HIGHER INTEGRABILITY FOR VARIATIONAL INTEGRALS WITH
NON-STANDARD GROWTH

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ABSTRACT. We consider autonomous integral functionals of the form
\[ \mathcal{F}[u] := \int_{\Omega} f(Du) \, dx \quad \text{with} \quad u : \Omega \to \mathbb{R}^N, \ N \geq 1, \]
where the convex integrand \( f \) satisfies controlled \((p, q)\)-growth conditions. We establish higher gradient integrability and partial regularity for minimizers of \( \mathcal{F} \) assuming \( \frac{q}{p} < 1 + \frac{2}{n-1} \), \( n \geq 3 \). This improves earlier results valid under the more restrictive assumption \( \frac{q}{p} < 1 + \frac{2}{n} \).

1. INTRODUCTION

In this note, we study regularity properties of local minimizers of integral functionals
\[ \mathcal{F}[u] := \int_{\Omega} f(Du) \, dx, \]
where \( \Omega \subset \mathbb{R}^n \), \( n \geq 3 \), is a bounded domain, \( u : \Omega \to \mathbb{R}^N, \ N \geq 1 \) and \( f : \mathbb{R}^{N \times n} \to \mathbb{R} \) is a sufficiently smooth integrand satisfying \((p, q)\)-growth of the form

Assumption 1. There exist \( 0 < \nu \leq L < \infty \) such that \( f \in C^2(\mathbb{R}^{N \times n}) \) satisfies for all \( z, \xi \in \mathbb{R}^{N \times n} \)
\[ \begin{cases} \nu |z|^p \leq f(z) \leq L (1 + |z|^q), \\ \nu |z|^{p-2} |\xi|^2 \leq (\partial^2 f(z) \xi, \xi) \leq L (1 + |z|^2)^{2+\frac{2}{q}} |\xi|^2. \end{cases} \]

Regularity properties of local minimizers of (1) in the case \( p = q \) are classical, see, e.g., [23]. A systematic regularity theory in the case \( p < q \) was initiated by Marcellini in [25, 26], see [27] for an overview. In particular, Marcellini [26] proves (among other things):

(A) If \( N = 1, 2 \leq p < q \) and \( \frac{q}{p} < 1 + \frac{2}{n-2} \) if \( n \geq 3 \), then every local minimizer \( u \in W^{1,q}_{loc}(\Omega) \) of (1) satisfies \( u \in W^{1,\infty}_{loc}(\Omega) \).

Local boundedness of the gradient implies that the non-standard growth of \( f \) and \( \partial^2 f \) in (1) becomes irrelevant and higher regularity (depending on the smoothness of \( f \)) follows by standard arguments, see e.g. [25, Chapter 7]. However, the \( W^{1,q}_{loc}(\Omega) \)-assumption on \( u \) in (A) is problematic: a priori we can only expect that minimizers of (1) are in the larger space \( u \in W^{1,p}_{loc}(\Omega) \). Hence, a first important step in the regularity theory for integral functionals with \((p, q)\)-growth is to improve gradient integrability for minimizers of (1). In [17], Esposito, Leonetti and Mingione showed

(B) If \( 2 \leq p < q \) and \( \frac{q}{p} < 1 + \frac{2}{n} \), then every local minimizer \( u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N) \) of (1) satisfies \( u \in W^{1,\infty}_{loc}(\Omega, \mathbb{R}^N) \).

The combination of (A) and (B) yields unconditional Lipschitz-regularity for minimizers of (1) in the scalar case under assumption \( \frac{q}{p} < 1 + \frac{2}{n} \), see [3] for a recent extension which includes in an optimal way a right-hand side. Only very recently, Bella and the author improved in [6] the results (A) and (B) (in the case \( N = 1 \)) in the sense that \( 'n' \) in the assumption on the ratio \( \frac{q}{p} \) can be replaced by \( 'n-1' \) for \( n \geq 3 \) (to be precise, [6, 25, 26] consider the non-degenerate version (4) of (2)). The argument in [6] relies on scalar techniques, e.g., Moser-iteration type arguments, and thus cannot be extended to the vectorial case \( N > 1 \). In this paper, we extend the gradient integrability result of [6] to the vectorial
case $N > 1$. Before we state the results, we recall a standard notion of local minimizer in the context of functionals with $(p, q)$-growth

**Definition 1.** We call $u \in W^{1,1}_{\text{loc}}(\Omega)$ a local minimizer of $\mathcal{F}$ given in (1) iff

$$f(Du) \in L^1_{\text{loc}}(\Omega)$$

and

$$\int_{\text{supp } \varphi} f(Du) \, dx \leq \int_{\text{supp } \varphi} f(Du + D\varphi) \, dx$$

for any $\varphi \in W^{1,1}(\Omega, \mathbb{R}^N)$ satisfying $\text{supp } \varphi \subseteq \Omega$.

The main result of the present paper is

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and suppose Assumption 1 is satisfied with $2 \leq p < q < \infty$ such that

$$\frac{q}{p} < 1 + \frac{2}{n - 1}.$$  \hspace{1cm} (3)

Let $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, $u \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^N)$.

As mentioned above, higher gradient integrability is a first step in the regularity theory for integral functionals with $(p, q)$-growth, see [11, 18, 19, 7] for further higher integrability results under $(p, q)$-conditions. Clearly, we cannot expect to improve from $W^{1,q}_{\text{loc}}$ to $W^{1,\infty}_{\text{loc}}$ for $N > 1$, since this even fails in the classic setting $p = q$, see [30]. Direct consequences of Theorem 1 are higher differentiability and a further improvement in gradient integrability in the form:

(i) (Higher differentiability). In the situation of Theorem 1 it holds $|\nabla u|^{\frac{2}{p} - 2} \nabla u \in W^{1,2}_{\text{loc}}(\Omega)$, see Theorem 3.

(ii) (Higher integrability). Sobolev inequality and (i) imply $\nabla u \in L^{\kappa p}_{\text{loc}}(\Omega, \mathbb{R}^{N \times n})$ with $\kappa = \frac{2}{n - 2}$.

Note that $\kappa p > q$ provided $\frac{q}{p} < 1 + \frac{2}{n - 2}$.

A further, on first glance less direct, consequence of Theorem 1 is partial regularity of minimizers of (1), see, e.g., [1, 7, 10, 28], for partial regularity results under $(p, q)$-conditions. For this, we slightly strengthen the assumptions on the integrand and suppose

**Assumption 2.** There exist $0 < \nu \leq L < \infty$ such that $f \in C^2(\mathbb{R}^{N \times n})$ satisfies for all $z, \xi \in \mathbb{R}^{N \times n}$

$$\begin{cases}
\nu |z|^p \leq f(z) \leq L(1 + |z|^q), \\
\nu (1 + |z|^2)^{\frac{2}{p} - 2} |\xi|^2 \leq \langle \partial^2 f(z) \xi, \xi \rangle \leq L(1 + |z|^2)^{\frac{2}{p} - 2} |\xi|^2.
\end{cases}$$ \hspace{1cm} (4)

In [7], Bildhauer and Fuchs prove partial regularity under Assumption 2 with $\frac{q}{p} < 1 + \frac{2}{n}$ ([7] contains also more general conditions including, e.g., the subquadratic case). Here we show

**Theorem 2.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, and suppose Assumption 2 is satisfied with $2 \leq p < q < \infty$ such that (3). Let $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, there exists an open set $\Omega_0 \subset \Omega$ with $|\Omega \setminus \Omega_0| = 0$ such that $\nabla u \in C^{0,\alpha}(\Omega_0, \mathbb{R}^{N \times n})$ for each $0 < \alpha < 1$.

We do not know if (3) in Theorem 1 and 2 is optimal. Classic counterexamples in the scalar case $N = 1$, see, e.g., [22, 26], show that local boundedness of minimizers can fail if $\frac{q}{p}$ is too large depending on the dimension $n$. In fact, [26, Theorem 6.1] and the recent boundedness result [24] show that $\frac{4}{p} - \frac{1}{q} \leq \frac{1}{n - 1}$ is the sharp condition ensuring local boundedness in the scalar case $N = 1$ (for sharp results under additional structure assumptions, see, e.g., [14, 21]).

For non-autonomous functionals, i.e., $\int_{\Omega} f(x, Du) \, dx$, rather precise sufficiently & necessary conditions are established in [19], where the conditions on $p, q$ and $n$ has to be balanced with the (Hölder)-regularity in space of the integrand. However, if the integrand is sufficiently smooth in space, the regularity theory in the non-autonomous case essentially coincides with the autonomous case, see [10]. Currently, regularity theory for non-autonomous integrands with non-standard growth, e.g. $p(x)$-Laplacian or double phase functionals are a very active field of research, see, e.g., [2, 12, 13, 15, 16, 29].
Coming back to autonomous integral functionals: In [11] higher gradient integrability is proven assuming so-called 'natural' growth conditions, i.e., no upper bound assumption on $\partial^2 f$, under the relation $\frac{q}{p} < 1 + \frac{1}{n-1}$. Moreover, in two dimensions we cannot improve the previous results on higher differentiability and partial regularity of, e.g., [7, 17], see [8] for a full regularity result under Assumption 2 with $n = 2$ and $\frac{q}{p} < 2$.

Let us briefly describe the main idea in the proof of Theorem 1 and from where our improvement compared to earlier results comes from. The main point is to obtain suitable a priori estimates for minimizers that may already be in $W^{1,q}_\text{loc}(\Omega, \mathbb{R}^N)$. The claim then follows by a known regularization and approximation procedure, see, e.g., [17]. For minimizers $v \in W^{1,q}_\text{loc}(\Omega, \mathbb{R}^N)$ a Caccioppoli-type inequality

$$\int_\Omega \eta^2 |D((1 + |Dv|^{1/2} Dv)|^2 \lesssim \int |\nabla \eta|^2 (1 + |Dv|^q)$$

is valid for all sufficiently smooth cut-off functions $\eta$, see Lemma 1. Very formally, the Caccioppoli inequality (5) can be combined with Sobolev inequality and a simple interpolation inequality to obtain

$$\|Dv\|_{L^p}^p \lesssim \|D((1 + |Dv|^{1/2} Dv)|^2 \|_{L^q}^q \lesssim \|Dv\|_{L^q}^q \lesssim \|Dv\|_{L^q}^q \|Dv\|_{L^q}^{(1-\theta)q},$$

where $\theta = \frac{1}{p} - \frac{1}{q} \in (0, 1)$ and $\kappa = \frac{n}{n-2}$. The $\|Dv\|_{L^q}$-factor on the right-hand side can be absorbed provided we have $\frac{q}{p} < 1$, but this is precisely the 'old' $(p, q)$-condition $\frac{q}{p} < 1 + \frac{2}{n}$, this type of argument was previously rigorously implemented in, e.g., [7, 18]. Our improvement comes from choosing a cut-off function $\eta$ in (5) that is optimized with respect to $v$, which enables us to use Sobolev inequality on $n - 1$-dimensional spheres which gives the desired improvement, see Section 3. This idea has its origin in joint works with Bella [4, 5] on linear non-uniformly elliptic equations.

With Theorem 1 at hand, we can follows the arguments of [7] almost verbatim to prove Theorem 2. In Section 4, we sketch (following [7]) a corresponding $\varepsilon$-regularity result from which Theorem 2 follows by standard methods.

2. Preliminary results

In this section, we gather some known facts. We begin with a well-known higher differentiability result for minimizers of (1) under the assumption that $u \in W^{1,q}_\text{loc}(\Omega, \mathbb{R}^N)$:

**Lemma 1.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and suppose Assumption 1 is satisfied with $2 \leq p < q < \infty$. Let $v \in W^{1,q}_\text{loc}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, $|Dv|^\frac{q-2}{2} Dv \in W^{1,2}_\text{loc}(\Omega, \mathbb{R}^{n \times n})$ and there exists $c = c(n, \mathcal{F}, N, p, q) \in [1, \infty)$ such that for every $Q \in \mathbb{R}^{n \times n}$

$$\int_\Omega \eta^2 |D((1 + |Dv|^\frac{q-2}{2} Dv)|^2 dx \leq c \int_\Omega (1 + |Dv|^q)^\frac{q}{q-2} |Dv - Q|^2 |\nabla \eta|^2 dx \quad \text{for all } \eta \in C_0^1(\Omega).$$

The Lemma 1 is known, see e.g. [7, 17, 26]. Since we did not find a precise reference for estimate (6), we included a prove here essentially the argument of [17].

**Proof of Lemma 1.** Without loss of generality, we suppose $\nu = 1$ the general case $\nu > 0$ follows by replacing $f$ with $f/\nu$ (and thus $L$ with $L/\nu$). Throughout the proof, we write $\lesssim$ if $\leq$ holds up to a multiplicative constant depending only on $n, N, p$ and $q$.

Thanks to the assumption $v \in W^{1,q}_\text{loc}(\Omega, \mathbb{R}^N)$, the minimizer $v$ satisfies the Euler-Largrange equation

$$\int_\Omega (\partial f(Dv), D\varphi) dx = 0 \quad \text{for all } \varphi \in W^{1,q}_0(\Omega, \mathbb{R}^N)$$

(for this we use that the convexity and growth conditions of $f$ imply $|\partial f(z)| \leq c(1 + |z|^q - 1)$ for some $c = c(L, \mathcal{F}, N, q) < \infty$). Next, we use the difference quotient method, to differentiate the above equation: For $s \in \{1, \ldots, n\}$, we consider the difference quotient operator

$$\tau_s, h v := \frac{1}{h}(v(\cdot + h\xi_s) - v) \quad \text{where } v \in L^{1,\text{loc}}(\mathbb{R}^n, \mathbb{R}^N),$$

where $\xi_s$ is a unit vector in the $s$-direction.
Fix \( \eta \in C^1_c(\Omega) \). Testing (7) with \( \varphi := \tau_{s,h}(\eta^2(\tau_{s,h}(v - \ell_Q))) \in W^{1,q}_0(\Omega) \), where \( \ell_Q(x) = Qx \), we obtain
\[
(\text{I}) := \int_\Omega \eta^2 \langle \tau_{s,h} \partial f(Dv), \tau_{s,h} Dv \rangle \, dx \\
= -2 \int_\Omega \eta \langle \tau_{s,h} \partial f(Dv), \tau_{s,h} (v - \ell_Q) \otimes \nabla \eta \rangle \, dx =: (\text{II}).
\]

Writing \( \tau_{s,h} \partial f(Dv) = \frac{1}{\eta} \partial f(Dv + th_{s,h} Dv) \bigg|_{t=0} \), the fundamental theorem of calculus yields
\[
\int_0^1 \int_\Omega \eta^2 \langle \partial^2 f(Dv + th_{s,h} Dv) \rangle \tau_{s,h} Dv, \tau_{s,h} Dv \rangle \, dt \, dx = (\text{I}) \\
(\text{II}) = -2 \int_0^1 \int_\Omega \eta \langle \partial^2 f(Dv + th_{s,h} Dv) \rangle \tau_{s,h} Dv, (\tau_{s,h} v - Qe_s) \otimes \nabla \eta \rangle \, dt \, dx,
\]
where we use \( \tau_{s,h} \ell_Q = Qe_s \). Youngs inequality yields
\[
\| (\text{II}) \| \leq \frac{1}{4} (\text{I}) + 2(\text{III}),
\]
where
\[
(\text{III}) := \int_\Omega \int_0^1 \langle \partial^2 f(Dv + th_{s,h} Dv) \rangle (\tau_{s,h} v - Qe_s) \otimes \nabla \eta, (\tau_{s,h} v - Qe_s) \otimes \nabla \eta \rangle \, dt \, dx.
\]
Combining (8), (9) with the assumptions on \( \partial^2 f \), see (2), with the elementary estimate
\[
|\tau_{s,h}(|Dv|^{\frac{p}{q-2}} Dv)|^2 \lesssim \int_0^1 |Dv + th_{s,h} Dv|^{\frac{p}{q-2}} |\tau_{s,h} Dv|^2 \, dt
\]
for \( h > 0 \) sufficiently small (see e.g. [17, Lemma 3.4]), we obtain
\[
\int_\Omega \eta^2 |\tau_{s,h}(|Dv|^{\frac{p}{q-2}} Dv)|^2 \, dx \\
\lesssim \int_\Omega \int_0^1 \eta^2 |Dv + th_{s,h} Dv|^{\frac{p}{q-2}} |\tau_{s,h} Dv|^2 \, dt \, dx \leq (\text{I}) \\
\leq 4(\text{III}) \leq 4L \int_\Omega \int_0^1 (1 + |Dv + th_{s,h} Dv|^{q-2}) |\nabla \eta|^2 |\tau_{s,h} v - Qe_s|^2 \, dt \, dx.
\]
Estimate (10), the fact \( v \in W^{1,q}_{\text{loc}}(\Omega) \) and the arbitrariness of \( \eta \in C^1_c(\Omega) \) and \( s \in \{1, \ldots, n\} \) yield \( |Dv|^{\frac{p}{q-2}} Dv \in W^{1,2}_{\text{loc}}(\Omega) \). Sending \( h \) to zero in (10), we obtain
\[
\int_\Omega \eta^2 |\partial_s(|Dv|^{\frac{p}{q-2}} Dv)|^2 \, dx \lesssim L \int_\Omega (1 + |Dv|^{q-2}) |\nabla \eta|^2 |\partial_s v - Qe_s|^2 \, dx
\]
the desired estimate (6) follows by summing over \( s \).

Next, we state a higher differentiability result under the more restrictive Assumption 2 which will be used in the proof of Theorem 2.

**Lemma 2.** Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), and suppose Assumption 2 is satisfied with \( 2 \leq p < q < \infty \). Let \( v \in W^{1,q}_{\text{loc}}(\Omega, \mathbb{R}^N) \) be a local minimizer of the functional \( F \) given in (1). Then, \( h := (1 + |Dv|^2)^{\frac{q}{q-2}} \in W^{1,2}_{\text{loc}}(\Omega) \) and there exists \( c = c(\frac{1}{p}, n, N, p, q) \in [1, \infty) \) such that for every \( Q \in \mathbb{R}^{N \times n} \)
\[
\int_\Omega \eta^2 \nabla h^2 \, dx \leq c \int_\Omega (1 + |Dv|^2)^{\frac{q}{q-2}} |Dv - Q|^2 |\nabla \eta|^2 \, dx \quad \text{for all } \eta \in C^1_c(\Omega).
\]
A variation of Lemma 2 can be found in [7] and we only sketch the proof.
Proof of Lemma 2. With the same argument as in the proof of Lemma 1 but using (4) instead of (2), we obtain $v \in W_{\text{loc}}^{2,2}(\Omega, \mathbb{R}^N)$ and the Caccioppoli inequality

\begin{equation}
\int_{\Omega} \eta^2(1 + |Dv|^2) \frac{1}{1 + t^2} |D^2v|^2 \, dx \leq c \int_{\Omega} (1 + |Dv|^2) \frac{1}{1 + t^2} |Dv - Q|^2|\nabla\eta|^2 \, dx \quad \text{for all } \eta \in C_c^0(\Omega),
\end{equation}

where $c = c\left(\frac{c}{2}, n, N, p, q\right) < \infty$. Formally, the chain-rule implies

\begin{equation}
|\nabla h|^2 \leq c(1 + |Dv|^2) \frac{1}{1 + t^2} |D^2v|^2,
\end{equation}

where $c = c(n, p) < \infty$, and the claimed estimate (11) follows from (12) and (13). In general, we are not allowed to use the chain rule, but the above reasoning can be made rigorous: Consider a truncated version $h_m$ of $h$, where $h_m := \Theta_m(|Dv|)$ with

\[\Theta_m(t) := \begin{cases} 
(1 + t^2)^{\frac{r}{2}} & \text{if } 0 \leq t \leq m \\
(1 + m^2)^{\frac{r}{2}} & \text{if } t \geq m
\end{cases}.
\]

For $h_m$ we are allowed to use the chain-rule and (12) together with (13) with $h$ replaced by $h_m$ imply (11) with $h$ replaced by $h_m$. The claimed estimate follows by taking the limit $m \to \infty$, see [7, Proposition 3.2] for details.

The following technical lemma is contained in [6] (see also [4, proof of Lemma 2.1, Step 1]) and plays a key role in the proof of Theorem 1.

Lemma 3 ([6], Lemma 3). Fix $n \geq 2$. For given $0 < \rho < \sigma < \infty$ and $v \in L^1(B_\sigma)$, consider

\[J(\rho, \sigma, v) := \inf \left\{ \int_{B_\sigma} |v||\nabla\eta|^2 \, dx \mid \eta \in C^1_0(B_\sigma), \eta \geq 0, \eta(1) = 1 \text{ in } B_\rho \right\}.
\]

Then for every $\delta \in (0, 1]$ \n
\begin{equation}
J(\rho, \sigma, v) \leq (\sigma - \rho)^{-(1+\frac{r}{2})} \left( \int_{\rho}^\sigma \left( \int_{\partial B_r} |v| \, d\mathcal{H}^{n-1} \right)^\delta \, dr \right)^{\frac{1}{\delta}}.
\end{equation}

For convenience of the reader we include a short proof of Lemma 3.

Proof of Lemma 3. Estimate (14) follows directly by minimizing among radial symmetric cut-off functions. Indeed, we obviously have for every $\varepsilon \geq 0$

\[J(\rho, \sigma, v) \leq \inf \left\{ \int_{\rho}^\sigma \eta'(r)^2 \left( \int_{\partial B_r} |v| + \varepsilon \right) \, dr \mid \eta \in C^1(\rho, \sigma), \eta(\rho) = 1, \eta(\sigma) = 0 \right\} =: J_{1d,\varepsilon}.
\]

For $\varepsilon > 0$, the one-dimensional minimization problem $J_{1d,\varepsilon}$ can be solved explicitly and we obtain

\begin{equation}
J_{1d,\varepsilon} = \left( \int_{\rho}^\sigma \left( \int_{\partial B_r} |v| \, d\mathcal{H}^{n-1} + \varepsilon \right)^{-1} \, dr \right)^{-1}.
\end{equation}

To see (15), we observe that using the assumption $v \in L^1(B_\sigma)$ and a simple approximation argument we can replace $\eta \in C^1(\rho, \sigma)$ with $\eta \in W^{1,\infty}(\rho, \sigma)$ in the definition of $J_{1d,\varepsilon}$. Let $\tilde{\eta} : [\rho, \sigma] \to [0, \infty)$ be given by

\[\tilde{\eta}(r) := 1 - \left( \int_{\rho}^\sigma b(r)^{-1} \, dr \right)^{-1} \int_{\rho}^r b(r)^{-1} \, dr \quad \text{where } b(r) := \int_{\partial B_r} |v| + \varepsilon.
\]

Clearly, $\tilde{\eta} \in W^{1,\infty}(\rho, \sigma)$ (since $b \geq \varepsilon > 0$), $\tilde{\eta}(\rho) = 1$, $\tilde{\eta}(\sigma) = 0$, and thus

\[J_{1d,\varepsilon} \leq \int_{\rho}^\sigma \tilde{\eta}'(r)^2 b(r) \, dr = \left( \int_{\rho}^\sigma b(r)^{-1} \, dr \right)^{-1}.
\]
The reverse inequality follows by Hölder’s inequality. Next, we deduce (14) from (15): For every $s > 1$, we obtain by Hölder inequality $\sigma - \rho = \int_0^s \left( \frac{1}{b} \right)^{\frac{1}{s-1}} \leq \left( \int_0^\sigma \left( \frac{1}{b} \right)^{\frac{1}{s-1}} \right)^{\frac{s-1}{s}}$ with $b$ as above, and by (15) that

$$J_{1d, \varepsilon} \leq (\sigma - \rho)^{-\frac{s}{s-1}} \left( \int_0^\sigma \left( \int_{\partial B_r} |v| + \varepsilon \right)^{s-1} \, dr \right)^{\frac{1}{s-1}}.$$

Sending $\varepsilon$ to zero, we obtain (14) with $\delta = s - 1 > 0$. $\square$

3. Higher integrability - Proof of Theorem 1

In this section, we prove the following higher integrability and differentiability result which clearly contains Theorem 1.

**Theorem 3.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and suppose Assumption 1 is satisfied with $2 \leq p < q < \infty$ such that $\frac{2}{q} < 1 + \min\{\frac{2}{n-1}, 1\}$. Let $u \in W^{1,1}_{{\text{loc}}} (\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, $u \in W^{1,q}_{{\text{loc}}} (\Omega, \mathbb{R}^N)$ and $|Du|^{\frac{p-2}{q}} Du \in W^{1,2}_{{\text{loc}}} (\Omega, \mathbb{R}^{N \times n})$. Moreover, for

$$(16) \quad \chi = \frac{n-1}{n-3} \quad \text{if} \ n \geq 4 \quad \chi \in \left( \frac{1}{2 - \frac{4}{p}}, \infty \right) \quad \text{if} \ n = 3 \quad \text{and} \quad \chi := \infty \quad \text{if} \ n = 2.$$

there exists $c = c(L, p, n, p, q, \chi) \in [1, \infty)$ such that for every $B_R(x_0) \subset \Omega$

$$(17) \quad \int_{B_{\frac{r}{\chi}}(x_0)} |Du|^q \, dx + R^2 \int_{B_{\frac{r}{\chi}}(x_0)} |D(|Du|^{\frac{p-2}{q}} Du)|^2 \, dx \leq c \left( \int_{B_R(x_0)} 1 + f(Du) \, dx \right)^{\frac{\alpha}{n}}$$

where

$$(18) \quad \alpha := \frac{1 - \frac{q}{\chi}}{2 - \frac{4}{p} - \frac{1}{\chi}}.$$

**Proof of Theorem 3.** Without loss of generality, we suppose $\nu = 1$ the general case $\nu > 0$ follows by replacing $f$ with $f/\nu$. Throughout the proof, we write $\lesssim$ if $\leq$ holds up to a multiplicative constant depending only on $L, n, N, p$ and $q$.

Following, e.g., [7, 17, 18], we consider the perturbed integral functionals

$$(19) \quad \mathcal{F}_\lambda(w) := \int_\Omega f_\lambda(Dw) \, dx, \quad \text{where} \quad f_\lambda(z) := f(z) + \lambda |z|^q \quad \text{with} \quad \lambda \in (0, 1).$$

We then derive suitable a priori higher differentiability and integrability estimates for local minimizers of $\mathcal{F}_\lambda$ that are independent of $\lambda \in (0, 1)$. The claim then follows with help of a by now standard double approximation procedure in spirit of [17].

**Step 1.** One-step improvement.

Let $v \in W^{1,1}_{{\text{loc}}} (\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}_\lambda$ defined in (19), $B_1 \subset \Omega$, and let $\chi > 1$ be defined in (16). We claim that there exists $c = c(L, n, p, q, \chi) \in [1, \infty)$ such that for all $\frac{2}{q} \leq \rho < \sigma \leq 1$ and every $\lambda \in (0, 1]$

$$\int_{B_1} 1 + f_\lambda(Dv) + \int_{B_2} |D(|Du|^{\frac{p-2}{q}} Du)|^2 \, dx \leq c \left( \int_{B_1} 1 + f_\lambda(Dv) + \int_{B_2} |D(|Du|^{\frac{p-2}{q}} Du)|^2 \, dx \right)^{\frac{\alpha}{n}}$$

with the understanding $\frac{\alpha}{n} = 1$ and

$$(20) \quad \int_{B_2} |D(|Du|^{\frac{p-2}{q}} Du)|^2 \, dx \lesssim \frac{1}{(\sigma - \rho)^{\frac{q}{p}} \lambda} \int_{B_1} 1 + f_\lambda(Dv) \, dx.$$

We next introduce a suitably chosen auxiliary function $\tilde{u} \in C^1(\Omega)$ such that $\tilde{u} \equiv 0$ on $\partial \Omega$ and $\tilde{u} \equiv 1$ on $\partial B_1$. This will be done in a moment. We claim that $u - \tilde{u}$ is a local minimizer of the functional $\mathcal{F}_\lambda$ and $\tilde{u}$ is a local minimizer of the functional $\mathcal{F}$.

**Step 2.** Application of the standard double approximation procedure.

...
The growth conditions of $f_\lambda$ and the minimality of $v$ imply $v \in W^{1,q}_\text{loc}(\Omega, \mathbb{R}^N)$ and thus by Lemma 1
\begin{equation}
\int_\Omega |D(|Du|^{\frac{q}{2}}Dv)|^2 \eta^2 \, dx \leq \int_\Omega (1 + |Du|^2)^{\frac{q}{2}} |Dv|^2 |\nabla \eta|^2 \, dx \quad \text{for all } \eta \in C_c^1(\Omega).
\end{equation}

Estimate (21) follows directly from (22) for $\eta \in C_c^1(B_\sigma)$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_\rho$ and $|\nabla \eta| \leq \frac{2}{\sigma - \rho}$, combined with $|z|^q \leq \frac{1}{\lambda} f_\lambda(z)$ and $\lambda \in (0, 1)$.

Hence, it is left to show (20). For this, we use a technical estimate which follows from Lemma 3 and Hölder's inequality: For given $0 < \rho < \sigma < \infty$ and $w \in L^q(B_\sigma)$ it holds
\begin{equation}
J(\rho, \sigma, |w|^q) \leq \left( \int_{B_\sigma \setminus B_\rho} |w|^p \right)^{\frac{1}{p}} \left( \int_{\sigma}^\rho \|w\|_{L^q(\partial B_r)}^p dr \right)^{\frac{1}{q}} \left( \frac{1}{\sigma - \rho} \right)^{1 + \frac{\sigma}{p}},
\end{equation}
where $J$ is defined as in Lemma 3. We postpone the derivation of (23) to the end of this step.

Combining (22) with $(1 + |Du|^2)^{\frac{q}{2}} |Dv|^2 \leq (1 + |Du|)^q$ and estimate (23) with $w = 1 + |Du|$, we obtain
\begin{equation}
\int_{B_\rho} |D(|Du|^{\frac{q}{2}}Dv)|^2 \, dx \leq \left( \int_{B_\sigma \setminus B_\rho} (1 + |Du|)^p \right)^{\frac{1}{p}} \left( \int_{\sigma}^\rho \|1 + |Du|\|_{L^q(\partial B_r)}^p dr \right)^{\frac{1}{q}} \left( \frac{1}{\sigma - \rho} \right)^{1 + \frac{\sigma}{p}}.
\end{equation}

Next, we use the Sobolev inequality on spheres to estimate the second factor on the right-hand side in (24): For $n \geq 2$ there exists $c = c(n, N, \chi) \in [1, \infty)$ such that for all $r > 0$
\begin{equation}
\|Du\|_{L^q(\partial B_r)}^p \leq c r^{(n-1)\left(\frac{1}{\chi} - 1\right)} \left( \int_{\partial B_r} |Du|^p \, dH^{n-1} + r^2 \int_{\partial B_r} |D(|Du|^{\frac{q}{2}}Dv)|^2 \, dH^{n-1} \right).
\end{equation}

Combining (25) with elementary estimates and assumption $\frac{1}{2} \leq \rho < \sigma \leq 1$, we obtain
\begin{equation}
\int_{\sigma}^\rho \|1 + |Du|\|_{L^q(\partial B_r)}^p dr \lesssim \int_{\sigma}^\rho 1 + \|Du\|_{L^q(\partial B_r)}^p dr \lesssim \int_{\sigma}^\rho 1 + \left( \int_{\partial B_r} |Du|^p + |D(|Du|^{\frac{q}{2}}Dv)|^2 \, dH^{n-1} \right) dr \lesssim \int_{B_\sigma \setminus B_\rho} 1 + |Du|^p + |D(|Du|^{\frac{q}{2}}Dv)|^2 \, dx.
\end{equation}

Combining (24) and estimate (26), we obtain
\begin{equation}
\int_{B_\rho} |D(|Du|^{\frac{q}{2}}Dv)|^2 \, dx \leq \frac{c \left( \int_{B_1} (1 + |Du|)^p \right)^{\frac{1}{p}} \left( \int_{B_\sigma \setminus B_\rho} 1 + |Du|^p + |D(|Du|^{\frac{q}{2}}Dv)|^2 \, dx \right)^{\frac{1}{q}} \left( \frac{1}{\sigma - \rho} \right)^{1 + \frac{\sigma}{p}}}{\left( \sigma - \rho \right)^{\frac{1}{p}} \left( \chi - 1 \right)\left( \frac{1}{\chi} - 1 \right)\left( \frac{1}{\chi} - 1 \right)\left( \frac{1}{\chi} - 1 \right)}.
\end{equation}

The claimed estimate (20) now follows since $|z|^p \leq f(z) \leq f_\lambda(z)$, $\frac{1}{\chi} - 1 = \frac{q}{p} - 1 = \frac{q}{p} \geq 1$ and $\int_{B_1} 1 + f_\lambda(Dv) \, dx \geq |B_1|$.

Finally, we present the computations regarding (23): Lemma 3 yields
\begin{equation}
J(\sigma, \rho, |w|^q) \leq \left( \int_{\sigma}^\rho \|w\|_{L^q(\partial B_r)}^q \, dr \right)^{\frac{q}{p}} \left( \frac{1}{\sigma - \rho} \right)^{1 + \frac{\sigma}{p}} \quad \text{for every } \delta > 0.
\end{equation}
Using two times the Hölder inequality, we estimate
\[
\left( \int_\rho^\sigma \|u\|_{L^q(\partial B_r)}^{q\delta} \, dr \right)^{\frac{1}{q\delta}} \leq \left( \int_\rho^\sigma \|u\|_{L^\theta(\partial B_r)}^{\theta q\delta} \|u\|_{L^{\frac{\theta q\delta}{\chi}}(\partial B_r)}^{\frac{1}{\chi}} \, dr \right)^{\frac{1}{\theta q\delta}} \quad \text{where } \frac{\theta}{p} + \frac{1 - \theta}{\chi} = \frac{1}{q}
\]
\[
\leq \left( \int_\rho^\sigma \|u\|_{L^\theta(\partial B_r)}^{\theta q\delta} \, dr \right)^{\frac{1}{\theta q\delta}} \left( \int_\rho^\sigma \|u\|_{L^{\frac{\theta q\delta}{\chi}}(\partial B_r)}^{\frac{1}{\chi}} \, dr \right)^{\frac{1}{\chi}} \quad \text{for every } s > 1.
\]
Inequality (23) follows with the admissible choice
\[
\delta = \frac{p}{q} \quad \text{and } \quad s = \frac{1}{1 - \theta} \quad \text{(recall } 1 - \theta = \frac{1}{p} - \frac{q}{\chi} \text{ and } p < q)
\]
which ensures \( \theta q\delta \frac{1}{\chi} = (1 - \theta)q\delta s = p. \)

**Step 2.** Iteration.
We claim that there exists \( c = c(L, n, N, p, q, \chi) \in [1, \infty) \) such that
\[
\int_{B_{\frac{3}{4}}} |Dv|^p + |D(|Dv|^{\frac{p-2}{2}} Dv)|^2 \, dx \leq c \left( \int_{B_1} 1 + f_\lambda(Dv) \, dx \right)^\alpha,
\]
where \( \alpha \) is defined in (18). For \( k \in \mathbb{N} \cup \{0\} \), we set
\[
\rho_k = \frac{3}{4} - \frac{1}{4^{1+k}} \quad \text{and } \quad J_k := \int_{B_1} 1 + f_\lambda(Dv) + \int_{B_{\rho_k}} |D(|Dv|^{\frac{p-2}{2}} Dv)|^2 \, dx.
\]
Estimate (21) and the choice of \( \rho_k \) imply for \( \lambda \in (0, 1] \)
\[
\sup_{k \in \mathbb{N}} J_k \leq \int_{B_1} 1 + f_\lambda(Dv) + \int_{B_{\frac{3}{4}}} |D(|Dv|^{\frac{p-2}{2}} Dv)|^2 \, dx \leq \frac{1}{\lambda} \int_{B_1} 1 + f_\lambda(Dv) \, dx < \infty.
\]
From (20) we deduce the existence of \( c = c(L, n, N, p, q, \chi) \in [1, \infty) \) such that for every \( k \in \mathbb{N} \)
\[
J_{k-1} \leq c4^{(1 + \frac{p}{n})k} \left( \int_{B_1} 1 + f_\lambda(Dv) \right)^{\frac{\chi}{\chi - 1}} \left( \int_{B_1} 1 + f_\lambda(Dv) \right)^{\frac{\chi}{\chi - 1}} J_k^{\frac{\chi}{\chi - 1}}.
\]
Assumption \( \frac{q}{p} < 1 + \min\{1, \frac{2}{n-1}\} \) and the choice of \( \chi \) yield
\[
\frac{\chi}{\chi - 1} \frac{q - p}{p} = \left\{ \begin{array}{ll}
\frac{q - p}{p} - 1 & \text{if } n = 2 \\
\frac{q - p}{p} \frac{\chi}{\chi - 1} & \text{if } n = 3 < 1, \\
\frac{q - p}{p} - 1 & \text{if } n \geq 4
\end{array} \right.
\]
where we use for \( n = 3 \) that \( \chi_{(16)} > \frac{1}{\chi - 1} > 0 \) and
\[
\frac{\chi}{\chi - 1} \frac{q - p}{p} < 1 \quad \Leftrightarrow \quad \frac{q - p}{p} < 1 - \frac{1}{\chi} \quad \Leftrightarrow \quad \frac{1}{\chi} < 2 - \frac{q}{p}.
\]
Hence, iterating (29) we obtain (using the uniform bound (28) on \( J_k \) and \( \frac{\chi - 1}{\chi - 1} \frac{q - p}{p} < 1 \))
\[
\int_{B_{\frac{3}{4}}} |Dv|^p + |D(|Dv|^{\frac{p-2}{2}} Dv)|^2 \, dx \leq J_0 \leq \left( \int_{B_1} 1 + f_\lambda(Dv) \right)^{\frac{\chi}{\chi - 1}} \left( \sum_{k=0}^{\infty} \left( \frac{\chi - 1}{\chi - 1} \frac{q - p}{p} \right)^k \right)
\]
and the claimed estimate (27) follow from
\[
\alpha = \frac{\chi}{\chi - 1} (1 - \frac{q}{\chi p}) \sum_{k=0}^{\infty} \left( \frac{\chi - 1}{\chi - 1} \frac{q - p}{p} \right)^k.
\]
**Step 3.** Conclusion.
We assume \( B_1 \subseteq \Omega \) and show that there exists \( c = c(L, n, N, p, q, \chi) \in [1, \infty) \)

\[
\int_{B_1^\#} |Du|^q \, dx \leq c \left( \int_{B_1} 1 + f(Du) \, dx \right)^{\frac{2q}{p}},
\]

where \( \alpha \) is given as in (18) above. Clearly, standard scaling, translation and covering arguments yield

\[
\int_{B_R(x_0)} |Du|^q \, dx \leq c \left( \int_{B_R(x_0)} 1 + f(Du) \, dx \right)^{\frac{2q}{p}}
\]

for all \( B_R(x_0) \subseteq \Omega \) and \( c = c(L, n, N, p, q, \chi) \in [1, \infty) \). The claimed estimate (17) then follows from Lemma 1.

Following [17], we introduce in addition to \( \lambda \in (0, 1) \) a second small parameter \( \varepsilon > 0 \) which is related to a suitable regularization of \( u \). For \( \varepsilon \in (0, \varepsilon_0) \), where \( 0 < \varepsilon_0 \leq 1 \) is such that \( B_{1+\varepsilon_0} \subseteq \Omega \), we set \( u_\varepsilon := u * \varphi_\varepsilon \) with \( \varphi_\varepsilon := \varepsilon^{-n} \varphi(\cdot/\varepsilon) \) and \( \varphi \) being a non-negative, radially symmetric mollifier, i.e. it satisfies

\[
\varphi \geq 0, \quad \text{supp } \varphi \subseteq B_1, \quad \int_{\mathbb{R}^n} \varphi(x) \, dx = 1, \quad \varphi(\cdot) = \tilde{\varphi}(|\cdot|) \quad \text{for some } \tilde{\varphi} \in C^\infty(\mathbb{R}).
\]

Given \( \varepsilon, \lambda \in (0, \varepsilon_0) \), we denote by \( v_{\varepsilon, \lambda} \in u_\varepsilon + W_0^{1,q}(B_1) \) the unique function satisfying

\[
\int_{B_1} f_\lambda(Dv_{\varepsilon, \lambda}) \, dx \leq \int_{B_1} f_\lambda(Dv) \, dx \quad \text{for all } v \in u_\varepsilon + W_0^{1,q}(B_1).
\]

Combining Sobolev inequality with the assumption \( \frac{q}{p} < 1 + \frac{2}{n-2} \) and estimate (27), we have

\[
\left( \int_{B_1^\#} |Dv_{\varepsilon, \lambda}|^q \, dx \right)^{\frac{2}{q}} \lesssim \int_{B_1^\#} |Dv_{\varepsilon, \lambda}|^p + |D(Dv_{\varepsilon, \lambda})|^2 \, dx \lesssim \left( \int_{B_1} 1 + f_\lambda(Dv_{\varepsilon, \lambda}) \, dx \right)^{\alpha} \leq \left( \int_{B_1} 1 + f(Du_\varepsilon) + \lambda |Du_\varepsilon|^q \, dx \right)^{\alpha} \leq \left( |B_1| + \int_{B_{1+\varepsilon}} f(Du) \, dx + \lambda \int_{B_1} |Du_\varepsilon|^q \, dx \right)^{\alpha},
\]

where we used Jensen’s inequality and the convexity of \( f \) in the last step. Similarly,

\[
\int_{B_1} |Dv_{\varepsilon, \lambda}|^p \, dx \leq \int_{B_1} f(Dv_{\varepsilon, \lambda}) \, dx \leq \int_{B_1} f(Du_\varepsilon) + \lambda |Du_\varepsilon|^q \, dx \leq \int_{B_{1+\varepsilon}} f(Du) \, dx + \lambda \int_{B_1} |Du_\varepsilon|^q \, dx.
\]

Fix \( \varepsilon \in (0, \varepsilon_0) \). In view of (33) and (34), we find \( u_\varepsilon \in u_\varepsilon + W_0^{1,p}(B_1) \) such that as \( \lambda \to 0 \), up to subsequence,

\[
v_{\varepsilon, \lambda} \rightharpoonup u_\varepsilon \quad \text{weakly in } W^{1,p}(B_1),
\]

\[
Dv_{\varepsilon, \lambda} \rightharpoonup Dw_\varepsilon \quad \text{weakly in } L^q(B_1^\#).
\]

Hence, a combination of (33), (34) with the weak lower-semicontinuity of convex functionals yield

\[
\| Dw_\varepsilon \|_{L^q(B_1^\#)} \leq \liminf_{\lambda \to 0} \| Dv_{\varepsilon, \lambda} \|_{L^{q\nu}(B_1^\#)} \lesssim \left( \int_{B_{1+\varepsilon}} f(Du) \, dx + 1 \right)^{\frac{q}{p}} \leq \int_{B_1} |Du_\varepsilon|^p \, dx \leq \int_{B_{1+\varepsilon}} f(Du) \, dx.
\]
Since \( w_\varepsilon \in u_\varepsilon + W^{1,q}_0(B_1) \) and \( u_\varepsilon \to u \) in \( W^{1,p}(B_1) \), we find by (36) a function \( w \in u + W^{1,p}_0(B_1) \) such that, up to subsequence, \( Dw_\varepsilon \rightharpoonup Dw \) weakly in \( L^p(B_1) \).

Appealing to the bounds (35), (36) and lower semicontinuity, we obtain

\[
\|Dw\|_{L^q(B_1)} \lesssim \left( \int_{B_1} f(Du) \, dx + 1 \right)^{\frac{q}{p}} \tag{37}
\]

\[
\int_{B_1} f(Dw) \, dx \leq \int_{B_1} f(Du) \, dx. \tag{38}
\]

Inequality (38), strict convexity of \( f \) and the fact \( w \in u + W^{1,p}_0(B_1) \) imply \( w = u \) and thus the claimed estimate (31) is a consequence of (37).

\[\square\]

4. Partial regularity - Proof of Theorem 2

Theorem 2 follows from the higher integrability statement Theorem 1, the \( \varepsilon \)-regularity statement of Lemma 4 below and a well-known iteration argument.

**Lemma 4.** Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 3 \), and suppose Assumption 2 is satisfied with \( 2 \leq p < q < \infty \) such that \( \frac{2}{q} < 1 + \frac{1}{n-1} \). Fix \( M > 0 \). There exists \( C^* = C^*(n, N, p, q, \frac{2}{p}, \frac{2}{q}, M) \in [1, \infty) \) such that for every \( \tau \in (0, \frac{1}{4}) \) there exists \( \varepsilon = \varepsilon(M, \tau) > 0 \) such that the following is true: Let \( u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N) \) be a local minimizer of the functional \( F \) given in (1). Suppose for some ball \( B_r(x) \subset \Omega \)

\[
|Du| \leq M,
\]

where we use the shorthand \( (w)_{x,r} := \int_{B_r(x)} w \, dy \), and

\[
E(x, r) := \int_{B_r(x)} |Du - (Du)_{x,r}|^2 \, dy + \int_{B_r(x)} |Du - (Du)_{x,r}|^q \, dy \leq \varepsilon,
\]

then

\[
E(x, r\tau) \leq C^* \tau^2 E(x, r).
\]

With the higher integrability of Theorem 3 and the Caccioppoli inequality of Lemma 2 at hand, we can prove Lemma 4 following almost verbatim the proof of the corresponding result [7, Lemma 4.1], which contain the statement of Lemma 4 under the assumption \( \frac{2}{p} < 1 + \frac{2}{n} \) (note that in [7] somewhat more general growth conditions including also the case \( 1 < p < q \) are considered). Thus, we only sketch the argument.

**Proof of Lemma 4.** Fix \( M > 0 \). Suppose that Lemma 4 is wrong. Then there exists \( \tau \in (0, \frac{1}{4}) \), a local minimizer \( u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N) \), which in view of Theorem 1 satisfies \( u \in W^{1,\frac{3}{2}}_{\text{loc}}(\Omega, \mathbb{R}^N) \), and a sequence of balls \( B_{r_m}(x_m) \subset B_R \) satisfying

\[
|(Du)_{x_m, r_m}| \leq M, \quad E(x_m, r_m) =: \lambda_m \quad \text{with} \quad \lim_{m \to \infty} \lambda_m = 0,
\]

\[
E(x_m, \tau r_m) > C^* \tau^2 \lambda_m^2,
\]

where \( C^* \) is chosen below. We consider the sequence of rescaled functions given by

\[
v_m(z) := \frac{1}{\lambda_m r_m}(u(x_m + r_m z) - a_m - r_m A_m z),
\]

where \( a_m := (u)_{x_m, r_m} \) and \( A_m := (Du)_{x_m, r_m} \). Assumption (39) implies \( \sup_m |A_m| \leq M \) and thus, up to subsequence,

\[
A_m \to A \in \mathbb{R}^{N \times n}.
\]

The definition of \( v_m \) yields

\[
Dv_m(z) = \lambda_m^{-1} (Du(x_m + r_m z) - A_m), \quad (v_m)_{0,1} = 0, \quad (Dv_m)_{0,1} = 0
\]
Assumptions (39) and (40) imply
\[(42) \quad \int_{B_1} |Dv_m|^2 \, dz + \lambda_m^{-2} \int_{B_1} |Dv_m|^q \, dz = \lambda_m^{-1} E(x_m, r_m) = 1,\]
\[(43) \quad \int_{B_r} |Dv_m - (Dv_m)_{0,r}|^2 \, dz + \lambda_m^{-2} \int_{B_r} |Dv_m - (Dv_m)_{0,r}|^q \, dz = \lambda_m^{-1} E(x_m, rr_m) > C_* r^2.\]
The bound (42) together with (41) imply the existence of \(v \in W^{1,2}(B_1, \mathbb{R}^N)\) such that, up to extracting a further subsequence,
\[v_m \to v \quad \text{in } W^{1,2}(B_1, \mathbb{R}^N), \quad \lambda_m Dv_m \to 0 \quad \text{in } L^2(B_1, \mathbb{R}^{N \times n}) \text{ and almost everywhere} \]
\[\lambda_m^{-2} v_m \to 0 \quad \text{in } W^{1,q}(B_1, \mathbb{R}^N).\]
The function \(v\) satisfies the linear equation with constant coefficients
\[\int_{B_1} (\partial^2 f(A) Dv, D\varphi) \, dz = 0 \quad \text{for all } \varphi \in C^1_0(B_1),\]
see, e.g., [20] or [7, Proposition 4.2]. Standard estimates for linear elliptic systems with constant coefficients imply \(v \in C^\infty_{\text{loc}}(B_1, \mathbb{R}^N)\) and existence of \(C^{**} < \infty\) depending only on \(n, N\) and the ellipticity contrast of \(\partial^2 f(A)\) (and thus on \(\frac{L}{\lambda}, p, q,\) and \(M\)) such that
\[(44) \quad \int_{B_r} |Dv - (Dv)_{0,r}|^2 \leq C^{**} r^2.\]
Choosing \(C^{*} = 2C^{**}\) we obtain a contradiction between (43) and (44) provided we have as \(m \to \infty\)
\[(45) \quad Dv_m \to Dv \quad \text{in } L^2_{\text{loc}}(B_1), \quad \lambda_m^{-2} Dv_m \to 0 \quad \text{in } L^2_{\text{loc}}(B_1).\]
Exactly as in [7, Proposition 4.3] (with \(\mu = 2 - p\), see also [9, Section 3.4.3.2] for a more detailed presentation of the proof), we have for all \(\rho \in (0, 1),\)
\[(47) \quad \lim_{m \to \infty} \int_{B_\rho} \int_0^1 (1 - s) \left(1 + |A_m + \lambda_m (Dv + s Dw_m)|^2\right)^{\frac{q-2}{2}} |Dw_m|^2 \, dz = 0,\]
where \(w := v_m - v,\) and thus the local \(L^2\)-convergence (45) follows. It is left to prove (46). For this, we introduce for \(\rho \in (0, 1)\) and \(T > 0\) the sequence of subsets
\[U_m := U_m(\rho, T) := \{ z \in B_\rho : \lambda_m |Dv_m| < T \}.\]
The local Lipschitz regularity of \(v, q > 2\) and (45) imply for all \(\rho \in (0, 1)\) and \(T > 0\)
\[\limsup_{m \to \infty} \int_{U_m(\rho, T)} \lambda_m^{q-2} |Dv_m|^q \, dz \leq \limsup_{m \to \infty} \int_{U_m(\rho, T)} \lambda_m^{q-2} |Dw_m|^q \, dz \leq \limsup_{m \to \infty} \int_{B_\rho} (M^{q-2} + \lambda_m^{q-2} |Dv|^{q-2}) |Dw_m|^2 \, dz = 0,\]
where here and for the rest of the proof \(\leq\) means \(\leq\) up to a multiplicative constant depending only on \(L, n, N, p\) and \(q\). Hence, it is left to show that there exists \(T > 0\) such that
\[\limsup_{m \to \infty} \int_{B_\rho \setminus U_m(\rho, T)} \lambda_m^{q-2} |Dv_m|^q \, dz \leq 0 \quad \text{for all } \rho \in (0, 1).\]
As in [7], we introduce a sequence of auxiliary functions
\[\psi_m := \lambda_m^{-1} \left[ (1 + |A_m + \lambda_m Dv_m|^2)^{\frac{q}{2}} - (1 + |A_m|^2)^{\frac{q}{2}} \right].\]
which satisfy
\[
\limsup_{m \to \infty} \| \psi_m \|_{W^{1,2}(B_\rho)} \lesssim c(\rho) \in [1, \infty) \quad \text{for all } \rho \in (0, 1).
\]
Indeed, by Theorem 1 and Lemma 2, we have for every \( \rho \in (0, 1) \) and every \( Q \in \mathbb{R}^{N \times n} \)
\[
\int_{B_{r_m}(x_m)} |\nabla (1 + |Du(x)|^2)^{\frac{q}{2}}| dx \lesssim r_m^{-2} c(\rho) \int_{B_{r_m}(x_m)} (1 + |\nabla u(x)|)^{q-2} |Du(x) - Q|^2 dx
\]
and thus by rescaling and setting \( Q = A_m \)
\[
\int_{B_1} |\nabla \psi_m|^2 dx \lesssim c(\rho) \int_{B_1} (1 + |A|^{q-2} + |\lambda_m Dv_m|^{q-2}) |Dv_m|^2 dx \quad (42)
\]
The identity \( \psi_m = \lambda_m^{-1} \int_0^1 \frac{d}{dt} \Theta(A_m + t\lambda_m v_m) dt \) with \( \Theta(F) := (1 + |F|^2)^{\frac{q}{2}} \) implies
\[
|\psi_m| \leq c(|Dv_m| + \lambda_m^{\frac{q-2}{q}} |Dv_m|^\frac{q}{2})
\]
(see [7, p. 555] for details) and thus with help of (47), we obtain
\[
\limsup_{m \to \infty} \int_{B_\rho} |\psi_m|^2 dx \lesssim c(\rho).
\]
For \( T \) sufficiently large (depending on \( M \)) there exists \( c > 0 \) such that for all \( z \in B_\rho \setminus U_m(\rho, T) \)
\[
\psi_m(z) \geq c \lambda_m^{-1} \lambda_m^{\frac{q}{2}} |Dv_m(z)|^{\frac{q}{2}} \quad \text{and thus } \quad \lambda_m^{2(1 + \frac{q}{2})} \psi_m^{\frac{2q}{p}}(z) \geq c \lambda_m^{-2} |Dv_m(z)|^{q}
\]
Estimate (48) and Sobolev embedding imply \( \limsup_{m \to \infty} \| \psi_m \|_{L^{\infty}(B_\rho)} \lesssim c(\rho) \in [1, \infty) \). Hence, using assumption \( \frac{q}{p} < 1 + \frac{2}{n-2} \) (and thus \( \lambda_m^{\frac{2q}{p}} < \lambda_m^{\frac{2q}{n-2}} \)), we obtain for every \( \rho \in (0, 1) \)
\[
\limsup_{m \to \infty} \int_{B_\rho \setminus U_m(\rho, T)} |Dv_m|^q dx \lesssim \lambda_m^{-2} |Dv_m|^q \lesssim M^{2(1 + \frac{q}{2})} \int_{B_\rho} |\psi_m|(z) dz \lesssim c(\rho) \limsup \lambda_m^{2(1 + \frac{q}{2})} = 0,
\]
which finishes the proof.

\[\square\]

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