The Dynamical equations for $\mathfrak{gl}(n|m)$

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Abstract

In this note we propose a compatible set of equations which commutes with the Knizhnik-Zamolodchikov equations based on the $\mathfrak{gl}(n|m)$ symmetry algebra and establish the Matsuo-Cherednik correspondence for them.

1 Introduction

There is a relation between the solutions of the (quantum) Knizhnik-Zamolodchikov equations ((q)KZ for short) and the spectral problem of quantum systems Calogero-Moser type, proposed in [3] and later in [4], [5]. The recent discussion of this correspondence [15], [11] was originated in the quantum-classical correspondence [13], [12].

It was shown in [1] and generalized in [6], that for twisted KZ equations, based on simple finite dimensional Lie algebra $\mathfrak{g}$, there is a compatible set of equations with respect to twist parameters, which commutes with KZ. This set of equations was called Dynamical equations. The joint system could be viewed as a ‘vector’ analogue of the bispectral problem of [2]. In fact they are related through the Matsuo-Cherednik map (MC for brevity): if one has a solution of joint system of KZ and Dynamical equations, then one can cook-up the solution of the bispectral problem for the corresponding system of Calogero-Moser type. For the proof one can consult with [17] (section 7), [18] and [14]. It is worth to mention that in [7] the KZ and Dynamical equations were defined for a wide class of algebras and it will be very interesting to study the MC map in this generality.

Integrable systems with $\mathbb{Z}/2\mathbb{Z}$ graded symmetry algebras (mostly of type $A$) were studied in many papers, for example one can consult with [16]. Also recently superchains were embedded in the context of the Bethe/Gauge correspondence in [8]. And, maybe, it might be interesting to derive the Dynamical equations, proposed in this note using approach of [7].

The outline of the work is the following: in the section 2 we will proof compatibility of the proposed Dynamical equations and in the section 3 we will establish the MC map. The generalization of the proposed equations to the difference case as well as issues about the monodromy we will leave for the future work [10].

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2 The Dynamical equations

Let \( \bar{z} \in \mathbb{C}^k \setminus \Pi_{i<j} z_i = z_j \) and \( \bar{\lambda} \in \mathbb{C}^{n+m} \setminus \Pi_{i<j} \lambda_i = \lambda_j \). Let \( V = \bigotimes_{i=1}^k \mathbb{C}^{n_i} \) be the tensor product of the vector representations of \( \mathfrak{gl}(n|m) \). And let \( |\Psi\rangle : (\bar{z}, \bar{\lambda}) \to V \) be the flat section, namely

\[
\left( \kappa \partial z_i - \sum_c \lambda_c e_c^{(i)}(j) - \sum_{j, \neq i} P_{ij} \right) |\Psi\rangle = 0, \tag{1}
\]

where graded permutation \( P_{ij} = \sum_{a,b} (-1)^{p(b)} e_{ab}^{(i)} E_{ba} - E_{aa} \), and \( p(a) \) is a parity function, defined in the Appendix. \( e_{ab}^{(j)} \) is the generator of \( \mathfrak{gl}(n|m) \) which non-trivially acts only in the \( j \)th tensor component of \( V \) as matrix unit in some basis (see Appendix).

**Theorem 2.1.** The following system of equations is compatible and commute with (1)

\[
\left( \kappa \partial a - \sum_j z_j e_{aa}^{(j)} - \sum_{b, \neq a} (-1)^{p(b)} \frac{E_{ab} E_{ba} - E_{aa}}{\lambda_a - \lambda_b} \right) |\Psi\rangle = 0, \tag{2}
\]

where \( E_{ab} = E_{ab} = \sum_{j=1}^k e_{ab}^{(j)} \).

**Proof.** Let us prove first, that (2) is compatible system. For showing this one has to check that the following commutator vanishes

\[
\kappa \partial a - \sum_j z_j e_{aa}^{(j)} - \sum_{c, \neq a} (-1)^{p(c)} \frac{E_{ac} E_{ca} - E_{aa}}{\lambda_a - \lambda_c}, \kappa \partial b - \sum_l z_l e_{bb}^{(l)} - \sum_{d, \neq b} (-1)^{p(d)} \frac{E_{bd} E_{db} - E_{bb}}{\lambda_b - \lambda_d} = 0. \tag{3}
\]

There are three types of terms

\[
(1) \quad \left[ \partial a, \sum_{d, \neq b} (-1)^{p(d)} \frac{E_{bd} E_{db} - E_{bb}}{\lambda_b - \lambda_d} \right] + \left[ \sum_{c, \neq a} (-1)^{p(c)} \frac{E_{ac} E_{ca} - E_{aa}}{\lambda_a - \lambda_c}, \partial b \right] = \]

\[
= (-1)^{p(a)} \frac{E_{ba} E_{ab} - E_{bb}}{\lambda_b - \lambda_a} - (-1)^{p(b)} \frac{E_{ab} E_{ba} - E_{aa}}{\lambda_a - \lambda_b} = 0. \tag{4}
\]

The last equation holds due to relations (29).

\[
(2) \quad \left[ \sum_j z_j e_{aa}^{(j)} - \sum_{d, \neq b} (-1)^{p(d)} \frac{E_{bd} E_{db} - E_{bb}}{\lambda_b - \lambda_d} \right] + \left[ \sum_{c, \neq a} (-1)^{p(c)} \frac{E_{ac} E_{ca} - E_{aa}}{\lambda_a - \lambda_c}, \sum_l z_l e_{bb}^{(l)} \right] =
\]

\[
= (-1)^{p(a)} \frac{E_{ba} E_{ab} - E_{bb}}{\lambda_b - \lambda_a} + (-1)^{p(b)} \frac{E_{ab} E_{ba} - E_{aa}}{\lambda_a - \lambda_b} = 0. \tag{5}
\]

The last equation is true due to (31)

\[
(3) \quad \left[ \sum_{c, \neq a} (-1)^{p(c)} \frac{E_{ac} E_{ca} - E_{aa}}{\lambda_a - \lambda_c}, \sum_{d, \neq b} (-1)^{p(d)} \frac{E_{bd} E_{db} - E_{bb}}{\lambda_b - \lambda_d} \right] =
\]

\[
= \sum_{c, \neq a, b} \left[ (-1)^{p(b)} \frac{E_{ab} E_{ba}}{\lambda_{ab}}, (-1)^{p(c)} \frac{E_{bc} E_{cb}}{\lambda_{bc}} \right] + \left[ \frac{E_{ac} E_{ca}}{\lambda_{ac}}, \frac{E_{bc} E_{cb}}{\lambda_{bc}} \right] +
\]

\[
+ \left[ (-1)^{p(c)} \frac{E_{ac} E_{ca}}{\lambda_{ac}}, (-1)^{p(a)} \frac{E_{ba} E_{ab}}{\lambda_{ba}} \right] + (-1)^{p(a)+p(b)} \left[ \frac{E_{ba} E_{ba}}{\lambda_{ab}}, \frac{E_{ba} E_{ba}}{\lambda_{ba}} \right] \tag{6}
\]
A short calculation shows that the last commutator vanishes. Let us focus on the middle row in the above expression, which is more complicated. To prove that it is zero one has to consider the following four cases

\[
p(a) = p(b) = 0, \quad p(c)= 1
\]

\[
- \left[ \frac{E_{ab}E_{ba}}{\lambda_{ab}}, \frac{E_{bc}E_{cb}}{\lambda_{bc}} \right] + \left[ \frac{E_{ac}E_{ca}}{\lambda_{ac}}, \frac{E_{bc}E_{cb}}{\lambda_{bc}} \right] - \left[ \frac{E_{ac}E_{ca}}{\lambda_{ac}}, \frac{E_{ba}E_{ab}}{\lambda_{ba}} \right] = 0.
\]

(7)

\[
p(a) = p(c) = 0, \quad p(b)= 1
\]

\[
- \left[ \frac{E_{ab}E_{ba}}{\lambda_{ab}}, \frac{E_{bc}E_{cb}}{\lambda_{bc}} \right] + \left[ \frac{E_{ac}E_{ca}}{\lambda_{ac}}, \frac{E_{bc}E_{cb}}{\lambda_{bc}} \right] + \left[ \frac{E_{ac}E_{ca}}{\lambda_{ac}}, \frac{E_{ba}E_{ab}}{\lambda_{ba}} \right] = 0.
\]

(8)

\[
p(b) = p(c) = 1, \quad p(a)= 0
\]

\[
\frac{E_{ab}E_{ba}}{\lambda_{ab}E_{bc}} - \frac{E_{bc}E_{bc}E_{ab}}{\lambda_{bc}} + \left[ \frac{E_{ac}E_{ca}}{\lambda_{ac}}, \frac{E_{bc}E_{cb}}{\lambda_{bc}} \right] - \frac{E_{ac}E_{ca}}{\lambda_{ac}E_{ba}} = 0.
\]

(9)

\[
p(a) = p(b) = 1, \quad p(c)= 0
\]

\[
- \left[ \frac{E_{ab}E_{ba}}{\lambda_{ab}}, \frac{E_{bc}E_{cb}}{\lambda_{bc}} \right] + \left[ \frac{E_{ac}E_{ca}}{\lambda_{ac}}, \frac{E_{bc}E_{cb}}{\lambda_{bc}} \right] - \left[ \frac{E_{ac}E_{ca}}{\lambda_{ac}}, \frac{E_{ba}E_{ab}}{\lambda_{ba}} \right] = 0.
\]

(10)

The two cases where parities of \(a, b\) and \(c\) are equal are not considered because the three corresponding generators of the \(gl(n|m)\) are bosonic.

From the above computation one sees that the system (2) is compatible and the first part of the theorem is proven.

Now let us show that systems (1) and (2) are compatible. Again one needs to check that the following commutator vanishes

\[
\left[ \kappa \partial z_i - \sum_c \lambda_c e^{(i)}_{cc}, -\sum_{j, \neq i} \frac{P_{ij}}{z_i - z_j}, \kappa \partial a - \sum_j z_j e^{(j)}_{aa}, -\sum_{b, \neq a} (-1)^{p(b)} \frac{E_{ab}E_{ba} - E_{aa}}{\lambda_a - \lambda_b} \right] = 0.
\]

(11)

Let is consider the most complicated parts of the above commutator

\[
1. \left[ \sum_c \lambda_c e^{(i)}_{cc}, -\sum_{b, \neq a} (-1)^{p(b)} \frac{E_{ab}E_{ba} - E_{aa}}{\lambda_{ab}} \right] + \left[ \sum_{j, \neq i} \frac{P_{ij}}{z_i - z_j}, \sum_j z_j e^{(j)}_{aa} \right] = \sum_{b, \neq a} (-1)^{p(b)} \left( e^{(i)}_{ab}E_{ba} - e^{(i)}_{ab}E_{ba} - e^{(i)}_{ab}E_{ba} + E_{ab}e^{(i)}_{ba} \right) = 0.
\]

(12)
2.
\[
\sum_{j \neq i} \frac{P_{ij}}{z_i - z_j} \sum_{b \neq a} (-1)^{p(b)} \frac{E_{ab} E_{ba} - E_{aa}}{\lambda_a - \lambda_b} = 0. \tag{13}
\]
Indeed one has to check that
\[
[P_{ij}, E_{ab} E_{ba}] = 0. \tag{14}
\]
Simple calculation shows that this is true.

3. The Matsuo-Cherednik map

In this section we fix \( k = n + m \) in the definition of \( V \).

**Theorem 3.1.** The following covectors provide the Matsuo-Cherednik map for Dynamical equations (2)

\[
\langle \Omega^0 \rangle = \sum_{\sigma \in S_{n+m}} \langle e_1 \otimes \ldots \otimes e_{n+m} | P_\sigma, \tag{15a}
\]

\[
\langle \Omega^1 \rangle = \sum_{\sigma \in S_{n+m}} \langle e_1 \otimes \ldots \otimes e_{n+m} | (-1)^{\text{sgn}(\sigma)} P_\sigma, \tag{15b}
\]

where \( P_\sigma = P_{s_1} \ldots P_{s_i}, \) where \( P_{s_{ij}} \) is a permutation, that corresponds to the reflection \( s_{ij} \) and \( \sigma = s_{i_1} s_{i_2} \ldots s_{i_l} \) is a reduced decomposition. Because \( P_{s_{ij}} \) satisfy braid relation the element \( P_\sigma \) is correctly defined.

The appearance of (15b) is a property of \( V[1] \) on which we are projecting the Dynamical equation. The \( V[1] \subset V \) is a subspace, such that \( E_{aa} | V[1] \rangle = 1 \) for all \( a \leq n + m \). Such condition is consistent with equations (1) and (2) because the Cartan generators commute with them. And presumably one can not consider the other weight subspaces different to \( V[1] \), because the symmetry between \( z' \)s and \( \lambda' \)s is broken. Now let us prove a simple and useful

**Lemma 3.2.** The covectors (15a) and (15b) are eigenvectors of the following operators

\[
(E_{ab} E_{ba} - E_{aa}) \langle \Omega^i \rangle = (-1)^{p(b)+i} \langle \Omega^i \rangle, \tag{16}
\]

where \( i = 0, 1 \).

**Proof.** One can consider vectors \( |\Omega^i \rangle \) and prove for them the analogue properties (16). Let us consider the symmetrized vector as an example

\[
(E_{ab} E_{ba} - E_{aa}) |\Omega^0 \rangle = \sum_{\sigma \in S_{n+m}} P_\sigma (E_{ab} E_{ba} - E_{aa}) |e_1 \otimes \ldots \otimes e_{n+m} \rangle, \tag{17}
\]

thanks to (14). A simple computation shows that

\[
(E_{ab} E_{ba} - E_{aa}) |e_1 \otimes \ldots \otimes e_{n+m} \rangle = (-1)^{p(b)} P_{ab} |e_1 \otimes \ldots \otimes e_{n+m} \rangle. \tag{18}
\]

Then applying to the result the symmetrizer, or skew symmetrizer one obtains (16) immediately. \( \square \)

Now we are ready to proof the following
Theorem 3.3. Let $|\Psi\rangle$ be the solution of joint system (1), (2) with values in $V[1]$, then

\[
\begin{align*}
\left(\sum_{a=1}^{n+m} \kappa^2 \sum_{a \neq b} \frac{(-1)^i \kappa - 1}{(z_{ab})^2}\right) \langle \Omega^i | \Psi \rangle &= \left(\sum_{i=1}^{n+m} \lambda^2_i\right) \langle \Omega^i | \Psi \rangle, \quad (19a) \\
\left(\sum_{a=1}^{n+m} \kappa^2 \sum_{a \neq b} \frac{(-1)^i \kappa - 1}{(\lambda_{ab})^2}\right) \langle \Omega^i | \Psi \rangle &= \left(\sum_{i=1}^{n+m} z_i^2\right) \langle \Omega^i | \Psi \rangle, \quad (19b)
\end{align*}
\]

where $i = 0, 1$

Proof. The first equation was proven in [11], so we are to prove the second. Let $D_a$ be the $a^{th}$ dynamical operator. So now we are consider the following sum

\[
\langle \Omega^i | \sum_{a=1}^{n+m} D_a^2 | \Psi \rangle = \langle \Omega | \left(\sum_{a=1}^{n+m} \kappa^2 \frac{\partial^2}{\partial \lambda_a^2} - \sum_{i,a=1}^{n+m} \frac{z_i \epsilon_a(i)}{\lambda_a - \lambda_b} \right) \psi - \left(\sum_{i,a=1}^{n+m} \frac{z_i \epsilon_a(i)}{\lambda_a - \lambda_b} \right) \psi
\]

where the (2) was used. After using (16) and the identity $\sum_{b \neq c \neq a} \frac{1}{(\lambda_a - \lambda_b)(\lambda_a - \lambda_c)} = 0$ the above expression simplifies to

\[
\langle \Omega^i | \sum_{a=1}^{n+m} \kappa^2 \frac{\partial^2}{\partial \lambda_a^2} - \sum_{i,a=1}^{n+m} \frac{z_i \epsilon_a(i)}{\lambda_a - \lambda_b} \rangle - \sum_{i} z_i^2 + \sum_{b \neq a} \frac{(-1)^i \kappa - 1}{(\lambda_a - \lambda_b)^2} |\Psi\rangle = 0. \quad (21)
\]

To prove the theorem one has to show that

\[
\langle \Omega^i | \left\{ \sum_{i,a=1}^{n+m} \frac{z_i \epsilon_a(i)}{\lambda_a - \lambda_b} \right\} |\Psi\rangle = 0. \quad (22)
\]

In fact one has to prove, that for any distinct $a$ and $b$

\[
\langle \Omega^i | \left\{ \sum_{i=1}^{n+m} \frac{z_i \epsilon_a(i)}{\lambda_a - \lambda_b} \right\} - \sum_{i=1}^{n+m} \frac{z_i \epsilon_b(i)}{\lambda_a - \lambda_b} \rangle = 0. \quad (23)
\]

After opening the brackets one obtains the following

\[
\langle \Omega^i | \left( \sum_{i=1}^{n} \frac{z_i \epsilon_a(i)}{\lambda_a - \lambda_b} \right) + (-1)^i \langle \Omega^i | \sum_{i=1}^{n} \frac{z_i \epsilon_a(i)}{\lambda_a - \lambda_b} \rangle - (-1)^i \langle \Omega^i | \sum_{i=1}^{n} \frac{z_i \epsilon_a(i)}{\lambda_a - \lambda_b} \rangle - (-1)^i \langle \Omega^i | \sum_{i=1}^{n} \frac{z_i \epsilon_b(i)}{\lambda_a - \lambda_b} \rangle \quad(24)
\]

One can write the projectors (15a), (15b) in the following form

\[
\langle \Omega^i | = \sum_{a_1 \neq \ldots \neq a_{n+m}} (e_{a_1} \otimes \ldots \otimes e_{a_{n+m}}^i \otimes \ldots \otimes e_{b} \otimes \ldots \otimes e_{a_{n+m}}^j) | f_i(a_1, \ldots, a_{n+m}) \rangle. \quad (25)
\]
Because of \( \sum \) where other value of \( i \) lies in the interval strictly between \( i \).

Here we give a brief summary of some definitions of the Lie superalgebra \( \mathfrak{gl}(n|m) \)

\[
\sum_{i=1}^{n+m} \left( \sum_{j \neq i} \sum_{\{a_i=a,j=b\}} \left( e_{a_1} \otimes \ldots \otimes e_{a_i}^{(i)} \otimes \ldots \otimes e_{a_{n+m}} \right) f_0(a_1, \ldots, a_{n+m}) \right) \times z_i \left( (-1)^{p(b)} (E_{ab}E_{ba} - E_{aa}) + 1 \right) =
\]

\[
\sum_{i=1}^{n+m} \left( \sum_{j \neq i} \sum_{\{a_i=a,j=b\}} \left( \langle e_{a_1} \otimes \ldots \otimes e_{a_i}^{(i)} \otimes \ldots \otimes e_{a_{n+m}} \rangle f_0(a_1, \ldots, a_{n+m}) \right) z_i, \right)
\]

where \( \sum_{\{a_i=a,j=b\}} = \sum_{a_1 \neq \ldots \neq b \neq \ldots \neq a_{n+m}} \) and \( \sum_{k \in \{i,j\}} \) means that summation runs over all \( k \), that lies in the interval strictly between \( i \) and \( j \). Then the second line gives the following

\[
\sum_{i=1}^{n+m} \left( \sum_{j \neq i} \sum_{\{a_i=a,j=b\}} \left( \langle e_{a_1} \otimes \ldots \otimes e_{a_i}^{(i)} \otimes \ldots \otimes e_{a_{n+m}} \rangle f_0(a_1, \ldots, a_{n+m}) \right) z_i. \right)
\]

Because of (16) one has the following relation

\[
f_0(a_1, \ldots, a_{n+m}) = (-1)^{\sum_{k \in \{i,j\}} p(a_k)} f_0(a_1, \ldots, b, \ldots, a_{n+m}).
\]

From which one sees that difference (24) vanishes. The other cases of parity of \( a \) and \( b \) as well as other value of \( i \) considered similarly.

\[
\square
\]

4 Summary

To conclude we have proved the compatibility of proposed system of Dynamical equations and its commutativity with 'twisted' KZ equations. The structure of the formula (2) is very much like in [1], but not quite because of fermionic roots. It will be interesting to generalize the work of [6] to the Lie superalgebra case as well as compute the monodromy of the proposed Dynamical equations. It will be addressed elsewhere [10].

5 Appendix

Here we give a brief summary of some definitions of the Lie superalgebra \( \mathfrak{gl}(n|m) \)

Let \( \mathcal{J} = \{1, \ldots, n+m\} \) and let \( p : \mathcal{J} \to \{0,1\} \)

\[
\begin{cases}
p(a) = 0, & a \leq n, \text{(bosons)}, \\
p(a) = 1, & a > n \text{(fermions)}. \end{cases}
\]

The \( \mathfrak{gl}(n|m) \) algebra is generated by \( e_{ab} \) where \( a,b \in \mathcal{J} \) with the following relations

\[
e_{ab}e_{cd} - (-1)^{p(e_{ab})p(e_{cd})} e_{cd} e_{ab} = \delta_{bc} e_{ad} - (-1)^{p(e_{ab})p(e_{cd})} \delta_{da} e_{cb},
\]

\[
\sum_{i=1}^{n+m} \left( \sum_{j \neq i} \sum_{\{a_i=a,j=b\}} \left( e_{a_1} \otimes \ldots \otimes e_{a_i}^{(i)} \otimes \ldots \otimes e_{a_{n+m}} \right) f_0(a_1, \ldots, a_{n+m}) \right) \times z_i \left( (-1)^{p(b)} (E_{ab}E_{ba} - E_{aa}) + 1 \right) =
\]

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\sum_{i=1}^{n+m} \left( \sum_{j \neq i} \sum_{\{a_i=a,j=b\}} \left( \langle e_{a_1} \otimes \ldots \otimes e_{a_i}^{(i)} \otimes \ldots \otimes e_{a_{n+m}} \rangle f_0(a_1, \ldots, a_{n+m}) \right) z_i, \right)
\]

where \( \sum_{\{a_i=a,j=b\}} = \sum_{a_1 \neq \ldots \neq b \neq \ldots \neq a_{n+m}} \) and \( \sum_{k \in \{i,j\}} \) means that summation runs over all \( k \), that lies in the interval strictly between \( i \) and \( j \). Then the second line gives the following

\[
\sum_{i=1}^{n+m} \left( \sum_{j \neq i} \sum_{\{a_i=a,j=b\}} \left( \langle e_{a_1} \otimes \ldots \otimes e_{a_i}^{(i)} \otimes \ldots \otimes e_{a_{n+m}} \rangle f_0(a_1, \ldots, a_{n+m}) \right) z_i. \right)
\]

Because of (16) one has the following relation

\[
f_0(a_1, \ldots, a_{n+m}) = (-1)^{\sum_{k \in \{i,j\}} p(a_k)} f_0(a_1, \ldots, b, \ldots, a_{n+m}).
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The \( \mathfrak{gl}(n|m) \) algebra is generated by \( e_{ab} \) where \( a,b \in \mathcal{J} \) with the following relations

\[
e_{ab}e_{cd} - (-1)^{p(e_{ab})p(e_{cd})} e_{cd} e_{ab} = \delta_{bc} e_{ad} - (-1)^{p(e_{ab})p(e_{cd})} \delta_{da} e_{cb},
\]
where

\[ p(e_{ab}) = p(a) + p(b) \mod 2. \tag{30} \]

The \( \otimes \) product of the superalgebra representations is defined in such a way, that for operators, which has definite parity, and acting non-trivially only in the \( i^{th} \) and \( j^{th} \) component of the tensor product the following holds

\[ A^{(i)}B^{(j)} = (-1)^{p(A)p(B)}B^{(j)}A^{(i)}. \tag{31} \]

In the \( \mathbb{C}^{n|m} \) there is a basis \( e_a \) such that \( e_{ab}(e_c) = \delta_{bc}e_a \), meaning that \( e_{ab} \) are just matrix units. Let \( x, y \in \mathbb{C}^{n|m} \) with definite \( p(x) \) and \( p(y) \) then the graded permutation acts as follows

\[ P_{12}(x \otimes y) = (-1)^{p(x)p(y)}y \otimes x. \tag{32} \]

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