RG Fixed Points and Flows in SQCD with Adjoints

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We map out and explore the zoo of possible 4d $\mathcal{N} = 1$ superconformal theories which are obtained as RG fixed points of $\mathcal{N} = 1$ SQCD with $N_f$ fundamental and $N_a$ adjoint matter representations. Using “a-maximization,” we obtain exact operator dimensions at all RG fixed points and classify all relevant, Landau-Ginzburg type, adjoint superpotential deformations. Such deformations can be used to RG flow to new SCFTs, which are then similarly analyzed. Remarkably, the resulting 4d SCFT classification coincides with Arnold’s ADE singularity classification. The exact superconformal R-charge and the central charge $a$ are computed for all of these theories. RG flows between the different fixed points are analyzed, and all flows are verified to be compatible with the conjectured $a$-theorem.

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1. Introduction

Asymptotically free gauge theories have a variety of interesting possible IR phases. For example, they can flow to theories with a mass gap for the gauge fields and confinement, or to theories with no mass gap and an interacting “non-Abelian Coulomb phase,” i.e. an interacting 4d CFT. Perturbative analysis suggested the possibility of the latter phase long ago, via the apparent weakly coupled RG fixed points in theories designed to be just barely asymptotically free [1,2]. There might have then been some lingering doubts about whether or not interacting 4d CFTs really existed (theoretically) at the non-perturbative level. But, over the past decade, the study of supersymmetric gauge theories (see e.g. [3]) has provided strong evidence for the existence of a zoo of non-perturbatively exact renormalization group fixed points.

Not only do interacting RG fixed points exist, but they are in some sense generic: general gauge theories with enough matter fields (but not too many so as to spoil asymptotic freedom) are believed to flow to interacting CFTs in the IR. A well-known example of this is ordinary (non-supersymmetric) $SU(N_c)$ gauge theory with $N_f$ fundamental flavors, which flows to an interacting CFT in the range $N_f^{min} < N_f < \frac{11}{2}N_c$, where it has been estimated that $N_f^{min} \approx 4N_c$ [4]. Here the conformal range is somewhat narrow, but it becomes wider when other representations are included. We will here focus on the case of supersymmetric theories, where there are some powerful tools available.

The vast collection of possible interacting 4d $\mathcal{N} = 1$ SCFTs and RG flows among them remains relatively unexplored. One well-studied case is SQCD, which flows to interacting RG fixed points when the number $N_f$ of fundamental flavors is in the range $\frac{3}{2}N_c < N_f < 3N_c$ [4]. A generalization of this is to consider theories with other matter representations; these theories provide interesting testing grounds for exploring ideas in quantum field theory. One idea which will be of particular interest for the present paper is the conjecture that there is a 4d analog of Zamolodchikov’s 2d c-theorem [6]: that there exists a “central charge,” which counts the number of degrees of freedom of a quantum field theory and monotonically decreases along RG flows to the IR, as degrees of freedom are integrated out. It is further conjectured [6], with much supporting evidence e.g. [8,9,10,11], that an appropriate such central charge (at least at RG fixed points) is the coefficient $\frac{1}{2}$ “a” of a

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1 Non-asymptotically free theories, on the other hand, flow to free field theories in the IR (at least when one starts the RG flow at weak coupling).

2 Unfortunately the name “$a$” is standard for this term. The name “$c$” reserved for a different curvature-square term, which is known to not obey the 4d version of the c-theorem [5].
certain curvature-squared term (the Euler density) of the conformal anomaly on a curved space-time background\textsuperscript{3}. The conjectured a-theorem is then that all RG flows satisfy $a_{IR} < a_{UV}$. We will often refer to “a-theorem predictions,” but please keep in mind that there is presently no generally accepted proof of the a-theorem – it is a conjecture.

The superconformal algebra implies that $a$ can be exactly computed simply in terms of the ’t Hooft anomalies for the superconformal R-symmetry, $a = 3\text{Tr} R^3 - \text{Tr} R$ \textsuperscript{9}. We will here rescale our definition of $a$ to eliminate a conventional factor of $3/32$. Because ’t Hooft anomalies can be exactly computed in terms of the original UV spectrum (’t Hooft anomaly matching), $a$ can be exactly determined provided that the superconformal R-symmetry is exactly determined. Determining the exact superconformal R-symmetry had been a stumbling block in analyzing theories with more than one type of matter representation. In \textsuperscript{10} we recently showed that the superconformal $U(1)_R$ symmetry is simply determined by “a-maximization”: letting $R_t$ be the most general trial R-symmetry (incorporating mixing with all global flavor symmetries), the superconformal $U(1)_R \subset SU(2,2|1)$ is the unique choice of $R_t$ which locally maximizes

$$a_{\text{trial}}(R_t) = 3\text{Tr} R_t^3 - \text{Tr} R_t. \quad (1.1)$$

The value of $a_{\text{trial}}$ at the local maximum is then the central charge $a$. Using this result, it is now possible to apply the powerful constraints of the superconformal algebra to explore these more general 4d $\mathcal{N} = 1$ SCFTs. The result also almost proves the a-theorem for SCFTs, up to some caveats \textsuperscript{10}. For a brief review of the a-maximization procedure, see Appendix B.

In \textsuperscript{11}, Kutasov, Parnachev, and Sahakyan applied and extended the method of \textsuperscript{10} to study $\mathcal{N} = 1$ SQCD with $N_f$ fundamentals, $Q_i$ and $\bar{Q}_i$, and $N_a = 1$ additional adjoint chiral superfield $X$. Our work here was inspired by their detailed analysis of the various possible RG fixed points and flows, and the many striking ways in which the conjectured a-theorem was found to be indeed always satisfied: for every RG flow, $a_{IR} < a_{UV}$. An important part of the analysis of \textsuperscript{11} was determining when gauge invariant chiral primary operators $M$ appear to violate the unitarity bound $R(M) = \frac{2}{3} \Delta(M) \geq \frac{2}{3}$. The belief

\textsuperscript{3} There is another quantity which has been conjectured to always decrease in the IR: the thermal c-function of \textsuperscript{12}. We will not discuss this latter proposal here, because supersymmetry does not provide a known way to exactly compute the thermal c-function at interacting RG fixed points (there is a known general expression only for free field theories).
is that such operators are actually free fields, with $R(M)$ corrected to be $2/3$ by an accidental $U(1)_M$ symmetry under which only the free field $M$ is charged. It was shown in [11] that the effect of any such accidental symmetries must be included in the $a$-maximization method, and that they non-trivially affect the value for the superconformal $R$-charges and the central charge $a$. This will be important in some of our examples.

In this paper, we study the larger family of RG fixed points and flows which can be obtained from $SU(N_c)$ SQCD with $N_f$ fundamentals together with $N_a = 2$ adjoint matter chiral superfields, $X$ and $Y$. $N_a = 2$ is the maximum number of adjoints compatible with asymptotic freedom, so this is a fairly complete study of the full family of possible RG fixed points and flows which can be obtained via SQCD with fundamental and adjoint representations.

The possible RG fixed points which we find and analyze can be summarized as follows:

\[
\begin{align*}
\hat{O} & : W_{\hat{O}} = 0 \\
\hat{A} & : W_{\hat{A}} = \text{Tr} Y^2 \\
\hat{D} & : W_{\hat{D}} = \text{Tr} XY^2 \\
\hat{E} & : W_{\hat{E}} = \text{Tr} Y^3 \\
A_k & : W_{A_k} = \text{Tr} (X^{k+1} + Y^2) \\
D_{k+2} & : W_{D_{k+2}} = \text{Tr} (X^{k+1} + XY^2) \\
E_6 & : W_{E_6} = \text{Tr} (Y^3 + X^4) \\
E_7 & : W_{E_7} = \text{Tr} (Y^3 + YX^3) \\
E_8 & : W_{E_8} = \text{Tr} (Y^3 + X^5). 
\end{align*}
\]

There are additional RG fixed points associated with adding Yukawa-type superpotentials involving the quarks; we will briefly mention some of these possibilities in a later section. The names given to the fixed points in (1.2) are motivated by Arnold’s $ADE$ classification of singularities, which precisely coincides with the possible relevant deformation superpotentials, listed as $A_k$, $D_{k+2}$ and $E_k$ in (1.2). Our method for obtaining the above superpotentials was based on a detailed analysis of the anomalous dimensions of operators at each of the RG fixed points, and when the associated Landau-Ginzburg superpotentials can be relevant and drive the theory to a new RG fixed point. On the face of it, this has no obvious connection to any of the other known ways in which Arnold’s singularities have appeared in mathematics or physics.

The possible RG flows between the above fixed points are as shown in Figure 1. In this terminology, the work [11] studied the $\hat{A}$ fixed points and RG flows to the $A_k$ fixed

\footnote{Of course, there are also RG fixed points for $N_a = 3$ adjoints and $N_f = 0$ flavors, provided that one adds the cubic superpotential of the $\mathcal{N} = 4$ theory or generalizations [16].}
points, as well as RG flows from $A_k$ fixed points to $A_{k'}$ fixed points with $k' < k$. We will extend this analysis to consider all of the RG fixed points and flows of Figure 1.

![Diagram](image)

**Figure 1:** The map of possible flows between fixed points.

Dotted lines indicate flow to a particular value of $k$.

The outline of this paper is as follows. In sect. 2 we discuss $SU(N_c)$ gauge theory with $N_f$ fundamental flavors, two adjoints, and $W_{\text{tree}} = 0$. For all $N_f$ in the asymptotically free range, $0 \leq N_f < N_c$, these theories flow to interacting RG fixed points in the IR. We refer to these 4d $\mathcal{N} = 1$ SCFTs as $\hat{O}(N_c, N_f)$, or simply $\hat{O}$. We obtain the exact superconformal $U(1)_R$ charge and the exact central charge $a_{\hat{O}}(x)$ in the limit $N_c \gg 1$, for arbitrary fixed $x \equiv N_c/N_f$. None of our methods actually depends on large $N_c$; it is merely employed to simplify the exact expressions. As $x$ increases, the RG fixed point is at stronger coupling. As might have been expected from the negative anomalous dimensions of gauge theories, we find that the superconformal $U(1)_R$ charges are monotonically decreasing functions of $x$. For all $x$, we find no apparent unitarity violations. Therefore it is not necessary for there to be any free-field operators or associated accidental symmetries for the $\hat{O}$ RG fixed points. We verify that several a-theorem predictions are indeed satisfied. Finally, we classify and discuss all of the possible relevant superpotential deformations of the $\hat{O}$ RG fixed point which involve only the adjoint fields $X$ and $Y$. The interesting possibilities are the superpotentials labeled $\hat{A}$, $\hat{D}$, and $\hat{E}$ in (1.2).

In sect. 3 we discuss the $\hat{E}$ SCFTs, which are the IR endpoints of an RG flow obtained by perturbing the $\hat{O}$ SCFT by the relevant superpotential $W_{\hat{E}} = \text{Tr} Y^3$. We obtain the exact superconformal R-charge and central charge $a_{\hat{E}}(x)$ of the $\hat{E}$ SCFTs. There are no
apparent unitarity violations for all \( x \), and thus there is no need for any free-field operators or associated accidental symmetries. We verify the a-theorem prediction, \( a^O(x) > a^E(x) \), for all \( x > 1 \). We also classify the relevant superpotential deformations of the \( \hat{E} \) RG fixed point which involve only the adjoint field. The possibilities are indicated in Fig. 1. The \( E_6 \), \( E_7 \), and \( E_8 \) deformations are only relevant provided that \( x \) is sufficiently large: \( x > x_{E_6}^{\text{min}} \approx 2.44 \), \( x > x_{E_7}^{\text{min}} \approx 4.12 \), and \( x > x_{E_8}^{\text{min}} \approx 7.28 \), respectively.

In sect. 4 we discuss the \( \hat{D} \) SCFTs, which are the IR endpoints of a RG flow obtained by perturbing the \( \hat{O} \) SCFT by the relevant superpotential \( W^D_{\hat{D}} = \text{Tr}XY^2 \). Unlike the \( \hat{O} \) and \( \hat{E} \) RG fixed points, here we do find apparent unitarity violations, indicating that various mesons necessarily become free fields as \( x \) increases. We account for the effect of the associated accidental symmetries by the procedure of [11] (reviewed in appendix B) to obtain the exact superconformal \( U(1)_R \) charges and exact central charge \( \tilde{a}^D_{\hat{D}}(x) \). We verify (numerically) the a-theorem prediction that \( a^O(x) > a^D_{\hat{D}}(x) \) for all \( x \geq 1 \). We also classify and discuss the various relevant deformations of the \( \hat{D} \) RG fixed point. A class of such deformations is \( \Delta W = \text{Tr}X^{k+1} \) which, for any \( k \), is a relevant deformation of \( \hat{D} \) provided that \( x \) is sufficiently large: \( x > x_{D_{k+2}}^{\text{min}} \). We discuss how to compute the lower bounds \( x_{D_{k+2}}^{\text{min}} \) and find e.g. that \( x_{D_{k+2}}^{\text{min}} \rightarrow \frac{9}{8}k \) for \( k \gg 1 \).

In sect. 5 we discuss the \( D_{k+2} \) SCFTs, which are the IR endpoints of the RG flow starting from the \( \hat{D} \) SCFT in the UV, upon perturbing \( W^D_{\hat{D}} \) by \( \Delta W = \text{Tr}X^{k+1} \). The \( D_{k+2} \) SCFT exists if \( x > x_{D_{k+2}}^{\text{min}} \), when \( \Delta W \) is a relevant \( \hat{D} \) deformation. The superconformal \( U(1)_R \) charges at the \( D_{k+2} \) fixed point are determined by \( W_{D_{k+2}} \), and it is thus seen that a variety of mesons apparently violate the unitarity bound and hence must be free. The central charge \( a_{D_{k+2}}(x) \) must be corrected, as in [11], to account for the free mesons. We numerically verify the a-theorem prediction, \( a^D_{\hat{D}}(x) > a_{D_{k+2}}(x) \) for all \( x > x_{D_{k+2}}^{\text{min}} \), plotting the example of \( a^D_{\hat{D}}(x) > a_{D_{2}}(x) \). At the \( D_{k+2} \) RG fixed point there is a relevant deformation by \( \Delta W = \text{Tr}X^{k'+1} \), for \( k' < k \), which leads to the RG flow \( D_{k+2} \rightarrow D_{k'+2} \). We discuss this flow in Section 5 and the a-theorem prediction that \( a_{D_{k+2}}(x) > a_{D_{k'+2}}(x) \) for all \( k' < k \) and \( x > x_{D_{k+2}}^{\text{min}} \). There are apparent violations of the a-theorem, but we verify that they always occur for \( x < x_{D_{k+2}}^{\text{min}} \), which is outside of the range of validity needed for the \( D_{k+2} \) RG fixed point to exist.

In sect. 6 we discuss a magnetic dual description of the \( D_{k+2} \) SCFTs due to Brodie [18]. We determine the exact superconformal R-charges and central charge \( \tilde{a}_{D_{k+2}}(x) \) in the magnetic dual. Our analysis sheds light on the meaning of this duality, and when the various terms in the magnetic superpotential are or are not relevant. We show that for all \( k \)
there is a conformal window \( x_{D_k+2}^{\text{min}} < x < 3k - \bar{x}_{D_k+2}^{\text{min}} \) where the \( D_{k+2} \) RG fixed point is fully interacting, with the full \( W_{D_{k+2}} \) present in both the electric and magnetic descriptions. For large \( k \) this conformal window is approximately given by \( \frac{9}{8}k < x < (2.062)k \).

In sect. 7 we discuss the \( E_6 \), \( E_7 \), and \( E_8 \) SCFTs, which arise as the IR limits of relevant deformations of the \( \hat{E} \) SCFTs upon perturbing the \( \hat{E} \) theory by the superpotentials \( \Delta W = \text{Tr} X^4 \), \( \Delta W = \text{Tr} Y X^3 \), and \( \Delta W = \text{Tr} X^5 \), respectively. These SCFTs only exist if \( x \) is sufficiently large: \( x > x_{E_6}^{\text{min}} \approx 2.44, x > x_{E_7}^{\text{min}} \approx 4.12, \) and \( x > x_{E_8}^{\text{min}} \approx 7.28 \), respectively, as mentioned above. There are also RG flows between these theories: \( E_8 \to E_7, E_8 \to E_6, E_7 \to E_6 \). Thus the a-theorem prediction is \( a_{E_6}(x) > a_{E_8}(x) > a_{E_7}(x) > a_{E_6}(x) \), at least for the range of \( x \) where each RG fixed point exists. We find that there are apparent violations of some of these a-theorem predictions for some ranges of \( x \), but these ranges are always outside of the range of validity \( x > x_{\text{min}} \) required for the UV SCFT to exist.

In sect. 8 we briefly discuss additional RG flows and fixed points which can be obtained from those of (1.2) by perturbing by Yukawa-type interactions.

In sect. 9 we discuss the possibility that our procedure which led to (1.2), starting at the \( \hat{O} \) SCFT and branching out to new SCFTs, as in fig. 1, could miss some additional SCFTs. This could happen if these hypothetical new SCFTs do not have \( \hat{O} \) in their domain of attraction. As a concrete example, we consider the theory with \( W = \lambda \text{Tr} Y^{k+1} \) for \( k > 2 \). For small \( \lambda \), this superpotential is an irrelevant deformation of \( \hat{O} \) and does not lead to a new SCFT. But perhaps there is nevertheless an RG fixed point for some large critical value of \( \lambda_* \). We find this to be unlikely but, anyway, discuss the properties that such hypothetical new SCFTs should have, in order that their RG flows down to our other SCFTs in (1.2) be compatible with the a-theorem conjecture.

In sect. 10 we make some closing comments.

Finally, we have several appendices. In appendix A, we discuss the \( \hat{O}, \hat{D}, \hat{E} \), and \( D_4 \) RG fixed points in the perturbative regime, where the theory is just barely asymptotically free: \( x = 1 + \epsilon \), with \( 0 < \epsilon \ll 1 \). In appendix B, we review a-maximization and the necessary modification when there are accidental symmetries associated with operators becoming free fields. In appendix C, we discuss baryon operators and whether or not they ever potentially violate the unitarity bound in our various SCFTs. In appendix D, we discuss the magnetic duals of the \( D_{k+2} \) RG fixed points in an asymptotic regime in order to determine the upper bound for the \( D_{k+2} \) interacting conformal window: \( \frac{9}{8}k < x < (2.062)k \) for large \( k \).
2. The $\hat{O}$ RG fixed points: $W(X,Y) = 0$.

The theories with $W_{\text{tree}} = 0$ are expected\footnote{An argument\cite{13} for this is to deform the theory by $W = m\text{Tr}Y^2 + \lambda \sum_i \text{Tr}\tilde{Q}_i XQ_i$, making the IR theory $\mathcal{N} = 2$ SQCD. That theory has a moduli space with massless monopoles and dyons at various places. Upon taking $m \to 0$ and $\lambda \to 0$, the massless monopole and dyon locations all collapse to the origin of the moduli space. The presence of massless fields which are not mutually local at the origin is believed to signify the presence of an interacting RG fixed point\cite{13}.} to flow to an interacting RG fixed point, which we call $\hat{O}$, for all $N_f$ in the asymptotically free range: $0 \leq N_f < N_c$.

To simplify the formulae of this paper, we take $N_c$ large, with $x \equiv N_c/N_f$ fixed. Large $N_c$ is not essential to any of the methods, it just simplifies the expressions for the results. It would be straightforward, though tedious, to work with arbitrary $N_c$.

In the limit where the theory is just barely asymptotically free, $x = 1 + \epsilon$ with $0 < \epsilon \ll 1$, the $\hat{O}$ RG fixed point is at weak coupling; $\beta(g_*) = 0$ for $g_*^2N \ll 1$. The $\hat{O}$ SCFT can then be analyzed in perturbation theory, as will be done in appendix A. As we increase $x$, the $\hat{O}$ RG fixed point is at stronger and stronger coupling. Powerful methods associated with supersymmetry will be used to obtain exact results for all $x$.

In the extreme case of $N_f = 0$, where $x \to \infty$, the Lagrangian has a unique candidate superconformal R-symmetry which is anomaly free and commutes with the flavor symmetries: $R(X) = R(Y) = \frac{1}{2}$. The true superconformal R-symmetry could potentially be modified by accidental symmetries, though there are no unitarity bound violations which would require this to happen. For the case $N_c = 2$, this RG fixed point has known electric, magnetic, and dyonic dual descriptions \cite{20,21}. We might expect analogous dual descriptions for higher $N_c$, corresponding to the various $Z_N \times Z_N$ electric and magnetic center phases, but this is not presently known.

2.1. The chiral ring of operators

The set of “observables” of SCFTs is the spectrum of gauge invariant operators and their correlation functions. A special set of such operators for SCFTs are the chiral primary operators, whose dimension is related to their R-charge by $\Delta = \frac{3}{2}R$.

In our theories, the microscopic chiral fields are the two adjoints $X$ and $Y$, the quarks $Q_i$ in the fundamental and $\tilde{Q}_i$ in the anti-fundamental, and the chiral gauge field strength $W_\alpha$. The chiral ring of operators is the set of all gauge invariant composites of these fields. For example, we can form

$$\mathcal{O}_{I_1 \ldots I_n} = \text{Tr}X_{I_1} \ldots X_{I_n}, \quad (2.1)$$
where $I_i = 1, 2$ labels the two adjoints, which we also call $X$ and $Y$. We can form arbitrary such products of operators, with $n$ arbitrarily large, though there will be some relations among them for $n$ of order $N_c$: the space of expectation values of these operators, subject to their classical relations, parameterize the $N_c^2 - 1$ dimensional classical moduli space of vacua where $X$ and $Y$ have D-flat expectation values, with $\langle Q_i \rangle = \langle \tilde{Q}_i \rangle = 0$, along which $SU(N_c)$ is completely Higgsed.

Another class of gauge invariant operators is the set of generalized “mesons”

\[(M_{I_1...I_n})_{ii} = \tilde{Q}_i X_{I_1} \cdots X_{I_n} Q_i. \tag{2.2}\]

Finally, there is another class of operators, the generalized baryons, which can be formed from the various dressed quarks:

\[(Q_{(I_1...I_n)})_i \equiv (X_{I_1} \cdots X_{I_n} Q)_i. \tag{2.3}\]

We make baryons by contracting $N_c$ dressed quarks with the $SU(N_c)$ epsilon tensor:

\[B = Q_{(I_1...I_{n_1})}^{n(I_1...I_{n_1})} Q_{(J_1...J_{n_2})}^{n(J_1...J_{n_2})} \cdots Q_{(K_1...K_{n_k})}^{n(K_1...K_{n_k})}, \tag{2.4}\]

with $N_c = n(I_1...I_{n_1}) + \cdots + n(K_1...K_{n_k})$ and all $n(I_1...I_{n_1}) \leq N_f$.

2.2. Finding the exact superconformal $U(1)_R$ by $a$-maximization

The theory with $N_f = 2$ fundamentals and $N_a = 2$ adjoints has an $SU(N_f) \times SU(N_f) \times U(1)_B \times SU(2)_a \times U(1)_X \times U(1)_R$ global flavor symmetry. As reviewed in Appendix B, we find the superconformal $U(1)_R$ symmetry by maximizing the combination of ’t Hooft anomalies \([1,4]\) using a general trial R-symmetry. We’ll parameterize the general anomaly-free trial R-symmetry as

\[R(Q_i) = R(\tilde{Q}_i) \equiv y, \quad R(Y) \equiv z, \quad R(X) = 1 - z + \frac{1 - y}{x}, \tag{2.5}\]

with $x \equiv N_c/N_f$. $a$-maximization always yields a superconformal R-symmetry which commutes with all non-Abelian flavor symmetries, as well as charge conjugation \([10]\). Thus, we could have restricted our trial $U(1)_R$ \([2.3]\) to commute with $SU(2)_a$, i.e. $R(X) = R(Y)$ and hence $z = \frac{1}{2}(1 + (1 - y)/x)$, at the outset. We chose to not impose this condition in \([2.3]\) for later reference, when we deform by superpotential terms which break $SU(2)_a$. 

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Computing $a_{trial}$ (1.1) with the R-charges (2.5) yields (taking $N_c$ large, $x$ fixed)

$$\frac{a}{N_f^2} = 2x^2 + 3x^2(z-1)^3 + 3x^2\left(\frac{1-y}{x} - z\right)^3 - x^2(z-1) \quad (2.6)$$

$$- x^2\left(\frac{1-y}{x} - z\right) + 6x(y-1)^3 - 2x(y-1).$$

Maximizing this with respect to $y$ and $z$ yields

$$R(Q) = R(\tilde{Q}) \equiv y(x) = 1 + \frac{3x - 2x\sqrt{26x^2 - 1}}{3(8x^2 - 1)} \quad (2.7)$$

and

$$R(X) = R(Y) \equiv z(x) = \frac{1}{2}\left(1 + \frac{-3 + 2\sqrt{26x^2 - 1}}{3(8x^2 - 1)}\right). \quad (2.8)$$

Plugging these R-charges back into the central charge gives

$$\frac{a_{\hat{O}}(x)}{N_f^2} = \frac{2x^2(18 + 648x^4 - 2\sqrt{26x^2 - 1} + x^2(-279 + 52\sqrt{26x^2 - 1}))}{9(1 - 8x^2)^2}. \quad (2.9)$$

Both $y(x)$ and $z(x)$ are strictly decreasing functions of $x$. In the limit $x \to \infty$,

$$y(x \to \infty) \to 1 - \frac{\sqrt{26}}{12} \approx 0.575, \quad z(x \to \infty) \to \frac{1}{2}. \quad (2.10)$$

We have plotted these functions in fig. 2 and fig. 3, respectively.

![Figure 2: y(x) = R(Q) for the \(\hat{O}\) theory.](image)
In fig. 4 we plot $N_f^{-2} a\hat{O}(x)$, given by (2.9), along with that of the $g = 0$ gauge coupling free field theory, where $R(Q) = R(\tilde{Q}) = R(X) = R(Y) = \frac{2}{3}$:

$$\frac{a_{\text{free}}}{N_f^2} = \frac{22}{9} x^2 + \frac{4}{9} x. \quad (2.11)$$

Because RG flow connects the free theory in the UV to the $\hat{O}$ SCFT in the IR, the conjectured a-theorem prediction is $a\hat{O}(x) < a_{\text{free}}(x)$ for all $x \geq 1$; as seen from fig. 4, this is indeed satisfied.
large $x$ (large coupling) which would modify the above results. But we will see that our results are consistent with various checks, such as the conjectured $a$-theorem, for all $x \geq 1$, without any such modifications. So we will tentatively propose that the above expressions are exactly correct as given for all $1 < x < \infty$.

One can easily generalize the above analysis to all $SU(N)$, $SO(N)$, and $Sp(N)$ gauge theories, with $N_f$ fundamentals and $N_a$ matter fields in a representation having quadratic index of order $N$ (e.g. for $SU(N)$ we could replace each adjoint with a $\frac{1}{2}N(N+1) + \frac{1}{2}N(N-1)$). As long as $N_a = 2$, in all such cases the unitarity bound is found to be satisfied, without any need for accidental symmetries.

2.3. Some checks of the conjectured $a$ theorem

Let’s do some more checks of the conjectured $a$-theorem, $a_{UV} > a_{IR}$, in these examples. Let $a_{\hat{O}}(N_c, N_f)$ be the central charge of the $\hat{O}$ RG fixed point for general $N_c$ and $N_f$, which is given by (2.9) in our limit of large $N_c$ and $N_f$ with $x \equiv N_c/N_f$ fixed.

The $a$-theorem conjecture requires

$$a_{\hat{O}}(N_c, N_f) > a_{\hat{O}}(N_c, N_f - 1) \quad \text{and} \quad a_{\hat{O}}(N_c, N_f) > a_{\hat{O}}(N_c - 1, N_f + 1).$$

The first comes from the RG flow associated with giving a mass to one of the fundamental quarks, $\Delta W = mQ_{N_f}\tilde{Q}_{N_f}$, and integrating it out. The second prediction in (2.12) comes from Higgsing, going along a flat direction where we give a vev to one of the fundamental flavors, or to some components of either adjoint. When we Higgs $SU(N_c)$ to $SU(N_c - 1)$, $N_f \rightarrow N_f - 1 + 2$, with one fundamental eaten, but two more coming from the two adjoints.

In the limit of large $N_c$ and $N_f$, we write the flow of the first prediction in (2.12) as

$$N_f^{-2} \rightarrow N_f^{-2}(1 + \frac{2}{N_f}), \quad x \rightarrow x(1 + \frac{1}{N_f}).$$

Then the first case in (2.12) can be written as:

$$x^{-2}\frac{a_{\hat{O}}(x)}{N_f^2} \quad \text{must be a monotonically decreasing function of } x.$$  

Likewise, the flow of the second prediction in (2.12) is

$$N_f^{-2} \rightarrow N_f^{-2}(1 - \frac{2}{N_f}), \quad x \rightarrow x(1 - \frac{2}{N_f}).$$

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The second prediction in (2.12) is then

\[ x^{-1}\frac{a_O(x)}{N_f^2} \] must be a monotonically increasing function of \( x \). \hfill (2.16)

Both of these predictions are indeed satisfied by our expression (2.9) for all \( x \geq 1 \).

Another \( a \)-theorem prediction is obtained by Higgsing \( SU(N \rightarrow U(1)^n \times \prod_{i=1}^n SU(N_i) \), for arbitrary \( N \) and \( N_i \) such that \( \sum_{i=1}^n N_i = N \), by an \( \langle X \rangle \) adjoint vev. Each \( SU(N_i) \) theory has two adjoints and \( N_f \) fundamentals, much as the original theory. But, in addition, there is a bi-fundamental flavor in the \( (N_i, N_j) \) for every pair of gauge groups \( SU(N_i) \) and \( SU(N_j) \), which come from decomposing \( Y \) (the corresponding components of \( X \) are eaten). With so many bi-fundamentals, the \( SU(N_i) \) theories generally are not asymptotically free; \( SU(N_i) \) is asymptotically free only if \( N_f + \sum_{j \neq i} N_j < N_i \), which generally is not satisfied, e.g. if all \( N_i \) are of the same order. When the \( SU(N_i) \) are not asymptotically free in the UV, they are free in the IR, and then \( a_{IR} \) is simply given in terms of the free-field contributions:

\[ \frac{a_{IR}(x)}{N_f^2} = \frac{16}{9} \sum_i x_i^2 + \frac{4}{9} x^2 + \frac{4}{9} x, \] \hfill (2.17)

where \( x_i \equiv N_i/N_f \). As an example, we check the \( a \)-theorem prediction, that \( a_O(x) > a_{IR}(x) \), for the special case \( N_i = N_c/l \), where

\[ \frac{a_{IR}(x)}{N_f^2} = \frac{1}{9} \left( 4 + \frac{16}{l} \right) x^2 + \frac{4}{9} x. \] \hfill (2.18)

Comparing (2.9) and (2.18), we see that \( a_O(x) > a_{IR}(x) \) is indeed satisfied for all \( x > 1 \). Figure 5 shows the result for the case \( l = 2 \).

Figure 5: \( N_f^{-2}a_{free}(x) \) (top, blue), \( N_f^{-2}a_O(x) \) (middle, green), and \( l = 2 \) broken \( N_f^{-2}a_{IR}(x) \) (bottom, red) theories.
2.4. Relevant superpotential deformations of the \( \hat{O} \) fixed points

We now classify the relevant superpotential deformations of the \( \hat{O} \) RG fixed points, of the form \( W = \text{Tr}X^nY^p \). This is relevant if \( \Delta(W) < 3 \), i.e. if \( R(W) = (n + p)z(x) < 2 \). Because \( z(x) \) is a monotonically decreasing function of \( x \), asymptoting to \( z(x \to \infty) \to \frac{1}{2} \), the only relevant possibilities for all \( x \geq 1 \) are quadratic and cubic superpotentials. The only such independent relevant deformations are

\[
\begin{align*}
W &= \text{Tr}XY : \hat{O} \to \text{SQCD}, \\
W &= \text{Tr}Y^2 : \hat{O} \to \hat{A}, \\
W &= \text{Tr}XY^2 : \hat{O} \to \hat{D}, \\
W &= \text{Tr}Y^3 : \hat{O} \to \hat{E}.
\end{align*}
\]

(2.19)

All of these are relevant for all \( x \) in the range \( 1 < x \leq \infty \). The deformation \( W = \text{Tr}XY \) gives a mass to both adjoints, taking the theory to SQCD with no adjoints (which does not lead to any RG fixed point in the \( x > 1 \) range). The deformation \( W = \text{Tr}Y^2 \) gives a mass to one of the adjoints, driving the theory to the \( \hat{A} \) RG fixed points of SQCD with one adjoint \( X \) and \( W(X) = 0 \); this \( \hat{A} \) RG fixed point was analyzed in detail in [11]. The remaining relevant deformations in (2.19) drive the \( \hat{O} \) RG fixed points to new RG fixed points, which we name \( \hat{D} \) and \( \hat{E} \) and discuss further in the following sections.

3. The \( \hat{E} \) RG fixed points: \( W_{\hat{E}} = \text{Tr}Y^3 \)

We have seen that \( W_{\hat{E}} = \text{Tr}Y^3 \) is a relevant deformation of the \( \hat{O} \) SCFTs for all \( x > 1 \), driving \( \hat{O} \) to some new RG fixed points which we name \( \hat{E} \). The \( \hat{E} \) SCFTs exist for all \( x > 1 \). When the theory is just barely asymptotically free, \( x = 1 + \epsilon \) with \( 0 < \epsilon \ll 1 \), the \( \hat{O} \to \hat{E} \) RG flow can be analyzed perturbatively; this is discussed in Appendix A. Upon increasing \( x \) the coupling of these SCFTs becomes stronger. We use a-maximization to find the exact superconformal \( U(1)_R \) symmetry for all \( x \geq 1 \).

3.1. The chiral ring of the \( \hat{E} \) theory.

The gauge invariant chiral operators of the \( \hat{E} \) theory are a subset of those of the \( \hat{O} \) theory, where we impose chiral ring relations coming from the superpotential \( W = \text{Tr}Y^3 \). For \( U(N) \), this yields \( Y^2 = 0 \) in the chiral ring, while for \( SU(N) \) it gives \( Y^2 = \alpha 1 \); \( \alpha \) is
a Lagrange multiplier used to set $\text{Tr}Y = 0$. For convenience, we’ll consider the simpler
$U(N)$ ring relation; for large $N$ there isn’t much difference in any case.

Imposing $Y^2 = 0$ in the ring, we can form operators such as

$$O_{I_1...I_n} = \text{Tr}X_{I_1} \cdots X_{I_n},$$

where, $I_i=1,2$ labels the two adjoints, as long as no two adjacent (including via trace
cyclicity) adjoints are $Y$’s. Such operators can still have many net $Y$’s, e.g. $\text{Tr}(XY)^5$.

Similarly, the meson and baryon operators are the subset of those in sect. 2. for which
there are no two adjacent $Y$’s, e.g. $\tilde{Q}_iY(XY)^{29}Q_i$.

### 3.2. Finding the superconformal $U(1)_R$ via $a$-maximization

At the eventual IR fixed point controlled by the superpotential $W = \text{Tr}Y^3$, we impose
$z = 2/3$ in (2.5), to ensure $R(W) = 2$, yielding the 1-parameter family

$$R(Y) = \frac{2}{3}, \quad R(Q) = R(\tilde{Q}) \equiv y \quad R(X) = \frac{1 + x - y}{x} - \frac{2}{3}.$$  \hfill (3.2)

The superconformal $U(1)_R$ is determined by maximizing the central charge $a$, given by
(2.6) with $z = 2/3$, with respect to $y$; the result is

$$y(x) = 1 + \frac{x(2 - \sqrt{10x^2 - 1})}{3(2x^2 - 1)}. \hfill (3.3)$$

This result is monotonically decreasing with $x$, with asymptotic value as $x \to \infty$

$$y(x \to \infty) \to 1 - \frac{\sqrt{10}}{6} \approx .4730, \quad R(X) \to \frac{1}{3}. \hfill (3.4)$$

In figs. 6 and 7, we plot $y(x)$ and $R(X)(x)$ for this theory.
Since \( y > 0.47 \) and \( R(X) > 1/3 \) for all \( x \), no gauge invariant chiral operator ever violates the unitarity bound \( R \geq 2/3 \) (aside from \( \text{Tr}X \) and \( \text{Tr}Y \) in the \( U(N) \) version, but they make negligible contribution for large \( N \) and \( N_f \).) So unitarity does not require any accidental symmetries, and we tentatively propose that the above results are correct as they stand for all \( x > 1 \). Again, it is possible that for sufficiently large \( x \) the strongly coupled theory actually does have some accidental symmetries. But we do not see any indication of this possibility in our results.

Plugging \( y(x) \) given by (3.3) and \( z = 2/3 \) into the expression (2.6) yields

\[
\frac{a_{\hat{E}}(x)}{N_f^2} = \frac{2x^2(17 + 36x^4 - 66x^2 + (10x^2 - 1)\frac{2}{3})}{9(1 - 2x^2)^2},
\]

which we have plotted in fig. 8, together with the \( N_f^{-2}a_{\hat{O}}(x) \) found in the previous section. Since perturbing \( \hat{O} \) by \( W_{\hat{E}} \) induces the RG flow \( \hat{O} \rightarrow \hat{E} \), the conjectured a-theorem prediction is \( a_{\hat{O}}(x) > a_{\hat{E}}(x) \) for all \( x \); this is indeed seen to be satisfied in fig. 8. We could have anticipated this because \( a_{\hat{O}}(x) \) involved maximizing with respect to \( z \), whereas in \( a_{\hat{E}}(x) \) it was constrained to \( z = \frac{2}{3} \) (though one must generally be careful with this argument, because the maxima are only local ones) [10].
3.3. Relevant deformations of the $\hat{E}$ RG fixed points

We consider deforming the superpotential as $W = \text{Tr}Y^3 \rightarrow W = \text{Tr}Y^3 + \Delta W$, with $\Delta W$ of the general form $\Delta W = \text{Tr}X^nY^p$ (allowing for various inequivalent $X$ and $Y$ orderings). This $\Delta W$ will be relevant, and can then lead to new SCFTs, if

$$R(\Delta W = X^nY^p) = \frac{2p}{3} + n\left(\frac{1}{3} + \frac{1-y(x)}{x}\right) < 2.$$  \hspace{1cm} (3.6)

Since $R(X)$ is monotonically decreasing, the highest possible $p$ and $n$ are found by considering the $x \rightarrow \infty$ limit of (3.6), where $R(X) \rightarrow \frac{1}{3}$:

$$2p + n < 6.$$ \hspace{1cm} (3.7)

This leads to the several possibilities, which we now discuss.

First, any quadratic $\Delta W$ superpotential is relevant. The possibilities, and where they drive the $\hat{E}$ SCFTs, are

$$\Delta W = \text{Tr}Y^2 : \hat{E} \rightarrow \hat{A},$$  
$$\Delta W = \text{Tr}X^2 : \hat{E} \rightarrow A_1,$$  
$$\Delta W = \text{Tr}XY : \hat{E} \rightarrow \text{SQCD}.$$ \hspace{1cm} (3.8)

These flows are all consistent with the a-theorem; e.g. in fig. 9 we see that $a_{\hat{E}}(x) > a_{\hat{A}}(x)$ is satisfied.
Figure 9: $a/N_f^2$ for the $\hat{E}$ (top, blue) and $\hat{A}$ (bottom, red) theories.

At the level of cubic $\Delta W$, the only independent, relevant possibility is

$$\Delta W = \text{Tr}X^2Y : \hat{E} \to D_4,$$

in the terminology of (1.2). Deforming $\text{Tr}Y^3$ by $\Delta W = \text{Tr}X^3$ is equivalent to (3.9) via a change of variables, and $\text{Tr}XY^2$ is eliminated by the $\hat{E}$ chiral ring relation. Using (3.6) and the results of the previous subsection, we can see that $\text{Tr}X^2Y$ is relevant for all $x > 1$ (since $y(x) > 1 - \frac{1}{3}x$ for all $x > 1$). Though $\Delta W = \text{Tr}X^2Y$ is relevant, we do not expect that it ever wins out over the original $W_{\hat{E}} = \text{Tr}Y^3$ term; both are important in determining the eventual RG fixed point. This will be further discussed in Appendix A.

Finally, we have the higher-degree $\Delta W$ solutions of (3.6). These are only relevant if $x$ is sufficiently large, and the independent possibilities (easily seen from (3.7)) for $W = W_{\hat{E}} + \Delta W$ are:

$$\hat{E} \to E_6 : W_{E_6} = \text{Tr}(Y^3 + X^4) \quad \text{if} \quad x \geq x_{E_6}^{\text{min}} \approx 2.55,$$

$$\hat{E} \to E_7 : W_{E_7} = \text{Tr}(Y^3 + YX^3) \quad \text{if} \quad x \geq x_{E_7}^{\text{min}} \approx 4.12,$$

$$\hat{E} \to E_8 : W_{E_8} = \text{Tr}(Y^3 + X^5) \quad \text{if} \quad x \geq x_{E_8}^{\text{min}} \approx 7.28.$$

The values of $x_{E_6}^{\text{min}}$, $x_{E_7}^{\text{min}}$, and $x_{E_8}^{\text{min}}$ are obtained by plotting, as in fig. 10, the R-charge (3.6) of the corresponding deformation, $\Delta W_{E_6} = \text{Tr}X^4$, $\Delta W_{E_7} = \text{Tr}YX^3$, and $\Delta W_{E_8} = \text{Tr}X^5$, at the $\hat{E}$ RG fixed point, and seeing when $R(\Delta W)$ just drops below $R = 2$, i.e. when the inequality in (3.6) is saturated.
If \( x > x_{E_{6,7,8}}^{\text{min}} \), these \( \Delta W \) drive \( \hat{E} \) to new SCFTs, which we call \( E_6, E_7, \) and \( E_8 \) and will analyze further in sect. 7.

4. The \( \hat{D} \) RG fixed points, \( W_{\hat{D}} = \text{Tr}XY^2 \)

We have seen that \( W_{\hat{D}} = \text{Tr}XY^2 \) is a relevant deformation of the \( \hat{O} \) SCFTs, driving them to new SCFTs, which we name \( \hat{D}, \) for all \( x \geq 1. \) When the theory is just barely asymptotically free, \( x = 1 + \epsilon \) with \( 0 < \epsilon \ll 1, \) the flow from the \( \hat{O} \) RG fixed points to the \( \hat{D} \) RG fixed points can be studied in perturbation theory, as will be done in Appendix A. For larger \( x \) the \( \hat{D} \) SCFT becomes more and more strongly coupled. In this section we exactly determine the superconformal R-charges and central charge \( a \) for all \( x. \)

4.1. The chiral ring for the \( \hat{D} \) theory

The chiral ring of gauge invariant operators is now subject to the relations coming from the \( W_{\hat{D}} \) EOM. For \( U(N_c) \) these are \( \partial_Y W_{\hat{D}} = \{X, Y\} = 0 \) and \( \partial_X W_{\hat{D}} = Y^2 = 0; \) again, for \( SU(N_c) \) there would be Lagrange multiplier unit matrices on the RHS, which we ignore in any case. The relation \( \{X, Y\} = 0 \) in the chiral ring is particularly convenient, since it allows us to freely re-order \( X \) and \( Y \) superfields (up to a minus sign). Using these ring relations, the only non-zero operators of the form (2.1) are

\[
\text{Tr}X^l, \quad l \geq 0, \tag{4.1}
\]

(for \( U(N) \) we also should include \( \text{Tr}Y \)). Note that \( \text{Tr}X^nY = 0 \) using \( \{X, Y\} = 0 \) and cyclicity of the trace. The non-zero mesons are

\[
M_{\ell,j} = \tilde{Q}_i \tilde{X}^{\ell} Y^j Q_i \quad \text{for} \quad \ell \geq 0, \quad j = 0, 1. \tag{4.2}
\]
The non-zero baryons are

\[ Q^{n(0,0)}_{(0,0)} Q^{n(1,0)}_{(1,0)} \cdots Q^{n(l,0)}_{(l,0)} Q^{n(0,1)}_{(0,1)} Q^{n(1,1)}_{(1,1)} \cdots Q^{n(k,1)}_{(k,1)} \]  

(4.3)

where

\[ Q_{(l,j)} = X^l Y^j Q, \quad l \geq 0, \quad j = 0, 1, \]  

(4.4)

and

\[ \sum_{j=0}^{l} n_{(j,0)} + \sum_{j=0}^{k} n_{(k,1)} = N_c, \quad n_{(l,j)} \leq N_f, \quad l, k \geq 0 \]  

(4.5)

4.2. \textit{a-maximization, this time with accidental symmetries}

At the \( \hat{D} \) RG fixed point, by \( W_{\hat{D}} = \text{Tr}XY^2 \), there is a one-parameter family of anomaly free R-charges satisfying \( R(W_{\hat{D}}) = 2 \):

\[ R(Q) = R(\tilde{Q}) = y, \quad R(Y) = \frac{y-1}{x} + 1, \quad R(X) = \frac{2-2y}{x}. \]  

(4.6)

Plugging these into (1.1), we see that \( y \) is determined by maximizing

\[ \frac{a^{(0)}_{\hat{D}}(x)}{N_f^2} = 2x^2 + 3x^2 \left( \frac{y-1}{x} \right)^3 + 3x^2 \left( \frac{2-2y}{x} - 1 \right)^3 - x^2 \left( \frac{y-1}{x} \right) - x^2 \left( \frac{2-2y}{x} - 1 \right) + 6x(y-1)^3 - 2x(y-1). \]  

(4.7)

This is maximized for

\[ y^{(0)}(x) = 1 + \frac{x(12 - \sqrt{11 + 38x^2})}{3(2x^2 - 7)}. \]  

(4.8)

The superscript on \( a^{(0)} \) and \( y^{(0)} \) will be explained presently.

Unlike the previous cases, now we do see that \( y^{(0)}(x) \), as given by (1.8), would lead to a unitarity bound violation if it were extrapolated to large \( x \). Indeed, \( y^{(0)}(x) \) (1.8) approaches a negative number at large \( x \): \( y^{(0)}(x \to \infty) \to 1 - \frac{1}{6}\sqrt{38} \approx -0.027 \). The expressions (1.7) and (1.8) are correct only in the range of \( x \) given by \( 1 \leq x \leq x_1 \), where \( x_1 \) is where \( R(\tilde{Q}Q) = 2y = \frac{2}{3} \), where the first meson crosses the unitarity bound. \( x_1 \) is thus the solution of \( y^{(0)}(x_1) = 1/3 \), giving \( x_1 \approx 3.67 \). For \( x > x_1 \), we need to re-work the above procedure, taking into account the free meson \( M \) and corresponding accidental symmetry; how to do this was found in \( \hat{A} \) SCFTs) and is reviewed in appendix B.
As we continue to increase $x$, more and more generalized mesons, $(M_{p,0})^i_i = \tilde{Q}^i X^{p-1}Q_i$, hit the unitarity bound and then become free fields. Mesons involving the operator $Y$, $M_{p,1} \equiv \tilde{Q} X^{p-1} Y Q$ never violate the unitarity bound since, as seen from (4.6), $R(Y)$ is always rather large: $R(Y) \geq 1$ and $R(X) \geq 0$, with $R(Y) \to 1$ for $x \to \infty$. As will be shown in appendix C, for all $x$, no baryons ever hit the unitarity bound. The chiral ring elements (4.1) will hit the unitarity bound and become free, but we can ignore their contribution in the large $N_f$ limit, since they are down by a factor of $N_f^2$ as compared with the meson contributions. So, for all $x$, we only need to account for the mesons $(M_{p})^i_i = \tilde{Q}^i X^{p-1}Q_i$ hitting the unitarity bound and becoming free; this happens successively in $p$ as we increase $x$.

Let $x_p$ be the value of $x$ where the $N_f^2$ mesons $M_p = \tilde{Q} X^{p-1}Q$ hit the unitarity bound:

$$2y(x_p) + (p - 1) \frac{2 - 2y(x_p)}{x_p} = \frac{2}{3}. \quad (4.9)$$

For $x$ in the range $x_p \leq x \leq x_{p+1}$, the mesons $M_{\ell,0}$ with $\ell = 1 \ldots p$ are free fields, while those with $\ell > p$ are interacting. We account for the accidental symmetries of the free mesons $M_1 \ldots M_p$ by using the modified central charge $a^{(p)}$, which is given as in (4.10) by

$$a^{(p)} = \frac{2x^2 + 3x^2 \left(\frac{y - 1}{x}\right)^3 + 3x^2 \left(\frac{2 - 2y}{x} - 1\right)^3}{2} - x^2 \left(\frac{y - 1}{x}\right) - x^2 \left(\frac{2 - 2y}{x} - 1\right) + 6x(y - 1)^3 - 2x(y - 1) + \frac{1}{9} \sum_{j=0}^{p-1} \left[2 - 3(2y + j \frac{2 - 2y}{x})\right]^2 \left[5 - 3(2y + j \frac{2 - 2y}{x})\right]. \quad (4.10)$$

Maximizing the function (4.10) with respect to $y$ yields the function $y^{(p)}(x)$.

The R-charge $R(Q) \equiv y(x)$ is given by patching together these various functions:

$$y(x) = y^{(p)}(x) \quad \text{for} \quad x_p < x < x_{p+1} \quad (4.11)$$

and the central charge is given by patching together the maximal values of the (4.10):

$$a_D(x) = \frac{a^{(p)}(x)}{N_f^2} \quad \text{for} \quad x_p < x < x_{p+1}. \quad (4.12)$$

When we solve for $x_p$, we use (4.9) with $y(x_p) = y^{(p-1)}(x_p)$, and iterate this procedure to all $x$. The functions $y(x)$ and $a_D(x)$ defined by (4.11) and (4.12) are continuous and smooth,
as in \[11\], despite the patching. We plot the resulting $y(x)$ and $N_f^{-2}a_{\tilde{D}}(x)$ (obtained numerically) in figs. 11 and 12. In fig. 12, we plot $N_f^{-2}a_{\tilde{D}}(x)$ along with $N_f^{-2}a_{\tilde{O}}(x)$ so that one may verify that the a-theorem is satisfied: $a_{\tilde{O}}(x) > a_{\tilde{D}}(x)$ for all $x > 1$.

![Figure 11: $y(x)$ for the $\tilde{D}$ theory.](image1)

![Figure 12: $a_{\tilde{O}}/N_f^2$ (top, blue) and $a_{\tilde{D}}/N_f^2$ (bottom, red).](image2)

We can analytically solve for the asymptotic $x \gg 1$ behavior of the $y(x)$ obtained by this procedure (in analogy with the $\tilde{A}$ case in \[11\]). In the large $x$ limit, we can replace the sum over $j$ with an integral over $v$, defined by

$$v = 2 - 3 \left( 2y + (j - 1) \frac{2 - 2y}{x} \right).$$

(4.13)

In the $x \gg 1$ limit (where $v$ becomes a continuous variable), this yields

$$\frac{a_{\tilde{D}}(x, y)}{N_f} \approx \frac{a_{\tilde{D}}^{(0)}(x, y)}{N_f^2} + \int_{0}^{2-6y} dv \frac{x v^2 (3 + v)}{54(1 - y)} \approx x \left( 6(y - 1)^3 - 19(y - 1) + \frac{2}{9} (1 - 3y)^3 \right).$$

(4.14)
which is maximized for 
\[ y(x \to \infty) = -1/8. \] (4.15)

The asymptotic value for the central charge is then
\[ \frac{a_{D}(x)}{N_{f}^{2}} \approx \frac{1931}{144} x \approx 13.41x \quad \text{for large } x. \]

### 4.3. Relevant deformations

In the limit \( x \to \infty \), we see from (4.6) and (4.15) that \( R(X) \to 0 \). Thus a deforming superpotential \( \Delta W = \text{Tr} X^{k+1} \) of the \( \hat{D} \) RG fixed point will be relevant for any \( k \), provided that \( x \) is chosen sufficiently large. This is because
\[ R(X^{k+1}) = 2(k + 1) \frac{(1 - y(x))}{x} \leq 2 \] (4.16)
will always be satisfied for \( x \) larger than some critical value \( x_{\min D}^{\text{min}} \) where the inequality (4.16) is saturated. Thus
\[ \Delta W = \text{Tr} X^{k+1} : \hat{D} \to D_{k+2} \quad \text{if} \quad x > x_{\min D}^{\text{min}}. \] (4.17)

The superpotential \( \text{Tr} X^{k+1} \) for the case \( k = 2 \) is a relevant deformation of the \( \hat{D} \) SCFTs, driving them to the \( D_{4} \) SCFT for all \( x \geq 1 \). In particular, this flow can be analyzed in the perturbative regime \( x = 1 + \epsilon \), with \( 0 < \epsilon \ll 1 \), as is discussed in Appendix A. Increasing \( k \) leads to larger and larger values of \( x_{\min D}^{\text{min}} \), so we need to be careful to use the appropriate \( y^{(p)}(x) \), determined via (1.10), in (4.16). E.g. for \( k \) sufficiently small so that \( x_{D_{k}}^{\text{min}} \) is below the value \( x_{1} \approx 3.67 \) where the meson \( \tilde{Q}Q \) becomes free, we can use (4.8). This gives
\[ x_{\min D_{k+2}}^{\text{min}} = \frac{\sqrt{10 - 34k + 19k^2}}{3\sqrt{2}} \quad \text{for} \quad k < 5. \] (4.18)
The first few are
\[ x_{D_{5}}^{\text{min}} = 2.09, \quad x_{D_{6}}^{\text{min}} = 3.14, \quad x_{D_{7}}^{\text{min}} = 4.24, \quad x_{D_{8}}^{\text{min}} = 5.37, \] (4.19)
for \( k = 3, 4, 5, 6 \).

For large \( k \), the \( x_{D_{k+2}}^{\text{min}} \) become large and we can use the asymptotic value \( y(x \to \infty) = -1/8 \) to get
\[ x_{D_{k+2}}^{\text{min}} \to \frac{9}{8}k \quad \text{for} \quad k \gg 1. \] (4.20)
Also (in analogy with the $A_k$ case discussed in [11]), we have the general inequality

\[
x_{D_{k+3}}^{\min} / (k + 2) > x_{D_{k+2}}^{\min} / (k + 1),
\]

which follows from (4.16), which gives $x_{D_{k+2}}^{\min} / (k + 1) = 1 - y(x_{D_{k+2}}^{\min})$, together with the fact that $y(x)$ is monotonically decreasing in $x$. Using $y(x \to \infty) = -1/8$ we get the inequality

\[
x_{D_{k+2}}^{\min} / (k + 1) < 1 - y(x \to \infty) = \frac{9}{8},
\]

which is saturated (4.20) for $k \to \infty$. Even for low $k$, this estimate isn’t too far off, e.g. it would give for $k = 6$: $x_{D_8}^{\min} < 7.875$, which isn’t so far off from (4.19).

5. The $D_{k+2}$ RG fixed points: $W = \text{Tr}(X^{k+1} + XY^2)$.

Consider deforming the $\hat{O}$ RG fixed points by the superpotential

\[
W = \lambda_1 \text{Tr} X^{k+1} + \lambda_2 \text{Tr} XY^2.
\]

If we were to start with $\lambda_2 = 0$, we have already seen that the $\lambda_1$ deformation would only be relevant for $k \leq 2$. But if we take $\lambda_2 \neq 0$, the theory first flows to be near the $\hat{D}$ RG fixed point. Then, starting at the $\hat{D}$ RG fixed point, we have seen in the previous section that $\text{Tr} X^{k+1}$ is relevant for all $k$, provided that $x > x_{D_{k+2}}^{\min}$. When $x > x_{D_{k+2}}^{\min}$ we thus expect that the RG flow is as in Fig. 13. If $x < x_{D_{k+2}}^{\min}$, the flow is instead as in Fig. 14.

\begin{center}
\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure13}
\caption{RG flow in the $\lambda_1 - \lambda_2$ plane when $x > x_{D_{k+2}}^{\min}$.}
\end{figure}
\end{center}
At the $D_{k+2}$ SCFT fixed point, $\lambda_1 \to \lambda_1^*$ and $\lambda_2 \to \lambda_2^*$. We will rescale $X$ and $Y$ to absorb these coefficients into the Kahler potential, taking $W_{D_{k+2}}$ as in (1.2).

5.1. The chiral ring, and the stability bound

As before, the classical chiral ring is that of the $\hat{O}$ theory, subject to the additional relations coming from the EOM of $W_{D_{k+2}}$: $\{X, Y\} = 0$ and $X^k + Y^2 = 0$ in the chiral ring. The result for $k$ odd is very different from the $k$ even case.

For $k$ odd, these imply that $Y^3 = 0$ in the chiral ring. To see this (see also [18,22]), multiply the second equation of motion by $Y$ on the left and add this to the same equation multiplied by $Y$ on the right, to get $YX^k + X^kY = -2Y^3$. Anticommuting all $Y$ fields to the right then gives $((-1)^k + 1)X^kY = -2Y^3$ and hence $Y^3 = 0$ for $k$ odd. The independent non-zero products of $X$ and $Y$ are then truncated to

$$X^{\ell - 1}Y^{j - 1}, \quad \ell = 1 \ldots k, \quad j = 1 \ldots 3,$$

where the order of $X$ and $Y$ does not matter because of the chiral ring relation $\{X, Y\} = 0$. When we take the traces to form gauge invariant operators, the only non-zero ones (due to $\{X, Y\} = 0$ and cyclicity of the trace) are the $k + 2 + \frac{1}{2}(k - 1)$ operators

$$\text{Tr}X^{\ell - 1} \quad (\text{for } \ell = 1 \ldots k), \quad \text{Tr}Y, \quad \text{Tr}Y^2, \quad \text{Tr}X^{2n}Y^2 \quad (\text{for } n = 1 \ldots \frac{1}{2}(k - 1)).$$

(5.3)

We can also form the $3kN_f^2$ mesons

$$M_{ij} = \bar{Q}X^{l - 1}Y^{j - 1}Q; \quad l = 1, \ldots, k; \quad j = 1, 2, 3,$$

(5.4)
and the baryons

\[ B^{(n_1, n_2, \ldots, n_k, m)} = Q_{(1,1)}^{n_1,1} \cdots Q_{(k,3)}^{n_k,3}, \quad \sum_{l=1}^{k} \sum_{j=1}^{3} n_{l,j} = N_c; \quad n_{l,j} \leq N_f \]  

(5.5)

formed from the dressed quarks

\[ Q_{(l,j)} = X^{l-1}Y^{j-1}Q; \quad l = 1, \ldots, k; \quad j = 1, 2, 3. \]  

(5.6)

We saw already that the \( D_{k+2} \) SCFTs can only exist if \( x > x_{\min}^{D_{k+2}} \). It turns out that these SCFTs, at least for \( k \) odd, also have an upper bound on the allowed value of \( x \):

\[ x < x_{\max}^{D_{k+2}} = 3k. \]  

(5.7)

(For \( A_k \) theories the analogous stability condition is \( x < x_{\max}^{A_k} = k \) \[13, 14\].) For \( x > x_{\max}^{D_{k+2}} \), the theory is rendered unstable by developing a dynamically generated superpotential, which will spoil conformal invariance and drive the theory away from the origin of the moduli space of vacua. The previously discussed RG fixed point theories, \( \hat{O}, \hat{D}, \) and \( \hat{E} \), as well as \( \hat{A} \), are stable for all \( x \), so for those SCFTs there is no upper bound on \( x \).

The stability bound is related to the truncation to the \( 3k \) independent products in (5.2). To see the stability bound, deform \( W_{D_{k+2}} \) by lower order terms, e.g. to

\[ W = \text{Tr}(F_{k+1}(X) + XY^2 + \alpha Y), \]  

(5.8)

where \( F_{k+1}(X) \) is a degree \( k+1 \) polynomial in \( X \). The classical chiral ring relations are the EOM

\[ XY + YX = -\alpha \quad \text{and} \quad Y^2 + F'_{k+1}(X) = 0. \]  

(5.9)

The irreducible representations of this algebra were actually discussed recently in \[22\] in the context of string theory realizations of related SUSY gauge theories. The first relation in (5.9) implies that \( X^2 \) and \( Y^2 \) are Casimirs, \([X^2, Y] = 0\) and \([Y^2, X] = 0\), so we write \( X^2 = x^21 \) and \( Y^2 = y^21 \). It is then seen \[22\] that the second equation in (5.3), for \( k \) odd, admits \( k+2 \) different one-dimensional representations, with \( X = x \) and \( Y = y \) for \( k+2 \) different values of \( x \) and \( y \). There are also \( \frac{1}{2}(k-1) \) two dimensional representations of the form \( X = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \) and \( Y = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} \). This is seen by writing \( F'_{k+1}(X) = XP(X^2) + Q(X^2) \), with the 2d reps specified by the \( \frac{1}{2}(k-1) \) roots of \( P(ab) = 0 \).
The \( \langle X \rangle \) and \( \langle Y \rangle \) vacua solutions of (5.9) can have \( N_i \) copies of the \( i \)-th one-dimensional representation, for \( i = 1 \ldots k + 2 \), and \( M_n \) copies of the \( n \)-th two-dimensional representation for \( n = 1 \ldots \frac{1}{2}(k - 1) \). In such a vacuum the gauge group is Higgsed as

\[
U(N_c) \rightarrow \prod_{i=1}^{k+2} U(N_i) \prod_{n=1}^{\frac{1}{2}(k-1)} U(M_n) \quad \text{with} \quad \sum_{i=1}^{k+2} N_i + \sum_{n=1}^{\frac{1}{2}(k-1)} 2M_n = N_c. \tag{5.10}
\]

Each \( U(N_i) \) factor in (5.10) has \( N_f \) flavors, while each \( U(M_n) \) factor has \( 2N_f \) flavors. (It’s \( 2N_f \) because \( U(M_n) = U(M_n)_{\text{diag}} \subset U(M_n) \times U(M_n) \subset U(N_c) \); the \( U(M_n) \times U(M_n) \) part is related to an example considered in [23,18].) All of the adjoint matter fields get a mass for the generic deforming superpotential (5.8). Since each of these low-energy theories is SQCD, the stability condition is that there is at least one choice of the \( N_i \) and \( M_n \) such that they all satisfy \( N_i < N_f \) and \( M_n < 2N_f \). For this to be the case requires

\[
N_c = \sum_{i=1}^{k+2} N_i + \sum_{j=1}^{\frac{1}{2}(k-1)} 2M_j < (k+2)N_f + \frac{1}{2}(k-1)2(2N_f) = 3kN_f. \tag{5.7}
\]

This is the stability bound (5.7).

The above chiral ring truncation and stability bound were for \( k \) odd. Classically, there is no such truncation or bound for \( k \) even. Though the classical vacua and chiral ring analysis differ for \( k \) even vs odd, we expect that \( k \) even and \( k \) odd are actually qualitatively similar at the quantum level. For example, we expect that the \( D_{k+2} \) SCFTs for even \( k \) have a stability bound similar to (5.7). The reason is that we can RG flow from \( D_{k+2} \) fixed points in the UV to \( D_{k'+2} \) fixed points in the IR, for \( k' < k \), by deforming the \( W_{D_{k+2}} \) superpotential by \( \Delta W = \text{Tr}X^{k'+1} \); in particular we can flow from \( k \) odd to \( k' \) even.

On general grounds, we expect that RG flows always reduce the stability bound:

\[
x_{IR}^{\text{max}} < x_{UV}^{\text{max}}, \tag{5.11}
\]

because the added tree-level superpotential terms of the IR theory can only make it easier to form a dynamically generated superpotential which could destabilize the origin of the moduli space of vacua. So we must have \( x_{D_{k'+2}}^{\text{max}} < x_{D_{k+2}}^{\text{max}} \) for any \( k' < k \). In particular, if we take \( k \) odd and \( k' \) even, we see that the \( D_{k'+2} \) theory does have a stability bound. The simplest possibility compatible with (5.11) is if (5.7) applies for all \( k \). For what follows we will mostly specialize to the case of \( k \) odd but, for the reason described above, we expect the \( k \) even case to be qualitatively similar at the quantum level.
5.2. Computing the central charge \(a_{D_{k+2}}(x)\)

We now compute the central charge \(a\) for the \(D_{k+2}\) theories. There is no need to employ \(\alpha\)-maximization to determine the superconformal \(R\)-charges; they are entirely fixed by the superpotential \(W_{D_{k+2}}\) to be

\[
R(X) = \frac{2}{k+1}; \quad R(Y) = \frac{k}{k+1}; \quad R(Q) = R(\tilde{Q}) = 1 - \frac{x}{k+1}. \tag{5.12}
\]

As in the \(A_k\) case we do, however, still need to account for the effect of apparent unitarity violations and the associated free fields in computing the central charge \(a\). Since (5.12) implies that \(R(Q) < 0\) for \(x > k + 1\), we will clearly need to take into account accidental symmetries for the mesons. It naively appears that baryons could also violate the unitarity bound for sufficiently large \(x\); in appendix C we show that, much as in the \(\hat{A}\) case \([1]\), no baryons ever actually violate this bound.

Computing \(a\) is straightforward but tedious. Since all mesons will acquire negative \(R\)-charges for large enough \(x\), we must figure out where each of the \(3kN_f^2\) mesons (5.4) hits the unitarity bound. Since the \(R\)-charges are fixed, this is easily done: The \(R\) charge of the meson \(M_{lj} = \tilde{Q}X^{l-1}Y^{j-1}Q\) is given by

\[
R(M_{lj}) = 2 \left(1 - \frac{x}{k+1}\right) + (l-1) \frac{2}{k+1} + (j-1) \frac{k}{k+1}, \tag{5.13}
\]

and this equals 2/3 when

\[
x = x_{lj} \equiv \frac{1}{6}(-2 + k + 6l + 3k j). \tag{5.14}
\]

As one can see, the meson with the largest \(R\)-charge, \(M_{k3}\), becomes free at

\[
x_{k3} = \frac{8k - 1}{3} < 3k = x_{D_{k+2}}^{\text{max}}, \tag{5.15}
\]

so, as \(x\) increases in the range \(x_{D_{k+2}}^{\text{min}} < x < x_{D_{k+2}}^{\text{max}}\) eventually all of these mesons become free and we must account for their accidental symmetries.

As an example, consider the \(k = 3\) case, with superpotential \(W = \text{Tr}X^4 + \text{Tr}XY^2\). From our results in Section 4.2, we know that in the range \(x < x_{D_5}^{\text{min}} \approx 2.09\), the \(X^4\) term is not relevant, and we should use our results for the \(\hat{D}\) theory. At the point \(x_{D_5}^{\text{min}} \approx 2.09\), the \(X^4\) becomes relevant, and we can use (5.12) for the \(R\)-charges. We can then compute the central charge \(a\), being careful to account for the accidental symmetries. We have plotted
$a$ for the $\hat{D}$ and $D_5$ theories in Figure 15. The two curves touch exactly at the point $x_{D_5}^{\min}$ as they must, since the central charge is a continuous function of $x$.

![Figure 15: The central charge $a$ for $\hat{D}$ (top, blue) and $D_5$ (bottom, red). The curves touch at $x_{D_5}^{\min} \approx 2.09$.](image)

**5.3. Flow from $D_{k+2}$ to $D_{k'+2}$**

We now add a term $X^{k'+1}$ with $k' < k$ to the $D_{k+2}$ superpotential. If $x > x_{D_{k+2}}^{\min}$, this is clearly a relevant deformation, since we then use the R-charges in (5.12) to get

$$R(X^{k'+1}) = 2 \frac{k'+1}{k+1} < 2.$$  \hspace{1cm} (5.16)

If $x < x_{D_{k+2}}^{\min}$ the $X^{k+1}$ term is not relevant, so the $X^{k'+1}$ is a deformation of the $\hat{D}$ theory. As such, it is only relevant if $x > x_{D_{k'+2}}^{\min}$, again driving the theory to the $D_{k'+2}$ SCFT. For $x < x_{D_{k'+2}}$ both terms are irrelevant and the theory flows back to the $\hat{D}$ SCFT.

As in [9,11], there is a range of $x$ for which the a-theorem is potentially violated:

$$a_{D_{k+2}}(x) < a_{D_{k'+2}}(x) \text{ for } 1 < x < x_{\text{int}}(k+2,k'+2).$$  \hspace{1cm} (5.17)

For several pairs $(k+2,k'+2)$, we have computed this value of $x_{\text{int}}(k+2,k'+2)$:

- $x_{\text{int}}(8,6) \approx 4.08$
- $x_{\text{int}}(7,6) \approx 3.64$
- $x_{\text{int}}(8,5) \approx 3.23$
- $x_{\text{int}}(7,5) \approx 2.94$
- $x_{\text{int}}(6,5) \approx 2.56$.  \hspace{1cm} (5.18)

However, as in the $A_k$ cases [11], in no case is the a-theorem ever actually violated, because all of the above potential violations occur for $x$ outside of the range where the $D_{k+2}$ SCFT exists. Recall from Section 4.2 that $W_{D_{k+2}}$ is only relevant for $x > x_{D_{k+2}}^{\min}$ with

$$x_{D_5}^{\min} = 2.09, \quad x_{D_6}^{\min} = 3.14, \quad x_{D_7}^{\min} = 4.24, \quad x_{D_8}^{\min} = 5.37.$$  \hspace{1cm} (5.19)
Since all of the apparent violations (5.18) occur for $x < x_{\text{int}}(k+2,k'+2)$ with $x_{\text{int}}(k+2,k'+2) < x_{D_{k+2}}^{\text{min}}$ none of them should be taken seriously: since the $D_{k+2}$ RG fixed point does not exist for this range of $x$, there is no $a$-theorem violating RG flow after all. In Figure 16, we have plotted what the central charges would have been if $D_6$ and $D_5$ were relevant for small $x$; one can see that the $a$-theorem would potentially be violated for $x < 2.56$, but the $D_6$ RG fixed point exists only for $x > x_{D_6}^{\text{min}} \approx 3.14$.

![Figure 16: The central charge $a$ for $D_6$ (blue) and $D_5$ (red).](image)

The potential violation of the $a$-theorem is for $1 < x < 2.56$.

6. Duality for the $D_{k+2}$ theories.

A magnetic dual description of the $D_{k+2}$ theories was proposed in [18]. We will here discuss and clarify the meaning of this duality. We also use the duality to determine the behavior of the $D_{k+2}$ RG fixed points at large $x$, where there are some accidental symmetries which are not manifest in the strongly coupled electric description but are more easily seen in the weakly coupled magnetic dual.

The magnetic dual [18] of the $D_{k+2}$ RG fixed point is an $SU(3kN_f - N_c)$ gauge theory with adjoints $\tilde{X}$ and $\tilde{Y}$, $N_f$ magnetic quarks $q_i$ and $\tilde{q}_i$, and $3kN_f^2$ singlets $(M_{\ell j})_{\tilde{X}}$, $\ell = 1 \ldots k$, $j = 1, 2, 3$, with the tree-level superpotential

$$ W = \text{Tr} \tilde{X}^{k+1} + \text{Tr} \tilde{X} \tilde{Y}^2 + \sum_{\ell=1}^{k} \sum_{j=1}^{3} M_{\ell j} \tilde{q} \tilde{X}^{k-\ell} \tilde{Y}^{3-j} q, \quad (6.1) $$

where we omit the RG fixed point coefficients in front of the various terms in (6.1). We will define $N_{\ell j} \equiv \tilde{q} X^{\ell-1} Y^{j-1} q$, which are the magnetic mesons. The superpotential (6.1) implies that

$$ M_{\ell j} \leftrightarrow N_{k+1-\ell,3-j} \quad (6.2) $$
are Legendre-transform conjugate variables. Thus, provided that the corresponding term in (6.1) is relevant at the RG fixed point, we should include either $M_{\ell j}$ or $N_{k+1-\ell,3-j}$ in the spectrum of independent operators at the RG fixed point, but not both.

The phases of the electric and magnetic theories can be summarized as

$$
\begin{align*}
x \leq 1 & \quad \text{free electric} \\
1 < x < x_{D_{k+2}}^{\min} & \quad \hat{D} \text{ electric} \\
x_{D_{k+2}}^{\min} < x < 3k - x_{D_{k+2}}^{\min} & \quad D_{k+2} \text{ conformal window} \\
3k - x_{D_{k+2}}^{\min} < x < 3k - 1 & \quad \hat{D} \text{ magnetic} \\
3k - 1 \leq x & \quad \text{free magnetic.}
\end{align*}
$$

(6.3)

For $x \leq 1$ the electric theory is not asymptotically free, so it flows to a free theory in the IR. In this case, we should definitely use the free-electric description! To see the analogous free-magnetic phase of the magnetic dual, it’s useful to introduce a dual variable to $x$:

$$
\bar{x} \equiv \frac{\bar{N}_c}{N_f} = 3k - x.
$$

(6.4)

The magnetic theory is asymptotically free if $\bar{x} > 1$. When the magnetic theory is not asymptotically free, i.e. $\bar{x} \leq 1$ and thus $x \geq 3k - 1$, the magnetic theory becomes free in the IR. In this case, we should definitely use the magnetic description. Within the range $1 < x < 3k - 1$, where both electric and magnetic theories are asymptotically free, we still have three possibilities. If $1 < x < x_{D_{k+2}}^{\min}$ the $\text{Tr} X^{k+1}$ on the electric side is irrelevant, and the electric theory flows back to the $\hat{D}$ SCFT. In this case the electric description is again definitely better, since it is easier to see the enhanced symmetries associated with the fact that $\text{Tr} X^{k+1}$ is irrelevant. Likewise, in the magnetic theory, the $\text{Tr} \tilde{X}^{k+1}$ superpotential is irrelevant if $\bar{x} < \bar{x}_{D_{k+2}}^{\min}$ (quantities which we’ll discuss shortly) and the magnetic theory then flows to a magnetic version of the $\hat{D}$ SCFT. In this case, the magnetic description is definitely better, since it’s easier there to see the enhanced symmetries associated with the fact that part of $W_{mag}$ is irrelevant. Finally, there is a “conformal window,” where $\text{Tr} X^{k+1}$ is relevant on the electric side, $x > x_{D_{k+2}}^{\min}$, and $\text{Tr} \tilde{X}^{k+1}$ is relevant on the magnetic side, $\bar{x} > \bar{x}_{D_{k+2}}^{\min}$. In the conformal window, both the electric and the magnetic theories flow to the same $D_{k+2}$ SCFT. Either the electric or the magnetic description is a useful description in the conformal window.

The computation of the R-charges for the magnetic theory proceeds similarly to the analogous computation in [11], although here it is complicated somewhat by the presence of additional fields $M_{\ell j}$. We consider first the situation where $\bar{x} = 1 + \bar{\epsilon}$, with $0 < \bar{\epsilon} \ll 1,$
so that the magnetic dual is just barely asymptotically free. The magnetic dual theory is then very weakly coupled and is a very useful description of the IR physics. In this small $\tilde{x}$ limit only the cubic terms in (5.1), $\text{Tr} \tilde{X} \tilde{Y}^2 + M_{k3} \tilde{q} \tilde{q}$ are relevant; all of the other terms in (6.1) are irrelevant and can be ignored in the far IR limit. In particular, for $k > 2$, the $\text{Tr} \tilde{X}^{k+1}$ term in (6.1) is irrelevant, so the magnetic theory actually flows to a $\tilde{D}$ RG fixed point rather than a $D_{k+2}$ RG fixed point in the IR.

So for $\tilde{x}$ not too far above 1, the superconformal $U(1)_R$ charge is given by the magnetic analog of the $\tilde{D}$ results (4.6):

$$R(q) = R(\tilde{q}) \equiv \tilde{y}, \quad R(\tilde{Y}) = \frac{\tilde{y} - 1}{\tilde{x}} + 1, \quad R(\tilde{X}) = \frac{2 - 2\tilde{y}}{\tilde{x}}.$$ (6.5)

The $(3k - 1)N_f^2$ fields $M_{lj}$ for all $l \leq k$ and $j \leq 3$, except for $M_{k3}$ are all decoupled free fields, with $R(M_{lj}) = 2/3$. The $N_f^2$ fields $M_{k3}$ couple via the relevant term in the superpotential (6.1) and are the Legendre transform conjugate variable to $N_{11} = \tilde{q} \tilde{q}$; the superpotential (6.1) then fixes $R(M_{k3}) = 2 - 2\tilde{y}$.

As we continue to increase $\tilde{x}$ from 1, until we reach some upper bound $\tilde{x}_1$, the only relevant terms in (6.1) are $\text{Tr} \tilde{X}^2 \tilde{Y} + M_{k3} \tilde{q} \tilde{q}$. We thus compute

$$\tilde{a}^{(0)} = \frac{\tilde{a}^{(0)}}{N_f^2} = 2\tilde{x}^2 + 3\tilde{x}^2 \left( \frac{\tilde{y} - 1}{\tilde{x}} \right)^3 + 3\tilde{x}^2 \left( \frac{2 - 2\tilde{y}}{\tilde{x}} - 1 \right)^3$$

$$- \tilde{x}^2 \left( \frac{\tilde{y} - 1}{\tilde{x}} \right) - \tilde{x}^2 \left( \frac{2 - 2\tilde{y}}{\tilde{x}} - 1 \right) + 6\tilde{x}(\tilde{y} - 1)^3 - 2\tilde{x}(\tilde{y} - 1) + \frac{1}{9}(2 - 3R(N_{11}))^2(5 - 3R(N_{11})) + \frac{2}{9}(3k - 2),$$ (6.6)

where we define $N_{1,1} \equiv \tilde{q} \tilde{q}$ to be the magnetic meson which is Legendre transform dual to the interacting meson $M_{3k}$ and $R(N_{11}) = 2\tilde{y}$. The first two lines in (6.6) are simply the magnetic version of $N_f^{-2}a^{(0)}_D(\tilde{x}, \tilde{y})$, found in (4.7). The last line in (6.6) includes the additional contributions of the $(3k - 1)N_f^2$ free field mesons, i.e. all $M_{l \ell j}$ aside from $M_{3k}$, each of which contributes $2/9$ to $a$, along with the contribution of the interacting meson $M_{k3}$, with R-charge $2 - 2\tilde{y}$. The first term on the last line looks similar to how we would correct $\tilde{a}$ if the meson $N_{11}$ were a free field, but that is a fake: we actually should not even include the magnetic meson $N_{11}$ as an independent field, because the relevant term in (6.1) makes it the Legendre transform of $M_{k3}$. It just happens that the $M_{l \ell j}$ contributions can be written in this similar form to the electric side, though the interpretation is different. We now maximize (6.6) with respect to $\tilde{y}$ to obtain $\tilde{y}^{(0)}(\tilde{x})$ and the central charge $N_f^{-2}a^{(0)}(\tilde{x}) = N_f^{-2}a^{(0)}(\tilde{x}, \tilde{y}^{(0)}(\tilde{x}))$. 31
As we continue to increase $\tilde{x}$, more of the previously irrelevant terms in

$$
\sum_{l=1}^{k+3} \sum_{j=1}^{3} M_{lj} \tilde{g} \tilde{X}^{k-l}\tilde{Y}^{3-j} q \tag{6.7}
$$
eventually become relevant. We continue by an iterative procedure. We compute $N_f^{-2}\tilde{a}(\tilde{x}, \tilde{y})^{(p-1)}$ for the theory where $p$ such terms are relevant, with the remaining $3k-p$ irrelevant. The $M_{lj}$ entering the relevant terms have R-charge determined by the superpotential to be

$$
R(M_{lj}) = 2 - 2R(q) - (k-l)R(X) - (3-j)R(Y) \\
= 2 - 2\tilde{y} - (k-l) \left( \frac{2 - 2\tilde{y}}{\tilde{x}} \right) - (3-j) \left( \frac{\tilde{y} - 1}{\tilde{x}} + 1 \right), \tag{6.8}
$$

while the $M_{ij}$ for which the term in (6.7) is irrelevant are free fields, with $R(M_{lj}) = 2/3$. We then maximize the corresponding $N_f^{-2}\tilde{a}(\tilde{x}, \tilde{y})^{(p-1)}$ with respect to $\tilde{y}$ to to find $\tilde{y}^{(p-1)}(\tilde{x})$ and $N_f^{-2}\tilde{a}^{(p-1)}(\tilde{x})$.

These results for $\tilde{y}^{(p-1)}(\tilde{x})$ and $\tilde{a}^{(p-1)}(\tilde{x})$ are applicable until $\tilde{x} > \tilde{x}_p$, where the next previously irrelevant term in (6.7) becomes relevant, which is when the corresponding

$$
R(N_{k+1-l,3-j}) = 2\tilde{y} + (k-l) \left( \frac{2 - 2\tilde{y}}{\tilde{x}} \right) + (3-j) \left( \frac{\tilde{y} - 1}{\tilde{x}} + 1 \right) = \frac{4}{3} \tag{6.9}
$$

(using $\tilde{y} = \tilde{y}^{(p-1)}(x)$) for some new values of $(\ell, j)$. When this happens, we switch to

$$
\frac{\tilde{a}(p)(\tilde{x}, \tilde{y})}{N_f^2} = \frac{\tilde{a}^{(p-1)}(\tilde{x}, \tilde{y})}{N_f^2} + \frac{1}{9}[(2 - 3R(N_{k+1-l,3-j}))^2(5 - 3R(N_{k+1-l,3-j})) - \frac{4}{9}], \tag{6.10}
$$

which accounts for the newly interacting field $M_{lj}$ having R-charge given by (6.8) rather than the free-field value $R(M_{kj}) = 2/3$. We then maximize $N_f^{-2}\tilde{a}^{(p)}(\tilde{x}, \tilde{y})$ to obtain $\tilde{y}^{(p)}(\tilde{x})$ and $N_f^{-2}\tilde{a}^{(p)}(\tilde{x})$ in this next $\tilde{x}$ range, $\tilde{x}^{(p)} \leq \tilde{x} \leq \tilde{x}^{(p+1)}$. Note that by the time any of the magnetic mesons $N_{k+1-l,3-j} = q\tilde{X}^{k-l}\tilde{Y}^{3-j}q$ hit their unitarity bound $R(N_{k+1-l,3-j}) = 2/3$, the term involving them in (6.7) is already relevant, so $N_{k+1-l,3-j}$ should not be included as an independent operator, it is to be already be eliminated in favor of its Legendre transform field $M_{lj}$.

As we continue to increase $\tilde{x}$, eventually we hit $\tilde{x} = \tilde{x}_{D_{k+2}}^{\min}$, where the term $\text{Tr} \tilde{X}^{k+1}$ becomes relevant:

$$
(k+1) \frac{2 - 2\tilde{y}(X_{D_{k+2}}^{\min})}{\tilde{x}^{\min}_{D_{k+2}}} = 2. \tag{6.11}
$$
The R-charges are then determined without any need for a-maximization, as in (5.12). One must still watch out for when mesons become interacting, since their R-charges are now linearly increasing functions of $x$. In fig. 17, we have plotted the central charges for the magnetic $D_5$ theory, for both relevant $\tilde{X}^4$ and irrelevant $\tilde{X}^4$. They touch at the point $\tilde{x}_{D_5}^{\text{min}} \approx 1.86$; for $\tilde{x} < 1.86$ the term $\text{Tr} \tilde{X}^4$ is irrelevant, while for $\tilde{x} > 1.86$ it is relevant. Recall that, in the electric $D_5$ theory, we found (1.19) that $\text{Tr} \tilde{X}^4$ is relevant for $x > x_{D_k+2}^{\text{min}} \approx 2.09$, so we see that the critical value for the superpotential to become relevant differs between the electric and magnetic theories, $\tilde{x}_{D_k+2}^{\text{min}} \neq x_{D_k+2}^{\text{min}}$, with the magnetic value of $\tilde{x}_{D_k+2}^{\text{min}}$ somewhat smaller; they are a little different because the magnetic theory contains the extra singlet fields $M_{\ell j}$ and the additional superpotential terms.

![Figure 17: The central charge for irrelevant (top, blue) and relevant (bottom, red) $\tilde{X}^4$.](image)

We can get an upper bound on $\tilde{x}_{D_k+2}^{\text{min}}$ as in (4.22), by using the fact that the $\tilde{y}(\tilde{x})$ obtained by the above procedure is a monotonically decreasing function of $\tilde{x}$. This implies

$$\frac{\tilde{x}_{D_k+2}^{\text{min}}}{k+1} < 1 - \tilde{y}_{\text{asympt}} \approx 1.1038.$$  \hfill (6.12)

Here $\tilde{y}_{\text{asympt}}$ is the value that $\tilde{y}$ plateaus at for a while, when $\tilde{x}$ is large but still below $\tilde{x}_{D_k+2}^{\text{min}}$. We compute $\tilde{y}_{\text{asympt}}$ in Appendix D. From (6.12), we can then calculate $\tilde{x}_{D_k+2}^{\text{min}}$ in the large $k$ limit:

$$\tilde{x}_{D_k+2}^{\text{min}} \approx 1.1038k.$$  \hfill (6.13)

Again, we see that $\tilde{x}_{D_k+2}^{\text{min}}$ (6.13) of the magnetic theory is slightly smaller than that of the corresponding electric theory (4.20).

Figure 18 shows, for the magnetic $D_5$ example, the R-charges $R[q] \equiv \tilde{y}(\tilde{x})$ as computed by the above a-maximization procedure. Though it was obtained by patching together
the $\tilde{y}^{(p)}(\tilde{x})$, it is continuous. In fig. 19 we plot $R[X]$ for this same theory, given as in (5.3). $\text{Tr}\tilde{X}^4$ becomes relevant when $R(\tilde{X}) \rightarrow \frac{1}{2}$, which is found from fig. 19 to occur for $\tilde{x} > \tilde{x}_{D^5}^{\text{min}} \approx 1.86$.

![Figure 18: $\tilde{y}(\tilde{x}) = R(q)$ for magnetic $D_5$.](image1)

![Figure 19: $R(\tilde{X})$ for magnetic $D_5$.](image2)

We see (as in the $A_k$ case discussed in [11]) that there is a conformal window in which the $D_{k+2}$ superpotential $\text{Tr}(X^{k+2} + XY^2)$ and its magnetic analog are both relevant:

$$x_{D_{k+2}}^{\text{min}} < x < 3k - \tilde{x}_{D_{k+2}}^{\text{min}}.$$ (6.14)

For example, for $k = 3$, $x_{D_5}^{\text{min}} \approx 2.09$ and $\tilde{x}_{D_5}^{\text{min}} \approx 1.86$. For the case of large $k$, we can use our results (4.20) and (6.13) to show that this window always exists:

$$\frac{9}{8}k = x_{D_{k+2}}^{\text{min}} < x < 3k - \tilde{x}_{D_{k+2}}^{\text{min}} \approx 2.062k.$$ (6.15)

Within this window the central charges agree, as they should:

$$a^{cl}(x) = \tilde{a}^m(3k - x).$$ (6.16)
Outside the window, the charges do not agree. Indeed they should not have been expected to agree, because one or the other side does not readily exhibit the accidental symmetries of the IR theory. In fig. 20, we have plotted the difference between the (numerically computed) electric and magnetic central charges for the $D_5$ theory and its dual. As one can see, they agree in the range $2.09 < x < 9 - 1.86 = 7.14$ but disagree outside. The correct $a$ to use is the larger of $a^{el}$ or $\tilde{a}^{mag}$, and it’s larger because of maximizing $a$ over a bigger symmetry group.

![Figure 20: $a_{D_5}^{el}(x)/N_f^2 - \tilde{a}_{D_5}^{mag}(9-x)/N_f^2$.](image)

7. The $E_{6,7,8}$ RG fixed point theories

7.1. Chiral rings and stability bounds

Before discussing the $E_{6,7,8}$ RG fixed points in detail, we make some general comments about their chiral rings and the possible existence of a stability bound. We’ll consider the $E_6$ example; the $E_7$ and $E_8$ cases are similar.

The superpotential $W_{E_6} = \text{Tr}(Y^3 + X^4)$ leads to the chiral ring relations

$$\partial_X W_{E_6} = X^3 = 0, \quad \partial_Y W_{E_6} = Y^2 = 0.$$  \hfill (7.1)

(For simplicity we consider $U(N)$ to avoid having to impose tracelessness). These do not truncate the chiral ring; e.g. they do not allow us to eliminate $(XY)^n$ for arbitrary $n$.

The stability bound is related to the counting of vacua upon deformation, which is related to the number of independent products in the chiral ring. Roughly $x^{\text{max}}$ equals the number of independent products of $X$ and $Y$ in the chiral ring. To be more precise, upon deforming the superpotential by the generic, lower order terms, the equations of motion will have representations of various dimensions. In the $A_k$ case, the chiral ring reps are all

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one dimensional, because we can diagonalize the single adjoint. For the $D_{k+2}$ case we saw, following [22], that the deformed ring relations (5.9) have $k + 2$ different one-dimensional reps and $\frac{1}{2}(k - 1)$ different two-dimensional representations.

More generally, suppose that some deformed superpotential has $n_d$ different $d$-dimensional representations, for $d = 1, 2, 3, \ldots$. We can take the adjoints to have $N_{d,i_d}$ copies of the $i_d$'th $d$-dimensional representation, with $i_d = 1 \ldots n_d$, breaking the gauge group as

$$U(N_c) \to \prod_d \prod_{i_d=1}^{n_d} U(N_{d,i_d}), \quad \text{with} \quad N_c = \sum_d \sum_{i_d=1}^{n_d} dN_{d,i_d}. \quad (7.2)$$

The $U(N_{d,i_d})$ theory has $dN_f$ massless flavors and no massless adjoints, and will be stable if we can find a solution such that all $N_{d,i_d} < dN_f$. It thus follows that

$$N_c < \sum_d n_dd^2N_f, \quad \text{i.e.} \quad x_{\text{max}} = \sum_{d=1}^{\infty} n_dd^2. \quad (7.3)$$

For the $A_k$, $D_{k+2}$, and $E_{6,7,8}$ Landau-Ginzburg superpotentials, the number of 1d representations is always the rank of the corresponding ADE group, i.e. $n_{d=1} = r$. In particular, for $E_6$, we have $n_1 = 6$, which would be the dimension of the chiral ring if $X$ and $Y$ were not matrices. From (7.3) it follows that $x_{\text{max}} \geq r$, e.g. for $E_6$, $x_{E_6}^{\text{max}} \geq 6$. We can see that if the chiral ring does not truncate, corresponding to having arbitrarily many different representations of the deformed superpotential EOM, then the sum in (7.3) will be infinite and so $x_{\text{max}} = \infty$, i.e. no stability bound. This is what our classical analysis (7.1) suggests for the $E_6$ SCFT: an infinite classical result for $x_{E_6}^{\text{max}}$.

But perhaps the $E_6$ chiral ring truncates at the quantum level, as we already suggested should be the case for the even $k$ case of the $D_{k+2}$ chiral ring. In this case, there would be a quantum stability bound $x_{E_6}^{\text{max}}$ which is finite. Our numerical analysis of this section, combined with our belief in the a-theorem, suggest that there is indeed a finite quantum stability bound $x_{\text{max}}$ for the $E_6$, and $E_7$ and $E_8$ RG fixed points (where, similar to the $E_6$ case, the classical ring relations do not suffice to truncate the ring).

Finally, as we discussed in (5.11), we expect that RG flows always reduce the stability bound $x_{IR}^{\text{max}} < x_{UV}^{\text{max}}$. E.g. we can flow from $E_6$ to $D_5$, so we expect $x_{E_6}^{\text{max}} > x_{D_5}^{\text{max}} = 9$. 36
7.2. $E_6$: $W = \text{Tr}(X^4 + Y^3)$

As seen in sect. 3.2, for $x > x_{E_6}^{\text{min}} \approx 2.55$, there is a new RG fixed point associated with $W = \lambda_1 \text{Tr}X^4 + \lambda_2 \text{Tr}Y^3$. Starting e.g. near the $\tilde{O}$ RG fixed point and perturbing by this superpotential, the $\lambda_1$ term is initially irrelevant while the $\lambda_2$ term is relevant, driving the theory near the $\tilde{E}$ RG fixed point. Then, for $x > x_{E_6}^{\text{min}}$, the $\lambda_1$ term becomes relevant and drives the theory to the new $\tilde{E}_6$ RG fixed point. The flow picture is analogous to fig. 13. The superconformal $U(1)_R$ charges at the $E_6$ fixed point are determined by the superpotential and the anomaly free condition to be

$$R(Q) = R(\tilde{Q}) = 1 - \frac{x}{6}, \quad R(X) = \frac{1}{2}, \quad R(Y) = \frac{2}{3}. \quad (7.4)$$

We can compute the central charge $a(x)$, using the R-charges (7.4), correcting it to account for the gauge invariant operators which hit the unitarity bound and then become free fields. This procedure is straightforward (but tedious). In addition to the mesons which hit the unitarity bound, we will argue in Appendix C that baryons would also hit the unitarity bound if we continued to $x$ sufficiently large. For the present discussion it suffices to take $x \leq 14$, for which one can check that no baryon has yet hit the unitarity bound. So we only need to account for the mesons which hit the bound, along with their multiplicities; e.g. one must account for the fact the two mesons $\tilde{Q}XYQ$ and $\tilde{Q}YXQ$ are distinct. Accounting for all this, we numerically obtained the central charge $a_{E_6}(x)$ for $x \leq 14$.

A prediction of the a-theorem is that $a$ should monotonically decrease as a function of $N_f$, for fixed $N_c$. As discussed in sect. 2.3, this implies that $a(x)x^{-2}N_f^{-2}$ must be a strictly decreasing function of $x$. We plotted $a_{E_6}(x)N_f^{-2}x^{-2}$ in fig. 21

![Figure 21: $a(x)x^{-2}N_f^{-2}$ for the $E_6$ theory.](image-url)
Contrary to the a-theorem prediction for a monotonically decreasing function, the curve in fig. 21 flattens out at $x \approx 13.80$. There is a similar flattening out for the $D_k+2$ theories right above the stability bound $x^\text{max}_{D_k+2} = 3k$. Since we, by now, have much faith in the a-conjecture, and we also suspected anyway that the $E_6$ RG fixed point might have a quantum truncated chiral ring and stability bound, this is what we think the flattening in fig. 21 is showing: that there is indeed a quantum stability bound at $x^\text{max}_{E_6} < 13.80$ (which is, fortunately, consistent with our earlier statements that $x^\text{max}_{E_6} > 6$ and, via the RG flow to $D_5$, $x^\text{max}_{E_6} > 9$). We leave a deeper understanding of the quantum stability bounds such as $x^\text{max}_{E_6}$ as an open question for future work.

We verified the a-theorem prediction that $a_{E_6}(x) > a_{D_5}(x)$ for all $x$ in the range where the $E_6$ fixed point exists: $x > x^\text{min}_{E_6} \approx 2.55$. One can see this behavior in fig. 22.

![Figure 22](image1.png)

**Figure 22:** $a_{E_6}/N_f^2$ (top, blue) and $a_{E_6}/N_f^2$ (bottom, red). The curves touch at $x^\text{min}_{E_6} \approx 2.55$.

Because there is a relevant $\Delta W$ deformation taking $E_6 \rightarrow D_5$, the a-theorem predicts that $a_{E_6}(x) > a_{D_5}(x)$ for the range of $x$, $x > x^\text{min}_{E_6}$ where both RG fixed points exist. We have plotted $a/N_f^2$ for both $D_5$ and $E_6$ in Figure 23. The a-theorem is potentially violated in the region $x < 1.50$, but is indeed satisfied in the entire region $x > x^\text{min}_{E_6} \approx 2.55$, where the $E_6$ SCFT actually exists.

![Figure 23](image2.png)

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Figure 23: $a_{E_6}/N_f^2$ (top, blue) and $a_{D_5}/N_f^2$ (bottom, red). The curves cross at $x \approx 1.50$.

7.3. $E_7$: $W = \text{Tr}(YX^3 + Y^3)$

Starting with $W = \lambda_1 \text{Tr}YX^3 + \lambda_2 \text{Tr}Y^3$, we have a RG flow that goes first to the vicinity of the $\hat{E}$ fixed point and then, provided that $x > x_{E_7}^{\text{min}} \approx 4.12$, the $\lambda_1$ term takes over and drives the theory to the new $E_7$ RG fixed point. At the $E_7$ RG fixed point the superconformal R-charges are determined to be

$$R(Q) = R(\tilde{Q}) = 1 - \frac{x}{9}, \quad R(X) = \frac{4}{9}, \quad R(Y) = \frac{2}{3}.$$ (7.5)

In computing the central charge $a_{E_7}(x)$, we must account for all of the independent mesons and baryons which hit the unitarity bound. Again, doing so is quite tedious, so we only carried out the analysis to relatively low $x$. In the range analyzed, we verified the a-theorem predictions. For example there is a RG flow $\hat{E} \rightarrow E_7$, and we verified the a-theorem prediction that $a_{E_7}(x) > a_{E_7}(x)$ for all $x > x_{E_7}^{\text{min}}$; see fig. 24.

Figure 24: $a_{\hat{E}}/N_f^2$ (top, blue) and $a_{E_7}/N_f^2$ (bottom, red). The curves touch at $x_{E_7}^{\text{min}} \approx 4.12$.

At the $E_7$ RG fixed point, the superpotential $\Delta W = \text{Tr}X^4$ is relevant, since (7.3) gives $R(\Delta W) = 16/9 < 2$, and it leads to the RG flow $E_7 \rightarrow E_6$. The a theorem thus requires that $a_{E_7}(x) > a_{E_6}(x)$ for all $x > x_{E_7}^{\text{min}}$. Looking at fig. 25, we see that this inequality is indeed satisfied. Again there is an apparent violation, since $a_{E_6}$ is larger than $a_{E_7}$ for $x < 3.16$, but actually no contradiction with the a-theorem because the $E_7$ fixed point does not exist for $x < x_{E_7}^{\text{min}} \approx 4.12$, so there is no a-theorem violating RG flow.
As in the $E_6$ case discussed above, we suspect that there is a stability bound upper limit $x_{E_7}^{max}$ for the $E_7$ fixed points. Since we can RG flow from $E_7 \rightarrow E_6$, it should satisfy $x_{E_7}^{max} > x_{E_6}^{max}$.

7.4. $E_8$: $W = \text{Tr}(X^5 + Y^3)$

Taking $W = \lambda_1 \text{Tr}X^5 + \lambda_2 \text{Tr}Y^3$, there is an RG flow first in $\lambda_2$ to the vicinity of the $\hat{E}$ RG fixed points, and then the $\lambda_1$ term drives the theory to the new $E_8$ RG fixed points provided that $x > x_{E_8}^{min} \approx 7.28$. At the $E_8$ superconformal fixed point, the R-chargers are

$$R(Q) = R(\bar{Q}) = 1 - \frac{x}{15}, \quad R(X) = \frac{2}{5}, \quad R(Y) = \frac{2}{3}. \quad (7.6)$$

(We note from (5.12), (7.4), (7.5), (7.6) that the $A_k$, $D_{k+2}$ and $E_{6,7,8}$ RG fixed points have $R(Q) = R(\bar{Q}) = 1 - \frac{2x}{h}$ where $h$ is the dual Coxeter number of the corresponding $A_k$, $D_{k+2}$, or $E_{6,7,8}$ group.)

The central charge $a_{E_8}(x)$ can thus be computed, where we again must account for the accidental symmetries associated with mesons and baryons which hit the unitarity bound. We carried out this process to $x = 16$. As seen in fig. 26, the a-theorem prediction $a_{\hat{E}}(x) > a_{E_8}(x)$ is satisfied for $x > x_{E_8}^{min}$, where the $E_8$ fixed point exists.
Figure 26: $a_E/N_f^2$ (top, blue) and $a_{E_8}/N_f^2$ (bottom, red). The curves touch at $x_{E_8}^{min} \approx 7.28$.

Starting from the $E_8$ fixed point, the deformation $\Delta W = \text{Tr} Y X^3$ is relevant, leading to the RG flow $E_8 \to E_7$. We can also deform the $E_8$ fixed point by the relevant deformation $\Delta W = \text{Tr} X^4$, which leads to the RG flow $E_8 \to E_6$. So the a-theorem predicts that $a_{E_8}(x) > a_{E_7}(x)$ and $a_{E_8}(x) > a_{E_6}(x)$, for all $x$ in the range where the $E_8$ fixed point exists, $x > x_{E_8}^{min}$. We see from fig. 27 that the first of these inequalities is indeed satisfied; the second works similarly. The inequality $a_{E_8}(x) > a_{E_7}(x)$ is satisfied for $x > 5.25$ and the inequality $a_{E_8}(x) > a_{E_6}(x)$ is satisfied for $x > 3.79$; so both are satisfied for $x > x_{E_8}^{min} \approx 7.28$.

![Figure 26](image)

Figure 27: $a_{E_8}/N_f^2$ (blue) and $a_{E_7}/N_f^2$ (red). The curves cross at $x \approx 5.25$.

As in the $E_6$ and $E_7$ cases discussed above, we suspect that there is a quantum stability bound upper limit $x_{E_8}^{max}$ for the $E_8$ fixed points. It should satisfy $x_{E_8}^{max} > x_{D_7}^{max} = 15$ since we can RG flow from $E_8 \to D_7$.

8. New RG fixed points, with mesonic superpotentials

We briefly mention some new SCFTs, which can be obtained from our previously discussed ones by relevant superpotential deformations involving the meson chiral operators.

8.1. Flowing from $\hat{O}$

Starting at the $\hat{O}$ SCFT, the mesonic gauge invariant operators (2.2), with $n$ powers of $X$ or $Y$, will be relevant if $2y(x) + nz(x) < 2$. Using the fact that the expressions (2.7) and (2.8) for $y(x)$ and $z(x)$ are monotonically decreasing with $x$, with asymptotic values
(2.10), we see that we must have $2(0.575) + \frac{1}{7}n < 2$. Thus, the only solution is $n = 1$, i.e. the superpotential

$$\Delta W = \lambda \tilde{Q}_i Y Q^i, \quad (8.1)$$

where we chose to break $SU(N_f) \times SU(N_f) \to SU(N_f)$ by including all meson flavor components diagonally. The superpotential (8.1) is relevant for all $x \geq 1$. This relevant deformation drives $\hat{O}$ to a new RG fixed point, which we’ll call $\hat{O}_M$. We expect it to flow to an interacting SCFT by the argument outlined in footnote 5.

At the $\hat{O}_M$ SCFT, the R-charges are given by the general expression (2.3) with the additional constraint $2y + z = 2$. Using this to eliminate $z$, the a-maximization procedure can be used to solve for $y(x)$. We leave analysis of this to the interested reader.

8.2. Flowing from $\hat{D}$

Starting at the $\hat{D}$ RG fixed point, we can see from the asymptotic values of the R-charges (4.15) that all of the mesons (4.2) can be relevant operators:

$$\Delta W_{M_{\ell j}} = \tilde{Q}_i X^\ell Y^j Q^i, \quad \ell \geq 0, \quad j = 0, 1, \quad (8.2)$$

will be relevant for any $\ell$ and $j = 0, 1$, provided that $x$ is sufficiently large, $x > x_{\min}^{\hat{D}_{M_{\ell j}}}$. The interested reader can analyze the resulting RG fixed point theories, where the superconformal R-charge is completely fixed by the condition that it respect the two terms in the superpotential.

8.3. Flowing from $\hat{E}$

Using the asymptotic values in (3.4) for $R(Q)$ and $R(X)$, together with $R(Y) = 2/3$, we see that the relevant mesonic deformations are

$$\Delta W = \tilde{Q}_i X^\ell Q^i, \quad \text{for} \quad 0 < \ell \leq 3, \quad \text{and} \quad \Delta W = \tilde{Q}_i XY Q^i. \quad (8.3)$$

Again, the interested reader can analyze the resulting RG fixed point theories, where the superconformal R-charge is completely fixed by the condition that it respect the two terms in the superpotential.
9. New SCFTs, with $\hat{O}$ not in their domain of attraction?

Our process of starting at $\hat{O}$ and then following RG flows to new SCFTs, as in fig. 1, could potentially miss SCFTs, if they do not have $\hat{O}$ in their domain of attraction. As a concrete example we ask if there might be $W = \text{Tr}Y^{k+1}$ RG fixed points for $k > 2$.

As seen sect. 2, $W = \lambda \text{Tr}Y^3$ is a relevant deformation of the $\hat{O}$ RG fixed point, driving the $\hat{O}$ RG fixed points to the $\hat{E}$ RG fixed points for all $N_f$. We have also seen that $W = \text{Tr}Y^{k+1}$ for $k > 2$ is an irrelevant deformation of the $\hat{O}$ RG fixed point. That is, if we add $W = \lambda \text{Tr}Y^{k+1}$ for $k > 2$, then $\beta(\lambda) > 0$, at least for small $\lambda$, and $\lambda \to 0$ in the IR. We might wonder, though, if $\beta(\lambda)$ could perhaps look like that of fig. 28, and nevertheless have a zero at some critical value $\lambda_*$ corresponding to some new hypothetical RG fixed points, which we name $G_k$, which do not have $\hat{O}$ in their domain of attraction.

![Figure 28: A $G_k$ fixed point?](image)

Our personal prejudice is that such a hypothetical new RG fixed point, requiring large Yukawa coupling as in fig. 28, seems unlikely. But we’ll here take an unbiased view and see what constraints could be placed on such hypothetical additional RG fixed points.

Assuming that the superpotential $W = \text{Tr}Y^{k+1}$ does control some hypothetical $G_k$ RG fixed point, the anomaly free R-symmetry with $R(W) = 2$ is the one parameter family

$$R(Y) = \frac{2}{k+1}, \quad R(Q) = R(\tilde{Q}) \equiv y, \quad R(X) = \frac{1+x-y}{x} - \frac{2}{k+1}. \quad (9.1)$$

We can now maximize the central charge $a$, given by (2.6) with $z = 2/(k+1)$, with respect to $y$; the result is

$$y = 1 + \frac{x(6 - \sqrt{-1 + 74x^2 + k^2(-1 + 2x^2) + k(-2 + 4x^2)})}{3(1 + k)(-1 + 2x^2)}. \quad (9.2)$$

For all $k$, this $y$ is such that no gauge invariant chiral operator ever violates the unitarity bound $R \geq 2/3$ (since $y > 1/3$ for all $x$, if $k \geq 2$). So unitarity does not require the
existence of accidental symmetries, and the above results could be correct as they stand, without modification. We can now plug (9.2) back into (2.6) to get $a_{G_k}(x)$.

We now consider the deformation of the hypothetical $G_k$ fixed points by $\Delta W = \text{Tr} Y X^2$ which, if relevant, will lead to RG flows either as $G_k \to \hat{D}$, or as $G_k \to D_{k+2}$, depending on whether $\Delta W$ wins over $\text{Tr} Y^{k+1}$, or if both remain important in the IR, respectively. To determine when $\Delta W$ is relevant, consider its superconformal R-charge:

$$R(YX^2) = 2 \left( \frac{1-y}{x} + 1 \right) - \frac{2}{k+1}. \quad (9.3)$$

For $x$ close to 1, we see from (9.2) that $R(YX^2) > 2$ and the deformation is irrelevant. On the other hand, for $x \to \infty$, we get $R(YX^2) \to 2k/(k+1) < 2$ and $\Delta W$ is relevant, so there is some critical value of $x$ where this deformation becomes relevant. We can now check if the hypothetical $G_k \to \hat{D}$ and $G_k \to D_{k+2}$ RG flows are compatible with the conjectured a-theorem, which would require $a_{G_k}(x) > a_{\hat{D}}(x)$, or $a_{G_k}(x) > a_{D_{k+2}}(x)$ for all $x$ such that the RG fixed points exist.

We did some checks of this, hoping to use the conjectured a-theorem to rule out the hypothetical $G_k$ SCFTs, or at least place some stringent bounds on them. What we found, however, is that all potential a-theorem violations only occur for low values of $x$. So, assuming the a-theorem, we can at best rule out the hypothetical $G_k$ fixed points for low $x$, but we were unable to rule them out entirely.

We could, similarly, consider the possibility that our $A_k$, $D_{k+2}$ or $E_{6,7,8}$ RG fixed points exist for values of $x$ below the $x_{\text{min}}$ found for $A_k$ in [11] and $D_{k+2}$ and $E_{6,7,8}$ here. All of these values of $x_{\text{min}}$ were based on having $\hat{O}$ in the domain of attraction. But perhaps, for example, the $D_{k+2}$ RG fixed points could actually also exist for $x$ in some range $x_{D_{k+2}}^{\text{min}} < x < x_{\hat{D}}^{\text{min}}$, which is outside of $\hat{O}$’s domain of attraction. If the hypothetical new lower bound for existence of the $D_{k+2}$ SCFT, $x_{D_{k+2}}^{\text{min}}$, extended too far below the values for $x_{D_{k+2}}^{\text{min}}$ found in sect. 4, we would have $D_{k+2} \to D_{k'}+2$ RG flows violating the a-theorem conjecture, as seen e.g. in fig. 16 if $x_{D_6}^{\text{min}} < 2.56$. Given our belief in the validity of the a-theorem, we expect that any hypothetical $D_6$ RG fixed points outside of $\hat{O}$’s domain of attraction, for example, could only exist for the $x$ range $2.56 \leq x_{D_6}^{\text{min}} < x < x_{D_6}^{\text{min}} \approx 3.14$.  

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10. Comments

For the a-maximization method to be useful, we must know the full symmetry group of the IR fixed point, including all accidental symmetries. The class of such accidental symmetries having to do with operators hitting the unitarity bound and becoming free are easy to spot and account for, as in [11] and the present paper. But there can be other accidental symmetries, which (to quote [11]) did not have such an obvious “smoking gun” characteristic as apparent unitarity violation. For example, in SQCD, the unitarity bound would tell us that the meson becomes free for \( N_f \leq \frac{3}{2} N_c \). But we know from Seiberg duality [5] that that’s not the whole story: the entire theory is in a free magnetic phase for \( N_f \leq \frac{3}{2} N_c \). It would be nice if there were some other tractable smoking gun tests which would reveal, even to someone who did not know about the Seiberg dual, that free mesons alone do not suffice for \( N_f \leq \frac{3}{2} N_c \).

Likewise, in sect. 6, we saw that the \( D_{k+2} \) conformal window does not extend above \( x > 3k - \bar{x}_{D_{k+2}}^{min} \); knowing the dual [15], we found that the IR theory is then governed by the magnetic version of the \( \bar{D} \). Because we do not know the duals of any of our other RG fixed points (1.2), we can not exclude the possibility that they too might have some new symmetries at sufficiently large \( x \) (stronger coupling), which could be very obscure in the original Lagrangian descriptions – but perhaps obvious in some as-yet-unknown dual descriptions. Short of knowing such dual descriptions, it would be nice if there were a tractable test to alert us to the possible presence of such accidental symmetries, which go beyond those required by the unitarity bound.

It would also be nice to connect the occurrence of Arnold’s ADE singularities found here to other ways in which the ADE series arises in physics and mathematics. E.g. in 2d \( \mathcal{N} = 2 \) theories the ADE superpotentials were special because they led to the \( \hat{c} < 1 \) minimal models [24,25], which could be characterized by the requirement that all elements of the chiral ring be relevant. Another occurrence of the ADE polynomials is via geometry, e.g. the ALE singularities and generalizations, which could connect with our SCFTs via string-engineering our SCFTs via D-branes at the singularities. It is indeed possible to string-engineer some variants of our 4d \( A_k \) and \( D_{k+2} \) SCFTs (as well as non-conformal variants) via 4d spacetime filling D3 branes at points (plus D5’s wrapped on cycles for the non-conformal variants) in a suitable local Calabi-Yau geometry [26,27,28,22,29]. These constructions do not yield precisely our SCFTs, but rather these theories deformed by the superpotential \( \Delta W = \bar{Q}_i X Q^i \) or \( \Delta W = \bar{Q}_i Y Q^i \); these are present in the string constructions because they are based on a deformation from \( \mathcal{N} = 2 \).
It would also be interesting to connect our results about 4d SCFTs to properties of these theories upon breaking conformal invariance by generic relevant superpotential deformations, where some new techniques are available for analyzing the effective glueball superpotentials and properties of the chiral ring e.g. [27,30,31,32]. Perhaps there is some generalization of some of the fascinating results in 2d $\mathcal{N} = 2$ theories connecting properties of SCFTs, such as the Poincare polynomial of superconformal R-charges, and properties of the critical points and solitons upon massive deformations, see e.g. [33].

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Appendix A. Some of the RG fixed points and flows, in the perturbative regime

We can now study the $\hat{O}$, $\hat{D}$ and $\hat{E}$ RG fixed points of (1.2) in a perturbative regime, along the lines of [1,2]. To do this, we take $SU(N_c)$ with $N_c$ large and $N_f$ such that the theory is just barely asymptotically free. Defining $x \equiv N_c/N_f$, the asymptotic freedom bound for the theory with $N_a = 2$ adjoint chiral matter fields is $x > 1$. The perturbative regime is $x = 1 + \epsilon$, with $0 < \epsilon \ll 1$; the RG fixed point couplings will be of order $\epsilon$, and we will here work only to leading order in $\epsilon$. These results should be qualitatively accurate provided that we tune $\epsilon$ to be sufficiently small by our choice of $N_c$ and $N_f$.

A.1. The $\hat{O}$ RG fixed points: $W = 0$.

In the limit $x = 1 + \epsilon$, with $0 \leq \epsilon \ll 1$, the gauge coupling beta function $\beta(g)$ has a negative one loop part and a positive two loop part. This leads as in [2] to a perturbatively accessible RG fixed point, with $\beta(\gstar) = 0$ solved by $\gstar$ small. A general expression for the RG fixed point coupling of SUSY gauge theories in this limit is [10]

$$\frac{g^2 |G|}{8\pi^2} \approx (6h - \sum_i n_i \mu(r_i)) \left(\sum_j n_j \mu(r_j)^2 |r_j|^{-1}\right)^{-1},$$

(A.1)

with $|G|$ the dimension of the gauge group, $2h$ the quadratic Casimir index of the adjoint representation, $n_i$ the number of matter chiral superfields in representation $r_i$, and $\mu(r_j)$ the quadratic Casimir index of $r_j$. For our present case this yields

$$\frac{g^2 N_c}{8\pi^2} \approx \frac{x - 1}{1 + 4x} \approx \frac{\epsilon}{4},$$

(A.2)
We can thus make the ’t Hooft coupling arbitrarily small by our choice of $\epsilon$. Because the fixed point coupling is small for sufficiently small $\epsilon$, we expect that perturbation theory is a reliable indicator of the phase of the gauge theory and that the theory is indeed in the “non-Abelian Coulomb phase,” which is an interacting conformal field theory.

A.2. The $\hat{E}$ RG fixed points: $W = \frac{1}{6}\lambda \text{Tr} Y^3$.

In our perturbative regime $x = 1 + \epsilon$, with $0 \leq \epsilon \ll 1$, the superpotential deformation does not affect the gauge coupling beta function to leading order in $\epsilon$ (i.e. to one-loop, but there are higher loop effects). So the gauge coupling remains at the same fixed point value as in the $\hat{O}$ case (A.2). The beta function for $\lambda$ is $\beta(\lambda) = \frac{3}{2}\lambda\gamma(Y)$, which we can compute to leading order in $\epsilon$. The one-loop anomalous dimension is

$$\gamma(\lambda) = -\frac{g_e^2 N_c}{8\pi^2} + \frac{\lambda^2 d^2}{32\pi^2}. \quad (A.3)$$

The $\lambda \text{Tr} Y^3$ interaction vertex is $\lambda d^{abc}$ with $d^{abc} = \frac{1}{2} \text{Tr}[T^a \{T^b, T^c\}]$ (with $a$ in the adjoint, and $T^a$ in the fundamental representation of $SU(N_c)$). It is straightforward to show that

$$d^{abc}d^{bce} = \frac{(N_c^2 - 4)}{4N_c} \delta^{ae} \equiv d^2 \delta^{ae}. \quad (A.4)$$

Plugging (A.3) and (A.2) into the $\beta$ function yields

$$\beta(\lambda) = \frac{3}{2}\lambda\left(-\frac{g_e^2 N_c}{8\pi^2} + \frac{\lambda^2 d^2}{32\pi^2}\right) = \frac{3}{2}\lambda\left(\frac{\lambda^2 d^2}{32\pi^2} - \frac{\epsilon}{4}\right). \quad (A.5)$$

The negative sign of (A.5) for small non-zero $\lambda$ shows that $\text{Tr} Y^3$ is a relevant deformation of the $\hat{O}$ RG fixed point theory, which drives the theory to a new RG fixed point. At this new RG fixed point, which we name $\hat{E}$, we can find the RG fixed point coupling for the superpotential interaction:

$$\lambda_*^2 = \frac{8\pi^2 \epsilon}{d^2} \quad (A.6)$$

to leading order in the $\epsilon \ll 1$ expansion.
A.3. The $\hat{D}$ RG fixed point theories: $W = \frac{1}{2} \lambda \text{Tr} X Y^2$

Again, the added superpotential interaction $W = \frac{1}{2} \lambda \text{Tr} X Y^2$ does not affect the gauge coupling beta function to leading order in the $\epsilon$ expansion. So to leading order in $\epsilon$ the RG fixed point coupling stays at the same value as in the $\hat{O}$ and $\hat{E}$ cases, (A.2). The beta function for $\lambda$ is

$$\beta(\lambda) = \lambda \left( -3 \frac{g^2 N_c}{16 \pi^2} + \frac{5 d^2}{64 \pi^2} \lambda^2 \right) = \lambda \left( -\frac{3 \epsilon}{8} + \frac{5 d^2}{64 \pi^2} \lambda^2 \right).$$

(A.7)

Again, this is negative at $\lambda = 0$, showing that $W = \lambda \text{Tr} X Y^2$ is a relevant perturbation of the $\hat{O}$ RG fixed point, which drives the theory to a new RG fixed point. At that new RG fixed point, which we name $\hat{D}$, the superpotential interaction has the fixed point value

$$\lambda^2_* = \frac{24 \pi^2 \epsilon}{5 d^2}$$

(A.8)

to leading order in the $\epsilon$ expansion.

A.4. Combining the $\hat{D}$ and $\hat{E}$ interactions: the $D_4$ SCFT

Consider the theory with superpotential

$$W = \frac{1}{6} \lambda_1 \text{Tr} X^3 + \frac{1}{2} \lambda_2 \text{Tr} X Y^2.$$  

(A.9)

We have seen that either $\lambda_1$ or $\lambda_2$ zero, with the other non-zero, drives the $\hat{O}$ RG fixed points to the $\hat{D}$ and $\hat{E}$ RG fixed points, respectively. We now consider the situation where both $\lambda_1$ and $\lambda_2$ are taken to be non-zero. Depending on the initial values of $\lambda_1$ and $\lambda_2$, we could study these flows by e.g. starting in the vicinity of the $\hat{O}$, $\hat{D}$, or $\hat{E}$ RG fixed points. From all of these initial RG fixed points, the added perturbation of (A.9) is seen to be a relevant deformation. We expect that $\hat{O}$, $\hat{D}$, and $\hat{E}$ are all in the same basin of attraction for a theory with both terms in the superpotential (A.9) important; we name this new RG fixed point SCFT $D_4$.

We can quantify and confirm this picture in the limit where $x = 1 + \epsilon$ with $0 < \epsilon \ll 1$, where perturbation theory is valid. Again, to leading order, the superpotential (A.3) does not affect the gauge beta so the gauge coupling fixed point value remains at the value (A.2). The interesting flow is in the couplings $\lambda_1$ and $\lambda_2$. The beta functions for $\lambda_1$ and $\lambda_2$ are given by

$$\beta(\lambda_1) = \frac{3}{2} \lambda_1 \gamma(X), \quad \beta(\lambda_2) = \lambda_2 \left( \frac{1}{2} \gamma(X) + \gamma(Y) \right).$$

(A.10)
Computing the anomalous dimensions to one loop yields

\[
\beta(\lambda_1) = \lambda_1 \left( -3g_2^2 N_c + \frac{3d^2}{64\pi^2} (\lambda_1^2 + \lambda_2^2) \right)
\]

\[
\beta(\lambda_2) = \lambda_2 \left( -3g_2^2 N_c + \frac{d^2}{64\pi^2} (\lambda_1^2 + 5\lambda_2^2) \right).
\]

These lead to flows which are attracted to the RG fixed point, at

\[
\lambda_{*1}^2 = \lambda_{*2}^2 = \frac{4\pi^2 \epsilon}{d^2}.
\]

as sketched in fig. 29.

\[
\text{Figure 29: The flow between fixed points in the } \lambda_1-\lambda_2 \text{ plane.}
\]

**Appendix B. \(a\)-maximization and the effect of accidental symmetries**

In [10], we proved that the unique exact superconformal R-symmetry is the one that (locally) maximizes the central charge \(a\). The idea is to write the most general possible R-symmetry, taken to be anomaly free and respected by relevant superpotential terms, as

\[
R_t = R_0 + \sum_I s_I J_I,
\]

where \(R_0\) is an arbitrary candidate R-symmetry, \(J_I\) are the various non-R flavor symmetries, and \(s_I\) are real parameters. The superconformal R-symmetry corresponds to some particular values of the parameters, \(\hat{R} = R_0 + \sum_I \hat{s}_I J_I\), which need to be determined. The method of [10] for determining \(\hat{R}\) is to locally maximize the quantity

\[
a_{\text{trial}}(s_I) = 3\text{Tr}R_t^3 - \text{Tr}R_t,
\]

\[(B.1)\]
i.e. find the unique values of $\hat{s}_I$ where

$$\frac{\partial a(s)}{\partial s_I} = 0 \quad \text{and} \quad \frac{\partial^2 a}{\partial s_I \partial s_J} < 0. \quad (B.2)$$

The above a-maximization procedure requires knowing the full set of flavor symmetries $J_I$. A subset of the flavor symmetry group can be determined immediately from the classical Lagrangian and anomaly considerations. But, if the RG fixed point is at relatively strong coupling, it is possible that it has additional accidental flavor symmetries, which are not easily visible in a weak coupling analysis of the Lagrangian. One situation where such accidental flavor symmetries are known to be present is when a chiral gauge invariant composite operator, $M$, appears otherwise to violate the unitarity bound $R(M) \geq 2/3$ for all chiral operators (this follows from the unitarity bound $\Delta(M) \geq 1$). The believed resolution is that $M$ is actually a free field, with $R(M) = 2/3$. There is then an accidental $U(1)_M$ symmetry under which only the operator $M$ is charged. $U(1)_M$ mixes with the superconformal $U(1)_R$ symmetry to make $R(M) = 2/3$, without directly affecting the superconformal R-charge of the other operators.

The a-maximization procedure, however, is non-trivially affected by such accidental symmetries $[\Pi]$, because the trial $U(1)_R$ symmetry should now include the possibility of mixing with $U(1)_M$: $R' = R_t + s_M J_M$, where $R_t$ is the old trial $U(1)_R$ symmetry; $R'$ includes mixing with $U(1)_M$, whereas $R_t$ did not. $J_M$ is the current of the accidental $U(1)_M$ symmetry, which gives charge $+1$ to the composite gauge invariant operators $M$ (suppose that there are $\text{dim}(M)$ of them), with all other gauge invariant operators neutral under $J_M$. The correct a-maximization procedure is to maximize the combination of 't Hooft anomalies $[B.2]$ for the most general trial R-symmetry $R'$, including mixing with $U(1)_M$: $a'(s_I, s_M) = 3\text{Tr} R'^3_t - \text{Tr} R'_t$. We can use 't Hooft anomaly matching to account for the difference between $R'_t$ and $R_t$, which only comes from the contribution to the 't Hooft anomalies of the $\text{dim}(M)$ free fields $M$:

$$a'(s_I, s_M) = a_t(s_I) + \text{dim}(M) \left[ 3(R_t(M) + s_M - 1)^3 - 3(R_t(M) - 1)^3 \right] - \text{dim}(M) \left[ (R_t(M) + s_M - 1) - (R_t(M) - 1) \right]. \quad (B.3)$$

$a'(s_I, s_M = 0) = 3\text{Tr} R'^3_t - \text{Tr} R'_t \equiv a_t(s_I)$ is the trial a-function which we would have maximized had there not been the accidental $U(1)_M$ symmetry.

Upon maximizing $a_t(s_I, s_M)$ in $(B.3)$ with respect to $s_M$, the solution $\hat{s}_M$ is immediately found to be given by

$$R'_{t'}(M) \equiv R_t(M) + \hat{s}_M = 2/3, \quad (B.4)$$
which is the expected result that $U(1)_M$ mixes with the superconformal $U(1)_R$ symmetry precisely so as to make $\hat{R}(M) = 2/3$. We can now maximize $a'_\nu(s_I, \hat{s}_M)$ with respect to the other $s_I$, to determine their values $\hat{s}_I$. Using (B.3) and (B.4), together with

$$3(R(M) - 1)^3 - (R(M) - 1) = \frac{2}{9} - \frac{1}{9}(2 - 3R(M)) \frac{2}{9}(5 - 3R(M)), \quad (B.5)$$

the quantity which we have to maximize to find the $\hat{s}_I$, is now

$$a'_\nu(s_I, \hat{s}_M) = a_I(s_I) + \frac{1}{9} \text{dim}(M)(2 - 3R_t(M))^2(5 - 3R_t(M)). \quad (B.6)$$

The presence of the second term in (B.4) leads to a different maximizing solution $\hat{s}_I$, which non-trivially affects the result for the superconformal $U(1)_R$ charge of all fields. It also affects the value of the central charge $a'_\nu(\hat{s}_I, \hat{s}_M)$; maximizing with respect to $s_M$, rather than setting $s_M = 0$, leads to a larger value for the maximal central charge. The second term in (B.4) vanishes when $R(M) = 2/3$, so the central charge is continuous when $M$ hits the unitarity bound and becomes free.

More generally, as in [11], the quantity to maximize will be as in (B.6), but with a sum over every operator $M$ which is a free field.

Appendix C. Baryons and the unitarity bound

In our large $N_c$ limit, baryons can only potentially violate the unitarity bound $R(B) \geq 2/3$ when $y(x) \equiv R(Q)$ is zero or negative for some $x$. This is the case for the $\hat{D}$, $D_{k+2}$, and $E_{6,7,8}$ RG fixed points. We discuss each of these now. If our results had led to baryons violating the unitarity bound, we would have concluded that those baryons are free fields, which would modify our results for the R-charges and central charge $a$.

C.1. $\hat{D}$: no baryons violate the unitarity bound

For large $x$, $y$ becomes negative and one might worry about baryons thus having negative R-charge. The worst-case scenario in terms of potentially violating the unitarity bound is a baryon of the form

$$\prod_{i=0}^{[x]-1} (X^i Q)^{N_f}$$
where, to minimize the R-charge, we put in as few $X$’s as possible and no $Y$’s, bearing in mind that we need $[x] = N_c/N_f$ different dressed quarks to antisymmetrize the $N_c$ gauge indices. The R-charge of the above baryon is

$$N_c y + N_f \sum_{i=1}^{[x]-1} i \frac{2 - 2y}{x} = N_c y + N_f \left(\frac{1 - y}{x}\right) [x]([x] - 1) \approx N_c,$$

with the terms involving $y$ canceling in the large $x$ limit. So no baryons actually ever violate the unitarity bound.

**C.2. $D_{k+2}$: again no baryons violate the unitarity bound.**

The R-charges of the baryons (5.5) are given by

$$R(B^{n_1, \cdots, n_{k,3}}) = \frac{1}{k+1} \left[-N_c(1 + x) + \sum_{l=1}^{k} \sum_{j=1}^{3} n_{l,j}(2l + kj)\right]. \quad \text{(C.1)}$$

We need to show that this is positive in the range $k+1 < x < 3k$; for $x < k+1$ the baryons certainly do not violate unitarity (all R-charges in (5.12) are positive), and for $x > 3k$ the baryons do not exist (moreover, there is no stable vacuum).

The baryon with smallest R-charge (the worst-case scenario for unitarity violation) is obtained by minimizing $\sum_{l=1}^{k} \sum_{j=1}^{3} n_{l,j}(2l + kj)$ in (C.1), subject to the constraint that $\sum_{l=1}^{k} n_{l,j} = N_c$ and $n_{l,j} \leq N_f$. This is achieved by taking $n_{l,j} = N_f$ for the $[x]$ different choices of $l$ and $j$ for which $2l + kj$ is as small as possible, with the other $n_{l,j}$ zero. The ordered list of possibilities for $2l + kj$ is as follows:

$$2l + kj = \begin{cases} 
    k+2, k+4, k+6, \ldots, 2k-1, 2k+1 & \text{taking } l = 1 \ldots \frac{1}{2}(k+1) \text{ with } j = 1 \\
    2k+2, 2k+3, 2k+4, \ldots, 4k, 4k+1 & \text{from the others} \\
    4k+3, 4k+5, 4k+7, \ldots 5k-2, 5k & \text{from } l = \frac{1}{2}(k+3) \ldots k \text{ with } j = 3.
\end{cases} \quad \text{(C.2)}$$

We thus have for our worst-case scenario

$$\sum_{l=1}^{k} \sum_{j=1}^{3} n_{l,j}(2l + kj) = \sum_{i=1}^{\frac{1}{2}(k+1)} N_f(2i + k) + \sum_{i=1}^{\text{min}([x]-\frac{1}{2}(k+1),2k)} N_f(2k+1+i) + \sum_{i=1}^{[x]-\frac{1}{2}(5k+1)} N_f(4k+1+2i), \quad \text{(C.3)}$$

where the last line of (C.3) is included only if $[x] > \frac{1}{2}(5k+1)$. Since $[x] > k+1$ in our range of interest, we must use all of the $\frac{1}{2}(k+1)$ terms in the first row of (C.2) and at least
some of the second row, which yield the first two sums in (C.3). If \( x > \frac{1}{2}(5k + 1) \) then the second row of (C.2) is also used up, and we need to use the third, giving the last line of (C.3). Doing the sums (C.3) and plugging into (C.1) yields the minimal R-charge of a baryon. The result is a complicated function of \( x \), which starts off positive at \( x = k + 1 \), initially increases with \( x \), and then monotonically decreases until it reaches 0 at \( x = 3k \). Thus no baryons ever violate the unitarity bound for the \( \tilde{D}_{k+2} \) theories, at least for \( k \) odd. We expect \( k \) odd to be qualitatively similar, since we can flow from \( k \) odd to a lower value of \( k' \), which could be even by adding a superpotential deformation.

C.3. \( E_6 \): baryons would violate the unitarity bound for large \( x \).

We now discuss the \( E_6 \) case; the \( E_7 \) and \( E_8 \) cases can be similarly analyzed. As we have seen, at the classical level the chiral ring does not appear to truncate and it is unclear whether or not there is an upper stability bound on \( x, x < x_{E_6}^{\text{max}} \). If we ignore the possible quantum truncation of the chiral ring, along with the possible stability bound upper limit on \( x \), we can see that there will be baryons which hit the unitarity bound, and thus become free fields, for sufficiently large \( x \). We would then have to correct the \( a \) to be maximized, along the lines of the general discussion in \([11]\), by terms analogous to the additional term in (B.6), but for the baryons.

However, our numerical evidence in sect. 7.1, along with assuming the a-theorem conjecture is correct, suggests that there is actually a quantum stability bound requiring \( x < x_{E_6}^{\text{max}} \approx 13.80 \). And it can be verified that no baryons have yet hit the unitarity bound if \( x \leq 13.80 \). So, if there is indeed such a stability bound, the discussion of this subsection is a moot point.

To see that baryons would hit the unitarity bound for \( x \) sufficiently large (ignoring the possible stability bound mentioned above), consider the \( 2^nN_f \) dressed fundamentals

\[
X^{r_1}YX^{r_2}Y \ldots X^{r_{n-1}}YX^{r_n}YQ_i,
\]

(C.4)

where each \( r_i = 1 \) or \( 2 \). Forming a baryon from these, we require \([x]\) types of such dressed quarks, with \( n \) ranging from zero to \( q \), given by \( \sum_{n=1}^{q} 2^n = [x] \), i.e. \( q \approx \log_2 x \) for large \( x \). The R-charge of such a baryon is

\[
\frac{R(B)}{N_f} = xR(Q) + \sum_{n=1}^{q} n2^nR(Y) + \sum_{n=1}^{q} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) (i + n)R(X)
\]

\[
= x \left( 1 - \frac{x}{6} \right) + \frac{2}{3} \sum_{n=1}^{q} n2^n + \frac{1}{2} \sum_{n=1}^{q} \sum_{i=0}^{n} \left( \begin{array}{c} n \\ i \end{array} \right) (i + n),
\]

(C.5)

\[
= x \left( 1 - \frac{x}{6} \right) + \frac{2}{3} (2 + (q - 1)2^{q+1}) + \frac{1}{2} (3 + 3(q - 1)2^q).
\]

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Taking $x$ large for simplicity, this gives $q \approx \log_2 x$, and then (C.3) yields

$$\frac{R(B)}{N_f} = -\frac{x^2}{6} + \frac{17x \log x}{12 \log 2} + x,$$  

which becomes negative for $x > 55$ (a slight overestimate, due to our approximations).

Appendix D. Magnetic Asymptotics

As at the end of Section 4.1, we can compute the asymptotic value of $\tilde{y}$ for the magnetic dual of the $D_{k+2}$ theory. For large $k$, $\tilde{x}_{D_{k+2}}^{\min}$ also becomes large, and $\tilde{y}(\tilde{x})$ flattens out before the $\tilde{X}^{k+1}$ term becomes relevant; it is this asymptotic value of $\tilde{y}$ we wish to compute. The situation is complicated here by the preponderance of mesons, $3k$ in all. It is convenient to divide these into 3 groups of $k$, each with different powers of $Y$.

The central charge $a$ for this theory can then be written as

$$\frac{a_{D_{k+2}}(\tilde{x}, \tilde{y})}{N_f^2} = \frac{a_{D}(\tilde{x}, \tilde{y})}{N_f^2} + \frac{1}{9} \sum_{l=1}^{p_1} [(2 - 3R(N_{l1}))^2(5 - 3R(N_{l1}))] + \frac{2}{9}(k - 2p_1)$$

$$+ \frac{1}{9} \sum_{l=1}^{p_2} [(2 - 3R(N_{l2}))^2(5 - 3R(N_{l2}))] + \frac{2}{9}(k - 2p_2)$$

$$+ \frac{1}{9} \sum_{l=1}^{p_3} [(2 - 3R(N_{l3}))^2(5 - 3R(N_{l3}))] + \frac{2}{9}(k - 2p_3),$$  

where

$$R(N_{lj}) = 2\tilde{y} + (l - 1)\frac{2 - 2\tilde{y}}{\tilde{x}} + (j - 1)\left(\frac{\tilde{y} - 1}{\tilde{x}} + 1\right)$$

and $a_{D}(\tilde{x}, \tilde{y})/N_f^2$ is the same as in (4.7). The sums run over the Legendre-transform partners $N_{lj}$ of the interacting mesons $M_{k+1-l,3-j}$, and we have split the sum over the remaining free mesons into three parts; the total number of free mesons is $3k - p_1 - p_2 - p_3$.

We would now like to approximate these sums by integrals. However, it is not immediately clear which sums we should include. We know that the meson $M_{k,3}$ is interacting for all $\tilde{x} > 1$, so we should definitely include the $p_1$ sum. But it is not obvious whether or not to include the $p_2$ and $p_3$ sums. We use a self-consistency check to give us the answer: The upper limit of the sums is the value at which the meson with the smallest R-charge becomes interacting,

$$2\tilde{y} + (p_j - 1)\left(\frac{2 - 2\tilde{y}}{\tilde{x}}\right) + (j - 1)\left(\frac{\tilde{y} - 1}{\tilde{x}} + 1\right) = \frac{4}{3}.$$  

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(Note: This looks like it describes when the Legendre-transform partner with the largest possible R-charge $N_{lj}$ becomes free, except for the $4/3$ on the RHS.) Since we expect $\tilde{y}$ to approach a constant, we can easily solve this for $p_j$ in the large $\tilde{x}$ limit:

$$
p_j = \left(\frac{4}{3} - j + 1 - 2\tilde{y}\right) \left(\frac{\tilde{x}}{2 - 2\tilde{y}}\right).
$$

To approximate these sums by integrals, we can integrate over the variable

$$
t_j = 2\tilde{y} + (l - 1) \left(\frac{2 - 2\tilde{y}}{\tilde{x}}\right) + (j - 1) \left(\frac{\tilde{y} - 1}{\tilde{x}} + 1\right);
$$

the limits of integration on the $n$th sum will be from $2\tilde{y} + n - 1$ to $4/3$. Performing these integrals and substituting the answer back into (D.4) gives an expression for $\tilde{a}$ which may be maximized as a function of $\tilde{y}$.

The self-consistency argument is easily done: First, assume that only the $p_1$ sum contributes. Including its contribution to $\tilde{a}$ and maximizing yields a number for $\tilde{y}$ which may then be substituted into (D.4). Doing so yields positive values for both $p_1$ and $p_2$, which is inconsistent with our assumption that only the first sum contributes. One may then perform this procedure for any combination of the sums; the only self-consistent answer turns out to be that one must include both the $p_1$ and $p_2$ sum, but not the $p_3$ sum. Maximizing $\tilde{a}$ with these two sums (and, of course, their accompanying sums over the free mesons) gives the asymptotic value for $\tilde{y}$

$$\tilde{y}_{asymp} \approx -0.1038.$$  \hspace{1cm} (D.6)

One may then easily compute the asymptotic value of the central charge

$$\frac{\tilde{a}_{D_{k+2}}(\tilde{x})}{N_f^2} \approx 13.1186\tilde{x} + \frac{6k}{9}$$

and also where the $\tilde{X}^{k+1}$ term becomes relevant,

$$\tilde{x}_{D_{k+2}} \approx 1.1038k.$$  \hspace{1cm} (D.8)
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