Topological invariant in terms of the Green functions for the Quantum Hall Effect in the presence of varying magnetic field

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Abstract

Recently the Wigner - Weyl formalism has been applied to the lattice models of solid state physics and to the lattice regularized quantum field theory. This allows to demonstrate that the electric current of intrinsic Anomalous Quantum Hall effect is expressed through the momentum space topological invariant composed of the Green functions both for the two - and the three - dimensional systems. Here we extend this consideration to the case of the Quantum Hall Effect existing in the presence of arbitrarily varying external magnetic field. The corresponding electric current appears to be proportional to the topological invariant in phase space composed of the Wigner transformed Green function that depends both on coordinates and momenta.

1. Introduction

Relation between topology and the Quantum Hall effect including the Anomalous Quantum Hall Effect (AQHE) follows the discovery of the so - called TKNN invariant \cite{1, 2, 3}; the Hall conductivity appears to be proportional to the integral of Berry curvature over the occupied branches of spectrum. The corresponding three - dimensional constructions were discussed, for example, in \cite{4, 5}. This formalism allows to calculate the Quantum Hall Effect (QHE) of electrons in the two - dimensional periodic potentials (or in the 2D lattice models) in the presence of constant magnetic field. However, it does not give the transparent calculation method for the QHE of Bloch electrons in the presence of varying external magnetic field. Besides, although by construction the TKNN invariant is defined for the noninteracting system, it is well - known that the total AQHE conductivity is robust to the introduction of disorder and weak interactions. The representation of this conductivity as the topological invariant composed of the Green functions was proposed for the two dimensional system in \cite{6, 7} (for more details see Chapter 21.2.1 in \cite{12}, see also \cite{8, 10, 11}). Using this approach the interacting system may be considered: the full two - point Green function of the interacting system should be substituted to the corresponding expression. It is usually believed, that the higher order full Green functions do not give a contribution to the QHE although there is still no direct proof of this statement. Such topological invariants were further discussed in the series of papers (see, for example, \cite{13, 14} and references therein). In \cite{15, 16} the similar relation between Hall conductivity and the topological invariants composed of the Green functions was deduced for the three - dimensional systems with the AQHE (in particular, for the discovered recently Weyl semimetals \cite{17, 18, 19, 20, 21, 22}). The role of disorder in the QHE has been discussed within several different approaches (see \cite{6, 23, 24, 25} and references therein). However, the introduction of disorder within the formalism of \cite{7, 8, 12, 16, 15} remains problematic: the corresponding topological invariants formally describe the idealized systems without impurities, where the Hall current remains in the bulk. Disorder pushes the Hall current to the boundary. But its total amount integrated over all space remains given by the topological invariants discussed in \cite{7, 8, 12, 16, 15}. In the present paper we will present the alternative proof of this conjecture for the case of the sufficiently weak disorder. Moreover, we extend the consideration of \cite{16} from the AQHE to the case of the Quantum Hall Effect in the presence of arbitrarily varying external magnetic field. Technically its consideration is more

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complicated than that of the AQHE in periodic systems without external magnetic field because the two-point Green function $G(p_1, p_2)$ now depends on two momenta $p_1, p_2$ instead of one unlike the case of the AQHE. We hope, that the topological invariants proposed in the present paper may become useful for the consideration of the models with essential inhomogeneity since they are composed of the Green functions and inherit the beautiful mathematical structure of the topological invariants of \[7, 8, 12, 13, 14, 16\]. It is worth mentioning that the Hall conductance is believed to be robust to the introduction of weak interactions while strong interactions may bring the system to the phase with the fractional QHE (see, for example, \[25\]). Its consideration, however, remains out of the scope of the present paper. For the general review of the topological aspects of the QHE we refer to \[3, 25, 24\] and references therein.

We explore the Wigner - Weyl formalism \[27, 28, 29, 30\] developed in \[14, 15\] for the lattice models of condensed matter physics (as well as for the lattice regularized quantum field theory). For the ordered review of this formalism see \[31\]. We demonstrate that the Hall conductance integrated over the whole space is given by the topological invariant in phase space (that consists of both coordinates and momenta). This topological invariant is expressed through the Wigner transformation of the two-point Green functions. Its structure repeats that of the topological invariants discussed in \[7, 8, 12, 13, 14, 16\] with the ordinary product substituted by the star product and the extra integration over the whole space added\[2\]. The value of our topological invariant responsible for the Hall conductance is not affected by the sufficiently weak disorder. We also find indications that it remains robust to the Electromagnetic interactions with sufficiently small effective coupling constant.

The paper is organized as follows. In Sect. 2 we introduce the notations that are useful for the investigation of various lattice models in momentum space. In Sect. 3 we recall the application of the Wigner - Weyl formalism to the lattice models proposed in \[14, 15\]. This formalism results, in particular, in the expression for the current of the AQHE through the topological invariants in momentum space. In Sect. 4 we present the extension of this result to the case of the lattice model in the presence of varying external magnetic field. Our main result is the expression for the QHE current through the topological invariant in phase space given in Eqs. \[10, 17, 18\]. In Sect. 5 we check the obtained results applying our expressions to the consideration of the problem with known solution - the QHE in the noninteracting system in the presence of constant external magnetic field. In Sections 2 – 5 we consider the idealized systems without disorder and with the interactions neglected. In Sect. 6 we give the arguments that suggest that the introduction of weak disorder does not influence our expressions. Also in Sect. 6 we present the indications that weak electromagnetic interactions do not affect them as well. In Sect. 7 we end with the conclusions.

In Appendixes we accumulate various results used in the main text of the paper. The majority of those results are not original and were proved in the other publications. We feel this important to collect the derivation of those results in the Appendixes to our present paper for completeness.

2. The description of the tight - binding models in momentum space

Following \[16, 13, 31\] we start from the lattice tight - binding fermionic model with the partition function of the following form

$$ Z = \int D\bar{\Psi} D\Psi \exp \left( \sum_{x,y} \bar{\Psi}^T(x) \left(-iD_{x,y}\right)\Psi(y) \right). \tag{1} $$

Here $D_{x,y}$ is a matrix that depends on the discrete lattice coordinates $x, y$. $\Psi, \bar{\Psi}$ are the multi - component Grassmann - valued fields defined on the lattice sites. The corresponding indices are omitted here and below for brevity. Such a partition function may be rewritten in momentum space as follows

$$ Z = \int D\bar{\psi} D\psi \exp \left( \int_{\mathcal{M}} \frac{d^Dp}{|\mathcal{M}|} \bar{\psi}^T(p)Q(p)\psi(p) \right). \tag{2} $$

\[2\] The topological invariants of \[7, 8, 12, 13, 14, 16\] have the same structure as the degree of mapping of the three - dimensional manyfold to a group of matrices.
where integration is over the fields defined in momentum space $\mathcal{M}$. $|\mathcal{M}|$ is its volume, $D$ is the dimensionality of space-time, $\psi$ and $\bar{\psi}$ are the anticommuting multi-component Grassmann variables defined in momentum space. Without loss of generality we assume that time is discretized, so that momentum space is compact, and its volume is finite. In condensed matter physics time typically is not discretized, the corresponding partition function may be obtained easily as the limit of Eq. (1) when the time spacing tends to zero. The partition function of Eq. (3) allows to describe the non-interacting fermion systems corresponding to matrix $\mathcal{Q}(p)$ (that is the Fourier transform of the lattice tight-binding matrix $D_{xy}$). The meaning of $\mathcal{Q}(p)$ for the lattice models of electrons in crystals is the inverse propagator of Bloch electron. The interactions are not taken into account at this stage. Their effect as well as the effect of disorder will be discussed later.

Introduction of the external gauge field $A(x)$ defined as a function of coordinates effectively leads to the Peierls substitution (see, for example, [16, 15, 31]):

$$Z = \int D\bar{\psi}D\psi \exp \left( \int_{\mathcal{M}} \frac{d^Dp}{|\mathcal{M}|} \bar{\psi}^T(p) \mathcal{Q}(p-A(i\partial_p))\psi(p) \right),$$

(3)

where the products of operators inside expression $\mathcal{Q}(p-A(i\partial_p))$ are symmetrized.

We relate operator $\mathcal{Q} = Q(p-A(i\partial_p))$ and its inverse $\mathcal{G} = \mathcal{Q}^{-1}$ defined in Hilbert space $\mathcal{H}$ of functions (on $\mathcal{M}$) with their matrix elements $\mathcal{Q}(p,q)$ and $\mathcal{G}(p,q)$ correspondingly:

$$\mathcal{Q}(p,q) = \langle p|\mathcal{Q}|q \rangle, \quad \mathcal{G}(p,q) = \langle p|\mathcal{Q}^{-1}|q \rangle.$$ 

Here the basis elements of $\mathcal{H}$ are normalized as $\langle p|q \rangle = \delta^{(D)}(p-q)$. Those operators obey the following equation

$$\langle p|\mathcal{Q}\mathcal{G}|q \rangle = \delta(p-q).$$

Eq. (3) may be rewritten as follows

$$Z = \int D\bar{\psi}D\psi \exp \left( \int_{\mathcal{M}} \frac{d^Dp_1}{\sqrt{|\mathcal{M}|}} \int_{\mathcal{M}} \frac{d^Dp_2}{\sqrt{|\mathcal{M}|}} \bar{\psi}^T(p_1) \mathcal{Q}(p_1,p_2)\psi(p_2) \right),$$

(4)

while the Green function of Bloch electron is given by

$$\mathcal{G}_{ab}(k_2,k_1) = \frac{1}{Z} \int D\bar{\psi}D\psi \exp \left( \int_{\mathcal{M}} \frac{d^Dp_1}{\sqrt{|\mathcal{M}|}} \int_{\mathcal{M}} \frac{d^Dp_2}{\sqrt{|\mathcal{M}|}} \bar{\psi}^T(p_1) \mathcal{Q}(p_1,p_2)\psi(p_2) \right) \frac{\bar{\psi}_b(k_2)}{\sqrt{|\mathcal{M}|}} \frac{\psi_a(k_1)}{\sqrt{|\mathcal{M}|}}.$$ 

(5)

Here indices $a, b$ enumerate the components of the fermionic fields. In the following we will omit those indices for brevity.

3. Wigner - Weyl formalism for the lattice models

The Wigner transformation of $\mathcal{G}$ is defined as the Weyl symbol of $\mathcal{G}$:

$$G_W(x, p) \equiv \int dq e^{ixq/2} \mathcal{G}(p+q/2, p-q/2).$$

(6)

It is assumed here that $\mathcal{Q}(p_1,p_2)$ is nonzero for the values of $|p_1 - p_2|$ much smaller than the size of the Brillouin zone. (The values of $|p_1 + p_2|$ may be arbitrary.) This occurs if the external electromagnetic field is slowly varying, i.e. its variation on the distance of the order of the interatomic distance may be neglected. Such fields correspond to the magnitudes of magnetic fields much smaller than thousands Tesla.
and wavelengths much larger than several Angstroms. Under these conditions the Wigner transformed Green function obeys the Groenewold equation (see \cite{10,13} and Appendix H of the present paper:

\[ G_W(x_n,p) \ast Q_W(x_n,p) = 1 , \tag{7} \]

that is

\[ 1 = G_W(x_n,p) e^{\frac{i}{\hbar} \left( p_n - \frac{\hbar}{2} \partial_n \right) q_n} Q_W(x_n,p) . \tag{8} \]

By \( x_n \) we denote the lattice points. Although the lattice points are discrete, the differentiation over \( x_n \) may be defined following \cite{10,13} because the functions of coordinates may be extended to their continuous values.

Let us define the (Grassmann - valued) Wigner function as

\[ W(p,q) = \frac{\bar{\psi}(p) \psi(q)}{\sqrt{|M|} \sqrt{|M|}} . \]

We may also define the operator \( \hat{W}|\psi, \bar{\psi} \rangle \), whose matrix elements are equal to \( W(p,q) = \langle p|\hat{W}|\psi, \bar{\psi}||q \rangle \). The functional trace of an operator \( \hat{U} \) is defined as

\[ \text{Tr} \hat{U} = \int dp \text{Tr} \langle p|\hat{U}|p \rangle . \]

Then the partition function receives the form:

\[ Z = \int D\bar{\psi} D\psi \exp \left( \text{Tr} \hat{W}[\psi, \bar{\psi}][\hat{Q}] \right) . \tag{9} \]

In the following by \( W_W \) we denote the Weyl symbol of \( \hat{W} \).

In general case the calculation of the Weyl symbol \( Q_W \) of an operator \( \hat{Q} = Q(p - A(i\partial_p)) \) is a rather complicated problem. It has been solved in \cite{31} for the particular case of lattice Wilson fermions. The result is represented in Appendix A. It follows from these results that if the field \( A \) is slowly varying, i.e. it almost does not vary on the distance of the order of the lattice spacing, we are able to substitute the sum over the lattice points by the integral, and take \( Q_W(p, x) = Q_W(p - A(x)) \). Of course, this refers not only to the lattice Wilson fermions, but to any lattice fermion model.

4. Equilibrium electric current

Variation of partition function gives the following expression for the average total current, i.e. the time average of the integral over the whole space of the electric current density (the derivation is given in Appendix B):

\[ \langle J^k \rangle = -T \int d^Dx \int \frac{d^Dp}{(2\pi)^D} \text{Tr} G_W(p, x) \ast \partial_p Q_W(p - A(x)) \]

\[ = -T \text{Tr} \hat{G} \partial_p Q(p - A(i\partial_p)) . \tag{10} \]

This expression is the topological invariant, i.e. the total current is not changed when the system is modified continuously.

In Appendix C we remind the reader the consideration of the anomalous QHE and represent the results obtained in \cite{10,13}. It appears that the response of the electric current to weak external field strength \( A_{ij} = \partial_i A_j - \partial_j A_i \) is the topological invariant in momentum space. This invariant is composed of the one - particle Green function (that is the function of the only momentum \( p \) for \( A = 0 \)). In the present paper we propose the generalization of this representation to the case, when in addition to the weak external field strength \( A_{ij} \) there is the strong external field strength \( B_{ij} = \partial_i B_j - \partial_j B_i \). The former gives rise to the external electric field while the latter corresponds to strong essentially varying external magnetic field that provides the system with the gaps and the topological structure responsible for the QHE. In this situation the one - particle Green function at \( A = 0 \) depends on two momenta because of the presence of the field \( B(x) \). So, we discuss the case, when the fermions are in the presence of two Abelian gauge fields \( A_i \) and \( B_i \).
Let us first consider the case of constant external field strength $A_{ij}$ and arbitrary field $B(x)$. We are going to expand $\langle J \rangle$ in powers of $A_{ij}$ and to keep the linear term only. At the same time the field $B(x)$ is taken into account completely. We start from

$$
\langle J^k \rangle = -T \int d^D x \int \frac{d^D p}{(2\pi)^D} \text{Tr} G_W(p, x) \ * \partial_{p_k} Q_W(p - A(x) - B(x))
$$

(11)

and define $G_W$ and $G^{(0)}_W$ that obey

$$
G_W(p, x) * Q_W(p - A(x) - B(x)) = 1
$$

and

$$
G^{(0)}_W(p, x) * Q_W(p - B(x)) = 1.
$$

We also denote $Q_W(p, x) = Q_W(p - A(x) - B(x))$. The Wigner transformed inverse propagator in the presence of external magnetic field (without external electric field) is $Q^{(0)}_W(p, x) = Q_W(p - B(x))$. Let us represent $Q_W$ as

$$
Q_W(p, x) \approx Q^{(0)}_W(p, x) - \partial_{p_m} Q^{(0)}_W(p, x) A_m(x)
$$

and $G_W$ as follows

$$
G_W(p, x) \approx G^{(0)}_W(p, x) + \partial_{p_m} G^{(0)}_W(p, x) A_m(x) * G^{(0)}_W(p, x)
$$

(12)

\[ y = x. \]

Here

$$
G_W(p, x | y) = G^{(0)}_W(p, x) + G^{(0)}_W(p, x)
$$

\[ * \partial_{p_m} Q^{(0)}_W(p, x) A_m(y) * G^{(0)}_W(p, x), \]

and

\[ * = \circ * \]

where $* = e^{\frac{i}{2} \left( \overline{\partial}_x \circ \overline{\partial}_p - \overline{\partial}_p \circ \overline{\partial}_x \right)}$ while $\circ = e^{\frac{i}{2} \left( \overline{\partial}_y \circ \overline{\partial}_p - \overline{\partial}_p \circ \overline{\partial}_y \right)}$. The direct check shows that in Eq. (11) the terms proportional to $A$ (that do not contain the derivatives of $A$) cancel each other, so that the first relevant terms are proportional to $\partial_i A_j$. We also may represent Eq. (11) as

$$
\langle J^k \rangle = -T \int d^D x \int \frac{d^D p}{(2\pi)^D} \text{Tr} \left( G_W(p, x | y) \circ * \partial_{p_k} Q_W(p - A(y) - B(x)) \right)_{y=x}.
$$

(13)

$G_W(p, x) = G_W(p, x | x)$ obeys equation

$$
G_W(p, x | y) \circ * Q_W(p - A(y) - B(x)) \big|_{y=x} = 1.
$$

Up to the terms linear in the derivatives of $A$ we get:

$$
\partial_{y_k} G_W(p, x | y) = \partial_{y_k} G^{(0)}_W(p, x) * \partial_{p_m} Q^{(0)}_W(p, x),
$$

$$
A_m(y) * G^{(0)}_W(p, x) \approx G^{(0)}_W(p, x) * \partial_{p_m} Q^{(0)}_W(p, x)
$$

$$
\partial_{y_k} A_m(y) * G^{(0)}_W(p, x)
$$
This allows to rewrite

\[ G_W(p, x|y) e^{-\frac{i}{2} \tilde{\partial}_{p_j} \tilde{\partial}_{p_i} A_{ij}} * Q_W(p - A(y) - B(x)) \big|_{y=x} = 1. \]

Up to the terms linear in \( A_{ij} \) this equation receives the form

\[ G_W(p, x|y) \left( 1 - \frac{i}{2} \tilde{\partial}_{p_j} \tilde{\partial}_{p_i} A_{ij} \right) * Q_W(p - A(y) - B(x)) \big|_{y=x} = 1. \]

Its solution is

\[ G_W(p, x|x) = G_W^{(0)} + G_W^{(0)} \partial_{p_m} Q_W^{(0)} * G_W^{(0)} A_m \]

\[ - \frac{i}{2} G_W^{(0)} \frac{\partial Q_W^{(0)}}{\partial p_i} * G_W^{(0)} \frac{\partial Q_W^{(0)}}{\partial p_j} * G_W^{(0)} A_{ij}. \]

We rewrite Eq. (10) as follows:

\[
\langle J^k \rangle = -T \int d^D x \int \frac{d^D p}{(2\pi)^D} \text{Tr} \left( G_W(p, x|y) e^{-\frac{i}{2} \tilde{\partial}_{p_j} \tilde{\partial}_{p_i} A_{ij}} * \partial_{p_k} Q_W(p - A(y) - B(x)) \right) \big|_{y=x},
\]

and obtain:

\[
\langle J^k \rangle \approx -T \int d^D x \int \frac{d^D p}{(2\pi)^D} \text{Tr} G_W^{(0)}(p, x) * \partial_{p_k} Q_W^{(0)}(p, x)
\]

\[
+ \frac{iT}{2} \int d^D x \int \frac{d^D p}{(2\pi)^D} \text{Tr} \left( G_W^{(0)} * \frac{\partial Q_W^{(0)}}{\partial p_i} \right) \frac{\partial Q_W^{(0)}}{\partial p_j} * G_W^{(0)} A_{ij}.
\]

The first term here is the equilibrium ground state current in the absence of electric field that according to the Bloch theorem vanishes (see [49] and references therein). It is worth mentioning that the general proof of this statement is absent in the framework of quantum field theory/condensed matter theory with relativistic spin - orbit interactions taken into account. We are aware of the clear proof of the Bloch theorem in the nonrelativistic quantum mechanics only. There may, in principle, be some marginal exclusions. However, we omit here consideration of such marginal cases and assume that the Bloch theorem in its conventional form is valid. In Appendix D we present the proof that the first term in Eq. (15) is the topological invariant, i.e. it is not changed when the system is modified smoothly. It complements the mentioned Bloch theorem and shows that this term remains constant (and is equal to zero according to the Bloch theorem) when the system is modified smoothly. Anyway, this term cannot contribute to the electric current of the QHE because it does not contain external electric field.

For \( D = 4 \) the second term in this expression may be rewritten as

\[
\langle J^k \rangle \approx \frac{N}{4\pi^2} \epsilon^{ijkl} M_i A_{ij}, \quad (16)
\]

\[
M_i = \frac{1}{T} \int d^4 x \int \text{Tr} \nu_l \text{d}^4 p \text{d}^4 x
\]

\[
= \frac{1}{V} \int \text{Tr} \nu_l \text{d}^4 p \text{d}^4 x
\]

\[
\nu_l = -\frac{i}{3! 8\pi^2} \epsilon^{ijkl} \left[ \frac{\partial G_W}{\partial p_i} * \frac{\partial G_W}{\partial p_j} * \frac{\partial G_W}{\partial p_k} \right]_{A=0},
\]
where $V$ is the spatial volume of the system. It is worth mentioning, that in the present paper the star product of several functions $f_1(p, x), f_2(p, x), \ldots, f_n(p, x)$ should be understood as

$$f_1 \ast f_2 \ast \ldots \ast f_n = \left[\ldots \left[ f_1 \ast f_2 \ast f_3 \right] \ast \ldots \ast f_n \right].$$

Here each operation $\ast$ acts within the nearest brackets only, that is the derivatives entering the star act only within those brackets. At the same time the property $\left[\ldots \left[ f_1 \ast f_2 \ast f_3 \right] \right] = \left[ f_1 \ast \left[ f_2 \ast f_3 \right] \right]$ (the proof is given in Appendix H) allows to move the brackets within the multiple products, and to write the product without brackets at all. For the two-dimensional systems ($D = 3$) we have:

$$\langle J^k \rangle \approx \frac{S}{4\pi} \mathcal{M} \epsilon^{ijk} A_{ij}, \quad (17)$$

where $S$ is the area of the system. Here $\mathcal{M}$ is the topological invariant in phase space

$$\mathcal{M} = \frac{1}{d^3 x} \int \text{Tr} \nu d^3 p d^3 x,$$

$$\nu = -\frac{i}{3!4\pi^2} \epsilon_{ijk} \left[ G_W(p, x) \ast \frac{\partial Q_W(p, x)}{\partial p_i} \right] \ast \frac{\partial G_W(p, x)}{\partial p_j} \ast \frac{\partial Q_W(p, x)}{\partial p_k} \bigg|_{A=0}. \quad (18)$$

The proof that Eq. (18) is indeed the topological invariant is given in Appendix D. In the similar way it may be proved that Eq. (10) represents the topological invariant. As it was mentioned above, for the validity of Eqs. (18) and (10) we need slowly varying potentials $A(x)$. This gives the two conditions: the magnitude of the external magnetic field is much smaller than thousands of Tesla, and the wavelength is much larger than the interatomic distance.

5. The alternative derivation of Hall conductivity for the case of constant magnetic field

Eqs. (10) and (18) consist the main result of our paper. It is worth mentioning that although we derived those expressions for lattice models, they remain valid for the continuous models as well. Notice, that Eq. (18) may be reduced to the TKNN invariant for the case of the electrons in crystals without interactions and with $Q(p) = i\omega - \mathcal{H}(p_x, p_y)$, where $\mathcal{H}$ is the one-particle Hamiltonian. This occurs if either there is no external magnetic field or if it is constant. The case of continuum model (say, with $\mathcal{H}(p_x, p_y) = \frac{p_x^2 + p_y^2}{2M}$) in the presence of constant magnetic field $\mathcal{B}$ (orthogonal to the plane of the given 2D system) is the simplest case, when Eq. (18) is not reduced to the TKNN invariant. Below we illustrate our result by the consideration of such a system.

We choose the gauge, in which the gauge potential is

$$B_x = 0, \quad B_y = B x.$$

External electric field $E_y$ corresponding to the gauge potential $A$ is directed along the axis $y$. The above derived expressions give the following expression for the electric current averaged over the area of the system:

$$\langle j_x \rangle \approx \frac{S}{2\pi} \mathcal{N} E_y. \quad (19)$$

Here the average Hall conductivity is given by $\sigma_{xy} = -\mathcal{N}/2\pi$ (recall that $j_x = -\sigma_{xy} E_y$) while $\mathcal{N} = i\mathcal{M}$ is the following topological invariant in phase space

$$\mathcal{N} = \frac{T}{3!4\pi^2 S} \int \text{Tr} d^3 p d^3 x \epsilon_{ijk} \left[ G_W(p, x) \ast \frac{\partial Q_W(p, x)}{\partial p_i} \right] \ast \frac{\partial G_W(p, x)}{\partial p_j} \ast \frac{\partial Q_W(p, x)}{\partial p_k} \bigg|_{A=0}. \quad (20)$$
For the practical calculations in the given particular case it is more useful to represent this expression in terms of the Green function written in momentum representation:

\[ N = \frac{T(2\pi)^3}{3!4\pi^2S} \int \text{Tr} \, d^3p^{(1)} \, d^3p^{(2)} \, d^3p^{(3)} \, d^3p^{(4)} \, \epsilon_{ijk} \]

\[ \left[ G(p^{(1)}, p^{(2)}) \left( [\partial_{p^{(2)}} + \partial_{p^{(3)}}] Q(p^{(3)}, p^{(4)}) \right) \right. \]

\[ \left. \left( [\partial_{p^{(4)}} + \partial_{p^{(1)}}] G(p^{(3)}, p^{(4)}) \right) \right]_{A=0}. \]  

(21)

This representation has been deduced from Eq. (20) in Appendix E. There also the transformation of Eq. (21) to the following form has been derived:

\[ N = -\frac{2\pi i}{S} \sum_{n,k} \epsilon_{ij} \left[ \frac{\langle n | [H, \hat{x}_i] | k \rangle \langle k | [H, \hat{x}_j] | n \rangle}{(E_k - E_n)^2} \right]_{A=0} \theta(-E_n) \theta(E_k). \]  

(22)

By operator \( \hat{x} \) we understand \( i\partial_{p} \) acting on the the wavefunction written in momentum representation:

\[ \hat{x}_i \Psi(p) = \langle p | \hat{x}_i | \Psi \rangle = i\partial_{p} \langle p | \Psi \rangle = i\partial_{p} \Psi(p). \]

Eq. (22) (divided by \( 2\pi \)) is the standard expression for the Hall conductance that follows from Kubo formula. For completeness we present the derivation of Eq. (22) from the Kubo formula in Appendix F (see also [23, 33]). The further calculation of Hall conductance using Eq. (22) is also standard (see, for example, [32]). We give it in Appendix G. The result reads

\[ N = N \text{sign}(-B), \]  

(23)

while \( N \) is the number of the occupied branches of spectrum. This way we came to the conventional expression for the Hall resistivity of a system of electrons with charge \(-|e|\) in the presence of constant magnetic field \( B = -|e|B_z \) (directed along the \( z \) axis) and constant electric field:

\[ \rho_{xy} = \frac{e}{\sigma_{xy}} = \frac{2\pi \hbar}{e^2N} \text{sign}(B_z). \]

Here we restore the conventional units with \( \hbar \neq 1 \). Recall, that the resistivity tensor is related to the conductivity tensor as follows:

\[ \begin{pmatrix} 0 & -\sigma_{xy} \\ \sigma_{xy} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \rho_{xy} \\ -\rho_{xy} & 0 \end{pmatrix}^{-1}. \]

### 6. Introduction of disorder and weak interactions

Let us discuss briefly the quantum Hall effect in the presence of disorder. The total electric current integrated over the volume/area of the system remains the same as without impurities. Here we present the arguments in favor of this statement based on the considered above Wigner - Weyl formalism.

We consider the case of variable external field strength \( A_0 \) corresponding to electric field \( E \), and nonzero field \( B(x) \) corresponding to a magnetic field. \( D - 1 \) dimensional space of the given system is divided by a plane/line into two equal half - infinite pieces. In one of those two pieces \( R^D_+ \) the electric field is disordered while in the other piece \( R^D_- \) it is perfectly constant and orthogonal to the mentioned dividing plane/line (let it be \( y = 0 \) plane for definiteness). At the same time the electric potential vanishes at the space infinity. Magnetic field corresponding to \( B(x) \) may be arbitrarily inhomogeneous within \( R^D_+ \), but its structure is
repeated within \( R^D \), i.e. \( B(-L + y) = B(y) \), where space infinity at \( y = -\infty \) is regularized and given by \( y = -L \).

The total current (i.e. the current density integrated over the volume of the system) is given by

\[
\langle J^k \rangle = -T \int d^Dx \int \frac{d^dp}{(2\pi)^D} \text{Tr} G_W(p, x) * \partial_p Q_W(p - A(x) - B(x)) = \langle J^k \rangle_{\text{disordered}} + \langle J^k \rangle_{\text{constant}},
\]

and is equal to the sum of electric current in the first piece \( R^D_0 \) of the system \( \langle J^k \rangle_{\text{constant}} \) and the total electric current in the disordered piece \( R^D_+ : \langle J^k \rangle_{\text{disordered}} \). Notice, that Eq. (24) works as long as the fields \( A(x), B(x) \) do not vary too fast, i.e. when their variations on the distances of the order of lattice spacing may be neglected. For the external magnetic field this results in two bounds: its magnitude is much smaller than about 1000 Tesla while its wavelength is much larger than several Angstroms. Electric potential should also not vary fast, and its variation is to be neglected on the distances of the order of lattice spacing. The electric potential of impurities typically satisfies this requirement. In the absence of screening we deal with the long - range Coulomb potential. The typical change of this potential on the distance of one lattice spacing may be neglected. The Debye screening introduces a correlation length that remains much larger than the interatomic distance.

We consider the total electric current as the sum Eq. (25) of the currents at \( y < 0 \) and at \( y > 0 \). At \( y < 0 \) (in \( R^D_0 \)) we deal with the system in the presence of constant electric field, no impurities. In systems without disorder there is the duality between the description of the IQHE in the bulk and along the boundary (see for example, [39]). Namely, we may calculate the total bulk Hall current, and it will be equal precisely to the total boundary Hall current provided that the voltage is not strong enough. The bulk current density may be calculated as if there is no boundary at all. Now the system at \( y < 0 \) does not have disorder, but it also does not have an ordinary boundary at \( y = 0 \). As a result one calculates the total Hall current in it through the bulk expression. It is distributed homogeneously, and therefore one may neglect possible change in the current density close to the line \( y = 0 \).

At \( y > 0 \) the situation is drastically different. The disorder pushes the current density from the bulk. In principle at very weak disorder the part of the Hall current density may remain in the bulk. But the sufficiently strong disorder eliminates this part completely. The whole current density of the second piece of the system will be concentrated along the line \( y = 0 \) and along the distant boundary at \( y = +L \rightarrow +\infty \).

Let us take for simplicity the two dimensional system. The consideration of the three - dimensional system is similar. As it was mentioned above, considering the piece of space with constant electric field we may neglect boundary effects, and obtain

\[
\langle J^k \rangle_{\text{constant}} \approx \frac{\mathcal{V}}{2\pi} \mathcal{N} \epsilon^{kj} E_j
\]

where \( \mathcal{N} \) is given by Eq. (21) with the integral over the corresponding half of space. \( \mathcal{V} \) is its volume. Eq. (26) is the linear response to external electric field that remains valid since at \( y < 0 \) there is no disorder.

The key point is that Eq. (24) (that is the sum of currents at \( y < 0 \) and \( y > 0 \)) is the topological invariant. This allows us to relate the total current at \( y > 0 \) to the external electric field without consideration of the linear response in this region. Since Eq. (24) is the topological invariant, we may change smoothly the external electric field, and the value of Eq. (24) will not be changed. This way we may turn off completely the electric field. At vanishing electric field the Hall current is absent. This proves that Eq. (24) gives vanishing total current. As a result, when the electric field is on, the sum of the currents at \( y < 0 \) and \( y > 0 \) is zero. We come to the following expression for the electric current in the disordered piece of space \( R^D_+ \):

\[
\langle J^k \rangle_{\text{disordered}} \approx \frac{\mathcal{V}}{2\pi} \mathcal{N} \epsilon^{kj} \langle E_j \rangle.
\]

Here \( \mathcal{N} \) is given by Eq. (21), where integration is over \( R^D_+ \). It is equal precisely to the similar expression for \( R^D \). The value of \( \langle E_j \rangle \) is the average value of electric field in \( R^D \) that is equal to the constant electric field in \( R^D \) with the sign minus. Eq. (27) looks like the linear response to the electric field, but it is not
the linear response to the total electric field including the electric field created by impurities. This is the
response to the external constant electric field.

These simple considerations give strong arguments in favor of the statement that although due to disorder
the local current density may be concentrated to a large extent close to the boundary, the total integrated
value of Hall conductivity remains the same as without disorder and is given by the topological invariant in
phase space defined above. For the more detailed discussion of the role of disorder see \cite{6, 23, 24, 25} and
references therein.

Finally, let us discuss briefly the role of interactions. As an example let is consider exchange by virtual
photons in the three-dimensional systems. They may be taken into account as the fluctuations $\delta A$ of the
gauge field $A$ within Euclidean theory. Let us explore here the following hypothesis: at the sufficiently weak
$\alpha$ (effective fine structure constant in medium) we are able to integrate in the functional integral only over
the weak configurations of $\delta A$, which may be considered on the same grounds as the weak disorder in the
above Eq. (27). The electric current is then given by

$$\langle j^k \rangle = \mathcal{V} \langle j^k \rangle = \frac{\mathcal{V}}{4\pi^2} \epsilon^{ijkl} \int D\delta A e^{-S[\delta A]} \mathcal{M}_t \int \frac{d^3x}{\beta \mathcal{V}} \left(i2\delta^k_4 E_j + \partial_\alpha \delta A_{ij}(x)\right)$$

(28)

where $\mathcal{M}$ is given by Eq. (10) while $\mathcal{V}$ is the total volume of the system, $\beta = 1/T$. (We assume the limit of
large $\mathcal{V}$ and $\beta$.) $S[\delta A]$ is the effective action in medium for the dynamical photon field $\delta A$. The integration
measure $D\delta A$ is chosen in such a way that $\int D\delta A e^{-S[\delta A]} = 1$. Without loss of generality we may assume,
that the periodical boundary conditions are applied to $A$. This gives

$$\langle j^k \rangle \approx \frac{i}{2\pi^2} \epsilon^{kij} \mathcal{M}_t \langle E_j \rangle$$

(29)

Thus modulo the above assumption about the smallness and smoothness of $\delta A$ in the functional integral, the
exchange by virtual photons does not affect Hall conductance. The consideration of the other interactions like
the phonon exchange is more complicated and requires modification of the above formalism. This remains
out of the scope of the present paper. We refer here to the previous studies of the role of interactions in the
Quantum Hall Effect: \cite{40, 41, 42, 43, 44, 45, 46, 47}. The role of interactions in the intrinsic Anomalous
Quantum Hall effect has been discussed in \cite{48}.

7. Conclusions

In the present paper we derive the representation of Hall conductivity as the topological invariant in
phase space composed of the Wigner transformed Green functions. Our main result is given by the following
expressions for the average Hall current (i.e. the current density integrated over the whole volume divided
by this volume and averaged with respect to time) in the presence of the external electric field $E_i$:

$$\langle j^k \rangle = \frac{1}{2\pi^2} \epsilon^{kij} \mathcal{N}_i E_j, \mathcal{N}_t = -\frac{T\epsilon^{ijkl}}{3! \pi^2} \int d^4x d^4p \text{Tr} \left[ G_W(p,x) \ast \frac{\partial Q_W(p,x)}{\partial p_i} \ast \frac{\partial G_W(p,x)}{\partial p_j} \ast \frac{\partial Q_W(p,x)}{\partial p_k} \right]$$

(30)

for the $3 + 1D$ system (here $\mathcal{V}$ is the overall volume), and

$$\langle j^k \rangle = \frac{1}{2\pi^2} \mathcal{N} \epsilon^{kij} E_j, \mathcal{N} = \frac{T\epsilon^{ijkl}}{3! \pi^2} \int d^3p \int d^3x \text{Tr} \left[ G_W(p,x) \ast \frac{\partial Q_W(p,x)}{\partial p_i} \ast \frac{\partial G_W(p,x)}{\partial p_j} \ast \frac{\partial Q_W(p,x)}{\partial p_k} \right]$$

(31)

for the $2 + 1D$ system (here $\mathcal{S}$ is the area of the system). Those expressions may be used both for the
description of the intrinsic AQHE and for the description of the QHE in the presence of magnetic field.
These expressions represent the average conductivity and are robust to the introduction of weak disorder.
More explicitly, when a certain measure of disorder $\eta$ is increased smoothly starting from zero, the average
Hall conductivity is not changed until a phase transition. Direct definition of $\eta$, and establishing of its critical
value remains out of the scope of the present paper. We also find indications that the total conductance
is robust to the introduction of weak electromagnetic interactions. The mathematical structure of Eqs. (30), (31) repeats that of the topological invariants discussed in [12] with the ordinary product of matrices substituted by the star (Moyal) product of Wigner - Weyl formalism.

The main advantage of the obtained expressions for the average conductivity is that they are written in terms of the Green functions. We present strong arguments in favor of the point of view that disorder does not affect those expressions while the only effect of (weak) interactions is the modification of those two - point Green functions - we should substitute the ones of the interacting system. Earlier this result has been obtained for the homogeneous systems (see [7, 8, 12, 16]) like the ones with the AQHE. Here it is extended to the inhomogeneous system in the presence of the arbitrarily varying magnetic field. Our formulas allow to show, in particular, that the average Hall conductivity is not changed when this magnetic field is modified smoothly (unless the topological phase transition is encountered). Therefore, the calculation of Hall conductance for the complicated configuration of magnetic field may be reduced to that of the more simple magnetic field profile.

It is worth mentioning, that the Coulomb interactions in the case of sufficiently pure systems give rise to the fractional version of the QHE, which is out of the scope of the present paper. However, we expect, that the Wigner - Weyl formalism may be relevant for its description as well (see, e.g. [52, 53]).

The approach of [16] has been applied also for the investigation of the other non - dissipative currents (for the review see [34, 35]). In this way the Chiral Separation Effect [39], the Chiral Vortical Effect [40], and the Chiral Torsional Effect [41] were related to the topological invariants. The methodology of the present paper may also be extended to those non - dissipative transport effects for the non - homogeneous systems, i.e. the systems with the arbitrarily varying magnetic field, rotation velocity, emergent torsion correspondingly. Finally, we would like to mention that another formulations of Wigner - Weyl formalism in lattice models have been proposed by various authors (see, for example, [54, 55, 56, 57] and references therein).

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Appendix A. Weyl symbol of Wilson Dirac operator

In [31] the Weyl symbol of $\hat{Q} = Q(p - A(i\partial_p))$ (for the field $A(x)$ that varies slowly, i.e. if its variation at the distance of the order of lattice spacing is negligible) was calculated for the case of the so - called Wilson fermions (in four space - time dimensions) with

$$\hat{Q}(p) = \sum_{k=1,2,3,4} \gamma^k g_k(p) - im(p)$$

(32)

where $\gamma^k$ are Euclidean $4 \times 4$ Gamma - matrices. The corresponding spinors are four - component. Functions $g_k$ and $m$ are given by

$$g_k(p) = \sin(p_k) \quad m(p) = m^{(0)} + \sum_{\nu=1}^4 \cos(p_\nu).$$

(33)

We obtain

$$\left[Q(p - A(i\partial_p))\right]_W = \sum_{k=1,2,3,4} \gamma^k \sin(p_k - A_k(x))$$

$$-i(m^{(0)} + \sum_{\nu=1}^4 (1 - \cos(p_\nu - A_\nu(x))))$$

(34)

where $A$ is the following transformation of electromagnetic field:

$$A_\mu(x) = \int \left[\frac{\sin(k_\mu/2)}{k_\mu/2} \tilde{A}_\mu(k)e^{ikx} + c.c.] \right] dk$$

(35)

while the original electromagnetic field had the form:

$$A_\mu(x) = \int [\tilde{A}_\mu(k)e^{ikx} + c.c.] dk.$$

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that is

\[ \tilde{A}(p) = \frac{1}{|\mathcal{M}|} \sum_{x_n} A(x_n) e^{-ikx_n} \]

In coordinate space we have

\[
A_j(x) = \int \left[ \sin\left(\frac{k_j}{2}\right) \tilde{A}_j(k) e^{ikx} \right] dk \\
= \frac{1}{|\mathcal{M}|} \sum_{y_n} \int \left[ \sin\left(\frac{k_j}{2}\right) A_j(y_n) e^{-iky_n} e^{ikx} \right] dk \\
= \sum_{y_n} A_j(y_n) \int_{-1/2}^{1/2} dz \int \left[ e^{i(k_j z + kx - ky_n)} \right] \frac{dk}{|\mathcal{M}|} \\
= \sum_{y_n} A_j(y_n) \mathcal{D}(x - y_n)
\]

(36)

where

\[ \mathcal{D}(x) = \int_{-1/2}^{1/2} dz \int \left[ e^{i(k_j z + kx)} \right] \frac{dk}{|\mathcal{M}|} \]

One may check via the direct substitution, that

\[
A_j(x) = \int_{-1/2}^{1/2} A_j(x + e^{(j)} u) du
\]

(37)

where \(e^{(j)}\) is the unity vector in the \(j\)-th direction:

\[
A_j(p) = \frac{1}{|\mathcal{M}|} \sum_{x_n} A_j(x_n) e^{-ipx_n} \\
= \frac{1}{|\mathcal{M}|} \int_{-1/2}^{1/2} du \sum_{x_n} A_j(x_n + e^{(j)} u) e^{-ipx_n} \\
= \int_{-1/2}^{1/2} du \frac{dq}{|\mathcal{M}|} \sum_{x_n} A_j(q) e^{-ipx_n + iq(x_n + e^{(j)} u)} \\
= \int_{-1/2}^{1/2} du dq \delta(p - q) \tilde{A}_j(q) e^{i(qe^{(j)} u)} \\
= \int_{-1/2}^{1/2} du \tilde{A}_j(p) e^{ip(e^{(j)} u)} \\
= \tilde{A}^{(j)}(p) \frac{\sin(pe^{(j)}/2)}{(pe^{(j)}/2)}
\]

(38)

Therefore, in coordinate space we have

\[
A_\mu(x_1, ..., x_\mu, ..., x_D) = \int_{x_{\mu-1/2}}^{x_{\mu+1/2}} A_\mu(x_1, ..., y_\mu, ..., x_D) dy_\mu.
\]

(39)

Here coordinates \(x_n\) are taken in lattice units, in which the lattice spacing is equal to unity. Since the field \(A\) is slowly varying, i.e. it almost does not vary on the distance of the order of the lattice spacing, in practical calculations we may substitute to our expressions \(A\) instead of \(\tilde{A}\). For the real solid state systems the meaning of the slow variation of \(A\) is that the magnitude of the external magnetic field is much smaller than thousands of Tesla (the condition that is always fulfilled in practice), and that the wavelength of the external field is much smaller than the lattice spacing. The latter condition corresponds to the wavelengths much larger than several Angstroms, i.e. the above expression for the Weyl symbol (and its extensions to more realistic lattices) cannot be valid for matter interacting with the X rays.
Appendix B. Electric current expressed through the Wigner transformed Green function

Here we present the proof of the expression for the electric current in the considered lattice model. A slightly different derivation of this expression may be found in [16]. Let us consider the variation of the partition function

$$\delta \log Z = - \frac{1}{Z} \int D\bar{\psi}D\psi \exp \left( \text{Tr} \hat{W}[\psi, \bar{\psi}] \hat{Q}(p - A(i\frac{\partial}{\partial p})) \right)$$
$$\int dx \int \frac{dp}{(2\pi)^D} \text{Tr} W[p, \bar{\psi}] \star \partial_{\bar{p}} W(p - A(x)) \delta A_k(x).$$ (40)

The total average current (i.e. the current density integrated over the whole volume of the system) appears as the response to the variation of $A$ that does not depend on coordinates:

$$\langle J^k \rangle = - \frac{T}{Z} \int D\bar{\psi}D\psi \exp \left( \text{Tr} \hat{W}[\psi, \bar{\psi}] \hat{Q}(p - A(i\frac{\partial}{\partial p})) \right)$$
$$\int d^Dx \int d^Dp \text{Tr} W[p, \bar{\psi}](p, x) \star \partial_{\bar{p}} W(p - A(x))$$
$$= -T \int d^Dx \int \frac{d^Dp}{(2\pi)^D} \text{Tr} G W(p, x) \star \partial_{\bar{p}} W(p - A(x)).$$ (41)

The properties of the star product allow to rewrite the last equation in the following way:

$$\langle J^k \rangle = -T \int d^Dx \frac{d^Dp}{(2\pi)^D} \text{Tr} G W(p, x) \partial_{\bar{p}} W(p - A)$$
$$= -T \text{Tr} \hat{G} \partial_{\bar{p}} \hat{Q}(p - A(i\bar{p})) \partial_{\bar{p}}.$$ (42)

The last three expressions for the total current are the topological invariants, i.e. the total current is not changed when the system is modified continuously. This may be checked via the consideration of the variation of Eq. (41) corresponding to the variation of $Q$.

Appendix C. Anomalous QHE. Factorization of topological invariants

Let us consider the case, when the fermions are in the presence of Abelian gauge field $A_i$. Let us then expand the expression for the electric current density $\langle j^k \rangle$ (the average is understood as an integral over the fermion field) in powers of the external field strength $A_{ij}$ of $A_i$ and its derivatives. Next, if the coefficient in front of the field strength $A_{ij}$ does not depend on coordinates, then this coefficient is in itself the topological invariant because the integral $\int d^dx A_{ij}$ is not changed, when the gauge potential $A$ is modified smoothly in a finite region of space:

$$\delta \int d^D x A_{ij} = \int d^D x \delta A_{ij} = 0.$$ Here $\delta A \rightarrow 0$ at infinity.

This way for $D = 3$ we come to the following result for the part of the current proportional to $A_{ij}$:

$$\langle j^k \rangle \approx \frac{1}{4\pi} \mathcal{M}^{ijk} A_{ij}.$$ (43)

Tensor $\mathcal{M}^{ijk}$ is the topological invariant in momentum space, which was calculated in [16]. The obtained result is

$$\mathcal{M}^{ijk} = \epsilon^{ijk} \mathcal{M}, \quad \mathcal{M} = \int \text{Tr} \nu d^3p$$
$$\nu = -\frac{i}{3! 4\pi^2} \epsilon^{ijk} \left[ G^{-1}(p) \frac{\partial G}{\partial p_i} \frac{\partial G(p)}{\partial p_j} \frac{\partial G^{-1}(p)}{\partial p_k} \right].$$ (44)
For $D = 4$ we obtain
\[
\langle j^k \rangle \approx \frac{1}{4\pi^2} \epsilon^{ijkl} M_l A_{ij},
\]
\[
M_l = \int \text{Tr} \nu_l d^4 p.
\]
\[
\nu_l = -\frac{i}{3! 8\pi^2} \epsilon^{ijkl} \left[ G \frac{\partial G^{-1}}{\partial p_l} \frac{\partial G}{\partial p_j} \frac{\partial G^{-1}}{\partial p_k} \right].
\]

Here $G(p)$ is the one-particle Green function of the system without external electric field. It depends on the conserved momentum $p$. This expression has been used in [16] in order to derive the expression for the AQHE in certain topological insulators. The result is expressed through the vectors of the reciprocal lattice. (It also has been obtained earlier in [50] using another method.)

**Appendix D. Topological invariant in phase space**

In [16] the proof that Eq. (44) is the topological invariant was presented. Here we extend this proof to the case of the topological invariant in phase space discussed in the main text of the present paper. Let us first give the proof for the simplest case of the first term in Eq. (15). We denote it $J^{(0)}$. Let us consider an arbitrary variation of the Green function: $G \to G + \delta G$. Then using Eq. (72), Eq. (75), and Eq. (76) we obtain
\[
\delta J^{(0)k} = -T \delta \int d^D x \int \frac{d^D p}{(2\pi)^D} \text{Tr} (G_W^{(0)}(p, x) \ast \partial_{p_k} Q_W^{(0)}(p, x))
\]
\[
= -T \int d^D x \int \frac{d^D p}{(2\pi)^D} \text{Tr} G_W \ast \partial_{p_k} Q_W
\]
\[
= -T \int d^D x \int \frac{d^D p}{(2\pi)^D} \text{Tr} (G_W \ast \partial_{p_k} Q_W + G_W \ast \partial_{p_k} \delta Q_W)
\]
\[
= -T \int d^D x \int \frac{d^D p}{(2\pi)^D} \text{Tr} (G_W \ast \partial_{p_k} (G_W \ast \partial_{p_k} Q_W + G_W \ast \partial_{p_k} Q_W))
\]
\[
= -T \int d^D x \int \frac{d^D p}{(2\pi)^D} \partial_{p_k} \text{Tr} (\delta Q_W \ast G_W) = 0,
\]

The last step follows because of the compactness of $p$ space.

Similarly the expression for $\mathcal{M}$ of the main text is changed as follows:
\[
\delta \mathcal{M} = \frac{3iT}{24\pi^2 V} \int \text{Tr} \left( \left[ [\delta G_W] \ast dQ_W + G_W \ast d[\delta Q_W] \right] \right)
\]
\[
\ast \wedge G_W \ast dQ_W \ast \wedge G_W \ast dQ_W \right) d^3 x
\]
\[
= -\frac{3iT}{24\pi^2 V} \int \text{Tr} \left( \left[ -G_W \ast [\delta Q_W] \ast G_W \ast dQ_W + G_W \ast d[\delta Q_W] \right] \right)
\]
\[
\ast \wedge G_W \ast dQ_W \ast \wedge G_W \ast dQ_W \right) d^3 x
\]
\[
= \frac{3iT}{24\pi^2 V} \int \text{Tr} \left( \left[ [\delta Q_W] \ast [dG_W] + d[\delta Q_W] \ast G_W \right] \right)
\]
\[
\ast \wedge dQ_W \ast \wedge dG_W \right) d^3 x
\]
\[
= \frac{3iT}{24\pi^2 V} \int d^3 x \left( \left[ [\delta Q_W] \ast G_W \right] \ast dQ_W \ast \wedge dG_W \right) d^3 x = 0.
\]

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The exterior differentiation is defined here as usual antisymmetric differentiation, which is not affected by the $\star$ product:

$$dA_W \star \wedge dB_W \star \wedge \ldots \wedge dC_W = \frac{\partial A_W}{\partial p_\mu} \star \frac{\partial B_W}{\partial p_\nu} \star \ldots \star \frac{\partial C_W}{\partial p_\rho} dp_\mu \wedge dp_\nu \wedge \ldots \wedge dp_\rho$$  \hspace{1cm} (47)

**Appendix E. Ordinary Quantum Hall effect in the 2D noninteracting system with a one-particle Hamiltonian**

Here we discuss the Hall conductance that according to the main text is given by $\sigma_{xy} = N/2\pi$ where $N$ is the topological invariant in phase space

$$N = \frac{T}{3!4\pi^2 S} \int Tr d^3p d^3x \epsilon_{ijk} \left[ G_W(p,x) \star \frac{\partial Q_W(p,x)}{\partial p_i} \right]_{A=0} \cdot \frac{\partial G_W(p,x)}{\partial p_j} - \frac{\partial G_W(p,x)}{\partial p_k} \frac{\partial Q_W(p,x)}{\partial p_i} \right]_{A=0}. \hspace{1cm} (48)$$

Our aim is to derive starting from Eq. (48) the following representation for $N$ in terms of the Green function written in momentum representation:

$$N = \frac{T(2\pi)^3}{3!4\pi^2 S} \int Tr d^3p d^3x \epsilon_{ijk} \left[ G(p^{(1)},p^{(2)}) \left( \frac{\partial Q(p^{(2)},p^{(3)})}{\partial p^{(3)}} \right) + \left( \frac{\partial Q(p^{(3)},p^{(4)})}{\partial p^{(4)}} \right) \right]_{A=0}. \hspace{1cm} (49)$$

In order to derive this representation we first represent $\partial_{p^i} G_W$ as follows:

$$\frac{\partial}{\partial p^i} G_W(p,x) = \frac{\partial}{\partial p^i} \int d^3P e^{iPx} G(p+\frac{P}{2},p-\frac{P}{2}) = \int d^3P e^{iPx} \left( \frac{\partial}{\partial K_1^j} + \frac{\partial}{\partial K_2^j} \right) G(K_1,K_2) \bigg|_{K_2=p/2} \bigg|_{K_1=p^i/2}, \hspace{1cm} (50)$$

where we used

$$\frac{\partial}{\partial p^i} = \frac{\partial K_1^j}{\partial p^i} \frac{\partial}{\partial K_1^j} + \frac{\partial K_2^j}{\partial p^i} \frac{\partial}{\partial K_2^j} = \frac{\partial}{\partial K_1^j} + \frac{\partial}{\partial K_2^j}.$$

We denote

$$G'_i(p+\frac{P}{2},p-\frac{P}{2}) = \left( \frac{\partial}{\partial K_1^j} + \frac{\partial}{\partial K_2^j} \right) G(K_1,K_2) \bigg|_{K_2=p/2} \bigg|_{K_1=p^i/2},$$

for convenience. So $\partial_{p^i} G_W$ becomes

$$\partial_{p^i} G_W = (G'_i)_W. \hspace{1cm} (51)$$

Next we can compute $G_W \star \partial_{p^i} Q_W$.

$$Q_W \star \partial_{p^i} G_W = (QG'_i)_W = \int d^3P e^{iPx} (QG'_i) \left( p+\frac{P}{2},p-\frac{P}{2} \right),$$

in which

$$(QG'_i)(p+\frac{P}{2},p-\frac{P}{2}) = \int \frac{d^3P'}{(2\pi^3)} G \left( p+\frac{P}{2},p' \right)$$
Therefore, we can write

\[
\left. \left( \frac{\partial}{\partial p_i^*} + \frac{\partial}{\partial k_j} \right) Q(p', K_2) \right|_{K_2 = p - \frac{p}{2}}.
\]

Using the associativity of *-product, similarly we have

\[
Q_W(p, x) * \frac{\partial G_W(p, x)}{\partial p_i} * \frac{\partial Q_W(p, x)}{\partial p_j} * \frac{\partial G_W(p, x)}{\partial p_k}
= (QG_i'Q_j'G_k') W
= \int d^3P e^{iPx} (QG_i'Q_j'G_k') \left( p + \frac{P}{2}, p - \frac{P}{2} \right).
\]

Integrating over \( x \) and \( P \), inserting the completeness relation \( 1 = \int \frac{d^3p_1}{(2\pi)^3} |p_1\rangle \langle p_1| \), we transform the expression for \( \mathcal{N} \) into the form of Eq. (13).

Our next purpose is to bring Eq. (13) to the conventional expression for the Hall conductance in the case, when the non-interacting charged fermions with Hamiltonian \( \mathcal{H}(p_x, p_y) \) are in the presence of constant external magnetic field \( \mathcal{B} \) orthogonal to the plane of the given 2D system. We use the gauge, in which the gauge potential is

\[
B_x = 0, \quad B_y = \mathcal{B}x.
\]

External electric field \( E_y \) corresponding to the gauge potential \( A \) is directed along the axis \( y \).

Function \( Q(p^{(1)}, p^{(2)}) \) in momentum space has the form:

\[
Q(p^{(1)}, p^{(2)}) = (p^{(1)}|Q|p^{(2)}) = \delta^{(3)}(p^{(1)} - p^{(2)})i\omega^{(1)}
- (p^{(1)}|\mathcal{H}|p^{(2)})\delta(\omega^{(1)} - \omega^{(2)}),
\]

where \( p = (p_1, p_2, p_3) = (p, \omega). \) At the same time

\[
G(p^{(1)}, p^{(2)}) = \sum_n \frac{1}{i\omega^{(1)} - \xi_n} (p^{(1)}|n\rangle \langle n|p^{(2)})\delta(\omega^{(1)} - \omega^{(2)}).
\]

This way we obtain:

\[
\mathcal{N} = -\frac{i(2\pi)^2}{8\pi^2 S} \sum_{n, k} \int d\omega d^2p^{(2)} d^2p^{(3)} d^2p^{(4)} e_{ij} \text{Tr} \left[ \frac{1}{(i\omega - \xi_n)^2} (p^{(1)}|n\rangle \langle n|p^{(2)}) \right.
\left. \left( \frac{\partial}{\partial p_j^{(3)}} + \frac{\partial}{\partial p_j^{(4)}} \right) (p^{(2)}|\mathcal{H}|p^{(3)}) \sum_k \frac{1}{(i\omega - \xi_k}) (p^{(3)}|k\rangle \langle k|p^{(4)}) \left( \frac{\partial}{\partial p_j^{(4)}} + \frac{\partial}{\partial p_j^{(5)}} \right) (p^{(4)}|\mathcal{H}|p^{(5)}) \right] A = 0.
\]

One may represent

\[
[\partial_{p_j^{(4)}} + \partial_{p_j^{(5)}}] (p^{(4)}|\mathcal{H}|p^{(5)}) = i(p^{(4)}|\mathcal{H}\hat{x}_j - \hat{x}_j\mathcal{H}|p^{(5)})
= i(p^{(4)}|[\mathcal{H}, \hat{x}_j]|p^{(5)}) = i(p^{(4)}|[\mathcal{H}, \hat{x}_j]|p^{(1)})
\]

By operator \( \hat{x}_i \) we understand \( i\partial_p \) acting on the the wavefunction written in momentum representation:

\[
\hat{x}_i \Psi(p) = (p|\hat{x}_i|\Psi) = i\partial_p (p|\Psi) = i\partial_p \Psi(p).
\]

Then, for example,

\[
\hat{x}_i \delta(q - p) = (p|\hat{x}_i|q) = i\partial_p (p|q) = i\partial_p \delta(p - q) = -i\partial_q (p|q).
\]

Therefore, we can write

\[
\hat{x}_i |p\rangle = -i\partial_p |p\rangle.
\]
Notice, that the sign minus here is counter-intuitive because the operator $\hat{x}$ is typically associated with $+i\partial_p$. We should remember, however, that with this latter representation the derivative acts on $p$ in the bra-vector $\langle p|$ rather than on $p$ in $|p\rangle$. Above we have shown that the sign is changed when the derivative is transmitted to $p$ of $|p\rangle$.

Thus we have

$$\mathcal{N} = \frac{i(2\pi)^2}{8\pi^2S} \sum_{n,k} \int d\omega \epsilon_{ij} \left[ \frac{1}{(i\omega - \mathcal{E}_n)^2} \langle n | [\mathcal{H}, \hat{x}_i] | k \rangle \right]$$

$$= \frac{1}{(i\omega - \mathcal{E}_k)} \langle k | [\mathcal{H}, \hat{x}_j] | n \rangle \bigg|_{A=0}$$

$$= -\frac{2i(2\pi)^3}{8\pi^2S} \sum_{n,k} \epsilon_{ij} \left[ \frac{1}{(\mathcal{E}_k - \mathcal{E}_n)^2} \langle n | [\mathcal{H}, \hat{x}_i] | k \rangle \langle k | [\mathcal{H}, \hat{x}_j] | n \rangle \right] \bigg|_{A=0} \theta(-\mathcal{E}_n) \theta(\mathcal{E}_k).$$

The last expression is just the conventional expression for the Hall conductance (multiplied by $2\pi$) for the given system [25].

Appendix F. Hall conductance in the $2 + 1$ D systems from the Kubo formula

According to our previous considerations, in the particular case, when the external gauge field $A$ represents constant electric field $E_k$, we get $A_k = -iE_k$. Therefore, Eq. (53) reads

$$\langle j^k \rangle \approx \frac{N}{2\pi} e^{3kj} E_j.$$ 

Here $N$ is the topological invariant in phase space

$$\mathcal{N} = \frac{T}{S} \int \text{Tr} \nu d^3p d^3x,$$

$$\nu = \frac{1}{3!4\pi^2} \epsilon_{ijk} \left[ G_W(p,x) * \frac{\partial Q_W(p,x)}{\partial p_i} \right]$$

$$= \frac{1}{(\mathcal{E}_k - \mathcal{E}_n)^2} \langle n | [\mathcal{H}, \hat{x}_i] | k \rangle \langle k | [\mathcal{H}, \hat{x}_j] | n \rangle \bigg|_{A=0} \theta(-\mathcal{E}_n) \theta(\mathcal{E}_k).$$

Above we demonstrated that this expression is reduced to Eq. (52). Here for completeness we present the standard derivation of Eq. (52). We will use the Kubo formula for Hall conductance (see, for example, [23]). Let us start from the consideration of one fermionic particle that may occupy the states enumerated by numbers $n = 0, 1, \ldots$. We have

$$\sigma_{xy} = \frac{1}{\omega S} \int_0^\infty dt \ e^{i\omega t} \langle 0 | [\hat{J}_y(0), \hat{J}_x(t)] | 0 \rangle.$$ 

Here

$$\hat{J}_x(t) = \hat{V}^{-1}(t) \hat{J}_x(0) \hat{V}(t) = e^{i\mathcal{H}_0 t} \hat{J}_x(0) e^{-i\mathcal{H}_0 t}$$

is the total electric current in the Heisenberg picture, while $\hat{V}(t)$ is the time evolution operator. Averaging is over the the occupied state $|0\rangle$. Substituting it into the Kubo formula of Eq. (55), inserting a complete set of energy eigenstates, we get

$$\sigma_{xy} = \frac{1}{\omega S} \int_0^\infty dt \ e^{i\omega t} \sum_n \left( \langle 0 | \hat{J}_y | n \rangle \langle n | \hat{J}_x(0) e^{i(\mathcal{E}_n - \mathcal{E}_0) t} \right.$$

$$- \langle 0 | \hat{J}_x(0) e^{-i(\mathcal{E}_n - \mathcal{E}_0) t} \rangle.$$
where we write \( J_i = J_i(0) \) for simplicity, and the \( n = 0 \) term vanishes. \( \mathcal{E}_n \) is the energy of state \( |n\rangle \). Next we integrate over time, and get

\[
\sigma_{xy} = \frac{1}{i \omega S} \sum_{n \neq 0} \left( \frac{\langle 0 | \hat{J}_y | n \rangle \langle n | \hat{J}_x | 0 \rangle}{\omega + i \epsilon + \mathcal{E}_n - \mathcal{E}_0} - \frac{\langle 0 | \hat{J}_x | n \rangle \langle n | \hat{J}_y | 0 \rangle}{\omega + i \epsilon - (\mathcal{E}_n - \mathcal{E}_0)} \right).
\]

To get the limit \( \omega \to 0 \) corresponding to constant background field, we expand the above expression in powers of \( \omega \)

\[
\sigma_{xy} = \frac{1}{i \omega S} \sum_{n \neq 0} \left( \langle 0 | \hat{J}_y | n \rangle \langle n | \hat{J}_x | 0 \rangle \left( \frac{1}{\mathcal{E}_n - \mathcal{E}_0} - \frac{\omega}{(\mathcal{E}_n - \mathcal{E}_0)^2} + \ldots \right) + \langle 0 | \hat{J}_x | n \rangle \langle n | \hat{J}_y | 0 \rangle \left( \frac{1}{\mathcal{E}_n - \mathcal{E}_0} + \frac{\omega}{(\mathcal{E}_n - \mathcal{E}_0)^2} + \ldots \right) \right).
\]

(57)

For the Hall effect

\[
\sigma_{xy} = -\sigma_{yx},
\]

which leads to vanishing of the first order term. In the second order, the Hall conductance becomes

\[
\sigma_{xy} = \frac{i}{S} \sum_{n \neq 0} \frac{(\langle 0 | \hat{J}_y | n \rangle \langle n | \hat{J}_x | 0 \rangle - \langle 0 | \hat{J}_x | n \rangle \langle n | \hat{J}_y | 0 \rangle)}{(\mathcal{E}_n - \mathcal{E}_0)^2}.
\]

(58)

Electric current may be written as

\[
\hat{J}_i = \frac{1}{i} \langle \hat{x}_i, \mathcal{H} \rangle,
\]

(59)

Thus the Hall conductance of this quantum - mechanical system may be written as

\[
\sigma_{xy} = \frac{1}{i S} \sum_{n \neq 0} \frac{(\langle 0 | \hat{y}, \mathcal{H} | n \rangle \langle n | \hat{x}, \mathcal{H} | 0 \rangle - \langle 0 | \hat{x}, \mathcal{H} | n \rangle \langle n | \hat{y}, \mathcal{H} | 0 \rangle)}{(\mathcal{E}_n - \mathcal{E}_0)^2}.
\]

This equation represents the result of the second order in the perturbation expansion that corresponds to the transition between the states: from \( |0\rangle \) to \( |n\rangle \), and back from \( |n\rangle \) to \( |0\rangle \).

The next step is the consideration of the second quantized system. Fermi statistics provides, that in vacuum all one particle states with energy (counted from the Fermi level) \( \mathcal{E}_k < 0 \) are occupied. The Hall conductance in the second order of perturbation theory is given by the contributions of the transitions from the occupied one - particle states to the vacant ones and the corresponding returns. The final answer reads:

\[
\sigma_{xy} = \frac{i}{S} \sum_{k} \sum_{n \neq k} \frac{\theta(-\mathcal{E}_k)\theta(\mathcal{E}_n)}{(\mathcal{E}_n - \mathcal{E}_k)^2} \langle k | \hat{x}, \mathcal{H} | n \rangle \langle n | \hat{y}, \mathcal{H} | k \rangle - \langle k | \hat{y}, \mathcal{H} | n \rangle \langle n | \hat{x}, \mathcal{H} | k \rangle - \langle k | \hat{y}, \mathcal{H} | n \rangle \langle n | \hat{x}, \mathcal{H} | k \rangle - \langle k | \hat{x}, \mathcal{H} | n \rangle \langle n | \hat{y}, \mathcal{H} | k \rangle.
\]

(60)

This expression gives rise to the Hall conductance of the form \( \sigma_{yx} = \mathcal{N}/2\pi \) where \( \mathcal{N} \) is the topological invariant given by Eq. (52). The latter expression, in turn, was derived from the phase space topological invariant of Eq. (51). Thus the consideration of the present section gives an alternative proof of Eq. (54) in the given particular case.

Appendix G. Calculation of the Hall conductance for the noninteracting 2D system in the presence of constant magnetic field

The average Hall conductivity may be represented as \( \mathcal{N}/(2\pi) \), where

\[
\mathcal{N} = -\frac{2i(2\pi)^3}{8\pi^2 S} \sum_{n,k} \theta(\mathcal{E}_k)\theta(-\mathcal{E}_n) \epsilon_{ij} \left[ \frac{1}{(\mathcal{E}_k - \mathcal{E}_n)^2} \langle n | [\mathcal{H}, \hat{x}_i] | k \rangle \langle k | [\mathcal{H}, \hat{x}_j] | n \rangle \right]_{A=0}.
\]

(61)
Following [32] (see also [51]) in order to calculate the value of $N$ we decompose the coordinates $x_1 = x, x_2 = y$ as follows:

$$\hat{x}_1 = -\frac{\hat{p}_y - Bx}{B} + \hat{X}_1 = \hat{\xi}_1 + \hat{X}_1,$$

$$\hat{x}_2 = \frac{\hat{p}_x}{B} + \hat{X}_2 = \hat{\xi}_2 + \hat{X}_2.$$  

The commutation relations follow:

$$[\hat{\xi}_1, \hat{\xi}_2] = \frac{i}{B}, \quad [\hat{X}_1, \hat{X}_2] = -\frac{i}{B},$$

$$[\mathcal{H}, \xi_1] = -i \frac{\partial}{\partial p_x} \mathcal{H}, \quad [\mathcal{H}, \xi_2] = -i \frac{\partial}{\partial p_y} \mathcal{H},$$

$$[\mathcal{H}, X_1] = [\mathcal{H}, X_2] = 0.$$  

Here we use that the Hamiltonian contains the following dependence on $x$:

$$\mathcal{H}(\hat{p}_x, \hat{p}_y - Bx)$$

and assume $\frac{\partial^2}{\partial p_x \partial p_y} \mathcal{H} = 0$. We use those relations to obtain:

$$S\mathcal{N} = \frac{-2i(2\pi)^3}{8\pi^2} \sum_{n,k} \left[ \frac{1}{(\mathcal{E}_k - \mathcal{E}_n)^2} \langle n| [\mathcal{H}, \hat{\xi}_j]|k\rangle \langle k| [\mathcal{H}, \hat{\xi}_i]|n\rangle \right]_{A=0}$$

$$\epsilon_{ij} \theta(-\mathcal{E}_n) \theta(\mathcal{E}_k)$$

$$= \frac{2i(2\pi)^3}{8\pi^2} \sum_{n,k} \epsilon_{ij} \left[ \langle n| \hat{\xi}_i|k\rangle \langle k| \hat{\xi}_j|n\rangle \right]_{A=0} \theta(-\mathcal{E}_n) \theta(\mathcal{E}_k)$$

$$= \frac{2}{8\pi^2 B} \sum_{n} \langle n|\hat{\xi}_1|n\rangle \theta(-\mathcal{E}_n).$$  

(62)

Momentum $p_y$ is a good quantum number, and it enumerates the eigenstates of the Hamiltonian:

$$\mathcal{H}|n\rangle = \mathcal{H}(\hat{p}_x, \hat{p}_y - Bx)|p_y, m\rangle = \mathcal{E}_m(p_y)|p_y, m\rangle, m \in \mathbb{Z}.$$  

We assume that the size of the system is $L \times L$. This gives

$$S\mathcal{N} = -(2\pi) \sum_{m} \int \frac{dp_y L}{2\pi} \frac{1}{B} \theta(-\mathcal{E}_m(p_y)).$$  

(63)

$\langle x \rangle = p_y/B$ plays the role of the center of orbit, and this center should belong to the interval $(-L/2, L/2)$. This gives

$$\mathcal{N} = N \text{sign}(-B),$$  

where $N$ is the number of the occupied branches of spectrum. This way we came to the conventional expression for the Hall conductance of the fermionic system in the presence of constant magnetic field and constant electric field.
Appendix H. Properties of Moyal (star) product

In this Appendix, we list and prove the identities of Wigner - Weyl formalism used throughout the text of the paper. Wigner transformation of a function $B(p_1, p_2)$ (where $p_1, p_2 \in \mathcal{M}$) is defined here as

$$B_W(x, p) = \int_{q\in\mathcal{M}} dq e^{ixq} B(p + q/2, p - q/2). \tag{65}$$

Identifying $B(p_1, p_2)$ with the matrix elements of an operator $\hat{B}$, we come to the definition of the Weyl symbol of operator (we denote it by $B_W$):

$$B_W(x, p) = \int_{q\in\mathcal{M}} dq e^{ixq} \langle p + q/2 | \hat{B} | p - q/2 \rangle. \tag{66}$$

Here integral is over the Brillouin zone. Moyal product of the two functions in phase space $f(x, p)$ and $g(x, p)$ is defined as

$$f(x, p) \ast g(x, p) \equiv f(x, p) e^{\frac{i}{2} \langle \vec{\partial}_x \vec{\partial}_p - \vec{\partial}_p \vec{\partial}_x \rangle} g(x, p)$$

Let us consider the case, when operators $\hat{A}$ and $\hat{B}$ are almost diagonal, i.e. $\langle p + q/2 | \hat{A} | p - q/2 \rangle$ and $\langle p + q/2 | \hat{B} | p - q/2 \rangle$ may be nonzero for arbitrary $p$ and small $q$ (compared to the size of momentum space). This occurs when the variation of $A_W(x, p)$ (and $B_W(x, p)$) as a function on $x$ may be neglected on the distances of the order of the lattice spacing. Below we assume that the considered operators satisfy this requirement. Then the following expression follows

$$(\hat{A} \hat{B})_W(x, p) = A_W(x, p) \ast B_W(x, p) = A_W(x, p) e^{\frac{i}{2} \langle \vec{\partial}_x \vec{\partial}_p - \vec{\partial}_p \vec{\partial}_x \rangle} B_W(x, p). \tag{67}$$

The proof is given in [31]. We repeat it here briefly:

$$(AB)_W(x, p) = \int_{\mathcal{M}} dP \int_{\mathcal{M}} dR e^{ixP} \langle p + \frac{P}{2} | \hat{A} | R \rangle \langle R | \hat{B} | p - \frac{P}{2} \rangle$$

$$= \frac{1}{2^D} \int_{\mathcal{M}} dP \int_{K/2 \in \mathcal{M}} dK e^{ixP} \langle p + \frac{P}{2} | \hat{A} | p - \frac{K}{2} \rangle \langle p - \frac{K}{2} | \hat{B} | p - \frac{P}{2} \rangle$$

$$= \frac{2^D}{2^D} \int_{\mathcal{M}} dqdk e^{ixq} \langle p + \frac{q}{2} | \hat{A} | p - \frac{q}{2} \rangle e^{i\frac{q}{2} \vec{\partial}_x - \frac{q}{2} \vec{\partial}_p} \langle p + \frac{k}{2} | \hat{B} | p - \frac{k}{2} \rangle$$

$$= \left[ \int_{\mathcal{M}} dqdk e^{ixq} \langle p + \frac{q}{2} | \hat{A} | p - \frac{q}{2} \rangle e^{\frac{i}{2} \langle \vec{\partial}_x \vec{\partial}_p - \vec{\partial}_p \vec{\partial}_x \rangle} \left[ \int_{\mathcal{M}} dk e^{ikx} \langle p + \frac{k}{2} | \hat{B} | p - \frac{k}{2} \rangle \right] \right]$$

Here the bra- and ket- vectors in momentum space are defined modulo vectors of reciprocal lattice. In the second line we change variables

$$P = q + k, \quad K = q - k$$

$$q = \frac{P + K}{2}, \quad k = \frac{P - K}{2}$$

with the Jacobian

$$J = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2^D$$

This results in the factor $2^D$ in the third line. Here $D$ is the dimension of space. The transition between the second and the third lines of Eq. (66) requires that the operators are almost diagonal. This allows to substitute the region of the values of $q$ and $k$ (that corresponds to $P, K/2 \in \mathcal{M}$) by $\mathcal{M} \otimes \mathcal{M}$.

In the present paper we apply the Wigner - Weyl technique to the lattice Dirac operator (the inverse Green function) and the Green function $G(p, q)$ itself (to be considered as matrix elements of an operator $\hat{G}$: $G(p + q/2, p - q/2) = \langle p + q/2 | \hat{G} | p - q/2 \rangle$. Both are almost diagonal if the external electromagnetic field
varies slowly, i.e. if its variation on the distance of the order of lattice spacing may be neglected. This occurs for the magnitudes of external magnetic field much smaller than thousands Tesla, and for the wavelengths much larger than 1 Angstrom. One has

\[ G_W(x,p) \star Q_W(x,p) = (\hat{G}Q)_W(x,p) = \int_{q \in \mathcal{M}} dq e^{i\pi q}(p + q/2)[\hat{G}q - q/2) \]

\[ = \int_{q \in \mathcal{M}} dq e^{i\pi q}\delta(q) = 1. \quad (69) \]

Thus we have

\[ Q_W(x,p) \star G_W(x,p) = 1. \quad (70) \]

From the definition the associativity of the Moyal product follows: \([A_W \star B_W] \star C_W = [A_W \star [B_W \star C_W]].\)

The proof is as follows:

\[ [[A_W(x,p) \star B_W(x,p)] \star C_W(x,p)] \]
\[ = [(AB)_W(x,p) \star B_W(x,p)] \]
\[ = (ABC)_W(x,p) \]
\[ = [A_W(x,p) \star (BC)_W(x,p)] \]
\[ = [A_W(x,p) \star [B_W(x,p) \star C_W(x,p)]]. \quad (71) \]

The Leibnitz product rule is valid for the Weyl symbols of considered operators. Let \(d\) be a differentiation operator \(d = dx^\mu \frac{\partial}{\partial x^\mu} + dp_\mu \frac{\partial}{\partial p_\mu}\), then

\[ d(A_W(x,p) \star B_W(x,p)) = dA_W(x,p) \star B_W(x,p) + A_W(x,p) \star dB_W(x,p). \quad (72) \]

The proof is straightforward. First, notice that

\[ d(A_W(x,p) \star B_W(x,p)) = dA_W(x,p) \star B_W(x,p) \]
\[ + A_W(x,p) \star dB_W(x,p). \quad (73) \]

Then we expand the star

\[ d(A_W(x,p) \star B_W(x,p)) = d\sum_n n! \left( \frac{i}{2} \right)^n \left( \frac{\partial x \partial p - \partial p \partial x}{2} \right)^n B_W(x,p) \]
\[ = dA_W(x,p) \sum_n n! \left( \frac{i}{2} \right)^n \left( \frac{\partial x \partial p - \partial p \partial x}{2} \right)^n B_W(x,p) \]
\[ + \sum_n A_W(x,p) \frac{1}{n!} \left( \frac{i}{2} \right)^n \left( \frac{\partial x \partial p - \partial p \partial x}{2} \right)^n dB_W(x,p). \quad (74) \]

From the Leibnitz product rule and the Groenewold equation we derive

\[ Q_W \star dG_W = -dQ_W \star G_W, \]
\[ dG_W = -G_W \star dQ_W \star G_W. \quad (75) \]
Since the Weyl symbols of the considered operators vary slowly as functions of coordinates, we may change everywhere the sum over the lattice points $x$ to the integral over $x$. Under this condition it follows from the definition of the Weyl symbol of an operator that

$$\int d^Dp \langle p | \hat{A} | p \rangle = \int d^Dx \frac{d^Dp}{(2\pi)^D} A_W(x, p)$$

The integral of a Moyal product over phase space has the following commutativity property

$$\int d^Dx \frac{d^Dp}{(2\pi)^D} \text{Tr} A_W(x, p) \ast B_W(x, p) = \int d^Dx \frac{d^Dp}{(2\pi)^D} \text{Tr} B_W(x, p) \ast A_W(x, p).$$

(76)

We prove it as follows

$$\int d^Dx \frac{d^Dp}{(2\pi)^D} \text{Tr} A_W(x, p) \ast B_W(x, p) = \int d^Dx \frac{d^Dp}{(2\pi)^D} \text{Tr}(AB)_W(x, p)$$

$$= \int d^Dp \text{Tr}(AB)(p, p)$$

$$= \int d^Dp d^Dp' \text{Tr} A(p, p') B(p', p)$$

$$= \int d^Dp d^Dp' \text{Tr} B(p', p) A(p, p')$$

$$= \int d^Dp' \text{Tr}(BA)(p', p')$$

$$= \int d^Dx \frac{d^Dp}{(2\pi)^D} \text{Tr} B_W(x, p) \ast A_W(x, p).$$

(77)

References

[1] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. 49, 405 (1982).
[2] B. Simon, PRL 51, 2167 (1983)
[3] F.D.M. Haldane, PRL 61, 2015 (1988)
[4] Mahito Kohmoto, Bertrand I. Halperin, and Yong-Shi Wu, "Diophantine equation for the three-dimensional quantum Hall effect", Phys. Rev. B 45 (1992), 13488
[5] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Physical Review B 78, 195424 (2008).
[6] Eduardo Fradkin, "Field Theories of Condensed Matter Physics", 1991, Addison Wesley Publishing Company,
[7] T. Matsuyama, “Quantization of Conductivity Induced by Topological Structure of Energy Momentum Space in Generalized QED in Three-dimensions," Prog. Theor. Phys. 77 (1987) 711.
[8] G.E. Volovik, "An analog of the quantum Hall effect in a superfluid 3He film", JETP, Vol. 67, No. 9 (1988), ZhETF, Vol. 94, No. 3(9), (1988), 123
[9] H. So, Prog. Theor. Phys. 74, 585 (1985)
[10] M.F.L. Golterman, K. Jansen and D.B. Kaplan, Chern-Simons currents and chiral fermions on the lattice, Phys.Lett. B 301, 219–223 (1993): arXiv: hep-lat/9209003
[11] K. Ishikawa and T. Matsuyama, Magnetic field induced multi component QED in three-dimensions and quantum Hall effect, Z. Phys. C 33, 41–45 (1986).
[12] G.E. Volovik, *The Universe in a Helium Droplet*, Clarendon Press, Oxford (2003).

[13] V. Gurarie, Single-particle Green's functions and interacting topological insulators, Phys. Rev. B 83, 085426 (2011).

[14] A.M. Essin and V. Gurarie, Bulk-boundary correspondence of topological insulators from their Green's functions, Phys. Rev. B 84, 125132 (2011).

[15] M. A. Zubkov, “Absence of equilibrium chiral magnetic effect,” Phys. Rev. D 93 (2016) no.10, 105036 doi:10.1103/PhysRevD.93.105036 [arXiv:1605.08724 [hep-ph]].

[16] M. A. Zubkov, “Wigner transformation, momentum space topology, and anomalous transport,” Annals Phys. 373, 298 (2016) [arXiv:1603.03665 [cond-mat.mes-hall]].

[17] M. Vazifeh and M. Franz, “Electromagnetic response of weyl semimetals”, Phys. Rev. Lett. 111, 027201 (2013) [arXiv:1305.7377].

[18] Y. Chen, S. Wu, and A. Burkov, “Axion response in Weyl semimetals”, Phys. Rev. B 88, 125105 (2013) [arXiv:1306.5344].

[19] Y. Chen, D. Bergman, and A. Burkov, “Weyl fermions and the anomalous Hall effect in metallic ferromagnets”, Phys. Rev. B 88, 125110 (2013) [arXiv:1305.0183].

[20] S. T. Ramamurthy and T. L. Hughes, “Patterns of electro-magnetic response in topological semi-metals”, arXiv:1405.7377.

[21] A. A. Zyuzin and A. A. Burkov, “Topological response in Weyl semimetals and the chiral anomaly,” Phys. Rev. B 86 (2012) 115133 [arXiv:1206.1868 [cond-mat.mes-hall]].

[22] Pallab Goswami, Sumanta Tewari, Axionic field theory of (3+1)-dimensional Weyl semi-metals, Phys. Rev. B 88, 245107 (2013), arXiv:1210.6352.

[23] Qian Niu, D. J. Thouless, and Yong-Shi Wu, ”Quantized Hall conductance as a topological invariant”, Phys. Rev. B 31 (1985) 3372.

[24] Hiroyuki Shiba, Kunihiko Kanada, Hiroshi Hasegawa, and Hidetoshi Fukuyama, ”Galvanomagnetic Effects in Impurity Band Conductions”, J. Phys. Soc. Jpn. 30, pp. 972-987 (1971).

[25] David Tong “Lectures on the Quantum Hall Effect”, arXiv:1606.06687 [hep-ph].

[26] Y Hatsugai, ”Topological aspects of the quantum Hall effect”, J. Phys. Condens. Matter 9, 2507-2549 (1997).

[27] H. J. Groenewold, ”On the Principles of elementary quantum mechanics”, Physica,12 pp. 405-460 (1946), doi:10.1016/S0031-8914(46)80059-4.

[28] J. E. Moyal, ”Quantum mechanics as a statistical theory”, Proceedings of the Cambridge Philosophical Society, 45 (1949) pp. 99-124. doi:10.1017/S0305004100000487.

[29] Berezin, F.A. and M.A. Shubin, 1972, in: Colloquia Mathematica Societatis Janos Bolyai (North-Holland, Amsterdam) p. 21.

[30] Curtright, T. L.; Zachos, C. K. (2012). "Quantum Mechanics in Phase Space". Asia Pacific Physics Newsletter. 01: 37. arXiv:1104.5269.

[31] M. Suleymanov and M. A. Zubkov, “Wigner-Weyl formalism and the propagator of Wilson fermions in the presence of varying external electromagnetic field,” Nucl. Phys. B 938 (2019) 171 [arXiv:1811.08233 [hep-lat]].
[32] Ryogo Kubo, Hiroshi Hasegawa, Natsuki Hashitsume, ”Quantum Theory of Galvanomagnetic Effect I. Basic Considerations”, Journal of the Physical Society of Japan 14(1) (1959) 56-74 DOI: 10.1143/JPSJ.14.56

[33] K. Landsteiner, E. Megias and F. Pena-Benitez, “Anomalous Transport from Kubo Formulae,” Lect. Notes Phys. 871 (2013) 433 [arXiv:1207.5808 [hep-th]].

[34] M. A. Zubkov, Z. V. Khaidukov and R. Abramchuk, “Momentum Space Topology and Non-Dissipative Currents,” Universe 4 (2018) no.12, 146 doi:10.3390/universe4120146 [arXiv:1812.05855 [hep-ph]].

[35] M. Zubkov and Z. Khaidukov, “Anomalous transport phenomena and momentum space topology,” EPJ Web Conf. 191 (2018) 05007 doi:10.1051/epjconf/201819105007 [arXiv:1811.07778 [hep-ph]].

[36] Z. V. Khaidukov and M. A. Zubkov, “Chiral Separation Effect in lattice regularization,” Phys. Rev. D 95 (2017) no.7, 074502 doi:10.1103/PhysRevD.95.074502 [arXiv:1701.03368 [hep-lat]].

[37] M. A. Zubkov and Z. V. Khaidukov and M. A. Zubkov, “Anatomy of the chiral vortical effect,” Phys. Rev. D 98 (2018) no.7, 076013 doi:10.1103/PhysRevD.98.076013 [arXiv:1806.02605 [hep-ph]].

[38] Z. V. Khaidukov and M. A. Zubkov, ”Chiral torsional effect,” doi:10.1134/S0021364018220046 [arXiv:1812.00970 [cond-mat.mes-hall]].

[39] Yasuhiro Hatsugai, Takahiro Fukui, and Hideo Aoki, ”Topological analysis of the quantum Hall effect in graphene: Dirac-Fermi transition across van Hove singularities and edge versus bulk quantum numbers”, Phys. Rev. B 74 (2006), 205414

[40] K. A. Muttalib and P. Wolfe, Phys. Rev. B 76, 204415 (2007).

[41] B. L. Altshuler, D. Khmel’nit’zkii, A. I. Larkin and P. A. Lee, ”Magnetoresistance and Hall effect in a disordered two-dimensional electron gas”, Phys.Rev.B 22, 5142 (1980).

[42] B.L. Altshuler and A.G. Aronov,Electron-electron inter-action in disordered systems(A.L. Efros, M. Pollak, Ams-terdam, 1985).

[43] M. Laubach, C. Platt, R. Thomale, T. Neupert and S. Rachel ”Density wave instabilities and surface state evolution in interacting Weyl semimetals”, Phys. Rev. B 94, 241102(R) (2016).

[44] J. Carlstrom and E.J. Bergholtz, ”Strongly interacting Weyl semimetals: Stability of the semimetallic phase and emergence of almost free fermions”, Physical Review B, 98, 241102 (2018).

[45] B. Roy, P. Goswami and V. Juricic, ”Interacting Weyl fermions: Phases, phase transitions, and global phase diagram”, Physical Review B, 95, 201102(R) (2017).

[46] S. Rachel, ”Interacting topological insulators: a review”, Reports on Progress in Physics, 81, 116501 (2018).

[47] H.K.Tang, J. N. Leaw, J. N. B. Rodrigues, I. F. Herbut, P. Sengupta, F. F. Assaad, and S. Adam, ”The role of electron-electron interactions in two-dimensional Dirac fermions”, Science, 361, 570 (2018).

[48] C. X. Zhang and M. A. Zubkov, ”Influence of interactions on the anomalous quantum Hall effect,” [arXiv:1902.06545 [cond-mat.mes-hall]].

[49] N. Yamamoto, ”Generalized Bloch theorem and chiral transport phenomena,” Phys. Rev. D 92 (2015) no.8, 085011 doi:10.1103/PhysRevD.92.085011 [arXiv:1502.01547 [cond-mat.mes-hall]].

[50] F.R. Klinkhamer, G.E. Volovik, Internat. J. Modern Phys. A 20 (2005) 2795. http://dx.doi.org/10.1142/S0217751X05020002 [hep-th/0403037]

[51] P. Maraner, ”Adiabatic motion of a quantum particle in a two-dimensional magnetic field”, J. Phys. A: Math. Gen. 29, 2199 (1996)
[52] L. Susskind, arXiv: hep-th/ 0101029

[53] A.P. Polychronakos, Non-commutative Fluids. In: Duplantier B. (eds) Quantum Spaces. Progress in Mathematical Physics, vol 53. Birkhauser Basel (2007). arXiv:0706.1095

[54] Felix A. Buot, Nonequilibrium Quantum Transport Physics in Nanosystems: Foundation of Computational Nonequilibrium Physics in Nanoscience and Nanotechnology (World Scientific Publishing, NJ, 2009)

[55] Felix A. Buot, Quantum Superfield Theory and Lattice Weyl Transform in Nonequilibrium Quantum Transport Physics, Quantum Matter, 2, 247-288 (2013)

[56] Felix A. Buot, ”Method for calculating TrH^n in solid – state theory”, Physical Review B 10 (1974) 3700

[57] P.Kasperkovitz, M.Peev, Annals of Physics 230 (1994) 21