Hölder properties of perturbed skew products and Fubini regained

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Abstract
In 2006, Gorodetski proved that central fibres of perturbed skew products are Hölder continuous with respect to the base point. In this paper, we give an explicit estimate of this Hölder exponent. Moreover, we extend Gorodetski’s result from the case when the fibre maps are close to the identity to a much wider class of maps that satisfy the so-called modified dominated splitting condition. In many cases (for example, in the case of skew products over the solenoid or over linear Anosov diffeomorphisms of the torus), the Hölder exponent is close to 1. This allows one to overcome the so-called Fubini nightmare, in some sense. Namely, we prove that the union of central fibres that are strongly atypical from the point of view of ergodic theory, has Lebesgue measure zero despite the lack of absolute continuity of the holonomy map for the central foliation. This result is based on a new kind of ergodic theorem, which we call special. To prove our main result, we revisit the theory of Hirsch, Pugh and Shub, and estimate the contraction constant of the graph transform map.

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1. Introduction
This paper is part of an ample program that was introduced in [5]. The program is aimed at finding new, robust properties of dynamical systems according to the following procedure:
1. A new property is found for random dynamical systems, also known as skew products over the Bernoulli shift.
2. This property is then transported to an open set in a space of skew product diffeomorphisms over a hyperbolic map (e.g. the Smale–Williams solenoid).

3. Finally, one proves that the property persists under $C^1$ perturbations of such skew products, and is thus open in the space of diffeomorphisms of a certain manifold.

The papers [5–17] and [19–22] have been written in the above framework. In this paper, we will focus on the third step by providing certain tools that we will now describe.

The first tool we provide is the Hölder continuity theorem. It claims that perturbations of certain partially hyperbolic skew products with compact fibres produce invariant central foliations whose fibres are Hölder continuous with respect to the initial conditions, with the exponent $\alpha$ close to 1: the smaller the perturbation, the smaller $1 - \alpha$ becomes. In theorem 2, we extend a partial case of this result to central-stable fibres. Note that the Hölder continuity of noncompact central fibres of partially hyperbolic maps is still an open problem.

The second tool we develop is the special ergodic theorem. It claims that the Hausdorff dimension of the set on which the time average of a continuous function is uniformly different from the space average, is strictly smaller than the Hausdorff dimension of the phase space. The theorem was proved during the preparation of this paper; different versions of it were published in [14, 30], see also [19].

The third tool is the study of Hausdorff dimension under the conjugacy provided by the Hirsch–Pugh–Shub ‘foliation stability theorem’. This conjugacy is not absolutely continuous, and therefore gives rise to the ‘Fubini nightmare’ described in [31], thereby producing a serious obstacle to estimating the measure of sets saturated by central fibers of perturbed maps.

We go around this nightmare using the previous two tools: the Hölder continuity of the central fibres and the special ergodic theorem allow us to prove in theorem 4 that certain dynamically important sets saturated by central fibres have Lebesgue measure zero, in spite of the Fubini nightmare. Hence the last part of the title of this paper.

The complete statements of theorems 1–4 are given in section 2, and the latter sections are devoted to proofs. In sections 3 and 4 we work with laminations and graph transform operators, culminating with the proof of our main theorem 1. Several of the results we obtain on the way will help us in section 5, where we prove theorem 2. In section 6, we deal with a special ergodic theorem and sort of a Fubini property of compact central leaves, thus proving theorems 3 and 4. Finally, section 6.2 consists of an appendix where several technical results are proved.

2. Main results

2.1. Persistence and the Hölder property for skew products

Throughout this paper, a $C^r$-morphism will refer to a $C^r$ map with a $C^r$ inverse. We will use this notion for maps whose domain and range are manifolds (with or without boundary). If the map is not surjective, but its inverse is defined on the whole image, then we will still use this name.

Given a linear operator $A$ on a normed vector space $V$, when we write $a \leq |A| \leq b$ we mean that

$$a|v| \leq |A(v)| \leq b|v|, \quad \forall v \in V.$$  

This notation will be used repeatedly throughout the paper.

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6 This space will be defined rigorously in section 2.1
We consider a smooth compact manifold $B$ and a $C^2$-morphism $h : B \to B$ with a hyperbolic attractor $\Lambda$. We restrict attention to only two cases:

- $h$ is surjective and $\Lambda = B$,
- $h$ is injective and $\Lambda$ is its maximal attractor:

$$\Lambda = \bigcap_{n \geq 0} h^n(B).$$

Being hyperbolic, the map $h$ will have contracting and expanding directions in the vicinity of $\Lambda$. Thus there exist a Riemannian metric $d$ on $B$ and real numbers $0 \leq \lambda_- \leq \lambda < 1$ and $0 \leq \mu_- \leq \mu < 1$, as well as a decomposition of the tangent bundle:

$$TB|_\Lambda = E^s \oplus E^u, \quad (1)$$

such that

$$dh : E^s \to E^s \quad \text{and} \quad \lambda_- \leq |dh| \leq \lambda,$$

$$dh : E^u \to E^u \quad \text{and} \quad \mu_- \leq |dh^{-1}| \leq \mu. \quad (2)$$

The standard notion of hyperbolicity only cares about the inequalities on the right. For our needs, we will also need to control the inequalities on the left, but those come for free by compactness of $\Lambda$.

We assume that the bundles $E^s$ and $E^u$ are trivialized, i.e. that there exist isomorphisms

$$\varphi^s : B \times \mathbb{R}^k \to E^s, \quad \varphi^u : B \times \mathbb{R}^l \to E^u \quad (3)$$

for some positive integers $k, l \geq 0$. The above is a technical condition necessary for our proof, but we conjecture that all our results hold without it. We note that it holds when $h$ is the Smale–Williams solenoid map or any linear Anosov diffeomorphism of the torus.

**Definition 1.** An invariant set $\Lambda$ of a map $h$ with the above properties will be called $(\lambda_-, \lambda, \mu_-, \mu)$-hyperbolic.

Take another compact manifold $M$, called the fibre, and form the Cartesian product $X = B \times M$. By definition, a skew product over $h$ is a $C^1$-map of the form

$$F : X \to X, \quad F(b, m) = (h(b), f_b(m)), \quad (4)$$

where $f_b(m) : M \to M$ is a $C^1$ family of $C^1$-morphisms.

**Definition 2.** We say that the skew product $(4)$ satisfies the modified dominated splitting condition if

$$\max \left( \max(\lambda, \mu) + \left\| \frac{\partial f_b^{\pm 1}}{\partial b} \right\|_{C^0(X)}, \left\| \frac{\partial f_b^{\pm 1}}{\partial m} \right\|_{C^0(X)} \right) =: L < \min(\lambda^{-1}, \mu^{-1}). \quad (5)$$

In the classical dominated splitting condition, the off-diagonal terms $\| \cdot \|_{C^0(X)}$ are not included in the left-hand side of (5). In our modified version, the main result is slightly weaker than it might be, though this does not affect the applications of this result. The reason we use the modified assumption above is to keep the proofs shorter.

**Definition 3.** Given $\rho > 0$, a $\rho$-perturbation of the skew product $(4)$ is a $C^1$-morphism $G : X \to X$ such that

$$d(G^{\pm 1}, F^{\pm 1})_{C^1(X)} < \rho. \quad (6)$$
We make a notational convention. In this paper, we will consider a fixed skew product $\mathcal{F}$ and a neighbourhood $\Omega \ni \mathcal{F}$ in the $C^1$-norm, as in definition 3. Therefore, we will often be concerned with small perturbations $\mathcal{G} \in \Omega$ of $\mathcal{F}$. The leaves of central foliations of the perturbed maps $\mathcal{G}$ are graphs of parameter dependent maps $\beta_b : M \rightarrow B$, or in other words, parameter dependent perturbations of the central fibres of $\mathcal{F}$. Whenever we write $||\beta_b||_{C^0} = O(\rho)$, we mean that there exists a constant $C$ depending only on $\Omega$, such that for any $\rho$-perturbation $\mathcal{G} \in \Omega$, the maps corresponding to all central leaves satisfy the inequality $||\beta_b||_{C^0} \leq C \rho$. Thus the operator that maps a diffeomorphism $\mathcal{G}$ to its leaf $\beta_b$ is Lipshitz at $\mathcal{F}$ with constant $C$ (uniformly in $b$). We will consider other (parameter dependent) operators and functionals defined on $\Omega$; the expression $O(\rho)$ has the same meaning for them.

Small perturbations of skew products are not necessarily skew products anymore. However, the following theorem shows that they are conjugated to skew products, and moreover the conjugation map satisfies a Hölder continuity property.

**Theorem 1 (The main theorem).** Consider a $C^2$-morphism $h$ with a $(\lambda, \mu)$-hyperbolic attractor $\Lambda$, as in definition 1. Consider a skew product $\mathcal{F}$ as in (4) over $h$ that satisfies the modified dominated splitting condition (5). Then for small enough $\rho > 0$, any $\rho$-perturbation $G$ of $\mathcal{F}$ has the following properties:

(a) There exists a $G$-invariant set $Y \subset X$ and a continuous map $p : Y \rightarrow \Lambda$ such that the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\mathcal{G}} & Y \\
\downarrow p & & \downarrow p \\
\Lambda & \xrightarrow{h} & \Lambda
\end{array}
\]

commutes. Moreover, the map

\[
H : Y \rightarrow \Lambda \times M, \quad H(b, m) = (p(b, m), m)
\]

is a homeomorphism.

(b) The fibres $W_b = p^{-1}(b)$ are Lipschitz close to vertical (constant) fibres, and Hölder continuous in $b$. This means that $W_b$ is the graph of a Lipschitz map $\tilde{\beta}_b : M \rightarrow B$ such that

\[
\begin{align*}
d(\tilde{\beta}_b, b)_{C^0} &\leq O(\rho), \\
\text{Lip } \tilde{\beta}_b &\leq O(\rho)
\end{align*}
\]

\[
\begin{align*}
d(\tilde{\beta}_b, \tilde{\beta}_{b'})_{C^0} &\leq \frac{d(b, b')^{\alpha - O(\rho)}}{O(\rho)^\alpha},
\end{align*}
\]

where

\[
\alpha = \min \left( \frac{\ln \lambda}{\ln \lambda - \ln \mu}, \frac{\ln \mu}{\ln \mu - \ln \lambda} \right).
\]

Moreover, the map $H^{-1}$ is also Hölder continuous, with the same $\alpha$.

**Remark 1.** Part (a) is a refinement of a classical result of [10]. In addition to [10], it provides an explicit upper bound for the contraction constant of the graph transform operator. This bound is crucial to our proof of part (b), and cannot be obtained by a mere reference to [10]. See lemma 1 for the precise statement of this bound, and section 2.6 for historical comments.
Remark 2. We first make a note about the exponent $\alpha$. In many cases (e.g., when $h$ is the solenoid map or a linear Anosov diffeomorphism of a torus), it may happen that $\lambda_- = \lambda$ and $\mu_- = \mu$. In that case, in the above theorem we have $\alpha = 1$, and thus the Hölder exponent can be made arbitrarily close to 1 by making $\rho$ small enough.

Remark 3. If $h$ is surjective and $\Lambda = B$, the invariant set $Y$ equals the entire phase space $X$. This may be proven in similar fashion to proposition 3.

Remark 4. In the particular case of Anosov diffeomorphisms of tori, a related result can be found in [9] (lemma 6.11), where it is shown that the holonomy of certain foliations is Hölder continuous.

Let us explain the usefulness of this theorem with respect to step 3 of the program outlined in section 1. The task is to take a property that is known to hold for certain $C^1$ skew products, and to prove that it holds for small perturbations thereof. The problem is that a priori, such a small perturbation $G$ might not be a skew product anymore. However, using theorem 1, the map

$$G = H \circ G|_Y \circ H^{-1} : \Lambda \times M \longrightarrow \Lambda \times M$$

is indeed a skew product:

$$G(b, m) = (h(b), g_b(m)).$$

One can then study the dynamical properties of the more mysterious map $G|_Y$ by studying the dynamical properties of its conjugate skew product $G$. The fibre maps $g_b$ of the skew product $G$ are $C^1$-close to those of the skew product $F|_Y$, in the following sense:

$$d(g_b^{\pm1}, f_b^{\pm1})_{C^1} \leq O(\rho).$$

But what can be said about the fibre maps $g_b$ for different $b$? Since $F$ is a $C^1$-morphism, the fibre maps $f_b$ depend in a $C^1$ fashion on the point $b \in \Lambda$. Such a result fails for the fibre maps $g_b$, but statement (b) of theorem 1 implies that the fibre maps $g_b$ depend Hölder continuously on the point $b \in \Lambda$:

$$d(g_b^{\pm1}, g_b'^{\pm1})_{C^0} \leq O(d(b, b')^\alpha),$$

where $\alpha$ is given by (11). A skew product $G$ whose fibre maps satisfy (13) will be called a Hölder skew product. Thus theorem 1 can be summarized as follows:

Let $G$ be any small perturbation of a $C^1$ skew product $F$ over a $C^2(\lambda_-, \lambda, \mu_-; \mu)$-hyperbolic map $h$, that satisfies the modified dominated splitting condition (5). Then $G$ has an invariant set $Y$ such that the restriction of $G|_Y$ is conjugated to a Hölder skew product close to $F|_Y \Lambda \times M$, in the sense of (11), (12) and (13).

2.2. The solenoid case

In this section we will present a partial improvement of theorem 1 in a special case inspired by the Smale–Williams solenoid. To recall this well-known construction, fix constants $R \geq 2$ and $\lambda < 0.1$ (their particular values will not be important). Let $B$ denote the solid torus:

$$B = S^1 \times D, \quad \text{where } S^1 = \{y \in \mathbb{R}/\mathbb{Z}\}, \quad D = \{z \in \mathbb{C}||z| \leq R\}.$$

The solenoid map is defined as

$$h : B \rightarrow B, \quad h(y, z) = (2y, e^{2\pi i y} + \lambda z).$$

(14)
The maximal attractor of the solenoid map:
\[ \Lambda = \bigcap_{k=0}^{\infty} h^k(B) \]
is called the **Smale–Williams solenoid**. It is a hyperbolic invariant set with contraction coefficient \( \lambda \) and expansion coefficient \( \mu^{-1} = 2 \) (we take the sup norm in \( T_B B \) in the coordinates \( y, z \)). Moreover, the estimates in (2) hold with \( \lambda = \lambda_\ast \) and \( \mu = \mu_\ast \).

We can generalize the above to the following setup: let \( B = Z \times F \) be the product of two compact Riemannian manifolds, with or without boundary. Suppose that \( h : B \to B \) is a skew product itself, i.e. there exists a \( C^2 \)-morphism \( \zeta : Z \to Z \) such that the following diagram commutes:

\[
\begin{array}{ccc}
B & \xrightarrow{h} & B \\
\vert & \searrow & \vert \\
Z & \xrightarrow{\zeta} & Z,
\end{array}
\]

(15)

where \( \pi \) is the standard projection. We assume that the map \( \zeta \) downstairs is expanding, and that the fibres \( \{z\} \times F \) are the stable manifolds of \( h \):

\[
\begin{align*}
\lambda_\ast & \leq |d\zeta^{-1}| \leq \mu, \\
\lambda_\ast & \leq |dh| \leq \lambda \\
& \quad \text{on } T(\{z\} \times F), \quad \forall z \in Z.
\end{align*}
\]

(16)

Again, for technical reasons we assume that \( E^s = TF \) is trivialized as in (3). In this setup, theorem 1 can be partially improved by the following result.

**Theorem 2.** Consider a map \( h \) as in (15), and a skew product:

\[
\begin{array}{ccc}
X = B \times M & \xrightarrow{\mathcal{F}} & X = B \times M \\
\vert & \searrow & \vert \\
B & \xrightarrow{h} & B,
\end{array}
\]

that also satisfies the modified dominated splitting condition. Then for small enough \( \rho > 0 \), any \( \rho \)-perturbation \( \mathcal{G} \) of \( \mathcal{F} \) has the following properties:

(a) There exists a continuous map \( q : X \to Z \) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\mathcal{G}} & X \\
\vert & \searrow & \vert \\
Z & \xrightarrow{\zeta} & Z
\end{array}
\]

(17)

commutes. Moreover, the commutative diagrams (7) and (17) must be compatible, in the sense that \( q|_Y = \pi \circ p \).

(b) The fibres \( W^s_z = q^{-1}(z) \) are Lipschitz close to the vertical (constant) fibres, and H"older continuous in \( z \). This means that \( W^s_z \) is the graph of a Lipschitz map \( \beta^s_z : F \times M \to Z \) such that

\[
\begin{align}
d(\beta^s_z, z)_{C^0} & \leq O(\rho), \\
\text{Lip } \beta^s_z & \leq O(\rho), \\
d(\beta^s_z, \beta^s_z')_{C^0} & \leq \frac{d(z, z')^{\alpha - O(\rho)}}{O(\rho)^\alpha},
\end{align}
\]

(18) (19)

where \( \alpha = \frac{\ln \mu}{\ln \mu_\ast} \).

As was mentioned above, a particularly important case of this theorem is the Smale–Williams solenoid: \( Z = S^1 \), \( F = D \) and \( h \) given by (14).
2.3. Special ergodic theorems

Recall the classical ergodic theorem: for a given ergodic map and a continuous (maybe even $L^1$) function $\varphi$, the set of points where the time average of $\varphi$ differs from the space average has measure zero. Special ergodic theorems are concerned with ‘strongly atypical points’ for which the difference between the time and space averages is uniformly separated from zero in magnitude. Our claim is that for certain classes of maps, the set of strongly atypical points has Hausdorff dimension strictly smaller than that of the phase space. Concretely, the following theorem is proved in section 6:

**Theorem 3.** Consider the circle doubling map:

$$s: S^1 \to S^1, \quad y \mapsto 2y,$$

an arc $\gamma \subset S^1$, and a number $\kappa > 0$. Let $\varphi = \chi_\gamma$ be the characteristic function of $\gamma$. Denote by $K_{\gamma, \kappa}$ the set

$$K_{\gamma, \kappa} = \{x \in S^1 | \liminf \bar{\varphi}_n(x), \limsup \bar{\varphi}_n(x) \not\subset (I - \kappa, I + \kappa)\},$$

(20)

where $I = \int_{S^1} \varphi$ and $\bar{\varphi}_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \varphi(s^k x)$. Then we have $\dim_H K_{\gamma, \kappa} < 1$.

The sketch of the proof of this theorem is given in [14], and the complete proof in section 6.

An analogous theorem for Anosov diffeomorphisms of a two torus was proved by Saltykov in [30]. The most general existing version of the special ergodic theorem is stated and proved in [19].

2.4. Fubini regained

We will now put the above pieces together. Consider a skew product $\mathcal{F}$ over the solenoid map $h$ from (14). For any small $\rho$-perturbation $G$ of $\mathcal{F}$, theorem 2 applies and in particular gives us a map $q: X \to Z = S^1$ that makes (17) commute.

**Theorem 4.** Consider an arbitrary positive number $\kappa$, an arc $\gamma \subset S^1$, and the set $K_{\gamma, \kappa}$ from (20). Then for $\rho$ small enough, and any $\rho$-perturbation $G$ of $\mathcal{F}$, the set $q^{-1}(K_{\gamma, \kappa})$ has Lebesgue measure 0.

This theorem is proved in section 6. Since the publication of this paper as a preprint, it has been generalized to a large class of hyperbolic maps $h$, via special ergodic theorems analogous to theorem 3 (see [19]).

2.5. Further developments

In this section, we give a brief survey of the various applications of the above results, as well as developments that have been obtained since the first version of this paper was published as a preprint ([16]). The results within are applied to the study of *attractors with intermingled basins*, *thick* and *bony* attractors, and gave rise to new versions of special ergodic theorems: [13, 14, 19–22]. We now write a few words about these types of attractors.

In [18], Kan discovered a striking new property of *boundary preserving endomorphisms* of the annulus, called *attractors with intermingled basins*. This property was later studied in [2] and [1], section 11.1.1. In [14] this property was transported to an open set of endomorphisms of the annulus (albeit in a weakened form). This was based on those results of this paper that were already known at that time. In [11], intermingled basins were observed for skew product diffeomorphisms, and in [20], they were observed for an open set in the space of all boundary preserving diffeomorphisms of $S^1 \times S^1 \times [0, 1]$. The following theorem was proved:
Theorem 5 ([14, 20]). The set of maps of the annulus $S^1 \times [0, 1]$ that have intermingled attracting basins is open in the set of all $C^2$ maps of the annulus into itself that keep the boundary invariant. The complement to the union of the attracting basins in the perturbed Kan example has Hausdorff dimension smaller than 2.

In [13], thick attractors of boundary preserving maps were discovered. These are attractors that have positive Lebesgue measure in the phase space, and so does their complement. The prototypes of such attractors for step skew products over the Bernoulli shift were constructed in [12]. In [13], the property of having thick attractors was transported to an open set in the space of all boundary preserving diffeomorphisms of $S^1 \times S^1 \times [0, 1]$, again using the results of this paper.

An attractor of a dynamical system is called bony if it has a thick intersection with a $k$-dimensional invariant submanifold (called the bone). By analogy to the previous paragraph, the intersection is called thick if it has positive Lebesgue measure in the bone, and so does its complement. In [21], bony attractors were studied for $k = 1$, in the case of skew products over a Bernoulli shift with the fibre a segment. In [22], this was generalized to an open set in the space of all diffeomorphisms, once again using the results of this paper. Bony attractors for $k > 1$ are subject to future investigations.

2.6. Historical comments

Our primary result, theorem 1, closely follows [10] (for statement (a)) and [8] (for statement (b)). Statement (a) of theorem 1 is a detailed presentation of a result from [10]. Our proof, however, is an improved version of the one in loc. cit., namely an explicit upper bound for the contraction constant of the graph transform operator (lemma 1, relation (34)). This improvement is crucial to the proof of statement (b) of theorem 1. After a consultation of the first author with Pugh and Shub in Toronto, September 2009, it was decided that this bound cannot be obtained by a mere reference to [10], but a new refined version must be written. This is the content of our section 3.

As for [8], it contains a general result on the Hölder continuity of compact centre fibres of perturbed partially hyperbolic skew products. A particular case of this was obtained earlier in [25]. Statement (b) of theorem 1 is a refinement of this result, in the sense that the Hölder exponent is estimated.

Another Hölder continuity result for partially hyperbolic systems appears in [27]: under a certain dominated splitting condition, a centre subbundle $E^c$ is shown to be Hölder continuous. Note that this does not imply the Hölder continuity of central fibres. Furthermore, since the appearance of this work as a preprint, theorem C of [28] has improved our theorem 1.

3. Rate of contraction of the graph transform map

In this section we prove statement (a) of theorem 1, and establish the rate of contraction of the graph transform map (see lemma 1). There are two ways to prove statement (a): the first one is to prove that $F$ is partially hyperbolic, and then refer to the Hirsch–Pugh–Shub theory. This theory implies the semiconjugacy statement (a), but gives no estimate of the rate of contraction of the graph transform map. The second way is to revisit the graph transform map and at the same time prove the fixed point theorem and the rate of contraction estimate for the graph transform. The latter is the path we will follow in this section.
3.1. Laminations

Let $B$, $h$ and $\Lambda$ be as in Section 2.1. In the fibres of the bundles $E^s$ and $E^u$ we have the abstract Riemannian metric, while in the fibres of the trivial bundles $B \times \mathbb{R}^k$ and $B \times \mathbb{R}^l$ we have the standard Euclidean metric. The isomorphism $\varphi^s$ of (3) implies that there exist $k$ linearly independent sections of $E^s$. By applying Gram–Schmidt orthonormalization to these sections, it follows that there exist $k$ orthonormal sections of $E^s$. Sending a fixed orthonormal basis of $\mathbb{R}^k$ to these orthonormal sections will give us a metric-preserving isomorphism $B \times \mathbb{R}^k \to E^s$, and it is this isomorphism that we will henceforth denote by $\varphi^s$. The same discussion applies to $\varphi^u$.

For any $\delta > 0$, we define $Q^s(\delta)$ and $Q^u(\delta)$ to be the open balls of radius $\delta$ around the origin of $\mathbb{R}^k$ and $\mathbb{R}^l$, respectively. The metric-preserving isomorphisms $\varphi^s$ and $\varphi^u$ induce metric-preserving isomorphisms in each fibre:

$$
\varphi^s_b(\delta) : Q^s(\delta) \to Q^s_b(\delta), \quad \varphi^u_b(\delta) : Q^u(\delta) \to Q^u_b(\delta),
$$

where $Q^s_b(\delta) \subset E^s$ and $Q^u_b(\delta) \subset E^u$ are the open balls of radius $\delta$ around the origin in the respective fibres.

The number $\delta$ must be chosen small enough such that for any $b \in B$, the exponential map gives us an open embedding $Q^s_b(\delta) \times Q^u_b(\delta) \hookrightarrow B$. We will write $B_b(\delta)$ for the image of this map. Composing this embedding with the isomorphism $\varphi^s_b(\delta) \times \varphi^u_b(\delta)$ gives us an open embedding (coordinate chart):

$$
\varphi_b(\delta) : Q^s(\delta) \times Q^u(\delta) \hookrightarrow B.
$$

We take $C > \max(\lambda^{-1}, \mu^{-1})$, and consider the above constructions for radius $C\delta$. Then we can express the map $h : B \to B$ locally around $b$ in the domain and around $h(b)$ in the target. Therefore, in coordinates given by the chart (22), the map $h$ takes the form

$$
h_b(\delta) = (\varphi_{h(b)}(C\delta))^{-1} \circ h \circ \varphi_b(\delta), \quad (h^{-1})_b(\delta) = (\varphi_b(C\delta))^{-1} \circ h^{-1} \circ \varphi_{h(b)}(\delta). \tag{23}
$$

For various values of $\delta$, the maps $h_b(\delta)$ will represent the same germ at 0, but will have different domains. Similarly, the maps $(h^{-1})_b(\delta)$ are representatives of the same germ at 0, but have different domains.

Up to section 4, we will be working with a single, fixed $\delta$. Therefore, we will often simply write $Q^s, Q^u, Q^s_b, Q^u_b, B_b, \varphi^s_b, \varphi^u_b, h_b, (h^{-1})_b$ for the notions introduced in the previous paragraphs. By (2) and the fact that diffeomorphisms (21) are metric preserving, $dh_b$ has block-diagonal form at 0

$$
dh_b(0) = \text{diag}(A^s, A^u),
$$

where $\lambda_- \leq |A_s| \leq \lambda$ and $\mu_- \leq |A_u^{-1}| \leq \mu$. Because the coordinate charts $\varphi_b$ are smooth functions, we have the following estimate throughout $Q^s \times Q^u$:

$$
||dh_b - \text{diag}(A^s, A^u)||_{C^0} \leq O(\delta). \tag{24}
$$

Now consider another compact manifold $M$, as in the statement of theorem 1. For any domain $A$ and any $\beta : A \to B$, we will denote by $\gamma(\beta)$ the map from $A$ onto the graph of $\beta$:

$$
\gamma(\beta) : A \to A \times B, \quad \gamma(\beta) : a \mapsto (a, \beta(a)) \in A \times B. \tag{25}
$$

Statement (a) of theorem 1 provides a correspondence between leaves and base points, so it’s about time we define these. The leaves of centre-stable, centre-unstable and centre foliations corresponding to $b \in \Lambda$ are represented by Lipschitz maps:

$$
\beta^s_b : Q^s \times M \to Q^s, \quad \beta^u_b : Q^u \times M \to Q^u, \quad \beta_b : M \to Q^u \times Q^s. \tag{26}
$$
Then we define the leaves to be simply the graphs of these Lipschitz maps, embedded in $B \times M$ via (22):

$$W_b^* = \text{Im}(\varphi_b \times \text{Id}) \circ \gamma(\beta_b^*).$$  \hfill (27)

Here and below, * stands for s, u or blank space.

Intuitively, $W_b^s$ denotes a centre-stable leaf, $W_b^u$ denotes a central-unstable leaf, while $W_b$ denotes a central leaf. We will never consider strongly stable or unstable leaves.

We now define certain functional spaces $B^*$ of maps $\beta^*$. These are, by definition, the spaces of Lipschitz maps (26) that satisfy the condition:

$$\max \left\{ \| \beta^* \|_{C^0}, \frac{\text{Lip} \beta^*}{D} \right\} \leq \frac{\delta}{2}. \hfill (28)$$

Here, $D$ is a constant that will be chosen in the proof of lemma 1. The norm on the spaces $B^*$ will always be the $C^0$ norm, and will be denoted by $\| \cdot \|$.

Intuitively speaking, a central-stable, central-unstable or central lamination is a continuous assignment of leaves as $b$ runs over $\Lambda$. Rigorously speaking, a lamination is a continuous map:

$$S^*: \Lambda \to B^*. \hfill (29)$$

The map $S^*$ is completely determined by the continuous collection of maps $\beta^*_b = S^*(b)$, as $b$ ranges over $\Lambda$. Equivalently, $S^*$ is completely determined by the leaves $W_b^*$ of these maps.

The space of continuous sections $S^*$ as above is denoted by $\Gamma^*$. The norm in this space is again the $C^0$ norm:

$$\| S^* \| = \max_{b \in \Lambda} \| S^*(b) \|.$$  

For any $\delta > 0$ small enough, the metric space $\Gamma^*$ with the distance $\rho(S_1^*, S_2^*) = \| S_1^* - S_2^* \|$ is complete. Indeed, if $\beta^*_b \to \beta^*$ and $\text{Lip} \beta^*_b \leq D\delta/2$, then $\text{Lip} \beta^* \leq D\delta/2$.

Now consider a map $G: B \times M \to B \times M$, as in the setup of theorem 1. A central-stable, central-unstable or central lamination is called $G$-invariant if its leaves $W_b^*$ satisfy

$$G(W_b^s) \subset W_b^s, \quad W_b^u \subset G(W_b^u)$$  \hfill (30)

or

$$W_b = G(W_b).$$  \hfill (31)

These conditions can all be written in terms of the maps $\beta^*_b$ defining these leaves, and thus in terms of laminations $S^*$ themselves. This will be done in the beginning of section 3.2.

Our plan for the proof of statement (a) of theorem 1 is the following: we will use the graph transform method described in the following subsection to find $G$-invariant central-stable and central-unstable laminations. Then the central lamination will be given by

$$W_b = W_b^s \cap W_b^u.$$  

Property (31) will follow from (30), so the central lamination will also be invariant under $G$. Once we have the central lamination, we will define

$$Y = \bigsqcup_{b \in \Lambda} W_b.$$  

Sending $W_b$ to $b$ defines the desired projection map $p: Y \to \Lambda$ of (7). Then the $G$-invariance of the central lamination is precisely equivalent to the commutativity of diagram (7). We will follow this plan in the next subsections.
3.2. The graph transform map

Here we will deal with the \( * = s \) case only, since the \( * = u \) case is treated similarly. After that, the central case will be treated as described at the end of the previous subsection. We will introduce first a 'pointwise' graph transform map:

\[
\varphi_b : B^s \to B^s
\]

that acts on single leaves, and then a 'global' graph transform map:

\[
\varphi : \Gamma^s \to \Gamma^s
\]

that acts on entire laminations. In both cases, the geometric idea is the same: start with a map \( \beta^s : Q^s \times M \to Q^u \) as in (26). Take the corresponding leaf \( W^s_b \subset B \times M \), and take its inverse image under \( \varphi \). The claim is that we obtain a different leaf \( W^s_b \subset B \times M \), corresponding to a map \( \beta^s : Q^s \times M \to Q^u \). Then we define the graph transform map as

\[
\varphi_b(\beta^s) = \beta^s.
\]

In other words, the graph transform is implicitly defined by the following relation:

\[
\{ G^{-1}(\varphi_{h(b)}(x_s, \beta^s(x_s, m)), m) \} = \{ (\varphi_b(x_s, \beta^s(x_s, m)), m) \}. \tag{32}
\]

We will prove in the appendix that the above correctly defines \( \beta^s \) (in other words, that the implicit function theorem applies). The above definition also works for families. For a lamination \( S^s \in \Gamma^s \) with leaves that are graphs of \( \beta^s = S^s(h(b)) \), define its graph transform by

\[
\varphi_b(S^s) = (S^s),
\]

where \( S^s(b) = \beta^s \) is defined by relation (32).

Comparing with (30), we see that a lamination \( S^s \) is \( G \)-invariant if and only if it is a fixed point of the graph transform map \( \varphi \). Therefore, to show that there exists a unique \( G \)-invariant central-stable lamination, we will use the fixed point principle: it is enough to show that \( \varphi \) is well defined and contracting.

Lemma 1. For \( \rho \) small enough and any \( F, G \) as in theorem 1, there exists \( \delta = O(\rho) \) so that the graph transform \( \varphi \) maps \( \Gamma^s \) into itself and is contracting with Lipschitz constant \( \mu + O(\delta) \).

In other words, for any \( S^s_0, S^s_1 \in \Gamma^s \) we have

\[
\| \varphi(S^s_0) - \varphi(S^s_1) \| \leq (\mu + O(\delta)) \| S^s_0 - S^s_1 \|.
\]

In the pointwise situation, for any \( b \in \Lambda \), we claim that \( \varphi_b \) maps \( B^s \) into itself. Furthermore, for any \( \beta^s_0, \beta^s_1 \in B^s \), we have

\[
\| \varphi_b(\beta^s_0) - \varphi_b(\beta^s_1) \| \leq (\mu + O(\delta)) \| \beta^s_0 - \beta^s_1 \|.
\]

Corollary 1. For \( \delta = O(\rho) \) small enough, the graph transform map \( \varphi \) has a unique fixed point in \( \Gamma^s \).

Proof. The statements about the global graph transform immediately follow from the corresponding statements in the pointwise case. So we start by proving that \( \varphi_b \) maps \( B^s \) to itself. Take \( b \in \Lambda \), \( \beta^s \in B^s \) and let \( \beta^s = \varphi_b(\beta^s) \). We need to prove that

\[
\| \beta^s \| \leq \frac{\delta}{2}, \tag{35}
\]

\[
\text{Lip} \beta^s \leq \frac{D\delta}{2}. \tag{36}
\]
Recall that \( \gamma_\beta \) is the map of \( Q^s \times M \) onto the graph of \( \beta^s \), see (25). In the appendix we prove that for any \( \beta = \beta^s \) that satisfies (28), there exists a Lipschitz homeomorphism \( G_{\beta,b} : Q^s \times M \to Q^s \times M \), see (63), such that

\[
\overline{\beta}^s = \pi_u \circ G_{\beta,b}^{-1} \circ \gamma(\beta^s) \circ G_{\beta,b},
\]

where the map \( \pi_u : Q^u \cdot M \to Q^u \) is the standard projection and \( G_{\beta,b} = (\psi_{h(b)} \times \text{Id})^{-1} \circ \mathcal{G} \circ (\psi_b \times \text{Id}) \). Note that

\[
||\overline{\beta}^s|| \leq ||\pi_u \circ G_{\beta,b}^{-1} \circ \gamma(\beta^s)||,
\]

because the shift in the argument of the right-hand side does not change the \( C^0 \) norm. Therefore, by (6), we have

\[
||\overline{\beta}|| \leq ||\pi_u \circ \mathcal{F}_b^{-1} \circ \gamma(\beta^s)|| + O(\rho) = ||\pi_u \circ (h^{-1})_b \circ \gamma(\beta^s)|| + O(\rho).
\]

By (24), we can further estimate the above

\[
||\overline{\beta}|| \leq (\mu + O(\delta))||\beta^s'|| + O(\rho).
\]

Since \( \mu < 1 \) and \( ||\beta^s'|| \leq \delta/2 \), for appropriately chosen \( \rho = O(\delta) \), the above can be made \( \leq \delta/2 \). This proves (35). As for (36), note that

\[
\text{Lip } \overline{\beta}^s \leq \text{Lip } (\pi_u \circ G_{\beta,b}^{-1} \circ \gamma(\beta^s)) \cdot \text{Lip } G_{\beta,b}.
\]

We need to show that the right hand side of the above is \( \leq D^2/2 \). It is enough to do this for \( \beta^s \) and \( \overline{\beta}^s \) of class \( C^1 \), since these maps are dense in \( B^s \). In the \( C^1 \) case, we have

\[
d(\pi_u \circ G_{\beta,b}^{-1} \circ \gamma(\beta^s)) = d(\pi_u \circ G_{\beta,b}^{-1}) \circ \gamma(\beta^s) \cdot dy(\beta^s)
\]

\[
\leq \left[d(\pi_u \circ \mathcal{F}_b^{-1}) \circ \gamma(\beta^s) + O(\rho)\right] \cdot \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right] + \left[\begin{array}{c} \frac{\partial \beta^s}{\partial x} \\ \frac{\partial \beta^s}{\partial x} + \mu \cdot \frac{\partial \beta^s}{\partial m} \\ 0 \end{array}\right] + O(\rho) \leq \mu \cdot \text{Lip } \beta^s + O(\delta),
\]

since \( \rho = O(\delta) \). Combining this estimate with proposition 5 of the appendix, we see that

\[
\text{Lip } \overline{\beta}^s \leq (\mu \cdot \text{Lip } \beta^s + O(\delta)) \cdot (L + O(\delta)) \cdot (1 + \text{Lip } \overline{\beta})
\]

Since \( \text{Lip } \beta^s \leq D^2/2 \), the above calculation gives us

\[
\text{Lip } \overline{\beta}^s \leq \mu L \cdot D^2/2 + L \cdot O(\delta) + O(\delta^2).
\]

By assumption (5), we have \( \mu L < 1 \). Therefore, if we pick the constant \( D \) large enough (but still requiring that \( D^2 \ll 1 \)), the right-hand side of the above will be \( \leq D^2/2 \). This proves (36).

Now that we have proved that \( g \) and \( g_b \) are well defined, we move on to the proof of (33) and (34). As we said before, the second inequality implies the first, so we will only prove the second one. As above, write \( \overline{\beta}^s_0 = g_b(\beta^s_0) \) and \( \overline{\beta}^s_1 = g_b(\beta^s_1) \). From (64), we see that

\[
||\overline{\beta}^s_0 - \overline{\beta}^s_1|| \leq T_1 + T_2.
\]
where
\[ T_1 = ||\pi_\alpha \circ G_{\beta, b}^{-1} \circ \gamma(\beta_0^b) \circ G_{\beta_0, b} \circ \pi_\alpha \circ G_{\beta, b}^{-1} \circ \gamma(\beta_1^b) \circ G_{\beta_1, b}||. \]
\[ T_2 = ||\pi_\alpha \circ G_{\beta, b}^{-1} \circ \gamma(\beta_1^b) \circ G_{\beta_0, b} \circ \pi_\alpha \circ G_{\beta, b}^{-1} \circ \gamma(\beta_1^b) \circ G_{\beta_1, b}||. \]
As it will soon be clear, \( T_1 \) is the dominant term:
\[ T_1 \leq \text{Lip}(\pi_\alpha \circ G_{\beta, b}^{-1}) \cdot ||\gamma(\beta_0^b) - \gamma(\beta_1^b)||. \]
The second factor in the right-hand side is \( \leq ||\beta_0^b - \beta_1^b||. \) As for the first factor, we see that
\[ \text{Lip}(\pi_\alpha \circ G_{\beta, b}^{-1}) \leq \text{Lip}(\pi_\alpha \circ \mathcal{F}_{\beta, b}^{-1}) + O(\rho) = \text{Lip}(\pi_\alpha \circ h_{\beta, b}^{-1}) + O(\rho) \leq \mu + O(\rho). \]
Since \( \rho = O(\delta) \), we conclude that
\[ T_1 \leq (\mu + O(\delta)) \cdot ||\beta_0^b - \beta_1^b||. \quad (40) \]
As for \( T_2 \), we see that
\[ T_2 \leq \text{Lip}(\pi_\alpha \circ G_{\beta, b}^{-1} \circ \gamma(\beta_1^b)) \cdot ||G_{\beta_0, b} - G_{\beta_1, b}||. \]
In (38), we saw that
\[ \text{Lip}(\pi_\alpha \circ G_{\beta, b}^{-1} \circ \gamma(\beta_1^b)) \leq O(\delta). \]
In proposition 6 of the appendix, we will prove that
\[ ||G_{\beta_0, b} - G_{\beta_1, b}|| \leq O(1) \cdot ||\beta_0^b - \beta_1^b||. \]
Therefore, we obtain
\[ T_2 \leq O(\delta) \cdot ||\beta_0^b - \beta_1^b||. \]
Together with (40), this implies
\[ ||\beta_0^b - \beta_1^b|| \leq (\mu + O(\delta)) \cdot ||\beta_0^b - \beta_1^b|| + O(\delta) \cdot ||\beta_0^b - \beta_1^b|| \]
\[ \Rightarrow ||\beta_0^b - \beta_1^b|| \leq (\mu + O(\delta)) \cdot ||\beta_0^b - \beta_1^b||. \]
This is precisely the desired inequality (34).

### 3.3. The central lamination

Corollary 1 tells us that there exists a unique \( G \)-invariant central-stable lamination \( S^s \in \Gamma^s \). This can be presented either via the maps \( \beta_0^b \) or via the leaves \( W^s_b \) (as \( b \) ranges over \( \Lambda \)). Similarly, there exists a unique \( G \)-invariant central-unstable lamination \( S^u \in \Gamma^u \). We define the central lamination \( S \) via its leaves \( W_b \), which we define by
\[ W_b = W^s_b \cap W^u_b. \quad (41) \]
This lamination will be \( G \)-invariant, in the sense of (31). We describe \( W_b \) more explicitly. By definition,
\[ W^s_b = \text{Im}(\phi_b \times \text{Id}) \{(x_s, \beta_0^b(x_s, m), m) | x_s \in Q^s, m \in M \}, \]
\[ W^u_b = \text{Im}(\phi_b \times \text{Id}) \{(x_u, \beta_0^u(x_u, m), m) | x_u \in Q^u, m \in M \}, \]
where \( \beta_0^b, \beta_0^u \) have Lipschitz norms at most \( D\delta/2 \ll 1 \). We claim that for any \( m \in M \), the system of equations
\[ x_s = \beta_0^b(x_u, m), \]
\[ x_u = \beta_0^u(x_s, m) \]
(42)
has a unique solution \((x_s, x_u) = \beta_b(m) \in Q^s \times Q^u\). Indeed, for any fixed \(m\) the maps 
\[
\beta^a_b \circ \beta^b_u : Q^a \to Q^a \quad \text{and} \quad \beta^b_u \circ \beta^a_b : Q^u \to Q^u
\]
are Lipshitz with constant \(\leq (L\delta)^2 \ll 1\). Therefore, each of the two maps is contracting and has a unique fixed point: call these \(x_u\) and \(x_s\), respectively. Then the pair \((x_s, x_u)\) is the solution of (42), and the above map \(\tilde{\beta}_b\) is well defined. If we define the map \(\tilde{\beta}_b = \psi_b(\beta_b) : M \to B\), then its graph is precisely \(W_b\):
\[
W_b = \{(\tilde{\beta}_b(m), m) | m \in M\}.
\]
Because it is the intersection of an invariant central-stable lamination with an invariant central-unstable lamination, \(S = (\tilde{\beta}_b) = (W_b)\) is an invariant central lamination.

It is not hard to see from (42) that
\[
\|\beta_b\|_{C^0} \leq \frac{\delta}{2} \quad \text{and} \quad \text{Lip } \beta_b \leq \delta. \tag{43}
\]
Because the chart \(\varphi_b\) is metric preserving at 0 and smooth in the domain \(Q^s \times Q^u\) (which has diameter of order \(\delta\)), we have
\[
d(\tilde{\beta}_b, b)_{C^0} \leq O(\delta) = O(\rho), \quad \text{Lip } \tilde{\beta}_b \leq O(\delta) = O(\rho). \tag{44}
\]

**Proof of statement (a) of theorem 1.** Start from the \(G\)-invariant central lamination \(S\) constructed above, and define \(Y = \bigcup_{b \in \Lambda} W_b\). This union is obviously an invariant set of \(G\), and moreover the following proposition implies that it is actually a disjoint union.

**Proposition 1.** For all \(b \neq b' \in \Lambda\), the corresponding central leaves are disjoint:
\[
W_b \cap W_{b'} = \emptyset.
\]

**Proof.** We assume by contraposition that \(W_b \cap W_{b'} \neq \emptyset\). By the \(G\)-invariance of the lamination,
\[
W_{\beta^k(b)} \cap W_{\beta^k(b')} \neq \emptyset,
\]
for all \(k \in \mathbb{Z}\). Pick a point \((\tilde{b}, m)\) in the above non-empty intersection. Then
\[
\tilde{b} = \tilde{\beta}_{\beta^k(b)}(m) = \tilde{\beta}_{\beta^k(b')}(m). \tag{45}
\]
By (44), the point \(\tilde{\beta}_{\beta^k(b)}(m)\) is at distance at most \(O(\rho)\) from \(h^k(b)\). Similarly, \(\tilde{\beta}_{\beta^k(b')}(m)\) is at distance at most \(O(\rho)\) from \(h^k(b')\). This implies that \(h^k(b)\) and \(h^k(b')\) are at most \(2 \cdot O(\rho)\) apart, for all \(k \in \mathbb{Z}\). This is obviously impossible for \(\rho\) small enough, because for such \(\rho\), the quantity \(O(\rho)\) is smaller than the expansivity constant of \(h\). □

Therefore, the map \(p : Y \to \Lambda\) that sends \(W_b\) to \(b\) is well defined. Moreover, the \(G\)-invariance of the lamination \(S = (W_b)\) is precisely equivalent to the commutativity of the diagram (7). The continuity of \(p\) follows from the continuity of our laminations, and this also implies that the map \(H\) of (8) is continuous.

Note that the map \(H\) is bijective, with inverse given by \(H^{-1}(b, m) = (\tilde{\beta}_b(m), m)\). The map \(H^{-1}\) is clearly continuous in \(m\), and continuity in \(b\) follows from the H"older continuity statement (10), which will be proved in the next subsection. Therefore \(H\) is a homeomorphism, thus concluding the proof of statement (a). □
4. H"older continuity of the central lamination

This section is concerned with the proof of statement (b) of theorem 1. By definition, we have \( p^{-1}(b) = W_b = \text{Graph}(\bar{\beta}_b) \), where \( \bar{\beta}_b \) satisfies relations (44). This is precisely the requirement (9). In this section we will prove the rest of statement (b), which refers to H"older continuity.

First, for any \( b \in \Lambda \), we will define its local central-stable and central-unstable manifolds as

\[
V^s_b = \{ b' \in B | d(h^{n_s}(b'), h^{s}(b)) \leq \delta, \forall n \geq 0 \},
\]

\[
V^u_b = \{ b' \in B | d(h^{n_u}(b'), h^{-s}(b)) \leq \delta, \forall n \geq 0 \}.
\]

**Proposition 2.** Let \( h, \Lambda \) and \( d \) be the same as at the beginning of section 2.1. Then the following statements hold for all \( b, b' \in \Lambda \):

1. if \( b' \in V^u_b \) and \( d(h^{-1}(b), h^{-1}(b')) \leq \delta \Rightarrow h^{-1}(b') \in V^u_{h(b)} \)
2. if \( b' \in V^u_b \) and \( d(h(b), h(b')) \leq \delta \Rightarrow h(b') \in V^u_{h(b)} \)
3. if \( b' \in V^u_b \Rightarrow \lambda - O(\delta) \leq \frac{d(h(b), h(b'))}{d(b, b')} \leq \lambda + O(\delta) \)
4. if \( b' \in V^u_b \Rightarrow \mu - O(\delta) \leq \frac{d(h^{-1}(b), h^{-1}(b'))}{d(b, b')} \leq \mu + O(\delta) \)

These properties are elementary and well known, and we omit the proofs.

We further ask that \( h \) has the following local product structure: for all \( b, b' \in \Lambda \) such that \( d(b, b') \leq \delta \), there exists a unique \( b^* \in B \) such that

\[
V^u_b \cap V^s_{b'} = \{ b^* \},
\]

and moreover:

\[
d(b, b^*) + d(b', b^*) \leq O(d(b, b')).
\]

This property is easily seen to hold for linear Anosov diffeomorphisms of the torus, because in this case \( V^u_b \) and \( V^s_{b'} \) are just straight lines that meet transversely under a fixed angle independent of \( b, b' \). It also holds for the Smale–Williams solenoid, because then \( V^u_b = \{ y(b) \} \times D \) and \( V^s_{b'} \) is a curve that intersects \( V^u_b \) transversely (such that the angle between \( V^u_b \) and \( V^s_{b'} \) is separated from zero by a constant).

**Proof of statement (b) of theorem 1.** We have already proved the closeness property (9) in relation (44). As for the H"older property (10), it is enough to prove it for \( b, b' \) which are at most \( \delta \) apart. Indeed, for any \( \alpha > 0 \) and any \( b, b' \) with \( d(b, b') > \delta \), we have by default:

\[
d(h(b), h(b')) \leq C d^\alpha(b, b'),
\]

where \( C = \frac{\text{diam} B}{\delta} \). Therefore, we can restrict attention to \( b, b' \) that are such that the unique point \( b^* \) of (46) satisfies:

\[
d(b, b^*) \leq \delta, d(b', b^*) \leq \delta, d(b, b') \leq \delta.
\]

For such nearby \( b, b' \), we essentially need to estimate the distance between the maps \( \bar{\beta}_b, \bar{\beta}_{b'} : M \to B \). These maps were defined by the condition that their graphs coincide with \( W^s_b \cap W^u_{b'} \) and \( W^u_b \cap W^s_{b'} \), respectively.

However, this is a bit of an issue: **different** leaves \( W^s_b \) and \( W^u_{b'} \) (and also their \( u \) counterparts) are defined using **different** coordinate charts \( \varphi_b \) and \( \varphi_{b'} \). To resolve this problem in the \( s \)-case (the \( u \)-case is treated in the same way), we write \( W^s_b \) as the graph of a function \( \bar{\beta}_b : Q^s \times M \to Q^u \) in the coordinate chart \( \varphi_{b'} \):

\[
[(\varphi_b \times \text{Id}_M)(x, \beta_b(x, m), m)] \equiv W^s_b = \{ (\varphi_{b'}, \text{Id}_M)(x, \beta_b(x, m), m) \}.
\]
The function $\overline{\beta}_b$ is defined uniquely and implicitly by the above relation, but we must require the inclusion $\text{Im}(\varphi_b) \subset \text{Im}(\varphi_{b^*})$. We certainly cannot ensure this if we define the charts $\varphi_b$ and $\varphi_{b^*}$ with respect to the same $\delta$ in (22). But if we define $\varphi_{b^*}$ with respect to $3\delta$ instead of $\delta$ (i.e. define the chart on a neighbourhood 3 times bigger), then the desired inclusion becomes a consequence of (49).

**Definition 4.** Under the assumption (49), we define the distance between the leaves corresponding to $b$ and $b^*$, to be

$$d(W_b^s, W_{b^*}^u) := ||\beta_{b^*}^s - \overline{\beta}_b^s||.$$  

Implicit in the definition is the fact that the right-hand side only makes sense on the domain of $\overline{\beta}_b^s$, which as was said before, is strictly contained in the domain of $\beta_{b^*}^s$. Note that the above definition is not symmetric in $b$ and $b^*$.

Now we must look at what happens with these leaves under the graph transform. Take two leaves $W_{h(b)}^s$ and $W_{h(b^*)}^s$, given in the coordinate chart $\varphi_{h(b^*)}$ by maps $\overline{\beta}_{h(b)}^s$ and $\beta_{h(b^*)}^s$, respectively. Then take their images under the graph transform $W_b^s$ and $W_{b^*}^s$, given in the coordinate chart $\varphi_{b^*}$ by maps $\overline{\beta}_b^s$ and $\beta_{b^*}^s$. The inequality (34) of lemma 1 precisely says that

$$d(W_b^s, W_{b^*}^u) \leq (\mu + O(\delta)) \cdot d(W_{h(b)}^s, W_{h(b^*)}^s).$$  

(50)

Doing the analogous computations for central-unstable foliations, we see that

$$d(W_b^u, W_{b^*}^u) \leq (\lambda + O(\delta)) \cdot d(W_{h^{-1}(b)}^u, W_{h^{-1}(b^*)}^u).$$  

(51)

Now recall that we fixed points $b$, $b^*$ satisfying relation (49). We consider the positive integers:

$$k = \left\lceil \log_{\lambda - O(\rho)} \frac{d(b, b^*)}{\delta} \right\rceil, \quad l = \left\lceil \log_{\lambda - O(\rho)} \frac{d(b^*, b^*)}{\delta} \right\rceil.$$

Iterating relation (50) $k$ times gives us

$$d(W_b^s, W_{b^*}^s) \leq (\mu + O(\delta))^k \cdot d(W_{h(b)}^s, W_{h(b^*)}^s).$$

By the definition of $k$ (and property 4 of proposition 2), $k$ is the biggest positive integer which would ensure that the points $h^k(b)$ and $h^k(b^*)$ remain at most distance $\delta$ apart. But since the distance between $h^k(b)$ and $h^k(b^*)$ is at most $\delta$, we infer that the distance between the corresponding leaves is also at most $O(\delta)$. Therefore, the above inequality implies

$$d(W_b^s, W_{b^*}^s) \leq (\mu + O(\delta))^k \cdot O(\delta) \leq d(b, b^*) \frac{\ln \delta}{\delta} \cdot O(1).$$

The analogous discussion with $l$, $\lambda$, $b^*$, $u$ instead of $k$, $\mu$, $b$, $s$ gives us

$$d(W_b^u, W_{b^*}^u) \leq (\lambda + O(\delta))^l \cdot O(\delta) \leq d(b^*, b^*) \frac{\ln \delta}{\delta} \cdot O(1).$$

Letting $\alpha$ be defined as in (11), the above relations give us

$$d(W_b^s, W_{b^*}^u) \leq d(b, b^*)^{\alpha - O(\delta)} \cdot O(1), \quad d(W_b^u, W_{b^*}^s) \leq d(b^*, b^*)^{\alpha - O(\delta)} \cdot O(1).$$  

(52)

We now prove that

$$W_b^s \subset W_{b^*}^u, \quad \text{and analogously} \quad W_b^u \subset W_{b^*}^s.$$  

(53)

Relation (50) for $b$ replaced with $b^*$ becomes

$$d(W_b^s, W_{b^*}^s) \leq (\mu + O(\delta)) \cdot d(W_{h(b)}^s, W_{h(b^*)}^s).$$

However, since $b^* \in V_{b^*}^u$, the map $h$ actually brings the points $b^*$ and $b^*$ closer together (by property 3 of proposition 2). So we can iterate the above inequalities as many times as we want. We see that

$$d(W_b^s, W_{b^*}^s) \leq (\mu + O(\delta))^i \cdot d(W_{h(i(b))}^s, W_{h(i(b^*))}^s),$$

(54)

(55)
for any $i > 0$. As $i \to \infty$, this implies $d(W_b, W_{b'}) = 0$. This proves (53) in the $s$-case. The proof in the $u$-case is similar.

We can now turn to the proof of (10), thus completing the proof of theorem 1. Recall that for any $b \in B$, $W_b = W_b^s \cap W_b^u$. We first prove that

$$d(W_b, W_{b'}) \leq d(b, b_s)^{\alpha-O(\delta)} \cdot O(1), \quad d(W_b, W_{b'}) \leq d(b', b_u)^{\alpha-O(\delta)} \cdot O(1).$$

By (53) we have

$$W_b = W_b^s \cap W_b^u, \quad W_{b'} = W_{b'}^s \cap W_{b'}^u.$$  \hspace{1cm} (55)

In the chart $\varphi_{b'} \times \text{Id}$, the leaves $W_{b'}^s, W_{b'}^u, W_b^s, W_b^u$ are given by maps $\beta_b^s, \beta_b^u, \beta_{b'}^s, \beta_{b'}^u$. Then (52) gives us

$$||\beta_{b'}^s - \tilde{\beta}_{b'}^s||, \quad ||\beta_{b'}^u - \tilde{\beta}_{b'}^u|| \leq d(b, b')^{\alpha-O(\delta)} \cdot O(1),$$

while (53) gives us

$$\beta_{b'}^s = \tilde{\beta}_{b'}^s, \quad \beta_{b'}^u = \tilde{\beta}_{b'}^u.$$  \hspace{1cm} (56)

Of course, when one reads the above inequalities, one should keep in mind that the maps $\beta_{b'}^s, \beta_{b'}^u$ are defined on a neighbourhood 3 times bigger than the maps $\tilde{\beta}_{b'}^s, \tilde{\beta}_{b'}^u$. Actually, the domain of the maps $\beta_{b'}^s, \beta_{b'}^u$ strictly contains the domain of the maps $\tilde{\beta}_{b'}^s, \tilde{\beta}_{b'}^u$. The above relations should therefore be understood on the smaller domain, on which the maps $\tilde{\beta}_{b'}^s, \tilde{\beta}_{b'}^u$ are actually defined.

Relation (55) is equivalent to $\tilde{\beta}_b(m) = (x_s, x_u)$ and $\beta_{b'}(m) = (x_s^*, x_u^*)$, where

$$\begin{align*}
x_s &= \tilde{\beta}_b(x_a, m) \quad \quad \quad x_s^* = \beta_{b'}^s(x_s^*, m) \\
x_u &= \tilde{\beta}_b(x_a, m) \quad \quad \quad x_u^* = \beta_{b'}^u(x_u^*, m).
\end{align*}$$  \hspace{1cm} (56)

For fixed $m$, the solutions $(x_s, x_u)$ and $(x_s^*, x_u^*)$ are fixed points of the contracting maps $\tilde{\beta}_b \circ \tilde{\beta}_b \times \tilde{\beta}_b \circ \tilde{\beta}_b : Q^s \times Q^u \to Q^s \times Q^u$ and $\beta_{b'} \circ \beta_{b'}^u \times \beta_{b'}^u \circ \beta_{b'}^u : Q^s \times Q^u \to Q^s \times Q^u$, respectively. The contraction coefficient is $< 1$, uniformly in $m$ and $b$. Therefore, the systems (56) have a unique solution for each $m$.

As was shown in (52) and (53), the maps $\beta_{b'}^s, \beta_{b'}^u$ of (56) are Hölder continuous in $b$. Therefore, the unique solutions of systems (56) are also Hölder continuous in $b$, and thus so are the maps $\beta_{b'}$ and $\tilde{\beta}_{b'}$. Therefore, we have the following analogues of (52):

$$||\tilde{\beta}_b - \beta_{b'}|| \leq d(b, b')^{\alpha-O(\delta)} \cdot O(1), \quad ||\tilde{\beta}_b - \beta_{b'}|| \leq d(b', b_u')^{\alpha-O(\delta)} \cdot O(1).$$

By the triangle inequality, this implies

$$||\tilde{\beta}_b - \beta_{b'}|| \leq 2d(b, b')^\alpha + d(b', b_u')^{\alpha-O(\delta)} \cdot O(1) \leq d(b, b')^{\alpha-O(\delta)} \cdot O(1),$$

where the last inequality follows from (47). This proves the desired inequality in the chart $\varphi_{b'} \times \text{Id}$ (that is, for the maps $\tilde{\beta}_b, \tilde{\beta}_{b'} : M \to Q^s \times Q^u$). On the manifold (that is, for the maps $\beta_b, \beta_{b'} : M \to B$), the analogous relation follows from the fact that the derivative of $\varphi_{b'}$ at 0 is the identity.

Therefore, relation (10) is proved. Note that $O(1)$ is a constant that does not depend on $b$ and $b'$, but it does depend on $\delta$, as in (48). We were able to put $\rho$ in the denominator of (53) instead of $\delta$, because $\rho = O(\delta)$. Finally, the inverse map $H^{-1}$ of (8) is explicitly given as

$$H^{-1}(b, m) = (\tilde{\beta}_b(m), m).$$

This map is Lipschitz in the variable $m$, and Hölder continuous in the variable $b$ by (10). Therefore, $H^{-1}$ is Hölder continuous. This concludes the proof of theorem 1.
5. Hölder continuity of the centre-stable foliation

In this section we complete the proof of theorem 2. Recall the setup: the map \( h \) is a skew product itself (see (15)) whose fibres are globally defined stable manifolds for \( h \). In this case, we will see that the central-stable leaves of \( \mathcal{G} \) can also be globally defined.

By analogy with section 3.1, a global central-stable leaf for \( z \in \mathcal{Z} \) is defined as a Lipschitz function
\[
\beta^s_z : F \times M \to \mathcal{Z}, \tag{57}
\]
and its graph is defined as
\[
W^s_z = \gamma(\beta^s_z) = \{ (\beta^s_z(f, m), f, m) \mid (f, m) \in F \times M \}.
\]
We ask that our leaves be Lipschitz close to the constant function \( z \), in the sense that
\[
\max \left\{ \frac{d(\beta^s_z, z)}{C_0}, \frac{\operatorname{Lip} \beta^s_z}{D} \right\} \leq \frac{\delta}{2}, \tag{58}
\]
Finally, a global central-stable lamination is defined as a continuous assignment
\[
\mathcal{G}(W^s_z) = W^s_{\mathcal{Z}(z)}, \quad \forall z \in \mathcal{Z}, \tag{59}
\]
where \( D \) is so chosen that the estimates in the (sketch of the) proof below work out. All these constructions are analogous to the ones in section 3.1. Moreover, the entire machinery of lemma 1 applies to our situation and produces a unique \( \mathcal{G} \)-invariant lamination \( \mathcal{S}^s = (\beta^s_z) \) satisfying (58) for \( D \) properly chosen. We will henceforth focus solely on this lamination. In particular, since \( \xi \) is expanding we obtain
\[
d(\beta^s_z, \beta^s_{\hat{z}}) \leq d(z, \hat{z})^{\alpha - O(\rho)} \frac{O(\rho)^\mu}{\ln \mu}, \tag{60}
\]
This is proven in analogous fashion to statement (b) of theorem 1, which was proved in the previous section.

**Proof of theorem 2.**

**Proposition 3.** The leaves \( W^s_z = \operatorname{Graph}(\beta^s_z) \) are disjoint and they cover the whole of \( X \):
\[
X = \bigsqcup_{z \in \mathcal{Z}} W^s_z.
\]

**Proof.** The fact that the leaves are disjoint is proven by analogy with proposition 1. The fact that their union is the whole of \( X \) is equivalent to the following claim: for any \( z \in \mathcal{Z} \) and \( y \in F \times M \), there exists \( \tilde{z} \in \mathcal{Z} \) such that \( \beta^s_{\tilde{z}}(y) = z \). We fix \( y \) and \( z \), and prove this claim.

Fix a coordinate neighbourhood of radius \( 2\delta \) of \( z \) inside \( \mathcal{Z} \). Let \( D(z, \delta), D(z, 2\delta), S(z, \delta), S(z, 2\delta) \) be the balls/spheres centered at \( z \) of radii \( \delta \) and \( 2\delta \) in \( \mathcal{Z} \), respectively. The map \( f : D(z, \delta) \to D(z, 2\delta) \) given by \( f(\tilde{z}) = \beta^s_{\tilde{z}}(y) \) is well defined, because (58) implies that \( d(f(\tilde{z}), \tilde{z}) \leq \delta/2 \). Moreover, (60) implies that the map \( f \) is continuous. Therefore, sliding points along a straight line segment gives us a homotopy between the identity map of \( D(z, \delta) \) and \( f \):
\[
h_t(\tilde{z}, y) = (\tilde{z}, y) + t(\beta^s_{\tilde{z}}(y) - \tilde{z}), 0).
\]
If the centre \( z \) did not lie in the image of \( f \), this would imply the absurd claim that the sphere \( S(z, \delta) \) is contractible inside \( D(z, 2\delta) \setminus \{z\} \). Indeed, one could simply homotope \( S(z, \delta) \) under \( f \) (because of the inequality \( d(f(\tilde{z}), \tilde{z}) \leq \delta/2 \), the image of the sphere never touches \( z \) during
the homotopy), and then contract it through the ball Im \( f \) which does not contain \( z \) anymore. Therefore, we conclude that 
\[ z \in \text{Im}(f) \Rightarrow \exists \tilde{z} \text{ such that } \beta^*_s(y) = z. \]

By proposition 3, the map \( q : X \to Z \) given by sending \( W^s_z \) to \( z \) is well-defined. Moreover, the \( G \)-invariance condition (59) implies that \( q \) makes the diagram (17) commute. Part (b) of theorem 2 follows immediately from (58) and (60).

Finally, we prove the relation \( q|_Y = \pi \circ p \). Take any point \( b = (f, z) \in F \times Z \), and recall that we denote \( z = \pi(b) \). If we take the map \( \beta^*_s \pi(b) \) that defines the global lamination (see (57)), and restrict it to the \( \delta \) neighbourhood of \( f \in F \), we obtain a map \( \tilde{\beta}^*_b \) as in (26). In other words, restricting the leaves of the global lamination \( W^s_{\pi(b)} \) is \( G \)-invariant, it is easily seen that the local lamination \( W^s_b \) will also be \( G \)-invariant.

But local laminations are unique, as proved in corollary 1. Therefore, the local leaves \( W^s_b \) coincide with the central-stable leaves \( W^s_\pi(b) \) of section 3. By the very definition of \( W^s_b \), this implies that \( W^s_b \subset W^s_{\pi(b)} \). Since \( W^s_b \subset W^s_\pi(b) \) by construction, we conclude that 
\[ W^s_b \subset W^s_{\pi(b)}, \quad \forall b. \]

Now take any point \( x \in Y = \bigsqcup_{b \in A} W^s_b \) and assume \( x \in W^s_b \). By the very definition of \( p \), we have \( p(x) = b \). But the above inclusion implies that \( x \in W^s_{\pi(b)} \), and then the definition of \( q \) implies that \( q(x) = \pi(b) \). This precisely amounts to saying that \( q|_Y = \pi \circ p \). \( \square \)

6. The simplest special ergodic theorem and Fubini revisited

In this section we will prove theorems 3 and 4.

6.1. Special ergodic theorem

We begin by recalling the definition of Hausdorff dimension, denoted by \( \dim_H \).

**Definition 5.** Let \( A \) be a subset of Euclidean space. A cover \( U \) of \( A \) is a finite or countable collection of balls \( Q_j \) of radii \( r_j \) whose union contains \( A \). The \( d \)-dimensional volume of \( U \), denoted by \( V_d(U) \), is defined as 
\[ V_d(U) = \sum_j r_j^d. \]

The Hausdorff dimension of \( A \) is defined as the infimum of those \( d \) for which there exists a cover of \( A \) with arbitrarily small \( d \)-dimensional volume:
\[ \dim_H A = \inf \{ d \mid \forall \varepsilon > 0 \exists \text{ a cover } U \text{ of } A \text{ such that } V_d(U) < \varepsilon \}. \]

Note that a compact manifold of dimension \( d \) also has Hausdorff dimension \( d \). The same holds for a set of a positive Lebesgue measure on the Riemannian manifold of dimension \( d \).

**Proof of theorem 3.** Consider first the special case when \( \gamma \) is an arc of the form:
\[ \gamma = [0.w, 0.w + 2^{-n}], \quad (61) \]
where \( w \) is a binary word of length \( n \). We will say that a finite word \( W \) is \((\kappa, w)\)-atypical if the frequency of appearances of the subword \( w \) in the word \( W \) lies outside the segment

\[ \text{The frequency is simply the number of occurrences divided by the length of } W. \]
The point \( y = 0.\omega^+ \), where \( \omega^+ \) is a one-sided binary sequence, belongs to \( K_{\kappa, w} \) if and only if infinitely many of initial subwords of the sequence \( \omega^+ \) are \((\kappa, w)\)-atypical; the sequence itself will be called \((\kappa, w)\)-atypical. Thus for any \( N_0 \), we have the following inclusion:

\[
\{ (\kappa, w) \text{-atypical sequences} \} \subset \bigcup_{N \geq N_0} \bigcup_{\text{length } v = N} \{ \text{ball of radius } 2^{-N} \text{ around } 0.v \}.
\]

This produces a covering \( U \) of the set \( K_{\kappa, w} \), as in definition 5. We compute the \( 1 - \varepsilon \) dimensional volume of this covering:

\[
V_{1-\varepsilon}(U) \leq \sum_{N \geq N_0} 2^{-N(1-\varepsilon)} \cdot \# \{ (\kappa, w) \text{-atypical words of length } N \}.
\]

The last factor is estimated in the following theorem.

**Theorem 6 (Large deviation theorem [33])**. There exists \( \nu = \nu(\kappa,w) \) such that for any \( N \) greater than some \( N_0 \), the number of \((\kappa, w)\)-atypical words of length \( N \) is at most \( 2^{N(1-\nu)} \).

If we take \( \varepsilon < \nu \), the above theorem implies

\[
V_{1-\varepsilon}(U) \leq \sum_{N \geq N_0} 2^{-N(1-\varepsilon)} = \frac{2^{-N_0(1-\varepsilon)}}{1 - 2^{1-\nu}}.
\]

Therefore, the Hausdorff dimension of the set \( K_{\kappa, w} \) is at most \( 1 - \varepsilon \). This completes the proof of theorem 3 for the arc \( \gamma \) of (61). From here, the same statement can be generalized to any arc with binary rational endpoints, and then finally to any arc.

**6.2. Measure zero and incomplete Hausdorff dimension**

**Proof of theorem 4**. The theorem is an immediate consequence of the following proposition:

**Proposition 4.** Recall the general setup of theorem 2. If \( A \subset Z = S^1 \) satisfies

\[
dim_H A < \frac{\ln \mu}{\ln \mu_-},
\]

then for \( \rho \) small enough, the set \( q^{-1}(A) \) has Lebesgue measure 0 in \( X \).

Since \( X = Z \times F \times M \), the classical Fubini theorem states that

\[
\text{mes}(q^{-1}(A) \cap Z \times \{x\}) = 0, \quad \forall x \in F \times M \Rightarrow \text{mes}(q^{-1}(A)) = 0.
\]

So all we need to do is to show that for any fixed \( x \in F \times M \), the intersection \( q^{-1}(A) \cap Z \times \{x\} \) has measure 0 in \( Z \). By the very definition of the map \( q \) of (17), this intersection is nothing but the set \( \{ \beta^z(x) | z \in A \} \subset Z \). Moreover, by statement (b) of theorem 2 the map

\[
\varphi : Z \to Z, \quad \varphi(z) = \beta^z(x)
\]

is Hölder continuous with exponent \( \alpha = \frac{\ln \mu}{\ln \mu_-} - O(\rho) \). All that we need to prove is that the set \( \varphi(A) \) has measure 0 in \( Z \). The following general lemma will do the trick:
Lemma 2 (Falconer). Let $Z$ be any Riemannian manifold, and $A \subset Z$ a subset. If $\varphi : Z \to Z$ is a Hölder map with exponent $\alpha$, then
\[
\dim_H \varphi(A) \leq \frac{\dim_H A}{\alpha}.
\]

The proof of this lemma can be found in [4]; the proof is straightforward. The above lemma, together with theorem 2 and the assumptions of proposition 4, imply that for small enough $\rho$, we will have $\dim_H \varphi(A) < \dim Z$. Therefore, $\varphi(A)$ has Lebesgue measure 0 in $Z$, and as we have seen above this implies that $q^{-1}(A)$ has Lebesgue measure 0 in $X$. This concludes the proof of proposition 4, and hence also that of theorem 4. \qed

Appendix

A.1. The graph transform map made explicit

Recall that the graph transform map $g_b : B^s \to B^s$ was defined by
\[
\beta \mapsto (\varphi_{h(b)}(x_s, \beta((x_s, m))), m).
\]

We now want to turn this implicit definition into an explicit formula. Recall our notation $\gamma(\beta)$, under which the above becomes:
\[
\operatorname{Im} G^{-1} \circ (\varphi_{h(b)} \times \text{Id}) \circ \gamma(\beta) \supset \{ (\varphi_{b}(x_s, \beta((x_s, m))), m) \}.
\]

If we write $G_b = (\varphi_{h(b)}(C\delta) \times \text{Id})^{-1} \circ G \circ (\varphi_{b}(\delta) \times \text{Id})$ as in (23), then our relation takes the form:
\[
\operatorname{Im} G^{-1} \circ \gamma(\beta) \supset \operatorname{Im} \gamma(\beta).
\]

Write $\pi_u : Q^s \times Q^u \times M \to Q^u$ and $\pi_{sc} : Q^s \times Q^u \times M \to Q^s \times M$ for the standard projections, and define
\[
G_{\beta,b} = \pi_{sc} \circ G_b \circ \gamma(\beta) : Q^s \times M \to Q^s \times M.
\]

Then (62) is equivalent to
\[
\pi_u \circ G_{\beta,b}^{-1} \circ \gamma(\beta) \circ G_{\beta,b} = \beta_s.
\]

Proposition 5. The composition (63) is well defined and
\[
\text{Lip } G_{\beta,b} \leq (L + O(\delta)) \cdot (1 + \text{Lip } \beta_s),
\]
where $L$ is the constant from definition 2. A similar estimate holds in the central-unstable case.

Proof. Define the composition
\[
F_{0,b} = \pi_{sc} \circ F_b \circ \gamma(0),
\]
in analogy with (63), with $G$ replaced by $F$ and $\beta_s$ replaced by the zero map $0 : Q^s \times M \to Q^u$. Since $d(G, F)_{C^1} \leq \rho$, we see that
\[
\text{Lip } G_{\beta,b} \leq (\text{Lip } F_{0,b} + O(\rho)) \cdot (1 + \text{Lip } \beta_s).
\]

But one can simply unravel the definition of $F_{0,b}$ when $F$ is a skew product, and obtain
\[
F_{0,b}(x_s, m) = (\pi_s \circ h_b(x_s, 0), f(x_s, 0)(m)).
\]

From this it is clear that
\[
\text{Lip } F_{0,b} \leq L + O(\delta).
\]

Recalling that we always choose $\delta = O(\rho)$, (65) implies
\[
\text{Lip } G_{\beta,b} \leq (L + O(\delta)) \cdot (1 + \text{Lip } \beta_s).
\]

\qed
Proposition 6. For any two central-stable leaves $\beta_0, \beta_1 \in B$, we have
$$||G_{\beta_0, b} - G_{\beta_1, b}|| \leq O(1) \cdot ||\beta_0 - \beta_1||.$$ A similar result holds in the central-unstable case.

Proof. We have
$$||G_{\beta_0, b} - G_{\beta_1, b}|| = ||\pi_{sc} \circ G_b \circ \gamma(\beta_0) - \pi_{sc} \circ G_b \circ \gamma(\beta_1)||$$
$$\leq \text{Lip}(\pi_{sc} \circ G_b) \cdot ||\gamma(\beta_0) - \gamma(\beta_1)|| \leq O(1) \cdot ||\beta_0 - \beta_1||.$$

A.2. Persistence of Hölder skew products

The second, independent technical result that we will prove concerns the setup of theorem 1: we have a small $\rho$-perturbation $G$ of the skew product $F$ from theorem 1. This theorem tells us that $G$ is conjugated to a skew product $G$:
$$G(b, m) = (h(b), g_b(m)).$$

In this subsection, we will prove formulae (12) and (13). To this end, from the very definition of $G$ we have the following explicit formula for the fibre maps $g_b$:
$$g_b(m) = \pi_m(G(\tilde{\beta}_b(m), m)), g_b^{-1}(m) = \pi_m(G^{-1}(\tilde{\beta}_b(h(b)(m), m)).$$

Since $d(G^{\pm 1}, F^{\pm 1})_{C^1} < \rho$, it follows from the above formulae that
$$d(g_b, f_b)_{C^1} \leq d(G(\tilde{\beta}_b(m), m), F(b, m))_{C^1} + \rho \leq ||G||_{C^1} \cdot d(\tilde{\beta}_b, b)_{C^1} + \rho = O(\rho),$$
and similarly for $d(g_b^{-1}, f_b^{-1})_{C^1}$. This proves (12). As for the Hölder property, we have that
$$d(g_b, g_{b'})_{C^\alpha} \leq ||G||_{C^1} \cdot d(\tilde{\beta}_b, \tilde{\beta}_{b'})_{C^\alpha} \leq O(d(b, b')^\alpha),$$
by (10). The statement concerning $d(g_b^{-1}, g_{b'}^{-1})_{C^\alpha}$ is proved analogously, thus concluding the proof of (13).

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