ALMOST POLYNOMIAL-LIKE MAPS IN POLYNOMIAL DYNAMICS

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Abstract. Holomorphic renormalization plays an important role in complex polynomial dynamics. We consider a situation, when a polynomial is not immediately renormalizable but admits an invariant continuum on which it is topologically conjugate to a lower degree polynomial. This invariant continuum may contain extra critical or parabolic points of the original polynomial that are not visible in the dynamical plane of the conjugate polynomial. Thus, we extend the notions of holomorphic renormalization and polynomial-like maps and describe a setup where these generalized notions are applicable and yield useful topological conjugacies.

1. Introduction

Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d > 1$ with connected filled Julia set $K_P$. Clearly, $P$ acts on the Riemann sphere $\tilde{\mathbb{C}} = \mathbb{C}P^1$ so that $P(\infty) = \infty$. In contrast to rational dynamics, the point at infinity plays a special role in the dynamics of $P$. A classical theorem of Böttcher states that $P$ is conjugate to $z \mapsto z^d$ near infinity. Since $K_P$ is connected, the conjugacy can be defined on $\tilde{\mathbb{C}} \setminus K_P$ as follows. We will write $D = \{z \in \mathbb{C} ||z| < 1\}$ for the open unit disk in $\mathbb{C}$ and $\overline{D}$ for its closure. Without loss of generality we may assume that $P$ is monic, i.e., the highest term of $P$ is $z^d$. Let $\psi_P : \mathbb{D} \to \mathbb{C} \setminus K_P$ be a conformal isomorphism normalized so that $\psi_P(0) = \infty$ and $\psi'_P(0) > 0$. Then $\psi_P^{-1} \circ P \circ \psi_P$ is a degree $d$ holomorphic self-covering of $\mathbb{D}$. The only option for such a holomorphic self-covering is $z \mapsto \lambda z^d$ with $|\lambda| = 1$. By the chosen normalization of $P$ and $\psi_P$, the coefficient $\lambda$ must be equal to $1$. Thus, $P(\psi_P(z)) = \psi_P(z^d)$ for any $z \in \mathbb{C}$.

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In other words, if we use the polar coordinates \((\theta, \rho)\) on \(D\) and identify \(D\) with \(\mathbb{C} \setminus K_P\) by \(P\), then the action of \(P\) will look like \(\theta, \rho \mapsto (d\theta, \rho^d)\). Here \(\theta\) is the angular coordinate; it takes values in \(\mathbb{R}/\mathbb{Z}\) (elements of \(\mathbb{R}/\mathbb{Z}\) are called *angles*). The coordinate \(\rho\), the radial coordinate, is the distance to the origin. On \(D\) (hence, after the transfer, on \(\mathbb{C} \setminus K_P\)) it takes values in \((0, 1)\). External rays of \(P\) are defined as the \(\psi_P\)-images of radial straight intervals in \(D\). More details are given in Section 2.1; the external ray of \(P\) with argument \(\theta \in \mathbb{R}/\mathbb{Z}\) is denoted by \(R_P(\theta)\).

1.1. **Generalized renormalization.** Suppose that two different rays \(R_P(\theta^R)\) and \(R_P(\theta^L)\) land at the same point \(a\). The superscripts “R” and “L” stand for Right and Left. Then the union \(\Gamma_P(\theta^R, \theta^L) = R_P(\theta^R) \cup R_P(\theta^L) \cup \{a\}\) is called a *cut*. The corresponding *wedge* \(W_P(\theta^R, \theta^L)\) is by definition the component of \(\mathbb{C} \setminus \Gamma_P(\theta^R, \theta^L)\) that includes the external rays with arguments \(\theta\) such that \(\theta^R < \theta < \theta^L\) in the counterclockwise order. The point \(a\) is called the *root point* of the wedge \(W_P(\theta^R, \theta^L)\).

**Definition 1.1 (Renormalizable collection of wedges).** Consider finitely many disjoint wedges \(W_1, \ldots, W_n\) in the dynamical plane of \(P\). Let us write \(W\) for this collection of wedges and \(\Gamma_1, \ldots, \Gamma_n\) for the corresponding cuts. The *avoiding set* \(A_P(W)\) of \(W\) is defined as the set of all \(z \in K_P\) such that \(P^i(z) \notin W_j\) for all \(i \geq 0\) and \(j \in \mathbb{1}, \ldots, \mathbb{n}\). Here \(\mathbb{1}, \ldots, \mathbb{n}\) denotes the set \(\{1, \ldots, n\}\). In other words, \(A_P(W)\) consists of points whose forward \(P\)-orbits stay bounded and avoid the specified wedges. We say that the collection \(W\) is

- **non-disconnecting:** if \(A_P(W)\) is connected;
- **non-separating:** if \(A_P(W) \setminus P(\Gamma_j)\) is connected for all \(j \in \mathbb{1}, \ldots, \mathbb{n}\);
- **essential:** if the root point of \(W_j\) is in \(A_P(W)\) for \(j \in \mathbb{1}, \ldots, \mathbb{n}\).

We say that \(W\) is a *renormalizable* collection of wedges of \(P\) if it satisfies these three properties. That is, if \(W\) is non-disconnecting, non-separating, and essential.

The requirement that the wedges be disjoint is included in the definition of a renormalizable collection. Formally, the definition of \(A_P(W)\) is applicable to the case \(W = \emptyset\). In this case we have \(A_P(\emptyset) = K_P\). Otherwise, \(A_P(W)\) is a proper subset of \(K_P\). Root points of wedges in \(W\) will also be referred to as *root points* of \(W\). A root point \(a\) of a wedge \(W \in W\) is called *outward parabolic* if \(a\) is a parabolic periodic point, and there is a Fatou component in \(W\) containing an attracting petal of \(a\).

We want to compare the dynamics of \(P|_{A_P(W)}\) with that of a lower degree polynomial on its filled Julia set. There are many cases, in which such a comparison can be made precise. Classical arguments
yield Theorem 1.2. We write $U \subset V$ if $U \subset V$; for polynomial-like maps and hybrid equivalence [DH85] see Definitions 2.8 and 2.9.

**Theorem 1.2.** Let $P$ be a degree $d > 1$ polynomial with connected Julia set. Consider a renormalizable collection $W$ of wedges in the dynamical plane of $P$. Suppose that no root point of $W$ is critical or outward parabolic. Then there exist Jordan domains $U \subset V$ such that $P : U \rightarrow V$ is polynomial-like, and $A_P(W)$ is the filled Julia set of this polynomial-like map. In particular, $P|_{A_P(W)}$ is hybrid equivalent to $Q|_{K_Q}$, where $Q$ is some polynomial of degree $< d$.

Recall that $P$ is said to be immediately renormalizable if the conclusion of Theorem 1.2 holds. The objective of this paper is to generalize Theorem 1.2. Indeed, it is useful to know if $P|_{A_P(W)}$ is just topologically conjugate to some polynomial on its connected filled Julia set. Since this conclusion is weaker than that of Theorem 1.2, it is natural to expect that it can be achieved with weaker assumptions than those of Theorem 1.2. In this paper we meet these expectations. The following is our main result.

**Main Theorem.** Let $P$ be a degree $d > 1$ polynomial with connected Julia set. Consider a renormalizable collection $W$ of wedges of $P$. Suppose that every critical root point of $W$ is eventually mapped to a repelling periodic orbit. Then $P|_{A_P(W)}$ is topologically conjugate to $Q|_{K_Q}$, where $Q$ is some polynomial of degree $< d$.

The following Corollary follows immediately from the Main Theorem and, in the irrational neutral case, results by Perez-Marco [P-M97].

**Corollary 1.3.** Assuming the conditions of the Main Theorem, let $\varphi$ be the topological conjugacy between $P|_{A_P(W)}$ and $Q|_{K_Q}$ and let $C$ be a periodic cycle in $A_P(W)$. Consider the multiplier $\lambda_P$ of $C$ and the multiplier $\lambda_Q$ of $\varphi(C)$. Then there are the following possibilities:

1. either both $\lambda_P$ and $\lambda_Q$ are in $\mathbb{D}$ or both are in $\mathbb{C} \setminus \overline{\mathbb{D}}$;
2. $\lambda_P = e^{2\pi i \alpha}$ and one of the two cases holds:
   (a) $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda_P = \lambda_Q$;
   (b) $\alpha \in \mathbb{Q}$ and either $\lambda_P = \lambda_Q$ or $|\lambda_Q| > 1$.

Figure shows $W$ and $A_P(W)$ for a specific cubic polynomial $P$.

1.2. **Almost polynomial-like maps.** Our principal tool is a certain generalization of the concept of polynomial-like maps. This notion, due to Douady and Hubbard [DH85], deals with holomorphic maps. We will use a more general setup of quasi-regular maps [Ric93], see Definition 2.3.
The set $A_P(W)$ for $P(z) = z(z+2)^2$ is shown in dark grey. Here $W$ consists of a single wedge $W$ (highlighted on the left) whose boundary is mapped to $R_P(0)$. The boundary rays of $W$ are $R_P(1/3)$ and $R_P(2/3)$, and the root point $a = -2$ of $W$ maps to the fixed point $0$. By the Main Theorem, the filled Julia set $K_P$ consists of a copy of $K_Q$, where $Q(z) = -z + z^2$, and countably many decorations. The parabolic point $0$ of $Q$ corresponds to the parabolic point $-1$ of $P$ of the same multiplier. The maps $P|_{A_P(W)}$ and $Q|_{K_Q}$ are topologically conjugate, but $A_P(W)$ is not a PL filled Julia set.

Consider a topological branched covering $f : U \to V$, where $U \subset V$ are Jordan domains in $\mathbb{C}$. We always assume that $f$ preserves orientation. Define the filled Julia set $K(f)$ of $f$ as the set of all points $z \in K(f)$ with $f^n(z) \in U$ for all $n = 0, 1, 2, \ldots$. It is easy to see that

$$K(f) = \bigcap_{n=0}^{\infty} f^{-n}(U).$$

It follows that $K(f)$ is a compact subset of $U$.

**Definition 1.4** (Almost polynomial-like maps). Let $f : U \to V$ be as above. We say that $f$ is almost polynomial-like (APL) if $f$ is quasi-regular on $U$ and $\overline{\partial}f = 0$ on $K(f)$.

Our Theorem 1.5 is an analog of the Straightening Theorem [DH85].

**Theorem 1.5.** Let $f : U \to V$ be an APL map. Then $f : K(f) \to K(f)$ is topologically conjugate to the restriction of a polynomial to its
Theorem 1.5 provides a technical tool for proving the Main Theorem. We will perform a certain surgery on \( P \) to transform it into an APL map \( f \) to which Theorem 1.5 is directly applicable.

1.3. Invariant continua similar to \( A_P(W) \). In Section 1.1, we associated an invariant continuum \( A_P(W) \) to a renormalizable collection \( W \) of wedges. We now suggest an alternative description of such invariant continua not referring to \( W \). Theorem 1.6 is proved in [BOPT16a].

**Theorem 1.6** (Theorem B [BOPT16a]). Let \( P : \mathbb{C} \to \mathbb{C} \) be a polynomial, and \( Y \subset \mathbb{C} \) be a non-separating \( P \)-invariant continuum. The following assertions are equivalent:

1. the set \( Y \) is the filled Julia set of some polynomial-like map \( P : U \to V \) of degree \( k \),
2. the set \( Y \) is a component of the set \( P^{-1}(P(Y)) \) and, for every attracting or parabolic point \( y \) of \( P \) in \( Y \), the attracting basin of \( y \) or the union of all parabolic domains at \( y \) is a subset of \( Y \).

We will show that a slightly weaker consequence than (1) holds under more general assumptions than (2). Let \( Y \) be a \( P \)-invariant continuum. We say that \( P : Y \to Y \) is *almost \( k \)-to-one* if there is a finite subset \( S_Y \subset Y \) such that \( P : Y \setminus P^{-1}(S_Y) \to Y \setminus S_Y \) is precisely \( k \)-to-one. Elements of the smallest such \( S_Y \) are called *singular values* of \( P \) on \( Y \). A point \( y \in Y \) can be a singular value only if several preimages in \( Y \) of a point \( y' \to y \) collide as \( y' \) merges with \( y \). Thus, \( y \) must be the image of a critical point from \( Y \). Points of \( P^{-1}(Y) \setminus Y \cap Y \) are called *irregular points* of \( Y \).

**Theorem 1.7.** Let \( P : \mathbb{C} \to \mathbb{C} \) be a polynomial. Consider a full \( P \)-invariant continuum \( Y \subset \mathbb{C} \) and an integer \( k > 1 \) such that:

1. the map \( P : Y \to Y \) is almost \( k \)-to-one;
2. any irregular point is eventually mapped to a repelling periodic point;
3. if an immediate attracting or parabolic basin of \( P \) intersects \( Y \), then it is contained in \( Y \).

Under these assumptions, \( P : Y \to Y \) is topologically conjugate to \( Q|_{K(Q)} \), where \( Q \) is a polynomial of degree \( k \).

It can be shown that if \( P \) has a connected Julia set, and \( Y = A_P(W) \) for some renormalizable \( W \), then \( Y \) satisfies the assumptions of Theorem 1.7. Thus Theorem 1.7 can be understood as a generalization of
the Main Theorem. In particular, Theorem 1.7 is applicable to polynomials $P$ with disconnected Julia sets.

**Conjecture 1.8.** If $P$ and $Y$ satisfy the assumptions of Theorem 1.7, the degree of $P$ is at least two, and the Julia set of $P$ is connected, then $Y = A_P(W)$ for some renormalizable collection of wedges $W$.

This conjecture is open even in the particular case of a cubic polynomial with no cuts at all. For if true, it would imply that a cubic polynomial $P$ like that cannot have an invariant quadratic-like connected Julia set $J^* \subset J_P$. However this case is hard to rule out.

**1.4. Plan of the paper.** Section 2 provides some background from complex polynomial dynamics and quasi-conformal geometry. We also recall the notion of polynomial-like maps and discuss its extension — the notion of APL maps. In particular, we prove Theorem 1.5. In Section 3, we discuss combinatorial aspects of renormalizable collections of wedges. A method to replace a renormalizable collection with another — simpler and more convenient — collection is explained. Section 4 gives a general strategy of the proof of the Main Theorem. Sections 5–7 are devoted to detailed study of special types of wedges that can occur in a renormalizable collection. We follow the same general strategy outlined Section 4 even though details may differ. Section 8 contains the proof of Theorem 1.7.

2. Almost polynomial-like maps

2.1. **Background in complex polynomial dynamics.** Consider a straight radial interval $R(\theta) = \{e^{2\pi i\theta} \rho \mid \rho \in (0, 1)\}$ from 0 to the point $e^{2\pi i\theta}$. Let $P$ be a degree $d > 1$ polynomial with connected Julia set. The external ray of $P$ of argument $\theta \in \mathbb{R}/\mathbb{Z}$ is the set $R_P(\theta) = \psi_P(R(\theta))$. External rays are useful to study the dynamics of $P$. In particular, it is important to know when different rays land at the same point.

**Definition 2.1** (Ray landing). A ray $R_P(\theta)$ lands at $a \in K_P$ if $a = \lim_{\rho \to 1^-} \psi_P(e^{2\pi i\theta} \rho)$ is the only accumulation point of $R_P(a)$ in $\mathbb{C}$.

By the Douady–Hubbard–Sullivan landing theorem, if $\theta$ is rational then $R_P(\theta)$ lands at a (pre)periodic point that is eventually mapped to a repelling or parabolic periodic point. Conversely, any point that eventually maps to a repelling or parabolic periodic point is the landing point of at least one and at most finitely many external rays.

An equipotential curve of $P$ (or simply an equipotential) is the $\psi_P$-image of a circle $\{z \in \mathbb{C} \mid |z| = \rho\}$ of radius $\rho \in (0, 1)$ centered at 0. External rays and equipotentials form a net that is the $\psi_P$-image of the polar coordinate net.
2.2. **Quasi-regular and quasi-symmetric maps.** Let us recall the definition of quasi-regular [Ric93] and quasi-conformal maps [Ahl66].

**Definition 2.2** (Quasi-regular maps). Let \( U \) and \( V \) be open subsets of \( \mathbb{C} \), and let \( \kappa \geq 1 \) be a real number. A map \( f : U \to V \) is said to be \( \kappa \)-quasi-regular if it has distributional partial derivatives in \( L^2_{\text{loc}} \), and 
\[
|df|^2 \leq \kappa \text{Jac}_f \text{ in } L^1_{\text{loc}}.
\]
Here \( df \) is the first differential of \( f \), and \( \text{Jac}_f \) is the Jacobian determinant of \( f \). Note that any holomorphic map is \( \kappa \)-quasi-regular with \( \kappa = 1 \). We say that \( f \) is quasi-regular if it is \( \kappa \)-quasi-regular for some \( \kappa \geq 1 \). A quasi-conformal map is by definition a quasi-regular homeomorphism.

The inverse of a \((\kappa-)\)quasi-conformal map is \((\kappa-)\)quasi-conformal. Quasi-conformal maps admit a number of analytic and geometric characterizations. They can be characterized in terms of Beltrami differentials and in terms of moduli of annuli or similar conformal invariants. See 4.1.1 and 4.5.16 — 4.5.18 in [Hub06]. A metric characterization of quasi-conformal maps is based on the following notion applicable to general metric spaces, cf. [TVS80].

**Definition 2.3** (Quasi-symmetric maps). Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces, and let \( \eta : [0,\infty) \to [0,\infty) \) be an increasing onto homeomorphism. A continuous embedding \( f : X \to Y \) is said to be quasi-symmetric of modulus \( \eta \) (or \( \eta \)-quasi-symmetric) if
\[
\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta \left( \frac{d_X(x,y)}{d_X(x,z)} \right) \quad \forall x \neq y \neq z \in X
\]
and \( x, y \) and \( z \) are sufficiently close to each other. We will sometimes abbreviate quasi-symmetric as QS. The inverse of a QS embedding (defined on \( f(X) \)) is \( \eta' \)-QS, where \( \eta'(t) = 1/\eta^{-1}(1/t) \). The composition of QS embeddings is also QS. A continuous embedding \( f : X \to Y \) is \( \kappa \)-weakly QS for some \( \kappa > 0 \) if
\[
d_X(x,y) \leq d_X(x,z) \implies d_Y(f(x), f(y)) \leq \kappa d_Y(f(x), f(z)).
\]
Weakly QS embeddings are \( \kappa \)-weakly QS for some \( \kappa > 0 \). Clearly, QS embeddings are weakly QS. The converse is not true in general, however, by Theorem 10.19 of [Hei00], weakly QS embeddings are QS in a lot of cases. In particular, a weakly QS embedding of a connected subset of \( \mathbb{R}^n \) to \( \mathbb{R}^n \) is QS. Occasionally we will talk about “QS maps” which will always mean “QS embeddings”.

The following theorem establishes a relationship between QS embeddings and quasi-conformal maps.
Theorem 2.4 (A special case of Theorems 2.3 and 2.4 of [Vai81]).

An $\eta$-QS embedding between domains in $\mathbb{C}$ is $\kappa$-quasi-conformal ($\kappa \geq 1$ is a constant depending only on $\eta$). Conversely, consider a $\kappa$-quasi-conformal map $f : U \to V$, where $U, V \subset \mathbb{R}^2$ are open. Then, for any $z \in U$ and $\varepsilon > 0$ such that the $2\varepsilon$-neighborhood of $z$ lies in $U$, the map $f$ is $\eta$-QS on the $\varepsilon$-neighborhood of $z$, where $\eta$ depends only on $\kappa$.

Quasi-conformal images of circle arcs, circles, and disks can be described explicitly.

Definition 2.5 (Quasi-arc, quasi-circle, quasi-disk). A simple arc in $\mathbb{C}$ is a homeomorphic image of $[0, 1]$ under a map $\xi : [0, 1] \to \mathbb{C}$. A simple arc $I$ is a quasi-arc if for any such $\xi$ and any $x \leq y \leq z$ we have

$$|\xi(x) - \xi(z)| \geq C|\xi(x) - \xi(y)|,$$

where $C > 0$ is a constant independent of $x, y, z$ and $\xi$. A quasi-circle is a Jordan curve such that any arc of it is a quasi-arc. For quasi-arcs and quasi-circles in the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C}P^1$, we use the spherical distance between $a$ and $b$ instead of $|a - b|$, etc. A quasi-disk is a Jordan disk bounded by a quasi-circle. A quasi-conformal reflection in a Jordan curve is an orientation-reversing quasi-conformal map of period 2 of the sphere onto itself which switches the inside and the outside of the curve fixing points on the curve.

The following theorem is due to L. Ahlfors, see [Ahl66] or 4.9.8, 4.9.12, and 4.9.15 in [Hub06]:

Theorem 2.6. Properties (1) – (3) of a Jordan curve $S$ are equivalent:

1. the curve $S$ is a quasi-circle;
2. there is a quasi-conformal reflection in $S$;
3. there is a quasi-conformal map $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $S = h(\overline{\mathbb{R}})$.

Observe also that QS embeddings of quasi-arcs are quasi-arcs; moreover preimages of quasi-arcs under QS-embeddings are quasi-arcs too.

Quasi-symmetric maps between quasi-circles can be extended inside the corresponding quasi-disks as quasi-conformal maps.

Theorem 2.7. If $U$ and $V$ are quasi-disks in $\overline{\mathbb{C}}$, and $f : \text{Bd}(U) \to \text{Bd}(V)$ is a quasi-symmetric map then there is a continuous map $F : U \to V$ such that $F = f$ on $\text{Bd}(U)$, and $F$ is quasi-conformal in $U$.

Proof. By Theorem 2.6 there are quasi-conformal maps $h_U, h_V : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that take the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} | \text{Im} z > 0\}$ onto $U, V$, respectively. Then the map $\varphi = h_V^{-1} \circ f \circ h_U : \mathbb{R} \to \mathbb{R}$ is quasi-symmetric as a composition of quasi-symmetric maps. Let $\eta$ be a modulus of $\varphi$
(so that \( \varphi \) is \( \eta \)-quasi-symmetric). Setting \( y = x + t \) and \( z = x - t \) in the definition of an \( \eta \)-quasi-symmetric map, we see that

\[
M^{-1} \leq \frac{\varphi(x + t) - \varphi(x)}{\varphi(x) - \varphi(x - t)} \leq M,
\]

where \( M = \eta(1) \). Maps \( \varphi \) that satisfy the above condition for some \( M > 0 \) are called \( \mathbb{R} \)-quasi-symmetric in [Hub06]. The constant \( M \) is called the modulus of a \( \mathbb{R} \)-quasi-symmetric map. By a theorem of Ahlfors and Beurling [AB56] (see also [Ahl66] and 4.9.3 and 4.9.5 of [Hub06]), a \( \mathbb{R} \)-quasi-symmetric map of modulus \( M \) admits a \( \kappa \)-quasi-conformal extension in \( \mathbb{H} \), where \( \kappa \) depends only on \( M \). More precisely, there is a continuous map \( \Phi : \mathbb{H} \to \mathbb{H} \) such that \( \Phi = \varphi \) on \( \mathbb{R} \), and \( \Phi|_{\mathbb{H}} \) is \( \kappa \)-quasi-conformal. Then \( F = h_V \circ \Phi \circ h_U^{-1} \) has the desired property.

2.3. APL maps. Let \( U \) and \( V \) be Jordan disks such that \( U \Subset V \). Recall the following classical definition of Douady and Hubbard [DH85].

**Definition 2.8** (Polynomial-like maps). Let \( f : U \to V \) be a proper holomorphic map. Then \( f \) is said to be **polynomial-like** (PL).

The difference between Definitions 2.8 and 1.4 is that a PL map \( f \) satisfies \( \partial f = 0 \) everywhere on \( U \), in contrast to an APL map that has to satisfy this property only on \( K(f) \). Thus, the notion of APL maps is an extension of the notion of PL maps. We extend the notion of hybrid equivalence, defined in [DH85] for PL maps, onto APL maps.

**Definition 2.9** (Hybrid equivalence). Let \( f_1 : U_1 \to V_1 \) and \( f_2 : U_2 \to V_2 \) be two APL maps. Consider Jordan neighborhoods \( W_1 \) of \( K(f_1) \) and \( W_2 \) of \( K(f_2) \). A quasi-conformal homeomorphism \( \phi : W_1 \to W_2 \) is called a **hybrid equivalence** between \( f_1 \) and \( f_2 \) if \( f_2 \circ \phi = \phi \circ f_1 \) whenever both parts are defined, and \( \partial \phi = 0 \) on \( K(f_1) \).

Similarly to polynomials and PL maps, we have Proposition 2.10.

**Proposition 2.10.** Let \( f : U \to V \) be an APL map. The set \( K(f) \) is connected if and only if all critical points of \( f \) are in \( K(f) \).

The main result of this section is the following theorem.

**Theorem 2.11.** Every APL map with connected filled Julia set is hybrid equivalent to a PL map.

The proof of Theorem 2.11 is based on a classical mapping theorem (see, e.g., [Hub06] Theorem 4.6.1]).

**Theorem 2.12** (Morrey, Ahlfors–Bers). Let \( \mu \) be a bounded measurable Beltrami differential on an open subset \( V \subset \mathbb{C} \). Then there exists a
quasi-conformal embedding $\Psi : V \to \mathbb{C}$ such that $\Psi^*\mu$ is the standard complex structure. In other words, $\Psi$ satisfies the Beltrami equation

$$\frac{\partial \Psi}{\partial z} = \mu \frac{\partial \Psi}{\partial \bar{z}}.$$ 

It follows from Weyl’s lemma that $\Psi$ is well defined up to a post-composition with a conformal map.

Proof of Theorem 2.11. Consider an APL map $f : U \to V$, and assume that $K(f)$ is connected. Replacing $V$ with a suitable smaller neighborhood $\tilde{V}$ of $K(f)$ and $U$ with $f^{-1}(\tilde{V})$, we may assume that $U$ and $V$ have smooth boundaries. Then $U_n = f^{-n}(U)$ is a disk for every $n$. Define a sequence of annuli by setting $A_0 = V \setminus U$ and $A_n = U_{n-1} \setminus U_n$ for $n \geq 1$. Let $\nu$ be the Beltrami differential corresponding to the standard conformal structure on $K(f)$. Choose a bounded measurable Beltrami differential $\nu_0$ on $A_0$. Define $\nu_n$ on $A_n$ for $n \geq 1$ inductively as the pullback of $\nu_{n-1}$ under $f$. Finally, let $\mu$ be the Beltrami differential on $V$ equal to $\nu$ on $K(f)$ and to $\nu_n$ on $A_n$. (Since $V = K(f) \cup A_n$ then $\mu$ is defined everywhere on $V$). By construction, $\mu$ is bounded, measurable, and $f$-invariant. By Theorem 2.12, there is a quasi-conformal embedding $\Psi : V \to \mathbb{C}$ that satisfies the Beltrami equation for $\mu$. It follows that the map $\Psi \circ f \circ \Psi^{-1} : \Psi(U) \to \Psi(V)$ is conformal; clearly, it is a PL map. \hfill $\Box$

Recall the following classical theorem of Douady and Hubbard [DH85].

**Theorem 2.13 (PL Straightening Theorem).** A polynomial-like map $f : U \to V$ is hybrid equivalent to a polynomial of the same degree restricted on a Jordan neighborhood of its filled Julia set.

Corollary 2.14 easily follows from Theorems 2.11 and 2.13.

**Corollary 2.14.** Every APL map $f$ with connected filled Julia set is hybrid equivalent to a PL restriction of a polynomial $Q$ on a Jordan neighborhood of its filled Julia set. In particular, $f|_{K(f)}$ is topologically conjugate to $Q|_{K_Q}$.

Observe that this proves Theorem 1.5 stated in the Introduction.

2.4. **Transversality.** Consider two simple arcs $R, L \subset \mathbb{C}$ sharing an endpoint $a$ and otherwise disjoint. The arcs $R, L$ are transverse at $a$ if, for any sequences $u_n \in R$ and $v_n \in L$ converging to $a$,

$$\frac{u_n - a}{v_n - a} \not\to 1.$$ 

Transversality is related with the notion of a quasi-arc as the following easy lemma explicates.
Lemma 2.15. Let simple arcs $R$, $L$ share an endpoint $a$ and be otherwise disjoint. If $R \cup L = I$ is a quasi-arc, then $R$ and $L$ are transverse.

Proof. By way of contradiction, suppose that

$$\frac{u_n - a}{v_n - a} \to 1.$$ 

for some $u_n \in R$ and $v_n \in L$ such that $u_n, v_n \to a$. It follows that

$$\frac{u_n - v_n}{v_n - a} \to 0.$$ 

This contradicts the inequality $|v_n - u_n| \geq C|v_n - a|$ with $C > 0$ from the definition of a quasi-arc.

Proposition 2.16. Consider a simple arc $R$ such that the image $R'$ of $R$ under $z \mapsto z^k$ with $k > 1$ is a simple arc. Moreover, assume that $0$ is an endpoint of $R$, and $\lambda R' \supset R'$ for some $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Then $R$ is transverse to $\zeta R$ at 0, for every $k$-th root of unity $\zeta \neq 1$.

Proof. The map $z \mapsto z^k$ is injective on the arc $R$ from Proposition 2.16 as otherwise $z^k$ is not 1-to-1 at a point $z \in R, z \neq 0$, a contradiction. Thus, $R$ and $\zeta R$ share only the endpoint 0. Assume the contrary: $v_n/u_n \to \zeta$, where $u_n, v_n \in R$ and $u_n, v_n \to 0$. Passing to a subsequence and choosing positive integers $m_n$ properly, we may assume that $\lambda^{m_n} u_n^k \to u \neq 0$, where $u \in R'$ is not an endpoint of $R'$. Then also $\lambda^{m_n} v_n^k \to u$. Let $I_n$ be the segment of $R$ connecting $u_n$ and $v_n$. Then the corresponding segment $I_n'$ of $R'$ connects $u_n^k$ with $v_n^k$. Consider the arc $T_n = \lambda^{m_n} I_n' \subset R'$; its endpoints $\lambda^{m_n} u_n^k$ and $\lambda^{m_n} v_n^k$ converge to $u$ but the arc itself has diameter bounded away from 0 as it makes one or several loops around 0 (if $S_n$ is the union of $\lambda^{m_n} I_n'$ and the straight segment connecting its endpoints then, since $v_n/u_n \to \zeta$, the loop $S_n$ has a nonzero winding number with respect to 0). Since sets $T_n$ are subarcs of $R'$ in the limit they converge to a non-degenerate loop in $R'$, a contradiction.

The following is a typical application of Proposition 2.16. Let $P$ be a polynomial; consider a repelling fixed point $a$ of $P$ and an invariant ray $R_P(\theta')$ landing at $a$. Set $\lambda = P'(a)$, that is, $\lambda$ is the multiplier of the fixed point $a$. Then, in some local coordinate $u$ near $a$, we have $u = 0$ at $a$, and $P$ coincides with multiplication by $\lambda$. On the other hand, suppose that a ray $R_P(\theta)$ maps to $R_P(\theta')$ under $P$. Write $b$ for the landing point of $R_P(\theta)$, and assume that $P$ has local degree $k$ at $b$. Then, in some local coordinate $z$ near $b$ combined with the local coordinate $u$ near $a$, the map $P$ looks like $z \mapsto z^k = u$, and $b$ is the point where $z = 0$. We can now define $R$ as an arc of $R_P(\theta)$ connecting
that is applicable to arcs \( R, R' \) and the chosen local coordinates. It claims that \( R \) is transverse to all other \( P \)-pullbacks of \( R' \) originating at \( b \).

**Lemma 2.17.** Let arcs \( R, L \) be as in Proposition 2.16 and let \( R', L' \) be their images under \( z \mapsto z^k \). If \( R' \cap L' = \{0\} \) and \( R', L' \) are transverse, then the restriction of \( z \mapsto z^k \) to \( R \cup L \) is QS.

If \( R' \cup L' \) is a quasi-arc, then, by Lemma 2.17, the arc \( R \cup L \) is also a quasi-arc. This observation will be useful in the sequel.

**Proof.** We will prove that \( z \mapsto z^k \) restricted to \( R \cup L \) is weakly QS. Assume, by way of contradiction, that there are three sequences \( u_n, v_n, w_n \in R \cup L \) such that, if we set \( \bar{v}_n = v_n/u_n, \bar{w}_n = w_n/u_n \), we will have

\[
\Delta_n = \frac{u_n^k - v_n^k}{w_n^k - u_n^k} \rightarrow \infty, |u_n - v_n| \leq |u_n - w_n| \quad \text{(and so } |1 - \bar{v}_n| \leq |1 - \bar{w}_n|)\]

Assume that \( u_n \rightarrow u, v_n \rightarrow v \) and \( w_n \rightarrow w \). If \( u \neq w \), then \( \Delta_n \) is bounded since \( z \mapsto z^k \) is injective on \( R \cup L \), a contradiction. Thus \( u = w \). If \( u \neq v \), then \( |u_n - v_n| > |u_n - w_n| \) for large \( n \), hence \( u = v = w \).

It is enough to consider the case when \( u = 0 \). We may also assume that \( \delta_n = (u_n - v_n)/(u_n - w_n) = (1 - \bar{v}_n)/(1 - \bar{w}_n) \rightarrow \delta \) with \( |\delta| \leq 1 \).

Assume that \( \bar{v}_n \rightarrow \bar{v} \) and \( \bar{w}_n \rightarrow \bar{w} \), where \( \bar{v} \) and \( \bar{w} \) are complex numbers or \( \infty \). Observe that

\[
\Delta_n = \frac{1 - \bar{v}_n^k}{1 - \bar{w}_n^k} = \bar{w}_n^{-k} - \bar{v}_n^{-k} = \frac{1 + \bar{v}_n + \cdots + \bar{v}_n^{k-1}}{1 + \bar{w}_n + \cdots + \bar{w}_n^{k-1}} \rightarrow \infty
\]

If \( \bar{v} = \bar{w} = 1 \), then \( \Delta_n \rightarrow \delta \), a contradiction. If \( \bar{w} = \infty \), then \( \bar{v}_n/\bar{w}_n \rightarrow \delta \) and \( \Delta_n \rightarrow \delta^k \), a contradiction. Since \( \bar{v} = \infty \) implies \( \bar{w} = \infty \) and \( \bar{w} = 1 \) implies \( \bar{v} = 1 \) (because \( |1 - \bar{v}| \leq |1 - \bar{w}| \)), both \( \bar{v}, \bar{w} \) are finite, and \( \bar{w} \neq 1 \). Now \( \delta_n \rightarrow \infty \) implies that \( \bar{w} \neq 1 \) is a \( k \)-th root of unity. By Proposition 2.16 and since \( w_n/u_n \rightarrow \bar{w} \), it is impossible that \( w_n, u_n \) are both in \( R \) or both in \( L \) for infinitely many values of \( n \). Thus we may assume that \( w_n \in R \) and \( u_n \in L \). However, in this case \( R' \) and \( L' \) are not transverse, a contradiction.

Consider two simple arcs \( R', L' \) with common endpoint at 0 and disjoint otherwise. Suppose that \( \lambda \) is a complex number with \( |\lambda| > 1 \). Furthermore, suppose that \( \lambda R' \supset R' \) and \( \lambda L' \supset L' \).

**Theorem 2.18.** If \( R', L' \) are as above, and \( R' \cup L' \) is smooth except possibly at 0, then \( R' \cup L' \) is a quasi-arc.

**Proof.** Assume the contrary: there are three sequences \( x_n, y_n, z_n \in R' \cup L' \) such that
• the point \( y_n \) is always between \( x_n \) and \( z_n \) in the arc \( R' \cup L' \) (in particular, the three points \( x_n, y_n, z_n \) are always different);
• we have \( \delta_n = |x_n - z_n|/|x_n - y_n| \to 0 \) as \( n \to \infty \).

It follows from the second assumption that \( |x_n - z_n| \to 0 \) since the denominator is bounded. We can now make a number of additional assumptions on \( x_n, y_n, z_n \) by passing to subsequences. Assume that \( x_n \) and \( z_n \) converge to the same limit. If this limit is different from 0, then straightforward geometric arguments yield a contradiction (it is obvious that every closed subarc of \( R' \cup L' \) not containing 0 is a quasi-arc). Thus we may assume that \( x_n, z_n \to 0 \). Since \( y_n \) is between \( x_n \) and \( z_n \), we also have \( y_n \to 0 \). From now on, we rely on the assumption that all three sequences \( x_n, y_n, z_n \) converge to 0.

Take \( r > 0 \) sufficiently small, and let \( A \) be the annulus \( \{ z \in \mathbb{C} \mid r < |z| < |\lambda| r \} \). Assume that \( x_n \neq 0 \) for all \( n \) (otherwise, for a suitable subsequence, \( z_n \neq 0 \) for all \( n \), and we may interchange \( x_n \) and \( z_n \)). For every \( n \), there exists a positive integer \( m_n \) such that \( \lambda^{m_n} x_n \in \mathbb{C} \). Set \( x'_n \), \( y'_n \), \( z'_n \) to be \( \lambda^{m_n} x_n, \lambda^{m_n} y_n, \lambda^{m_n} z_n \), respectively. We may assume that \( x'_n \in R' \) rather than \( L' \). (By the invariance property of \( R' \cup L' \), we must have \( x'_n \in R' \cup L' \).) Passing to a subsequence, arrange that \( x'_n \to x \in \mathbb{C} \) as \( n \to \infty \). Since \( |x'_n - z'_n|/|x'_n - y'_n| = \delta_n \to 0 \), then \( z'_n \to x \). Since the intersections of \( R' \) and \( L' \) with an open neighborhood of \( \mathbb{C} \) are smooth open arcs, it follows that \( y'_n \to x \) and that \( \delta_n \to 0 \). A contradiction.

\[ \square \]

3. Renormalizable collections of wedges: combinatorics

In this section, we fix a degree \( d > 1 \) complex polynomial \( P \) with connected Julia set and study combinatorics associated with renormalizable collections of wedges (of \( P \)). Consider a collection \( \mathcal{W} \) of disjoint wedges. Different claims will depend on different properties of \( \mathcal{W} \). For the first proposition, we assume nothing special of \( \mathcal{W} \). Let \( X_P(\mathcal{W}) \) be the union of \( K_P \) and all rays \( R_P(\theta) \) which land in \( A_P(\mathcal{W}) \). The following result follows immediately.

**Proposition 3.1.** A ray \( R_P(\theta) \) lies in \( X_P(\mathcal{W}) \) if and only if for any \( n \in \mathbb{Z}_{\geq 0} \) either \( R_P(3^n \theta) \not\subset \bigcup \mathcal{W} \), or \( R_P(3^n \theta) \) lands at a root point of \( \mathcal{W} \).

Now we introduce a certain equivalence relation on collections of wedges.

**Definition 3.2** (Equivalence on collections of wedges). Two collections of disjoint wedges \( \mathcal{W} \) and \( \mathcal{W}' \) are said to be equivalent if \( A_P(\mathcal{W}) = A_P(\mathcal{W}') \). We write \( \mathcal{W} \sim \mathcal{W}' \). A collection \( \mathcal{W} \) of wedges is said to be
reduced if for every $W \in \mathcal{W}$, there are no external rays of $P$ contained in $W$ and landing at the root point of $W$.

The following lemma is almost immediate.

**Lemma 3.3.** Any collection of disjoint wedges is equivalent to a reduced collection.

**Proof.** Let $\mathcal{W}$ be any collection of disjoint wedges. Replace any $W \in \mathcal{W}$ with the set of wedges into which $W$ is divided by all rays in $W$ landing at the root point of $W$. The new collection of wedges thus obtained is reduced and equivalent to $\mathcal{W}$. □

We now consider a reduced collection $\mathcal{W}$ of wedges. Note that in this case Proposition 3.1 takes a simpler form. Namely, $R_P(\theta) \subset X_P(\mathcal{W})$ if and only if $R_P(\theta)$ is never mapped to $\bigcup \mathcal{W}$ under the iterates of $P$. Complementary components of $X_P(\mathcal{W})$ are called planar holes of $X_P(\mathcal{W})$. Boundaries of holes of $X_P(\mathcal{W})$ are called planar edges of $X_P(\mathcal{W})$. We will not discuss what any planar hole or edge of $X_P(\mathcal{W})$ looks like; rather, we concentrate on specific planar holes and edges.

**Lemma 3.4.** Suppose that $\mathcal{W}$ is a reduced essential collection of disjoint wedges. Then every $W \in \mathcal{W}$ is a planar hole of $X_P(\mathcal{W})$.

**Proof.** Since $\mathcal{W}$ is reduced, $W$ lies in a planar hole $H$ of $X_P(\mathcal{W})$. On the other hand, since $W$ is essential, the root point $a$ of $W$ belongs to $A_P(\mathcal{W})$, and then the boundary rays of $W$ belong to $X_P(\mathcal{W})$ by definition. Therefore, $W = H$, as claimed. □

The next lemma uses the property of $\mathcal{W}$ being non-separating.

**Lemma 3.5.** Suppose that $\mathcal{W}$ is reduced and non-separating. Consider a wedge $W \in \mathcal{W}$ and the corresponding cut $\Gamma$. Then $P(\Gamma)$ does not separate $X_P(\mathcal{W})$.

**Proof.** Let $a$ be the root point of $W$. Since $\mathcal{W}$ is non-separating, $P(\Gamma)$ does not separate $A_P(\mathcal{W})$. If it separates $X_P(\mathcal{W})$, then an external ray $R_P(\theta) \subset W$ lands at $a$ so that $R_P(3\theta)$ is separated from $A_P(\mathcal{W})$ by $P(\Gamma)$. However, since $\mathcal{W}$ is reduced, such a ray cannot exist. □

Although we do not describe all planar edges of $X_P(\mathcal{W})$, we say precisely what iterated images of $\text{Bd}(W)$ with $W \in \mathcal{W}$ may look like.

**Lemma 3.6.** Let $\mathcal{W}$ be reduced, essential and non-separating. Consider a wedge $W \in \mathcal{W}$, the corresponding cut $\Gamma = \text{Bd}(W)$, and an iterated $P$-image $\Gamma^n = P^n(\Gamma)$ of $\text{Bd}(W)$. Then either $\Gamma^n$ is a planar edge of $X_P(\mathcal{W})$, or $\Gamma^n$ is a landing external ray with its landing point.
Proof. Let $\Lambda$ be a cut; let $Z$ and $Y$ be components of $\mathbb{C} \setminus \Lambda$. Suppose that $X_P(W) \subset Z$. If $X_P(W)$ intersects two wedges created by $P(\Lambda)$ then $\Lambda$ is not the boundary of a wedge from $W$ (for $W$ is non-separating), and some points from $Y$ map to points of $X_P(W)$. However, since $W$ is essential and reduced, any point $y \in Y$ with $P(y) \in X_P(W)$ is such that $y \in X_P(W) \cap Y$, a contradiction. Hence only one component of $\mathbb{C} \setminus P(\Lambda)$ contains $X_P(W)$ in its closure. It follows that if $\Gamma^*_n$ is a planar edge of $X_P(W)$ while $\Gamma^{n+1}$ is not, then $\Gamma_n = \text{Bd}(W)$ for a wedge $W \in \mathcal{W}$ which contradicts the assumption that $W$ is non-separating.  

Consider a wedge $W \in \mathcal{W}$, the cut $\Gamma = \text{Bd}(W)$, and an iterated $P$-image $\Gamma^*_n = P^n(\Gamma)$ of $\text{Bd}(W)$. Call $\Gamma^*_n$ degenerate if it is the closure of a single external ray and non-degenerate otherwise. If $\Gamma^*_n$ is degenerate, set $W(\Gamma^*_n) = \Gamma^*_n$; otherwise $W(\Gamma^*_n)$ is the planar hole of $X_P(W)$ bounded by $\Gamma^*_n$. Say that $\Gamma^*_n$ is an iterated planar $W$-edge if it is non-degenerate or the first image of a non-degenerate iterated planar $W$-edge, or if it is eventually periodic. If $\Gamma^*_n$ is an iterated planar $W$-edge, then $W(\Gamma^*_n)$ is an iterated planar $W$-wedge; $W(\Gamma^*_n)$ can be degenerate or non-degenerate. An iterated planar $W$-edge is not necessarily a genuine edge of $X_P(W)$ as there might exist some (pre)periodic degenerate iterated planar $W$-edges. A planar non-degenerate edge $\Gamma$ of $X_P(W)$ is critical if $P(\Gamma)$ is degenerate; otherwise $\Gamma$ is non-critical. By Lemma 3.6 if a non-critical planar edge of $X_P(W)$ is an iterated planar $W$-edge, then it is mapped to an iterated planar $W$-edge.

A generalized wedge is a wedge or the closure of a landing external ray. Denote by $\Gamma(\alpha, \beta)$ a planar cut formed by the external rays with arguments $\alpha$ and $\beta$ that land at the same point.

Set $W^*$ to be the collection of all iterated planar $W$-wedges. Then all elements of $W^*$ are by definition generalized wedges. Recall some relevant combinatorial notions introduced in [BOPT16b].

**Definition 3.7** (Semi-laminational set). The map $\sigma_d : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is defined by the formula $\sigma_d(z) = z^d$. Let $G \subset \overline{\mathbb{D}}$ be the convex hull of $G \cap \mathbb{S}^1$ in $\overline{\mathbb{D}}$. Suppose also that $G$ is compact and bounded by chords (called disk edges of $G$) and arcs of $\mathbb{S}^1$; there might exist degenerate disk edges of $G$. Circle holes of $G$ are defined as components of $\mathbb{S}^1 \setminus G$ (in which case they are associated with non-degenerate chords of $G$ in a natural way) or degenerate disk edges of $G$. Say that $G$ is $(\sigma_d)$-semi-laminational if, for every circle hole $(x, y)$ of $G$, either $\sigma_d(x) = \sigma_d(y)$, or the open arc $(\sigma_d(x), \sigma_d(y))$ is a circle hole of $G$. The length of a circle hole of $G$ is measured on $\mathbb{S}^1$ with respect to the usual Lebesgue measure normalized so that the length of $\mathbb{S}^1$ is 1. The length of a disk
edge is defined as the length of the circle hole connecting the same endpoints. A disk edge of length $\geq 1/d$ is said to be major.

**Lemma 3.8** (Lemma 2.12 of [BOPT16b]). A disk edge of a semi-laminational set $G$ is major if and only if the closure of its circle hole contains a fixed point. Any non-degenerate disk edge of $G$ is eventually mapped to a major edge of $G$.

Consider a reduced renormalizable collection $\mathcal{W}$ of wedges and the associated collection $\mathcal{W}^*$. If $\Gamma = \Gamma(\theta^R, \theta^L)$ is an iterated planar $\mathcal{W}$-edge, associate with it the chord $\ell_\Gamma$ connecting $e^{2\pi i \theta_R}$ with $e^{2\pi i \theta_L}$. Observe that if $\Gamma$ is degenerate, then $\theta^R = \theta^L$. Define the set $G_{\mathcal{W}}$ bounded by $S^1$ and the chords $\ell_\Gamma$, where $\Gamma$ runs through all iterated planar $\mathcal{W}$-edges.

**Corollary 3.9.** The set $G_{\mathcal{W}}$ is semi-laminational and has finitely many edges.

**Proof.** Follows from definitions, Lemma 3.6 and Lemma 3.8. □

Some iterated planar $\mathcal{W}$-edges correspond to degenerate chords, i.e., to points of $S^1$. Assume that all chords $\ell_\Gamma$ associated to iterated planar $\mathcal{W}$-edges are distinguished as disk edges of $G_{\mathcal{W}}$ even if they are degenerate. Call $G_{\mathcal{W}}$ the semi-laminational set associated with $\mathcal{W}$. There is a dictionary between $\mathcal{W}^*$ and $G_{\mathcal{W}}$: iterated planar $\mathcal{W}$-edges correspond to disk edges of $G_{\mathcal{W}}$ and iterated planar $\mathcal{W}$-holes correspond to circle holes of $G_{\mathcal{W}}$. An iterated planar $\mathcal{W}$-edge $\Gamma$ and the corresponding iterated planar $\mathcal{W}$-hole are said to be major if $\ell_\Gamma$ is a major disk edge of $G_{\mathcal{W}}$. Thus, every critical iterated planar $\mathcal{W}$-edge is major.

**Proposition 3.10.** If $\mathcal{W}$ is reduced and renormalizable, then any non-degenerate iterated planar $\mathcal{W}$-hole is eventually mapped onto a major iterated planar $\mathcal{W}$-hole. Every major iterated planar $\mathcal{W}$-edge is either eventually critical or eventually periodic.

**Proof.** This follows directly from Lemma 3.8 applied to the associated semi-laminational set $G_{\mathcal{W}}$. □

By Corollary 3.9 the family of the iterated planar non-degenerate $\mathcal{W}$-holes, and the family $\mathcal{W}^*$, are finite. Define a map $\Sigma$ on $\mathcal{W}^*$ as follows: $\Sigma(H) = H'$ for an iterated planar $\mathcal{W}$-hole $H$ if $P(Bd(H)) = Bd(H')$. Then $P(H) = H'$ unless $H$ is a major hole. The map $\sigma_d$ extends to the set of chords of $S^1$: a chord with endpoints $x$ and $y$ is mapped to the chord with endpoints $\sigma_d(x)$ and $\sigma_d(y)$. In this sense, the action of $\sigma_d$ on the edges of $G_{\mathcal{W}}$ corresponds to the action of $\Sigma$ on $\mathcal{W}^*$.

Definition 3.2 naturally extends to collections of generalized wedges. In fact, degenerate wedges of $\mathcal{W}$ do not contribute to the formation of the set $A_P(\mathcal{W})$. 
Lemma 3.11. Let $W$ be reduced and renormalizable. Then $W$ is equivalent to $W^*$.

Proof. Since $W \subset W^*$ (a priori, $W^*$ imposes more restrictions), we have $A_P(W^*) \subset A_P(W)$. To prove the converse, take any $z \in A_P(W)$ and assume that $P^n(z) \in W^* \in W^*$. The point $P^n(z)$ is separated from $A_P(W^*)$ by the $W$-edge $\Gamma^* = \text{Bd}(W^*)$. By Lemma 3.6, the cut $\Gamma^*$ is an edge of $X_P(W)$, hence it also separates $A_P(W)$ from $P^n(z)$. A contradiction with the forward invariance of $A_P(W)$. \qed

A reduced renormalizable collection of generalized wedges $W$ is said to be canonical if $W^* = W$. Lemma 3.11 states that any reduced renormalizable collection of wedges is equivalent to a canonical collection. Thus, when studying $A_P(W)$, we may assume that $W$ is canonical. The advantage of a canonical collection is that it carries a well-defined action of $\Sigma$ on all non-degenerate iterated planar $W$-wedges. Another advantage is terminological: when talking about a canonical collection $W$ of wedges we remove the term “iterated” and simply talk about planar $W$-wedges, planar $W$-edges, disk edges of $G_W$ and circle holes of $G_W$. Observe also that speaking of elements of $W$ we do not say planar $W$-holes, rather planar $W$-wedges (a planar hole of $X_P(W)$ is a more general concept not needed here).

4. Carrots: a general strategy

Let $P$ be a degree $d > 1$ polynomial with connected Julia set. Consider a canonical collection $W$ of generalized wedges in the dynamical plane of $P$ (within this section we always make these assumptions about $P$ and $W$). Recall that some wedges in $W$ may be degenerate. However, by definition the collection $W$ is always finite. Moreover, under the assumptions of the Main Theorem, the collection $W$ is forward-invariant. Our current objective is to roughly sketch the proof of the Main Theorem. We will fill missing details later.

First we define a Jordan neighborhood $V$ of $A_P(W)$ and fix it for the duration of the construction. This allows us to ignore dependence of other objects upon $V$ and omit $V$ from the notation. The choice of $V$ is made by defining its boundary $E_V$. In some cases, $E_V$ can be defined as an equipotential of $P$. This choice of $E_V$ (and $V$) will be referred to later as the standard choice. In some other cases, the definition of $E_V$ will be more involved. All other choices, modifications etc will be made inside $V$. The choice of $V$ is far from unique.

Next we define, for every $W \in W$, a pair of simple arcs $R_W$ and $L_W$ connecting $E_V$ with the root point of $W$. Let $\Gamma(\theta^R, \theta^L)$ be the
boundary of the wedge $W$. In the simplest cases, $R_W$ and $L_W$ will be defined as
\[ R_W = R_P(\theta^R) \cap V, \quad L_W = R_P(\theta^L) \cap V. \]
This choice of $R_W$ and $L_W$ will be referred to later as the \textit{standard choice}. In other cases, the definition of $R_W$ and $L_W$ is more involved. In any case, the arcs $R_W$ and $L_W$, once chosen, will stay the same for the duration of the construction.

Desired properties of $R_W$ and $L_W$ are stated below. Before being able to establish these properties, we will give a precise definition of $R_W$ and $L_W$. However, once the properties of $R_W$ and $L_W$ are established, we will rely upon these properties rather than the definitions.

\textbf{Property 4.1.} The curves $R_W$ and $L_W$ connect $E_V$ with the root point $a$ of $W$ and satisfy the following:

1. they lie in $V \setminus K_P$ except the endpoints, and $R_W \cap L_W = \{a\}$;
2. they are smooth except possibly at the common landing point $a$;
3. they are transverse to $E_V$;
4. for every wedge $W \in \mathcal{W}$, we have $P(R_W) \cap \overline{\mathcal{V}} = R_{\Sigma(W)}$ and $P(L_W) \cap \overline{\mathcal{V}} = L_{\Sigma(W)}$.

The next step is to define another Jordan disk $U \Subset V$ bounded by a Jordan curve $E_U \subset V$. Desired properties of $E_U$ will be stated below.

\textbf{Definition 4.2 (Carrots).} For a wedge $W \in \mathcal{W}$ the carrot $C_W(V)$ is defined as the bounded “triangular” region bounded by $R_W$, $L_W$ and the corresponding arc of $E_V$. Carrots $C_W(U)$ are defined similarly, with $E_U$ instead of $E_V$ (i.e., the choice of $E_U$ must guarantee that the carrots $C_W(U)$ are well defined). By definition, carrots are open Jordan disks.

We will need some properties of carrots. Formally, these are additional properties of the curves $R_W$, $L_W$, $E_V$ and $E_U$ listed below.

\textbf{Property 4.3.} Distinct carrots $C_W(V)$ are disjoint. The sets $K_P \cap W$ and $K_P \cap C_W(V)$ coincide in a neighborhood of the root point of $W$.

It follows from Property 4.3 that carrots are disjoint from $A_P(W)$. Indeed, $K_P \cap C_W(V)$ coincides with $K_P \cap W$ in some small neighborhood of $a$, the root point of $W$. Since there are no points of $A_P(W)$ in $W$, there are no points of $A_P(W)$ in $C_W(V)$ (recall that $A_P(W) \subset K_P$).

We will then define a certain modification $P^c$ of $P$, which will be called a \textit{carrot modification}. The desired properties of $P^c$ are listed below and are sufficient for the proof of the Main Theorem. They will be deduced from the rigorous definition of $P^c$.

\textbf{Property 4.4.} The map $P^c: U \to V$ is continuous. It satisfies the following.
(1) We have \( P^c(z) = P(z) \) unless \( z \in C_W(U) \), where \( W \in \mathcal{W} \).

(2) For every \( W \in \mathcal{W} \), the map \( P^c : C_W(U) \to C_{\Sigma(W)}(V) \) is a quasi-conformal homeomorphism.

(3) If \( W \) is \( \Sigma \)-periodic with minimal period \( m \), then, for every \( z \in C_W(U) \) there exists \( k \geq 0 \) such that \( (P^c)^mk(z) \) is either undefined or outside \( U \).

Part (3) of Property [4.4] is called the escaping property. Now, using Properties [4.1], [4.3], and [4.4], we can prove that \( P^c \) is APL. In order to lighten the notation, we set \( f = P^c \).

**Theorem 4.5.** If \( f \) has Properties [4.1], [4.3], and [4.4], then \( f \) is APL. Moreover, we have \( K(f) = A_P(W) \).

**Proof.** By construction, \( f \) is well defined and continuous.

To prove that the map \( f \) is quasi-regular, we need to show that \( ||d_x f||^2 \leq \kappa \text{Jac}_f(x) \), where \( \text{Jac}_f(x) \) is the Jacobian of \( f \), and \( \kappa \geq 1 \) is some constant. (The inequality should hold almost everywhere.) This inequality holds at all points outside of carrots \( C_W(U) \). Indeed, the map \( f = P \) is holomorphic there, and we may even take \( \kappa = 1 \). On the other hand, consider a wedge \( W \in \mathcal{W} \). The inequality also holds in \( C_W(U) \) since the map \( f \) is quasi-conformal there by Property [4.4]. Since the boundaries of \( C_W(U) \) for \( W \in \mathcal{W} \) have measure zero, we conclude that the desired inequality holds almost everywhere with suitable \( \kappa \).

Let us prove that \( f \) is proper. Since \( f \) extends continuously to the boundary of \( U \), it suffices to prove that \( f(E_U) \subset E_V \). However, this is true by construction. Clearly, \( A_P(W) \subset K(f) \), and \( f = P \) on \( A_P(W) \), hence \( \partial f = 0 \) on \( A_P(W) \). It remains to show that \( K(f) \subset A_P(W) \). By way of contradiction, we assume that \( z \in K(f) \setminus A_P(W) \). Then there exists the smallest non-negative integer \( i \) such that \( f^i(z) \in C_W(U) \), where \( W \) is \( \Sigma \)-periodic. Let \( m \) be the minimal period of \( W \) under \( \Sigma \). Then, by item (3) of Property [4.4], we have \( f^{i+mj}(z) \notin U \) for some \( j \). A contradiction with our assumption that \( z \in K(f) \). \( \square \)

To compute the degree of \( f \), we first prove the following lemma.

**Lemma 4.6.** The map \( P : A_P(W) \to A_P(W) \) is almost \( k \)-to-1, for some \( k \).

**Proof.** Set \( Y = A_P(W) \) and define \( S_Y \) as the union of the set of all critical values of \( P \) in \( A_P(W) \) and the set of all root points of \( W \). We claim that \( P : Y \setminus P^{-1}(S_Y) \to Y \setminus S_Y \) is a covering of degree \( k \), for some integer \( k > 0 \). Clearly, this map is proper. To prove that it is a local homeomorphism, take a point \( y \in Y \setminus P^{-1}(S_Y) \). There is an open disk \( D \) around \( y \) such that \( P : D \to P(D) \) is a homeomorphism. Indeed,
by our assumption, $y$ is not a critical point of $P$; otherwise, $P(y) \in S_Y$ is a critical value. We need to show that $P : D \cap Y \to P(D) \cap Y$ is also a homeomorphism, i.e., it is surjective. Otherwise there are points of $D \setminus Y$ mapped to $Y$. By definition of $A_P(W)$, then $D$ must intersect some wedge from $W$. Since $D$ can be chosen arbitrarily small, it follows that $y$ is a root point of $W$. This is again a contradiction with the definition of $S_Y$ and our assumption that $y \notin P^{-1}(S_Y)$. □

The following general definition applies to any almost $k$-to-1 map.

**Definition 4.7 (The $Y$-multiplicity).** If $Y \subset K_P$ is a continuum such that $P : Y \to Y$ is almost $k$-to-1 then, for every $y \in Y$, there exists an integer $\mu_Y(y)$ with the following property. For all $z \neq P(y)$ sufficiently close to $P(y)$, exactly $\mu_Y(y)$ points of $P^{-1}(z) \cap Y$ are near $y$. The function $\mu_Y : Y \to \mathbb{Z}_{\geq 1}$ thus defined is called the $Y$-multiplicity. Let $S_Y \subset Y$ be a finite subset such that $P : Y \setminus P^{-1}(S_Y) \to Y \setminus S_Y$ is a degree $k$ covering. Then $\mu_Y(y) = 1$ for all $y \notin P^{-1}(S_Y)$. Moreover, irregular points of $Y$ are precisely the points $y \in Y$ with $\mu_Y(y) > 1$. The singular values of $Y$ are then the $P$-images of the irregular points.

**Lemma 4.8.** In a small neighborhood of a point $y \in A_P(W)$, the map $f$ is a branched covering of degree $\mu_{A_P(W)}(y)$. The degree of $f$ equals

$$\deg(f) = 1 + \sum_{a \in A_P(W)} \mu_{A_P(W)}(a) - 1.$$ 

The sum in the right hand side should be interpreted as the sum of the finitely many terms that are different from 0.

**Proof.** The second statement of the lemma follows from the first one and the Riemann–Hurwitz formula. Let us prove the first statement.

Suppose that a critical point $a \in A_P(W)$ of $P$ is a root point of one or several wedges from $W$. Then the desired claim follows from the observation that a point $z \in A_P(W)$ near $P(a)$ does not have $f$-preimages in carrots attached to $a$. Indeed, by Property 4.4 every carrot is mapped to another carrot, and carrots are disjoint from $A_P(W)$. □

**Corollary 4.9.** If $W$ is nonempty, then we have $\deg(f) < \deg(P)$.

**Proof.** Set $Y = A_P(W)$. If $W$ is nonempty, then there is at least one major wedge in $W$. Let $a$ be the root point of $W$. By Lemma 4.8 it is enough to show that $\mu_Y(a) < \mu(a)$, where $\mu(a)$ is the usual local degree of $P$ at $a$ (that is, $P$ is a branched covering of degree $\mu(a)$ in a small neighborhood of $a$). Note that a point $z \in A_P(W)$ close to $P(a)$ has at least one preimage in $W$. This preimage is not accounted for
in $\mu_Y(a)$ since we count only the preimages in $A_P(\mathcal{W})$. The inequality $\mu_Y(a) < \mu(a)$ follows. □

We can now complete the proof of the Main Theorem modulo Properties 4.1, 4.3, and 4.4.

**Proof of the Main Theorem modulo Propositions 4.1, 4.3, and 4.4.** By Theorem 4.5, the map $f = P^c : U \to V$ is APL. By Corollary 2.14, the map $f$ is hybrid equivalent to a PL restriction of some polynomial $Q$. In particular, $f|_{AP(\mathcal{W})} = P|_{AP(\mathcal{W})}$ is topologically conjugate to $Q|_{K_Q}$. The degree of $Q$ is equal to $\deg(f) < \deg(P)$ by Corollary 4.9. □

We conclude this section with a remark on our subsequent construction of $P^c$. Note that $P^c$ can be defined independently on each $\Sigma$-orbit of wedges from $\mathcal{W}$. For this reason, we may assume for simplicity that there is only one $\Sigma$-orbit. The construction of carrots and of the corresponding carrot extension $P^c$ will depend on what this orbit looks like.

We classify finite $\Sigma$-orbits $\mathcal{O}$ in $\mathcal{W}$ into the following types:

**Regular critical type:** There are no periodic major wedges in the given $\Sigma$-orbit $\mathcal{O}$. Then, under the assumptions of the Main Theorem, all periodic wedges are degenerate; they coincide with external rays of $P$ landing at repelling periodic points.

**Hyperbolic type:** Root points of periodic major wedges in $\mathcal{O}$ are repelling periodic points of $P$.

**Parabolic type:** Root points of periodic major wedges in $\mathcal{O}$ are parabolic periodic points of $P$.

Throughout Sections 5 – 7 we make the following assumptions:

1. the polynomial $P$ has degree $d > 1$ and connected Julia set;
2. a canonical collection $\mathcal{W}$ of generalized wedges is fixed for $P$;
3. the collection $\mathcal{W}$ is just one $\Sigma$-orbit.

Recall that $\mathcal{W}$ being canonical means in particular that it is reduced and renormalizable. Assumption (3) may not be always the case but, as noted above, can be safely made without modifying the arguments.

## 5. Hyperbolic Type

In this section, we assume that $\mathcal{W}$ is of hyperbolic type.

### 5.1. Periodic wedges

The argument is standard, thus, we only provide a sketch. Suppose that all $\Sigma$-periodic wedges in $\mathcal{W}$ are $W_0, \ldots, W_{m-1}$. The numbering can be arranged so that $\Sigma(W_j) = W_{j+1} \ (\text{mod} \ m)$ for $j = 0, \ldots, m-1$. Since $\mathcal{W}$ is of hyperbolic type, the root points $a_j$ of $W_j$ are repelling periodic points. Let $p$ be the minimal period of $a_0$. This number divides $m$ but a priori may be smaller.
Let $\Gamma(\theta^R_j, \theta^L_j)$ be the cut corresponding to $W_j$, i.e., the boundary of $W_j$. Then both rays $R_P(\theta^R_j)$ and $R_P(\theta^L_j)$ land at $a_j$. Choose a tight equipotential $E_0$ that intersects $R_P(\theta^R_j)$ and $R_P(\theta^L_j)$ at points $r_j \in R_P(\theta^R_j)$ and $l_j \in R_P(\theta^L_j)$ close to $a_j$, $j = 0, 1, \ldots, m - 1$. Let $T_j = W_j \cap E_0$. Let $R_{W_j} = R_j$ be a subarc of $R_P(\theta^R_j)$ from $r_j$ to $a_j$ and let $L_{W_j} = L_j$ be a subarc of $R_P(\theta^L_j)$ from $l_j$ to $a_j$. Clearly $R_j \cup L_j \cup T_j$ is a smooth curve that encloses a Jordan disk $Q_j$. Connect $r_j$ and $l_j$ by a smooth curve $Y_j = Y_{W_j} \subset Q_j$ transversal to both $R_j$ and $L_j$.

Since $a_j$ is repelling, a standard argument allows one to arrange for those curves to be such that $P(Y_j) \subset W_{j+1}$ is separated from $a_{j+1}$ in $Q_{j+1}$ by $Y_{j+1}$ (i.e., $P$ repels $Y_j$’s farther and farther away from $a_j$). The Jordan curve $E_V$ that consists of pieces of $E_0$ outside the wedges from $W$ union all the curves $Y_j$ inside the wedges from $W$ encloses a Jordan disk $V$. By construction, $P(E_V) \cap E_V = \emptyset$. Evidently, there exists a pullback $U$ of $V$ containing $A_P(W)$. If $E_U$ is the boundary of $U$ then $P(E_U) \cap E_U = \emptyset$ as otherwise we can apply $P$ and get that $P(E_V) \cap E_V \neq 0$, a contradiction. This defines carrots $C_W(V)$. Properties 4.1 and 1.3 are immediate for the choices made above. We set $P^c = P$. All required properties of the carrot extension are obviously satisfied.

Note that the step just made is enough to prove Theorem 1.2.

5.2. Carrots are quasi-disks. Let $W \in \mathcal{W}$ be a $\Sigma$-periodic wedge. As above, we write $m$ for the minimal period of $W$ and $a$ for the root point of $W$. Recall that $Y_W$ is a segment of $E_V$ between $E_V \cap R_W$ and $E_V \cap L_W$. We will need the following geometric property of carrots.

**Proposition 5.1.** Let $W$ be a $\Sigma$-periodic wedge as above. The curve $R_W \cup L_W \cup Y_W = Z_W$ is a quasi-circle, and $C_W(V)$ is a quasi-disk.

**Proof.** By our construction, it is enough to prove that $R_W \cup L_W$ is a quasi-arc locally near $a$. Consider a local holomorphic coordinate $u$ near $a$ such that $u = 0$ at $a$, and $P^m$ takes the form $u \mapsto \lambda u$. Here $\lambda$ is the derivative of $P^m$ at $a$, hence $|\lambda| > 1$. A local coordinate $u$ with the properties stated above exists by the classical Königs linearization theorem. Set $R', L'$ to be the images of $R, L$ in the $u$-plane and apply Theorem 2.18 to $R', L'$. Since a holomorphic local coordinate change takes quasi-arcs to quasi-arcs, we obtain the desired. 

5.3. Strictly preperiodic wedges. Let $W \in \mathcal{W}$ be a strictly preperiodic wedge. By induction, we can assume that the following objects are already defined: the carrots and the carrot extension and $\Sigma(W)$ for all $i > 0$ (e.g., $L_{\Sigma_i(W)}$ and $R_{\Sigma_i(W)}$), and the disks $U$ and $V$. Let us now define $R_W$ and $L_W$. By definition, these are the pullbacks of
$R_{\Sigma(W)}$ and $L_{\Sigma(W)}$, respectively, that separate $K_P \cap W$ from $K_P \setminus W$ in $V$. This property defines $R_W$ and $L_W$ uniquely. By induction, assume that Property 4.1 holds for $\Sigma^i(W)$ for all $i > 0$. Then it is clear that it also holds for $W$. Property 4.3 is also immediate from the definition. Thus the carrot $C_W(U)$ is well defined. It remains to modify $P$ in $C_W(U)$ to satisfy Property 4.4. In fact, since $W$ is not periodic, it is enough to satisfy properties (1) and (2) from Property 4.4.

If $W$ is not major, set $P^c = P$ on $W$, and the corresponding claim of Property 4.4 is immediate. We now assume that $W$ is a major wedge. 

Lemma 5.2. Let $W$ be a major strictly preperiodic wedge as above. Then $R_W$ and $L_W$ are transverse at the root point of $W$.

Proof. Let $a$ be the root point of $W$. We know that $R_{\Sigma(W)}$ and $L_{\Sigma(W)}$ are transverse at $P(a)$ since $C_{\Sigma(W)}(V)$ is a quasi-circle and by Lemma 2.15. Choosing a suitable local coordinate $z$ near $a$, we may assume that $a = 0$ and $P$ is given by $z \mapsto z^k$. For a pair of sequences $u_n, v_n \to 0$, assume by way of contradiction that $u_n/v_n \to 1$. Then $u_n^k/v_n^k \to 1$, contradicting the transversality of $R_{\Sigma(W)}$ and $L_{\Sigma(W)}$.

We now want to apply Lemma 2.17. Set $R = R_W$ and $L = L_W$. Suppose that $p > 0$ is the smallest integer with $W' = \Sigma^p(W)$ periodic. As above, $m$ denotes the minimal $\Sigma$-period of $W'$. Choose a local holomorphic coordinate $u$ near the root point of $W'$ so that $P^m$ is given by $u \mapsto \lambda u$. Here $\lambda$ is the derivative of $P^m$ at the root point of $W'$; it satisfies $|\lambda| > 1$. Making $R$ and $L$ smaller if necessary, assume that the $u$-coordinate is defined on $R' = P^p(R)$ and $L' = P^p(L)$. There is a local coordinate $z$ near $R \cap L$ in which the pullback of $u$ under $P^p$ takes the form $z \mapsto z^k$. Lemma 2.17 is now applicable; it implies that $R \cup L$ is a quasi-arc, hence $C_W(U)$ is a quasi-disk. It also implies that the map $P : C_W(U) \to C_{\Sigma(W)}(V)$ is QS. Now, by Theorem 2.7, there is a continuous map $f : C_W(U) \to C_{\Sigma(W)}(V)$ that coincides with $P$ on the boundary of $C_W(U)$ and that is quasi-conformal in $C_W(U)$. Parts (1) and (2) of Property 4.4 now follow for $f$.

6. Regular critical type

In this section, we consider the case when $W$ is of regular critical type. We adopt the assumptions of the Main Theorem. Then $\Sigma$-periodic wedges from $W$ are periodic external rays landing at repelling periodic points of $P$.

We first consider periodic wedges from $W$. Fix one such degenerate wedge $W$, which in fact coincides with an external ray $R_P(\theta_0)$ landing at $a$. Here $a$ is a repelling periodic point of $P$ of minimal period $m$. 


Definition 6.1 (Prototype carrot). The prototype carrot (of level \( \rho_0 \)), or simply proto-carrot, is the “triangular” region in \( \mathbb{D} \) given by the inequalities \( \rho_0 \leq \rho \leq e^{-|\theta|} \) in the polar coordinates \((\theta, \rho)\). Here \( \theta \) is the angular coordinate that can be either positive or negative, and \( \rho \) is the radial coordinate, i.e., the distance to the origin. We assume that the parameter \( \rho_0 < 1 \) is close to 1. A proto-carrot is bounded by a circle arc and two symmetric segments of logarithmic spirals, see Figure 2.

All proto-carrots are homeomorphic. For a fixed angle \( \theta \), define the set \( C(\theta) \), the proto-carrot of argument \( \theta \), as \( \{e^{2\pi i \theta} z \mid z \in \mathbb{C}\} \); the proto-carrot \( C(\theta) \) is obtained if we rotate \( C \) by angle \( \theta \) (i.e., \( 2\pi \theta \) radians). We suppress dependence on \( \rho_0 \) in our notation. The point \( e^{2\pi i \theta} \) is called the root point of a proto-carrot \( C(\theta) \).

A part of the boundary of \( C \) near point 1 (at that point \( \theta = 0, \rho = 1 \)) is given by \( \rho = e^{-|\theta|} \) and consists of two analytic curves meeting at 1 and invariant under the map \( z \mapsto z^d \) (regardless of \( d \)). Since rotation by \( \theta \) composed with \( z^d \) equals \( z^d \) composed with the rotation by \( d \cdot \theta \), the next proposition follows.

Proposition 6.2. Take any \( \theta_0 \in \mathbb{R}/\mathbb{Z} \). The map \( z \mapsto z^d \) takes a neighborhood of \( e^{2\pi i \theta_0} \) in \( C(\theta_0) \) to a neighborhood of \( e^{2\pi i d \theta_0} \) in \( C(d \theta_0) \).

Let us recall the following concept.

Definition 6.3 (Stolz angle). A Stolz angle in \( \mathbb{D} \) at a point \( u \in \mathbb{S}^1 \) is by definition a convex cone with apex at \( u \) bisected by the radius and with aperture strictly less than \( \pi \).

The proto-carrot \( C(\theta) \) approaches the unit circle within some Stolz angle at \( e^{2\pi i \theta} \) (the Implicit Function Theorem shows that the aperture
of such Stolz angles can be made arbitrarily close to $\pi/2$). The following theorem proved in [CG92, Theorem 2.2 on page 8] describes an important property of Stolz angles.

**Theorem 6.4.** Consider a simply connected domain $D \subset \mathbb{C}$ that is not the sphere minus a singleton, and let $\psi : \mathbb{D} \to D$ be a conformal isomorphism. Suppose that a point $z_0 \in \text{Bd}(D)$ is accessible from $D$. Then there is a point $u_0 \in \mathbb{S}^1$ with the following property: $\psi(u) \to z_0$ as $u \to u_0$ inside any Stolz angle with apex at $u_0$.

We now proceed with the construction of carrots and the carrot extension in our case. Let $E_V$ be a tight equipotential curve around $K_P$ that encloses the Jordan domain $V$. Recall that $\psi_P : \mathbb{D} \to \mathbb{C} \setminus K_P$ is the uniformization defined in the Introduction. It conjugates $z \mapsto z^d$ with the restriction of $P$ to the basin of infinity. Then $E_V$ is the image under $\psi_P$ of a circle whose center is 0 and whose radius is $\rho_0 \in (0, 1)$. The equipotential $E_V$ being tight means that $\rho_0$ is close to 1. Set $U = P^{-1}(V)$; then $E_U = P^{-1}(E_V)$ is also an equipotential. Consider a degenerate wedge $W = R_P(\theta_0)$ from $\mathcal{W}$. Take the $\psi_P$-images of the two edges of $C(\theta)$ landing at $e^{2\pi i \theta_0}$. Define $R_W$ and $L_W$ as the closures of these images. It follows from Theorem 6.4 that $R_W$ and $L_W$ are simple arcs landing at $a$, the root point of $W$. As soon as we defined $E_V$, $E_U$, $R_W$, and $L_W$, we can talk about the carrots $C_W(V)$ and $C_W(U)$. We now need to verify Properties 4.1 and 4.3 of carrots.

The main property we need to verify is that sets $C_W(V)$ are pairwise disjoint. Other parts of Properties 4.1 and 4.3 are immediate from the definition. To establish the disjointness property, we choose $\rho_0$ very close to 1. It suffices to prove that the proto-carrots $C(d^n \theta_0)$ are all disjoint. Indeed, the sequence $d^n \theta_0 \in \mathbb{R}/\mathbb{Z}$ is periodic, thus it corresponds to finitely many points of $\mathbb{S}^1$. If $\rho_0$ is close to 1, then the corresponding proto-carrots are in small pairwise disjoint neighborhoods of their root points. Clearly, the proto-carrots are disjoint.

Recall that, by our assumption, $W$ is a periodic degenerate wedge with a repelling root point $a$. Set $f = P$ on $C_W(U)$. With this choice, Property 4.4 clearly holds. (Since, by definition, $C_W(U)$ lies in the basin of infinity except the root point, the escaping property follows).

Finally, we define the carrots and the carrot modification for strictly preperiodic wedges from $\mathcal{W}$ similarly to how it was done in Section 5.3.

7. **Parabolic type**

In this section, we consider the case where $W$ is of parabolic type.
7.1. **Periodic wedges.** We first consider a parabolic periodic point $a_0$ of $P$. Let $m$ be the smallest positive integer with $P^m(a_0) = a_0$ and $(P^m)'(a_0) = 1$. Near $a_0$, we have $P^m(z) = z + \eta(z-a_0)^{m+1} + \ldots$, where dots denote higher order terms as $z \to a_0$. Clearly, $m$ is the same for all points in the cycle of $a_0$.

Recall that a complex number (=vector in the plane) $v$ is **repelling** (for the germ of $P^m$ at $a_0$) if $v$ and $\eta v^{m+1}$ have the same direction. This means that $\eta v^m > 0$. Similarly, $v$ is said to be **attracting** if $v$ and $\eta v^{m+1}$ have opposite directions, i.e., $\eta v^m < 0$. Repelling vectors form $m$ straight rays emanating from 0 so that the angles between consecutive rays are equal. Similarly, attracting vectors form $m$ rays alternating with rays of repelling vectors. Set $\zeta = e^{2\pi i/m}$; two consecutive repelling (or attracting) directions are represented by vectors $v$ and $\zeta v$. External rays of $P$ landing at $a_0$ are tangent to some repelling vectors.

The general strategy. In the parabolic case, we change (for convenience) the usual order of definitions. Before defining the carrots, we define some smaller sets called pseudo-carrots. Then we modify $P$ in pseudo-carrots; let $P^c$ be the modified map. After that, we define the curve $E_V$ and, for each periodic wedge $W \in W$, the curves $R_W$ and $L_W$. The curve $E_U$ is, as always, the pullback of $E_V$ under $P^c$ (rather than under $P$!). The carrots $C_W(U)$, $C_W(V)$ are then defined in the usual way. We establish Properties 4.1 and 4.3 and show that pseudo-carrots are indeed contained in carrots. The map $P^c$ differs from $P$ only in pseudo-carrots, hence only in carrots. Finally, we verify, for $P^c$, Property 4.4.

**Pseudo-carrots.** Let us now define pseudo-carrots. By our assumptions, every periodic wedge in $W$ includes a periodic parabolic domain associated with the root point, and all such parabolic domains form one cycle $\Omega_0, \ldots, \Omega_{m-1}$ so that $P(\Omega_j) = \Omega_{j+1}$ (mod $m$). Write $a_j$ for the parabolic point to which all points of $\Omega_j$ converge under iterates of $P^m$. Let $W_j \in W$ be the wedge containing $\Omega_j$; these wedges are different since $W$ is reduced. Let $T_m$ be a small open connected neighborhood of $a_0$ in $\operatorname{Bd}(\Omega_0)$. Then $P^m$ is injective on $T_m$. Moreover, the $P^m$-pullback of $T_m$ containing $a_0$ is compactly contained in $T_m$. For $j \in 0, m-1$, we consider the pullback $\tilde{T}_j$ of $T_{j+1}$ containing $a_j$. Define $T_j$ as a sufficiently tight neighborhood of $\tilde{T}_j$ in $\operatorname{Bd}(\Omega_j)$. Then $\tilde{T}_j \subset T_j$ are two neighborhoods of $a_j$ in $\operatorname{Bd}(\Omega_j)$. We also have $T_0 \subset T_m$.

Any parabolic domain $\Omega_j$ is conformally isomorphic to the upper half-plane $\mathbb{H}$. A conformal isomorphism can be extended to a homeomorphism between the closures. Indeed, the boundary of $\Omega_j$ is a Jordan
Figure 3. The uniformization of a parabolic domain \( \Omega = \Omega_0 \) of period 1. The action of the map \( \Pi = \Pi_0 \) on the set \( \text{Re}(z) \in (-1, 1), \text{Im}(z) > 0 \) is shown on the left. The right figure shows the modification \( \Pi^c \), for which 0 repels nearby points.

curve [RY08]. Let \( \xi_j : \overline{\Omega}_j \rightarrow \overline{\mathbb{H}} \) denote a homeomorphism that is holomorphic on \( \Omega_j \). Any triple of different points in \( \mathbb{R} \) can be mapped to any other such triple by the action of \( \text{PSL}(2, \mathbb{R}) \). Recall also that the action of \( \text{PSL}(2, \mathbb{R}) \) extends to \( \mathbb{H} \) as the orientation preserving isometry group. Therefore, post-composing \( \xi_j \) with a suitable element of \( \text{PSL}(2, \mathbb{R}) \), we can arrange that \( \xi_j(a_j) = 0 \) and \( \xi_j(\tilde{T}_j) = (-1, 1) \).

Define the maps \( \Pi_j : \mathbb{H} \rightarrow \mathbb{H} \) as \( \xi_j^{j+1} \pmod{m} \circ P \circ \xi_j^{-1} \). By the Schwartz reflection principle, it follows that \( \Pi_j \) extends to a rational self-map of \( \overline{\mathbb{C}} \). This map is injective on \((-1, 1)\) although globally the degree of this map is the same as that of \( P|_{\Omega_j} \). Note that \( \Pi_j(-1, 1) \supset (-1, 1) \) for all \( j \in \mathbb{N} \). Since \( \Pi_j(0) = 0 \), we can set \( \Pi_j(z) = z \chi_j(z) \), where \( \chi_j \) is a rational function with \( \chi_j(0) \neq 0 \).

Define the new map \( \Pi^c_j \) as follows:

\[
\Pi^c_j(z) = z(2\theta \chi_j(-\rho) + (1 - 2\theta) \chi_j(\rho)), \quad z = \rho e^{2\pi i \theta}
\]

Here \( \theta \) and \( \rho \) are polar coordinates; \( \theta \) takes values in \([0, 1/2] \). It is easy to see that \( \Pi^c_j = \Pi_j \) on \( \mathbb{R} \). We also have \( \Pi^c_j(\mathbb{D} \cap \mathbb{H}) \supset \mathbb{D} \cap \mathbb{H} \). Finally, \( \Pi^c_j \) is quasi-symmetric on \( \mathbb{D} \cap \mathbb{H} \). Indeed, \( \Pi^c_j(z) = \chi_j(0)z + o(z) \) near 0, and \( \Pi^c_j \) is smooth at all other points. The extension \( \Pi^c_j \) is illustrated in Figure 3. Set \( \mathcal{C}^c j = \xi_j^{-1}(\mathbb{D} \cap \mathbb{H}) \). These are pseudo-carrots.

We also need much smaller sets of the same kind. Let \( \mathbb{D}(r) = r \mathbb{D} \) stand for the disk of radius \( r \) around the origin. Set \( \mathcal{C}^c_j(r) = \xi_j^{-1}(\mathbb{D}(r) \cap \mathbb{H}) \). To lighten the notation, we now write \( j+1 \pmod{m} \) meaning \( j+1 \) (mod \( m \)). The set \( \overline{\mathcal{C}^c_{j+1}(r)} \cap \text{Bd}(\Omega_{j+1}) \) is always a strictly smaller segment of \( \text{Bd}(\Omega_{j+1}) \) than \( P(\overline{\mathcal{C}^c_{j}(r)}) \cap \text{Bd}(\Omega_{j+1}) \), for every \( r \leq 1 \). If \( \varepsilon > 0 \) is sufficiently small, then additionally the following properties hold.

1. We have \( \mathcal{C}^c_{j+1}(\varepsilon) \subset P(\mathcal{C}^c_j) \) for each \( j \in \mathbb{N} \).
2. The set \( P^m(\mathcal{C}^c_j(\varepsilon)) \) is a subset of \( \mathcal{C}^c_j \).
The map $P^c$. Take $\varepsilon > 0$ sufficiently small. We set $P^c = \xi_j^{-1} \pmod{m} \circ \Pi_j \circ \xi_j$ on $C_{j}^c(\varepsilon)$ and $P^c = P$ outside of $C_{j}^c$. Since $\Pi_j$ coincides with $\Pi_j$ on $(-1, 1)$, the map $P^c$ coincides with $P$ on $C_{j}^c(\varepsilon) \cap \text{Bd}(\Omega_j)$. Also, $P^c$ is quasi-conformal on $C_{j}^c(\varepsilon)$ since $\Pi_j$ is quasi-conformal, and $\xi_j$, $\xi_j+1$ are conformal. It remains to define $P^c$ on $C_{j}^c \setminus C_{j}^c(\varepsilon)$. The map is already defined on the boundary of this set (on $\text{Bd}(C_{j}^c) \setminus \text{Bd}(\Omega_j)$ the map $P^c$ must coincide with $P$); we extend it inside as a quasi-conformal homeomorphism. The following property holds.

Lemma 7.1. The root points $a_j$ are repelling periodic points for $P^c$.

Proof. It suffices to prove this property for $a_0$. There are several wedges $W_j \in \mathcal{W}$ having $a_0$ as the root point. Each wedge contains a pseudo-carrot $C_{j}^c$. There is a Jordan disk $D$ around $a_0$ such that $D \cap \Omega_j = C_{j}^c(\varepsilon)$. Let $p$ be the minimal period of $a_0$; a priori it can be smaller than $m$. Note that $P^p(C_{j}^c(\varepsilon)) \subset C_{j}^c$ for some $j' = j + p \pmod{m}$ not necessarily equal to $j$. It follows that the $P^p$-image of the arc $I_j^c = \text{Bd}(C_{j}^c(\varepsilon)) \cap \overline{\Omega}_j$ is further away from $a_0$ in $\overline{\Omega}_{j'}$ than $I_j^c$. Since $P$ is repelling outside of parabolic domains at $a_0$, we may arrange that $D \subset P^p(D)$ (possibly modifying $D$ outside of all $\Omega_j$). It is now easy to see that all points of $D$ converge to $a_0$ under backward iteration of $P^p$. 

The situation is now almost reduced to the case considered in Section 5.1. We only need to verify that our pseudo-carrots lie in carrots.

Consider the orbit of $a_0$. The minimal period $p$ of $a_0$ divides $m$ but can be smaller. Thus the cycle of $a_0$ consists of points $a_0, \ldots, a_{p-1}$. Similarly to the hyperbolic case and starting with the disk $D$ from the proof of Lemma 7.1, we can now construct disks $D_i$ around $a_i$, for every $i \in 0, p-1$ such that $D_{i+1} \pmod{p} \subset P(D_i)$ for all $i \in 0, p-1$. We can also arrange that the escaping property holds in $D_i$.

We now proceed as in Section 5.1 with $P^c$ instead of $P$. In this way we define $E_V$, $E_U$, $R_{W_j}$, $L_{W_j}$ and $C_j(U)$. Let us once again stress however that $E_V$ is defined as the $P^c$-pullback of $E_V$ rather than the $P$-pullback. It follows from our construction that $C_j^c \subset C_j(U)$. Therefore, the modification $P^c$ is different from $P$ only in the carrots $C_j(U)$. Properties 4.1, 4.3 and 4.4 are verified as in Section 5.1.

7.2. Strictly preperiodic wedges. We now consider the case of a strictly preperiodic wedge $W \in \mathcal{W}$. In our construction, the curves $R_W$ and $L_W$ are defined as suitable segments of the external rays on the boundary of $W$, together with the landing point. The curves $E_V$ and $E_U$ encircling $A_{\scriptscriptstyle W}(\mathcal{W})$ were already defined. Therefore, we can talk about carrots $C_{\scriptscriptstyle W}(U)$ and $C_{\scriptscriptstyle \Sigma(W)}(V)$. We only need to define a
new quasi-conformal map from $C_W(U)$ to $C_{\Sigma(W)}(V)$ so that it coincides with $P$ on the boundary. The problem that we face here in contrast to other cases is that $C_{\Sigma(W)}(V)$ may fail to be a quasi-disk.

On the other hand, we can use what we know about geometry of parabolic domains. We know that the boundary of a parabolic domain is tangent at the parabolic point to certain repelling vectors, not necessarily different. If the boundary of $\Omega_j$ is tangent to only one repelling direction and forms a cusp like that of a cardioid, then $\Omega_j$ and $W_j \cap V$ fail to be quasi-disks, which is problematic. Consider now some periodic wedge $W_j$. Then there are two smooth arcs in $W_j$ landing at $a_j$ and forming a nonzero angle at $a_j$ (in the sense that there are well-defined tangent lines to these arcs at $a_j$, and these tangent lines are distinct). These arcs may approach $a_j$ inside $\Omega_j$; we may even assume that they approach $a_j$ as straight line segments. We will write $R_j^m$ and $L_j^m$ for these curves. We may assume that $R_j^m$, $L_j^m$ are disjoint and form, together with the curve $E_V$, some curved triangle $C_j^m(V)$ that is a quasi-disk. Now suppose that $W$ is strictly preperiodic but $\Sigma(W) = W_j$. Let $R_j^m$, $L_j^m$ be pullbacks of $R_j^m$, $L_j^m$ chosen so that all pullbacks of $C_j^m$ in $W$ are between $R_j^m$ and $L_j^m$. Then the curves $R_j^m$, $L_j^m$ and $E_U$ form a curved triangle $C_j^m(U)$. The triangle $C_j^m(U)$ is clearly a quasi-disk as $R_j^m$, $L_j^m$ also meet at a nonzero angle. By Theorem 2.7, we can modify $P$ in $C_j^m(U)$ so that the modification is continuous, and its restriction to $C_j^m(U)$ is quasi-conformal. This will be the carrot modification for $W$.

We defined carrots and the carrot modification for strictly preperiodic $W \in W$ that are mapped to periodic wedges under $P$. By induction, it is straightforward to extend this definition to all strictly preperiodic wedges $W \in W$. Since $C_j^m(U) \subset C_W(U)$, all properties of carrot modifications are satisfied.

8. Proof of Theorem 1.7

Let $P$ be a polynomial. Consider a $P$-invariant continuum $Y \subset K_P$ satisfying the assumptions of Theorem 1.7 (e.g., $P : Y \to Y$ is almost $k$-to-one). For such maps, the $Y$-multiplicity function $\mu_Y$ is defined, see Definition 4.7. The proof of Theorem 1.7 splits into several steps.

8.1. Reduction to the case when $K_P$ is connected. Suppose first that $K_P$ is disconnected. Let $K_P(Y)$ be the component of $K_P$ containing $Y$. Clearly, $K_P(Y)$ is a $P$-invariant continuum. Choose a tight equipotential $E_V$ around $K_P(Y)$ so that the disk $V$ bounded by $E_V$ does not contain escaping critical points of $P$. Then $P : U \to V$ is
a PL map with filled Julia set $K_P(Y)$, where $U$ is the component of $P^{-1}(V)$ containing $Y$. By Theorem 2.13 $P : U \to V$ is hybrid equivalent to a PL restriction of a polynomial, say, $\tilde{P}$. Let $\tilde{Y}$ be the subset of $K_P$ corresponding to $Y \subset K_P(Y)$. Evidently, $\tilde{P}$ and $\tilde{Y}$ satisfy the assumptions of Theorem 1.7. Thus, we can consider only polynomials with connected Julia sets.

8.2. A candidate collection of wedges. From now on, assume that the Julia set of $P$ is connected. Let us define a certain renormalizable collection $W$ of wedges for which conjecturally $A_P(W) = Y$. Even though we will not prove this conjecture here, the collection $W$ will play a role in the proof of Theorem 1.7.

Start by defining a collection of wedges whose root points are irregular points of $Y$. Let $a$ be an irregular point; it is necessarily a critical point of $P$. By the assumptions of Theorem 1.7, the point $a$ is eventually mapped to a repelling periodic point. It follows from the Landing Theorem that there are preperiodic external rays landing at $a$.

Lemma 8.1. There are one or several wedges $W_1, \ldots, W_s$ with root point $a$ and with the following properties. The boundary rays of each $W_i$ are $P$-pullbacks of the same external ray for $P$ landing at the point $P(a)$. For any $w \in Y \setminus \{P(a)\}$ close to $P(a)$, all $P$-preimages of $w$ near $a$ but not in $Y$ belong to $W_1 \cup \cdots \cup W_s$.

There are precisely $\mu_{K_P}(a) - \mu_Y(a) > 0$ such preimages. Let $W^{irr}$ be the collection of all wedges associated in this way with all irregular points of $Y$. Here the superscript “irr” stands for “irregular”.

Proof of Lemma 8.1. Let $R$ be any external ray of $P$ landing at $P(a)$. Then the $P$-pullbacks of $R$ landing at $a$ divide $\mathbb{C}$ into several sectors. Since $P : Y \to Y$ is almost $k$-to-one, for each sector $W$ there are two options: either $W \cap Y = \emptyset$ or $P : (W \cap Y) \cup \{a\} \to Y$ is a local homeomorphism near $a$. In the former case say that $W$ is $Y$-empty. It is now enough to define $W_1, \ldots, W_s$ as all $Y$-empty sectors.

Next, we deal with parabolic points in $Y$. Let $a \in Y$ be a parabolic point, and $\Omega$ a parabolic domain at $a$ with $\Omega \cap Y = \emptyset$. By Theorem 6.6 of [McM94], two external rays $R^R$ and $R^L$ of $P$ land at $a$ so that the intersection of $K_P$ with the corresponding wedge $W_\Omega \supset \Omega$ is connected.

Lemma 8.2. The wedge $W_\Omega$ has no points of $Y$.

Proof. We claim that $\text{Bd}(\Omega) \cap Y = \{a\}$. Recall that $\text{Bd}(\Omega)$ is a Jordan curve by [RY08]. If $Y \cap \Omega \neq \{a\}$, then there is an open arc $I$ of $\text{Bd}(\Omega)$ all whose points either belong to $Y$ or are shielded from $\infty$ by $Y$. The
latter is impossible since \( I \subset J_P \); thus \( I \subset Y \). Since \( Y \) is invariant, it follows that \( \text{Bd}(\Omega) \subset Y \). A contradiction with \( Y \) being full.

Set \( L = K_P \cap W_\Omega \). By construction, \( L \) is connected. Assume that \( W_\Omega \cap Y \neq \emptyset \). By the first paragraph there is a component \( L_Y \) of \( L \setminus \overline{\Omega} \) containing points of \( Y \). Since \( Y \) is connected and by the first paragraph we have \( Y \cap W_\Omega \subset L_Y \). If \( L_Y \cap \overline{\Omega} = \{a\} \), then \( L_Y \) is closed and open in \( L \), a contradiction with connectivity of \( L \). Thus \( L_Y \cap [\Omega \setminus \{a\}] \neq \emptyset \), a contradiction with the first paragraph. \( \Box \)

The wedges \( W_\Omega \) constructed for all pairs \( (a, \Omega) \) as above form a collection \( W^\text{par} \). Here the superscript “par” stands for “parabolic”. The collection \( W = W^\text{irr} \cup W^\text{par} \) is called the candidate collection of wedges for \( Y \). We conjecture that \( Y = A_P(W) \).

8.3. **The case of empty candidate collection.** We keep the notation introduced above. Lemmas 8.1 and 8.2 imply that \( Y \subset A_P(W) \). By the Main Theorem applied to \( P \) and \( W \), there is a polynomial \( \tilde{P} \) such that \( \tilde{P} : K_P \to K_P \) is topologically conjugate to \( P : A_P(W) \to A_P(W) \). Let \( \tilde{Y} \) be the \( \tilde{P} \)-invariant continuum corresponding to \( Y \) under this conjugacy. We claim that the candidate collection of wedges for \( \tilde{Y} \) is empty. That is, firstly, there are no irregular points in \( \tilde{Y} \) and, secondly, parabolic domains at parabolic points from \( Y \) lie in \( Y \).

If \( a \in \tilde{Y} \) is an irregular point, then \( \mu_{\tilde{Y}}(a) < \mu_{K_P}(a) \). In other words, there are preimages of a point \( v \neq \tilde{P}(a) \) close to \( \tilde{P}(a) \) that are near \( a \) but not in \( \tilde{Y} \). Suppose that \( b, w \) are the points of \( Y \) corresponding to \( a, v \). Then the corresponding \( P \)-preimages of \( w \) near \( b \) must be in the wedges from \( W^\text{irr} \) by definition of \( W^\text{irr} \). A contradiction. Thus all points of \( \tilde{Y} \) are regular. Similarly, it is impossible that a parabolic domain at \( a \in \tilde{Y} \) is disjoint from \( \tilde{Y} \). We conclude that the candidate collection of wedges for \( \tilde{Y} \) is empty. From now on, we may assume that \( W \) was empty from the very start.

However, if the candidate collection of wedges is empty, then \( P : Y \to Y \) satisfies the assumptions of Theorem 1.6. (Observe that the absence of irregular points is equivalent to the condition that \( Y \) is a component of \( P^{-1}(Y) \).) The conclusion of Theorem 1.7 now follows from Theorem 1.6.

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