STRUCTURE RELATIONS IN SPECIAL \( A_\infty -\)BIALGEBRAS

RONALD UMBLE

Abstract. We compute the structure relations in special \( A_\infty -\)bialgebras whose operations are limited to those defining the underlying \( A_\infty -(co)\)algebra substructure. Such bialgebras appear as the homology of certain loop spaces. Whereas structure relations in general \( A_\infty -\)bialgebras depend upon the combinatorics of permutahedra, only Stasheff’s associahedra are required here.

1. Introduction

A general \( A_\infty -\)infinity bialgebra is a DG module \((H, d)\) equipped with a family of structurally compatible operations \(\omega_{j,i} : H^{\otimes i} \to H^{\otimes j}\), where \(i, j \geq 1\) and \(i + j \geq 3\) (see [3]). In special \( A_\infty -\)bialgebras, \(\omega_{j,i} = 0\) whenever \(i, j \geq 2\), and the remaining operations \(m_i = \omega_{1,i}\) and \(\Delta_j = \omega_{j,1}\) define the underlying \( A_\infty -(co)\)algebra substructure. Thus special \( A_\infty -\)bialgebras have the form \((H, d, m_i, \Delta_j)\) subject to the appropriate structure relations involving \(d\), the \(m_i\)’s and \(\Delta_j\)’s. These relations are much easier to describe than those in the general case, which require the S-U diagonal \(\Delta_P\) on permutahedra. Instead, the S-U diagonal \(\Delta_K\) on Stasheff’s associahedra \(K = \sqcup K_n\) is required here (see [5]).

\( A_\infty -\)bialgebras are fundamentally important structures in algebra and topology. In general, the homology of every \( A_\infty -\)bialgebra inherits an \( A_\infty -\)bialgebra structure [7]; in particular, this holds for the integral homology of a loop space. In fact, over a field, the \( A_\infty -\)bialgebra structure on the homology of a loop space specializes to the \( A_\infty -(co)\)algebra structures observed by Gugenheim [2] and Kadeishvili [3].

The main result of this paper is the following simple formulation of the structure relations in special \( A_\infty -\)bialgebras that do not involve \(d\): Let \( TH \) denote the tensor module of \( H \) and let \( e^{n-2} \) denote the top dimensional face of \( K_n \). There is a “fraction product” on \( M = \text{End}(TH) \) (denoted here by “\( \bullet \)” and certain cellular cochains \( \xi, \zeta \in C^* (K; M) \) such that for each \( i, j \geq 2 \),

\[
\Delta_j \bullet m_i = \xi^i \left( e^{i-2} \right) \bullet \zeta^j \left( e^{j-2} \right),
\]

where the exponents indicate certain \( \Delta_K \)-cup powers.

I must acknowledge the fact that many of the ideas in this paper germinated during conservations with Samson Saneblidze, whose openness and encouragement led to this paper. For this I express sincere thanks.

Date: June 22, 2005.
1991 Mathematics Subject Classification. Primary 55U05, 52B05, 05A18, 05A19; Secondary 55P35.

Key words and phrases. Associahedron, S-U diagonal, biderivative, \( A_\infty -\)bialgebra.

\[1\] This research was funded in part by a Millersville University faculty research grant.
2. Matrix Considerations

We begin with a brief review of the algebraic machinery we need; for a detailed exposition see \[8\]. Let \( M = \bigoplus_{m,n \geq 1} M_{m,n} \) be a bigraded module over a commutative ring \( R \) with identity \( 1_R \) and consider the module \( TTM \) of tensors on \( TM \).

Given matrices \( X = [x_{ij}] \) and \( Y = [y_{ij}] \in \mathbb{N}^{p \times q} \), \( p, q \geq 1 \), consider the submodule

\[
M_{Y,X} = (M_{y_{11},x_{11}} \otimes \cdots \otimes M_{y_{pq},x_{pq}}) \otimes \cdots \otimes (M_{y_{q1},x_{q1}} \otimes \cdots \otimes M_{y_{qp},x_{qp}})
\]

\( \subset (M^\oplus p)^\otimes q \subset TTM \).

Represent a monomial \( A = (\theta_{y_{11},x_{11}} \otimes \cdots \otimes \theta_{y_{pq},x_{pq}}) \otimes \cdots \otimes (\theta_{y_{q1},x_{q1}} \otimes \cdots \otimes \theta_{y_{qp},x_{qp}}) \in M_{Y,X} \) as the \( q \times p \) matrix \( [A] = [a_{ij}] \) with \( a_{ij} = \theta_{y_{ij},x_{ij}} \). Then \( A \) is the \( q \)-fold tensor product of the rows of \( [A] \) thought of as elements of \( M^\oplus p \); we refer to \( A \) as a \( q \times p \) monomial and often write \( A \) when we mean \( [A] \). The matrix submodule of \( TTM \) is the sum

\[
\mathbb{M} = \bigoplus_{x,y \in \mathbb{N}^p \times \mathbb{N}^q} M_{x,y} = \bigoplus_{p,q \geq 1} (M^\oplus p)^\otimes q.
\]

Given \( x \times y = (x_1, \ldots, x_p) \times (y_1, \ldots, y_q) \in \mathbb{N}^p \times \mathbb{N}^q \), set \( X = [x_{ij} = x_j]_{1 \leq i \leq q} \), \( Y = [y_{ij} = y_i]_{1 \leq j \leq p} \) and denote \( \mathbb{M}_{x} = M_{x,Y} \). The essential submodule of \( TTM \) is

\[
\mathbb{M} = \bigoplus_{x \times y \in \mathbb{N}^p \times \mathbb{N}^q} \mathbb{M}_{x}
\]

and a \( q \times p \) monomial \( A \in \mathbb{M} \) has the form

\[
A = \begin{bmatrix}
\theta_{y_{11},x_{11}} & \cdots & \theta_{y_{1p},x_{1p}} \\
\vdots & \ddots & \vdots \\
\theta_{y_{q1},x_{11}} & \cdots & \theta_{y_{qp},x_{1p}}
\end{bmatrix}.
\]

Graphically represent \( A \) as \( [\theta_{y_{ij},x_{ij}}] \in \mathbb{M}_{x} \) two ways: (1) as a matrix of “double corollas” in which \( \theta_{y_{ij},x_{ij}} \) is pictured as two corollas joined at the root—one opening downward with \( x_i \) leaves and the other opening upward with \( y_j \) leaves—and (2) as an arrow in the positive integer lattice \( \mathbb{N}^2 \) from \( (|x|, q) \) to \( (p, |y|) \), where \( |u| = u_1 + \cdots + u_k \) (see Figure 1).

![Figure 1. Graphical representations of a typical monomial.](image)

Each pairing \( \gamma : \bigoplus_{r,s \geq 1} M^\otimes r \otimes M^\otimes s \to \mathbb{M} \) induces an \( \upsilon \)-product \( \Upsilon : \mathbb{M} \otimes \mathbb{M} \to \mathbb{M} \) supported on “block transverse pairs,” which we now describe.
Definition 1. A monomial pair $A^{q \times s} \otimes B^{t \times p} = [\theta_{y_{k}, v_{k}}] \otimes [\eta_{u_{j}, x_{j}}] \in \mathbb{M} \otimes \mathbb{M}$ is a

(i) **Transverse Pair (TP)** if $s = t = 1$, $u_{1, j} = q$ and $v_{k, 1} = p$ for all $j, k$, i.e., setting $x_{j} = x_{1,j}$ and $y_{k} = y_{k,1}$ gives

$$A \otimes B = \begin{bmatrix} \theta_{y_{1}, p} \\ \vdots \\ \theta_{y_{q}, p} \end{bmatrix} \otimes \begin{bmatrix} \eta_{q, x_{1}} & \cdots & \eta_{q, x_{p}} \end{bmatrix} \in \mathbb{M}_{q}^{y} \otimes \mathbb{M}_{q}^{x}.
$$

(ii) **Block Transverse Pair (BTP)** if there exist $t \times s$ block decompositions $A = [A'_{i \ell}]$ and $B = [B'_{i \ell}]$ such that $A'_{i \ell} \otimes B'_{i \ell}$ is a TP for all $i, \ell$.

Unlike the blocks in a standard block matrix, the blocks $A'_{i \ell}$ (or $B'_{i \ell}$) in a general BTP may vary in length within a given row (or column). However, when $A \otimes B \in \mathbb{M}_{q}^{y} \otimes \mathbb{M}_{q}^{x}$ is a BTP with $u = (q_{1}, \ldots, q_{t})$, $v = (p_{1}, \ldots, p_{s})$, $x = (x_{1}, \ldots, x_{s})$ and $y = (y_{1}, \ldots, y_{q})$, the TP $A'_{i \ell} \otimes B'_{i \ell} \in \mathbb{M}_{q_{i}}^{x_{i}} \otimes \mathbb{M}_{p_{j}}^{y_{j}}$ so that for a fixed $i$ (or $\ell$) the blocks $A'_{i \ell}$ (or $B'_{i \ell}$) have constant length $q_{i}$ (or $p_{j}$); furthermore, $A \otimes B$ is a BTP if and only if $y \in \mathbb{N}^{[q]}$ and $x \in \mathbb{N}^{[p]}$ if and only if the initial point of arrow $A$ coincides with the terminal point of arrow $B$. Note that BTP block decomposition is unique.

**Example 1.** A pairing of monomials $A^{4 \times 2} \otimes B^{2 \times 3} \in \mathbb{M}_{2,1}^{1,5,4,3} \otimes \mathbb{M}_{1,2,3}^{3,1}$ is a $2 \times 2$ BTP per the block decompositions

$$A = \begin{bmatrix} \theta_{1,1} & \theta_{1,2} \\ \theta_{5,1} & \theta_{5,2} \\ \theta_{4,1} & \theta_{4,2} \\ \theta_{3,1} & \theta_{3,2} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \eta_{1,1} & \eta_{1,2} & \eta_{1,3} \\ \eta_{3,1} & \eta_{3,2} & \eta_{3,3} \end{bmatrix}.
$$

Given a pairing $\gamma = \sum_{x \times y} \gamma_{x}^{y} : \mathbb{M}_{p}^{y} \otimes \mathbb{M}_{x}^{y} \rightarrow \mathbb{M}_{[x]}^{y}$, extend $\gamma$ to an upsilon product $\Upsilon : \mathbb{M} \otimes \mathbb{M} \rightarrow \mathbb{M}$ via

$$\Upsilon (A \otimes B)_{i \ell} = \begin{cases} \gamma (A'_{i \ell} \otimes B'_{i \ell}), & \text{if } A \otimes B \text{ is a BTP} \\ 0, & \text{otherwise.} \end{cases}
$$

Then $\Upsilon$ sends a BTP $A^{q \times s} \otimes B^{t \times p} \in \mathbb{M}_{p}^{y} \otimes \mathbb{M}_{x}^{y}$ with $A'_{i \ell} \otimes B'_{i \ell} \in \mathbb{M}_{p_{j}}^{y_{j}} \otimes \mathbb{M}_{x_{i}}^{y_{i}}$ to a $t \times s$ monomial in $\mathbb{M}_{[x_{1}], \ldots, [x_{s}]}^{y_{1}, \ldots, [y_{t}]}$. We denote $A \cdot B = \Upsilon (A \otimes B)$; when $[\theta_{j}] \otimes [\eta_{j}]$ is a TP we denote $\gamma (\theta_{1}, \ldots, \theta_{q}; \eta_{1}, \ldots, \eta_{p}) = (\theta_{1} \otimes \cdots \otimes \theta_{q}) \cdot (\eta_{1} \otimes \cdots \otimes \eta_{p})$. As an arrow, $A \cdot B$ runs from the initial point of $B$ to the terminal point of $A$. Note that $\mathbb{M} \cdot \mathbb{M} \subseteq \mathbb{M}$ so that $\Upsilon$ restricts to an upsilon product on $\mathbb{M}$.

**Example 2.** Continuing **Example 1**, the action of $\Upsilon$ on $A^{4 \times 2} \otimes B^{2 \times 3} \in \mathbb{M}_{2,1}^{1,5,4,3} \otimes \mathbb{M}_{1,2,3}^{3,1}$ produces a $2 \times 2$ monomial in $\mathbb{M}_{3,3}^{10,3}$.
double corollas typically produces a matrix of non-planar graphs (see Figure 2). The applications below relate to the following special case: Let $H$ be a graded module over a commutative ring with unity and view $M = \text{End}(TH)$ as a bigraded module via $M_{n,m} = \text{Hom}(H^\otimes m, H^\otimes n)$. Then a $q \times p$ monomial $A \in M^n_x$ admits a representation as an operator on $M$ via

$$\left( H^\otimes |x| \right)^{\otimes q} \approx (H^\otimes x_1 \otimes \cdots \otimes H^\otimes x_t)^{\otimes q} \overset{\sigma_{x_1,p} \otimes \cdots \otimes \sigma_{x_t,p}}{\longrightarrow} (H^\otimes y_1)^{\otimes p} \otimes \cdots \otimes (H^\otimes y_s)^{\otimes p}$$

where $(s,t) \in \mathbb{N}^2$ is identified with $(H^\otimes s)^{\otimes t}$ and $\sigma_{x,t} : (H^\otimes s)^{\otimes t} \approx (H^\otimes t)^{\otimes s}$ is the canonical permutation of tensor factors $\sigma_{x,p} : ((a_{11} \cdots a_{q_1}) \cdots (a_{1p} \cdots a_{qp})) \mapsto ((a_{11} \cdots a_{1p}) \cdots (a_{q_1} \cdots a_{qp}))$. The canonical structure map is

$$\gamma = \sum \gamma_{x} : M^n_x \otimes M^p_x \overset{\iota_p \otimes \iota_q}{\longrightarrow} M^{|y|} \otimes M^{pq}_x \overset{id \otimes \sigma_{y,p}^*}{\longrightarrow} M^{|y|} \otimes M^{pq}_x \overset{\sigma}{\longrightarrow} M^{|y|},$$

where $\iota_p$ and $\iota_q$ are the canonical isomorphisms and $\sigma^*_{y,p}$ is induced by $\sigma_{q,p}$ (c.f. [1], 41), induces a canonical associative $\gamma$ product on $M$ whose action on matrices of double corollas typically produces a matrix of non-planar graphs (see Figure 2).

$$\begin{bmatrix}
\begin{array}{c}
\theta_{1,2} \\
\theta_{5,2} \\
\theta_{4,2} \\
\theta_{3,2}
\end{array}
\end{bmatrix}
\cdot
\begin{bmatrix}
\begin{array}{c}
\gamma(\theta_{1,2},\theta_{5,2},\theta_{4,2};\gamma_{3,1},\gamma_{3,2}) \\
\gamma(\theta_{1,2},\theta_{5,1},\theta_{4,1};\gamma_{3,3}) \\
\gamma(\theta_{3,2};\gamma_{1,1},\gamma_{1,2}) \\
\gamma(\theta_{3,1};\gamma_{1,1},\gamma_{1,2})
\end{array}
\end{bmatrix} =
\begin{bmatrix}
\begin{array}{c}
\gamma(\theta_{1,2},\theta_{5,2},\theta_{4,2};\gamma_{3,1},\gamma_{3,2}) \\
\gamma(\theta_{1,2},\theta_{5,1},\theta_{4,1};\gamma_{3,3}) \\
\gamma(\theta_{3,2};\gamma_{1,1},\gamma_{1,2}) \\
\gamma(\theta_{3,1};\gamma_{1,1},\gamma_{1,2})
\end{array}
\end{bmatrix}.$$
3. Cup products

The two pairs of dual cup products defined in this section play an essential role in the theory of structure relations. Let \((H,d)\) be a DG module over a commutative ring with unity. For each \(i,j \geq 2\), choose operations \(m_i : H^\otimes i \to H\) and \(\Delta_j : H \to H^\otimes j\) thought of as elements of \(M = \text{End}(TH)\). Recall that planar rooted trees (PRT's) parametrize the faces of Stasheff’s associahedra \(K = \bigsqcup_{n \geq 2} K_n\) and provide module generators for cellular chains \(C_*(K)\) \([4]\). Whereas top dimensional faces correspond with corollas, lower dimensional faces correspond with more general PRT’s. Now given a face \(a \subset K\), consider the class of all planar rooted trees with levels (PLT’s) representing \(a\) and choose a representative with exactly one node in each level. In this way, we obtain a particularly nice set of module generators for \(C_*(K)\), denoted by \(K\). Note that the elements of a class of PLT’s represent the same function obtained by composing in various ways. The results obtained here are independent of choice since they depend only on the function.

Let \(G\) be a DGA concentrated in degree zero and consider the cellular cochains on \(K\) with coefficients in \(G\):

\[
C^p(K;G) = \text{Hom}^{-p}(C_p(K);G).
\]

A diagonal \(\Delta\) on \(C_*(K)\) induces a cup product \(\smile\) on \(C^*(K;G)\) via

\[
f \smile g = (f \otimes g) \Delta,
\]

where \(\cdot\) denotes multiplication in \(G\).

The essential submodule \(\mathcal{M}\), which serves as our coefficient module, is canonically endowed with dual associative wedge and Čech cross products defined on a monomial pair \(A \otimes B \in \mathcal{M}_x^y \otimes \mathcal{M}_u^v\) by

\[
A \hat{\times} B = \begin{cases} A \otimes B, & \text{if } v = x, \\ 0, & \text{otherwise,} \end{cases} \quad A \check{\times} B = \begin{cases} A \otimes B, & \text{if } u = y, \\ 0, & \text{otherwise.} \end{cases}
\]

Denote \(\hat{\mathcal{M}} = \left(\mathcal{M}, \hat{\times}\right)\) and \(\check{\mathcal{M}} = \left(\mathcal{M}, \check{\times}\right)\) and note that \(\mathcal{M}_x^y \hat{\times} \mathcal{M}_u^v \subseteq \mathcal{M}_x^u \otimes \mathcal{M}_y^v\) and \(\mathcal{M}_x^y \check{\times} \mathcal{M}_u^v \subseteq \mathcal{M}_y^v \otimes \mathcal{M}_x^u\). Thus non-zero cross products concatenate matrices:

\[
A \hat{\times} B = [A^\hat{\times}] \quad \text{and} \quad A \check{\times} B = [A^\check{\times}].
\]

As arrows, \(A \hat{\times} B\) runs from vertical \(x = |x|\) to vertical \(x = p\), whereas \(A \check{\times} B\) runs from horizontal \(y = q\) to horizontal \(y = |y|\). In particular, if \(A \in \mathcal{M}_{a,b}^{|x|}\), then \(A \hat{\times} n \in \mathcal{M}_{a,b}^{|x|\cdot n}\) is an arrow from \((a,n)\) to \((1,nb)\) and \(A \check{\times} n \in \mathcal{M}_{a\cdots a}^{|x|\cdot n}\) is an arrow from \((na,1)\) to \((n,b)\). These cross products together with the S-U diagonal \(\Delta_K\) \([5]\) induce wedge and Čech cup products \(\wedge\) and \(\vee\) in \(C^*(K;\hat{\mathcal{M}})\) and \(C^*(K;\check{\mathcal{M}})\), respectively.

The modules \(C^*(K;\hat{\mathcal{M}})\) and \(C^*(K;\check{\mathcal{M}})\) are equipped with second cup products \(\wedge_t\) and \(\vee_t\) arising from the T-product on \(\mathcal{M}\) together with the “leaf coproduct” \(\Delta_t : C_*^t(K) \to C_*^t(K) \otimes C_*^t(K)\), which we now define. Let \(T = T^t \in \mathcal{K}\) be a \(k\)-level PLT. Prune \(T\) immediately below the first (top) level, trimming off a single corolla with \(n_1\) leaves and \(r_1 - 1\) stalks. Numbering from left-to-right, let \(i_1\) be the position of the corolla. The \((\text{first})\) leaf sequence of \(T\) is the \(r_1\)-tuple \(x_{i_1}(n_1) = (1 \cdots n_1 \cdots 1)\) with \(n_1\) in position \(i_1\) and 1’s elsewhere. Label the pruned tree \(T^2\); inductively, the \(j^{th}\) leaf sequence of \(T\) is the leaf sequence of \(T^j\). The induction terminates when
\( j = k \), in which case \( i_k = r_k = 1 \) and \( x_{i_k}(n_k) = n_k \). The descent sequence of \( T \) is the \( k \)-tuple \((x_{i_1}(n_1), \ldots, x_{i_k}(n_k))\).

**Definition 2.** Let \( T \in \mathcal{K} \) and identify \( T \) with its descent sequence \( n = (n_1, \ldots, n_k) \).

The leaf coproduct of \( T \) is given by

\[
\Delta_\ell(T) = \begin{cases}
\sum_{2 \leq i \leq k} (n_1, \ldots, |n_i|) \otimes (n_i, n_{i+1}, \ldots, n_k), & k > 1 \\
0, & k = 1.
\end{cases}
\]

Define the leaf cup products \( \wedge_\ell \) and \( \vee_\ell \) on \( C^*(\mathcal{K}; \wedge M) \) and \( C^*(\mathcal{K}; \vee M) \) by

\[
f \wedge_\ell g = (f \otimes g) \tau \Delta_\ell \quad \text{and} \quad f \vee_\ell g = (f \otimes g) \Delta_\ell,
\]

where \( \tau \) interchanges tensor factors and \( \cdot \) denotes the \( \Upsilon \)-product.

Note that all cup products defined in this section are non-associative and non-commutative. Unless explicitly indicated otherwise, iterated cup products are parenthesized on the extreme left, e.g., \( f \vee g \vee h = (f \vee g) \vee h \).

#### 4. Special \( A_\infty \)-bialgebras

Structural compatibility of \( d \), the \( m_i \)'s and \( \Delta_j \)'s is expressed in terms of the (restricted) biderivative \( d_\omega \) and the “fraction product” \( \cdot \) by the equation

\[
d_\omega \cdot d_\omega = 0.
\]

We begin with a construction of the biderivative in our restricted setting. Let \( \varphi \in C^*(\mathcal{K}; \wedge M) \) and \( \psi \in C^*(\mathcal{K}; \vee M) \) be the cochains with top dimensional support such that

\[
\varphi(e^{i-2}) = m_i \quad \text{and} \quad \psi(e^{j-2}) = \Delta_j.
\]

We think of \( \varphi \) and \( \psi \) as acting on uprooted and downrooted trees, respectively (see Figure 3).

\[
\varphi \begin{bmatrix} \end{bmatrix} = m_5 \quad \psi \begin{bmatrix} \end{bmatrix} = \Delta_5
\]

Figure 3: The actions of \( \varphi \) and \( \psi \).

Let \( T^cH \) denote the tensor coalgebra of \( H \). The coderivation cochain of \( \varphi \) is the cochain \( \varphi^c \in C^*(\mathcal{K}; \wedge M) \) that extends \( \varphi \) to cells of \( K \) in codim 1 such that

\[
\sum_{\text{codim } e = 0, 1} \varphi^c(e) \in \text{Coder} (T^cH)
\]

is the cofree linear coextension of \( \varphi(K) = \sum_{i \geq 2} \varphi(e^{i-2}) \) as a coderivation. Thus if \( T \in \mathcal{K} \) is an uprooted 2-level tree with \( n + k \) leaves and leaf sequence \( x_i(k) \),

\[
\varphi^c(T) = 1 \otimes \cdots \otimes m_k \otimes 1 \otimes n^{-i+1} = [1 \cdots m_k \cdots 1] \in M^1_{x_i(k)}
\]

and is represented by the arrow from \((n + k, 1)\) to \((n + 1, 1)\) on the horizontal axis in \( \mathbb{N}^2 \). Dually, let \( T^a(H) \) denote the tensor algebra of \( H \). The derivation cochain of
ψ is the cochain \( \psi^a \in C^* (K; \mathbf{M}) \) that extends \( \psi \) to cells of \( K \) in \( \text{codim} \ 1 \) such that
\[
\sum_{\text{codim} \ e = 0, 1} \psi^a ( e ) \in \text{Der} ( T^a H )
\]
is the free linear extension of \( \psi ( K ) = \sum_{i \geq 2} \psi ( e^{i-2} ) \) as a derivation. Thus if \( T \in K \) is a downrooted 2-level tree with \( n + k \) leaves and leaf sequence \( y_i ( k ) \),
\[
\psi^a ( T ) = 1^\otimes i - 1 \otimes \Delta_k \otimes 1^\otimes n - i + 1 = [ 1 \cdots \Delta_k \cdots 1 ]^T \in \mathbf{M}_1^{T ( k )}
\]
and is represented by the arrow from \( ( 1, n + 1 ) \) to \( ( 1, n + k ) \) on the vertical axis.

Evaluating leaf cup powers of \( \varphi^c \) (respt. \( \psi^a \)) generates a representative of each class of compositions involving the \( m_i \)'s (respt. \( \Delta_i \)'s). So let
\[
\zeta = \varphi^c + \varphi^c \wedge \varphi^c + \cdots + ( \varphi^c )^\wedge k + \cdots
\]
and note that if \( e \) is a cell of \( K \), each non-zero component of \( \zeta ( e ) \) (respt. \( \zeta ( e ) \)) is represented by a left-oriented horizontal (respt. upward-oriented vertical) arrow.

Furthermore, evaluating wedge and Čech cup powers of \( \xi \) (respt. \( \zeta \)) generates the components of the cofree coextension of \( \xi ( K ) \) as a \( \Delta_K \)-coderivation (respt. free extension of \( \zeta ( K ) \) as a \( \Delta_K \)-derivation). So let
\[
\hat{\varphi} = \xi + \xi \wedge \xi + \cdots + \xi^k + \cdots
\]
and note that the component \( \xi^k ( e^{i-2} ) : ( H^\otimes i )^\otimes k \to ( H^\otimes 1 )^\otimes k \) is represented by a left-oriented horizontal arrow from \( ( i, k ) \) to \( ( 1, k ) \) while the component \( \zeta^k ( e^{i-2} ) : ( H^\otimes 1 )^\otimes k \to ( H^\otimes i )^\otimes k \) is represented by an upward-oriented vertical arrow from \( ( k, 1 ) \) to \( ( k, i ) \).

Let \( M_0 = M_{1,1} \). For reasons soon to become clear, the only structure relations involving the differential \( d \) are the classical quadratic relations in an \( A_\infty \)-(co)algebra. Note that \( d \in M_0 \) and let \( 1^* = ( 1, \ldots, 1 ) \in \mathbb{N}^* \). Given \( \theta \in M_0 \) and \( p, q \geq 1 \), consider the monomials \( \theta_i^q \times 1 \in M_1^{1 \times 1} \) and \( \theta_j^1 \times p \in M_1^{1 \times p} \) all of whose entries are the identity except the \( i^{th} \) in \( \theta_i^q \times 1 \) and the \( j^{th} \) in \( \theta_j^1 \times p \), both of which are \( \theta \). Define \( B d_0 : M_0 \to \mathbf{M} \) by
\[
B d_0 ( \theta ) = \sum_{1 \leq i \leq q, 1 \leq j \leq p, \# \{ 1 \leq r \leq q : \theta_r^q \} \leq 1} \theta_i^q \times 1 + \theta_j^1 \times p.
\]
Then \( B d_0 ( \theta ) \) is the (co)free linear (co)extension of \( \theta \) as a (co)derivation. Note that each component of \( B d_0 ( \theta ) \) is represented by an arrow of “length” zero.

Let \( M_1 = ( M_{1,1} \oplus M_{*,1} ) / M_{1,1} \) and define \( B d_1 : M_1 \to \mathbf{M} \) by
\[
B d_1 ( \theta ) = \sum_{\text{codim} \ e = 0} e ( \hat{\varphi} + \psi ) ( e ) + \sum_{\text{codim} \ e = 1} ( \varphi^c + \psi^a ) ( e ).
\]

Note that the components of \( B d_1 ( \theta ) \) are represented by upward-oriented vertical arrows and left-oriented horizontal arrows; the right-hand component of (1.1) is given by Gerstenhaber’s \( \triangleright \)-(co)operation.

Let \( \rho_0 : \mathbf{M} \to M_0 \) and \( \rho_1 : \mathbf{M} \to M_1 \) denote the canonical projections.
Definition 3. The restricted biderivative is the (non-linear) map \( d : M \rightarrow M \) given by
\[
d = Bd_0 \circ \rho_0 + Bd_1 \circ \rho_1.
\]
The symbol \( d_\theta \) denotes the restricted biderivative of \( \theta \).

Finally, the composition
\[
\bullet : M \times M \xrightarrow{d \otimes d} M \times M \xrightarrow{\Upsilon} M
\]
defines the fraction product. Special \( A_\infty \)-bialgebras are defined in terms of the fraction product as follows:

Definition 4. Let \( \omega = d + \sum_{i,j \geq 2} (m_i + \Delta_j) \in M_0 \oplus M_1 \). Then \((H, d, m_i, \Delta_j)_{i,j \geq 2}\)
is a special \( A_\infty \)-bialgebra provided
\[
d_\omega \bullet d_\omega = 0.
\]

Note that one recovers the classical quadratic relations in an \( A_\infty \)-algebra when \( \omega = d + \sum_{i \geq 2} m_i \).

5. Structure Relations

The structure relations in a special \( A_\infty \)-bialgebra \((H, d, m_i, \Delta_j)_{i,j \geq 2}\) follow easily from the following two observations:

1. If \( \theta, \eta \in M \), then \( \theta \bullet \eta = 0 \) whenever the projection of \( \theta \) or \( \eta \) to \( M_0 \oplus M_1 \) is zero.
2. Each non-zero component in the projections of \( \theta \) and \( \eta \) is represented by a horizontal, vertical or zero length arrow.

By (1), each component of \( d_\omega \bullet d_\omega \) is a “transgression” represented by a “2-step” path of arrows from the horizontal axis \( M_1 \) to the vertical axis \( M_1 \); and by (2), each such 2-step path follows the edges of a (possibly degenerate) rectangle positioned with one of its vertices at \((1,1)\).

Now relations involving \( d \) arise from degenerate rectangles since arrows of length zero represent components in the (co)extensions of \( d \). Hence \( d \) interacts with the \( m_i \)'s or the \( \Delta_j \)'s exclusively and the relations involving \( d \) are exactly the classical quadratic relations in an \( A_\infty \)-(co)algebra.

On the other hand, relations involving the \( m_i \)'s and \( \Delta_j \)'s arise from non-degenerate rectangles since \( m_i \) and \( \Delta_j \) are represented by the arrows \((i,1) \rightarrow (1,1)\) and \((1,1) \rightarrow (i,j)\). While the two-step path \((i,1) \rightarrow (1,1) \rightarrow (1,j)\) represents the (usual) composition \( \Delta_j \bullet m_i \), the two-step path \((i,1) \rightarrow (i,j) \rightarrow (1,j)\) represents \( \xi^j \left( e^{i-2} \right) \bullet \xi^i \left( e^{j-2} \right) \). Thus we obtain the relation
\[
\Delta_j \bullet m_i = \xi^j \left( e^{i-2} \right) \bullet \xi^i \left( e^{j-2} \right).
\]

For example, by setting \( i = j = 2 \) we obtain the classical bialgebra relation
\[
\Delta_2 \bullet m_2 = \begin{bmatrix} m_2 \\ m_2 \end{bmatrix} \bullet [\Delta_2 \Delta_2].
\]

And with \((i,j) = (3,2)\) we obtain
\[
\Delta_2 \bullet m_3 = \left\{ \begin{bmatrix} m_3 \\ m_2 (1 \otimes m_2) \end{bmatrix} + \begin{bmatrix} m_2 (m_2 \otimes 1) \\ m_3 \end{bmatrix} \right\} \bullet [\Delta_2 \Delta_2 \Delta_2]
\]
(see Figure 4).
We summarize the discussion above in our main theorem:

**Theorem 1.** \((H,d,m_i,\Delta_j)_{i,j\geq 2}\) is a special \(A_\infty\)-bialgebra if \((H,d,m_i)_{i\geq 2}\) is an \(A_\infty\)-algebra, \((H,d,\Delta_j)_{j\geq 2}\) is an \(A_\infty\)-coalgebra and for all \(i,j \geq 2\),

\[
\Delta_j \cdot m_i = \xi^j \left( e^{i-2} \right) \cdot \eta^i \left( e^{j-2} \right).
\]

**References**

[1] J. F. Adams, “Infinite Loop Spaces,” Annals of Mathematical Studies, 90, Prinston University Press, Prinston, New Jersey (1978).

[2] V.K.A.M. Gugenheim, On a Perturbation Theory for the Homology of the Loop Space, *J. Pure Appl. Algebra* 25 (1982), 197-205.

[3] T. Kadeishvili, On the Homology Theory of Fibre Spaces, *Russian Math. Survey* 35 (1980), 131-138.

[4] M. Markl, S. Shnider and J. Stasheff, “Operads in Algebra, Topology and Physics,” Mathematical Surveys and Monographs, 96 (2002).

[5] S. Saneblidze and R. Umble, Diagonals on the Permutahedra, Multiplihedra and Associahedra, *J. Homology, Homotopy and Appl.*, 6 (1) (2004), 363-411.

[6] ————–, The Biderivative and \(A_\infty\)-bialgebras, *J. Homology, Homotopy and Appl.*, 7 (2) (2005), 161-177.

[7] ————–, Matrons and the Category of \(A_\infty\)-bialgebras, in preparation.

[8] B. Shoikhet, The CROCs, Non-commutative Deformations, and (Co)associative Bialgebras, preprint, math. QA/0306143.

Department of Mathematics, Millersville University of Pennsylvania, Millersville, PA. 17551.