The dipole coupling of atoms and light in gravitational fields

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Abstract

The dipole coupling term between a system of N particles with total charge zero and the electromagnetic field is derived in the presence of a weak gravitational field. It is shown that the form of the coupling remains the same as in flat space-time if it is written with respect to the proper time of the observer and to the measurable field components. Some remarks concerning the connection between the minimal and the dipole coupling are given.

1 Introduction

Although modern theories of quantized matter in curved space make predictions for extreme situations like, e.g., the very early universe still very few experiments are done to shed light on the connection between general relativity and quantum mechanics. The recent progress in atomic interferometry may lead to new contributions on this topic. For example, the Sagnac phase [1] and the influence of the earth’s acceleration [2] were measured showing the behaviour of atoms in non inertial frames of reference. These experiments, which use the interaction between lasers and atoms to split and recombine the atomic beam, may also be of use to measure the influence of space-time curvature on atoms [3]. It is therefore of interest to study the behaviour of such devices in weak gravitational fields like that of the earth. In order to do so it is necessary to generalize the theoretical methods which are used for the description of atoms and lasers from a flat space-time to a curved one.

In quantum optics it is often convenient to use the dipole coupling \(-\vec{d} \cdot \vec{E}\) instead of the minimal coupling scheme in the calculations. While the latter is invariant under Lorentz transformations the dipole coupling has the advantage that it is directly related to the physical electric field \(\vec{E}\) and not to the gauge dependent vector potential \(A_\mu\). The (non relativistic) equivalence of the two approaches was first demonstrated by M. Göppert-Mayer [4]. She showed that the classical Lagrangians of the two theories are related by a canonical transformation. Power and Zienau [5] have extended her work by use of a unitary transformation in quantum theory and derived, in an approximation, a multipolar Hamiltonian where the dipole coupling is only the first order interaction term of a multipole series. This transformation was made exact by Woolley (see, e.g., Ref. [6]). All these derivations are formulated in a fixed non-covariant gauge, mostly the Coulomb gauge. The generalization to an arbitrary gauge fixing was done by Power and Thirunamachandran [7] by adding a total time derivative to the classical Lagrangian and by Woolley [8].

Despite the formal equivalence of both couplings there has been a long discussion in the literature which one is better suited to describe the interaction between matter and light. Lamb [9] stated that the dipole interaction is preferable because the results of certain calculations fit better to the experiment. The minimal coupling scheme on the other hand is covariant and connected with a gauge symmetry, facts which seem to imply that this coupling is more fundamental. Ackerhalt and Milonni [10] have pointed out that this controversy may arise from the necessity to transform also the states if one applies a unitary transformation. This leads to the question for which coupling scheme the textbook wavefunctions are the right choice. We will argue in this paper that the textbook wavefunctions seem to belong to the dipole coupling.

The main subject however is to study the modification of the dipole coupling in weak gravitational fields. Our strategy is to begin with a general relativistic Lagrangian for a multiparticle system in an electromagnetic field, to consider the limit of small velocities, and to perform the Power-Zienau transformation along the
lines of Ref. [7]. This approach has two advantages: one can avoid to use the Coulomb gauge which is not covariant, and one circumvents the inclusion of gauge constraints like, e.g., the Gupta-Bleuler condition which are not common in quantum optics. One disadvantage is that the spin has to be treated separately. To the knowledge of the author the dipole coupling in gravitational fields was addressed only by Bordé et al. [11, 12] by replacing the space-indices of the non-covariant expression by tetrad indices. In this paper it is the aim to derive its structure from an underlying multi-particle theory.

We use the conventions of Ref. [13] for general relativity, i.e. $\text{sgn } g_{\mu\nu} = +2$. Greek indices run from 0 to 3 and latin ones from 1 to 3. Summation is understood whenever an index appears twice. Tetrad indices are underlined. We use natural units ($\hbar = c = 1$) and Heaviside-Lorentz conventions for the electromagnetic field.

\section{The derivation of the Hamiltonian}

We begin with the covariant action $S$ of a system of particles and the electromagnetic field,

$$S = \int \mathcal{L} \sqrt{-g} dt \equiv \int L dt,$$

where

$$L = -\sum_{\alpha=1}^{N} \left\{ m_\alpha \left[ -g_{\mu\nu}(x_{(\alpha)}) \dot{x}_{(\alpha)}^\mu \dot{x}_{(\alpha)}^\nu \right]^{-1/2} + q_\alpha A_\mu(x_{(\alpha)}) \dot{x}_{(\alpha)}^\mu \right\} - \frac{1}{4} \int d^3 y \sqrt{-g(y)} F_{\mu\nu}(y) F^{\mu\nu}(y).$$

(2)

$\alpha$ labels the particles which travel on the trajectories $x_{(\alpha)}^\mu$. The dot denotes the derivative with respect to the coordinate time $t^0 = x_{(\alpha)}^0 = y^0$. Following Refs. [14, 15] we perform the limit of small velocities, $|\dot{x}_{(\alpha)}| \ll 1$, and consider only weak gravitational fields, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. This leads to the new Lagrangian

$$L = \sum_{\alpha} \left\{ \frac{m_\alpha}{2} h_{00} + m_\alpha h_{0i} \dot{x}_{(\alpha)}^i + \frac{m_\alpha}{2} \left[ (1 + \frac{1}{2} h_{00}) \delta_{kl} + h_{kl} \right] \dot{x}_{(\alpha)}^k \dot{x}_{(\alpha)}^l \right\} + q_\alpha A_0(x_{(\alpha)}) + q_\alpha A_i(x_{(\alpha)}) \dot{x}_{(\alpha)}^i + \frac{1}{2} \int d^3 y \left\{ (1 + \frac{1}{2} h_{\lambda\lambda}) \delta_{kl} - h_{kl} \right\} \times$$

$$\left( A_{k,0} - A_{0,k} \right)(A_{l,0} - A_{0,l}) - \left[ 1 - \frac{1}{2} h_{\lambda\lambda} \delta_{kl} + h_{kl} \right] B_k B_l - 2(A_{i,0} - A_{0,i}) \varepsilon_{ijk} B_j h_{0k} \right\}.$$

Each factor of $h_{\mu\nu}$ in the sum over $\alpha$ has to be taken at the point $x_{(\alpha)}$. Throughout the paper all expressions are calculated to first order in $h_{\mu\nu}$ only. In the derivation of the Lagrangian we have subtracted the total rest energy $M = \sum_{\alpha} m_\alpha$ and have defined $B_i := \varepsilon_{ijk} A_{k,j}$ where $\varepsilon_{ijk}$ is the total antisymmetric symbol with $\varepsilon_{123} = 1$. A comma denotes the derivative with respect to the following coordinate. Here it is necessary to give a remark concerning the position of the indices. In general relativity an index which appears twice has to appear as one upper and one lower index. Any expression which does not fulfill this requirement cannot be invariant under coordinate transformations. Although the starting point of this calculation was a covariant expression we will use extensively differential geometric methods in flat three-dimensional space. It is therefore convenient to switch to a three-space notation where the indices of any three-vector $\vec{V} \equiv \{ V_i \}$ are lower case indices except for all coordinates $x_{(a)}, y, \ldots$ where, e.g., $\vec{y} \equiv \{ y^i \}$. This can be done without making errors as long as no index is moved with the space-time metric, and as long as we do not transform to another coordinate system. The resulting Hamiltonian then describes the time evolution of the particles and fields in this coordinate system.

Eq. (3) is the Lagrangian of non relativistic particles moving in weak gravitational field. If one defines a Hamiltonian $H = p \dot{x} - L$ one ends up with the minimal coupling scheme. As will be shown below this non-covariant way to define $H$ is only justified in the case of time independent gravitational fields. In order to derive the dipole approximation of the coupling we follow Ref. [16] and add a total time derivative to the
With Eq. (8) we arrive at

$$L' = L - \frac{d}{dx^0} \int d^3y \vec{P} \cdot \vec{A}$$

with

$$\vec{P}(y) = \sum_\alpha q(\alpha)(\vec{x}(\alpha) - \vec{R}_{cm}) \int_0^1 \delta(\vec{y} - \vec{R}_{cm} - \lambda(\vec{x}(\alpha) - \vec{R}_{cm})) d\lambda.$$  \hspace{1cm} (5)$$

$$\vec{R}_{cm} := \frac{1}{M} \sum_{\alpha=1}^N m_\alpha \vec{x}(\alpha)$$  \hspace{1cm} (6)

is the center of mass of the particles. This is the point where covariance is explicitly broken by adding a non-covariant term to $L$. This step is necessary as the aim, the dipole coupling, is also non-covariant. Terms like $\sqrt{-g}$ or $\sqrt{g_\Sigma}$ (where $g_\Sigma$ is the determinant of the metric on the hypersurface $x^0 = \text{const.}$) are not included in the integrand of Eq. (4) because this would destroy the derivation if the definition of $\vec{P}$ is not appropriately modified. Such alternative derivations should lead to the same result if measurable quantities are considered.

It is not difficult to show that the (flat space) polarization field $\vec{P}$ is related in the sense of distributions to the (flat space) magnetization

$$\vec{M}(y) = \sum_\alpha q_\alpha (\vec{x}(\alpha) - \vec{R}_{cm}) \times (\vec{x}(\alpha) - \vec{R}_{cm}) \int_0^1 \lambda\delta(\vec{y} - \vec{R}_{cm} - \lambda(\vec{x}(\alpha) - \vec{R}_{cm})) d\lambda$$  \hspace{1cm} (7)

and the Röntgen current $\vec{J}_{\text{Rö}} = \text{rot}(\vec{P} \times \vec{R}_{cm})$ via

$$\dot{\vec{P}} + \text{rot}\vec{M} + \vec{J}_{\text{Rö}} = \sum_\alpha q_\alpha (\dot{\vec{x}}(\alpha) - \dot{\vec{R}}_{cm}) \cdot \delta(\vec{y} - \vec{x}(\alpha)) - \dot{Q}\vec{R}_{cm} \delta(\vec{y} - \vec{R}_{cm}).$$  \hspace{1cm} (8)

In the remainder we assume the system to be neutral, $Q \equiv \sum_\alpha q_\alpha = 0$. In addition, we decompose vectors $\vec{V} = \vec{V}^\parallel + \vec{V}^\perp$ related to the electromagnetic field into its longitudinal ($\text{rot} \vec{V}^\parallel = 0$) and transverse ( $\text{div} \vec{V}^\perp = 0$) part, for example

$$\vec{A} = \vec{A}^\parallel + \vec{A}^\perp, \text{ rot}\vec{A}^\parallel = \text{div}\vec{A}^\perp = 0.$$  \hspace{1cm} (9)

Note that the spatial integral over the scalar product of any transverse vector with any longitudinal vector vanishes and that the longitudinal part of the polarization for $Q = 0$ is given by

$$\vec{P}^\parallel = \nabla V_0$$  \hspace{1cm} (10)

where

$$V_0(\vec{x}) = -\frac{1}{4\pi} \int d^3y \frac{\text{div} \vec{P}(\vec{y})}{|\vec{x} - \vec{y}|} = \frac{1}{4\pi} \sum_\alpha \frac{q_\alpha}{|\vec{x} - \vec{x}(\alpha)|}.$$  \hspace{1cm} (11)

With Eq. (8) we arrive at

$$L' = \sum_\alpha \left\{ \frac{m_\alpha}{2} h_{00} + m_\alpha \vec{h}_0 \cdot \dot{\vec{x}}(\alpha) + \frac{m_\alpha}{2} \left[ (1 + \frac{1}{2}h_{00})\delta_{kl} + h_{kl} \right] \dot{x}_{l}(\alpha) \right\} + \frac{1}{2} \int d^3y \left\{ \left[ (1 + \frac{1}{2}h_{\lambda\lambda})\delta_{kl} - h_{kl} \right] (A_{k,0} - A_{0,k}) (A_{l,0} - A_{0,l}) - \left[ (1 - \frac{1}{2}h_{\lambda\lambda})\delta_{kl} + h_{kl} \right] B_k B_l + 2(\vec{A} - \nabla A_0) \cdot (\vec{B} \times \vec{h}_0) \right\} + \int d^3y \left\{ \vec{A}^\perp \cdot [\text{rot}\vec{M} + \vec{J}_{\text{Rö}}] - \dot{\vec{A}}^\perp \cdot \vec{P}^\perp + \vec{P}^\parallel \cdot (\vec{A}^\parallel) \right\}.$$  \hspace{1cm} (12)

For notational convenience we have introduced the vector $\vec{h}_0 \equiv \{ h_{0j} \}$. The new Lagrangian $L'$ has the feature that it depends only on $\vec{A}^\parallel$, not on $A_i^\parallel$, which is therefore a cyclic variable. The corresponding Routhian (see, e.g., Ref. [16]) is given by

$$R = L' - \int d^3y A_i^\parallel \frac{\partial L'}{\partial A_i^\parallel}.$$  \hspace{1cm} (13)
The derivative in the r.h.s. is given by
\[ \frac{\partial \mathcal{L}'}{\partial A_i^\parallel} = -\Delta_i^\parallel \]
which is a constant of motion. The vector \( \Delta \) is defined by
\[ \Delta_i := - \left[ \left( 1 + \frac{1}{2} h_{\lambda \lambda} \right) \delta_{ij} - h_{ij} \right] (A_{j,0} - A_{0,j}) + (\vec{B} \times \vec{h}_0)_i + P_i. \]
and agrees in absence of a gravitational field with the electric displacement. Using Eq. (15) to eliminate the cyclic variables from \( R \) one finds an expression with no time derivative of \( A_0 \) and which depends linearly on it. Hence, \( A_0 \) plays the role of a Lagrangian multiplier. Solving the corresponding constraint leads to \( \vec{P}^\parallel - \vec{A}^\parallel = \nabla V_0 \) which implies \( \Delta^\parallel = 0 \). We thus find for the Routhian
\[ R = \sum_\alpha \left\{ \frac{m(\alpha)}{2} h_{00} + m(\alpha) \vec{h}_0 \cdot \vec{x}(\alpha) + \frac{m(\alpha)}{2} \left[ (1 + \frac{1}{2} h_{00}) \delta_{kl} + h_{kl} \right] \dot{x}^k_{(\alpha)} \dot{x}^l_{(\alpha)} \right\} - V_{coul} \]
\[ + \frac{1}{2} \int d^3y \left\{ \dot{\vec{A}}^\perp \right\}^2 - 2 \vec{B} \cdot (\vec{M} + \vec{P} \times \vec{R}) - \int d^3y (\vec{h}_\perp \cdot \vec{h}_\perp) \]
\[ - 2(\vec{P}^\parallel + \dot{\vec{A}}^\perp) \cdot (\vec{B} \times \vec{h}_0) + \left[ \frac{1}{2} h_{\lambda \lambda} \delta_{kl} - h_{kl} \right] (\vec{P}^\parallel + \dot{\vec{A}}^\perp)_k (\vec{P}^\parallel + \dot{\vec{A}}^\perp)_l \]
where
\[ V_{coul} = \frac{1}{8\pi} \sum_{\alpha,\beta} \frac{q_{\alpha} q_{\beta}}{|\vec{x}(\alpha) - \vec{x}(\beta)|} = \frac{1}{2} \int d^3y (\vec{P}^\parallel)^2 \]
is the total Coulomb interaction between all charges. One may interpret this explicit occurence of the Coulomb interaction between the particles as an indication that the textbook wavefunctions for atomic electrons, which are usually derived by assuming the interaction to be of the Coulomb-type, belong to the dipole coupling.

The Hamiltonian is defined by the usual relation
\[ H = \sum_{\alpha=1}^N \vec{p}(\alpha) \cdot \dot{\vec{x}}(\alpha) + \int d^3y \vec{\Pi}^\perp \cdot \dot{\vec{A}}^\perp - R \]
(18)
where \( \vec{p}(\alpha) \) and \( \vec{\Pi}^\perp \) are the canonical momenta of the particles and the transverse electromagnetic field, respectively. Performing the calculations and neglecting terms of the order \( h_{\mu \nu}/m_\alpha \) we arrive at
\[ H = \sum_{\alpha=1}^N m_\alpha \left( 1 - \frac{1}{2} h_{00}(x_{(\alpha)}) \right) + \sum_{\alpha=1}^N \frac{1}{2m_\alpha} \left( \vec{p}(\alpha) - m_\alpha \vec{h}_0(x_{(\alpha)}) - \int d^3y \vec{B} \times \vec{n}_\alpha \right)^2 \]
\[ + \frac{1}{2} \int d^3y \left\{ \left[ (1 + \frac{1}{2} h_{\lambda \lambda}) \delta_{kl} + h_{kl} \right] \left[ (\Delta^\perp_k \Delta^\perp_l + B_k B_l) - 2 \Delta^\perp_k P_l + P_k P_l \right] \right\} \]
\[ + 2(\vec{P} - \vec{A}^\perp) \cdot (\vec{B} \times \vec{h}_0) \]
(19)
where the vectors \( \vec{n}_\alpha \) are defined by
\[ \vec{n}_\alpha(\vec{y}) := \frac{m_\alpha}{M} \left[ \vec{P}(\vec{y}) - \int \sum_{\beta} \frac{q_{\beta} \vec{r}_{(\beta)}(\vec{y})}{\lambda(\vec{y}) - \vec{R}_{cm} - \lambda \vec{r}_{(\beta)}} \right] + q_{\alpha} \vec{r}_{(\alpha)} \]
(20)
In the last equation we have introduced the nonrelativistic relative coordinates and the center of mass position
\[ \vec{r}_{(\alpha)} := \vec{x}_{(\alpha)} - \vec{R}_{cm} \quad \vec{R}_{cm} := \frac{1}{M} \sum_{\alpha=1}^N m_\alpha \vec{x}_{(\alpha)} \]
(21)
which will be of use in the derivation of the dipole approximation in the following section.

Note that this Hamiltonian contains explicitly the (modified) Coulomb potential. This can be seen by using Eq. (17) for the evaluation of the term proportional to \( P_k P_l \) in Eq. (19); we will discuss the modifications to the Coulomb potential below. The term proportional to \( \Delta_k^i P_l i \) describes the modified dipole coupling between the transverse electric displacement and the total dipole momentum of the atom.

3 The dipole approximation

To gain more physical insight into the result (19) it is of advantage to perform the dipole approximation which is valid for many quantum optical applications. In our context it amounts in the restriction to terms which are at most linear in the relative coordinates \( \vec{r} \). In this limit the polarization and the magnetization become

\[
\vec{P}(y) = \vec{d} \delta(\vec{y} - \vec{R}_{cm}) , \quad \vec{M}(y) = \frac{1}{2} \sum_{\alpha} q_{(\alpha)} \vec{r}_{(\alpha)} \times \vec{r}_{(\alpha)} \delta(\vec{y} - \vec{R}_{cm})
\]

where

\[
\vec{d} := \sum_{\alpha=1}^{N} q_{\alpha} \vec{r}_{(\alpha)}
\]

is the dipole operator of the atom. It is convenient to exploit the fact that the Hamiltonian transforms as a scalar under a change of the dynamical variables. We take as new variables of the particles the center of mass position \( \vec{R}_{cm} \) and the relative coordinates \( \vec{r}_{(\alpha)} \) of the first \( N - 1 \) particles. Denoting the corresponding canonical momenta as \( \vec{p}_{cm} \) and \( \hat{\vec{p}}_{(\alpha)} \) it is not difficult to show that these are related to the old momenta by

\[
\hat{\vec{p}}_{(\alpha)} = \frac{m_{\alpha}}{M} \left( \vec{P}_{cm} - \sum_{\beta=1}^{N-1} \hat{\vec{p}}_{(\beta)} \right) + \hat{\vec{p}}_{(\alpha)} , \quad \alpha = 1, \ldots, N - 1
\]

\[
\hat{\vec{p}}_N = \frac{m_N}{M} \left( \vec{P}_{cm} - \sum_{\beta=1}^{N-1} \hat{\vec{p}}_{(\beta)} \right)
\]

Inserting this into Eq. (19) and performing the dipole approximation we can write the Hamiltonian as the sum

\[
H = H_{at} + H_{cm} + H_{rot} + H_{rad} + H_{int}
\]

of five parts. The first is the internal Hamiltonian

\[
H_{at} = \sum_{\alpha=1}^{N-1} \left( \hat{\vec{p}}_{(\alpha)} - \frac{m_{(\alpha)}}{M} \sum_{\beta=1}^{N-1} \hat{\vec{p}}_{(\beta)} \right)^2 + V_{coul} - h_{0k,l}(\vec{R}_{cm}) \sum_{\alpha=1}^{N-1} r_{(\alpha)}^l \vec{\hat{p}}_{(\alpha)k}
\]

\[
+ \frac{1}{2} \int \left( \frac{1}{2} h_{\lambda\lambda} \delta_{kl} - \delta_{kl} \right) \left( \vec{P}_k^l \vec{P}_l^k + \vec{R}_k^l \vec{R}_l^k + 2 \vec{R}_k^l \vec{P}_l^k \right)
\]

which does not depend on the transversal electromagnetic field and \( \vec{P}_{cm} \). The first sum is the kinetic energy term in which the subtraction of the sum over \( \beta \) describes the generalization of the reduced mass for more than two particles. The last term in the second line contains modifications to the Coulomb potential and to the dipole energy. It should be noted that the Coulomb potential contains the self-energy of the particles and is therefore divergent. This is also the case for the term proportional to \( \langle \vec{P}_{\perp} \rangle^2 \) which describes the dipole self-energy. The last sum in the first line describes the coupling of the internal angular momentum to a rotation. This can be seen by switching to the Fermi coordinates of an rotating observer in the weakly curved space [17]. In this coordinate system the components \( h_{0i}(\vec{x}) \) are essentially given by \( \varepsilon_{ijk} \omega^j x^k \) where \( \omega^j \) is the angular velocity of the observer. Inserting this into Eq. (26) shows that the corresponding term is of the form \(-\hat{\vec{\omega}} \cdot \sum_{\alpha} \vec{L}_{(\alpha)} \) if \( \vec{L}_{(\alpha)} \) is the orbital angular momentum of particle \( \alpha \).
The center of mass contribution
\[ H_{cm} = \frac{1}{2M} \tilde{p}^2_{cm} - \frac{M}{2} h_{00}(\tilde{R}_{cm}) - \tilde{p}_{cm} \cdot \tilde{h}_0(\tilde{R}_{cm}) \] (27)
has the same structure as the Hamiltonian for a free particle in a weakly curved space, comp. Refs. [14, 15].

The internal and external degrees of freedom of the atom are coupled via the Röntgen term
\[ H_{R\beta} = -\frac{1}{M} \tilde{p}_{cm} \cdot (\tilde{B} \times \tilde{d}) . \] (28)

This is the same expression as in Minkowski space because we have neglected all terms of the order \( h_{\mu\nu}/m_{(\alpha)} \).

The radiative part of \( H \) is found to be
\[ H_{rad} = \frac{1}{2} \int d^3y \left\{ \left( 1 - \frac{1}{2} h_{\lambda\lambda} \right) \delta_{kl} + h_{kl} \right\} \left( \Delta^k \Delta^l + B_k B_l \right) - 2 h_{00} \cdot (\Delta^\perp \times \tilde{B}) \right\} \] (29)
and contains a coupling between the Poynting vector and the rotation. Here we have used the relation \( \Pi^\perp = -\Delta^\perp \).

The interaction between matter and radiation is described by

\[ H_{int} = -\left\{ \left( 1 - \frac{1}{2} h_{\lambda\lambda} \right) \delta_{kl} + h_{kl} \right\} \Delta^k d_l - \tilde{B} \cdot \left\{ \sum_{\alpha=1}^{N-1} \frac{q_{(\alpha)}}{2m_{(\alpha)}} \left( \tilde{r}_{(\alpha)} \times \left( \tilde{p}_{(\alpha)} - \frac{m_{(\alpha)}}{M} \sum_{\beta=1}^{N-1} \tilde{p}_{(\beta)} \right) \right) \right\} . \] (30)

The first term describes the dipole coupling between the atom and the transverse electromagnetic field. The second term is the well known coupling between the magnetic field and the angular momentum of the particles. As in Eq. (28) the mass reduction has to be taken into account. Again this term is the same as in flat space since we have neglected terms of the order of \( O(h_{\mu\nu}/m_{(\alpha)}) \).

4 The dipole coupling between measurable quantities

To make contact with the measurable quantities of the theory it is necessary in general relativity to consider the components of each tensor with respect to a tetrad field \( e^\mu_\alpha \) which fulfills \( e^{\mu}_\alpha e^\beta_\mu = \eta_\alpha^\beta \) at each point in space. Here \( \eta_\alpha^\beta \) is the Minkowski metric. The measured components of, e.g., the electric field are then given by
\[ E^\lambda_\mu = F^\mu_\nu = e^\nu_\mu e^{\nu}_\lambda F_{\mu\nu} \] (31)
(Tetrad indices are raised and lowered with the Minkowski metric). We now focus on the dipole term in Eq. (30). The vector \( e^\mu_\alpha \) of the tetrad is assumed to be orthogonal to the hypersurfaces \( x^\alpha = \text{const.} \) and is therefore given by \( e^\mu_\alpha = 1 + h_{00}/2, \) the rest of its components vanish. The orthogonality of the tetrad then implies \( e^\alpha_0 = 0 \) for all three space-like vectors \( e_\beta \).

Consider now the dipole term in Eq. (30) and insert Eq. (31). The dipole coupling then has the form
\[ -\left( -F_{0l} + (\tilde{B} \times \tilde{h}_0)_l + \left( 1 - \frac{1}{2} h_{\lambda\lambda} \right) \delta_{kl} + h_{kl} \right) \tilde{P}_k \right\}^\perp d_l . \] (32)
Since the longitudinal part of the vector \( \Delta \) is zero we can omit in this expression the index \( \perp \). First we examine the term proportional to \( F_{0l} \). Recalling that the dipole moment is the sum of relative coordinates \( r_{(\alpha)}^\perp \) it is not difficult to find
\[ F_{0l} \sum_{\alpha} q_{(\alpha)} r_{(\alpha)}^l = \left(1 - h_{00}/2 \right) F_{0l} \sum_{\alpha} q_{(\alpha)} r_{(\alpha)}^l = -\left(1 - h_{00}/2 \right) E_0 \sum_{\alpha} q_{(\alpha)} r_{(\alpha)}^l . \] (33)

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3 To achieve true invariant quantities it would be necessary to work with the derivative \( \partial_{\mu}(s^2_{(\alpha)})/2 \) of the geodesic distance \( s_{(\alpha)} \) between the center of mass of the atom and the particle \( \alpha \) instead of the relative coordinate \( r_{(\alpha)}^\perp \) of the particle. But since we are working in the dipole approximation (up to linear terms in \( r_{(\alpha)}^\perp \)) we have \( r_{(\alpha)}^\perp \approx g_{\mu\nu}(R_{cm}) \partial_{\mu}(s^2_{(\alpha)})/2 \) so that the difference is unimportant.
If we interpret
\[ P_i^G := (1 + h_{00}/2) \left[ \left(1 - \frac{1}{2}h_{\lambda\lambda}\right) \delta_{kl} + h_{kl} \right] P_k + (\vec{B} \times \vec{h}_0)_l \] (34)
as the actual polarization field modified by the gravitational field we find for the dipole term the expression
\[ H_{dip} = -(1 - h_{00}/2)D^l_\perp d^l. \] (35)
Compared to flat space the dipole coupling seems to be modified by the factor \((1 - h_{00}/2)\) which is identical to \(\sqrt{-g_{00}}\). But even this term disappears if we take into account that the Hamiltonian \((25)\) describes the time evolution of the system with respect to the coordinate time \(x^0\). If we switch to the proper time \(\tau\) of a family of observers with four-velocity \(u^\mu = e^\mu_0\) the Hamiltonian density \(H\), defined by \(H = \int \mathcal{H} d^3y\), has to be transformed according to
\[ \mathcal{H}' = \mathcal{H} \frac{dx^0}{d\tau} = \mathcal{H} e^0_\perp = \mathcal{H} \frac{1}{\sqrt{-g_{00}}}. \] (36)
This implies that the dipole coupling between the atom and the electric field takes the form
\[ H_{dip}' = -D^l_\perp d^l \] (37)
even in a weakly curved space when it is expressed with respect to the measured quantities and the proper time of an observer. This result is in agreement with a recent work of Lämmerzahl \([19]\) in which he derives the dipole coupling for a quantum mechanical particle moving in a PPN space-time.

We now discuss briefly the modifications to the Coulomb potential as this is of interest for the decision to which form of coupling the textbook wavefunctions belong (see the remark below Eq. (17)). It is of advantage to start with the expression
\[ V := \frac{1}{2} \int d^3y \left[ (1 - \frac{1}{2}h_{\lambda\lambda}) \delta_{kl} + h_{kl} \right] P_k P_l \] (38)
in Eq. (19) which contains the Coulomb as well as the dipole energy of the atom. In a first step we replace the flat space polarization \(P_l\) by its gravitational counterpart \(P_i^G\) and arrive at
\[ V = \frac{1}{2} \int d^3y \left( 1 - h_{00} \right) \left[ (1 + \frac{1}{2}h_{\lambda\lambda}) \delta_{kl} - h_{kl} \right] P_k^G P_l^G - 2 \vec{P}^G \cdot (\vec{B} \times \vec{h}_0) \right\}. \] (39)
The last term can be interpreted as a gravitational induced interaction between the atom and the magnetic field and is therefore not a part of the Coulomb or dipole energy; we will omit this part. Performing the same steps as for the dipole coupling we find that the first term has, after the insertion of the tetrad vectors, the form
\[ \frac{1}{2} \int d^3y (1 + h_{\lambda\lambda}/2) P^G_\perp P^G_\perp. \] (40)
It only left to replace the coordinate time by the proper time of the observers according to Eq. (36). This leads to the result that the sum of Coulomb and dipole energy is given by
\[ \frac{1}{2} \int dV P_\perp^G P_\perp^G \] (41)
where \(dV = (1 + h_{ii}/2) d^3y = \sqrt{g_{00}} d^3y\) is the three-volume element of the hypersurface \(x^0 = \text{constant}\). It has therefore the same form as in flat space if it is written with respect to the measurable quantities and with respect to the proper time.

We thus find the result that in connection with the dipole coupling the modifications of the Coulomb potential are related to the modifications of the (longitudinal part of) the polarization field. If the calculation of the modifications in the minimal coupling scheme are different from the present result this would, in principle, open a new way to test experimentally to which coupling the textbook wavefunctions belong. It should be noted, however, that the magnitude of the modifications in atomic systems is far too small to be measurable.
This section will be closed with some general remarks concerning the use of the family of observers to define the tetrads and therefore the measurable quantities. In principle it would be more convenient to work in the atom’s frame of reference instead of including the family of observers in the description. But in general relativistic situations a frame of reference cannot be unambiguously defined. Usually one uses Fermi coordinates as a local approximation for the reference frame, but this construction has some shortcomings. A modification of it may circumvent these problems, but the true form of a reference frame, if it exists, has to be determined by experiments.

A second remark concerns the fact that the Hamiltonian is changed if we switch to the proper time of the family of observers. This is due to the fact that the Hamilton operator is the time evolution operator of the system. If we change the time coordinate then the evolution operator is also changed. The use of a proper time is of advantage compared to the coordinate time since it is directly measurable by the clocks of the observers.

5 The spin interaction

In the derivation of the Hamiltonian the inclusion of the spin was not possible because only classical particles were considered. But the heart of the Power-Zienau transformation, the subtraction of a total time derivative from the Lagrangian, can be made without any reference to the spin. Hence, it should be possible to include the spin by deriving the minimal coupling Hamiltonian and by taking over the spin terms into the Hamiltonian from the Lagrangian, e.g., Ref. [18] with the unitary operator exp(iS).

The corrections to the Pauli equation can be found by performing a Foldy-Wouthuysen transformation (see, e.g., Ref. [8]) with the unitary operator exp(iS) where the Hermitean operator S is given by

$$S = \frac{1}{2m} \gamma_l(-i\partial_t - qA_i) - \frac{1}{4m} \Sigma_{ik} \gamma_k(-i\partial_t - qA_i) + \frac{i}{8m} \gamma_j h_{ki,j}$$

The Hamiltonian for the Schrödinger field is then found to be

$$H = \frac{1}{2m}(-i\partial_t - qA_i - mh_{0i})^2 - qA_0 + m \left(1 - \frac{1}{2} h_{00}\right) - \frac{1}{4} h_{0l,i} \epsilon_{ilk} \sigma_k - \frac{q}{2m} B_k \sigma_k$$

This is the same result as in Refs. [14, 15] except for the inclusion of the spin-gravity interaction

$$- \frac{1}{4} h_{0l,i} \epsilon_{ilk} \sigma_k.$$  

Here σ_k are the Pauli matrices of the particle under consideration, say particle α. In the discussion of the result [23] we have seen that the term h_{0l,i} is essentially given by ε_{lmi}ω^m with the angular velocity ω^m of the observer. It is therefore obvious that this term describes the $-\vec{\omega} \cdot \vec{S}$ coupling of the spin part $\vec{S}$ of the total angular momentum to the rotation $\vec{\omega}$.

The derivation along the lines of Ref. [3] shows also that the Hamiltonian is only related to a relativistic Hermitian energy operator if the gravitational field is time independent. Since the spinless part of it is exactly the same as the canonical Hamiltonian, and since the canonical procedure is not manifest covariant, we conclude that the same condition must hold for the Hamiltonian [23].
6 Discussion

In this paper we have shown that the interaction between atomic particles and the electromagnetic field can be described with a dipole coupling term also if space-time is weakly curved. The form of the coupling remains the same as in Minkowski space if it is written with respect to the proper time of an observer and the measurable quantities of the theory. The same is true for the Coulomb and the dipole energy of the atom.

The central assumptions in this derivation are the smallness of the velocity of each particle, the weakness of the gravitational field, and the validity of the dipole approximation. A further assumption was tacitly made by using the non-relativistic form of the center-of-mass coordinates. Although this should in general be a good approximation, Fischbach et. al. \cite{22} have shown that the use of different relative coordinates (center-of-energy, e.g.) can result in different perturbational contributions of the weak gravitational field which do not vanish even when the mass of the nucleus is very large. We assume that these contributions are small enough to be neglected.

It is interesting to make a comparison of the present results with the well known formal equivalence between the Maxwell-field in a curved space and a dielectric medium \cite{23}. In this approach one defines a formal dielectric displacement vector to describe the influence of gravity on the Maxwell field. In absence of particles, i.e. for vanishing polarisation $\vec{P}$, the formal electric displacement agrees with the vector $\Delta$ defined above (in Ref. \cite{23} the presence of charged matter was not considered). Also the coupling of the Poynting vector to the rotation occurs in the energy density of the formal Maxwell field.

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