Eigenvalues of Laplacian with constant magnetic field on noncompact hyperbolic surfaces with finite area

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Abstract

We consider a magnetic Laplacian $-\Delta_A = (id + A)^*(id + A)$ on a noncompact hyperbolic surface $\mathcal{M}$ with finite area. $A$ is a real one-form and the magnetic field $dA$ is constant in each cusp. When the harmonic component of $A$ satisfies some quantified condition, the spectrum of $-\Delta_A$ is discrete. In this case we prove that the counting function of the eigenvalues of $-\Delta_A$ satisfies the classical Weyl formula, even when $dA = 0$.\textsuperscript{1}

1 Introduction

We consider a smooth, connected, complete and oriented Riemannian surface $(\mathcal{M}, g)$ and a smooth, real one-form $A$ on $\mathcal{M}$. We define the magnetic Laplacian, the Bochner Laplacian

$$-\Delta_A = (id + A)^*(id + A),$$

(1.1)

$$((id + A)u = i du + uA, \forall u \in C_0^\infty(\mathcal{M}; \mathbb{C}).$$

The magnetic field is the exact two-form $\rho_B = dA$.

If $dm$ is the Riemannian measure on $\mathcal{M}$, then

$$\rho_B = \tilde{b} dm, \quad \text{with} \quad \tilde{b} \in C^\infty(\mathcal{M}; \mathbb{R}).$$

(1.2)

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The magnetic intensity is \( b = |\tilde{b}| \).

It is well known, (see [Shu] ), that \( -\Delta A \) has a unique self-adjoint extension on \( L^2(M) \), containing in its domain \( C_0^\infty(M; \mathbb{C}) \), the space of smooth and compactly supported functions. The spectrum of \( -\Delta A \) is gauge invariant : for any \( f \in C^1(M; \mathbb{R}) \), \( -\Delta A \) and \( -\Delta A + df \) are unitarily equivalent, hence they have the same spectrum.

We are interested in constant magnetic fields on \( M \) in the case when \( (M, g) \) is a non-compact geometrically finite hyperbolic surface of finite area; (see [Per] or [Bor] for the definition and the related references). More precisely

\[
M = \bigcup_{j=0}^{J} M_j
\]

(1.3)

where the \( M_j \) are open sets of \( M \), such that the closure of \( M_0 \) is compact, and (when \( J \geq 1 \)) the other \( M_j \) are cuspidal ends of \( M \).

This means that, for any \( j, 1 \leq j \leq J \), there exist strictly positive constants \( a_j \) and \( L_j \) such that \( M_j \) is isometric to \( S \times ]a_j^2, +\infty[ \), equipped with the metric

\[
ds_j^2 = y^{-2} (L_j^2 d\theta^2 + dy^2);
\]

(1.4)

\( S = \mathbb{S}^1 \) is the unit circle and \( M_j \cap M_k = \emptyset \) if \( j \neq k \).

Let us choose some \( z_0 \in M_0 \) and let us define

\[
d : M \to \mathbb{R}^+ ; \quad d(z) = d_g(z, z_0);
\]

(1.5)

\( d_g(., .) \) denotes the distance with respect to the metric \( g \).

For any \( b \in \mathbb{R}^J \), there exists a one-form \( A \), such that the corresponding magnetic field \( dA \) satisfies

\[
dA = \tilde{b}(z)dm \quad \text{with} \quad \tilde{b}(z) = b_j \forall \, z \in M_j.
\]

(1.6)

The following statement on the essential spectrum is proven in [Mo-Tr1] :

**Theorem 1.1** Assume (1.3) and (1.6). Then for any \( j, 1 \leq j \leq J \) and for any \( z \in M_j \) there exists a unique closed curve through \( z \), \( C_{j,z} \) in \( (M_j, g) \), not contractible and with zero \( g \)-curvature. (\( C_{j,z} \) is called an horocycle of \( M_j \)). The following limit exists and is finite:

\[
[A]_{M_j} = \lim_{d(z) \to +\infty} \int_{C_{j,z}} A.
\]

(1.7)
If \( J^A = \{ j \in \mathbb{N} : 1 \leq j \leq J \text{ s.t. } [A]_{M_j} \in 2\pi \mathbb{Z} \} \neq \emptyset \), then

\[
\text{sp}_{\text{ess}}(-\Delta_A) = \left[ \frac{1}{4} + \min_{j \in J^A} b_j^2, +\infty \right].
\] (1.8)

If \( J^A = \emptyset \), then \( \text{sp}_{\text{ess}}(-\Delta_A) = \emptyset ; \)

\(-\Delta_A\) has purely discrete spectrum, (its resolvent is compact).

When the magnetic Laplacian \(-\Delta_A\) has purely discrete spectrum, it is called a magnetic bottle, (see [Col2]).

If \( A = df + A^H + A^\delta \) is the Hodge decomposition of \( A \) with \( A^H \) harmonic, \((dA^H = 0 \text{ and } d^*A^H = 0)\), then \( \forall j, [A]_{M_j} = [A^H]_{M_j} \), so the discreteness of the spectrum of \(-\Delta_A\) depends only on the harmonic component of \( A \). So one can see the case \( J^A = \emptyset \) as an Aharonov-Bohm phenomenon [Ah-Bo], a situation where the magnetic field \( dA \) is not sufficient to describe \(-\Delta_A\) and the use of the magnetic potential \( A \) is essential: we can have magnetic bottle with null intensity.

### 2 The Weyl formula in the case of finite area with a non-integer class one-form

Here we are interested in the pure point part of the spectrum. We assume that \( J^A = \emptyset \), then the spectrum of \(-\Delta_A\) is discrete. In this case, we denote by \((\lambda_j)_j\) the increasing sequence of eigenvalues of \(-\Delta_A\), (each eigenvalue is repeated according to its multiplicity). Let

\[
N(\lambda, -\Delta_A) = \sum_{\lambda_j < \lambda} 1.
\] (2.1)

We will show that the asymptotic behavior of \( N(\lambda) \) is given by the Weyl formula:

**Theorem 2.1** Consider a geometrically finite hyperbolic surface \((M, g)\) of finite area, and assume (1.6) with \( J^A = \emptyset \), (see (1.7 for the definition). Then

\[
N(\lambda, -\Delta_A) = \frac{\lambda |M|}{4\pi} + O(\sqrt{\lambda} \ln \lambda).
\] (2.2)
Remark 2.2 As $J^A$ depends only on the harmonic component of $A$, $J^A$ is not empty when $M$ is simply connected. In [Go-Mo] there are some results close to Theorem 2.1, but for simply connected manifolds. The cases where the magnetic field prevails were studied in [Mo-Tr1] and in [Mo-Tr2].

Proof of Theorem 2.1. Any constant depending only on the $b_j$ and on $\min_{1 \leq j \leq J} \inf_{k \in \mathbb{Z}} |[A]_{M_j} - 2k\pi|$ will be denoted invariably $C$.

Consider a cusp $M = M_j = S \times ]\alpha^2, +\infty[\ equipped with the metric $ds^2 = L^2 e^{-2t} d\theta^2 + dt^2$ for some $\alpha > 0$ and $L > 0$.

Let us denote by $-\Delta^M_A$ the Dirichlet operator on $M$, associated to $-\Delta_A$.

The first step will be to prove that

$$N(\lambda, -\Delta^M_A) = \lambda \frac{|M|}{4\pi} + O(\sqrt{\lambda} \ln \lambda).$$

(2.3)

Since $-\Delta^M_A$ and $-\Delta^M_{A+\phi + k\theta}$ are gauge equivalent for any $\phi \in C^\infty(M; \mathbb{R})$ and any $k \in \mathbb{Z}$, we can assume that

$$-\Delta^M_A = L^{-2} e^{2t}(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}, \text{ with } A_1 = -\xi \pm bL e^{-t}, \xi \in ]0, 1[, (b = b_j, 2\pi \xi - [A]_M \in 2\pi \mathbb{Z}).$$

Then we get that

$$\text{sp}(-\Delta^M_A) = \bigcup_{\ell \in \mathbb{Z}} \text{sp}(P_\ell) ; \ P_\ell = D_t^2 + \frac{1}{4} + \left( e^{\ell(\xi/2)} e^{-t} \right)^2,$$

for the Dirichlet condition on $L^2(I; dt) ; \ I = ]\alpha^2, +\infty[\.$ This implies that

$$N(\lambda, -\Delta^M_A) = \sum_{\ell \in X_{\lambda}} N(\lambda, P_\ell) = \sum_{\ell \in X_{\lambda}} N(\lambda, P_\ell)$$

(2.4)

with $X_{\lambda} = \{ \ell / e^{a^2 \ell(\xi/2) / L} < \sqrt{\lambda - 1/4} - b \}.$

Denoting by $Q_\ell$ the Dirichlet operator on $I$ associated to

$$Q_\ell = D_t^2 + \frac{1}{4} + \frac{(\ell + \xi)^2}{L^2} e^{2t},$$

we easily get that

$$Q_\ell - C \sqrt{Q_\ell} \leq P_\ell \leq Q_\ell + C \sqrt{Q_\ell}.$$

(2.5)
Therefore one can find a constant $C(b)$, depending only on $b$, such that, for any $\lambda >> 1 + C(b)$,

$$N(\lambda - \sqrt{\lambda} C(b), Q_\ell) \leq N(\lambda, P_\ell) \leq N(\lambda + \sqrt{\lambda} C(b), Q_\ell). \quad (2.6)$$

Following Titchmarsh’s method ([Tit], Theorem 7.4) we establish the following bounds

**Lemma 2.3** There exists $C > 1$ so that for any $\mu >> 1$ and any $\ell \in X_\mu$,

$$w_\ell(\mu) - \pi \leq \pi N(\mu - \frac{1}{4}, Q_\ell) \leq w_\ell(\mu) + \frac{1}{12} \ln \mu + C, \quad (2.7)$$

with

$$w_\ell(\mu) = \int_{\alpha^2}^{+\infty} \left[ \mu - \frac{(\ell + \xi)^2}{L^2} e^{2t} \right]^{1/2} \frac{1}{\sqrt{2}} dt$$

$$= \int_{T_{\mu,\ell}}^{T_{\mu,\ell}} \left[ \mu - \frac{(\ell + \xi)^2}{L^2} e^{2t} \right]^{1/2} \frac{1}{\sqrt{2}} dt;$$

$$(e^{T_{\mu,\ell}} = L \sqrt{\mu} / (\inf_{k \in \mathbb{Z}} |\xi - k|)) .$$

**Proof of Lemma 2.3**

The lower bound is easily obtained (see [Tit], Formula 7.1.2 p 143) so we focus on the upper bound.

Let us define $V_\ell = \frac{(\ell + \xi)^2}{L^2} e^{2t}$ and denote by $\phi_\mu^\ell$ a solution of $Q_\ell \phi = (\mu - \frac{1}{4}) \phi$. Consider $x_\ell$ and $y_\ell$ so that $V_\ell(x_\ell) = \mu$ and $V_\ell(y_\ell) = \nu$, for a given $0 < \nu < \mu$ to be determined later. We denote by $m$ the number of zeros of $\phi_\mu^\ell$ on $]\alpha^2, y_\ell[.$

Recall that the number $n$ of zeros of $\phi_\mu^\ell$ on $]\alpha^2, x_\ell[.$ is equal to $N(\mu - \frac{1}{4}, Q_\ell)$. Applying Lemma 7.3 p 146 in [Tit] we deduce that

$$m\pi = \int_{\alpha^2}^{y_\ell} \left[ \mu - V_\ell \right]^{1/2} dt + R_\ell$$

with $R_\ell = \frac{1}{4} \ln(\mu - V_\ell(\alpha^2)) - \frac{1}{4} \ln(\mu - V_\ell(y_\ell)) + \pi$, hence

$$|n\pi - \int_{\alpha^2}^{x_\ell} \left[ \mu - V_\ell \right]^{1/2} dt| \leq (x_\ell - y_\ell)(\mu - \nu)^{1/2} + R_\ell + (n - m)\pi$$

According to the Sturm comparison theorem ([Tit], p 107-108), we have

$$(n - m)\pi \leq (x_\ell - y_\ell)(\mu - \nu)^{1/2}$$

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and
\[ |n\pi - \int_{\alpha^2}^{2\ell} [\mu - V\ell]^{1/2} dt| \leq \ln(\mu/\nu)(\mu - \nu)^{1/2} + \frac{1}{4} \ln \mu - \frac{1}{4} \ln(\mu - \nu) + 2\pi \]

Now taking \( \nu = \mu - \mu^{2/3} \) we get the desired estimate.

In view of (2.4) we now compute \( \sum_{\ell \in \mathbb{Z}} w_\ell(\mu) \). We first get the following

**Lemma 2.4** There exists \( C > 1 \) such that, for any \( \mu >> 1 \) and any \( t \in [\alpha^2, T_{\mu,L}] \),
\[
\left| \int_{\mathbb{R}} \left[ \mu - \frac{(x + \xi)^2}{L^2} e^{2t} \right]^{1/2} dx - \sum_{\ell \in \mathbb{Z}} \left[ \mu - \frac{(\ell + \xi)^2}{L^2} e^{2t} \right]^{1/2} \right| \leq C(\sqrt{\mu} + e^t/L) .
\]

This leads to

**Lemma 2.5** There exists \( C > 1 \) such that, for any \( \mu >> 1 \),
\[
\left| \int_{\alpha^2}^{T_{\mu,L}} \int_{\mathbb{R}} \left[ \mu - \frac{(x + \xi)^2}{L^2} e^{2t} \right]^{1/2} dxdt - \sum_{\ell \in \mathbb{Z}} w_\ell(\mu) \right| \leq C\sqrt{\mu} \ln \mu .
\]

We now compute the integral in the left-hand side.
Making the change of variables \( y^2 = \frac{(x + \xi)^2}{L^2} e^{2t} \) we obtain that it is equal to
\[
\mu L \int_{\alpha^2}^{T_{\mu,L}} e^{-t} dt \int_{\mathbb{R}} \left[ 1 - x^2 \right]^{1/2} dx,
\]
so we get

**Lemma 2.6** There exists \( C > 1 \) such that, for any \( \mu >> 1 \),
\[
\left| \int_{\alpha^2}^{T_{\mu,L}} \int_{\mathbb{R}} \left[ \mu - \frac{(x + \xi)^2}{L^2} e^{2t} \right]^{1/2} dxdt - \mu L e^{-a^2} \int_{\mathbb{R}} \left[ 1 - x^2 \right]^{1/2} dx \right| \leq C\sqrt{\mu} .
\]

Noticing that \( |M| = 2\pi Le^{-a^2} \) and using Lemmas 2.5 and 2.6 we have

**Lemma 2.7**
\[
\frac{1}{\pi} \sum_{\ell} w_\ell(\mu) = \frac{|M|}{4\pi} \mu + O(\sqrt{\mu} \ln \mu) , \quad \text{as} \quad \mu \to +\infty .
\]
In view of (2.4), (2.6) and (2.7) Lemma 2.7 ends the proof of formula (2.3). Now it remains to consider the whole surface $M$.

We have: $M = \bigcup_{j=0}^{J} M_j$

where the $M_j$ are open sets of $M$, such that the closure of $M_0$ is compact, and the other $M_j$ are cuspidal ends of $M$ and $M_j \cap M_k = \emptyset$, if $j \neq k$. We denote $M_0^0 = M \setminus (\bigcup_{j=1}^{J} M_j)$, then

$$M = M_0^0 \cup \left( \bigcup_{j=1}^{J} M_j \right). \quad (2.9)$$

Let us denote respectively by $-\Delta_{A,D}^0$ and by $-\Delta_{A,N}^0$ the Dirichlet operator and the Neumann-like operator on an open set $\Omega$ of $M$ associated to $-\Delta_A$. The minimax principle and (2.9) imply that

$$N(\lambda, -\Delta_{A,D}^{M_j^0}) + \sum_{1 \leq j \leq J} N(\lambda, -\Delta_{A,D}^{M_j^0}) \leq N(\lambda, -\Delta_A) \quad (2.10)$$

$$\leq N(\lambda, -\Delta_{A,N}^{M_j^0}) + \sum_{1 \leq j \leq J} N(\lambda, -\Delta_{A,N}^{M_j^0}).$$

The Weyl formula with remainder, (see [Hor] for Dirichlet boundary condition and [Sa-Va] p. 9 for Neumann-like boundary condition), gives that

$$\left\{ \begin{array}{l} N(\lambda, -\Delta_{A,D}^{M_j^0}) = (4\pi)^{-1}|M_0^0|\lambda + O(\sqrt{\lambda}) \\ N(\lambda, -\Delta_{A,N}^{M_j^0}) = (4\pi)^{-1}|M_0^0|\lambda + O(\sqrt{\lambda}) \end{array} \right\}. \quad (2.11)$$

The asymptotic formula for $N(\lambda, -\Delta_{A,N}^{M_j})$,

$$N(\lambda, -\Delta_{A,N}^{M_j}) = \lambda \frac{|M_j|}{4\pi} + O(\sqrt{\lambda} \ln \lambda), \quad (2.12)$$

is obtained as for the Dirichlet case (2.3) (with $M = M_j$), by noticing that $N(\lambda, P_{\ell,D}) \leq N(\lambda, P_{\ell,N}) \leq N(\lambda, P_{\ell,D}) + 1$, where $P_{\ell,D}$ and $P_{\ell,N}$ are Dirichlet and Neumann operators on a half-line $I = ]a^2, +\infty[$, associated to the same differential Schrödinger operator $P_{\ell} = D_{\ell}^2 + \frac{1}{4} + (\epsilon(\frac{\ell + \xi}{L}) \pm b)^2$.

We get (2.2) from (2.3) with $M = M_j$, (2.12), (for any $j = 1, \ldots, J$), (2.10) and (2.11). □
Remark 2.8 Theorem 2.1 still holds if the metric of $M$ is modified in a compact set.

When $A = 0$, $-\Delta = -\Delta_0$ has embedded eigenvalues in its essential spectrum, $(\text{sp ess}(-\Delta) = \left[\frac{1}{4}, +\infty\right])$. If $N_{\text{ess}}(\lambda, -\Delta)$ denotes the number of these eigenvalues in $\left[\frac{1}{4}, \lambda\right]$, then it is well known that one has an upper bound $N_{\text{ess}}(\lambda, -\Delta) \leq \frac{|M|}{4\pi}$; see [Col1] and [Hej] for the history and related improvement of the upper bound.

Recently [Mul] established a sharp asymptotic formula, similar to our case,

$$N_{\text{ess}}(\lambda, -\Delta) = \frac{\lambda |M|}{4\pi} + O(\sqrt{\lambda} \ln \lambda),$$

for some particular $M$.

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