Electromagnetic Power Radiated by an Accelerating Point Charge

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Abstract

We derive the electromagnetic power radiated by an accelerating point charge, with the acceleration and velocity in the result being taken at the present time in the motion of the accelerating charge. This contrasts with the usual derivation which calculates the power radiated through the surface of a large sphere, and gives the radiated power in terms of the acceleration and velocity at an arbitrary retarded time.

1 Introduction

Larmor’s formula\[1\]

\[
\frac{dW_{\text{rad}}}{dt} = \frac{2}{3}q^2a^2, \quad (1)
\]

for the power radiated by an accelerating point charge, was first derived over 100 years ago by Joseph Larmor [1]. The non-relativistic Larmor formula was extended for large velocities to the relativistic Liénard formula,

\[
\frac{dW_{\text{rad}}}{dt} = \frac{2}{3}q^2\gamma^0[a^2 - (v \times a)^2], \quad (2)
\]

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\[\text{1We are using Gaussian units with } c = 1.\]
by Alfred-Marie Liénard [2]. Recent derivations\(^2\) have generally been based on the rate,

\[
\frac{dW_{\text{rad}}}{dt} = \frac{1}{4\pi} \oint_S \mathbf{dS} \cdot [\mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)] = \frac{1}{4\pi} \int \mathbf{\hat{r}} \cdot [\mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)] R_{\text{rad}}^2 d\Omega,
\]

at which radiated energy passes through a spherical surface at a large radius, \(R_{\text{rad}}\), from the radiating particle.

However, a major problem arises if Eq. (3) is used to calculate the electromagnetic power. The fields in the surface integral are to be evaluated at the present time and the point \(\mathbf{r}\) at which the fields are observed, but the fields for an accelerating particle are given in terms of variables given at the retarded time by the Liénard-Wiechert field equations.

Thus the Liénard formula of Eq. (2) (and the non-relativistic Larmor formula) would be given in terms of the acceleration and velocity at the retarded time, \(t_r\), and not at the present time, \(t\). The retarded acceleration and velocity would depend on the arbitrary radius picked for evaluating Eq. (3). That means that the acceleration and velocity appearing in Eq. (2) could be any acceleration and velocity from the past motion of the accelerating particle. That makes Eq. (2) almost useless, since the acceleration and velocity would be arbitrary, depending on what was chosen as the radius of observation, \(R_{\text{rad}}\), of the radiation.

Another problem arises when the retarded variables in the Larmor formula (as derived) are used as if they were given at the present time to suggest derivations of the Abraham-Lorentz radiation reaction force \(^3\).

We resolve these problems in the next section by a new derivation of Eq. (2) that gives the emission of electromagnetic power at the present time in terms of all variables at the present time.

\(^2\)See, for instance, Chapter 14 of [3] or Chapter 11 of [4].

\(^3\)Also see [5], which also uses the Larmor formula for a derivation of the Abraham-Lorentz force, and includes some earlier references.
2 Electromagnetic Power Emitted by an Accelerating Point Charge

The electric and magnetic fields appearing in Eq. (3) are given by the Liénard-Wiechert field equations,

\[
E(r, t) = \left\{ \frac{q(\hat{r}_r - \hat{v}_r)}{r^2\gamma^2(1 - \hat{r}_r \cdot \hat{v}_r)^3} \right\} + \left\{ \frac{\hat{r}_r \times [(\hat{r}_r - \hat{v}_r) \times a_r]}{r^2(1 - \hat{r}_r \cdot \hat{v}_r)^3} \right\},
\]

(4)

\[
B(r, t) = \hat{r}_r \times E(r, t),
\]

(5)

where the variables, \(r_r, v_r, \gamma_r = 1/\sqrt{1 - v^2_r}\), and \(a_r\) are all evaluated at the retarded time,

\[
t_r = t - r_r.
\]

The radius vector, \(r_r\), is the distance from the charged particle’s position at the retarded time to the point of observation of the electromagnetic fields at the present time.

The retarded radius, \(r_r\), is related to the radius, \(r\), which is directed from the present position of the accelerating charge to the point of observation, by

\[
r_r - r = \langle v \rangle (t_r - t) = -r_r \langle v \rangle,
\]

(7)

where \(\langle v \rangle\) is the average velocity in the interval from \(t_r\) to \(t\). This means that, using Eq. (3) at a radius, \(R_{rad}\), the radiated power would depend not only on \(R_{rad}\), but also on the average velocity in the past motion of the accelerating charge. In our derivation, we will take the limit \(R_{rad} \to 0\), which means that the acceleration and all the variables in Eq. (2) will be taken at the present time.

We consider a point charge \(q\) at a position \(r(t)\) with a velocity \(v(t)\) and acceleration \(a(t)\). We make a Lorentz transformation to the rest frame of the point charge where \(v' = 0\) and

\[
a'_{\parallel} = a_{\parallel} \gamma^3,
\]

(8)

\[
a'_{\perp} = a_{\perp} \gamma^2.
\]

(9)

\(a'_{\parallel}\) is the rest frame acceleration parallel to \(v\), and \(a'_{\perp}\) is the rest frame acceleration perpendicular to \(v\).

We now evaluate the rest frame surface integral

\[
\frac{dW'_{rad}}{dt'} = \frac{1}{4\pi} \int \hat{r}' \cdot [(E'(r', t') \times B'(r', t'))] R_{rad}^2 d\Omega',
\]

(10)
not at a large radius, but in the limit $R'_{\text{rad}} \to 0$. In this limit, $t'_r \to t'$ and $a'_r \to a'$, so the electric field is given by

$$E'(r', t') = \frac{q r'}{r'^2} + \frac{[\hat{r}'(\hat{r}' \cdot a') - a']}{r'}$$

(11)

with all variables evaluated at the present time in the rest frame.

Then, the surface integral in Eq. (10) for the radiated power reduces to

$$\frac{dW'_{\text{rad}}}{dt'} = \frac{1}{4\pi} \oint \hat{r}' \cdot [a' \times (\hat{r} \times a')] d\Omega'$$

$$= \frac{1}{4\pi} \oint [a'^2 - (\hat{r} \cdot a')^2] d\Omega'$$

$$= \frac{2}{3} a'^2.$$  

(12)

The radiated power can be put back in terms of the original acceleration, using Eqs. (8) and (9) to give

$$\frac{dW'_{\text{rad}}}{dt'} = \frac{2}{3} (a^2 \parallel \gamma^6 + a^2 \perp \gamma^4)$$

$$= \frac{2}{3} \gamma^6 [a^2 - (v \times a)^2].$$  

(13)

The variables in Eq. (13) are in the original moving frame, but the rate of energy emission on the left hand side of the equation is still in the rest frame. However, the right-hand side of Eq. (13) has been shown to be a Lorentz invariant$^4$, so Eq. (13) can be Lorentz transformed to the moving frame, finally giving

$$\frac{dW_{\text{rad}}}{dt} = \frac{2}{3} \gamma^6 [a^2 - (v \times a)^2].$$  

(14)

This result has the same form as Lénard’s relativistic extension of Larmor’s formula, but is given here with all variables at the present time, and not an arbitrary retarded time. Thus, there is no uncertainty in the radiated power given by Eq. (14).

$^4$See, for instance, page 666 of [3].
3 Conclusion

Our main conclusion is that the power radiated by an accelerating point charge is given by

\[
\frac{dW_{\text{rad}}}{dt} = \frac{2}{3} \gamma^6 [a^2 - (v \times a)^2].
\]  

(15)

This result has the same form as Liénard’s relativistic extension of Larmor’s formula, but is given here with all variables at the present time, and not at an arbitrary retarded time, with arbitrary acceleration and velocity.

References

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[5] https//en.wikipedia.or < wiki > Abraham–Lorentz_force