A Polynomial Invariant Of Twisted Graph Diagrams

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Abstract

Twisted graph diagrams are virtual graph diagrams with bars on edges. A bijection between abstract graph diagrams and twisted graph diagrams is constructed. Then a polynomial invariant of Yamada-type is developed which provides a lower bound for the virtual crossing number of virtual graph diagrams.

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1 Introduction

Let \( G \) be a finite graph considered as a topological space. An embedding of \( G \) into three-dimensional space is called a spatial graph. A regular projection of \( G \) onto a surface \( S \) is a continuous map \( G \to S \) whose multiple points are finitely many transverse double points away from the vertices of \( G \). The image of \( G \) under a regular projection together with over/under information given to the double points is called a (regular) graph diagram on \( S \). In [3] regular graph diagrams are extended to virtual (regular) graph diagrams motivated by L. Kauffman’s theory of virtual links, see [5]. A one-to-one correspondence between virtual links and so called abstract link diagrams is presented in [4]. In the first part of this note the notion of an abstract link diagram is extended to an abstract graph diagram. Differently from [4] we allow the disk/band surfaces to be non-orientable. This enables us to construct a bijection from abstract graph diagrams to so called twisted graph diagrams. These diagrams are generalisations of virtual graph diagrams by adding bars to edges. Geometrically a bar corresponds to a twist of a band of the surface. Concerning links this idea can be found in [2].
In chapter 5 we interpret the polynomial of B. Bollobás and O. Riordan which is defined for possibly non-orientable disk/band surfaces, see [1], as a polynomial for pure twisted graph diagrams via their abstract graph diagrams. This leads to a polynomial invariant for twisted graph diagrams. The definition is similar to that of the Yamada polynomial in [9]. As an application we obtain a lower bound for the virtual crossing number of a virtual graph diagram.

2 Abstract Graph Diagrams

In this paper the underlying graph of a regular graph diagram may have several components. In addition, components without vertices, so called circle components, are allowed.

Definition 2.1 A pair \((S, D)\) is called an abstract graph diagram if \(S\) is a two-dimensional disk/band surface, \(D\) is a regular graph diagram on \(S\) and (as a subset of \(S\)) a strong deformation retract of \(S\).

The crossings and the vertices of an abstract graph diagram are contained in the disks of the surface. Two examples for orientable surfaces are shown in figure 1.

![Figure 1](image)

Definition 2.2 An abstract graph diagram \((S, D)\) is obtained from another abstract graph diagram \((S', D')\) by an abstract Reidemeister move of type I, II, III, IV, V or VI if there exist embeddings \(f : S \rightarrow F\), \(f' : S' \rightarrow F\) for a closed surface \(F\), so that \(f(D)\) is obtained from \(f'(D')\) by a Reidemeister move resp. of type I to VI on \(F\).

Reidemeister moves are shown in [3], figure 2.

Definition 2.3 Two abstract graph diagrams are said to be abstract Reidemeister move equivalent or equivalent if one is transformed into the other by a finite sequence of abstract Reidemeister moves.

We denote the set of abstract graph diagrams by \(\mathcal{AG}\) and the corresponding set of equivalence classes by \(\mathcal{AG}\).
3 Twisted Graph Diagrams

Extending classical graph diagrams by virtual crossings and virtual Reidemeister moves I* to V* we get virtual graph diagrams and virtual graphs. For definitions see [3], chapter 2 and figure 4.

We denote the set of virtual graph diagrams by $\mathcal{VG}$. The set of equivalence classes of $\mathcal{VG}$ generated by Reidemeister moves I to VI and virtual Reidemeister moves I* to V* is denoted by $\mathcal{VG}$. Following [2] we define twisted graph diagrams as virtual graph diagrams with bars on edges. The set of twisted graph diagrams is denoted by $\mathcal{TG}$. The set of equivalence classes generated by Reidemeister moves I to VI, I* to V* and the twisted moves $T_1$, $T_2$, $T_3$ and $T_4$ of figure 2 is called $\mathcal{TG}$.

4 Abstract vs. Twisted Graph Diagrams

As in [4] we define a map $\phi: \mathcal{TG} \to \mathcal{AG}$. In our setting, for a twisted graph diagram $E$ we have 2-disks as regular neighborhoods for the crossings and the vertices. In figure 3 it is shown how the classical resp. virtual crossings are replaced by a surface $S \subset \mathbb{R}^3$ and a diagram $D$ on $S$.

1. classical crossing

2. virtual crossing

3. vertex

Figure 2

Figure 3
Note that up to homeomorphism in 2. and 4. the surface does not depend on the sign of the crossing of the bands resp. the twist. We define $\phi(E) := (S,D)$.

**Theorem 4.1** The map $\Phi: TG \to AG$ defined by $\Phi([E]) := [\phi(E)]$ is a bijection.

Before we give a proof of the theorem we construct a map $\psi: AG \to TG$ and define $\Psi: AG \to TG$ to be $\Psi([(S,D)]) := \psi((S,D))$.

We remind the reader of the following notion from [10]: Let $P \subset \mathbb{R}^3$ be a plane and $p: \mathbb{R}^3 \to P$ a projection. The projection $p$ is *regular* for a disk/band surface $S \subset \mathbb{R}^3$ if the following conditions are satisfied:

1. For each $y \in p(S)$, $p^{-1}(y) \cap S$ consists of either one, two or infinitely many points.

2. If $p^{-1}(y) \cap S$ consists of two points, then there are two band parts $B_i, B_j$ of $S$ with $y \in p(B_i) \cap p(B_j)$ such that $p(B_i)$ and $p(B_j)$ meet as in figure 4.

3. If $p^{-1}(y) \cap S$ consists of infinitely many points, then there is exactly one band part $B$ of $S$ with $y \in p(B)$ such that $p(B)$ is as in figure 5.

Let $(S,D) \in AG$, $g : S \to \mathbb{R}^3$ an embedding and $p$ a regular projection for the disk/band surface $g(S)$. Consider $p \circ g(S)$ as a virtual graph diagram as follows: those double points of $p \circ g(D)$ belonging to the images of crossings of $D$ on $S$ are labelled with the corresponding over/under information. The remaining double points are considered as virtual crossings. Now we define a twisted graph diagram $E$ by adding a bar for every singularity like figure 5 coming from the image of $S$ under $p \circ g$. Then we set $\psi((S,D)) := [E]$. In the following propositions 4.2 to 4.12 it is shown that the maps $\phi$, $\Phi$, $\psi$ and $\Psi$ are well-defined.
Proposition 4.2 $\phi$ is well-defined.

Proof. By construction we have nothing to prove.

Proposition 4.3 $\Phi$ is well-defined.

Proof. Let $D, E \in TG$. We have to show that $\phi(D)$ is equivalent to $\phi(E)$ as abstract graph diagrams for $[D] = [E] \in TG$. Suppose $D$ and $E$ differ by Reidemeister move VI. Thus they are identical outside a 2-disk $\Sigma \subset \mathbb{R}^2$. Abstract graph diagrams $(S_D, G_D)$ and $(S_E, G_E)$ embedded in three-dimensional space and being identical outside $\Sigma$ can be constructed. This is indicated in figure 6. As $S_D \cup \Sigma$ is homeomorphic to $S_E \cup \Sigma$ they are contained in a closed surface constructed by gluing 2-disks to their boundary components. By definition of $\phi$ we have $\phi(D) = (S_D, G_D)$ and $\phi(E) = (S_E, G_E)$ since abstract graph diagrams are considered up to homeomorphism. Hence $\phi(D)$ is obtained from $\phi(E)$ by an abstract Reidemeister move.

Figure 6

The remaining Reidemeister moves I, II, III, IV and V can be treated in an analogue manner.

Now suppose $D$ and $E$ differ by Reidemeister move IV*. It is shown in figure 7 how the abstract graph diagrams can be obtained with respect to the disk $\Sigma$. There are several possible ways to choose the over/under behaviour of the bands inside a suitable neighborhood of the disk, but this does not affect the type of the surface up to homeomorphism. Thus $\phi(D) \approx \phi(E)$, i.e. $[\phi(D)] = [\phi(E)] \in AG$.

Figure 7

The remaining Reidemeister moves I*, II*, III* and V* can be treated in an analogue manner.

Now suppose $D$ and $E$ differ by a twisted move $T2$ inside the disk $\Sigma$. Obviously the corresponding abstract graph diagrams are homeomorphic by
the definition of \( \phi \) in \[4\], since two half-twists either cancel or become a full-twist. If \( D \) and \( E \) differ by \( T1 \), we argue just as in the case of pure virtual moves: one possible result of constructing the abstract graph diagrams is shown in figure 8.

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=3cm]{figure8a.png} \\
\includegraphics[width=3cm]{figure8b.png}
\end{array}
\end{array}
\]

Figure 8

In figure 9 we see how a homeomorphism may be obtained in the case of a \( T3 \)-move. Rotate the surface around an horizontal axis and keep it fixed outside a suitable neighborhood of \( \Sigma \).

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=3cm]{figure9a.png} \\
\includegraphics[width=3cm]{figure9b.png}
\end{array}
\end{array}
\]

Figure 9

In the same way we treat the \( T4 \)-move, i.e. flipping the surface around an appropriate vertical axis.

**Remark 4.4** For Reidemeister move \( VI^* \) of \[3\], figure 5 the proof of Proposition 4.3 does not work, because the corresponding surfaces may not be homeomorphic. An example is shown in figure 10.

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=3cm]{figure10a.png} \\
\includegraphics[width=3cm]{figure10b.png}
\end{array}
\end{array}
\]

Figure 10

**Definition 4.5** \[4\] Let \( D \) and \( E \) be virtual graph diagrams of the same underlying graph such that they are identical inside regular neighborhoods \( N_1, \ldots, N_m \) of the crossings and the vertices. For \( X \in \{D, E\} \) put \( W_X := X \cap (\mathbb{R}^2 \setminus \bigcup N_i) \). Then the set \( W_X \) is a union of immersed arcs. The diagrams \( D \) and \( E \) have the same Gauss data, if there is a 1-1-correspondence between their immersed arcs \( W_D \) and \( W_E \) with respect to their boundary points in the union of the neighborhoods.
Proposition 4.6  Two virtual graph diagrams represent the same equivalence class in VG if they have the same Gauss data.

Proof. As in the proof of Lemma 4.3 in [4] the immersed arcs can be transformed into one another by a finite sequence of virtual Reidemeister moves up to isotopy. Apparently the forbidden move VI* is not required.

Definition 4.7  Let $S$ be a disk/band surface with an orientation chosen for every disk of $S$. Let $B$ be a band of $S$ with (possibly equal) incident disks $D_1$ and $D_2$. The sign of $B$ is defined to be $+1$ if the orientation of $D_1$ is equal to that of $D_2$ after moving it along $B$. Otherwise it is defined to be $-1$. By an orientation of a band we mean an orientation chosen for the topological disk belonging to the band.

Proposition 4.8  Let $S$ be a disk/band surface with oriented disks and bands. For a band $B$ let $D_1, D_2$ be the incident disks with $B \cap D_j = \partial B \cap \partial D_j =: I_j \approx [0, 1]$. Then $\text{sign}(B) = 1$ if and only if $\partial B$ induces the same orientation on $I_j$ as $\partial D_j$ for $j \in \{1, 2\}$ or $\partial B$ induces the opposite orientation on $I_j$ than $\partial D_j$ for $j \in \{1, 2\}$.

Proof. To show the if-part, suppose the orientation of $\partial B, \partial D_1$ and $\partial D_2$ correspond to each other as in figure 11.

We conclude orientation $D_2 = -$orientation $B = - (-$orientation $D_1) =$ orientation $D_1$. Now suppose the orientation of $\partial B, \partial D_1$ and $\partial D_2$ do not correspond to each other as in figure 12. Then orientation $D_2 = $ orientation $B = $ orientation $D_1$, i.e. $\text{sign}(B) = 1$ in both cases.

We show the only-if-part in the same way assuming the negation of the statement about the induced orientations in the proposition and conclude $\text{sign}(B) = -1$.

Definition 4.9  A system of oriented disks (sod) consists of the following data: Let $D$ be a finite collection of oriented disks. Every disk $D \in D$ comes with distinct points $n_1, \ldots, n_k$ on $\partial D$ along the orientation of $\partial D$, where $n_j \in \mathbb{Z} \setminus \{0\}$, see figure 13.
In addition any number appears exactly twice in $D$. A disk/band surface of a sod is constructed by connecting each pair of equal numbers by a band $B$ with sign $(B)$ is the sign of the number.

**Proposition 4.10** Let $(S,D)$ an abstract graph diagram, $g : S \to \mathbb{R}^3$ an embedding, $p : \mathbb{R}^3 \to \mathbb{R}^2$ a regular projection for $g(S)$, $f := p \circ g$ and $pr : \mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \mapsto (x,y)$ the standard projection. Moreover let $E$ be the twisted graph diagram coming from the image of $S$ under $f$ and $\phi(E) = (S',D')$ for some choice of $S' \subset \mathbb{R}^3$ according to the definition of $\phi$. Then there is an embedding $f_\phi : S \to \mathbb{R}^3$ such that $f_\phi(S) = S'$ and

$$
\begin{array}{ccc}
S & \xrightarrow{g} & \mathbb{R}^3 \\
\downarrow{f_\phi} & & \downarrow{p} \\
\mathbb{R}^3 & \xrightarrow{pr} & \mathbb{R}^2
\end{array}
$$

is a commutative diagram.

**Proof.** First, choose an orientation of the disks and the bands of $S$. Then there is an sod, such that $S$ is a disk/band surface of that sod. Via the orientation-preserving homeomorphisms $f|_D : D \to f(D) \subset \mathbb{R}^2$ for every disk $D$ of $S$, we get another sod consisting of the disks $f(D)$ of the surface $S'$. The pairs of numbers on the boundaries of the disks define a one-to-one correspondence between the bands of $S$ and $S'$. It follows from Proposition 4.8 that those corresponding bands have the same sign, as $f$ preserves the orientation of the boundaries of the disks and the bands. Therefore $S$ and $S'$ have to be homeomorphic, as they are disk/band surfaces of homeomorphic sod with the same signs on the bands. We conclude that $S'$ is an image of an embedding $f_\phi$ of $S$ into $\mathbb{R}^3$. From the definition of $\phi$ it follows that the diagram commutes. \qed

To show, that $\psi$ is well-defined, we have

**Proposition 4.11** Let $(S,D)$ be an abstract graph diagram, $g, g' : S \to \mathbb{R}^3$ embeddings, $p, p' : \mathbb{R}^3 \to \mathbb{R}^2$ regular projections for $g(S)$ resp. $g'(S)$, $f := p \circ g$ and $f' := p' \circ g'$. Let $E, E'$ be the twisted graph diagrams coming from the image of $S$ under $f$ resp. $f'$. Then $E$ is equivalent to $E'$ in $TG$.  

8
Proof. First, choose an orientation of the disks and the bands of $S$. Suppose $D$ is a disk of $S$ such that $f'(D) \subset \mathbb{R}^2$ has the opposite orientation of $f(D) \subset \mathbb{R}^2$. Depending on whether there is a real crossing or a vertex inside the disk we get a diagram $C$ equivalent to $E'$ by performing a T3 resp. a T4-move at $E'$ for all such disks. As in the proof of proposition 4.3, there is a homeomorphism $H : \mathbb{R}^3 \to \mathbb{R}^3$ coming from rotating that disks around 2\pi such that $H(\phi(E')) = \phi(C)$. (To keep the notation short, by $\phi(\cdot)$ we mean only the surface-part of the abstract graph diagram.) As a result the disks $f(D) \subset \mathbb{R}^2$ of $\phi(E)$ and the disks $H \circ f'(D) \subset \mathbb{R}^2$ of $\phi(C)$ have the same orientation. Moreover, the diagrams $E$ and $C$ have the same Gauss data. With $f_\phi$ and $f'_\phi$ being the embeddings introduced in proposition 4.10 the composition

$$h : \phi(E) \xrightarrow{f_\phi} S \xrightarrow{f'_\phi} \phi(E') \xrightarrow{H} \phi(C)$$

maps the disks and bands of $\phi(E)$ to the disks and bands of $\phi(C)$. It follows from Proposition 4.8 that the bands mapped onto each other via $h$ have the same sign, because the disks have the same orientation. In the sense of definition 4.7, i.e. moving an orientation along the band, those bands must have the same number of twists modulo 2. Therefore the corresponding arcs of the diagrams $E$ and $C$ have the same number of bars modulo 2. Combining this with Proposition 4.6 and the twisted Reidemeister moves we see that $E \approx C$, thus $E \approx E'$.

**Proposition 4.12** $\Psi$ is well-defined.

Proof. We have to show $\Psi([[(S, D)]]) = \Psi([[(S', D')]])$ for equivalent abstract graph diagrams $(S, D)$ and $(S', D')$. Assume that $(S, D)$ and $(S', D')$ differ by an abstract Reidemeister move. Then there are embeddings $f : S \to F$ and $f' : S' \to F$ into a closed surface $F$, such that $f(D)$ and $f'(D')$ differ by a Reidemeister move inside a disk $\Sigma \subset F$ of type I, II, III, IV, V or VI. Outside the disk the diagrams are identical, i.e. $f(D) \cap F \setminus \Sigma = f'(D') \cap F \setminus \Sigma$. Hence we may choose regular neighborhoods $N$ and $N'$ of $f(D)$ resp. $f'(D')$ satisfying $N \approx f(S) \approx S$, $N' \approx f'(S') \approx S'$, $N \setminus \Sigma = N' \setminus \Sigma$ and $N \cup \Sigma = N' \cup \Sigma$. Applying $\psi$ to the abstract graph diagram $(N, f(D))$ we get an embedding $g : N \to \mathbb{R}^3$ and a regular projection $p$ for $g(N)$. As $N$ and $N'$ are equal outside $\Sigma$ it is easy to construct an embedding $h : N \cup \Sigma = N' \cup \Sigma \to \mathbb{R}^3$ and a projection $\tilde{p}$ regular for $h(N \cup \Sigma)$ with $h$ equal to $g$ when restricted to $N$, such that the twisted graph diagram $E$ belonging to $p \circ g(N)$ resp. $E'$ coming from $\tilde{p} \circ h|_{N'}$ ($N'$) differ by the same Reidemeister move mentioned above. Therefore we calculate

$$\Psi([[(S, D)]] = \Psi([[(N, f(D))]]) = \psi(N, f(D)) = [E] = [E']$$
\[ \psi(N', f'(D')) = \Psi(\{(N', f'(D'))\}) = \Psi(\{(S', D')\}). \]

**Proof of Theorem.** \( \Phi \) injective: Let \( D', E' \in TG \), \( \phi(D') = (F_D, D) \), \( \phi(E') = (F_E, E) \) and

\[
\Phi([D']) = \Phi([E']).
\]

The projection \( pr : \mathbb{R}^3 \to \mathbb{R}^2, (x, y, z) \mapsto (x, y) \) is regular for \( F_E \) and \( F_D \), and \( pr(D) = D', pr(E) = E' \) by the definition of \( \phi \). This implies \( [D'] = \psi((F_D, D)) \) and \( [E'] = \psi((F_E, E)) \). Thus

\[
[D'] = \Psi(\{(F_D, D)\}) = \Psi(\{(F_E, E)\}) = [E']
\]

as \( \Psi \) is well-defined.

To show that \( \Phi \) is surjective let \( (S, D) \in AG \) and \( [E] := \Psi(\{(S, D)\}) \). Then \( E \) is constructed via an embedding \( g : S : \mathbb{R}^3 \to \mathbb{R}^3 \) and a regular projection \( p \) for \( g(S) \). Because of proposition 4.10, the disk/band surface of the abstract graph diagram \( \phi(E) \) is homeomorphic to \( S \). As the over/under informations of \( D \) on \( S \) correspond to those of \( E \), we get \( \phi(E) \approx (S, D) \) and from that \( \Phi([E]) = [\phi(E)] = [(S, D)] \).

## 5 Pure Twisted Graph Diagrams

**Definition 5.1** A twisted graph diagram \( E \) is called pure if it has only virtual crossings.

**Definition 5.2** Let \( E \) be a twisted graph diagram without circle components \( (c.c.) \), \( \phi(E) = (S, D) \) the corresponding abstract graph diagram and

1. \( k(S) := \# \) connected components of \( S \),
2. \( n(S) := \) first betti-number of \( S \),
3. \( b(S) := \# \) boundary components of \( S \),
4. \( t(S) := 0 \), if \( S \) is orientable, otherwise \( t(S) := 1 \).

Define \( M(\emptyset) := 1 \) and

\[
M(E)(y, z, w) := (-1)^{k(S)} y^{n(S)} z^{k(S) - b(S) + n(S)} w^{t(S)}
\]

as a polynomial in \( \mathbb{Z}[y, z, w] \) modulo \( (w^2 - w) \). Let \( F \) be a twisted graph diagram possibly with \( c.c. \) For the number of \( c.c. \) having an odd number of
bars we write $o(F)$, for those with no or an even number of bars $e(F)$. Then define $Q(\emptyset) := 1$ and

$$Q(F)(y, z, w) := (-1 - y)^{e(F)} (-1 - yzw)^{o(F)} \sum_{E \subset F} M(E).$$

Here by $E \subset F$ we mean a twisted graph (sub-)diagram $E$ (of $F$) belonging to a spanning subgraph of $F$ ignoring the c.c.

**Remark 5.3** The polynomial $M$ is that of [1] for $X = 0$. As $(S, D)$ is defined up to homeomorphism, so are $M$ and $Q$.

**Remark 5.4** From the previous section we know that Reidemeister moves $I^*, II^*, III^*$ and $IV^*$ do not change the abstract graph diagrams. Hence $Q$ is invariant under those moves.

**Remark 5.5** From the previous section we know that Reidemeister moves $T1, T2, T3$ and $T4$ do not change the abstract graph diagrams. Hence $Q$ is invariant under those moves as well.

**Example 5.6**

1. For a vertex we calculate $Q(\bullet) = M(\bullet) = -1$.

2. For a pure twisted graph diagram $F$ without c.c. we have $Q(F) = \sum_{E \subset F} M(E)$.

3. $Q(\bigcirc) = M(\bullet) + M(\bigcirc) = -1 - y = Q(\bigcirc)$.

4. $Q(\bigtriangledown) = M(\bullet) + M(\bigtriangledown) = -1 - yzw = Q(\bigtriangledown)$.

**Definition 5.7** Let $E$ be a twisted graph diagram looking like figure 14 inside a disk. We call the twisted graph diagram $E/e$ the contraction of $E$ along a twisted edge $e$ and define it to be identical with $E$ outside the disk and to look like figure 15 inside the disk.

**Remark 5.8** The deletion $E - e$ is defined in the usual way no matter if $e$ has a bar or not. That is, we omit the edge $e$ in the diagram. If $e$ has no bar, then the contraction $E/e$ is the usual one as well.
Remark 5.9 Contracting along an arbitrary edge is always possible, because with Reidemeister moves IV and IV* a situation like figure 14 can be obtained.

Remark 5.10 By definition, contracting along a twisted edge is the same as contracting along an ordinary edge after performing a T4-move.

Remark 5.11 Note that even though the disk/band surfaces of the abstract graph diagrams belonging to $E$ and $E/e$ are homeomorphic, the abstract graph diagrams are not, because their diagrams are different. Nevertheless we have $M(E) = M(E/e)$.

Definition 5.12 A twisted graph diagram $E$ in $\mathbb{R}^2$ is split into subdiagrams $E_1$ and $E_2$ if there is a simple close curve in $\mathbb{R}^2 \setminus E$ seperating $\mathbb{R}^2$ into a disk $\Sigma$ and $\mathbb{R}^2 \setminus \Sigma$ containing $E_1$ resp. $E_2$. We write $E = E_1 \sqcup E_2$.

If $E_1$ and $E_2$ share exactly one vertex $v$, $E$ is a union of $E_1$ and $E_2$ and there is a simple closed curve in $(\mathbb{R}^2 \setminus E) \cup v$ meeting $v$ and seperating $\mathbb{R}^2$ into a disk $\Sigma$ and $\mathbb{R}^2 \setminus \Sigma$ with $E_1 \subset \Sigma$, $E_2 \subset (\mathbb{R}^2 \setminus \Sigma) \cup v$, we call $E$ a vertex connected sum and name it $E = E_1 \vee E_2$.

An edge $e$ of $E$ is a cut-edge if $E - e$ is a split diagram.

Proposition 5.13 Let $E$ be a pure twisted graph diagram and $e$ a non-loop edge which is not a c.c. Then $Q(E) = Q(E/e) + Q(E - e)$.

Proof. As $(-1 - y)^{e(E)} = (-1 - y)^{e(E/e)} = (-1 - y)^{e(E - e)} =: \alpha$ and $(-1 - yzw)^{o(E)} = (-1 - yzw)^{o(E/e)} = (-1 - yzw)^{o(E - e)} =: \beta$ we calculate

$$Q(E) = \alpha \beta \sum_{F \subset E} M(F) = \alpha \beta \left[ \sum_{\{F \subset E \mid e \notin F\}} M(F) + \sum_{\{F \subset E \mid e \in F\}} M(F) \right] \overset{5.11}{=} \alpha \beta \left[ \sum_{F \subset E - e} M(F) + \sum_{F \subset E/e} M(F) \right] = Q(E - e) + Q(E/e).$$

Proposition 5.14 We obtain $Q(E_1 \sqcup E_2) = Q(E_1) Q(E_2)$ for pure twisted graph diagrams $E_1$ and $E_2$.

Proof. Before we proof the proposition we note that

$$o(E_1 \sqcup E_2) = o(E_1) + o(E_2), e(E_1 \sqcup E_2) = e(E_1) + e(E_2) \quad (2)$$
for the c.c. , and there is a one-to-one correspondence between the sets
\[
\{ F \subset E_1 \cup E_2 \} \longleftrightarrow \{ F_1 \subset E_1 \} \times \{ F_2 \subset E_2 \}.
\] (3)
Moreover from [1] we know that the proposition is true for the polynomial \( M \). To abbreviate the notation let \( A := -1 - y \) and \( B := -1 - yzw \) in the following calculation:

\[
Q(E_1)Q(E_2) = \left[ A^{e(E_1)}B^{o(E_1)} \sum_{F_1 \subset E_1} M(F_1) \right] \left[ A^{e(E_2)}B^{o(E_2)} \sum_{F_2 \subset E_2} M(F_2) \right]
\] (2)

\[ = A^{e(E_1 \cup E_2)}B^{o(E_1 \cup E_2)} \sum_{F_1 \subset E_1} \left[ \sum_{F_2 \subset E_2} M(F_2) \right] M(F_1)
\]

\[ = A^{e(E_1 \cup E_2)}B^{o(E_1 \cup E_2)} \sum_{F_1 \times F_2 \subset \{ F_1 \subset E_1 \} \times \{ F_2 \subset E_2 \}} M(F_1)M(F_2)
\] (3)

\[ = A^{e(E_1 \cup E_2)}B^{o(E_1 \cup E_2)} \sum_{F \subset E_1 \cup E_2} M(F) = Q(E_1 \cup E_2).
\]

**Proposition 5.15** We have \( Q(E_1 \cup E_2) = -Q(E_1)Q(E_2) \) for pure twisted graph diagrams \( E_1 \) and \( E_2 \).

**Proof.** First we note that there is a 1-1-correspondence between the sets
\[
\{ F \subset E_1 \cup E_2 \} \longleftrightarrow \{ F_1 \subset E_1 \} \times \{ F_2 \subset E_2 \}.
\] (4)
For the number of c.c. we have
\[
o(E_1 \cup E_2) = o(E_1) + o(E_2), \quad e(E_1 \cup E_2) = e(E_1) + e(E_2).
\] (5)
Moreover from [1] we know that the proposition is true for the polynomial \( M \). Using \( A := -1 - y \) and \( B := -1 - yzw \) we calculate

\[
Q(E_1)Q(E_2) = \left[ A^{e(E_1)}B^{o(E_1)} \sum_{F_1 \subset E_1} M(F_1) \right] \left[ A^{e(E_2)}B^{o(E_2)} \sum_{F_2 \subset E_2} M(F_2) \right]
\] (2)

\[ = A^{e(E_1 \cup E_2)}B^{o(E_1 \cup E_2)} \sum_{F_1 \subset E_1} \left[ \sum_{F_2 \subset E_2} M(F_2) \right] M(F_1)
\]

\[ = A^{e(E_1 \cup E_2)}B^{o(E_1 \cup E_2)} \sum_{F_1 \times F_2 \subset \{ F_1 \subset E_1 \} \times \{ F_2 \subset E_2 \}} M(F_1)M(F_2)
\] (3)

\[ = A^{e(E_1 \cup E_2)}B^{o(E_1 \cup E_2)} \sum_{F \subset E_1 \cup E_2} M(F) = -Q(E_1 \cup E_2).
\]
Proposition 5.16 If a pure twisted graph diagram $E$ has a cut edge $e$ then $Q(E) = 0$.

Proof. We may write $E - e = E_1 \sqcup E_2$ and $E/e = E_1 \vee E_2$ for appropriate subdiagrams $E_1$ and $E_2$. Note that we need not bother if $e$ has a bar or not because of remark 5.10. Thus

$$Q(E) = Q(E - e) + Q(E/e) = Q(E_1 \sqcup E_2) + Q(E_1 \vee E_2) = Q(E_1)Q(E_2) - Q(E_1)Q(E_2) = 0.$$  

□

The next proposition shows that $Q$ is a topological invariant in the sense that it does not care about vertices of degree 2.

Proposition 5.17 Let $E$ be a pure twisted graph diagram looking like figure 16 inside a disk $\Sigma$ and $E'$ the pure twisted graph diagram being identical with $E$ outside and looking like figure 17 inside $\Sigma$. Then $Q(E) = Q(E')$.

![Figure 16](image1.png) ![Figure 17](image2.png)

Proof. If the two segments of figure 16 belong to the same edge, we use example 5.6 together with proposition 5.14 to show the assertion. Now suppose those segments belong to different edges $e$ and $f$. Then $f$ is a cut edge for $E - e$ and $E'$ is equivalent to $E/e$ as $E$ is pure. Using the above propositions we calculate $Q(E) = Q(E - e) + Q(E/e) = 0 + Q(E')$.

6 An Invariant for Twisted Graph Diagrams

Definition 6.1 Let $E$ be a twisted graph diagram. For a crossing $c$ of $E$ we define the spin of $c$ to be $1, -1$ or $0$ as shown in figure 18. The pure twisted graph diagram obtained by replacing each crossing with a spin is called a state of $E$. The set of states will be denoted by $S(E)$. For $S \in S(E)$ put $\{E \mid S\} := a^{p-q}$, where $p$ and $q$ are the numbers of crossings with spin $+1$ and resp. $-1$ in $S$. Now define a polynomial

$$R(E)(a, z, w) := \sum_{S \in S(E)} \{E \mid S\} Q(S) \left(-a - 2 - a^{-1}, z, w\right).$$ (6)
Remark 6.2 If $E$ is pure we have $R(E) (a, z, w) = Q(E) (-a - 2 - a^{-1}, z, w)$.

Proposition 6.3 The contraction/deletion formula is valid for the polynomial $R$, i.e. $R(\overline{\textbullet}) = R(\textbullet) + R(\langle \rangle)$.

Proof. First we note
\[
\{\overline{\textbullet} | S\} = \{\langle \rangle | S\} = \{\textbullet | S\}
\] (7)
for any state $S$. Hence we calculate
\[
R(\overline{\textbullet}) = \sum_{S \in S(\overline{\textbullet})} \{\overline{\textbullet} | S\} Q(S) = \sum_{S \in S(\overline{\textbullet})} \{\textbullet | S\} [Q(\langle \rangle) + Q(\textbullet)]
\]
\[
= \sum_{S \in S(\langle \rangle)} \{\langle \rangle | S\} Q(\langle \rangle) + \sum_{S \in S(\textbullet)} \{\textbullet | S\} Q(\textbullet)
\]
\[
= R(\langle \rangle) + R(\textbullet).
\]

Proposition 6.4 $R(\textbullet) = aR(\langle \rangle) + a^{-1}R(\overline{\textbullet}) + R(\textbullet)$.

Proof. Let $S$ be a state. We write $p = p(S, \cdot)$ and $q = q(S, \cdot)$. Then $p(S, \langle \rangle) = p(S, \textbullet) - 1$ and $q(S, \langle \rangle) = q(S, \textbullet)$, hence
\[
\{\langle \rangle | S\} = a^{p(S, \textbullet) - q(S, \textbullet)} = a^{p(S, \langle \rangle) - q(S, \textbullet)} = a^{-1} \{\textbullet | S\}.
\]
In an analogue manner we obtain $\{\overline{\textbullet} | S\} = a \{\textbullet | S\}$ and $\{\textbullet | S\} = \{\textbullet | S\}$, therefore $R(\textbullet)
\[
= \sum_{S \in S(\langle \rangle)} \{\langle \rangle | S\} Q(S) + \sum_{S \in S(\textbullet)} \{\textbullet | S\} Q(S) + \sum_{S \in S(\overline{\textbullet})} \{\overline{\textbullet} | S\} Q(S)
\]
\[
= aR(\langle \rangle) + a^{-1}R(\overline{\textbullet}) + R(\textbullet).
\]

Proposition 6.5 We obtain $R(E_1 \sqcup E_2) = R(E_1)R(E_2)$ for twisted graph diagrams $E_1$ and $E_2$. 

Figure 18
Proof. Let \( E = E_1 \sqcup E_2 \) and \( S \in S(E) \). Then \( S = S_1 \sqcup S_2 \) for unique \( S_i \in S(E_i) \). We write \( p = p(S, \cdot) \) and \( q = q(S, \cdot) \). Hence \( p(E_1 \sqcup E_2, S) = p(E_1, S_1) + p(E_2, S_2) \), \( q(E_1 \sqcup E_2, S) = q(E_1, S_1) + q(E_2, S_2) \) and therefore \( \{E_1 \sqcup E_2 \mid S_1 \sqcup S_2\} = \{E_1 \mid S_1\} \{E_2 \mid S_2\} \). We check the equation as in the proof of proposition 5.14 using the assertion of that proposition.

**Proposition 6.6** We have \( R(E_1 \vee E_2) = -R(E_1) R(E_2) \) for twisted graph diagrams \( E_1 \) and \( E_2 \).

**Proof.** Replace \( E_1 \sqcup E_2 \) with \( E_1 \vee E_2 \) and 5.14 with 5.15 in the proof of proposition 6.5.

**Proposition 6.7** \( R(E) = 0 \) if a twisted graph diagram \( E \) has a cut-edge.

**Proof.** Let \( e \) be a cut-edge of \( E \), \( E - e = E_1 \sqcup E_2 \), \( E_1 \subset \Sigma \) and \( E_2 \subset \mathbb{R}^2 \setminus \Sigma \). Then the components of a state \( S \) not containing the arc \( a \) of \( S \) coming from the edge \( e \) are either contained in \( \Sigma \) or in \( \mathbb{R} \setminus \Sigma \). Therefore \( S - a \) is split, hence \( a \) is a cut-edge for \( S \). Now the assertion follows by means of proposition 5.16.

**Example 6.8** Let \( y = -a - 2 - a^{-1} \).

1. \( R(\bullet)(a, z, w) = Q(\bullet)(-a - 2 - a^{-1}, z, w) = -1 \).
2. \( R(\bigoten)(a, z, w) = Q(\bigoten)(-a - 2 - a^{-1}, z, w) = -1 + (a + 1 + a^{-1}) = \sigma = R(\bigoten) \).
3. \( R(\avert) = -1 - (-a - 2 - a^{-1}) zw = -1 + (\sigma + 1) zw = R(\avert) \).
4. \( R(\bigpentagon) = Q(\bigpentagon) = M(\bullet) + 2M(\bigoten) + M(\bigpentagon) = -1 - 2y - y^2 z^2 \).
5. \( R(\bigpentagon) = -R(\bigpentagon) R(\bigpentagon) = -(-1 - y)^2 \).

Because of propositions 6.3, 6.4, 6.5, 6.6 and example 6.82 the propositions 4 and 5 as well as theorem 5 of [9] are valid in our setting. We sum it up in

**Proposition 6.9** The polynomial \( R \) in \( \bigoten \) is invariant under Reidemeister moves II, III, IV and up to multiplication with some \((-a)^n \) invariant under I and V.

**Proposition 6.10** The polynomial \( R \) in \( \bigoten \) is invariant under Reidemeister moves I*, II*, III* and IV*.
Proof. If twisted graph diagrams $E$ and $E'$ differ by one of the moves mentioned in the assertion then for each state $S \in S(E)$ there is a unique state $S' \in S(E')$ differing by the same Reidemeister move. As $E$ and $E'$ have the same crossings, we obtain $\{E \mid S\} = \{E' \mid S'\}$. From remark 5.4 we know $Q(S) = Q(S')$. Thus the proof is finished by the definition of $R$.

**Proposition 6.11** The polynomial $R$ in (6) is invariant under Reidemeister move $V^*$.

**Proof.** Because of propositions 6.4 and 6.10 we may calculate $R(\includegraphics{X}) = aR(\includegraphics{X}) + a^{-1}R(\includegraphics{X}) + R(\includegraphics{X}) = aR(\includegraphics{X}) + a^{-1}R(\includegraphics{X}) + R(\includegraphics{X}) = R(\includegraphics{X})$.

**Proposition 6.12** The polynomial $R$ in (6) is invariant under Reidemeister moves $T1$, $T2$, $T4$.

**Proof.** The proof is exactly the same as in proposition 6.10 except for replacing remark 5.4 by remark 5.5.

**Proposition 6.13** The polynomial $R$ in (6) is invariant under Reidemeister move $T3$.

**Proof.** We calculate

\[
R(\includegraphics{X\includegraphics{X\includegraphics{X}}}) = aR(\includegraphics{X\includegraphics{X\includegraphics{X}}}) + a^{-1}R(\includegraphics{X\includegraphics{X\includegraphics{X}}}) + R(\includegraphics{X\includegraphics{X\includegraphics{X}}}) \quad (8)
\]

\[
= aR(\includegraphics{X\includegraphics{X\includegraphics{X}}}) + a^{-1}R(\includegraphics{X\includegraphics{X\includegraphics{X}}}) + R(\includegraphics{X\includegraphics{X\includegraphics{X}}}) \quad (9)
\]

\[
= R(\includegraphics{X\includegraphics{X\includegraphics{X}}}) \quad (10)
\]

using proposition 6.4 in (8) resp. (10) and Reidemeister moves I*, II*, T2 and T4 in (9).

7 Relations to other polynomials

Let $S$ be a state. As usual we regard $S$ as a pure twisted graph diagram as well as the underlying abstract graph. Define $k(S) = \#$ components of $S$, $n(S) = \text{first betti number of } S$, $E(S) = \text{set of edges of } S$, $V(S) = \text{set of vertices of } S$, $u(S) = \# \text{ of circle components of } S$ and $\hat{F} = \text{spanning subgraph/subdiagram of } S$ with edge set $F \subset E(S)$.

For a classical graph diagram $D$ the Yamada polynomial is defined to be

\[
Y(D)(a) = \sum_{S \in S(D)} \{D \mid S\} h(S)(-1,y), \quad y := -a - 2 - a^{-1} \quad \text{where}
\]
\[ h(S)(-1, y) = \sum_{F \subseteq E(S)} (-1)^{k(S-F)} y^{n(S-F)} = \sum_{F \subseteq S} (-1)^{k(F)} y^{n(F)}. \]

Note that we consider each c.c. of \( S \) as a loop with one vertex of degree 2. In the last summation \( F \) raises over all spanning subgraphs/subdiagrams of \( S \). From [9] we know \( h(\bigcirc) (-1, y) = -1 - y. \)

**Proposition 7.1** Let \( D \) be a classical graph diagram possibly with circle components. Then \( R(D)(a, 1, 1) = Y(D)(a). \)

**Proof.** For \( z = w = 1 \) we get \( R(D)(a, 1, 1) = \sum_{S \in S(D)} \{ D \mid S \} Q(S)(y, 1, 1) \) with \( y = -a - 2 - a^{-1} \). Thus we have to show \( h(S)(-1, y) = Q(S)(y, 1, 1): \)

\[
Q(S)(y, 1, 1) = (-1 - y)^{e(S)} (-1 - y)^{o(S)} \sum_{E \subseteq S} M(E)(y, 1, 1)
= (-1 - y)^{u(S)} \sum_{E \subseteq S} (-1)^{k(E)} y^{n(E)}
= h(\bigcirc) (-1, y) \hat{\#} h(S \setminus \text{c.c.}) (-1, y) \tag{11}
= h(S)(-1, y).
\]

Note that in [11] we identify each c.c. of \( D \) with a loop \( \bigcirc \).

Let \( E \) be a virtual graph diagram possibly with c.c. In [7] a polynomial is defined as follows:

\[
H_E(a, 1) = \sum_{S \in S(E)} \{ E \mid S \} Z_S (a + 2 + a^{-1}),
\]

\[
Z_S(-y) = (-1 - y)^{u(S)} y^{-\# V(S)} \sum_{F \subseteq E(S)} (-y)^{k(F)} y^{n(F)} \tag{12}
\]

where \( y = -a - 2 - a^{-1} \). It turns out that this is the Yamada polynomial for virtual graphs introduced in [3]. For the convenience of the reader we proof this fact in this context.

**Proposition 7.2** Let \( E \) be a virtual graph diagram possibly with circle components. Then \( H_E(a, 1) = R(E)(a, 1, 1). \)

**Proof.** Set \( y = -a - 2 - a^{-1} \). It is sufficient to show \( Z_S(-y) = h(S)(-1, y) \) for a state \( S \) because of proposition 7.1. From (12) we get

\[
Z_S(-y) = (-1 - y)^{u(S)} \sum_{F \subseteq E(S)} (-1)^{k(F)} y^{-\# V(F) + k(F) + \# E(F)}
= (-1 - y)^{u(S)} \sum_{F \subseteq S} (-1)^{k(F)} y^{n(F)} = h(S)(-1, y).
\]
8 Applications

By contrast with the Yamada polynomial our polynomial distinguishes certain diagrams. The reason is that the Yamada polynomial ignores the virtual crossings of a virtual bouquet and the R-polynomial does not, see example 6.8. The following two diagrams from [3], figure 20 have the same Yamada polynomial but different R-polynomials.

Example 8.1 Let \( y = -a - 2 - a^{-1} \).

1. \( R(\ominus) = R(\bigcirc) + R(\bigcirc) = R(\bigcirc) - R(\bigcirc)^2 = -1 - y - (-1 - y)^2 \)
2. \( R(\bigcirc) = R(\bigcirc) + R(\bigcirc) = Q(\bigcirc) + Q(\bigcirc) = M(\bullet \bullet) + 2M(\rightarrow \rightarrow) + M(\bigcirc) + Q(\bigcirc) = -2 - 3y - y^2z^2. \)

Definition 8.2 Let \( S \) be an orientable, connected disk/band surface. The minimum genus among all closed orientable surfaces in which \( S \) is embeddable is called the supporting genus of \( S \).

Remark 8.3 Glueing 2-disks to the boundary components of \( S \) in definition 8.2 we obtain a closed orientable surface \( \sigma(S) \) realizing the supporting genus of \( S \).

Proposition 8.4 Suppose \( S \) is an orientable, connected disk/band surface. Then \( n(\sigma(S)) = 2 - \chi(\sigma(S)) = 1 - b(S) + n(S) \).

Proof. The surface \( S \) is homeomorphic to a disk with bands attached, see [6] figure 6.1. We write \( n_1(S) \) for the number of generators of \( H_1S \) coming from the 'handles' and \( n_2(S) \) for the number of generators belonging to boundary components. Then \( n(S) = n_1(S) + n_2(S) \), \( n_2(S) = b(S) - 1 \) and \( n(\sigma(S)) = n_1(S) - n_2(S) = n(S) - b(S) + 1 \). As \( \sigma(S) \) is connected and orientable, the rank of \( H_0S \) and \( H_2S \) is 1. The first equation follows immediately.

Consider the maximal degree of \( z \) in the polynomial \( R \) resp. \( Q \). Because of propositions 6.9 to 6.11 we will call it the \( z \)-degree.

Proposition 8.5 For a classical graph diagram \( D \) the \( z \)-degree of \( R(D) \) is zero.

Proof. Each state \( S \) of \( D \) has neither virtual nor real crossings. Hence \( S \) is a planar embedding. Let \( y = -a - 2 - a^{-1} \). As \( D \) is not twisted, we have \( Q(S)(y, z, w) = (-1 - y)^{n(S)} \sum_{E \in S}(-1)^{k(F)}y^{n(\bigcirc)}z^{k(F)} + 1 \) where \( F \) denotes the surface-part of the abstract graph diagram corresponding to the
subdiagram $E$ of $S$. Let $F_i$ be the components of $F$. Then $k(F) - b(F) + n(F) = \sum k(F_i) - b(F_i) + n(F_i) = \sum 1 - b(F_i) + n(F_i) = \sum 2 - \chi(\sigma(F_i))$. Each component of $S$ is a planar embedding, hence $F_i$ is homeomorphic to a planar embedding of a disk/band surface. Thus $\sigma(F_i) \approx S^2$, i.e. $\chi(\sigma(F_i)) = 2$ finishing the proof.

□

As an immediate corollary we have

**Proposition 8.6** If the degree of $z$ in $R(D)$ is not zero, then $D$ is not a classical graph diagram.

Suppose $E$ is a virtual graph diagram. For the number of virtual crossings of $E$ we write $\#\text{vcr}(E)$.

**Definition 8.7** The virtual crossing number $\text{vcr}(E)$ of a virtual graph diagram $E \in \mathcal{VG}$ is defined to be $\min \{ \#\text{vcr}(E') \mid E' \sim E \text{ in } \mathcal{VG} \}$.

**Proposition 8.8** For a virtual graph diagram the $z$-degree of $R$ is bounded above by the virtual crossing number as follows: $z$-degree $R(E) \leq 2\text{vcr}(E)$.

**Proof.** Let $E_1, \ldots, E_n$ be the components of $E$. Firstly suppose $E$ is pure. The surface $S$ of the abstract graph diagram $\phi(E)$ consists of components $S_i$ coming from $\phi(E_i)$. Consider each $E_i$ as a diagram in $S^2$. Instead of modifying $S_i$ as in 2. of figure 3 we add a handle and let the surface $S_i$ pass it. Then $S_i$ is embedded in a closed orientable surface $F_{g_i}$ of genus $\#\text{vcr}(E_i) = g_i$. In an analogue manner $S$ is embedded in $F_g$ where $g = \#\text{vcr}(E)$. Thus $\sum g_i = \sum \#\text{vcr}(E_i) \leq \#\text{vcr}(E) = g$. Now attach disks to the boundary components of $S_i$ to obtain $\sigma(S_i)$ having genus $\tilde{g}_i$. From remark 8.3 we know $\tilde{g}_i \leq g_i$. Then $z$-degree $M(E) = k(S) - b(S) + n(S) = \sum 1 - b(S_i) + n(S_i) = \sum n(\sigma(S_i)) = \sum 2\tilde{g}_i \leq \sum 2g_i \leq 2g$. As $E$ has no twists, $Q$ has the form $Q(E)(y, z, w) = (-1 - y)^{\chi(S)} \sum_{D \subset E} M(D)(y, z, w)$. Therefore we calculate $z$-degree $Q(E) = \max \{ z$-degree $M(D) \mid D \subset E \} \leq \max \{ 2\#\text{vcr}(D) \mid D \subset E \} \leq 2\#\text{vcr}(E)$.

Now suppose $E$ is a virtual graph diagram not necessarily pure. Abbreviating $y = -a - 2 - a^{-1}$ we have

$$z\text{-degree } R(E)(a, z, w) = \max \{ z\text{-degree } Q(S)(y, z, w) \mid S \in \mathcal{S}(E) \} \leq \max \{ 2\#\text{vcr}(S) \mid S \in \mathcal{S}(E) \} \leq 2\#\text{vcr}(E).$$

Taking the minimum over all diagrams equivalent to $E$ in $\mathcal{VG}$ finishes the proof.
Example 8.9 Consider a virtual diagram $E$ of the handcuff graph shown in figure 19.

![Figure 19](image)

We use the algorithm [3] to determine the polynomial of $E$. The result is

$$R(E) = -a^{-5} \left(r_4(a)z^4 + r_2(a)z^2 + r_0(a)\right)$$

with

$$r_4(a) = a^9 + 8a^8 + 28a^7 + 56a^6 + 70a^5 + 56a^4 + 28a^3 + 8a^2 + a,$$

$$r_2(a) = -15a^8 - 43a^7 - 70a^6 - 81a^5 - 70a^4 - 37a^3 - 6a^2 - 2a^9 + 3a + 1,$$

$$r_0(a) = 6a^8 + 14a^7 + 13a^6 + 11a^5 + 14a^4 + 10a^3 - a^2 - 3a.$$

We conclude $4 = z\text{-degree} \ R(E) \leq 2vcr(E)$. Thus $vcr(E) = 2$.

Example 8.10 Consider a diagram of the handcuff graph of order 4 depicted below. Let $S$ be the state with zero spin at each crossing.

![Figure 20](image)
Then \( k(S) = 1, b(S) = 1 \) and \( n(S) = 8 \). We conclude \( 8 = k(S) + n(S) - b(S) \leq z - \text{degree } R(E) \leq 2vcr(E) \). Hence \( vcr(E) = 4 \).

**Proposition 8.11** For every \( n \in \mathbb{N} \) there is a virtual graph diagram with virtual crossing number \( n \).

**Proof.** For \( n = 1 \) see example 6.8.4, for \( n = 2 \) example 8.9. For \( n \geq 3 \) consider the handcuff graph of order \( n \) as in example 8.10.

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