Phase transitions for the Lifshitz black holes

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Abstract

We study possibility of phase transitions between Lifshitz black holes and other configurations by using free energies explicitly. A phase transition between Lifshitz soliton and Lifshitz black hole might not occur in three dimensions. We find that a phase transition between Lifshitz and BTZ black holes unlikely occurs because they have different asymptotes. Similarly, we point out that any phase transition between Lifshitz and black branes unlikely occurs in four dimensions since they have different asymptotes. This is consistent with a necessary condition for taking a phase transition in the gravitational system, which requires the same asymptote.

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1 Introduction

Recently, the Lifshitz-type black holes \[1 \, 2 \, 3 \, 4 \, 5 \, 6 \, 7\] have received considerable attentions since these may provide a model of generalizing AdS/CFT correspondence to non-relativistic condensed matter physics as the AdS/CMT correspondence \[8 \, 9 \, 10\]. Although their asymptotic spacetimes are apparently known to be Lifshitz, obtaining an analytic solution seems to be a nontrivial task. The known solutions include a four-dimensional topological black hole which is asymptotically Lifshitz with the dynamical exponent \(z = 2\) \[11\]. Analytic black hole solution with \(z = 2\) that asymptotes planar Lifshitz spacetimes was found in the Einstein-scalar-massive vector theory \[12\] and in the Einstein-scalar-Maxwell theory \[13\]. Another analytic solution has been recently found in the Lovelock gravity \[14\]. The \(z = 3\) Lifshitz black hole \[15\] was derived from the new massive gravity (NMG) in three-dimensional spacetimes \[16\]. Numerical solutions to Lifshitz black hole and thermodynamic property of this system were also explored in \[17, 18\].

Their thermodynamic studies was limited because it was difficult to compute their conserved quantities in asymptotic Lifshitz exactly. Recently, there was a progress on computation of mass and related thermodynamic quantities by using the ADT method \[19, 20\] and Euclidean action approach \[21\]. Concerning the mass of Lifshitz black hole in three dimensions, however, there is a discrepancy between \(M = \frac{7r^4}{4\ell^2}\) obtained from the ADT method \[19\] and \(M = \frac{r^4}{4\ell^2}\) from other approaches \[22, 23, 21\]. In this work, we use the latter expression because it respects the first-law of thermodynamics and the ADT mass is not reliable to use for a thermodynamic study of the Lifshitz black hole \[20\].

On the other hand, the Schwarzschild black hole is in an unstable equilibrium with the heat reservoir of the temperature \(T\) \[24\]. Its fate under small fluctuations will be either to decay to hot flat space by Hawking radiation or to grow without limit by absorbing thermal radiations in the infinite heat reservoir \[25\]. This means that an isolated black hole is never in thermal equilibrium in asymptotically flat spacetimes. Thus, one has to find a way of achieving a stable black hole which is in an equilibrium with the finite heat reservoir. A black hole could be rendered thermodynamically stable by placing it in four-dimensional anti-de Sitter (AdS\(_4\)) spacetimes because AdS\(_4\) spacetimes plays the role of a confining box. An important point to understand is to know how a stable black hole with positive specific heat could emerge from thermal radiation through a phase transition. This was known to be the Hawking-Page phase transition between thermal AdS space and Schwarzschild-AdS black hole \[26, 27\], which shows a typical example of the first-order phase transition in the gravitational system. See Appendix for a detail description of how the first-order phase transition occurs.
Witten \[28\] has extended this four-dimensional transition to arbitrary dimension and provided a natural explanation of a confinement/deconfinement transition on the boundary field theory via the AdS/CFT correspondence. On later, it was proposed that a transition between black hole with scalar hair (Martinez-Troncoso-Zanelli black hole \[29\]) and a topological black hole is possible to occur as a second-order phase transition in asymptotically AdS\(_4\) spacetimes \[30\]. We have shown that the phase transition between these is second-order when employing the temperature matching and using the difference of free energies \[31\].

Concerning the Lifshitz black holes, it was firstly proposed that the transition between Lifshitz and black branes may occur because the anisotropic background of Lifshitz brane is favored at low temperature, while the AdS background of black brane is favored in high temperature \[13\]. If it is possible to occur, it corresponds to a phase transition between two different asymptotes and inquiring its order is a curious question. Also, it was suggested that the Lifshitz black hole found numerically in string theory may appear from thermal Lifshitz via the Hawking-Page phase transition at critical temperature \[32\]. Recently, three U(1) fields extension of Einstein-scalar-Maxwell theory provided a charged Lifshitz black hole and phase transitions between Lifshitz black hole and thermal Lifshitz were discussed in \[33\]. Very recently, the Lifshitz soliton was proposed to be a ground state of Lifshitz spacetimes in three dimensions \[21\], implying that a phase transition between Lifshitz soliton and Lifshitz black hole may occur, as the transition between thermal AdS\(_3\) and BTZ black hole was possible to occur \[34\]. However, a necessary condition for a phase transition between two configurations to take place is that they should have the same asymptotic structure.

In this work, in order to answer to the above questions partly, we wish to study possibility of phase transitions between Lifshitz black holes and other configurations explicitly. Thermodynamics study is a key analysis for all Lifshitz black holes. If phase transitions between Lifshitz black holes and other configurations really occur, these would provide some information on phase transitions in condensed matter physics via the presumed AdS/CMT correspondence.

The organization of our work is as follows. In section 2, we first review a well-known phase transition between thermal AdS\(_3\) and BTZ black hole briefly. Then, we investigate a transition between Lifshitz soliton with \(z = 1/3\) and Lifshitz black hole with \(z = 3\). In section 3, we study possibility of transitions between Lifshitz brane with \(z = 2\) and black brane with \(z = 1\). It turns out that these all transitions are not allowed because two black holes have different asymptotes. Finally, we discuss our results in section 4.
2 Phase transitions in three dimensions

The NMG action \([16]\) composed of the Einstein-Hilbert action with a cosmological constant \(\lambda\) and higher-order curvature terms is given by

\[
S^{(3)}_{NMG} = -\left[ S^{(3)}_{EH} + S^{(3)}_{HC} \right], \tag{1}
\]

\[
S^{(3)}_{EH} = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} \left( R - 2\lambda \right), \tag{2}
\]

\[
S^{(3)}_{HC} = -\frac{1}{16\pi G_3 m^2} \int d^3x \sqrt{-g} \left( R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right), \tag{3}
\]

where \(G_3\) is a three-dimensional Newton constant and \(m^2\) a parameter with mass dimension 2. We mention that to avoid negative mass and entropy, it is necessary to take “−” sign in the front of \([S^{(3)}_{EH} + S^{(3)}_{HC}]\). The field equation is given by

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} = 0, \tag{4}
\]

where

\[
K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} R - \frac{1}{2} \Box g_{\mu\nu} + 4 R_{\mu\rho\sigma} R^{\rho\sigma} - \frac{3}{2} R R_{\mu\nu} - R_{\rho\sigma} R^{\rho\sigma} g_{\mu\nu} + \frac{3}{8} R^2 g_{\mu\nu}. \tag{5}
\]

In order to obtain Lifshitz black hole solution with dynamical exponent \(z\), it is convenient to introduce dimensionless parameters

\[
y = m^2 \ell^2, \quad w = \lambda \ell^2, \tag{6}
\]

where \(y\) and \(w\) are proposed to take

\[
y = -\frac{z^2 - 3z + 1}{2}, \quad w = -\frac{z^2 + z + 1}{2}. \tag{7}
\]

In order to obtain the \(z = 1\) BTZ black hole \([35, 36]\), one has \(y = \frac{1}{2}\) and \(w = -\frac{3}{2}\), while \(y = -\frac{1}{2}\) and \(w = -\frac{13}{2}\) are chosen for getting the \(z = 3\) Lifshitz black hole.

Explicitly, we find the \(z\)-dependent black hole solution \([13]\) as

\[
ds_z^2 = g_{\mu\nu} dx^\mu dx^\nu = -\left( \frac{r^2}{\ell^2} \right)^z \left( 1 - \frac{M \ell^2}{r^2} \right) dt^2 + \frac{dr^2}{\left( \frac{r^2}{\ell^2} - M \right)} + r^2 d\phi^2, \tag{8}
\]

where \(M\) is an integration constant related to the mass of black hole. The horizon radius \(r_+\) is determined by the relation of \(g^{rr} = 0\) and \(\ell\) denotes the curvature radius of Lifshitz (AdS) spacetimes. This line element is invariant under the anisotropic scaling of

\[
t \rightarrow \tilde{\lambda}^2 t, \quad \phi \rightarrow \tilde{\lambda} \phi, \quad r \rightarrow \frac{r}{\tilde{\lambda}}, \quad M \rightarrow \frac{M}{\tilde{\lambda}^2}. \tag{9}
\]
For the BTZ black hole, the ADM mass is determined to be
\[ M = r^2 + \ell^2. \]

Before we proceed, it is necessary to derive all thermodynamic quantities. First of all, we mention that the Hawking temperature can be determined from the metric by itself as
\[ T_z = \frac{1}{4\pi} \sqrt{-g} \left| g''_{tt} \right| \bigg|_{r=r_+} = \frac{r_+}{2\pi \ell^{\pm 1}}, \]
irrespective of knowing other conserved quantities. According to the Euclidean action approach in [21], one has to use the original Bergshoeff-Hohm-Townsend action together with Gibbons-Hawking and counter terms to obtain the \( z = 3 \) regularized action
\[ I_{\text{reg}} = I_{\text{BHT}} + I_{\text{GH}} + I_{\text{ct}}. \]

Here \( I_{\text{BHT}} \) is the Euclidean version of Bergshoeff-Hohm-Townsend action as [16]
\[ I_{\text{BHT}} = -\frac{1}{16\pi G_3} \int_{\mathcal{M}} d^3x \sqrt{-g} \left[ R - 2\lambda + f^{\mu\nu} G_{\mu\nu} + \frac{m^2}{4} \left( f_{\mu\nu} f^{\mu\nu} - f^2 \right) \right], \]
where \( f^{\mu\nu} \) is an auxiliary field and \( G_{\mu\nu} = R_{\mu\nu} - R g_{\mu\nu} / 2 \). The remaining terms are boundary terms. The Gibbons-Hawking term \( I_{\text{GH}} \)
\[ I_{\text{GH}} = -\frac{1}{16\pi G_3} \int_{\partial\mathcal{M}} d^2x \sqrt{-h} \left[ -2K - \hat{f}^{ij} K_{ij} + \hat{f}K \right], \]
is required to have a well-defined variational problem for the graviton. Here \( h_{ij} \) is an induced metric on the boundary, \( K_{ij} \) is an extrinsic curvature tensor and \( \hat{f}^{ij} = f^{ij} + 2f^r N^i N^j + f^{rr} N^i N^j \) with the shift \( N^j \). Finally, the counter-term \( I_{\text{ct}} \) is necessary to regularize the divergence on the boundary at infinity. It is given by
\[ I_{\text{ct}} = \frac{1}{32\pi G_3} \int_{\partial\mathcal{M}} d^2x \sqrt{-h} \left[ 15 + \frac{\hat{f}}{2} - \frac{\hat{f}^2}{16} \right]. \]
In this approach, the Euclidean time \( (\tau = it) \) is periodic as \( 0 \leq \tau \leq \beta \) and \( 0 \leq \phi \leq 2\pi \ell \). These are computed to be
\[ I_{\text{BHT}} = \frac{\beta r_+^2}{G_3 \ell^4} \left( -r^2 + r_+^2 \right), \quad I_{\text{GH}} = \frac{\beta r_+^2}{G_3 \ell^4} \left( 2r^2 - r_+^2 \right), \quad I_{\text{ct}} = \frac{\beta r_+^2}{G_3 \ell^4} \left( -r^2 + \frac{3}{4} r_+^2 \right). \]
Adding these and thus, divergences of \( r \to \infty \) cancel out so that the Euclidean action (11) is finite as
\[ I_{\text{of}} = \frac{3\beta r_+^4}{4G_3 \ell^4}, \]
which corresponds to the off-shell free energy. In order to obtain the on-shell free energy, we replace $\beta$ by $\beta_H = 3H(r_+)$ the inverse of Hawking temperature (10) in (16) which leads to

$$I_{\text{reg}}^{\text{on}}(r_+) = \frac{3}{4G_3} \frac{1}{(2\pi \ell)^{4/3}} \frac{1}{(\beta_H^{z=3})^{1/3}}.$$  (17)

It could be rewritten as

$$I_{\text{reg}}^{\text{on}}(r_+) = -\beta_H^{z=3}(r_+) F_{\text{Lif}}^{\text{on}}(r_+), \quad F_{\text{Lif}}^{\text{on}}(r_+) = \mathcal{M}(r_+) - T_{\text{H}}^{z=3}(r_+) S_{\text{BH}}^{z=3}(r_+).$$  (18)

By requiring that the action possess an extremum, the mass $\mathcal{M}$ is determined by

$$\mathcal{M}(r_+) = -\frac{\partial}{\partial \beta_H^{z=3}} I_{\text{reg}}^{\text{on}}.$$  (19)

On the other hand, replacing $T_{\text{H}}^{z=3}$ by $T$ in the second of (18) leads to the off-shell free energy

$$F_{\text{Lif}}^{\text{off}}(r_+, T) = \mathcal{M} - TS_{\text{BH}}^{z=3}.$$  (20)

Here $T$ is an independent control parameter for the study of phase transition. The on-shell (extremum) condition of $\frac{d}{dr_+} F_{\text{Lif}}^{\text{off}} = 0$ (equivalently, requiring the first-law of $d\mathcal{M}/dr_+ = TdS_{\text{BH}}^{z=3}/dr_+$) determines $T$ to be the Hawking temperature

$$T \rightarrow T_{\text{H}}^{z=3}$$  (21)

which shows how the off-shell free energy reduces to the on-shell free energy:

$$F_{\text{Lif}}^{\text{off}}(r_+, T)\big|_{T \rightarrow T_{\text{H}}^{z=3}} \rightarrow F_{\text{Lif}}^{\text{on}}(r_+).$$  (22)

For the $z = 3$ Lifshitz black hole, its mass, heat capacity ($C = \frac{d\mathcal{M}}{dT_{\text{H}}}$), Bekenstein-Hawking entropy, and on-shell (Helmholtz) free energy are given by

$$\mathcal{M} = \frac{r_+^4}{4G_3 \ell^4}, \quad C = \frac{4\pi r_+}{3G_3}, \quad S_{\text{BH}} = \frac{2\pi r_+}{G_3}, \quad F_{\text{Lif}}^{\text{on}} = -\frac{3r_+^4}{4G_3 \ell^4}. $$  (23)

Here we check that the first-law of thermodynamics is satisfied as

$$d\mathcal{M} = T_{\text{H}}^{z=3} dS_{\text{BH}}^{z=3}.$$  (24)

On the other hand, the Lifshitz soliton has its thermodynamic quantities [21]

$$T_{\text{H}}^{TL} = 0, \quad \mathcal{M}_{TL} = -\frac{3}{4G_3}, \quad S_{\text{BH}} = 0, \quad F_{TL} = -\frac{3}{4G_3}. $$  (25)

because of the absence of horizon. It corresponds to the manifold with the dynamical exponent $z = 1/3$. 

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We mention the $z = 1$ BTZ black hole case. For this case, thermodynamic quantities are expressed in terms of a different radial coordinate $\rho_+$

\[
M_{BTZ} = \frac{\rho_+^2}{8G_3\ell^2}, \quad C = \frac{\pi \rho_+}{2G_3}, \quad S_{BH}^{z=1} = \frac{\pi \rho_+}{2G_3}, \quad F_{BTZ} = M_{BTZ} - T_{H}^{z=1}S_{BH}^{z=1} = -\frac{\rho_+^2}{8G_3\ell^2}.
\] (26)

Also the first-law of thermodynamics is given by

\[
dM_{BTZ} = T_{H}^{z=1}dS_{BH}^{z=1}.
\] (27)

The thermal AdS$_3$ has its thermodynamic quantities

\[
M_{T,AdS} = -\frac{1}{8G_3}, \quad F_{T,AdS} = -\frac{1}{8G_3}.
\] (28)

because of the absence of horizon. At this stage, we discuss global structures of Lifshitz and BTZ black holes. Their Penrose diagram are similar as $\otimes$ except that the BTZ (Lifshitz) black holes are regular (singular) at $r = 0$ (top and bottom), while at $r = \infty$ (two sides) the BTZ (Lifshitz) black hole spacetimes have asymptotically AdS (Lifshitz).

Now, we introduce five parameters to study phase transition in black hole physics [25]:

- $r_+ \rightarrow$ order parameter,
- $T_H(r_+) \rightarrow$ order parameter (onshell temperature),
- $T \rightarrow$ control parameter (offshell temperature),
- $F^\text{on}(r_+) \rightarrow$ increasing (decreasing) black hole via equilibrium process,
- $F^\text{off}(r_+, T) \rightarrow$ increasing (decreasing) black hole via nonequilibrium process,

where off-shell (on-shell) mean equilibrium (non-equilibrium) configurations. In general, the equilibrium process implies a reversible process, while the non-equilibrium process implies a irreversible process. The off-shell free energy means a generalized free energy which is similar to a temperature-dependent scalar potential $V(\varphi, T)$ for a simple model of thermal phase transition.

Generally, increasing black hole ($\rightarrow$) is induced by absorbing radiations in the heat reservoir, while decreasing black hole ($\leftarrow$) is done by Hawking radiations as evaporation process.

In studying the phase transition, two important quantities are the heat capacity $C$ which shows thermal stability (instability) for $C > 0 (C < 0)$ and free energy $F^\text{on}$ which indicates the global stability (instability) for $F^\text{on} < 0 (F^\text{on} > 0)$. For the case of positive heat capacity ($C > 0$), one relevant quantity is just the free energy “$F^\text{on}$”. Here we would like to mention
that all black holes under consideration are thermally stable because all their heat capacities are positive.

Hence, the free energy plays a key role in studying phase transition between two gravitational configurations if they have the same asymptote. If they have different asymptotics, comparing free energies does not make sense.

### 2.1 Transition between thermal AdS$_3$ and BTZ black hole

In order to discuss the phase transition, we first compare two free energies of thermal AdS$_3$ (TAdS) and BTZ black hole. One finds from the left panel of Figure 1 that TAdS is favored at small black hole (low temperature), while BTZ black hole is favored at large black hole (high temperature). This observation suggests a phase transition between TAdS space and BTZ black hole. The Horowitz-Myers conjecture for the AdS soliton [37] implies that the soliton with a negative energy could be taken as the thermal background (ground state) in any dimensions. We note that the three-dimensional AdS soliton is just the thermal AdS$_3$ space (TAdS) [38]. Then, we can calculate the difference of free energy with respect to TAdS as

$$
\Delta F_{BTZ}(\rho_+) = F_{BTZ}^{on} - F_{TAdS} = \frac{1}{8G_3} \left[ 1 - \frac{\rho_+^2}{\ell^2} \right].
$$

\(29\)
Figure 2: Left: free energy of Lifshitz soliton [horizontal line] and Lifshitz black hole [bold curve] with $G_3 = 1/2$ and $\ell = 1$. At $r_+ = 1$, $F_{mL}^m = F_{mLBH}^m = -1$. For $r_+ < 1(r_+ > 1)$, the ground state is Lifshitz soliton (Lifshitz black hole). Right: The bold curve denotes the difference in on-shell free energy $\Delta F_{onL}^m(r_+)$, while three solid curves show the off-shell free energy $\Delta F_{offL}(r_+, T)$ for three different temperatures: from top to bottom, $T = 0.059(< T_c)$, $T_c = 0.159$, $0.259(> T_c)$.

Also we introduce the difference in off-shell free energies

$$\Delta F_{BTZ}^{off}(\rho_+, T) = F_{BTZ}^{off}(\rho_+, T) - F_{TAdS} = \frac{1}{8G_3} \left[ \frac{\rho_+^2}{\ell^2} + 1 \right] - T \cdot \frac{\pi \rho_+}{2G_3}.$$  \hspace{1cm} (30)

At the minimum point of $\rho_+ = \rho_m$ defined by the condition of $dF_{BTZ}^{off}(\rho_+, T)/d\rho_+ = 0$, we have the relation of $\Delta F_{BTZ}^m(\rho_m) = \Delta F_{BTZ}^{off}(\rho_m, T)$. This is depicted in the right panel of Fig. 1. At $T = T_c$, the transition from TAdS to BTZ black hole is possible to occur. For $T < T_c$, the TAdS dominates because of $\Delta F_{BTZ}^{off}(\rho_m, T) > 0$, while for $T > T_c$, the BTZ black hole dominates because of $\Delta F_{BTZ}^{off}(\rho_m, T) < 0$. This indicates a change of dominance at the critical temperature $T = T_c = \frac{1}{2\pi} = 0.159$ [34]. However, we wish to mention that this transition is not considered as a truly Hawking-Page transition because an unstable small black hole with negative heat capacity was missed in this transition [39]. See Appendix for the Hawking-page transition.

### 2.2 Lifshitz soliton and Lifshitz black hole

We observe the similarity between AdS systems of (26) and (28) and Lifshitz systems of (23) and (25). Hence, we would like to compare two free energies of Lifshitz soliton with $z = 1/3$ [21] and Lifshitz black hole with $z = 3$. One finds from the left of Figure 2 that Lifshitz soliton is favored at small black hole (low temperature), while Lifshitz black hole is favored at large black hole (high temperature). This observation may allow a phase transition
between Lifshitz soliton and Lifshitz black hole. The Lifshitz soliton with a negative energy could be taken as the ground state, suggesting that the Lifshitz soliton may be considered as the thermal Lifshitz (TL).

Now, we can calculate the difference of free energy with respect to the Lifshitz soliton background as

\[ \Delta F_{\text{Lif}}(r_+) = F_{\text{LBH}}^\text{on} - F_{\text{TL}} = \frac{3}{4 G_3} \left[ 1 - \frac{r_+^4}{\ell^4} \right]. \]  

(31)

We define the difference in off-shell free energies

\[ \Delta F_{\text{Lif}}^\text{off}(r_+, T) = F_{\text{Lif}}^\text{off}(r_+, T) - F_{\text{TL}} = \frac{1}{4 G_3} \left[ \frac{r_+^4}{\ell^4} + 3 \right] - T \cdot \frac{2 \pi r_+}{G_3}. \]  

(32)

At the minimum point of \( r_+ = r_m \), we have \( \Delta F_{\text{Lif}}^\text{on}(r_m) = \Delta F_{\text{Lif}}^\text{off}(r_m, T) \). This is depicted in right of Figure 2.

At the critical temperature \( T = T_c \), a transition from the Lifshitz soliton to Lifshitz black hole is possible to occur. For \( T < T_c \), the Lifshitz soliton dominates because of \( \Delta F_{\text{Lif}}^\text{off}(r_m, T) > 0 \), whereas for \( T > T_c \), the Lifshitz black hole dominates because of \( \Delta F_{\text{Lif}}^\text{off}(r_m, T) < 0 \). This may indicate a change of dominance at the critical temperature \( T = T_c = \frac{1}{2 \pi \ell} = 0.159 \), as the transition occurs from thermal AdS\(_3\) space to BTZ black hole. However, this transition seems not to occur because their asymptotes are different [21]. Explicitly, the Lifshitz soliton takes the asymptote of \( -r^2 dt^2 + dr^2/r^2 + r^2 \, d\phi^2 \), while the Lifshitz black hole takes its asymptote of \( -r^2 dt^2 + dr^2/r^2 + r^2 \, d\phi^2 \). They are the same only for \( z = 1 \) AdS\(_3\) spacetimes, but they are different for \( z = 3 \) Lifshitz spacetimes. Hence, in order to define the phase transition properly, we have to find a new ground state of thermal Lifshitz with the Lifshitz asymptote \( -r^2 dt^2 + dr^2/r^2 + r^2 \, d\phi^2 \).

Similarly, one may consider that a phase transition between Lifshitz and BTZ black holes occur because their free energies in Figure 3 imply that Lifshitz black hole is favored at small black hole (low temperature), while BTZ black hole is favored at large black hole (high temperature). However, there is no transition between Lifshitz and BTZ black holes in three dimension because of their different asymptotes of Lifshitz and AdS\(_3\) spaces.

3 Lifshitz brane and black brane

In order to confirm “no phase transition between Lifshitz black hole and other configuration in three dimensions”, we wish to study possibility of a phase transition between Lifshitz brane and black brane in four dimensions. For this purpose, we introduce the effective action
Figure 3: Two free energies of Lifshitz [dashed] and BTZ [solid] black holes with temperature matching $T_{H}^{BTZ} = T_{H}^{RH}$. At $T_{H} = T_{c} = 0.29$, one finds that $F_{LH}^{on}(T_{H}) = F_{BTZ}^{on}(T_{H}) = -3.375$. It is suggested that the ground state is Lifshitz black hole (BTZ black hole) for $T_{H} < T_{c}$ ($T_{H} > T_{c}$).

in four-dimensional spacetimes [13]

$$S = \frac{1}{16\pi G_{4}} \int d^{4}x \sqrt{-g} [R - 2\Lambda - \frac{1}{2} \partial_{\mu}\phi \partial^{\mu}\phi - \frac{1}{4} e^{\lambda\phi} F_{\mu\nu} F^{\mu\nu}], \quad (33)$$

where $\Lambda$ is the cosmological constant and two fields are a massless scalar and a Maxwell field. It admits the Lifshitz (black) brane with dynamical exponent $z = 2$ as solution to equations of motion [40]

$$ds_{LB}^{2} = L^{2} \left[ - r^{2z} f(r) dt^{2} + \frac{dr^{2}}{r^{2} f(r)} + r^{2} \sum_{i=1}^{2} dx_{i}^{2} \right],$$

$$f(r) = 1 - \frac{r^{z+2}}{r^{+2}}, \quad e^{\lambda\phi} = \frac{1}{r^{4}}, \quad \lambda^{2} = \frac{4}{z - 1},$$

$$F_{rt} = qr^{z+1}, \quad \Lambda = -\frac{(z + 1)(z + 2)}{2L^{2}},$$

$$q^{2} = 2L^{2}(z - 1)(z + 2), \quad (34)$$

where the event horizon is located at $r = r_{+}$. This line element is invariant under the anisotropic scaling of $t \rightarrow \lambda^{z} t, x_{i} \rightarrow \lambda x_{i}, r \rightarrow r/\lambda$, and $r_{+} \rightarrow r_{+}/\lambda$. It is important to note from the last relation that the charge $q$ is not an independent charge hair because it is determined by the curvature radius $L$ of Lifshitz black brane and its dynamic exponent $z$, in compared to the Reissner-Nordström-AdS black hole [41]. A similar case was found in the charged MTZ black hole [42, 43].
On the other hand, the black brane (BB) solution is obtained for $z = 1$ as
\begin{equation}
\begin{split}
    ds_{BB}^2 &= L^2 \left[ -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 \sum_{i=1}^{2} dx_i^2 \right], \\
    f(r) &= 1 - \frac{r^3}{r^3}, \quad \phi = \phi_0 = \text{const}, \\
    F_{rt} &= 0, \quad \Lambda = \frac{3}{L^2},
\end{split}
\end{equation}
where the scalar and Maxwell field play no role. Hence, the scalar field $\phi$ and the Maxwell field $F_{\mu\nu}$ play the essential role in modifying asymptotic geometry from AdS$_4$ spacetimes to Lifshitz spacetimes.

The temperature and entropy are determined by
\begin{equation}
    T_H = \frac{1}{\beta_H} = \left[ \frac{z + 2}{4\pi} \right] r_+^z, \quad S_{BH} = \frac{L^2 V_2}{4G_4} r_+^z,
\end{equation}
where $V_2$ denotes the volume of two-dimensional spatial directions. In order to use the Euclidean action approach, we need the Euclidean version $I_E$ of (33),
\begin{equation}
    I_E = -\frac{1}{16\pi G_4} \int_\mathcal{M} d^4x \sqrt{-g} [R - 2\Lambda - \frac{1}{2} \partial_{\mu}\phi \partial^{\mu}\phi - \frac{1}{4} e^{\lambda\phi} F_{\mu\nu} F^{\mu\nu}],
\end{equation}
and the Gibbons-Hawking term (extrinsic boundary term) \[13\]
\begin{equation}
    I\text{GH} = -\frac{1}{8\pi G} \int d^3x \sqrt{\tilde{h}} K.
\end{equation}
Also the counter term (intrinsic boundary term) is given by
\begin{equation}
    I\text{ct} = \frac{z + 1}{8\pi G} \int d^3x \sqrt{h}.
\end{equation}
Then, the on-shell action of $I^{on} = I_E + I\text{GH} + I\text{ct}$ with $\beta = \beta_H$ leads to
\begin{equation}
    I^{on} = -\frac{V_2 L^2 r_+^{z+2}}{16\pi G} \beta_H^z = \beta_H^z F^{on}_{LB}.
\end{equation}
Mass and heat capacity are obtained as
\begin{equation}
    M^z = \frac{2L^2 V_2 r_+^{z+2}}{16\pi G_4}, \quad C^z = \frac{dM^z}{dT_H^z} = \frac{2V_2 r_+^2}{4zG_4} = \frac{2S_{BH}}{z} > 0.
\end{equation}
Because of positiveness of heat capacity for $z = 1$ and $2$, it is natural to define an on-shell free energy.
According to the Euclidean action approach, the on-shell free energy can be read off from (40) as
\[
F_{\text{on}}^{\text{LB}}(r) = -\frac{L^2 V_2 r_+^{z+2}}{16\pi G_4},
\]
which is consistent with the Gibbs free energy
\[
\tilde{F}_{\text{on}}^{\text{LB}} = M^z - \Phi Q - T_\text{BH}\tilde{S}_{\text{BH}} = \frac{L^2 V_2}{16\pi G_4} \left[ 2r_+^{z+2} + (z - 1)r_+^{z+2} - (z + 2)r_+^{z+2} \right] = -\frac{L^2 V_2 r_+^{z+2}}{16\pi G_4},
\]
with charge \( Q = qV_2/32\pi G_4 \) and potential \( \Phi = -qr_+^{z+2}/(z + 2) \). Here \( q^2 = 2L^2(z - 1)(z + 2) \) was used to obtain \( \tilde{F}_{\text{on}}^{\text{LB}} \). We note that the first-law of thermodynamics is satisfied to be
\[
dM^z = T dS_{\text{BH}} + \Phi dQ \rightarrow dM^z = T dS_{\text{BH}}
\]
because of \( dQ = 0 \) for the \( z = 2 \) Lifshitz brane.

In this case, their off-shell free energies are defined as
\[
F_{\text{off}}^{\text{LB}}(r, T) = M^z - \Phi Q - T S_{\text{BH}} = 3r_+^4 - 4\pi r_+^2 T,
\]
\[
F_{\text{off}}^{\text{BB}}(\rho, T) \big|_{\rho \rightarrow \frac{4}{3}r_+^2} = \left[ M^z = 1 - T S_{\text{BH}} \right]_{\rho \rightarrow \frac{4}{3}r_+^2} = \left( \frac{4}{3} \right)^3 \left[ 2r_+^4 - 3\pi r_+^2 T \right],
\]
where we used the normalization of \( \frac{L^2 V_2}{16\pi G_4} = 1 \) for numerical computation and temperature matching of \( T_{\tilde{H}}^{z=2} = T_{\tilde{H}}^{z=1} \), implying that \( \rho \rightarrow (4/3)r_+^2 \) in the black brane sector. However, the isotropic scaling (\( z = 1 \)) is transformed to the anisotropic scaling (\( z = 2 \)) when working with the temperature matching. As is shown in Figure 4, it is clear that \( dF_{\text{off}}^{\text{LB}} / dr_+ = 0 \rightarrow T = T_{\tilde{H}}^{z=2} \), while \( dF_{\text{off}}^{\text{BB}} / dr_+ = 0 \rightarrow T = T_{\tilde{H}}^{z=1} \). In other words, three minimum points of off-shell Lifshitz free energy are not crossed by on-shell free energy, while three minimum points of off-shell black brane free energy are crossed by on-shell free energy.

In order to have a better picture, on may use the Helmholtz free energy \( \tilde{F}_{\text{on}}^{\text{LB}}(r) \) defined by
\[
\tilde{F}_{\text{on}}^{\text{LB}}(r) = M^z - T \tilde{S}_{\text{BH}} = -\frac{zL^2 V_2 r_+^{z+2}}{16\pi G_4}
\]
because \( q^2 \) is no longer an independent charge hair. In this case, the first-law is satisfied as
\[
dM^z = T dS_{\text{BH}}.
\]
Thus, the corresponding off-shell free energy
\[
\tilde{F}_{\text{off}}^{\text{LB}}(r, T) = M^z - T S_{\text{BH}} = 2r_+^4 - 4\pi r_+^2 T
\]
implies that
\[
\frac{d}{dr_+} \tilde{F}_{\text{off}}^{\text{LB}}(r, T) = 0 \rightarrow T = \frac{dM^z}{dS_{\text{BH}}} \rightarrow T = T_{\tilde{H}}.
\]
Figure 4: Left: free energy of Lifshitz brane [on-shell: bold curve; off-shell: solid]. Three minimum points are not crossed by on-shell free energy. This means that $F_{\text{off}}^\text{LB}$ is not suitable for describing the Lifshitz brane. Right: free energy of black brane [on-shell: bold curve; off-shell: solid]. Here three minimum points are crossed by on-shell free energy. This implies that $F_{\text{off}}^\text{BB}$ is suitable for describing for the black brane.

We wish to mention why the correct definition of the free energy is (46) but not (43). Following Ref. [44], the reason is just to have a well defined variational problem at the boundary. As was pointed out after (34), the charge $q^2$ associated to the U(1) gauge field is fixed by $L$ and $z$. This means that $q^2$ does not correspond to the value of the gauge field at the boundary, but it corresponds to gradient of gauge field which must be kept fixed when doing variations of the action to find the equations of motion. Hence, choosing the Helmholtz free energy amounts to changing boundary conditions from the Dirichlet to the Neumann. This could be taken into account by Legendre transformation which leads to the definition (46) for the free energy.

On the other hand, it is well-known that one uses the Gibbs free energy when working in the grand canonical ensemble with fixed potential, while one uses the Helmholtz free energy working in the canonical ensemble with fixed charge. Hence there is a problem in studying thermodynamics of the Lifshitz brane, which states that fixing charge is not compatible with grand canonical ensemble.

As was suggested by Ref. [13], the anisotropic background of Lifshitz brane is favored at small brane (low temperature), while the isotropic background of black brane is favored at large brane (high temperature). It is shown in Figure 5. This is mainly because $F_{\text{on}}^\text{LB} < F_{\text{on}}^\text{BB}$ for $r_+ < r_c = 0.65$ and $F_{\text{on}}^\text{LB} > F_{\text{on}}^\text{BB}$ for $r_+ > r_c = 0.65$. However, the suggested transition could not occur because their asymptotes are different.
Figure 5: Comparison of two free energies [Lifshitz brane (dashed) and black brane (solid)] with temperature matching $T^z_H = 2 = T^z_H = 1$. At $T^z_H = T_c = 0.13$, one finds that $F_{LB}^{on}(T^z_H) = F_{BB}^{on}(T^z_H) = -0.18$. It is suggested that the ground state is Lifshitz brane (black brane) for $T^z_H < T_c$ ($T^z_H > T_c$).

4 Discussions

We have discussed phase transitions between Lifshitz black holes and other configurations by using free energy. We summarize the phase transitions: thermal AdS$_3 \rightarrow$ BTZ black hole, but Lifshitz soliton $\not\rightarrow$ Lifshitz black hole, Lifshitz black hole $\not\rightarrow$ BTZ black hole, and Lifshitz brane $\not\rightarrow$ black brane.

We remind the reader that a necessary condition for a phase transition to take place is to have the same asymptote. In this sense, the above non-occurrences ($\not\rightarrow$) are consistent with the necessary condition for a phase transition in the gravitational system. In other words, the non-occurrence is mainly due to different asymptotes: asymptotically AdS and Lifshitz.

Consequently, if two configurations have the same asymptotes, the free energy analysis is necessary to test whether does the phase transition between two configurations occur. If not, the conventional free energy analysis seems to be meaningless.
Appendix: Hawking-page transition

In this appendix, we present the Hawking-page transition as the first-order phase transition in the Schwarzschild-AdS black hole (SAdS) \[24\]. The ADM mass, Hawking temperature, and the Bekenstein-Hawking entropy are given by

\[
M_{SAdS}(r_+)=\frac{1}{2}\left(r_+ + \frac{r_+^3}{l^2}\right), \quad T_H(r_+) = \frac{1}{4\pi}\left(\frac{1}{r_+^2} + \frac{3r_+}{l^2}\right), \quad S_{BH} = \pi r_+^2.
\] (50)

As is shown in Figure 6, the shape of Hawking temperature is \(\sim\). The minimum temperature is \(T_0 = 0.027\) for fixed \(l = 10\) and thus, it is impossible for a phase transition to take place for \(T < T_0\). The critical temperature is determined by the condition of on-shell free energy.

The heat capacity and on-shell free energy are given by \[45\]

\[
C_{SAdS}(r_+) = 2\pi r_+^2 \frac{3r_+^4 + l^2 r_+^2}{3r_+^4 - l^2 r_+^2},
\] (51)

\[
F_{SAdS}^{on}(r_+) = M_{SAdS} - T_H S_{BH} = \frac{r_+}{4} \left(1 - \frac{r_+^2}{l^2}\right),
\] (52)

where \(C_{SAdS}\) blows up at \(r_+ = r_0 = l/\sqrt{3}\) (heat capacity is changed from \(-\infty\) to \(\infty\) at \(r_+ = r_0\)) and \(F_{SAdS}^{on} = 0\) for \(r_+ = r_c = l\). The thermal radiation is located at \(r_+ = 0\) in this black hole picture. A SAdS is globally stable only if \(C_{SAdS} > 0\) and \(F_{SAdS}^{on} < 0\). We observe that the free energy is maximum at \(r_+ = r_0\) and zero at \(r_+ = r_c\) which determines the critical temperature as shown in Figure 7. For \(r_+ > r_c\), one finds negative free energy. The corresponding temperatures are given by the minimum \(T_0 = T_H(r_0)\) and the critical one...
Figure 7: Phase transition for the SAdS: the solid curve represents the on-shell free energy $F_{SAdS}^{on}(r_+)$, while five dashed curves denote the off-shell free energy $F_{SAdS}^{off}(r_+, T)$ with five temperatures $T = T_1, T_0, T_2, T_c, \text{ and } T_3$.

$T_c = T_H(r_c)$, respectively. Introducing the off-shell free energy as a function of $r_+$ and $T$

$$F_{SAdS}^{off}(r_+, T) = M_{SAdS}(r_+) - T \cdot S_{BH}(r_+)$$

we find the phase transition. Here $T$ plays a role of control parameter for taking a phase transition.

Before we proceed, we point out an important relation

$$\frac{d}{dr_+} F_{SAdS}^{off}(r_+, T) = 0 \rightarrow T = T_H \rightarrow F_{SAdS}^{off}(r_+, T)|_{T \rightarrow T_H} = F_{SAdS}^{on}(r_+),$$

which shows clearly how the off-shell free energy goes to the on-shell free energy. In other words, summing over all extremum (saddle and minimum points) in off-shell free energy $F_{SAdS}^{off}(r_+, T)$ provides the on-shell free energy $F_{SAdS}^{on}(r_+)$.

For $T = T_3 > T_c$, the process of phase transition is shown in Figure 7 explicitly. In this case, one starts with thermal radiation ($r_+ = 0$) in AdS space, a small black hole (SBH$_-$) appears at $r_+ = r_u$ [solution to $F_{SAdS}^{on}(r_u) = F_{SAdS}^{off}(r_u, T_3)]$. Here the SBH$_-$ denotes the unstable small black hole with $C_{SAdS} < 0$ and $F_{SAdS}^{on} > 0$. Then, since the heat capacity changes from $-\infty$ to $\infty$ at $r_+ = r_0$, the large black hole (LBH$_+$) finally comes out as a stable object at $r_+ = r_s$ [solution to $F_{SAdS}^{SAdS}(r_s) = F_{SAdS}^{off}(r_s, T_3)]$. Here the LBH$_+$ denotes a globally stable black hole because of $C_{SAdS} > 0$ and $F_{SAdS}^{on} < 0$.

Actually, there is a change of the dominance at the critical temperature $T = T_c$ from thermal radiation to black hole [26] as seen from $F_{SAdS}^{off}(0, T_c) = F_{SAdS}^{off}(r_c, T_c) = 0$. This is called the Hawking-Page phase transition as a typical example of the first-order transition in the gravitational system: thermal gas $\rightarrow$ SBH$_-$ $\rightarrow$ LBH$_+$. For two temperatures $T = T_1$
and $T_2(< T_c)$, the free energy $F_{SAdS}^m(0) = 0$ of thermal gas is the lowest one, while for the
$T = T_3(> T_c)$ case, the lowest one is the free energy $F_{SAdS}^m(r_s) < 0$ for the large black hole.
Hence, for $T < T_c$, the ground state is thermal gas, whereas for $T > T_c$, the ground state
is a large black hole. This dictates a thermal phase transition which is controlled by the
temperature $T$ in the gravitational system.

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