Shrinking optical devices

W H Wee\textsuperscript{1} and J B Pendry

Imperial College London, Department of Physics, Condensed Matter Theory Group, London SW7 2AZ, UK
E-mail: w.wee07@imperial.ac.uk

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\textbf{Abstract.} Much of optics depends on objects being much larger than the wavelength of light: shadows of opaque objects are sharp only if free of diffraction effects, and ‘cat’s eye’ retroreflectors function only if they are large. Here, we show how to make theoretically arbitrarily small versions of these devices by exploiting the power of a negatively refracting lens to magnify objects that are smaller than the wavelength, thus creating the effect of a large object while keeping all physical dimensions small. We also give a new perspective on the ‘perfect lens theorem’ on which the paper is based.

\textsuperscript{1} Author to whom any correspondence should be addressed.
1. Introduction

The question is this: can we shrink optical devices indefinitely in a systematic way? The answer we hope to demonstrate in this paper is we can. The aim of this paper therefore is to demonstrate the use of negative refractive index materials (NRIM) [1]–[3] to create super-scatterers (SSs) [4]—scatterers with such an enhanced optical scattering cross section that they are larger than the physical size of the scatterer. A normal conventional scatterer can have an enlarged scattering cross section by using conventional lens to capture light rays that impact on the lens. However, such an endeavor can never generate an image bigger than the physical dimension of the scatterer (that is, the lens). In contrast, an SS can do much better, by capturing light rays that do not impact the lens, to form an image larger than the physical size of the lens (figure 1). Following on from the work of Pendry and Ramakrishna [5], related work on SS has been published by Yang et al [4], who considered a perfect electrical conductor (PEC) enclosed in a cylinder made of NRIM. More recently, a transparent type of SS has also been shown to create the effect of cloaking of dielectric objects at a distance [6].

In this paper, we shall demonstrate the SS enhancement of scattering cross section, by examining three specific SSs: the transparent SS [6], the absorber SS and the retro-reflector SS (RR SS). Firstly, through the transparent SS, we will show how an SS can map an internal space to an external space much larger than even the SS itself. Secondly, we will show how SSs can cast a larger shadow than the physical size of the scatterer through the absorber SS and, finally, we will demonstrate the use of SSs to create a super all-angle retroreflector (or an enhanced cat’s eye). With this intent, the paper is divided into three main parts. In section 2, we will present some background material and examine how to create an SS by using a combination

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In principle, indefinitely. In practice, the extent of shrinking depends on realistic limitations of the material such as losses, defects and inhomogeneity of the metamaterial.
of the perfect lens theorem (PLT) [5] and transformation optics [7]–[10], [11]. Following this, in section 3 we shall discuss the methods we used to analyze the scattering of electromagnetic waves from these SSs. Finally, in section 4, using the methods discussed in section 3 we shall examine scattering of electromagnetic waves from these three types of SSs.

2. How to create an SS

In this section, we will begin by introducing two basic recipes to creating an SS: the PLT (section 2.1) and transformation optics (section 2.2). After this, we will demonstrate the general strategy that one might adopt to create an SS by using these two concepts to create a cylindrical SS (section 2.3).

2.1. PLT

A lens made of a slab of negative refractive index material ($n = -1$) has been shown to focus both the propagating and evanescent modes of light, forming images with unprecedented resolution [12], limited only by the material losses and the inhomogeneity of the lens. Because of this property, such a lens has been coined by Pendry in [12] as the ‘perfect lens’.

Now the PLT is a generalization of this simple flat perfect lens to a spatially inhomogeneous anisotropic case [5].

Suppose that the half-space $z > 0$ is filled with a material with a spatially varying permittivity (similarly permeability) given by $\epsilon^\ast(x)$ (similarly $\mu^\ast(x)$) tensors,

$$
\epsilon^\ast(x, y, z > 0) = \begin{bmatrix}
\epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\
\epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\
\epsilon_{31} & \epsilon_{32} & \epsilon_{33}
\end{bmatrix} \tag{1}
$$

(where $\epsilon_{ij}$ are spatially varying functions with the indices $i, j = 1, 2, 3$ corresponding to the $x, y, z$-directions. Similarly, for $\mu^\ast(x)$ with $\epsilon_{ij} \rightarrow \mu_{ij}$). If the solution to Maxwell’s equations in this medium is known and given by the electric and magnetic fields $\{\mathbf{E}^\ast, \mathbf{H}^\ast\}$ and the displacement field and magnetic flux density $\{\mathbf{D}^\ast, \mathbf{B}^\ast\}$, then according to the PLT, when the
Figure 2. Illustration and summary of PLT. The yellow and blue regions are complementary and defined by the electric permittivity tensor $\epsilon^+$ and $\epsilon^-$ given by (1) and (2), respectively.

Other half-space $z \leq 0$ is filled with a material with permittivity (similarly permeability) given by $\epsilon^-(x)$ (similarly $\mu^-(x)$),

$$\epsilon^-(x, y, z < 0) = \begin{bmatrix} -\epsilon_{11} & -\epsilon_{12} & \epsilon_{13} \\ -\epsilon_{21} & -\epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & -\epsilon_{33} \end{bmatrix}, \tag{2}$$

the solutions to Maxwell’s equations in this medium will be given by $\{E^-, H^-\}$ and $\{D^-, B^-\}$, where $\{E^-, H^-\}$ is a mirror image of $\{E^+, H^+\}$ (about the $z = 0$ plane), while $\{D^-, B^-\}$ is a reflection with (local) inversion of $\{D^+, B^+\}$. That is, $\forall z \leq 0$,

$$E^- = [E^-_x(x, y, z), E^-_y(x, y, z)] = [E^+_x(x, y, -z), -E^+_z(x, y, -z)], \tag{3}$$
$$D^- = [D^-_x(x, y, z), D^-_y(x, y, z)] = [-D^+_x(x, y, -z), D^+_z(x, y, -z)] \tag{4}$$

(where the Greek indices $\alpha, \beta = 1, 2$ correspond to the $x, y$-direction).

A medium with permittivity and permeability given by $\epsilon^-, \mu^-$ is otherwise known as a complementary medium to one with $\epsilon^+, \mu^+$.

Now the PLT can be understood simply as a consequence of the symmetry property of the Maxwell equations (in the absence of free currents and charges) under spatial reflections. That is, Maxwell’s equations would be invariant when $E$ and $H$ fields transform as polar vectors (vector), whereas $D$ and $B$ fields transform as axial vectors (pseudo-vectors) under reflections. Since $\epsilon$ and $\mu$ are maps, which map $E$ and $H$ fields to $D$ and $B$ fields, $\epsilon$ and $\mu$ would transform into (2) under reflection. Equivalently, we can view the complementary medium as a means of performing a spatial reflection transformation on all the fields.

Before we proceed to see the implications of the PLT, it is instructive to examine how the propagating modes of light transform in a complementary medium. Now, since the Poynting vector defined by $S = E \times H$ is also a pseudo-vector, the Poynting vector would transform like $D$ and $B$ fields (from (4)), while the wavevector $k$, being a vector, would transform like $E$ and $H$ fields (from (3)) under reflection. Figure 2 shows a summary of the PLT.

Now the symmetries of the $\{E^\pm, H^\pm, k^\pm\}$ vectors and $\{D^\pm, B^\pm, S^\pm\}$ pseudo-vectors have the following implications: firstly, the symmetry for the Poynting vector $S^\pm$ implies that diverging rays from a source in a material will converge to a focal point in a complementary
medium equidistant from the interface. Thus we can see that such a complementary medium behaves as a flat lens at least for the propagating far fields. Secondly, because $E^-, H^-$ and $k^-$ are mirror images of $E^+, H^+$ and $k^+$, we note that the fields $E^+, H^+$ and the phase/wave vector $k^+$ reverse themselves once they pass through the interface into the complementary medium. So, decaying fields that approach the interface would appear to be amplified in the opposite manner as they pass through and move away from the interface into the complementary medium. Similarly, the phase accumulated within a medium is canceled out by the phase in the complementary medium so that the net phase gain is zero. Thus we see that not only does a complementary medium restore the phases of the propagating modes, it also restores all the evanescent modes so that the image formed within the complementary medium is indeed an exact replica or a perfect image of the object. Because the complementary medium reverses both the amplitude and the phases of the fields, exactly nullifying the effect of propagation in a medium, one can therefore view a complementary medium as optical antimatter.

Finally, a material is said to be a positive refractive index medium (PRIM) when $k^+ \cdot S^+ > 0$ and/or eigenvalues of $\epsilon^+, \mu^+ > 0$. Now it can be seen that according to PLT, a complementary medium to the PRIM would have $k^- \cdot S^- < 0$, and/or eigenvalues of $\epsilon^-, \mu^- < 0$, which is therefore an NRIM. Although NRIM properties cannot be found naturally, they can be accomplished using resonance effects in metamaterials [1]–[3].

2.2. Transformation optics

With the PLT, we can only create perfect lenses that are flat in geometry. Now we can generalize and extend the perfect lens to one with an arbitrary geometry using transformation optics [7]–[10], [11].

To begin with, we note that the $E$ and $H$ fields can be written using the language of differential geometry [13] as 1-forms $E, H$ given by

$$ E = E_i dx^i, \quad H = H_i dx^i, $$

whereas $D, B$ are 2-forms $D, B$ written as

$$ D = D_{ij} dx^i \wedge dx^j, \quad B = B_{ij} dx^i \wedge dx^j, $$

where $E_i, H_i, D_{ij}$ and $B_{ij}$ are the field components that we measure, while $dx^i$ and $dx^i \wedge dx^j$ are the basis.

Assuming linearity, the 1-forms are related to the 2-forms by the constitutive equations given by

$$ D = (\epsilon^{ij} E_i) \partial_j \partial^3, \quad B = (\mu^{ij} H_i) \partial_j \partial^3, $$

where $\partial^3$ is the spatial 3-form given by $\sqrt{g} dx^1 \wedge dx^2 \wedge dx^3$ and $g = \det(g_{ij})$ is the determinant of the metric.

Now, since the forms $E, H, D$ and $B$ are invariant/covariant objects under coordinate transformations, when we transform from one set of coordinates $\{\tilde{x}\}$ to another $\{x\}$, the 1-forms will transform under a chain rule as

$$ E = E_i dx^i = \tilde{E}_i d\tilde{x}^i = \left( E_j \frac{\partial x^j}{\partial \tilde{x}^i} \right) d\tilde{x}^i $$

where, by comparing the terms, we can see that in the new coordinates the measured field component is $\tilde{E}_i = E_j \frac{\partial x^j}{\partial \tilde{x}^i}$. Similarly, we can perform the same procedure for the 1-forms $H$ and
2-forms $D$, $B$. Using this fact of invariance/covariance and the transformations of 1-forms and 2-forms in equation (7), we can then show that the permittivity and permeability tensors $\epsilon_{ij}$ and $\mu_{ij}$ transform under coordinate transformation as

$$
\tilde{\epsilon}_{ij} = \epsilon_{\ell k} \frac{\partial x^\ell}{\partial \tilde{x}^j} \frac{\partial x^k}{\partial \tilde{x}^i} \left| \frac{\partial \tilde{x}^i}{\partial x^j} \right|^{-1},
$$

(9)

$$
\tilde{\mu}_{ij} = \mu_{\ell k} \frac{\partial x^\ell}{\partial \tilde{x}^j} \frac{\partial x^k}{\partial \tilde{x}^i} \left| \frac{\partial \tilde{x}^i}{\partial x^j} \right|^{-1},
$$

(10)

where $|\frac{\partial \tilde{x}^i}{\partial x^j}|$ is the Jacobian of the transformation$^3$.

So, in summary, if we have a map $f$ that relates a set of coordinates $\{\tilde{x}\}$ to $\{x\}$, then we say that the two systems are electromagnetically equivalent when the fields are transformed from $\{E_i, H_i\} \rightarrow \{\tilde{E}_i, \tilde{H}_i\}$ (8) and the permittivity and permeability tensors transform as $\{\epsilon_{ij}, \mu_{ij}\} \rightarrow \{\tilde{\epsilon}_{ij}, \tilde{\mu}_{ij}\}$ (according to (9) and (10)).

### 2.3. How to make an SS

Now, equipped with the tools of transformation optics and PLT, we can discuss the general strategy of creating an SS from a normal scatterer. Before we launch into a discussion of our method, it should be noted that our method is not the only method to create an SS. A folding method, it should be noted that our method is not the only method to create an SS. A folding map $f$ is used to enclose a normal scatterer with a map. The composite structure of the object (with $\epsilon_{ij}, \mu_{ij}$) is therefore our required SS.

To accomplish this, we need to use a suitable map $f$ that can map the compact regions 1, 2, 3 (with coordinates $\{x, y, z\}$ in figure 3) to planar slabs that we denote as regions $1', 2', 3'$ (with coordinates $\{u, v, w\}$). If such a map exists, we can use the concepts of transformation optics, to transform $\epsilon_1, \mu_1$ of region 1 (using (9) and (10)) to $\tilde{\epsilon}_1, \tilde{\mu}_1$ of region $1'$ (a spatially inhomogeneous PRIM slab)$^4$. Next, we can apply the concepts of PLT (using (2)) to determine $\tilde{\epsilon}_2, \tilde{\mu}_2$ of region $2'$ so that region $2'$ can be filled with a suitable NRIM that is complementary to the PRIM in region $1'$. Finally, we can apply transformation optics in reverse to determine the required $\epsilon_2, \mu_2$ from $\tilde{\epsilon}_2, \tilde{\mu}_2$. The composite structure of the object (with $\epsilon_3, \mu_3$) surrounded by a layer of NRIM (with $\epsilon_2, \mu_2$) is therefore our required SS.

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$^3$ We note that the permittivity and permeability tensors transform as second-rank tensor densities.

$^4$ The permittivity and permeability of regions 1, 2 and 3 are denoted by tensors $\tilde{\epsilon}_1$ and $\tilde{\mu}_1$, where the subscript $I = 1, 2, 3$ denotes the respective regions.
Figure 3. Illustration of the creation of SSs. A map $f$ is used to map regions 1, 2 and 3 to regions $1'$, $2'$ and $3'$. Transformation optics (TO) is used to determine the permittivity and permeability $\bar{\epsilon}_I$ and $\bar{\mu}_I$ in regions $1'$, $2'$ and $3'$ from the $\epsilon_I$ and $\mu_I$ in regions 1, 2 and 3, (where $I = 1, 2, 3$ is the index for the respective regions). PLT is used to determine the complementary medium.

Now, suppose that the permittivity and permeability of region 3 are given by $\epsilon_3(x)$ and $\mu_3(x)$ (general spatially inhomogeneous tensors). If $\Gamma'$ is a function that enlarges region 3 to fill up the optically canceled region (regions 1+2). Then, the effective permittivity and permeability, $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$, of the SS we have obtained in the procedure above would (from (9) and (10)) be given by

$$\epsilon_{\text{eff}}(\Gamma(x)) = \frac{\Gamma' \epsilon_3(x) \Gamma' \top}{|\Gamma'|}, \quad \mu_{\text{eff}}(\Gamma(x)) = \frac{\Gamma' \mu_3(x) \Gamma' \top}{|\Gamma'|},$$  \hfill (11)

where $\Gamma' = \frac{\partial \Gamma}{\partial x}$, and $|\Gamma'|$ is the determinant of $\Gamma'$.

If this scaling is isotropic (that is, $\Gamma(x) = \gamma x$, where $\gamma$ is the scaling factor to enlarge region 3), then (11) can be reduced to

$$\epsilon_{\text{eff}}(\gamma x) = \frac{\epsilon_3(x)}{\gamma}, \quad \mu_{\text{eff}}(\gamma x) = \frac{\mu_3(x)}{\gamma}. \hfill (12)$$

Now suppose that, on the other hand, we want to design an SS with an effective size of $\gamma \Omega_3$ (where $\Omega_3$ is the volume of region 3) with an effective permittivity and permeability $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$, we can reverse (12) to determine what permittivity and permeability $\epsilon_3$ and $\mu_3$ we should fill region 3 with.

5 If $\partial \Omega_3$ and $\partial \Omega_1$ are the outer boundaries of regions 3 and 1, respectively, then $\Gamma : \partial \Omega_3 \rightarrow \partial \Omega_1$. 

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2.4. Example—a cylindrical SS

Now that we have seen the general principles of how to create an SS in section 2.3, we will show an explicit example of how to create a cylindrical SS [8] using these principles.

Let \( \{ x^i \} = (x, y, z) \) be the Cartesian coordinates to describe a set of annular regions, where region \( I = 1, 2, 3 \) is defined by \( r_{i-1} < r \leq r_i \). Let \( \{ \bar{x}^i \} = (u, v, w) \) be the curvilinear coordinates describing a set of planar regions \( I' = 1', 2', 3' \) defined by \( u_{j-1} < u \leq u_j \). If we choose a map \( f : x \to \bar{x} \) such that

\[
  x = r_0 \exp(u) \cos(v), \quad y = r_0 \exp(u) \cos(v), \quad z = w,
\]

we can map annular regions \( I \) to planar regions \( I' \) as shown in figure 4.

Let the optical material parameters in region \( I \) be denoted by \( \epsilon_i, \mu_i \), and the electric and magnetic fields be \( \mathbf{E}^i \) and \( \mathbf{H}^i \) (where the index \( I = 1, 2, 3 \) denotes the region). Similarly, in the new coordinates let \( \bar{\epsilon}_i, \bar{\mu}_i \) and \( \bar{\mathbf{E}}^i, \bar{\mathbf{H}}^i \) correspond to the optical material parameters and fields in the equivalent region \( I' \).

Using (8) from transformation optics, and the map \( f \), defined by (13), we can map the electric and magnetic fields \( \mathbf{E}^i \) and \( \mathbf{H}^i \) in region \( I \) to the equivalent electric and magnetic fields \( \bar{\mathbf{E}}^i \) and \( \bar{\mathbf{H}}^i \) in region \( I' \). If the components of \( \mathbf{E}^i \) are given by \( \{ E^i_j \} = (E^i_x, E^i_y, E^i_z) \) and \( \bar{\mathbf{E}}^i \) by \( \{ \bar{E}^i_j \} = (\bar{E}^i_u, \bar{E}^i_v, \bar{E}^i_w) \) (where the index \( i, j \) denotes the field orientation) then:

\[
  \bar{E}^i_u = x.E^i_x + y.E^i_y, \quad \bar{E}^i_v = -y.E^i_x + x.E^i_y, \quad \bar{E}^i_w = E^i_z. \tag{14}
\]

Similarly, the \( \bar{\mathbf{H}}^i \) fields can be expressed in terms of the \( \mathbf{H}^i \) fields by \( E^i_j \to H^i_j \) and \( \bar{E}^i_j \to \bar{H}^i_j \) in (14).

Similarly, if the permittivity and permeability in Cartesian coordinates are given by

\[
  \epsilon_i = \mu_i = \text{diag} \left[ \epsilon^{(I)}_x, \epsilon^{(I)}_y, \epsilon^{(I)}_z \right], \quad r_{i-1} < r \leq r_i, \tag{15}
\]

then using the results from the previous section, we get the material parameters for the cylindrical SS (with physical size \( r_2 \)) as

\[
  \epsilon^{(1)}_x = +1, \quad \epsilon^{(1)}_y = +1, \quad \epsilon^{(1)}_z = +1, \quad r_2 \leq r,
\]

\[
  \epsilon^{(2)}_x = -1, \quad \epsilon^{(2)}_y = -1, \quad \epsilon^{(2)}_z = -\frac{r_3^4}{r_2^4}, \quad r_3 \leq r \leq r_2, \tag{16}
\]

\[
  \epsilon^{(3)}_x = +1, \quad \epsilon^{(3)}_y = +1, \quad \epsilon^{(3)}_z = +\frac{r_3^4}{r_2^4} \epsilon_{\text{eff}}, \quad r \leq r_3.
\]
where the effective permittivity (permeability) of the SS is given by:

\[
\epsilon_{\text{eff}} = \text{diag} [1, 1, \epsilon_{\text{eff}}] = \text{diag} \left[ 1, 1, \frac{\epsilon^{(3)}}{\gamma^2} \right], \quad r \leq r_1
\]  

(17)

(where \(\gamma = r_2^2/r_3^2\), and \(\epsilon^{(3)}_z\) is the permittivity of the core of SS) with an apparent effective size given by \(r_1\)—when in the new coordinates, region 1' is designed to optically cancel region 2', this means that the width of region 1' is equal to the width of region 2' (or \(u_2 - u_3 = u_1 - u_2\)) so that in Cartesian coordinates this implies that

\[
r_1 = \frac{r_2}{r_3}.
\]  

(18)

Hence, depending on what type of SS we want to design (that is, what \(\epsilon_{\text{eff}}\)) and what effective size of the SS we want (that is, what \(r_1\)), we can use (16) above to design such an SS.

Now, for the discussion of the later parts of the paper, we will examine the scattering of light from three particular types of cylindrical SS, given by three different \(\epsilon_{\text{eff}}\), namely the (i) transparent SS (with \(\epsilon_{\text{eff}} = 1\)), (ii) absorber SS (with \(\epsilon^{(3)}_x = \epsilon^{(3)}_y = \epsilon_{\text{eff}} = 1 + i \alpha\), where \(\alpha > 0\) is a measure of loss) and (iii) RR SS (with \(\epsilon_{\text{eff}} = 4\)).

3. Methods of analyzing scattering

In this section, we shall examine how electromagnetic waves interact with the cylindrical SS we have derived in section 2.4. We will first examine how the SS increases the scattering cross section by using simple ray-optics. Following this, we will show how a full-wave solution can be obtained using cylindrical wave expansion.

3.1. Ray optics

In this section, we will use a simple ray tracing technique to find out how a set of rays coming in from infinity will interact with a transparent SS. Again the labeling of the various regions in this discussion follows from the previous section (figure 4).

The trajectory of a ray in free space (region 1) is given by

\[
r = \frac{b}{\sin(\phi)}, \quad r \geq r_2.
\]  

(19)

where \(b\) is the impact parameter (\(r\) and \(\phi\) are the usual cylindrical polar coordinates). Now we know that region 1' is complementary to region 2', and as we have seen in section 2.1, the symmetry of the Poynting vector implies that the ray trajectories in region 1' are a mirror reflection of those in region 2'.

Transforming the ray trajectory in region 1 (from (19)) into the new coordinates, we have the following trajectory in region 1':

\[
r_0 \exp(u) = \frac{b}{\sin(v)}, \quad u \geq u_2.
\]  

(20)

Now invoking the symmetry about the interface of regions 1' and 2' that we have discussed in section 2.1, this implies that the ray trajectory in region 2' in the new coordinate system would be a mirror reflection of region 1' or:

\[
r_0 \exp(2u_2 - u) = \frac{b}{\sin(v)}, \quad u_3 \leq u \leq u_2.
\]  

(21)
Figure 5. (a) Ray trajectories for the transparent SS. The black solid lines are the boundaries of the NRIM lens, whereas the black dashed line is the annular region of vacuum that is optically canceled by the lens. The blue lines are the set of incident rays that impacts the lens, whereas the green lines are rays that do not. The red lines are self-closed loops that are localized optical modes evanescently coupled to the green lines. (b) Full-wave solution of the same transparent SS. The plot shows the color contour plot of field magnitude $|E_z|$ in the $x, y$-plane.

which we can then transform back to the Cartesian coordinates as

$$\frac{r^2}{r} = \frac{b}{\sin(\phi)}, \quad r_3 \leq r \leq r_2$$

the trajectories of rays in region 2.

For the innermost region, since the refractive index according to (16) is constant, we know that the trajectories are straight lines, the same as those in region 1 except that they appear compressed. Another way of seeing this follows from the perfect lens theorem, where regions 1 and 2 (or $1'$ and $2'$) cancel each other, and region 3 is designed to be optically matched to the rest of the vacuum; this implies that the trajectory in region $3'$ is merely a translation of the trajectory in region $1'$ (from (20)) by the distance of the canceled region ($u_1 - u_3$). Thus we get

$$r_0 \exp(u_1 - u_3 + u) = \frac{b}{\sin(v)}, \quad u \leq u_3,$$

which we can then transform back to the old Cartesian coordinates as

$$r = \frac{r_3}{r_1} \cdot \frac{b}{\sin(\phi)} = \frac{r^2_3}{r^2_2} \cdot \frac{b}{\sin(\phi)}, \quad r \leq r_3,$$

where we used (18) for the last step to express the trajectory as a function of the inner and outer radii of the NRIM annulus. Now, with equations (19), (22) and (24), we can completely describe the ray trajectories for our structure, which we plot in figure 5.

We note from these equations that the trajectories in all the regions are connected (or continuous) as long as $b \leq r_2$. This can be seen in figure 5. This forms the first set of trajectories.
that on impact with the NRIM lens are refracted and channeled within to the inner core. We also note that the trajectories within the inner core are compressed by a factor of $r_1/r_3$ just as expected.

The second set of trajectories in region 1 are those that do not impact the NRIM and appear to escape (the green lines in figure 5). These trajectories belong to the same set as trajectories in regions 2 and 3 that form self-closed loops (the red lines in figure 5). These self-closed loops are localized optical modes that are evanescently coupled to the trajectories in region 1 that seem to escape. In other words, this set of self-closed loops are an exact replica of the second set of trajectories in region 1. This implies that if we place an object in region 3, which interacts with these self-closed loops, the information on this interaction is transmitted to the rays outside in region 1, effecting a change. Conversely, any changes to the ray paths in these parts of region 1 would be reflected by changes to these self-closed loops. It is this unique feature that gives the cylindrical SS its ‘edge’ over conventional optics.

Now we note that the self-closed loops that are larger are coupled to trajectories in region 1 that are nearer to the NRIM lens. These trajectories are more strongly coupled as they are physically nearer than smaller self-closed loops that are coupled to trajectories further away. As such, we would expect any losses present in the NRIM to kill off these smaller self-closed loops first. Eventually, with a sufficient amount of losses one would expect all the self-closed loops to be killed off and the cylindrical perfect lens can only then capture only the rays that impact on the lens itself (the blue lines in figure 5(a)).

We also note that this evanescent coupling is due to surface plasmon resonances on the NRIM, so it is entirely a resonance phenomenon—in other words, such a coupling takes a finite amount of time—the same amount of time as it takes to build up the resonance [15]. This implies that any time-varying interactions with the light rays in region 3 will take a finite amount of time to be effected in region 1, so fluctuations that are faster than the time it takes to build up the resonance are not captured by this coupling. The time taken for a non-impact ray to become established as a resonance can be estimated from figure 6(a), where we show the effects of loss. If losses dissipate energy faster than the rate at which the resonance is being replenished, then the space where the resonance sits appears dark in the figure. It will be seen that the dark region creeps down towards the central region in figure 6(a) as losses are increased. The time taken to establish this resonance for a particular non-impact ray is therefore inversely proportional to the critical rate of energy loss that destroys this resonance. In general, it can therefore be shown that the smaller the self-closed loops, the longer it takes for this resonance to build up and therefore the harder it is to capture rapid fluctuations at the edges of region 3.

### 3.2. Full-wave solution

The discussion in the previous section on rays is simple enough for us to understand the reason why SSs can have a larger scattering cross-sectional area than conventional scatterers. However, we cannot determine precisely how losses will affect the performance of such a scatterer. In addition, we need a more general model that works not only for the transparent SS but also for a general class of SSs.

With this in mind, in order to be more precise and rigorous in our examination of electromagnetic interaction with our SS structure, we need to move beyond the ray analysis to the full-wave solution of Maxwell’s equations.

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Figure 6. (a) Contour plots of the $E$-field strength $\Re(E_z)$ ($a_1$ to $c_1$), and intensity ($|E_z|$) ($a_2$ to $c_2$) showing the effects of increasing losses, $\delta = 5 \times 10^{-4}$, $5 \times 10^{-3}$ and $5 \times 10^{-2}$, respectively. (b) Plot of the transmitted scattering coefficient $d_m$ as a function of the angular order $m$, for losses $\delta$ of various magnitudes in the NRIM lens ($r_2 = 20$, $r_3 = 10$). The two solid black lines define the boundaries of region 1 with $m_2 = k_1 r_2 = 15$ and $m_1 = k_1 r_1 = 30$.

In the same way as we have derived the trajectories in the ray model, we need to work out what the wave solutions are in the various regions. Following this, we need to match the wave solutions at each region using the continuity of fields boundary conditions. After that, we can then investigate how losses within the NRIM lens and changes to the permittivity (permeability) of the core of the SS would affect the scattering of the incident electromagnetic wave.
Now suppose that an incident plane wave (in polar coordinates) of the following form impacts the SS:

\[
E_z^{\text{inc}} = E_{z0} \exp(ik_1 r \cos \phi - i\omega t) = E_{z0} \sum_{m=-\infty}^{m=\infty} J_m(k_1 r) i^m \exp(im\phi - i\omega t),
\]  

(25)

where we assume that the electric field vector is parallel to the cylinder axis (or \(z\)-axis) and that the wave vector of magnitude \(k_1\) lies along the \(x\)-axis and is given by

\[
k_1 = \omega/c. \tag{26}
\]

In the second line of equation (25), the incident plane wave is decomposed into a sum of cylindrical waves using the Jacobi–Anger expansion [16] where \(J_m\) is a cylindrical Bessel function of the order of \(m\).

Now solving for Maxwell’s equations, given the above initial conditions, it can be shown that the field solutions for region 1 are given by

\[
E_z^{(1)} = E_z^{\text{inc}} + E_z^{\text{scatt}} = E_{z0} \sum_{m=-\infty}^{m=\infty} \left[ J_m(k_1 r) + a_m H_m^{(1)}(k_1 r) \right] i^m e^{i(m\phi - i\omega t)}, \quad r \geq r_2, \tag{27}
\]

where the first term is the incident cylindrical wave and the second term is the outgoing one that satisfies radiation conditions (given by a sum of Hankel functions of the 1st kind, \(H_m^{(1)}\)). Here \(a_m\) is the amplitude of the scattered cylindrical wave.

Now, to determine the field solution for region 2, we can solve Maxwell’s equations in cylindrical coordinates again with the permittivity and permeability given by (16) but a much simpler way is to make use of the symmetries in PLT. Using the symmetry in PLT, between \(E^\pm\) (that is, \(E_2^{(2)}\) is a mirror image of \(E_1^{(1)}\), the fields in region 2’ are related to that in region 1’ by

\[
E_z^{(1)}(u, v) = E_z^{(2)}(2u_2 - u, v), \quad u \leq u_2. \tag{28}
\]

Therefore the fields in region 2 would then be given by

\[
E_z^{(2)} = E_{z0} \sum_{m=-\infty}^{m=\infty} \left[ b_m J_m \left( k_2 \left[ \frac{r_2^2}{r} \right] \right) + c_m H_m^{(1)} \left( k_2 \left[ \frac{r_2^2}{r} \right] \right) \right] i^m e^{i(m\phi - i\omega t)}, \quad r_3 \leq r \leq r_2. \tag{29}
\]

where \(k_2 = k_1, b_m = 1\) and \(c_m = a_m\), in the lossless case.

To generalize to the lossy NRIM case, we see that a lossy region 2 is no longer complementary to region 1 (vacuum), but is complementary to a pseudoregion 1—which is a region 1 that has gain, where \(\epsilon^{(1)} = \mu^{(1)} = (1 - i\delta), \forall r > r_2\) and \(\delta > 0\) is a measure of the gain of the PRIM in this pseudoregion 1. This implies that for a lossy NRIM, the material properties become

\[
\epsilon^{(2)}(r) = (1 - i\delta)\epsilon_{\text{ideal}}^{(2)}(r), \quad r_3 \leq r \leq r_2, \tag{30}
\]

\[
\mu^{(2)}(r) = (1 - i\delta)\mu_{\text{ideal}}^{(2)}(r),
\]

where \(\epsilon_{\text{ideal}}^{(2)}\) and \(\mu_{\text{ideal}}^{(2)}\) are the ideal lossless material parameters given by (16), and \(\delta > 0\) is now a measure of losses in the NRIM. Using (30), the wave vector in region 2 would then be given by

\[
k_2 = (1 - i\delta)k_1. \tag{31}
\]

Consequently, \(b_m \neq 1\) and \(c_m \neq a_m\) in the general lossy case.
Finally, for region 3 with $\epsilon_f^{(3)} = \epsilon_{\gamma}^{(3)} = 1$, $\epsilon_z^{(3)} = \frac{r^d}{r^3} \epsilon_{\mathrm{eff}}$, ($\forall r \leq r_3$ and $\epsilon^{(3)} = \mu^{(3)}$)

$$E_z^{(3)} = E_{z0} \sum_{m=-\infty}^{m=\infty} [d_m J_m(k_3r)] i^m \exp(i m \phi - i w t), \quad r \leq r_3, \quad (32)$$

where

$$k_3 = \left(\frac{r_2}{r_3}\right)^2 \sqrt{\epsilon_z^{(3)} \epsilon_{\mathrm{eff}} \cdot k_1}. \quad (33)$$

(Note that for the case of an absorber SS, we have $\epsilon_f^{(3)} = \epsilon_{\gamma}^{(3)} = \epsilon_{\mathrm{eff}} = 1 + i \alpha$, and $\alpha > 0$ is a measure of loss.) We note that the fields given by (32) in region 3 are composed of only Bessel functions, so that the fields are finite at $r = 0$. Physically, $d_m$ is the amplitude of the transmitted wave of the order of $m$.

Now, with equations (27), (29) and (32), we have determined the general solutions of the fields in each of the regions. To have a consistent overall solution, we still need to determine the unknown scattering coefficients $a_m, b_m, c_m$ and $d_m$, subjected to boundary conditions at the interfaces of the various regions. That is, the tangential $E$ and $H$ fields must be matched and continuous across each interface

$$E_\parallel^I(r) = E_\parallel^{I+1}(r), \quad I = 1, 2, \quad (34)$$

$$H_\parallel^I(r) = H_\parallel^{I+1}(r), \quad I = 1, 2, \quad (35)$$

where $E_\parallel^I$ and $H_\parallel^I$ are the respective $E$- and $H$-field vectors projected onto the surface of the interface and the superscript again represents the regions that the fields are in. Using the $E$-field solution given by (27), (29) and (32), we can easily determine the $H$-field vectors using Maxwell’s equation:

$$\nabla \times E = -\mu \cdot H. \quad (36)$$

Since $E_\parallel = E_z \hat{z}$, using (36), we get

$$E_\parallel^I = E_z^I \hat{z}, \quad (37)$$

$$H_\parallel^I = H_\phi = \frac{-1}{\mu \mu_\parallel \omega} \cdot \frac{\partial}{\partial \phi} E_z^I, \quad (38)$$

which can then be used in (34) and (35) to determine the scattering coefficients.

Now, with the scattering coefficients $a_m, b_m, c_m, d_m$ and (27), (29), (32), we can fully determine the electric field and intensity distribution of an SS in the presence of an incident plane wave. Moreover, with these scattering coefficients, we can determine how each cylindrical wave component (of order $m$) individually would be affected by the presence of different scatterers. Of particular importance is the scattering coefficients $a_m$ and $d_m$, which are linked to the reflection and transmission coefficients of the cylindrical wave mode (of order $m$). So, when a wave mode of order $m$ scatters from a scatterer, it will have $a_m \neq 0$. Similarly, $|d_m|^2$ gives us a measure of the power transmitted into region 3 (for the ideal transparent SS, $|d_m|^2 = 1$).

Now if we know $a_m$, we can also determine the scattered field distribution large distances away. So if we take the asymptotic limits ($r \to \infty$) of the Hankel functions in (27), this would give us

$$E_z^{\mathrm{scatt}}(r \to \infty, \phi) = \left(\sqrt{\frac{2}{\pi k_1 r}} e^{i(kr - \pi/4)}\right) g(\phi), \quad (39)$$

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where \( g(\phi) \) is the angular distribution of scattered field amplitude at infinity, otherwise known as the scattering amplitude

\[
g(\phi) = a_0 + 2 \sum_{m=1}^{\infty} a_m \cos(m\phi) \]  \tag{40}

Equation (40) will be useful later when we determine the back-scattered intensity distribution \( \sim |g(\pi)|^2 \) of an RR SS in section 4.3.

### 3.3. The localization principle

Before we make use of the tools from the previous sections to analyze the scattering of an SS, it is instructive to note that the full-wave analysis from section 3.2 is related to the ray analysis in section 3.1. This connection is given by the localization principle [17], which states that a term of order \( m \) in the cylindrical wave expansion is related to a ray with an impact parameter \( b = m/k_1 \).

This can be understood if we examine the Bessel function \( J_m(k_1 r) \) of the cylindrical wave expansion of a plane wave (from (25)). We see that because

\[
|J_m(k_1 r)| \approx \begin{cases} 1 & k_1 r \geq m, \\ 0 & k_1 r < m, \end{cases}
\]  \tag{41}

it implies that a cylindrical wave of order \( m \), will not (normally) interact with the scatterer with dimensions \( d \), when \( d < m/k_1 \), in exactly the same way as a ray will not interact with a scatterer when its impact parameter \( b > d \).

Another way to understand this correspondence between rays and waves is the analogy with quantum (wave) mechanics. A light ray with momentum \( \hbar k_1 \) and impact parameter \( b \) has an angular momentum \( \hbar k_1 b \). Correspondingly, a cylindrical wave of the order of \( m \) has an angular momentum \( i\hbar \partial_\phi \psi = \hbar m \); so this means that a ray corresponds to a cylindrical wave mode \( m = k_1 b \).

Now we can apply the localization principle to relate the discussion on ray modeling of SS with the wave model. Previously, we have seen from section 3.1 (figure 5(a)) that there are two sets of rays: one set of (green) rays in region 1 (\( r_2 < r < r_1 \)) that do not impact the SS and a second set of (blue) rays that do. We claimed the set of (green) non-impact rays to be evanescently coupled to the internal optical modes (red rays); and we claim that it is this coupling that gives the SS a scattering cross section enhancement over the conventional scatterer. If such a coupling exists, then according to the localization principle, examining the corresponding cylindrical wave modes of order \( m_2 < m < m_1 \) \( (m_1 = k_1 r_1, m_2 = k_1 r_2) \), we would find that \( a_m, d_m \neq 0 \) \( (d_m = 1, \) for an ideal lossless transparent SS). In addition, we also claimed previously that increasing the material losses will kill the coupling for non-impact rays that are further away (with a larger impact parameter \( b \)). Correspondingly, we would then expect that increasing the losses would extinguish these cylindrical wave modes (that is, scattering coefficients \( a_m, d_m \rightarrow 0 \)), starting from modes with larger \( m \).

### 4. Scattering of an SS

Using the machinery from the previous sections, we can now study the full-wave interaction of an electromagnetic wave with our SS. In this section, we will examine the electromagnetic
interaction with three different SSs—the transparent SS, the absorber SS and the RR SS—and through this, we will demonstrate the capability of NRIM lens to enhance the scattering cross section beyond conventional optics.

4.1. Transparent SS

The cylindrical transparent SS that we will study here is defined by a scatterer with material parameters given by \((\varepsilon, \mu)\) with \(\varepsilon_{\text{eff}} = 1\) and \(\mu_{\text{eff}} = 1\), with an effective size \(r_1 = r_2^2/r_3\) (\(r_2\) and \(r_3\) refer to the outer and inner radii of the NRIM annulus). Hence, ideally, in the absence of losses, such an SS would be completely transparent to incoming electromagnetic waves. By examining how losses in the NRIM layer would affect the transparency of such an SS, we can understand the mechanism behind super-enhancement and how material losses can affect the performances of a general SS.

Using the results from the previous section, figure 5(b) shows a contour plot of \(\Re(E_z)\) as a function of \(x, y\) of an incident electromagnetic plane wave in the presence of an ideal lossless transparent SS.

Comparing the full-wave analysis in figure 5(b) with the ray traced results in figure 5(a), we can see that the rays in figure 5(a) are normal to the wave or phase fronts of figure 5(b) just as expected. We can also see that the magnitude of the electric fields are uniform in all the regions, so there is no distinction between the set of rays that impact the NRIM and the set that do not. In particular, the self-closed loops have the same intensity as the rays that are guided in by refraction.

In addition, we note that the number of wavelengths in the effective volume of the SS (that is, region 1, delineated by the dashed line in figure 5(b)) is the same as the number of wavelengths in region 3, demonstrating that region 3 is an optical equivalent of this effective volume albeit a compressed one.

Figure 6 shows a contour plot of \(\Re(E_z)\) and \(|E_z|\) for losses of various magnitudes in the NRIM of the transparent SS. Further to our discussion in section 3.1, we can see in figure 6(a) that as we increase the amount of losses in the NRIM \(\delta\) from \(\delta = 5 \times 10^{-4}\) to \(5 \times 10^{-2}\), the \(E\)-field strength and intensity in the core region decrease in extent. This effect is most acute at the edges near \(y = r_3\), which, as explained in section 3.1, is due to the killing of the coupling between the propagating modes (the green trajectory) in vacuum with the localized optical modes (the red trajectory) in the NRIM (see figure 5(a)).

Now applying the localization principle from section 3.3, we can discuss more quantitatively about the effects of losses on each cylindrical wave mode (of order \(m\)) by examining the effects of losses on the scattering coefficient \(d_m\) (the transmitted field amplitude).

In figure 6(b), we plot a graph of the transmitted field amplitude \(d_m\) in region 3 for the various orders of \(m\). As we have seen, optical modes with \(m_2 \leq m \leq m_1\) (\(m_1 = k_1r_1\) and \(m_2 = k_1r_2\)) correspond to the rays in region 1 that do not impact the SS. Now \(d_m(m = 1)\) indicates that the SS can perfectly capture light rays with impact parameter \(m/k_1\), whereas \(d_m(m) = 0\) indicates that the SS does not interact with the non-impact light rays at all. The effective size of the transparent SS can thus be defined by the value of \(m = m_{0.5}\) where \(d_m(m_{0.5}) = 0.5\), and the effective size is \(\sim m_{0.5}/k_1\). So from figure 6(b), for a loss \(\delta = 5 \times 10^{-4}\), the effective size of the SS is \(\approx 31\). As expected we can see that increasing the losses in the SS reduces the number nonzero \(d_m\) or number of optical modes captured; thus for \(\delta \lesssim 5 \times 10^{-2}\), only the rays that impact the SS are captured.
4.2. Absorber SS

When region 3 of the transparent SS is lossy—that is, \( \epsilon_r^{(3)} = \epsilon_r^{(SS)} = \epsilon_r^{eff} = (1 + i\alpha) \) in (16) (where \( \alpha > 0 \) is a measure of losses)—we have an absorber SS.

Such an SS (in the absence of losses in the NRIM layer) has an effective permittivity and permeability, \( \epsilon_r^{eff} = \mu_r^{eff} = 1 + i\alpha \), and an effective size \( r_1 = r_2^2/r_3 \) (\( r_2 \) and \( r_3 \) refer to the outer and inner radii of the NRIM annulus). Because the absorber SS is impedance matched to vacuum (since \( \sqrt{\epsilon_r^{eff}}/\mu_r^{eff} = 1 \)), this means that such an SS has minimal reflections and maximum absorption of the incident electromagnetic waves. Ideally, in the absence of losses in the NRIM, such an SS would behave like a perfect cylindrical absorber.

In this section, we will study the ability of this absorber SS to cast an enhanced shadow. To avoid confusion, the losses in the core region 3 are denoted by \( \alpha \) (\( \alpha = 0.23 \) in this study), whereas the losses in the NRIM lens are denoted by \( \delta > 0 \). An ideal absorber SS is given by an SS that has a lossless NRIM layer (\( \delta = 0 \)).

Now, from figures 7(A) and (B), we can see that an ideal absorber SS (NRIM layer with \( r_1 = 10 \) and \( r_2 = 20 \)) is completely electromagnetically equivalent (for \( r \gg r_1 \)) to a larger lossy cylinder (\( \alpha = 0.23 \)) with radius \( r_1 = 40 = r_2^2/r_3 \). We can see in the figure that the shadow cast by the smaller absorber SS has the same extent as the larger dielectric cylinder.

Now we can examine the effect of losses in the NRIM layer of the SS, by increasing \( \delta \). As we can see from figures 7(B) and (C), as we increase \( \delta \) from \( \delta = 0 \) to \( 5 \times 10^{-3} \), the extent of the shadow cast by the NRIM structure decreases as expected.

Again we can be more precise in examining the losses in the NRIM by looking at the relative scattering coefficient \( a_m^{\alpha} \), which is defined as

\[
 a_m^{(r)} = \frac{a_m^{(SS)}}{a_m^{(cylinder)}},
\]

where \( a_m^{(SS)} \) is the scattering coefficient of the SS with a lossy NRIM lens (with inner and outer radii defined as \( r_3 \) and \( r_2 \), respectively), and \( a_m^{(cylinder)} \) is the reflected scattering coefficient of the equivalent normal lossy dielectric cylinder of radius \( r_1 = r_2^2/r_3 \). Figure 8(a) is the graph of \( a_m^{(r)} \) as a function of the order \( m \), for an NRIM lens with \( r_3 = 10 \) and \( r_2 = 20 \). The relative scattering coefficient is plotted for various amounts of losses in the NRIM given by the loss tangent \( \delta \), ranging from \( \delta = 0 \) to \( 5 \times 10^{-2} \). In a similar manner to the previous section, \( a_m^{(r)} = 1 \) indicates that the SS scatters an optical mode of order \( m \) in the same way as a normal lossy cylinder. Conversely, when \( a_m^{(r)} = 0 \) it indicates that the optical mode of order \( m \) does not interact with the SS. Unlike the previous section, \( a_m^{(r)} \) can be greater than 1, and this implies that the SS does not scatter the optical mode \( m \) in the same way as a lossy cylinder does. Similarly to the previous section, the effective size of the SS (or size of the shadow cast) is approximately determined by the value of \( m = m_{0.5} \) where \( a_m^{(r)}(m_{0.5}) = 0.5 \), so the effective extent of the shadow cast is \( \sim m_{0.5}/k_1 \). So for a loss \( \delta = 5 \times 10^{-4} \) the effective size of the SS is \( \approx 28.7 \). A complete electromagnetic equivalence between an ideal absorber SS as compared with the lossy cylinder can be seen from \( a_m^{(r)} = 1, \forall m \), which can be seen in the \( \delta = 0 \) curve. With an increase in the amount of losses, the number of nonzero \( a_m^{(r)} \) terms for \( m_2 < m < m_1 \) decreases, showing once again that the losses in the NRIM layer diminish the ability of the SS to cast a bigger shadow. Again we can see from figure 8(a) that for \( \delta = \delta_c \approx 5 \times 10^{-2} \) the lossy absorber SS will only cast a shadow as big as the physical dimension of the SS (\( r = r_2 \)), which is as expected. Note once again that this critical loss \( \delta_c \) is consistent with the results from the previous section.

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Figure 7. Contour plot of field strength $\Re[E_z]$ (A1 to C1) and field magnitude $|E_z|$ (A2 to C2). The solid lines denote the physical boundaries, whereas the dashed line denotes the effective size of the ideal lossless SS ($r_1 = 40$). (A1 and A2) The contour plots for a normal lossy cylinder ($\alpha = 0.23$), of radius $r_1 = 40$. (B1 and B2) The extent of shadow cast by the ideal absorber SS (C1 and C2) for $\delta = 5 \times 10^{-3}$. For (B and C) the field for $r \leq r_1$ have been filtered out, as the field in this region diverges very rapidly, which would saturate the color plots.

Now we can also intuitively guess that as the amount of lossy NRIM material decreases, the absorber SS would also be less sensitive to losses. This is seen in figure 8(b) where the outer radius of the NRIM is now reduced to $r_2 = 15$. We can see that even when we have losses at $\delta_c = 5 \times 10^{-1}$ the lens can still capture light rays that do not impact the lens.

4.3. RR SS

4.3.1. Principles of retroreflection. An RR is a scatterer that has a strong directed back-scatter (backward reflection) so that a significant portion of the incident plane wave is reflected.
Figure 8. Plot of the relative reflected scattering coefficient $a_m^r$ as a function of the order $m$ of the Bessel functions, for increasing amount of the loss tangent $\delta$ in the NRIM lens. The two solid black lines define the boundaries of region 1. (a) $m_2 = k_1 r_2 = 15$ and $m_1 = k_1 r_1 = 30$ and (b) $m_2 = k_1 r_2 = 18$ and $m_1 = k_1 r_1 = 27$.

backwards in exactly the opposite direction. The quality of an RR is measured by the magnitude of the back-scattered intensity and how divergent the back-scattered radiation is. The higher the back-scattered intensity and the narrower the back-scattered beam, the better the RR is. Now a mirror for instance can be considered as an RR, but only when the incident plane wave is normal to the mirror. All RRs have an angle of acceptance, that is, a range of incident angles such that the incident plane wave can be retro-reflected backwards. So in the case of a mirror, the angle of acceptance is zero, while a corner cube (two perpendicular mirrors) has an angle of incidence $\approx \pm 45^\circ$. An all-angle RR is an RR that has a full $360^\circ$ (or $\pm 180^\circ$) angle of acceptance.

It can be shown that a cylinder with refractive index $n = 2$ behaves like such an all-angle RR $[18]$. The principle of operation is based on that of a backward (primary) rainbow. It can be shown from geometrical optics (ray tracing) $[19]$ that when the refractive index of the cylinder $n = 2$, the position of the rainbow caustics lies directly in the opposite direction to that of the incident light rays. That is, a bundle of incident rays with (close to) zero impact parameters are reflected backwards from the back surface of the cylinder.

Such an RR has a number of properties that are useful for our discussion later. Firstly, the back-scattered field magnitude $|E|_{\infty}$ (at $r \to \infty$, $\phi = \pi$) should increase monotonically with the radius, $d$, of the retro-reflecting cylinder, so larger RRs have better retroreflection. If geometrical optics is applied (for sufficiently large $d$), we would expect the back-scattered field magnitude, $|E|_{\infty}$ (at $r \to \infty$, $\phi = \pi$), to be approximately linearly proportional to $d$. This is because the incident flux (of rays that will be back-scattered) is linearly proportional to $d$, and the angle of divergence of the retroreflected rays is inversely proportional to $d$ (for sufficiently large $d$). So the combined effect is that the intensity of the back-scattered wave $I_{\infty}$ (at $r \to \infty$, $\phi = \pi$) is proportional to $d^2$. For the range of $d$ in this paper, more accurate numerical result shows $I_{\infty} \propto d^{1.5}$ and $|E|_{\infty} \propto d^{0.73}$.

Secondly, the magnitude of the back-scattered field $|E|_{\infty}$ is also dependent on the interference effects between the light rays reflected from the front and the back of the

$\text{It is also important that the cylinder is impedance mismatched to allow for reflections.}$

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Figure 9. (a) Plot of the back-scattered field magnitude at infinity $|E|_\infty$, as a function of the radius $d$ of a dielectric cylinder with refractive index $n = 2$ and 1.6, respectively. The cylinder with $n = 2$ behaves like an RR and $\langle |E|_\infty \rangle \propto d^{0.73}$. (b) Plot of $\langle |E|_\infty \rangle$ as a function of the radius $d_2$ of the various scattering cylinder where the oscillations due to interference are filtered out. The green dashed curve is the scattering due to an ideal lossless RR SS, whereas the red dashed curve is due to an ordinary retroreflecting cylinder of the same physical size. All other curves are for the lossy SS. The inset shows the structure of the SS.

cylinder. This means that when we plot $|E|_\infty$ as a function of $d$, other than the overall monotonic dependence on $d$, we will also get oscillations (with spatial frequency $2k_1$) due to the constructive and destructive interference of these rays (see figure 9(a)). Here $\langle |E|_\infty \rangle$ refers to the back-scattered field magnitude with oscillations due to interference effects averaged out.

Finally, because only the bundle of rays with (close to) zero impact parameters are eventually retroreflected, this implies that from the localization principle only cylindrical wave modes with low orders of $m \sim 0$ are important in retroreflection.

4.3.2. Results. When region 3 of the SS with radius $d_2$ has material parameters given by (16) with $\epsilon_{\text{eff}} = 4$, then the SS has an effective refractive index of $n_{\text{eff}} = 2$. Such an SS will become a retroreflecting cylinder with a physical radius $d_2$ but an effective size $d_1 = d_2^2/d_3$ ($d_2$ and $d_3$ are the outer and inner radii of the NRIM annulus), and this we will refer to as an RR SS (figure 10).

Now since we know from section 4.3.1 that any retro-reflecting cylinder has an averaged back-scattered field magnitude that scales monotonically with the radius of the cylinder ($\langle |E|_\infty \rangle \propto d^{0.73}$), an ideal lossless RR SS would give an averaged back-scattered intensity $\langle I_\infty \rangle$ that is approximately $(d_2/d_3)^{1.5}$ larger than a normal retroreflecting cylinder of the same radius.

Using the results from section 3.2 (or (39)), we can plot a graph of $\langle |E|_\infty \rangle$ as a function of $d_2$ (figure 9(b)) for the ideal lossless RR SS (with $d_2/d_3 = 2$) and show its superiority over an ordinary retroreflecting cylinder of the same physical size. The effective size of the (ideal/lossy) RR SS is defined as the size of a normal retroreflecting cylinder, which would give the same field magnitude $\langle |E|_\infty \rangle$ as that of the (ideal/lossy) RR SS. As can be seen from figure 9(b), an ideal RR SS has an effective size $d_1 = d_2^2/d_3$ (for all values of $d_2$). Intuitively, we would expect losses in the NRIM layer to degrade the performance of the RR SS, so for a lossy RR SS with

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Figure 10. Contour plot of $|E_z(r, \phi)|$ showing the retroreflection of an RR SS (with $d_3 = 12.35$, $d_2 = 21.525$) that has an effective size of $d_1 = d_2^2/d_3 = 37.5$. This scattering pattern is exactly equivalent to a normal RR with a radius 37.5.

$\delta = 5 \times 10^{-4}$ the effective size of the RR SS (with $d_2 = 20$ and an effective radius $d_1 = 40$) is reduced to $d_1 = 35$. Indeed, we would expect that further losses would eventually degrade the performance of the RR SS. However, it is not always the case that losses would degrade performance. As shown in figure 9(b), a small amount of losses in the NRIM (loss tangent $\delta = 10^{-9}$) actually enhances the retroreflection instead! So for a lossy RR SS with $\delta = 1 \times 10^{-9}$ the effective size of the RR SS (with $d_2 = 20$) is approximately $d_1 = 47$, which is even larger than the effective size of an ideal RR SS ($d_1 = 40$)!

Now we can see this more explicitly by considering the gain, $G(\phi)$, of a scatterer—defined as the ratio of the scattered intensity of a scatterer as compared with an isotropic scatterer (at $r \to \infty$). Using (40) and this definition of gain, we can plot a graph of $G(\phi)$ as shown in figure 11. We can see that a PEC cylinder has a gain $G_{\text{PEC}} \approx 1$, while a normal retroreflecting cylinder, which has a gain that scales monotonically as $d_2$, outperforms a PEC. In turn, it can be seen that the ideal RR SS has a gain $(d_2/d_1)^{1.5}$ better than a normal retroreflecting cylinder of the same size. Once again it is remarkable and counter intuitive to note that when we have a small amount of loss, the lossy RR SS has a gain that scales even better than the ideal case. Of course there is no magic and, as expected, a further increase in losses would eventually degrade the performance of the RR SS, as can be seen in the reduction in gain in figure 11.

Now we can be more quantitative about the effects of losses when we plot a graph of the back-scattered intensity $I_\infty(r \to \infty, \phi = \pi)$ (or equivalently gain) as a function of loss $\delta$ in the NRIM (figure 12). We can clearly see that for loss $\delta \lesssim 10^{-4}$ a lossy RR SS has an $I_\infty$ equal to or greater than the ideal RR SS. As losses increase beyond $\delta \gtrsim 10^{-4}$, the gain of the lossy RR SS falls below the ideal RR SS, and for loss $\delta \gtrsim 10^{-2}$, the lossy RR SS would eventually lose its enhanced retroreflection completely.

As can be seen from these results, the RR SS is more robust to loss than the absorber SS in section 4.2, where the extent of shadow cast for the latter is always decreasing with increasing losses. In addition, if we compare the performance of an absorber SS with an RR SS with approximately the same structure ($d_2$ and $d_3$) and loss $\delta = 5 \times 10^{-4}$, a lossy absorber SS has a reduced effective radius $\approx 1.4d_2$, whereas a lossy RR SS has a larger effective radius $\approx 1.7d_2$ (an ideal SS has an effective radius $2d_2$). This robustness to loss for an RR SS can be partially understood in the following way. Firstly, retroreflection relies only on rays with low impact parameters (or optical modes with small $m$). Second, from geometrical optics, the retroreflection is strictly a result of reflection of these rays from the back surface of the
Figure 11. Gain as a function of back-scattered angle where 0 rad is the exact opposite direction to the incoming plane wave for various scatterers. (a) The gain for an RR SS with $d_2/d_3 = 2$, where a loss $\delta = 1 \times 10^{-9}$ RR SS has the best gain. Similarly, figure (b) is for an RR SS with $d_2/d_3 = 1.5$, where a loss $\delta = 7 \times 10^{-5}$ RR SS has the best gain.

Figure 12. Graph of back-scattered intensity $I_\infty (r \to \infty, \phi = \pi)$ as a function of loss $\delta$ in the NRIM. The structure of the RR SS is given in the inset with $d_2/d_3 = 2$. The green dashed line is $I_\infty$ for the ideal RR SS, whereas the red dashed line is $I_\infty$ for an ordinary retroreflecting cylinder of the same radius.

This means that only paraxial or $m \approx 0$ modes are involved; higher order $m$ modes are transmitted through the cylinder. As we can see from figures 6(b) and 8 and the discussion in the previous sections, losses in the NRIM layer would kill off rays/modes with higher order $m$ first, leaving the lower order $m$ modes otherwise unaffected; this explains why the RR SS is more robust to loss. In addition, losses in the NRIM layer improve this back reflection, by making the back surface of the retroreflecting cylinder more reflective; hence a
slightly lossy RR SS can capture more slightly higher order $m$ modes than would otherwise be transmitted through. This explains why for losses less than some critical value, a lossy RR SS retroreflects better than an ideal RR SS.

5. Conclusion

Through the use of PLT and transformation optics, we have shown how to create and analyze SSs—scatterers with an effective scattering size much larger than the physical size of the scatterer. In this paper, we have examined the scattering enhancement of three types of SS—the transparent SS, the absorber SS and the RR SS—and the effects of loss in the NRIM layer on the scattering enhancement.

The transparent SS is the simplest SS possible, and through it we can understand the mechanism of scattering enhancement, and how losses degrade this enhancement. Through the absorber SS, we can see the effects of this enhancement explicitly when we observe that the shadow cast by the SS is larger than even the physical extent of the SS. Similarly, we observe that losses in the NRIM layer of the absorber SS reduce this enhanced shadow casting. Finally, we have shown how to create an all-angle-super-retroreflector using the RR SS. Notably, the RR SS has an interesting counter-intuitive property that small amounts of losses actually enhance retroreflection such that a slightly lossy RR SS performs better than even an ideal lossless RR SS. This tolerance to loss can be understood to be the result of retroreflection being dependent only on low impact parameter rays (waves with low angular order $m$) and these low $m$ optical modes are more tolerant to loss.

The concept of the SS as demonstrated in this paper is useful for any application where we want a physically small object to create an image much larger than its physical size. Such applications could include making ultracompact subwavelength antennas or allowing an otherwise small object to appear larger on the radar. The RR SS, in particular, can be used to create an ultracompact, energy-efficient passive transmitter—that is, the power source for transmission of signals comes mainly from the receiver, and the only energy required is in modulating the retroreflecting property of the RR SS. Such an application could be useful for communication or for tracking a distant satellite or explorer (Mars rover, etc) where energy efficiency and compactness are important.

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