Noncommutative Orlicz spaces over $W^*$-algebras

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Abstract

Using the Falcone–Takesaki theory of noncommutative integration, we construct a family of noncommutative Orlicz spaces that are canonically associated to an arbitrary $W^*$-algebra without any choice of weight involved, and we show that this construction is functorial over the category of $W^*$-algebras with $*$-isomorphisms as arrows.

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1 Introduction

Construction of standard representation by Connes, Araki and Haagerup and its further canonical refinement by Kosaki allows to assign a canonical $L_2(\mathcal{N})$ space to every $W^*$-algebra $\mathcal{N}$, without any choice of a weight on $\mathcal{N}$ involved. This leads to a question: is it possible to develop the theory of noncommutative integration and $L_p(\mathcal{N})$ spaces for arbitrary $W^*$-algebras $\mathcal{N}$ along the lines of analogy between Hilbert–Schmidt space as a member $\mathfrak{S}_2(\mathcal{H})$ of the family of von Neumann–Schatten $\mathfrak{S}_p(\mathcal{H})$ spaces and a Hilbert $L_2(\mathcal{X}, \mathfrak{U}(\mathcal{X}), \tilde{\mu})$ space as a member of a family of Riesz–Radon $L_p(\mathcal{X}, \mathfrak{U}(\mathcal{X}), \tilde{\mu})$ spaces? Falcone and Takesaki [24] answered this question in the affirmative by an explicit construction, showing that to any $W^*$-algebra $\mathcal{N}$ there is a corresponding noncommutative $L_p(\mathcal{N})$ space, and, furthermore, this assignment is functorial.

In this paper we extend their result by constructing a canonical family of noncommutative Orlicz spaces $\mathcal{Y}_p(\mathcal{N})$ for arbitrary $W^*$-algebras $\mathcal{N}$ that is associated to every $W^*$-algebra $\mathcal{N}$ for any choice of an Orlicz function $\mathfrak{Y}$. This association is functorial: every $*$-isomorphism of $W^*$-algebras induces a corresponding isometric isomorphism of noncommutative Orlicz spaces. We also infer basic properties of these spaces. Further study of these spaces will be carried in another paper. Section 2 contains a brief introduction to main notions of theory of $W^*$-algebras and integration over semi-finite $W^*$-algebras. Section 3 provides an overview of the Falcone–Takesaki integration theory over arbitrary $W^*$-algebras. The discussion of commutative and noncommutative Orlicz spaces together with our construction is contained in Section 4. In Section 5 we show that our construction of noncommutative Orlicz spaces is functorial. For an extended discussion of all notions related to $W^*$-algebras and noncommutative integration we refer to [39] as a recent overview (close to the spirit of [76]) which has precisely the same notation and terminology as we use here, as well as to [83, 88] as standard references.

2 Integration on semi-finite $W^*$-algebras

A $*$-algebra is defined as an algebra over $\mathbb{C}$ equipped with an operation $*: \mathcal{C} \to \mathcal{C}$ satisfying $(xy)^* = y^*x^*$, $(x + y)^* = x^* + y^*$, $x^{**} = x$, $(\lambda x)^* = \lambda^*x^*$, where $\lambda^*$ is a complex conjugation of $\lambda \in \mathbb{C}$. A Banach space $\mathcal{C}$ over $\mathbb{C}$ is also a $*$-algebra with unit $1$ called: a Banach $*$-algebra iff $|x| = |x^*|$ and $\|xy\| \leq \|x\\|\|y\| \forall x, y \in \mathcal{C}$; a $C^*$-algebra iff $\|x^*x\| = \|x\|^2$. Given any $C^*$-algebra $\mathcal{C}$ the opposite algebra $\mathcal{C}^\circ$ is defined as a $C^*$-algebra which has the same elements and norm as $\mathcal{C}$, but the opposite multiplication (maps that is, if $x, y \in \mathcal{C}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, then $x^\circ, y^\circ \in \mathcal{C}^\circ$ satisfy: $(\lambda_1 x + \lambda_2 y)^\circ = \lambda_1 x^\circ + \lambda_2 y^\circ$, $(x^\circ)^\circ = (x^\circ)^*$, $(xy)^\circ = y^\circ x^\circ$). A $W^*$-algebra is defined as such $C^*$-algebra that has a Banach predual. If a predual of $C^*$-algebra exists then it is unique. Given a $W^*$-algebra $\mathcal{N}$, we will denote its predual by $\mathcal{N}_*$. Moreover, $\mathcal{N}_+ := \{\phi \in \mathcal{N}_* \mid \phi(x^*x) \geq 0 \forall x \in \mathcal{N}\}$,
\(N_{x0}^+ := \{ \phi \in N^+ \mid \omega(x^*x) = 0 \Rightarrow x = 0 \ \forall x \in N \}, \ N_{x1}^+ := \{ \phi \in N^+ \mid \|\phi\| = 1 \}, \ N_{x\text{unii}}^+ := \{ x \in N \mid xx^* = x^*x = 1 \}, \ N^\text{sa} := \{ x \in N \mid x = y^*y \}, \ \text{Proj}(N) := \{ x \in N^\text{sa} \mid xx = x \}. \)

An element \(x \in N\) is called: \textit{partial isometry} iff \(x^*x \in \text{Proj}(N); \ \textit{absolute value} of \ y \in N, \ \text{denoted} \ x = |y|, \ \text{iff} \ y^*y = x^2. \) The elements of \(N_{x+}^+\) will be called \textit{quantum states} or \textit{states}. For \(N = \mathcal{B}(H)\), where \(\mathcal{B}(H)\) is defined as the space of all bounded linear operators on the Hilbert space \(H, N_\omega = \mathcal{B}_1(H) := \{ x \in \mathcal{B}(H) \mid x|x\}_{\mathcal{B}_1(H)} := \text{tr}(\sqrt{x^*x}) < \infty \}. \)

\(\text{If} (X, \mathcal{U}(X), \tilde{\mu}) \) is a localisable measure space, then \(L_\infty(X, \mathcal{U}(X), \tilde{\mu}) \) is a commutative \(W^*\)-algebra, and \(L_1(X, \mathcal{U}(X), \tilde{\mu}) \) is its predual. Every commutative \(W^*\)-algebra can be represented in this form. This indicates how the theory of \(W^*\)-algebras generalises both the localisable measure theory and the theory of bounded operators over Hilbert spaces.

A \textit{weight} on a \(W^*\)-algebra \(N\) is defined as a function \(\omega : N^+ \rightarrow [0, +\infty] \) such that \(\omega(0) = 0, \ \omega(x + y) = \omega(x) + \omega(y), \) and \(\lambda \geq 0 \Rightarrow \omega(\lambda x) = \lambda \omega(x), \) with the convention \(0 \cdot (+\infty) = 0. \) A weight is called: \textit{faithful} iff \(\omega(x) = 0 \Rightarrow x = 0; \ \textit{finite} iff \omega(\|x\|) < \infty; \ \textit{semi-finite} iff a left ideal in \(N\) given by

\[ n_\phi := \{ x \in N \mid \phi(x^*x) < \infty \} \]  

is weakly-\(\ast\) dense in \(N; \ \textit{trace} iff \omega(xx^*) = \omega(x^*x) \forall x \in N; \ \textit{normal} iff \omega(\text{supp}(x)) = \text{supp}(\omega(x)) \) for any uniformly bounded increasing net \(\{ x_i \} \subseteq N^+. \) A space of all normal semi-finite weights on a \(W^*\)-algebra \(N\) is denoted \(W(N), \) while the subset of all faithful elements of \(W(N)\) is denoted \(W_0(N)\).

Every state is a finite normal weight, and every faithful state is a finite faithful normal state, hence the diagram

\[ \begin{array}{ccc}
N_{x0}^+ & \longrightarrow & W_0(N) \\
\downarrow & & \downarrow \\
N_{x\text{unii}}^+ & \longrightarrow & W(N)
\end{array} \]  

commutes. The domain of a weight \(\omega\) can be extended by linearity to the topological \(\ast\)-algebra

\[ m_\omega := \text{span}_\mathbb{C}\{ x^*y \mid x, y \in N, \ \omega(x^*x) < \infty, \ \omega(y^*y) < \infty \} \]  

(3)

\[ = \text{span}_\mathbb{C}\{ x \in N^+ \mid \omega(x) < \infty \} \subseteq N, \]  

(4)

while \(\omega\) can be extended to a positive linear functional on \(m_\omega, \) which coincides with \(\omega\) on \(m_\omega \cap N^+. \)

\(W^*\)-algebras for which there exists at least one faithful normal state are called \textit{countably finite}. W*-algebras for which there exists at least one faithful normal semi-finite trace are called \textit{semi-finite}. Every \(W^*\)-algebra admits faithful normal weight. A \(W^*\)-algebra is called: \textit{type I} iff it is \(\ast\)-isomorphic to \(\mathcal{B}(H)\) for some Hilbert space \(H; \ \textit{type III} if it is not semi-finite; \ \textit{type II} iff it is neither type I nor type III. \)

For \(\psi \in W(N), \)

\[ \text{supp}(\psi) = I - \text{supp}(P \in \text{Proj}(N) \mid \psi(P) = 0). \]  

(5)

For \(\omega, \phi \in N_{x+}^+ \) we will write \(\omega \ll \phi \) iff \(\text{supp}(\omega) \subseteq \text{supp}(\phi). \)

An element \(\omega \in N^{B+}\) is faithful iff \(\text{supp}(\omega) = I. \) If \(\phi\) is a normal weight on a \(W^*\)-algebra \(N\) (which includes \(\omega \in N_{x+}^+\) as a special case), then the restriction of \(\phi\) to a \textit{reduced} \(W^*\)-algebra,

\[ N_{\text{supp}(\phi)} := \{ x \in N \mid \text{supp}(\phi)x = x = x\text{supp}(\phi) \} = \bigcup_{x \in N} \{ \text{supp}(\phi)x\text{supp}(\phi) \}, \]  

is a faithful normal weight (respectively, an element of \(N_{\text{supp}(\phi)}^{+\text{unii}}\)). If \(\phi\) is semi-finite, then \(\phi|_{N_{\text{supp}(\phi)}} \in W_0(N_{\text{supp}(\phi)}). \) Hence, given \(\psi \in W(N)\) and \(P \in \text{Proj}(N), \)

\[ P = \text{supp}(\psi) \iff \psi|_{N_{\text{supp}(\phi)}} \in W_0(N_{\text{supp}(\phi)}). \]  

and \(\psi(P) = \psi(PxP) \forall x \in N^+. \) In particular, for \(\omega, \phi \in N_{x+}^+\) and \(\omega \ll \phi, \) we have \(\omega|_{N_{\text{supp}(\phi)}} \in W_0(N_{\text{supp}(\phi)}). \)

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\footnote{If \(N = \mathcal{B}(H)\) and \(\omega = \text{tr}(\rho_\omega)\) for \(\rho_\omega \in \mathcal{G}_1(H)^+, \) then \(\text{supp}(\omega) = \text{ran}(\rho_\omega), \) so for any \(\phi = \text{tr}(\rho_\phi)\) with \(\rho_\phi \in \mathcal{G}_1(H)^+\) one has \(\omega \ll \phi \) iff \(\text{ran}(\rho_\phi) \subseteq \text{ran}(\rho_\omega). \)}
One of the most important properties of a $W^*$-algebra is the existence of unique polar decompositions of their elements, as well as of elements of their preduals (when considered only for elements of an algebra, unique polar decompositions exist for all Rickart $C^*$-algebras [3, 30]). For any $x \in \mathcal{N}$ there exists a unique partial isometry $v \in \mathcal{N}'$ and a unique $y \in \mathcal{N}^+$ such that $x = vy$, where $y = (x^*x)^{1/2}$, while $v$ satisfies $v^*v = \text{supp}(|x|)$ and $vv^* = \text{supp}(|x^*|)$ [95]. On the other hand, if $\phi \in \mathcal{N}$, then there exists a unique partial isometry $v \in \mathcal{N}$ and a unique $\omega \in \mathcal{N}^+$ such that $\phi(\cdot) = \omega(\cdot, v)$, where $\|\omega\| = \|\phi\|$, $\text{supp}(\phi) = v^*v$, and $\text{supp}(\phi^*) = vv^*$, with $|\phi| := \omega$ [75, 90]. Moreover, $|\phi| = \phi(v^*\cdot)$. The equations $x = v|x|$ and $\phi = |\phi|(\cdot, v)$ are called polar decompositions of, respectively, $x$ and $\phi$. A $*$-homomorphism of $C^*$-algebras $\mathcal{C}_1$ and $\mathcal{C}_2$ is defined as a map $\zeta: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ such that $\zeta(\lambda_1x_1 + \lambda_2x_2) = \lambda_1\zeta(x_1) + \lambda_2\zeta(x_2)$, $\zeta(x_1x_2) = \zeta(x_1)\zeta(x_2)$, $\zeta(x^*) = \zeta(x^*)^*$ for all $x, x_1, x_2 \in \mathcal{C}_1$. A $*$-homomorphism $\zeta: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ of $C^*$-algebras $\mathcal{C}_1$ and $\mathcal{C}_2$ is called: unital if $\zeta(1) = 1$; normal iff it is continuous with respect to the weak-$*$ topologies of $\mathcal{C}_1$ and $\mathcal{C}_2$; a $*$-isomorphism iff $0 = \ker(\zeta) := \{x \in \mathcal{C}_1 \mid \zeta(x) = 0\}$. A representation of a $C^*$-algebra $\mathcal{C}$ is defined as a pair $(\mathcal{H}, \pi)$ of a Hilbert space $\mathcal{H}$ and a $*$-homomorphism $\pi: \mathcal{C} \rightarrow \mathcal{B}(\mathcal{H})$. A representation $\pi: \mathcal{C} \rightarrow \mathcal{B}(\mathcal{H})$ is called: nondegenerate iff $\{\pi(x)\xi \mid (x, \xi) \in \mathcal{C} \times \mathcal{H}\}$ is dense in $\mathcal{H}$; faithful iff $\ker(\pi) = \{0\}$. An element $\xi \in \mathcal{H}$ is called cyclic for a $C^*$-algebra $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H})$ iff $\overline{\mathcal{C}}\xi := \bigcup_{x \in \mathcal{C}}\{x\xi\}$ is norm dense in $\mathcal{B}(\mathcal{H})$. A representation $\pi: \mathcal{C} \rightarrow \mathcal{B}(\mathcal{H})$ of a $C^*$-algebra $\mathcal{C}$ is called cyclic iff there exists $\Omega \in \mathcal{H}$ that is cyclic for $\pi(\mathcal{C})$. According to the Gel’fand–Naimark–Segal theorem [28, 78] for every pair $(\mathcal{C}, \omega)$ of a $C^*$-algebra $\mathcal{C}$ and $\omega \in \mathcal{C}^{B^+}$ there exists a triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of a Hilbert space $\mathcal{H}_\omega$ and a cyclic representation $\pi_\omega: \mathcal{C} \rightarrow \mathcal{B}(\mathcal{H})$ with a cyclic vector $\Omega_\omega \in \mathcal{H}_\omega$, and this triple is unique up to unitary equivalence. It is constructed as follows. For a $C^*$-algebra $\mathcal{C}$ and $\omega \in \mathcal{C}^{B^+}$, one defines the scalar form $\langle \cdot, \cdot \rangle_\omega$ on $\mathcal{C}$, 

$$
\langle x, y \rangle_\omega := \omega(x^*y) \quad \forall x, y \in \mathcal{C},
$$

and the Gel’fand ideal

$$
\mathcal{I}_\omega := \{x \in \mathcal{C} \mid \omega(x^*x) = 0\} = \{x \in \mathcal{C} \mid \omega(x^*y) = 0 \ \forall y \in \mathcal{C}\},
$$

which is a left ideal of $\mathcal{C}$, closed in the norm topology (it is also closed in the weak-$*$ topology if $\omega \in \mathcal{C}^{B^+}$). The form $\langle \cdot, \cdot \rangle_\omega$ is hermitian on $\mathcal{C}$ and it becomes a scalar product $\langle \cdot, \cdot \rangle_\omega$ on $\mathcal{C}/\mathcal{I}_\omega$. The Hilbert space $\mathcal{H}_\omega$ is obtained by the completion of $\mathcal{C}/\mathcal{I}_\omega$ in the topology of norm generated by $\langle \cdot, \cdot \rangle_\omega$. Consider the morphisms

$$
\begin{align*}
[\cdot]_\omega: \mathcal{C} & \ni x \mapsto [x]_\omega \in \mathcal{C}/\mathcal{I}_\omega, \\
\pi_\omega: \mathcal{C} & \ni y \mapsto [xy]_\omega.
\end{align*}
$$

The element $\omega \in \mathcal{C}^{B^+}$ is uniquely represented in terms of $\mathcal{H}_\omega$ by the vector $[1]_\omega := \Omega_\omega \in \mathcal{H}_\omega$, which is cyclic for $\pi_\omega(\mathcal{C})$ and satisfies $[\Omega_\omega] = [\omega]$. Hence

$$
\omega(x) = \langle \Omega_\omega, \pi_\omega(x)\Omega_\omega \rangle_\omega \quad \forall x \in \mathcal{C},
$$

An analogue of this theorem for weights follows the similar construction, but lacks cyclicity. If $\mathcal{N}$ is a $W^*$-algebra, and $\omega$ is a weight on $\mathcal{N}$, then there exists the Hilbert space $\mathcal{H}_\omega$, defined as the completion of $n_\omega/\ker(\omega)$ in the topology of a norm generated by the scalar product $\langle \cdot, \cdot \rangle_\omega: n_\omega \times n_\omega \ni (x, y) \mapsto \omega(x^*y) \in \mathbb{C}$,

$$
\mathcal{H}_\omega := n_\omega/\ker(\omega) = \{x \in \mathcal{N} \mid \omega(x^*x) < \infty\}/\{x \in \mathcal{N} \mid \omega(x^*x) = 0\} = n_\omega/\mathcal{I}_\omega,
$$

and there exist the maps

$$
\begin{align*}
[\cdot]_\omega: n_\omega & \ni x \mapsto [x]_\omega \in \mathcal{H}_\omega, \\
\pi_\omega: \mathcal{N} & \ni x \mapsto ([y]_\omega \mapsto [xy]_\omega) \in \mathcal{B}(\mathcal{H}_\omega),
\end{align*}
$$

such that $[\cdot]_\omega$ is linear, $\text{ran}([\cdot]_\omega)$ is dense in $\mathcal{H}_\omega$, and $(\mathcal{H}_\omega, \pi_\omega)$ is a representation of $\mathcal{N}$. If $\omega \in \mathcal{W}(\mathcal{N})$ then $(\mathcal{H}_\omega, \pi_\omega)$ is nondegenerate and normal. It is also faithful if $\omega \in \mathcal{W}_0(\mathcal{N})$. 


The **commutant** of a subalgebra $\mathcal{N}$ of any algebra $\mathcal{C}$ is defined as
\[
\mathcal{N}^\bullet := \{ y \in \mathcal{C} \mid xy = yx \ \forall x \in \mathcal{N} \},
\]
while the **center** of $\mathcal{N}$ is defined as $\mathfrak{Z}_\mathcal{N} := \mathcal{N} \cap \mathcal{N}^\bullet$. A unital $*$-subalgebra $\mathcal{N}$ of an algebra $\mathcal{B}(\mathcal{H})$ is called: **factor** iff $\mathfrak{Z}_\mathcal{N} = \mathbb{C}1$; the **von Neumann algebra** \[94, 58\] iff $\mathcal{N} = \mathcal{N}^{\bullet\bullet}$. An image $\pi(\mathcal{N})$ of any representation $(\mathcal{H}, \pi)$ of a $W^*$-algebra $\mathcal{N}$ is a von Neumann algebra iff $\pi$ is normal and nondegenerate.

Recall that any weight on a $W^*$-algebra $\mathcal{N}$ can be uniquely extended to a linear functional on $m_\omega$ which coincides with $\phi$ on $\mathcal{N}^+ \cap m_\phi$. Given a semi-finite trace $\tau: \mathcal{N}^+ \to [0, \infty]$ on a semi-finite $W^*$-algebra $\mathcal{N}$, its extension to a two-sided ideal $m_\tau$ of $\mathcal{N}$ satisfies
\[
\tau(yx) = \tau(xy) \ \forall x \in m_\tau, \forall y \in \mathcal{N}.
\]
In addition, if $\tau$ is normal, then for any $x \in m_\tau$ the map
\[
y \mapsto \omega_x(y) := \tau(xy)
\]
is an element of $\mathcal{N}_\tau^+$ \[21\]. Moreover,
\[
\tau(yx) = \tau(x^{1/2}yx^{1/2}) = \tau(y^{1/2}xy^{1/2}) \ \forall x \in m_\tau^+ \forall y \in \mathcal{N}^+.
\]
So, the formula
\[
\omega_x(y) := \tau(x^{1/2}yx^{1/2}) \ \forall y \in \mathcal{N}
\]
gives rise to $\omega_x \in \mathcal{N}_\tau^+$ with $\|\omega_x\| = \tau(|x|)$ for each $x \in m_\tau$. Let $\tau$ be a faithful normal semi-finite trace on a $W^*$-algebra $\mathcal{N}$. The map
\[
\|\cdot\|_p: \mathcal{N} \ni x \mapsto \|x\|_p := \tau(|x|^p)^{1/p} \in [0, \infty]
\]
for $p \in [1, \infty]$ is a norm on a vector space $\{ x \in \mathcal{N} \mid \|x\|_p < \infty \}$. Denote the Cauchy completion of this normed vector space by $L_p(\mathcal{N}, \tau)$. Equivalently, $L_p(\mathcal{N}, \tau)$ can be defined as a Cauchy completion of $\{ x \in \mathcal{N} \mid \tau(|x|) < \infty \}$ in the norm given by $\|\cdot\|_p$ \[64\], or as a Cauchy completion of $\text{span}_{\mathbb{C}} \{ x \in \mathcal{N}^+ \mid \tau(\text{supp}(x)) < \infty \}$ in $\|\cdot\|_1$ \[67\]. The space $L_1(\mathcal{N}, \tau)$ can be equivalently defined also as a Cauchy completion of $m_\omega$ in $\|\cdot\|_1$, while $L_2(\mathcal{N}, \tau)$ as a Cauchy completion of $m_\tau$ in $\|\cdot\|_2$ \[18, 88\]. The property $|\tau(x)| \leq \|x\|_1 \forall x \in m_\tau$ allows the unique continuous extension of $\tau$ from a linear functional on $m_\tau$ to a linear functional on $L_1(\mathcal{N}, \tau)$. This extends a bilinear form
\[
m_\tau \times \mathcal{N} \ni (h, x) \mapsto \tau(h^{1/2}xh^{1/2}) \in \mathbb{C}
\]
to the bilinear form $L_1(\mathcal{N}, \tau) \times \mathcal{N} \to \mathbb{C}$, which defines a duality between $L_1(\mathcal{N}, \tau)$ and $\mathcal{N}$, and makes $L_1(\mathcal{N}, \tau)$ isometrically isomorphic to $\mathcal{N}_\tau^*$ \[18\]. Extending the notation $\omega_x$ of \((17)\) to all elements of $\mathcal{N}_\tau^*$ corresponding to $x \in L_1(\mathcal{N}, \tau)$, we have
\[
\omega_x(y) = \tau(yx) = \tau(xy) \ \forall y \in \mathcal{N} \ \forall x \in L_1(\mathcal{N}, \tau),
\]
and \[21, 79\]
\[
\forall \omega \in \mathcal{N}_\tau^*: \exists! x \in L_1(\mathcal{N}, \tau)^+ \forall y \in \mathcal{N} \ \omega(y) = \tau(xy) = \tau(x^{1/2}yx^{1/2}).
\]
Such $x$ will be called a **Dye–Segal density** of $\omega$ with respect to $\tau$.

A closed densely defined linear operator $x: \text{dom}(x) \to \mathcal{H}$ with $\text{dom}(x) \subseteq \mathcal{H}$ and polar decomposition $x = v|x|$ will be called **affiliated** with a von Neumann algebra $\mathcal{C}$ acting on $\mathcal{H}$ iff $v \in \mathcal{C}$ and all spectral projections $P^{||x||}(\mathcal{Y})$ of $|x|$ belong to $\mathcal{C}$ (where $\mathcal{Y}$ is an element of a countably additive algebra of Borel subsets of a spectrum of $|x|$). Let $\tau$ be a fixed faithful normal semi-finite trace on a $W^*$-algebra $\mathcal{N}$. Using the notion of measurability with respect to a trace $\tau$, the above range of $L_p(\mathcal{N}, \tau)$ spaces can be represented in terms of operators affiliated to a von Neumann algebra $\pi(\mathcal{N})$ acting on $\mathcal{H}_\tau$, where $(\mathcal{H}_\tau, \pi_\tau)$ is the GNS Hilbert space of $(\mathcal{N}, \tau)$. A closed densely defined linear operator $x: \text{dom}(x) \to \mathcal{H}$ is called $\tau$-**measurable** \[79, 64\] iff $\exists \lambda > 0$ $\tau(P^{||x||}(\lambda, +\infty])) < \infty$. The space of all $\tau$-measurable
operators affiliated with $\pi_\tau(\mathcal{N})$ will be denoted by $\mathcal{M}(\mathcal{N}, \tau)$. For $x, y \in \mathcal{M}(\mathcal{N}, \tau)$ the algebraic sum $x + y$ and algebraic product $xy$ may not be closed, hence in general they do not belong to $\mathcal{M}(\mathcal{N}, \tau)$. However, their closures (denoted with the abuse of notation by the same symbol) belong to $\mathcal{M}(\mathcal{N}, \tau)$. See [57] for further discussion of $\mathcal{M}(\mathcal{N}, \tau)$ and its topologies. Consider the extension of a trace $\tau$ from $\mathcal{N}^+$ to $\mathcal{M}(\mathcal{N}, \tau)^+$ given by
\begin{equation}
\tau : \mathcal{M}(\mathcal{N}, \tau)^+ \ni x \mapsto \tau(x) := \sup_{n \in \mathbb{N}} \left\{ \tau \left( \int_0^n P^x(\lambda) \lambda \right) \right\} \in [0, \infty],
\end{equation}
the map
\begin{equation}
\| \cdot \|_p : \mathcal{M}(\mathcal{N}, \tau) \ni x \mapsto \| x \|_p := (\tau(|x|^p))^{1/p} \in [0, \infty],
\end{equation}
and the family of vector spaces
\begin{equation}
L_p(\mathcal{N}, \tau) := \{ x \in \mathcal{M}(\mathcal{N}, \tau) \mid \| x \|_p < \infty \},
\end{equation}
where $p \in [1, \infty]$. The map (25) is a norm on (26) [97], and $L_p(\mathcal{N}, \tau)$ are Cauchy complete with respect to the topology of this norm. In addition, one defines $L_\infty(\mathcal{N}) := \mathcal{N}$. The Banach spaces $L_p(\mathcal{N}, \tau)$ defined this way coincide with the $L_p(\mathcal{N}, \tau)$ spaces defined before. The spaces $L_p(\mathcal{N}, \tau)$ embed continuously into $\mathcal{M}(\mathcal{N}, \tau)$ [64]. For all $\gamma \in [0, 1]$, the duality
\begin{equation}
L_{1/\gamma}(\mathcal{N}, \tau) \times L_{1/(1-\gamma)}(\mathcal{N}, \tau) \ni (x, y) \mapsto \| x, y \| := \tau(xy) \in \mathbb{R}
\end{equation}
determines an isometric isomorphism of Banach spaces
\begin{equation}
L_{1/\gamma}(\mathcal{N}, \tau)^B \cong L_{1/(1-\gamma)}(\mathcal{N}, \tau).
\end{equation}

3 Falcone–Takesaki integration on arbitrary $W^*$-algebras

The Falcone–Takesaki approach to noncommutative integration relies on the construction and properties of the core algebra $\overline{\mathcal{N}}$ and on Masuda’s [53] reformulation of Connes’ noncommutative Radon–Nikodým type theorem. We will first briefly recall the key notions from the relative modular theory, and then move to the construction of the core algebra and integration over it.

A subspace $\mathcal{D} \subseteq \mathcal{H}$ of a complex Hilbert space $\mathcal{H}$ is called a cone iff $\lambda \xi \in \mathcal{D}$ $\forall \xi \in \mathcal{D} \forall \lambda \geq 0$. A cone $\mathcal{D} \subseteq \mathcal{H}$ is called self-polar iff
\begin{equation}
\mathcal{D} = \{ \zeta \in \mathcal{H} \mid \langle \zeta, \zeta \rangle_{\mathcal{H}} \geq 0 \forall \xi \in \mathcal{D} \}.
\end{equation}
Every self-polar cone $\mathcal{D} \subseteq \mathcal{H}$ is pointed ($\mathcal{D} \cap (-\mathcal{D}) = \{0\}$), spans linearly $\mathcal{H}$ (span$_{\mathbb{C}}\mathcal{D} = \mathcal{H}$), and determines a unique conjugation $J$ in $\mathcal{H}$ such that $J\xi = \xi \forall \xi \in \mathcal{H}$ [32], as well as a partial order on the set $\mathcal{H}^{sa} := \{ \xi \in \mathcal{H} \mid J\xi = \xi \}$ given by,
\begin{equation}
\xi \leq \zeta \iff \xi - \zeta \in \mathcal{D} \forall \xi, \zeta \in \mathcal{H}^{sa}.
\end{equation}
If $\mathcal{N}$ is a $W^*$-algebra, $\mathcal{H}$ is a Hilbert space, $\mathcal{H}^\square \subseteq \mathcal{H}$ is a self-polar cone, $\pi$ is a nondegenerate faithful normal representation of $\mathcal{N}$ on $\mathcal{H}$, and $J$ is conjugation on $\mathcal{H}$, then the quadruple $(\mathcal{H}, \pi, J, \mathcal{H}^\square)$ is called standard representation of $\mathcal{N}$ and $(\mathcal{H}, \pi(\mathcal{N}), J, \mathcal{H}^\square)$ is called standard form of $\mathcal{N}$ iff the conditions
[33]
\begin{equation}
J\pi(\mathcal{N})J = \pi(\mathcal{N})^*, \quad \xi \in \mathcal{H}^\square \Rightarrow J\xi = \xi, \quad \pi(\xi)J\pi(\xi)J\mathcal{H}^\square \subseteq \mathcal{H}^\square,
\end{equation}
\begin{equation}
\pi(x) \in \mathcal{F}(\pi(\mathcal{N})) \Rightarrow J\pi(x)J = \pi(x)^*.
\end{equation}
hold. For any standard representation
\begin{equation}
\forall \phi \in \mathcal{N}_+^s \exists! \xi_\pi(\phi) \in \mathcal{H}^\square \forall x \in \mathcal{N} \phi(x) = (\xi_\pi(\phi), \pi(x)\xi_\pi(\phi))_{\mathcal{H}}
\end{equation}
\footnote{A linear operator $J : \text{dom}(J) \to \mathcal{H}$, where $\text{dom}(J) \subseteq \mathcal{H}$, is called a conjugation iff it is antilinear, isometric, and involutive ($J^2 = \mathbb{1}$).}
holds. The map $$\xi_x : \mathcal{N}^+ \rightarrow \mathcal{H}^2$$ is order preserving.

For a given $$\mathbb{W}^*$$-algebra $$\mathcal{N}$$, $$\phi \in \mathbb{W}(\mathcal{N})$$, and $$\omega \in \mathbb{W}_0(\mathcal{N})$$ the map

$$R_{\phi,\omega} : [x]_\omega \mapsto [x^*]_\phi \quad \forall x \in n_\omega \cap n^*_\phi$$

is a densely defined, closable antilinear operator. Its closure admits a unique polar decomposition

$$\overline{R}_{\phi,\omega} = J_{\phi,\omega} \Delta_{\phi,\omega}^{1/2}$$

where $$J_{\phi,\omega}$$ is a conjugation operator, called relative modular conjugation, while $$\Delta_{\phi,\omega}$$ is a positive self-adjoint operator on $$\text{dom}(\Delta_{\phi,\omega}) \subseteq \mathcal{H}_\omega$$ with $$\text{supp}(\Delta_{\phi,\omega}) = \text{supp}(\phi)\mathcal{H}_\omega$$, called a relative modular operator [4, 15, 17]. For any $$\phi \in \mathbb{W}_0(\mathcal{N})$$, we define $$\Delta_{\phi} := \Delta_{\phi,\phi}$$ and $$\sigma^0_\phi(x) = \pi^{-1}_\omega(\Delta_{\phi,\omega}^{it}\pi_\omega(x)\Delta_{\phi,\omega}^{-it})$$. If $$\mathcal{N} \cong \mathcal{B}(\mathcal{H})$$, $$\phi = \text{tr}(\rho_\phi \sigma^0_\phi)$$, $$\omega = \text{tr}(\rho_\omega \sigma^0_\omega)$$, $$\mathcal{L}_\rho$$ denotes left multiplication by $$\rho$$, $$\mathcal{R}^{-1}_\rho$$ denotes right multiplication by $$\rho^{-1}$$, then $$\Delta_{\phi,\omega} = \mathcal{L}_{\rho_\phi}\mathcal{R}^{-1}_{\rho_\omega}$$. The relative modular operators allow to define a one-parameter family of partial isometries in $$\text{supp}(\phi)\mathcal{N}$$, called Connes’ cocycle [14],

$$\mathbb{R} \ni t \mapsto [\phi : \omega]_t := \Delta_{\phi,\omega}^{it}\Delta_{\phi,\omega}^{-it} = \Delta_{\phi,\omega}^{it}\Delta_{\phi,\omega}^{-it} \in \text{supp}(\phi)\mathcal{N},$$

where $$\psi \in \mathbb{W}_0(\mathcal{N})$$ is arbitrary, so it can be set equal to $$\omega$$. Connes showed that $$[\phi : \omega]_t$$ can be characterised as a canonical object associated to any pair $$(\phi, \omega) \in \mathbb{W}(\mathcal{N}) \times \mathbb{W}_0(\mathcal{N})$$ on any $$\mathbb{W}^*$$-algebra $$\mathcal{N}$$, independently of any representation. As shown by Araki and Masuda [6] (see also [53]), the definition of $$\Delta_{\phi,\omega}$$ and $$[\phi : \omega]_t$$ can be further extended to the case when $$\phi, \omega \in \mathbb{W}(\mathcal{N})$$, by means of a densely defined closable antilinear operator

$$R_{\phi,\omega} : [x]_\omega + (\overline{1 - \text{supp}(n_\phi|_\omega)\zeta} + \text{supp}(\omega)[x^*]_\phi \quad \forall x \in n_\omega \cap n^*_\phi \forall \zeta \in \mathcal{H},$$

where $$\mathcal{H}, \pi, J$$ are a standard representation of a $$\mathbb{W}^*$$-algebra $$\mathcal{N}$$, and $$\mathcal{H}_\phi \subseteq \mathcal{H} \supseteq \mathcal{H}_\omega$$. For $$\phi, \omega \in \mathcal{N}^+$$ this becomes a closable antilinear operator [5, 38]

$$R_{\phi,\omega} : x\pi(\omega) + \zeta \mapsto \text{supp}(\omega)x^*\pi(\phi) \quad \forall x \in \pi(N) \forall \zeta \in (\pi(N)\pi(\omega))^\perp,$$

acting on a dense domain $$(\pi(N)\pi(\omega)) \cup (\pi(N)\pi(\omega))^\perp \subseteq \mathcal{H}$$, where $$(\pi(N)\pi(\omega))^\perp$$ denotes a complement of the closure in $$\mathcal{H}$$ of the linear span of the action $$\pi(N)$$ on $$\pi(\omega)$$. In both cases, the relative modular operator is determined by the polar decomposition of the closure $$\overline{R}_{\phi,\omega}$$ of $$R_{\phi,\omega}$$,

$$\Delta_{\phi,\omega} := R^{*}_{\phi,\omega} \overline{R}_{\phi,\omega}.$$ 

If (37) or (38) is used instead of (34), then the formula (36) has to be replaced by

$$\mathbb{R} \ni t \mapsto [\phi : \omega]_t\text{supp}(n_\phi|_\omega) := \Delta_{\phi,\omega}^{it}\Delta_{\phi,\omega}^{-it},$$

and $$[\phi : \omega]_t$$ is a partial isometry in $$\text{supp}(\phi)\mathcal{N}$$supp($$\omega$$) whenever $$[\text{supp}(\phi), \text{supp}(\omega)] = 0$$. For $$\phi, \psi \in \mathbb{W}_0(\mathcal{N})$$ Connes’ theorem [13, 14] states that the following conditions are equivalent:

i) $$\exists \lambda > 0 \quad 0 \leq \psi \leq \lambda \phi,$$

ii) $$x \in n_\phi \Rightarrow x \in n_\psi,$$

iii) $$t \mapsto [\psi : \phi]_t$$ can be extended to a map that is valued in $$\mathcal{N}$$, bounded (by $$\lambda^{1/2}$$) and weakly-$$\star$$ continuous on a strip $$\{z \in \mathbb{C} \mid \text{im}(z) \in [-\frac{1}{2}, 0]\}$$, holomorphic in interior of this strip, and satisfying the boundary condition

$$\psi(x) = \phi \left([\psi : \phi]_{-i/2}\star [\psi : \phi]_{-i/2}\right) \forall x \in m_\psi.$$
This theorem extends to \( \psi \in \mathcal{W}(\mathcal{N}) \), with \( \mathbb{R} \ni t \mapsto [\psi : \phi]_t \in \text{supp}(\psi)\mathcal{N} \forall t \in \mathbb{R} [38] \). Thus, whenever the condition i) is satisfied, the analytic continuation of Connes’ cocycle

\[
h^{1/2} = [\psi : \phi]_{-1/2}
\]

plays the role of a noncommutative (square root of) Radon–Nikodým quotient.

There is also another closely related operator. Consider a von Neumann algebra \( \mathcal{N} \), acting on a Hilbert space \( \mathcal{H} \), a weight \( \phi^* \in \mathcal{W}_0(\mathcal{N}^*) \), a lineal set \([81, 82]\)

\[
\mathcal{D}(\mathcal{H}, \phi) := \left\{ \zeta \in \mathcal{H} \mid \exists \lambda \geq 0 \forall x \in n_{\phi^*} \subseteq \mathcal{N}^* \|x\zeta\|^2 \leq \lambda \phi^*(x^*x) \right\},
\]

and a bounded operator

\[
R_{\phi^*}(\xi) : \mathcal{H}_{\phi^*} \ni [x]_{\phi^*} \mapsto x\xi \in \mathcal{H} \ \forall x \in n_{\phi^*} \ \forall \xi \in \mathcal{D}(\mathcal{H}, \phi),
\]

where \((\mathcal{H}_{\phi^*}, \pi_{\phi^*})\) is a GNS representation of \( \mathcal{N}^* \) with respect to \( \phi^* \). The map \( \xi \mapsto R_{\phi^*}(\xi) \) is linear, and \( R_{\phi^*}(\xi)R_{\phi^*}(\xi)^* \in \mathcal{N}^* \). For any \( \psi \in \mathcal{W}(\mathcal{N}) \) the function

\[
\xi \mapsto \psi(R_{\phi^*}(\xi)R_{\phi^*}(\xi)^*)
\]

is closable and bounded on

\[
\mathcal{D}(\mathcal{H}, \phi, \psi) := \{ \xi \in \mathcal{D}(\mathcal{H}, \phi) \mid R_{\phi^*}(\xi) \in n_\psi \}.
\]

The Connes spatial quotient\(^3\) [16] is defined as the largest positive self-adjoint operator \( \frac{\psi}{\phi^*} : \mathcal{D}(\mathcal{H}, \phi, \psi) \to \mathcal{H} \) satisfying

\[
\left\| \left( \frac{\psi}{\phi^*} \right)^{1/2} \xi \right\|^2 \mathcal{D}(\mathcal{H}, \phi, \psi).
\]

The operator \( \left( \frac{\psi}{\phi^*} \right)^{1/2} \) is unbounded, but it is essentially self-adjoint on \( \mathcal{D}(\mathcal{H}, \phi, \psi) \).

Consider now a \( W^* \)-algebra \( \mathcal{N} \) and a relation \( \sim_t \) on \( \mathcal{N} \times \mathcal{W}_0(\mathcal{N}) \) defined by \([24]\)

\[
(x, \psi) \sim_t (y, \phi) \iff y = x[\psi : \phi]_t \ \forall x, y \in \mathcal{N} \ \forall \psi, \phi \in \mathcal{W}_0(\mathcal{N}).
\]

The cocycle property of \([\psi : \phi]_t \) implies that \( \sim_t \) is an equivalence relation in \( \mathcal{N} \times \mathcal{W}_0(\mathcal{N}) \). The equivalence class \((\mathcal{N} \times \mathcal{W}_0(\mathcal{N}))/ \sim_t \) is denoted by \( \mathcal{N}(t) \), and its elements are denoted by \( x\psi^t \). The operations

\[
x\psi^t + y\psi^t := (x + y)\psi^t,
\]

\[
\lambda(x\psi^t) := (\lambda x)\psi^t \ \forall \lambda \in \mathbb{C},
\]

\[
\|x\psi^t\| := |x|
\]

equip \( \mathcal{N}(t) \) with the structure of the Banach space, which is isometrically isomorphic to \( \mathcal{N} \), considered as a Banach space. By definition, \( \mathcal{N}(0) \) a \( W^* \)-algebra that is trivially \( * \)-isomorphic to \( \mathcal{N} \). However, for \( t \neq 0 \) the spaces \( \mathcal{N}(t) \) are not \( W^* \)-algebras.

The operations

\[
* : \mathcal{N}(t_1) \times \mathcal{N}(t_2) \ni (x\psi^{t_1}, y\psi^{t_2}) \mapsto x\sigma_{t_1}(y)\psi^{t_1+t_2} \in \mathcal{N}(t_1 + t_2),
\]

\[
* : \mathcal{N}(t) \ni x\psi^t \mapsto \sigma_{-t}(x)\psi^{-t} \in \mathcal{N}(-t),
\]

\(^3\)We use the term ‘quotient’ instead of ‘derivative’, in order to reserve the term ‘derivative’ for the notions defined by means of differential or smooth structures. By the same reason we use the term ‘Radon–Nikodým quotient’ instead of ‘Radon–Nikodým derivative’. The Radon–Nikodým quotient and Connes’ spatial quotient are defined in the absence of any differential structure, and they are not derivatives (apart from very special cases), so they shall not be called ‘derivatives’.
equip the disjoint sum $F(\mathcal{N}) := \bigsqcup_{t \in \mathbb{R}} \mathcal{N}(t)$ over $\mathcal{N} \times \mathbb{R}$ with the structure of $*$-algebra. The bijections

$$\mathcal{N}(t) \ni x \psi^t \mapsto (x, t) \in \mathcal{N} \times \mathbb{R}$$

(54)

allow to endow $F(\mathcal{N})$ with the topology induced by (54) from the product topology on $\mathcal{N} \times \mathbb{R}$ of the weak-$*$ topology on $\mathcal{N}$ and the usual topology on $\mathbb{R}$. This provides the Fell’s Banach $*$-algebra bundle structure on $F(\mathcal{N})$ (see [25, 26, 27] for a general theory of the Fell bundles). One can consider the Fell bundle $F(\mathcal{N})$ as a natural algebraic structure which enables to translate between elements of $\mathcal{N}(t)$ at different $t \in \mathbb{R}$. In order to recover an element of $F(\mathcal{N})$ at a given $t \in \mathbb{R}$, one has to select a section $\tilde{x} : \mathbb{R} \to F(\mathcal{N})$ of $F(\mathcal{N})$:

$$t \mapsto x(t) \psi^t := \tilde{x}(t).$$

(55)

Consider the set $\Gamma^1(F(\mathcal{N}))$ of such cross-sections of $F(\mathcal{N})$ that are integrable in the following sense:

i) for any $\epsilon > 0$ and any bounded interval $I \subseteq \mathbb{R}$ there exists a compact subset $Y \subseteq I$ such that $|I - Y| < \epsilon$ and the restriction $Y \ni t \mapsto x(t) \in F(\mathcal{N})$ is continuous relative to the topology induced in $F(\mathcal{N})$ by (54),

ii) $\int_{\mathbb{R}} dr \| x(r) \| < \infty$.

The set $\Gamma^1(F(\mathcal{N}))$ can be endowed with a multiplication, involution, and norm,

$$\langle \tilde{x} \tilde{y} \rangle(t) := \int_{\mathbb{R}} dr \langle x(r) y(t - r) \rangle = \left( \int_{\mathbb{R}} dr \langle x(r) \sigma^t_1(y(t - r)) \rangle \right) \psi^t,$$

(56)

$$\tilde{x}^*(t) := \tilde{x}(-t)^* = \sigma^t_1(x(-t)^*) \psi^t,$$

(57)

$$\| \tilde{x} \| := \int_{\mathbb{R}} dr \| x(r) \|,$$

(58)

thus forming a Banach $*$-algebra, denoted by $B(\mathcal{N})$. Falcone and Takesaki [23, 24] constructed also a suitably defined ‘bundle of Hilbert spaces’ over $\mathbb{R}$. Let $\mathcal{N} = \pi(\mathcal{C})$ be a von Neumann algebra representing a $W^*$-algebra $\mathcal{C}$ in terms of a standard representation $(\mathcal{H}, \pi, J, \mathcal{H}^2)$. The space $\mathcal{H}$ can be considered as a $\mathcal{N} \cdot (\mathcal{N}^\circ)^\circ$ bimodule, with the left action of $\mathcal{N}$ on $\mathcal{H}$ given by ordinary multiplication from the left, and with the right action of $(\mathcal{N}^\circ)^\circ$ on $\mathcal{H}$ defined by

$$\xi x^\circ := x \xi \quad \forall \xi \in \mathcal{H} \quad \forall x^\circ \in (\mathcal{N}^\circ)^\circ.$$  

(59)

Thus, the left action of $\mathcal{N}$ on $\mathcal{H}$ is just an action of a standard representation of the underlying $W^*$-algebra $\mathcal{C}$, while the right action of $(\mathcal{N}^\circ)^\circ$ is provided by the corresponding antirepresentation of $\mathcal{C}$ (that is, by commutant of a standard representation of $\mathcal{C}^\circ$). Given arbitrary $r_1, r_2 \in \mathbb{R}$, $\zeta_1, \zeta_2 \in \mathcal{H}$, $\phi_1, \phi_2 \in \mathcal{W}_0(\mathcal{N})$, and $\varphi_1, \varphi_2 : \mathcal{W}_0((\mathcal{N}^\circ)^\circ)$ the condition

$$\left( \begin{array}{c} \phi_1 \\ \varphi_1 \end{array} \right)^{ir_1} \zeta_1 = \left( \begin{array}{c} \phi_2 \\ \varphi_2 \end{array} \right)^{ir_2} \zeta_2[\varphi_2 : \varphi_1]_t,$$

(60)

defines an equivalence relation

$$\left( r_1, \phi_1, \zeta_1, \varphi_1 \right) \sim_t \left( r_2, \phi_2, \zeta_2, \varphi_2 \right)$$

(61)

on the set $\mathbb{R} \times \mathcal{W}_0(\mathcal{N}) \times \mathcal{H} \times \mathcal{W}_0((\mathcal{N}^\circ)^\circ)$. The equivalence class of the relation (61) is denoted by $\mathcal{H}(t)$, and its elements have the form

$$\phi^it \xi = \left( \begin{array}{c} \phi \\ \varphi \end{array} \right)^{it} \xi \varphi^it,$$

(62)

which is equivalent to

$$\phi^it \xi \varphi^{-it} = \left( \begin{array}{c} \phi \\ \varphi \end{array} \right)^{it} \xi.$$

(63)

\footnotetext[4]{That is, a Hilbert space $\mathcal{H}$ equipped with a normal representation $\pi_1 : \mathcal{N} \to \mathfrak{B}(\mathcal{H})$ and a normal representation $\pi_2 : \mathcal{N}^\circ \to \mathfrak{B}(\mathcal{H})$ such that $\pi_1(\mathcal{N})$ and $\pi_2(\mathcal{N}^\circ)$ commute.}
Falcone and Takesaki show that $\mathcal{H}(t)$ is a Hilbert space independent of the choice of weights $\phi_1, \phi_2, \varphi_1, \varphi_2$ and of the choice of $r \in \mathbb{R}$. This enables to form the Hilbert space bundle over $\mathbb{R}$, $\prod_{t \in \mathbb{R}} \mathcal{H}(t)$, and to form the Hilbert space of square-integrable cross-sections of this bundle,

$$\tilde{\mathcal{H}} := \Gamma^2 \left( \prod_{t \in \mathbb{R}} \mathcal{H}(t) \right).$$

(64)

The Hilbert space bundle $\prod_{t \in \mathbb{R}} \mathcal{H}(t)$ is homeomorphic to $\mathcal{H} \times \mathbb{R}$ for any choice of weight $\psi \in \mathcal{W}_0(N)$. The left action of $B(\mathcal{N})$ on $\tilde{\mathcal{H}}$ generates a von Neumann algebra $\tilde{\mathcal{N}}$ called standard core [24]. For type $\text{III}_1$ factors $\mathcal{N}$ the standard core $\tilde{\mathcal{N}}$ is a type $\text{II}_\infty$ factor, but in general case $\tilde{\mathcal{N}}$ is not a factor. The structure of $\tilde{\mathcal{N}}$ is independent of the choice of weight on $\mathcal{N}$. However, for any choice of $\psi \in \mathcal{W}_0(\mathcal{N})$ there exists a unitary map

$$u_\psi : \tilde{\mathcal{H}} = \Gamma^2 \left( \prod_{t \in \mathbb{R}} \mathcal{H}(t) \right) \to L_2(\mathbb{R}, dt; \mathcal{H}) \cong \mathcal{H} \otimes L_2(\mathbb{R}, dt),$$

(65)

such that

$$u_\psi(\xi)(t) = \psi^{-i}t \xi(t) \in \mathcal{H} \quad \forall \xi \in \tilde{\mathcal{H}}.$$

(66)

It satisfies

$$(u_\psi xu\psi_\ast)(\xi)(t) = \sigma^\psi_{-t}(x)\xi(t),$$

(67)

$$(u_\psi^is\psi\chi)(\xi)(t) = \xi(t - s),$$

(68)

$$(u_\psi^\phi^{-is}\phi\chi)(\xi)(t) = \left( \frac{\psi}{\phi} \right)^i \chi.$$

(69)

for all $\xi \in L_2(\mathbb{R}, dt; \mathcal{H})$, $x \in \mathcal{N}$, $\phi, \psi \in \mathcal{W}_0(\mathcal{N})$, $s, t \in \mathbb{R}$. This map provides a $\ast$-isomorphism between the standard core $\tilde{\mathcal{N}}$ on $\tilde{\mathcal{H}}$ and the crossed product $\mathcal{N} \rtimes_{\sigma, \psi} \mathbb{R}$ on $\mathcal{H} \otimes L_2(\mathbb{R}, dt)$.

$$u_\psi^*\tilde{\mathcal{N}}u_\psi = \mathcal{N} \rtimes_{\sigma, \psi} \mathbb{R}. 

(70)$$

Using the uniqueness of the standard representation up to unitary equivalence, Falcone and Takesaki [24] proved that the map $\mathcal{N} \to \tilde{\mathcal{N}}$ extends to a functor VNCore from the category VNIso of von Neumann algebras with $\ast$-isomorphisms to its own subcategory VN$_d$Iso of semi-finite von Neumann algebras with $\ast$-isomorphisms.

The one-parameter automorphism group of $F(\mathcal{N})$,

$$\tilde{\sigma}_s(x\phi^it) := e^{-its}x\phi^it \quad \forall x\phi^it \in \tilde{\mathcal{N}}(t),$$

(71)

corresponding to the unitary group $\tilde{u}(s)$ on $\tilde{\mathcal{H}}$ given by

$$(\tilde{u}(s)\xi)(t) = e^{-its}\xi(t) \quad \forall t, s \in \mathbb{R} \forall \xi \in \tilde{\mathcal{H}},$$

(72)

extends uniquely to a group of automorphisms $\tilde{\sigma}_s : \tilde{\mathcal{N}} \to \tilde{\mathcal{N}}$. The automorphism $\tilde{\sigma}_i$ provides a weight-independent replacement for a dual automorphism $\tilde{\sigma}_i^\psi$ used in Haagerup’s theory [34, 89]. The triple $(\tilde{\mathcal{N}}, \mathbb{R}, \tilde{\sigma})$ is a $W^\ast$-dynamical system.

There exist canonical isomorphisms

$$\tilde{\mathcal{N}} \rtimes_{\sigma} \mathbb{R} \cong \mathcal{N} \otimes \mathcal{B}(L_2(\mathbb{R}, d\lambda)), 

(77)$$

$$\tilde{\mathcal{N}}_\sigma \cong \mathcal{N}. 

(78)$$

---

5See [39] for the notions of, and further references on, crossed product, operator valued weight, dual weight, and $\mathcal{N}_{\text{ext}}$.

6A weak-$\ast$ topology on a group $\text{Aut}(\mathcal{N})$ of $\ast$-isomorphisms $\mathcal{N} \to \mathcal{N}$ of a $W^\ast$-algebra $\mathcal{N}$ is defined by the collection of neighbourhoods

$$N_{(x_i), \epsilon}(\alpha) := \{ \xi \in \text{Aut}(\mathcal{C}) \mid |\xi(x_i) - \alpha(x_i)| < \epsilon \},$$

(73)

where $\{x_i\} \subseteq \mathcal{C}$, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$, $\epsilon > 0$. A triple $(\mathcal{N}, G, \alpha)$ of a $W^\ast$-algebra, locally compact group $G$, and a representation $\alpha : G \to \text{Aut}(\mathcal{N})$ is called a $W^\ast$-dynamical system iff $\alpha$ is continuous in the weak-$\ast$ topology of $\text{Aut}(\mathcal{N})$. A unitary implementation of a representation $\alpha : G \to \text{Aut}(\mathcal{C})$ in a given representation $\pi : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ is
The action of $\tilde{\sigma}_s$ on $\tilde{\mathcal{N}}$ is integrable over $s \in \mathbb{R}$, and

$$T_{\tilde{\sigma}} : \tilde{\mathcal{N}}^+ \ni x \mapsto T_{\tilde{\sigma}}(x) := \int_{\mathbb{R}} ds \tilde{\sigma}_s(x) \in \mathcal{N}^{\text{ext}},$$  \hspace{1cm} (79)

is an operator valued weight from $\tilde{\mathcal{N}}$ to $\tilde{\mathcal{N}}_\phi \cong \mathcal{N}$. For any $\phi \in \mathcal{W}(\mathcal{N})$, its dual weight over $\tilde{\mathcal{N}}$ is given by

$$\hat{\phi} := \tilde{\phi} \circ T_{\tilde{\sigma}} \in \mathcal{W}(\tilde{\mathcal{N}}).$$  \hspace{1cm} (80)

Every $\phi \in \mathcal{W}_0(\mathcal{N})$ can be considered as an analytic generator of the one parameter group of unitaries \{${\phi}^t \mid t \in \mathbb{R}$\} $\subseteq \tilde{\mathcal{N}}$ acting on $\mathcal{H}$ from the right, given by

$$\phi = \exp \left(-i \frac{d}{dt} (\phi^t) \big|_{t=0} \right).$$  \hspace{1cm} (81)

This allows to equip $\tilde{\mathcal{N}}$ with a faithful normal semi-finite trace $\tilde{\tau}_\phi : \tilde{\mathcal{N}}^+ \to [0, \infty]$, where

$$\tilde{\tau}_\phi(x) := \lim_{\epsilon \to +0} \tilde{\phi}((\phi^{-1} + \epsilon \phi^{-1})^{-1})^{1/2} x ((\phi^{-1} + \epsilon \phi^{-1})^{-1})^{1/2}$$

$$= \lim_{\epsilon \to +0} \tilde{\phi}(\phi^{-1/2} (1 + \epsilon \phi^{-1})^{-1/2} x \phi^{-1/2} (1 + \epsilon \phi^{-1})^{-1/2})$$

$$= \lim_{\epsilon \to +0} \tilde{\phi}((\phi + \epsilon)_{-1/2} x (\phi + \epsilon)_{-1/2}).$$  \hspace{1cm} (82)

This definition is independent of the choice of weight (e.g., $\tilde{\tau}_\phi = \tilde{\tau}_\psi \forall \phi, \psi \in \mathcal{W}_0(\mathcal{N})$), which follows from the fact that

$$[\tilde{\tau}_\phi : \tilde{\tau}_\psi]_t = [\tilde{\tau}_\phi : \tilde{\phi}]_t \tilde{\psi} : \tilde{\tau}_t \tilde{\phi} \tilde{\psi} = \varphi^{-it}[\phi : \psi]_t \psi^{-it} = \varphi^{-it} \varphi^{-it} \psi^{-it} = 1$$  \hspace{1cm} (83)

for all $\phi, \psi \in \mathcal{W}_0(\mathcal{N})$ and for all $t \in \mathbb{R}$. This allows to write $\tilde{\tau}$ instead of $\tilde{\tau}_\phi$. Moreover, $\tilde{\tau}$ has the scaling property

$$\tilde{\tau} \circ \tilde{\sigma}_s = e^{-s \tilde{\tau}} \forall s \in \mathbb{R}.$$  \hspace{1cm} (84)

This allows to call $\tilde{\tau}$ a **canonical trace** of $\tilde{\mathcal{N}}$. It will play the role analogous to a Haagerup’s trace $\tilde{\tau}_\psi$ on a crossed product algebra $\tilde{\mathcal{N}} = \mathcal{N} \ltimes_{\sigma, \psi} \mathbb{R}$ [34, 89]. Nevertheless, the definition (82) is not a straightforward generalisation of Haagerup’s trace. It is another type of ‘perturbed’ construction of a weight, which is designed in this case for the purpose of direct elimination of the dependence of $\tilde{\tau}_\phi$ on $\phi$.

### 4 Commutative and noncommutative Orlicz spaces

In this section we will first briefly recall the key notions from the theory of commutative Orlicz spaces, and then we will move to discussion of the noncommutative version of this theory. For a detailed treatment of the theory of commutative Orlicz spaces, as well as the associated theory of modular spaces, see [61, 63, 62, 99, 42, 48, 59, 51, 71, 11, 72].

If $\mathcal{X}$ is a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $\mathcal{Y} : \mathcal{X} \to [0, \infty]$ is a convex function satisfying

1) $\mathcal{Y}(0) = 0,$

defined as a map $u : G \ni g \mapsto u(g) \in \mathfrak{B}(\mathcal{H})^{\text{unr}}$ that determines a family \{${u(g) \mid g \in G}$\} of unitary operators satisfying the **covariance equation**

$$\pi(\alpha_g(x)) = u(g) \pi(x) u(g)^* \quad \forall x \in \mathcal{C} \forall g \in G.$$  \hspace{1cm} (74)

If $(\mathcal{H}, \pi, J, \mathcal{H}^\perp)$ is a standard representation of a $W^*$-algebra $\mathcal{N}$, then there exists a unique strongly continuous unitary implementation $V_\alpha (g)$ of $\alpha$ satisfying

$$V_\alpha(g) \mathcal{H}^\perp = \mathcal{H}^\perp,$$  \hspace{1cm} (75)

$$J V_\alpha(g) = V_\alpha(g) J.$$  \hspace{1cm} (76)

Such family \{${V_\alpha(g) \mid g \in G}$\} is called a **standard** unitary implementation of $\alpha$. 

2) \( \Upsilon(\lambda x) = 0 \forall \lambda > 0 \Rightarrow x = 0, \)

3) \( |\lambda| = 1 \Rightarrow \Upsilon(\lambda x) = \Upsilon(x), \)

then \( \Upsilon \) is called a pseudomodular function [60]. If 2) is replaced by

2') \( \Upsilon(x) = 0 \Rightarrow x = 0, \)

then \( \Upsilon \) is called a modular function [61, 63, 62]. If 2) is replaced by

2'') \( x \neq 0 \Rightarrow \lim_{\lambda \to +\infty} \Upsilon(\lambda x) = +\infty, \)

then \( \Upsilon \) is called a Young function [98, 9, 10]. A Young function \( f \) on \( \mathbb{R} \) is said to satisfy: local \( \triangle_2 \) condition iff [9, 10]

\[ \exists \lambda > 0 \exists x_0 \geq 0 \forall x \geq x_0 \ f(2x) \leq \lambda f(x); \quad (85) \]

global \( \triangle_2 \) condition iff \( x_0 \) in (85) is set to 0. A convex function \( f : \mathbb{R} \to \mathbb{R}^+ \) is called \( N \)-function [10] iff \( \lim_{x \to +0} \frac{f(x)}{x} = 0 \) and \( \lim_{x \to +\infty} \frac{f(x)}{x} = +\infty. \) Every Young function \( f \) allows do define a Young–Birnbaum–Orlicz dual [10, 52]

\[ f^Y : \mathbb{R} \ni y \mapsto f^Y(y) := \sup_{x \geq 0} \{ x|y| - f(x) \} \in [0, \infty]. \]

Every YBO dual is a nondecreasing Young function, and each pair \((f, f^Y)\) satisfies Young’s inequality [98]

\[ xy \leq f(x) + f^Y(y) \forall x, y \in \mathbb{R}. \]

If \( f \) is also an \( N \)-function, then \( f^{YY} = f. \) Every modular function \( \Upsilon : X \to [0, \infty] \) determines a modular space [61, 63]

\[ X_\Upsilon := \{ x \in X \mid \lim_{\lambda \to +0} \Upsilon(\lambda x) = 0 \} \]

and the Morse–Transue–Nakano–Luxemburg norm on \( X_\Upsilon \) [54, 63, 49, 96],

\[ \|\cdot\|_\Upsilon : X_\Upsilon \ni x \mapsto \|x\|_\Upsilon := \inf \{ \lambda > 0 \mid \Upsilon(\lambda^{-1} x) \leq 1 \} \in \mathbb{R}^+, \]

which allows to define a Banach space

\[ L_\Upsilon(X) := \overline{X_\Upsilon}^{\|\cdot\|_\Upsilon}. \]

An Orlicz function is defined as a function \( f : [0, \infty] \to [0, \infty] \) that is convex, continuous, nondecreasing, satisfying \( f(0) = 0, \lambda > 0 \Rightarrow f(\lambda) > 0, \lim_{\lambda \to +\infty} f(\lambda) = +\infty. \) By definition, an Orlicz function is a restriction to \( \mathbb{R}^+ \) of a modular Young function on \( \mathbb{R}, \) equipped with the additional conditions of continuity and nondecreasing.⁷ Every Orlicz function \( f \) defines a continuous modular function \( \Upsilon_f \) on \( L_0(\mathcal{X}, \mathcal{U}(\mathcal{X}), \mu; \mathbb{R}) \) where \((\mathcal{X}, \mathcal{U}(\mathcal{X}), \mu)\) is a localisable measure space, by the formula

\[ \Upsilon_f : L_0(\mathcal{X}, \mathcal{U}(\mathcal{X}), \mu; \mathbb{R}) \ni x \mapsto \Upsilon_f(x) := \int \mu f(|x|) \in [0, \infty]. \]

An Orlicz space [66] is defined as a Banach space

\[ L_\Upsilon(\mathcal{X}, \mathcal{U}(\mathcal{X}), \mu; \mathbb{R}) := L_\Upsilon(L_0(\mathcal{X}, \mathcal{U}(\mathcal{X}), \mu; \mathbb{R})), \]

an Orlicz class [65] is defined by

\[ \tilde{L}_\Upsilon(\mathcal{X}, \mathcal{U}(\mathcal{X}), \mu; \mathbb{R}) := \{ x \in L_0(\mathcal{X}, \mathcal{U}(\mathcal{X}), \mu; \mathbb{R}) \mid \int \mu f(|x|) < \infty \}, \]

⁷In all application discussed here, the condition of continuity of an Orlicz function \( f \) can be relaxed to continuity at \( 0, x_f \) with left continuity at \( x_f, \) where \( x_f := \sup \{ \lambda > 0 \mid f(\lambda) < \infty \}. \)
while a **Morse–Transue–Krasnosel’skii–Rutickii space** [54, 40, 41] is defined by

$$E_T(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R}) := \{x \in L_0(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R}) \mid \forall \lambda < 0 \int \hat{\mu} f(\lambda |x|) < \infty\}. \tag{94}$$

Every MTKR space is a Banach space with respect to the MTNL norm $\| \cdot \|_T$. By application of the Lebesgue dominated convergence theorem, it follows that

$$L_T(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R}) = \{x \in L_0(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R}) \mid \exists \lambda > 0 \int \hat{\mu} f(\lambda |x|) < \infty\}. \tag{95}$$

Moreover, $L_T(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R}) = \text{span}_{\mathbb{R}} B_T(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R})$, where

$$B_T(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R}) := \{x \in L_T(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R}) \mid \int \hat{\mu} f(|x|) \leq 1\} \tag{96}$$

is a unit closed ball in $L_T(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R})$ and a convex subset of $L_T(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R})$. The definitions of $L_T(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R})$, $\tilde{L}_T(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R})$, $E_T(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R})$, and $B_T(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R})$, as well as their above properties, can be extended by replacing $\mathbb{R}$ with $[-\infty, +\infty]$ [99, 49], and by replacing Orlicz function $f$ by an arbitrary Young function $\mathcal{Y} : \mathbb{R} \to [0, \infty]$ [71]. In the latter case, (91) does not define a modular function, but it is anyway a norm on $L_0(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R})$. Hence, the corresponding Orlicz space $L_f(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R})$ can be defined as a completion of $L_0(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; \mathbb{R})$ in the MTNL norm $\| \cdot \|_T$, and the same holds for $L_f(\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X}), \hat{\mu}; [-\infty, +\infty])$. In what follows, we will use indices $(\cdot)_T$ to refer to any of these two constructions: the one based on an arbitrary Young function $\mathcal{Y}$, and the one based on a modular function determined by an Orlicz function $\mathcal{Y}$. In order to keep the notation concise, in what follows we will also omit symbols $\mathcal{X}, \overline{\mathcal{U}}(\mathcal{X})$, and $(\mathbb{R} \text{ or } [-\infty, +\infty])$, whenever they will not make any difference.

As an example, the space $L_\infty(\hat{\mu})$ can be determined as an Orlicz space $L_T(\hat{\mu})$, where

$$\mathcal{Y}_\infty(x) := \begin{cases} 0 & : x \in [0, 1] \\ +\infty & : x > 1 \\ \mathcal{Y}_\infty(-x) & : x < 0 \end{cases} \tag{97}$$

is a Young function but not an Orlicz function, while the spaces $L_p(\hat{\mu})$ can be defined as Orlicz spaces $L_T(\hat{\mu})$ with $\mathcal{Y}(x)$ given by any of the Orlicz functions: $|x|^p$, $|x|^p$, $\frac{x^p}{p}$, or $x^p$.

In general, $E_T(\hat{\mu})$ is the largest vector subspace of $L_0(\hat{\mu})$ contained in $L_T(\hat{\mu})$, $\tilde{L}_T(\hat{\mu})$ is a convex subset of $L_T(\hat{\mu})$, while $L_T(\hat{\mu})$ is the smallest vector subspace of $L_0(\hat{\mu})$ containing $L_T(\hat{\mu})$. Moreover, for any Young function $\mathcal{Y}$, $L_T(\hat{\mu}) \subseteq (L_{T\mathcal{Y}}(\hat{\mu}))^\mathcal{B}$ and [65, 66, 54, 42, 68, 69]

$$(E_T(\hat{\mu}))^\mathcal{B} \ni L_{T\mathcal{Y}}(\hat{\mu}), \tag{98}$$

so that

$$\forall x \in L_{T\mathcal{Y}}(\hat{\mu}) \exists! y \in L_{T\mathcal{Y}}(\hat{\mu}) \forall z \in E_T(\hat{\mu}) \ x(z) = \int \hat{\mu} y z. \tag{99}$$

If $\mathcal{Y}$ and $\mathcal{Y}^\mathcal{Y}$ are Young N-functions, then [42, 71] (see also [8])

$$xy \in L_1(\hat{\mu}) \ \forall (x,y) \in \tilde{L}_T(\hat{\mu}) \times L_{T\mathcal{Y}}(\hat{\mu}), \tag{100}$$

$$|xy|_{L_1(\hat{\mu})} = \int \hat{\mu} |xy| \leq \|x\|_{T\mathcal{Y}} \|y\|_{T\mathcal{Y}} \ \forall (x,y) \in L_T(\hat{\mu}) \times L_{T\mathcal{Y}}(\hat{\mu}). \tag{101}$$

The space $\tilde{L}_T(\hat{\mu})$ is a vector space iff $E_T(\hat{\mu}) = L_T(\hat{\mu})$ as sets, and in such case also $\tilde{L}_T(\hat{\mu}) = L_T(\hat{\mu})$ holds. If a Young function $\mathcal{Y}$ satisfies (local $\Delta_2$ condition and $\hat{\mu}(\mathcal{X}) < \infty$) or (global $\Delta_2$ condition and $\hat{\mu}(\mathcal{X}) = \infty$) then:

i) $\tilde{L}_T(\hat{\mu})$ is a vector space,
ii) $E_Y(\hat{\mu}) \cong L_Y(\hat{\mu})$,

iii) $E_Y(\hat{\mu}) = \tilde{L}_Y(\hat{\mu}) = L_Y(\hat{\mu})$,

iv) $(L_Y(\hat{\mu}))^\mathcal{B} \cong L_{Y^Y}(\hat{\mu})$.

If $|x|_Y = 1 \Rightarrow \int \mu Y(x) = 1 \ \forall x \in L_Y(\hat{\mu})$ (which holds for example when $\hat{\mu}(\mathcal{X}) < \infty$, $\hat{\mu}$ is atomless, and $Y$ satisfies global $\Delta_2$ condition), and if $Y$ is strictly convex on $\mathbb{R}^+$, then $(L_Y(\hat{\mu}), \| \cdot \|_Y)$ is strictly convex [93]. If $\hat{\mu}$ is atomless, then $(L_Y(\hat{\mu}), \| \cdot \|_Y)$ is uniformly convex iff [49, 37] $(\hat{\mu}(\mathcal{X}) < \infty, Y$ is strictly convex on $\mathbb{R}^+$, satisfies local $\Delta_2$ condition with $x_0$, and is uniformly convex for $x \geq x_0$) or $(\hat{\mu}(\mathcal{X}) = \infty, Y$ is uniformly convex on $\mathbb{R}^+$, and satisfies global $\Delta_2$ condition). Finally, $L_Y(\hat{\mu})$ and $L_{Y^Y}(\hat{\mu})$ are reflexive if (both $Y$ and $Y^Y$ satisfy global $\Delta_2$ condition, and $\hat{\mu}(\mathcal{X}) = \infty$) or (both $Y$ and $Y^Y$ satisfy local $\Delta_2$ condition, and $\hat{\mu}(\mathcal{X}) < \infty$) [66, 71].

The noncommutative Orlicz spaces associated with the algebra $\mathfrak{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$ were implicitly introduced by Schatten [77] as ideals in $\mathfrak{B}(\mathcal{H})$ generated by the so-called symmetric gauge functions, and were studied in more details by Gohkberg and Krein [29] (see also [31]). First explicit study of those ideals which are direct noncommutative analogues of Orlicz spaces is due to Rao [70, 71], where $\mathcal{Y}$ is assumed to be a continuous modular function, $Y(x)$ for $x \in \mathfrak{B}(\mathcal{H})$ is understood in terms of the spectral representation, an analogue of the MNTL norm reads

$$\mathfrak{B}(\mathcal{H}) \ni x \mapsto |x|_Y := \inf \left\{ \lambda > 0 \mid \text{tr} \left( Y \left( \frac{|x|}{\lambda} \right) \right) \leq 1 \right\},$$

while the noncommutative Orlicz space is defined as

$$\mathcal{Y}(\mathcal{H}) := \{ x \in \mathfrak{B}(\mathcal{H}) \mid |x|_Y < \infty \}.$$  

The generalisation of Orlicz spaces to semi-finite $W^\ast$-algebras $\mathcal{N}$ equipped with a faithful normal semi-finite trace $\tau$ were proposed by Muratov [55, 56], Dodds, Dodds, and de Pagter [19], and Kunze [43]. Two latter constructions are based on the results of Fack and Kosaki [22]. Given any $y \in \mathcal{M}(\mathcal{N}, \tau)$, the rearrangement function is defined as [31] (see also [97])

$$R_y^\tau : [0, \infty] \ni t \mapsto R_y^\tau(t) := \inf \{ s \geq 0 \mid \tau(P^{[x]}(s, +\infty]) \leq t \} \in [0, \infty].$$  

If $x \in \mathcal{M}(\mathcal{N}, \tau)^+$ and $f : [0, \infty] \mapsto [0, \infty]$ is a continuous nondecreasing function, then [22]

$$\tau(f(x)) = \int_0^\infty dt f(R_y^\tau(t)), \quad R_y^\tau(f(x))(t) = f(R_y^\tau(t)) \ \forall t \in \mathbb{R}^+.$$  

Using this result, Kunze [43] defined a noncommutative Orlicz space associated with an arbitrary Orlicz function $Y$ as

$$L_Y(\mathcal{N}, \tau) := \text{span}_{\mathbb{C}} \{ x \in \mathcal{M}(\mathcal{N}, \tau) \mid \tau(Y(|x|)) \leq 1 \},$$

equipped with a quantum version of a MNTL norm,

$$\| x \|_Y : \mathcal{M}(\mathcal{N}, \tau) \ni x \mapsto \inf \{ \lambda > 0 \mid \tau(Y(\lambda^{-1}|x|)) \leq 1 \},$$

under which, as he proves, (108) is a Banach space. From linearity, it follows that

$$L_Y(\mathcal{N}, \tau) = \{ x \in \mathcal{M}(\mathcal{N}, \tau) \mid \exists \lambda > 0 \ \tau(Y(\lambda|x|)) < \infty \}. \tag{110}$$

---

*A function $f : \mathbb{R} \rightarrow [-\infty, +\infty]$ on a Banach space $X$ is called uniformly convex [47] iff $f \neq +\infty$, and there exists an increasing function $g : \mathbb{R}^+ \rightarrow [-\infty, +\infty]$ with $g(0) = 0$, such that

$$f(x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \lambda(1 - \lambda)g(|x - y|) \ \forall x, y \in \{ z \in X \mid f(z) \neq +\infty \} \ \forall t \in [0, 1].$$

---
On the other hand, Dodds, Dodds, and de Pagter [19] defined (implicitly) a noncommutative Orlicz space associated with \((\mathcal{N}, \tau)\) and an Orlicz function \(\Upsilon\) as

\[
L_\Upsilon(\mathcal{N}, \tau) := \{ x \in \mathcal{M}(\mathcal{N}, \tau) \mid \mathbf{R}_x^+ \in L_\Upsilon(\mathbb{R}^+, \mathcal{U}_{\text{Borel}}(\mathbb{R}^+), d\lambda) \}. \tag{111}
\]

By means of (106) and (107), these two definitions are equivalent. Kunze [43] showed that, for \(\Upsilon\) satisfying global \(\Delta_2\) condition,

\[
\begin{align*}
L_\Upsilon(\mathcal{N}, \tau) &= \{ x \in \mathcal{M}(\mathcal{N}, \tau) \mid \tau(\Upsilon(|x|)) < \infty \}, \tag{112} \\
L_\Upsilon(\mathcal{N}, \tau) &= E_\Upsilon(\mathcal{N}, \tau), \tag{113} \\
(L_\Upsilon(\mathcal{N}, \tau))^B &\cong L_{\Upsilon^Y}(\mathcal{N}, \tau), \tag{114}
\end{align*}
\]

where \(E_\Upsilon(\mathcal{N}, \tau)\) is defined for any Orlicz function \(\Upsilon\) as

\[
E_\Upsilon(\mathcal{N}, \tau) := \overline{\mathcal{N} \cap L_\Upsilon(\mathcal{N}, \tau)}^{\Upsilon}. \tag{115}
\]

In [7] it is shown that if \(\tau_1\) and \(\tau_2\) are faithful normal semi-finite traces on a semi-finite \(W^*\)-algebra \(\mathcal{N}\), and \(\Upsilon\) is an Orlicz function satisfying global \(\Delta_2\) condition, then \(L_\Upsilon(\mathcal{N}, \tau_1)\) and \(L_\Upsilon(\mathcal{N}, \tau_2)\) are isometrically isomorphic. Further analysis of the structure of \(L_\Upsilon(\mathcal{N}, \tau)\) spaces in the context of modular function was provided by Sadeghi [73], who showed that the map \(\mathcal{M}(\mathcal{N}, \tau) \ni x \mapsto \tau(\Upsilon(|x|)) \in [0, \infty]\) is a modular function for any Orlicz function \(\Upsilon\). He also notes that the results of [12] and [20] allow to infer, respectively, the uniform convexity and reflexivity of the spaces \((L_\Upsilon(\mathcal{N}, \tau), \| \cdot \|_\Upsilon)\) from the corresponding properties of the commutative Orlicz spaces \((L_\Upsilon(\mathbb{R}^+, \mathcal{U}_{\text{Borel}}(\mathbb{R}^+), d\lambda), \| \cdot \|_\Upsilon)\). This leads to conclusion that \((L_\Upsilon(\mathcal{N}, \tau), \| \cdot \|_\Upsilon)\) is: uniformly convex if \(\Upsilon\) is uniformly convex and satisfies global \(\Delta_2\) condition; reflexive if \(\Upsilon\) and \(\Upsilon^Y\) satisfy global \(\Delta_2\) condition.\(^\text{10}\)

Al-Rashed and Zegarliński [1, 2] proposed a construction of a family of noncommutative Orlicz spaces associated with a faithful normal state on a countably finite \(W^*\)-algebra. Ayupov, Chilin and Abdullaev [7] proposed the construction of a family of noncommutative Orlicz spaces \(L_\Upsilon(\mathcal{N}, \psi)\) for a semi-finite \(W^*\)-algebra \(\mathcal{N}\), a faithful normal locally finite weight \(\psi\), and an Orlicz function \(\Upsilon\) satisfying global \(\Delta_2\) condition. This construction extends Trunov’s theory of \(L_{\psi}(\mathcal{N}, \tau)\) spaces [91, 92, 100]. Labuschagne [44] provided a construction of the family of noncommutative Orlicz spaces \(L_\Upsilon(\mathcal{N}, \psi)\) associated with an arbitrary \(W^*\)-algebra and a faithful normal semi-finite weight \(\psi\). This construction uses Haagerup’s approach to noncommutative integration and is quite complicated, losing the direct structural analogy between commutative and noncommutative Orlicz spaces.\(^\text{11}\)

We propose here an alternative construction, based on the Falcone–Takesaki approach to noncommutative integration.

**Definition 4.1.** For an arbitrary \(W^*\)-algebra \(\mathcal{N}\) and arbitrary Orlicz function \(\Upsilon\), we define a noncommutative Orlicz space as a vector space

\[
L_\Upsilon(\mathcal{N}) := \{ x \in \mathcal{M}(\mathcal{N}, \tilde{\tau}) \mid \exists \lambda > 0 \ \tilde{\tau}(\Upsilon(\lambda |x|)) < \infty \}, \tag{116}
\]

equipped with the norm

\[
\| \cdot \|_\Upsilon : \mathcal{M}(\mathcal{N}, \tilde{\tau}) \ni x \mapsto \inf\{ \lambda > 0 \mid \tilde{\tau}(\Upsilon(\lambda^{-1}|x|)) \leq 1 \}. \tag{117}
\]

In addition, we define

\[
E_\Upsilon(\mathcal{N}) := \overline{\mathcal{N} \cap L_\Upsilon(\mathcal{N})}^{\| \cdot \|_\Upsilon}. \tag{118}
\]

---

\(^9\)See a discussion in [46, 44] of the case when continuity of \(\Upsilon\) is relaxed to continuity on \([0, x_\Upsilon]\) and left continuity at \(x_\Upsilon\) with \(x_\Upsilon \neq +\infty\).

\(^\text{10}\)The statement of the sufficient condition for reflexivity in Corollary 4.3 of [73] is missing the requirement of the global \(\Delta_2\) condition for \(\Upsilon^Y\).

\(^\text{11}\)In addition, various constructions of noncommutative Orlicz spaces associated with a Young function \(\Upsilon(x) = \cosh(x) - 1\) were given in [84, 35, 85, 45, 86, 87, 36, 50].
Because $\tilde{N}$ is a semi-finite von Neumann algebra, while $\tilde{\tau}$ is a faithful normal semi-finite trace on $\tilde{N}$, all above results on the Banach space structure of $L_T(\mathcal{N}, \tau)$ immediately apply to $L_T(\mathcal{N})$. In particular, if $\mathcal{Y}$ satisfies global $\triangle_2$ condition, then $L_T(\mathcal{N}) = E_T(\mathcal{N})$ and $L_T(\mathcal{N})^B \cong L_{\mathcal{Y}T}(\mathcal{N})$.

Moreover: if $\mathcal{Y}$ is also uniformly convex, then $L_T(\mathcal{N})$ is uniformly convex and $L_{\mathcal{Y}T}(\mathcal{N})$ is uniformly Fréchet differentiable; if $\mathcal{Y}$ also satisfies global $\triangle_2$ condition, then $L_T(\mathcal{N})$ is reflexive. A particular example of the space $E_T(\mathcal{N})$ is considered in [35, 36].

For any choice of a normal semi-finite weight $\psi$ on $\mathcal{N}$, $\tilde{N}$ can be represented as $\mathcal{N} \rtimes_{\sigma \psi} \mathbb{R}$ by means of (70). In such case our construction provides an alternative to Labuschagne’s. They do not coincide, because the representation of a canonical trace $\tilde{\tau}$ on $\mathcal{N} \rtimes_{\sigma \psi} \mathbb{R}$, given by $\tilde{\tau}(u_\psi \cdot u_\psi^*)$, differs from Haagerup’s trace $\tilde{\tau}_\psi$.

## 5 Functoriality

We finish this paper with the proof of functoriality of the above construction. We will first recall Kosaki’s construction of canonical representation of $W^*$-algebra. Then we will use it to extend functoriality of the Falcone–Takesaki noncommutative flow of weights. Finally, we will show that $\ast$-isomorphisms of $W^*$-algebras are canonically mapped to isometric isomorphisms of $L_T(\mathcal{N})$ spaces.

Following Kosaki [38], consider new addition and multiplication structure on $\mathcal{N}_+^\ast$,

\[
\lambda \sqrt{\phi} = \sqrt{\lambda^2 \phi} \quad \forall \phi \in \mathbb{R}^+ \forall \phi \in \mathcal{N}_+^\ast, \tag{119}
\]

\[
\sqrt{\phi} + \sqrt{\psi} = \sqrt{(\phi + \psi)(y^* \cdot y)} \quad \forall \phi, \psi \in \mathcal{N}_+^\ast, \tag{120}
\]

where

\[
y := [\phi : (\phi + \psi)]_{-1/2} + [\psi : (\phi + \psi)]_{-1/2}, \tag{121}\]

and $\sqrt{\phi}$ is understood as a symbol denoting the element $\phi$ of $\mathcal{N}_+^\ast$ whenever it is subjected to the above operations instead of ‘ordinary’ addition and multiplication on $\mathcal{N}_+^\ast$. A ‘noncommutative Hellinger integral’ on $\mathcal{N}_+^\ast$,

\[
\left( \sqrt{\phi} \sqrt{\psi} \right) := (\phi + \psi) \left[ [\psi : (\phi + \psi)]_{-1/2}^* [\phi : (\phi + \psi)]_{-1/2} \right] \tag{122}
\]

is a positive bilinear symmetric form on $\mathcal{N}_+^\ast$ with respect to the operations defined by (119) and (120). Consider an equivalence relation $\sim_\sqrt{}$ on pairs $(\sqrt{\phi}, \sqrt{\psi}) \in \mathcal{N}_+^\ast \times \mathcal{N}_+^\ast$,

\[
(\sqrt{\phi_1}, \sqrt{\psi_1}) \sim_\sqrt{} (\sqrt{\phi_2}, \sqrt{\psi_2}) \iff \sqrt{\phi_1} + \sqrt{\psi_1} = \sqrt{\phi_2} + \sqrt{\psi_2}. \tag{123}
\]

The set of equivalence classes $\mathcal{N}_+^\ast \times \mathcal{N}_+^\ast / \sim_\sqrt{}$ can be equipped with a real vector space structure, provided by

\[
\left( (\sqrt{\phi_1}, \sqrt{\psi_1}), (\sqrt{\phi_2}, \sqrt{\psi_2}) \right)_\sqrt{} := (\sqrt{\phi_1} + \sqrt{\psi_1}, \sqrt{\phi_2} + \sqrt{\psi_2})_\sqrt{}, \tag{124}
\]

\[
\lambda \cdot (\sqrt{\phi}, \sqrt{\psi})_\sqrt{} := \begin{cases} (\lambda \sqrt{\phi}, \lambda \sqrt{\psi})_\sqrt{} : \lambda \geq 0 \\ ((-\lambda) \sqrt{\phi}, (-\lambda) \sqrt{\psi})_\sqrt{} : \lambda < 0, \end{cases} \tag{125}
\]

where $(\sqrt{\phi}, \sqrt{\psi})_\sqrt{}$ denotes an element of $\mathcal{N}_+^\ast \times \mathcal{N}_+^\ast / \sim_\sqrt{}$. The real vector space $(\mathcal{N}_+^\ast \times \mathcal{N}_+^\ast / \sim_\sqrt{}, +, \cdot)$ will be denoted $V$. The map

\[
\mathcal{N}_+^\ast \ni \phi \mapsto (\sqrt{\phi}, 0)_\sqrt{} \in V \tag{126}
\]

is injective, positive and preserves addition and multiplication by positive scalars. Its image in $V$ will be denoted by $L_2(\mathcal{N})^\ast$. A function

\[
\langle \cdot, \cdot \rangle_\sqrt{} : V \times V \to \mathbb{R}, \tag{127}
\]

\[
\left\langle (\sqrt{\phi_1}, \sqrt{\psi_1}), (\sqrt{\phi_2}, \sqrt{\psi_2}) \right\rangle_\sqrt{} := \left( \sqrt{\phi_1} \sqrt{\psi_1} \right) + \left( \sqrt{\phi_2} \sqrt{\psi_2} \right) \tag{128}
\]

\[
+ \left( \sqrt{\phi_2} \sqrt{\psi_1} \right) + \left( \sqrt{\phi_2} \sqrt{\psi_2} \right), \tag{129}
\]

15
is an inner product on $V$, and $(V, \langle \cdot, \cdot \rangle)$ is a real Hilbert space with respect to it, denoted $L_2(N; \mathbb{R})$. The \textit{canonical Hilbert space} is defined as a complexification of the Hilbert space $L_2(N; \mathbb{R})$,

$$L_2(N) := L_2(N; \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}. \quad (130)$$

The space $L_2(N)^+$ is a self-polar convex cone in $L_2(N)$, and, by (126), it is an embedding of $N^+_1$ into $L_2(N)$. The elements of $L_2(N)^+$ will be denoted $\phi^{1/2}$, where $\phi \in N^*_+$. Every element of $L_2(N)$ can be expressed as a linear combination of four elements of $L_2(N)^+$. The antilinear conjugation $J_N : L_2(N) \to L_2(N)$ is defined by

$$J_N(\xi + i\zeta) = \xi - i\zeta \quad \forall \xi, \zeta \in L_2(N; \mathbb{R}). \quad (131)$$

The quadruple $(L_2(N), N, J_N, L_2(N)^+)$ is a standard form of $N$, called a \textit{canonical standard form} of $N$. A bounded generator $\partial_N(x) := \frac{d}{dt} (f(t^x)) |_{t=0}$ of the norm continuous one parameter group of automorphisms

$$\mathbb{R} \ni t \mapsto f(t^x) \in \mathcal{B}(L_2(N)) \quad \forall x \in N, \quad (132)$$

where $N \ni x \mapsto f(x) \in \mathcal{B}(L_2(N))$ is defined as a unique extension of the bounded linear function

$$L_2(N)^+ \ni \phi^{1/2} \mapsto \left( \phi \left( \sigma_{\phi+i/2}(x)x^* \cdot x\sigma_{\phi-i/2}(x^*) \right) \right)^{1/2} \in L_2(N)^+ \quad \forall x \in N \ \forall \phi \in N^*_+, \quad (133)$$

determines a map

$$\partial_N : N \ni x \mapsto \partial_N(x) \in \mathcal{B}(L_2(N)), \quad (134)$$

which is a homomorphism of real Lie algebras: for all $x, y \in N$ and for all $\lambda \in \mathbb{R}$,

$$[\partial_N(x), \partial_N(y)] = \partial_N([x, y]), \quad (135)$$

$$\lambda \partial_N(x) = \partial_N(\lambda x). \quad (136)$$

The faithful normal representation $\pi_N : N \to \mathcal{B}(L_2(N))$,

$$\pi_N(x) := \frac{1}{2} (\partial_N(x) - i\partial_N(ix)), \quad (137)$$

determines a standard representation $(L_2(N), \pi_N, J_N, L_2(N)^+)$ of a $W^*$-algebra $N$, called a \textit{canonical representation} of $N$. Every $*$-isomorphism $\zeta : N_1 \to N_2$ of $W^*$-algebras $N_1, N_2$ determines a unique unitary equivalence $u_\zeta : L_2(N_2) \to L_2(N_1)$ satisfying $u_\zeta(L_2(N_2)^+) = L_2(N_1)^+$, $u_\zeta^*J_Nu_\zeta = J_{N_1}$, and such that $Ad(u_\zeta^*)$ is a unitary implementation of $\zeta$. This means that Kosaki's construction of canonical representation defines a functor CanRep from the category $W^*\text{Iso}$ of $W^*$-algebras with $*$-isomorphisms to the category $\text{StdRep}$ of standard representations with standard unitary equivalences. It also allows to define a functor CanVN from the category $W^*\text{n}$ of $W^*$-algebras with normal $*$-homomorphisms to the category $\text{VVNn}$ of von Neumann algebras with normal $*$-homomorphisms. The functor CanVN assigns $\pi_N(N)$ to each $N$, and normal $*$-homomorphism $\pi_N \circ z \circ \pi_N^{-1} : N_1 \to N_2$ to each normal $*$-homomorphism $z : N_1 \to N_2$ (which is well defined due to faithfulness of $\pi_N$). Let FrgHlb : $\text{VVNn} \to W^*\text{n}$ be the forgetful functor which forgets about Hilbert space structure that underlies von Neumann algebras and their normal $*$-homomorphisms. Due to Sakai’s theorem [74], CanVN and FrgHlb form the equivalence of categories,

$$\text{FrgHlb} \circ \text{CanVN} \cong \text{id}_{W^*\text{n}}, \quad \text{CanVN} \circ \text{FrgHlb} \cong \text{id}_{\text{VVNn}}. \quad (138)$$

Consider the category $W^*\text{Cov}_R\text{Tr}$ of quadruples $(N, \mathbb{R}, \alpha, \tau)$, where $N$ is a semi-finite $W^*$-algebra, $\tau$ is a faithful normal semi-finite trace on $N$, and $(N, \mathbb{R}, \alpha)$ is a $W^*$-dynamical system, with morphisms

$$(N_1, \mathbb{R}, \alpha^1, \tau_1) \to (N_2, \mathbb{R}, \alpha^2, \tau_2) \quad (139)$$
given by such *-isomorphisms \( \varsigma : \mathcal{N}_1 \to \mathcal{N}_2 \) which satisfy
\[
\varsigma \circ \alpha^1_t = \alpha^2_t \circ \varsigma \quad \forall t \in \mathbb{R}, \tag{140}
\]
\[
\tau_1 = \tau_2 \circ \varsigma. \tag{141}
\]

Falcone and Takesaki call the quadruple \( (\hat{\mathcal{N}}, \mathbb{R}, \hat{\sigma}, \hat{\tau}) \) a \textit{noncommutative flow of weights}, and prove that every *-isomorphism \( \varsigma : \mathcal{N}_1 \to \mathcal{N}_2 \) of von Neumann algebras extends to a *-isomorphism \( \hat{\varsigma} : \hat{\mathcal{N}}_1 \to \hat{\mathcal{N}}_2 \) satisfying (140) and (141). This defines a functor
\[
\text{FTflow} : \text{VNIso} \to W^* \text{Cov}_R \text{Tr}, \tag{142}
\]
where \( \text{VNIso} \) is a category of von Neumann algebras with *-isomorphisms. The restriction of \( \hat{\sigma} \) to the center \( \mathcal{N}_0 \) is the Connes–Takesaki flows of weights \( (\mathcal{N}_0, \mathbb{R}, \sigma|_{\mathcal{N}_0}) \) [24]. Hence, the relationship between the Falcone–Takesaki noncommutative flow of weights and the Connes–Takesaki flow of weights can be summarised in terms of the commutative diagram
\[
\begin{array}{ccc}
W^* \text{Iso} & \xrightarrow{\text{CanVN}} & \text{VNIso} & \xrightarrow{\text{FTflow}} & W^* \text{Cov}_R \text{Tr} \\
\updownarrow & & \updownarrow & & \updownarrow \\
W^* \text{III} \text{Iso} & \xrightarrow{\text{CanVN}} & \text{VNIso}_\text{III} & \xrightarrow{\text{CTflow}} & W^* \text{Cov}_R,
\end{array}
\tag{143}
\]
where \( W^* \text{III} \text{Iso} \) (respectively, \( \text{VNIso}_\text{III} \)) is a category of type III factor \( W^* \)-algebras (respectively, von Neumann algebras) with *-isomorphisms, \( \text{ForgTr} \) denotes the forgetful functor that forgets about traces, while \( \mathfrak{3} : W^* \text{Cov}_R \to W^* \text{Cov}_R \) is an endofunctor that assigns an object \( (\mathcal{N}_0, \mathbb{R}, \alpha|_{\mathcal{N}_0}) \) to each \( (\mathcal{N}, \mathbb{R}, \alpha) \), and assigns a morphism \( \varsigma^1_{\mathcal{N}_1} \) such that
\[
\varsigma^1_{\mathcal{N}_1} \circ \alpha^1_t|_{\mathcal{N}_1} = \alpha^2_t|_{\mathcal{N}_2} \circ \varsigma_{\mathcal{N}_1}^1, \tag{144}
\]
to each \( \varsigma : (\mathcal{N}_1, \mathbb{R}, \alpha^1) \to (\mathcal{N}_2, \mathbb{R}, \alpha^2) \).

**Proposition 5.1.** Every *-isomorphism \( \varsigma : \mathcal{N}_1 \to \mathcal{N}_2 \) of \( W^* \)-algebras gives rise to a corresponding isometric isomorphism \( L_{\mathcal{T}}(\mathcal{N}_1) \to L_{\mathcal{T}}(\mathcal{N}_2) \).

**Proof.** By the Falcone–Takesaki construction, and its composition (143) with Kosaki’s construction, \( \varsigma : \mathcal{N}_1 \to \mathcal{N}_2 \) induces a *-isomorphism \( \hat{\varsigma} : \hat{\mathcal{N}}_1 \to \hat{\mathcal{N}}_2 \) of semi-finite von Neumann algebras and a mapping \( (\hat{\mathcal{N}}_1, \mathbb{R}, \hat{\sigma}^1, \hat{\tau}_1) \to (\hat{\mathcal{N}}_2, \mathbb{R}, \hat{\sigma}^2, \hat{\tau}_2) \) satisfying
\[
\hat{\varsigma} \circ \hat{\sigma}^1_t = \hat{\alpha}^2_t \circ \hat{\varsigma} \quad \forall t \in \mathbb{R}, \tag{145}
\]
\[
\hat{\tau}_1 = \hat{\tau}_2 \circ \hat{\varsigma}. \tag{146}
\]
By Collorary 38 in [89], every *-isomorphism of semi-finite von Neumann algebras satisfying (145) and (146) extends to a topological *-isomorphism of corresponding spaces of \( \tau \)-measurable operators affiliated with these algebras, and this extension preserves the property (146). The *-isomorphism \( \hat{\varsigma} \) extends to \( \hat{\varsigma} : \mathcal{M}(\hat{\mathcal{N}}_1, \hat{\tau}_1) \to \mathcal{M}(\hat{\mathcal{N}}_2, \hat{\tau}_2) \) by \( \hat{\varsigma}(\cdot) = u(\cdot)u^* \), where \( u \) is a unitary operator implementing \( \varsigma(\cdot) = u(\cdot)u^* \). It remains to show that \( \hat{\varsigma} \) is an isometric isomorphism. Using (117) as
\[
\|x\|_\mathcal{T} = \inf\{\lambda > 0 \mid \int_0^\infty dt\mathcal{T}(\lambda^{-1}R_{|x|}(t)) < \infty\}, \tag{147}
\]
where
\[
R_{|x|}(t) = \inf\{s \geq 0 \mid \hat{\tau}(P_{|x|}(s, +\infty)) \leq t\}. \tag{148}
\]
For \( \mathcal{M}(\hat{\mathcal{N}}_2, \hat{\tau}_2) = \mathcal{M}(\bar{\varsigma}(\hat{\mathcal{N}}_1), \bar{\tau}_1 \circ \bar{\varsigma}^{-1}) \) and \( x \in \mathcal{M}(\hat{\mathcal{N}}_1, \hat{\tau}_1) \) we have
\[
\bar{\tau}_1 \circ \bar{\varsigma}^{-1}(P_{|\bar{\varsigma}(x)|}(s, +\infty)) = \bar{\tau}_1 \circ \bar{\varsigma}^{-1} \circ \bar{\tau}(P_{|\bar{\varsigma}(x)|}(s, +\infty)) = \bar{\tau}_1(P_{|x|}(s, +\infty)). \tag{149}
\]
Hence, \( \hat{\varsigma} : L_{\mathcal{T}}(\mathcal{N}_1) \to L_{\mathcal{T}}(\mathcal{N}_2) \) is an isometric isomorphism. \( \square \)
Denoting the category of noncommutative Orlicz spaces $L_\Upsilon(\mathcal{N})$ with isometric isomorphisms by $ncL_\Upsilon Iso$, we conclude that our construction determines a functor

$$ncL_\Upsilon: W^* Iso \to ncL_\Upsilon Iso.$$ (150)

Following the results of Sherman [80], we end this section with an interesting problem: for which Orlicz functions $\Upsilon$ there exists a functor from $ncL_\Upsilon Iso$ to the category of $W^*$-algebras with surjective Jordan $*$-isomorphisms?

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References

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- For the Latin transliteration of the Cyrillic script (in references and surnames) we use the following modification of the system GOST 7.79-2000B:
- $\alpha = x$.
- $\beta = y$.
- $\gamma = z$.
- $\delta = c$.
- $\epsilon = y$.
- $\zeta = i$.
- $\eta = w$.
- $\theta = h$.
- $\iota = s$.
- $\kappa = k$.
- $\lambda = h$.
- $\mu = y$.
- $\nu = i$.
- $\xi = x$.
- $\pi = p$.
- $\rho = r$.
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