Time-Dependent Current Partition in Mesoscopic Conductors

M. Büttiker

IBM T. J. Watson Res. Ctr., P. O. Box 218, Yorktown Heights, N. Y. 10598, and
Université de Genève, Département de Physique Théorique, 24 quai E. -Ansermet,
CH-1211 Genève, Switzerland.

(Received November 12, 2018)

The currents at the terminals of a mesoscopic conductor are evaluated in the
presence of slowly oscillating potentials applied to the contacts of the sample. The
need to find a charge and current conserving solution to this dynamic current parti-
tion problem is emphasized. We present results for the electro-chemical admittance
describing the long range Coulomb interaction in a Hartree approach. For multiply
connected samples we discuss the symmetry of the admittance under reversal of an
Aharonov-Bohm flux.

PACS numbers: 72.10.Bg, 72.30.+9, 73.50.Td, 72.70.+m
I. INTRODUCTION

In this work we present results for the dynamic current partition in phase-coherent, mesoscopic conductors in the presence of oscillating voltages applied to the contacts of the sample. The time-dependent oscillating voltages cause small departures of the sample away from its equilibrium state. A specific configuration of conductors is shown in Fig. 1. The contacts are numbered consecutively, \( k = 1, 2, 3 \). The structure which connects contacts 2 and 3 is called the "conductor". A second structure, which interacts with the conductor only via long range Coulomb forces and is connected to a single contact \( k = 1 \), is called the "gate". The currents are generated by a small oscillating voltage \( \delta V_k(\omega) \) applied to the contact of the gate or one of the contacts of the conductor. The task is to find the oscillating current \( \delta I_k(\omega) \) at contact \( k \) of the structure. Below we emphasize the need to find a current conserving answer, \( \sum_k \delta I_k(\omega) = 0 \), to this problem.

For single charges moving in the electric field of an arrangement of capacitors a current conserving answer was given by Shockley [1] and Ramo [2]. Clearly, in the problem of interest here, we deal not with the motion of single charges but face a many electron problem. Despite this, a part of the theoretical literature takes the discussion of Shockley and Ramo to derive expressions for the current of a conductor in the form of a volume integral over the velocities of the carriers [3,4]. Any mutual interaction between the carriers is neglected. Another larger portion of the literature is even less perceptive and treats frequency dependent transport as a linear response problem to an electric field which is supposed to be known. In the most naive examples the electric field is taken to be uniform even for conductors which are strongly inhomogeneous [5,6]. The extent to which this approach is accepted is illustrated by the fact that this portion of the literature includes authors which are famous for their contributions to solid state physics [7].

The ac-response of non-interacting carriers is not current conserving: Application of a time-dependent field to the sample of Fig. 1 will give rise to currents at the terminals of the gate and the conductor which are unrelated to one another. In reality currents are necessarily
conserved \[8\]. Current conservation is closely related to the fact that the electrical response of a conductor depends only on voltage differences. Hence an interpretation of a linear response calculation of independent electrons in terms of dynamic conductance coefficients is not possible!

It is remarkable that the problem addressed here, which is so central and basic to solid state electronics, has found so little attention. We can cite only a few works where the authors express a similar perception \[110\]. Below we emphasize the need of a charge and current conserving approach to ac-conductance and exemplify this with a discussion of the ac-response of the conductor shown in Fig. 1. We restrict our attention to mesoscopic conductors: It is assumed that carriers inside the sample are subject to elastic scattering and are subject to (long range) Coulomb forces. There is no inelastic scattering within the sample. The discussion of the dc conductance of mesoscopic conductors has already taught us that we must focus on the entire conductor including its current and voltage contacts \[11\]. To discuss ac-conductance we must take an even a wider view and consider not only the conductor but also other nearby metallic bodies \[12\]. In the sample of Fig. 1 the ”gate” is such a nearby metallic body. Fortunately, there is currently a growing experimental interest in the ac-properties of small mesoscopic conductors. A recent experiment by Pieper and Price determines both the real and imaginary part of the ac-conductance of a metallic diffusive Aharonov-Bohm loop over a wide range of frequencies \[13\]. A theoretical discussion of these results is presented by Pieper and Price \[14\] using a diagramatic approach and by Liu et al. \[15\] using the results of Ref. \[8\]. These results should be contrasted with earlier work on metallic wires by Webb et al. and De Vegvar et al. \[16\] which found a non-linear response even at very low voltages and very low frequencies. For the work presented here the experiment by Chen et al. \[17\] which measures the magnetic field symmetry of capacitance coefficients \[12\] on a large two dimensional electron gas is of particular interest. The geometry of this experiment corresponds to that of Fig. 1. Below we state the results for the low frequency admittance of this geometry.

The relationship between the currents at the contacts of the sample and the time-
dependent potentials applied to the contacts of the sample is given by a dynamic conductance or admittance [8]

$$\delta I_k = \sum_l G_{kl}(\omega) \delta V_l(\omega).$$  \hspace{1cm} (1)

We emphasize that this conductance is a response to oscillating electro-chemical potentials \(\mu_k \equiv eV_k\) of the contacts. Current conservation implies that the rows and columns of the dynamic conductance matrix add up to zero,

$$\sum_l G_{kl}(\omega) = 0,$$  \hspace{1cm} (2)

$$\sum_k G_{kl}(\omega) = 0.$$  \hspace{1cm} (3)

The sum rules Eq. (2) and (3) are the same for the frequency dependent admittance as for the elements of the dc conductance matrix [11]. As in the dc-case these sum rules of the admittance matrix guarantee that the currents depend only on voltage differences.

We exemplify the structure of a current conserving answer with a discussion of the admittance to leading order in frequency. For small frequencies we find

$$G_{kl}(\omega) = G_{kl}(0) - i\omega E_{kl} + O(\omega^2).$$  \hspace{1cm} (4)

Here \(G_{kl}(0)\) is the dc-conductance. For the configuration of Fig. 1 the dc-conductances are related \(G_{22} = G_{33} = -G_{23} = -G_{32} \equiv G(0)\). All other elements of the dc-conductance matrix vanish since the configuration of Fig. 1 allows direct transmission only from contact 2 to contact 3. In Eq. (4) the term multiplying the frequency is called an emittance [12]. In general an element of the emittance matrix might be dominated by capacitive effects or kinetic inductive effects resulting from direct transmission between contacts \(k\) and \(l\). The gate in Fig. 1 is coupled to the conductor in a purely capacitive way and hence the emittances describing this coupling are just capacitance coefficients, \(E_{kl} = C_{kl}\) if either \(k = 1\) or \(l = 1\). There is direct transmission from contact 2 to 3, and therefore, the emittances \(E_{kl}\) which connect these contacts can in general also contain inductive effects. Thus the emittance matrix for the conductor of Fig. 1 is of the form,
\[
E = \begin{pmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & E_{22} & E_{23} \\
C_{31} & E_{22} & E_{33}
\end{pmatrix}.
\] (5)

Eq. (2) implies that the rows and columns of the emittance matrix add up to zero. Below we derive explicit expressions for the emittances and capacitances which are current conserving.

Consider next the configuration shown in Fig. 2. Here the "conductor" is a ring with an Aharonov-Bohm (AB) flux threading the hole [18] and connected via leads to electron reservoirs [19]. The dc-conductance exhibits sample specific AB-oscillations with fundamental period \(\Phi_0 = \frac{hc}{e}\). Below we show that all coefficients of the emittance matrix are similarly periodic functions of the AB-flux. Even so only electrons in the "conductor" have the possibility to circle the AB-flux long range Coulomb interactions induce an electron density in the gate which is also a periodic function of the AB-flux. As a consequence we predict that the capacitance coefficients in Eq. (4) are also periodic functions of the AB-flux [20]. Of particular interest is the symmetry of these transport coefficients under reversal of the AB-flux. We will show that the elements of the emittance matrix obey a reciprocity symmetry,

\[E_{kl}(\Phi) = E_{lk}(-\Phi).\]

This symmetry applies in particular also to the elements of the emittance matrix which are purely capacitive. Hence we predict that there are capacitance elements in Eq. (4) which are not even functions of flux. If we cut the conductor and consider a sample as shown in Fig. 3, which consists of capacitors only, all elements of the emittance matrix are purely capacitive. Again, long range Coulomb interactions cause all elements of the capacitance tensor to be periodic functions of the AB-flux. However, in contrast to the configuration of Fig. 2, all capacitance elements of the sample in Fig. 3 are even functions of the AB-flux. The key difference between the samples of Fig. 2 and 3 is that the "conductor" in Fig. 2 represents a capacitor plate with two contacts, whereas in Fig. 3 each capacitor plate is connected to a single contact. Microscopically these differing symmetries of the capacitance coefficients originate from the fact that the capacitance coefficients in Fig. 3 represent a long range Coulomb coupling between the full equilibrium densities of each conductor whereas in the sample of Fig. 2 the capacitance coefficients couple only to
II. CHARGE AND CURRENT CONSERVATION

In the presence of ac-transport current conservation is a consequence of the long range nature of the Coulomb interactions. In the absence of such interactions the continuity equation

\[ -i\omega \delta n(r, \omega) + \text{div}\delta j(r, \omega) = 0 \]  

(6)
can be applied separately to the conductor and to the gate. The volume \( \Omega \) delimited by the surface shown as a broken line in Fig. 1 defines surfaces \( S_k \) where it intersects the reservoirs. We can evaluate the currents across these intersections and can evaluate the charge on the gate and on the conductor inside the volume delimited by these surfaces. The current at the terminal of the gate is equal to the time derivative of the total charge on the gate

\[ -i\omega \delta Q_g(S_1, \omega) = \delta I_1(S_1, \omega) \]

The currents at the terminals of the conductor are related to the total charge \( \delta Q_c(S_2, S_3, \omega) \) on the conductor,

\[ -i\omega \delta Q_c(S_2, S_3, \omega) = \delta I_2(S_2, \omega) + \delta I_3(S_3, \omega) \]

In the absence of interactions the currents at the conductor are not conserved nor are they in any way related to the current at the gate. Furthermore, the currents and charges depend on the exact location of the surfaces \( S_k \). In an interacting system a time-dependent voltage applied to the contacts of the sample leads to an internal time dependent Coulomb potential \( \delta U(r, \omega) \) and to a displacement field \( \delta d(r, \omega) \) which is via the Poisson equation related to the piled up charge density \( \text{div}\delta d(r, \omega) = -4\pi\delta n(r, \omega) \). If the Poisson equation is combined with the continuity equation, Eq. (6), then the total current density is conserved,

\[ \text{div}\delta j(r, \omega) = 0. \]  

(7)

Here the total current density is the sum of the ”displacement” current and the ”particle” current, \( \delta j(r, \omega) = -i(\omega/4\pi)\delta d(r, \omega) + \delta j_p(r, \omega) \). The total current density has no sinks or sources. Consider the volume \( \Omega \) as shown in Fig. 1. The total current flowing across...
the surface $S$ into this volume is zero. For an arbitrary choice of the volume $\Omega$ the total current density is not spatially limited to the contacts of the conductors. But assume that the contacts are good metallic conductors which screen any electric fields over distances of a Thomas-Fermi screening length. This permits us to chose the volume $\Omega$ so large that no electric field lines will penetrate its surface. We can assume that the surface of $\Omega$ intercepts the electron reservoirs at a distance which is sufficiently far from the connection of the contact to a lead. At such distances electric fields are screened efficiently. Now if there are no electric field lines penetrating the surface of $\Omega$, the current must be confined to the reservoirs. Current conservation now means that the sum of all currents in the contacts of the sample must add up to zero,

$$
\sum_k \delta I_k(\omega) = 0.
$$

(8)

Since there are no electric field lines which penetrate the surface of $\Omega$ the total charge in $\Omega$, according to Gauss, is also zero,

$$
\delta Q(\omega) = 0.
$$

(9)

For the conductors of Fig. 1, Eq. (9) implies that any charge accumulation on the conductor is compensated by an equal charge of opposite sign on the gate. If the capacitance of the conductor vanishes, then the total charge on the conductor must vanish. In that case, application of an ac-voltage can polarize the conductor but cannot lead to a total fluctuating charge on the conductor. In the non-interacting case, the currents and charges depend on the exact location of the surfaces, irresepctively on how far out in the reservoir the surfaces are located. In contrast, in the interacting case, the currents are independent of the exact choice of $\Omega$, once the intersections are deep in the reservoirs.

The task at hand is not trivial since the electric field or potential must be known to find the correct answer. The potential is, however, not a single particle property, but depends on the location of all mobile carriers, their separation among themselves and their separation to the ionic lattice, sheets of donors etc. The solution of this problem depends, therefore,
on the approach taken to handle the many body problem. We presented solutions using three different simple schemes: (a) A discrete potential model approximates the potential landscape with the help of a finite number of discrete potentials. In the simplest case only one potential per conductor is introduced [8]. (b) For metallic conductors with efficient screening the local potential landscape is determined in Thomas-Fermi screening approximation [21]. (c) For conductors as shown in Fig. 1 for which long range Coulomb forces matter, the potential landscape is determined in a Hartree-like approximation [12,20]. This list should eventually be extended to a Hartree-Fock approach to include exchange and to density functional theory to include exchange and correlations.

Quite generally we approach the task as follows [8]: First we derive an external response: The currents are found in response to an oscillating chemical potential at a contact. During this step the Coulomb interaction is switched off but the response is calculated with the help of the wave functions of the self-consistent equilibrium problem. In a second step the Coulomb potential caused by the additional injected charges is evaluated. In a third step the current response due this internal potential is found.

III. EXTERNAL RESPONSE DUE TO OSCILLATING CHEMICAL POTENTIALS

In electrical conductors one has to distinguish electro-chemical potentials $\mu_k$, chemical potentials $E_{F_k}(r)$ and the electro-static potential $eU(r)$. At equilibrium these potentials are related, $\mu_k = E_{F_k}(r) + eU(r)$. In a first step we evaluate the response to an electro-chemical potential oscillation $edV_k(t) = d\mu_k(t)$ assuming that it is caused by an oscillation in the chemical potential $edV_k(t) = d\mu_k(t) = dE_{F_k}(t)$ in contact $k$. In reality, of course, the contact, except where it narrows into a lead, has to be charge neutral. To achieve this we will then in a second step calculate the potential and restore charge neutrality in the contacts. To obtain the leading order frequency dependent terms in the external response we proceed as described below.
We assume that the conductors of interest are described by scattering matrices \( s_{kl} \) which give the outgoing current amplitudes in the asymptotic states \([22]\) of contact \( k \) in terms of the incoming current amplitudes in contact \( l \). For the conductor of Fig. 1, the scattering matrices are,

\[
S = \begin{pmatrix}
    s_{11} & 0 & 0 \\
    0 & s_{22} & s_{23} \\
    0 & s_{32} & s_{33}
\end{pmatrix}.
\]

The matrix \( S \) is unitary and due to micro-reversibility \([22]\) we have \( S^\star(-\Phi) = S^{-1}(\Phi) \) and consequently \( S^T(\Phi) = S(-\Phi) \). The elements of the scattering matrix are explicit functions of the energy of the incident carriers, the AB-flux (except for the elements of \( s_{11} \)), and are a functional of the electrostatic potential \( U(r) \). (The self-consistent potential \( U \) also depends on the AB-flux and depends on the electro-chemical potentials). To brief we will not write s-matrix elements as a function or functional of all these quantities but only emphasize the argument most important for the immediate discussion. The density of states at the Fermi energy in the gate is

\[
\frac{dN_{11}}{dE} = \frac{1}{4\pi i} \text{Tr} \left[ s_{11}^\dagger(E) \frac{ds_{11}(E)}{dE} - \frac{ds_{11}^\dagger(E)}{dE} s_{11}(E) \right].
\]

The density of states in the conductor is a sum of four contributions,

\[
\frac{dN_{kl}}{dE} = \frac{1}{4\pi i} \text{Tr} \left[ s_{kl}^\dagger(E) \frac{ds_{kl}(E)}{dE} - \frac{ds_{kl}^\dagger(E)}{dE} s_{kl}(E) \right],
\]

where \( k, l = 2, 3 \). The scattering matrix \( s_{kl} \) determines the outgoing current amplitudes in contact \( k \) as a function of the current amplitudes of the incident waves. Thus the density of states given by Eq. (12) represent a partition of the total density of states according to the origin (injecting contact) and destination (emitting contact) of the sample. (A portion of the total density of states of the conductor might consist of localized states which are disconnected from any contact. At this stage the localized states do not play a role. Later on, to determine the screening properties of the sample, we will have to consider not only the mobile states, but also the localized states).
A small increment in the Fermi energy $dE_{F1}(t)$ at the gate injects an additional charge $dQ(t) = (e dN_{11}/dE) dE_{F1}(t)$ onto the gate. This charge flows to the gate through contact 1 and gives rise to a current at this contact given by $dI_1(t) = dQ/dt$. For a sinusoidal variation of the chemical potential $dE_{F1}(t) = dE_{F1} \exp(-i \omega t)$ we find thus at contact 1 a current

$$dI_1(\omega) = -i \omega e^2 (dN_{11}/dE) dV_1(\omega),$$

(13)

where $e dV_1(\omega) = dE_{F1}(\omega)$ has been used. Consider now the conductor with two contacts. A chemical potential variation $e dV_2(\omega) = dE_{F2}(\omega)$ at contact 2 leads to an additional charge $(e dN_{32}/dE + e dN_{22}/dE) e dV_2(\omega)$ on this conductor. Now the fact that we have scattering matrix expressions (see Eq. (12)) for these charges helps us further. Apparently it is only the additional charge $dQ_3(\omega) = e (dN_{32}/dE) dV_2(\omega)$ which leads to a current at contact 3, whereas $dQ_2(\omega) = e (dN_{22}/dE) dV_2(\omega)$ is determined by carriers which leave the conductor through contact 2. Therefore, charging of this conductor causes currents

$$dI_k(\omega) = -i \omega e^2 (dN_{kl}/dE) dV_l(\omega).$$

(14)

at its contacts. Since direct transmission between contact 2 and 3 is possible, an oscillating voltage causes in addition at these contacts a current determined by the dc-conductance. Thus the leading low frequency-current response to an oscillating chemical potential $e dV_k(\omega)$ is given by an external response [8]

$$G_{kl}^e(\omega) = G_{kl}(0) - i \omega e^2 (dN_{kl}/dE).$$

(15)

This external response is not current conserving. Since for the dc-conductances we have $\sum_k G_{kl}(0) = \sum_l G_{kl}(0) = 0$, we find that to leading order in frequency $\sum_k G_{kl}^e(\omega)$ is proportional to the total charge injected from contact $l$ into the conductor. This charge is determined by the injectance $\sum_k (dN_{kl}/dE)$ of contact $l$. Similarly we find that to leading order in frequency $\sum_l G_{kl}^e(\omega)$ is proportional to the total charge that is emitted by the conductor through contact $k$. This charge is determined by the (unscreened) emittance $\sum_l (dN_{lk}/dE)$ of contact $k$. Only if for some reason the injectances and emittances vanish would we have
current conservation. We mention only in passing that in the absence of a magnetic field the injectance and emittance of each contact is equal.

The injected charges create an internal, time-dependent Coulomb potential \( \delta U(r,t) \). This time-dependent electro-static potential will in turn cause additional currents. We next investigate the response to such an internal electro-static potential.

IV. RESPONSE TO AN OSCILLATING ELECTRO-STATIC POTENTIAL

We are interested in the currents generated in the contacts of a sample in the presence of an oscillating potential \( \delta U(r,t) \). We can Fourier transform this potential with respect to time and consider a perturbation of the form \( u(r)(U_{+\omega} \exp(-i\omega t) + U_{-\omega} \exp(+i\omega t)) \). Since the potential is real we have \( U_{-\omega} = U^{*}_{+\omega} \). The response to such a potential can be treated using a scattering approach [21]: Due to the oscillating internal potential a carrier incident with energy \( E \) can gain or loose modulation energy \( \hbar \omega \) during reflection at the sample or during transmission through the sample. The amplitude of an out going wave is a superposition of carriers incident at energy \( E \) and at the side-band energies, \( E \pm \hbar \omega \). In the low-frequency limit of interest here the amplitude of the out going waves can be obtained by considering the scattering matrix \( s_{kl}(U(r,t), E) \) to be a slowly varying function of the potential \( U(r,t) \). Since the deviations of the actual potential away from the (time-independent) equilibrium potential \( U_{eq}(r) \) are small, we can expand the scattering matrix in powers of \( \delta U(r,t) = U(r,t) - U_{eq}(r) \). To linear order the amplitudes of the out going waves are determined by \( s_{kl}(U(r,t), E) = s_{kl}(U_{eq}(r), E) + (\delta s_{kl}/\delta U(r)) \delta U(r,t) \). Evaluation of the current at contact \( k \) gives [21]

\[
\delta I_k(\omega) = i e^2 \omega \int d^3r (dn(k,r)/dE) u(r) U_{+\omega}.
\] (16)

Here we have introduced the (local) density of states

\[
dn(k,r)/dE = -\left(\frac{1}{4\pi i}\right) \sum_l Tr \left[ s^\dagger_{kl} \frac{\delta s_{kl}}{\delta eU(r)} - \frac{\delta s^\dagger_{kl}}{\delta eU(r)} s_{kl}(E) \right]
\] (17)
of carriers at point \( r \) which are emitted by the conductor at probe \( k \). We refer to this density of states as *emissivity* of the sample into contact \( k \). In Eq. (17) the summation over \( l \) invokes a single term \( l = 1 \) for the gate and invokes the terms \( l = 2 \) and \( l = 3 \) for the currents at the contacts of the conductor. The scattering matrices are evaluated at the equilibrium chemical potentials \( E = \mu_1 \) of the gate and \( E = \mu_2 = \mu_3 \) of the conductor. For later reference we also introduce the *injectivity*

\[
\frac{dn(r, l)}{dE} = -\frac{1}{4\pi i} \sum_k Tr \left[ s^\dagger_{kl} \left( \frac{\delta s_{kl}}{\delta e U(r)} \right) - \left( \frac{\delta s^\dagger_{kl}}{\delta e U(r)} \right) s_{kl} \right] \tag{18}
\]

where again the summation invokes just \( k = 1 \) for the gate and invokes both \( k = 2 \) and \( k = 3 \) for the conductor. A more detailed derivation of Eq. (16) can be found in Ref. [21]. It is useful to express the response to the internal potential in the form of a conductance defined as \( \delta I_k(\omega) = G^i_k(\omega) U_{+\omega} \). Comparison with Eq. (17) gives for the internal conductances

\[
G^i_k(\omega) = ie^2 \omega \int d^3 r (dn(k, r)/dE) u(r). \tag{19}
\]

We note that the internal conductance, in contrast to the external conductance, contains no dc-contribution. A static potential that vanishes far out in the leads cannot produce a dc-current [23,24]. A static potential variation in the interior of a conductor just brings us from a conductor with one equilibrium potential configuration to another conductor with another equilibrium potential configuration. The absence of a dc-term in Eq. (19) re-emphasizes that the dc-transport is a consequence of differences in electro-chemical potentials at the contacts rather than an acceleration of carriers due to an electric field.

In the presence of an oscillating voltage at contact \( l \) the current at contact \( k \) has two contributions determined by the external response \( G^e_{kl} \) and by the internal response \( G^i_k \). But the internal response given by Eq. (19) contains an as yet undetermined internal potential. To complete the calculation of the total response we must find the dependence of the internal potential on the oscillating electro-chemical potentials of the contacts. Fortunately, in the low-frequency limit of interest here, it is sufficient to find the dependence of the internal potential on the electro-chemical potentials in quasi-static approximation.
V. CHARACTERISTIC POTENTIALS

Let us first consider a steady state of the configuration of Fig. 1. We allow for dc-transport and will consider the time-dependence later. The electro-static potential \( U(\mu_1, \mu_2, \mu_3, \mathbf{r}) \) for these conductors is a function of the electro-chemical potentials, and a complicated function of position. Small increases in the electro-chemical potentials \( d\mu_1, d\mu_2, d\mu_3 \) will bring the conductor to a new state with an electro-static potential \( U(\mu_1 + d\mu_1, \mu_2 + d\mu_2, \mu_3 + d\mu_3) \). The difference \( dU \) between these two potentials can be expanded in powers of the increment in electro-chemical potential. To linear order we have

\[
edU(\mu_1, \mu_2, \mu_3, \mathbf{r}) = \sum_k u_k(\mathbf{r}) d\mu_k. \tag{20}
\]

Here \( u_k(\mathbf{r}) = edU(\mu_1, \mu_2, \mu_3, \mathbf{r})/d\mu_k|_{d\mu_k=0} \), with \( k = 1, 2 \) are the characteristic potentials \[12\]. These characteristic functions determine the electro-static potential inside the sample in response to a variation of an electro-chemical potential at a contact. These static characteristic potentials are what we need to complete our calculation.

The characteristic potentials are the solutions of a Poisson equation with a non-local screening kernel:

\[
-\Delta u_k(\mathbf{r}) + 4\pi e^2 \int d^3r' \Pi(\mathbf{r}, \mathbf{r'}) u_k(\mathbf{r'}) = 4\pi e^2 (dn_k(\mathbf{r})/dE)_U. \tag{21}
\]

The source term \((dn_k(\mathbf{r})/dE)_U\) is the injectivity of contact \( k \): The charge density injected into the conductor as a consequence of a small increment in the electro-chemical potential at contact \( k \). The injectivity is given by Eq. \( \text{[18]} \). The screening kernel is determined by the charge density response function \( \Pi \), which to be brief, we call Lindhard function. A potential acting on the sample, in the absence of long range Coulomb interactions, generates an induced charge density

\[
d_{\text{ind}}(\mathbf{r}) = -\int d^3r' \Pi(\mathbf{r}, \mathbf{r'}) edU(\mathbf{r'}). \tag{22}
\]

Note that the Lindhard function describes the variation of the charge density not only of the mobile electrons which can be reached from the contacts but also of the localized states.
which might exist inside the conductors or the gate or somewhere in the volume \( \Omega \). If the source term \( e(\text{dn}_k(\mathbf{r})/dE)_U \) in Eq. (21) is replaced by a test charge \( e\delta(\mathbf{r} - \mathbf{r}_0) \) which is concentrated at one point \( \mathbf{r}_0 \), the solution to Eq. (21) is Green’s function \( g(\mathbf{r}, \mathbf{r}_0) \). With the help of Green’s function we find for the characteristic function,

\[
    u_k(\mathbf{r}) = \int d^3r'g(\mathbf{r}, \mathbf{r}')\left(\text{dn}_k(\mathbf{r}')/dE\right)_U. \tag{23}
\]

Since an equal increment of the electro-chemical potential at all contacts by \( d\mu \) only changes the energy scale but must leave all densities unchanged \[8\], the characteristic functions must add up to one at every point \[12\],

\[
    \sum_k u_k(\mathbf{r}) = 1. \tag{24}
\]

This implies for Green’s function, the property \[12\]

\[
    \int d^3r'g(\mathbf{r}, \mathbf{r}')\sum_k\left(\text{dn}_k(\mathbf{r}')/dE\right)_U = 1. \tag{25}
\]

The same relationship follows from the condition that the sum of all induced charge densities plus the test charge is zero. Eq. (23) is an important result of this section: It will be used to demonstrate charge and current conservation of the admittance.

VI. THE ELECTRO-CHEMICAL EMITTANCE MATRIX

The characteristic potentials determine the quasi-static potential inside the conductor. In response to an oscillating voltage \( edV_k(\omega)\exp(-i\omega t) = d\mu_k(\omega)\exp(-i\omega t) \) the internal potential is given by \( edU(\mu_1, \mu_2, \mu_3, \omega, \mathbf{r}) = \sum_k u_k(\mathbf{r})d\mu_k(\omega) \). Inserting this into the expression for the internal conductance we find upon comparison with Eq. (3) the following expression for the emittance:

\[
    E_{kl} = e^2(\text{dN}_{kl}/dE) - e^2 \int d^3r \int d^3r'\left(\text{dn}(k, \mathbf{r})/dE\right)g(\mathbf{r}, \mathbf{r}')\left(\text{dn}(l, \mathbf{r}')/dE\right). \tag{26}
\]

To understand Eq. (26) it is useful to consider for a moment the configuration in Fig. 3. Here we have cut the "conductor". Thus in this configuration there is no transmission. The
emittance coefficients given by Eq. (26) determine the electro-chemical capacitance matrix of this configuration, $E_{kl} = C_{kl}$. In this configuration the first term in Eq. (26) is non-vanishing only in the diagonal terms. The non-diagonal terms are negative. Furthermore, since each capacitor is coupled to a single contact the injectivity and emissivity in Eq. (26) are identical, $dn(k, r)/dE = dn(r, k)/dE$. Hence the capacitance matrix has the symmetry $C_{kl} = C_{lk}$.

For the conductors shown in Fig. 1 and 2 the topmost row and the first column of the emittance matrix consists of purely capacitive elements since the first term in Eq. (26) vanishes. However the emittance elements $E_{22}, E_{32}, E_{23}$ and $E_{33}$ represent a competition between two effects: The first term in Eq. (26) is a kinematic term due to the fact that we have direct transmission from contact $l$ to $k$. It is a kinetic inductance which is counterbalanced by the second, capacitive term in Eq. (26). Depending on the transmission properties of the sample and the screening properties of the conductor either the kinetic inductance can dominate and give a positive emittance or the capacitive effects can dominate and the non-diagonal elements of the emittance are negative.

VII. MAGNETIC-FLUX SYMMETRY OF THE EMITTANCE MATRIX

In this section we discuss briefly the symmetry of the low frequency transport coefficients under reversal of the magnetic flux. The electro-chemical emittances are periodic functions of the AB-flux $\Phi$ with fundamental period $hc/e$. Here we are concerned with the symmetry of these coefficients as the polarity of the flux is reversed. First consider the geometry of Fig. 3 in which all conductors are coupled purely capacitively. In this case the emissivities and injectivities in each conductor are identical, $dn(k, r)/dE = dn(r, k)/dE$ and represent a change in the equilibrium density of the conductor. But the equilibrium density of the conductor is an even function of flux, $dn(\Phi, r)/dE = dn(-\Phi, r)/dE$. The same conclusion follows immediately from the scattering matrix expressions for these densities. Similarly, since the Linhard function gives the change in density as we go from one equilibrium state
to another, the Lindhard function is also an even function of flux. Consequently, in the purely capacitive geometry of Fig. 3 the electro-chemical capacitance coefficients are all even functions of the flux, $C_{kl}(\Phi) = C_{kl}(-\Phi)$. We have shown earlier that the capacitance coefficients in this configuration are also symmetric, $C_{kl}(\Phi) = C_{lk}(\Phi)$.

Consider next the conductor of Fig. 2. The Lindhard function is again an even function of flux. But now the emissivities and injectivities of the "conductor" are not identical. Instead they obey a reciprocity relationship, $dn(\Phi, r, k)/dE = dn(-\Phi, k, r)/dE$. This is seen most directly by using the scattering matrix expressions Eqs. (15) and (18) for these densities. As a consequence the electro-chemical emittance matrix has the reciprocity symmetry, $E_{kl}(\Phi) = E_{lk}(-\Phi)$. Only the diagonal elements of the emittance matrix are even functions of the flux. Of particular interest is the fact that this symmetry also applies to the purely capacitive elements of the emittance matrix. Thus for the conductor in Fig. 2 the capacitance elements obey $C_{kl}(\Phi) = C_{kl}(-\Phi)$. There are thus purely capacitive elements of the emittance matrix of the conductor in Fig. 2 which are not even functions of magnetic flux. This is in stark contrast to the capacitance matrix of the conductor in Fig. 2.

In the geometry of Fig. 3, the injected charge is determined by a single density of states $dn_k(\Phi, r)/dE$ which is equal to the injectance and is equal to the emittance of this conductor. In contrast in the conductor of Fig. 2 the capacitance coefficients and the emittances are determined by the injectivities and emittances of differing contacts. Each of these densities represents only a portion of the total local density of states (of mobile carriers). The total local density in the conductor is equal to the sum of the injectivities of both contacts, $dn(2, r)/dE + dn(3, r)/dE$ and is equal to the sum of the emissivities into both contacts $dn(2, r)/dE + dn(3, r)/dE$. The capacitance elements in the geometry of Fig. 3 are even functions of flux since the total local density of states counts. In contrast the differing properties of the capacitance and emittance elements of the conductor of Fig. 2 are due to the fact that the emissivities and injectivities represent only a portion of the total local density of states.

This novel prediction of the symmetry of capacitance elements was tested in an experi-
ment by Chen et al. [17]. The symmetry predictions for a conductor in a uniform magnetic field are the same as those discussed above for an AB-flux. The conductor (a high mobility two-dimensional electron gas) used by Chen et al. has the same topology as the conductor in Fig. 1. A gate was made which only partially overlaps the edge of the conductor. The capacitance elements of the gate with respect to the contacts of the two dimensional electron gas are shown to exhibit a very dramatic asymmetry: The full geometrical capacitance is obtained for one polarity of the field whereas the capacitance elements nearly vanish for the other field polarity. The emittance elements which describe dynamic corrections to the \( \text{dc-conductance} \) are harder to measure than the capacitance coefficients.

VIII. DISCUSSION

We have emphasized the need for a current conserving approach to frequency dependent conductances. An illustration of such a theory for low frequencies has led to novel transport coefficients which are determined by effective, local density of states. Measurements of these transport coefficients provides information on the charge and potential distribution inside the sample.

Clearly, experiments which demonstrate an AB-effect in an electro-chemical capacitance would be fascinating. Such experiments would highlight that these coefficients should not be treated as geometrical constants, but like conductances, represent information on the physical properties of samples. An electro-chemical emittance has been measured in the experiment by Pieper and Price [13] and has been shown to exhibit AB-oscillations over the entire accessible range of frequencies [13]. AB-oscillations in a capacitance coefficient are likely to be an even more challenging task since the AB effect in the capacitance is likely only a small correction to a geometrical capacitance.
REFERENCES

[1] W. Shockley, J. Appl. Phys. 9, 639 (1938).

[2] S. Ramo, Proc. IRE 27, 584 (1939).

[3] F. N. H. Robinson, "Noise and Fluctuations in Electronic Devices and Circuits", (Cal-rendon, Oxford, 1974). Sec. 5.3.

[4] O. M. Bulashenko and V. A. Kochelap, J. Phys. -Cond. Matter 5, L469 (1993).

[5] R. C. Albers and J. E. Gubernatis, Phys. Rev. B17, 4487 (1978).

[6] T. Saso, J. Phys. C17, 2905 (1984).

[7] N. F. Mott, Phil. Mag. 17, 1259 (1968); Phil. Mag. 22, 7 (1970).

[8] M. Büttiker, A. Prêtre, H. Thomas, Phys. Rev. Lett. 70, 4114 (1993); M. Büttiker, M., H. Thomas, and A. Prêtre, Phys. Lett. A180, 364 (1993).

[9] B. Pellegrini, Phys. Rev. B34, 5921 (1986).

[10] P. G. N. deVegvar, Phys. Rev. Lett. 70, 837 (1992); C. Bruder and H. Schoeller, Phys. Rev. Lett. 72, 1076 (1994).

[11] M. Büttiker, Phys. Rev. Lett. 57, 1761 (1986); IBM J. Res. Develop. 32, 317 (1988).

[12] Büttiker, M., J. Phys.: Condens. Matter 5, 9631, (1993).

[13] J. B. Pieper and J. C. Price, Phys. Rev. Lett. 72, 3586 (1994).

[14] J. B. Pieper and J. C. Price, Phys. Rev. 49, 17059 (1994).

[15] D. Z. Liu, B. Y.-K. Hu, A. C. Stafford and S. Das Sarma, "Dynamic Magneto Conductance Fluctuations and Oscillations in Mesoscopic Wires and Rings", (unpublished).

[16] R. A. Webb, S. Washburn, and C. P. Umbach, Phys. Rev. B37, 8455 (1988); P. G. N. DeVegvar, et al., Phys. Rev. B40, 3491 (1988).
[17] W. Chen, T. P. Smith, M. Büttiker and M. Shayegan, Phys. Rev. Lett. 73, 146 (1994).

[18] M. Büttiker, Y. Imry, R. Landauer, Phys. Lett. A96, 365 (1983).

[19] Y. Gefen, Y. Imry and M. Ya. Azbel, Phys. Rev. Lett. 52, 1761 (1984): M. Büttiker, Y. Imry and M. Ya. Azbel, Phys. Rev. A30, 1982 (1984): A. D. Stone and Y. Imry, Phys. Rev. Lett. 56, 189 (1986): M. Büttiker, Phys. Rev. B32, 1846 (1985): P. Mello, Phys. Rev. B47, 16358 (1993). A. M. Jayannavar, D. P. and Singha, Phys. Rev. B49, 13685 (1994). I. E. Aronov, A. Grincwajg, M. Jonson, R. I. Shekhter, E. N. Bogachek, Solid State Communic. 91, 75 (1994): B. Reulet and H. Bouchiat, Phys. Rev. B50, 2259 (1994).

[20] M. Büttiker, Physica Scripta, T, (unpublished).

[21] M. Büttiker, A. Prêtre, and H. Thomas, Z. Phys. - Condens. Matter B94, 133 (1994).

[22] For a detailed discussion of the scattering approach to dc-conductance see M. Büttiker, in ”Nanostructured Systems”, edited by Mark Reed, Semiconductor and Semimetals, Vol. 35, 191 (1991).

[23] C. L. Kane, P. A. Lee and D. P. DiVincenzo, Phys. Rev. B38, 2995 (1988).

[24] J. U. Noeckel, A. D. Stone, H. U. Baranger, Phys. Rev. B48, 17569 (1993)
FIGURES

FIG. 1. A gate at electro-chemical potential $\mu_1$ coupled capacitively to a conductor with contacts at electro-chemical potentials $\mu_2$ and $\mu_3$. A Gauss volume $\Omega$ is chosen such that no electric field lines penetrate its surface.

FIG. 2. A gate coupled to a doubly connected conductor. An AB-flux $\Phi$ threads the hole of the loop. As a consequence of the long range Coulomb interaction the densities of the conductor and of the gate depend on the AB-flux.

FIG. 3. Arrangement of capacitors each connected to a single electron reservoir. One capacitor plate is a loop threaded by an AB-flux $\Phi$. All elements of the electro-chemical capacitance matrix are (even) functions of flux.