STRONG OSCILLATIONS OF CUMULANTS
OF PHOTON DISTRIBUTION FUNCTION
IN SLIGHTLY SQUEEZED STATES

V.V.Dodonov, I.M.Dremin and P.G.Polynkin
Lebedev Physics Institute,
Leninsky Prospect, 53, 117924, Moscow, Russia
and
V.I.Man’ko
Lebedev Physics Institute and University of Naples "Federico II",
Mostra d’Oltremare, Pad.20, 80125 Naples, Italy

Abstract

The cumulants and factorial moments of photon distribution for squeezed and correlated light are calculated in terms of Chebyshev, Legendre and Laguerre polynomials. The phenomenon of strong oscillations of the ratio of the cumulant to factorial moment is found.
1 INTRODUCTION

The coherent light [1],[2],[3] has the photon distribution function described by the standard Poisson distribution. The nonclassical states of light, for example squeezed states [4],[5],[6], Schrödinger cat states [7],[8],[9], correlated states [10] have the photon distribution functions which differ essentially from poissonian ones demonstrating either superpoissonian or sub-poissonian behaviour which distinguishes the nonclassical types of light from the coherent light considered by definition as classical one.

One of the important differences of photon distribution function of the nonclassical types of light from the poissonian distribution of the coherent light is the possible strong oscillations of the photon distribution function. For the squeezed light these oscillations have been found in [11],[12]. For the correlated light the existence of such oscillations has been demonstrated in [13]. For even and odd coherent states (Schrödinger cat states) the fast oscillations of the photon distribution function are connected with the absence of the states with even numbers of photons in odd cat states and the states with odd numbers of photons in even cat states [7],[14],[15]. Such properties of the photon distribution oscillations for the nonclassical types of light are preserved also for the multimode electromagnetic radiation. It was shown for two-mode squeezed light in [16],[17],[18] and for even and odd cat states in [19],[20].

On the other hand, in high energy physics another characteristics of the multiparticle distribution function has been widely used, namely, factorial moments and cumulants (for the review see [21]). It was shown that the cumulants and the functions of these quantities are very sensitive to the details of the particle distribution demonstrating the oscillating behaviour [22],[23],[24]. The properties of the cumulants of the integer rank for the known distributions as poissonian and binomial ones are well known and for these distributions the
oscillatory behaviour is absent [24].

The behaviour of the cumulants (in particular, their oscillations) for the squeezed states, correlated states or other nonclassical states has not been studied till now in all details. At the same time, for the most general photon distribution function corresponding to squeezed and correlated one-mode light at finite temperature the explicit expression in terms of Hermite polynomials of two variables has been obtained in [25] and it was widely used in [26],[27],[28],[29].

The aim of this work is to obtain the explicit expressions for the cumulants and factorial moments of the photon distribution function for the squeezed and correlated light at finite temperature. We demonstrate that the cumulants possess the strongly oscillating behaviour in the region of slight squeezing where the photon distribution function itself has no oscillations. And vice versa in the region of large squeezing, where the photon distribution function strongly oscillates, the cumulants behave smoothly. Thus the behaviour of the cumulants may provide a very sensitive method of detecting very small squeezing and correlation phenomena due to presence of strong oscillations.

2 PHOTON DISTRIBUTION FUNCTION

AND ITS MOMENTS

Since in what follows it will be necessary to use some characteristics of one-mode squeezed light, we will briefly review the main results obtained (see, for example [26]).

Let us consider the most general mixed squeezed state of one-mode light described by the Wigner function $W(p,x)$ of the generic Gaussian form with five real parameters, $\langle x \rangle$, $\langle p \rangle$, $\sigma_{xx}$, $\sigma_{pp}$, $\sigma_{px}$ (first two parameters are means of position and momentum, others are matrix
elements of the dispersion matrix for the position and momentum),

\[
W(p, x) = d^{-1/2} \exp \left\{ -(2d)^{-1} \left[ \sigma_{xx}(p - \langle p \rangle)^2 \\
+ \sigma_{pp}(x - \langle x \rangle)^2 - 2\sigma_{px}(p - \langle p \rangle)(x - \langle x \rangle) \right] \right\},
\]

(2.1)

where

\[
d = \sigma_{pp}\sigma_{xx} - \sigma_{px}^2
\]

is determinant of the dispersion matrix.

For the photon distribution function the following formula was obtained [25],[26]:

\[
P_n = P_0 \frac{H_{nn}^R(y_1, y_2)}{n!},
\]

(2.2)

where

\[
P_0 = \left( d + \frac{1}{2}T + \frac{1}{4} \right)^{-1/2} \times \exp \left[ -\langle p \rangle^2(2\sigma_{xx} + 1) + \langle x \rangle^2(2\sigma_{pp} + 1) - 4\sigma_{px}\langle p \rangle\langle x \rangle \right] \frac{1 + 2T + 4d}{1 + 2T + 4d}
\]

is the probability to have no photons,

\[
T = \sigma_{pp} + \sigma_{xx}
\]

is the trace of the dispersion matrix, \( H_{nn}^R \) – Hermite polynomials of two variables.

Elements of the symmetric matrix

\[
R = \begin{pmatrix}
R_{11} & R_{12} \\
R_{12} & R_{22}
\end{pmatrix}
\]

determining the Hermite polynomials are given by the formulas:

\[
R_{11} = R_{22} = \frac{2(\sigma_{pp} - \sigma_{xx} - 2i\sigma_{px})}{1 + 2T + 4d},
\]

\[
R_{12} = \frac{1 - 4d}{1 + 2T + 4d},
\]
and two arguments of the polynomials are defined by the equation:

\[ y_1 = y_2^* = \frac{2 \left( (T - 1) z^* + (\sigma_{pp} - \sigma_{xx} + 2i\sigma_{px}) z \right)}{2T - 4d - 1} \]

The complex parameter \( z \) is given by the relation:

\[ z = 2^{-1/2} (\langle x \rangle + i\langle p \rangle) \]  

The generating function for the photon distribution function was also obtained in [26]:

\[ G(u) = P_0 \left[ \left( 1 - \frac{u}{\lambda_1} \right) \left( 1 - \frac{u}{\lambda_2} \right) \right]^{-1/2} \exp \left[ \frac{u\xi_1}{u - \lambda_1} + \frac{u\xi_2}{u - \lambda_2} \right] \]

where

\[ \lambda_1 = \left( \sqrt{R_{11}R_{22}} - R_{12} \right)^{-1}, \quad \lambda_2 = -\left( \sqrt{R_{11}R_{22}} + R_{12} \right)^{-1}, \]

\[ \xi_1 = \frac{1}{4} \left( 1 - \frac{R_{12}}{\sqrt{R_{11}R_{22}}} \right) \left( y_1^2 R_{11} + y_2^2 R_{22} - 2\sqrt{R_{11}R_{22}} y_1 y_2 \right), \]

\[ \xi_2 = \frac{1}{4} \left( 1 + \frac{R_{12}}{\sqrt{R_{11}R_{22}}} \right) \left( y_1^2 R_{11} + y_2^2 R_{22} + 2\sqrt{R_{11}R_{22}} y_1 y_2 \right). \]

The photon distribution function is related to \( G(u) \) as follows:

\[ P_n = \left. \frac{1}{n!} \frac{d^n G(u)}{du^n} \right|_{u=0} \]

The above-mentioned normalized cumulants and factorial moments are defined as

\[ K_q = \left. \frac{1}{\langle n \rangle^q} \frac{d^q \ln G(u)}{du^q} \right|_{u=1}, \]

\[ F_q = \left. \frac{1}{\langle n \rangle^q} \frac{d^q G(u)}{du^q} \right|_{u=1}, \]

respectively, and related by the following recursion relation:

\[ F_q = \sum_{m=0}^{q-1} C_{q-1}^m K_{q-m} F_m \]
are the binomial coefficients. Some formulas for the cumulants of photon distribution function can be found in [30],[31]. The factorial moments have been widely used in particle physics [21] to analyze intermittency properties [32] of fluctuations. However, they are less instructive than the cumulants or their functions. It was shown in [22],[24] that the ratio of cumulant to factorial moments, i.e. the function

$$H_q = K_q / F_q,$$ (2.8)

is a very sensitive measure of tiny details of the multiplicity distribution. In particular, it can be used to distinguish between different distributions which otherwise look quite similar.

3 ANALYSIS OF $K_q$, $F_q$ AND $H_q$

It was already mentioned in the Introduction that the photon distribution function exhibits an oscillatory behaviour if we deal with highly squeezed states $(T = \sigma_{pp} + \sigma_{xx} \gg 1)$ for large values of the parameter $z$ (2.3). A question arises: is it possible to obtain a similar ”abnormal” behaviour of other characteristics of the photon distribution, namely, cumulants, factorial moments and the function $H_q$ defined in (2.8)? If yes, then in what region of parameters of the function (2.1) such anomalies can take place? The present section is dedicated to the solution of this problem.

The direct differentiation of the function $\ln G(u)$ at $u = 1$ yields:

$$K_q = \frac{(q - 1)!}{(\langle n \rangle^q)} \left[ \frac{1}{(\lambda_1 - 1)^q} \left( \frac{1}{2} + q \frac{\xi_1 \lambda_1}{1 - \lambda_1} \right) + \frac{1}{(\lambda_2 - 1)^q} \left( \frac{1}{2} + q \frac{\xi_2 \lambda_2}{1 - \lambda_2} \right) \right],$$ (3.1)
where \[26\]

\[\langle n \rangle = \frac{T - 1}{2} + |z|^2.\]

High oscillation are obtained at strong squeezing (large \(T\)) or at large values of \(|z|^2\).

It is known \[27\] that

\[T \geq 1, \quad d \geq \frac{1}{4}. \tag{3.2}\]

Using (3.2) the following inequalities can be easily obtained:

\[\lambda_1 > 1 \quad \text{and} \quad \lambda_2 < 0 \quad \text{or} \quad \lambda_2 > 1. \tag{3.3}\]

Expression in the square brackets in (3.1) consists of two terms:

\[\frac{1}{(\lambda_1 - 1)q} \left( \frac{1}{2} + q \frac{\xi_1 \lambda_1}{1 - \lambda_1} \right), \tag{3.4}\]

\[\frac{1}{(\lambda_2 - 1)q} \left( \frac{1}{2} + q \frac{\xi_2 \lambda_2}{1 - \lambda_2} \right). \tag{3.5}\]

The first term is of constant sign. The second one is oscillating in the case \(\lambda_2 < 0\). With aim to obtain the oscillations of the whole function \(K_q\) we will treat only this case:

\[\lambda_2 < 0.\]

Then

\[\frac{1}{(\lambda_2 - 1)q} = \frac{(-1)^q}{(1 + |\lambda_2|)^q}.\]

However, then it follows that

\[|\lambda_2| \geq \lambda_1 > 1,\]
and the alternating term diminishes faster than the constant sign term. The terms $1/|\lambda_2 - 1|$ and $1/(\lambda_1 - 1)$ are most close to one another if

$$d = \frac{1}{4} \quad \text{(the pure state)},$$

that is used in the following.

First of all we consider the simplest case when the value of $z$ as given by (2.3) equals to zero. Then

$$K_q = \frac{\beta^{q/2}}{\langle n \rangle^q} (q - 1)! T_q(\alpha),$$

$$F_q = \frac{\beta^{q/2}}{\langle n \rangle^q} q! P_q(\alpha), \quad (3.6)$$

where

$$\beta = d + \frac{1}{4} - \frac{T}{2}, \quad \alpha = \frac{T - 1}{\sqrt{4d + 1 - 2T}}.$$

$T_q(\alpha)$ and $P_q(\alpha)$ are the Chebyshev polynomials of the first kind and Legendre polynomials, respectively. Let us note that the arguments of polynomials are purely imaginary but the whole expressions for moments are real, surely.

For $H_q$ we obtain the expression:

$$H_q = \frac{T_q(\alpha)}{q P_q(\alpha)}. \quad (3.7)$$

Taking $d = 1/4$ we have:

$$K_q = \left(\frac{2}{1 - T}\right)^{q/2} (q - 1)! T_q(\alpha),$$

$$F_q = \left(\frac{2}{1 - T}\right)^{q/2} q! P_q(\alpha),$$

$$\alpha = \sqrt{\frac{1 - T}{2}}.$$

In this case the curve $H_q$ has step-like shape at $(T - 1) \to 0$; steps become smoothed as $T$ grows (fig.1). We should note that direct limit $T \to 1$ shows the discontinuous character
of the function $H_q(T)$ at $T = 1$. The point is that at $T = 1$ we are dealing with the usual Poisson distribution (let us remind that we treat a case $d = 1/4$, $|z| = 0$), where $H_q = \delta_{q1}$, i.e. $H_1 = 1$, $H_q = 0$ at $q \neq 1$, which differs from the behaviour of $H_q$ at $(T - 1) = 10^{-5}$, depicted in the fig 1. The particular values of the second rank moments are very high what reveals extremely wide distribution (so wide distributions are unknown in particle multiproduction, for example). Namely, one can easily show that $F_2 > 3$, $K_2 > 2$, $H_2 > 2/3$.

Consider now the case $|z| \neq 0$. Since the photon distribution function is invariant with respect to rotation in a phase space, without loss of generality we can consider $\sigma_{xx} = \sigma_{pp}$ ($\sigma_{px} \neq 0$ -- correlated state). By appropriate choice of the phase of (2.3) ($\langle x \rangle = -\langle p \rangle$) we cancel the linearly growing term $q\xi_1\lambda_1/(1 - \lambda_1)$ in (3.4). Moreover, the analogous linear term $q\xi_2\lambda_2/(1 - \lambda_2)$ in (3.5) becomes maximal at fixed $|z|$. Thus we have left only two variable parameters $T$ and $|z|$, and formula (3.1) has the following final form:

$$K_q = \frac{(q - 1)!}{(\frac{T - 1}{2} + |z|^2)^q} \left[ \frac{1}{2(\lambda_1 - 1)^q} + \frac{(-1)^q}{(1 + |\lambda_2|)^q} \left( \frac{1}{2} - q \xi_2|\lambda_2| \right) \right],$$

(3.8)

where

$$\lambda_1 = -\lambda_2 = \sqrt{\frac{T + 1}{T - 1}},$$

$$\xi_2 = 2\left( \frac{T}{\sqrt{T^2 - 1}} + 1 \right) |z|^2.$$

In the case of large $T$ (highly squeezed state) we can obtain the finite number of oscillations of $K_q$ taking large value of $|z|$. However, the average number of photons in corresponding states is large, and the amplitude of the oscillations decreases exponentially due to the factor $1/\langle n \rangle^q$. Remind that in this very case the strong oscillations of the photon distribution function can be observed.

Now let us consider the case of the slightly squeezed state, $y = (T - 1) \ll 1$, when photon
distribution function does not oscillate. Impose also an additional condition

$$\gamma = \frac{|z|^2}{\sqrt{y/2}} \gg 1 ,$$

that makes possible to obtain approximate formulas for the functions $K_q$, $F_q$ and $H_q$. For $K_q$ we have the following approximate expression:

$$K_q = q!(-1)^{q-1}\gamma^{1-q} . \quad (3.9)$$

Then recursion relation (2.7) yields:

$$F_q = q!(-1)^q\gamma^{-q}L_q^{-1}(\gamma) , \quad (3.10)$$

where $L_q^{-1}(x)$ are generalized Laguerre polynomials. For $H_q$ with $q \ll \gamma$ we have:

$$H_q = K_q/F_q = -\frac{\gamma}{L_q^{-1}(\gamma)} \approx (-1)^{q+1}q!\gamma^{1-q} \ll 1 . \quad (3.11)$$

(If $\gamma \gg q$, the term with the highest power of $\gamma$ dominates over the rest of the sum in $L_q^{-1}(\gamma)$, and $F_q \to 1$ as for Poisson distribution). The exact shape of the function $H_q$ is shown in fig.2. The distribution function $P_n$ does not oscillate (fig.2a).

However, the most abrupt oscillations of the functions $K_q$ and $H_q$ have been obtained when $(T - 1) \ll 1$, but condition $\gamma \gg 1$ is not valid. The corresponding curves are shown in the figs.3, 3a. Note that the photon distribution function is smooth again being approximately equal to zero at $q \neq 1$.

The most regular oscillating patterns of $K_q$ and $H_q$ are seen at $(T - 1) \sim 0.1$, $|z| \sim 1$ (figs.4, 4a).

The alternating sign cumulants are typical also for the fixed multiplicity distribution, i.e., for $P_n = \delta_{n_0} (n_0 = \text{const})$ [24]. Let us note that there exist smooth multiplicity distributions which give rise to cumulants oscillating with larger period (see [22]).
Finally we consider the opposite case when the photon distribution function $P_n$ exhibits strong oscillations while $K_q$ and $H_q$ behave smoothly. Such a behaviour is typical at $T \sim 100$, $|z| \sim 1$ when $K_q$ exponentially grows while $H_q$ monotonically decreases with $q$ (fig.5).

4 CONCLUSION

We have shown that cumulants of the photon distribution function and their ratio to the corresponding factorial moments exhibit oscillating behaviour in the case of slightly squeezed states: $(T - 1) = (\sigma_{pp} + \sigma_{qq}) - 1 \ll 1$. We have considered also the pure state with $d = 1/4$. Oscillations of the photon distribution function are absent in that case.

We are yet unable to establish direct correspondence between the oscillations of cumulants and the behaviour of the photon distribution function. Somehow it should depend on the range of parameters considered. For example, it is known [26] that parameter $d = \sigma_{pp}\sigma_{xx} - \sigma_{px}^2$ characterizes the temperature of the system and the temperature equals to zero at $d = 1/4$. Is it possible to observe oscillations of cumulants and of the function $H_q$ at other parameters of the function (2.1), for example, at $d > 1/4$? This problem has not been studied in all details yet. However, evidently, the oscillations must be smoothed as $d$ grows since it is analogous to the behaviour of the function $P_n$, whose oscillations disappear with increase of temperature. The extension of the above procedure to non-integer ranks is straightforward. We hope to consider these problems in subsequent publications.

ACKNOWLEDGEMENTS

This work was supported by Russian State Science and Technology Program ”Fundamental Nuclear Physics”. One of us (V.I.M) thanks the University of Naples ”Federico II” for kind
hospitality.
References

[1] R.J.Glauber, Phys.Rev.Lett. 10 (1963) 84.

[2] E.C.G.Sudarshan, Phys.Rev.Lett. 10 (1963) 277.

[3] J.R.Klauder, J.Math.Phys. 4 (1963) 1055.

[4] J.N.Hollenhorst, Phys.Rev.D 19 (1979) 1669.

[5] D.Stoler, Phys.Rev.D 1 (1970) 3217.

[6] H.P.Yuen, Phys.Rev.A 13 (1976) 2226.

[7] V.V.Dodonov, I.A.Malkin, and V.I.Man’ko, Physica 72 (1974) 597.

[8] J.Perina in Quantum statistics of linear and nonlinear optical phenomena (Reidel, Dor-drecht 1984).

[9] B.Yurke and D.Stoler, Phys.Rev.Lett. 57 (1986) 13.

[10] V.V.Dodonov, E.V.Kurmyshev, and V.I.Man’ko, Phys.Lett.A 79 (1980) 150.

[11] W.Schleich and J.A.Wheeler, J.Opt.Soc.Am.B 4 (1987) 1715.

[12] A.Vourdas and R.M.Weiner, ibid. 36 (1987) 5866.

[13] V.V.Dodonov, A.B.Klimov, and V.I.Man’ko, Phys.Lett.A 134 (1989) 211.

[14] A.Vourdas, Opt.Comm. 91 (1992) 236.

[15] V.Bužek, A.Vidiella-Barranco, and P.L.Knight, Phys.Rev.A 45 (1992) 6750.

[16] M.Artoni, V.P.Ortiz, and J.L.Birman, Phys.Rev.A 43 (1993) 3954.

[17] C.M.Caves, Chang Zhu, G.L.Milburn, and W.Schleich, Phys.Rev.A 43 (1991) 3854.
[18] G.Shrade, V.M.Akulin, V.I.Man’ko, and W.Schleich, Phys.Rev.A 48 (1993) 2398.

[19] N.A.Ansari and V.I.Man’ko, ”Photon statistics of multimode even and odd coherent light” Preprint of the University of Naples INFN-NA-IV-93/34, DSF-T-93/34 (1993) (Phys.Rev.A, in press).

[20] V.V.Dodonov, D.E.Nikonov, and V.I.Man’ko, ”Even and odd coherent states (Schrödinger cat states) for multimode parametric systems” Preprint of the University of Naples N INFN-NA-IV-93/49, DSF-T-93/49 (1993) (submitted to Phys.Rev.A).

[21] E.A.DeWolf, I.M.Dremin, and W.Kittel, Uspekhi Fiz. Nauk 163 (1993) 3; Phys. Rep. (to be published).

[22] I.M.Dremin, Phys.Lett. B313 (1993) 209.

[23] I.M.Dremin, JETP Lett. 59 (1994) 561.

[24] I.M.Dremin, Modern Phys. Lett.A Vol.8 No29 (1993) 2747.

[25] V.V.Dodonov, V.I.Man’ko, and V.V.Semjonov, Nuovo Cim.B 83 (1984) 145.

[26] V.V.Dodonov, O.V.Man’ko, and V.I.Man’ko, Phys.Rev.A 49 (1994) 2993.

[27] V.V.Dodonov and V.I.Man’ko, Proceedings of Lebedev Physics Institute, Vol. 183 (Nauka, Moscow, 1987) [Translation by Nova Science, N.Y., Commack, 1989].

[28] G.S.Aggarwal, J.Mod.Opt. 34 (1987) 909.

[29] P.Marian and T.A.Marian, Phys.Rev.A 47 (1993) 4474.

[30] G.S.Aggarwal and G.Adam, Phys.Rev.A 39 (1989) 6259.

[31] S.Chaturvedi and V.Srinivasan, Phys.Rev.A 40 (1989) 6095.
[32] A.Bialas and R.Peschanski, Nucl.Phys. B273 (1986) 703.
Figure 1: The behaviour of the function $H_q$ defined in (2.8) at $d = 1/4$, $|z| = 0$; parameter $T$ is varied: (1) $(T - 1) = 10^{-5}$, (2) $T = 1.1$, (3) $T = 1.2$.

Figure 2: The behaviour of the function $H_q$ at $d = 1/4$, $T = 1.01$; the curves (1) and (2) correspond to the values $|z|^2 = 1.01$, 0.8, respectively.

Figure 2a: The photon distribution function at $d = 1/4$, $T = 1.01$; the curves in order of the lowering maxima correspond respectively to the values $|z|^2 = 1.01$, 0.8.

Figure 3: The cumulants of the photon distribution function at $d = 1/4$, $(T - 1) = 10^{-5}$, $|z|^2 = 0.01$.

Figure 3a: The behaviour of the function $H_q$ at $d = 1/4$, $(T - 1) = 10^{-5}$; parameter $|z|$ is varied: (1) $|z|^2 = 0.01$, (2) $|z|^2 = 0.005$, (3) $|z|^2 = 0.01$.

Figure 4: The cumulants of the photon distribution function at $d = 1/4$, $T = 1.1$, $|z|^2 = 1.1$.

Figure 4a: The behaviour of the function $H_q$ at $d = 1/4$, $T = 1.1$; parameter $|z|$ is varied: (1) $|z|^2 = 2$, (2) $|z|^2 = 1.1$.

Figure 4b: The photon distribution function at $d = 1/4$, $T = 1.1$; the curves in order of the lowering maxima correspond to the values $|z|^2 = 2$, 1.1, respectively.

Figure 5: The smooth curve for the function $H_q$ and the oscillating photon distribution function $P_n$ at $d = 1/4$, $T = 100$, $|z| = 1$. 
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9406143v1
This figure "fig2-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9406143v1
This figure "fig3-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9406143v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9406143v1
This figure "fig2-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9406143v1
This figure "fig3-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9406143v1
This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9406143v1
This figure "fig2-3.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9406143v1
This figure "fig3-3.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9406143v1