Differential Calculi on h-deformed Bosonic and Fermionic Quantum Planes

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Abstract

We study differential calculus on h-deformed bosonic and fermionic quantum space. It is shown that the fermionic quantum space involves a parafermionic variable as well as a classical fermionic one. Further we construct the classical $su(2)$ algebra on the fermionic quantum space and discuss a mapping between the classical $su(2)$ and the h-deformed $su(2)$ algebras.
1. Introduction

Quantum groups and quantum algebras [1-4] have attracted much attention in theoretical physics and mathematics, such as statistical models, integrable models, conformal field theories, knot theory and so on [5-8]. Quantum space was introduced to represent quantum groups [9]. Differential calculus on the quantum space was also studied [10-15]. It is intriguing also from the viewpoint of non-commutative geometry. The quantum differential calculus is closely related with q-oscillators, which have been applied to various fields [16-18]. Further, we can obtain quantum groups $Sp_{q}(2n)$ and $SO_{q}(2n)$ using the q-deformed phase space which is defined through the differential calculus on the quantum space for $SL_{q}(n)$ [19]. Furthermore, on the quantum space quantum deformed algebras have been constructed, e.g., q-deformed Lorentz, Poincaré, conformal and superconformal algebras [20-25]. These analyses imply that the quantum space is interesting as applications of quantum groups and useful to show some aspects of quantum groups.

Recently other than the conventional q-deformation and its multi-parametric extension, another deformation was discovered in [26-29] and is called h-deformation. Further, differential calculus on an h-deformed quantum bosonic space has been considered in [30, 31]. The purpose of this article is to investigate in detail bosonic and fermionic differential calculus on h-deformed quantum planes, extending the above analyses on the q-deformed differential calculus to the h-deformed case. The differential calculus on the bosonic quantum space is studied and a comment is obtained from the viewpoint of a constrained system. In addition, we discuss a differential calculus on an h-deformed fermionic space. Ref. [25] shows that q-deformed fermionic coordinates and derivatives represent $su_{q}(2)$ algebra, which is related with Drinfeld-Jimbo basis by some mapping. Following the approach, we investigate algebra which is constructed in terms of the h-deformed fermionic elements.

This paper is organized as follows. In section two we review on general non-commutative differential calculus including the quantum space. Then using a new solution of the Yang-Baxter equation, we derive h-deformed bosonic differential calculus. Further some comments on the h-deformed space are given. In section three, similarly we derive h-deformed differential calculus on the fermionic quantum space. It is shown that we can represent the classical Lie algebra $su(2)$, using their fermionic variables and derivatives. Also we obtain a mapping between the classical algebra
2. Differential calculus on h-deformed bosonic space

A quantum space is a non-commutative space representing the corresponding quantum group. Ref. [11] clarified general non-commutative differential calculus including the quantum space and its analysis was extended to superspaces in [13]. We set up commutation relations between coordinates $x^i$ and derivatives $\partial_{x^i}$ as follows,

$$x^i x^j = B^{ij}_{k\ell} x^k x^\ell, \quad \partial_{x^k} x^i = \delta^i_k + C^{ij}_{k\ell} x^\ell \partial_{x^j}. \quad (2.1)$$

These matrices should satisfy the following relations:

$$B^{ij}_{pr} B^{pq}_{tn} B^{qt}_{lm} = B^{jk}_{pr} B^{ip}_{lt} B^{tr}_{mn}, \quad (\delta^i_k \delta^j_l - B^{ij}_{k\ell}) (\delta^k_m \delta^\ell_n + C^{k\ell}_{mn}) = 0. \quad (2.2)$$

The former equation is called Yang-Baxter equation. We consider the case where $B^{ij}_{k\ell}$ is proportional to $C^{ij}_{k\ell}$. In this case we can write commutation relations of derivatives by $B$-matrix as follows,

$$\partial_{x^k} \partial_{x^i} = B^{ij}_{k\ell} \partial_{x^j} \partial_{x^i}. \quad (2.3)$$

Recently a new solution of the Yang-Baxter equation was discovered in [27] as follows,

$$\hat{R} = \begin{pmatrix}
1 & -h' & h & hh' \\
0 & 0 & 1 & h \\
0 & 1 & 0 & -h \\
0 & 0 & 0 & 1
\end{pmatrix}. \quad (2.4)$$

Also in Ref. [26] and [28], the same type of the $\hat{R}$-matrix was discovered in the case where $h = h'$ and $h = h' = -1$, respectively. New type of a quantum group $SU_h(2)$ was studied in [26-28] and the corresponding deformed algebra $su_h(2)$ has been constructed in [29]. We identify $\hat{R}$ with $B$ in order to obtain h-deformed differential calculus. The $\hat{R}$-matrix has $\pm 1$ as eigenvalues. This fact leads to a condition that we should identify $C$ with $\hat{R}$, too. Namely we hereafter study the case where

$$B^{ij}_{k\ell} = C^{ij}_{k\ell} = \hat{R}^{ij}_{k\ell}. \quad (2.5)$$
We restrict ourselves to the case where \( h = h' \). It is easy to extend the following analysis to the case with two independent parameters, \( h \) and \( h' \). We substitute eq.(2.4) into (2.1) and (2.3), so that we obtain the following commutation relations;

\[
[x^1, x^2] = h(x^2)^2, \quad [\partial_x x^1, \partial_x x^2] = h(\partial_x x^1)^2,
\]

\[
[\partial_x x^2, x^1] = h x^1 \partial_x x^2 + h x^2 \partial_x x^2 + h^2 x^2 \partial_x x^1, \quad (2.6)
\]

\[
[x^i \partial_x x^i, x^i] = 1 - h x^2 \partial_x x^1, \quad [\partial_x x^1, x^2] = 0.
\]

The above algebra becomes classical in the limit, i.e., \( h \to 0 \). We can introduce \( h \)-deformed quantum group \( T^i_j \) such that the elements transform the coordinates and the derivatives like \( T^i_j x^j \). Covariance of the commutation relations (2.6) under the transformation requires the following commutation relation of \( T^i_j \):

\[
\hat{R}^{ij}_{\; kl}T^k_m T^l_n = T^i_k T^j_l \hat{R}^{kl}_{\; mn}. \quad (2.7)
\]

Next we consider conjugation consistent with (2.6). Suppose that \( h \) is pure imaginary, i.e., \( \bar{h} = -h \). Then we obtain the consistent conjugation as follows,

\[
\overline{x^1} = x^1 + h x^2, \quad \overline{x^2} = x^2,
\]

\[
\overline{\partial_x x^1} = -\partial_x x^1, \quad \overline{\partial_x x^2} = -\partial_x x^2 - h \partial_x x^1.
\]

(2.8)

Through the conjugation we can define real coordinates and momenta as follows,

\[
\hat{x}^1 = x^1 + \frac{h}{2} x^2, \quad \hat{x}^2 = x^2,
\]

\[
\hat{p}_1 = -i \partial_x x^1, \quad \hat{p}_2 = -i (\partial_x x^2 + \frac{h}{2} \partial_x x^1).
\]

(2.9)

They satisfy the following commutation relations;

\[
[\hat{x}^1, \hat{x}^2] = h(\hat{x}^2)^2, \quad [\hat{p}_1, \hat{p}_2] = h(\hat{p}_1)^2,
\]

\[
[\hat{p}_j, \hat{x}^i] = -i - h \hat{x}^2 \hat{p}_1, \quad [\hat{p}_1, \hat{x}^2] = 0, \quad (2.10)
\]

\[
[\hat{p}_2, \hat{x}^1] = h(-i + \hat{x}^1 \hat{p}_1 + \hat{x}^2 \hat{p}_2 - h \hat{x}^2 \hat{p}_1).
\]

The similar phase space algebra as well as (2.6) has been obtained in \([30]\). These \( h \)-deformed coordinates and momenta can be represented in terms of classical ones \( \hat{x} \) and \( \hat{p} \). Suppose that \( \hat{x}^2 = \bar{x}^2 \) and \( \hat{p}_1 = \bar{p}_1 \), then we have

\[
\hat{x}^1 = \hat{x}^1 (1 - ih \bar{p}_1 \bar{x}^2) + ih (\bar{x}^2)^2 \bar{p}_2, \quad \hat{p}_2 = \bar{p}_2 (1 - ih \bar{p}_1 \bar{x}^2) + ih (\bar{p}_1)^2 \bar{x}^2. \quad (2.11)
\]
The deformed phase space algebra such as eq.(2.10) might remind us of Dirac brackets of a constrained system [32]. Actually we can find a “constraint” which we can make identically equal to zero through the above commutation relations (2.10), as follows,

\[ f = 1 - 2i\hbar \hat{p}_1 \hat{x}^2. \]  

(2.12)

The function \( f \) commutes with \( \hat{x}^2 \) and \( \hat{p}_1 \) and satisfies commutation relations with the other elements as follows,

\[ [f, \hat{x}] = i\hbar \hat{x} f, \quad [\hat{p}_2, f] = i\hbar \hat{p}_1 f. \]  

(2.13)

These relations seems to be somewhat different from the Dirac bracket, whose right hand side vanishes completely. But we can make \( f \) equal to zero identically. The above fact seems to show that \( SU_h(2) \) is a ‘group’ which transforms the “constraint” \( f \) and its relations (2.13) covariantly.

3. Differential calculus on h-deformed fermionic space

In this section we discuss differential calculus on an h-deformed quantum fermionic space. Commutation relations of h-deformed coordinates \( \theta^\alpha \) and derivatives \( \partial_\alpha \) are obtained by replacing \( \hat{R} \) in (2.1) and (2.3) in terms of \( -\hat{R} \) as follows,

\[ \theta^\alpha \theta^\beta = -\hat{R}^{\alpha\beta}_{\mu\nu} \theta^\mu \theta^\nu, \quad \partial_\nu \partial_\mu = -\hat{R}^{\alpha\beta}_{\mu\nu} \partial_\beta \partial_\alpha, \]

\[ \partial_\mu \theta^\alpha = \delta_\mu^\alpha - \hat{R}^{\alpha\beta}_{\mu\nu} \theta^\nu \partial_\beta. \]  

(3.1)

These relations are written explicitly as follows,

\[ (\theta^1)^2 = h\theta^1 \theta^2, \quad \{\theta^1, \theta^2\} = (\theta^2)^2 = 0, \]

\[ (\partial_1)^2 = \{\partial_1, \partial_2\} = 0, \quad (\partial_2)^2 = h\partial_1 \partial_2, \]

\[ \{\partial_\alpha, \theta^\alpha\} = 1 + h\theta^2 \partial_1, \quad \{\partial_1, \theta^2\} = 0, \]

\[ \{\partial_2, \theta^1\} = -h\theta^1 \partial_1 - h\theta^2 \partial_2 - h^2 \theta^2 \partial_1. \]  

(3.2)

Further eq.(3.2) leads to the following trilinear relations;

\[ (\theta^1)^3 = (\partial_2)^3 = 0. \]  

(3.3)
The coordinate $\theta^1$ and the derivative $\partial_2$ are parafermionic \cite{33}, while its derivatives $\partial_1$ and its coordinates $\theta^2$ are nilpotent.

As discussed in the previous section, we can introduce h-deformed quantum group elements $T^{\alpha \beta}_\theta$ which transform $\theta^\alpha$ into $\theta'^\alpha = T^{\alpha \beta}_\theta \theta^\beta$. The elements satisfy the same commutation relation as (2.7). This h-deformed quantum group is interesting also from the aspect that it transforms fermionic and parafermionic variables, i.e., $\theta^1$ and $\theta^2$. Further, we can define a determinant of $T^{\alpha \beta}_\theta$ using the h-deformed fermionic space as follows,

$$\theta^1 \theta^2 = \det T \cdot \theta^1 \theta^2. \quad (3.4)$$

The definition leads to

$$\det T = T^1_1 T^2_2 - T^1_2 T^2_1 + hT^1_1 T^2_1. \quad (3.5)$$

Eq. (3.5) coincides with the definition of the determinant in \cite{28}.

Ref. \cite{25} shows that q-deformed fermionic coordinates and derivatives can represent q-deformed quantum algebra, e.g., $su_q(2)$. Here we apply the similar analysis to the h-deformed case. In the similar way to \cite{25}, we define the following generators;

$$L_+ = \theta^2 \partial_1, \quad L_- = \theta^1 \partial_2. \quad (3.6)$$

Using their commutation relation, we introduce a Cartan generator as follows,

$$L_0 = [L_+, L_-] = \theta^2 \partial_2 - \theta^1 \partial_1. \quad (3.7)$$

In addition to (3.7), these generators satisfy the following relations;

$$[L_0, L_\pm] = \pm 2L_\pm. \quad (3.8)$$

Eqs. (3.7) and (3.8) are nothing but the classical $su(2)$ algebra up to the normalization factor, although the coordinates and the derivatives are deformed. The generators act on the coordinates and the derivatives as follows,

$$[L_0, \theta^\alpha] = (-1)^\alpha \theta^\alpha, \quad [L_0, \partial_\alpha] = -(-1)^\alpha \partial_\alpha,$$

$$[L_+, \theta^1] = \theta^2, \quad [L_+, \partial_2] = -\partial_1,$$

$$[L_+, \theta^2] = [L_+, \partial_1] = 0,$$

$$[L_-, \theta^1] = -h\theta^1 L_0 - h^1 \theta^2 L_+, \quad (3.9)$$
\[ [L_-, \theta^2] = \theta^1 + h\theta^2 L_0, \]
\[ [L_-, \partial_1] = -\partial_2 + h\partial_1 L_0 + h\partial_1 L_0, \]
\[ [L_-, \partial_2] = -h\partial_2 L_0 - 3h^2 \partial_1 + 3h^2 \partial_2 L_+. \]

The actions of \( L_+ \) and \( L_0 \) are never deformed, while those of \( L_- \) are deformed.

In Ref. [29] a \( h \)-deformed \( su(2) \) algebra has been constructed as follows,
\[ [H, X] = \frac{2\sinh(hX)}{h}, \]
\[ [H, Y] = -\{ Y, \cosh(hX) \}, \quad (3.10) \]
\[ [X, Y] = H. \]

The \( h \)-deformed algebra \( su_h(2) \) can be related with the classical algebra \( su(2) \) through the following mapping;
\[ X = \frac{\log \bar{L}_+}{h}, \]
\[ H = (\bar{L}_+ - (\bar{L}_+)^{-1})\bar{L}_0, \quad (3.11) \]
\[ Y = \frac{h}{2}(\bar{L}_+ - (\bar{L}_+)^{-1})(2\bar{L}_+\bar{L}_- - h\bar{L}_0), \]
where \( \bar{L}_0 = L_0/2 \) and \( \bar{L}_\pm = L_\pm/\sqrt{2} \).

In Ref. [13] Zumino derived quantum groups \( Sp_q(2n) \) and \( SO_q(2n) \) from \( q \)-deformed bosonic and fermionic phase space algebra for \( SL_q(n) \). That was extended to the supersymmetric case in [15]. In the similar way, here we discuss \( h \)-deformation of \( SO(4) \).

At first, we define \( h \)-deformed gamma matrices \( \gamma^\alpha \) as
\[ \gamma^\alpha = \partial_\alpha, \quad \gamma^{5-\alpha} = \theta^\alpha. \quad (3.12) \]

They satisfy the following \( h \)-deformed Clifford algebra;
\[ \gamma^\alpha \gamma^\beta + \tilde{B}^{\alpha\beta}_{\mu\nu} \gamma^\mu \gamma^\nu = \eta^{\alpha\beta}, \quad (3.13) \]
where \( \eta^{\alpha\beta} \) is an \( SO(4) \) metric. The matrix \( \tilde{B} \) is composed of \( \tilde{R} \)-matrix (2.4) as follows,
\[ \tilde{B}^{\alpha'}_{\mu'}^{\beta'} = -\tilde{R}^{\alpha\beta}_{\mu\nu}, \quad \tilde{B}^{\alpha\beta}_{\mu\nu} = -\tilde{R}^{\mu\nu}_{\beta\alpha}, \]
\[ \tilde{B}^{\alpha\beta}_{\mu\nu} = -\tilde{R}^{\beta\nu}_{\alpha\mu}, \quad \tilde{B}^{\alpha'}_{\mu'}^{\beta'} = -(\tilde{R}^{-1})^{\alpha\mu}_{\beta\nu}, \quad (3.14) \]
where $\alpha, \beta, \mu, \nu = 1, 2$ and $\alpha' = 5 - \alpha$. Ref. [13] shows that $\tilde{B}$ constructed through the above procedure satisfy the Yang-Baxter equation, if $\tilde{R}$ is the solution of the Yang-Baxter equation. We could introduce h-deformed $SO(4)$ group which transform the relation (3.13) covariantly. Instead of deriving explicitly $SO_h(4)$, we here introduce h-deformed $SO(4)$ quantum space $X^i \ (i = 1 \sim 4)$ whose commutation relations are written as $X^i X^j = \tilde{B}^{ij}_{kl} X^k X^\ell$. We have explicitly

$$[X^1, X^2] = h(X^1)^2, \quad [X^1, X^3] = 0,$$

$$[X^1, X^4] = [X^2, X^3] = -hX^3 X^1,$$

$$[X^2, X^4] = hX^4 X^1 + hX^3 X^2 + h^2 X^3 X^1, \quad [X^3, X^4] = -h(X^3)^2.$$  

The algebra has a center element $C \equiv X^4 X^1 + X^3 X^2$. Differential calculus on the above h-deformed space $X^i$ could be similarly obtained. Their commutation relations are covariant under the $SO_h(4)$ transformation, as said the above. Elements of $SO_h(4)$ satisfy the same relation as (2.7) except $\tilde{R}$-matrix replacing $\tilde{B}$-matrix.

4. Conclusion

We have studied here the differential calculi on the h-deformed bosonic and fermionic spaces. We have constructed the classical $su(2)$ algebra on the h-deformed fermionic space. The algebra is related with the h-deformed algebra $su_h(2)$. It is shown that $SU_h(2)$ is a ‘group’ which transforms fermionic and parafermionic variables into each other. This fact is very interesting in applications to the parastatistics. Also $SO_h(4)$ was discussed. Supersymmetric extension is also intriguing. It is easy to introduce h-deformed oscillators in the similar way to the above h-deformed differential calculus. For example, we can define a sort of deformed oscillators by identifying the h-deformed coordinates and derivatives with deformed annihilation and creation operators, respectively. Their applications to various fields are very interesting.

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