ON THE RANK OF COMPACT $p$-ADIC LIE GROUPS

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Abstract. The rank of a profinite group $G$ is the basic invariant $$\text{rk}(G) := \sup \{ d(H) \mid H \leq G \},$$ where $H$ ranges over all closed subgroups of $G$ and $d(H)$ denotes the minimal cardinality of a topological generating set for $H$. A compact topological group $G$ admits the structure of a $p$-adic Lie group if and only if it contains an open pro-$p$ subgroup of finite rank.

For every compact $p$-adic Lie group $G$ one has $\text{rk}(G) \geq \text{dim}(G)$, where $\text{dim}(G)$ denotes the dimension of $G$ as a $p$-adic manifold. In this paper we consider the converse problem, bounding $\text{rk}(G)$ in terms of $\text{dim}(G)$.

Every profinite group $G$ of finite rank admits a maximal finite normal subgroup, its periodic radical $\pi(G)$. One of our main results is the following. Let $G$ be a compact $p$-adic Lie group such that $\pi(G) = 1$, and suppose that $p$ is odd. If $\{ g \in G \mid g^{p-1} = 1 \}$ is equal to $\{1\}$, then $\text{rk}(G) = \text{dim}(G)$.

1. Introduction

In the 1960s Lazard developed a sophisticated theory of $p$-adic Lie groups. One of the high points of his work was the algebraic characterisation of such groups, providing a solution to the $p$-adic analogue of Hilbert’s 5th problem. A more recent approach to the theory of compact $p$-adic Lie groups, inspired by Lazard’s work and focusing on group-theoretic aspects, is given in [1].

The rank of a profinite group $G$ is the basic invariant $$\text{rk}(G) := \sup \{ d(H) \mid H \leq G \},$$ where $H$ ranges over all closed subgroups of $G$ and $d(H)$ denotes the minimal cardinality of a topological generating set for $H$. According to [1] Corollary 8.34, a topological group $G$ admits the structure of a compact $p$-adic Lie group if and only if it is a profinite group of finite rank which contains an open pro-$p$ subgroup. Moreover, the analytic structure which makes $G$ into a $p$-adic Lie group, whenever it exists, is unique. See [1] Interlude A for a wide range of alternative characterisations of pro-$p$ groups of finite rank.

Another key invariant of a compact $p$-adic Lie group $G$ is its dimension, $\text{dim}(G)$, as a $p$-adic manifold. Algebraically, $\text{dim}(G)$ can be
described as $d(U)$ where $U$ is any uniformly powerful open pro-$p$ subgroup of $G$. Consequently, for every compact $p$-adic Lie group $G$ one has $\text{rk}(G) \geq \dim(G)$. Uniformly powerful pro-$p$ groups are rather special $p$-adic analytic groups which are torsion-free and in many ways behave like lattices. For instance, one has $\text{rk}(U) = \dim(U)$ for every uniformly powerful pro-$p$ group $U$.

Straightforward examples show that, in general, to bound the rank $\text{rk}(G)$ of a compact $p$-adic Lie group $G$ in terms of $\dim(G)$, one needs to restrain the torsion of $G$. In the first instance, it is thus natural to consider groups which are torsion-free.

Question 1.1. Is it true that for every torsion-free compact $p$-adic Lie group $G$ one has $\text{rk}(G) = \dim(G)$?

In Section 2 we show that a result of Laffey [8], which bounds the number of generators of finite $p$-groups, readily implies

**Proposition 1.2.** Let $G$ be a torsion-free compact $p$-adic Lie group, and suppose that $p$ is odd. Then $\text{rk}(G) = \dim(G)$.

It is remarkable that a $p$-group argument yields such a result, but unfortunately the method of proof does not appear to lead further. Proposition 1.2 is restricted to groups without any torsion and falls short of dealing with 2-adic Lie groups. An analogue of Laffey’s theorem for $p = 2$ is given in [3, Corollary 3.10]; it has, however, rather weak implications in the present context.

In the present paper we take a new approach which leads to the more general Theorems 1.3 and 1.4 below and which, we think, will be more suitable for dealing with 2-adic Lie groups. Indeed, our results are formulated for compact $p$-adic Lie groups which possess no non-trivial finite normal subgroups, a condition which also arises in the study of lattices. Every profinite group $G$ of finite rank admits a maximal finite normal subgroup $\pi(G)$, the periodic radical of $G$. If $G$ is $p$-adic analytic, so is $G/\pi(G)$ and $\dim(G) = \dim(G/\pi(G))$. These facts provide a natural reduction to groups $G$ satisfying $\pi(G) = 1$. For every prime $\ell$, we define the $\ell$-rank of a profinite group $G$ to be $\text{rk}_{\ell}(G) := \text{rk}(S)$ where $S$ is any Sylow pro-$\ell$ subgroup of $G$. Our main result is

**Theorem 1.3.** Let $G$ be a compact $p$-adic Lie group with $\pi(G) = 1$, and suppose that $p$ is odd. Then one has

$$\max\{\text{rk}_{\ell}(G) \mid \ell > 2 \text{ prime}\} = \text{rk}_p(G) = \dim(G)$$

and

$$\text{rk}_2(G) \leq \begin{cases} \lfloor 3\dim(G)/2 \rfloor & \text{if } p \equiv 1, \\ \dim(G) & \text{if } p \equiv 3. \end{cases}$$

A theorem of Guralnick and Lucchini, generalising work of Kovács, shows that every profinite group $G$ satisfies $\text{rk}(G) \leq \sup\{\text{rk}_{\ell}(G) \mid \ell \text{ prime}\} + 1$; see [3, 10]. This and related results allow us to prove
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Theorem 1.4. Let $G$ be a compact $p$-adic Lie group with $\pi(G) = 1$, and suppose that $p$ is odd. Then one has

1. $\text{rk}(G) \leq \max\{\dim(G), \text{rk}_2(G)\} + 1$;
2. $\text{rk}(G) \leq \max\{\dim(G) + 1, \text{rk}_2(G)\}$ if $G$ is prosoluble;
3. $\text{rk}(G) \leq \dim(G) + 1$ if $p \equiv 4$;
4. $\text{rk}(G) = \dim(G)$ if $G$ has trivial $(p-1)$-torsion, i.e. if $\{g \in G \mid g^{p-1} = 1\}$ is equal to $\{1\}$.

Remark 1.5. Relatively simple examples show that Theorems 1.3 and 1.4 are to some extent best possible. Indeed, for $p \equiv 4$ the matrix group $S := \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rangle \leq \text{GL}_2(Z_p)$ has order 16, acts irreducibly on the abelian lattice $Z_p^2$ and requires a minimum of 3 generators. Thus for any $r \in \mathbb{N}$ and any prime $p \equiv 4$, the 2r-dimensional compact $p$-adic Lie group $G = (S \rtimes Z_p^2)^r$ has $\pi(G) = 1$ and $\text{rk}_2(G) = 3r = 3 \dim(G)/2$.

For any prime $p$, any divisor $m \neq 1$ of $p - 1$ and any $d \in \mathbb{N}$, the group $G = T \rtimes Z_p^d$, where $T \cong C_m$ consists of the scalar matrices in $\text{GL}_d(Z_p)$ corresponding to $m$th roots of unity in $Z_p$, has $\pi(G) = 1$ and $\text{rk}(G) = d + 1 = \dim(G) + 1$.

While the focus of the present paper is on $p$-adic Lie groups for odd primes $p$, it remains a challenging open problem to describe the precise relation between rank and dimension for compact 2-adic Lie groups. Basic examples indicate that things are somewhat different for the prime 2. For instance, for every $r \in \mathbb{N}$ the $r$-fold direct power of the pro-2 dihedral group $D = C_2 \rtimes Z_2$ satisfies $\pi(D^r) = 1$ and $\text{rk}(D^r) = 2r = 2 \dim(D^r)$.

We recall that a pro-$p$ group $G$ is said to be $d$-maximal if $d(H) < d(G)$ for every proper open subgroup $H$ of $G$. It is known that, for odd $p$, every $d$-maximal finite $p$-group is nilpotent of class at most 2; cf. [3] and references therein. Our proof of Theorem 1.3 indicates that Question 1.1 for $p = 2$ is linked to the long standing problem of understanding the structure of $d$-maximal finite 2-groups and pro-2 groups. Of particular relevance would be a positive answer to

Question 1.6. Is it true that every $d$-maximal pro-2 group is soluble?

Besides using the concept of $d$-maximal groups, we employ in our proof of Theorem 1.3 standard techniques from the theory of $p$-adic analytic pro-$p$ groups. Furthermore we take advantage of the description of maximal $\ell$-subgroups of general linear groups over $\mathbb{Q}_p$, given in [9], and the classification of indecomposable $Z_p C$-modules for cyclic groups $C$ of order $p$, provided in [5]. Of independent interest is our
elementary proof of the following auxiliary result, which one may compare with more general, asymptotic estimates for the minimal number of generators of finite linear groups; for instance, see [6, 2, 7, 12].

**Theorem 1.7.** Suppose that \( p \) is odd. Let \( d \in \mathbb{N} \), and let \( \ell \) be prime with \( \ell \neq p \). Let \( d_0 \in \{0, 1\} \) such that \( d \equiv_2 d_0 \), and set \( m(p, \ell) := \min\{n \in \mathbb{N} \mid \ell \text{ divides } p^n - 1\} \). Then

\[
\text{rk}_\ell(\text{GL}_d(\mathbb{Z}_p)) = \text{rk}_\ell(\text{GL}_d(\mathbb{F}_p)) = \begin{cases} 
\lfloor d/m(p, \ell) \rfloor & \text{if } p \not\equiv 1, \\
(3d - d_0)/2 & \text{if } \ell = 2 \text{ and } p \equiv 1, \\
d & \text{in all other cases.}
\end{cases}
\]

**Notation.** Throughout, \( p \) and \( \ell \) denote primes. The field of \( p \)-adic numbers and the ring of \( p \)-adic integers are denoted by \( \mathbb{Q}_p \) and \( \mathbb{Z}_p \). The commutator subgroup of a group \( G \) is denoted by \([G, G]\). The centre of \( G \) is denoted by \( Z(G) \), the centraliser in \( G \) of a subset \( S \subseteq G \) by \( C_G(S) \). The minimal cardinality of a generating set for a group \( G \) is denoted by \( d(G) \). Likewise the minimal cardinality of a generating set for a module \( M \) over a ring \( R \) is denoted by \( d_R(M) \). For \( n \in \mathbb{N}_0 \) it is customary to denote by \( G^n \) the \( n \)-fold direct power of \( G \); in a few places, we use the same notation, \( G^n \) for \( n \) a power of \( p \), to denote the characteristic subgroup of \( G \) generated by all \( n \)th powers of elements of \( G \). All wreath products considered are permutational wreath products.

When \( G \) is a topological group, subgroups \( H \) of \( G \) are tacitly taken to be closed, \( d(G) \) tacitly refers to the minimal number of topological generators, etc. A key concept in the theory of \( p \)-adic Lie groups is that of a powerful \( p \)-group. We refer to [1, Part I] for standard properties of uniformly powerful \( p \)-groups. A profinite group \( G \) is said to be just infinite if it is infinite and if every proper quotient of \( G \) is finite.

2. A short proof using Laffey’s bound

Clearly, Proposition 1.2 is a consequence of Theorem 1.3. But it also admits a short independent proof, based on a result of Laffey [8].

**Proof of Proposition 1.2.** For every prime \( \ell \) with \( \ell \neq p \), the Sylow pro-\( \ell \) subgroup of \( G \) is finite. Since \( G \) is torsion-free, we conclude that \( G \) is a pro-\( p \) group. Let \( U \) be a uniformly powerful open normal subgroup of \( G \). Then \( d(U) = \dim(G) \) and descending to an open subgroup, if necessary, it suffices to show that \( d(G) \leq \dim(G) \). The open subgroups \( U^{p^n} \), \( n \in \mathbb{N} \), provide a base for the neighbourhoods of 1 in \( G \). Consequently, since \( G \) is torsion-free, we find \( m \in \mathbb{N} \) such that \( \{x \in G \mid x^p \in U^{p^m}\} \subseteq U \). (In fact, a short argument shows that already \( m = 1 \) works.) As \( U \) is uniformly powerful, this implies that for all \( n \in \mathbb{N} \) with \( n \geq m \),

\[
\Omega_1(G/U^{p^n}) = \langle xU^{p^n} \in G/U^{p^n} \mid x^p \in U^{p^n} \rangle = U^{p^n-1}/U^{p^n}
\]
The symmetric group $\text{Sym}(n)$, where $n$ is $\prod$ iterated wreath products: each of them is isomorphic to $c$, furthermore, we recall that for $p$.

We define the following numerical invariants:

$$d(G) = \limsup_{n \in \mathbb{N}} d(G/U^{p^n}) \leq \limsup_{n \in \mathbb{N}} \log_p|\Omega_1(G/U^{p^n})| = \dim(G).$$

\[\square\]

3. Maximal $\ell$-subgroups of $\text{GL}_{d}(\mathbb{Q}_p)$

For $r \in \mathbb{N}_0$ we denote by $W_r(\ell)$ the $r$-fold iterated permutational wreath product $C_\ell \wr \cdots \wr C_\ell$ of cyclic groups of order $\ell$. This finite permutation group of degree $\ell^r$ has order $|W_r(\ell)| = (\ell^r - 1)/(\ell - 1)$ and requires $d(W_r(\ell)) = r$ generators. Recall that the Sylow $\ell$-subgroups of a symmetric group $\text{Sym}(n)$ of finite degree $n$ can be described in terms of iterated wreath products: each of them is isomorphic to $\prod_{i=1}^{\ell} W_i(\ell)^{n_i}$, where $n = \sum_{i=0}^{\ell} n_i\ell^i$ is the $\ell$-adic expansion of $n$ with coefficients $0 \leq n_i < \ell$.

The Sylow $\ell$-subgroups of finite symmetric groups feature in the classification of maximal $\ell$-subgroups of general linear groups; see [9]. We define the following numerical invariants:

$$m(p, \ell) := \begin{cases} \min\{n \in \mathbb{N} \mid \ell \text{ divides } p^n - 1\} & \text{if } p \neq \ell, \\ p - 1 & \text{if } p = \ell, \end{cases}$$

$$a(p, \ell) := \begin{cases} \max\{b \in \mathbb{N}_0 \mid \ell^b \text{ divides } p^{m(p, \ell)} - 1\} & \text{if } p \neq \ell, \\ 1 & \text{if } p = \ell. \end{cases}$$

In addition, we define for $p \equiv 3 \text{ (and } \ell = 2)$,

$$c(p, 2) := \max\{b \in \mathbb{N} \mid 2^b \text{ divides } p^2 - 1\}.$$  

Furthermore, we recall that for $c \in \mathbb{N}$ with $c \geq 3$ the semidihedral group of order $2^{c+1}$ is given by the presentation

$$D^{\text{semi}}_{2^{c+1}} = \langle x, y \mid x^2 = y^{2^c} = 1, y^x = y^{-(1+2^{c-1})} \rangle.$$  

We require the following information about maximal $\ell$-subgroups of general linear groups over the $p$-adic field $\mathbb{Q}_p$.

**Proposition 3.1.** Let $d \in \mathbb{N}$ and suppose that $(p, \ell) \neq (2, 2)$. Then the group $\text{GL}_{d}(\mathbb{Q}_p)$ has exactly one conjugacy class of maximal $\ell$-subgroups. Let $L$ be one of the maximal $\ell$-subgroups of $\text{GL}_{d}(\mathbb{Q}_p)$.

1. If $\ell$ is odd, then $L \cong C_p \wr S$, where $a = a(p, \ell)$ and $S$ is a Sylow $\ell$-subgroup of $\text{Sym}(\lfloor d/m(p, \ell) \rfloor)$.

2. Suppose that $\ell = 2$ and that $p$ is odd. If $p \equiv 1$, then $L \cong C_{2^c} \wr S$, where $a = a(p, 2)$ and $S$ is a Sylow 2-subgroup of $\text{Sym}(d)$. If $p \equiv 3$ and $d \equiv 2 \mod 4$, then $L \cong D^{\text{semi}}_{2^{c+1}} \wr S$, where $c = c(p, 2)$ and $S$ is a Sylow 2-subgroup of $\text{Sym}(d/2)$. If $p \equiv 3$ and $d \equiv 1 \mod 4$, then $L \cong C_2 \times (D^{\text{semi}}_{2^{c+1}} \wr S)$, where $c = c(p, 2)$ and $S$ is a Sylow 2-subgroup of $\text{Sym}((d-1)/2)$.
Proof. The proposition is a special instance of Theorems II.4, IV.2 and IV.3 in [9]. Indeed, \( m(p, \ell) \) is equal to the degree of the \( \ell \)-th cyclotomic field \( \mathbb{Q}_p(\zeta_\ell) \) over \( \mathbb{Q}_p \), where \( \zeta_\ell \) denotes a primitive \( \ell \)-th root of unity, and \( a(p, \ell) = \max\{b \in \mathbb{N} \mid \mathbb{Q}_p(\zeta_\ell) \text{ contains a primitive } \ell \text{-th root of 1} \} \).

Moreover, if \( \ell = 2 \) and \( p \equiv 4 \mod 3 \), then \( c = c(p, 2) \) takes the same values as the invariant \( \gamma(\mathbb{Q}_p) \) used in [9, Section IV]. We observe that Theorem IV.4 in op. cit. is not used, as we do not discuss the case \( p = \ell = 2 \).

Our aim is to prove Theorem \[14\]. In view of Proposition \[31\], we consider for \( r \in \mathbb{N}_0 \) the groups

\[
X_{a,r}(\ell) := C_{\ell^a} \wr W_r(\ell), \quad a \in \mathbb{N},
Y_{c,r}(2) := D_{2^{c+1}} \wr W_r(2), \quad c \in \mathbb{N} \text{ with } c \geq 3.
\]

We require the following lemma.

**Lemma 3.2.** Let \( n \in \mathbb{N} \), and suppose that \( \sigma \in \text{Sym}(n) \) is a permutation of \( \ell \)-power order acting without fixed points. Let \( h \in \text{GL}_n(\mathbb{Z}_\ell) \) be a monomial matrix with permutation part corresponding to \( \sigma \). Let \( M \) be a \( \mathbb{Z}_\ell(\langle h \rangle) \)-module acting on \( V = \mathbb{Z}_\ell^n \). Then \( d_{\mathbb{Z}_\ell(\langle h \rangle)}(M) \leq 2n/\ell \).

**Proof.** We observe that \( n \) is a multiple of \( \ell \), and we argue by induction on \( n \). First suppose that \( n = \ell \) so that \( \sigma \) is an \( \ell \)-cycle. We are to show that \( d_{\mathbb{Z}_\ell(\langle h \rangle)}(M) \leq 2 \). Clearly, this is true if \( \ell = 2 \). Now suppose that \( \ell > 2 \). Then we write \( \det(h) = \lambda^\ell(1 + \mu) \), where \( \lambda \in \mathbb{Z}_\ell^* \) and either \( \mu = 0 \) or \( \mu \in \ell\mathbb{Z}_\ell \setminus \ell^2\mathbb{Z}_\ell \). After a change of basis, we may assume that the action of \( h \) on the standard \( \mathbb{Z}_\ell \)-basis \( e_1, \ldots, e_\ell \) of \( V \) is given by

\[
e_i^h = \lambda e_{i+1} \quad \text{for } i \in \{1, \ldots, \ell - 1\} \quad \text{and} \quad e_\ell^h = \lambda(1 + \mu)e_1.
\]

For the purpose of bounding \( d_{\mathbb{Z}_\ell(\langle h \rangle)}(M) \) we may as well assume that \( \lambda = 1 \), and we treat the two possibilities for \( \mu \) separately.

First consider the case \( \mu = 0 \). Then \( h \) is the permutation matrix associated to the \( \ell \)-cycle \( \sigma \) and \( V \) the corresponding integral permutation module. The \( \mathbb{Z}_\ell(\langle h \rangle) \)-module \( M \) is free as a \( \mathbb{Z}_\ell \)-module and decomposes as a direct sum of indecomposable \( \mathbb{Z}_\ell(\langle h \rangle) \)-modules. Each indecomposable summand \( I \) of \( M \) satisfies \( d_{\mathbb{Z}_\ell(\langle h \rangle)}(I) = 1 \) and \( \dim_{\mathbb{Z}_\ell}(I) \in \{1, \ell - 1, \ell\} \); see [5, Theorem 2.6]. Since \( \dim_{\mathbb{Z}_\ell}(M) \leq n = \ell \) and since \( \langle h \rangle \) does not act trivially on \( M \), it follows that \( M \) is the sum of at most two indecomposable \( \mathbb{Z}_\ell(\langle h \rangle) \)-modules, and hence \( d_{\mathbb{Z}_\ell(\langle h \rangle)}(M) \leq 2 \), as wanted.

Next consider the case \( \mu \in \ell\mathbb{Z}_\ell \setminus \ell^2\mathbb{Z}_\ell \). Then the minimum polynomial of \( h - 1 \) over \( \mathbb{Q}_\ell \) is \( (X + 1)^\ell - (1 + \mu) \), an Eisenstein polynomial. Thus the ring \( \mathbb{Z}_\ell(h) \) is isomorphic to the ring of integers \( \mathcal{O} \) of a totally ramified extension \( K \) of \( \mathbb{Q}_\ell \) of degree \( \ell \). Furthermore, the \( \mathbb{Z}_\ell(\langle h \rangle) \)-module \( V \) corresponds to the \( \mathcal{O} \)-module \( \mathcal{O} \), and \( M \) to an ideal of the principal ideal domain \( \mathcal{O} \). Thus \( d_{\mathbb{Z}_\ell(\langle h \rangle)}(M) \leq 1 \), finishing the argument for \( n = \ell \).
Now suppose that \( n > \ell \) and that the permutation group \( \langle \sigma \rangle \) is intransitive: \( \langle \sigma \rangle \leq \text{Sym}(n_1) \times \text{Sym}(n_2) \), where \( n = n_1 + n_2 \) with positive summands. Write \( \sigma_1 \) and \( \sigma_2 \) for the images of \( \sigma \) under projection into \( \text{Sym}(n_1) \) and \( \text{Sym}(n_2) \) respectively, and note that each of these permutations acts without fixed points. The monomial matrix \( h \) admits a corresponding block decomposition \( h = h_1 \oplus h_2 \), with the permutation part of the monomial matrix \( h_i \) corresponding to \( \sigma_i \) for \( i \in \{1, 2\} \).

Clearly, the module \( V \) admits a \( \mathbb{Z}_\ell(h_i) \)-submodule \( W \) such that \( V_1 := V/W \cong \mathbb{Z}_\ell^{n_1} \) and \( V_2 := W \cong \mathbb{Z}_\ell^{n_2} \) are naturally modules for \( \mathbb{Z}_\ell(h_1) \) and \( \mathbb{Z}_\ell(h_2) \). Setting \( M_1 := (M + W)/W \) and \( M_2 := M \cap W \), we deduce by induction that

\[ d_{\mathbb{Z}_\ell(h_1)}(M) \leq d_{\mathbb{Z}_\ell(h_1)}(M_1) + d_{\mathbb{Z}_\ell(h_2)}(M_2) \leq 2(n_1 + n_2)/\ell = 2n/\ell. \]

Finally, suppose that \( n > \ell \) and that the permutation group \( \langle \sigma \rangle \) is transitive. Then \( n \) is equal to the order of \( \sigma \), hence an \( \ell \)-power: \( n = \ell^k \) with \( k \geq 2 \). Clearly, \( \sigma^\ell \in \text{Sym}(n) \) acts with \( \ell \) orbits of size \( \ell^{k-1} \) and has no fixed points. Applying the argument given before, we deduce that \( d_{\mathbb{Z}_\ell(h)}(M) \leq d_{\mathbb{Z}_\ell(h^\ell)}(M) \leq 2n/\ell. \)

\[ \square \]

**Proposition 3.3.** Let \( a \in \mathbb{N} \) and \( r \in \mathbb{N}_0 \). Then

\[ \text{rk}(X_{a,r}(\ell)) = \begin{cases} 3 \cdot 2^{r-1} & \text{if } \ell = 2, a \geq 2 \text{ and } r \geq 1, \\ \ell^r & \text{in all other cases.} \end{cases} \]

**Proof.** We write \( X := X_{a,r}(\ell) \) and argue by induction on \( r \). If \( r = 0 \), the group \( X \cong C_{\ell^a} \) is cyclic of order \( \ell^a \) and the assertion holds. Now suppose that \( r \geq 1 \).

First we derive a lower bound for \( \text{rk}(X) \). Let \( E := E_{a,r}(\ell) \) denote the homocyclic group of exponent \( \ell^a \) which arises as the base subgroup of \( X = C_{\ell^a} \wr W_r(\ell) \). Clearly, \( \text{rk}(X) \geq \text{rk}(E) = \ell^r \). Next we derive for \( \ell = 2, a \geq 2 \) and \( r \geq 1 \) the stronger lower bound \( \text{rk}(X) \geq 3 \cdot 2^{r-1} \).

Indeed, the 2-group \( S \) displayed in Remark 3.3 has \( d(S) = 3 \) and is easily embedded into \( C_4 \wr C_2 \). Since \( a \geq 2 \), the group \( C_4 \wr C_2 \) embeds into \( C_{2^a} \wr C_2 \), and the decomposition \( X \cong (C_{2^a} \wr C_2) \wr W_{r-1}(2) \) shows that the \( 2^{r-1} \)-fold direct power of \( S \) embeds into \( X \). We conclude that \( \text{rk}(X) \geq d(S^{2^{r-1}}) = 3 \cdot 2^{r-1} \).

It remains to give an upper bound for \( \text{rk}(X) \). Consider an arbitrary subgroup \( H \leq X \), and decompose \( X \cong X_{a,r-1}(\ell) \wr C_\ell \) as \( X = B \rtimes C \), where \( B = B_1 \times \ldots \times B_\ell \) with \( B_i \cong X_{a,r-1}(\ell) \) for \( i \in \{1, \ldots, \ell\} \) and \( C \cong C_\ell \). In addition, we decompose the homocyclic base group \( E \) of \( X \) into \( E = E_1 \times \ldots \times E_\ell \), where \( E_i = E \cap B_i \cong (C_{\ell^a})^{\ell-r} \) is the homocyclic base group of the wreath product \( B_i \cong C_{\ell^a} \wr W_{r-1}(\ell) \) for \( i \in \{1, \ldots, \ell\} \).
Indeed, by projecting $H$ into the first factor $B_1$ of $B$, then projecting the kernel $N_i$ of the first projection into the second factor $B_2$ of $B$, and so on, we find a descending subnormal series $H = N_0 \supseteq N_1 \supseteq \ldots \supseteq N_\ell = 1$ such that, for $i \in \{1, \ldots, \ell\}$, the quotient $N_{i-1}/N_i$ embeds into $B_i$. Clearly, $d(H) \leq \sum_{i=1}^\ell d(N_{i-1}/N_i) \leq \sum_{i=1}^\ell \text{rk}(B_i)$ and (3.1) is justified.

It remains to consider the case where $H$ is not contained in $B$, and it will be enough to prove that under this hypothesis $d(H) \leq 3\ell r^{-1}$. Decompose $X \cong C_{\ell r} \wr W_r(\ell)$ as $X = E \rtimes W$, where $E = E_1 \times \ldots \times E_\ell$ with $E_i = E \cap B_i$ for $i \in \{1, \ldots, \ell\}$ as before and $W \cong W_r(\ell) = X_1, r-1(\ell)$. By induction on $r$, we have $d(HE/E) \leq \text{rk}(W) = \ell r^{-1}$. Furthermore, if $\ell = 2$, we have $d(H \cap E) \leq \text{rk}(E) = \ell r = 2 \cdot \ell r^{-1}$, thus $d(H) \leq d(HE/E) + d(H \cap E) \leq 3\ell r^{-1}$, as wanted.

Now suppose that $\ell \geq 3$. Choose $h \in H \setminus B$ and regard $H \cap E$ as a $\mathbb{Z}_\ell \langle h \rangle$-module. In view of the inequality $d(H) \leq d(HE/E) + d_{\mathbb{Z}_\ell \langle h \rangle}(H \cap E)$, it suffices to prove that $d_{\mathbb{Z}_\ell \langle h \rangle}(H \cap E) \leq 2\ell r^{-1}$. Note that $h$ permutes cyclically the $\ell$ factors $E_i$ of $E$. Working with a pre-image $M$ of $H \cap E$ in the base group of the $\ell$-adic group $\mathbb{Z}_\ell \wr W$, we apply Lemma 3.2 to deduce that $d_{\mathbb{Z}_\ell \langle h \rangle}(H \cap E) \leq 2\ell r / \ell = 2\ell r^{-1}$. \hfill \Box

We remark that, for $\ell \geq 5$, one can easily extend the argument given in the proof of Proposition 3.3 to show that every subgroup $H$ of $X := X_{a,r}(\ell) = C_{\ell r} \wr W_r(\ell)$ with $d(H) = \ell r = \text{rk}(X)$ is contained in the homocyclic base group $E \cong C_{\ell r}$. It remains a curious open problem to classify subgroups witnessing the rank of $X_{a,r}(\ell)$ for the small primes $\ell \in \{2,3\}$.

**Proposition 3.4.** Let $c \in \mathbb{N}$ with $c \geq 3$, and let $r \in \mathbb{N}_0$. Then
\[
\text{rk}(Y_{c,r}) = 2^{r+1}.
\]

**Proof.** We reason similarly as in the proof of Proposition 3.3. Write $Y := Y_{c,r}$ and argue by induction on $r$. If $r = 0$, the group $Y \cong D_{2c+1}^{\text{semi}}$ is metacyclic but not cyclic, and the assertion follows. From now on suppose that $r \geq 1$.

By considering the base subgroup of $Y = D_{2c+1}^{\text{semi}} \wr W_r(2)$, we obtain
\[
\text{rk}(Y) \geq \text{rk}((D_{2c+1}^{\text{semi}})^{2^r}) = 2^r \text{rk}(D_{2c+1}^{\text{semi}}) = 2^{r+1},
\]
and it remains to show that $\text{rk}(Y) \leq 2^{r+1}$. Consider an arbitrary subgroup $H \leq Y$, and decompose $Y \cong Y_{c,r-1} \wr C_2$ as $Y = B \rtimes C$, where
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$B = B_1 \times B_2$ with $B_i \cong Y_{c,r-1}$, $i \in \{1, 2\}$, and $C \cong C_2$. If $H \leq B$, then induction on $r$ yields

$$d(H) \leq \text{rk}(B) = 2 \text{rk}(Y_{c,r-1}) = 2 \cdot 2^r = 2^{r+1}.$$ 

Thus it remains to consider the case where $H$ is not contained in $B$. Decompose $Y \cong D_{2e+1} \wr W_r(2)$ as $Y = E \rtimes W,$ where $E \cong (D^\text{semi}_{2e+1})^{2^r}$ and $W \cong W_r(2)$. Since $D^\text{semi}_{2e+1}$ is metacyclic, $E$ is metabelian and, in fact, decomposes via a characteristic subgroup $F$ into homocyclic parts $E/F \cong (C_2)^{2^r}$ and $F \cong (C_{2^e})^{2^r}$.

Choose $h \in H \setminus B$ and regard $(H \cap E)F/F$ and $H \cap F$ as $\mathbb{Z}_2\langle h \rangle$-modules. In view of the inequality

$$d(H) \leq d(HE/E) + d_{\mathbb{Z}_2\langle h \rangle}((H \cap E)F/F) + d_{\mathbb{Z}_2\langle h \rangle}(H \cap F),$$

it suffices to establish: (i) $d(HE/E) \leq 2^{r-1}$, (ii) $d_{\mathbb{Z}_2\langle h \rangle}((H \cap E)F/F) \leq 2^{r-1}$ and (iii) $d_{\mathbb{Z}_2\langle h \rangle}(H \cap F) \leq 2^r$.

Since $HE/E$ embeds into $W \cong X_{1,r-1}(2)$, Proposition 3.3 yields (i). Considering (ii), we write $E = E_1 \times E_2$, where $E_i = E \cap B_i \cong (D^\text{semi}_{2e+1})^{2^{r-1}}$ for $i \in \{1, 2\}$. It is convenient to set $K := H \cap E$ and to use $\tau$ to denote reduction modulo $F$. Note that $h$ transposes the two factors in the decomposition $E = E_1 \times E_2$ of the elementary 2-group $E$. Thus $h$ conjugates $K \cap E_2$ injectively into $E_1 \cong E/E_2$, and we conclude that

$$d_{\mathbb{Z}_2\langle h \rangle}(K) \leq (d(K/E_2) - d(K \cap E_2)) + d(K \cap E_2) = d(K/E_2).$$

As $E/E_2 \cong E_1$ is an elementary 2-group of rank $2^{r-1}$, this proves (ii). Finally, $h$ also transposes the two factors $F_i := F \cap E_i$, $i \in \{1, 2\}$, of $F = F_1 \times F_2$. Working with the pre-image $M$ of $H \cap F$ in the base group of the 2-adic group $(\mathbb{Z}_2 \wr \mathbb{Z}_2) \wr W$, we apply Lemma 6.2 to deduce (iii).

Again, it would be interesting to give a complete description of all subgroups witnessing the rank of $Y_{c,r}$. We are ready to prove Theorem 1.7 and the following complementary result.

**Proposition 3.5.** Suppose that $p$ is odd, and let $d \in \mathbb{N}$. Let $G$ be a maximal finite $p$-subgroup of $GL_d(\mathbb{Q}_p)$. Then $\text{rk}(G) = \lceil d/(p-1) \rceil$.

**Proof of Theorem 1.7 and Proposition 3.5.** The claims are direct consequences of Propositions 3.1, 3.3 and 3.4 combined with the description of Sylow subgroups of finite symmetric groups in terms of wreath products.

4. **Bounding the $\ell$-rank**

**Proposition 4.1.** Suppose that $p$ is odd. Let $G$ be a finite $p$-group, and let $M$ be a faithful $\mathbb{Z}_pG$-module which is free and of finite rank as a $\mathbb{Z}_p$-module. Then

$$d(G) + d_{\mathbb{Z}_pG}(M) \leq \dim_{\mathbb{Z}_p}(M).$$
Proof. Clearly, we may suppose that $G$ is not trivial. Let $C = \langle x \rangle$ be a central subgroup of $G$ of order $p$. According to [3] Theorem 2.6, there are three types of indecomposable $\mathbb{Z}_pC$-modules: they are the trivial module $\mathbb{Z}_p$ of dimension 1, the free module $\mathbb{Z}_pC$ of dimension $p$, and the $(p - 1)$-dimensional module $\mathbb{Z}_pC/\Phi_p(x)\mathbb{Z}_pC$ where $\Phi_p = 1 + X + \ldots + X^{p-1}$. As a $\mathbb{Z}_p$-$C$-module, $M$ is the direct sum of indecomposable modules of these three types.

Set $N := C_M(C) = \{ m \in M \mid m^x = m \}$. Since $C$ is central in $G$ and since $C$ acts faithfully on $M$, the set $N$ constitutes a proper $\mathbb{Z}_pG$-submodule of $M$. We distinguish two cases: $N = \{0\}$ and $N \neq \{0\}$.

First suppose that $N \neq \{0\}$. In this case, $G$ acts naturally on the quotient module $M_1 := M/N$ with image $G_1 \leq \text{GL}(M/N)$ and kernel $G_2$, say. The finite group $G_2$ acts faithfully on $M_2 := N$; indeed, the $\mathbb{Q}_pG$-module $\mathbb{Q}_p \otimes \mathbb{Z}_p M$ is completely reducible and $\mathbb{Q}_p \otimes \mathbb{Z}_p M_2$ admits a complement isomorphic to $\mathbb{Q}_p \otimes \mathbb{Z}_p M_1$. Clearly, $d(G) \leq d(G_1) + d(G_2)$ and $d_{\mathbb{Z}_pG}(M) \leq d_{\mathbb{Z}_pG_1}(M_1) + d_{\mathbb{Z}_pG_2}(M_2)$. Therefore induction gives

$$d(G) + d_{\mathbb{Z}_pG}(M) \leq d(G_1) + d(G_2) + d_{\mathbb{Z}_pG_1}(M_1) + d_{\mathbb{Z}_pG_2}(M_2) \leq \dim_{\mathbb{Z}_p}(M_1) + \dim_{\mathbb{Z}_p}(M_2) = \dim_{\mathbb{Z}_p}(M).$$

Now suppose that $N = \{0\}$. Then, as a $\mathbb{Z}_p$-$C$-module, $M$ necessarily decomposes into a direct sum of indecomposable $\mathbb{Z}_pC$-modules each of which is isomorphic to $\mathbb{Z}_pC/\Phi_p(x)\mathbb{Z}_pC$. Clearly,

$$d_{\mathbb{Z}_pG}(M) \leq d_{\mathbb{Z}_pC}(M) = (p - 1)^{-1} \dim_{\mathbb{Z}_p}(M).$$

On the other hand, Proposition [3] shows that $d(G) \leq (p - 1)^{-1} \dim_{\mathbb{Z}_p}(M)$ and thus

$$d(G) + d_{\mathbb{Z}_pG}(M) \leq 2(p - 1)^{-1} \dim_{\mathbb{Z}_p}(M) \leq \dim_{\mathbb{Z}_p}(M),$$

as wanted. □

Lemma 4.2. Let $G$ be an insoluble pro-$p$ group, and suppose that $p$ is odd. Then for every open normal subgroup $N$ of $G$ there exists an open normal subgroup $K$ of $G$ such that $K \nsubseteq N$ and $d(K) \geq d(N)$.

Proof. Clearly, the assertion holds if $G$ is not finitely generated. Now suppose that $d(G) < \infty$, and let $N$ be an open normal subgroup of $G$. Let $\Phi(N) = N^p[N, N]$ denote the Frattini subgroup of $N$. Since $G$ is insoluble, we have $N := [\Phi(N), N] \neq 1$. Observe that $N$ is normal in $G$, but not necessarily open. Nevertheless, working modulo an open normal subgroup $U$ of $G$ such that $U \subseteq M \subseteq \Phi(N)$, we find an open normal subgroup $L$ of $G$ such that $L \subseteq MU \subseteq \Phi(N)$ and $MU/L \cong C_p$. Now an argument similar to the one given in the proof of [3] Theorem 3.3] shows that

$$K := \{ g \in N \mid [\Phi(N), g] \subseteq L \}$$

satisfies $d(K) \geq d(N)$. One checks easily that $K \leq G$ and $K \nsubseteq N$. □
Recall that $\pi(G)$ denotes the periodic radical, viz. the unique maximal finite normal subgroup, of a profinite group $G$ of finite rank.

**Lemma 4.3.** Let $G$ be a profinite group of finite rank and $H$ a subgroup of $G$. If $H$ is normal or open in $G$, then $\pi(H) \leq \pi(G)$.

**Proof.** In both cases, the union of all conjugates of $\pi(H)$ in $G$ is a finite normal subset of $G$ consisting of elements of finite order. By Dicman’s Lemma, this set generates a finite normal subgroup of $G$. □

For the next two results, we denote by $d_\ell(G)$ the minimal cardinality of a generating set for a Sylow pro-$\ell$ subgroup of a profinite group $G$.

**Proposition 4.4.** Let $G$ be a compact $p$-adic Lie group such that $\pi(G) = 1$, and suppose that $p$ is odd. Then $d_p(G) \leq \dim(G)$.

**Proof.** By Lemma 4.3, we may assume without loss of generality that $G$ is a pro-$p$ group. If $\dim(G) = 0$, then $G = \pi(G) = 1$ and there is nothing further to prove. Hence suppose that $\dim(G) \geq 1$. Choose a normal subgroup $N$ of $G$ such that $G/N$ is just infinite. Note that $\pi(G/N) = 1$ and $\pi(N) = 1$, by Lemma 4.3. Thus, if $N \neq 1$ we apply induction to deduce that

$$
d(G) \leq d(G/N) + d(N) \leq \dim(G/N) + \dim(N) = \dim(G).
$$

It remains to consider the case that $N = 1$, i.e. that $G$ is just infinite. First suppose that $G$ is soluble. Then $G$ is virtually abelian; we find an abelian open normal subgroup $A$ of $G$. Put $C := C_G(A)$. Then $|C : Z(C)| \leq |C : A| < \infty$, and hence $[C, C]$ is finite. Since $G$ is just infinite, we conclude that $C$ is abelian and self-centralising. Thus Proposition 4.1 shows that $d(G) \leq \dim(C) = \dim(G)$.

Next suppose that $G$ is insoluble. For a contradiction assume that $d(G) > \dim(G)$. By Lemma 1.2, we find a strictly descending chain

$$
G = N_1 \supset N_2 \supset N_3 \supset \ldots
$$

of open normal subgroups $N_i$ of $G$ such that $d(N_i) \geq d(G)$ for all $i \in \mathbb{N}$. Fix a uniformly powerful open normal subgroup $U$ of $G$. Since $\text{rk}(U) = \dim(U) < d(G)$, we have $N_i \not\subseteq U$ for each $i \in \mathbb{N}$, witnessed by $g_i \in N_i \setminus U$, say. Since $G$ is compact, there is a subsequence of $g_i$, $i \in \mathbb{N}$, which converges to an element $g \in G \setminus U$. Clearly, we also have $g \in \bigcap\{N_i \mid i \in \mathbb{N}\} =: M$. Since $M$ is a normal subgroup of infinite index in $G$ and since $G$ is just infinite, we conclude $g \in M = \{1\} \subseteq U$, a contradiction. □

**Proposition 4.5.** Let $G$ be a compact $p$-adic Lie group such that $\pi(G) = 1$, and let $d := \dim(G)$. Then for every prime $\ell$ with $\ell \neq p$,

$$
d_\ell(G) \leq \text{rk}_\ell(\text{GL}_d(\mathbb{Z}_p)) = \text{rk}_\ell(\text{GL}_d(\mathbb{F}_p)).
$$
Proof. Arguing as at the beginning of the proof of Proposition 4.4, we may suppose that $G$ is just infinite. Let $S$ be a Sylow pro-$\ell$ subgroup of $G$, and choose a uniformly powerful open normal pro-$p$ subgroup $U$ of $G$. Then $S$ is a finite $\ell$-group, and we claim that $S$ acts faithfully on $U$ by conjugation. Indeed, since $G$ is just infinite, $C_G(U)$ is either trivial or open in $G$. In the former case there is nothing further to show. In the latter case, writing $C := C_G(U)$, we note that $|C:Z(C)| \leq |C:C \cap U| < \infty$ so that $[C,C]$ is a finite normal subgroup of $G$. Since $G$ is just infinite, we conclude that $C$ is abelian and pro-$p$. Hence $S \cap C = 1$, and again $S$ acts faithfully on $U$.

Since $U$ is uniformly powerful, the set $U$ admits the structure of a $\mathbb{Z}_p$-Lie lattice with additive group isomorphic to $\mathbb{Z}_p^d$, where $d := \dim(U) = \dim(G)$, and the faithful action of $S$ on $U$ translates into an embedding of $S$ into $\text{GL}_d(\mathbb{Z}_p)$. Dividing out the first principal congruence subgroup $\text{GL}_1(\mathbb{Z}_p)$, we obtain an embedding of $S$ into $\text{GL}_d(\mathbb{F}_p)$. □

Proof of Theorem 1.3. Let $G$ be a compact $p$-adic Lie group such that $\pi(G) = 1$, and suppose that $p$ is odd. The assertion about $\text{rk}_2(G)$ follows directly from Proposition 4.5 and Theorem 1.7. For the assertion about $\text{rk}_\ell(G)$, $\ell > 2$, it suffices to show that

(i) $\text{rk}_p(G) \geq \dim(G),$

(ii) $\text{rk}_\ell(G) \leq \dim(G)$ for every odd prime $\ell$.

For every uniformly powerful open pro-$p$ subgroup $U$ of $G$ one has $d(U) = \dim(U) = \dim(G)$, and this implies (i). Let $\ell$ be an odd prime. It is known that

$$\text{rk}_\ell(G) = \sup\{d(H) \mid H \leq G \text{ an open subgroup}\};$$

cf. [11, Proposition 3.11]. Thus (ii) follows from Lemma 4.3 and Propositions 4.4 and 4.5, in conjunction with Theorem 1.7. □

5. Bounding the rank

We will make use of the following theorem which was proved for solvable groups by Kovács, and for all finite groups (using the classification of finite simple groups) by Lucchini [10] and Guralnick [4].

Theorem 5.1. If every Sylow subgroup of a finite group $G$ can be generated by $d$ elements, then $d(G) \leq d + 1$.

Corollary 5.2. For every profinite group $G$ one has

$$\text{rk}(G) \leq \sup\{\text{rk}_\ell(G) \mid \ell \text{ prime}\} + 1.$$  

Proof of Theorem 1.4. Claims (1) and (3) are direct consequences of Corollary 5.2 and Theorem 1.3. To deduce claims (2) and (4) we use the additional analysis given in [11], which distinguishes between groups of zero and non-zero presentation rank.

First suppose that $G$ is prosoluble. Then all its finite quotients have zero presentation rank. Hence [11, Theorem 1] shows that equality in
(5.1) implies \( \operatorname{rk}(G) \leq \operatorname{rk}_\ell(G) + 1 \) for some odd prime \( \ell \). Thus claim (2) follows from Corollary 5.2 and Theorem 1.3.

Finally suppose that \( G \) has trivial \((p-1)\)-torsion and without loss of generality assume that \( \dim(G) \geq 1 \). Then \( \operatorname{rk}_p(G) = \dim(G) \) and \( \operatorname{rk}_\ell(G) = 0 \leq \dim(G) - 1 \) for primes \( \ell \) with \( p \equiv \ell \mod 1 \). Furthermore, Proposition 4.5 and Theorem 1.7 show that \( \operatorname{rk}_\ell(G) \leq \lfloor \dim(G)/2 \rfloor \leq \dim(G) - 1 \) for all remaining primes \( \ell \). Corollary 5.2 gives \( \operatorname{rk}(G) \in \{\dim(G), \dim(G) + 1\} \), and [11] Theorems 1 and 2 rule out the possibility \( \operatorname{rk}(G) = \dim(G) + 1 \). This proves claim (4). \( \square \)

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