Phonon-induced artificial magnetic fields in optical lattices

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Abstract – We investigate the effect of a rotating Bose-Einstein condensate on a system of
immersed impurity atoms trapped by an optical lattice. We analytically show that for a one-
dimensional, ring-shaped setup the coupling of the impurities to the Bogoliubov phonons of the
condensate leads to a non-trivial phase in the impurity hopping. The presence of this phase can
be tested by observing a drift in the transport properties of the impurities. These results are
quantitatively confirmed by a numerically exact simulation of a two-mode Bose-Hubbard model.
We also give analytical expressions for the occurring phase terms for a two-dimensional setup.
The phase realises an artificial magnetic field and can, for instance, be used for the simulation of
the quantum Hall effect using atoms in an optical lattice.

There are several phenomena in physics, such as the quantum Hall effect [1] or high-temperature superconductivity [2], that have been known for more than two decades, but where a full theoretical understanding of the underlying mechanisms is still lacking. This is due to the fact that a complicated many-body system consisting of electrons in a crystal has to be described on a quantum mechanical level, which quickly exceeds the capability of modern computers. As Feynman suggested [3], it is therefore worthwhile to search for alternative quantum systems that allow for a clean realisation of the underlying models and that are easy to manipulate. With such a quantum simulator, we would be able to do simulations that by far exceed what is currently possible on classical computers.

Ultracold atoms in optical lattices are a top candidate when it comes to the simulation of condensed matter phenomena [4]. It has been shown theoretically that for a suitable experimental setup the atoms in the lattice indeed realise typical quantum Hall states [5–7]. In a solid-state setup, these states are realised by the application of a strong magnetic field. Since the motional state of the neutral atoms in the optical lattice does not couple to a magnetic field, the effect of this field has to be simulated by other means. Proposals include using special lattice setups and Raman-assisted hopping [8], exploiting light with orbital angular momentum [9,10], or simply rotating the lattice [11]. Whereas the first proposals are rather complicated to implement, the rotation of the lattice has already been demonstrated in experiments [12]. However, in order to observe the quantum Hall states the centrifugal term caused by the rotation has to be carefully balanced by an additional harmonic trapping potential confining the gas, which is difficult to achieve in an experiment.

In this work, we therefore propose a way of exploiting a rotating Bose-Einstein condensate (BEC) to create an artificial magnetic field in an optical lattice system which is submerged into the condensate. This has the advantage that the system in which the quantum Hall states are to be observed, i.e. the optical lattice, does not need to be rotated and thus there is no need of employing a balancing potential. The BEC, on the other hand, only provides the artificial magnetic field and we are not interested in a direct observation of its states. Hence, we can employ a harmonic trapping potential which overcompensates the centrifugal force instead of exactly canceling it, which would not be possible if we wanted to observe the quantum Hall states in the BEC directly.

The presence of the condensate can not only be used to provide an artificial magnetic field, it also allows for the simulation of other effects which are present in condensed-matter physics [13]. For example, it has been shown that for suitably chosen parameters the coupling to the BEC phonons may change the transport behaviour of the lattice atoms from coherent to incoherent [14]
and that an attractive interaction between the lattice atoms can be mediated by the condensate which leads to clustering effects [15]. By including an artificial magnetic field, the whole setup can therefore be used as a well-suited simulator for phenomena encountered in condensed-matter physics.

**Theoretical description.** In the following, we will derive an effective Hamiltonian for the atoms in the optical lattice. A similar derivation has been given in ref. [13], where it was assumed that the wave function describing the condensate is real. In general, this will no longer be the case if the condensate is rotated or translated, leading to an additional phase in the hopping terms of the lattice Hamiltonians. The Hamiltonian describing the whole system consists of three parts, $H = H_B + H_I + H_a$. Here, $H_a$ determines the free dynamics of the atoms of species $b$ trapped in the optical lattice, which we will call impurities in the following. In the laboratory frame, the BEC Hamiltonian $H_B$ describing the atoms of species $b$ and the density-density interaction Hamiltonian $H_I$ are

$$
H_B = \int \text{d}r \hat{\phi}^\dagger(r) \left[ \hat{H}_0 + \frac{g}{2} \hat{\phi}^\dagger(r) \hat{\phi}(r) \right] \hat{\phi}(r),
$$

$$
H_I = \kappa \int \text{d}r \hat{\chi}^\dagger(r) \hat{\chi}(r) \hat{\phi}^\dagger(r) \hat{\phi}(r),
$$

where $\hat{\chi}(r)$ is the impurity field operator, $\hat{\phi}(r)$ is the condensate atom field operator satisfying the usual bosonic commutation relations, and $\hat{H}_0$ contains the kinetic energy term $-\hbar^2 \nabla^2 / 2m_b$, an external trapping potential $V_{\text{ext}}(r)$, the chemical potential $\mu$ and any other terms that correspond to rotations or translations of the BEC as will be detailed later. The coupling constants $g > 0$ and $\kappa$ account for the boson-boson and impurity-boson interaction, respectively, and $m_b$ is the mass of a condensate atom.

In the tight-binding approximation, the impurity field operator can be expanded as $\hat{\chi}(r) = \sum_j \eta_j(r) \hat{a}_j$, where $\hat{a}_j^\dagger$ creates an impurity atom in lattice site $j$ and $\eta_j$ is the corresponding Wannier function [16]. The probability densities of these functions have a negligible overlap, i.e., \[ \int \eta_j(r)^2 \eta_{j'}(r)^2 \text{d}r \approx 0 \] for $j \neq j'$, which will be important later. By using the above expansion for the field operator $\hat{\chi}$ the Hamiltonian for the impurities is given by $H_a = -J_a \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + (U_{ab}/2) \sum_j \hat{n}_j (\hat{n}_j - 1) - \mu_a \sum_j \hat{n}_j$, where $\mu_a$ describes the chemical potential, $J_a$ is the hopping matrix element between two neighbouring sites of the lattice, and $U_{ab}$ is the on-site interaction strength.

To investigate the influence of the rotating BEC on the impurities we employ the Bogoliubov approximation to simplify the dynamics of the condensate. To this end, we define $\hat{\phi}(r) = \phi_0(r) + \hat{\phi}(r)$, where $\phi_0$ is a solution of the Gross-Pitaevskii equation (GPE) [17] for $\kappa = 0$,

$$
0 = (\hat{H}_0 + g |\phi_0(r)|^2) \phi_0(r).
$$

In contrast to [13] we here do not assume that the wave function $\phi_0$ is real. To proceed we require a weak impurity-boson coupling $\kappa$ such that $|\kappa|/\hbar g n_0(r) |\phi_0(r)|^2 \ll 1$, with $\xi_k(r) = \hbar \sqrt{m_b g n_0(r)}$ the healing length, $n_0(r) = |\phi_0(r)|^2$ the density of the condensate and $D$ the number of spatial dimensions. In this case, the deviation of $\hat{\phi}(r)$ from $\phi_0(r)$ is of order $\kappa$, i.e., $(\delta \hat{\phi}(r)) / (\phi_0(r)) \ll \kappa$, where $(\cdot)$ stands for the expectation value. Inserting $\hat{\phi}(r) = \phi_0(r) + \delta \hat{\phi}(r)$ into the Hamiltonian $H_B + H_I$ and keeping terms up to second order in $\kappa$, we obtain the linear term $\kappa \int \text{d}r \chi^\dagger(r) \chi(r) \phi_0(r) \delta \hat{\phi}(r) + \phi_0(r) \delta \hat{\phi}(r)$, in addition to the standard constant and quadratic terms in $\delta \phi(r)$ and $\delta \hat{\phi}(r)$.

The quadratic terms in $\delta \phi(r)$ are diagonalised by making use of the Bogoliubov transformation. The perturbation $\delta \hat{\phi}(r)$ is expanded in terms of Bogoliubov phonons $u_q$ and $v_q$, $\delta \hat{\phi}(r) = \sum_q [u_q(r) \hat{b}_q - v_q^\dagger(r) \hat{b}_q^\dagger]$, where $\hat{b}_q$ creates a Bogoliubov phonon in mode $q$, the prime at the sum indicates that the mode corresponding to the ground state is not included, and the functions $u_q$ and $v_q$ solve the Bogoliubov-de Gennes (BdG) equations

$$
\hat{H}_0 + 2g |\phi_0(r)|^2 u_q(r) - g (\phi_0(r))^2 v_q(r) = E_q u_q(r),
$$

$$
\hat{H}_0 + 2g |\phi_0(r)|^2 v_q(r) - g (\phi_0(r))^2 u_q(r) = -E_q v_q(r).
$$

Putting this expansion for the condensate field operator into the interaction Hamiltonian and using the fact that the overlaps of two different impurity modes is negligible allows us to rewrite the total Hamiltonian as a Hubbard-Holstein model [18,19]

$$
\hat{H} = \hat{H}_0 + \sum_j \sum_q \hbar \omega_q (M_{j,q} \hat{b}_q + M^*_{j,q} \hat{b}_q^\dagger) \hat{n}_j
$$

$$
+ \sum_j E_j \hat{n}_j + \sum_q \hbar \omega_q \hat{b}_q \hat{b}_q^\dagger,\]

with $\hbar \omega_q = E_q$ the energies of the Bogoliubov excitations, the number operator $\hat{n}_j = \hat{b}_j^\dagger \hat{b}_j$, the dimensionless matrix elements $M_{j,q} = (\kappa / \hbar g n_0) \int \text{d}r [\phi_0(r) u_q(r) - \phi_0(r) v_q(r)] \eta_j(r)^2$, and the mean field shift $E_j = \kappa \int \text{d}r n_0(r) |\eta_j(r)|^2$.

The total Hamiltonian can be brought into a more intuitive form by applying the unitary Lang-Firsov transformation [18] $\hat{U} = \hat{U} \hat{H} \hat{U}^\dagger$, with $\hat{U} = \exp \left[ \sum_j \sum_q \left( M_{j,q} \hat{b}_q - M^*_{j,q} \hat{b}_q^\dagger \right) \hat{n}_j \right]$, which yields

$$
\hat{U} \hat{H} \hat{U}^\dagger = \hat{U} \hat{H}_0 \hat{U}^\dagger + \sum_j \left( E_j - E_j \right) \hat{n}_j - \sum_j E_j \hat{n}_j (\hat{n}_j - 1)
$$

$$
\frac{1}{2} \sum_{j,j'} V_{j,j'} \hat{n}_j \hat{n}_j' + \sum_q \hbar \omega_q \hat{b}_q \hat{b}_q^\dagger.\]

The presence of the BEC mediates a non-retarded interaction $V_{j,j'} = \sum_q \hbar \omega_q (M_{j,q} M^*_{j',q} + M^*_{j,q} M_{j',q})$ between impurities in different lattice sites $j$ and $j'$. The characteristic potential energy of an impurity in the deformed condensate is described by the polaronic level shift $E_j = \sum_q \hbar \omega_q |M_{j,q}|^2$.
The transformation of the impurity Hamiltonian $\hat{H}_a$ yields

$$\tilde{\hat{H}}_a \tilde{\hat{U}}^\dagger = -J_a \sum_{\langle i,j \rangle} (\hat{X}_i \hat{a}_i) \hat{X}_j \hat{a}_j + \frac{U_a}{2} \sum_j \hat{n}_j (\hat{n}_j - 1) - \mu_a \sum_j \hat{n}_j,$$

where $\hat{X}_i$ is a Glauber displacement operator that creates a coherent phonon cloud around the impurity. Thus, the Hamiltonian $\tilde{\hat{H}}_{\text{eff}}$ describes the behaviour of polarons according to an extended Hubbard model [4] provided that $c \gg a J_a / \hbar$, with $c \sim \sqrt{g n_0 / m_b}$ the phonon velocity and $a$ the lattice spacing [13].

In ref. [13] it has been shown that for low temperatures $k_B T \ll E_j$, the behaviour of the polarons is essentially coherent. Here, $k_B$ is Boltzmann’s constant. If also $J_a \ll E_j$, we can treat the hopping term in eq. (8) as a perturbation and trace over the condensate degrees of freedom [13]. This allows us to derive the Hamiltonian

$$\tilde{\hat{H}}^{(1)} = \sum_{\langle i,j \rangle} \tilde{J}_{i,j} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2} \sum_j \tilde{U}_j \hat{n}_j (\hat{n}_j - 1) - \mu_0 \sum_j \hat{n}_j,$$

with

$$\tilde{\mu}_j = \mu_a - \kappa n_0 + E_j,$$

where $\kappa$ is related to the number occupancy of mode $q$. As was discussed in ref. [13], the first exponential term leads to a suppression of the coherent hopping $\tilde{\mu}_j$. The second exponential, which does not occur for the cases presented in [13] since there $\phi_0$ was assumed to be real, has always an absolute value of 1 and causes a phase whenever an atom hops from one lattice site $j$ to a neighbouring one $i$. By defining the reduced hopping constant

$$\tilde{J}_a = J_a \exp \left( -\sum'_{q} \left( \hat{M}_{i,q} - \hat{M}_{j,q} \right)^2 (2N_q(T) + 1) \right),$$

where $N_q(T) = [\exp(\hbar \omega_q / k_B T) - 1]^{-1}$ is the number occupation of mode $q$, we find

$$\langle \hat{X}_i^\dagger \hat{X}_j \rangle \approx \exp \left( -\frac{1}{2} \sum_q \left( \hat{M}_{i,q} - \hat{M}_{j,q} \right)^2 (2N_q(T) + 1) \right) \times \exp \left( -\frac{1}{2} \sum_q \left( \hat{M}_{i,q} \hat{M}_{j,q}^\dagger - \hat{M}_{j,q} \hat{M}_{i,q}^\dagger \right) \right).$$

This phase depends on the properties of the order parameter $\phi_0$ and drastically affects the dynamics of the impurities. For example, for a quasi-one-dimensional setup the presence of such a phase twist in nearest-neighbour hopping corresponds to a rotation of the ring and thus causes persistent currents [20]. This is similar to the behaviour of an electron in a superconducting ring subject to a magnetic flux. In a two-dimensional setup a suitable phase factor realises an artificial magnetic field, which enables, for instance, the investigation of quantum Hall effects [6]. In the remainder of this paper, we will characterise the occurring phase term for different setups and discuss ways of detecting its influence on the dynamics of the impurity atoms.

**BEC in a ring.** – We first consider the case where the BEC is loaded into a quasi–one-dimensional trap of length $L$ with periodic boundary conditions, which corresponds to a ring of radius $R = L/2\pi$ and has already been achieved in experiments [21, 22]. The impurities are correspondingly trapped by a one-dimensional optical lattice which is ring-shaped, see fig. 1(a). We assume that the ring trapping the condensate is rotated with an angular speed $\Omega$. The Hamiltonian $\hat{H}_0$ describing the interaction-free part of the BEC is then given by $\hat{H}_0 = -(\hbar^2 / 2m_b) \partial^2 / \partial x^2 + i \hbar v \partial / \partial x - \mu$, where $v = R \Omega$. The full GPE is solved by the wave function $\phi_0 = \sqrt{n_0} \exp(\imath q_0 x)$, where $q_0 = 2\pi j / L$ for some integer $j$ such that $q_0 = m_b v / \hbar - \Delta q$ with $\Delta q \in [-\pi / L, \pi / L]$. This choice of $q_0$ ensures that the BEC is in its ground state with a chemical potential $\mu = \hbar^2 q_0^2 / 2m_b + gn_0 - \hbar \omega_0$ and that the phase of the BEC fulfills the periodic boundary conditions.

The BdG equations are solved by the mode functions $u_q(x) = A_q \exp(i (q + q_0) x) / \sqrt{L}$ and $v_q(x) = B_q \exp(i (q - q_0) x) / \sqrt{L}$, where the coefficients are given by the relation $A_q \pm B_q = (E_q / \epsilon_0) \pm 1/2$, and the quasi-momenta fulfill the condition $q = 2\pi j / L$ for some integer $j$. The energies of the Bogoliubov modes are given by $\hbar \omega_q = E_q^B - \hbar^2 \Delta q / m_b$, with $E_q^B = \sqrt{\epsilon_0^2 (\epsilon_0^2 + 2g n_0)}$ and $\epsilon_0^2 = \hbar^2 q_0^2 / 2m_b$. In order to calculate the coupling matrix elements $M_{i,q}$ we assume that the lattice trapping the impurities is sufficiently deep.
such that the Wannier functions $\eta_j$ are well approximated by Gaussians of width $\sigma$ centered at lattice site $j$, i.e., $\eta_j(x) \approx \exp(-|x-x_j|^2/2\sigma^2)/\sqrt{\pi\sigma}$, where $x_j$ is the position of the lattice site $j$. We, furthermore, note that due to the translational invariance the phase $\alpha_{i+1} = \alpha_a$ does not depend on the index $i$ and we thus find

$$\alpha_a = \frac{1}{2\pi} \sum_{q \neq 0} \frac{2\sigma_0^0}{E_0^B} \frac{1}{(\hbar\omega_q)^2} e^{-q^2\sigma^2/2} \sin(qa). \quad (11)$$

We see that the finite width of the Wannier functions $\sigma$ leads to an effective cut-off of the sum for large quasi-momenta $q$ and thus large energies $\hbar\omega_q$. Furthermore, the expression for $\alpha_a$ contains the Bogoliubov excitation energies $\hbar\omega_q$ of the rotating condensate as well as the static structure factor $E_0^B$ of the non-rotating BEC [23,24].

The value of $\alpha_a$ depends critically on the rotation speed $\Omega$. For zero rotation it can be shown from eq. (11) that $\alpha_a = 0$. If the rotation speed is increased, the phase twist $\alpha_a$ initially increases linearly with $\Omega$, as shown in fig. 1(b). However, for a critical rotation speed a jump occurs and the value of the phase twist changes from positive to negative. This behaviour can be explained as follows: due to the finite size of the ring, the possible quasi-momentum states of the BEC are restricted to values $q_0 = 2\pi j/L$ for some integer $j$. For rotation speeds close to zero, the ground state of the BEC exhibits zero quasi-momentum. Only if the rotation speed is above a critical value $\Omega_{crit} = \hbar/2m_b R^2$ will the ground state change to a non-zero quasi-momentum. Further quasi-momentum jumps occur at odd multiples of this critical angular speed. Together with the change in the quasi-momentum of the BEC, also the jump in the induced phase twist $\alpha_a$ occurs. To some extent one could thus say that the induced phase is caused by a mismatch of the angular rotation speed $\Omega$ and the quasi-momentum $q_0$ of the BEC, which is enforced to be quantised by the boundary conditions. It should be noted that at the critical points where the BEC changes its quasi-momentum, the ground state of the condensate is degenerate, which leads to a failure of the Bogoliubov approximation [25].

The parameters presented in fig. 1(b) fulfill the conditions for our derivation to be valid and can be achieved using present experimental techniques [13–15,26,27]. The precise value of the coupling constant $g$ can be changed via a Feshbach resonance. The temperature chosen for our calculations is zero, but the discussed effects will be observable as long as the temperature is lower than the polaron energy, which is typically on the order of a few tens of nanokelvins [13]. This also ensures that the first factor in eq. (10) is close to one leading to sufficiently large hopping. In the above examples the size of the induced phase twist can reach values as large as $\alpha_a \approx 0.03$. This will significantly affect the phase of the atomic wave function and make effects of the induced artificial magnetic field observable even if the lattice only consists of a few tens of sites in each direction.

![Fig. 2: (a) Plot of the impurity distribution vs. dimensionless time $t_d = t/J_a/\hbar$. A darker colour corresponds to a higher density. (b) Mean position of the impurity after an evolution time of $t_d = 6$. We have chosen an initial state $|\psi\rangle_{ini}$ of width $w_{ini} = \sqrt{2}$ centered at lattice site $j_0 = 15$. In (a), the chosen phase twist was $\alpha_a = 0.03$.](image-url)

**Drift of the impurities.** The presence of the phase twist $\alpha_a$ can be measured by observing its influence on the transport properties of the impurities. We consider the following case: a single impurity is initially trapped in a finite spatial region of the lattice. This can be achieved by applying an additional trapping potential $V_{ini}(j) = V_0(j - j_0)^2$, in which case the single impurity starts in the state $|\psi\rangle_{ini} = \sum_j \exp[-(j - j_0)^2/2w_{ini}^2] |j\rangle_{\text{vac}}$, where $w_{ini}$ is the width of the initial state. Without a phase twist $\alpha_a$ after turning the initial potential off, the impurity expands symmetrically around the lattice site $j_0$ and no overall drift to either higher or lower lattice sites occurs.

This behaviour changes if a finite phase twist $\alpha_a$ is taken into account as shown in fig. 2(a). For the chosen parameters and a relatively short time of $t = 6h/J_a$, which under standard experimental conditions corresponds to a few milliseconds, a drift towards higher lattice site numbers is clearly visible. Such drifts can be measured in experiments [21] and thus allow to probe the presence of an induced phase twist. The mean position of the impurity after an evolution time of $t = 6h/J_a$ is shown in fig. 2(b). It exhibits a periodic behaviour in $\alpha_a$ with a period of 1, which is caused simply by the fact that the integer part of $\alpha_a$ only contributes a trivial phase. In total, the behaviour is akin to an electron in a ring subject to a magnetic flux. The electron will also drift towards a preferential direction and thus will cause a current as known from superconducting rings.

**Numerical simulations of the full ring system.** We now compare our prediction to a system where a numerically exact solution is possible, namely a small two-mode Bose-Hubbard model. We assume that the condensate is represented by a mode $c$ that experiences a rotation whereas the other mode $a$ representing a single impurity does not. The system is described by the Hamiltonian

$$\hat{H}_{BH} = -J_a \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + U_{I} \sum_j \hat{a}_j^\dagger \hat{a}_j \hat{c}_j^\dagger \hat{c}_j$$

$$-J_c \sum_j \left( \alpha^{\text{tria}} \hat{c}_j^\dagger \hat{c}_{j+1} + \text{h.c.} \right) + \frac{U_c}{2} \sum_j \hat{c}_j^\dagger \hat{c}_j^\dagger \hat{c}_j \hat{c}_j.$$  

(12)
Hamiltonian describing the condensate subject to a harmonic trapping potential $\omega_\text{tr}$ thus reads

$$
\hat{H}_0 = -(\hbar^2/2m_b)\nabla^2 + m_b\omega_\text{tr}^2(r')^2/2 - \mu - \Omega e_z \cdot r' \times \hat{p}_z,
$$

where $e_z$ is the unit vector in $z$-direction and $\hat{p}_z = -i\hbar \nabla'$. It can be shown that the solutions for this Hamiltonian are given by

$$
\psi_{n,l}(r, \varphi) = \sqrt{n!/(\pi (n+l)!)} e^{i\varphi - \xi^2/2} L_n^l(\xi^2)/a_0
$$

with energies $E_{n,l} = (2n+l+1)\hbar\omega - \hbar\Omega$. Here, $\xi = r/a_0$ with $a_0 = \sqrt{\hbar m_b\omega_\text{tr}}$ the harmonic oscillator length and $(r, \varphi)$ are polar coordinates. The quantum numbers $n$ and $l$ are restricted to $n = 0, 1, 2, \ldots$ and $l = -n, -n+1, \ldots$. From this we see that for high rotation speeds $|\Omega| > \omega_\text{tr}$ the system is unstable since for suitable $(n, l)$ the energy can be made infinitely small which corresponds to the atoms being driven infinitely far away from the origin due to the centrifugal term. We therefore restrict the rotation speed to moderate values $|\Omega| < \omega_\text{tr}$, for which the ground state is always given by $(n, l) = (0, 0)$.

For brevity, let us introduce the multi-index $\nu = (n, l)$. If we assume that the condensate is in a state $\nu_0$, we can expand the field operator of the condensate as

$$
\psi(r, \varphi) = \sqrt{N_0} \psi_{\nu_0}(r, \varphi) + \sum_{\nu \neq \nu_0} \psi_{\nu}(r, \varphi) b_\nu^\dagger b_\nu,
$$

where $N_0$ is the number of atoms in state $\nu_0$, and we assume that $b_\nu^\dagger b_\nu = N_0$ for $\nu \neq \nu_0$. As we consider the interaction-free case this is exactly the same as an expansion in terms of Bogoliubov phonons, which would yield $\psi_{\nu_0} = 0$ and $\psi_{\nu_0} = \psi_{\nu_0}(r, \varphi)$. The chemical potential is determined by solving the Gross-Pitaevskii equation for the corresponding state $\nu_0$, which gives $\mu_{\nu_0} = -E_{n_0,l_0}$. This allows us to derive the phase twist $\alpha_{\nu_0}$ in the same way as for the ring optical lattice. In order to solve the integral in the expression for the coupling matrix elements $M_{\nu',\nu}$ we assume that the Wannier functions are strongly localised and can be approximated by a Dirac delta function. If the condensate is in the ground state $\nu_0 = (0, 0)$, we find for the induced phase twist

$$
\alpha^{(0)}_{\nu_0} = \frac{\kappa^2 N_0}{2\pi^3 a_0^3} \sum_{\nu \neq \nu_0} \frac{1}{(\hbar\omega)^2 (n+l)!} e^{-\xi^2 - \xi_0^2} \times L_n^l(\xi_0^2) \sin(l(\varphi_i - \varphi_j)),
$$

where $(\xi_i, \varphi_i)$ and $(\xi_j, \varphi_j)$ are the coordinates of the two lattice sites before and after the jump of the impurity. We immediately see that if $\varphi_i = \varphi_j$, which corresponds to a hopping in radial direction, the induced phase is zero. Furthermore, the induced phase vanishes as $\xi_i$ or $\xi_j$ go to infinity, which can be explained by the vanishing density of the condensate for large distances.

Our theory is also applicable if the condensate is initially in an excited state, for example in the state $\nu_0 = (0, 1)$ that exhibits a vortex. In this case, we derive for the phase twist

$$
\alpha^{(0,1)}_{\nu_0} = \frac{\kappa^2 N_0}{2\pi^3 a_0^3} \sum_{\nu \neq \nu_0} \frac{1}{(\hbar\omega)^2 (n+l)!} e^{-\xi^2 - \xi_0^2} \times L_n^l(\xi_0^2) L_n^l(\xi_0^2) \sin(l(\varphi_i - \varphi_j)(l+1)).
$$

The behaviour of the two phase twists for varying angular rotation speed $\Omega$ is shown in fig. 4(a). In both

Fig. 3: Expansion of a single impurity atom initially in state $|\psi\rangle_{i\text{ai}}$ with $j_0 = 11$ and width $w_{i\text{ai}} = \sqrt{2}$. (a) Density distribution after a time $t_\text{d} = tJ_0/\hbar = 4$ for $\alpha_s = 0.0227$ (dashed line) and $\alpha_s = 0.0251$ (solid line). (b) Mean position of the impurity after the evolution time $t_\text{d} = 4$. The two crosses mark the positions of the densities in (a). The other parameters are $U_c = J_c = J_a, U_1 = 2J_c$ with three atoms in mode $c$ and a lattice consisting of $N_\text{s} = 21$ sites.
cases, the induced phase twists increase as the rotation speed $\Omega$ is increased and diverge as $\Omega$ approaches the trapping frequency $\omega_{r}$. Here, $\xi_{i} = \xi_{j} = 0.5$ and $\varphi_{i} - \varphi_{j} = 0.1$. (b) Reduced phase twists vs. $\xi$. The angular velocity is given by $\Omega = 0.25\omega_{r}$ and $\varphi_{i} - \varphi_{j} = 0.1$.

Summary. – We have investigated an alternative way to create artificial magnetic fields in a system of ultracold atoms in an optical lattice. We have shown analytically that for impurities trapped in a ring-shaped optical lattice a drift in the transport of the impurities occurs. This drift was qualitatively confirmed by a full numerical simulation of a two-mode Bose-Hubbard model. We also showed that an artificial magnetic field can be created in a two-dimensional optical lattice by submerging it into a moderately rotating, harmonically trapped condensate. With this, our system lends itself as a well-suited quantum simulator for investigating phenomena encountered in condensed-matter physics.

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