Gauge Interaction as Periodicity Modulation

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Abstract
The paper is devoted to a geometrical interpretation of gauge invariance in terms of the formalism of field theory in compact space-time dimensions [1]. In this formalism, the kinematic information of an interacting elementary particle is encoded on the relativistic geometrodynamics of the boundary of the theory through local transformations of the underlying space-time coordinates. Therefore, gauge interaction is described as invariance of the theory under local deformations of the boundary, the resulting local variations of field solution are interpreted as internal transformations, and the internal symmetries of the gauge theory turn out to be related to corresponding local space-time symmetries. In the case of local infinitesimal isometric transformations, Maxwell’s kinematics and gauge invariance are inferred directly from the variational principle. Furthermore we explicitly impose periodic conditions at the boundary of the theory as semi-classical quantization condition in order to investigate the quantum behavior of gauge interaction. In the abelian case the result is a remarkable formal correspondence with scalar QED.

Introduction

In 1918 Weyl [2] introduced the idea of gauge invariance in field theory in an attempt to describe electromagnetic interaction as invariance under local transformations of space-time coordinates. In particular he tried to extend the principle of relativity of the choice of reference frame to the choice of local units of length. For this reason the idea was named gauge invariance. The requirement of gauge invariance in fact necessitates the introduction of a new field in the theory, named compensating field [3]. It cancels all the unwanted effects of the local transformations of variables such as the related internal transformation of the matter fields, and enable the existence of a local symmetry. Weyl noticed that such a compensating field has important analogies with the electromagnetic potential. His proposal was very appealing because of its deep analogies with the geometrodynamical description of General Relativity (GR). Even
though the compensating field associated with local Weyl transformations can be successfully used to represent gravitational interactions, further developments soon showed that such geometrodynamical description of electromagnetism (EM) was not possible. Contrary to the experimental evidence, the compensating field associated with Weyl’s gauge interacts in the same manner with particle and antiparticles. The possibility of a geometrodynamical description of EM in terms of local transformations of the underlying space-time coordinates was abandoned, though the terms gauge invariance still remain for historical reasons. In order to reproduce the correct gauge field (i.e. compensating fields with an imaginary unit in front with respect to the original Weyl compensating fields) an “internal” symmetry under local phase transformations of the matter fields was postulated. These internal transformations of the fields originate ordinary gauge interaction.

Another important attempt for a geometrodynamical description of gauge interaction is represented by the so-called Kaluza’s miracle. In [5] Kaluza showed that classical EM actually has a well defined geometrodynamical interpretation, but this interpretation involves an eXtra-Dimension (XD) — at least as a “mathematical trick”. The gauge field appears as entries in the space-time components on XD Kaluza’s metric and the Maxwell equations are retrieved from the fifth components of the five-dimensional (5D) Einstein equations. This represent a de facto 5D geometrodynamical unification of electromagnetic and gravitational interaction. Also to be considered is Klein’s original proposal, which was to impose Periodic Boundary Conditions (PBCs) at the ends of a compact XD (cyclic XD with topology $S^1$) in the attempt of a semi-classical interpretation of Quantum Mechanics (QM). In [6] he actually noticed that such PBCs provide an analogy with the Bohr-Sommerfeld quantization condition — in particular he used this hypothetical cyclic XD to interpret the quantization of the electric charge. Similarly it is important to mention Wheeler’s program of reduction of every physical phenomenon to a purely geometric aspect, as summarized by his slogan “Physics is Geometry!” [7] or the Rainich’s “already unified field theory” where the (“square”) of the electromagnetic field strength is reinterpreted in terms of the Ricci tensor [8].

In this paper we will investigate these attempts for a geometrical description of gauge invariance in terms of the formalism of field theory in compact space-time dimensions (compact 4D). This formalism, defined in [1] and summarized in sec.(1), see also [9, 10, 11, 12, 13, 14], can be regarded as the natural realization of the de Broglie assumption at the base of wave mechanics (wave-particle duality). In fact, by using de Broglie’s words, the formalism is based on the fundamental assumption “of existence of a certain periodic phenomenon of a yet to be determined character, which is to be attributed to each and every isolated energy parcel elementary particle” [15, 16]. This so-called “de Broglie periodic phenomenon” [17] or “de Broglie internal clock” [18] has been implicitly tested by 80 years of successes of QFT and indirectly observed in a recent experiment [18]. We will realize this assumption by imposing the de Broglie periodicity as a constrain to a free field. That is to say, as the solution of a bosonic action in compact 4D and PBCs. This means that the compactification lengths as well as the boundary of the theory, are explicitly related to the de Broglie periodicity. Therefore the resulting field solution will be a sum over harmonic modes similarly to a string vibrating in compact dimensions. That is, the PBCs (allowed by the variational principle) will be used as quantization condition similarly to the semi-classical quantization of a particle in a box. As noticed in [1] our description can be regarded as the full relativistic generalization of
sound waves. A sound source is a vibrating string in compact spatial dimension within a classical framework. Similarly, an elementary particle will be described as a vibrating string in compact 4D within a relativistic framework. Indeed, as we will mention later, the theory can be regarded as a particularly simple type of string theory.

In the first part of the paper we will mainly investigate classical gauge invariance, in particular classical EM. In par. 2, we will investigate in a geometric way how the de Broglie spatial and temporal periodicities of an elementary boson, i.e. of a corresponding single mode of a Klein-Gordon (KG) field, varies dynamically under local transformations of reference frame or interactions. The temporal and spatial de Broglie periodicities can be used to describe the four-momentum of a particle, according to the de Broglie phase harmony. They can be represented as a four-vector. Such a de Broglie four-periodicity is derived by Lorentz transformations from the invariant periodicity of the proper time associated with the mass of the particle. Indeed it describes a “periodic phenomenon” of topology $S^1$. According to de Broglie, the local and retarded variations of four-momentum (or four-frequency) of a particle occurring during a given relativistic interaction scheme can be equivalently described in terms of local and retarded variations (modulations) of four-periodicity of a corresponding periodic phenomenon. In the formalism of field theory in compact 4D the de Broglie four-periodicity is described by the boundary of the theory. Thus the kinematics of a particle in a given interaction scheme turns out to be encoded in corresponding geometrodynamics of the boundary of the theory — in a manner of a holographic principle. Furthermore, deformations of the boundary of the action can be obtained from corresponding transformations of the space-time variables, i.e. diffeomorphisms of the compact 4D of the theory. Hence this description of interaction mimics linearized gravity. In fact gravitational interaction can be described in terms of modulations of periodicity of reference clocks encoded in corresponding geometrodynamics of the underlying 4D. In [20] we have noticed that such a geometrical description applied to the Quark-Gluon-Plasma (QGP) exponential freeze-out actually provides an interesting parallelism with phenomenological aspects of the AdS/QCD correspondence.

Thus, similarly to GR, we want to describe interaction in terms of invariance of the theory under a corresponding local transformations of variables. In ordinary field theory simple isometric transformations of the underlying space-time dimensions have no effect. This is essentially because in ordinary field theory the BCs have a very marginal role. The KG field used for practical computations is the most generic solution of the KG equation. However, as evident in our formalism, a generic transformation of variables implies a corresponding deformation of the boundary of the theory and in turn, through the PBCs, of the de Broglie periodicity of the field solution of the theory. Indeed we have a variation of field solutions even in the case in which the transformation of variables is a simple isometry. In fact, even if the transformation is from a flat space-time to a flat space-time, leaving the structure of the action invariant, it could involve a corresponding deformation (rotation) of the boundary of the theory. We will show that classical gauge interactions can be derived by requiring invariance of the theory in compact 4D under local isometries and by applying the variational principle at the boundary. To figure out the idea we can imagine to describe the trembling motion [21, 22] (we will use the german term *zitterbewegung* for the analogy of the idea to the Scrödinger’s *zitterbewegung* model [23]) of a charged particle interacting with an electromagnetic field in terms of local transformations of reference frame. For this reason, in this paper we limit our study to the approximation of local isometries; that is, particular isomorphisms where the lengths
of the four-vectors are in a first approximation preserved (contrary to the Weyl invariance we do not consider scale transformations). The case of local scale (conformal) invariance has been partially investigated in [20] through the dualism with XD theories.

More in detail, in sec.(3) we will interpret the variation of the field solution associated to a local isometry in terms of internal symmetries of the field. A local isometry induces a minimal substitution formally equivalent to the one of classical EM. From the variational principle it is possible to find out that this description formally reproduces the Noether current of ordinary gauge interactions. The symmetry of the gauge transformation turns out to be the symmetry of the isometry which originates it. The resulting gauge field describes the local deformation of periodicity of a matter field under a local transformation of variables.

In sec.(4), we will see the possibility to write fields with different periodicities in an action with persistent boundary by using gauge invariant terms. The requirement of gauge invariance is therefore derived from the variational principle. Gauge transformations allow one to tune the periodicity of the different fields of a theory in order to minimize the action at the common boundary. For the same reason we will see that only particular types of local isometries, which we will call polarized, are allowed by the variational principle. These polarized local isometries reproduce Maxwell dynamics for the gauge field. Thus the geometrodynamics associated with these particular local isometries reproduce formally classical gauge theory.

In [20] we have shown that field theory in compact 4D is dual to the Kaluza-Klein (KK) field theory. Under this dualism the geometrodynamical description of gauge interaction proposed here can be regarded as a purely 4D formulation of Kaluza’s original proposal.

The formalism of field theory in compact 4D also provides an interesting analogy with Klein’s original proposal. In fact, in [1, 20] we have shown that the PBCs at the geometrical boundary of the theory represent a semi-classical quantization condition. This can be regarded as the relativistic generalization of the quantization of a particle in a box. This idea is inspired to the ’t Hooft determinism and the stroboscopic quantization where QM is interpreted as an emerging phenomenon associated to some underlying cyclic dynamics. In our theory the Feynman Path Integral (FPI) formulation arises in a semi-classical way as interference between classical paths with different winding numbers associated with the underlying cyclic geometry $S^1$ of the compact 4D. Moreover the theory has implicit commutation relations. Generalizing this description, in sec.(5) we will finally find that the modulation of four-periodicity of an interacting cyclic field is formally described in local Hilbert spaces by the ordinary Scattering Matrix of QM. The space-time evolution associated to such local isometries of the compact 4D is formally described by the ordinary FPI of scalar QED.

1. Compact space-time formalism

In relativistic mechanics every isolated elementary system (free elementary particle) has associated persistent four-momentum $\hat{p}_\mu = \{E/c, -\vec{p}\}$. On the other hand, the de Broglie-Planck formulation of QM prescribes that to every particle with four-momentum there is a corresponding “periodic phenomenon” with four-angular-frequency $\hat{\omega}_\mu = \hat{p}_\mu c/\hbar$, i.e. with corresponding de Broglie four-periodicity $T^\mu = \{T_t, \vec{T}_x/c\}$. The topology of
this so-called “de Broglie periodic phenomenon” is $S^1$. In fact it is fully characterized by the proper time periodicity $T_\tau$ \[15, 16\]. In a generic frame, the spatial and temporal components of the de Broglie four-periodicity $T^\mu$ are obtained through Lorentz transformations: $c T_\tau = c \gamma T_t - \gamma \beta \cdot \vec{X}_x$. The energy and the momentum of a particle with mass $\bar{M}$ in the new reference frame is $E = \gamma \bar{M} c^2$ and $\vec{p} = \gamma \beta \bar{M} c$, respectively. In the “de Broglie periodic phenomenon”, also known as “de Broglie internal clock”, the proper time periodicity is fixed by the mass of the particle, according to $T_\tau \bar{M} c^2 = \hbar$. Therefore, in a generic reference frame, we have de Broglie-Planck relation (de Broglie phase harmony)

\[ c \bar{p}_\mu T^\mu = \hbar. \] (1)

Similarly to the de Broglie assumption of “periodic phenomenon”, in the compact 4D formalism \[1\] we assume that every elementary bosonic particle with four-momentum $\bar{p}_\mu$ is described by an intrinsically periodic bosonic field with de Broglie four-periodicity $T^\mu$ fixed dynamically through PBCs \[4\]. This means that, as long as we describe free particles, i.e. persistent $\bar{p}_\mu$, the intrinsic four-periodicity $T^\mu$ of the fields must be assumed to be persistent. Thus we describe such a free bosonic field with persistent four-periodicity $T^\mu$ as the field solution of a relativistic bosonic action in compact 4D with PBCs (PBCs are represented by the circle in the volume integral $\oint$)

\[ S = \oint_0^{T^\mu} d^4x \mathcal{L}(\partial_\mu \Phi(x), \Phi(x)). \] (2)

Roughly speaking, in a “de Broglie periodic phenomenon” the whole physical information is contained in a single four-period $T^\mu$, \[1\]. In fact, “By a clock we understand anything characterized by a phenomenon passing periodically through identical phases so that we must assume, by the principle of sufficient reason, that all that happens in a given period is identical with all that happens in an arbitrary period” A. Einstein \[39\]. Indeed, under this assumption of intrinsic periodicity, every isolated elementary particle can be regarded as a reference clock. This aspect will play a central role since, in analogy with GR, we will describe interactions in terms of modulations of periodicity of these “de Broglie internal clocks”.

It is important to note that the PBCs minimize the above bosonic action at the boundary. That is, every bosonic field (solution of the Euler-Lagrange equations) with four-periodicity $T^\mu$ automatically minimizes the above action \[1\]. This is the fundamental reason why the theory turns out to be fully consistence with Special Relativity (SR) \[1\]. Indeed PBCs have the same formal validity of the usual Stationary (or Synchronous) BCs (SBCs) of ordinary field theory (i.e fixed values of the fields at the boundary). The Lorentz covariance of relativistic bosonic actions is preserved by PBCs.

The theory satisfies time ordering and relativistic causality. In fact, just as Newton’s law of inertia does not imply that every point particle moves on a straight line (persistent $\bar{p}_\mu$), our assumption of intrinsic periodicities does not mean that our field solutions have always persistent periodicities $T^\mu$. Indeed, the four-periodicity $T^\mu$ must be regarded as local and dynamical as the four-momentum $\bar{p}_\mu$, according to \[1\]. The retarded and local variations of four-momentum occurring during interactions imply retarded and local variations (modulations) of the intrinsic four-periodicity of the particles. Events in time (i.e. interactions) are characterized by variations of periodic regimes of the fields. In this theory we must interpret relativistic causality and time ordering in terms of variations
(modulations) of four-periodicity rather than of variations of four-momentum (roughly speaking, they are two faces of the same coin).

To see explicitly that the theory is Lorentz invariant we consider a global Lorentz transformation

\[ x'^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu \]  

(3) as a global and isometric substitution of variables in the action (2). The resulting action

\[ S = \oint_0^{T^\tau} d^4x' L(\partial'_\mu \Phi'(x'), \Phi'(x')) \]  

(4) describes the same elementary system of (2) but in a new reference frame. It is important to note that, even though we are considering an isometry and the form of the bosonic Lagrangian is unchanged, the transformation of variables induces a correspondent transformation (rotation) of the boundary of the theory

\[ T^\mu \rightarrow T'^\mu = \Lambda^\mu_\nu T^\nu . \]  

(5)

Indeed we find that \( T^\mu \) — being an ordinary space-time interval — transforms as a contravariant four-vector.

This implies that the field solution \( \Phi'(x') \) minimizing the transformed action (4) is in general different from the field solution \( \Phi(x) \) of the original action (2). They have the same equations of motion but they have different BCs. This aspect will play a central role in the rest of the paper. Besides the naive substitution of the 4D variable \( x \rightarrow x' \), to minimize the transformed action (4), the transformed field solution \( \Phi'(x') \) must have transformed de Broglie four-periodicity \( T'^\mu \). Thus, according to (1), the transformed four-momentum associated with the transformed field \( \Phi'(x') \) is

\[ \bar{p}_\mu \rightarrow \bar{p}'_\mu = \Lambda^\mu_\nu \bar{p}_\nu . \]  

(6)

This actually describes the four-momentum of our elementary particle in the new reference frame, as expected from the Lorentz transformation (3). The de Broglie phase harmony (1) prescribes that \( T^\mu \) is such that the phase of the field is invariant under four-periodic translations. But the phase of the field is also invariant (scalar) under transformation of variables. Thus

\[ e^{-\frac{i}{\hbar} x^\mu \bar{p}_\mu} = e^{-\frac{i}{\hbar} (x^\mu + cT^\mu) \bar{p}_\mu} \rightarrow e^{-\frac{i}{\hbar} (x'^\mu + cT'^\mu) \bar{p}_\mu} = e^{-\frac{i}{\hbar} x'^\mu \bar{p}_\mu} . \]

The intrinsic four-periodicity of the field transforms from reference frame to reference frame in a relativistic way as in (5). This description is equivalent to the relativistic Doppler effect. By using the useful notation \( \bar{p}_\mu = h/T^\mu c \) (12) the relativistic constrain on the variations of the temporal and spatial components of the de Broglie periodicity can be expressed as

\[ \frac{1}{T^\tau} \equiv \frac{1}{T^\mu} \frac{1}{T^\nu} . \]  

(7)

This constrain is induced by the underlying Minkowski metric \( c^2 d\tau^2 = dx^\mu dx_{\mu} \). In fact, if multiplied by the Planck constant, (17) is nothing but the geometrical description in terms of four-periodicity of the relativistic constraint

\[ \bar{M}^2 c^2 = \bar{p}^\mu \bar{p}_\mu , \]  

(8)
in agreement with (1). The mass $\bar{M}$ is described by the proper time periodicity $T_\tau$ of the field: $\bar{M} = \hbar / T_\tau c^2$. This corresponds to the time for light to travel across the Compton wavelength of the elementary system $\lambda_s = c / T_\tau$. The heavier the mass, the faster the periodicity. A massless elementary system has infinite (or frozen) proper time periodicity. A hypothetical bosonic particle with the mass of an electron has proper time periodicity of the order of $\sim 10^{-20}$s. This is extremely fast even if compared with the modern resolution in time which is of the order of $\sim 10^{-17}$s, or with the characteristic periodicity of the Cesium atom which by definition is $\sim 10^{-10}$s.

The geometric constraint (7) describes the relativistic dispersion relation of the energy of our elementary system

$$\bar{E}(\bar{p}) = \sqrt{\bar{p}^2 c^2 + \bar{M}^2 c^4}. \tag{9}$$

In the first part of the paper we will consider only the fundamental field solution $\bar{\Phi}(x)$ of the action in compact 4D (2). This is the single mode of de Broglie four-angular-frequency $\bar{\omega}_\mu$ associated with the PBCs at the boundary $T^\mu$. We use the normalization of a string vibrating in compact 4D (similar to the case of “a particle in a box”), so that $\bar{N}$ depends only on the volume of the compact 4D and is invariant under isometries. We will denote by the bar sign the quantities related to the fundamental mode of a cyclic field. That is,

$$\Phi(x) = \bar{N}\bar{\phi}(x) = \bar{N}e^{-i\bar{p}_\mu x^\mu}. \tag{10}$$

As we will see more explicitly in sec.(5), this fundamental mode corresponds to the non-quantum (tree-level) limit of the theory. Since its dispersion relation is (9), it describes the relativistic behavior of a corresponding classical particle with mass $\bar{M}$. In fact, the fundamental solution $\bar{\Phi}(x)$, extended to the whole Minkowskian space-time $\mathbb{R}^4$, formally corresponds to a single mode of an ordinary non-quantized free Klein-Gordon (KG) field $\Phi_{KG}(x) = \bar{\Phi}(x)$ with the same mass $M_{KG} = \bar{M}$ and energy $E_{KG}(\bar{p}) = \bar{E}(\bar{p})$. In other words, they have the same four-momentum $\bar{p}_\mu$, four periodicity $T_\mu$ and dispersion relation (9). Thus, in terms of the invariant mass $\bar{M}$, the fundamental mode $\bar{\Phi}(x)$ is described by the following action in compact 4D

$$\bar{S} = \frac{1}{2} \int^{T_\mu} d^4x \left[ \partial_\mu \bar{\Phi}^* (x) \partial^\mu \bar{\Phi}(x) - \bar{M}^2 \bar{\Phi}^* (x) \bar{\Phi}(x) \right]. \tag{11}$$

This action is formally the KG action with boundary $T^\mu$. In this case the PBCs are replaced by suitable SBCs in order to select the single fundamental mode with persistent four-periodicity $T^\mu$ — we have eliminated the circle from the integral symbol in order to distinguish its solution from more general field solution of the action with PBCs. As we will see in sec.(2.1.1), the study of the variations of BCs proposed here for field theory in compact 4D can be extended to the SBCs of ordinary field theory.

When in the last part of the paper we will investigate the quantum behavior of the theory, we will need to use the most generic field solution with intrinsic four-periodicity $T^\mu$. We will address this generic solution as cyclic field. Similarly to a vibrating string or a particle in a box it is easy to figure out that such a generic cyclic field solution (topology $S^1$) will be a sum of the eigenmodes associated with a quantized energy-momentum spectrum.
2. Interaction

According to the de Broglie-Planck relation (1) the local and retarded variations four-momentum occurring during interactions can be equivalently written as local and retarded modulations of de Broglie four-periodicity. In GR, modulations of periodicity of the reference clocks are expressed in terms of deformations of the underlying 4D. Similarly to GR, we will describe the modulation of periodicity of the cyclic fields during interaction in terms of deformations of the compact 4D of the theory. For these reasons we will generically denote this description with the term “geometrodynamics”.

2.1. Geometrodynamics

In relativistic mechanics a generic interaction scheme can be formalized in terms of corresponding variations of the four-momentum $\bar{p}_\mu(x)$ with respect to the non interacting case $\bar{p}_\mu$. That is,

$$\bar{p}_\mu \rightarrow \bar{p}_\mu(x) = e^a_\mu(x)\bar{p}_a.$$  \hfill (12)

With this notation we mean that the persistent four-momentum $\bar{p}_\mu$ in any given interaction point $x = X$ deforms into $\bar{p}_\mu(x)|_{x=X}$ when we switch on interaction. Hence the interaction scheme (12) is locally encoded by the tetrad $e^a_\mu(x)$.

Similarly to GR where interaction is encoded in a corresponding deformation of the underlying 4D, we will formalize interactions in terms of local diffeomorphisms of the compact 4D. This means that we will generalize the case of global transformation of variable (3) to the case of local transformations. We will therefore use field theory in curved 4D.

To describe this interaction scheme in terms of cyclic fields we must apply the de Broglie-Planck condition (1) locally. It must be noted however that when interaction are concerned, the local four-periodicity of the interacting field which we will denote from now on by $\tau^\mu(x)$, in general does not coincide with the boundary of the theory at $T^\mu(x)$. That is, in general $\tau^\mu(x) \neq T^\mu(x)$. As we will see, $T^\mu(x)$ transforms as a finite and contravariant four-vector, i.e. as $x^\mu$, whereas $\tau^\mu(x)$ transforms as an tangent and contravariant four-vector, i.e. as $dx^\mu$. The former describes the recurrence period $\Phi'(x) = \Phi'(x + T)$ whereas the latter describes the local or instantaneous periodicity in a given point, similarly to the formalism of modulated signals. The local periodicity $\tau^\mu(x)$ of an interacting field varies from point to point, according to the relativistic retarded potentials. In this case the local de Broglie phase harmony is

$$c\bar{p}_\mu(x)\tau^\mu(x) = h.$$  \hfill (13)

Therefore the local variation of four-momentum in the interaction scheme (12) corresponds to the contravariant local variation of the four-periodicity

$$T^\mu \rightarrow \tau^\mu(x) = T^\mu e^a_\mu(x).$$  \hfill (14)

Similarly to the variation of four-momentum induced by a given interaction scheme with respect to the free case, the persistent four-periodicity $T^\mu = \tau^\mu$ of the free field $\Phi(x)$ turns out to be modulated to the local four-periodicity $\tau^\mu(x)$ for the interacting field $\Phi'(x)$.

The deformation of the local four-periodicity $\tau^\mu(x)$ of the cyclic fields is associated with the corresponding stretching of the compactification four-vector $T^\mu(x)$ of the theory.
through the PBCs. This actually induces a deformation of the original Minkowskian metric according to the following relation

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab}.$$  \hspace{1cm} (15)

This resulting curved space-time encodes the interaction scheme (12) locally. To check this geometrodynamical description we consider a local transformation of variables

$$x^\mu \rightarrow x'^\mu(x) = x^a\Lambda^\mu_a(x),$$  \hspace{1cm} (16)

whose tetrad matches the one used in (13) to encode our interaction scheme. That is

$$e^a_\mu(X) = \left(\frac{\partial x^a}{\partial x'^\mu}\right)_{x'=X}.$$  \hspace{1cm} (17)

For the scope of this paper we will work in the approximation $e^a_\mu(x') \simeq e^a_\mu(x)$ (for the sake of simplicity we neglect Christoffel symbols). Since $x^a\Lambda^\mu_a(x)$ is the primitive of the tetrad $e^a_\mu(x)$ we can use the following notation (omitting prime indexes in the integrands)

$$x^a\Lambda^\mu_a(x) \simeq \int x^a dx^a e^\mu_a(x).$$  \hspace{1cm} (18)

The transformation (16) relates locally the inertial frame $x \in S$ of the free cyclic field solution $\Phi$ to the generic frame $x' \in S'$ associated with the interacting cyclic field solution $\Phi'$. Its Jacobian is $\sqrt{-g(x)} = \det[e^a_\mu(x)]$. The Latin letters describe the free field in an inertial frame $S$ while the Greek letters refer to the locally accelerated frame $S'$ of the interacting field $\Phi'$. Finally, by using (16) as a substitution of variables in the compact 4D action (2), we find that the interaction scheme (12) is described by the following action in locally deformed compact 4D

$$S \simeq \int d^4x \sqrt{-g(x)} [\mathcal{L}(e^\mu_a(x)\partial_\mu \Phi'(x), \Phi'(x))].$$  \hspace{1cm} (19)

It is important to point out that (16) induces the local deformation (or stretching) of the compactification four-vector

$$T'^\mu(X) \simeq T^a\Lambda^\mu_a|_X(T) \simeq \int_{X^a=T^a} dx^a e^\mu_a(x).$$  \hspace{1cm} (20)

This is the local deformation of the boundary associated with the modulation of local periodicity $\tau'^\mu(x)$, i.e. with the interaction scheme (12).

As we well see later the resulting formalism will be the four-dimensional analogous of the formalism of modulated signals. Indeed our interacting system is described by the cyclic field solution $\Phi'(x)$ of the transformed action (19) in curved space-time (15) whose four-frequency or four-momentum spectrum is modulated point by point. According to the diffeomorphism (16) or to the phase harmony (13), if the free cyclic field $\Phi(x)$ has four-momentum $p_\mu$, the transformed cyclic field $\Phi'(x)$ has corresponding modulated four-momentum $\bar{p}'_\mu(x)$. The four-momentum of the field is in fact described by the derivative operator $\partial_\mu$ which transforms as the tangent four-vector $\bar{p}'_\mu(x)$, i.e. $\partial_\mu \rightarrow \partial'_\mu = e^a_\mu(x)\partial_a$. 

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The explicit form of the cyclic field solution $\Phi'(x)$ will be written later for the particular transformations where the normalization $\bar{N}$ is left invariant. To study the tree-level behavior of the system under the generic interaction scheme \((12)\) in this first part of the paper will be sufficient to consider only the fundamental modes $\bar{\Phi}'(x)$. Similarly to \((11)\), its interaction is described by the transformed action

$$S \simeq \int_{T^4} \Lambda_\mu |x(T)\rangle \, d^4x \sqrt{-g(x)} \mathcal{L}(e_\mu^a (x) \partial_\mu \bar{\Phi}'(x), \bar{\Phi}'(x)). \tag{21}$$

This is formally the KG action in curved space-time $g_{\mu\nu}(x)$ with finite integration region $T^4(X)$ and suitable SBCs to select $\bar{\Phi}'(x)$ as solution.

2.1.1. Generalization to ordinary field theory

In the practical applications of ordinary field theory the role of the BCs is marginal. The ordinary fields used for computations are the more generic solution of the KG equation, i.e. an integral over all the possible energy eigenmodes. In this paper we will see that the formalism of fields in compact 4D has interesting properties that are not manifest in ordinary field theory. The variation of the boundary of the action describes different field solutions, and in turns different kinematic configurations of the same particles. The action \((21)\) actually allows one to investigate the dynamical behavior of a single KG mode under transformations of reference frame. Its variation of four-momentum $\bar{p}'_\mu(x)$, i.e. its kinematics, is therefore encoded in the deformation of the boundary, in a manner of a holographic principle.

Although in ordinary field theory BCs are not explicitly used in practical computations, a boundary $\Sigma^\mu$ is implicitly assumed (typically of infinite spatial lengths). Suitable SBCs at $\Sigma^\mu$ can also be applied to ordinary KG action in order to select a particular single mode which locally matches a KG mode: $\Phi'(x)|_{x=X} \equiv \Phi_{KG}(X)$. The generalization to the ordinary field theory with generic boundary $\Sigma^\mu$ and suitable SBCs is formally obtained from \((2)\) through the following formal substitution $\int_{T^4} \rightarrow \int_{\Sigma^\mu}$. By analogy to field theory in compact 4D, through the transformation of variable \((16)\) it is easy to find that — in the approximation used in this paper — this generic integration region $\Sigma^\mu(x)$ transforms as \((20)\). That is, $\Sigma^\mu \rightarrow \Sigma^\mu(X) \sim \Sigma^\mu \Lambda_\mu |x(\Sigma)\rangle$. With this we wanted to point out that the analysis of gauge interactions that we will carry on in the next sections for the fundamental mode, can be extended to single modes of ordinary non-quantized KG fields.

2.2. Applications

We have introduced interactions for cyclic fields in terms of the local diffeomorphisms \((16)\) from a flat manifold $S$ to a generic manifold $S'$. Thus, under the interaction scheme \((12)\), our system is described in terms of the geometry of the compact 4D.

In order to illustrate this in a heuristic way, we will briefly discuss two examples: the weak Newtonian interaction and the QGP logarithmic freeze-out. The complete analysis

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2 Typically $\Sigma^\mu$ has infinite size along the spatial directions, at whose boundary the field is imposed to vanish, whereas along the time dimension the boundaries are the generic initial and final time where SBCs are supposed. For a KG theory, given the mass as a parameter, the description of the deformation of the time boundary with SBCs is sufficient to described kinematical variations of the field or interactions.
of the former case will be left to forthcoming papers whereas the case of the QGP has
been investigated in [20]. In this preliminary section we will work in the approximation
in which the local four-periodicity can be identified with the compactification four-vector
of the theory \( \tau^\mu(x) \sim T^\mu(x) \). This approximation is valid in the cases where the metric
varies sufficiently smoothly, i.e. \( e^\mu_a(x) \sim \Lambda^\mu_a(x) \) (or logarithmically, see [20]).

In the next section, by writing (12) as a minimal substitution, we will show that such
a geometric approach to interactions can be also used to describe gauge interactions.

2.2.1. Weak Newtonian potential

The geometric approach to interactions described above is interesting because it ac-
tually mimics the usual geometrodynamical approach of linearized gravity. To show
this we consider a weak Newtonian potential \( V(x) = -GM_\odot/|x| \ll c^2 \). Under this
interaction scheme the energy in a gravitational well varies with respect to the free
case as \( E \rightarrow E' \sim (1 + GM_\odot/|x|c^2) E \). According to the geometrodynamical
description of interaction (14), this means that the de Broglie clocks in a gravita-
tional well (the periodicity of cyclic fields) are slower with respect to the free clocks
\( T_t \rightarrow \tau'_t \sim (1 - GM_\odot/|x|c^2) T_t \). With the schematization of interactions based on the de
Broglie phase harmony (1), we have retrieved two important predictions of GR such as
time dilatation and gravitational red-shift \( \bar{\omega} \rightarrow \bar{\omega}' \sim (1 + GM_\odot/|x|c^2) \bar{\omega} \).

Besides this we may also consider the analogous variation of spatial momentum and
the corresponding variation of spatial periodicities of cyclic fields in a gravitational well
[4]. Thus, according to (15), we find that the weak Newtonian interaction turns out to
be encoded in the linearized Schwarzschild metric.

Indeed, the geometrodynamical description of interaction in the formalism of compact
4D actually mimics linearized gravity. In the formalism of compact 4D with PBCs,
i.e. under the assumption of intrinsic periodicity, every cyclic field can be regarded
as a relativistic reference clock (see again Einstein’s definition [39]), namely the “de
Broglie internal clock”. The diffeomorphism (16) encodes the modulation of periodicity
of this clocks occurring during interaction. This is similar to GR where gravitational
interaction can be interpreted in terms of modulations of periodicity of reference clocks
encoded in a corresponding deformed 4D background. Furthermore, it is well known
that GR can be derived from the linearized formulation by considering self-interactions
[4] — for instance by relaxing the assumption of smooth interactions. Nevertheless it
is important to mention that “what is fixed at the boundary of the action principle of
GR” is not uniquely defined [28]. More considerations about this aspect has been given in
[12, 13]. SR and GR fix the differential framework of the 4D without giving any
particular prescription about the BCs. The only requirement for the BCs is to minimize
a relativistic action at the boundary. For this aspect both SBCs and PBCs have the same
formal validity and consistence with relativity. With this analysis we have provided an
evidence of the consistence of our formalism of compact 4D with GR.

2.2.2. AdS/CFT interpretation

In [20] we have applied the geometrodynamical description of interaction in compact
4D to a simple Bjorken Hydrodynamical Model for QGP logarithmic freeze-out [29]. In a
first approximation the fields constituting the QGP can be supposed massless and their
four-momentum can be supposed to decay exponentially during the freeze-out. According
to (12), this interaction scheme is therefore encoded by the conformal warped tetrad
\[ e^a_\mu(|x|) = \delta^a_\mu e^{-k|x|/c}, \] where \(|x|/c = s/c = \tau\) and \(k\) are the proper time and the gradient of the QGP freeze-out, respectively. The time periodicity \(T_t(s) = e^{ks}/k = h/E(s)\) is the conformal parameter which describes naturally the inverse of the energy of the fields during the freeze-out, according to de Broglie. The resulting variations of normalization of the cyclic field solutions during the freeze-out reproduce formally the logarithmic running of the coupling constant typical of QCD \cite{20}. In this paper we will not explore further the running associated with gauge interactions. This means that we will limit our investigation to transformations of variables which (in a first approximation) preserve the lengths.

Moreover, in \cite{20} we have also shown that a field in deformed compact 4D is dual to a massless XD field in a corresponding 5D metric. For instance they have the same topology \(S^1\). The dualism is explicit if we assume that our cyclic world-line parameter plays the role of a cyclic XD. To address this identification we say that the theory has a virtual XD, see also \cite{1}. In the virtual XD formalism, the QGP exponential freeze-out turns out to be encoded in a virtual AdS metric and the classical configurations of cyclic fields in such a deformed background reproduces basic aspects of AdS/QCD.

3. Rotating the boundary

In order to give a mental picture of the description of gauge interaction that we want to achieve in next sections we can imagine a charged particle (for simplicity a boson) interacting electromagnetically. Typically, the particle will be characterized by dynamics (similarly to the trembling motion of the \textit{zitterbewegung}) induced by the interaction. Intuitively we will describe such dynamics in terms of local transformations of flat reference frame (avoiding the use of creation and annihilation operators). The resulting modulations of de Broglie space-time periodicity will be used to reproduce gauge interaction. The formalism that we will adopt has analogies with that modulated signals. In analogy with an antenna, the EM field can characterized by the dynamics of the charged particle generating it.

Indeed, in the particular approximation of local isometries, only the boundary of the theory is deformed without affecting the underlying flat metric. The coefficients of these local isometries can be described in terms of vectorial fields which therefore encode the transformation. The resulting variation of four-momentum and modulation four-periodicity of the field solution turns out to be written formally as the \textit{minimal substitution} and the \textit{parallel transport} of ordinary electrodynamics, respectively.

3.1. Minimal substitution

We now want to apply the geometrodynamical description of interaction \cite{12} to the following local infinitesimal transformation of variables

\[ x^\mu(x) \rightarrow x'^\mu \sim x^\mu - e \xi a_\mu \Omega^\mu(x). \] \hspace{1cm} (22)

The coefficient of the expansion is denoted by \(e\) and address as \textit{gauge coupling}. In this paper we will work in the approximation in which this transformation is a local isometries, \textit{i.e.} we limit our study to the case in which the Jacobian is \(\sqrt{-g'} \approx 1\).

\footnote{More exactly “it is natural to attribute the origin of the \textit{zitterbewegung} to the self-interaction of the electron with its own electromagnetic field.” \cite{21}}
In terms of the formalism (16), a local isometry is described by
\[ \Lambda_\mu^a(x) \simeq \delta_\mu^a - e\Omega_\mu^a(x), \] (23)
whereas the tangent transformation (19) is described by the local tetrad
\[ e_\mu^a(x) \simeq \delta_\mu^a - e\omega_\mu^a(x). \] (24)
As we will discuss in sec. (4.4), these isometries can be regarded as belonging to some local subgroups of the Lorentz transformations. Moreover the requirement of local isometries implies that \( \omega_\mu^a(x) \) is antisymmetric (Killing equations). For the sake of simplicity we assume that such an isometry is a unitary transformation
\[ \omega_\mu^a(x) \in U(1). \]
According to (18) the finite and tangent transformations are related by
\[ x^a \Omega_\mu^a(x) = \int x^a \omega_\mu^a(x). \] (25)
To each point \( x = X \) of the inertial frame \( S \) we are associating a local Lorentz reference frame \( S' \), as represented by the orthogonal tetrad \( e_\mu^a(X) \). As already noticed, the tetrad \( e_\mu^a(x) \) encodes the information, point by point, of a corresponding interaction scheme. In this case the information is contained in \( \omega_\mu^a(x) \) which in turn can be used to define a vectorial field \( \bar{A}_\mu(x) \) as
\[ \bar{A}_\mu(x) \equiv \omega_\mu^a(x)\bar{p}_a. \] (26)
Thus the vectorial fields \( \bar{A}_\mu(x) \) can be used to encode the interaction scheme (22). In particular, the variation of four-momentum (12) associated with this local isometry is in this case given by
\[ p'_\mu(x) \sim p_\mu - e\bar{A}_\mu(x), \] (27)
which formally is the minimal substitution of the vectorial field \( \bar{A}_\mu(x) \).
Since we are assuming that the local isometry (22) is a unitary transformation \( \omega(x) \in U(1) \), we say that the vectorial field \( \bar{A}_\mu(x) \) is an abelian field. In sec. (4.4) we will discuss the geometrical meaning of this assumption as well as the generalization to non-abelian isometries.
Under such a local isometric change of flat manifold \( g_{\mu\nu}(x) \simeq \eta_{\mu\nu} \), the resulting transformed action (21) has the same structure of the original one, i.e. the equations of motion remain unchanged. Nevertheless the boundary of the transformed action turns out to be locally rotated with respect to the original action (11). Therefore its fundamental solution \( \bar{\Phi}'(x) \) in the point \( x = X \) turns out to be different from the original fundamental solution \( \bar{\Phi}(x) \). The free solution \( \bar{\Phi}(x) \) has the persistent periodicity \( \tau^\mu = T^\mu \) of an isolated particle. The transformed solution \( \bar{\Phi}'(x) \) has modulated periodicity \( \tau'^\mu(X) \) varying from point to point in order to describe interaction.
3.1.1. Global Isometry

To formulate interactions in terms of gauge fields we need to express the rotations of the boundary in terms of internal transformations of the field.

Here, as in par. 11, we consider the simple case of a global isometry

\[ A_{\mu}^a(x) = e_{\mu}^a(x) \equiv e_{\mu}^a. \]  

(29)

For reasons that will be clear later we address this case as pure gauge — this terminology is not completely equivalent to ordinary QFT.

In this case the tetrad is homogeneous, it does not depend on \( x \). The resulting four-periodicity \( \tau^\mu = e_{\mu}^a T^a \) and four-momentum \( p^\mu = e_{\mu}^a \tilde{p}_a \) of the transformed field \( \bar{\Phi}' \) vary globally. The transformed four-periodicity (5) coincides with the compactification four-vector of the transformed action \( \tau^\mu = T^\mu \).

In every interaction point \( x = X \) the fundamental solution \( \bar{\Phi}'(x') \) associated with the transformed action (28) is related to the original solution \( \Phi(x) \) by the following transformation

\[ \Phi(x) = N e^{-\frac{\hbar}{2} x^\mu \tilde{p}_\mu} \rightarrow \Phi'(x') = N e^{-\frac{\hbar}{2} x'^\mu \tilde{p}'_\mu}. \]  

(30)

It is easy to see that, as long as \( \bar{\Phi}(x) \) is a fundamental solution of the free action, under this transformation \( \bar{\Phi}'(x') \) is automatically the correct fundamental solution of the transformed action (28). In fact, it has transformed four-periodicity \( T^\mu = e_{\mu}^a T^a \), according to the de Broglie phase harmony.

Now we expand the global transformation \( e_{\mu}^a \) as in (22), so that the interaction scheme is formally the minimal substitution (27); \( \bar{A}_\mu \) and \( \tilde{p}_a \) are homogeneous (constant). The physical effect of this pure gauge is a global transformation of reference frame. Thus, under this global isometry the field transforms as

\[ \Phi(x) = N e^{-\frac{\hbar}{2} x^\mu \tilde{p}_\mu} \rightarrow \Phi'(x') = N e^{-\frac{\hbar}{2} x'^\mu (\tilde{p}'^a - e^{A^a}_\mu)}. \]  

(31)

This also means that to our transformation of variables there is associated an internal transformation of the field described by

\[ V(x) = e^{\frac{\hbar}{2} x^\mu A^a_\mu}. \]  

(32)

We call \( A_\mu \) gauge connection and \( V(x) \) parallel-transport. Since we are assuming abelian transformation of variables, the resulting internal transformation of the fundamental solution (31) generates an abelian parallel-transport \( \bar{V}(x) \in U(1) \).

We also introduce the covariant derivative of the transformed field \( \bar{\Phi}' \) as

\[ \partial_\mu \Phi(x) = \partial_\mu [V^{-1}(x) \bar{\Phi}(x)] = V^{-1}(x) D_\mu \bar{\Phi}(x). \]  

(33)

Thus the covariant derivative associated with a global isometry is

\[ \partial_\mu \rightarrow D_\mu = \partial_\mu - \frac{ie}{\hbar} \bar{A}_\mu. \]  

(34)

It is important to note that, even though the transformed fundamental solution \( \bar{\Phi}'(x) \) has a transformed four-periodicity \( T'^\mu \), the terms \( \bar{V}^{-1}(x) \bar{\Phi}(x) \) and \( \bar{V}^{-1}(x) D_\mu \bar{\Phi}(x) \) have the same persistent four-periodicity \( T^\mu \) as the original fundamental mode \( \Phi(x) \) and its
derivative $\partial_\mu \Phi(x)$, respectively. We will address this important aspect by saying that the inverse of the parallel transport, together with the covariant derivative in derivative terms, tunes the periodicity $T'_{\mu}$ of the transformed fundamental mode $\Phi'(x)$ to the periodicity $T'_{\mu}$ of the free fundamental mode $\Phi(x)$. We will next generalize these considerations to local isometries.

3.2. Local Isometry

In the general case of local isometries, the tetrad $e^a_\mu(x)$ varies from point to point. Thus we must take into account that, at every point $x = X$, the transformed fundamental mode $\Phi'(x)$ must have local four-periodicity $\tau'_{\mu}(x)|_{x = X}$ in order to be a solution of the transformed action (28) with boundary at $T'_{\mu}(x)|_{x = X}$ given by (20). This also means that the transformed fundamental solution $\Phi'(x)$ must have four-momentum $p'_\mu(x)|_{x = X}$ in order to satisfy (13) in $x = X$. In the approximation $e^a_\mu(x) \sim e^a_\mu(x')$ and considering that the normalization factor $N$ is invariant under isometries, the fundamental mode $\Phi'(x')$ solution of the transformed action (28) can be written, according to our notation (18), as

$$\Phi(x) = \bar{N} e^{-i e^a_\mu \cdot A^a(x)} \rightarrow \Phi'(x') = \bar{N} e^{-i e^a_\mu \cdot A^a(x)} e^{-i e^a_\mu \cdot p'_\mu(x)}.$$ 

To check this, besides the de Broglie phase harmony, the analogy with the CKM formalism and the modulated signals formalism, we may note that the derivative operator $ih\partial_\mu$ actually gives the correct transformed four-momentum $p'_\mu(x)$ in the new reference frame

$$ih\partial_\mu \Phi(x) = \bar{p}_\mu \Phi(x) \rightarrow ih\partial_\mu \Phi'(x') = \bar{p}'_\mu(x') \Phi'(x').$$ 

According to the definition of the local tetrad $e^a_\mu(x)$ in (24), the fundamental solution $\Phi'(x')$ of (28) is obtained from the free fundamental solution $\Phi(x)$ by the following internal transformation of the field

$$\Phi(x) = \bar{N} e^{-i e^a_\mu \cdot p} \rightarrow \Phi'(x') = \bar{N} e^{-i e^a_\mu \cdot \int dx' A^a(x')} e^{-i e^a_\mu \cdot p} = \bar{V}(x') \Phi(x').$$

Hence the parallel-transport $\bar{V}(x)$ describing the internal transformation of fundamental solution under this local abelian transformation of variables is formally a Wilson line of the gauge connection

$$\bar{V}(x) = e^{i e^a_\mu \cdot \int dx' A^a(x')}.$$ 

This allows one to pass from a fundamental solution with persistent periodicity $\tau_{\mu}$ to a fundamental solution with transformed local periodicity $\tau'_{\mu}(x)$. In this way, as long as $\Phi(x)$ is solution of the free action, the modulated field $\Phi'(x')$ is automatically solution of the transformed action (28). The vectorial field $\bar{A}_\mu(x)$ describes the resulting modulation of periodicity under the local transformation of reference frame.

The generalization to local isometry of the covariant derivative (33) of the transformed field $\Phi'(x)$ is

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - \frac{ie}{\hbar} \bar{A}_\mu(x).$$ 

Also in this more general case we find that the inverse of the parallel-transport, together with the covariant derivative in derivative terms, tunes the locally varying periodicity $\tau'_{\mu}(x)$ of the transformed fundamental field $\Phi'$ to the persistent periodicity $\tau_{\mu} = T'_{\mu}$ of the free fundamental field $\Phi$. This tuning through parallel transport will be used to allow a field with locally varying four-periodicity $\tau'_{\mu}(x)$ to fulfill the variational principle in an action with persistent boundary $\tau_{\mu} = T'_{\mu}$.
3.3. Noether Currents

The formalism of field theory in compact 4D allows one to relate, through the variational principle, transformation of variables $\delta x$ and internal transformations of the field $\delta \Phi(x)$. This can be easily seen from the role of the parallel-transport (38). The internal transformation associated to the local abelian isometry (22) is indeed

$$
\delta \Phi(x) = \Phi'(x) - \Phi(x) = ie\bar{A}_\mu(x)\Phi(x)\delta x^\mu.
$$

(40)

To understand the role of these internal transformations of the field we consider the role of the boundary terms in the variation of the action. In the approximation of local isometries, the original and transformed action differ by an explicit variation of the boundary

$$
\delta \bar{S} = \int^{T'_\mu} d^4x' \bar{L}'(\Phi'^i, x') - \int^{T_\mu} d^4x \bar{L}(\Phi^i, x).
$$

(41)

As represented diagrammatically in fig. 1, the stress-energy-momentum of the fundamental mode $\Phi$ is not manifestly conserved because of the local transformation of reference frame. In fact, it is easy to see from (41) that the conservation of the stress-energy-momentum tensor involves the contribution of the internal transformation of the field (40), that is, of the current

$$
\bar{J}_\mu = ie[\Phi^* D_\mu \Phi - D_\mu \Phi^* \Phi].
$$

(42)

The interesting aspect of this analysis is that actually the current $\bar{J}_\mu$ has the same form as the Noether current of an abelian gauge invariant theory with internal transformation (40), as long as we identify the connection $A_\mu(x)$ with an abelian gauge field.
4. Gauge interaction

The formalism introduced so far is very useful to describe the geometrodynamics associated with a given interaction scheme of an elementary particle. However it does not explicitly take into account the conservation of four-momentum. This is because it involves a transformation of reference frame. As we have pointed out in the Noether analysis, this is related to the local variations of the boundary. From a analytic point of view it would be easier to describe the same interaction scheme in terms of an action whose boundary is invariant under isometric transformation of variables. In this way the currents related to the transformation of the boundary will be directly expressed in terms of symmetries of the Lagrangian. The possibility of such a formalism is offered by the fact that the periodicities of the fields can be tuned through parallel transport. The result will be an ordinary gauge invariant theory.

4.1. Tuned action

We want to define a new formalism to describe $\bar{\Phi}'(x)$ under the interaction scheme (27) such that there is an explicit conservation of four-momentum. Our strategy will be to write a new action, which we will call tuned action, containing the same physical information of the transformed action (28) but with persistent boundary at $T^{\mu}$. Similarly to the transformed action, this tuned will be obtained directly from the free action (11).

To build the tuned action we need to know that to a change in the periodicity of the fundamental field solution there is an associated internal transformation (37), i.e a parallel-transport. If we want to vary the four-periodicity of a field in an action with given boundary, and at the same time fulfills the variational principle at the boundary, we must use the parallel-transport to tune the four-periodicity of the field. Since the only terms involved in the BCs are the derivative terms (through integration by parts), the tuned action can be obtained from the free action by modifying only derivative terms. We have already noticed, for instance, that the corresponding covariant derivative (39) allows one to tune the periodicity of $\bar{\Phi}'(x)$ to the periodicity of $\bar{\Phi}(x)$.

In the tuned action the interaction will be described in terms of symmetries of the Lagrangian rather than in terms of the variations of the boundary. Hence the interacting field $\bar{\Phi}'(x)$ will be described by different equations of motion with respect to the free case and the currents (42) will be the conserved currents associated with the symmetries of the tuned Lagrangian.

At a mathematical level this can be easily achieved through the parallel-transport $\bar{V}(x)$, by explicitly writing in the free action (11) the fundamental solution $\bar{\Phi}(x)$ as a function of the transformed fundamental solution $\bar{\Phi}'(x)$, i.e. by using (37). In this way we find

$$\int_{T^{\mu}} d^4x \mathcal{L}(\partial_{\mu} \bar{\Phi}, \bar{\Phi}) = \int_{T^{\mu}} d^4x \mathcal{L}(\partial_{\mu} \bar{V}' \bar{\Phi}' \bar{V}'^{-1} \bar{V}'^{-1} \bar{\Phi}', \bar{V}'^{-1} \bar{\Phi}') = \int_{T^{\mu}} d^4x \mathcal{L}(D_{\mu} \bar{\Phi}', \bar{\Phi}').$$

(43)

According to the definition (39), the derivative terms $\partial_{\mu} \bar{\Phi}(x)$ of the original action must be replaced by the covariant derivative $D_{\mu} \bar{\Phi}'(x)$ in order to tune locally the periodicity $\tau'^{\mu}(x)$ to $T^{\mu}$. We have also used the fact that non-derivative terms, such as the mass term, contributes only to the equations of motion, but not to the BCs (the periodicity of the solution in such terms can be arbitrarily varied without compromising the variational principle at the boundary).
Therefore the tuned Lagrangian can be directly obtained from the free Lagrangian by replacing the ordinary derivatives with covariant derivatives

$$\tilde{L}_{\text{tuned}}(\partial_\mu \tilde{\Phi}', \tilde{\Phi}', A_\mu) = \tilde{L}(D_\mu \tilde{\Phi}', \tilde{\Phi}') .$$ (44)

In the specific interaction scheme (27) for the fundamental scalar mode of a cyclic field with mass $\tilde{M}$, the tuned action is

$$\mathcal{S} = \frac{1}{2} \int T^\mu d^4x \left[ D_\mu \tilde{\Phi}'(x) D^\mu \tilde{\Phi}'(x) - \tilde{M}^2 \tilde{\Phi}'(x) \tilde{\Phi}'(x) \right] .$$ (45)

The tuned action is related to the original free action by parallel-transport. If $\tilde{\Phi}(x)$ is the fundamental solution of the original free action, then the transformed fundamental solution $\tilde{\Phi}'(x)$ automatically minimizes (45). On the other hand $\tilde{\Phi}'(x')$ is also the solution of the transformed action (28). Hence the tuned action (45) contains the same physical information of the transformed action (28).

The vectorial field $\tilde{A}_\mu(x)$ encodes the modulation of periodicity of an interacting system giving rise to covariant derivatives in the tuned action. If we identify the gauge connection $\tilde{A}_\mu(x)$ with an ordinary gauge field, the tune action (45) is formally a gauged KG action with boundary.

In this way we have given a geometrical meaning to covariant derivatives, parallel-transports, and gauge connections, in terms of variations of periodicities. This picture can be intuitively interpreted by imagining the transporting of the arms of a clock with given periodicity on a closed path in a curved manifold. At the end of the loop we must either retune the arms of the clock through parallel-transport or to assume that the clock has varied its characteristic periodicity. Indeed the parallel-transport allows one to describe the modulation of periodicity “associated with deformation of the underlying manifold in a background independent way”.

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4.2. Gauge Invariance

At a mathematical level the parallel-transport $\bar{V}(x)$ allows one the possibility to tune periodicity of terms of the Lagrangian to the periodicity imposed by the boundary of the action. However it is easy to note that such a parallel-transport, as well as the gauge connection, is not uniquely defined: the tuned action (45) has a manifest invariance which we call — for obvious reasons — gauge invariance.

It is well known that (45) is invariant under the following transformation

$$\bar{A}_\mu(x) \rightarrow \bar{A}'_\mu(x) = \bar{A}_\mu(x) - e\partial_\mu \bar{\theta}(x).$$

(46)

In the parallel-transport $\bar{V}(x)$, which is formally a Wilson line, this transformation generates boundary terms which can be absorbed by the following local phase transformation of the fundamental scalar mode

$$\bar{\Phi}'(x') \rightarrow \bar{\Phi}''(x') = \tilde{N} e^{-i\frac{\hbar}{e} \bar{\theta}(x)} \bar{\Phi}'(x).$$

(47)

The resulting local phase transformation of the field is

$$\bar{U}(x) = e^{-i\frac{\hbar}{e} \bar{\theta}(x)}.$$  (48)

This local phase turns out to be of the same kind as the parallel-transport $\bar{V}(x)$ or of the local isometry generating it. Therefore $\bar{U}(x) \in U(1)$ in the case of abelian isometry. We have finally shown that a fundamental mode subject to a local abelian isometry is described by an abelian gauge invariant action.

The local gauge invariance $U(1)$ is now a symmetry of the tuned Lagrangian. Therefore gauge transformations do not affect the boundary of the action. In other words we have found that gauge invariance identifies particular class of isometries describing the same interaction scheme. For this reason we call them gauge orbits (we only mention that such a gauge orbit can be regarded as an holonomy, since it corresponds to an isometry whose parameter $\omega(x)$ is a total derivative and “the boundary of a boundary is zero”).

The meaning of the tuning of the field described so far can be generalized. Gauge invariant terms allow one to tune the periodicity of the fundamental field solution to the periodicity imposed, through the variational principle, by the boundary of the action. This is essentially because the parallel-transport $\bar{V}(x)$ tunes the periodicity of the bosonic field solutions.

At this point we can repeat the variational analysis of par. 3.3. Noether’s theorem can be applied directly to the tuned Lagrangian instead of the transformed action. Since the tuning formalism allows a description in which the boundary of the action does not vary under isometric transformation of variables, it is easy to show that the Noether currents (42) are directly associated with the symmetries of the tuned Lagrangian. Indeed the Noether analysis is completely parallel to the one of ordinary gauge theory.

The remaining step to prove that the interaction scheme associated with such a local abelian isometry is nothing but the usual gauge interaction, is to find that the dynamics of the abelian gauge field $A_\mu(x)$ is actually described by the Maxwell equations.

4.3. Yang-Mills action

The tuned action (45) is formally a $U(1)$ gauge invariant action. The tuning of the interacting field has been obtained at the expense of the introduction of a new field in
the theory. This is the gauge connection $\bar{A}_\mu(x)$. It compensates the variation of four-periodicity of the interacting field in order to have a tuning to the persistent boundary of the action. Therefore it is natural to interpret $\bar{A}_\mu(x)$ as a new dynamical field with given four-momentum, in general different from $\bar{p}_\mu$ or $\bar{p}'\mu(x)$, and thus with given local four-periodicity, in general different from $T^\mu$ or $\tau'^\mu(x)$. This is illustrated in fig. (2). This requires one to introduce a kinetic term for $\bar{A}_\mu(x)$. Similarly to the interacting fundamental field solution $\Phi'(x)$ of the tuned action (45), this kinetic term may appear in an action with persistent integration region $T^\mu$, in order to describe explicitly the conservation of four-momentum. That is, we want to add a kinetic term for $A_\mu(x)$ to the tuned action (45) — a similar analysis can be done for the transformed action.

From the correspondence between gauge invariance and the periodicity tuning, we require that such a kinetic term must be gauge invariant. In fact, only in this way the periodicity of $A_\mu(x)$ can be tuned with the BCs imposed by the tuned action. In particular we have already noticed that such a tuning of the periodicity of the field in derivative terms, such as in the kinetic terms, is possible by using covariant derivatives. Through suitable covariant derivatives the four-periodicity of $\bar{A}_\mu(x)$ in the kinetic term can be tuned to $T^\mu$.

Therefore, by following the same requirement of gauge invariance as in ordinary field theory, we infer that the correct kinetic terms allowed for $\bar{A}_\mu(x)$ by the variational principle is the gauge invariant term $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, where $F^{\mu\nu}$ is the field strength. In fact, it is well known that the derivatives in the field strength can be replaced by arbitrary covariant derivatives of the gauge field itself $F^{\mu\nu}(x) = \bar{p}'_\mu A_\nu(x) - \bar{p}'_\nu A_\mu(x)$. This means that in such a kinetic term the periodicity of the gauge field is tuned to the characteristic periodicity of the action through an appropriate covariant derivative (in general different from the one of the matter field $\Phi'$).

Finally, consistently with the variational principle, the full action describing the interaction scheme (24) is

$$S_{YM} = \int_{T^\mu} d^4x \left\{ -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{1}{2} [D_\mu \Phi'(x) D^\mu \Phi'(x) - M^2 \Phi'(x) \Phi'(x)] \right\}. \tag{49} $$

We have obtained nothing but the usual non-quantum (tree-level) abelian Yang-Mills action (in a finite volume) describing a matter field solution with four-momentum $\bar{p}'_\mu(x)$ interacting with a gauge field $\bar{A}_\mu(x)$ and total four-momentum $\bar{p}_\mu$.

It must be noticed that the simultaneous minimization of the above action at the boundary for both the fields $\Phi'(x)$ and $\bar{A}_\mu(x)$, or equivalently the requirement of gauge invariance of the action, constrains the general form of the gauge field. In turn the so far generic form of the abelian transformation of variables (24) is constrained to a particular subclass, modulo gauge orbits. In fact the fundamental field solution $\bar{A}_\mu$ is no more a generic vectorial field. That is, according to the variational principle, it must be solution of the equations of motion associated with (49), i.e. it has Maxwell dynamics. For instance this means that $\bar{A}_\mu(x)$ is massless and has only two transversal d.o.f. On the other hand, through (25), the Maxwell dynamics of gauge field $\bar{A}_\mu(x)$ corresponds to related

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4For instance, in a given gauge $\bar{A}_\mu(x)$, the correct covariant derivative to tune the gauge field is given by combining the inverse parallel-transport $\bar{V}^{-1}(x)$ and the gauge transformation $\bar{U}(x)$ where $e\theta(x) = \bar{p}_\mu x^\mu$. 20
dynamics of the coefficients $\omega^\mu_a(x)$ of the transformation of variables. Only isometries satisfying this dynamics and which we will address as transversally polarized are allowed by the variational principle. The result is a geometrodynamical description of ordinary gauge interactions.

The kinetic term of the gauge field has been inferred by noticing that the variational principle requires gauge invariant terms. The requirement of gauge invariance is actually the usual way to introduce such kinetic terms in ordinary YM theory. The same argument can be used to introduce the kinematic terms for the gauge field in the formalism of the transformed action (28). In this case we must assume covariant derivatives only in the field strength such that the periodicity of the gauge field is tuned to the locally varying periodicity $\tau^\mu(x)$ of the interacting field.

“The modern viewpoint”, [30, 3], to introduce gauge interaction in ordinary field theory is to postulate a parallel-transport — sometimes called connection — to a matter field, i.e. to postulate internal symmetries. In this way the derivative terms generate covariant derivatives and the Lagrangian is gauge invariant. Thus, in ordinary field theory, “the covariant derivative and the transformation law of the connection $\bar{A}_\mu$ follow from the postulate of local phase rotation symmetry” [30]. From the viewpoint of field theory in compact 4D the same gauge invariant Lagrangian is obtained from the geometrodynamics allowed by the variational principle, without postulating it. In fact we have seen that the parallel-transport of a matter field arises naturally to describe the modulation of periodicity associated with local transformation of variables and to tune the periodicity of the field solution to the one imposed by the minimization of the action at the boundary. The invariance of the action under transformation of variables induces an internal transformation of the field solutions. This reveals an intuitive geometrodynamical nature of gauge interactions. This important and non-trivial result is in the spirit of Weyl’s original proposal of a geometric interpretation of gauge interactions.

The formalism of compact 4D makes manifest this geometrodynamical interpretation of gauge interaction, because it explicitly relates internal transformations of the field solution to variations of the BCs. The same arguments can be in principle repeated in ordinary field theory by replacing the compact integration region $T^\mu$ with $\Sigma^\mu$ and the PBCs with SBCs.

4.4. Non-abelian case

We conclude this section by giving a generalization of our interaction scheme to non-abelian local transformations of variables and discussing the relation to space-time symmetries.

To generalize our result to the non-abelian case we must assume that the transformation of variables (23) originating our interaction scheme (27) is an element of a non-abelian group $H$. This implies that in the equation obtained so far we must perform the following substitution from an abelian vectorial field to a non-abelian one

$$\bar{A}_\mu(x) \to \bar{A}_\mu^a(x)\tau^a,$$

where $\tau^a$ are generators of $H$. As a consequence of the commutation relations of the generators the parallel-transport must be written as an path-ordered Wilson line

$$\bar{V}(x') = e^{i\frac{\hbar}{\epsilon} \int_{x'}^{x'} dx . A^a(x) \tau^a} \to \bar{V}(x') = P[e^{i\frac{\hbar}{\epsilon} \int_{x'}^{x'} dx . \bar{A}^a(x) \tau^a}].$$

(50)
A similar redefinition must be considered for the covariant derivatives. In this way it turns out to describe a Yang-Mills theory with non-abelian gauge invariance \( H \).

We have described gauge interaction in the approximation where the transformation of variables \( (22) \) is a local isometry. In this approximation the lengths of the four vector are preserved and we do not consider variations of the normalizations of the fields. Moreover the variational principle allows only a particular subclass of polarized local isometries, modulo gauge orbits. In this way the dynamics for the related vectorial field turns out to be the usual Maxwell’s dynamics.

We can now associate the polarized isometries \( (26) \) describing our interaction scheme to corresponding polarized local Lorentz transformations. By representing the Lorentz group as \( SU_L(2) \otimes SU_R(2) \), the most general global isometry which we may consider is \( \omega \in SU_L(2) \otimes SU_R(2) \). In this case \( e \omega^a_\mu = g w^a_\mu \tau^i + g' y^a_\mu \tau^i \) where \( w^a_\mu \) and \( y^a_\mu \), \( g \) and \( g' \) are the coefficients and the coupling of \( SU_L(2) \) and \( SU_R(2) \), respectively. The index \( i = 1, 2, 3 \) is associated with the generators \( \tau^i \) of \( SU(2) \). The global components of this isometry describe pure-gauge transformations (global Lorentz transformations) whereas the local polarized components correspond to the gauging of the corresponding subgroup \( H \subset SU_L(2) \otimes SU_R(2) \). For instance the abelian gauge theory describing ordinary classical electrodynamics can be obtained by assuming that only the corresponding polarized and abelian Lorentz transformations with generator in \( U_{em}(1) \subset SU_L(2) \otimes SU_R(2) \) are local.

Similarly we can imagine to describe electroweak interactions by assuming that the local polarized isometries are only those associated with the Lorentz subgroups \( SU_L(2) \otimes U_Y(1) \subset SU_L(2) \otimes SU_R(2) \) (the gauging the electroweak group from a larger global group \( SU_L(2) \otimes SU_R(2) \) is typical in technicolor models and useful to describe the custodial symmetry of the Standard Model of electroweak interactions, see for instance \( (51) \)). As for the abelian case \( (26) \) we can define gauge fields associated with \( SU_L(2) \) and \( U_Y(1) \) as \( W_\mu(x) = W_\mu^a(x) \tau^i = w_\mu^a(x) \tau^i \bar{p}_a \) and \( Y_\mu(x) = Y_\mu^a(x) \tau^3 = y_\mu^a(x) \tau^3 \bar{p}_a \). Indeed we have the remarkable possibility to relate the electroweak gauge group to a local Lorentz subgroup.

A possible generalization of this geometrodynamical description of gauge interaction to fermionic fields has a natural realization on the Zitterbewegung models. Originally proposed by Schrödinger this idea provides a semi-classical interpretation of the spin and of the magnetic momentum in terms of cyclic dynamics whose periodicity is actually the de Broglie periodicity of the fermions, see for instance \( (22) \). Such a trembling motion of the fermions can be derived from the Dirac equation.

Finally, it is interesting to mention that the geometrodynamical description of gauge interaction described so far has a deep motivation in the so called Kaluza’s miracle \( (5) \). This can be seen by considering the dualism of the theory to an XD theory \( (1, 20) \). Under this dualism, the metric \( (15) \) associated to the substitution of variable \( (22) \) turns out to be a Kaluza-like XD metric. This point out interesting algebraic properties of both the curvature tensor and the electromagnetic field tensor as already noticed by Rainich \( (8) \) and then improved by Misner and Wheeler \( (3) \).

Interesting aspects of the geometrical interpretation of gauge interaction given here,
such as parallel transport, holonomy and the relation with space-time symmetries, have a similar description in the “Higher Gauge Theory” of Baez, see for instance [42]. In an appealing formalism it is in fact shown how gauge interaction can be described as “change in phase of a quantum particle”.

5. Quantum Behavior

So far we have limited our study to the fundamental mode $\Phi(x)$ of the cyclic field $\Phi(x)$. This corresponds to study the theory at tree-level. In fact the fundamental mode can be matched with a corresponding single mode of a non-quantized KG field with corresponding four-frequency. Because of this matching with a KG mode, the fundamental mode can be in principle quantized in the usual way by imposing explicitly commutation relations and obtaining ordinary scalar QED. However, as shown in [1] and summarized in this section, the classical evolution of a cyclic field (with all its energy excitations) has remarkable correspondences with the quantum evolution of ordinary second quantized fields. This correspondence has been checked explicitly in both the canonical formulation of QM (the theory has implicit commutation relations) and the Feynman Path Integral (FPI) formulation, as well as for many other peculiar quantum phenomena and problems (including Schrodinger problems [1, 9, 32]). The correspondence with the FPI formulated for free systems in [1], will be generalized to the interacting system studied so far. In fact, as for the free case, an interacting cyclic field also has Markovian evolution and an Hilbert space can be defined locally. The assumption of PBCs can be regarded as a semi-classical quantization condition for relativistic fields.

5.1. Mode expansion

To investigate the quantum behavior of the theory we need to use the most generic cyclic field solutions of the action in compact 4D and PBCs. By considering the discrete Fourier transform associated with the cyclic 4D of the theory, it is easy to figure out that such a periodic field has a quantized energy-momentum spectrum. For the sake of simplicity we consider the simple topology $S^1$ so that the spectrum is described by a single quantum number $n$.

From the relation $\bar{E}(\bar{p}) = \hbar \bar{\omega}(\bar{p})$, the intrinsic time periodicity $T_t(\bar{p})$ of cyclic field in a given reference frame denoted by $\bar{p}$ implies a quantized energy spectrum $E_n(\bar{p}) = \hbar \omega_n(\bar{p})$. In the free case PBCs yield $\omega_n(\bar{p}) = n\bar{\omega}(\bar{p})$, which is nothing but the harmonic frequency spectrum of a vibrating string with the characteristic time periodicity of the field. Thus this formulation can be regarded as the full relativistic analogous to the semi-classical quantization of a “particle” in a box. It also shares deep analogies with the Matsubara and the Kaluza-Klein (KK) theories [33, 20]. Indeed, from this harmonic spectrum and from [4] we see that free cyclic fields reproduce the same quantized energy spectrum of ordinary second quantized fields (after normal ordering)

$$E_n(\bar{p}) = n\sqrt{\bar{p}^2c^2 + M^2c^4}.$$  

6The theory can be regarded as a particular kind of string theory. The compact world-line parameter plays the role the compact world-sheet parameter of ordinary string theory. Therefore it would be more appropriate to speak about strings rather than fields.

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A free cyclic field $\Phi(x) = \{x, t\}$ — with intrinsic periodicity $T(t, p)$, solution of the action in compact 4D $S^4$, is a tower of energy eigenmodes $\phi_n(x)$ with eigenvalues $[51]$. Furthermore, the time periodicity induces a periodicity $\lambda_x$ to the modulo of the spatial dimensions according to $[7]$. Hence, similarly to the quantization of the energy spectrum, there is a quantization of the modulo of the spatial momentum $|p_n| = n|p| = n\hbar/\lambda_x$.

The fact that $T(t, p)$ is a four-vector means that there is a single periodicity induced between the time dimension and the modulo of the spatial dimensions. In the simple case of topology $S^1$, this implies that the resulting quantization is to be expressed in terms of the single quantum number $n$. It is important to note, however, that together with this single fundamental periodicity $S^1$, in spherical problems (such as the Hydrogen atom, the 3D harmonic oscillator, or problems with particular bounded geometries), two other cyclic variables (or their deformations) must be considered: the spherical angles $\theta \in [0, 2\pi)$ and $\psi \in [0, \pi)$. As well-known from ordinary QM they lead to the ordinary quantization of angular momentum. In this case the cyclic field would be written as a sum over two additional quantum numbers, typically denoted by $(m, l)$, and the topology of the fields would be $S^1 \otimes S^2$ (PBCs for the transverse mode of a field, as in the front-light-quantization, can be used to calculate semi-classically important predictions of perturbative QED such as the anomalous-magnetic-momentum, see for instance [27]). For simplicity we will not consider the expansion in spherical harmonics (or their deformations). We will investigate only the quantization of the four-momentum spectrum associated with $S^1$, which in the free case is harmonic

$$p_{\mu} = n\bar{p}_\mu.$$  

Depending whether we want to make explicit the normalization factor of the energy eigenmodes, we write the cyclic field solution by using the following notations,

$$\Phi(x) = \sum_n \Phi_n(x) = \sum_n N_n \alpha_n(p) \phi_n(x) = \sum_n N_n \alpha_n(p) e^{-\frac{\pi p_n \cdot x}{\lambda_x}}. $$  

As already mentioned we assume that the normalization factor $N_n$ is invariant under isometries. The coefficients of the Fourier expansion are represented by $\alpha_n(p)$. The non-quantum limit corresponds to the case where the cyclic field solutions can be approximated with the fundamental mode $\Phi(x) \sim \Phi(x) = \bar{N} \phi(x)$, i.e. when only the fundamental level is largely populated: $\bar{\alpha}(p) \sim 1$ and $\alpha_n \neq 1(p) \sim 0$, see [1, 9].

5.2. Correspondence with Quantum Mechanics

Next we summarize the correspondence of field theory in compact 4D with QFT in [1, 9]. We note that the evolution of the free cyclic field [53] along the compact time dimension is described by the so-called “bulk” equations of motion $(\partial_t^2 + \omega_n^2)\phi_n(x, t) = 0$. For the sake of simplicity in this section we assume a single spatial dimension $x$. Thus the time evolution of the energy eigenmodes $\phi_n(x, t)$ can be written as first order differential equations

$$i\hbar \partial_t \phi_n(x, t) = E_n \phi_n(x, t).$$  

The cyclic field [53] is a sum of eigenmodes of an harmonic system. Actually this harmonic system is the typical classical system which can be described in a Hilbert space.
In fact, the energy eigenmodes of a cyclic field form a complete set with respect to the inner product
\[ \langle \phi_n'(t')|\phi_n(t) \rangle \equiv \int_0^{\lambda_x} \frac{dx}{\lambda_x} \phi_n^*(x, t') \phi_n(x, t), \]  
(55)
They energy eigenmodes can also be used to define Hilbert eigenstates
\[ \langle x, t|\phi_n \rangle \equiv \frac{\phi_n(x, t)}{\sqrt{\lambda_x}}. \]  
(56)

As obvious in the non-interacting case where the cyclic fields have persistent periodicity, the integration region \( \lambda_x \) can be extended to an arbitrary large integer number of periods \( V_x = N' \lambda_x \). That is, by assuming \( N' \to \infty \), it can be extended to an infinite region.

In this Hilbert space we can formally build an Hamiltonian operator defined as
\[ \mathcal{H}|\phi_n \rangle \equiv \hbar \omega_n |\phi_n \rangle. \]  
(57)
Similar considerations hold for the spatial dimension and the momentum operator is defined as
\[ \mathcal{P}|\phi_n \rangle \equiv -i\hbar k_n |\phi_n \rangle, \]  
(58)
where \( k_n = nk = n\hbar/\lambda_x \). An Hilbert state is defined as generic superposition of energy eigenmodes
\[ |\phi \rangle \equiv \sum_n a_n |\phi_n \rangle. \]  
(59)
This means that cyclic fields can be represented in an Hilbert space. With this Hilbert notation the time evolution of a generic Hilbert state, as well as of a generic cyclic field, turns out to be described by the familiar Schrödinger equation
\[ i\hbar \frac{\partial}{\partial t} |\phi(x, t) \rangle = \mathcal{H}|\phi(x, t) \rangle. \]  
(60)
As can be easily seen in the free case, \( i.e. \) homogeneous Hamiltonian (we will generalize later to interactions), the finite time evolution is given by the operator
\[ \mathcal{U}(t_f; t_i) = e^{-\frac{i}{\hbar} \mathcal{H}(t_f - t_i)} \]  
(61)
which turns out to be a Markovian (unitary) operator:
\[ \mathcal{U}(t_f; t_i) = \prod_{m=0}^{N-1} \mathcal{U}(t_i + t_m; t_i + t_m - \epsilon) \]  
(62)
where \( N \epsilon = t'' - t' \). Similarly considerations can be applied to the spatial evolution. The finite classical space-time evolution in the Schrödinger representation is
\[ |\phi(t, x) \rangle = \mathcal{U}(x, t; 0, 0)|\phi \rangle = e^{-\frac{i}{\hbar} (\mathcal{H}t - \mathcal{P}x)}|\phi \rangle = \sum_n a_n e^{-\frac{i}{\hbar} \hat{p}_n \cdot \hat{x}}|\phi_n \rangle. \]  
(63)
In order to allow an easy generalization of the following result to the interaction case, we will describe the finite space-time evolutions in terms of the infinitesimal space-time
evolutions of the Markovian operator. In particular this will be useful in the interaction case where in every point a different inner product must be considered. As a result we will obtain the integral product $\int \mathcal{V} \, d\mathbf{x}$.

All the elements necessary to build a FPI are already present in the theory without any further assumption than PBCs. In fact, we can plug the completeness relation of the energy eigenmodes in between the elementary time Markovian evolutions obtaining

$$
\mathcal{Z} = \int \mathcal{V} \left( \prod_{m=1}^{N-1} dx_m \right) U(x_f, t_f; x, t) \times \ldots \times U(x_2, t_2; x_1, t_1) U(x_1, t_1; x', t').
$$

From the notation (63), the elementary space-time evolutions of a free system can be written as

$$
U(x_{m+1}, t_{m+1}; x_m, t_m) = \langle \phi | e^{-\frac{i}{\hbar} (\mathcal{H} \Delta t_m - P \Delta x_m)} | \phi \rangle,
$$

where $\Delta x_m = x_{m+1} - x_m$ and $\Delta t_m = t_{m+1} - t_m$. Thus, proceeding in a completely standard way we formally obtain the ordinary Feynman Path Integral (FPI) for the time-independent Hamiltonian (57),

$$
\mathcal{Z} = \lim_{N \to \infty} \int \mathcal{V} \left( \prod_{m=1}^{N-1} dx_m \right) \prod_{m=0}^{N-1} \langle \phi | e^{-\frac{i}{\hbar} (\mathcal{H} \Delta t_m - P \Delta x_m)} | \phi \rangle.
$$

This remarkable result has been obtained by using classical-relativistic mechanics and without any further assumption than intrinsic four-periodicity. As in the usual FPI formulation in phase-space we are assuming on-shell elementary space-time evolutions. Though we started with homogeneous $\mathcal{H}$, this derivation, being based on elementary evolutions of a Markovian operator, can be generalized to the interacting case as we will see in sec. (5.3.4).

Proceeding in complete analogy with the ordinary derivation of the FPI in configuration space we note that the infinitesimal products of (65) in (66) can be generically written in terms of the action of the corresponding classical particle

$$
\mathcal{S}_{cl}(t_f, t_i) \equiv \int_{t_i}^{t_f} dt L_{cl} = \int_{t_i}^{t_f} dt (P \dot{x} - \mathcal{H}).
$$

Finally the FPI in (66) can be written in the familiar form

$$
\mathcal{Z} = \int \mathcal{D}x e^{i \mathcal{S}_{cl}(t_f, t_i)}.
$$

This important result can be intuitively interpreted by considering that a cyclic field has topology $S^1$. Indeed, for a field in such a cyclic geometry there is an infinite set of possible classical paths with different winding numbers linking every given initial and final configuration. Contrary to ordinary fields, a cyclic field can self-interfere and this is described by the ordinary FPI (1).

We may also mention that, by evaluating the expectation value of the observable $\partial_x F(x)$ associated with the inner product (55) of our Hilbert space, integrating by parts, and considering that the boundary term vanishes because of the periodicity of spatial coordinate, we find

$$
\langle \phi_f | \partial_x F(x) | \phi_i \rangle = \frac{i}{\hbar} \langle \phi_f | P F(x) - F(x) P | \phi_i \rangle.
$$
Finally, by assuming that the observable is a spatial coordinate \( F(x) = x \) (Feynman used a similar demonstration to show the correspondence of the FPI with canonical QM, \([34]\)), the above equation for generic initial and final Hilbert states \( |\phi_irangle \) and \( |\phi_frangle \), is nothing but the commutation relation of ordinary QM: \([x, P] = i\hbar \) — or more in general \([F, P] = i\hbar \partial_x F \). With this result we have also checked the correspondence with canonical QM. The commutation relations, as well as the Heisenberg uncertain relation \([1]\), can be regarded as implicit (and not imposed) in this theory. This is a consequence of the intrinsically cyclic (undulatory) nature of elementary particles \([1]\) conjectured by de Broglie in 1924 \([16]\), implicitly tested by 80 years of successes of QFT and indirectly observed in a recent experiment \([18]\).

5.3. Semi-classical interactions

In this section we want to generalize our geometrodynamical description of interaction to all the possible harmonic modes of the cyclic field. In this way we will find a formal correspondence with the ordinary FPI of interacting systems (Feynman diagrams and perturbative calculations will be discussed elsewhere \([7]\)).

We have already seen that the classical propagation of a free cyclic field is described by a FPI \((66)\) written in terms of the Lagrangian of a corresponding free particle with mass \( \bar{M} \) \([1]\). On the other hand, we have also seen that local transformations of variables induce internal transformations of field solutions can be used to describe gauge interaction. Next we will formally extend this correspondence to the quantum limit. In particular we will see that the propagation of the cyclic bosonic field - with all its harmonics - is described by the usual FPI and Scattering Matrix of an ordinary gauge interacting bosonic fields. The Lagrangian in FPI \((66)\) will turn out to be the usual Lagrangian of classical electrodynamics of bosonic particles. Here we assume again three spatial dimensions \( x \).

Next we will extend our semi-classical description of quantum systems to electrodynamics and we will find a formal correspondence to the corresponding QFT.

5.3.1. Bohr-Sommerfeld quantization

The assumption of PBCs in field theory in compact 4D can be regarded as the quantization condition. It is easy to see that PBCs reproduce the Bohr-Sommerfeld (BS) quantization condition. Intuitively the BS condition is a periodicity condition because it says that the only possible orbits of the system are those with an integer number of cycles.

The correspondence with BS is immediate in the case of free fields, since it corresponds to an isochronous system \( T'^\mu(x) = T''\mu \) (the analogous of to the Galilean isochronism of the pendulum where the orbits at different energies have the same periodicity). The PBCs applied to the free cyclic field \((53)\) give the following harmonic quantization condition of the phase of the fields

\[
\oint dx \cdot p_n = \int dx \cdot p_n = T \cdot p_n = n\hbar. \tag{70}
\]

\footnote{The Fourier coefficients \( \bar{a} = a_1 \) and \( \bar{a}^* = a_{-1}/\sqrt{-1} \) in \((66)\) can be regarded as the annihilating and creation operators. The anti-foundamental \((n = -1)\) modes can be associated to an anti-particle. Similarly, \( a_n = a_n/\sqrt{n} \) can be regarded as Virasoro operators.}
This is nothing but the harmonic spectrum of the four-momentum obtained in (52). The quantization of a free field through PBCs is thus immediate because both the four-periodicity and four-momentum are homogeneous.

The interacting cyclic field $\Phi'(x)$ is a sum of eigenmodes with the same analytical form of the fundamental mode (35),

$$\bar{\Phi}(x') = \sum_n \bar{N}_n e^{-\frac{i}{\hbar} \int x' \cdot \bar{p}_n(x')}.$$  \hfill (71)

The quantized spectrum of the cyclic interacting field in $x = X$ is given by the PBCs at $T^\mu (X)$, i.e. $\Phi'(X) \equiv \Phi'(X + cT(X))$, which can be written as

$$e^{-\frac{i}{\hbar} \int T^\mu (X) \cdot \bar{p}_n(x')} = e^{-\frac{i}{\hbar} \oint X \cdot \bar{p}_n(x')} = e^{-i2\pi n}.$$ \hfill (72)

Therefore the quantization condition in the interacting case is

$$\oint_X dx \cdot \bar{p}_n(x') = nh,$$ \hfill (73)

according to (12) and (70). This is nothing but the generalization of the BS quantization condition in 4D.

We now define a four-momentum operator $\mathcal{P}_\mu = \{\mathcal{H}/c, -\mathcal{P}_t\}$, and a number operator $\hat{n}|\phi_n\rangle \equiv n|\phi_n\rangle$. In the Hilbert space associated with this interacting system the non-homogeneous operator $\mathcal{P}'_\mu (x)$ can be obtained from the homogeneous one similarly to (12),

$$\mathcal{P}_\mu \rightarrow \mathcal{P}'_\mu (x) = e^{a(x)} \mathcal{P}_\mu.$$ \hfill (74)

In fact the quantization condition (73) of an interacting cyclic system can be expressed as

$$\oint_X dx \cdot \mathcal{P}'(x) = \hat{n}h.$$ \hfill (75)

Therefore the quantization of interacting cyclic field through PBCs can be also regarded as a generalization of the familiar BS quantization and described in terms of Hilbert operators.

5.3.2. Dirac quantization and Symmetry Breaking

We now consider the specific case of gauge interaction. For the cyclic field solution $\Phi'(x)$ associated with the minimal substitution (27), the quantization condition (72) turns out to be

$$e^{-\frac{i}{\hbar} \oint X \cdot \bar{p}_n - eA_n (x)} = e^{-i2\pi n}.$$ \hfill (76)

As it can be seen from (70), in the case of pure gauge orbits $A_\mu(x) = \partial_\mu \theta(x)$, this condition leads to a quantization condition for the Wilson loop

$$\oint_X dx \cdot eA_n (x) = hn.$$ \hfill (77)

\footnote{The Morse coefficient of the ordinary BS quantization can be retrieved by assuming a global twist factor in the PBCs and it can be interpreted as the vev (see also Gauge-Higgs unification and Hosotani mechanism). This aspect has been shortly discussed in [1].}
Similarly to the Hamiltonian and momentum operator we define the operator $eA$ such that

$$eA |\phi_n\rangle \equiv eA_n |\phi_n\rangle .$$

(78)

In this case the assumption of PBCs reproduces formally quantization condition of a Dirac string.

$$\oint_X dx \cdot eA(x) = \hat{n}h .$$

(79)

This confirms the result of the related paper [35] where the Dirac quantization condition (79) has been obtained in an indirect way just by assuming time periodicity for an abelian gauge field. It has been used to interpret phenomenological aspects of superconductivity such as the quantization of the magnetic flux, the penetration length, the Meissner effect, the Josephson effect. In this description, according to [36, 35], superconductivity is a phenomenological consequence of the breaking of the electromagnetic gauge invariance associated with $S^1 \to \mathbb{Z}_2$. In fact, in a pure gauge, because of the PBCs on the matter field $\Phi$, the phase of the gauge transformation is periodic and defined modulo factor $hn$. This can be seen also from (79). Thus the “Goldstone” $\theta$ of the related gauge transformation can vary only by finite amounts and the electromagnetic gauge invariance is broken — without involving a vev. Notice however that this breaking of the gauge invariance is a quantum effect since in the classical limit $h \to 0$ the quantization condition (79) yields a non quantized spectrum. Thus, field theory in compact 4D not only provides a geometrodynamical description of the gauge invariance; PBCs provides also a mechanism of gauge symmetry breaking with interesting analogy with the ones typically used in XD Higgsless and Gauge-Higgs-Unification models.

As discussed in sec.(4.4), the generalization of the quantization condition (79) to the case of a non-abelian gauge $SU_L(2) \otimes SU_Y(1)$ is obtained through the following substitution $eA(x) \to gW(x) + g'Y(x)$. According to [37], the resulting quantization condition for the neutral component

$$\oint_X dx \cdot [gW(x) + g'Y(x)] = \hat{n}h$$

(80)

could provide a realistic electroweak symmetry breaking mechanism which can be interpreted as induced by monopole condensations.

---

9To the l.h.s. of this equation can be added a factor $1/2$ by noticing that, according to the inner product [80], only the modulo of the field has a physical meaning. In analogy with the XD formalism it arises naturally in the orbifold notation $s \in S^1/\mathbb{Z}_2$.

10This condition can also be regarded as a quantization for the electric charge. The PBCs allows the possibility to use Noether’s theorem to directly describe the quantized variables of the theory. This idea will be expanded in future works.

11In the Hosotani mechanism [40, 41], the shift of the mass eigenvalue of the KK fundamental mode is $M'(y) = M - g_5 A_5(y)$ where $A_5(y)$ and $g_5$ are the fifth component and the coupling of an gauge field with an XD $y$. Under the dualism with XD studied [21] this is corresponds with the minimal substitution $\tilde{p}_\mu \to \tilde{p}_\mu - eA_\mu(x)$. On the other hand, in complete parallelism with our 4D description, a Hilbert space can also be introduced to describe the KK mode of an XD field. The Wilson line of the $A_5(y)$ can be written as an operator which takes integer number, similarly to [79]. This corresponds to generalize the Hosotani mechanism from to winding number $n = 1$ to all the possible $n$. As a results we find a deformation of the whole KK tower, mode by mode, which can be equivalently described as induced by a corresponding deformation of the XD.
5.3.3. Scattering Matrix

The Hilbert formalism (63) used to describe the evolution of a free field $\Phi(x)$ can be easily extended to interacting fields $\Phi'(x')$.

The transformation under local abelian (polarized) isometries reproduces ordinary gauge interaction for the fundamental mode. Similarly to (3 7), the corresponding transformation of a generic mode $\phi$ of a free cyclic field to an corresponding mode $\phi'$ of a gauge interacting cyclic field is described by the following internal transformation

$$\phi'(x) = e^{i\Xi^\mu dx A(x)}\phi(x).$$

This describes the modulation of periodicity (tuning) of the generic mode $\phi'$ with respect the free mode $\phi$, as a function of the generic gauge mode $A_\mu(x)$.

Thus, from the definitions of the Hilbert operator $A$ in (78), we find that the internal transformation of the cyclic field corresponds to pass from the Schrödinger representation to the interaction representation of perturbation theory. In fact we find

$$|\phi'(x)\rangle = e^{i\Xi^\mu dx eA}|\phi(x)\rangle = \sum_n \alpha_n e^{i\Xi^\mu dx eA_n}|\phi_n(x)\rangle.$$  (82)

Now we define a tuning operator, Hilbert analogous of the parallel transport, as

$$S(x) = e^{i\Xi^\mu dx eA}.$$  (83)

This is nothing but the ordinary scattering matrix of ordinary perturbation theory. In fact, it turns out to define formally the interaction term $L_{\text{int}}(x)$ of the ordinary Lagrangian of classical electrodynamics

$$e \int^{x^\mu} dx^\mu A_\mu(x) = e \int_{x^\mu_i}^{x^\mu_f} d\tau A_\mu(x)J^\mu(x) = \int_{x^\mu_i}^{x^\mu_f} d^4x L_{\text{int}}(x).$$

In writing this equation we have explicitly written the integration region as $x^\mu = x^\mu_f - x^\mu_i$ (modulo periods) and we have defined the current $J^\mu = dx^\mu/d\tau$.

From a formal point of view, such a representation of interaction for a cyclic field as modulation of periodicity actually matches the ordinary interaction representation written in terms of the scattering matrix of ordinary QFT

$$|\phi'(x)\rangle = S(x)|\phi(x)\rangle.$$  (85)

It is important to note that, as in ordinary QFT, this interaction representation is formally sufficient to describe QED in terms of the Feynman diagrams.

5.3.4. Feynman Path Integral and Scalar QED

With this formalism at hand it is finally possible to describe the evolution of an cyclic field (including all its possible harmonics) under a given interaction scheme. We will find that its evolution will be formally described by the ordinary FPI of the corresponding quantum interacting system. In the specific case of a cyclic field transforming under abelian polarized local isometries, the result will be the ordinary FPI of QED (for bosons).
The description of the FPI given in (66), being written in terms of elementary space-time evolutions, can be easily generalized to the non-interacting case. In case of interactions the four-momentum operator $P_i$ of (74) is non-homogeneous. Nevertheless the space-time evolutions are Markovian even in case of interactions. This can be seen for instance by considering the scattering matrix (53). In fact the time evolution in the interaction representation is described by the exponential operator $S(t) = e^{-\frac{i}{\hbar}\int_{t_{0}}^{t}H_{int}(t')dt'}$. Similarly to the evolution of a free cyclic field (61), the contribution associated to interactions is Markovian $S(t+d) = e^{-\frac{i}{\hbar}\int_{t}^{t+d}H(t')dt'}$. Hence, in case of interactions the resulting evolution is Markovian: $U'(t+dt;t) = e^{-\frac{i}{\hbar}\int_{t}^{t+dt}H(t')dt'}$. The generalization of the elementary space-time evolutions (65) of an interacting cyclic field in terms of the non-homogeneous four-momentum operator $P_i(x)$ is

$$U'(x_{m+1}, t_{m+1}; x_m, t_m) = \langle \phi | e^{-\frac{i}{\hbar}\int_{t_{m}}^{t_{m+1}}(H'\Delta t_m - P_i'\Delta x_{m})} | \phi \rangle .$$ (86)

Another difference with respect to the free case is that interaction deforms point by point the completeness relation. That is, the integration volume $V_x$ of the inner product (55) varies with $x = X$: that is, $\int_{V'_X(X)} dx'(X)/V'_X(X)$ (the number of period $N'$ remains fixed). The Markovian evolution of our cyclic system allows us to write the the evolution in terms of elementary evolutions (53). This guarantees the possibility to use a different inner product at every point $x = x_m$ of (64). Moreover, in order to avoid a different integration volume $V'_X(X)$ in every integration point, the integration region of the inner-product can be extended to a very large or infinite volume $V'_X(X)$ (large or infinite number of periods $N'$), much bigger than the (finite) interaction region $T$. In this way the volume $V'_X(X)$, as well as the normalization of the field, is overall not affected by the local deformations: $V'_X(X) \cong V_x$. Thus the correct mathematical tool to represent this non trivial evolution is actually the integral product $\int_{V_x} Dx$.

At this point, by following the same generic demonstration used in (64) (plugging locally the completeness relation in the elementary Markovian evolutions), we find formally the ordinary FPI in phase-space of an interacting particle

$$Z = \lim_{N \to \infty} \int_{V_x} \left( \prod_{m=1}^{N-1} dx_m \right)^{-1} \sum_{m=0}^{N-1} \langle \phi | e^{-\frac{i}{\hbar}\int_{t_{m}}^{t_{m+1}}(H'\Delta t_m - P_i'\Delta x_{m})} | \phi \rangle .$$ (87)

Similarly to (67), it is possible to define a Lagrangian $L'_{cl} = P_i'^{\dagger}d - H'$. Since $P_i'$ in (74) transforms as the four-momentum $P_i$ of the corresponding classical particle (12), this lagrangian defines formally the action $S'_{cl}(t_f, t_i) \equiv \int_{t_i}^{t_f} dt L'_{cl}$ of the corresponding interacting classical particle — written in terms of operators. Therefore the classical evolution of an interacting cyclic field is formally described by the ordinary FPI in configuration space associated with the interaction scheme,

$$Z = \int_{V_x} Dx e^{\frac{i}{\hbar}S'_{cl}(t_f, t_i)} .$$ (88)

Finally, from the analysis of the scattering matrix (53) we find that in the approximation of the local isometries investigated in this paper, the resulting evolution is formally given by the ordinary FPI of scalar QED. In fact the resulting Lagrangian associated with the action $S'$ in the exponential is formally the classical Lagrangian of a charged
bosonic particle interacting electromagnetically: $\mathcal{L}_c' = \mathcal{L}_c + \mathcal{L}_{\text{int}}$. With this formal correspondence to the ordinary FPI formulation at hand we can invoke Feynman’s saying that “the same equations have the same solutions” [38]. Hence the assumption of intrinsic periodicity can in principle be used for a geometrodynamical semi-classical description of QED.

This formal correspondence must be however tested in explicit computations of QED observables. This will be the subject of a dedicated paper. Nevertheless we mention that recent studies seem to show the liability of these semi-classical computations. We may note that light-front quantization, a semiclassical theory which similarly to our theory uses the assumption of PBCs as quantization condition, shows that it is actually possible to reproduce semi-classically the electron anomalous magnetic momentum in term of harmonics expansion of the fields [44]. Similar results pointing in the same direction are the computation of quantum behavior through the AdS/CFT, see discussion in sec. 2.2.2 and [20], and the calculation of Feynman diagrams in Twistor Theory [45] or other integrable theories.

5.4. Further motivations

Here we present a digression about the physical meaning of the assumption of intrinsic periodicity for elementary quantum systems. The motivations go beyond the de Broglie periodic phenomenon and involve interesting aspects of modern physics, as described in detail in [1, 2, 10, 11, 12, 13, 14, 20, 38] and summarized here.

We may consider the recent attempts to interpret QM as an emerging theory, such as the ‘t Hooft determinism [24] and the stroboscopic quantization [25]. According to ‘t Hooft [24], there is a “close relationship between the quantum harmonic oscillator and a particle moving on a circle”, both with extremely fast periodicity $T$. Our field theory in compact 4D can be intuitively derived by noticing that, as well known, the quantum harmonic oscillator is the basic element of ordinary second quantized KG fields. A cyclic field can be intuitively derived from the ‘t Hooft determinism (in the continuos limit of the lattice used by ‘t Hooft) by considering that the characteristic periodicity $T_i$ varies in a relativistic way, as described in sec. 1. This cyclic behavior of a “particle on a circle” of the ‘t Hooft determinism has motivated the “stroboscopic quantization” [25]. In this case we explicitly find the idea of dimensions compactified in a torus and, even more interesting, the fact that the “ticks” resulting from ergodic dynamics yield an effective description of the arrow of time. Similarly, in our theory every instant in time can be characterized by a different combination of the phases of all the de Broglie clocks constituting an isolated system of elementary particles. This is similar to a calendar or a stopwatch which allows us to fix events in time in terms of combinations of the phases of periods that we call years, months, days, hours, minutes, and so on. If the elementary cycles constituting our systems of periodic phenomena have irrational periods, the total evolution will result in an ergodic evolution; if we also allow interactions between elementary periodic phenomena, i.e. exchange energy or equivalently variation of periodicity, the resulting evolution of the non elementary system will be chaotic. Indeed, the assumption of intrinsic periodicity, realized in terms of field theory in compact 4D with PBCs, has important motivations in the interpretation of the notion of time in physics [1, 13]. In particular, as Galileo taught us with the experiment of the pendulum in the Pisa dome, or as explicitly stated in the Einstein [39] definition of relativistic clock, or according to the operative definition of a second through the Cs-133 atom, time can be
only defined by counting the number of periods of a phenomenon supposed to be periodic
in order to guarantee the constancy of the unit of time. Thus every free elementary parti-
cle can be regarded as a reference clock\footnote{\textit{In this way the local character of the relativistic time turns out to be enforced.}}\footnote{\textit{We can say that LHC is exploring indirectly time scale of the order of $10^{-27}$ s corresponding to energy scale of the order of the TeV.}} the so-called “de Broglie internal clock”. As we
have seen, the modulation of periodicity of these internal clocks can be used to describe
interactions similarly to GR. We also mention the geometric quantization\footnote{\textit{This brings elements of supersymmetry in the theory and it could be investigated in a dedicated paper.}}\footnote{\textit{which is an attempt to reproduce quantum behavior by introducing two grassmannian partners of the physical time. This could be interesting for a possible semi-classical description of spinning particle typical of the \textit{zitterbewegung} models}} which is
an attempt to reproduce quantum behavior by introducing two grassmannian partners
of the physical time. This could be interesting for a possible semi-classical description
of spinning particle typical of the \textit{zitterbewegung} models\footnote{\textit{this brings elements of
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supersymmetry in the theory and it could be investigated in a dedicated paper).

The time periodicity of the de Broglie internal clock is bounded by the inverse of
the mass, $T_{\tau} \leq T_{\tau} = \hbar/Mc^2$, \textit{i.e.} by the Compton wavelength divided by the seed
of light $T_{\tau} = \lambda_s/c$. In this way it is easy to see that these de Broglie intrinsic clocks of
elementary particles are typically extremely fast (except neutrinos). The heavier the
particle, the faster the periodicity. As already mentioned, a light particle such as the
electron has a periodicity faster than $10^{-20}$ s which means that for every “tick” of the
Cs-133 atomic clock ($T_{Cs} \sim 10^{-10}$ s) an electron does a number of cycles of the order
of the age of the universe expressed in years. Even with the modern time resolution
it is not yet possible to resolve such small time scales, though the internal clock of the
electron has been indirectly observed in a recent interference experiment\footnote{\textit{Thus the
observation of such a fast de Broglie internal clock is similar to the observation of a “clock
under a stroboscopic light”, [23]. That is, at every observation the particle appears to
be in an aleatory phase of its cyclic evolution and, similarly to a dice rolling too fast (de
Broglie deterministic dice) with respect to our resolution in time, the outcomes can be
only described in a statistical way [12]. Similarly to the deterministic models mentioned
above, the results of the preview section show that the statistics associated to these cyclic
behavior have formal correspondences to ordinary QM. They also suggest that the direct
experimental exploration of microscopical time scales (smaller than $10^{-20}$ s in the case of the
electron) is of primary interest in understanding the inner nature of the elementary
systems\footnote{\textit{Another motivation is given by the variational analysis of the BCs discussed in [1] and
mentioned in sec. [1]. Roughly speaking we may say that relativity fixes the differential
structure of the continuous space-time of a theory without giving particular prescriptions
about BCs, [1, 28]. The important requirement for the BCs is that they must fulfill the
variational principle. As already discussed in this paper, the role of the BCs is marginal
in ordinary QFT. On the other hand, BCs have played an important role since the earliest
days of QM, according to de Broglie, Bohr, Sommerfeld, et.al.. We have also noticed
that the non-quantum limit of a massive cyclic field corresponds to the limit where
only the fundamental mode $\Phi(x)$ is exited, see [1] and [2] for more detail. This also is
the non-relativistic limit of a massive particle ($|\vec{p}| \ll \bar{M}c$). Thus, the classical particle
description, corresponding to the limit of large mass ($\bar{M} \to \infty$) or equivalently the spatial
momentum to zero ($\vec{p} \to 0$), the cyclic field reduces to $\Phi(x) \sim \exp[-i \bar{M}c^2 \frac{\hbar}{\bar{M}} t + i \frac{\hbar}{\bar{M}} x^2]$, see [1]. Neglecting the de Broglie rest clock represented by the first term, the modulo square
of a massive cyclic field is a distribution centered along the path of the corresponding}}.
classical particle. Its width is of the order or smaller of the Compton wavelength $\lambda_s$, as can be easily shown by performing an explicit plot. Thus in the classical limit ($\lambda_s \to 0$) this distribution reproduces the ordinary non-relativistic limit of the FPI, i.e. a Dirac delta distribution. In this limit the spatial compactification lengths tend to infinity ($\bar{p} \to 0$) whereas the compactification length along the time dimensions tends to zero ($\bar{M} \to \infty$). Indeed a classical particle turns out to be actually described by a point like distribution in $\mathbb{R}^3$. Similar arguments can be used to interpret other interesting aspects of the wave-particle duality of QM. The assumption of periodic phenomenon enforces the wave-particle dualism, giving rise to implicit commutations relations and Heisenberg uncertain relations, $[1]$. A massless field, i.e. a field leaving in the light-cone ($ds = 0$), is always relativistic. Since its Compton wavelength is infinite we say that the rest de Broglie clock of a massless field, such as the EM field, is frozen. Thus the energy spectrum of a massless cyclic field can be approximated to a continuous in the IR region where the compactification length tend to infinity ($T_t \to \infty$) and the PBCs can be neglected. In the UV limit however the PBCs are important because the compactification length are very small ($T_t \to 0$). Therefore, the quantized nature of the energy spectrum becomes manifest avoiding the UV catastrophe of the black-body radiation. Indeed at high frequencies the field theory has an effective corpuscular description, $[1]$.

A further conceptual motivation to use BCs as quantization condition is that we have the remarkable property that QM emerges without involving any (local) hidden variable in the theory. Since the hypothesis of existence of local hidden variable is not realized, the Bell’s theorem can not be applied to our theory. The assumption of intrinsic periodicity introduces an element of non locality which, however, can be regarded as consistent with SR since the periodicity varies in a relativistic way. Thus the theory can in principle violate the Bell’s inequality (if we try to adapt the Bell’s theorem to our case, i.e. to evaluate the expectation values of an observable in the Hilbert space described above, we find again a formal parallelism with ordinary QM). For this reason we speak about — mathematically — deterministic theories $[1]$.

We have seen that the assumption of intrinsic periodicity of elementary systems provides a semi-classical description of scalar QED. The study of the limit where quantum corrections become relevant for gravitational interaction is beyond the scope of this paper. In particular quantum gravity represents another important subject where to test the consistence and the validity of the theory. Some more detail about this point is given in $[1, 13]$.

6. Conclusions

Field theory in compact 4D represents a natural realization of the “periodic phenomenon” associated to every elementary particle, as conjectured by de Broglie in 1924 $[15, 16]$, at the base of the wave-particle duality, implicitly tested by 80 years of QFT and indirectly observed in a recent experiment $[18]$ (Schrödinger used a similar assumption in his zitterbewegung model of the electron).

In this formalism the kinematical information of an interaction scheme is encoded in the relativistic geometrodynamics of the boundary of the theory — in a sort of holographic description. The resulting description has shown remarkable relationships with the following fundamental approaches to interactions:
• the approach typical of classical-relativistic mechanics in which interaction is described in terms of retarded and local variations of four-momentum;

• the approach typically used in QM to describe systems in generic potentials in terms waves and BCs. The dynamical boundary of the theory reproduces the retarded and local modulation of de Broglie four-periodicity and thus the local and retarded variation of four-momentum of an interacting quantum system;

• in GR, gravitational interaction can be described as local modulations of periodicity of reference clocks encoded in corresponding deformations of the underlying space-time coordinates. Similarly, in our description, local modulations of the four-periodicity are described as deformations of the compact 4D.

• the typical approach to interaction of gauge theory is obtained because the variations of field solution associated with the variations of the boundary of the theory defines an internal transformations which, in the approximation of the local isometries described in the paper, formally matches classical electrodynamics.

Remarkably, we have found that gauge interaction can be derived from the invariance of the theory under local transformations of variables as gravitational interaction can be derived by requiring invariance under diffeomorphisms. Gauge symmetries are related to space-time symmetries. This can be regarded as in the spirit of Weyl’s, Kaluza’s and Wheeler’s original proposal of a geometrodynamical description of gauge invariance.

On the other hand the assumption of PBCs (or similar BCs such as anti-PBCs, N-BCs, D-BCs) provides a semi-classical quantization condition for fields. In fact the theory can be regarded as the full relativistic generalization of the quantization of a “particle in a box”. This geometric quantization method, without introducing hidden-variables, reproduces formal correspondences between well established quantization methods:

• the correspondence to canonical formulation of QM, arises from the fact that a cyclic field is naturally described by an Hilbert space, that it evolves according to the Schrödinger equation and its cyclic variables implicitly satisfy commutation relations and Heisenberg uncertain relations;

• the Feynman Path Integral is obtained as interference of the classical paths with different winding numbers associated with the intrinsically cyclic geometry of fields in compact 4D [1, 2];

• The Bohr-Sommerfeld quantization condition is nothing but a periodicity condition. It simply states that the allowed orbits are those with an integer number of cycles (close orbits) and can be described in terms of PBCs;

• the Dirac quantization condition is obtained as a result of the periodicity induced by the matter cyclic field on pure gauge transformations.

• the Scattering Matrix naturally describes the modulations of periodicity between a free cyclic field (Schrödinger representation) and an interacting cyclic field (interaction representation).
Remarkably, field theory in cyclic 4D, without any further assumption than intrinsic periodicity, provides the possibility of a geometrodynamical and semi-classical description of scalar QED.

To the above list of well established quantization methods it should be added that, as shown in [20] through the dualism of cyclic fields to XD massless fields, the theory yields interesting analogies with the classical XD geometry to quantum behavior correspondence typical of AdS/CFT. The dualism to XD theory will be used in future papers to investigate the geometrodynamical description of gauge invariance in terms of Kaluza’s original proposal.

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