R(x^{-1}, g, g')
\]
with two independent solutions
\[
g_{\pm}(x) = x^{-1/2} \exp(\pm x \sqrt{\delta/2}) (1 + O(x^{-1}))
\]
where we see that it is convenient to take $\delta = 2$. 

Let \( w(x) \) satisfying the assumptions of Theorem 5. To normalize the equation, substitute

\[
w(x) = -1 + \frac{\gamma}{x} + u(x)
\]

so that \( u(x) = O(x^{-2}) \) and satisfies an equation of the form

\[
u'' = \left( 1 - \frac{1}{x} \right) u + f(x^{-1}, u, u')
\]

where \( f \) is analytic at \((0, 0, 0)\) and \( f = O(x^{-2}) + O(u^2) + O(u'2) + O(uu') \). This is a normal form for a second order equation, by the argument in §4.1: the normalizing transformation is (21) with \( q = -1/2 \) and the normal form as a first order system is of the type (22) with \( q = -1/2 \). Theorem 9 applies, and reverting the substitutions, Theorem 5 follows.

5.1. The first array of poles: proof of Theorem 7. Setting \( \xi = Ce^{-x}x^{-1/2}, \) plugging in \( P(\alpha, \beta, \gamma, 2) \) an expansion (39) and expanding, it follows that \( F_0(\xi) \) must satisfy

\[
\xi^2 F_0'' + \xi F_0' = \frac{\xi^2 (3 F_0 - 4) F_0^2}{2 (F_0 - 2) (F_0 - 1)} + 2 \frac{F_0 (F_0 - 1)}{F_0 - 2}
\]

whose unique solution analytic at \( \xi = 0 \) with \( F_0(\xi) = \xi + O(\xi^2) \) \((\xi \to 0)\) is

\[
F_0(\xi) = \frac{\xi}{(\xi/4 + 1)^2}
\]

Let \( x_n \) be solutions of \( \xi(x)/4 + 1 = 0 \). By Theorem 11 if \( x_n \) solve \( Ce^{-x}x^{-1/2} = -4 \), then solutions \( u(x) \) have singularities located at \( x_n + o(1) \) for large \( n \) and Theorem 7 follows.

Remark. While \( F_0(\xi) \) has a pole of order two, it is known that the poles of \( P_V \) have order one if \( \alpha \neq 0 \). This suggests that there are two array of poles within \( O(n^{-1}) \) distance from each other, as the expansion in \( x^{-1} \) would collapse them into a double pole. The author has no rigorous argument at this time that this is indeed the case, and it is formulated here as a conjecture.

6. Appendix: results used

6.1. Summation of transseries formal solutions.

Theorem 8. [18] Consider the (nonlinear) system of first order differential equations:

\[
y' + \left( \Lambda - \frac{1}{x}A \right) y = g(x^{-1}, y)
\]

with \( g = O(x^{-2}) + O(|y|^2) \) for \( x \to \infty, \ |y| \to 0 \)

and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_d), \ A = \text{diag}(\alpha_1, \ldots, \alpha_d) \)

Assume the non-resonance condition: any collection among \( \lambda_1, \ldots, \lambda_d \) which are in the same open half-plane are linearly independent over \( \mathbb{Z} \).

Assume, for simplicity, the system is scaled so that \( \lambda_1 = 1 \).

(i) Then (31) has formal solutions

\[
\tilde{y} = \tilde{y}(x; C) = \tilde{y}_0(x) + \sum_{k \in \mathbb{N}^d \setminus \{0\}} C^k e^{-x\lambda k} \tilde{y}_k(x) \quad \text{with} \quad \tilde{y}_k(x) = x^{\alpha k} \tilde{s}_k(x)
\]
A formal solution \((32)\) is a transseries for \(x \to \infty\) along any direction along which the terms can be well ordered; this means along any direction along which all the exponentials present are decaying, i.e. along any direction in the sector

\[
S_{\text{trans}} = \{ x \in \mathbb{C} \mid \text{Re} (\lambda_j x) > 0 \text{ for all } j \text{ with } C_j \neq 0 \} \tag{33}
\]

(ii) A transseries is Borel summable along any direction of argument \(\phi\), where the sector \(a_1 < \arg x < a_2\) does not contain another Stokes or antistokes line besides \(\arg x = 0\) and the Borel sum is an actual solution:

\[
y(x) = \begin{cases} 
\mathcal{L}_\phi \tilde{y}_0(x) + \sum_{k \in \mathbb{N}_0} C_+^k \mathcal{L}_\phi \tilde{y}_k(x) \\
\mathcal{L}_\phi \tilde{y}_0(x) + \sum_{k \in \mathbb{N}_0} C_-^k \mathcal{L}_\phi \tilde{y}_k(x)
\end{cases} \quad \text{for } -\phi = \arg x \in (0, a_2)
\]

where \(\tilde{y}_k = \mathcal{B}_\phi \tilde{y}_k\) is the analytic continuation of the Borel transform of \(\tilde{y}_k\) along the direction of argument \(\phi\).

Along \(\arg x = 0\) balanced averages of \(\mathcal{L}_\phi \tilde{y}_k\) sum to the solution \(y(x)\).

(iii) The first component \(C_1\) of the constant beyond all orders in \((31)\) changes when \(\arg x\) crosses the Stokes line \(\arg x = 0\), corresponding to \(\lambda_1 = 1\). The change of this constant depends only on the equation: it is a multiple of the first Stokes constant.

We use this theorem in the particular case when \(d = 2\), \(\lambda_1 = 1\), \(\lambda_2 = -1\), In this case the sectors \((33)\) become the right (respectively left) half plane and then the constants have the form \(C = (C, 0)\) (respectively \(C = (0, C)\)); Theorem \(8\) takes the following simpler formulation:

**Theorem 9.** Consider the two-dimensional system of first order differential equations \((31)\) with \(\Lambda = \text{diag}(1, -1)\), \(\Lambda = \text{diag}(\alpha_1, \alpha_2)\). Then:

(i) Then system has formal transseries solutions

\[
\tilde{y} = \tilde{y}(x; C) = \tilde{y}_0(x) + \sum_{k \geq 1} C^k e^{-k x} \tilde{y}_k(x) \quad \text{for } |\arg x| < \frac{\pi}{2}
\]

where \(\tilde{y}_k(x) = x^{k \alpha_1} \tilde{s}_k(x) = x^{k \alpha_1} \sum_{n=0}^{\infty} \tilde{s}_{k,n} x^{-n}\) and \(\tilde{y}_0(x) = O(x^{-2}) \tag{35}\)

and

\[
\hat{y} = \hat{y}(x; C) = \hat{y}_0(x) + \sum_{k \geq 1} C^k e^{k x} \hat{y}_k(x) \quad \text{for } \frac{\pi}{2} < |\arg x| < \frac{3\pi}{2} \tag{36}\]

where \(\hat{y}_k(x) = x^{k \alpha_2} \hat{s}_k(x)\) with \(\hat{s}_k(x)\) are integer power series and \(\tilde{y}_0(x) = O(x^{-2})\).

(ii) The formal solution \((33)\) is Borel summable along any direction of argument \(\phi = -\arg x\) in the sector \(-\frac{\pi}{2} < \arg x < 0\), and in the sector \(0 < \arg x < \frac{\pi}{2}\), and it is generalized Borel summable along \(\arg x = 0\). The sum is an actual solution:

\[
y(x) = \begin{cases} 
\mathcal{L}_\phi y_0(x) + \sum_{k \geq 1} C_+^k e^{-k x} \mathcal{L}_\phi y_k(x) \\
\mathcal{L}_\phi y_0(x) + \sum_{k \geq 1} C_-^k e^{-k x} \mathcal{L}_\phi y_k(x)
\end{cases} \quad \text{for } -\phi = \arg x \in (0, \frac{\pi}{2})
\]

Furthermore (cf. \[11\], Theorem 2) \(\mathcal{L}_\phi y_k\) are analytic for large \(x\) with \(\arg x \in (-\pi/2, 3\pi/2)\), \(\mathcal{L}_\phi y_k = O(x^{\alpha_k})\) and the series converge for \(|x|\) large enough with \(0 < |\arg x| < \frac{\pi}{2}\) and, if \(\text{Re} \alpha_1 < 0\) converges for \(0 < |\arg x| \leq \frac{\pi}{2}\).

(iii) Similarly, the formal solution \((36)\) is Borel summable along any direction of argument \(\phi\), where the sector \(\frac{\pi}{2} < \arg x < \pi\) and in the sector \(\pi < |\arg x| < \frac{3\pi}{2}\), generalized...
Borel summable along arg $x = \pi$, and the sum is an actual solution:

$$y(x) = \begin{cases} 
\mathcal{L}_\phi Y_0(x) + \sum_{k \geq 1} C_{1,+}^k e^{kx} \mathcal{L}_\phi Y_k(x) & \text{for } -\phi = \arg x \in \left(\frac{\pi}{2}, \pi\right) \\
\mathcal{L}_\phi Y_0(x) + \sum_{k \geq 1} C_{1,-}^k e^{kx} \mathcal{L}_\phi Y_k(x) & \text{for } -\phi = \arg x \in \left(\pi, \frac{3\pi}{2}, 0\right). \end{cases} \tag{38}$$

Furthermore $\mathcal{L}_\phi Y_k$ are analytic for large $x$ with arg $x \in (\pi/2, 5\pi/2)$, $\mathcal{L}_\phi Y_k = O(x^{\varpi_k})$ and the series converge for $|x|$ large enough with $\pi < \arg x < \frac{3\pi}{2}$ and, if $\Re \varpi > 0$ converges for in the closed sector.

6.2. Arrays of singularities bordering the sector of analyticity. Theorem 8 establishes existence of solutions, in one-to-one correspondence with formal transseries solutions, and which are analytic for large $x$ in the sector $\left(33\right)$ where these formal solutions are defined (i.e. they are well-ordered with respect to $\gg$). In $\left[29\right]$ it is further shown that on the boundary of the sector of analyticity, these solutions develop arrays of singularities.

In the particular case when $d = 2$, $\lambda_1 = 1$, $\lambda_2 = -1$ of this paper the results in $\left[29\right]$ are as follows.

Denote

$$\xi = \xi(x) = Ce^{-x}x^{-\alpha_1}$$

For $x$ near $i\mathbb{R}_+$, i.e. when $\xi \gg x^{-k}$ for all $k > 0$, the transseries $\left[35\right]$ can be formally reordered as

$$\sum_{k \geq 0} \xi^k s_{k,0}(x) + \frac{1}{x} \sum_{k \geq 0} \xi^k s_{k,1} + \frac{1}{x^2} \sum_{k \geq 0} \xi^k s_{k,2} + \ldots$$

It turns out that the series in $\xi$ are convergent, and the resulting expansion is asymptotic to the solution the original transseries summed to:

**Theorem 10.** $\left[29\right]$ Let $\delta, c > 0$. There exists $\delta_1 > 0$ so that for $|\xi| < \delta_1$ the power series

$$F_m(\xi) = \sum_{k=0}^{\infty} \xi^k s_{k,m}, \quad m = 0, 1, 2 \ldots$$

converge.

Furthermore

$$y(x) \sim \sum_{m=0}^{\infty} x^{-m} F_m(\xi(x)) \quad \text{for } x \to \infty \text{ with } \arg x \in \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right], \quad |\xi(x)| < \delta_1 \tag{39}$$

The asymptotic series is uniform, it is differentiable and satisfies Gevery-like estimates.

**Note:** $F_0(0) = 0$ and $F'_0(0) = 1$.

In fact $\left[39\right]$ is valid in a larger domain, up to distance $o(1)$ of the singularities of $F_0$ ($F_m$ with $m > 0$ can have no other singularities). In general $F_0$ has branch point singularities, and a Riemann surface needs to be considered. But Painlevé equations have no movable branch points, so the singularities of $F_0$ can only be poles. Then for simplicity we state here the general result of $\left[29\right]$ in this case only.

Let $\rho_{1,2}$ so that the small term $g$ in $\left[31\right]$ is analytic in the polydisk $|x^{-1}| < \rho_1$, $|y| < \rho_2$.

Let $\Xi$ be a finite set (possibly empty) of poles of $F_0$. Let $D \subset \mathbb{C} \setminus \Xi$ be open, connected, relatively compact, containing $|\xi| < \delta_1$, so that $F_0$ is analytic in an $\epsilon$-neighborhood of $D$ ($\epsilon > 0$), so that sup$_D |F_0(\xi)| = \rho_3 < \rho_2$.

We need the counterpart of $D$ in the $x$-plane: let $X = \xi^{-1}(D) \cap \{|x| > R, \arg x \in \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right]\}$.

**Theorem 11.** $\left[29\right]$

All $F_m$ with $m \geq 1$ are analytic on $D$ and for $R$ large enough the asymptotic expansion $\left(39\right)$ holds for $x \to \infty$ with $x \in X$. 
Furthermore, assume that $F_0$ is singular at $\xi_s \in \mathcal{D}$. Then $y(x)$ is singular near points $x$ with $\xi(x) = \xi_s$, more precisely at

$$x_n = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C - \ln \xi_s + o(1) \quad \text{as} \quad n \to \infty$$

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