Numerical analysis of Lane Emden–Fowler equations
Atta Ullah and Kamal Shah
Department of Mathematics, University of Malakand, Chakadara Dir(L), Khyber Pakhtunkhwa, Pakistan

ABSTRACT
This paper is devoted to the numerical solutions of Lane Emden–Fowler partial differential equations. For the numerical analysis, we apply Laplace transform coupled with the Adomian decomposition method known as the Laplace Adomian decomposition method (LADM). We also compare our numerical results with some already existing methods such as homotopy perturbation method, which reveals that the LADM provides the same solutions without any need of perturbation or collocation. Some numerical test problems are also provided by using Maple software.

1. Introduction
Partial differential equations (PDEs) play significant roles in the modelling of many physical, biological and dynamical phenomena. Therefore in the last few decades, this area has been given much attention and plenty of research work were done. Also, PDEs can be used to model large numbers of phenomena such as sound, heat, electrostatics, electrodynamics, fluid dynamics, elasticity or quantum mechanics. The respective research work was related to the analytical and numerical solutions of the considered PDEs, see [1–3]. The well-known PDEs such as wave, heat and Laplace equations have been studied very well and plenty of work is available on these, see [4–6].

One of the most interesting and important equations is known as the Lane Emden–Fowler-type [7] partial differential equation. A variety of problems in the dynamics and physics can be described by using the Lane–Emden–type differential equation. The aforesaid equation has many applications in astrophysics. The Lane–Emden equation is a dimensionless form of Poisson’s equation describing the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid. Solving such type of problems gives some particular solutions which lead to the Lane Emden-type equations. This type of problems has been used in various fields such as physics, engineering and dynamics, see [8]. Now the question is how to obtain the solutions for the mentioned problems. At times, it is difficult to find exact solutions to such type of PDEs. Therefore, various numerical techniques were developed to obtain numerical solutions for such type of PDEs. This is our aim to obtain the analytical solutions of the aforesaid problems for physical understanding. Many methods such as the homotopy perturbation method (HPM) [9, 10] and semi-inverse method [11] were used to find the approximate analytical solutions of the Lane Emden–Fowler equation. In this paper, we use the Laplace Adomian decomposition method (LADM) for the approximate analytical solutions of the Lane Emden–Fowler-type equation given by

\[ \Psi_{xx}(x, t) + \frac{r}{x} \Psi(x, t) + f(x, t) \Psi(x, t) = g(x) + \Psi_{tt}(x, t) \]

with initial conditions,

\[ \Psi(x, 0) = \alpha_1(x), \quad \Psi_t(x, 0) = \alpha_2(x). \]  (1)

We will use the LADM, because perturbation methods are restricted by the parameter. Similarly, collocation methods are based on the discretization of the data which need extra memory and time-consuming. Therefore, we will apply the LADM which needs no extra parameter nor discretization of data and neither needs extra memory. We also compare our results for the first two problems with the solutions of the Lane Emden-Fowler equation obtained by the HPM as well as with their exact answers. For the computation purposes, we use Maple 16.

2. Preliminaries and procedure of LADM
In this section, we first recall some basic definitions needed throughout this paper.
Definition 2.1: For the function $\Psi(x, t)$ defined for $x \in [a, b]$ and $t > 0$, the Laplace transform of $\Psi(x, t)$ is defined as

$$L[\Psi(x, t)] = \int_0^\infty \exp(-st)\Psi(x, t)\,dt, \quad s > 0.$$

Taking Laplace transform to both sides and applying the initial conditions, Equation (1) yields

$$L[\Psi(x, t)] = \frac{1}{s}L[\alpha_1(x)] + \frac{1}{s^2}L[\alpha_2(x)] + \frac{1}{s^2}L_x[\Psi(x, t)] + f(x, t)\Psi(x, t) - g(x) + \Psi_{xx}(x, t). \quad (2)$$

Applying inverse Laplace transform to both sides of Equation (2), we obtain

$$\Psi(x, t) = \alpha_1(x) + t\alpha_2(x) - s^{-1}\left[\frac{1}{s}L[g(x)]\right] + \frac{s}{x}L^{-1}\left[\frac{1}{s^2}L[\Psi(x, t)]\right] + \frac{1}{s}L^{-1}\left[\frac{1}{s^2}L[f(x, t)\Psi(x, t)]\right] + \frac{1}{s}L^{-1}\left[\frac{1}{s^2}L[\Psi_{xx}(x, t)]\right]. \quad (3)$$

The required solution $\Psi(x, t)$ can be obtained in the form of infinite series as

$$\Psi(x, t) = \sum_{n=0}^\infty \Psi^{(n)}(x, t). \quad (4)$$

After putting Equation (4) in Equation (3), we obtain

$$\sum_{n=0}^\infty \Psi^{(n)}(x, t) = \alpha_1(x) + t\alpha_2(x) - s^{-1}\left[\frac{1}{s}L[g(x)]\right] + \frac{s}{x}L^{-1}\left[\frac{1}{s^2}L\left[\sum_{n=0}^\infty \Psi^{(n)}(x, t)\right]\right] + \frac{1}{s}L^{-1}\left[\frac{1}{s^2}L[f(x, t)\sum_{n=0}^\infty \Psi^{(n)}(x, t)]\right] + \frac{1}{s}L^{-1}\left[\frac{1}{s^2}L\left[\sum_{n=0}^\infty \Psi^{(n)}_{xx}(x, t)\right]\right]. \quad (5)$$

Comparing the terms on both sides of Equation (5), we obtain

$$\Psi^{(0)}(x, t) = \alpha_1(x) + t\alpha_2(x)$$

and

$$\Psi^{(1)}(x, t) = -s^{-1}\left[\frac{1}{s}L[g(x)]\right] + \frac{s}{x}L^{-1}\left[\frac{1}{s^2}L[\Psi^{(0)}(x, t)]\right] + \frac{1}{s}L^{-1}\left[\frac{1}{s^2}L[f(x, t)\Psi^{(0)}(x, t)]\right] + \frac{1}{s}L^{-1}\left[\frac{1}{s^2}L[\Psi^{(0)}_{xx}(x, t)]\right]. \quad (6)$$

In the same way, we obtain from Equation (5)

$$\Psi^{(2)}(x, t) = \frac{r}{x}L^{-1}\left[\frac{1}{s^2}L[\Psi^{(1)}(x, t)]\right] + \frac{1}{s}L^{-1}\left[\frac{1}{s^2}L[f(x, t)\Psi^{(1)}(x, t)]\right] + \frac{1}{s}L^{-1}\left[\frac{1}{s^2}L[\Psi^{(1)}_{xx}(x, t)]\right]. \quad (7)$$

and so on. Generally, we can obtain the following relation:

$$\Psi^{(n+1)}(x, t) = \frac{r}{x}L^{-1}\left[\frac{1}{s^2}L[\Psi^{(n)}(x, t)]\right] + \frac{1}{s}L^{-1}\left[\frac{1}{s^2}L[f(x, t)\Psi^{(n)}(x, t)]\right] + \frac{1}{s}L^{-1}\left[\frac{1}{s^2}L[\Psi^{(n)}_{xx}(x, t)]\right]. \quad (8)$$

After evaluating the expression in the above equations, we obtain the solution $\Psi(x, t)$ in the form of infinite series as

$$\Psi(x, t) = \Psi^{(0)} + \Psi^{(1)} + \Psi^{(2)} + \cdots. \quad (9)$$

3. Numerical examples

Example 3.1: Consider the linear Lane Emden–Fowler equation [7]

$$\Psi_{xx} + \frac{2}{x}\Psi_x - (5 + 4x^2)\Psi = \Psi_x + (6 - 5x^2 - 4x^4), \quad (10)$$

with the initial condition

$$\Psi(x, 0) = x^2 + \exp(x^2).$$

Taking Laplace transform of Equation (10), and applying the initial condition, Equation (10) yields,

$$L[\Psi(x, t)] = \frac{1}{s}L[\Psi_x + (6 - 5x^2 - 4x^4)] - \frac{1}{s}L[\Psi_x] + \frac{1}{s}L[\Psi_x] - \frac{1}{s}L[\Psi_x] = \frac{1}{s}L[\Psi_x] - \frac{1}{s}L[\Psi_x]. \quad (11)$$

Applying inverse Laplace transform to both sides of Equation (11), we obtain

$$\Psi(x, t) = x^2 + \exp(x^2) + s^{-1}\left[\frac{1}{s}L^{-1}[\Psi_{xx}]\right] + \frac{s}{x}L^{-1}\left[\frac{1}{s}L[\Psi^{(0)}(x, t)]\right] + \frac{1}{s}L^{-1}\left[\frac{1}{s^2}L[f(x, t)\Psi^{(0)}(x, t)]\right] + \frac{1}{s}L^{-1}\left[\frac{1}{s^2}L[\Psi^{(0)}_{xx}(x, t)]\right]. \quad (12)$$

The proposed solution $\Psi(x, t)$ can be received in the form of infinite series as

$$\Psi(x, t) = \sum_{n=0}^\infty \Psi_n(x, t). \quad (13)$$
Using Equation (13) in Equation (12), we obtain

\[
\sum_{n=0}^{\infty} \Psi(n)(x, t) = x^2 + \exp(x^2) + L^{-1}\left[\frac{1}{5}L^{-1}\left[\sum_{n=0}^{\infty} \Psi(n)\right]\right]
\]

\[
+ L^{-1}\left[\frac{1}{5}L\left[2\sum_{n=0}^{\infty} \Psi(n)\right]\right]
\]

\[
- L^{-1}\left[\frac{1}{5}L\left[(5 + 4x^2) \sum_{n=0}^{\infty} \Psi(n)\right]\right]
\]

\[
- L^{-1}\left[\frac{1}{5}L\left[(6 - 5x^2 - 4x^4)\right]\right].
\] (14)

Comparing the terms on both sides of Equation (14), we have

\[
\Psi(0)(x, t) = x^2 + \exp(x^2)
\]

and

\[
\Psi(1)(x, t) = L^{-1}\left[\frac{1}{5}L^{-1}[\Psi(0)]\right] + L^{-1}\left[\frac{1}{5}L\left[2\Psi(0)\right]\right]
\]

\[
- L^{-1}\left[\frac{1}{5}L\left[(5 + 4x^2)\Psi(0)\right]\right]
\]

\[
- L^{-1}\left[\frac{1}{5}L\left[(6 - 5x^2 - 4x^4)\right]\right].
\] (15)

Putting the values of \(\Psi(0), \Psi_x(0)\) and \(\Psi_{xx}(0)\) in Equation (15), then after simplification, we obtain

\[
\Psi(1)(x, t) = t \exp(x^2).
\]

Similarly, we obtain

\[
\Psi(2)(x, t) = L^{-1}\left[\frac{1}{5}L^{-1}[\Psi(1)]\right] + L^{-1}\left[\frac{1}{5}L\left[2\Psi(1)\right]\right]
\]

\[
- L^{-1}\left[\frac{1}{5}L\left[(5 + 4x^2)\Psi(1)\right]\right].
\] (16)

Using the values of \(\Psi(1), \Psi_x(1)\) and \(\Psi_{xx}(1)\) in Equation (16), then after simplification, we obtain

\[
\Psi(2)(x, t) = \frac{t^2}{2!}\exp(x^2).
\]

In the closed form, the solution is obtained as (Table 1 and Figure 1)

\[
\Psi(x, t) = x^2 + \exp(x^2) + t \exp(x^2) + \frac{t^2}{2!}\exp(x^2)
\]

\[
+ \frac{t^3}{3!}\exp(x^2) + \frac{t^4}{4!}\exp(x^2) + \frac{t^5}{5!}\exp(x^2).
\]

**Example 3.2:** Consider the linear Lane Emden–Fowler equation [7]

\[
\Psi_{xx} + \frac{2}{x} \Psi_x - (5 + 4x^2)\Psi = \Psi_{tt} + (12x - 5x^3 - 4x^5),
\] (17)

with initial conditions

\[
\Psi(0) = x^2 + \exp(x^2), \Psi_t(0, x) = 0.
\]

Taking Laplace transform of Equation (17), and applying the initial conditions, Equation (17) yields

\[
L[\Psi(x, t)] = \frac{1}{s^2}L[x^2 + \exp(x^2)] + \frac{1}{s}L[\Psi_{xx}]
\]

\[
+ \frac{1}{s^2}L[2\Psi_x] - \frac{1}{s^2}L[(5 + 4x^2)\Psi]
\]

\[
- \frac{1}{s^2}L[(12x - 5x^3 - 4x^5)].
\] (18)

Applying inverse Laplace transform to both sides of Equation (18), we obtain

| \(x\) | \(t\) | \(HPM\) | \(LADM\) | Exact answer |
|---|---|---|---|---|
| 0.5 | 0 | 1.53402 | 1.53402 | 1.53402 |
| 0.5 | 0.2 | 1.81822 | 1.81822 | 1.81831 |
| 0.5 | 0.4 | 2.16405 | 2.16405 | 2.16554 |
| 0.5 | 0.6 | 2.58179 | 2.58179 | 2.58964 |
| 0.5 | 0.8 | 3.08170 | 3.08170 | 3.10765 |
| 0.5 | 1.0 | 3.67406 | 3.67406 | 3.74034 |

**Table 1.** Comparison between six terms HPM and four terms LADM with exact answer for Example (3.1).

![Figure 1. Plot of four terms approximate solution for Example 3.1.](image-url)
Equation (18), we obtain
\[
\Psi(x, t) = x^3 + \exp(x^2) + L^{-1}\left[\frac{1}{s^2}L^{-1}[\Psi_{xx}]\right]
+ L^{-1}\left[\frac{1}{s^2}L\left[\frac{2}{x}\Psi_x\right]\right] - L^{-1}\left[\frac{1}{s^2}L((5 + 4x^2)\Psi)\right]
- L^{-1}\left[\frac{1}{s^2}L[(12x - 5x^3 - 4x^5)]\right].
\] (19)

Using \(\Psi(x, t)\) in the form of infinite series as
\[
\Psi(x, t) = \sum_{n=0}^{\infty} \Psi_n(x, t).
\] (20)

After putting Equation (20) in Equation (19), we obtain
\[
\sum_{n=0}^{\infty} \Psi^{(n)}(x, t) = x^3 + \exp(x^2) + L^{-1}\left[\frac{1}{s^2}L^{-1}\left[\sum_{n=0}^{\infty} \Psi^{(n)}_{xx}\right]\right]
+ L^{-1}\left[\frac{1}{s^2}L\left[\sum_{n=0}^{\infty} \Psi^{(n)}_x\right]\right]
- L^{-1}\left[\frac{1}{s^2}L\left[\sum_{n=0}^{\infty} (5 + 4x^2)\Psi^{(n)}\right]\right]
- L^{-1}\left[\frac{1}{s^2}L[(12x - 5x^3 - 4x^5)]\right].
\] (21)

Comparing the terms on both sides of Equation (21) yields
\[
\Psi^{(0)}(x, t) = x^3 + \exp(x^2),
\]
and
\[
\Psi^{(1)}(x, t) = L^{-1}\left[\frac{1}{s^2}L^{-1}[\Psi^{(0)}_{xx}]\right] + L^{-1}\left[\frac{1}{s^2}L\left[\frac{2}{x}\Psi^{(0)}_x\right]\right]
- L^{-1}\left[\frac{1}{s^2}L((5 + 4x^2)\Psi^{(0)})\right]
- L^{-1}\left[\frac{1}{s^2}L[(12x - 5x^3 - 4x^5)]\right].
\] (22)

Plugging the values of \(\Psi^{(0)}\), \(\Psi^{(0)}_x\) and \(\Psi^{(0)}_{xx}\) in Equation (22), then after simplification, we obtain
\[
\Psi^{(1)}(x, t) = \frac{t^2}{2!}\exp(x^2).
\]

Also
\[
\Psi^{(1)}(x, t) = L^{-1}\left[\frac{1}{s^2}L^{-1}[\Psi^{(1)}_{xx}]\right]
+ L^{-1}\left[\frac{1}{s^2}L\left[\frac{2}{x}\Psi^{(1)}_x\right]\right]
- L^{-1}\left[\frac{1}{s^2}L((5 + 4x^2)\Psi^{(1)})\right]
\] (23)

using the values of \(\Psi^{(1)}\), \(\Psi^{(1)}_x\), and \(\Psi^{(1)}_{xx}\) in Equation (22), then after simplification, we obtain
\[
\Psi^{(2)}(x, t) = \frac{t^4}{4!}\exp(x^2).
\]

Similarly, we obtain the other terms as
\[
\Psi^{(3)}(x, t) = \frac{t^6}{6!}\exp(x^2),
\]
\[
\Psi^{(4)}(x, t) = \frac{t^8}{8!}\exp(x^2),
\]
\[
\Psi^{(5)}(x, t) = \frac{t^{10}}{10!}\exp(x^2),
\]
and so on.

In the closed form, the solution is obtained as (Figure 2 and Table 2)
\[
\Psi(x, t) = x^3 + \exp(x^2) + \frac{t^2}{2!}\exp(x^2) + \cdots
= x^3 + \exp(x^2)\left[1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right].
\]

**Example 3.3:** Consider the linear Lane Emder–Fowler equation [7]
\[
\Psi_{xx} + \frac{4}{x}\Psi_x - (18 + 9x^4)\Psi = \Psi_{tt} - 2 - (18x + 9x^4)t^2,
\] (24)
with initial conditions
\[
\Psi(x, 0) = x^3, \Psi_t(x, 0) = 0.
\]
Taking Laplace transform of Equation (24), and

![Figure 2](image-url). Plot of four terms approximate solution for Example 3.2.

| x     | t   | HPM  | LADM | Exact answer |
|-------|-----|------|------|--------------|
| 0.5   | 0   | 1.40902 | 1.40902 | 1.40902 |
| 0.5   | 0.2 | 1.69322 | 1.43479 | 1.17627 |
| 0.5   | 0.4 | 2.03905 | 1.51311 | 0.98570 |
| 0.5   | 0.6 | 2.45679 | 1.64708 | 0.82968 |
| 0.5   | 0.8 | 2.95670 | 1.84182 | 0.70194 |
| 0.5   | 1.0 | 3.54906 | 2.10453 | 0.59736 |
applying the initial conditions, Equation (25) yields
\[
\mathcal{L}[\Psi(x, t)] = \frac{1}{s^2} \mathcal{L}[x^3] + \frac{1}{s^2} \mathcal{L}[2] + \frac{1}{s^2} \mathcal{L}[\Psi_{xx}]
\]
\[
+ \frac{1}{s^2} \mathcal{L}\left[\frac{4}{x} \Psi_x\right] - \frac{1}{s^2} \mathcal{L}[18 + 9x^4] \Psi
\]
\[
+ \frac{1}{s^2} \mathcal{L}[(18x + 9x^4)t^2].
\]
(25)

Applying inverse Laplace transform to both sides of equation (25), we obtain
\[
\Psi(x, t) = x^3 + t^2 + \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}^{-1}[\Psi_{xx}]\right]
\]
\[
+ \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}\left[\frac{4}{x} \sum_{n=0}^{\infty} \Psi_x^{(n)}\right]\right]
\]
\[
- \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}\left[(18 + 9x^4) \sum_{n=0}^{\infty} \Psi_x^{(n)}\right]\right]
\]
\[
+ \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}[(18x + 9x^4)t^2]\right].
\]
(26)

Taking \(\Psi(x, t)\) in the form of infinite series as
\[
\Psi(x, t) = \sum_{n=0}^{\infty} \Psi_{x}^{(n)}(x, t).
\]
(27)

Using Equation (27) in Equation (26), we obtain
\[
\sum_{n=0}^{\infty} \Psi_{x}^{(n)}(x, t) = x^3 + t^2 + \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}^{-1}\left[\sum_{n=0}^{\infty} \Psi_{xx}^{(n)}\right]\right]
\]
\[
+ \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}\left[\frac{4}{x} \sum_{n=0}^{\infty} \Psi_x^{(n)}\right]\right]
\]
\[
- \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}\left[(18 + 9x^4) \sum_{n=0}^{\infty} \Psi_x^{(n)}\right]\right]
\]
\[
+ \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}[(18x + 9x^4)t^2]\right].
\]
(28)

Comparing terms on both sides of Equation (28), we obtain
\[
\Psi_{x}^{(0)}(x, t) = x^3 + t^2
\]
and
\[
\Psi_{x}^{(1)}(x, t) = \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}^{-1}[\Psi_{x}^{(0)}]\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}\left[\frac{4}{x} \Psi_x^{(0)}\right]\right]
\]
\[
- \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}[(18 + 9x^4)\Psi^{(0)}]\right]
\]
\[
+ \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}[(18x + 9x^4)t^2]\right].
\]
(29)

Putting the values of \(\Psi_{x}^{(0)}, \Psi_{x}^{(0)}\) and \(\Psi_{x}^{(0)}\) in Equation (29), then after simplification, we obtain
\[
\Psi_{x}^{(1)}(x, t) = 0.
\]

Further, we have
\[
\Psi_{x}^{(1)}(x, t) = \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}^{-1}[\Psi_{xx}^{(1)}]\right]
\]
\[
+ \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}\left[\frac{4}{x} \Psi_x^{(1)}\right]\right]
\]
\[
- \mathcal{L}^{-1}\left[\frac{1}{s^2} \mathcal{L}[(5 + 4x^2)\Psi^{(1)}]\right].
\]
(30)

Putting the values of \(\Psi_{x}^{(1)}, \Psi_{x}^{(1)}, \) and \(\Psi_{xx}^{(1)}\) in Equation (29), then upon simplification, one has
\[
\Psi_{x}^{(2)}(x, t) = 0.
\]

All the other terms become zero for \(n \geq 3\). Hence, we obtain the solution as
\[
\Psi(x, t) = x^3 + t^2,
\]
which is the exact solution of the given problem (Figure 3).

4. Conclusion
In this paper, we have successfully solved three kinds of the Lane Emden–Fowler differential equation with initial conditions by the LADM. From numerical point of view, we observed that the approximate solutions obtained by the LADM up to four terms achieve accurate answer compared with the HPM for six terms with the exact solution. Maple 2016 has been used for tabulation and simulation of the values.

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