TOWARD A SCHURIFICATION OF PARKING FUNCTION FORMULAS VIA
BIJECTIONS WITH YOUNG TABLEAUX

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Abstract. This paper contains a partial answer to the open problem 3.11 of [Hag08]. That is to
find an explicit bijection on Schröder paths that inverts the statistic area and bounce. This paper
started as an attempt to write the sum over $m$-Schröder paths with a fix number of diagonal steps
into Schur functions in the variables $q$ and $t$. Some of the results have been generalized to parking
functions and some bijections where made with standard Young tableaux giving way to a partial
combinatorial formulas in the basis $s_\mu(q,t)s_\lambda(X)$ for $\nabla(e_n)$ (respectively, $\nabla^m(e_n)$), when $\mu$ and $\lambda$
are hooks (respectively, $\mu$ is of length one). We also give an explicit algorithm that gives all the
Schröder paths related to a Schur function $s_\mu(q,t)$ when $\mu$ is of length one. In a sense it is a partial
decomposition of Schröder paths into crystals.

1. Introduction

In this paper, Proposition 1 gives a partial answer to the open problem 3.11 of [Hag08]. The
problem asks for an explicit bijection on Schröder paths that inverts the statistic area and bounce.
But the aim of this paper is to decompose parking functions and Schröder path in terms of the
basis $s_\mu(q,t)s_\lambda(X)$. It is then used in [Wal] to give explicit combinatorial formulas for the modules
of multivariate diagonal harmonics. In other word, the combinatorics of Parking functions is used
to better understand the structure of the modules of diagonal harmonics. This combinatorial
representation was first known as the Shuffle Conjecture. It was introduced in [HHL+05] and
proved by Carlson and Mellit [CM18], [Mel16]. Beforehand, it was shown in [GH96] and [Hai02],

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that Frobenius transformation of its graded characters may be expressed as $\nabla^m(e_n)$, where $\nabla$ is the Macdonald eigenoperator introduced in [BG99] and $e_n$ is the $n$-th elementary symmetric function both recalled in Section 2, along with classical combinatorial tools.

More precisely, we will give a partial decomposition of parking functions and Schröder path in terms of the basis $s_\mu(q,t)s_\lambda(X)$. That is to say, we prove the following:

**Theorem 1.** If $\mu \in \{(d,1^{n-d}) \mid 1 \leq d \leq n\}$ and $\nu \vdash n$, then:

1. \[ \langle \nabla(e_n), s_\mu \rangle_{\text{hooks}} = \sum_{\tau \in \text{SYT}(\mu)} s_{\text{maj}(\tau)}(q,t) + \sum_{i=2}^{\text{des}(\tau)} s_{\text{maj}(\tau)-i,1}(q,t), \]

2. \[ \langle \nabla^m(e_n), s_\nu \rangle_{\text{Part}} = \sum_{\tau \in \text{SYT}(\nu)} s_{m(\frac{n}{2})-\text{maj}(\nu)}(q,0) = \sum_{\tau \in \text{SYT}(\nu)} s_{m(\frac{n}{2})-\text{maj}(\nu)}(0,t), \]

and:

3. \[ \langle \nabla^m(e_n), e_n \rangle_{\text{hooks}} = s_{m(\frac{n}{2})}(q,t) + \sum_{i=2}^{n-1} s_{m(\frac{n}{2})-i,1}(q,t). \]

This will be done by characterizing certain parking functions, in Section 5, which leads to Equation (2). In Section 6 we restrict the characterization on Schröder paths. In Section 7 we give bijections between subsets of Schröder paths and Standard Young tableaux, and use them to prove Equation (1) and Equation (3).

Moreover, in Section 4, we exhibit an explicit algorithm that gives all the Schröder paths associated to a Schur function in the variables $q$ and $t$ when $\mu$ is of length one. We will briefly explain, in section 9, what it means in term of Crystal decomposition. We end with a list of problem to solve in Section 10. Finally, Section 3 will recall notions on path combinatorics.

## 2. Combinatorial Tools

The notion discussed in this section are classical and are recalled to set notations.

An alphabet, $A$, is a set, the elements of that set are **letters**. A **word** is a finite sequence of elements of $A$, we, usually, omit the parentheses and the commas. The empty word is denoted $\varepsilon$. The number of letters in a word $w$ is called the **length**, denoted $|w|$, the number of occurrences of the letter $a$ in $w$ is denoted $|w|_a$. The set of word of length $n$ in the alphabet $A$ is denoted $A^n$, we denote $A^*$ the set $\cup_n A^n$. A **factor** of $w$ is a consecutive subsequence of $w$. Additionally, if we are interested in word ending with a certain factor $u$, we will denote the set $A^*u$ and $u$ is called a suffix. If we want those word to be of length $n + |u|$ we will denote the set $A^n u$. Like wise a factor at the beginning of a word is called a prefix and the set of word of prefix $u$ is denoted $uA^*$. For a word $w = w_1w_2 \cdots w_k$, $w^n$ is the concatenation on $m$ copies of $w$ and $w^{-1} = w_k \cdots w_2w_1$. For two word $u$ and $w$ the set $u \sqcup w$ is the set containing all words such that $u$ and $w$ are subsequences. We call these words **shuffles**.

A **permutations** of $n$ can be seen as words of $\{1, \ldots, n\}^n$ with all distinct letters. The **descent set of a permutation** $w = w_1 \cdots w_n$, denoted $\text{Des}(w)$, is the set of $i$’s such that $w_i > w_{i+1}$. The cardinality of the set will be denoted $\text{des}(w)$. The **major index of a permutation**, denoted $\text{maj}(w)$, is by definition $\text{maj}(w) = \sum_{i \in \text{Des}(w)} i$.

A **partition** of $n$ is a decreasing sequence of positive integers it can be represented by a Ferrer’s diagram (see Figure 1). Each number in the sequence is called a **part** and if it has $k$ parts it is of **length** $k$ denoted $\ell(\lambda) = k$. If $\lambda = \lambda_1, \cdots, \lambda_k$ and $n = \sum_i \lambda_i$ we say $\lambda$ is of **size** $n$, denoted
A tableau is a filling of a diagram by positive integer, the number in each box is called an entry. The size of a tableau relates to the size of the diagram it fills. It is said to be a semi-standard Young tableau if all entries are strictly increasing in rows and columns. A standard Young tableau is a tableau size \( n \) all numbers from 1 to \( n \) appear exactly once and if all entries are strictly increasing in rows and columns. If a tableau is a filling of the diagram associated to the partition \( \lambda \) it is said to be of shape \( \lambda \). The set of standard Young tableaux of shape \( \lambda \) is denoted \( \text{SYT}(\lambda) \). The descent set of a tableau \( \tau \), denoted \( \text{Des}(\tau) \) is the set of entries \( i \) such that \( i + 1 \) lies in a higher row. The cardinality of the descent set of \( \tau \) is denoted \( \text{des}(\tau) \) and the sum of the elements in the descent set is the major index denoted \( \text{maj}(\tau) \) (see Figure 3). Again it will be clear by context if the descent set and the major index is used on words or tableaux. Since each box of \( \tau \) is associated to its own entry, we will write \( c \in \tau \) when we refer to the entry \( c \) in the tableau \( \tau \). We will use the notation \( x_\tau \) for the monomial \( \prod_{c \in \tau} x_c \).

For a possibly infinite set of variables, \( X = \{x_1, \ldots, x_n\} \), the elementary symmetric functions \( e_n(X) \) are the sum of all square free monomial of degree \( n \) in the set of variables \( X \). The symmetric function \( e_\lambda \) is simply \( e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}} \). The elementary symmetric functions form a basis of the symmetric functions. An other basis is the Schur functions. For \( \lambda \) a partition the Schur function \( s_\lambda(X) = \sum x_\tau \), where the sum over all semi-standard Young tableau of shape \( \lambda \). The Schur basis in the \( X \) variables is self-dual for the modified Macdonald polynomial scalar product, denoted \( \langle -,- \rangle \). We will use this notation when we want to display the coefficient of a particular Schur function. Note that the Schur functions in the variables \( q \) and \( t \) are coefficients and can go in and out of the scalar product. We will sometime call Schur functions index by partitions that are hooked-shaped, hook-shaped Schur functions of simply hook Schur functions. It will also be useful to remember that \( e_n = s_1^n \). Furthermore, the complete homogeneous symmetric functions are a basis such that \( h_n(X) = s_n(X) \) and \( h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}} \). We simply write \( e_\lambda \) for \( e_n(X) \), \( s_\lambda \) for \( s_\lambda(X) \) and \( h_\lambda \) for \( h_\lambda(X) \) (not for \( e_\lambda(q,t), h_\lambda(q,t) \) or \( s_\lambda(q,t) \)). A curious reader could look at [Mac95]. The modified Macdonald polynomials \( \tilde{H}_\mu(X; q, t) \) form an other base of the ring of symmetric functions (some authors write \( \tilde{H}_\mu(X; q, t) \)). In [BG99] Bergeron and Garsia introduce the operator \( \nabla \) defined with the modified Macdonald polynomials as eigenfunctions. The Shuffle
Theorem proven by Carlson and Mellit (see [CM18] and [Mel16]) gives a combinatorial formula for $\nabla^m(e_n)$. This formula uses path combinatorics.

### 3. Path Combinatorics

Before we can state the Shuffle Theorem, we need more classical definition, relating to path combinatorics. More details on these classical notions can be found in [Hag08].

The following $q$-analogues happen to be very useful:

\[
[n]_q := 1 + q + q^2 + \cdots + q^{n-1},\quad [n]!_q := \prod_{i=1}^{n} [i]_q, \quad \text{and} \quad \left[\frac{n}{k}\right]_q = \frac{[n]!_q}{[n-k]!_q[k]!_q}.
\]

Let $C^n_k$ be the set of paths composed of north and east steps, in an $(n-k) \times k$ grid starting at the bottom left corner. The area of a path is the number of boxes under the path. A classical result relates the Gaussian Polynomials to path combinatorics. Indeed, $[n]_q! = \sum_{\gamma \in C^n_k} q^{\text{area}(\gamma)}$. A path $\gamma \in C^n_k$ can be identified with word, $w_\gamma$ in $\{N,E\}^n$ such that $|w_\gamma|_E = k$. To facilitate reading we will frequently refer to $\gamma$ when we talk about $w_\gamma$ (see Figure 4 for an example).

**Figure 4.** Path of area 6 in an $5 \times 4$ grid, with word representation EENNENNNE

In an $n \times m$ grid the main diagonal is the diagonal starting at the bottom left corner and finishing at the top right corner. A Dyck path of size $n,m$ is a path composed of north and east steps, starting at the bottom left corner in an $n \times m$ grid such that the path always stay over the main diagonal. The set of such paths is denoted $\mathcal{D}_{n,m}$. A classical result makes it possible to represent Dyck paths by words in $\{N,E\}^*$ such that for all $\gamma_i$ prefix of $\gamma$ we have $|\gamma_i|_N \geq \frac{m}{n}|\gamma_i|_E$. The lines of the grid are numbered from bottom to top. A line $i$ is said to contain an east step if the factor starting with the $i$-th north step and ending the letter before the $i+1$-th north step contains an east step. A column of a path, $\gamma$, is a factor $N^jE^k$ such that $\gamma = uN^jE^kw$, where $u,w \in N\{N,E\}E \cup \{\varepsilon\}$. For example in Figure 5 the path has 3 columns.

The area of a Dyck path will be the number of boxes under the path and over the main diagonal (see Figure 5 for an example). The area at line $i$ noted $a_i$ is the number of boxes in the line $i$ that are between the path and the main diagonal. Obviously the area of a path is the sum of the $a_i$'s. The path $\gamma$ is said to have a return to the main diagonal if there is $\gamma_i$ a non trivial prefix of $\gamma$ such that $\gamma_i$ is a Dyck path and the end point of $\gamma_i$ lies on the main diagonal of $\gamma$. The touch sequence of a path $\gamma$, denoted $\text{Touch}(\gamma)$, is defined as a sequence $(\gamma_1, \ldots, \gamma_k)$ of factors of $\gamma$ such that $\gamma = \gamma_1 \cdots \gamma_k$, all $\gamma_i$ are Dyck paths and all $\gamma_i$ contain no return to the main diagonal. Usually, the sequence $(\frac{1}{2}|\gamma_1|, \ldots, \frac{1}{2}|\gamma_k|)$ defines the touch vector $\text{touch}(\gamma)$. The touch vector contains all
the touch points. For example in Figure 5 \( \text{Touch}(NNENNEEENE) = (NNENNEEEE, NE) \) and \( \text{touch}(NNENNEEENE) = (4, 1) \). The bounce path of a path \( \gamma \in D_{n,n} \) will be the path that remains under the path \( \gamma \) and changes direction only when it touches the path \( \gamma \) or the main diagonal. The bounce vector is the vector containing the positions of the return to the main diagonal, starting from the top, of the bounce path. For the bounce vector the lines are numbered from the top starting at 0. Finally, the bounce statistic is the sum of the integer in the bounce vector minus \( n \). It is, usually, simply referred to as bounce (see Figure 6 for an example). Note that the bounce statistic is not defined for Dyck paths in an \( n \times m \) grid with \( m \neq n \). In these cases we use the diagonal inversion statistic which will be discussed at the end of this section.

A Schröder path of size \( n, r n \) is a path composed of north, east and diagonal steps in an \( n \times rn \) grid such that the path always stay over the main diagonal starting at the bottom left corner. The set of paths containing \( d \) diagonal steps is denoted \( \text{Sch}_{n,d}^{(r)} \). These paths can also be seen as word in the alphabet \( \{N, E, D\}^* \) such that for all prefix \( \gamma_i \) of \( \gamma \) we have \( |\gamma_i|_N \geq r |\gamma|_E \). Clearly \( D_{n, rn} = \text{Sch}_{n,0}^{(r)} \). Moreover, the path obtained by deleting all diagonal steps in a Schröder path is a Dyck path. For a Schröder path \( \pi \) this new path will be denoted \( \Gamma(\pi) \). For example the path \( \pi \) in Figure 7 is such that \( \Gamma(\pi) \) is the path seen in Figure 5. We will also frequently use an other subset of Schröder paths:

\[
\text{Sch}_{n,d} = \{ \gamma \in \text{Sch}_{n,d} | \gamma = wNE, w \in \{D, N, E\}^* \} = \text{Sch}_{n,d} \cap \{D, N, E\}^*NE
\]

The area statistic of a Schröder path is fairly the same as the other definitions of the area statistic. Instead of counting the squares we count the number of lower triangles under the path and over the main diagonal. Where a lower triangle is the lower half of a square cut in two starting by the left lower corner and ending at the top right corner (see Figure 7 for an example).

In [Hag08] Haglund defines a bounce statistic for Schröder paths in an \( n \times n \) grid. We first define the set of peaks of the path, \( \Gamma(\gamma) \). These are the set of lattice points at the beginning of an east step such that the bounce path of \( \Gamma(\gamma) \) switches from a north step to an east step. By extension the peaks of \( \gamma \) are the lattice points found by reinserting the diagonal steps in \( \Gamma(\gamma) \). The number of peaks of the path \( \gamma \), with multiplicity, that lie under each diagonal step is the statistic \( \text{numph} \), denoted \( \text{numph}(\gamma) \). The bounce statistic will be extended to a Schröder paths, \( \gamma \), by the formula (see Figure 8 for an example):

\[
bounce(\gamma) = bounce(\Gamma(\gamma)) + \text{numph}(\gamma)
\]

Finally, touch points can be defined for Schröder path, simply change Dyck path for Schröder paths in the definition.
Figure 7. Schröder path of area 9, and word representation NDNENNEDDEEDNE

Figure 8. For this path, \( \gamma \), \( \text{numph}(\gamma) = 4 \) and \( \text{bounce}(\gamma) = 8 \).

The generating function of the Schröder paths and the generating function of the Schröder paths ending with \( NE \) are defined by:

\[
\text{Sch}_{n,d}(q,t) = \sum_{\gamma \in \text{Sch}_{n,d}} q^{\text{bounce}(\gamma)} t^{\text{area}(\gamma)}
\]

\[
\widetilde{\text{Sch}}_{n,d}(q,t) = \sum_{\gamma \in \widetilde{\text{Sch}}_{n,d}} q^{\text{bounce}(\gamma)} t^{\text{area}(\gamma)}
\]

Since the subset \( \widetilde{\text{Sch}} \) is chosen to work with the bounce statistic which is not defined for \( n \times nm \) grids when \( m \neq 1 \), we will not define \( \widetilde{\text{Sch}}_{n,d}^{(m)} \). We will define \( \widetilde{\text{Sch}}_{n,d}^{(m)}(q,t) \) as follows:

\[
\widetilde{\text{Sch}}_{n,d}^{(m)}(q,t) = \sum_{k=d}^{n} (-1)^{k-d} \text{Sch}_{n,k}^{(m)}(q,t)
\]

The reason is due to the fact that \( s_{d+1,1^{n-d-1}} = \sum_{k=d}^{n} (-1)^{k-d} e_{n-d} h_d \), which is used in Equation (4) due to Haglund and Equation (5) due to Mellit.

An \((n,mn)\)-parking function is a pair consisting of and a \((n,mn)\)-Dyck path and a permutation of \( n, w \), for which we write \( w_i \) on line \( i \) of the Dyck path. Moreover, all factors of \( w \) in a column of the path must contain no descents (see Figure 9 and Figure 10 for examples). The set of all \((n,mn)\)-parking function is denoted \( \mathcal{P}_{n,mn} \).

The reading word is obtained by reading the letters of \( w \) (which are written immediately to the right of each north step) in regard to the diagonals parallel to the main diagonal starting from top right corner to the bottom left corner and starting with the diagonal that is the farthest from the main diagonal. For example the reading word in Figure 9 is read(\( \gamma, 183457692 \)) = 675438291. The reading word of the parking function \((\gamma,w)\) is denoted \( \text{read}(\gamma,w) \).

The area of a parking function is the area of its Dyck path. The diagonal inversion statistic (sometimes called dinv for short) is given by the formula \( \sum_{i<j} d_i(j) \), where:

\[
d_i(j) = \begin{cases} 
\chi(w_i < w_j) \max(0, r - |a_i - a_j|) + \chi(w_i > w_j) \max(0, r - |a_j - a_i + 1|) & \text{if } i < j \\
0 & \text{if } i \geq j.
\end{cases}
\]

The diagonal inversion statistic of the parking function \((\gamma,w)\) is denoted \( \text{dinv}(\gamma,w) \). Note that all the definition work if \( w \) is not a permutation (some authors use words but these can be regrouped
with permutation as representatives). Equivalently for a \((\gamma, w) \in \mathcal{P}_{n,mn}\) we can consider the diagonal inversion of \((\tilde{\gamma}, \tilde{w})\), where \(\tilde{\gamma}\) is the \((mn, mn)\)-Dyck path obtained by repeating all north steps \(m\) times and for \(w = w_1 \cdots w_n\) we have \(\tilde{w} = w_1^m \cdots w_n^m\) (here \(\tilde{w}\) is not a permutation). In this case we can consider \(d_i(j) = \sum_{t=1}^{m} d^t_i(j)\), where \(d^t_i(j)\) is calculated with \(\tilde{\gamma}\).

A visual representation of the diagonal inversion statistic for \((\gamma, w) \in \mathcal{P}_{n,n}\) is to consider one diagonal parallel to the main diagonal on each north step. For the north step on line \(j\) if the diagonal crosses the north step at line \(i\), with \(i < j\) and \(w_i < w_j\), then the pair \((i, j)\) contributes one to the diagonal inversion statistic. If the diagonal immediately over the diagonal crossing the north step at line \(j\) crosses the line \(i\), with \(i < j\) and \(w_i > w_j\), then the pair \((i, j)\) contributes one to the diagonal inversion statistic (see Figure 11 and Figure 12).

The Schröder paths in an \(n \times mn\) grid with \(d\) diagonal steps can be seen as parking functions \((\gamma, w)\) such that \(\text{read}(\gamma, w) \in \{n-d+1, \cdots, n\} \sqcup \{n-d, \cdots, 1\}\). As a matter of fact, by definition of parking functions, if \(w_i\) is in \(\{n-d+1, \cdots, n\}\), then the north step at line \(i\) is followed by an east step. Therefore, for all \(w_i\) in \(\{n-d+1, \cdots, n\}\) one can change the factor \(NE\) in line \(i\) for a \(D\) and unlabelled the path. This procedure gives us a Schröder path with \(d\) diagonal steps. Conversely, all \(D\) steps of a Schröder path can be changed for \(NE\) factors and tagged in the reading order by the letters in \(\{n-d+1, \cdots, n\}\) and all the north steps can be tag in the reading order by letters in \(\{n-d, \cdots, 1\}\). This bijection will be mostly used for proofs. Hence, we will often refer to Schröder using there parking function description.
In [TW18], Thomas and Williams proved that the zeta map, denoted \( \zeta \), is a bijection on rational parking functions such that \( \text{dinv}(\gamma, w) = \text{area}(\zeta(\gamma, w)) \) and \( \text{area}(\gamma, w) = \text{bounce}(\zeta(\gamma, w)) \). This will be used implicitly in the following way: if one as a decomposition of Schröder paths with \( d \) diagonal steps in Schur functions, in the variables \( q \) and \( t \), in term of area and bounce, the decomposition in terms of diagonal inversions and area is the same.

For more on \( m \)-Schröder paths see [Hag08] and [Son05].

In this paper we will give explicit decomposition in Schur function in the variables \( q \) and \( t \) for \( \langle \nabla^m(e_n), e_n \rangle_{\text{hook}}, \langle \nabla^m e_n, s_\mu \rangle_{1\text{Part}} \) and \( \langle \nabla e_n, s_{d+1,1^{n-d-1}} \rangle_{\text{hook}} \) by using Corollary 2.4 in [Hag04]:

**Theorem** (Haglund). Let \( n, d \) be positive integer such that \( n \geq d \). Then:

\[
\Sch_{n,d}(q, t) = \langle \nabla e_n, s_{d+1,1^{n-d-1}} \rangle,
\]

and:

\[
\Sch_{n,d}(q, t) = \langle \nabla e_n, e_{n-d} h_d \rangle.
\]

and the following equalities. It can be inferred from Mellit’s proof found in [Mel16] of the compositional shuffle conjecture of [BGLX16]. Let \( n, d, m \) be positive integer such that \( n \geq d \). Then:

\[
\Sch_{n,d}^{(m)}(q, t) = \langle \nabla^m e_n, s_{d+1,1^{n-d-1}} \rangle
\]

and:

\[
\nabla^m(e_n) = \sum_{(\gamma, w) \in \mathcal{P}_{n,m}} t^{\text{dinv}(\gamma, w)} q^{\text{area}(\gamma)} F_{\text{co}(\text{Des}(\text{read}(w)^{-1}))}(X),
\]

where \( F_c \) is the fundamental quasisymmetric function index by the composition \( c \) and for \( S \) a subset of \( \{1, \ldots, n-1\} \), \( \text{co}(S) \) is the composition associated to \( S \).

The last result we need can be inferred from [Sta79] and [Hai02]:

\[
\nabla(e_n)|_{q=0} = \sum_{\tau \in \text{SYT}(n)} t^{\text{maj}(\tau)} s_{\lambda(\tau)}
\]

4. **Algorithm on Schröder Paths Related to Schur Functions Index By One Part**

It was proven in [Hai02] that \( \nabla(e_n) \) is the character of the \( GL_2 \times S_n \)-module of diagonal harmonics. Hence, the polynomials \( \Sch_{n,d}(q, t) \) and \( \Sch_{n,d}(q, t) \) are symmetric in \( q, t \) and can be written as a sum of Schur functions evaluated in \( q, t \).

The restriction of a symmetric function to the sum of Schur functions indexed by only one part (respectively, hook-shaped Schur functions) will be denoted by \( |_{1\text{Part}} \) (respectively, \( |_{\text{hooks}} \)). For example, if \( f = \sum_{\lambda \subseteq C} c_{\lambda} s_{\lambda} \), then the restriction to one part is \( f|_{1\text{Part}} = \sum_{\lambda \subseteq C, \ell(\lambda) = 1} c_{\lambda} s_{\lambda} \) (respectively, the restriction to hooks is \( f|_{\text{hooks}} = \sum_{\lambda \subseteq C, \ell(\lambda) = 1} c_{\lambda} s_{\lambda} \)).

In this section, we will give a simple formula for the Schur functions indexed by one part partitions contained in the development of \( \Sch_{n,d}(q, t) \). This will be done by proving an algorithm that allows us to describe all the paths of \( \Sch_{n,d} \) relating to the restrictions to Schur functions indexed by one part in the Schur function decomposition of \( \Sch_{n,d}(q, t) \).

Let’s first notice that Schur functions on a set of \( k \) ordered variables is indexed by a partition with length smaller or equal to \( k \). This follows from the combinatorial definition of Schur functions, since the filling of the first column of the semi-standard Young tableau must be strictly increasing and the first column as the same number of boxes to fill than the number of parts of the partition.
Hence, Schur function on two variables have at most two parts. Furthermore, a Schur function in two variables, is such that \( s_{n,b}(q,t) = q^{a-b}t^{a-b}(q^b + q^{b-1}t + \cdots + qt^{b-1} + t^b) \). Ergo, for \( c \) an integer, the monomial \( q^c \) as a non zero coefficient in the decomposition of a symmetric function \( f(q,t) \) if and only if the decomposition in Schur function contains the term \( s_c(q,t) \).

This is equivalent to stating that the Schur functions in the variables \( q \) and \( t \) appearing in \( \text{Sch}_{n,d}(q,t)_{|\text{Part}} \) are the same than the Schur functions in the variable \( q \) appearing in \( \text{Sch}_{n,d}(q,0) \). The only paths that contribute to monomial with non zero coefficients in \( \text{Sch}_{n,d}(0,t) \) are the paths \( \gamma \) such that \( \text{area}(\gamma) = 0 \). Hence, these are the set of paths \( \{NE,D\}^n \).

The algorithm \( \varphi \) takes a path, \( \gamma \) in \( \{NE,D\}^* \) for input, and returns a sequence of paths \( (\gamma_0, \gamma_1, \ldots, \gamma_{\text{bounce}(\gamma)}) \):

First set \( \varphi(\gamma) = (\gamma_0), \ k = |\gamma|_E + |\gamma|_N + |\gamma|_D \).

For \( v = 1 \) to \( \text{bounce}(\gamma) \):

Let \( \gamma_{v-1} = w_1 w_2 \cdots w_k \).

Let \( i \) be such that \( w_i = E, w_{i+1} \neq E \) and \( w_j = E \) implies \( w_{j+1} = E \) or \( j \leq i \).

Set \( \gamma_v = w_1 \cdots w_{i-1} w_{i+1} w_i w_{i+2} \cdots w_k \).

Append \( \gamma_v \) to the sequence \( \varphi(\gamma) \).

repeat ;

return \( \varphi(\gamma) \).

Figure 13. The sequence \( \varphi(DDNEDNENE) \).

We first need to prove this algorithm actually provides us with a sequence of Schröder paths.

**Lemma 1.** For all \( \gamma \in \{NE,D\}^* \), the elements of the sequence \( \varphi(\gamma) \) are Schröder paths. Moreover, if \( \gamma \in \text{Sch}_{n,d} \), then \( \varphi(\gamma) \subseteq \text{Sch}_{n,d} \).

**Proof.** Recall that for a path \( \gamma \) to be a Schröder path we must have \( |\omega|_N \geq |\omega|_E \) for all prefix \( \omega \) of \( \gamma \). Because \( \gamma_0 \) is a Schröder path, it is sufficient to show that if \( \gamma_i \) is a Schröder path, then the path \( \gamma_{i+1} \), obtained by parsing one time through the algorithm, is also a Schröder path. Let \( \gamma_i = w_1 w_2 \cdots w_k \), then \( \gamma_{i+1} = w_1 \cdots w_{i-1} w_{i+1} w_i w_{i+2} \cdots w_k \) and the only prefixes that are different are \( w_1 \cdots w_{i-1} w_{i+1} \) compared to \( w_1 \cdots w_{i-1} w_i \) and \( w_1 \cdots w_{i-1} w_{i+1} w_i \) compared to \( w_1 \cdots w_{i-1} w_{i+1} \). The last pair have the same letters ordered in a different way, so \( |w_1 \cdots w_{i-1} w_{i+1} w_i|_N \geq |w_1 \cdots w_{i-1} w_{i+1} w_i|_E \). Now for the first pair of prefixes we have:

\[
|w_1 \cdots w_{i-1} w_{i+1}|_N \geq |w_1 \cdots w_{i-1} w_i|_N, \text{ since } w_i = E \text{ and } w_{i+1} \in \{N,D\},
\]

\[
\geq |w_1 \cdots w_{i-1} w_i|_E, \text{ since } \gamma_i \text{ is a Schröder path},
\]

\[
> |w_1 \cdots w_{i-1} w_{i+1}|_E, \text{ since } w_i = E \text{ and } w_{i+1} \neq E.
\]

Therefore, \( \gamma_{i+1} \) is indeed a Schröder path.
Finally, for \( \gamma \) a Schröder path we can move east steps to the left at least a number of times equal to bounce. Indeed, the peaks are associated to an east step and the numph statistic gives the number of diagonal steps to the right of an east step. Because the bounce path associated to \( \text{bounce}(\Gamma(\gamma)) \) changes direction when it hits an east step, with the exception of the last entry, the vector associated to \( \text{bounce}(\Gamma(\gamma)) \) gives to number of north steps that are to the right of an east step.

\[\square\]

We give an example of \( \varphi(\gamma) \) for \( \gamma \) a Schröder path not in \( \{NE, D\} \).

![Figure 14. The sequence \( \varphi(DDN|DEN|NE) \).](image)

In Figure 14 the bounce statistic doesn’t decrease evenly throughout the iterations of the algorithm. In Figure 13 each iteration increases the area by exactly one and decreases the bounce statistic by exactly one. We will show in Lemma 3 that this is not a coincidence if \( \gamma_0 \) is a Schröder path of area 0. But first, we need to show a result on the prefixes of \( \varphi \).

**Lemma 2.** Let \( \gamma \) be in \( \{NE, D\}^* \) and \( \varphi(\gamma) = (\gamma_0, \ldots, \gamma_{\text{bounce}(\gamma)}) \). If \( \gamma_i \) as a prefix \( \alpha \) in \( \{NE, D\} \), then \( \alpha \) is a prefix of \( \gamma \). Moreover, if \( \alpha \) is the longest prefix of \( \gamma_i \) such that \( \alpha \) is in \( \{NE, D\} \), then for \( \beta \) such that \( \alpha \beta = \gamma_i \) we have \( \beta = \omega E|\beta|\varepsilon^{-1} \), where \( \omega \) is a word in the alphabet \( \{N, E, D\}^* \). Finally, if \( \gamma_i = w_1 \cdots w_{j-2} w_j w_{j-1} w_{j+1} \cdots w_k \) and \( \gamma_{i-1} = w_1 \cdots w_{j-2} w_j w_{j-1} w_j w_{j+1} \cdots w_k \), then \( w_{j-1} \) is a letter in \( \omega \).

**Proof.** By induction on \( i \). If \( i = 0 \), then \( \alpha = \gamma_0 = \gamma \) and \( \beta = \varepsilon \). For \( i > 0 \) let \( \alpha \) (respectively, \( \alpha' \)) be the longest prefix of \( \gamma_i \) (respectively, \( \gamma_{i-1} \)) such that \( \alpha \) is in \( \{NE, D\}^* \) (respectively, \( \alpha' \in \{NE, D\}^* \)) and \( \beta \) (respectively, \( \beta' \)) be such that \( \alpha \beta = \gamma_i \) (respectively, \( \alpha' \beta' = \gamma_{i-1} \)). By induction we know that \( \alpha' \) is a prefix of \( \gamma_i \) and \( \beta' = \omega' E|\beta'|\varepsilon^{-1} \). By definition of each iteration of the algorithm there is \( j \) such that \( \gamma_i = w_1 \cdots w_{j-2} w_j w_{j-1} w_{j+1} \cdots w_k \) and \( \gamma_{i-1} = w_1 \cdots w_{j-2} w_j w_{j-1} w_j w_{j+1} \cdots w_k \).

If \( |\alpha'| \leq j-2 \), then \( \beta' = w_{l+1} \cdots w_{j-2} w_j w_{j+1} \cdots w_k \) and \( \alpha = \alpha' \). By definition of the algorithm \( w_j \in \{N, D\} \). On account of \( \beta' = \omega' E|\beta'|\varepsilon^{-1} \) we must have that \( w_j \) and \( w_{j-1} \) are both letters of \( \omega' \). Ergo, the suffix \( E|\beta'|\varepsilon^{-1} \) of \( \beta' \) is unchanged in \( \beta \). Hence, \( \beta = \omega E|\beta|\varepsilon^{-1} = \omega E|\beta|\varepsilon^{-1} \) and \( |\omega'| = |\omega| \).

If \( |\alpha'| > j-2 \) by definition of the algorithm \( w_{j-1} = E \). On account of \( \alpha' \in \{NE, D\}^* \), \( w_{j-2} = N \). Consequently, \( \gamma_i = w_1 \cdots w_{j-3} N w_j E w_{j+1} \cdots w_k \) and \( \alpha = w_1 \cdots w_{j-3} \) is in \( \{NE, D\}^* \) and is a prefix of \( \alpha' \). Thus, it is a prefix of \( \gamma \). This means \( \beta = N w_j E w_{j+1} \cdots w_k \). Each iterations of the algorithm swaps the rightmost east step that is not followed by an east step to the right. In consequence, \( w_j \in \{N, D\} \) and we must have \( \omega \) such that \( \beta = \omega E|\beta|\varepsilon^{-1} \), with \( w_{j-1} \) is a letter of \( \omega \), otherwise the letters \( w_j \) and \( w_{j-1} \) would not have been swapped.

\[\square\]
Lemma 3. Let $\gamma$ be in $\{NE,D\}^*$ and $\varphi(\gamma) = (\gamma_0, \ldots, \gamma_{\text{bounce}(\gamma)})$. Then, for all $i$, $0 \leq i < \text{bounce}(\gamma)$, the following equalities hold:

$$
\begin{align*}
\text{area}(\gamma_i) + 1 &= \text{area}(\gamma_{i+1}), \\
\text{bounce}(\gamma_i) &= \text{bounce}(\gamma_{i+1}) + 1, \\
\text{area}(\gamma_i) &= \text{bounce}(\gamma_{\text{bounce}(\gamma) - i}), \quad \text{and}, \\
\text{bounce}(\gamma_i) &= \text{area}(\gamma_{\text{bounce}(\gamma) - i}).
\end{align*}
$$

Proof. Let's first notice that the algorithm changes $EN$ for $NE$ or $ED$ for $DE$. In both cases, this adds exactly one lower triangle under the path. Therefore, $\text{area}(\gamma_i) + 1 = \text{area}(\gamma_{i+1})$.

For the second condition, let $\gamma_i = w_1 \cdots w_{j-1} w_j w_{j+1} w_{j+2} \cdots w_k$. By definition of the algorithm $\varphi$, we know that $\gamma_{i+1} = w_1 \cdots w_{j-1} w_{j+1} w_{j+2} \cdots w_k$, $w_j = E$ and $w_{j+1} \in \{N,D\}$. By Lemma 2, we know $\gamma_{i+1} = \alpha \beta$ with $\alpha \in \{NE,D\}^*$, so there is a return to the main diagonal of the bounce path between $\alpha$ and $\beta$. Hence, the following east step is associated to a peak. By Lemma 2, $\beta = \omega E^{|\beta|} E^{-1}$, $w_j$ is a letter of $\omega$ and $\omega$ contains exactly one east step. Consequently, there is a peak at $w_j$ in $\gamma_{i+1}$. Let $\gamma_i = \alpha' \beta'$, if $w_j$ is a letter in $\omega'$, the same reasons puts peak at $w_j$ in $\gamma_i$. If $w_j$ is a letter in $\alpha'$, there is a peak at $w_j$ in $\gamma_i$, since $\alpha' \in \{NE,D\}^*$. Thus, $NE = w_{j-1} w_j \in \text{Touch}(\alpha)$.

If $w_{j+1} = D$, then $\text{bounce}(\Gamma(\gamma_{i+1})) = \text{bounce}(\Gamma(\gamma_{i+1}))$ because $\Gamma$ discards the diagonal steps. Recall that the peaks of a Schröder path, $\gamma_i$ are also obtained from $\Gamma(\gamma_i)$, in consequence, the peaks in $\gamma_i$ and $\gamma_{i+1}$ are associated to the same east steps. Recall that numph is the number of diagonal steps, with multiplicity, positioned after a peak (higher if you consider the path itself rather than the word representation). The diagonal step $w_{j+1}$ is after the peak at $w_j$ in $\gamma_i$ and before the peak at $w_j$ in $\gamma_{i+1}$. All other peak and diagonal steps remain unchanged. Hence, $\text{numph}(\gamma_i) = \text{numph}(\gamma_{i+1}) + 1$.

If $w_{j+1} = N$, then the peak at $w_j$ moves one position to the right in the word representation (one line higher if you consider the path itself). By definition, at this point, the bounce path returns to the main diagonal and goes north to the next east step, which, by Lemma 2, are all after $\omega$. Therefore, the peak at $w_j$ does not move to a line already containing a peak unless $\omega$ ends with $w_j D^1$. In this case, the peak at $w_j$ contributed to $1$ in $\gamma_i$. In any case, the peak on the first line of $\Gamma(\gamma_{i+1})$, contributes $0$ to bounce. Consequently, in both cases, $\text{bounce}(\Gamma(\gamma_{i})) = \text{bounce}(\Gamma(\gamma_{i+1})) + 1$. Additionally, all the east steps keep the same number of diagonal steps positioned after them. Hence, all the peaks keep the same number of diagonal steps positioned after them and $\text{numph}(\gamma_i) = \text{numph}(\gamma_{i+1})$.

Considering the path $\gamma$ in $\{NE,D\}^*$ has area equal to zero, the third and forth conditions follows from the first two conditions. \hfill $\square$

We now present a map that will be useful for the discussion on crystals in Section 9.

For $\gamma$ in $\{NE,D\}^*$ we have $\varphi(\gamma) = (\gamma_0, \ldots, \gamma_{\text{bounce}(\gamma)})$. With this notation we define the map:

$$
\varphi : \{ \gamma \in \text{Sch}_{n,d-1} \mid \text{area}(\gamma) = 0 \} \to \{ \gamma \in \text{Sch}_{n,d-1} \mid \text{area}(\gamma) = 1 \}
$$

$$
\gamma = \gamma_0 \mapsto \gamma_1
$$

This next lemma will be used in the proof of Theorem 1.

Lemma 4. Let $d$ be an integer such that $1 \leq d \leq n - 1$, then the image of the map $\varphi$ is given by the set $\{ uD^j NNEE, vNDE^j D^i NE \mid u \in \{NE,D\}^{n-d-2}, v \in \{NE,D\}^{n-d-1} \}$.

Proof. Follows from the definition of $\varphi$. \hfill $\square$
In order to give the decomposition in Schur functions evaluated in the variables $q$ and $t$, for $\langle \nabla(e_n), s_{d+1,n-d-1} \rangle_{\text{Part}}$, we will actually show that for a path $\gamma$ in $(NE,D)^*$, the sum
$$\sum_{\pi \in \varphi(\gamma)} q^{\text{bounce}(\pi)} t^{\text{area}(\pi)}$$
is a Schur function in the variables $q$ and $t$. For the general result to hold we need the intersection of these sets to be empty.

**Lemma 5.** Let $\gamma$, $\pi$ be in $(NE,D)^*$ such that $\gamma \neq \pi$ then $\varphi(\gamma) \cap \varphi(\pi) = \emptyset$.

**Proof.** Let’s first notice that the algorithm changes $EN$ for $NE$ or $ED$ for $DE$. Therefore, the relative order of the north and diagonal steps do not change for all paths in $\varphi(\gamma)$ and $\varphi(\pi)$. Hence, $\gamma_0$ and $\pi_0$ have the same relative order for the north and diagonal steps which uniquely determine paths of $(NE,D)^*$. Consequently, $\varphi(\gamma) \cap \varphi(\pi) = \emptyset$. \qed

We can now display a bijection that inverts the statistics area and bounce. This partially solves open problem 3.11 of [Hag08].

**Proposition 1.** Let $n$ be a positive integer and $\varphi((NE,D)^n)$ be the set $\cup_{\gamma \in (NE,D)^n} \varphi(\gamma)$. There is a bijection, $\Omega_n$, of $\varphi((NE,D)^n)$ onto itself. For all $n \geq 1$ we have, $\text{area}(\gamma_i) = \text{bounce}(\Omega_n(\gamma_i))$ and $\text{bounce}(\gamma_i) = \text{area}(\Omega_n(\gamma_i))$.

**Proof.** By Lemma 5, for $\gamma \in \varphi((NE,D)^n)$ there is a unique $\gamma_0$ and a unique $i$ such that $\gamma \in \varphi(\gamma_0)$ and $\gamma = \gamma_i$. Thus, we can define $\Omega_n(\gamma) = \gamma_{\text{bounce}(\gamma_0)-\text{area}(\gamma_0)}$. The result is a consequence of Lemma 3. \qed

The following Lemma gives us a full set representatives of Schur functions indexed by one part.

**Lemma 6.** Let $A_d = \{ \gamma \in (NE,D)^n \ |\ |\gamma|_D = d \}$, there is a bijection $\theta : A_d \rightarrow C^n_d$ such that $\theta(NE) = N$ and $\theta(D) = E$. Moreover, for $\gamma \in A_d$ we have $\text{bounce}(\gamma) = \text{area}(\theta(\gamma)) + \binom{n-d}{2}$.

**Proof.** In $A_d$, the factor $NE$, can be seen as a letter. A path $\gamma$ in $A_d$ is a word of length $n$ with $d$ occurrences of one letter and $n-d$ occurrences of the other letter. A path in $C^n_d$ can be seen as a word with $d$ occurrences of the letter $E$ and $n-d$ occurrences of the letter $N$. Hence, $\theta$ merely relabels the letters and is a bijection.

Furthermore, in $A_d$ all east steps are associated to a peak; therefore, the $i$-th diagonal steps contributes the number of factors $NE$ before the $i$-th diagonal steps to numph. But that number also happens to be the number of boxes under the $i$-th north step in $\theta(\gamma)$. Thus, numph$(\gamma) = \text{area}(\theta(\gamma))$. Finally, $\text{bounce}(\Gamma(\gamma)) = \binom{n-d}{2}$ for all paths $\gamma$ in $A_d$ because all the $n-d$ east steps return to the main diagonal. \qed

The next proposition will be generalized for parking functions by Proposition 5 and generalized for the restriction to Schur functions indexed by a hook-shaped partition, evaluated in the variables $q$ and $t$ by Theorem 1. Although the generalizations will not account for all the paths related to each Schur functions.

**Proposition 2.** For $\gamma$ in $(NE,D)^*$, we have:

$$\sum_{\gamma_i \in \varphi(\gamma)} q^{\text{bounce}(\gamma_i)} t^{\text{area}(\gamma_i)} = s_{\text{bounce}(\gamma)}(q,t),$$

$$\langle \nabla e_n, e_{n-d} h_d \rangle_{\text{Part}} = \sum_{\gamma \in (NE,D)^n} s_{\text{bounce}(\gamma)}(q,t) = \sum_{\gamma \in C^n_d} s_{\text{area}(\gamma)+\binom{n-d}{2}}(q,t), \text{ and,}$$
\begin{equation}
\langle \nabla e_n, s_{d+1,1^{n-d-1}} \rangle_{\text{Part}} = \sum_{\gamma \in \{NE,D\}^{n-1}NE, |\gamma|_{D} = d} s_{\text{bounce}}(\gamma)(q,t) = \sum_{\gamma \in C_d^{n-1}} s_{\text{area}(\gamma)+\binom{n-d}{2}}(q,t).
\end{equation}

**Proof.** Equation (7) follows from Lemma 3. For the first equality of Equation (8), we notice that $s_a(q,t) = q^a + q^{a-1}t + \cdots + qt^{a-1} + t^a$, and, thus, by Haglund’s Theorem a one part term in $\langle \nabla e_n, e_{n-d}h_d \rangle$ can be associated to a path $\gamma$ in $\text{Sch}_{n,d}$ such that area($\gamma$) = 0. But all of these are in $\{NE,D\}^n$ and have $d$ diagonals steps. For this reason, by Equation (7), the equality holds. The second equality of Equation (8) follows from Lemma 6. Finally, for Equation (9) we only need to notice that paths of $C_d^{n-1}$ ending with a north step are in bijection with paths of $C_d^{n-1}$ and have the same area. The result is a consequence of Haglund’s Theorem, Lemma 6 and Equation (7). □

We end this section with a result needed for the generalization of Theorem 1.

**Corollary 1.** Let $d$ be an integer and $\gamma$ be a path of the set $\text{Sch}_{n,d}$. Then, $\gamma = \gamma'NDED^jNE$ or $\gamma = \gamma'NEDN^{j}NNEE$, with $\gamma' \in \{NE,D\}^*$ if and only if area($\gamma$) = 1 and $\gamma$ contributes to a Schur function indexed by a partition of length 1 in $\langle \nabla(e_n), s_{d+1,1^{n-d-1}} \rangle$.

**Proof.** If $\gamma = \gamma'NDED^jNE$ or $\gamma = \gamma'NEDN^{j}NNEE$, with $\gamma' \in \{NE,D\}^*$, then, by Lemma 4, $\gamma$ is in the image of $\varphi$ and the result follows from Proposition 2.

If area($\gamma$) = 1 and $\gamma$ contributes to a Schur function indexed by a partition of length 1, by Lemma 4, $\gamma$ is in the image of $\varphi$ and the result follows from Proposition 2. □

5. From Parking Functions Formulas to Schur Functions

The aim of this section is to give a combinatorial formula for $\nabla^m(e_n)$ restricted to Schur functions indexed by one part partitions in the variables $q$ and $t$. We will denote this restriction $\nabla^m(e_n)_{\text{Part}}$. In this section we will be using the diagonal inversion statistic, since bounce is not defined for parking functions. This is the main obstacle to knowing all path related to each Schur functions in the variables $q$ and $t$ in the formula.

In order to obtain a formula for $\nabla^m(e_n)_{\text{Part}}$, we will give the necessary and sufficient conditions for a parking function to have a diagonal inversion statistic of 0. We will also determine the necessary and sufficient conditions for a parking function to have a diagonal inversion statistic of 1 if $m$ is greater or equal to 2.

The next three results are technical and will be mostly used in order to quickly discard some recurring cases.

**Claim 1.** Let $(\gamma,w)$ be in $\mathcal{P}_{n,m}$ such that $\gamma$ has a factor $\gamma' = NE^pN$, with $1 \leq p \leq m$, and the north steps associated to $w_i$ and $w_{i+1}$. Then, if $w_i > w_{i+1}$, $d_i(i+1) = p$ and if $w_i < w_{i+1}$, $d_i(i+1) = p - 1$.

**Proof.** By definition:

\[d_i(i+1) = \chi(w_i < w_{i+1}) \max(0,m - |a_i - a_{i+1}|) + \chi(w_i > w_{i+1}) \max(0,m - |a_{i+1} - a_i + 1|)\]

Hence, by hypothesis, $a_i = a_{i+1} + p - m$. Therefore:

\[d_i(i+1) = \chi(w_i < w_{i+1})(p) + \chi(w_i > w_{i+1})(p - 1)\]

□

**Lemma 7.** Let $(\gamma,w)$ be in $\mathcal{P}_{n,m}$ such that $\gamma$ has a factor $\gamma' = NE^pN$, with $p > m$. Then, dinv$(\gamma,w) \geq m - 1$.
Proof. Let \( w_i \) and \( w_{i+1} \) be the letters of \( w \) associated to the factor \( \gamma' \). Since the path is continuous and over the main diagonal, there exists \( j_1, \ldots, j_m \) such that the \( k \)-th copy (from the top) of \( w_{i+1} \) is to the north on the same diagonal than the north step associated to the letter \( w_{j_k} \) in \( w \). Note that the \( j_k \)’s are not necessarily distinct, ergo, \( d_{j_k}(i+1) = d_s^k(i+1) \) for \( s = j_k \) and \( j_k \) is the \( t \)-th copy (from the top) of \( w_s \). Which means that for \( 1 \leq k \leq m-1 \) when \( w_{j_k} > w_{i+1} \) we get \( d_{j_k}(i+1) \geq 1 \) (see Figure 15) and when \( w_{j_k} < w_{i+1} \) the copy \( k+1 \) of \( w_{i+1} \) is to the north and one diagonal lower than \( w_{j_k} \). Therefore, \( d_{j_k}(i+1) \geq 1 \) (see Figure 16). Hence:

\[
\text{dinv}(\gamma, w) = \sum_{s=1}^{n-1} \sum_{t=1}^{m} \sum_{r>t} d_s^k(r) \geq \sum_{k=1}^{m-1} d_{j_k}(i+1) \geq m-1.
\]

\[\square\]

**Figure 15.**

**Figure 16.**

**Lemma 8.** Let \((\gamma, w)\) be in \( P_{n,mn} \). If there is a factor \( \gamma, \gamma' = NEPN \) associated to the lines \( i-1 \) and \( i \) such that \( p \geq 2 \), then there is \( k \) such that \( d_k(i) = 1 \), if \( m = 1 \) and \( d_k(i) \geq 1 \), if \( m > 1 \).

**Proof.** We will work with \( \tilde{\gamma} \) and \( \tilde{w} \). Therefore we can use a Dyck path in an \( mn \times mn \) grid. Lets suppose there is no such \( k \). Dyck paths have the property of always having more north step than east steps for all prefixes. Hence, there is a line \( j \) in \( \gamma \) such that \( j < i \) and the north step on line \( j_s \) is on the same diagonal than the north step on line \( i_1 \). We can assume \( j_s \) is the upper bound of such lines.

By hypothesis, \( d_j(i) = 0 \) for \((\gamma, w)\), in consequence, \( d_{j_s}(i_1) = 0 \) and we must have \( \tilde{w}_{j_s} > \tilde{w}_{i_1} \), the contrary would lead to \( d_{j_s}(i_1) = 1 \). Since \( p \geq 2 \), there is \( l \) such that \( j \leq l < i \) and the line \( l_r \) is one diagonal over the diagonal passing trough the north step at line \( i_1 \) in \((\tilde{\gamma}, \tilde{w})\). We can assume \( l \) is the smallest line satisfying theses properties. Note that if \( j = l \), then \( r = s-1 \) and if \( j \neq l \), then \( s = 1 \) (see Figure 17 and Figure 18). Again, \( \tilde{w}_{i_r} < \tilde{w}_{i_1} \) or else we would have \( d_{l_r}(i_1) = 1 \). This means \( l \neq j \) and \( l \neq j+1 \), since \( w_j > w_i > w_l \). So there must be at least one east step between the lines \( j_1 \) and \( j+1 \). If there is just one, then the line \( j+1 \) is on the same diagonal than the lines \( i_1 \) and \( j_1 \) contradicting that \( j \) is the upper bound. If there are two or more east steps between
the lines $j_1$ and $j + 1_m$, then the path goes under the diagonal passing through the north steps at line $j_1$ and at line $i_1$. But it must cross it again before the line $l_r$ because the path is continuous and the line $l_r$ is over the diagonal. Which contradicts again that $j$ is the upper bound. \hfill\qed

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure17.png}
\caption{Figure 17.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure18.png}
\caption{Figure 18.}
\end{figure}

We can now state a first condition.

**Lemma 9.** Let $(\gamma, w)$ be in $\mathcal{P}_{n,n}$. If $\text{dinv}(\gamma, w) = 0$, then $\text{read}(\gamma, w) = w^{-1}$. Additionally, all non trivial factors of $\gamma$, $\gamma' = NE^pN$ are such that $p = 1$.

**Proof.** If $p \leq 1$, then by Claim 1, for all factors $NE^pN$ of $\gamma$ we must have $p = 1$ when $w_i > w_{i+1}$ and $p = 0$ when $w_i < w_{i+1}$, since $\text{dinv}(\gamma, w) = 0$.

If $\gamma' = NE^pN$ is a factor of $\gamma$ associated to lines $i$ and $i + 1$ such that and $p > 1$, then, by Lemma 8, there is $k$ such that $d_k(i + 1) = 1$. Therefore, $p \neq 1$ because $\text{dinv}(\gamma, w) = 0$. Finally, $\text{read}(\gamma, w) = w^{-1}$, is a direct consequence of $p \leq 1$. \hfill\qed

The same result is also true for general parking functions when $m \geq 2$. The following gives somewhat of a generalization.

**Lemma 10.** Let $a$ and $m$ be an integers such that $2 \leq a \leq m$ and $(\gamma, w)$ be in $\mathcal{P}_{n,nm}$. If $\text{dinv}(\gamma, w) = a - 2$, then $\text{read}(\gamma, w) = w^{-1}$. Additionally, all factor of $\gamma$, $\gamma' = NE^pN$ are such that $p \leq m$.

**Proof.** By hypothesis $\text{dinv}(\gamma, w) = a - 2$. If $p > m$, then, by Lemma 7, $a - 2 \geq m - 1$, which contradicts $a \leq m$. Hence, all factors $\gamma' = NE^pN$ of $\gamma$ are such that $p \leq m$. Thereafter, all factors $\gamma_{i,j} = NE^{p_i}NE^{p_{i+1}} \cdots NE^{p_{j-1}}N$, with $i < j$, satisfy $|\gamma_{i,j}|_E = \sum_{k=i}^{j-1}p_k \leq (j - i)m = m|\gamma_{i,j}|_N$. Consequently, for all $i < j$ we read $w_j$ before $w_i$ in read$(\gamma, w)$ and read$(\gamma, w) = w^{-1}$ as stated. \hfill\qed

Obviously, $\text{read}(\gamma, w) \neq w^{-1}$ in general. But sometimes $\text{read}(\gamma, w) = w^{-1}$ even without the condition on diagonal inversion. The last part of the proof gave us a weaker yet more general statement.

**Claim 2.** Let $m$ be an integer and $(\gamma, w)$ be in $\mathcal{P}_{n,nm}$. If all factor of $\gamma$, $\gamma' = NE^pN$ are such that $p \leq m$, then $\text{read}(\gamma, w) = w^{-1}$. 
Sadly Lemma 10 dose not apply for $m = 2$, when the diagonal inversion statistic has value 1. Therefore, we have to prove it separately.

**Lemma 11.** Let $(\gamma, w)$ be in $\mathcal{P}_{n, 2n}$. If $\text{dinv}(\gamma, w) = 1$, then $\text{read}(\gamma, w) = w^{-1}$. Additionally, all factor of $\gamma$, $\gamma' = NE^pN$ are such that $p \leq 2$.

**Proof.** Suppose there is a factor $NE^pN$ of $\gamma$ such that $p > 2$, associated to lines $j - 1$ and $j$. We can assume $j$ being the smallest line satisfying that property. Let’s consider $\tilde{\gamma}$ the path found by doubling each north step in $\gamma$ and $\tilde{w}$ the word $w_{1,1}w_{1,2}w_{2,1}w_{2,2}\cdots w_{n,1}w_{n,2}$. Considering the path is continuous and $p > 2$, the path goes over the diagonal passing through $w_{j,1}$. The path finishes under that diagonal, ergo there must be $i$ such that $w_{j,2}$ is on the same diagonal as $w_{i,1}$ or $w_{i,2}$ and $w_{i+1,1}$ is strictly over the diagonal passing through $w_{j,1}$. The three cases possible are illustrated by Figure 19, Figure 20 and Figure 21.

For the first case, if $w_i < w_j$, then the pairs $(i_1, j_1)$ and $(i_2, j_2)$ both contribute to dinv. If $w_i > w_j$, then the pairs $(i + 1_2, j_1)$ and $(i_1, j_2)$ both contribute to dinv. Thus, the diagonal inversion statistic cannot be equal to 1.

For the second case, if $w_{i+1} < w_j$, then the pairs $(i + 1_2, j_1)$ and $(i_1, j_2)$ both contribute to dinv, since $w_i < w_{i+1}$. If $w_{i+1} > w_j$, then the pairs $(i + 1_1, j_1)$ and $(i + 1_2, j_2)$ both contribute to dinv. Hence, the diagonal inversion statistic cannot be equal to 1.

For the last case, if $w_i < w_j$, then the pairs $(i_1, j_1)$ and $(i_2, j_2)$ both contribute to dinv. If $w_i > w_j$, then the pairs $(i + 1_2, j_2)$ and $(i_1, j_2)$ both contribute to dinv because $w_i < w_{i+1}$. Ergo, the diagonal inversion statistic cannot be equal to 1.

Therefore, $p \leq 2$ for all factor $NE^pN$ of $\gamma$ and, by Claim 2, we get $\text{read}(\gamma, w) = w^{-1}$.

---

By definition, it is fairly easy to see that for $(\gamma, w)$ the descent set of $w$ is related to the number of columns of $\gamma$. We state the following claim is order to avoid repetition.

**Claim 3.** Let $(\gamma, w)$ be in $\mathcal{P}_{n,m}$, $1 \leq m$. Then, the number of descents of $w$ plus 1 is smaller or equal to the number of distinct columns. Additionally, the descents are at the top of a column.

**Proof.** If $w_i$ and $w_{i+1}$ are in the same columns, then by definition of parking functions we must have $w_i < w_{i+1}$. Therefore, descents must be at the top of a column. The last column can’t have a descent, since the top of that column is $w_n$ and we know the last letter of a permutation can’t be a descent. □
The last result relates the number of distinct columns to the descent set of \( w \). But the following relates the number of distinct columns to the descent set of \( \text{read}(\gamma, w)^{-1} \).

**Lemma 12.** Let \( m \) and \( n \) be integers, \( (\gamma, w) \) be in \( P_{n,m} \) and \( T(\gamma) \) be the number of distinct columns. Let \( \sigma \) be the permutation such that \( \sigma . (\text{read}(\gamma, w)^{-1}) = w \). Then:

\[
\text{dinv}(\gamma, w) \geq \begin{cases} 
T(\gamma) - \text{des}(\text{read}(\gamma, w)^{-1}) - 1 & \text{if } \sigma(n) = n, \\
T(\gamma) - \text{des}(\text{read}(\gamma, w)^{-1}) - 2 & \text{if } \sigma(n) \neq n.
\end{cases}
\]

**Proof.** We will actually show that the letter at the top of a column contribute at least 1 to \( \text{dinv} \) unless they are in the descent set of \( \text{read}(\gamma, w)^{-1} \), in the last column, or the last letter of \( \text{read}(\gamma, w)^{-1} \). Notice that if \( \sigma(n) = n \), then last letter of \( \text{read}(\gamma, w)^{-1} \) is in the last column, so we only need to subtract it once.

Let \( \text{read}(\gamma, w)^{-1} = v_1v_2 \cdots v_n \). Let \( v_i \) be a the top of a column. If \( v_i \) is not in the last column and \( i \neq n \), then by definition of the reading word and because the path is continuous, we have, in \( (\tilde{\gamma}, \tilde{w}) \), these three cases: the last copy from the top of \( v_{i+1} \) is to the north and on the same diagonal than a copy of \( v_i \), lets say the \( k \)-th copy from the top (see Figure 22), the letter \( v_{i+1} \) is to south and on one of diagonals crossing one of the \( m-1 \) first copies of \( v_i \) (see Figure 23), lets say the \( p \)-th copy, or \( v_{i+1} \) is to the south one diagonal higher than the first copy of \( v_i \) (see Figure 24).

Let \( \sigma \) be the permutation that send \( \text{read}(\gamma, w)^{-1} \) to \( w \). When \( i \) is not in the descent set of \( \text{read}(\gamma, w)^{-1} \), our first case yields \( d_{\sigma(i)}(\sigma(i+1)) = k \), by definition of the diagonal inversion statistics. The same reasoning shows \( d_{\sigma(i+1)}(\sigma(i)) = p + 1 \) for the second case and \( d_{\sigma(i+1)}(\sigma(i)) = 1 \) for the last case.

\[\square\]

**Figure 22.**  \[\text{Figure 23.}\]

**Figure 24.**

We can now state necessary and sufficient conditions for the diagonal inversion statistic to be equal to zero.

**Proposition 3.** Let \( (\gamma, w) \) be in \( P_{n,m} \), \( 1 \leq m \). Then, \( \text{dinv}(\gamma, w) = 0 \) if and only if the following conditions apply:

- The path \( \gamma \) can be written as \( \gamma'E^j \) where all factors of \( \gamma' \) of length 2 have at most one east step.
- If \( \{i_1, \ldots, i_k, n\} \) is the set of all lines containing an east step, then \( \{i_1, \ldots, i_k\} \) is the descent set.
- \( \text{read}(\gamma, w) = w^{-1} \).
Proof. If \( \text{dinv}(\gamma, w) = 0 \), then by Claim 1 and Lemma 7, we have the first condition. If \( \gamma' = NE^pN \) is a factor of \( \gamma \) associated to \( w_i \) and \( w_{i+1} \), then by now proven first condition and Claim 1, \( w_i > w_{i+1} \). Hence, the position \( i \) is a descent. But, by Claim 3, we know that the number of descents plus 1 is greater or equal to the number of columns and \( w_n \) contains an east step, ergo the second condition. The last condition follows from Lemma 9 and Lemma 10. Therefore, \( \text{dinv}(\gamma, w) = 0 \) does imply the stated conditions.

Conversely, by Claim 3, the descents are at the top of each column, since \( n \) can’t be a descent. Therefore, by the first condition and by Claim 1, we know that for all line \( i \) with an east step we are at the top of a column and \( d_i(i + 1) = 0 \). For all line \( i \) with an east step and all line \( j \) such that \( j > i + 1 \) we know that \( a_i + (j - i)m \geq a_j \geq a_i + (j - i)(m - 1) \) considering there is at most one east step between each north step. This leads to:

\[
(j - i)m + 1 \geq |a_j - a_i + 1| \geq (j - i)(m - 1) + 1 \geq 2(m - 1) + 1 \geq m,
\]

and:

\[
(j - i)m \geq |a_j - a_i| \geq (j - i)(m - 1) \geq 2(m - 1) \geq m,
\]

Therefore, \( d_i(j) = 0 \) for all \( i \) and \( j \). Hence, \( \text{dinv}(\gamma, w) = 0 \).

Note that the second statement of the previous proposition implies that the number of descents in \( w \) is equal to the number of distinct columns plus 1.

Looking at the specialization \( q = 0 \) is equivalent to looking only at the parking functions \( (\gamma, w) \) such that \( \text{dinv}(\gamma, w) = 0 \), and, thus, we need the area of theses parking functions.

Proposition 4. Let \((\gamma, w)\) be in \(\mathcal{P}_{n, nm}, 1 \leq m \) such that \( \text{dinv}(\gamma, w) = 0 \), then:

\[
\text{area}(\gamma, w) = m\left(\frac{n}{2}\right) - \text{des}(w)n + \text{maj}(w)
\]

Proof. Let \( \{i_1, \ldots, i_k, n\} \) be the lines with east steps. We know that these are tops of columns and by the previous proposition we know that \( \{i_1, \ldots, i_k\} \) is the descent set. By the previous proposition we also have that:

\[
\text{area}(\gamma, w) = m\left(\frac{n}{2}\right) - \sum_{j=1}^{k} (n - i_j)
= m\left(\frac{n}{2}\right) - n\sum_{j=1}^{k} 1 + \sum_{j=1}^{k} i_j
= m\left(\frac{n}{2}\right) - \text{des}(w)n + \text{maj}(w)
\]

\[ \square \]

This last proposition allows us to give a proper formula for \( \nabla^m(e_n)|_{q=0} \). Which is just an extension of the Stanley-Lusztig formula.

Proposition 5. For integers \( n, m \) such that \( 1 \leq n, m \)

\[
\nabla^m(e_n)|_{\text{Part}} = \sum_{\tau \in \text{SYT}(n)} s_{\text{maj}(\tau) + (m-1)(\frac{n}{2})}(q, t)s_{\lambda(\tau)}(X), \text{ and,}
\]

\[
\nabla^m e_n|_{q=0} = t^{(m-1)(\frac{n}{2})} \sum_{w \in S_n} t^{(\frac{n}{2}) - \text{des}(w)n + \text{maj}(w)} F_{\tau(\text{Des}(w^{-1}))}(X) = t^{(m-1)(\frac{n}{2})} \sum_{\tau \in \text{SYT}(n)} t^{\text{maj}(\tau)} s_{\lambda(\tau)}.
\]
Proof. The first equality of Equation (11) follows from Proposition 4. Consequently, by the equation inferred from [Sta79] and [Hai02] (see Equation (6)), we have:

\[ \sum_{w \in S_n} t(w)_{\text{des}(w)\text{maj}(w)} F_{\text{comp}(\text{Des}(w^{-1}))}(X) = \sum_{\tau \in \text{SYT}(n)} t^{\text{maj}(\tau)} s_{\lambda(\tau)}(X). \]

Therefore, the second equality of Equation (11) holds. For Equation (10), we only need to notice that \( \nabla^m(e_n) \) is symmetric in \( q,t \) and \( s_{\lambda}(q,t) = 0 \) if \( \ell(\lambda) > 2 \) and \( s_{a,b}(q,t) = (qt)^b(q^{a-b} + q^{a-b-1}t + \ldots + qt^{a-b-1} + t^{a-b}) \). Hence, \( s_{a,b}(0,t) = 0 \) if \( b \neq 0 \) and \( s_a(0,t) = t^a \). Ergo, we have the stated result by Equation (11).

From this last proposition and Proposition 2 we can obtain the following \( q \)-analogues that are used in [Wal19].

**Corollary 2.** Let \( n, m, d \) be integers, then:

\[ \langle \nabla^m(e_n), s_{d+1,1^{n-d-1}} \rangle_{|t=0} = q^{m(1)_{(n)} + \binom{n-d}{2}} \left[ \begin{array}{c} n-1 \\ d \end{array} \right]_q = q^{m(1)_{(n)} - \binom{d+1}{2}} \left[ \begin{array}{c} n-1 \\ d \end{array} \right]_q^{-1}. \]

Proof. We recall that \( \left[ \begin{array}{c} n \\ k \end{array} \right]_q = \sum_{\gamma \subseteq C_k} q^{\text{area}(\gamma)}. \) In consequence, by Proposition 2 and Proposition 5, we have the first equality. The second equality follows from \( \binom{n}{2} - \binom{d+1}{2} = d(n-d-1) + \binom{n-d}{2} \) and \( q^{d(n-d-1)} \left[ \begin{array}{c} n-1 \\ d \end{array} \right]_q^{-1} = \left[ \begin{array}{c} n-1 \\ d \end{array} \right]_q^{-1}. \]

This next lemma emulates Proposition 3 for diagonal inversion statistics values of one.

**Lemma 13.** Let \((\gamma, w)\) be in \( \mathcal{P}_{n,m} \), \( 2 \leq m \) and \( T(\gamma) \) be the number of distinct columns. Then, \( \text{dinv}(\gamma, w) = 1 \) if and only if one of the following conditions apply:

- All factors, \( NE^{\circ} N \) of \( \gamma \) are such that \( p \leq 1 \) and \( T(\gamma) = \text{des}(w) + 2 \).
- Exactly one factor, \( NE^{\circ} N \) of \( \gamma \) is such that \( p = 2 \) all other such factors satisfy \( p \leq 1 \) and \( T(\gamma) = \text{des}(w) + 1 \).

Proof. We will start by proving the statement for \( m \geq 3 \).

If \( \text{dinv}(\gamma, w) = 1 \), by Lemma 10, we have \( \text{read}(\gamma, w) = w^{-1} \). Thus, by Lemma 12 and Claim 3, \( \text{des}(w) + 1 \leq T(\gamma) \leq \text{des}(w) + 2 \).

By Claim 3, if \( T(\gamma) = \text{des}(w) + 1 \), then \( w_i > w_{i+1} \) for all \( i \) at the top of a column. Hence, by Lemma 10, \( p \leq m \). In consequence, by Claim 1, there is exactly one factor, \( NE^{\circ} N \) of \( \gamma \) is such that \( p = 2 \), all other such factors satisfy \( p \leq 1 \).

Again, by Claim 3, if \( T(\gamma) = \text{des}(w) + 2 \), there is exactly one line \( i \) containing an east step such that \( w_i < w_{i+1} \). Ergo, by Claim 1, all factors, \( NE^{\circ} N \) of \( \gamma \) are such that \( p \leq 1 \).

If all factors, \( NE^{\circ} N \) of \( \gamma \) are such that \( p \leq 1 \) and \( T(\gamma) = \text{des}(w) + 2 \), then, by Claim 3, we know there is exactly one \( i \) at the top of a column such that \( w_i < w_{i+1} \). Moreover, \( p \leq 1 \) for all factors \( NE^{\circ} N \) of \( \gamma \), so, by Claim 1, \( \text{dinv}(\gamma, w) = 1 \) if \( d_i(j) = 0 \) for all \( j \geq i + 2 \). But \( p \leq 1 \) means \( m \geq a_{i+1} - a_i \geq m - 1 \). Hence, for \( j > i \), \( |a_j - a_i| \geq (m-1)(j-i) \geq m \) and \( |a_j - a_i + 1| \geq (m-1)(j-i) + 1 \geq m \), since \( j - i \geq 2 \) and \( m \geq 3 \). Consequently, \( d_i(j) = 0 \) for all \( j \geq i + 2 \) and \( \text{dinv}(\gamma, w) = 1 \).

If exactly one factor, \( NE^{\circ} N \) of \( \gamma \) is such that \( p = 2 \) all other such factors satisfy \( p \leq 1 \) and \( T(\gamma) = \text{des}(w) + 1 \). By Claim 3, for all \( i \) at the top of a column \( w_i > w_{i+1} \). Additionally, exactly one factor, \( NE^{\circ} N \) of \( \gamma \) is such that \( p = 2 \), all other such factors satisfy \( p \leq 1 \). In consequence, by Claim 1, \( \text{dinv}(\gamma, w) = 1 \) if \( d_i(j) = 0 \) for all \( j \geq i + 2 \). But \( p \leq 1 \) means \( m \geq a_{i+1} - a_i \geq m - 1 \).
and \( p = 2 \) means \( m \geq a_{i+1} - a_i \geq m - 2 \). Ergo, for \( j > i \), \(|a_j - a_i + 1| \geq (m - 1)(j - i) + 1 \geq m\) because \( j - i \geq 2 \) and \( m \geq 3 \). Thus, \( d_i(j) = 0 \) for all \( j \geq i + 2 \) and \( \text{dinv}(\gamma, w) = 1 \).

For \( m = 2 \) the proof is the same we only need to change references of Lemma 10 to Lemma 11. □

Note that for \( m = 1 \) nothing holds (see Figure 25, Figure 26 and Figure 27) but, in Section 7, we manage to obtain a Proposition 5 type formula for the restriction to hook Schur functions in the variables \( q \) and \( t \), by using Schröder paths and the bounce statistic.

\[ \text{Figure 25. The diagonal inversion statistic is 1 yet } p > 2. \]

\[ \text{Figure 26. The diagonal inversion statistic is 2 yet } p \leq 1 \text{ and } T(\gamma) = \text{des}(1243) + 2 = 3. \]

\[ \text{Figure 27. The diagonal inversion statistic is 2 yet exactly one factor } NEpN \text{ is such that } p = 2 \text{ all others are such that } p \leq 1 \text{ and } T(\gamma) = \text{des}(1432) + 1 = 3. \]

6. Restriction to \( m \)-Schröder Paths

This section is dedicated to the restriction to Schröder paths. With this restriction we can actually give the necessary and sufficient conditions for a path to have a diagonal inversion statistic value of one. Proposition 3 can be restated in terms of Schröder paths.

**Corollary 3.** Let \((\gamma, w)\) be in \( \text{Sch}^{(m)}_{n,d} \), \(1 \leq m\). Then, \( \text{dinv}(\gamma, w) = 0 \) if and only the following conditions apply:

- The path \( \gamma \) can be written as \( \gamma' E^j \) where all factors of \( \gamma' \) of length 2 have at most one east step.
- If \( \{i_1, \ldots, i_k, n\} \) is the set of all lines containing an east step, then \( \{w_{i_1}, \ldots, w_{i_k}\} \subseteq \{n - d + 1, \ldots, n\} \).
- \( \text{read}(\gamma, w) = w^{-1} \).

**Proof.** Condition one and three are consequences of Proposition 3. By definition of Schröder paths \( \text{read}(\gamma, w) \in \{n - d + 1, \ldots, n\} \sqcup \{n - d, \ldots, 1\} \). Therefore, \( \text{read}(\gamma, w)^{-1} = w \in \{n, \ldots, n - d + 1\} \sqcup \{1, \ldots, n - d\} \) and the descents of \( w \) are the positions of \( n - d + 1, \ldots, n \) in \( w \). Hence, the result follows from Proposition 3. □

The restriction to \( m \)-Schröder paths allows us to write the formula of Proposition 5 in terms of paths in a rectangular grid as we did for \( m = 1 \) in Proposition 2.
Corollary 4. Let \( n, d \) and \( m \) be a positive integer such that \( n \geq d \) and \( m > 1 \). Then:

\[
\text{Sch}^m_{n,d}(q,t)|_{1\text{Part}} = \langle \nabla^m e_n, e_{n-d} d^d \rangle |_{1\text{Part}} = \sum_{\gamma \in C^m_{n-1}} s_{(n) - (d+1) - \text{area}(\gamma)}(q,t) + \sum_{\gamma \in C^m_{d}} s_{(m) - (d+1) - \text{area}(\gamma)}(q,t)
\]

Additionally:

\[
(12) \quad \text{Sch}^m_{n,d}(q,t)|_{1\text{Part}} = \langle \nabla^m e_n, s_{d+1,n-d-1} \rangle |_{1\text{Part}} = \sum_{\gamma \in C^m_{d}} s_{(m) - (d+1) - \text{area}(\gamma)}(q,t)
\]

Proof. On account of \( \sum_{\gamma \in C^m_{d}} \text{area}(\gamma) = \sum_{\gamma \in C^m_{d}} (n-1-d) - \text{area}(\gamma) \) and \( (n) = (d+1) = (n-1-d) + (n-1-d)d \), the result follows from Proposition 2 and Proposition 5.

The main proof of this section is very technical case by case proof. It will be used to prove the main theorem via Corollary 6.

Proposition 6. Let \( (\gamma, w) \) be in \( \text{Sch}_{n,d} \) and let \( T(\gamma) \) be the number of distinct columns. Then, \( \text{dinv}(\gamma, w) = 1 \) if and only if one of the following conditions apply:

- All factors, \( NE^p N \) of \( \gamma \) are such that \( p \leq 1 \) and \( T(\gamma) = \text{des}(w) + 2 \).
- Exactly one factor, \( \gamma' = NE^p N \) of \( \gamma \) is such that \( p = 2 \), and all other such factors satisfy \( p \leq 1 \) and \( T(\gamma) = \text{des}(w) + 1 \).
- Exactly one factor, \( \gamma' = NE^p N \) of \( \gamma \) is such that \( p > 2 \) and \( \gamma' \) is associated to lines \( n-1, n, w_n \in \{n-d+1, \ldots, n\} \) and all other such factors satisfy \( p \leq 1 \) and \( T(\gamma) = \text{des}(w) + 2 \) if \( w_n \in \{n-d-1, \ldots, 1\} \) or \( T(\gamma) = \text{des}(w) + 1 \) if \( w_n \in \{n-d+1, \ldots, n\} \).

Proof. If \( \text{dinv}(\gamma, w) = 1 \) and all factors \( \gamma' = NE^p N \) are such that \( p = 1 \), then, by Claim 1, there is exactly one line \( i \) a the top of a column such that \( w_i < w_{i+1} \). Additionally, by Claim 3, we know that \( T(\gamma) \geq \text{des}(w) + 1 \). By Claim 2 and Lemma 12, we have \( T(\gamma) \leq \text{des}(w) + 2 \). Since \( w_n \) is the top of its own column, \( i \) and \( n \) are not descents and the top of all the other columns are descents. Thus, \( T(\gamma) = \text{des}(w) + 2 \).

If \( \text{dinv}(\gamma, w) = 1 \), then, by Lemma 8, we know there is at most one factor of \( \gamma \) say \( \gamma' = NE^p N \) associated to the lines \( i \) and \( i+1 \) such that \( p > 1 \) and there is \( k \) such that \( d_k(i+1) = 1 \).

If \( w_k > w_{i+1} \), then the north step at line \( k \) is on the diagonal above the north step at line \( i+1 \). Moreover, the path is continuous and the north step at line \( i \) is over the diagonal passing trough the north steps at line \( k \) and \( i+1 \), and, thus, there exist \( l < k \) such that the north step at line \( i+1 \) and the north step at line \( l \) are on the same diagonal. Let’s assume \( l \) is the biggest such \( l \). We know, \( w_l > w_{i+1} \) because \( d_l(i+1) = 0 \). This means \( w_k \) is read before \( w_{i+1} \) and \( w_l \) is read before \( w_l \). By definition of Schr¨oder paths read(\( \gamma, w \) \) \( \in \{n-d+1, \ldots, n\} \cup \{n-d, \ldots, 1\} \). Hence, \( w_l \in \{n-d+1, \ldots, n\} \). There is at most one east step between the line \( l \) and the line \( l+1 \), so \( w_{i+1} \) is read before \( w_l \). Ergo, \( w_l > w_{i+1} \). Therefore, \( w_{i+1} \) is not in the same column as \( w_l \). Consequently, \( w_{i+1} \) is on the same diagonal as \( w_l \) on account of there being at most one factor \( NE^p N \) with \( r > 1 \). This contradicts \( l \) is the highest line such that \( l < k, l \) and \( i+1 \) are on the same diagonal. (See Figure 28.)

If \( w_k < w_{i+1} \), \( p \geq 2 \) and there is an east step between the lines \( k \) and \( k+1 \), then \( d_k(i+1) = 1 \) implies the lines \( k \) and \( i+1 \) are crossed by the same diagonal. By Lemma 8, there is exactly one east step between the lines \( k \) and \( k+1 \), and, thus, they are on the same diagonal and \( k \neq i \). Considering \( \text{dinv}(\gamma, w) = 1 \) we need \( d_k(k+1) = 0 \) and \( d_{k+1}(i+1) = 0 \). Therefore, \( w_k > w_{k+1} \) and \( w_{k+1} > w_{i+1} \) which is absurd. (See Figure 29.)
For the case $w_k < w_{i+1}$, $p \geq 2$, $k = i - 1$ and there is no east step between the lines $k$ and $k + 1$. Notice that if $k = i - 1$, there are $i + 1 - k = 2$ north steps. Since $d_k(i + 1) = 1$, we know $k$ and $i + 1$ are on the same diagonal. Hence, there is 2 east step between the north step at line $k$ and the north step at line $i + 1$. In addition, letters on the same diagonal are separated by the same number of east steps than north step. Thus, $p = 2$.

Additionally, by Claim 3, $T(\gamma) \geq \text{des}(w) + 1$. Moreover, $w_i$ is read before $w_{i+1}$, since they are separated by more than one east step and $d_i(i + 1) = 0$. Thus, $w_i > w_{i+1}$. Furthermore, all descent in $w$ contribute to a different column. Hence, if $T(\gamma) > \text{des}(w) + 1$ we must have a change of column at a line $l$, $l \neq i$, such that $l$ is not a descent. Ergo $w_l < w_{l+1}$ and because there is at most one east step between $w_l$ and $w_{l+1}$, by Lemma 8, we must have $d_l(l + 1) = 1$ which is absurd. Therefore, $T(\gamma) = \text{des}(w) + 1$.

If $w_k < w_{i+1}$, $p \geq 2$, $k \neq i - 1$ and there is no east step between the lines $k$ and $k + 1$, then $k \neq i$ and $w_k < w_{k+1}$. Moreover, $d_k(i + 1) = 1$ implies the lines $k$ and $i + 1$ are crossed by the same diagonal. Consequently, the north step at line $k + 1$ is on the diagonal over the north step at line $i + 1$. Hence, $w_{k+1} < w_{i+1}$, since $d_{k+1}(i + 1) = 0$. Thus, $w_{k+1}$ is read before $w_{i+1}$ and $w_{i+1}$ is read before $w_k$ in read($\gamma, w)$. We know read($\gamma, w) \in \{n - d + 1, \ldots, n\} \cup \{n - d, \ldots, 1\}$, ergo, $w_{i+1} \in \{n - d + 1, \ldots, n\}$ and $w_{k+1}, w_k \in \{n - d, \ldots, 1\}$. If $i + 1 \neq n$ and $w_{i+2}$ is in the same column as $w_{i+1}$, then $w_{i+2}$ is read before $w_{i+1}$ and $w_{i+1} < w_{i+2}$. But this is impossible because $w_{i+1}$ is in the set $\{n - d + 1, \ldots, n\}$. Therefore, $w_{i+2}$ and $w_{i+1}$ are on the same diagonal and $w_{i+1} > w_{i+2}$, since $d_{i+1}(i + 2) = 0$. On account of div($\gamma, w) = 1$ and $d_k(i + 1) = 1$, we have $d_k(i + 2) = 0$. The north step at line $i + 2$ is on the same diagonal as the north step at line $k$, ergo $w_k > w_{i+2}$. For this reason, $w_{i+2}$ is read before $w_k$. But, $w_k \in \{n - d, \ldots, 1\}$ means $w_{i+2} > w_k$ which is impossible. So, $i + 1 = n$. (See Figure 30.)

Figure 28.  

Figure 29.  

Figure 30.
them and \(d_j(k) = 0\), unless the north step on lines \(j\) and \(k\) are on the same diagonal. But when the north step a line \(k\) and the north step at line \(j\) are on the same diagonal, \(k > j\) and \(p \leq 1\) we know the line \(j\) is associated to a factor \(NE\) of \(\gamma\) and is at the top of a column. By Claim 2, we have \(\text{read}(\gamma, w) = w^{-1}\), and, thus, \(w_j \in \{n-d+1, \ldots, n\}\) and \(w_k\) is read before \(w_j\). Consequently, \(w_j > w_k\) and \(\text{dinv}(\gamma, w) = 1\).

If exactly one factor, \(\gamma' = NEp^0\) of \(\gamma\) is such that \(p = 2\), and all other such factors satisfy \(p \leq 1\) and \(T(\gamma) = \text{des}(w) + 1\), then by Claim 3, all lines \(j\) at the top of a column are such that \(j\) is in the descent set of \(w\). Hence, if \(w_i\) and \(w_{i+1}\) are associated to the factor \(\gamma' = NE^2N\) of \(\gamma\) \(w_i\) is on the diagonal above \(w_{i+1}\) and \(w_i > w_{i+1}\), so \(d_i(i+1) = 1\). For the same reasons as in the previous case for all \(j\) and \(k\) such that \(j \neq i\) and \(k \neq i+1\), then \(d_j(k) = 0\). Therefore, \(\text{dinv}(\gamma, w) = 1\).

Let consider the case when exactly one factor, \(\gamma' = NEp^0\) of \(\gamma\) is such that \(p > 2\) and \(\gamma'\) is associated to lines \(n-1\), \(n\) and all other such factors satisfy \(p \leq 1\) and \(T(\gamma) = \text{des}(w) + 2\). If we take out the last north step and the last east step and call the new path \(\tilde{\gamma}\), then, by Proposition 3, \(\text{dinv}(\gamma, w_i \cdots w_{n-1}) = 0\). Hence, for all \(1 \leq j < k \leq n-1\) we have \(d_j(k) = 0\). Because \(d_j(k)\) is a local property, it is also true for \((\gamma, w)\). By continuity of the path, since \(p > 2\), there is a line \(k\) such that \(w_k\) and \(w_{k+1}\) are in the same column and the north step a line \(n\) is on the same diagonal than the north step at line \(k\). Thus, we read \(w_{k+1}\) before \(w_n\) and \(w_n\) before \(w_k\) in \(\text{read}(\gamma, w)\). Moreover, \(w_{k+1} > w_k\), and, therefore, \(w_k \in \{n-d, \ldots, 1\}\) and \(w_n > w_{k+1} > w_k\), is a consequence of \(w_n \in \{n-d+1, \ldots, n\}\). So, \(d_k(n) = 1\) and \(d_{k+1}(n) = 0\). All other factors \(\gamma'' = NEp'\) satisfy \(p' \leq 1\), if there is \(j\) distinct from \(k\) such that \(w_j\) is on the same diagonal than \(w_n\), then \(w_j, w_{j+1}, \ldots, w_k\) are all on the same diagonal and \(w_j, w_{j+1}, \ldots, w_{k-1}\) are at the top of their column. Furthermore, the condition \(T(\gamma) = \text{des}(w) + 2\) if \(w_{i+1} \in \{n-d, \ldots, 1\}\) or \(T(\gamma) = \text{des}(w) + 1\) if \(w_{i+1} \in \{n-d+1, \ldots, n\}\) forces all tops of column except for \(w_{n-1}\) and \(w_n\) to be in \(\{n-d+1, \ldots, n\}\). Consequently, \(w_j > w_{j+1} > \cdots > w_{k-1} > w_n > w_k\) and \(d_i(n) = 0\), for all \(j \leq l \leq k - 1\). Therefore, \(\text{dinv}(\gamma, w) = 1\).

The next corollary can also be deduced from the more general Lemma 13. We only state it here, so one can notice that Proposition 6 is hiding a general statement for \(\text{Sch}_{n,d}^{(m)}\).

**Corollary 5.** Let \(m\) be an integer such that \(m \geq 2\), \((\gamma, w)\) be in \(\text{Sch}_{n,d}^{(m)}\) and let \(T(\gamma)\) be the number of distinct columns. Then, \(\text{dinv}(\gamma, w) = 1\) if and only if one of the following conditions apply:

- All factors, \(NEp^0\) of \(\gamma\) are such that \(p \leq 1\) and \(T(\gamma) = \text{des}(w) + 2\).
- Exactly one factor, \(\gamma' = NEp^0\) of \(\gamma\) is such that \(p = 2\), and all other such factors satisfy \(p \leq 1\) and \(T(\gamma) = \text{des}(w) + 1\), then \(\text{dinv}(\gamma, w) = 1\).

**Proof.** The proof of the previous proposition can be extended to \(\tilde{\gamma}\), since \(w_i = w_j\) only if they are in the same column.

The following is the restriction to unlabelled Dyck paths.

**Corollary 6.** Let \(m\) be an integer such that \(m \geq 1\), \((\gamma, w)\) be in \(\text{Sch}_{n,0}^{(m)}\) and let \(T(\gamma)\) be the number of distinct columns. Then, \(\text{dinv}(\gamma, w) = 1\) if and only if one of the following conditions apply:

- All factors, \(NEp^0\) of \(\gamma\) are such that \(p \leq 1\) and \(T(\gamma) = \text{des}(w) + 2\).
- Exactly one factor, \(\gamma' = NEp^0\) of \(\gamma\) is such that \(p = 2\), and all other such factors satisfy \(p \leq 1\) and \(T(\gamma) = \text{des}(w) + 1\), then \(\text{dinv}(\gamma, w) = 1\).

**Proof.** For \(m = 1\), the proof is a direct consequence of \(\text{read}(\gamma, w) = n \cdots 1\) and Proposition 6. For \(m > 1\) the proof follows from the last corollary.
7. Bijections With Tableaux

From Equation (11) of Proposition 5, one could wonder what tableau is associated to what path. In this section, we will first show a bijection between standard Young tableaux of shape \((d, 1^{n-d})\) and the subset of Schröder paths \(\{\gamma \in \widetilde{\text{Sch}}_{n,d-1} \mid \text{area}(\gamma) = 0\}\). Afterwards, we exhibit a bijection between the set of paths \(\{\gamma \in \text{Sch}_{n,d-1} \mid \text{area}(\gamma) = 1\}\) and a standard Young tableau \((d, 1^{n-d})\) and a number \(i, 0 \leq i \leq n - d\). This last bijection will allow us to write the sum over these paths with the area and bounce statistics in terms of hook shaped Schur functions in the variables \(q\) and \(t\). In other word we will obtain an explicit combinatorial formula for the expansion in Schur functions of \(\langle \nabla(e_n), s_{\mu}\rangle|_{\text{hooks}}\).

Before we start let's also notice that these bijections could easily be extended to paths ending with a diagonal step, by using the bijection between paths with \(d\) diagonal steps that end with the factor \(\text{NE}\) and paths with \(d + 1\) diagonal steps that end with a \(D\) step.

Recall in Section 3 we defined the touch points of a path. Notice that for a path \(\gamma\) if \(\text{area}(\gamma) = 0\) and \(\text{Touch}(\gamma) = (\gamma_1, \gamma_2, \ldots, \gamma_k)\), then for all \(i, \gamma_i\) is in \(\{\text{NE}, D\}\). Let define the sets \(\widetilde{\text{Sch}}_{n,d,(i)}\) by:

\[
\widetilde{\text{Sch}}_{n,d,(i)} = \{\gamma \in \widetilde{\text{Sch}}_{n,d} \mid \text{area}(\gamma) = i\}.
\]

Let \(\{M_{n,d}\}\) be a family of maps:

\[
M_{n,d} : \text{SYT}(d, 1^{n-d}) \rightarrow \widetilde{\text{Sch}}_{n,d-1,(0)} \quad \tau \mapsto \gamma_1\gamma_2\cdots\gamma_n,
\]

with \(\gamma_n = \text{NE}, \gamma_{n-i} = \text{NE}\) if \(i \in \text{Des}(\tau)\) and \(\gamma_{n-i} = \text{D}\) otherwise (see Figure 31 for an example).

Let \(\{R_{n,d}\}\) be a family of maps:

\[
R_{n,d} : \widetilde{\text{Sch}}_{n,d-1,(0)} \rightarrow \text{SYT}(d, 1^{n-d}) \quad \gamma \mapsto \tau,
\]

with \(\text{Des}(R_{n,d}(\gamma)) = \{n - i \mid 1 \leq i \leq n - 1, \gamma_i = \text{NE} \in \text{Touch}(\gamma)\}\) (see Figure 32 for an example).

\[
\begin{align*}
\tau &= \begin{array}{cccc}
4 & 3 \\
1 & 2 & 5 & 6
\end{array} & \mapsto & \begin{array}{cccc}
\gamma_1 \\
\gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6
\end{array} \\
\text{Des}(\tau) &= \{2, 3\} & \text{Touch}(\gamma) &= \{D, D, \text{NE}, D, \text{NE}, \text{NE}\}
\end{align*}
\]

\[
\begin{align*}
\gamma &= \begin{array}{cccc}
\gamma_1 \\
\gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6
\end{array} & \mapsto & \begin{array}{cccc}
4 \\
2 & 1 & 3 & 5 & 6
\end{array}
\end{align*}
\]

**Figure 31.** Example of the application of map \(M_{6,4}\).

**Figure 32.** Example of the application of map \(R_{6,4}\).

**Lemma 14.** The families of maps \(\{M_{n,d}\}\) and \(\{R_{n,d}\}\) are well defined.

**Proof.** We have already seen that \(\gamma\) in \(\widetilde{\text{Sch}}_{n,d-1,(0)}\) is represented by a word in \(\{\text{NE}, D\}^{n-1}\text{NE}\) such that \(|\gamma|_D = d - 1\). Moreover, \(M_{n,d}(\tau)\) is in \(\{\text{NE}, D\}^{n-1}\text{NE}\) by construction. For \(\tau \in \text{SYT}(d, 1^{n-d})\) the descent set of the tableau \(\tau\) is a subset of \(n - d\) elements in \(\{1, \ldots, n - 1\}\), since for \(j \neq 1\) in
the first column $j - 1$ is lower or to the right of $j$ by definition of hook-shaped standard tableaux. Hence, there are $n - d + 1$, $i$'s such that $\gamma_{n-i} = NE$ and $|M_{n,d}(\tau)|_D = d - 1$.

For a map $R_{n,d}$, notice that hooked-shaped tableaux are uniquely defined by their descent set. Furthermore, there are $n - d + 1$ north steps in $\gamma$ and $\gamma_n = NE \in \text{Touch}(\gamma)$. Thus, for $1 \leq i \leq n - 1$ there are $n - d$ factors $\gamma_i$ such that $\gamma_i = NE$ and $\text{des}(R_{n,d}(\tau)) = n - d$. Ergo, $R_{n,d}(\gamma) \in \text{SYT}(d, 1^{n-d})$.

**Proposition 7.** Let $n, d$ be integers such that $0 \leq d \leq n$. The map $M_{n,d}$ is a bijection from the set of standard tableaux of shape $(d, 1^{n-d})$ to the subset of Schröder paths $\widehat{\text{Sch}}_{n,d-1,0}$, of inverse $R_{n,d}$. Moreover, the map $M_{n,d}$ is such that $\text{maj}(\tau) = \text{bounce}(M_{n,d}(\tau))$ and the map $R_{n,d}$ is such that $\text{maj}(R_{n,d}(\gamma)) = \text{bounce}(\gamma)$.

**Proof.** For the first statement we only need to show that $M_{n,d}$ and $R_{n,d}$ are inverse maps. Let $\tau$ be in $\text{SYT}(d, 1^{n-d})$ if $i$ is in $\text{Des}(\tau)$ (respectively, $i$ is not in $\text{Des}(\tau)$), then the map $M_{n,d}$ sends $i$ to $\gamma_{n-i} = NE$ in $\text{Touch}(M_{n,d}(\tau))$ (respectively, to $\gamma_{n-i} = D$) and $R_{n,d}$ sends $\gamma_{n-i} = NE$ in $\text{Touch}(M_{n,d}(\tau))$ to $i$ in $\text{Des}(R_{n,d}(M_{n,d}(\tau)))$ (respectively, to $i$ not in $\text{Des}(R_{n,d}(M_{n,d}(\tau)))$). Hence, $R_{n,d}(M_{n,d}(\tau)) = \tau$, since hooked-shaped tableaux are uniquely determined by their descent set. For $\gamma$ in $\{\gamma \in \text{Sch}_{n,d-1} \mid \text{area}(\gamma) = 0\}$, the proof of $M_{n,d}(R_{n,d}(\gamma)) = \gamma$ is quite similar.

Additionally, in $\text{Sch}_{n,d-1,0}$ all east steps are associated to a peak. For $\gamma = \gamma_1\gamma_2 \cdots \gamma_n$, if $\gamma_{n-i} = NE$ and there are $k$ diagonal steps after the factor $\gamma_{n-i}$, then the peak associated to that factor contributes $k$ to numph and there is a return to the main diagonal after the factor $\gamma_{n-i}$ that contributes $i - k$ to $\text{bounce}(\Gamma(\gamma))$. Moreover, the peak $\gamma_n = NE$ contributes nothing to $\text{bounce}(\gamma)$ because it is the end of the path. Hence, $\text{maj}(\tau) = \text{bounce}(M_{n,d}(\tau))$ and $\text{maj}(R_{n,d}(\gamma)) = \text{bounce}(\gamma)$, since both maps associate the factor $\gamma_{n-i} = NE$ in the touch sequence to $i$ in the descent set. $\square$

This last proposition gives a combinatorial formula for $\langle \nabla e_n, s_{d, 1^{n-d}} \rangle$ in terms of the major index as in Proposition 5. Although we now know to which paths the top weight are associated to (see Section 9 for more on this).

**Corollary 7.** Let $n, d$ be positive integer such that $n \geq d$. Then:

$$\langle \nabla e_n, s_{d, 1^{n-d}} \rangle |_{\text{Part}} = \sum_{\tau \in \text{SYT}(d, 1^{n-d})} s_{\text{maj}(\tau)}(q, t)$$

**Proof.** By Proposition 2 and Proposition 7. $\square$

In order to get the same type of formula for the restriction on hook-shaped partitions, we will partition the set $\text{Sch}_{n,d-1,1}$. We need maps to do so. Let $\tau$ be in $\text{SYT}(d, 1^{n-d})$, then we define the path $\gamma_\tau = \gamma_1\gamma_2 \cdots \gamma_{n-1}$ where:

$$\gamma_{n-i} = \begin{cases} NE & \text{if } i = 1 \text{ or } i \in \text{Des}(\tau) \setminus \{\max(\text{Des}(\tau))\} \\ D & \text{if } i \notin \text{Des}(\tau) \cup \{1\} \\ \text{NNEE} & \text{if } i = \max(\text{Des}(\tau)) \text{ and } 1 \in \text{Des}(\tau) \\ \text{NDE} & \text{if } i = \max(\text{Des}(\tau)) \text{ and } 1 \notin \text{Des}(\tau) \end{cases}$$

Let $V_{n,d}$ be a collection of sets defined by:

$$V_{n,d} = \{\gamma \in \widehat{\text{Sch}}_{n,d-1} \mid \gamma = D^j \text{NNEE}u \text{ or } \gamma = D^j \text{NDE}u, j \geq 0, u \in \{NE, D\}^*NE\}.$$
Our first family of maps $S_{n,d}$ is defined as follows, for $n - d \geq 1$:

$$S_{n,d} : \text{SYT}(d, 1^{n-d}) \to V_{n,d}$$

$$\tau \mapsto \gamma_\tau$$

The second family of maps is defined as follows, for $n - d \geq 1$:

$$T_{n,d} : V_{n,d} \to \text{SYT}(d, 1^{n-d})$$

$$\gamma \mapsto \tau_\gamma$$

Where for $U = \{1\}$ if $\gamma = D^iNDEv$, $U = \emptyset$ if $\gamma = D^iNNEEu$, $u \in \{NE, D\}^*$, $v \in \{NE, D\}^*NE$ we have $\text{Des}(\tau_\gamma) = \{n - i : \gamma_i = Nw \in \text{Touch}(\gamma), w \in \{NEE, E, DE\}^*\}\{U\}$.

**Figure 33. Example of the map $S_{6,4}$**  

**Figure 34. Example of the map $T_{6,4}$**

**Lemma 15.** For $n, d$ be positive integers such that $n - d \geq 1$, the maps $S_{n,d}$ and $T_{n,d}$ are well defined.

**Proof.** The factor $\gamma_{n-1}$ of $\gamma_\tau$ contains a north step and all the other north steps are related to an element in the descent set. Hence, we have $n - d + 1$ north steps and $\gamma_\tau$ is an element of $\text{Sch}_{n,d-1}$. Moreover, the construction of $\gamma_\tau$ is based on four mutually exclusive conditions that define $\gamma_{n-i}$. If $i = \max(\text{Des}(\tau))$, then $n - i \leq n - k$ for all $k \in \text{Des}(\tau)$. Thus, by construction the path starts with $D^iNNE$ or $D^iNDE$. Since there is only one maximum of the descent set the map $S_{n,d}$ is well defined.

Let's notice that for all path in $V_{n,d}$ there is exactly one factor in $\{NNEE, NDE\}$ and all other are in $\{NE, D\}$. Hence, $\text{Touch}(\gamma)$ has $n - 1$ factors because it is equivalent to counting the numbers of north and diagonal steps minus one. Therefore, the descent set is included in $\{1, \ldots, n - 1\}$. Moreover, there are $n - d + 1$ north steps and one is in the factor $\gamma_{n-1}$. Ergo, $1 \in \{n - i : \gamma_i = Nw \in \text{Touch}(\gamma), w \in \{NEE, E, DE\}^*\}$.

If $\gamma = D^iNNEEu$, $u \in \{NE, D\}^*$, then there are $n - d + 1$ north steps for $n - d$ factors containing a north step. Consequently, $\text{des}(\tau_\gamma) = n - d$ and it uniquely determines a hooked-shaped tableau in $\text{SYT}(d, 1^{n-d})$.

If $\gamma = D^iNDEv$, $v \in \{NE, D\}^*NE$, then there are $n - d + 1$ north steps for $n - d + 1$ factors containing a north step. But $U$ takes out one from the descent set. Hence, the descent set has $n - d$ elements and it uniquely determines a hooked-shaped tableau in $\text{SYT}(d, 1^{n-d})$. Consequently, $T_{n,d}$ is well defined.

The interesting thing about these maps is that they preserve statistics as shown in the next lemma.
Lemma 16. Let \( n, d \) be positive integers such that \( n - d \geq 1 \). The maps \( S_{n,d} \) and \( T_{n,d} \) are bijective maps and \( T_{n,d} = S_{n,d}^{-1} \). Moreover, the maps \( T_{n,d} \) and \( S_{n,d} \) preserve statistics in the following way:

\[
\text{bounce}(\gamma) = \text{maj}(T_{n,d}(\gamma)) - \text{des}(T_{n,d}(\gamma)),
\]

\[
\text{bounce}(S_{n,d}(\tau)) = \text{maj}(\tau) - \text{des}(\tau).
\]

Proof. Let \( \tau \) be a Standard Young tableau in of shape \((d, 1^{n-d})\), \( S_{n,d}(\gamma) = \gamma \), \( \gamma = \gamma_1 \gamma_2 \cdots \gamma_{n-1} \). For \( i \geq 2 \), if the factor \( \gamma_{n-i} = \text{NE} \) then the map \( T_{n,d} \) sends that factor to \( i \in \text{Des}(T_{n,d}(\gamma_\tau)) \). But the map \( S_{n,d} \) gives us \( \gamma_{n-i} = \text{NE} \) when \( i \in \text{Des}(\tau) \). If \( \gamma_{n-i} = \text{NNEE} \), then the map \( T_{n,d} \) send that factor to \( 1, i \in \text{Des}(T_{n,d}(\gamma_\tau)) \) and the map \( S_{n,d} \) gives \( \gamma_{n-i} = \text{NNEE} \) when \( 1, i \in \text{Des}(T_{n,d}(\gamma_\tau)) \). Finally, if \( \gamma_{n-i} = \text{NDE} \), then the map \( T_{n,d} \) send that factor to \( i \in \text{Des}(T_{n,d}(\gamma_\tau)) \) and the image of map \( S_{n,d} \) is \( \gamma_{n-i} = \text{NDE} \) when \( i \in \text{Des}(T_{n,d}(\gamma_\tau)) \). Thus, \( T_{n,d}(S_{n,d}(\tau)) = \tau \). The proof of \( S_{n,d}(T_{n,d}(\gamma)) = \gamma \) is similar.

The proof that \( \text{bounce}(S_{n,d}(\tau)) = \text{maj}(\tau) - \text{des}(\tau) \) is almost the same as the proof in Proposition 7. Since the factor go only up to \( \gamma_{n-1} \), we need to subtract one to each contribution to \( \text{bounce}(\Gamma(\gamma)) \). Moreover, if \( \gamma_{n-i} = \text{NNEE} \) and there are \( k \) diagonal steps after the factor \( \gamma_{n-i} \), then the peak associated to that factor contributes \( k \) to numph and there is a return to the main diagonal after the factor \( \gamma_{n-i} \) that contributes \( i - k - 1 \) to \( \text{bounce}(\Gamma(\gamma)) \). Finally, if \( \gamma_{n-i} = \text{NDE} \) and there are \( k \) diagonal steps after the factor \( \gamma_{n-i} \), then the peak associated to that factor contributes \( k \) to numph and there is a return to the main diagonal after the factor \( \gamma_{n-i} \) that contributes \( i - k - 1 \) to \( \text{bounce}(\Gamma(\gamma)) \). But \( 1 \notin \text{Des}(T_{n,d}(\gamma)) \) which mean we do not add \( 1 - 1 = 0 \) to \( \text{Des}(T_{n,d}(\gamma)) \). Hence, \( \text{maj}(\tau) - \text{des}(\tau) = \text{bounce}(S_{n,d}(\tau)) \) and \( \text{maj}(T_{n,d}(\gamma)) - \text{des}(T_{n,d}(\gamma)) = \text{bounce}(\gamma) \) because both maps associate the factor \( \gamma_{n-i} = \text{NE} \) to \( i \) in the descent set. Consequently, \( \text{bounce}(S_{n,d}(\tau)) = \text{maj}(\tau) - \text{des}(\tau) \). The map \( S_{n,d} \) is a bijection of inverse \( T_{n,d} \), and, thus, we also have \( \text{bounce}(\gamma) = \text{maj}(T_{n,d}(\gamma)) - \text{des}(T_{n,d}(\gamma)) \). \( \square \)

In order to extend the maps \( S_{n,d} \) and \( T_{n,d} \) we need to partition the paths of \( \text{Sch}_{n,d}(1) \). Notice that \( \text{Sch}_{n,n}(1) = \text{Sch}_{n,n-1}(1) = \emptyset \) and \( \text{Sch}_{n,n-2}(1) = \{D^{n-2}NNEE, D^iNDED^jNE \mid i + j = n - 3\} \).

For \( d = n - 2 \), we define \( \Pi_{n,d-1} \) to be the identity map. For \( n - d + 1 \geq 3 \), let

\[
\Pi_{n,d-1} : \text{Sch}_{n,d-1}(1) \rightarrow \text{Sch}_{n,d-1}(1)
\]

\[
\begin{align*}
uNNEED^jNEv & \mapsto uNED^jNNEEv \\
uNDED^jNEv & \mapsto uNED^jNDEv \\
uNNEED^jNE & \mapsto uNED^jNNEE \\
D^iNEuD^jNNEE & \mapsto D^iNNEEuD^jNE \\
D^iNEuD^jNDED^jNE & \mapsto D^iNDEuNED^jNE
\end{align*}
\]

For \( u \) in \( \{NE, D\}^* \) and \( v \) in \( \{NE, D\}^*NE \).

Proposition 8. For all integers \( n \) and \( d \) such that \( n - d \geq 1 \). The sets \( \{\Pi_{n,d-1}^k(\gamma_\tau)\} \) over all \( \tau \in \text{SYT}(d, 1^{n-d}) \) are a partition of the set \( \text{Sch}_{n,d-1}(1) \). Additionally, \( \Pi_{n,d-1} \) is cyclic of order \( n - d \).

Proof. Let us first notice that the map is well defined, since there is exactly one factor \( NNEE \) or \( NDE \) in path of area 1. Secondly, \( \gamma_\tau \) is of shape \( D^iNNEED^jNEuNE \) or \( D^iNDED^jNEuNE \) and the action of \( \Pi_{n,d-1} \) is to exchange the factor \( NNEE \) (respectively, \( NDE \)) with the factor next \( NE \).
factor. Because there are \( n - d + 1 \) north steps in \( \gamma_r \), if \( NNEE \) (respectively, \( NDE \)) is a factor of \( \gamma_r \), there are \( n - d - 1 \) (respectively, \( n - d \)) factors \( NE \) above the factor \( NNEE \) (respectively, \( NDE \)) and \( \Pi^k_{n,d-1}(\gamma_r) = D^j NED^j NEEuNNEE \) (respectively, \( \Pi^k_{n,d-1}(\gamma_r) = D^j NED^j NEEu'NDED^kNE \), where \( u = u'NED^k \)). Therefore, \( \Pi^k_{n,d-1}(\gamma_r) = \gamma_r \).

Let \( \gamma \) be a path in \( \tilde{\text{Sch}}_{n,d-1,1} \). Then, there is a unique factor \( NNEE \) or \( NDE \) which corresponds to the line of area 1. Hence, \( \gamma = D^j NEEu'NNEE u''NE \) (respectively, \( \gamma = D^j NEEu'NDEu''NE \), \( \gamma = D^j NEEu'NNEE \)), with \( u' \) and \( u'' \) in \( \{NE, D\} \). Let \( k \) be the number of north step before the factor \( NNEE \) (respectively, \( NDE, NNEE \), with \( k = n - d - 1 \)). Ergo, for \( \gamma_0 = D^j NEEu'NNEE u''NE \) (respectively, \( \gamma_0 = D^j NEEu'NDEu''NE \), \( \gamma_0 = D^j NEEu''NE \)) we have \( \Pi^k_{n,d-1}(\gamma_0) = \gamma \). Thus, \( \tau_{n,d}(\gamma_0) = \tau_{n,d}(\gamma) \). So, \( \gamma \) is in the set \( \{\Pi^k_{n,d-1}(\gamma_r)\} \).

Finally, let \( \gamma \) be in \( \{\Pi^k_{n,d-1}(\gamma_r)\} \cap \{\Pi^k_{n,d-1}(\gamma_\rho)\} \), then there is \( k \) such that \( \gamma = \Pi^k_{n,d-1}(\gamma_\rho) \) and \( l \) such that \( \gamma = \Pi^k_{n,d-1}(\gamma_\rho) \). Hence, \( \Pi^{n-k+l}_{n,d-1}(\gamma_\rho) = \gamma_\tau \). By previous statements we know \( k = l \); it is the number of north step before the factor \( NNEE \) of \( NDE \) in \( \gamma \). Consequently, \( \gamma_\rho = \gamma_\tau \). Thus, by Lemma 16, \( \rho = \tau_{n,d}(\gamma_\rho) = \tau_{n,d}(\gamma_\tau) = \tau \). Therefore \( \{\Pi^k_{n,d-1}(\gamma_r)\} \cap \{\Pi^k_{n,d-1}(\gamma_\rho)\} = \emptyset \) if \( \rho \neq \tau \).

One might see similarities between the next lemma and Lemma 3, since the bounce statistic increases by exactly one with each iteration of \( \Pi_{n,d-1} \). But it is worth mentioning that in this case the area remains one. Thus, this is not an extension of the algorithm seen in Section 4 for paths with a \( B \) statistic value of zero.

**Lemma 17.** Let \( U = \{uNED^jNNEE, uNDED^jNE \mid u \in \{NE, D\}^*\} \). The map \( \Pi_{n,d-1} \) is such that \( \text{bounce}(\Pi_{n,d-1}(\gamma)) = \text{bounce}(\gamma) + 1 \) for all \( \gamma \in \tilde{\text{Sch}}_{n,d-1,1} \backslash U \).

**Proof.** If \( \gamma \) as a factor \( NNEE \), then the map \( \Pi_{n,d-1} \) swaps the factor \( NNEE \) with the next factor \( NE \) and all factors return to the main diagonal in \( \Pi_{n,d-1}(\gamma) \). Hence, the number of peak under each diagonal step is left unchanged and \( \text{numph}(\gamma) = \text{numph}(\Pi_{n,d-1}(\gamma)) \). Moreover, area(\( \gamma \)) = 1, and, therefore, there is some \( k \) such that \( \Gamma(\gamma) = (NE)^{n-d-k} NNEE(NE)^{k-1} \). Thus, \( \text{bounce}(\Gamma(\gamma)) = \frac{(n-2)}{2} - k \) and \( \text{bounce}(\Pi_{n,d-1}(\gamma)) = \frac{(n-2)}{2} - k + 1 \).

If \( \gamma \) as a factor \( NDE \), then \( \text{bounce}(\Pi_{n,d-1}(\Gamma(\gamma))) = \text{bounce}(\Gamma(\gamma)) \). Since the map \( \Pi_{n,d-1} \) swaps the factor \( NDE \) with the next factor \( NE \) and all factors return to the main diagonal, in \( \Pi_{n,d-1}(\gamma) \) there is one more peak below the diagonal step coming from the factor \( NDE \) than in \( \gamma \). Moreover, the number of peaks below all the other diagonal steps remain unchanged. Ergo, \( \text{numph}(\gamma) + 1 = \text{numph}(\Pi_{n,d-1}(\gamma)) \).

With this partition we can put forward a bijection between tableaux and Schröder paths.

**Proposition 9.** Let \( Q_{n,d} \) be a map between the product set \( \text{SYT}(d, 1^{n-d}) \times \{0, 1, \cdots, n - d - 1\} \) and the set \( \tilde{\text{Sch}}_{n,d-1,1} \), defined by \( Q_{n,d}(\tau, i) = \Pi_{n,d-1}(S_{n,d}(\tau)) \). Then, the map \( Q_{n,d} \) is a bijection. Moreover, \( \text{bounce}(Q_{n,d}(\tau, i)) = \text{maj}(\tau) - \text{des}(\tau) + i \) for all \( i \) in \( \{0, 1, \cdots, n - d - 2\} \).
Proof. By Lemma 16 and Proposition 8, this map is well defined. Furthermore, for $\gamma$ in $\tilde{\text{Sch}}_{n,d-1,(1)}$ there is a unique $\tau$ in SYT($d,1^{n-d}$) an a unique integer $i$ such that $0 \leq i \leq n - d - 1$ and $\Pi_{n,d-1}^i(\gamma) = \gamma$. Moreover, $S_{n,d}$ is a bijection of inverse $T_{n,d}$. Hence, $T_{n,d}(\gamma) = \tau$ and the pre-image of $\gamma$ is unique. Ergo $Q_{n,d}$ is a bijection.

By the previous lemma bounce($\Pi_{n,d-1}^i(\gamma)$) = $i$ = bounce($\gamma$) if $\Pi_{n,d-1}^i(\gamma)$ is not an element of $U = \{uNED^iNNEE, uNDE^dNIE | u \in \{NE, D\}^*\}$. The proof of Proposition 8 shows that $\Pi_{n,d-1}^{i-1}(\gamma)$ is an element of $U$ if and only if $i = n - d - 1$. Hence, we only need to show bounce($\gamma$) = maj($\tau$) - des($\tau$). But this is true by Lemma 16.

We now restate and prove our main theorem.

**Theorem (1).** If $\mu \in \{(d, 1^{n-d}) \mid 1 \leq d \leq n\}$ and $\nu \vdash n$, then:

\[
\langle \nabla(e_n), s_\mu \rangle_{\text{hooks}} = \sum_{\tau \in \text{SYT}(\mu)} s_{\text{maj}(\tau)}(q,t) + \sum_{i=2}^{\text{des}(\tau)} s_{\text{maj}(\tau)-i,1}(q,t),
\]

\[
\langle \nabla^m(e_n), s_\nu \rangle_{\text{Part}} = \sum_{\tau \in \text{SYT}(\nu)} s_{m(\nu)}(2^{\text{maj}(\tau)},0) = \sum_{\tau \in \text{SYT}(\nu)} s_{m(\nu)}(1^{\text{maj}(\tau)'},0,t),
\]

and:

\[
\langle \nabla^m(e_n), e_n \rangle_{\text{hooks}} = s_{m(\nu)}(q,t) + \sum_{i=2}^{n-1} s_{m(\nu)}(1,1)(q,t).
\]

Note that maj($1^n$) = $\binom{n}{2}$, $\binom{n}{2} - \text{maj}(\tau') = \text{maj}(\tau)$ and des($\tau$) = $n - 1 - \text{des}(\tau)$.

**Proof.** Equation (2) is true by Proposition 5. For $m = 1$ Equation (3) is a direct consequence of Equation (1). Notice that if $\gamma$ is in $\text{Sch}_{n,d}^{(1)}$, then $\gamma E^{(m-1)n}$ is in $\text{Sch}_{n,d}^{(m)}$. Furthermore, the difference in the area of $\gamma$ and the area of $\gamma E^{(m-1)n}$ is exactly $(m - 1)\binom{n}{2}$. Ergo, for $m > 1$ Equation (3) follows from Corollary 6.

The first sum on the right side of Equation (1) follows from Proposition 5. For the second sum on the right side of Equation (1), let’s first notice that, by Proposition 9, there is a bijection between paths of area one with $d$ diagonal steps ending with $NE$ or $NNEE$ and the product set of standard Young tableaux of shape ($d,1^{n-d}$) and the set $\{0, \ldots, n-d-1\}$.

Moreover, one can clearly see that the cyclic action of the map $\Pi_{n,d-1}$, proved in Proposition 8, puts $\Pi_{n,d-1}^{n-d-1}(S_{n,d}(\tau))$ in $\{uNED^jNNEE, vNDE^dNIE | u \in \{NE, D\}^{n-d-2}, v \in \{NE, D\}^{n-d-1}\}$, since it is of order $n - d$. Hence, by Corollary 1 and Proposition 5, the set $\{\Pi_{n,d-1}^{n-d-1}(S_{n,d}(\tau))\}$ are the only paths that contribute to Schur functions indexed by partition of length 1. Consequently, we need only to consider the set $\{0, \ldots, n-d-2\}$ because the second sum relates to partition of length 2. Additionally, des($\tau$) = $n - d$ for $\tau \in \text{SYT}(d,1^{n-d})$, so $2 \leq i \leq \text{des}(\tau)$ if and only if $n - d - 2 \geq \text{des}(\tau) - i \geq 0$. Therefore, we sum from 2 to des($\tau$). With the exception of the paths already contributing to Schur function having only one part, the restriction to a value of one for the area correspond to hooked-shaped Schur functions. Indeed, $s_{a,b}(q,t) = (qt)^b(q^{a-b} + q^{a-b+1}t + \cdots + qt^{a-b} + t^{a-b})$, and, thus, the monomial $q^t$ can only be found when $b \in \{0,1\}$. Ergo, by Proposition 9, we have the stated result.

We actually conjecture this is true for all $\mu$ when $m = 1$. We also know, by Lemma 13, that this is not true for $m > 1$. 

Conjecture 1. For all \( \mu \vdash n \):

\[
\langle \nabla (e_n), s_\mu \rangle_{\text{hooks}} = \sum_{\tau \in \text{SYT}(\mu)} s_{\text{maj}}(\tau)(q,t) + \sum_{i=2}^{\text{des}(\tau)} s_{\text{maj}}(\tau-i,1)(q,t).
\]

8. Inclusion Exclusion

In this section we will see that half the paths in \( \widetilde{\text{Sch}}_{n,d}(1) \) are related to \( \widetilde{\text{Sch}}_{n,d-1}(1) \) and the other half is related to \( \widetilde{\text{Sch}}_{n,d+1}(1) \). Thereafter, this will be used to find a positive formula for an alternating sum. The outcome is needed to prove results on multivariate diagonal harmonics in [Wal].

Recall that \( \widetilde{\text{Sch}}_{n,d}(1) = \{ \gamma \in \widetilde{\text{Sch}}_{n,d} \mid \text{area}(\gamma) = 1 \} \) and \( \widetilde{\text{Sch}}_{n,n}(1) = \widetilde{\text{Sch}}_{n,n-1}(1) = \emptyset \). Now, let:

\[
\text{Sch}_{n,d} = \{ \gamma \in \widetilde{\text{Sch}}_{n,d}(1) \mid \gamma = D^j \text{NNEE}\gamma'\text{NENE} \text{ or } \gamma = \gamma'\text{NED}\text{D}\text{NNEE}\gamma'', j \geq 0 \},
\]

and:

\[
\text{Sch}_{n,d} = \{ \gamma \in \widetilde{\text{Sch}}_{n,d}(1) \mid \gamma = D^j \text{NNEE}\gamma'\text{DNE} \text{ or } \gamma = \gamma'\text{NED}\text{D}\text{NNEE}\gamma'', j \geq 0 \}.
\]

Lemma 18. For \( 1 \leq d \leq n - 4 \) we have the following equality \( \widetilde{\text{Sch}}_{n,d}(1) = \text{Sch}_{n,d} \cup \text{Sch}_{n,d} \). Additionally, \( \text{Sch}_{n,0}(1) = \text{Sch}_{n,0} \), \( \text{Sch}_{n,n-3}(1) = \text{Sch}_{n,n-3} \cup \text{Sch}_{n,n-3} \cup \{D^{n-3}\text{NNEE}\text{NE}\} \) and \( \text{Sch}_{n,n-2}(1) = \text{Sch}_{n,n-2} \cup \{D^{n-2}\text{NNEE}\} \). Furthermore, for all \( d \), \( \text{Sch}_{n,d} \cap \text{Sch}_{n,d} = \emptyset \).

Proof. A simple check shows that the four cases of \( \text{Sch}_{n,d} \) and \( \text{Sch}_{n,d} \) are mutually exclusive. Hence, \( \text{Sch}_{n,d} \cap \text{Sch}_{n,d} = \emptyset \). The cases \( d = 0, d = n - 2 \) and \( d = n - 3 \) are related to the maximal number of north steps and diagonal steps. For \( d \) general, \( 1 \leq d \leq n - 4 \), let \( \gamma \) be in \( \widetilde{\text{Sch}}_{n,d}(1) \). When a Schröder path has an area value of 1, there is a factor \( \pi = \text{NNEE} \text{ or } \pi = \text{DNE} \) such that \( \gamma = u\pi v \), \( u \in \{\text{NE}, D\}^* \) and \( v \in \{\text{NE}, D\}^*\text{NE} \cup \{\varepsilon\} \). By definition of \( \widetilde{\text{Sch}}_{n,d}(1) \), if \( v = \varepsilon \), then \( \pi = \text{NNEE} \) and \( \gamma \) is in \( \text{Sch}_{n,d} \). Moreover, if \( \pi = \text{DNE} \), then there is a factor \( \text{NE} \) in \( v \) and \( \gamma \) is in \( \text{Sch}_{n,d} \). If \( \pi = \text{NNEE} \) and \( u = D^j \), then \( v \) has at least two factors \( \text{NE} \), since \( d \leq n - 4 \), and \( v \) can end with \( \text{NNEE} \), in which case \( \gamma \) is in \( \text{Sch}_{n,d} \) or \( v \) end with \( \text{DNE} \) and \( \gamma \) is in \( \text{Sch}_{n,d} \).

Let \( d \) be an integer such that \( 0 \leq d \leq n - 1 \). For each \( d \) let:

\[
\rho_{n,d} : \text{Sch}_{n,d} \rightarrow \text{Sch}_{n,d+1} \quad \gamma'\text{NED}\text{D}\text{NNEE}\gamma'' \mapsto \gamma'\text{NED}\text{D}\text{NE}\gamma''
\]

\[
D^j\text{NNEE}\gamma'\text{NENE} \mapsto D^j\text{NNEE}\gamma'\text{DNE}
\]

Lemma 19. Let \( d \) be an integer such that \( 0 \leq d \leq n - 1 \). Then, for all \( d \), \( \rho_{n,d} \) is a bijection such that \( \text{bounce}(\rho(\gamma)) = \text{bounce}(\gamma) - 1 \).

Proof. Notice that \( \gamma'\text{NED}\text{D}\text{NNEE}\gamma'' \neq D^j\text{NNEE}\gamma'\text{NENE} \), since one has a \( \text{NE} \) factor before it’s \( \text{NNEE} \) factor and not the other. The path as an area value of one, ergo there is only one factor \( \text{NNEE} \) in \( \gamma \). Moreover, the map \( \rho_{n,d} \) increases the number of diagonal steps by one. Therefore, for all \( d \) the map \( \rho_{n,d} \) is well defined. For the same reasons the map \( \rho_{n,d}^{-1} \) defined by \( \rho_{n,d}^{-1}(\gamma'\text{NED}\text{D}\text{NE}\gamma'') = \gamma'\text{NED}\text{D}\text{NNEE}\gamma'' \) and \( \rho_{n,d}^{-1}(D^j\text{NNEE}\gamma'\text{DNE}) = D^j\text{NNEE}\gamma'\text{NENE} \) is well defined. Thus, it is the inverse of \( \rho_{n,d} \) and we have a bijection.

Recall that the numph statistic is related to the number of peaks that are in a lower row that the diagonals. When the area value is one, all factors \( \text{NE} \) and \( \text{NNEE} \) contain a peak; they always
return to the diagonal. If we compare the path $\gamma = \gamma'NEDJNNEE\gamma''$ and the path $\rho_{n,d}(\gamma)$ we see that only one diagonal step was added in $\rho_{n,d}(\gamma)$ and the factor $NDE$ in which it was added has its peak above the diagonal step. Hence, $\text{numph}(\rho_{n,d}(\gamma))$ is equal to $\text{numph}(\gamma)$ plus the number of peaks in $\gamma'$. Considering that $\Gamma(\gamma'NEDJNNEE\gamma'') = (NE)^{|\gamma''|+1}NNEE(NE)^{|\gamma''|}$. The value of $\text{bounce}(\Gamma(\gamma))$ is $\left(\begin{array}{c} n-d \\ 2 \end{array}\right) - |\gamma''|_N - 1$. Moreover, $\Gamma(\gamma'NEDJNNEE\gamma'') = (NE)^{n-d-1}$; therefore, $\text{bounce}(\Gamma(\rho_{n,d}(\gamma))) = \left(\begin{array}{c} n-d-1 \\ 2 \end{array}\right)$. Because the number of peaks in $\gamma'$ is equal to $|\gamma'|_E$ which is equal to $|\gamma'|_N$ and $\gamma |\gamma'|_N + |\gamma'|_N = n - d - 3$, we obtain $\text{bounce}(\gamma) = \text{bounce}(\rho_{n,d}(\gamma)) = 1$.

Now comparing the paths $\gamma = D^jNNEE\gamma'\gamma''\gamma''$ and $\rho_{n,d}(\gamma)$ we see that $\text{nymph}(\rho_{n,d}(\gamma))$ is equal to $\text{nymph}(\gamma)$ plus the number of peaks in $D^jNNEE\gamma'$. This is equivalent to counting the number of non consecutive east steps in $D^jNNEE\gamma'$, that is $n - d - 3$, since there are $n - d$ east steps in $\gamma$. We know $\Gamma(D^jNNEE\gamma'\gamma''\gamma'') = NNEE(NE)^{n-d-2}$. The value of $\text{bounce}(\Gamma(\gamma))$ is $\left(\begin{array}{c} n-d \\ 2 \end{array}\right) - (n - d - 1)$. Moreover, $\Gamma(D^jNNEE\gamma'\gamma''\gamma'') = NNEE(NE)^{n-d-3}$, ergo $\text{bounce}(\Gamma(\rho_{n,d}(\gamma))) = \left(\begin{array}{c} n-d-1 \\ 2 \end{array}\right) - (n - d - 2)$. Hence, $\text{bounce}(\gamma) = \text{bounce}(\rho_{n,d}(\gamma)) = 1$. \hfill \Box

We now know some Schröder paths of area 1 are associated to Schur functions in the variables $q$ and $t$ indexed by partitions of length one. This corollary is merely to show that the bijection in the previous lemma sends paths associated to Schur functions indexed by a length one partition to another path associated to Schur functions indexed by a length one partition.

**Corollary 8.** Let $\gamma$ be a path of $\text{Sch}_{n,d}$. Then, $\gamma$ contributes to a Schur function indexed by a partition of length 1 in $\langle \nabla(e_n), s_{d+1,1,n-d-1} \rangle$ if and only if $\rho_{n,d}(\gamma)$ contributes to a Schur function indexed by a partition of length 1 in $\langle \nabla(e_n), s_{d+1,1,n-d-2} \rangle$. Moreover, if $\gamma$ satisfies this property, then $\gamma = \gamma'NEDJNNEE$ and $\rho_{n,d}(\gamma) = \gamma'NEDJNNEE$, with $\gamma' \in \{NE,D\}^\ast$.

**Proof.** By Corollary 1, the definition of $\rho_{n,d}$ and the definition of $\text{Sch}_{n,d}$. \hfill \Box

**Lemma 20.** Let $d$ and $n$ be integers such that $0 \leq d \leq n - 2$. Then, $T_{n,d+2} \circ \rho_{n,d} \circ S_{n,d+1}$ is a bijection between the sets of tableaux:

$$\{\tau \in \text{SYT}(d + 1, 1^n-3) \mid \{1, 2\} \subseteq \text{Des}(\tau)\} \simeq \{\tau \in \text{SYT}(d + 2, 1^n-4) \mid 1 \in \text{Des}(\tau), 2 \notin \text{Des}(\tau)\},$$

such that $\text{maj}(\tau) - \text{des}(\tau) = \text{maj}(T_{n,\tau+2} \circ \rho_{n,d} \circ S_{n,d+1}(\tau)) - \text{des}(T_{n,\tau+2} \circ \rho_{n,d} \circ S_{n,d+1}(\tau)) + 1$.

Additionally, $Q_{n,d+1}^{-1} \circ \rho_{n,d} \circ Q_{n,d+1}$ is a bijection between the set :

$$\{(\tau, i) \in \text{SYT}(d + 1, 1^n-3) \times \{1, \ldots, n - d - 3\} \mid 1 \in \text{Des}(\tau)\},$$

and the set :

$$\{(\tau, i) \in \text{SYT}(d + 2, 1^n-4) \times \{0, \ldots, n - d - 4\} \mid 1 \notin \text{Des}(\tau)\},$$

such that $\text{maj}(\tau) - i = \text{maj}(Q_{n,d+2}^{-1} \circ \rho_{n,d} \circ Q_{n,d+1}(\tau, i)) - (i - 1)$. \hfill \Box

**Proof.** By Corollary 1 we know paths of shape $\gamma'NEDJNNEE$ or $\gamma'NEDJNNEE$ are associated to a Schur function of length one. Notice that by definition $\rho_{n,d}$ sends paths of these shapes to other paths of these shapes. Hence, $Q_{n,d+1}^{-1} \circ \rho_{n,d} \circ Q_{n,d}(\tau, n - d - 2) = (\tau', n - d - 3)$, where $\text{Des}(\tau') = \text{Des}(\tau) \backslash \{1\}$. At this point one only needs to notice that $Q_{n,d+2}^{-1} \circ \rho_{n,d} \circ Q_{n,d+1}(\tau, i) = (\tau', i - 1)$, $\text{Des}(\tau') = \text{Des}(\tau) \backslash \{1\}$ and $\text{Des}(T_{n,\tau+2} \circ \rho_{n,d} \circ S_{n,d+1}(\tau)) = \text{Des}(\tau) \backslash \{2\}$. The rest of the proof follows Proposition 9 from the definition of the maps $\rho_{n,d}, Q_{n,d}, S_{n,d}$ and $T_{n,d}$.

\hfill \Box
Proposition 10. Let \( k \) be an integer such that \( 0 \leq k \leq n-3 \) and \( \psi : \Lambda Q \to \mathbb{Q}[q,t] \) be a linear map defined by \( \psi(s_{\lambda}) = q^{\lambda^\text{t}} t^{\ell(\lambda)-1} \). If:

\[
h_k(q) := \psi \left( \sum_{d=0}^{k} (-1)^{k-d} \langle \nabla(e_n), s_{d+1,1^{n-d-1}} \rangle_{\text{pure hook}} \right) q^{-k+d} t^{-1},
\]

then we have:

\[
h_k(q) = \sum_{\tau \in \text{SYT}(k+1,1^{n-k-1})} q^{\text{maj}(\tau) - \text{des}(\tau)} + \sum_{\tau \in \text{SYT}(k+1,1^{n-k-1})} q^{\text{maj}(\tau) - i},
\]

where the restriction \( |_{\text{pure hook}} \) is the restriction to Schur function indexed by partitions \((a,1)\), with \( a \geq 1 \).

Proof. Let \( \tilde{h}_k(q) = \psi \left( \sum_{d=0}^{k} (-1)^{k-d} \langle \nabla(e_n), s_{d+1,1^{n-d-1}} \rangle_{\text{hook}} \right) q^{-k+d} t^{-1} \). On account of Haglund’s theorem, Lemma 19 and Lemma 18, for an integer \( k \) we have:

\[
\tilde{h}_k(q) = \sum_{d=0}^{k} (-1)^{k-d} \sum_{\gamma \in \text{Sch}_{n,d}} q^{\text{bounce}(\gamma) - k + d} + \sum_{\gamma \in \text{Sch}_{n,d}} q^{\text{bounce}(\gamma) - k + d},
\]

\[
= \sum_{d=0}^{k} (-1)^{k-d} \sum_{\gamma \in \text{Sch}_{n,d}} q^{\text{bounce}(\rho_{n,d}(\gamma)) + 1 - k + d} + \sum_{d=1}^{k} (-1)^{k-d} \sum_{\gamma \in \text{Sch}_{n,d}} q^{\text{bounce}(\gamma) - k + d}, \text{ since } \text{Sch}_{n,0} = \emptyset,
\]

\[
= \sum_{d=0}^{k} (-1)^{k-d} \sum_{\gamma \in \text{Sch}_{n,d+1}} q^{\text{bounce}(\gamma) + 1 - k + d} - \sum_{d=1}^{k} (-1)^{k-d+1} \sum_{\gamma \in \text{Sch}_{n,d}} q^{\text{bounce}(\gamma) - k + d},
\]

\[
= \sum_{d=0}^{k} (-1)^{k-d} \sum_{\gamma \in \text{Sch}_{n,d+1}} q^{\text{bounce}(\gamma) + 1 - k + d} - \sum_{d=0}^{k-1} (-1)^{k-d} \sum_{\gamma \in \text{Sch}_{n,d+1}} q^{\text{bounce}(\gamma) - k + d+1},
\]

\[
= \sum_{\gamma \in \text{Sch}_{n,k+1}} q^{\text{bounce}(\gamma) + 1},
\]

\[
= \sum_{\gamma \in \text{Sch}_{n,k}} q^{\text{bounce}(\gamma)}.
\]

The polynomial \( h_k(q) \) only take into account pure hooks, so we only need to consider the paths \( \gamma^\prime \text{NED}^j \text{NNEE} \gamma'' \) and \( D^j \text{NNEE} \gamma^\prime \text{NENE} \), with \( \gamma', \gamma'' \in \{NE,D\} \), as shown in Corollary 8. Thus, by Lemma 16, the map \( T_{n,k+1} \), for \( \gamma = D^j \text{NNEE} \gamma^\prime \text{NENE} \), gives us \( T_{n,k+1}(\gamma) \) is a tableau containing \( \{1,2\} \) in its descent set and such that \( \text{bounce}(\gamma) = \text{maj}(T_{n,k+1}(\gamma)) - \text{des}(T_{n,k+1}(\gamma)) \). Additionally, for \( \gamma = \gamma^\prime \text{NED}^j \text{NNEE} \gamma'' \), \( Q_{n,k+1}(\gamma) \) is a tableau containing \( \{1\} \) in its descent set. By Proposition 9, \( \gamma \) is such that \( \text{bounce}(\gamma) = \text{maj}(Q_{n,k+1}(\gamma)) - i, 2 \leq i \leq \text{des}(Q_{n,k+1}(\gamma)) \). Moreover, if \( \{1\} \) is in the descent set of \( \tau \), then the map \( Q_{n,k+1} \) send \( (\tau,0) \) to \( D^j \text{NNEE} \gamma^\prime \text{NENE} \) or \( D^j \text{NNEE} \gamma^\prime \text{DNE} \) the first one was already considered and the last one is in \text{Sch}_{n,d}. Finally, if
\{1\} is in the descent set of \(\tau\), then the map \(Q_{n,k+1}\) send \((\tau, i)\) to \(\text{Sch}_{n,d}\) for all \(1 \leq i \leq n - d - 3\). Hence, we sum over \(2 \leq i \leq \text{des}(Q_{n,k+1}^{-1}(\gamma)) - 1\). The second sum is a consequence of Lemma 20.

Considering, what is known about multivariate diagonal harmonics it should be possible to extend the results of this section to the case for \(\text{Sch}_{n,d,(i)}\). This generalization would lead to more results on multivariate diagonal harmonics.

9. Partial Crystal Decomposition

This section is mainly to explain the underlying idea throughout this paper. We can see this as finding the crystal decomposition of the Schröder paths and the parking function. We basically found some of the top weight and for some of them gave a map that gives the remainder of the crystal. In that setting we can say that for \(m = 1\), we can describe all the crystals in the case where the Schur functions are indexed by length one partitions. When \(m > 1\), we can characterize only the top weights. For hooked-shaped Schur functions, we can only depict the top weight, when \(m = 1\).

More precisely the maps \(R_{n,d}\) and \(T_{n,d}\) determine in which crystal the paths lie. The maps \(\tilde{\varphi}\), defined by the map \(\varphi\) in Section 4, give the decomposition according to the top weight. This is also well defined, since for all \(\gamma \in \{NE, D\}^{n-1}NE\) we have \(T_{n,d} \circ \Pi \circ \tilde{\varphi}(\gamma) = R_{n,d}(\gamma)\) (see Figure 37 for an example). Notice that map \(M_{n,d}\) (respectively, \(Q_{n,d}\)) tells us in which crystal component are the Schröder paths of area value 0 (respectively, area value 1).

![Figure 37.](image)

Using the zeta map, so far we know the top weights for all crystals containing a parking function having a diagonal inversion statistic value of 0, and for all hook-shaped partitions we know all top weights for all crystals containing a parking function having a diagonal inversion statistic value of 1. For crystals containing a parking function having a diagonal inversion statistic value of 0 that are not associated to a hook-shaped partition, we do not know the exact paths. Although, we do know in which subset of parking functions the lowest weight lie. Figure 38 gives an overview of what is known so far.
Figure 38. The nodes represent paths. Each chain is associated to a Schur function in the variables $q$ and $t$. The height of the first node determines which Schur function. The partitions determine the Schur function in the variables $X$. Each chain can be associated to a Standard Young tableau corresponding to the shape of the partition. More than one chain can be associated with the same tableau. When nodes are in black we know which paths they relate to, in red we don’t.

10. CONCLUSION AND FURTHER QUESTIONS

Proving Conjecture 1 would be a great start. Moreover, can one describe in a nice way the algorithm described in Section 4, in terms of diagonal inversions and extend it to $m$-Schröder paths. Here we are looking for more than just applying the zeta map. This would allow us to know exactly what paths contribute to each Schur function, even when $m \neq 1$. It might be easier to start by the following problem:

Problem 1. Using the bounce statistic, generalize the algorithm in Section 4 to all Schröder paths.

This would actually give the Schröder paths associated to all the Schur functions and not only the one with one part. It would also answer completely Haglund’s open problem 3.11 of [Hag08]. One could also generalize the algorithm for labelled Dyck relating to the Delta conjecture and get a partial decompositions Schur functions in the variables $q$ and $t$ indexed by partitions of length one.

Using Corollary 5 it should be possible to decompose $\nabla^m(e_n)$ into the basis $s_\mu(q,t)s_\lambda(X)$. The following problems could lead to finding a partial decomposition $\nabla^m(e_n)$ in to Schur functions in the $X$, $s_\lambda(X)$, when $\lambda$ is not a hook. Which is a known hard problem.
Problem 2. Using Lemma 13 decompose $\nabla^m(e_n)$ into the basis $s_\mu(q,t)F_c(X)$, for $\mu$ a hook and $c$ a composition.

Even if the decomposition is in fundamental quasisymmetric functions it would help get a partial decomposition $\nabla^m(e_n)$ into Schur functions, since it should be easier to regroup the fundamental quasisymmetric functions into Schur functions because there will be less coefficients.

Extending, the maps in this paper from $m$-Schröder paths to tableaux with the multiplicity of the descent set, would help decompose completely the Schröder paths into crystals. Of course the extended map must somewhat preserves the area and diagonal inversion statistic or area and bounce statistic through the major index and the number of descents. A related problem is:

Problem 3. Find a bijection between parking functions $(\gamma, w)$, in an $n \times n$, grid, having diagonal inversion statistic value of 1 and Standard Young tableaux, $\tau$, with a multiplicity related to the major index of the tableau $(\text{maj}(\tau, i) - i - 1)$ such that $\text{maj}(\tau, i) - i = \text{area}(B(\tau, i))$.

Using the zeta map we already have a bijections for such $n \times n$ parking functions $(\gamma, w)$ such that $\text{read}(\gamma, w) \in \{n - d + 1, \ldots, n\} \sqcup \{n - d, \ldots, 1\}$. The idea here is to ”extend” that bijection, with multiplicity. We need the multiplicity because there are actually more than just the parking functions $(\gamma, w)$ such that $\text{read}(\gamma, w) \in \{n - d + 1, \ldots, n\} \sqcup \{n - d, \ldots, 1\}$ that contribute to $\nabla(e_n)$ seen as a sum of parking functions. The said paths are merely representatives. If the solution to Problem 3 is indeed an extension of $\zeta \circ Q_{n,d}$ this would actually solve Conjecture 1.

As mentioned in Section 8 the insight coming from multivariate diagonal harmonics foresees a solution to the problem:

Problem 4. Find a general maps $\rho_{n,d}^{(i)}$ that partitions $\widetilde{\text{Sch}}_{n,d,(i)}$.

This generalization would lead to more results on combinatorial formulas for multivariate diagonal harmonics, like the one in [Wal19]. Actually, any explicit decomposition in terms of Schur functions of parking functions could be lifted with the tools discussed in [Wal] (the long version of [Wal19]) and give an explicit combinatorial formula for a partial Schur decomposition of the multivariate diagonal harmonics.

Finally, Proposition 5 suggests a bijection between permutations and tableaux with a multiplicity such that $(\binom{n}{2}) - \text{des}(w)n + \text{maj}(w) = \text{maj}(\tau)$. Research on this last problem could lead to a decomposition of $\nabla^m(e_n)$ altogether. Since, it further our knowledge of how fundamental quasi symmetric functions index by permutations relate to Schur functions.

Problem 5. Find a combinatorial proof of Proposition 5.

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