Self-organized critical behavior and marginality in Ising spin glasses

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Abstract.

We have studied numerically the states reached in a quench from various temperatures in the one-dimensional fully-connected Kotliar, Anderson and Stein Ising spin glass model. This is a model where there are long-range interactions between the spins which falls off as a power $\sigma$ of their separation. We have made a detailed study in particular of the energies of the states reached in a quench from infinite temperature and their overlaps, including the spin glass susceptibility. In the regime where $\sigma \leq 1/2$, where the model is similar to the Sherrington-Kirkpatrick model, we find that the spin glass susceptibility diverges logarithmically with increasing $N$, the number of spins in the system, whereas for $\sigma > 1/2$ it remains finite. We attribute the behavior for $\sigma \leq 1/2$ to self-organized critical behavior, where the system after the quench is close to the transition between states which have trivial overlaps and those with the non-trivial overlaps associated with replica symmetry breaking. We have also found by studying the distribution of local fields that the states reached in the quench have marginal stability but only when $\sigma \leq 1/2$.

\textit{Keywords}: spin glasses, self-organized criticality, marginal stability

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1. Introduction

In this paper we have studied three topics related to deterministic quenches in a spin glass system. The spin glass system is that of the one-dimensional long-range Ising Hamiltonian introduced by Kotliar, Anderson and Stein (KAS) [1], which serves as a proxy model for the short-range $d$-dimensional Edwards-Anderson model [2]. The KAS model has long-range interactions between the spins which fall off with a power $\sigma$ of their separation distance. In a quench we start from an initial state, such as the fully equilibrated state at a temperature $T$ and then apply a deterministic algorithm, such as the “greedy”, “polite” or sequential algorithm [3] until a state is reached in which the energy cannot be lowered further by flipping just a single spin. Much of our investigation has been of the case where the initial state is at infinite temperature so that spins are randomly $\pm 1$ with the quench being performed with the sequential algorithm.

The first study is of the nature of the state reached in the quench, as revealed by the form of the Parisi overlap function $P(q)$. Our finding here is that its form is determined by the nature of the initial state at temperature $T$. If $T > T_c$, where $T_c$ is the equilibrium transition temperature of the spin glass system, then the final state has the trivial overlap of the paramagnetic state, $P(q) = \delta(q)$. When $T < T_c$ its form after the quench resembles that of the initial state. We conclude that in a deterministic quench, the form of the initial state is imprinted onto the final quenched state.

The second study which we make is of the distribution of local fields $p(h)$ in the quenched state. We review the argument of Anderson, reported in Ref. [4], for the form of $p(h)$ at small fields and find that our numerical data is consistent with the state generated in the quench having marginal stability in the regime where mean-field applies, that is for $\sigma \leq 1/2$, but that outside this regime the quenched state is not marginal. The mean-field limit includes the Sherrington-Kirkpatrick (SK) model which corresponds to the case $\sigma = 0$. The form of $p(h)$ after a quench has been much studied for the Ising SK model [5, 6, 7].

The third topic studied is that of Self-Organized Criticality (SOC) which is the phenomenon where some large dissipative systems can be in a scale-invariant critical state but without any parameter being tuned to a critical value [8]. It is believed that it is behind the fractal features [9] associated with many phenomena, such as earthquakes, the meandering of sea coasts and the structure of galactic clusters. For equilibrium systems, scale invariant behavior is usually only found at critical points where some parameter e.g. temperature is at its critical value $T_c$. However, over the years a number of examples have been found of SOC behavior in models which have very artificial dynamical rules such as in the sandpile model [10] and the forest fire model [11]. More recently, Andresen et al. [12] have found SOC features in the Sherrington-Kirkpatrick (SK) model of Ising spin glasses which were absent in the $d$-dimensional Edwards-Anderson (EA) spin glass models. The signature of SOC behavior for them was the size of the spin avalanches following a change in the applied field; only when there was a diverging number of neighbors as in the SK model were the avalanche sizes
limited by the number of spins $N$ in the system.

Most studies of SOC behavior focus on dynamical features such as the size of avalanches etc. [13, 12, 14]. In this paper we have studied entirely static features: in particular the spin glass susceptibility calculated via the overlaps of the quenched states obtained from different initial states. We have found that it diverges logarithmically with the number of spins $N$ in the system, provided that the exponent $\sigma \leq 1/2$. We thus conclude that when $\sigma \leq 1/2$ the system reaches a set of quenched states which are close to a critical energy. When $\sigma > 1/2$ the divergence of the susceptibility goes away, indicating that the quench does not then take the system to a critical state. It is thought [15] that systems with $\sigma \leq 1/2$ behave just as the SK model. Our results therefore complement those in Ref. [12], where they found that SOC behavior was only present for the SK model, but was lacking in the $d$-dimensional EA models, which correspond to values of $\sigma > 1/2$ in the KAS model [16, 17, 18, 19, 20]. Furthermore, Gonçalves and Boettcher [21] studied avalanche sizes as a function of $\sigma$ in the KAS model and concluded that $\sigma = 1/2$ was indeed the borderline value above which the avalanches changed their behavior as a function of system size $N$. An extensive study of avalanches in the SK model itself is in Refs. [13, 22].

Our interest in the static aspects of SOC behavior was triggered by our previous studies [23] of vector spin glasses in the Sherrington-Kirkpatrick (SK) model. We found that the quench in those models reached metastable minima whose energy per spin $E_c$ was very close to that calculated for the energy which separates minima with zero overlap with each other from those which have a full replica symmetry overlap with each other [24]. In other words the quench takes one close to the critical energy which separates states with a trivial $P(q)$ from those with a non-trivial $P(q)$. The same type of mean-field calculation fails in the Ising case as the states reached in a quench are quite atypical of the set of all the metastable states of energy $E$. In the thermodynamic limit the energy per spin reached after the quench from a random initial state tends to a well-defined limit, dependent on the method used to flip the spins (e.g. ‘polite’ or ‘greedy’ or ‘sequential’ algorithm etc. [3]). These observations are consistent with the rigorous arguments of Newman and Stein [25]. The states reached in the quench have a distribution $p(h)$ of their local fields $h_i = \sum_j J_{ij} S_j$ with interactions $J_{ij}$ among the spins $S_i$, which is linear in $h$ at small fields, whereas for the totality of metastable states of energy $E$, $p(0)$ is finite [26].

Our main finding is that for Ising spin glasses in the SK region $\sigma \leq 1/2$, the energy of the system after the quench is close to the critical energy $E_c$ which separates the metastable states of the kind produced in the quench which have no overlap with each other from those which would exist at lower energy which would have full replica symmetry breaking overlaps. That is Ising spin glasses with $\sigma \leq 1/2$ behave very similarly to vector spin glasses, except that for Ising spin glasses the definition of $E_c$ is not that for the set of all states of energy $E$ as for the case of vector spin glasses but instead it is the critical energy for those states produced in the quench (which have a distribution of local fields $p(h) \sim h$ at small $h$). In Ref. [12] the nature of the ordering
associated with the SOC behavior was not specified. If the quench is close to this critical energy one would expect there to be a divergent spin glass susceptibility; the definition and the study of this susceptibility is one of the main topics of this paper. It is a purely static quantity: the study of avalanches alone does not provide insights into the nature of the incipient ordering associated with the SOC.

There is an important distinction between Ising and vector spin glasses. Edwards hypothesized (for a review see [27]) that systems like powders or sand piles etc. could be understood not by solving the full dynamics of the system from its initial state to its final resting state (which is hard) but instead by determining for these systems the analogue of the number of states in spin glasses in which the spins are parallel to their local fields, (which is easy) [24]. For vector spin glasses in the SK limit, his hypothesis has utility. It fails completely for modelling quenches in the Ising SK spin glass as it is only by a full dynamical treatment that one can obtain a $p(h)$ which is linear in $h$ [5, 6, 7].

In Sec. 2 we introduce the KAS model. In Sec. 3 we investigate how the quenched state depends on the initial state. In Sec. 4 we study the distribution of the local fields $p(h)$ of the quenched state, and from its form deduce that marginality only exists when $\sigma \leq 1/2$. The existence of SOC behavior is deduced from a study of the $N$ dependence of the spin glass susceptibility and the energy of the quenched state in Sec. 5.

2. The Model

The Kotliar, Anderson and Stein (KAS) [1] Hamiltonian is

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j,$$

where the Ising spins $S_i$ ($i = 1, 2, \cdots, N$), taking values $\pm 1$, are arranged in a circle of perimeter $N$. The geometric distance between sites $i$ and $j$ is $r_{ij} = \frac{N}{\pi} \sin \left[ \frac{\pi}{N} (i - j) \right]$, the length of the chord between the sites $i,j$. The interactions $J_{ij}$ are long-ranged and depend on the distance $r_{ij}$ as $J_{ij} = c(\sigma, N) \varepsilon_{ij} / r_{ij}^\sigma$, where $\varepsilon_{ij}$ is a Gaussian random variable of mean zero and unit variance. The coefficient $c(\sigma, N)$ is chosen to make the mean-field transition temperature $T_{c, MG}$ equal to unity for all values of $\sigma$:

$$[T_{c, MG}^{MF}(c)]^2 = \frac{1}{N} \sum_{i \neq j} [J_{ij}^2]_{av} = c(\sigma, N)^2 \sum_{j=2}^{N} \frac{1}{r_{1j}^{2\sigma}} = 1. \quad (2)$$

The sum over $j$ can be done for large $N$ and gives

$$\frac{1}{c(\sigma, N)^2} = 2\zeta[2\sigma] + \frac{\Gamma[1/2 - \sigma]^2}{2^{2\sigma} \Gamma[1 - 2\sigma]} \left( \frac{N}{\pi} \right)^{1-2\sigma} + O\left( \frac{1}{N^2} \right). \quad (3)$$

Thus when $\sigma < 1/2$, $c(\sigma, N) \sim [1/N^{1/2-\sigma}](1 + O(1/N^{1-2\sigma}))$, while for $\sigma > 1/2$, $c(\sigma, N) \sim (1 + O(1/N^{2\sigma-1}))$. We have found when studying the energy per spin $E$ reached in the quench that there is a finite size correction for $\sigma > 1/2$ of $O(1/N^{2\sigma-1})$, whose origin is that $E \propto c(\sigma, N)$. Such corrections to scaling are large, especially
Figure 1. The distribution of overlaps $P(q)$ of the states reached when the initial state is equilibrated at a temperature $T$. A range of temperatures $T$ is shown for three values of $\sigma = 0, 0.75, 1.50$. The system size is $N = 256$. For each sample of disorder, $N_{\text{min}} = 15$ minima are found and overlapped with each other, thus obtaining 105 different values of $q$ from which an histogram $P(q)$ is obtained. This $P(q)$ is then averaged over $N_{\text{samp}} = 200$ samples of disorder, thus also extracting error-bars. The SK limit has a phase transition at $T_c = 1$, while at $\sigma = 0.75$, $T_c = 0.62$ [16]. No phase transition exists for $\sigma = 1.5$. Whether the final state displays a non-trivial $P(q)$ depends on whether the initial temperature $T$ is less than $T_c$ or not. For $\sigma = 1.5$, a trivial $P(q)$, that is one approaching $\delta(q)$ as $N \to \infty$, is obtained even for quenches from very low temperatures.
when $\sigma$ is close to $1/2$. Note that this correction to scaling is associated with the zero temperature fixed point, rather than the critical fixed point. It is only the discovery of this form for the leading correction to scaling that has enabled us to analyze our data.

There is a mapping between $\sigma$ and an effective dimensionality $d_{\text{eff}}$ of the EA model \cite{16, 17, 18, 19, 20}. For $1/2 < \sigma < 2/3$, it is $d_{\text{eff}} = 2/(2\sigma - 1)$; thus $\sigma = 2/3$ corresponds to an effective dimensionality of 6.

We generated one spin flip stable states by a quenching procedure that involves repeatedly flipping spins to orient them with their local fields, according to the sequential (as opposed to the ‘greedy’ or ‘polite’) algorithm. Previous work \cite{3} suggests that although a difference in the energy of the final state can be seen based on the precise algorithm employed, the nature of the final state is independent of the algorithm. In the sequential algorithm used here, sites are scanned sequentially from 1 through $N$, and at each of them the spin is aligned to its local field, thus monotonically reducing the energy of the system. When a spin is flipped the local fields $h_i$ are immediately updated. The protocol of repeatedly aligning spins is carried out until convergence is obtained. The initial state was either a random spin state, which corresponds to infinite temperature, or one of the spin configurations of an equilibrated system at temperature $T$.

3. Dependence of the Parisi overlap $P(q)$ on the initial state

The overlap between two minima $A$ and $B$ obtained after a quench is defined as

$$ q \equiv \frac{1}{N} \sum_i S_i^A S_i^B. $$

(4)

Its distribution $P(q)$ contains crucial information about the nature of the final state reached by the quench. In Fig. 1 we have plotted the sample averaged distribution function $P(q)$ for systems of $N = 256$ spins obtained in quenches from various temperatures $T$. For a quench which starts at a temperature $T > T_c$, the resulting $P(q)$ is trivial, in that it reduces in the large $N$ limit to $P(q) = \delta(q)$ \cite{25}. Such a form indicates that the states obtained in the quench are completely uncorrelated from each other. However, for quenches which start from a temperature $T < T_c$ a non-trivial $P(q)$ was found. One would expect that the $P(q)$ obtained from a quench starting from $T < T_c$ has the replica symmetry breaking or replica symmetry features expected for that $\sigma$ value, whatever that might be. For $\sigma > 1 T_c$ is expected to be zero, and our results at $\sigma = 1.5$, shown in the third panel, are consistent with the final state always being that expected from a quench which starts in the paramagnetic region.
4. Marginality and the distribution of local fields $p(h)$

In this section we discuss the distribution of local fields $h_i$ after the quench from infinite temperature. The magnitude of $h_i$ after the quench is given by

$$h_i = S_i \sum_j J_{ij} S_j,$$

(5)

where $S_i$ are the spins at the end of the quench. Notice that $h_i > 0$. What will interest us mostly is whether the form of $p(h)$ provides any evidence for marginality, in the sense that the state which is reached is just on the edge of stability [28]. Our conclusion will be that the quenched state has indeed marginal stability if $\sigma \leq 1/2$, but not if $\sigma > 1/2$.

We shall use the argument of P. W. Anderson (as reported in Ref. [4]) to obtain a “bound” on the local field distribution $p(h)$ for small $h$. For $\sigma < 1/2$, it is expected that $p(h) = h/H^2$ at small fields in the thermodynamic limit (see Fig. 2). In the state reached in the quench, relabel the sites in order of their increasing local field $h_i$ and consider the first $n$ of these sites, where $1 \ll n \ll N$. Suppose one flips all $n$ of the...
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![Figure 3](image_url)

Figure 3. Plot of $p(h) - p(0)$ versus $h$ for values of $\sigma$, 0.0, 0.1, 0.2, 0.3, 0.4, and 0.5 for $N = 8192$ after a quench from a random initial state. The subtraction of $p(0)$ is done to reduce finite size effects. The red line is a line of slope 0.89 which is the value expected if the quenched states are just marginal.

Spins at these low-field sites: the consequent energy change is

$$\Delta E = 2 \sum_{i=1}^{n} h_i - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} J_{ij} S_i S_j.$$  \hspace{1cm} (6)

$\Delta E$ would be non-negative if the initial state were the ground state. Biroli and Monasson \cite{29} gave an argument that for the SK model any state (and not just the ground state) is stable against flipping a finite number of spins. Their argument was that $\Delta E_{ij} = 2h_i + 2h_j - 2J_{ij} S_i S_j$ which corresponds to flipping just two spins, reduces to $2h_i + 2h_j$ in the large $N$ limit as $J_{ij}$ goes to zero as $1/N^{1/2}$ when $N \to \infty$ for the SK model. But $h_i$ and $h_j$ could themselves be of order $O(1/\sqrt{N})$ and excluding this possibility will give us a bound on the value of the coefficient $H$.

The value of $h_n$ can be obtained from solving

$$n = N \int_0^{h_n} dh \, \frac{h}{H^2} = \frac{Nh_n^2}{2H^2}. \hspace{1cm} (7)$$

The first term, $\Delta E_1$ in Eq. (6) is similarly

$$\Delta E_1 = 2 \sum_{i=1}^{n} h_i = 2N \int_0^{h_n} dh \, \frac{h^2}{H^2} = \frac{2Nh_n^3}{3H^2} = \frac{4\sqrt{2}}{3} Hn \sqrt{\frac{n}{N}}. \hspace{1cm} (8)$$
For the case $\sigma > 1/2$, it is expected that marginality [28] requires
\[ p(h) = \frac{1}{H} \left( \frac{h}{H} \right)^{1/(\sigma-1)}. \] (9)

Eq. (7) becomes
\[ n = \sigma N \left( \frac{h_n}{H} \right)^{1/\sigma}, \] (10)

while Eq. (8) becomes
\[ \Delta E_1 = 2n \left( \frac{n}{N} \right)^{\sigma} \frac{H}{(\sigma+1)\sigma^\sigma}. \] (11)

The second term in Eq. (6) can be re-written as \( \Delta E_2 = -2 \sum_{i=1}^n S_i \Delta_i \), where \( \Delta_i = \sum_{j=1}^n J_{ij} S_j \). \( \Delta_i \) is a quantity which on average is zero. Its variance is
\[ \frac{1}{n} \sum_{i=1}^n \Delta_i^2 = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n J_{ij} S_j \sum_{k=1}^n J_{ik} S_k = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n J_{ij}^2. \] (12)

Suppose now that the positions of the spins \( S_i, i = 1, 2, \cdots, n \) are equally spaced so that \( R_i = iN/n \), then the variance equals \( (n^{2\sigma}/N^{2\sigma})c(\sigma, N)^2/c(\sigma, n)^2 \) which reduces to \( n/N \) when \( \sigma < 1/2 \) and to \( (n/N)^{2\sigma} \) when \( \sigma > 1/2 \). \( (c(\sigma, N) \) was defined in Eq. (3) and we have used its large \( N \) and \( n \) form). Since \( S_i \) and \( \Delta_i \) are correlated in sign, we have for \( \sigma < 1/2, \Delta E_2 = -2n\sqrt{n/N}. \) For \( \sigma > 1/2, \Delta E_2 = -2n(n/N)^{\sigma}. \)

Then the total energy \( \Delta E = \Delta E_1 + \Delta E_2 \) becomes for \( \sigma < 1/2, \)
\[ \Delta E = 2n \left( \frac{n}{N} \right)^{1/2} \left[ \frac{2\sqrt{2}}{3} H - 1 \right], \] (13)
so the quenched state would be just marginal (i.e. has \( \Delta E = 0 \)) if
\[ H = \frac{3}{2\sqrt{2}}. \] (14)

For \( \sigma > 1/2 \)
\[ \Delta E = 2n \left( \frac{n}{N} \right)^{\sigma} \left[ \frac{1}{(\sigma+1)\sigma^\sigma} H - 1 \right]. \] (15)
Thus if the system is just marginal
\[ H = (\sigma+1)\sigma^\sigma. \] (16)

In Fig. 3 we have plotted \( p(h) - p(0) \) (the subtraction of \( p(0) \) is to reduce the consequences of the finite size intercept on the y-axis) as a function of \( h \) for some \( \sigma \) values less than 1/2. The slope of the red line which is drawn using the value of \( H \) which makes the system just marginal agrees quite well with the data for \( \sigma < 1/2. \)

In Fig. 4 we have repeated the exercise for \( \sigma = 0.7. \) The green line is the line which would be expected if the system is just marginal. The data points are not close to this expectation at all and indicate that the state reached in the quench is stable (i.e. \( \Delta E > 0 \)) by the Anderson criterion. We have examined other \( \sigma \) values which are greater
than 0.5 and have found that the size of the discrepancy increases steadily as $\sigma$ rises above 0.5.

Another confirmation that the quenched state is not marginal for $\sigma > 1/2$ is provided by Fig. 5. An assumption behind marginality is that Eq. (9) should hold. If that is the case, then in the large $N$ limit the smallest value of $h$, $h_{\text{min}}$ should decrease as $1/N^\sigma$ (see Eq. (10) with $n = 1$). The results for $\sigma = 0.75$ in Fig. 5 indicate that a better fit to the data is as $h_{\text{min}} \sim 1/N^{0.70}$. However, the discrepancy is modest for the exponent.

The states generated in the quench for $\sigma > 1/2$ seem to be stable according to the Anderson argument, where one examines the stability against flipping the spins in the first $n$ smallest fields, (see Fig. 4), as $H$ is larger than the just marginal value $(\sigma + 1)\sigma^\sigma$ if one determines it from the slope of the data at small $h$. However, we suspect that they are unstable by the argument of Biroli and Monasson against flips of two spins where the fields of the flipped spins are not restricted to be small as in the Anderson argument. In other words, the states generated by the quench are just one spin flip stable states. In the SK limit a state generated by the quench will be a pure state [29].
stable against flipping an arbitrary number of spins.

In the next section we show that the existence of marginality for $\sigma \leq 1/2$ seems to be associated with self-organized criticality as we can only find that when $\sigma \leq 1/2$.

5. Self-organized criticality

We have made a finite size scaling study of the spin-glass susceptibility $\chi_{SG}$:

$$\chi_{SG} = \frac{1}{N} \sum_{i,j} \langle (S_i S_j)^2 \rangle_{\text{av}},$$

(17)

where the angular brackets represent an average over the metastable minima for a given sample of disorder. The minima in this case were obtained from a random initial state, so that we are studying the case where $P(q)$ is trivial. Note that $\chi_{SG} = N \text{Variance}(q^2)$.

In the regime $0 \leq \sigma \leq 1/2$, $\chi_{SG}$ appears to diverge as $\ln(N)$, whereas for the region $\sigma > 1/2$, $\chi_{SG}$ saturates to a finite value at large $N$, the form of this dependence being
Figure 6. The first two panels show the $N$-dependence of $\chi_{SG}$ as defined in Eq. (17) for a variety of $\sigma$. The number of samples used was 1000 and the number of initial infinite temperature (random) configurations was 15. For $\sigma \leq 0.5$, $\chi_{SG}$ diverges as a logarithm of the system-size. The data points for $\sigma = 0.0, 0.1, 0.2, 0.3$ are so close as to be barely distinguishable. For $\sigma > 0.5$, there is a clear tendency for $\chi_{SG}$ to saturate at the largest sizes we are able to study: we are able to find very good fits to the saturating functional form $\chi_{SG} = a - b/N^{2\sigma-1}$ (see the second panel). The third panel shows the fit parameters $a, b$ as a function of $\sigma$. In the first panel VB refers to the Viana-Bray model which is a diluted version of the SK limit, $\sigma = 0$, of the KAS model in which each spin is only coupled to six others: its size independent $\chi_{SG}$ shows that it lacks SOC. Ref. [12] also reached the same conclusion based on their study of avalanches.
Figure 7. Energy per spin $E$ as a function of system size $N$ after quenches, for $\sigma \leq 0.5$ in the left panel and $\sigma > 0.5$ in the right panel. In the quenches from the infinite temperature (random) initial state 1000 samples were used with 15 different initial starts. The quenches from $T_c$ were done only for the SK limit ($\sigma = 0$ where $T_c = 1$) and for $\sigma = 0.75$ where $T_c = 0.62$ [16] and for these values of $\sigma$ straight lines have been drawn through the data points for both types of quench as a guide to the eye. For the quenches from $T_c$ only 200 samples were averaged. For $0 \leq \sigma \leq 0.5$, the energy saturates to a characteristic energy $E_c$ as $1/\ln N$ for both the quench from infinite temperature and from $T_c$. For $\sigma > 0.5$, the energy of quenches fits well to the form $c + d/N^{2\sigma-1}$. The coefficients $a, b$ appear to approach each other and diverge as $\sigma \to 0.5^+$ (see Fig. 6). Note that

$$
\frac{1}{2\sigma - 1} \left[ 1 - \frac{1}{N^{2\sigma-1}} \right] \to \ln(N)
$$

in the limit $\sigma \to 0.5$. Thus the divergence of $\chi_{SG}$ for $\sigma \leq 1/2$ as $\ln N$ seems to be natural if one takes the Mori argument [15, 30] that all systems for $\sigma \leq 1/2$ behave in the same way, and just as in the SK limit of $\sigma = 0$. Consistent with this finding for $\chi_{SG}$, Fig. 7 shows that the energy reached by the quench $E(N)$ goes as $E_c + \text{const}/\ln(N)$ for $\sigma < 0.5$, almost independent of $\sigma$, at least for $\sigma = 0.0, 0.1, 0.2$ and 0.3: only the data points for $\sigma = 0.4$ and 0.5 differ significantly and for them the finite size corrections are very large. The Mori argument says that in the thermodynamic limit quantities such as the energy should be independent of the value of $\sigma$ when it is less than 0.5. However, for $\sigma > 1/2$ the energy $E(N)$ behaves quite differently and the right panel of Fig. 7 shows that it goes as $c + d/N^{2\sigma-1}$, just as could have been anticipated from the $N$ dependence of $c(\sigma, N)$.

We next explain why these results are consistent with SOC behavior for $\sigma \leq 1/2$. There is an energy $E_c$ in the large $N$ limit which separates minima which are just at the brink of having a non-trivial form for $P(q)$ from those at higher energy which have trivial overlaps. This has been established for the SK model when the average is taken over all
one spin-flip stable states [31]. $E_c$ marks the transition to a state with broken replica symmetry. We expect that there will be a similar critical energy for states prepared by quenches from infinite temperature, and that its numerical value will depend on the quench procedure.

At $E_c$ massless modes are present: that is near $E_c$ the system has marginal stability. We learned in Sec. III that for $\sigma < 1/2$ the state reached in the quench had marginal stability so we would expect that the energy reached in the quench is close to $E_c$. $E_c$ is the analogue of the transition temperature $T_c$ in studies of the thermal spin glass susceptibility [31] as the spin glass susceptibility diverges for $\sigma \leq 1/2$ as $\chi_{SG} \sim 1/\tau$, the usual mean-field form, where $\tau = (1 - T_c/T)$. Our quenches take us close to $E_c$ but miss by an amount of $O(1/\ln N)$ due to finite size effects; our analogue of $\tau$ is $\sim 1/\ln N$, so $\chi_{SG} \sim \ln N$. This result is also consistent with our argument by continuity from $\sigma > 1/2$ in Eq. (18).

For quenches from a temperature $T > T_c$ one would expect that the extrapolated energy $E_c(T)$ would be slightly different from that obtained in the quench from infinite temperature. Fig. 7 shows that the quench from $T_c$ for the SK model goes indeed to a somewhat lower value of the energy by an amount ($\sim 1\%$) from that at infinite temperature, (and which is incidentally very close to the Parisi type estimates of the true ground state per spin: $E_g = -0.76316677265(6)\ldots$ [32]). The quench from $T_c$ is from an initial state where there are long-range correlations which are absent for the initial state at infinite temperature, which is probably why their associated values of $E_c$ differ.

For $\sigma > 1/2$ we did not see marginal stability and so the energy reached in the quenched state is probably not close to any critical value below which replica symmetry breaking effects might become visible. For $\sigma > 2/3$ we doubt even the existence of any states with broken replica symmetry [33]. Because of the absence of marginality it is no surprise really that $\chi_{SG}$ shows no sign of diverging with increasing $N$. Figure 6 indicates that it is approaching a finite value as $1/N^{2\sigma-1}$.

We suspect that the existence of SOC behavior and marginality only for $\sigma \leq 1/2$ might be reflected by differences in the nature of spin avalanches for $\sigma$ above and below $1/2$. Horner [6] found that in the SK model the number of spin flips per site before the final quenched state was reached increased $\approx \ln N$. However, Andresen and others [12, 22] found avalanches on the scale of the system size $N$ only when $z$, the number of neighbors of a given site increases with $N$, as happens in the SK model. We would imagine such behavior would extend up to $\sigma = 1/2$. For $\sigma > 1/2$ the effective number of neighbors is finite (even though the critical behavior remains mean-field like up to $\sigma = 2/3$; the KAS model with $1/2 < \sigma < 2/3$ maps to the nearest-neighbor Edwards-Anderson model in a dimension $d > 6$). Thus the dynamics would be expected to change at $\sigma = 1/2$ along with the disappearance of SOC behavior and marginality.
6. Conclusions

In the KAS model we have discovered that there is a connection between SOC behavior and marginality. Because the states reached in the quenches are marginal when $\sigma \leq 1/2$, they are near the energy at which the states have massless modes i.e. are becoming critical. In this case, the criticality is that associated with the onset of replica symmetry breaking.

It would be interesting to know whether in the many systems which are thought to have marginal behavior [23], there is a similar connection with self-organized critical behavior. What is striking about the KAS model is that the transition which is self-organized can be identified; it is the transition to states with correlations between them due to the onset of broken replica symmetry.

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