Abstract

The effective action of a (1+2)-dimensional defect is obtained as an expansion in powers of the thickness. Considering non-straight solutions as the zero order term, the corrections to the Nambu action are found to depend on the curvature scalar and on the gaussian curvature.
1 Introduction

Topological defects can arise in field theories with a non-trivial vacuum manifold. The dynamical evolution of these objects can be derived from the field theory, following the time evolution of the fields. However, in the absence of explicit solutions, effective actions can be constructed for strongly localized defects, describing directly the geometrical evolution of the defect world-surface.

Considering the straight and static solution as the zero order limit in the thickness $\epsilon$, and expanding the equations in powers of $\epsilon$, it was proved that defects have generalized Nambu actions [1] in the lower order of the characteristic width of the defect. Higher order terms were expected to include corrections to the Nambu action [2].

However, R. Gregory showed, recently, that if the zero order solutions are consistently taken into account in the second order corrections, then these corrections to the Nambu action are proportional to $R$, the Ricci curvature [3,4]. This term vanishes for particles and it is a topological constant for strings.

Only bubbles and higher-dimensional defects would have a non-trivial contribution to the dynamical equation.

One important ingredient in R. Gregory’s analysis is the starting point: the static and straight solution as the solution to represent the limit $\epsilon \to 0$. This means that two limits are taken simultaneously $\epsilon \to 0$ and $\rho \to \infty$, in which $\rho$ stands for an average or local curvature of the defect surface.

Nevertheless, there is no reason to reject a treatment in which $\epsilon \to 0$, or better $\frac{\epsilon}{\rho} \to 0$ as the zero order limit, without requiring the $\rho \to \infty$ limit. This alternative procedure would be more suitable to describe, for example, the evolution of circular strings and spherical bubbles, that may be thin, but are neither static nor straight.

Roughly, the importance of the curved features of the solutions will depend on the relative sizes of terms like $\phi''$ and $\frac{1}{\sqrt{-g}} (\sqrt{-g})' \phi'$, in which $\phi$ is the scalar field and ' stands for derivative in a direction perpendicular to the defect surface. This contribution may be important or not, but it is independent of the defect width.

In what follows, we consider the simplest case of a scalar field with two vacuum states. We start with a non-straight solution with a domain wall, and, in the zero order limit, we recover the Nambu action. The next order
correction is also calculated and Gregory’s term is obtained together with extra terms.

2 Gaussian Coordinates

We consider a scalar field $\phi$ and a potential $V(\phi)$ with symmetry breakdown and two (or more) vacuum states $\phi_1$ and $\phi_2$. We select solutions with a topological defect, and they are considered highly concentrated on a well-defined surface. If $V(\phi)$ is the usual $\lambda(\phi^2 - v^2)^2$ potential, then this surface may be identified with $\phi = 0$.

A new coordinate system based on the defect surface is introduced. In the general case, a p-dimensional defect spans a (p+1)-dimensional manifold in the space-time. This world surface is localized by $X^\mu(\sigma^A)$ with $\mu = 0, 1...n$, and $\sigma^A$ are the coordinates on the surface with $A = 0, ...p$. $X^\mu_A = \frac{\partial X^\mu}{\partial \sigma^A}$ are tangent vectors. On each point of the surface, there is a normal plane, spanned by the normal vectors $N^\mu_i$ with $i = 1, ...m$ and $p + m = n$. In each normal direction, we choose coordinates $\xi^i$. Normal vectors are normalized and orthogonal to the tangent vectors. Near the surface, the points are localized by:

$$Z^\mu(\sigma^A, \xi^i) = X^\mu(\sigma^A) + \xi^i N^\mu_i(\sigma^A)$$  \(1\)

We start with $G_{\mu\nu}$, a diagonal metric of the flat (n+1)-dimensional space that may be Minkowski metric. In the new coordinate system, we have $g_{\mu\nu}$ with:

$$G_{\mu\nu}dZ^\mu dZ^\nu = g_{AB}d\sigma^A d\sigma^B + 2g_{Aj}d\sigma^A d\xi^j + g_{ij}d\xi^i d\xi^j$$  \(2\)

From (1), we have $dZ^\mu$ given by:

$$dZ^\mu = (X^\mu_A + \xi^i N^\mu_i) d\sigma^A + N^\mu_i d\xi^i$$  \(3\)

and replacing (3) in the left-hand side of (2), we get $g_{\mu\nu}$:

$$g_{AB} = \tilde{g}_{AB} + \xi^i N^\mu_i X_{A\mu,B} + \xi^j N^\nu_j X_{\nu,A} + \xi^i \xi^j N^\mu_i N^\nu_j$$  \(4\)

$$g_{iB} = \xi^j N^\mu_j N_{i\mu,B}$$  \(5\)
\[ g_{ij} = \xi_i N_{i,A} N_{\mu j} \quad (6) \]
\[ g_{ij} = \delta_{ij} \quad (7) \]

for \( \tilde{g}_{AB} = G_{\mu \nu} X_{\mu}^A X_{\nu}^B \) and \( N^i N_{\mu j} = \delta_{ij} \). The inverse metric \( g^{\mu \nu} \) can be computed in powers of \( \xi^i \). Requiring that \( g_{\mu \nu} g^{\nu \lambda} = \delta_{\lambda \mu} \), we find:

\[ g^{AB} = \tilde{g}^{AB} - \xi^i N^{\mu A}_{i} X_{\mu}^B - \xi^i N^{\mu B}_{i} X_{\mu}^A + O(\xi^2) \quad (8) \]
\[ g^{iB} = -\xi^j N^{\mu i} N^{B}_{\mu j} + O(\xi^2) \quad (9) \]
\[ g^{Aj} = -\xi^i N^{\mu, A}_{i} N^{j}_{\mu} + O(\xi^2) \quad (10) \]
\[ g^{ij} = \delta^{ij} + \xi^k \xi^l N_{\mu k : A} N^{\mu i} N^{A}_{\mu j} + O(\xi^3) \quad (11) \]

The crossed terms \( g_{iB} \) and \( g_{Aj} \) are related to \( A_{ijB} = N^{\mu i}_{i} N_{\mu j} B \), which is the torsion or twisting vector and vanishes in the case of hypersurfaces \( n = p+1 \) [5], making:

\[ g_{iB} = g_{Aj} = 0 \]
\[ g^{iB} = O(\xi^2), g^{Aj} = O(\xi^2), g^{ij} = O(\xi^3) \quad (12) \]

3 Solutions in powers of the defect width

The action of a field configuration is given by:

\[ S = \int \sqrt{-G} \mathcal{L} \, d^{n+1}Z = \int \sqrt{-g} \mathcal{L} \, d^{p+1} \sigma \, d^{n} \xi \quad (13) \]

with \( \mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \).

To construct the effective action, we consider solutions for \( \phi \) with a topological defect and replace it in (13). This new action is, then, understood as function of the surface coordinates, and the dynamical evolution is studied directly for \( X^\mu(\sigma^A) \).
The defect surface is characterized by a constant value of $\phi$, namely $\phi = 0$ and the coordinates are chosen with $\xi^i = 0$ on this surface. The equation of motion from (13) is then written as:

$$\frac{1}{\sqrt{-g}} \partial_i(\sqrt{-g} g^{ij} \partial_j \phi) + \frac{1}{\sqrt{-g}} \partial_i(\sqrt{-g} g^{iA} \partial_A \phi) + \frac{1}{\sqrt{-g}} \partial_A(\sqrt{-g} g^{Aj} \partial_j \phi) +$$

$$+ \frac{1}{\sqrt{-g}} \partial_A(\sqrt{-g} g^{AB} \partial_B \phi) + \frac{\partial V}{\partial \phi} = 0 \quad (14)$$

Since the solutions we are interested on are concentrated around $\xi^i = 0$, both $\phi - v$ and $\partial^\mu \phi$ decrease fast with $\xi^i$. Because of this, it is reasonable to solve (14) in powers of the typical thickness of the defect core ($\epsilon$). We consider solutions of the form:

$$\phi = \phi_0(\xi^i) + \phi_1(\xi^i) + \phi_2$$

with $\phi_1$ of the order of $\epsilon$ and $\phi_2$ of the order of $\epsilon^2$. Because of the fast decreasing behavior of $\partial_i \phi$ for $\xi > \epsilon$, the terms like $\xi \partial_i \phi$ are one order of correction higher than $\partial_i \phi$ and, in the expansion of (14) in powers of $\epsilon$, we must also include the expansion of the metric terms in powers of $\xi$. From (4-7) and (8-11), we can compute $K_i$ defined by:

$$K_i = \frac{1}{\sqrt{-g}} \partial_i(\sqrt{-g}) =$$

$$= \frac{1}{2} \left( g^{AB} \partial_i g_{AB} + g^{Aj} \partial_i g_{Aj} + g^{kB} \partial_i g_{kB} + g^{kj} \partial_i g_{kj} \right)$$

We find:

$$K_i = K_i^0 + \xi^j K_{ij}^1 \quad (15)$$

with

$$K_i^0 = g^{AB} b_{ABi} \quad (16)$$

$$K_{ij}^1 = -b_{ABi} b_{ABj} \quad (17)$$
and from Gauss-Weingarten’s equation:

$$N_{i,A}^\mu = g^{CD}b_{ACi}X_D^\mu + g^{kij}A_{kiA}N_j^\mu$$

where $K^0_i$ is the mean curvature, $K^1_{ij}$ is the gaussian curvature, $b_{ACi}$ is the second fundamental form and $A_{kiA}$ is the twisting vector. Their values depend on the surface we are dealing with and they must be compared with the $\partial_i \partial^i \phi$ - terms. They may be large and important or not, depending on how bent the surface is. Anyway, they are not related to the orders of correction of the defect width.

Now, using (15) and (12), equation (14) splits into two pieces to describe the zero and the first order equations. Up to zero order of $\epsilon$ we have:

$$\partial_i \partial^i \phi_0 + K^0_i \partial^i \phi_0 + \frac{\partial V}{\partial \phi} \bigg|_0 = 0$$

where $\big|_0$ indicates evaluation at $\phi = \phi_0$. The first order correction, in $\epsilon$, of (14) is:

$$\partial_i \partial^i \phi_1 + K^0_i \partial^i \phi_1 + K^1_{ij} \partial^i \phi_0 + \phi_1 \frac{\partial^2 V}{\partial \phi^2} \bigg|_0 = 0$$

From (19), we note that $\phi_1$ can not be identically zero as long as $K^1_{ij}$ and $\partial^i \phi_0$ are not zero. Up to the order of $\epsilon^2$, we would get an equation for $\phi_2$. However, this contribution will not be important for our calculations.

In the next section we will compute the effective action, integrating out $\xi$ in the original action (13). To do so, it is important that both $\phi_0$ and $\phi_1$ are $\sigma^A$ - independent. In the standard derivation of Nambu action, the defect is locally approximated by the static and straight solution. The world surface has, locally, constant normal vectors making $b_{ACi} = 0$. Consequently, both $K^0_i$ and $K^1_{ij}$ are $\sigma^A$ - independent (equal to zero) and, from (18-19), $\phi_0$ and $\phi_1$ are also $\sigma^A$ - independent.

In this paper, we consider the defect locally approximated by a non-plane solution, adjusting the values of $K^0_i$ and $K^1_{ij}$. However, to maintain the $\sigma^A$- independence of $\phi_0$ and $\phi_1$, we restrict ourselves to solutions with locally constant values of $K^0_i$ and $K^1_{ij}$. We are approximating the defect world surface by a local solution surface with constant mean and gaussian curvatures. This would be satisfied, for example, by a spherical or cylindrical surface.
with constant radius. These surfaces have \( b_{ABi} \neq 0 \) and \( X_{\nu}^A N_{\nu B} G_{\mu \nu} = b_{ABi} \), \( \sigma^A \)- independent. In this case, both \( \phi_0 \) and \( \phi_1 \) remain, through (18-19), \( \sigma^A \)- independent.

Finally, we should note that the expansion in powers of \( \epsilon \) would not reproduce the results (18) and (19) if it were done directly in the action. The Euler-Lagrange equations introduce the derivatives on \( \xi^i \) that change the power counting. We claim that, as long as we are dealing with the classical dynamical evolution of a function of \( \xi^i \), namely \( \phi(\xi^i) \), the correct place to make the expansion in powers of \( \xi^i \) is the equation of motion.

4 The effective action

The next step is to look for an effective action that will describe the evolution of the surface \( \xi^i = 0 \) as a geometrical object. Starting from (13):

\[
S = \int \sqrt{-g} L(\phi) \, d^{p+1} \sigma \, d^m \xi
\]  

(20)

the idea is to integrate out the \( \xi^i \)-dependence, leaving an action of the form \( S = \int f(\hat{g}(\sigma)) d\sigma \). This means that the Euler-Lagrange equations for this effective action involves neither \( \xi^i \) nor its derivatives and, in order to get the equation of the defect from the effective action, we may safely expand in powers of \( \xi^i \) directly in the action (20).

We expand \( L(\phi) = L_0 + L_1 + L_2 \) and replace in (20) to find:

\[
S = \int \sqrt{-g} \left[ \frac{1}{2} \partial_i \phi_0 \partial^i \phi_0 - V(\phi_0) \right] \, d^{p+1} \sigma \, d^m \xi +
\]

\[
+ \int \sqrt{-g} \left[ \partial_i \phi_1 \partial^i \phi_0 + \partial_i \phi_2 \partial^i \phi_0 - \frac{\partial V}{\partial \phi} \bigg|_0 \phi_1 - \frac{\partial V}{\partial \phi} \bigg|_0 \phi_2 \right] \, d^{p+1} \sigma \, d^m \xi +
\]

\[
+ \frac{1}{2} \int \sqrt{-g} \left[ \partial_i \phi_1 \partial^i \phi_1 - \frac{\partial^2 V}{\partial \phi^2} \bigg|_0 \phi_1^2 \right] \, d^{p+1} \sigma \, d^m \xi,
\]  

(21)

Integrating by parts the first and second order correction terms and using equations (18),(19), we get:

\[
S = \int \sqrt{-g} \left[ \left( \frac{1}{2} \partial_i \phi_0 \partial^i \phi_0 - V(\phi_0) \right) - \frac{1}{2} \xi^i K_{ij}^1 \partial^j \phi_0 \phi_1 \right] \, d^{p+1} \sigma \, d^m \xi
\]  

(22)

7
To be consistent, we must also expand $\sqrt{-g}$:

$$\sqrt{-g} = \sqrt{-\tilde{g}} + \partial_i(\sqrt{-g})\xi^i + \frac{1}{2}\partial_i\partial_j(\sqrt{-g})\xi^i\xi^j$$  \tag{23}

From (4-7), (8-11) and (16-17), we find:

$$\partial_i(\sqrt{-g}) = \sqrt{-\tilde{g}}K_i^0$$  \tag{24}

$$\partial_i\left[\partial_j(\sqrt{-g})\right] = \partial_i\left[\sqrt{-g}(K_j^0 + \xi^k K_{kj}^{1})\right] =$$

$$= \sqrt{-\tilde{g}}(K_j^0 K_i^0 + K_{ij}^{1})$$  \tag{25}

Up to zero order, we have the action:

$$S = \int \sqrt{-\tilde{g}} \left[ \frac{1}{2} \partial_i\phi_0 \partial^i\phi_0 - V(\phi_0) \right] \, dp^{p+1} \, d^m\xi$$  \tag{26}

With $\phi_0$ independent of $\sigma^A$ and $\sqrt{-g}$ independent of $\xi^i$, the integration in $\xi^i$ can be performed to reproduce the Nambu action:

$$S = \mu_0 \int \sqrt{-\tilde{g}} \, dp^{p+1}\sigma$$  \tag{27}

and $\mu_0 = \int d^m\xi (\frac{1}{2} \partial_i\phi_0 \partial^i\phi_0 - V(\phi_0))$. As it is well known, this describes a minimal surface, with equation of motion given by:

$$K_i^0 = \tilde{g}^{AB} b_{ABi} = 0$$  \tag{28}

The first order correction does not vanish on integration over $\xi^i$ because $\phi_0$, solution of(18), is not an even function of $\xi^i$ for $K_i^0 \neq 0$. The second order has two contributions. The last term in the right hand side of (22) is not zero, due to the $\phi_1$ contribution. Rewriting these terms, and adding the contribution from (23), we have:

$$S = \mu_0 \int \sqrt{-\tilde{g}} \left[ \frac{1}{\mu_0} K_i^0 + \frac{\mu_1^{ij}}{\mu_0} (K_i^0 K_j^0 + K_{ij}^{1}) - \frac{\mu_2^{ij}}{\mu_0} K_{ij}^{1} \right] \, dp^{p+1}\sigma$$  \tag{29}

with:
\[ \mu_1 = \int d^m \xi \xi^i \left[ \frac{1}{2} \partial_j \phi_0 \partial^j \phi_0 - V(\phi_0) \right] \]

\[ \mu_2^{ij} = \frac{1}{2} \int d^m \xi \xi^i \xi^j \left[ \frac{1}{2} \partial_i \phi_0 \partial^j \phi_0 - V(\phi_0) \right] \]

\[ \tilde{\mu}_2^{ij} = \frac{1}{2} \int d^m \xi \xi^i \partial_j \phi_0 \phi_1 \]

Both \( \mu_2^{ij} \) and \( \tilde{\mu}_2^{ij} \) are second order contributions due to \( \xi^2 \) and \( \xi \phi_1 \).

In the specific example we are dealing with, there is only one direction \( \xi^i \) and \( i \) takes just one value. We write \( \tilde{\mu}_2^{ij} K_1^{ij} = -\mu_2 K \) with \( K = b_{AB} b^{AB} \). Using the Gauss-Codazzi relations:

\[ \sum_i ((K_1^0)^2 + K_1^{ii}) = -R \] (30)

we get

\[ S = \mu_0 \int \sqrt{-\tilde{g}} \left[ 1 + \frac{\mu_1}{\mu_0} K^0 - \frac{\mu_2}{\mu_0} R + \frac{\tilde{\mu}_2}{\mu_0} K \right] d^{p+1} \sigma. \] (31)

### 5 Conclusion

So, starting with an arbitrary configuration, we calculate the first non-zero corrections to the generalized Nambu action for a bubble. One term of this correction agrees with Gregory’s result. The others come in only if the original defect is not plane. They depend on the odd part of \( \phi_0 \) and on the field correction \( \phi_1 \). It should be noted that \( \phi_1 \) remains non zero even if we use the zero order condition \( K_1^0 = 0 \) (28) in the equation of motion of \( \phi_1 \), (19). This procedure is advocated by R. Gregory [4] on the basis of consistency, and for plane zero order solutions it implies identically zero solutions for \( \phi_1 \). In fact, if the zero order solution is plane, not only \( K_1^0 = 0 \) but also \( K_1^{ij} = 0 \) because the normal vectors are, up to zero order, constants and satisfy \( N_{i,A} = 0 \). In this case, equation (19) has solutions with \( \phi_1 = 0 \).
We should also point out that the choice of perturbations in the form \( \phi = \phi_0(\xi^i) + \phi_1(\xi^i) \) was crucial to allow the factorization of the \( \xi \)-dependence from the action integral. If we had \( \phi = \phi(\xi^i, \sigma^A) \), it would be impossible to extract the geometrical part of the dynamics, expressed by an action integrated over \( \sigma^A \) only, as (31). This restriction means that, if the zero order is a spherical bubble, the first order correction will not violate the spherical symmetry.

The procedure described here can be applied to other defects. In particular, for strings, the \( R \) term is a topological constant, and the non-trivial correction are \( K \)-like terms which we expect to be important near cusps. We should also mention that the existence of a similar term and its influence on the rigidity of the string were considered before by Polyakov [6]. The evolution of strings with rigidity produced by extrinsic curvature terms was also studied in [7]. In these references, an extra contribution to the action proportional to \((K^0)^2\) is proposed. The major difference from our work is that instead of putting by hand the extra term, we obtain this contribution directly from the field equations of the theory and, besides, we get an additional first order contribution, proportional to the mean curvature \( K^0 \) and connected with the odd part of the zero order solution \( \phi_0 \).

6 Acknowledgments

The authors would like to thank the CNPq for the financial support.

References

1 H. B. Nielsen and P. Olesen, Nucl. Phys. B 61, 45 (1973); D. Foster, Nucl. Phys. B 81, 84 (1974); P. K. Townsend, Phys. Lett. B 202, 53 (1988);

2 K. I. Maeda and N. Turok, Phys. Lett. B 202, 376 (1988); R. Gregory, Phys. Lett. B 206, 199 (1988)

3 R. Gregory, D. Haws, and D. Garfinkle, Phys. Rev. D 42, 343 (1990)

4 R. Gregory, Phys. Rev. D 43, 520 (1991)
5 L. P. Eisenhart, Riemannian Geometry, Princeton University Press

6 A. Polyakov, Nucl. Phys. B 268, 406 (1986);

7 T. L. Curtright, G. I. Ghandour, C. B. Thorn, and C. K. Zachos Phys. Rev. Lett. vol 57, 799 (1986);