Long-time asymptotic behavior of the nonlocal nonlinear Schrödinger equation with initial potential in weighted Sobolev space

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January 12, 2021

Abstract

In this paper, we are going to investigate Cauchy problem for nonlocal nonlinear Schrödinger equation with the initial potential \( q_0(x) \) in weighted Sobolev space \( H^{1/1}(\mathbb{R}) \),

\[
\begin{align*}
    iq_t(x,t) + q_{xx}(x,t) + 2\sigma q^2(x,t)\bar{q}(-x,t) &= 0, \quad \sigma = \pm 1, \\
    q(x,0) &= q_0(x).
\end{align*}
\]

We show that the solution can be represented by the solution of a Riemann-Hilbert problem (RH problem), and assuming no discrete spectrum, we majorly apply \( \partial \)-steepest descent method on analyzing the long-time asymptotic behavior of it.

Key words: Nonlocal nonlinear Schrödinger equation, weighted Sobolev space, long-time asymptotic behavior, Riemann-Hilbert problem, \( \partial \)-steepest descent method.

2010 Mathematics Subject Classification Numbers: 35Q15, 35Q58, 35B40.

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1 Introduction

The nonlocal nonlinear Schrödinger (NNLS) equation

\[ iq_t(x,t) + q_{xx}(x,t) + 2\sigma q^2(x,t)\bar{q}(-x,t) = 0, \quad \sigma = \pm 1, \]  

was first introduced by Ablowitz and Musslimani in 2013 [1]. It is an integrable system with Lax pair, and its inverse scattering transformations with zero and nonzero boundary condition have been completed by Ablowitz et al [2,3]. The long-time asymptotic analysis for the NNLS equation with rapidly decaying initial data
was given by Rybalko and Shepelsky [4]. In 2018, Feng et al have got the general soliton for the NNLS equation by Hirota’s bilinear method and the Kadomtsev-Petviashvili hierarchy reduction method [5]. We note that the NNLS have the \( \mathcal{PT} \)-symmetry \[6\] potential 
\[
V(x,t) = q(x,t)\bar{q}(-x,t): \quad V(x,t) = V(-x,t).
\]
Recent years, there are many works for the \( \mathcal{PT} \)-symmetric equations [7–11]. \( \mathcal{PT} \) symmetry is also an important conception in optics [12–14].

In this paper, we apply a systematic dbar-steepest descent method to analyze the long-time asymptotic behavior of the solution for the NNLS equation \[1\] with the initial potential
\[
q(x,0) = q_0(x) \in H^{1,1}(\mathbb{R}),
\]
where \( H^{1,1}(\mathbb{R}) \) is a weighted Sobolev space defined by
\[
H^{1,1}(\mathbb{R}) = L^{2,1}(\mathbb{R}) \cap H^1(\mathbb{R}), \quad L^{2,1}(\mathbb{R}) = \{(1 + | \cdot |^2)^{1/2} f \in L^2(\mathbb{R})\}, \\
H^1(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) | f' \in L^2(\mathbb{R}) \}.
\]
For \( f \in L^{2,1}(\mathbb{R}) \), its norm is defined by \( \| f \|_{2,1} = \|(1 + | \cdot |^2)^{1/2} f \|_2 \).

The dbar-steepest descent method, developed from the Deift-Zhou steepest descent method, is very powerful in analyzing the long-time asymptotic behavior with potential in weighted Sobolev space [15–19]. Cuccagna et al also use it to analyze the asymptotic stability for the soliton solutions of the NLS equation [20, 21].

As shown in [23], there is a bijection map between space of initial potential and space of reflection coefficients:
\[
H^{1,1}(\mathbb{R}) \rightarrow H^{1,1}(\mathbb{R}): \quad q(x) \mapsto \{ r(z), \bar{r}(z) \},
\]
then, we can set the initial data in \( H^{1,1}(\mathbb{R}) \). Indeed, in this paper, the proof in this article only require \( \{ r(z), \bar{r}(z) \} \subset H^1(\mathbb{R}) \); however, by simply calculation,
\{r(z), \bar{r}(z)\} do not belong to \(H^1(\mathbb{R})\) as time evolving, but persist in \(H^{1,1}(\mathbb{R})\) as time evolving, seeing Remark 2.2, so, by (3), we restrict our initial potential \(q_0(x) \in H^{1,1}(\mathbb{R})\). We also restrict the potential \(q_0(x)\) to be generic: for the direct scattering, \(q(x) \mapsto \{r(z), \bar{r}(z)\}\), the reflection coefficient do not possess any singular point along the continuous spectrum. Moreover, we also assume that the scattering coefficient \(a(z)\) possesses no zero on \(\mathbb{C}_+\) while \(\bar{a}(z)\) has not any zero on \(\mathbb{C}_-\).

This article is organized as follows. At section 2 we simply display the main result of the direct scattering. At section 3 we construct the Riemann-Hilbert (RH) problem based on the Lax pair (4). At section 4 we establish the map \(q(x) \mapsto \{r(z), \bar{r}(z)\}\). At section 5 we carry out a series of RH problem transformations: \(M \sim M^{(1)} \sim M^{(2)}\) and factorizing \(M^{(2)}\) into a product of one model RH problem \(M^{(2)}_{\text{RH}}\) and one pure \(\bar{\partial}\) problem \(M^{(3)}\). At section 6 we get the long-time asymptotic behavior of the NNLS equation.

## 2 Direct Scattering Problem

In this section, we state the main result of the direct scattering transformation. Based on the Lax pair (4), we obtain the Jost solution, modified Jost solution, the scattering matrix, their symmetric properties and their asymptotic properties as \(z \to \infty\).

The NNLS equation admits the Lax pair:

\[
\begin{align*}
\phi_x + i z \sigma_3 \phi &= Q \phi, \\
\phi_t + 2 i z^2 \sigma_3 \phi &= P \phi,
\end{align*}
\]

\[
Q \equiv Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -\sigma q(-x,t) & 0 \end{pmatrix}, \quad (4a)
\]

\[
P = i \sigma_3 (Q_x - Q^2) + 2 z Q, \quad (4b)
\]

where \(\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\) is a Pauli matrix, \(\{x,t\} \subset \mathbb{R}\) and \(z \in \mathbb{C}\) denotes the spectrum.
The Lax pair (4) admits the Jost solution $\phi^\pm \equiv \phi^\pm(z;x,t)$:

$$
\phi^\pm \sim e^{-t\varphi^3}, \quad \text{as} \quad x \to \pm \infty,
$$

(5)

where $\varphi \equiv \varphi(z;x,t) = i(zt + 2z^2)$ is the phase function. Then, it’s natural for us to introduce the modified Jost solution $\Phi^\pm \equiv \Phi^\pm(z;x,t)$:

$$
\Phi^\pm = \phi^\pm e^{t\varphi^3},
$$

(6)

such that

$$
\Phi^\pm \sim I, \quad \text{as} \quad x \to \pm \infty,
$$

(7)

We can derive that $\Phi^\pm(z,x) = \Phi^\pm(z;x,t)$ satisfy the following Volterra integral equations associated with (4):

$$
\Phi^\pm(z,x) = I + \int_{\pm\infty}^{x} e^{iz(y-x)^{\sigma_3}}[Q(y)\Phi^\pm(z,y)]dy, \quad z \in \mathbb{C}.
$$

(8)

Because $q(x) \in L^{2,1}(\mathbb{R})$, and by Schwartz inequality,

$$
\|q\|_{1,\infty} \leq \|q\|_{2,1} \left(\int_{\mathbb{R}} (1 + x^2)^{-1}\right)^{1/2} \leq \sqrt{\pi} \|q\|_{2,1},
$$

the $L^1$-norm of $q(x)$ is bounded. By taking Neumann series of $\Phi^\pm$ in the Volterra integral, we naturally obtain analytic properties of $\Phi^\pm$ that are given in the following Proposition 2.1. We write $\Phi^\pm = (\Phi_1^\pm, \Phi_2^\pm)$.

**Proposition 2.1.** For the potential $q(x) \in L^{2,1}(\mathbb{R})$, $(\Phi_1^\pm(z,x), \Phi_2^\pm(z,x))$ is uniquely defined and analytic on $\mathbb{C}_+ = \{\text{Im} z > 0\}$, and continuously extended to $z \in \mathbb{C}_+ \cup \mathbb{R}$; In the meanwhile, $(\Phi_1^+(z,x), \Phi_2^-(z,x))$ is uniquely defined and analytic on $\mathbb{C}_- = \{\text{Im} z < 0\}$, and continuously extended to $z \in \mathbb{C}_- \cup \mathbb{R}$. See $\mathbb{C}_+$ and $\mathbb{C}_-$ at Figure [4].

Noticing in Lax pair (4) that

$$
\text{tr}(Q - iz\sigma_3) = 0,
$$

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Figure 1: $\mathbb{C}_+ = \{\text{Im} z > 0\}$, $\mathbb{C}_- = \{\text{Im} z < 0\}$ and the real line $\mathbb{R}$ considered as the continuous spectrum

since $\phi^+$ and $\phi^-$ both solve the Lax pair (4), we have that $\det \phi^\pm \equiv 1$ and there exists a unique $2 \times 2$ matrix $S \equiv S(z)$ independent on $(x, t)$ such that

$$\phi^-(z; x, t) = \phi^+(z; x, t)S(z), \quad S(z) = \begin{pmatrix} a(z) & \tilde{b}(z) \\ b(z) & \tilde{a}(z) \end{pmatrix},$$

where $S(z)$ is well known as scattering matrix and $a(z), \tilde{a}(z), b(z), \tilde{b}(z)$ are so called scattering coefficients. By basic linear algebra, we derive from (6) and (9)

$$a(z) = \det(\Phi_1^-, \Phi_2^+) \quad b(z) = \det(\Phi_1^+, \Phi_1^-)e^{-2t\varphi} ,$$

$$\tilde{a}(z) = \det(\Phi_1^+, \Phi_2^-) \quad \tilde{b}(z) = \det(\Phi_2^-, \Phi_2^+)e^{2t\varphi} ,$$

which implies, by Proposition 2.1, that $a(z)$ and $\tilde{a}(z)$ are analytic on $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, and continuously extended to $\mathbb{C}_+ \cup \mathbb{R}$ and $\mathbb{C}_- \cup \mathbb{R}$ respectively; in the meanwhile, both $b(z)$ and $\tilde{b}(z)$ are continuous on $\mathbb{R}$.

By WKB expansion, someone can get the asymptotic property of the modified Jost solutions:

$$\Phi^\pm(z; x, t) \sim I + O(z^{-1}) , \quad \text{as } z \to \infty ,$$

which with (10) implies that as $z \to \infty$,

$$a(z) \sim 1 + O(z^{-1}) , \quad b(z) \sim O(z^{-1}) ,$$

$$\tilde{a}(z) \sim 1 + O(z^{-1}) , \quad \tilde{b}(z) \sim O(z^{-1}) .$$
in addition, we can write the asymptotic property of $\Phi_{1,2}^\pm$ more precisely,

$$\Psi_{1,2}^\pm(z; x, t) \sim -\frac{i}{2} q(x, t)z^{-1} + O(z^{-2}), \quad \text{as } z \to \infty. \quad (13)$$

Defining the reflection coefficients,

$$r \equiv r(z) = \frac{b(z)}{a(z)}, \quad \tilde{r} \equiv \tilde{r}(z) = \frac{\tilde{b}(z)}{\tilde{a}(z)}, \quad \text{on } z \in \mathbb{R}, \quad (14)$$

we can see that if $q_0(x) \in L^2(\mathbb{R})$, these reflection coefficients belong to $H^1(\mathbb{R})$. See more detail in Section 4. Then, both $r(z)$ and $\tilde{r}(z)$ are $\frac{1}{2}$-Hölder continuous on the real line and satisfy

$$|r(z)|, |\tilde{r}(z)| \lesssim (1 + z^2)^{-\frac{1}{4}}, \quad z \in \mathbb{R}, \quad (15)$$

therefore, both $r(z)$ and $\tilde{r}(z)$ are continuous and bounded on $z \in \mathbb{R}$; moreover, recalling (9) and that $\det \phi^\pm \equiv 1$, we have

$$\det S(z) = a(z)\tilde{a}(z) - b(z)\tilde{b}(z) = 1,$$

then,

$$1 - r(z)\tilde{r}(z) = \frac{1}{a(z)\tilde{a}(z)},$$

and, with the generic assumption, we obtain that $1 - r(z)\tilde{r}(z)$ is also bounded, continuous and non-vanishing on the real line.

By basic calculation, $\phi^\pm$ admits symmetry:

$$\Lambda \phi^\pm(-\bar{z}; -x, t)\Lambda^{-1} = \phi^\mp(z; x, t), \quad \Lambda = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}, \quad z \in \mathbb{C}, \quad (16)$$

which implies the symmetry for the correspondent scattering coefficients:

$$a(z) = \bar{a}(-z), \quad \tilde{a}(z) = -\bar{a}(-z), \quad b(z) = -\sigma \bar{b}(-z), \quad \tilde{b}(z) = -\bar{b}(-z), \quad z \in \mathbb{R}. \quad (17)$$
Remark 2.2. By (6), (10a) and (14), sure we have
\[ r = \frac{\det(\phi_1^+, \phi_1^-)}{\det(\phi_1^-, \phi_2^+)} \] (18)

If we set \( q|_{z=t_0} \) as the initial data for \( t_0 > 0 \) and \( \tilde{\phi}^\pm \equiv \tilde{\phi}^\pm(z; x, t) \) as the correspondent Jost solution such that
\[ \tilde{\phi}^\pm \sim e^{-i(zx + 2z^2(t-t_0))\sigma_3}, \quad x \to \pm\infty, \] (19)
then, there is a correspondent reflection coefficient \( \tilde{r} \) satisfying:
\[ \tilde{r} = \frac{\det(\tilde{\phi}_1^+, \tilde{\phi}_1^-)}{\det(\tilde{\phi}_1^-, \tilde{\phi}_2^+)} \] (20)
Comparing (5), (18) with (19), (20) respectively, we obtain the relation of \( r(z) \) and \( \tilde{r}(z) \) by uniqueness of the Jost solution:
\[ \tilde{r}(z) = r(z)e^{4t_0z^2}, \]
which means that \( r(z) \) persists in \( H^{1,1}(\mathbb{R}) \) by simple computation. Of course, \( \tilde{r}(z) \) also does possess this property.

3 RH problem

In this section, we construct the correspondent RH problem for the Lax pair and the reconstructed formula (22) for \( q(x) \).

Introducing
\[ M \equiv M(z; x, t) = \begin{cases} \left( \frac{\Phi_1^-(z; x, t)}{\Phi_1^-(z; x, t)}, \frac{\Phi_2^+(z; x, t)}{\Phi_2^+(z; x, t)} \right), & z \in \mathbb{C}_+, \\ \left( \frac{\Phi_1^-(z; x, t)}{\Phi_1^-(z; x, t)}, \frac{\Phi_2^+(z; x, t)}{\Phi_2^+(z; x, t)} \right), & z \in \mathbb{C}_-, \end{cases} \] (21)
and the jump contour \( \mathbb{R} \), we observe from Section 2 that \( M \) admits the following RH problem.
RH problem 3.1. Find a $2 \times 2$ matrix function on $\mathbb{C} \setminus \mathbb{R}$ such that:

- **Analyticity:** $M$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$.
- **Normalization:**
  
  $$M \sim I + O(z^{-1}), \quad \text{as} \quad z \to \infty.$$  

- **Jump condition:**
  
  $$M_+ = M_- V, \quad \text{on} \quad \mathbb{R},$$

  where

  $$V = e^{t \sigma_3} \begin{pmatrix} 1 - r \hat{r} & -\hat{r}^* \\ r & 1 \end{pmatrix}.$$  

In addition, we get the reconstructed formula for the potential by (12a), (13) and (21).

$$q(x,t) = 2i \lim_{z \to \infty} z M_{1,2}(z;x,t). \quad (21)$$  

4 Analysis on scattering maps: $q(x) \mapsto \{ r(z), \hat{r}(z) \}$

In this section, we focus on the map from initial data to reflection coefficients. Here, we denote $q_0(x)$ by $q(x)$ without confusion of notation.

**Proposition 4.1.** If $q(x) \in L^2(\mathbb{R})$, then the reflection coefficient $r(z)$ and $\hat{r}(z)$ both belong to $H^1(\mathbb{R})$.

4.1 $r(z) \in H^1(\mathbb{R})$

By (10), (12) and Proposition 2.1, we learn that $a(z), b(z), \hat{a}(z)$ and $\hat{b}(z)$ are continuous and bounded on $z \in \mathbb{R}$. Since the assumption that $q(x)$ is generic and
the fact that
\[ r(z) = \frac{b(z)}{a(z)}, \quad r'(z) = \frac{1}{a(z)}(b'(z)a(z) - a'(z)b(z)), \]
by the boundedness of \(a(z)\) and \(b(z)\), we learn that \(r(z) \in H^1(\mathbb{R})\) only if
\[ a'(z), b(z), b'(z) \in L^2(\mathbb{R}). \] (23)

Introducing
\[ Y^\pm \equiv Y^\pm(z, x) = e^{iz\sigma_3} \Phi^\pm(z; x, 0) = e^{iz\sigma_3} \phi^\pm(z; x, 0), \] (24)
by Proposition and (11), \(Y^\pm\) is bounded on \(z \in \mathbb{R}\). By (8) and (24), we get the Volterra integral for \(Y^\pm\):
\[ Y^\pm(z, x) = I + \int_{\pm\infty}^x e^{iy\sigma_3} Q(y) Y^\pm(z, y) dy, \] (25)
and by (10a), (16) and (24), we have
\[ a(z) = Y^-_{1,1}(z, x) Y^-_{1,1}(-z, -x) + Y^-_{2,1}(z, x) Y^-_{2,1}(-z, -x), \] (26a)
\[ a'(z) = \partial_z Y^-_{1,1}(z, x) Y^-_{1,1}(-z, -x) + \partial_z Y^-_{2,1}(z, x) Y^-_{2,1}(-z, -x) \]
\[ - Y^-_{1,1}(z, x) \partial_z Y^-_{1,1}(-z, -x) - Y^-_{2,1}(z, x) \partial_z Y^-_{2,1}(-z, -x), \] (26b)
\[ b(z) = Y^+_{1,1}(z, x) Y^+_{2,1}(z, x) - Y^-_{1,1}(z, x) Y^+_{2,1}(z, x), \] (26c)
\[ b'(z) = \partial_z Y^+_{1,1}(z, x) Y^+_{2,1}(z, x) - \partial_z Y^-_{1,1}(z, x) Y^+_{2,1}(z, x) \]
\[ + Y^+_{1,1}(z, x) \partial_z Y^-_{2,1}(z, x) - Y^-_{1,1}(z, x) \partial_z Y^+_{2,1}(z, x). \] (26d)
Therefore, seeing from (26), (23) is the consequence of the boundedness of \(Y^\pm\) on \(z \in \mathbb{R}\) and Lemma 4.2

**Lemma 4.2.** If \(q(x) \in L^{2-1}(\mathbb{R})\), then \(\left\{ Y^\pm_1 - \frac{1}{0}, \partial_z Y^\pm_1 \right\} \subset \mathcal{A} = L^\infty(\mathbb{R}, L^2(\mathbb{R}))\),
where for \(f(z, x) = (f_1(z, x), f_2(z, x)) \in \mathcal{A}\),
\[ \| f \|_\mathcal{A} = \sup_{x \in \mathbb{R}} \| f(\cdot, x) \|_2. \]
In this article, without confusion of notation, we will denote $L^2(\mathbb{R})$ and $(L^2(\mathbb{R}))^2$ uniformly by $L^2(\mathbb{R})$, and the norm of $f(\cdot, x) \in (L^2(\mathbb{R}))^2$ is denoted by
\[
\| f(\cdot, x) \|_2 = (\| f_1(\cdot, x) \|_2^2 + \| f_1(\cdot, x) \|_2^2)^{\frac{1}{2}}.
\]

Before the proof of Lemma 4.2 we introduce integral operators $T^\pm$ such that:
\[
(T^\pm f)(z, x) = \int_{\pm\infty}^{x} e^{iyz \sigma_3} Q(y) f(z, y) dy.
\]

Then, it can be derived from (25) that
\[
Y_1^\pm(z, x) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sum_{n=1}^{\infty} \left[ (T^\pm)^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] (z, x), \tag{27a}
\]
\[
\partial_z Y_1^\pm(z, x) = \sum_{n=1}^{\infty} \left[ (T^\pm)^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]' (z, x). \tag{27b}
\]

To obtain the result in Lemma 4.2 we estimate $A$-norm for each element of the summation appearing in the right hand of (27), which is shown in Proposition 4.3 and 4.4.

**Proposition 4.3.** If $q(x) \in L^{2, 1}(\mathbb{R})$, some ones obtain estimates for the $A$-norm of $(T^\pm)^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$:
\[
\left\| (T^\pm)^{2n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_A \leq \sqrt{\pi} \| q \|_2 \| q \|_1^{2n-2} \frac{1}{(n-1)!},
\]
\[
\left\| (T^\pm)^{2n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_A \leq \sqrt{\pi} \| q \|_2 \| q \|_1^{2n-1} \frac{1}{(n-1)!},
\]
where $n = 2, 3, \ldots$.

**Proposition 4.4.** If $q(x) \in L^{2, 1}(\mathbb{R})$, some ones obtain estimates for the $A$-norm
\[ \left[ (T^\pm)^n \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right]'_z : \]

\[
\| (T^\pm) \left( \begin{array}{c} 1 \\ 0 \end{array} \right)'_z \|_A \leq C \| q \|_{2,1},
\]

\[
\| (T^\pm)^2 \left( \begin{array}{c} 1 \\ 0 \end{array} \right)'_z \|_A \leq C \| q \|_{2,1} \| q \|_1,
\]

\[
\| (T^\pm)^{2n-1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right)'_z \|_A \leq C \| q \|_{2,1} \frac{\| q \|_{2n-2}}{(n-2)!},
\]

\[
\| (T^\pm)^{2n} \left( \begin{array}{c} 1 \\ 0 \end{array} \right)'_z \|_A \leq C \| q \|_{2,1} \frac{\| q \|_{2n-1}}{(n-2)!},
\]

where \( n = 2, 3, \ldots \) and \( C \) is some fixed positive number.

**proof of Proposition 4.3.** In functional analysis, there is an important fact that for any function \( g \in L^2(\mathbb{R}) \), the \( L^2 \)-norm of \( g \) can be written as

\[
\| g \|_2 = \sup_{h \in L^2} \int_{\mathbb{R}} g(s)\overline{h(s)}ds,
\]

which is very useful in our proof. Without loss of generality, we only check the result for \( T^- \) in detail, and remaining results is similarly obtained. By definition of \( T^- \), we have

\[
(T^-)^{2n-1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) (z, x) = \int_{-\infty}^{x} \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_{2n-2}} \left( K_{2n-1} \right) dy_{2n-1} \cdots dy_1,
\]

\[
K_{2n-1} = \prod_{k=1}^{n} (-\bar{q}(-y_{2k-1})) \prod_{k=1}^{n-1} q(y_{2k})e^{2izA_{2n-1}},
\]

\[
(T^-)^{2n} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) (z, x) = \int_{-\infty}^{x} \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_{2n-1}} \left( K_{2n} \right) dy_{2n} \cdots dy_1,
\]

\[
K_{2n} = \prod_{k=1}^{n} (-\bar{q}(-y_{2k-1})) \prod_{k=1}^{n} q(y_{2k})e^{-2izA_{2n}},
\]

where

\[
A_n = \sum_{k=1}^{n} (-1)^k y_k, \quad n = 1, 2, \ldots
\]
By (28), Schwartz inequality and Fourier transformation, we have

\[
\left\| (T^-)^{2n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_A = \sqrt{\pi} \sup_{x \in \mathbb{R}, \|f\|_2 \leq 1} \int_{-\infty}^{\infty} \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_{2n-2}} \prod_{k=1}^{n-1} (\bar{q}(-y_{2k-1})q(y_{2k})) \times
\]

\[
(-1)^n (\bar{q}(-y_{2n-1})) \hat{f}(-A_{2n-1}) dy_{2n-1} \cdots dy_1
\]

\[
\leq \sqrt{\pi} \sup_{x \in \mathbb{R}, \|f\|_2 \leq 1} \int_{-\infty}^{x} \int_{-\infty}^{y_{2n-3}} \cdots \int_{-\infty}^{y_2} \prod_{k=1}^{n-1} |\bar{q}(-y_{2k-2})q(y_{2k})| \times
\]

\[
\| q \|_2 \| \hat{f} \|_2 dy_{2n-2} \cdots dy_1
\]

\[
\leq \sqrt{\pi} \| q \|_2 \sup_{x \in \mathbb{R}} \int_{-\infty}^{x} \int_{-\infty}^{y_{n-2}} \cdots \int_{-\infty}^{y_2} \prod_{k=1}^{n-1} |q(-y_k)| dy_{n-1} \cdots dy_1 \times
\]

\[
\int_{-\infty}^{x} \cdots \int_{-\infty}^{y_{n-2}} \prod_{k=1}^{n-1} |q(y_k)| dy_{n-1} \cdots dy_1
\]

\[
\leq \sqrt{\pi} \| q \|_2 \frac{\| q \|^{2n-2}}{(n-1)!},
\]

where \( \hat{f} \) is the Fourier transformation of \( f \):

\[
\hat{f}(\zeta) = \pi^{-\frac{1}{2}} \int_{\mathbb{R}} f(z) e^{-2iz \zeta} dz.
\]

The result for \( (T^-)^{2n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) can be obtained similarly, and then the proposition is confirmed. \( \square \)

**Proof of Proposition 4.4.** Without loss of generality, we also only give the proof of this proposition for \( T^- \). Taking the derivative in (29a), it follows that

\[
\left[ (T^-)^{2n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]'_z (z, x) = \int_{-\infty}^{x} \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_{2n-2}} \begin{pmatrix} 0 \\ K'_{2n-1} \end{pmatrix} dy_{2n-1} \cdots dy_1,
\]

\[
K'_{2n-1} = 2i A_{2n-1} \left( \prod_{k=1}^{n} (\bar{q}(-y_{2k-1})) \right) \left( \prod_{k=1}^{n-1} q(y_{2k}) \right) e^{2iz A_{2n-1}},
\]

13
where \( n = 1, 2, \ldots \). We split \([ (T^-)^{2n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} ]'z\) into \(2n-1\) terms:

\[
[T^{-})^{2n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} ]'z = \sum_{j=1}^{2n-1} \begin{pmatrix} 0 \\ g_j \end{pmatrix} .
\]

(31a)

\[
g_j(z, x) = \int_{-\infty}^{x-y} \cdots \int_{-\infty}^{y_{2n-2}} \int_{-\infty}^{y_{2n-1}} 2i(-1)^j y_j \left( \prod_{k=1}^{n} q(-y_{2k-1}) \right) \times 
\]

\[
\left( \prod_{k=1}^{n-1} q(y_{2k}) \right) e^{2izA} e^{2n-1} y_j \cdots dy_{2n-1} \cdots dy_1, \quad j = 1, \ldots, 2n-1,
\]

(31b)

and \( \| g_j \|_A \) is bounded by \( \| q \|_{2,1} \) and \( \| q \|_1 \):

\[
\| g_j \|_A \leq \sqrt{\pi} \| q \|_{2,1} \frac{\| q \|_{2n-2}}{(n-1)!},
\]

which can be verified by strictly applying the technique to bound \( \| (T^-)^{2n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \|_A \) in Proposition 4.3. Then, the result for \( \| \left[ (T^-)^{2n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]'z \|_A \) immediately follows.

The result for \( \| \left[ (T^-)^{2n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]'z \|_A \) is obtained similarly.

According to control convergence theorem and the fact that

\[
\| q \|_1 \leq \| q \|_{2,1} \left( \int_{\mathbb{R}} (1 + x^2)^{-1} dx \right)^{\frac{1}{2}} \leq \sqrt{\pi} \| q \|_{2,1},
\]

we conclude from Proposition 4.3 and 4.4 that \( Y_1^+ - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \partial_2 Y_1^+ \) exist in \( \mathcal{A} \), and the \( \mathcal{A} \)-norm of them satisfy

\[
\| Y_1^+ - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \|_A \leq \sqrt{\pi} \| q \|_2 (1 + \| q \|_1) e^{\| q \|_2^2} = \sqrt{\pi} \| q \|_2 (1 + \sqrt{\pi} \| q \|_{2,1}) e^{\pi \| q \|_{2,1}^2},
\]

\[
\| \partial_2 Y_1^+ \|_A \leq C \| q \|_{2,1} \| q \|_1 e^{\| q \|_2^2} (2 + \| q \|_1 + \| q \|_2^2)
\]

\[
\leq \| q \|_{2,1}^2 (2 + \sqrt{\pi} \| q \|_{2,1} + \pi \| q \|_{2,1}^2) e^{\pi \| q \|_{2,1}^2}.
\]

Then, the result of Lemma 4.2 is verified.
4.2 $\hat{r}(z) \in H^1(\mathbb{R})$

Because of the assumption that $q(x)$ is generic and the fact that for $z \in \mathbb{R}$, by (17),
\[
\hat{r}(z) = \frac{\hat{b}(z)}{\hat{a}(z)} = -\sigma \frac{b(-z)}{a(z)},
\]
$\hat{r}(z) \in L^2(\mathbb{R})$ follows immediately after that $b(z) \in L^2(\mathbb{R})$. Also, for $\hat{r}'(z)$, we have
\[
\hat{r}'(z) = \frac{\hat{b}(-z)}{\hat{a}(z)} + \frac{\hat{a}'(z)\hat{b}(z)}{\hat{a}^2(z)},
\]
then, because of that $\hat{a}(z)$ and $\hat{b}(z)$ are bounded on $\mathbb{R}$, $b'(z) \in L^2(\mathbb{R})$ and that $q(x)$ is generic, $\hat{r}'(z) \in L^2(\mathbb{R})$ only if $\hat{a}'(z) \in L^2(\mathbb{R})$. By (16) and (24), we have
\[
\hat{a}'(z) = \partial_z Y_{1,1}^+(z, x) Y_{1,1}^+(-z, -x) + \partial_x Y_{2,1}^+(z, x) Y_{2,1}^+(-z, -x) - Y_{1,1}^+(z, x) \partial_x Y_{1,1}^+(-z, -x) - Y_{2,1}^+(z, x) \partial_x Y_{2,1}^+(-z, -x).
\]
Therefore, $\hat{a}'(z) \in L^2(\mathbb{R})$ is the consequence of (32), the boundedness of $Y^\pm$ and Lemma 4.2. To sum up, we have completed the proof of $\hat{r}(z) \in H^1(\mathbb{R})$.

5 Deformations for RH problem

In this section, we deform $M$ several time such that the final RH problem satisfies a model RH problem. From $\varphi = i(zx/t + 2z^2)$, we can get stationary phase point $\xi = -\frac{2}{4t}$, which satisfies
\[
\partial_z \varphi(\xi) = 0, \quad \partial^2_z \varphi(\xi) \neq 0.
\]

5.1 The first RH problem transformation

Introducing $\delta$:
\[
\delta \equiv \delta(z) = e^{i \int_{-\infty}^{\xi} \frac{\nu(s)}{s} ds}, \quad \nu(s) = -\frac{\log(1 - r(s)\hat{r}(s))}{2\pi},
\]
\[
\delta(z) = \frac{\hat{a}(z)}{\hat{a}(\xi)}, \quad \hat{a}(\xi) = \hat{a}(z) e^{i \int_{\xi}^{z} \frac{\nu(s)}{s} ds}, \quad \hat{a}(z) = \hat{a}(\xi) e^{-i \int_{\xi}^{z} \frac{\nu(s)}{s} ds}.
\]

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and assuming that \(|\arg(1 - r(s)\bar{r}(s))| < \pi\) for \(s \in \mathbb{R}\) to secure that \(\log(1 - r(s)\bar{r}(s))\) is single-valued, since \(\{r(z), \bar{r}(z)\} \subset H^1(\mathbb{R})\), we find it possessing properties listed in proposition 5.1.

**Proposition 5.1.** Function \(\delta\) admits the following properties:

1. \(\delta\) is analytic and non-zero on \(\mathbb{C} \setminus (-\infty, \xi]\).

2. \(\delta(z)\) and \(\delta(z)^{-1}\) is bounded on \(\mathbb{C} \setminus (-\infty, \xi]\).

3. On \((-\infty, \xi]\), \(\delta\) satisfies the jump condition:

\[
\delta_+ = (1 - r\bar{r})\delta_-.
\]

4. For any positive number \(c < \pi\), as \(z \to \infty\) and \(|\arg(z - \xi)| \leq c\),

\[
\delta(z) \sim 1 - \frac{i}{z} \int_{-\infty}^{\xi} \nu(s) ds + O(z^{-2}). \tag{34}
\]

5. If we write

\[
\delta(z) = e^{i\beta(z, \xi)}(z - \xi)^{\nu(\xi)},
\]

\[
\beta(z, \xi) = \int_{-\infty}^{\xi} \frac{\nu(s) - \chi(x)\nu(\xi)}{z - s} ds - \nu(\xi) \log(z - \xi + 1),
\]
where $\chi$ is the characterized function of the interval: $[\xi - 1, \xi]$, then, at the neighborhood of $z = \xi$, $\beta(z, \xi)$ possesses the asymptotic property:

$$|\beta(z, \xi) - \beta(\xi, \xi)| \leq C(\|r\|_{H^1} + \|r\|_{H^1})|z - \xi|^\frac{1}{2}, \quad z \to \xi.$$  

**Proof.** Property 1, 2, and 3 is trivial seeing from the definition (33). For property 4., by taking the Laurent expansion of $(1 - s/z)^{-1/2}$ at $z \to \infty$, we derive that

$$\delta(z) = e^{-\frac{i}{2} \int_{-\infty}^{\xi} \nu(s)ds + O(z^{-2})} = I_1 + I_2 + I_3,$$

For $I_1$, we have

$$\log(z - \xi + 1) = (z - \xi) + O((z - \xi)^2).$$

For $I_2$, recalling that both $r(z)$ and $\tilde{r}(z)$ are continuous and bounded on the real line, we have

$$|I_2| = |2\pi (C\nu(z) - C\nu(\xi))| = 2\pi \int_{\xi}^{z} \left(\frac{d}{ds} C\nu\right) (s)ds = 2\pi \int_{\xi}^{z} C\nu'(s)ds \leq 2\pi \|C\nu'\|_2 |z - \xi| \lesssim (\|r\|_{H^1} + \|\tilde{r}\|_{H^1})|z - \xi|^{\frac{1}{2}},$$

where $C$ is the Cauchy integral operator that is bounded from $L^2$ to $L^2$ on interval $(-\infty, \xi - 1)$:

$$Cf(z) = \frac{1}{2\pi i} \int_{-\infty}^{\xi - 1} \frac{f(s)}{s - z} ds.$$
Refer to Chapter 7 in [22] for more information about Cauchy integral operators.

For $I_3$, recalling that $r(z)$ is $\frac{1}{2}$-Hölder continuous on the real line, we apply the Cauchy integral operator on the interval $[\xi - 1, \xi]$ and have similar estimate for $I_3$:

$$|I_3| \lesssim (\|r\|_{H^1} + \|\tilde{r}\|_{H^1})|z - \xi|^{\frac{1}{2}}.$$

Finally, we complete the proof. □

Defining a $2 \times 2$ matrix function $M^{(1)} = M\delta^{-\sigma_3}$, observing RH problem 3.1 and Proposition 5.1 we obtain that $M^{(1)}$ solve the following RH problem.

**RH problem 5.2.** Find a $2 \times 2$ matrix function on $\mathbb{C} \setminus \mathbb{R}$, such that:

- **Analyticity:** $M^{(1)}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$.

- **Normalization:**
  
  $$M^{(1)} \sim I + O(z^{-1}), \quad as \quad z \to \infty.$$

- **Jump condition:**
  
  $$M_+^{(1)} = M_-^{(1)} e^{\tau \varphi \delta_3} V^{(1)}, \quad on \quad \mathbb{R},$$

  where

  $$V^{(1)} = \begin{cases} 
  \begin{pmatrix}
  1 - r\tilde{r} & -\tilde{r}\delta^2 \\
  r\delta^{-2} & 1 
  \end{pmatrix} = \begin{pmatrix}
  1 & -\tilde{r}\delta^2 \\
  0 & 1 \end{pmatrix} \begin{pmatrix}
  1 & 0 \\
  r\delta^{-2} & 1 
  \end{pmatrix} & \text{on} \ (\xi, +\infty), \\
  \begin{pmatrix}
  1 & -\tilde{r}\delta^2 \\
  r\delta^{-2} & 1 - r\tilde{r} 
  \end{pmatrix} = \begin{pmatrix}
  1 & 0 \\
  r\delta^{-2} & 1 
  \end{pmatrix} \begin{pmatrix}
  1 & -\tilde{r}\delta^2 \\
  1 & 1 \end{pmatrix} & \text{on} \ (-\infty, \xi).
  \end{cases}$$

**5.2 $\bar{\partial}$-RH problem**

Then, we make another transformation: $M^{(1)} \rightarrow M^{(2)}$, where $M^{(2)}$ admits a $\bar{\partial}$-RH problem, and deform the jump contour $\mathbb{R}$ into the contour $\Sigma$ consisting of four rays: $\Sigma_j = \xi + e^{i\pi(2j-1)}\mathbb{R}^\pm$, $j = 1, \ldots, 4$. See more detail of $\Sigma$ at Figure 3.
Lemma 5.3. We can find scalar functions $R_j : \Omega_j \to \mathbb{C}$ with the boundary condition:

\begin{align*}
R_1(z) &= \begin{cases}
r(z)\delta^{-2}(z), & z \in (\xi, +\infty), \\
r(\xi)\delta_0(\xi)^{-2}(z - \xi)^{-2i\nu(\xi)}, & z \in \Sigma_1,
\end{cases} \\
R_3(z) &= \begin{cases}
\tilde{r}(z)\delta^2(z) \\
\tilde{r}(\xi)\delta_0^2(\xi)(z - \xi)^{-2i\nu(\xi)}
\end{cases} \\
R_4(z) &= \begin{cases}
r(z)\delta^{-2}(z) \\
\tilde{r}(\xi)\delta_0^2(\xi)(z - \xi)^{-2i\nu(\xi)}
\end{cases} \\
R_6(z) &= \begin{cases}
\tilde{r}(z)\delta^2(z), & z \in (\xi, +\infty), \\
\tilde{r}(\xi)\delta_0^2(\xi)(z - \xi)^{-2i\nu(\xi)}, & z \in \Sigma_4,
\end{cases}
\end{align*}

such that for $j = 1, 3, 4, 6$,

\begin{align*}
|R_j(z)| &\leq C(\sin^2(\arg(z - \xi)) + \langle \text{Re}z \rangle^{-1/2}), \\
|\partial_j R_j(z)| &\leq C(|r'(\text{Re}z)| + |p'(\text{Re}z)| + |z - \xi|^{-1/2}),
\end{align*}

where $\langle \cdot \rangle = \sqrt{1 + (\cdot)^2}$ and $\delta_0(\xi) = e^{i\beta(\xi, \xi)}$.

Proof. In [36], without loss of generality, we only check the case of $j = 1, 3$ and the
estimates for other cases similarly follow. For $j = 1$, rewriting $z = x + iy = se^{i\psi} + \xi$ and defining
\[ f_1(z) = r(\xi)\delta^2(z)\delta_0(\xi)^{-2}(z - \xi)^{-2i\nu(\xi)}, \]
\[ R_1(z) = (r(Rez)\cos(2\psi) + f_1(z)(1 - \cos(2\psi)))\delta^{-2}(z), \]
which obviously satisfies the boundary condition (35), we can see from Proposition 5.1 that
\[ f_1(z) = r(\xi)e^{i(\beta(z,\xi) - \beta(\xi,\xi))} \]
is bounded on $\Omega_1$; therefore, with the fact that $|r(x)| \lesssim |x|^{-\frac{1}{2}}$ in (15), the inequality (36a) is verified. Because of the fact that $\bar{\partial}z = \frac{1}{2}(\partial_x + i\partial_y) = e^{i\psi}(\partial_s + is^{-1}\partial_\psi)$, it follows that
\[ |\bar{\partial}R_1(z)| = \frac{1}{2}r'(x)\cos(2\psi) + \frac{e^{i\psi}}{2} (r(x) - f_1(z))\frac{\sin(2\psi)}{|z - \xi|} ||\delta^{-2}(z)|| \]
\[ \lesssim |r'(x)| + \frac{|r(Rez) - r(\xi)| + |r(\xi) - f_1(z)|}{|z - \xi|}. \quad (37) \]
Noticing that by Proposition 5.1
\[ |r(x) - R(\xi)| = \Big| \int_x^\infty r'(s)ds \Big| \leq \| r \|_{H^1} |z - \xi|^{\frac{1}{2}}, \]
\[ |r(\xi) - f_1(z)| = |r(\xi)||1 - e^{i(\beta(z,\xi) - \beta(\xi,\xi))}| \lesssim r(\xi)||\beta(z,\xi) - \beta(\xi,\xi)| \lesssim |z - \xi|^{\frac{1}{2}}, \]
we consequently obtain the estimate for $|\bar{\partial}R_1(z)|$ from (37):
\[ |\bar{\partial}R_1(z)| \lesssim |r(x)| + |z - \xi|^{-\frac{1}{2}}. \]
For $j = 3$, defining
\[ f_3(z) = \frac{\bar{r}(\xi)}{1 - r(\xi)\bar{r}(\xi)} \delta^{-2}(z)\delta_0(\xi)^2(z - \xi)^{2i\nu(\xi)}, \]
\[ R_3(z) = \frac{\bar{r}(Rez)}{1 - r(Rez)\bar{r}(Rez)} \cos(2\psi) + f_3(z)(1 - \cos(2\psi)), \]
by similar computation, we obtain the estimate (36) for $j = 3$. \[ \square \]
Now, we construct a $2 \times 2$ matrix function $\mathcal{R}$ on $\bigcup_{j=1}^{6} \Omega_j$:

$$
\mathcal{R}(z) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & z \in \Omega_2 \cup \Omega_5, \\
\begin{pmatrix} 1 & 0 \\ -R_1(z)e^{-2t\varphi(z)} & 1 \\ R_3(z)e^{2t\varphi(z)} & 0 \\ 1 & \end{pmatrix}, & z \in \Omega_1, \\
\begin{pmatrix} 1 & 0 \\ R_4(z)e^{-2t\varphi(z)} & 0 \\ 1 & \end{pmatrix}, & z \in \Omega_3, \\
\begin{pmatrix} 1 & 0 \\ -R_6(z)e^{2t\varphi(z)} & 0 \\ 1 & \end{pmatrix}, & z \in \Omega_4, \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \end{pmatrix}, & z \in \Omega_6, 
\end{cases}
$$

and introduce the second RH problem transformation:

$$
M^{(2)} = M^{(1)} \mathcal{R},
$$

where we can see from Figure 2 that $\mathcal{R}$ decays to the unit matrix as $t \to \infty$. Seeing from RH problem 5.2, Lemma 5.3 and (39), someone obtains that $M^{(2)}$ admits the following $\bar{\partial}$-RH problem.

\textbf{$\bar{\partial}$-RH problem 5.4.} Find a $2 \times 2$ matrix function on $\mathbb{C} \setminus \Sigma$ such that:

- \textit{Continuity:} $M^{(2)} \in C^1(\mathbb{C} \setminus \Sigma)$.

- \textit{Jump condition:} On $\Sigma$,

$$
M^{(2)}_+ = M^{(2)}_- V^{(2)}, \quad V^{(2)} = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ R_1(z)e^{-2t\varphi(z)} & 1 \\ 1 & \end{pmatrix}, & z \in \Sigma_1, \\
\begin{pmatrix} 1 & 0 \\ -R_3(z)e^{2t\varphi(z)} & 0 \\ 1 & \end{pmatrix}, & z \in \Sigma_2, \\
\begin{pmatrix} 1 & 0 \\ R_4(z)e^{-2t\varphi(z)} & 0 \\ 1 & \end{pmatrix}, & z \in \Sigma_3, \\
\begin{pmatrix} 1 & 0 \\ -R_6(z)e^{2t\varphi(z)} & 0 \\ 1 & \end{pmatrix}, & z \in \Sigma_4.
\end{cases}
$$
• Normalization: \( M^{(2)} \sim I + O(z^{-1}) \) as \( z \to \infty \).

• \( \bar{\partial} \)-condition: For \( \mathbb{C} \setminus \Sigma \), we have that
  \[
  \bar{\partial}M^{(2)} = M^{(2)}R.
  \]

### 5.3 The factorization of \( \bar{\partial} \)-RH problem

For the sake of that asymptotic analysis for the \( \bar{\partial} \)-RH problem is fairly complicated, we shall factorize it into the product of \( M^{(2)}_{\text{RHP}} \) and \( M^{(3)} \)

\[
M^{(2)} = M^{(3)}M^{(2)}_{\text{RHP}} \tag{40}
\]

where \( M^{(2)}_{\text{RHP}} \equiv M^{(2)}_{\text{RHP}}(z; x, t) \) admits RH problem 5.5 and \( M^{(3)} \equiv M^{(3)}(z; x, t) \) is the solution of \( \bar{\partial} \)-problem 5.6.

**RH problem 5.5.** Find a \( 2 \times 2 \) matrix function \( M^{(2)}_{\text{RHP}} \) holomorphic on \( \mathbb{C} \setminus \Sigma \) and satisfying the normalization and jump condition of \( \bar{\partial} \)-RH problem 5.4.

**\( \bar{\partial} \)-problem 5.6.** Find a \( 2 \times 2 \) matrix function on \( \mathbb{C} \) such that:

- **Continuity:** \( M^{(3)} \in C^0(\mathbb{C}) \cap C^1(\mathbb{C} \setminus \Sigma) \).

- **Normalization:** \( M^{(3)} \sim I + O(z^{-1}) \) as \( z \to \infty \).

- **\( \bar{\partial} \)-condition:**

  \[
  \bar{\partial}M^{(3)} = M^{(3)}W,
  \]

  where \( W = M^{(2)}_{\text{RHP}} \bar{\partial}R(M^{(2)}_{\text{RHP}})^{-1} \).

**Remark 5.7.** To see the well-definedness of factorization 40, we assume the solvability of RH problem 5.5 and the existence of \( M^{(2)}_{\text{RHP}} \) that will be proven at section 5.4; then

\[
M^{(3)} = M^{(2)}(M^{(2)}_{\text{RHP}})^{-1}
\]
is well-defined. Seeing from $\bar{\partial}$-RH problem $5.4$ and RH problem $5.5$, $M^{(3)}$ does satisfy $\bar{\partial}$ problem $5.6$.

![Diagram](image)

Figure 4: The jump contour $\hat{\Sigma} = \bigcup_{j=1}^{6} \hat{\Sigma}_j$ for RH problem $5.8$ and regions $\hat{\Omega}_j$, $j = 1, \ldots, 6$

### 5.4 the model RH problem

To solve RH problem $5.5$, we first make a scaling transformation for $M^{(2)}_{\text{RHP}}$:

$$\hat{M} \equiv \hat{M}(\zeta) = M^{(2)}_{\text{RHP}}(z), \quad z = \zeta/\sqrt{8t} + \xi,$$

then, by RH problem $5.5$ and (41), $\hat{M}$ solves the RH problem:

**RH problem 5.8.** Find a $2 \times 2$ matrix function on $\mathbb{C} \setminus \hat{\Sigma}$, where $\hat{\Sigma} = \Sigma - \xi$ and see more detail about $\hat{\Sigma}$ at Figure 4, such that:

- **Analyticity:** $\hat{M}$ is holomorphic on $\mathbb{C} \setminus \hat{\Sigma}$.

- **Normalization:** $\hat{M}(\zeta) \sim I + O(1/\zeta)$, as $\zeta \to \infty$.

- **Jump condition:** On $\hat{\Sigma}$, we have

$$\hat{M}_+ = \hat{M}_- \hat{V},$$
where \( \hat{V} \equiv \hat{V}(\zeta) = V^{(2)}(\zeta/\sqrt{8t} + \zeta) \) and by basic computation, \( \hat{V} \) can be written explicitly:

\[
\hat{V}(\zeta) = e^{\frac{ix^2}{4t}} \sigma_3 \zeta^{i\nu(\xi)} \hat{V}_0,
\]

where

\[
\hat{V}_0 = \begin{cases}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, & \zeta \in \hat{\Omega}_1, \\
\begin{pmatrix}
r(\xi)(\delta_0(\xi))^{-2}(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}} & 0 \\
1 & 1 - r(\xi)\delta_0(\xi)/(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}}
\end{pmatrix}, & \zeta \in \hat{\Omega}_2, \\
\begin{pmatrix}
1 & 0 \\
-\frac{r(\xi)(\delta_0(\xi))^{-2}(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}}}{1 - r(\xi)\delta_0(\xi)/(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}}} & 1
\end{pmatrix}, & \zeta \in \hat{\Omega}_3, \\
\begin{pmatrix}
r(\xi)(\delta_0(\xi))^{-2}(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}} & 0 \\
0 & 1
\end{pmatrix}, & \zeta \in \hat{\Omega}_4, \\
\begin{pmatrix}
1 & 0 \\
r(\xi)(\delta_0(\xi))^{-2}(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}} & 1
\end{pmatrix}, & \zeta \in \hat{\Omega}_5, \\
\begin{pmatrix}
0 & 1 \\
r(\xi)(\delta_0(\xi))^{-2}(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}} & 0
\end{pmatrix}, & \zeta \in \hat{\Omega}_6.
\end{cases}
\]

Set

\[
\Psi \equiv \Psi(\zeta) = \hat{M}(\zeta)e^{\frac{ix^2}{4t}} \sigma_3 \zeta^{i\nu(\xi)} P_0,
\]

where \( P_0 \) is a 2 \times 2 matrix function that is constant on each \( \hat{\Omega}_j \) for \( j = 1, \ldots, 6 \):

\[
P_0 = \begin{cases}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, & \zeta \in \hat{\Omega}_2 \cup \hat{\Omega}_5, \\
\begin{pmatrix}
r(\xi)(\delta_0(\xi))^{-2}(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}} & 0 \\
1 & 1 - r(\xi)\delta_0(\xi)/(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}}
\end{pmatrix}, & \zeta \in \hat{\Omega}_1, \\
\begin{pmatrix}
1 & 0 \\
-\frac{r(\xi)(\delta_0(\xi))^{-2}(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}}}{1 - r(\xi)\delta_0(\xi)/(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}}} & 1
\end{pmatrix}, & \zeta \in \hat{\Omega}_3, \\
\begin{pmatrix}
r(\xi)(\delta_0(\xi))^{-2}(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}} & 0 \\
0 & 1
\end{pmatrix}, & \zeta \in \hat{\Omega}_4, \\
\begin{pmatrix}
1 & 0 \\
r(\xi)(\delta_0(\xi))^{-2}(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}} & 1
\end{pmatrix}, & \zeta \in \hat{\Omega}_5, \\
\begin{pmatrix}
0 & 1 \\
r(\xi)(\delta_0(\xi))^{-2}(8t)^{i\nu(\xi)}e^{\frac{ix^2}{4t}} & 0
\end{pmatrix}, & \zeta \in \hat{\Omega}_6.
\end{cases}
\]

then, it’s trivial to verify that \( \Psi \) admits RH problem 5.9.

**RH problem 5.9.** Find a 2 \times 2 matrix function on \( \mathbb{C} \setminus \mathbb{R} \) such that
• **Analyticity:** $Ψ$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$.

• **Normalization:** $Ψ(ζ)e^{-\frac{iπ}{4}ζ^2}ζ^{-iν(ζ)σ_3} \sim I$ as $ζ \to \infty$.

• **Jump condition:** The boundary value of $Ψ$ on $\mathbb{R}$ satisfies:

$$Ψ_+ = Ψ_- V_ξ,$$

$$V_ξ = \begin{pmatrix} 1 - r(ξ)\bar{r}(ξ) & -\bar{r}(ξ)\bar{δ}_0(ξ)^2(8t)^{-iν(ξ)}e^{-\frac{iπ}{4}ζ^2} \\ r(ξ)\bar{δ}_0(ξ)^{-2}(8t)^{iν(ξ)}e^{\frac{iπ}{4}ζ^2} & 1 \end{pmatrix}.$$

Finally, like Step 8 in [16], we get a kind of RH problem that we can change it into Weber equation and we obtain the solution in terms of parabolic cylinder functions.

**Proposition 5.10.** RH problem 5.9 is solvable and the solution $Ψ$ is given by

$$Ψ = \begin{pmatrix} Ψ_{1,1} \\ Ψ_{2,1} \\ Ψ_{1,2} \\ Ψ_{2,2} \end{pmatrix},$$

where

$$Ψ_{1,1}(ζ) = \begin{cases} e^{\frac{πν}{4} D_{-iν}(e^{-\frac{π}{4}ζ})}, & \text{Im}ζ > 0, \\ e^{-\frac{πν}{4} D_{-iν}(e^{\frac{π}{4}ζ})}, & \text{Im}ζ < 0, \end{cases} \quad Ψ_{1,2}(ζ) = \begin{cases} e^{-\frac{3πν}{4} (β_1)^{-1}}[\partial_ζ D_{iν}(e^{-\frac{3π}{4}ζ})] + \frac{ίν}{2} D_{iν}(e^{-\frac{3π}{4}ζ}), & \text{Im}ζ > 0, \\ e^{\frac{3πν}{4} (β_1)^{-1}}[\partial_ζ D_{iν}(e^{\frac{3π}{4}ζ})] + \frac{ίν}{2} D_{iν}(e^{\frac{3π}{4}ζ}), & \text{Im}ζ < 0, \end{cases}$$

$$Ψ_{2,1}(ζ) = \begin{cases} e^{\frac{πν}{4} (β_2)^{-1}}[\partial_ζ D_{-iν}(e^{-\frac{π}{4}ζ})] - \frac{ίν}{2} D_{-iν}(e^{-\frac{π}{4}ζ}), & \text{Im}ζ > 0, \\ e^{-\frac{πν}{4} (β_2)^{-1}}[\partial_ζ D_{-iν}(e^{\frac{π}{4}ζ})] - \frac{ίν}{2} D_{-iν}(e^{\frac{π}{4}ζ}), & \text{Im}ζ < 0, \end{cases} \quad Ψ_{2,2}(ζ) = \begin{cases} e^{-\frac{3πν}{4} D_{iν}(e^{-\frac{3π}{4}ζ})}, & \text{Im}ζ > 0, \\ e^{\frac{3πν}{4} D_{iν}(e^{\frac{3π}{4}ζ})}, & \text{Im}ζ < 0, \end{cases}$$

$$β_1 = \frac{\sqrt{2πe^{\frac{π}{4}ζ}e^{-\frac{π}{4}ζ}}}{σ_0Γ(-iν)}, \quad β_2 = \frac{ν}{β_1}, \quad ρ_0 = -\bar{r}(ξ)\bar{δ}_0(ξ)^2(8t)^{iν(e^{-\frac{π}{4}ζ})}e^{-\frac{iπ}{4}ζ^2}, \quad ν \equiv ν(ξ),$$

and $D_a(η)$ is a solution of the Weber equation

$$\partial^2_η D_a(η) + \left[\frac{1}{2} - \frac{η^2}{4} + a\right] D_a(η) = 0.$$
Remark 5.11. In addition to Proposition 5.10, if we write \( \hat{M} \) as

\[
\hat{M}(\zeta) = I + \zeta^{-1} \hat{M}_{-1} + \mathcal{O}(\zeta^{-2}),
\]

then

\[
\beta_1 = i(\hat{M}_{-1})_{1,2}, \quad \beta_2 = -i(\hat{M}_{-1})_{2,1}.
\]

5.5 Analysis on a \( \bar{\partial} \)-problem

In this section, according to \( \bar{\partial} \)-problem 5.6, we obtain an integral equation (44) for it; then, we make some estimates based on the integral operator \( \hat{j} \) defined by (46).

\( \bar{\partial} \)-problem 5.6 is equivalent to the integral equation:

\[
M^{(3)}(z) = I + \frac{1}{\pi} \iint_{\mathbb{C}} \frac{M^{(3)}(s)W(s)}{z - s} d^2s, \quad z \in \mathbb{C},
\]

which is equivalent to

\[
(I - \hat{j})M^{(3)} = I,
\]

where \( \hat{j} \) is an integral operator such that for a \( 2 \times 2 \) matrix function \( f \),

\[
\hat{j}(f)(z) = \frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(s)W(s)}{z - s} d^2s, \quad z \in \mathbb{C}.
\]

To derive the solvability of \( \bar{\partial} \)-problem 5.6 we prove that when \( t \) is sufficiently large, \( \| \hat{j} \|_{L^\infty \rightarrow L^\infty} \) is small and the resolvent operator \( (1 - \hat{j})^{-1} \) exists on \( L^\infty(\mathbb{C}) \).

Proposition 5.12. For \( t > 0 \), there is a constant \( C > 0 \), such that

\[
\| \hat{j} \|_{L^\infty \rightarrow L^\infty} \leq Ct^{-1/4}.
\]
Proof. We only consider the case of matrix function with support region in \( \hat{\Omega}_1 \), and cases for other regions follow in the similar way. Setting \( f(s) \in L^\infty(C) \) and \( s = u + iv \), we obtain the modular estimate for \( z = x + iy \in C \),

\[
|\mathcal{J}[f](z)| \leq \frac{1}{\pi} \iint_{\hat{\Omega}_1} \left| f(s)M^{(2)}_{\text{RHP}}(s)\partial R(s)M^{(2)}_{\text{RHP}}(s)^{-1} \right| \frac{|s - z|}{|z - s|} ds \\
\leq \| f \|_{L^\infty} \| M^{(2)}_{\text{RHP}} \|_{L^\infty} \| M^{(2)}_{\text{RHP}}^{-1} \|_{L^\infty} \iint_{\hat{\Omega}_1} \left| \partial R_1(s)|e^{-8tv(u-\xi)}|s - z|^{-1} \right| d^2s. \tag{47}
\]

Seeing from RH problem 5.5 we find that \( \det M^{(2)}_{\text{RHP}} \equiv 1 \), i.e., \( M^{(2)}_{\text{RHP}} \) is invertible; then, since \( M^{(2)}_{\text{RHP}} \) has no pole on the complex plane, \( \| M^{(2)}_{\text{RHP}} \|_{L^\infty} \) and \( \| M^{(2)}_{\text{RHP}}^{-1} \|_{L^\infty} \) are bounded; therefore, by (36) and (47), we obtain that

\[
|\mathcal{J}[f](z)| \lesssim (I_4 + I_5) \| f \|_{L^\infty}, \tag{48}
\]

where

\[
I_4 = \iint_{\hat{\Omega}_1} \frac{(|r'(u)| + |\tilde{r}'(u)|)|e^{-8tv(u-\xi)}|}{|s - z|} d^2s, \\
I_5 = \iint_{\hat{\Omega}_1} \frac{|s - \xi|^\frac{1}{2} e^{-8tv(u-\xi)} |s - z|^{-1}}{|s - z|} d^2s.
\]

Since \( r(u), \tilde{r}(u) \in H^1(\mathbb{R}) \), we have

\[
I_4 \leq \int_0^\infty e^{-tv^2} \int_{v+\xi}^\infty \frac{|r'(u)| + |\tilde{r}'(u)|}{|s - z|} ds \, du \\
\leq (\| r' \|_{L^2} + \| \tilde{r}' \|_{L^2}) \int_0^\infty e^{-tv^2} ds \left( \int_{v+\xi}^\infty |s - z|^{-2} du \right)^\frac{1}{2} \\
\lesssim \int_0^\infty \frac{e^{-tv^2}}{|v - y|^2} dv \lesssim \int_0^\infty \frac{e^{-tv^2}}{v^2} \lesssim t^{-\frac{1}{2}}. \tag{49}
\]
For the boundedness of $I_2$, by Hölder’s inequality, we have

$$I_5 \leq \int_0^\infty e^{-8tv^2} \int_0^\infty \frac{dudv}{|v^\frac{1}{2}| |s-z|} = \int_0^\infty e^{-8tv^2} \left( \int_0^\infty |s-\xi|^{-\frac{q}{2}} \right)^{\frac{1}{p}} \left( \int_0^\infty |s-z|^{-q} \right)^{\frac{1}{q}} \leq \int_0^\infty e^{-8tv^2} v^{\frac{1}{p}-\frac{1}{2}} |v-y|^{-\frac{1}{q}-1} dv. \tag{50}$$

On the one hand, the right side of (50) is obviously less than $\int_0^\infty e^{-8tv^2} dv \lesssim t^{-\frac{1}{4}}$ as $y \leq 0$; on the other hand, when $y > 0$,

$$\int_0^\infty e^{-8tv^2} v^{\frac{1}{p}-\frac{1}{2}} |v-y|^{-\frac{1}{q}-1} dv = \int_0^y e^{-8tv^2} v^{\frac{1}{p}-\frac{1}{2}} (y-v)^{\frac{1}{q}-1} dv + \int_y^\infty e^{-8tv^2} v^{\frac{1}{p}-\frac{1}{2}} (v-y)^{\frac{1}{q}-1} dv 
\leq t^{-\frac{1}{4}} \int_0^1 e^{-8tv^2(y^2+g^2)^{\frac{1}{2}}} v^{\frac{1}{p}-1} (1-v)^{\frac{1}{q}-1} dv + \int_0^\infty e^{-8tv^2} v^{\frac{1}{p}-\frac{1}{2}} dv \lesssim t^{-\frac{1}{4}},$$

which yield

$$I_5 \lesssim t^{-\frac{1}{4}}. \tag{51}$$

With (48, 49, 51), the result is verified. \qed

It’s a consequence of (45) and Proposition 5.12 that for $t > 0$ large enough, the $L^\infty$-norm of $M^{(3)}$ is bounded. Since we have confirmed the solvability of $\partial$-problem for large $t$, we now determine the long-time asymptotic behavior of the second coefficient in the Laurent expansion for $M^{(3)}$:

$$M^{(3)}(z) = I + M_{-1}^{(3)} z^{-1} + O(z^{-2}),$$
$$M_{-1}^{(3)} = \frac{1}{\pi} \int_C M^{(3)}(s) W(s) d^2 s. \tag{52}$$

**Proposition 5.13.** For $t > 0$, the coefficient $M_{-1}^{(3)}$ satisfies that

$$|M_{-1}^{(3)}| \lesssim t^{-\frac{3}{2}}.$$
**Proof.** To bound the norm, without loss of generality, we consider the integral region in (52) as \( \Omega_1 \); then, by the boundedness of \( \| M^{(3)} \|_{L^\infty} \), \( \| M^{(2)} \|_{L^\infty} \) and \( \| M^{(2)}_{\text{RHP}} \|^{-1}_{L^\infty} \), we obtain

\[
|M^{(3)}_{-1}| \leq \frac{1}{\pi} \| M^{(3)} \|_{L^\infty} \| M^{(2)} \|_{L^\infty} \| M^{(2)}_{\text{RHP}} \|^{-1}_{L^\infty} \int_{\Omega_1} |\partial R(s)| d^2 s
\]

\[\lesssim \int_{\Omega_1} |\partial R_1(s) e^{-2t\phi(s)}| d^2 s \leq I_6 + I_7, \tag{53}\]

where

\[I_6 = \int_{\Omega_1} (||r'(u)| + |r''(u)||) e^{-8t\phi(u-\xi)} dudu, \quad I_7 = \int_{\Omega_1} |s - \xi|^{-\frac{1}{2}} e^{-8t\phi(u-\xi)} dudu.\]

For the boundedness of \( I_3 \) and \( I_4 \), by Schwartz inequality and variable substitutions,

\[I_6 = \int_0^\infty e^{-8tuv} \int_{\xi + v}^\infty (||r'(u)| + |r''(u)||) e^{-8t\phi(u-\xi)} dudu \]

\[\leq \int_0^\infty e^{-8tuv} (||r'||_{L^2} + ||r''||_{L^2}) \left( \int_0^\infty e^{-16tuv} dudv \right)^{\frac{1}{2}} dv \]

\[\lesssim \int_0^\infty e^{-8tuv} \frac{1}{4\sqrt{tv}} dv \lesssim t^{-\frac{1}{4}},\]

\[I_7 = \int_0^\infty e^{-8tuv} \int_{\xi + v}^\infty ((u - \xi)^2 + v^2)^{-\frac{1}{2}} e^{-8t\phi(u-\xi)} dudu \]

\[\leq \int_0^\infty e^{-8tuv} \left( \int_v^\infty (u^2 + v^2)^{-1} du \right)^{\frac{1}{2}} \left( \int_0^\infty e^{-16tuv} du \right)^{\frac{1}{2}} dv \]

\[\lesssim \int_0^\infty e^{-8tuv} t^{-\frac{1}{2}} v^{-1} dv \lesssim t^{-\frac{1}{4}},\]

which combined with (53) yield the result. \( \square \)

6 Long-time asymptotics of the NNLS equation

By the matrix transformation in section 5 we learn that

\[M(z) = M^{(3)}(z) M(\sqrt{\delta}(z - \xi)) R^{-1} \delta(z)^{\sigma_3},\]

29
then, taking $z \to \infty$ for $z \in \Omega_2$ and using (34), (38) and (43), we get the asymptotic property of $M$

$$M \sim I + \left( M^{(3)}_{-1} + \frac{\dot{M}_{-1}}{\sqrt{8t}} - \left( i \int_{-\infty}^{\xi} \nu(s) ds \right)^{\sigma_3} \right) z^{-1} + O(z^{-2}),$$

which combined with (22), Remark 5.11 and Proposition 5.13 yields that when

$$\arg(1 - r(\bar{r}(\xi))) > -\frac{\pi}{2},$$

i.e., $\operatorname{Im}\nu(\xi) < \frac{1}{4}$,

$$q(x,t) = 2 \left( \beta_1 \frac{1}{\sqrt{8t}} + i(M^{(3)}_{1,2}) \right) = \alpha(\xi) t^{\frac{1-\nu}{2}} + O(t^{-\frac{3}{2}}),$$

$$\alpha(\xi) = -\frac{\sqrt{\pi} e^{-\frac{\pi}{4} + \frac{\pi}{4} + 4it \xi^2}}{\bar{r}(\xi) 8^{-\nu} \delta_0(\xi) 2 \Gamma(-i\nu)}.$$
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