A REPRESENTATION PROBLEM FOR SMOOTH SUMS OF RIDGE FUNCTIONS

Rashid A. Aliev
Institute of Mathematics and Mechanics, NAS of Azerbaijan, Baku, Azerbaijan
Baku State University, Baku, Azerbaijan
e-mail: aliyevrashid@mail.ru

Vugar E. Ismailov
Institute of Mathematics and Mechanics, NAS of Azerbaijan, Baku, Azerbaijan
e-mail: vugaris@mail.ru

Abstract. In this paper we prove that if a function of a certain smoothness class is represented by a sum of arbitrarily behaved ridge functions, then it can be represented by a sum of ridge functions of the same smoothness class. This provides an answer to the question of A. Pinkus raised in his monograph “Ridge Functions, Cambridge Tracts in Mathematics 205, Cambridge University Press, 2015”.

Mathematics Subject Classification: 26B40, 39B22.
Keywords: ridge function; Cauchy functional equation; difference property; polynomial function.

1. Introduction

This paper gives an answer to the following open question raised in Buhmann and Pinkus [5], and Pinkus [30, p. 14]. Assume we are given a function \( f(x) = f(x_1, ..., x_n) \) of the form

\[
f(x) = \sum_{i=1}^{r} f_i(a^i \cdot x),
\]

(1.1)

where the \( a^i, i = 1, ..., r, \) are pairwise linearly independent vectors (directions) in \( \mathbb{R}^n \), \( f_i \) are arbitrarily behaved univariate functions and \( a^i \cdot x \) are standard inner products. Assume, in addition, that \( f \) is of a certain smoothness class, that is, \( f \in C^k(\mathbb{R}^n) \), where \( k \geq 0 \) (with the convention that \( C^0(\mathbb{R}^n) = C(\mathbb{R}^n) \)). Is it true that there will be always exist \( g_i \in C^k(\mathbb{R}) \) such that

\[
f(x) = \sum_{i=1}^{r} g_i(a^i \cdot x)?
\]

(1.2)

Functions of the form \( g(a \cdot x) \), involved in the right hand sides of (1.1) and (1.2), are called ridge functions. These functions appear in various fields and under various guises. They appear in partial differential equations (where they are called plane waves, see, e.g., [16]), in computerized tomography (see, e.g., [23, 29]), in statistics (especially,
in the theory of projection pursuit and projection regression; see, e.g., [10, 11]. Ridge functions are also the underpinnings of many central models in neural networks which has become increasingly more popular in the last few decades in many fields of science and engineering (see [32] and a great deal of references therein). Finally, these functions are used in modern approximation theory as an effective and convenient tool for approximating complicated multivariate functions (see, e.g., [13, 14, 15, 19, 24, 26]). We refer the reader to the monograph by Pinkus [30] for a detailed and systematic study of ridge functions.

Note that the functions involved in (1.4) are bivariate ridge functions with the directions \((1,0), (0,1)\) and \((1,1)\), respectively. This example shows that for \(r \geq 3\) the functions \(f_i\) in (1.1) may not inherit smoothness properties of the function \(f\). Thus the above problem arises naturally.

However, it was shown by some authors that, additional conditions on \(f\) or the directions \(a^i\) guarantee smoothness of the representation (1.1). It was first proved by Buhmann and Pinkus [5] that if in (1.1) \(f \in C^k(\mathbb{R}^n), k \geq r - 1\) and \(f_i \in L^1_{\text{loc}}(\mathbb{R})\) for each \(i\), then \(f_i \in C^k(\mathbb{R})\) for \(i = 1, \ldots, r\). Later Pinkus [31] found a strong relationship between CFE and the problem of smoothness in ridge function representation. He generalized extensively the previous result of Buhmann and Pinkus [5]. He showed that the solution is quite simple and natural if the functions \(f_i\) are taken from a certain class \(\mathcal{B}\) of real-valued functions \(u\) defined on \(\mathbb{R}\). By definition, \(u\) is in \(\mathcal{B}\) if for any function \(v \in C(\mathbb{R})\) for which \(u - v\) satisfies CFE, \(u - v\) is linear, i.e. \(u(x) - v(x) = cx\), where \(c \in \mathbb{R}\) (see [31]). The result of Pinkus states that if in (1.1) \(f \in C^k(\mathbb{R}^n)\) and each \(f_i \in \mathcal{B}\), then necessarily \(f_i \in C^k(\mathbb{R})\)
for \( i = 1, \ldots, r \). Severe restrictions on the directions \( a^i \) also guarantee smoothness of the representation \((1.1)\). For example, in \([17]\) it was easily proven that in \((1.1)\) the inclusions 
\[ f_i \in C^k(\mathbb{R}), \quad i = 1, \ldots, r, \]
are automatically valid if the directions \( a^i \) are linearly independent and if these directions are not linearly independent, then there exists \( f \in C^k(\mathbb{R}^n) \) of the form \((1.1)\) such that the \( f_i \notin C^k(\mathbb{R}) \), \( i = 1, \ldots, r \).

The above result of Pinkus was a starting point for further research on continuous sums of ridge functions. Much work in this direction was done by Konyagin and Kuleshov \([17, 18]\), and Kuleshov \([22]\). They mainly analyze the continuity of \( f_i \), that is, the question of if and when continuity of \( f \) guarantees the continuity of \( f_i \). There are also other results concerning different properties, rather than continuity, of \( f_i \). Most results in \([17, 18, 22]\) involve certain subsets (convex open sets, convex bodies, etc.) of \( \mathbb{R}^n \) instead of only \( \mathbb{R}^n \) itself.

In \([3]\), we gave a partial solution to the above representation problem. Our solution comprises the cases in which \( k \geq 1 \) and \( r - 1 \) directions of the given \( r \) directions are linearly independent. For bivariate functions having degree of smoothness \( k \geq r - 2 \), the problem was completely solved in \([4]\).

Kuleshov \([21]\) generalized our result \([3, \text{Theorem 2.3}]\) to all possible cases of \( k \). That is, he proved that if a function \( f \in C^k(\mathbb{R}^n) \), where \( k \geq 0 \), is of the form \((1.1)\) and \((r - 1)\)-tuple of the given set of \( r \) directions \( a^i \) forms a linearly independent system, then there exist \( g_i \in C^k(\mathbb{R}) \), \( i = 1, \ldots, r \), such that \((1.2)\) holds (see \([21, \text{Theorem 3}]\)). In \([2]\), we reproved this result using completely different ideas. Note that our proof contains a theoretical method for constructing the functions \( g_i \in C^k(\mathbb{R}) \) in \((1.2)\) (see \([2, \text{Theorem 2.1, Theorem 2.2}]\)). Using this method, we also estimated the modulus of continuity of \( f_i \) in terms of the modulus of continuity of \( f \) (see \([2, \text{Remark 2}]\)).

In this paper, based on the theory of polynomial functions (see \([20, \text{Section 15.9}]\)), we give a complete solution to the above representation problem. That is, we show that if \((1.1)\) holds for \( f \in C^k(\mathbb{R}^n) \) and arbitrarily behaved \( f_i \), then there exist \( g_i \in C^k(\mathbb{R}) \) such that \((1.2)\) is valid.

2. Polynomial functions of k-th order

Given \( h_1, \ldots, h_k \in \mathbb{R} \), we define inductively the difference operator \( \Delta_{h_1 \ldots h_k} \) as follows

\[
\Delta_{h_1} f(x) : = f(x + h_1) - f(x),
\Delta_{h_1 \ldots h_k} f : = \Delta_{h_k} (\Delta_{h_1 \ldots h_{k-1}} f), \quad f : \mathbb{R} \to \mathbb{R}.
\]

If \( h_1 = \ldots = h_k = h \), then we write briefly \( \Delta_{h^n} f \) instead of \( \Delta_{h \ldots h} f \). For various properties of difference operators see \([20, \text{Section 15.1}]\).
Definition 2.1 (see [20]). A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is called a polynomial function of order \( k \) \((k \in \mathbb{N})\) if for every \( x \in \mathbb{R} \) and \( h \in \mathbb{R} \) we have
\[
\Delta^{k+1}_h f(x) = 0.
\]

It can be shown that if \( \Delta^{k+1}_h f = 0 \) for any \( h \in \mathbb{R} \), then \( \Delta^{h_1 \cdots h_{k+1}} f = 0 \) for any \( h_1, \ldots, h_{k+1} \in \mathbb{R} \) (see [20, Theorem 15.3.3]). A polynomial of degree at most \( k \) is a polynomial function of order \( k \) (see [20, Theorem 15.9.4]). The polynomial functions generalize ordinary polynomials, and reduce to the latter under mild regularity assumptions. For example, if a polynomial function is continuous at one point, or bounded on a set of positive measure, then it continuous at all points (see [8, 23]), and therefore is a polynomial of degree \( k \) (see [20, Theorem 15.9.4]).

Basic results concerning polynomial functions are due to S. Mazur-W. Orlicz [27], McKiernan [28], Djoković [9]. The following theorem, which we will use in the sequel, yield implicitly the general construction of polynomial functions.

Theorem 2.1 (see [20, Theorems 15.9.1 and 15.9.2]). A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a polynomial function of order \( k \) if and only if it admits a representation
\[
f = f_0 + f_1 + \ldots + f_k,
\]
where \( f_0 \) is a constant and \( f_i : \mathbb{R} \rightarrow \mathbb{R}, 1, \ldots, k, \) are diagonalizations of \( i \)-additive symmetric functions \( F_i : \mathbb{R}^i \rightarrow \mathbb{R}, i.e.,
\[
f_i(x) = F_i(x, \ldots, x).
\]

Note that a function \( F_p : \mathbb{R}^p \rightarrow \mathbb{R} \) is called \( p \)-additive if for every \( i, 1 \leq i \leq p, \) and for every \( x_1, \ldots, x_p, y_i \in \mathbb{R} \)
\[
F(x_1, \ldots, x_i + y_i, \ldots, x_p) = F(x_1, \ldots, x_p) + F(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_p),
\]
i.e., \( F \) is additive in each of its variables \( x_i \) (see [20, p. 363]). A simple example of a \( p \)-additive function is given by the product
\[
f_1(x_1) \times \ldots \times f_p(x_p),
\]
where the univariate functions \( f_i, i = 1, \ldots, p, \) are additive.

Following de Bruijn, we say that a class \( \mathcal{D} \) of real functions has the difference property if any function \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \Delta_h f \in \mathcal{D} \) for all \( h \in \mathbb{R} \) admits a decomposition \( f = g + S, \) where \( g \in \mathcal{D} \) and \( S \) satisfies the Cauchy Functional Equation (1.3). Several classes with the difference property are investigated in de Bruijn [6, 7]. Some of these classes are:

1) \( C^k(\mathbb{R}) \), functions with continuous derivatives up to order \( k \);
2) \( C^\infty(\mathbb{R}) \), infinitely differentiable functions;
3) analytic functions;
4) functions which are absolutely continuous on any finite interval;
5) functions having bounded variation over any finite interval;
6) algebraic polynomials;
7) trigonometric polynomials;
8) Riemann integrable functions.

A natural generalization of classes with the difference property are classes of functions
with the difference property of \( k \)-th order.

**Definition 2.2** (see [12]). A class \( \mathcal{F} \) is said to have the difference property of \( k \)-th order if any function \( f : \mathbb{R} \to \mathbb{R} \) such that \( \Delta_h^k f \in \mathcal{F} \) for all \( h \in \mathbb{R} \), admits a decomposition \( f = g + H \), where \( g \in \mathcal{F} \) and \( H \) is a polynomial function of \( k \)-th order.

It is not difficult to see that the class \( \mathcal{F} \) has the difference property of first order if and only if it has the difference property in de Bruijn’s sense. There arises a natural question: which of the classes above have difference properties of higher orders? Gajda [12] considered this question in its general form, for functions defined on a locally compact Abelian group and showed that for any \( k \in \mathbb{N} \), continuous functions have the difference property of \( k \)-th order (see [12, Theorem 4]). The proof of this result is based on several lemmas, in particular, on the following lemma, which we will also use in the sequel.

**Lemma 2.1.** (see [12, Lemma 5]). For each \( k \in \mathbb{N} \) the class of all continuous functions defined on \( \mathbb{R} \) has the difference property of \( k \)-th order.

In fact, Gajda [12] proved this lemma for Banach space valued functions, but the simplest case with the space \( \mathbb{R} \) has all difficulties. Unfortunately, the proof of the lemma has an essential gap. The author of [12] tried to reduce the proof to mod 1 periodic functions, but made a mistake in proving the continuity of the difference \( \Delta_{h_1 \ldots h_{k-1}} (f - f^*) \).

Here \( f^* : \mathbb{R} \to \mathbb{R} \) is a mod 1 periodic function defined on the interval \([0, 1)\) as \( f^*(x) = f(x) \) and extended to the whole \( \mathbb{R} \) with the period 1. That is, \( f^*(x) = f(x) \) for \( x \in [0, 1) \) and \( f^*(x + 1) = f^*(x) \) for \( x \in \mathbb{R} \). In the proof, the author of [12] takes a point \( x \in [m, m + 1) \) and writes that

\[
\Delta_{h_1 \ldots h_{k-1}} (f - f^*)(x) = \Delta_{h_1 \ldots h_{k-1}} (f(x) - f(x - m)),
\]

which is not valid. Even though \( f^*(x) = f(x - m) \) for any \( x \in [m, m + 1) \), the differences \( \Delta_{h_1 \ldots h_{k-1}} f(x) \) and \( \Delta_{h_1 \ldots h_{k-1}} f(x - m) \) are completely different, since the latter may involve values of \( f \) at points outside \([0, 1)\), which have no relationship with the definition of \( f^* \).

In the next section, we give a new proof for Lemma 2.1 (see Theorem 3.1). We wish to hope that our proof is free from mathematical errors and thus the above lemma itself is valid.
3. Some auxiliary results on polynomial functions

In this section, we do further research on polynomial functions and prove some auxiliary results.

**Lemma 3.1.** If \( f : \mathbb{R} \to \mathbb{R} \) is a polynomial function of order \( k \), then for any \( p \in \mathbb{N} \) and any fixed \( \xi_1, ..., \xi_p \in \mathbb{R} \), the function
\[
g(x_1, ..., x_p) = f(\xi_1 x_1 + ... + \xi_p x_p),
\]
considered on the \( p \) dimensional space \( \mathbb{Q}^p \) of rational vectors, is an ordinary polynomial of degree at most \( k \).

**Proof.** By Theorem 2.1,
\[
f = \sum_{m=0}^{k} f_m,
\]
where \( f_0 \) is a constant and \( f_m : \mathbb{R} \to \mathbb{R} \), \( 1, ..., m \), are diagonalizations of \( m \)-additive symmetric functions \( F_m : \mathbb{R}^m \to \mathbb{R} \), i.e.,
\[
f_m(x) = F_m(x, ..., x).
\]
For a \( m \)-additive function \( F_m \) the equality
\[
F_m(\xi_1, ..., \xi_{i-1}, r\xi_i, \xi_{i+1}, ..., \xi_m) = rF_m(\xi_1, ..., \xi_m)
\]
holds for all \( i = 1, ..., m \) and any \( r \in \mathbb{Q} \), \( \xi_i \in \mathbb{R} \), \( i = 1, ..., m \) (see [20, Theorem 13.4.1]). Using this, it is not difficult to verify that for any \( (x_1, ..., x_p) \in \mathbb{Q}^p \),
\[
f_m(\xi_1 x_1 + ... + \xi_p x_p) = F_m(\xi_1 x_1 + ... + \xi_p x_p, ..., \xi_1 x_1 + ... + \xi_p x_p) = \sum_{0 \leq s_i \leq m, s_1 + ... + s_p = m} A_{s_1...s_p} F(\xi_1, ..., \xi_1, ..., \xi_p, ..., \xi_p) x_1^{s_1} ... x_p^{s_p}.
\]
Here \( A_{s_1...s_p} \) are some coefficients, namely \( A_{s_1...s_p} = m!/(s_1!...s_p!) \). Considering the last formula in (3.1), we conclude that the function \( g(x_1, ..., x_p) \), restricted to \( \mathbb{Q}^p \), is a polynomial of degree at most \( k \).

**Lemma 3.2.** Assume \( f \) is a polynomial function of order \( k \). Then there exists a polynomial function \( H \) of order \( k + 1 \) such that \( H(0) = 0 \) and
\[
f(x) = H(x + 1) - H(x).
\]

**Proof.** Consider the function
\[ H(x) := xf(x) + \sum_{i=1}^{k} (-1)^i \frac{x(x+1)...(x+i)}{(i+1)!} \Delta_i^1 f(x). \] (3.3)

Clearly, \( H(0) = 0 \). We are going to prove that \( H \) is a polynomial function of order \( k + 1 \) and satisfies (3.2).

Let us first show that for any polynomial function \( g \) of order \( m \) the function \( G_1(x) = xg(x) \) is a polynomial function of order \( m + 1 \). Indeed, for any \( h_1, ..., h_{m+2} \in \mathbb{R} \) we can write that

\[ \Delta_{h_1...h_{m+2}} G_1(x) = (x + h_1 + ... + h_{m+2}) \Delta_{h_1...h_{m+2}} g(x) + \sum_{i=1}^{m+2} h_i \Delta_{h_1...h_{i-1}h_{i+1}...h_{m+2}} g(x). \] (3.4)

The last formula is verified directly by using the known product property of differences, that is, the equality

\[ \Delta (g_1g_2) = g_1 \Delta g_2 + g_2 \Delta g_1 + \Delta g_1 \Delta g_2. \] (3.5)

Now since \( g \) is a polynomial function of order \( m \), all summands in (3.4) is equal to zero; hence we obtain that \( G_1(x) \) is a polynomial function of order \( m + 1 \). By induction, we can prove that the function \( G_p(x) = x^p g(x) \) is a polynomial function of order \( m + p \). Since \( \Delta_1^1 f(x) \) in (3.3) is a polynomial function of order \( k - i \), it follows that all summands in (3.3) are polynomial functions of order \( k + 1 \). Therefore, \( H(x) \) is a polynomial function of order \( k + 1 \).

Now let us prove (3.2). Considering the property (3.5) in (3.3) we can write that

\[ \Delta_1 H(x) = [f(x) + (x + 1) \Delta_1 f(x)] \]

\[ + \sum_{i=1}^{k} (-1)^i \left[ \frac{(x+1)...(x+i+1)}{(i+1)!} \Delta^{i+1} f(x) + \Delta_1 \left( \frac{x(x+1)...(x+i)}{(i+1)!} \right) \Delta_i^1 f(x) \right]. \] (3.6)

Note that in (3.6)

\[ \Delta_1 \left( \frac{x(x+1)...(x+i)}{(i+1)!} \right) = \frac{(x+1)...(x+i)}{i!}. \]

Considering this and the assumption \( \Delta_1^{k+1} f(x) = 0 \), it follows from (3.6) that

\[ \Delta_1 H(x) = f(x), \]

that is, (3.2) holds.

The next lemma is due to Gajda [12].
Lemma 3.3 (see [12, Corollary 1]). Let \( f : \mathbb{R} \to \mathbb{R} \) be a mod 1 periodic function such that, for any \( h_1, \ldots, h_k \in \mathbb{R} \), \( \Delta_{h_1 \ldots h_k} f \) is continuous. Then there exist a continuous function \( g : \mathbb{R} \to \mathbb{R} \) and a polynomial function \( H \) of \( k \)-th order such that \( f = g + H \).

The following theorem generalizes de Bruijn’s theorem (see [6, Theorem 1.1]) on the difference property of continuous functions and shows that Gajda’s above lemma (see Lemma 2.1) is valid. Note that the main result of [12] also uses this theorem.

**Theorem 3.1.** Assume for any \( h_1, \ldots, h_k \in \mathbb{R} \), the difference \( \Delta_{h_1 \ldots h_k} f(x) \) is a continuous function of the variable \( x \). Then there exist a function \( g \in C(\mathbb{R}) \) and a polynomial function \( H \) of \( k \)-th order with the property \( H(0) = 0 \) such that

\[ f = g + H. \]

**Proof.** We prove this theorem by induction. For \( k = 1 \), the theorem is the result of de Bruijn: if \( f \) is such that, for each \( h \), \( \Delta_h f(x) \) is a continuous function of \( x \), then it can be written in the form \( g + H \), where \( g \) is continuous and \( H \) is additive (that is, satisfies the Cauchy Functional Equation). Assume that the theorem is valid for \( k - 1 \). Let us prove it for \( k \). Without loss of generality we may assume that \( f(0) = f(1) \). Otherwise, we can prove the theorem for \( f_0(x) = f(x) - [f(1) - f(0)] x \) and then automatically obtain its validity for \( f \).

Consider the function

\[ F_1(x) = f(x + 1) - f(x), \quad x \in \mathbb{R}. \quad (3.7) \]

Since for any \( h_1, \ldots, h_k \in \mathbb{R}, \Delta_{h_1 \ldots h_k} f(x) \) is a continuous function of \( x \) and \( \Delta_{h_1 \ldots h_{k-1}} F_1 = \Delta_{h_1 \ldots h_{k-1}} f \), the difference \( \Delta_{h_1 \ldots h_{k-1}} F_1(x) \) will be a continuous function of \( x \), as well. By assumption, there exist a function \( g_1 \in C(\mathbb{R}) \) and a polynomial function \( H_1 \) of \((k - 1)\)-th order with the property \( H_1(0) = 0 \) such that

\[ F_1 = g_1 + H_1. \quad (3.8) \]

It follows from Lemma 3.2 that there exists a polynomial function \( H_2 \) of order \( k \) such that \( H_2(0) = 0 \) and

\[ H_1(x) = H_2(x + 1) - H_2(x). \quad (3.9) \]

Considering (3.9) in (3.8) we obtain that

\[ F_1(x) = g_1(x) + H_2(x + 1) - H_2(x). \quad (3.10) \]

It follows from (3.7) and (3.10) that

\[ g_1(x) = [f(x + 1) - H_2(x + 1)] - [f(x) - H_2(x)]. \quad (3.11) \]

Consider the function
\[ F_2 = f - H_2. \] (3.12)

Since \( H_2 \) is a polynomial function of order \( k \) and for any \( h_1, \ldots, h_k \in \mathbb{R} \) the difference \( \Delta_{h_1 \ldots h_k} f(x) \) is a continuous function of \( x \), we obtain that \( \Delta_{h_1 \ldots h_k} F_2(x) \) is also a continuous function of \( x \). In addition, since \( f(0) = f(1) \) and \( H_2(0) = H_2(1) = 0 \), it follows from (3.12) that \( F_2(0) = F_2(1) \). We will use these properties of \( F_2 \) below.

Let us write (3.11) in the form

\[ g_1(x) = F_2(x + 1) - F_2(x), \] (3.13)

and define the following mod 1 periodic function

\[
F^*(x) = F_2(x) \text{ for } x \in [0, 1), \\
F^*(x + 1) = F^*(x) \text{ for } x \in \mathbb{R}.
\]

Consider the function

\[ F = F_2 - F^*. \] (3.14)

Let us show that \( F \in C(\mathbb{R}) \). Indeed since \( F(x) = 0 \) for \( x \in [0, 1) \), \( F \) is continuous on \((0, 1)\). Consider now the interval \([1, 2)\). For any \( x \in [1, 2) \) by the definition of \( F^* \) and (3.13) we can write that

\[ F(x) = F_2(x) - F_2(x - 1) = g_1(x - 1). \] (3.15)

Since \( g_1 \in C(\mathbb{R}) \), it follows from (3.15) that \( F \) is continuous on \((1, 2)\). Note that by (3.13) \( g_1(0) = 0 \); hence \( F(1) = g_1(0) = 0 \). Since \( F \equiv 0 \) on \([0, 1)\), \( F(1) = 0 \) and \( F \in C([1, 2)) \), we obtain that \( F \) is continuous on \((0, 2)\). Consider the interval \([2, 3)\). For any \( x \in [2, 3) \) we can write that

\[ F(x) = F_2(x) - F_2(x - 2) = g_1(x - 1) + g_1(x - 2). \] (3.16)

Since \( g_1 \in C(\mathbb{R}) \), \( F \) is continuous on \((2, 3)\). Note that by (3.15) \( \lim_{x \to 2^-} F(x) = g_1(1) \) and by (3.16) \( F(2) = g_1(1) \). We obtain from these arguments that \( F \) is continuous on \((0, 3)\). By the same way, we can prove that \( F \) is continuous on \((0, m)\) for any \( m \in \mathbb{N} \).

Similar arguments can be used to prove the continuity of \( F \) on \((-m, 0)\) for any \( m \in \mathbb{N} \). We show it for the first interval \([-1, 0)\). For any \( x \in [-1, 0) \) by the definition of \( F^* \) and (3.13) we can write that

\[ F(x) = F_2(x) - F_2(x + 1) = -g_1(x). \]

Since \( g_1 \in C(\mathbb{R}) \), it follows that \( F \) is continuous on \((-1, 0)\). Besides, \( \lim_{x \to 0^-} F(x) = -g_1(0) = 0 \). This shows that \( F \) is continuous on \((-1, 1)\), since \( F \equiv 0 \) on \([0, 1)\). Combining all the above arguments we conclude that \( F \in C(\mathbb{R}) \).
Since \( F \in C(\mathbb{R}) \) and \( \Delta_{h_1...h_k}F_2(x) \) is a continuous function of \( x \), we obtain from (3.14) that \( \Delta_{h_1...h_k}F^*(x) \) is also a continuous function of \( x \). By Lemma 3.3, there exist a function \( g_2 \in C(\mathbb{R}) \) and a polynomial function \( H_3 \) of order \( k \) such that

\[
F^* = g_2 + H_3. \tag{3.17}
\]

It follows from (3.12), (3.14) and (3.17) that

\[
f = F + g_2 + H_2 + H_3. \tag{3.18}
\]

Introduce the notation

\[
H(x) = H_2(x) + H_3(x) - H_3(0),
\]
\[
g(x) = F(x) + g_2(x) + H_3(0).
\]

Obviously, \( g \in C(\mathbb{R}) \) and \( H(0) = 0 \). It follows from (3.18) and the above notation that

\[
f = g + H.
\]

This completes the proof of the theorem.

4. Ridge function representation

We start this section with the following lemma.

**Lemma 4.1.** Assume we are given pairwise linearly independent vectors \( a^i, i = 1, ..., k \), and a function \( f \in C(\mathbb{R}^n) \) of the form

\[
f(x) = \sum_{i=1}^k f_i(a^i \cdot x), \tag{4.1}
\]

where \( f_i \) are arbitrarily behaved univariate functions. Then for any \( h_1, ..., h_{k-1} \in \mathbb{R} \), and all indices \( i = 1, ..., k \), \( \Delta_{h_1...h_{k-1}}f_i \in C(\mathbb{R}) \).

**Proof.** We prove this lemma for the function \( f_k \). It can be proven for the other functions \( f_i \) by the same way. Let \( h_1, ..., h_{k-1} \in \mathbb{R} \) be given. Since the vectors \( a^i \) are pairwise linearly independent, for each \( j = 1, ..., k-1 \), there is a vector \( b^j \) such that \( b^j \cdot a^i = 0 \) and \( b^j \cdot a^k \neq 0 \). It is not difficult to see that for any \( \lambda \in \mathbb{R} \), \( \Delta_{\lambda b^j}f_j(a^j \cdot x) = 0 \). Therefore, for any \( \lambda_1, ..., \lambda_{k-1} \in \mathbb{R} \), we obtain from (4.1) that

\[
\Delta_{\lambda_1 b^1...\lambda_{k-1} b^{k-1}}f(x) = \Delta_{\lambda_1 b^1...\lambda_{k-1} b^{k-1}}f_k(a^k \cdot x). \tag{4.2}
\]

Note that in multivariate setting the difference operator \( \Delta_{h_1...h_k}f(x) \) is defined similarly as in Section 2. If in (4.2) we take
\[ x = \frac{a^k}{\|a^k\|^2} t, \quad t \in \mathbb{R}, \]
\[ \lambda_j = \frac{h_j}{a^k \cdot b^j}, \quad j = 1, ..., k - 1, \]
we will obtain that \( \Delta_{h_1 \ldots h_{k-1}}f_k \in C(\mathbb{R}) \).

Our main result is the following theorem.

**Theorem 4.1.** Assume a function \( f \in C(\mathbb{R}^n) \) is of the form (4.1). Then there exist continuous functions \( g_i : \mathbb{R} \to \mathbb{R}, \ i = 1, ..., k, \) such that

\[ f(x) = \sum_{i=1}^{k} g_i(a^i \cdot x). \]  

(4.2)

**Proof.** By Lemma 4.1 and Theorem 3.1, for each \( i = 1, ..., k, \) there exist a function \( g^*_i \in C(\mathbb{R}) \) and a polynomial function \( H_i \) of \((k - 1)\)-th order with the property \( H_i(0) = 0 \) such that

\[ f_i = g^*_i + H_i. \]  

(4.3)

Consider the function

\[ F(x) = f(x) - \sum_{i=1}^{k} g^*_i(a^i \cdot x). \]  

(4.4)

It follows from (4.1), (4.3) and (4.4) that

\[ F(x) = \sum_{i=1}^{k} H_i(a^i \cdot x). \]

Denote the restrictions of the functions \( H_i(a^i \cdot x) \) to the space \( \mathbb{Q}^n \) by \( P_i(a^i \cdot x), \) respectively. By Lemma 3.1, the functions \( P_i(a^i \cdot x) \) are polynomials of degree at most \( k - 1. \) Since the space \( \mathbb{Q}^n \) is dense in \( \mathbb{R}^n, \) and the functions \( F(x), \ P_i(a^i \cdot x), \ i = 1, ..., k, \) are continuous on \( \mathbb{R}^n, \) and the equality

\[ F(x) = \sum_{i=1}^{k} P_i(a^i \cdot x). \]  

(4.5)

holds for all \( x \in \mathbb{Q}^n, \) we obtain that (4.5) holds also for all \( x \in \mathbb{R}^n. \) Now (4.2) follows from (4.4) and (4.5) by putting \( g_i = g^*_i + P_i, \ i = 1, ..., k. \)
Now we generalize Theorem 4.1 from $C(\mathbb{R}^n)$ to any space $C^k(\mathbb{R}^n)$ of $k$-th order continuously differentiable functions.

**Theorem 4.2.** Assume $f \in C^k(\mathbb{R}^n)$ is of the form (4.1). Then there exist functions $g_i \in C^k(\mathbb{R})$, $i = 1, \ldots, k$, such that (4.2) holds.

The proof is based on Theorem 4.1 and the following result of A. Pinkus [31].

**Theorem 4.3** (Pinkus [31]). Assume $f \in C^k(\mathbb{R}^n)$ is of the form (4.1). Assume, in addition, that each $f_i \in \mathcal{B}$. Then necessarily $f_i \in C^k(\mathbb{R})$ for $i = 1, \ldots, r$.

In Theorem 4.3, $\mathcal{B}$ denotes any linear space of real-valued functions $u$ defined on $\mathbb{R}$, closed under translation, such that if there is a function $v \in C(\mathbb{R})$ for which $u - v$ satisfies the Cauchy Functional Equation, then $u - v$ is necessarily linear, i.e. $u(x) - v(x) = cx$, for some constant $c \in \mathbb{R}$.

Now the proof of Theorem 4.2 becomes obvious. Indeed, on the first hand, it follows from Theorem 4.1 that $f$ can be expressed as (4.2) with continuous $g_i$. On the other hand, since the class $\mathcal{B}$ in Theorem 4.3, in particular, can be taken as $C(\mathbb{R})$, it follows that $g_i \in C^k(\mathbb{R})$.

**Remark 1.** Theorem 4.2 solves the problem posed in Buhmann and Pinkus [5] and Pinkus [30, p. 14].

**Remark 2.** In addition to the above $C^k(\mathbb{R})$, Theorem 4.1 can be restated also for some other subclasses of the space of continuous functions. These are $C^\infty(\mathbb{R})$ functions; analytic functions; algebraic polynomials; trigonometric polynomials. More precisely, assume $\mathcal{H}(\mathbb{R})$ is any of these subclasses and $\mathcal{H}(\mathbb{R}^n)$ is the $n$-variable analog of the $\mathcal{H}(\mathbb{R})$. If under the conditions of Theorem 4.1, we have $f \in \mathcal{H}(\mathbb{R}^n)$, then this function can be represented also in the form (4.2) with $g_i \in \mathcal{H}(\mathbb{R})$. This follows, similarly to the case $C^k(\mathbb{R})$ above, from Theorem 4.1 and Remark 2.2 in the book by Pinkus [30]. In that remark, it was shown that, Theorem 4.3 can be restated for several classes of functions, in particular, for the above mentioned classes.

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