ENTROPY RIGIDITY FOR THREE DIMENSIONAL VOLUME PRESERVING ANOSOV FLOWS

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Abstract. We show that for every $C^r$ ($r \geq 2$) volume preserving three dimensional Anosov flows, if the unique maximal entropy measure coincides to the volume measure, then up to a finite covering, this flow is $C^{r-\epsilon}$ conjugate to an algebraic Anosov flow. This proves a conjecture of Foulon [10].

1. Introduction

Throughout this paper, assume $M$ to be a three dimensional $C^\infty$ Riemannian manifold without boundary, $\phi_t$ to be a $C^r$ ($r \geq 1$) flow generated by a vector field $X$. $\phi_t$ is an Anosov flow if there exist a flow invariant splitting of the tangent bundle

$$TM = \mathbb{R}X \oplus E^s \oplus E^u$$

with respect to a Riemannian metric on $M$ and positive real numbers $c_1, c_2$ and $\gamma$ such that

$$\|D\phi_t(v)\| \leq c_1 e^{-t\gamma} \|v\| \text{ for any } v \in E^s \text{ and } t \geq 0$$

$$\|D\phi_t(v)\| \geq c_2 e^{-t\gamma} \|v\| \text{ for any } v \in E^u \text{ and } t \geq 0.$$ 

Anosov flows can be only defined on manifolds with dimension larger than or equal to 3.

For discrete systems, a similar definition can be given for the Anosov diffeomorphisms. It is well known that, 2 or 3 dimensional Anosov diffeomorphisms have very well description: they are topologically conjugate to algebraic ones—the linear Anosov automorphism over torus (\cite{12, 23}). But the situation is much different for Anosov flows, they have much richer phenomenon. For instance, by the conjugacy, every 3 dimensional Anosov diffeomorphism must be transitive, but the example of Franks and Williams \cite{13} shows that 3 dimensional Anosov flows are not necessarily transitive (for more examples see \cite{5}). There are infinitely many types of transitive Anosov flows which are not conjugate to algebraic ones \cite{13, 16, 5}; these Anosov flows can even be chosen to be volume preserving \cite{8} or preserve contact form \cite{11}. More non-algebraic examples see \cite{17}. Many non-algebraic examples come from surgeries of algebraic models.

A 3 dimensional Anosov flow is algebraic if it belongs to one of the following two categories:

(a) Suspension of hyperbolic automorphism of torus $\mathbb{T}^2$;  
(b) Geodesic flow on a closed surface with a metric of constant negative curvature.

Remark 1.1. P. Tomter \cite{32} proved that, there are the only two algebraic Anosov flows in dimension 3. By Plante \cite{27}, if $M$ supports a suspension of an Anosov diffeomorphism, then, up to orbit equivalence, $M$ supports at most two Anosov flows.
flows: the suspension of a linear automorphism and the suspension of its inverse. And Ghys [14] showed that, all 3 dimensional Anosov flows on closed manifolds which are circle bundles, up to a finite covering, are topologically equivalent to the geodesic flow of a surface of constant negative curvature.

Although the topological classification of 3 dimensional Anosov flows are far from satisfying, the equilibrium states for Anosov flows have been well studied. For instance, from Margulis [21, 22] and Bowen [9], every transitive Anosov flows admits a unique maximal entropy measure, which is named Margulis measure or Bowen measure depending on their different constructions.

It was observed first by Katok [19] that, the coincidence of maximal entropy measure and the volume measure implies rigidity: for geodesic flow on compact surface with negative Euler characteristic, the Liouville measure coincide with the maximal entropy measure if and only if the metric has constant negative curvature. He further conjectured that the equivalence of Bowen-Margulis and Liouville measure only occurs for locally symmetric spaces in any dimension.

Around two decades later, Foulon extends this result beyond the realm of geodesic flows: for any $C^\infty$ 3 dimensional contact Anosov flow, if the volume measure coincides with the maximal entropy measure, then this flow is $C^\infty$ conjugate to an algebraic one, up to finite cover. He further conjectured that this should be a more general situation:

**Conjecture 1.2.** Let $\phi_t$ be a $C^2$ Anosov flow on a 3-manifold $M$ which preserves a smooth volume vol. If $h_{\text{top}}(\phi_t) = h_{\text{vol}}(\phi_t)$, then up to a finite covering, $\phi_t$ is smoothly conjugate to an algebraic flow.

We give a positive answer of the Foulon’s conjecture in this paper:

**Theorem A.** Suppose $\phi_t$ is a $C^r$ ($r \geq 2$) three dimensional volume preserving Anosov flow, and the volume measure is the maximal entropy measure. Then up to finite covers, this flow is $C^{r-\epsilon}$ conjugate to an algebraic Anosov flow for any $\epsilon > 0$.

**Remark 1.3.** The regularity is essential for the proof. Indeed, the main result fails in the $C^{1+\alpha}$ category, see the construction of [24, 25]. For instance, for any given $C^\infty$ volume preserving Anosov flow, there always exists a time change such that the Bowen-Margulis measures of the new flow is the Lebesgue measure, and the reparametrized flow is not in the $C^2$ class.

2. Preliminary

2.1. Conjugation.

**Definition 2.1.** Two flows $\phi_t$ and $\psi_t$ are $C^r$ orbit equivalent ($r \geq 0$) if there is $h \in \text{Diff}^r(M)$ (Diff$^0(M)$ denotes homeomorphism) which sends every orbit of flow $\phi_t$ to the orbit of flow $\psi_t$. Two flows $\phi_t$ and $\psi_t$ are $C^r$ conjugate if there is $h \in \text{Diff}^r(M)$ such that

$$h \circ \phi_t = \psi_t \circ h.$$ 

The following result is a well known criteria that two Anosov flows are conjugate, a proof can be found for instance [18][Theorem 19.2.9]:

**Proposition 2.2.** Suppose $\phi_t$ and $\psi_t$ are two orbit equivalent $C^2$ Anosov flows such that, the periods of corresponding periodic orbits agree, then $\phi_t$ and $\psi_t$ are conjugate.

For two 3 dimensional Anosov flows that are conjugate, if the corresponding periodic orbits under conjugacy have the same Lyapunov exponents, then the conjugacy is indeed smooth:
Proposition 2.3. [23/Theorem 1.1] Suppose $\phi_t$ and $\psi_t$ are two 3 dimensional $C^r$ ($r \geq 2, \infty, w$) Anosov flow which are conjugate:

$$h \circ \phi_t = \psi_t \circ h.$$ 

Suppose the Lyapunov exponents at corresponding periodic orbits of $\phi_t$ and $\psi_t$ are coincide, then $\phi_t$ and $\psi_t$ are $C^r$ conjugate for any $s < r$.

2.2. Regularity of foliations. We need the following version of Journé regular lemma for foliations, which can be found in [25].

Lemma 2.4. Suppose that $W$ and $L$ are foliations of the manifold $M$, and $W$ is transversal to $L$. Let $\mathcal{H}$ be the holonomy along the leaves of $W$, between the leaves of $L$. If the foliations $W$ and $L$ have uniformly $C^2$ leaves, and $\mathcal{H}$ is uniformly $C^2$ (resp. $C^r$), then $W$ is a $C^1,1$ foliation.

2.3. Smooth center stable foliation and center unstable foliation. The following beautiful result is due to Ghys.

Proposition 2.5. [18/Theorem 4.6] Suppose $\phi_t$ is a 3 dimensional volume preserving $C^r$ ($r \geq 2$) Anosov flow, with $C^1,1$ weak stable and weak unstable foliations. Then $\phi_t$ is $C^r$ orbit equivalent to a suspension of linear hyperbolic automorphism on $T^2$ or up to finite lift to conjugate to the hyperbolic geodesic flow.

2.4. Disintegrations. Let $\mathcal{F}$ be a foliation, $\mu$ a Borel measure on $M$, and $B$ be a foliation chart of $\mathcal{F}$ with $\mu(B) > 0$. Let $D$ be a transverse disk of the foliation chart, then we have a local projection map $\pi : B \to D$ along the $\mathcal{F}$ leaves, and a corresponding quotient measure $\overline{\mu} = \pi_*(\mu)$ supported on $D$. Then $D$ can be identified as the space of $\mathcal{F} \mid B$ leaves and the measure $\overline{\mu}$ is a measure on the set of leaves. By [29], the disintegration of $\mu$ along the foliation $\mathcal{F}$ on $B$ consists of the probability measures $\{\mu^F_x\}$, supported on $\mu$ almost every leaf $F(x)$, such that:

$$d\mu(y) = \int d\mu^F_x(y)d\mu(x) = \int_D d\mu^F_x(y)d\mu(x).$$

The measures $\{\mu^F_x\}$ exist and are unique (up to $\mu$-zero measure), and are called the disintegrations of $\mu$ along the plaques $F(x)$.

2.4.1. Margulis measure. Margulis measure is a classification of maximal entropy measure for topologically mixing Anosov flow (see [21, 22]). Different with the disintegration which are defined only on local foliation chart, the Margulis measure $\mu$, which is the unique maximal entropy measure, has good disintegration on global leaves. The following properties can be found in [21, 22] (see also [18]). For every $x \in M$, there are measures $\mu^M_u, \mu^M_s, \mu^{M,cs}$ defined on $F^u(x)$ and $F^{cs}(x)$ respectively, such that denote by $h$ the topological entropy of $f_0 = \phi_1$, then

(a) $(\phi_t)_*(\mu^M_u) = e^{th} \mu^M_u$;

(b) Denote by $\mathcal{H}^u$ the local holonomy map between two center stable leaves $F^s_{t_1}(x_i) (i = 1, 2)$ induced by the unstable foliation such that $\mathcal{H}^u(x_1) = x_2$, then

$$\mathcal{H}^u(\mu^{M,cs}_{t_1} | F^s_{t_1}(x_1)) = \mu^{M,cs}_{t_2} | F^s_{t_2}(x_2).$$

Although the holonomy map induced by the center stable foliation in general does not preserve the Margulis measure along the unstable leaves, they do preserve the scaled Margulis measure along the unstable leaves. More precisely, by the local product structure, for $x_0 \in M$ and the sets $D^u_{x_0} \subset F^u_{loc}(x_0)$ and $D^{cs}_{x_0} \subset F^{cs}_{loc}(x_0)$, we may define local product set:

$$[D^u_{x_0}, D^{cs}_{x_0}] = \{F^u_{loc}(y) \cap F^{cs}_{loc}(z); y \in D^u_{x_0} \text{ and } z \in D^{cs}_{x_0}\}. $$
For any \( x \in [D_{x_0}^u, D_{x_0}^{cs}] \), let \( D^u_x = F^u_{loc}(x) \cap [D_{x_0}^u, D_{x_0}^{cs}] \) and \( D^{cs}_x = F^{cs}_{loc}(x) \cap [D_{x_0}^u, D_{x_0}^{cs}] \).

**Lemma 2.6.** For any \( x_1, x_2 \in [D_{x_0}^u, D_{x_0}^{cs}] \), denote by \( H^u : D_{x_1}^u \to D_{x_2}^u \) the holonomy map induced by the center stable foliation, then

\[
H^u(M_{x_1}^u | D_{x_1}^u) = M_{x_2}^{cs u(D_{x_2}^{cs})}.
\]

**Proof.** Since the disintegrations should be probabilities, we have:

(i) the disintegration of \( \mu ||[D_{x_0}^u, D_{x_0}^{cs}] \) along the foliations \( D^{cs}_x \) is

\[
\frac{M_{x_1}^{cs}}{M_{x}^{cs u(D_{x}^{cs})}}
\]

(ii) the disintegration of \( \mu ||[D_{x_0}^u, D_{x_0}^{cs}] \) along the foliations \( D^u_x \) is

\[
\frac{M_{x_1}^{cs}}{M_{x}^{cs u(D_{x}^{cs})}}
\]

By (1), the disintegration of \( \mu ||[D_{x_0}^u, D_{x_0}^{cs}] \) along the foliations \( D^{cs}_x \) are invariant under the holonomy map \( H^u \). By (31) [Lemma 5.2], it is equivalent to say that the disintegration of \( \mu ||[D_{x_0}^u, D_{x_0}^{cs}] \) along the foliations \( D^{cs}_x \) are invariant under the holonomy map \( H^u \). The proof is finished.

\( \square \)

3. Proof

The proof of Theorem [3] occupies the whole section. We assume that \( \phi_t \) is a \( C^r (r \geq 2) \) three dimensional volume preserving Anosov flow, and the volume measure is the maximal entropy measure. The proof consists of three parts: first in Section 3.1 we consider the trivial case that the flow is not topologically mixing; in Section 3.2 we use the fact that Margulis measure has smooth holonomy to show that both center stable and center unstable foliations are in fact \( C^{1,1} \); this enables us to use the result of Ghys (Proposition 2.5) to get smooth orbit equivalence between \( \phi_t \) and an algebraic Anosov flow \( \psi_t \). In Section 3.3 we show that the orbit equivalence \( h \) smoothly conjugates the Poincare return map of both flows restricted to the center unstable leaves. This implies that corresponding periodic orbits have the same period and unstable Lyapunov exponent, showing that \( \phi_t \) is smooth conjugate to an algebraic Anosov flow.

3.1. Non-topologically mixing Anosov flow. In this Section we will prove the Theorem in the case the flow \( \phi_t \) is not topologically mixing. Because \( \phi_t \) is a volume preserving Anosov flow, hence is transitive. It is shown by Plante in [26]:

**Lemma 3.1.** Let \( \phi_t \) be a transitive Anosov flow, then either \( \phi_t \) is topologically mixing, or it is the suspension of a Anosov diffeomorphism over \( \mathbb{T}^2 \) with constant return time.

We take a smooth torus \( \mathbb{T}^2 \) transverse to the flow, denote by the induced Poincare return map by \( f \). Then by the previous lemma, the return time is constant. To prove the Theorem, it deserves to show that \( f \) is \( C^{r-\varepsilon} \) conjugate to a hyperbolic automorphism on \( \mathbb{T}^2 \).

We denote by \( \pi^* \) the projection from the ambient manifold to \( \mathbb{T}^2 \) along the flow orbit of the suspension, then the projection of the Lebesgue measure \( m = (\pi^*)_* (\text{vol}) \) is a volume measure which is preserved by \( f \). Moreover, because the root function
of the suspension flow is constant, and vol is also the maximal entropy measure of $\phi$, $m$ is the maximal entropy measure of $f$, which is unique, by [9].

Denote by $h$ the conjugacy between $f$ and a linear hyperbolic automorphism $L$ over $\mathbb{T}^2$. Then $h_*$ maps the maximal entropy measure of $f$ to the maximal entropy measure of $L$. Because the unique maximal measure of $L$ is the Lebesgue measure, and by our previous discussion, $m$ is also a volume measure of $\mathbb{T}^2$, thus, $h_*(m) = \text{vol } |\tau|$ implies that $h$ is absolutely continuous. By the following lemma we may conclude that $h$ is $C^{r-\varepsilon}$.

**Lemma 3.2.** Let $f, g$ be two $C^r$ ($r = 2, 3, \cdots, w$) Anosov diffeomorphisms of a compact two dimensional manifold satisfying

$$f \circ h = h \circ g$$

where $h$ is a homeomorphism. If $h, h^{-1}$ are absolutely continuous with respect to Lebesgue measure then, $h$, $h^{-1}$ are $C^{r-\varepsilon}$.

3.2. Smooth center stable and center unstable foliations. From now on, we assume $\phi_t$ is always topologically mixing. This hypothesis enable us to use Margulis measures.

The main result of this section is the following proposition:

**Proposition 3.3.** The foliations $F^{cs}$ and $F^{cu}$ have uniformly $C^{1,1}$ (resp. $C^{s}$ for $s < r$).

**Remark 3.4.** There is a classical method from measure theory to show the smoothness of the one dimensional map $f$ between segments $I_1$ and $I_2$: if there exist probability measures $\mu_i$ on $I_i$, absolutely continuous with respect to the Lebesgue measures on $I_i$, with $C^{r-1}$ densities $\rho_i$ bounded away from zero, and such that $H_*(\mu_1) = \mu_2$, then $h$ is $C^r$. This observation has been widely used in series of works, see for instance [20, 41] and [40].

**Proof of proposition 3.3.** It is suffic to prove for the foliation $F^{cs}$, the proof for the foliation $F^{cu}$ is similar. Because for any $C^r$ Anosov flow, the center stable foliation has uniform $C^r$ leaves, by Lemma 2.4 it remains to show the holonomy map induced by $F^{cs}$ is uniformly $C^r$.

Fix $x_0 \in M$ and the sets $D^u \subset F_{\text{loc}}^{cu}(x)$ and $D^{cs} \subset F_{\text{loc}}^{cs}(x)$, we may define local product set:

$$B = [D^u, D^{cs}] = \{x : \{x\} = F_{\text{loc}}^{cu}(y) \cap F_{\text{loc}}^{cs}(z) ; y \in D^u, z \in D^{cs}\}.$$ We denote by $H^*$ the holonomy map between unstable disks inside $B$ along the center-stable foliation $F^{cs}$. For any $x \in B$, denote by $D^u(x)$ the connected component of $F^u(x) \cap B$ which contains $x$.

We need to consider the disintegration of the Lebesgue measure along the strong unstable foliations; this disintegration is equivalent to the Lebesgue measure on the leaves, see for instance [11, 24, 33]. The regularity of the disintegration can be found from [20] (see also [17]).

**Lemma 3.5.** Suppose $\phi_t$ is a volume preserving $C^r$ Anosov flow $r > 1$. The volume measure has $C^{r-1}$ density of disintegration along the strong unstable leaves. More precisely, let $B$ be any foliation chart of $F^u$, then for vol $|B$ almost every $x$, the disintegration of vol along $F^u_{\text{loc}}(x)$ equals to

$$\text{vol}^u_x = \rho(z) d\text{vol} |F^u_{\text{loc}}(x) (z)$$

where $\rho$ is continuous on $B$ and uniformly $C^{r-1}$ along the plaques of $F^u$, and is given by the formulas

$$\rho(z) = \frac{\Delta(x, z)}{\int \Delta(x, z) d\text{vol} |F^u_{\text{loc}}(x) (z)$$
and
\[ \Delta(x, z) = \lim_{n \to \infty} \frac{\det(Df^{-n}|_{T_xF(z)})}{\det(Df^{-n}|_{T_xF(z)})}. \]

Since we assume the volume measure and Margulis measure coincide, then by the uniqueness of disintegration, the disintegration of volume measure along the unstable plaques inside \( B = [D^u, D^cs] \) given by Lemma 3.4, \( \text{vol}_x^u \), and the normalization of Margulis measure \( \mu_{M,u} = \frac{\mu_{M,u} |D^u(x)|}{\text{vol}_x^u} \) should be the same. Thus, the Margulis measure
\[ \mu_{M,u} |D^u(x)| = \mu_{M,u}(D^u_x) = \mu_{M,u}(D^u_x) \times \text{vol}_x^u \]

has \( C^{r-1} \) density with respect to \( \text{vol} |D^u(x)| \).

Because the Margulis measure \( \mu_{M,u} \) is preserved by holonomy map \( \mathcal{H}^{cs} \), then for any \( y, z \in B \), denote by \( \mathcal{H}^{cs}_{y,z} \) the holonomy map from \( D^u_y \) to \( D^u_z \) induced by foliation \( F^{cs} \), we have
\[ \mathcal{H}^{cs}_{y,z}(\mu_{M,u} |D^u(y)|) = \mu_{M,u} |D^u(z)|. \]
Hence by Remark 3.4 the holonomy map is \( C^r \). The proof is complete. \( \square \)

3.3. Periodic data. By Lemma 3.3 and Proposition 2.5, \( \phi_t \) is \( C^r \) orbit equivalent to a suspension of linear hyperbolic automorphism on \( \mathbb{T}^2 \) or up to finite lift to orbit equivalent to the hyperbolic geodesic flow \( \psi_t \). Replacing by a finite lift, we may assume \( \phi_t \) is orbit equivalent to an algebraic Anosov flow \( \psi_t \). Denote by \( h \) the \( C^r \) diffeomorphism which maps orbit of \( \phi_t \) to \( \psi_t \). After time changing, we may assume the two flows \( \phi_t \) and \( \psi_t \) both have the same topological entropy:
\[ h_{top}(\phi^t) = h_{top}(\psi^t) = h_0. \]

Fix \( p \) any point belongs to a periodic orbit of \( \phi_t \) with period \( \pi(p) \), then \( h(p) \) belongs to a periodic orbit of \( \psi_t \) with period \( \pi(h(p)) \). We use \( \tilde{F}^i \) \((i = s, c, u, cs, cu)\) the corresponding foliations for the flow \( \psi_t \).

One may consider the Poincare return map \( P^{cs} : F^s_{loc}(x) \to F^s(x) \) where \( F^s(x) \) is parameterized by length and denote by \( \gamma(s) \). Then we have
\[ P^{cs}(\mu_{M,u} |D^u(x)|) = e^{h_0 \pi(p)} \mu_{M,u}(D^u_x). \]
Because the Margulis measure and the volume measure coincide with each other (see (10)), and observe that the Jacobian \( P^{cs} \) respect to \( \text{vol}_{F^s(x)} \) on the point \( x \) is exactly \( \left. \frac{d(P^{cs})}{ds} \right|_{s=0} = e^{h_0 \pi(p)} \).

A similar argument applies to flow \( \psi_t \) and the point \( h(p) \) shows that the Poincare return map for the flow \( \psi_t \) on the unstable leaf of \( h(p) \) has derivative \( e^{h_0 \tilde{\pi}(h(p))} \), where \( \tilde{\pi} \) denotes the period of periodic orbit of \( \psi_t \).

Because the orbit conjugacy \( h \) is \( C^r \), the curve \( \tilde{\gamma}(s) = h \circ \gamma(s) \subset \tilde{F}^{cs}(h(p)) \) is a curve transverse to the flow direction and \( \tilde{\gamma}(0) = h(p) \). Denote by \( \tilde{P}^{cs} : \tilde{\gamma} \to \tilde{\gamma} \) the Poincare return map induced by the flow \( \psi_t \). It is easy to see that \( \tilde{P}^{cs} |_{\tilde{\gamma}} \) is conjugate to the corresponding Poincare return map on the unstable leaf of \( h(p) \) induced by the flow of \( \psi_t \). Then, by the previous discussion,
\[ \left. \frac{d(\tilde{P}^{cs})}{ds} \right|_{s=0} = e^{h_0 \tilde{\pi}(p)}. \]

Since \( h \) is orbit conjugacy between flows \( \phi_t \) and \( \psi_t \), and \( h(\gamma) = \tilde{\gamma} \), it induces a conjugacy between \( P^{cs} \) and \( \tilde{P}^{cs} \). Since \( h \) is smooth, then \( \left. \frac{d(P^{cs})}{ds} \right|_{s=0} = \left. \frac{d(\tilde{P}^{cs})}{ds} \right|_{s=0} \), by
(13) and (14), we have
\[ \pi(p) = \tilde{\pi}(h(p)). \]

Thus, by Proposition 2.2, the two Anosov flows \( \phi_t \) and \( \psi_t \) are conjugate to each other. Moreover, observe that \( \log \frac{P^{cs}(t)}{dt} \big|_{t=0}/\pi(p) = h_0 \) is exactly the Lyapunov exponent of \( \phi_t \) on the orbit of \( p \) along the unstable direction, and
\[ \log \frac{\tilde{P}^{cs}(t)}{dt} \big|_{t=0}/\tilde{\pi}(h(p)) = h_0 \]
is exactly the Lyapunov exponent of \( \psi_t \) on the orbit of \( h(p) \) along the unstable direction, they are the same and both equal to \( h_0 \). Similar results apply on the stable bundle if one considers the inverse of flow; this enables us to show that the Lyapunov exponents at corresponding periodic orbits of \( \phi_t \) and \( \psi_t \) coincides. By Proposition 2.3 these two flows are \( C^s \) conjugate for any \( s < r \). The proof is complete.

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