Temperate holomorphic solutions and regularity of holonomic D-modules on curves

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Abstract

Let $X$ be a complex manifold. In [7] M. Kashiwara and P. Schapira made the conjecture that a holonomic $\mathcal{D}_X$-module $M$ is regular holonomic if and only if $RIhom_{\beta X\mathcal{D}_X}(\beta X M, \mathcal{O}_X)$ is regular (in the sense of [7]), the “only if” part of this conjecture following immediately from [7]. Our aim is to prove this conjecture in dimension one.

1 Introduction

Let $X$ be a complex manifold. In [7] the authors defined the notions of microsupport and regularity for ind-sheaves and applied the results to

$$Sol^t(M) := RHom_{\beta X\mathcal{D}_X}(\beta X M, \mathcal{O}_X),$$

the ind-sheaves of tempered holomorphic solutions of $\mathcal{D}_X$-modules. They proved that

$$SS(Sol^t(M)) = Char(M),$$

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where $\text{Char}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$, and that $\text{Sol}^t(\mathcal{M})$ is regular if $\mathcal{M}$ is regular holonomic. In fact, M. Kashiwara and P. Schapira made the following conjecture:

**(K-S)-Conjecture.** Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then $\mathcal{M}$ is regular holonomic if and only if $R\text{Hom}_{\mathcal{D}_X}(\beta_X \mathcal{M}, \mathcal{O}_X^t)$ is regular.

In this paper, we prove that, in dimension one, the regularity of $\text{Sol}^t(\mathcal{M})$ implies the regularity of the holonomic $\mathcal{D}_X$-module $\mathcal{M}$.

We start, in Section 2, with a quick review on sheaves, ind-sheaves, microsupport and regularity for ind-sheaves and we recall the results on the microsupport and regularity of $\text{Sol}^t(\mathcal{M})$, proved in [7]. We include an unpublished Lemma of M. Kashiwara and P. Schapira that will be essential in our proof (see the (K-S)-Lemma).

Section 3 is dedicated to the proof of the irregularity of $\text{Sol}^t(\mathcal{M})$, when $\mathcal{M}$ is an irregular holonomic $\mathcal{D}$-module on $\mathbb{C}$. We may reduce our proof to the case $\mathcal{M} = \mathcal{D}_X^m / \mathcal{D}_X^m P$, where $P$ is a matrix of differential operators of the form $z^N \partial_z I_m + A(z)$, with $m, N \in \mathbb{N}$, $I_m$ the identity matrix of order $m$ and $A$ a $m \times m$ matrix of holomorphic functions on a neighborhood of the origin. We also show that it is enough to prove the irregularity of $\mathcal{S}^t := \text{H}^0(\text{Sol}^t(\mathcal{M}))$ and we prove that $\mathcal{S}^t$ is irregular at $(0; 0) \in T^*\mathbb{C}$.

As an essential step we recall a classical result that gives the characterization of the holomorphic solutions of the differential operator $z^N \partial_z I_m + A(z)$ in some open sectors (see Theorem 3.2). As a consequence of this result we obtain a characterization of $\mathcal{S}^t$, the ind-sheaf of temperate holomorphic solutions of the differential operator $z^N \partial_z I_m + A(z)$, in some open sectors (see Corollary 3.4).

Using the characterization above we find an open sector $S$, with $0 \in \overline{S}$, a small and filtrant category $I$ and a functor $I \to D([a,b]; k_S); i \mapsto F_i$ such that

$$\text{“lim}_{i \to} F_i \simeq \mathcal{S}^t|_S.$$
Moreover, we prove that for each open neighborhood $V$ of $(0; 0)$ there exist a morphism $i \rightarrow i'$ in I and a section $u$ of $\text{H}^0(\mu_{\text{hom}}(F_i, F_{i'}))$ such that for any morphism $i' \rightarrow i''$ in I, denoting by $u'$ the image of $u$ in $\text{H}^0(\mu_{\text{hom}}(F_i, F_{i''}))$, one has $\text{supp}(u') \cap V \not\subset SS(S^t)$. By the (K-S)-Lemma, this is enough to conclude the irregularity of $S^t$ at $(0; 0)$.

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## 2 Notations and review

We will follow the notations in [7].

**Sheaves.** Let $X$ be a real $n$-dimensional manifold. We denote by $\pi : T^*X \rightarrow X$ the cotangent bundle to $X$. We identify $X$ with the zero section of $T^*X$ and we denote by $\check{T}^*X$ the set $T^*X \setminus X$.

Let $k$ be a field. We denote by $\text{Mod}(k_X)$ the abelian category of sheaves of $k$-vector spaces on $X$ and by $D^b(k_X)$ its bounded derived category.

We denote by $\mathbb{R} - C(k_X)$ the abelian category of $\mathbb{R}$-constructible sheaves of $k$-vector spaces on $X$ and by $D^b_{\mathbb{R} - c}(k_X)$ the full subcategory of $D^b(k_X)$ consisting of objects with $\mathbb{R}$-constructible cohomology.

For an object $F \in D^b(k_X)$, we denote by $SS(F)$ the *microsupport of $F$*, a closed $\mathbb{R}^+$-conic involutive subset of $T^*X$. We refer [3] for details.

**Ind-sheaves on real manifolds.** Let $X$ be a real analytic manifold. We denote by $I(k_X)$ the abelian category of ind-sheaves on $X$, that is, the category of ind-objects of the category $\text{Mod}^c(k_X)$ of sheaves with compact
support on $X$ (see [6]).

Recall the natural faithful exact functor

$$\iota_X : \text{Mod}(k_X) \to \text{I}(k_X); F \mapsto \lim_{U \subset X} F_U.$$  

We usually don’t write this functor and identify $\text{Mod}(k_X)$ with a full abelian subcategory of $\text{I}(k_X)$ and $D^b(k_X)$ with a full subcategory of $D^b(\text{I}(k_X))$.

The category $\text{I}(k_X)$ admits an internal hom denoted by $\text{I}hom$ and this functor admits a left adjoint, denoted by $\otimes$. If $F \simeq \lim_i F_i$ and $G \simeq \lim_j G_j$, then:

$$\text{I}hom(G, F) \simeq \lim_j \lim_i \text{Hom}(G_j, F_i),$$

$$G \otimes F \simeq \lim_j \lim_i (G_j \otimes F_i).$$

The functor $\iota_X$ admits a left adjoint

$$\alpha_X : \text{I}(k_X) \to \text{Mod}(k_X); F = \lim_i F_i \mapsto \lim_i F_i.$$  

This functor also admits a left adjoint $\beta_X : \text{Mod}(k_X) \to \text{I}(k_X)$. Both functors $\alpha_X$ and $\beta_X$ are exact. We refer [6] for the description of $\beta_X$.

Let $X$ be a real analytic manifold. We denote by $\mathbb{R} - C^c(k_X)$ the full abelian subcategory of $\mathbb{R} - C(k_X)$ consisting of $\mathbb{R}$-constructible sheaves with compact support. We denote by $\mathbb{I}\mathbb{R} - c(k_X)$ the category $\text{Ind}(\mathbb{R} - C^c(k_X))$ and by $D^b_{\mathbb{I}\mathbb{R} - c}(\text{I}(k_X))$ the full subcategory of $D^b(\text{I}(k_X))$ consisting of objects with cohomology in $\mathbb{I}\mathbb{R} - c(k_X)$.

**Theorem 2.1 ([6]).** The natural functor $D^b(\mathbb{I}\mathbb{R} - c(k_X)) \to D^b_{\mathbb{I}\mathbb{R} - c}(\text{I}(k_X))$ is an equivalence.

Recall that there is an alternative construction of $\mathbb{I}\mathbb{R} - c(k_X)$, using Grothendieck topologies. Denote by $\text{Op}_{X_{sa}}$ the category of open subanalytic subsets of $X$. We may endow this category with a Grothendieck topology by
deciding that a family \( \{ U_i \}_i \) in Op\(_{X_{sa}} \) is a covering of \( U \in \text{Op}_{X_{sa}} \) if for any compact subset \( K \) of \( X \), there exists a finite subfamily which covers \( U \cap K \).

One denotes by \( X_{sa} \) the site defined by this topology and by Mod\((\mathbb{k}_{X_{sa}})\) the category of sheaves on \( X_{sa} \) (see [6]). We denote by Op\(_{X_{sa}}^c \) the subcategory of Op\(_{X_{sa}} \) consisting of relatively compact open subanalytic subsets of \( X \).

Let \( \rho : X \to X_{sa} \) be the natural morphism of sites. We have functors

\[
\text{Mod}(\mathbb{k}_X) \xrightarrow{\rho_*} \text{Mod}(\mathbb{k}_{X_{sa}}),
\]

and we still denote by \( \rho_* \) the restriction of \( \rho_* \) to \( \mathbb{R} - c(\mathbb{k}_X) \) and to \( \mathbb{R} - C^c(\mathbb{k}_X) \).

We may extend the functor \( \rho_* : \mathbb{R} - C^c(\mathbb{k}_X) \to \text{Mod}(\mathbb{k}_{X_{sa}}) \) to \( \mathbb{R} - c(\mathbb{k}_X) \), by setting:

\[
\lambda : \mathbb{R} - c(\mathbb{k}_X) \to \text{Mod}(\mathbb{k}_{X_{sa}})
\]

\[
\text{"lim"}_i F_i \mapsto \lim_{i} \rho_* F_i.
\]

For \( F \in \mathbb{R} - c(\mathbb{k}_X) \), an alternative definition of \( \lambda(F) \) is given by the formula

\[
\lambda(F)(U) = \text{Hom}_{\mathbb{R} - c(\mathbb{k}_X)}(\mathbb{k}_U, F).
\]

**Theorem 2.2 ([6]).** The functor \( \lambda \) is an equivalence of abelian categories.

Most of the time, thanks to \( \lambda \), we identify \( \mathbb{R} - c(\mathbb{k}_X) \) with \( \text{Mod}(\mathbb{k}_{X_{sa}}) \).

We denote by \( C_X^\infty \) the sheaf of complex-valued functions of class \( C^\infty \) and by \( D_X \) the sheaf of Schwartz’s distributions.

Let \( U \) be an open subset of \( X \) and let us denote \( \Gamma(U, C_X^\infty) \) by \( C_X^\infty(U) \).

**Definition 2.3.** Let \( f \in C_X^\infty(U) \). One says \( f \) has polynomial growth at \( p \in X \) if for a local coordinate system \( (x_1, ..., x_n) \) around \( p \), there exist a sufficiently small compact neighborhood \( K \) of \( p \) and a positive integer \( N \) such that

\[
\sup_{x \in K \cap U} (\text{dist}(x, K \setminus U))^N |f(x)| < \infty.
\]

We say that \( f \) is tempered at \( p \) if all its derivatives have polynomial growth at \( p \). We say that \( f \) is tempered if it is tempered at any point.
For each open subanalytic subset $U \subset X$, we denote by $\mathcal{C}^\infty_{X,t}(U)$ the subspace of $\mathcal{C}^\infty_X(U)$ consisting of tempered functions and by $\mathcal{D}^b_{X}(U)$ the sheaf of tempered distributions on $U$. Recall that $\mathcal{D}^b_{X}(U)$ is defined by the exact sequence

$$0 \to \Gamma_{X\setminus U}(X; \mathcal{D}b_X) \to \Gamma(X; \mathcal{D}b_X) \to \mathcal{D}^b_{X}(U) \to 0.$$ 

It follows from the results of Lojasiewicz [10] that $U \mapsto \mathcal{C}^\infty_{X,t}(U)$ and $U \mapsto \mathcal{D}^b_{X}(U)$ are sheaves on the subanalytic site $X_{sa}$, hence define ind-sheaves. We call $\mathcal{C}^\infty_{X,t}(U)$ (resp. $\mathcal{D}^b_{X}(U)$) the ind-sheaf of tempered $\mathcal{C}^\infty$-functions (resp. tempered distributions). These ind-sheaves are well-defined in the category $\operatorname{Mod}(\beta_X \mathcal{D}_X)$, where $\mathcal{D}_X$ now denotes the sheaf of analytic finite-order differential operators.

Let us now recall the definition of the ind-sheaf $\mathcal{O}^l_X$ of tempered holomorphic functions in a complex manifold $X$. By definition,

$$\mathcal{O}^l_X := R\mathcal{I}hom_{\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{D}^b_{X_R}),$$

where $\overline{X}$ denotes the complex conjugate manifold, $X_R$ the underlying real analytic manifold, identified with the diagonal of $X \times \overline{X}$ and $\mathcal{D}_{\overline{X}}$ the sheaf of rings of holomorphic differential operators of finite order over $\overline{X}$. $\mathcal{O}^l_X$ is actually an object of $D^b(\beta_X \mathcal{D}_X)$ and it is not concentrated in degree 0 if $\dim X > 1$. When $X$ is a complex analytic curve, $\mathcal{O}^l_X$ is concentrated in degree 0. Moreover, $\mathcal{O}^l_X$ is $\rho_*$-acyclic and $\mathcal{O}^l_X$ is a sub-ind-sheaf of $\rho_* \mathcal{O}_X$.

We end this section by recalling two results of G. Morando, which will be useful in our proof.

**Theorem 2.4 ([1]).** Let $X$ be an open subset of $\mathbb{C}$, $f \in \mathcal{O}_C(X)$, $f(X) \subset Y \subset \mathbb{C}$. Let $U \in \text{Op}_{X_{sa}}$ such that $f|_U$ is an injective map. Let $h \in \mathcal{O}_X(f(U))$. Then $h \circ f \in \mathcal{O}_X^l(U)$ if and only if $h \in \mathcal{O}_X^l(f(U))$.

**Proposition 2.5 ([1]).** Let $p \in z^{-1}\mathbb{C}[z^{-1}]$ and $U \in \text{Op}_{X_{sa}}$ with $0 \notin U$. The conditions below are equivalent.

1. $\exp(p(z)) \in \mathcal{O}^l_X(U)$.
2. There exists $A > 0$ such that $\operatorname{Re}(p(z)) < A$, for all $z \in U$. 

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Microsupport and regularity for ind-sheaves. We refer [7] for the equivalent definitions for the microsupport $SS(F)$ of an object $F \in D^b(I(k_X))$. We shall only recall the following useful properties of this closed conic subset of $T^*X$.

**Proposition 2.6.** (i) For $F \in D^b(I(k_X))$, one has $SS(F) \cap T^*_X X = \text{supp}(F)$.

(ii) Let $F \in D^b(k_X)$. Then $SS(\iota_X F) = SS(F)$.

(iii) Let $F \in D^b(I(k_X))$. Then $SS(\alpha_X(F)) \subset SS(F)$.

(iv) Let $F_1 \to F_2 \to F_3 \xrightarrow{+1} \to$ be a distinguished triangle in $D^b(I(k_X))$. Then $SS(F_i) \subset SS(F_j) \cup SS(F_k)$, for $i, j, k \in \{1, 2, 3\}$.

Let $J$ denotes the functor $J : D^b(I(k_X)) \to (D^b(\text{Mod}^c(k_X)))^\wedge$ (where $(D^b(\text{Mod}^c(k_X)))^\wedge$ denotes the category of functors from the $D^b(\text{Mod}^c(k_X))^\text{op}$ to Set) defined by:

$$J(F)(G) = \text{Hom}_{D^b(I(k_X))}(G, F),$$

for every $F \in D^b(I(k_X))$ and $G \in D^b(\text{Mod}^c(k_X))$.

**Definition 2.7 ([7]).** Let $F \in D^b(I(k_X))$, $\Lambda \subset T^*X$ be a locally closed conic subset and $p \in T^*X$. We say that $F$ is regular along $\Lambda$ at $p$ if there exists $F'$ isomorphic to $F$ in a neighborhood of $\pi(p)$, an open neighborhood $U$ of $p$ with $\Lambda \cap U$ closed in $U$, a small and filtrant category $I$ and a functor $I \to D^{[a,b]}(k_X); i \mapsto F_i$ such that $J(F') \simeq \underset{i \in I}{\text{lim}} J(F_i)$ and $SS(F_i) \cap U \subset \Lambda$. Otherwise, we say that $F$ is irregular along $\Lambda$ at $p$.

We say that $F$ is regular at $p$ if $F$ is regular along $SS(F)$ at $p$. If $F$ is regular at each $p \in SS(F)$, we say that $F$ is regular.

**Proposition 2.8 ([7]).** (i) Let $F \in D^b(I(k_X))$. Then $F$ is regular along any locally closed set $S$ at each $p \notin SS(F)$.

(ii) Let $F_1 \to F_2 \to F_3 \xrightarrow{+1} \to$ be a distinguished triangle in $D^b(I(k_X))$. If $F_j$ and $F_k$ are regular along $S$, so is $F_i$, for $i, j, k \in \{1, 2, 3\}, j \neq k$.

(iii) Let $F \in D^b(k_X)$. Then $\iota_X F$ is regular.
The next result is an unpublished Lemma of M. Kashiwara and P. Schapira and it is very useful in the study of regularity in $D^b(I(k_X))$.

**K-S-Lemma.** Let $F \in D^b(I(k_X))$, $\Lambda \subset T^*X$ be a locally closed conic subset and $p \in T^*X$. Assume that there exist an open subset $S$ of $X$, with $\pi(p) \in S$, a small and filtrant category $I$ and a functor $F : I \to D^{[a,b]}(k_S); i \mapsto F_i$ such that $J(F|_S) \simeq \lim_{i} J(F_i)$ and, for all open neighborhood $V$ of $p$, with $\Lambda \cap V$ closed in $V$, there exist a morphism $i \to i'$ in $I$ and a section $u$ of $H^0(\muhom(F_i,F_{i'}))$ such that, for any morphism $i' \to i''$ in $I$, denoting by $u'$ the image of $u$ in $H^0(\muhom(F_i,F_{i''}))$, one has $\text{supp}(u') \cap V \not\subset \Lambda$. Then $F$ is irregular along $\Lambda$ at $p$.

**Proof.** We argue by contradiction. Assume that the hypothesis are satisfied and also that $F$ is regular along $\Lambda$ at $p$. Then there exist an open neighborhood $U$ of $p$ with $\Lambda \cap U$ closed in $U$, a small and filtrant category $L$ and a functor $G : L \to D^{[a,b]}(k_X); l \mapsto G_l$ such that $J(F|_{\pi(U)}) \simeq \lim_{i} G_l|_{\pi(U)}$ and $SS(G_l) \cap U \subset \Lambda$, for all $l \in L$.

Since, by hypothesis, one has the following isomorphism on $\pi(U) \cap S$:

$$\lim_{i} F_i \simeq \lim_{i} G_l,$$

for each $i \in I$, there exists $l(i) \in L$ and a morphism $\rho_i : F_i \to G_{l(i)}$ on $\pi(U) \cap S$ and, for each $l \in L$, there exist $i(l) \in I$ and a morphism $\sigma_l : G_l \to F_{i(l)}$ on $\pi(U) \cap S$. Moreover, for each $i \in I$ there exist $k \in I$ and morphisms $f : i \to k$ and $g : i(l(i)) \to k$ in $I$ such that $F(g) \circ \sigma_{l(i)} \circ \rho_i = F(f)$.

Let $i \to i'$ be a morphism in $I$ and $u$ a section of $H^0(\muhom(F_i,F_{i'}))$ such that for any morphism $i' \to i''$ in $I$, denoting by $u'$ the image of $u$ in $H^0(\muhom(F_i,F_{i''}))$, one has $\text{supp}(u') \cap U \not\subset \Lambda$. Let $k \in I$ and $f : i' \to k$, $g : i(l(i')) \to k$ morphisms in $I$ such that $F(g) \circ \sigma_{l(i')} \circ \rho_{i'} = F(f)$. Then $\rho_{i'}$ and $F(g) \circ \sigma_{l(i')}$ induce, respectively, the following morphisms

$$H^0(\muhom(F_i,F_{i'})) \to H^0(\muhom(F_i,G_{l(i')})) \to H^0(\muhom(F_i,F_k)),$$

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that send the section $u$ of $H^0(\mu hom(F_i, F_{i'}))$ to a section $u'$ of $H^0(\mu hom(F_i, F_k))$.

Since $\text{supp}(H^0(\mu hom(F_i, G_{l(i')}) \cap U \subset \Lambda$, one has $\text{supp}(u') \cap U \subset \Lambda$, which is a contradiction. \[ \text{q.e.d.} \]

**Temperate holomorphic solutions of a $\mathcal{D}$-module.** Let $X$ be a complex manifold and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Set

$$Sol(\mathcal{M}) = Rp_* R\text{Hom}_{\mathcal{D}_X} (\mathcal{M}, \mathcal{O}_X),$$

$$Sol^t(\mathcal{M}) = R\text{IHom}_{\beta_X, \mathcal{D}_X} (\beta_X \mathcal{M}, \mathcal{O}_X).$$

It is proved in [7] the equality:

$$SS(Sol^t(\mathcal{M})) = \text{Char}(\mathcal{M}), \quad (3)$$

and that the natural morphism $Sol^t(\mathcal{M}) \to Sol(\mathcal{M})$ is an isomorphism, when $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module, which proves the “only if” part of the (K-S)-Conjecture.

### 3 Proof of (K-S)-Conjecture in dimension one

In this section, we consider $X = \mathbb{C}$ endowed with the holomorphic coordinate $z$ and we shall prove that, for every irregular holonomic $\mathcal{D}_X$-module $\mathcal{M}$, $Sol^t(\mathcal{M})$ is irregular.

We shall first reduce the proof to the case where $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$, for some $P \in \mathcal{D}_X$.

Let $\mathcal{M}$ be an irregular holonomic $\mathcal{D}_X$-module. Since $\mathcal{M}$ is holonomic it is locally generated by one element and we may assume $\mathcal{M}$ is of the form $\mathcal{D}_X/\mathcal{I}$, for some coherent left ideal $\mathcal{I}$ of $\mathcal{D}_X$. We may also assume that $\text{Char}(\mathcal{M}) \subset T^*_X \times T^*_0 X.$ Moreover, we may find $P \in \mathcal{I}$ such that the kernel of the surjective morphism

$$\mathcal{D}_X/\mathcal{D}_X P \to \mathcal{M} \to 0,$$
is isomorphic to a regular holonomic $\mathcal{D}_X$-module $\mathcal{N}$ (see, for example, Chapter VI of [9]). Therefore, we have an exact sequence

$$0 \to \mathcal{N} \to \mathcal{D}_X/\mathcal{D}_X P \to \mathcal{M} \to 0,$$

and we get the distinguished triangle

$$\text{Sol}^t(\mathcal{M}) \to \text{Sol}^t(\mathcal{D}_X/\mathcal{D}_X P) \to \text{Sol}^t(\mathcal{N}) \overset{+1}{\to}.$$

Since $\text{Sol}^t(\mathcal{N})$ is regular, by Proposition 2.8, $\text{Sol}^t(\mathcal{M})$ will be regular if and only if $\text{Sol}^t(\mathcal{D}_X/\mathcal{D}_X P)$ is.

Let us now recall the following result, due to G. Morando:

**Theorem 3.1 ([1])**. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. The natural morphism

$$H^1(\text{Sol}^t(\mathcal{M})) \to H^1(\text{Sol}(\mathcal{M})),$$

is an isomorphism.

Let $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$, with $P \in \mathcal{D}_X$, with an irregular singularity at the origin. The Theorem above together with the results in [3] entails that:

$$H^1(\text{Sol}^t(\mathcal{M})) \simeq H^1(\text{Sol}(\mathcal{M})) \simeq \mathbb{C}^m_{\{0\}},$$

for some $m \in \mathbb{N}$. Then $H^1(\text{Sol}^t(\mathcal{M}))$ is regular and $SS(H^1(\text{Sol}^t(\mathcal{M}))) = T^*_{\{0\}} X$.

As in [7], let us set for short

$$\mathcal{S}^t := H^0(\text{Sol}^t(\mathcal{M})) \simeq \mathcal{I}hom_{\mathcal{D}_X}(\beta_X \mathcal{M}, \mathcal{O}_X^t),$$

$$\mathcal{S} := H^0(\text{Sol}(\mathcal{M})) \simeq \rho_*\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq \ker(\rho_*\mathcal{O}_X \overset{P}{\to} \rho_*\mathcal{O}_X).$$

Remark that, since $\dim X = 1$, one has a monomorphism $\mathcal{S}^t \to \mathcal{S}$. Moreover, we have the following distinguished triangle:

$$\mathcal{S}^t \to \text{Sol}^t(\mathcal{M}) \to H^1(\text{Sol}^t(\mathcal{M}))[−1] \overset{+1}{\to}.$$
Therefore, one has
\[ SS(S^t) \subset \text{Char}(M) \cup T^*_\{0\}X \subset T^*_XX \cup T^*_\{0\}X, \]
and \( S^t \) will be irregular if and only if \( \text{Sol}^t(M) \) is.

The problem is then reduced to study the irregularity of \( S^t \), for an irregular holonomic \( D_X \)-module of the form \( M = D_X/D_XP \), with \( P \in D_X \), with an irregular singularity at the origin.

We shall prove that \( S^t \) is not regular at \( p = (0; 0) \). The plan of the proof is to find an open subset \( S \) of \( \mathbb{C}\{0\} \), with \( 0 \in S \), a small and filtrant category \( I \) and a functor \( I \to D^\\{a,b\}(k_S); i \mapsto F_i \) such that

\[ \text{"lim}_{i} F_i \simeq S^t|_S, \]

and for all open neighborhood \( V \) of \( p \) there exist a morphism \( i \to i' \) in \( I \) and a section \( u \) of \( H^0(\mu\text{hom}(F_i, F_{i'})) \) such that for any morphism \( i' \to i'' \) in \( I \), denoting by \( u' \) the image of \( u \) in \( H^0(\mu\text{hom}(F_{i'}, F_{i''})) \), one has \( \text{supp}(u') \cap V \not\subset SS(S^t) \). This proves that we may apply the (K-S)-Lemma to conclude that \( S^t \) is irregular at \( p \).

Let \( U \) be an open neighborhood of the origin in \( \mathbb{C} \). The problem of finding the solutions of the differential equation \( Pu = 0 \) in \( O_X(U) \) is equivalent to the one of finding the solutions in \( O_X(U)^m \) of a system of ordinary differential equations defined by a matrix of differential operators of the form

\[ z^N\partial_z I_m + A(z), \]

where \( m, N \in \mathbb{N}, I_m \) is the identity matrix of order \( m \) and \( A \in M_m(O_X(U))^1 \).

In fact, setting

\[ P = z^N\partial_z I_m + A(z), \tag{4} \]

we have

\[ S \simeq \rho_*\text{Hom}_{D_X}(D^\mathbb{R}_X/D^\mathbb{R}_XP, O_X), \]

\(^{1}\text{For a ring } R \text{ we denote by } M_m(R) \text{ the ring of } m \times m \text{ matrices and by } \text{GL}_m(R) \text{ the group of invertible } m \times m \text{ matrices.} \)
\[ S^t \simeq \text{Thom}_{\beta_X} \mathcal{D}_X (\beta_X (\mathcal{D}_X^m / \mathcal{D}_X^m P), \mathcal{O}_X), \]

and we may replace \( \mathcal{M} \) by \( \mathcal{D}_X^m / \mathcal{D}_X^m P \).

Let \( \theta_0, \theta_1, R \in \mathbb{R} \), with \( \theta_0 < \theta_1 \) and \( R > 0 \). We denote the open set

\[ \{ z \in \mathbb{C}; \theta_0 < \arg z < \theta_1, 0 < |z| < R \}, \]

by \( S(\theta_0, \theta_1, R) \) and call it open sector of amplitude \( \theta_1 - \theta_0 \) and radius \( R \).

Let \( l \in \mathbb{N} \). We consider \( z_{1/l} \) as a holomorphic function on subsets of open sectors of amplitude smaller than \( 2\pi \), by choosing the branch of \( z_{1/l} \) which has positive real values on \( \arg z = 0 \).

The next Theorem will be fundamental in our proof. It gives a characterization of the holomorphic solutions of the differential system \( Pu = 0 \), where \( P \) is the operator \( \mathcal{H} \).

**Theorem 3.2 (see [11]).** Let \( P \) be the matrix of differential operators \( \mathcal{H} \). There exist \( l \in \mathbb{N} \) and a diagonal matrix \( \Lambda(z) \in M_m(z^{-1/l} \mathbb{C}[z^{-1/l}]) \) such that, for any real number \( \theta \), there exist \( R > 0, \theta_1 > \theta > \theta_0 \) and \( F_\theta \in \text{GL}_m(\mathcal{O}_X(S(\theta_0, \theta_1, R)) \cap C^0(S(\theta_0, \theta_1, R) \setminus \{0\})) \), such that the \( m \)-columns of the matrix \( F_\theta(z) \exp(-\Lambda(z)) \) are \( \mathbb{C} \)-linearly independent holomorphic solutions of the system \( Pu = 0 \). Moreover, for each \( \theta \) there exist constants \( C, M > 0 \) so that \( F_\theta \) has the estimate

\[ C^{-1}|z|^M < |F_\theta(z)| < C|z|^{-M}, \text{ for any } z \in S(\theta_0, \theta_1, R). \quad (5) \]

If there is no risk of confusion we shall write \( F(z) \) instead of \( F_\theta \).

**Definition 3.3.** We call the matrix \( F(z) \exp(-\Lambda(z)) \), given in Theorem 3.2, a fundamental solution of \( P \) on \( S(\theta_0, \theta_1, R) \).

Let \( l \in \mathbb{N} \) and \( \Lambda(z) \) be the diagonal matrix given in Theorem 3.2. For each \( 1 \leq j \leq m \), let \( \Lambda_j(z) = \sum_{k=1}^{n_j} a_{kj}^j z^{-k/l} \) be the \( (j, j) \) entry of \( \Lambda(z) \), with \( n_j \in \mathbb{N}, a_{1j}^j, ..., a_{nj}^j \in \mathbb{C} \).
Corollary 3.4. Let \( V \in \text{Op}_{X_{sa}} \) and let us suppose \( P \) has a fundamental solution \( F(z) \exp(-\Lambda(z)) \) on \( V \). Then, \( \Gamma(V; S^t) \simeq \mathbb{C}^n(V) \), where \( n(V) \) is the cardinality of the set:

\[
J(V) := \{ j \in \{ 1, \ldots, m \}; \exp(-\Lambda_j(z))|_V \in \mathcal{O}_X^t(V) \}.
\]

Proof. By hypothesis, \( \Gamma(V; S) \) is the \( m \)-dimensional \( \mathbb{C} \)-vector space generated by the \( m \)-columns of the matrix \( F(z) \exp(-\Lambda(z)) \). For each \( j = 1, \ldots, m \), let us denote by \( e_j \) the \( j \)-th column of the matrix \( F(z) \exp(-\Lambda(z)) \). We have:

\[
\Gamma(V; S^t) = \Gamma(V; S) \cap \Gamma(V; \mathcal{O}_X^{t,m}) = \{ u \in \Gamma(V; \mathcal{O}_X^{t,m}); \}
\]

\[
\quad u = F(z) \exp(-\Lambda(z)) C, \text{ for some scalar column matrix } C \}.
\]

Let \( k \) be the dimension of the \( \mathbb{C} \)-vector space \( \Gamma(V; S^t) \) and let us prove that \( k = n(V) \).

For each \( j = 1, \ldots, m \), we have \( e_j = F(z) \exp(-\Lambda(z)) C_j \), where \( C_j \) is the scalar column matrix with the \( j \)-th entry equal to 1 and all the other entries equal to zero. Since \( F(z) \) is a matrix of tempered holomorphic functions on \( V \), we get \( e_j \in \Gamma(V; S^t) \), for each \( j \in J(V) \), and the \( \mathbb{C} \)-vector space generated by the family \( \{ e_j \}_{j \in J(V)} \) is a vector subspace of \( \Gamma(V; S^t) \). It follows that \( n(V) \leq k \). On the other hand, we may find a subset \( K \) of \( \{ 1, \ldots, m \} \), with cardinality \( k \), such that \( e_j \in \Gamma(V; S^t) \), for all \( j \in K \). Since \( F^{-1}(z) \) is a matrix of tempered holomorphic functions on \( V \), we get \( \exp(-\Lambda_j(z)) \in \mathcal{O}_X^{t}(V) \), for each \( j \in K \). This entails that \( n(V) \geq k \) and completes our proof. q.e.d.

Lemma 3.5. Let \( S \) be an open sector of amplitude smaller than \( 2\pi \), \( p \in z^{-1}\mathbb{C}[z^{-1}] \), \( l \in \mathbb{N} \) and \( V \in \text{Op}_{X_{sa}} \), with \( V \subset S \). Then \( \exp(p(z^{1/l})) \in \mathcal{O}_X^{t}(V) \) if and only if there exists \( A > 0 \) such that \( \text{Re}(p(z^{1/l})) < A \), for all \( x \in V \).

Proof. Let \( \theta_0, \theta_1, R \in \mathbb{R} \) such that \( 0 < \theta_1 - \theta_0 < R \) and \( S = S(\theta_1, \theta_0, R) \), and let us denote by \( U \) the open sector \( S(\frac{\theta_0}{R}, \frac{\theta_1}{R}, R^{1/l}) \). Let \( f : X \rightarrow X \) be the holomorphic function defined by \( f(z) = z^l \). Since \( \theta_1 - \theta_0 < 2\pi \), we may easily check that \( f|\mathcal{D} \) is an injective map. Moreover, \( f(U) = S \) and
\( f \mid_U : U \to S \) is bijective. Set \( V' = f^{-1}(V) \cap U \) and let \( h \) denotes the holomorphic function defined for each \( z \in S \) by \( h(z) = \exp(p(z^{1/l})) \). By Theorem 2.4 we have \( h \circ f \in O_X(V') \) if and only if \( h \in O_X(V) \). On the other hand, one has \( p \mid_{V'} = h \circ f \mid_{V'} \) and, by Proposition 2.6 \( h \circ f \in O_X(V') \) if and only if there exists \( A > 0 \) such that \( \Re(p(z)) < A \), for all \( z \in V' \). Combining these two facts, we conclude that \( \exp(p(z^{1/l})) \in O_X(V') \) if and only if there exists \( A > 0 \) such that \( \Re(p(z^{1/l})) < A \), for all \( z \in V' \), as desired. q.e.d.

**Proposition 3.6.** With the notation above, there exist an open sector \( S \), with amplitude smaller than \( 2\pi \) and radius \( R > 0 \), and a non-empty subset \( I \) of \( \{1, ..., m\} \) such that, for each \( j \in I \) and each open subanalytic subset \( V \subset S \), the conditions below are equivalent:

(i) there exists \( A > 0 \) such that \( \Re(-\Lambda_j(z)) < A \) for all \( z \in V \),

(ii) there exists \( 0 < \delta < R \) such that \( V \subset \{ z \in S; |z| > \delta \} \).

Moreover, for each \( j \in \{1, ..., m\} \setminus I \), there exists \( A > 0 \) such that, for every \( z \in S \), \( \Re(-\Lambda_j(z)) < A \).

We shall need the following Lemma:

**Lemma 3.7.** Let \( m, l \in \mathbb{N} \), \( \phi_1, ..., \phi_m \in [0, 2\pi[ \) and \( n_1, ..., n_m \in \mathbb{N} \). For each \( j = 1, ..., m \), \( 0 < C < 1 \) and \( \theta \in \mathbb{R} \), let us consider the following two conditions:

(i) \( j, C, \theta \)  
\[ 1 \geq \cos(\phi_j - n_j/l\theta) \geq C, \]

(ii) \( j, C, \theta \)  
\[ -C \geq \cos(\phi_j - n_j/l\theta) \geq -1. \]

Then we may find \( \theta_0, \theta_1 \in \mathbb{R} \), with \( 0 < \theta_1 - \theta_0 < 2\pi \), and positive real numbers \( C_1, ..., C_m \) such that, for each \( j = 1, ..., m \), one of the two conditions (i)\( j, C, \theta \) or (ii)\( j, C, \theta \) holds, for every \( \theta_0 \leq \theta \leq \theta_1 \). Moreover, condition (i)\( j, C, \theta \) is satisfied, for every \( \theta_0 \leq \theta \leq \theta_1 \), for some \( j = 1, ..., m \).

**Proof.** Let us prove the result by induction on \( m \). If \( m = 1 \), set \( C_1 = 1/2 \), \( \theta_0 = 0 \) and \( \theta'_1 = \frac{\theta_0}{n_1} \) if \( \phi_1 = 0 \), and \( \theta_0 = \max\{0, \frac{\phi_1}{n_1} - \frac{\pi}{4}\} \), \( \theta'_1 = \frac{\theta_0}{n_1} \) if \( \phi_1 \neq 0 \). Then, for every \( \theta_0 \leq \theta \leq \theta'_1 \), we have \( 1 \geq \cos(\phi_1 - n_1/l\theta) \geq C_1 \).
Since $0 \leq \theta_0 < 2l\pi$, we may find $0 \leq k < l$ such that $2k\pi \leq \theta_0 < 2(k+1)\pi$ and $\theta_1 \in \mathbb{R}$ such that $\theta_0 < \theta_1 < \theta_1'$ and $\theta_1 < 2(k+1)\pi$. Therefore, condition $(i)_{1,C_1,\theta}$ is satisfied, for every $\theta_0 \leq \theta \leq \theta_1$, where $0 < \theta_1 - \theta_0 < 2\pi$.

Let us now assume that the result is true for some $m \geq 1$ and let us consider $\phi_1, ..., \phi_{m+1} \in [0, 2\pi]$ and $n_1, ..., n_{m+1} \in \mathbb{N}$. By hypothesis, there exist $\theta_0', \theta_1' \in \mathbb{R}$, with $0 < \theta_1' - \theta_0' < 2\pi$, and positive real numbers $C_1, ..., C_m$ such that, for each $j = 1, ..., m$, one of the two conditions $(i)_{j,C_j,\theta}$ or $(ii)_{j,C_j,\theta}$ holds, for every $\theta_0' \leq \theta \leq \theta_1'$. Moreover, condition $(i)_{j,C_j,\theta}$ is satisfied, for every $\theta_0 \leq \theta \leq \theta_1'$, for some $j = 1, ..., m$. Let us prove that there exist $\theta_0, \theta_1 \in \mathbb{R}$, with $\theta_0' < \theta_0 < \theta_1 < \theta_1'$, and $0 < C_{m+1} < 1$ such that one of the two conditions $(i)_{m+1,C_{m+1},\theta}$ or $(ii)_{m+1,C_{m+1},\theta}$ holds, for every $\theta_0 \leq \theta \leq \theta_1$.

For each $j \in \mathbb{Z}$, set:

\[ I_j = [2j\pi, \pi/2 + 2j\pi[, \quad J_j = ]3\pi/2 + 2j\pi, 2\pi(1 + j)[, \]

\[ K_j = ]\pi/2 + 2j\pi, \pi + 2j\pi[, \quad L_j = ]\pi + 2j\pi, 3\pi/2 + 2j\pi[, \]

If $[\phi_{m+1} - n_{m+1}/l\theta_1', \phi_{m+1} - n_{m+1}/i\theta_0'] \subset A_j$, for some $A \in \{I, J, K, L\}$ and $j \in \mathbb{Z}$, set:

\[ C_{m+1} = \begin{cases} 
\cos(\phi_{m+1} - n_{m+1}/l\theta_0'), & A = I, \\
\cos(\phi_{m+1} - n_{m+1}/l\theta_1'), & A = J, \\
-\cos(\phi_{m+1} - n_{m+1}/l\theta_1'), & A = K, \\
-\cos(\phi_{m+1} - n_{m+1}/l\theta_0'), & A = L,
\end{cases} \]

Then condition $(i)_{m+1,C_{m+1},\theta}$ holds, for all $\theta_0' \leq \theta \leq \theta_1'$, if $A \in \{I, J\}$ and condition $(ii)_{m+1,C_{m+1},\theta}$ holds, for all $\theta_0' \leq \theta \leq \theta_1'$, if $A \in \{K, L\}$.

If $[\phi_{m+1} - n_{m+1}/l\theta_1', \phi_{m+1} - n_{m+1}/l\theta_0'] \not\subset A_j$, for every $A \in \{I, J, K, L\}$ and $j \in \mathbb{Z}$, we may choose $\theta_0, \theta_1 \in \mathbb{R}$ such that $\theta_0' < \theta_0 < \theta_1 < \theta_1'$ and $[\phi_{m+1} - n_{m+1}/l\theta_1, \phi_{m+1} - n_{m+1}/l\theta_0] \subset A_j$, for some $A \in \{I, J, K, L\}$ and $j \in \mathbb{Z}$. Then the result follows from the previous cases. \( \text{q.e.d.} \)

**Proof of Proposition 3.6** For each $j = 1, ..., m$, if $z = \rho \exp(i\theta)$, one
has:
\[
\text{Re}(-\Lambda_j(z)) = \sum_{k=1}^{n_j} \alpha_k^j \rho^{-k/l} \cos(\phi_k^j - k/l\theta),
\]
where \(\alpha_k^j = |a_k^j|\) and \(\phi_k^j = \arg(-a_k^j)\), for every \(k = 1, \ldots, n_j\).

For each \(j = 1, \ldots, m\), \(1 > C > 0\) and \(\theta \in \mathbb{R}\), let us consider the following two conditions:
\[
(i)_{j,C,\theta} \quad 1 \geq \cos(\phi_{n_j}^j - n_j/l\theta) \geq C,
(ii)_{j,C,\theta} \quad -C \geq \cos(\phi_{n_j}^j - n_j/l\theta) \geq -1.
\]

By Lemma 3.7, we may find \(\theta_0, \theta_1 /in \mathbb{R}\), with \(0 < \theta_1 - \theta_0 < 2\pi\), and positive real numbers \(C_1, \ldots, C_m\) such that, for each \(j = 1, \ldots, m\), one of the two conditions \((i)_{j,C,\theta}\) or \((ii)_{j,C,\theta}\) holds, for every \(\theta_0 \leq \theta \leq \theta_1\). Moreover, condition \((i)_{j,C,\theta}\) is satisfied, for every \(\theta_0 \leq \theta \leq \theta_1\), for some \(j = 1, \ldots, m\).

Let us set:
\[
J := \{j \in \{1, \ldots, m\} \mid \text{condition \((ii)_{j,C,\theta}\) is satisfied, for every } \theta_0 \leq \theta \leq \theta_1\}.
\]

For each \(j \in J\), \(\theta_0 \leq \theta \leq \theta_1\) and \(\rho > 0\), one has:
\[
\text{Re}(-\Lambda_j(\rho \exp(i\theta))) = \rho^{-n_j/l}[\sum_{k=1}^{n_j-1} \alpha_k^j \rho^{(n_j-k)/l} \cos(\phi_k^j - k/l\theta) + \alpha_{n_j}^j \cos(\phi_{n_j}^j - n_j/l\theta)] \leq \rho^{-n_j/l} \sum_{k=1}^{n_j-1} \alpha_k^j \rho^{(n_j-k)/l} - \alpha_{n_j}^j C_j,
\]
and
\[
\lim_{\rho \to 0^+} \rho^{-n_j/l}[\sum_{k=1}^{n_j-1} \alpha_k^j \rho^{(n_j-k)/l} - \alpha_{n_j}^j C_j] = -\infty.
\]

It follows that there exists \(0 < R_j\) such that \(\text{Re}(-\Lambda_j(\rho \exp(i\theta))) < 0\), for every \(0 < \rho < R_j\) and \(\theta_0 \leq \theta \leq \theta_1\). Therefore, setting \(R = \min\{R_j; j \in J\}\), one gets that \(\text{Re}(-\Lambda_j(z)) < A\), for every \(A > 0\), \(z \in S(\theta_0, \theta_1, R)\) and \(j \in J\).

Let us now set
\[
I := \{j \in \{1, \ldots, m\} \mid \text{condition \((i)_{j,C,\theta}\) is satisfied, for every } \theta_0 \leq \theta \leq \theta_1\}.
\]
Let \( j \in I \) and \( V \) be an open subanalytic subset of the sector \( S(\theta_0, \theta_1, R) \). Suppose that there exists \( A > 0 \) such that \( \text{Re}(-\Lambda_j(z)) < A \) for every \( z \in V \) and that, for each \( 0 < \delta < R \), there exists \( z_{\delta} \in V \) with \( |z_{\delta}| \leq \delta \). For each \( 0 < \delta < R \), let us denote: \( \rho_{\delta} = |z_{\delta}| \) and \( \theta_{\delta} = \text{arg}(z_{\delta}) \). The sequence \( \{\rho_{\delta}\}_{\delta} \) converges to 0 and since \( \{\theta_{\delta}\}_{\delta} \) is a bounded sequence it admits a convergent subsequence. Replacing these two sequences by convenient subsequences, we may assume that \( \{\theta_{\delta}\}_{\delta} \) converges to some \( \theta_2 \in [\theta_0, \theta_1] \). Then:

\[
\lim_{\delta \to 0^+} \text{Re}(-\Lambda_j(\rho_{\delta} \exp(i\theta_{\delta}))) = \lim_{\delta \to 0^+} \rho_{\delta}^{-n_j/l} \left[ \sum_{k=1}^{n_j-1} \alpha_k^j \rho_{\delta}^{(n_j-k)/l} \cos(\phi_k^j - k/l\theta_{\delta}) + \alpha_{n_j}^j \cos(\phi_{n_j}^j - n_j/l\theta_{\delta}) \right] \geq \lim_{\delta \to 0^+} \rho_{\delta}^{-n_j/l} \left[ - \sum_{k=1}^{n_j-1} \alpha_k^j \rho_{\delta}^{(n_j-k)/l} + \alpha_{n_j}^j C_j \right] = +\infty,
\]

which contradicts the fact that \( \text{Re}(-\Lambda_j(\rho_{\delta} \exp(i\theta_{\delta}))) < A \), for every \( 0 < \delta < R \). Conversely, if \( V \) is an open subanalytic subset of \( \{z \in S(\theta_0, \theta_1, R) ; |z| > \delta \} \), for some \( 0 < \delta < R \), then \( V \) is contained on the compact set \( \{z \in \mathbb{C} ; \theta_0 \leq \text{arg} z \leq \theta_1, \delta \leq |z| \leq R \} \), and \( \text{Re}(-\Lambda_j(z)) \) is obviously bounded on \( V \). We conclude that \( I \) is the desired subset of \( \{1, \ldots, m\} \), with \( \{1, \ldots, m\} \setminus I = J \).

q.e.d.

Let \( S(\theta_0', \theta_1', R') \) and \( I \) be, respectively, the open sector and the subset of \( \{1, \ldots, m\} \) given by Proposition 3.6 and let us choose \( \theta_0, \theta_1, R \in \mathbb{R} \), with \( \theta_0' < \theta_0 < \theta_1 < \theta_1' \) and \( R > 0 \), such that the matrix of differential operators \( P \) admits a fundamental solution \( F(z) \exp(-\Lambda(z)) \) on the open sector \( S(\theta_0, \theta_1, R) \). Let us denote \( S = S(\theta_0, \theta_1, R) \), \( S_{\delta} = \{z \in S ; |z| > \delta\} \), for each \( 0 < \delta < R \), and let \( n \) be the cardinality of the set \( I \). Remark that, \( I \neq \emptyset \) and so, \( n > 0 \).

Proposition 3.8. One has the following isomorphism on \( S \),

\[
\lim_{R \to 0^+} \lim_{\delta \to 0^+} (\mathbb{C}^n_{S_{\delta}} \oplus \mathbb{C}^{m-n}_{S}) \to \mathcal{I} \text{hom}_{\beta_X \mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X)_{|S}. \quad (6)
\]
**Proof.** By Lemma 3.5, for each \( j = 1, \ldots, m \) and \( V \in \text{Op}_{X_{sa}} \), with \( V \subset S \), \( \exp(-\Lambda_j(z)) \in O_X^k(V) \) if and only if there exists \( A > 0 \) such that \( \text{Re}(\Lambda_j(z)) < A \) for each \( z \in V \). Let \( V \in \text{Op}_{X_{sa}} \), with \( V \subset S \). Thus, by Proposition 3.6, for all \( j \in \{1, \ldots, m\} \setminus I \), one has \( \exp(-\Lambda_j(z)) \in O_X^k(V) \) and, for \( j \in I \) one has \( \exp(-\Lambda_j(z)) \in O_X^k(V) \) if and only if \( V \subset S_\delta \), for some \( 0 < \delta < R \). Therefore, by Corollary 3.4, either \( V \subset S_\delta \), for some \( 0 < \delta < R \), and in this case \( \Gamma(V; S_X) \simeq \mathbb{C}^m \), or else \( \Gamma(V; S_X) \simeq \mathbb{C}^{m-n} \). By Theorem 2.2, we get the desired isomorphism. q.e.d.

For each \( 0 < \delta < R \), set \( F_\delta = \mathbb{C}^n_{S_\delta} \oplus \mathbb{C}^{m-n}_{S} \) and let us prove that, for every open neighborhood \( U \) of \( p = (0;0) \), \( SS(F_\delta) \cap U \not\subset SS(S^l) \).

Let us set \( z = x + iy \). For each \( 0 < \delta < R \), we have,

\[
SS(F_\delta) = SS(\mathbb{C}_{S_\delta}) \cup T^*_S S = \{ (x, y; 0) \in T^*S; x^2 + y^2 \geq \delta^2 \} \cup \{ (x, y; \lambda x, \lambda y) \in T^*S; \lambda < 0, x^2 + y^2 = \delta^2 \} \cup T^*_S S.
\]

Hence, for each open neighborhood \( U \) of \( p \), we may find \( 0 < \delta < R \) and \((x, y) \in S \) such that \( x^2 + y^2 = \delta^2 \) and \((x, y; -x, -y) \in (SS(F_\delta) \cap U) \setminus SS(S^l) \).

To finish the proof that \( S^l \) is irregular at \( p \), let us recall that, for each \( 0 < \delta < R \), one has the following natural morphisms:

\[\text{Hom}(F_\delta, F_\delta) \simeq H^0(X; R\text{Hom}(F_\delta, F_\delta)) \simeq H^0(T^*X; \mu\text{hom}(F_\delta, F_\delta)) \to \Gamma(T^*X; H^0(\mu\text{hom}(F_\delta, F_\delta))),\]

and we shall denote by \( u_\delta \) the image of \( \text{id}_{F_\delta} \in \text{Hom}(F_\delta, F_\delta) \) in \( \Gamma(T^*X; H^0(\mu\text{hom}(F_\delta, F_\delta))) \).

For each \( 0 < \epsilon < \delta < R \), \( S_\delta \) is an open subset of \( S_\epsilon \) and we have an exact sequence

\[0 \to F_\delta \to F_\epsilon \to F_{\delta, \epsilon} \to 0,\]  

where \( F_{\delta, \epsilon} \) denotes the sheaf \( \mathbb{C}^{m-n}_{S_\delta \setminus S_\delta} \oplus \mathbb{C}^n_{S} \).

Applying the functor \( \mu\text{hom}(F_\delta, \cdot) \) to the exact sequence \( \text{1} \), we obtain the distinguished triangle:

\[\mu\text{hom}(F_\delta, F_\delta) \to \mu\text{hom}(F_\delta, F_\epsilon) \to \mu\text{hom}(F_{\delta, \epsilon}) \xrightarrow{+1}.\]
Since $\mu_{hom}(F_\delta, F_\delta), \mu_{hom}(F_\delta, F_{\varepsilon}), \mu_{hom}(F_\delta, F_{\delta, \varepsilon}) \in D^{\geq 0}(k_T, X)$, it follows from (8) that we have the exact sequence

$$0 \to H^0(\mu_{hom}(F_\delta, F_\delta)) \to H^0(\mu_{hom}(F_\delta, F_{\varepsilon})).$$

Hence, $\text{supp}(u_\delta) = \text{supp}(u')$, where $u'$ is the image of $u_\delta$ in $H^0(\mu_{hom}(F_\delta, F_{\varepsilon}))$. Moreover, by Corollary 6.1.3 of [8], one has $\text{supp}(u_\delta) = \text{SS}(F_\delta)$. This proves that we may apply the (K-S)-Lemma to conclude that $S^l$ is irregular at $p$.

We finish with an example.

**Example 3.9.** Let us consider the $\mathcal{D}_X$-module

$$\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X (z^2 \partial_z + 1).$$

In this case, $\exp(1/z)$ is a fundamental solution of the differential operator $z^2 \partial_z + 1$ in $X \setminus \{0\}$. Arguing as in the proof of Proposition 3.6, we find $R > 0$ with the following property: given an open subanalytic subset $V$ of the sector $S = S(0, \pi/4, R)$, then there exists $A > 0$ such that $\text{Re}(-1/z) < A$, for every $z \in V$, if and only if there exists $0 < \delta < R$ such that $V \subset \{z \in S; |z| > \delta\}$. Moreover, by Proposition 3.8, one has the isomorphism below:

$$\text{"lim"}_{R > \delta > 0} C_{S_\delta} \simeq \text{"lim"}_{\varepsilon > 0} C_{U_\varepsilon \cap S}.$$

In [7], M. Kashiwara and P. Schapira proved the following isomorphism on $X$,

$$\text{"lim"}_{\varepsilon > 0} C_{U_\varepsilon} \simeq \text{Hom}_{\mathcal{D}_X} (\beta_X \mathcal{M}, \mathcal{O}_X^l)|_S,$$

where $U_\varepsilon = X \setminus B_\varepsilon(0, \varepsilon)$, and $B_\varepsilon(0, \varepsilon)$ denotes the open ball with center at $(0, \varepsilon)$ and radius $\varepsilon$, for every $\varepsilon > 0$.

Let us check that

$$\text{"lim"}_{R > \delta > 0} C_{S_\delta} \simeq \text{"lim"}_{\varepsilon > 0} C_{U_\varepsilon \cap S}.$$

It is enough to prove that for every $0 < \delta < R$, there exists $\varepsilon > 0$ such that $S_\delta \subset U_\varepsilon \cap S$ and that for every $\varepsilon > 0$ there exists $0 < \delta < R$ such that $U_\varepsilon \cap S \subset S_\delta$. 

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For each $0 < \delta < R$, we have $S_\delta \subset U_{\delta/4} \cap S$. In fact, given $x + iy \in S_\delta$ we have $x^2 + y^2 > \delta^2$ and $x > y > 0$. It follows that $2x^2 > \delta^2$ and hence, $x^2 + y^2 > x^2 > 2x(\delta/4)$, i.e., $(x - \delta/4)^2 + y^2 > (\delta/4)^2$. Conversely, given $\varepsilon > 0$ and $x + iy \in U_\varepsilon \cap S$, we have $x^2 + y^2 > 2\varepsilon x$ and $x > y > 0$. This gives $x > \varepsilon$. Hence, $x^2 + y^2 > \varepsilon^2$ and $x + iy \in S_\varepsilon$.

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