Quantum computation via Floquet topological edge modes

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Floquet topological matter has emerged as one exciting platform to explore rich physics and game-changing applications of topological phases. As one remarkable and recently discovered feature of Floquet symmetry protected topological (SPT) phases, in principle a simple periodically driven system can host an arbitrary number of topological protected zero edge modes and π edge modes, with Majorana zero modes and Majorana π modes as examples protected by the particle-hole symmetry. This work advocates a new route to holonomic quantum computation by exploiting the co-existence of many Floquet SPT edge modes, all of which have trivial dynamical phases during a computation protocol. As compelling evidence supporting this ambitious goal, three pairs of Majorana edge modes, hosted by a periodically driven one-dimensional (1D) superconducting superlattice, are shown to suffice to encode two logical qubits, realize quantum gate operations, and execute two simple quantum algorithms through adiabatic lattice deformation. When compared with early studies on quantum computation based on Majorana zero modes of topological quantum wires, significant resource saving is now made possible by use of Floquet SPT phases. This paper is thus hoped to motivate a series of future studies on the potential of Floquet topological matter in quantum computation.

I. INTRODUCTION

Fault-tolerant quantum computation has been sought as a long term goal towards the development of quantum computers. Potential candidates for this purpose are Majorana zero modes (MZMs) emerging at the vortices or edges of topological superconductors [1–8], which possess topological protection at the hardware level. In such systems, a qubit is encoded nonlocally from a pair of MZMs separated far apart from each other, and quantum gate operations are achieved by braiding them around each other [4]. Due to the constraints put in place by fermionic parity conservation and the number of MZMs that can be generated in a given system, Majorana-based quantum computation usually requires intricate geometry [5–8] to initialize qubits and facilitate braiding between a pair of MZMs, posing some difficulties in scaling it up to solve heavy computational tasks.

It is therefore of fundamental interest to seek innovative and alternative quantum computation schemes with considerable error tolerance on the hardware level. In this work we advocate to exploit an unusual feature of the so-called Floquet topological matter to realize holonomic quantum computation. In recent years Floquet topological matter has emerged as one exciting platform to explore rich physics and potentially game-changing applications of topological phases. In periodically driven systems, energy is no longer a conserved quantity and is replaced by the so-called quasienergy which is only defined modulo $2\pi/T$, with $T$ the driving period. As one recently discovered feature of Floquet symmetry protected topological (SPT) phases, in principle a simple periodically driven system can host an arbitrary number of topological zero edge modes and π edge modes (with quasi-energy 0 and $\pi/T$ respectively) [9–11], with MZMs and Majorana π modes as examples in the presence of the particle-hole symmetry [12–16]. Given that both MZMs and Majorana π modes yield a zero dynamical phase at even multiples of $T$, their coexistence presents a motivating case in reconsidering holonomic quantum computation. As to other parts of a system not directly hosting the edge modes, they can be deemed as auxiliary components, necessary to ensure topology-based fault tolerance inherent in the edge modes and also serving as temporary information storage.

To advocate such a promising marriage between Floquet topological matter and quantum computation, one naturally starts with a one-dimensional (1D) prototype system capable of hosting multiple Floquet Majorana modes. One also hopes that these Floquet Majorana modes are manipulable in order to accomplish braiding between them and consequently quantum gate operations without the need of introducing branched geometries of a quantum wire. To this end, our previous study [16] has moved the first encouraging step by considering a periodically driven topological superconducting wire. In particular, though a 1D static topological superconductor typically hosts only a single MZM at each end, the application of periodic driving can add another pair of Majorana π edge modes, thus yielding the minimal number of Majorana modes required to encode a single qubit [16]. Braiding between the Floquet MZM and Floquet π mode therein and hence single-qubit gate operations were indeed shown to be feasible by using adiabatic lattice deformation alone. It thus becomes necessary and significant to explore the full potential of the coexistence of multiple or even many Floquet topological edge modes hosted by one single quantum wire.

Models with the particle-hole symmetry such as the Kitaev model [1] naturally hosts MZMs and a periodically
driven version may add the Majorana \( \pi \) modes. SPT edge modes due to other symmetries are also of great interest \[17\], but are not directly useful for topologically protected quantum computation. Take, for example, the edge modes in the 1D Su-Schrieffer-Heeger (SSH) \[18\] model. Despite that SSH edge modes are also pinned at zero energy, they are protected by the chiral instead of particle-hole symmetry. Because each SSH edge mode is already fermionic (rather than half-fermionic) in nature, one cannot combine two such edge modes to form a qubit sharing the same feature of Majorana qubits. Nevertheless, the other side of the story is stimulating. That is, a single SSH-like edge mode can be broken down into two Majorana fermions, each of which carries zero energy and is thus an MZM. The same philosophy applies to SSH-like \( \pi \) edge modes. This being the case, a single SSH-like edge mode afforded by the chiral symmetry can be used to encode a (local) qubit. Two such edge modes localized at two opposite ends of a 1D wire are however far apart and their MZM constituents cannot be braided.

Given our general insights above, we construct a working model here with a periodically driven superconducting superlattice with both chiral and particle-hole symmetries. In the absence of periodic driving, such type of quantum wires can host either SSH- or Kitaev-like edge modes \[19, 20\] (that is, SSH- or Kitaev-like edge modes cannot coexist for any given set of system parameters). In the presence of periodic driving, it becomes possible for the SSH- and Kitaev-like edge modes to coexist, one of which is pinned at quasienergy zero, whereas the other is pinned at quasienergy \( \pi/T \). Intriguing quantum gate operations can then be anticipated. Indeed, one pair of SSH-like edge modes, viewed as two pairs of constituent Majorana modes, can now be exploited for information encoding and gate operations because of the possibility of braiding one of the Majorana constituents of a SSH-like edge mode with the other isolated Kitaev-like edge mode at quasienergy \( \pi/T \).

To demonstrate the feasibility of the quantum computation scheme outlined above, we restrict ourselves to the situation where in total three pairs of Majorana edge modes are hosted by a driven quantum wire. After taking into account the fermionic parity conservation, two logical qubits can be constructed. This would be the first time to obtain two logical qubits using the topological edge modes of one single quantum wire, thus considered by us as a theoretical jump from our previous work \[10\] where only a single qubit was obtained with Kitaev-like edge states only. In addition to the explicit construction of the logical qubits based on Floquet topological edge modes, the protocols to accomplish the braiding between different pairs of Majorana modes are one main focus of this paper. We outline a proposal to readout the qubits by breaking the system’s chiral symmetry. We also demonstrate how our quantum computation scheme can be applied to implement two simple quantum algorithms with one single quantum wire. At the end of this work we also discuss how to scale up our quantum computation scheme by explicitly showing how controlled-not (CNOT) gates can be realized with the use of two quantum wires.

This paper is structured as follows. We start in Sec. II A with a short review of Floquet theory to describe time-periodic (Floquet) systems and discuss the emergence of symmetry protected topological edge modes at quasienergy zero and \( \pi/T \). In Sec. II B we adapt the theory of adiabatic processes and holonomy to Floquet systems. We present our model in Sec. III along with its symmetry properties and \( Z \times Z \) topological invariants characterizing the emergence of zero and \( \pi \) edge modes. In Sec. IV we show how the two different species of zero and \( \pi \) edge modes can be written in terms of Majorana operators, which can in turn be used to encode two qubits. In Sec. V we present the application of such edge modes in holonomic quantum computation. In particular, we explicitly develop protocols to realize various single-gate operations by adiabatically deforming the system’s Hamiltonian in various closed cycles, propose a means to readout qubits, demonstrate the implementation of two simple quantum algorithms with our system, and discuss the possibility to scale up our system to generate more logical qubits and construct entangling gates. Section VI discusses possible experimental realization, the feasibility of our proposal with respect to some experimental parameters, and a subtle comparison between our computation protocols with topological quantum computing (TQC). Finally, we conclude our work in Sec. VII.

II. BACKGROUND

A. Floquet Formalism and Edge Modes

Consider a time-periodic (Floquet) Hamiltonian with period \( T \), such that \( H(t + T) = H(t) \). Since energy is no longer conserved, the spectral properties of the system are instead captured by an analogous quantity called quasienergy \[21, 22\], defined from the eigenphase of the one-period propagator (Floquet operator) \( U \equiv \mathcal{T} \text{exp} \left( \int_0^T \frac{-iH(t)}{\hbar} dt \right) \), i.e.,

\[
U|\varepsilon\rangle = \exp \left( -i\varepsilon T \right) |\varepsilon\rangle ,
\]

where \( \mathcal{T} \) is the time-ordering operator, \( \varepsilon \) is the quasienergy, and \( |\varepsilon\rangle \) is the associated Floquet eigenstate. Since \( \varepsilon T \) is only defined up to a modulus of \( 2\pi \), i.e., \( \varepsilon/T \) and \( \varepsilon + 2\pi n/T \) where \( n \in \mathbb{Z} \) represent the same solution. As a result, quasienergy is usually defined in \( (-\pi/T, \pi/T) \) and forms the so-called Floquet Brillouin zone, which is analogous to quasimomentum Brillouin zone in spatially periodic systems. The periodicity of the quasienergy Brillouin zone is mainly responsible for the existence of edge modes at quasienergy \( \pi/T \) \[12, 14–16, 23–25\] and anomalous edge states \[26, 28\]. The former is especially relevant to this work, and will thus be elaborated further.
There are two types of edge modes, namely, fermionic and Majorana (half-fermionic) edge modes. In the second quantization language, we define $\Psi_\pi$ as a fermionic mode associated with quasienergy $\varepsilon/T$. Namely, given a reference state $|R\rangle$ satisfying $U|R\rangle = |R\rangle$, a Floquet eigenstate with quasienergy $\varepsilon/T$ can be constructed as $|\varepsilon\rangle = \Psi_\pi |R\rangle$.

In systems possessing chiral symmetry [9, 29, 30] with $U|\Gamma\rangle = U\Gamma$ for some unitary chiral operator $\Gamma$, quasienergies are guaranteed to come in pairs. That is, associated with a fermionic mode $\Psi_\pi$ at quasienergy $\varepsilon/T$, there exists another fermionic mode $\Psi_{-\pi} = \Psi_\pi^\dagger \Gamma$ at quasienergy $-\varepsilon/T$. In particular, when $\varepsilon = 0 (\pi/T)$, chiral symmetry dictates that the quasienergy becomes degenerate, i.e., there must exist two fermionic zero ($\pi$) modes $\Psi_0^A$ and $\Psi_0^B$ ($\Psi_\pi^A$ and $\Psi_\pi^B$) related to each other by $\Psi_0^A = \Psi_0^B \Gamma$ ($\Psi_\pi^A = \Psi_\pi^B \Gamma$).

On the other hand, superconducting systems usually also possess an inherent particle-hole symmetry. This associates a fermionic mode $\Psi_\pi$ at quasienergy $\varepsilon/T$ with the conjugate of another fermionic mode $\Psi_{-\pi}$ at quasienergy $-\varepsilon/T$, i.e., $\Psi_{-\pi} = \Psi_\pi^\dagger$. As a direct consequence, $\gamma_0 \equiv \Psi_0$ and $\gamma_\pi \equiv \Psi_\pi$ become Hermitian, and are thus termed Majorana zero and $\pi$ modes respectively. Since the Floquet operator $U$ (when expanded) can only contain terms of the form $\Psi^\dagger \Psi$, where $\Psi$ is a complex fermion, Majorana zero ($\pi$) modes should come in pairs as $\gamma_0^{(1)}$ and $\gamma_0^{(2)}$ ($\gamma_\pi^{(1)}$ and $\gamma_\pi^{(2)}$) so as to be able to form a complex fermion $\Psi_0^{(1)} = \gamma_0^{(1)} + i \gamma_0^{(2)} (\Psi_\pi^{(1)} = \gamma_\pi^{(1)} + i \gamma_\pi^{(2)})$. In this sense, Majorana zero and $\pi$ modes are clearly half-fermions and fundamentally different from the fermionic zero and $\pi$ modes induced by the less subtle chiral symmetry alone.

Figure 1 depicts zero and $\pi$ edge modes when a gapped system is subject to open boundaries. In particular, fermionic and Majorana modes highlighted above are localized near the systems’ left or right boundaries. By definition, one quickly arrives that fermionic or Majorana zero modes commute with the Floquet operator $U$, whereas fermionic or Majorana $\pi$ modes anticommute with the Floquet operator $U$. Though not pursued in this work, we note that the $\pi$ edge modes being anticommuting with $U$ offers a dynamical-decoupling scenario from within the system dynamics itself and they are thus expected to be even more robust than zero edge modes against certain noise. It should be also noted that while fermionic and Majorana zero modes can also emerge in static systems by the same mechanism elucidated above, fermionic and Majorana $\pi$ modes can only exist in Floquet systems due to the periodicity of quasienergy.

**B. Floquet adiabatic process and holonomy**

Let $H(t, \lambda)$ be time-periodic with period $T$ and depending also on a tunable parameter $\lambda$. If Floquet eigenstates are not degenerate, then a Floquet adiabatic process is accomplished by slowly tuning $\lambda$ from a certain initial value $\lambda_0$ to a final value $\lambda_f$ at time $t = M T$, such that a state initially prepared in a Floquet eigenstate with quasienergy $\varepsilon_n(\lambda_0)$ will evolve with $\lambda$ as an instantaneous Floquet eigenstate with quasienergy $\varepsilon_n(\lambda)$ [31]. It is convenient to assume that $\lambda$ is only tuned stroboscopically at the beginning of each new driving period, such that $\lambda \equiv \lambda(s)$ when $st \leq t < (s + 1) T$. Adiabaticity then requires $\tau/T = M \gg 1$ as well as other conditions involving the gap of the Floquet states versus $\hbar/T$ [31].

Floquet adiabatic holonomy arises from a Floquet adiabatic process in which $H(t, \lambda_f) = H(t, \lambda_0)$ and its associated Floquet operator $U(\lambda)$ always possesses degenerate Floquet states throughout the adiabatic cycle. For each quasienergy $\varepsilon_n$, we can thus define a column vector containing all of its degenerate Floquet eigenstates as $|\varepsilon_n\rangle \equiv (|\varepsilon_{n,1}\rangle, \cdots, |\varepsilon_{n,k_n}\rangle)^T$, where $k_n$ is the number of degeneracy associated with $\varepsilon_n$. As detailed in Appendix A the evolution of a Floquet eigenstate $|\varepsilon_n\rangle$ of $U(\lambda_0)$ after one adiabatic cycle is given by

$$|\varepsilon_n(\lambda_f)\rangle = P \exp \left( -i \int [A_n + \Omega_n + \varepsilon_n T] d\lambda \right) |\varepsilon_n(\lambda_0)\rangle ,$$

where $P$ is the path ordering operator, $A_n$ and $\Omega_n$ are defined in Appendix A and the closed integration is used since the Hamiltonian returns to itself after one adiabatic cycle.
The first term in the exponential of Eq. (2) is the non-Abelian Berry matrix, while the second term represents the explicit monodromy \[ \exp \left( \sum \frac{\pi}{T} \epsilon_n \sigma_3 \right) \], i.e., permutation/braiding in the degenerate subspace, induced by the holonomy. The summation of the first two terms gives rise to the total non-Abelian geometric phase of the system, whereas the last term denotes the dynamical phase contribution. In particular, since the geometric phase appears as a matrix, Eq. (2) may in general induce a nontrivial rotation of \( |\epsilon_n(\lambda_0)\rangle \) within the degenerate subspace, so that \( |\epsilon_n(\lambda_0)\rangle \) and \( |\epsilon_n(\lambda_1)\rangle \) are not simply related by an overall phase as in the nondegenerate (Abelian) case. This is the basic idea behind holonomic and topological quantum computation (HQC and TQC), which we have now extended to Floquet systems. Equation (2) also makes it clear why holonomic quantum computation with topologically zero modes and \( \pi \) modes are of special interest: the dynamical phase contribution can be clearly separated out if the zero or \( \pi \) modes persist throughout the adiabatic process. That is, the dynamical phase \( -\int_{\lambda_0}^{\lambda} \epsilon_n \tau d\lambda = M\epsilon_n T \) is equivalent to zero given that \( \epsilon_n = 0 \) or \( \pi / T \) and that \( M \) is even (that is, if the adiabatic process takes even multiples of driving periods).

III. DESCRIPTION OF THE MODEL

The general model we will be using throughout this work describes a 1D time-periodic \( p \)-wave superconducting superlattice with alternating real and imaginary hopping as well as pairing at every half period, i.e.,

\[
H(t) = \begin{cases} 
H_1 & \text{for } (m - 1)T < t \leq (m - 1/2)T \\
H_2 & \text{for } (m - 1/2)T < t \leq mT 
\end{cases},
\]

\[
H_1 = \sum_i \left( -J_{\text{intra},i} c_{A,i}^\dagger c_{A,i} - J_{\text{inter},i} c_{A,i+1}^\dagger c_{B,i} + \Delta_{\text{intra},i} c_{A,i}^\dagger c_{A,i} + \Delta_{\text{inter},i} c_{A,i+1}^\dagger c_{B,i} + h. c. \right),
\]

\[
H_2 = \sum_i \left( -\tilde{J}_{\text{intra},i} c_{B,i}^\dagger c_{A,i} - \tilde{J}_{\text{inter},i} c_{A,i+1}^\dagger c_{B,i} + \tilde{\Delta}_{\text{intra},i} c_{B,i}^\dagger c_{A,i} + \tilde{\Delta}_{\text{inter},i} c_{A,i+1}^\dagger c_{B,i} + h. c. \right),
\]

where \( c_{A,i} \) (\( c_{B,i} \)) and \( c_{A,i}^\dagger \) (\( c_{B,i}^\dagger \)) denote the fermion creation and annihilation operators at sublattice A (B) of lattice site \( i \) respectively, \( J_{\text{intra},i}, J_{\text{inter},i}, \tilde{J}_{\text{intra},i}, \) and \( \tilde{J}_{\text{inter},i} \) denote intra- and inter-lattice hopping strength at site \( i \) at different half of the period, \( \Delta_{\text{intra},i}, \Delta_{\text{inter},i}, \delta_{\text{intra},i}, \) and \( \delta_{\text{inter},i} \) are the intra- and inter-lattice pairing strength at site \( i \) at different half of the period. The total number of lattice sites is denoted as \( N \), which is finite in our actual calculations under open boundary conditions. By construction, \( T \) is the time period of the above periodically-quenched Hamiltonian. Unless otherwise specified, we take \( J_{\text{intra},i} = J_1, J_{\text{inter},i} = J_2, \Delta_{\text{intra},i} = \Delta_1, \Delta_{\text{inter},i} = \Delta_2, \delta_{\text{intra},i} = \delta_1, \) and \( \delta_{\text{inter},i} = \delta_2 \) for all \( i = 1, \ldots, N \). Each of \( H_1 \) or \( H_2 \) itself depicted in Eq. (4) represents a static dimerized Kitaev chain. In the absence of sublattice degree of freedom, i.e., by taking \( J_1 = J_2, \Delta_1 = \Delta_2, J_1 = j_2, \) and \( \delta_1 = \delta_2 \), Eq. (3) reduces to a time-periodic Kitaev Hamiltonian, which is known to possess Majorana zero edge modes [1] under suitable parameter values. Due to the sublattice degree of freedom, the SSH-like zero (quasi) energy edge modes [18] are also expected.

In general, Kitaev- and SSH-like zero edge modes will compete with each other, and only one of them can exist for a given set of system parameters. This competition can be well understood in terms of an integer topological invariant [15, 19, 20]. On the other hand, since our system is periodically quenched, Kitaev- or SSH-like edge modes at quasienergy \( \pi / T \) may also exist, which are governed by a separate integer topological invariant [43]. As a result, while only one type of edge modes can emerge at quasi-energy zero or \( \pi / T \), it is possible to find certain parameter windows for which two different types of edge modes coexist, one at quasienergy zero, while the other at quasienergy \( \pi / T \) [14, 15].

A. Symmetry analysis

To gain more insights into our working model, we first rewrite Eq. (4) in the Nambu-momentum representation as follows:

\[
H_1 = \sum_{k \geq 0} \Psi_k^\dagger h_{1,k} \Psi_k ,
\]

\[
h_{1,k} = -\tau_x J(k) \cdot \sigma + \tau_y \Delta(k) \cdot \sigma ,
\]

\[
h_{2,k} = -\tilde{J}(k) \cdot \sigma + \tau_x \delta(k) \cdot \sigma .
\]

where \( \Psi_k \) is given by \( \left( c_{A,k}^\dagger, c_{B,k}^\dagger, c_{A,-k}, c_{B,-k} \right) \), \( l \in \{1, 2\} \), \( \sigma_i \) and \( \tau_i \) are Pauli matrices in the sublattice and particle-hole degrees of freedom respectively. Other terms used above are given by

\[
J(k) \cdot \sigma = (J_1 + J_2 \cos k) \sigma_x - J_2 \sin k \sigma_y ,
\]

\[
j(k) \cdot \sigma = (j_1 - j_2 \cos k) \sigma_y - j_2 \sin k \sigma_x ,
\]

\[
\Delta(k) \cdot \sigma = (\Delta_1 - \Delta_2 \cos k) \sigma_y - \Delta_2 \sin k \sigma_x ,
\]

\[
\delta(k) \cdot \sigma = (\delta_1 - \delta_2 \cos k) \sigma_y - \delta_2 \sin k \sigma_x .
\]

To analyze the symmetry, it is convenient to consider the momentum space Floquet operator in a symmetric time frame [39, 40, 41, 42] as

\[
U_k = \hat{F}_k \hat{G}_k ,
\]

\[
\hat{F}_k = \exp (-ih_{1,k} T/4) \times \exp (-ih_{2,k} T/4) ,
\]

\[
\hat{G}_k = \exp (-ih_{2,k} T/4) \times \exp (-ih_{1,k} T/4) .
\]
It can be checked that Eq. (7) possesses sublattice chiral symmetry since \( \Gamma \hat{F}_k \Gamma^\dagger = \hat{G}_k^\dagger \) with \( \Gamma = \sigma_z \). As expected for a typical superconducting system, Eq. (7) also possesses particle-hole symmetry given by \( \mathcal{P} U_k \mathcal{P}^\dagger = U_{-k} \), where \( \mathcal{P} = \tau_x \mathcal{K} \) and \( \mathcal{K} \) is the complex conjugation operator [15, 34]. The presence of both chiral and particle-hole symmetries also implies the existence of time reversal symmetry dictated by the operator \( T = \sigma_z \tau_x \mathcal{K} \), which is easily verified in the symmetric time frame according to \( T h(k,t)T^{-1} = h(-k, T - t) \), where \( h(k,t) \) is the full time-dependent Hamiltonian depicted by Eq. (4) in the momentum space [15, 34]. Our working system thus belongs to the BDI class according to the Altland-Zirnbauer classification scheme [35], which is characterized by a \( Z \times Z \) topological invariant [34].

**B. \( Z \times Z \) topological invariant**

As a result of the chiral symmetry, we can identify the \( Z \times Z \) topological invariants by combining some techniques from Ref. [19, 29, 30]. First, we change Eq. (5) to a canonical basis [19] by applying a unitary transformation with

\[
U = \frac{1 + \sigma_x}{2} (1 - \sigma_x) \times \frac{1 + \tau_x}{2} (1 - \tau_x) ,
\]

so that \( U^\dagger \Gamma U = \tau_z \). Next, we follow Ref. [30] and write \( \hat{F}_k \) in this basis as a block matrix, i.e.,

\[
\hat{F}_k \doteq \begin{pmatrix} a(k) & b(k) \\ c(k) & d(k) \end{pmatrix} ,
\]

where each block is a \( 2 \times 2 \) matrix.

The number of edge states at quasienergy zero and \( \pi \) can then respectively be found by calculating the topological invariants [30]

\[
\nu_0 = \frac{1}{2\pi i} \int_{-\pi}^{\pi} dk \text{Tr} \left( b^{-1} \frac{d}{dk} b \right) ,
\]

\[
\nu_\pi = \frac{1}{2\pi i} \int_{-\pi}^{\pi} dk \text{Tr} \left( d^{-1} \frac{d}{dk} d \right) .
\]

The topological invariants \( \nu_0 \) and \( \nu_\pi \) under some representative parameter values are depicted in Fig. 2 along with their associated Floquet eigen-spectrum under open boundary conditions (OBC). There, \( \nu_0 = 1 \) (\( \nu_\pi = 1 \)) is associated with the presence of Kitaev-like edge states at quasienergy zero (\( \pi/T \)), which predicts only one Majorana zero (\( \pi \)) edge mode at each edge. On the other hand, SSH-like edge states, being complex-fermionic in nature, can be broken down into two Majorana zero (\( \pi \)) modes and thus emerge whenever \( \nu_0 = 2 \) (\( \nu_\pi = 2 \)). The parameter window for which \( \nu_0 = 2 \) and \( \nu_\pi = 1 \) will be used in this work, because the coexistence of Kitaev- and SSH-like edge states will prove to be essential for our encoding and manipulation of the logical qubits we obtain.

**IV. EDGE-MODES BASED QUBIT ENCODING**

**A. Edge modes in the Majorana representation**

In Sec. III B, we have shown that for certain parameter windows, two different species of edge modes, originating from sublattice and particle-hole symmetry-protected topology respectively, may coexist on one single quantum wire. To elucidate on the application of these edge modes for quantum computation, it is convenient to first define (Hermitian) Majorana operators as follows,

\[
\gamma_{s,i}^\alpha = c_{s,i} + c_{s,i}^\dagger ,
\]

\[
\gamma_{s,i}^\beta = i \left( c_{s,i} - c_{s,i}^\dagger \right) ,
\]

where \( s \in \{ A, B \} \) and \( i = 1, \cdots , N \). Moreover, to simplify our analysis, we will take the following parameter values: \( j_1 = \delta_1 = \delta_2 = 0 \), \( J_1 T = J_2 T = \Delta_1 T = \Delta_2 T = \pi/4 \), and \( j_3 T = 2\pi \), which from here onwards shall be referred to as the **ideal case**. It should be stressed however that such fine tuning is not necessary in the actual implementation, and the results we present in the following still hold under small deviations from the ideal case.

In the ideal case, Eq. (4) can be written in terms of Majorana operators as
where \( n_\pi, n_0^L, n_0^R \in \{0, 1\} \). There are now 8 simultaneous eigenstates of \( \mathcal{P}_\pi, \mathcal{P}_0^L, \) and \( \mathcal{P}_0^R \). The total parity conservation divides this eight-dimensional Hilbert space into two four-dimensional parity preserving subspaces. The odd and even parity subspaces are respectively spanned by \( \{001\}, \{010\}, \{100\}, \{111\} \) and \( \{000\}, \{011\}, \{110\}, \{101\} \). Without loss of generality, in this work we assume that the system is initialized in the even parity subspace. This allows us to define the four qubit basis states with \( |00\rangle \equiv |000\rangle, |01\rangle \equiv |011\rangle, |10\rangle \equiv |110\rangle, \) and \( |11\rangle \equiv |101\rangle \). These four qubit states, which represent the basis states of two logical qubits, are related to each other by

\[
|01\rangle = \gamma_0^L \gamma_0^R |00\rangle,
|10\rangle = \gamma_0^L \gamma_0^R |10\rangle,
|11\rangle = \gamma_0^L \gamma_0^R |11\rangle.
\]  

(17)

V. HOLONOMIC QUANTUM COMPUTATION WITH EDGE MODES

Having shown how logical qubits can be encoded in our system, we now investigate which logical gate operations can be implemented. For Majorana-based qubits, topologically protected gate operations can be carried out through braiding between a pair of Majorana modes\(^4\). Assuming that all pairs of Majorana modes in a given system can be braided, all Clifford, i.e., Hadamard, CNOT, and phase, gates can in principle be implemented\(^{11,37,38}\). However, in many proposed systems hosting Majorana modes, especially those in 1D setups, braiding some pairs of Majorana modes may be challenging, especially if they are separated too far apart. For example, in 1D systems such as those studied in Ref.\(^{10,39}\), a single qubit requires two pairs of Majorana modes located at two opposite edges. As such, braiding one Majorana mode from one edge with that from the other edge may be quite difficult to carry out in practice, which in turn hinders the realization of universal quantum computation.

Recognizing that relying exclusively on nonlocal Majorana qubits is still a big challenge for quantum computation purposes, our qubit encoding scheme outlined in the previous subsection represents a hybrid scenario with both local and nonlocal fermions. The advantage of involving local fermions in our encoding is that it allows more pairs of Majorana modes to be easily braided. As seen below, this feature leads to the implementation of a larger set of gate operations, at least in principle. In the
FIG. 3: (color online). Schematic of the holonomic protocol to braid $\gamma_{0,1}^L$ and $\gamma_{0,2}^L$. Only the first two lattice sites are shown. Red and blue ellipses represent sublattice A and B respectively, with two circles at each ellipse are the associated Majorana operators. Coloured circles denote the Majorana modes as described in the inset. Some Majorana modes are superposition of two Majorana operators, which are represented by half-coloured circles. Black solid and gray dashed lines denote the coupling between two Majorana operators due to $H_2$ and $H_1$ respectively.

following, we explicitly present the protocols to implement some gate operations by braiding between different pairs of Majorana modes. This is done by adiabatically deforming the system’s Hamiltonian in closed cycles, in the spirit of holonomic quantum computation [40, 41].

A. Phase gate and Pauli $Z$ gate

With the two-qubit encoding introduced in Sec. IV B, single phase gate and Pauli $Z$ gate (up to a global phase factor) on the first or second qubit individually can be obtained by braiding $\gamma_{0,1}^L$ and $\gamma_{0,2}^L$ or braiding $\gamma_{0,1}^R$ and $\gamma_{0,2}^R$ once and twice respectively. In terms of unitaries $P_s \equiv U_s = \exp \left[ \left( \pi/4 \right) \gamma_{0,2}^s \gamma_{0,1}^s \right]$ and $Z_s \equiv U_s^2 = \exp \left[ \left( \pi/2 \right) \gamma_{0,2}^s \gamma_{0,1}^s \right]$, where $s = L$ ($s = R$). This can be verified by applying $P_s$ and $Z_s$ directly to Eq. (17), with the obvious identity

$$\exp \left( \theta \gamma_{0,1}^s \gamma_{0,2}^s \right) = \cos \theta + \sin \theta \gamma_{0,1}^s \gamma_{0,2}^s \cdot \quad (18)$$

Indeed, identity Eq. (18) is also needed to verify that $U_s^\dagger \gamma_{0,1}^s U_s = -\gamma_{0,2}^s$ and $U_s^\dagger \gamma_{0,2}^s U_s = \gamma_{0,1}^s$, which is precisely a braiding unitary.

We now present below the details of our protocol to realize the braiding unitary $U_s$. In order to simplify our discussion, we focus on the ideal case, which allows us to keep track of the analytical solutions at the end of each step. As will be shown in our numerics later on, however, the result of our protocol still holds even if we tune the system parameters away from the ideal case. Furthermore, we will only present the protocol to braid $\gamma_{0,1}^L$ and $\gamma_{0,2}^L$ (to find $U_L$). Braiding $\gamma_{0,1}^R$ and $\gamma_{0,2}^R$ can be accomplished in the same fashion, by applying our considerations to the right edge instead. For each step elaborated below, the adiabatic parameter $\phi$ is slowly increased at the beginning of each driving period, starting from 0 and ending at $\pi/2$ after a total of even number of driving periods. We only briefly elucidate the output of each step, thus leaving more technical details in Appendix B.

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**Step 1.**—Adiabatic deformation of $H_1$ and $H_2$ is introduced by setting $(J_{\text{inter},1} + \delta_{\text{inter},1}) T = 2\pi$, $$(J_{\text{intra},1} + \Delta_{\text{intra},1}) T = \pi$$ and slowly tuning $$(J_{\text{inter},1} - \delta_{\text{inter},1}) T = 2\pi \cos \phi, \quad J_{\text{intra},1} T = -\delta_{\text{intra},1} T =$$
\[ \pi \sin \phi, \quad J_{\text{inter}, 1} T = \Delta_{\text{inter}, 1} T = \frac{\pi}{2} \cos \phi, \quad \text{and} \quad (J_{\text{intra}, 1} - \Delta_{\text{intra}, 1}) T = -\pi \sin \phi \] with \( \phi \) always being the adiabatic parameter in all the steps. The net outcome of this step is to move \( \gamma_{1,0,1}^L \) to the second lattice site, i.e., changing \( \gamma_{A,1}^L \) to \( \gamma_{A,2}^L \).

**Step 2.**—With \( H_1 \) untouched, adiabatic deformation is applied to \( H_2 \) by letting \( J_{\text{inter}, 1} T = \delta_{\text{inter}, 1} T = \pi \cos \phi \), \( (J_{\text{inter}, 1} + \delta_{\text{intra}, 1}) T = 2\pi \sin \phi \) under \( (J_{\text{inter}, 1} - \delta_{\text{intra}, 1}) T = 2\pi \). This results in moving \( \gamma_{0,2,1}^L \) and \( \gamma_{\pi}^L \) to the second lattice site, i.e.,
\[
\frac{1}{\sqrt{2}} \left( \gamma_{A,1}^L \pm \gamma_{B,1}^L \right)
\]
to
\[
\frac{1}{\sqrt{2}} \left( \gamma_{A,2}^L \pm \gamma_{B,2}^L \right).
\]

**Step 3.**—With \( (J_{\text{intra}, 1} + \delta_{\text{intra}, 1}) T = 2\pi \), adiabatic manipulation is applied to \( (J_{\text{intra}, 1} - \delta_{\text{intra}, 1}) T = 2\pi \cos \phi \), \( J_{\text{inter}, 1} T = -\delta_{\text{inter}, 1} T = -i\pi \sin \phi \). At the end of the step, \( \gamma_{\pi}^L = \frac{1}{\sqrt{2}} \left( \gamma_{A,1}^L - \gamma_{B,1}^L \right) \) and \( \gamma_{0,2,1}^L = \frac{1}{\sqrt{2}} \left( \gamma_{A,1}^L + \gamma_{B,1}^L \right) \). That is, \( \gamma_{\pi}^L \) and \( \gamma_{0,2,1}^L \) return to the first lattice site, but they have transformed to different superpositions of Majorana operators.

**Step 4.**—This step amounts to separating \( \gamma_{0,2,1}^L \) from \( \gamma_{\pi}^L \), which is accomplished by tuning \( (J_{\text{intra}, 1} + \Delta_{\text{intra}, 1}) T = \pi \), and \( (J_{\text{intra}, 1} - \Delta_{\text{intra}, 1}) T = -\pi \cos \phi \), such that \( \gamma_{\pi}^L = -\gamma_{B,1}^L \) and \( \gamma_{0,2,1}^L = \gamma_{A,1}^L \) at the end of the step.

**Step 5.**—In this step, \( \gamma_{0,1,1}^L \) and \( \gamma_{\pi}^L \) are turned into superpositions of two Majorana operators. This is done by tuning \( J_{\text{intra}, 1} T = -\delta_{\text{inter}, 1} T = -\pi \exp \left[ i(\pi/2 + \phi) \right] \) and \( J_{\text{inter}, 1} T = \Delta_{\text{inter}, 1} T = \frac{\pi}{2} \sin \phi \), which leads to
\[
\gamma_{\pi}^L = \frac{1}{\sqrt{2}} \left( \gamma_{A,2}^L - \gamma_{B,2}^L \right) \quad \text{and} \quad \gamma_{0,1,1}^L = -\frac{1}{\sqrt{2}} \left( \gamma_{A,2}^L + \gamma_{B,2}^L \right)
\]
at the end of the step.

**Step 6.**—Finally, \( H_1 \) and \( H_2 \) are returned to their original forms. This is done by tuning \( j_{\text{inter}, 1} T = \delta_{\text{inter}, 1} T = \pi \cos \phi \), \( (J_{\text{inter}, 1} + \delta_{\text{inter}, 1}) T = 2\pi \sin \phi \), and \( (J_{\text{inter}, 1} - \delta_{\text{inter}, 1}) T = 2\pi \), which results in
\[
\gamma_{\pi}^L = \frac{1}{\sqrt{2}} \left( \gamma_{A,1}^L - \gamma_{B,1}^L \right) \quad \text{and} \quad \gamma_{0,1,1}^L = -\frac{1}{\sqrt{2}} \left( \gamma_{A,1}^L + \gamma_{B,1}^L \right)
\]
at the end of the step.

In the Majorana representation, the above six steps, as depicted in Fig. 3, result in the braiding transformation \( \gamma_{0,1,1}^L \rightarrow -\gamma_{0,2}^L \) and \( \gamma_{0,2,1}^L \rightarrow \gamma_{0,1,1}^L \), while leaving the other Majorana modes invariant. We have thus achieved the braiding unitary \( U_L \) necessary to construct \( P_L \) and \( Z_L \) gates as claimed above. Figures 4(a) and 4(b) depict computational examples via the evolution of Majorana correlation functions between the three involved Majorana modes in the protocol. There, the initial state is chosen to be \(|+\rangle = 1/\sqrt{2} \left( |01\rangle + |10\rangle \right) \), so that \( \langle \gamma_{0,1,1}^L \psi' \rangle = \langle \gamma_{0,2}^L \gamma_{0,1,1}^L \rangle = 1 \), with any other cross correlation functions found to be zero. The success of the protocol is signified by the change in the cross correlations \( \langle \gamma_{0,1,1}^L \gamma_{0,2,1}^L \rangle \) and \( \langle \gamma_{0,2,1}^L \gamma_{0,1,1}^L \rangle \), which become 1 or -1 at the end of the protocol. The shown correlation functions in the computational example confirm the successful implementation of the braiding unitaries \( U_L \) and \( U_R \). It should be emphasized that the system parameters used in the computational example have been tuned away from the ideal case, so fine tuning of the system parameters is indeed unnecessary. In addition, Fig. 4(c) verifies that zero and \( \pi \) edge modes maintain a large quasienergy gap from the bulk states throughout the computation protocol. This spectral feature is necessary to guarantee the adiabatic processes needed without leaking population to the bulk states.

---

**B. Hadamard gate and Pauli X gate**

Upon implementation of phase gate and Pauli Z gate, let us now attempt to realize Hadamard gate (H) gate and Pauli X gate with braiding operations again. The braiding is now between \( \gamma_{\pi}^L \) and \( \gamma_{0,2}^L \) or between \( \gamma_{\pi}^R \) and \( \gamma_{0,1}^R \). It is again straightforward to verify, by using the encoding relations in Eq. 17, that \( V_s = H_s Z_s = \exp \left[ i\left(\pi/4\right) \gamma_{A,1,0}^L \right] \) and \( X_s = V_s^L = \exp \left[ i\left(\pi/2\right) \gamma_{A,1,0}^L \right] \), where \( s = L \) (\( s = R \)) for the first (second) qubit. That is, the braiding unitary \( V_s \) realizes the product of the Hadamard gate and the Z gate, which can be further used to realize the X gate.

In order to braid \( \gamma_{\pi}^L \) and \( \gamma_{0,2}^L \) with different quasienergy values, we adopt one remarkable technique introduced by us earlier in Ref. 10. In short, both \( \gamma_{\pi}^L \) and \( \gamma_{0,2}^L \) modes can be regarded as topological zero edge modes of \( U^L \), namely, the Floquet operator associated with every two periods. This insight suggests that if we introduce adiabatic deformation of the quantum wire Hamiltonian every other period, then it is possible to induce rotation in the “degenerate” subspace formed by \( \pi \) and zero edge modes. As before, here we only present our protocol to braid \( \gamma_{0,2}^L \) and \( \gamma_{\pi}^L \) in the ideal case. Braiding \( \gamma_{\pi}^R \) and \( \gamma_{0,1}^R \) can be likewise achieved by tuning instead the appropriate hopping and pairing strength near the right end of the quantum wire. At each step below, except for steps 3 and 5, the adiabatic parameter \( \phi \) is slowly tuned at the beginning of each period from 0 to \( \pi/2 \). We again leave more technical details in Appendix C.

**Step 1.**—In this step, \( \gamma_{0,2}^L \) and \( \gamma_{\pi}^L \) are moved to the \( n + 1 \)-th lattice site. In order to reduce unwanted non-Abelian rotation between the two degenerate modes \( \gamma_{0,1}^L \) and \( \gamma_{0,2}^L \), it is better to take large \( n \) > 2. Certainly the value of \( n \) is also limited by the actual lattice size in order to avoid potential overlap with Majorana modes at the right edge. Thanks to our analytical solution of the edge modes for the ideal case (detailed in Appendix B), we find that this step can be easily carried out by adiabatically tuning \( J_{\text{inter}, k} + \delta_{\text{inter}, k} T = 2\pi \cos \phi \), and \( j_{\text{intra}, k} T = \delta_{\text{intra}, k} T = 2\pi \sin \phi \), with
In the subspace spanned by zero and π edge modes, both regarded as zero modes of $U^\pi$, this is accomplished by adiabatic manipulation of the system Hamiltonian every other period. The final outcome of this step is that $\gamma_0^{\pi,2}$ and $\gamma_2^{\pi}$ become superpositions of Majorana zero and π modes.

System parameters to be tuned in this manner are $VT = \pi (1 - f(a))$, $j_{\text{intra,n}}T = \delta_{\text{intra,n}}T = \frac{\pi}{2} (1 - f(a))$, and $j_{\text{inter,2}}T = \pi (1 + f(a))$, where $VT$ is the potential bias introduced in step 2, $f(a)$ is a rather arbitrary function which increases from $-1$ to $1$ as $a$ is adiabatically tuned. Similar to the protocol in our early work [10], such adiabatic protocol induces a $\pi/4$ rotation between zero and π edge modes, resulting in $\gamma_0^{\pi} = \gamma_A^{\beta,n}$ and $\gamma_0^{\pi} = \gamma_B^{\alpha,n}$. While we keep the same notations as before, it should be noted that $\gamma_0^{\pi}$ and $\gamma_2^{\pi}$ are no longer Majorana π and zero modes with respect to $U$. Instead, at the end of this step they are only Majorana zero modes of $U^\pi$ [10].

Step 4.—We further tune the system according to $(j_{\text{inter,n}} + \delta_{\text{inter,n}})T = 2\pi \cos \phi$, $(j_{\text{inter,n}} - \delta_{\text{inter,n}})T = 2\pi$, and $j_{\text{intra,n}}T = \delta_{\text{intra,n}}T = \pi \sin \phi$. This results in moving $\gamma_0^{\pi}$ and $\gamma_2^{\pi}$ to $\gamma_{A,n+1}^{\alpha}$ and $\gamma_{B,n+1}^{\alpha}$ respectively.
FIG. 5: (color online). Schematic of the holonomic protocol to braid $\gamma_\pi^L$ and $\gamma_0^L$. Only the first three lattice sites are shown. Blue coloured circles denote the two-period Majorana modes due to the superposition of Majorana zero and $\pi$ modes. The meaning of the other symbols are the same as those in Fig. 3.

**Step 5.—** Step 5 is identical to step 2, which moves $\gamma_\pi^L$ and $\gamma_0^L$ to $\gamma_A^L$ and $-\gamma_A^L$, respectively.

**Step 6.—** Step 6 is identical to step 3. Because $\gamma_\pi^L$ and $\gamma_0^L$ are already superposition Majorana zero and $\pi$ modes, another $\pi/4$ rotation between the zero and $\pi$ modes leads to $\gamma_\pi^L = \frac{1}{\sqrt{2}} (\gamma_A^L + \gamma_B^L$ ) and $\gamma_0^L = \frac{1}{\sqrt{2}} (\gamma_B^0 - \gamma_A^0)$. Interestingly, one also finds that at the end of this step, $\gamma_\pi^L$ and $\gamma_0^L$ have again become Majorana zero and $\pi$ modes of $\mathcal{U}$.

The seven steps above are depicted in Fig. 5 with the net outcome $\gamma_0^L \rightarrow -\gamma_0^L$ and $\gamma_\pi^L \rightarrow \gamma_\pi^L$. This is a highly nontrivial and encouraging result because the braiding is accomplished while the rest of the Majorana zero modes remain unaffected. Using again computation examples with the same initial state and parameter values as those in Sec. IV A, we show the Majorana correlation functions in Figs. 4(c) and (d). There, we take $n = 4$ in step 1 and $f(a) = \cos (a\pi)$ in step 3 and step 6, where $a$ decreases slowly from 1 to 0. The success of the protocol is signified by the change in cross correlations $\langle i\gamma^R_{\pi,0,0,2} \rangle$ and $\langle i\gamma^L_{\pi,0,0,2} \rangle$ (where $\langle \cdot \rangle$ denotes average over lattice sites) to 1 or $-1$ for braiding between $\gamma_0^L$ and $\gamma_\pi^L$ ($\gamma_0^R$ and $\gamma_\pi^R$). Note also that Fig. 4(e) (which is the initial $\gamma_0^L$ multiplied by $-1$).

**Step 7.—** As the final step, we need to return the Hamiltonian to its original form. This is done by tuning $(\delta_{\text{inter},k} + \delta_{\text{intr},k})T = 2\pi \sin \phi$, $(\delta_{\text{inter},k} - \delta_{\text{intr},k})T = 2\pi$, and $\delta_{\text{intr},k}T = \delta_{\text{intr},k}T = 2\pi \cos \phi$, where $k = 1, 2, \ldots, n$. This step also moves $\gamma_\pi^L$ and $\gamma_0^L$ back to the first site. In particular, we find that $\gamma_\pi^L \rightarrow \frac{1}{\sqrt{2}} (\gamma_A^L + \gamma_B^L$ ) (which is the initial $\gamma_0^L$ and $\gamma_0^L \rightarrow \frac{1}{\sqrt{2}} (\gamma_A^0 - \gamma_B^0$ ) (which is the initial $\gamma_\pi^L$ multiplied by $-1$).

**C. Qubit readout**

The last step in a typical quantum computation task is to readout qubits, which allows one to confirm that a sequence of gate operations applied on an input qubit indeed gives the intended outcome. Our system uses three physical qubits to encode two logical qubits. As elucidated in Sec. IV B, two of these three physical qubits originate from the chiral symmetry protected edge states at both ends of the lattice. By systematically introducing a chiral symmetry breaking term in the Hamiltonian, the degeneracy of these two edge states can then be lifted, which thus allows one to distinguish between the four logical-qubit states according to their quasienergy values.

To be more explicit, we may add the following symmetry-breaking terms to the Hamiltonian in Eq. (3),

$$H_{\text{break}} = \sum_i \left[ (\mu_1 + \mu_2) c_{A, i}^d c_{A, i} + (\mu_1 - \mu_2) c_{B, i}^d c_{B, i} \right].$$

(19)

It can be easily verified that $H_{\text{break}}$ violates the chiral symmetry defined in Sec. III A. In particular, $\mu_1$ shifts the quasienergy of both edge states by an equal amount. As a result, qubit states associated with occupied edge states, such as $|01\rangle$, $|10\rangle$, and $|11\rangle$, will have different quasienergies (modulo $\pi/T$) as compared with $|00\rangle$, which has nei-
ther fermionic nor Majorana excitations. Moreover, \(|11\rangle\) will have different quasienergies (modulo \(\pi/T\)) as compared with \(|01\rangle\) and \(|10\rangle\) since the former has both edge states occupied. Finally, \(\mu_2\) introduces a quasienergy difference between the two edge states, which results in \(|01\rangle\) and \(|10\rangle\) having different quasienergy values. Thus, in the presence of \(H_{\text{break}}\), all four qubit states now have different quasienergy values (modulo \(\pi/T\)), as illustrated in Fig. 6. In practice, the difference in quasienergy can be indirectly probed by, for example, irradiating the system with electromagnetic waves, which results in qubit-state dependent resonant frequency [12][14].

D. Implementation of simple quantum algorithms

To demonstrate the application of our results presented in Sec. V A and V B, we now illustrate two simple quantum algorithms realized by the gate operations developed in Sec. V A and V B. The first one is a simplified version of the Grover’s search algorithm [23], which is capable of finding a particular object from a certain database. Unlike Grover’s algorithm which can be applied for any given database, our algorithm assumes a special structure of the database which maps a number \(z \in \{1, 2 \cdots 2^n\}\) to \(\bar{z} = 2^n - z\). In other words, our simple algorithm amounts to solving \(z\), given \(\bar{z}\), quantum mechanically. By employing the quantum circuit in Fig. 7(a), where the oracle operator is to be defined below, this can be accomplished in just a single step, similar to its classical counterpart. While it does not demonstrate the advantage of quantum over classical computation, this simple example illustrates how quantum computation works.

To be more explicit, let \(\bar{z} = (z_1, \cdots, z_n)\) be a column vector representing the binary expansion of \(z\), i.e., \(z = z_1 \times 2^n + \cdots + z_n \times 2^{n-1}\), and define \(|\bar{z}\rangle = |z_1, \cdots, z_n\rangle\). Next, define the oracle operator as \(O = \prod_{i=1}^{n} Z_i^2\), where \(Z_i\) is the Pauli Z gate acting on qubit \(i\), \(\bar{z}_i = z_i \oplus 1\), and \(\oplus\) is the addition operation modulo two. It is now straightforward to show that Fig. 7(a) indeed maps an input \(|\bar{z}\rangle\) to the desired output \(|\bar{z}\rangle\).

To illustrate this, consider a function \(g(x)\) as in Sec. V A, which by definition is constant for all \(x\) in a certain database of size \(2^n\). In terms of braiding operations, our circuit and its associated oracle operator are depicted in Figs. 7(b) and (c). Assuming that all Majorana modes are initialized in \(|00\rangle\) state, protocol described in Sec. V B is first carried out to implement \(H_L Z_L\) and \(H_R Z_R\) gate operations, which brings our qubit state to an equal-weight superposition of all qubit basis states. Next, depending on the input we supply to the black box, the oracle operator will execute one of the four sets of Pauli \(Z_L\) and \(Z_R\) gates as illustrated in Fig. 7, all of which are achievable through the protocol developed in Sec. V A. This flips the sign of the weight of some qubit basis states. Lastly, another \(H_L Z_L\) and \(H_R Z_R\) gates are applied to bring our qubit state to the desired output. This output is then measured by implementing the readout process described in Sec. V B.

It can be seen that the same oracle can be used to implement the Deutsch-Jozsa algorithm [24], capable of identifying whether a particular function is constant, i.e., \(g(x) = 0\) or \(g(x) = 1\) for any input \(x \in \{1, \cdots, 2^n\}\), or balanced, i.e., \(g(x) = 0\) for half the inputs and \(g(x) = 0\) for the other half. To proceed, note that any balanced or constant function can be expressed as \(g(x) = \bar{x} \cdot \bar{z} \oplus k\) for

\[
|0\rangle \begin{array}{c}
(HZ \otimes^n) \\
O \\
(HZ \otimes^n)
\end{array} \sum_{\bar{x}} |\bar{x}\rangle
\begin{array}{c}
\sum_{\bar{y}} (-1)\bar{y} |\bar{y}\rangle
\end{array}
\begin{array}{c}
\sum_{\bar{y}} \delta_{\bar{x}, \bar{y}} |\bar{y}\rangle
\end{array}
\begin{array}{c}
|\bar{z}\rangle,
\end{array} \tag{20}
\]

where we have suppressed any normalization constant for brevity, \(\bar{x} \cdot \bar{y} = x_1y_1 \oplus \cdots \oplus x_ny_n\), and we have used the fact that \(\sum_{\bar{y}} (-1)\bar{y} = \delta_{\bar{x}, \bar{y}}\).

Going back to our system, We have shown that a single superlattice is already capable of hosting two logical qubits, and the two gate operations needed to implement our algorithm, i.e., the \(Z\) and \(HZ\) gates, can be implemented in this 1D system by braiding Majorana modes according to the protocols outlined in Sec. V A and V B respectively. In the two-qubit case, our algorithm is capable of finding an object from a database of size \(2^n = 4\).
a fixed but unknown $z \in \{1, \ldots, 2^n\}$ and $k = 0, 1$. Indeed, it can be checked that $g(x)$ is constant if and only if $\vec{z} = \vec{0}$, otherwise it is balanced. Therefore, Deutsch-Jozsa algorithm proceeds in the same way as above, i.e., as depicted in Fig. 7(a)-(c), with $\vec{x} \cdot \vec{z}$ being now identified as the function $g(x)$. The latter being constant is thus identified when $|1 \cdots 1\rangle$ appears as output; any other output implies $g(x)$ being balanced. In fact, similar braiding-based oracle has also been used in Ref. [39] for exactly this purpose, although a minimum of three wires is required to construct an oracle of size $N = 2^2 = 4$ in the setup of Ref. [39]. By contrast, here we only require a single wire after exploiting the coexistence of two pairs of MZMs and one pair of Majorana $\pi$ modes.

### E. Scalability and implementation of entangling gates

Given that two logical qubits are encoded and manipulated in a 1D setup, it is important to examine the possibility of scaling up our proposal. There are two routes to scale up. The first route is to consider many zero modes and $\pi$ modes in one single quantum wire. In principle, their coexistence can be used to encode multi-qubit quantum information and it is not hard to imagine that certain quantum information processing becomes possible. This is an exciting target but we yet need to investigate how to braid two particular edge modes out of many without affecting the rest. The other route for scaling up is to add more wires arranged in parallel with each other, as shown in Fig. 8. Edge modes belonging to different wires can also be braided by turning on hopping and/or pairing between the wires. The actual braiding protocols between two such Majorana modes from different wires can be designed by slightly modifying the protocols introduced in Sec. V A and Sec. V B. For example, braiding Majorana modes marked by blue and red circles in Fig. 8 can be obtained by directly applying the protocol of Sec. V A on wire labelled $(l)$, with step 2 and step 6 being slightly modified by introducing interwire hopping and pairing in order to move two Majorana modes from wire $(l + 1)$ to wire $(l)$, as shown in Fig. 8.

As a promising side finding, in the following we show that by considering only the two wires $(l)$ and $(l + 1)$ illustrated in Fig. 8 entangling gates such as CNOT and other controlled-Pauli gates can be implemented through...
a series of braiding and measurement operations only. For brevity, we will only present the construction of a CNOT gate with the first and second qubits being the target and control qubits respectively, encoded in wire (l), with its Majorana modes denoted as $\gamma_{0,1}^{(l),s}$, $\gamma_{0,2}^{(l),s}$, and $\gamma_1^{(l),s}$, where $s = L, R$. The additional six Majorana modes in wire ($l + 1$) give rise to additional two logical qubits, but for the purpose of implementing controlled-
Pauli gates, only a single qubit encoded by $\gamma_{0,1}^{(l+1),L}$ and $\gamma_{0,2}^{(l+1),L}$ will be used as ancilla, whereas the other two qubits can be used as additional stabilizer operators. It is further assumed that the ancilla is prepared in $|1\rangle_a$, which can be done by following the protocol of Sec. V C.

We start by writing the CNOT unitary as $U(X_L) = \exp \left[ i \pi/4 (1 - Z_L) (1 - X_L) \right]$, which can be written in terms of Majorana modes as

$$U(X_L) = \exp \left[ i \pi/4 \right] \times \exp \left[ i \pi/4 \left( \gamma_{0,1}^{(l),R} \gamma_{0,2}^{(l),L} \gamma_{0,1}^{(l),\pi} \right) \right] \times \exp \left[ i \pi/4 \gamma_{0,1}^{(l),L} \gamma_{0,2}^{(l),R} \gamma_{0,1}^{(l),\pi} \right].$$

The third and fourth exponentials of $U(X_L)$ are simply the braiding unitaries discussed in Sec. V A and Sec. V B.

On the other hand, the second exponential can be implemented by performing projective measurements on $\Pi_1 = \gamma_{0,1}^{(l),R} \gamma_{0,2}^{(l),L} \gamma_{0,1}^{(l),\pi}$ and $\Pi_2 = \gamma_{0,1}^{(l+1),L} \gamma_{0,2}^{(l+1),\pi}$, followed by measurement dependent corrections, which are realizable through braiding [47, 48].

To be more explicit, we can write $\Pi_1 = \frac{1}{2} \left( 1 + p_1 \gamma_{0,1}^{(l),R} \gamma_{0,2}^{(l),L} \gamma_{0,1}^{(l),\pi} \right)$ and $\Pi_2 = \frac{1}{2} \left( 1 + p_2 \gamma_{0,1}^{(l+1),L} \gamma_{0,1}^{(l),\pi} \right)$, where $p_1, p_2 = \pm 1$ are the measurement results of $\Pi_1$ and $\Pi_2$ respectively. The effect of the two measurements can then be written as

$$\Pi_2 \Pi_1 = \frac{1}{4} \left( 1 - ip_1 \gamma_{0,1}^{(l),R} \gamma_{0,2}^{(l),L} \gamma_{0,1}^{(l),\pi} \right) + \frac{1}{4} \left( 1 + ip_1 \gamma_{0,1}^{(l),R} \gamma_{0,2}^{(l),L} \gamma_{0,1}^{(l),\pi} \right) + \frac{1}{4} \left( 1 - ip_2 \gamma_{0,1}^{(l+1),L} \gamma_{0,1}^{(l),\pi} \right) + \frac{1}{4} \left( 1 + ip_2 \gamma_{0,1}^{(l+1),L} \gamma_{0,1}^{(l),\pi} \right).$$

where we have used $i \gamma_{0,1}^{(l+1),L} \gamma_{0,1}^{(l),L} |1 \rangle_a = -|1 \rangle_a$. By further applying $U_1(p_2) = \exp \left[ -\frac{i}{4} \gamma_{0,2}^{(l+1),L} \gamma_{0,1}^{(l),\pi} \right]$, Eq. 22 becomes

$$U_1(p_2) \Pi_2 \Pi_1 = \frac{1}{2 \sqrt{2}} \left( 1 + ip_1 \gamma_{0,1}^{(l),R} \gamma_{0,2}^{(l),L} \gamma_{0,1}^{(l),\pi} \right).$$

Note that Eq. 23 is equal to the second exponential of $U(X_L)$, up to a constant, provided $p_1 p_2 = 1$. If $p_1 p_2 = -1$, further unitary $U_2 = \exp \left[ \frac{i}{2} \gamma_{0,1}^{(l+1),L} \gamma_{0,2}^{(l),\pi} \right]$ is applied to Eq. 23, which leads also to the desired result.

In our system, $\Pi_1$ can be carried out by first braiding $\gamma_{0,1}^{(l),\pi}$ and $\gamma_{0,2}^{(l+1),\pi}$, measuring $\Pi_1 = \gamma_{0,1}^{(l),R} \gamma_{0,2}^{(l),L} \gamma_{0,1}^{(l+1),L}$, via the introduction of chiral symmetry breaking terms on the left half of wires ($l$) and ($l + 1$), then finally undoing the braiding between $\gamma_{0,1}^{(l),\pi}$ and $\gamma_{0,2}^{(l+1),\pi}$. Likewise, $\Pi_2$ is carried

FIG. 8: Generalization of our single-wire braiding scheme to an array of wires. Majorana modes can be moved to another site belonging to the same (blue circle) or different wires (red-magenta circle) by appropriately tuning intra- and inter-wire hopping and pairing strengths.
out by first braiding $\gamma_{0,1}^{(l)}$ and $\gamma_{0,2}^{(l+1)}$, measuring $\Pi_2 = \exp[i\gamma_{0,1}^{(l+1)}\gamma_{0,2}^{(l+1)}]$ by introducing chiral symmetry breaking terms on wire $(l+1)$, then undoing the braiding $\Pi_1 = \exp[-i\gamma_{0,1}^{(l)}\gamma_{0,2}^{(l)}]$. After some algebra, $U(X_L)$ can finally be expressed as

$$U(X_L) = 2 \exp \left[ \frac{\pi}{4} (2 - p_1 p_2) \right] \times \exp \left[ \frac{\pi}{4} (2 - p_1 p_2) \gamma_{0,1}^{(l)} \gamma_{0,2}^{(l+1)} \right] \times \exp \left[ \frac{\pi}{4} (p_2 - 1) \gamma_{0,1}^{(l)} \gamma_{0,2}^{(l+1)} \right] \times \Pi_2 \times \exp \left[ \frac{\pi}{4} \gamma_{0,1}^{(l)} \gamma_{0,2}^{(l+1)} \right] \times \Pi_1 \times \exp \left[ \frac{\pi}{4} \gamma_{0,1}^{(l)} \gamma_{0,2}^{(l+1)} \right],$$

where $p_1, p_2 = \pm 1$ are now the measurement results of $\Pi_1$ and $\Pi_2$ respectively. Other controlled-Pauli gates $U(P_L) = \exp[i\pi/4 (1 - Z_R)(1 - P_L)]$ can be implemented similarly, as $P_L$ can be expressed as a product of two Majorana modes.

### VI. DISCUSSION

#### A. Experimental consideration

Similar to other topological superconducting wires, our model Eq. (3) can be engineered in either cold-atom [12] or proximitized semiconductor [49] [50] platforms. In a cold-atom setup, such a 1D model is formed by embedding optically trapped fermions inside a dimensional Bose-Einstein condensate (BEC). The hopping and pairing terms are provided respectively by the two Raman lasers forming the optical lattice and the radio frequency (rf) field coupling the fermions with the surrounding BEC reservoir [12]. In this context the pairing and the hopping are in principle highly controllable. Sub-lattice degree of freedom can then be realized by using spatially periodic Raman lasers and rf field, which then allow two adjacent fermions to experience different hopping and pairing strength. Manipulation of the hopping and pairing strength to carry out the protocols described in Sec. V A and Sec. V B should be feasible by tuning the Rabi frequencies of the Raman lasers and rf field respectively. In particular, switching between real and imaginary hopping and pairing parameters, i.e., between $H_1$ and $H_2$, can be done through switching between real and imaginary Rabi frequencies, which can be realized by appropriately setting the electric field profiles of the Raman lasers and rf field. Alternatively, by fixing the electric field profiles of the Raman lasers and rf field, one could also switch the phase of the hopping and pairing parameters by rapidly shaking the optical lattice at every integer multiple of $T/2$ [21].

Following the discussion of Ref. [12], the coherence time-scale of Majorana modes in such cold atom setup can be extendable to the order of seconds. Meanwhile, given that the system parameters can be of the order of tens of kHz [12], a single period of the system is typi-
B. Comparison with TQC

At first sight, our holonomic braiding-based protocols to realize quantum gate operations are very similar to typical approach in TQC. Though TQC is also usually implemented through adiabatic holonomy, one striking difference between TQC and HQC lies in the robustness of the non-Abelian geometric phase against arbitrary path deformation. A sufficient condition to accomplish this is to ensure that the non-Abelian Berry phase contribution in Eq. (2) is zero during the holonomic cycle, so that the total geometric phase arises solely from the so-called explicit monodromy [32, 33]. By writing

\[ \gamma^s_{0,a} = \sum_{D \in \{A,B\}, j \in \{1,2,\ldots,N\}, \nu \in \{a,\beta\}} C^s_{a,\gamma^c_{D,j}} \gamma^c_{D,j}, \]

\[ \gamma^s_\pi = \sum_{D \in \{A,B\}, j \in \{1,2,\ldots,N\}, \nu \in \{a,\beta\}} C^s_{a,\gamma^c_{D,j}} \gamma^c_{D,j}, \]

where \( a \in \{1,2\} \) and \( s \in \{L,R\} \), it can be verified that for all steps involved in Sec. V A (at least in the ideal case), \( \gamma^s_\pi \) indeed contains no Berry phase contribution, and thus qualifies as TQC when implemented alone to carry out computational tasks. Indeed, it can also be verified that replacing \( \cos \phi \) with \( \sin \phi \) any function decreasing from 1 to 0 (increasing from 0 to 1) at each step in the protocol does not change the net result.

On the other hand, the protocol elucidated in Sec. V B would have also qualified as TQC if not for its step 3. In these two processes, non-Abelian Berry phase is necessarily introduced between \( \gamma^R_\pi \) and \( \gamma^L_0 \), or between \( \gamma^R_\pi \) and \( \gamma^L_0 \), to induce rotation between Majorana zero and \( \pi \) modes. It is thus these two steps that inhibit our protocol to possess the full robustness of braiding operations. However, we do not view this feature as a genuine weakness of our quantum computation protocols, because in actual physical implementation of the braiding of Majorana modes in any platform so far, certain degree of control of the system is always needed, and this allows the implementation of the adiabatic paths to a certain precision. As the other side of the story, the non-TQC nature of our quantum computation protocols can also be exploited to realize a \( T \) gate required for universal quantum computation [34][35][36], which is otherwise impossible to construct via topologically protected braiding operations alone. To appreciate this point we can skip steps 4-6 in the protocol described in Sec. V B leading to the net outcome \( \gamma^\pi_\pi - \gamma^L_0 \rightarrow \gamma^L_0 \) and \( \gamma^R_0 \rightarrow \gamma^R_0 - \gamma^L_\pi \). This outcome is equivalent to the unitary \( T_{\pi} = \exp\left[(\pi/8)\gamma^L_0\gamma^L_0\right] \), i.e., the \( T \) gate acting on the first qubit. Similar approach can also be applied to realize \( T_{R} \), the \( T \) gate acting on the second qubit.

Aside from examining the robustness of our quantum scheme versus TQC, it is also important to point out that the novelty of our quantum computation scheme lies in the use of edge modes. Because our qubits are made of edge states, they do possess topological protection against some variations in the system parameters. This important advantage renders perfect fine tuning unnecessary and thus in principle provides advantages over other holonomic quantum computation proposals that do not rely on topological phases at all [50][50].

VII. CONCLUSION

This work aims to advocate an innovative avenue of quantum computation by use of symmetry-protected edge modes of topological matter. A periodically driven quantum wire may host many zero and \( \pi \) edge modes [9][11][13] being either as Majorana or fermionic excitations. Their dynamical phase contributions are trivial and hence adiabatic manipulations of these multiple edge modes associated with Floquet topological matter can be used for quantum information processing. This is an exciting possibility not foreseen before. As the first step along this avenue, we exploit the coexistence of three pairs of Majorana edge modes in one single periodically driven quantum wire, equivalent to obtaining two local fermions and one nonlocal fermions as topologically protected edge modes. The three pairs of Majorana edge modes can be used to encode two logical qubits, protected by both particle-hole and chiral symmetries. Adiabatic protocols are designed to simulate the braiding between various pairs of Majorana modes, which then realizes several gate operations. A means to readout these qubits is also proposed through introducing chiral-symmetry breaking terms into the system. As an encouraging side result, we have also shown that our system can be scaled up, at least by adding more parallel quantum wires. This then allows the implementation of entangling quantum gates. To demonstrate the application of our quantum computation schemes, we have also constructed a quantum circuit to implement two simple quantum algorithms, which requires much less hardware resources as compared with previous work. We have also briefly discussed potential realizations of our proposal in experiments. Understanding that there can be experimental challenges ahead but not yet identified, we do not claim that any experimental realizations of this theoretical work would be straightforward at this point. A comparison between our approach with that of TQC is also made.

This paper indicates a possible new paradigm for realizing many logical qubits with minimal amount of physical resources on the hardware level. Such kind of possibility, even still on the theoretical level, is always stimulating towards the realization of a scalable quantum computer. As another consideration to scale up our quantum computation protocols, we call for future studies to explore the feasibility of using one single quantum wire to host and individually address more than two
logical qubits. A good starting point to achieve this is to consider systems capable of hosting many Majorana zero and π modes, such as that considered in Ref. [13]. More follow-up studies to that end will certainly enhance the marriage of two timely research topics as of today, namely, quantum computation and Floquet topological matter. Indeed, this work should also serve as the first step to extend the idea of TQC to periodically driven systems. Following our discussion in Sec. VI B, a possible future study is to devise computation protocols that can braid Majorana zero and π modes purely through explicit monodromy, so as to unleash the full topological protection offered by braiding operations. It is expected that the combination of scalability of our proposal and the fault-tolerance nature of TQC approach may eventually lead to a full-fledged quantum computer based on topological edge modes.

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Appendix A: Derivation of Floquet non-Abelian Berry phase

Following the notation in Sec. [11B] we consider the application of $U(\lambda(s))$ on $|\varepsilon_n[\lambda(s-1)]\rangle$ as

$$
U(\lambda(s))|\varepsilon_n[\lambda(s-1)]\rangle = U(\lambda(s))U(\lambda(s-1))U(\lambda(s-1))|\varepsilon_n[\lambda(s-1)]\rangle
$$

$$
= \exp(-i\varepsilon_n[\lambda(s-1)]T)U(\lambda(s))U(\lambda(s-1))|\varepsilon_n[\lambda(s-1)]\rangle
$$

$$
\approx \exp(-i\varepsilon_n[\lambda(s-1)]T) \left( I + \frac{dU}{d\lambda} \right) |\varepsilon_n[\lambda(s-1)]\rangle
$$

$$
\approx \exp(-i\varepsilon_n[\lambda(s-1)]T) \exp\left( \frac{dU}{d\lambda} \right) |\varepsilon_n[\lambda(s-1)]\rangle ,
$$

(A1)

where we have used the fact that $U(\lambda(s))$ serves as a one-period propagator, combined with the adiabaticity condition that the state remains in a Floquet eigenstate associated with quasienergy $\varepsilon_n[\lambda(s)]$. $V_n(\lambda) = \exp(i\Omega_n(\lambda))$ is a $k_n \times k_n$ path dependent unitary matrix which potentially rotates $|\varepsilon_n[\lambda(s)]\rangle$ in the degenerate subspace, thus generalizing the appearance of a global phase in the nondegenerate (Abelian) case.

Next, we expand

$$
|\varepsilon_n[\lambda(s)]\rangle = \sum_m \exp(-i\varepsilon_m(\lambda)T) V_m(\lambda)C_m(\lambda)|\varepsilon_n[\lambda(s-1)]\rangle ,
$$

(A2)

where $C_m(\lambda)$ is another $k_n \times k_n$ matrix that generalizes the spectral coefficients in the nondegenerate case. The left hand side of Eq. (A1) can be rewritten as

We can then combine Eqs. (A2) and (A3) with Eq. (A1), and apply both sides with \langle\varepsilon_n[\lambda(s-1)]| from the left to obtain

$$
\langle\varepsilon_n[\lambda(s-1)]| \exp\left( \frac{dU}{d\lambda} \right) |\varepsilon_n[\lambda(s-1)]\rangle = C_n ,
$$

(A4)

where we have used the fact that matrix $C_n$ is only non-diagonal within a degenerate subspace, so that $\langle\varepsilon_n[\lambda(s-1)]|C_m|\varepsilon_m[\lambda(s-1)]\rangle = 0$ if $m \neq n$. By spectral decomposing $\frac{dU}{d\lambda}$ and explicitly expanding the column vector defined in Sec. [11B] we can derive the matrix coefficient of $C_n$ as

$$
C_{n,\alpha\beta} = \exp\left( \langle\varepsilon_n,\alpha| \frac{d}{d\lambda}|\varepsilon_n,\beta\rangle \right) .
$$

(A5)

Finally, by recursively combining Eqs. (A2) and (A5), we arrive at

$$
|\varepsilon_n(\lambda)\rangle = \mathcal{P} \exp\left( -i \int_{\lambda_0}^{\lambda} [\mathcal{A}_n + \Omega_n + \varepsilon_n T] \, d\lambda \right) |\varepsilon_n(\lambda_0)\rangle ,
$$

(A6)

where $\mathcal{P}$ is the path ordering operator, and $\mathcal{A}_{n,\alpha\beta} = i[\varepsilon_n,\alpha] \frac{d}{d\lambda}|\varepsilon_n,\beta\rangle$ is the non-Abelian Berry connection.

Appendix B: Evolution of Majorana modes during $\gamma_{1,1}^t$, and $\gamma_{1,2}^t$ braiding protocol

For all the steps presented in Sec. [1A] we are able to analytically keep track the evolution of all Majo-
where \( \Gamma_{1} = \gamma_{B,2;A,2} + \gamma_{A,3;B,2} \), \( \Gamma_{2} = \gamma_{B,2;A,3} + \gamma_{B,2;A,3} \), \( \gamma_{\mu, 3;D, k}^{\alpha, \beta} \) stands for \( \gamma_{C,1;D, k}^{\alpha, \beta} \). \( C, D \in \{A, B\} \), \( \mu, \nu \in \{\alpha, \beta\} \), \( j, k \in \{1, 2, 3\} \) denote the lattice site, \( s(\phi) \) and \( c(\phi) \) stand for \( \sin(\phi) \) and \( \cos(\phi) \) respectively.

Rather than showing the full derivation of the Majorana modes from the recurrence relation, we will instead show the form of the Majorana modes affected by the deformation at each step, and briefly verify them by commuting with \( \mathcal{U}^{(S)}(\phi) \). The latter can be done analytically by using the following two facts.

1. Most of the terms in \( H_{1}^{(S)} \) and \( H_{2}^{(S)} \) commute with one another. This allows us to write Eq. (B1) as products of many exponentials.

2. Each exponential can be expanded using the identity Eq. (18).

**Step 1:** \( \gamma_{0,1}^{\alpha}(\phi) = s(\phi)\gamma_{A,2}^{\alpha} + c(\phi)\gamma_{A,1}^{\alpha} \). This is easily verified by noting that it commutes with both \( H_{1}^{(1)} \) and \( H_{2}^{(1)} \).

**Step 2:**

\[
\begin{align*}
\gamma_{1}^{L}(\phi) &= \left[ c(\phi)\gamma_{A,1}^{\beta} + s(\phi)\gamma_{A,2}^{\beta} \right] - \left[ c(\phi)\gamma_{B,1}^{\alpha} + s(\phi)\gamma_{B,2}^{\alpha} \right], \\
\gamma_{0,2}^{L}(\phi) &= \left[ c(\phi)\gamma_{A,1}^{\beta} + s(\phi)\gamma_{A,2}^{\beta} \right] + \left[ c(\phi)\gamma_{B,1}^{\alpha} + s(\phi)\gamma_{B,2}^{\alpha} \right],
\end{align*}
\]

where we have suppressed the normalization factor for brevity here and for the rest of the steps. This can be verified by first noting, by using the two facts above, that \( -iH_{2}^{(2)}T/2 \) interchanges \( \gamma_{A,1}^{\beta}, \gamma_{A,2}^{\beta} \) → \( -\gamma_{A,1}^{\beta}, \gamma_{A,2}^{\beta} \) and \( \gamma_{B,1}^{\beta}, \gamma_{B,2}^{\beta} \) → \( \gamma_{B,1}^{\beta}, \gamma_{B,2}^{\beta} \). On the other hand, \( -iH_{2}^{(2)}T/2 \) flips the sign of \( \gamma_{B,1}^{\beta} \) and
\[ U^{(2)} \gamma_{\pi}^L U^{(2)^*} = -\gamma_{\pi}^L \text{ and } U^{(2)} \gamma_{0.2}^L U^{(2)^*} = \gamma_{0.2}^L. \]

**Step 3:**

\[
\begin{align*}
\gamma_{L}^L(\phi) &= [c(\phi)\gamma_{A,2}^\beta + s(\phi)\gamma_{A,1}^\alpha] - [c(\phi)\gamma_{B,2}^\alpha + s(\phi)\gamma_{B,1}^\beta], \\
\gamma_{0,2}^L(\phi) &= [c(\phi)\gamma_{A,2}^\beta + s(\phi)\gamma_{A,1}^\alpha] + [c(\phi)\gamma_{B,2}^\alpha + s(\phi)\gamma_{B,1}^\beta].
\end{align*}
\]

(B4)

This can be verified following the same argument as before. That is, \(\exp\left(-i H_1^{(3)} T/2 \right)\) interchanges \((\gamma_{A,2}^\alpha, \gamma_{A,1}^\beta) \rightarrow -(\gamma_{A,2}^\alpha,\gamma_{B,1}^\beta)\) and \((\gamma_{B,2}^\alpha, \gamma_{B,1}^\beta) \rightarrow (\gamma_{B,2}^\alpha, \gamma_{A,1}^\beta)\), \(\exp\left(-i H_2^{(3)} T/2 \right)\) flips the sign of \(\gamma_{B,2}^\alpha\) and \(\gamma_{B,1}^\beta\), resulting in \(U^{(3)} \gamma_{\pi}^L U^{(3)^*} = -\gamma_{\pi}^L\) and \(U^{(3)} \gamma_{0.2}^L U^{(3)^*} = \gamma_{0.2}^L\).

**Step 4:**

\[
\begin{align*}
\gamma_{\pi}^L(\phi) &= s \left(\frac{\pi}{4} c(\phi)\right) \gamma_{A,1} + c \left(\frac{\pi}{4} c(\phi)\right) \gamma_{B,1}, \\
\gamma_{0.2}^L(\phi) &= c \left(\frac{\pi}{4} c(\phi)\right) \gamma_{A,1} + s \left(\frac{\pi}{4} c(\phi)\right) \gamma_{B,1}.
\end{align*}
\]

(B5)

This can be verified by noting that \(\exp\left(-i H_1^{(4)} T/2 \right)\) maps \(\gamma_{\pi}^L \rightarrow -s \left(\frac{\pi}{4} c(\phi)\right) \gamma_{A,1} - c \left(\frac{\pi}{4} c(\phi)\right) \gamma_{B,1}\) and \(\gamma_{0.2}^L(\phi) \rightarrow c \left(\frac{\pi}{4} c(\phi)\right) \gamma_{A,1} + s \left(\frac{\pi}{4} c(\phi)\right) \gamma_{B,1}\). On the other hand, \(\exp\left(-i H_2^{(4)} T/2 \right)\) flips the sign of \(\gamma_{B,1}^\beta\).

**Step 5:**

\[
\begin{align*}
\gamma_{\pi}^L(\phi) &= [s(\phi) \left(\gamma_{A,2}^\beta - \gamma_{B,2}^\alpha\right) - c(\phi) \gamma_{B,1}^\beta] s \left(\frac{\pi}{4} s(\phi)\right) - c(\phi) \gamma_{B,1} \left(\frac{\pi}{4} s(\phi)\right), \\
\gamma_{0,1}^L(\phi) &= [c(\phi) \gamma_{A,2}^\beta - s(\phi) \left(\gamma_{A,2}^\beta + \gamma_{B,2}^\alpha\right)] c \left(\frac{\pi}{4} s(\phi)\right) - c(\phi) \gamma_{B,1} \left(\frac{\pi}{4} s(\phi)\right).
\end{align*}
\]

This can be verified by noting that \(\exp\left(-i H_1^{(5)} T/2 \right)\) maps

\[
\gamma_{\pi}^L(\phi) \rightarrow \left[-s(\phi) \left(\gamma_{A,2}^\beta + \gamma_{B,2}^\alpha\right) + c(\phi) \gamma_{A,2}^\beta\right] s \left(\frac{\pi}{4} s(\phi)\right) - c(\phi) \gamma_{B,1} \left(\frac{\pi}{4} s(\phi)\right), \\
\gamma_{0,1}^L(\phi) \rightarrow \left[c(\phi) \gamma_{A,2}^\beta - s(\phi) \left(\gamma_{A,2}^\beta - \gamma_{B,2}^\alpha\right)\right] c \left(\frac{\pi}{4} s(\phi)\right) + c(\phi) \gamma_{B,1} \left(\frac{\pi}{4} s(\phi)\right).
\]

On the other hand, \(\exp\left(-i H_2^{(5)} T/2 \right)\) flips the sign of \(\gamma_{B,2}^\alpha\) and \(\gamma_{B,1}^\beta\).

**Step 6:**

\[
\begin{align*}
\gamma_{\pi}^L(\phi) &= [c(\phi) \gamma_{A,2}^\beta + s(\phi) \gamma_{A,1}^\beta - c(\phi) \gamma_{B,2}^\beta + s(\phi) \gamma_{B,1}^\beta] s \left(\frac{\pi}{4} s(\phi)\right), \\
\gamma_{0,1}^L(\phi) &= [c(\phi) \gamma_{A,2}^\beta + s(\phi) \gamma_{A,1}^\beta + c(\phi) \gamma_{B,2}^\beta + s(\phi) \gamma_{B,1}^\beta] c \left(\frac{\pi}{4} s(\phi)\right).
\end{align*}
\]

(B8)

This can be verified by noting that \(\exp\left(-i H_1^{(6)} T/2 \right)\) interchanges \((\gamma_{A,2}^\beta, \gamma_{A,1}^\beta) \rightarrow -(\gamma_{B,2}^\beta, \gamma_{B,1}^\beta)\) and \((\gamma_{B,2}^\alpha, \gamma_{B,1}^\beta) \rightarrow (\gamma_{A,2}^\beta, \gamma_{A,1}^\beta)\), whereas \(\exp\left(-i H_2^{(6)} T/2 \right)\) flips the sign of \(\gamma_{B,2}^\alpha\) and \(\gamma_{B,1}^\beta\).

**Appendix C:** Evolution of Majorana modes during \(\gamma_{L}^L\) and \(\gamma_{L}^0\) braiding protocol

In the protocol described in Sec. V B, only \(H_1^{(S)}\) is adiabatically deformed, whereas \(H_{1}^{(S)} \equiv H_1\) is kept constant, where \(S = 1, 2, \cdots 7\). In Majorana basis, \(H_{1}^{(S)}\) can be expressed as (keeping only terms in the first \(n\) lattice sites for brevity)
where $\xi_n = \sum_{k=1}^{n-1} \left( \gamma^\beta_{A,k:B,k} + \gamma^\beta_{B,k:A,k+1} + \gamma^\alpha_{B,k:A,k+1} \right)$, $C = (1 - f(s))/2$, $S = (1 + f(s))/2$, and $f(s)$ is defined in Sec. [1]. Following the same discussion as Appendix [1], we will now present the evolution of Majorana modes under the aforementioned adiabatic deformation in step 1, 2, 4, and 7. As elucidated in Sec. [1] step 3 and 6 involve a special two-period adiabatic deformation which is difficult to keep track analytically, whereas step 5 is the same as step 2. Note that throughout the steps, only $\gamma^L_{A}$ and $\gamma^L_{B,2}$ are affected, while the other Majorana modes stay intact.

**Step 1:**

\[
\gamma^L_{\pi}(\phi) = \sum_{k=1}^{n+1} \left( \gamma^\beta_{A,k} - \gamma^\beta_{B,k} \right) \cos^{n+1-k} \phi \sin^{k-1} \phi , \\
\gamma^L_{0,2}(\phi) = \sum_{k=1}^{n+1} \left( \gamma^\beta_{A,k} + \gamma^\beta_{B,k} \right) \cos^{n+1-k} \phi \sin^{k-1} \phi ,
\]

(C2)

This is verified by noting that $\exp(-iH_1T/2)$ interchanges $\gamma^\beta_{A,k} \rightarrow -\gamma^\beta_{B,k}$ and $\gamma^\beta_{B,k} \rightarrow -\gamma^\beta_{A,k}$, whereas $\exp(-iH_2^{(1)}T/2)$ flips the sign of $\gamma^\alpha_{B,n+1}$. 

**Step 2:**

\[
\gamma^L_{\pi}(\phi) = \left[ c(\phi)\gamma^\beta_{A,n+1} + s(\phi)\gamma^\beta_{B,n} \right] + \left[ c(\phi)\gamma^\beta_{B,n+1} + s(\phi)\gamma^\beta_{A,n} \right] , \\
\gamma^L_{0,2}(\phi) = \left[ c(\phi)\gamma^\beta_{A,n+1} + s(\phi)\gamma^\beta_{B,n} \right] + \left[ c(\phi)\gamma^\beta_{B,n+1} - s(\phi)\gamma^\beta_{A,n} \right] .
\]

(C3)

This is verified by noting that $\exp(-iH_1T/2)$ interchanges $\left( \gamma^\beta_{A,n+1}, \gamma^\beta_{B,n} \right) \rightarrow \left( -\gamma^\beta_{B,n+1}, -\gamma^\beta_{A,n} \right)$ and $\left( \gamma^\beta_{B,n+1}, \gamma^\beta_{A,n} \right) \rightarrow \left( -\gamma^\beta_{A,n+1}, -\gamma^\beta_{B,n} \right)$, whereas $\exp(-iH_2^{(1)}T/2)$ flips the sign of $\gamma^\alpha_{B,n+1}$ and $\gamma^\beta_{A,n}$. 

**Step 4:**

\[
\gamma^L_{\pi}(\phi) = c(\phi)\gamma^\beta_{A,n} + s(\phi)\gamma^\beta_{A,n+1} , \\
\gamma^L_{0,2}(\phi) = c(\phi)\gamma^\beta_{B,n} + s(\phi)\gamma^\beta_{B,n+1} .
\]

(C4)

Note however that $\gamma^L_{\pi}$ and $\gamma^L_{0,2}$ are no longer Majorana π and zero modes in this step. As a result of step 3, $\gamma^L_{\pi} = \gamma_{\pi} + \gamma_{0}$ and $\gamma^L_{0,2} = \gamma_{0} - \gamma_{\pi}$, where $\gamma_{\pi}$ and $\gamma_{0}$ are instantaneous Majorana π and zero modes of $U^{(4)}$, which are given by

\[
\gamma_{\pi}(\phi) = \left[ c(\phi)\gamma^\beta_{A,n} + s(\phi)\gamma^\beta_{A,n+1} \right] - \left[ c(\phi)\gamma^\beta_{B,n} + s(\phi)\gamma^\beta_{B,n+1} \right] , \\
\gamma_{0}(\phi) = \left[ c(\phi)\gamma^\beta_{A,n} + s(\phi)\gamma^\beta_{A,n+1} \right] + \left[ c(\phi)\gamma^\beta_{B,n} + s(\phi)\gamma^\beta_{B,n+1} \right] ,
\]

(C5)

and can be verified in the same way as step 2.

**Step 7:**

\[
\gamma^L_{\pi}(\phi) = \sum_{k=1}^{n+1} \left( \gamma^\beta_{A,k} + \gamma^\alpha_{B,k} \right) \cos^{n+1-k} \phi \sin^{k-1} \phi , \\
\gamma^L_{0,2}(\phi) = -\sum_{k=1}^{n+1} \left( \gamma^\beta_{A,k} - \gamma^\alpha_{B,k} \right) \sin^{n+1-k} \phi \cos^{k-1} \phi .
\]

(C6)

This is verified in the same way as step 1.

[1] A. Y. Kitaev, *Unpaired majorana fermions in quantum wires*, Phys. Usp 44, 131 (2001).

[2] A. Kitaev, *Anyons in exactly solved model and beyond*,
L. W. Zhou and J. B. Gong, L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, M. N. Chen, F. Mei, W. Shu, H.-Q. Wang, S.-L. Zhu, Q.-J. Tong, J.-H. An, J. B. Gong, H.-G. Luo, and Floquet topological phases in
L. W. Zhou and J. B. Gong, H.-Q. Wang, M. N. Chen, R. W. Bomantara, J. B. Gong, Simulation of Non-Solitons in
W. P. Su, J. R. Schrieffer, and A. J. Heeger, Observation of photonic anomalous Floquet topological insulators, Nat. Commun. 8, 13756 (2017).
J. K. Asbóth and H. Obuse, Bulk-boundary correspondence for chiral symmetric quantum walks, Phys. Rev. B 88, 121406(R) (2013).
J. K. Asbóth, B. Tarasinski, and P. Delpace, Chiral symmetry and bulk-boundary correspondence in periodically driven one-dimensional systems, Phys. Rev. B 90, 125143 (2014).
H. L. Wang, L. W. Zhou, J. B. Gong, Interband coherence induced correction to adiabatic pumping in periodically driven systems, Phys. Rev. B 91, 085420 (2015).
V. Gurarie and C. Nayak, A plasma analogy and Berry matrices for non-Abelian quantum Hall states, Nucl. Phys. B 506, 685 (1997).
A. Stern, F. von Oppen, and E. Mariani, Geometric phases and quantum entanglement as building blocks for non-Abelian quasiparticle statistics, Phys. Rev. B 70, 205338 (2004).
R. Roy and F. Harper, Periodic table for Floquet topological insulators, Phys. Rev. B 96, 155118 (2017).
M. Thakurathi, A. A. Patel, D. Sen, and A. Dutta, Floquet generation of Majorana end modes and topological invariants, Phys. Rev. B 88, 155133 (2013).
A. Altland and M. R. Zirnbauer, Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures, Phys. Rev. B 55, 1142 (1997).
S. Bravyi and A. Kitaev, Fermionic quantum computation, Ann. Phys. 298, 210 (2002).
A. Ahlbrecht, L. S. Georgiev, and R. F. Werner, Implementation of Clifford gates in the Ising-anyon topological quantum computer, Phys. Rev. A 79, 032311 (2009).
C. V. Kraus, P. Zoller, and M. A. Baranov, Braiding of Atomic Majorana Fermions in Wire Networks and Implementation of the Deutsch-Jozsa Algorithm, Phys. Rev. Lett. 111, 203001 (2013).
P. Zanardi and M. Rasetti, Holonomic quantum computation, Phys. Lett. A 264, 94 (1999).
J. K. Pachos, Introduction to Topological Quantum Com-
putation (Cambridge University Press, New York, 2012).

42] J. Tuorila, M. Silveri, M. Sillanpää, E. Thuneberg, Y. Makhlin, and P. Hakonen, Stark Effect and Generalized Bloch-Siegert Shift in a Strongly Driven Two-Level System, Phys. Rev. Lett. 105, 257003 (2010).

43] C. Deng, J.-L. Orgiazzi, F. Shen, S. Ashhab, and A. Lupascu, Observation of Floquet States in a Strongly Driven Artificial Atom, Phys. Rev. Lett. 115, 133601 (2015).

44] M. Silveri, J. Tuorila, M. Kemppainen, and E. Thuneberg, Probe spectroscopy of quasiequilibrium states, Phys. Rev. B 87, 134505 (2013).

45] L. K. Grover, Quantum Mechanics Helps in Searching for a Needle in a Haystack, Phys. Rev. Lett. 79, 325 (1997).

46] D. Deutsch and R. Jozsa, Rapid solution of problems by quantum computation, Proc. R. Soc. A 439, 553 (1992).

47] T. Hyart, B. van Heck, I. C. Fulga, M. Burrello, A. R. Akhmerov, and C. W. J. Beenakker, Flux-controlled quantum computation with Majorana fermions, Phys. Rev. B 88, 035121 (2013).

48] S. Bravyi, Universal quantum computation with the \( \nu = \frac{5}{2} \) fractional quantum Hall state, Phys. Rev. A 73, 042313 (2006).

49] R. M. Lutchyn, J. D. Sau, and S. D. Sarma, Majorana Fermions and a Topological Phase Transition in Semiconductor-Superconductor Heterostructures, Phys. Rev. Lett. 105, 077001 (2010).

50] Y. Oreg, G. Refael, and F. von Oppen, Helical Liquids and Majorana Bound States in Quantum Wires, Phys. Rev. Lett. 105, 177002 (2010).

51] C. E. Creffield and F. Solis, Controlled Generation of Coherent Matter Currents Using a Periodic Driving Field, Phys. Rev. Lett. 100, 250402 (2008).

52] D. Rainis and D. Loss, Majorana qubit decoherence by quasiparticle poisoning, Phys. Rev. B 85, 174533 (2012).

53] D. J. van Woerkom, A. Geresdi, and L. P. Kouwenhoven, One minute parity lifetime of a NbTiN Cooper-pair transistor, Nat. Phys. 11, 547 (2015).

54] A. P. Higginbotham, S. M. Albrecht, G. Kiršanskas, W. Chang, F. Kuemmeth, P. Krogstrup, T. S. Jespersen, J. Nygård, K. Flensberg, and C. M. Marcus, Parity lifetime of bound states in a proximitized semiconductor nanowire, Nat. Phys. 11 1017 (2015).

55] S. M. Albrecht, E. B. Hansen, A. P. Higginbotham, F. Kuemmeth, T. S. Jespersen, J. Nygård, P. Krogstrup, J. Danon, K. Flensberg, and C. M. Marcus, Transport Signatures of Quasiparticle Poisoning in a Majorana Island, Phys. Rev. Lett. 118, 137701 (2017).

56] S. Bravyi and A. Kitaev, Universal quantum computation with ideal Clifford gates and noisy ancillas, Phys. Rev. A 71, 022316 (2005).

57] M. Freedman, C. Nayak, and K. Walker, Towards universal topological quantum computation in the \( \nu = \frac{3}{2} \) fractional quantum Hall state, Phys. Rev. B 73, 245307 (2006).

58] P. Bonderson, D. J. Clarke, C. Nayak, and K. Shtengel, Implementing Arbitrary Phase Gates with Ising Anyons, Phys. Rev. Lett. 104, 180505 (2010).

59] T. Karzig, Y. Oreg, G. Refael, and M. H. Freedman, Universal Geometric Path to a Robust Majorana Magic Gate, Phys. Rev. X 6, 031019 (2016).

60] L. Faoro, J. Siwetz, and R. Fazio, Non-Abelian Holonomies, Charge Pumping, and Quantum Computation with Josephson Junctions, Phys. Rev. Lett. 90, 028301 (2003).

61] P. Zhang, Z. D. Wang, J. D. Sun, and C. P. Sun, Holonomic quantum computation using rf superconducting quantum interference devices coupled through a microwave cavity, Phys. Rev. A 71, 042301 (2005).

62] X. D. Zhang, Q. Zhang, and Z. D. Wang, Physical implementation of holonomic quantum computation in decoherence-free subspaces with trapped ions, Phys. Rev. A 74, 034302 (2006).

63] I. Kamleitner, P. Solinas, C. Müller, A. Shnirman, and M. Möttönen, Geometric quantum gates with superconducting qubits, Phys. Rev. B 83, 214518 (2011).

64] V. V. Albert, C. Shu, S. Krastanov, C. Shen, R.-B. Liu, Z.-B. Yang, R. J. Schoelkopf, M. Mirrahimi, M. H. Devoret, and L. Jiang, Holonomic Quantum Control with Continuous Variable Systems, Phys. Rev. Lett. 116, 140502 (2016).

65] K. S. Kumar, A. Vepsäläinen, S. Danilin, and G. S. Paraoanu, Stimulated Raman adiabatic passage in a three-level superconducting circuit, Nat. Commun. 7, 10628 (2016).

66] B.-J. Liu, Z.-H. Huang, Z.-Y. Xue, and X.-D. Zhang, Superadiabatic holonomic quantum computation in cavity QED, Phys. Rev. A 95, 062308 (2017).