Some upper bounds for the signless Laplacian spectral radius of digraphs *

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Abstract

Let \( G = (V(G), E(G)) \) be a digraph without loops and multiarcs, where \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and \( E(G) \) are the vertex set and the arc set of \( G \), respectively. Let \( d^+_i \) be the outdegree of the vertex \( v_i \). Let \( A(G) \) be the adjacency matrix of \( G \) and \( D(G) = \text{diag}(d^+_1, d^+_2, \ldots, d^+_n) \) be the diagonal matrix with outdegrees of the vertices of \( G \). Then we call \( Q(G) = D(G) + A(G) \) the signless Laplacian matrix of \( G \). The spectral radius of \( Q(G) \) is called the signless Laplacian spectral radius of \( G \), denoted by \( q(G) \). In this paper, some upper bounds for \( q(G) \) are obtained. Furthermore, some upper bounds on \( q(G) \) involving outdegrees and the average 2-outdegrees of the vertices of \( G \) are also derived.

Key Words: Digraph, Signless Laplacian spectral radius, Upper bounds.

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1 Introduction

Let \( G = (V(G), E(G)) \) be a digraph without loops and multiarcs, where \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and \( E(G) \) are the vertex set and the arc set of \( G \), respectively. If \((v_i, v_j)\) be an arc of \( G \), then \( v_i \) is called the initial vertex of this arc and \( v_j \) is called the terminal vertex of this arc. For any vertex \( v_i \) of \( G \), we denote \( N^+_i = N^+_{v_i}(G) = \{v_j : (v_i, v_j) \in E(G)\} \) and \( N^-_i = N^-_{v_i}(G) = \{v_j : (v_j, v_i) \in E(G)\} \) the set of out-neighbors and in-neighbors of \( v_i \), respectively. Let \( d^+_i = |N^+_i| \) denote the outdegree of the vertex \( v_i \) and \( d^-_i = |N^-_i| \) denote

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the indegree of the vertex $v_i$ in the digraph $G$. The maximum vertex outdegree is denoted by $\Delta^+$, and the minimum outdegree by $\delta^+$. If $\delta^+ = \Delta^+$, then $G$ is a regular digraph. Let $t_i^+ = \sum_{v_j \in N_i^+} d_j^+$ be the 2-outdegree of the vertex $v_i$, $m_i^+ = \frac{t_i^+}{d_i^+}$ the average 2-outdegree of the vertex $v_i$. A digraph is strongly connected if for every pair of vertices $v_i, v_j \in V(G)$, there exists a directed path from $v_i$ to $v_j$ and a directed path from $v_j$ to $v_i$. In this paper, we consider finite digraphs without loops and multiarcs, which have at least one arc.

For a digraph $G$, let $A(G) = (a_{ij})$ denote the adjacency matrix of $G$, where $a_{ij} = 1$ if $(v_i, v_j) \in E(G)$ and $a_{ij} = 0$ otherwise. Let $D(G) = \text{diag}(d_1^+, d_2^+, \ldots, d_n^+)$ be the diagonal matrix with outdegrees of the vertices of $G$ and $Q(G) = D(G) + A(G)$ the signless Laplacian matrix of $G$. However, the signless Laplacian matrix of an undirected graph $D$ can be treated as the signless Laplacian matrix of the digraph $G'$, where $G'$ is obtained from $D$ by replace each edge with pair of oppositely directed arcs joining the same pair of vertices. Therefore, the research of the signless Laplacian matrix of a digraph has more universal significance than undirected graph.

The eigenvalues of $Q(G)$ are called the signless Laplacian eigenvalues of $G$, denoted by $q_1, q_2, \ldots, q_n$. In general $Q(G)$ are not symmetric and so its eigenvalues can be complex numbers. We usually assume that $|q_1| \geq |q_2| \geq \ldots \geq |q_n|$. The signless Laplacian spectral radius of $G$ is denoted and defined as $q(G) = |q_1|$, i.e., the largest absolute value of the signless Laplacian eigenvalues of $G$. Since $Q(G)$ is a nonnegative matrix, it follows from Perron Frobenius Theorem that $q(G) = q_1$ is a real number.

For the Laplacian spectral radius and signless Laplacian spectral radius of an undirected graph are well treated in the literature, see $[12, 13, 14, 16]$ and $[3, 4, 6, 7, 8, 14]$, respectively. Recently, there are some papers that give some lower or upper bounds for the spectral radius of a digraph, see $[2, 5, 15]$. Now we consider the signless Laplacian spectral radius of a digraph $G$. For application it is crucial to be able to computer or at least estimate $q(G)$ for a given digraph.

In 2014, Hong and You in $[9]$ obtained the following bounds for signless Laplacian spectral radius of a digraph.

$$q(G) \leq \max\{d_i^+ + d_j^+ : (v_i, v_j) \in E(G)\}. \quad (1)$$

$$q(G) \leq \max\{d_i^+ + m_i^+ : v_i \in V(G)\}. \quad (2)$$

$$q(G) \leq \max\left\{ \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4m_i^+m_j^+}}{2} : (v_i, v_j) \in E(G) \right\}. \quad (3)$$

$$q(G) \leq \max\left\{ d_i^+ + \sum_{v_j : (v_j, v_i) \in E(G)} d_j^+ : v_i \in V(G) \right\}. \quad (4)$$

In 2013, S.B. Bozkurt and D. Bozkurt in $[1]$ obtained the following bounds for signless Laplacian spectral radius of a digraph.

In 2014, Hong and You in $[9]$ gave a sharp bound for the signless Laplacian spectral radius of a digraph:

$$q(G) \leq \min_{1 \leq i \leq n} \left\{ \frac{d_1^+ + 2d_i^+ - 1 + \sqrt{(2d_i^+ - d_1^+ + 1)^2 + 8 \sum_{k=1}^{i-1} (d_k^+ - d_i^+)}}{2} \right\}. \quad (5)$$
Remark 1.1. Note that $G$ is a strongly connected digraph for bounds (1), (3), (4), respectively.

In this paper, we study on the signless Laplacian spectral radius of a digraph $G$. We obtain some upper bounds for $q(G)$, and we also show that some upper bounds on $q(G)$ involving outdegrees and the average 2-outdegrees of the vertices of $G$ can be obtained from our bounds.

2 Preliminaries Lemmas

In this section, we give the following lemmas which will be used in the following study.

Lemma 2.1. ([10]) Let $M = (m_{ij})$ be an $n \times n$ nonnegative matrix with spectral radius $\rho(M)$, i.e., the largest eigenvalues of $M$, and let $R_i = R_i(M)$ be the $i$-th row sum of $M$, i.e., $R_i(M) = \sum_{j=1}^{n} m_{ij} \ (1 \leq i \leq n)$. Then

$$\min\{R_i(M) : 1 \leq i \leq n\} \leq \rho(M) \leq \max\{R_i(M) : 1 \leq i \leq n\}. \tag{6}$$

Moreover, if $M$ is irreducible, then any equality holds in (6) if and only if $R_1 = R_2 = \ldots = R_n$.

Lemma 2.2. ([10]) Let $M$ be an irreducible nonnegative matrix. Then $\rho(M)$ is an eigenvalue of $M$ and there is a positive vector $X$ such that $MX = \rho(M)X$.

Lemma 2.3. ([11]) Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, $r_i = \sum_{j \neq i} |a_{ij}|$ for each $i = 1, 2, \ldots, n$, $S_{ij} = \{z \in \mathbb{C} : |z - a_{ii}| \cdot |z - a_{jj}| \leq r_iri_j\}$ for all $i \neq j$. Also let $E(A) = \{(i, j) : a_{ij} \neq 0, 1 \leq i \neq j \leq n\}$. If $A$ is irreducible, then all eigenvalues of $A$ are contained in the following region

$$\Omega(A) = \bigcup_{(i, j) \in E(A)} S_{ij}. \tag{7}$$

Furthermore, a boundary point $\lambda$ of (7) can be an eigenvalue of $A$ only if $\lambda$ locates on the boundary of each oval region $S_{ij}$ for $e_{ij} \in E(A)$.

3 Some upper bounds for the signless Laplacian spectral radius of digraphs

In this section, we present some upper bounds for the signless Laplacian spectral radius $q(G)$ of a digraph $G$ and also show that some bounds involving outdegrees, the average 2-outdegrees, the maximum outdegree and the minimum outdegree of the vertices of $G$ with $n$ vertices and $m$ arcs can be obtained from our bounds.
Theorem 3.1. Let $G$ be a strongly connected digraph with $n \geq 3$ vertices, $m$ arcs, the maximum vertex outdegree $\Delta^+$ and the minimum outdegree $\delta^+$. Then

$$q(G) \leq \max\{\Delta^+ + \delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+}, \delta^+ + 1 + \frac{m - \delta^+(n-1)}{2}\}. \quad (8)$$

Moreover, if $G(\neq C_n)$ is a regular digraph or $G \cong K_{1,n-1}$, where $K_{1,n-1}$ denotes the digraph on $n$ vertices which replace each edge in star graph $K_{1,n-1}$ with the pair of oppositely directed arcs, then the equality holds in (8).

Proof. From (8), we know that $q(G) \leq \max\{d^+_i + m^+_i : v_i \in V(G)\}$. So we only need to prove that $\max\{d^+_i + m^+_i : v_i \in V(G)\} \leq \max\{\Delta^+ + \delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+}, \delta^+ + 1 + \frac{m - \delta^+(n-1)}{2}\}$.

Suppose $\max\{d^+_i + m^+_i : v_i \in V(G)\}$ occurs at vertex $u$. Two cases arise $d^+_u = 1$, or $2 \leq d^+_u \leq \Delta^+$.

**Case 1.** $d^+_u = 1$. Suppose that $N^+_u = \{w\}$. Since $m^+_w = d^+_w \leq \Delta^+$, thus $d^+_u + m^+_w \leq 1 + \Delta^+$. Since $\sum_{v_i \in V(G)} d^+_i = m$, let $d^+_j = \Delta^+$, then $\sum_{i \neq j} d^+_i = m - \Delta^+ \geq (n-1)\delta^+$, so $m - (n-1)\delta^+ \geq \Delta^+$. Therefore $\delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+} \geq \delta^+ - 1 + \frac{\Delta^+}{\Delta^+} = \delta^+ \geq 1$. Thus $d^+_u + m^+_w \leq 1 + \Delta^+ \leq \Delta^+ + \delta^+ - 1 + \frac{m - \delta^+(n-1)}{\Delta^+}$, the result follows.

**Case 2.** $2 \leq d^+_u \leq \Delta^+$. Note that $m - (n-1)\delta^+ \geq d^+_u \geq 2$, and

$$m = \sum_{v : (u,v) \in E(G)} d^+_v + \sum_{v : (u,v) \notin E(G)} d^+_v \geq \sum_{v : (u,v) \in E(G)} d^+_v + d^+_u + (n - d^+_u - 1)\delta^+,$$

thus

$$\sum_{v : (u,v) \in E(G)} d^+_v \leq m - d^+_u - (n - d^+_u - 1)\delta^+$$

$$= m - (n-1)\delta^+ + (\delta^+ - 1)d^+_u$$

$$m^+_u = \frac{\sum_{v : (u,v) \in E(G)} d^+_v}{d^+_u} \leq \frac{m - (n-1)\delta^+}{d^+_u} + \delta^+ - 1.$$ ($8$)

This follows that $m^+_u + d^+_u \leq d^+_u + \frac{m - (n-1)\delta^+}{d^+_u} + \delta^+ - 1$. Let $f(x) = x + \frac{m - (n-1)\delta^+}{x} + \delta^+ - 1$, where $x \in [2, \Delta^+]$. It is easy to see that $f'(x) = 1 - \frac{m - (n-1)\delta^+}{x^2}$. Let $a = m - (n-1)\delta^+$, then $\sqrt{a}$ is the unique positive root of $f'(x) = 0$. We consider the next three Subcases.

**Subcase 1.** $\sqrt{a} < 2$. When $x \in [2, \Delta^+]$, since $f'(x) > 0$, then $f(x) \leq f(\Delta^+)$. $f(x) \leq f(\Delta^+)$.

**Subcase 2.** $2 \leq \sqrt{a} \leq \Delta^+$. Then $f'(x) < 0$ for $x \in [2, \sqrt{a}]$, and $f'(x) \geq 0$, for $x \in [\sqrt{a}, \Delta^+]$. Thus, $f(x) \leq \max\{f(2), f(\Delta^+)\}$.

**Subcase 3.** $\Delta^+ < \sqrt{a}$. When $x \in [2, \Delta^+]$, since $f'(x) < 0$, then $f(x) \leq f(2)$.

Recall that $2 \leq d^+_u \leq \Delta^+$, thus

$$m^+_u + d^+_u \leq \max\{f(2), f(\Delta^+)\}.$$
If $G(\neq C_n)$ is a regular digraph, then $d^+_i + m^+_i = 2d^+_i = 2\Delta^+$ for all $v_i \in V(G)$. We can get $q(G) = 2\Delta^+$. Since $G(\neq C_n)$ is a strongly connected digraph, then we may assume that $\Delta^+ \geq 2$, this implies that $\delta^+ + 1 + \frac{m - \delta^+ (n-1)}{2} = \Delta^+ + 1 + \frac{\Delta^+}{2} \leq 2\Delta^+ = \Delta^+ + \delta^+ - 1 + \frac{m - \delta^+ (n-1)}{2\Delta^+}$. So $\max\{\Delta^+ + \delta^+ - 1 + \frac{m - \delta^+ (n-1)}{2\Delta^+}, \delta^+ + 1 + \frac{m - \delta^+ (n-1)}{2}\} = 2\Delta^+$. Thus, the equality also holds. By combining the above discussion, the result follows. \hfill \Box

**Corollary 3.2.** Let $G$ be a strongly connected digraph with $n \geq 3$ vertices, $m$ arcs, the maximum outdegree $\Delta^+$ and the minimum outdegree $\delta^+$. If $\Delta^+ \geq \frac{m - \delta^+ (n-1)}{2}$ and $\delta^+ = 1$, then

$$q(G) \leq \Delta^+ + 2. \tag{9}$$

**Proof.** Because $\Delta^+ + \delta^+ - 1 + \frac{m - \delta^+ (n-1)}{2} \leq \Delta^+ + 2$, $\delta^+ + 1 + \frac{m - \delta^+ (n-1)}{2} \leq \Delta^+ + 2$, therefore by Theorem 3.1 we have $q(G) \leq \Delta^+ + 2$. \hfill \Box

Let $G^*(m, n, \frac{m - \delta^+ (n-1)}{2}, 1)$ be a class of strongly connected digraphs with $\Delta^+ \geq \frac{m - \delta^+ (n-1)}{2}$, $\delta^+ = 1$, and there exists a vertex $v_0 \in V(G)$ such that $d_{v_0} = \Delta^+$ and there exists a vertex $v_k \in N^+_{v_0}$, $d^+_v \geq 2$.

**Remark 3.3.** For $G \in G^*(m, n, \frac{m - \delta^+ (n-1)}{2}, 1)$, we have $\Delta^+ + 2 \leq \max\{d^+_i + d^+_j : (v_i, v_j) \in E(G)\}$, thus the upper bound (9) is better than the upper bound (11) for the class of digraphs $G \in G^*(m, n, \frac{m - \delta^+ (n-1)}{2}, 1)$. But for general digraphs, the upper bound (11) is incomparable with the upper bound (1).  

**Example 3.4.** Let $G$ be the digraph of order 4, as shown in Figure 1. Since it has 9 arcs, and the maximum outdegree $\Delta^+ = 3 = \frac{9 - (4-1)}{2}$, the minimum outdegree $\delta^+ = 1$, and there exists a vertex $v_4 \in N^+_{v_4}$, $d^+_v = 3 > 2$, therefore $G = G^*(9, 4, 3, 1)$.

![Figure 1: Graph $G^*(9, 4, 3, 1)$](image-url)
Lemma 2.3, there at least exists \((G)\). Moreover if \((10)\) holds in \(G\), it is easy to see that \(P\) is irreducible and nonnegative. Now the \((i, j)\)-th element of \(P = D^{-\frac{1}{2}}Q(G)D^\frac{1}{2}\) is

\[
p_{ij} = \begin{cases} 
    d_i^+ & \text{if } i = j, \\
    \frac{\sqrt{d_j^+}}{\sqrt{d_i^+}} & \text{if } (v_i, v_j) \in E(G), \\
    0 & \text{otherwise.}
\end{cases}
\]

Let \(R_i(P)\) be the \(i\)-th row sum of \(P\) and \(R_i'(P) = R_i - d_i^+\). Then by Cauchy-Schwarz inequality, we have

\[
R_i'(P)^2 = \left( \sum_{v_j: (v_i, v_j) \in E(G)} \frac{\sqrt{d_j^+}}{\sqrt{d_i^+}} \right)^2 \leq \sum_{v_j: (v_i, v_j) \in E(G)} 1^2 \sum_{v_j: (v_i, v_j) \in E(G)} \frac{d_j^+}{d_i^+} = \sum_{v_j: (v_i, v_j) \in E(G)} d_j^+ = d_i^+ m_i^+.
\]

Since \(P\) is irreducible and nonnegative, \(\rho(P)\) denotes the spectral radius of \(P\). Then by Lemma 2.3, there at least exists \((v_i, v_j) \in E(G)\) such that \(\rho(P)\) is contained in the following oval region

\[
|\rho(P) - d_i^+| |\rho(P) - d_j^+| \leq R_i'(P) R_i(P) \leq \sqrt{d_i^+ m_i^+} \sqrt{d_j^+ m_j^+}.
\]

Obviously, \(\rho(P) = q(G) > \max\{d_i^+ : v_i \in E(G)\}\), and \((\rho(P) - d_i^+) (\rho(P) - d_j^+) \leq |\rho(P) - d_i^+| |\rho(P) - d_j^+|\). Therefore, solving the above inequality we obtain

\[
q(G) \leq \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4 d_i^+ m_i^+ d_j^+ m_j^+}}{2}.
\]

Table 1: Values of the upper bounds for example 1.

| \(G^*(9, 4, 3, 1)\) | 4.7321 | 6 | 5 |

**Theorem 3.5.** Let \(G\) be a strongly connected digraph with vertex set \(V(G) = \{v_1, v_2, \ldots, v_n\}\) and arc set \(E(G)\). Then

\[
q(G) \leq \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4 d_i^+ m_i^+ d_j^+ m_j^+}}{2} : (v_i, v_j) \in E(G) \}. \quad (10)
\]

Moreover if \(G\) is a regular digraph or a bipartite semiregular digraph, then the equality holds in \((10)\).

**Proof.** From the definition of \(D = D(G)\) we get \(D^\frac{1}{2} = \text{diag} (\sqrt{d_i^+} : v_i \in V(G))\), and consider the similar matrix \(P = D^{-\frac{1}{2}}Q(G)D^\frac{1}{2}\). Since \(G\) is a strongly connected digraph, it is easy to see that \(P\) is irreducible and nonnegative. Now the \((i, j)\)-th element of \(P = D^{-\frac{1}{2}}Q(G)D^\frac{1}{2}\) is

\[
p_{ij} = \begin{cases} 
    d_i^+ & \text{if } i = j, \\
    \frac{\sqrt{d_j^+}}{\sqrt{d_i^+}} & \text{if } (v_i, v_j) \in E(G), \\
    0 & \text{otherwise.}
\end{cases}
\]

Let \(R_i(P)\) be the \(i\)-th row sum of \(P\) and \(R_i'(P) = R_i - d_i^+\). Then by Cauchy-Schwarz inequality, we have

\[
R_i'(P)^2 = \left( \sum_{v_j: (v_i, v_j) \in E(G)} \frac{\sqrt{d_j^+}}{\sqrt{d_i^+}} \right)^2 \leq \sum_{v_j: (v_i, v_j) \in E(G)} 1^2 \sum_{v_j: (v_i, v_j) \in E(G)} \frac{d_j^+}{d_i^+} = \sum_{v_j: (v_i, v_j) \in E(G)} d_j^+ = d_i^+ m_i^+.
\]

Since \(P\) is irreducible and nonnegative, \(\rho(P)\) denotes the spectral radius of \(P\). Then by Lemma 2.3, there at least exists \((v_i, v_j) \in E(G)\) such that \(\rho(P)\) is contained in the following oval region

\[
|\rho(P) - d_i^+| |\rho(P) - d_j^+| \leq R_i'(P) R_i(P) \leq \sqrt{d_i^+ m_i^+} \sqrt{d_j^+ m_j^+}.
\]

Obviously, \(\rho(P) = q(G) > \max\{d_i^+ : v_i \in E(G)\}\), and \((\rho(P) - d_i^+) (\rho(P) - d_j^+) \leq |\rho(P) - d_i^+| |\rho(P) - d_j^+|\). Therefore, solving the above inequality we obtain

\[
q(G) \leq \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4 d_i^+ m_i^+ d_j^+ m_j^+}}{2}.
\]
Then for $1 \leq i \leq n$

$$q(G)x_i = d_i^+x_i + \sum_{v_k : (v_i, v_k) \in E(G)} x_k = \sum_{v_k : (v_i, v_k) \in E(G)} (x_i + x_k).$$

By (12), we have

$$q(G)(x_i + x_j) = \sum_{v_k : (v_i, v_k) \in E(G)} (x_i + x_k) + \sum_{v_k : (v_j, v_k) \in E(G)} (x_j + x_k).$$
For convenience we use \( f(i, j) \) denote \( f(v_i, v_j) \). Set \( g(i, j) = \frac{x_i + x_j}{f(i, j)} \). If \( (v_i, v_j) \in E(G) \), then

\[
q(G)f(i, j)g(i, j) = \sum_{v_k: (v_i, v_k) \in E(G)} f(i, k)g(i, k) + \sum_{v_k: (v_j, v_k) \in E(G)} f(j, k)g(j, k).
\]

By (13), we get

\[
|q(G)f(i, j)g(i, j)| = q(G)f(i, j)|g(i, j)| \leq \sum_{v_k: (v_i, v_k) \in E(G)} f(i, k)|g(i, k)| + \sum_{v_k: (v_j, v_k) \in E(G)} f(j, k)|g(j, k)|.
\]

Now choose \( i_1, j_1 \) such that \( (v_i, v_j) \in E(G) \) and \( |g(i_1, j_1)| = \max\{|g(i, j)| : (v_i, v_j) \in E(G)\} \). If \( g((i_1, j_1)) = 0 \), then \( |g(i, j)| = 0 \) for all arcs \( (v_i, v_j) \in E(G) \), i.e., \( x_i + x_j = 0 \) for all arcs \( (v_i, v_j) \in E(G) \). By (12), we have \( q(G) = 0 \) which is impossible, since \( G \) has at least one arc. So \( |g(i_1, j_1)| > 0 \). Then

\[
q(G)f(i_1, j_1)|g(i_1, j_1)| \leq \sum_{v_k: (v_i, v_k) \in E(G)} f(i_1, k)|g(i_1, k)| + \sum_{v_k: (v_j, v_k) \in E(G)} f(j_1, k)|g(j_1, k)|.
\]

Therefore, we obtain

\[
q(G) \leq \sum_{v_k: (v_i, v_k) \in E(G)} \frac{f(i_1, k)}{f(i_1, j_1)}|g(i_1, k)| + \sum_{v_k: (v_j, v_k) \in E(G)} \frac{f(j_1, k)}{f(i_1, j_1)}|g(j_1, k)|
\]

i.e.,

\[
q(G) \leq \sum_{v_k: (v_i, v_k) \in E(G)} \frac{f(i_1, k)}{f(i_1, j_1)} + \sum_{v_k: (v_j, v_k) \in E(G)} \frac{f(j_1, k)}{f(i_1, j_1)}, \text{ where } (v_i, v_j) \in E(G).
\]

This proves the desired result.

**Corollary 3.7.** Let \( G = (V(G), E(G)) \) be a digraph. Then

\[
q(G) \leq \max \left\{ d_i^+ \sqrt{m_i^+ \over d_i^+} + d_j^+ \sqrt{m_j^+ \over d_j^+} : (v_i, v_j) \in E(G) \right\}.
\]

**Proof.** Setting \( f(v_i, v_j) = \sqrt{d_i^+ d_j^+} \) in (11), by Cauchy-Schwarz inequality,

\[
\sum_{v_k: (v_i, v_k) \in E(G)} f(v_i, v_k) = \sum_{v_k: (v_i, v_k) \in E(G)} \sqrt{d_i^+ d_k^+} = \sum_{v_k: (v_i, v_k) \in E(G)} \left( \sqrt{d_i^+} \sqrt{d_k^+} \right)
\]

\( 8 \)
\[
\sum_{v_k : (v_i, v_k) \in E(G)} d_k^+ \sum_{v_k : (v_i, v_k) \in E(G)} d_k^+ = \sqrt{d_i^+} \sum_{v_k : (v_i, v_k) \in E(G)} d_k^+ = d_i^+ \sqrt{d_i^+ m_i^+}.
\]

By (11), we get
\[
q(G) \leq \max \left\{ \frac{\sum_{v_k : (v_i, v_k) \in E(G)} f(v_i, v_k) + \sum_{v_k : (v_i, v_k) \in E(G)} f(v_j, v_k)}{f(v_i, v_j)} : (v_i, v_j) \in E(G) \right\}
\]
\[
\leq \max \left\{ \frac{d_i^+ \sqrt{d_i^+ m_i^+} + d_j^+ \sqrt{d_j^+ m_j^+}}{\sqrt{d_i^+ d_j^+}} : (v_i, v_j) \in E(G) \right\}
\]
\[
= \max \left\{ d_i^+ \sqrt{\frac{m_i^+}{d_j^+}} + d_j^+ \sqrt{\frac{m_j^+}{d_i^+}} : (v_i, v_j) \in E(G) \right\}.
\]

\begin{proof}
Setting \( f(v_i, v_j) = d_i^+ + d_j^+ \) in (11), since
\[
\sum_{v_k : (v_i, v_k) \in E(G)} f(v_i, v_k) = \sum_{v_k : (v_i, v_k) \in E(G)} (d_i^+ + d_k^+) = d_i^+ (d_i^+ + m_i^+) \text{, so we get the desired result.}
\end{proof}

\textbf{Corollary 3.8.} Let \( G = (V(G), E(G)) \) be a digraph. Then
\[
q(G) \leq \max \left\{ \frac{d_i^+ (d_i^+ + m_i^+) + d_j^+ (d_j^+ + m_j^+)}{d_i^+ + d_j^+} : (v_i, v_j) \in E(G) \right\}. \tag{15}
\]

\begin{proof}
Setting \( f(v_i, v_j) = d_i^+ + d_j^+ \) in (11), since
\[
\sum_{v_k : (v_i, v_k) \in E(G)} f(v_i, v_k) = \sum_{v_k : (v_i, v_k) \in E(G)} (d_i^+ + d_k^+) = d_i^+ (d_i^+ + m_i^+) \text{, so we get the desired result.}
\end{proof}

\textbf{Corollary 3.9.} Let \( G = (V(G), E(G)) \) be a digraph. Then
\[
q(G) \leq \max \left\{ \frac{d_i^+ \sqrt{d_i^+ m_i^+} + d_j^+ \sqrt{d_j^+ m_j^+}}{\sqrt{d_i^+ d_j^+}} : (v_i, v_j) \in E(G) \right\}. \tag{16}
\]

\begin{proof}
Setting \( f(v_i, v_j) = \sqrt{d_i^+} + \sqrt{d_j^+} \) in (11), since
\[
\sum_{v_k : (v_i, v_k) \in E(G)} f(v_i, v_k) = \sum_{v_k : (v_i, v_k) \in E(G)} (1 \cdot \sqrt{d_i^+} + \sqrt{d_k^+}) \leq \sqrt{d_i^+} \sum_{v_k : (v_i, v_k) \in E(G)} (d_i^+ + d_k^+)
\]
\[
= \sqrt{d_i^+} (d_i^+ + d_i^+ m_i^+) = d_i^+ \sqrt{d_i^+ m_i^+} \text{ by Cauchy-Schwarz inequality.}
\]
Thus by (11) we get the desired result.
\end{proof}

\textbf{Corollary 3.10.} Let \( G = (V(G), E(G)) \) be a digraph. Then
\[
q(G) \leq \max \left\{ \frac{d_i^+ (\sqrt{d_i^+} + \sqrt{m_i^+}) + d_j^+ (\sqrt{d_j^+} + \sqrt{m_j^+})}{\sqrt{d_i^+} + \sqrt{d_j^+}} : (v_i, v_j) \in E(G) \right\}. \tag{17}
\]

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Proof. Setting \( f(v_i, v_j) = \sqrt{d_i^+} + \sqrt{d_j^+} \) in (11), since
\[
\sum_{v_k : (v_i, v_k) \in E(G)} f(v_i, v_k) = \sum_{v_k : (v_i, v_k) \in E(G)} (\sqrt{d_i^+} + \sqrt{d_k^+}) \leq d_i^+ \cdot \sqrt{d_k^+} + \sum_{v_k : (v_i, v_k) \in E(G)} d_k^+ = d_i^+ \cdot (\sqrt{d_i^+} + \sqrt{m_i^+})
\]
by Cauchy-Schwarz inequality. By (11) the result follows. \( \square \)

Notice that (16) and (17) can be viewed as adding square roots to (15) at different places.

4 Example

Let \( G_1, G_2 \) be the digraphs of order 4,6, respectively, as shown in Figure 2.

![Figure 2](image)

Table 2: Values of the various bounds for example 1.

| \( q(G) \) | (1) | (2) | (3) | (4) | (5) | (8) | (10) | (14) | (15) | (16) | (17) |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( G_1 \) | 3.0000 | 4.0000 | 3.5000 | 3.3028 | 3.4142 | 3.5616 |
|          | 3.5000 | 3.5651 | 3.4495 | 3.3333 | 3.6029 | 3.5731 |
| \( G_2 \) | 4.1984 | 5.0000 | 4.6667 | 4.6016 | 5.0000 | 4.7321 |
|          | 5.0000 | 4.7913 | 4.5644 | 4.6000 | 4.7956 | 4.7866 |

Remark 4.1. Obviously, from Table 1, the bound (3) is the best in all known upper bounds for \( G_1 \), and the bound (14) is the best for \( G_2 \). Finally bound (15) is the second-best bounds for \( G_1 \) and \( G_2 \). In general, these bounds are incomparable.

References

[1] S.B. Bozkurt, D. Bozkurt, On the signless Laplacian spectral radius of digraphs, Ars Combinatoria, 108, (2013) 193–200

[2] R. Brualdi, Spectra of digraphs, Linear Algebra Appl. 432 (2013) 193–200
[3] Y.Q. Chen, L.G. Wang, Sharp bounds for the largest eigenvalue of the signless Laplacian of a graph, Linear Algebra Appl. 433 (2010) 908–913

[4] S.Y. Cui, C.X. Tian, J.J. Guo, A sharp upper bound on the signless Laplacian spectral radius of graphs, Linear Algebra Appl. 439 (2013) 2442–2447

[5] A.D. Güngör, K.C. Das, Improved upper and lower bounds for the spectral radius of digraphs, Appl. Math. Comput. 216 (2010) 791–799

[6] A.D. Güngör, K.C. Das, A.S. Çevik, Sharp upper bounds on the spectral radius of the signless Laplacian matrix of a graph, Appl. Math. Comput. 219 (2013) 5025–5032

[7] P. Hansen, C. Lucas, Bounds and conjectures for the signless Laplacian index of graphs, Linear Algebra Appl. 432 (2010) 3319–3336

[8] B. He, Y.L. Jin, X.D. Zhang, Sharp bounds for the signless Laplacian spectral radius in term of clique number, Linear Algebra Appl. 438 (2013) 3851–3861

[9] W.X. Hong, L.H. You, Spectral radius and signless Laplacian spectral radius of strongly connected digraphs, Linear Algebra Appl. 457 (2014) 93–113

[10] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, New York, (1985)

[11] L.L. Li, A simplified Brauer’s theorem on matrix eigenvalues, Appl. Math. J. Chin. Uni. Ser. B. 14(3) (1999) 259–264

[12] T.F. Wang, Several sharp upper bounds for the largest Laplacian eigenvalue of a graph, Science in China Series A: Mathematics, 50(12) (2007) 1755–1764

[13] T.F. Wang, J. Yang, B. Li, Improved upper bounds for the Laplacian spectral radius of graph, Electron. J. Comb. 18 (2011) #P35

[14] F.Y. Wei, M.H. Liu, A sharp upper bound on the Laplacian and Quasi-Laplacian spectral radius of a graph, Fuzzy Engineering and Operations Research, 147 (2012) 525–531

[15] G.H. Xu, C.Q. Xu, Sharp bounds for the spectral radius of digraphs, Linear Algebra Appl. 430 (2009) 1607–1612

[16] D.M. Zhu, On upper bounds for Laplacian graph eigenvalues, Linear Algebra Appl. 432 (2010) 2764–2772