ANOTHER PROOF OF WILMES’ CONJECTURE

SAM HOPKINS

Abstract. We present a new proof of the monomial case of Wilmes’ conjecture, which gives a formula for the coarsely-graded Betti numbers of the $G$-parking function ideal in terms of maximal parking functions of contractions of $G$. Our proof is via poset topology and relies on a theorem of Gasharov, Peeva, and Welker [6] that connects the Betti numbers of a monomial ideal to the topology of its lcm-lattice.

1. Introduction: the $G$-parking function ideal

Wilmes’ conjecture concerns the Betti numbers of a certain polynomial ideal closely related to the chip-firing game on a graph. Chip-firing on a graph has been studied in various contexts and under various names, including graphical parking functions [15], the Abelian sandpile model [14], and discrete Riemann-Roch theory [1]. For a comprehensive introduction to the sandpile theory behind the problem, see [14], especially §7. We state the conjecture here concisely and without broader context; however, a few definitions are required first:

Let $G$ be an unoriented, connected graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E$. Loops are not permitted, but multiple edges are allowed; formally, we represent the edges of $G$ by a finite set of labels $E$ together with a map $\varphi: E \to (V^2)$. For a subset $W \subseteq V$, let $G_W$ denote the subgraph induced on $W$ by $G$. A connected partition $\pi = \{W_1, \ldots, W_k\}$ of $G$ is a partition of $V$ such that $G_{W_i}$ is connected for all $i$. We use $|\pi| := k$ to denote the number of parts. Note that the set of connected partitions of $G$ is independent of the multiplicity of edges: it depends only on which vertices are adjacent. For a connected partition $\pi$ of $G$, let $G|_{\pi}$ be the contraction of $G$ to $\pi$; that is, let $G|_{\pi}$ be the graph obtained from $G$ by collapsing all the vertices in each part of $\pi$ into one vertex. A cut of $G$ is a partition of $V$ into two parts. A cut is connected if it is a connected partition.

From now on, we fix $v_n$ as the sink vertex of $G$. Let $R := k[x_1, \ldots, x_{n-1}]$ be a polynomial ring in $n-1$ variables over a field $k$. For a cut $C = \{U, W\}$ with $v_n \in W$, define

$$x^C := \prod_{i=1}^{n-1} x_i^{d_U(v_i)},$$

2010 Mathematics Subject Classification. 05E40, 05C25, 13D02, 06B30.
Key words and phrases. Chip-firing, G-parking function, lcm-lattice.
where
\[
d_U(u) := \begin{cases} 
\{|e \in E: \varphi(e) = \{u, w\} \text{ with } w \in V \setminus U\| & \text{if } u \in U, \\
0 & \text{otherwise.}
\end{cases}
\]

Define the monomial ideal \( I \) of \( R \) by
\[
I := \langle x^C : C \text{ is a connected cut of } G \rangle.
\]

We call \( I \) the \( G \)-parking function ideal \(^1\), because a monomial \( x^c \) is nonvanishing on \( R/I \) exactly when \( \sum_{i=1}^{n-1} c_i v_i \) is a \( G \)-parking function:

**Definition 1.** Set \( \tilde{V} := V \setminus \{v_n\} \). A \( G \)-parking function \( c = \sum_{i=1}^{n-1} c_i v_i \) with respect to \( v_n \) is an element of \( \mathbb{Z} \tilde{V} \) such that for every non-empty subset \( U \subseteq \tilde{V} \), there exists \( v_i \in U \) with \( 0 \leq c_i < d_U(v_i) \). The parking functions inherit a partial order from \( \mathbb{Z} \tilde{V} \). A parking function \( c' \) is maximal if \( c' \leq c \) for any parking function \( c \) implies \( c = c' \).

The number of \( G \)-parking functions, and the number of maximal \( G \)-parking functions, are independent of the choice of sink vertex \(^2\). We use \( \text{mpf}(\Gamma) \) to denote the number of maximal parking functions of any connected graph \( \Gamma \). The following conjecture about the coarsely graded Betti numbers \( \beta_i \) of \( R/I \) was stated in \([18]\) and again in \([14]\):

**Conjecture 2.** (Wilmes’ conjecture) For all \( i \geq 1 \), we have
\[
\beta_i(R/I) = \sum_{|\pi| = i + 1} \text{mpf}(G|_{\pi}),
\]
where the sum is over all connected partitions \( \pi \) of \( G \) with \( i + 1 \) parts.

Actually, Wilmes originally conjectured that this sum gives the Betti numbers of the topppling ideal, a binomial ideal of which \( I \) is a distinguished monomial initial ideal. The \( G \)-parking function ideal was introduced by Cori, Rossin, and Salvy \([4]\) and its resolutions have been studied by Postnikov and Shapiro \([15]\) and Manjunath and Sturmfels \([9]\). At any rate, Wilmes’ conjecture has now been proven several times: by Mohammadi and Shokrieh \([12]\) (for both the monomial and binomial cases), by Manjunath, Schreyer, and Wilmes \([8]\) (again for both the monomial and binomial cases), and by Dochtermann and Sanyal \([5]\) (in the monomial case). Also, Mania \([7]\) has a proof for \( \beta_1 \) in the binomial case. We present a new proof here (in the monomial case) because we believe the ideas are significantly novel.

The proof, via poset topology \([17]\), rests on a result by Gasharov, Peeva, and Welker \([6]\) that connects the Betti numbers of a monomial ideal to the topology of the lcm-lattice of the ideal. The idea is to construct an ideal \( J \), the connected cut-set ideal of \( G \), whose lcm-lattice is dual to the connected

---

\(^1\)The \( G \)-parking function ideal can be constructed in various other, equivalent ways. See \([7]\), \([12]\), \([8]\), or \([5]\) for a proof that the \( x^C \) are in fact the minimal generators of \( I \).

\(^2\)The Betti numbers are topological invariants of an \( R \)-module that can be read off from a minimal free resolution of that module. See \([10]\) for an introduction to Betti numbers.
partition lattice of $G$, and then to show that $I$ and $J$ have the same Betti numbers. The dual connected partition lattice, $L^*_G$, is useful for our purposes because for any $\pi \in L^*_G$, the Möbius function evaluated at $\pi$ is equal (up to sign) to the number of maximal parking functions on $G|_\pi$. We use one further ideal $K$, the oriented connected cut-set ideal of $G$, to connect $I$ and $J$. Example 3 illustrates all of the objects we attach to the graph $G$ in order to prove Conjecture 2: the dual connected partition lattice $L^*_G$, its Möbius function, and the three ideals $I$, $J$, and $K$.

**Example 3.** Let $G$ be the “kite graph” pictured below:

```
   v2
  /|
 / \
 v1 | b
 / \
  c
 / \
 v4
```

$G$

Figure 1 shows the dual connected partition lattice $L^*_G$ (§2), whose atoms are labeled by generators of the $G$-parking function ideal $I$ (§1), the connected cut-set ideal $J$ (§3), and the oriented connected cut-set ideal $K$ (§4), and all of whose elements are labeled by their Möbius function values. (For $K$, we orient the edges so that if $f \in E$ with $f_1 = v_i$ and $f_2 = v_j$, then $i < j$.) □

**Remark 4.** Many of the ideas presented here are similar to those found in a preprint of Mohammadi and Shokrieh [11] that was posted to the arXiv shortly after the first version of this note. Mohammadi and Shokrieh explain the matroidal structure behind the ideals considered in §3 and §4: the ideal $J$ is the (unoriented) matroid ideal of $G$ and $K$ is the graphic oriented matroid ideal. The content of [11] is much more comprehensive; for instance, it addresses also the binomial toppling ideal. Nevertheless, the emphasis on the lcm-lattice here differentiates our approach from that of [11].

**Acknowledgements:** We thank David Perkinson for the helpful discussion, comments, and proofreading. We also thank the anonymous reviewer for help with the exposition and references.

2. The connected partition lattice and its dual

**Definition 5.** The connected partition lattice $L_G$ is the set of connected partitions of $G$ partially ordered by refinement: $\pi < \pi'$ if $\pi$ refines $\pi'$. We use $L^*_G$ to denote the lattice that is the order dual of $L_G$. We use $\hat{0}$ and $\hat{1}$ to denote the minimal and maximal elements of a lattice, respectively. So we have $\hat{0} = \{\{v_1\}, \ldots, \{v_n\}\}$ and $\hat{1} = \{V\}$ for $L_G$, and vice-versa for $L^*_G$.

---

3For a direct translation between this note and [11]: our ideal $I$ is their ideal $M^c_n$; our ideal $J$ is their ideal $\text{Mat}_G$; and our ideal $K$ is their ideal $O^c_n$. 
Figure 1. The dual connected partition lattice $L_G^*$ for $G$ the “kite graph” of Example 3. For each $\pi \in L_G^*$, the value of $\mu(\hat{0}, \pi)$ appears in a square box beside $\pi$.

Note that $L_G$ is in fact the lattice of flats of the graphic matroid associated to $G$ [13, §5] and is thus geometric. The lattice $L_G^*$ is also geometric: dualizing preserves semi-modularity and that it is atomic follows from Proposition 8 below. The M"obius function of the dual connected partition lattice can be computed in terms of maximal parking functions, as Proposition 6 explains. This proposition is [13, Proposition 5.3].

**Proposition 6.** The M"obius function of $L_G^*$ is given by

$$\mu(\hat{0}, \pi) = (-1)^{|\pi| - 1} \text{mpf}(G|_\pi)$$

for all $\pi \in L_G^*$. 
3. The connected cut-set ideal

Definition 7. For a cut \( \{U,W\} \), we define its cut-set to be
\[ \{e \in E: \varphi(e) = \{u,w\} \text{ with } u \in U, w \in W\}. \]

A connected cut-set of \( G \) is the cut-set of a connected cut.

Let \( S := k[y^e: e \in E] \) be a polynomial ring in \(|E|\) variables over \( k \). For \( F \subseteq E \), define
\[ y^F := \prod_{e \in F} y^e. \]

Define the squarefree monomial ideal \( J \) of \( R \) by
\[ J := \langle y^F : F \subseteq E \text{ is a connected cut-set of } G \rangle. \]

We call \( J \) the connected cut-set ideal of \( G \). It is clear that the generators above are minimal. Let \( L_J \) be the lcm-lattice of \( J \): the lattice of least common multiples of minimal generators \( y^F \) ordered by divisibility (\( y \leq y' \) if \( y \) divides \( y' \)).

Proposition 8. We have the isomorphism of lattices \( L_J \cong L_G^* \).

Proof: For a connected partition \( \pi \in L_G^* \), define
\[ F(\pi) := \{e \in E: \varphi(e) = \{u,v\} \text{ with } u \text{ and } v \text{ not in the same part of } \pi\}. \]

We claim that \( \phi: \pi \mapsto y^{F(\pi)} \) is a bijection between \( L_G^* \) and \( L_J \). An atom \( \pi \) of \( L_G^* \) is just a connected cut, and in this case \( F(\pi) \) is a connected cut-set, so clearly \( \phi \) bijects between the atoms of \( L_G^* \) and atoms of \( L_J \).

Let \( \pi, \pi' \in L_G^* \). Their join can be obtained as follows. First, find their common refinement as partitions:
\[ \pi'' := \{W \cap W' : W \subseteq \pi, W' \subseteq \pi', W \cap W' \neq \emptyset\}. \]

But \( \pi'' \) in general is not a connected partition because \( G_W \) for \( W \in \pi'' \) may be disconnected. Thus,
\[ \pi \vee \pi' = \{U : G_U \text{ is a connected component of some } G_W \text{ for } W \in \pi''\}. \]

It is easy to see that \( F(\pi \vee \pi') = F(\pi) \cup F(\pi') \): if \( e \in F(\pi) \) or \( e \in F(\pi') \) with \( \varphi(e) = \{u,w\} \), then \( u \) and \( v \) will not be in the same part of \( \pi \vee \pi'' \); on the other hand, if \( e \notin F(\pi) \) and \( e \notin F(\pi') \), then \( u \) and \( v \) will still be in the same part of \( \pi \vee \pi'' \). And then observe that \( y^{F(\pi)} \vee y^{F(\pi')} = y^{F(\pi) \cup F(\pi')}. \)

Let us use \( \text{rk}(x) \) to denote the rank of an element \( x \) of a lattice \( L \). Since \( L_J \) is a geometric lattice, each interval \((0,y) \) in \( L_J \) has the homotopy type of a wedge of spheres of dimension equal to \( \text{rk}(y) - 2 \)\(^4\). Thus, the following theorem of Gasharov, Peeva, and Welker [9 Theorem 2.1] lets us compute the Betti numbers of \( S/J \) from the Möbius function of \( L_J \). Actually these Betti numbers were computed and given a combinatorial interpretation already in [13]; but the lcm-lattice makes this computation straightforward.

---

\(^4\)When we say that a poset has a topological property, we mean that the order complex of that poset has that property. See [17] for the definition of the order complex of a poset.
Theorem 9. The $i$th coarsely graded Betti numbers of $S/J$ are given by
\[ \beta_i(S/J) = \sum_{y \in L_J \setminus \{0\}} \dim \tilde{H}_{i-2}((\hat{0}, y); k) \]
for all $i \geq 1$. Here $\tilde{H}_{i-2}((\hat{0}, y); k)$ is the reduced homology of $(\hat{0}, y)$.

Corollary 10. By the above theorem, the Betti numbers are
\[ \beta_i(S/J) = \sum_{y \in L_J} |\mu(\hat{0}, y)| = \sum_{\pi \in L_G |\pi| = i+1} \text{mpf}(G|\pi) \]
for all $i \geq 1$.

Proof: For any $y \in L_J$, the interval $(\hat{0}, y)$ has vanishing reduced homology in every dimension except dimension $\text{rk}(y) - 2$. Then by the Euler characteristic [17] we get $\dim \tilde{H}_{\text{rk}(y) - 2}((\hat{0}, y); k) = |\mu(\hat{0}, y)|$. The second equality follows from Propositions 6 and 8. □

4. The oriented connected cut-set ideal

Let $T := k[z^e_1, z^e_2 : e \in E]$ be a polynomial ring in $2|E|$ variables over $k$. Recall that we have fixed $v_n$ as the sink of $G$. Let us also choose $e_1, e_2 \in V$ for each $e \in E$ so that $\varphi(e) = \{e_1, e_2\}$. In other words, let us fix an orientation of each edge of $G$. For a cut $C = \{U, W\}$ with $v_n \in W$, define
\[ z^C := \prod_{e_1 \in U, e_2 \in W} z^e_1 \cdot \prod_{e_2 \in U, e_1 \in W} z^e_2. \]

Define the squarefree monomial ideal $K$ of $T$ by
\[ K := \langle z^C : C \text{ is a connected cut of } G \rangle. \]

We call $K$ the oriented connected cut-set ideal of $G$. Again, these generators are minimal. The oriented connected cut-set ideal serves as the bridge between the connected cut-set ideal and the $G$-parking function ideal, as Propositions 11 and 12 demonstrate.

Proposition 11. The sequence
\[ A := \{z^e_i - z^f_j : e, f \in E, 1 \leq i, j \leq 2, \text{ and } e_i = f_j\} \]
is a permutable regular sequence on $T/K$. Further, $T/K \otimes_T T/(A) \simeq R/I$.

Proof: The isomorphism is clear from construction. We now prove that $A$ is a permutable regular sequence. Our strategy is similar to the proof of the analogous fact for the polarization of a monomial ideal given in [16]. This proof depends crucially on a monotonicity property of the generators of $K$ (so observe the connection with [15 §5]). Let $\bar{A}$ be any subsequence of $A$. It suffices for us to show that if $z^e_i - z^f_j \notin \bar{A}$ with $z^e_i - z^f_j \in A$, then $z^e_i - z^f_j$
is non-zerodivisor of $T/K \otimes_T T/\langle A \rangle$. Without loss of generality suppose that $i = j = 1$; so in particular, $e_1 = f_1$.

Let $K$ be the ideal of $T/\langle A \rangle$ obtained from the generators of $K$ by identifying all the $z^0_k$ with $z^1_k$ for $z^0_k - z^1_k \in A$. Write the ideal $K$ as

$$K = z^0_1 K_1 + z^1_1 K_2 + z^0_1 z^1_1 K_3 + K_4,$$

where the minimal monomial generators of $K_1$ do not involve $z^1_1$, the minimal monomial generators of $K_2$ do not involve $z^1_1$, and the minimal monomial generators of $K_3$ and of $K_4$ involve neither of these. Suppose there exists some $r \in T$ such that $r(z^1_1 - z^0_1) \in K$. Then $rz^0_1 \in K$ and $rz^1_1 \in K$, since $K$ is a monomial ideal, which gives us $r \in K_1 + z^1_1 K_2 + z^0_1 K_3 + K_4$ and $r \in z^1_1 K_1 + K_2 + z^0_1 K_3 + K_4$. So we have,

$$r \in (K_1 + z^1_1 K_2 + z^0_1 K_3 + K_4) \cap (z^1_1 K_1 + K_2 + z^0_1 K_3 + K_4) = K + K_1 \cap K_2 + K_1 \cap z^1_1 K_3 + K_2 \cap z^0_1 K_3.$$

Minimal monomial generators of $K_1 \cap K_2$ are given by $\text{lcm}(z', z'')$, where $z'$ ranges over minimal monomial generators of $K_1$ and $z''$ ranges over minimal monomial generators of $K_2$. Let $z'$ be a minimal generator of $K_1$ and $z''$ be a minimal generator of $K_2$. Note that the monomial $z^1_1 z'$ corresponds to some cut $C_1 = \{U_1, W_1\}$ with $e_1, f_2 \in U_1$ and $e_2, v_n \in W_1$, and similarly $z^1_1 z''$ corresponds to a cut $C_2 = \{U_2, W_2\}$ with $e_1, e_2 \in U_2$ and $f_2, v_n \in W_2$. Set $W := W_1 \cap W_2$ and $U := U_1 \cup U_2$. Let $W_3 \subseteq W$ be such that $G_{W_3}$ is the connected component of $G_W$ that contains $v_n$. Set $C_3 := \{V \setminus W_3, W_3\}$. Note that $C_3$ is connected: by definition, $G_{W_3}$ is connected, and $G_U$ is connected since $e_1$ and $e_2$ and $e_1$ and $f_2$ are adjacent. Finally, there must be an edge between a vertex in each connected component of $G_W$ and a vertex in $G_U$, since $G$ as a whole is connected and these connected components are disconnected from one another. Next we claim that $z^0_1 z' z''$ (with the identifications of $z^0_1$ and $z^1_1$ as above) divides $\text{lcm}(z', z'')$: let $F_1, F_2, F_3$ be the cut-sets of $C_1, C_2, C_3$; then we have $F_3 \subseteq (F_1 \cup F_2)$ and $e, f \notin F_3$. So $K_1 \cap K_2 \subseteq K$. Similarly one can show $K_1 \cap z^1_1 K_3$ and $K_2 \cap z^0_1 K_3$ are subsets of $K$, and therefore $r \in K$. Thus, $z^1_1 - z^1_1$ cannot be a zerodivisor. \hfill $\Box$

**Proposition 12.** The sequence

$$B := \{z^0_1 - z^0_2 : e \in E\}$$

is a permutable regular sequence on $T/K$. Further, $T/K \otimes_T T/\langle B \rangle \simeq S/J$.

**Proof:** This follows from [13 Corollary 2.7]. Alternatively, essentially the same proof as in the last proposition works again. \hfill $\Box$

**Theorem 13.** For all $i \geq 1,$

$$\beta_i(R/I) = \beta_i(T/K) = \beta_i(S/J) = \sum_{\pi \in L_G, |\pi|=i+1} \text{mpf}(G|\pi).$$
Proof: Taking the quotient of \( T/K \) modulo the ideal generated by a regular sequence preserves homological information: if \( \mathcal{F} \) is a minimal free resolution of \( T/K \) and \( \mathcal{F} \otimes T/(A) \) is a minimal free resolution of \( R/I \) and \( \mathcal{F} \otimes T/(B) \) is a minimal free resolution of \( S/J \) (see [3, Proposition 1.1.5]). Thus, these modules all have the same Betti numbers, and by Corollary [10] the Betti numbers are given as above. □

References

[1] Matthew Baker and Serguei Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. Adv. Math., 215:766–788, 2007.

[2] Brian Benson, Deeparnab Chakrabarty, and Prasad Tetali. G-parking functions, acyclic orientations and spanning trees. Discrete Math., 310(8):1340–1353, 2010.

[3] Winfried Bruns and Jürgen Herzog. Cohen-Macaulay Rings, volume 39 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1993.

[4] Robert Cori, Dominique Rossin, and Bruno Salvy. Polynomial ideals for sandpiles and their Gröbner bases. Theor. Comput. Sci., 276(1):1–15, 2002.

[5] Anton Dochtermann and Raman Sanyal. Laplacian ideals, arrangements, and resolutions. Forthcoming in J. Algebr. Comb. eprint, arXiv:1212.6244, December 2012.

[6] Vesselin Gasharov, Irena Peeva, and Volkmar Welker. The lcm-lattice in monomial resolutions. Math. Res. Lett., 5–6:521–532, 1999.

[7] Horia Mania. Wilmes’ conjecture and boundary divisors. eprint, arXiv:1210.8109, October 2012.

[8] Madhusudan Manjunath, Frank-Olaf Schreyer, and John Wilmes. Minimal free resolutions of the G-parking function ideal and the toppling ideal. Forthcoming in Trans. Amer. Math. Soc. eprint, arXiv:1210.7569, October 2012.

[9] Madhusudan Manjunath and Bernd Sturmfels. Monomials, binomials and Riemann-Roch. J. Algebr. Comb., 37(4):737–756, 2013.

[10] Ezra Miller and Bernd Sturmfels. Combinatorial Commutative Algebra, volume 227 of Graduate Texts in Mathematics. Springer, New York, 2005.

[11] Fatemeh Mohammadi and Farbod Shokrieh. Divisors on graphs, binomial and monomial ideals, and cellular resolutions. eprint, arXiv:1305.5351, June 2013.

[12] Fatemeh Mohammadi and Farbod Shokrieh. Divisors on graphs, connected flags, and syzygies. Int. Math. Res. Notices, 2013.

[13] Isabella Novik, Alexander Postnikov, and Bernd Sturmfels. Syzygies of oriented matroids. Duke Math. J., 111(2):287–317, 2002.

[14] David Perkinson, Jacob Perlman, and John Wilmes. Primer for the algebraic geometry of sandpiles. eprint, arXiv:1112.6163, December 2011.

[15] Alexander Postnikov and Boris Shapiro. Trees, parking functions, syzygies, and deformations of monomial ideals. Trans. Amer. Math. Soc., 356(8):3109–3142 (electronic), 2004.

[16] Irena Swanson. Polarization (of monomial ideals). Available online at http://people.reed.edu/~iswanson/polarization.pdf, November 2006.

[17] Michelle L. Wachs. Poset topology: Tools and applications. In Geometric combinatorics, volume 13 of IAS/Park City Math. Ser., pages 497–615, Providence, RI, 2007. Amer. Math. Soc.

[18] John Wilmes. Algebraic invariants of the sandpile model. Reed College, 2010. Bachelor’s thesis.

E-mail address: shopkins@mit.edu

Massachusetts Institute of Technology, Cambridge MA, 02139