Further remarks on local discriminants

Chandan Singh Dalawat

Harish-Chandra Research Institute
Chhatnag Road, Jhunsi, Allahabad 211019, India
Email: dalawat@gmail.com

Abstract. Using Kummer theory for a finite extension $K$ of $\mathbb{Q}_p(\zeta)$ (where $p$ is a prime number and $\zeta$ a primitive $p$-th root of 1), we compute the ramification filtration and the discriminant of an arbitrary elementary abelian $p$-extension of $K$. We also develop the analogous Artin-Schreier theory for finite extensions of $\mathbb{F}_p((\pi))$ and derive similar results for their elementary abelian $p$-extensions.

1. Review of local Kummer theory. Before stating the main results of this paper in §2, let us briefly review Kummer theory of local fields as expounded in [2]. This review is justified on the grounds that §4 consists largely of applications of this theory, and that §6 gives the analogous Artin-Schreier theory for local function fields, which is summarised in §5.

We fix a prime number $p$ and denote by $\zeta$ a primitive $p$-th root of 1. Let $K|\mathbb{Q}_p(\zeta)$ be a finite extension, $k$ the residue field, $e_1$ the ramification index, and $f = [k: \mathbb{F}_p]$ the residual degree; the ramification index of $K|\mathbb{Q}_p$ is $e = (p - 1)e_1$ (and the residual degree is $f$).

The filtration $(U_n)_{n>0}$ on $K^\times$ by units of various levels induces a filtration on the $\mathbb{F}_p$-space $\overline{K}^\times = K^\times/K^\times_p$ denoted by $(\overline{U}_n)_{n>0}$; we have $\overline{U}_{pe_1+1} = \{1\}$, and the codimension at each step is given by

$$\{1\} \subset_1 \overline{U}_{pe_1} \subset_1 \overline{U}_{pe_1-1} \subset_f \cdots \subset_f \overline{U}_{pe_1+1} = \overline{U}_{pi} \subset_f \cdots \subset_f \overline{U}_1 \subset_1 \overline{K}^\times.$$

Here, $i$ is any integer in the interval $[1, e_1]$ (which is empty when $e_1 = 1$), and an inclusion $E \subset_f E'$ means that $E$ is a codimension-$r$ subspace of $E'$.

We have $\overline{a^\times} = \overline{U}_1$, and the valuation gives $\overline{K}^\times/\overline{U}_1 = \mathbb{Z}/p\mathbb{Z}$. Moreover, the choice of $\zeta$ leads to an isomorphism $\overline{U}_{pe_1} \to \mathbb{F}_p$ sending $a$ to $S_k|\mathbb{F}_p(\hat{c})$, where $\hat{c}$ is the image of $c = (1 - b)/p(1 - \zeta)$ in $k/\varphi(k)$ and $b \in \overline{U}_{pe_1}$.

Keywords: Local fields, Kummer theory, ramification filtration, Artin-Schreier theory, elementary abelian $p$-extensions, discriminants.
represents $a$; the isomorphism $k/\wp(k) \rightarrow F_p$ is induced by the trace map $S_k|_{F_p}$. (Question: How does the isomorphism $\bar{U}_{pe_1} \rightarrow F_p$ change when we replace $\zeta$ by $\zeta^a$ for some $a \in F_p^\times$?)

The unramified degree-$p$ extension of $K$ is $K(\sqrt{U_{pe_1}})$. For an $F_p$-line $D \neq \bar{U}_{pe_1}$ in $K^\times$ such that $D \subset \bar{U}_m$ but $D \not\subset \bar{U}_{m+1}$ (with the convention that $\bar{U}_0 = K^\times$), the unique ramification break of the (cyclic, degree-$p$) extension $K(\sqrt{D})$ occurs at $pe_1 - m$. This break is thus an integer prime to $p$, unless $m = 0$.

Let $M = K(\sqrt{K^\times})$ be the maximal elementary abelian $p$-extension of $K$, and $G = \text{Gal}(M/K)$, endowed with the ramification filtration $(G^u)_{u \in [-1, +\infty[}$ in the upper numbering. It follows from the foregoing (see Part IX of [2], cf. [3]), that $G^u = G^1$ for $u \in ]-1, 1]$ and that, for $u \in [1, pe_1 + 1]$, we have the “orthogonality relation” $(G^u)^\perp = \bar{U}_{pe_1 - [u] + 1}$, where the orthogonal is taken with respect to the Kummer pairing $K^\times \times G \rightarrow p\mu$. In particular, $G^u = \{\text{Id}_M\}$ for $u > pe_1$.

This shows that the upper ramification breaks of $M|K$ occur precisely at $-1$, at the $e$ integers in $[1, pe_1]$ which are prime to $p$, and at $pe_1$.

2. The main results. We begin with a brief account (§3) of the congruence satisfied by the absolute norm of a $p$-primary unit, as worked out by S. Pisolkar [12]. Although not a direct consequence of the orthogonality relation, her result was inspired by these ideas.

Next, we provide some applications (§4) of the orthogonality relation. These include the computation of the discriminant of elementary abelian $p$-extensions, the existence of such extensions which given ramification breaks, their possible degrees and their total number.

Guided by the fecund analogy with local function fields, we then look for an orthogonality relation for the Artin-Schreier pairing. The results are proved in §6 and summarised in §5, which should be compared with the review of Kummer theory in §1. We end (§7) with a few remarks about our approach.

For an opinionated presentation of most of the background, see [2], which is freely available online.

3. Absolute norms of $p$-primary numbers. At the Journées arithmétiques in Exeter (1980), J. Martinet generalised the congruence $D \equiv 0, 1 \pmod{4}$ for the absolute discriminant of a number field to a congruence for the absolute norm of the relative discriminant of an extension of number fields. One of his results [9, p. 198] suggested the following local
version: if \( K \mid Q_2 \) is a finite extension containing the \( 2^m \)-th roots of 1, and if \( L \mid K \) is a finite unramified extension, then the discriminant \( d_{L \mid K} \in \mathfrak{o}_K^{\times} \) of any \( \mathfrak{o}_K \)-basis of \( \mathfrak{o}_L \) satisfies \( N_{K \mid Q_2}(d_{L \mid K}) \equiv 1 \pmod{2^{m+1}} \).

More generally, it suggested that the absolute norm of any \( p \)-primary unit in a finite extension \( K \) of \( Q_p \) containing a primitive \( p^m \)-th root of 1 (for some \( m > 0 \)) should be \( \equiv 1 \pmod{p^{m+1}} \), where a unit \( \alpha \) is called \textquoteleft\textquoteleft \( p \)-primary\textquoteright\textquoteright\ if the extension \( K(\sqrt[p]{\alpha}) \) is unramified. This has been verified by S. Pisolkar; we present a variant of her proof.

**Theorem 1** ([12]). — Let \( K \mid Q_p \) be a finite extension for which \( K^{\times} \) has an element of order \( p^m \) \((m > 0)\), and let \( \alpha \in \mathfrak{o}_K^{\times} \) be a unit such that the extension \( K(\sqrt[p]{\alpha}) \mid K \) is unramified. Then \( N_{K \mid Q_p}(\alpha) \equiv 1 \pmod{p^{m+1}} \).

The proof has four ingredients. First, we may assume that \( K = Q_p(\xi_m) \), where \( \xi_m \in K^{\times} \) has order \( p^m \). Assuming this, \( \alpha \) can be written \( \alpha = \beta \gamma^p \), where \( \beta \in U_{p^m} \) and \( \gamma \in \mathfrak{o}_K^{\times} \). Thirdly, \( N_{K \mid Q_p}(\gamma) \equiv 1 \pmod{p^m} \). Finally, \( N_{K \mid Q_p}(\beta) \equiv 1 \pmod{p^{m+1}} \). Granting these, the theorem follows because \( N_{K \mid Q_p}(\gamma)^p \equiv 1 \pmod{p^{m+1}} \) (cf. [2, prop. 27]).

Reduction to the case \( K = Q_p(\xi_m) \). We shall prove that if \( F \) be a finite extension of \( Q_p(\xi_m) \), \( E \) a finite extension of \( F \), and \( a \in \mathfrak{o}_E^{\times} \) a \( p \)-primary unit, then so is \( \mathfrak{N}_{E \mid F}(a) \in \mathfrak{o}_E^{\times} \). We may assume that \( E \mid F \) is either unramified or totally ramified.

If \( E \mid F \) is unramified, then ([2, prop. 37]) the image of \( N_{E \mid F}(a) \) in \( F^{\times}/F^{\times p} \) lies in the \( F_p \)-line which gives the unramified degree-\( p \) extension of \( F \) ([2, prop. 17]).

Suppose that \( E \mid F \) is totally ramified, and let \( E' \mid E \) be an unramified extension such that \( a = b^p \) for some \( b \in E'^{\times} \). There exists an unramified extension \( E' \mid F \) such that \( E' = EF' \); it suffices to show that \( N_{E \mid F}(a) \in F'^{\times p} \).

Indeed, \( N_{E' \mid F'}(a) = N_{E' \mid F'}(b)^p \); the first equality holds because \( N_{E' \mid F'}(a) \) (resp. \( N_{E' \mid F'}(b)^p \)) is the determinant of the multiplication-by-\( a \) automorphism of the \( F' \)-space \( E \) (resp. of the \( F' \)-space \( E' = E \otimes_F F' \)).

From now on, we may and do assume that \( K = Q_p(\xi_m) \) and denote by \( (U_n)_n \) (resp. \( (V_n)_n \)) the filtration on \( K^{\times} \) (resp. \( Q_p^{\times} \)).

One may write \( \alpha = \beta \gamma^p \) \((\beta \in U_{p^m}, \gamma \in \mathfrak{o}_K^{\times})\). This follows from the fact that \( \alpha \in \mathfrak{U}_{pe^1} \) ([2, prop. 17]) and the fact that \( e = \varphi(p^m) = p^{m-1}(p-1) \) ([2, prop. 23]).

Proof that \( N_{K \mid Q_p}(\mathfrak{o}_K^{\times}) = V_m \). We borrow the argument from [1, p. 208]; see also [11, p. 45]. As \( K \mid Q_p \) is a totally ramified abelian extension of degree \( \varphi(p^m) \) (cf. [2, prop. 23]), the subgroup \( N_{K \mid Q_p}(\mathfrak{o}_K^{\times}) \subset Z_p^{\times} \) has
index \( \varphi(p^m) \) [13, p. 196]. So has the subgroup \( V_m \). It thus suffices to prove the inclusion \( V_m \subseteq N_{K|Q_p}(\sigma_K^x) \) to show their equality.

If \( p \neq 2 \), the raising-to-the-exponent-\( p \) map \( (\cdot)^p \) is an isomorphism \( V_r \rightarrow V_{r+1} \) for every \( r > 0 \) ([2, prop. 30]), so \( V_m = V_1^{\varphi(p^m)} \) and the inclusion \( V_m \subseteq N_{K|Q_p}(\sigma_K^x) \) is clear.

When \( p = 2 \), we may assume that \( m > 1 \) (since \( N_{Q_2|Q_1}(\mathbb{Z}_2^x) = V_1 \)). Squaring is an isomorphism \( V_r \rightarrow V_{r+1} \) for \( r > 1 \) ([2, prop. 30]), so \( V_2^{m-2} = V_m \). But notice that

\[
V_2 = V_3 \cup 5V_3 = V_2^2 \cup 5V_2^2.
\]

Raising to the exponent \( 2^{m-2} \) gives \( V_m = V_2^{\varphi(2^m)} \cup 5^{2^{m-2}}V_2^{\varphi(2^m)} \). Clearly, \( V_2^{\varphi(2^m)} \subseteq N_{K|Q_2}(\sigma_K^x) \). To get the inclusion \( V_m \subseteq N_{K|Q_2}(\sigma_K^x) \), it remains to show that \( 5^{2^{m-2}} \in N_{K|Q_2}(\sigma_K^x) \). Indeed, putting \( i = \xi_m^{2^{m-2}} \), so that \( i^2 = -1 \), we have

\[
N_{K|Q_2}(2 + i) = N_{Q_2|Q_1}(2 + i)^{2^{m-2}} = 5^{2^{m-2}}.
\]

(More generally, let \( F \) is a finite extension of \( Q_p \), let \( \pi \) be a uniformiser of \( F \), and let \( m \) be a positive integer. There is a unique abelian extension \( E|F \) such that \( \pi \in N_{E|F}(E^x) \) and \( N_{E|F}(\sigma_E^x) = U_m \cdot E \) [11, p. 45]. When \( F = Q_p \) and \( \pi = p \), then \( E = Q_p(\xi_m) \), in view of the fact that \( N_{E|Q_p}(1 - \xi_m) = p \), and, as we have just seen, \( N_{E|Q_p}(\sigma_E^x) = V_m \).)

**Proof that** \( N_{K|Q_p}(U_{p^m}) \subseteq V_{m+1} \). We adopt the notation \( \phi = \phi_{K|Q_p} \) and \( \psi = \phi^{-1} \) ("Hasse-Herbrand") for the piecewise-linear increasing bijections of \([-1, +\infty[ \) relative to the (galoisian) extension \( K|Q_p \) [13, p. 73] and use the fact that \( N_{K|Q_p}(U_{\psi(m)+1}) \subseteq V_{m+1} \) [13, p. 91]. It thus suffices to show that \( \psi(m) = p^m - 1 \), or, equivalently, that \( \phi(p^m - 1) = m \).

The upper ramification subgroups of \( G = (\mathbb{Z}/p^m\mathbb{Z})^x = \text{Gal}(K|Q_p) \) are given by \( G^w = \text{Ker}((\mathbb{Z}/p^m\mathbb{Z})^x \rightarrow (\mathbb{Z}/p^w\mathbb{Z})^x) \) for \( w \in [0, m] \) [13, p. 79]; notice that \( G^1 = G^0 \) when \( p = 2 \) because \( F_2^x \) is trivial. The orders are \( g^0 = p^m - 1 \) and \( g^w = p^m - w \) for \( w \in [1, m] \). The lower indexing is given by

\[
G_u = \begin{array}{cccc}
0 & 1, p & [p, p^2] & \cdots & [p^{m-1}, +\infty[ \\
\end{array}
\]

(Incidentally, this gives the valuation of the absolute discriminant of \( K \) as \( (p^m - p^{m-1} - 1) + (p - 1)(p^{m-1} - 1) + \cdots + (p^{m-1} - p^{m-2})(p - 1) \), which equals \( mp^m - p^{m-1} \) and vanishes precisely when \( m = 1, p = 2 \). Notice that when \( m > 1 \), \( K \) is a ramified degree-\( p \) kummerian extension of \( F = Q_p(\xi_m^p) \); its unique ramification break occurs at \( p^{m-1} - 1 \) (cf. [2],
This can also be seen directly by remarking that 
\[ o_K = o_F[\zeta_m] \] ([2], prop. 23), that, \( \sigma \) being a generator of \( \text{Gal}(K|F) \), one has \( \sigma(\zeta_m) = \zeta \zeta_m \) for some order-\( p \) element \( \zeta \in K^\times \), and that the valuation of \( 1 - \zeta \) in \( K \) is \( p^{m-1} \). Therefore

\[ v(\sigma(\xi_m) - \xi_m) = v(\zeta \xi_m - \xi_m) = v(1 - \zeta) = p^{m-1} \]

and the ramification break occurs at \( p^{m-1} - 1 \) [13, p. 61]. Equivalently (cf. [2] prop. 60) in view of the fact that \( e_1 = p^{m-1} \), the image of \( \xi_m \) in \( K^\times/K^{\times p} \) is in \( \overline{U}_1 \) but not in \( \overline{U}_2 \).

Let \( g_u \) be the order of \( G_u \). Recall that \( g_0 \phi(n) = g_1 + \cdots + g_n \) [13, p. 73]. We thus have, for every integer \( n \in [1, m] \),

\[ g_0.\phi(p^n - 1) = (g_1 + \cdots + g_{p-1}) + \cdots + (g_{p^{n-1}} + \cdots + g_{p^n}) \]

\[ = (p - 1).g^1 + \cdots + (p^n - p^{n-1}).g^n \]

\[ = n.(p^n - p^{n-1}) = n.g_0 \]

and hence \( \phi(p^n - 1) = n \); in particular \( \phi(p^m - 1) = m \), as was to be proved.

(The same result can also be derived directly from the integral expression for \( \psi \) ([13, p. 74] or (3) below), which gives \( \psi(1) = p - 1 \), and, recursively, \( \psi(n + 1) = \psi(n) + p^n(p - 1) \) for \( n \in [1, m[ \).

This completes the proof of Pisolkar’s result saying that the absolute norm of a \( p \)-primary unit in a finite extension \( K|Q_p \) containing a primitive \( p^m \)-th root of 1 for some \( m > 0 \) is \( \equiv 1 \) (mod. \( p^{m+1} \)). The case \( p = 2 \) of th. 1 provides a purely local proof of Martinet’s generalisation [9] of Stickelberger’s congruence.

4. Elementary abelian \( p \)-extensions. Let \( K \) be a finite extension of \( Q_p \) containing a primitive \( p \)-th root of 1, of ramification index \( e \) and residual degree \( f \); put \( e_1 = e/(p - 1) \) and \( q = p^f \). Let \( M = K(\sqrt[p^m]{K^\times}) \) be the maximal kummerian extension of \( K \) of exponent \( p \). Let us first show, using the orthogonality relation (§1), that the valuation of the different of \( M|K \) is

\[ v_M(\mathcal{D}_{M|K}) = (1 + pe_1)pq^e - b(e+1) - 1, \]

where \( b(e+1) \) is the biggest break in the ramification filtration in the lower numbering on \( G = \text{Gal}(M|K) \); the lower breaks \( b(i) \) are computed in prop. 3.

The orthogonality relation basically says that the filtration in the upper numbering \( (G^n)_{n \in [-1, +\infty[} \) on \( G \) is given by \( G^{-1} = G, G^0 = G^1, \)

\[ G^n = \bigcup_{pe_1 - n + 1} \]

\[ (n \in [1, pe_1 + 1]) \]
with the convention that \( \hat{U}_0 = K^\times / K^{\times p} \); in particular, \( G^{pe_1 + 1} = \{ \text{Id}_M \} \). Here the orthogonal is taken with respect to the Kummer pairing \( K^\times / K^{\times p} \times G \to \mu_p \).

It follows from (2), the fact that the pairing is perfect (which implies that \( \dim_{\mathbb{F}_p} \hat{U}_m + \dim_{\mathbb{F}_p} \hat{U}_m^\perp = 2 + ef \) for every \( m \)), and our knowledge of \( \text{Card} \ \hat{U}_m \) (\cite{2}, prop. 42) that, for \( n \in [-1, +\infty[ \),

\[
(G^n : G^{n+1}) = \begin{cases} 
1 & \text{if } n > pe_1, \\
p & \text{if } n = pe_1, \\
1 & \text{if } n < pe_1 \text{ and } p \mid n, \\
p & \text{if } n = -1, \\
q & \text{otherwise.}
\end{cases}
\]

In the notation from §1, this information can be summarised in one line:

\[
\{1\} \subset_1 G^{pe_1} \subset_f \cdots \subset_f G^{pi+1} = G^{pi} \subset_f \cdots \subset_f G^1 = G^0 \subset_1 G,
\]

where \( i \) is any integer in \([1, e_1[\), and “\( \subset \)” means “codimension-\( r \)”.

Thus, for \( n \in [1, pe_1[ \), we have \( \dim_{\mathbb{F}_p} G^n = 1 + \left( e - n + 1 + \left\lfloor \frac{n-1}{p} \right\rfloor \right) f \); cf. \cite[prop. 43]{2}. Here \( e = (p - 1)e_1 \) is the ramification index, and \( f \) the residual degree, of \( K|Q_p \). In particular, \( G^1 \) (resp. \( G^{pe_1} \)) has order \( pq^e \) (resp. \( p \)).

The upper ramification breaks occur therefore at \(-1\), at the \( e \) integers \( 1 = b^{(1)} < b^{(2)} < \cdots < b^{(e)} = pe_1 - 1 \) in \([1, pe_1[ \) which are prime to \( p \), and at \( b^{(e+1)} = pe_1 \). The order of the group drops by a factor of \( p \) at \(-1\), by a factor of \( q = p^f \) at each of the \( b^{(i)} \) \( (i \in [1, e[) \), and by a factor of \( p \) at \( pe_1 \). Consequently, we have the following table for the index of \( G^n \) in \( G^0 \):

\[
\begin{array}{cccccc}
  n \in & \{0, 1\} & [b^{(1)}, b^{(2)}] & [b^{(2)}, b^{(3)}] & \cdots & [b^{(e-1)}, b^{(e)}] & \{pe_1\} \\
(G^0 : G^n) = & 1 & q & q^2 & \cdots & q^{e-1} & q^e
\end{array}
\]

The lower ramification breaks occur therefore at \(-1\) and at the \( e + 1 \) integers \( b_{(i)} = \psi(b^{(i)}) \) \( (i \in [1, e + 1[) \), where \( \psi = \psi_{M|K} \) is the function on \([-1, +\infty[ \) satisfying

\[
\psi(w) = \int_0^w (G^0 : G^t) \, dt
\]

\cite[p. 74]{13}. In view of the above table, it follows that \( b_{(1)} = 1 \) and that \( b_{(i+1)} = b_{(i)} + (b^{(i+1)} - b^{(i)})q^i \) for \( i \in [1, e[ \). This may also be verified using the formula \( g_0 \psi(r) = g_1 + g_2 + \cdots + g_r \) \cite[p. 73]{13}, where \( g_n = \text{Card} \ G_n \).
(Notice that the $b_{(i)}$ are all $\equiv 1 \pmod{p}$, cf. [13, p. 70]).

The $b_{(i)}$ can be computed recursively, starting from $b_{(1)} = 1$. Explicitly, for $i \in [1, e]$, we have

$$b_{(i+1)} - b_{(i)} = \begin{cases} 1 & \text{if } i = e \text{ or } i \not\equiv 0 \pmod{(p - 1)} \\ 2 & \text{if } i \not= e \text{ and } i \equiv 0 \pmod{(p - 1)}. \end{cases}$$

(Notice that $(p - 1) \mid i$ is equivalent to $p \mid (b_{(i)} + 1)$.) Therefore we have

$$b_{(i)} = (1 + q + \cdots + q^{i-1}) + (q^{p-1} + \cdots + q^{a(i)(p-1)}),$$

with $a(i)$ the integral part of $(i - 1)/(p - 1)$, for $i \in [1, e]$, and $b_{(e+1)} = b_{(e)} + q^e$. These are the expressions obtained in [3, p. 287], albeit in the special case when $K = F(\sqrt[p]{F})$ for some (finite) extension $F|Q_p$.

To compute the valuation of the different $\mathcal{D}_{M|K}$ of $M|K$, it now suffices to recall that the order $g_n$ of the ramification subgroup $G_n$ is

$$g_n = \begin{array}{cccc} n \in & \{0, 1\} & |b_{(1)}, b_{(2)}| & |b_{(2)}, b_{(3)}| & \cdots & |b_{(e-1)}, b_{(e)}| & |b_{(e)}, b_{(e+1)}| \\ & pq^e & pq^{e-1} & pq^{e-2} & \cdots & pq & p \end{array}$$

As the valuation of the different $\mathcal{D}_{M|K}$ is $\sum_{n \in N} (g_n - 1)$ [13, p. 64], we get

$$v_M(\mathcal{D}_{M|K}) = (1 + p e_1) pq^e - b_{(e+1)} - 1,$$

the expression claimed in (1). This expression follows from — and indeed led to — the following lemma.

**Lemma 2.** Let $E|F$ be a finite galoisian extension of local fields, of group $G = \text{Gal}(E|F)$. Suppose that the filtration $(G^w)_{w \in [-1, +\infty[}$ has $m > 0$ positive breaks $b^{(1)}, \ldots, b^{(m)}$. Let $b_{(i)} = \wp_{E|F}(b^{(i)})$, for $i \in [1, m]$, be the breaks in the lower numbering. Then the valuation of the different of $E|F$ is $v_E(\mathcal{D}_{E|F}) = (1 + b^{(m)}) g_0 - (1 + b^{(m)})$, where $g_0 = \text{Card} G_0$ is the order of the inertia subgroup.

For $w \in [0, +\infty[$, denote the index $(G^0 : G^w)$ of $G^w$ in $G^0$ as follows:

$$w \in [0, b^{(1)}] \quad |b^{(1)}, b^{(2)}| \quad \cdots \quad |b^{(m-1)}, b^{(m)}| \quad (G^0 : G^w) = 1 \quad h^{(1)} \quad \cdots \quad h^{(m-1)}$$

We have $b_{(i+1)} = b_{(i)} + (b_{(i+1)} - b_{(i)}) h^{(i)}$ for every $i \in [1, m[$. The cardinality of the subgroups $G_t$ ($t \in [0, +\infty]$) of $G_0$ in the lower numbering is:

$$t \in [0, b^{(1)}] \quad |b^{(1)}, b^{(2)}| \quad \cdots \quad |b^{(m-1)}, b^{(m)}| \quad \text{Card} G_t = g_0 \quad g_0/h^{(1)} \quad \cdots \quad g_0/h^{(m-1)}$$
Now, for \( i \in [1, m[ \), the contribution of the interval \([b_{(i)}, b_{(i+1)}]\) to the sum
\[\sum_{n \in \mathbb{N}} (\text{Card } G_n - 1)\]
is
\[(b_{(i+1)} - b_{(i)}) \left( \frac{g_0 - h^{(i)}}{h^{(i)}} \right) = (b_{(i+1)} - b_{(i)}) h^{(i)} \left( \frac{g_0 - h^{(i)}}{h^{(i)}} \right)\]
\[= (b_{(i+1)} - b_{(i)}) \left( g_0 - h^{(i)} \right),\]
and hence \( v_E(\mathfrak{D}_{E|F}) \), the sum over the contributions of these \( m - 1 \) intervals
\([b_{(i)}, b_{(i+1)}]\) \( (i \in [1, m[ \) and of the initial segment \([0, b_{(1)}]\) of \( 1 + b_{(1)} \) points
is given by
\[(1 + b_{(1)} + b^{(m)} - b^{(1)}) g_0 - b_{(m)} - 1 = (1 + b^{(m)}) g_0 - (1 + b_{(m)}),\]
proving lemma 2. (The case \( m = 1 \) implies [2, prop. 60], in view of fact
that for an \( F_p \)-line \( D \subset \hat{U}_c, D \not\subset \hat{U}_{c+1}, \) the unique ramification break of the
kummerian extension \( K(\sqrt[n]{D})|K \) occurs at \( b^{(1)} = b_{(1)} = pe_1 - c \) if \( c \neq pe_1 \).

Example. — Take \( F = Q_p \) and \( E = Q_p(\xi_a) \), where \( \xi_a \) is a primitive
\( p^a \)-th root of 1 for some \( a > 0 \). When \( p \neq 2 \), we have \( m = a, \)
\( b^{(1)} = b_{(1)} = 0, b^{(m)} = a - 1, b_{(m)} = p^{a-1} - 1 \) and \( g_0 = \varphi(p^a) \). Therefore
\( v_E(\mathfrak{D}_{E|F}) = a \varphi(p^a) - p^{a-1} \). Consider now \( p = 2 \). If \( a = 1 \), the extension
\( E|F \) is trivial and the lemma does not apply; nor do we need to apply it.
For \( a > 1 \), the only change is that \( m = a - 1, b^{(1)} = b_{(1)} = 1 \), leading to
the same result: \( v_E(\mathfrak{D}_{E|F}) = a \varphi(2^a) - 2^{a-1} = (a - 1)2^{a-1} \).

Let us summarise what we have learnt about our maximal kummerian extension \( M|K \) of exponent \( p \). Let \( a(i) = \left\lfloor \frac{i - 1}{p - 1} \right\rfloor \).

Proposition 3. — The \( e + 1 \) positive ramification breaks of \( M|K \) occur at
\( b^{(i)} = i + a(i) \) for \( i \in [1, e[ \), and at \( b^{(e+1)} = pe_1 \), in the upper numbering.
In the lower numbering, they occur at
\[(4) \quad b_{(i)} = (1 + q + \cdots + q^{i-1}) + (q^{p-1} + \cdots + q^{a(i)(p-1)}) \quad (i \in [1, e[)\]
and at \( b_{(e+1)} = b_{(e)} + q^e \). We have \( v_M(\mathfrak{D}_{M|K}) = (1 + pe_1)pq^e - (1 + b_{(e+1)}) \)
and \( v_K(d_{M|K}) = p.v_M(\mathfrak{D}_{M|K}) \).

The statement about the discriminant follows from the fact that the
residual degree of \( M|K \) is \( p \). Notice that \( a(e) = e_1 - 1 \), so that \( b_{(e+1)} = \)
\((1 + q + q^2 + \cdots + q^e) + (q^{p-1} + q^{2(p-1)} + \cdots + q^{(e-1)(p-1)}) \), with \( q = p^f \).

Example. — Take \( K = Q_p(\zeta) \), where \( \zeta \) is a primitive \( p \)-th root of 1.
Then \( e = p - 1, e_1 = 1, \) and \( f = 1 \). The \( p \) ramification breaks of \( M|K \) are
1, 2, ... , p in the upper numbering; 1, 1 + p, ... , 1 + p + p^2 + · · · + p^{p-1} in the lower numbering. Therefore \( v_M(\mathfrak{M}|K) = p^{p+1} + p^p - p^{p-1} - \cdots - p - 2 \).

**Example.** — The last result of [3, p. 287] can be recovered by taking 
\( K = F(\sqrt[p]{F} \times) \), where \( F \) is any finite extension of \( \mathbb{Q}_p \). Keep the notation \( e = (p-1)e_1 \) and \( q = p^j \) relative to \( K \). We have \( v_K(\mathfrak{D}|F) = p-2 \), so the valuation of \( \mathfrak{D}|F = \mathfrak{D}|K \mathfrak{D}|K|F \) [13, p. 51] is

\[
v_M(\mathfrak{D}_M|F) = v_M(\mathfrak{D}_M|K) + v_M(\mathfrak{D}_K|F)
= ((1 + pe_1)pq^e - b_{e+1}) - 1 + pq^e(p - 2)
= (e_1 + 1)p^2q^e - pq^e - (1 + b_{e+1}).
\]

It is also possible to deduce the following result of J.-M. Fontaine.

**Corollary 4 ([5, p. 362]).** — Let \( F|Q_p \) be a finite extension, \( E|F \) a totally ramified elementary abelian \( p \)-extension. Every upper ramification break \( u \) of \( E|F \) is in \([1, pe_1]\) and is prime to \( p \), unless \( u = pe_1 \). The order of \( \text{Gal}(E|F)^{pe_1} \) is 1 or \( p \).

One may add that if \( pe_1 \) occurs for some \( E|F \), then \( F \) contains a primitive \( p \)-th root \( \zeta \) of 1 (cf. [2, prop. 63]) and there is a uniformiser \( \pi \) of \( F \) such that \( \pi \in E^{\times p} \).

One may ask for a converse: which sequences do occur as the upper ramification breaks of an elementary abelian \( p \)-extension \( E|F \) ? We may ask for the maximal degree \( [E : F] \) when there is a single break. We may ask for the number of extensions with given ramification breaks. If \( \zeta \in F \), these questions can be answered by Kummer theory; see below. If \( \zeta \notin F \), we may reduce to the previous case by considering the extension \( E(\zeta) | F(\zeta) \), as in the proof of [2, prop. 63]. Alternatively, one may appeal to local class field theory, to which we turn in a moment.

**The existence of exponent-\( p \) kummerian extensions with given upper ramification breaks.** Suppose that \( K \) is a finite extension of \( Q_p(\zeta) \). Every strictly increasing sequence \( u_1 < u_2 < \cdots < u_n (n > 0) \) of numbers which are in \([1, pe_1]\), with the possible exception of \( u_1 \), which can be \(-1\), and which are all prime to \( p \), with the possible exception of \( u_n \), which can be \( pe_1 \), is the sequence of upper ramification breaks of some exponent-\( p \) kummerian extension \( L \) of \( K \).

Note first that we need only consider the case \( u_1 \neq -1 \). For if \( L_1 \) is the unique unramified degree-\( p \) extension of \( K \), and if \( L_2 \) is a (totally ramified) exponent-\( p \) kummerian extension with upper ramification breaks \( u_2, \ldots, u_n \), then \( L = L_1L_2 \) has ramification breaks \(-1, u_2, \ldots, u_n \) in the upper numbering. Assume therefore that \( u_1 > 0 \).
We may look for $L$ inside the maximal exponent-$p$ kummerian extension $M = K(\sqrt[p]{K^\times})$ of $K$. Equivalently, we look for a subgroup $H$ of $G = \text{Gal}(M|K)$ such that

$$G/H = (G/H)^{-1} = \cdots = (G/H)^{u_1} \neq (G/H)^{u_1+1} = \cdots = (G/H)^{u_2}$$
$$\neq (G/H)^{u_2+1} = \cdots = (G/H)^{u_3}$$
$$\cdots\cdots$$
$$\neq (G/H)^{u_{n-1}+1} = \cdots = (G/H)^{u_n}$$
$$\neq (G/H)^{u_n+1} = \{\bar{1}\}$$

and take $L = M^H$. In view of the compatibility of the upper-numbering filtration with the passage to the quotient, we have $(G/H)^i = G^iH/H$, and we are led to require

$$G = G^{-1}H = \cdots = G^{u_1}H \neq G^{u_1+1}H = \cdots = G^{u_2}H$$
$$\neq G^{u_2+1}H = \cdots = G^{u_3}H$$
$$\cdots\cdots$$
$$\neq G^{u_{n-1}+1}H = \cdots = G^{u_n}H$$
$$\neq G^{u_n+1}H = H.$$

If we identify $G$ with $\tilde{U}_0 = K^\times/K^{\times p}$ using the reciprocity isomorphism, and recall the structure of the filtered $\mathbb{F}_p$-space $\tilde{U}_0$ ([2, prop. 42]), we may conclude that such a subspace $H \subset G$ exists always.

However, this appeal to local class field theory can be avoided when $\zeta \in K$, as here. Appeal can be made instead to the orthogonality relation (2), $G^{n\perp} = \tilde{U}_{pe_1-n+1}$ for $n \in [1, pe_1+1]$. So we look for a subspace $D \subset \tilde{U}_0$ such that, writing $D_i$ for $\tilde{U}_i \cap D$, we have

$$\{\bar{1}\} = D_{pe_1} = \cdots = D_{pe_1-u_1+1} \neq D_{pe_1-u_1} = \cdots = D_{pe_1-u_2+1}$$
$$\neq D_{pe_1-u_2} = \cdots = D_{pe_1-u_3+1}$$
$$\cdots\cdots$$
$$\neq D_{pe_1-u_{n-1}} = \cdots = D_{pe_1-u_n+1}$$
$$\neq D_{pe_1-u_n} = \cdots = D_0 = D$$

and take $H = D^{\perp}$. In view of [2, prop. 42], it is clear that such a $D$ exists always. It is also clear how to get every such $D$, and how to count the number of such $D$. (Notice that every such $D$ is an $\mathbb{F}_p$-point of a certain open subvariety of the $\mathbb{F}_p$-variety of all subspaces of $\tilde{U}_0$.)

More explicitly, choose, for every $i \in [1, n]$, a $d_i$-dimensional ($d_i > 0$) sub-$\mathbb{F}_p$-space $E_i \subset \tilde{U}_{pe_1-u_i}$ whose intersection with $\tilde{U}_{pe_1-u_i+1}$ is $\{\bar{1}\}$; such
choices are possible in view of [2, prop. 42]. Finally take \( D = E_1 E_2 \cdots E_n \),
so that \( D_{pe_1-u_n} = E_1 E_2 \cdots E_i \). The dimension of \( E_i \) is between 1 and \( f \),
except when \( i = n \) and \( u_n = pe_1 \), where \( \dim_{E_{pe_1}} E_{pe_1} = 1 \).

The degree of a totally ramified exponent-\( p \) kummerian extension with
a single ramification break. Let \( L/K \) be such an extension, and let \( u \) be
the unique ramification break. As we have seen, \( [L : K] = p^j \), where \( j = 1 \) if
\( u = pe_1 \) and \( j \in [1, f] \) if \( u \neq pe_1 \); in the latter case, there are extensions
with any preassigned \( j \) in \([1, f]\). The valuation of the discriminant is
\( v_K(d_{L|K}) = (p^j - 1)(1 + u) \); the case \( j = 1 \) is [2, cor. 64].

The number of exponent-\( p \) kummerian extensions with a given sequence
of upper ramification breaks. It is clear that every such extension \( L/K \) with
given upper ramification breaks \( u_1 < u_2 < \ldots < u_n \) as above arises as
\( L = K(\sqrt[p]{E_1 E_2 \cdots E_n}) \) for some choice of subspaces \( E_i \). For a different
choice \( E'_i \) of the subspaces, we get the same extension if and only if
\( E_1 E_2 \cdots E_n = E'_1 E'_2 \cdots E'_n \). This leads to a mildly complicated
problem — how many subspaces \( D \subset \bar{U}_0 \) are there such that \( D_i = D \cap \bar{U}_i \)
satisfy the conditions displayed above? — which can be solved in any
given instance; see [2, cor. 66] for the degree-\( p \) cyclic case. At the other
extreme, when \( n = e + 1 \) and \( d_i \) are as large as they can be, namely \( d_i = f \)
for \( i \in [1, e] \) and \( d_{e+1} = 1 \), a subspace \( D \subset \bar{U}_0 \) is a solution if and only if
it is a hyperplane not containing the line \( \bar{U}_{pe_1} \).

Remark. — Let \( D, D' \) be two distinct lines in \( \bar{U}_b \) neither of which is
contained in \( \bar{U}_{b+1} \). If the plane \( DD' \) meets \( \bar{U}_{b+1} \) only at the origin \( \{1\} \),
then the extension \( L = K(\sqrt[p]{DD'}) \) has a single ramification break, namely
\( pe_1 - b \). Otherwise, let a be the largest integer such that \( DD' \cap \bar{U}_a \neq \{1\} \); we have \( a \in [b, pe_1] \). The extension \( L/K \) now has two ramification breaks,
namely \( pe_1 - a \) and \( pe_1 - b \) if \( a \neq pe_1 \), and \(-1 \) and \( pe_1 - b \) if \( a = pe_1 \). For
an extreme example, where \( b = 0 \) and \( a = pe_1 \), see [2, ex. 51].

The valuation of the different of an exponent-\( p \) kummerian extension
with given upper ramification breaks. Let \( u_1 < u_2 < \ldots < u_n \) be a
strictly increasing sequence of numbers in \([1, pe_1]\) which are all prime
to \( p \) except possibly \( u_n \), which can be \( pe_1 \); choose \( E_i \) as above and let
\( L = K(\sqrt[p]{E_1 E_2 \cdots E_n}) \); the upper ramification breaks of \( L/K \) occur at
\( u_1, u_2, \ldots, u_n \). Let \( d_i \) be the dimension of \( E_i \), so that \( \text{Card} E_i = p^{d_i} \). The
valuation of the different of the extension \( L/K \) is given by
\( v_L(\mathfrak{D}_{L|K}) = (1 + u_n)p^{d_1 + d_2 + \cdots + d_n} - (1 + \psi_{L|K}(u_n)) \) (lemma 2), where
\[
\psi_{L|K}(u_n) = u_1.1 + (u_2 - u_1).p^{d_1} + \cdots + (u_n - u_{n-1}).p^{d_1 + d_2 + \cdots + d_{n-1}},
\]
as follows from the definition of \( \psi_{L|K} \) (3) and the following piece of

11
information about the ramification filtration on $G = \text{Gal}(L|K)$:

$$j \in [0, u_1] \cup [u_1, u_2] \cup [u_2, u_3] \cdots \cup [u_{n-1}, u_n]$$

$$(G^0 : G^j) = 1 \quad p^{d_1} \quad p^{d_1 + d_2} \cdots \quad p^{d_1 + d_2 + \cdots + d_{n-1}}$$

Notice that $v_L(\mathfrak{D}_{L|K})$ depends on the subspaces $E_1, E_2, \ldots, E_n$ only via the breaks $u_i$ and the dimensions $d_i$.

**Example.** — For an example of the smallest possible degree having every possible upper ramification break, take $n = e + 1$, $u_i = b(i) = i + a(i)$ for $i \in [1, e]$, $u_{e+1} = pe_1$, and $d_i = 1$ for every $i$. In the lower numbering, the breaks occur at

$$l_i = (1 + p + \cdots + p^{i-1}) + (p^{p-1} + \cdots + p^{a(i)(p-1)}) \quad (i \in [1, e])$$

and at $l_{e+1} = l_e + p^e$. We have $v_L(\mathfrak{D}_{L|K}) = (1 + pe_1)p^{e+1} - (1 + l_{e+1})$ (lemma 2).

Let us summarise a part of our discussion of exponent-$p$ kummerian extensions with given ramification breaks.

**Proposition 5.** — Every strictly increasing sequence $u_1 < u_2 < \ldots < u_n$ of integers in $[1, pe_1]$ which are all prime to $p$, with the possible exception of $u_n$, which can be $pe_1$, is the sequence of upper ramification breaks of some (totally ramified) exponent-$p$ kummerian extension $L$ of $K$. If a break occurs at $pe_1$, then there is a uniformiser $\pi$ of $K$ such that $\pi \in L^{\times p}$ and conversely. When there is a single break $u$, there are $f$ possibilities for the degree $[L : K]$ if $u \neq pe_1$, namely $p, p^2, \ldots, p^f$, but only one possibility, namely $[L : K] = p$, if $u = pe_1$.

***

Now suppose that $F|Q_p$ is a finite extension for which $F^{\times}$ does not have an element of order $p$ (and hence $p \neq 2$), let $N$ be the maximal abelian extension of $F$ of exponent $p$, and let $G = \text{Gal}(N|F)$. Local class field theory provides an isomorphism $F^{\times}/F^{\times p} \to G$ under which $U_n$ surjects onto $G^n$ for every $n > 0$; thus the inertia group $G^0 = G^1$ has order $q^e$ and index $p$ in $G$. There are now only $e$ positive ramification breaks [2, prop. 42]; in the upper numbering, they occur at the $e$ integers $b(i) = i + a(i)$ ($i \in [1, e]$) in the interval $[1, pe_1]$ which are prime to $p$. At each break, the order of the group drops by a factor of $q$, so the $e$ lower ramification breaks are as given in (4). The valuation of the different is $v_N(\mathfrak{D}_{N|F}) = (1 + b(e))q^e - (1 + b(e))$ (lemma 2).

Here and elsewhere, we have made use of the following elementary fact: for every prime $p$ and every integer $m > 0$, the $c(m) = m - \lfloor m/p \rfloor$ integers
in \([1, m]\) which are prime to \(p\) constitute the image of the strictly increasing function \(b^{(i)} : [1, c(m)] \rightarrow [1, m]\), \(b^{(i)} = i + a(i)\).

**Example.** — Take \(F = \mathbb{Q}_p\) \((p \neq 2)\). There is a unique positive ramification break, at \(b^{(1)} = b^{(2)} = 1\) (see the parenthetical remark before cor. 64 of [2]), so \(v_N(\mathcal{D}_{N|F}) = 2(p - 1)\). (When \(p = 2\), \(N\) is the maximal kummerian extension of \(\mathbb{Q}_2\), which has been treated earlier.)

Let \(K\) be a finite extension of \(\mathbb{Q}_p\). In principle, the determination of the functions \(\phi_{L|K}, \psi_{L|K}\), for any finite galoisian extension \(L|K\) can be reduced to the case treated in [2, prop. 63] (cyclic of degree \(p\)), and indeed to the case treated in [2, cor. 62] (kummerian of degree \(p\)). But a little bit of local class field theory makes life simpler.

**Example.** — Let \(K\) be any finite extension of \(\mathbb{Q}_p\), let \(m > 0\) be an integer, let \(\pi\) be a uniformiser of \(K\), let \(L\) be the unique totally ramified abelian extension of \(K\) such that \(\pi \in N_{L|K}(L^\times)\) and \(N_{L|K}(\varpi_L) = U_m\), and let \(G = \text{Gal}(L|K) = \varpi_K/U_m\). (If \(K = \mathbb{Q}_p\) and \(\pi = p\), then \(L = \mathbb{Q}_p(\xi_m)\), which we have treated in \(\S\ 1\).) Then \(G^{-1} = G^0\) has order \(q^{m-1}(q - 1)\), and the index of \(G^0\) in \(G^0\) is:

| \(n\) | 0 | 1 | 2 | \(\cdots\) | \(m\) |
|-------|---|---|---|--------|-----|
| \(G^0 : G^n\) | 1 | \(q-1\) | \(q(q-1)\) | \(\cdots\) | \(q^{m-1}(q-1)\) |

If the residual cardinality \(q\) of \(K\) is \(\neq 2\), there are \(m\) ramification breaks, \(b^{(i)} = i - 1\) for \(i \in [1, m]\), in the upper numbering; in the lower numbering, they are \(b^{(1)} = 0, b^{(2)} = q - 1, \ldots, b^{(m)} = q^{m-1} - 1\). In particular, \(v_K(d_{L|K}) = mq^{m-1}(q - 1) - q^{m-1}\), which is independent of \(\pi\).

If \(q = 2\) and \(m > 1\), there are only \(m - 1\) breaks, at \(b^{(i)} = i\) for \(i \in [1, m]\). If \(q = 2\) and \(m = 1\), the extension is trivial. The expression for the valuation of the discriminant remains the same in all cases; cf. [7, p. 110].

**Remark.** — Take \(m = 1\), so that \(L|K\) is cyclic of degree \(q - 1\), and hence, being totally ramified, obtained by adjoining \(\sqrt[q]{\varpi}\) for some uniformiser \(\varpi\), uniquely determined up to 1-units. Which uniformiser? The answer for \(K = \mathbb{Q}_p\) and \(\pi = p\) is \(\varpi = -p\) [2, prop. 24]. We have \(\varpi = -\pi\) in general, for \(L\) is the splitting field of \(T^q + \pi T\) [11, p. 61]. Turning things around, we may say that if \(L = K(\sqrt[q]{\varpi})\) for some uniformiser \(\varpi\) of \(K\), then \(N_{L|K}(L^\times)/K^\times q^{-1}\) is the subgroup of \(K^\times/K^\times q^{-1}\) generated by the image of \(-\varpi\) (which is the same as the image of \(\varpi\) when \(q\) is even, for then \(-1 = (-1)^{q-1}\)). It reflects the fact that the product of all elements in the multiplicative group \(k^\times\) of the residue field is \(-1\); indeed, the conjugates of \(\sqrt[q]{\varpi}\) are \(u \sqrt[q]{\varpi}\), as \(u\) runs through \(k^\times\).
Recall that the Hilbert symbol $K^×/K^q× → q−1μ$ has the property that if $L = K(q−1√D)$ for some subgroup $D ⊂ K^×/K^q−1$, then $D⊥ = NL|K(L×)/K^q−1$ [4, p. 144]. This means that $D⊥ = D^{−1}$, where $D^{−1} ⊂ K^×/K^q−1$ is the subgroup generated by $a ∈ K^×$. It might be added that for $D = o×/o^q−1 = k×$, we have $D⊥ = D$, because $K(q−1√D)$ is the unramified extension of $K$ of degree $q − 1$.

An application of the Führerdiskriminantenproduktformel. — Let us show how the discriminant of the maximal kummerian extension $M|K$ of exponent $p$ could have been computed by an application of this formula after we had determined the possible ramification breaks $t$ of a degree-$p$ kummerian extension $L|K$ [2, cor. 62] and the number of extensions for which a given break occurs [2, cor. 66], if we knew that the exponent of the conductor of $L|K$ is $t + 1$. Class field theory provides this last bit of knowledge, for it says that, under the reciprocity map, $U_t$ surjects onto $Gal(L|K)$ whereas the image of $U_{t+1}$ is $\{Id_M\}$, so $t + 1$ is the smallest integer $m$ such that $U_m ⊂ NL|K(L×)$.

Recall that the formula in question, applied to an abelian extension $E|F$ of local fields, says that the discriminant ideal $d_{E|F}$ equals $\prod χ|f(χ)$, where the product is taken over all characters $χ : Gal(E|F) → C×$ and $f(χ)$ is the conductor of $χ$ [7, p. 113], [13, p. 104]; of course, only ramified characters need be considered.

To a ramified character $χ$ of $G = Gal(M|K)$ corresponds a ramified degree-$p$ cyclic extension $L = M^{Ker(χ)}$, and each ramified degree-$p$ cyclic extension $L$ arises from $p−1 = Card Aut(pμ)$ characters $χ$. In view of this, it is sufficient to compute $(p−1) ∑_{i=1}^{e+1} (t_i + 1).n_i$, where

$$t_i = i + a(i) \quad (i ∈ [1, e]) \quad ; \quad t_{e+1} = pe$$

are the possible positive ramification breaks [2, cor. 62] and, as shown in [2, cor. 66],

$$n_i = p^{(i−1)f+1} + \cdots + p^{if} = \frac{p(q^i − q^{i−1})}{p−1} \quad (i ∈ [1, e]) \quad ; \quad n_{e+1} = pq^e$$

with $q = p^f$, is the number of ramified degree-$p$ cyclic extensions of $K$ whose ramification break occurs at $t_i$. Now, it is easily seen that

$$(p−1) ∑_{i=1}^{e} n_i = p.(q^e − 1), \quad (p−1) ∑_{i=1}^{e} in_i = p.(eq^e − q^e−1), \quad$$

$$(p−1) ∑_{i=1}^{e} a(i)n_i = p.(e_1 q^e − \frac{q^e−1}{q−1} + 1).$$
where, to compute the last sum, recall that \( a(i) \) is the integral part of \( (i - 1)/(p - 1) \), and hence

\[
a(i + 1) - a(i) = \begin{cases} 
0 & \text{if } i \not\equiv 0 \mod (p - 1) \\
1 & \text{if } i \equiv 0 \mod (p - 1).
\end{cases}
\]

Therefore \( (p - 1) \sum_{i=1}^{e}(t_i + 1)n_i = p. \left( e_1pq^e - \frac{q^e - 1}{q - 1} - \frac{q^e - 1}{q^{p - 1} - 1} \right) \) and, by the F"uhrerdiskriminantenproduktformel, \( v_K(d_{M|K}) \) equals

\[
(5) \quad (p - 1) \sum_{i=1}^{e+1}(t_i + 1)n_i = p. \left( (e_1p^2 + p - 1)q^e - \frac{q^e - 1}{q - 1} - \frac{q^e - 1}{q^{p - 1} - 1} \right),
\]

which is the same as in prop. 3. This computation can be encapsulated in the following lemma.

**Lemma 6.** — Let \( p > 1 \), \( e_1 > 0 \) be integers and \( q > 1 \) real; put \( e = e_1(p - 1) \). For \( i \in [1, e] \), let \( a(i) \) be the integral part of \( (i - 1)/(p - 1) \); define \( t_i = i + a(i) \), \( n_i = p(q^i - q^{i-1})/(p - 1) \) \( (i \in [1, e]) \) and \( t_{e+1} = pe_1 \), \( n_{e+1} = pq^e \). Then \( (p - 1) \sum_{i=1}^{e+1}(t_i + 1)n_i \) is given by (5).

Maximal elementary abelian \( p \)-extensions can be treated in like manner.

It is not surprising that the F"uhrerdiskriminantenproduktformel can compute the discriminant without reference to the lower ramification filtration. Indeed, information about this filtration goes into the proof of the formula [7, p. 113].

**Totally ramified finite abelian \( p \)-extensions and the endomorphism of raising to the exponent \( p \).** Let \( F \) be a finite extension of \( \mathbb{Q}_p \) and denote by \( U_n \) \( (n > 0) \) the groups of higher principal units of \( F \). As always, \( e \) is the absolute ramification index and \( e_1 = e/(p - 1) \).

Let \( G \) be a finite commutative \( p \)-group. A result of Fontaine [5, p. 362] about totally ramified \( G \)-extensions \( E|F \) follows from the study of the raising-to-the-exponent-\( p \) map \( ( )^p \) on the \( \mathbb{Z}_p \)-modules \( U_n \) and the fact that the reciprocity map \( F^\times \to G \) carries \( U_n \) onto \( G^n \). As \( E|F \) is totally ramified, \( G = G^0 \), and as \( G \) is a \( p \)-group, \( G^0 = G^1 \), so \( G \) is essentially a quotient of \( U_1 \) by a (closed) subgroup of finite index.

Recall that the map \( ( )^p \) carries \( U_n \) into \( U_{\lambda(n)} \) [2, prop. 27], where \( \lambda(n) = \inf(pn, n + e) \), and that \( U_{\lambda(n) + 1}^p = U_{\lambda(n)} \) in all cases except when \( F^\times \) has an element of order \( p \) and \( n = e_1 \), in which case \( U_{e_1}^p U_{pe_1 + 1} \) has index \( p \) in \( U_{pe_1} \) [2, prop. 29]. Hence the following result, which sharpens [5, p. 362]. For a subgroup \( A \) of \( G \), denote by \( A^{(p)} \) the image of \( A \) under the endomorphism \( ( )^p \), because the notation \( A^p \) is in conflict with the upper numbering.
Proposition 7. — Let \( E|F \) be a totally ramified finite abelian \( p \)-extension and let \( G = \text{Gal}(E|F) \). Then equality holds in \((G^n)^{(p)}G^{\lambda(n)+1} \subset G^{\lambda(n)}\) in all cases except possibly when \( n = e_1 \); for \( n = e_1 \), the index can be 1 or \( p \). The index-\( p \) case occurs precisely when \( H^{p\lambda_1} \neq \{1\} \), where \( H = G/G^{(p)} \) is the maximal elementary abelian quotient of \( G \).

Write \( L = E^{G^{(p)}} \), so that \( H = \text{Gal}(L|F) \). As \( H \) is an elementary abelian \( p \)-group, we always have \( H^{p\lambda_1+1} = \{1\} \) (cor. 4). Also, if \( H^{p\lambda_1} \neq \{1\} \), then \( \zeta \in F \) and there is a uniformiser \( \pi \) of \( F \) such that \( \pi \in L^{x_\pi} \), and conversely (prop. 5).

Remark finally that, going modulo \( G^{(p)} \), we may assume that \( G \) is elementary abelian, in which case \( H = G \), and we are reduced to cor. 4, and ultimately to [2, prop. 42], whether we use the orthogonality relation (2) or the reciprocity isomorphism.

5. Summary of local Artin-Schreier theory. Let us first summarise our results in the Artin-Schreier theory for local function fields of characteristic \( p \). These results were arrived at by analogy with the Kummer theory of local number fields as recalled in \( \S 1 \), and they may be considered as a refinement of the theory presented in standard textbooks such as [4, Chapter III, \( \S 2 \), p. 74]. The proofs will be given in \( \S 6 \).

Let \( k|F \) be a finite extension, \( f \) its degree, and let \( K \) be a local field with \( k \) as the field of constants (and the residue field). Denote by \( \mathfrak{o} \) the ring of integers of \( K \), and by \( \mathfrak{p} \subset \mathfrak{o} \) the unique maximal ideal of \( \mathfrak{o} \).

The filtration \((\mathfrak{p}^n)_{n\in\mathbb{Z}}\) on the additive group \( K \) by powers of \( \mathfrak{p} \) induces a filtration on the \( F_p \)-space \( \overline{K} = K/\mathfrak{p}(K) \), where \( \mathfrak{p} \) is the endomorphism \( x \mapsto x^p - x \) of \( K \). We denote the induced filtration by \((\overline{\mathfrak{p}^n})_{n\in\mathbb{Z}}\); we have \( \overline{\mathfrak{p}} = \{0\} \) (lemma 8), and the codimension at each step is given by

\[
\{0\} \subset_1 \overline{\mathfrak{o}} \subset_f \overline{\mathfrak{p}^{-1}} \subset_f \cdots \subset_f \overline{\mathfrak{p}^{-mi+1}} = \overline{\mathfrak{p}^{-mi}} \subset_f \cdots \subset \overline{K}
\]

(prop. 9 and 11). Here, \( i \) is any integer \( > 0 \), and, as before, an inclusion of \( F_p \)-spaces \( E \subset \mathfrak{e} \) means that \( E \) is a codimension-\( r \) subspace of \( \mathfrak{e} \).

There is a canonical isomorphism \( \overline{\mathfrak{o}} \to F_p \) sending \( a \) to \( S_{k|F_p}(\hat{b}) \), where \( \hat{b} \) is the image in \( k/\mathfrak{p}(k) \) of a representative \( b \in \mathfrak{o} \) of \( a \); the isomorphism \( k/\mathfrak{p}(k) \to F_p \) is induced, as before, by the trace map \( S_{k|F_p} \).

The unramified degree-\( p \) extension of \( K \) is \( K(\mathfrak{p}^{-1}(\mathfrak{o})) \) (prop. 12). For an \( F_p \)-line \( D \neq \overline{\mathfrak{o}} \) in \( \overline{K} \) such that \( D \subset \overline{\mathfrak{p}^{-m}} \) but \( D \not\subset \overline{\mathfrak{p}^{-m+1}} \) for some \( m > 0 \), the unique ramification break of the (cyclic, degree-\( p \)) extension \( K(\mathfrak{p}^{-1}(D)) \) occurs at \( m \) (prop. 14), which is an integer prime to \( p \).
The extension $M = K(\wp^{-1}(K))$ is the maximal elementary abelian $p$-extension of $K$. Denote by $G = \text{Gal}(M|K)$ the profinite group of $K$-automorphism of $M$; it comes equipped with the ramification filtration $(G^u)_{u \in [-1, +\infty[}$ in the upper numbering.

We have $G^u = G^1$ for $u \in ]-1, 1]$, and, for $u > 0$, we have the “orthogonality relation” $(G^u)^\perp = p^{-\left\lfloor u \right\rfloor + 1}$ (prop. 17), where the orthogonal is with respect to the Artin-Schreier pairing $G \times K \to \mathbf{F}_p$. This is the function-fields analogue of the relation $(G^u)^\perp = \overline{\lambda}^{p-\left\lfloor u \right\rfloor + 1}$ ($\S$1).

The orthogonality relation implies that the upper ramification breaks of $G$ occur precisely at $-1$ and at the integers $>0$ which are prime to $p$. Given this, it is tantamount to $K(\wp^{-1}(p^{-m})) = M^{G_{m+1}}$ for every $m \in \mathbf{N}$ (cor. 18). This last relation allows us to compute the discriminant (over $K$) of these intermediate finite extensions (prop. 19); the result should be compared with prop. 3.

6. Justifications. We now prove the statements of $\S$5. As there, $k$ is a finite extension of $\mathbf{F}_p$ of degree $f$, $K$ is a local field with field of constants $k$, $\mathfrak{o}$ is the ring of integers of $K$ and $\mathfrak{p}$ is the unique maximal ideal of $\mathfrak{o}$; we have $k = \mathfrak{o}/\mathfrak{p}$.

We denote by $\wp$ the endomorphism $x \mapsto x^p - x$ of the additive group of any $\mathbf{F}_p$-algebra, such as $\mathfrak{o}$, $K$, $k$. For any subset $E \subset K$, denote by $K(\wp^{-1}(E))$ the extension of $K$ obtained by adjoining all $\alpha$ (in an unspecified algebraic closure of $K$) such that $\wp(\alpha) \in E$.

Denote by $(\mathfrak{p}^n)_{n \in \mathbf{Z}}$ the filtration on $\overline{K} = K/\wp(K)$ induced by the filtration $(\mathfrak{p}^n)_{n \in \mathbf{Z}}$ on (the $\mathbf{F}_p$-space) $K$ (where $\mathfrak{p}^0 = \mathfrak{o}$).

**Lemma 8.** — We have $\overline{\mathfrak{p}} = \{0\}$. In fact, $\wp : \mathfrak{p} \to \mathfrak{p}$ is an isomorphism.

For every $a \in \mathfrak{p}$, the reduction $T^p - T$ of the polynomial $T^p - T - a$ has the $p$ roots $0, 1, \ldots, p - 1$ making up the subfield $\mathbf{F}_p \subset k$ (“Fermat’s little theorem”). Hensel’s lemma then implies that there is a unique root $x \in \mathfrak{o}$ of $T^p - T - a$ whose reduction is $0 \in k$, so $x \in \mathfrak{p}$ is the unique element such that $\wp(x) = a$. (The $p$ roots of $T^p - T - a$ are $x + b$, for $b \in \mathbf{F}_p$)

For $n \in \mathbf{Z}$, let $\lambda(n) = \inf(n, pn)$, so that $\lambda(n) = n$ (resp. $pn$) if $n \in \mathbf{N}$ (resp. if $-n \in \mathbf{N}$). It is clear that $\wp(\mathfrak{p}^n) \subset \mathfrak{p}^{\lambda(n)}$ and $\wp(\mathfrak{p}^{n+1}) \subset \mathfrak{p}^{\lambda(n)+1}$. There is therefore an induced map $\wp_n : \mathfrak{p}^n/\mathfrak{p}^{n+1} \to \mathfrak{p}^{\lambda(n)}/\mathfrak{p}^{\lambda(n)+1}$; it is the function-field analogue of the map $(\ )^p : U_n/U_{n+1} \to U_{\lambda(n)}/U_{\lambda(n)+1}$ in the case of local number fields, where $\lambda$ was defined only for $n > 0$ (as $\inf(pn, n + e)$, $e$ being the absolute ramification index; cf. [2], Part III.3).
Lemma 9. — The image \( \bar{o} \) of \( o \) in \( \overline{K} \) is the same as the cokernel of \( \varphi_0 : o/p \to o/p \).

Notice first that \( \varphi(o) = o \cap \varphi(K) \). We have \( \bar{o} = o/\varphi(o) \). Now, \( p \subset \varphi(o) \) (lemma 8), so \( \bar{o} \) is also the quotient of \( o/p \) by \( \varphi(o)/p = \text{Im} \varphi_0 \).

To see that \( \bar{o} \) is canonically isomorphic to \( F_p \), consider the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & F_p & \to & o/p & \overset{\varphi_0}{\to} & o/p & \to & \bar{o} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & F_p & \to & k & \overset{\varphi}{\to} & k & \overset{S_k|F_p}{\to} & F_p & \to & 0
\end{array}
\]

which is the analogue of the diagram in [2], Part III.3 (where the choice of a primitive \( p \)-th root of 1, or a \( (p - 1) \)-th root of \( -p \), was necessary).

To bring out the analogy further, consider, for \( n \neq 0 \) in \( \mathbb{Z} \), and for every choice of a uniformiser \( \pi \) for \( K \), the commutative diagram

\[
\begin{array}{ccc}
p^n/p^{n+1} & \overset{\varphi_n}{\to} & p^{\lambda(n)}/p^{\lambda(n)+1} \\
\downarrow & & \downarrow \\
k & \overset{h}{\to} & k,
\end{array}
\]

where \( h(x) = x \) (resp. \( x^p \)) if \( n > 0 \) (resp. \( n < 0 \)). The vertical maps are the isomorphisms induced by the \( o \)-bases \( \pi^n \), \( \pi^{\lambda(n)} \) of \( p^n \), \( p^{\lambda(n)} \). In particular, \( \varphi_n \) is an isomorphism for \( n \neq 0 \).

Lemma 10. — For every integer \( n > 0 \), we have \( p^{-n} \cap \varphi(K) = \varphi(p^{-\lfloor n/p \rfloor}) \).

Consider \( n = pi \) (\( i > 0 \)); we have to show that \( p^{-pi} \cap \varphi(K) = \varphi(p^{-i}) \). One inclusion follows from \( \varphi(p^{-i}) \subset p^{-pi} \). For the converse, let \( x \in K \) be such that \( \varphi(x) \in p^{-pi} \); we have to show that \( x \in p^{-i} \). If \( v(x) \geq 0 \), then clearly \( x \in p^{-i} \). Suppose that \( v(x) < 0 \). Then \( v(\varphi(x)) = pv(x) \), but by assumption \( v(\varphi(x)) \geq -pi \). It follows that \( v(x) \geq -i \) and \( x \in p^{-i} \).

Consider next \( n = pi + j \) (\( i \in \mathbb{N} \), \( 0 < j < p \)); we have to show that \( p^{-n} \cap \varphi(K) = \varphi(p^{-i}) \). As before, we have \( \varphi(p^{-i}) \subset p^{-pi} \subset p^{-n} \). If \( x \in K \) is such that \( v(x) < 0 \) and \( \varphi(x) \in p^{-n} \), then \( v(\varphi(x)) = pv(x) \), and, as before, \( v(x) \geq -i - (j/p) \). But \( v(x) \) is in \( \mathbb{Z} \), so \( v(x) \geq -i \) and \( x \in p^{-i} \).
Proposition 11. — Let $m > 0$ be an integer. If $m = pi$ is a multiple of $p$, then $\mathbb{p}^{-pi+1} = \mathbb{p}^{-pi}$, whereas if $m = pi + j$ ($0 < j < p$) is prime to $p$, then $\mathbb{p}^{-pi+1} \subset \mathbb{p}^{-m}$ is a subspace of codimension $f$ (over $\mathbb{F}_p$). In particular, the group $K$ is infinite.

This is the analogue of [2, prop. 42], the major difference being that for a local number field $F$, the group $F^\times / F^\times p$ is finite, and that $\bar{U}_{pe_1+1} \subset \bar{U}_{pe_1}$ has codimension 1 when $F$ contains a primitive $p$-th root of 1.

The proof runs along the same lines. As there, the source of the dichotomy between multiples of $p$ and integers prime to $p$ lies in lemma 10, which implies that $\mathbb{p}^{-pi} \cap \varphi(K) = \varphi(p^{-i})$ but $\mathbb{p}^{-pi+1} \cap \varphi(K) = \varphi(p^{-i+1})$, whereas $\mathbb{p}^{-m} \cap \varphi(K) = \varphi(p^{-i})$ and $\mathbb{p}^{-m+1} \cap \varphi(K) = \varphi(p^{-i})$.

Consider first multiples of $p$. In the commutative diagram

\[
\begin{array}{cccccccc}
0 & \rightarrow & \mathbb{p}^{-i+1} & \rightarrow & \mathbb{p}^{-i} & \rightarrow & \mathbb{p}^{-i}/\mathbb{p}^{-i+1} & \rightarrow & 0 \\
\varphi & & \downarrow & & \varphi & & \varphi_{-i} & & \\
0 & \rightarrow & \mathbb{p}^{-pi+1} & \rightarrow & \mathbb{p}^{-pi} & \rightarrow & \mathbb{p}^{-pi}/\mathbb{p}^{-pi+1} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \mathbb{p}^{-pi+1} & \rightarrow & ? & \rightarrow & \mathbb{p}^{-pi} & & \\
\end{array}
\]

the rows and columns are exact and $\varphi_{-i}$ is bijective (see above), because $i \neq 0$. Thus, the arrow marked “?” is an isomorphism, by the snake lemma.

By contrast, for the integer $m$ (prime to $p$), the rows and columns in the commutative diagram

\[
\begin{array}{cccccccc}
0 & \rightarrow & \mathbb{p}^{-i} & \rightarrow & \mathbb{p}^{-i} & \rightarrow & \mathbb{p}^{-i}/\mathbb{p}^{-i} & \rightarrow & 0 \\
\varphi & & \downarrow & & \varphi & & 0 & & \downarrow \\
0 & \rightarrow & \mathbb{p}^{-m+1} & \rightarrow & \mathbb{p}^{-m} & \rightarrow & \mathbb{p}^{-m}/\mathbb{p}^{-m+1} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \mathbb{p}^{-m+1} & \rightarrow & \mathbb{p}^{-m} & & & & \\
\end{array}
\]
are as exact as before, but one of the arrows is now 0, instead of being an isomorphism. Therefore the induced map \( p^{-m}/p^{-m+1} \to p^{-m}/p^{-m+1} \) is an now isomorphism, instead of being 0. As the space \( p^{-m}/p^{-m+1} \) is of dimension \( f = [k : \mathbb{F}_p] \), the proof of prop. 11 is complete.

Remark. — The same method can be used to determine the filtration on \( \overline{K^x} = K^x/K^{xp} \). Indeed, we have \( U_p^n \subset U_p m \), just as \( \mathcal{W}(p^{-n}) \subset p^{-p n} \) here. The result can be expressed succinctly as

\[
\cdots \subset_f \bar{U}_p i+1 = \bar{U}_p i \subset_f \cdots \subset_f \bar{U}_1 \subset_1 \overline{K^x},
\]

with \( \overline{K^x}/\overline{U}_1 = \mathbb{Z}/p\mathbb{Z} \). This has the appearance of being the mirror image of (6); the phenomenon will be explained further on.

**Proposition 12.** — The unramified degree-\( p \) extension of \( K \) is \( K(\mathcal{W}^{-1}(a)) \).

Let \( a \in \mathfrak{o} \) be such that its image \( \bar{a} \) in \( \bar{\mathfrak{o}} \) generates \( \bar{\mathfrak{o}} \). We have to show that, \( \alpha \) being a root of \( T^p - T - a \) (in an algebraic closure of \( K \)), the extension \( K(\alpha) \) is unramified.

But this follows from the fact that the reduction \( T^p - T - \bar{a} \in k[T] \) is an irreducible polynomial. Indeed, \( S_{k|F_p}(\bar{a}) \), being the image of \( \bar{a} \) under the isomorphism \( \bar{\mathfrak{o}} \to F_p \), generates the latter group, and hence \( \bar{a} \notin \mathcal{W}(k) \).

Remark. — Writing \( G_0 = \text{Gal}(K(\alpha)|K) \), we also have the identification \( G_0 \to \mathbb{Z}/p\mathbb{Z} \) sending \( \varphi \) to 1, where \( \varphi \) ("Frobenius") is the unique element of \( G_0 \) whose restriction to \( k(\alpha) \) is the \( k \)-automorphism \( \varphi(x) = x^p \). In terms of these identifications, the Artin-Schreier pairing \( G_0 \times \bar{\mathfrak{o}} \to F_p \) gets identified with the mapping \( (\sigma, c) \mapsto \sigma.c \) from \( F_p \times F_p \) to \( F_p \).

This amounts to showing that \( \varphi(\beta) - \beta = S_{k|F_p}(b) \) for every \( b \in k \), where \( \beta \in k(\alpha) \) is a root of \( T^p - T - b \). Successively raising the relation \( \beta^p - \beta = b \) to the exponents \( 1, p, p^2, \ldots, p^{f-1} \), we get

\[
\beta^p - \beta = b, \quad \beta^{p^2} - \beta^p = b^p, \quad \ldots, \quad \beta^{p^f} - \beta^{p^f-1} = b^{p^f-1},
\]

and adding these \( f \) equations together gives

\[
\varphi(\beta) - \beta = \beta^{p^f} - \beta = b + b^p + \cdots + b^{p^f-1} = S_{k|F_p}(b),
\]

which was to be proved.

Let us fix some notation. Let \( D \neq \mathfrak{o} \) be an \( F_p \)-line in \( \overline{K} \), \( m \) the integer such that \( D \subset p^{-m} \) but \( D \not\subset p^{-m+1} \); we have seen that \( m \) is > 0 and prime to \( p \) (prop. 11). Fix an element \( a \in p^{-m} \) whose image generates \( D \),
also fix a uniformiser \(\pi\) of \(K\). Let \(\alpha\) be a root of \(T^p - T - a\) (in an algebraic closure of \(K\)), and let \(L = K(\alpha) = K(\varphi^{-1}(D))\).

Our first task is to find a uniformiser for \(L\) (in the analogous case of a degree-\(p\) kummerian extension of local number fields, see [2], prop. 61). We denote the normalised valuations of \(K, L\) by \(v_K, v_L\); as the extension \(L|K\) is totally ramified (prop. 12) of degree \(p\), we have \(v_L(x) = pv_K(x)\) for every \(x \in K\). Let \(\pi\) be any uniformiser of \(K\).

**Proposition 13.** — Let \(x, y \in \mathbb{Z}\) be such that \(-mx + py = 1\). Then \(\alpha^x \pi^y\) is a uniformiser of \(L\), and the ring of integers of \(L\) is \(\mathfrak{o}_L = \mathfrak{o}[\alpha^x \pi^y]\).

Notice first that \(v_L(\alpha) < 0\), for otherwise \(\alpha^p - \alpha = a\) would be in \(\mathfrak{o}\). Therefore \(v_L(\alpha^p - \alpha) = v_L(\alpha^p) = pv_L(\alpha)\). But we also have \(v_L(a) = pv_K(a) = -pm\). Therefore \(v_L(\alpha) = -m\).

It follows that \(v_L(\alpha^x \pi^y) = -mx + py = 1\), and, because \(L|K\) is totally ramified, that \(\mathfrak{o}_L = \mathfrak{o}[\alpha^x \pi^y]\).

**Proposition 14.** — The unique ramification break of the degree-\(p\) cyclic extension \(L|K\) occurs at \(m\).

Let \(H = \text{Gal}(L|K)\) and let \(\sigma \in H\) be such that \(\sigma(\alpha) - \alpha = 1\); as \(\sigma\) generates \(H\), we must show that \(\sigma \in H_m\) but \(\sigma \notin H_{m+1}\).

For this, it is enough [13, p. 61] to show that \(v_L(\sigma(\varpi) - \varpi) = m + 1\) for some uniformiser \(\varpi\) of \(L\). We choose \(\varpi = \alpha^x \pi^y\) (prop. 13) and compute

\[
\frac{\sigma(\varpi)}{\varpi} = \frac{\sigma(\alpha^x \pi^y)}{\alpha^x \pi^y} = \left(\frac{\sigma(\alpha)}{\alpha}\right)^x = \left(1 + \alpha^{-1}\right)^{x} \equiv 1 + x\alpha^{-1} \pmod{\varpi^{m+1}},
\]

recalling that \(v_L(\alpha^{-1}) = m\) and that \(x\) is prime to \(p\) (as \(-mx + py = 1\). This shows that \(v_L(\sigma(\varpi) - \varpi) = m + 1\), hence \(\sigma \in H_m\) but \(\sigma \notin H_{m+1}\), hence \(H_m = H\) but \(H_{m+1} = \{1d_L\}\), and the lower (as well as the upper) ramification break of \(H\) occurs at \(m\).

**Corollary 15.** — The valuation \(v_L(\mathfrak{D}_{L|K})\) of the different \(\mathfrak{D}_{L|K}\) of \(L|K\), as well as the valuation \(v_K(d_{L|K})\) of the discriminant, equals \((p-1)(1+m)\).

Indeed, \(v_L(\mathfrak{D}_{L|K}) = \sum_{n \in \mathbb{N}} (\text{Card} H_n - 1) = (p-1)(1+m)\), and \(v_K(d_{L|K})\) is the same because \(L|K\) is totally ramified.

The determination of the ramification of a degree-\(p\) cyclic extension of \(K\) goes back to Hasse [6]. I haven’t checked if he uses the uniformiser \(\alpha^x \pi^y\). It wouldn’t be surprising if he does, because \(\alpha^x \pi^y\) is just the function-field analogue of \((\xi - \sqrt{\mu})^x \lambda^y\), which can be found in his *Klassenkörperbericht*, and even in Hilbert’s *Zahlbericht* (the second \(\Omega\) in the proof of Satz 148).
Remark. — This allows us — in principle — to compute the discriminant of any finite extension of global function fields. Briefly, one reduces first to the local case, then to the galoisian case, then to the case of a $p$-extension, and finally to the case of a degree-$p$ extension, where cor. 15 can be applied.

Now let $M_m = K(\wp^{-m})$ for every $m \in \mathbb{N}$, and $M = K(\wp^{-1}(K))$, which is the maximal elementary abelian $p$-extension of $K$. It is the increasing union

$$K \subset M_0 \subset M_1 \subset M_2 \cdots \subset M_{pi-1} = M_{pi} \subset \cdots \subset M$$

($i > 0$ being arbitrary), where an inclusion of fields $E \subset E'$ means that $E'$ is a degree-$p^r$ extension of $E$.

**Corollary 16.** — For every $m \in \mathbb{N}$, the degree of the extension $M_m|K$ is $p^{1+c(m)f}$, where

$$c(m) = m - \left\lfloor \frac{m}{p} \right\rfloor$$

is the number of integers in $[1, m]$ which are prime to $p$.

Indeed, the $\mathbb{F}_p$-dimension of $\bar{p}^{-m}$ is $1 + c(m)f$, by prop. 11.

Define $a(i) = \left\lfloor \frac{i-1}{p-1} \right\rfloor$ as before. Notice that the strictly increasing map defined by $b^{(i)} = i + a(i)$ establishes a bijection between $[1, c(m)]$ and the set of integers in $[1, m]$ which are prime to $p$. In other words, the set in question is

$$b^{(1)} < b^{(2)} < \cdots < b^{(c(m))}.$$  

Put $G_m = \text{Gal}(M_m|M)$ and $G = \text{Gal}(M|K)$; we are going to think of these groups as $\mathbb{F}_p$-spaces. Our next task is to determine the ramification filtrations on $G_m$ (upper and lower) and on $G$ (upper) in terms of the Artin-Schreier pairings

$$G_m \times \bar{p}^{-m} \rightarrow \mathbb{F}_p, \quad G \times \bar{K} \rightarrow \mathbb{F}_p.$$  

The case $m = 0$ is easy: $M_0|K$ is the unramified degree-$p$ extension (prop. 12), so $G_0^{-1} = G_0$, $G_0^u = \{\text{Id}_{M_0}\}$ for $u > -1$.

**Proposition 17.** — We have $G^u = G^1$ for $u \in [-1, 1]$, and, for $u > 0$,

$$G^u \perp = \bar{p}^{-u+1}$$
under \( G \times \overline{K} \to \mathbf{F}_p \). The positive ramification breaks in the filtration on \( G \) occur precisely at the integers prime to \( p \), namely \( b^{(i)} (i > 0) \).

Let \( u \in ] - 1, 1] \). Notice first that \( G^u \neq G \), for otherwise the unique ramification break of \( G/\delta^\perp \) would be \( > u \), which it is not (prop. 12). Now let \( H \) be a hyperplane containing \( G^u \), so that \( G/H \) is cyclic of order \( p \). As the filtration on \( G/H \) is the quotient of the filtration on \( G \), the ramification break of \( G/H \) occurs somewhere \( < u \) (because \( G^u \subset H \)). But the only degree-\( p \) cyclic extension of \( K \) whose ramification break is \( < 1 \) is \( K(\wp^{-1}(\sigma)) \) (prop. 84). So \( H = \delta^\perp \) is the only hyperplane containing \( G^u \). This implies that \( G^u = H = G^1 = \overline{\delta}^\perp \).

It remains to show the orthogonality relation \( G^u^\perp = \mathbf{p}^{-[u]+1} \) for \( u > 1 \).

The principle of the proof is simplicity itself: two subspaces are the same if they contain the same lines. We show that, for a line \( D \subset \overline{K} \), we have \( D \subset G^u^\perp \) if and only if \( D \subset \mathbf{p}^{-[u]+1} \).

Take a line \( D \) and denote by \( m \) be the unique ramification break of \( G/D^\perp \). Then

\[
D \subset G^u^\perp \Leftrightarrow (G/D^\perp)^u = 0 \Leftrightarrow m < u \Leftrightarrow m < [u] \Leftrightarrow D \subset \mathbf{p}^{-[u]+1}.
\]

It now follows from prop. 11 that the positive ramification breaks of \( G \) occur precisely at the integers \( b^{(i)} (i > 0) \) prime to \( p \). In fact, the ramification filtration on \( G \) looks like

\[
\cdots \subset_f G^{pi+1} = G^{pi} \subset_f \cdots \subset_f G^1 = G^0 \subset_1 G,
\]

\((i > 0)\), where an inclusion \( H \subset_r H' \) means that \( H \) is a codimension-\( r \) subspace (an index-\( p^r \) subgroup) of \( H' \).

**Corollary 18.** — *For every \( m \in \mathbf{N} \), we have \( K(\wp^{-1}(\mathbf{p}^{-m})) = M^{G^m+1} \).*

Indeed, \( (G^{m+1})^\perp = \mathbf{p}^{-m} \), by prop. 17.

Prop. 17 allows us to determine the ramification filtration on \( \text{Gal}(L/K) \) for any elementary abelian \( p \)-extension \( L/K \), provided we know the subspace \( \text{Ker}(\overline{K} \to \overline{L}) \), where \( \overline{L} = L/\wp(L) \). Let us do this exercise for \( L = K(\wp^{-1}(\mathbf{p}^{-m})) \), in which case the subspace in question is \( \mathbf{p}^{-m} \).

**Proposition 19.** — *The upper ramification breaks of \( \text{Gal}(L/K) \) occur at \( -1 \) and at \( b^{(1)} < b^{(2)} < \cdots < b^{c(m)} \), the \( c(m) \) integers in \([1, m]\) which are prime to \( p \). In the lower numbering, they occur at \( -1 \) and at

\[
b^{(i)} = (1 + q + \cdots + q^{i-1}) + (q^{p-1} + \cdots + q^{a(i)(p-1)}), \quad i \in [1, c(m)],
\]

23
where \( q = p^f \). We have \( v_L(\mathcal{D}_{L|K}) = (1 + b(c(m)))q^{c(m)} - (1 + b(c(m))) \), and

\[ v_K(d_{L|K}) = pv_L(\mathcal{D}_{L|K}) \].

This is very similar to what we saw in prop. 3 in the case of kummerian extensions of local number fields. To compute the valuation \( v_L(\mathcal{D}_{L|K}) \) of the different of \( L|K \), we appeal to lemma 2, noting that the order of the inertia group \( \text{Gal}(L|K)^0 \) is \( q^{c(m)} \), by cor. 16. The valuation \( v_K(d_{L|K}) \) of the discriminant is \( pv_L(\mathcal{D}_{L|K}) \) because the residual degree of \( L|K \) is \( p \).

**Remark.** — Notice that \( L = K(\varphi^{-1}(p^{-m})) \) is the maximal elementary abelian \( p \)-extension of \( K \) with ramification breaks in \([-1, m]\). Notice also that the kernel of the projection \( G \to \text{Gal}(L|K) \) is \( G^{m+1} \) (cor. 18), which corresponds to \( \hat{U}_{m+1} \) under the reciprocity isomorphism \( K^\times/K^{\times p} \to G \). But the kernel of \( K^\times \to \text{Gal}(L|K) \) is the group of norms \( N_{L|K}(L^\times) \). It follows that \( N_{L|K}(L^\times) = U_{m+1}^{m+1}K^{\times p} \).

The analogue for a local number field \( K \) (containing a primitive \( p \)-th root of 1) would say that \( N_{L|K}(L^\times) = U_{pe_1-m+1-1}^{m+1}K^{\times p} \) for \( L = K(\varphi^{-1}(p^{-m})) \), where \( m \in [0, pe_1 + 1] \) and \( U_0 = K^\times \), which goes back to [8].

The orthogonality relation of prop. 17 has other applications as in the case of local number fields (§4). Thus we can determine the ramification filtration of an arbitrary elementary abelian \( p \)-extension \( L|K \), the valuation of its discriminant if \( L|K \) is finite, and so on. We can specify the sequences which can occur as the ramification breaks of some \( L|K \), and determine the possible degrees, and the total number, of such \( L|K \). *N’insistons pas.*

**7. Discussion.** Our main theme here has been the compatibility of the Kummer (resp. the Artin-Schreier) pairing with the filtration on the multiplicative group \( K^\times \) (resp. the additive group \( K \)) on the one hand and the ramification filtration — in the upper numbering — of \( G = \text{Gal}(M|K) \) on the other, where \( M \) is the maximal elementary abelian \( p \)-extension of a local number field \( K \) containing a primitive \( p \)-th root of 1 (resp. a local function field \( K \) of residual characteristic \( p \)).

This compatibility is expressed by the orthogonality relation as recalled in §1 (resp. proved in §6). In essence, it says that \( M^G \) is the same as \( K(\varphi^{-1}(p^{-m})) \) whenever \( m+n = pe_1 + 1 \) (resp. \( m+n = 1 \)). The proof is entirely elementary and purely conceptual.

The orthogonality relation has many consequences, some of which we have discussed above. It implies, without invoking Hasse-Arf, that the ramification breaks of \( M|K \) occur at integers. It allows us to compute the discriminant of any elementary abelian \( p \)-extension of local fields, without invoking class field theory and the *Führerdiskriminantenproduktformel.*
When the two approaches are combined, one can compute the norm group of the extension \( K(\sqrt[p]{U_m}) \) (resp. \( K(\varphi^{-1}(p^{-m})) \)), as explained in §6.

Bibliographic References

[1] Artin (E.). — Algebraic numbers and algebraic functions, AMS Chelsea Publishing, Providence, 2006. xiv+349 pp.

[2] Dalawat (C.). — Local discriminants, kummerian extensions, and elliptic curves, to appear in the Journal of Ramanujan Mathematical Society, cf. arXiv:0711.3878v1.

[3] Del Corso (I.) and Dvornicich (R.). — The compositum of wild extensions of local fields of prime degree, Monatsh. Math. 150 (2007), no. 4, pp. 271–288.

[4] Fesenko (I.) and Vostokov (S.). — Local fields and their extensions, American Mathematical Society, 2002. xii+345 pp.

[5] Fontaine (J.-M.). — Groupes de ramification et représentations d’Artin, Ann. sci. École norm. sup. (4) 4, 1971, 337–392.

[6] Hasse (H.). — Theorie der relativ-zyklischen algebraischen Funktionenkörper, insbesondere bei endlichem Konstantenkörper, J. f. d. reine und angewandte Math., 172, 1934, pp. 37-54.

[7] Iwasawa (K.). — Local class field theory, Oxford University Press, 1986, 155 pp.

[8] Nguyen-Quang-Do (T.). — Filtration de \( K^*/K^{*p} \) et ramification sauvage, Acta Arith., 30, 1976, no. 4, 323–340.

[9] Martinet (J.). — Les discriminants quadratiques et la congruence de Stickelberger, Sém. Théor. Nombres Bordeaux (2) 1, 1989, no. 1, pp. 197–204.

[10] Maus (E.). — Existenz \( p \)-adischer Zahlkörper zu gegebenem Verzweigungsverhalten, Dissertation, Hamburg, 1965.

[11] Neukirch (J.). — Class Field Theory, Springer-Verlag, Berlin, 1986, 140 pp.

[12] Pisolkar (S.). — Absolute norms of \( p \)-primary numbers, to appear in the Journal de Théorie de Nombres de Bordeaux, cf. arXiv:0807.1174.

[13] Serre (J.-P.). — Corps locaux, Publications de l’Université de Nancago, No. VIII, Hermann, Paris, 1968, 245 pp.