KP solitons in shallow water

Yuji Kodama

Department of Mathematics, Ohio State University, Columbus, OH 43210, USA
E-mail: kodama@math.ohio-state.edu

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Abstract
The main purpose of the paper is to provide a survey of our recent studies on soliton solutions of the Kadomtsev–Petviashvili (KP) equation. The KP equation describes weakly dispersive and small amplitude wave propagation in a quasi-two-dimensional framework. Recently, a large variety of exact soliton solutions of the KP equation has been found and classified. These solutions are localized along certain lines in a two-dimensional plane and decay exponentially everywhere else, and are called line-solitons. The classification is based on the far-field patterns of the solutions which consist of a finite number of line-solitons. Each soliton solution is then defined by a point of the totally non-negative Grassmann variety which can be parametrized by a unique derangement of the symmetric group of permutations. Our study also includes certain numerical stability problems of those soliton solutions. Numerical simulations of the initial value problems indicate that certain classes of initial waves asymptotically approach to these exact solutions of the KP equation. We discuss an application of our theory to the Mach reflection problem in shallow water. This problem describes the resonant interaction of solitary waves appearing in the reflection of an obliquely incident wave onto a vertical wall, and it predicts an extraordinary fourfold amplification of the wave at the wall. There are several numerical studies confirming the prediction, but all indicate disagreements with the KP theory. Contrary to those previous numerical studies, we find that the KP theory actually provides an excellent model to describe the Mach reflection phenomena when the higher order corrections are included in the quasi-two-dimensional approximation. We also present laboratory experiments of the Mach reflection recently carried out by Yeh and his colleagues, and show how precisely the KP theory predicts this wave behavior.

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(Some figures in this article are in colour only in the electronic version)
1. Introduction

It is a quite well-known story that in August 1834, John Scott Russell observed a large solitary wave in a shallow water channel in Scotland. He noted in his first paper (1838) on the subject that

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed . . . .

This solitary wave is now known as an example of a soliton, and is described by a solution of the Korteweg–de Vries (KdV) equation. The KdV equation describes one-dimensional wave propagation such as beach waves parallel to the coast line or waves in a narrow canal, and is obtained in the leading-order approximation of an asymptotic perturbation theory under the assumptions of weak nonlinearity (small amplitude) and weak dispersion (long waves). The KdV equation has rich mathematical structure including the existence of $N$-soliton solutions and the Lax pair for the inverse scattering method, and it is a prototype equation of the $(1 + 1)$-dimensional integrable systems. In particular, the initial value problem of the KdV equation
The KP equation has been extensively studied by means of the method of inverse scattering transform (IST). It is well known that general initial data decaying rapidly in the spatial variable evolve to a number of individual solitons and weakly dispersive wave trains separate from the solitons (see for examples [1, 33, 36, 50]).

In 1970, Kadomtsev and Petviashvilli [17] proposed a (2 + 1)-dimensional dispersive wave equation to study the stability of the one-soliton solution of the KdV equation under the influence of weak transverse perturbations. This equation is now referred to as the Kadomtsev–Petviashvili (KP) equation. It turns out that the KP equation has much richer structure than the KdV equation, and might be considered as the most fundamental integrable system in the sense that many known integrable systems can be derived as special reductions of the so-called KP hierarchy which consists of the KP equation together with its infinitely many symmetries. The KP equation can also be represented in the Lax form, that is, there exists a pair of linear equations associated with an eigenvalue problem and an evolution of the eigenfunction, which enables the method of IST. However, unlike the case of the KdV equation, the IST for the KP equation does not seem to provide a practical method of solving the initial value problem for initial waves consisting of line-solitons in the far field.

It is quite important to recognize that the resonant interaction plays a fundamental role in a multi-dimensional wave phenomenon. The original description of the soliton interaction for the KP equation was based on a two-soliton solution found in the Hirota bilinear form, which has the shape of ‘X’, describing the intersection of two lines with an oblique angle and a phase shift at the intersection point. This X-shape solution is referred to as the ‘O’-type soliton, where ‘O’ stands for original. In his study of 1977 on an oblique interaction of two line-solitons, Miles [29, 30] pointed out that the O-type solution becomes singular if the angle of the intersection is smaller than a certain critical value depending on the amplitudes of the solitons. Miles then found that at the critical angle, the two line-solitons of the O-type solution interact resonantly, and a third wave is created to make a ‘Y-shaped’ wave form. Indeed, it turns out that such Y-shaped resonant wave forms are exact solutions of the KP equation (see also [34, 37]). Miles applied his theory to study the Mach reflection of an incident wave onto a vertical wall, and predicted that the third wave, called the Mach stem, created by the resonant interaction can reach fourfold amplification of the incidence wave. Several laboratory and numerical experiments were attempted to validate his prediction of fourfold amplification, but with no definitive success (see for examples [15, 21, 46] for numerical experiments, and [28, 39] for laboratory experiments).

After the discovery of the resonant phenomena in the KP equation, several numerical and experimental studies were performed to investigate resonant interactions in other physical two-dimensional equations such as the ion-acoustic and shallow water wave equations under the Boussinesq approximation (see for examples [13, 15, 18, 19, 31, 35, 38, 46, 49]). However, apart from these activities, no significant progress has been made in the study of the solution space or real applications of the KP equation. It would appear that the general perception was that there were not many new and significant results left to be uncovered in the soliton solutions of the KP theory.

Over the past several years, we have been working on the classification problem of the soliton solutions of the KP equation and their applications to shallow water waves. Our studies have revealed a large variety of solutions that were totally overlooked in the past [4, 6–9, 23], and we found that some of those exact solutions can be applied to study the Mach reflection problem [8, 24, 52]. Our numerical study [20] indicates that the solution to the initial value

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1 The lower dimensional solutions, called (2, 2)-soliton solutions, have been found by the binary Darboux transformation in [5].
problem of the KP equation with a certain class of initial waves associated with the Mach reflection problem converges asymptotically to some of these new exact solutions, that is, a separation of dispersive radiations from the soliton solution similar to the case of the KdV soliton.

The main purpose of this paper is to present a survey of our studies on the soliton solutions of the KP equation. The paper also presents several results for recent laboratory experiments done by Harry Yeh and his colleagues at Oregon State University.

The paper is organized as follows.

In section 2, we present the derivation of the Boussinesq-type equation from the three-dimensional Euler equation for the irrotational and incompressible fluid under the assumptions of weak nonlinearity and weak dispersion. The purpose of this section is to give a precise physical meaning to those assumptions and to explain the existence of a solitary wave solution in the form of the KdV soliton.

In section 3, we explain the quasi-two-dimensional approximation to derive the KP equation, and discuss physical interpretation of the KP soliton in terms of the KdV soliton. In order to describe the general soliton solutions, we here introduce the \( \tau \)-function which is expressed by a Wronskian determinant for a set of \( N \) linearly independent functions \( \{ f_i : i = 1, \ldots, N \} \). Each function \( f_i \) is a linear combination of the exponential functions \( \{ E_j : j = 1, \ldots, M \} \), where \( E_j = e^{\theta_j} \) with \( \theta_j = k_j x + k_j^2 y - k_j^3 t \) for some \( k_j \in \mathbb{R} \). The \( \tau \)-function in the Wronskian form was found in \([14, 26, 45]\) (see also [16]). Setting \( f_i = \sum_{j=1}^{M} a_{ij} E_j \), each solution is parametrized by the \( N \times M \) coefficient matrix \( A = (a_{ij}) \) of rank \( N \). This representation naturally leads to the notion of the Grassmann variety \( \text{Gr}(N, M) \), the set of \( N \)-dimensional subspaces given by \( \text{Span}_\mathbb{R} \{ f_i : i = 1, \ldots, N \} \) of \( \mathbb{R}^M = \text{Span}_\mathbb{R} \{ E_j : j = 1, \ldots, M \} \), and each point of \( \text{Gr}(N, M) \) is marked by this \( A \)-matrix [8, 23, 43].

In section 4, we provide a brief summary of the totally non-negative (TNN) Grassmann variety, denoted by \( \text{Gr}^+(N, M) \), which provides the foundation of the classification theorem for the regular soliton solutions of the KP equation discussed in the next section. The main purpose of this section is to show that the \( \tau \)-function in the Wronskian determinant form described in section 3 can be identified as a point of the TNN Grassmannian cell. That is, a classification of the regular soliton solutions of the KP equation is equivalent to a parametrization of TNN Grassmannian cells [6, 8, 9].

In section 5, we present a classification theorem which states that the \( \tau \)-function identified as a point of \( \text{Gr}^+(N, M) \) generates a soliton solution of the KP equation that has asymptotically \( M - N \) line-solitons for \( y \ll 0 \) and \( N \) line-solitons for \( y \gg 0 \). Moreover, these solutions can be parametrized by the derangements (the permutations without fixed points) of the symmetric group \( S_M \). This type of solutions is called the \( (M - N, N) \)-soliton solution. The derangements then give a parametrization of the TNN Grassmannian cells, and each derangement is expressed by a unique chord diagram. The chord diagram is particularly useful to describe the far-field structure of the corresponding soliton solution. (See [6, 8, 9].)

In section 6, we explain all the soliton solutions generated by the \( \tau \)-functions on \( \text{Gr}^+(2, 4) \) (see also [5]). Some of these solutions are useful to describe the Mach reflection problem in shallow water. We show that the \( A \)-matrix determines the detailed structure of those solutions, such as the asymptotic locations of solitons and local interaction patterns. (See [8, 9, 20].)

In section 7, we present the numerical study of the KP equation for certain types of initial waves. In particular, we consider an initial value problem where the initial wave consists of two semi-infinite line-solitons forming a V-shape pattern. Those initial waves were considered in the study of the generation of large amplitude waves in shallow water [41, 49]. The main result of this section is to show that the solutions of this particular initial value
problem converge asymptotically to some of the exact (2,2)-soliton solutions. These results
demonstrate a separation of the (exact) soliton solution from dispersive radiation in the manner
similar to the KdV case. (See [8, 20, 24].)

In section 8, we discuss the Mach reflection problem in terms of the KP solitons, which
is equivalent to Miles’ theory (assuming quasi-two-dimensionality). We first show that the
previous numerical results (see for examples [15, 46]), which reported a large discrepancy
with the theory, are actually in good agreement with the predictions given by the KP theory.
However, here one needs to give a proper physical interpretation of the theory when one
compares it with the numerical results. We also present some laboratory experiments of
shallow water waves [52]. We show that the experimental results are all in good agreement
with the predictions of the KP theory which can describe the evolution of the wave profile.
Finally we demonstrate that the most complex (2,2)-soliton solution associated with the
τ-function on Gr+(2, 4), referred to as the T-type solution, can be realized in an experiment.

John Scott Russell continued on to say in his book (1865) that

This is a most beautiful and extraordinary phenomenon: the first day I saw it was the
happiest day of my life. Nobody has ever had the good fortune to see it before or, at
all events, to know what it meant. It is now known as the solitary wave of translation.
No one before had fancied a solitary wave as a possible thing.

I hope that this survey is successful to convince the readers that the two-dimensional wave
pattern generated by the soliton solutions of the KP equation is the most beautiful and
extraordinary phenomena of two-dimensional shallow water waves, and one should have
no doubt that observing these patterns at a beach will bring the happiest moment of one’s life.

2. Shallow water waves: basic equations

Let us start with the physical background of the KP equation: we consider a surface wave
on water which is assumed to be irrotational and incompressible (see for examples [1, 50]).
We are mainly interested in a long-wave phenomena, and assume zero surface tension, that
is, we ignore the gravity-capillary waves. Then the surface wave may be described by the
three-dimensional Euler equation in the potential form,

\[
\begin{align*}
\Delta \phi &= 0, & \text{for } 0 < \tilde{z} < h_0 + \tilde{\eta}, \\
\tilde{\phi}_z &= 0, & \text{at } \tilde{z} = 0, \\
\tilde{\phi}_t + \frac{1}{2} |\tilde{\nabla} \tilde{\phi}|^2 + g \tilde{\eta} &= 0, & \text{at } \tilde{z} = \tilde{\eta} + h_0, \\
\tilde{\eta}_t + \tilde{\nabla} \tilde{\phi} \cdot \tilde{\nabla} \tilde{\eta} &= \tilde{\phi}_z, \\
\end{align*}
\]  

(2.1)

where \(\tilde{\phi}\) is the velocity potential with the Laplacian \(\Delta = \partial_z^2 + \partial_x^2 + \partial_y^2\), \(g = 980 \text{ cm s}^{-2}\) is the
gravitational constant and \(h_0\) is the average depth. The first two equations of (2.1) imply

\[
\tilde{\phi}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = \cos(\tilde{z} \sqrt{\Delta}) \psi(\tilde{x}, \tilde{y}, \tilde{t}) \quad \text{with} \quad \Delta_\perp = \partial_x^2 + \partial_y^2,
\]

where \(\psi(\tilde{x}, \tilde{y}, \tilde{t}) = \phi(\tilde{x}, \tilde{y}, 0, \tilde{t})\). In the linear limit, the system can be written in the form

\[
\begin{align*}
\cos(h_0 \sqrt{\Delta_\perp}) \tilde{\psi}_t + g \tilde{\eta} &= 0 \\
\tilde{\eta}_t &= \sqrt{\Delta_\perp} \sin(h_0 \sqrt{\Delta_\perp}) \tilde{\psi}.
\end{align*}
\]

This then gives the dispersion relation

\[
\omega^2 = gk \tanh h_0 k = c_0^2 k^2 \left(1 - \frac{1}{3} h_0^2 k^2 + \cdots\right),
\]  

(2.2)
where \( k := \sqrt{k_x^2 + k_y^2} \) and the speed of the surface wave \( c_0 = \sqrt{gh_0} \) (e.g. \( c_0 = 70 \text{ cm s}^{-1} \) when \( h_0 = 5 \text{ cm} \)).

Let us denote the following scales:

- \( \lambda_0 \sim \) horizontal length scale = typical wavelength
- \( h_0 \sim \) vertical length scale = asymptotic water depth
- \( a_0 \sim \) nonlinear scale = typical wave amplitude.

The non-dimensional variables \( \{x, y, z, t, \eta, \phi\} \) are defined as

\[
\begin{align*}
\tilde{x} &= \lambda_0 x, \quad \tilde{y} = \lambda_0 y, \quad \tilde{z} = h_0 z, \quad \tilde{t} = \frac{\lambda_0}{c_0} t, \\
\tilde{\eta} &= a_0 \eta, \quad \tilde{\phi} = \frac{a_0}{h_0} \tilde{\phi}.
\end{align*}
\]

Then the shallow water equation in the non-dimensional form is given by

\[
\begin{align*}
\phi_{zz} + \beta \Delta \phi &= 0, \quad \text{for } 0 < z < 1 + \alpha \eta, \\
\phi_z &= 0, \quad \text{at } z = 0, \\
\eta_t + \frac{1}{\beta} \phi_z &= 0, \quad \text{at } z = 1 + \alpha \eta,
\end{align*}
\]

where the parameters \( \alpha \) and \( \beta \) are given by

\[
\begin{align*}
\alpha &= \frac{a_0}{h_0}, \quad \beta = \left( \frac{h_0}{\lambda_0} \right)^2.
\end{align*}
\]

The weak nonlinearity implies \( \alpha \ll 1 \), and the weak dispersion (or the long-wave assumption) implies \( \beta \ll 1 \). With a small parameter \( \epsilon \ll 1 \), we assume

\[
\alpha \sim \beta = \mathcal{O}(\epsilon).
\]

As in the previous manner, \( \psi \) can be written formally in the form

\[
\phi(x, y, z, t) = \cos(z \sqrt{\beta \Delta}) \psi(x, y, t),
\]

which leads to the expansion

\[
\phi = \psi - \beta \frac{\psi_z^2}{2} \Delta \psi + \mathcal{O}(\epsilon^2).
\]

Then the equations at the surface give the following system of equations, sometimes called the Boussinesq-type equation,

\[
\begin{align*}
\eta + \psi_t + \frac{\alpha}{2} |\nabla \psi|^2 - \frac{\beta}{2} \Delta \psi_t &= \mathcal{O}(\epsilon^2) \\
\eta_t + \Delta \psi - \alpha \nabla \cdot (\psi_t \nabla \psi) - \frac{\beta}{6} \Delta^2 \psi &= \mathcal{O}(\epsilon^2).
\end{align*}
\]

Here we have omitted the subscript \( \perp \). Eliminating \( \eta \) in (2.4), we obtain the so-called isotropic Benney–Luke equation [2].

\[
\psi_{tt} - \Delta \psi + \alpha (\nabla \psi \cdot \nabla \psi_t + \nabla \cdot (\psi_t \nabla \psi)) - \frac{\beta}{2} \left( \Delta \psi_{tt} - \frac{1}{3} \Delta^2 \psi \right) = \mathcal{O}(\epsilon^2).
\]

One can then write this equation in the following form up to the same order:

\[
\left( 1 - \frac{\beta}{3} \Delta \right) \psi_{tt} - \Delta \psi + \alpha (\nabla \psi \cdot \nabla \psi_t + \nabla \cdot (\psi_t \nabla \psi)) = \mathcal{O}(\epsilon^2).
\]

(2.5)
This is a regularized form of the two-dimensional Boussinesq-type equation for the shallow water waves. Note here that the dispersion relation of this equation is given by
\[ \omega^2 = \frac{k^2}{1 + \frac{2}{5}k^2} = k^2 \left( 1 - \frac{\beta}{3}k^2 + \cdots \right) \]
which agrees with (2.2) up to \( O(\epsilon^2) \).

It is also well known that the Boussinesq-type equation can be reduced to the KdV equation for a far field with a unidirectional approximation: let \( \chi \) be the coordinate which is perpendicular to the wave crest of a linear shape solitary wave, i.e.
\[ \chi = \tilde{x} \cos \Psi_0 + \tilde{y} \sin \Psi_0, \]
where \((\cos \Psi_0, \sin \Psi_0)\) is the unit vector in the propagation direction. Then using the far-field coordinates in the propagation direction, i.e.
\[ \xi := \chi - t, \quad \tau = \epsilon t, \]
the Boussinesq-type equation (2.5) becomes the KdV equation
\[ \epsilon \psi_t + \frac{3\alpha}{2} \psi_x \psi_{xx} + \frac{\beta}{6} \psi_{xxxx} = 0. \]
From the first equation of (2.4), we have \( \eta = \psi_x + O(\epsilon) \). Then the KdV equation can be expressed in the form with physical coordinates:
\[ \hat{\eta}_t + c_0 \hat{\eta}_x + \frac{3c_0}{2h_0} \hat{\eta}_{xx} + \frac{c_0 h_0^2}{6} \hat{\eta}_{xxx} = 0. \] (2.6)
The one-soliton solution of the KdV equation is then given by
\[ \hat{\eta} = \hat{a}_0 \text{sech}^2 \left( \sqrt{\frac{3\alpha}{4h_0}} \left[ \chi - c_0 \left( 1 + \frac{\hat{a}_0}{2h_0} \right) \tau - \chi_0 \right] \right), \] (2.7)
where \( \hat{a}_0 > 0 \) and \( \chi_0 \) are arbitrary constants. One should note that any line-solitary wave in the Euler equation (2.1) can be (at least locally) approximated by this soliton under the assumption of weak dispersion and weak nonlinearity. This remark will be important when we compare any numerical results of the Euler equation or the Boussinesq-type equation with those of the KP equation.

3. The KP equation

In this section, we give some basic properties of the KP equation, and introduce the soliton solutions relevant to the shallow water wave problem. In particular, we discuss the physical aspect of the KP equation which is derived by a further assumption called the quasi-two-dimensional approximation (see for example [1, 17]).

3.1. Quasi-two-dimensional approximation and one-soliton solution

Let us now assume a quasi-two-dimensionality with a weak dependence in the \( y \)-direction, and we introduce a small parameter \( \gamma \) so that the \( y \)-coordinate is scaled as
\[ \zeta := \sqrt{\gamma} y, \quad \text{with} \quad \gamma = O(\epsilon). \] (3.1)
Then the system of equations (2.4) becomes
\[
\begin{cases}
\eta_t + \frac{\alpha}{2} \psi_x^2 - \frac{\beta}{2} \psi_{xxt} = O(\epsilon^2) \\
\eta_x + \psi_x - \alpha (\psi_x \psi_x)_x - \frac{\beta}{6} \psi_{xxxx} + \gamma \psi_{\xi\xi} = O(\epsilon^3).
\end{cases}
\]
Now we consider a far field expressed with the scaling,
\[ \xi = x - t \quad \text{and} \quad \tau = \epsilon t. \] (3.2)

Then the above equations have the expansions
\[
\begin{align*}
\eta - \psi_\xi + \epsilon \psi_\tau + \frac{\alpha}{2} \psi_\xi^2 + \frac{\beta}{2} \psi_{\xi\xi\xi} &= \mathcal{O}(\epsilon^2) \\
-\eta_\xi + \psi_{\xi\xi} + \epsilon \eta_\tau + \alpha \left(\psi_\xi^2\right)_\xi - \frac{\beta}{6} \psi_{\xi\xi\xi\xi} + \gamma \psi_{\xi\xi} &= \mathcal{O}(\epsilon^2).
\end{align*}
\]

Eliminating \( \eta \), we obtain
\[
2\epsilon \psi_{\xi\xi} + 3\alpha \psi_\xi \psi_{\xi\xi} + \frac{\beta}{3} \psi_{\xi\xi\xi\xi} + \gamma \psi_{\xi\xi} = \mathcal{O}(\epsilon^2).
\]

Noting \( \eta = \psi_\xi + \mathcal{O}(\epsilon) \), we have the KP equation for \( \eta \) at the leading order:
\[
\left(2\epsilon \eta_\tau + 3\alpha \eta \eta_\xi + \frac{\beta}{3} \eta_{\xi\xi\xi}\right)_{\xi} + \gamma \eta_{\xi\xi} = \mathcal{O}(\epsilon^2). \tag{3.3}
\]

In terms of physical coordinates, the KP equation is given by
\[
\left(\tilde{\eta}_\xi^0 + c_0 \tilde{\eta}_t^0 + \frac{3c_0}{2h_0} \tilde{\eta}_\xi^0 + \frac{c_0 h_0^2}{6} \tilde{\eta}_{\xi\xi}\right)_{\xi} + \gamma \eta_{\xi\xi} = 0.
\]

As a particular solution, we have one line-soliton solution in the form
\[
\tilde{\eta} = a_0 \operatorname{sech}^2 \sqrt{\frac{3a_0}{4h_0}} \left[ \bar{x} + \bar{y} \tan \psi_0 - c_0 \left(1 + \frac{a_0}{2h_0} + \frac{1}{2} \tan^2 \psi_0\right) \tilde{t} - \tilde{x}_0 \right], \tag{3.4}
\]

where \( a_0 > 0, \psi_0 \) and \( \tilde{x}_0 \) are the arbitrary constants. One should note that the KP equation is derived under the assumption of quasi-two dimensionality, that is, the angle \( \Psi_0 \) should be small of order \( \mathcal{O}(\epsilon^{\frac{1}{2}}) \), i.e. \( \gamma = \tan^2 \psi_0 = \mathcal{O}(\epsilon) \), and solution (3.4) becomes unphysical for the case with a large angle. This can be seen explicitly by writing it in the following form in the coordinate perpendicular to the wave crest, i.e. \( \chi = \bar{x} \cos \psi_0 + \bar{y} \sin \psi_0 \),
\[
\tilde{\eta} = a_0 \operatorname{sech}^2 \sqrt{\frac{3a_0}{4h_0 \cos^2 \psi_0}} \left[ \chi - c_0 \cos \psi_0 \left(1 + \frac{a_0}{2h_0} + \frac{1}{2} \tan^2 \psi_0\right) \tilde{t} - \chi_0 \right]. \tag{3.5}
\]

Noting that \( \cos \psi_0 = 1 - \frac{1}{2} \tan^2 \psi_0 + \mathcal{O}(\epsilon^{2}) \) with \( \psi_0 = \mathcal{O}(\epsilon^{\frac{1}{2}}) \), the velocity of the soliton has the corrected form up to \( \mathcal{O}(\epsilon) \), i.e.
\[
\cos \psi_0 \left(1 + \frac{a_0}{2h_0} + \frac{1}{2} \tan^2 \psi_0\right) = 1 + \frac{a_0}{2h_0} + \mathcal{O}(\epsilon^2),
\]

which does not depend on the angle up to \( \mathcal{O}(\epsilon) \). This is consistent with the assumption of the quasi-two-dimensionality.

We also note that the line-soliton of (3.4) does not satisfy the KdV equation (2.6) except the case with \( \Psi_0 = 0 \). Now comparing the KP soliton (3.4) with the KdV soliton (2.7), one can find the correction to the quasi-two-dimensional approximation, that is, the amplitude \( a_0 \) in (3.4) is now corrected to
\[
a_0 = \frac{a_0}{\cos^2 \psi_0} = a_0 (1 + \tan^2 \psi_0). \tag{3.6}
\]

With this correction, the KP soliton (3.4) now satisfies the KdV equation (2.6) up to \( \mathcal{O}(\epsilon) \). This correction becomes quite important when we compare our KP results with numerical results of the Euler- or Boussinesq-type equations, that is, (3.6) gives the relation between the amplitudes of the KP soliton and the KdV soliton (see section 8 for the details).
3.2. Soliton solutions in the Wronskian determinant

In order to give a general scheme to discuss the soliton solutions, we first put the KP equation (3.3) in the standard form

\[(4u_T + 6uu_X + u_{XXX})_X + 3u_{YY} = 0,\]  

where the new variables \((X, Y, T)\) and \(u\) are related to the physical ones with

\[\tilde{\eta} = \frac{2h_0}{3}u, \quad \tilde{x} - c_0\tilde{t} = h_0X, \quad \tilde{y} = h_0Y, \quad \tilde{t} = \frac{3h_0}{2c_0}T.\]  

Hereafter we use the lower case letters \((x, y, t)\) for \((X, Y, T)\) (we do not use the non-dimensional variables in (2.3), and the KP variables can be converted to the physical variables directly through relations (3.8)). We write the solution of the KP equation (3.7) in the \(\tau\)-function form

\[u(x, y, t) = 2\partial_x^2 \ln \tau(x, y, t),\]  

where the \(\tau\)-function is assumed to be the Wronskian determinant with \(N\) functions \(f_i\)'s (see for examples [14, 16, 26, 45]),

\[\tau = \text{Wr}(f_1, f_2, \ldots, f_N) := \begin{vmatrix} f_1 & f_1^{(1)} & \cdots & f_1^{(N-1)} \\ f_2 & f_2^{(1)} & \cdots & f_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_N^{(1)} & \cdots & f_N^{(N-1)} \end{vmatrix}, \]  

with \(f_i^{(n)} = \partial^n_x f_i\). The functions \(\{f_i : i = 1, \ldots, N\}\) satisfy the linear equations \(\partial_y f_i = \partial_x^2 f_i\) and \(\partial_t f_i = -\partial_x^3 f_i\), and for the soliton solution, we take

\[f_i = \sum_{j=1}^M a_{ij} E_j, \quad \text{with} \quad E_j = e^\phi := \exp \left(k_j x + k_j^2 y - k_j^3 t \right).\]  

Here \(A := (a_{ij})\) defines an \(N \times M\) matrix \(A = (a_{ij})\) of rank \(N\), and we assume the real parameters \(\{k_j : j = 1, \ldots, M\}\) to be ordered:

\[k_1 < k_2 < \cdots < k_M.\]

One should emphasize here that we have a parametrization of each soliton solution of the KP equation in terms of the \(k\)-parameters and the \(A\)-matrix. Then the classification of the soliton solutions is to give a complete characterization of the \(\tau\)-function (3.10) with the exponential functions in (3.11). This is the main theme in [3, 8, 9, 23] and will be discussed in the following sections.

In terms of the \(\tau\)-function, the KP equation (3.7) is written in the bilinear form

\[4(\tau_{xy} - \tau_x \tau_y) + \tau_{xxxx} - 4\tau_x \tau_{xxx} + 3\tau_{xx}^2 + 3(\tau_{yy} - \tau_y^2) = 0.\]  

To show that the \(\tau\)-function (3.10) satisfies this equation, we express the derivatives of the \(\tau\)-function using the Young diagram. Let \(Y\) be given by the partition \(Y = (\lambda_1 \geq \cdots \geq \lambda_n)\), where \(\lambda_j\)'s represent the numbers of boxes in \(Y\), and \(|Y|\) denote the total number of boxes, i.e. \(|Y| = \sum_{j=1}^n \lambda_j\). Denote \(\tau\) as the \(N\)-tuple,

\[\tau = \tau_\beta = (0, 1, 2, \ldots, N - 1),\]

where the numbers describe the orders of the derivative of the column vector \((f_1, \ldots, f_N)^T\) in the \(\tau\)-function (3.10). Then the number of boxes in each row of \(Y\) can be found by counting
For the τ-representation

\[ \tau = (0, 1, 2, \ldots, N - 3, N - 1, N + 1) . \]

For the number \( N + 1 \), two numbers \( N - 2 \) and \( N \) are missing, and this gives \( \tau \). For the number \( N - 1 \), one number \( N - 2 \) is missing, and this gives \( \tau \). In terms of the determinant, \( \tau \) represents

\[
\tau = \begin{vmatrix}
 f_1 & \cdots & f_1^{(N-3)} & f_1^{(N-1)} & f_1^{(N+1)} \\
 \vdots & \ddots & \vdots & \vdots & \vdots \\
 f_N & \cdots & f_N^{(N-3)} & f_N^{(N-1)} & f_N^{(N+1)} \\
 0 & \cdots & 0 & f_1^{(N-3)} & f_1^{(N-1)} \\
 \vdots & \ddots & \vdots & \vdots & \vdots \\
 0 & \cdots & 0 & f_N^{(N-3)} & f_N^{(N-1)} \\
\end{vmatrix}
\]

With this notation and using the equations for \( f_i \), i.e. \( \partial_i f_i = f_i^{(2)} \), \( \partial_i f_i = -f_i^{(3)} \), the derivatives of the \( \tau \)-function (3.10) are given by

\[
\tau_\sigma = \tau, \quad \tau_\infty = \frac{1}{2} (\tau_{xx} + \tau_y), \quad \tau_\Box = \frac{1}{2} (\tau_{xx} - \tau_y),
\]

\[
\tau_\Box = \frac{1}{2} (\tau_{xxx} + 3\tau_{yy} - 4\tau_{xy}), \quad \tau_\Box = \frac{1}{2} (\tau_{xxx} - \tau_{yy}).
\]

Then the bilinear equation (3.12) can be written in the form

\[
\tau_\theta \tau_\Box - \tau_\theta \tau_\Box + \tau_\Box \tau_\Box = 0. \tag{3.13}
\]

This equation is nothing but the Laplace expansion of the following \( 2N \times 2N \) determinant identity:

\[
\begin{vmatrix}
 f_1 & \cdots & f_1^{(N-2)} & f_1^{(N-1)} & f_1^{(N+1)} \\
 \vdots & \ddots & \vdots & \vdots & \vdots \\
 f_N & \cdots & f_N^{(N-2)} & f_N^{(N-1)} & f_N^{(N+1)} \\
 0 & \cdots & 0 & f_1^{(N-3)} & f_1^{(N-1)} \\
 \vdots & \ddots & \vdots & \vdots & \vdots \\
 0 & \cdots & 0 & f_N^{(N-3)} & f_N^{(N-1)} \\
\end{vmatrix} = 0,
\]

so that (3.13) is identically satisfied. Relation (3.13) is the simplest example of the Plücker relations (see below), and it can be symbolically written by

\[
\xi(1, 2)\xi(3, 4) - \xi(1, 3)\xi(2, 4) + \xi(1, 4)\xi(2, 3) = 0, \tag{3.14}
\]

where the Young diagrams are expressed by \( Y = (j - 2, i - 1) \) for the symbol \( \xi(i, j) \). These symbols \( \xi(j_1, \ldots, j_N) \) are the so-called Plücker coordinates of the Grassmannian manifold Gr(\( N, M \)). In the next section, we outline the basic information for the Grassmannian Gr(\( N, M \)) which will provide the foundation of the classification theory for the soliton solutions of the KP equation [8, 23].

**Example 3.1.** Let us express one line-soliton solution (3.4) in our setting. Here we also introduce some notations to describe the soliton solutions. The soliton solution (3.4) is obtained by the \( \tau \)-function with \( M = 2 \) and \( N = 1 \), i.e.

\[
\tau = f_1 = a_{11} E_1 + a_{12} E_2.
\]

Since the solution \( u \) is given by (3.9), one can assume \( a_{11} = 1 \) and denote \( a_{12} = a > 0 \). Then

\[
\tau = E_1 + a E_2 = 2\sqrt{a} e^{i(\theta_1 + \theta_2)} \cosh \frac{1}{2} (\theta_1 - \theta_2 - \ln a),
\]
with the $1 \times 2$ $A$-matrix of the form $A = (1 \ a)$. The parameter $a$ in the $A$-matrix must satisfy $a \geq 0$ for a non-singular solution and it determines the location of the soliton solution. Since $a = 0$ leads to a trivial solution, we consider only $a > 0$. Then the solution $u = 2\theta_1^2(\ln \tau)$ gives

$$u = \frac{1}{2}(k_1 - k_2)^2 \operatorname{sech}^2 \left( \frac{1}{2}(\theta_1 - \theta_2 - \ln a) \right).$$

Thus the solution is localized along the line $\theta_1 - \theta_2 = \ln a$; hence we call it line-soliton solution. We emphasize here that the line-soliton appears at the boundary of two regions where either $E_1$ or $E_2$ is the dominant exponential term, and because of this we also call this soliton a $[1, 2]$-soliton solution. In section 5, we construct more general line-soliton solutions which separate into a number of one-soliton solutions asymptotically as $|y| \to \infty$. We refer to each of these asymptotic line-solitons as the $[i, j]$-soliton. The $[i, j]$-soliton solution with $i < j$ has the same (local) structure as the one-soliton solution, and can be described as follows:

$$u = A_{[i, j]} \operatorname{sech}^2 \left( \frac{1}{2}(K_{[i, j]} \cdot x - \Omega_{[i, j]} t + \Theta_0^{[i, j]}) \right)$$

with some constant $\Theta_0^{[i, j]}$. The amplitude $A_{[i, j]}$, the wave vector $K_{[i, j]}$ and the frequency $\Omega_{[i, j]}$ are defined by

$$A_{[i, j]} = \frac{1}{2}(k_j - k_i)^2,$$

$$K_{[i, j]} = (k_j - k_i, k_i^2 - k_j^2) = (k_j - k_i)(1, k_i + k_j),$$

$$\Omega_{[i, j]} = k_j^3 - k_i^3 = (k_j - k_i)(k_i^2 + k_i k_j + k_j^2).$$

The direction of the wave vector $K_{[i, j]} = (K_{[i, j]}^x, K_{[i, j]}^y)$ is measured in the counterclockwise sense from the $y$-axis, and it is given by

$$\frac{K_{[i, j]}^x}{K_{[i, j]}^y} = \tan \Psi_{[i, j]} = k_i + k_j,$$

that is, $\Psi_{[i, j]}$ gives the angle between the line $K_{[i, j]} \cdot x = \text{const}$ and the $y$-axis. Then one line-soliton can be written in the form with three parameters $A_{[i, j]}$, $\Psi_{[i, j]}$ and $x_0^{[i, j]}$:

$$u = A_{[i, j]} \operatorname{sech}^2 \sqrt{\frac{A_{[i, j]}}{2}} \left( x + y \tan \Psi_{[i, j]} - C_{[i, j]} t - x_0^{[i, j]} \right), \quad (3.15)$$

with $C_{[i, j]} = k_i^2 + k_i k_j + k_j^2 = \frac{1}{2} A_{[i, j]} + \frac{3}{4} \tan^2 \Psi_{[i, j]}$. In figure 1, we illustrate one line-soliton solution of $[i, j]$-type. In the right panel of this figure, we show a chord diagram which represents this soliton solution. Here the chord diagram indicates the permutation of the dominant exponential terms $E_i$ and $E_j$ in the $\tau$-function, that is, with the ordering $k_i < k_j$, $E_j$ dominates in $x < 0$, while $E_j$ dominates in $x > 0$ (see section 4 for the precise definition of the chord diagram).

For each soliton solution of (3.15), the wave vector $K_{[i, j]}$ and the frequency $\Omega_{[i, j]}$ satisfy the soliton–dispersion relation (see (2.2))

$$4\Omega_{[i, j]} K_{[i, j]}^x = \left( K_{[i, j]}^x \right)^4 + 3 \left( K_{[i, j]}^y \right)^2. \quad (3.16)$$

The soliton velocity $V_{[i, j]}$ is along the direction of the wave vector $K_{[i, j]}$, and is defined by $K_{[i, j]} \cdot V_{[i, j]} = \Omega_{[i, j]}$, which yields

$$V_{[i, j]} = \frac{\Omega_{[i, j]}}{|K_{[i, j]}|^2} K_{[i, j]} = \frac{k_i^2 + k_i k_j + k_j^2}{1 + (k_i + k_j)^2} (1, k_i + k_j).$$

Note in particular that since $C_{[i, j]} = k_i^2 + k_i k_j + k_j^2 > 0$, the $x$-component of the soliton velocity is always positive, i.e. any soliton propagates in the positive $x$-direction. In the
Figure 1. One line-soliton solution of \([i,j]\)-type and the corresponding chord diagram. The amplitude \(A_{[i,j]}\) and the angle \(\Psi_{[i,j]}\) are given by

\[ A_{[i,j]} = \frac{1}{2}(k_i - k_j)^2 \quad \text{and} \quad \tan\Psi_{[i,j]} = k_i + k_j. \]

The upper-oriented chord represents the part of the \([i,j]\)-soliton for \(y \gg 0\) and the lower one for \(y \ll 0\).

physical coordinates (see (3.8)), this implies that soliton propagates with super-sonic speed (i.e. the speed of soliton is faster than \(c_0 = \sqrt{gh_0}\), because of its nonlinear effect with \(\tilde{\eta} > 0\), see section 2). On the other hand, one should note that any small perturbation propagates in the negative \(x\)-direction, i.e. the \(x\)-component of the group velocity is always negative. This can be seen from the dispersion relation of the linearized KP equation for a plane wave \(\phi = \exp(ik \cdot x - i\omega t)\) with the wave vector \(k = (k_x, k_y)\) and the frequency \(\omega\):

\[ \omega = -\frac{k_x^3}{4} + \frac{3}{4} \frac{k_y^2}{k_x}, \]

from which the group velocity of the wave is given by

\[ v = \nabla \omega = \left( \frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y} \right) = \left( -\frac{3}{4} \left( k_x^2 + \frac{k_y^2}{2 k_x} \right) \right). \]

Physically, this means that the radiations disperse with sub-sonic speeds. This is similar to the case of the KdV equation, and we expect that asymptotically, the soliton separates from small radiations. We further discuss this issue in section 7 where we numerically observe the separation.

Remark 3.2. In formulas (3.11), if we include the higher times \(t_n\) in the exponential functions, i.e.

\[ E_j = \exp \left( \sum_{n=1}^{\infty} k_j^n t_n \right), \]

then the \(\tau\)-function (3.10) gives a solution of the KP hierarchy. The equation for the \(t_n\)-flow is a symmetry of the KP equation, and the \(\tau\)-function with those higher times also satisfies the other Plücker relations which are expressed with the Young diagrams having larger numbers of boxes [32].

4. Totally non-negative Grassmannian \(Gr^+(N, M)\)

In the previous section, we considered a class of solutions which are expressed by the \(\tau\)-functions (3.10) with the exponential functions (3.11). Those solutions are determined by
the $k$-parameters and the $A$-matrix. Fixing the $k$-parameters, we have a set of $M$ exponentials \( \{ E_j = e^{\theta_j} : j = 1, \ldots, M \} \) which spans $\mathbb{R}^M$. Then the set of functions $\{ f_i : i = 1, \ldots, N \}$ of (3.11) defines an $N$-dimensional subspace of $\mathbb{R}^M$. This leads naturally to the notion of Grassmannian $\text{Gr}(N, M)$, the set of all $N$-dimensional subspaces in $\mathbb{R}^M$, and each point of $\text{Gr}(N, M)$ can be parametrized by the $A$-matrix in (3.11). Here we give a brief review of the Grassmann manifold $\text{Gr}(N, M)$; in particular, we describe the TNN part of $\text{Gr}(N, M)$.

The main purpose of this section is to explain a mathematical background of regular soliton solutions of the KP equation.

4.1. The Grassmannian $\text{Gr}(N, M)$

Recall that the set of the functions $f_i$ spans an $N$-dimensional subspace which is parametrized by an $N \times M$ matrix $A$ of rank $N$, i.e.

\[
(f_1, f_2, \ldots, f_N) = (E_1, E_2, \ldots, E_M)A^T.
\]

Since the other set of functions $(g_1, \ldots, g_N) = (f_1, \ldots, f_N)H$ for some $H \in \text{GL}_N(\mathbb{R})$ gives the same subspace, the $A$-matrix can be canonically chosen in the reduced row echelon form (RREF). This then gives an explicit definition of the Grassmannian,

\[
\text{Gr}(N, M) = \text{GL}_N(\mathbb{R}) \setminus \mathcal{M}_{N \times M}(\mathbb{R}),
\]

where $\mathcal{M}_{N \times M}(\mathbb{R})$ denotes the set of $N \times M$ matrices of rank $N$. The canonical form of $A$ is distinguished by a set of pivot columns labeled by $I = \{ i_1, i_2, \ldots, i_N \}$, $1 \leq i_1 < i_2 < \cdots < i_N \leq M$ such that the $N \times N$ sub-matrix $A_I$ formed by the column set $I$ is the identity matrix. Each $N \times M$ matrix $A$ in RREF uniquely determines an $N$-dimensional subspace, thus providing a coordinate for a point of $\text{Gr}(N, M)$. The set $W_I$ of all points in $\text{Gr}(N, M)$ represented by RREF matrices $A$ which have the same pivot set $I$ is called a Schubert cell which gives the decomposition of the Grassmannian, the Schubert decomposition,

\[
\text{Gr}(N, M) = \bigsqcup_{1 \leq i_1 < i_2 < \cdots < i_N \leq M} W_I, \quad I = \{ i_1, i_2, \ldots, i_N \}.
\]

For example, if $I = \{ 1, 2, \ldots, N \}$, then the Schubert cell $W_I$ contains all $A$ matrices whose RREF is given by

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & * & \cdots & * \\
0 & 1 & \cdots & 0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & * & \cdots & *
\end{pmatrix},
\]

where the $N(M - N)$ entries of the right-hand block are arbitrary real numbers. This particular Schubert cell is often referred to as the top cell which has the maximum number of free parameters marked by *. It follows from this that the dimension of $\text{Gr}(N, M)$ is $N(M - N)$. The number of free parameters for an $A$-matrix in RREF with a given pivot set $I = \{ i_1, \ldots, i_N \}$ is equal to the the dimension of the cell $W_I$, and is given by

\[
\dim W_I = N(M - N) - \sum_{n=1}^{N} (i_n - n).
\]

Note here that the index set $I = \{ i_1, \ldots, i_N \}$ can be expressed by the Young diagram with $Y_I = (i_N - N, \ldots, i_2 - 2, i_1 - 1)$, and then $\text{codim } W_I = |Y|$. 

13
**Example 4.1.** The Schubert decomposition of $\text{Gr}(2, 4)$ has the form

$$\text{Gr}(2, 4) = \bigsqcup_{1 \leq i < j \leq 4} W_{[i,j]}.$$  

There are six cells $W_{[i,j]}$ with $\dim W_{[i,j]} = 7 - (i + j)$, and are listed below:

(a) $W_{[1,2]} = W_\emptyset = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix} \right\}$,  

(b) $W_{[1,3]} = W_\emptyset = \left\{ \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \right\}$,  

(c) $W_{[1,4]} = W_\emptyset = \left\{ \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$,  

(d) $W_{[2,3]} = W_\emptyset = \left\{ \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \right\}$,  

(e) $W_{[2,4]} = W_\emptyset = \left\{ \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$,  

(f) $W_{[3,4]} = W_\emptyset = \left\{ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$.

The top cell $W_{[1,2]}$ has four free parameters which gives $\dim \text{Gr}(2, 4)$, while the bottom cell $W_{[3,4]}$ is zero dimensional and corresponds to a single point of the Grassmannian.

We also note that each cell in the Schubert decomposition can be parametrized by a unique element of $S_M$, the symmetric group of permutations for $M$ letters. The group $S_M$ is generated by the adjacent transpositions $s_j := (j, j + 1)$, i.e.

$$S_M = \langle s_1, s_2, \ldots, s_{M-1} \rangle,$$

with $s_1^2 = e$, the identity element, $s_j s_j = s_j s_i$ if $|i - j| > 1$ and $(s_i s_{i+1})^3 = e$. Let $P_N$ be a maximal parabolic subgroup of $S_M$ generated by $s_j$’s without the element $s_{M-N}$, i.e.

$$P_N := \langle s_1, \ldots, s_{M-N-1}, s_{M-N+1}, \ldots, s_{M-1} \rangle \cong S_{M-N} \times S_N.$$

Then the pivot set $I = \{i_1, i_2, \ldots, i_N\}$ parametrizing the Schubert cell $W_I$ can be uniquely labeled by a minimal length representative of the quotient:

$$S^{(N)}_M := \frac{S_M}{P_N} = \{\text{the reduced words of mod}(P_N) \text{ ending in } s_{M-N}\}.$$  

Namely, we have the Schubert decomposition of $\text{Gr}(N, M)$ in terms of the quotient $S^{(N)}_M$:

$$\text{Gr}(N, M) = \bigsqcup_{\pi \in S^{(N)}_M} W_\pi,$$

where the dimension of the cell $W_\pi$ is given by the length of the permutation, i.e. $\dim W_\pi = \ell(\pi)$. For example, in the case of $\text{Gr}(1, 3)$, we have $S_3^{(1)} = \{s_1, s_2\}/\{s_1\} = \{e, s_2, s_1 s_2\}$:

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{s_2} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \xrightarrow{s_1} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$  

Here $[1]$ represents a pivot, so we have

$$W_e = \{(0, 0, 1)\}, \quad W_{s_2} = \{(0, 1, *)\}, \quad W_{s_1 s_2} = \{(1, *, *)\}.$$  

Also in the case of $\text{Gr}(2, 3)$, we have $S_3^{(2)} = \{s_1, s_2\}/\{s_2\} = \{e, s_1, s_2 s_1\}$,

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{s_1} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{s_2} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

and the Schubert cells $W_\pi$ are given by

$$W_e = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad W_{s_1} = \left\{ \begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad W_{s_2 s_1} = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & \ast \end{pmatrix} \right\}.$$
Note in particular that the last elements in the above examples have no fixed points, and they are called derangements. We show that each derangement of $S_M$ parametrizes a unique line-soliton solution generated by the $\tau$-function of the form (3.10). It is important for our purposes to remark that each permutation $\pi \in S_M$ with marked pivot positions can be uniquely expressed by the chord diagram. This permutation is the decorated permutation defined in [42] for a parametrization of the TNN Grassmann cells.

**Definition 4.2.** A chord diagram associated with $\pi \in S_M$ is defined as follows: consider a line segment with $M$ marked points by the numbers $\{1, 2, \ldots, M\}$ in the increasing order from the left.

- (a) If $i < \pi(i)$ (excedance), then draw a chord joining $i$ and $\pi(i)$ on the upper part of the line.
- (b) If $j > \pi(j)$ (deficiency), then draw a chord joining $j$ and $\pi(j)$ on the lower part of the line.
- (c) If $l = \pi(l)$ (fixed point), then
  - (i) if $l$ is a pivot, then draw a loop on the upper part of the line at this point.
  - (ii) if $l$ is a non-pivot, then draw a loop on the lower part of the line at this point.

The dimension of each Schubert cell of $Gr(N, M)$ can also be found from the chord diagram, and it is given by

$$\dim W_\pi = N + \{\text{# of crossings}\} + \{\text{# of cusps in the lower part}\} - \{\text{# of loops in the upper part}\}.$$  

Here we say that the point marked by $j$ is a ‘cusp’, if $\pi(j) < j = \pi(k) < k$ or $k < \pi(k) = j < \pi(j)$ for some $k$. In particular, the point $j$ is a cusp in the lower part of the diagram, if $\pi(j) < j = \pi(k) < k$ (see [10, 51]).

**Example 4.3.** Consider the case of $Gr(2, 4)$. The Schubert cells $W_{\{i,j\}}$ are marked by the pivots $\{i, j\}$ with $1 \leq i < j \leq 4$, and the permutation representations are given by

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4
\end{pmatrix} \xrightarrow{s_2} \begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 3 & 2 & 4
\end{pmatrix} \xrightarrow{s_1} \begin{pmatrix}
1 & 3 & 2 & 4 \\
1 & 2 & 3 & 4
\end{pmatrix} \\
\downarrow s_3 \quad \downarrow s_3 \\
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 3 & 4 & 2
\end{pmatrix} \xrightarrow{s_1} \begin{pmatrix}
1 & 3 & 2 & 4 \\
1 & 4 & 3 & 2
\end{pmatrix} \\
\downarrow s_2 \quad \downarrow s_2 \\
\begin{pmatrix}
1 & 3 & 2 & 4 \\
3 & 4 & 1 & 2
\end{pmatrix}.
\]
The chord diagrams are shown below, and the points with filled circle indicate the pivots for those cells:

One should note here that each fixed point corresponds to a loop of the diagrams, and the diagrams without loops are associated with the derangements of the permutation group. Also note that each pivot gives an (weak) excedance, i.e. \( i < \pi(i) \) for the pivot index \( i \). In the chord diagrams in the rest of the paper, each cord will be oriented, and the pivots are identified as the starting points of the upper chords.

As we will show, each chord (not loop) identifies a line-soliton for \( y \gg 0 \) (or \( y \ll 0 \)) corresponding to the location of the chord in the upper (or lower) part of the chord diagram. For example, in the case \( \pi = (1^2 3 4) \), we have the \([1, 3]\)- and \([3, 4]\)-solitons in \( y \gg 0 \) and the \([1, 2]\)- and \([2, 4]\)-solitons in \( y \ll 0 \).

4.2. The Plücker coordinates and total non-negativity

We here describe the TNN Grassmannian \( \text{Gr}^+ (N, M) \) as a subspace of \( \text{Gr}(N, M) \). Then we will show that the \( \tau \)-function associated with \( \text{Gr}^+ (N, M) \) is necessary and sufficient conditions for the solution generated by the \( \tau \)-function to be regular.

We first note that the coordinates of \( \text{Gr}(N, M) \) are given by the Plücker embedding into the projectivization of the wedge product space \( \wedge^N \mathbb{R}^M \), i.e.

\[
\text{Gr}(N, M) \hookrightarrow \mathbb{P}(\wedge^N \mathbb{R}^M),
\]

which maps each frame given by \( \{ f_1, \ldots, f_N \} \in \text{Gr}(N, M) \) to the point on \( \mathbb{P}(\wedge^N \mathbb{R}^M) \), i.e.

\[
f_1 \wedge \cdots \wedge f_N = \sum_{1 \leq j_1 < \cdots < j_N \leq M} \xi(j_1, \ldots, j_N) E_{j_1} \wedge \cdots \wedge E_{j_N}.
\] (4.2)

Here the coefficients \( \xi(j_1, \ldots, j_N) \) are the \( N \times N \) minors of the \( A \)-matrix defined by

\[
\xi(j_1, \ldots, j_N) := \det | (a_{n, j_k})_{1 \leq n \leq N} |.
\]

Note here that \( \xi(j_1, \ldots, j_N) = 1 \) where the set \( \{ j_1, \ldots, j_N \} \) is the pivot set. Those minors \( \xi(j_1, \ldots, j_N) \) are called the Plücker coordinates, which give a coordinate system for the linear space \( \wedge^N \mathbb{R}^M \) with the basis,

\[
\{ E_{j_1} \wedge \cdots \wedge E_{j_N} : 1 \leq j_1 < \cdots < j_N \leq M \}.
\]

Then the Grassmannian structure is determined by certain relations on the Plücker coordinates, called the Plücker relations: for any two index sets \( \{ \alpha_1, \ldots, \alpha_{N-1} \} \) and \( \{ \beta_1, \ldots, \beta_{N+1} \} \) with \( 1 \leq \alpha_i, \beta_j \leq M \), they are given by

\[
\sum_{j=1}^{N+1} \xi(\alpha_1, \ldots, \alpha_{N-1}, \beta_j) \xi(\beta_1, \ldots, \beta_j, \ldots, \beta_{N+1}) = 0,
\] (4.3)
where \( \hat{\beta}_j \) implies the deletion of \( \beta_j \). The Plücker relations can be derived using elementary linear algebra from the Laplace expansion of the following \( 2N \times 2N \) determinant formed by the columns \( A_i \) of the matrix \( A \), i.e.

\[
\begin{vmatrix}
A_{a_1} & \cdots & A_{a_{N-1}} & A_{\hat{\beta}_1} & \cdots & A_{\hat{\beta}_{N-1}} \\
0 & \cdots & 0 & A_{\beta_1} & \cdots & A_{\beta_{N-1}}
\end{vmatrix} = 0.
\]

The Plücker coordinates, modulo the Plücker relations, give the correct dimension of \( \text{Gr}(N, M) \) which is typically less than that of \( \mathbb{P}(\wedge^N \mathbb{R}^M) \).

**Example 4.4.** For \( \text{Gr}(2, 4) \), the Plücker coordinates are given by the maximal minors, 

\[
\xi(1.2), \quad \xi(1.3), \quad \xi(1.4), \quad \xi(2.3), \quad \xi(2.4), \quad \xi(3.4).
\]

Taking \( \{\alpha_i\} = \{1\}, \{\beta_1, \beta_2, \beta_3\} = \{2, 3, 4\} \) in (4.3) gives the only Plücker relation in this case, 

\[
\xi(1,2)\xi(3,4) - \xi(1,3)\xi(2,4) + \xi(1,4)\xi(2,3) = 0,
\]

which is the same as (3.14). Since \( \dim(\wedge^2 \mathbb{R}^4) = 6 \), the projectivization gives \( \dim(\mathbb{P}(\wedge^2 \mathbb{R}^4)) = 6 - 1 = 5 \). Then with one Plücker relation, the dimension of \( \text{Gr}(2, 4) \) turns out to be 4, which is consistent with the dimension of the top cell \( W_{1,2} \) as shown in example 3.2 (case (a)).

Since each point of \( \text{Gr}(N, M) \) is expressed by (4.2), the TNN Grassmannian \( \text{Gr}^*(N, M) \) is defined by the set of \( N \times M \) matrices of rank \( N \) whose minors, the Plücker coordinates, are all non-negative, i.e.

\[
\text{Gr}^*(N, M) := \{ A \in \text{Gr}(N, M) : \xi(j_1, \ldots, j_N) \geq 0, \forall 1 \leq j_1 < \cdots < j_M \leq M \}.
\]

Then the most interesting question is to find a parametrization of all the cells in \( \text{Gr}^*(N, M) \). This question has been solved by Postnikov and his colleagues (see [42, 51]), and our classification theorem of the soliton solutions provides an alternative proof based on a simple asymptotic analysis as described in section 5 (see also [6, 8]).

### 4.3. The \( \tau \)-function as a point on \( \text{Gr}^*(N, M) \)

Expanding the \( \tau \)-function in the Wronskian determinant (3.10) by the Binet–Cauchy formula, we have

\[
\tau = \text{Wr}(f_1, f_2, \ldots, f_N) = \sum_{1 \leq i_1 < \cdots < i_N \leq M} \xi(i_1, i_2, \ldots, i_N) E(i_1, i_2, \ldots, i_N),
\]

where \( \xi(i_1, i_2, \ldots, i_N) \) are the Plücker coordinates given by the maximal minors of the \( A \)-matrix and \( E(i_1, i_2, \ldots, i_N) = \text{Wr}(E_{i_1}, E_{i_2}, \ldots, E_{i_N}) \). Here each \( E(i_1, i_2, \ldots, i_N) \) can be identified with \( E_{i_1} \wedge E_{i_2} \wedge \cdots E_{i_N} \) which is the basis element for \( \wedge^N \mathbb{R}^M \), i.e.

\[
\text{Span}_\mathbb{R} \{ E(i_1, i_2, \ldots, i_N) : 1 \leq i_1 < i_2 < \cdots < i_N \leq M \} \equiv \bigwedge^N \mathbb{R}^M.
\]

Note here that the sum \( k_{i_1} + k_{i_2} + \cdots + k_{i_N} \) should be distinct for distinct sets \( \{i_1, i_2, \ldots, i_N\} \) in order for the functions \( E(i_1, i_2, \ldots, i_N) \) to be linearly independent. It is then clear that the \( \tau \)-function given by the Wronskian determinant (3.10) can be identified with a point on \( \text{Gr}(N, M) \), and the Wronskian map \( \text{Wr} : [f_1, \ldots, f_N] \mapsto \tau \) gives the Plücker embedding. With the ordering \( k_1 < \cdots < k_M \), the Wronskian \( \text{Wr}(E_{i_1}, \ldots, E_{i_N}) > 0 \) for \( i_1 < \cdots < i_N \). Then \( \tau \in \text{Gr}^*(N, M) \) implies that the \( \tau \)-function is positive definite and the solution \( u(x, y, t) = 2\beta (\ln \tau) \) is regular for all \((x, y, t) \in \mathbb{R}^3\). In order to prove a converse of this statement, we first show the following: let \((t_1, t_2, \ldots, t_M)\) be the higher times for the
KP equation (see remark 3.2, and here the first three times \( t_1 = x, t_2 = y \) and \( t_3 = -t \) give the KP variables).

**Proposition 4.1.** Suppose that the \( \tau \)-function is regular for all \((t_1, t_2, \ldots, t_M)\). Then \( \tau \in Gr^+(N, M) \).

**Proof.** Let us first write the exponential terms

\[
E_j = \exp \left( \sum_{n=1}^{M} k_j^n t_n + \theta^0_j \right) =: \hat{E}_j e^{\theta^0_j} \quad \text{for} \quad j = 1, \ldots, M,
\]

with \( \theta^0_j \in \mathbb{R} \), i.e. the shifts of \( t_n \)'s in the exponential functions. Because the \( k \)-parameters are all distinct, one can take the coordinates \((\theta^0_1, \ldots, \theta^0_M)\) instead of \((t_1, \ldots, t_M)\). The \( \tau \)-function is then given by

\[
\tau = \sum_{1 \leq j_1 < \cdots < j_N \leq M} \xi(j_1, \ldots, j_N) \hat{E}(j_1, \ldots, j_N) \prod_{k=1}^{N} e^{\theta^0_k},
\]

where \( \hat{E}(j_1, \ldots, j_N) = \text{Wr}(\hat{E}_{j_1}, \ldots, \hat{E}_{j_N}) \). Then one can choose the parameters \((\theta^0_{j_1}, \ldots, \theta^0_{j_N})\) so that

\[
\sum_{k=1}^{N} \theta^0_k \gg \sum_{k=1}^{N} \theta^0_k,
\]

for any other choice of the parameters \((\theta^0_1, \ldots, \theta^0_M)\). This means that the exponential term having this index set \{\(j_1, \ldots, j_N\}\} is the dominant one in the \( \tau \)-function, while all other parameters are of \( O(1) \). Suppose the minor \( \xi(j_1, \ldots, j_N) \) associated with this index set is negative, that is, \( \tau \approx \xi(j_1, \ldots, j_N) E(j_1, \ldots, j_N) < 0 \). Now note that for \( x \ll 0 \), the dominant exponential in the \( \tau \)-function is \( E(e_1, \ldots, e_N) > 0 \) with the pivot set \{\(e_1, \ldots, e_N\)\} and the ordering \( k_1 < \cdots < k_M \), so that \( \tau \approx E(e_1, \ldots, e_N) > 0 \). This implies that the \( \tau \)-function vanishes at some point in the \((x, y)\)-plane, and therefore the solution \( u(x, y, t) \) is not regular. \( \square \)

It is then clear from the proof that the total non-negativity is not only sufficient but necessary for the regularity of the solution. Namely we have the following.

**Corollary 4.1.** The solution of the KP equation generated by the \( \tau \)-function in the form (3.10) with (3.11) is non-singular for any initial data if and only if \( \tau \in Gr^+(N, M) \).

Thus the classification of the regular soliton solutions is equivalent to a study of the TNN Grassmannian.

**Remark 4.5.** Since each \( \tau \)-function can be identified as a point on \( Gr(N, M) \), one can define a moment map, \( \mu : Gr(N, M) \rightarrow h^*_R \) [25],

\[
\mu(\tau) = \frac{\sum_{1 \leq j_1 < \cdots < j_N \leq M} \xi(j_1, \ldots, j_N) E(j_1, \ldots, j_N)^2 (L_{j_1} + \cdots + L_{j_N})}{\sum_{1 \leq j_1 < \cdots < j_N \leq M} \xi(j_1, \ldots, j_N) E(j_1, \ldots, j_N)^2},
\]

where \( L_j \) are the weights of the standard representation of \( SL(M) \) and \( h^*_R \) is the real part of the dual of the Cartan subalgebra of \( sl(M) \) defined by

\[
h^*_R = \text{Span}_R \left\{ L_1, \ldots, L_M : \sum_{j=1}^{M} L_j = 0 \right\} \cong \mathbb{R}^{M-1}.
\]

Then the closure of the image of the moment map is a convex polytope whose vertices are the fixed points of the \( S_M \) orbit, that is, the dominant exponentials.
5. Classification of soliton solutions

In this section, we now show the asymptotic behavior of the $\tau$-function in (4.4) and then present a classification scheme for the regular line-soliton solutions of KP based on the $\tau$-function asymptotics (see also [3]).

5.1. Asymptotic line-solitons

The $\tau$-function of (4.4) is given explicitly by the sum of exponential terms with the Wronskians:

$$\tau(x, y, t) = \sum_{1 \leq m_1 < \cdots < m_N \leq M} \xi(m_1, \ldots, m_N) E(m_1, \ldots, m_N),$$

(5.1)

where $E(m_1, \ldots, m_N) = \text{Wr}(E_{m_1}, \ldots, E_{m_N})$ with $E_m = e^{\theta_m}$, and $\theta_m(x, y, t) = k_m x + k_m^2 y - k_m^3 t$. Since $u = 2 \partial_x^2 \ln \tau$, the regularity condition on the line-soliton solutions requires that the $\tau$-function does not vanish for all values of $x, y$ and $t$. To ensure that it is sign-definite, the following necessary and sufficient conditions are imposed on the $\tau$-function in (5.1).

(i) The parameters $k_1, k_2, \ldots, k_M$ are ordered as $k_1 < k_2 < \cdots < k_M$, and the sums $k_i + k_j$ are all distinct.

(ii) $A$ is a TNN matrix, that is, all its maximal minors are $\xi(m_1, \ldots, m_N) \geq 0$.

The asymptotic spatial structure of the solution $u(x, y, t)$ is determined from the consideration of dominant exponentials $E(m_1, \ldots, m_N)$ in the $\tau$-function at different regions of the $(x, y)$-plane for large $|y|$. The solution $u = 2 \partial_x^2 \ln \tau$ is localized at the boundaries of two distinct regions where a balance exists between two dominant exponentials in the $\tau$-function (5.1), whereas the solution is exponentially small in the interior of each of these regions where only one exponential $E(m_1, \ldots, m_N)$ with a specific index set $\{m_1, \ldots, m_N\}$ is dominant. Before discussing a general theorem, let us first consider the following simple examples which illustrate the resonant interactions among the line-solitons. As discussed in the introduction, the resonant interaction is one of the most important features of the KP equation (see e.g. [18, 30, 34, 37]).

Example 5.1. We consider the case with $N = 1$ and $M = 3$, where the $1 \times 3$ coefficient matrix $A$ is given by

$$A = (1 \quad a \quad b).$$

The parameters $a$ and $b$ in the matrix are positive constants; meaning that this $A$-matrix marks a point on $\text{Gr}^+ (1, 3)$, and the positivity implies the regularity of the KP solution. The $\tau$-function is simply given by

$$\tau = f = E_1 + a E_2 + b E_3.$$

Now let us determine the dominant exponentials and analyze the structure of the solution in the $xy$-plane. First we consider the function $f$ along the line $x = -cy$ with $c = \tan \Psi$ where $\Psi$ is the angle measured counterclockwise from the $y$-axis (see figure 1). Then along $x = -cy$, we have the exponential function $E_j = \exp \left[ \eta_j(c) y - k_j^3 t \right]$ with

$$\eta_j(c) = k_j (k_j - c).$$

(5.2)

It is then seen from figure 2 that for $y \gg 0$ and a fixed $t$, the exponential term $E_1$ dominates when $c$ is large positive ($\Psi \approx \frac{\pi}{2}$, i.e. $x \to -\infty$). Decreasing the value of $c$ (rotating the line clockwise), the dominant term changes to $E_3$. Thus we have

$$w := \partial_x \ln f \to \begin{cases} k_1 & \text{as } x \to -\infty, \\ k_3 & \text{as } x \to \infty. \end{cases}$$
The transition of the dominant exponentials $E_1 \rightarrow E_3$ is characterized by the condition $\eta_1 = \eta_3$, which corresponds to the direction parameter value $c = \tan \Psi_{[1,3]} = k_1 + k_3$. In the neighborhood of this line, the function $f$ can be approximated as

$$f \approx E_1 + bE_3,$$

which implies that there exists a $[1, 3]$-soliton for $y \gg 0$. The constant $b$ can be used to choose a specific location of this soliton.

Next consider the case of $y \ll 0$. The dominant exponential corresponds to the least value of $\eta_j$ for any given value of $c$. For large positive $c$ ($\Psi \approx \frac{\pi}{2}$, i.e. $x \to \infty$), $E_3$ is the dominant term. Decreasing the value of $c$ (rotating the line $x = -cy$ clockwise), the dominant term changes to $E_2$ when $k_2 + k_3 > c > k_1 + k_2$, and $E_1$ becomes dominant for $c < k_1 + k_2$. Hence, we have for $y \ll 0$

$$w = \partial_x \ln f \rightarrow \begin{cases} k_1 & \text{as } x \to -\infty, \\ k_2 & \text{for } -(k_1 + k_2)y < x < -(k_2 + k_3)y, \\ k_3 & \text{as } x \to \infty. \end{cases}$$

In the neighborhood of the line $x + (k_1 + k_2)y =$constant,

$$f \approx E_1 + aE_2,$$

which corresponds to a $[1, 2]$-soliton and its location is fixed by the constant $a$. The solution in $y \ll 0$ also consists of a $[2, 3]$-soliton in the neighborhood of the line $x + (k_2 + k_3)y =$constant whose location is determined by the locations of other line-solitons. Therefore, we need only two parameters $a$ and $b$ (besides the $k$-parameters) to specify the solution uniquely, and those parameters fix the locations of the line $x + y \tan \Psi_{[i,j]} = C_{[i,j]}t = x^0_{[i,j]}$ for the $[i, j]$-soliton. For $[i, j] = [1, 3]$ and $[2, 3]$ in $x \gg 0$, we have

$$x_{[1,3]} = -\frac{1}{k_3 - k_1} \ln b, \quad x_{[2,3]} = -\frac{1}{k_3 - k_2} \ln \frac{b}{a}.$$
This feature is common even for the general case. which are trivially satisfied with \( K_{[i,j]} = (k_j - k_i, k_j^2 - k_i^2) \) and \( \Omega_{[i,j]} = k_j^3 - k_i^3 \).

The resonant condition may be symbolically written as
\[
[1, 3] = [1, 2] + [2, 3].
\]

One can also represent this line-soliton solution by a permutation of three indices: \([1, 2, 3]\) which is illustrated by a (linear) chord diagram shown below. Here, the upper chord represents the \([1, 3]\)-soliton in \( y \gg 0 \) and the lower two chords represent the \([1, 2]\)- and \([2, 3]\)-solitons in \( y \ll 0 \). Following the arrows in the chord diagram, one recovers the permutation,
\[
\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{or simply} \quad \pi = (312).
\]

From now on, we use one-line notation for permutations, i.e. \( \pi = (\pi(1), \pi(2), \ldots, \pi(M)) \). In general, each line-soliton solution of the KP equation can be parametrized by a unique permutation corresponding to a chord diagram (see the next subsection).

The results described in this example can easily be extended to the general case where \( f \) has an arbitrary number of exponential terms (see also [4, 27]).

**Proposition 5.1.** If \( f = a_1 E_1 + a_2 E_2 + \cdots + a_M E_M \) with \( a_j > 0 \) for \( j = 1, 2, \ldots, M \), then the solution \( u \) consists of \( M - 1 \) line-solitons for \( y \ll 0 \) and one line-soliton for \( y \gg 0 \).

Such solutions are referred to as the \((M - 1, 1)\)-soliton solutions, meaning that \((M - 1)\) line-solitons for \( y \ll 0 \) and one line-soliton for \( y \gg 0 \). Note that the line-soliton for \( y \gg 0 \) is labeled by \([1, M]\), whereas the other line-solitons in \( y \ll 0 \) are labeled by \([k, k + 1]\) for \( k = 1, 2, \ldots, M - 1 \), counterclockwise from the negative to the positive \( x \)-axis, i.e. increasing \( \Psi \) from \(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\). As in the previous examples one can set \( a_1 = 1 \) without any loss of generality, then the remaining \( M - 1 \) parameters \( a_2, \ldots, a_M \) determine the locations of the \( M \) line-solitons. Also note that the \( xy \)-plane is divided into \( M \) sectors for the asymptotic region with \( x^2 + y^2 \gg 0 \), and the boundaries of those sectors are given by the asymptotic line-solitons. This feature is common even for the general case.

![Figure 3. Example of the (2, 1)-soliton solution and the chord diagram. The k-parameters are chosen as (k_1, k_2, k_3) = (−\frac{\pi}{2}, −\frac{\pi}{4}, \frac{\pi}{2}). The right panel shows the corresponding chord diagrams for the permutation which is represented in one-line notation as \( \pi = (\pi(1), \pi(2), \pi(3)) \). We take the A-matrix \( A = (1 \quad 1 \quad 1) \) so that at \( t = 0 \) three line-solitons meet at the origin. Each \( E(j) \) with \( j = 1, 2 \) or 3 indicates the dominant exponential term \( E_j \) in that region. The boundaries of any two adjacent regions give the line-solitons indicating the transition of the dominant terms \( E_j \). The k-parameters are the same as those in figure 2, and the line-solitons are determined from the intersection points of the \( \eta_j \)'s in figure 2. Here \( a = b = 1 \) (i.e. \( \tau = E_1 + E_2 + E_3 \)) so that the three solitons meet at the origin at \( t = 0 \).](image-url)
Figure 4 illustrates the case for a \((3, 1)\)-soliton solution with \(f = E_1 + E_2 + E_3 + E_4\). The chord diagram for this solution represents the permutation \(\pi = (4123) = s_1s_2s_3 \in S_{4(1)}^3\).

**Example 5.2.** Let us now consider the case with \(N = 2\) and \(M = 3\): we take the A-matrix in (3.11) of the form

\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & a & b \end{pmatrix},
\]

where \(a\) and \(b\) are positive constants, that is, \(A\) marks a point on \(\text{Gr}^+(2, 3)\). Then the \(\tau\)-function is given by

\[
\tau = E(1, 2) + a E(1, 3) + b E(2, 3).
\]

In order to carry out the asymptotic analysis in this case one needs to consider the sum of two \(\eta_j(c)\), i.e. \(\eta_{ij} = \eta_i + \eta_j\) for \(1 \leq i < j \leq 3\). This can still be done using figure 2, but a more effective way is described below (see the graph of \(\eta(k, c)\) in figure 6).

For \(y \gg 0\), the transitions of the dominant exponentials are given by following scheme:

\[
E(1, 2) \rightarrow E(1, 3) \rightarrow E(2, 3),
\]

as \(c\) varies from large positive (i.e. \(x \rightarrow -\infty\)) to large negative values (i.e. \(x \rightarrow \infty\)). The boundary between the regions with the dominant exponentials \(E(1, 2)\) and \(E(1, 3)\) defines the \([2, 3]\)-soliton solution since here the \(\tau\)-function can be approximated as

\[
\tau \approx E(1, 2) + a E(1, 3) = 2(k_2 - k_1) e^{\frac{1}{2}(\theta_1 + \theta_2 - \theta_3)} \cosh \frac{1}{2}(\theta_2 - \theta_3 + \theta_{23}),
\]

so that we have

\[
u = 2\theta_{23}^2 \ln \tau \approx \frac{1}{2}(k_2 - k_3)^2 \text{sech}^2 \frac{1}{2}(\theta_2 - \theta_3 + \theta_{23}),
\]

where \(\theta_{23}\) is related to the parameter of the A-matrix (see below). A similar computation as above near the transition boundary of the dominant exponentials \(E(1, 3)\) and \(E(2, 3)\) yields

\[
\tau \approx 2(k_3 - k_1) a e^{\frac{1}{2}(\theta_1 + \theta_2 - \theta_{12})} \cosh \frac{1}{2}(\theta_1 - \theta_2 + \theta_{12}).
\]
The phases $\theta_{12}$ and $\theta_{23}$ are related to the parameters of the $A$-matrix:

$$a = \frac{k_2 - k_1}{k_3 - k_1} e^{-\theta_{23}}, \quad b = \frac{k_2 - k_1}{k_3 - k_2} e^{-\theta_{12}}.$$

For $y \ll 0$, there is only one transition, namely

$$E(2, 3) \rightarrow E(1, 2),$$

as $c$ varies from a large positive value (i.e. $x \rightarrow \infty$) to a large negative value (i.e. $x \rightarrow -\infty$).

In this case, a $[1, 3]$-soliton is formed for $y \ll 0$ at the boundary of the dominant exponentials $E(2, 3)$ and $E(1, 2)$. The contour plot of the line-soliton solution is shown in figure 5. Note that this figure can be obtained from figure 3 by changing $(x, y) \rightarrow (-x, -y)$. This solution can be represented by the chord diagram corresponding to the permutation $\pi = (231)$ shown below. Note that this diagram is the $\pi$-rotation of the chord diagram in example 5.1 whose permutation $\pi = (312)$ is the inverse of $\pi = (231)$.

As shown in those examples, it is now clear that each line-soliton appears as a boundary of two dominant exponentials, and with the condition that $k_i + k_j$ are all distinct for $i \neq j$, we have the following proposition.

**Proposition 5.2.** Two dominant exponentials of the $\tau$-function in adjacent regions of the $xy$-plane are of the forms $E(i, m_2, \ldots, m_N)$ and $E(j, m_2, \ldots, m_N)$ for some $N - 1$ common indices $m_2, \ldots, m_N$.

As a consequence of proposition 5.2, the KP solution behaves asymptotically like a single line-soliton

$$u(x, y, t) \approx \frac{1}{2} (k_j - k_i)^2 \text{sech}^2 \frac{1}{2} (\theta_j - \theta_i + \theta_{ij}),$$

in the neighborhood of the line $x + (k_i + k_j)y = \text{constant}$, which forms the boundary between the regions of dominant exponentials $E(i, m_2, \ldots, m_N)$ and $E(j, m_2, \ldots, m_N)$. Equation (5.3) defines an asymptotic line-soliton, i.e. $[i, j]$-soliton, as a result of those two dominant exponentials. In order to identify the set of asymptotic line-solitons associated with a given solution, we need to determine which exponential terms $E(m_1, m_2, \ldots, m_N)$ are actually dominant along each line $[i, j] : x = -(k_i + k_j)y$ as $|y| \rightarrow \infty$. For this purpose, first note that along a line $x = -cy$ each exponential term $E(m_1, m_2, \ldots, m_N)$ has the form

$$E(m_1, m_2, \ldots, m_N) \propto \exp \left( \sum_{n=1}^{N} \eta_{m_n}(c)y \right), \quad \eta_{m_n}(c) = k_m(k_m - c).$$
Thus for $y \gg 0$ (or $\ll 0$), the dominant exponential corresponds to the largest (or least) value of the sum of $\eta_m(c)$ for each $c$. When two dominant exponentials $E(i, m_2, \ldots, m_N)$ and $E(j, m_2, \ldots, m_N)$ are in balance along the direction of the $[i, j]$-soliton, we have $\eta_i(c) = \eta_j(c)$ which implies that $c = k_i + k_j$. Since $\eta_m(c) - \eta_i(c) = (k_m - k_i)(k_m + k_i - c)$ and the $k$-parameters are ordered as $k_1 < k_2 < \cdots < k_M$, we have the following order relations among the other $\eta_m(c)$'s along $c = k_i + k_j$:

$$
\begin{align*}
\eta_i &= \eta_j < \eta_m & \text{if } m < i \text{ or } j < m, \\
\eta_i &= \eta_j > \eta_m & \text{if } i < m < j.
\end{align*}
$$

The relations among the phases $\eta_j(c)$ can be seen easily from the plots of $\eta_j(c)$ versus $c$ as well as $\eta(k, c) = k(k - c)$ versus $k$ for a fixed value of $c$ illustrated by figure 6. Proposition 5.2 and relations (5.4) are particularly useful in order to find the asymptotic line-solitons from a given KP $\tau$-function as demonstrated by the example below.

**Example 5.3.** Let us consider the $2 \times 4$ matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & -a \\ 0 & 1 & b & c \end{pmatrix},$$

where $a$, $b$, and $c$ are the positive real numbers. In this case, there are six maximal minors, five of which are positive, namely

$$\xi(1, 2) = 1, \quad \xi(1, 3) = b, \quad \xi(1, 4) = c, \quad \xi(2, 4) = a, \quad \xi(3, 4) = ab$$

and $\xi(2, 3) = 0$. Then from (5.1) the $\tau$-function has the form

$$\tau = (k_2 - k_1)E(1, 2) + b(k_3 - k_1)E(1, 3) + c(k_4 - k_1)E(1, 4) + a(k_2 - k_4)E(2, 4) + ab(k_4 - k_3)E(3, 4).$$

Proposition 5.2 implies that the line-solitons are localized along the lines $x + cy = \text{constant}$ with $c = k_i + k_j = \tan \Psi_{i,j}$. Hence, we look for dominant exponential terms in the $\tau$-function along those directions. For $y \gg 0$, the $c$-values decrease as we sweep clockwise from negative to positive $x$-axis starting with the largest value $c = k_3 + k_4$. We have $\eta_1, \eta_2 > \eta_3 = \eta_4$ from the order relations (5.4) for $c = k_3 + k_4$. This means that $\eta_1 + \eta_2$ is the dominant phase combination along this direction. Since $\xi(1, 2) \neq 0$, $\tau(x, y, t) \approx (k_2 - k_1)E(1, 2)$ implying

![Figure 6](image-url)
that \( u \approx 0 \) along the line \([3, 4]\), so there is no \([3, 4]\) line-soliton. By similar reasoning one can verify that the \([1, 4]\)- and \([1, 2]\)-solitons are also impossible. Let us consider the direction \( c = k_2 + k_4 \) to check for the \([2, 4]\)-soliton. From (5.4) (see also figure 6), \( \eta_3 < \eta_2 = \eta_4 < \eta_1 \), and since both \( \xi(1, 2) \) and \( \xi(1, 4) = a \) are non-zero, the \( \tau \)-function in (5.1) corresponds to a dominant balance of exponentials: \( \tau \approx (k_2 - k_1)E(1, 2) + c(k_4 - k_1)E(1, 4) \) along the line \([2, 4]\). Therefore, \([2, 4]\) corresponds to an asymptotic line-soliton as \( y \to \infty \). The \([1, 3]\)-soliton also exists by a similar argument. Thus, we have two asymptotic line-solitons of \([1, 3]\)- and \([2, 4]\)-types for \( y \gg 0 \).

We next look for the asymptotic solitons for \( y \ll 0 \) by sweeping from the negative \( x \)-axis. Recall that in this case the dominant exponential \( E(i, j) \) corresponds to the least value of the sum \( \eta_i(c) + \eta_j(c) \). It is easy to see that the \([1, 2]\)- and \([3, 4]\)-solitons are impossible since \( E(3, 4) \) and \( E(1, 2) \) are, respectively, the only dominant exponentials along those directions. Then consider the \([1, 3]\)-soliton. Along \( c = k_1 + k_3 \), (5.4) implies that \( \eta_2 < \eta_1 = \eta_3 < \eta_4 \), and so the exponentials \( E(1, 2) \) and \( E(2, 3) \) would give the dominant balance. But \( E(2, 3) \) is not present in the above \( \tau \)-function because \( \xi(2, 3) = 0 \). So we conclude that the \([1, 3]\)-soliton does not exist as \( y \ll 0 \), and for similar reasons, the \([2, 4]\)-soliton is also impossible. Next, checking for the \([1, 4]\)-soliton, we have \( \eta_2, \eta_3 < \eta_1 = \eta_4 \) from (5.4). But as seen earlier, the dominant exponential \( E(2, 3) \) is not present in the \( \tau \)-function. However, there does exist a balance between the next dominant exponential pairs \( \{E(1, 3), E(3, 4)\} \) or \( \{E(1, 2), E(2, 4)\} \) depending on whether \( \eta_2 > \eta_3 \) or \( \eta_2 < \eta_3 \). In either case, there exists an asymptotic line-soliton along \([1, 4]\). A similar argument applies along the line \([2, 3]\) which corresponds to the other asymptotic line-soliton as \( y \ll 0 \).

In summary, the \( \tau \)-function corresponding to the \( A \)-matrix given above generates a KP solution with the asymptotic line-solitons \([1, 3]\) and \([2, 4]\) as \( y \gg 0 \), and asymptotic line-solitons \([1, 4]\) and \([2, 3]\) as \( y \ll 0 \). This line-soliton solution with the parameters \( a = b = c = 1 \) in the \( A \)-matrix is shown in figure 7.

We note that the line-solitons associated with the resonant \((1, 2)\)- and \((2, 1)\)-soliton solutions can be determined in the same way as in the above example by applying for the dominant balance conditions given by propositions 5.2 and (5.4). We now proceed to discuss a
more general characterization of all line-soliton solutions of the KP equation whose $\tau$-functions are given in the Wronskian form (3.10).

5.2. Characterization of the line-solitons

It should be clear from the above examples that a dominant exponential term determined by relations (5.4) is actually present in the given $\tau$-function if its coefficient term given by a maximal minor of the $A$-matrix is non-zero. Thus, in order to obtain a complete characterization of the asymptotic line-solitons, it is necessary to consider the structure of the $N \times M$ coefficient $A$-matrix in some detail. We consider the matrix $A$ to be in RREF, and we will also assume that $A$ is \textit{irreducible} as defined below.

\textbf{Definition 5.4.} An $N \times M$ matrix $A$ is irreducible if each column of $A$ contains at least one non-zero element, or each row contains at least one non-zero element other than the pivot once $A$ is in RREF.

If an $N \times M$ matrix $A$ is not irreducible, then the corresponding $\tau$-function gives the same KP solution $u$ which is obtained from another $\tau$-function associated with a smaller size matrix $\tilde{A}$ derived from $A$. One can note from the determinant expansion in (5.1) that

(a) if the $m$th column of $A$ has only zero elements, then $\xi(m_1, \ldots, m_N) = 0$ if $m_k = m$ for some $k$, that is, the exponential $E_m$ will never appear in the $\tau$-function; in terms of the chord diagram, this corresponds to a loop in the lower part of the diagram ($m$ is a non-pivot index);

(b) if the $n$th row of $A$ has the pivot as the only non-zero element, then all $\xi(m_1, \ldots, m_N) \neq 0$ contain the index $n$, that is, the exponential $E_n$ can be factored out from the $\tau$-function; in terms of the chord diagram, this corresponds to a loop in the upper part of the diagram.

So the irreducibility implies that we consider only derangements (i.e. no fixed points) of the permutation.

We now present a classification scheme of the line-soliton solutions by identifying the asymptotic line-solitons as $y \to \pm \infty$. We denote a line-soliton solution by the $(N, N)^-$-soliton whose asymptotic form consists of $N$ line-solitons as $y \to -\infty$ and $N$ line-solitons as $y \to \infty$ in the $xy$-plane as shown in figure 8. The next proposition provides a general result characterizing the asymptotic line-solitons of the $(N, N)^-$-soliton solutions (the proof can be found in [8]).

\textbf{Proposition 5.3.} Let $\{e_1, e_2, \ldots, e_N\}$ and $\{g_1, g_2, \ldots, g_{M-N}\}$ denote, respectively, the pivot and non-pivot indices associated with an irreducible, $N \times M$, TNN $A$-matrix. Then the soliton solution obtained from the $\tau$-function in (5.1) with this $A$-matrix has the following structure.

(a) For $y \gg 0$, there are $N$ asymptotic line-solitons of $[j_n]$-type for some $j_n$.

(b) For $y \ll 0$, there are $(M - N)$ asymptotic line-solitons of $[i_m]$-type for some $i_m$.

An important consequence of proposition 5.3 is that it defines the \textit{pairing} map $\pi : [M] \to [M]$ on the integer set $[M] := \{1, 2, \ldots, M\}$ according to

$$
\begin{align*}
\pi(e_n) &= j_n, & n &= 1, 2, \ldots, N, \\
\pi(g_m) &= i_m, & m &= 1, 2, \ldots, M - N.
\end{align*}
$$

Recall that $\{e_n\}_{n=1}^N$ and $\{g_m\}_{m=1}^{M-N}$ are, respectively, the pivot and non-pivot indices of the $A$-matrix and form a disjoint partition of $[M]$. Then the unique index pairings in proposition 5.3 imply that the map $\pi$ is a \textit{permutation} of $M$ indices. More precisely,
\( \pi \in S_M \) where \( S_M \) is the group of permutations of the index set \([M]\). Furthermore, since \( \pi(e_n) = j_n > e_n, n = 1, \ldots, N \), and \( \pi(g_m) = i_m < g_m, m = 1, \ldots, M - N, \pi \) defined by (5.5) is a permutation with no fixed point, i.e. derangements. Yet another feature of \( \pi \) is that it has exactly \( N \) excedances defined as follows: an element \( l \in [M] \) is an excedance of \( \pi \) if \( \pi(l) > l \). The excedance set of \( \pi \) in (5.5) is the set of pivot indices \( \{e_1, e_2, \ldots, e_N\} \).

The above results can be summarized to deduce the following characterization for the line-soliton solution of the KP equation [8].

**Theorem 5.5.** Let \( A \) be an \( N \times M \), TNN, irreducible matrix which corresponds to a point in the non-negative Grassmannian \( \text{Gr}^+ (N, M) \subset \text{Gr}(N, M) \). Then the \( \tau \)-function (5.1) associated with this \( A \)-matrix generates an \((M - N, N)\)-soliton solution. The \( M \) asymptotic line-solitons associated with each of these solutions can be identified via a pairing map \( \pi \) defined by (5.5). The map \( \pi \in S_M \) is a derangement of the index set \([M]\) with \( N \) excedances given by the pivot indices \( \{e_1, e_2, \ldots, e_N\} \) of the \( A \)-matrix in RREF.

As explained in section 4, the derangements \( \pi \in S_M \) are represented by the chord diagrams with the arrows above the line point from \( e_n \) to \( j_n \) for \( n = 1, 2, \ldots, N \), while the arrows below the line point from \( g_m \) to \( i_m \) for \( m = 1, 2, \ldots, M - N \). Figure 9 illustrates the time evolution of an example of the \((3, 3)\)-soliton solution. The chord diagram shows all asymptotic line-solitons for \( y \to \pm \infty \).

Theorem 5.5 provides a unique parametrization of each TNN Grassmannian cell in terms of the derangement of \( S_M \). This agrees with the result obtained by Postnikov et al in [42, 51]. One should, however, note that theorem 5.5 does not give the indices \( j_n \) and \( i_m \) in the \([e_n, j_n]\) and \([i_m, g_m]\) line-solitons. The specific conditions for finding those asymptotic line-solitons are given by identifying the dominant exponential in each domain in the \( xy \)-plane. The example below illustrates how to apply theorem 5.5, and identify all the asymptotic line-solitons for a given irreducible TNN \( A \)-matrix.

**Example 5.6.** Let us consider the \( 3 \times 5 \) matrix

\[
A = \begin{pmatrix}
1 & 0 & -a & 0 & b \\
0 & 1 & c & 0 & -d \\
0 & 0 & 0 & 1 & e
\end{pmatrix}
\]

with \( ad - bc = 0 \).
where $a, b, c, d$ and $e$ are the positive constants, that is, the $A$-matrix marks a point on $\text{Gr}^*(3, 5)$. Then the purpose is to find asymptotic line-solitons generated by the $\tau$-function (3.10) associated with this $A$-matrix. From proposition 5.3, one can see that the $\tau$-function with this matrix will produce a $(3, 2)$-soliton solution since $N = 3$ and $M = 5$. Moreover, the asymptotic line-solitons for this solution are labeled by $[1, j_1], [2, j_2]$ and $[4, j_3]$ for $y \gg 0$ for some $j_1 > 1, j_2 > 2$ and $j_3 > 4$. Similarly, the line-solitons for $y \ll 0$ are labeled by $[i_1, 3]$ and $[j_2, 5]$ for some $i_1 < 3$ and $i_2 < 5$. The basic idea to determine those indices $j_1, j_2, j_3$ and $i_1, i_2$ is to apply proposition 5.3 and the dominant relations (5.4).

Let us first consider the case for $y \gg 0$. Starting with the last pivot $e_3 = 4$, it is immediate to find $j_3 = 5$, because of $j_3 > 4$ (just proposition 5.3). We now take the next pivot $e_2 = 2$ and find the index $j_2$. Since index 5 is already taken as the pair index of $e_3 = 4$, we need to check only the cases $[2, 4]$ and $[2, 3]$. For the existence of the $[2, 4]$-soliton, the dominant relation (5.4) requires that both $\xi(1, 2, 5)$ and $\xi(1, 4, 5)$ are not zero. Calculating those minors for our $A$-matrix, we have

$$\xi(1, 2, 5) = e \neq 0, \quad \xi(1, 4, 5) = d \neq 0,$$

and hence the $[2, 4]$-soliton exists. Now we consider the case with $e_1 = 1$, that is, we only have the $[1, 2]$ and $[1, 3]$ possibilities. In the case of $[1, 3]$, we use again the dominant relation (5.4), and check the minors $\xi(1, 4, 5)$ and $\xi(3, 4, 5)$ which correspond to the dominant exponentials. We then find $\xi(1, 4, 5) = d \neq 0$ but $\xi(3, 4, 5) = bc - ad = 0$. This implies that the $[1, 3]$-soliton is impossible for $y \gg 0$. So the last one is of $[1, 2]$-type, which can be confirmed by the conditions $\xi(1, 4, 5) = d \neq 0$ and $\xi(2, 4, 5) = b \neq 0$.

Now we consider the case for $y \ll 0$. Theorem 5.5 states that for the non-pivot index $g_1 = 3$, only the pair $[1, 3]$ is possible (index 2 is already taken because the $[1, 2]$-soliton exists, i.e. $\pi(1) = 2$). Then the final soliton must be of $[3, 5]$-type from the non-pivot index.
$g_2 = 5$. The last one can be confirmed by the least condition in (5.4) with $\xi(2, 3, 4) = a \neq 0$ and $\xi(2, 4, 5) = b \neq 0$.

Thus we have a $(2, 3)$-soliton solution of $\pi = (24153)$-type for the $\tau$-function (3.10) with the $A$-matrix considered. The photos in figure 10 show some interacting shallow water waves, which we think are a realization of this example. We demonstrate an exact solution whose parameters are given by $(k_1, k_2, \ldots, k_5) = (-2, -1, 0, 0.5, 2)$ and the $A$-matrix with $(a, b, c, d, e) = (1, 2, 1, 2, 1)$.

6. $(2, 2)$-soliton solutions

Here we give a summary of all soliton solutions of the KP equation generated by the $2 \times 4$ irreducible, TNN $A$-matrices. Proposition 5.3 implies that each of the soliton solutions consists of two asymptotic line-solitons as $y \to \pm \infty$. That is, they are $(2, 2)$-soliton solutions. We outline below the classification scheme for the $(2, 2)$-soliton solutions, and discuss some of the exact solutions in detail for the applications discussed in the following sections. First note that for $2 \times 4$ matrices, there are only two types given by

$$
\begin{pmatrix}
1 & 0 & -c & -d \\
0 & 1 & a & b
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & a & 0 & -c \\
0 & 0 & 1 & b
\end{pmatrix}.
$$

The fact that $A$ is TNN implies that the constants $a, b, c$ and $d$ must be non-negative. For the first type, one can easily see that $ad = 0$ is impossible because then either $\xi(3, 4) < 0$ or $A$ is not irreducible. Then there are five possible cases with $ad \neq 0$, namely

$$
\begin{align*}
(1) \quad & ad - bc > 0, \\
(2) \quad & ad - bc = 0, \\
(3) \quad & b = 0, c \neq 0, \\
(4) \quad & c = 0, b \neq 0, \\
(5) \quad & b = c = 0.
\end{align*}
$$

Figure 10. Example of shallow water waves. The upper photos are taken at a beach in Mexico. The lower figures show the evolution of the corresponding exact $(3, 2)$-soliton solution of $(24153)$-type shown in the chord diagram (see example 5.6): the left one at $t = 1.5$ and the right one at $t = 10$. (Photographs by courtesy of Mark J Ablowitz.)
Figure 11. The chord diagrams for seven different types of (2, 2)-soliton solutions. Each diagram corresponds to a TNN Grassmannian cell in $\text{Gr}(2, 4)$.

For the second type, $ab \neq 0$ due to irreducibility. Hence, we have only two cases:

(6) $c \neq 0,$  \hspace{1cm} (7) $c = 0.$

Thus we have total seven different types of $A$-matrices, and using theorem 5.5, we can show that each $A$-matrix gives a different (2, 2)-soliton solution which can be enumerated according to the seven derangements of the index set $\{1, 2, 3, 4\}$ with two excedances. Namely, for those cases from (1) to (7) we have

(1) $\pi = (3412), \hspace{1cm} (2) \pi = (2413), \hspace{1cm} (3) \pi = (4312), \hspace{1cm} (4) \pi = (3421), \hspace{1cm} (5) \pi = (4321), \hspace{1cm} (6) \pi = (3142), \hspace{1cm} (7) \pi = (2143).$

In figure 11, we show the chord diagrams for all those seven cases. One should note that any derangement of $S_4$ with exactly two excedances should be one of the graphs. This uniqueness in the general case has been used to count the number of TNN Grassmann cells [42, 51].

Let us now summarize the results for all those seven cases of the (2, 2)-soliton solutions.

(1) $\pi = (3412)$: this case corresponds to the $T$-type two-soliton solution which was first obtained as the solution of the Toda lattice hierarchy [4]. This is why we call it 'T-type' (see also [23]). The asymptotic line-solitons are of $[1, 3]$- and $[2, 4]$-types for $|y| \to \infty$. The $A$-matrix is given by

$$A = \begin{pmatrix} \frac{1}{\xi(3,4)} & 0 & -c & -d \\ 0 & 1 & a & b \end{pmatrix},$$

where $a, b, c, d > 0$ are the free parameters with $ad - bc > 0$. This is the generic solution on the maximum dimensional cell of $\text{Gr}^{+}(2, 4)$, and the corresponding line-soliton has the most complicated pattern due to the fully resonant interactions among all line-solitons.

(2) $\pi = (2413)$: the asymptotic line-solitons are given by the $[1, 2]$- and $[2, 4]$-solitons for $y \gg 0$, and the $[1, 3]$- and $[3, 4]$-solitons for $y \ll 0$. The $A$-matrix is given by

$$A = \begin{pmatrix} 1 & 0 & -c & -d \\ 0 & 1 & a & b \end{pmatrix},$$

where $a, c, d > 0$ with $\xi(3, 4) = ad - bc = 0$. Note the change of the solution structure by imposing just one constraint $\xi(3, 4) = 0$ to the previous case (1).

(3) $\pi = (4312)$: the asymptotic line-solitons for this case are the $[1, 4]$- and $[2, 3]$-solitons for $y \gg 0$, and the $[1, 3]$- and $[2, 4]$-solitons for $y \ll 0$. The $A$-matrix is given by

$$A = \begin{pmatrix} 1 & 0 & -b & -c \\ 0 & 1 & a & 0 \end{pmatrix},$$

where $a, b, c > 0$ are the free parameters. Note that two line-solitons for $y \ll 0$ are the same as in the T-type solution (see the crossing in the lower chords in figure 11).
\( \pi = (3421) \): the asymptotic line-solitons are given by \([1, 3]\) and \([2, 4]\) for \( y \gg 0 \), and for \( y \ll 0 \), these are the \([1, 4]\)- and \([2, 3]\)-solitons. The \( A \)-matrix is given by

\[
A = \begin{pmatrix}
1 & 0 & 0 & -c \\
0 & 1 & a & b
\end{pmatrix},
\]

where \( a, b, c > 0 \) are the positive free parameters. This solution can be considered as a dual of the previous case (3), that is, two sets of line-solitons for \( y \gg 0 \) and \( y \ll 0 \) are exchanged (also note the duality in the chord diagrams in figure 11). The example discussed after proposition 5.3 corresponds to this solution (see figure 7).

\( \pi = (4321) \): the solution in this case is called the \textit{P-type} two-soliton solution which has asymptotic line-solitons of \([1, 4]\)- and \([2, 3]\)-types as \(|y| \to \infty\). This type of solutions fits better with the \textit{physical} assumption of quasi-two-dimensionality with weak \( y \)-dependence underlying the derivation of the KP equation. This is why we call it ‘P-type’ (see [23]). The \( A \)-matrix is given by

\[
A = \begin{pmatrix}
1 & 0 & 0 & -b \\
0 & 1 & a & 0
\end{pmatrix}.
\]

The chord diagram indicates that those two line-solitons must have different amplitudes, i.e. \( A[1, 4] > A[2, 3] \), but they can propagate in the same direction, which correspond to the two-soliton solution of the KdV equation.

\( \pi = (3142) \): the asymptotic line-solitons are given by the \([1, 3]\)- and \([3, 4]\)-solitons for \( y \gg 0 \), and the \([1, 2]\)- and \([2, 4]\)-solitons for \( y \ll 0 \). The \( A \)-matrix is given by

\[
A = \begin{pmatrix}
1 & a & 0 & -c \\
0 & 0 & 1 & b
\end{pmatrix},
\]

where \( a, b, c > 0 \). This solution is dual to case (2) in the sense that the two sets of asymptotic line-solitons for \( y \gg 0 \) and \( y \ll 0 \) are switched, as well as the missing minors are switched by \( \xi(3, 4) \leftrightarrow \xi(1, 2) \). Also note the duality between the corresponding chord diagrams.

\( \pi = (2143) \): this case is called the \textit{O-type} two-soliton solution. The asymptotic line-solitons are of \([1, 2]\)- and \([3, 4]\)-types as \(|y| \to \infty\). The letter ‘O’ for this type is due to the fact that this solution was \textit{originally} found to describe the two-soliton solution of the KP equation (see for example [14, 44]). The \( A \)-matrix for the O-type two-soliton solution is given by

\[
A = \begin{pmatrix}
1 & a & 0 & 0 \\
0 & 0 & 1 & b
\end{pmatrix}.
\]

Note that this \( A \)-matrix is obtained as a limit \( c \to 0 \) in the previous one of case (6), i.e. the \((3142)\)-soliton solution.

Now let us describe the details of some of the \((2, 2)\)-soliton solutions, which will be important for an application of those solutions to the shallow water problem discussed in the next section. In particular, we explain how the \( A \)-matrix uniquely determines the structure of the corresponding soliton solution such as the location of the solitons and their phase shifts.

\textbf{6.1. O-type soliton solutions}

This is the original two-soliton solution, and the solutions correspond to the chord diagram of \( \pi = (2143) \). A solution of this type consists of two full line-solitons of \([1, 2]\) and \([3, 4]\) (see figure 12). Note here that they have phase shifts due to their collision. Let us describe
Figure 12. The time evolution of an O-type soliton solution. Each $E(i, j)$ indicates the dominant exponential in that region. The parameters are chosen as $A_{[1, 2]} = A_{[3, 4]} = 0.1$ and $\Psi_{[1, 2]} = -\Psi_{[1, 2]} = 30^\circ$.

explicitly the structure of the solution of this type: the $\tau$-function defined in (5.1) for this case is given by

$$\tau = E(1, 3) + bE(1, 4) + aE(2, 3) + abE(2, 4),$$

where $a, b > 0$ are the free parameters given in the $A$-matrix listed above. As we will show that those two parameters can be used to fix the locations of those solitons, that is, they are determined by the asymptotic data of the solution for large $|y|$.

For the latter application of the solution, we assume that the $[1, 2]$-soliton has a ‘negative’ $y$-component in the wave vector (i.e. $\tan \Psi_{[1, 2]} < 0$), and the $[3, 4]$-soliton has a ‘positive’ $y$-component, (i.e. $\tan \Psi_{[3, 4]} > 0$, see figure 1). Then for the region with large positive $x$, we have the $[1, 2]$-soliton in $y > 0$ and the $[3, 4]$-soliton in $y < 0$.

For the $[1, 2]$-soliton in $x > 0$ (and $y \gg 0$), we have the dominant balance between $E(1, 4)$ and $E(2, 4)$. Then the $\tau$-function can be written in the following form;

$$\tau \approx bE(1, 4) + abE(2, 4)$$

which leads to the $[1, 2]$-soliton solution in the region near $\theta_1 \approx \theta_2$ for $x \gg 0$,

$$u \approx 2\theta_1 \ln \tau \approx \frac{1}{2}(k_2 - k_1)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_1 - \theta_2 + \theta_{12}^+).$$

Here the shift $\theta_{12}^+$ (+ indicates $x > 0$) is related to the parameter $a$ in the $A$-matrix (see below).

For the $[3, 4]$-soliton in $x > 0$ (and $y \ll 0$), from the balance $\tau \approx aE(2, 3) + abE(2, 4)$, we have

$$u \approx \frac{1}{2}(k_4 - k_3)^2 \operatorname{sech}^2 \frac{1}{2}(\theta_3 - \theta_4 + \theta_{34}^+).$$

The shifts $\theta_{12}^+$ and $\theta_{34}^+$ are related to the parameters in the $A$-matrix:

$$a = \frac{k_4 - k_1}{k_4 - k_2} e^{-\theta_{12}^+}, \quad b = \frac{k_3 - k_2}{k_4 - k_2} e^{-\theta_{34}^+}. \quad (6.1)$$

Thus the parameters in the $A$-matrix can be determined by the asymptotic data of the locations of those $[1, 2]$- and $[3, 4]$-solitons for $x \gg 0$ and $|y| \gg 0$.

The most important feature of the O-type solution is the phase shift due to the interaction of those two oblique line-solitons. The phase shift for the $[i, j]$-soliton is defined by $\theta_{ij} = \theta_{ij} - \theta_{ij}^+$. 
The positive phase shifts \( \Delta_1x \) (see for example [8, 12, 47]). Since \( 0 \leq \Delta_1x \leq \pi \), one can find the formula of the maximum amplitude which occurs at the center of the interaction point. Figure 13 illustrates an O-type interaction of two solitons which have the same amplitude, \( A_{1,2} = A_{3,4} = \frac{1}{2} \), and are symmetric with respect to the \( y \)-axis, \( \Psi_{3,4} = -\Psi_{1,2} \approx 45^\circ \). Since the solution is close to the resonance, we have the large phase shifts \( \Delta x_{1,2} = \Delta x_{3,4} \approx 7.8 \) and the maximum value of the soliton \( u_{\text{max}} \approx 1.96 \) (almost four times larger than \( A_{1,2} \)).

The O-type soliton solution has a steady X-shape with phase shifts in both line-solitons. One can also find the formula of the maximum amplitude which occurs at the center of the intersection point (center of the X-shape), which is given by

\[
\Delta \Omega := \frac{(k_3 - k_2)(k_4 - k_1)}{(k_4 - k_2)(k_3 - k_1)} = 1 - \frac{(k_2 - k_1)(k_4 - k_3)}{(k_4 - k_2)(k_3 - k_1)} < 1.
\]

This implies that \( \theta_{12} \equiv \theta_{34} > 0 \), and each \([i, j] \)-soliton shifts in \( x \) with

\[
\Delta x_{i,j} = \frac{1}{k_j - k_i} \theta_{ij}.
\]

(6.2)

The positive phase shifts \( \Delta x_{1,2} > 0 \) and \( \Delta x_{3,4} > 0 \) indicate an attractive force in the interaction. Figure 13 illustrates an O-type interaction of two solitons which have the same amplitude, \( A_{1,2} = A_{3,4} = \frac{1}{2} \), and are symmetric with respect to the \( y \)-axis, \( \Psi_{3,4} = -\Psi_{1,2} \approx 45^\circ \). Since the solution is close to the resonance, we have the large phase shifts \( \Delta x_{1,2} = \Delta x_{3,4} \approx 7.8 \) and the maximum value of the soliton \( u_{\text{max}} \approx 1.96 \) (almost four times larger than \( A_{1,2} \)).

Figure 13. O-type interaction for two equal amplitude solitons. The parameters \( k_i \)'s are taken to be \((k_1, k_2, k_3, k_4) = (-1 - 10^{-4}, 10^{-4}, 10^{-4}, 1 + 10^{-4})\), which give \( A_{1,2} = A_{3,4} = \frac{1}{2} \) and \( \tan \Psi_{3,4} = -\tan \Psi_{1,2} = 1 + 2 \times 10^{-4} \) (i.e. \( \Psi_{3,4} \approx 45.0057 \)). The constants \( a \) and \( b \) in the \( A \)-matrix are chosen so that the center of the interaction point is located at the origin, and \( u_{\text{max}} = u(0, 0, 0) \approx 1.96 \) and the phase shift \( \Delta x_{1,2} = \Delta x_{3,4} \approx 7.8 \).
the other hand, for the case \( k_2 = k_3 \) (i.e. \( \Delta_O = 0 \)), the \( \tau \)-function has only three terms, which corresponds to a solution showing a Y-shape interaction (i.e. the phase shift becomes infinity and the middle portion of the interaction stretches to infinity). This limit has been discussed in [30, 34] as a resonant interaction of three waves to make a Y-shape soliton. This limit gives a critical angle between those solitons which can be found as follows: first let us express each \( k_j \) parameter in terms of the amplitude and the slope:

\[
k_{1,2} = \frac{1}{2} \left( \tan \Psi_{1,2} + \sqrt{2A_{1,2}} \right),
\]

\[
k_{3,4} = \frac{1}{2} \left( \tan \Psi_{3,4} + \sqrt{2A_{3,4}} \right),
\]

where the angle \( \Psi_{i,j} \) is measured in the counterclockwise direction from the \( y \)-axis (see figure 13). In particular, we have

\[
\tan \Psi_{1,2} = -\sqrt{2A_{1,2}} + 2k_2, \quad \tan \Psi_{3,4} = \sqrt{2A_{3,4}} + 2k_3.
\]

For simplicity, let us consider the special case when both solitons are of equal amplitude and symmetric with respect to the \( y \)-axis, i.e. \( A_{1,2} = A_{3,4} = A_0 \) and \( \Psi_{3,4} = -\Psi_{1,2} = \Psi_0 > 0 \). This corresponds to setting \( k_1 = -k_4 \) and \( k_2 = -k_3 \). Then, for the fixed amplitude \( A_0 \), the angle \( \Psi_0 \) has a lower bound given by

\[
\tan \Psi_0 = \sqrt{2A_0} + 2k_3 \geq \sqrt{2A_0} := \tan \psi_c.
\]

The lower bound is achieved in the limit \( k_3 = k_4 = 0 \), and the critical angle \( \psi_c \) is given by

\[
\psi_c = \tan^{-1} \sqrt{2A_0}.
\] (6.4)

In [30], Miles introduced the following parameter to describe the interaction properties for the O-type solution:

\[
\kappa := \frac{\tan \Psi_0}{2A_0} = \frac{\tan \Psi_0}{\tan \psi_c}.
\] (6.5)

With this parameter, the maximum amplitude of (6.3) for this symmetric case is given by

\[
u_{\text{max}} = \frac{4A_0}{1 + \sqrt{\Delta_O}}, \quad \text{with} \quad \Delta_O = 1 - \frac{1}{\kappa^2}.
\] (6.6)

Thus, at the critical angle \( \Psi_0 = \psi_c \) (i.e. \( \kappa = 1 \)), we have \( \nu_{\text{max}} = 4A_0 \) and the phase shift \( \theta_{12} \to \infty \), leading to the resonant Y-shape interaction (see also [12, 30, 47]).

One should note that if we use the form of the O-type solution even beyond the critical angle, i.e. \( k_3 < k_2 \), then the solution becomes singular (note that the sign of \( E(2, 3) \) changes). In earlier works, this was considered to be an obstacle for using the KP equation to describe an interaction of two line-solitons with a smaller angle. In contrast, the KP equation should give a better approximation to describe oblique interactions of solitons with smaller angles. Thus one should expect to have explicit solutions of the KP equation describing such phenomena. It turns out that the new types of (2, 2)-soliton solutions discussed above can indeed serve as good models for describing line-soliton interactions of solitons with small angles. We show in section 8 how these solutions are related to the Mach reflections in shallow water waves.

6.2. (3142)-type soliton solutions

We consider a solution of this type which consists of two line-solitons for large positive \( x \) and two other line-solitons for large negative \( x \). We then assume that the slopes of two solitons in each region have opposite signs, i.e. one in \( y > 0 \) and the other in \( y < 0 \) (see figure 14). The line-solitons for the (3142)-type solution are determined from the balance between two appropriate exponential terms in its \( \tau \)-function which has the form

\[
\tau = E(1, 3) + bE(1, 4) + aE(2, 3) + abE(2, 4) + cE(3, 4).
\]
The solution contains three free parameters $a$, $b$, and $c$, which can be used to determine the locations of three (out of four) asymptotic line-solitons (e.g. two in $x \gg 0$ and one in $x \ll 0$). Thus, the parameters are completely determined from the asymptotic data on large $|y|$.

Let us first consider the line-solitons in $x \gg 0$: there are two line-solitons which are the $[1,3]$-soliton in $y \gg 0$ and the $[2,4]$-soliton in $y \ll 0$. The $[1,3]$-soliton is obtained by the balance between the exponential terms $bE(1,4)$ and $cE(3,4)$, and the $[2,4]$-soliton is by the balance between $aE(2,3)$ and $cE(3,4)$. Consequently, the phase shifts of the $[1,3]$- and $[2,4]$-solitons for $x \gg 0$ are given by

$$\theta_{13}^+ = \ln \frac{k_4 - k_1}{k_4 - k_3} + \ln \frac{b}{c}, \quad \theta_{24}^+ = \ln \frac{k_3 - k_2}{k_4 - k_3} + \ln \frac{a}{c}.$$  
(6.7)

Now we consider the line-solitons in $x \ll 0$: they are the $[3,4]$-soliton in $y \gg 0$ and the $[1,2]$-soliton in $y \ll 0$. The phase shifts are given, respectively, by

$$\theta_{34}^- = \ln \frac{k_3 - k_1}{k_4 - k_1} - \ln b, \quad \theta_{12}^- = \ln \frac{k_3 - k_1}{k_3 - k_2} - \ln a.$$  

We then define the parameter $s$ (representing the total phase shifts $\theta_{13}^+ + \theta_{34}^- = \theta_{24}^+ + \theta_{12}^-$):

$$s := \exp \left( -\theta_{13}^+ - \theta_{34}^- \right),$$  
(6.8)

which leads to

$$a = \frac{k_3 - k_1}{k_3 - k_2} s e^{\theta_{13}^+}, \quad b = \frac{k_3 - k_1}{k_4 - k_1} s e^{\theta_{34}^-}, \quad c = \frac{k_3 - k_1}{k_4 - k_3} s.$$  
(6.9)

The parameter $s$ represents the relative locations of the intersection point of the $[1,3]$- and $[3,4]$-solitons with the $x$-axis, in particular, $\theta_{13}^+ + \theta_{34}^- = 0$ when $s = 1$ (see figure 15).

Thus the parameters $a$, $b$, and $c$ are related to the locations of the $[1,3]$-soliton (with $\theta_{13}^+$), of the $[2,4]$-soliton (with $\theta_{24}^+$) and the intersection point of the $[1,3]$- and $[3,4]$-solitons (with $s$).

Now we consider the case where the $[1,3]$- and $[2,4]$-solitons have the same amplitude ($A_{[1,3]} = A_{[2,4]} = A_0$) and they are symmetric with respect to the $x$-axis ($\Psi_{[2,4]} = -\Psi_{[1,3]} = \Psi_0$). Then in terms of the $k$-parameters, we have

$$k_3 - k_1 = k_4 - k_2 = \sqrt{2A_0}.$$
Figure 15. (3142)-type soliton solution with the parameter $s$. The line-solitons are given by $A_{[1,3]} = A_{[2,4]} = 0.5$ and $\Psi_{[2,4]} = -\Psi_{[1,3]} = 25^\circ$ (these then give the other two solitons uniquely). The parameters in the $A$-matrix are chosen as (6.9) with $\theta_{[1,3]} = \theta_{[2,4]} = 0$. Then at $s = 1$, all the solitons meet at the origin, i.e. the parameter $s$ shifts the $[1, 2]$- and $[3, 4]$-solitons.

Also the symmetry of the wave vectors, i.e. $\Psi_{[2, 4]} = \Psi_0 = -\Psi_{[1, 3]}$, gives

$$k_2 + k_4 = -(k_1 + k_3) = \tan \Psi_0.$$  

This implies that we have

$$k_4 = -k_1 > 0, \quad k_3 = -k_2 > 0.$$  

The angle $\Psi_0$ takes the value in $(0, \Psi_c)$, where the critical angle is given by the condition $k_2 = k_3 = 0$, i.e.

$$\Psi_c = \tan^{-1} \sqrt{2A_0}.$$  

Note that this formula is the same as that of the O-type soliton solution (see (6.4)), and the (3142)-type exists when the parameter $\kappa$ is less than 1, i.e. for (3142)-type, we have

$$\kappa = \tan \Psi_0 \sqrt{2A_0} < 1.$$  

From (6.10), one can easily deduce the following facts for the $[1, 2]$- and $[3, 4]$-solitons in $x < 0$.

(a) Those solitons have the same amplitude, i.e.

$$A_{[1,2]} = A_{[3,4]} = \frac{1}{2}(k_4 - k_3)^2 = \frac{1}{2}(k_4 + k_2)^2 = \frac{1}{4} \tan^2 \Psi_0 = k^2 A_0.$$  

Thus, if the $[1, 3]$- and $[2, 4]$-solitons in $x > 0$ are close to the y-axis (i.e. a small $\Psi_0$), then the amplitudes of the solitons in $x < 0$ are small, whereas at the critical angle $\Psi_0 = \Psi_c$, the solitons $[1, 2]$ and $[3, 4]$ in $x < 0$ take the maximum amplitude $A_{[1,2]} = A_{[3,4]} = A_0$.

(b) The directions of the wave vectors for the $[1, 2]$ and $[3, 4]$-solitons are also symmetric, i.e.

$$\tan \Psi_{[3,4]} = -\tan \Psi_{[1,2]} = k_3 + k_4.$$  

Moreover, symmetry (6.10) implies that $\tan \Psi_{[3,4]} = k_4 - k_2 = \sqrt{2A_{[2,4]}} = \sqrt{2A_0}$, so

$$\Psi_{[3,4]} = \Psi_c = \tan^{-1} \sqrt{2A_0}.$$  

Thus the directions of the wave vectors for the $[1, 2]$ and $[3, 4]$-solitons in $x < 0$ depend only on the amplitude of the solitons in $x > 0$ but not on their directions (i.e. angle of their V-shape).
Let us choose the parameters in the \( A \)-matrix for the \((3142)\)-soliton solution appropriately, so that at \( t = 0 \) all the solitons intersect at the origin (see figure 14). Then for \( t < 0 \), the resonant interaction between \([1,3]\)- and \([3,4]\)-solitons (as well as \([2,4]\)- and \([1,2]\)-solitons) generates an intermediate line-soliton (called a ‘stem’ soliton) which is the \([1,4]\)-soliton. The amplitude of this soliton is given by

\[
A_{[1,4]} = \frac{1}{2}(k_4 - k_1)^2 = \frac{1}{2}(\sqrt{2A_0} + \tan \Psi_0)^2 = A_0(1 + \kappa)^2. \tag{6.11}
\]

Note here that at the critical angle \( \Psi_0 = \Psi_c \), the amplitude takes the maximum \( A_{[1,4]} = 4A_0 \) (see [41, 49]).

For \( t > 0 \), the resonant interaction between \([1,3]\)- and \([1,2]\)-solitons (as well as \([2,4]\)- and \([3,4]\)-solitons) generates an intermediate line-soliton of the \([2,3]\)-soliton. The amplitude of the \([2,3]\)-soliton is given by

\[
A_{[2,3]} = \frac{1}{2}(k_3 - k_2)^2 = \frac{1}{2}(\sqrt{2A_0} - \tan \Psi_0)^2 = A_0(1 - \kappa)^2.
\]

Because of symmetry (6.10), both \([1,4]\) - and \([2,3]\)-solitons are parallel to the \( y \)-axis, i.e. \( \tan \Psi_{[1,4]} = \tan \Psi_{[2,3]} = 0 \).

6.3. T-type soliton solutions

There are four parameters in the \( A \)-matrix for the T-type soliton solution. Here we explain that those parameters give the information of the locations of those line-solitons, the phase shift and onset of the opening of a box. Thus three of those four parameters are determined by the asymptotic data on large \( |y| \), and we need internal data for the other one.

Following the arguments in the previous section, one can find the phase shifts of the line-solitons of \([1,3]\) and \([2,4]\). For the \([1,3]\)-soliton in \( x > 0 \) (and \( y \gg 0 \)), the phase shift is calculated as

\[
\theta_{13}^+ = \ln \frac{k_4 - k_1}{k_4 - k_3} - \ln \frac{D}{b},
\]

where \( D = ad - bc = \xi(3,4) \). For the same soliton in \( x < 0 \) (and \( y \ll 0 \)), we have

\[
\theta_{13}^- = \ln \frac{k_2 - k_1}{k_3 - k_2} - \ln c.
\]

So the total phase shift \( \theta_{13} := \theta_{13}^+ - \theta_{13}^- \) depends on the \( A \)-matrix unlike the cases of O- and P-types, and it can take any value.

For the \([2,4]\)-soliton in \( x > 0 \) (and \( y \ll 0 \)), we have

\[
\theta_{24}^+ = \ln \frac{k_3 - k_2}{k_4 - k_3} - \ln \frac{D}{c}
\]

and for the same one in \( x < 0 \) (and \( y \gg 0 \)), we have

\[
\theta_{24}^- = \ln \frac{k_2 - k_1}{k_4 - k_1} - \ln b.
\]

Note that the total phase shift \( \theta_{24} = \theta_{24}^+ - \theta_{24}^- \) is the same as that for the \([1,3]\)-soliton, i.e. the phase conservation along the \( y \)-axis \( \theta_{13}^+ + \theta_{24}^- = \theta_{13}^- + \theta_{24}^+ \) holds. Then as in (6.8) for the case of \((3142)\)-type, we define the parameter \( s \):

\[
s := \exp(-\theta_{13}^+ - \theta_{24}^-),
\]

which represents the intersection point of the \([1,3]\) - and \([2,4]\)-soliton. With the parameter \( s \), we have

\[
b = \frac{k_2 - k_1}{k_4 - k_1} s e^{i\theta_5}, \quad c = \frac{k_2 - k_1}{k_3 - k_2} s e^{i\theta_2^*}, \quad D = \frac{k_2 - k_1}{k_4 - k_3} s. \tag{6.12}
\]
Figure 16. T-type interaction with the parameter \( s \). The \( k \)-parameters are chosen as \((k_1, k_2, k_3, k_4) = (-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2})\). The \( A \)-matrix is chosen as (6.12) and (6.13) with \( \theta'_{13} = \theta'_{14} = 0 \) and \( r = 1 \). The parameter \( s \) gives the phase shift for the [1, 2]- and [3, 4]-solitons in \( x < 0 \).

Figure 17. T-type interaction with the parameter \( r \). The \( k \)-parameters are the same as those in figure 23. The \( A \)-matrix is chosen as (6.12) with \( \theta'_{13} = \theta'_{14} = 0 \) and \( s = 1 \). The parameter \( r \) gives the onset of the box, and it does not affect the locations of all four line-solitons, that is, \( r \) is an internal parameter.

Figure 16 illustrates the solution pattern of T-type with the different \( s \). Namely, the three parameters \( b, c \) and \( D = ad - bc \) determine the locations and the phase shift (i.e. the intersection point of the [1, 3]- and [2, 4]-solitons). One other parameter is then related to the onset of a box at the intersection point (see figure 17).

In order to characterize this parameter, let us consider the intermediate solitons of [1, 4] and [2, 3]. First note that for \( t \gg 0 \), the [1, 4]-soliton appears as the dominant balance between \( E(1, 2) \) and \( E(2, 4) \). Then one can find the phase shift \( \theta'_{14} \) (here + indicates \( t > 0 \)):

\[
\theta'_{14} = \ln \frac{k_2 - k_1}{k_4 - k_2} - \ln d.
\]

Similarly one can get the phase shift \( \theta_{14}^- \) for \( t \ll 0 \) as

\[
\theta_{14}^- = \ln \frac{k_3 - k_1}{k_4 - k_3} - \ln \frac{D}{a}.
\]

Now consider the sum of \( \theta_{14}^+ \), i.e.

\[
\theta_{14}^+ + \theta_{14}^- = \ln \frac{(k_2 - k_1)(k_3 - k_1)}{(k_4 - k_2)(k_4 - k_3)} - \ln \frac{d D}{a}.
\]
Also, for the \([2, 3]\)-soliton, one can get

\[
\theta^+_2 + \theta^-_3 = \ln \frac{(k_2 - k_1)(k_4 - k_2)}{(k_3 - k_1)(k_4 - k_3)} - \ln \frac{aD}{d}.
\]

Now we introduce a parameter \(r\) in the form

\[
aD = r \frac{k_4 - k_2}{k_3 - k_1},
\]

so that we have

\[
\theta^+_4 + \theta^-_4 = \ln \frac{r}{s}, \quad \theta^+_2 + \theta^-_3 = -\ln (rs).
\]

Suppose that at \(t = 0\), \([1, 3]\)- and \([2, 4]\)-solitons in \(x > 0\) are placed so that they meet at the origin, that is, we choose \(\theta^+_1 = \theta^-_4 = 0\). Also if there is no phase shift for those solitons, i.e. \(s = 1\), then the sums become

\[
\theta^+_4 + \theta^-_4 = \ln r = -\left(\theta^+_3 + \theta^-_3\right).
\]

This implies that at \(t = 0\) (and \(s = 1\)) if \(r = 1\), then the T-type soliton solution has an exact shape of ‘X’ without any opening of a box at the intersection point on the origin. Moreover, at \(t = 0\) if \(r > 1\), then the \([1, 4]\)-soliton appears in \(x > 0\) and the \([2, 3]\)-soliton in \(x < 0\), whereas if \(0 < r < 1\), then the \([1, 4]\)-soliton appears in \(x < 0\) and the \([2, 3]\)-soliton in \(x > 0\). Figure 17 illustrates those cases with \(s = 1\). The parameter \(r\) determines the exponential term that is dominant in the region inside the box. When \(r < 1\), \(E(2, 4)\) is the dominant exponential term, and when \(r > 1\) the dominant exponential is \(E(1, 3)\). One should note that the parameter \(r\) cannot be determined by the asymptotic data, that is, \(r\) is considered as an ‘internal’ parameter.

7. Numerical simulation and the stability of the soliton solutions

In this section, we present some numerical simulations of the KP equation with ‘V-shape’ initial wave form related to a physical situation (see for examples [15, 41, 49]). The main purpose of the numerical simulation is to study the interaction properties of line-solitons, and we show that the solutions of the initial value problems with V-shape incident waves approach asymptotically to some of the exact soliton solutions of the KP equation discussed in the previous section. This implies a stability of those exact solutions under the influence of certain deformations (note that the deformation in our cases is not so small).

The initial value problem considered here is essentially an infinite energy problem in the sense that each line-soliton in the initial wave is asymptotically supported in either \(y \gg 0\) or \(y \ll 0\), and the interactions occur only in a finite domain in the \(xy\)-plane. In the numerical scheme, we consider the rectangular domain \(D = \{(x, y) : |x| \leq L_x, |y| \leq L_y\}\), and each line-soliton is matched with a KdV soliton at the boundaries \(y = \pm L_y\). The details of the numerical scheme and the results can be found in [20].

We consider the initial data given in the shape of ‘V’ with the amplitude \(A_0\) and the oblique angle \(\Psi_0 > 0\):

\[
u(x, y, 0) = A_0 \text{sech}^2 \sqrt{\frac{A_0}{2}} (x - |y| \tan \Psi_0).
\]

Figure 18 illustrates the initial wave profile of V-shape. Note here that two semi-infinite line-solitons are propagating toward each other into the positive \(x\)-direction, so that they interact
strongly at the corner of the V-shape. At the boundaries $y = \pm L_y$ of the numerical domain, those line-solitons are patched to the KdV one-soliton solutions given by

$$u(x, \pm L_y, t) = A_0 \text{sech}^2 \left( \frac{A_0}{2} (x \mp L_y \tan \Psi_0 - vt) \right),$$

with $v = \frac{1}{2} \tan^2 \Psi_0 + \frac{1}{2} A_0$. Note here that these solitons correspond to the exact one-soliton solution of the KdV equation with the velocity shift due to the oblique propagation of the line-soliton, i.e. $\partial^2 u / \partial y^2 = \tan^2 \Psi_0 \partial^2 u / \partial x^2$. The numerical simulations are based on a spectral method with the window technique similar to the method used in [49] (see [20] for the details).

The V-shape initial wave was first considered by Oikawa and Tsuji [41, 49] in order to study the generation of freak (or rogue) waves (see for example [22]). They noticed generations of different types of asymptotic solutions depending on the initial oblique angle $\Psi_0$, and found the resonant interactions which create localized high-amplitude waves. In this section, we present the results for the cases corresponding to $A_0 = 2$ and two different angles, $\Psi_1$ and $\Psi_2$ with $\Psi_1 < \Psi_c < \Psi_2$ where the critical angle is given by $\Psi_c = \tan^{-1} \sqrt{2A_0} \approx 63.4^\circ$. Then we explain these results in terms of certain $(2, 2)$-soliton solutions discussed in the previous section, and in particular, we describe the connection with the Mach reflection (this will be further discussed in section 8).

The main idea here is to consider the V-shape initial wave as the part of some $(2, 2)$-soliton solutions listed in the previous section. In order to identify those soliton solutions from the V-shape initial wave form, let us first denote them as the $[i_1, j_1]$-soliton for $y \gg 0$ and the $[i_2, j_2]$-soliton for $y \ll 0$. Then using the relations, $k_j - k_i = \sqrt{2A_0} = 2$ and $k_j + k_i = \tan \Psi_0$, for the $[i, j]$-soliton and the Miles parameter $\kappa = \tan \Psi_0 / \sqrt{2A_0}$ of (6.5), we have

$$\begin{align*}
{k_i} & = -(1 + \kappa), & {k_j} & = 1 - \kappa, \\
{k_{i_1}} & = -(-1 + \kappa), & {k_{j_1}} & = 1 + \kappa.
\end{align*}$$

(7.2)

Note that $k_{j_1} = -k_{i_1}$ and $k_{i_2} = -k_{j_2}$ because of the symmetry in the initial wave. Moreover, at the critical angle $\Psi_0 = \Psi_c$ (i.e. $\kappa = 1$), we have $k_{i_1} = k_{j_1} = 0$. We also note $k_{i_1}$ as the smallest parameter and $k_{j_1}$ as the largest one, so that depending on the angle $\Psi_0$, we obtain the following ordering in the $k$-parameters.

For $0 < \Psi_0 < \Psi_c$ (i.e. $\kappa < 1$), we have

$$k_{i_1} < k_{i_2} < 0 < k_{j_1} < k_{j_2},$$

implying that the corresponding chords of the $[i_1, j_1]$- and the $[i_2, j_2]$-solitons overlap. That is, the $[1, 3]$-chord appears on the upper side of the diagram, and the $[2, 4]$-chord on the lower side. This means that the two solitons can be identified as part of either the $(3412)$-type ($T$-type) or the $(3142)$-type solution (see the chord diagrams in figure 11).
For $\Psi_c < \Psi_0 < \frac{\pi}{2}$ (i.e. $\kappa > 1$), we have
\[ k_{i_1} < k_{j_1} < 0 < k_{i_2} < k_{j_2}. \]
In this case, the corresponding chords are separated, and the two solitons form part of either (2413)- or (2143)-type (O-type) solution. Here [1, 2]- and [3, 4]-chords appear on the upper and lower sides of the chord diagram, respectively.

Then the numerical simulations show that we have the following types of the asymptotic solutions depending on the values $\Psi_0$.

(a) If the angle satisfies $\Psi_0 < \Psi_c$ (i.e. $\kappa < 1$), then the solution converges asymptotically to the (3142)-type soliton solution (not T-type).

(b) If the angle satisfies $\Psi_c < \Psi_0$ (i.e. $\kappa > 1$), then the solution converges asymptotically to an O-type soliton solution (not (2413)-type).

The convergence here is in a locally defined $L^2$-sense with the usual norm:
\[ \|f\|_{L^2(D)} := \left( \int \int_D |f(x, y)|^2 \, dx \, dy \right)^{\frac{1}{2}}, \]
where $D \subset \mathbb{R}^2$ is a compact set which covers the main structure of the interactions in the solution. To confirm the convergence statements, we define the (relative) error function
\[ E(t) := \|u' - u^\text{exact}'\|_{L^2(D)}^2 / \|u^\text{exact}'\|_{L^2(D')}^2, \]
with the solution $u'(x, y) := u(x, y, t)$ and an exact solution $u_{\text{exact}}'(x, y)$, where $D'$ is the circular disk given by
\[ D' := \{(x, y) \in \mathbb{R}^2 : (x - x_0(t))^2 + (y - y_0(t))^2 \leq r^2 \}. \]

The center $(x_0(t), y_0(t))$ of the circular domain $D'$ is chosen as the intersection point of two lines determined from the corresponding exact solution. We find the exact solution $u_{\text{exact}}'(x, y)$ by minimizing $E(t)$ at a certain large time $t = T_0$. In the minimization process, we assume that the $k$-parameters remain the same as those given by (7.2), and vary the corresponding $A$-matrix to adjust the solution pattern (recall that the $A$-matrix determines the locations of the line-solitons in the solution, see section 5). After minimizing $E(t)$, that is finding the corresponding exact solution, we check that $E(t)$ further decreases for a larger time $t > T_0$ up to a time $t = T_1 > T_0$, just before the effects of the boundary enter the disk $D'$ (those effects include the periodic condition in $x$ and a mismatch on the boundary patching). We take the radius $r$ in $D'$ large enough so that the main interaction area is covered for all $t < T_0$, but $D'$ should be kept away from the boundary to avoid any influence coming from the boundaries. The time $T_1 > T_0$ gives an optimal time to develop a pattern close to the corresponding exact solution, but it is also limited to avoid any disturbance from the boundaries for $t < T_1$. Thus, our convergence implies the separation of the radiations from the soliton solution, just like the case of the KdV equation (see the end of section 3).

We also note that the convergence here implies a completion of the partial chord diagram consisting of only two chords which corresponds to the semi-infinite solitons in the initial V-shape wave. Namely, the asymptotic solution of the initial value problem with the V-shape initial wave is given by an exact solution parametrized by a unique chord diagram, and the initial (partial) chord diagram is completed by adding two other solitons (chords) generated by the interaction. The completion may not be unique, and in [24], we proposed a concept of minimal completion in the sense that the complete diagram has the minimum total length of the chords and the corresponding TNN Grassmannian cell has the minimum dimension. However, this problem is still open, and we need to make a precise statement of the minimal completion of the partial chord diagram given by the initial wave profile.
7.1. Regular reflection: $\kappa > 1$

We consider the V-shape initial wave with $A_0 = 2$ and $\tan \Psi_0 = \frac{12}{5} \approx 67.3^\circ$ which gives $\kappa = 1.2$. Here the critical angle is $\Psi_c = \tan^{-1}(2) \approx 63.4^\circ$, and we expect asymptotically an O-type soliton solution. The corresponding $k$-parameters are obtained from (7.2), i.e.

$$(k_1, k_2, k_3, k_4) = \left( -\frac{11}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{11}{5} \right).$$

Figure 19 illustrates the result of the numerical simulation. The top figures show the direct simulation of the KP equation. The wake behind the interaction point has a large negative amplitude, and it disperses and decays in the negative $x$-direction. This shows a separation of the radiations from the exact solution similar to the case of the KdV soliton. The steady pattern left after shedding the radiations can be identified as an O-type solution. The middle figures
show the corresponding O-type exact solution whose $A$-matrix is determined by minimizing the error function $E(t)$ at $t = 6$:

$$A = \begin{pmatrix} 1 & 1.91 & 0 & 0 \\ 0 & 0 & 1 & 0.17 \end{pmatrix}.$$ 

Using (6.1), we obtain the shift of the initial line-solitons:

$$x_{[1,2]} = x_{[3,4]} = -0.020.$$ 

(Note here that because of the symmetric profile, the shifts for initial solitons are the same.)

The negative shifts imply the slowdown of the incidence waves due to the generation of the solitons extending the initial solitons in the negative $x$-direction. The phase shifts $\Delta x_{[i,j]}$ for the O-type exact solution are calculated from (6.2), and they are

$$\Delta x_{[1,2]} = \Delta x_{[3,4]} = 0.593.$$ 

The positivity of the phase shifts is due to the attractive force between the line-solitons, and this explains the slowdown of the initial solitons, i.e. the small negative shifts of $x_{[i,j]}$. The bottom graph in figure 19 shows $E(t)$ of (7.3), where we take $r = 12$ for the domain $D_r$. One can see a rapid convergence of the solution to the O-type exact solution with those parameters. One should however remark that when $\Psi_0$ is close to the critical one, i.e. $k_2 \approx k_3$, there exists a large phase shift in the soliton solution, and the convergence is very slow. Note that in the limit $k_2 = k_3$, the amplitude of the intermediate soliton generated at the intersection point reaches four times larger than that of the initial solitons. This large amplitude wave generation has been considered as the Mach reflection problem of the shallow water wave [8, 24, 29, 30, 40, 41, 48] (see also section 8). The chord diagram in figure 19 shows a completion of the (partial) chord diagram: the solid chords indicate the initial solitons forming V-shape, and the dotted chords correspond to the solitons generated by the interaction (see [24] for further discussion).

### 7.2. The Mach reflection: $\kappa < 1$

We consider the initial V-shape wave with $A_0 = 2$ and $\Psi_0 = 45^\circ$ (i.e. $\kappa = 0.5$). The angle $\Psi_0$ is now less than the critical angle $\Psi_c \approx 63.4^\circ$. The asymptotic solution is expected to be of $(3142)$-type whose $k$-parameters are obtained from (7.2), i.e.

$$(k_1, k_2, k_3, k_4) = \left( -\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \right).$$ 

Figure 20 illustrates the result of the numerical simulation. The top figures show the direct simulation of the KP equation. We again observe a bow-shape wake behind the interaction point. The wake expands and decays, and then we see the appearance of new solitons which form resonant interactions with the initial solitons. One should note that the solution generates a large amplitude intermediate soliton at the interaction point, and this soliton is identified as the $[1, 4]$-soliton with the amplitude $A_{[1,4]} = 4.5$. This $[1, 4]$-soliton is called the Mach stem in the Mach reflection [8, 24] (see also section 8).

The middle figures in figure 20 show the corresponding exact solution of $(3142)$-type whose $A$-matrix is found by minimizing $E(t)$ at $t = 10$:

$$A = \begin{pmatrix} 1 & 1.92 & 0 & -1.96 \\ 0 & 0 & 1 & 0.64 \end{pmatrix}.$$ 

Using (6.9), we obtain the phase shifts $x_{[i,j]}$ for the initial solitons of $[1, 3]$- and $[2, 4]$-type in $x > 0$, and the parameter $s$:

$$x_{[1,3]} = x_{[2,4]} = -0.01, \quad s = 0.980.$$
Figure 20. Numerical simulation of the V-shape initial wave for $\kappa < 1$ (Mach reflection): the initial wave consists of the $[1, 3]$-soliton in $y > 0$ and the $[2, 4]$-soliton in $y < 0$ with $A_0 = 2$ and $\Psi_0 = 45^\circ$. The top figures show the result of the direct simulation, and the middle figures show the corresponding exact solution of $(3142)$-type. Note that a large amplitude intermediate soliton is generated at the intersection point, and it corresponds to the $[1, 4]$-soliton with the amplitude $A_{1, 4} = 4.5$. The circles in the middle figures show $D_r$ with $r = 12$, which cover well the main part of the interaction regions up to $t = 12$. The bottom graph of the error function $E(t)$ is minimized at $t = 10$. The solid chords in the diagram indicate the incident solitons, and the dotted ones show the reflected solitons.

Those values indicate that the solution is very close to the exact solution for all the time. The negative value of the shifts $x_{i,j}$ is due to the generation of a large amplitude soliton $[1, 4]$-type (i.e. the initial solitons slow down), and $s < 1$ implies that the $[1, 4]$-soliton is generated after $t = 0$. Also note that the $[1, 4]$-soliton now resonantly interacts with the $[1, 3]$- and $[2, 4]$-solitons to create new $[1, 2]$- and $[3, 4]$-solitons (called the reflected waves in the Mach reflection problem [8, 24, 30]). This process then seems to compensate the shifts of incident waves, even though we observe a large wake behind the interaction point.

The bottom graph in figure 20 shows a rapid convergence of the initial wave to the $(3142)$-type soliton solution with those parameters of the $A$-matrix and $k$ values given above, and for the error function $E(t)$, we minimize it at $t = 10$ for $D_r$ with $r = 12$.

7.3. T-type interaction with the X-shape initial wave

In this example, we consider an X-shape initial wave given by the sum of two line-solitons. For simplicity, we consider a symmetric initial wave with $A_0 = 2$ and $\Psi_0 = 45^\circ$ (i.e. extend
Figure 21. Numerical simulation for the X-shape initial wave with $A_0 = 2$ and $\Psi_0 = 45^\circ$ (i.e. $\kappa = 0.5 < 1$): the initial wave is the sum of the [1, 3]- and [2, 4]-solitons. The top figures show the numerical simulation, and the middle figures show the corresponding exact solution of (3412)-type, i.e. T-type. Note that the circle showing $D^r_t$ with $r = 22$ covers well the box generated by the resonant interaction up to $t = 7$. The bottom graph shows the error function $E(t)$ of (7.3) which is minimized at $t = 6$. The four solid chords in the diagram show the asymptotic solitons in the incident wave, and they form a T-type soliton solution.

Although the T-type soliton appears for a smaller angle $\Psi_0$, one should not take so small a value. For an example of the symmetric case, if we take $\Psi_0 = 0$ giving twice higher amplitude than one-soliton case, we obtain the KdV two-soliton solution with different amplitudes (as can be shown by the method of IST). So for the case with a very small angle $\Psi_0$, we expect to see those KdV solitons near the intersection point. However, the solitons expected from the chord diagram have almost the same amplitude as the incidence solitons for the case with a small angle. The detailed study also shows that near the intersection point for the T-type solution at the time when all four solitons meet at this point (i.e. X-shape), the solution at the intersection point has a small amplitude due to the repulsive force similar to the KdV solitons. Then the initial X-shape wave with a small angle generates a large soliton at the intersection point. This then implies that our initial wave given by the sum of two line-solitons creates a large dispersive perturbation at the intersection point, and one may need to wait a long time to see the convergence.

The top figures in figure 21 illustrate the numerical simulation, which clearly shows an opening of a resonant box as expected by the feature of T-type. The corresponding exact
solution is illustrated in the middle figures, where the \( A \)-matrix of the solution is obtained by minimizing the error function \( E(t) \) of (7.3) at \( t = 6 \):

\[
A = \begin{pmatrix}
1 & 0 & -0.368 & -0.330 \\
0 & 1 & 1.198 & 0.123
\end{pmatrix}.
\]

In the minimization process, we take \( x_{[1,3]} = x_{[2,4]} \) due to the symmetric profile of the solution, and adjust the onset of the box (see subsection 6.3). Note that the symmetry reduces the number of free parameters to 3. We obtain

\[
x_{[1,3]} = x_{[2,4]} = 0.025, \quad r = 3.63, \quad s = 0.350.
\]

The positive shifts of those \([1, 3]\)- and \([2, 4]\)-solitons in the wavefront also indicate the positive shift of the newly generated soliton of \([1, 4]\)-type at the front. This is due to the repulsive force which exists in the KdV-type interaction as explained above, that is, the interaction part in the initial wave has a larger amplitude than that of the exact solution, so that this part of the solution moves faster than that in the exact solution. This difference may result in a shift of the location of the \([1, 4]\)-soliton. The relatively large value \( r > 1 \) indicates that the onset of the box is actually much earlier than \( t = 0 \), and \( s < 1 \) shows the positive phase shifts as calculated from (6.12).

The bottom graph in figure 21 shows the evolution of the error function \( E(t) \) of (7.3) which is minimized at \( t = 6 \). Note here that the circular domain \( D_r^t \) with \( r = 22 \) covers well the main feature of the interaction patterns for all the time computed for \( t \leq 7 \). The chord diagram in the figure shows four asymptotic solitons in the initial wave which form a T-type soliton solution (see [20] for further discussion).

### 8. Shallow water waves: the Mach reflection

In this last section, we discuss a real application of the exact soliton solutions of the KP equation described in the previous sections to the Mach reflection phenomena in shallow water. In [30], Miles considered an oblique interaction of two line-solitons using O-type solutions. He observed that resonance occurs at the critical angle \( \Psi_c \), and when the initial oblique angle \( \Psi_0 \) is smaller than \( \Psi_c \), the O-type solution becomes singular (recall that at the critical angle \( \Psi_c \), one of the exponential term in the \( \tau \)-function vanishes, see subsection 6.1). He also noticed a similarity between this resonant interaction and the Mach reflection found in the shock wave interaction (see for example [11, 50]). This is illustrated by the left figure of figure 22, where an incidence wave shown by the vertical line is propagating to the right, and it hits a rigid wall with the angle \(-\Psi_0\) measured counterclockwise from the axis perpendicular to the wall (see also [15]). If the angle \( \Psi_0 \) (equivalently the inclination angle of the wall) is large, the reflected wave behind the incidence wave has the same angle \( \Psi_0 \), i.e. a regular reflection occurs. However, if the angle is small, then an intermediate wave called the Mach stem appears as illustrated in figure 22. The Mach stem, the incident wave and the reflected wave interact resonantly, and those three waves form a resonant triplet. The right panel in figure 22 illustrates the wave propagation which is equivalent to that in the left panel, if one ignores the effect of viscosity on the wall (i.e. no boundary layer). At the point \( O \), the initial wave has V-shape with the angle \( \Psi_0 \), which forms the initial data for our simulation discussed in the previous section. Then as we presented, the numerical simulation describes the reflection of the line-soliton with an inclined wall, and these results explain well the Mach reflection phenomena in terms of the exact soliton solutions of the KP equation.
8.1. Previous numerical results of the Boussinesq-type equations

One of the most interesting features of the Mach reflection is that the KP theory predicts an extraordinary fourfold amplification of the stem wave at the critical angle [30]. We recall the formulas of the maximum amplitudes which are given by (6.6) for the O-type solution ($\kappa > 1$) and (6.11) for the (3142)-type solution ($\kappa < 1$). Let $\alpha$ denote the amplification factor in terms of the Miles parameter $\kappa$ of (6.5), i.e.

$$
\alpha = \begin{cases} 
(1 + \kappa)^2, & \text{for } \kappa < 1, \\
\frac{4}{1 + \sqrt{1 - \kappa^{-2}}}, & \text{for } \kappa > 1.
\end{cases}
$$

(8.1)

Several laboratory and numerical experiments tried to confirm formula (8.1), in particular, the fourfold amplification at the critical value $\kappa = 1$ (see for example [15, 28, 39, 46]). In [15], Funakoshi made a numerical simulation of the Mach reflection problem using the system of equations,

$$
\begin{align*}
\eta_t + \Delta \psi + \alpha \nabla \cdot (\eta \nabla \psi) - \frac{\beta}{6} \Delta^2 \psi &= 0, \\
\left(\psi - \frac{\beta}{2} \Delta \psi\right)_t + \eta + \frac{\alpha}{2} |\nabla \psi|^2 &= 0,
\end{align*}
$$

which is equivalent to the Boussinesq-type equation (2.4) up to this order. He considered the initial wave to be the KdV soliton with higher order corrections up to $O(\epsilon)$. In his paper, he mainly presented the results for the incidence waves with the amplitude $a_i = 0.05 = \hat{a}_0/h_0$ and the angles $\frac{\pi}{10} \leq \Psi_0 \leq \frac{\pi}{7}$. He concluded that his results agree very well with the resonantly interacting solitary wave solution predicted by Miles. However, his results on the amplification parameter $\alpha$ are slightly shifted to the lower values of the Miles parameter $\kappa$. Tanaka in [46] then re-examined Funakoshi’s results for higher amplitude incidence waves with $a_i = 0.3$ using the high-order spectral method. He noted that the effect of a large amplitude tends to prevent the Mach reflection to occur, and all the parameters such as the critical angle $\Psi_c$ are shifted toward the values corresponding to the regular reflection (i.e. O-type). For example, he obtained the maximum amplification $\alpha = 2.897$ at $\kappa = 0.695$. 

Figure 22. The Mach reflection. The left panel illustrates a semi-infinite line-soliton (incidence wave) propagating parallel to the wall with the mirror image. The right panel illustrates an equivalent system to the left one when we ignore the viscous effect on the wall. The incident wave then forms a V-shape wave at $t = 0$ as discussed in section 7. The resulting wave pattern shown here is a (3142)-soliton solution.
However, we claim in our recent paper [52] (see also subsection 3.1) that those previous results did not properly interpret their comparisons with the theory, and in fact their results are in good agreement with the predictions given by the KP theory except for the cases near $\kappa = 1$. One should emphasize that the KP equation is derived under the assumptions of quasi-two-dimensionality and weak nonlinearity. Thus the key ingredient is to include higher order corrections to those assumptions when we compare the numerical or experimental results with the theory. In particular, the quasi-two-dimensionality can be corrected by comparing the KP soliton with the KdV soliton in the propagation direction as mentioned in section 3. More precisely, we have the amplitude correction (3.6), i.e.

$$\hat{a}_0 = \frac{a_0}{\cos^2 \Psi_0} = \frac{2h_0A_0}{3\cos^2 \Psi_0},$$

where $\hat{a}_0$ is the amplitude observed in the numerical computation of the Boussinesq-type equation (which has rotational symmetry in $\mathbb{R}^2$). This then suggests that the parameter $\kappa$ should be evaluated by the following formula using the experimental amplitude $\hat{a}_0$:

$$\kappa := \tan \frac{\Psi_0}{\sqrt{2A_0}} = \frac{\tan \Psi_0}{\sqrt{3(\hat{a}_0/h_0)^2 \cos \Psi_0}}. \quad (8.2)$$

Because of the quasi-two-dimensional approximation, i.e. $|\Psi_0| \ll 1$, Miles in his paper [30] replaced $\tan \Psi_0$ by $\Psi_0$, and then in [15, 46], the authors continued on to use this replacement. Then their computations with rather large values of $\Psi_0$ gave significant shifts of the parameter $\kappa$. We then re-evaluate their results with our formula (8.2), and the new results are shown in figure 23. Since Funakoshi’s simulations are based on small amplitude incidence waves, his results agree quite well with the KP predictions. Tanaka’s results are also in good agreement with the KP theory except for the cases near the critical angle (i.e. $\kappa = 1$), where the amplification parameter $\alpha$ gets close to 3. This region clearly violates the assumption of the weak nonlinearity. Although one needs to make higher order corrections to weak nonlinearity, the original plots of Tanaka’s are significantly improved with formula (8.2). The black dots in figure 23 indicate the results of recent laboratory experiments done by Yeh and his colleagues. We discuss their experimental results in the next section.
Before closing this section, we remark on the length of the Mach stem (i.e. the intermediate soliton of $[1,4]$-type). From the $(3142)$-soliton solution, one can find the point $(x_*, y_*)$ of the interaction of the triplet with the $[1,3]$, $[3,4]$- and $[1,4]$-solitons [8].

\[
\begin{align*}
\frac{1}{4} (\tan \Psi_c + \tan \Psi_0) t = \frac{A_0}{2}(1 + \kappa)^2 t, \\
\frac{1}{2} (\tan \Psi_c - \tan \Psi_0) t = \frac{\sqrt{A_0}}{2}(1 - \kappa)t,
\end{align*}
\]

(see figure 22). In the physical coordinates with $(\tilde{x}_*, \tilde{y}_*, \tilde{t})$, we have

\[
\begin{align*}
\tilde{x}_* &= c_0 \left( 1 + \frac{\hat{a}_0 \cos^2 \Psi_0}{h_0} (1 + \kappa)^2 \right) \tilde{t}, \\
\tilde{y}_* &= c_0 \sqrt{ \frac{\hat{a}_0 \cos^2 \Psi_0}{3h_0} (1 - \kappa) \tilde{t}}.
\end{align*}
\]

The angle $\Phi$ in figure 22 is then given by $\tan \Phi = \frac{\tilde{y}_*}{\tilde{x}_*}$, which is approximated in [15, 30] by

\[
\tan \Phi \approx \sqrt{\frac{\hat{a}_0}{3}} (1 - \kappa).
\]

Using the corrected formula $\tan \Phi = \frac{\tilde{y}_*}{\tilde{x}_*}$, one can see again good agreement with the KP theory (see figure 8 in [15]).

8.2. Experiments

Recently, Yeh and his colleagues performed several laboratory experiments on the Mach reflection phenomena using a 7.3 m long and 3.6 m wide wave tank with a water depth of 6.0 cm. Here we briefly describe their results and show that our KP theory can predict very well the evolution of the waves observed in the experiments.

The wave tank is equipped with the 16 axis directional-wave maker system along the 3.6 m long side wall marked by $x = 0$. An oblique incident solitary wave is created by driving those 16 paddles synchronously along the sidewall, and the wave maker is designed to generate a KdV soliton with any heights before the breaking. The temporal and spatial variations of water-surface profiles are measured by the laser-induced fluorescent (LIF) method (a highly accurate measurement technique). The water dyed with fluorescein (green) fluoresces when excited by the laser sheet. The illuminated image of the water profiles are recorded by a high-speed and high-resolution video camera.

8.2.1. The Mach reflection. In table 1, their experimental results of the amplification factors $\alpha$ are compared with those obtained from the exact solutions of the KP equation (i.e. O-type for $\kappa > 1$ and $(3142)$-type for $\kappa < 1$). The waves were measured at $\tilde{x} = 4.27$ m $(x = \tilde{x} / h_0 = 71.1)$ which is the farthest measuring location in the experiments, except for the case with $A_0 = 0.390$. In the latter case, the $\alpha$ values in the brackets are measured at $x = 50.8$ because of the wave-breaking (note that at this point $\alpha = 2.48$ implies $a_i = 1.02$). We calculate the corresponding KP exact solution at $t = 41.05$ (recall here that relations (3.8) gives $\tilde{x} - c_0 \tilde{t} = h_0 x$ and $c_0 \tilde{t} = \frac{2h_0}{3} t$). The amplification factor $\alpha$ is still growing along the propagation direction, and the values obtained from the exact solutions are in good agreement with the measurements. We note here that near the critical case (i.e. $\kappa = 1$) the growth of the stem amplitude is very slow and at $x = 71.1$ ($\tilde{x} = 4.27$ m) the amplification factor is only achieved about 65%. Also for the cases with a small oblique angle, i.e. $\Psi_0 = 20^\circ$ in table 1, the amplification factor $\alpha$ grows slower when the incidence wave amplitude is smaller, that is, at $x = 71.1$, $\alpha$ is almost constant (slightly decreases) as $\kappa$ increases. However the
The lower panels show the corresponding realizations, in which the incident and reflected waves separate away from the wall by the stem of the spatial profiles (100 slices per second) with approximately 3000 pixel resolution in the solution of well separated from the main part of the wave profile as predicted in the numerical simulation.

This choice of the most complex and interesting soliton solution associated with the pattern generated in the same tank by Yeh and his collaborators. The T-type solution is the Figure 25 shows a preliminary result for the T-type interaction 8.2.2. T-type interaction. Table 1. Amplification factor \( \alpha \) for different values of \( \kappa = \tan \Psi_0 / \sqrt{\kappa A_0} \); \( \alpha_{x=71.1}^{(Exp.)} \) are the laboratory data at \( x = 71.1 \) (\( \Psi = 4.27 \) m), \( \alpha_{x=71.1}^{(KP)} \) are calculated from the corresponding KP exact solutions at \( t = 41.05 \) and \( \alpha_{x=\infty}^{(KP)} \) are from (6.6) and (6.11). In the row of \( A_0 = 0.390 \) with \( \Psi_0 = 30^\circ \), the values of \( \alpha \) in the brackets are obtained at \( x = 50.8 \), because of the wave breaking immediately after this point; hence, the greater amplification cannot be realized [52].

| \( \kappa \) | \( A_0 \) | \( \Psi_0 \) | \( \alpha_{x=71.1}^{(Exp.)} \) | \( \alpha_{x=71.1}^{(KP)} \) | \( \alpha_{x=\infty}^{(KP)} \) |
|---|---|---|---|---|---|
| 1.417 | 0.083 | 30\(^\circ\) | 2.10 | 2.34 | 2.34 |
| 1.272 | 0.103 | 30\(^\circ\) | 2.13 | 2.47 | 2.47 |
| 1.051 | 0.151 | 30\(^\circ\) | 2.24 | 3.06 | 3.06 |
| 0.917 | 0.198 | 30\(^\circ\) | 2.33 | 2.61 | 2.67 |
| 0.753 | 0.294 | 30\(^\circ\) | 2.52 | 2.57 | 3.07 |
| 0.722 | 0.127 | 20\(^\circ\) | 1.89 | 1.84 | 2.96 |
| 0.654 | 0.390 | 30\(^\circ\) | (2.48) | (2.47) | 2.73 |
| 0.591 | 0.189 | 20\(^\circ\) | 1.95 | 1.93 | 2.53 |
| 0.516 | 0.249 | 20\(^\circ\) | 1.99 | 2.08 | 2.30 |
| 0.425 | 0.367 | 20\(^\circ\) | 2.01 | 1.99 | 2.03 |

asymptotic value of \( \alpha \) for large \( x \) increases as \( \kappa \) increases. This means that the observed waves are still in the transient stage, and a longer tank is necessary to observe further growth of the amplification factor (see [52] for the details). In figure 24, we show the image of the wave profile at \( x = 71.1 \) for the case when the incident wave amplitude has \( A_0 = 0.198 \) and \( \Psi_0 = 30^\circ \). The corresponding exact solution with those parameters is of (3142)-type (i.e. \( \kappa = 0.917 < 1 \)). The upper panels show two views in different angles of the temporal variation of the wave profile of the experiment at \( x = 71.1 \), which is made from 250 slices of the spatial profiles (100 slices per second) with approximately 3000 pixel resolution in the y-direction. As expected from the (3142)-type exact solution, the stem-wave formation is realized, in which the incident and reflected waves separate away from the wall by the stem wave. The lower panels show the corresponding (3142)-type exact solution at \( t = 41.05 \) (the \( x \)-coordinate is converted to the \( t \)-coordinate, using (3.8)). Here the \( k \)-parameters are \((k_1, k_2, k_3, k_4) = (-0.603, -0.026, 0.026, 0.603)\) from \( A_0 = 0.198 \) and \( \Psi_0 = 30^\circ \). Then we calculate the \( A \)-matrix using (6.9) with \( s = 1 \), and take

\[
A = \begin{pmatrix} 1 & 12.10 & 0 & -1.153 \\ 0 & 0 & 1 & 0.522 \end{pmatrix}.
\]

This choice of the \( A \)-matrix places the incidence wave crossing at the origin at \( t = 0 \), i.e. \( \theta_{1,3}^{(1)} = \theta_{1,4}^{(1)} = 0 \) in (6.9). We see good agreement between the experiment and the KP theory. At \( x = 71.1 \), the wave profile observed in the experiment is close to the corresponding exact solution of (3142)-type, that is, the radiations generated at the beginning stage dispersed and well separated from the main part of the wave profile as predicted in the numerical simulation (see subsection 7.2).

8.2.2. T-type interaction. Figure 25 shows a preliminary result for the T-type interaction pattern generated in the same tank by Yeh and his collaborators. The T-type solution is the most complex and interesting soliton solution associated with the \( \tau \)-function on \( \text{Gr}^r(2, 4) \). The initial wave has the V-shape (half of the X-shape); then other half of the X-shape is generated by the line-soliton with opposite angle. The upper panels of figure 25 show the evolution of the wave pattern with \( A_0 = 0.431 \) and \( \Psi_0 = 25^\circ \). Behind the crossing wave form (the
Figure 24. Two views of the temporal variation of the water-surface profile in the y-direction (perpendicular to the wall) at x = 71.1 [52]. The top panels show the experimental result, and the bottom ones show the corresponding (3142)-type exact soliton solution of the KP equation. The incident wave amplitude $A_0 = 0.198$, and the angle $\Psi_0 = 30^\circ$. The amplification factors obtained from the experiment and the exact solutions are close, and they are $\alpha_{x=71.1}^{\text{Exp.}} = 2.33$ and $\alpha_{x=71.1}^{\text{KP}} = 2.61$ (see table 1). Reproduced from [52]. With kind permission of The European Physical Journal (EPJ).

right side in the figure), the large wakes are generated at the early stage of the evolution, but they eventually separate from the main pattern of the T-type interaction, as we observed in the numerical simulation. The figures clearly show the formation of a box pattern as expected by the KP theory. The lower panels show the corresponding T-type exact soliton solution whose parameters are $(k_1, k_2, k_3, k_4) = (-0.697, -0.231, 0.231, 0.697)$ and the $A$-matrix given by (6.12) and (6.13) with $\theta_{1,3}^{[1]} = \theta_{2,4}^{[2]} = 0$, $s = 40$ and $r = 1$, i.e.

$$A = \begin{pmatrix} 1 & 0 & -40.34 & -24.07 \\ 0 & 1 & 24.07 & 13.37 \end{pmatrix}$$

The large value of the parameter $s$ indicates a large phase shift of the incident line-solitons (see figure 16), that is, the parts of line-solitons on the right side (i.e. behind the interaction
Figure 25. T-type solutions generated in the water tank. The experimental results are shown in the upper panels. Those figures are made by combining the real image of the wave profiles with their mirror images of the wall at $y = 0$ (the center horizontal line). The lower panels show the corresponding T-type exact solution of the KP equation. The initial wave has $a_1 = 2.1$ cm ($A_0 = 0.43$) and $\Psi_0 = 25^\circ$ (i.e., $\kappa = 0.503$). The exact solution is plotted in the $xy$-plane, and the solitons are propagating to the left. (The experimental figures by courtesy of Harry Yeh.)

point) were created with some delayed time. In finding those parameter values, we did not make a precise minimization of a certain error function, like the one given in (7.3). In a future communication, we hope that we will be able to develop a method to determine the exact solution from the experimental data, that is, the inverse problem of the KP equation.

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References

[1] Ablowitz M J and Segur H 1981 Solitons and the Inverse Scattering Transform (SIAM Studies in Applied Mathematics) (Philadelphia: SIAM)
[2] Benney J and Luke J C 1964 Interactions of permanent waves of finite amplitude J. Math. Phys. 43 309–13
[3] Biondini G and Chakravarty S 2006 Soliton solutions of the Kadomtsev–Petviashvili II equation J. Math. Phys. 47 033514
[4] Biondini G and Kodama Y 2003 On a family of solutions of the Kadomtsev–Petviashvili equation which also satisfy the Toda lattice hierarchy J. Phys. A: Math. Gen. 36 10519–36
[5] Boiti M, Pempinelli F, Pogrebkov A K and Prinari B 2001 Towards an inverse scattering theory for non-decaying potentials of the heat equation Inverse Problems 17 937–57
[6] Chakravarty S and Kodama Y 2008 Classification of the line-solitons of KP II J. Phys. A: Math. Theor. 41 275209
[7] Chakravarty S and Kodama Y 2008 A generating function for the N-soliton solutions of the Kadomtsev–Petviashvili II equation Contemp. Math. 471 47–67
[8] Chakravarty S and Kodama Y 2009 Soliton solutions of the KP equation and application to shallow water waves Stud. Appl. Math. 123 83–151
[9] Chakravarty S and Kodama Y 2010 Line-soliton solutions of the KP equation AIP Conf. Proc. 1212 312–41
[10] Corteel S 2007 Crossing and alignments of permutations Adv. Appl. Math. 38 149–63
[11] Courant R and Friedrichs K O 1948 Supersonic Flow and Shock Waves (New York: Intersciences)
[12] Duan W-S, Shi Y-R and Hong X-R 2004 Theoretical study of resonance of the Kadomtsev–Petviashvili equation Phys. Lett. A 323 89–94
[13] Folkes P A, Ikezi H and Davis R 1980 Two-dimensional interaction of ion-acoustic solitons Phys. Rev. Lett. 45 902–4
[14] Freeman N C and Nimmo J C J 1983 Soliton-solutions of the Korteweg–deVries and Kadomtsev–Petviashvili equations: the Wronskian technique Phys. Lett. A 95 1–3
[15] Funakoshi M 1980 Reflection of obliquely incident solitary waves J. Phys. Soc. Japan 49 2371–9
[16] Hirota R 2004 The Direct Method in Soliton Theory (Cambridge: Cambridge University Press)
[17] Kadomtsev B B and Petviashvili V I 1970 On the stability of solitary waves in weakly dispersive media Sov. Phys.—Dokl. 15 539–41
[18] Kako F and Yajima N 1980 Interaction of ion-acoustic solitons in two-dimensional space J. Phys. Soc. Japan 49 2063–71
[19] Kako F and Yajima N 1982 Interaction of ion-acoustic solitons in multi-dimensional space II J. Phys. Soc. Japan 51 311–22
[20] Kao C-Y and Kodama Y 2010 Numerical study on the KP equation for non-periodic waves Math. Comput. Simul. at press (arXiv:1004.0407)
[21] Kato S, Takagi T and Kawahara M 1998 A finite element analysis of Mach reflection by using the Boussinesq equation Int. J. Numer. Meth. Fluids 28 617–31
[22] Kharif C, Pelinovsky E and Slunyaev A 2009 Rogue Waves in the Ocean (Advances in Geophysical and Environmental Mechanics and Mathematics) (Berlin: Springer)
[23] Kodama Y 2004 Young diagrams and N-soliton solutions of the KP equation J. Phys. A: Math. Gen. 37 11169–90
[24] Kodama Y, Oikawa M and Tsuji H 2009 Soliton solutions of the KP equation with V-shape initial waves J. Phys. A: Math. Theor. 42 312001
[25] Kodama Y and Shipman B 2008 The finite non-periodic Toda lattice: a geometric and topological viewpoint arXiv:0805.1389
[26] Matveev V B 1979 Darboux transformation and explicit solutions of the Kadomtsev–Petviashvili equation, depending on functional parameters Lett. Math. Phys. 3 213–6
[27] Medina E 2002 An N soliton resonance for the KP equation: interaction with change of form and velocity Lett. Math. Phys. 62 91–9
[28] Melville W K 1980 On the Mach reflection of a solitary wave J. Fluid Mech. 98 285–97
[29] Miles J W 1977 Obliquely interacting solitary waves J. Fluid Mech. 79 157–69
[30] Miles J W 1977 Resonantly interacting solitary waves J. Fluid Mech. 79 171–9
[31] Milewski P A and Keller J B 1996 Three dimensional water waves Stud. Appl. Math. 37 149–66
[32] Miwa T, Jimbo M and Date E 2000 Solitons: Differential Equations, Symmetries and Infinite-Dimensional Algebras (Cambridge: Cambridge University Press)
[33] Newell A C 1985 Solitons in Mathematics and Physics (CBMS-NSF Regional Conference Series in Applied Mathematics 45) (Philadelphia: SIAM)
[34] Newell A C and Redekopp L G 1977 Breakdown of Zakharov–Shabat theory and soliton creation Phys. Rev. Lett. 38 377–80
[35] Nagasawa T and Nishida Y 1983 Virtual states in strong interactions of plane ion-acoustic solitons Phys. Rev. A 28 3043–50
[36] Novikov S, Manakov S V, Pitaevskii L P and Zakharov V E 1984 Theory of Solitons: The Inverse Scattering Method (Contemporary Soviet Mathematics) (New York and London: Consultants Bureau)
[37] Ohkuma K and Wadati M 1983 The Kadomtsev–Petviashvili equation, the trace method and the soliton resonance J. Phys. Soc. Japan 52 749–60
[38] Oikawa M and Tsuji H 2006 Oblique interactions of weakly nonlinear long waves in dispersive systems Fluid Dyn. Res. 38 868–98
[39] Perroud P H 1957 The solitary wave reflection along a straight vertical wall at oblique incidence Tech. Rep. 99/3 Institute of Engineering Research, Wave Research Laboratory, University of California, Berkeley, 93pp
[40] Peterson P, Soomere T, Engelbrecht J and van Groesen E 2003 Interaction solitons as a possible model for extreme waves in shallow water Nonlinear Proc. Geophys. 10 503–10
[41] Porubov A V, Tsuji H, Lavrenov I L and Oikawa M 2005 Formation of the rogue wave due to non-linear two-dimensional waves interaction Wave Motion 42 202–10
[42] Postnikov A 2006 Total positivity, Grassmannians, and networks arXiv:math.CO/0609764
[43] Sato M 1981 Soliton equations as dynamical systems on an infinite dimensional Grassmannian manifold RIMS Kokyuroku (Kyoto University) 439 30–46
[44] Satsuma J 1976 N-soliton solution of the two-dimensional Korteweg–de Vries equation J. Phys. Soc. Japan 40 286–90
[45] Satsuma J 1979 A Wronskian representation of N-soliton solutions of nonlinear evolution equations J. Phys. Soc. Japan 46 359–60
[46] Tanaka M 1993 Mach reflection of a large-amplitude solitary wave J. Fluid Mech. 248 637–61
[47] Soomere T 2004 Interaction of Kadomtsev–Petviashvili solitons with unequal amplitudes Phys. Lett. A 332 74–8
[48] Soomere T and Engelbrecht J 2005 Extreme evaluation and slopes of interacting solitons in shallow water Wave Motion 41 179–92
[49] Tsuji H and Oikawa M 2007 Oblique interaction of solitons in an extended Kadomtsev–Petviashvili equation J. Phys. Soc. Japan 76 8401–8
[50] Whitham G B 1974 Linear and Nonlinear Waves (Pure and Applied Mathematics) (New York: Wiley)
[51] Williams L K 2005 Enumeration of totally positive Grassmann cells Adv. Math. 190 319–42
[52] Yeh H, Li W and Kodama Y 2010 Mach reflection and KP solitons in shallow water Eur. Phys. J. (Special Topics) 185 97–111 (arXiv:1004.0370)