A method for constructing random matrix models of disordered bosons

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Abstract
Random matrix models of disordered bosons consist of matrices in the Lie algebra \( g = \mathfrak{sp}_n(\mathbb{R}) \). Assuming dynamical stability, their eigenvalues are required to be purely imaginary. Here, a method is proposed for constructing ensembles \((\mathcal{E}, P)\) of \( G \)-invariant sets \( \mathcal{E} \) of such matrices with probability measures \( P \). These arise as moment map direct images from phase spaces \( X \) which play an important role in complex geometry and representation theory.

In the toy-model case of \( n = 1 \), where \( X \) is the complex bidisk and \( P \) is the direct image of the uniform measure, an explicit description of the spectral measure is given.

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The goal of this paper is to point out a method for constructing ensemble probability spaces in a bosonic setting. We do this in the context of the model in [LSZ] where due to the requirement of dynamical stability the generators

\[
S = \begin{pmatrix} A & -B \\ C & -A^t \end{pmatrix}
\]

of the time evolution are symplectic matrices of elliptic type, i.e. their eigenvalues come in pairs \( \pm i \lambda_j \) with \( \lambda_j > 0 \). We denote the set of these elements of the symplectic Lie algebra \( g = \mathfrak{sp}_2n(\mathbb{R}) \) by \( \mathcal{E} \).

To put this in perspective it is perhaps useful to recall that in the case of fermions the basic model spaces consist of the Hermitian operators \( H \) such that \( iH \) is, for example, an element of one of the Lie algebras \( \mathfrak{su}_n \), \( \mathfrak{so}_n \) or \( \mathfrak{usp}_{2n} \) of the classical compact groups. For example, in the case of \( \mathfrak{su}_n \) the basic model space becomes the vector space of all Hermitian matrices. In that case, the model probability distribution can be chosen to be Gaussian, up to constants the density being \( f(H) = e^{-\|H\|^2} \).

Since one is usually most interested in associated eigenvalue distributions, it is entirely appropriate that the model densities are invariant with respect to the action by conjugation of the compact group \( K \) at hand, i.e. the adjoint representation of \( K \).
In the case of $\mathfrak{su}_n$ this is just the action of the special unitary group $K = \text{SU}_n$. Reducing the symmetry by this action, one obtains associated classical Weyl-group-invariant distributions for the eigenvalues. One then computes various associated distributions, e.g. eigenvalue spacing distributions and their limits. For example, in the case of the Hermitian matrices the famous GUE density appears as a limiting distribution in this way.

At the very outset, one observes two major differences between the setting of disordered bosons and that of fermions. First, the necessity of dynamic stability imposes a strong condition on the operators being considered. Secondly, unlike the fermionic case where the model probability densities can be chosen to be invariant with respect to the full compact group of symmetries, in the bosonic case the group $G = \text{Sp}_{2n}(\mathbb{R})$ is noncompact and $G$-invariant densities are not available.

In [LSZ], a method for constructing bosonic ensembles $(\mathcal{E}, P)$ is described and the program which we sketched above for fermions is carried out in one particular example. In this work, as in [LSZ], ensemble densities which are invariant with respect to the (unique up to conjugation) maximal compact subgroup $K$ of $G$ are proposed (see section 5 for a comparison of the two methods). The usual choice is $K = U_n$. Our hope here is that $K$-invariant constructions which are closely related to the geometry and representation theory of the symplectic group $G$ will also produce bosonic ensembles of quantum mechanical interest.

We begin by explaining this approach for arbitrary $n$. Calculations are carried out in what might seem to be a toy model, i.e. the case of $n = 1$. Due to the fact that we are dealing with a group of Hermitian type, where the Poincaré disk plays a fundamental role in understanding the associated symmetric space, this is an important special case in our future work.

1. Generalities

From the mathematical point of view, there is a very interesting canonically defined neighborhood $U(G)$ of the 0-section of the cotangent bundle $T^*M$ of the associated symmetric space $M = G/K$. In the terminology of [FHW], to which we refer for background, it is called the universal domain associated with $G$. Its importance in representation theory, where it is most often called the crown of the symmetric space, has been underlined in numerous works.

In the case at hand, where $G$ is of Hermitian type, this is $G$-equivariantly and holomorphically isomorphic to the product $B \times \overline{B}$ of the associated Hermitian bounded domain and its complex conjugate $\overline{B}$ [BHH]. It should be noted that $B$ and $\overline{B}$ are biholomorphically equivalent as complex manifolds, but are not $G$-equivariantly biholomorphic.

1.1. Background on Hermitian symmetric spaces

In order to emphasize the concrete nature of this situation let us provide a sketch of some basic information about the Hermitian symmetric space $B$ (see, e.g., [H] for details). A symmetric space is simply a (finite-dimensional) Riemannian manifold $(M, g)$ with the additional property that at every $p \in M$ there exists an isometry $\sigma_p$ which is a symmetry at $p$ in the sense that $\sigma_p^2 = \text{Id}_M$ and $d\sigma_p = -\text{Id}_{T_pM}$. One shows that the connected component of the identity of the isometry group is a Lie group $G$ which acts transitively on $M$ and that the isotropy subgroup at a neutral point under consideration is a compact subgroup $K$.

Based on curvature conditions and canonical decompositions of Lie groups, symmetric spaces break up into products of symmetric spaces of three irreducible types. One such type is $M = G/K$ where $G$ is a simple (real) noncompact Lie group and $K$ is a (unique up to conjugation) maximal compact subgroup. In some cases, these symmetric spaces $M$ have the structure of a complex manifold such that the group $G$ acts as a group of holomorphic
transformations. This situation can be group-theoretically characterized by the condition that $K$ has a positive-dimensional center which in fact turns out to be $S$. In this case, the metric can be chosen to be Hermitian and therefore one refers to $M$ as a Hermitian symmetric space and $G$ being of Hermitian type. Note that the maximal compact subgroup $K = U_p$ of $G = Sp_n(\mathbb{R})$ satisfies this condition.

Noncompact Hermitian symmetric spaces of the type discussed above are naturally realized as distinguished open $G$-orbits in compact Hermitian symmetric spaces $Z$ which are referred to as the compact duals of the noncompact Hermitian symmetric spaces. A simple example is the case where $G = SU(n, 1)$ acts on the space $\mathbb{P}_n(\mathbb{C})$ of lines in $\mathbb{C}^{n+1}$. Here, an associated symmetric space is the open $G$-orbit of negative lines. In general, there are always exactly two $G$-invariant complex structures on a Hermitian symmetric space. In this case, the ‘other’ structure is the space of positive $n$-planes in $\mathbb{P}_n$.

Using a precise description of the action of the complex Lie group $G^\mathbb{C}$ on $Z$, one observes that the noncompact Hermitian symmetric space is contained as a bounded domain in a canonically determined (dense) open subset of $Z$ which is biholomorphically equivalent to $\mathbb{C}^m$ where $m := \dim_{\mathbb{C}} M$. Conversely, every (irreducible) bounded domain $D$ in $\mathbb{C}^m$ which is symmetric with respect to holomorphic symmetries arises in this way. In our case, we may view the Hermitian symmetric spaces $B$ and $\overline{B}$ as open $G$-orbits in the space $Z$ of Lagrangian subspaces of $(\mathbb{C}^{2n}, \omega)$, where $\omega = dp \wedge dq$ is the standard (holomorphic) symplectic structure.

As a complex manifold $X = B \times \overline{B}$ can also be $G$-equivariantly realized as a $G$-invariant open neighborhood of the (totally real) orbit $G_{20}$ of the neutral point in the affine symmetric space $G^\mathbb{C}/K^\mathbb{C}$. As mentioned above, it is $G$-equivariantly identifiable with a neighborhood of the $0$-section in $T^*M$. These identifications allow us to consider natural $G$-invariant symplectic structures on $X$ and the resulting moment maps $\mu : X \to g^*$. The most obvious of these are the restriction to $X$ of the canonical symplectic structure on $T^*M$ and structures of the type $\omega := i\partial \overline{\partial} \rho$, where $\rho$ is a strictly plurisubharmonic $G$-exhaustion of $X$ defined, e.g., by representation theory. Below we introduce the basics of momentum geometry, in particular in settings where tools of complex analysis are available.

One can show that with respect to any of these structures the generic $G$-orbits are coisotropic, i.e. the $\mu$-fibers are contained in the $G$-orbits. Thus, an associated moment map $\mu$ transports a setting of geometric and representation importance to its image $\mu(X)$ in an optimally controllable way. Our goal is to construct bosonic ensembles by equipping such an image with $\mu$-direct image measures. However, for the symplectic structures mentioned above, $\mu(X)$ is never contained in $E$.

The fact that this happens, which was initially surprising to us, should be further investigated. Here, however, we move to a slightly different complex geometric situation by implementing the $G$-equivariant complex conjugation $\kappa : \overline{B} \to B$ and thereby redefining $X = B \times B$. Thus, we still have the advantage of the symplectic geometry and representation theory of the original setup. In addition, in this new situation the moment map $\mu$ associated with any $G$-invariant Kählerian structure on $X$ satisfies $\mu(X) \subset E$. Furthermore, $\mu$ is generically an open map and consequently its image is not pathological. The discussion of the $\mu$-images, both before and after complex conjugation, is contained in section 2.

1.2. Background on momentum geometry

Most of our work here is carried out by direct calculations. We view these in the framework of a certain kind of momentum geometry which we now review (see, e.g., [GS] for details). For this, one begins with a symplectic manifold $(M, \omega)$. At least locally there are canonical coordinates so that $\omega = \sum dq_i \wedge dp_i$. However, in most cases $\omega$ reflects global geometric
phenomena which may not be visible in such coordinates. In our setup, $M$ is equipped with an action $G \times M \to M$ of a connected Lie group of symplectic transformations, i.e. $g^*\omega = \omega$ for all $g \in G$. Elements $\xi$ of the Lie algebra $\mathfrak{g}$ define vector fields $\xi$ on $M$ by differentiating along the associated 1-parameter groups:

$$\dot{\xi}(f)(p) = \frac{d}{dt}|_{t=0} f(\exp(t\xi),p).$$

These fields are symmetries of the symplectic structure in the sense that the associated Lie derivative annihilates the symplectic form $L_\xi \omega = 0$. In this formalism, such a field is Hamiltonian if there is a function $H$ on $M$ which satisfies $dH = \omega(\xi, \cdot)$. This means that the ODE defined by $\dot{\xi}$ is given in the expected way by $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $-\dot{p}_i = \frac{\partial H}{\partial q_i}$. All fields which satisfy $L_X \omega = 0$ are locally Hamiltonian, e.g., on contractible open subsets, but globally this may not be the case.

In the case at hand where $g$ is semisimple, every $\xi$ is Hamiltonian in a canonical way. We refer to the associated Hamiltonian as a momentum or energy function and denote it by $\mu_\xi$. These functions can be bundled together in a moment map which is defined by $\mu(x)(\xi) = \mu_\xi(x)$. In the semisimple case, it is automatically $G$-equivariant. This means that $\mu(g(x))(\xi) = \text{Ad}^*(g)(\mu(x))$ where the coadjoint representation $\text{Ad}^*$ is defined by $\text{Ad}^*(g)(\alpha) := \alpha \circ \text{Ad}(g^{-1})$.

The symplectic manifolds under consideration in this paper are Kählerian. A Kähler manifold $(X, \omega)$ is in particular symplectic. In addition, $X$ is a complex manifold such that holomorphic coordinates define a complex structure tensor $\hat{J}$ on the (real) tangent bundle of $X$ which is an isometry of $\omega$, i.e. $\omega(Jv, Jw) = \omega(v, w)$. Furthermore, it is required that $J$ defines a (positive-definite) Hermitian metric $h(v, w) = \omega(Jv, w)$. It is a fundamental fact that $\omega$ possesses (local) potentials $\rho$. This means that locally $\omega = i\partial \bar{\partial} \rho$. The fact that $h$ is positive-definite is equivalent to $\rho$ being strictly plurisubharmonic, i.e. the complex Hessian $\left(\frac{\partial^2 h}{\partial \overline{z_i} \partial z_j}\right)$ is positive-definite. These functions play a central role in the subject of several complex variables.

In our setting, $G$ is acting as a group of holomorphic transformations which preserves a globally defined Kählerian potential $\rho$. In that case

$$\mu_\xi := -\hat{J}\dot{\xi}(\rho)$$

defines an equivariant moment map.

In the following section, we compute $\mu$ for a particularly beautiful symplectic structure on the product $X = \Delta \times \Delta$, where $\Delta$ is the unit disk in the complex plane. Here, we use the identification of $G$ with SU$(1, 1)$. This is certainly a toy model. However, the bounded domain $B$ contains a (flat) $r$-dimensional complex polydisk $\Delta_r$ defined by a choice of a maximal (abelian) subalgebra $a$ in the p-part of a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $K.\Delta_r = B$. Therefore, certain results can be immediately carried over to $B$ by arguing one variable at a time in $\Delta_r$. On the other hand, our goal of understanding asymptotic information concerning spectral invariants is of course not touched here. Nevertheless, we do indicate how to compute the distribution function of the eigenvalue $i\omega$ with $\omega > 0$, and we compute this explicitly in the concrete case of the direct image of the uniform measure.

2. The SU$(1, 1)$-toy model

Define the mixed signature Hermitian form $(\cdot, \cdot)_{1,1}$ on $\mathbb{C}^2$ by $(z, w)_{1,1} := \bar{z}_1 w_1 - \bar{z}_2 w_2$ and denote by $G = \text{SU}(1, 1)$ its group of complex linear isometries. As usual $\mathfrak{g} := \text{Lie}(G)$. The group has exactly three orbits on the projective space $\mathbb{P}_1(\mathbb{C})$: $\Delta := G[0 : 1]$, $\hat{\Delta} = G[1 : 0]$. 4
and the real projective space $P_1(\mathbb{R}) = G.[1 : 1]$. Note that the involution $\kappa : \mathbb{P}_1 \to \mathbb{P}_1$, $[z_1, z_2] \mapsto [\bar{z}_2 : \bar{z}_1]$ defines a $G$-equivariant, antiholomorphic isomorphism $\kappa : \Delta \to \Delta$.

For moment-map considerations it is convenient to define the symmetric bilinear form $b : g \times g \to \mathbb{R}$ by $b(x, y) := \frac{1}{2} \text{Tr}(xy)$ which is invariant under the adjoint representation of $G$. Using the duality defined by $b$ we will regard the moment map as having image in $g$. We will use the following $b$-orthogonal basis of $g$:

$$
\xi = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \zeta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

If a moment map with values in $g^*$ is given by $\mu = \mu_\xi \xi^* + \mu_\eta \eta^* + \mu_\zeta \zeta^*$, then the corresponding map with values in $g$ is given by $\mu = -\mu_\xi \xi + \mu_\eta \eta + \mu_\zeta \zeta$.

### 2.1. Kählerian structure

Here, we make two observations which make a case for considering the $B \times B$ instead of $B \times \overline{B}$ as the appropriate phase space.

#### Proposition 2.1

The image of a moment map $\mu : \Delta \times \overline{\Delta} \to g^*$ which is defined either by the restricted coadjoint structure or by a Kähler form $\omega = i\partial \bar{\partial} \rho$ of a $G$-invariant potential is contained in the complement of the cone $E$ of elliptic elements.

For a complementary statement for $X = \Delta \times \Delta$ we denote by $\hat{E}$ the augmentation of either the positive or negative cone of elliptic elements by adding $0 \in g$.

#### Proposition 2.2

The image of a moment map $\mu : \Delta \times \Delta \to g^*$ which is defined by a Kähler form $\omega = i\partial \bar{\partial} \rho$ of a $G$-invariant potential is contained in an augmented cone $\hat{E}$ with the diagonal being mapped to 0 and the complement of the diagonal being mapped to the component of $E$ which is contained in $\hat{E}$.

It should be noted that these observations can be translated to the higher dimensional setting by using strongly orthogonal roots and the polydisk slice mentioned above.

Turning to the proofs of these remarks, we first note that in the case where the Kähler form is defined by an invariant potential, an associated moment map is given by

$$
\mu(x)(\xi) = \mu_\xi(x) = -J\hat{\xi}(x)(\rho).
$$

Here, $J$ is the complex structure and $\hat{\xi}$ denotes the vector field on the manifold which is associated with $\xi \in g$. Recall that since $G$ is semisimple, the moment map is the unique equivariant map to $g^*$ with the Hamiltonian property $d\mu_\xi = i\xi \omega$. Of course, $\hat{\xi}$ is tangent to the level sets of the invariant function $\rho$ and the moment map is measuring how $\rho$ is growing along $-J\hat{\xi}$. In this situation it is therefore important to determine the complex tangent spaces $T^c_x[\rho = \rho(x)]$ of the $\rho$-level sets. Assuming $d\rho(x) \neq 0$, this Cauchy–Riemann tangent space is the one-codimensional maximal complex subspace of the (real) tangent space $T_x[\rho = \rho(x)]$.

#### Proof of proposition 2.1

Let $z_0 \in \Delta \times \overline{\Delta}$ be the base point so that $M = G.z_0 = G/K$ is the unique symmetric space orbit in $\Delta \times \Delta$. In the standard homogeneous coordinates this is defined by $\langle z, \omega \rangle_{1,1} = 0$. Choose $\alpha \subset \mathfrak{p}$ to be spanned by the matrix $\eta$ above. Let $\Sigma$ be the image of the half-open interval $[0, \frac{1}{2})$ by the map $i \mapsto \exp(i\eta)$. Then, $\Sigma$ is an exact slice for the $G$-action on $\Delta \times \Delta$ (see [FHW] for this and other background used here). This means that every $G$-orbit intersects $\Sigma$ in exactly one point. The orbits $M_p := G.p$ for $p \in \Sigma \setminus \{z_0\}$ are real hypersurfaces. The complex tangent spaces $T^c_p[M_p]$ are calculated in a general setting in [FHW] (see page 103). Applying this to the case at hand, we observe that these spaces are
corresponding to \( \mu(p) \). The image \( \mu(\Sigma) \) is \( 0 \):

\[
\mu(\Sigma) = 0.
\]

For this observe that in this case \( g \) consists of those matrices of the form

\[
A = \begin{pmatrix} 0 & c \bar{t}^2 \\ c & 0 \end{pmatrix}
\]

with \( c \in \mathbb{C} \). (2)

It is then immediate that \( \eta \) and \( \zeta \) are linearly dependent modulo \( \mathbb{g}^C \) when regarded as elements of \( \mathbb{g}^C \). Thus, the real subspace of \( T_pZ \) which is generated by \( \tilde{h}(p) \) and \( \tilde{\xi}(p) \) is a complex line which by dimension arguments is the complex tangent space \( T^*M \).

Finally, exactly the same argument as in the proof of the previous theorem it follows that \( \mu(\Sigma) = 0 \) for all \( p \in \Sigma \). Since \( \mu = -\mu_\xi \eta + \mu_\eta \eta + \mu_\xi \zeta \), it follows that \( \mu(\Sigma) = -\mu_\xi \xi \). Since 0 is necessarily a critical point of \( \mu \), it follows that \( \mu(0) = 0 \). Standard computations of rank\( (\mu_\eta) \) show that \( \mu(p) \neq 0 \) for \( p \in \Sigma \setminus \{0\} \) and the desired result follows.

Actually, we proved more than what was stated in the above proposition. For future reference we state this here.

**Zusatz.** The image \( \mu(\Sigma) \) is contained in the \( \xi \)-axis \( \{ \eta^* = \zeta^* = 0 \} \).

### 3. Poincaré moment map

Here, we let \( d_p : X \to \mathbb{R}_{\geq 0} \) be the distance function defined by the Poincaré metric on \( \Delta \).

We temporarily remove the diagonal from \( X \) where \( d_p \) is no longer smooth, show that \( d_p \) is strictly plurisubharmonic so that \( \omega = i\partial \bar{\partial} d_p \) is a \( G \)-invariant Kähler form and compute the restriction to the slice \( \Sigma \) of the resulting moment map. As would be expected, this map can be (continuously) extended to \( X \) with the diagonal being mapped to 0.

#### 3.1. Strict plurisubharmonicity of \( d_p \)

It is convenient to change coordinates on \( X \), letting \( u = z + w \) and \( v = z - w \) where \( (z, w) \) are the affine coordinates which are used above. Thus, the complex disk \( L := \{ u = 0 \} \) contains the slice \( \Sigma \) as a radius. In particular, \( L \) is transversal to the \( G \)-orbit of any point \( p \in \Sigma \setminus \{0\} \).
Note that $L$ is invariant by the diagonal $S^1$-action defined by the element $\xi \in g$. It therefore follows that a function $f$ on $L$ is strictly plurisubharmonic if and only if $f(\epsilon^x)$ is strictly convex. Let us begin with this step which of course requires an explicit computation with $d_P$.

If
$$d_S(z, w) := \left| \frac{z - w}{1 - \bar{w}z} \right|$$
is the Schwarz distance function, then
$$d_P = \log \frac{1 + d_S}{1 - d_S} = \tanh^{-1}(d_S).$$

Since $\tanh^{-1}$ is strictly convex, in order to prove the following it is enough to show that $f(\epsilon^x)$ is strictly convex where $f$ is the restriction of $d_S$ to $L$.

**Lemma 3.1.** The restriction of $d_P$ to $L$ is strongly subharmonic.

**Proof.** Using the obvious coordinates,
$$f(\epsilon^x) = \frac{2\epsilon^x}{1 + e^{2\epsilon^x}}.$$

Now we turn to the ‘other’ direction, namely the coordinate $u$. For this at each point of $L$ we consider the complex curve $\gamma$ defined by $\gamma(u) = (t + u, -t + u)$.

**Lemma 3.2.** Near $u = 0$ the pullback $\delta(u) = d_P(\gamma(u)) - d_P(\gamma(0))$ is nonnegative with $\delta(0) = 0$ and $\delta(u) > 0$ otherwise. Furthermore,
$$\frac{d^2\delta}{du^2}(0) > 0.$$

**Proof.** As above, it is enough to prove the positivity for $d_P$ replaced by $d_S$. This is equivalent to
$$|1 - (u + t)(\bar{u} - t)|^2 < (1 + t^2)^2$$
for $|u|$ sufficiently small and nonzero. This is done by elementary manipulations (see [S], pages 66–7). One can also prove (3) by making several estimates (also see [S], page 67), but this can be proved in a more conceptual way. Since the curve $\gamma$ touches the $d_P$ level set from outside, it is immediate that $\frac{d^2\delta}{du^2}(0) \geq 0$. If this derivative vanished at $p$, then the Levi form restricted to the complex tangent bundle of the orbit $G.p$ would vanish identically. It would then follow that $G.p$ would be foliated by complex curves. This is for numerous reasons impossible, e.g. continuously moving the leaf through $p$ away from $p$ so that it no longer intersects the curve $\gamma$ yields a contradiction to Hurwitz’s theorem.

**Proposition 3.3.** The Poincaré distance function is plurisubharmonic on $X$ and is strictly plurisubharmonic outside of the diagonal.

**Proof.** The above two lemmas show that
$$\frac{\partial^2 d_P}{\partial^2 u} > 0 \quad \text{and} \quad \frac{\partial^2 d_P}{\partial^2 v} > 0$$
at every point of $L \setminus \{0\}$. The fact that the curves $\gamma$ are tangent to the $d_P$-level sets at every point of $L \setminus \{0\}$ implies that $\frac{d^2 P}{du^2}$ vanishes along $L$ and therefore the mixed derivatives $\frac{\partial^2 d_P}{\partial u \partial v}$ vanish identically there as well. In other words, the full complex Hessian is diagonalized along $L \setminus \{0\}$ with positive entries along the diagonal. Since $d_P$ is only continuous along the
diagonal of $X$, to show that it is plurisubharmonic there, we must prove that if $h : \Delta \to X$ is holomorphic with $h(0) = 0$, then $d_P \circ h$ is subharmonic. If the image of $h$ is contained in the diagonal, then this is just the identically zero function. Otherwise, we may assume that 0 is the only point of the diagonal in the image and the result follows from the plurisubharmonicity of $d_P$ outside the diagonal and that, since $d_P(0) = 0$, the mean value property is fulfilled at 0.

3.2. Restricted moment map

To keep the notation straight let $\Sigma_X$ denote the slice $\Sigma$ in $X$. The sign of the moment map is chosen so that its image is contained in the augmented cone $\hat{E}$, where $E$ is the positive cone defined as $G.\Sigma_E$ where $\Sigma_E := \mathbb{R}^{>0} \xi$. By the above zusatz we know that $\mu|\Sigma_X : \Sigma_X \to \Sigma_E$. Since $\Sigma_X$ and $\Sigma_E$ are slices for the respective $G$-actions, for our purposes here it is enough to have an explicit description of the restricted moment map $\mu|\Sigma_X$. This amounts to a description of the function $\mu_\xi$ along $\Sigma_X$.

**Proposition 3.4.** If $L$ is parameterized by $z \to (z, -z)$ and the ordered basis $\{\xi, \eta, \zeta\}$ is used for $g$, then

$$\mu|L(z) = \left(\frac{8|z|}{1 - |z|^2}, 0, 0\right).$$

**Proof.** In this parameterization the restriction of the Poincaré distance function to $L$ is given by

$$d_P(z) = \log\left(\frac{1 + |z|}{1 - |z|}\right).$$

By definition

$$\mu_\xi(z) = -J_\hat{E}(z) d_P = -\frac{d}{ds}\bigg|_{s=0} d_P(e^{i \xi s}z) = -\frac{d}{ds}\bigg|_{s=0} \log\left(\frac{1 + e^{-2s|z|}}{1 - e^{-2s|z|}}\right)^2$$

and the desired result follows by a direct computation of the derivative. □

**Corollary 3.5.** The moment map $\mu : X \to \hat{E}$ is surjective. The diagonal in $X$ is the $\mu$-preimage of the vertex and the restriction of $\mu$ to the complement of the diagonal is a proper $S^1$-bundle. Furthermore, $\mu$ has the coisotropic property that $\mu^{-1}(\mu(x)) \subset G.x$ for all $x \in X$.

4. Direct image of the uniform measure

Let us conclude this paper by explicitly computing the spectral density for the direct image measure $P$ on $E$ defined by the uniform measure $\nu$ on $X$. It appears that other natural densities, e.g. those related to Gaussian-type distributions, will only be numerically computable. Closed form descriptions of the behavior of these at the vertex of the cone should be possible.

Note here that the matrices on the slice $\Sigma_E$ are of the form

$$S = \omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $\omega > 0$. The relevant distribution is therefore $F(x) := P(\omega < x)$. We will sketch the explicit computation of $F$ here, referring to [S] for details.

The mapping $\mu|\Sigma_X : \Sigma_X \to \Sigma_E$ is given by $x = \mu(t) = \frac{\omega}{\omega - t}$. Let $t = t(x)$ be the corresponding inverse value. Thus, $F(x)$ is the Euclidean volume of the region $\{d_P < t(x)\} = R(x)$. We compute this via fiber integration using the projection $\pi : X \to \Delta$ on the first coordinate. For $a \in \Delta$ let $A(a, x)$ be the Euclidean area of the fiber
\[ \pi^{-1}(a) \cap R(x) =: D(a, x) \] so that up to a constant
\[ F(x) = \int_{\Delta} A(a, x) \, da \wedge \dabar. \]

Let us begin by computing \( A(0, x) \).

**Lemma 4.1.** The fiber \( R(x) \cap \pi^{-1}(0) \) is a disk of radius \( u = u(t(x)) = \frac{2}{1+t^2} \) in the \( \pi \)-fiber \( [0] \times \Delta \).

**Proof.** Since the projection \( \pi \) is \( G \)-equivariant, \( R(x) \cap \pi^{-1}(0) \) is a region in the \( \pi \)-fiber which is bounded by an orbit of the \( G \)-isotropy group at 0 in the \( \pi \)-image space. Since this isotropy group is acting on the fiber by rotations, this is a disk. Now the region \( R(x) \cap L \) has the point \( p_u = (u, -u) \) on its boundary. Thus, if we apply the transformation \( g(z, w) = (Tu(z), Tu(w)) \) with
\[ T_\zeta(z) = z - \zeta \frac{1}{1 - \overline{\zeta}w}, \]
then \( p_u \) is mapped to the point \( (0, Tu(-u)) \) which is on the boundary of the disk in question and the desired result follows. \( \square \)

Since we are only dealing with the uniform measure on \( X \), it follows that
\[ F(x) = \frac{i}{2\pi} \int_{a \in \Delta} A(a, x) \, da \wedge \dabar. \]

Now \( D(a, x) \) is the image of the disk \( D(0, x) \) by the transformation \( T_u \). Since \( T_u \) preserves the Poincaré metric, this is a hyperbolic disk, i.e. a disk with respect to \( dP \). But hyperbolic disks are Euclidean disks. Thus,
\[ F(x) = \frac{i}{2} \int_{a \in \Delta} r^2(a, x) \, da \wedge \dabar, \]
where \( r = r(a, x) \) is the Euclidean radius of \( D(a, x) \) which we now compute.

By rotational invariance it is enough to compute \( r(a, x) \) for \( a = s \in [0, 1) \) on the positive real axis of \( \Delta \). Thus, the transformation \( T_s \) stabilizes the interval \( I \) of real points of \( \Delta \). Since \( I \) is orthogonal to the boundary of \( D(0, x) \) and \( T_s \) is conformal, it follows that \( I \) is orthogonal to the boundary of \( D(s, x) \). Consequently, the length of \( I \cap D(s, x) \) is the diameter of \( D(s, x) \). This leads to the following fact.

**Lemma 4.2.** For \( s \in [0, 1) \) the radius of \( D(s, x) \) is
\[ r(s, x) = \frac{1}{2} \left( T_s(u) - T_s(-u) \right) = \frac{u(1-s^2)}{1-s^2 u^2}. \]

Introducing polar coordinates and using rotational symmetry, one shows that
\[ F(x) = 2 \int_0^1 \left( \frac{u(1-s^2)}{1-s^2 u^2} \right)^2 \, ds \]
for \( u = u(t(x)) \) as above. This integral can actually be computed in closed form:
\[ F(x) = 2 \frac{1-u^2}{u^2} \log(1-u^2) + 2 - u^2. \]

Using the simple dilation \( \tilde{x} = \frac{x}{2} \) to clean up the numbers, one computes that
\[ u^2 = \frac{\tilde{x}^2}{1+\tilde{x}^2}. \]
Thus, one has a rather simple closed form description of the distribution function $F$ and the associated density $f$.

**Proposition 4.3.** In the variable $\tilde{x} = \frac{1}{x}$ the eigenvalue distribution function $F$ and its density $f$ associated with the direct image of the uniform measure by the Poincaré moment map $\mu : X \to \hat{E}$ are given by

$$F = -\frac{2}{\tilde{x}^2} \log(1 + \tilde{x}^2) + \frac{1}{1 + \tilde{x}^2}$$

and

$$f = \frac{4}{\tilde{x}^3} \log(1 + \tilde{x}^2) - \frac{6\tilde{x}^2 + 4}{\tilde{x}(1 + \tilde{x}^2)^2}.$$

Numerical computation of $f$ yields the picture shown in figure 1.

Near 0 one has the power series representation

$$f(x) = \sum_{k=2}^{\infty} \frac{(-1)^k k(k-1)}{2} \frac{1}{k+1} \left(\frac{x}{4}\right)^{2k-1}.$$

In particular, $f \sim x^3$ near 0. For $x \sim \infty$ the expression in the above proposition implies that $f \sim \frac{\log(x)}{x}$. The expected value $E(X) = \int_0^\infty x f(x) \, dx$ can actually be explicitly computed as $E(X) = \frac{3}{2}\pi$. However, as one observes from the estimate of $f$ near $\infty$, the second moment $E(X^2)$ is infinite. This phenomenon does occur in nature, but here it is probably due to the fact that the original uniform measure on the phase space $X$ is physically unrealistic. Gaussian densities, e.g., of the type $e^{-\alpha x^2}$ would perhaps be of greater interest, but the resulting integrals are more complicated.

5. Comparisons

Although the major part of the work in [LSZ] is focused on computations for one particular case, the method which is proposed for constructing bosonic ensembles is quite general. Let us sketch this construction.
Let $g := \mathfrak{sp}_{2N}(\mathbb{R})$ and define
\[
J := \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}.
\]
Let $\text{Sym}_{2N}(\mathbb{R})$ be the space of symmetric matrices and observe that the linear map
\[
\varphi : g \to \text{Sym}_{2N}(\mathbb{R}), \quad X \mapsto XJ,
\]
is a $G$-equivariant isomorphism. Note that $X \in E$ if and only if $h = XJ$ defines a positive definite quadratic form. In fact, $h$ is simply the quadratic Hamiltonian function on the standard symplectic space $(\mathbb{R}^{2N}, dq \wedge dp)$.

Now $\text{Sym}_{2N}(\mathbb{R})$ (equipped with the standard action of the orthogonal group $SO_{2N}(\mathbb{R})$) is one of the main symmetry classes of fermionic random matrix theory. In [LSZ], the restriction of the standard fermionic density $e^{-\frac{1}{2}Tr(h)}$ to the open subset of positive-definite matrices is normalized to provide a probability density. The resulting bosonic ensemble $(E, P)$ is then analyzed in detail. One could say that the underlying principle is to study bosonic ensembles which come from a related fermionic setting. Although not all ensembles which are invariant with respect to a maximal compact subgroup $K$ in $G$ are constructed in this way, it is a very natural viewpoint.

Our procedure is more general in the sense that every $K$-invariant ensemble arises as a direct image measure in our construction. This is a consequence of the following facts.

1. Since $g$ is semisimple, one may regard moment maps as having values in $g$.
2. The $G$-orbits in the domain space $X$ are coisotropic. Consequently, the fibers of $\mu : X \to E \to g$ are contained in $G$-orbits.
3. As in the case of the above toy model, in general a slice $\Sigma_X$ which parameterizes the $G$-orbits in $X$ is mapped diffeomorphically by $\mu$ onto a slice $\Sigma_E$ which parameterizes the $G$-orbits in a region in $E$. Thus, lifting a $K$-invariant measure on such a region only requires lifting density functions along orbits.

As a final remark we would like to emphasize that we are not particularly interested in proposing a method that is more general than some other. Our main goal here is to introduce a natural complex analytic, symplectic geometric viewpoint which seems to lead to natural constructions of bosonic ensembles which otherwise might not be visible.

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