On drift parameter estimation in models with fractional Brownian motion

Y. Kozachenko\textsuperscript{a}, A. Melnikov\textsuperscript{b,*} and Y. Mishura\textsuperscript{a}

\textsuperscript{a}Department of Probability, Statistics and Actuarial Mathematics, Mechanics and Mathematics Faculty, Taras Shevchenko National University of Kyiv, Volodymyrska, 60, 01601 Kyiv, Ukraine; \textsuperscript{b}Department of Mathematical and Statistical Sciences, University of Alberta, 632 Central Academic Building, Edmonton, AB, Canada T6G 2G1

(Received 15 September 2011; accepted 14 March 2014)

We consider a stochastic differential equation involving standard and fractional Brownian motion with unknown drift parameter to be estimated. We investigate the standard maximum likelihood estimate of the drift parameter, two non-standard estimates and three estimates for the sequential estimation. Model strong consistency and some other properties are proved. The linear model and Ornstein–Uhlenbeck model are studied in detail. As an auxiliary result, an asymptotic behaviour of the fractional derivative of the fractional Brownian motion is established.

Keywords: fractional Brownian motion; Brownian motion; parameter estimation; stochastic differential equation; sequential estimation

AMS Subject Classification: 60G22; 60J65; 60H10; 62F05

1. Introduction

Modern mathematical statistics tends to shift away from the standard statistical schemes based on independent random variables; besides, these days many statistical models are based on continuous time. Therefore, the corresponding statistical problems (e.g. parameter estimation) can be handled by methods of the theory of stochastic processes in addition to the standard statistical methods. Statistics for stochastic processes is well developed for diffusion processes and even for semimartingales (see, for instance [1]) but is still developing for the processes with long-range dependence. The latter is an integral part of stochastic processes, featuring a wide spectrum of applications in economics, physics, finance and other fields. The present paper is devoted to the parameter estimation in such models involving fractional Brownian motion (fBm) with Hurst parameter $H > \frac{1}{2}$ which is a well-known long-memory process. The paper also studies a mixed model based on both standard and fractional Brownian motion which turns out to be more flexible. One of the reasons to consider such model comes from the modern mathematical finance where it has become very popular to assume that the underlying random noise consists of two parts: the fundamental part, describing the economical background for the stock price, and the

\*Corresponding author. Email: melnikov@ualberta.ca

© 2014 Taylor & Francis
trading part, related to the randomness inherent to the stock market. In our case the fundamental part of the noise has a long memory while the trading part is a white noise.

Statistical aspects of models involving fractional Brownian motion were studied in many sources. One of the important problems in particular is the drift parameter estimation. In this regard, let us mention papers,[2,3] where the fractional Ornstein–Uhlenbeck process with unknown drift parameter originally was studied, books [4–6] and the references therein, and papers,[7–10] where the estimate was constructed via discrete observations. We shall also use the results for sequential estimates for semimartingales from Melnikov and Novikov.[11] In the present paper, we consider stochastic differential equations involving fractional Brownian motion along with equations involving both standard and fractional Brownian motion. We derive the standard maximum likelihood estimate and propose non-standard estimates for the unknown drift parameter. Several non-standard estimates for the drift parameter were proposed in [2] for the fractional Ornstein–Uhlenbeck process. We go a step ahead and propose non-standard estimates for the drift parameter in a general stochastic differential equation involving fBm. For the models involving only fractional Brownian motion, we compare properties of the estimates. In the mixed models the standard maximum likelihood estimate does not exist but the non-standard estimate works. To formulate the conditions for strong consistency of the non-standard estimates, we need to investigate the asymptotic behaviour of the fractional derivative of the fractional Brownian motion using the general growth results for Gaussian processes.

The paper is organized as follows. In Section 2 we introduce the models and the estimates: the maximum likelihood estimate, two non-standard estimates and three sequential estimates. Section 3 contains the main results concerning the strong consistency of all estimates and some additional properties of sequential estimates. The linear model and Ornstein–Uhlenbeck model are studied in detail. We generalize the result of strong consistency of the drift parameter estimate in the Ornstein–Uhlenbeck model from [3] to the model with variable coefficients. Auxiliary results for the asymptotic growth of Gaussian processes and, in particular, asymptotic growth of the fractional derivative of fBm is established in Section 4.

2. Model description and preliminaries

2.1. Model description

Let $(\Omega, \mathcal{F}, \tilde{\mathcal{F}}, P)$ be a complete probability space with filtration $\tilde{\mathcal{F}} = \{\mathcal{F}_t, t \in \mathbb{R}^+\}$ satisfying the standard assumptions. It is assumed that all processes under consideration are adapted to filtration $\tilde{\mathcal{F}}$.

**Definition 2.1** Fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ is a Gaussian process $B^H = \{B^H_t, t \in \mathbb{R}^+\}$ on $(\Omega, \mathcal{F}, P)$ featuring the properties

(a) $B^H_0 = 0$;
(b) $EB^H_t = 0$, $t \in \mathbb{R}^+$;
(c) $EB^H_t B^H_s = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H})$, $s, t \in \mathbb{R}^+$.

We consider the continuous modification of $B^H$ whose existence is guaranteed by the classical Kolmogorov theorem.

To describe the statistical model, we need to introduce the pathwise integrals w.r.t. fBm. Consider two non-random functions $f$ and $g$ defined on some interval $[a, b] \subset \mathbb{R}^+$. Suppose also that the following limits exist: $f(u+) := \lim_{\delta \to 0} f(u + \delta)$ and $g(u-) := \lim_{\delta \to 0} g(u - \delta)$, $a \leq u \leq b$. 

Let

$$f_{a+}(x) := (f(x) - f(a+))I_{(a,b)}(x), \quad g_{b-}(x) := (g(b-) - g(x))I_{(a,b)}(x).$$

Suppose that $f_{a+} \in L^p_+(L_p[a,b]), g_{b-} \in L^{1-a}_-(L_q[a,b])$ for some $p \geq 1$, $q \geq 1, 1/p + 1/q \leq 1, 0 \leq \alpha \leq 1$ (for the standard notation and statements concerning fractional analysis, see [12]).

Introduce the fractional derivatives

$$(D^\alpha_{a+} f_{a+})(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f_{a+}(s)}{(s-a)^\alpha} + \alpha \int_a^s \frac{f_{a+}(u) - f_{a+}(a)}{(s-u)^{1+\alpha}} \, du \right) I_{(a,b)}(x),$$

and

$$(D^{1-\alpha}_{b-} g_{b-})(x) = \frac{\alpha}{\Gamma(\alpha)} \left( \frac{g_{b-}(s)}{(s-b)^{1-\alpha}} + (1-\alpha) \int_s^b \frac{g_{b-}(u) - g_{b-}(b)}{(s-u)^{2-\alpha}} \, du \right) I_{(a,b)}(x).$$

It is known that $D^\alpha_{a+} f_{a+} \in L^p_+(L_p[a,b])$, $D^{1-\alpha}_{b-} g_{b-} \in L^q_-(L_q[a,b])$.

**Definition 2.2** [13,14] Under above assumptions, the generalized (fractional) Lebesgue–Stieltjes integral $\int_a^b f(x) \, dg(x)$ is defined as

$$\int_a^b f(x) \, dg(x) := e^{i\pi \alpha} \int_a^b (D^\alpha_{a+} f_{a+})(x)(D^{1-\alpha}_{b-} g_{b-})(x) \, dx + f(a+)(g(b-) - g(a+)),$$

and for $\alpha p < 1$ it can be simplified to

$$\int_a^b f(x) \, dg(x) := e^{i\pi \alpha} \int_a^b (D^\alpha_{a+} f)(x)(D^{1-\alpha}_{b-} g_{b-})(x) \, dx.$$

As follows from Samko et al., [12] for any $1 - H < \alpha < 1$ there exist fractional derivatives $D^{1-\alpha}_{b-} B^H_{b-}$ and $D^{1-\alpha}_{b-} B^H_{b-} \in L_\infty[a,b]$ for any $0 \leq a < b$. Therefore, for $f \in L^\alpha_{a+}(L_1[a,b])$ we can define the integral w.r.t. fBm in the following way.

**Definition 2.3** [13–15] The integral with respect to fBm is defined as

$$\int_a^b f \, dB^H := e^{i\pi \alpha} \int_a^b (D^\alpha_{a+} f)(x)(D^{1-\alpha}_{b-} B^H_{b-})(x) \, dx. \quad (1)$$

An evident estimate follows immediately from Equation (1):

$$\left| \int_a^b f \, dB^H \right| \leq \sup_{a \leq s \leq b} |(D^{1-\alpha}_{b-} B^H_{b-})(x)| \int_a^b |(D^\alpha_{a+} f)(x)| \, dx. \quad (2)$$

Let us take a Wiener process $W = \{W_t, t \in \mathbb{R}^+\}$ on probability space $(\Omega, \mathcal{F}, \mathcal{F}, P)$, possibly correlated with $B^H$. Assume that $H > \frac{1}{2}$ and consider a one-dimensional mixed stochastic differential equation involving both the Wiener process and the fractional Brownian motion

$$X_t = x_0 + \theta \int_0^t a(s, X_s) \, ds + \int_0^t b(s, X_s) \, dB^H_s + \int_0^t c(s, X_s) \, dW_s, \quad t \in \mathbb{R}^+, \quad (3)$$

where $x_0 \in \mathbb{R}$ is the initial value, $\theta$ is the unknown parameter to be estimated, the first integral in the right-hand side of Equation (3) is the Lebesgue–Stieltjes integral, the second integral is the generalized Lebesgue–Stieltjes integral introduced in Definition 2.3, and the third one is the Itô integral. From now on, we shall assume that the coefficients of Equation (3) satisfy the following assumptions on any interval $[0, T]$:
(A1) Linear growth of $a$ and $b$: for any $s \in [0, T]$ and any $x \in \mathbb{R}$

$$|a(s,x)| + |b(s,x)| \leq K(1 + |x|).$$

(A2) Lipschitz continuity of $a,c$ in space: for any $t \in [0, T]$ and $x,y \in \mathbb{R}$

$$|a(t,x) - a(t,y)| + |c(t,x) - c(t,y)| \leq K|x-y|.$$

(A3) Hölder continuity in time: function $b(t,x)$ is differentiable in $x$ and there exists $\beta \in (1 - H, 1)$ such that for any $s, t \in [0, T]$ and any $x \in \mathbb{R}$

$$|a(s,x) - a(t,x)| + |b(s,x) - b(t,x)| + |c(s,x) - c(t,x)| + |\partial_x b(s,x) - \partial_x b(t,x)| \leq K|s-t|^\beta.$$

(A4) Lipschitz continuity of $\partial_x b$ in space: for any $t \in [0, T]$ and any $x,y \in \mathbb{R}$

$$|\partial_x b(t,x) - \partial_x b(t,y)| \leq K|x-y|.$$

(A5) Boundedness of $c$ and $\partial_x b$: for any $s \in [0, T]$ and $x \in \mathbb{R}$

$$|c(s,x)| + |\partial_x b(s,x)| \leq K.$$

Here $K$ is a constant independent of $x, y, s$ and $t$. For an arbitrary interval $[0,T]$, $\alpha > 0$ and $\kappa = \frac{1}{2} \land \beta$ define the following norm:

$$\|f\|_{\infty,\alpha,[0,T]} = \sup_{s \in [0,T]} \left( |f(s)| + \int_0^s |f(s) - f(z)| (s-z)^{-1-\alpha} \, dz \right).$$

It was proved in [16] that under assumptions (A1)–(A5) there exists solution $X = \{X_t, \mathcal{F}_t, t \in [0, T]\}$ for Equation (3) on any interval $[0,T]$ which satisfies

$$\|X\|_{\infty,\alpha,[0,T]} < \infty \quad \text{a.s.}$$

for any $\alpha \in (1 - H, \kappa)$. This solution is unique in the class of processes satisfying Equation (4) for some $\alpha > 1 - H$.

**Remark 1** In case when components $W$ and $B^H$ are independent, assumptions for the coefficients can be relaxed, as it has been shown in [17]. More specifically, coefficient $c$ can be of linear growth, and $\partial_x b$ can be Hölder continuous up to some order less than 1.

### 2.2. Construction of drift parameter estimates: the standard maximum likelihood estimate

To start with, consider the case $c(t,x) \equiv 0$ which was studied, for instance, in [3,5]. Recall some facts from the theory of drift parameter estimation in this case. Consider the equation

$$X_t = x_0 + \theta \int_0^t a(s, X_s) \, ds + \int_0^t b(s, X_s) \, dB^H_s, \quad t \in \mathbb{R}^+.$$  \(5\)

Let assumptions (A1) and (A3) with $c \equiv 0$ hold on any interval $[0,T]$, together with the following assumptions:

(A5) Lipschitz continuity of $a,b$ in space: for any $t \in [0, T]$ and $x,y \in \mathbb{R}$

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq K|x-y|.$$
(A′) Hölder continuity of $\partial_x b(t,x)$ in space: there exists such $\rho \in \left(\frac{3}{2} - H, 1\right)$ that for any $t \in [0,T]$ and $x, y \in \mathbb{R}$

$$|\partial_x b(t,x) - \partial_x b(t,y)| \leq D|x - y|^\rho.$$ 

Then, according to Nualart and Rascănul,[15] solution for Equation (5) exists on any interval $[0,T]$ and is unique in the class of processes satisfying Equation (4) for some $\alpha > 1 - H$.

In addition, suppose that the following assumption holds:

(B1) $b(t,X_t) \neq 0, t \in [0,T]$ and $a(t,X_t)/b(t,X_t)$ is a.s. Lebesgue integrable on $[0,T]$ for any $T > 0$.

Denote $\psi(t,x) = a(t,x)/b(t,x)$, $\varphi(t) := \psi(t,X_t)$. Also, let the kernel

$$l_H(t,s) = c_H s^{1/2-H}(t-s)^{1/2-H} I_{\{0 < s < t\}},$$

with $c_H = (\Gamma(3 - 2H)/2H \Gamma(\frac{3}{2} - H) \Gamma(H + \frac{1}{2}))^{1/2}$, and introduce the integral

$$J_t = \int_0^t l_H(t,s) \varphi(s) \, ds = c_H \int_0^t (t-s)^{1/2-H} s^{1/2-H} \varphi(s) \, ds.$$ (6)

Finally, let $M^H_t = \int_0^t l_H(t,s) \, dB^H_s$ be Gaussian martingale with square bracket $\langle M^H \rangle_t = t^{-2H}$ (Molchan martingale, see [18]).

Consider two processes:

$$Y_t = \int_0^t b^{-1}(s,X_s) \, dX_s = \theta \int_0^t \varphi(s) \, ds + B^H_t$$

and

$$Z_t = \int_0^t l_H(t,s) \, dY_s = \theta J_t + M^H_t.$$ 

Remark 2 Note that the transformation from $X$ to $Z$ does not lead to loss of information since we can present $Y$ (consequently, $X$) via $Z$ and Volterra kernel introduced in Theorem 5.2.[18] So, these processes generate the same filtration.

Also, note that we can rewrite process $Z$ as

$$Z_t = \int_0^t l_H(t,s) b^{-1}(s,X_s) \, dX_s,$$

so $Z$ is a functional of the observable process $X$. The following smoothness condition for the function $\psi$ [5, Lemma 6.3.2] ensures the semimartingale property of $Z$. 
Lemma 2.4 Let $\psi = \psi(t,x) \in C^1(\mathbb{R}^+) \times C^2(\mathbb{R})$. Then for any $t > 0$

\[ J'(t) = (2 - 2H)C_H \psi(0,x_0)t^{1-2H} + \int_0^t l_H(t,s)(\psi'_x(s,X_s) + \theta \psi'_x(s,X_s)a(s,X_s)) \, ds \]

\[ - \left( H - \frac{1}{2} \right) c_H \int_0^t s^{-1/2-H}(t - s)^{1/2-H} \int_0^s (\psi'_x(u,X_u) + \theta \psi'_x(u,X_u)a(u,X_u)) \, du \, ds \]

\[ + (2 - 2H)c_H t^{1-2H} \int_0^t s^{2H-3} \int_0^s u^{3/2-H}(s - u)^{1/2-H} \psi'_x(u,X_u)b(u,X_u) \, dB_u^H \, ds \]

\[ + c_H t^{-1} \int_0^t u^{3/2-H}(t - u)^{1/2-H} \psi'_x(u,X_u)b(u,X_u) \, dB_u^H, \quad (7) \]

where

\[ C_H = B \left( \frac{3}{2} - H, \frac{3}{2} - H \right) c_H = \left( \frac{\Gamma(3/2 - H)}{2H\Gamma(H + 1/2)\Gamma(3 - 2H)} \right)^{1/2}, \]

and all of the involved integrals exist a.s.

Remark 3 Suppose that $\psi(t,x) \in C^1(\mathbb{R}^+) \times C^2(\mathbb{R})$ and limit $\zeta(0) = \lim_{s \to 0} \zeta(s)$ exists a.s., where $\zeta(s) = s^{1/2-H} \varphi(s)$. In this case $J(t)$ can be presented as

\[ J(t) = c_H \int_0^t (t - s)^{1/2-H} \zeta(s) \, ds = \frac{c_H 3/2 - H}{3/2 - H} \zeta(0) + c_H \int_0^t (t - s)^{3/2 - H} \zeta'(s) \, ds, \]

and $J'(t)$ from Equation (7) can be simplified to

\[ J'(t) = c_H t^{1/2-H} \zeta(0) + \int_0^t l_H(t,s) \left( \frac{1}{2} - H \right) s^{-1/2} \varphi(s) + \psi'_x(s,X_s) + \theta \psi'_x(s,X_s)a(s,X_s) \right) \, ds \]

\[ + \int_0^t l_H(t,s)\psi'_x(s,X_s)b(s,X_s) \, dB_s^H. \]

Same way as $Z$, processes $J$ and $J'$ are functionals of $X$. It is more convenient to consider process $\chi(t) = (2 - 2H)^{-1}J'(t)t^{2H-1}$, so that

\[ Z_t = (2 - 2H)\theta \int_0^t \chi(s)s^{1-2H} \, ds + M^H_t = \theta \int_0^t \chi(s) \, d\langle M^H \rangle_s + M^H_t. \]

Suppose that the following conditions hold:

\[(B_2) \quad E\langle T \rangle := E \int_0^T \chi_s^2 \, d\langle M^H \rangle_s < \infty \text{ for any } T > 0, \]

\[(B_3) \quad I_\infty := \int_0^\infty \chi_s^2 \, d\langle M^H \rangle_s = \infty \text{ a.s.} \]

Then we can consider the maximum likelihood estimate

\[ \theta_T^{(1)} = \frac{\int_0^T \chi_s \, dZ_s}{\int_0^T \chi_s^2 \, d\langle M^H \rangle_s} = \theta + \frac{\int_0^T \chi_s \, dM^H_s}{\int_0^T \chi_s^2 \, d\langle M^H \rangle_s}. \]

Condition (B_2) ensures that process $\int_0^t \chi_s \, dM^H_s, t > 0$ is a square-integrable martingale, and condition (B_3) alongside with the law of large numbers for martingales ensure that $\int_0^T \chi_s \, dM^H_s / \int_0^T \chi_s^2 \, d\langle M^H \rangle_s \to 0$ a.s. as $T \to \infty$. Summarizing, we arrive at the following result.[5]

Proposition 2.5 Let $\psi(t,x) \in C^1(\mathbb{R}^+) \times C^2(\mathbb{R})$ and assumptions (A_1), (A_3), (A_5), (A_4) and (B_1)-(B_3) hold. Then estimate $\theta_T^{(1)}$ is strongly consistent as $T \to \infty$. 

2.3. Construction of drift parameter estimates: two non-standard estimates

In case when \( c = 0 \), it is possible to construct another estimate for parameter \( \theta \), preserving the structure of the standard maximum likelihood estimate. Similar approach was applied in [2] to the fractional Ornstein–Uhlenbeck process with constant coefficients. We shall use process \( Y \) to define the estimate as

\[
\theta_T^{(2)} = \frac{\int_0^T \varphi_s \, dY_s}{\int_0^T \varphi_s^2 \, ds} = \theta + \frac{\int_0^T \varphi_s \, dB^H_s}{\int_0^T \varphi_s^2 \, ds}. \tag{8}
\]

If we return to general equation (3) with non-zero \( c \), then it is impossible to construct reason-
able maximum likelihood estimate of parameter \( \theta \). Therefore, we construct the estimate of the
same type as above. More exactly, suppose that the following assumption holds:

\( (C_1) \ c(t, X_t) \neq 0, t \in [0, T], a(t, X_t)/c(t, X_t) \) is a.s. Lebesgue integrable on \([0, T]\) for any \( T > 0 \)
and there exists generalized Lebesgue–Stieltjes integral \( \int_0^T (b(t, X_t)/c(t, X_t)) \, dB^H_t \).

Define functions \( \psi_1(t, x) = a(t, x)/c(t, x) \) and \( \psi_2(t, x) = b(t, x)/c(t, x) \), processes \( \psi_i(t) = \psi_i(t, X_t), i = 1, 2 \) and process

\[
Y_t = \int_0^t c^{-1}(s, X_s) \, dX_s = \theta \int_0^t \varphi_1(s) \, ds + \int_0^t \varphi_2(s) \, dB^H_s + W_t.
\]

Evidently, \( Y \) is a functional of \( X \) and is observable. Assume additionally that the generalized
Lebesgue–Stieltjes integral \( \int_0^T \varphi_1(t) \varphi_2(t) \, dB^H_t \) exists and

\( (C_2) \) for any \( T > 0 \) \( \int_0^T \varphi_1^2(s) \, ds < \infty \).

Denote \( \vartheta(s) = \varphi_1(s) \varphi_2(s) \). We can consider the following estimate of parameter \( \theta \)

\[
\theta_T^{(3)} = \frac{\int_0^T \varphi_1(s) \, dY_s}{\int_0^T \varphi_1^2(s) \, ds} = \theta + \frac{\int_0^T \vartheta(s) \, dB^H_s}{\int_0^T \varphi_1^2(s) \, ds} + \frac{\int_0^T \varphi_1(s) \, dW_s}{\int_0^T \varphi_1^2(s) \, ds}. \tag{9}
\]

Estimate \( \theta_T^{(3)} \) preserves the traditional form of maximum likelihood estimates for diffusion
models. The right-hand side of Equation (9) provides a stochastic representation of \( \theta_T^{(3)} \). We shall
use it to investigate the strong consistency of this estimate.

2.4. Construction of drift parameter estimates: sequential estimates

Return to model (5) and suppose that conditions \((B_1)–(B_3)\) hold. For any \( h > 0 \) consider the
stopping time

\[
\tau(h) = \inf \left\{ t > 0 : \int_0^t \chi_s^2 \, d\langle M^H \rangle_s = h \right\}.
\]

Under conditions \((B_1)–(B_2)\) we have \( \tau(h) < \infty \) a.s. and \( \int_0^{\tau(h)} \chi_s^2 \, d\langle M^H \rangle_s = h \). The sequential
maximum likelihood estimate has a form

\[
\theta_T^{(1)} = \frac{\int_0^{\tau(h)} \chi_s \, dZ_s}{h} = \theta + \frac{\int_0^{\tau(h)} \chi_s \, dM^H_s}{h}. \tag{10}
\]

Sequential versions of estimates \( \theta_T^{(2)} \) and \( \theta_T^{(3)} \) have a form

\[
\theta_T^{(2)} = \theta + \frac{\int_0^{\tau(h)} \varphi_s \, dB^H_s}{h},
\]

\[
\theta_T^{(3)} = \theta + \frac{\int_0^{\tau(h)} \vartheta(s) \, dB^H_s}{\int_0^{\tau(h)} \varphi_1^2(s) \, ds}. \tag{11}
\]
where
\[ \nu(h) = \inf \left\{ t > 0 : \int_0^t \varphi^2(s) \, ds = h \right\} \]
and
\[ \theta_{\nu_1}(h) = \theta + \frac{\int_0^{\nu_1(h)} \vartheta(s) \, dB_s^H}{h} + \frac{\int_0^{\nu_1(h)} \varphi_1(s) \, dW_s}{h}, \]
where
\[ \nu_1(h) = \inf \left\{ t > 0 : \int_0^t \varphi_1^2(s) \, ds = h \right\}. \]

To provide an exhaustive study of the introduced estimates, we will need a number of auxiliary facts about Gaussian processes. These facts are presented in the next section. Technical proofs may be found in appendix.

3. Main results

3.1. General results on strong consistency

In this section we shall establish conditions for strong consistency of \( \theta_2(T) \) and \( \theta_3(T) \).

**Theorem 3.1** Let assumptions \( (A_1), (A_3), (A'_2), (A'_4), (B_1) \) and \( (B_2) \) hold and let function \( \varphi \) satisfy the following assumption:

(B_4) There exists such \( \alpha > 1 - H \) and \( p > 1 \) that

\[ \frac{T^{H+\alpha-1}(\log T)^p \int_0^T |(D_0^\alpha \varphi)(s)| \, ds}{\int_0^T \varphi_1^2 \, ds} \to 0 \text{ a.s. as } T \to \infty. \quad (11) \]

Then estimate \( \theta_2(T) \) is correctly defined and strongly consistent as \( T \to \infty \).

**Proof** We must prove that \( \int_0^T \varphi_s \, dB_s^H / \int_0^T \varphi_s^2 \, ds \to 0 \) a.s. as \( T \to \infty \). According to Equation (2)

\[ \left| \int_0^T \varphi_s \, dB_s^H \right| \leq \sup_{0 \leq s \leq T} |(D_0^\alpha \varphi)(s)| \int_0^T |(D_0^\alpha \varphi)(s)| \, ds. \]

Furthermore, according to Theorem 4.4, for any \( p > 1 \) there exists a random variable \( \xi = \xi(p) \) independent of \( T \) such that for any \( T > 0 \)

\[ \sup_{0 \leq t \leq T} |(D_0^\alpha \varphi)(s)| \leq \xi(p) T^{H+\alpha-1}(\log T)^p, \]

which concludes the proof. ■

Relation (11) ensures convergence \( \int_0^T \varphi_s \, dB_s^H / \int_0^T \varphi_s^2 \, ds \to 0 \) a.s. in the general case. In a particular case when function \( \varphi \) is non-random and integral \( \int_0^T \varphi_s \, dB_s^H \) is a Wiener integral w.r.t. the fractional Brownian motion, conditions for existence of this integral are simpler since assumption (11) can be simplified.
Theorem 3.2 Let assumptions \((A_1), (A_3), (A'_3) \), \((A'_4) \), \((B_1)\) and \((B_2)\) hold and let function \(\varphi\) be non-random and satisfy the following assumption:

\((B_5)\) There exists such \(p > 0\) that

\[
\lim_{T \to \infty} \sup_{r} \frac{T^{2H-1+p}}{\int_0^T \varphi^2(t) \, dt} < \infty.
\]

Then estimate \(\theta_T^{(2)}\) is strongly consistent as \(T \to \infty\).

Proof It follows from [19] and the Hölder inequality that for any \(r > 0\)

\[
E \left| \int_0^T \varphi(s) \, dB^H_s \right|^r \leq C(H, r) \|\varphi\|_{L^r_{[0, T]}} \leq C(H, r) \|\varphi\|_{L^r_{[0, T]}} T^{(H-1/2)r}. \]

Denote \(F_T = \left| \int_0^T \varphi(t) \, dB^H_t \right| / \int_0^T \varphi^2(t) \, dt\). Also, for any \(N > 1\) and any \(\varepsilon > 0\) define event \(A_N = \{F_N > \varepsilon\}\). Then

\[
P(A_N) \leq \varepsilon^{-r} E \left| \int_0^N \varphi(s) \, dB^H_s \right|^r \leq \varepsilon^{-r} C(H, r) \|\varphi\|_{L^r_{[0, N]}} \|\varphi\|_{L^r_{[0, N]}} \leq \varepsilon^{-r} C(H, r) N^{(H-1/2)r}. \]

Under condition \((B_5)\) we have \(P(A_N) \leq C(H, r, p) N^{-rp/2}\). If \(r > 2/p\), then it follows immediately from the Borel–Cantelli lemma that series \(\sum P(A_N)\) converges, whence \(F_N \to 0\) a.s. as \(N \to \infty\).
Now estimate the residual

\[
R_N = \sup_{T \in [N, N+1]} |F_T - F_N|. \]

Evidently,

\[
R_N \leq \sup_{T \in [N, N+1]} \left| \int_N^T \varphi(t) \, dB^H_t \right| + F_N, \]

and it is sufficient to estimate

\[
R_N^{(1)} = \sup_{T \in [N, N+1]} \left| \int_N^T \varphi(t) \, dB^H_t \right| \leq \sup_{T \in [N, N+1]} \left| \int_N^T \varphi(t) \, dB^H_t \right| / \int_0^T \varphi^2(t) \, dt := R_N^{(2)}. \]

According to Theorem 1.10.3 from [5] and the Hölder inequality

\[
E \left( \sup_{T \in [N, N+1]} \left| \int_N^T \varphi(t) \, dB^H_t \right| \right)^r \leq C(H, r) \|\varphi\|_{L^r_{[N, N+1]}} \leq C(H, r) \|\varphi\|_{L^r_{[0, N+1]}}, \]

Now we can use condition \((B_5)\) to conclude that for any \(\varepsilon > 0\)

\[
P(R_N^{(2)} > \varepsilon) \leq C(H, r) \varepsilon^{-r} \|\varphi\|_{L^r_{[N, N+1]}} \|\varphi\|_{L^r_{[0, N]}} \leq C(H, r) \varepsilon^{-r} N^{-r(H-1+p)}. \]

We can set \(r > 1/(2H - 1 + p)\) and apply the Borel–Cantelli lemma again. Then we obtain that \(R_N^{(2)} \to 0\) a.s. as \(N \to 0\), which means that \(\theta_T^{(2)}\) is strongly consistent.

Theorem 3.3 Let assumptions \((C_1)\) and \((C_2)\) hold, and, in addition,
(C₃) \( \int_0^T \varphi_1^2(s) \, ds = \infty \) a.s.

(C₄) There exist such \( \alpha > 1 - H \) and \( p > 1 \) that

\[
\frac{T^{H+\alpha-1}(\log T)^p \int_0^T |(D_{\alpha_{0\theta}}^\varphi)(s)| \, ds}{\int_0^T \varphi_1^2(s) \, ds} \to 0 \quad \text{a.s. as } \ T \to \infty. \tag{12}
\]

Then estimate \( \theta_T^{(3)} \) is strongly consistent as \( T \to \infty \).

**Proof** The last term in the right-hand side of Equation (9) tends to zero under condition (C₃). The proof of convergence of the second term repeats the proof of Theorem 3.1. \( \square \)

Similarly to Theorem 3.2, conditions stated in Theorem 3.3 can be simplified in case when function \( \vartheta \) is non-random.

**Theorem 3.4** Let assumptions (C₁) and (C₂) hold. Then, if functions \( \varphi_1 \) and \( \varphi_2 \) are non-random, function \( \varphi_1 \) satisfies condition (B₃), function \( \varphi_2 \) is bounded, then estimate \( \theta_T^{(3)} \) is strongly consistent as \( T \to \infty \).

Now we shall take a look at the properties of sequential estimates.

**Theorem 3.5** (a) Let assumptions (B₁)–(B₃) hold. Then estimate \( \theta_T^{(1)} \) is unbiased, efficient, strongly consistent, \( E(\theta_T^{(1)} - \theta)^2 = 1/h \), and for any estimate of the form

\[
\theta = \frac{\int_0^\tau \chi_s \, dZ_s}{\int_0^\tau \chi_s^2 \, d\langle M^H \rangle_s} = \theta + \frac{\int_0^\tau \chi_s \, dM^H_s}{\int_0^\tau \chi_s^2 \, d\langle M^H \rangle_s},
\]

with \( \tau < \infty \) a.s. and \( E\int_0^\tau \chi_s^2 \, d\langle M^H \rangle_s \leq h \) we have that

\[
E(\theta_T^{(1)} - \theta)^2 \leq E(\theta_T - \theta)^2.
\]

(b) Let function \( \varphi \) be separated from zero, \( |\varphi(s)| \geq c > 0 \) a.s. and satisfy the assumption: for some \( 1 - H < \alpha < 1 \) and \( p > 0 \)

\[
\int_0^{(v(h))} |(D_{0\varphi}^\varphi)(s)| \, ds \to 0 \quad \text{a.s.} \tag{13}
\]

as \( h \to \infty \). Then estimate \( \theta_T^{(2)} \) is strongly consistent.

(c) Let function \( \varphi_1 \) be separated from zero, \( |\varphi_1(s)| \geq c > 0 \) a.s. and let function \( \vartheta \) satisfy the assumption: for some \( 1 - H < \alpha < 1 \) and \( p > 0 \)

\[
\int_0^{(v_1(h))} |(D_{0\varphi}^\varphi)(s)| \, ds \to 0 \quad \text{a.s.} \tag{14}
\]

as \( h \to \infty \). Then estimate \( \theta_T^{(3)} \) is strongly consistent.

(d) Let function \( \vartheta \) be non-random, bounded and positive, \( \varphi_1 \) be separated from zero. Then estimate \( \theta_T^{(3)} \) is consistent in the following sense: for any \( p > 0 \), \( E|\theta - \theta_T^{(3)}|_p \to 0 \) as \( h \to \infty \).

**Proof** (a) Process \( \int_0^\tau \chi_s \, dM^H_s \) is a square-integrable martingale which implies that estimate \( \theta_T^{(1)} \) is unbiased. Besides, the results from Liptser and Shiryayev [1, Chapter 17] can be applied
to Equation (10) directly, therefore estimate $\theta_{r(h)}^{(1)}$ is efficient, $E(\theta_{r(h)}^{(1)} - \theta)^2 = 1/h$, and for any estimate of the form $\theta_r = \int_0^t \chi_s dZ_s / \int_0^t \chi_s^2 d(M^H)_s = \theta + \int_0^t \chi_s dM^H_s / \int_0^t \chi_s^2 d(M^H)_s$ with $\tau < \infty$ a.s. and $E \int_0^\tau \chi_s^2 d(M^H)_s \leq h$ we have that $E(\theta_{r(h)}^{(1)} - \theta)^2 \leq E(\theta_r - \theta)^2$. Strong consistency is also evident.

(b) We have that $\int_0^{\tau(h)} \varphi(s) d(B^H_s) \leq (\nu(h))^{H+\alpha-1+p} \int_0^{\tau(h)} |(D^\alpha \varphi)(s)| ds$. It is sufficient to note that $h = \int_0^{\tau(h)} \varphi^2(s) ds \geq c^2 \nu(h)$. The proof of statement (b) is now evident. Statement (c) is proved similarly.

(d) It was proved in [5] that in case of non-random bounded positive function $0 \leq \vartheta(s) \leq \vartheta^\gamma$, for any stopping time $\nu$

$$
\left( E \left( \sup_{0 \leq t \leq \nu} \left| \int_0^t \vartheta(s) d(B^H_s) \right| \right)^p \right)^{1/p} \leq C(H, p) \vartheta^\gamma \left( E(\nu^H) \right)^{1/p}.
$$

Furthermore, same as before, $|\varphi_1(s)| \geq c$ and $h = \int_0^{\tau(h)} \varphi^2_1(s) ds \geq c^2 \nu_1(h)$. These inequalities together with the Burkholder–Gundy inequality yield

$$
E|\theta - \theta_{r(h)}^{(3)}|^p \leq C(H, p) \left( \frac{\vartheta^\gamma}{c^2 h^{H-1} + h^{-p/2}} \right) \to 0 \quad \text{as } h \to \infty.
$$

\[\square\]

Remark 4 Another proof of statement (a) is contained in [6]. Assumptions (13) and (14) hold, for example, for bounded and Lipschitz functions $\varphi$ and $\vartheta$ correspondingly.

3.2. **Linear models and strong consistency**

(I) Consider the linear version of model (5)

$$
dX_t = \theta a(t) X_t dt + b(t) X_t d\lambda^H_t,
$$

where $a$ and $b$ are locally bounded non-random measurable functions. In this case solution $X$ exists, is unique and can be presented in the integral form

$$
X_t = x_0 + \theta \int_0^t a(s) X_s ds + \int_0^t b(s) X_s d\lambda^H_s = x_0 \exp \left\{ \theta \int_0^t a(s) ds + \int_0^t b(s) d\lambda^H_s \right\}.
$$

Suppose that function $b$ is non-zero and note that in this model

$$
\varphi(t) = \frac{a(t)}{b(t)}.
$$

Suppose that $\varphi(t)$ is also locally bounded and consider maximum likelihood estimate $\theta_{r(h)}^{(1)}$. According to Equation (6), to guarantee existence of process $J'$, we have to assume that the fractional derivative of order $\frac{3}{2} - H$ for function $\zeta(s) := \varphi(s)^{1/2-H}$ exists and is integrable. The sufficient conditions for the existence of fractional derivatives can be found in [12]. One of these conditions states:

$(B_6)$ Functions $\varphi$ and $\zeta$ are differentiable and their derivatives are locally integrable.

So, it is hard to conclude what is the behaviour of the maximum likelihood estimate for an arbitrary locally bounded function $\varphi$. Suppose that condition $(B_6)$ holds and limit $\zeta_0 = \lim_{s \to 0} \zeta(s)$
exists. In this case, according to Lemma 2.4 and Remark 3, process $J'$ admits both of the following representations:

$$J'(t) = (2 - 2H)C_H \varphi(0)t^{1 - 2H} + \int_0^t l_H(t, s)\varphi'(s)\,ds$$

$$- \left(H - \frac{1}{2}\right) c_H \int_0^t s^{1/2-H}(t - s)^{1/2-H} \int_0^s \varphi'(u)\,du\,ds$$

$$= c_H \xi_0 t^{1/2-H} + c_H \int_0^t (t - s)^{1/2-H} \xi'(s)\,ds,$$

and assuming $(B_3)$ also holds true, the estimate $\theta_T^{(1)}$ is strongly consistent. Let us formulate some simple conditions sufficient for the strong consistency. The proof is obvious and therefore is omitted.

**Lemma 3.6** If function $\varphi$ is non-random, locally bounded, satisfies $(B_0)$, limit $\xi(0)$ exists and one of the following assumptions hold:

(a) function $\varphi$ is not identically zero and $\varphi'$ is non-negative and non-decreasing;
(b) derivative $\xi'$ preserves the sign and is separated from zero;
(c) derivative $\xi'$ is non-decreasing and has a non-zero limit,

then the estimate $\theta_T^{(1)}$ is strongly consistent as $T \to \infty$.

**Example 3.7** Let the coefficients are constant, $a(s) = a \neq 0$ and $b(s) = b \neq 0$, then the estimate has a form $\theta_T^{(1)} = \theta + bM_T/aC_H T^{2-2H}$ and is strongly consistent. In this case assumption (a) holds. In addition, power functions $\varphi(s) = s^p$ are appropriate for $p > H - 1$: this can be verified directly from Equation (6).

Let us now apply estimate $\theta_T^{(2)}$ to the same model. It has a form (8). We can use Theorem 3.2 directly and under assumption $(B_3)$ estimate $\theta_T^{(2)}$ is strongly consistent. Note that we do not need any assumptions on the smoothness of $\varphi$, which is a clear advantage of $\theta_T^{(2)}$. We shall consider two more examples.

**Example 3.8** If the coefficients are constant, $a(s) = a \neq 0$ and $b(s) = b \neq 0$, then the estimate has a form $\theta_T^{(2)} = \theta + bB_T^H/aT$. We can refer to Theorem 3.2 and conclude that $\theta_T^{(2)}$ is strongly consistent. Alternatively, we can use Remark 7 which states that $|B_T^H| \leq \xi T^H (\log T)^p$ for any $p > 1$ and some random variable $\xi$, therefore $B_T^H/T \to 0$ a.s. as $T \to \infty$. In this case both estimates $\theta_T^{(1)}$ and $\theta_T^{(2)}$ are strongly consistent and $E(\theta - \theta_T^{(1)})^2 = \gamma^2 T^{2H-2}/a^2 C_H^2$ has the same asymptotic behaviour as $E(\theta - \theta_T^{(2)})^2 = \gamma^2 T^{2H-2}/a^2$.

**Example 3.9** If non-random functions $\varphi$ and $\xi$ are bounded on some fixed interval $[0, t_0]$ but $\xi$ is sufficiently irregular on this interval and has no fractional derivative of order $3/2 - H$ or higher then we cannot even calculate $J'(t)$ on this interval and it is hard to analyse the behaviour of the maximum likelihood estimate. However, if we assume that $\varphi(t) \sim t^{H-1+\rho}$ at infinity with some $\rho > 0$, then assumption $(B_3)$ holds and estimate $\theta_T^{(2)}$ is strongly consistent as $T \to \infty$. In this sense estimate $\theta_T^{(2)}$ is more flexible. Estimate $\theta_T^{(1)}$ was considered in [20].
(II) Consider a mixed linear model of the form
\[
dX_t = X_t(\theta a(t) \, dt + b(t) \, dB_t^H + c(t) \, dW_t), \tag{15}
\]
where \(a, b\) and \(c\) are non-random measurable functions. Assume that they are locally bounded. In this case solution \(X\) for Equation (15) exists, is unique and can be presented in the integral form
\[
X_t = x_0 \exp \left\{ \theta \int_0^t a(s) \, ds + \int_0^t b(s) \, dB_t^H + \int_0^t c(s) \, dW_s - \frac{1}{2} \int_0^t c^2(s) \, ds \right\}.
\]
In what follows assume that \(c(s) \neq 0\). We have that \(\varphi_1(t) = a(t)/c(t)\) and \(\varphi_2(t) = b(t)/c(t)\). Estimate \(\theta_T^{(3)}\) has a form
\[
\theta_T^{(3)} = \frac{\int_0^T \varphi_1(s) \, dY_s}{\int_0^T \varphi_1^2(s) \, ds} = \theta + \frac{\int_0^T \varphi_1(s) \varphi_2(s) \, dB_s^H}{\int_0^T \varphi_1^2(s) \, ds} + \frac{\int_0^T \varphi_1(s) \, dW_s}{\int_0^T \varphi_1^2(s) \, ds}. \tag{16}
\]
In accordance with Theorem 3.4, assume that function \(\varphi_1\) satisfies \((B_s)\) and \(\varphi_2\) is bounded. Then estimate \(\theta_T^{(3)}\) is strongly consistent. Evidently, these assumptions hold for the constant coefficients.

### 3.3. The fractional Ornstein–Uhlenbeck model and strong consistency

(I) Consider the fractional Ornstein–Uhlenbeck, or Vasicek, model with non-constant coefficients. It has a form
\[
dX_t = \theta(\alpha(t)X_t + b(t)) \, dt + \gamma(t) \, dB_t^H, \quad t \geq 0,
\]
where \(a, b\) and \(\gamma\) are non-random measurable functions. Suppose they are locally bounded and \(\gamma = \gamma(t) > 0\). The solution for this equation is a Gaussian process and has a form
\[
X_t = e'^{(A(t))} \left( x_0 + \theta \int_0^t b(s) \, e^{-\theta A(s)} \, ds + \int_0^t \gamma(s) \, e^{-\theta A(s)} \, dB_s^H \right) = E(t) + G(t),
\]
where \(A(t) = \int_0^t a(s) \, ds, E(t) = e'^{(A(t))} (x_0 + \theta \int_0^t b(s) \, e^{-\theta A(s)} \, ds)\) is a non-random function, \(G(t) = e'^{(A(t))} \int_0^t \gamma(s) \, e^{-\theta A(s)} \, dB_s^H\) is a Gaussian process with zero mean.
Denote \(c(t) = a(t)/\gamma(t), \ d(t) = b(t)/\gamma(t)\). Now we shall state the conditions for strong consistency of the maximum likelihood estimate.

**Theorem 3.10** Let functions \(a, c, d\) and \(\gamma\) satisfy the following assumptions:

\[(B_7) \ -a_1 \leq a(s) \leq -a_2 < 0, \ -c_1 \leq c(s) \leq -c_2 < 0, \ 0 < \gamma_1 \leq \gamma(s) \leq \gamma_2, \text{ functions } c \text{ and } d \text{ are continuously differentiable, } c' \text{ is bounded, } c'(s) \geq 0 \text{ and } c'(s) \to 0 \text{ as } s \to \infty.\]

Then estimate \(\theta_T^{(3)}\) is strongly consistent as \(T \to \infty\).
Proof. We shall check the conditions of Proposition 2.5. Obviously, \( \psi(t, x) = c(t)x + d(t) \in C^1(\mathbb{R}^+) \times C^2(\mathbb{R}) \) and

\[
J(t) = \int_0^t l_H(t, s)(d(s) + c(s)E(s)) \, ds + \int_0^t l_H(t, s)c(s)G(s) \, ds := F(t) + H(t).
\]

Furthermore, assumptions (A1), (A3), (A2'), (A4') and (B1) hold. Note that the trajectories of process \( G \) are a.s. Hölder up to order \( H \), whence

\[
\lim_{t \to 0} s^{1/2-H} c(s)G(s) = 0.
\]

Therefore

\[
J'(t) = F'(t) + H'(t) = F'(t) + \int_0^t l_H(t, s)f(s)G(s) \, ds + \int_0^t l_H(t, s)c(s) \gamma(s) \, dB^H_s,
\]

where \( f(s) = (\frac{1}{2} - H)s^{-1}c(s) + \dot{c}(s) + \theta a(s)c(s) \). Evidently, \( J' \) is Gaussian process with mean and variance that are bounded on any bounded interval. Therefore, condition (B2) holds. As for condition (B3), we must verify that \( I_\infty = \int_0^\infty (J'_t)^2 t^{2H-1} \, dt = \infty \) a.s. For any \( \lambda > 0 \) consider the moment generation function

\[
\Theta_T(\lambda) = E \exp(-\lambda I_T) = E \exp \left\{ -\lambda \int_0^T (J'_t)^2 t^{2H-1} \, dt \right\}
\]

and

\[
\Theta_\infty(\lambda) = E \exp(-\lambda I_\infty) = E \exp \left\{ -\lambda \int_0^\infty (J'_t)^2 t^{2H-1} \, dt \right\},
\]

so that \( \Theta_\infty(\lambda) = \lim_{T \to \infty} \Theta_T(\lambda) \). Evidently,

\[
\int_0^T (J'_t)^2 t^{2H-1} \, dt \geq T^{-1} \left( \int_0^T J'_t t^{H-1/2} \, dt \right)^2,
\]

whence

\[
\Theta_T(\lambda) \leq \Theta_T^{(1)}(\lambda) := E \exp \left\{ -\frac{\lambda}{T} \left( \int_0^T J'_t t^{H-1/2} \, dt \right)^2 \right\}.
\]

Random variable \( \int_0^T J'_t t^{H-1/2} \, dt \) is Gaussian with mean \( M(T) \) and variance \( \sigma^2(T) \), say. Note that for a Gaussian random variable \( \xi = m + \sigma N(0, 1) \) we can easily calculate

\[
E \exp(-a\xi^2) = (2a\sigma^2 + 1)^{-1/2} \exp \left\{ -\frac{am^2}{2a\sigma^2 + 1} \right\}.
\]

This value attains its maximum at the point \( m = 0 \). Hence, it is sufficient to prove that

\[
\lim_{T \to \infty} \Theta_T^{(2)}(\lambda) := \lim_{T \to \infty} E \exp \left\{ -\frac{\lambda}{T} \left( \int_0^T H'_t t^{H-1/2} \, dt \right)^2 \right\} = 0.
\]

However, it follows from Equation (17) that \( \Theta_T^{(2)}(\lambda) = (2\lambda\sigma^2_T/T + 1)^{-1/2} \), therefore to prove the strong consistency of the maximum likelihood estimate \( \hat{\theta}^{(1)}_T \), we only need to analyse the asymptotic behaviour of \( \sigma^2_T \). More specifically, we need to prove that \( \sigma^2_T/T \to \infty \) as \( T \to \infty \).
In what follows we apply the following formulae from [18,19] for Wiener integrals w.r.t. the fractional Brownian motion
\[
E \int_0^t g(s) \, dB_s^H \int_0^t h(s) \, dB_s^H = H(2H - 1) \int_0^t \int_0^t g(s_1) h(s_2) |s_1 - s_2|^{2H-2} \, ds_1 \, ds_2 \\
\leq C(H) \|g\|_{L^1([0,t])} \|h\|_{L^1([0,t])}.
\]

(a) Let \( \theta < 0 \). Divide \( \int_0^T H_I(t)^{H-1/2} \, dt \) into two parts:
\[
H_I^{(1)} = \int_0^T t^{H-1/2} \int_0^t l_H(t,s)f(s)G(s) \, ds \, dt
\]
and
\[
H_I^{(2)} = \int_0^T t^{H-1/2} \int_0^t l_H(t,s)c(s) \gamma(s) \, dB_s^H \, dt.
\]

Since functions \( c \) and \( \gamma \) are bounded from below and from above
\[
E(H_I^{(2)})^2 = C(H) \int_0^T \int_0^T \int_0^t \int_0^t \Pi_{i=1,2} l_H(t_i,s_i) \gamma(s_i) \, ds_1 \, ds_2 \, dt_1 \, dt_2
\times |s_1 - s_2|^{2H-2} ds_1 \, ds_2 \, dt_1 \, dt_2 \leq C(H) \int_0^T \int_0^t \Pi_{i=1,2} l_H(t_i,s_i) \, ds_1 \, ds_2 \, dt_1 \, dt_2 \times C(H) T^3
\]
as \( T \to \infty \).

Consider the behaviour of \( f \). Under assumption (B7) terms \( s^{-1} c(s) + c'(s) \) vanish at infinity, \( \theta a(s)c(s) \) is negative and separated from zero. Therefore, there exist \( C_i > 0, i = 1, 2 \) and \( s_0 > 0 \) such that \( -C_1 \leq f(s) \leq -C_2 \) for all \( s > s_0 \). Boundedness of \( f \) implies that \( E(H_I^{(1)})^2 \) has the same asymptotic behaviour as
\[
\int_0^T \int_0^T \int_0^t \int_0^t (\Pi_{i=1,2} l_H(t_i,s_i) (-f(s_i)))
\times \left( \int_{s_0}^{s_1} \int_{s_0}^{s_2} \gamma(u_1) \gamma(u_2) \exp \left\{ \theta \left( \int_{u_1}^{s_1} + \int_{u_2}^{s_2} \right) a(v) \, dv \right\} |u_1 - u_2|^{2H-2} \, du_1 \, du_2 \right) \, ds_1 \, ds_2 \, dt_1 \, dt_2
\geq C(H) \int_0^T \int_0^T \int_0^t \int_0^t (\Pi_{i=1,2} l_H(t_i,s_i))
\times \left( \int_{s_0}^{s_1} \int_{s_0}^{s_2} |u_1 - u_2|^{2H-2} \, du_1 \, du_2 \right) \, ds_1 \, ds_2 \, dt_1 \, dt_2 \times C(H) T^5.
\]

Relations (18) and (19) mean that the asymptotic behaviour of \( \sigma_I^2 \) is \( \sigma_I^2 \propto C(H) T^5 \) and \( \sigma_I^2 / T \to \infty \) as \( T \to \infty \).

(b) Let \( \theta > 0 \). The asymptotic behaviour of \( E(H_I^{(2)})^2 \) is the same as before, \( C(H) T^3 \), since it does not depend on \( \theta \). In what follows we denote the constants as \( C \) and it can change the value from line to line. As for \( E(H_I^{(1)})^2 \), at first note that for \( 0 \leq u < t \leq T \) \( l_H(t,u) \geq C_t^{1/2-H} T^{1/2-H} \). Moreover, form \( 0 \leq z \leq u \) \( A(u) - A(z) \geq -a_1 (u-z) \). Taking into account that any integration “increases” the power of the integral for one point, we can conclude that the asymptotic
behaviour of

\[
E(H_T^{(1)})^2 = \int_0^T \int_0^T \int_0^{H-1/2} \int_0^{H-1/2} \int_0^t \int_0^s l_H(t, u)l_H(s, v) e^{\theta A(u) + \theta A(v)} f(u)\]
\[
	imes f(v) \int_0^u \int_0^v e^{-\theta A(z) - \theta A(w)} y(z) y(w) |z - w|^{2H-2} \, dz \, dw \, du \, dv \, ds \, dt
\]

is the same as of the integral

\[
\int_0^T \int_0^T \int_0^{H-1/2} \int_0^{H-1/2} \int_0^t \int_0^s l_H(t, u)l_H(s, v) e^{\theta A(u) + \theta A(v)} f(u)\]
\[
	imes f(v) \int_0^u \int_0^v e^{-\theta A(z) - \theta A(w)} y(z) y(w) |z - w|^{2H-2} \, dz \, dw \, du \, dv \, ds \, dt
\]

On one hand,

\[
\int_0^T \int_0^T \int_0^{H-1/2} \int_0^{H-1/2} \int_0^t \int_0^s l_H(t, u)l_H(s, v) e^{\theta A(u) + \theta A(v)} f(u)\]
\[
	imes f(v) \int_0^u \int_0^v e^{-\theta A(z) - \theta A(w)} y(z) y(w) |z - w|^{2H-2} \, dz \, dw \, du \, dv \, ds \, dt
\]
\[
\geq CT^{1-2H} \int_0^T \int_0^T \int_0^{H-1/2} \int_0^{H-1/2} \int_0^t \int_0^s e^{-a_1(u-z) - a_1(v-w)} |z - w|^{2H-2} \, dz \, dw
\]

Furthermore,

\[
\int_0^u \int_0^v e^{-a_1(u-z) - a_1(v-w)} |z - w|^{2H-2} \, dz \, dw = e^{-a_1u - a_1v} \int_0^u \int_0^v e^{a_1z + a_1w} (z - w)^{2H-2} \, dw \, dz
\]
\[
= e^{-a_1u - a_1v} \int_0^u \int_0^v e^{2a_1z} e^{-a_1x} x^{2H-2} \, dx \, dz.
\]

Now, applying l’Hopital’s rule several times to the integral in the right-hand side of Equation (22), we get

\[
\lim_{T \to \infty} T^{-4} \int_0^T \int_0^T \int_0^{H-1/2} \int_0^{H-1/2} \int_0^t \int_0^s e^{-a_1(u-z) - a_1(v-w)} |z - w|^{2H-2} \, dz \, dw
\]
\[
\geq \lim_{T \to \infty} T^{-2} \int_0^T \int_0^T e^{-a_1u - a_1v} \int_0^u \int_0^v e^{2a_1z} e^{-a_1x} x^{2H-2} \, dx \, dz
\]
\[
= \lim_{T \to \infty} e^{-2a_1T} \int_0^T \int_0^v e^{2a_1z} e^{-a_1x} x^{2H-2} \, dx \, dz = \lim_{T \to \infty} \int_0^T e^{-a_1x} x^{2H-2} \, dx = C.
\]
So, \( \liminf_{T \to \infty} T^{-5+2H} E(H_T^{(1)})^2 \geq C(H) \). On the other hand
\[
\int_{s_0}^{T} \int_{s_0}^{T} H^{1/2} S^{1/2} \int_{s_0}^{T} \int_{s_0}^{T} l_H(t, u) l_H(s, v) e^{\theta(u)+\theta(v)} f(u) \\
\times f(v) \int_{0}^{u} \int_{0}^{v} e^{-\theta(z)-\theta(w)} \gamma(z) \gamma(w) |z - w|^{2H-2} dz dw du dv \, ds \, dt
\]
\[
\leq C \int_{s_0}^{T} \int_{s_0}^{T} l_H^{1/2} S^{1/2} \int_{s_0}^{T} \int_{s_0}^{T} l_H(t, u) l_H(s, v) \int_{0}^{u} \int_{0}^{v} |z - w|^{2H-2} dz dw du dv \, ds \, dt
\]
\[
\leq CT^5 \int_{0}^{1} l_H^{1/2} S^{1/2} \int_{0}^{1} l_H(t, u) l_H(s, v) \int_{0}^{u} \int_{0}^{v} |z - w|^{2H-2} dz dw du dv \, ds \, dt.
\]
(25)

It means that asymptotically \( E(H_T^{(1)} H_T^{(2)}) \) is between \( CT^{4-H} \) and \( CT^4 \). If we compare the asymptotical behaviour of \( E(H_T^{(2)})^2 \sim C(H) T^3 \), we can conclude that \( \frac{\sigma_T^2}{T} \geq C(H) T^{4-2H} \to \infty \) as \( T \to \infty \).

(c) Let \( \theta = 0 \). Then it is easy to verify that \( E(H_T^{(1)})^2 \sim C(H) T \) and we can refer to the case \( \theta > 0 \).

**Remark 5** At first sight a reader might believe that \( \sigma_T^2 / T \) may not converge to infinity because of [3], but in fact it follows directly from Equations (18)–(25). The assumptions of the theorem are fulfilled, for example, if \( a(s) = -1, b(s) = b \in \mathbb{R} \) and \( \gamma(s) = \gamma > 0 \). In this case we deal with a standard Ornstein–Uhlenbeck process \( X \) with constant coefficients that satisfies the equation
\[
dX_t = \theta(b - X_t) \, dt + \gamma \, dB^H_t, \quad t \geq 0.
\]

This model with constant coefficients was studied in [3] where the Laplace transform \( \Theta_T(\lambda) \) was calculated explicitly and strong consistency of \( \theta_T^{(1)} \) was established. Our idea was to generalize their result for the coefficients that are not constant, so that the method from [3] does not work however for the case when the coefficients are sufficiently close to constants so that the result \( \frac{\sigma_T^2}{T} \to \infty \) still holds. Therefore, our results generalize the statement of strong consistency from [3] to the case of variable coefficients.

(II) Consider a simple version of the Ornstein–Uhlenbeck model where \( a = \gamma = 1, b = x_0 = 0 \). Corresponding stochastic differential equation has a form \( dX_t = \theta X_t \, dt + dB^H_t, \ t \geq 0 \) with evident solution \( X_t = e^{\theta t} \int_0^t e^{-\theta s} dB^H_s \). Consider estimate \( \theta_T^{(2)} \)
\[
\theta_T^{(2)} = \frac{\int_0^T X_s \, dX_s}{\int_0^T X_s^2 \, ds} = \theta + \frac{\int_0^T X_s \, dB^H_s}{\int_0^T X_s^2 \, ds}.
\]

**Theorem 3.11** Let \( \theta > 0 \). Then estimate \( \theta_T^{(2)} \) is strongly consistent as \( T \to \infty \).

**Proof** Applying Remark 7 yields
\[
\left| \int_0^T e^{-\theta s} \, dB^H_s \right| \leq e^{-\theta T} |B_T^H| + \int_0^T e^{-\theta s} |B_t^H| \, ds \leq \xi \left( e^{-\theta T} T^{H+p} + \int_0^T e^{-\theta s} s^{H+p} \, ds \right) \leq \xi(\theta),
\]
where \( \xi(\theta) \) is a random variable independent of \( T \). Moreover, in a similar way we obtain that for any \( s < t \)
\[
\left| \int_s^t e^{-\theta u} \, dB^H_u \right| \leq \xi(\theta)(t - s).
\]
that the pseudometric space
where

Theorem 3.10, we can consider the moment generation function

\[ \eta(\theta) \]

and

\[ \int T \]

Using these bounds, we can multiply numerator and denominator in \( \theta_T^{(2)} \) by \( e^{-\theta T} \) and estimate

the integral \( \int_0^T X_s \, dB_s^H \) with the help of the bound (2) and Theorem 4.4 for any \( 1 - H < \alpha < \frac{1}{2} \) and \( q > 0 \) as

\[ \left| \int_0^T X_s \, dB_s^H \right| \leq \xi \zeta(\theta) T^{H + \alpha - 1 + q} \left( \int_0^T e^{\theta t} \left( \int_0^t e^{-\theta u} \, dB_u^H - \int_0^t e^{-\theta u} \, dB_u^H \right) \, dt \right) \]

\[ \leq \xi \zeta(\theta) T^{H + \alpha - 1 + q} \left( e^{\theta T} C(\theta, \alpha) + \int_0^T \int_0^T e^{\theta t} \left( \int_0^t e^{-\theta u} \, dB_u^H \right) \, dt \, ds \right) \]

\[ \leq \eta(\theta) T^{H + \alpha - 1 + q} e^{\theta T}, \]

where the random variable \( \eta(\theta) \) does not depend on \( T \). So, to prove the strong consistency of \( \theta_T^{(2)} \) it is sufficient to establish that \( T^{-H - \alpha + 1 + q} e^{-\theta T} \int_0^T X_s^2 \, ds \rightarrow \infty \) a.s. Similarly to the proof of Theorem 3.10, we can consider the moment generation function

\[ E \exp \left\{ -\lambda T^{-H - \alpha + 1 + q} e^{-\theta T} \int_0^T X_s^2 \, ds \right\} \leq E \exp \left\{ -\lambda T^{-H - \alpha + 1 + q} e^{-\theta T} \right\}

\times \left( \int_0^T e^{\theta s} \int_0^s e^{-\theta u} \, dB_u^H \, ds \right)^2 \]

\[ = (2\lambda T^{-H - \alpha + 1 + q} e^{-\theta T} \sigma_T^2 + 1)^{-1/2}, \]

where

\[ \sigma_T^2 = E \left( \int_0^T e^{\theta s} \int_0^s e^{-\theta u} \, dB_u^H \, ds \right)^2 = \int_0^T \int_0^T e^{\theta s + \theta t} \int_0^s e^{-\theta u} \, du \, dv \, ds \, dt \]

\[ \geq T^{2H-2} C(\theta) e^{2\theta T}, \]

whence the proof follows.

4. Auxiliary results for Gaussian processes related to the fractional Brownian motion

We start with the exponential maximal bound for a Gaussian process defined on an abstract pseudometric space, expressed in terms of the metric capacity of this space. This result is a particular case of the general theorem proved in [21, p.100].

Lemma 4.1 Let \( T \) be a non-empty set, \( X = \{X(t), \ t \in T\} \) be centred Gaussian process. Suppose that the pseudometric space \( (T, \rho) \) with pseudometric

\[ \rho(t, s) = (E(X(t) - X(s))^2)^{1/2} \]

is separable and process \( X \) is separable on this space. Also, let the following conditions hold:

\[ a := \sup_{t \in T} \left( E|X(t)|^2 \right)^{1/2} < \infty \]

and

\[ \int_0^a \left( \log N_T(u) \right)^{1/2} du < \infty, \]
where $N_T(u)$ is the number of elements in the minimal $u$-covering of space $(T, \rho)$. Then for any $\lambda > 0$ and any $\theta \in (0, 1)$ the following inequality holds:

$$E \exp \left\{ \lambda \sup_{t \in T} |X(t)| \right\} \leq 2Q(\lambda, \theta),$$

where

$$Q(\lambda, \theta) = \exp \left\{ \frac{\lambda^2 a^2}{2(1-\theta)^2} + \frac{2\lambda}{\theta(1-\theta)} \int_0^{\theta a} (\log(N_T(u)))^{1/2} \, du \right\}.$$

Consider set $T = \{ t = (t_1, t_2) \in \mathbb{R}_+^2 : 0 \leq t_2 \leq t_1 \}$ supplied with the distance

$$m(t, s) = |t_1 - s_1| \lor |t_2 - s_2|.$$

Assume random process $X = \{ X(t), t \in T \}$ satisfies the following conditions.

- $(D_1)$ Process $X$ is a centred Gaussian process on $T$, separable on metric space $(T, m)$.
- $(D_2)$ There exist $\beta > 0, \gamma > 0$ and a constant $C(\beta, \gamma)$ independent of $X$, $t$ and $s$ such that for any $t, s \in T$

$$E(X(t) - X(s))^2)^{1/2} \leq C(\beta, \gamma)(t_1 \lor s_1)^\beta (m(t, s))^{\gamma}.$$  \hspace{1cm} (26)

- $(D_3)$ There exist $\delta > 0$ and a constant $C(\delta)$ independent of $X$ and $t$ such that for any $t \in T$

$$E(X(t))^2)^{1/2} \leq C(\delta)t_1^\delta.$$  \hspace{1cm} (27)

Let us introduce the following notations. Let $A(t) > 1, t \geq 0$ be an increasing function such that $A(t) \to \infty$, $t \to \infty$. Consider an increasing sequence $b_0 = 0, b_\ell < b_{\ell+1}, \ell \geq 1$ and suppose that $b_\ell \to \infty, \ell \to \infty$. For $\delta_\ell = A(b_\ell)$ and $\kappa > 0$ we denote

$$S(\delta) = \sum_{\ell=0}^{\infty} b_{\ell+1}^\delta \delta_\ell^{-1}, \quad k_1 = \frac{\kappa}{2} \left( 1 + \frac{\beta}{\gamma} - \frac{\delta}{\gamma} \right), \quad B_1 = C(\delta)S(\delta),$$

$$C_1 = C_2 k^{-1/2} S(\delta + k_1) \quad \text{and} \quad C_2 = \frac{2^{(1-\kappa)/2}}{1-\kappa/2\gamma} (C(\delta))^{1-\kappa/2\gamma} (C(\beta, \gamma))^{\kappa/2\gamma}.$$

Now we shall present the auxiliary exponential maximal bound for a Gaussian process defined on $(T, m)$.

**Theorem 4.2** Let $\{ X(t), t \in T \}$ be a random process satisfying assumptions $(D_1)$–$(D_3)$. Let $0 \leq a < b$, set $T_{a,b} = \{ t = (t_1, t_2) \in T : a \leq t_1 \leq b, 0 \leq t_2 \leq t_1 \}$. Then for any $0 < \theta < 1, \lambda > 0$ and $0 < \kappa < 1$ and $2\gamma$ the following inequality holds:

$$E \exp \left\{ \lambda \sup_{t \in T_{a,b}} |X(t)| \right\} \leq 2\tilde{Q}(\lambda, \theta),$$

where

$$\tilde{Q}(\lambda, \theta) = \exp \left\{ \frac{\lambda^2(b^\delta C(\delta))^2}{2(1-\theta)^2} + \frac{2\lambda}{1-\theta} b^{\delta + k_1} \frac{C_2}{\theta^{\kappa/2\gamma} \kappa^{1/2}} \right\}.$$
Theorem 4.3

Define $J$

Proof. It follows from Equations (26) and (27) that

$$d := \sup_{t \in T_{a,b}} (E|X(t)|^2)^{1/2} \leq C(\delta) b^\delta$$

(28)

and

$$\sup_{m(t,s) \leq h, t,s \in T_{a,b}} (E(X(t) - X(s))^2)^{1/2} \leq \sigma(h) := C(\beta, \gamma) b^\beta h^\gamma.$$  

(29)

In turn, it follows from Equation (29) that

$$N_{T_{a,b}}(v) \leq \left( \frac{b - a}{2\sigma^{(-1)}(v)} + 1 \right) \left( \frac{b}{2\sigma^{(-1)}(v)} + 1 \right) \leq \left( \frac{(C(\beta, \gamma))^{1/\gamma} b^{1+\beta/\gamma}}{2v^{1/\gamma}} + 1 \right)^2.$$  

(30)

Define $J(\theta d) := \int_0^{\theta d} (\log N_{T_{a,b}}(u))^{1/2} du$. It follows from Equation (30) that

$$J(\theta d) \leq \int_0^{\theta d} \sqrt{2} \left[ \log \left( \frac{(C(\beta, \gamma))^{1/\gamma} b^{1+\beta/\gamma}}{2v^{1/\gamma}} + 1 \right) \right]^{1/2} dv.$$  

(31)

For any $0 < \kappa \leq 1$,

$$\log(1 + x) = \frac{1}{\kappa} \log(1 + x)^\kappa \leq \frac{x^\kappa}{\kappa}.$$  

Now, let $\kappa \in (0, 1 \wedge 2\gamma)$. Then it follows from Equations (28) and (31) that

$$J(\theta d) \leq \frac{\sqrt{2}}{\kappa^{1/2}} \int_0^{\theta d} \frac{(C(\beta, \gamma))^{1/\gamma} b^{1+\beta/\gamma})^{\kappa/2}}{(2v^{1/\gamma})^{\kappa/2}} dv$$

$$= \frac{\sqrt{2}}{\kappa^{1/2}(1 - \kappa/2\gamma)} \left( \frac{(C(\beta, \gamma))^{1/\gamma} b^{1+\beta/\gamma}}{2} \right)^{\kappa/2} (\theta d)^{1-\kappa/2\gamma} \leq b^{\delta + \kappa} \frac{1-\kappa/2\gamma}{\kappa^{1/2}} C_2.$$  

Separability of $X$ on $(T, m)$ and relation (29) ensure separability of $X$ on $(T, \rho)$ with $\rho(t,s) = (E(X(t) - X(s))^2)^{1/2}$. Hence the statement of the theorem follows from Lemma 4.1.\Halmos

Now we are ready to state the general result concerning the asymptotic maximal growth of a Gaussian process defined on $(T, m)$.

Theorem 4.3. Let $X = \{X(t), t \in T\}$ satisfy assumptions $(D_1) - (D_3)$. Suppose that function $A(t)$ is chosen in such a way that series $S(\delta)$ converges. In case when $1 + \beta/\gamma - \delta/\gamma > 0$, assume additionally that there exists such $0 < \kappa < 1$ that series $S(\delta + \kappa_1)$ converges with $\kappa_1 = (\kappa/2)(1 + \beta/\gamma - \delta/\gamma)$.

Then there exists such random variable $\xi > 0$ that on any $\omega \in \Omega$ and for any $t \in T$

$$|X(t)| \leq A(t_1)\xi,$$

and $\xi$ satisfies the following assumption:

$(D_4)$ for any $\epsilon > (2C_1 + 1)^{2\gamma/(2\gamma + \kappa)}$

$$P[\xi > \epsilon] \leq 2 \exp \left\{ - \frac{(\epsilon - \epsilon^{2\gamma/(2\gamma + \kappa)}) (2C_1 + 1)^2}{2B^2_1} \right\}.$$
Here the value of $\kappa < 2\gamma$ is chosen to ensure the convergence of series $S(\delta + \kappa_1)$ in case when $1 + \beta/\gamma - \delta/\gamma > 0$, and we set $\kappa = \frac{1}{2} \wedge \gamma$ in case when $1 + \beta/\gamma - \delta/\gamma \leq 0$.

**Proof** It is easy to check that

$$I := E \exp \left\{ \lambda \sup_{t \in T} \frac{|X(t)|}{A(t_1)} \right\} \leq E \exp \left\{ \lambda \sum_{\ell=0}^{\infty} (\delta_\ell)^{-1-N_{t_1 \in (b_\ell, b_{\ell+1})}} |X(t)| \right\}. \tag{32}$$

Let $\ell \geq 0$, $r_\ell > 1$ be such integers that $\sum_{\ell=0}^{\infty} (1/r_\ell) = 1$. Then it follows from Equation (32), Theorem 4.2 and Hölder inequality that for any $\theta \in (0, 1)$ and $0 < \kappa < 1 + 2\gamma$

$$I \leq \prod_{\ell=0}^{\infty} \left( E \exp \left\{ \frac{\lambda r_\ell}{\delta_\ell} \sup_{t_1 \in (b_\ell, b_{\ell+1})} |X(t)| \right\} \right)^{1/r_\ell} \leq \prod_{\ell=0}^{\infty} (2Q_\ell(\lambda, \theta))^{1/r_\ell} = 2 \prod_{\ell=0}^{\infty} (Q_\ell(\lambda, \theta))^{1/r_\ell},$$

where

$$Q_\ell(\lambda, \theta) = \exp \left\{ \frac{\lambda_2^2 r_\ell^2 (b_\ell^2 + C(\delta))^2}{2\delta_\ell^2 (1-\theta)^2} + \frac{2\lambda r_\ell}{(1-\theta)\delta_\ell} b_\ell^2 \kappa^{1/2} \right\}. \tag{33}$$

Therefore, if we take such value of $\kappa < 2\gamma$ that series $S(\delta + \kappa_1)$ converges in case when $1 + \beta/\gamma - \delta/\gamma > 0$ and set $\kappa = \frac{1}{2} \wedge \gamma$ in case when $1 + \beta/\gamma - \delta/\gamma \leq 0$, we obtain

$$I \leq 2 \exp \left\{ \frac{\lambda_2^2 (C(\delta))^2}{2(1-\theta)^2} \sum_{\ell=0}^{\infty} \frac{r_\ell (b_\ell^2)^2}{\delta_\ell^2} + \frac{2\lambda C_2 k^{1/2} S(\delta + \kappa_1)}{(1-\theta)\theta^{\kappa/2}\gamma} \right\}. \tag{33}$$

Now we can substitute $r_\ell = S(\delta) b_\ell^{-2} e_\ell^2$ into Equation (33)

$$I \leq 2 \exp \left\{ \frac{\lambda_2^2 (S(\delta) C(\delta))^2}{2(1-\theta)^2} + \frac{2\lambda C_2 k^{1/2} S(\delta + \kappa_1)}{(1-\theta)\theta^{\kappa/2}\gamma} \right\}. \tag{33}$$

Therefore,

$$E \exp \left\{ \lambda \sup_{t \in T} \frac{|X(t)|}{A(t_1)} \right\} \leq 2 \exp \left\{ \frac{\lambda_2^2}{2} \hat{B}^2 + 2\lambda \hat{C} \right\}. \tag{34}$$

where

$$\hat{B} = \frac{S(\delta) C(\delta)}{1-\theta} \quad \text{and} \quad \hat{C} = \frac{C_2 k^{1/2} S(\delta + \kappa_1)}{(1-\theta)\theta^{\kappa/2}\gamma}.$$ It follows immediately from Equation (34) that for any $\lambda > 0$, $\varepsilon > 0$

$$P \left\{ \sup_{t \in T} \frac{|X(t)|}{A(t_1)} > \varepsilon \right\} \leq \exp\{-\lambda \varepsilon\} E \exp \left\{ \lambda \sup_{t \in T} \frac{|X(t)|}{A(t_1)} \right\} \leq 2 \exp \left\{ \frac{\lambda_2^2}{2} \hat{B}^2 + 2\lambda \hat{C} - \lambda \varepsilon \right\}. \tag{35}$$

If we minimize the right-hand side of Equation (35) w.r.t. $\lambda$, then we obtain that for any $\varepsilon > 2\hat{C}$

$$P \left\{ \sup_{t \in T} \frac{|X(t)|}{A(t_1)} > \varepsilon \right\} \leq 2 \exp \left\{ \frac{-(\varepsilon - 2\hat{C})^2}{2\hat{B}^2} \right\} = 2 \exp \left\{ \frac{-(\varepsilon(1-\theta) - 2\theta^{-\kappa/2}\gamma C_1)^2}{2\hat{B}_1^2} \right\}. \tag{36}$$
Finally, we can insert \( \theta = \varepsilon^{-2\gamma/(2\gamma + 2)} \) into Equation (36) and derive that for \( \varepsilon > (2C_1 + 1)^{2\gamma/(2\gamma + 2)} \)

\[
P\left( \sup_{t \in T} \frac{|X(t)|}{A(t)} > \varepsilon \right) \leq 2 \exp\left\{ -\frac{(\varepsilon - \varepsilon^{1/(2\gamma + 2)}(1 + 2C_1))^2}{2B_1^2} \right\}.
\]

Denote \( \xi : = \sup_{t \in T} (|X(t)|/A(t_1)). \) Then \( \xi \) satisfies assumption \((D_4)\), and on any \( \omega \in \Omega \)

\[
X(t) \leq A(t_1)\xi,
\]

which concludes the proof.

**Theorem 4.4** Let \( 0 < H < 1, 1 - H < \alpha < 1, \ T = \{t = (t_1, t_2), 0 \leq t_2 < t_1\}, \)

\[
X(t) = \frac{B_{t_1}^H - B_{t_2}^H}{(t_1 - t_2)^{1-\alpha}} + \int_{t_2}^{t_1} \frac{B_{u}^H - B_{t_2}^H}{(u - t_2)^{2-\alpha}} \, du.
\]

Then for any \( p > 1 \) there exists random variable \( \xi = \xi(p) \) such that for any \( t \in T \)

\[
|X(t)| \leq ((t_1^{H+\alpha-1}(\log(t_1))^p) \lor 1)\xi(p),
\]

where \( \xi(p) \) satisfies assumption \((D_4)\) with some constants \( B_1 \) and \( C_1 \), and with \( \kappa = \gamma = \frac{1}{2} \) in the case when \( \alpha + H > \frac{3}{2} \) and \( \kappa = \gamma = \alpha + H - 1 - \nu \) with any \( 0 < \nu < (\alpha + H - 1) \land \frac{1}{2} \) in the case when \( \alpha + H \leq \frac{3}{2} \).

**Remark 6** Equality \( \gamma = \kappa \) means that condition \((D_4)\) can be rewritten as

\[
P\{\xi > \varepsilon\} \leq 2 \exp\left\{ -\frac{(\varepsilon - \varepsilon^{1/3}(2C_1 + 1))^2}{2B_1^2} \right\}
\]

for any \( \varepsilon > (2C_1 + 1)^{2/3}. \)

To apply Theorem 4.3 to the fractional derivative of the fractional Brownian motion and to prove Theorem 4.4, we need an auxiliary result. In what follows we denote by \( C(H, \alpha) \) a constant depending only on \( H \) and \( \alpha \) and not on other parameters.

**Lemma 4.5** Let \( z_i > 0 \) for \( i = 1, 2. \) In addition, let \( 0 < H < 1, 1 - H < \alpha < 1 \) and

\[
I = z_2^{2(H+\alpha-1)} + z_1^{2(H+\alpha-1)} + \frac{|z_2 - z_1|^{2H}}{(z_1 z_2)^{1-\alpha}}.
\]

Then \( I \leq C(H, \alpha)|z_2 - z_1|^{2(H+\alpha-1)}. \)
Proof Let \( z_2 > z_1 > 0 \) (the case \( z_1 > z_2 > 0 \) can be dealt with in a similar way). We can rewrite \( I \) as
\[
I = (z_2^{H+\alpha-1} - z_1^{H+\alpha-1})^2 + 2(z_1z_2)^{H+\alpha-1} \\
+ ((z_2 - z_1)^{2H} - (z_2 - z_1^H)^2 - 2(z_1z_2)^H(z_1z_2)^{\alpha-1}) \\
= (z_2^{H+\alpha-1} - z_1^{H+\alpha-1})^2 + \frac{(z_2 - z_1)^{2H} - (z_2^H - z_1^H)^2}{(z_1z_2)^{1-\alpha}} = I_1 + I_2.
\]
Recall a simple inequality \( b' - a' \leq (b - a)' \) for \( b > a, \ 0 < r \leq 1 \). Since \( 0 < H + \alpha - 1 < 1 \), we can estimate \( I_1 \) by \( (z_2 - z_1)^{2(H+\alpha-1)} \). Furthermore, \( I_2 \) can be rewritten as
\[
I_2 = (z_2 - z_1)^{2(H+\alpha-1)} \left| \frac{z_2 - z_1}{z_1z_2} \right|^{2H} - (\frac{z_2^H - z_1^H}{z_1z_2})^2 \\
= (z_2 - z_1)^{2(H+\alpha-1)} f(u),
\]
where \( u = \frac{z_2}{z_1} > 1, \ f(u) = (\frac{u-1)^{2H} - (u^H - 1)^2}{u^{1-\alpha}(u-1)^{2(H+\alpha-1)}} \geq 0. \)
Calculate the limit of function \( f \) at 1
\[
\lim_{u \to 1} f(u) = \lim_{u \to 1} \frac{(u-1)^{2H} - (u^H - 1)^2}{(u-1)^{2(H+\alpha-1)}} = \lim_{u \to 1} (u - 1)^{2-2\alpha} = 0.
\]
Here
\[
\lim_{u \to 1} \frac{(u-1)^{2H}}{(u-1)^{2(H+\alpha-1)}} = \lim_{u \to 1} (u - 1)^{2-2\alpha} = 0
\]
and
\[
\lim_{u \to 1} \frac{(u^H - 1)^2}{(u-1)^{2(H+\alpha-1)}} = H^2 \lim_{u \to 1} (u - 1)^{4-2H-2\alpha} = 0,
\]
since \( \lim_{u \to 1} (u^H - 1)/(u - 1) = H \). Calculate the limit of the function \( f \) at infinity
\[
0 \leq \lim_{u \to \infty} f(u) = \lim_{u \to \infty} \frac{(u-1)^{2H} - (u^H - 1)^2}{u^{1-\alpha}(u-1)^{2(H+\alpha-1)}} \\
\leq \lim_{u \to \infty} \frac{u^{2H} - (u^H - 1)^2}{u^{2H+\alpha-1}} = \lim_{u \to \infty} \frac{2u^H - 1}{u^{2H+\alpha-1}} = 0.
\]
This implies that function \( f \) is bounded, i.e. there exists \( C(H, \alpha) > 0 \) such that
\[
I_2 \leq C(H, \alpha)(z_2 - z_1)^{2(H+\alpha-1)},
\]
and the proof follows if we combine the bounds for \( I_1 \) and \( I_2 \). \( \blacksquare \)

We are now ready to check conditions \( (D_2) \) and \( (D_3) \) for the fractional derivative of the fractional Brownian motion.

**Lemma 4.6** Let
\[
X(t) = \frac{B_{H_1} - B_{H_2}}{(t_1 - t_2)^{1-\alpha}} + \int_{t_2}^{t_1} \frac{B_{H_1} - B_{H_2}}{(u - t_2)^{2-\alpha}} \, du,
\]
where \( 0 \leq t_2 < t_1, \ 0 < H < 1, \ 1 - H < \alpha < 1. \)
Then the following bounds hold:

(1) for any \( 0 \leq t_2 < t_1 \)
\[
(E(X(t))^2)^{1/2} \leq C(H, \alpha)(t_1 - t_2)^{H+\alpha-1};
\]
Let \( H + \alpha \leq \frac{3}{2} \). Then for any \( 0 \leq t_2 < t_1, \ 0 \leq s_2 < s_1 \) and any \( 0 < \nu < (H + \alpha - 1) \wedge \frac{1}{2} \)

\[
(E|X(t) - X(s)|^2)^{1/2} \leq C(H, \alpha)(1 + \nu^{-1})(|t_1 - s_1| \vee |t_2 - s_2|)^{H+\alpha-1-\nu}(t_1 \vee s_1)\nu
\]

with \( C(H, \alpha) \) not depending on \( X \), its arguments and \( \nu \).

Let \( H + \alpha > \frac{3}{2} \). Then for any \( 0 \leq t_2 < t_1, \ 0 \leq s_2 < s_1 \)

\[
(E|X(t) - X(s)|^2)^{1/2} \leq C(H, \alpha)(|t_1 - s_1| \vee |t_2 - s_2|)^{1/2}(t_1 \vee s_1)^{H+\alpha-3/2}.
\]

**Proof** The first statement follows immediately from the Minkowski’s integral inequality

\[
(E(X(t))^2)^{1/2} \leq \left( E\left( \frac{B_{t_1}^H - B_{t_2}^H}{(t_1 - t_2)^{1-\alpha}} \right)^2 \right)^{1/2} + \left( E\left( \int_{t_2}^{t_1} \frac{B_{u}^H - B_{t_2}^H}{(u - t_2)^{2-\alpha}} \, du \right)^2 \right)^{1/2}
\]

\[
\leq \left( \frac{(t_1 - t_2)^{2H}}{(t_1 - t_2)^{2(1-\alpha)}} \right)^{1/2} + \int_{t_2}^{t_1} \left( \frac{B_{u}^H - B_{t_2}^H}{(u - t_2)^{2-\alpha}} \right)^2 \, du = (t_1 - t_2)^{H+\alpha-1} +
\]

\[
+ \int_{t_2}^{t_1} \left( \frac{(u - t_2)^{2H}}{(u - t_2)^{2(2-\alpha)}} \right)^{1/2} \, du = \frac{\alpha + H}{\alpha + H - 1} (t_1 - t_2)^{H+\alpha-1}.
\]

In order to prove the second statement, denote

\[
X_1(t) = \frac{B_{t_1}^H - B_{t_2}^H}{(t_1 - t_2)^{1-\alpha}} \quad \text{and} \quad X_2(t) = \int_{t_2}^{t_1} \frac{B_{u}^H - B_{t_2}^H}{(u - t_2)^{2-\alpha}} \, du.
\]

Evidently,

\[
(E|X(t) - X(s)|^2)^{1/2} \leq (E|X_1(t) - X_1(s)|^2)^{1/2} + (E|X_2(t) - X_2(s)|^2)^{1/2}. \quad (37)
\]

Let \( t_1 > s_1 \), the opposite case can be considered in a similar way. Then

\[
(E|X_1(t) - X_1(s)|^2)^{1/2} = \left( E\left( \frac{B_{t_1}^H - B_{t_2}^H}{(t_1 - t_2)^{1-\alpha}} - \frac{B_{t_1}^H - B_{s_2}^H}{(t_1 - s_2)^{1-\alpha}} \right)^2 \right)^{1/2}
\]

\[
+ \left( E\left( \frac{B_{t_1}^H - B_{s_2}^H}{(t_1 - s_2)^{1-\alpha}} - \frac{B_{s_1}^H - B_{s_2}^H}{(s_1 - s_2)^{1-\alpha}} \right)^2 \right)^{1/2}
\]

\[
\leq \left( E\left( \frac{B_{t_1}^H - B_{t_2}^H}{(t_1 - t_2)^{1-\alpha}} - \frac{B_{t_1}^H - B_{s_2}^H}{(t_1 - s_2)^{1-\alpha}} \right)^2 \right)^{1/2}
\]

\[
+ \left( E\left( \frac{B_{t_1}^H - B_{s_2}^H}{(t_1 - s_2)^{1-\alpha}} - \frac{B_{s_1}^H - B_{s_2}^H}{(s_1 - s_2)^{1-\alpha}} \right)^2 \right)^{1/2} = I_3 + I_4. \quad (38)
\]

It is more convenient to estimate the squares \((I_3)^2\) and \((I_4)^2\) from Equation (38) instead of \(I_3\) and \(I_4\). As for \((I_3)^2\), we can calculate it explicitly and then estimate it with the help of Lemma 4.5;
We derive from Equation (39) that $
abla$ can be evaluated similarly.

\[
(I_3)^2 = (t_1 - t_2)^2t_{H+\alpha-1} + (t_1 - s_2)^2t_{H+\alpha-1} - \frac{2}{(t_1 - t_2)^{1-\alpha}(t_1 - s_2)^{1-\alpha}} E(B_t^H - B_{s_2}^H)(B_{t_2}^H - B_{s_2}^H) - \frac{2}{(t_1 - t_2)^{1-\alpha}(t_1 - s_2)^{1-\alpha}} \]

\[
\times \left[ t_{1}^{2H} - \frac{1}{2}(t_{2}^{2H} + t_{1}^{2H} - (t_1 - t_2)^{2H}) - \frac{1}{2}(t_{1}^{2H} + s_{2}^{2H} - (t_1 - s_2)^{2H}) + \frac{1}{2}(t_{2}^{2H} + s_{2}^{2H} - |t_2 - s_2|^2H) \right]
\]

\[
= (t_1 - t_2)^2t_{H+\alpha-1} + (t_1 - s_2)^2t_{H+\alpha-1} + \frac{|t_2 - s_2|^2H - (t_1 - t_2)^2H - (t_1 - s_2)^2H}{(t_1 - t_2)^{1-\alpha}(t_1 - s_2)^{1-\alpha}} \leq C(H, \alpha)|t_2 - s_2|^2t_{H+\alpha-1}.
\]  

(39)

We derive from Equation (39) that

\[
I_3 \leq C(H, \alpha)|t_2 - s_2|^H+\alpha-1,
\]  

(40)

and similarly,

\[
I_4 \leq C(H, \alpha)|t_1 - s_1|^H+\alpha-1.
\]  

(41)

It follows immediately from Equations (40) and (41) that

\[
(E|X_1(t) - X_1(s)|^2)^{1/2} \leq C(H, \alpha)(|t_1 - s_1| \vee |t_2 - s_2|)^{H+\alpha-1}.
\]  

(42)

Now estimate

\[
F(t, s) = (E|X_2(t) - X_2(s)|^2)^{1/2} = \left( E \left( \int_{t_1}^{s_2} \frac{B_{u}^H - B_{t_2}^H}{(u-t_2)^{2-\alpha}} du - \int_{s_2}^{s_1} \frac{B_{u}^H - B_{s_2}^H}{(u-s_2)^{2-\alpha}} du \right) \right)^{1/2}.
\]

Let, for instance, $0 \leq t_2 < s_2 < s_1 < t_1$ (other types of relation between these points can be handled similarly). Then

\[
F(t, s) \leq \left( E \left( \int_{t_1}^{s_2} \frac{B_{u}^H - B_{t_2}^H}{(u-t_2)^{2-\alpha}} du \right) \right)^{1/2} + \left( E \left( \int_{s_2}^{s_1} \frac{B_{u}^H - B_{t_2}^H}{(u-t_2)^{2-\alpha}} du - \frac{B_{u}^H - B_{s_2}^H}{(u-s_2)^{2-\alpha}} du \right) \right)^{1/2} + \left( E \left( \int_{s_1}^{t_1} \frac{B_{u}^H - B_{t_2}^H}{(u-t_2)^{2-\alpha}} du \right) \right)^{1/2} =: I_5 + I_6 + I_7.
\]  

(43)

Using the Minkowski’s integral inequality we immediately obtain

\[
I_5 \leq \int_{t_2}^{s_2} \left( E \left( \frac{B_{u}^H - B_{t_2}^H}{(u-t_2)^{2-\alpha}} \right) \right)^{2} \frac{1}{H + \alpha - 1} \left( s_2 - t_2 \right)^{H+\alpha-1}.
\]  

(44)
Similarly,

\[ I_7 \leq \frac{1}{H + \alpha - 1} (t_1 - s_1)^{H+\alpha-1}. \]  

(45)

Again, using the Minkowski’s integral inequality and Lemma 4.5 we conclude that

\[ I_6 \leq \int_{s_2}^{S_1} \left( E \left( \frac{B^H_B - B^H_{I}}{(u - t_2)^2 - \alpha} - \frac{B^H_B - B^H_{I}}{(u - s_2)^2 - \alpha} \right)^2 \right)^{1/2} du \]

\[ = \int_{s_2}^{S_1} \left( (u - t_2)^2 (u - s_2)^2 - (u - t_2)^2 (u - s_2)^2 - \frac{(s_2 - t_2)^2 H - (u - t_2)^2 H - (u - s_2)^2 H}{(u - t_2)^2 - \alpha (u - s_2)^2 - \alpha} \right)^{1/2} du \]

\[ = \int_{s_2}^{S_1} (u - s_2)^{-1/2} (u - t_2)^{-1/2} \left[ (u - t_2)^{2(H+\alpha-2)} (u - s_2) (u - t_2) \right. \]

\[ + (u - s_2)^{2(H+\alpha-2)} (u - s_2) (u - t_2) \]

\[ + \frac{(s_2 - t_2)^2 H - (u - t_2)^2 H - (u - s_2)^2 H}{(u - t_2)^2 - \alpha (u - s_2)^2 - \alpha} \right]^{1/2} du \]

\[ \leq \int_{s_2}^{S_1} (u - s_2)^{-1/2} (u - t_2)^{-1/2} \left[ (u - t_2)^{2(H+\alpha-1)} + (u - s_2)^{2(H+\alpha-1)} \right. \]

\[ + (u - s_2)^{2(H+\alpha-2)} (s_2 - t_2) + \frac{(s_2 - t_2)^2 H - (u - t_2)^2 H - (u - s_2)^2 H}{(u - t_2)^2 - \alpha (u - s_2)^2 - \alpha} \right]^{1/2} du \]

\[ \leq C(H, \alpha) \int_{s_2}^{S_1} (u - s_2)^{-1/2} (u - t_2)^{-1/2} (s_2 - t_2)^{H+\alpha-1} du \]

\[ + C(H, \alpha) \int_{s_2}^{S_1} (u - s_2)^{H+\alpha-2} (u - t_2)^{-1/2} (s_2 - t_2)^{1/2} du =: I_8 + I_9. \]  

(46)

Evidently,

\[ I_8 = (s_2 - t_2)^{H+\alpha-1} \int_{s_2}^{S_1} (u - s_2)^{-1/2} (u - t_2)^{-1/2} du = (s_2 - t_2)^{H+\alpha-1} I_{10} \]

up to the constant multiplier and for any $0 < \rho < \frac{1}{2}$ and $0 < \nu < \frac{1}{2} - \rho$ integral $I_{10}$ can be rewritten as

\[ I_{10} = \int_{s_2}^{S_1} (u - s_2)^{-1/2} (u - t_2)^{-1/2} du \]

\[ = \int_0^{(s_1-t_2)/(s_2-t_2)} (y + 1)^{-1/2} y^{-1/2} dy \leq \left( \frac{s_1 - s_2}{s_2 - t_2} \right)^\nu \int_0^{(s_1-s_2)/(s_2-t_2)} (y + 1)^{-1/2} y^{-1/2 - \nu} dy \]

\[ \leq \left( \frac{s_1 - s_2}{s_2 - t_2} \right)^\nu \int_0^{\infty} (y + 1)^{-1/2} y^{-1/2 - \nu} dy \leq C(1 + \nu^{-1}) \left( \frac{s_1 - s_2}{s_2 - t_2} \right)^\nu. \]

Therefore, for any $0 < \nu < (H + \alpha - 1) \land \left( \frac{1}{2} - \rho \right)$

\[ I_8 \leq C(H, \alpha, \nu)(1 + \nu^{-1})(s_2 - t_2)^{H+\alpha-1-\nu}(s_1 - s_2)^\nu. \]  

(47)
Furthermore,
\[ I_9 = (s_2 - t_2)^{1/2} \int_{s_2}^{s_1} (u - s_2)^{H+\alpha-2} (u - t_2)^{-1/2} \, du = (s_2 - t_2)^{1/2}I_{11} \]
up to a constant multiplier. In the case when \( H + \alpha < \frac{3}{2} \) the integral \( I_{11} \) can be rewritten as
\[
I_{11} = \int_{s_2}^{s_1} (u - s_2)^{H+\alpha-2} (u - t_2)^{-1/2} \, du \\
= \int_{0}^{(s_1-s_2)/(s_2-t_2)} y^{H+\alpha-2} (1 + y)^{-1/2} (s_2 - t_2)^{H+\alpha-2+1/2} \, dy \\
\leq (s_2 - t_2)^{H+\alpha-3/2} \int_{0}^{\infty} y^{H+\alpha-2} (1 + y)^{-1/2} \, dy \leq C(H, \alpha)(s_2 - t_2)^{H+\alpha-3/2}.
\]
In case when \( H + \alpha > \frac{3}{2} \) integral \( I_{11} \) admits an obvious bound
\[
I_{11} \leq \int_{s_2}^{s_1} (u - s_2)^{H+\alpha-2} (u - s_2)^{-1/2} \, du \leq C(H, \alpha)(s_1 - s_2)^{H+\alpha-3/2}.
\]
Finally, for \( H + \alpha = \frac{3}{2} \) integral \( I_{11} \) admits the same bound as \( I_{10} \). Therefore,
\[
I_9 \leq C(H, \alpha)(s_2 - t_2)^{H+\alpha-1} \tag{48}
\]
for \( H + \alpha < \frac{3}{2} \),
\[
I_9 \leq C(H, \alpha)(s_2 - t_2)^{1/2}(s_1 - s_2)^{H+\alpha-3/2} \tag{49}
\]
for \( H + \alpha > \frac{3}{2} \), and
\[
I_9 \leq C(H, \alpha, v)(s_2 - t_2)^{1/2-v}(s_1 - s_2)^v \tag{50}
\]
for \( H + \alpha = \frac{3}{2} \).

This implies that
\[
F(t, s) \leq C(H, \alpha)(1 + v^{-1})(|t_1 - s_1| \vee |t_2 - s_2|)^{H+\alpha-1-v}(s_1 \vee t_1)^v \tag{51}
\]
for \( H + \alpha \leq \frac{3}{2} \). In case \( H + \alpha > \frac{3}{2} \) we can put \( v = H + \alpha - \frac{3}{2} \in (0, \frac{1}{2}) \) in Equation (47) and conclude that
\[
F(t, s) \leq C(H, \alpha) (|t_1 - s_1| \vee |t_2 - s_2|)^{1/2}(s_1 \vee t_1)^{H+\alpha-3/2}. \tag{52}
\]

The proof follows immediately from Equations (37) and (42)–(52).

**Proof of Theorem 4.4** First of all we should verify conditions \((D_1) - (D_3)\). Condition \((D_1)\) is evident, since \( X \) is continuous in both variables. According to the second statement of Theorem 4.6, condition \((D_2)\) holds with \( \beta = \nu, 0 < \nu < (H + \alpha - 1) \wedge \frac{1}{2} \) and \( \gamma = H + \alpha - 1 - \nu \) in case when \( \alpha + H \leq \frac{3}{2} \), and with \( \beta = H + \alpha - \frac{3}{2} \) and \( \gamma = \frac{1}{2} \) in case when \( \alpha + H > \frac{3}{2} \).

According to the first statement of Theorem 4.6, condition \((D_3)\) holds with \( \delta = H + \alpha - 1 \).

Let \( A(t) = (t^{H+\alpha-1} | \log t|^p) \vee 1 \) for some \( p > 1 \) and for any \( t > 0 \) and let \( b_i = e^l, l \geq 0 \). Then \( \delta_l = (e^{l(H+\alpha-1)}|p) \vee 1 \) and \( b_i^\delta = e^{l(H+\alpha-1)}|p) \). Therefore, in this case series \( S(\delta) \) converges since
\[
S(\delta) = e^{H+\alpha-1} + \sum_{l=1}^{\infty} \frac{e^{(l+1)(H+\alpha-1)}|p)}{e^{(H+\alpha-1)|p)} = e^{H+\alpha-1} \left( 1 + \sum_{l=1}^{\infty} l^{-p} \right) < \infty.
\]
Moreover, it is easy to check that \( 1 + \beta/\gamma - \delta/\gamma = 0 \) for any values of \( \alpha + H \), hence \( \kappa_1 = 0 \), \( \kappa = \frac{1}{2} \) in the case when \( \alpha + H > \frac{3}{2} \) and \( \kappa = \alpha + H - 1 - \nu \) with any \( 0 < \nu < (\alpha + H - 1) \wedge \frac{1}{2} \).
in the case when $\alpha + H \leq \frac{3}{2}$. This implies that all conditions of Theorem 4.3 hold true and we can apply the theorem with $A(t) = (t^{H+\alpha-1} |\log t|^p) \lor 1$ which concludes the proof.

\[ \square \]

**Remark 7** Instead of the fractional derivative, we can consider the fractional Brownian motion $B^H_t$ itself and apply the same reasoning to it. This case is much simpler and we immediately obtain that $\sup_{0 \leq s \leq t} |B^H_s| \leq ( (t^{H} (\log(t))^p) \lor 1 ) \xi(p)$ for any $p > 1$.

### Acknowledgements

This paper was partially supported by NSERC grant 261855. We are thankful to Ivan Smirnov and Georgii Shevchenko for the assistance in the preparation of the manuscript. We are thankful to the anonymous referees for their helpful suggestions.

### References

[1] Liptser R, Shiryaev A. Statistics of random processes: II. Applications. Berlin: Springer; 1978.
[2] Hu Y, Nualart D. Parameter estimation for fractional Ornstein–Uhlenbeck processes. Stat Probab Lett. 2010;80(10–12):1030–1038.
[3] Kleptsyna ML, Le Breton A. Statistical analysis of the fractional Ornstein–Uhlenbeck type process. Stat Inference Stoch Process. 2002;5:229–248.
[4] Bishwal JPN. Parameter estimation in stochastic differential equations. Lecture notes in mathematics, vol. 1923. Berlin: Springer; 2008.
[5] Mishura Y. Stochastic calculus for fractional Brownian motion and related processes. Lecture notes in mathematics, vol. 1929. Berlin: Springer; 2008.
[6] Prakasa Rao BLS. Statistical inference for fractional diffusion processes. Chichester: John Wiley & Sons; 2010.
[7] Berciu E, Tudor C. Drift parameter estimation in fractional diffusions driven by perturbed random walks. Stat Probab Lett. 2011;81:243–249.
[8] Hu Y, Xiao W, Zhang W. Exact maximum likelihood estimators for drift fractional Brownian motions. Arxiv preprint arXiv:0904.4186; 2009 – arxiv.org
[9] Xiao W, Zhang W, Xu W. Parameter estimation for fractional Ornstein–ÜUhlenbeck processes at discrete observation. Appl Math Model. 2011;35:4196–4207.
[10] Xiao W-L, Zhang W-G, Zhang X-L. Maximum-likelihood estimators in the mixed fractional Brownian motion. Statistics. 2011;45:73–85.
[11] Melnikov A, Novikov A. Sequential inferences with prescribed accuracy for semimartingales. Theory Probab Appl. 1988;33:446–459.
[12] Samko S, Kilbas A, Marichev O. Fractional integrals and derivatives. Theory and applications. New York: Gordon and Breach Science Publishers; 1993.
[13] Zähle M. Integration with respect to fractal functions and stochastic calculus. I. Probab Theory Related Fields. 1998;111:333–374.
[14] Zähle M. On the link between fractional and stochastic calculus. In: Cramel H, Gundlach M, editors. Stochastic dynamics. New York: Springer; 1999. p. 305–325.
[15] Nualart D, Rascanu A. Differential equation driven by fractional Brownian motion. Collect Math. 2002;53:55–81.
[16] Mishura Yu, Shevchenko G. Stochastic differential equation involving Wiener process and fractional Brownian motion with Hurst index $H > \frac{1}{4}$. Comm Stat Theory Methods. 2011;40(19–20):3492–3508.
[17] Guerra J, Nualart D. Stochastic differential equations driven by fractional Brownian motion and standard Brownian motion. Stoch Anal Appl. 2008;26(5):1053–1075.
[18] Norros I, Valkeila E, Virtamo J. An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. Bernoulli. 1999;5(4):571–587.
[19] Memin J, Mishura Y, Valkeila E. Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion. Stat Probab Lett. 2001;51:197–206.
[20] Alain Le Breton. Filtering and parameter estimation in a simple linear system driven by a fractional Brownian motion. Stat Probab Lett. 1998;38(3):263–274.
[21] Buldygin VV, Kozachenko YuV. Metric characterization of random variables and random processes. Translations of mathematical monographs, vol. 188. Providence, RI: American Mathematical Society; 2000. 257 p.