The average number of elements in the 4-Selmer groups of elliptic curves is 7

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1 Introduction

Any elliptic curve $E$ over $\mathbb{Q}$ is isomorphic to a unique curve of the form $E_{A,B} : y^2 = x^3 + Ax + B$, where $A, B \in \mathbb{Z}$ and for all primes $p$: \( p^6 \mid B \) whenever $p^4 \mid A$. Let $H(E_{A,B})$ denote the (naive) height of $E_{A,B}$, defined by $H(E_{A,B}) := \max\{4|A^3|, 27B^2\}$.

In previous papers ([6] and [7]), we showed that the average size of the 2-Selmer group of all elliptic curves over $\mathbb{Q}$, when ordered by height, is 3; meanwhile the average size of the 3-Selmer group is 4. The purpose of this article is to prove an analogous result for the average size of the 4-Selmer group of all elliptic curves over $\mathbb{Q}$. Specifically, we prove the following theorem:

**Theorem 1** When all elliptic curves $E/\mathbb{Q}$ are ordered by height, the average size of the 4-Selmer group $S_4(E)$ is equal to 7.

We will in fact prove a stronger version of Theorem 1 where we compute the average size of the 4-Selmer group of elliptic curves satisfying any finite set of congruence conditions:

**Theorem 2** When elliptic curves $E : y^2 = x^3 + Ax + B$ over $\mathbb{Q}$, in any family defined by finitely many congruence conditions on the coefficients $A$ and $B$, are ordered by height, the average size of the 4-Selmer group $S_4(E)$ is 7.

We will also prove an analogue of Theorem 2 for certain families of elliptic curves defined by infinitely many congruence conditions (e.g., the family of all semistable elliptic curves).

Since we have shown in [6] that the average number of elements in the 2-Selmer groups of elliptic curves over $\mathbb{Q}$ is 3, we may use Theorem 1 to prove that a positive proportion of 2-Selmer elements of elliptic curves do not lift to 4-Selmer elements:

**Theorem 3** For an elliptic curve $E$ over $\mathbb{Q}$, let $\times 2 : S_4(E) \to S_2(E)$ denote the multiplication-by-2 map. Then, when elliptic curves $E$ over $\mathbb{Q}$ are ordered by height, the average number elements in the 2-Selmer group of $E$ that have no preimage under $\times 2$ is at least $3/5 > 0$.

It follows, in particular, that a positive proportion (in fact, at least one fifth) of all 2-Selmer elements of elliptic curves $E$ over $\mathbb{Q}$, when such $E$ are ordered by height, correspond to nontrivial 2-torsion elements of the Tate–Shafarevich group $\text{III}_E$ of $E$. Another consequence is that there exist infinitely many elliptic curves $E$ over $\mathbb{Q}$ with trivial rational 2-torsion for which the 2-primary part of the group $\text{III}_E$ contains $\mathbb{Z}/2\mathbb{Z}$ as a factor.

As we will explain, Theorems 1 and 2 and the methods of their proofs, lead naturally to the following conjecture on the average size of the $n$-Selmer group of elliptic curves for general $n$:
Conjecture 4 Let $n$ be any positive integer. Then, when all elliptic curves $E$ are ordered by height, the average size of the $n$-Selmer group $S_n(E)$ is $\sigma(n)$, the sum of the divisors of $n$.

Thus the conjecture is proven for $n = 2$, $n = 3$, and $n = 4$ (and also for $n = 1$!). We will prove Conjecture 4 for $n = 5$ in [8]. This paper represents the first time that the average size of the $n$-Selmer group has been determined for a composite value of $n$.

Conjecture 4 also has consequences for the distribution of ranks of elliptic curves. Since $\epsilon n^2$ grows faster than $\sigma(n)$, as a function of $n$, for any $\epsilon > 0$, we obtain:

Proposition 5 Suppose that Conjecture 4 is true for all $n$, or indeed, any infinite sequence of positive integers $n$. Then when all elliptic curves over $\mathbb{Q}$ are ordered by height, a density of 100% have rank $\leq 1$.

The parity conjecture states that an elliptic curve has even rank if and only if its root number is 1. Hence the above proposition has the following consequence:

Corollary 6 Suppose that Conjecture 4 is true for all $n$, or any infinite sequence of positive integers $n$. Further assume that the root numbers of elliptic curves are equidistributed and that the parity conjecture holds. Then when elliptic curves are ordered by height, 50% have rank 0 and 50% have rank 1.

Thus our results on Selmer groups above give independent theoretical evidence for the elliptic curve rank distribution conjecture, due to Goldfeld [22] and Katz–Sarnak [24] (see also [2] for a nice survey), which states that 50% of all elliptic curves have rank 0 and 50% rank 1.

Our method for proving Theorem 1 is as follows. We view $n$-Selmer elements of an elliptic curve $E$ as locally soluble $n$-coverings of $E$. Here, an $n$-covering of $E$ is a genus one curve $C/\mathbb{Q}$ together with maps $\phi : C \to E$ and $\theta : C \to E$, where $\phi$ is an isomorphism defined over $\mathbb{C}$, and $\theta$ is a degree $n^2$ map defined over $\mathbb{Q}$, such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{P}^n & \overset{[n]}{\longrightarrow} & \mathbb{P}^n \\
\phi \downarrow & & \downarrow \theta \\
C & \rightarrow & E
\end{array}
$$

An $n$-covering $C$ is said to be locally soluble if $C$ has points defined over $\mathbb{R}$ and over $\mathbb{Q}_p$ for all primes $p$. Cassels [14] proved that any locally soluble $n$-covering has a degree $n$ divisor defined over $\mathbb{Q}$, yielding an embedding of $C$ into $\mathbb{P}^{n-1}$ defined over $\mathbb{Q}$. We may thus represent $n$-Selmer elements of elliptic curves as genus one normal curves in $\mathbb{P}^{n-1}$. When $n = 4$, as is well known, any such genus one curve in $\mathbb{P}^{n-1} = \mathbb{P}^3$ arises naturally as the complete intersection of a pair of quadrics in $\mathbb{P}^3$, where the two quadrics are well-defined up to appropriate changes-of-basis. Indeed, it turns out that 4-Selmer elements of an elliptic curve $E_{A,B}$ over $\mathbb{Q}$ may naturally be viewed in terms of the “locally soluble” orbits of $G_\mathbb{Q}$ on $V_\mathbb{Q}$, where $G$ is the algebraic group such that

$$
G_R := \{(g_2, g_4) \in \text{GL}_2(R) \times \text{GL}_4(R) : \det(g_2) \det(g_4) = 1\}/\{(\lambda^{-2}I_2, \lambda I_4) : \lambda \in R^\times\}
$$

for all rings $R$, and $V$ is the representation $2 \otimes \text{Sym}^2(4)$ of pairs of quadrics (see [5, §4.3] for the reasons behind this choice of group $G_R$). The invariant ring for the representation of $G_\mathbb{C}$ on $V_\mathbb{C}$ turns out to be freely generated by two invariants, which naturally correspond to the invariants $A$ and $B$ of
the Jacobian elliptic curve \( E_{A,B} \) of the associated genus one curve in \( \mathbb{P}^3 \). These classical connections among orbits on pairs of quadrics, genus one normal curves in \( \mathbb{P}^3 \), and explicit 4-descent on elliptic curves over global fields were fully developed in recent years in a series of beautiful works by An, Kim, Marshall, Marshall, McCallum, and Perlis \[1\], Siksek \[30\], Merriman, Siksek, and Smart \[24\], Womack \[32\], and Fisher \[20\] \[21\]. Furthermore, it is a theorem of Cremona, Fisher, and Stoll \[15\] that the orbit of \( G_\mathbb{Q} \) on \( V_\mathbb{Q} \) corresponding to any 4-Selmer element of \( E_{A,B} \) always contains an element of \( V_\mathbb{Z} \) having invariants exactly \( A \) and \( B \) (up to bounded powers of 2 and 3).

To prove Theorem 1 we are thus reduced to counting suitable orbits of \( G_\mathbb{Z} \) on \( V_\mathbb{Z} \), where a counting method involving the geometry-of-numbers, developed in \[3\], \[4\], and \[6\], may be applied. The method involves counting lattice points, in fundamental domains for the action of \( G_\mathbb{Z} \) on \( V_\mathbb{R} \), corresponding to elliptic curves of bounded height. The difficulty, as in \[6\], lies in dealing with the cusps of these fundamental domains. In the case at hand, a number of suitable adaptations to the method of \[6\] are required. For example, the geometry of the cusps of the fundamental domains is considerably more complicated than that in \[6\]. In addition, the method requires a count of elements having squarefree discriminant, which again necessitates a technique that is quite different than that used in \[6\] (but is closer to that used in \[7\]); this is perhaps the most technical ingredient of the paper.

The end result of the method, however, is quite simple to state. Namely, we show that the average occurring in Theorem 1 arises naturally as the sum of two contributions. One comes from the main body of the fundamental domains, which corresponds to the average number of elements in the 4-Selmer group having exact order 4; we show that this average is given by the Tamagawa number \( \tau(G_\mathbb{Q}) = \tau(\text{PGL}_4(\mathbb{Q})) = 4 \). The other comes from the cusps of the fundamental domains, which corresponds to the average number of elements in the 4-Selmer group having order strictly less than 4. This latter contribution is equal to the average size of the 2-Selmer group, which is 3 by the work of \[6\]. The sum \( 4 + 3 = 7 \) then yields the average size of the 4-Selmer group, as stated in Theorem 1. (This also explains why, in general, we expect the average size of the \( n \)-Selmer group to be \( \sigma(n) \). Namely, by the analogous reasoning, we expect the average number of order \( n \) elements in the \( n \)-Selmer group to equal \( n \), the Tamagawa number of \( \text{PGL}_n \); summing over the divisors of \( n \) yields Conjecture \[4\].)

In Section 2, we recall the parametrization of elements of the 4-Selmer groups of elliptic curves by orbits of \( G_\mathbb{Z} \) on \( V_\mathbb{Z} = \mathbb{Z}^2 \otimes \text{Sym}^2(\mathbb{Z}^4) \), collecting the necessary results from \[1\], \[5\], and especially \[15\]. In Section 3, we then adapt the methods of \[4\] and \[6\] to count the number of \( G_\mathbb{Z} \)-orbits on \( V_\mathbb{Z} \) of bounded height. In Section 4, by developing a suitable sieve, we then count just those elements that correspond to 4-Selmer elements of exact order 4 in appropriate congruence families of elliptic curves having bounded height. Combined with the average size of the 2-Selmer group in such congruence families as determined in \[6\], this is then used to deduce Theorems 1, 2, and 3.

2 Pairs of quaternary quadratic forms and 4-coverings of elliptic curves

Let \( E : y^2 = x^3 + Ax + B \) be an elliptic curve over \( \mathbb{Q} \), where \( A \) and \( B \) are integers such that, for all primes \( p \), we have \( p^6 \nmid B \) if \( p^4 \mid A \). We define the quantities \( I(E) \) and \( J(E) \) by

\[
I(E) := -3A, \quad J(E) := -27B.
\]

In this section, we collect results relating 4-coverings of elliptic curves to certain orbits on pairs
of quaternary quadratic forms. For our applications, we need to consider not just elliptic curves over \( \mathbb{Q} \), but also elliptic curves over other fields such as \( \mathbb{R} \) and \( \mathbb{Q}_p \). For any ring \( R \) of characteristic 0 (or prime to 6), let \( V_R \) denote the space of pairs of quaternary quadratic forms with coefficients in \( R \). We always identify quadratic forms with their Gram-matrices, and write elements \((A, B) \in V_R\) as pairs of \(4 \times 4\) symmetric matrices with

\[
2 \cdot (A, B) = \begin{pmatrix}
2a_{11} & a_{12} & a_{13} & a_{14} \\
a_{12} & 2a_{22} & a_{23} & a_{24} \\
a_{13} & a_{23} & 2a_{33} & a_{34} \\
a_{14} & a_{24} & a_{34} & 2a_{44}
\end{pmatrix}, \quad
\begin{pmatrix}
b_{11} & b_{12} & b_{13} & b_{14} \\
b_{12} & 2b_{22} & b_{23} & b_{24} \\
b_{13} & b_{23} & 2b_{33} & b_{34} \\
b_{14} & b_{24} & b_{34} & 2b_{44}
\end{pmatrix}
\]

(2)

where \(a_{ij}\) and \(b_{ij}\) are elements of \( R \).

The group \( \text{GL}_2(R) \times \text{GL}_4(R) \) acts naturally on \( V_R \): an element \( g_2 = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \text{GL}_2(R) \) acts via \( g_2 \cdot (A, B) = (rA + sB, tA + uB) \) while an element \( g_4 \in \text{GL}_4(R) \) acts via \( g_4 \cdot (A, B) = (g_4A_1g_4^t, g_4B_1g_4^t) \). It is clear that the actions of \( g_2 \) and \( g_4 \) commute. Also note that the element \((\lambda^{-2}I_2, \lambda I_4)\) acts trivially on \( V_R \), where \( \lambda \in R^\times \) and \( I_n \) denotes the identity element in \( \text{GL}_n(R) \). We thus obtain a faithful action of \( G_R \) on \( V_R \), where \( G_R \) is the group

\[
G_R := \{(g_2, g_4) \in \text{GL}_2(R) \times \text{GL}_4(R) : \det(g_2) \det(g_4) = 1\}/\{\lambda^{-2}I_2, \lambda I_4 : \lambda \in R^\times\}. \tag{3}
\]

We now describe the ring of invariants for the action of \( G_C \) on \( V_C \). If \((A, B) \in V_C\), we define the binary quartic resolvent form \( f_{A,B} \) of \((A, B)\) by

\[
f_{A,B}(x, y) := 2^4 \det(Ax + By). \tag{4}
\]

If \((A', B') = (g_2, g_4) \cdot (A, B)\) for \((g_2, g_4) \in G_C\), then one checks the identity

\[
f_{A',B'}(x, y) = \det(g_4)^2 f_{A,B}((x, y) \cdot g_2) = \frac{f_{A,B}((x, y) \cdot g_2)}{\det(g_2)^2}. \tag{5}
\]

The action of \( \text{PGL}_2(\mathbb{C}) \) on the space of binary quartic forms over \( \mathbb{C} \), defined by [5], has a free ring of invariants, generated by two elements traditionally denoted by \( I \) and \( J \). (See, e.g., [6] Equation (4) for the definitions of \( I \) and \( J \).) Thus the quantities \( I \) and \( J \) defined by

\[
I(A, B) := I(f_{A,B}), \quad J(A, B) := J(f_{A,B}) \tag{6}
\]

are also invariant, under the action of \( G_C \) on \( V_C \), and in fact they freely generate the full ring of invariants for this action. We may use the above definitions of \( f_{A,B}, I(A, B), \) and \( J(A, B) \) for elements \((A, B) \in V_R\), where \( R \) is any ring. Note that since \( I(f) \) and \( J(f) \) are polynomials having degrees 2 and 3, respectively, in the coefficients of \( f \), the polynomials \( I(A, B) \) and \( J(A, B) \) have degrees 8 and 12, respectively, in the coefficients of \((A, B)\).

The significance of the action of \( G_R \) on \( V_R \) may be seen from the following three propositions. For a field \( K \), we say that \((A, B) \in V_K\) is \( K \)-soluble if the quadrics defined by \( A \) and \( B \) have a \( K \)-rational point of intersection in \( \mathbb{P}^3 \). Then we have:

**Proposition 7** Let \( K \) be a field having characteristic not 2 or 3. Let \( E : y^2 = x^3 - \frac{t}{44}x - \frac{t}{17} \) be an elliptic curve over \( K \). Then there exists a bijection between elements in \( E(K)/4E(K) \) and \( G_K \)-orbits of \( K \)-soluble elements in \( V_K \) having invariants equal to \( I \) and \( J \). Under this bijection, a
$G_K$-orbit $G_K \cdot (A, B)$ corresponds to an element in $E(K)/4E(K)$ having order less than 4 if and only if the binary quartic resolvent form of $(A, B)$ has a linear factor over $K$.

Furthermore, the stabilizer in $G_K$ of any (not necessarily $K$-soluble) element in $V_K$, having nonzero discriminant and invariants $I$ and $J$, is isomorphic to $E(K)[4]$, where $E$ is the elliptic curve defined by $y^2 = x^3 - \frac{1}{3}x - \frac{1}{27}$.

**Proof:** The first and third assertions of the proposition, concerning the bijection and the stabilizer, follow immediately from [1] and [5, §4.3]. For the second assertion, regarding the elements of $E(K)/4E(K)$ having order less than 4, [1, §3.3] states that if $C_4 \rightarrow E$ is the 4-covering of $E$ corresponding to $(A, B)$, then it factors through a 2-covering $C_2$ of $E$, i.e., we have maps $C_4 \rightarrow C_2 \rightarrow E$, where $C_2 \rightarrow E$ is the 2-covering corresponding to the binary quartic resolvent form of $(A, B)$ via [1, §3.1]. Hence $(A, B)$ corresponds to an element having order less than 4 if and only if its binary quartic resolvent form corresponds to a trivial element in $E(K)/2E(K)$, i.e., it has a linear factor over $\mathbb{Q}$ [16, Proposition 2.2].

An element $(A, B) \in V_\mathbb{Q}$ is said to be *locally soluble* if it is $\mathbb{R}$-soluble and $\mathbb{Q}_p$-soluble for all primes $p$. We similarly then obtain the following proposition:

**Proposition 8** Let $E : y^2 = x^3 - \frac{1}{3}x - \frac{1}{27}$ be an elliptic curve over $\mathbb{Q}$. Then there exists a bijection between elements in the 4-Selmer group of $E$ and $G_\mathbb{Q}$-orbits on locally soluble elements in $V_\mathbb{Q}$ having invariants equal to $I$ and $J$.

Furthermore, if $(A, B)$ has invariants $I$ and $J$, then the $G_\mathbb{Q}$-orbit $G_\mathbb{Q} \cdot (A, B)$ corresponds to an element in $S_4(E)$ having order less than 4 if and only if the binary quartic resolvent form of $(A, B)$ has a rational linear factor.

By the work of Cremona, Fisher, and Stoll [15, Theorem 1.1], any locally soluble element $(A, B) \in V_\mathbb{Q}$ having integral invariants $I$ and $J$ is $\text{GL}_2(\mathbb{Q}) \times \text{GL}_4(\mathbb{Q})$-equivalent to an integral element $(A', B') \in V_\mathbb{Z}$ having the same invariants $I$ and $J$. In particular, it follows that such an $(A, B)$ is $G_\mathbb{Q}$-equivalent to either $(A', B')$ or $(A', -B')$. Since $(A', B')$ and $(A', -B')$ have the same invariants, we obtain the following proposition:

**Proposition 9** Let $E/\mathbb{Q}$ be an elliptic curve. Then the elements in the 4-Selmer group of $E$ are in bijective correspondence with $G_\mathbb{Q}$-equivalence classes on the set of locally soluble elements in $V_\mathbb{Z}$ having invariants equal to $I(E)$ and $J(E)$.

Furthermore, under this correspondence, elements of exact order 4 correspond to the $G_\mathbb{Q}$-equivalence classes whose binary quartic resolvent forms have no rational linear factor.

Motivated by Propositions 7–9, we say that an element of $V_\mathbb{Z}$ (or $V_\mathbb{Q}$) is *strongly irreducible* if its binary quartic resolvent form does not possess a rational linear factor. Thus to count the number of 4-Selmer elements of elliptic curves having bounded invariants, we wish to count the number of $G_\mathbb{Q}$-equivalence classes of strongly irreducible elements in $V_\mathbb{Z}$ having bounded invariants. In the next section, we begin by first determining the asymptotic number of $G_\mathbb{Z}$-equivalence classes.

### 3 The number of $G_\mathbb{Z}$-classes of strongly irreducible pairs of integral quaternary quadratic forms having bounded invariants

For $i \in \{0, 1, 2\}$, let $V_\mathbb{R}^{(i)}$ denote the set of elements $(A, B) \in V_\mathbb{R}$ such that the binary quartic resolvent form $f_{A,B}(x,y) := 2^4 \det(Ax + By)$ has nonzero discriminant, $i$ pairs of complex conjugate
roots in \( \mathbb{P}^1 \mathbb{C} \), and thus \( 4 - 2i \) roots in \( \mathbb{P}^1 \mathbb{R} \). It follows from [30] Lemma 6.2.2 that every element in \( V^{(1)}_\mathbb{Z} \) and \( V^{(2)}_\mathbb{Z} \) is \( \mathbb{R} \)-soluble. However, this is not the case for all elements in \( V^{(0)}_\mathbb{Z} \); we denote the set of \( \mathbb{R} \)-soluble elements in \( V^{(0)}_\mathbb{R} \) by \( V^{(0#)}_\mathbb{R} \). Let \( V^{(i)}_\mathbb{Z} := V^{(0)}_\mathbb{Z} \cap V^{(i)}_\mathbb{R} \) for \( i \in \{0, 1, 2, 0#\} \). Then the action of \( G^\mathbb{Z} \) on \( V^\mathbb{Z} \) preserves also the sets \( V^{(i)}_\mathbb{Z} \).

The invariants \( I(A, B) \) and \( J(A, B) \) of \( (A, B) \in V^\mathbb{Z} \) are as defined in [3]. We then define the discriminant and the height of \( (A, B) \) having invariants \( I \) and \( J \) as follows:

\[
\Delta(A, B) := \Delta(f_{A,B}) = \Delta(I, J) := (4I^3 - J^2)/27; \\
H(A, B) := H(f_{A,B}) := H(I, J) := \max\{|I^3|, J^2/4\}.
\]

Equation (7) yields an expression for the discriminant \( \Delta(A, B) \) that is an integer polynomial of degree 24 in the entries of \( A \) and \( B \). We use (7) as the definition of the discriminant of elements in \( V^\mathbb{R} \) for any ring \( R \), and as the definition of the height of elements in \( V^\mathbb{R} \).

Our purpose in this section is to count the number of strongly irreducible \( G^\mathbb{Z} \)-orbits on \( V^{(i)} \) having bounded height for \( i \in \{0#, 1, 2\} \). To state the precise result we need some further notation. For any \( G^\mathbb{Z} \)-invariant set \( S \subseteq V^\mathbb{Z} \), let \( N(S; X) \) denote the number of \( G^\mathbb{Z} \)-equivalence classes on \( S^{irr} \) having height less than \( X \), where \( S^{irr} \) is used to denote the set of strongly irreducible elements of \( S \). Let \( N^+(X) \) (resp. \( N^-(X) \)) denote the number of integer pairs \((I, J)\) satisfying \( \Delta(I, J) > 0 \) (resp. \( \Delta(I, J) < 0 \)) and \( H(I, J) < X \). By [3] Proposition 2.10], we have

\[
N^\pm(X) = \frac{8}{5} X^{5/6} + O(X^{1/2}); \\
N^{\pm}(X) = \frac{32}{5} X^{5/6} + O(X^{1/2}).
\]

Let \( \omega \) be a fixed algebraic nonzero top-degree left-invariant differential form on \( G \) such that, for every prime \( p \), the measure of \( G^\mathbb{Z}_p \) computed with respect to \( \omega \) is \( \#G^\mathbb{Z}_p/p^{\dim G} = \#G^\mathbb{F}_p/p^{18} \). There is a natural map \( G^\mathbb{R} \times R^{(i)} \to V^{(i)}_\mathbb{R} \) given by \((\gamma, x) \mapsto \gamma \cdot x \), where the sets \( R^{(i)} \subseteq V^\mathbb{R} \) are defined just after (10). We will see in Section 3.3 that the Jacobian change of variables of this map (computed with respect to the measure on \( G^\mathbb{R} \) obtained from \( \omega \), the measure \( dIdJ \) on \( R^{(i)} \), and the Euclidean measure on \( V^\mathbb{R} \) normalized so that \( V^\mathbb{Z} \) has covolume 1) is a nonzero rational constant independent of \( i \). Henceforth, we will denote this constant by \( J \).

The aim of this section is to prove the following theorem:

**Theorem 10** We have:

(a) \( N(V^{(1)}_\mathbb{Z}; X) = \frac{1}{4} |J| \cdot \Vol(G^\mathbb{Z}\setminus G^\mathbb{R})N^-(X) + o(X^{5/6}) \);

(b) \( N(V^{(i)}_\mathbb{Z}; X) = \frac{1}{8} |J| \cdot \Vol(G^\mathbb{Z}\setminus G^\mathbb{R})N^+(X) + o(X^{5/6}) \) for \( i = 0# \) and 2,

where the volume of \( G^\mathbb{Z}\setminus G^\mathbb{R} \) is computed with respect to the measure obtained from \( \omega \).

The value of \( J \) is not difficult to compute, but is irrelevant for the proofs of Theorems 1 and 2 because of its cancellation in (31).
3.1 Reduction theory

In this subsection, we construct certain finite covers of fundamental domains for the action of $G_Z$ on $V_R^{(i)}$ for $i \in \{0\# , 1, 2\}$. We start by constructing fundamental sets for the action of $G_R$ on $V_R^{(i)}$, for $i \in \{0\# , 1, 2\}$. The following result is a consequence of Proposition 7 along with the fact that every element in $V_R^{(0\#)}$, $V_R^{(1)}$ and $V_R^{(2)}$ is $R$-soluble.

Proposition 11 Let $(I, J)$ be an element of $\mathbb{R} \times \mathbb{R}$ such that $\Delta(I, J) \neq 0$. Then

1. If $\Delta(I, J) < 0$, then the set of elements in $V_R$ having fixed invariants $I$ and $J$ consists of one $R$-soluble $G_R$-orbit. The size of the stabilizer in $G_R$ of any element in this orbit is 4.

2. If $\Delta(I, J) > 0$, then the set of $R$-soluble elements in $V_R$ having fixed invariants $I$ and $J$ consists of two $G_R$-orbits. There is one such orbit from each of $V_R^{(0\#)}$ and $V_R^{(2)}$. The size of the stabilizer in $G_R$ of any element in either of these orbits is 8.

For $i = 0\# , 1, 2$, we choose fundamental sets $R^{(i)} \subset V_R^{(i)}$ for the action of $G_R$ on $V_R^{(i)}$ as follows. Let $f^{(i)}_{I, J}$ be the forms constructed in [6, Table 1], for $i = 0, 1, 2$. Then for each $(I, J) \in \mathbb{R} \times \mathbb{R}$ with $\Delta(I, J) > 0$ (resp. $\Delta(I, J) < 0$) and $H(I, J) = 1$, we obtain two binary quartic forms $f^{(0)}_{I, J}$ and $f^{(2)}_{I, J}$ (resp. one binary quartic form $f^{(1)}_{I, J}$) having invariants $I$ and $J$. The coefficients of all these forms $f^{(i)}_{I, J}$ are bounded independently of $I$ and $J$. We write

$$f^{(0)}_{I, J} = \kappa y(x + \lambda_1 y)(x + \lambda_2 y)(x + \lambda_3 y),$$
$$f^{(1)}_{I, J} = \kappa y(x + \lambda y)(x^2 + r y^2),$$
$$f^{(2)}_{I, J} = \kappa(x^2 + r_1 y^2)(x^2 + r_2 y^2),$$

with $\kappa > 0$, $\lambda_1 > \lambda_2 > \lambda_3$, $r > 0$ and $r_1 > r_2 > 0$.

Consider the sets

$$L^{(0\#)} = \left\{ \kappa^{1/4} \begin{pmatrix} 0 & -1 \\ -1 & \frac{\lambda_1}{\lambda_2} \\ \frac{1}{\lambda_3} & \frac{1}{\lambda_4} \end{pmatrix} \right\},$$

$$L^{(1)} = \left\{ \kappa^{1/4} \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} \right\},$$

$$L^{(2)} = \left\{ \kappa^{1/4} \begin{pmatrix} 1 & \frac{1}{r_1} \\ \frac{1}{r_2} & \frac{1}{r_3} \end{pmatrix} \right\}.$$

Since the coefficients of the forms $f^{(i)}_{I, J}$ are bounded independently of $I$ and $J$, the coefficients of the elements in $L^{(0\#)}$, $L^{(1)}$, and $L^{(2)}$ are also bounded independently of $I$ and $J$.

Let $R^{(i)}$ be defined to be $\mathbb{R}_{>0} \cdot L^{(i)}$. The sets $R^{(i)}$ then satisfy the following two properties that we use throughout this section:
1. The sets $R^{(i)}$ are subsets of $V_{\mathbb{R}}^{(i)}$ for $i = 0\#, 1,$ and 2. Furthermore, $R^{(0\#)}$ and $R^{(2)}$ (resp. $R^{(1)}$) contain exactly one point having invariants $I$ and $J$ for each pair $(I, J) \in \mathbb{R} \times \mathbb{R}$ with $\Delta(I, J) > 0$ (resp. $\Delta(I, J) < 0$).

2. For $i \in \{0\#, 1, 2\}$, the coefficients of all the elements of height $X$ in $R^{(i)}$ are bounded by $O(X^{1/24})$.

To verify that $R^{(i)} \subseteq V_{\mathbb{R}}^{(i)}$, it suffices to show that the elements in $L^{(i)}$ are soluble over $\mathbb{R}$. For $(A, B) \in L^{(0\#)}$, this follows by applying [30] Theorem 6.3.1 on $(A + \epsilon B, B)$ for sufficiently small $\epsilon$, and for $(A, B) \in L^{(i)}$ with $i = 1, 2$ this follows from [30] Lemma 6.2.2. The second part of the first assertion is immediate from our choices of the $f_{I, J}$’s. The second assertion follows from the fact that the height of $(A, B)$ is a homogeneous function of degree 24 in the coefficients of $A$ and $B$.

Let $F$ denote a fundamental domain in $G_{\mathbb{R}}$ for the left action of $G_{\mathbb{Z}}$ on $G_{\mathbb{R}}$ that is contained in a standard Siegel set [11, §2]. We may assume that $F = \{nak : n \in N'(a), a \in A', k \in K\}$, where

$$
K = \{\text{subgroup of orthogonal transformations } \text{SO}_2(\mathbb{R}) \times \text{SO}_4(\mathbb{R}) \subset G_{\mathbb{R}}\};
A' = \{a(s_1, s_2, s_3, s_4) : s_1 > c_1; s_2, s_3, s_4 > c_2\},
$$

where

$$
a(s_1, s_2, s_3, s_4) = \begin{bmatrix} s_1^{-1} & 0 \\ 0 & s_1 \end{bmatrix}, \begin{bmatrix} s_2^2 s_3^2 & s_3^{-1} s_4^{-1} \\ s_3 s_4 & s_2^{-2} s_3^2 \end{bmatrix}, \begin{bmatrix} s_2 s_3 s_4 & 0 \\ 0 & s_2 s_3 s_4 \end{bmatrix};
$$

$$
N' = \{n(u_1, \ldots, u_7) : (u_i) \in \nu(a)\},
$$

where

$$
n(u) = \begin{bmatrix} 1 & 0 \\ u_1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & u_2 & 1 \\ u_3 & u_4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & u_5 & u_6 & u_7 & 1 \end{bmatrix};
$$

here $\nu(a)$ is a bounded and measurable subset of $[-1/2, 1/2]^7$ depending only on $a \in A'$, and $c_1, c_2 > 0$ are absolute constants.

Fix $i \in \{0\#, 1, 2\}$. For $h \in G_{\mathbb{R}}$, we regard $Fh \cdot R^{(i)}$ as a multiset, where the multiplicity of an element $v \in V_{\mathbb{R}}$ is equal to $\#\{g \in F : v \in gh \cdot R^{(i)}\}$. As in [6, §2.1], it follows that for any $h \in G_{\mathbb{R}}$ and any $v \in V_{\mathbb{R}}^{(i)}$, the $G_{\mathbb{Z}}$-orbit of $v$ is represented $m(v)$ times in $Fh \cdot R^{(i)}$, where

$$
m(v) := \#\text{Stab}_{G_{\mathbb{Z}}}(v)/\#\text{Stab}_{G_{\mathbb{Z}}}(v).
$$

That is, the sum of the multiplicity in $Fh \cdot R^{(i)}$ of $v'$, over all $v'$ that are $G_{\mathbb{Z}}$-equivalent to $v$, is equal to $m(v)$.

The set of elements in $V_{\mathbb{R}}^{(i)}$ that have a nontrivial stabilizer in $G_{\mathbb{Z}}$ has measure 0 in $V_{\mathbb{R}}^{(i)}$. Thus, by Proposition [11] for any $h \in G_{\mathbb{R}}$ the multiset $Fh \cdot R^{(i)}$ is an $n_1$-fold cover of a fundamental domain for the action of $G_{\mathbb{Z}}$ on $V_{\mathbb{R}}^{(i)}$, where $n_1 = 4$ and $n_{0\#} = n_2 = 8$.

It follows that if we let $R^{(i)}(X)$ denote the set of elements in $R^{(i)}$ having height bounded by $X$, then for any $G_{\mathbb{Z}}$-invariant set $S \subset V_{\mathbb{Z}}$, the product $n_1 N(S^{\text{irr}}; X)$ is equal to the number of elements in $Fg \cdot R^{(i)}(X) \cap S^{\text{irr}}$, with the slight caveat that the (relatively rare—see Proposition [21]) elements with $G_{\mathbb{Z}}$-stabilizers of size $r$ ($r > 1$) are counted with weight $1/r$.

Counting strongly irreducible integer points in a single such region $Fg \cdot R^{(i)}(X)$ is difficult because it is an unbounded region. As in [6], we simplify the counting by suitably averaging over a continuous range of elements $g$ lying in a compact subset of $G_{\mathbb{R}}$.
3.2 Averaging and cutting off the cusp

Throughout this section, we let $dg$ denote the Haar measure on $G_{\mathbb{R}}$ obtained from its Iwasawa decomposition $G_{\mathbb{R}} = NAK$ normalized in the following way: for $g = n a k$ with $n = n(u_1, \ldots, u_7) \in N$, $a = a(s_1, \ldots, s_4) \in A$, and $k \in K$, we set

$$dg = s_1^{-2} s_2^{-12} s_3^{-8} s_4^{-12} \prod_i du_i d^x s_1 d^x s_2 d^x s_3 d^x s_4 dk,$$

where $d^x s$ denotes $s^{-1} ds$ and $dk$ is Haar measure on $K$ normalized so that $\int_K dk = 1$.

Let $G_0 \subset G_{\mathbb{R}}$ be a compact, semialgebraic, left $K$-invariant subset that is the closure of some nonempty open set in $G_{\mathbb{R}}$. Fix $i$ to be equal to $0\#$, $1$, or $2$. Then, by the arguments of \[3.1\] we may write

$$N(S; X) = \frac{\int_{g \in G_0} \#(Fg \cdot R^{(i)}(X) \cap S^{\text{irr}}) dg}{C_{G_0}}, \tag{11}$$

where $C_{G_0} = n_i \int_{g \in G_0} dg$. We use the right hand side of (11) to define $N(S; X)$ also for sets $S \subset V_Z$ that are not necessarily $G_{\mathbb{Z}}$-invariant.

Identically as in [6, Theorem 2.5], the right hand side of (11) is equal to

$$\frac{1}{C_{G_0}} \int_{g \in N(a)A'} \#(S^{\text{irr}} \cap B(n, a; X)) s_1^{-2} s_2^{-12} s_3^{-8} s_4^{-12} du d^x s \tag{12}$$

where $B(n, a; X) := n a G_0 \cdot R^{(i)}(X)$ and $d^x s := d^x s_1 d^x s_2 d^x s_3 d^x s_4$.

To estimate the number of integer points in the bounded multiset $B(n, a; X)$, we use the following proposition due to Davenport.

**Proposition 12** ([17]) Let $R$ be a bounded, semi-algebraic multiset in $\mathbb{R}^n$ having maximum multiplicity $m$, and that is defined by at most $k$ polynomial inequalities each having degree at most $\ell$. Then the number of integer lattice points (counted with multiplicity) contained in the region $R$ is

$$\text{Vol}(R) + O(\max\{\text{Vol}(\bar{R}), 1\}),$$

where $\text{Vol}(\bar{R})$ denotes the greatest $d$-dimensional volume of any projection of $R$ onto a coordinate subspace obtained by equating $n - d$ coordinates to zero, where $d$ takes all values from $1$ to $n - 1$. The implied constant in the second summand depends only on $n$, $m$, $k$, and $\ell$.

Proposition 1[2] yields a good estimate on the number of integer points in $B(n, a; X)$ when the $s_i$’s ($a = a(s_1, s_2, s_3, s_4)$) are bounded by a small power of $X$ (we shall make this more precise in what follows). Our next aim is to show that when one of the $s_i$’s is large relative to $X$, the set $B(n, a; X)$ has very few strongly irreducible integer points. To this end, we first give conditions that guarantee that an element in $V_Z$ is not strongly irreducible.

**Lemma 13** Let $(A, B)$ be a point in $V_Z$ expressed in the form (2), and suppose that one of the following four conditions is satisfied:

1. $a_{11} = a_{12} = a_{13} = a_{14} = 0$;
2. $a_{11} = a_{12} = a_{13} = a_{22} = a_{23} = 0$;
(3) \( a_{11} = a_{12} = a_{13} = b_{11} = b_{12} = b_{13} = 0; \)

(4) \( a_{11} = a_{12} = a_{22} = b_{11} = b_{12} = b_{22} = 0. \)

Then \((A, B)\) is not strongly irreducible.

**Proof:** In the first two cases, we see that \(\det(A) = 0\). This implies that the \(x^4\)-coefficient of \(f(x, y)\) is equal to zero; hence \(f(x, y)\) is reducible over \(\mathbb{Q}\) and \((A, B)\) is not strongly irreducible.

In the last two cases, the binary quartic resolvent form \(f(x, y)\) of \((A, B)\) has a multiple root over \(\mathbb{Q}\). Thus, the discriminant \(\Delta(A, B) = \Delta(f)\) of \((A, B)\) is equal to zero and so again \((A, B)\) is not strongly irreducible. \(\square\)

Next, note that the action of \(a(s_1, s_2, s_3, s_4)\) on \((A, B) \in V_{\mathbb{R}}\) scales each coordinate \(t = a_{ij}\) or \(b_{ij}\) of \(V_{\mathbb{R}}\) by a rational function \(w(t)\) in the \(s_i\)’s. We define the weight of a product of such coordinates to be the product of the weights of these coordinates. Then evidently the size of the coordinate \(t\) of an element in \(B(n, a; X)\) is \(O(X^{1/24}w(t))\). For example, we have \(w(a_{11}) = s_1^{-1}s_2^{-6}s_3^{-2}s_4^{-2}\), and so if \((A, B) = ((a_{ij}), (b_{ij})) \in B(n, a; X)\), then \(a_{11} = O(X^{1/24}w(a_{11}))\). We now have the following lemma:

**Lemma 14** Let \(na(s_1, s_2, s_3, s_4) \in N'(a)A'\) be such that \(V_{Z}^{\text{irr}} \cap B(n, a; X)\) is nonempty. Then \(s_i = O(X^{1/24})\) for \(i \in \{1, \ldots, 4\}\).

**Proof:** Let \(na\) be an element satisfying the hypothesis of the lemma. Since \(B(n, a; X)\) contains an integral point \((A, B)\) not satisfying any of the four conditions of Lemma 13, we see that \(X^{1/24}w(t) \gg 1\) for \(t = a_{14}, a_{23}, b_{13}, \) and \(b_{22}\). Thus, we obtain the following four estimates:

\[
(1) \frac{s_1s_2^2}{s_4^3} = O(X^{1/24}), \quad (2) \frac{s_1s_4^2}{s_2} = O(X^{1/24}), \quad (3) \frac{s_2s_3^2}{s_1} = O(X^{1/24}), \quad (4) \frac{s_2s_4^2}{s_1s_2} = O(X^{1/24}).
\]

Multiplying the first two estimates immediately yields \(s_1 = O(X^{1/24})\). Using this bound on \(s_1\) and the third estimate then gives \(s_2 = O(X^{1/24})\) and \(s_4 = O(X^{1/24})\). Finally, multiplying the first and fourth estimates gives \(s_3 = O(X^{1/24})\), completing the proof of the lemma. \(\square\)

We now prove the following estimate which bounds the number of strongly irreducible points in \(\mathcal{F}g \cdot R^{(4)}(X) \cap V_Z\) that have \(a_{11} = 0\), as we average over \(g \in G_0\). More precisely:

**Lemma 15** We have

\[
\int_{g \in N'(a)A'} \# \{(A, B) \in \mathcal{V}^{\text{irr}}_Z \cap B(n, a; X) : a_{11} = 0\} s_1^{-2}s_2^{-12}s_3^{-8}s_4^{-12}du \, d^sX = O(X^{19/24}).
\]

**Proof:** The proof of this lemma is very similar to that of [1] Lemma 11]. We partition the set \(\{(A, B) \in \mathcal{V}^{\text{irr}}_Z : a_{11} = 0\}\) into fourteen subsets defined by setting certain coordinates of \((A, B) \in \mathcal{V}^{\text{irr}}_Z\) equal to zero and certain other coordinates to be nonzero. These sets are listed in the second column of Table 1 and it follows from Lemma 13 that they do indeed form a partition.

For any subset \(T \subset V_Z\), let us define \(N^*(T, X)\) by

\[
N^*(T, X) := \int_{g \in N'(a)A'} \# \{T \cap B(n, a; X)\} dg.
\]

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Then Lemma 14, together with the bound $N^*(T, X) = O(X^{19/24})$ for the fourteen sets $T$ listed in Table 1, imply Lemma 15.

We now describe how the required bound on $N^*(T, X)$ may be obtained for Cases 1, 2a, and 3a of Table 1. In Case 1, we have

$$N^*(T, X) = O\left(\int_{g \in N'(a)A'} \frac{X^{20/24}}{X^{1/24}w(a_{11})} s_1^{-2} s_2^{-12} s_3^{-8} s_4^{-12} du \, d^x s \right)$$

$$= O\left(X^{19/24} \int_{g \in N'(a)A'} s_1^{-1} s_2^{-6} s_3^{-6} s_4^{-10} du \, d^x s \right).$$

Since the $s_i$'s are bounded from below, we obtain the required bound.

| Case | The set $T \subset V_{2\pi}^*$ defined by | $N^*(T, X) \ll$ | Use factor |
|------|-----------------------------------------------|------------------|-----------|
| 1    | $a_{11} = 0$  \hspace{1cm} \hspace{1cm} $a_{12}, b_{11} \neq 0$ | $X^{19/24}$ | -         |
| 2a   | $a_{11}, a_{12} = 0$  \hspace{1cm} \hspace{1cm} $a_{13}, a_{22}, b_{11} \neq 0$ | $X^{18/24+\varepsilon}$ | -         |
| 2b   | $a_{11}, b_{11} = 0$  \hspace{1cm} \hspace{1cm} $a_{12} \neq 0$ | $X^{18/24+\varepsilon}$ | -         |
| 3a   | $a_{11}, a_{12}, a_{13} = 0$  \hspace{1cm} \hspace{1cm} $a_{14}, a_{22}, b_{11} \neq 0$ | $X^{18/24+\varepsilon}$ | $a_{14}$ |
| 3b   | $a_{11}, a_{12}, a_{22} = 0$  \hspace{1cm} \hspace{1cm} $a_{13}, b_{11} \neq 0$ | $X^{18/24+\varepsilon}$ | $a_{13}$ |
| 3c   | $a_{11}, a_{12}, b_{11} = 0$  \hspace{1cm} \hspace{1cm} $a_{13}, a_{22}, b_{12} \neq 0$ | $X^{18/24+\varepsilon}$ | $a_{13}^2$ |
| 4a   | $a_{11}, a_{12}, a_{13}, a_{22} = 0$  \hspace{1cm} \hspace{1cm} $a_{14}, a_{23}, b_{11} \neq 0$ | $X^{18/24+\varepsilon}$ | $a_{14}^3$ |
| 4b   | $a_{11}, a_{12}, a_{13}, b_{11} = 0$  \hspace{1cm} \hspace{1cm} $a_{14}, a_{22}, b_{12} \neq 0$ | $X^{18/24+\varepsilon}$ | $a_{14}^2 b_{12}$ |
| 4c   | $a_{11}, a_{12}, a_{22}, b_{11} = 0$  \hspace{1cm} \hspace{1cm} $a_{13}, b_{12} \neq 0$ | $X^{18/24+\varepsilon}$ | -         |
| 4d   | $a_{11}, a_{12}, b_{11}, b_{12} = 0$  \hspace{1cm} \hspace{1cm} $a_{13}, a_{22} \neq 0$ | $X^{18/24+\varepsilon}$ | $a_{13}^2$ |
| 5a   | $a_{11}, a_{12}, a_{13}, a_{22}, b_{11} = 0$  \hspace{1cm} \hspace{1cm} $a_{14}, a_{23}, b_{12} \neq 0$ | $X^{18/24+\varepsilon}$ | $a_{14}$ |
| 5b   | $a_{11}, a_{12}, a_{13}, b_{11}, b_{12} = 0$  \hspace{1cm} \hspace{1cm} $a_{14}, a_{22}, b_{13} \neq 0$ | $X^{18/24+\varepsilon}$ | $a_{14}^2 b_{13}$ |
| 5c   | $a_{11}, a_{12}, a_{22}, b_{11}, b_{12} = 0$  \hspace{1cm} \hspace{1cm} $a_{13}, b_{22} \neq 0$ | $X^{18/24+\varepsilon}$ | $a_{13}^2 b_{22}$ |
| 6    | $a_{11}, a_{12}, a_{13}, a_{22}, b_{11}, b_{12} = 0$  \hspace{1cm} \hspace{1cm} $a_{14}, a_{23}, b_{13}, b_{22} \neq 0$ | $X^{18/24+\varepsilon}$ | $a_{14}^2 b_{13} b_{22}$ |

Table 1: Estimates on the number of strongly irreducible points in cuspidal regions
Similarly, in Case 2a, we have

\[ N^*(T, X) = O\left( \int_{g \in \mathcal{N}'(a)A'} \frac{X^{20/24} \cdot X^{1/24} w(a_{14})}{X^{3/24} w(a_{11}) w(a_{12}) w(a_{13})} s_1^{-2}s_2^{-12}s_3^{-8}s_4^{-12} du d^x s \right) \]

\[ = O\left( X^{18/24} \int_{g \in \mathcal{N}'(a)A'} s_2^{-4}s_3^{-4}s_4^{-4} du d^x s \right). \]

Again, since the \( s_i \) are bounded by \( O(X^{1/24}) \), we obtain the required bound \( N^*(T, X) = O(X^{18/24+\epsilon}) \).

Finally, in Case 3a, we have

\[ N^*(T, X) = O\left( \int_{g \in \mathcal{N}'(a)A'} \frac{X^{20/24}}{X^{3/24} w(a_{11}) w(a_{12}) w(a_{13})} s_1^{-2}s_2^{-12}s_3^{-8}s_4^{-12} du d^x s \right) \]

\[ = O\left( X^{18/24} \int_{g \in \mathcal{N}'(a)A'} s_2^{-4}s_3^{-4}s_4^{-4} du d^x s \right), \]

where the second equality follows by multiplying the integrand by \( X^{1/24} \) times the weight of the “factor” listed in the fourth column of Table 1. This yields an upper bound for the desired integral since the factor (which is an integer) was assumed to be nonzero, and therefore is at least 1 in absolute value; hence the corresponding weight must also be bounded from below by an absolute positive constant. As in Case 2a, we obtain the required bound \( N^*(T, X) = O(X^{18/24+\epsilon}) \).

The proof of the bound for the other eleven cases are identical. This concludes the proof of Lemma 15. \( \square \)

We have proven that the number of irreducible elements in the “cuspidal regions” of the fundamental domain is negligible. The next lemma states that the number of reducible elements in the “main body” of the fundamental domain is negligible:

**Lemma 16** With notation as above, we have

\[ \int_{g \in \mathcal{N}'(a)A'} \#\{(A, B) \in V^\text{red}_Z \cap B(n, a; X) : a_{11} \neq 0\} dg = o(X^{5/6}), \]

where \( V^\text{red}_Z \) denotes the set of elements in \( V_Z \) that are not strongly reducible.

Therefore, in order to estimate \( N(V_Z; X) \), it suffices to count the total number of (not necessarily strongly irreducible) integral points in the main body of the fundamental domain. We do this in the following proposition:

**Proposition 17** With notation as above, we have

\[ \frac{1}{C_{G_0}} \int_{g \in \mathcal{N}'(a)A'} \#\{(A, B) \in V_Z \cap B(n, a; X) : a_{11} \neq 0\} dg = \frac{1}{n_i} \text{Vol}(\mathcal{F} \cdot R^{(i)}(X)) + o(X^{5/6}), \]

where the volume of sets in \( V_Z \) is computed with respect to Euclidean measure normalized so that \( V_Z \) has covolume 1.
**Proof:** The proof of Proposition 17 is similar to that of [4, Proposition 12]. If \( v \in B(n, a; X) \), then we know that \( a_{12}(v) = O(X^{1/60} w(a_{12})) \). Thus, from Propositions 15 and 16 we obtain

\[
\frac{1}{C_{G_0}} \int_{n a \in F} \# \{ B(n, a; X) \cap V_{Z}^{irr} \} dI dJ = \frac{1}{C_{G_0}} \int_{n a \in F} \# \{ B(n, a; X) \cap V_{Z} \} dI dJ + o(X^{5/6}).
\]

Since \( a_{11} \) has minimal weight, and the projection of \( B(n, a; X) \) onto the \( a_{11} \)-axis has length greater than an absolute positive constant when \( X^{1/24} w(a_{11}) \gg 1 \), Proposition 12 implies that the main term on the right hand side of (13) is equal to

\[
\frac{1}{C_{G_0}} \int_{X^{1/24} w(a_{11}) \gg 1} \text{Vol}(B(n, a; X)) dI dJ. \tag{14}
\]

Since the region \( \{ n a \in F : w(a_{11}) \ll X^{\epsilon} \} \) has volume \( o(1) \) for any fixed \( \epsilon \), (14) is equal to

\[
\frac{1}{C_{G_0}} \int_{n a \in F} \text{Vol}(B(n, a; X)) dI dJ + o(X^{5/6}).
\]

The proposition follows since

\[
\frac{1}{C_{G_0}} \int_{n a \in F} \text{Vol}(B(n, a; X)) dI dJ = \frac{1}{C_{G_0}} \int_{h \in G_0} \text{Vol}(F h \cdot R^{\pm}(X)) dh,
\]

and the volume of \( F h \cdot R^{\pm}(X) \) is independent of \( h \). ∎

Lemmas 15 and 16 and Proposition 17 imply that, up to an error of \( o(X^{5/6}) \), the quantity \( n_i \cdot N(V_{Z}^{irr}; X) \) is equal to the volume of \( F \cdot R^{(i)}(X) \) for \( i = 0\#, 1, \) and 2. In the next section, we obtain a useful expression for this volume.

### 3.3 Computing the volume

Recall that at the beginning of Section 3, we fixed an algebraic nonzero top-degree left invariant differential form \( \omega \) on \( G \) such that for all primes \( p \), the measure of \( G_{Z_{p}} \) with respect to \( \omega \) is \#\( G_{Z_{p}} / P_{18} \). Let \( dv \) denote Euclidean measure on \( V_{R} \) normalized so that \( V_{Z} \) has covolume 1. Finally, note that for \( i = 0\#, 1, \) and 2, the sets \( R^{(i)} \) contain at most one point \( p_{I, J} \) having invariants \( I \) and \( J \) for any pair \( (I, J) \in R \times R \). We choose \( dIdJ \) to be the measure on \( R^{(i)} \).

With these measure normalizations, we have the following proposition whose proof is identical to that of [6, Proposition 2.8]:

**Proposition 18** For any measurable function \( \phi \) on \( V_{R} \), we have

\[
|J| \cdot \int_{p_{I, J} \in R^{(i)}} \int_{h \in G_{R}} \phi(h \cdot p_{I, J}) \omega(h) dIdJ = \int_{G_{R} \cdot R^{(i)}} \phi(v) dv = n_i \int_{V_{R}^{(i)}} \phi(v) dv, \tag{15}
\]

where \( J \) is a nonzero constant in \( Q \) independent of \( i \).

Using Proposition 18, it is easy to compute the volume of the multiset \( F \cdot R^{(i)}(X) \):

\[
\int_{F \cdot R^{(i)}(X)} dv = |J| \cdot \int_{p_{I, J} \in R^{(i)}(X)} \int_{F} \omega(h) dI dJ = |J| \cdot \text{Vol}(F) \int_{R^{(i)}(X)} dI dJ. \tag{16}
\]
Up to an error of $O(X^{1/2})$, the quantity $\int_{R^{(i)}} (X)\ dI\ dJ$ is equal to $N^+(X)$ when $i = 0# \text{ or } 2$, and $N^-(X)$ when $i = 1$ (see the proof of [6, Proposition 2.10] for details).

We conclude that

$$N(V^{(i)}_Z; X) = \frac{1}{4} |J| \cdot \text{Vol}(G_Z \backslash G_{\mathbb{A}}) N^-(X) + o(X^{5/6}),$$

$$N(V^{(i)}_Z; X) = \frac{1}{8} |J| \cdot \text{Vol}(G_Z \backslash G_{\mathbb{A}}) N^+(X) + o(X^{5/6}),$$

(17)

for $i = 0# \text{ and } 2$. We thus obtain Theorem 10.

### 3.4 Congruence conditions

In this subsection, we present a version of Theorem 10 in which we count pairs of integral quaternary quadratic forms satisfying any finite set of congruence conditions.

For any set $S$ in $V_Z$ that is definable by congruence conditions, let us denote by $\mu_p(S)$ the $p$-adic density of the $p$-adic closure of $S$ in $V_{Z_p}$, where we normalize the additive measure $\mu_p$ on $V_{Z_p}$ so that $\mu_p(V_{Z_p}) = 1$. We then have the following theorem whose proof is identical to that of [6, Theorem 2.11].

**Theorem 19** Suppose $S$ is a subset of $V^{(i)}_Z$ defined by congruence conditions modulo finitely many prime powers. Then we have

$$N(S \cap V^{(i)}_Z; X) = N(V^{(i)}_Z; X) \prod_p \mu_p(S) + o(X^{5/6}),$$

(18)

where $\mu_p(S)$ denotes the $p$-adic density of $S$ in $V_Z$, and where the implied constant in $o(X^{5/6})$ depends only on $S$.

We furthermore have the following weighted version of Theorem 19 whose proof is identical to that of [6, Theorem 2.12].

**Theorem 20** Let $p_1, \ldots, p_k$ be distinct prime numbers. For $j = 1, \ldots, k$, let $\phi_{P_j} : V_Z \to \mathbb{R}$ be bounded $G_Z$-invariant functions on $V_Z$ such that $\phi_{P_j}(v)$ depends only on the congruence class of $v$ modulo some power $p_j^{\alpha_j}$ of $p_j$. Let $N_{\phi}(V^{(i)}_Z; X)$ denote the number of strongly irreducible $G_Z$-orbits in $V^{(i)}_Z$ having height bounded by $X$, where each orbit $G_Z \cdot v$ is counted with weight $\phi(v) := \prod_{j=1}^k \phi_{P_j}(v)$. Then we have

$$N_{\phi}(V^{(i)}_Z; X) = N(V^{(i)}_Z; X) \prod_{j=1}^k \int_{v \in V_{Z_{p_j}}} \tilde{\phi}_{P_j}(v) \, dx + o(X^{5/6}),$$

(19)

where $\tilde{\phi}_{P_j}$ is the natural extension of $\phi_{P_j}$ to $V_{Z_{p_j}}$ by continuity, $dv$ denotes the additive measure on $V_{Z_{p_j}}$ normalized so that $\int_{v \in V_{Z_{p_j}}} dv = 1$, and where the implied constant in the error term depends only on the local weight functions $\phi_{P_j}$.

### 3.5 The number of reducible points and points with large stabilizers in the main bodies of the fundamental domains is negligible

In this section we prove Lemma 16, which states that the number of integral elements that are not strongly irreducible in the main body of the fundamental domain is negligible. We then prove that
the number of strongly irreducible $G_\mathbb{Z}$-orbits on elements with a nontrivial stabilizer in $G_\mathbb{Q}$ having bounded height is negligible.

**Proof of Lemma 16:** An element $(A, B) \in V_\mathbb{Z}$ with $\Delta(A, B) \neq 0$ fails to be strongly irreducible if and only if the binary quartic resolvent form $f_{A,B}(x, y) = 16 \det(Ax + By)$ has a root in $\mathbb{P}^1_\mathbb{Q}$.

Let $p > 3$ be prime. If $f(x, y)$ has a root in $\mathbb{P}^1_\mathbb{Q}$, then the reduction of $f(x, y)$ modulo $p$ has a root in $\mathbb{P}^1_\mathbb{F}_p$. We construct elements $(A, B) \in V_\mathbb{F}_p$, for a positive density family of primes $p$, such that $f_{A,B}(x, y)$ has no root in $\mathbb{F}^1(\mathbb{F}_p)$.

Let $p$ be a prime congruent to 3 modulo 4 such that there exists an element $s \in \mathbb{F}_p$ satisfying $s^2 = -2$. Consider the pair

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}. \tag{20}$$

We have $\det(Ax + By) = x^4 + y^4$, implying that $f_{A,B}(x, y)$ has no root defined over $\mathbb{F}_p$. Therefore, if the reduction modulo $p$ of $(A, B) \in V_\mathbb{Z}$ is $G_\mathbb{F}_p$-equivalent to any $\mathbb{F}_p^\times$-multiple of the right hand side of (20), then $(A, B)$ is not strongly irreducible. Since $\# \{ g \cdot \lambda \cdot (A, B) : g \in G_\mathbb{F}_p, \lambda \in \mathbb{F}_p^\times \} \gg V_\mathbb{F}_p/p$, we obtain

$$\int_{g \in N' \lambda(a) A'} \# \{ (A, B) \in V_\mathbb{F}_p^{\text{red}} \cap B(n, a; X) : a_{11} \neq 0 \} dg = O \left( X^{5/6} \prod_{p \equiv 3 \pmod{4}} (1 - p^{-1}) \right)$$

for any $Y > 0$. Letting $Y \to \infty$ yields Lemma 16. \qed

**Proposition 21** The number of $G_\mathbb{Z}$-orbits on elements on $V_\mathbb{Z}$ that are strongly irreducible, have height bounded by $X$, and have a nontrivial stabilizer in $G_\mathbb{Q}$ is $o(X^{5/6})$.

**Proof:** Proposition 17 implies that an element $(A, B) \in V_\mathbb{Z}$ having invariants $I$ and $J$ has a nontrivial stabilizer in $G_\mathbb{Q}$ if and only if the elliptic curve $E : y^2 = g_{A,B}(x) = x^3 - \frac{I}{3} - \frac{J}{27}$ contains a nontrivial 4-torsion point over $\mathbb{Q}$, which happens exactly when $g(x)$ has a rational root.

Let $p$ be a prime congruent to 1 modulo 3. Let $t \in \mathbb{F}_p$ be an element having no solution $a^3 = t$ for $a \in \mathbb{F}_p$. Consider the pair $(A, B)$ given by

$$2(A, B) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -t \end{pmatrix}. \tag{21}$$

We have $16 \det(Ax + By) = x^3y - ty^4$, implying that $g_{A,B}(x, y) = x^3 - ty^3$ is irreducible over $\mathbb{F}_p$.

Therefore, if the reduction modulo $p$ of $(A, B) \in V_\mathbb{Z}$ is $G_\mathbb{F}_p$-equivalent to the right hand side of (21) for any prime $p$, then $(A, B)$ has a trivial stabilizer in $G_\mathbb{Q}$. Proposition 21 now follows from Lemma 15 and an argument identical to the proof of Lemma 16. \qed
3.6 Tail estimates and a squarefree sieve

In order to prove Theorems 1 and 2, we require a stronger version of Theorem 20: one which counts weighted $G_2$-orbits where the weights are defined by congruence conditions modulo infinitely many prime powers. In this subsection, we use the methods and results of [9] to prove the necessary result.

We start with the following two definitions. A function $\phi : V \to [0, 1] \subset \mathbb{R}$ is said to be defined by congruence conditions if, for all primes $p$, there exist functions $\phi_p : V_{zp} \to [0, 1]$ satisfying the following conditions:

(1) For all $(A, B) \in V$, the product $\prod_p \phi_p(A, B)$ converges to $\phi(A, B)$.

(2) For each prime $p$, the function $\phi_p$ is locally constant outside some closed set $S_p \subset V_{zp}$ of measure zero.

Such a function $\phi$ is called acceptable if, for sufficiently large primes $p$, we have $\phi_p(A, B) = 1$ whenever $p^2 \nmid \Delta(A, B)$.

Then we will prove the following theorem:

**Theorem 22** Let $\phi : V \to [0, 1]$ be an acceptable function that is defined by congruence conditions via the local functions $\phi_p : V_{zp} \to [0, 1]$. Then, with notation as in Theorem 20, we have:

$$N_{\phi}(V(i)_p; X) = N(V(i)_p; X) \prod_p \int_{v \in V_{zp}} \phi_p(v) \, dv + o(X^{5/6}). \quad (22)$$

For a prime $p$, let $W_p$ denote the set of elements in $V$ whose discriminants are divisible by $p^2$.

The key ingredient needed to prove Theorem 22 is the following tail estimate:

**Theorem 23** Let $\epsilon > 0$ be fixed. Then for any $i \in \{0#, 1, 2\}$ we have:

$$N(\cup_{p > M} W_p, X) = O(\epsilon X^{5/6}/(M \log M) + X^{19/24}) + O(\epsilon X^{5/6}). \quad (23)$$

**Proof:** Let $W_p^{(1)}$ denote the set of elements in $(A, B) \in V$ whose discriminants are divisible by $p^2$ for (mod $p$) reasons; i.e., $p^2$ divides the discriminant of $(A, B) + p(A', B')$ for every $(A', B') \in V$. For $\epsilon > 0$, let $\mathcal{F}^{(\epsilon)} \subset \mathcal{F}$ denote the subset of elements $na(s_1, s_2, s_3, s_4)k \in \mathcal{F}$ such that the $s_i$ are bounded above by an appropriate constant to ensure that $\text{Vol}(\mathcal{F}^{(\epsilon)}) = (1 - \epsilon)\text{Vol}(\mathcal{F})$. Then $\mathcal{F}^{(\epsilon)} \cdot R^{(i)}(X)$ is a bounded domain in $V_\mathbb{R}$ that expands homogeneously with $X$. By [9, Theorem 3.3], we obtain

$$\#\{\mathcal{F}^{(\epsilon)} \cdot R^{(i)}(X) \cap (\cup_{p > M} W_p^{(1)})\} = O(X^{5/6}/(M \log M) + X^{19/24}). \quad (24)$$

Also, the results of (3.1) and (3.2) imply that

$$\#\{(\mathcal{F} \setminus \mathcal{F}^{(\epsilon)}) \cdot R^{(i)}(X) \cap V_{\mathbb{Z}}^{\text{irr}}\} = O(\epsilon X^{5/6}). \quad (25)$$

Combining the estimates (24) and (25) yields (23) with $W_p$ replaced with $W_p^{(1)}$.

Next, let $(A, B)$ be an element of $W_p^{(2)} := W_p \setminus W_p^{(1)}$ for some prime $p > 2$. By definition, $v_p(\Delta(A, B)) = v_p(\Delta(f))$, where $f = f_{A,B}$ is the binary quartic resolvent form of $(A, B)$. Thus $p^2$ divides the discriminant of $f$, and since $(A, B) \notin W_p^{(1)}$ we may assume that the reduction of $f$ modulo $p$ contains the square of a linear factor. By replacing $(A, B)$ with a $G_2$-translate, if necessary, we may further assume that $p^2$ divides the $x^2$-coefficient of $f(x, y)$ and $p$ divides the $x^3 y$-coefficient of $f(x, y)$. This condition (along with the fact that $(A, B) \notin W_p^{(1)}$) implies that we may assume $(A, B) = ((a_{ij}), (b_{ij}))$ satisfies the following conditions:
Lemma 24 The map $\phi$ from $G_Z$-orbits on $W_p^{(2)}$ to $G_Z$-orbits on $W_p^{(1)}$ is at most 2 to 1.

Proof: Let $(A, B) \in W_p^{(1)}$ be any element in the image of $\phi$ and let $(\overline{A}, \overline{B})$ denote its reduction modulo $p$. It is easy to see that $\overline{\phi^{-1}}(A, B)$ is integral if and only if $b_{22} \equiv b_{23} \equiv b_{24} \equiv b_{33} \equiv b_{34} \equiv b_{44} \equiv 0 \pmod{p}$. Therefore, the $G_Z$-orbits on $\phi^{-1}(A, B)$ give rise to elements $[r : s] \in \mathbb{P}_p^1$ along with a linear factor of the quadratic form corresponding to $r \overline{A} + s \overline{B}$. If there are two elements in $\mathbb{P}_p^1$ such that the corresponding quadratic forms factor, then $(A, B)$ is $G_Z$-equivalent to $(A_1, B_1)$, where the reductions of $A_1$ and $B_1$ modulo $p$ both factor over $\mathbb{F}_p$. We may thus assume that the bottom $3 \times 3$ submatrix of $B_1$ is congruent to zero modulo $p$. If $(A_2, B_2)$ is $\gamma_p^{-1}(A_1, B_1)$, then we see that the reduction of $A_2$ modulo $p$ also factors over $\mathbb{F}_p$, implying that $(A_2, B_2) \in W_p^{(1)}$. Thus, $(A, B)$ cannot lie in the image of $\phi$ contradicting our hypothesis. Therefore, if $(A, B)$ is in the image of $\phi$, then there is exactly one element $[r : s] \in \mathbb{P}_p^1$ such that the quadratic form corresponding to $r \overline{A} + s \overline{B}$ factors.

We assume without loss of generality that $[r : s] = [0 : 1]$. If $\overline{B}$ has more than two linear factors, then $\overline{B} \equiv 0 \pmod{p}$. Then it is easy to see that $\gamma_p^{-1}(A, B) \in W_p^{(1)}$ because its binary quartic resolvent form is congruent to zero modulo $p$, again contradicting the hypothesis that $(A, B)$ is in the image of $\phi$. This concludes the proof of the lemma. □

Therefore, we obtain

$$N(\cup_{p \geq M} W_p^{(2)}(V), X) \leq 2N(\cup_{p \geq M} W_p^{(1)}(V), X) = O_e(X^{5/6}/(M \log M) + X^{19/24}) + O(eX^{5/6}), \quad (26)$$

and Theorem 23 follows. □

Theorem 22 follows from Theorem 23 just as [6] Theorem 2.21] followed from [6] Theorem 2.13].

4 The average number of elements in the 4-Selmer group of elliptic curves

In this section, we prove Theorems 1 and 2 by computing the average size of the 4-Selmer group of elliptic curves over $\mathbb{Q}$, when these curves are ordered by height. In fact, we prove a generalization of these theorems that allows us to average the size of the 4-Selmer group of elliptic curves whose defining equations satisfy certain acceptable sets of local conditions.
To state the theorem, we need the following definitions. For any elliptic curve \( E \) over \( \mathbb{Q} \), we defined the invariants \( I(E) \) and \( J(E) \) as in (1). Let us denote the elliptic curve having invariants \( I \) and \( J \) by \( E^{I,J} \). Throughout his section we work with the slightly different height \( H' \) on elliptic curves \( E \), defined by

\[
H'(E) := H(I(E), J(E)) = \max\{|I(E)^3|, J(E)^2/4\},
\]

so that the height on elliptic curves agrees with the height on \( V_Z \) defined in (7). Note that since \( H \) and \( H' \) differ by a constant factor of 4/27, they induce the same ordering on the set of (isomorphism classes of) elliptic curves.

For each prime \( p \), let \( \Sigma_p \) be a closed subset of \( Z_p^2 \backslash \{ \Delta = 0 \} \) whose boundary has measure 0. To this collection \( \Sigma = (\Sigma_p)_p \), we associate the family \( F_\Sigma \) of elliptic curves, where \( E^{I,J} \in F_\Sigma \) if and only if \( (I, J) \in \Sigma_p \) for all \( p \). Such a family of elliptic curves over \( \mathbb{Q} \) is said to be defined by congruence conditions. We may also impose “congruence conditions at infinity” on \( F_\Sigma \) by insisting that an elliptic curve \( E^{I,J} \) belongs to \( F_\Sigma \) if and only if \( (I, J) \) belongs to \( \Sigma_\infty \), where \( \Sigma_\infty \) is equal to \( \{(I, J) \in \mathbb{R}^2 : \Delta(I, J) > 0\}, \{(I, J) \in \mathbb{R}^2 : \Delta(I, J) < 0\}, \) or \( \{(I, J) \in \mathbb{R}^2 : \Delta(I, J) \neq 0\} \).

For such a family \( F \) of elliptic curves defined by congruence conditions, let \( \text{Inv}(F) \) denote the set \( \{(I, J) \in \mathbb{Z} \times \mathbb{Z} : E^{I,J} \in F\} \), and let \( \text{Inv}_p(F) \) be the \( p \)-adic closure of \( \text{Inv}(F) \) in \( Z_p^2 \backslash \{ \Delta = 0 \} \). Similarly, we define \( \text{Inv}_\infty(F) \) to be \( \{(I, J) \in \mathbb{R}^2 : \Delta(I, J) > 0\}, \{(I, J) \in \mathbb{R}^2 : \Delta(I, J) < 0\}, \) or \( \{(I, J) \in \mathbb{R}^2 : \Delta(I, J) \neq 0\} \) in accordance with whether \( F \) contains only curves of positive discriminant, negative discriminant, or both, respectively. Such a family \( F \) of elliptic curves is said to be large if, for all but finitely many primes \( p \), the set \( \text{Inv}_p(F) \) contains at least those pairs \( (I, J) \in \mathbb{Z}_p \times \mathbb{Z}_p \) such that \( p^2 \nmid \Delta(I, J) \). Our purpose in this section is to prove the following theorem which generalizes Theorems 1 and 2.

**Theorem 25** Let \( F \) be a large family of elliptic curves. When elliptic curves \( E \) in \( F \) are ordered by height, the average size of the 4-Selmer group \( S_4(E) \) is 7.

### 4.1 Computation of \( p \)-adic densities

Throughout the rest of this section, we fix \( F \) to be a large family of elliptic curves. Proposition 9 asserts that elements in the 4-Selmer group of the elliptic curve \( E^{I,J} \) over \( \mathbb{Q} \) are in bijection with \( G_{\mathbb{Q}} \)-equivalence classes on the set of locally soluble elements in \( V_Z \) having invariants \( I \) and \( J \). Furthermore, elements of exact order 4 in the 4-Selmer group of \( E^{I,J} \) are in bijection with strongly irreducible \( G_{\mathbb{Q}} \)-equivalence classes in the set of locally soluble elements in \( V_Z \) having invariants \( I \) and \( J \).

In Section 2, we computed the asymptotic number of \( G_{\mathbb{Z}} \)-equivalence classes of strongly irreducible elements in \( V_Z \) having bounded height. In order to use this to compute the number of \( G_{\mathbb{Q}} \)-equivalence classes of strongly irreducible locally soluble elements of \( V_Z \) having bounded height and invariants in \( \text{Inv}(F) \), we count each strongly irreducible \( G_{\mathbb{Z}} \)-orbit \( G_{\mathbb{Z}} \cdot x \) weighted by \( \phi(x) \), where \( \phi : V_Z \rightarrow \mathbb{R} \) is a \( G_{\mathbb{Z}} \)-invariant function that we now define.

For \( x \in V_Z \), let \( B(x) \) denote a set of representatives for the action of \( G_{\mathbb{Z}} \) on the \( G_{\mathbb{Q}} \)-equivalence class of \( x \) in \( V_Z \). We define our weight function \( \phi \) to be:

\[
\phi(x) := \begin{cases} 
\left( \sum_{x' \in B(x)} \frac{\# \text{Aut}_\mathbb{Q}(x')} {\# \text{Aut}_\mathbb{Z}(x')} \right)^{-1} & \text{if } x \text{ is locally soluble and } (I(x), J(x)) \in \text{Inv}_p(F) \text{ for all } p; \\
0 & \text{otherwise},
\end{cases}
\]

(27)
where $\text{Aut}_Q(x)$ and $\text{Aut}_G(x)$ denote the stabilizers of $x \in V_Z$ in $G_Q$ and $G_Z$, respectively.

Note that if $x \in V_Z$ has a trivial stabilizer in $G_Q$, it is locally soluble, and satisfies $(I(x), J(x)) \in \text{Inv}(F)$, then $\phi(x) = \#B(x)^{-1}$. Thus, Proposition 21 implies the following result:

**Proposition 26** Let $F$ be a large family of elliptic curves. Following the notation of Theorem 20 and 22, we have

$$\sum_{E \in F, H'(E) < X} \#\{\sigma \in S_4(E) : \sigma^2 \neq 1\} = N_\phi(V_Z; X) + o(X^{5/6}).$$

To evaluate the right hand side of the above equation using Theorem 22, we need to show that the weight function $\phi$ is acceptable in the sense of Section 3.6. To this end, we define local functions $\phi_p : V_{Z_p} \to \mathbb{R}$ as follows. For $x \in V_{Z_p}$, let $B_p(x)$ denote a set of representatives for the action of $G_{Z_p}$ on the $G_{Q_p}$-equivalence class of $x$ in $V_{Z_p}$. Then we define

$$\phi_p(x) := \begin{cases} \left( \sum_{x' \in B_p(x)} \frac{\#\text{Aut}_{Q_p}(x')}{\#\text{Aut}_{Z_p}(x')} \right)^{-1} & \text{if } x \text{ is soluble over } \mathbb{Q}_p \text{ and } (I(x), J(x)) \in \text{Inv}_p(F); \\ 0 & \text{otherwise}, \end{cases} \quad (28)$$

where $\text{Aut}_{Q_p}(x)$ and $\text{Aut}_{Z_p}(x)$ denote the stabilizer of $x \in V_{Z_p}$ in $G_{Q_p}$ and $G_{Z_p}$, respectively. Before we prove that $\phi$ is acceptable, we need the following lemma:

**Lemma 27** For sufficiently large primes $p$, if $(A, B) \in V_{Z_p}$ satisfies $\phi_p(A, B) \neq 1$, then the discriminant of $(A, B)$ is divisible by $p^2$.

**Proof:** Since $F$ is a large family of elliptic curves, we know that for large enough primes $p$, if $(I, J) := (I(A, B), J(A, B)) \notin \text{Inv}_p(F)$, then $p^2 | \Delta(A, B)$.

Now suppose that $(I, J) \in \text{Inv}_p(F)$ but $\phi_p(A, B) \neq 1$. Then either $(A, B)$ is not soluble over $\mathbb{Q}_p$, $\text{Aut}_{Q_p}(A, B)$ is not trivial, or $B_p(A, B)$ has size at least two. Let $C \in \mathbb{P}_F^3$ be the curve cut out by the intersection of the quadrics defined by the reductions of $A$ and $B$ modulo $p$. The Lang-Weil estimates [25] imply that, for sufficiently large primes $p$, either $C$ is geometrically reducible or $C$ has a smooth $\mathbb{F}_p$-point. Thus either $p^2$ divides the discriminant of $(A, B)$ or $(A, B)$ is locally soluble.

Finally, [28, Corollary 2.2] implies that if $(A, B)$ is soluble and either $\text{Aut}_{Q_p}(A, B)$ is nontrivial or $\#B_p(A, B) > 1$, then the reduction type of the elliptic curve $E^I,J$ over $\mathbb{Q}_p$ is not $I_0$ or $I_1$. This implies that $p^2 | \Delta(E^I,J) = \Delta(A, B)$, as desired. $\square$

This leads us to the following proposition:

**Proposition 28** The function $\phi : V_Z^{\text{HT}} \to \mathbb{R}$ is acceptable.

**Proof:** The local weight functions $\phi_p$ are supported and locally constant outside the set of elements in $V_{Z_p}$ having discriminant zero. That $\phi(A, B) = \prod_p \phi_p(A, B)$, for $(A, B) \in V_Z$, follows from an argument identical to the proof of [6, Proposition 3.6] and the fact that the class number of $G_Q$ is 1. Lemma 27 then implies that $\phi$ is acceptable. $\square$

We end the section with a proposition that evaluates $\int_{V_{Z_p}} \phi_p(x)dx.$
Proposition 29 We have
\[ \int_{x \in \mathbb{Z}_p} \phi_p(x) dx = |J_p| \cdot \text{Vol}(G_{\mathbb{Z}_p}) \cdot \int_{(I,J) \in \text{Inv}_p(F)} \frac{\#(E_{I,J}^{p}(\mathbb{Q}_p))/4E_{I,J}^{p}(\mathbb{Q}_p)}{\#(E_{I,J}^{p}(\mathbb{Q}_p)[4])} \]
\[ = \begin{cases} |J_p| \cdot \text{Vol}(G_{\mathbb{Z}_p}) \cdot \text{Vol}(\text{Inv}_p(F)) & \text{if } p \neq 2; \\ 4 \cdot |J_p| \cdot \text{Vol}(G_{\mathbb{Z}_p}) \cdot \text{Vol}(\text{Inv}_p(F)) & \text{if } p = 2, \end{cases} \]

where the volume of \( \text{Inv}_p(F) \subset \mathbb{Z}_p \times \mathbb{Z}_p \) is taken with respect to the additive Haar measure on \( \mathbb{Z}_p \times \mathbb{Z}_p \) normalized so that \( \text{Vol}(\mathbb{Z}_p \times \mathbb{Z}_p) = 1 \).

The first equality in Proposition 29 follows from an argument identical to [6, Proposition 3.9]. The second follows from an argument identical to the proof of [13, Lemma 3.1], yielding that \( \#(E_{I,J}^{p}(\mathbb{Q}_p))/4E_{I,J}^{p}(\mathbb{Q}_p)) \) is equal to \( \#(E_{I,J}^{p}(\mathbb{Q}_p)[4]) \) when \( p \neq 2 \) and equal to \( 4 \#(E_{I,J}^{p}(\mathbb{Q}_p)[4]) \) when \( p = 2 \).

### 4.2 The proof of the main theorem (Theorem 25)

We first state a theorem, proved in [6, Theorem 3.17], that counts the number of elliptic curves having bounded height in a large family \( F \).

**Theorem 30** Let \( F \) be a large family of elliptic curves and let \( N(F; X) \) denote the number of elliptic curves in \( F \) that have height bounded by \( X \). Then

\[ N(F; X) = \text{Vol}(\text{Inv}_\infty(F; X)) \prod_p \text{Vol}(\text{Inv}_p(F)) + o(X^{5/6}), \tag{29} \]

where \( \text{Inv}_\infty(F; X) \) denotes the set of elements in \( \text{Inv}_\infty(F) \) that have height bounded by \( X \).

For any large family \( F \) of elliptic curves over \( \mathbb{Q} \), it follows from Proposition 28 that

\[ \lim_{X \to \infty} \frac{\sum_{E \in F \text{ s.t. } H'(E) < X} \#\{\sigma \in S_4(E) : \sigma^2 \neq 1\}}{\sum_{E \in F \text{ s.t. } H'(E) < X} 1} = \lim_{X \to \infty} \frac{N_\phi(V; X)}{N(F; X)}. \tag{30} \]

Proposition 28 states that \( \phi \) is acceptable. Thus, the right hand side of (30) can be evaluated using Theorems 22 and 30

\[ \lim_{X \to \infty} \frac{N_\phi(V; X)}{N(F; X)} = \lim_{X \to \infty} \frac{\frac{1}{4} |J| \cdot \text{Vol}(G_{\mathbb{Z}} \setminus G_{\mathbb{R}}) \text{Vol}(\text{Inv}_\infty(F; X)) \prod_p \int_{\mathbb{Z}_p} \phi_p(x) dx}{\text{Vol}(\text{Inv}_\infty(F; X)) \prod_p \text{Vol}(\text{Inv}_p(F))} \]
\[ = \frac{|J| \cdot \text{Vol}(G_{\mathbb{Z}} \setminus G_{\mathbb{R}}) \prod_p (|J_p| \cdot \text{Vol}(G_{\mathbb{Z}_p}) \cdot \text{Vol}(\text{Inv}_p(F)))}{\prod_p \text{Vol}(\text{Inv}_p(F))}, \tag{31} \]
where the second equality follows from Proposition 29. Since \( \text{Vol}(G_{\mathbb{Z}_p}) \prod_p \text{Vol}(G_{\mathbb{Z}_p}) \) is equal to the Tamagawa number of \( G_{\mathbb{Q}} \) which is 4 (see \([26]\)), we obtain that

\[
\lim_{X \to \infty} \frac{\sum_{E \in F} \# \{ \sigma \in S_4(E) : \sigma^2 \neq 1 \}}{\sum_{E \in F} 1} = 4.
\]

(32)

Now, for any elliptic curve \( E \) over \( \mathbb{Q} \), the short exact sequence

\[
0 \to E[2] \to E[4] \to E[2] \to 0
\]

yields the long exact sequence

\[
0 \to E[2](\mathbb{Q}) \to E[4](\mathbb{Q}) \to E[2](\mathbb{Q}) \to H^1(\mathbb{Q}, E[2]) \to H^1(\mathbb{Q}, E[4]).
\]

Therefore, if \( E \) has no nontrivial rational 2-torsion points, then the group \( H^1(\mathbb{Q}, E[2]) \) injects into \( H^1(\mathbb{Q}, E[4]) \). This implies that \( S_2(E) \) injects into \( S_4(E) \), and thus

\[
\#S_4(E) = \# \{ \sigma \in S_4(E) : \sigma^2 \neq 1 \} + \#S_2(E).
\]

The number of elliptic curves over \( \mathbb{Q} \) having nontrivial rational 2-torsion and height less than \( X \) is negligible, i.e., is \( o(X^{5/6}) \). That the sum of the sizes of the 4-Selmer groups of such elliptic curves is negligible follows from Proposition 21. Since we have shown in [6, Theorem 3.1] that the average size of the 2-Selmer group of elliptic curves in any large family \( F \) is equal to 3, we obtain from (32) that

\[
\lim_{X \to \infty} \frac{\sum_{E \in F} \#S_4(E)}{\sum_{E \in F} 1} = 4 + 3 = 7.
\]

This concludes the proof of Theorem 25 (and hence also of Theorems 1 and 2).

Finally, to obtain Theorem 3 we note that for an elliptic curve \( E \) over \( \mathbb{Q} \) with no rational 2-torsion, if the 4-Selmer group \( S_4(E) \) is isomorphic to \((\mathbb{Z}/4\mathbb{Z})^a \times (\mathbb{Z}/2\mathbb{Z})^b \), then the 2-Selmer group \( S_2(E) \) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^{a+b}; \) the number of 2-Selmer elements that are not in the image of the \( \times 2 \) map from \( S_4(E) \) to \( S_2(E) \) is thus \( 2^{a+b} - 2^a \) in this case. To prove Theorem 3 we wish to determine a lower bound on the liminf of the average of \( 2^{a+b} - 2^a \) over all elliptic curves \( E \) over \( \mathbb{Q} \) (having trivial rational 2-torsion), when these elliptic curves are ordered by height. Equivalently, we wish to determine an upper bound on the limsup of the average size of \( 2^a \).

We have proven that the average number of order 4 elements in the 4-Selmer groups of these elliptic curves is 4, i.e., the average size of \((4^a - 2^a)2^b \) is 4. It follows that the limsup of the average size of \( 4^a - 2^a \) is at most 4. Since \( 5 \cdot 2^a - 8 \leq 4^a - 2^a \) for all integers \( a > 0 \), we conclude that the limsup of the average size of \( 2^a \) is at most \( 12/5 \). Hence the liminf of the average size of \( 2^{a+b} - 2^a \) is at least \( 3 - 12/5 = 3/5 \); this completes the proof of Theorem 3. (We note that the proof also naturally yields a distribution of 2- and 4-Selmer groups—for which the average sizes of
these groups are given by 3 and 7, respectively—that achieves the bound of 3/5; hence the bound of 3/5 in Theorem 3 is in fact the best possible given these two constraints.)

As a consequence, we see that a proportion of at least \((3/5)/3 = 1/5\) of 2-Selmer elements of elliptic curves \(E\) over \(\mathbb{Q}\), when ordered by height, do not lift to 4-Selmer elements; i.e., we have proven that at least a fifth of all 2-Selmer elements yield nontrivial 2-torsion elements in the corresponding Tate–Shafarevich groups.

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