Flow Equations
for the Higgs Top System

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Abstract:
The flow equations or exact RG equations for the Higgs Top System are solved to leading order in $1/N_c$. This allows to relate arbitrary bare actions with this field content continuously to effective low energy theories, and we find the flow converging towards general renormalizable models. The assumption of a bare action of the generalized Nambu-Jona-Lasinio type does not restrict the parameters of the low energy theory.

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1 Introduction

The experimental lower bound on the top quark mass has revived speculations about dynamical electroweak symmetry breaking in the form of a top quark condensate [1]-[3]. A large Higgs-top Yukawa coupling might be a residual effect of compositeness in the corresponding channel, i.e. the Higgs field of the standard model could reveal itself as a $t\bar{t}$ bound state. A concrete dynamical model can be formulated [3] by assuming the existence of non-renormalizable four Fermi interactions as in the Nambu-Jona-Lasinio (NJL) model [4] and rely on an analysis to leading order in $1/N_c$, where $N_c$ is the number of colours. The effective low energy theory of such a model has the form of the Higgs top sector of the standard model with restricted parameters such that predictions on the physical Higgs and top quark masses can be made [3]. Subsequently it was observed that these constraints are relaxed if the original model is generalized [5]-[7]. Whereas the original model contains just a single pointlike four Fermi interaction (and a specified form of the cutoff), the generalizations include additional irrelevant operators in the form of four Fermi interactions involving derivatives and general cut-off procedures.

The underlying problem is the typical question in quantum field theory of the relation between a “bare” high energy action and an effective low energy theory. Here the bare action does not necessarily have to be of the renormalizable type, but could again be the result of technicolour, grand unification, supergravity or even string theory. Nevertheless the effective low energy action will always be of the renormalizable type in the sense that the coefficients of all irrelevant operators (or all one-particle-irreducible Green functions with dimensions larger than four) are computable, in perturbation theory, in terms of the finite number of coefficients of marginal or relevant operators (or renormalizable couplings).

An intuitive picture of the relation between the bare high energy action and the effective low energy theory is provided by a scale-dependent effective action $S_{\text{int}}(\mu)$, which is obtained by integrating out all physical degrees of freedom with momenta $p$ with $|p| > \mu$, and which interpolates continuously between the high energy and low energy action. The scale dependence of $S_{\text{int}}$ can be described by means of exact renormalization group or flow equations [8], [9], which have been introduced to continuum quantum field theory by Polchinski [10]. These flow equations describe the RG flow of relevant, marginal, and irrelevant couplings simultaneously and are of particularly simple form, albeit they constitute an infinite set of coupled differential equations. The equations can be useful for proofs of renormalizability [11]-[14], and can be obtained within the context of an average action [15]. Recently it was proposed to apply these equations to the formation of bound states and dynamical symmetry breaking by combining them with the introduction of collective fields [16], and the generalized NJL models are an appropriate testing ground for this formalism. In the context of a composite Higgs field within the Standard Model, the use of flow equations has also been advertised (but not exploited) in [17]. Below we will present the solution of the flow equations,
to leading order in $1/N_c$, for the Higgs top system. Since it allows to relate effective high and low energy theories in an explicit and continuous way, it can be used both as a new method to compute the effective low energy theory, but also to gain insight into the physics of this relation. In particular the convergence of the RG flow, starting with an infinite number of nonrenormalizable couplings (or irrelevant operators) at high scales, towards an effective low energy theory parametrized by only a finite number of renormalizable couplings can be studied explicitly. This way we can discuss whether the assumption of a bare action of the generalized NJL type restricts the parameters of the low energy theory. The corresponding negative answer is not new $[5]-[7]$, but in any case every exactly solvable system as the present one constitutes a useful terrain for testing approximations to these flow equations, which have to be developed in the future if they want to be applied to more complicated theories as non-abelian gauge theories.

In the next section we will present the flow equations for the Higgs top system to leading order in $1/N_c$, their solutions and implications for the effective low energy theory. We concentrate mainly on the effective potential, whose scale dependence within different models has also recently been studied in [18]. Section 3 is devoted to generalized NJL models, which correspond to just a particular set of boundary conditions of the flow equations. Finally we close with discussions and an outlook in section 4.

## 2 The Flow Equations

Let us discuss, for simplicity, the general features of the flow equations in the context of a model with a single scalar field $\phi$, and we will work in Euclidean space. The flow equations rely on a cutoff propagator $P_{\Lambda}(p^2)$, which is defined by

$$P_{\Lambda}(p^2) = \frac{f(\Lambda, p)}{p^2 + m^2}$$

with

$$f(\Lambda, p) = \tilde{\theta}(\Lambda^2/p^2) - \tilde{\theta}(\Lambda^2/p^2).$$

$\tilde{\theta}$ denotes a smeared $\infty$-differentiable $\theta$-like function

$$\tilde{\theta}(\Lambda^2/p^2) = \begin{cases} 1, & p^2 < \Lambda^2/2 \\ \text{smooth}, & \Lambda^2/2 < p^2 < 2\Lambda^2 \\ 0, & 2\Lambda^2 < p^2. \end{cases}$$

Hence $P_{\Lambda}(p^2)$ is non-vanishing for $\Lambda^2/2 < p^2 < 2\Lambda^2$.

Let us define $S_{int}(\varphi, \Lambda)$ by

$$e^{-S_{int}(\varphi, \Lambda)} = e^{D_{\Lambda}} e^{-S_{int}(\varphi, \Lambda)}$$

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with
\[ D^\Lambda_0 = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} P^\Lambda_0(p^2) \frac{\delta}{\delta \varphi(p)} \frac{\delta}{\delta \varphi(-p)}. \] (2.5)

\[ S_{\text{int}}(\varphi, \Lambda_0) \text{ consists of all Feynman diagrams generated by the interactions present in } S_{\text{int}}(\varphi, \Lambda_0), \text{ which is to be identified with the bare action of the theory. The internal propagators, however, involve both an ultraviolet cutoff } \Lambda_0 \text{ and an infrared cutoff } \Lambda. \text{ It can be shown that } S_{\text{int}}(\varphi, \Lambda = 0) \text{ is related to the generating functional } G(J) \text{ of connected Green functions of the theory [13], [16]:} \]
\[ S_{\text{int}}(\varphi, \Lambda = 0) = G(J) - \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} J(p) P^\Lambda_0(p^2) J(-p). \] (2.6)

Furthermore \( S_{\text{int}}(\varphi, \Lambda) \) satisfies the flow equations [10]-[16]
\[ \partial_\Lambda S_{\text{int}}(\varphi, \Lambda) = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \partial_\Lambda P^\Lambda_0(p^2) \{ \frac{\delta^2 S_{\text{int}}(\varphi, \Lambda)}{\delta \varphi(p) \delta \varphi(-p)} - \frac{\delta S_{\text{int}}(\varphi, \Lambda)}{\delta \varphi(p)} \frac{\delta S_{\text{int}}(\varphi, \Lambda)}{\delta \varphi(-p)} \}. \] (2.7)

In principle the integration of the flow equations (2.7) with specified boundary conditions at \( \Lambda = \Lambda_0 \) allows the computation of \( S_{\text{int}}(\varphi, 0) \), which unites the information about all physical Green functions according to (2.6). Generally, however, \( S_{\text{int}}(\varphi, \Lambda) \) contains apart from the relevant and marginal operators all irrelevant operators allowed by the symmetries of the theory. After a decomposition of eq. (2.7) into this infinite set of operators one thus obtains an infinite set of flow equations for the associated couplings. In contrast to standard \( \beta \) functions, at least, the right-hand side of these flow equations is at most quadratic in the couplings. Nevertheless we will see that for the Higgs top system the flow equations can be solved exactly to leading order in \( 1/N_c \).

The field content of the Higgs top system is given by a four-component Dirac spinor \( \psi_a \) and a complex scalar \( \phi = \sigma + i\pi \) with \( \sigma \) and \( \pi \) real. \( a \) denotes the colour index, hence \( a = 1, \ldots, N \) with \( N = N_c = 3 \) for QCD. The right-handed components of \( \psi_a \) are identified with the SU(2)-singlet right-handed top quark with hypercharge 2/3, whereas the left-handed components of \( \psi_a \) denote the left-handed top quark, which transforms as a \( I_3 = 1/2 \) component of an (incomplete) SU(2) multiplet with isospin 1/2 and hypercharge 1/6. The complex scalar \( \phi \) has the quantum numbers of the composite operator \( \bar{\psi}_L \psi_R \), thus it transforms as the component of the Higgs doublet whose VEV breaks the weak SU(2) and U(1) hypercharge down to electromagnetism.

The set of operators to be taken into account in \( S_{\text{int}}(\phi, \psi, \Lambda) \) has to be general enough for our purposes, and to respect the above-mentioned symmetries (which can be expressed as linear combinations of the vector and axial symmetries of the NJL model [4]). Defining
\[ D_p^n = \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_n}{(2\pi)^4} \delta^4(\sum_{i=1}^n p_i)(2\pi)^4 \] (2.8)
the following decomposition of $S_{\text{int}}(\phi, \psi, \Lambda)$ turns out to be sufficient:
\[ S_{\text{int}}(\phi, \psi, \Lambda) = \sum_{n=1}^{\infty} \int D_{p}^{2n} F_{2n}(\Lambda; p_{1}, ..., p_{2n}) \phi^*(p_{1})\phi(p_{2})...\phi^*(p_{2n-1})\phi(p_{2n}) \]
\[ + \sum_{n=1}^{\infty} \int D_{p}^{2n+1} \phi^*(p_{1})\phi(p_{2})...\phi^*(p_{2n-3})\phi(p_{2n-2}) \]
\[ \cdot \bar{\psi}_{a}(p_{2n})G_{2n-1}(\Lambda; p_{1}, ..., p_{2n+1})(\sigma(p_{2n-1}) + i\gamma_{5}\pi(p_{2n-1}))\psi_{a}(p_{2n+1}) \]
\[ + \sum_{n=1}^{\infty} \int D_{p}^{2n+2} \phi^*(p_{1})\phi(p_{2})...\phi^*(p_{2n-1})\phi(p_{2n}) \]
\[ \cdot \bar{\psi}_{a}(p_{2n+1})H_{2n}(\Lambda; p_{1}, ..., p_{2n+2})(-i\dot{\phi}_{2n+2}\psi_{a}(p_{2n+2}) \quad (2.9) \]

(The index of the generalized couplings \( F, G \) and \( H \) corresponds to the number of scalars contained in the associated operator, and the couplings \( G \) and \( H \) may be matrices with spinor indices.) This ansatz has to be inserted into the flow equations (2.7), which generally involve contractions of both scalars with a propagator \( P_{\phi} \) and fermions with a propagator \( P_{\psi} \). Then both sides have to be ordered according to their field content, after which flow equations for the individual couplings \( F_{2n}, G_{2n-1} \) and \( H_{2n} \) are obtained. In the case of \( F_{2n} \) these equations have the following schematic form:

\[ \partial_{\Lambda}F_{2n} = N\partial_{\Lambda}P_{\psi}H_{2n} + \sum_{m} \partial_{\Lambda}P_{\phi}F_{2m}F_{2n-2m+2} + \partial_{\Lambda}P_{\phi}F_{2n+2}. \quad (2.10) \]

The factor \( N \) in front of the first term is due to \( N \) pairs of fermions, which are contained in the operator of the type \( H \) and which have to be contracted in order to turn it into an operator of the type \( F \). Let us, in order to study the \( N \to \infty \) limit, perform the standard rescaling of fields and couplings:

\[ \phi \to \sqrt{N}\phi, \quad F_{2n} \to N^{1-n}F_{2n}, \quad G_{2n-1} \to N^{1/2-n}G_{2n-1}, \]
\[ H_{2n} \to N^{-n}H_{2n}. \quad (2.11) \]

Now eq. (2.10) becomes

\[ \partial_{\Lambda}F_{2n} = \partial_{\Lambda}P_{\psi}H_{2n} + \sum_{m} \partial_{\Lambda}P_{\phi}F_{2m}F_{2n-2m+2} + N^{-1}\partial_{\Lambda}P_{\phi}F_{2n+2} \quad (2.12) \]

and we see that the last term can be dropped. \( N \)-power counting now also justifies the omission of operators involving more than two fermionic fields.

Next we have to remember that the couplings \( F_{2n} \) describe connected Green functions for \( 2n \) scalars including one-particle-reducible diagrams, cf. the general relation (2.9). Finally we will be interested, however, in the effective action or the generating functional of one-particle-irreducible diagrams. In principle we have to perform a Legendre transformation in order to get there; in terms of diagrams, however, we simply have to omit the one-particle-reducible ones.

Generally it is not easy to modify the flow equation such that they describe the flow of the couplings of the effective action. In the present case, however, this can
be achieved by simply omitting the second term on the right-hand side of equation (2.12). Since the last term in (2.12) has been dropped already, no self-contractions or tadpole-like diagrams involving the φ-fields are included any more, and it is true only under these circumstances that the omission of the second term corresponds only to the omission of one-φ-particle-reducible diagrams in the final expression \( S_{\text{int}}(\phi, \psi, \Lambda = 0) \) and not more.

Thus, after omitting the second term in (2.12), we have not made any approximation, but changed the physical interpretation of the couplings \( F_{2n} \), which now describe the coefficients of the effective action for the fields \( \phi \).

Generally a local effective action can be expanded in powers of derivatives or, in momentum space, in powers of \( p \). Subsequently we will be interested in the effective potential, which corresponds to the zeroth order in such an expansion, and the wave function normalization of the field \( \phi \), for which one needs the second order of the effective action in \( p \). From the structure of the flow equation one then finds that it is sufficient to know all the couplings \( F, G, H \) only up to second order in an expansion in the momenta associated with the fields \( \phi \), since no \( \phi \) contractions appear any more.

To start with, however, we will concentrate on the effective potential and thus deal with the couplings \( F, G, H \) at vanishing bosonic momenta only. Therefore we define

\[
F_{2n}(\Lambda; 0, \ldots, 0) \equiv F_{2n}^{(0)},
G_{2n-1}(\Lambda; 0, \ldots, 0, p_{2n} = p, p_{2n+1} = -p) \equiv G_{2n-1}^{(0)}(p),
H_{2n}(\Lambda; 0, \ldots, 0, p_{2n+1} = p, p_{2n+2} = -p) \equiv H_{2n}^{(0)}(p).
\]

(2.13)

In the case of the restricted couplings of the form (2.13) the ansatz (2.9) for \( S_{\text{int}} \) can most conveniently be expressed in terms of fields \( \hat{\phi} \) of the form

\[ \phi(p) = \delta^4(p)\hat{\phi} \]

(2.14)

and

\[
\hat{F}(\Lambda; \hat{\phi}) = \sum_{n=1}^{\infty} F_{2n}^{(0)}(|\hat{\phi}|^2)^n,
\hat{G}(\Lambda; \hat{\phi}, p) = \sum_{n=1}^{\infty} G_{2n-1}^{(0)}(p)(|\hat{\phi}|^2)^{n-1},
\hat{H}(\Lambda; \hat{\phi}, p) = \sum_{n=1}^{\infty} H_{2n}^{(0)}(p)(|\hat{\phi}|^2)^n.
\]

(2.15)

Now (2.9) becomes

\[
S_{\text{int}}(\phi, \psi, \Lambda) = \hat{F}(\Lambda; \hat{\phi})\delta^4(0)(2\pi)^4 + \int \frac{d^4p}{(2\pi)^4}
\{ \bar{\psi}_a(p)\hat{G}(\Lambda; \hat{\phi}, p)(\hat{\sigma} + i\gamma_5\hat{\pi})\psi_a(-p) + \bar{\psi}_a(p)\hat{H}(\Lambda; \hat{\phi}, p)(-i\hat{\phi})\psi_a(-p) \}.
\]

(2.16)
As usual $(2\pi)^4\delta^4(0)$ is to be interpreted as the space-time volume. Note that $\hat{F}(\Lambda; \hat{\phi})$ corresponds to nothing but the $\Lambda$-dependent effective potential.

The flow equations have the form of eq. (2.7) where, after $1/N$ counting and the previous discussion, only fermionic contractions occur. The regularized fermionic propagator has the form

$$P_{\psi,\Lambda}(p^2) = -\frac{if(\Lambda, p)}{\hat{p}}$$  \hspace{1cm} (2.17)

with $f(\Lambda, p)$ as in eq. (2.2). Then, after inserting the ansatz (2.9) with (2.13) into (2.7), performing the necessary traces over $\gamma$ matrices and ordering the resulting equation according to their field content, one finds the following flow equations for the couplings of (2.13):

$$\partial_\Lambda F^{(0)}_{2n} = -4 \int \frac{d^4p}{(2\pi)^4} \partial_\Lambda f(\Lambda, p) H^{(0)}_{2n}(p) ,$$

$$\partial_\Lambda G_{2n-1}^{(0)}(p) = -2 \sum_{m=1}^{n} \partial_\Lambda f(\Lambda, p) G_{2m-1}^{(0)}(p) H_{2n-2m}^{(0)}(p) ,$$

$$\partial_\Lambda H_{2n}^{(0)}(p) = - \sum_{m=0}^{n} \partial_\Lambda f(\Lambda, p) H_{2m}^{(0)}(p) H_{2n-2m}^{(0)}(p)$$

$$+ \sum_{m=1}^{n} \frac{\partial_\Lambda f(\Lambda, p)}{p^2} G_{2m-1}^{(0)}(p) G_{2n-2m+1}^{(0)}(p).$$  \hspace{1cm} (2.18)

This infinite set of coupled differential equations can actually be handled by use of the quantities defined in (2.15). In terms of $\hat{F}, \hat{G}$ and $\hat{H}$ and after a rearrangement of indices the flow equations (2.18) simply become

$$\partial_\Lambda \hat{F}(\Lambda; \hat{\phi}) = -4 \int \frac{d^4p}{(2\pi)^4} \partial_\Lambda f(\Lambda, p) \hat{H}(\Lambda; \hat{\phi}, p) ,$$

$$\partial_\Lambda \hat{G}(\Lambda; \hat{\phi}, p) = -2 \partial_\Lambda f(\Lambda, p) \hat{G}(\Lambda; \hat{\phi}, p) \hat{H}(\Lambda; \hat{\phi}, p) ,$$

$$\partial_\Lambda \hat{H}(\Lambda; \hat{\phi}, p) = -\partial_\Lambda f(\Lambda, p) \left[ \hat{H}^2(\Lambda; \hat{\phi}, p) - \hat{G}^2(\Lambda; \hat{\phi}, p) \frac{|\hat{\phi}|^2}{p^2} \right] .$$  \hspace{1cm} (2.19)

After an expansion of eqs. (2.19) in powers of $\hat{\phi}$ one recovers that they coincide with the infinite system (2.18). On the other hand, the solutions to eqs. (2.19) with arbitrary boundary conditions at $\Lambda = \Lambda_0$,

$$\hat{F}(\Lambda_0; \hat{\phi}) \equiv \hat{F}_0(\hat{\phi}) ,$$

$$\hat{G}(\Lambda_0; \hat{\phi}, p) \equiv \hat{G}_0(\hat{\phi}, p) ,$$

$$\hat{H}(\Lambda_0; \hat{\phi}, p) \equiv \hat{H}_0(\hat{\phi}, p) ,$$  \hspace{1cm} (2.20)

are easily found:
\[ \hat{F}(\Lambda; \hat{\phi}) = -2 \int \frac{d^4p}{(2\pi)^4} \ln \left[ 1 + 2f(\Lambda, p) \hat{H}_0 + f^2(\Lambda, p)(\hat{H}_0^2 + \hat{G}_0^2 |\hat{\phi}|^2 / p^2) \right] + \hat{F}_0, \]

\[ \hat{G}(\Lambda; \hat{\phi}, p) = \frac{\hat{G}_0}{(1 + f(\Lambda, p) \hat{H}_0)^2 + f^2(\Lambda, p) \hat{G}_0^2 |\hat{\phi}|^2 / p^2}, \]

\[ \hat{H}(\Lambda; \hat{\phi}, p) = \frac{\hat{H}_0 + f(\Lambda, p)(\hat{H}_0^2 + \hat{G}_0^2 |\hat{\phi}|^2 / p^2)}{(1 + f(\Lambda, p) \hat{H}_0)^2 + f^2(\Lambda, p) \hat{G}_0^2 |\hat{\phi}|^2 / p^2}. \]  

(2.21)

Let us recall the physical interpretation of the quantities \( \hat{F}(\Lambda; \hat{\phi}), \hat{G}(\Lambda; \hat{\phi}, p) \) and \( \hat{H}(\Lambda; \hat{\phi}, p) \). Whereas \( \hat{F} \) is related to the effective potential for \( \hat{\phi} \) (up to a factor \( N \) implied by the rescaling (2.11)), \( \hat{G} \) and \( \hat{H} \) appear in the \( \psi \)-dependent part of \( S_{int} \), see (2.16). In order to relate \( \hat{G} \) and \( \hat{H} \) to the full fermionic propagator \( S_F \) in the background \( \hat{\phi} \), we have to take into account that \( S_{int} \) is related to the generating functional of connected Green functions, cf. (2.6). In the present case, where \( S_{int} \) is just quadratic in the fermionic fields, the analog of (2.6) reads

\[ S_{int}(\psi = P_{\psi, 0} \eta, \Lambda = 0) = \int \frac{d^4p}{(2\pi)^4} \left\{ -\bar{\eta}S_F \eta + \bar{\eta}P_{\psi, 0} \eta \right\}. \]  

(2.22)

Here \( P_{\psi, 0} \) is the free cutoff propagator as in (2.17), and the full propagator \( S_F \) can now be determined by inverting (2.22) and using (2.16) with (2.21):

\[ S_F(\hat{\phi}, p) = \frac{f(0, p)}{i\hat{\phi}(1 + f(0, p) \hat{H}_0(\hat{\phi}, p)) + f(0, p) \hat{G}_0(\hat{\phi}, p)(\hat{\sigma} + \gamma_5 \hat{\pi})}. \]  

(2.23)

For dimensional reasons the boundary conditions \( \hat{F}_0, \hat{G}_0 \) and \( \hat{H}_0 \) of eq. (2.20) have to be of the following form:

\[ \hat{F}_0(\hat{\phi}) = \alpha \Lambda_0^4 + \frac{\mu}{2N} \Lambda_0^2 \hat{\phi}^2 + \frac{\lambda_0}{2N} \hat{\phi}^4 + O \left( \frac{\hat{\phi}^2}{\Lambda_0^2} \right), \]

\[ \hat{G}_0(\hat{\phi}, p) = g \left( \frac{p^2}{\Lambda_0^2} \right) + O \left( \frac{\hat{\phi}^2}{\Lambda_0^2} \right), \]

\[ \hat{H}_0(\hat{\phi}, p) = h \left( \frac{p^2}{\Lambda_0^2} \right) + O \left( \frac{\hat{\phi}^2}{\Lambda_0^2} \right) \]  

(2.24)

with arbitrary dimensionless parameters \( \alpha, \mu \) and \( \lambda_0 \) and arbitrary functions \( g \) and \( h \). Here we assumed that the bare action is local (i.e. \( \hat{G}_0 \) and \( \hat{H}_0 \) are finite for \( p^2 \to 0 \)) and involves only positive powers of the field \( \phi \).

Subsequently we will be interested in the situation where the scale of physics, represented by the Higgs VEV \( \hat{\phi} \), is much smaller than the scale \( \Lambda_0 \) of the bare
action. This allows us to neglect the last terms in \((2.24)\), and the effective potential given by \(N\hat{F}(\Lambda = 0; \hat{\phi})\) can be written as

\[
V_{\text{eff}} = \frac{\mu}{2} \Lambda_0^2 \hat{\phi}^2 + \frac{\lambda_0}{2} \hat{\phi}^4 - \frac{N}{8\pi^2} \int_0^2 x dx \ln \left[ 1 + \frac{|\hat{\phi}|^2}{x\Lambda_0^2} k(x) \right]
\]

(2.25)

with

\[
k(x) = \frac{f^2(\Lambda = 0, p^2 = x\Lambda_0^2)g^2(x)}{(1 + f(\Lambda = 0, p^2 = x\Lambda_0^2)h(x))^2}.
\]

(2.26)

We have dropped a \(\hat{\phi}\)-independent constant piece, and the upper limit 2 of the \(x\) integration is due to the fact that \(f(x)\) and hence \(k(x)\) vanish for \(x > 2\) according to eqs. (2.2) and (2.3).

Note that, before putting \(\Lambda = 0\) in (2.21), \(N\hat{F}(\Lambda; \hat{\phi})\) interpolates continuously between the bare potential \(N\hat{F}_0\) and the full effective potential (2.25). In the limit of a sharp cutoff, \(\theta \to \theta\) in (2.2), and for the particularly simple boundary conditions \(g(x) = \text{const.} = g\) and \(h(x) = 0\) in (2.24) the “flowing” potential \(V_{\text{eff}}(\Lambda, \hat{\phi})\) becomes

\[
V_{\text{eff}}(\Lambda, \hat{\phi}) = \frac{\mu}{2} \Lambda_0^2 \hat{\phi}^2 + \frac{\lambda_0}{2} \hat{\phi}^4 + \frac{N}{16\pi^2} \left\{ \frac{g^2 \hat{\phi}^2 (\Lambda^2 - \Lambda_0^2)}{2} + g^4 \hat{\phi}^4 \ln \left( \frac{\Lambda^2 + g^2 \hat{\phi}^2}{\Lambda^2 + g^2 \hat{\phi}^2} \right) + \Lambda^4 \ln \left( 1 + \frac{g^2 \hat{\phi}^2}{\Lambda^2} \right) - \Lambda_0^4 \ln \left( 1 + \frac{g^2 \hat{\phi}^2}{\Lambda_0^2} \right) \right\}.
\]

(2.27)

(A similar formula for different models can be found in [18].)

One finds that \(V_{\text{eff}}(\Lambda, \hat{\phi})\) develops a nontrivial minimum in \(\hat{\phi}\) as soon as the infrared scale \(\Lambda\) satisfies

\[
\Lambda^2 \leq \Lambda_0^2 \left( 1 - \frac{4\pi^2 \mu}{Ng^2} \right).
\]

(2.28)

In the case of general boundary conditions the formula (2.25) for \(V_{\text{eff}}\) can still be evaluated, neglecting powers of \(\hat{\phi}/\Lambda_0\). After some algebra one obtains

\[
V_{\text{eff}} = \frac{\mu}{2} \Lambda_0^2 \hat{\phi}^2 + \frac{\lambda_0}{2} \hat{\phi}^4 + \frac{N}{8\pi^2} \left[ b_0 + b_1 \ln \left( \frac{\Lambda_0^2}{\hat{\phi}^2 k(0)} \right) \right]
\]

(2.29)

with

\[
a = \int_0^2 dx \, k(x),
\]

\[
b_0 = \frac{\mu}{2} \Lambda_0^2 \hat{\phi}^2 + \frac{\lambda_0}{2} \hat{\phi}^4 + \frac{N}{8\pi^2} \left[ b_0 + b_1 \ln \left( \frac{\Lambda_0^2}{\hat{\phi}^2 k(0)} \right) \right]
\]

\[
b_1 = k^2(0).
\]

(2.30)

Note that \(V_{\text{eff}}\), according to eq. (2.29), has the same form as in the renormalization group-improved standard model to leading order in \(1/N_c\) (where \(g_t\) denotes the top quark Yukawa coupling):

\[
V_{\text{eff}}^{\text{SM}} = \frac{m^2}{2} \hat{\phi}^2 + \frac{\lambda}{2} \hat{\phi}^4 + \lambda(\mu) + \frac{N_c g_t^4}{8\pi^2} \ln \left( \frac{\mu^2}{\hat{\phi}^2} \right)
\]

(2.31)
This correspondence has been achieved for virtually arbitrary boundary conditions at \( \Lambda_0 \), we only used \( \hat{\phi}/\Lambda_0 \ll 1 \). The explicit relations between the parameters are

\[
m^2 = \Lambda_0^2 \left( \mu - \frac{Na}{4\pi^2} \right),
\]

\[
\lambda(\mu) = \lambda_0 + \frac{N}{8\pi^2} \left[ b_0 + b_1 \ln \left( \frac{\Lambda_0^2}{\mu^2k(0)} \right) \right],
\]

and, for the top quark Yukawa coupling \( g_t \),

\[
g_t^2 = k(0).
\]

Of course fine tuning between the parameters \( \mu \) and \( a \), which are specified by the boundary conditions at \( \Lambda_0 \), is required in order to achieve \( \hat{\phi} \ll \Lambda_0 \) at the minimum of \( V_{\text{eff}} \) of eq. (2.29). Whereas the fine tuning problem is automatically obvious within the present framework, there is no conceptual difficulty in assuming \( \hat{\phi} \ll \Lambda_0 \) and studying the consequences.

In this limit and for momenta \( p \) with \( p^2/\Lambda_0^2 \ll 1 \) the fermion propagator \( S_F \) of (2.23) simplifies as well; with \( f(0, p \ll \Lambda_0) = 1 \) and taking \( \hat{\pi} =< \pi >= 0 \) by definition one finds

\[
(1 + h(0))S_F(\hat{\phi}, p) = \frac{1}{i\hat{\phi} + \frac{g(0)\hat{\sigma}}{1+h(0)}}.
\]

The constant factor \((1 + h(0))\) on the left-hand side can be absorbed by a rescaling of the fields \( \psi_a \), and the right-hand side is the top quark propagator of the standard model with \( g_t = g(0)/(1 + h(0)) \) or \( g_t^2 = k(0) \) in agreement with eq. (2.34).

In order to relate the Higgs VEV \( \hat{\phi} = \hat{\sigma} \) to the gauge boson masses, and the second derivative of the effective potential to the mass of the physical Higgs fields \( \sigma \) (with the mass defined by the inverse propagator at zero momentum), the wave function normalization \( Z_{\text{eff}} \) of the real component \( \sigma \) of the complex field \( \phi \) has to be known as well. \( Z_{\text{eff}} \) appears in the effective action in the form

\[
\frac{1}{2} Z_{\text{eff}}(\hat{\phi}) \int d^4p \sigma(p)p^2\sigma(-p).
\]

This term can be obtained from the couplings \( F_{2n} \) of (2.9), once they are expanded to second order in the momenta \( p_i \) and to second order in the deviation of the field \( \sigma \) from its VEV \( \hat{\phi} \). From the structure of the flow equations, eqs. (2.18), it follows that the knowledge of \( F_{2n} \), expanded to second order in the momenta \( p_i \), requires also the knowledge of \( H_{2n} \) and \( G_{2n-1} \) expanded to second order in the momenta associated with the scalars \( \phi \). Unfortunately the number of possible Lorentz invariants which can be formed out of two “bosonic” momenta \( p_i \) and \( p_j \), a momentum \( p \) of a fermion (e.g. the one associated with the fields \( \psi_a \) in eqs. (2.3)) and \( \gamma \) matrices is quite large. Thus, instead of eq. (2.13), we now have the following
expansion (assuming the couplings to be symmetric in the bosonic momenta):

\[
F_{2n} = F^{(0)}_{2n} + p_i^2 F^{(1)}_{2n} + p_i \cdot p_j F^{(2)}_{2n} + O(p_i^4)
\]

\[
G_{2n-1} = G^{(0)}_{2n-1} + p_i \cdot p G^{(1)}_{2n-1} + p \hat{p} G^{(2)}_{2n-1} + p_i^2 G^{(3)}_{2n-1} + p_i \cdot p_i G^{(4)}_{2n-1} + (p_i \cdot p)^2 G^{(5)}_{2n-1} + p_i \cdot p_j G^{(6)}_{2n-1} + p_i \cdot p \hat{p} G^{(7)}_{2n-1} + (p_i \cdot p)(p_j \cdot p) G^{(8)}_{2n-1} + O(p_i^2)
\]

\[
H_{2n-1} = H^{(0)}_{2n-1} + p_i \cdot p H^{(1)}_{2n-1} + p \hat{p} H^{(2)}_{2n-1} + p_i^2 H^{(3)}_{2n-1} + p_i \cdot p_i H^{(4)}_{2n-1} + (p_i \cdot p)^2 H^{(5)}_{2n-1} + p_i \cdot p_j H^{(6)}_{2n-1} + p_i \cdot p \hat{p} H^{(7)}_{2n-1} + (p_i \cdot p)(p_j \cdot p) H^{(8)}_{2n-1} + O(p_i^2)
\]  

\[ (2.37) \]

The couplings \( F^{(i)} \), \( G^{(i)} \) and \( H^{(i)} \) do not depend on any “bosonic” momenta by definition, but \( G^{(i)} \) and \( H^{(i)} \) depend on the “fermionic” momentum \( p \). After some combinatorics one finds that the wave function normalization \( Z_{\text{eff}} \) of (2.36) is related to the couplings \( F^{(2)}_{2n} \) and \( F^{(2)}_{2n} \) as

\[
Z_{\text{eff}}(\hat{\phi}) = \sum_{n=1}^{\infty} (|\hat{\phi}|^2)^{n-1} \{ (2n-1) F^{(1)}_{2n} - F^{(2)}_{2n} \}. \quad (2.38)
\]

It is straightforward, though not very illuminating, to derive the system of flow equations for the couplings \( F^{(i)} \), \( G^{(i)} \) and \( H^{(i)} \) and to solve it along the line of eqs. (2.18), (2.15), and (2.19). More details will be presented in a forthcoming publication [49], here we just present the result for \( Z_{\text{eff}} \) for the following class of boundary conditions at \( \Lambda_0 \): Defining, in analogy to (2.18),

\[
\hat{F}^{(i)}(\Lambda; \hat{\phi}) = \sum_{n=1}^{\infty} F^{(i)}_{2n} (|\hat{\phi}|^2)^n \quad (2.39)
\]

and \( \hat{G}^{(i)} \) and \( \hat{H}^{(i)} \) correspondingly, we take at \( \Lambda = \Lambda_0 \):

\[
\hat{F}^{(i)}(\Lambda_0; \hat{\phi}) \equiv \hat{F}^{(i)}_0(\hat{\phi}), \\
\hat{G}^{(i)}(\Lambda_0; \hat{\phi}, p) \equiv \hat{G}^{(i)}_0(\hat{\phi}, p) = g \left( \frac{p^2}{\Lambda_0^2} \right) + O \left( \frac{\hat{\phi}^2}{\Lambda_0^2} \right), \\
\hat{G}^{(i)}(\Lambda_0; \hat{\phi}, p) \equiv \hat{G}^{(i)}_0(\hat{\phi}, p) = g_i \left( \frac{p^2}{\Lambda_0^2} \right) + O \left( \frac{\hat{\phi}^2}{\Lambda_0^2} \right) \quad \text{for} \quad i = 1, 3, 5, \\
\hat{G}^{(i)}(\Lambda_0; \hat{\phi}, p) \equiv \hat{G}^{(i)}_0(\hat{\phi}, p) = 0 \quad \text{for} \quad i \neq 0, 1, 3, 5, \\
\hat{H}^{(0)}(\Lambda_0; \hat{\phi}, p) \equiv \hat{H}^{(0)}_0(\hat{\phi}, p) = h \left( \frac{p^2}{\Lambda_0^2} \right) + O \left( \frac{\hat{\phi}^2}{\Lambda_0^2} \right), \\
\hat{H}^{(i)}(\Lambda_0; \hat{\phi}, p) \equiv \hat{H}^{(i)}_0(\hat{\phi}, p) = 0 \quad \text{for} \quad i \neq 0. \quad (2.40)
\]
With \( f \equiv f(0, p^2/\Lambda_0^2) \) and primes denoting \( \partial/\partial p^2 \) we obtain for \( Z_{\text{eff}} \) in the limit \( |\hat{\phi}| \ll \Lambda_0 \):

\[
Z_{\text{eff}}(|\hat{\phi}|) = \frac{N}{4\pi^2} \int_0^{2\Lambda_0^2} dp^2 \left\{ \frac{3p^4}{2((1+h)^2p^2+g^2|\phi|^2)^2} - \frac{p^6(1+h)^2}{((1+h)^2p^2+g^2|\phi|^2)^4} \right\} g^2 f^2 (1+h)^4 + 2gg g^2 f^2 (1+h)^2 + g^2 f^2 h' \frac{2(1+h)^2}{2(1+h)^3} + \frac{gg g^2 f^2}{(1+h)^2} \left( -gg g^2 f^2 + \frac{gg g^2 f^2}{4(1+h)^3} - \frac{gg g^2 f^2}{2(1+h)^3} + \frac{gg g^2 f^2}{(1+h)^3} \right) \left( -gg g^2 f^2 + \frac{gg g^2 f^2}{4(1+h)^3} - \frac{gg g^2 f^2}{2(1+h)^3} + \frac{gg g^2 f^2}{(1+h)^3} \right) \} + Z_{\text{eff}}(|\hat{\phi}|)_0.
\]

Here \( Z_{\text{eff}}(|\hat{\phi}|)_0 \) is determined by the \( \hat{F}_0^{(i)} \) of (2.40) in analogy to (2.38). The first term on the right hand side contains a logarithmically divergent part leading to

\[
Z_{\text{eff}}(|\hat{\phi}|) = \frac{N}{8\pi^2} k(0) \ln \left( \frac{\Lambda_0^2}{|\hat{\phi}|^2} \right) + \text{finite}
\]

with \( k(0) \) as in (2.26). Note that the finite part contains a term involving \( f'' \) which leads to a divergence in the limit where \( \hat{\theta} \) approaches the sharp \( \theta \) function. This has nothing to do with the use of the flow equations themselves, but with the use of cutoff propagators as (2.1) or (2.17). As discussed in the appendix of [16], the effect can already be seen by computing \( Z_{\text{eff}} \) with standard methods using cutoff propagators.

It is thus clear that the physical predictions of the model depend not only on the bare action in the form of the functions \( g_i \) and \( h \), but also on the form of the cutoff. This is quite plausible, though, since at least the present way of implementing the cutoff can be considered as a regularization through higher derivatives and thus the addition of higher-dimensional operators to the action, which has similar physical effects as the addition of higher dimensional operators in the form of the functions \( g_i \) and \( h \).

3 The Generalized NJL Models

In the NJL model the only field, to start with, is a Dirac spinor \( \psi_\alpha \). The action is invariant under vector and axial vector symmetries, and contains a pointlike four-Fermi interaction [4]. It has been proposed to apply the model to the Higgs-top sector of the standard model [3]: Then the Higgs scalar can be interpreted as a top-antitop bound state, and predictions for the Higgs and top quark masses arise.

According to [3]-[7], however, these predictions get lost if the model is generalized by the addition of higher derivative couplings (which have to be expected if the model is considered as an effective low energy theory of something else).
The class of generalized NJL models we will consider is defined by a bare action of the form

\[
\tilde{S}^0 = \int \frac{d^4p}{(2\pi)^4} (1 + \tilde{h}(p)) \bar{\psi}_a(p)(-i\not{p})\psi_a(p) \\
- \int \frac{d^4p}{(2\pi)^4} \frac{d^4p_1}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} N \tilde{g}(p, p_1) \tilde{g}(p_2) \frac{1}{2\Lambda_0^2} \left\{ \bar{\psi}_a(p) \psi_a(-p - p_1) \bar{\psi}_b(p) \psi_b(p - p_2) \right\}
\]  \tag{3.1}

where, on dimensional grounds, the functions \(\tilde{h}\) and \(\tilde{g}\) depend on \(p\) only via the dimensionless ratio \(p/\Lambda_0\). The functions \(\tilde{h}\) and \(\tilde{g}\) generate, once expanded in powers of the momenta, derivative couplings which are more general than the ones considered in [5], [6].

The action (3.1) can be rewritten by the introduction of the complex scalar field \(\phi = \sigma + i\pi\). The action

\[
\tilde{S}^0 = \int \frac{d^4p}{(2\pi)^4} (1 + \tilde{h}(p)) \bar{\psi}_a(p)(-i\not{p})\psi_a(p) \\
+ \int \frac{d^4p}{(2\pi)^4} \frac{\Lambda_0^2}{2N} \phi^*(p)\phi(-p) \\
+ \int \frac{d^4p}{(2\pi)^4} \frac{d^4p_1}{(2\pi)^4} \tilde{g}(p, p_1) \bar{\psi}_a(p)(\sigma(p_1) + i\pi(p_1)\gamma_5)\psi_a(-p - p_1) \tag{3.2}
\]

is easily seen to coincide with the one of eq. (3.1) once the real scalars \(\sigma\) and \(\pi\) are integrated out or replaced by their classical equations of motion.

On the other hand, the action (3.2) belongs to the class of actions treated in sect. 2, cf. eq. (2.9). Note that the action (3.2) is to be considered as a bare action, which has to be used as boundary condition for the running action \(S_{\text{int}}(\psi, \phi, \Lambda)\) at \(\Lambda = \Lambda_0\). The general boundary conditions for those couplings, which are relevant for the effective potential and the full fermionic propagator for \(\phi \ll \Lambda_0\), have previously been formulated in eqs. (2.24). They involved two parameters \(\mu\) and \(\lambda_0\) and two arbitrary functions \(g(p)\) and \(h(p)\).

In order to coincide with the bare action (3.2) of a generalized NJL model, these boundary conditions have to be specified as

\[
\mu = 1, \\
\lambda_0 = 0, \\
g(p) = \tilde{g}(p, 0), \\
h(p) = \tilde{h}(p). \tag{3.3}
\]

Since we have already computed the effective potential and the fermionic propagator of the low energy theory in the most general case, we can investigate whether the restrictions (3.3) for \(\mu\) and \(\lambda_0\) result in constraints on the three parameters \(m^2, \lambda(\mu)\)
and \( g_i \) in (2.31). From eqs. (2.32), (2.33), and (2.34) together with (2.30) and (2.26), however, it is easily seen that for general functions \( \tilde{g} \) and \( \tilde{h} \) no constraints emerge. Actually, even if the function \( \tilde{g} \) and \( \tilde{h} \) would be fixed (e.g. by \( \tilde{g} = \text{const.} = g \) and \( \tilde{h} = 0 \)) predictions arise only if the detailed form of the cutoff, i.e. the function \( f(\Lambda, p) \) or the \( \tilde{\theta} \) function in eq. (2.2), is fixed as well.

The same result emerges in the case of the wave function normalization \( Z_{\text{eff}} \) of eq. (2.39), which is required in order to relate the parameters of \( V_{\text{eff}} \) to physical quantities. The bare action (3.2) corresponds to the boundary condition

\[
Z_{\text{eff}}(\hat{\phi}, \Lambda = \Lambda_0) \equiv Z_{\text{eff}0} = 0.
\]  

Again this constraint is not sufficient to restrict \( Z_{\text{eff}} \) of eq. (2.39) unless the functions \( f, g \) and \( h \) are further specified.

Thus we have rederived the results of [5]-[7], though in an even more general framework and by a different method, that generalized NJL models are not restrictive enough to allow for predictions of the parameters of the effective low energy theory. On the other hand we can easily check that in the limit \( \ln(\Lambda_0^2/\hat{\phi}^2) \gg 1 \) the relation \( m_\sigma = 2m_t \) is obtained independently of the boundary conditions [20],[21].

4 Discussion and Outlook

The flow equations have allowed us to relate arbitrary bare actions for the Higgs top sector to effective low energy theories, which have as many free parameters as a general renormalizable model, as it should be [10]. We could investigate whether the constraints imposed on the bare action by requiring it to be of the generalized NJL-type (3.2) lead to relations among the low energy parameters. The negative answer was known before [3]-[7], just our ansatz (3.2) is even more general than the one of [3],[5]. Also we have verified the additional dependence of the low energy parameters on the form of the cutoff [7]. This affects the original version of the NJL model even if the couplings of the bare action are fixed.

The emphasis of the present paper is, however, on the method by which our results have been obtained. The flow equations make the intuitive picture of scale-dependent effective actions manifest. Here we have studied an example, where the \( 1/N \) expansion allows us to find non-perturbative solutions to these equations (which have been presented in detail, however, only for the couplings relevant for the effective potential). With these solutions we can study the convergence and irreversibility of the flow of actions towards the infrared; this phenomenon essentially keeps us from gaining knowledge about a possible underlying theory valid at some large scale \( \Lambda_0 \), unless we perform measurements with a precision of \( \sim O(Q^2/\Lambda_0^2) \) where \( Q^2 \) denotes the energy scale of the process.

The flow equations, furthermore, can possibly be used to study more complicated strongly interacting systems. Then the infinite set of coupled differential equations will necessarily have to be approximated, typically by restricting the set of operators
in \( S_{int} \). It will thus be of great help that a solvable system of these equations exists; this can then be used to test approximation schemes. To our knowledge the system present here is the first one satisfying these requirements.
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