Conformal Carroll groups

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Abstract

Conformal extensions of Lévy–Leblond’s Carroll group, based on geometric properties analogous to those of Newton–Cartan space-time are proposed. The extensions are labeled by an integer \( k \). This framework includes and extends our recent study of the Bondi–Metzner–Sachs (BMS) and Newman–Unti (NU) groups. The relation to conformal Galilei groups is clarified. Conformal Carroll symmetry is illustrated by ‘Carrollian photons’. Motion both in the Newton–Cartan and Carroll spaces may be related to that of strings in the Bargmann space.

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1. Introduction

Inönü–Wigner contraction of the Poincaré group yields the Galilei group [1]. The possibility of having another, rather unusual contraction was pointed out some time ago by Lévy–Leblond, who named the result the ‘Carroll group’ [2, 3]. Despite occasional attempts [4–7], this has long remained a sort of mathematical curiosity, though, owing to the difficulty of finding interesting physical systems which might carry such a symmetry: a massive classical mechanical system with Carrollian symmetry, for example, cannot move [4, 7, 8].

The situation started to change when attention was drawn to the rôle of Carrollian symmetry for tachyons [9, 10].

Another recent line of research concerns the so-called conformal Galilean algebras (CGA), initiated by a failed attempt to derive the then new Schrödinger symmetry by...
extending the Inönü–Wigner contraction from Poincaré to the conformal group [11], and yielding instead another ‘non-relativistic conformal group’ with ‘relativistic’ dynamical exponent \( z = 1 \). now identified as ‘the’ conformal Galilean group. But this mysterious result has long been thought uninteresting as it is not a symmetry of any ‘reasonable’ physical system, let alone of a free particle. This negative outcome disqualified the subject for a long time, which has only arisen again in recent times [12, 13].

Searching for physical realizations of CGA lead to general relativity [14] and in particular to BMS (Bondi–Metzner–Sachs) symmetry [15, 16]. Even more recently [17], it was recognized that the natural context of BMS symmetry is to study the conformal extensions of the Carroll group, and the CGA symmetry found in [16] is just a coincidence in one space dimension [17]. This is because the Carroll and Galilei groups in \( 1 + 1 \) dimension are isomorphic, the isomorphism being effected by interchanging space and time.

In this paper is to study the various conformal extensions of the Carroll group systematically, along lines analogous to those for CGA [13].

Our paper is organized as follows. To make our paper self contained and to motivate the various conformal extensions, we first summarize once again the geometric framework of the Carroll group, cf [8].

Then we proceed to a systematic discussion of various conformal extensions, insisting on the analogy between the Galilean and the Carrollian cases.

In our previous paper [8] we constructed massive particle models with Carroll symmetry using Souriau’s method [18]. The disappointing result was that the resulting dynamics is very poor: a ‘motion’ of such a particle is just one fixed point in space independently of its constant Carrollian ‘velocity’; see also [4, 7].

Now we show that massless particle models constructed by the same method, i.e., as Carroll coadjoint orbits behave better as their motions take place on strings. Their ‘space of motions’ is identified with the space of oriented geodesics of Euclidean 3-space. The latter also carry a conformal Carroll symmetry with \( k = 0 \).

Most interestingly, motion in both in Galilean (Newton–Cartan) and in Carroll spacetimes may be derived by considering strings in one higher dimension called Bargmann space [19, 20]. Our massless Carrollian ‘photons’ correspond, in particular, to null (also called tension-less) strings considered by Schild in 1977 [21].

2. Spacetime structures

2.1. Newton–Cartan manifolds

The geometric definition of Carroll manifolds and transformations proposed in [8] and [17] is dual to that of a Newton–Cartan manifold and its generalized symmetries [13, 22]; therefore we first remind the reader how the latter are defined.

The weak definition of Newton–Cartan (NC) manifold is that it is a triple \( (N, \gamma, \theta) \), where \( N \) (for Newton) is a smooth \((d + 1)\)-dimensional manifold and \( \gamma \) a twice-symmetric, contravariant, non-negative tensor field, whose kernel is generated by the nowhere vanishing 1-form \( \theta \).

The strong definition requires, in addition, a symmetric affine connection \( \nabla \) with Christoffel symbols \( \Gamma_{ij}^k \), which parallel-transport both \( \gamma \) and \( \theta \) [23], extending the previous triple to a quadruple \( (N, \gamma, \theta, \nabla) \).

Note that \( \nabla \) is not uniquely defined by the \( (N, \gamma, \theta) \) and it is precisely this ambiguity which is responsible for the multiplicity of conformal structures discussed in this paper.
The ‘clock’ one-form is closed, \( d\theta = 0 \), so \( \ker \theta \) is thus a Fröbenius-integrable distribution, whose leaves are \( d \)-dimensional and are endowed with a Riemannian structure inherited from \( \gamma \) [23]. The quotient \( N/\ker \theta \) is 1-dimensional: it is the absolute Newtonian time-axis (which can be either compact or non-compact).

The standard flat NC structure is given, in an adapted coordinate system \((x')\), by

\[
N^{d+1} = \mathbb{R}^d \times \mathbb{R}, \quad \gamma = \delta^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, \quad \theta = dt, \quad \Gamma^i_{ij} = 0 \tag{2.1}
\]

for all \( i, j, k = 0, 1, \ldots, d \), where \( A, B = 1 \ldots d \), and \( t = x^{d+1} \) is the Galilean time-coordinate. (In the weak case, the Christoffel symbols are ignored). Other non-trivial NC structures are presented [13].

The automorphisms, i.e., transformations which preserve all geometrical ingredients of the theory provide us with [generalized] Galilei algebras. In the weak case, these are typically infinite-dimensional, but become finite dimensional when the strong definition is used [22].

For the flat NC structure (2.1), and without considering the flat NC connection, we get the Coriolis Lie algebra

\[
X = \left( \omega^A(t)x^A + \eta^A(t) \right) \frac{\partial}{\partial x^i} + \tau \frac{\partial}{\partial t} \tag{2.2}
\]

where \( \omega(t) \in \mathfrak{so}(d) \) and \( \eta(t) \) are arbitrary functions of time and \( t \in \mathbb{R} \) [22], while the strong definition leaves us with the usual Galilei group \( \text{Gal}(d+1) \), represented by the matrices,

\[
a = \begin{pmatrix} R & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \in \text{Gal}(d+1), \tag{2.3}
\]

where \( R \in O(d), b, c \in \mathbb{R}^d, \) and \( e \in \mathbb{R}, \) cf [18, 24]. Then the Galilei Lie algebra \( \mathfrak{gal}(d+1) \) is isomorphic to the vector fields of \( N^{d+1} \),

\[
X = \left( \omega^A x^A + \beta^A t + \gamma^A \right) \frac{\partial}{\partial x^i} + \tau \frac{\partial}{\partial t} \in \mathfrak{gal}(d+1), \tag{2.4}
\]

where \( \omega \in \mathfrak{so}(d), \beta, \gamma \in \mathbb{R}^d \) and \( \tau \in \mathbb{R} \).

2.2. Carroll manifolds

Analogous (but ‘dual’) definitions of a Carroll manifold can be proposed [8, 17]. The weak definition requires having a triple \((C, g, \xi)\) where \( C \) (for Carroll) is again a smooth \((d+1)\)-dimensional manifold, endowed with a twice-symmetric covariant, positive, tensor field \( g \), whose kernel is generated by the nowhere vanishing, complete vector field \( \xi \). The strong definition requires, in addition, a symmetric affine connection \( \nabla \) that parallel-transports both \( g \) and \( \xi \), extending the triple to a quadruple \((C, g, \xi, \nabla)\). Note that, just as in the Galilei framework, the degeneracy of the ‘metric’ \( g \) implies that the connection \( \nabla \) is not uniquely defined by the pair \((g, \xi)\).

The standard weak Carroll structure is given, in an adapted coordinate system \((x')\), by

\[
C^{d+1} = \mathbb{R}^d \times \mathbb{R}, \quad g = \delta_{AB} \, dx^A \otimes dx^B, \quad \xi = \frac{\partial}{\partial s}, \tag{2.5}
\]

completed, in the strong case, with

\[
\Gamma^k_{ij} = 0 \tag{2.6}
\]

for all \( i, j, k = 0, 1, \ldots, d \), where \( s = x^{d+1} \) is now the ‘Carrollian time’ coordinate. The coordinate \( s \) has the dimension of (action)/(mass).
Further examples of a Carroll manifolds were discussed in [17]; see also section 4 below. The isometry group of the weak Carrollian structure \( (C, g, \xi) \) is infinite-dimensional, since it is invariant under the mappings

\[
x'^A = x^A, \quad x' = x + f(x^0, \ldots, x^d)
\]

for an arbitrary function \( f \). However, requiring the preservation of the connection \( V \) implies that the automorphisms of the flat Carroll structure (2.5) form the finite-dimensional Carroll group \([2, 3, 20]\), which we will denote by \( \text{Carr}(d + 1) \). The latter is represented by the matrices,

\[
a = \begin{pmatrix} R & 0 & c \\ -b^T & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \in \text{Carr}(d + 1),
\]

where \( R \in O(d), \ b, c \in \mathbb{R}^d, \) and \( f \in \mathbb{R} \) represent rotations, Carrollian boosts, space translations and Carrollian time translations, respectively. The superscript \( [\cdot]^T \) denotes transposition.

Note the dual aspect of the matrix representation (2.8) when compared to the Galilean case, (2.4).

The Carroll Lie algebra, \( \text{carr}(d + 1) \), is isomorphic to the Lie algebra of the following vector fields of \( C, \) i.e.,

\[
X = \left( \omega^{\dot{A}} x^b + \gamma^{\dot{A}} \right) \frac{\partial}{\partial x^A} + \left( \sigma - \beta A x^A \right) \frac{\partial}{\partial s} \in \text{carr}(d + 1),
\]

where \( \omega \in so(d), \ \beta, \gamma \in \mathbb{R}^d, \) and \( \sigma \in \mathbb{R} \).

2.3. Unification: Bargmann, Newton–Cartan, Carroll

Recall that a Bargmann manifold is a triple \((B, G, \xi)\), where \( B \) (for Bargmann) is a \((d + 2)\)-dimensional manifold with \( G \) a metric of Lorentz signature \((1, 1)\), and the ‘vertical’ vector, \( \xi \), a nowhere vanishing, complete, null vector, which is parallel-transported by the Levi–Civita connection \( V \) of \( G \)[19, 20].

The flat Bargmann structure is given, in an adapted coordinate system \((x') = (x^A, t, s)\), by

\[
B = \mathbb{R}^{d+1} \times \mathbb{R} \times \mathbb{R}, \quad G = \sum_{A, \beta = 1}^{d+1} \delta_{A\beta} dx^A \otimes dx^\beta + dt \otimes ds + ds \otimes dt, \quad \xi = \frac{\partial}{\partial s}.
\]

Note that both \( s \) and \( t \) are light-cone, i.e. null, coordinates. The automorphisms of \((B, G, \xi)\), i.e., the \( \xi \)-preserving isometries \( a \) of the flat Bargmann structure (2.10),

\[
a^* G = G, \quad a_\xi \xi = \xi
\]

form the Bargmann group (also called extended Galilei group) \( \text{Barg}(d + 1) \)[19, 20], i.e., the group of those matrices of the form

\[
a = \begin{pmatrix} R & b & 0 & c \\ 0 & 1 & 0 & e \\ -b^T & -\frac{1}{2}b^2 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{Barg}(d + 1, 1),
\]

where \( R \in O(d), \ b, c \in \mathbb{R}^d, \) and \( e, f \in \mathbb{R} \).
The Bargmann Lie algebra \( \mathfrak{barg} \) is hence isomorphic to the Lie algebra of the vector fields of \( B \),

\[
X = \left( \omega^d \frac{\partial}{\partial x^d} + \beta^d t + \gamma^d \right) \frac{\partial}{\partial x^d} + \tau \frac{\partial}{\partial t} + \left( \sigma - \beta^d x^d \right) \frac{\partial}{\partial s} \in \mathfrak{barg}(d + 1),
\]

(2.13)

where \( \omega \in \mathfrak{so}(d) \), \( \beta, \gamma \in \mathbb{R}^d \), and \( \tau, \sigma \in \mathbb{R} \).

In the flat case, the Bargmann group (2.12) is a non-trivial central extension of the Galilei group with ‘vertical’ translations generated by \( \xi = \partial / \partial s \).

Factoring out flat Bargmann space, \( B \), by the ‘vertical’ translations generated by \( \xi \), the quotient, \( N = B / R \), acquires a (flat) NC structure \([19, 20]\). Let us call \( \theta = G(\xi) \) the pull-back to \( B \) of a ‘clock’ 1-form \( \theta \) on the quotient.

Finally, it has been shown that the Levi–Civita connection, \( \nabla \), of \( B \) naturally defines an affine connection \( V^\gamma \) on \( N \) that transports \( (\gamma, \theta) \). A Bargmann structure thus projects onto the strong NC structure (2.5). In what follows the superscript \( \cdot \) \( C \) will be dropped.

3. Conformal extensions

3.1. Conformal Galilei groups

Recall that in the relativistic case, where the metric \( G \) of space-time (e.g., of Minkowski space-time \( E^{1,\gamma} \)), is invertible, an infinitesimal transformation which preserves the conformal class \( [G] \) of the metric \( G \) is a vector field \( X \) such that

\[
L_X[G] = 0 \iff L_X[G \otimes G^{-1}] = 0.
\]

Below, we will adapt this defining property of infinitesimal conformal transformations to both the Galilean and Carrollian cases.

Let us first remind the reader of the Galilean case \([8, 13]\). There we start with a (weak) Newton–Cartan space-time \((N, \gamma, \theta)\) with degenerate ‘metric’ \( \gamma \) and ‘clock’ \( \theta \). We choose a positive integer \( k \) and call a (local) diffeomorphism, \( a \), of \((N, \gamma, \theta)\) a conformal Galilei transformation of level \( k \in \mathbb{N} \) if

\[
a^k (\gamma \otimes \theta^i) = \gamma \otimes \theta^i,
\]

(3.2)

where \( \theta^i \) is a shorthand for the \( k \)th tensor power \( \theta^\otimes_k \).
Infinitesimally, the \textit{conformal Galilei algebra} (CGA) of level \( k \), we denote by \( \mathfrak{cgal}_\gamma \) such that

\[
L_X \left( \gamma \otimes \theta^j \right) = 0. \tag{3.3}
\]

Spelt out in separate terms, (3.3) amounts to requiring

\[
L_X \gamma = \lambda \gamma \quad \text{and} \quad L_X \theta = \mu \theta \quad \text{such that} \quad \lambda + \mu k = 0. \tag{3.4}
\]

In the flat case, the corresponding Lie algebra is spanned by the vector fields

\[
X = \left( \omega^A_\mu(t)x^\mu + k_T(t)x^A + \eta^A(t) \right) \frac{\partial}{\partial x^A} + T(t)\frac{\partial}{\partial t} \in \mathfrak{cgal}_\gamma(d + 1), \tag{3.5}
\]

where \( \omega(t) \in \mathfrak{so}(d) \), \( \eta(t) \) and \( T(t) \), interpreted as (generalized) rotations, translations and time-reparametrizations, depend arbitrarily on time, \( t \). This Lie algebra is infinite-dimensional, and generates \( \text{CGal}_\gamma \), the conformal Galilei group of level \( k \).

Demanding the strong definition, i.e., that these transformations should also preserve the projective structure associated with some NC connection \( \nabla \) or, if the transformation acts improperly, with a much larger pseudo-group. See [13, 26] and references therein.

In particular, the choice \( k = 1 \) yields the infinite-dimensional \textit{Schrödinger–Virasoro algebra}, [27, 28] denoted by \( \mathfrak{so}(d + 1) \) in the flat NC case; see also [13]. Its intersection with the (infinitesimal) projective group (i.e., (local) diffeomorphisms of \( (N, \gamma, \theta, \nabla) \) that permute the unparametrized geodesics of \( \nabla \) ) is the (finite dimensional) \textit{Schrödinger Lie algebra} of \( (N, \gamma, \theta, \nabla) \). In the flat NC case, this algebra,

\[
sch(d + 1) = \mathfrak{cgal}_\gamma(d + 1) \cap \mathfrak{sl}^\infty \tag{3.6}
\]

features a dynamical exponent \( z = 2 \). For \( k = 2 \), which, as said above, extends the Wigner–Inönü contraction from Poincaré to Galilei we get what is called simply ‘the’ \textit{conformal Galilei Lie Algebra},

\[
\mathfrak{cgal}_\gamma(d + 1) = \mathfrak{cgal}_\gamma(d + 1) \cap \mathfrak{sl}(d + 2);
\]

it has dynamical exponent \( z = 1 \) [12, 13].

Let us spell out, for further use, the \( (1 + 1) \)-dimensional flat case,

\[
N = \mathbb{R} \times S^1 \quad \gamma = \frac{\partial}{\partial s} \otimes \frac{\partial}{\partial x}, \quad \theta = dt, \quad \Gamma^t = 0, \tag{3.6}
\]

where \( t \) is an affine time-coordinate on \( S^1 \). (If we want to produce the Virasoro cocycle, the ‘time axis’ must be compact.) The CGA is spanned by the vector fields

\[
X = (\alpha'(t)x + \eta(t)) \frac{\partial}{\partial x} + \alpha(t)\frac{\partial}{\partial t} \in \mathfrak{cgal}_\gamma(1 + 1). \tag{3.7}
\]

This Lie algebra admits the centre-free Virasoro algebra as the Lie subalgebra generated by the vector field \( \alpha(t)\partial/\partial t \) on \( S^1 \). If \( X_i = [X_1, X_2] \), we find

\[
\alpha_3 = \alpha_1 \alpha_2 - \alpha_2 \alpha_1 \quad \text{and} \quad \eta_3 = \left( \alpha_1 \eta_2 - \eta_2 \alpha_1 \right) + \left( \eta_1 \alpha_2 - \alpha_2 \eta_1 \right);
\]

this corresponds to equation (2.4) of Bagchi \textit{et al} in [14]. This implies that

\[
\mathfrak{cgal}_\gamma(1 + 1) \cong \text{Vec}(S^1) \ltimes C^\infty(\mathbb{R}). \tag{3.8}
\]

In [14] it has been shown how \( \mathfrak{cgal}_\gamma(1 + 1) \) arises as a ‘non-relativistic contraction’ of two (centre-free) Virasoro algebras, i.e. of holomorphic vector fields of the complex plane.
The Lie algebra \(\mathfrak{gal}(1 + 1)\) clearly admits a non-trivial central extension by \(\mathbb{R}\), the Virasoro cocycle of the Lie subalgebra \(\text{Vect}(S^1)\) being constructed via the infinitesimal Schwarzian derivative \(s(\alpha(t)\partial/\partial t) = \alpha''(t)dt^2\). The reader is referred to [12, 13] for the general finite-dimensional case with prescribed dynamical exponent \(z = 2/k\). Central extensions of the Virasoro algebra were classified in [30].

It is worth mentioning that requiring in addition that the clock \(\theta\) be preserved, the conformal Galilei group with \(k = 0\) is the Coriolis group (2.2) of Galilean isometries. The affine Coriolis transformations, i.e., such that \(a^*\mathcal{V} = \mathcal{V}\), then form the Galilei group of \((N, \gamma, \theta, \mathcal{V})\) [24].

### 3.2. The (extended) Schrödinger group

The ‘\(k = 1\) conformal Galilei group’ is just the (extended) Schrödinger group, which can also be treated in the Bargmann framework outlined in section 2.3. In fact it is the group of all conformal transformations \(a\) of \((B, G, \xi)\) that also preserve the ‘vertical’ vector field \(\xi\),

\[
a^*G = \Omega^2G \quad \& \quad a_\xi = \xi
\]

for some strictly positive function \(\Omega\) of \(B\) depending on \(a\). The (pseudo-)group, \(\text{Sch}(B, G, \xi)\), of those transformations satisfying (3.9) projects down to NC space-time as the Schrödinger group \(\text{Sch}(N, \gamma, \theta, \mathcal{V})\), and turns out to be a one-dimensional central extension of the latter [20]. As a group of conformal transformations, the (extended) Schrödinger group is finite-dimensional.

In the flat case, this group descends to NC space-time as the group of matrices,

\[
a = \begin{pmatrix} R & b & c \\ 0 & d & e \\ 0 & f & g \end{pmatrix} \in \text{Sch}(d + 1),
\]

where \(R \in O(d)\), \(b, c \in \mathbb{R}^d\), and \(d, e, f, g \in \mathbb{R}\) with \(dg - ef = 1\). We record for further reference that its Lie algebra \(\mathfrak{sch}(d + 1, 1)\) is spanned by the vector fields \(X\) solutions of (7.7), namely

\[
X = \left(\omega^\chi x^\chi + \beta^e t + \gamma^\gamma + 2\alpha t x^\chi + \gamma x^\chi\right)\frac{\partial}{\partial x^\chi} + \left(2\chi t^2 + 2\chi t + \tau\right)\frac{\partial}{\partial t}
+ \left(-\beta \cdot x - \kappa x^2 + \varphi\right)\frac{\partial}{\partial s} \in \mathfrak{sch}(d + 1, 1),
\]

where \(\omega \in \mathfrak{so}(d)\), \(\beta, \gamma \in \mathbb{R}^d\), \(\kappa, \chi, \tau, \varphi \in \mathbb{R}\).

Equation (3.9) is clearly a conformal extension of the definition of Bargmann group (2.11). See also (3.10) and (2.12) for the flat case.

Let us mention that this treatment of the Schrödinger case is unique in that the other conformal Galilei groups, i.e., those with \(k > 1\), admit no lift to Bargmann space.

### 3.3. Conformal Carroll transformations

Now we define conformal Carroll transformations of level \(k\), we denote by \(\text{ccarr}_k(d + 1)\). Our definition relies on proper Carroll structures and the ‘dual’ nature of the latter to those of Newton–Cartan. Trading the ‘clock’ 1-form \(\theta\) in (3.2) for its ‘dual’ object, i.e., the ‘vertical’ vector \(\xi\), we consider the transformations \(a\) of \(C\) which satisfy
The (pseudo-)group of conformal Carroll transformation of level \( k \) will be denoted by \( \text{C Carr}(C, g, \xi) \).

Infinitesimally, we translate (3.12) as

\[
L\xi \left( g \otimes \xi^i \right) = 0,
\]

which amounts to requiring

\[
L\xi g = \lambda g \quad \& \quad L\xi \xi = \mu \xi \quad \text{with} \quad \lambda + k\mu = 0, \tag{3.14}
\]

cf (3.4). The Lie algebra of infinitesimal conformal Carroll transformation of level \( k \), i.e., of all solutions \( X \) of the system (3.13) will be denoted by \( \text{c Carr}(C, g, \xi) \).

For the flat Carroll structure (2.5), i.e., \( C = \mathbb{R}^{d+1} \), with \( g = \delta_{ab} \, dx^a \otimes dx^b, \xi = \partial/\partial s \), the vector fields \( X = X^a \partial/\partial x^a + T \partial/\partial s \) satisfying (3.14) are such that

\[
\partial_s X_A + \partial_{\varphi} X_A = \lambda \delta_{ab}, \quad \partial_s X^A = 0, \quad \partial_T = -\mu
\]

for all \( A, B = 1, \ldots, d \). For \( k \neq 0 \), our vector fields are of the form

\[
X = \left( \omega^A \right) x^A + \chi x^A + \kappa x^A = 2\kappa x^A \frac{d}{dx^A} - F(x^A) \frac{d}{ds} \in \text{c Carr}_d(d+1) \tag{3.15}
\]

with \( \omega \in \mathfrak{so}(d), \chi, \kappa \in \mathbb{R}^d, \chi \in \mathbb{R}, \) and \( F \in C^\infty(\mathbb{R}^d, \mathbb{R}) \); they generate the Lie algebra we have already denoted by \( \text{c Carr}_d(d+1) \) in (3.15). The conformal factor in (3.14) is expressed as

\[
\lambda = 2(\chi - 2\kappa x^A). \tag{3.16}
\]

Our Lie algebra is infinite-dimensional, since \( F \) is an arbitrary function of the ‘space’ variables \( x^A \). The Carroll Lie algebra \( \text{carr}(d+1) \) itself corresponds to the Lie subalgebra obtained by choosing \( \chi = \kappa = 0, \quad F = \sigma - \beta_A x^A \), as in (2.9).

The two following extreme cases are particularly interesting.

(i) For \( k = 0 \), the defining condition (3.12) becomes

\[
L\xi g = 0 \quad \& \quad L\xi \xi = \mu \xi, \tag{3.17}
\]

and we end up with the algebra we call \( \text{c Carr}_d(d+1) \), which is spanned by the vector fields

\[
X = \left( \omega^A \right) x^A + \chi x^A + T \left( x^A, s \right) \frac{d}{ds} \in \text{c Carr}_d(d+1), \tag{3.18}
\]

where \( (\omega, \chi) \in \mathfrak{e}(d), \) and \( T \in C^\infty(\mathbb{R}^{d+1}, \mathbb{R}) \).

It follows that \( \text{c Carr}_d(d+1) \) is the semi-direct product of the Euclidean Lie algebra with ‘supertranslations’ of Carollian ‘time’,

\[
\text{c Carr}_d(d+1) \cong \mathfrak{e}(d) \ltimes C^\infty(\mathbb{R}^{d+1}, \mathbb{R}). \tag{3.19}
\]

The terminology will be explained in example 3 of section 4.

(ii) Taking instead the limit \( k \to \infty \), from (3.14) we infer, using (3.15), the explicit expression valid in the flat case,
\[ X = \left( \omega^a b \, x^b + \gamma^a + \chi x^a + \kappa^a x b x^b - 2 \kappa b x^b x^a \right) \frac{\partial}{\partial x^a} + T \left( x^a \right) \frac{\partial}{\partial s} \in \mathfrak{ccarr}_\omega(d + 1). \]  

Hence

\[ \mathfrak{ccarr}_\omega(d + 1) \cong \mathfrak{so}(d + 1, 1) \ltimes C = \mathfrak{R}^d, \mathfrak{R} \]  

which is infinite-dimensional because of the presence of ‘supertranslations’, given by the function \( T(x^a) \).

Let us observe that the transformations \(a^a c\) of \(C\) generated by (3.20) satisfy formally the same conditions (3.9) as those defining the Schrödinger group, but for mappings of the Carroll manifold \((C, g, \xi)\), and with the Bargmann metric \(G\) replaced by the degenerate Carroll ‘metric’ \(g, \xi\), namely

\[ a^a b = \Omega^2 g \quad \& \quad a^a _g = \xi \]  

for some strictly positive function \(\Omega\) of \(C\), depending on \(a\). The infinitesimal version of (3.22) is

\[ L x^a g = \lambda g \quad \& \quad L x^a _g = 0 \]  

for some function \(\lambda\) on \(C\). The solutions of (3.23) reproduce in fact (3.20).

### 3.4. Conformal Carroll subgroups of the Schrödinger group

The (pseudo-)group of conformal transformations of Bargmann space is finite-dimensional and the subgroup which preserves the ‘vertical’ vector \(\xi\) form the Schrödinger group \([19, 20, 26]\). But the Carroll manifold can be embedded into Bargmann space as a \(t = \text{const.}\) slice and one may wonder what part of the Schrödinger symmetry is consistent with the constraint \(t = \text{const.}\). Then a calculation (not detailed here) shows that, in the flat case, we are left with the Schrödinger-Carroll algebra \(\mathfrak{scarr}(d + 1)\)

\[ X = \left( \omega^a b \, x^b + \gamma^a + \chi x^a + \kappa^a x b x^b - \frac{1}{2} \kappa^a x b x^a \right) \frac{\partial}{\partial x^a} \]  

where \(\omega \in \mathfrak{so}(d)\), \(\beta, \gamma \in \mathfrak{R}^d\) and \(\chi, \kappa, \sigma \in \mathfrak{R}\), which is clearly an extension of the Carroll action (2.9) with space but no ‘time’ (i.e., \(s\) dilations \(\chi\). The supertranslations are time-independent and also involve Schrödinger expansions with \(\kappa\), namely

\[ T \left( x^a \right) = \sigma - \beta^a x^a - \frac{1}{2} \kappa x^a x^a. \]  

A straightforward computation shows that the Lie algebra (3.24) is

\[ \mathfrak{scarr}(d + 1) = \left\{ X \in \mathfrak{ccarr}_\omega(d + 1) \mid \frac{\partial}{\partial x^a} X^a + \alpha g \xi_\omega = 0, \quad \alpha \in \mathfrak{R} \right\}. \]  

For a general Carroll manifold \((C, g, \xi)\) embedded into Bargmann space, our algebra satisfies \(\mathfrak{scarr}(C, g, \xi, V) = \left\{ X \in \mathfrak{ccarr}_\omega(C, g, \xi) \mid L x V + \alpha g \otimes \xi = 0, \quad \alpha \in \mathfrak{R} \right\} \),

where \(V\) is the induced Levi–Civita connection from Bargmann space.

### 4. Examples of Carroll spacetimes & symmetries

We now illustrate our general theory by some selected examples.
1. Let us first consider \((1 + 1)\)-dimensional flat Carroll space-time,

\[ C = \mathbb{R} \times \mathbb{R}, \quad g = dx^2, \quad \xi = \frac{\partial}{\partial s}, \quad \Gamma^k_{ij} = 0, \quad (4.1) \]

to find the \(k\)-conformal Carroll algebra spanned by the vector fields

\[ X = \alpha(x) \frac{\partial}{\partial x} + \left( \frac{k}{2} \alpha'(x) s + \eta(x) \right) \frac{\partial}{\partial s} \in \mathfrak{ccarr}_k(1 + 1), \quad (4.2) \]

where \(\alpha\) and \(\eta\) are arbitrary smooth functions of ‘space’ \(\mathbb{R}\). For the ‘relativistic’ value \(k = 2\), this reduces precisely to the conformal Galilei case \((3.7)\), whenever \textit{Newtonian time is changed into position, and position into Carrollian time},

\[ t \to x \quad \& \quad x \to s, \quad (4.3) \]

confirming that the two algebras do indeed have the same structure \([14, 16]\). Note that \((4.3)\) also swaps Galilean and Carrollian boosts.

2. CFT-type applications envisaged in \([14, 16]\) require to work in the plane, \(d = 2\), e.g., with

\[ C = \mathbb{R}^2 \times \mathbb{R}, \quad g = dx^2 + dy^2, \quad \xi = \frac{\partial}{\partial s}, \quad \Gamma^k_0 = 0. \quad (4.4) \]

In geometric terms, the conformal \(k = \infty\) Carroll Lie algebra is spanned by the vector fields

\[ X = h(z) \partial_x + \overline{h(z)} \partial_y + f(z) \partial_s \in \mathfrak{ccarr}_\infty(1 + 1), \quad (4.5) \]

where \(h(z)\) is a holomorphic, and \(f(z)\) a smooth real valued function, on the complex plane. Our \(k\)-conformal Carroll vectors fields are, in turn,

\[ X = h(z) \partial_x + \overline{h(z)} \partial_y + \left( \frac{k}{2} \partial h + \partial \overline{h} \right) \partial_s \in \mathfrak{ccarr}_k(1 + 1). \quad (4.6) \]

Letting \(k \to \infty\) the conformal Carroll algebra \(\mathfrak{ccarr}_\infty\) is recovered. Again, we have a 1-dimensional central extension of \(\mathfrak{ccarr}_1(1 + 1)\) governed by the infinitesimal Schwarzian derivative \(s(h(z) \partial \partial z) = h''(z)dz^2\).

3. More general Carroll manifolds are given by

\[ C = \Sigma \times \mathbb{R}, \quad g = \mathfrak{g}_{AB}(x)dx^A \otimes dx^B, \quad \xi = \frac{\partial}{\partial s}, \quad \Gamma^C_{AB} = \Gamma^{C}_{AB}, \quad \Gamma^1_{AB} = \Gamma^{1}_{AB} \quad (4.7) \]

where \((\Sigma, \mathfrak{g})\) is a \(d\)-dimensional Riemannian manifold, with Christoffel symbols \(\Gamma^C_{AB}\), and where the symmetric quantities \(\Gamma^1_{AB}\) remain arbitrary. Computing the general expression for a conformal Carroll vector field of level \(k\) yields

\[ X = \tilde{X} + \left( \frac{2\lambda}{k} s + F(x^i) \right) \frac{\partial}{\partial s}, \quad (4.8) \]

where \(\tilde{X} = \tilde{X}^A(x)dx^A\) is a conformal vector field of \((\Sigma, \mathfrak{g})\), i.e., such that

\[ L_{\tilde{X}} \mathfrak{g} = \lambda \mathfrak{g} \quad \text{with} \quad \lambda = \frac{2}{d} \tilde{V}_A \tilde{X}^A, \quad (4.9) \]

and \(F\) is a density on \(\Sigma\) of weight \(-2/(k d)\), i.e., is a real function \(F\) which transforms as \(F \to \Omega^{-2} F\) under a rescaling \(\mathfrak{g} \to \Omega^2 \mathfrak{g}\) of the metric. Integration of the vector field \((4.8)\) yields the group action \((x, s) \mapsto (x', s')\), where

\[ x' = \varphi(x), \quad s' = \Omega^{2N}(x)(s + \alpha(x)), \quad (4.10) \]

with \(\varphi \in \text{Conf}(\Sigma, \mathfrak{g})\), and \(\alpha \in C^\infty(\Sigma, \mathbb{R})\).
For example, $\Sigma$ could be the circle, $\Sigma = S^1$. Then we would get, for level-$k$ conformal Carroll transformations, the semi-direct product of the conformal transformations of $S^1$, namely $\text{Diff}(S^1)$, with supertranslations of weight $-2/k$. They are generated by the vector fields
\[
X = \tilde{X} (\theta) \frac{\partial}{\partial \theta} + \left( \frac{2}{k} \tilde{X}^* (\theta) s + F(\theta) \right) \frac{\partial}{\partial s},
\]
(4.11)
Similarly, consider the two-sphere, $\Sigma = S^2$, endowed with its standard round metric. The conformal Carroll vector fields of level $k$ is given by (4.8) with $d = 2$. Recalling that the conformal transformations of the two-sphere constitute the identity component of the Lorentz group, $\text{PSL}(2, C) = \text{SL}(2, C)/\mathbb{Z}_2$, we conclude that the conformal Carroll transformations are now the semi-direct product of $\text{SL}(2, C)$ with supertranslations, and hence
\[
\text{ccarr}_t \left( S^2 \times \mathbb{R}, g, \xi \right) \cong \mathfrak{s}(2, C) \rtimes C^\infty (S^2, \mathbb{R}).
\]
In particular, for $k = 2$ we get, using (4.9),
\[
X = \tilde{X} + \frac{1}{2} \hat{\nabla}_A \tilde{X}^A s + F(x^3) \frac{\partial}{\partial s},
\]
(4.12)
which is, indeed the BMS Lie algebra [15, 31]. We recall that the BMS group is an infinite dimensional extension of the Poincaré group which arises as the asymptotic symmetry group of asymptotically flat four dimensional spacetimes [15]. The Poincaré group is the semi-direct product of the Lorentz group with the four-dimensional abelian group of translations; the BMS group is the semi-direct product of the Lorentz group with an infinite dimensional abelian group, which may be realized as functions of a certain conformal weight on the 2-sphere. Since ordinary translations are realized as functions spanned by the lowest four harmonics on the 2-sphere, members of this infinite dimensional abelian group are referred to as super-translations.

4. Now we choose $C$ to be the punctured future light-cone in Minkowski space $\mathbb{R}^{d+1,1}$. The ‘metric’ $\hat{g}$ is simply the induced Minkowski metric, and $\xi$ the restriction to $C$ of the Euler vector field of $\mathbb{R}^{d+1,1}$. We have $C \cong S^d \times \mathbb{R}^{d,0}$, which is described by those $t(u, 1) \in \mathbb{R}^{d+1,1}$ where $u \in S^d$ and $t > 0$. Then $g = r^2 \hat{g}$ where $\hat{g}$ is the round metric of $S^d$, and $\xi = \partial t$. We will also put $s = \log t$ for convenience. Hence the light-cone is an intrinsically defined Carroll manifold in the weak sense. A remarkable fact is that it carries no compatible connection.

To see this, choose the coordinate system $(x^A, s)$ on $C$, such that
\[
g_{AB} = e^{x^A} \hat{g}_{AB}, \quad g_{As} = 0, \quad g_{sa} = 0, \quad \xi^A = 0, \quad \xi^s = 1,
\]
(4.13)
for all $A, B = 1, \ldots, d$. Then if $V$ were such a connection, then we would have
\[
\begin{align*}
(V_A g)_{AB} &= \partial_A (e^{x^A} \hat{g}_{AB}) - \Gamma^C_{AB} g_{CB} - \Gamma^C_{BC} g_{AC} \\
&= 2g_{AB} - \Gamma^C_{BA} g_{CB} - \Gamma^C_{BC} g_{AC} = 0,
\end{align*}
\]
(4.14)
as well as
\[
\begin{align*}
(V_A g)_{AB} &= \partial_A g_{AB} - \Gamma^C_{AB} g_{CB} - \Gamma^C_{AC} g_{CB} \\
&= 0 - \Gamma^C_{BA} g_{CB} - 0 = 0.
\end{align*}
\]
(4.15)
Equation (4.15) yields trivially $\Gamma_{AB}^C g_{CB} = 0$ for all $A, B, C = 1, \ldots, d$. Then from equation (4.14) we infer, using the symmetry of the connection, that $g_{AB} = 0$, a contradiction.

Luckily enough, our weak definition of a conformal Carroll transformation does not involve any connection, and we find that $L_x g = \lambda g$ requires that $X = X^A \partial / \partial x^A + T \partial / \partial s$ be such that $\partial X^A = 0$ as well as $L_{\tilde{X}} \hat{g} = (\lambda - 2X^A) \hat{g}$ where $\tilde{X} = X^A(x) \partial / \partial x^A$. Using that equation (3.14) implies $L_x \xi = -(\lambda/k) \xi$ allows us to deduce that the conformal Carroll algebra of the punctured future light cone is for $k = 2$, the BMS algebra (4.12). As to Carroll isometries, the condition $\lambda = 0$ would fix the supertranslations as

$$T = -\frac{L_{\tilde{X}} \hat{g}}{2 \hat{g}}$$

while leaving us with the vector field $\hat{X}$ as a conformal vector field of $(\Sigma, \hat{g})$; the Carroll ‘isometries’ of the light-cone therefore span the conformal group, $O(d + 1, 1)$, of the celestial sphere, $S^d$, with rigidly fixed ‘compensating’ supertranslations.

5. Newman–Unti group

The Newman–Unti (NU) group [32] is spanned by those (local) diffeomorphisms $a$ of $C$ which preserve the degenerate ‘metric’, $g$, up to a conformal factor, namely

$$a^* [g] = [g].$$

This implies that the direction of $\xi$ is automatically preserved (since $a^* \xi$ lies again in the kernel of $g$). Its Lie algebra consists, hence, of all vector fields $X$ on $C$ such that

$$L_x g = \lambda g,$$

the condition $L_x \xi = \mu \xi$ being automatically satisfied.

In the product case $C = \Sigma \times \mathbb{R}$ of example 3. In section 4, we find that

$$X = \tilde{X} + T \left( x^A, s \right) \frac{d}{ds},$$

with $\tilde{X} = \tilde{X}^A(x) \partial / \partial x^A$ a conformal vector field of $\Sigma$, and $T \in C^\infty(C, \mathbb{R})$ now an arbitrary function of $x^A$ and $s$. Therefore the Newman–Unti group is

$$\text{NU}(C, G, \xi) \equiv \text{Conf}(\Sigma) \ltimes C^\infty(C, \mathbb{R}).$$

Choose an integer $n = 0, 1, \ldots$. An ‘intermediate’ Lie subalgebra $\nu_n (d+1)$ of the NU Lie algebra, $\nu(d+1)$, defined by

$$L_x g = \lambda g, \quad \text{and} \quad (L_\lambda)^n X = 0$$

consists of vector fields $X = \tilde{X}^A(x) \partial / \partial x^A + T \partial / \partial s$ such that

$$\tilde{X} \in \text{conf}(\Sigma) \quad \text{and} \quad (\partial)^n T = 0,$$

i.e., such that the supertranslation is a polynomial of degree $n - 1$ in $s$,

$$T \left( x^A, s \right) = \sum_{m=1}^{n-1} T_m \left( x^A \right),$$

where the $T_m$ remain arbitrary functions on $\Sigma$.

Referring to equations (4.8) and (5.3), which give the generators of the conformal Carroll Lie algebras, we can highlight the interesting array of nested Lie groups
We finally notice that $\text{NU}_1 = \text{CCarr}_{\omega}$.

6. Carrollian ‘photons’

In special relativity the massless representations of the Poincaré group may describe massless particles, e.g., photons. Massless representations have already been investigated for the Galilei group, leading to classical models of both spinning and spinless light rays [18, 33]. These models were constructed using coadjoint orbits of the Euclidean group, which is in fact a subgroup of the Galilei group. Below we pursue this idea for the Carroll group. We begin by reviewing its coadjoint orbits.

The general idea [18] is to consider an ‘evolution space’, i.e., a manifold $V$ equipped with a closed 2-form $\sigma$ whose kernel $\sigma_{\ker}$ has constant rank. Now, $\sigma$ being closed, the distribution $\sigma_{\ker}$ is integrable; its space of leaves, $U$, when a manifold is endowed with a symplectic 2-form $\omega$, given by the projection of $\sigma$. Souriau calls $(U, \omega)$ the ‘space of motions’.

Given a vector field $X$ on $V$ such that $L_X \sigma = 0$, we may define a function $\mu_X$ by

$$d\mu_X = -\sigma(X, \cdot)$$  \hspace{1cm} (6.1)

provided $\sigma(X, \cdot)$ is globally exact. It follows that for all $Y \in \ker \sigma$ we find that $Y(\mu_X) = 0$. Thus $\mu_X$ is a constant of the motion in the sense that it is constant along the leaves of $\ker \sigma$; this is the presymplectic Noether theorem [18].

In the special case where $\sigma$ is exact, i.e., $\sigma = d\varpi$, and $L_X \varpi = 0$, then we obtain

$$\mu_X = \varpi(X)$$  \hspace{1cm} (6.2)

thus fixing the overall additive constant present in the general definition (6.1) of the momentum mapping $\mu$, defined by $\mu_X = \mu \cdot X$.

A particular case of this general procedure is to take $V$ to be a Lie group $G$ whose Lie algebra is denoted $\mathfrak{g}$. For each $\mu_0 \in \mathfrak{g}^*$ we define $\sigma$ by

$$\sigma = d\varpi \quad \text{with} \quad \varpi = \mu_0 \cdot \Theta,$$  \hspace{1cm} (6.3)

where $\Theta = \theta^{-1}d\theta$ is the left-invariant $\mathfrak{g}$-valued Maurer–Cartan 1-form on $G$. The space of motions $U$ is then given by the coadjoint orbit $O_{\mu_0} = \text{Coad}(G)\mu_0$ passing through $\mu_0$.

Applied to our case, we represent the Carroll Lie algebra $\text{carr}(d + 1)$ by the matrices

$$Z = \begin{pmatrix} \omega & 0 & \gamma \\ -\beta^T & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (6.4)

with $\omega \in \mathfrak{so}(d)$, $\beta, \gamma \in \mathbb{R}^d$, and $\alpha \in \mathbb{R}$, cf equation (2.8). Bearing in mind that Carrollian time, $s$, has dimension $L^2/T$, i.e., action/mass, we find that $[\omega] = 1$, $[\beta] = L/T$, $[\gamma] = L$, and $[\alpha] = [s]$. Then an element in the dual of the Lie algebra is $\mu = (\ell, \mathbf{k}, \mathbf{p}, m) \in \text{carr}(d + 1)^*$, where the pairing between the Lie algebra and its dual is defined as

$$\mu \cdot Z = \frac{1}{2} \text{Tr} (\ell \omega) - \mathbf{k} \cdot \beta - \mathbf{p} \cdot \gamma + ma.$$  \hspace{1cm} (6.5)
Then the adjoint action of $\text{Carr}(d + 1)$ in the representation (2.8) is

$$
\text{Ad}(a^{-1})(\omega, \beta, \gamma, \alpha) = (R^{-1} \omega R, R^{-1}(\omega \mathbf{b} + \beta), R^{-1}(\omega \mathbf{c} + \gamma), \alpha + \mathbf{b} \cdot (\omega \mathbf{c} + \gamma) - \beta \cdot \gamma),
$$

(6.6)

and yields the coadjoint action, defined by $\text{Coad}(a) \mu \equiv \mu \circ \text{Ad}(a^{-1})$, as given by $\text{Coad}(a)(\ell, \mathbf{k}, \mathbf{p}, m) = (\ell', k', \mathbf{p}', m')$, where

$$
\ell' = R(\ell R^{-1} + (Rk \mathbf{b}' - \mathbf{b} (Rk)^{\mathbf{c}'}) + (Rp \mathbf{c}' - \mathbf{c} (Rp)^{\mathbf{y}'})
$$

$$
+ m (\mathbf{c} \mathbf{b}' - \mathbf{b} \mathbf{c}')
$$

(6.7)

$$
k' = Rk + mc
$$

(6.8)

$$
\mathbf{p}' = Rp - mb
$$

(6.9)

$$
m' = m
$$

(6.10)

showing that $m$ is a Casimir invariant with the dimension of mass and $[m/\hbar] = [s^{-1}]$ has the dimension of Carrollian frequency.

The Euclidean group is a subgroup of the Carroll group, and its Lie algebra $e(d)$ can be identified with (6.4) when $\beta = 0$ and $\alpha = 0$. The dual space $e(d)^*$ consists of pairs $(\ell, \mathbf{p})$ and the coadjoint action if given by (6.8) and (6.9) with $k = 0$ and $m = 0$. If $d = 3$, the coadjoint orbits are labelled by the invariants $p = |p|$ and $\ell \cdot \mathbf{p}$ where $\ell$ is viewed now as a 3-vector. If $p \neq 0$, then it is interpreted as the color, and $j = \ell \cdot \mathbf{p}/p$ as the spin. The coadjoint orbits are identified with $TS^2$ for all values of $s$; for $j = 0$ they consist of oriented straight lines in $E^3$ [18, 33, 34].

The Carrollian massive case $m \neq 0$ was studied in [4, 7, 8]. Therefore we consider now the massless case $m = 0$. Then we find three invariants, namely

$$
p = |p| \quad \& \quad k = |k| \quad \& \quad w = k \cdot \mathbf{p}. \quad (6.11)
$$

By analogy with the Galilean/Euclidean cases, we call $p > 0$ the ‘color’ [18, 33]. These Carroll invariants have physical dimension $[p] = AL^{-1}$, $[k] = ML$, $[w] = MA$, where $[A] = [\hbar]$.

In view of the form (6.7)–(6.10) of the coadjoint action of the Carroll group, we choose first the origin of our massless coadjoint orbit as

$$
\mu_0 = (0, 0, \mathbf{p}_0, 0) \quad \text{with} \quad \mathbf{p}_0 = p \mathbf{u}_0, \quad (6.12)
$$

where $\mathbf{u}_0 \in S^{d-1} \subset \mathbb{R}^d$ is a fixed direction. This massless orbit has $k = w = 0$, see (6.11). The associated 1-form (6.3) on the Carroll group is $\varpi = p \mathbf{u}_0 \cdot R^{-1}d\mathbf{x}$ and descends to the new evolution space

$$
V = \{ (x, u, s) \in \mathbb{R}^{1+4} \times \mathbb{R} \mid u \cdot u = 1 \} \quad \text{as} \quad \varpi = p \delta_{ab} u^a dx^b, \quad (6.13)
$$

where $u = R \mathbf{u}_0$. $V$ is the direct product of the tangent bundle of the unit sphere in $d$ dimensions carrying its canonical one-form with (Carrollian) time. Computing the characteristic foliation of

$$
\sigma = d\varpi = p \delta_{ab} du^a \wedge dx^b, \quad (6.14)
$$

in the $(2d)$-dimensional evolution space $(V, \sigma)$ we find it to be two-dimensional, tangent to the distribution $\text{ker} \sigma$, and such that
\[ Y \in \ker \sigma \iff Y = \alpha u^a \frac{\partial}{\partial x^a} + \beta \frac{\partial}{\partial s}, \tag{6.15} \]

where \( \alpha \) is a real Lagrange multiplier to enforce the constraint \(|u|^2 = 1\), and \( \beta \in \mathbb{R} \). The space of motions \( V / \ker \sigma \) now consists of straight lines (with \( u = \text{const.} \)), i.e., oriented geodesics of Euclidean space \( \mathbb{E}^d \). It is hence symplectomorphic to the cotangent bundle

\[ \mathcal{O}_{d_0} \cong T^{d-1}_p \tag{6.16} \]

of the \((d - 1)\)-sphere of radius \( p \).

We note that the 1-form \( \sigma \) in (6.13) describes a ‘Fermat’ particle whose motions are oriented line rays in Euclidean 3-space (with no spin) studied in geometrical optics [33].

The Fermat particle model can be obtained by a fixed-energy reduction of either a Galilean or a Poincaré-invariant particle with vanishing mass and spin [33]. This is explained by the fact that the space of motions, \( T^{d-1} \), endowed with the projection of the two-form (6.14) is in fact a spin-zero coadjoint orbit of the Euclidean group; but the latter is a subgroup of both the Galilei and the Poincaré groups [33].

What we have just found shows that this same statement is valid for the Carroll group.

Lifting the Carroll vector field \( X \) in (2.9) to the evolution space \( V \) as \( X_V \), allows us to evaluate \( \sigma(X_V) \); see equation (6.2). These quantities are constant on the leaves of the characteristic foliation of \( \sigma \), and therefore provide with the help of (6.5) the components of the Carrollian momentum mapping

\[ \mu = \ell, k, p, m \] (6.17)

In particular, \( m = 0 \) is the conserved quantity associated to Carrollian time translations, \( s \to s + f \). This follows immediately from that the variational form \( \sigma \) in (6.13) has no component for \( ds \).

### 6.1. Conformal symmetries of massless Carrollian models

We now show that the zero-level conformal Carroll algebra,

\[ \mathfrak{c} \mathfrak{c} \mathfrak{a} \mathfrak{r} \mathfrak{r}_0(d + 1) \cong \mathfrak{c}(d) \ltimes \mathbb{C}^{\infty} \left( \mathbb{R}^{d+1}, \mathbb{R} \right) \]

in (3.18), is a symmetry algebra of the Carroll massless particle model above. To see this, we lift the \( \mathfrak{c} \mathfrak{c} \mathfrak{a} \mathfrak{r} \mathfrak{r}_0(d + 1) \) generators (3.18) to \( (V, \sigma) \), as

\[ X_V = \left( \omega^A \chi^B + \gamma^A \right) \frac{\partial}{\partial x^A} + \omega^A \chi^B \frac{\partial}{\partial u^A} + T \left( x^A, s \right) \frac{\partial}{\partial s} \tag{6.18} \]

and we trivially check that \( L_{X_V} \sigma = 0 \) and hence \( L_{X_V} \sigma = 0 \) for all \( X \in \mathfrak{c} \mathfrak{c} \mathfrak{a} \mathfrak{r} \mathfrak{r}_0(d + 1) \).

A curious fact is that although supertranslations extend the finite-dimensional symmetry (2.9) to an infinite-dimensional one, they do not contribute to the conserved quantities. Again this follows from the absence of any \( ds \)-term in the ‘variational’ 1-form (6.13). To put it in another way, we can say that any (super or not) translation has \( m = 0 \) as associated conserved quantity. This also explains why Carrollian boosts \( (x, s) \mapsto (x, s - b \cdot x) \) have a vanishing conjugate momentum, \( k = 0 \); they are just supertranslations of a particular form.

So far, we discussed ‘Fermat photons’ with vanishing spin. Spin can also be included. Restricting ourselves to \( d = 3 \) spatial dimensions, it is sufficient to consider, instead of (6.12), the basepoint
with \( j \in \mathbb{R} \), which yields the 1-form on the (neutral) Carroll group
\[
\sigma_{p,j} = pu \cdot dx - jv \cdot dw,
\]
where \((u, v, w) \in SO(3)\) and \(x \in \mathbb{R}^3\). The associated 2-form differs only from the spinless expression (6.14) by an extra term,
\[
\sigma_{p,j} = du \wedge dx - j \text{surf},
\]
where \(\text{surf} = \frac{1}{2} \epsilon_{ABC} u^A du^B \wedge du^C\), the canonical surface form of \(S^2\) represents the spin 2-form.

The characteristic foliation of (6.21) is still two-dimensional; any tangent vector \(Y \in \ker \sigma_{p,j}\) is again given by
\[
Y = \alpha u^A \frac{\partial}{\partial x^A} + \beta \frac{\partial}{\partial s},
\]
where \(\alpha, \beta \in \mathbb{R}\). We notice that the motions of these spinning particles are independent of the spin, \(j\). The associated space of motions is again \(T \mathbb{S}^2\) with the new twisted symplectic form
\[
\omega_{p,j} = \omega_{p,0} - j \text{surf}.
\]
Therefore \(CCarr(4)\) is a symmetry group of spinning Carrollian photons also. The only effect of the new term in (6.20) (resp. in (6.21)) is to change the angular momentum (viewed as a 3-vector) in (6.17) to
\[
\ell = x \times p + ju,
\]
where the notations introduced in (6.19) and below (6.20) were used.

Our Carrollian ‘photons’ are reminiscent of Galilean photons, studied before in [13, 33]. The latter are described by the 2-form
\[
\sigma_{p,j} = \sigma_{p,0} - dE \wedge dt,
\]
defined on the Galilean extension of the evolution space with coordinates \((u, x, E, t)\), obtained by adding \(E\) and \(t\), the energy and Galilean time, respectively, to the Euclidean evolution space. Their motions are hence instantaneous, i.e., ‘move’ at \(t = \text{const.}\) along straight lines in \(E^3\).

Spinless Galilean photons were shown [13] to carry an infinite-dimensional conformal Galilean symmetry, generated by the vector field
\[
X = \left( \omega^A(t) x^A + \eta^A(t) \right) \frac{\partial}{\partial x^A} + T(t) \frac{\partial}{\partial t} \in \mathfrak{cgal}_\infty(4),
\]
where \(\omega(t) \in \mathfrak{so}(3)\), \(\eta(t) \in \mathbb{R}^3\), and \(T(t) \in \mathbb{R}\) depend arbitrarily on time \(t\), cf equation (5.58) in [13]. Galilean boosts and space translations correspond respectively to \(\eta(t) = \beta t + \gamma\), and time translations to \(T(t) = \tau \in \mathbb{R}\). The time-independent vector fields (6.26) generate Souriau’s ‘Aristotle group’, i.e., the Euclidean group extended with time-translations.

Viewed as an extension of the Euclidean model, the extra term \(-dE \wedge dt\) in (6.25) is consistent with arbitrary time dependence of the coefficients (since \(t = t_0\) is now itself a constant of the motion), but eliminates any position-dependence of the supertranslations \(T\).

For the sake of comparison, we recall the conserved quantities found in [13].

Define first a natural pairing between the infinite-dimensional Lie algebra \(\mathfrak{cgal}_\infty(4)\), see (6.26), and its (formal) dual spanned by \(\mu = (\ell(t), p(t), H(t))\) by
ω · H = u ℓ t + j \mu · \omega(t_0) - p(t_0) \cdot \eta(t_0) + H(t_0)T(t_0) \quad (6.27)

for some \( t_0 \). Then the associated constants of the motion,

\[
\begin{align*}
\ell(t_0) &= x \times p + j u \text{ angular momentum,} \\
p(t_0) &= p \text{ linear momentum,} \\
H(t_0) &= E \text{ energy,} \\
\end{align*}
\]

(6.28)

are actually independent of the choice of \( t_0 \).

7. Strings and particles

Now we relate our particles to strings in the \((d + 2)\)-dimensional Bargmann space \((B, G, \xi)\), introduced in section 2.3. This new approach will enable us to reveal distinguished, finite-dimensional, Lie subalgebras of the conformal Galilei and Carroll Lie algebras discussed in section 3.4.

We consider only the spinless case in flat Bargmann space, \((2.10)\). Let us start with the \((2d + 2)\)-dimensional manifold \( T^*B \), with canonical coordinates \((p^1, \ldots, p^{d+2}, x^1, \ldots, x^{d+2})\), endowed with the Liouville 1-form and symplectic 2-form

\[
\omega = \omega_A dx^A \quad \text{and} \quad \Omega = \Omega_A dp_A \wedge dx^A, \quad (7.1)
\]

respectively.

7.1. Massive systems

We first consider the massive case, already discussed in [20]. Let us consider the \((2d + 1)\)-dimensional submanifold \( V \) of \( T^*B \) defined by two constraints,

\[
V = \{ y = (p, x) \in T^*B \mid p^i x^i = m, \quad G^i p_j p_j = 0 \} \quad (7.2)
\]

with \( m = \text{const.} \neq 0 \). In our usual coordinate system \((2.10)\), the first constraint says that the momentum conjugate to the ‘vertical’ variable, \( s \), is \( p_s = m \), and then the second relation identifies the momentum conjugate to \( t \) as (minus) the kinetic energy of a non-relativistic particle of mass \( m \) in \( \mathbb{R}^d \), namely \( p_t = -H = -|p_t|^2/(2m) \). Call \( \sigma \) the 2-form on induced on \( V \) by \( \Omega \); we find

\[
\sigma = \Omega|_V = dp_A \wedge dx^A - dH \wedge dt. \quad (7.3)
\]

Then \((V, \sigma)\) will be our evolution space.

The characteristic distribution of the closed 2-form \( \sigma \) is two-dimensional since

\[
Y \in \ker \sigma \iff Y = \alpha p^A \frac{\partial}{\partial y^A} + \beta \frac{\partial}{\partial s}, \quad (7.4)
\]

where \( p^A = p_A, \ p = m, \ p_s = -H \) are constants of the motion, with \( \alpha, \beta \in \mathbb{R} \). By abuse of notation we will often identify \( \ker \sigma \) given by \((7.4)\) with its pointwise projection to Bargmann space \( B \). In view of \((7.4)\), its leaves project to Bargmann space as two-dimensional surfaces denoted by \( \Sigma \) which may be viewed as the world sheets of a string. The metric induced on the world sheet has Lorentz signature \(+−\) since \( G(Y, Y) = 2m \alpha \beta \) at each \( x \in \Sigma \).

The quotient space \( U = V/\ker \sigma \) is symplectomorphic to \( T^*\mathbb{R}^d \), i.e., to the space of motions of spinless & massive Galilean particles. In fact, we recover the ‘Bargmannian’ description of such a particle, considered in [19, 20, 35].
These strings $\Sigma$ project onto Newton–Cartan space-time $N$ as future-pointing time-like worldlines [in fact, as straight lines] since their tangent vectors are of the form
\[
\pi_{\alpha}Y = \alpha \left( p^\alpha \frac{\partial}{\partial x^\alpha} + m \frac{\partial}{\partial t} \right)
\]  
(7.5)
with $p = \text{const.}$ and $\alpha \in \mathbb{R}$. In other words, they are worldlines of massive Galilean particles.

The intersection of the above strings $\Sigma$ with the Carroll manifold $C \equiv B|_{t=0}$ yields again world lines. To see this we observe that requiring that the distribution (7.4) also be tangent to $C$ implies $\alpha = 0$ because $m \neq 0$. The corresponding distribution restricted to $C$ is therefore spanned by
\[
Y|_C = \beta \frac{\partial}{\partial s}
\]  
(7.6)
with $\beta \in \mathbb{R}$.

The associated worldlines in $C$ are thus null with respect to the Carroll metric $g$ in (2.5). Moreover (7.6) shows that free massive spin-0 Carrollian particles do not move in Carroll absolute $d$-space, cf [4, 7, 8].

Now we turn to symmetries.

In the massive case $m \neq 0$, the constraints in (7.2) are clearly invariant under the conformal transformation generated by the vector fields $X$ on $B$ verifying
\[
L_XG = \lambda G \quad \& \quad L_X\xi = 0
\]  
(7.7)
for some function $\lambda$ of $B$. But this is precisely the infinitesimal version of (3.9) and defines the Schrödinger Lie algebra (3.11).

By construction, $\mathfrak{sch}(d + 1, 1)$ is a symmetry Lie algebra of $(V, \sigma)$: if $X_\nu$ is the canonical lift of $X$ to $V \subset T^*B$, then
\[
L_{X_\nu}\sigma = 0, \quad \text{for all } X \in \mathfrak{sch}(d + 1, 1).
\]  
(7.8)

The projected vector fields generate the centre-free Schrödinger Lie algebra of Newton–Cartan space-time $N$.

The vector fields in (3.11) which also preserve $C$, i.e., Carroll space $t = 0$, form the Lie algebra $\mathfrak{sch}carr \cong \mathfrak{c}(d+2)$ in (3.24), i.e.,
\[
X = \left( \omega^\delta_{\beta} x^\beta + \gamma^\gamma + \chi x^s \right) \frac{\partial}{\partial x^\alpha} + \left( \sigma - \beta^s_{\lambda} x^s - \frac{1}{2} \kappa x^s x^s \right) \frac{\partial}{\partial s},
\]  
which is thus a finite dimensional infinitesimal conformal symmetry of massive Carroll systems. (Here $\mathfrak{c}(d)$ denotes the Lie algebra of the conformal Euclidean group, i.e., the Euclidean group augmented with dilations.)

7.2. Massless systems

Now consider the new $(2d + 1)$-dimensional manifold
\[
V = \left\{ \gamma = (p, x) \in T^*B \mid p_\xi = 0, \quad G|_{p_\xi} = p^2 \right\},
\]  
(7.9)
where $p = \text{const.} > 0$ will be interpreted as the color of massless ($m = 0$) systems. We have hence $p_\xi = 0$ and $|p|^2 = p^2$. The pull-back to $V$ of the canonical symplectic form on $T^*B$ is
\[
\sigma = d\sigma|_V = \left[ \Omega|_V = dp_\lambda \wedge dx^\lambda + dp_\nu \wedge dt. \right]
\]  
(7.10)
The pair $(V, \sigma)$ is our new evolution space for a massless particle with color $p = |p|$. The kernel of $\sigma$ is two-dimensional, and pointwise spanned by the vectors...
\[ Y = \alpha p^t \frac{\partial}{\partial x^t} + \beta \frac{\partial}{\partial s} \]  

(7.11)

with \( \alpha, \beta \in \mathbb{R} \); this can be read off from (7.4).

The restriction of the leaves of \( \ker \sigma \) to Carroll space are clearly two-dimensional manifolds \( \Sigma \) which may be thought of as string worldsheets. The ‘metric’ induced on these sheets from the Bargmann metric has signature \((+, 0)\) since \( G(Y, Y) = \alpha^2 \) at each \( x \in \Sigma \); these strings are null strings. In other words, our ‘particles’ with both vanishing spin and mass are delocalized in Carroll space-time; their ‘history’ is a two-dimensional sheet, and not a curve.

Let us then investigate the conformal Carroll symmetries inherited from the Bargmann automorphisms.

The constraints in (7.9) describing massless systems are invariant under the transformations generated by the vector fields \( X \) on \( B \) satisfying

\[ L_X G = 0 \quad \& \quad L_X \xi = \mu \xi \]  

(7.12)

for some function \( \mu \) on \( B \). The solutions \( X \) of the system (7.12) are of the form

\[ X = \left( \omega^\beta_\gamma x^\beta + \beta^\tau t + \gamma^\lambda \right) \frac{\partial}{\partial x^\lambda} + (at + \tau) \frac{\partial}{\partial t} \]

\[ + \left( \sigma - \beta_\tau x^\tau - \alpha \right) \frac{\partial}{\partial s} \in \text{opt}(d + 1, 1), \]  

(7.13)

where \( \omega \in \mathfrak{so}(d), \beta, \gamma \in \mathbb{R}^d, \alpha, \tau, \sigma \in \mathbb{R} \); they are the generators of a Lie algebra that we propose to call the optical Lie algebra of the Bargmann space.

By construction, \( \text{opt}(d + 1, 1) \) is a symmetry of \( \sigma \): if \( X_V \) is the canonical lift of \( X \) to \( V \subset T^*B \), then

\[ L_{X_V} \sigma = 0, \quad \text{for all} \quad X \in \text{opt}(d + 1, 1). \]  

(7.14)

The associated moment maps, i.e., conserved quantities are those of the Carroll group listed in (6.17). Note, once again, that both the ‘\( t \)’ and ‘\( s \)’ supertranslations have vanishing contribution to the conserved quantities. This follows at once from the particular form of \( \sigma \) in (7.10).

Projecting the vector fields (7.13) onto Newton–Cartan space-time yields

\[ \pi_X X = \left( \omega^\beta_\gamma x^\beta + \beta^\tau t + \gamma^\lambda \right) \frac{\partial}{\partial x^\lambda} + (at + \tau) \frac{\partial}{\partial t} \]

(7.15)

which generate the centre-free optical Lie algebra, i.e., the Galilei Lie algebra augmented with time dilations, found earlier as a symmetry of the Chaplygin gas with viscosity [36].

Finally, the vector fields (7.13) preserving the Carroll space \( C \) viewed as the \( t = 0 \) slice of \( B \), are of the form

\[ X|_C = \left( \omega^\beta_\gamma x^\beta + \gamma^\lambda \right) \frac{\partial}{\partial x^\lambda} + (\sigma - \beta \cdot \mathbf{x} - \alpha s) \frac{\partial}{\partial s} \]

(7.16)

and span, as anticipated, a Lie subalgebra of the conformal Carroll Lie algebra of level \( k = 0 \). It is isomorphic to the semi-direct product of the Euclidean Lie algebra in dimension \( d \), and of the space of polynomials of degree 1 on Carroll space \( C \equiv \mathbb{R}^{d+1} \),

\[ \epsilon(d) \ltimes \text{Pol}_1(C) \subset \text{ccarr}_0(d + 1). \]

(7.17)

It is worth noting that the symmetry (7.16) actually extends to the full \( \text{ccarr}_0(d + 1) \) algebra (6.26) since the supertranslations do not contribute.
7.3. Schild strings

As noted in section 7.2, for representations with vanishing mass and spin the space of motions (i.e., the set of leaves of \( \ker \sigma \)) consists two-dimensional world sheets moving in Bargmann spacetime and such that the metric induced from the Bargmann metric is degenerate and has rank 1. The dynamics of such strings, which are often referred to as ‘null strings’ or ‘tensionless strings’, was first discussed by Schild in a posthumous paper in 1977 [21].

With the development of modern string theory Schild’s work has been extensively revisited in the theoretical physics literature from a variety of perspectives, both classical and quantum. Just as null geodesics may be thought of as a limiting case of timelike geodesics, Schild strings may be thought of as limiting case of Nambu–Goto strings. The analogy goes deeper, in that just as the usual proper time action whose variation yields the equations of motion for timelike geodesics is not suitable for null geodesics, the analogous Nambu–Goto action for timelike strings is not suitable for null strings. Schild was able to provide a convenient replacement. Below we provide a summary of the formalism and compare our work with some existing treatments of the conformal symmetries with those in the literature.

Here we briefly summarize Schild’s 1977 formalism [21] for null strings which differs somewhat from more recent treatments. Schild considers a two-dimensional submanifold \( \Sigma \)

\[
x^i = x^i(u^a), \quad i = 1, 2, ..., n, \quad a = 1, 2,
\]

of an \( n \)-dimensional Lorentzian \((M, g)\) spacetime which satisfy the Schild variational equations. These obtained by considering the bi-vector

\[
\sigma^{ij} = -\sigma^{ji} = \frac{1}{2} \epsilon^{ab} \frac{\partial x^i}{\partial u^a} \frac{\partial x^j}{\partial u^b}, \quad \epsilon^{ab} = -\epsilon^{ba}, \quad \epsilon^{12} = 1,
\]

and its magnitude squared

\[
\sigma^2 = \frac{1}{2} g_{ab} g^{ij} \sigma^{ij} \sigma^{ij} \quad \text{and requiring that}
\]

\[
\delta \int_{\Sigma} \frac{1}{2} \sigma^2 du^i du^j = 0.
\]

Note that from the point of view of the world sheet \( \Sigma \), the quantity \( \sigma^2 \) is a scalar density of weight 2 and hence Schild’s action is not invariant under all diffeomorphism of the world sheet \( \Sigma \). In more recent treatments compensating fields are included to restore full diffeomorphism invariance. This means that the world sheet symmetries may differ. However it does not affect the spacetime symmetries.

The Euler–Lagrange equations imply that

\[
\sigma^2 = \text{const.}
\]

and we obtain null strings by taking the constant to vanish. One may check that then the induced metric

\[
\gamma_{ab} = g_{ij} \frac{\partial x^i}{\partial u^a} \frac{\partial x^j}{\partial u^b}
\]

is tangent to the local light cone and is therefore degenerate, i.e. has signature \((+, 0)\). The null direction is the kernel of \( \sigma_0 \) and defines null vector field on \( \Sigma \) whose integral curves, called null generators, fibre \( \Sigma \).
Schild’s equations of motion imply that these null curves are null geodesics of the ambient spacetime manifold \((M, g)\). Thus the general solution of Schild’s equations of motion for null strings is obtained by taking an arbitrary spacelike curve \(\gamma\) in \((M, g)\) and a one-parameter family of null vectors along it. One then pushes \(\gamma\) along the null geodesics of \((M, g)\) with these initial tangent vectors. If \((M, g)\) is flat then these null geodesics are straight lines with a null tangent vector. In the case of Carroll strings lifted to Bargmann space one takes \(\gamma\) to be a straight line, i.e., a geodesic in \(E^n\) and the null geodesics to be straight lines with \(x \in \gamma\) and \(t\) both constant. The coordinate \(s\) then serves as an affine parameter along the null generators.

As written by Kar [37], the partially gauge-fixed Schild equations are

\[
\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \quad g_{ij} \ddot{x}^i = 0, \quad g_{ij} \dot{x}^i = 0,
\]

(7.24)

where \(x^i = \partial_t x^i\) and \(x^i = \partial_s x^i\). Thus at \(\sigma = u^i\) constant we have null geodesics with affine parameter \(\tau = u^2\) and we have the induced metric

\[
ds^2 = N^2(\sigma) d\sigma^2.
\]

(7.25)

Kar claims that these equations are invariant under the remaining gauge freedom

\[
\tau \rightarrow \tau(\sigma, \sigma), \quad \sigma \rightarrow \sigma(\sigma),
\]

(7.26)

which are generated by

\[
A(\tau, \sigma) \partial_\tau + G(\sigma) \partial_\sigma.
\]

(7.27)

This is a Newman–Unti group in 1 + 1 dimensions, discussed in section 5.

By contrast Bagchi [38], following earlier work [39], claims invariance under a group generated by

\[
\left(\sigma'(\sigma) + g(\sigma)\right) \partial_\sigma + g(\sigma) \partial_\tau.
\]

(7.28)

This preserves the vector half-density

\[
V^a = \tilde{\delta}^a_\tau.
\]

(7.29)

His CGA is in turn the same as \(\mathfrak{ccarr}_2(1 + 1)\), as pointed out in section 4, since we are in 1 + 1 dimensions.

8. Conclusion

In this paper we introduced systematically a family of infinite dimensional conformal Carroll groups \(\mathfrak{ccarr}_k(d + 1)\) labelled by an integer \(k = 0, 1, \ldots\), which acts upon \((d + 1)\)-dimensional Carrollian space-time. When \(d = 1\), they are the same as the conformal Galilei groups \(\mathfrak{cgal}_1\). These generalize in a natural way to more general Carroll manifolds and in particular give rise to the Bondi–Metzner–Sachs (BMS) groups and their Newman–Unti (NU) generalizations. The case \(k = 0\) is shown to apply to massless Carrollian systems. The latter are in fact strings, which lift to Schild’s null strings in \((d + 2)\)-dimensional Bargmann space.

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