The spin structure of the proton from lattice QCD

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Abstract. This proceeding discusses recent achievements in lattice QCD calculations of the longitudinal and transverse spin structure of the nucleon. Starting from the latest lattice results on moments of generalized parton distributions, we focus on the decomposition of the proton spin $1/2$ in terms of quark (and gluon) spin and orbital angular momentum contributions. Recent results on the transverse nucleon spin structure obtained from lattice studies of transverse momentum distributions will be discussed and illustrated in the form of spin densities of quarks in the proton. We compare the results with expectations from phenomenological and model studies, and point out the most important systematic uncertainties and remaining challenges in the lattice approach.

1. Introduction
In this contribution, we illustrate the advances that have been made in lattice QCD calculations of hadron structure by presenting and discussing a small number of results from recent dynamical lattice studies on the longitudinal and transverse spin structure of the nucleon. For more detailed accounts of the progress that has been made on the lattice, in particular with respect to the quark and gluon structure of the pion and the nucleon, we refer to [1, 2, 3, 4, 5, 6].

We begin with a presentation of moments of generalized parton distributions (GPDs) obtained by the LHP collaboration, and then concentrate on the impact of the results on the nucleon spin sum rule. The final section deals with new results on the transverse spin structure of the nucleon obtained from first lattice studies of transverse momentum distribution (TMDs).

2. Longitudinal spin structure of the proton
Since the famous EMC measurement [7] of the spin structure function $g_1^p$ more than two decades ago at the latest, the quest for a solid understanding of the origin of the nucleon spin $1/2$ has been at the center of countless experimental, theoretical and phenomenological investigations. These studies provided remarkable insights into the quark and gluon structure of the nucleon, often on the basis of progress made with respect to elastic and deeply inelastic lepton-hadron- and related scattering processes.

Generalized parton distributions (GPDs)\textsuperscript{2} turned out to provide a particularly successful framework for the understanding of hadron structure, which not only encompasses the well

\textsuperscript{1} that gave rise to what is known as the proton spin crisis

\textsuperscript{2} for reviews, we refer to Refs. [8, 9, 10]
known hadron form factors and the PDFs, but which also provides a solid basis for the decomposition of the nucleon spin in terms of spin and (orbital) angular momentum contributions of quarks and gluons. According to [11], the nucleon spin can be given in terms of \( x \)-moments of the GPDs \( H(x,\xi,t) \) and \( E(x,\xi,t) \),

\[
\frac{1}{2} = \sum_{q,g} J_{q,g} = \sum_{q,g} \frac{1}{2} \int_{-1}^{1} dx x \{ H^{q,g}(x,\xi,t) + E^{q,g}(x,\xi,t) \}_{t=0} = \sum_{q,g} \frac{1}{2} (A^{q,g}_{20}(t) + B^{q,g}_{20}(t))_{t=0},
\]

where \( A_{20} \) and \( B_{20} \) are generalized form factors (GFFs) that will be discussed in greater detail below. It is important for the following discussion to keep in mind that the total angular momentum of quarks can be further decomposed in a manifestly gauge-invariant manner in terms of quark spin and orbital angular momentum contributions, \( J_q = \Delta \Sigma/2 + L_q \) [11].

While GPDs can be measured for example in deeply virtual Compton scattering employing QCD factorization, their extraction from experimental data turns out to be very challenging. This holds in particular for their dependence on \( x \) at fixed longitudinal momentum transfer \( \xi \), and therefore also for the \( x \)-moments in the decomposition in Eq. (1). As it turns out, this is different in the framework of lattice QCD, which represents at least in principle a straightforward (but in practice still challenging) approach from first principles to the lowest \( x \)-moments of PDFs and GPDs.

2.1. Moments of GPDs

Significant advances in particular with respect to moments of GPDs have been made since the pioneering calculations by the LHPC and QCDSF collaborations in 2003 [12, 13]. A comprehensive lattice study of GPDs by LHPC using a mixed action approach was presented in Ref. [14]. The discussions below will be based on a recent update of this study [15], which includes an additional data set with a pion mass of \( m_\pi \approx 300 \) MeV, a factor of 8 increased statistics, and a strongly improved statistical analysis. With respect to isosinglet observables, we note that only the quark line connected contributions have been taken into account (see also footnote 3). The \( x \)-moments of generalized parton distributions can be written in terms of polynomials in powers of \( x \) with the GFFs as coefficients. For the \( n=2-(x^2) \)-moments in the unpolarized case, the relevant GFFs \( A_{20}, B_{20}, C_{20} \) parametrize the off-forward matrix element.
of the (symmetric, traceless) energy momentum tensor, e.g. for quarks

\[
\langle P'|\overline{q}\gamma^{\mu}D^{\nu}q|P \rangle = \langle P'|\mathcal{T}_{\mu\nu}^{q}|P \rangle = \overline{U}(P') \left\{ \gamma^{\mu \nu} A_{20}(t) - \frac{i\Delta_{\mu \nu}}{2m_N} B_{20}(t) + \frac{\Delta_{\mu} \Delta_{\nu}}{m_N} C_{20}(t) \right\} U(P),
\]

where \( \mathcal{T}_{\mu\nu}^{q} = (P' - P)/2, \Delta = P' - P, \) and \( t = \Delta^2 \). Exemplary lattice results from LHPC [15] for the \( t \)-dependence of these GFFs in the isovector and isosinglet channels, for a comparatively low pion mass of \( \approx 300 \) MeV, are displayed in Fig. 1. We note at least two characteristic features: First, while the \( B_{20} \)-GFF is dominant in the isovector case, it is very small and fully compatible with zero within the uncertainties for all accessible values of \( t \) in the isosinglet channel. Conversely, the GFF \( C_{20} \) (which is directly related to the \( n = 2 \)-moment of the so-called \( D \)-term[16]) is zero within relatively small errors in the isovector channel, but clearly non-zero and negative for \( u + d \)-quarks. From the sum of \( A_{20} \) and \( B_{20} \) at zero momentum transfer, we can compute the quark angular momentum \( J_q \), cf. Eq. (1). The above findings for \( B_{20} \) obviously have a significant impact on the decomposition of the nucleon spin in terms of quark and gluon degrees of freedom.

2.2. Decomposition of the nucleon spin

Instead of calculating matrix elements of the quark orbital angular momentum (OAM) operator directly on the lattice, we can employ the relation \( L_q = J_q - \Delta \Sigma/2 \), together with corresponding results for the quark spin fraction \( \Delta \Sigma \), to compute the OAM carried by the quarks. The numerical results for \( u + d \)-quarks are shown on the left in Fig. 2 as a function of the squared pion mass. For the extrapolation of \( \Delta \Sigma_{u+d} \) to the physical pion mass, we employed the heavy baryon ChPT results of Ref. [17]. Although HBChPT at the given order is most probably not applicable at the accessible pion masses, the two-parameter fit to the lattice data points, indicated by the error band, shows a very good (but likely accidental) agreement with the values from HERMES (indicated by the cross) and COMPASS [18, 19] at \( m_{\pi}^{\text{phys}} \). We used the results of a covariant baryon chiral perturbation theory calculation [20] for the simultaneous extrapolation of the GFFs \( A_{20}, B_{20}, C_{20} \) in \( m_{\pi} \) and in \( t \), in particular to obtain \( B_{20}(t = 0, m_{\pi}^{\text{phys}}) \). The results of the fits to the lattice data were subsequently used to compute \( L_q \), which is shown on the left in Fig. 2 for \( q = u + d \) by the diamonds and the error band. Most remarkably, following
this procedure, we find a very small OAM contribution of only $L_{u+d} \approx (6 \pm 3)\%$ of 1/2 at the physical pion mass. Note that we obtain a slightly larger value, $L_{u+d} \approx (11 \pm 3)\%$, on the basis of a HBCChPT extrapolation of $J_{u+d}$ employing the results of Ref. [21]. These results may be surprising and at first sight even appear to be in clear contradiction to expectations from relativistic quark models, where the quark OAM contributes about $\approx 30 - 40\%$ to the total nucleon spin. Moreover, it has been frequently pointed out in the past that non-zero quark orbital angular momentum is strictly required with respect to certain non-vanishing single spin asymmetries related to, e.g., the Sivers effect, as well as for the Pauli form factor $F_2(Q^2)$ to be non-zero [22, 23, 24, 25].

Concerning the latter, it is important to be precise in the use of the term quark orbital angular momentum: The lattice results on the one hand correspond to the proton matrix element of the manifestly gauge invariant quark OAM operator as given in, e.g., Ref. [11], which is part of Jis nucleon spin sumrule, see Eq. (1) and the adjacent discussion. On the other hand, what is required with respect to a non-vanishing $F_2(Q^2)$ and the Sivers effect are light cone wave functions with non-zero orbital angular momentum in an overlap representation of the corresponding matrix elements. Also, the more heuristic approach to explain single spin asymmetries, e.g. the Sivers effect, proposed in Ref. [24], requires a generically non-zero quark orbital motion. The notions of OAM and orbital motion in these contexts are, however, not necessarily in one-to-one correspondence with the gauge invariant $L_q$ introduced above.

Furthermore, we note that $L_q$ is in general not positive definite. To fully appreciate the potential consequences of this basic observation, it is useful to study the individual up and down quark OAM contributions to the proton spin, which are shown on the right in Fig. 2 as functions of $m_\pi^2$, together with $\Delta \Sigma_{u,d}/2$ and the corresponding chiral extrapolations (given by the error bands). It indeed turns out that the orbital angular momentum of up and of down quarks is in each case large and of similar magnitude, but opposite in sign, and therefore nearly cancels in the sum over a wide range of pion masses. From the chiral extrapolations, we find that the individual quark OAM contributions are also substantial at the physical pion mass, amounting to $|L_{u,d}| \approx 33\%$. For an understanding of the mechanism behind the observed approximate cancellation, one probably has to study more complex correlation functions on the lattice, or get inspiration from model calculations. We also note that cancellations of this sort in the isosinglet channel are not unusual. A well known example is the anomalous magnetic moment, which is large both for the proton and for the neutron, $\kappa_p = F_2^p(0) \sim 1.79$ and $\kappa_n = F_2^n(0) \sim -1.91$, respectively. Using isospin symmetry, however, one finds that the sizeable, individual contributions from up and down quarks to $\kappa_p$ largely cancel out in the sum, $\kappa_p^{u+d} \approx -0.36$. Keeping in mind that $F_2(Q^2 = -t)$ correspond to the first, and the GFF $B_{20}(t)$ to the second $x$-moment of the GPD $E(x, \xi = 0, t)$, we also find this to be perfectly consistent with the cancellation observed above in Fig. 1, where in particular $B_{20}^{u+d}(t) \approx 0$, while $B_{20}^{u-d}(t)$ is large and positive.

In the following, we will briefly address the apparent discrepancy between the common expectations from relativistic quark model calculations of $L_q$ and the lattice QCD results in Fig. 2. To avoid additional systematic uncertainties on the side of the lattice, we will concentrate on the isovector channel $(u-d)$, where contributions from disconnected diagrams would cancel out. While lattice studies point towards a sizeable negative $L_{u-d} \sim -0.38$, relativistic quark models typically predict a sizeable positive $L_{u+d}^{\text{RQM}} \approx 0.2$. However, as has been pointed already out some time ago [26, 27], the model calculations generically correspond to a low hadronic scale $\mu \approx 1 \text{ GeV}$, while a typical renormalization scale for which the lattice results are given is $\mu \sim 2 \text{ GeV}$ (e.g. in the $\overline{\text{MS}}$-scheme). Since OAM is a scale and scheme dependent quantity [28], a naive, direct comparison of lattice and model values at the respective scales is therefore in general meaningless. Nevertheless, one might try to use QCD evolution equations to evolve the lattice values down to the lower model scale, where they may be compared to the model
Figure 3. Scale evolution of the orbital angular momentum of $u - d$ quarks [29].

results. To this extent, we show in Fig. 3 the evolution of $L_{u-d}$ at LO, NLO and NNLO [29]. One finds the interesting result that the OAM for $u - d$ quarks indeed bends upwards and eventually becomes large and positive at lower $\mu^2$. Remarkably, this trend persist at least to NNLO accuracy, although the size of the corrections going from LO to NNLO shows that the evolution equations are quantitatively not applicable at very low scales. In any case, the initial suspicion about a contradiction of lattice and model results appears to be premature.

Finally, we show a comparison of the results for the total quark angular momentum $J_q = (A_{20}^q + B_{20}^q)_{u+d}/2$, as obtained from the chiral extrapolations of $A_{20}$ and $B_{20}$ mentioned above, with earlier lattice as well as model- and phenomenological calculations in Fig. 4 in the $J_u, J_d$-plane. It turns out that the numerical values [15], $J_u = 0.236(6)$ and $J_d = 0.0018(37)$, are in surprisingly good agreement not only with the previous lattice computations, but also with most of the phenomenological and model results, at least when they are evolved to the common renormalization point of 4 GeV$^2$. We note in passing that in the lattice calculations, the smallness of $J_d$ can be traced back to a cancellation of the separately sizeable spin- and OAM contributions of down quarks, as can be inferred directly from Fig. 2. It will be very interesting to see if the slight discrepancy to the cloudy bag model (CBM) study of Ref. [27] is due to systematic effects, missing contributions or similar on the side of the lattice and the other approaches, or if it points towards the need for a further improvement of the CBM calculation.

3. Transverse spin structure of the nucleon

In this section, we briefly discuss lattice QCD results on the transverse spin structure of the nucleon. A basic but important observable with respect to the transverse spin structure is the transversity distribution function $h_1(x) = \delta q(x)$. Its lowest $x$-moments have been studied in lattice QCD by different groups over many years, and for an overview we refer to Ref. [5]. In [36], it has been shown that the twist-2 tensor (or quark helicity flip) GPDs $H_T, E_T, ...$ provide a deeper insight into the transverse spin structure of the nucleon by giving access to interesting correlations between transverse spin and coordinate (impact parameter, $b_\perp$) degrees of freedom, based on the probabilistic interpretation of GPDs in impact parameter [37]. The lowest moments of the tensor GPDs were subsequently computed on the lattice and found to be sizeable in general, giving rise to characteristically deformed "tomographic" density distributions of transversely polarized quarks in an unpolarized or transversely polarized nucleon [38]. A

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3 Most notable issues are: discretization and finite volume effects, missing disconnected contributions, and the so far neglected mixing of quark and gluon operators under renormalization and evolution in the singlet sector.
Figure 4. Comparison of quark angular momentum contributions to the nucleon spin, obtained in different lattice (for uncertainties, see footnote 3), model and phenomenological studies [30, 14, 31, 32, 33, 27], displayed in the $J_u, J_d$-plane in the style of Refs. [34, 35], for the $\overline{\text{MS}}$ scheme at a scale of $\mu = 2$ GeV. Note that the shaded bands are mainly given for illustration purposes and only approximately represent the constraints obtained in Refs. [34, 35], which are still strongly model dependent.

A closely related question concerns possible correlations between the transverse spin DOFs and the intrinsic transverse momentum, $k_\perp$, carried by the quarks and gluons inside the nucleon. It can be addressed on the basis of another important class of observables, the so-called transverse momentum dependent parton distribution functions (TMDs), generically denoted by $[f, g, h](x, k_\perp^2)$. Keeping in mind that the transverse momentum, $k_\perp$, and the coordinate, $b_\perp$, are not Fourier-conjugated variables, one finds that TMDs encode fundamental information about hadron structure that is mostly complementary to the physics content of GPDs.

TMDs play a key role in the phenomenology of semi-inclusive deep inelastic scattering (SIDIS) and the Drell-Yan-process, where correlations between the intrinsic transverse momenta of the partons, the hadron momenta, and their spins lead to a variety of interesting asymmetries. Considerable attention has been attracted by the Sivers and the Collins effect [40, 41], which give rise to single spin azimuthal asymmetries in SIDIS, and which have been studied experimentally at HERMES, COMPASS and BELLE [42, 43, 44].

First exploratory lattice QCD studies of $k_\perp$-distributions have been presented in [45, 46, 47]. They were based on (nucleon matrix elements of) manifestly non-local, gauge invariant quark operators, which provide direct access to a set of invariant amplitudes. The numerical results for the amplitudes can in turn be parametrized and then Fourier-transformed to obtain the $k_\perp$-distributions. Up to this point, the published lattice results correspond to a particular choice of “process-independent” non-local lattice operators, and therefore should not be quantitatively...
Figure 5. Transverse momentum densities of up (on the left) and down (on the right) quarks in the proton (from [46]).

compared to phenomenological TMD-extractions as discussed in, e.g., [48, 49].

Analogously to GPDs in \( b_\perp \)-space, the various TMDs have particularly useful interpretations in form of quark densities in the transverse momentum plane\(^4\). The density of, e.g., longitudinally polarized quarks with helicity \( \lambda \) in a nucleon with helicity \( \Lambda \) or transverse spin \( S_\perp \) is given by [36]

\[
\rho_L = \frac{1}{2} \left( f_1 + \lambda \Lambda g_1 + \left[ \frac{S_\perp}{m_N} \epsilon_{ij} k_i f_{T1} \right] + \lambda \frac{k_\perp \cdot S_\perp}{m_N} g_{1T} \right),
\]

where all TMDs depend on \( x \) and \( k_\perp^2 \), and where the term in square brackets is absent for our choice of non-local lattice operators [46]. A corresponding expression for the density \( \rho_T \) of transversely polarized quarks in a longitudinally polarized nucleon can be found in Ref. [36]. It turns out that the densities have the form of a multipole-expansion, with monopole terms \( \propto f_1, g_1, h_1 \), dipole structures \( \propto f_{1T}, g_{1T}, \ldots \), and a quadrupole term (in \( \rho_T \)). The lattice results for the lowest \( x \)-moment of the densities are illustrated in Fig. 5 for up- and down-quarks on the left and the right, respectively. The density \( \rho_L(x, k_\perp) \) is displayed in the upper part for \( \lambda = +1 \) and \( S_\perp = (1, 0) \), while the lower part of Fig. 5 shows \( \rho_T(x, k_\perp) \) for \( s_\perp = (1, 0) \) and \( \Lambda = +1 \). As a direct consequence of the non-zero lattice results for the distributions, e.g. \( g_{1T} > 0, g_{2T} < 0, \) we observe sizeable correlations between the quark and nucleon spins and the intrinsic quark transverse momentum, leading to clearly visible dipole deformations in opposite directions for up- and for down-quarks in Fig. 5. In contrast to others TMDs, the distributions \( g_{1T}, h_{1L} \), which are responsible for the deformations in Fig. 5, do not have analogs in the framework of GPDs due to time reversal symmetry [36]. The observed distortions therefore cannot be related

\(^4\) Note that the TMDs as discussed in the literature strictly speaking do not have a probabilistic interpretation (cf. [50]), in contrast to GPDs in impact parameter space.


to deformed impact parameter densities and may be seen as a genuine effect of intrinsic $k_\perp$ of quarks in the nucleon.

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