A multiparameter family of irreducible representations of the quantum plane and of the quantum Weyl algebra

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Abstract

We construct a family of irreducible representations of the quantum plane and of the quantum Weyl algebra over an arbitrary field, assuming the deformation parameter is not a root of unity. We determine when two representations in this family are isomorphic, and when they are weight representations, in the sense of [1].

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1 Introduction

Assume throughout that \( F \) is a field of arbitrary characteristic, not necessarily algebraically closed, with group of units \( F^* \). Fix \( q \in F^* \) with \( q \neq 1 \). The quantum plane is the unital associative algebra

\[
F_q[x, y] = F\{x, y\}/(yx - qxy)
\]

with generators \( x \) and \( y \) subject to the relation \( yx = qxy \).

Consider the operators \( \tau_q \) and \( \partial_q \) defined on the polynomial algebra \( F[t] \) by

\[
\tau_q(p)(t) = p(qt), \quad \text{and} \quad \partial_q(p)(t) = \frac{p(qt) - p(t)}{qt - t}, \quad \text{for} \ p \in F[t].
\]

Then the assignment \( x \mapsto \tau_q \), \( y \mapsto \partial_q \) yields a (reducible) representation \( F_q[x, y] \rightarrow \text{End}_F(F[t]) \) of \( F_q[x, y] \), which is faithful if and only if \( q \) is not a root of unity. The operators \( \tau_q \) and \( \partial_q \) are central in the theory of linear \( q \)-difference equations and \( \partial_q \) is also known as the Jackson derivative, as it appears in [3]. See e.g. [6], [5, Chap. IV] and references therein for further details.

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The irreducible representations of the quantum plane $\mathbb{F}_q[x, y]$ have been classified in [1] using results from [2]. Following [1] we say that a representation of $\mathbb{F}_q[x, y]$ is a weight representation if it is semisimple as a representation of the polynomial subalgebra $\mathbb{F}[H]$ generated by the element $H = xy$. When $q$ is a root of unity all irreducible representations of $\mathbb{F}_q[x, y]$ are finite-dimensional weight representations, and these are well understood. For example, if $\mathbb{F}$ is algebraically closed and $q$ is a primitive $n$-th root of unity then the irreducible representations of $\mathbb{F}_q[x, y]$ are either 1 or $n$ dimensional. When $q$ is not a root of unity there are irreducible representations of $\mathbb{F}_q[x, y]$ that are not weight representations, and in particular are not finite dimensional. These turn out to be the $\mathbb{F}[H]$-torsionfree irreducible representations of $\mathbb{F}_q[x, y]$, as they remain irreducible (i.e. nonzero) upon localizing at the nonzero elements of $\mathbb{F}[H]$. In [1, Cor. 3.3] the torsionfree representations of $\mathbb{F}_q[x, y]$ are classified in terms of elements satisfying certain conditions, but no explicit construction of these representations is given.

We assume $q$ is not a root of unity, and we give an explicit construction of a 3-parameter family $V_{m,n}^f$ of infinite-dimensional representations of $\mathbb{F}_q[x, y]$ having the following properties (compare Propositions 2.4, 2.10, and 2.13):

- $m$ and $n$ are positive integers, and $f : \mathbb{Z} \to \mathbb{F}^*$ satisfies condition (2.1) below, which essentially encodes $n$ independent parameters from $\mathbb{F}^*$;
- $V_f^{m,n}$ is irreducible if and only if $\gcd(m, n) = 1$;
- if $(m, n) \neq (m', n')$ then $V_f^{m,n}$ and $V_{f'}^{m',n'}$ are not isomorphic;
- $V_f^{m,n}$ is a weight representation if and only if $m = n$;
- if $\mathbb{F}$ is algebraically closed and $V$ is an irreducible weight representation of $\mathbb{F}_q[x, y]$ that is infinite dimensional, then $V \simeq V_{f}^{1,1}$ for some $f : \mathbb{Z} \to \mathbb{F}^*$.

Thus, in some sense weight and non-weight representations of $\mathbb{F}_q[x, y]$ are rejoined in the family $V_{m,n}^f$.

The localization of $\mathbb{F}_q[x, y]$ at the multiplicative set generated by $x$ contains a copy of the $q$-Weyl algebra, which is the algebra

$$\mathbb{A}_1(q) = \mathbb{F}\{X,Y\}/(YX - qXY - 1)$$

(1.3)

with generators $X$ and $Y$ subject to the relation $YX - qXY = 1$ (see (2.15) for details about this embedding). This is used in Subsection 2.2 to regard the representations $V_f^{m,n}$ as infinite-dimensional irreducible representations of $\mathbb{A}_1(q)$. In contrast with the action of $\mathbb{F}_q[x, y]$ on $V_f^{m,n}$ when $m = n$, it turns out that $V_f^{m,n}$ is never a weight representation of $\mathbb{A}_1(q)$.

2 A family $V_f^{m,n}$ of infinite-dimensional irreducible representations of $\mathbb{F}_q[x, y]$ for $q$ not a root of unity

Assume $q \in \mathbb{F}^*$ is not a root of unity. We introduce a family $V_f^{m,n}$ of infinite-dimensional representations of $\mathbb{F}_q[x, y]$ which are not in general weight representations in the sense
of \([1]\), but which includes all irreducible infinite-dimensional weight representations of \(\mathbb{F}_q[x,y]\) if we further assume \(\mathbb{F}\) to be algebraically closed.

Fix positive integers \(m,n \in \mathbb{Z}_{>0}\) and a function \(f : \mathbb{Z} \rightarrow \mathbb{F}^*\) satisfying

\[
f(i + n) = qf(i), \quad \text{for all } i \in \mathbb{Z}.
\]

(2.1)

Such functions are in one-to-one correspondence with elements of \((\mathbb{F}^*)^n\). Let \(V_{f}^{m,n}\) denote the representation of \(\mathbb{F}_q[x,y]\) on the space \(\mathbb{F}[t^{\pm 1}]\) of Laurent polynomials in \(t\) given by

\[
x.t^i = t^{i+n}, \quad y.t^i = f(i)t^{i-m}, \quad \text{for all } i \in \mathbb{Z}.
\]

(2.2)

Condition (2.1) ensures that the expressions (2.2) do define an action of \(\mathbb{F}_q[x,y]\) on \(\mathbb{F}[t^{\pm 1}]\) as, for all \(i \in \mathbb{Z},\)

\[(yx - qxy).t^i = (f(i + n) - qf(i))t^{i+n-m} = 0.
\]

Example 2.3. Fix \(\mu \in \mathbb{F}^*\) and \((m,n) \in \mathbb{Z}_{>0}\). For \(i \in \mathbb{Z}\) let \(f(i) = \mu q^{|\frac{i}{d}|}\), where \(\lfloor \frac{i}{d}\rfloor\) denotes the largest integer not exceeding \(\frac{i}{d}\). Then \(f : \mathbb{Z} \rightarrow \mathbb{F}^*\) satisfies condition (2.1) and thus there is a representation \(V_{f}^{m,n}\) of \(\mathbb{F}_q[x,y]\) on \(\mathbb{F}[t^{\pm 1}]\) with action

\[
x.t^i = t^{i+n}, \quad y.t^i = \mu q^{|\frac{i}{n}|}t^{i-m}, \quad \text{for all } i \in \mathbb{Z}.
\]

We begin the study of the representations \(V_{f}^{m,n}\) by determining their structure in terms of the parameters \(m\) and \(n\).

**Proposition 2.4.** Let \((m,n) \in \mathbb{Z}_{>0}\) be arbitrary, with \(d = \gcd(m,n)\), and assume \(f : \mathbb{Z} \rightarrow \mathbb{F}^*\) satisfies (2.1). Then there is a direct sum decomposition

\[
V_{f}^{m,n} \simeq \bigoplus_{k=0}^{d-1} V_{f_k}^{m/d,n/d}
\]

(2.5)

into irreducible representations, where \(f_k(i) = f(k + id), \) for \(0 \leq k < d\) and \(i \in \mathbb{Z}\).

Moreover, suppose \((m',n') \in \mathbb{Z}_{>0}\), and \(f' : \mathbb{Z} \rightarrow \mathbb{F}^*\) satisfies (2.1) (with \(n\) replaced by \(n'\)). If \(V_{f}^{m,n} \simeq V_{f'}^{m',n'}\) then \(m = m'\) and \(n = n'\).

**Proof.** For \(0 \leq k < d\), the subspace \(t^k\mathbb{F}[t^\pm d]\) of \(V_{f}^{m,n}\) is readily seen to be invariant under the actions of \(x\) and \(y\), and we have \(V_{f}^{m,n} = \bigoplus_{k=0}^{d-1} t^k\mathbb{F}[t^\pm d]\). Thus, next we argue that the subrepresentation \(t^k\mathbb{F}[t^\pm d]\) is isomorphic to \(V_{f_k}^{m/d,n/d}\), where \(f_k(i) = f(k + id)\) for all \(i \in \mathbb{Z}\). First notice that \(f_k(i + n/d) = f(k + id + n) = qf(k + id + qf_k(i))\), so \(V_{f_k}^{m/d,n/d}\) is defined. Consider the map \(\phi : V_{f_k}^{m/d,n/d} \rightarrow t^k\mathbb{F}[t^\pm d]\) given by \(\phi(p)(t) = t^kp(t^d)\), for all \(p \in \mathbb{F}[t^\pm 1]\). In particular, \(\phi(t^i) = t^{k+id}\) for \(i \in \mathbb{Z}\). Still viewing \(t^k\mathbb{F}[t^\pm d]\) as a subrepresentation of \(V_{f}^{m,n}\), we have:

\[
\phi(x.t^i) = \phi(t^{i+n/d}) = t^{k+id+n} = x.t^{k+id} = x.\phi(t^i),
\]

\[
\phi(y.t^i) = \phi(f_k(i)t^{i-m/d}) = f(k+id)t^{k+id-m} = y.t^{k+id} = y.\phi(t^i).
\]
Since \( \phi \) is clearly bijective, the calculations above show that \( \phi \) is an isomorphism of representations, and \( V_f^{m,n} \cong \bigoplus_{d=1}^{k-1} V_{f_d}^{m/d,n/d} \). The fact that each summand \( V_f^{m/d,n/d} \) is irreducible follows from \( \gcd(m/d, n/d) = 1 \) and Proposition 2.8, which will be established independently.

Finally, assume \( V_f^{m,n} \cong V_{f'}^{m',n'} \) for positive integers \( m' \) and \( n' \), and \( f' : \mathbb{Z} \to \mathbb{F}^* \) satisfying \( f'(i + n') = qf'(i) \), for all \( i \in \mathbb{Z} \). Then, up to isomorphism, \( V_f^{m,n} \) and \( V_{f'}^{m',n'} \) have the same composition factors, and in particular the same composition length. This proves that \( d = \gcd(m, n) = \gcd(m', n') \) and that \( V_{f_0}^{m/d,n/d} \cong V_{f_k'}^{m'/d,n'/d} \) for some \( k \). By Proposition 2.10, which will also be established independently, we have \( m/d = m'/d \) and \( n/d = n'/d \), so \( m = m' \) and \( n = n' \).

In view of the previous result, we henceforth assume that the positive integers \( m \) and \( n \) are coprime. The following consequence of (2.1) will be helpful.

**Lemma 2.6.** Assume \( \gcd(m, n) = 1 \) and \( f : \mathbb{Z} \to \mathbb{F}^* \) satisfies (2.1). For \( k \in \mathbb{Z} \) define

\[
s_f(k) = \prod_{i=0}^{n-1} f(k - im).
\]

Then \( s_f(k) = s_f(0)q^k \).

**Proof.** For \( j \in \mathbb{Z} \) let \( 0 \leq j < n \) be the unique integer such that \( j \equiv j \mod n \). Then the formula \( f(j) = f(j)q^{j \mod n} \) can be verified by induction on \( |j| \mod n \). Thus,

\[
s_f(k) = \prod_{i=0}^{n-1} f(k - im) = \prod_{i=0}^{n-1} f ( k - im ) \prod_{i=0}^{n-1} q^{k-1 \frac{-im - k+im}{n}}.
\]

Since \( m \) and \( n \) are coprime, the set \( \{ k - im \mid 0 \leq i < n \} \) consists of all the integers from 0 to \( n - 1 \), and is thus independent of \( k \). Moreover,

\[
\sum_{i=0}^{n-1} \frac{k - im - k - im}{n} = \sum_{i=0}^{n-1} \frac{-im - k - im}{n} = \sum_{i=0}^{n-1} \frac{-im + (-im)}{n}.
\]

Hence,

\[
s_f(k) = q^k \prod_{i=0}^{n-1} f(-im) \prod_{i=0}^{n-1} q^{-im} = q^k s_f(0).
\]

**Proposition 2.8.** Assume \( \gcd(m, n) = 1 \) and \( f : \mathbb{Z} \to \mathbb{F}^* \) satisfies (2.1). Then the representation \( V_f^{m,n} \) defined by (2.2) is an irreducible representation of \( \mathbb{F}_q[x, y] \).
Proof. We begin with a computation: for \( k \in \mathbb{Z} \) we have, by Lemma \[2.3\]

\[ x^m y^n t^k = x^m \left( \prod_{i=0}^{n-1} f(k - im) \right) t^{k-nm} = s_f(k) t^k = s_f(0) q^k t^k. \] (2.9)

Hence, \( x^m y^n \cdot p(t) = s_f(0) p(qt) \) for all \( p \in \mathbb{F}[t^\pm 1] \).

Let \( W \subseteq V_{f,m,n} \) be a nonzero subrepresentation. If \( p(t) \in W \) then also \( p(qt) \in W \), by (2.9). As \( q \) is not a root of unity, the latter implies that \( t^\ell \in W \) for some \( \ell \in \mathbb{Z} \). The coprimeness of \( m \) and \( n \) shows the existence of integers \( a \) and \( b \) so that \( an - bm = 1 \). By replacing \( a \) and \( b \) with \( a + jm \) and \( b + jn \) for a sufficiently large integer \( j \), we can assume \( a, b \in \mathbb{Z}_{>0} \). Then \( x^a y^b t^k = \lambda_k t^{k+1} \) for some \( \lambda_k \in \mathbb{F}^* \), showing that \( t^k \in W \) for all \( k \geq \ell \). A similar argument shows that \( t^k \in W \) for all \( k \leq \ell \). Hence \( W = V_{f,m,n} \), establishing the irreducibility of \( V_{f,m,n} \).

Next we describe \( V_{f,m,n} \) in terms of a maximal left ideal of \( \mathbb{F}_q[x, y] \). Recall that for a representation \( V \) of \( \mathbb{F}_q[x, y] \) and an element \( v \in V \), the annihilator of \( v \) in \( \mathbb{F}_q[x, y] \) is \( \text{ann}_{\mathbb{F}_q[x, y]}(v) = \{ r \in \mathbb{F}_q[x, y] \mid r.v = 0 \} \), a left ideal of \( \mathbb{F}_q[x, y] \).

**Proposition 2.10.** Assume \( \gcd(m, n) = 1 \) and \( f : \mathbb{Z} \to \mathbb{F}^* \) satisfies (2.1).

(a) For \( 1 \in V_{f,m,n} \), \( \text{ann}_{\mathbb{F}_q[x, y]}(1) = \mathbb{F}_q[x, y] / (x^m y^n - s_f(0)) \) and

\[ V_{f,m,n} \simeq \mathbb{F}_q[x, y] / (x^m y^n - s_f(0)). \]

(b) For positive integers \( m', n' \), and \( f' : \mathbb{Z} \to \mathbb{F}^* \) satisfying (2.1) (with \( n \) replaced by \( n' \)), we have \( V_{f,m,n} \simeq V_{f',m',n'} \) if and only if \( m = m', n = n' \) and \( s_f(0) = q^k s_f(0) \) for some \( k \in \mathbb{Z} \).

**Proof.** (a) Let \( \theta = x^m y^n \). First we show that

\[ \text{ann}_{\mathbb{F}_q[x, y]}(1) = \mathbb{F}_q[x, y] (\mathbb{F}[\theta] \cap \text{ann}_{\mathbb{F}_q[x, y]}(1)). \] (2.11)

The inclusion \( \supseteq \) is clear, so suppose \( u \in \text{ann}_{\mathbb{F}_q[x, y]}(1) \). Write \( u = \sum_{i \geq 0} \mu_i x^{a_i} y^{b_i} = \sum_{k \in \mathbb{Z}} u_k \), where \( u_k = \sum_{n a_i - m b_i = k} \mu_i x^{a_i} y^{b_i} \). Since \( u_k.1 \) is in \( \mathbb{F} t^k \), it follows that \( u_k \in \text{ann}_{\mathbb{F}_q[x, y]}(1) \) for all \( k \in \mathbb{Z} \), and it suffices to prove \( u_k \in \mathbb{F}_q[x, y] (\mathbb{F}[\theta] \cap \text{ann}_{\mathbb{F}_q[x, y]}(1)) \).

If \( n a_i - m b_i = n a_j - m b_j \) then, as \( \gcd(m, n) = 1 \), we deduce that \( (a_i, b_i) = (a_j, b_j) + \xi(m, n) \) for some \( \xi \in \mathbb{Z} \). Thus, by the normality of \( x \) and \( y \), there are \( a, b \geq 0 \) with \( na - mb = k \) such that \( u_k = x^a y^b u_0 \), where \( u_0 = \sum_{j \geq 0} \nu_j x^{\xi j} y^{\xi j} \in \mathbb{F}[\theta] \). Notice that for any \( \ell \in \mathbb{Z} \), \( x^a y^b t^\ell \) is a nonzero scalar multiple of \( t^{\ell+k} \), so \( x^a y^b u_0 = u_k \in \text{ann}_{\mathbb{F}_q[x, y]}(1) \) implies that \( u_0 \in \text{ann}_{\mathbb{F}_q[x, y]}(1) \). Hence, \( u_k \in \mathbb{F}_q[x, y] (\mathbb{F}[\theta] \cap \text{ann}_{\mathbb{F}_q[x, y]}(1)) \) and (2.11) is established.

Now (2.3) implies that \( \theta - s_f(0) \in \mathbb{F}[\theta] \cap \text{ann}_{\mathbb{F}_q[x, y]}(1) \). Since \( \mathbb{F}[\theta] (\theta - s_f(0)) \) is a maximal ideal of \( \mathbb{F}[\theta] \) it follows that \( \mathbb{F}[\theta] \cap \text{ann}_{\mathbb{F}_q[x, y]}(1) = \mathbb{F}[\theta] (\theta - s_f(0)) \) and \( \text{ann}_{\mathbb{F}_q[x, y]}(1) = \mathbb{F}_q[x, y] (\theta - s_f(0)) \). This proves (a) as \( 1 \in V_{f,m,n} \) generates \( V_{f,m,n} \).
Moreover, for \( i \) not on the particular \( V \) of interest, we have
\[
\mathbb{V}_f^{m,n} \simeq \mathbb{F}_q[x,y]/(x^m y^n - q^k s_f(0)),
\]
for any \( k \in \mathbb{Z} \). This establishes the \( \text{if} \) part of (b). For the direct implication, suppose \( \mathbb{V}_f^{m,n} \simeq \mathbb{V}_{f'}^{m',n'} \). We have, for \( a,b \geq 0 \) and \( t^k \in \mathbb{V}_f^{m,n} \),
\[
x^a y^b, t^k = \prod_{i=0}^{b-1} f(k - im)
\]
and \( \prod_{i=1}^{b-1} f(k - im) \neq 0 \). This implies that \( x^a y^b \) is diagonalizable on \( \mathbb{V}_f^{m,n} \) if and only if \( na = mb \). As \( \gcd(m,n) = 1 \) this amounts to having \( (a,b) = \xi(m,n) \) for some \( \xi \geq 0 \).

Since \( \mathbb{V}_f^{m,n} \simeq \mathbb{V}_{f'}^{m',n'} \), then \( x^m y^n \) is diagonalizable on \( \mathbb{V}_f^{m,n} \) and similarly \( x^a y^b \) is diagonalizable on \( \mathbb{V}_{f'}^{m',n'} \). By the relation above we conclude that \( (m,n) = (m',n') \).

Moreover, the eigenvalues of \( x^m y^n \) on \( \mathbb{V}_f^{m,n} \) are of the form \( q^k s_f(0) \), whereas \( s_f(0) \) is an eigenvalue of \( x^a y^b = x^m y^n \) on \( \mathbb{V}_{f'}^{m',n'} \). Hence \( s_f(0) = q^k s_f(0) \) for some \( k \in \mathbb{Z} \), which concludes the proof. \( \square \)

**Remark 2.12.** By Proposition 2.10 above, for \( \gcd(m,n) = 1 \) and \( f : \mathbb{Z} \to \mathbb{F}^* \) satisfying (2.41), the isomorphism class of \( \mathbb{V}_f^{m,n} \) depends only on \( m, n \) and \( s_f(0) \in \mathbb{F}^* \).

Fix \( \lambda \in \mathbb{F}^* \). Since \( \gcd(m,n) = 1 \) there is a unique \( f_\lambda : \mathbb{Z} \to \mathbb{F}^* \) such that (2.1) holds and \( f_\lambda(km) = \lambda \) if \( k = 0 \) and \( f_\lambda(km) = 1 \) if \( -n \leq k \leq 1 \). Then \( s_f(0) = \lambda \), \( \mathbb{V}_f^{m,n} \simeq \mathbb{F}_q[x,y]/\mathbb{F}_q[x,y](x^m y^n - \lambda) \) and, for \( \lambda' \in \mathbb{F}^* \), \( \mathbb{V}_{f_\lambda}^{m,n} \simeq \mathbb{V}_{f_{\lambda'}}^{m,n} \) if and only if \( \lambda/\lambda' \in \langle q \rangle \), where \( \langle q \rangle \) is the subgroup of \( \mathbb{F}^* \) generated by \( q \).

If \( \mathbb{F} \) contains an \( n \)-th root of \( \lambda \), say \( \mu \), there is a more natural construction for the irreducible representation \( \mathbb{F}_q[x,y]/\mathbb{F}_q[x,y](x^m y^n - \lambda) \). Define \( f^\mu(i) = \mu q^{i/\lambda} \), as in Example 2.3. Then \( s_f(0) = q^k \mu = q^k \lambda \), for some \( k \in \mathbb{Z} \). It follows from Proposition 2.10 that \( \mathbb{V}_{f^\mu}^{m,n} \simeq \mathbb{F}_q[x,y]/\mathbb{F}_q[x,y](x^m y^n - \lambda) \) and \( \mathbb{V}_f^{m,n} \) depends only on \( m, n \) and \( \lambda \), and not on the particular \( n \)-th root of \( \lambda \) that was chosen.

### 2.1 Weight representations of the form \( \mathbb{V}_f^{m,n} \)

Let us now determine when \( \mathbb{V}_f^{m,n} \) is a weight representation in the sense of \( [1] \). Recall that this occurs when \( \mathbb{V}_f^{m,n} \) is semisimple as a representation over the polynomial subalgebra \( \mathbb{F}[H] \), where \( H = xy \). Assume first that \( m = n = 1 \) and fix \( \lambda \in \mathbb{F}^* \). The map \( f_\lambda \) defined in Remark 2.12 is given by \( f_\lambda(i) = \lambda q^i \) for all \( i \in \mathbb{Z} \), and the corresponding representation \( \mathbb{V}^{1,1}_{f_\lambda} \simeq \mathbb{F}_q[x,y]/\mathbb{F}_q[x,y](H - \lambda) \) is irreducible. Since \( H, t^i = xy, t^i = q^i t^i \) for all \( i \), the decomposition \( \mathbb{V}^{1,1}_{f_\lambda} \simeq \bigoplus_{i \in \mathbb{Z}} \mathbb{F} t^i \) shows that \( \mathbb{V}^{1,1}_{f_\lambda} \) is semisimple over \( \mathbb{F}[H] \).

Moreover, for \( \nu \in \mathbb{F}^* \), \( \mathbb{V}^{1,1}_{f_\lambda} \simeq \mathbb{V}^{1,1}_{f_\nu} \) if and only if \( \lambda/\nu \in \langle q \rangle \), the multiplicative subgroup of \( \mathbb{F}^* \) generated by \( q \), by Proposition 2.10. In case \( \mathbb{F} \) is algebraically closed, these are all the infinite-dimensional irreducible weight representations of \( \mathbb{F}_q[x,y] \), by [1] Cor. 3.2.
Combined with Proposition 2.10(b) the above yields the classification of irreducible weight representations in the family $V_f^{m,n}$.

**Proposition 2.13.** Assume $\gcd(m, n) = 1$ and $f : \mathbb{Z} \to \mathbb{F}^*$ satisfies (2.1). Then $V_f^{m,n}$ is a weight representation if and only if $m = n = 1$.

For completeness, we include a brief and direct proof of Proposition 2.13 not assuming that $\mathbb{F}$ is algebraically closed, a condition that was used implicitly at the end of the previous paragraph.

**Proof.** Assume first that $m = n = 1$. Then since $f$ satisfies (2.1) we have $f = f_\lambda$ for $\lambda = f(0)$ and the discussion above shows that $V_f^{m,n}$ is a weight representation of $\mathbb{F}_q[x, y]$. Conversely, suppose $V_f^{m,n}$ is a weight representation of $\mathbb{F}_q[x, y]$. Then clearly $\dim_{\mathbb{F}} \mathbb{F}[H].v < +\infty$ for any $v \in V_f^{m,n}$. Notice that, for all $i \in \mathbb{Z}$, $H.t^i = xy.t^i = f(i)t^{i+n-m}$. Thus, for $\ell \in \mathbb{Z}$, $H^\ell.t^i = \zeta t^{i+\ell(n-m)}$ for some $\zeta \in \mathbb{F}^*$. But then the condition $\dim_{\mathbb{F}} \mathbb{F}[H].1 < +\infty$ immediately implies $m = n$, and hence $m = n = 1$, as $\gcd(m, n) = 1$. \hfill \Box

**Remark 2.14.** Given arbitrary positive integers $m$ and $n$, and $f$ satisfying (2.1), the representation $V_f^{m,n}$ is a weight representation if and only if $m = n$. The direct implication follows from the proof of Proposition 2.13. For the converse implication, recall that $V_f^{m,n}$ is the direct sum of $m$ representations of the form $V_f^{1,1}$, for $0 \leq k < m$, by Proposition 2.4 so the claim follows as each of these is a weight representation.

### 2.2 Extension of the action to the q-Weyl algebra $A_1(q)$

We continue to assume $q \in \mathbb{F}^*$ is not a root of unity, and apply the results above to the q-Weyl algebra $A_1(q)$ given by (13). It is straightforward to show that $\{x^k \mid k \geq 0\}$ is a right and left Ore set consisting of regular elements of $\mathbb{F}_q[x, y]$, and we denote the corresponding localization by $\mathbb{F}_q[x^{\pm 1}, y]$. The calculation

$$(x^{-1}(y - 1))x - qx(x^{-1}(y - 1)) = x^{-1}yx - q(y - 1) - 1 = qy - q(y - 1) - 1 = q - 1$$

demonstrates that there is an algebra map

$$A_1(q) \to \mathbb{F}_q[x^{\pm 1}, y], \quad x \mapsto x, \quad y \mapsto \frac{1}{q-1}x^{-1}(y - 1). \quad (2.15)$$

To see that the map in (2.15) is injective we can argue as follows. The multiplicative subset $\{X^k \mid k \geq 0\}$ of $A_1(q)$ is a right and left Ore set of regular elements and we denote the corresponding localization by $\hat{A}_1(q)$. Then the map in (2.15) extends to a map $\hat{A}_1(q) \to \mathbb{F}_q[x^{\pm 1}, y]$, which has an inverse $\mathbb{F}_q[x^{\pm 1}, y] \to \hat{A}_1(q)$ with $x^{\pm 1} \mapsto X^{\pm 1}$ and $y \mapsto (q - 1)XY + 1$. It follows that (2.15) induces an isomorphism $\hat{A}_1(q) \simeq \mathbb{F}_q[x^{\pm 1}, y]$, and in particular (2.15) is injective. In view of the above we will identify $X$ with $x$, $Y$ with $\frac{1}{q-1}x^{-1}(y - 1)$ and $A_1(q)$ with the corresponding subalgebra of $\mathbb{F}_q[x^{\pm 1}, y]$. Since $y = (q - 1)XY + 1$, we have the embeddings

$$\mathbb{F}_q[x, y] \subseteq A_1(q) \subseteq \mathbb{F}_q[x^{\pm 1}, y] = \hat{A}_1(q). \quad (2.16)$$
Our aim in this subsection is to extend the action of $\mathbb{F}_q[x, y]$ on $V_{f}^{m,n}$ to an action of the $q$-Weyl algebra $A_1(q)$. Assume thus that $m, n$ are positive integers and $f : \mathbb{Z} \rightarrow \mathbb{F}^*$ satisfies (2.1). If $\rho_{f}^{m,n} : \mathbb{F}_q[x, y] \rightarrow \text{End}_{\mathbb{F}_q}(V_{f}^{m,n})$ is the representation of $\mathbb{F}_q[x, y]$ on $V_{f}^{m,n}$, we first observe that $\rho_{f}^{m,n}(x)$ is an invertible linear map on $V_{f}^{m,n}$, a fact which is clear from (2.2). Therefore $\rho_{f}^{m,n}$ extends to the localization $\mathbb{F}_q[x^{\pm 1}, y]$, and $V_{f}^{m,n}$ can be seen as a representation of $\mathbb{F}_q[x^{\pm 1}, y]$ with $x^{-1}.t^i = t^{i-n}$ for all $i \in \mathbb{Z}$. Now we get an action of $A_1(q)$ on $V_{f}^{m,n} = \mathbb{F}[t^{\pm 1}]$ by restricting $\rho_{f}^{m,n}$:

$$X.t^i = x.t^i = i^{i+n},$$

$$Y.t^i = \frac{1}{q - 1}x^{-1}(y - 1).t^i = \frac{1}{q - 1}(f(i)i^{i-m-n} - i^{i-n}), \quad \text{for all } i \in \mathbb{Z}. \quad (2.17)$$

In our next result we view $V_{f}^{m,n}$ as a representation of $A_1(q)$, as above.

**Proposition 2.18.** Assume $\gcd(m, n) = 1$ and $f : \mathbb{Z} \rightarrow \mathbb{F}^*$ satisfies (2.1). Then:

(a) $V_{f}^{m,n}$ defined by (2.17) is an irreducible representation of $A_1(q)$.

(b) For positive integers $m', n'$, and $f' : \mathbb{Z} \rightarrow \mathbb{F}^*$ satisfying (2.1), with $n$ replaced by $n'$, we have $V_{f}^{m,n} \cong V_{f'}^{m',n'}$ as representations of $A_1(q)$ if and only if $m = m'$, $n = n'$ and $s_{f'}(0) = q^k s_{f}(0)$ for some $k \in \mathbb{Z}$.

(c) $V_{f}^{m,n}$ is not semisimple as a representation over the polynomial subalgebra of $A_1(q)$ generated by $XY$; hence, $V_{f}^{m,n}$ is not a weight representation of $A_1(q)$ in the sense of [I].

**Proof.** Part (a) and the direct implication in (b) follow from the embedding (2.10), and from Propositions 2.8 and 2.10.

Suppose now $f' : \mathbb{Z} \rightarrow \mathbb{F}^*$ satisfies (2.1), and there is $k \in \mathbb{Z}$ so that $s_{f'}(0) = q^k s_{f}(0)$. Then by Proposition 2.10 there is an isomorphism $\phi : V_{f}^{m,n} \rightarrow V_{f'}^{m,n}$ as representations of $\mathbb{F}_q[x, y]$. For $v \in V_{f}^{m,n}$ we have $\phi(v) = \phi(xx^{-1}.v) = x.\phi(x^{-1}.v)$, thus $\phi(x^{-1}.v) = x^{-1}.\phi(v)$. Whence $\phi$ is an isomorphism of representations of $\mathbb{F}_q[x^{\pm 1}, y]$. The other implication in (b) now follows from (2.16).

Observe that $XY = \frac{1}{q - 1}(y - 1)$, so the polynomial subalgebra of $A_1(q)$ generated by $XY$ is just $\mathbb{F}[y]$. Given $0 \neq v \in V_{f}^{m,n}$, the formula $y.t^i = f(i)t^{i-m}$ for $i \in \mathbb{Z}$ implies $\dim_{\mathbb{F}} \mathbb{F}[y].v = +\infty$. Hence, $V_{f}^{m,n}$ is not semisimple over $\mathbb{F}[y] = \mathbb{F}[XY]$, and therefore it is not a weight representation of $A_1(q)$ in the sense of [I].

**Remark 2.19.** In [3] the authors introduce Whittaker representations for generalized Weyl algebras. For the cases covered in this note, a representation $V$ is a Whittaker representation for $\mathbb{F}_q[x, y]$ (respectively, for $A_1(q)$) if $V$ is generated by an element $v \in V$ which is an eigenvector for the action of $x \in \mathbb{F}_q[x, y]$ (respectively, for the action of $X \in A_1(q)$). Since $m, n \geq 1$, it is immediate that the operators $x, y \in \mathbb{F}_q[x, y]$ (respectively, $X, Y \in A_1(q)$) have no eigenvectors in $V_{f}^{m,n}$, so $V_{f}^{m,n}$ is not a Whittaker representation for the quantum plane (respectively, for the $q$-Weyl algebra).
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