Time operators for continuous-time and discrete-time quantum walks

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Abstract

We study time operators for both continuous and discrete-time quantum walks. Because the continuous-time quantum has a Hamiltonian, we can define its time operators by the canonical commutation relation to the Hamiltonian. On the other hand, the discrete-time quantum walk has no Hamiltonian and its evolution is described by a unitary operator. Hence, it is natural to define the time operators for the discrete-time quantum walk by a commutation relation to the unitary operator.

In this paper, we construct concrete examples of the time operators for both continuous and discrete-time quantum walks. We also separate time operators for a unitary operator into two classes and give their basic properties.

1 Introduction

Quantum walks are universal computational primitives [7, 8, 14] and efficient tools for building quantum algorithms [3]. They can also simulate quantum systems such as Dirac particles [2, 16, 17]. It is hence important to know the dynamics of the quantum walk. In particular, many authors have been studied the long-time behavior in various ways [9, 10, 12, 13, 15, 24, 25, 26]. Here we employ time operators to obtain time decay estimates of transition probabilities between states, which will play important roles in implementing quantum walk based algorithms and simulations.
A time operator of a Hamiltonian $H$ is formally defined as a Hermitian operator which satisfies the canonical commutation relation with $H$, i.e., $TH - HT = i$. It was widely believed for a long time that in quantum theory there exists no time operator, as Pauli pointed out in his famous textbook [20, p. 63, footnote 2]. On the other hand, Aharonov and Bohm constructed a concrete time operator of the one-dimensional free Hamiltonian [1]. This apparently contradicts to Pauli’s claim. The key to solve the contradiction is that Pauli’s argument can not apply to Araronov and Bohm’s time operator because it is quite formal. This suggests that we must study time operators in a mathematically rigorous way. It is important to pay enough attention to domains of time operators. Such a study was initiated by Miyamoto [18]. He expand his theory in a functional analytic context. One of the most important result therein is that a time operator is related to “time” in the sense that the decay of the survival probability is estimated by the amount which is determined only from the time operator [18, Theorem 4.1].

In the paper, we construct time operators for concrete examples of quantum walks as a first step. There are two kinds of quantum walks, one of which is called a continuous-time quantum walk, and the other is called a discrete-time quantum walk. The time evolution of the former is described by a self-adjoint operator, hence we can use Miyamoto’s theory of time operators of self-adjoint operators. However, since the time evolution of the later is described by a unitary operator, we need a notion of time operators of unitary operators. Such a notion is recently posted by Sambou and Tiedra de Aldecoa [21]. They call a Hermitian operator $T$ with the commutation relation $TU - UT = U$ a time operator of a unitary operator $U$, although their definition is formal. In the sequel, we shall give a rigorous definition for our purpose.

The paper is organized as follows. In Section 2, we first recall time operators of self-adjoint operators. Next we introduce a definition of time operators of unitary operators. Their basic properties are investigated here. In Section 3, examples of the time operators for continuous-time quantum walks are given. In Section 4, we construct examples of the time operators for discrete-time quantum walks. The main idea behind our construction of the time operators is the Fourier transformation, which diagonalizes time evolution operators.

**Definitions and Notations**

Let $\mathcal{H}$ be a complex Hilbert space. We denote the inner product and the norm of $\mathcal{H}$ by $(\cdot, \cdot)$ (complex linear in the second variable) and $\| \cdot \|$, respectively.

For an operator $A$, we denote by $D(A)$ the domain of $A$, and by $\sigma(A)$ the spectrum of $A$. If $A$ is densely defined, and if $A \subset A^*$ holds, then $A$ is said to be symmetric. In addition, if $A = A^*$ holds, then $A$ is said to be self-adjoint.

We denote by $\bar{A}$ the closure of a closable operator $A$. A symmetric operator is called essentially self-adjoint if $\bar{A}$ is self-adjoint.

For two operators $A, B$, we denote by $[A, B]$ the commutator of $A$ and $B$, i.e., $[A, B] := AB - BA$.

For the reader who is not familiar with functional analysis, we refer to the excellent textbook [23].
2 Definition of a time operator

In this section, we first recall the definitions of two classes of time operators of a self-adjoint operator. Next, we introduce definitions of two classes of time operators of a unitary operator, as unitary analogs of time operators of a self-adjoint operator. We also discuss some basic properties of them.

2.1 Time operators of a self-adjoint operator

Let $H$ be a self-adjoint operator on $\mathcal{H}$.

Definition 2.1. Let $T$ be a symmetric operator.

(1) We say that $T$ is a time operator of $H$, if there exists a subspace $\mathcal{D} \neq \{0\}$ of $\mathcal{H}$ such that $\mathcal{D} \subset D(TH) \cap D(HT)$ and $[T, H] = i$ holds on $\mathcal{D}$.

(2) We say that $T$ is a strong time operator of $H$, if

$$e^{itH}Te^{-itH} = T + t$$

holds for any $t \in \mathbb{R}$.

The notion of strong time operator is introduced by Miyamoto [18] in the context of quantum theory. A purely mathematical study of such pairs $(T, H)$ is done by Schmüdgen [22]. The following remarks are fundamental. In the sequel, we will discuss their analogs for time operators of a unitary operator.

Remark 2.2. Let $T$ be a strong time operator of $H$. Then the following hold.

(1) $T$ is a time operator of $H$ with a dense subspace $\mathcal{D} = D(TH) \cap D(HT)$ [18 Proposition 2.1]. Note that the converse is not true, i.e., not every time operator of $H$ is a strong time operator of $H$. Such concrete examples can be found in [6].

(2) Its closure $\bar{T}$ is a strong time operator of $H$ as well. This follows from a basic limiting argument.

(3) If $T$ is essentially self-adjoint, then $\sigma(H) = \mathbb{R}$.

Remark 2.3. For any non-scalar self-adjoint operator, its time operator does exist [27, Theorem 2.2].

2.2 Time operators of a unitary operator

We are now ready to give definitions of time operators of a unitary operator $U$. It will be done by replacing $e^{itH}$ in the definition of strong time operators of a self-adjoint operator $H$ by $U^*$. Let $U$ be a unitary operator on $\mathcal{H}$.

Definition 2.4. Let $T$ be a symmetry operator.

(1) We say that $T$ is a time operator of $U$, if there exists a subspace $\mathcal{D} \neq \{0\}$ of $\mathcal{H}$ such that $\mathcal{D} \subset D(TU) \cap D(T)$ and $[T, U] = U$ holds on $\mathcal{D}$.
We say that $T$ is a strong time operator of $U$, if

$$U^*TU = T + 1$$

holds.

Let us compare fundamental properties of time operators of self-adjoint operators with that of unitary operators.

**Remark 2.5.** Let $T$ be a strong time operator of $U$. Then the following hold.

1. $T$ is a time operator of $U$ with a dense subspace $D = D(TU) \cap D(T)$. Note that the converse is not true, i.e., not every time operator of $U$ is a strong time operator of $U$ as we will see in the sequel.

2. Its closure $\bar{T}$ is a strong time operator of $U$ as well. This follows from a basic limiting argument.

3. $\sigma(T) = \sigma(T + 1)$ holds. In particular, $T$ is unbounded.

**Proposition 2.6.** Let $T$ be a strong time operator of $U$. If $T$ is essentially self-adjoint, then $\sigma(U) = \{z \in \mathbb{C} \mid |z| = 1\}$.

**Proof.** We denote by $\bar{T}$ the closure of $T$. Then we can show that

$$U^* e^{it\bar{T}} U = e^{it(T+1)} = e^{it} e^{it\bar{T}}$$

for any $t \in \mathbb{R}$. Thus, we obtain $e^{itT}U e^{-itT} = e^{itU}$. This means that $\sigma(U) = \sigma(e^{itU})$ for all $t \in \mathbb{R}$. Hence, $\sigma(U) = \{z \in \mathbb{C} \mid |z| = 1\}$ holds.

The following is a toy example, but it makes a difference between that of self-adjoint operators and that of unitary operators stand out clearly.

**Example 2.7.** Let $\mathcal{H} = \ell^2(\mathbb{Z})$. We define a unitary operator $V$ and a symmetric operator $X$ by

$$(V\psi)(n) := \psi(n - 1), \quad (X\psi)(n) := n\psi(n), \quad \psi \in \mathcal{H}, \ n \in \mathbb{Z},$$

where the domain of $X$ is the subspace spanned by the standard basis vectors of $\mathcal{H}$. A direct calculation shows that $X$ is a strong time operator of $V$. Since $X$ is essentially self-adjoint, we have $\sigma(V) = \{z \in \mathbb{C} \mid |z| = 1\}$ by Proposition 2.6.

Note that the spectrum of $X$ coincides with the set of integers, each element of which is an eigenvalue of $X$ as well. In particular, $X$ does have an eigenvalue. This phenomena is in contrast to the case of time operators of self-adjoint operators, because time operators of self-adjoint operators have no eigenvalues [18, Corollary 4.2].

A strong time operator is related to the dynamics of a quantum system in such a way that if its time evolution is described by a unitary operator $U$, then the transition probability between states is bounded by the amount which is determined only from the time operator of $U$ and states. More precisely, we have the following.
Theorem 2.8. Let $T$ be a strong time operator of $U$, and let $S$ be a bounded operator commuting with $U$. Then for any $\psi \in D(T)$, $\phi \in D(T^*)$ and $t \in \mathbb{Z} \setminus \{0\}$,
\[
|\langle \phi, U^t \psi \rangle| \leq \frac{\|[(T^* + S^*)\phi]\|\|\psi\| + \|\phi\|\|T + S\|\|\psi\|}{|t|} \tag{1}
\]
holds.

Proof. Since we have
\[
U^t(T + S) = (T + S - t)U^t
\]
for any $t \in \mathbb{Z}$, we obtain
\[
|\langle \phi, U^t \psi \rangle| = \left| \frac{1}{t} \langle (T^* + S^*)\phi, U^t \psi \rangle - \frac{1}{t} \langle U^{-t}\phi, (T + S)\psi \rangle \right|
\leq \frac{\|[(T^* + S^*)\phi]\|\|\psi\| + \|\phi\|\|T + S\|\|\psi\|}{|t|},
\]
which is the desired result. \qed

Remark 2.9. Theorem 2.8 is a unitary analog of [4, Proposition 3.1].

For an operator $A$ and a unit vector $\psi \in D(A)$, we define the uncertainty $(\Delta A)_\psi$ by
\[
(\Delta A)_\psi := \|A\psi - \langle \psi, A\psi \rangle \psi\|.
\]

Corollary 2.10. Let $T$ be a strong time operator of $U$, and let $\psi \in D(T)$ be a unit vector. Then we have
\[
|\langle \psi, U^t \psi \rangle|^2 \leq \frac{4(\Delta T)_\psi^2}{t^2}
\]
for any $t \in \mathbb{Z} \setminus \{0\}$.

Proof. This follows from Theorem 2.8 with $S = -\langle \psi, T\psi \rangle$. \qed

Remark 2.11. Corollary 2.10 is a unitary analog of [18, Theorem 4.1].

We close this section with the following theorem, which guarantees the existence of a time operator.

Theorem 2.12. Let $U$ be a non-scalar unitary operator on $\mathcal{H}$. Then there exists a time operator $T$ of $U$.

Proof. We split the proof into two cases.

Case 1. $U$ has at least two distinct eigenvalues.
In this case, we modify the proof of [27, Lemma 2.1] to construct a time operator of $U$. Let $\lambda, \mu$ be two distinct eigenvalues of $U$, and let $\xi, \eta$ be normalized eigenvectors
corresponding to \(\lambda, \mu\), respectively. Note that \(\xi\) and \(\eta\) are mutually orthogonal. Define the bounded self-adjoint operator \(T\) on \(\mathcal{H}\) by

\[
T\xi := \frac{i\mu}{\lambda - \mu}\eta, \quad T\eta := \frac{i\lambda}{\lambda - \mu}\xi, \quad T\psi := 0, \quad \psi \in \text{span}\{\xi, \eta\}^\perp.
\]

Let \(\mathcal{D}\) be the one-dimensional subspace spanned by the vector \(\xi + i\eta\). We show that \(T\) satisfies \([T, U] = U\) on \(\mathcal{D}\). By direct computations, we see that

\[
TU(\xi + i\eta) = T(\lambda\xi + i\mu\eta) = \frac{i\lambda\mu}{\lambda - \mu}\eta - \frac{\lambda\mu}{\lambda - \mu}\xi,
\]

and

\[
UT(\xi + i\eta) = U\left(\frac{i\mu}{\lambda - \mu}\eta + i\frac{i\lambda}{\lambda - \mu}\xi\right) = \frac{i\mu^2}{\lambda - \mu}\eta - \frac{\lambda^2}{\lambda - \mu}\xi,
\]

hence we get

\[
[T, U](\xi + i\eta) = i\mu\eta + \lambda\xi = U(\xi + i\eta).
\]

Therefore \(T\) is a time operator of \(U\).

**Case 2.** \(U\) has at most one eigenvalue.

In this case, we can choose a real number \(\theta \in \mathbb{R}\) for which \(1 - e^{i\theta}U\) is injective. We define the self-adjoint operator \(H\) by the Cayley transformation of \(e^{i\theta}U\). Note that we have

\[
e^{i\theta}U = (H - i)(H + i)^{-1}.
\]

We now use [27, Theorem 2.2] to get a symmetric operator \(T'\) of \(H\) and a subspace \(\mathcal{D}\) of \(\mathcal{H}\) with \(\mathcal{D} \subset D(T'H) \cap D(HT')\), \([T', H] = i\) on \(\mathcal{D}\), and \(\mathcal{D} \cap D(HT'H) \neq \{0\}\). The last statement is not explicitly written in the theorem, but it follows from the proof of [27, Theorem 2.2]. Let

\[
T := -(H - i)T'(H + i)/2.
\]

By the definitions of \(T\) and \(H\), we have \(U(\mathcal{D} \cap D(HT'H)) \subseteq D(T)\). In the sequel, we have to pay attention to domains of operators. For all \(\psi \in \mathcal{D} \cap D(HT'H)\), we see that

\[
[T, U]\psi = e^{-i\theta}[T, e^{i\theta}U]\psi = e^{-i\theta}[T, (H - i)(H + i)^{-1}]\psi
\]

\[
= -\frac{e^{-i\theta}}{2}\left((H - i)T'(H + i)(H - i)(H + i)^{-1} - (H - i)(H + i)^{-1}(H - i)T'(H + i)\right)\psi
\]

\[
= -\frac{e^{-i\theta}}{2}\left((H - i)T'(H - i) - (H - i)(H + i)^{-1}(H - i)T'(H + i)\right)\psi
\]

\[
= -\frac{e^{-i\theta}}{2}\left((H - i)(H + i)^{-1}(H + i)T'(H - i) - (H - i)T'(H + i)\right)\psi
\]

\[
= -\frac{e^{-i\theta}}{2}\left((H - i)(H + i)^{-1}(-2iHT' + 2iT'H)\right)\psi
\]

\[
= -ie^{-i\theta}(H - i)(H + i)^{-1}[T', H]\psi = e^{-i\theta}(H - i)(H + i)^{-1}\psi
\]

\[
= U\psi.
\]

Hence \(T\) is a time operator of \(U\). This completes the proof. \(\square\)
3 Time operators for continuous-time quantum walks

In this section, we consider a continuous-time quantum walk on the lattice $\mathbb{Z}^d$ with $d \in \mathbb{N}$. The Hamiltonian of the system is the bounded self-adjoint operator $L_d$ on the Hilbert space $\mathcal{H}_d := \ell^2(\mathbb{Z}^d)$, where $L_d$ is defined by

$$(L_d \psi)(x) := \frac{1}{2d} \sum_{|y-x|=1} \psi(y), \quad \psi \in \mathcal{H}_d, \ x \in \mathbb{Z}^d.$$  

We shall find a time operator of $L_d$. Let $\mathbb{T}^d := [0, 2\pi]^d$ be the $d$-dimensional torus, and let $\mathcal{K}_d := L^2(\mathbb{T}^d, dk/(2\pi)^d)$ be the Hilbert space of square-integrable functions on $\mathbb{T}^d$. We define the Fourier transform $\mathcal{F}_d : \mathcal{H}_d \to \mathcal{K}_d$ as a unitary operator so that

$$(\mathcal{F}_d \psi)(k) := \sum_{x \in \mathbb{Z}^d} \psi(x) e^{-ix \cdot k}, \ k \in \mathbb{T}^d$$  

for $\psi \in \mathcal{H}_d$ with finite support. From now on, the multiplication operator on $\mathcal{K}_d$ associated to a function $g : \mathbb{T}^d \to \mathbb{R}$ will be denoted by the same symbol $g$. Then we can diagonalize $L_d$ as

$$\hat{L}_d := \mathcal{F}_d L_d \mathcal{F}_d^{-1} = \frac{1}{2d} \sum_{j=1}^d \cos k_j,$$

where $k = (k_1, \ldots, k_d) \in \mathbb{T}^d$. Thus, to construct a time operator of $L_d$, it is sufficient to find time operators of multiplication operators on $\mathcal{K}_d$. We begin with the case $d = 1$.

For this, let $AC[0, 2\pi]$ be the set of absolutely continuous functions on the interval $[0, 2\pi]$. Define an operator $P$ on $\mathcal{K}_1 = L^2(\mathbb{T})$ by $Pf := -if'$ with domain

$$D(P) := \{f \in AC[0, 2\pi] \mid f' \in L^2(\mathbb{T}), \ f(0) = f(2\pi)\}.$$  

It is known that $P$ is a self-adjoint operator. We use the following theorem.

**Theorem 3.1.** Let $g : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable periodic function with period $2\pi$ whose derivative $g'$ has at most finitely many zeros in $(0, 2\pi)$. Then the operator

$$\hat{T} := -\frac{1}{2} \left( \frac{1}{g'(k)} P + P \frac{1}{g'(k)} \right)$$

on $\mathcal{K}_1$ with natural domain is a strong time operator of the self-adjoint operator $g(k)$.

**Proof.** We first show that $\hat{T}$ is a symmetric operator. Let $Z$ be the set of zeros of $g'$ in $(0, 2\pi)$, which is a finite set by assumption. Let $\Omega := (0, 2\pi) \setminus Z$. Then $C_0^\infty(\Omega)$ is dense in $\mathcal{K}_1$, thus the domain of $\hat{T}$ is dense. On the other hand, we have

$$\hat{T}^* \supset -\frac{1}{2} \left( P \frac{1}{g'(k)} + \frac{1}{g'(k)} P \right) = \hat{T},$$

which means that $\hat{T}$ is symmetric.
Next we show that $\hat{T}$ is a strong time operator of $g(k)$. Let $t \in \mathbb{R}$ be arbitrary. Since $D(P)$ is invariant under $e^{itg(k)}$, so is $D(\hat{T})$. Thus for any $f \in D(\hat{T})$, we have $e^{itg(k)} f \in D(\hat{T})$, and we see that

$$e^{itg(k)} \hat{T} f = e^{itg(k)} \cdot \frac{i}{2} \left\{ \frac{1}{g'(k)} f' + \frac{d}{dk} \left( \frac{1}{g'(k)} f \right) \right\}$$

and

$$\hat{T} e^{itg(k)} f = \frac{i}{2} \left\{ i t e^{itg(k)} f + \frac{1}{g'(k)} e^{itg(k)} f' + \frac{d}{dk} \left( \frac{1}{g'(k)} f \right) \right\} - t e^{itg(k)} f$$

whence we get

$$e^{itg(k)} \hat{T} f = (\hat{T} + t) e^{itg(k)} f.$$  

Summing up the above arguments, we obtain the inequality $e^{itg(k)} \hat{T} \subset (\hat{T} + t) e^{itg(k)}$. The converse inequality immediately follows by replacing $t$ by $-t$. This finishes the proof. \hfill \Box

**Remark 3.2.** Theorem 3.1 is an analog of [11, Theorem 1.9]. See also [5, Theorem 2.4].

By Theorem 3.1, we get a strong time operator

$$\hat{T}_1 := \frac{1}{2} \left( \frac{1}{\sin k} P + P \frac{1}{\sin k} \right)$$

of $\hat{L}_1 = \cos k$. To compute the inverse Fourier transform of $\hat{T}_1$, let $X, V$ be the operators on $\mathcal{H}_1$ defined in Example 2.7. Direct calculations show that we have $\mathcal{F}_1^{-1} P \mathcal{F}_1 = -X$ and $\mathcal{F}_1^{-1} e^{-ik} \mathcal{F}_1 = V$. Hence we conclude that

$$T_1 := \mathcal{F}_1^{-1} \hat{T}_1 \mathcal{F}_1 = \frac{1}{2} \left( \text{Im}(V)^{-1} X + X \text{Im}(V)^{-1} \right)$$

is a strong time operator of $L_1 = \text{Re}(V)$, where

$$\text{Re}(V) := \frac{V + V^*}{2}, \quad \text{Im}(V) := \frac{V - V^*}{2i}.$$ 

Let us consider the case $d \geq 2$. We identify $\mathcal{H}_d = \ell^2(\mathbb{Z}^d)$ with $\otimes^d \mathcal{K}_1 = \otimes^d \ell^2(\mathbb{Z})$. Under the identification, we have

$$L_d = \frac{1}{d} \sum_{j=1}^{d} 1 \otimes \cdots \otimes 1 \otimes \hat{L}_1 \otimes 1 \otimes \cdots \otimes 1.$$ 

Therefore the operator

$$T_d := \sum_{j=1}^{d} 1 \otimes \cdots \otimes 1 \otimes T_1 \otimes 1 \otimes \cdots \otimes 1.$$ 

with domain $\otimes_{\text{alg}}^d D(T_1)$ is a strong time operator of $L_d$. Here, $\otimes_{\text{alg}}$ denotes the algebraic tensor product.
4 Time operators for discrete-time quantum walks

We consider a discrete-time quantum walk. The model we will analyze in the section is the one-dimensional homogeneous quantum walk. Let $V$ be the unitary operator on $\ell^2(\mathbb{Z})$ defined in Example 2.7, and let $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a $2 \times 2$ unitary matrix. The time evolution of the system is described by the unitary operator $U$ on the Hilbert space $H := \ell^2(\mathbb{Z}; \mathbb{C}^2)$, where $U$ is defined by

$$U := \begin{pmatrix} V^* & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} V^*a & V^*b \\ Vc & Vd \end{pmatrix}.$$ 

We will construct a time operator of $U$.

For this, let $K$ be the Hilbert space of square integrable functions $f : [0, 2\pi] \to \mathbb{C}^2$ with norm

$$\|f\| := \left( \int_0^{2\pi} \|f(k)\|^2_{\mathbb{C}^2} \frac{dk}{2\pi} \right)^{\frac{1}{2}}.$$ 

We define the Fourier transform $\mathcal{F} : H \to K$ as a unitary operator so that

$$(\mathcal{F}\psi)(k) := \sum_{x \in \mathbb{Z}} e^{-ikx}\psi(x), \quad k \in [0, 2\pi]$$

for $\psi \in H$ with finite support. Then we can show that

$$\mathcal{F}U\mathcal{F}^{-1} = \int_{[0,2\pi]} \hat{U}(k) \frac{dk}{2\pi}, \quad \hat{U}(k) := \begin{pmatrix} e^{ik}a & e^{ik}b \\ e^{-ik}c & e^{-ik}d \end{pmatrix}, \quad k \in [0, 2\pi].$$

For each $k \in \mathbb{R}$, the unitary matrix $\hat{U}(k)$ has exactly two eigenvalues $\lambda_1(k), \lambda_2(k)$. Let $v_1(k), v_2(k)$ be corresponding normalized mutually orthogonal eigenvectors. By a direct computation, we may assume that the map

$$\mathbb{R} \ni k \mapsto \lambda_j(k) \in \mathbb{C}$$

is an analytic periodic function with period $2\pi$ for each $j = 1, 2$. Similarly, we may assume that the map

$$\mathbb{R} \ni k \mapsto v_j(k) \in \mathbb{C}^2$$

is a smooth periodic function with period $2\pi$ for each $j = 1, 2$. Define the $2 \times 2$ unitary matrix $W(k)$ by $W(k) := \begin{pmatrix} v_1(k) & v_2(k) \end{pmatrix}$. Then we obtain

$$W^{-1}(k)\hat{U}(k)W(k) = \begin{pmatrix} \lambda_1(k) & 0 \\ 0 & \lambda_2(k) \end{pmatrix}.$$ 

Thus, to construct a time operator of $U$, it is sufficient to find a time operator of the multiplication operator $\lambda_j(k)$ on the Hilbert space $K_1 = L^2(\mathbb{T}, dk/2\pi)$.

To see this this, let $S^1$ be the set of complex numbers of modulus 1, and let $P$ be the operator on $K_1$ defined in Section 3. Note that for any continuous periodic function $g : \mathbb{R} \to S^1$ with period $2\pi$, there are an integer $n \in \mathbb{Z}$ and a continuous periodic function $\theta : \mathbb{R} \to \mathbb{R}$ with period $2\pi$ so that $g(k) = e^{i(nk+\theta(k))}$ holds for any $k \in \mathbb{R}$. The integer $n$ is called the winding number of $g$. For the proof, see e.g., [19] Lemma 3.5.14.
Theorem 4.1. Let $g : \mathbb{R} \to S^1$ be a twice continuously differentiable periodic function with period $2\pi$ whose derivative $g'$ has at most finitely many zeros in $(0, 2\pi)$. Then the operator
\[ \hat{T} := \frac{i}{2} \left( \frac{g(k)}{g'(k)} P + P \frac{g(k)}{g'(k)} \right) \]
on $\mathcal{K}_1$ with natural domain is a strong time operator of the unitary operator $g(k)$.

Proof. Take an integer $n \in \mathbb{Z}$ and a continuous periodic function $\theta : \mathbb{R} \to \mathbb{R}$ with period $2\pi$ so that $g(k) = e^{i(nk + \theta(k))}$ holds for any $k \in \mathbb{R}$. Then $\theta$ is a twice continuously differentiable function, which follows from the following two facts. First, differentiability and continuity are both local properties. Second, the logarithm function is locally well-defined in $\mathbb{C} \setminus \{0\}$.

We compute that
\[ \hat{T} = \frac{i}{2} \left( \frac{g(k)}{g'(k)} P + P \frac{g(k)}{g'(k)} \right) = \frac{1}{2} \left( \frac{1}{n + \theta'(k)} P + P \frac{1}{n + \theta'(k)} \right). \]

Hence, Theorem 4.1 can now be proved by the same manner as Theorem 3.1. \qed

Before applying Theorem 4.1 to $\lambda_j(k)$, we note that $\lambda'_j(k)$ has at most finitely many zeros in $(0, 2\pi)$ because the map
\[ \mathbb{R} \ni k \mapsto \lambda'_j(k) \in \mathbb{C} \]
is analytic. Hence, by Theorem 4.1, we get a strong time operator
\[ \hat{T}_j := \frac{i}{2} \left( \frac{\lambda_j(k)}{\lambda'_j(k)} P + P \frac{\lambda_j(k)}{\lambda'_j(k)} \right) \]
of $\lambda_j(k)$ for each $j = 1, 2$. Let
\[ W := \int_{[0,2\pi]} W(k) \frac{dk}{2\pi}, \quad T := \mathcal{F}^{-1} W \begin{pmatrix} \hat{T}_1 & 0 \\ 0 & \hat{T}_2 \end{pmatrix} W^{-1} \mathcal{F}. \]

Then we conclude that $T$ is a strong time operator of $U$.

Example 4.2. We consider the Hadamard walk, which is the case $C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then we have
\[ \lambda_1(k) = \frac{\sqrt{1 + \cos^2 k} + i \sin k}{\sqrt{2}}, \quad \lambda_2(k) = \frac{-\sqrt{1 + \cos^2 k} + i \sin k}{\sqrt{2}}, \]
whence
\[ \hat{T}_1 = \frac{1}{2} \left( \frac{\sqrt{1 + \cos^2 k}}{\cos k} P + P \frac{\sqrt{1 + \cos^2 k}}{\cos k} \right), \quad \hat{T}_2 = -\hat{T}_1 \]
hold.
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