Generalised $G_2$–manifolds

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Abstract

We define new Riemannian structures on 7–manifolds by a differential form of mixed degree which is the critical point of a (possibly constrained) variational problem over a fixed cohomology class. The unconstrained critical points generalise the notion of a manifold of holonomy $G_2$, while the constrained ones give rise to a new geometry without a classical counterpart. We characterise these structures by means of spinors and show the integrability conditions to be equivalent to the supersymmetry equations on spinors in type II supergravity theory with bosonic background fields. In particular, this geometry can be described by two linear metric connections with skew–symmetric torsion. Finally, we construct explicit examples by introducing the device of $T$–duality.

1 Introduction

Over a 7–manifold, a topological reduction to a principal $G_2$–bundle is achieved by the existence of a certain 3–form. The fact that this 3–form is generic or stable (following the language of Hitchin) enables one to set up a variational principle over a fixed cohomology class whose critical points are precisely the manifolds of holonomy $G_2$ [1].

In this paper we are concerned with a new type of Riemannian geometry over 7–manifolds which generalises this notion. Topologically speaking, it is defined by an even or odd form which we think of as a spinor for the orthogonal bundle $T \oplus T^*$ with its natural inner product of split signature. This construction is perfectly general and works in all dimensions, but there are special cases where $\mathbb{R}^* \times \text{Spin}(n, n)$ acts with an open orbit in its spin representations $\Lambda^{ev,od} T^*$. An example in dimension 6 are the so–called generalised Calabi–Yau manifolds associated with an $SU(3, 3)$–invariant spinor [12]. In dimension 7, there are stable spinors whose stabiliser is either conjugate to $G_2(\mathbb{C})$ or $G_2 \times G_2$. In this paper, we shall deal with the latter case and define a generalised $G_2$–structure as a (topological) reduction from $\mathbb{R}^* \times \text{Spin}(7, 7)$ to $G_2 \times G_2$. Stability allows us to consider a generalised variational problem which provides us with various integrability conditions.

We begin by introducing the algebraic setup. A reduction to $G_2 \times G_2$ gives rise to various objects. Firstly, it induces a metric on $T$, and moreover, a 2–form $b$ which we refer to as the $B$–field. As an element of the Lie algebra $\mathfrak{so}(7)$ inside $\mathfrak{so}(7, 7)$, it acts on any $\text{Spin}(7, 7)$–representation by exponentiation. Secondly, we obtain two unit spinors $\Psi_+$ and $\Psi_-$ in the irreducible spin representation of $\text{Spin}(7, 7)$.
Spin(7). Tensoring the spinors yields an even or odd form $[\Psi_+ \otimes \Psi_-]^{ev,od}$ by projection on $\Lambda^{ev,od}$. The first important result we prove states that any $G_2 \times G_2$–invariant spinor $\rho$ can be expressed as

$$\rho = e^{-\phi} \exp(b/2) \wedge [\Psi_+ \otimes \Psi_-]^{ev,od}. \quad (1)$$

In physicists’ terminology, the scalar $\phi$ represents the *dilaton* – here it appears as a scaling factor.

Moving on to global issues, we see that up to a $B$–field transformation, a generalised $G_2$–structure is essentially a pair of principle $G_2$–fibre bundles inside the orthonormal frame bundle determined by the metric. In particular, any topological $G_2$–manifold trivially induces a generalised $G_2$–manifold, and consequently, any spinnable seven–fold carries such a structure. Over a compact manifold, we can classify generalised $G_2$–structures up to vertical homotopy by an integer which effectively counts (with an appropriate sign convention) the number of points where the two $G_2$–structures inside the $\text{Spin}(7)$–principal bundle coincide.

Over a closed manifold, we can then set up a variational problem along the lines of [12]. In close analogy to the classical case, the condition for a critical point is that both the even and the odd $G_2 \times G_2$–invariant spinor, now regarded as a form, have to be closed. We shall adopt this as the condition of *strong* integrability for any (not necessarily compact) generalised $G_2$–structure. Interpreting this integrability condition in terms of the right–hand side of (1) leads to our main result. Theorem 4.3 characterises strongly integrable generalised $G_2$–structures in terms of two linear metric connections $\nabla^\pm$ with skew–symmetric, closed torsion $\pm T$ such that

$$(d\phi \pm \frac{1}{2} T) \cdot \Psi_\pm = 0$$

holds. Connections with skew–symmetric torsion have gained a lot of attention in the recent mathematical literature (see, for instance, [1], [5], [6] and [13]) due to their importance in string theory. Eventually, our reformulation yields the supersymmetry equations arising in type II supergravity with bosonic background fields [8].

The spinorial picture is also useful for deriving geometrical properties. In particular, we compute the Ricci tensor which is given by

$$\text{Ric}(X,Y) = -2H^\phi(X,Y) + \frac{1}{4} g(X,T,Y,T),$$

with $H^\phi$ denoting the Hessian of the dilaton. A further striking consequence is a no–go theorem for compact manifolds (generalising similar statements in [14] and [8]): Here, the torsion of any strongly integrable generalised $G_2$–structure must vanish, that is, the underlying spinors $\Psi_+$ and $\Psi_-$ are parallel with respect to the Levi–Civita connection. In this sense, only “classical” solutions can be found for the variational problem. However, local examples with non–vanishing torsion exist in abundance. Using again the form definition of a generalised $G_2 \times G_2$–structure, we will describe a systematic construction method known in string theory as $T$–duality. It consists of changing the topology by replacing a fibre isomorphic to $S^1$ (or more generally to an $n$–torus) without destroying integrability. This allows us to pass from a trivial generalised structure coming from a classical $S^1$–invariant $G_2$–structure with vanishing torsion (for which we can easily find examples) to a trivial $S^1$–bundle with a non–trivial generalised $G_2$–structure.

The lack of interesting compact examples motivated us to consider a *constrained* variational problem following ideas in [11]. This gives rise to *weakly* integrable structures of either even or odd type. For these the no–go theorem does not apply, but the construction of examples, let alone compact
ones, remains an open problem. Although arising out of a similar constraint as manifolds of weak holonomy $G_2$, this notion gives rise to a new geometry without a classical counterpart which renders a straightforward application of $T$–duality impossible. We investigate its properties along with those of the strongly integrable case.

The algebraic theory outlined above not only makes sense for $G_2 \times G_2$–structures, but also for $Spin(7) \times Spin(7)$ inside $Spin(8,8)$, leading to the notion of a generalised $Spin(7)$–manifold. It is also defined by an invariant form of mixed degree, which, however, is not stable. Comparison with classical $Spin(7)$–geometry suggests closeness of this form as a natural notion of integrability.

Section 5 briefly explores the theory of these structures which is developed in full detail in [18].

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2 The linear algebra of generalised $G_2$–structures

2.1 Generalised metrics

We consider the vector bundle $T \oplus T^*$, where $T$ is a real, $n$–dimensional vector space. It carries a natural orientation and the inner product of signature $(n,n)$, defined for $v \in T$ and $\xi \in T^*$ by

$$(v + \xi, v + \xi) = -\frac{1}{2} \xi(v),$$

singles out a group conjugate to $SO(n,n)$ inside $GL(2n)$. Note that $GL(n) \leq SO(n,n)$. As a $GL(n)$–space, the Lie algebra of $SO(n,n)$ decomposes as

$$\mathfrak{so}(n,n) = \Lambda^2(T \oplus T^*) = End(T) \oplus \Lambda^2 T^* \oplus \Lambda^2 T.$$ 

In particular, any 2–form $b$ defines an element in the Lie algebra $\mathfrak{so}(n,n)$. We will refer to such a 2–form as a $B$–field. Exponentiated to $SO(n,n)$, its action on $T \oplus T^*$ is

$$\exp(b)(v \oplus \xi) = v \oplus (v \lrcorner b + \xi).$$

Next we define an action of $T \oplus T^*$ on $\Lambda^* T^*$ by

$$(v + \xi) \bullet \tau = v \lrcorner \tau + \xi \wedge \tau.$$ 

As this squares to minus the identity it gives rise to an isomorphism $Cliff(T \oplus T^*) \cong End(\Lambda^* T^*)$. The exterior algebra $S = \Lambda^* T^*$ becomes thus the pinor representation space of $Cliff(T \oplus T^*)$ and splits into the irreducible spin representation spaces $S^\pm = \Lambda^{ev,od} T^* \otimes (\Lambda^n T)^{1/2}$.

Remark: There is a canonical embedding $GL_+(n) \hookrightarrow Spin(n,n)$ of the identity component of $GL(n)$ into the spin group of $T \oplus T^*$. As a $GL_+(n)$–module we have

$$S^\pm = \Lambda^{ev,od} T^* \otimes (\Lambda^n T)^{1/2},$$

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in analogy to the complex case. There $U(n) \rightarrow Spin^C(2n) = Spin(2n) \times \mathbb{Z}_2 S^1$, and the even and odd forms get twisted with the square root of the canonical line bundle. As long as we are doing linear algebra this is a mere notational issue but in the global situation we cannot trivialise $\Lambda^n T$ unless the manifold is orientable. In fact, a more refined analysis reveals that we can always choose a spin structure for $T \oplus T^* –$ whether the manifold is orientable or not – such that the spinor bundle is isomorphic, as a $GL(n)$–space, to the exterior form bundle, albeit in a non–canonical way (see, for instance, the discussion in Section 2.8 in [9]). We will neglect these subtleties for we will consider orientable manifolds only, and therefore omit the twist to ease notation. However it is important to bear it in mind when we set up the variational formalism in Section 4.1.

Let $\sigma$ be the Clifford algebra anti–automorphism defined on any element of degree $p$ by $\sigma(\alpha^p) = \epsilon(p)\alpha^p$ where $\epsilon(p) = 1$ for $p \equiv 0, 3 \mod 4$ and $-1$ for $p \equiv 1, 2 \mod 4$. The bilinear form

$$\langle \alpha, \beta \rangle = (\alpha \wedge \sigma(\beta))_n,$$

where the subscript $n$ indicates taking the top degree component, is non–degenerate and invariant under the action of $Spin(n,n)$. It is symmetric if $n \equiv 0, 3 \mod 4$ and skew if $n \equiv 1, 2 \mod 4$. Moreover, $S^+$ and $S^\perp$ are non–degenerate and orthogonal if $n$ is even and totally isotropic if $n$ is odd. Finally, we note that the action of a $B$–field $b$ on a spinor $\tau$ is given by

$$\exp(b) \cdot \tau = (1 + b + \frac{1}{2}b \wedge b + \ldots) \wedge \tau = e^{b} \wedge \tau.$$

In this paper we shall be concerned with special structures on $T \oplus T^*$, which we describe in terms of reductions to special subgroups of $O(n,n)$ or $\mathbb{R}^* \times O(n,n)$.

**Definition 2.1.** A generalised metric structure is a reduction from $O(n,n)$ to $O(n) \times O(n)$.

Figure 1 suggests how to characterise a metric splitting algebraically. If we think of the coordinate axes $T$ and $T^*$ as a lightcone, choosing a subgroup conjugate to $O(n) \times O(n)$ inside $O(n,n)$ boils down to the choice of a spacelike $V^+$ and a timelike orthogonal complement $V^\perp$. Interpreting $V^+$ as the graph of a linear map $P_+ : T \rightarrow T^*$ yields a metric $g$ and a 2–form $b$ as the symmetric and the skew part of the corresponding bilinear form $P_+ \in T^* \otimes T^*$. Indeed we have

$$g(t,t) = (t,P_+t) = (t \oplus P_+t, t \oplus P_+t)/2 > 0$$

Figure 1: Metric splitting of $T \oplus T^*$
so that $g$ is positive definite. As $V_+$ and $V_-$ are orthogonal, taking $V_-$ instead of $V_+$ yields the same 2–form $b$ but the metric $-g$. Conversely, assume we are given a metric $g$ and a 2–form $b$ on $T$. If we transform the diagonal $D_\pm = \{ t \oplus \mp t \cdot g \mid t \in T \}$ by $\exp(b)$, we obtain a splitting $V_+ \oplus V_-$ inducing $g$ and $b$.

**Proposition 2.1.** The choice of an equivalence class in the space $O(n, n)/O(n) \times O(n)$ is equivalent to either set of the following data:

(i) a metric splitting

$$T \oplus T^* = V_+ \oplus V_-$$

into subbundles $(V_+, g_+)$ and $(V_-, g_-)$ with positive and negative definite metrics $g_\pm = (\cdot, \cdot)_{|V_\pm}$.

(ii) a Riemannian metric $g$ and a 2–form $b$ on $T$.

Note that a reduction to $O(n) \times O(n)$ determines $g$ and $b$ up to a common scalar which, however, is fixed by the explicit choice of the inner product.

**Corollary 2.2.**

$$O(n, n)/O(n) \times O(n) = \{ P : T \to T^* \mid (Pt, t) > 0 \text{ for all } t \neq 0 \}.$$ 

In the same vein, we call an element of $SO(n, n)/SO(n) \times SO(n)$ a generalised oriented metric structure, which corresponds to a metric $g$, a $B$–field $b$ and an orientation on $T$. Since the bundle $T \oplus T^*$ is always spinnable, we can also lift the discussion to the group $Spin(n, n)$. Moreover, in some situations it is natural to introduce an additional scalar, that is, we enhance the structure group $Spin(n, n)$ to the conformal spin group $\mathbb{R}^* \times Spin(n, n)$ so that a reduction to $Spin(n) \times Spin(n)$ gives a further degree of freedom. This provides the right framework within which we can discuss generalised $G_2$–structures, to which we turn next.

### 2.2 Generalised $G_2$–structures and stability

**Definition 2.2.** A generalised $G_2$–structure is a reduction from the structure group $\mathbb{R}^* \times Spin(7, 7)$ of $T \oplus T^*$ to $G_2 \times G_2$.

We want to characterise generalised $G_2$–structures along the lines of Proposition 2.1. To get things rolling, we first look at the tensorial invariants on $T$ which are induced by such a reduction. Since $G_2 \times G_2$ determines some group $SO(V_+) \times SO(V_-)$ conjugate to $SO(7) \times SO(7)$, it induces a generalised oriented metric structure $(g, b)$. The isomorphisms

$$\pi_{b \pm} : x \in (T, \pm g) \mapsto -x \oplus -xL(g \pm b)$$

allow us to transport the respective $G_2$–structure on $V_+$ and $V_-$ to the tangent bundle. The resulting $G_2$–structures $G_2^+$ and $G_2^-$ inside $SO(T, g) = SO(7)$ give rise to two unit spinors $\Psi^+$ and $\Psi^-$ in the irreducible spin representation $\Delta = \mathbb{R}^8$ of $Spin(7) = Spin(T, g)$.

On the other hand, $G_2 \times G_2$ also acts on $\Lambda^* T^*$ as a subgroup of $Spin(7, 7)$. To relate these two actions we consider the following construction which is basically the classical identification of $\Delta \otimes \Delta$ with $\Lambda^2$ followed by a twist with $\exp(b/2)$. We write the Clifford algebra $\text{Cliff}(T \oplus T^*)$ as a $\mathbb{Z}_2$–graded tensor product $\text{Cliff}(V_+) \otimes \text{Cliff}(V_-) \cong \text{Cliff}(T \oplus T^*)$ where the isomorphism is given by extension of
On the other hand, there exists an algebra isomorphism $\kappa : \text{Cliff}(T) \rightarrow \text{Cliff}(V_+, g_\pm)$ mapping $\text{Spin}(7) = \text{Spin}(T, \pm g)$ isomorphically onto $\text{Spin}(V_+, g_\pm)$. Consequently, the compounded algebra isomorphism

$$\iota_b : \text{Cliff}(T, g) \otimes \text{Cliff}(T, -g) \rightarrow \text{Cliff}(T \oplus T^*)$$

maps $\text{Spin}(7) \times \text{Spin}(7)$ onto $\text{Spin}(V_+) \times \text{Spin}(V_-)$ inside $\text{Spin}(7, 7)$.

Let $\omega$ denote the $\text{Spin}(7)$–invariant inner product on $\Delta$. For two spinors $\Psi_+$ and $\Psi_-$ the pinor product $\Psi_+ \cdot \Psi_-$ is the endomorphism of $\Delta$ defined by

$$\Psi_+ \cdot \Psi_-(\Phi) = q(\Psi_-, \Phi) \Psi_+.$$

On the other hand, there exists an algebra isomorphism $\kappa : \text{Cliff}(T) \rightarrow \text{End}(\Delta) \oplus \text{End}(\Delta)$. Projection on the first summand induces a matrix representation for Clifford multiplication. For concrete computations, we realise this representation by

$$e_1 \mapsto E_{1,2} - E_{3,4} - E_{5,6} + E_{7,8},$$
$$e_2 \mapsto E_{1,3} + E_{2,4} - E_{5,7} - E_{6,8},$$
$$e_3 \mapsto E_{1,4} - E_{2,3} + E_{5,8} - E_{6,7},$$
$$e_4 \mapsto E_{1,5} + E_{2,6} + E_{3,7} + E_{4,8},$$
$$e_5 \mapsto E_{1,6} - E_{2,5} + E_{3,8} - E_{4,7},$$
$$e_6 \mapsto E_{1,7} - E_{2,8} - E_{3,5} + E_{4,6},$$
$$e_7 \mapsto E_{1,8} + E_{2,7} - E_{3,6} - E_{4,5},$$

where $E_{ij} = (\delta_{jk}\delta_{li} - \delta_{jl}\delta_{ki})_b$ if $i < j$ is the standard basis of skew–symmetric endomorphisms of $\Delta$, taken with respect to an orthonormal basis $\Psi_1, \ldots, \Psi_8$. This relates to the pinor product by

$$(x \cdot \Psi_+) \cdot \Psi_- = \kappa(x) \circ (\Psi_+ \cdot \Psi_-)$$

and $\Psi_+ \cdot (x \cdot \Psi_-) = (\Psi_+ \cdot \Psi_-) \circ \kappa(\sigma(x))$. (4)

Let $j$ denote the canonical vector space isomorphism between $\text{Cliff}(T)$ and $\Lambda^* T^*$. We consider the map

$$[\cdot, \cdot] : \Delta \otimes \Delta \xrightarrow{\delta = 0} \text{End}(\Delta) \oplus \text{End}(\Delta) \xrightarrow{\kappa} \text{Cliff}(T, g) \xrightarrow{j} \Lambda^* T^*$$

and think of any element in $\Delta \otimes \Delta$ as a form. We denote by $[\Psi_+ \otimes \Psi_-]^{\text{ev,od}}$ the projection on the even or odd part and add a subscript $b$ if we wedge with the exponential $\exp(b/2)$. The following result states that up to a sign twist, the action of $T$ on $\Delta \otimes \Delta$ and $\Lambda^{\text{ev,od}}$ commute.

**Proposition 2.3.** For any $x \in T$ and $\Psi_+, \Psi_- \in \Delta$ we have

$$[x \cdot \Psi_+ \otimes \Psi_-]^{\text{ev,od}}_b = \iota_b(x \otimes 1) \cdot [\Psi_+ \otimes \Psi_-]^{\text{od,od}}_b$$
$$[\Psi_+ \otimes x \cdot \Psi_-]^{\text{ev,od}}_b = \pm \iota_b(1 \otimes x) \cdot [\Psi_+ \otimes \Psi_-]^{\text{od,ev}}_b.$$

In particular, the forms $[\Psi_+ \otimes \Psi_-]^{\text{ev,od}}_b$ are $G_2 \times G_2$–invariant.

**Proof:** First we assume that $b \equiv 0$. By convention, we let $T$ act through the inclusion

$$T \rightarrow \text{Cliff}(T) \cong \text{End}(\Delta) \oplus \text{End}(\Delta).$$
followed by projection on the first summand. Thus

$$[x \cdot \Psi_+ \otimes \Psi_-] = j(\kappa^{-1}(x \cdot \Psi_+ \otimes \Psi_-))$$

$$= j(x \cdot \kappa^{-1}(\Psi_+ \otimes \Psi_-))$$

$$= -x_{(V \otimes \Psi_-)} + x \wedge (\Psi_+ \otimes \Psi_-)$$

$$= \iota_0(x \wedge 1) \cdot (\Psi_+ \otimes \Psi_-),$$

where we have used (4) and the identity

$$\text{Cliff}(c)$$

and

$$\text{Ad}$$

where

The adjoint representation

$$
\text{Ad}$$

exponential map from

$$\mathbb{B}$$

so

$$\text{Proposition 2.4.}$$

the computation of the normal form is straightforward.

$$\text{Now let } b \text{ be an arbitrary } B\text{-field. For the sake of clarity we will temporarily denote by } \hat{\exp} \text{ the exponential map from } \mathfrak{so}(7, 7) \text{ to } \text{Spin}(7, 7), \text{ while the untilded exponential takes values in } SO(7, 7). \text{ The adjoint representation } \text{Ad} \text{ of the group of units inside a Clifford algebra } \text{Cliff}(V) \text{ restricts to the double cover } \text{Spin}(V) \to SO(7, 7) \text{ still denoted by } \text{Ad}. \text{ As a transformation in } SO(V) \text{ we then have } \text{Ad} \circ \hat{\exp} = \exp \circ \text{ad}. \text{ Since } \text{ad}([v, w]) = 4v \wedge w \text{ and the vectors } e_1, \ldots, e_7 \text{ are isotropic with respect to the inner product } \langle \cdot, \cdot \rangle, \text{ in our situation we get } \text{ad}(e_i \bullet e_j) = 2e_i \wedge e_j \text{ and thus}$$

$$e^b = \text{Ad}(\sum b_{ij} e_i \bullet e_j / 2)$$

for the } B\text{-field } b = \sum_{i<j} b_{ij} e_i \wedge e_j. \text{ Hence}

$$[x \cdot \Psi_+ \otimes \Psi_-]^{\text{ev,od}}_b = \left(\hat{\exp}(x \wedge 1) \cdot (\Psi_+ \otimes \Psi_-)^{\text{ev,od}}_b \right) = \text{Ad}(\hat{\exp}(x \wedge 1)) \cdot (\Psi_+ \otimes \Psi_-)^{\text{ev,od}}_b$$

$$= \iota_0(x \wedge 1) \cdot (\Psi_+ \otimes \Psi_-)^{\text{ev,od}}_b.$$

Similarly, the claim is checked for $[\Psi_+ \otimes x \cdot \Psi_-]^{\text{ev,od}}_b$ which completes the proof. \qed

The identification $\Delta \otimes \Delta \cong \Lambda^{\text{ev,od}}$ also enables us to derive a normal form for $[\Psi_+ \otimes \Psi_-]^{\text{ev,od}}_b$ by choosing a suitable orthonormal frame. The coefficients of the homogeneous components are given up to a scalar by $q(\kappa(e_i) \cdot \Psi_+, \Psi_-)$. Since the action of $\text{Spin}(7)$ on the Stiefel variety $V_2(\Delta)$, the set of pairs of orthonormal spinors, is transitive, we may assume that $\Psi_+ = \Psi_1$ and $\Psi_- = c \Psi_1 + s \Psi_2$, where $c$ and $s$ is shorthand for $\cos(a)$ and $\sin(a)$ with $a = \angle(\Psi_+, \Psi_-)$. Using the representation (4), the computation of the normal form is straightforward.

**Proposition 2.4.** There exists an orthonormal basis $e_1, \ldots, e_7$ such that

$$[\Psi_+ \otimes \Psi_-]^{\text{ev}} = \cos(a) + \sin(a)(-e_{23} - e_{45} + e_{67}) +$$

$$+ \cos(a)(e_{1247} + e_{1256} + e_{1346} + e_{1357} - e_{2345} + e_{2367} + e_{4567}) +$$

$$+ \sin(a)(e_{1246} + e_{1257} + e_{1347} - e_{1356}) - \sin(a)e_{234567}$$

and

$$[\Psi_+ \otimes \Psi_-]^{\text{od}} = \sin(a)e_1 + \sin(a)(e_{247} - e_{256} - e_{346} - e_{357}) +$$

$$+ \cos(a)(e_{123} - e_{145} - e_{167} + e_{246} + e_{257} + e_{447} - e_{356}) +$$

$$+ \sin(a)(-e_{12345} + e_{12367} + e_{14567}) + \cos(a)e_{1234567}. $$
If the spinors $\Psi_+$ and $\Psi_-$ are linearly independent, we can express the structure form in terms of the invariants associated with $SU(3)$, the stabiliser of the pair $(\Psi_1, \Psi_2)$.

**Corollary 2.5.** If $\alpha$ denotes the dual of the unit vector in $T$ which is stabilised by $SU(3)$, $\omega$ the Kähler form and $\psi_+$ the real and imaginary parts of the holomorphic volume form, then

\[ [\Psi_+ \otimes \Psi_-]^\text{ev} = c + s\omega + c(\alpha \land \psi_- - \frac{1}{2}\omega^2) - s\alpha \land \psi_+ - \frac{1}{6}s\omega^3 \]

and

\[ [\Psi_+ \otimes \Psi_-]^\text{od} = s\alpha - c(\psi_+ + \alpha \land \omega) - s\psi_- - \frac{1}{2}s\alpha \land \omega^2 + \text{cvol}_g. \]

Moreover, $\Psi_- = s\alpha \cdot \Psi_+$. If $a = 0$, then both $G_2$–structures coincide and

\[ [\Psi_+ \otimes \Psi_-]^\text{ev} = 1 - \ast \varphi, \quad [\Psi_+ \otimes \Psi_-]^\text{od} = -\varphi + \text{vol}_g, \]

where $\varphi$ is the stable 3–form associated with $G_2$.

**Remark:** The stabiliser of the forms $1 + \ast \varphi$ and $\varphi + \text{vol}_g$ is isomorphic to $G_2(\mathbb{C})$ [13].

The 63 degrees of freedom which parametrise a reduction from $Spin(7,7)$ to $G_2 \times G_2$ are exhausted by 28 degrees of freedom for the choice of a metric $g$, 21 for a 2–form $b$ and twice 7 degrees for two unit spinors $\Psi_+$ and $\Psi_-$. However, this data does not achieve a full description of $G_2 \times G_2$–invariant spinors yet as these are stable, a notion due to Hitchin [12].

**Definition 2.3.** A spinor $\rho$ in $\Lambda^{\text{ev,od}}T^* n$ is said to be stable if $\rho$ lies in an open orbit under the action of $R^* \times Spin(n,n)$.

In [10] Sato and Kimura classified the representations of complex reductive Lie groups which admit an open orbit. Apart from dimension 7, stable spinors in the sense of the definition above only occur in dimension 6 and give rise to generalised Calabi–Yau–structures associated with the group $SU(3,3)$ [12]. The key point here is that in both cases we obtain an invariant volume form. In our case, this generalises the concept of stability for a $G_2$–invariant 3–form $\varphi$ with associated volume form $\varphi \land \ast \varphi$ [11].

To begin with, we note that the spin representation $\Delta$ of $Spin(7)$ is real, and so is the tensor product $\Delta \otimes \Delta$. Consequently, there is (up to a scalar) a unique invariant in $\Lambda^\text{ev} \otimes \Lambda^\text{od}$, or equivalently, $Spin(7) \times Spin(7)$–equivariant maps $\Lambda^{\text{ev,od}}T^* n \rightarrow \Lambda^{\text{od,ev}}T^* n$. Morally these are given by the Hodge $\ast$–operator twisted with the $B$–field and the anti–automorphism $\sigma$.

**Definition 2.4.** The box–operator or generalised Hodge $\ast$–operator $\Box_{g,b} : \Lambda^{\text{ev,od}}T^* n \rightarrow \Lambda^{\text{od,ev}}T^* n$ associated with the generalised metric $(g,b)$ is defined by

\[ \Box_{g,b}\rho = e^{b/2} \land \ast_g \sigma(e^{-b/2} \land \rho). \]

If $g$ and $b$ are induced by $\rho$ we will also use the sloppier notation $\Box_\rho$ or drop the subscript altogether. For present and later use, we note the following lemma whose proof is immediate from the definitions.

**Lemma 2.6.** Let $\rho \in \Lambda^{\text{ev,od}}T^* n$. Then $\sigma(\ast \rho) = \ast \sigma(\rho)$ and $\sigma(e^b \land \rho) = e^{-b} \land \sigma(\rho)$.
Proposition 2.7. For any $\Psi_+, \Psi_- \in \Delta \otimes \Delta$

$$\Box_{g,b}[(\Psi_+ \otimes \Psi_-)]_b = [(\Psi_+ \otimes \Psi_-)]_b$$

or equivalently,

$$\Box_{g,b}[(\Psi_+ \otimes \Psi_-)]_{b^{ev,od}} = [(\Psi_+ \otimes \Psi_-)]_{b^{ev,od}}.$$

Proof: According to Lemma 2.6,

$$\Box_{g,b}[(\Psi_+ \otimes \Psi_-)]_b = e^{b/2} \wedge \ast \sigma [\Psi_+ \otimes \Psi_-]$$

where we used the general identity $\ast j(x) = j(\sigma(x) \cdot \text{vol})$. But $\kappa^{-1}((\Psi_+ \hat{\circ} \Psi_- \oplus 0) \cdot \text{vol})$ is just $\kappa^{-1}(\Psi_+ \hat{\circ} \Psi_- \oplus 0)$, whence the assertion.

Let $U$ denote the space of $G_2 \times G_2$–invariant spinors in $\Lambda^{ev}$ or $\Lambda^{od}$. With any element of $U$ we associate a volume form $Q$:

$$Q : \rho \in U \mapsto q(\Box \rho, \rho) \in \Lambda^7 T^*.$$

Now the $\Box$–operator transforms naturally under the lift $\tilde{A} \in \text{Pin}(7,7)$ of any element $A \in O(7,7)$ which means that

$$\Box_{\tilde{A} \rho, \rho} \tilde{A} \rho = \tilde{A} \cdot \Box \rho.$$

Therefore, we immediately conclude

Proposition 2.8. $Q$ is homogeneous of degree 2 and $\text{Spin}(7,7)$–invariant.

Remark: An explicit coordinate description of the complexified invariant was given in [10]. However, with the aim of setting up the variational principle, this formulation proved to be rather cumbersome for our purposes, which motivated our approach in terms of $G$–structures.

We will also need the differential of this map. Since the form $\langle \cdot, \cdot \rangle$ is non–degenerate, we can write

$$DQ_{\rho}(\dot{\rho}) = \langle \dot{\rho}, \dot{\rho} \rangle.$$

for a unique $\dot{\rho} \in \Lambda^{od} T^*$, the companion of $\rho$, which is also a $G_2 \times G_2$–invariant spinor. By rescaling $Q$ appropriately, we conclude that $\dot{\rho} = \Box \rho$.

From $Q$ we derive a further invariant attached to a $G_2 \times G_2$–invariant spinor $\rho$, namely a real scalar $\phi_{\rho}$ which we refer to as the dilaton. It is defined by

$$Q(\rho) = 8 e^{-2\phi_{\rho} \text{vol}_g}.$$

Using the normal form description of Proposition 2.4 we obtain $Q([\Psi_+ \otimes \Psi_-]_{b^{ev,od}}) = 8 \text{vol}_g$, hence

$$\rho = e^{-\phi_{\rho}} [\Psi_+ \otimes \Psi_-]_{b^{ev,od}}.$$

We summarise the results of this section in the following

Theorem 2.9. Generalised $G_2$–structures are in 1–1 correspondence with lines of spinors $\rho$ in $\Lambda^{ev} T^*$ (or $\Lambda^{od} T^*$) whose stabiliser under the action of $\text{Spin}(7,7)$ is isomorphic to $G_2 \times G_2$. We refer to $\rho$ as the structure form of the generalised $G_2$–structure. This form can be uniquely written (modulo a simultaneous sign change for $\Psi_+$ and $\Psi_-$) as

$$\rho = e^{-\phi} [\Psi_+ \otimes \Psi_-]_{b^{ev}}$$

for a 2–form $b$, two unit spinors $\Psi_\pm \in \Delta$ and a real scalar $\phi$. In particular, the space of $G_2 \times G_2$–invariant spinors is open.
3 Topological generalised $G_2$–structures

**Definition 3.1.** A topological generalised $G_2$–structure over a 7–manifold $M$ is a topological $G_2 \times G_2$–reduction of the $\mathbb{R}^* \times \text{Spin}(7, 7)$–principal bundle associated with $T \oplus T^*$. It is characterised by a stable even or odd spinor $\rho$ which we view as a form. Consequently, we will denote this structure by the pair $(M, \rho)$ and call $\rho$ the structure form.

We will usually drop the adjective “topological” and simply refer to a generalised $G_2$–structure. As we saw earlier, a generalised $G_2$–structure induces a generalised metric. In particular there exists a metric $g$ or equivalently an $\text{SO}(7)$–principal fibre bundle which admits two $G_2$–subbundles.

The inclusion $G_2 \pm \subset \text{Spin}(7)$ implies that the underlying manifold is spinnable and distinguishes a preferred spin structure for which we consider the associated spinor bundle $\Delta$. Using Theorem 2.9 we can now assert the following statement.

**Theorem 3.1.** A topological generalised $G_2$–structure $(M, \rho)$ is characterised by the following data:

- an orientation
- a metric $g$
- a 2–form $b$
- a scalar function $\phi$
- two unit spinors $\Psi_+, \Psi_- \in \Delta$ such that

$$e^{-\phi}[\Psi_+ \otimes \Psi_-]_b = \rho + \Box_{g, b} \rho.$$

A trivial example of a generalised $G_2$–structure is a topological $G_2$–manifold with associated spinor $\Psi = \Psi_\pm$. A generalised structure arising in this way (possibly with a non–trivial $B$–field and dilaton) is said to be straight. The existence of a nowhere vanishing spinor field in dimension 7 is a classical result [15] and implies

**Corollary 3.2.** A 7–fold $M$ carries a topological generalised $G_2$–structure if and only if $M$ is spinnable.

Next we discard the $B$–field and the dilaton and focus on the $G_2$–structures induced by $\Psi_+$ and $\Psi_-$. Our aim is to classify the $G_2$–structures up to equivalence under a $\text{Spin}(7)$–fibre bundle isomorphism. Since the classification of principal fibre bundles is a problem of homotopy theory, this boils down to deform homotopically $\Psi_+$ into $\Psi_-$ through sections. More concretely, we regard $G_2$–structures as being defined by (continuous) sections of the sphere bundle $p_\mathbb{S} : \mathbb{S} \to M$ associated with $\Delta$. On the space of sections $\Gamma(\mathbb{S})$ we introduce the following equivalence relation. Two spinors $\Psi_+$ and $\Psi_-$ are considered to be equivalent (denoted $\Psi_+ \sim \Psi_-$) if and only if there exists a continuous map $G : M \times I \to \mathbb{S}$ such that $G(x, 0) = \Psi_+(x)$, $G(x, 1) = \Psi_-(x)$ and $p_\mathbb{S} \circ G(x, t) = x$. An equivalence class will be denoted by $[\Psi]$. If two sections are vertically homotopic, then the corresponding $G_2$–structures are isomorphic as principal $G_2$–bundles over $M$. In particular the generalised structure defined by the pair $(\Psi_+, \Psi_-)$ is equivalent to a straight structure if and only if $\Psi_+ \sim \Psi_-$. What we aim to determine is the set of generalised structures with fixed $\Psi_+$, i.e. $\text{Gen}(M) = \Gamma(\mathbb{S})/\sim$, the set of isomorphism classes of principal $G_2$–fibre bundles. If a generalised structure is defined by two inequivalent spinors, then it is said to be exotic. Here is an example.
**Example:** Consider $M = S^7$. Since the tangent bundle of $S^7$ is trivial, so is the sphere bundle of $\Delta$, i.e. $\mathbb{S} = S^7 \times S^7$. Consequently,

$$\text{Gen}(S^7) = \Gamma(\mathbb{S})/\sim = [S^7, S^7] = \pi_7(S^7) = \mathbb{Z}$$

and any map $\Psi_+ : S^7 \to S^7$ which is not homotopic to a constant gives rise to an exotic structure for $\Psi_+ \equiv \text{const}$.

In general, the question whether or not two sections are vertically homotopic can be tackled by using obstruction theory (see for instance [17]). Assume that we are given a fibre bundle with connected fibre over a not necessarily compact $n$–fold $M^n$, and two sections $s_1$ and $s_2$ which are vertically homotopy equivalent over the $q$–skeleton $M^{[q]}$ of $M$. The obstruction for extending the vertical homotopy to the $q + 1$–skeleton lies in $H^{q+1}(M, \pi_{q+1}(F))$. In particular, there is the first non–trivial obstruction $\delta(s_1, s_2) \in H^m(M, \pi_m(F))$ for the least integer $m$ such that $\pi_m(F) \neq 0$, called the primary difference of $s_1$ and $s_2$. It is a homotopy invariant of $s_1$ and $s_2$ and enjoys the additivity property

$$\delta(s_1, s_2) + \delta(s_2, s_1) = \delta(s_1, s_3). \quad (5)$$

Coming back to the generalised $G_2$–case we consider the sphere bundle $\mathbb{S}$ over $M^7$ with fibre $S^7$. Consequently, the primary difference of two sections lies in $H^7(M, \mathbb{Z})$ and this is the only obstruction for two sections to be vertically homotopy equivalent. The additivity property implies that $\delta(\Psi, \Psi_1) = \delta(\Psi, \Psi_2)$ if and only if $\delta(\Psi_1, \Psi_2) = 0$, that is, $\Psi_1 \sim \Psi_2$. Moreover, for any class $d \in H^7(M, \mathbb{Z})$ there exists a section $\Psi_d$ such that $d = \delta(\Psi, \Psi_d)$ [17]. As a consequence, we obtain the

**Proposition 3.3.** The set of generalised $G_2$–structures can be identified with

$$\text{Gen}(M) = H^7(M, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } M \text{ is compact} \\ 0, & \text{if } M \text{ is non–compact} \end{cases}.$$  

Modulo $(b, \phi)$, generalised $G_2$–structures are therefore classified by an integer invariant which over a compact $M^7$ has the natural interpretation as the number of points (counted with an appropriate sign convention) where the two $G_2$–structures coincide. To see this we associate with every equivalence class $[\Psi_-]$ the intersection class $\#(\Psi_+^{\ast}(M), \Psi_-(M)) \in H_{14}(\mathbb{S}, \mathbb{Z})$ of the 7–dimensional oriented submanifolds $\Psi_+^{\ast}(M)$ and $\Psi_-(M)$ inside $\mathbb{S}$. Since the total space of the sphere bundle is 14–dimensional, the intersection class counts the number of points in $M$ where the two spinors $\Psi_+$ and $\Psi_-$ coincide. Taking the cup product of the Poincaré duals of $\Psi_-(M)$ and $\Psi_+^{\ast}(M)$ sets up a map

$$[\Psi_-] \in \text{Gen}(M) \mapsto PD(\Psi_+^{\ast}(M)) \cup PD(\Psi_-(M)) \in H^{14}(\mathbb{S}, \mathbb{Z}).$$

On the other hand, integration along the fibre defines an isomorphism $\pi_{3*} : H^{14}(\mathbb{S}, \mathbb{Z}) \to H^7(M, \mathbb{Z})$ by the Gysin sequence. Therefore, any generalised $G_2$–structure induced by the equivalence class $[\Psi_-]$ over a compact 7–manifold $M$ gives rise to a well–defined cohomology class $d(\Psi_+, \Psi_-) \in H^7(M, \mathbb{Z})$. The following theorem shows this class to coincide with the primary difference. For the proof, I benefited from discussions with W. Sutherland and M. Crabb.

**Theorem 3.4.** If $M$ is compact, then

$$d(\Psi_+, \Psi_-) = \delta(\Psi_+, \Psi_-).$$

In particular generalised $G_2$–structures are classified by the number of points where the two underlying $G_2$–structures coincide.
Proof: We regard the spinor bundle $\Delta$ as an 8–dimensional oriented real vector bundle over $M$ and consider the two sections $\Psi_+$ and $\Psi_-$ of the sphere bundle. Deform the section $(x, t) \mapsto (1 - t)\Psi_+(x) + t\Psi_-(x)$ of the pullback of $\Delta$ to $M \times \mathbb{R}$ to be transverse to the zero–section. The primary difference $\delta(\Psi_+, \Psi_-)$ can then be represented geometrically by the (signed) zero–set of this deformation. In particular, if $\Psi_+$ and $-\Psi_-$ never coincide, then the primary difference is 0. Therefore the intersection number, defined geometrically by making $\Psi_+(M)$ and $\Psi_-(M)$ transverse and taking the coincidence set, will be $\delta(\Psi_+, -\Psi_-)$ (with appropriate sign conventions). By virtue of (4), we have $\delta(\Psi_+, -\Psi_-) = \delta(\Psi_+, \Psi_-) + \delta(\Psi_-, -\Psi_-)$. The difference class $\delta(\Psi_-, -\Psi_-)$ corresponds to the self–intersection number $\#(\Psi_-(M), \Psi_-(M))$ which is 0 since $M$ is 7–dimensional, whence the assertion. 

4 Integrable generalised $G_2$–structures and supersymmetry

4.1 A variational principle

Assume $M$ to be a closed and oriented manifold which carries a topological generalised $G_2$–structure defined by the stable spinor $\rho$. We shall set up a variational problem along the lines of [11] and [12]. From a $GL(7)$–point of view, $\rho$ is a section of the vector bundle whose fibre is $\Lambda^{ev,od}T^* \otimes (\Lambda^7 T)_{1/2}$ (cf. the remark in Section 2.1). Untwisting by the line bundle $(\Lambda^7 T$)–exact, we obtain a corresponding open set still denoted by $U$, on which we can consider the induced volume functional $Q$ as described in Proposition 2.1. It is homogeneous of degree 2 and therefore, it defines a $GL(7)+$–equivariant function $Q: U \subset \Lambda^{ev,od}T^* \rightarrow \Lambda^7 T$ since

$$Q(A^* \rho) = (\det A)^{-1} Q(A \bullet \rho) = (\det A)^{-1} Q(\rho).$$

Associated with the $GL(7)$–principal fibre bundle over $M$, $Q$ thus takes values in $\Omega^7(M)$ and we obtain the volume functional

$$V(\rho) = \int_M Q(\rho)$$

defined over stable forms. As stability is an open condition, we can differentiate this functional and consider its variation over a fixed cohomology class. Instead of working with ordinary cohomology only we will allow for an extra twist by a closed 3–form $H$. This means that we replace the differential operator $d$ by the twisted operator $d_H = d + H \wedge$. Closeness of $H$ guarantees that $d_H$ still defines a differential complex. Moreover, we will also consider the following constraint. The bilinear form $\langle \cdot, \cdot \rangle$ induces a non–degenerate pairing between $\Omega^ev(M)$ and $\Omega^od(M)$ defined by integration of $\langle \eta, \tau \rangle$ over $M$. If $\eta = d_H \gamma$ is $H$–exact, Stokes’ theorem and the definition of the involution $\sigma$ imply that

$$\int_M \langle d_H \gamma, \tau \rangle = \int_M \langle \gamma, d_H \tau \rangle.$$

This vanishes for all $\gamma$ if and only if $\tau$ is $H$–closed. Consequently, we can identify

$$\Omega^ev_{H-exact}(M)^* \cong \Omega^od(M)/\Omega^od_{H-closed}(M).$$

The exterior differential $d_H$ maps the latter space isomorphically onto $\Omega^ev_{H-exact}(M)$ so that

$$\Omega^ev_{H-exact}(M)^* \cong \Omega^ev_{H-exact}(M).$$
Finally, we obtain the non–degenerate quadratic form on \( \Omega_{H,\text{exact}}^{ev}(M) \) given by

\[
Q(d_H \gamma) = \int_M \langle \gamma, d_H \gamma \rangle.
\]

The same conclusion holds for odd instead of even forms.

**Theorem 4.1.** Let \( H \) be a closed 3–form.

(i) A \( d_H \)-closed stable form \( \rho \in \Omega^{ev,od}(M) \) is a critical point in its cohomology class if and only if

\[
d_H \hat{\rho} = 0.
\]

(ii) A \( d_H \)-exact form \( \rho \in \Omega^{ev,od}(M) \) defines a critical point subject to the constraint \( Q(\rho) = \text{const} \) if and only if there exists a real constant \( \lambda \) with \( d_H \rho = \lambda \hat{\rho} \).

**Proof:** The first variation of \( V \) is

\[
\delta V_{\rho}(\dot{\rho}) = \int_M DQ_{\rho}(\dot{\rho}) = \int_M \langle \hat{\rho}, \dot{\rho} \rangle.
\]

To find the unconstrained critical points we have to vary over a fixed \( d_H \)-cohomology class, i.e. \( \dot{\rho} = d_H \gamma \). As we saw in (6), we have

\[
\delta V_{\rho}(\dot{\rho}) = \int_M \langle \dot{\rho}, d_H \gamma \rangle = \int_M \langle d_H \dot{\rho}, \gamma \rangle,
\]

and this vanishes for all \( \gamma \) if and only if \( \dot{\rho} \) is \( d_H \)-closed. In the constrained case the differential of \( Q \) at \( \rho \) is

\[
(\delta Q)_{\rho}(d_H \gamma) = 2 \int_M \langle \rho, \gamma \rangle.
\]

By Lagrange’s theorem, we see that for a critical point we need \( d_H \dot{\rho} = \lambda \rho \). \( \square \)

We adopt these various conditions for defining a critical point as integrability condition of a topological generalised \( G_2 \)-structure.

**Definition 4.1.** Let \( H \) be a closed 3–form and \( \lambda \) be a real, non–zero constant.

(i) A topological generalised \( G_2 \)-structure \( (M, \rho) \) is said to be strongly integrable with respect to \( H \) if and only if

\[
d_H \rho = 0, \quad d_H \hat{\rho} = 0.
\]

(ii) A topological generalised \( G_2 \)-structure \( (M, \rho) \) is said to be weakly integrable with respect to \( H \) and with Killing number \( \lambda \) if and only if

\[
d_H \rho = \lambda \hat{\rho}.
\]

We call such a structure even or odd according to the parity of the form \( \rho \). If we do not wish to distinguish the type, we will refer to both structures as weakly integrable.

Similarly, the structures in (i) and (ii) will also be referred to as integrable if a condition applies to both weakly and strongly integrable structures.
As we shall see in Corollary 4.6, the number \( \lambda \) represents the 0–torsion form of the underlying \( G_2 \)–structures which is why we refer to it as the Killing number \([7]\).

**Example:**

Consider a straight topological \( G_2 \)–manifold \((M, \varphi, b, \phi)\) equipped with an additional closed 3–form \( H \). According to Corollary 2.5 and Theorem 2.9, the corresponding structure form is \( \rho = e^{-\varphi} e^{b/2} \wedge (1 - * \varphi) \) with companion \( \bar{\rho} = e^{-\phi} e^{b/2} \wedge (-\varphi + vol_g) \). Writing \( T = db/2 + H \), we want to solve the equations of strong integrability

\[
\begin{align*}
d T e^{-\varphi}(1 - * \varphi) &= 0, \\
&d T e^{-\varphi}(-\varphi + vol_g) = 0.
\end{align*}
\]

It follows immediately that this is equivalent to \( d \phi = 0 \), \( T = 0 \) and \( d \varphi = 0 \), \( d * \varphi = 0 \), that is, the holonomy is contained in \( G_2 \). If we ask for weak integrability the question only makes sense in the even case as \( \cos(a) = 0 \) for structures of odd type. The equation of weak integrability becomes

\[
d T (1 - * \varphi) - d \phi \wedge (1 - * \varphi) = \lambda (\varphi + vol_g),
\]

implying \( d \phi = 0 \), \( T = -\lambda \varphi \) and \( -\lambda \varphi \wedge * \varphi = \lambda vol_g \). Since \( \varphi \wedge * \varphi = 7 vol_g \), we have \( \lambda = 0 \). A straight structure can therefore never induce a weakly integrable structure – in this sense, weak integrability has no classical counterpart.

### 4.2 Spinorial solution of the variational problem and supergravity

Next we want to interpret the integrability conditions in Definition 4.1 in terms of the data of Theorem 3.1. For a vanishing \( B \)–field, we have
\[
|\mathcal{D}(\Psi_+ \otimes \Psi_-)_{b}^{ev,od}| = \sum e_i \cdot \nabla e_i \Psi_+ \otimes \Psi_- + e_i \cdot \Psi_- \otimes \nabla e_i \Psi_+,
\]
\[
|\bar{\mathcal{D}}(\Psi_+ \otimes \Psi_-)_{b}^{ev,od}| = \sum \nabla e_i \Psi_- \otimes e_i \cdot \Psi_+ + \Psi_- \otimes e_i \cdot \nabla e_i \Psi_+
\]

into the Dirac operators on \( p \)–forms \( d + d^* \) and \((-1)^p(d \pm d^*) \). Here and in the sequel, \( \nabla \) designates the Levi–Civita connection on the tangent bundle \( T \) or its lift to \( \Delta \). The transformation under a non–trivial \( B \)–field is given in the next proposition.

**Proposition 4.2.** We have

\[
|\mathcal{D}(\Psi_+ \otimes \Psi_-)_{b}^{ev,od}| = d[\Psi_+ \otimes \Psi_-]_{b}^{od,ev} + d^c[\Psi_+ \otimes \Psi_-]_{b}^{od,ev} + \frac{1}{2} e^{b/2} \wedge (db_b[\Psi_+ \otimes \Psi_-]_{b}^{od,ev} - db \wedge [\Psi_+ \otimes \Psi_-]_{b}^{od,ev})
\]

\[
|\bar{\mathcal{D}}(\Psi_+ \otimes \Psi_-)_{b}^{ev,od}| = \pm d[\Psi_+ \otimes \Psi_-]_{b}^{od,ev} + d^c[\Psi_+ \otimes \Psi_-]_{b}^{od,ev} + \frac{1}{2} e^{b/2} \wedge (db_b[\Psi_+ \otimes \Psi_-]_{b}^{od,ev} + db \wedge [\Psi_+ \otimes \Psi_-]_{b}^{od,ev})
\]

where \( d^c \rho = \square d \rho \).

**Proof:** For \( b = 0 \) this is just the classical case mentioned above. For an arbitrary \( B \)–field \( b \) we have

\[
|\mathcal{D}(\Psi_+ \otimes \Psi_-)_{b}^{ev,od}| = e^{b/2} \wedge [\mathcal{D}(\Psi_+ \otimes \Psi_-)]_{ev,od} = e^{b/2} \wedge [\mathcal{D}(\Psi_+ \otimes \Psi_-)]_{ev,od} + e^{b/2} \wedge d^*[\Psi_+ \otimes \Psi_-]^{od,ev}.
\]
The first term on the right hand side equals
\[
e^{b/2} \wedge d[\Psi_+ \otimes \Psi_-]^{\text{od, ev}} = d[\Psi_+ \otimes \Psi_-]^{\text{od, ev}} - \frac{1}{2} e^{b/2} \wedge db \wedge [\Psi_+ \otimes \Psi_-]^{\text{od, ev}}
\]
while for the second term, we have
\[
d^*[\Psi_+ \otimes \Psi_-]^{\text{od, ev}} = \mp d^*[\Psi_+ \otimes \Psi_-]^{\text{od, ev}}.
\]

Proposition 2.7 and Lemma 2.6 give
\[
e^{b/2} \wedge d^*[\Psi_+ \otimes \Psi_-]^{\text{od, ev}} = \mp \frac{1}{2} e^{b/2} \wedge \phi d[\Psi_+ \otimes \Psi_-]^{\text{ev, od}}
\]
hence
\[
[D(\Psi_+ \otimes \Psi_-)]^{\text{ev, od}}_b = \frac{1}{2} e^{b/2} \wedge (db_b [\Psi_+ \otimes \Psi_-]^{\text{od, ev}} - db \wedge [\Psi_+ \otimes \Psi_-]^{\text{od, ev}}).
\]

Similarly, we obtain
\[
[D(\Psi_+ \otimes \Psi_-)]^{\text{ev, od}}_b = \pm e^{b/2} \wedge (d[\Psi_+ \otimes \Psi_-]^{\text{od, ev}} \pm d^*[\Psi_+ \otimes \Psi_-]^{\text{od, ev}})
\]
\[
= \pm d[\Psi_+ \otimes \Psi_-]^{\text{od, ev}}_b \pm d^2 [\Psi_+ \otimes \Psi_-]^{\text{od, ev}}_b
\]
\[
+ \frac{1}{2} e^{b/2} \wedge (db_b [\Psi_+ \otimes \Psi_-]^{\text{od, ev}} + \phi d[\Psi_+ \otimes \Psi_-]^{\text{od, ev}}).
\]

We now come to our main result.

**Theorem 4.3.** A generalised $G_2$–structure $(M, \rho)$ is weakly integrable with respect to $H$ and Killing number $\lambda$ if and only if $e^{-\phi}[\Psi_+ \otimes \Psi_-]_b = \rho + \Box_{g,b} \rho$ satisfies (with $T = db/2 + H$)

\[
\nabla_X \Psi_{\pm} \pm \frac{1}{4} (X \cdot T) \cdot \Psi_{\pm} = 0
\]
and

\[
(d \phi \pm \frac{1}{2} T \cdot \Psi_{\pm}) = 0
\]
in case of an even structure and

\[
(d \phi \pm \frac{1}{2} T \cdot \Psi_{\pm}) = 0
\]
in case of an odd structure.

The structure $e^{-\phi}[\Psi_+ \otimes \Psi_-]_b = \rho + \Box_{g,b} \rho$ is strongly integrable if and only if these equations hold for $\lambda = 0$. 

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We refer to the equation involving the covariant derivative of the spinor as the \textit{generalised Killing equation} and to the equation involving the differential of $\phi$ as the \textit{dilatino equation}. The generalised Killing equation basically states that we have two metric connections $\nabla_G$ and $\nabla_K$ preserving the underlying $G_{2\pm}$–structures whose torsion (as it is to be defined in the next section) is \textit{skew–symmetric}. The dilatino equation then serves to identify the components of the torsion with respect to the decomposition into irreducible $G_{2\pm}$–modules with the additional data $d\phi$ and $\lambda$. The generalised Killing and the dilatino equation occur in physics as solutions to the supersymmetry variations in type II superstring theory with bosonic background fields \[8\].

**Proof:** Assume that $\rho$ is even and satisfies $d\rho = -H \wedge \rho + \lambda \rho$. The odd case is dealt with in a similar fashion. Applying the $\Box$–operator and using $\Box \rho = \tilde{\rho}$, we obtain

$$d^3 e^{-\phi} \tilde{[\Psi_+ \otimes \Psi_-]}_b^{od} = e^{b/2} \wedge (H_{\lambda}[e^{-\phi} \Psi_+ \otimes \Psi_-]^{od} + \lambda[e^{-\phi} \Psi_+ \otimes \Psi_-]^{ev}).$$

Consequently, Proposition 4.2 implies

$$[D(e^{-\phi} \Psi_+ \otimes \Psi_-)]^{ev} = T_{\lambda}[e^{-\phi} \Psi_+ \otimes \Psi_-]^{od} - T \wedge [e^{-\phi} \Psi_+ \otimes \Psi_-]^{od} + \lambda[e^{-\phi} \Psi_+ \otimes \Psi_-]^{ev}. \quad (7)$$

As a corollary to Proposition 2.3, we see that

$$T_{\lambda}[\Psi_+ \otimes \Psi_-]^{ev,od} = \frac{1}{8} \left( -T \cdot \Psi_+ \otimes \Psi_+ \pm T \cdot \Psi_- \mp \sum_i (e_{i\lambda} T) \cdot \Psi_+ \otimes e_i \cdot \Psi_- + \sum_i e_i \cdot \Psi_+ \otimes [e_{i\lambda} T] \cdot \Psi_- \right)^{od,ev}$$

and

$$T \wedge [\Psi_+ \otimes \Psi_-]^{ev,od} = \frac{1}{8} \left( T \cdot \Psi_+ \otimes \Psi_+ \pm T \cdot \Psi_- \mp \sum_i (e_{i\lambda} T) \cdot \Psi_+ \otimes e_i \cdot \Psi_- - \sum_i e_i \cdot \Psi_+ \otimes (e_{i\lambda} T) \cdot \Psi_- \right)^{od,ev}.$$

Hence \[7\] entails

$$[D(e^{-\phi} \Psi_+ \otimes \Psi_-)]^{ev} = \frac{1}{4} e^{-\phi} [T \cdot \Psi_+ \otimes \Psi_- + \sum_i e_i \cdot \Psi_+ \otimes (e_{i\lambda} T) \cdot \Psi_- + \lambda \Psi_+ \otimes \Psi_-]^{ev}. \quad (8)$$

Let $D$ denote the Dirac operator associated with the Clifford bundle $(\Delta, q)$. Contraction with $q(\cdot, e_m \cdot \Psi_+)$ yields

$$q(D e^{-\phi} \Psi_+, e_m \cdot \Psi_+ \Psi_- + e^{-\phi} \nabla e_m \Psi_- = -\frac{1}{4} e^{-\phi} q(T \cdot \Psi_+, e_m \cdot \Psi_+ \Psi_- + \frac{1}{4} e^{-\phi} (e_{m\perp} T) \cdot \Psi_-),$$

and therefore

$$e^{-\phi} \nabla e_m \Psi_- = -\frac{1}{4} q(4D e^{-\phi} \Psi_+ + e^{-\phi} T \cdot \Psi_+, e_m \cdot \Psi_+ \Psi_-) + \frac{1}{4} e^{-\phi} (e_{m\perp} T) \cdot \Psi_-.$$

From this expression we deduce

$$e_m q(\Psi_-, \Psi_-) = -\frac{1}{2} q(4e^\phi D e^{-\phi} \Psi_+ + T \cdot \Psi_+, e_m \cdot \Psi_+) = 0,$$

since $q((e_{m\perp} T) \cdot \Psi_-) = 0$. It follows

$$\nabla e_m \Psi_- = \frac{1}{4} (e_{m\perp} T) \cdot \Psi_- = 0.$$
We derive the corresponding expression for the spinor $\Psi_+$ by using $\hat{D}$ which accounts for the minus sign.

Next we turn to the dilatino equation. From $\nabla_X \Psi_+ = -\frac{1}{4}(X \cdot T) \cdot \Psi_+$ we deduce

$$D\Psi_+ = -\frac{3}{4} T \cdot \Psi_+.$$ 

On the other hand, contracting (8) with $q(\cdot, \Psi_-)$ yields

$$De^{-\phi} \Psi_+ = e^{-\phi} \lambda \Psi_+ - \frac{1}{4} e^{-\phi} T \cdot \Psi_+.$$ 

Putting the last two equations together results in the dilatino equation for $\Psi_+$. We can perform the same calculation with $\hat{D}$ instead of $D$ to derive the dilatino equation for $\Psi_-$. Conversely assume that the generalised Killing and dilatino equations of an even structure hold – the odd case, again, is analogous. We note that

$$d_H e^{-\phi} [\Psi_+ \otimes \Psi_-]^{ev}_b = e^{-\phi} \lambda [\Psi_+ \otimes \Psi_-]^{od}_b$$

is equivalent to

$$d[\Psi_+ \otimes \Psi_-]^{ev} - d\phi \wedge [\Psi_+ \otimes \Psi_-]^{ev} = \lambda [\Psi_+ \otimes \Psi_-]^{od} - T \wedge [\Psi_+ \otimes \Psi_-]^{ev}. \quad (9)$$

Now

$$d[\Psi_+ \otimes \Psi_-]^{ev} - d\phi \wedge [\Psi_+ \otimes \Psi_-]^{ev} = \frac{1}{2} [e_i \cdot \nabla e_i [\Psi_+ \otimes \Psi_-]^{ev} +
\sum_i e_i \cdot \nabla e_i [\Psi_+ \otimes \Psi_-]^{ev} +
\lambda [\Psi_+ \otimes \Psi_-]^{od} + \frac{1}{4} [T \cdot \Psi_+ \otimes \Psi_- + \Psi_+ \otimes T \cdot \Psi_-]^{od} +
\frac{1}{2} \sum_i (e_i \cdot \nabla e_i [\Psi_+ \otimes \Psi_- - \Psi_+ \otimes e_i \cdot \Psi_- +
+ e_i \cdot \Psi_- \otimes e_i \Psi_- - \Psi_+ \otimes e_i \cdot \nabla e_i [\Psi_+ \otimes \Psi_-]^{od}
\lambda [\Psi_+ \otimes \Psi_-]^{od} - \frac{1}{8} [T \cdot \Psi_+ \otimes \Psi_- + \Psi_+ \otimes T \cdot \Psi_- -
\sum_i (e_i \cdot T) \cdot \Psi_+ \otimes e_i \cdot \Psi_- - \sum_i e_i \cdot \Psi_+ \otimes (e_i \cdot T) \cdot \Psi_-]^{od}
\lambda [\Psi_+ \otimes \Psi_-]^{od} - T \wedge [\Psi_+ \otimes \Psi_-]^{ev}$$

which proves (9) and thus the theorem. \[\square\]

**Remark:** The theorem holds more generally for any 3–form $T$, closed or not. Similarly, we can introduce a 1–form $\alpha$ and consider the twisted differential operator $d_\alpha$. This substitutes $dF$ by $dF + \alpha$ in the dilatino equation.
4.3 The Ricci curvature and a no–go theorem

In the light of the spinorial formulation of integrability, we shall from now on always consider the twisted differential \( d_T \) applied to a \( B \)-field free form \( e^{-\phi}[\Psi_+ \otimes \Psi_-]^{ev.od} \). We refer to the 3–form \( T \) as the torsion of the generalised structure. Generally speaking, the torsion tensor \([5]\) is defined by

\[
g(\tilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2} \text{Tor}(X, Y, Z)
\]

and measures the difference between an arbitrary metric connection \( \tilde{\nabla} \) and the Levi–Civita connection \( \nabla \). In our situation the spinors \( \Psi^+ \) and \( \Psi^- \) induce two \( G_2 \)-subbundles which carry metric connections \( \nabla^+ \) and \( \nabla^- \) such that

\[
\nabla^\pm_X Y = \nabla_X Y \pm \frac{1}{2} T(X, Y, \cdot).
\]

It therefore makes sense to consider the broader class of geometries defined by two linear metric connections \( \nabla^\pm \) with skew–symmetric and closed torsion \( \pm T \). This class encapsulates all the structures we obtain by applying the variational principle and conveniently avoids distinguishing between “internal” torsion \( db \) coming from the ubiquitous \( B \)-field (corresponding to the untwisted variational problem) and “external” torsion \( H \) which might or might not be present. Consequently, we regard \( T \) to be part of the intrinsic data of an integrable structure. To see how the torsion encodes the geometry, we state the following proposition (see for instance \([6]\)).

**Proposition 4.4.** For any \( G_2 \)-structure with stable form \( \varphi \) there exist unique differential forms \( \lambda \in \Omega^0(M) \), \( \theta \in \Omega^1(M) \), \( \xi \in \Omega^2(M, \varphi) \), and \( \tau \in \Omega^3(M, \varphi) \) so that the differentials of \( \varphi \) and \( \star \varphi \) are given by

\[
\begin{align*}
d\varphi &= -\lambda \star \varphi + \frac{3}{4} \theta \wedge \varphi + \star \tau \\
d\star \varphi &= \theta \wedge \star \varphi + \xi \wedge \varphi.
\end{align*}
\]

Here, \( \Omega^2(M, \varphi) \) and \( \Omega^3(M, \varphi) \) denote the bundles associated with the irreducible \( G_2 \)-modules \( \Lambda^2_{14} \leq \Lambda^2 \) and \( \Lambda^3_{27} \leq \Lambda^3 \[[4]\].

To specify the torsion tensor of a connection is in general not sufficient to guarantee its uniqueness. However, this is true for \( G_2 \)-connections with skew–symmetric torsion. Using the notation of the previous proposition, we can assert the following result.

**Proposition 4.5.**\textsuperscript{[5], [6]} For a \( G_2 \)-structure with stable form \( \varphi \) the following statements are equivalent:

(i) the \( G_2 \)-structure is integrable, i.e. \( \xi = 0 \).

(ii) there exists a unique linear connection \( \tilde{\nabla} \) whose torsion tensor \( \text{Tor} \) is skew and which preserves the \( G_2 \)-structure, i.e. \( \tilde{\nabla} \varphi = 0 \).

The torsion can be expressed by

\[
\text{Tor} = -\frac{1}{6} \lambda \cdot \varphi + \frac{1}{4} \star (\theta \wedge \varphi) - \star \tau.
\]  

Moreover, the Clifford action of the torsion 3–form on the induced spinor \( \Psi \) is

\[
\text{Tor} \cdot \Psi = \frac{7}{6} \lambda \Psi - \theta \cdot \Psi.
\]  

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Using additional subscripts $\pm$ to indicate the torsion forms of $\nabla^{\pm}$, equations (10) and (11) read

$$\text{Tor}_\pm = \pm T - \frac{1}{6} \lambda_\pm \cdot \varphi_\pm + \frac{1}{4} * (\theta_\pm \wedge \varphi_\pm) - * r_\pm$$

and

$$\text{Tor}_\pm \cdot \Psi_\pm = \pm T \cdot \Psi_\pm = \frac{7}{6} \lambda_\pm \Psi_\pm - \theta_\pm \cdot \Psi_\pm.$$

In view of Theorem 4.3 we can use the dilatino equation to relate the torsion components to the additional parameters $d\phi$ and $\lambda$. We have

$$\pm T \cdot \Psi_\pm = \mp 2\lambda \Psi_\pm - 2d\phi \cdot \Psi_\pm$$

if the structure is even and

$$\pm T \cdot \Psi_\pm = -2\lambda \Psi_\pm - 2d\phi \cdot \Psi_\pm$$

if the structure is odd.

**Corollary 4.6.** If the generalised structure is weakly integrable, then there exist two linear connections $\nabla^{\pm}$ preserving the $G_2^{\pm}$–structure with skew torsion $\pm T$. These connections are uniquely determined. If the structure is weakly integrable and of even type, then

$$d\varphi_+ = \frac{12}{7} \lambda \ast \varphi_+ + \frac{3}{2} d\phi \wedge \varphi_+ + T_{27+}$$

$$d*\varphi_+ = 2d\phi \wedge *\varphi_+$$

and

$$d\varphi_- = -\frac{12}{7} \lambda \ast \varphi_- + \frac{3}{2} d\phi \wedge \varphi_- + T_{27-}$$

$$d*\varphi_- = 2d\phi \wedge *\varphi_-,$$

where $T_{27\pm}$ denotes the projection of $T$ onto $\Omega^3_{27}(M,\varphi_{\pm})$. Moreover, the torsion can be expressed by the formula

$$\text{Tor}_\pm = \pm T = - e^{2\phi} \ast de^{-2\phi} \varphi_\pm \pm 2\lambda \cdot \varphi_\pm. \quad (12)$$

If the structure is weakly integrable and of odd type, then

$$d\varphi_+ = \frac{12}{7} \lambda \ast \varphi_+ + \frac{3}{2} d\phi \wedge \varphi_+ + T_{27+}$$

$$d*\varphi_+ = 2d\phi \wedge *\varphi_+$$

and

$$d\varphi_- = \frac{12}{7} \lambda \ast \varphi_- + \frac{3}{2} d\phi \wedge \varphi_- + T_{27-}$$

$$d*\varphi_- = 2d\phi \wedge *\varphi_-.$$

The torsion can be expressed by the formula

$$\text{Tor}_\pm = \pm T = - e^{2\phi} \ast de^{-2\phi} \varphi_\pm \pm 2\lambda \cdot \varphi_\pm. \quad (13)$$

we obtain the formulae for strongly integrable case if we set $\lambda = 0$.

Conversely, if we are given two $G_2$–structures defined by the stable forms $\varphi_+$ and $\varphi_- inducing the same metric, a constant $\lambda$ and a function $\phi$ such that (12) or (13) defines a closed 3–form $T$, then the corresponding spinors $\Psi_\pm$ satisfy the integrability condition of Theorem 4.3 and hence define a (weakly) integrable generalised $G_2$–structure (of even or odd type).
The previous discussion has a striking consequence. Assume $M$ to be compact and endowed with a weakly integrable structure of even type. Then (12) and Stokes’ Theorem imply

$$
\int_M e^{-2\phi} T \wedge *T = \mp \int_M T \wedge de^{-2\phi} \varphi_{\pm} + 2\lambda \int_M T \wedge e^{-2\phi} \star \varphi_{\pm} = \frac{4}{\lambda^2} \int_M e^{-2\phi} vol_M.
$$

Here we have used that $dT = 0$ and that the projection of $T$ on $\varphi_{\pm}$ is given by $T_{\bar{i} \pm} = 2\lambda \varphi_{\pm}$. The same identity holds for odd structures. Since the left hand side is strictly positive unless $T \equiv 0$, we need $\lambda \neq 0$. As a result, we obtain the following no–go theorem, generalising a similar statement in [3].

**Theorem 4.7.** If $M$ is compact and carries an integrable generalised $G_2$–structure, then $T = 0$ if and only if $\lambda = 0$. In this case the spinors $\Psi_{\pm}$ are parallel with respect to the Levi–Civita connection. Moreover, a weakly integrable structure has necessarily non–trivial torsion.

**Remark:** In case of strong integrability, it follows in particular that the underlying topological generalised $G_2$–structure cannot be exotic. If the spinors were to be linearly dependent at one point, covariant constancy would imply global linear dependency and we would have an ordinary manifold of holonomy $G_2$. If the two spinors are linearly independent at some (and hence at any) point, then the holonomy reduces to an $SU(3)$–principal fibre bundle which is the intersection of the two $G_2$–structures. In this case, $M$ is locally isometric to $CY^3 \times S^1$ where $CY^3$ is a Calabi–Yau 3–fold.

Next we compute the Ricci tensor. To begin with, let $\text{Ric}$ and $\text{Ric}^\pm$ denote the Ricci tensors associated with the Levi–Civita connection and the connections $\nabla^\pm$. Generally speaking, the Ricci tensor $\text{Ric}$ associated with a $G$–invariant spinor $\Psi$ and a $G$–preserving, metric linear connection $\nabla$ with skew torsion is determined by the following relation.

**Proposition 4.8.** [5] The Ricci tensor associated with $\nabla$ is determined by

$$
\text{Ric}(X) \cdot \Psi = \nabla_X \text{Tor} \cdot \Psi + \frac{1}{2} (X \cdot d\text{Tor}) \cdot \psi
$$

and relates to the metric Ricci tensor through

$$
\text{Ric}(X,Y) = \text{Ric}(X,Y) + \frac{1}{2} d^* \text{Tor}(X,Y) + \frac{1}{4} g(X \cdot \text{Tor}, Y \cdot \text{Tor}).
$$

**Theorem 4.9.** The Ricci–tensor of an integrable generalised $G_2$–structure is given by

$$
\text{Ric}(X,Y) = -2H^\phi(X,Y) + \frac{1}{4} g(X \cdot T, Y \cdot T),
$$

where $H^\phi(X,Y) = X \cdot Y \cdot \phi - \nabla_X Y \cdot \phi$ is the Hessian of the dilaton $\phi$. It follows that the scalar curvature of the metric $g$ is

$$
\text{Scal} = \text{Tr}(\text{Ric}) = 2\Delta \phi + \frac{3}{4} \|T\|^2,
$$

where $\Delta(\cdot) = -\text{Tr}_g H^{(\cdot)}$ is the Riemannian Laplacian.

**Proof:** According to the previous proposition we obtain

$$
\text{Ric}(X,Y) = \frac{1}{2} (\text{Ric}^+(X,Y) + \text{Ric}^-(X,Y)) + \frac{1}{4} g(X \cdot T, Y \cdot T)
$$
and it remains to compute the Ricci–tensors $\text{Ric}^+$ and $\text{Ric}^-$. Since $\nabla^\pm$ preserves the $G_{2\pm}$–structure, we derive
\[ \pm(\nabla_X^\pm T) \cdot \Psi^\pm = -2(\nabla_X^\pm d\phi) \cdot \Psi^\pm \]
from the dilatino equation, and therefore
\[ \text{Ric}^\pm(X) = -2\nabla_X^\pm d\phi. \]
Now consider a frame that satisfies $\nabla_{e_i}e_j = 0$ at a fixed point, or equivalently, $\nabla_{e_i}^\pm e_j = \pm \frac{1}{2} \sum_k T_{ijk}e_k$.

As the connections $\nabla^\pm$ are metric, we obtain
\[ \text{Ric}^\pm(e_i, e_j) = -2g(\nabla_{e_i}^\pm d\phi, e_j) \]
\[ = -2e_i, g(d\phi, e_j) + g(d\phi, \nabla_{e_i}^\pm e_j) \]
\[ = -2e_i, e_j, \phi \pm \sum_k T_{ijk}e_k, \phi. \]
The Hessian evaluated in this basis is just $H^\phi(e_i, e_j) = e_i, e_j, \phi$, whence the result. ■

If the dilaton is constant, then $T_{1\pm} = 0$ and the underlying $G_{2\pm}$–structures are co–calibrated, i.e. $d\ast \phi^\pm = 0$. Consequently, the Ricci tensors $\text{Ric}^\pm$ of $\nabla^\pm$ vanish. We then appeal to Theorem 5.4 of [5] which translated to our context asserts that if $T_{1\pm}$ vanishes, the Levi–Civita connection reduces to the underlying $G_{2\pm}$–structure.

**Proposition 4.10.** The metric $g$ of a strongly integrable generalised $G_2$–structure is Ricci–flat if and only if the dilaton $\phi$ is constant. In particular, any strongly integrable generalised $G_2$–structure which is homogeneous is Ricci–flat.

The following example illustrates how restrictive the assumption $dT = 0$ really is. It defines a compact generalised $G_2$–structure with constant dilaton and non–trivial $T$ such that $dT\rho = 0$, $dT\hat{\rho} = 0$, but $dT \neq 0$.

**Example:** Consider the 6–dimensional nilmanifold $N$ associated with the Lie algebra $g$ spanned by the orthonormal basis $e_2, \ldots, e_7$ and determined by the relations
\[ de_i = \begin{cases} 
  e_{37}, & i = 4 \\
  -e_{35}, & i = 6 \\
  0, & \text{else}.
\end{cases} \]

We let $M = N \times S^1$ and endow $M$ with the product metric $g = g_N + dt \otimes dt$. On $N$ we choose the $SU(3)$–structure coming from
\[ \omega = -e_{23} - e_{45} + e_{67}, \quad \psi_+ = e_{356} - e_{347} - e_{257} - e_{246} \]
which induces a generalised $G_2$–structure on $M$ with $\alpha = e_1 = dt$ (cf. Section 2). We put $T = e_{167} + e_{145}$. Obviously, $dT \neq 0$ holds. Writing $\rho = \omega + \psi_+ \wedge \alpha - \omega^3/6$ and $\hat{\rho} = \alpha - \psi_- - \omega^2 \wedge \alpha/2$ the equations $dT\rho = 0$ and $dT\hat{\rho} = 0$ are equivalent to
\[ d\omega = 0, \quad d\psi_+ \wedge \alpha = -T \wedge \omega, \quad T \wedge \alpha = d\psi_, \quad T \wedge \psi_- = 0. \]
By design, $T \wedge \alpha = 0$ and $T \wedge \psi_- = 0$. Moreover,
\[ d\omega = -de_4 \wedge e_5 + de_6 \wedge e_7 = 0 \]
and
\[ d\psi_- = -e_3 \wedge de_4 \wedge e_6 + e_{34} \wedge de_6 + e_{25} \wedge de_6 + e_2 \wedge de_4 \wedge e_7 = 0. \]

Finally, we have
\[ d\psi_+ \wedge \alpha = -e_{12367} - e_{12345} = -T \wedge \omega. \]

5 Generalised $Spin(7)$–structures

In the same vein as generalised $G_2$–structures we can develop a theory of generalised $Spin(7)$–structures associated with $Spin(7) \times Spin(7)$ in $\mathbb{R}^* \times Spin(8,8)$. We content ourselves with an outline of this theory. Mutatis mutandis the proofs translate without too much difficulty from the generalised $G_2$–case – the only new feature to take into account is the chirality of the spinors. A more detailed exposition can be found in [18].

As before, a generalised $Spin(7)$–structure gives rise to two unit spinors $\Psi_+$ and $\Psi_-$ in the associated irreducible $Spin(8)$–representations $\Delta_+$ and $\Delta_-$ (the subscript does not indicate the chirality of the spinors). Tensoring those induces a $Spin(7) \times Spin(7)$–invariant spinor which is given by either an even or an odd form according to the chirality of the spinors $\Psi_+$ and $\Psi_-$. This leads to the notion of generalised $Spin(7)$–structures of even or odd type for spinors of equal or opposite chirality. The box–operator is now a map $\Box_{g,b} : \Lambda^{even,odd} \to \Lambda^{even,odd}$ for which the (anti–)self–duality property
\[ \Box_{g,b}[\Psi_+ \otimes \Psi_-]_b = (-1)^{even,odd} \Box_{g,b}[\Psi_+ \otimes \Psi_-]_b \]
holds. To any $Spin(7) \times Spin(7)$–invariant spinor we can then associate the volume form $q(\Box_{g} \rho, \rho)$ and define the dilaton $\phi$ by $Q(\rho) = q(\rho, \Box_{g} \rho) = 16e^{-2\phi} \text{vol}_g$. Note that such a spinor is not stable.

**Theorem 5.1.** A topological generalised $Spin(7)$–structure over an 8–manifold $M$ is a topological $Spin(7) \times Spin(7)$–reduction of the $\mathbb{R}^* \times Spin(8,8)$–principal fibre bundle associated with $T \oplus T^*$. It is said to be of even or odd type according to the parity of the $Spin(7) \times Spin(7)$–invariant spinor $\rho$. We will denote this structure by the pair $(M, \rho)$ and call $\rho$ the structure form. Equivalently, such a structure can be characterised by the following data:

- an orientation
- a metric $g$
- a 2–form $b$
- a scalar function $\phi$
- two unit half spinors $\Psi_+, \Psi_- \in \Delta$ of either equal (even type) or opposite chirality (odd type) such that
\[ e^{-\phi}[\Psi_+ \otimes \Psi_-]_b = \rho. \]

The normal form description is computed as in the $G_2$–case.
Proposition 5.2. There exists an orthonormal basis $e_1, \ldots, e_8$ such that

$$[\Psi_+ \otimes \Psi_-] = \cos(a) + \sin(a)(e_{12} - e_{34} - e_{56} + e_{78}) +$$

$$+ \cos(a)(e_{134} + e_{1246} - e_{1278} + e_{1357} + e_{1368} + e_{1458} - e_{1467} -$$

$$- e_{2358} + e_{2378} + e_{2457} + e_{2468} - e_{3467} + e_{4578} + e_{5678}) +$$

$$+ \sin(a)(e_{1358} - e_{1457} - e_{1468} + e_{2357} + e_{2368} + e_{2458} - e_{2467}) +$$

$$+ \sin(a)(-e_{123456} + e_{123478} + e_{125678} - e_{345678} + \cos(a)e_{12345678},$$

where $a = \langle \psi_+, \psi_- \rangle$ and the spinors are of equal chirality (even case). In the odd case, we have

$$[\Psi_+ \otimes \Psi_-] = -e_1 + e_{234} + e_{256} - e_{278} + e_{357} + e_{368} + e_{458} - e_{467} +$$

$$+ e_{12358} - e_{12367} - e_{12457} - e_{12468} + e_{13456} - e_{13478} - e_{15678} + e_{2345678}.$$

If the spinors are not parallel, we can express the homogeneous components of $[\Psi_+ \otimes \Psi_-]$ in terms of the invariant forms of the intersection $\text{Spin}(7)_+ \cap \text{Spin}(7)_- = SU(4)$ (even case) or $\text{Spin}(7)_+ \cap \text{Spin}(7)_- = G_2$ (odd case). We find

$$[\Psi_+ \otimes \Psi_-] = c + s\omega + c(\psi_+- \frac{1}{2}\omega^2) - s\psi_- \frac{s}{6}\omega^3 + \text{cvol}_g$$

where $\omega$ is the Kähler form and $\psi_\pm$ the real and imaginary parts of the holomorphic volume form stabilised by $SU(4)$. In the odd case, we have

$$[\Psi_+ \otimes \Psi_-] = -\alpha + \varphi - \alpha \wedge \psi + \frac{1}{i} \varphi \wedge \psi$$

where $\alpha$ denotes the dual of the unit vector in $T$ which is stabilised by $G_2$, $\varphi$ the invariant stable 3-form on the complement and $\psi = \ast 7\varphi$.

Existence of generalised $\text{Spin}(7)$–structures follows from the existence of either a unit spinor (even structures) or a unit spinor and a unit vector field (odd structure), both of which is classical [15].

Proposition 5.3.

(i) An 8–manifold $M$ carries an even topological generalised $\text{Spin}(7)$–structure if and only if $M$ is spin and $8\chi(M) + p_1(M)^2 - 4p_2(M) = 0$.

(ii) A differentiable 8–manifold $M$ carries an odd topological generalised $\text{Spin}(7)$–structure if and only if $M$ is spin, has vanishing Euler class and satisfies $p_1(M)^2 = 4p_2(M)$.

For a generalised $\text{Spin}(7)$–structure of even type we can discuss classification issues as in Section 3. The sphere bundle associated with $\Delta_+ \cup \Delta_-$ has fibre isomorphic to $S^7$ and an 8–dimensional base, so that two transverse sections will intersect in a curve. We meet the first obstruction for the existence of a vertical homotopy in $H^7(M, \mathbb{Z})$, which by Poincaré duality trivially vanishes if $H_1(M, \mathbb{Z}) = 0$ (e.g. if $M$ is simply connected). Since $\pi_8(S^7) \cong \mathbb{Z}_2$, the second obstruction lies in the top cohomology module

$$H^8(M, \pi_8(S^7)) \cong \begin{cases} \mathbb{Z}_2, & \text{if } M \text{ compact} \\ 0, & \text{if } M \text{ non-compact} \end{cases}.$$

Remark: The stable homotopy group $\pi_{n+k}(S^n)$ is isomorphic to the framed cobordism group of $k$–manifolds. It is conceivable that the $\mathbb{Z}_2$–class is the framed (or spin) cobordism class of the 1–manifold where the two sections coincide.
Example: The tangent bundle of the 8–sphere is stably trivial and therefore all the Pontrjagin classes vanish. Since the Euler class is non–trivial, there exists no generalised $\text{Spin}(7)$-structure on $S^8$. However, they do exist on manifolds of the form $M = S^1 \times N^7$ for $N^7$ spinnable. For instance, take $N^7$ to be the 7–sphere $S^7$. Then the tangent bundle of $M$ is trivial and so is the sphere bundle $\mathbb{S}$ associated with the spinor bundle. Hence $\text{Gen}(M) = [S^1 \times S^7, S^7]$ which contains the set $[S^7, S^7] = \pi_7(S^7) = \mathbb{Z}$. Choosing a non–trivial homotopy class in $\pi_7(S^7)$ which we extend trivially to $S^1 \times S^7$ defines an exotic generalised $\text{Spin}(7)$-structure.

Since the structure form is not stable, we cannot setup a variational problem. We therefore follow the analogy of the classical case and impose ad–hoc the strong integrability condition.

**Definition 5.1.** A topological generalised $\text{Spin}(7)$-structure $(M, \rho)$ is said to be integrable with respect to a closed 3–form $H$ if and only if

$$d_H \rho = 0.$$ 

Example:

(i) Consider an 8–manifold $M$ endowed with a $\text{Spin}(7)$–invariant 4–form $\Phi$ and associated Riemannian volume form $\text{vol}_g$. The structure form of the induced straight structure (necessarily of even type) is given by the $B$–field transform of

$$\rho = e^{-\phi} e^{b/2} \wedge (1 - \Phi + \text{vol})$$

which follows from the normal form description above. We want to solve

$$d_T e^{\phi}(1 - \Omega + \text{vol}_g) = 0$$

which is equivalent to $d\phi = 0$, $T = 0$ and $d\Omega = 0$. Consequently, the holonomy of $(M, g)$ reduces to $\text{Spin}(7)$.

(ii) Examples of odd type are provided by product manifolds of the form $M = N^7 \times S^1$, where $N^7$ carries a $G_2$–structure induced by the stable 3–form $\varphi$. Let $T = \eta + \xi \wedge dt$ be a closed 4–form on $M$. The spinor

$$\rho = e^{-\phi} e^b \wedge (dt \wedge (-1 + *N\varphi) - \varphi + \text{vol}_N)$$

defines a generalised $\text{Spin}(7)$-structure of odd type which is $dT$–closed if and only if $d\phi = 0$, $T = 0$ and $d\varphi = 0$, $d *_N \varphi = -\xi \wedge \varphi$, i.e. the $G_2$–structure on $N$ is calibrated.

(iii) Manifolds with holonomy contained in $\text{Spin}(7)$ can be easily built out of a trivial $S^1$–bundle over a 7–manifold with holonomy $G_2$. This easily generalises to our context where a strongly integrable generalised $G_2$–structure $(M, \rho, T)$ induces an integrable generalised $\text{Spin}(7)$–structure of even type $(M^7 \times S^1, dt \wedge \hat{\rho} + \rho, T)$.

**Theorem 5.4.** A generalised $\text{Spin}(7)$–structure $(M, \rho)$ is integrable if and only if $e^{-\phi}[\Psi_+ \otimes \Psi_-]_b = \rho$ satisfies (with $T = db/2 + H$)

$$\nabla_X \Psi_\pm \pm \frac{1}{4} (X \cdot T) \cdot \Psi_\pm = 0$$

and

$$(d\phi \pm \frac{1}{2} T) \cdot \Psi_\pm = 0.$$
Using Theorem 5.4 we can discuss the torsion of the underlying Spin(7)–structures defined by the invariant 4–forms Ω± in the same way as in the G2–case. Using results from [13], we obtain:

**Proposition 5.5.** If the generalised Spin(7)–structure is integrable, then
\[ ±T = e^{2\phi} \ast d(e^{-2\phi} \Omega_\pm) \] (14)
and
\[ d\Omega_\pm = \frac{12}{7} d\phi \pm \ast T_{48\pm}, \] (15)

where \( T_{48\pm} \) denotes the projection of \( T \) onto \( \Omega^3_{48}(M, \Omega_\pm) \), the bundle associated with the irreducible \( \text{Spin}(7)_{\pm} \)–representation space \( \Lambda^3_{48} \) in \( \Lambda^3 \mathbb{R}^5 \).

Conversely, if we are given two \( \text{Spin}(7)_{\pm} \)–invariant forms inducing the same metric \( g \), a function \( \phi \) and a closed 3–form \( T \) such that (14) and (15) hold, then the corresponding spinors \( \Psi_\pm \) satisfy Theorem 5.4 and hence define an integrable generalised Spin(7)–structure.

As in the G2–case, we then deduce the following results.

**Corollary 5.6.** If \( M \) is compact and carries an integrable generalised Spin(7)–structure, then \( T = 0 \). Consequently, the spinors \( \Psi_\pm \) are parallel with respect to the Levi–Civita connection.

**Proposition 5.7.** The Ricci–tensor and the scalar curvature of an integrable generalised Spin(7)–structure are given by
\[ \text{Ric}(X,Y) = -2\nabla^\phi(X,Y) + \frac{1}{4}g(X,T,Y,T) \]
and
\[ \text{Scal} = \text{Tr}(\text{Ric}) = 2\Delta \phi + \frac{3}{4} \|T\|^2. \]

**Proposition 5.8.** An integrable generalised Spin(7)–structure is Ricci–flat if and only if the dilaton \( \phi \) is constant. In particular, any homogeneous integrable generalised Spin(7)–structure is Ricci–flat.

Again the condition \( dT = 0 \) is crucial.

**Example:** In conjunction with (iii) of the previous example, the compact \( G_2 \)–structure discussed at the end of Section 4.3 gives trivially rise to an instance of a compact generalised Spin(7)–structure of even type that satisfies \( d_T \rho = 0 \), but \( d_T \neq 0 \). For an odd example, just take a compact calibrated \( G_2 \)–manifold which, for instance, can be built out of the nilmanifold \( N \) considered above. It follows that \( d\ast\varphi = \xi \wedge \varphi \) (Proposition 4.4), so that \( d_T \rho = 0 \) for \( T = -\xi \wedge dt \) and \( \rho \) defined as in (ii) of the previous example.

**Corollary 5.9.** The torsion 3–form of a compact calibrated \( G_2 \)–manifold can never be closed.

6 \( T \)–duality

Type IIA and IIB string theory are interrelated by the so–called \( T \)–duality. Formally speaking, \( T \)–duality transforms the data \( (g, b, \phi) \) consisting of a generalised metric \( (g, b) \) and the dilaton \( \phi \), all living on a principal \( S^1 \)–bundle \( P \to M \) with connection form \( \theta \), into a generalised metric \( (g', b') \)
and a dilaton $\phi$ over a new principal $S^1$–fibre bundle $P^t \to M$ with connection form $\theta^t$. In the physics literature, the coordinate description of this dualising procedure is known under the name of Buscher rules. A neat mathematical formulation particularly apt for applications in the setting of generalised geometry was given in $\text{[3]}$.

Let $X$ denote the dual vertical vector field of $\theta$, i.e. $X_\theta = 1$. Consider its curvature 2–form $\mathcal{F}$ which we regard as a 2–form on $M$ so $d\theta = p^* \mathcal{F}$. We assume to be given a closed, integral and $S^1$–invariant 3–form $T$ such that the 2–form $\mathcal{F}^t$ defined by $p^* \mathcal{F}^t = -X_\theta T$ is also integral. In practice, we will assume $T = 0$ so that this condition is automatically fulfilled. Integrality of $\mathcal{F}^t$ ensures the existence of another principal $S^1$–bundle $P^t$, the $T$–dual of $P$ defined by the choice of a connection form $\theta^t$ with $d\theta^t = p^* \mathcal{F}^t$. Writing $T = \theta \wedge \mathcal{F}^t - T$ for a 3–form $T \in \Omega^3(M)$, we define the $T$–dual of $T$ by

$$T^t = -\theta^t \wedge \mathcal{F} + T.$$

Here and from now on, we ease notation and drop the pull–back $p^*$.

To make contact with our situation, consider an $S^1$–invariant structure form $\rho$ which we decompose into

$$\rho = \theta \wedge \rho_0 + \rho_1.$$

The $T$–dual of $\rho$ is defined to be

$$\rho^t = \theta^t \wedge \rho_1 + \rho_0.$$

In particular, $T$–duality reverses the parity of forms and maps even to odd and odd to even forms. It is enacted by multiplication with the element $X \oplus \theta \in \text{Pin}(n, n)$ on $\rho$ followed by the substitution $\theta \to \theta^t$.

The crucial feature of $T$–duality is that it preserves the $\text{Spin}(n, n)$–orbit structure on $\Lambda^{ev,od} TP^*$. To see this, we decompose

$$TP \oplus T^*P \cong TM \oplus \mathbb{R}X \oplus T^*M \oplus \mathbb{R}\theta, \quad TP \oplus T^*P \cong TM \oplus \mathbb{R}X^t \oplus T^*M \oplus \mathbb{R}\theta^t,$$

where $\mathbb{R}X$ denotes the vertical summand ker $\pi_*$ of $TP$ which is spanned by $X$, and similarly for $\theta$ and their $T$–duals. Then consider the map $\tau : TP \oplus T^*P \to TP^t \oplus T^*P^t$ defined with respect to this splitting by

$$\tau(V + uX \oplus \xi + v\theta) = -V + vX^t \oplus -\xi + u\theta^t.$$

It satisfies

$$(a \bullet \rho)^t = \tau(a) \bullet \rho^t$$

for any $a \in TP \oplus T^*P$ and in particular, $\tau(a)^2 = -(a, a)$. Hence this map extends to an isomorphism $\text{Cliff}(TP \oplus T^*P) \cong \text{Cliff}(TP^t \oplus T^*P^t)$, and any orbit of the form $\text{Spin}(TP \oplus T^*P)/G$ gets mapped to an equivalent orbit $\text{Spin}(TP^t \oplus T^*P^t)/G$ where $G$ and $G^t$ are isomorphic as abstract groups.

As an illustration of this, consider a generalised $G_2$–structure over $P$ with structure form $\rho = \theta \wedge \rho_0 + \rho_1$ and companion $\hat{\rho} = \theta \wedge \rho_0 + \rho_1$. These have $T$–duals $\rho^t$ and $\hat{\rho}^t$. Since $\rho$ and $\rho$ have the same stabiliser $G$ inside $\text{Spin}(TP \oplus T^*P) \cong \text{Spin}(7, 7)$, it follows that $\rho^t$ and $\hat{\rho}^t$ are stabilised by the same $G^t$ inside $\text{Spin}(TP^t \oplus T^*P^t) \cong \text{Spin}(7, 7)$, which is isomorphic to $G_2 \times G_2$. By invariance, $\hat{\rho}^t$ and $\hat{\rho}^t$ coincide up to a constant which we henceforth ignore. The integrability condition transforms as follows:

**Proposition 6.1.**

$$d_T \rho = \lambda \hat{\rho} \text{ if and only if } d_T \rho^t = -\lambda \hat{\rho}^t.$$
The proof is a straightforward computation using the definition of \( \rho \) and \( \rho^t \).

We put this machinery into action as follows. Start with a non-trivial principal \( S^1 \)-fibre bundle \((P, \theta)\) which admits a metric of holonomy \( G_2 \) or \( \text{Spin}(7) \) and let \( T = 0 \). The resulting straight structure is strongly integrable and so is its dual, but according to the \( T \)-duality rules, we acquire non-trivial torsion given by \( T^t = -\theta^t \wedge F \). Local examples of such \( G_2 \)-structures exist in abundance \([2]\). In conjunction with (iii) of the first example in Section 5, this gives an \( S^1 \)-invariant generalised \( \text{Spin}(7) \)-structure of even type with integral, \( S^1 \)-invariant torsion \( T \). Contracting with \( dt \) yields \( F^t = 0 \), hence the \( T \)-dual defines an integrable generalised \( \text{Spin}(7) \)-structure of odd type with \( T^t = T \).

As a further application of this formalism, we note that a principal \( S^1 \)-fibre bundle \( \pi : P \to M \) whose holonomy reduces to an \( S^1 \)-invariant \( G_2 \)- or \( \text{Spin}(7) \)-structure must be trivial if the base \( M \) is compact. Indeed, if \( \theta \) is a connection form on \( P \) and \( T = 0 \), then the \( T \)-dual defines a strongly integrable generalised structure with torsion \( T^t = -\theta^t \wedge F \). But this vanishes as a consequence of Corollaries 4.7 and 5.6, hence \( F = 0 \) which implies the triviality of \( P \).

**Corollary 6.2.** If a compact, simply-connected 7- or an 8-manifold admits an \( S^1 \)-invariant \( G_2 \)- or \( \text{Spin}(7) \)-structure to which the holonomy reduces, then the principal \( S^1 \)-fibre bundle is trivial.

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