ASYMPTOTICS AND INEQUALITIES FOR PARTITIONS INTO SQUARES

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Abstract. In this paper we prove that the number of partitions into squares with an even number of parts is asymptotically equal to that of partitions into squares with an odd number of parts. We further show that, for \( n \) large enough, the two quantities are different and which of the two is bigger depends on the parity of \( n \). This answers a recent conjecture formulated by Bringmann and Mahlburg (2012).

1. Introduction

A partition of a positive integer \( n \) is a non-increasing sequence of positive integers (called its parts), usually written as a sum, which add up to \( n \). The number of partitions of \( n \) is denoted by \( p(n) \). For example, \( p(5) = 7 \) as the partitions of 5 are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1 and 1 + 1 + 1 + 1 + 1. By convention, \( p(0) = 1 \). This is the case of the so-called unrestricted partitions, but one can consider partitions with various other properties, such as partitions into odd parts, partitions into distinct parts, etc.

Studying congruence properties of various partition functions fascinated many people and we limit ourselves to mentioning the famous congruences of Ramanujan [9], who proved that if \( n \geq 0 \), then
\[
\begin{align*}
p(5n + 4) &\equiv 0 \pmod{5}, \\
p(7n + 5) &\equiv 0 \pmod{7}, \\
p(11n + 6) &\equiv 0 \pmod{11}.
\end{align*}
\]

In this paper we study partitions based on their number of parts being in certain congruence classes. For \( r \in \mathbb{N} \), let \( p_r(a, m, n) \) be the number of partitions of \( n \) into \( r \)-th powers with a number of parts that is congruent to \( a \) modulo \( m \). Glaisher [7] proved (with different notation) that
\[
p_1(0, 2, n) - p_1(1, 2, n) = (-1)^n p_{\text{odd}}(n),
\]
where \( p_{\text{odd}}(n) \) denotes the number of partitions of \( n \) into odd parts without repeated parts.

It is as such of interest to ask what happens for partitions into \( r \)-th powers with \( r \geq 2 \), and a natural point to start by investigating partitions into squares. Based on computer experiments, Bringmann and Mahlburg [6] observed an interesting pattern and conjectured the following.

Conjecture 1 (Bringmann–Mahlburg, 2012).

(i) As \( n \to \infty \), we have
\[
p_2(0, 2, n) \sim p_2(1, 2, n).
\]

(ii) We have
\[
\begin{align*}
p_2(0, 2, n) &> p_2(1, 2, n) \quad \text{if } n \equiv 0 \pmod{2}, \\
p_2(0, 2, n) &< p_2(1, 2, n) \quad \text{if } n \equiv 1 \pmod{2}.
\end{align*}
\]

We build up on the initial work done by Bringmann and Mahlburg [6] towards solving Conjecture 1, the goal of this paper being to prove that the inequalities stated in part (ii) hold true asymptotically. In turn, this will show that part (i) of Conjecture 1 holds true as well.

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More precisely, we prove the following.

**Theorem 1.**

(i) As $n \to \infty$, we have
\[ p_2(0, 2, n) \sim p_2(1, 2, n). \]

(ii) Furthermore, for $n$ sufficiently large, we have
\[
\begin{cases}
    p_2(0, 2, n) > p_2(1, 2, n) & \text{if } n \equiv 0 \pmod{2}, \\
    p_2(0, 2, n) < p_2(1, 2, n) & \text{if } n \equiv 1 \pmod{2}.
\end{cases}
\]

In other words, we prove that the number of partitions into squares with an even number of parts is asymptotically equal to that of partitions into squares with an odd number of parts. However, for $n$ large enough, the two quantities are always different, which of the two is bigger depending on the parity of $n$.

Given that asymptotics for partitions into $r$-th powers (in particular, for partitions into squares) are known due to Wright [11], we can make the asymptotic value in part (i) of Theorem 1 precise. We will come back to this after we give the proof of Theorem 1.

The paper is organized as follows. In Sections 2 and 3 we do some preliminary work needed for the proof of Theorem 1, which we give in Section 4.

### 2. Notation and preliminaries

Before going into details, we recall some notation and well-known facts that will be used throughout. By $\Gamma(s)$, $\zeta(s)$ and $\zeta(s, k)$ we denote the usual Gamma, Riemann zeta and Hurwitz zeta functions. For reasons of space, we will sometimes use $\exp(z)$ for $e^z$. Whenever we take logarithms of complex numbers, we use the principal branch and denote it by $\text{Log}$.

By $\zeta_m = e^{2\pi i/m}$ we denote the standard primitive $m$-th root of unity.

If by $p_r(n)$ we denote the number of partitions of $n$ into $r$-th powers, then it is well-known (see, for example, Andrews [2, Ch. 1]) that
\[
\prod_{n=1}^{\infty} (1 - q^n)^{-1} = 1 + \sum_{n=1}^{\infty} p_r(n)q^n
\]
where, as usual, $q = e^{2\pi i\tau}$ and $\tau \in \mathbb{H}$ (the upper half-plane).

#### 2.1. A key identity.

Let
\[
\tilde{H}_r(w; q) = \sum_{m,n \geq 0} p_r(m, n)w^m q^n,
\]
where $p_r(0, n)$ denotes the number of partitions of $n$ into $r$-th powers with exactly $m$ parts and let
\[
H_{r,m,a}(q) = \sum_{n} p_r(a, m, n)q^n,
\]
where $p_r(a, m, n)$ denotes, as defined in the Introduction, the number of partitions of $n$ into $r$-th powers with a number of parts that is congruent to $a$ modulo $m$.

We obtain, by using the orthogonality of roots of unity, that
\[
H_{r,m,a}(q) = \frac{1}{m} H_r(q) + \frac{1}{m} \sum_{j=1}^{m-1} \zeta_m^{-aj} \tilde{H}_r(\zeta_m^j; q).
\]
2.2. The case $r = 2$. For the rest of the paper we will only deal with the case $r = 2$, which corresponds to partitions into squares. To prove Theorem 1 part (ii), it is enough to show that the series

$$H_{2,2,0}(-q) - H_{2,2,1}(-q) = \sum_{n=0}^{\infty} a_2(n)q^n$$

has positive coefficients for sufficiently large $n$, since

$$a_2(n) = \begin{cases} p_2(0, 2, n) - p_2(1, 2, n) & \text{if } n \equiv 0 \pmod{2}, \\ p_2(1, 2, n) - p_2(0, 2, n) & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$ (2)

Using, in turn, (1) and eq. (2.1.1) from Andrews [2, p. 16], we obtain

$$H_{2,2,0}(q) - H_{2,2,1}(q) = \tilde{H}_2(-1; q) = \prod_{n=1}^{\infty} \frac{1}{1 + q^{n^2}}.$$ Changing $q \mapsto -q$ gives

$$\tilde{H}_2(-1; -q) = \prod_{n=1}^{\infty} \frac{1}{1 + (-q)^{n^2}} = \prod_{n=1}^{\infty} \frac{1}{(1 + q^{4n^2})(1 - q^{4n^2})} = \prod_{n=1}^{\infty} \frac{(1 - q^{4n^2})^2}{(1 - q^{8n^2})(1 - q^{8n^2})}.$$ Therefore, by setting

$$G(q) = H_{2,2,0}(-q) - H_{2,2,1}(-q),$$

we obtain

$$G(q) = \prod_{n=1}^{\infty} \frac{(1 - q^{4n^2})^2}{(1 - q^{8n^2})(1 - q^{8n^2})} = \sum_{n=0}^{\infty} a_2(n)q^n$$

and we want to prove that the coefficients $a_2(n)$ are positive as $n \to \infty$. We will come back to this in the next section.

3. PREPARATIONS FOR THE PROOF

3.1. Meinardus’ asymptotics. Our approach is to some extent similar to that taken by Meinardus [8] in proving his famous theorem on asymptotics of certain infinite product generating functions and described by Andrews in [2, Ch. 6]. Our case is however slightly different and, whilst we can follow some of his steps, we cannot apply his result directly and need to make certain modifications. One of them pertains to an application of the circle method.

Under certain conditions on which we do not insist for the moment, as we shall formulate similar assumptions in the course of our proof, Meinardus gives an asymptotic formula for the coefficients $r(n)$ of the infinite product

$$f(\tau) = \prod_{n=1}^{\infty} (1 - q^n)^{-a_n} = 1 + \sum_{r=1}^{\infty} r(n)q^n, \quad (3)$$

where $a_n > 0$ and $q = e^{-\tau}$ with $\text{Re}(\tau) > 0$.

**Theorem 2** (Meinardus [8], cf. Andrews [2, Ch. 6]). As $n \to \infty$,

$$r(n) = C n^\kappa \exp \left( \frac{\alpha}{n+\tau} \left( 1 + \frac{1}{\alpha} \left( A\Gamma(\alpha + 1)\zeta(\alpha + 1) \frac{\tau^\alpha}{\alpha+1} \right) \right) (1 + O(n^{-\kappa_1})) \right),$$

where

$$C = e^{D'(0)} (2\pi(1 + \alpha))^{-\frac{1}{2}} (A\Gamma(\alpha + 1)\zeta(\alpha + 1))^{\frac{1-2\kappa+\alpha}{2\pi\alpha}},$$

$$\kappa = \frac{D(0) - 1 - \frac{1}{2}\alpha}{1 + \alpha},$$

$$\kappa_1 = \frac{\alpha}{\alpha + 1} \min \left\{ \frac{C_0}{\alpha} - \delta \frac{1}{4}, \delta \right\},$$

with $\delta$ an arbitrary real number.
Here, the Dirichlet series
\[ D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad (s = \sigma + it) \]
is assumed to converge for \( \sigma > \alpha > 0 \) and to possess an analytic continuation in the region \( \sigma > -c_0 \) (\( 0 < c_0 < 1 \)). In this region, \( D(s) \) is further assumed to be analytic except for a simple pole at \( s = \alpha \) with residue \( A \).

3.2. Partitions into squares. We now turn our attention to our problem. Let \( \tau = y - 2\pi ix \) and \( q = e^{-\tau} \), with \( y > 0 \) (so that \( \text{Re} \tau > 0 \) and \( |q| < 1 \)). Recall that, as defined in Section 2,
\[ G(q) = \prod_{n=1}^{\infty} \frac{(1 - q^{4n^2})^2}{(1 - q^{n^2})(1 - q^{8n^2})}. \] (4)
As one can easily see, unlike the product in (3), where all factors appear to negative powers, the factors \( (1 - q^{4n^2}) \) have positive exponents in the product from the right-hand side of (4). Therefore, we cannot directly apply Theorem 2 and obtain asymptotics for the coefficients \( a_2(n) \). We will, nevertheless, follow certain steps in the proof of Meinardus [8] as presented by Andrews [2, Ch. 6].

Let \( s = \sigma + it \) and
\[ D(s) = \sum_{n=1}^{\infty} \frac{1}{n^{2s}} + \sum_{n=1}^{\infty} \frac{1}{(8n^2)^s} - 2 \sum_{n=1}^{\infty} \frac{1}{(4n^2)^s} = (1 + 8^{-s} - 2^{1-2s})\zeta(2s), \]
which is convergent for \( \sigma > \frac{1}{2} = \alpha \), has an analytic continuation to \( \mathbb{C} \) (thus we may choose \( 0 < c_0 < 1 \) arbitrarily) and a simple pole at \( s = \frac{1}{2} \) with residue \( A = \frac{1}{4\sqrt{2}} \). From classical properties of the \( \zeta \)-function (see, for example, Titchmarsh [10, Ch. 5]) we know that, for some \( c_1 > 0 \),
\[ D(s) = O(|t|^{c_1}) \quad \text{as} \quad |t| \to \infty. \]

We have
\[ D(0) = 0, \]
\[ D'(0) = \zeta(0)(-3 \log 2 + 4 \log 2) = -\frac{\log 2}{2}. \]
By Cauchy’s Theorem we have, for \( n > 0 \),
\[ a_2(n) = e^{ny} \int_{-y}^{y} G(e^{-y+2\pi ix})e^{-2\pi i nx} dx. \]
We choose
\[ y = n^{-\frac{2}{3}} \left( \frac{\sqrt{\pi}}{8\sqrt{2}} \zeta \left( \frac{3}{2} \right) \right)^{\frac{2}{3}} > 0 \] (5)
and set
\[ m = n^{\frac{1}{4}} \left( \frac{\sqrt{\pi}}{8\sqrt{2}} \zeta \left( \frac{3}{2} \right) \right)^{\frac{1}{4}}, \]
so that \( ny = m \). (The reason for this choice of \( y \) will become apparent later and is originally motivated by the saddle-point method employed by Meinardus [8] in his proof). Moreover, let
\[ \beta = 1 + \frac{\alpha}{2} \left( 1 - \frac{\delta}{2} \right), \quad \text{with} \quad 0 < \delta < \frac{2}{3}, \]
so that
\[ \frac{7}{6} < \beta < \frac{5}{4}. \] (6)
We then obtain
\[ a_2(n) = e^{m} \int_{-y^\beta}^{y^\beta} G(e^{-y^{2\pi i x}})e^{-2\pi i x} dx + R(n), \] (7)
where
\[ R = e^m \int_{y^2 \leq |x| \leq \frac{1}{2}} G(e^{-y+2\pi i x}) e^{-2\pi i n x} dx. \]

We first prove the following estimate.

**Lemma 1.** If \(|\text{Arg } \tau| \leq \frac{\pi}{4}\), then
\[ G(e^{-\tau}) = \frac{1}{\sqrt{2}} \exp \left( \frac{\sqrt{\pi} \zeta \left( \frac{3}{2} \right)}{4\sqrt{2}\sqrt{\tau}} + O(y^{c_0}) \right) \]
holds uniformly in \(x\) as \(y \to 0\).

**Proof.** We have
\[ \log G(e^{-\tau}) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left( e^{-k n^2 \tau} + e^{-8 k n^2 \tau} - 2 e^{-4 k n^2 \tau} \right). \]

Using the Mellin inversion formula, for \(\text{Re } \tau > 0\) and \(\sigma_0 > 0\), we get
\[ e^{-\tau} = \frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \tau^{-\sigma} \Gamma(s) ds \]
and thus,
\[ \log G(e^{-\tau}) = \frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(s) \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left( (k n^2 \tau)^{-s} + (8 k n^2 \tau)^{-s} - 2(4 k n^2 \tau)^{-s} \right) ds \]
\[ = \frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Gamma(s) D(s) \zeta(s+1) \tau^{-s} ds. \]

Now, by assumption,
\[ |\tau^{-s}| = |\tau|^{-\sigma} e^{t \cdot \text{Arg } \tau} \leq |\tau|^{-\sigma} e^{\frac{\pi}{4} |t|}. \]

Classical results (see, e.g., [3] Ch. 1) tell us that the bounds
\[ D(s) = O(|t|^{c_1}), \]
\[ \zeta(s+1) = O(|t|^{c_2}), \]
\[ \Gamma(s) = O \left( e^{-\frac{\pi |t|}{2} |t|^{c_3}} \right) \]
hold uniformly in \(-c_0 \leq \sigma \leq \frac{\pi}{2}\) for \(|t| \to \infty\).

Thus we may shift the path of integration to \(\sigma = -c_0\). At \(s = \frac{1}{2}\) we have a simple pole and at \(s = 0\) a double pole. We compute the residues:
\[ \text{Res}_{s=\frac{1}{2}} \left( \Gamma(s) D(s) \zeta(1+s) \tau^{-s} \right) = \Gamma \left( \frac{1}{2} \right) A \zeta \left( \frac{3}{2} \right) \tau^{-\frac{1}{2}}, \]
\[ \text{Res}_{s=0} \left( \left( \frac{1}{s} + O(1) \right) \left( D'(0) s + O(s^2) \right) \left( \frac{1}{s} + O(1) \right) (1 + O(s)) \right) = D'(0) = -\frac{\log 2}{2}. \]

The remaining integral equals
\[ \frac{1}{2\pi i} \int_{-c_0-i\infty}^{-c_0+i\infty} \tau^{-s} \Gamma(s) \zeta(s+1) D(s) ds \ll |\tau|^{c_0} \int_0^{\infty} t^{c_1+c_2+c_3} e^{-\frac{\pi t}{4}} dt \ll |\tau|^{c_0} = |y - 2\pi i x|^{c_0} \leq (\sqrt{2} y)^{c_0} \]
since, again by the assumption,
\[ \frac{2\pi x}{y} = \tan(\text{Arg } \tau) \leq \tan \left( \frac{\pi}{4} \right) = 1. \]

We therefore obtain
\[ \log G(e^{-\tau}) = \left( \frac{\zeta \left( \frac{3}{2} \right) \sqrt{\pi}}{4\sqrt{2}\sqrt{\tau}} - \frac{\log 2}{2} \right) + O(y^{c_0}), \]
which completes the proof. ∎
3.3. **Wright’s modular transformation.** Like Wright [11], we want to use modular transformations. For this, consider

$$H(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-1} = \sum_{n=0}^{\infty} p_2(n)q^n.$$ 

In what follows, we choose the principal branch of the square root. The starting point is the modular transformation law obtained by Wright [11, Theorem 4], which in our case rewrites as

$$H(q) = H(e^{\frac{2\pi ia}{b}y}) = C_{a,b} \sqrt{y} \exp \left( \frac{\Lambda_{a,b}}{\sqrt{y}} \right) P_{a,b}(y),$$

where

$$\Lambda_{a,b} = \frac{\Gamma \left( \frac{3}{2} \right)}{b} \sum_{m=1}^{\infty} S_{a,m,b} \frac{1}{m^{\frac{3}{2}}},$$

$$S_{a,b} = \sum_{n=1}^{b} \exp \left( \frac{2\pi i n^2}{b} \right)$$

and

$$C_{a,b} = \frac{b_1}{2\pi},$$

with $b_1$ the least positive integer such that $b | b_1^2$ and $b = b_1 b_2$. Here, $0 \leq a < b$ are non-negative integers such that $(a,b) = 1$.

Furthermore, let

$$P_{a,b}(y) = \prod_{h=1}^{b} \prod_{s=1}^{2} \prod_{\ell=0}^{\infty} (1 - g(h, \ell, s))^{-1},$$

with

$$g(h, \ell, s) = \exp \left( \frac{(2\pi)^{\frac{3}{2}} \left( \ell + \mu_{h,s} \right) \frac{1}{b} e^{\frac{2\pi (2s+1)}{b \sqrt{y}}} - \frac{2\pi i h}{b} \right),$$

where $0 \leq d_h < b$ is defined by the congruence

$$ah^2 \equiv d_h \pmod{b}$$

and, for $d_h \neq 0$,

$$\mu_{h,s} = \begin{cases} \frac{d_h}{b} & \text{if } s = 1, \\ \frac{b-d_h}{b} & \text{if } s = 2. \end{cases}$$

If $d_h = 0$, we take $\mu_{h,s} = 1$.

3.4. **Circle method.** The proof of the upcoming Lemma 2 is similar in spirit with that of the second part of Lemma 6.1 from Andrews [2, Ch. 6]. However, our case is more subtle, in that it involves the factors $P_{a,b}$ and requires certain modifications. For this, we need a setup in which to apply the circle method as described by Wright [11, p. 172].

We consider the Farey dissection of order $\left\lfloor y^{-\frac{2}{3}} \right\rfloor$ and distinguish two kinds of arcs:

(i) **Major arcs**, $M$ or $M_{a,b}$, such that $b \leq y^{-\frac{1}{3}}$.

(ii) **Minor arcs**, $m$ or $m_{a,b}$, such that $y^{-\frac{2}{3}} < b \leq y^{-\frac{2}{5}}$.

We write any $\tau \in M \cup m$ as

$$\tau = y - 2\pi i x = \tau' - 2\pi i \frac{a}{b}$$

with $\tau' = y - 2\pi i x'$ and

$$|x'| \leq \frac{y^{\frac{2}{5}}}{b}.$$
Lemma 2. There exists \( \varepsilon > 0 \) such that

\[
G(e^{-\tau}) = O \left( \frac{\lambda_{a,b}}{e^{2\sqrt{2}y} - cy^{-\varepsilon}} \right)
\]

holds uniformly in \( x \) with \( y^3 \leq |x| \leq \frac{1}{2} \), as \( y \to 0 \), for some \( c > 0 \).

Recall that \( q = e^{-\tau} \), with \( y > 0 \) (so that \( \text{Re} \, \tau > 0 \) and \( |q| < 1 \)). From (11), (8) and (11) we have, for some positive constant \( C \) that can be made explicit,

\[
G(q) = \frac{H(q)H(q^8)}{H(q^3)^2} = C \exp \left( \frac{\lambda_{a,b}}{\sqrt{\tau}} \right) \frac{P_{a,b}(y)P'_{a,b}(8y)}{P''_{a,b}(4y)^2},
\]

where

\[
P'_{a,b} = P_{\frac{8a}{(8,8)}, \frac{b}{(8,8)}}, \quad P''_{a,b} = P_{\frac{8a}{(8,8)}, \frac{b}{(8,8)}}
\]

and

\[
\lambda_{a,b} = \Lambda_{a,b} + \frac{1}{2\sqrt{2}} \frac{\Lambda_{\frac{8a}{(8,8)}, \frac{b}{(8,8)}} - \Lambda_{\frac{8a}{(8,8)}, \frac{b}{(8,8)}}}{\Lambda_{\frac{8a}{(8,8)}, \frac{b}{(8,8)}}}.
\]

Additionally, set

\[
\Lambda_{a,b} = \frac{\Lambda_{a,b}}{\Gamma \left( \frac{3}{2} \right)} \quad \text{and} \quad \lambda_{a,b} = \frac{\lambda_{a,b}}{\Gamma \left( \frac{3}{2} \right)}.
\]

We want to study the behavior of \( P_{a,b}(y) \).

Lemma 3. If \( \tau \in \mathbb{M} \cup \mathfrak{m} \), then, as \( y \to \infty \),

\[
\log |P_{a,b}(y)| \ll b.
\]

Proof. Using (12) and letting \( y \to 0 \), we have

\[
|\tau|^\frac{3}{2} = (y^2 + 4\pi^2 x^2)^\frac{3}{2} \leq \left( y^2 + \frac{4\pi^2 y^2}{b^2} \right)^\frac{3}{2} \leq c_4 y^\frac{3}{2} = \frac{c_4 \text{Re} (\tau')}{b^\frac{3}{2}},
\]

for some \( c_4 > 0 \). Thus, [11] Lemma 4] gives

\[
|g(h, \ell, s)| \leq e^{-c_5 (\ell + 1)^\frac{1}{2}},
\]

with \( c_5 = \frac{2\sqrt{2\pi} e_4}{c_4} \), which in turn leads to

\[
|\log |P_{a,b}(y)|| \leq \sum_{h=1}^{b} \sum_{s=1}^{2} \sum_{\ell=1}^{\infty} \left| \log |1 - g(h, \ell, s)|| \leq 2b \sum_{\ell=1}^{\infty} \left| \log |1 - e^{-c_5 (\ell + 1)^\frac{1}{2}}| \right| \ll b,
\]

concluding the proof. \( \Box \)

3.5. Final lemmas. We first want to bound \( G(q) \) on the minor arcs.

Lemma 4. If \( \varepsilon > 0 \) and \( \tau \in \mathfrak{m}_{a,b} \), then

\[
|\log G(q)| \ll \varepsilon y^{\frac{1}{2} - \varepsilon}.
\]

Proof. In the proof and notation of [11] Lemma 17], replace \( a = \frac{1}{2}, b = \frac{1}{3}, c = 2, \gamma = \varepsilon \) and \( N = y^{-1} \). \( \Box \)

Before delving into the proof of Lemma 2 we need two final, though tedious, steps.

Lemma 5. For coprime integers \( 0 \leq a < b \) with \( b \geq 2 \), we have

\[
\max \{|\text{Re} (\lambda_{a,b})|, |\text{Im} (\lambda_{a,b})|\} < \frac{\zeta \left( \frac{3}{2} \right) \Gamma \left( \frac{3}{2} \right)}{1.14 \cdot 2\sqrt{2}}.
\]
Proof. A well-known result due to Gauss (for a proof see, e.g., [5] Ch. 1) says that, for \((a, b) = 1\), the sum \(S_{a, b}\) defined in [10] can be computed by the formula

\[
S_{a, b} = \begin{cases} 
0 & \text{if } b \equiv 2 \pmod{4}, \\
\varepsilon_b \sqrt{b} \left(\frac{a}{b}\right) & \text{if } b \text{ is odd}, \\
(1 + i) \varepsilon_a^{-1} \sqrt{b} \left(\frac{a}{b}\right) & \text{if } 4|b,
\end{cases}
\]

where \(\left(\frac{a}{b}\right)\) is the Jacobi symbol and

\[
\varepsilon_b = \begin{cases} 
1 & \text{if } b \equiv 1 \pmod{4}, \\
i & \text{if } b \equiv 3 \pmod{4}.
\end{cases}
\]

On recalling (9), (14) and (15), it is enough to prove that

\[
\max\{|\text{Re}(\lambda^*_{a, b})|, |\text{Im}(\lambda^*_{a, b})|\} < \frac{\zeta(\frac{3}{2})}{1.14 \cdot 2\sqrt{2}}.
\]

We explicitly evaluate \(\Lambda^*_{a, b}\). We have

\[
\Lambda^*_{a, b} = \frac{1}{b} \sum_{m=1}^{\infty} \frac{S_{m, a, b}}{m^{\frac{3}{2}}} = \frac{1}{b} \sum_{d|b} \sum_{m \geq 1} \frac{S_{m, a, b}}{m^{\frac{3}{2}}} = \frac{1}{b} \sum_{d|b} d^{-\frac{1}{2}} \sum_{m \geq 1} \frac{S_{m, a, b}}{m^{\frac{3}{2}}}
\]

\[
\quad = \frac{1}{b} \sum_{d|b} \left(\frac{b}{d}\right)^{-\frac{1}{2}} \sum_{m \geq 1} \sum_{(m, d) = 1} \frac{S_{m, a, b}}{m^{\frac{3}{2}}} = \frac{1}{b^2} \sum_{d|b} d^{-\frac{1}{2}} \sum_{m \geq 1} \sum_{(m, d) = 1} \frac{S_{m, a, b}}{m^{\frac{3}{2}}}.
\]

We distinguish several cases, in all of which we shall apply the following bound for divisor sums, which can be easily deduced. If \(\beta, L, \ell \in \mathbb{N}\) with \(\frac{\ell}{L} > 0.064\ldots\) and \(\gamma\) is the Euler-Mascheroni constant, then

\[
\sum_{d|\beta} \frac{1}{d} \leq \sum_{1 \leq \ell \leq \beta \leq \frac{\ell + L}{2}} \frac{1}{L \ell} \leq \frac{1}{\ell} + \frac{1}{L} \sum_{1 \leq d \leq \frac{\beta}{L}} \frac{1}{d} \leq \frac{1}{\ell} + \frac{1}{L} \left(\log \left(\frac{\beta}{L}\right) + \gamma + \frac{1}{2\gamma + 1}\right).
\]

(16)

Remark. We can apply this bound since in each of the following cases we only need to use values of \(\beta\) and \(L\) for which \(\frac{\beta}{L} \geq \frac{1}{4}\).

Case 1: \(b\) is odd. Then

\[
\lambda^*_{a, b} = \Lambda^*_{a, b} + \frac{1}{2\sqrt{2}} \Lambda^*_{a, b} - \Lambda^*_{a, b} = \frac{1}{b^2} \sum_{d|b} d^{-\frac{1}{2}} \sum_{m \geq 1} \frac{1}{m^{\frac{3}{2}}} \left(\frac{a}{m} + \frac{S_{m, a, b}}{2\sqrt{2}} - S_{m, a, b}\right)
\]

\[
\quad = \frac{1}{2\sqrt{2}b^2} \sum_{d|b} d^{-\frac{1}{2}} \sum_{m \geq 1} \frac{1}{m^{\frac{3}{2}}} \left(\frac{a}{m} + \frac{S_{m, a, b}}{2\sqrt{2}} - S_{m, a, b}\right).
\]

In case \(b \equiv 1 \pmod{4}\), we can bound both the real and imaginary part of \(\lambda^*_{a, b}\) (for \(j = 1, 3\) respectively) by

\[
\frac{1}{2\sqrt{2}b^2} \sum_{d \equiv j (\pmod{4})} d^{\frac{1}{2}} \left(\frac{3}{2}\right) = \frac{1}{2\sqrt{2}b^2} \sum_{d \equiv j (\pmod{4})} \frac{b}{d}^{\frac{1}{2}} \left(\frac{3}{2}\right) = \frac{1}{2\sqrt{2}b^2} \sum_{d \equiv j (\pmod{4})} \frac{1}{d}^{\frac{1}{2}} \left(\frac{3}{2}\right),
\]

whilst for \(b \equiv 3 \pmod{4}\) we can bound the two quantities by

\[
\frac{1}{2\sqrt{2}b^2} \sum_{d \equiv j + 2 (\pmod{4})} d^{\frac{1}{2}} \left(\frac{3}{2}\right) = \frac{1}{2\sqrt{2}b^2} \sum_{d \equiv j + 2 (\pmod{4})} \frac{b}{d}^{\frac{1}{2}} \left(\frac{3}{2}\right) = \frac{1}{2\sqrt{2}b^2} \sum_{d \equiv j + 2 (\pmod{4})} \frac{1}{d}^{\frac{1}{2}} \left(\frac{3}{2}\right).
\]
Using the bound (16) in the worst case possible (that is, \(d \equiv 1 \mod 4\)) gives

\[
\sum_{\substack{d \mid b \\ d \equiv 1 \mod 4}} \frac{1}{d} \leq 1 + \frac{1}{4} \left( \log \left( \frac{b}{4} \right) + \gamma + \frac{1}{2} + \frac{1}{3} \right).
\]

We checked with MAPLE that

\[
\frac{\zeta \left( \frac{3}{2} \right)}{2\sqrt{2} b^\frac{3}{2}} \left( 1 + \frac{1}{4} \left( \log \left( \frac{b}{4} \right) + \gamma + \frac{1}{6} \right) \right) \leq \frac{\zeta \left( \frac{3}{2} \right)}{1.14 \cdot 2\sqrt{2}}
\]

for \(b > 1\). Since the left-hand side above is a decreasing function, we are done in this case.

**Case 2:** \(2 \parallel b\). Then

\[
\lambda_{a,b}^* = \Lambda_{a,b}^* + \frac{1}{2\sqrt{2}} \Lambda_{4a,b}^* - \Lambda_{2a,b}^* = \frac{1}{b^\frac{3}{2}} \sum_{d \mid b} d^\frac{3}{2} \sum_{\substack{m \geq 1 \\ (m,d) = 1}} \frac{1}{m^\frac{3}{2}} \left( S_{ma,d} + S_{4ma,d} - 2\sqrt{2} S_{2ma,d} \right)
\]

\[
= \frac{2}{b^\frac{3}{2}} \sum_{d \mid b} d^\frac{3}{2} \sum_{\substack{m \geq 1 \\ (m,d) = 1}} \varepsilon_d \left( \frac{ma}{d} \right) \left( 1 - \sqrt{2} \left( \frac{2}{d} \right) \right) \frac{\sqrt{d}}{m^\frac{3}{2}}
\]

\[
= \frac{2}{b^\frac{3}{2}} \sum_{d \mid b} d^\frac{3}{2} \varepsilon_d \left( 1 - \sqrt{2} \left( \frac{2}{d} \right) \right) \sum_{\substack{m \geq 1 \\ (m,d) = 1}} \frac{(ma)}{m^\frac{3}{2}}.
\]

Taking the real and imaginary part gives (for \(j = 1, 3\) respectively, and some \(\ell = 1, 3\) depending on the congruence class of \(\frac{b}{2} \mod 8\))

\[
\frac{2}{b^\frac{3}{2}} \sum_{d \equiv j \mod 4} d^\frac{3}{2} \varepsilon_d \left( 1 - \sqrt{2} \left( \frac{2}{d} \right) \right) \sum_{\substack{m \geq 1 \\ (m,d) = 1}} \frac{(ma)}{m^\frac{3}{2}} \leq \frac{\zeta \left( \frac{3}{2} \right)}{b^\frac{3}{2}} \left( \sum_{d \equiv \ell \mod 8} \frac{1}{d} \left( \sqrt{2} - 1 \right) + \sum_{d \equiv \ell+4 \mod 8} \frac{1}{d} \left( \sqrt{2} + 1 \right) \right).
\]

We now use (16) in the worst case possible (that is, \(\ell + 4 \equiv 1 \mod 8\)) to obtain the bound

\[
\frac{\zeta \left( \frac{3}{2} \right)}{b^\frac{3}{2}} \left( \sqrt{2} - 1 \right) \left( \frac{5}{8} + \frac{1}{8} \left( \log \left( \frac{b}{16} \right) + \gamma + \frac{1}{8} \right) \right) + \left( \sqrt{2} + 1 \right) \left( \frac{1}{8} \left( \log \left( \frac{b}{16} \right) + \gamma + \frac{1}{8} \right) \right)
\]

This is a decreasing function and a computer check in MAPLE shows that it is bounded above by \(\frac{\zeta \left( \frac{3}{2} \right)}{1.14 \cdot 2\sqrt{2}}\) for \(b \geq 124\). For the remaining cases we use the well-known relation between a Dirichlet \(L\)-series and the Hurwitz zeta function (see, e.g., Apostol [11, Ch. 12]) to write

\[
\lambda_{a,b}^* = \frac{2}{b^\frac{3}{2}} \sum_{d \mid b} d^{-\frac{3}{2}} \varepsilon_d \left( 1 - \sqrt{2} \left( \frac{2}{d} \right) \right) \sum_{\ell=1}^{d} \left( \frac{\ell a}{d} \right) \zeta \left( \frac{3}{2} \left( \frac{\ell}{d} \right) \right).
\]

We checked that, for \(b \leq 124\),

\[
\max\{|\text{Re}(\lambda_{a,b}^*)|, |\text{Im}(\lambda_{a,b}^*)|\} < \frac{\zeta \left( \frac{3}{2} \right)}{1.14 \cdot 2\sqrt{2}}.
\]
Case 3: $4 \parallel b$. Then

$$
\lambda_{a,b}^* = \Lambda_{a,b}^* + \frac{1}{2\sqrt{2}} \Lambda_{2a,b}^* - \Lambda_{a,b}^* = \frac{1}{b^2} \sum_{d \mid b} d^{\frac{3}{2}} \sum_{m \geq 1} S_{ma,d} \frac{m^2}{m^2} + \frac{8}{b^2} \sum_{d \mid b} d^{\frac{3}{2}} \sum_{m \geq 1} \frac{1}{m^2} \left( S_{2ma,d} - S_{ma,d} \right) 
$$

$$
= \frac{1}{b^2} \sum_{d \mid b} d^{\frac{3}{2}} \sum_{m \geq 1} \frac{1}{m^2} \left( S_{ma,d} + 2\sqrt{2} S_{2ma,d} - 8S_{ma,d} \right) 
$$

$$
+ \frac{1}{b^2} \sum_{d \mid b} (4d)^{\frac{3}{2}} \sum_{m \geq 1} \frac{S_{ma,4d}}{m^2} 
$$

$$
= \frac{1}{b^2} \sum_{d \mid b} d^{\frac{3}{2}} \sum_{m \geq 1} \frac{\varepsilon_d \sqrt{d} \left( \frac{ma}{d} \right)}{m^2} \left( -7 + 2\sqrt{2} \left( \frac{3}{d} \right) \right) 
$$

$$
+ \frac{1}{b^2} \sum_{d \mid b} (4d)^{\frac{3}{2}} \sum_{m \geq 1} \frac{(1 + i)\varepsilon_{ma} \sqrt{d} \left( \frac{4d}{ma} \right)}{m^2} . 
$$

The real and imaginary parts of $\lambda_{a,b}^*$ can, in the same way as before (for some $j = 1, 3$ depending on the congruence class of $b \not\equiv \frac{1}{4} \pmod{4}$), be bounded by

$$
\zeta \left( \frac{3}{2} \right) \left( \sum_{d \mid b} d^{\frac{3}{2}} \left( 7 + 2\sqrt{2} \right) + \sum_{d \mid b} d^{\frac{3}{2}} \left( 7 - 2\sqrt{2} \right) \right) + 4 \left( 1 - 2^{-\frac{3}{2}} \right) \zeta \left( \frac{3}{2} \right) \sum_{d \mid b} d 
$$

which, by using (16) in the worst case (that is, $j + 4 \equiv 3 \pmod{4}$), is seen to be less than

$$
\zeta \left( \frac{3}{2} \right) \frac{4b^{\frac{3}{2}}}{4b^{\frac{3}{2}}} \left( 7 + 2\sqrt{2} \right) \left( 1 + \frac{1}{8} \left( \log \left( \frac{b}{32} \right) + \gamma + \frac{1}{\frac{16}{5} + \frac{1}{3}} \right) \right) + \left( 7 - 2\sqrt{2} \right) \left( \frac{1}{5} + \frac{1}{8} \left( \log \left( \frac{b}{32} \right) + \gamma + \frac{1}{\frac{16}{5} + \frac{1}{3}} \right) \right) 
$$

$$
+ 4 \left( 1 - \frac{1}{2^{\frac{3}{2}}} \right) \left( 1 + \frac{1}{2} \left( \log \left( \frac{b}{8} \right) + \gamma + \frac{1}{\frac{16}{5} + \frac{1}{3}} \right) \right) . 
$$

In turn, a computer check shows that this decreasing function is bounded above by $\zeta \left( \frac{3}{2} \right) \frac{1.14 - 2\sqrt{2}}{1.14 \cdot 2\sqrt{2}}$ for $b \geq 390$. For the remaining cases we rewrite

$$
\lambda_{a,b}^* = \frac{1}{b^2} \sum_{d \mid b} \left( d^{\frac{3}{2}} \varepsilon_d \left( -7 + 2\sqrt{2} \left( \frac{2}{d} \right) \right) \sum_{\ell=1}^{d} \left( \frac{\ell a}{d} \right) \zeta \left( \frac{3}{2} \right) \left( \frac{\ell}{d} \right) \right) + \left( 4d \right)^{-\frac{1}{2}} \left( 1 + i \right) \sum_{\ell=1}^{4d} \varepsilon_{\ell a} \left( \frac{4d}{\ell a} \right) \zeta \left( \frac{3}{2} \right) \left( \frac{\ell}{4d} \right) . 
$$

We checked that, for $b \leq 390$, we have

$$
\max \{ |\text{Re}(\lambda_{a,b}^*)|, |\text{Im}(\lambda_{a,b}^*)| \} < \zeta \left( \frac{3}{2} \right) \frac{1.14 \cdot 2\sqrt{2}}{1.14 - 2\sqrt{2}} .
$$
CASE 4: $8|b$. We write $b = 2^\nu b'$, with $b'$ odd. Then, if we define $\delta_{d,4} = 0$ for $4 \nmid d$ and $\delta_{d,4} = 1$ for $4|d$, we have

$$\lambda_{a,b}^* - \Lambda_{a,b}^* + \frac{1}{2\sqrt{2}} = \frac{1}{b^2} \sum_{d | b} d^\frac{1}{2} \sum_{m \geq 1 \atop (m,d) = 1} \frac{1}{m^2} \left( \varepsilon_d \left( \frac{4ma}{d} \right) \sqrt{d} + \delta_{d,4} \varepsilon_{ma}^{-1}(1+i) \sqrt{d} \left( \frac{d}{ma} \right) \right)$$

$$+ \frac{1}{b^2 \sqrt{2}} \sum_{d | b} d^\frac{1}{2} \sum_{m \geq 1 \atop (m,d) = 1} \frac{1}{m^2} \left( \varepsilon_d \left( \frac{4ma}{d} \right) \sqrt{d} + \delta_{d,4} \varepsilon_{ma}^{-1}(1+i) \sqrt{d} \left( \frac{d}{ma} \right) \right)$$

$$- \frac{1}{b^2 \sqrt{2}} \sum_{d | b} d^\frac{1}{2} \sum_{m \geq 1 \atop (m,d) = 1} \frac{1}{m^2} \left( \varepsilon_d \left( \frac{4ma}{d} \right) \sqrt{d} + \delta_{d,4} \varepsilon_{ma}^{-1}(1+i) \sqrt{d} \left( \frac{d}{ma} \right) \right)$$

$$= \frac{1}{b^2} \sum_{d | b'} d \sum_{m \geq 1 \atop (m,d) = 1} \varepsilon_d \left( \frac{ma}{d} \right) + \frac{1+i}{b^2} \sum_{d | b'} \sum_{\nu \leq j \leq \nu-1} \frac{d \cdot 2^j}{m^2} \sum_{m \geq 1 \atop (m,2d) = 1} \frac{\varepsilon_{ma}^{-1} \left( \frac{2^j d}{ma} \right)}{m^2}$$

$$- \frac{7(i+1)}{b^2} \sum_{d | b'} d \cdot 2^{\nu-2} \sum_{\nu \leq j \leq \nu-1} \frac{-\varepsilon_{ma}^{-1} \left( \frac{2^j d}{ma} \right)}{m^2} + \frac{1+i}{b^2} \sum_{d | b'} \sum_{\nu \leq j \leq \nu-1} \frac{d \cdot 2^j}{m^2} \sum_{m \geq 1 \atop (m,2d) = 1} \frac{\varepsilon_{ma}^{-1} \left( \frac{2^j d}{ma} \right)}{m^2}$$

Taking real and imaginary parts gives

$$\frac{\zeta \left( \frac{3}{2} \right)}{b^2} \sum_{d | b'} d + \frac{\zeta \left( \frac{3}{2} \right)}{b^2} \left( 1 - 2^{-\frac{3}{2}} \right) \sum_{d | b'} d \left( 3 \cdot 2^{\nu-1} + \sum_{2 \leq j \leq \nu} 2^j \right)$$

$$= \frac{\zeta \left( \frac{3}{2} \right)}{b^2} \sum_{d | b'} \frac{b}{d} + \frac{\zeta \left( \frac{3}{2} \right)}{b^2} \left( 1 - 2^{-\frac{3}{2}} \right) \sum_{d | b'} \frac{b}{2^\nu d} \left( 3 \cdot 2^{\nu-1} + \sum_{2 \leq j \leq \nu} 2^j \right)$$

as a bound for $\max \{ |\Re(\lambda_{a,b}^*)|, |\Im(\lambda_{a,b}^*)| \}$. The expression inside the brackets is then seen to be less than $7 \cdot 2^{\nu-1} - 4 < 7 \cdot 2^{\nu-1}$, and thus we obtain (for some $\ell = 1,3$ depending on the congruence class of $b'$ (mod 4)) the overall bound

$$\zeta \left( \frac{3}{2} \right) \left( \frac{1}{b^2} \sum_{d | b'} \frac{1}{d} + \frac{7 \left( 1 - 2^{-\frac{3}{2}} \right)}{2b^2} \sum_{d | b'} \frac{1}{d} \right)$$

which, in the worst case (that is, $d \equiv 1$ (mod 4)) equals

$$\zeta \left( \frac{3}{2} \right) \left( \frac{1}{b^2} \sum_{d | b'} \frac{1}{d} + \frac{7 \left( 1 - 2^{-\frac{3}{2}} \right)}{2b^2} \sum_{d | b'} \frac{1}{d} \right)$$

$$\leq \frac{\zeta \left( \frac{3}{2} \right)}{b^2} \left( \frac{1}{2} \left( 1 + \frac{1}{4} \log \left( \frac{b'}{4} \right) + \gamma + \frac{1}{b' + \frac{1}{2}} \right) \right) + \left( 1 - 2^{-\frac{3}{2}} \right) \frac{7}{2} \left( 1 + \frac{1}{2} \left( \log \left( \frac{b'}{2} \right) + \gamma + \frac{1}{b' + \frac{1}{3}} \right) \right)$$

$$\leq \frac{\zeta \left( \frac{3}{2} \right)}{b^2} \left( \frac{1}{8} \left( 1 + \frac{1}{4} \log \left( \frac{b}{32} \right) + \gamma + \frac{1}{b + \frac{1}{3}} \right) \right) + \left( 1 - 2^{-\frac{3}{2}} \right) \frac{7}{2} \left( 1 + \frac{1}{2} \left( \log \left( \frac{b}{16} \right) + \gamma + \frac{1}{b + \frac{1}{3}} \right) \right).$$
A computer check shows that this last expression, which is a decreasing function, is bounded above by \(\frac{\zeta(\frac{3}{2})}{1.14 \cdot 2 \sqrt{2}}\) for \(b \geq 527\). For the remaining cases we rewrite
\[
\lambda_{a,b}^* = \frac{1}{b^2} \sum_{d|b} d \varepsilon_d \sum_{m \geq 1} \frac{4ma}{d} \left( \frac{a+b}{m} \right) + \frac{1}{b^2} \sum_{d|\frac{b}{16}} 4d(1+i) \sum_{m \geq 1} \frac{\varepsilon_{ma}^{-1} \left( \frac{4d}{ma} \right)}{m^2} - 8 \sum_{d|\frac{b}{16}} 4d(1+i) \sum_{m \geq 1} \frac{\varepsilon_{ma}^{-1} \left( \frac{4d}{ma} \right)}{m^2}
\]
\[
= \frac{1}{b^2} \sum_{d|b} d^{-\frac{1}{2}} \varepsilon_d \sum_{\ell=1} d \left( \frac{4\ell a}{d} \right) \left( \frac{3}{2} - \frac{\ell}{d} \right) + \frac{1}{b^2} \sum_{d|\frac{b}{16}} (4d)^{-\frac{1}{2}} \sum_{\ell=1} (4d)^{-\frac{1}{2}} \varepsilon_{\ell a}^{-1} \left( \frac{4d}{\ell a} \right) \left( \frac{3}{2} - \frac{\ell}{4d} \right) - 8(i+1) \sum_{d|\frac{b}{16}} (4d)^{-\frac{1}{2}} \varepsilon_{\ell a}^{-1} \left( \frac{4d}{\ell a} \right) \left( \frac{3}{2} - \frac{\ell}{4d} \right)
\]
and check that
\[
\max\{ |\Re(\lambda_{a,b}^*)|, |\Im(\lambda_{a,b}^*)| \} < \frac{\zeta(\frac{3}{2})}{1.14 \cdot 2 \sqrt{2}}.
\]
This finishes the proof of the lemma. \(\square\)

**Lemma 6.** For some \(c \geq 0\), we have
\[
\frac{\lambda_{0,1}}{\sqrt{y}} - \Re \left( \frac{\lambda_{a,b}}{\sqrt{\tau'}} \right) \geq \frac{c}{\sqrt{y}}.
\]

**Proof.** We write \(\tau' = y + ity\) for some \(t \in \mathbb{R}\). We have
\[
\Re \left( \frac{\lambda_{a,b}}{\sqrt{\tau'}} \right) = \frac{1}{\sqrt{y}} \Re \left( \frac{\lambda_{a,b}}{\sqrt{1 + ity}} \right) = \frac{1}{\sqrt{y}} \Re \left( \frac{\lambda_{a,b}}{(1 + t^2)^{\frac{1}{4}} e^{\frac{1}{2} \arctan t}} \right)
\]
\[
= \frac{1}{\sqrt{y}(1 + t^2)^{\frac{1}{4}}} \left( \cos \left( \frac{\arctan t}{2} \right) \Re(\lambda_{a,b}) + \sin \left( \frac{\arctan t}{2} \right) \Im(\lambda_{a,b}) \right).
\]
We aim to find the maximal absolute value of
\[
f(t) = \frac{1}{(1 + t^2)^{\frac{1}{4}}} \left( \left| \cos \left( \frac{\arctan t}{2} \right) \right| + \left| \sin \left( \frac{\arctan t}{2} \right) \right| \right).
\]
Using the trigonometric identities
\[
\cos \left( \frac{\Theta}{2} \right) = \sqrt{\frac{1 + \cos \Theta}{2}}, \quad \sin \left( \frac{\Theta}{2} \right) = \sqrt{\frac{1 - \cos \Theta}{2}}, \quad \text{and} \quad \cos(\arctan t) = \frac{1}{\sqrt{1 + t^2}},
\]
as well as the fact that \(|\arctan t| < \frac{\pi}{4}\), we obtain
\[
f(t) = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{1}{1 + t^2}} + \frac{1}{1 + t^2} + \sqrt{\frac{1}{1 + t^2} - \frac{1}{1 + t^2}} \right),
\]
and an easy calculus exercise shows that
\[
f(t) < 1.139753528 \ldots < 1.14.
\]
On noting that \(\lambda_{0,1} = \frac{\lambda_{0,1}}{2\sqrt{2}} = \frac{\Gamma(\frac{3}{2})\zeta(\frac{3}{2})}{2\sqrt{2}}\) and that by Lemma 5 there exists a small enough \(c > 0\) such that
\[
\Re \left( \frac{\lambda_{a,b}}{\sqrt{\tau'}} \right) \leq \frac{\lambda_{0,1} - c}{\sqrt{y}},
\]
we conclude the proof. \(\square\)
Proof of Lemma 2. If we are on a minor arc, then it suffices to apply Lemma 4 (because, as $y \to 0$, a negative power of $y$ will dominate any positive power of $y$), so let us assume that we are on a major arc. We first consider the behavior near the cusp 0, corresponding to $y = 0$, we obtain, by letting $y \to 0$.

On using (17) to prove the first inequality below and expanding into Taylor series to prove the second

Theorem 1.

We begin by proving part (ii). By Lemma 2 and the fact that $\Lambda = 0$, ensures $|y| \leq 0$ for some $\Lambda > 0$.

By (13) we get

$$G(q) = C e^{\frac{\Lambda_0,1}{\sqrt{\gamma}}} \frac{P_{0,1}(y)P_{0,1}(8y)}{P_{0,1}(4y)^2}$$

for some $C > 0$ and thus, by Lemma 3

$$\log |G(q)| = \frac{\Lambda_{0,1}}{2\sqrt{2}\sqrt{\gamma}} + O(1).$$

On using (17) to prove the first inequality below and expanding into Taylor series to prove the second one, we obtain, by letting $y \to 0$,

$$\frac{1}{\sqrt{\gamma}} \frac{1}{\sqrt{\gamma}} \left(1 + \frac{4\pi^2 y^2}{y^2} \right)^{\frac{1}{4}} \leq \frac{1}{\sqrt{\gamma}} \left(1 + \frac{4\pi^2 y^2}{y^2} \right)^{\frac{1}{4}} \leq \frac{1}{\sqrt{\gamma}} \left(1 - c_6 y^2 - 2\varepsilon \right)$$

for some $c_6 > 0$, and this concludes the proof in this case.

To finish the claim we assume $2 \leq b \leq y^{-\frac{1}{2}}$. If $\tau \in \mathcal{M}_{a,b}$, then by (13), Lemma 3 and Lemma 6 we obtain that, as $y \to 0$,

$$\log |G(q)| = \text{Re} \left( \frac{\lambda_{a,b}}{\sqrt{\gamma}} \right) + O \left( y^{-\frac{1}{2}} \right) \leq \frac{\lambda_{0,1}}{\sqrt{\gamma}} - \frac{c_7}{\sqrt{\gamma}} + O \left( y^{-\frac{1}{2}} \right) \leq \frac{\lambda_{0,1}}{\sqrt{\gamma}} - \frac{c_8}{\sqrt{\gamma}},$$

and the proof is complete. □

4. PROOF OF THE MAIN THEOREM

We have now all the necessary ingredients to prove Theorem 1 whose statement we repeat for convenience.

Theorem 1.

(i) As $n \to \infty$, we have

$$p_2(0, 2, n) \sim p_2(1, 2, n).$$

(ii) Furthermore, for $n$ sufficiently large, we have

$$\begin{cases} p_2(0, 2, n) > p_2(1, 2, n) & \text{if } n \equiv 0 \pmod{2}, \\ p_2(0, 2, n) < p_2(1, 2, n) & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

Proof. We begin by proving part (ii). By Lemma 2 and the fact that $\Lambda_{0,1} = \Gamma \left( \frac{3}{2} \right) \zeta \left( \frac{3}{2} \right)$, we have the bound

$$R(n) = e^{ny} \int_{y^2 \leq |x| \leq \frac{1}{2}} G \left( e^{-y+2\pi i x} \right) e^{-2\pi i n x} dx \ll e^{ny} \int_{y^2 \leq |x| \leq \frac{1}{2}} e^{\frac{1}{\sqrt{2}} \Gamma \left( \frac{3}{2} \right) \zeta \left( \frac{3}{2} \right) - cy^{-\varepsilon}} dx$$

$$\leq e^{ny} + \frac{1}{\sqrt{2}} \Gamma \left( \frac{3}{2} \right) \zeta \left( \frac{3}{2} \right) - cy^{-\varepsilon} = e^{3n^2 \left( \Gamma \left( \frac{3}{2} \right) \zeta \left( \frac{3}{2} \right) \right)^{\frac{1}{2}} - Cn^2},$$

with $\varepsilon = \frac{2n}{3} > 0$ and some $C > 0$.

We next turn to the asymptotic main term integral. Let $n \geq n_1$ be large enough so that $y^2 \leq \frac{1}{n_1}$, which ensures $|x| \leq \frac{1}{2}$ throughout the interval of integration, and $n \geq n_2$ large enough so that $y^2 - 1 \leq \frac{1}{n_2}$, which
ensures \(|\text{Arg } \tau| \leq \frac{\pi}{4}\) and allows us to apply Lemma \([1]\). By choosing then \(n \geq \max\{n_1, n_2\}\) and recalling that \(\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}\), we obtain

\[
e^{ny} \int_{-y^\beta}^y G(e^{-y+2\pi ix})e^{-2\pi inx} dx = \frac{e^{ny}}{\sqrt{2}} \int_{-y^\beta}^y \frac{1}{\sqrt{\tau}} \Gamma\left(\frac{3}{2}\right) \frac{(\sqrt{\tau})}{\sqrt{\tau}} + O(y^\delta) e^{-2\pi inx} dx. \tag{19}\]

Splitting

\[
\frac{1}{\sqrt{\tau}} = \frac{1}{\sqrt{\gamma}} + \left(\frac{1}{\sqrt{\tau}} - \frac{1}{\sqrt{\gamma}}\right),
\]

we rewrite (19) as

\[
e^{ny} \int_{-y^\beta}^y G(e^{-y+2\pi ix})e^{-2\pi inx} dx = \frac{e^{ny}}{\sqrt{2}} \int_{-y^\beta}^y \frac{1}{\sqrt{\tau}} \Gamma\left(\frac{3}{2}\right) \frac{(\sqrt{\tau})}{\sqrt{\tau}} e^{-2\pi inx} + O(y^\delta) dx
= \frac{1}{\sqrt{2}} \int_{-y^\beta}^y \left(\frac{e^{ny}}{\sqrt{\tau}} \Gamma\left(\frac{3}{2}\right) \frac{(\sqrt{\tau})}{\sqrt{\tau}} e^{-2\pi inx} + O(y^\delta)\right) dx
= \frac{3n^\beta}{\sqrt{2}} \int_{-y^\beta}^y \frac{1}{\sqrt{\tau}} \Gamma\left(\frac{3}{2}\right) \frac{(\sqrt{\tau})}{\sqrt{\tau}} \frac{1}{\sqrt{1 - \frac{\sqrt{\tau}}{\sqrt{\gamma}}} - 1} e^{-2\pi inx} + O(y^\delta) dx.
\]

Putting \(u = -\frac{2\pi y}{\gamma}\), we get

\[
e^{ny} \int_{-y^\beta}^y G(e^{-y+2\pi ix})e^{-2\pi inx} dx = \frac{y^{3n^\beta}}{2\sqrt{2\pi}} \int_{-2\gamma y^\beta}^{-y^\beta} \frac{1}{\sqrt{\tau}} \Gamma\left(\frac{3}{2}\right) \frac{(\sqrt{\tau})}{\sqrt{\tau}} \frac{1}{\sqrt{1 - \frac{\sqrt{\tau}}{\sqrt{\gamma}}} - 1} du + inuy + O(y^\delta). \tag{20}\]

Set \(B = \frac{1}{2\sqrt{2}} \Gamma\left(\frac{3}{2}\right) \frac{(\sqrt{\tau})}{\sqrt{\tau}}\). We have the Taylor series expansion

\[
\frac{1}{\sqrt{1 + iu}} = 1 - \frac{iu}{2} - \frac{3iu^2}{8} + \frac{5iu^3}{16} + \cdots = 1 - \frac{iu}{2} - \frac{3iu^2}{8} + O(u^3),
\]

thus

\[
B \frac{1}{\sqrt{\gamma}} \left(\frac{1}{\sqrt{1 + iu}} - 1\right) + inuy = -\frac{Bi u}{2\sqrt{y}} + inuy - 3Bu^2 \frac{1}{8\sqrt{y}} + O\left(\frac{u^3}{\sqrt{y}}\right).
\]

However, an easy computation shows that, for \(y\) chosen as in \([5]\),

\[
y = \left(\frac{B}{2n}\right)^\frac{2}{3}, \quad \text{or equivalently,} \quad B = 2ny^3,
\]

which means that

\[
\frac{Bi u}{2\sqrt{y}} + inuy = 0,
\]

hence

\[
B \frac{1}{\sqrt{y}} \left(\frac{1}{\sqrt{1 + iu}} - 1\right) + inuy = -\frac{3Bu^2}{8\sqrt{y}} + O\left(\frac{u^3}{\sqrt{y}}\right).
\]

Thus we may change the integral from the right-hand side of (20) into

\[
\int_{|u| \leq 2\pi y^\beta - 1} e^{B \frac{1}{\sqrt{y}} \left(\frac{1}{\sqrt{1 + iu}} - 1\right) + inuy + O(y^\delta)} du = \int_{|u| \leq 2\pi \left(\frac{B}{2n}\right)^\frac{2}{3} (\beta - 1)} e^{-\frac{3Bu^2}{8\sqrt{y}}} \cdot O\left(y^\delta + \frac{u^3}{\sqrt{y}}\right) du
= \int_{|u| \leq 2\pi \left(\frac{B}{2n}\right)^\frac{2}{3} (\beta - 1)} e^{-\frac{3Bu^2}{8\sqrt{y}}} \cdot O\left(y^\delta + \frac{u^3}{\sqrt{y}}\right) du
= \int_{|u| \leq 2\pi \left(\frac{B}{2n}\right)^\frac{2}{3} (\beta - 1)} e^{-\frac{3Bu^2}{8\sqrt{y}}} \cdot O\left(y^\delta + u^3\right) du
= \int_{|u| \leq 2\pi \left(\frac{B}{2n}\right)^\frac{2}{3} (\beta - 1)} e^{-\frac{3Bu^2}{8\sqrt{y}}} \left(1 + \left(O\left(y^\delta + u^3\right) - 1\right)\right) du.
\]
Recall (3) and write $\beta = \frac{7}{6} + \frac{\epsilon}{2}$, with $\epsilon > 0$. Then

$$u^3 n^{\frac{1}{2}} \leq c' \frac{n^{\frac{3}{2}}}{n^{2(\beta - 1)}} = c' \frac{n^{\frac{3}{2}}}{n^\epsilon},$$

for some $c' > 0$, and thus

$$e^{O(n^{-\kappa} + an^{\frac{1}{2}})} - 1 = O(n^{-\kappa} + n^{-\epsilon}) = O(n^{-\kappa}),$$

where $\kappa = \min\{c_0, \epsilon\}$. We further get

$$\int_{|u| \leq 2\pi y^{\beta - 1}} e^{B \frac{1}{\sqrt{2}} \left( \frac{1}{2(n^{\beta})} - 1 \right) + i u y + O(y^{\alpha})} \, du = \int_{|u| \leq 2\pi(B \frac{1}{\sqrt{2}})^{\frac{1}{2}(\beta - 1)}} e^{-\frac{3\sqrt{\pi}}{\sqrt{2n}} \sqrt{\beta} u^2} (1 + O(n^{-\kappa})) \, du.$$  

On putting $v = \frac{\sqrt{3} \sqrt{2n} \sqrt{B} u}{2 \sqrt{2}}$ and $C = 2^\frac{1}{2} \frac{7}{6} \sqrt{3} \pi B^{\frac{1}{2} - \frac{1}{n^{\beta - \frac{1}{2}}} \frac{1}{2}}$, we obtain

$$\int_{|u| \leq 2\pi y^{\beta - 1}} e^{B \frac{1}{\sqrt{2}} \left( \frac{1}{2(n^{\beta})} - 1 \right) + i u y + O(y^{\alpha})} \, du = \int_{|u| \leq 2\pi(B \frac{1}{\sqrt{2}})^{\frac{1}{2}(\beta - 1)}} e^{-\frac{3\sqrt{\pi}}{\sqrt{2n}} \sqrt{\beta} u^2} (1 + O(n^{-\kappa})) \, du$$

$$= \frac{2\sqrt{2}}{\sqrt{3} \sqrt{2n} \sqrt{B}} \int_{|v| \leq C} e^{-v^2} (1 + O(n^{-\kappa})) \, dv. \quad (21)$$

Turning the integral from (21) into a Gauss integral and putting together (7), (18) and (20), we obtain

$$cn^{-\frac{1}{6}} e^{3n^{\frac{1}{2}} \left( \frac{1}{\sqrt{2}} \gamma\left( \frac{3}{2} \right) \zeta\left( \frac{3}{2} \right) \right) \frac{1}{6}}$$

to be the main contribution asymptotically for our coefficients $a_2(n)$, where $c > 0$ depends on $y$ which, in turn, depends on $n$. To make this precise, the Gauss-type integral above tends to $\sqrt{\pi}$ as $n \to \infty$, and we obtain

$$a_2(n) \sim \frac{y}{2\sqrt{2\pi}} \cdot \frac{2\sqrt{2}}{\sqrt{3} \sqrt{2n} \sqrt{B}} e^{3n^{\frac{1}{2}} \left( \frac{1}{\sqrt{2}} \gamma\left( \frac{3}{2} \right) \zeta\left( \frac{3}{2} \right) \right) \frac{1}{6}} \int_{-\infty}^{\infty} e^{-v^2} \, dv$$

$$= \frac{y\sqrt{\pi}}{\pi\sqrt{3} \sqrt{2n} \sqrt{B}} e^{3n^{\frac{1}{2}} \left( \frac{1}{\sqrt{2}} \gamma\left( \frac{3}{2} \right) \zeta\left( \frac{3}{2} \right) \right) \frac{1}{6}}$$

$$= \frac{\sqrt{B}}{\sqrt{3\pi \cdot (2n)^{\frac{1}{2}}} \epsilon} e^{3n^{\frac{1}{2}} \left( \frac{1}{\sqrt{2}} \gamma\left( \frac{3}{2} \right) \zeta\left( \frac{3}{2} \right) \right) \frac{1}{6}}. \quad (22)$$

This gives asymptotically the inequalities, hence part (ii) of Theorem 1 is proven.

We now turn to part (i). Clearly, $p_2(n) = p_2(0, 2, n) + p_2(1, 2, n)$ and we can compute $p_2(n)$ using [11, Theorem 2]. Keeping the notation from [11, pp. 144–145], we have

$$p_2(n) \sim B_0 n^{-\frac{z}{6}} e^{\Lambda n^{\frac{1}{2}}}$$

where

$$B_0 = \frac{\Lambda}{2 \cdot (3\pi)^{\frac{1}{2}}} \quad \text{and} \quad \Lambda = 3 \left( \frac{\Gamma\left( \frac{3}{2} \right) \zeta\left( \frac{3}{2} \right)}{2} \right)^{\frac{3}{2}} = 6 \left( \frac{1}{4\sqrt{2}} \Gamma\left( \frac{3}{2} \right) \zeta\left( \frac{3}{2} \right) \right)^{\frac{3}{2}}.$$  

We thus obtain

$$p_2(n) \sim B_0 n^{-\frac{z}{6}} e^{6n^{\frac{1}{2}} \left( \frac{1}{\sqrt{2}} \gamma\left( \frac{3}{2} \right) \zeta\left( \frac{3}{2} \right) \right) \frac{1}{6}}. \quad (23)$$

On recalling (2), adding (or subtracting) (22) and (23) yields

$$p_2(0, 2, n) \sim p_2(1, 2, n) \sim \frac{B_0}{2} n^{-\frac{z}{6}} e^{6n^{\frac{1}{2}} \left( \frac{1}{\sqrt{2}} \gamma\left( \frac{3}{2} \right) \zeta\left( \frac{3}{2} \right) \right) \frac{1}{6}} \quad \text{as} \ n \to \infty,$$

and the proof is complete. \qed
Remark. As promised at the beginning of this paper and already revealed by our proof, by plugging in the values of $B_0$ and $\Lambda$ we obtain the asymptotics

$$p_2(0, 2, n) \sim p_2(1, 2, n) \sim \frac{1}{2\pi \sqrt{3\pi}} \left( \frac{1}{4\sqrt{2}} \Gamma\left( \frac{3}{2} \right) \zeta\left( \frac{3}{2} \right) \right)^{\frac{3}{2}} n^{-\frac{7}{6}} e^{\frac{6}{n^2}} \frac{6\pi}{\sqrt{3}} \left( \frac{1}{\sqrt{2}} \Gamma\left( \frac{3}{2} \right) \zeta\left( \frac{3}{2} \right) \right)^{\frac{3}{4}}$$

as $n \to \infty$.

Remark. Note that, although we could not apply Meinardus’ Theorem to our product in (1), the asymptotic value we obtained for $a_2(n)$ in (22) agrees, surprisingly or not, precisely with that given for $r(n)$ in Theorem 2. This indicates that, even if it may not directly apply to certain generating products, Meinardus’ Theorem is a powerful enough tool to provide correct heuristics.

Remark. We note that, in its original formulation, part (ii) of Conjecture 1 is not entirely true since there are cases when $p_2(0, 2, n) = p_2(1, 2, n)$, as it happens, e.g., for $n \in \{4, 5, 6, 7, 13, 14, 15, 16, 22, 23, 24, 31, 39, 47, 48, 56, 64\}$. No other values of $n$ past 64 revealed such pattern and, based on the behavior we observed, we strongly believe that the inequalities hold true for $n \geq 65$. In particular, we checked this is the case up to $n = 50,000$.

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