TRAJECTORIES IN THE CONTEXT OF THE QUANTUM NEWTON’S LAW

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Abstract
In this paper, we apply the one dimensional quantum law of motion, that we recently formulated in the context of the trajectory representation of quantum mechanics, to the constant potential, the linear potential and the harmonic oscillator. In the classically allowed regions, we show that to each classical trajectory there is a family of quantum trajectories which all pass through some points constituting nodes and belonging to the classical trajectory. We also discuss the generalization to any potential and give a new definition for de Broglie’s wavelength in such a way as to link it with the length separating adjacent nodes. In particular, we show how quantum trajectories have as a limit when ℏ → 0 the classical ones. In the classically forbidden regions, the nodal structure of the trajectories is lost and the particle velocity rapidly diverges.

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1. Introduction

For a one-dimensional system of energy $E$ and potential $V(x)$, the quantum stationary Hamilton-Jacobi equation (QSHJE) is

$$\frac{1}{2m} \left( \frac{\partial S_0}{\partial x} \right)^2 + V(x) - E = \frac{\hbar^2}{4m} \left[ \frac{3}{2} \left( \frac{\partial S_0}{\partial x} \right)^{-2} \left( \frac{\partial^2 S_0}{\partial x^2} \right)^2 - \left( \frac{\partial S_0}{\partial x} \right)^{-1} \left( \frac{\partial^3 S_0}{\partial x^3} \right) \right].$$  \(1\)

The solution of this equation is investigated in Refs. [1, 2, 3, 4, 5, 6, 7, 8]. It is shown that it can be written as

$$S_0 = \hbar \arctan \left[ \frac{a \phi_1}{\phi_2} + b \right] + \hbar l, \quad (2)$$

where $(\phi_1, \phi_2)$ is a set of two real independent solutions of the Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2 \phi}{dx^2} + V(x) \phi = E \phi \quad (3)$$

and $(a, b, l)$ are real integration constants satisfying the condition $a \neq 0$. In Eq. (2), $S_0$ depends also on the energy $E$ through the solutions $\phi_1$ and $\phi_2$. Remark that both for the classical stationary Hamilton-Jacobi equation and the QSHJE, if $S_0$ is solution, $-S_0$ is also solution. Therefore, the conjugate momentum is given by

$$P = \frac{\partial S_0}{\partial x} = \pm \frac{\hbar a W}{\phi_2^2 + (a \phi_1 + b \phi_2)^2}, \quad (4)$$

where $W = \phi_1' \phi_2 - \phi_1 \phi_2'$ is a constant representing the Wronskian of $(\phi_1, \phi_2)$. As also observed in Refs. [6, 10, 11, 12], the ± sign in Eq. (4) indicates that the motion may be in either direction on the $x$ axis. In contrast with Bohm’s theory, it is shown in Refs. [1, 2, 3, 4, 12] that it is possible to relate the reduced action $S_0$ to the Schrödinger wave function in a unified form both for bound and unbound states so that the conjugate momentum never has a vanishing value.

Recently [9], by taking advantage of the fact that the solution of (1) is known, we constructed a Lagrangian from which we derived the fundamental relation

$$\dot{x} \frac{\partial S_0}{\partial x} = 2[E - V(x)]. \quad (5)$$

By using (4) in this last equation, we get

$$\frac{dx}{dt} = \pm \frac{2[E - V(x)]}{\hbar a W} \left[ \phi_2^2 + (a \phi_1 + b \phi_2)^2 \right]. \quad (6)$$

In what follows, we adopt the following convention: the sign of the parameter $a$ is chosen so that $aW > 0$. In this way, if the particle moves in the classically allowed region ($E > V$) in the positive direction, we must use the plus sign in the right hand side (RHS) of (6). When the particle gets to a turning point, we must use the minus sign whether the particle remains in the classically allowed region by changing its direction of motion or it enters the classically forbidden region ($E < V$) by keeping its direction of motion. In other words, when the particle changes the branch on its trajectory at the turning point, even if it passes to the classically forbidden region, the sign which precedes the RHS of (6) must be changed.
In Ref. [9], we showed that relation (5) leads to a third order differential equation representing the first integral of the quantum Newton’s law (FIQNL)

\[
\left( E - V \right)^4 - \frac{m\dot{x}^2}{2} \left( E - V \right)^3 + \frac{\hbar^2}{8} \left[ \frac{3}{2} \left( \frac{\ddot{x}}{x} \right)^2 - \frac{3}{2} \right] \left( E - V \right)^2 \\
- \frac{\hbar^2}{8} \left[ \ddot{x} \frac{d^2 V}{dx^2} + \dot{x} \frac{dV}{dx} \right] \left( E - V \right) - \frac{3\hbar^2}{16} \left[ \dot{x} \frac{dV}{dx} \right]^2 = 0 .
\]  
(7)

The solution \( x(t) \) of this equation will contain the two usual integration constants \( E \) and \( x_0 \) and two additional constants which we will call the non-classical integration constants. All these constants can be determined by the knowledge of \( x(t_0), \dot{x}(t_0), \ddot{x}(t_0) \) and \( \dot{\ddot{x}}(t_0) \).

Without appealing to the Lagrangian formulation, we emphasize that relation (5)
- is obtained by using the quantum version of Jacobi’s theorem [9];
- can be obtained by the Hamiltonian formulation.

In this paper, we apply respectively in Sections 2, 3 and 4 the quantum law of motion (5) or (7) in the cases of a constant potential, a linear potential and a harmonic oscillator. In Section 5, we comment on the generalization to any potential of the obtained results and give a new definition for de Broglie’s wavelength and its physical meaning in trajectory interpretation of quantum mechanics.

2. Constant potential

Let us consider the case in which the potential is constant \( V(x) = V_0 \) and set

\[ \epsilon = E - V_0 . \]  
(8)

We begin by the classically allowed case \( (\epsilon > 0) \). With the same procedure which we have used in Ref. [9] for the free particle, we can integrate (7) after having substituted \( V(x) \) by \( V_0 \). We obtain

\[ x(t) = \pm \frac{\hbar}{\sqrt{2m\epsilon}} \arctan \left[ a \tan \left( \frac{2t}{\hbar} \right) + b \right] + x_0 . \]  
(9)

Note that for the particular values \( a = 1 \) and \( b = 0 \) of the non-classical integration constants, expression (9) reduces to the classical relation

\[ x(t) = \pm \sqrt{\frac{2\epsilon}{m}} t + x_0 \]

whether the velocity is positive or negative.

Since the arctangent function is contained between \(-\pi/2\) and \(\pi/2\), it is necessary to readjust the additive integration constant \( x_0 \) after every interval of time in which the tangent function goes from \(-\infty\) to \(+\infty\). This readjustment must be made in such a way as to guarantee the continuity of \( x(t) \). For this reason, expression (9) must be rewritten as follows

\[ x(t) = \frac{\hbar}{\sqrt{2m\epsilon}} \arctan \left[ a \tan \left( \frac{2t}{\hbar} \right) + b \right] + \frac{\pi \hbar}{\sqrt{2m\epsilon}} n + x_0 \]  
(10)
with
\[ t \in \left[ \frac{\pi \hbar}{2 \epsilon} \left( n - \frac{1}{2} \right), \frac{\pi \hbar}{2 \epsilon} \left( n + \frac{1}{2} \right) \right] \]
for every integer number \( n \). In (10), the parameter \( a \) is assumed positive and we have considered only the case of positive velocity. In Fig. 1, we have plotted in \((t, x)\) plane for a free electron of energy \( E = 10 \text{ eV} \) some trajectories corresponding to different values of \( a \) and \( b \). All these trajectories, even the classical one \((a = 1, b = 0)\), pass through some points which we will call nodes and which correspond to the times
\[ t_n = \frac{\pi \hbar}{2 \epsilon} \left( n + \frac{1}{2} \right) \quad (11) \]
for which \( x(t) \) does not depend on \( a \) and \( b \). The distances between two adjacent nodes on the time axis
\[ \Delta t_n = t_{n+1} - t_n = \frac{\pi \hbar}{2 \epsilon} \quad (12) \]
and the space axis
\[ \Delta x_n = x(t_{n+1}) - x(t_n) = \frac{\pi \hbar}{\sqrt{2m\epsilon}} \quad (13) \]
are both proportional to \( \hbar \). This means that in the classical limit \( \hbar \to 0 \), the nodes become infinitely close and, then, all possible quantum trajectories tend to be identical to the classical one. In fact, let us consider an arbitrary point \( M(t_M, x_M) \) on any quantum trajectory between two adjacent nodes \((t_{n-1}, x_{n-1})\) and \((t_n, x_n)\). Considering that the variable on the \( t \) axis is a product of a unit velocity by the time, the distance between \( M \) and its orthogonal projection \( M_0(t_{M_0}, x_{M_0}) \) on the classical trajectory is
\[ MM_0 = \sqrt{\frac{m}{2\epsilon} + 1} |t_M - t_{M_0}|. \quad (14) \]
Note that this relation can be obtained without using the expression for \( x(t) \) corresponding to the trajectory on which \( M \) is located. From Eq. (6), we can see that for any potential and in any interval which does not contain turning points, the function \( x(t) \) is monotonous. In the case of Fig. 1, the function \( x(t) \) is increasing and, then, we have \( t_{n-1} \leq t_M \leq t_n \) and \( t_{n-1} \leq t_{M_0} \leq t_n \). This implies that \( |t_M - t_{M_0}| \leq t_n - t_{n-1} \) and, with the use of (14), it follows that \( MM_0 \to 0 \) in the limit \( \hbar \to 0 \). Of course, if \( x(t) \) is a decreasing function, we also get to the same conclusion. This is the fundamental reason why in problems for which the constant \( \hbar \) can be disregarded, quantum trajectories reduces to the classical one. This conclusion is not compatible with the finding of Floyd \[13\] who states that a residual indeterminacy subsists when we take the classical limit. It is not also compatible with our previous paper \[9\] in which we have not taken into account the presence of these nodes.

Finally, note that the solution (9) of (7) in the case where \( V(x) = V_0 \) can be also obtained from the differential equation (6),
\[ \frac{dx}{dt} = \pm \frac{1}{a} \sqrt{\frac{2\epsilon}{m}} \left[ \cos^2 \left( \frac{\sqrt{2m\epsilon}}{\hbar} x \right) + \left[ a \sin \left( \frac{\sqrt{2m\epsilon}}{\hbar} x \right) + b \cos \left( \frac{\sqrt{2m\epsilon}}{\hbar} x \right) \right]^2 \right]^{\frac{1}{2}}, \quad (15) \]
Classical trajectory \( (a = 1, b = 0) \)

\[ x(t) = \pm \frac{\hbar}{2\sqrt{2m\epsilon}} \ln \left| \frac{1}{a} \tan \left( \frac{2\epsilon}{\hbar}(t-t_0) \right) - \frac{b}{a} \right| , \]  

(17)

where \( a, b \) and \( t_0 \) are real integration constants satisfying the condition \( a \neq 0 \).

We also observe that this solution can also be obtained from (6) after having solved (3).

Relation (17) represents the quantum time equation for a particle moving in a constant potential in the classically forbidden region. Obviously, there are no nodes and no classical trajectory. The velocity is given by

\[ \dot{x}(t) = \pm \sqrt{\frac{\epsilon}{2m} \left( 1 + \tan^2 \left[ -2\epsilon(t-t_0)/\hbar \right] \right) - \frac{b}{\hbar} + \tan \left[ -2\epsilon(t-t_0)/\hbar \right] } , \]  

(18)
or, in term of $x$,

$$
\dot{x}(t) = \frac{1}{a} \sqrt{-\frac{\epsilon}{2m}} \left[ \exp(-2\rho x) + [a \exp(\rho x) + b \exp(-\rho x)]^2 \right].
$$

(19)

where $\rho = \sqrt{-2me/\hbar}$. Note that (19) is obtained from (17) and (18) by using the plus sign in the RHS of (17). It can be straightforwardly obtained from (6) if we take $\phi_1 = \exp(\rho x)$ and $\phi_2 = \exp(-\rho x)$. Of course, if we use the minus sign in (17), we can also reproduce the obtained result from (6). Relation (18) indicates that if the particle enters the classically forbidden region at any time belonging to the interval

$$
\left[t_0 - \frac{\pi\hbar}{2\epsilon} \left( n - \frac{1}{2} \right), t_0 - \frac{\pi\hbar}{2\epsilon} \left( n + \frac{1}{2} \right)\right],
$$

its velocity becomes infinite at the time $t_0 - (2n + 1)\pi\hbar/4\epsilon$ (we consider a non-relativistic theory). This means that the particle takes, at the very most, a time equal to $-\pi\hbar/2\epsilon$ before its velocity becomes infinite.

Let us now apply our result for the following rectangular potential barrier

$$
V(x) = \begin{cases} 
0, & x < 0 \\
V_0, & 0 \leq x \leq q \\
0, & x > q.
\end{cases}
$$

First, we mention that our goal is not to determine the conditions for which the particle traverses the barrier. This question requires in our point of view further investigations. Our task here consists in calculating the time delay in tunneling through this barrier and comparing it to earlier results. After we express $t$ in terms of $x$ in (17), we easily calculate the time necessary for the particle to go from $x = 0$ to any point $x$ located between 0 and $q$. We get

$$
T(x) \equiv t(x) - t(0) = -\frac{\hbar}{2\epsilon} \left[ \arctan[a \exp(2\rho x) + b] - \arctan(a + b) \right].
$$

(20)

In this relation, $a$ is assumed positive and we have considered the positive velocity case. In order to calculate the time delay in tunneling through the potential barrier, it is sufficient to substitute in (20) $x$ by $q$. For a thin barrier ($\rho q \ll 1$) and a thick one ($\rho q \gg 1$), the above result turns out to be

$$
T(q) = \frac{a}{1 + (a + b)^2} \sqrt{\frac{2m}{\epsilon} q},
$$

(21)

and

$$
T(q) = -\frac{\hbar}{2\epsilon} \left[ \frac{\pi}{2} - \arctan(a + b) \right],
$$

(22)

respectively. As for Fletcher’s [14] results, in the thin barrier case, the time $T(q)$ is proportional to the thickness $q$ and, in the thick barrier one, $T(q)$ becomes independent on the thickness. However, in contrast with the results of Refs. [14, 15], ours depends only on the difference $|\epsilon| = V_0 - E$ and not on $E$ and $V_0$. Furthermore, ours depends also on the parameters $a$ and $b$ which themselves depend on the initial conditions [9] and specify the particular microstate that
we considered, while the ones established in \cite{14,15} are obtained with the use of wave packets.

Now, let us compare our results to those of Floyd \cite{11}, obtained in the context of another formulation of trajectory representation. For simplicity, let us choose as independent solutions of Schrödinger’s equation the following functions
\begin{equation}
\phi_1 = \exp(-\rho x), \quad \phi_2 = \exp(\rho x),
\end{equation}
inside the barrier \((0 \leq x \leq q)\). Substituting these solutions in expression (2) for \(S_0\) and using Jacobi’s theorem,
\begin{equation}
t - t_0 = \frac{\partial S_0}{\partial E},
\end{equation}
as proposed by Floyd \cite{16}, the expression for \(T(x)\) as defined in (20) takes the form
\begin{equation}
T(x) = \frac{2ma}{\hbar \rho} \frac{x \exp(-2\rho x)}{1 + [a \exp(-2\rho x) + b]^2}.
\end{equation}
We indicate that in Floyd’s notation, \(a\) and \(b\) represent respectively \(b/(ab - c^2/4)^{1/2}\) and \(c/2(ab - c^2/4)^{1/2}\). First, we remark that for thick barriers \((q \to \infty)\), expression (25) leads to \(T(q) = 0\). This result is different from the one obtained by Floyd in [11] by using another couple of solutions of the Schrödinger equation. This means that the trajectories obtained from Eq. (24) depend on the choice of mathematical solutions of the Schrödinger equation. Another problem disclosed by relation (25) for thick barriers \((q \to \infty)\) is the fact that, after having calculated \(dT/dx\), we see that there exists a point \(x_0\) for which \(T(x)\) is a decreasing function for \(x \geq x_0\). Furthermore, the point \(x_0\) is not a turning point since \(dT/dx\) vanishes at \(x_0\) and, then, the velocity is infinite. So, we get to the conclusion that the evolution of time is reversed and, therefore, the causality of the theory is lost. We also indicate that after a tedious calculation, even if we use the solutions chosen by Floyd in [11] to calculate \(T(x)\), we get to the same conclusion for a set of allowed values for \(b\). However, in the context of our formulation of trajectory representation, there is no interval for \(x\) for which \(T(x)\), as defined in (20), is a decreasing function. Furthermore, as we will see in Section 5, our equations of motion do not depend on the choice of the solutions \(\phi_1\) and \(\phi_2\).

3. Linear potential

Let us consider now the linear potential
\begin{equation}
V(x) = gx,
\end{equation}
where \(g\) is a constant which we choose positive. First, remark that the Schrödinger equation can be written in the form of Airy equation
\begin{equation}
\frac{d^2\phi}{dy^2} - y\phi(y) = 0,
\end{equation}
where
\begin{equation}
y = \left(\frac{2m}{\hbar^2 g^2}\right)^{1/3} (gx - E).
\end{equation}
Fig. 2: Quantum trajectories for an electron of energy $E = 10$ eV moving in a linear potential $V(x) = gx$ ($g = 10^{-9}$ kg m s$^{-2}$) in the classically allowed region. For all the curves, we have chosen $x(t = 0) = 3.25405 \times 10^{-10}$ m. The maximum of the curves is located at $(t = 14.47545 \times 10^{-16}$ s, $x = 16.02189 \times 10^{-10}$ m).

The series method allows us to get for Eq. (27) two real independent solutions which can be related to Airy functions $Ai$ and $Bi$ as

$$\phi_1(y) = Ai(y) + \frac{1}{\sqrt{3}} Bi(y) ,$$  \hspace{1cm} (29)

$$\phi_2(y) = \sqrt{3} Ai(y) - Bi(y) .$$  \hspace{1cm} (30)

The equation of motion is obtained by substituting (29) and (30) in (6)

$$\frac{dx}{dt} = \pm \frac{2(E - gx)}{(2mg)^{1/3} aW} \left[ \left( a^2 + 3b^2 + 2\sqrt{3}ab + 3 \right) Ai^2(y) 
+ 2 \left( \frac{a^2}{\sqrt{3}} - \sqrt{3}b^2 - \sqrt{3} \right) Ai(y)Bi(y) + \left( \frac{a^2}{3} + b^2 - \frac{2ab}{\sqrt{3}} + 1 \right) Bi^2(y) \right] ,$$  \hspace{1cm} (31)
Fig. 3: Quantum trajectory for an electron of energy $E = 10$ eV moving in a linear potential $V(x) = gx$ ($g = 10^{-9}$ kg m s$^{-2}$) in the classically forbidden region. We have chosen $a = 10$, $b = 1/\sqrt{3}$ and $x(t = 14.47546 \times 10^{-16} \text{s}) = 16.02190 \times 10^{-16} \text{m}$.

where

$$W = \frac{d\phi_1}{dy} \phi_2 - \frac{d\phi_2}{dy} \phi_1 = 2 \left[ \frac{dB_i}{dy} A_i - \frac{dA_i}{dy} B_i \right] = \frac{2}{\pi}$$

is the Wronskian of $\phi_1$ and $\phi_2$. Eq. (31) is valid both in the classically allowed case and the forbidden one. It is a first order differential equation in which we see the presence of three integration constants $E$, $a$ and $b$. Since it does not have an exact solution, we have appealed to numerical methods. In Fig. 2, we have plotted from (31) in $(t, x)$ plane some trajectories corresponding to different values of $a$ and $b$ in the classically allowed case ($y \leq 0$). The considered system is an electron of energy $E = 10$ eV and we have chosen $g = 10^{-9}$ kg m s$^{-2}$. From the classical analogue of Eq. (31), given by

$$\frac{dx}{dt} = \pm \sqrt{\frac{2}{m}(E - gx)} ,$$

we have plotted in the same figure the classical trajectory. As in the constant potential case, the quantum trajectories oscillate about the classical one. We observe that all trajectories, even the classical one, pass through some points constituting nodes. In particular, for all possible trajectories, the velocity has a vanishing value at $y = 0$. As explained in Section 1, we indicate that for the trajectories plotted in Fig. 2, we have used the plus sign in the RHS of (31) in
the domain where $\dot{x} > 0$ and the minus sign in the domain where $\dot{x} < 0$. This is also the case in (32) for the classical trajectory.

In contrast with the constant potential case, the distance between two adjacent nodes is not constant. We remark that the two intervals starting from the node where the velocity vanishes (at $y = 0$) are the most long ones. The length of the intervals decreases gradually as the velocity increases along the trajectories. We will explain this observation in Section 5 and show that this length is proportional to $\bar{\hbar}$. This means that in the classical limit $\bar{\hbar} \to 0$, the adjacent nodes become infinitely close, and as we will see in Section 5, the quantum trajectories tend to be identical to the classical one.

We remark also that, in contrast with the constant potential case, there are no particular values for $a$ and $b$ for which the quantum trajectories reduce to the classical one. In fact, the RHS of (31) can be developed as a power series in $y$ while the RHS of (32) is proportional to $\sqrt{-y}$. However, it is peculiar to observe that for the particular values $a = 2$ and $b = -1/\sqrt{3}$, the quantum trajectory for $y < 0$ is quasi identical to the classical one. This result is in agreement with the fact that $Ai^2(y) + Bi^2(y)$ acts like $1/\sqrt{-y}$.

Now, let us consider the classically forbidden region ($y > 0$). As in the constant potential case, our investigations do not concern the conditions for which the particle enters this region. We suppose only that the particle is present in this region and we determine its trajectory from the equation of motion (31) by appealing to numerical methods. As an example, we have considered in Fig. 3 an electron with energy $E = 10$ eV for the particular values $a = 10$ and $b = 1/\sqrt{3}$. We see that as soon as the particle enters this region, its velocity increases quickly. We have checked that there are no nodes.

4. Harmonic oscillator

Without appealing to the usual axiomatic interpretation of the wave function, Faraggi and Matone showed [3, 17] that energy quantization is a consequence of the equivalence postulate [1, 2, 3]. The case of the harmonic oscillator is particularly studied in Ref. [3]. In one dimension, the potential is given by

$$V(x) = \frac{1}{2}m\omega^2x^2. \tag{33}$$

Let us begin by the fundamental state for which the physical wave function, up to a constant factor, is given by

$$\phi_2(x) = \exp(-\alpha x^2), \tag{34}$$

where $\alpha = m\omega/2\bar{\hbar}$. The relationship between the corresponding energy and the frequency is $E_0 = \hbar \omega/2$. A second independent solution of the Schrödinger equation can be obtained by using the fact that the Wronskian is constant

$$\phi_1(x) = \exp(-\alpha x^2) \int_{x_0}^x \exp(2\alpha q^2) \, dq. \tag{35}$$

Here, we have chosen the Wronskian $W(\phi_1, \phi_2) = \phi_2 \, d\phi_1/dx - \phi_1 \, d\phi_2/dx = 1$. Note that the lower boundary $x_0$ of the integral in (35) can be arbitrary chosen. Thus, in what follows, we set $x_0 = 0$ and, then, $\phi_1(x)$ represents the Dawsons integral. Substituting (34) and (35) in (6), the equation of motion takes the
form
\[ \frac{dx}{dt} = \pm \frac{2E_0}{\hbar a} (1 - 2\alpha x^2) \exp(-2\alpha x^2) \left[ 1 + \left( a \int_0^x \exp(2\alpha q^2) \, dq + b \right)^2 \right]. \tag{36} \]

Again, there is no exact solution for \( x \). Numerical methods allow us to plot some trajectories corresponding to different values of \( a \) and \( b \). In Fig. 4, we have considered in the classically allowed region (\( |x| \leq x_{M_0} \)) the motion of an electron of energy \( E_0 = 10 \text{ eV} \) over one period. Here, \( x_{M_0} \) represents the corresponding classical amplitude
\[ x_{M_0} = \sqrt{\frac{2E_0}{m\omega^2}} = \frac{\hbar}{\sqrt{2mE_0}}. \tag{37} \]

We observe the presence of nodes in the \((t,x)\) plane at the points \( x = -x_{M_0} \) and \( x = x_{M_0} \), corresponding to the vanishing values of the velocity. We notice that, even if we impose a node by choosing for trajectories the same initial condition \( x(t = 0) = x_0 \) at any point inside the interval \([-x_{M_0}, x_{M_0}]\), all the following nodes in the \((t,x)\) plane will be at the points \( x = \pm x_{M_0} \) where the velocity vanishes. On the other hand, we indicate that at the half-periods where the velocity is positive (negative), we have used the plus (minus) sign in
Fig. 5: Quantum trajectories for the first excited state of energy $E_1 = 30$ eV of a harmonic oscillator in the classically allowed region. For all the curves, we have chosen $x(t = 0) = -x_{M1} = -1.06911 \times 10^{-10}$ m. The first maximum of the curves is located at ($t = 1.04200 \times 10^{-16}$ s, $x = x_{M1} = 1.06911 \times 10^{-10}$ m).

the RHS of (36). In the classical limit $\hbar \to 0$, the oscillator becomes a point at rest because the classical amplitude vanishes.

The classical analogue of Eq. (36) is

$$\frac{dx}{dt} = \pm \sqrt{\frac{2}{m}(E_0 - \frac{1}{2}m\omega^2x^2)}.$$  \hspace{1cm} (38)

As in the linear potential case, there are no particular values for $a$ and $b$ for which the quantum equation (36) reduces to the classical equation (38). However, it is peculiar to observe that for $a = 10^{10}$ and $b = 0$, the quantum trajectory plotted from (36) resembles the classical one.

Now, consider the first excited state. The physical solution of Schrödinger’s equation is

$$\phi_2(x) = x \exp(-\alpha x^2).$$  \hspace{1cm} (39)

The relationship between the corresponding energy and the frequency is $E_1 = 3\hbar\omega/2$. It follows that the amplitude of the corresponding classical oscillator is $x_{M1} = 3\hbar/\sqrt{2mE_1}$. Its ratio with the corresponding amplitude of the fundamental state is $\sqrt{3}$. A second independent solution is obtained by using the fact
that the Wronskian is constant
\[ \phi_1(x) = x \exp(-\alpha x^2) \int_0^x \frac{\exp(2aq^2)}{q^2} dq. \] (40)

Here, we have chosen the Wronskian \( W(\phi_1, \phi_2) = 1 \). As in the fundamental state case, we substitute (39) and (40) in (6) to obtain the quantum equation of motion from which we plot some trajectories (Fig. 5) for different values of \( a \) and \( b \). The value of the energy, \( E_1 = 30 \) eV, that we take is three times that of the fundamental state. We remark that we have an additional node for every half-period of the oscillator motion compared to the fundamental state case. As we will explain in the next section, this additional node is a consequence of the zero of the function \( \phi_2(x) \) given by (39).

Concerning the classical forbidden case, both for the fundamental and the first excited states, we remark that as soon as the particle enters this region, the velocity increases quickly. The nodes do not appear. In Fig. 6, we plotted \( x(t) \) for the fundamental state with \( a = 8 \times 10^{10} \) and \( b = 1 \) in the case where \( x > x_{M_0} \) and with \( a = 8 \times 10^{10} \) and \( b = -1 \) in the case where \( x < -x_{M_0} \).

5. General potential and de Broglie’s wavelength

Concerning the classically forbidden region, we remark that for all the po-
tentials considered here, the velocity increases quickly. We think that this is the case for any another potential. This rapid divergence seems to be in agreement with the predictions of Copenhagen School. In fact, in a natural way, we can assume the existence of a link between the time the particle stays in an interval and the probability of finding this particle in it. The rapid divergence may then be explained by the fact that the probability density decreases rapidly in the classically forbidden regions. As an example, in the harmonic oscillator case, the probability density decreases as \( \exp(-2\alpha x^2) \).

Let us now consider the classically allowed region. The general idea which emerges from the previous sections is that, to each classical trajectory, we can associate a family of quantum trajectories which can be specified by the different values of the non-classical integration constants \( a \) and \( b \). These quantum trajectories oscillate about their corresponding classical one which contains some points called nodes through which pass all the trajectories of the family. Since the nodes are obtained in the \((t, x)\) plane, the time the particle takes to go from one node to another is the same for all possible trajectories, even for the classical one.

In the constant potential case, the existence of these nodes is shown with an analytical method. We have seen that they are strongly linked to the zeros of the function appearing in the denominator of the expression of the reduced action \( S_0 \). We can also check graphically that the obtained nodes in the linear potential and the harmonic oscillator cases correspond to turning points or to zeros of the Schrödinger solution used in the denominator appearing in the expression of \( S_0 \). This strongly suggests that for any potential, we will obtain nodes in these particular points. Furthermore, from Eq.(6), we see that for any potential, the velocity does not depend on the values of \( b \) at the zeros of \( \phi_2 \).

On the other hand, in the constant potential case, we showed that the distance on the \( x \) axis between two adjacent nodes is a constant given by expression (13). This distance is related to de Broglie’s wavelength

\[
\lambda = \frac{h}{p} \tag{41}
\]

by

\[
\Delta x_n = \frac{\lambda}{2} \tag{42}
\]

In (41), \( p \) is the classical momentum

\[ p = mv \tag{43} \]

Note that \( v \) can be considered as the classical velocity or as the mean velocity of any quantum trajectory between the two nodes. In fact, by using (12) and (13), we have

\[
v = \frac{\Delta x_n}{\Delta t_n} = \sqrt{\frac{2\epsilon}{m}} \tag{44}
\]

It is important to observe that \( p \) also represents the average of the quantum conjugate momentum along one interval separating two nodes. In fact, taking into account (10), (11) and (13), and using (2) to determine \( S_0 \) in the case where \( V(x) = V_0 \), we can deduce that

\[
\left\langle \frac{\partial S_0}{\partial x} \right\rangle = \frac{1}{\Delta x_n} \int_{x(t_n)}^{x(t_{n+1})} \frac{\partial S_0}{\partial x} \, dx = \frac{S_0(x(t_{n+1})) - S_0(x(t_n))}{\Delta x_n} = \sqrt{2m\epsilon} \tag{45}
\]
which is equal to \( p \) with the use of (43) and (44). This result suggests strongly and in a natural way, that for any potential we define a new wavelength associated to any interval between two adjacent nodes as in (41) except that \( p \) must be substituted by

\[
p = \left\langle \frac{\partial S_0}{\partial x} \right\rangle.
\]  

(46)

Therefore, by using expression (2) for \( S_0 \) to average \( \partial S_0/\partial x \) between two adjacent zeros of \( \phi_2 \), we obtain for any potential

\[
p = \frac{\pi \hbar}{\Delta x},
\]  

(47)

\( \Delta x \) being the length between the two zeros. It also represents the length between the two corresponding nodes. Substituting (47) in (41), we obtain

\[
\Delta x = \frac{\lambda}{2},
\]  

(48)

as it is for the constant potential case, Eq. (42). This relation gives the link between the length separating adjacent nodes and the new wavelength as defined by (41) and (46). We stress that we do not associate any wave to our particle motion but we just keep the terminology introduced by de Broglie.

Taking into account (41) and (48), the previous conclusion implies that the distance between adjacent nodes is also proportional to \( \hbar \), as it is in the constant potential case. We deduce therefore that for any potential in the classical limit \( \hbar \to 0 \), the adjacent nodes become infinitely close. As in the constant potential case, this finding implies that the quantum trajectories tend to be identical to their corresponding classical one. In fact, since

- the particular expression for \( x(t) \) is not used in our reasoning for the constant potential in Section 2,
- Eq. (6) indicates that the function \( x(t) \) is monotonous between two adjacent nodes for any potential,
- in the classical limit \( (\hbar \to 0) \), the classical trajectory between two adjacent nodes can be assimilated to an infinitesimal straight segment,

our reasoning in Section 2 can be easily generalized for any potential. Now, we can assert that, for any potential, the classical limit \( (\hbar \to 0) \) of any quantum trajectory is the classical trajectory. This conclusion is compatible with the fact that the quantum equations of motion, Eq. (5), the FIQNL (Eq. (7)) and even the QSHJE (Eq. (1)), become all identical to their corresponding classical equations in the limit \( \hbar \to 0 \). It will not be logical if the quantum time equations do not have as a limit the classical equations when \( \hbar \to 0 \), while the quantum equations of motion have as a limit the classical ones.

An important quantity to determine is the Ermakov invariant \[18, 19\]. In the context of Schrödinger’s equation, this invariant has been first introduced by Floyd \[13\] and later written by Faraggi-Matone \[14\] as

\[
I = \frac{1}{\sqrt{2m}} \left[ \frac{\partial S_0}{\partial x} \right]^2 \psi_E^2 + \hbar^2 \left[ \frac{1}{2} \left( \frac{\partial S_0}{\partial x} \right)^{-3/2} \frac{\partial^2 S_0}{\partial x^2} \psi_E + \left( \frac{\partial S_0}{\partial x} \right)^{-1/2} \frac{\partial \psi_E}{\partial x} \right]^2 \right],
\]  

(49)
where $\psi_E$ is the physical solution of Schrödinger's equation. Of course $\psi_E$ can be written as

$$
\psi_E = \alpha \phi_1 + \beta \phi_2 ,
$$

(50)

where $\alpha$ and $\beta$ are complex constants. With the use of (4) and (50), we can show that (49) leads to

$$
I = \frac{\hbar}{\alpha \sqrt{2m}} \left[ \alpha^2 + (a\beta - b\alpha)^2 \right].
$$

(51)

It is clear that $I$ is an invariant.

Another important question which we must investigate concerns the link between the nodes and the zeros of the function $\phi_2$: do quantum trajectories depend on the choice of $\phi_2$? In other words, we are afraid that the mathematical choices may affect our physics results.

In order to answer this crucial question, let us consider a new set of real solutions of Schrödinger’s equation, Eq. (3),

$$
\theta_1 = \mu \phi_1 + \nu \phi_2,
$$

(52)

$$
\theta_2 = \alpha \phi_1 + \beta \phi_2.
$$

(53)

We suppose that the real parameters $(\mu, \nu, \alpha, \beta)$ satisfy the condition $\mu \beta - \nu \alpha \neq 0$ in such a way as to guarantee the fact that $\theta_1$ and $\theta_2$ must be independent. Let us look for the existence of a couple of parameters $(\tilde{a}, \tilde{b})$ with which the reduced action takes the form

$$
S_0 = \hbar \arctan \left[ \frac{\theta_1}{\theta_2} + \tilde{b} \right] + \hbar \tilde{l},
$$

(54)

as in (2), and from which we deduce the same equation of motion, Eq. (6). For this purpose, let us apply the fundamental relation (5) in which we substitute $S_0$ by expression (54). Taking into account relations (52) and (53), we obtain

$$
\frac{dx}{dt} = \pm \frac{2[E - V(x)]}{\hbar W} \left[ \frac{\mu^2 \tilde{a}^2 + 2\mu \nu \tilde{a} \tilde{b} + \alpha^2(1 + \tilde{b}^2)}{(\mu \beta - \nu \alpha)\tilde{a}} \phi_1^2 + 2\mu \nu \tilde{a}^2 + (\mu \beta + \nu \alpha)\tilde{a} \tilde{b} + \alpha \beta(1 + \tilde{b}^2) \phi_1 \phi_2 \right. \\
+ \left. \frac{\nu^2 \tilde{b}^2 + 2\beta \nu \tilde{a} \tilde{b} + \beta^2(1 + \tilde{b}^2)}{(\mu \beta - \nu \alpha)\tilde{a}} \phi_2^2 \right],
$$

(55)

where we have used the fact that the Wronskian $W$ of $(\theta_1, \theta_2)$ is related to the one of $(\phi_1, \phi_2)$, $W$, by $W = (\mu \beta - \nu \alpha)W$. Equation of motion (55) is identical to (6) if and only if

$$
a = \frac{\mu^2 \tilde{a}^2 + 2\mu \nu \tilde{a} \tilde{b} + \alpha^2(1 + \tilde{b}^2)}{(\mu \beta - \nu \alpha)\tilde{a}},
$$

(56)

$$
b = \frac{\mu \nu \tilde{a}^2 + (\mu \beta + \nu \alpha)\tilde{a} \tilde{b} + \alpha \beta(1 + \tilde{b}^2)}{(\mu \beta - \nu \alpha)\tilde{a}},
$$

(57)
\[
\frac{1 + b^2}{a} = \frac{\nu^2 a^2 + 2 \beta a b + \beta^2 (1 + b^2)}{(\mu \beta - \nu \alpha a)}. \tag{58}
\]

The parameters \( \tilde{a} \) and \( \tilde{b} \) can be determined from (56) and (57). On the other hand, if we substitute expressions (56) and (57) for \( a \) and \( b \) in (58), we find that (58) represents an identity. This means that (58) is compatible with (56) and (57). Therefore, for any couple \((\theta_1, \theta_2)\) defined by \((\mu, \nu, \alpha, \beta)\), it is always possible to get parameters \((\tilde{a}, \tilde{b})\) with which we reproduce the same quantum motion as the one given by (6) which we deduce from the reduced action (2). In conclusion, the mathematical choices of \((\phi_1, \phi_2)\) do not affect the physics results.

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