Palatini Formalism of 5-Dimensional
Kaluza-Klein Theory

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Abstract

The Einstein field equations can be derived in $n$ dimensions ($n > 2$) by the variations of the Palatini action. The Killing reduction of 5-dimensional Palatini action is studied on the assumption that pentads and Lorentz connections are preserved by the Killing vector field. A Palatini formalism of 4-dimensional action for gravity coupled to a vector field and a scalar field is obtained, which gives exactly the same fields equations in Kaluza-Klein theory.

Keywords: Palatini action; Kaluza-Klein theory; Killing vector.

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1 Introduction

Palatini formalism is causing great interest in the study of non-perturbative quantum gravity[1][2][3], modified gravity theories[4][5] and their cosmological applications[6][7]. Also, spacetime reduction is very important in any high dimensional theory of physics such as Kaluza-Klein theory[8][9][10][11] and string theory[12][13][14]. The dimensional reduction can make a high dimensional theory contact with the 4-dimensional sensational world. The Campbell-Magaard theorem is generalized to study the embedding of an $n$-dimensional spacetime into an $(n+1)$-dimensional spacetime in high dimensional physics[14]. On the other hand, Killing reduction of 4-dimensional and 5-dimensional spacetimes have been studied by Geroch[15] and Yang et. al. [16]. As in Kaluza-Klein theory, the Killing reduction of 5-dimensional Einstein spacetimes gives the 4-dimensional gravity coupled to the electromagnetic field and a scalar field. The interesting scalar field may contribute to the explanation to the dark physics in current cosmology as well as the Higgs field in particle physics[17][18].

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Let \((M, g_{ab})\) be an \(n\)-dimensional spacetime with a Killing vector field \(\xi^a\), which is everywhere spacelike. Let \(S\) denote the collection of all trajectories of \(\xi^a\). A map \(\psi\) from \(M\) to \(S\) can be defined as follows: For each point \(p\) of \(M\), \(\psi(p)\) is the trajectory of \(\xi^a\) passing through \(p\). Assume \(S\) is given the structure of a differentiable \((n-1)\)-manifold such that \(\psi\) is a smooth mapping. It is natural to regard \(S\) as a quotient space of \(M\). The proof of Geroch about the following conclusion is independent of the dimension of \(M\) \[16\]: There is a one-to-one correspondence between tensor fields \(\hat{T}_{b...d}^{a...c}\) on \(S\) and tensor fields \(T_{b...d}^{a...c}\) on \(M\) which satisfy
\[
\xi^a T_{b...d}^{a...c} = 0, \quad \ldots, \quad \xi_d T_{b...d}^{a...c} = 0, \\
\mathcal{L}_\xi T_{b...d}^{a...c} = 0,
\]
where \(\mathcal{L}_\xi\) denotes the Lie derivative with respect to \(\xi^a\). The entire tensor field algebra on \(S\) is completely and uniquely mirrored by tensor field on \(M\) subject to Eq. (1). Thus, we shall speak of tensor fields being on \(S\) merely as a shorthand way of saying that the fields on \(M\) satisfy Eq. (1). The metric and the Kronecker delta on \(S\) are defined as
\[
h_{ab} = g_{ab} - \lambda^{-1}\xi_a \xi_b, \\
h_b^a = \delta_b^a - \lambda^{-1}\xi_a \xi^b,
\]
where \(\lambda \equiv \xi^a \xi_a\). Eq. (3) can also be interpreted as the projection operator onto \(S\). Note that in general \(S\) cannot be an embedded submanifold of \(M\) \[16\], hence the Campbell theorem is not valid for this treatment. Note also that the abstract index notation \[18\] \[19\] is employed for spacetime indices through this paper.

To study the Palatini formalism of 5-dimensional Kaluza-Klein theory, we first extremize the \(n\)-dimensional Palatini action and obtain the pure Einstein field equations. Then, we reduce the 5-dimensional Palatini action, assuming there is a spacelike Killing vector field in the 5-dimensional spacetime. Note that if the extra dimension is compactified as a circle \(S^1\) with a microscopic radius, a Killing vector field may arise naturally in low energy regime \[8\]. Since we are working in Palatini formalism, besides the assumption that the connection and pentad are preserved by the Killing vector field, we also have to assume certain relation of the 4-dimensional electromagnetic field as well as scalar field and some components of the underlying 5-dimensional connection. This is motivated by the pentad formalism. By the Killing reduction, we obtain a Palatini formalism of 4-dimensional action coupled to a vector field and a scalar field. The variations of this action give the coupled fields equation, which are as same as those in the 5-dimensional Kaluza-Klein theory.

2 Palatini action in \(n\) dimensions

In this section, following the approach in 4 dimensions \[12\], we will show in detail that the Palatini action reproduce Einstein’s equation in \(n\) dimensions.
(n > 2). Although this is a well-known result, to our knowledge the same proof for n dimensions has not appeared so far in the literature.

Consider an n-manifold $M$, on which the basic dynamical variables in the Palatini framework are n-bases $(e_\mu)^a$ and Lorentz connections $\omega^{\mu\nu}_a$, where the Greek indices $\mu, \nu$ denote the internal Lorentz group. The internal space is equipped with a Minkowskian metric $\eta_{\mu\nu}$ (of signature $- + \ldots +$), which is fixed once and for all. Consequently, one can freely raise and lower the internal indices; their position does not depend on the choice of dynamical variables. To raise or lower the spacetime indices $a, b, \ldots$, on the other hand, one needs a space-time metric $g_{ab}$ which is a dynamical variable, constructed from the duel bases $(e^\mu)_a$ via:

$$g_{ab} = \eta_{\mu\nu}(e^\mu)_a (e^\nu)_b.$$  

The connection 1-form $\omega^{\mu\nu}_a$ acts only on internal indices; it defines a generalized derivative operator $\tilde{\nabla}_a$ via:

$$\tilde{\nabla}_a K_\mu := \partial_a K_\mu + \omega_{\mu a}^\nu K_\nu, \quad (4)$$

where $\partial_a$ is a fiducial derivative operator. Since $\tilde{\nabla}_a$ annihilates the fiducial Minkowskian metric $\eta_{\mu\nu}$ on the internal space, the connection 1-forms $\omega^{\mu\nu}_a$ are antisymmetric in $\mu$ and $\nu$; they take values in the Lorentz Lie algebra. The curvature $\Omega^{\mu\nu}_{ab}$ of the connection $\omega^{\mu\nu}_a$ is given by:

$$\Omega^{\mu\nu}_{ab} = (d\omega^{\mu\nu})_{ab} + [\omega_a^\mu, \omega_b^\nu], \quad (8)$$

where $[,]$ stands for the commutator in the Lorentz Lie algebra.

The Palatini action is given by:

$$S_p[(e_\mu)^a, \omega^{\mu\nu}_a] = \int_M e(e_\mu)^a (e_\nu)^b \Omega^{\mu\nu}_{ab}, \quad (5)$$

where $e$ is the square root of the determinant of the n-metric $g_{ab}$. The field equations are obtained by varying this action with respect to $(e_\mu)^a$ and $\omega^{\mu\nu}_a$. To carry out the variation with respect to the connection, it is convenient to introduce the unique (torsion free) connection $\nabla_a$ on both space-time and internal indices determined by the bases $(e^\mu)_a$ via:

$$\nabla_a (e_\mu)^b = 0. \quad (6)$$

The difference between the actions of $\nabla_a$ and $\tilde{\nabla}_a$ on internal indices is characterized by a field $C_{a\mu}^\nu$:

$$(\nabla_a - \tilde{\nabla}_a)V_\mu = C_{a\mu}^\nu V_\nu. \quad (7)$$

The difference between their curvatures is given by:

$$\Omega^{\mu\nu}_{ab} - R^{\mu\nu}_{ab} = 2\nabla_{[a} C^{\mu\nu}_{b]} + 2C_{[a}^{\mu\rho} C_{b]}^\rho \nu, \quad (8)$$

where $R^{\mu\nu}_{ab}$ is the internal curvature of $\nabla_a$. Note that the variation of the action with respect to $\omega^{\mu\nu}_a$ (keeping the basis fixed) is the same as the variation
with respect to $C_{a}^{\mu \nu}$. Using Eq. (8), the Palatini action becomes:

$$S_{p}[(e_{\mu})^{a}, C_{a}^{\mu \nu}] = \int_{M} e_{\mu}^{a}(e_{\nu})^{b}(R_{ab}^{\mu \nu} + 2\nabla_{[a}C_{b]}^{\mu \nu} + 2C_{[a}^{\mu \rho}C_{b]}^{\nu \rho}).$$

By varying this action with respect to $C_{a}^{\mu \nu}$, one obtains:

$$((e_{\rho})^{[a}(e_{\sigma})^{b]}\delta_{\mu}^{\rho}\delta_{\nu}^{\sigma})C_{b}^{\sigma} = 0,$$

We now show that Eq. (10) implies:

$$C_{\alpha}^{\mu \nu} = 0.$$  
(11)

To see this, define a space-time tensor field $S_{abc} := C_{a}^{\mu \nu}(e_{\mu})_{b}(e_{\nu})_{c}$. Then the condition $C_{a}^{\mu \nu} = C_{a[\mu \nu]}$ is equivalent to $S_{abc} = S_{a[bc]}$. Now contracting Eq. (10) with $(e_{\mu})_{a}(e_{\nu})_{c}$, one obtains

$$(n - 2)S_{bc}^{b} + S_{ca}^{a} = 0$$

This yields $S_{bc}^{b} = 0$, when $n \neq 2$. Hence $S_{abc}$ is trace-free on its first and last indices. Using this result, Eq. (10) leads to

$$C_{b}^{\mu \rho}(e_{\rho})^{a}(e_{\nu})^{b} - C_{b}^{\nu \rho}(e_{\rho})^{b}(e_{\mu})^{a} = 0.$$  
(13)

If we now contract Eq. (13) with $(e_{\mu})_{c}(e_{\nu})_{d}$, we get

$$S_{cd}^{a} = S_{(cd)}^{a}$$

Thus, $S_{abc}$ is symmetric in its first two indices. Since $S_{abc} = S_{a[bc]}$ and $S_{abc} = S_{abc}$, we can successively interchange the first two indices to show $S_{abc} = 0$. Furthermore, since $(e_{\mu})_{a}$ are invertible, we get $C_{a}^{\mu \nu} = 0$. This is the desired result.

Thus, the equation of motion for the derivative operator $\tilde{\nabla}_{a}$ is simply that it equals $\nabla_{a}$ while acting on objects with only internal indices. Thus the connection $\nabla_{a}$ is completely determined by the bases. By carrying out the variation of action with respect to the bases, one obtains:

$$(e_{\mu})^{c}\Omega_{cb}^{\mu \nu} - \frac{1}{2} \Omega_{cd}^{\rho \sigma}(e_{\rho})^{c}(e_{\sigma})^{d}(e_{\nu})_{b} = 0$$  
(15)

Substitution of Eq. (11) in Eq. (8) implies that $\Omega_{ab}^{\mu \nu} = R_{ab}^{\mu \nu}$. Using the fact that the internal curvature of $\nabla_{a}$ is related to its space-time curvature by $R_{abcd} = R_{abc}^{d}(e_{\mu})^{(c}(e_{\nu})_{b}$ and multiplying Eq. (15) by $(e_{\nu})_{a}$ tells us that the Einstein tensor $G_{ab}$ of the metric $g_{ab}$ vanishes.

### 3 Palatini action in 5-dimensional Kaluza-Klein theory

To make Killing reduction of the Palatini action, we first generalize the reduction program in Refs. [15] and [16] to generalized tensor fields.
Suppose $S$ is the reduced manifold of $n$-dimensional spacetime $(M, g_{ab})$ with a spacelike Killing vector field $\xi^a$ as in Sec.1. Let $(e_\mu)^a$ ($\mu = 0, 1, \cdots, n - 1$) be orthonormal bases on $M$. To simplify the formalism, we make a partial gauge fixing by choosing

$$(e_{n-1})^a = \lambda^{-\frac{1}{2}} \xi^a,$$

and assume

$$\mathcal{L}_\xi \omega_{\mu a} = 0, \quad \mathcal{L}_\xi (e_i)^a = 0, \quad i = 0, 1, \cdots, n - 2.$$  \hspace{1cm} (16)

It is easy to see that Eq. (16) implies $\mathcal{L}_\xi (e_{n-1})^a = 0$ and $\xi_a (e_i)^a = 0$. Eq. (17) then means that $(e_i)^a$ are orthonormal bases on $S$, denoted by $(\hat{e}_i)^a$. It is natural to think $i, j, k = 0, 1, \cdots, n - 2$ as internal Lorentz indices on $S$. Using Eq. (17), one can define Lorentz connection 1-forms $\hat{\omega}^i_{ja}$ on $S$ as

$$\hat{\omega}^i_{ja} = h^b_a \omega^i_{jb}.$$  \hspace{1cm} (18)

It is easy to see that there is a one-to-one correspondence between generalized tensor fields $\hat{T}^{b \cdots di \cdots j}_{a \cdots ck \cdots l}$ on $S$ and generalized tensor fields $T^{b \cdots di \cdots j}_{a \cdots ck \cdots l}$ on $M$ which satisfy

$$\xi^a T^{b \cdots di \cdots j}_{a \cdots ck \cdots l} = 0, \quad \xi^a d_{b \cdots di \cdots j} = 0, \quad \mathcal{L}_\xi T^{b \cdots di \cdots j}_{a \cdots ck \cdots l} = 0.$$  \hspace{1cm} (19)

A generalized derivative on $S$ is defined by

$$\hat{D}_a T^{b_1 \cdots b_k i_1 \cdots i_m}_{c_1 \cdots c_l} = h^a_{c_1} h^{b_1}_{d_1} \cdots h^{b_k}_{d_k} h^{c_1}_{e_1} \cdots h^{c_l}_{e_l} \nabla_{a_1} T^{d_1 \cdots d_k i_1 \cdots i_m}_{c_1 \cdots c_l},$$

where $\nabla_a$ is the generalized derivative on $M$ defined by Eq. (4) and $T^{b_1 \cdots b_k i_1 \cdots i_m}_{c_1 \cdots c_l}$ is any generalized tensor field on $S$. Note that $\hat{D}_a$ satisfies all the conditions of a derivative operator and

$$\hat{D}_a V_i = \hat{\partial}_a V_i + \hat{\omega}^j_{ja} V_j,$$

where $\hat{\partial}_a$ is the fiducial derivative operator on $S$ defined by $\partial_a$ on $M$. The unique connection determined by $(\hat{e}_i)^a$ on $S$ reads

$$D_a V_i = h^b_a \nabla_b V_i,$$

where $\nabla_a$ is the connection on $M$ defined by Eq. (5).

We now consider the special case where $n = 5$. Let $\varepsilon_{abcde}$ be the volume element associated with the metric $g_{ab}$ on $M$. Then it can be shown that $\varepsilon_{abcde} \equiv |\lambda|^{-\frac{1}{2}} \varepsilon_{abcde} \xi^e$ is the volume element associated with the metric $h_{ab}$ on $S$. Let

$$F_{ab} \equiv -\frac{1}{2} \lambda^{-\frac{1}{2}} \varepsilon_{abcde} \omega^{cd},$$  \hspace{1cm} (20)

where the twist 2-form of $\xi^a$ is defined as $\omega_{ab} := \varepsilon_{abcde} \xi^c \nabla^d \xi^e$. Clearly we have $F_{ab} = F_{[ab]}$ and $F_{ab} \xi^a = 0$. It is also easy to see that

$$\mathcal{L}_\xi \lambda = 0, \quad \mathcal{L}_\xi F_{ab} = 0.$$  \hspace{1cm} (21)
Hence, $\lambda$ and $F_{ab}$ are fields on $S$. It is shown in Ref. [16] that there is at least locally a one-form $A_a$ on $S$ such that $F_{ab} = (dA)_{ab}$, which will be shown to play still the role of electromagnetic field on $S$.

Suppose the Lorentze connection $\omega^\mu_{\nu a}$ be compatible with the pentad $(e^\nu)_b$. It is then easy to see that the reduced connection $\tilde{\omega}^i_{ja}$ on $S$ would be compatible with the tetrad $(\tilde{e}^i)_b$. One thus has the structure equations [10]

\[(de^\mu)_{ab} = -\omega^\mu_{\nu a} \wedge (e^\nu)_b \] (22)

and

\[(de^i)_{ab} = -\tilde{\omega}^i_{ja} \wedge (\tilde{e}^j)_b. \] (23)

It is easy to check $\xi^a(\xi^a)_{ab} = 0$, which leads to

\[(de^i)_{ab} \equiv -\omega^i_{ja} \wedge (e^j)_b - \omega^i_{4a} \wedge (e^4)_b = (de^i)_{ab}. \] (24)

From Eq. (18), one has

\[\tilde{\omega}^{ij}_{a}(\tilde{e}^k)_a \equiv \tilde{\omega}^{ij}_{k} \equiv \omega^{ij}_{a}(e^k)_a. \] (25)

Substituting Eqs. (23) and (25) into Eq. (24), one gets

\[\omega^i_{4a} \wedge (e^4)_b = 0, \]

which leads to

\[\omega^i_{[4j]} = 0. \] (26)

Using Eq. (20), one obtains

\[\omega_{4ij} = -\omega_{ij4}, \quad \omega^4_{ij} = \omega^4_{[ij]}. \] (27)

The exterior derivative of the remaining basis is given by

\[d(e^4)_{ab} = -\omega^4_{ia} \wedge (e^i)_b. \] (28)

Using Eqs. (10) and (20), it can also be expressed as

\[(de^4)_{ab} = 2\nabla_{[a} \lambda^{-\frac{1}{2}} \xi_{b]} = \lambda^{-\frac{1}{2}} \xi_{[b} D_{a]} \lambda + \lambda^{\frac{1}{2}} F_{ab}. \] (29)

By using Eqs. (27), (28) and (29), one obtains

\[-\omega_{ij4} = \omega^4_{ij} = \frac{1}{2} \lambda^{\frac{1}{2}} F_{ij}, \] (30)

\[\omega^4_{i4} = \frac{1}{2} \lambda^{-\frac{1}{2}} D_i \lambda, \] (31)

where $D_i \lambda \equiv (e_i)^{a} D_a \lambda$, $F_{ij} \equiv F_{ab}(e_i)^a (e_j)^b$.

Although $\omega^\mu_{\nu a}$ is not necessarily compatible with $(e^\nu)_b$ in the Palatini formalism, we will still take Eqs. (20) and (31) as an assumption. Using Eqs. (25) and (31), one thus gets the relationship

\[\omega^i_{ja} = \tilde{\omega}^j_{ja} + \frac{1}{2} \lambda^{\frac{1}{2}} F_j^i (e^4)_a \] (32)
between the connections in \( M \) and \( S \). And using Eqs. (31) and (30), one obtains

\[
\omega^4_{ia} = \frac{1}{2} \lambda^2 F^i (e^j)_a + \frac{1}{2} \lambda^{-1} D_i \lambda (e^4)_a.
\]  

(33)

The curvature 2-forms of connections \( \omega^{\mu\nu} \) on \( M \) are defined by the structure equation

\[
\Omega_{ab}^{\mu\nu} = (d\omega^{\mu\nu})_{ab} + \omega^{\mu}_{\lambda a} \wedge \omega^{\lambda\nu}_b \equiv \frac{1}{2} \Omega_{\rho\sigma}^{\mu\nu} (e^\rho)_a \wedge (e^\sigma)_b.
\]

Using Eqs. (32) and (33), one gets

\[
\Omega_{ab}^{ij} = (d\omega^{ij})_{ab} + \omega^{j}_{ka} \wedge \omega^{k}_{ia} \wedge \omega^{ij}_b
\]

(34)

and

\[
\Omega_{ab}^{ij} = (d\omega^{ij})_{ab} + \omega^{j}_{ka} \wedge \omega^{k}_{ia} \wedge \omega^{ij}_b
\]

(35)

Here we have used the identity \((dF)_{abc} = 0\) and the 4-dimensional curvature 2-forms

\[
\tilde{\Omega}_{ab}^{ij} = (d\tilde{\omega}^{ij})_{ab} + \tilde{\omega}^{j}_{ka} \wedge \tilde{\omega}^{k}_{ia} \wedge \tilde{\omega}^{ij}_b \equiv \frac{1}{2} \tilde{\Omega}_{kl}^{ij} (e^k)_a \wedge (e^l)_b.
\]

(36)

These geometrical considerations become physically relevant when we postulate that gravitation in the five-dimensional space is governed by the corresponding Palatini action \( S \). Using Eqs. (34) and (35), the action \( S \) on \( M \) can be reduced to the following action on \( S \):

\[
S_p[(\hat{e}_i)^a, \tilde{\omega}^{ij}_a, \lambda, F_{ab}] = \int_S \lambda^2 \hat{e}[(\hat{e}_i)^a (\tilde{\omega}^{ij}_a) \tilde{\Omega}_{ab}^{ij} - \frac{1}{2} \lambda F_{ab} F^{ab} - 2 \lambda^{-2} (\hat{e}_i)^a \hat{D}_a D^i (\lambda^2)]
\]

(36)

where \( \hat{e} \) is the square root of the determinant of \( h_{ab} \). Note that \((\hat{e}_i)^a \hat{D}_a D^i (\lambda^2) \neq \hat{D}_a D^a (\lambda^2)\). Now we define

\[
\hat{C}_{ai}^j \equiv h^b_c C_{bi}^j,
\]

where \( C_{ai}^j \) comes from \( C_{ai}^{\mu\nu} \) defined by Eq. (7). It is then easy to see

\[
(\hat{D}_a - D_a) V_i = \hat{C}_{ai}^j V_j.
\]
Thus one has
\[ \hat{\Omega}_{ab}^{ij} - \hat{R}_{ab}^{ij} = 2 D_a [\hat{C}_b^{ij} + 2 \hat{C}^{ik}_{[a} \hat{C}^{j]}_{b k}], \]  
(37)
where \( \hat{R}_{ab}^{ij} \) denotes internal curvature of \( D_a \).

Then the action (36) becomes
\[
S_p[(\hat{e}_i)^a, \hat{C}_{ai}^j, \lambda, F_{ab}] = \int S[(\hat{e}_i)^a, \hat{C}_{ai}^j, \lambda, F_{ab}] = \int \lambda (\hat{e}_i)^a (\hat{e}_j)^b (\hat{R}_{ab}^{ij} + 2 D_a [\hat{C}_b^{ij} + 2 \hat{C}^{ik}_{[a} \hat{C}^{j]}_{b k}] - \frac{1}{4} \lambda F_{ab} F^{ab}) - 2 \lambda^2 (\lambda^2)^{ij} - 2 (\hat{e}_i)^a \hat{C}_{ai}^j D^j(\lambda^2), \]
(38)

where \( D^2 = D_a D^a \) is the four-dimensional d’Alembertian operator.

The sum of the second and the sixth terms reads:
\[
2 \lambda^2 (\hat{e}_i)^a (\hat{e}_j)^b D_a [\hat{C}_b^{ij} + 2 \hat{C}^{ik}_{[a} \hat{C}^{j]}_{b k}] - 2 (\hat{e}_i)^a \hat{C}_{ai}^j D^j(\lambda^2) = 2 D_a [\hat{C}_b^{ij} + 2 \hat{C}^{ik}_{[a} \hat{C}^{j]}_{b k}] (\hat{e}_i)^a (\hat{e}_j)^b. \]
(39)

Since \( D_a \) annihilates the tetrad, Eq. (39) and the fifth term are pure divergences and therefore do not contribute to the variation. Neglecting the boundary terms, action (38) becomes:
\[
S_p[(\hat{e}_i)^a, \hat{C}_{ai}^j, \lambda, F_{ab}] = \int S[(\hat{e}_i)^a, \hat{C}_{ai}^j, \lambda, F_{ab}] = \int \lambda (\hat{e}_i)^a (\hat{e}_j)^b (\hat{R}_{ab}^{ij} + 2 \hat{C}^{ik}_{[a} \hat{C}^{j]}_{b k}] - \frac{1}{4} \lambda F_{ab} F^{ab}), \]
(40)

The variation of this action with respect to \( \hat{C}_{ai}^j \) yields:
\[
((\hat{e}_k)^a (\hat{e}_j)^b) \delta^k_{[i} \delta^m_{j]} \hat{C}_{bm}^l = 0,
\]
which has the same form as Eq. (10), implying that:
\[
\hat{C}_{ai}^j = 0. \]
(41)

Hence the equation of motion for the connection \( \hat{D}_a \) is again that it equals \( D_a \). It is then straightforward to see that action (40) gives exactly the same fields equations for the dynamical variables \((\hat{e}_i)^a, \lambda \) and \( A_a \) as in 4-dimensional Kaluza-Klein theory [10]. One can also substitute Eq. (41) for action (40) and get the conventional reduced Kaluza-Klein action:
\[
S_p[(\hat{e}_i)^a, \lambda, F_{ab}] = \int \lambda (\hat{e}_i)^a (\hat{e}_j)^b (\hat{R}_{ab}^{ij} + 2 \hat{C}^{ik}_{[a} \hat{C}^{j]}_{b k} - \frac{1}{4} \lambda F_{ab} F^{ab}). \]
(42)

In conclusion, we have studied the reduction of the Palatini action in 5-dimensional spacetime with a spacelike Killing vector field. The 4-dimensional electromagnetic and scalar fields are assumed to be related to the 5-dimensional Lorentz connection by Eqs. (30) and (31). The reduced action (40) is in the 4-dimensional Palatini formalism of gravity coupled to the electromagnetic field and the scalar field. It gives the same equations of motion of Kaluza-Klein...
theory. The reduced Palatini action for 5-dimensional Kaluza-Klein theory is thus obtained. The reduction scheme might also be extended to the Palatini formalism of higher-dimensional Kaluza-Klein theories. Thus one may study Kaluza-Klein theory in Palatini formalism as well, where the scaler field might play an important role in the explanation to the dark physics in cosmology and the Higgs field in particle physics.

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