Fast- and Slow-Scale Bifurcations in an Interrupted Circuit with Multiple Inputs

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Abstract

Fast- and slow-scale bifurcations have been observed in electrical power circuits with multiple inputs. This paper studies the bifurcation phenomena in an electric circuit with multiple inputs. First, the circuit model and the waveform behavior are shown. Then, discrete maps are defined. Finally, the bifurcation phenomena in the circuit are analyzed using the discrete maps.

1. Introduction

Interrupted dynamical systems have both discrete and continuous variables. Typical examples of interrupted dynamical systems are electric circuits with a switching element and mechanical systems with an impact phenomenon. Various nonlinear phenomena have been observed in interrupted dynamical systems by changing the circuit and the physical parameters [1].

In recent years, the bifurcation phenomena observed in fast- and slow-scale dynamics, which we refer to fast- and slow-scale bifurcations, have been attracting interest from researchers. Fast- and slow-scale bifurcations can occur in dc/ac inverters and dc/dc converters. Most of the papers analyzed fast- and slow-scale bifurcations separately [2]. However, few studies have focused on the relationship between fast- and slow-scale bifurcations [3]. Therefore, we proposed a simple interrupted electric circuit with two inputs, which are a sinusoidal wave and a clock pulse, to initiate such studies [4]. Moreover, we proposed another circuit model, which has a sawtooth wave and a clock pulse, to clarify the dynamical effect of input signals with different shapes from an experimental point of view [5]. However, the mathematical analysis of the circuit proposed in Ref. [5] has been insufficient.

In this paper, we investigate the bifurcation phenomena in the circuit studied in Ref. [5]. First, we show the circuit model, and then we explain the waveform behavior. Next, we define discrete maps of the fast- and slow-scale dynamics with an exact solution. Finally, we study the bifurcation phenomena using the discrete maps.

2. Circuit Model and Behavior

Figure 1 shows the circuit model [5]. The circuit has a sawtooth wave and a clock pulse. The sawtooth wave and clock pulse oscillate with periods $T_s'$ and $T_f'$, respectively, where $T_s' = NT_f'$ with $N = 30$. The circuit parameters are as follows:

$$V_U = 150\,[\text{mV}], V_L = -150\,[\text{mV}], T_s' = 2.64\,[\text{ms}],$$
$$R = 10\,[\text{k}\Omega], C = 0.33\,[\text{µF}], E = 3.0\,[\text{V}], T_f' = 79.2\,[\text{ms}]$$

(1)

The circuit equations are as follows:

\[ RC \frac{dv}{dt} = \begin{cases} -v + E, & \text{system a} \\ -v + e(t), & \text{system b} \end{cases} \]

(2)

We define the sawtooth wave $e(t)$ as follows:

\[ e(t) = V_L + \frac{1}{T_f} (V_U - V_L) \left( t \mod T_s \right) \]

(3)

In addition, we use the following dimensionless variables:

\[ \tau = \frac{t}{RC}, T_s = \frac{T_s'}{RC}, T_f = \frac{T_f'}{RC} \]

(4)

![Figure 1: Circuit model](image-url)
Let the initial value of the capacitor voltage be \( v_k \). Discrete Maps

switch is connected to system a, and the clock pulse is ignored if the switch changes to system a again. The clock pulse is applied and the switch changes to system a again. The clock pulse is ignored if the switch is connected to system a.

3. Discrete Maps

The waveform behavior during a time interval \( T \) can be classified into two cases by using a border \( D \) that is defined as follows:

\[
D = (V_t - E)e^{T} + E \quad (5)
\]

Let the initial value of the capacitor voltage be \( v_{k+1} \), which is sampling data of the waveform at \( \tau = kT_s + iT \), where \( i = 1, 2, \ldots, N \). Figure 3 shows the waveform behavior during the clock interval. If the initial value satisfies \( v_{k+1} > D \), as shown in Fig. 3(a), the circuit remains in system a for a time of \( \tau_{a+i} \). The following map is defined:

\[
M_{ai} : \quad v_{k+1} \mapsto V_t = (v_{k+1} - E)e^{-\tau_{a+i}} + E \quad (6)
\]

Then, system b is maintained until the next clock pulse is applied. The following map is defined:

\[
M_{bi} : \quad V_t \mapsto v_{k+i+1} = e^{-\tau_{a+i}} \left\{ V_t - V_L - \frac{1}{T_s}(V_U - V_L)(\tau_{a+i} - 1) + \frac{1}{T_s}(V_U - V_L)(T_{f} - 1) + V_L \right\} \quad (7)
\]

Therefore, the discrete map of the fast-scale dynamics \( M_{fi} \) is defined as follows:

\[
M_{fi} : \quad v_{k+i} \mapsto v_{k+i+1}
\]

\[
= M_{b0} \circ M_{fi} \quad (8)
\]

On the other hand, if the waveform satisfies \( v_{k+i} \leq D \), as shown in Fig. 3(b), the discrete map \( M_{SI} \) is defined as follows:

\[
M_{SI} : \quad v_{k+i} \mapsto v_{k+i+1}
\]

\[
= (v_{k+i} - E)e^{-\tau_{f}} + E \quad (9)
\]

The derivative of the discrete map \( DM_{gi} \) is derived as follows:

\[
DM_{gi} = \begin{cases} e^{-\tau_{f}}, & v_{k+i} \leq D \\ \frac{e^{-\tau_{f}}}{V_t - E} \left( V_t - V_L - \frac{1}{T_s}(V_U - V_L) \log \frac{v_{k+i} - E}{V_t - E} \right), & v_{k+i} > D \end{cases} \quad (10)
\]

On the other hand, the discrete map of the slow-scale dynamics is defined on the basis of Eqs. (7) and (9) as follows:

\[
M_{s} : \quad V_k \mapsto V_{k+1}
\]

\[
= M_{1N} \circ \cdots \circ M_{12} \circ M_{11} \quad (11)
\]

where \( V_k \) and \( V_{k+1} \) are the sampling data of the waveform at times of \( \tau = kT_s \) and \( \tau = kT_s + NT_f \) with \( N = 30 \), respectively. The derivative of Eq. (11) is derived as follows:

\[
DM_{S} = \prod_{i=1}^{N} DM_{gi} \quad (12)
\]
where $DM_{fi}$ is given by Eq. (10).

4. Bifurcation Phenomena

Figure 4 shows the one-parameter bifurcation diagram for the slow-scale dynamics obtained by changing the reference value from $V_r = 0.5$ to $V_r = 3.0$. In addition, Fig. 5 shows the corresponding waveforms. We observe a slow-scale bifurcation in the figure. For example, it is clear that the period-1 solution bifurcates to an other period-1 solution at around $V_r = 2.0$. In addition, the period-1 solution bifurcates to a nonperiodic solution at around $V_r = 2.5$. Hereafter, we focus on the fast-scale bifurcation that occurs at around $V_r = 2.0$ and $V_r = 2.5$.

Figure 6 shows the waveform behaviors of the fast-scale dynamics at around $V_r = 2.0$. From the figure, we observe that a period-doubling bifurcation occurred at around $V_r = 2.0$. To verify this mathematically, we analyze the stability of the period-1 solution using Eqs. (10) and (12). Table 1 shows the stability of the fast- and slow-scale dynamics at around $V_r = 2.0$. Here, $DM_{f}$ denotes the stability of the slow-scale dynamics. Note that SFSD, UFDS, SSSD, and USSD mean the stable and unstable fast- and slow-scale dynamics, respectively. Moreover, we show the number of unstable fast-scale dynamics within one cycle of the slow-scale dynamics. It is clear that all of the fast-scale dynamics included in one cycle of the slow-scale dynamics maintain stable oscillation until $V_r = 2.02$. However, at $V_r = 2.03$, part of the fast-scale dynamics becomes unstable. Here, it is interesting that the slow-scale dynamics maintains stable oscillation until $V_r = 2.07$. Therefore, we conclude that the instability of the fast-scale dynamics may not directly affect the stability of the slow-scale dynamics.

Finally, we examine the fast-scale bifurcation that occurs at around $V_r = 2.6$. Figure 7 shows the waveform behavior of the fast- and slow-scale dynamics. We changed the reference value from $V_r = 2.65$ to $V_r = 2.60$ at exactly $\tau = 48$. Although the waveform is stable until $\tau = 48$, it changes to unstable oscillation after varying the reference value. Here, we focus on the waveform behavior at around $\tau = 48$ as shown in Fig. 7(b).

From the figure, it is observed that a border-collision bifurcation occurs in the fast-scale dynamics because the switching pattern is changed drastically by varying the reference value (see the gray areas in Fig. 7(b)). Figure 8 shows the discrete maps corresponding to Fig. 7. Note that (a) and (b) show discrete maps of the fast-scale dynamics, whereas (c) and (d) shows those of the slow-scale dynamics. We can observe that part of the discrete map in the fast-scale dynamics collides with the border $D$ and then the slow-scale dynamics becomes unstable. Therefore, it is concluded that a border-collision bifurcation occurring in the fast-scale dynamics strongly affects the slow-scale dynamics.
Table 1: Stability of the fast- and slow-scale dynamics

| $V_r$  | $DM_r$ | Remarks     | Num. of UFSD |
|-------|--------|-------------|--------------|
| ...   | ...    | ...         | ...          |
| 2.010 | -0.05152 | SFSD, SSSD | 0            |
| 2.020 | -0.08233 | SFSD, SSSD | 0            |
| 2.030 | -0.13189 | UFSD, SSSD | 2            |
| 2.040 | -0.21181 | UFSD, SSSD | 5            |
| 2.070 | -0.89157 | UFSD, USSD | 15           |
| 2.080 | -1.44762 | UFSD, USSD | 17           |


![Figure 7](image)

Figure 7: Border-collision bifurcation: (a) Slow-scale dynamics, (b) Fast-scale dynamics

5. Conclusion

This paper analyzed the bifurcation phenomena in a circuit with two inputs. We clarified that a period-doubling bifurcation and border-collision bifurcation occur in the fast-scale dynamics. The period-doubling bifurcation did not strongly affect the stability of the slow-scale dynamics, whereas the border-collision bifurcation did. This paper analyzed a circuit having a sawtooth wave expressed as Eq. (3), where $T'_s = NT'_f$ with $N = 30$. We predict that the bifurcation parameter of the slow-scale dynamics will be changed when $N \neq 30$. A detailed analysis of the circuit with $N \neq 30$ will be given in the near future. Moreover, we have to analyze the relationship between the instabilities in the fast- and slow-scale dynamics over a wide parameter space. In addition, as a practical application, we will design circuit parameters that are suitable for driving power converter circuits while taking the characteristics of the bifurcations into consideration.

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