Constraints on Inflation

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Abstract

A short introduction to structure formation is given, followed by a discussion of the possible characteristics of the initial perturbations assuming a generic inflationary origin. Observational data related to large-scale structure and the cosmic microwave background radiation is then used in an attempt to constrain the characteristics of such perturbations. Future directions are also explored.

The possibility of direct detection of a stochastic gravitational wave background produced during an inflationary phase in the early Universe is briefly discussed, as well as the available evidence regarding the present value of the total energy density in the Universe.

Key words: Cosmology; Inflation

1 Overview

The simplest models of inflation, those in which only one scalar field is present and is minimally coupled, lead to very simple and clear predictions: (1) the Universe is simply connected, (nearly) homogeneous and isotropic, with no detectable large-scale rotation, at least up to scales slightly larger than the present particle horizon; (2) the observable Universe is spatially flat; (3) the scalar perturbations that eventually originated the large-scale structures observed today, if generated during inflation, were primordial (i.e. passive), adiabatic and Gaussian distributed, with a nearly scale-invariant power spectrum.

However, in more complicated models, where more than one scalar field is present, with the possibility of the such fields being strongly coupled, most of the above predictions can be weakened. The exceptions are those of a trivial topology, the absence of large-scale rotation, and the primordial nature of the scalar perturbations. Nevertheless, these are still sufficient to provide tests...
for both hypothesis: of inflation as the event responsible for the present-day
large-scale Universe being nearly homogeneous, isotropic and spatially flat;
and of inflation as the most important mechanism behind the generation of
the scalar perturbations that eventually originated the large-scale structures
observed in the Universe today.

I will focus on the later hypothesis [see (55) for a general review of inflation],
 describes the observational tests, and present constraints, on the various char-
acteristics expected for scalar perturbations in the simplest inflationary mod-
els. These characteristics have an impact both on the nature and timescale for
the formation and evolution of structures on large-scale in the Universe, and
on the properties of the cosmic microwave background radiation (CMBR). I
will therefore spend most of this review on how the present observational data
regarding these two topics constrain the characteristics of the scalar pertur-
bations, and what in turn that tells us about inflation. I will also discuss the
prospects of detecting locally a possible stochastic background of gravitational
waves produced during inflation, and the present evidence regarding the geom-
etry of the Universe, and whether it supports the prediction of spatial flatness
associated with the simplest inflationary models. Though tensor perturbations
and spatial flatness cannot be used to test inflation, given that inflationary
models exist which do not predict them, they can offer strong support to the
hypothesis that an inflationary period did occur in the early Universe.

2 Structure formation and inflation

2.1 Introduction

In studies of structure formation one is particularly interested in the statistical
properties of the density contrast of the matter distribution,

\[ \delta(x, t) \equiv \frac{\rho(x, t) - \overline{\rho}}{\overline{\rho}}, \]

which is defined in terms of the density field \( \rho(x, t) \) and the comoving mean
background density \( \overline{\rho} \), where \( x \) is the comoving position.

A Fourier expansion can be made in a large enough box (of volume \( V \)) so that
\( \delta(x, t) \) is periodic within the box. We then have

\[ \delta(x, t) = \sum_k \delta_k(t) e^{ikx}, \]
where \( k \equiv |k| \) is a comoving wavenumber, with the Fourier coefficients being given by

\[
\delta_k(t) = \frac{1}{V} \int \delta(x, t) e^{-ik \cdot x} \, dx.
\]

We will define the \textit{power spectrum}, \( P(k, t) \), as

\[
P(k) \equiv \left( \frac{Vk^3}{2\pi^2} \right) |\delta_k|^2.
\]

The dispersion of the density contrast is then simply given by

\[
\sigma^2(t) \equiv \left\langle \delta^2(x, t) \right\rangle = \int_0^\infty P(k, t) \frac{dk}{k}.
\]

If it is assumed that for any given realization of the volume \( V \) the phases of the Fourier coefficients \( \delta_k \) are uncorrelated, the central limit theorem then guarantees that at any point the density contrast \( \delta(x, t) \) obeys Gaussian statistics. The probability distribution of \( \delta(x, t) \) at each point is then

\[
p(\delta) \, d\delta = \frac{1}{\sqrt{2\pi}\sigma} \exp\left( -\frac{\delta^2}{2\sigma^2} \right) \, d\delta.
\]

This equation implies that there is always some probability of having \( \delta(x, t) < -1 \), which, by definition, is not physically possible. Therefore, as a first approximation, it is only valid to consider \( \delta(x, t) \) as a Gaussian random field if there is only a very small probability of having \( \delta(x, t) < -1 \) by the above equation, i.e. if \( \sigma(t) \ll 1 \). This condition is also necessary if we wish to use linear perturbation theory to follow the evolution of \( \delta(x, t) \). Gaussian random fields are very special, since only the power spectrum is required to specify all of the statistical properties of the field, whereas for non-Gaussian fields the full hierarchy of probability distributions is needed.

After matter domination, the power spectrum of the density contrast, \( \delta(x, t) \), can be written as (56; 57; 67)

\[
P(k, t) = \frac{g^2(\Omega, \lambda)}{g^2(\Omega_0, \lambda_0)} \left( \frac{k}{aH} \right)^4 T^2(k, t) \delta_H^2(k),
\]

where the quantities \( aH, \Omega \) and \( \lambda \equiv \Lambda c^2/3H^2 \) are to be calculated at \( t \). The
function $g(\Omega, \lambda)$ accounts for the rate of growth of density perturbations relative to the Einstein-de Sitter case, whose growth is given by the $(aH)^4$ factor. The transfer function, $T(k, t)$, measures the change at $t$ in the amplitude of a perturbation with comoving wavenumber $k$ relative to a perturbation with infinite wavelength, thus in the limit $k \to 0$ (in practice $k \to k_{hor}$ due to gauge ambiguities), one has $T(k, t) \to 1$. The shape of the transfer function results mostly from the different behaviour of perturbations in the radiation and matter dominated eras, and from sub-horizon damping effects, like Silk damping, which affects baryons, and free-streaming (Landau damping), which acts on hot dark matter perturbations. An oscillatory pattern can also appear in the transfer function if baryons contribute significantly to the matter density in the Universe, due to acoustic oscillations of the photon-baryon fluid on scales below the horizon until decoupling occurs. The calculation of a transfer function not only depends on the type of mechanism responsible for the generation of the density perturbations, but also on the assumed matter and energy content in the Universe. It thus needs to be determined numerically, though nowadays there are several analytical prescriptions which approximate it for the most popular structure formation scenarios [see e.g. (24)].

The quantity $\delta^2_H(k)$, defined as

$$\delta^2_H(k) \equiv \left\langle \left( \frac{\delta \rho}{\rho} \right)^2 \right\rangle_{aH=k},$$

specifies the power spectrum of density perturbations at horizon re-entry. In the simplest inflationary models it can be well described by a single power-law,

$$\delta^2_H(k) = \delta^2_H(k_0) \left( \frac{k}{k_0} \right)^{n-1},$$

where $n$ is the so-called spectral index and $\delta_H(k_0)$ is a normalisation factor at an arbitrary comoving wavelength $k_0$. Since the COBE measurement of the amplitude of the large-angle anisotropies in the temperature of the CMBR became available, the value of $\delta_H(k_0)$ is usually set so as to reproduce it (though some previous assumption has to be made regarding the contribution of tensor perturbations, i.e. gravitational waves, to the anisotropies). When this is done, in the simplest inflationary models the value of $\delta_H(k_0)$ then depends essentially only on the values of $n$, $\Omega_0$ and $\lambda$ [see e.g. (11)]. A scale-invariant, or Harrison-Zel’dovich, power spectrum corresponds to $n = 1$. In general most inflationary models give $n \leq 1$, though in some it is possible to have $n > 1$. 

4
Perturbations in a multi-component system can be of the *entropy* or of the *adiabatic* types. The first correspond to fluctuations in the form of the local equation of state of the system, e.g. fluctuations in the relative number densities of the different particle types present in the system, while the second correspond to fluctuations in its energy density. In the case of a perfect fluid composed of matter and radiation, pure entropy perturbations are characterised by $\delta \rho_r = -\delta \rho_m$, while for pure adiabatic perturbations, $\delta \rho_r / \rho_r = (4/3)(\delta \rho_m / \rho_m)$. The entropy perturbations are also called *isocurvature*, given that the total density of the system remains homogeneous. In contrast, the adiabatic perturbations are also known as *curvature* perturbations, as they induce inhomogeneities in the spatial curvature. The two types of perturbations are orthogonal, in the sense that all other types of perturbations on a system can be written as a combination of both adiabatic and entropy modes.

On scales smaller than the Hubble radius any entropy perturbation rapidly becomes an adiabatic perturbation of the same amplitude, as local pressure differences, due to the local fluctuations in the equation of state, re-distribute the energy density. However, this change is slightly less efficient during the radiation dominated era than during the matter dominated era (and can only occur after the decoupling between photons and baryons, in the case of baryonic isocurvature perturbations). Causality precludes this re-distribution on scales bigger than the Hubble radius, and thus any entropy perturbation on these scales remains with constant amplitude. The end result is that initially scale-invariant power spectra of adiabatic and isocurvature perturbations give rise, after matter-radiation equality, to power spectra of density perturbations with almost the same shape.

Entropy perturbations are not affected by either Silk or Landau damping, contrary to adiabatic perturbations, thus potentially providing a means of baryonic density perturbations existing below the characteristic Silk damping scale after recombination (note that this could also have been achieved if there were cold dark matter adiabatic perturbations at such scales, with the dark matter necessarily being the dominant matter component in the Universe).

However, presently the amplitude of any primordial entropy perturbations is severely constrained by the level of anisotropy in the temperature of the CMBR as measured by *COBE*. In the case of an Universe with critical-density and scale-invariant perturbations, the total anisotropy on large angular scales is six times bigger in the case of pure entropy perturbations than in the case of pure adiabatic perturbations, for the same final matter density perturbation at those scales (54). We will later see that in a critical-density universe with initial scale-invariant adiabatic perturbations, the amplitude of the den-
sity perturbation power spectrum needed to generate observed structures, like rich galaxy clusters, and that needed to generate the temperature anisotropies measured by COBE, are roughly compatible. Therefore if one requires small-scale density perturbations with high enough amplitude to reproduce known structures in an universe with critical-density and initial scale-invariant isocurvature perturbations, one ends up with CMBR anisotropies on COBE scales with much larger amplitude than those which are measured.

Possible ways of escaping this handicap associated with entropy perturbations are: breaking the assumption of scale-invariance by assuming a steeper dependence with \(k\), i.e. decreasing the amount of large-scale power relatively to small-scale power; and decreasing the matter density in the Universe, i.e. assuming \(\Omega_0 < 1\). However, the first possibility leads to values for the spectral index which are in conflict with the constraints imposed by the COBE data [see e.g. (36)], while the second solution demands unrealistically small values for \(\Omega_0\), well below 0.1 (12). However, combinations of these two changes, together with the introduction of more exotic forms of dark matter, like decaying particles (36), may provide working models purely with isocurvature perturbations. Further, isocurvature perturbations need not have an inflationary origin, as cosmological defect models provide an alternative means of generating structure from isocurvature initial conditions.

In any case, clearly though the possibility of isocurvature perturbations is not yet ruled out from the point of view of structure formation, it is in much worse shape than the hypothesis of adiabatic perturbations, only surviving by appealing to a complex mixture of effects in the case of an inflationary origin, or by being associated with topological defects.

### 2.3 Passive vs. active perturbations

Perturbations can be **passive** (i.e. primordial) or **active**. Passive perturbations are those which are generated in the very early Universe (e.g. during inflation), and henceforth evolve passively, being changed only by the action of cosmic expansion and gravity. This means that the phases of such perturbations, in a Fourier expansion of the density field, will remain constant once the perturbations are generated, and as long as they evolve linearly. Therefore, if one adds them up over time, they add up coherently. It is in this sense that passive perturbations are usually also called coherent. Further, as soon as these perturbations enter the particle horizon, the photon-baryon fluid will try to flow into the potential wells associated with the perturbations, setting in motion acoustic oscillations in the fluid which will be in *in phase*. Primordial perturbations, adiabatic or isocurvature, always lead to such in-phase (or coherent) acoustic oscillations of the photon-baryon fluid.
On the contrary, (random) active perturbations, because they are constantly being produced \textit{randomly} across space, tend to produce an ensemble of perturbations whose phases will add up over time incoherently. Such perturbations necessarily lead to oscillations of the photon-baryon fluid which will be out-of-phase with each other, with the result that their effects will cancel themselves out. Active perturbations, like those produced by standard topological defect models, lead in general to incoherent acoustic oscillations in the photon-baryon fluid (though in a few models some coherence can temporarily exist on scales that have just entered the Hubble radius), as they are generated almost completely at random. However, active perturbations are not necessarily produced in such a way. Models of active perturbations have been proposed, though in a somewhat contrived manner, which seem to be able to produce perfectly coherent oscillations in the photon-baryon fluid, analogous to those produced by passive perturbations [see e.g. (76)]. However, clearly, the discovery that the photon-baryon fluid that existed before decoupling had coherent acoustic oscillations, would strongly boost the status of inflation as the best available explanation for the formation of structure.

In terms of large-scale structure, these oscillations only leave a detectable signature in the power spectrum of density perturbations if baryons comprise in excess of about 15 per cent of the total matter content in the Universe (61). The main problem is that non-linear evolution of the power spectrum washes out the signature of the baryon induced oscillations for large $k$, where the power spectrum can be better determined observationally, leaving out basically only the two peaks on the largest scales as a smoking gun, where observational errors are larger due to cosmic variance.

2.4 Gaussian vs. non-Gaussian perturbations

The two characteristics of the density perturbations generated in the simplest inflationary models that could, in principle, be more easily searched for in large-scale structure data are their initial Gaussian probability distribution and the near scale-invariance of their power spectrum.

In the simplest models of inflation, due to the nature of quantum fluctuations, the phases of the Fourier modes associated with the perturbations in the value of the scalar field are independent, drawn at random from a uniform distribution in the interval $[0,2\pi]$. Therefore, the density perturbations resulting from these perturbations will also be a superposition of Fourier modes with independent random phases. From the central limit theorem of statistics it then follows that the density probability distribution at any point in space is \textit{Gaussian}. As previously mentioned, this result is extremely important, since only one function, the power spectrum at horizon re-entry, is then required to
specify all of the statistical properties of the initial density distribution.

Present large-scale structure data cannot be used to say whether the the initial density perturbations followed a Gaussian distribution or not. One problem with the detection of Gaussian initial conditions is that as the density perturbations grow under gravity, their distribution increasingly deviates from the initial Gaussian shape. If the density contrast $\delta$ had a perfect Gaussian distribution, with dispersion $\sigma$, there would always be a non-zero probability of having $\delta < -1$ in some region of space, which is clearly unphysical. Therefore, the real distribution of density perturbations will always be at least slightly non-Gaussian, with a cut-off at $\delta = -1$, i.e. positively skewed. Further, once the (initially very rare) densest regions of the Universe start to turn-around and collapse, due to their own self-gravity, their density will increase much faster than that associated with most other regions at the same scale, which are still expanding with the Universe, i.e. evolving linearly. This leads to the development by the density distribution of a positive tail associated with high values for $\delta$, thus further increasing the skewness. These deviations do not matter as long as $\sigma \ll 1$, for then the probability that $\delta < -1$ or the existence of regions in the process of turn-around is extremely small. Thus, in the early stages of the gravitational evolution of non-correlated random density perturbations it is a very reasonable assumption to take their distribution as perfectly Gaussian. However, as the value of $\sigma$ approaches 1, the probability that $\delta < -1$ or of high values for $\delta$ cannot be neglected any longer, and assuming the density distribution to be Gaussian will induce significant errors in calculations.

In summary, gravitational evolution of density perturbations induces increasingly larger deviations from an initial Gaussian distribution, leading to a progressively more positively skewed distribution. Higher moments than the skewness, equally zero for a Gaussian distribution, are also generated in the process, though at increasingly later times for the same amplitude. This means that among these only the kurtosis (which compares the size of the side tails of some distribution against those of a Gaussian distribution) has developed significantly by today on scales larger than about 1 Mpc (on smaller scales astrophysical processes irremediably mess up the calculations).

In order to determine whether the initial probability distribution was Gaussian one then needs either to go to scales $R$ which are still evolving linearly, i.e. $\sigma(R) \ll 1$, or one needs to quantify, using gravitational perturbation theory [see e.g. (9)], the amount of skewness and kurtosis a Gaussian density field develops under gravitational instability for the values of $\sigma(R)$ observed.

If the values for the skewness and kurtosis were found to be well in excess of those expected, either the initial density distribution was non-Gaussian or structure did not form through gravitational instability. The problem is
that we presently do not have direct access to the density field, though this
will change in the near future by its reconstruction using data from large
areas of the sky searched for gravitational lenses [see e.g. (81, 62)].
Traditionally, the galaxy distribution has been used as a tracer of the density
distribution, being assumed that the skewness and kurtosis of the density field
should be the same as that of the galaxy field. However, it was soon realised
that the galaxies could have a biased distribution with relation to the matter,
i.e. \( \delta_{\text{gal}} = b \delta \) where \( b \) is the bias parameter. And a biased mass distribution
with respect to galaxies closely resembles one which is unbiased, but which has
a stronger degree of gravitational evolution. Dropping the assumption of linear
bias, by considering the possibility that the galaxy distribution might depend
on higher order terms of the density field, further complicates. Observationally
the situation is also not very clear, with often incompatible values for both
the skewness and kurtosis derived from the same galaxy surveys (41).

Another means of having access to the density field is by reconstructing it
using the galaxy velocity field, under the assumption that galaxies move solely
under the action of gravity. In fact, one can use directly the velocity field to
constrain the initial distribution of the density perturbations, through its own
skewness and kurtosis. The scaled skewness and kurtosis of the divergence
of the velocity field have the advantage of not depending on a possible bias
between the mass and galaxy distributions. But they depend on the value of
\( \Omega_0 \), in such a way that the velocity field of a low-density Universe is similar to
that of a high-density Universe whose density field has evolved further under
gravity. Nevertheless, by combining measurements of both the scaled skewness
and kurtosis it should in principle be possible to determine if the initial density
field was Gaussian (1). However, only the line-of-sight velocity component is
measurable, hence reconstruction methods, like POTENT (23), are used to
recover the full 3D velocity field based solely on it. But such methods have
their weaknesses, like for example the need for very good distance indicators,
which have up to today hindered the use of the velocity field to determine
whether the initial density perturbations followed a Gaussian distribution.

The topology of the galaxy distribution can also be used to determine whether
the initial density perturbations had a Gaussian distribution or not, if it is once
more assumed that it reflects the underlying properties of the density field.
The topological measure most widely used to distinguish between different
underlying distributions is the genus, which essentially gives the number of
holes minus the number of isolated regions, defined by a surface, plus one (e.g.
it is zero for a sphere, one for a doughnut). Up to today all measures of the
genus of the galaxy distribution seem compatible with it being Gaussian on
the largest scales, above about 10 h^{-1} Mpc [see e.g. (13)], with the prospects
of even tighter constraints coming from the big galaxy surveys under way like
the 2dF and the SDSS (15).
2.5 Scale-invariant perturbations?

The nearly scale-invariant nature of the initial perturbations, as expected from inflation, can only be probed using large-scale structure data insofar as one assumes a certain matter content in the Universe. This unfortunate situation results from the fact that after horizon re-entry the subsequent evolution of the density perturbations is greatly affected by both the type of matter present in the Universe and the cosmic expansion rate, which is in turn a function of the total quantity of matter present in the Universe.

The changes in the initial power spectrum of density perturbations, which in our notation is given by $\delta^2_H(k)$, are encapsulated in the transfer function, $T(k, t)$. Thus, unless one assumes a certain transfer function, we have no hope of recovering the shape of $\delta^2_H(k)$ from large-scale structure data. The evidence for $\delta^2_H(k) \propto k^{n-1}$, as expected in the simplest inflationary models (in particular $n \leq 1$), is therefore dependent on assumptions regarding the parameters that affect the transfer function. However, the choice of certain values for some of these parameters may come at a price, and be in conflict with observations not directly related with the power spectrum. For example, the observational data regarding light element abundances implies that the present cosmic baryon density, $\Omega_b$, is about $0.02 h^{-2}$, with an error at 95 per cent confidence of less than about 20 per cent, if homogeneous standard nucleosynthesis is assumed [see e.g. (77)]. Heavy tinkering with the value of $\Omega_b$ is tantamount to throwing away the standard nucleosynthesis calculation, thus requiring a viable alternative to be put in its place. It is also not possible to have any type of dark matter one might want to consider. If all the dark matter particles had high intrinsic velocities, i.e. if they were hot, like neutrinos, then the effect of free-streaming would completely erase all the perturbations on small scales, e.g. in the case of standard neutrinos on all scales smaller than about $4 \times 10^{14} (\Omega_0/\Omega_\nu) (\Omega_\nu h^2)^{-2} M_\odot$. Not only it would then be impossible to reproduce the abundance of high-redshift objects, like proto-galaxies, quasars, or damped Lyman-\alpha systems, but the actual present-day abundance and distribution of galaxies would be radically different from that observed. It seems very improbable that neutrinos presently contribute with more than about 30 per cent of the total matter density in the Universe [see e.g. (78; 18; 66)].

What is needed is to reformulate slightly the original question and ask instead whether for the simplest, best observationally supported assumptions one can make, the present-day power spectrum of density perturbations is compatible with an initial power-law shape, and in particular with one which is nearly scale-invariant. These assumptions are: the total energy density in the Universe is equal to or less than the critical density, i.e. $\Omega \leq 1$, and results from a matter component, plus a possible classical cosmological constant; the Hubble constant, in the form of $h$, has a value between 0.4 and 0.9; the baryon abun-
dance, in the form of $\Omega_b h^2$, is within 0.015 to 0.025; the (non-baryonic) dark matter is essentially cold, with the possibility of any of the 3 known standard neutrinos having a cosmologically significant mass. Among all these assumptions, the one which could be more easily changed without conflicting with non-large scale structure data is the nature of the (non-baryonic) dark matter. There could be warm, decaying, or even self-interacting, dark matter, though usually the existence of some contribution by cold dark matter is found to be required to fit all the available large-scale structure data.

In order to constrain the above free parameters in this simplest model, one needs at least the same number of independent observational constraints. The most widely used are: the slope of the galaxy/cluster power spectrum; the present number density of rich galaxy clusters; the high-redshift abundance of proto-galaxies, quasars and damped Lyman-\(\alpha\) systems; the amplitude of velocity bulk flows; the amplitude of CMBR temperature anisotropies, both on large-angular scales, as measured by COBE, and on intermediate-angular scales, as presently measured by balloon experiments. While the first constraint directly limits the slope of the density power spectrum, the next four constraints do such only indirectly, by imposing possible intervals for the amplitude of the density power spectrum at specific scales.

Other constraint that has started to be used recently, is the slope and normalisation of the density power spectrum on Mpc scales as inferred from the abundance and distribution of Lyman-\(\alpha\) forest absorption features in the spectra of distant ($z \sim 2.5$) quasars (19). However, these calculations depend on assuming a relatively simple physical picture for the formation of such features, being presently still unclear if such a picture provides a good approximation to reality.

The comparison of the above defined simplest structure formation models with CMBR anisotropy data is also presently not as clean as one would like. The two most important culprits are: the possibility of a gravity wave contribution at the COBE scale, thus allowing one to arbitrarily decrease the amplitude imposed by the COBE result on the density power spectrum; the possibility of re-ionization, which allows models with too much intermediate-scale power in the CMBR anisotropy angular power spectrum to evade the observational limits on such scales.

Again taking refuge in the simplicity assumption, the simplest models would be those with a negligible contribution of gravity waves to the large-angle CMBR anisotropy signal, together with no significant re-ionization. In any case, it should be mentioned that relaxing these two further assumptions does not open up a large region of parameter space (16).

Unfortunately, the comparison of these simplest structure formation models
with the observational constraints just described does not tell us much about the shape of the initial density power spectrum. The assumption of a power-law shape is perfectly compatible with the data, with the value of the spectral index being loosely constrained to be between 0.7 and 1.4 (78; 66). However, by imposing further restrictions on the type of structure formation model considered, stronger constraints can be obtained. For example, if the Universe was Einstein-de Sitter then the value of the Hubble constant would have to be smaller than about 0.55, in order for the Universe to be more than 12 Gyr old. This does not matter much, because high values for $h$ increase the amount of small-scale power relative to large-scale power and at the same time suppress power at intermediate angular scales on the CMBR, and for $h > 0.55$ these two effects join together to exclude almost all viable Einstein-de Sitter structure formation models in the context of the simplest assumptions laid down before (78). Restricting ourselves to $0.5 < h < 0.55$, then yields a preferred value for the spectral index $n$ roughly between 0.9 and 1.1, with a total neutrino density of $\Omega_\nu \sim 0.15$ [see e.g. (78; 31; 66). Another example, is the case of the $\Omega_0 = 0.3$ flat model, that preferred by the high-redshift type Ia supernova data of (68), for which values for the spectral index in excess of 1 tend to be preferred (78; 66). One should note however that both models are only marginally compatible with the observational data if $n = 1.0$.

Finally, it should be mentioned that, even within the simplified structure formation scenario we have been assuming, there is enough room for initial density power spectra with deviations from a power-law shape to be viable, as the survival of the broken scale-invariance model testifies (53). This only goes to show the still scarceness of good quality large-scale structure data at present.

In the near future the Sloan Digital Sky Survey (SDSS) will allow a much better constraint to be imposed on the value of $n$, by extending the measure of the slope of the galaxy power spectrum to larger scales, thus probing a region of the power spectrum which has not been in principle too much affected by the dark matter properties, retaining therefore more information about the shape of the initial density power spectrum (58).

3 The CMBR and inflation

The measurement of the characteristics of the anisotropies in the temperature of the CMBR can teach us a lot not only about the processes that gave rise to the density perturbations responsible for the appearance of the large-scale structures we observe today, but also about cosmological parameters, like the Hubble constant and the matter and energy content in the Universe, and physical events that happened between decoupling and the present, like re-ionization and the formation of structure (e.g. cluster formation through the
Sunyaev-Zel’dovich effect). However, there are so many possible ways in which the properties of the CMBR anisotropies can be altered, the most troublesome being by changing the nature of the initial perturbations and/or of the dark matter (energy), that in order to extract a tight interval of confidence for some parameter, e.g. the value of the Hubble constant, it is necessary to make \textit{a priori} some restrictions to the nature of the phenomena that might affect the characteristics of the CMBR temperature anisotropies.

Therefore, once again the strategy has to be to initially work within a simplified framework, and only if the observational data demands it, expand the initial assumptions so as to widen the allowed parameter space. At present, the simplified framework within which most analysis of the present CMBR anisotropy data are made has 10 associated parameters, which may or may not be allowed to vary freely, and assumes primordial, adiabatic perturbations with a Gaussian distribution. The dark matter is taken to be of only two possible types, cold or massive standard neutrinos, and any dark energy is assumed to behave dynamically as a classical cosmological constant. We will follow the choice of parameters of (73), as slightly different, in practice equivalent, sets of parameters can be worked with. We then have 5 cosmological parameters: $\Omega_k$, the spatial curvature, equal to $-k/a^2H^2$; $\Omega_{\Lambda}$, the energy density associated with a classical cosmological constant, $\Lambda$, which is equal to $\Lambda/3H^2$; $w_{cdm} \equiv \Omega_{cdm}h^2$, the matter density in the form of cold dark matter; $w_{\nu} \equiv \Omega_{\nu}h^2$, the matter density in the form of standard neutrinos; $w_{b} \equiv \Omega_{b}h^2$, the matter density in the form of baryons. There are also 2 parameters that characterise the primordial power spectrum of the scalar adiabatic perturbations, which is assumed to have a power-law dependence with scale, $\delta_H^2(k) \propto k^{n_s-1}$: $n_s$, the spectral index of the scalar perturbations; $A_s$, the contribution to the CMBR quadrupole anisotropy by scalar perturbations, which is in practice equivalent to using $\delta_H$, the amplitude of the scalar perturbation power spectrum. And 2 parameters that characterise the primordial power spectrum of tensor perturbations, i.e. gravity waves, which is also assumed to have a power-law dependence with scale, $P_t(k) \propto k^{n_t}$: $n_t$, the spectral index of the tensor perturbations; $A_t$, the contribution to the CMBR quadrupole anisotropy by tensor perturbations. Finally, there is 1 physical parameter: $\tau$, the optical depth to the surface of last scattering (equal to zero if there is no re-ionization after recombination occurred). From these quantities other well known parameters can be obtained, like the total matter density in the Universe, $\Omega_m = 1 - \Omega_k - \Omega_{\Lambda}$, and the value of the Hubble constant.

Again, it should be emphasised that any conclusions on the value of these parameters, obtained by comparing the theoretical expectation with the CMBR anisotropy data, is restricted to the tight framework imposed. The conclusions could be relaxed substantially if one was to consider introducing the added possibilities of part of the perturbations responsible for the CMBR anisotropies being active, incoherent, isocurvature, or with a non-Gaussian distribution,
or the existence of more exotic types of dark matter (warm, or decaying, or self-interacting) or dark energy which does not behave dynamically as a cosmological constant (e.g. an evolving scalar field).

Through the study of the distribution of CMBR temperature anisotropies in the sky at different angular scales, it is possible to confirm or disprove the assumption of a Gaussian distribution for the perturbations. If these have a Gaussian distribution, then the anisotropies should also have one. At present only the COBE data has been searched for evidence of deviations from Gaussianity. Early results [see e.g. (47, 30, 24)] were negative. However, more recently, there have been reports of strong evidence for deviations from a Gaussian distribution in the COBE data (28, 63, 65, 29, 60). However, some analysis continue to dispute the presence of a clear non-Gaussian signal (10, 4, 8).

Among the contradicting reports, it has become clear that the one-point distribution of the COBE anisotropies is indeed Gaussian, but the anisotropies seem not to be as randomly distributed across the sky as they should be under the Gaussian (random-phase) hypothesis, (11). The later is supported by the fact that the statistical techniques that find evidence for non-Gaussianity seem to be those which are most sensitive to phase-correlations. The source for these could be instrument noise, the method of extraction of the monopole, dipole or the Galactic contribution, unaccounted for systematic effects associated with the way COBE collected data (3), or of course real features in the microwave sky, which could be at the surface of last scattering or due to some unknown source of foreground contamination.

Observational evidence for a primordial and adiabatic origin for the perturbations that produced the CMBR temperature anisotropies is scarce at present, even after the release of the Antarctica long duration flight data from the BOOMERanG experiment (20, 51) and data gathered by the first flight of the MAXIMA balloon (32).

The evidence for the primordial nature of the perturbations will come from the signature that the acoustic oscillations induced in the photon-baryon fluid leave in the angular power spectrum of the CMBR anisotropies: a sequence of peaks in $\ell$ space. Their adiabatic nature is revealed by the spacing between the peaks: $\ell_1 : \ell_2 : \ell_3 : ...$ corresponds to ratios of 1 : 2 : 3 : .... Isocurvature primordial perturbations also generate a sequence of peaks in the CMBR temperature anisotropy angular power spectrum, but at different locations. The first peak due to the acoustic oscillations, generated by the potential wells that result from the isocurvature primordial perturbations after they enter the horizon, is moved to an higher $\ell$ value, and the ratios between the first three (real) peak locations are now 3 : 5 : 7 : .... The ratios between the peaks in both cases only depend weakly on the assumed cosmological parameters, and therefore provide an excellent test of the adiabatic or isocurvature nature of the fluctua-
tions (37, 38). Unfortunately, at present, though the BOOMERanG LDF and the MAXIMA-1 temperature anisotropy measurements go to sufficiently high \( \ell \) to probe the expected location of a second peak in the adiabatic case, the evidence is inconclusive. Nevertheless, purely isocurvature perturbations seem to be highly disfavoured just by the shape of the first peak (13). So, presently one cannot use solely CMBR anisotropy data to distinguish between an adiabatic or an isocurvature origin for the density perturbations. In any case, the present known location of the first acoustic peak (\( \ell \sim 200 \)) is sufficient to say that if the Universe is spatially flat then most probably the initial perturbations were adiabatic, as one then expects \( \ell_1 \sim 220 \). Isocurvature perturbations necessarily need the Universe to be closed, which moves the horizon scale at decoupling to larger angular scales, given that if the Universe was flat then the first acoustic peak for primordial isocurvature perturbations would be expected at \( \ell_1 \sim 340 \). If one admits the possibility that both adiabatic and isocurvature modes are present, then even with MAP it will be difficult to estimate the relative amplitude of both. One will have to wait for Planck to measure such a quantity with less than 10 per cent error (7).

As discussed before, the appearance of acoustic peaks in the angular power spectrum of the CMBR temperature anisotropies is not proof of an inflationary (or primordial) origin for the initial perturbations, though their absence could only be (realistically) explained by early and substantial (i.e. \( \tau \) close to unity) re-ionization of the intergalactic medium. However, the fact that at least one acoustic peak has been detected argues against the later scenario, and its presence is evidence for some degree of coherence of the initial perturbations (which does not mean their are primordial, as previously mentioned).

In summary, the primordial, adiabatic and Gaussian nature of the initial perturbations seems to be still a viable first assumption, providing the simplest coherent framework within which it is possible to explain the present CMBR temperature anisotropy data. Unfortunately, even under this framework the CMBR data presently does not tell us much about the parameters \( n_s, A_s, n_t, \) and \( A_t \), which characterise the power spectrum of the scalar and tensor perturbations in the simplest inflationary models.

Gravitational waves can only contribute significantly to the CMBR temperature anisotropies on scales above 1°, or \( \ell < 100 \), given that at the surface of last scattering any gravitational waves inside the horizon had their amplitude severely diminished by redshifting. This means that only the COBE and Tenerife data (73) might in principle be sensitive to the tensor spectral index \( n_t \). However, the effect of varying \( n_t \) can be easily masked by simultaneously varying the scalar spectral index \( n_s \), or by reducing the tensor contribution to the large-angle CMBR anisotropy to levels small enough that (almost) any variation of \( n_t \) could be accommodated within the measured errors. Therefore, at present there are no useful observational limits on the value of \( n_t \).
The ratio between $A_t$ and $A_s$, also sometimes called $T/S$, should in principle be a more readily measurable quantity, as it affects the height of the first acoustic peak relatively to the large-angle Sachs-Wolfe plateau, and also the CMBR anisotropy normalisation of the density perturbation power spectrum, thus affecting the formation of large-scale structure. By analysing these two effects, (84) concluded that values of $T/S$ as large as 4 are still viable, as long as $n_s$ is simultaneously allowed to go up to 1.2. Adding the initial data coming from the BOOMERanG and MAXIMA experiments seems to decrease the maximum possible value for $T/S$ to 2 (45). Restricting $n_s$ to be lower or equal to unity, given that this is what is predicted in the simplest inflationary models that produce a cosmologically significant background of gravitational waves, then the ratio $T/S$ would need to be lower than about 1. In particular, in the case of power-law inflation models the observational data seems to require $T/S < 0.5$.

Finally, $n_s$ is also not very well constrained at present by CMBR anisotropy data alone or in conjunction with structure formation data. The CMBR data by itself badly constrains $n_s$ if all the 10 parameters associated with the simplest inflationary motivated structure formation models are allowed to vary (73). By fixing the matter density in the form of baryons to be $\Omega_b h^2 = 0.02$ (as implied by observations if homogeneous standard nucleosynthesis is assumed), and a reasonable range for the Hubble constant, $0.55 < h < 0.75$, a interval for $n_s$ can be obtained, going from 0.8 to about 1.5 (3). However, values for $n_s$ higher than 1.3 can only be obtained at the cost of having a very large value for $T/S$ (73; 72). And in any case the the initial data from the BOOMERanG and MAXIMA experiments seem to disfavour values for $n_s$ above 1.2 (40; 43; 46). We have seen that structure formation observations do not constrain very well $n_s$ either, with values between 0.7 and 1.4 being possible. Joining all the data together would thus indicate that a value for $n_s$ close to 1 is preferable, with the possibility of going as low as 0.8 or as high as 1.3. The upper limit is also confirmed by bounds imposed on the production of primordial black holes (12).

Looking into the future, the prospects of measuring the inflationary parameters $n_s$, $A_s$, $n_t$, and $A_t$ from CMBR anisotropies are mixed. Data from balloon experiments like BOOMERanG and MAXIMA will not help much in improving these constraints, except for a better handle on the value of $n_s$. As mentioned, initial data from the two experiments has already further restricted the value of $n_s$ to being within about 10 per cent of unity (10; 13). The results from the MAP and Planck Surveyor satellites will however change the situation. All studies that have been done to date of the precision with which both MAP and Planck will be able to estimate different cosmological parameters use the so-called Fisher information matrix. This gives the manner in which experimental data depends upon a set of underlying theoretical parameters that one wishes to measure. Within this set of parameters, the Fisher matrix
yields a lower limit to error bars and hence an upper limit on the information
that can be extracted from such a data set, i.e. the Fisher matrix reveals the
best possible statistical error bars achievable from an experiment. With the
further assumption of Gaussian-distributed signal and noise, Fisher matrices
can be constructed from the specifications of CMBR experiments. However,
given that the Fisher matrix formalism in practice asks how well an experiment
can distinguish the true model of the Universe from other possible models, the
results for the error bars on the parameters of these models will depend on
which true model is chosen as input. Several groups have tried to estimate how
well MAP and Planck will be able to extract information on the cosmological
parameters from CMBR anisotropies using the Fisher matrix formalism [e.g.
[1, 8, 80]]. I will quote the results from the work by (25), who were proba-
bly those who more thoroughly explored, within the simplest model context
described before, the parameter degeneracies that may appear when the infor-
mation is extracted from the CMBR data. They even included the possibility
of a dependence with scale of the spectral index of density pertur-
bations [see also (17)]. Regarding $n_s$, (25) conclude that among the four inflationary quan-
tities it is the most readily measurable: just by using the CMBR temperature
anisotropy angular power spectrum it will be possible to estimate it with an
error of about 0.1 (1σ) through MAP and around 0.05 for Planck. Also using
the CMBR polarisation information the error tends to be halved in the case
of MAP and reduced by a factor of 5 for Planck. The other quantities, $A_s$, $n_t$, and $A_t$, will not be much better known after MAP and Planck, if just
the CMBR temperature anisotropy spectrum is considered. Only by including
polarisation information it will be possible to vastly improve the knowledge
we have about the values taken by $T/S$ and $n_t$.

The CMBR is expected to be partially linearly polarised [see (39) for a thor-
ough review]. The polarisation signal can be decomposed into two separate
and orthogonal components, the so-called $E$ and $B$ modes, with each mode
having their own associated power spectra, which will depend in a different
way on cosmological parameters. A cross-correlation signal between the tem-
perature anisotropy and $E$-mode polarisation maps will also be present [see
e.g. (44, 45)]. The knowledge of the polarisation signal will clearly contribute
to improving the CMBR constraints on cosmological and perturbation param-
eters, but not equally on all of them. Those parameters that most gain from
the use of the CMBR polarisation information are the ratio $T/S$, $n_t$ and the
optical depth $\tau$. The reason is that only vector or tensor perturbations can
give rise to $B$-mode polarisation, which can also arise from re-ionization of
the intergalactic medium. However, vector perturbations rapidly decay in an
expanding Universe, if all perturbations are primordial (e.g of inflationary ori-
gin). Consequently, while the total polarisation signal can help in the better
estimation of all parameters, the power spectrum information associated with
the $B$-mode polarisation will help particularly in bringing down the relative
errors in the estimation of $T/S$, $n_t$ and $\tau$, most noticeably if the first and
last are small (45, 23). However, from a practical point of view, it will not be easy to disentangle the CMBR polarisation signal from foreground polarisation sources, given the present lack of knowledge about their nature (42).

Interestingly, the polarisation signal can also provide one of the strongest tests known of the inflationary paradigm. Because polarisation is generated at the last scattering surface, models in which perturbations are causally produced, necessarily on sub-horizon scales, cannot generate a polarisation signal on angular scales larger than about $2^\circ$, approximately the horizon size at photon-baryon decoupling (71). Hence, polarisation correlations on such scales would indicate either a seemingly acausal mechanism for the generation of the perturbations or re-ionization at work. However, the two mechanisms can in principle be disentangled due to their different $\ell$ dependence on large-angular scales (82). If what seems like acausality at work is proven, then it has been argued (54) that inflation provides the only mechanism through which it can be generated, by expanding initialy sub-horizon quantum fluctuations to super-horizon sizes. Unless it is postulated the existence of super-horizon perturbations since the beginning of the Universe, which would give rise to a new initial conditions problem.

The measurement of the polarisation signal, by improving the estimation of $T/S$ and $n_t$, will also provide the possibility of checking whether the so-called inflationary consistency relation holds, $T/S \simeq -7n_t$ [e.g. (56)]. If this is shown than single-field slow-roll models will receive a tremendous boost. Note, however, that there are inflationary models which do not predict such a relation.

Finally, we have until now only discussed what future constraints can be imposed on the cosmological parameters and the nature of the perturbations solely through the CMBR temperature anisotropy and polarisation signals. Clearly it would help if some of the degeneracies inherent to CMBR analysis could be broken by using large-scale structure data and direct measurements of the geometry and expansion rate of the Universe. The most promising large-scale structure data to be expected in the near future is the SDSS data, which together with the already well known local rich cluster abundance, provides a means of constraining the slope and normalisation of the present-day density power spectrum. Supernova type Ia and direct measurements of the Hubble constant will enable independent estimates of respectively the geometry and expansion rate of the Universe. Further, the evolution with redshift of the number density of rich galaxy clusters is a powerful method for determining the present total matter content, $\Omega_0$, while gravitational lensing is able to impose interesting constraints on the value of a possible cosmological constant.
4 Direct detection of gravitational waves

Gravitons are the propagating modes associated with transverse, traceless tensor metric perturbations, and they behave as a superposition of two minimally coupled scalar fields, each corresponding to a polarisation state. As a result, the graviton field, which is massless, has a spectrum of quantum mechanical fluctuations similar to the one obtained for the scalar field $\phi$. For each polarisation state, the rms amplitude of the tensor metric perturbations associated with a given Fourier mode at horizon crossing is then \((\Delta h)^2\)

\[
(\Delta h)^2_k = \frac{V}{2\pi^2} k^3 |h_k|^2 = \frac{4}{\pi} \left( \frac{H}{m_{Pl}} \right)^2,
\]

where $V$ is the volume associated with the Fourier expansion, and the value of $H$ is to be calculated when the comoving scale $k$ crosses outside the Hubble radius during inflation. The phases of the Fourier modes $h_k$ are again independent and randomly distributed. Consequently, the power spectrum at horizon re-entry, $P_t(k)$, contains all the information necessary to describe the stochastic background of gravitational waves generated during inflation. Again, in general, $P_t(k)$ can be approximated as a single power-law,

\[ P_t(k) \propto k^{n_t}. \]

A scale-invariant power spectrum for the tensor perturbations then corresponds to $n_t = 0$. For the simplest single-field inflationary models, as long as the gravitational waves are produced during the slow-roll phase, necessarily $n_t \leq 0$. Further, under such conditions, and in the simplest models, the ratio between $P_g(k)$ and $\delta_H^2(k)$ is approximately equal to $-n_t/2$, which is another way of stating the so-called inflationary consistency relation (56). Note that in these models, if the expansion of the Universe during inflation is perfectly exponential, one gets $n_t = 0$, but then the amplitude of the tensor perturbation also effectively tends to zero compared with the amplitude of the (equally scale-invariant) scalar perturbation. However, in some slow-roll models with more complicated potentials, like intermediate inflation, it is possible to have an important tensor contribution to the perturbation spectrum at horizon re-entry though the power spectrum of density perturbations is scale-invariant.

If a gravitational wave background produced during inflation was detected using the CMBR, and it was possible to estimate both the amplitude and spectral index of the tensor perturbation contribution to the CMBR temperature anisotropy and polarisation signals, the next step would be to try to detect locally those gravitational waves. If successful, such a detection would
do much the same for the credibility of inflation as was achieved for the Hot Big Bang theory itself by the detection of the CMBR.

The simplest quantity to compare with the experimental sensitivity of gravitational wave detectors is the present-day contribution per logarithmic interval of the gravitational waves to the total energy density,

$$\Omega_{gw}(k) \equiv \frac{1}{\rho_c} \frac{d \rho_{gw}(k)}{d \log k},$$

where $\rho_c = 3H_0^2/8\pi G$ and $\rho_{gw}$ is the energy density of the stochastic background of gravitational waves with comoving wavenumber $k$. Given that

$$\Omega_{gw}(k) = \frac{1}{6} \left( \frac{k}{H_0} \right)^2 (\Delta h)^2_k,$$

we then get in the case of a initial scale-invariant power spectrum of gravitational waves

$$\Omega_{gw}(k) = \frac{2}{3\pi} \left( \frac{k}{H_0} \right)^2 \left( \frac{H}{m_{pl}} \right)^2.$$ 

The shape of the initial power spectrum is broken at the scale of matter-radiation equality, $k_{eq} = 6.22 \times 10^{-2} \Omega_0 h^2 \sqrt{3.36/g_*} \text{ Mpc}^{-1}$, where $g_*$ is the effective number of relativistic degrees of freedom (equals 3.36 for the standard cosmology with 3 massless neutrino species), as gravitational waves that enter the horizon prior to matter-radiation equality redshift more slowly with time. Given that we are only interested in gravitational waves that can be detected locally, we will concentrate on those which entered the horizon during radiation domination. These correspond to $k \ll 2 \times 10^{-24} \text{m}$, or $f \ll 10^{-16} \text{Hz}$, where $f = c k/2\pi$ is the frequency.

Working within the simplest inflationary models, those which obey the consistency relation $T/S \simeq -7n_t$, and requiring the CMBR temperature anisotropies detected on large-angular scales by COBE to be reproduced, one obtains (75)

$$\Omega_{gw}(k) = 5.1 \times 10^{-15} h^{-2} \left( \frac{g_*}{3.36} \right) \left( \frac{n_t}{n_t - 1/7} \right) \times \exp \left( n_t \ln \frac{k}{c H_0} \right).$$

If the power spectrum of tensor perturbations is not a perfect power-law there will be small corrections to this expression, which will be in principle more
important for the smallest scales, i.e. high $k$. A sensitivity of $\Omega_{gw}(k) h^2 \sim 10^{-15}$ is therefore needed for a serious search for local gravitational waves produced during inflation. With its initial strain detectors, the Earth-based LIGO (Laser Interferometer Gravitational Wave Observatory) should be able to identify a stochastic background of gravitational waves provided $\Omega_{gw} h^2$ is at least a few times $10^{-3}$, at its most sensitive operating frequency of roughly 100 Hz, with the limit dramatically improving by possibly 6 orders of magnitude with more advanced strain detectors installed in a later phase (59). Unfortunately, this misses the mark by six orders of magnitude.

Because the energy density in gravitational waves is proportional to the rms strain $\Delta h$ squared times frequency squared, a detector operating at lower frequency has better energy-density sensitivity for fixed strain sensitivity. Earth-based detectors cannot operate at frequencies below about 10 Hz because of seismic noise, but space-based operators can. The LISA (Laser Interferometer Space Antenna) mission has already been approved by ESA, with an initial predicted launch date for only around 2020, but which may be brought down to later this decade if NASA gets interested in a joint effort. It will have a peak sensitivity in terms of $\Omega_{gw} h^2$ of about $10^{-12}$ at a frequency close to $10^{-3}$ Hz (59), which is more promising, but still misses by at least three orders of magnitude the required sensitivity level for the detection of a local stochastic background of inflationary produced gravitational waves. Also, at frequencies above $10^{-4}$ Hz it is expected that the stochastic background of gravitational waves produced by compact white-dwarf binaries will swamp the inflationary signal. However, LISA may be able to disentangle the two backgrounds, given that it rotates in orbit and so it will be sensitive at different times to regions in the Galactic plane, where the binaries are, and outside (59).

5 Is the Universe flat?

One of the problems of standard cosmology is the near flatness of the Universe. For the present total energy density in the Universe to be within one order of magnitude of the critical density, i.e. $\Omega_{total} \sim 1$ today, at the Planck time ($10^{-43}$ s) the value of $\Omega_{total}$ could not deviate from unity by more than $10^{-60}$. This is one of the problems that can be solved by assuming the existence of an inflationary period in the very early Universe. The simplest models of inflation predict that $\Omega_{total}$ should in practice be equal to unity today, i.e. the Universe to be spatially flat. Therefore, these models would receive strong support if it was shown that presently $\Omega_{total} = 1$. Note that because there are inflationary models which predict the Universe not to be presently flat, then proof that $\Omega_{total}$ is different from unity today would not disprove the inflationary paradigm, but simply be evidence that if inflation indeed occurred than it did so in a more complicated fashion than it is generally assumed.
Recently, tantalising evidence has appeared that seem to indicate that the Universe is flat. There are several methods that directly or indirectly probe the geometry of the Universe. Those which presently provide the cleanest constraints on the geometry are the position of the first acoustic peak on the CMBR temperature anisotropy angular power spectrum and the magnitude-distance relation for Supernovae type Ia. Two other methods provide limits essentially on the total amount of non-relativistic matter in the Universe, \( \Omega_m \), the evolution with redshift of the abundance of rich galaxy clusters and deviations from Gaussianity measured either through the galaxy or the cluster velocity fields. Finally, the number of observed gravitational lensed high-redshift objects puts limits mainly on the possible contribution to the total matter density by a classical cosmological constant, \( \Omega_\Lambda \). Given that most analysis assume only these two possible contributions to the total energy density in the Universe, non-relativistic matter, \( p \approx 0 \), and a cosmological constant, \( p = -\rho \), I will not consider other eventual contributions with different equations of state, for example arising from an evolving scalar field.

Let me then summarise what we presently know about \( \Omega_{\text{total}} \), \( \Omega_m \) and \( \Omega_\Lambda \). This list does not pretend to be exhaustive, as some of the results cannot be expressed through a simple function of \( \Omega_m \) and \( \Omega_\Lambda \). Some of the limits were determined by (69) from results in the references given.

From the angular scale of the first acoustic peak in the CMBR anisotropy spectrum, under the assumption of Gaussian adiabatic initial perturbations: \( \Omega_{\text{total}} > 0.85 \) (10); \( \Omega_{\text{total}} = 1.15 \pm 0.20 \) (64); \( \Omega_{\text{total}} = 0.90 \pm 0.15 \) (2); \( \Omega_{\text{total}} = 1.11 \pm 0.07 \) (13). From the magnitude-distance relation for Supernovae type Ia: \( 0.8\Omega_m - 0.6\Omega_\Lambda = -0.2 \pm 0.1 \) (58). From the cluster abundance evolution with redshift, under the assumption of Gaussian initial perturbations: \( \Omega_m = 0.2^{+0.3}_{-0.1} \) (1); \( \Omega_m = 0.45 \pm 0.20 \) (20); \( \Omega_m > 0.3 \) (79); \( \Omega_m = 0.45 \pm 0.10 \) (33); \( \Omega_m = 0.75 \pm 0.20 \) (3). From the cosmic velocity field, \( \Omega_m > 0.3 \) (21) at more than 95 per cent confidence from the amplitude of diverging flows of galaxies from voids (22) and from the skewness of the velocity field assuming the initial density distribution to be Gaussian (64). From the gravitational lensing of objects at high-redshift: \( \Omega_\Lambda = 0.70 \pm 0.16 \) (14); \( \Omega_m > 0.26 \) (27); \( \Omega_m < 0.62 \) (16); \( -1.78 < \Omega_\Lambda - \Omega_m < 0.27 \) (27).

The quoted results indicate that the situation is still too confusing for one to be able to say with any degree of certainty which is the value of either \( \Omega_m \) or \( \Omega_\Lambda \). However, it seems clear that the best explanation for the combined data is an Universe which is spatially flat (69; 71).
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