Monte Carlo approach to turbulence

P. Düben\textsuperscript{a}, D. Homeier\textsuperscript{a}, K. Jansen\textsuperscript{b}, D. Mesterhazy\textsuperscript{c} and G. Münster\textsuperscript{a}\textsuperscript{†}

\textsuperscript{a} Westfälische Wilhelms-Universität, Institut für Theoretische Physik
Wilhelm-Klemm-Str. 9, 48149 Münster, Germany
\textsuperscript{b} NIC, DESY Zeuthen
Platanenallee 6, 15738 Zeuthen, Germany
\textsuperscript{c} Humboldt-Universität zu Berlin, Institut für Physik
Newtonstrasse 15, 12489 Berlin, Germany
E-mail: munsteg@uni-muenster.de

The behavior of the one-dimensional random-force-driven Burgers equation is investigated in the path integral formalism on a discrete space-time lattice. We show that by means of Monte Carlo methods one may evaluate observables, such as structure functions, as ensemble averages over different field realizations. The regularization of shock solutions to the zero-viscosity limit (Hopf-equation) eventually leads to constraints on lattice parameters required for the stability of the simulations. Insight into the formation of localized structures (shocks) and their dynamics is obtained.

The XXVII International Symposium on Lattice Field Theory
July 26 - 31, 2009
Peking University, Beijing, China

\textsuperscript{†}DESY 09-174, MS-TP-09-23
\textsuperscript{†}Speaker.
1. Introduction

Besides tremendous research having been done since Kolmogorov’s famous publication in 1941 [1], hydrodynamic turbulence essentially remains an unsolved problem of modern physics. This is especially remarkable as the fundamentals seem to be fairly easy – the Navier-Stokes equations for the velocity field $u_\alpha$ and pressure $p$

$$\partial_t u_\alpha + u_\beta \partial_\beta u_\alpha - \nu \nabla^2 u_\alpha + \frac{1}{\rho} \partial_\alpha p = 0$$  \hspace{1cm} (1.1)

with the additional constraint

$$\partial_\alpha u_\alpha = 0$$  \hspace{1cm} (1.2)

simply express the conservation of momentum in a classical, incompressible fluid of viscosity $\nu$ and density $\rho$. For laminar flows it is well known that the Navier-Stokes equations reproduce realistic flows very accurately; in the turbulent regime, it is still an open question how the universal characteristics of turbulent flow, characterized by the scaling exponents $\xi_p$ of structure functions $S_p$ of order $p$, defined by

$$S_p(x) := \frac{1}{\bar{u}(r+x) - \bar{u}(r)} \sim |x|^{\xi_p},$$  \hspace{1cm} (1.3)

can be extracted from first principles. Here the bar corresponds to a spatial averaging.

Monte Carlo simulations in the path integral formulation enable us to gain direct insight into the formation of localized structures and their behavior, and to measure observables as, e.g. structure functions and their scaling exponents [2 – 4].

2. Burgers’ Equation

We decided to elaborate the methods using the stochastically forced Burgers equation [5] in 1+1 dimensions

$$\partial_t u + u \partial_x u - \nu \partial_x^2 u = f,$$  \hspace{1cm} (2.1)

which may be interpreted as the flow equation for a fully compressible fluid. The stochastic force is modeled to be Gaussian with correlation

$$\chi(x,t;x',t') := \langle f(x,t) f(x',t') \rangle = \epsilon \delta(t-t') \exp \left(-\frac{|x-x'|}{\Lambda}\right),$$  \hspace{1cm} (2.2)

where $\Lambda$ defines the correlation length of the forcing and the $\langle \cdots \rangle$ denotes the ensemble average. A finite viscosity $\nu$ and energy dissipation $\epsilon$ provide a dissipation length scale $\lambda$ corresponding to the Kolmogorov-scale in Navier-Stokes turbulence:

$$\lambda := \left(\frac{v^3}{\epsilon}\right)^{\frac{1}{2}}.$$  \hspace{1cm} (2.3)

We can furthermore identify the Reynolds-number as

$$Re := (\epsilon \Lambda^4 / v^3)^{1/3}.$$  \hspace{1cm} (2.4)
The fundamental solutions to the Burgers equation are well-known – in the limit of vanishing viscosity (Hopf-equation) these form singular shocks. A finite dissipation scale $\lambda \sim \nu / U$, where $U$ is the characteristic velocity, provides an UV-regularization of the shock structures:

$$u = -U \tanh \frac{U}{2\nu} x.$$  \hspace{1cm} (2.5)

Most interestingly, the exponents $\xi_p$ as defined in (1.3) are non-trivial for Burgers turbulence; for the forcing (2.2) and in the regime $x / \lambda$ we have the analytic result [6]

$$\xi_p = \min(1, p).$$  \hspace{1cm} (2.6)

3. Path Integral Formulation

Following the method of Martin, Siggia and Rose [7], we established a path integral for Burgers’ equation

$$Z \propto \int D u \exp \left( -\frac{1}{2} \int dt dx \left( \partial_t u + u \partial_x u - \nu \partial_x^2 u \right) \chi^{-1} \ast \left( \partial_t u + u \partial_x u - \nu \partial_x^2 u \right) \right),$$  \hspace{1cm} (3.1)

where $\ast$ denotes the convolution.

It has been shown by Falkovich et al. [8, 9] on the basis of an equivalent sum of states that the fundamental solutions of Burgulence can be understood as instantons.

4. Monte Carlo Simulations

For 1+1 dimensional Burgulence, a large number of stable simulations could be performed; we are working on the final analysis. Typical lattice sizes range from $(N_x = 16) \times (N_t = 16)$ up to $(N_x = 4096) \times (N_t = 128)$ lattice points.

4.1 Boundary Conditions

To be in general agreement with literature and analytic calculations, we started with lattices periodic both in time and space direction. In an attempt to reduce autocorrelation times, we dropped these boundary conditions. While autocorrelation times did not change much, simulating with free boundaries effectively doubles the spatial lattice extent and gives access to excitations of the Burgers vacuum state.

4.2 Lattice Discretization

Once having discretized the path integral on a Euclidean lattice of spacings $\Delta x$ and $\Delta t$, we get for $\nu$:

$$\nu = \alpha (\Delta x)^2 / \Delta t.$$  \hspace{1cm} (4.1)

The continuum limit of our lattice theory is reached by holding $\nu$ and $Re$ constant while increasing the number of lattice sites. $\alpha$ is an a priori arbitrary constant that can be interpreted as a measure for $\Delta t$ depending on $\Delta x$ and also has to be kept fixed while performing the continuum limit.
4.3 Algorithms

We use a local heat bath algorithm with successive over-relaxation (SOR) for the Monte Carlo evaluation of the partition function \([11]\). The use of certain acceleration techniques with SOR, specifically Chebyshev acceleration \([12]\), significantly accelerates the thermalization process.

Though suiting our purposes so far, it poses certain restrictions on parallelization. We therefore started employing a Hybrid Monte Carlo algorithm that we expect to scale better with the number of parallel processors.

4.4 Autocorrelation Times

With \(\chi\) being a nonlocal operator one would expect long autocorrelation times in the simulation of stochastically forced differential equations. However, with the over-relaxed heat bath algorithm and an appropriate definition of structure functions on the lattice (where the reference point for evaluation is chosen randomly for each configuration) the integrated autocorrelation time is reduced to \(\tau \sim O(1)\).

4.5 Resources

For testing purposes small lattices may easily be simulated on desktop PCs. However, high resolution simulations on large lattices require massively parallel architectures. We have run our simulations on the IBM p690 cluster JUMP at FZ Jülich and on the Linux cluster at Humboldt University Berlin with up to 256 processors in parallel. In July 2009 we continued our simulations on the new supercomputer JUROPA at FZ Jülich.
5. First Results

First results include further constraints that have to be imposed in order to ensure stable numerics. Most constraining is the need to resolve the Kolmogorov-length scale $\lambda$ on the lattice. We can in this way show the effect of $\lambda$ as UV-regularization of the otherwise singular shocks. This translates into a relation for the Reynolds-number:

$$Re < \frac{\Lambda}{\Delta x}.$$  

(5.1)

This will become crucial for Navier-Stokes turbulence enforcing us to simulate big lattices.

5.1 Structure Functions

From analytic calculations \[8\] we have

$$S_p(x) \sim C_p |x|^p + C'_p |x|,$$  

(5.2)

for small separations in the inertial range.

Though our results are in general agreement with this, the extraction of scaling exponents is far from trivial and very sensitive to statistical errors.

5.2 Extended Self-Similarity (ESS)

Rather than measuring the scaling exponents $\xi_p$ directly, there have been attempts to measure the scaling behavior of ratios of structure functions \[10\]. It was shown that this greatly enhances
the inertial range not only at high but also moderate Reynolds numbers. However, we must stress that up to now it is not clear if there are any systematic effects in the evaluation of the structure function exponents via ESS.

5.3 Outlook

After completing the analysis of 1+1 dimensional Burgulence, we will proceed to 3+1 dimensions. The ultimate challenge will be the simulation and analysis of 2+1 and 3+1 dimensional Navier-Stokes turbulence.

5.4 Acknowledgements

We thank the John von Neumann-Institute for Computing (NIC), Jülich, for computing time and support. D. M. thanks the IRZ Physik at HU Berlin for computing time spent on the local Linux cluster and their staff for technical support.

References

[1] A. N. Kolmogorov, *The local structure of turbulence in incompressible viscous fluid for very large Reynolds number*, Dokl. Akad. Nauk SSSR 30 (1941) 299.

[2] P. Düben, D. Homeier, K. Jansen, D. Mesterhazy, G. Münster, C. Urbach, *Monte Carlo simulations of the randomly forced Burgers equation*, Europhys. Lett. 84 (2008) 40002.

[3] D. Mesterhazy, *Investigations of Burgers Equation in the Path Integral Formalism* (Bachelor thesis, 2008).

Figure 3: $\log[S_3(x)]$ as function of $\log[S_1(x)]$ clearly showing the linear ESS-dependence.
[4] P. Düben, *Numerische Anwendungen des Pfadintegralformalismus in hydrodynamischer Turbulenz* (Diploma thesis, 2009).

[5] J. M. Burgers, *The nonlinear Diffusion Equation* (Reidel, Dordrecht, 1974).

[6] J. Bec, K. Khanin, *Burgers Turbulence*, Phys. Rep. 447 (2007) 1.

[7] P. C. Martin, E. D. Siggia, H. A. Rose, *Statistical Dynamics of Classical Systems*, Phys. Rev. A 8 (1973) 423.

[8] G. Falkovich, I. Kolokolov, V. Lebedev and A. A. Migdal, *Instantons and intermittency*, Phys. Rev. E 54 (1996) 4896.

[9] E. Balkovsky, G. Falkovich, I. Kolokolov and V. Lebedev, *Intermittency of Burgers’ Turbulence*, Phys. Rev. Lett. 78 (1997) 1452.

[10] R. Benzi, S. Ciliberto, R. Tripiccione, C. Baudet, F. Massaioli, S. Succi, *Extended self-similarity in turbulent flows*, Phys. Rev. E 48 (1993) R29.

[11] S. L. Adler, *Over-relaxation method for the Monte Carlo evaluation of the partition function for multiquadratic actions*, Phys. Rev. D 23 (1981) 2901.

[12] R. S. Varga, *Matrix Iterative Analysis* (Prentice Hall, 1962).