On the mean average of integer partition as the sum of powers

Pengyong Ding\textsuperscript{1,*}

\textbf{Abstract}

This paper is concerned with the function $r_{k,s}(n)$, the number of (ordered) representations of $n$ as the sum of $s$ positive $k$-th powers, where integers $k, s \geq 2$. We examine the mean average of the function, or equivalently,

$$\sum_{m=1}^{n} r_{k,s}(m).$$

\textbf{Keywords:} mean average, partition, sum of powers, lattice points

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\section{1. Introduction}

This paper is concerned with the function $r_{k,s}(n)$, the number of (ordered) representations of $n$ as the sum of $s$ positive $k$-th powers, where integers $k, s \geq 2$:

$$r_{k,s}(n) = \sum_{\substack{x_1, x_2, \ldots, x_s \in \mathbb{Z} \setminus \{0\} \atop x_1^k + x_2^k + \cdots + x_s^k = n}} 1. \quad (1)$$

The function is important in the study of Waring’s problem, which is to find the least $s$ for a given $k$, such that $r_{k,s}(n) > 0$ for every sufficiently large $n$. Besides Waring’s problem, the function is also useful in the study of Diophantine equations. However, the property of $r_{k,s}(n)$ is only poorly understood even when $k$ and $s$ are small. For example, when both $k$ and $s$ are as small as 3, the asymptotic formula for the second moment sum

\textsuperscript{*}School of Mathematics, Shandong University, Jinan, Shandong 250100 China
of \( r_{3,3}(n) \) is still unknown. The best result that we have already known is provided by Vaughan [1]:

\[
\sum_{m=1}^{n} r_{3,3}(n)^2 \ll n^{\frac{7}{3}} (\log n)^{\frac{1}{3} - \frac{3}{4}},
\]

although we may conjecture that

\[
\sum_{m=1}^{n} r_{3,3}(n)^2 \sim Cn
\]

for some positive constant \( C \). The mean average of \( r_{k,s}(n) \), or equivalently, the sum

\[
\sum_{m=1}^{n} r_{k,s}(m), \quad (2)
\]

is not as badly known as the other properties. In fact, similar to the argument by Vaughan [1], if we let

\[
\Delta_{k,s}(n) = \sum_{m=1}^{n} r_{k,s}(m) - \frac{\Gamma\left(\frac{k+1}{k}\right)^s}{\Gamma\left(\frac{k+s}{k}\right)} n^{\frac{s}{k}},
\]

then it represents the difference between the number of lattice points in an \( s \)-dimensional convex body and its volume, which is bounded by its \((s - 1)\)-dimensional surface volume. So we have

\[
\Delta_{k,s}(n) \ll n^{\frac{s-1}{k}},
\]

and thus

\[
\sum_{m=1}^{n} r_{k,s}(m) = \frac{\Gamma\left(\frac{k+1}{k}\right)^s}{\Gamma\left(\frac{k+s}{k}\right)} n^{\frac{s}{k}} + O\left(n^{\frac{s-1}{k}}\right). \quad (3)
\]

However, we can improve that result by applying van der Corput’s method. For example, Vaughan [1] have shown that

\[
\sum_{m=1}^{n} r_{3,2}(m) = \frac{\Gamma\left(\frac{4}{3}\right)^2}{\Gamma\left(\frac{2}{3}\right)} n^{\frac{2}{3}} + O\left(n^{\frac{2}{3}} (\log n)^{\frac{1}{3}}\right),
\]

and

\[
\sum_{m=1}^{n} r_{3,3}(m) = \frac{\Gamma\left(\frac{4}{3}\right)^3}{\Gamma\left(\frac{2}{3}\right)^2} n^{\frac{2}{3}} + O\left(n^{\frac{2}{3}} (\log n)^{\frac{1}{3}}\right),
\]
which is more precise than (3) when \(k = 3\) and \(s = 2\) or 3. Hence we can follow a similar argument and calculate the mean average of \(r_{k,s}(n)\) when \(k \geq 4\) and \(s \geq 2\). The following theorem shows the result:

**Theorem 1.1.** If \(k \geq 4\) and \(s\) are integers, and \(2 \leq s \leq k + 1\), then

\[
\sum_{m=1}^{n} r_{k,s}(m) = \frac{\Gamma\left(\frac{k+1}{k}\right)^{s}}{\Gamma\left(\frac{k+s}{k}\right)} n^{\frac{s}{k}} - \frac{s}{2} \cdot \frac{\Gamma\left(\frac{k+1}{k}\right)^{s-1}}{\Gamma\left(\frac{k+s-1}{k}\right)} n^{\frac{s-1}{k}} + O\left(n^{\frac{(s-1)k-1}{k^2}}\right),
\]

or equivalently,

\[
\sum_{m \leq x} r_{k,s}(m) = \frac{\Gamma\left(\frac{k+1}{k}\right)^{s}}{\Gamma\left(\frac{k+s}{k}\right)} x^{\frac{s}{k}} - \frac{s}{2} \cdot \frac{\Gamma\left(\frac{k+1}{k}\right)^{s-1}}{\Gamma\left(\frac{k+s-1}{k}\right)} x^{\frac{s-1}{k}} + O\left(x^{\frac{(s-1)k-1}{k^2}}\right).
\]

2. The proof of Theorem 1.1

We state the following two results without proof. The first one is the van der Corput Lemma, which is Theorem 2.2 of Graham and Kolesnik [2].

**Lemma 2.1.** Suppose that \(a < b\) and \(f\) has a continuous second derivative on \([a, b]\). Suppose also that \(\mu > 0, \eta > 1\) and that for every \(\alpha \in [a, b]\), we have \(\mu \leq |f''(\alpha)| \leq \eta \mu\). Then

\[
\sum_{a < n \leq b} e(f(n)) \ll \mu^{1/2} + (b-a)\eta \mu^{1/2}.
\]

The second result is Lemma 2.2 of Vaughan [1]. For any \(\alpha \in \mathbb{R}\), define

\[
B_1(\alpha) = \alpha - \lfloor \alpha \rfloor - \frac{1}{2}, \quad B_2(\alpha) = \int_{0}^{\alpha} B_1(\beta) \, d\beta,
\]

then we have

**Lemma 2.2.** Let \(H, \alpha \in \mathbb{R}\) and \(H \geq 2\). Then

\[
B_1(\alpha) = -\sum_{0 < |h| \leq H} \frac{e(\alpha h)}{2\pi i h} + O\left(\min\left(1, \frac{1}{H\|\alpha\|}\right)\right)
\]

and

\[
\min\left(1, \frac{1}{H\|\alpha\|}\right) = \sum_{h=-\infty}^{\infty} c(h)e(\alpha h),
\]

where

\[
c(0) = \frac{2}{H} \left(1 + \log \frac{H}{2}\right), \quad c(h) \ll \min\left(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{H^2}\right)(h \neq 0).
\]
We can use these results to prove the following lemma.

**Lemma 2.3.** If \( k \geq 4 \) is an integer and \( x > 1 \), then

\[
\sum_{m \leq (x/2)^{1/k}} B_1\left(\frac{1}{x} \left(\frac{x}{m^k}\right)^{1/k}\right) \ll x^{(k-1)/k^2}. \tag{6}
\]

**Proof.** We use a similar method to the proof given by Vaughan \[1\]. First, for convenience, we define

\[
M' = \min\left(2M, \left(\frac{x}{2}\right)^{1/k}\right)
\]

for every \( M \). Then let \( \nu \) be a parameter such that \( 1 \leq \nu \leq (x/2)^{1/k} \), and define the set

\[
\mathcal{M} = \left\{ \nu \cdot 2^j \mid j \geq 0, \nu \cdot 2^j \leq \left(\frac{x}{2}\right)^{1/k}\right\}.
\]

So we have

\[
\sum_{m \leq (x/2)^{1/k}} B_1\left(\frac{1}{x} \left(\frac{x}{m^k}\right)^{1/k}\right) = \sum_{M \in \mathcal{M}} \sum_{M < m \leq M'} B_1\left(\frac{1}{x} \left(\frac{x}{m^k}\right)^{1/k}\right) + O(\nu). \tag{7}
\]

By Lemma 2.2 let \( \alpha = (x - m^k)^{1/k} \), we have

\[
\sum_{M < m \leq M'} B_1\left(\frac{1}{x} \left(\frac{x}{m^k}\right)^{1/k}\right) = -\sum_{0 < |h| \leq H} \frac{T(M, h)}{2\pi i h} + O\left( \sum_{M < m \leq M'} \min\left(1, \frac{1}{H \|x - m^k\|^1}\right) \right),
\]

where

\[
T(M, h) = \sum_{M < m \leq M'} e\left(\frac{1}{x} \left(\frac{x}{m^k}\right)^{1/k} h\right).
\]

Regarding the error term, still by Lemma 2.2 we have

\[
\sum_{M < m \leq M'} \min\left(1, \frac{1}{H \|x - m^k\|^1}\right) = \sum_{h = -\infty}^{\infty} c(h)T(M, h),
\]

so

\[
\sum_{M < m \leq M'} B_1\left(\frac{1}{x} \left(\frac{x}{m^k}\right)^{1/k}\right) = -\sum_{0 < |h| \leq H} \frac{T(M, h)}{2\pi i h} + O\left( \sum_{h = -\infty}^{\infty} c(h)T(M, h) \right). \tag{8}
\]
Now consider $T(M, h)$ when $h \neq 0$. First, for convenience let 

$$f(\alpha) = h(x - \alpha^k)^\frac{1}{k},$$

then for $\alpha < x^{1/k}$,

$$f''(\alpha) = -(k-1)hx\alpha^{k-2}(x - \alpha^k)^{\frac{1}{k} - 2}.$$ 

So when $\alpha \in [M, M']$, we have

$$(k-1)|h|x^\frac{1}{k}M^{k-2} \leq |f''(\alpha)| \leq 2^{k-\frac{1}{k}}(k-1)|h|x^\frac{1}{k}M^{k-2}.$$ 

Therefore, the function $f(\alpha)$ satisfies the conditions in Lemma 2.1, where $a, b, \mu$, and $\eta$ are replaced by $M, M', (k-1)|h|x^\frac{1}{k}M^{k-2}$, and $2^{k-\frac{1}{k}}$ respectively. By that lemma, when $h \neq 0$, we have

$$T(M, h) \ll (M' - M)2^{k-\frac{1}{k}}((k-1)|h|x^\frac{1}{k}M^{k-2})^{\frac{1}{2}}$$

$$\ll |h|^{\frac{1}{2}}x^{\frac{k-1}{2k}}M^{2-k} + |h|^{\frac{1}{2}}x^{\frac{1-k}{2k}}M^{\frac{k}{2}}.$$ 

Finally, if $h = 0$, then obviously $T(M, 0) \ll M$. Hence by (8) and Lemma 2.2 we have

$$\sum_{M < m \leq M'} B_1\left((x - m^k)^\frac{1}{k}\right)$$

$$\ll \sum_{0 < |h| \leq H} \frac{|T(M, h)|}{|h|} + \sum_{\substack{0 < |h| \leq H \\text{and} \ h \neq 0}} |c(h)||T(M, h)| + |c(0)||T(M, 0)|$$

$$\ll \sum_{0 < |h| \leq H} \left(|h|^{-\frac{1}{2}}x^{\frac{k-1}{2k}}M^{2-k} + |h|^{-\frac{1}{2}}x^{\frac{1-k}{2k}}M^{\frac{k}{2}}\right)$$

$$+ \sum_{\substack{h = -\infty \\text{and} \ h \neq 0}} \left(\frac{1}{|h|}|h|^{-\frac{1}{2}}x^{\frac{k-1}{2k}}M^{2-k} + \min\left(\frac{1}{|h|}, \frac{H}{k^2}\right)|h|^\frac{1}{2}x^{\frac{1-k}{2k}}M^{\frac{k}{2}}\right)$$

$$+ \frac{2}{H}\left(1 + \log\frac{H}{2}\right)M$$

$$\ll x^{\frac{k-1}{2k}}M^{2-k} + H\frac{1}{2}x^{\frac{1-k}{2k}}M^{\frac{k}{2}} + MH^{-1}\log(2H).$$

(9)

To minimize the size of the error terms, we need an optimal choice for $H$. A possible option is

$$H = \frac{x^{\frac{3}{2k}}}{2},$$

(10)
so that (9) becomes
\[
\sum_{M < m \leq M'} B_1 \left( (x - m^k)^k \right) \ll x^{k-1-\frac{1}{2k^2}} M^{\frac{1}{2k^2}} + x^{\frac{3+2k-2k^2}{4k^2}} M^{\frac{1}{2k^2}} + M^{-\frac{3}{2k^2}} \log x.
\]

Therefore, by (7),
\[
\sum_{m \leq (x/2)^{1/k}} B_1 \left( (x - m^k)^{\frac{1}{k}} \right)
\ll \sum_{M \in M} \left( x^{\frac{k-1}{2k}} M^{\frac{3}{2k^2}} + x^{\frac{3+2k-2k^2}{4k^2}} M^{\frac{1}{2k^2}} \right) + \nu
\ll x^{\frac{k-1}{2k}} \nu + x^{\frac{2k+3}{4k^2}} + x^{\frac{2k-3}{2k^2}} (\log x) + \nu.
\]
From the first and the last terms, the only optimal choice for \( \nu \) such that the error terms have the minimal size is
\[
\nu = x^{\frac{k-1}{2k^2}},
\]
which gives the conclusion. \( \square \)

We notice that in the proof of Lemma 2.3, the first and the last terms of (11) are independent from the choice for \( H \), so the optimal choice for \( \nu \) does not depend on \( H \) as well, and it is always supposed to be (12). However, we do have plenty of choices for \( H \) to replace (10). For example, if \( H = x^A \) where
\[
\frac{1}{k^2} < A < \frac{k - 2}{k^2},
\]
then we can also prove the lemma in a similar way.

Now we use Lemma 2.3 to prove Theorem 1.1. We start from the case when \( s = 2 \).

**Lemma 2.4.** If \( k \geq 4 \) is an integer and \( x > 1 \), then
\[
\sum_{n \leq x} r_{k,2}(n) = \frac{\Gamma \left( \frac{k+1}{k} \right)^2}{\Gamma \left( \frac{k+2}{k} \right)} x^{\frac{1}{k}} - x^{\frac{1}{k}} + O \left( x^{\frac{k-1}{k^2}} \right).
\]

**Proof.** The LHS of (13) is the number of lattice points in Quadrant I under the curve \( X^k + Y^k = x \). By separating the lattice points according to whether one or both coordinates are no more than \( (x/2)^{1/k} \), we have
\[
\sum_{n \leq x} r_{k,2}(n) = 2 \sum_{m \leq (x/2)^{1/k}} \left( (x - m^k)^{\frac{1}{k}} \right) - \left( \left( \frac{x}{2} \right)^{\frac{1}{k}} \right)^2.
\]
By definition of $B_1(\alpha)$ and Lemma 2.3, we have

\[
\sum_{n \leq x} r_{k,2}(n) = 2 \sum_{m \leq (x/2)^{1/k}} ((x - m^k)^{\frac{1}{k}} - \frac{1}{2} - B_1((x - m^k)^{\frac{1}{k}})) - \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{k}} \right\rfloor^2
\]

\[
= 2 \sum_{m \leq (x/2)^{1/k}} (x - m^k)^{\frac{1}{k}} - \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{k}} \right\rfloor - \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{k}} \right\rfloor^2 + O\left(x^{\frac{k-1}{k}}\right).
\]

(14)

We write $(x - m^k)^{1/k}$ as an integral, and change the order of summation. The first term above is

\[
2 \sum_{m \leq (x/2)^{1/k}} (x - m^k)^{\frac{1}{k}} = 2 \sum_{m \leq (x/2)^{1/k}} \int_{x/2}^{x^{1/k}} \alpha^{k-1}(x - \alpha^k)^{-\frac{k-1}{k}} d\alpha
\]

\[
= 2 \int_{0}^{x^{1/k}} \min\left(\left\lfloor \alpha \right\rfloor, \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{k}} \right\rfloor \right) \alpha^{k-1}(x - \alpha^k)^{-\frac{k-1}{k}} d\alpha
\]

\[
= \int_{0}^{(x/2)^{1/k}} 2\left(\alpha - \frac{1}{2}\right) \alpha^{k-1}(x - \alpha^k)^{-\frac{k-1}{k}} d\alpha
\]

\[
- \int_{0}^{(x/2)^{1/k}} 2B_1(\alpha)\alpha^{k-1}(x - \alpha^k)^{-\frac{k-1}{k}} d\alpha
\]

\[
+ 2 \int_{(x/2)^{1/k}}^{x^{1/k}} \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{k}} \right\rfloor \alpha^{k-1}(x - \alpha^k)^{-\frac{k-1}{k}} d\alpha.
\]

(15)

A straightforward calculation on the last integral shows that

\[
\int_{(x/2)^{1/k}}^{x^{1/k}} \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{k}} \right\rfloor \alpha^{k-1}(x - \alpha^k)^{-\frac{k-1}{k}} d\alpha = \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{k}} \right\rfloor \left(\frac{x}{2}\right)^{\frac{1}{k}}.
\]

(16)

Then we integrate the other two integrals by parts. The first integral is

\[
(1 - 2\alpha)(x - \alpha^k)^{\frac{1}{k}} \bigg|_{0}^{(x/2)^{1/k}} - \int_{0}^{(x/2)^{1/k}} (x - \alpha^k)^{\frac{1}{k}} d(1 - 2\alpha)
\]

\[
= -2\left(\frac{x}{2}\right)^{\frac{1}{k}} + \left(\frac{x}{2}\right)^{\frac{1}{k}} - x^{\frac{1}{k}} + 2 \int_{(x/2)^{1/k}}^{x^{1/k}} (x - \alpha^k)^{\frac{1}{k}} d\alpha,
\]

(17)
while the second one is

\[
\int_0^{(x/2)^{1/k}} 2\alpha^{k-1}(x - \alpha^k)^{-\frac{k-1}{k}} d(B_2(\alpha))
= 2B_2\left(\left(\frac{x}{2}\right)^{\frac{1}{k}}\right) - 2 \int_0^{(x/2)^{1/k}} B_2(\alpha) d\left(\alpha^{k-1}(x - \alpha^k)^{-\frac{k-1}{k}}\right).
\]

Since \(B_2(\alpha) \ll 1\), and

\[
\frac{d}{d\alpha}\left(\alpha^{k-1}(x - \alpha^k)^{-\frac{k-1}{k}}\right) = x\alpha^{k-2}(x - \alpha^k)^{-\frac{2k-1}{k}} > 0
\]

when \(0 < \alpha < (x/2)^{1/k}\), we have

\[
\int_0^{(x/2)^{1/k}} 2B_1(\alpha)\alpha^{k-1}(x - \alpha^k)^{-\frac{k-1}{k}} d\alpha = O(1). \tag{18}
\]

By (14), (15), (16), (17) and (18), and the fact that

\[
\left(\frac{x}{2}\right)^{\frac{1}{k}} - \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{k}} \right\rfloor \ll 1
\]

and its square

\[
\left(\frac{x}{2}\right)^{\frac{1}{k}} - 2\left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{k}} \right\rfloor \left(\frac{x}{2}\right)^{\frac{1}{k}} + \left\lfloor \left(\frac{x}{2}\right)^{\frac{1}{k}} \right\rfloor^2 \ll 1,
\]

we have

\[
\sum_{n \leq x} r_{k,2}(n) = 2 \int_0^{(x/2)^{1/k}} (x - \alpha^k)\frac{1}{k} d\alpha - \left(\frac{x}{2}\right)^{\frac{1}{k}} - x^{\frac{1}{k}} + O\left(x^{\frac{k-1}{k}}\right).
\]

Finally, the integral above represents the area of the set of points \((X, Y)\) in Quadrant I such that \(X^k + Y^k \leq x\) and \(X \leq (x/2)^{1/k}\), or equivalently, such that \(Y^k + X^k \leq x\) and \(Y \leq (x/2)^{1/k}\) when interchanging the variables \(X\) and \(Y\). Therefore, twice the integral represents the area of the whole region \(X^k + Y^k \leq x\) plus the area of the square \(0 \leq X \leq (x/2)^{1/k}, 0 \leq Y \leq (x/2)^{1/k}\). Hence

\[
2 \int_0^{(x/2)^{1/k}} (x - \alpha^k)\frac{1}{k} d\alpha - \left(\frac{x}{2}\right)^{\frac{1}{k}} = \int_0^{x^{1/k}} (x - \alpha^k)\frac{1}{k} d\alpha = \frac{1}{k}B\left(\frac{1}{k}, \frac{1}{k} + 1\right)x^{\frac{1}{k}}.
\]

The lemma is then proved according to the relationship between the beta function and the gamma function.
To finalize the proof of Theorem 1.1, we use induction on \( s \). By Lemma 2.4, the statement holds for the initial case \( s = 2 \). Hence we only need to prove that if the statement is true for a particular \( s \) where \( 2 \leq s \leq k \), then it is also true for \( s + 1 \). First, we notice that

\[
\sum_{m \leq x} r_{k,s+1}(m) = \sum_{l \leq x^{1/k}} \sum_{n \leq x-l^k} r_{k,s}(n),
\]

since both sides represent the number of \((s+1)\)-tuples \((x_1, x_2, \cdots, x_s, l)\) such that all coordinates are positive integers and

\[
x_1^k + x_2^k + \cdots + x_s^k + l^k \leq x.
\]

Hence from assumption, we have

\[
\sum_{m \leq x} r_{k,s+1}(m) = \frac{\Gamma(\frac{k+1}{k})^s}{\Gamma(\frac{k+s}{k})} \sum_{l \leq x^{1/k}} (x-l^k)^{\frac{s}{k}} - \frac{s}{2} \frac{\Gamma(\frac{k+1}{k})^{s-1}}{\Gamma(\frac{k+s-1}{k})} \sum_{l \leq x^{1/k}} (x-l^k)^{\frac{s-1}{k}} + O\left(x^{\frac{s^2-1}{k^2}}\right).
\]

(19)

Now we calculate the first sum on RHS. We have

\[
\sum_{l \leq x^{1/k}} (x-l^k)^{\frac{s}{k}} = \sum_{l \leq x^{1/k}} \int_{l}^{x^{1/k}} \frac{1}{\alpha^{k-1}} (x-\alpha^k)^{\frac{s-1}{k}} d\alpha
\]

\[
= \int_{0}^{x^{1/k}} \left[ \alpha \right] \cdot \frac{1}{\alpha^{k-1}} (x-\alpha^k)^{\frac{s-1}{k}} d\alpha
\]

\[
= \int_{0}^{x^{1/k}} \left( \alpha - \frac{1}{2} \right) \frac{1}{\alpha^{k-1}} (x-\alpha^k)^{\frac{s-1}{k}} d\alpha
\]

\[
- \int_{0}^{\eta} B_1(\alpha) \frac{1}{\alpha^{k-1}} (x-\alpha^k)^{\frac{s-1}{k}} d\alpha
\]

\[
- \int_{0}^{\eta} B_1(\alpha) \frac{1}{\alpha^{k-1}} (x-\alpha^k)^{\frac{s-1}{k}} d\alpha,
\]

(20)

where

\[
\eta = \left( x - x^{\frac{1}{k}} \right)^{\frac{1}{k}}.
\]

Integrating the first integral on RHS of (20) by parts, it is

\[
\int_{0}^{x^{1/k}} \left( \frac{1}{2} - \alpha \right) d((x-\alpha^k)^{\frac{s}{k}}) = -\frac{1}{2} x^{\frac{s}{k}} + \int_{0}^{x^{1/k}} (x-\alpha^k)^{\frac{s}{k}} d\alpha,
\]

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Similarly, we notice that if \( 0 \leq B \leq 1 \), we have
\[
\int_{0}^{x^{\frac{1}{k}}} (x - \alpha^k) \frac{d\alpha}{\alpha} = \frac{1}{k} B \left( \frac{1}{k}, \frac{s}{k} + 1 \right) x^{\frac{s+1}{k}} = \frac{\Gamma \left( \frac{k+1}{k} \right) \Gamma \left( \frac{k+s}{k} \right)}{\Gamma \left( \frac{k+s+1}{k} \right)} x^{\frac{s+1}{k}}.
\]
Since \( B \ll 1 \), we have
\[
\int_{\eta}^{x^{\frac{1}{k}}} B_1(\alpha)x^{\alpha-1} \frac{d\alpha}{\alpha} \ll \int_{\eta}^{x^{\frac{1}{k}}} d\left( -(x - \alpha^k) \frac{d\alpha}{\alpha} \right) = x^{\frac{\alpha(k-1)}{k}}.
\]
Finally, we integrate the last integral of (20) by parts. As \( B_1(\alpha) = d(B_2(\alpha)) \), the integral is
\[
B_2(\eta)x^{\frac{1}{k}}(x - \eta^k) \frac{\alpha}{\alpha} - \int_{0}^{\eta} B_1(\alpha)x^{\frac{1}{k}}(x - \alpha^k) \frac{d\alpha}{\alpha} = x^{\frac{\alpha(k-1)}{k}}.
\]
Since \( B_2(\alpha) \ll 1 \), we have
\[
B_2(\eta)x^{\frac{1}{k}}(x - \eta^k) \frac{\alpha}{\alpha} - \int_{0}^{\eta} B_1(\alpha)x^{\frac{1}{k}}(x - \alpha^k) \frac{d\alpha}{\alpha} = x^{\frac{\alpha(k-1)}{k}}
\]
and
\[
\int_{0}^{\eta} B_2(\alpha)d \left( x^{\frac{1}{k}}(x - \alpha^k) \frac{\alpha}{\alpha} \right) \ll \int_{0}^{\eta} \frac{d}{d\alpha} \left( x^{\frac{1}{k}}(x - \alpha^k) \frac{\alpha}{\alpha} \right) d\alpha.
\]
We notice that if \( 0 \leq \alpha \leq \eta, \ k \geq 4 \) and \( 2 \leq s \leq k \), then
\[
\frac{d}{d\alpha} \left( x^{\frac{1}{k}}(x - \alpha^k) \frac{\alpha}{\alpha} \right) = x^{\frac{s-1}{k}}(x - \alpha^k) \frac{\alpha}{\alpha} - (k - 1)x + (1 - s) \alpha^k \geq 0, \quad (21)
\]
so
\[
\int_{0}^{\eta} B_2(\alpha)d \left( x^{\frac{1}{k}}(x - \alpha^k) \frac{\alpha}{\alpha} \right) \ll \eta x^{\frac{1}{k}}(x - \eta^k) \frac{\alpha}{\alpha} \ll x^{\frac{\alpha(k-1)}{k}}.
\]
Hence
\[
\sum_{l \leq x^{\frac{1}{k}}} (x - l^k) \frac{\alpha}{\alpha} = \frac{\Gamma \left( \frac{k+1}{k} \right) \Gamma \left( \frac{k+s}{k} \right)}{\Gamma \left( \frac{k+s+1}{k} \right)} x^{\frac{s+1}{k}} - \frac{1}{2} x^{\frac{s}{k}} + O \left( x^{\frac{\alpha(k-1)}{k}} \right). \quad (22)
\]
Similarly,
\[
\sum_{l \leq x^{\frac{1}{k}}} (x - l^k) \frac{s-1}{k} = \frac{\Gamma \left( \frac{k+1}{k} \right) \Gamma \left( \frac{k+s-1}{k} \right)}{\Gamma \left( \frac{k+s}{k} \right)} x^{\frac{s-1}{k}} - \frac{1}{2} x^{\frac{s-2}{k}} + O \left( x^{\frac{s-1+1(k-1)}{k}} \right). \quad (23)
\]

Therefore, by (19), (22) and (23), we have

\[
\sum_{m \leq x} r_{k,s+1}(m) = \frac{\Gamma\left(\frac{k+1}{k}\right)^{s+1}}{\Gamma\left(\frac{k+s+1}{k}\right)} x^{\frac{k+1}{k}} - \frac{(s + 1)}{2} \cdot \frac{\Gamma\left(\frac{k+1}{k}\right)^{s}}{\Gamma\left(\frac{k+s}{k}\right)} x^{\frac{s+1}{k}} + O\left(x^{\frac{\alpha}{k^2}}\right). \tag{24}
\]

which means that the statement is also true for \(s + 1\) under the assumption. Hence Theorem 1.1 is proved by induction.

3. Several Notes

Sometimes, we are particularly interested in the case when \(s = k\). For convenience, we rewrite \(r_{k,k}(n)\) as \(r_k(n)\). Then we have

**Corollary 3.1.** If \(k \geq 4\) is an integer, then

\[
\sum_{m=1}^{n} r_k(m) = \Gamma\left(\frac{k+1}{k}\right)^{k} n^{\frac{k}{2}} \cdot \frac{\Gamma\left(\frac{k+1}{k}\right)^{k-1}}{\Gamma\left(\frac{2k-1}{k}\right)} n^{1-\frac{k}{2}} + O\left(n^{\frac{k^2-k-1}{k^2}}\right), \tag{25}
\]

or equivalently,

\[
\sum_{m \leq x} r_k(m) = \Gamma\left(\frac{k+1}{k}\right)^{k} x^{\frac{k}{2}} \cdot \frac{\Gamma\left(\frac{k+1}{k}\right)^{k-1}}{\Gamma\left(\frac{2k-1}{k}\right)} x^{1-\frac{k}{2}} + O\left(x^{\frac{k^2-k-1}{k^2}}\right). \tag{26}
\]

We also notice that the inequality (21) is not necessarily true when \(s \geq k + 1\) and \(0 \leq \alpha \leq \eta\). Therefore, we can not have any further induction from \(s\) to \(s + 1\) when \(s \geq k + 1\). Hence Theorem 1.1 may not be true when \(s\) is large.

References

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