QUANTUM CURRENT OPERATORS (III)
COMMUTATIVE QUANTUM CURRENT OPERATORS,
SEMI-INFINITE CONSTRUCTION AND FUNCTIONAL MODELS

JINTAI DING AND BORIS FEIGN

Abstract. We construct a commutative current operator \( \bar{x}^+(z) \) inside \( U_q(\hat{sl}(2)) \). With this operator and the condition of quantum integrability on the quantum current of \( U_q(\hat{sl}(2)) \), we derive the quantization of the semi-infinite construction of integrable modules of \( \hat{sl}(2) \) with the current operator \( e(z) \) of \( \hat{sl}(2) \). The quantization of the functional models for \( \hat{sl}(2) \) are also given.

1. Introduction.
For any integrable highest weight module of \( \hat{sl}(2) \) of level \( k \), the current operators \( e(z) \) and \( f(z) \) satisfy the following relations:

\[
e(z)^{k+1} = f(z)^{k+1} = 0,
\]

which we call the condition of integrability [LP]. For the case of quantum affine algebras, Drinfeld presented a formulation of affine quantum groups with generators in the form of current operators[Dr2], which, for the case of \( U_q(\hat{sl}(2)) \), give us the quantized current operators corresponding to \( e(z) \) and \( f(z) \) of \( \hat{sl}(2) \). In [DM], we derive the quantum integrable condition for \( U_q(\hat{sl}(2)) \). On any level \( k \) integrable module of \( U_q(\hat{sl}(2)) \) the matrix coefficients of \( x^+(z_1)x^+(z_2)\ldots x^+(z_{k+1}) \) zero at \( z_2/z_1 = z_3/z_2 = \ldots = z_{k+1}/z_k = q^2 \), and those of \( x^-(z_1)x^-(z_2)\ldots x^-(z_{k+1}) \) are zero at \( z_1/z_2 = z_2/z_3 = \ldots = z_k/z_{k+1} = q^2 \), where \( x^+(z) \) and \( x^-(z) \) are the quantized current operators of \( U_q(\hat{sl}(2)) \) corresponding to \( e(z) \) and \( f(z) \) of \( \hat{sl}(2) \), respectively.

In the case of \( \hat{sl}(2) \), the integrable condition was used by Feigin and Stoyanovsky [FS] to construct a level \( k \) module as a semi-infinite and describe the function models for the dual spaces. With the condition of the quantum integrability, still we can not simply derive the quantization of the semi-infinite construction, because of the noncommutativity of the current operator \( x^+(z) \). Thus we have to modify the current operator \( x^+(z) \) to “force” it to commute with itself. We use the subalgebra
coming from the Heisenberg algebra of $U_q(\hat{\mathfrak{sl}}(2))$ to construct a commutative current operator $\bar{x}^+(z) = \sum \bar{x}_i z^{-i}$, which commutes with itself and

$$\bar{x}^+(z_1)\bar{x}^+(z_1 q^2)\ldots \bar{x}^+(z_{k+1} q^{2k}) = 0.$$ 

Then the quantization of the semi-infinite construction, simply follows, namely the integrable modules of $U_q(\hat{\mathfrak{sl}}(2))$ can be identified with the space consisting of semi-infinite expressions $\bar{x}^+_{i_1} \ldots \bar{x}^+_{i_n} \ldots$, whose tails stabilize in certain way and the action of $\bar{x}^+_i$ acts by multiplication. Due to the introduction of the parameter $q$, we can describe the action of the operators rigorously, especially the action of the operator $a_{-1}$, which corresponds to the operator $b_{-1}$ of $\mathfrak{sl}(2)$. As in the case of [FS], the functional models for the dual spaces of the subspace generated by $\bar{x}^+(z)$ on the highest weight vector in any irreducible integrable module of level $k$ are derived by using symmetric functions $f(t_1, \ldots, t_n)$, which is zero when $t_1 = t_2 q^2 \ldots = t_{k+1} q^{2k}$.

2. $U_q(\hat{\mathfrak{sl}}(n))$ and commutative quantum current operators

For the case of affine quantum groups, Drinfeld gave a realization of those algebras in terms of operators in the form of current [Dr2]. We will first present such a realization for the case of $U_q(\hat{\mathfrak{sl}}(n))$.

Let $A = (a_{ij})$ be the Cartan matrix of type $A_{n-1}$.

**Definition 1.** The algebra $U_q(\hat{\mathfrak{sl}}(n))$ is an associative algebra with unit 1 and the generators: $\varphi_i(-m), \psi_i(m), x_i^\pm(l)$, for $i = 1, \ldots, n - 1$, $l \in \mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$ and a central element $c$. Let $z$ be a formal variable and $x_i^\pm(z) = \sum_{l \in \mathbb{Z}} x_i^\pm(l) z^{-l}$, $\varphi_i(z) = \sum_{m \in \mathbb{Z}_{\geq 0}} \varphi_i(m) z^{-m}$ and $\psi_i(z) = \sum_{m \in \mathbb{Z}_{\geq 0}} \psi_i(m) z^{-m}$. In terms of the formal variables,
the defining relations are
\[
\begin{align*}
\varphi_i(z)\varphi_j(w) &= \varphi_j(w)\varphi_i(z), \\
\psi_i(z)\psi_j(w) &= \psi_j(w)\psi_i(z), \\
\varphi_i(z)\psi_j(w)\varphi_i(z)^{-1}\psi_j(w)^{-1} &= \frac{g_{ij}(z/wq^{-c})}{g_{ij}(z/wq^c)}, \\
\varphi_i(z)x_j^\pm(w)\varphi_i(z)^{-1} &= g_{ij}(z/wq^{\pm\frac{1}{2}c})x_j^\pm(w), \\
\psi_i(z)x_j^\pm(w)\psi_i(z)^{-1} &= g_{ij}(w/zq^{\pm\frac{1}{2}c})x_j^\pm(w), \\
[x_i^+(z), x_j^-(w)] &= \frac{\delta_{ij}}{q-q^{-1}}\left\{\delta(z/wq^{-c})\psi_i(wq^{\frac{1}{2}c}) - \delta(z/wq^c)\varphi_i(zq^{\frac{1}{2}c})\right\}, \\
(z - q^{\pm a_{ij}}w)x_i^+(z)x_j^+(w) &= (q^{\pm a_{ij}}z - w)x_j^+(w)x_i^+(z), \\
[x_i^\pm(z), x_j^\pm(w)] &= 0 \quad \text{for } a_{ij} = 0, \\
x_i^+(z_1)x_i^+(z_2)x_j^+(w) - (q + q^{-1})x_i^+(z_1)x_j^+(w)x_i^+(z_2) + x_j^+(w)x_i^+(z_1)x_i^+(z_2) \\
+ \{z_1 \leftrightarrow z_2\} &= 0, \quad \text{for } a_{ij} = -1
\end{align*}
\]
where
\[
\delta(z) = \sum_{k \in \mathbb{Z}} z^k, \quad g_{ij}(z) = \frac{q^{a_{ij}}z - 1}{z - q^{a_{ij}}} \quad \text{about } z = 0.
\]

We define a grading on this algebra such that \(x_i^\pm(n), \varphi_i(n)\) and \(\psi_i(n)\) are of degree \(n\). We also always assume that \(q\) is generic.

Clearly, we have that \(x^+(z)\) does not commute with itself. In order to do modify this operator, we have to rewrite the operators \(\varphi_i(z)\) and \(\psi_i(z)\) with new operators \(a_{i,n}\) for \(n \in \mathbb{Z}\).

For the case of \(U_q(\hat{\mathfrak{sl}}_2)\), we have that
\[
\varphi(z) = \exp[-(q - q^{-1})\sum_{k \geq 0} a_{-k}z^k],
\]
\[
\psi(z) = \exp[(q - q^{-1})\sum_{k \leq 0} a_{-k}z^k].
\]

The new operators are defined as:
\[
\exp[-(q - q^{-1})a_0] = \varphi(0), \\
-(q - q^{-1})\sum_{k \geq 0} a_{-k}z^k = \log(1 + (\varphi(z)\varphi(0)^{-1} - 1)) = \sum_{n > 0}(\varphi(z)\varphi(0)^{-1} - 1)^n/n, \\
(q - q^{-1})\sum_{k \leq 0} a_{k}z^{-k} = \log(1 + (\psi(z)\psi(0)^{-1} - 1)) = \sum_{n > 0}(\psi(z)\psi(0)^{-1} - 1)^n/n.
\]
Proposition 2.

\[ [a_k, a_l] = \delta_{k+l,0}(q^{2k} - q^{-2k})(q^c - q^{-c})/(k(q - q^{-1})^2), \]
\[ [a_k, x^\pm(l)] = (q^{2k} - q^{-2k})q^{\mp|c|/2}x^\pm(k + l)/(k(q - q^{-1})). \]

Let \( k^{-}(z) \) be an current operator in \( U_q(\hat{sl}(2)) \), such that

\[
(1) k^{-}(z) = 1 + \sum_{n>0} k^{-}(n)z^{-n},
\]
\[
(II) k^{-}(z)x^+(w) = \frac{z - wq^2}{z - w}x^+(w)k^{-}(z),
\]

where \( k^{-}(n) \) are operators of degree \( n \), \( k^{-}(n)k^{-}(m) = k^{-}(m)k^{-}(n) \) and \( \frac{z - wq^2}{z - w} \) is expanded in \( w/z \).

Let \( \bar{x}^+(w) = x^+(w)k^{-}(w) \), then we have

Proposition 3.

\[
(z - w)\bar{x}^+(z)\bar{x}^+(w) = (z - w)\bar{x}^+(w)\bar{x}^+(z).
\]

The proof comes from the following calculation.

\[
(z-w)\bar{x}^+(z)\bar{x}^+(w) = x^+(z)\bar{x}^+(z)x^+(w)k^{-}(w) = (z-w)\frac{z - wq^2}{z - w}x^+(z)x^+(w)k^{-}(z)k^{-}(w) =
\]
\[
(qz^2 - w)x^+(w)x^+(z)k^{-}(w)k^{-}(z) = x^+(w)k^{-}(w)x^+(z)k^{-}(z)(zq^2 - w)(\frac{w - zq^2}{w - z})^{-1} =
\]
\[
(z - w)\bar{x}^+(w)\bar{x}^+(z)
\]

Theorem 4.

\[ \bar{x}^+(z)\bar{x}^+(w) = \bar{x}^+(w)\bar{x}^+(z). \]

Proof Let \( V_k \) be an integrable module of \( U_q(\hat{sl}(2)) \) and \( V_k^* \) be its dual. Let \( v \in V_k \) and \( v^* \in V_k^* \). From the commutation relation between \( x^+(z) \) and \( x^+(w) \), we have that as an analytic function, the matrix coefficient \(< v^*, x^+(z)x^+(w)v > \) is zero when \( z = w \), thus the correlation function \(< v^*, x^+(z)x^+(w)v > \) always has a factor \( z - w \).

This implies that the correlation function of \(< v^*, \bar{x}^+(z)\bar{x}^+(w)v > \) does not have a pole at the point \( z = w \), thus

\[ \bar{x}^+(z)\bar{x}^+(w) = \bar{x}^+(w)\bar{x}^+(z) \]

follows from

\[ (z - w)\bar{x}^+(z)\bar{x}^+(w) = (z - w)\bar{x}^+(w)\bar{x}^+(z). \]

Proposition 5. Let

\[ k^{-}(z) = \exp[(q - q^{-1})\Sigma_{n>0} q^{2(n+c)/2}/(1 + q^{2n})a_nz^{-n}]. \]

Then \( k^{-}(z) \) satisfies (I) and (II).
From now on, we will denote \( x^+(z) \exp[(q - q^{-1})\Sigma_{n>0} - q^{2(n+c)/2}/(1 + q^{2n})a_nz^{-n}] \) by \( \tilde{x}(z) \) throughout this paper.

Let \( k^+(z) \) be an current operator in \( U_q(\mathfrak{sl}(2)) \) such that

\[
(\text{I}')k^+(z) = 1 + \Sigma_{n<0}k^+(n)z^n,
\]

\[
(\text{II}')k^+(z)x^+(w) = \frac{z - w}{zq^2 - w}x^+(w)k^+(z),
\]

where \( k^+(n) \) are operators of degree \( n \), \( k^+(n)k^+(m) = k^+(m)k^+(n) \) and \( \frac{z - w}{zq^2 - w} \) is expanded in \( z/w \). Let \( \tilde{x}^+(w) = k^+(w)\tilde{x}^+(w) \), then we have

**Proposition 6.**

\[
(z - w)\tilde{x}^+(z)\tilde{x}^+(w) = (z - w)\tilde{x}^+(w)\tilde{x}^+(z).
\]

The proof comes from the following calculation.

\[
(z-w)\tilde{x}^+(z)\tilde{x}^+(w) = k^+(z)x^+(z)k^+(w)x^+(w) = (z-w)\frac{z - wq^2}{z - w}k^+(z)k^+(w)x^+(z)x^+(w) = (zq^2 - w)k^+(z)k^+(w)x^+(z)x^+(w) = k^+(w)x^+(w)k^+(z)x^+(z)(zq^2 - w)(\frac{z - w}{zq^2 - w}) = (z - w)\tilde{x}^+(w)\tilde{x}^+(z)
\]

**Theorem 7.**

\[
\tilde{x}^+(z)\tilde{x}^+(w) = \tilde{x}^+(w)\tilde{x}^+(z).
\]

The proof is the same as that if the theorem above.

**Proposition 8.** Let

\[
k^+(z) = \exp[-(q - q^{-1})\Sigma_{n<0} - q^{2(n+c)/2}/(1 + q^{2n})a_nz^{-n}].
\]

Then \( k^+(z) \) satisfies the condition \((\text{I}')\) and \((\text{II}')\).

From now on, we will denote the operator \( \exp[-(q - q^{-1})\Sigma_{n<0} - q^{2(n+c)/2}/(1 + q^{2n})a_nw^{-n}]x^+(w) \) by \( \tilde{x}^+(w) \).

For the case of \( U_q(\mathfrak{sl}_n) \), we have that

\[
\varphi_i(z) = \exp[-(q - q^{-1})\sum_{k \geq 0} a_{i,-k}z^k],
\]

\[
\psi_i(z) = \exp[(q - q^{-1})\sum_{k \leq 0} a_{i,-k}z^k].
\]

The new operators are defined as:

\[
\exp[-(q - q^{-1})a_{i,0}] = \varphi_i(0),
\]
\[-(q - q^{-1}) \sum_{k \geq 0} a_{i-k} z^k = \log(1 + (\varphi_i(z)\varphi_i(0)^{-1} - 1)) = \sum_{n \geq 0} (\varphi_i(z)\varphi_i(0)^{-1} - 1)^n/n,\]
\[(q - q^{-1}) \sum_{k \leq 0} a_{i+k} z^{-k} = \log(1 + (\psi_i(z)\psi_i(0)^{-1} - 1)) = \sum_{n \geq 0} (\psi_i(z)\psi_i(0)^{-1} - 1)^n/n.\]

Let
\[k^+_i(z) = \exp[-(q - q^{-1}) \sum_{n < 0} q^{2(n+c)/2}/(1 + q^{2n})a_{i,n}z^{-n}],\]
and
\[k^-_i(z) = \exp[(q - q^{-1}) \sum_{n > 0} q^{2(n+c)/2}/(1 + q^{2n})a_{i,n}z^{-n}].\]

Let
\[\bar{x}^+(z) = x^+(z)k^-_i(z),\]
and
\[\tilde{x}^+(z) = k^+_i(z)x^+(z).\]

**Theorem 9.**

\[\bar{x}^+_i(z)\bar{x}^+_i(w) = \bar{x}^+_i(w)\bar{x}^+_i(z),\]
\[\tilde{x}^+_i(z)\tilde{x}^+_i(w) = \tilde{x}^+_i(w)\tilde{x}^+_i(z).\]

It is obvious that the set of current operators \(\varphi_i(z), \psi_i(z), \bar{x}^+_i(z)\) and \(x^- (z)\) and the the set of the current operators \(\varphi_i(z), \psi_i(z), \bar{x}^+_i(z)\) and \(x^- (z)\) generate the quantum affine algebra \(U_q(\hat{\mathfrak{sl}}(n))\) respectively. The reformulation of the quantum affine algebra \(U_q(\hat{\mathfrak{sl}}(n))\) with current operators \(\varphi_i(z), \psi_i(z), \bar{x}^+_i(z)\) and \(x^- (z)\) is the key for the quantized semi-infinite construction in the next section, namely we need to use the kernel coming from the current operator \(\tilde{x}^+_i(z)\) to define the semi-infinite space. From now on, we will restrict ourselves to the case of \(U_q(\hat{\mathfrak{sl}}(2))\). The case for \(U_q(\hat{\mathfrak{sl}}(n))\) can be dealt with in a similar way [FS].

For the case of \(U_q(\hat{\mathfrak{sl}}(2))\), the relations between \(\psi(z), \bar{x}^+(z)\) is the same as that of \(\psi_3(z), \bar{x}^+(z)\), however the rest are changed, which we will write them down below.

**Proposition 10.**

\[\varphi(z)\bar{x}^+(w)\varphi(z)^{-1} = f_1(z/w)g(z/wq^{-\frac{1}{2}})^{1/2}\bar{x}^+(w),\]
\[(f_2(w/z)\bar{x}^+(z)x^-(w) - x^-(w)\bar{x}^+(z) = \frac{\delta_{i,j}}{q-q^{-1}} \left\{ \delta(z/wq^{-c})\psi(wq^{\frac{1}{2}})k^-(z) - \delta(z/wq^c)\varphi(zq^{\frac{1}{2}})k^-(z) \right\} ,\]
\[\bar{x}^+(z)\bar{x}^+(w) = \bar{x}^+(w)\bar{x}^+(z),\]

where \(f_1(z/w) = (1-z/wq^{-2}\sqrt{q(c)})^{-1}(1-z/wq^{-2}\sqrt{q(c)})^{-1}\) and \(f_2(w/z) = (q^2q^c w/z - 1)(w/zq^{-c} - 1).\)
Similarly, one can write down the relations between $\varphi(z)$, $\psi(z)$, $\bar{x}^+(z)$ and $x^-(z)$, which we will omit here. In the next section, we will $\bar{x}^+(z)$ and instead of $x^+(z)$ as the current operator for our semi-infinite construction of representations of $U_q(\mathfrak{sl}(2))$.

3. **Quantum integrability condition and semi-infinite construction**

The integrability condition of the current operator $e(z)$ induces the semi-infinite construction for the unquantized case. The quantum integrability condition was studied in [DM], which is stated as the following:

**Theorem 11.** For any level $k \geq 1$ integrable module of $U_q(\mathfrak{sl}(2))$, the correlation functions of $x^+(z_1)x^+(z_2)...x^+(z_k)x^+(z_{k+1})$ is zero if $z_2/z_1 = z_3/z_2 = \ldots = z_{k+1}/z_k = q^2$, the correlation functions of $x^-(z_1)x^-(z_2)...x^-(z_k)x^-(z_{k+1})$ is zero if $z_1/z_2 = z_2/z_3 = \ldots = z_k/z_{k+1} = q^2$

However this condition can not be directly used for the semi-infinite construction, because the noncommutativity of the current operator $x^+z$ of $U_q(\mathfrak{sl}(2))$. However the theorem above implies:

**Corollary 12.** For any level $k \geq 1$ integrable module of $U_q(\mathfrak{sl}(2))$,

$$\bar{x}^+(z_1)\bar{x}^+(z_2)...\bar{x}^+(z_k)\bar{x}^+(z_{k+1}) = 0$$

if $z_2/z_1 = z_3/z_2 = \ldots = z_{k+1}/z_k = q^2$, and

**Proof.** Let $F(z_1,\ldots,z_n)$ be the correlation function of a vector $v$ in any level $k \geq 1$ integrable module of $U_q(\mathfrak{sl}(2))$ and $v^*$ in the dual space of this level $K$ module,

$$<v^*,\bar{x}^+(z_1)\bar{x}^+(z_2)...\bar{x}^+(z_k)\bar{x}^+(z_{k+1})v>.$$

Then we have that

$$<v^*,\bar{x}^+(z_1)\bar{x}^+(z_2)...\bar{x}^+(z_k)\bar{x}^+(z_{k+1})v> =<v^*,\prod_{i,j}\frac{(z_i - z_j)}{z_i - z_j}x^+(z_1)x^+(z_2)...x^+(z_k)x^+(z_{k+1})k^+(z_{k+1}v>.$$

Because $<v^*,x^+(z_1)x^+(z_2)...x^+(z_k)x^+(z_{k+1})v_1>$ for for a vector $v_1$ in any level $k \geq 1$ integrable module of $U_q(\mathfrak{sl}(2))$ and the function $\prod_{i,j}(\frac{z_i - z_j}{z_i - z_j})$ is not zero if $z_2/z_1 = z_3/z_2 = \ldots = z_{k+1}/z_k = q^2$, we have

$$F(z_1,...,z_n) = 0.$$

Because of the commutativity of the operator $\bar{x}^+(z)$, we have $\bar{x}^+(z)\bar{x}^+(z)$ is a well-defined operator, thus

$$\bar{x}^+(z)\bar{x}^+(z) = 0.$$

With the preparation above, in this section, we will describe a quantized semi-infinite construction along the line of [FS]. Their starting point for the case of $\mathfrak{sl}(2)$
is the integrability condition for level $k$ integrable modules, namely on any level $K$ module from the category of representations with highest weight is a sum of irreducible integrable representations, if and only if $e(z)^{k+1}$ is zero.

Similarly we can make the following claim:

**Theorem 13.** Any level $K$ module of $U_q(\hat{\mathfrak{sl}}(2))$ from the category of representations with highest weight is a sum of irreducible integrable representations, if and only if $\bar{x}^+(z)\bar{x}((q^2)^k)\ldots x^q(2^k)$ is zero.

**Proof.** The theorem above already gives the proof for half of the theorem. The other half comes from the fact that if we quotient the relation $q = 1$, the condition $\bar{x}^+(z)\bar{x}((q^2)^k)\ldots x^q(2^k)$ is zero simply degenerates into the condition that $e(z)^{k+1}$ is zero. Thus, it is integrable as a module of $\mathfrak{sl}(2)$. From the theory of Lusztig, we know that all the integrable highest weight module must comes from the corresponding quantized module. Thus the module is also an integrable module when $q$ is generic.

We will start our semi-infinite construction with the irreducible integrable module $V_{0,1}$ with the highest weight vector $\bar{v}_{0,1}$, such that the weight of the highest weight vector is 0 and the central element $c$ acts as 1.

Let $\bar{x}^+(z) = \Sigma \bar{x}^+_iz^{-i}$ and $U(\bar{x})$ be the subalgebra generated by $\bar{x}^+_i$. We denote $U(\bar{x})^-$ the subalgebra generated by $\bar{x}^+(n), n \geq 0$ and and $U(\bar{x})^+$ the subalgebra generated by $\bar{x}^+(n), n < 0$ Let $W = U(\bar{x})v_{0,1}$, because $U(\bar{x})^+v_{0,1} = 0$, we have that $W$ is equivalent to $U(\bar{x})^+/I)v_{0,1}$, where $I$ is an ideal.

**Lemma 14.** The ideal $I$ is generated by $S_k^1 = \Sigma \bar{x}_i\bar{x}_{k-i}(q^{2i} + q^{2k-2i})$, for $k < -1$.

**Proof.** From the quantum integrability condition above, we know that the elements $S_k^1$ for $k < -1$ are inside the ideal $I$. We will denote the ideal generated by those elements by $I'$. The proof follows from that fact that $U(\bar{x})^+/I)v_{0,1}$ has the same character as the case, when quotient the relation $q - 1 = 0$. Thus $I = I'$.

**Definition 15.** $V_{0,1}$ is an vector space with the basis of infinite monomials $M$ in the form of $x_{i_1}x_{i_2}\ldots x_{i_n}$, where $\{i_1, i_2, \ldots\}$ is an infinite sequence of indices such that, for some $n$, $i_n$ is odd and $i_{p+1} = i_p + 2$, if $p > n$. Let $V$ be a quotient space of $\bar{V}$, the quotient is given by the following relations:

1. $\bar{x}_i$ and $\bar{x}_j$ commutes, if $i \neq j$,
2. if an element $m \in \bar{V}$ contains a part $x_{u}x_{2N+1}x_{2N+3}x_{2N+5}\ldots$ and $u > 2N - 1$, then $m = 0$.
3. The operator $S_k = \Sigma_{a+b=k}x_ax_b(q^{2b} + q^{2a})$ acts on $\bar{V}$ and $S_kv = 0$ for $v \in \bar{V}$.

We define the action of $\bar{x}_i$ simply by multiplication. The action of $a_i$ for $i > 0$ is given by

$$a_i \bar{x}_1, \bar{x}_{i+2}, \ldots = [a_i, \bar{x}_1, \bar{x}_{i+2}, \ldots + \bar{x}_1[a_i, \bar{x}_{i+2}, \bar{x}_{i+4}, \ldots +$$
This is an finite expression. We define the action of $a_0$ by that:

$$a_0(x_{2N+1}x_{2N+3}x_{2N+5}....) = -2N x_{2N+1}x_{2N+3}x_{2N+5}....$$

The action of $a_{-1}$ is defined as:

$$a_{-1}x_1\bar{x}_3\bar{x}_5.... = \left(x_0\bar{x}_3\bar{x}_5.... + \bar{x}_1\bar{x}_2\bar{x}_5.... + \bar{x}_1\bar{x}_3\bar{x}_4\bar{x}_7\bar{x}_9.... + \bar{x}_1\bar{x}_3\bar{x}_5\bar{x}_6\bar{x}_9.... + .... \right) =$$

$$\left(x_0\bar{x}_3\bar{x}_5.... - \frac{(q^6 + 1)}{(q^4 + q^2)}\bar{x}_0\bar{x}_3\bar{x}_5.... + \frac{(q^6 + 1)}{(q^4 + q^2)}\frac{(q^{10} + q^4)}{(q^6 + q^6)}x_0\bar{x}_3\bar{x}_5.... = \right)$$

$$(\bar{x}_0\bar{x}_3\bar{x}_5....)1/(1 + \frac{(q^6 + 1)}{(q^4 + q^2)}).$$

Thus it converges if $|\frac{(q^6 + 1)}{(q^4 + q^2)}| < 1$.

We would like to define the action of $a_{-2}$ as the following:

$$a_{-2}x_1\bar{x}_3\bar{x}_5.... =$$

$$(q^2 - q^{-2})(\bar{x}_{-1}\bar{x}_3\bar{x}_5.... + \bar{x}_1\bar{x}_1\bar{x}_5.... + \bar{x}_1\bar{x}_3\bar{x}_3\bar{x}_7\bar{x}_9.... + \bar{x}_1\bar{x}_3\bar{x}_5\bar{x}_6\bar{x}_9.... + ....) =$$

$$\left(q^2 - q^{-2}\right)(\bar{x}_{-1}\bar{x}_3\bar{x}_5.... - \frac{(q^4 + 1)}{2q}\bar{x}_0\bar{x}_2\bar{x}_5.... + \frac{(q^6 + q^{-2})}{2q}\bar{x}_{-1}\bar{x}_3\bar{x}_5.... - \right)$$

$$\left(\frac{(q^{10} + q^2)}{(q^6 + q^6)}\bar{x}_1\bar{x}_4\bar{x}_7.... + \frac{(q^8 + q^4)}{(q^6 + q^6)}\bar{x}_1\bar{x}_2\bar{x}_4\bar{x}_7.... \right)$$

To use the relation (1) (2) (3) to reduce this expression to prove the convergence of the expression is very complicated. Similar problems appears for the defining the action of $a_{-n}, n < -2$.

Thus we will use the same trick played in [FS]. Let $U(\bar{x}, r)$ be the subalgebra generated by $\bar{x}_i$. Let $\bar{V}_{0,1}(r)$ be the subspace $\bar{V}_{0,1}$, which consists of the element $\bar{x}_i....\bar{x}_n....$ and $i_j > r$.

**Lemma 16.** $\bar{V}_{0,1}(r)$ spans the whole space $\bar{V}_{0,1}$.

**Proof** The proof is the same as lemma 2.5.1 in [FS]. The way to prove it is to use the relation (3) to express any element in $\bar{V}_{0,1}$ with linear expression of elements in $\bar{V}_{0,1}$.

For any element expressed in a linear combination of elements in $\bar{V}_{0,1}(r)$, we define the action of $x^-(k)$, for $k + r > 0$, as that of $a_{-1}$ by using the commutation relations between $\bar{x}^+(z)$ and $\bar{x}^- (z)$. Because $k + r > 0$, we know that it is well defined. As in [FS], this is a well defined action, namely if we express an element in two different ways in $\bar{V}_{0,1}(r)$, the actions of $x^-(k)$ defined above coincide. Again, with the commutation relation between $\bar{x}^+(z)$ and $\bar{x}^-(z)$, we can define the action of $a_n, n < -1$, because $\bar{x}^+(z)$ and $\bar{x}^-(z)$ generate the whole algebra. Thus we have
**Theorem 17.** There exists an action of $U_q(\mathfrak{sl}(2))$ on the space $\bar{V}_{0,1}$, such that $\bar{V}_{0,1}$ is equivalent to $V_{0,1}$ as a representation of $U_q(\mathfrak{sl}(2))$ and the action of $\bar{x}_i$ acts by multiplication.

Let $\bar{W}$ be the set of the elements $\bar{x}_{i_1},...,\bar{x}_{i_n}$ in $\bar{V}_{0,1}$, such that $i_{j+1} - i_j > 1$.

**Proposition 18.** $\bar{W}$ forms a linear independent basis of the space $\bar{V}_{0,1}$.

The proof is the same as in [FS], which gives the character of the representation.

Similarly as in [FS], a functional model for the description of $W^*$, the dual space of $W$, can be derived from the lemma above.

As a commutative algebra, $U(\bar{x})$ can be identified with the space $C[t, t^{-1}]$. Let $U(\bar{x})^+ = \Sigma U(\bar{x})^+(n)$, where $U(\bar{x})^+(n)$ consists of the elements $\bar{x}_{i_1}^+\bar{x}_{i_2}^+...\bar{x}_{i_n}^+$. We identify any element $\bar{x}_{i_1}^+\bar{x}_{i_2}^+...\bar{x}_{i_n}^+$ in $U(\bar{x})^+(n)$ as $t_{i_1}^1...t_{i_n}^n$, where $t_i$ are variables. Similarly we can any element $\bar{x}_{i_1}^+\bar{x}_{i_2}^+...\bar{x}_{i_n}^+$ in $W$ as $t_{i_1}^1...t_{i_n}^m$. Let $S^n(\Omega^1 \mathbb{C})$ be the space of expressions $f(t_1,...,t_n)dt_1...dt_n$, such that $f(t_1,...,t_n)$ is a symmetric functions and different $dt_i$ commute. $S^n(\Omega^1 \mathbb{C})$ is also called the space of $n$ particles. We can pair $S^n(\Omega^1 \mathbb{C})$ with $U(\bar{x})^+(n)$ by the following:

\[
< f(t_1,...,t_n)dt_1...dt_n, t_{i_1}^1...t_{i_n}^n >= \text{RES}_{t_1=...=t_n=0}(< f(t_1,...,t_n)t_{i_1}^1...t_{i_n}^n dt_1...dt_n).
\]

Thus $W^* = \Sigma \oplus W^* \cap S^n(\Omega^1 \mathbb{C})$.

**Theorem 19.**

\[
W^* \cap S^n(\Omega^1 \mathbb{C}) = \{ f(t_1,...,t_n)dt_1...dt_n : f = 0, \text{ if } t_1t_2q^2 \}.
\]

Similarly we can present the semi-infinite constructions for the higher level cases.

Let $V_{i,k}$ be the irreducible highest weight representation of $U_q(\mathfrak{sl}(2))$ with the action of $c$ as $k$ and the highest weight is $l$ times the fundamental weight and $v_{0,1}$ be its highest weight vector. Let $W_{i,k} = U(\bar{x})v_{i,k}$, because $U(\bar{x})^+v_{i,k} = 0$, we have that $W_{i,k}$ is equivalent to $U(\bar{x})^+/I_{i,k})v_{i,k}$, where $I_{i,k}$ is an ideal.

**Lemma 20.** The ideal $I_{i,k}$ is generated by

\[
S_i^{k+1} = \Sigma a_i = -ix_{a_1}x_{a_2}...x_{a_{k+1}}(\Sigma \sigma \in S_{k+1}(q^{\Sigma_i=2,k+1,2i-1a_{k+1}})).
\]

$i < -k$ and $x_{-1}^{k+1}$.

**Definition 21.** Let $\bar{V}_{i,k}$ be a space spaned by the elements of the expressions in the form of $\bar{x}_{i_1}...\bar{x}_{i_n}\bar{x}_{2N+1}\bar{x}_{2N+2}^{k-l}$, such that

1. $\bar{x}_i$ commutes with $\bar{x}_j$,
2. if an element $m \in \bar{V}$ contains a part $\bar{x}_i\bar{x}_j\bar{x}_{2N+1}\bar{x}_{2N+2}^{k-l}$, $i > 2N - 1$ or $\bar{x}_i\bar{x}_{2N+1}\bar{x}_{2N+2}^{k-l}$, $i > 2N$, then $m = 0$.
3. The operator $S_k = \Sigma a_i = x_{a_1}x_{a_2}...x_{a_{k+1}}(\Sigma \sigma \in S_{k+1}(q^{\Sigma_i=2,k+1,2i-1a_{k+1}}))$ acts on $\bar{V}$ and $S_kv = 0$ for $v \in \bar{V}$, where $S_{k+1}$ is the permutation group on $k+1$ numbers.
Theorem 22. On the space $\tilde{V}_{l,k}$, there is an action of $U_q(\widehat{sl}(2))$, such that the action of $\bar{x}^+(n)$ is given by comultiplication. This representation is the irreducible highest weight representation of $U_q(\widehat{sl}(2))$ with the action of $c$ as $k$ and the highest weight is $l$ times the fundamental weight.

Let $\tilde{W}(l,k)$ be the set of the elements $\bar{x}_{i_1}, \ldots, \bar{x}_{i_n}$ in $\tilde{V}_{l,k}$, such that $i_{j+k} - i_j > 1$.

Proposition 23. $\tilde{W}(l,k)$ forms an linear independent basis of the space $\tilde{V}_{l,k}$.

As in the case of $V_{0,1}$, let $U(\bar{x}^+) = \Sigma \oplus U(\bar{x}^+)(n)$, where $U(\bar{x}^+)(n)$ consists of the elements $\bar{x}_{i_1}^+ \bar{x}_{i_2}^+ \ldots \bar{x}_{i_n}^+$. We identify any element $\bar{x}_{i_1}^+ \bar{x}_{i_2}^+ \ldots \bar{x}_{i_n}^+$ in $U(\bar{x}^+)(n)$ as $t_1^{i_1} \ldots t_n^{i_n}$, where $t_i$ are variables. Similarly we can any element $\bar{x}_{i_1}^+ \bar{x}_{i_2}^+ \ldots \bar{x}_{i_n}^+$ in $W_{l,k}$ as $t_1^{i_1} \ldots t_n^{i_n}$.

Let $\mathcal{S}(\Omega^1 \mathbb{C})$ be the space of expressions $f(t_1, \ldots, t_n)dt_1 \ldots dt_n$, such that $f(t_1, \ldots, t_n)$ is a symmetric functions and different $dt_i$ commute. $\mathcal{S}^n(\Omega^1 \mathbb{C})$ is also called the space of $n$ particles. We can pair $\mathcal{S}^n(\Omega^1 \mathbb{C})$ with $U(\bar{x}^+)(n)$ by the following:

$$< f(t_1, \ldots, t_n)dt_1 \ldots dt_n, t_1^{i_1} \ldots t_n^{i_n} > = \text{RES}_{t_1=\ldots=t_n=0}(< f(t_1, \ldots, t_n)t_1^{i_1} \ldots t_n^{i_n}dt_1 \ldots dt_n>.)$$

Thus $W^* = \Sigma \oplus W^* \cap \mathcal{S}^n(\Omega^1 \mathbb{C})$.

Theorem 24.

$$W^* \cap \mathcal{S}^n(\Omega^1 \mathbb{C}) = \{ f(t_1, \ldots, t_n)dt_1 \ldots dt_n : f = 0, \text{if } t_1 = t_2q^2 \ldots = t_{k+1}q^{2(k)}$$

$$\text{or } t_1 = \ldots t_{k-l+1} = 0 \}. $$

In Section 2, we define the operator $\bar{x}_i^+(z)$, it is clear that this can be also applied to other cases. Our next step is to extend the semi-infinite to the cases of $U_q(\hat{g})$, where $g$ is a simple-laced simple Lie algebra, for which we need to define the proper $x_i^+(z)$ associated to the roots of $g$. The simplest case $g = \widehat{sl}(3)$ will be the subject of a subsequent paper. On the other hand, this paper follows the algebraic theory developed in [FS] [FS]. The semi-infinite constructions can be geometrically understood according to the structure of the corresponding infinite dimensional flag manifold and the infinite Shubert cells. The geometric interpretation of the quantized semi-infinite construction is still an open problem. This is related to another immediate problem to extend the explicit construction of modular functor [FS] to the quantized case, which should leads us toward certain quantization of conformal field theory. One more possible application of such a construction is to generaliza such a construction to more general cases. From the point view of the functional realization of the dual space, one natural generalization is to substitute the condition $x_1 = x_2q^2$, which is generalization of the classical condition $x_1 = x_2$, with more general conditions, for example $x_1 = x_2q_1 = x_3q_3$. One would like ask a questions that what kind of structures those generalized spaces possibly represent? We believe it is related with the recent work about generalization of the quantum affine algebras [DI], where those kind of new conditions should be satisfied for the quantum current operators. We hope our
construction can help us to understand the structures of those new algebras in [DI],
for which we have not been able to give any concrete realization of the non-trivial
integrable representations yet.

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Jintai Ding, RIMS, Kyoto University

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