BERNSTEIN CENTER OF SUPERCUSPIDAL BLOCKS

MANISH MISHRA

Abstract. Let $G$ be a tamely ramified connected reductive group defined over a non-archimedean local field $k$. We show that the Bernstein center of a tame supercuspidal block of $G(k)$ is isomorphic to the Bernstein center of a depth zero supercuspidal block of $G_0(k)$ for some twisted Levi subgroup of $G$.

1. Introduction

Let $G$ be a connected reductive group defined over a non-archimedean local field $k$. Assume that $G$ splits over a tamely ramified extension $k'$ of $k$. We will denote the group of $k$-rational points of $G$ by $G$ and likewise for other algebraic groups. In [8], Jiu-Kang Yu gives a very general construction of a class of supercuspidal representations of $G$ which he calls tame. A tame supercuspidal representation $\pi = \pi_\Sigma$ of $G$ is constructed out of a depth zero supercuspidal representation $\pi_0$ of $G^0$ and some additional data, where $G^0$ is a twisted Levi subgroup of $G$. By twisted, we mean that $G^0 \otimes k'$ is a Levi factor of a parabolic subgroup of $G \otimes k'$. The additional data, together with $G^0$ and $\pi_0$ is what we are denoting by $\Sigma$ in the notation $\pi_\Sigma$. In [4], Kim showed that under certain hypothesis, which are met for instance when the residue characteristic is large, these tame supercuspidals exhaust all the supercuspidals of $G$.

The depth zero supercuspidal $\pi_0$ of $G^0$ is compactly induced from $(K^0, \varrho_0)$ where $K^0$ is a compact mod center open subgroup of $G^0$ and $\varrho_0$ is a representation of $K^0$. The constructed representation $\pi_\Sigma$ is compactly induced from $(K, \varrho)$, where $K$ is a compact mod center open subgroup of $G$ containing $K^0$ and $\varrho$ is a representation of $K$. The representation $\varrho$ is of the form $\varrho_0 \otimes \kappa$, where $\varrho_0$ is seen as a representation of $K$ by extending from $K^0$ “trivially” (see [8] Sec. 4) and $\kappa$ is a representation of $K$ constructed out of the part of $\Sigma$ which is independent of $\varrho_0$.

Let $Z^\pi$ (resp. $Z^\pi_0$) denote the Bernstein center of the Bernstein block (see Section II for these terms) of $G$ (resp. $G^0$) containing $\pi$ (resp. $\pi_0$). Under certain hypothesis $C(G)$ [3 Page 47], we show that:

Theorem. $Z^\pi \cong Z^\pi_0$. Thus, the Bernstein center of a tame supercuspidal block of $G$ is isomorphic to the Bernstein center of a depth zero supercuspidal block of a twisted Levi subgroup of $G$. 

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Let $\mathcal{H}(G, ^o \varrho)$ (resp. $\mathcal{H}(G^0, ^o \varrho_0)$) denote the Hecke algebra of the type constructed out of $(K, \varrho)$ (resp. $(K^0, \varrho_0)$) (see Sec. 1.2). As a consequence of the above theorem, we obtain

$$Z(\mathcal{H}(G, ^o \varrho)) \cong Z(\mathcal{H}(G^0, ^o \varrho_0)),$$

where $Z(\mathcal{H}(G, ^o \varrho))$ (resp. $Z(\mathcal{H}(G^0, ^o \varrho_0))$) denotes the center of $\mathcal{H}(G, ^o \varrho)$ (resp. $\mathcal{H}(G^0, ^o \varrho_0)$).

In [8] Conj. 0.2, Yu conjectures that $\mathcal{H}(G, ^o \varrho) \cong \mathcal{H}(G^0, ^o \varrho_0)$. This is a special case of his more general conjecture [3] Conj. 17.7. Assuming certain conditions on $\pi_{\Sigma}$ ([11 Sec. 5.5]) which are satisfied quite often, for instance whenever $\pi_{\Sigma}$ is generic, in Theorem [10] we show that

$$\mathcal{H}(G, ^o \varrho) \cong \mathcal{H}(G^0, ^o \varrho_0).$$

2. Notations

Throughout this article, $k$ denotes a non-archimedean local field. For an algebraic group $G$ defined over $k$, we will denote its $k$-rational points by $G$. We will follow standard abuses of notation and terminology and refer, for example, to parabolic subgroups of $G$ in place of $k$-points of $k$-parabolic subgroups of $G$. Center of $G$ will be denoted by $Z_G$. The category of smooth representations of $G$ will be denoted by $\mathcal{R}(G)$. If $K$ is a subgroup of $G$ and $g \in G$, we denote $gKg^{-1}$ by $^gK$. If $\rho$ is a complex representation of $K$, $^g\rho$ denotes the representation $x \mapsto \rho(g^{-1}xg)$ of $^gK$. For $g \in G$, we say that $g$ intertwines $\rho$ if the vector space $\text{Hom}_{K \cap K}(^g\rho, \rho)$ is non-zero.

3. Yu’s construction [8]

Let $G$ be a connected reductive group defined over a non-archimedean local field $k$. A twisted $k$-Levi subgroup $G'$ of $G$ is a reductive $k$-subgroup such that $G' \otimes_k \bar{k}$ is a Levi subgroup of $G \otimes_k \bar{k}$. Yu’s construction involves the notion of a generic $G$-datum. It is a quintuple $\Sigma = (\vec{G}, y, \vec{r}, \vec{\phi}, \rho)$ satisfying the following:

1. $\vec{G} = (G^0 \subseteq G^1 \subseteq \ldots \subseteq G^d = G)$ is a tamely ramified twisted Levi sequence such that $Z_{G^0}/Z_G$ is anisotropic.
2. $y$ is a point in the extended Bruhat-Tits building of $G^0$ over $k$.
3. $\vec{r} = (r_0, r_1, \ldots, r_{d-1}, r_d)$ is a sequence of positive real numbers with $0 < r_0 < \cdots < r_{d-2} < r_{d-1} \leq r_d$ if $d > 0$, $0 \leq r_0$ if $d = 0$.
4. $\vec{\phi} = (\phi_0, \ldots, \phi_d)$ is a sequence of quasi-characters, where $\phi_i$ is a $G^{i+1}$-generic quasi-character [8] Sec. 9 of $G^i$; $\phi_0$ is trivial on $G^0_{y, r_0}$, but nontrivial on $G^0_{y, r_i}$ for $0 \leq i \leq d-1$. If $r_{d-1} < r_d$, $\phi_d$ is nontrivial on $G^i_{y, r_d}$ and trivial on $G^i_{y, r_{d+1}}$. Otherwise, $\phi_d = 1$. Here $G^i_{y, r}$ denote the filtration subgroups of the parahoric at $y$ defined by Moy-Prasad (see [6] Sec. 2.6).
(5) \( \rho \) is an irreducible representation of \( G^0_{[y]} \), the stabilizer in \( G^0 \) of the image [\( y \)] of \( y \) in the reduced building of \( G^0 \), such that \( \rho|G_{y,0}^0 \) is isotypical and c-Ind_{\( G_{y}^0 \)}^G \( \rho \) is irreducible and supercuspidal.

Let \( K^0 = G^0_{[y]}, K^0+ = G^0_{y,0} \), \( K^i = G^0_{[y]} G^1_{y,s_0} \cdots G^i_{y,s_{i-1}} \) and \( K^{i+} = G^0_{[y]} G^1_{y,s_0} \cdots G^i_{y,s_{i-1}}+ \) where \( s_j = r_j/2 \) for \( i = 1, \ldots, d \). In [3, Sec. 11], Yu constructs certain representation \( \kappa \) of \( K^d = K^d(\Sigma) \) which is independent of \( \rho \) and constructed only out of \( (\overline{G}, y, \overline{r}, \overline{\phi}) \). He defines certain subgroups \( J^i = (G^i-1, G^i)(k)_{y,(r_i-1,s_i-1)} \) and \( J^{i+} = (G^{i+}, G^{i})(k)_{y,(r_i,s_i-1+)} \) for \( 1 \leq i \leq d \) (see [3, Sec. 3] for the meaning of notations used here). For these groups, one has

\[
K^{i-1} J^i = K^i, \quad K^{(i-1)+} J^{i+} = K^{i+}.
\]

Also, \( K^{i-1} \cap J^i \subset K^{(i-1)+} \). Since \( \rho \) is iso-trivial on \( K^0+ \), one can successively inflate the representation \( \rho \) of \( K^0 \) to a representation of \( K^d \), which we again denote by \( \rho \), via the maps

\[
K^i \rightarrow K^{i-1} J^i / J^i = K^{i-1} / (K^{i-1} \cap J^i)
\]

(see [3, Sec. 4] for details). Write \( \rho_\Sigma := \rho \otimes \kappa \).

**Theorem 1** (Yu). \( \pi_\Sigma := c\text{-}\text{Ind}_{\mathcal{K}^+}^{\mathcal{G}} \rho_\Sigma \) is irreducible and thus supercuspidal.

The following theorem of Kim [4] says that under certain hypothesis (which are met for instance when the residue characteristic is sufficiently large), the representations \( \pi_\Sigma \) for various generic \( G \)-datum \( \Sigma \) exhaust all the supercuspidal representations of \( G \).

**Theorem 2** (Ju-Lee Kim). *Suppose the hypothesis (H.k), (HB), (HGT) and (HN) in [4] are valid. Then all the supercuspidal representations of \( G \) arise through Yu’s construction.*

In [3, Theorem 6.6, 6.7] under certain hypothesis denoted by \( C(\overline{G}) \) [3] Page 47, Hakim and Murnaghan determine when two supercuspidal representations are equivalent:

**Theorem 3** (Hakim-Murnaghan). Let \( \Sigma = (\overline{G}, y, \overline{r}, \overline{\phi}, \rho) \) and \( \Sigma' = (\overline{G}', y', \overline{r}', \overline{\phi}', \rho') \) be two generic \( G \)-data. Set \( \phi = \Pi_{i=1}^d \phi_i|G^0 \), \( \phi' = \Pi_{i=1}^d \phi'_{i}|G^0' \), \( \pi_0 = c\text{-}\text{Ind}_{\mathcal{G}_{[y]}}^{G^0} \rho \) and \( \pi_0' = c\text{-}\text{Ind}_{\mathcal{G}_{[y]}}^{G^0'} \rho' \). Then \( \pi_\Sigma \cong \pi_{\Sigma'} \) if and only if there exists \( g \in G \) such that \( K^d(\Sigma) = g K^d(\Sigma') \) and \( \rho_\Sigma = g \rho_{\Sigma'} \) if and only if \( G^0 = g G^0' \) and \( \pi_0 \otimes \phi \cong g (\pi_0' \otimes \phi' \).
4. Bernstein center

4.1. Bernstein decomposition. Let $X_k(G) = \text{Hom}(G, \mathbb{G}_m)$, the lattice of $k$-rational characters of $G$. Let

$$G := \{ g \in G : \text{val}_k(\chi(g)) = 0, \forall \chi \in X_k(G) \}.$$

In [5, Section 7], Kottwitz defined a functorial homomorphism $\kappa_G : G \rightarrow X_*(Z_G)^{Fr}_{I_k}$. Here $X_*(Z_G)$ denotes the co-character lattice of $Z_G$, $(\cdot)^{Fr}$ (resp. $(\cdot)^{I_k}$) denotes taking invariant (resp. coinvariant) with respect to Frobenius $Fr$ (resp. inertia subgroup $I_k$). The map $\kappa_G$ induces a functorial surjective map:

$$(4.1) \quad \kappa_G : G \rightarrow X_*(Z_G)^{Fr}_{I_k}/\text{torsion}$$

and $\ker(\kappa_G)$ is precisely $G$ (see [2, Sec. 3.3.1]).

Let $X_{nr}(G) := \text{Hom}(G/\mathcal{O}_G, \mathbb{C}^\times)$ denote the group of unramified characters of $G$. For a smooth representation $\pi$ of $G$, the representations $\pi \otimes \chi$, $\chi \in X_{nr}(G(k))$ are called the unramified twists of $\pi$.

Consider the collection of all cuspidal pairs $(L, \sigma)$ consisting of a Levi subgroup $L$ of $G$ and an irreducible cuspidal representation $\sigma$ of $L$. Define an equivalence relation $\sim$ on the class of all cuspidal pairs by

$$(L, \sigma) \sim (M, \tau) \text{ if } ^gL = M \text{ and } ^g\sigma \cong \tau \nu,$$

for some $g \in G$ and some $\nu \in X_{nr}(M)$. Write $[L, \sigma]_G$ for the equivalence class of $(L, \sigma)$ and $\mathcal{B}(G)$ for the set of all equivalence classes. The set $\mathcal{B}(G)$ is called the Bernstein spectrum of $G$. We say that a smooth irreducible representation $\pi$ has inertial support $s := [L, \sigma]_G$ if $\pi$ appears as a subquotient of a representation parabolically induced from some element of $s$. Define a full subcategory $\mathcal{R}(G)^s$ of $\mathcal{R}(G)$ as follows: a smooth representation $\pi$ belongs to $\mathcal{R}(G)^s$ iff each irreducible subquotient of $\pi$ has inertial support $s$. The categories $\mathcal{R}(G)^s, s \in \mathcal{B}(G)$, are called the Bernstein Blocks of $G$.

**Theorem 4** (Bernstein). We have

$$\mathcal{R}(G) = \prod_{s \in \mathcal{B}(G)} \mathcal{R}(G)^s.$$

4.2. Hecke algebra. Let $J$ be a compact open subgroup of $G$ and let $(\tau, W)$ be an irreducible representation of $J$. We call $(J, \tau)$ a compact open datum. The Hecke algebra $\mathcal{H}(G, \tau)$ associated to a compact open datum $(J, \tau)$ is the space of compactly supported functions $f : G \rightarrow \text{End}_\mathbb{C}(W)$ such that

$$f(j_1 gj_2) = \tau(j_1)f(g)\tau(j_2), \quad j_1, j_2 \in J \text{ and } g \in G.$$
The standard convolution operation gives $H(G, \tau)$ the structure of an associative $C$-algebra with identity.

Let $\mathcal{R}_\tau(G)$ be the subcategory of $\mathcal{R}(G)$ consisting of smooth representations which are generated by their $\tau$-isotypic component. If $\mathcal{R}_\tau(G) = \mathcal{R}(G)^s$ for some $s \in \mathcal{B}(G)$, then we say that $(J, \tau)$ is an $s$-type. Let $\mathcal{H}(G, \tau) - \mathcal{M}od$ denote the category of non-degenerate $\mathcal{H}(G, \tau)$ modules. If $(J, \tau)$ is an $s$-type, then $\mathcal{R}(G)^s$ is equivalent to $\mathcal{H}(G, \tau) - \mathcal{M}od$.

4.3. The center of $\mathcal{R}(G)$. Let $C$ be an abelian category. The set $\text{End}_C(id)$ of natural transformations of the identity functor of $C$ is a ring which by definition is the center of $C$. Denote it by $Z(C)$. Explicitly, $z \in Z(C)$ is a collection of morphisms $z_A : A \to A$, one for each object $A$ in $C$, such that for any morphism $f : B \to C$, the diagram

$$
\begin{array}{ccc}
B & \xrightarrow{f} & C \\
\downarrow{z_B} & & \downarrow{z_C} \\
B & \xrightarrow{f} & C
\end{array}
$$

commutes.

Let $R$ be a ring with identity. Let $\mathfrak{Z}(R)$ (resp. $\mathfrak{Z}(R - \mathcal{M}od)$) denote the center of $R$ (resp. the center of the category of left $R$-modules). There is a canonical ring isomorphism

$$
(4.2) \quad c \in \mathfrak{Z}(R) \mapsto \mu_c \in \mathfrak{Z}(R - \mathcal{M}od),
$$

where $\mu_c$ acts on each left $R$-module $M$ by $\mu_c(m) = cm$, for all $m \in M$ (see [7, Sec. 1.6.2]).

Let $s \in \mathcal{B}(G)$. The center of $\mathfrak{Z}(G)$ (resp. $\mathfrak{Z}(G)^s$) of the category $\mathcal{R}(G)$ (resp. $\mathcal{R}(G)^s$) is called the Bernstein center. If $(J, \tau)$ is an $s$-type, then $\mathcal{R}(G)^s \cong \mathcal{H}(G, \tau) - \mathcal{M}od$, and therefore by Equation (4.2), there is a canonical isomorphism

$$
(4.3) \quad \mathfrak{Z}(G)^s \cong Z(\mathcal{H}(G, \tau)),
$$

where $Z(\mathcal{H}(G, \tau))$ denotes the center of $\mathcal{H}(G, \tau)$.

5. Supercuspidal block

Let $G$ be a connected reductive group over $k$. Let $\pi$ be an irreducible supercuspidal representation of $G$ of the form $\pi = c \cdot \text{Ind}_J^G(\tau)$, where $J$ is an open, compact mod center subgroup of $G$ and $\tau$ is a representation of $J$. Write $^o J = J \cap ^o G$ and let $^o \tau$ be some irreducible component of $\tau | ^o J$. Then

Proposition 5. [1] Sec. 5.4] The group $^o J$ is the unique maximal compact subgroup of $J$ and $(^o J, ^o \tau)$ is a $[G, \pi]_G$-type in $G$. 

5.1. **Commutativity conditions.** Assume that the representation \( \pi \) satisfies the following conditions:

1. The representation \( \tau|^{0}\mathcal{J} \) is irreducible, i.e., \( \tau|^{0}\mathcal{J} = \tau|^{0}\mathcal{J} \).
2. Any \( g \in G \) which intertwines the representation \( \tau|^{0}\mathcal{J} \) lies in \( \mathcal{J} \).

These conditions are quite frequently satisfied (see [1] Sec. 5.5), for instance if \( \pi \) admits a Whittaker model ([7] Remark 1.6.1.3)). Under these assumptions, we have:

**Proposition 6.** [1] Sec. 5.5 | The Hecke algebra \( \mathcal{H}(G, \tau|^{0}\mathcal{J}) \) associate to the type \((\tau|^{0}\mathcal{J}, \tau|^{0}\mathcal{J})\) is commutative.

6. **Main result**

We use the notations of Section 3. Fix a generic \( G \)-datum \( \Sigma = (\mathbf{G}, y, \mathbf{T}, \mathbf{F}, \phi, \rho) \). Then \( \pi_{\Sigma} := c\text{-Ind}_{K_{d}^{G}}^{G} \rho_{\Sigma} \) is an irreducible supercuspidal representation of \( G \), where \( \rho_{\Sigma} \) is of the form \( \rho \otimes \kappa \) and \( \kappa \) is a representation of \( K_{d}^{G} \), constructed only out of the data \( (\mathbf{G}, y, \mathbf{T}, \mathbf{F}, \phi, \rho) \). The representation \( \pi_{0} = c\text{-Ind}_{K_{d}^{0}}^{G_{0}} \rho \) of \( G_{0} \) is depth zero supercuspidal. Set \( s := [G, \pi_{\Sigma}]_{G} \) and \( s_{0} := [G_{0}, \pi_{0}]_{G_{0}} \). Let \( J(G) \) (resp. \( J(G)^{s} \), resp. \( J(G)^{s_{0}} \)) be the Bernstein center of the category \( \mathcal{R}(G) \) (resp. \( \mathcal{R}(G)^{s} \), resp. \( \mathcal{R}(G)^{s_{0}} \)). Let \( \text{Irr}^{s}(G) \) (resp. \( \text{Irr}^{s_{0}}(G^{0}) \)) denote the isomorphism classes of irreducible elements in \( \mathcal{R}(G)^{s} \) (resp. \( \mathcal{R}(G^{0})^{s_{0}} \)). We assume the hypothesis \( C(\mathbf{G}) \) in [3] Page 47 in the rest of this section.

By functoriality of the map \( \mathbf{G} \hookrightarrow G \) induces a map

\[
\chi \in X_{\text{nr}}(G) \mapsto \chi|_{G_{0}} \in X_{\text{nr}}(G_{0}).
\]

**Theorem 7.** The map \( f_{\Sigma} : \pi_{\Sigma} \otimes \nu \in \text{Irr}^{s}(G) \mapsto \pi_{0} \otimes \nu|_{G_{0}} \in \text{Irr}^{s_{0}}(G^{0}), \nu \in X_{\text{nr}}(G) \), is well defined and is a bijection. Consequently, there is an isomorphism \( f_{\Sigma} : J(G)^{s} \cong J(G_{0})^{s_{0}} \).

**Proof.** We first prove well definedness. Suppose \( \pi_{\Sigma} \otimes \chi \cong \pi_{\Sigma} \) for \( \chi \in X_{\text{nr}}(G) \). Then we want to show that \( \pi_{0} \otimes \chi|_{G_{0}} \cong \pi_{0} \). Define a new quintuple \( \Sigma_{\chi} = (\mathbf{G}, y, \mathbf{T}, \mathbf{F}, \phi, \rho \otimes (\chi|_{K_{d}^{G}})). \) We have \( \pi_{\Sigma} \otimes \chi \cong c\text{-Ind}_{K_{d}^{G}}^{G} (\rho \otimes \kappa \otimes (\chi|_{K_{d}^{G}})) \). Since \( \chi \) is unramified, it follows that \( \pi_{\Sigma} \otimes \chi \cong \pi_{\Sigma_{\chi}} \). By Theorem 3, \( \pi_{\Sigma} \cong \pi_{\Sigma_{\chi}} \) is equivalent to \( (K_{d}, \rho_{\Sigma}) \) being conjugate to \( (K_{d}, \rho_{\Sigma_{\chi}}) \) by an element \( g \in G \). Since \( \rho_{\Sigma}|_{K_{d}^{+}} = \rho_{\Sigma_{\chi}}|_{K_{d}^{+}} \), it follows that \( g \) intertwines \( \rho_{\Sigma} |_{K_{d}^{+}} \). By [3] Prop. 4.4 and 4.1], it implies that \( g \in K_{d}^{G_{0}} K_{d}^{G} \). Thus we can assume without loss of generality that \( g \in G_{0} \). Let \( \rho' = (\rho \otimes \chi|_{K_{d}^{0}}) \).

Then by Theorem 3 we get \( \pi_{0} \otimes \phi \cong (\pi_{0} \otimes \phi) \) as \( G_{0} \)-representations, where \( \phi \) is as in Theorem 3 and \( \pi_{0} := c\text{-Ind}_{K_{d}^{0}}^{G_{0}} \rho' \cong \pi_{0} \otimes (\chi|_{G_{0}}) \). It follows that \( \pi_{0} \otimes \chi|_{G_{0}} \cong \pi_{0} \) and therefore \( f_{\Sigma} \) is well defined. Now if \( \chi \in X_{\text{nr}}(G) \) is such that \( \pi_{0} \otimes \chi|_{G_{0}} \cong \pi_{0} \), then it follows from Theorem 3 or directly that \( \pi_{\Sigma_{\chi}} \cong \pi_{\Sigma} \), i.e., \( \pi_{\Sigma} \otimes \chi \cong \pi_{\Sigma} \). This shows that the map \( f_{\Sigma} \) is also injective.
We now prove surjectivity. Now given \( \nu \in X_{\text{nr}}(G^0) \), using notation similar to before, write \( \Sigma_{\nu} = (\mathcal{G}, y, \rightarrow, \phi, \rho \otimes (\nu|K^0)) \). Let \( \circ K^d, \circ \rho_{\Sigma} \) (resp. \( \circ K^d, \circ \rho_{\Sigma} \)) be a type constructed out of \( (K^d, \rho_{\Sigma}) \) (resp. \( (K^d, \rho_{\Sigma}) \)) as in Sec. 5. Then \( \circ K^0 = G^0 \) and \( \circ K^d = \circ K^0 G^1_y s_{\alpha_{\circ_{s}}} \cdots G^d_{y, s_{\alpha_{d-1}}} \) (see notations in Sec. 3) is the maximal compact subgroup of \( K^d \) (see [8 Cor. 15.3]). Since \( \rho_{\Sigma} \circ K^d = \rho_{\Sigma_0} \circ K^d \), we can assume that \( \circ K^d, \circ \rho_{\Sigma} = \circ K^0, \circ \rho_{\Sigma} \). Now since \( \circ K^d, \circ \rho_{\Sigma} \) is an \( s \)-type, it follows that \( \pi_{\Sigma_0} \cong \pi_{\Sigma} \otimes \chi \) for some \( \chi \in X_{\text{nr}}(G) \). By the argument used in the proof of the well-definedness of the map \( f_{\Sigma} \) in the previous paragraph, we get, \( \pi_0 \otimes \nu \cong \pi_0 \otimes (\chi|G^0) \), i.e., \( \pi_0 \otimes \nu \) is the image of \( \pi_{\Sigma} \otimes \chi \) under \( f_{\Sigma} \). Thus \( f_{\Sigma} \) is also surjective.

We thus have a bijection \( f_{\Sigma} : \pi_{\Sigma} \otimes \chi \in \text{Irr}^s(G) \mapsto \pi_0 \otimes (\chi|G^0) \in \text{Irr}^s(G^0), \chi \in X_{\text{nr}}(G) \). Since \( 3(G)^s \) (resp. \( 3(G^0)^{s_0} \)) is canonically the ring of regular functions on \( \text{Irr}^s(G) \) (resp. \( \text{Irr}^s(G^0) \)) [7 Prop. 1.6.4.1], the Theorem follows.

For each irreducible object \( \tau \in \mathcal{R}(G) \) and \( z \in 3(G) \), denote by \( \chi_z(\tau) \), the scalar by which \( z \) acts on \( \tau \).

**Corollary 8.** Let \( z \in 3(G)^s \) and \( \pi \in \text{Irr}^s(G) \). Then \( \chi_z(\tau) = \chi_{f_{\Sigma}(z)}(f_{\Sigma}(\pi)) \).

**Proof.** This follows from [7] Prop. 1.6.4.1 and Theorem 4.

For an algebra \( \mathcal{A} \), denote by \( Z(\mathcal{A}) \) the center of \( \mathcal{A} \). Let \( H(G, \circ \rho_{\Sigma}) \) (resp. \( H(G^0, \circ \rho) \)) denote the Hecke algebra associated to the compact open data \( (\circ K^d, \circ \rho_{\Sigma}) \) (resp. \( (\circ K^0, \circ \rho) \)) (see Sec. 3.2). Then by Proposition 6, the Hecke algebras \( H(G, \circ \rho_{\Sigma}) \) and \( H(G^0, \circ \rho) \) are commutative. Theorem then follows from Corollary 8.

**Theorem 10.** \( H(G, \circ \rho_{\Sigma}) \cong H(G^0, \circ \rho) \).

**Proof.** By assumption, \( \rho_{\Sigma} \circ K^d \) is irreducible. Since \( \rho_{\Sigma} = \rho \otimes \kappa \) in the notations of Sec. 3, it implies that \( \rho \circ K^0 \) is also irreducible. Now by [8 Corr. 15.5], \( g \in G^0 \) intertwines \( \circ \rho \) iff it intertwines \( \circ \rho_{\Sigma} \). But then by assumption, \( g \in K^d \). Thus any \( g \in G^0 \) which intertwines \( \circ \rho \) lies in \( K^0 \). This means that \( \pi_0 \) also satisfies the commutativity conditions of Sec. 5.1. Then by Proposition 6, the Hecke algebras \( H(G, \circ \rho_{\Sigma}) \) and \( H(G^0, \circ \rho) \) are commutative. The Theorem then follows from Corollary 8.
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E-mail address: manish.mishra@gmail.com
Current address: Im Neuenheimer Feld 288, D-69120, Heidelberg, Germany