An optimised stiffness reduction method for simulating infinite elastic space using commercial Finite Elements codes

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Abstract. A common goal when using Finite Element (FE) modelling in time domain wave scattering problems is to minimise model size by only considering a region immediately surrounding a scatterer or feature of interest. The model boundaries must simulate infinite space by minimising the reflection of incident waves. This is a significant and long-standing challenge that has only achieved partial success. Industrial companies wishing to perform such modelling are keen to use established commercial FE packages that offer a thorough history of validation and testing. Unfortunately, this limits the flexibility available to modellers preventing the use of popular research tools such as Perfectly Matched Layers (PML). Unlike PML, Absorbing Layers by Increasing Damping (ALID) have proven successful offering practical implementation into any solver that has representation of material damping. Despite good performance further improvements are desirable. Here, a Stiffness Reduction Method (SRM) has been developed and optimised to operate within a significantly reduced spatial domain. The technique is applied by altering damping and stiffness matrices, inducing decay of incident waves. Variables are expressed as a function of known model constants, easing implementation for generic problems. Analytical and numerical solutions have shown that SRM out performs ALID, with results approaching those of PML.

1. Introduction
Commercial Finite Element (FE) codes have been used increasingly to simulate elastic wave propagation and scattering problems \cite{1, 2}. Numerical techniques can offer solutions that extend to any desired level of complexity. However, the greatest limitation when using FE methods is the computational resource that can be allocated to solving the problem. To maintain a highly accurate solution, novel techniques have been developed that allow for the use of reduced spatial domains that do not compromise on having an accurate representation of the system.

Such methods include the use of domain linking algorithms \cite{3-5} and the simulation of infinite elastic space. These methods allow modelers to only consider a small area immediately surrounding a scatterer or feature of interest. Computational resource can therefore be focused on a highly accurate representation of the Area of Study (AoS). This is practically advantageous when considering three-dimensional problems where even a significantly reduced spatial domain can consist of tens of millions of degrees of freedom to be solved.
When using such techniques, a common problem is the removal of unwanted reflections from model boundaries manifesting themselves as additional remnants in ultrasonic scattering signals. The removal of unwanted reflections has proven to be a challenging problem. Innovative techniques have been developed which can be collated into the three categories: infinite element methods [6], non-reflecting boundary conditions [7], and absorbing layer methods [8-11].

Despite the advances made with these methods, they do not meet all of the bespoke requirements that are necessary for industrial Non-Destructive Evaluation (NDE) applications using commercial FE codes. The relative high performance and ease of implementation of absorbing layer methods has meant that they have become the most viable tool for this application. Two successful absorbing layer techniques are Absorbing Layers by Increasing Damping (ALID) [8], and Perfectly Matched Layers (PML) [9].

Rajagopal et al [11], performed a comparative study of absorbing layer methods and showed that PML generally achieves a smaller model size than ALID. However, PML is more suited to frequency domain simulations and requires specific programming and functionality; currently this is only available in a minority of commercial FE packages. Therefore, ALID is a more accessible tool for practical use, being directly compatible with explicit time domain solvers in commercial FE codes.

Despite the good performance of ALID, further improvements are desirable. The Stiffness Reduction Method (SRM) is an extension on current absorbing layer techniques that improves upon the efficiency of ALID, whilst still remaining accessible to implementation using commercial FE programs. The method which will be presented in the following sections, exploits benefits that can be gained by making gradual changes to the material stiffness, as well as increasing the damping, across the absorbing region.

2. Theory of SRM

We start by reviewing the underlying workings of absorbing layer methods, from which we will be able to identify the role of the stiffness term and thus develop the new stiffness reduction approach. A full derivation of this approach is given by Pettit et al [12].

The absorbing layer concept changes the physical properties of the elements within successive layers such that they absorb incident wave energy. The physical properties must be changed gradually, but at an optimum rate because of two competing mechanisms.

The first is that reflections are caused by impedance mismatches between successive absorbing layers. Thus, alterations in the physical properties of adjacent elements cause reflections that return energy back into the AoS at each interface.

The second is that the incident wave may not be fully absorbed within the total thickness of the absorbing region and reflect from the model boundary back into the AoS. An efficient absorbing boundary balances these two mechanisms to produce a net reflection below a pre-defined tolerable threshold from a minimal thickness of absorbing material.

The equation of dynamic equilibrium that forms the basis of the FE discretisation of the wave propagation problem is:

\[ [F] = [M]\ddot{u} + [C]\dot{u} + [K]u \quad (1) \]

where \([M]\), \([C]\) and \([K]\) are the respective mass, damping and stiffness matrices and \([F]\) represents external force.

The damping matrix can be expressed as two separate components proportional to the mass and stiffness matrices, consistent with most commercial FE packages:

\[ [C] = C_M [M] + C_K [K] \quad (2) \]

Combining equation (1) and equation (2) produces a new equation for dynamic equilibrium where damping is expressed solely as a function of coefficients \(C_M\), mass proportional damping and \(C_K\), stiffness proportional damping:
From the inspection of the coefficients of equation (3) it can be shown that instead of the damping matrix \([C]\), there are complex values of the density, \(\rho\), and Young's modulus, \(E\), that can be expressed as:

\[
\rho \to \rho \left(1 + \frac{iC_M}{\omega}\right)
\tag{4}
\]

and

\[
E \to E \left(1 - i\omega C_K\right)
\tag{5}
\]

Inside the AoS, damping is usually zero \((C_M = 0\) and \(C_K = 0\)). Inside the absorbing region, damping is non-zero, meaning that incident wave energy is absorbed. This is attributed to the imaginary components of terms introduced into the solution via \(\rho\) and \(E\). This can be illustrated by noticing that the wavenumber, \(k\), is related to the density and stiffness such that:

\[
k \propto \frac{\rho}{\sqrt{E}}
\tag{6}
\]

which becomes

\[
k(k_{\text{real}}, k_{\text{imag}}) \propto \frac{\rho \left(1 + \frac{iC_M}{\omega}\right)}{\sqrt{E(1-i\omega C_K)}}
\tag{7}
\]

Inside the absorbing layers the wavenumber has both real and imaginary parts such that:

\[
k(k_{\text{real}}, k_{\text{imag}}) = k_{\text{real}} + ik_{\text{imag}}
\tag{8}
\]

Substituting this into the standard harmonic solution reveals an exponential decay term, \(\exp[-k_{\text{imag}}x]\), that attenuates waves within the absorbing region.

\[
u(x, t) = \exp[i(k_{\text{real}}x - \omega t)]\exp[-k_{\text{imag}}x]
\tag{9}
\]

It is the imaginary component of the wavenumber which induces the decay of the waves inside the absorbing region and therefore that must be exploited. Commercial FE packages do not offer a means to alter the wavenumber directly. However, this can be achieved indirectly, by altering values of the stiffness, damping and mass matrices, to increase the value of \(k_{\text{imag}}\) inside the absorbing region. Consequently, this will also increase the value of \(k_{\text{real}}\) which must be managed carefully in order to minimise the reflection between adjacent elements.

2.1. Absorbing layer properties

Layer properties must be carefully selected such that successive values of \(k_{\text{real}}\) result in impedance mismatches that generate reflections which can be considered negligible. Progress made in ALID by Rajagopal et al [11], gives a good indication of how best to optimise initial parameters and forms the basis of the SRM technique.

Firstly, it is noted that introducing damping into the model can decrease the value of the stable time increment within explicit schemes, thereby reducing computational efficiency. However, this effect is only noticeable when using \(C_K\). Setting \(C_K\) to zero eliminates this issue and allows damping to be controlled solely by \(C_M\) in which case it is found that the stability is not compromised. The use of absorbing layer methods with a non-zero \(C_K\) is discussed by Semblat et al [10], relating to the Rayleigh/Caughey damping formulation.

Secondly, damping is defined as an increasing power law set by the power \(p\), so as to gradually alter the value across successive layers.
\[ C_M(x) = C_{M_{\text{max}}} X(x)^p \] (10)

\( C_{M_{\text{max}}} \) is a positive real number, where \( X(x) \) ranges from 0 at the AoS boundary to 1 at the end of the absorbing region. The thickness of each absorbing layer is equal to one element width, \( dx \).

From inspection of equation (10), it can be seen that the imaginary component of the wavenumber, \( k_{\text{imag}} \), can be incrementally increased by using \( C_M(x) \) but also by decreasing the value of the Young's Modulus, \( E \).

Therefore within the absorbing region \( E \) becomes a function of \( x \), decreasing gradually across the absorbing region, avoiding any dramatic changes in material properties. Equation (11) indicates how best to increment variables between successive layers by using an exponential decay function containing an attenuation factor such that:

\[ E(x) = E_0 \exp[-\alpha (x) k_{\text{inc}} x] \] (11)

where \( E_0 \) is the Young’s Modulus inside the AoS and \( k_{\text{inc}} \) is the incident wavenumber. However, the attenuation factor \( \alpha(x) \) cannot be a constant for each layer since this will induce impedance mismatches between adjacent layers. Therefore, it is necessary to gradually increase this value across the absorbing region as shown in equation (12).

\[ \alpha (x) = \alpha_{\text{max}} X(x)^p \] (12)

The SRM still maintains the mass proportional damping \( C_M(x) \), which is varied in the same manner as ALID, but now includes the additional reductions made to the stiffness matrix. The combination of these two effects allows for the attenuation of incident waves within a significantly reduced spatial domain.

2.2. Defining absorbing layer input parameters

Having derived functions to vary the stiffness and damping matrices, it is now necessary to calculate input parameters that define the absorbing region. The procedure outlined here is transferable to other absorbing boundary methods. The influence these variables have on attenuating an incident wave is dependent upon the angular frequency, \( \omega \), and FE mesh density, \( N \), which is given by:

\[ N = \frac{2\pi}{k_{\text{inc}} dx} \] (14)

From inspection of equation (11), it can be seen that \( C_M(x) \) is intrinsically linked to the angular frequency of the incident wavelength, \( \omega \). From trial and error methods carried out in this study, it has been observed that the optimal value selected for \( C_{M_{\text{max}}} \) is of the same order as \( \omega \).

This makes the application of the absorbing regions to different inspection scenarios more straightforward, since now variables can be expressed as a function of the frequency of the incident wave rather than resulting to trial and error methods.

The power to which successive layers are raised, \( p \), is best expressed as a cubic variation so that the interface between layers is continuous to the 1st and 2nd order. This is consistent with findings from Rajagopal et al [11], who stated that this is sufficient to minimise impedance mismatches between successive layers. The larger impedance mismatches that do occur happen deep within the absorbing region.

At this point the input parameters for SRM and ALID differ. For the SRM, it is necessary to define the maximum attenuation of the stiffness matrix, \( \alpha_{\text{max}} \), in the final absorbing layer where \( x=L \). This is done by rearranging equation (12) to make \( \alpha(x) \) the subject of the equation and defining a new variable \( \alpha_{\text{SRM}} \); the ratio of the Young's Modulus in the final absorbing layer and the AoS.

\[ \alpha_{\text{max}} = \alpha(L) = \frac{\ln(\alpha_{\text{SRM}})}{k_{\text{inc}} L} \] (15)
where

\[ \alpha_{SRM} = \frac{E(\lambda)}{E_0} = 0.01 \]  \hspace{1cm} (16)

Just as a generic value of \( C_{M\text{max}} \) has been chosen through inspection and best practice, so too has \( \alpha_{SRM} \). It has been observed that setting the Young’s Modulus in the final absorbing layer to be 1% of the value in the AoS will render a good performing absorbing layer. When using ALID, the stiffness is not varied, such that \( \alpha_{SRM} = 1 \).

Because the SRM can dampen out signals more readily than ALID, the SRM can operate in a reduced spatial domain. Work carried out by Rajagopal et al [11], indicates that an optimal thickness of ALID would be equal to \( 3\lambda_{\text{inc}} \). Due to the improved performance of SRM, that same level of performance can be achieved within a thickness of \( 1.5\lambda_{\text{inc}} \).

3. Comparison of SRM and ALID

To quantify the performance of a set of absorbing layers, it is necessary to calculate the magnitude of unwanted reflections that return from the model boundary. This is achieved through the use of an analytical approach based upon the Global Matrix method [13]. The analytical model is then validated by making comparisons from a FE model that contains the same absorbing boundary. The package ABAQUS/Explicit [14], has been chosen for the study, as it is widely available, developed and supported for wave propagation simulations.

The analytical and numerical models are non-dimensional and the material properties are defined such that the longitudinal and shear wavelengths are 1 and 2 respectively; each incident wave mode is considered separately.

To illustrate the advantages offered by the SRM, the performance is compared to conventional ALID. For both cases, the length of the absorbing region is equal to 1.5 times the incident wavelength, \( \lambda_{\text{inc}} \). The attenuation parameters are defined such that for the SRM, the \( \alpha_{SRM} = 0.01, C_{M\text{max}} = \omega \) and \( p = 3 \). For ALID, the attenuation parameters are defined according to work presented by Rajagopal et al [11], where the stiffness is not reduced and \( C_{M\text{max}} = 5\omega \).

3.1. Analytical model

The Global Matrix method establishes a matrix relationship which describes plane wave propagation across all the layers of a multi-layered system. Each layer contains a local description of the wave, where adjacent layers are coupled by applying Snell’s law for continuity of stress and displacement.

Rajagopal et al [11], successfully applied this technique to analyse the performance of the ALID by gradually implementing damping using the complex expression for density. The extension made here is to include the reduction in Young’s modulus, as well as the complex density used by ALID.

The complete global matrix for waves propagating into an absorbing region is illustrated in figure 1 and defined by equation (17).

\[
\begin{bmatrix}
[M^b_1] & [M^f_2] & 0 & \cdots & 0 \\
0 & [M^b_2] & [M^f_3] & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & [M^b_{\text{end}-1}] & [M^f_{\text{end}}] \\
0 & 0 & \cdots & 0 & [M^b_{\text{end}}]
\end{bmatrix}
\begin{bmatrix}
[neg_{1}^1] \\
pos_{2}^1 \\
\vdots \\
[neg_{\text{end}-1}^1] \\
pos_{\text{end}-1}^1
\end{bmatrix}
= 
\begin{bmatrix}
[M^b_{\text{pos}_{1}}] \\
[pos_{1}^1] \\
\vdots \\
[M^b_{\text{neg}_{\text{end}}}] \\
[neg_{\text{end}}^1]
\end{bmatrix}
\]  \hspace{1cm} (17)

where \([M^a_n]\) and \([M^b_n]\) are the coefficients defined by the material properties of each layer, \( n \), governed by Hooke’s law and the Helmholtz decomposition linking displacement, \( u \), with the compression and shear wave potentials, at the front boundary, \( f \), and back, \( b \) boundary in the positive, \( \text{pos} \), and negative, \( \text{neg} \), propagation directions, with the subscript ‘end’ denoting the final absorbing layer.
For a given model of known material properties the left hand matrix is known. Similarly, for a specified incident wave the initial amplitude propagating into the absorbing region [pos.] is also known. Matrix inversion is used to calculate the value of the wave amplitudes within each layer, thereby calculating the amplitude of the wave returning into the AoS, [neg.]. Since the amplitude of the incident wave mode is known, this gives the reflection coefficient of both reflected wave modes and for angles of incidence within the range 0° to 90°.

3.2. Numerical model

A harmonic bulk wave is incident on an absorbing region across a range of incident angles $\theta_{\text{inc}}$, using FE code ABAQUS/Explicit [14], see figure 2.

![Figure 2](image)

**Figure 2.** FE model of absorbing boundary for a) an incident compression wave, $\phi^{\text{inc}}$, and b) and incident shear wave, $\psi^{\text{inc}}$, entering an absorbing region of thickness L, from the AoS.

The amplitudes of the incident and reflected wave modes are monitored along their predicted lines of propagation and the reflection coefficient deduced.

Results for the analytical and numerical models are shown in figure 3 and figure 4 for an incident longitudinal wave and in figure 5 and figure 6 for an incident shear wave.
From these figures it is apparent that the SRM out performs ALID across all angles of incidence due to the attenuation of the terms in the stiffness matrix.

There is excellent agreement between the numerical and analytical solutions for both the ALID and SRM, confirming the reliability of the analyses. Performance at low angles of incidence is significantly better than at higher glancing angles; consistent with other absorbing layer techniques. This confirms that the angle of incidence plays a critical role in the selection of SRM variables. Performance at high angles will always render a less acceptable solution (however it is often the case that these angles are not of concern in simulations). The presence of critical angles generates evanescent waves propagating parallel to the back boundary of the SRM. The angle at which this
occurs is not necessarily obvious since it is due to a net effect from all the layers combined. This is illustrated in figure 5 whereby after the critical angle for an incident shear wave, the reflected longitudinal component is not radiated back into the AoS.

4. Optimising SRM performance

When building any absorbing region, there are a number of variables at the disposal of the user. Generic rules can be applied to absorbing region techniques that will ensure good performance. Despite the success of the absorbing boundary methods, it is noticed that for three-dimensional models, the additional contribution of the absorbing layers to the spatial domain is relatively large. The SRM helps to overcome this issue by significantly reducing the size of the spatial domain, however, through the use of optimisation functions, highly specialised absorbing regions can be designed that will result in further improvements.

This section will discuss methods to optimise the SRM absorbing layer. To do this, the requirements of the layer must first be defined. Here, the requirement is that the SRM absorbing layer will be optimised to absorb incident waves at a particular angle of incidence. This must be achieved within the smallest thickness of absorbing material possible.

Optimisation functions can be used to find the optimum input parameters that meet a given condition, for a given function. Here, the optimisation function 'fminsearch' provided by MATLAB is used, [15]. The function looks to calculate the input parameters that will achieve the smallest possible reflection coefficient at a given angle of incidence. This is achieved by varying all input values until the minimum reflection coefficient at the specified angle has been found.

If an absorbing layer is being built for a specific scenario, it may be necessary to put restrictions in place. This is done by fixing one or more of the variables, for example, $\omega$, N and L, which will be constants when using an explicit FE model.

**Figure 7.** Global Matrix Method (GMM) reflection coefficients as a function of incident angle for the reflected compression and shear wave modes (L and S) from a SRM absorbing boundary that is optimised for an incident longitudinal bulk wave at 0°.

**Figure 8.** Global Matrix Method (GMM) reflection coefficients as a function of incident angle for the reflected compression and shear wave modes (L and S) from a SRM absorbing boundary that is optimised for an incident longitudinal bulk wave at 45°.

Figure 7 and figure 8 show the reflection coefficients from an SRM absorbing region for a model which has longitudinal and shear wavelengths of 1 and 2 respectively. The SRM thickness is fixed to a length of $\lambda_{\text{min}}/4$ and the optimisation function has been used to calculate the input parameters such that a minimal reflection is achieved for two specific angles of incidence, 0° and 45° respectively.
From figure 7 and figure 8 it can be seen that the optimisation function can be used to build highly specialised absorbing boundaries that operate within significantly reduced spatial domains. This approach provides a means to significantly reduce the computational burden associated with three-dimensional models, by tailoring the high level of performance to within a narrow angular range.

To use the optimisation function correctly, it is important to avoid scenarios where local minima are found over the global minima. To achieve this, the optimisation function requires starting values for the inputs, before finding the optimal input values themselves. These should be equal to those given for the generic case described in section 2.2. This will help to avoid finding the local minima over the global values.

5. Conclusions
The SRM is a new absorbing boundary method that has been developed to operate within a significantly reduced spatial domain. Furthermore, the SRM can be easily implemented into commercial FE codes and does not require specific programming or functionality. All absorbing boundary variables can be controlled from the stiffness and damping matrices.

The reduction in stiffness is defined as a function of the known physical properties of the model, maximum damping values are a function of the incident frequency and the thickness of the absorbing region is of the order of $1.5\lambda_{inc}$.

A two dimensional analytical model based upon the Global Matrix method was shown to have excellent agreement with numerical simulations. Through use of an optimisation function, a highly specialised SRM can be developed for application towards specific scenarios. The optimisation function looks to reduce the magnitude of unwanted reflections that occur at a particular angle of incidence. This allows for a high performing absorbing boundary that can operate within a spatial domain that is further reduced; however, this is only applicable for the designed scenario.

By applying the SRM, computational resource can be focused on a highly accurate representation AoS. This is practically advantageous when considering three-dimensional problems where even a significantly reduced spatial domain can consist of tens of millions of degrees of freedom to be solved.

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