**Supplementary information:**

Spiral-based phononic plates: From wave beaming to topological insulators

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(A) Bragg limit approximation

One way a periodic medium can interact destructively with waves is by the unit cell size being a positive integer multiple of the wavelength. The lowest frequency around which this happens is referred to as the Bragg limit. For a given homogeneous material (density \( \rho = 1018 \text{ Kg/m}^3 \), Young’s Modulus \( E = 2 \text{ GPa} \) and Poisson’s ratio = 0.33), the speed of sound is usually fixed. Therefore, by having wave velocity and the order of spacial periodicity (i.e., the unit cell size), the Bragg limit can be calculated (unit cell size, \( a = 25 \text{ mm} \) and thickness, \( t \theta = 3 \text{ mm} \)). However, introducing cuts within the unit cell reduces the effective properties of the material, thus changing its dynamical characteristics. In addition, for flexural waves in plates, the wave velocity is not constant. The dispersion is rather parabolic [Fig. S1(a)], which adds complexity in determining the limit at which periodicity can no longer affect waves. Such parabolic nature shows an increase in the slope of the dispersion branch (i.e., phase velocity) as the wavenumber increases. By identifying the maximum of this phase velocity, we approximate the Bragg limit to be at the edge of the first Brillouin zone (i.e., the high-symmetry point \( X \)). In order to determine that phase velocity maximum, we apply a standard finite difference scheme to the dispersion bands. We utilize this scheme to determine the Bragg-limit for the designs presented in figure 2. The parameters are as follows: Fig. 2(a) \( r = 0 \text{ mm}, n = 0.25, R = 12.8 \text{ mm}, w = 0.6 \text{ mm}, \) and \( \alpha = 45^\circ \). Fig. 2(b) \( r = 0 \text{ mm}, n = 1.25, R = 12.8 \text{ mm}, w = 0.6 \text{ mm}, \) and \( \alpha = 45^\circ \). Fig. 2(c) \( r = 6 \text{ mm}, n = 1.25, R = 12.8 \text{ mm}, w = 0.6 \text{ mm}, \) and \( \alpha = 23^\circ \). Fig. 2(d) \( r = 10 \text{ mm}, n = 0.65, R = 13.25 \text{ mm}, w = 0.65 \text{ mm}, \) and \( \alpha = 37.7^\circ \). We numerically determine the point of maximum phase velocity (black dots) for flexural waves in the \( \Gamma - X \) direction. We use the tangent of the dispersion branch at these points to project the Bragg-limit frequency. The frequencies susceptible to periodicity (i.e., higher than the Bragg limit) are highlighted with a gray dashed background.

By applying the method outlined above, we compute the Bragg limit for the unit cells of figure 2 in the main text. We start by calculating the Bragg limit for the homogeneous material [Fig. S1(a)]. The maximum phase velocity coincides with the \( X \)-point and thereby the Bragg limit, which is well established for homoge-
FIG. S1. Bragg limit computation. (a-e) Dispersion curves for both attenuated (orange, b-e) and propagating waves (in-plane: blue, and out-of-plane: green). The effective wave velocity is tangential in the point of maximum phase velocity (black). The cells’ Bragg limit is where the tangent meets the Γ-point, frequencies beyond this limit are indicated by a gray background. The respective unit cell is inset on the left.

FIG. S2. Subwavelength Beaming and Dirac cones. Dispersion relation for out-of-plane waves of the unit cells (top). The maximum effective wave velocity is indicated by a black dot, frequencies that exceed the Bragg limit are indicated by a gray dashed background. The frequencies of operation for both demonstrations of beaming and topological insulator are indicated by black dashed lines. For the beaming unit cell in (a) Γ – X direction, and (b) in Γ – Y direction. (c) for the hexagonal TI cell in Γ – M direction.

(B) Design of spiral geometries

The parameter space for spiral-based phononic metamaterial is vast, with multiple design dimensions. To capture the influence of each parameter on the resulting pattern, we systematically vary some of these parameters and record the evolution of the first three flexural band gaps [Fig. S3]. All parameters are kept constant except for the one being analyzed. The reference parameters are: the unit cell size \( a = 25 \) mm, the plate thickness \( th = 3 \) mm, the spiral width \( w = 0.6 \) mm, the inside radius \( r = 6 \) mm, the outside radius \( R = 12.84 \) mm, the number of turns \( n = 1.25 \) and the rotation \( \alpha = 23^\circ \) [Fig. S3(a,b)]. These reference parameters result in a central mass connected to the plate frame by thin spiraling ligaments. Changing the dimensions of these ligaments, or its connected central mass, influences the frequency response of the metamaterial. For instance, varying the number of turns, while keeping both the inner and outer radius constant, affects both the length and the width of the connecting ligaments. Therefore, as the number of turns increases, the band gap frequencies decrease significantly [Fig. S3(c)]. Similarly, increasing either the inner radius [Fig. S3(d)] or the cutting width of the spirals [Fig. S3(e)] has a similar effect (i.e., lowers the frequencies) on the band gaps, as it increases the central mass or reduces the width of the connecting ligaments. By increasing the plate thickness, the band gap frequencies increase at the beginning, for very small thickness, but eventually saturates. Since variations in the ligaments’ thickness do not have a profound effect on in-plane waves, changing the thickness is an ideal tuning parameter to independently
FIG. S3. Parameter influence. (a) Geometric parameters of an Archimedean spiral and (b) the unit cell used as a reference for the parameter analysis. (c-h) Out-of-plane band gaps of the inertially amplified unit cell for various parameters. We vary (c) the inside radius of the spirals, $r$, (d) the number of turns, $n$, (e) the spiral width, $w$, (f) the plate thickness, $t_h$, (g) the unit cell size, $a$, and (h) the orientation, $\alpha$, of the spirals.

(C) Wave beaming as a function of the spiral orientation angle

As presented in figure 3 in the main text, spiral metamaterials can present desirable wave beaming within pass bands and band gaps. The wave beaming patterns can be tuned by varying the excitation frequency. For instance, at orientation angle $\alpha = -30^\circ$, the wave pattern is fundamentally different at $f = 490, 510, 560$ Hz [Fig. S4(c)]. At $f = 490$ Hz, the frequency contours are connected with no strong beaming present with a mostly radial propagation pattern. At $f = 510$ Hz, the frequency contours start to separate into two distinct domains resembling radial propagation with absence of propagative waves in the vertical direction. At $f = 560$ Hz, the frequency contours separate further into four islands, presenting two beaming lines. For the same orientation angle $\alpha$, at the vicinity of the second band gap, strong dependency on frequency is also evident [Fig. S4(b)]. For example, at $f = 1520$ Hz, the frequency contours are connected, which translates to radial propagation with a slight preference for waves to propagate along the vertical and horizontal directions. The separation of these connected contours takes place around $f = 1540$ Hz. More interestingly, between $f = 1560$ and 1600 Hz, the frequency contours take the shape of two isolated islands. At $f = 1560$ Hz, a single beaming line in the vertical direction is present. As the frequency increases, the beaming orientation exhibits a clockwise shift from the vertical direction.

Moreover, the properties of the beaming can be controlled further by tuning the orientation $\alpha$ of the spirals spanning a wide range of frequencies. For example, the isofrequency contours for rotated identical spiral-cuts are plotted in [Fig. S4]. The rotation angles vary from $-30^\circ$ to $45^\circ$. It is evident that the orientation of the spiral influences the pattern of wave beaming, even at the same frequency. For example, in the vicinity of the first band gap, waves excited at $f = 560$ Hz will have two different propagation patterns at orientation angle 0 and 30 [Fig. S4(c)]. At $\alpha = 0^\circ$, the wave will propagate in two perpendicular directions radiating from the four corners of the unit cell. While at $\alpha = 30^\circ$, the wave will propagate along a single direction passing through the center of the unit cell. Similarly, in the vicinity of the second band gap [Fig. S4(b)] at $f = 1600$ Hz and orientation angle $-20^\circ$ and $30^\circ$. When $\alpha = -20^\circ$ the beaming occurs along a straight line, while $\alpha = -20^\circ$ causes the wave to radiate from the corners of the unit cell in two perpendicular directions.

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FIG. S4. Beaming parametric tunability. Isofrequency plot for out-of-plane waves of the unit cell presented in Fig. 3(a) with changing orientations of the spiral (columns), for the two frequency ranges (rows, colorbar). The dashed square indicates the first Brillouin zone of the unit cell.