A note on fluctuations for internal diffusion limited aggregation*

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Abstract

We consider a cluster growth model on $\mathbb{Z}^d$, called internal diffusion limited aggregation (internal DLA). In this model, random walks start at the origin, one at a time, and stop moving when reaching a site not occupied by previous walks. It is known that the asymptotic shape of the cluster is spherical. Also, when dimension is 2 or more, and when the cluster has volume $n^d$, it is known that fluctuations of the radius are at most of order $n^{1/3}$. We improve this estimate to $n^{1/(d+1)}$, in dimension 3 or more. In so doing, we introduce a closely related cluster growth model, that we call the flashing process, whose fluctuations are controlled easily and accurately. This process is coupled to internal DLA to yield the desired bound. Part of our proof adapts the approach of Lawler, Bramson and Griffeath, on another space scale, and uses a sharp estimate (written by Blachère in our Appendix) on the expected time spent by a random walk inside an annulus.

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1 Introduction

The internal DLA cluster of volume $N$, say $A(N)$, is obtained inductively as follows. Initially, we assume that the explored region is empty, that is $A(0) = \emptyset$. Then, consider $N$ independent discrete-time random walks $S_1, \ldots, S_N$ starting from 0. Assume $A(k-1)$ is obtained, and define

$$\tau_k = \inf \{ t \geq 0 : S_k(t) \not\in A(k-1) \}, \quad \text{and} \quad A(k) = A(k-1) \cup \{S_k(\tau_k)\}.$$  \hspace{1cm} (1.1)

In such a particle system, we call explorers the particles. We say that the $k$-th explorer is \textit{settled} on $S_k(\tau_k)$ after time $\tau_k$, and is \textit{unsettled} before time $\tau_k$. The cluster $A(N)$ is the positions of the $N$ settled explorers. We study the growth of $A(N)$, as $N$ tends to infinity.

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The mathematical model of internal DLA was introduced first in the chemical physics literature by Meakin and Deutch [11]. There are many industrial processes that look like internal DLA (see the nice review paper [5]). The most important seems to be electropolishing, defined as the improvement of surface finish of a metal effected by making it anodic in an appropriate solution. There are actually two distinct industrial processes (i) anodic levelling or smoothing which corresponds to the elimination of surface roughness of height larger than 1 micron, and (ii) anodic brightening which refers to elimination of surface defects which are protruding by less than 1 micron. The latter phenomenon requires an understanding of atom removal from a crystal lattice. It was noted in [11], at a qualitative level, that the model produces smooth clusters, and the authors wrote “it is also of some fundamental significance to know just how smooth a surface formed by diffusion limited processes may be”.

Diaconis and Fulton [2] introduced internal DLA in mathematics. Their model is more general than ours: explorers can start on distinct sites, and the explored region at time 0 is not necessarily empty. They were interested in defining a random growth process by iterating simple operation. They introduced many variations, and treat, among other things, the special one dimensional case.

In dimensions two and more, Lawler, Bramson and Griffeath [8] prove that in order to cover, without holes, a sphere of radius $n$, we need about the number of sites of $\mathbb{Z}^d$ contained in this sphere. In other words, the asymptotic shape of the cluster is a sphere. Then, Lawler in [7] shows subdiffusive fluctuations. The latter result is formulated in terms of inner and outer errors, which we now introduce with some notation. We denote with $\| \cdot \|$ the euclidean norm on $\mathbb{R}^d$. For any $x$ in $\mathbb{R}^d$ and $r$ in $\mathbb{R}$, set

$$B(x, r) = \{ y \in \mathbb{R}^d : \| y - x \| < r \} \quad \text{and} \quad B(x, r) = B(x, r) \cap \mathbb{Z}^d.$$  \hspace{1cm} (1.2)

For $\Lambda \subset \mathbb{Z}^d$, $|\Lambda|$ denotes the number of sites in $\Lambda$. The inner error $\delta_I(n)$ is such that

$$n - \delta_I(n) = \sup \{ r \geq 0 : B(0, r) \subset A(|B(0, n)|) \}. \hspace{1cm} (1.3)$$

Also, the outer error $\delta_O(n)$ is such that

$$n + \delta_O(n) = \inf \{ r \geq 0 : A(|B(0, n)|) \subset B(0, r) \}. \hspace{1cm} (1.4)$$

The main result of [7] reads as follows.

**Theorem 1.1** [Lawler] Assume $d \geq 2$. With probability 1,

$$\lim_{n \to \infty} \frac{\delta_I(n)}{n^{1/3} \log(n)^2} = 0, \quad \text{and} \quad \lim_{n \to \infty} \frac{\delta_O(n)}{n^{1/3} \log(n)^4} = 0. \hspace{1cm} (1.5)$$

Since Lawler’s paper, published 15 years ago, no improvement of these estimates was achieved, but it is believed that fluctuations are on a much smaller scale than $n^{1/3}$. Computer simulations [12, 3] suggest indeed that fluctuations are logarithmic. In addition, Levine and Peres studied a deterministic analogue of internal DLA, the rotor-router model, introduced by J.Propp [4]. They bound, in [10], the inner error $\delta_I(n)$ by $\log(n)$, and the outer error $\delta_O(n)$ by $n^{1-1/d}$.

We present here an improvement on (1.5) in dimension $d \geq 3$. 

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**Fluctuations for internal DLA**

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Theorem 1.2 Assume $d \geq 3$. There is a positive constant $A_d$ such that, with probability 1,

$$\limsup_{n \to \infty} \frac{\delta_1(n)}{\frac{1}{n} \log(n)} \leq A_d \quad \text{and} \quad \lim_{n \to \infty} \frac{\delta_2(n)}{\frac{1}{n} \log^2(n)} = 0. \quad (1.6)$$

Let us now recall the approach of Lawler, Bramson and Griffeath in [8], explain our idea, and introduce a new growth model. The approach of [8] is based on estimating the number $W(z)$ of explorers that visit each site $z \in \mathbb{Z}^d$. It is based on the following observations. (i) If explorers would not settle, they would just be independent random walks; (ii) exactly one explorer occupies each site of the cluster. Thus, if we launch one explorer from each site of the cluster $A(N)$, and call $M(A(N), z)$ the number of crossings of site $z$, then $M(N\delta_0, z)$ which we define as $W(z) + M(A(N), z)$ would be equal in law to the number of walks crossing $z$ out of $N$ independent walks started on 0. Even though $M(A(N), z)$ and $M(N\delta_0, z)$ are dependent variables, some estimates on $P(W(z) = 0)$ can be extracted from estimating the means of $M(A(N), z)$ and of $M(N\delta_0, z)$.

Rather than thinking in terms of one single site, we observe that a site has good chances to lie inside the cluster if some large region, about this site, is crossed by many explorers. How large should be this region, say $C$, and how many should be these crossings, say $W(C)$, can be answered as follows. The size and location of $C$ should be such that (i) the expected number of crossings of $C$ is much larger than its standard deviation, and (ii) the number of crossings of $C$ needed to cover $C$ is of order $|C|$ (as suggested by the spherical shape result of [8])

$$\begin{align*}
(i) \quad & E[W(C)] \gg \sqrt{\text{var}(W(C))}, \\
(ii) \quad & E[W(C)] \geq |C|.
\end{align*} \quad (1.7)$$

Now, assume that $|\mathbb{B}(0, n)|$ explorers start at the origin. For a space-scale $h(n)$ and an integer $k > 1$, to be determined, assume that $\mathbb{B}(0, n - kh(n))$ is covered by settled explorers.

Partition the shell $S = \mathbb{B}(0, n - (k - 1)h(n)) \setminus \mathbb{B}(0, n - kh(n))$ into about $(n/h(n))^{d-1}$ cells, each of volume $h(n)^d$. Cells are brick-like domain, of side length the width of the shell. It is convenient to imagine that a cell $C \in S$ is the basis of a column of $k$ cells reaching the boundary of $\mathbb{B}(0, n)$. It is also convenient to stop the explorers as they reach the boundary of $\mathbb{B}(0, n - kh(n))$. Thus, with such a stopped process, explorers are either settled inside the $\mathbb{B}(0, n - kh(n))$ or unsettled but stopped on its boundary, that we denote by $\partial \mathbb{B}(0, n - kh(n))$. We have called earlier the number of explorers crossing $C$ is taken here to be the unsettled explorers stopped on $C \cap \partial \mathbb{B}(0, n - kh(n))$. In these heuristics, we make the simplifying assumption that only explorers stopped on $C \cap \partial \mathbb{B}(0, n - kh(n))$ can cover $C$ once released.

- On the average, $kh(n)^d$ explorers are stopped on each cell of $S$. Thus, for $k$ large enough (ii) of $(1.7)$ should be fulfilled.

- The standard deviation of $W(C)$ is a delicate issue not yet settled. Here follows a heuristic justification for an upper bound for the standard deviation. If each explorer had to choose uniformly at random a cell $C$ of $S$ and were constrained to perform a reflected random walk inside a cone issued from the origin and with base $C$, then we would obtain another growth model whose fluctuations are expectedly larger than those of
internal DLA. For such a model the standard deviation of the crossing of a cell is of order
\[ \sqrt{\mathbb{B}(0, n)|P_0(S(\tau) \in \mathcal{C})} \sim \sqrt{n^d (h(n)/n)^{d-1}}, \]  
where \( \tau \) is the exit time from \( \mathbb{B}(0, n - kh(n)) \), and \( P_0 \) is the law of a simple random walk, \( S \), starting at the origin. We expect (1.8) to be an upper bound for the standard deviation of \( W(\mathcal{C}) \).

At a heuristic level, (i) of (1.7) follows if
\[ kh(n)^d \gg n^d \times \left( \frac{h(n)}{n} \right)^{d-1} \iff h(n) \gg n^{\frac{1}{1+d}}. \]  

This discussion motivates a growth model associated from the start with the exponent \( \frac{1}{1+d} \). We call this model the flashing internal DLA process, or simply the flashing process. The flashing process looks like internal DLA on a large scale, but has a distinct covering mechanism, which makes it much simpler to analyze. To obtain this growth model, we generalize the rules described in (1.1), by enabling explorers to settle only at special times, called flashing times. Thus, each explorer is associated with a sequence of stopping times, and it is only at these times that it settles if outside the cluster. The precise definition of the chosen stopping times requires additional notation, which we postpone to Section 3. We describe here key features of the flashing process.

First, \( \mathbb{Z}^d \) is partitioned into concentric shells around the origin: a shell at a distance \( r \) from the origin has a width of order \( r^{\frac{1}{1+d}} \). Each shell is in turn partitioned into cells, which are brick-like domain, of side length the width of the shell. The key features are as follows.

\begin{enumerate}[label=P\arabic*.,leftmargin=*,itemsep=0pt]
  \item An explorer \textit{flashes} at most once in each shell.
  \item The \textit{flashing} position in a shell, is essentially uniform over the cell an explorer first hits upon entering the shell.
  \item When an explorer leaves a shell, it cannot afterward \textit{flash} in it.
\end{enumerate}

Feature \( P_b \) is the seed of a deep difference with internal DLA. The mechanism of covering a cell, for the flashing process, is very much the same as completing an album in the classical coupon-collector process. Thus, we need of the order of \( V \log(V) \) explorers to cover a cell of volume \( V \). For internal DLA, with explorers started at the origin, we need only of order \( V \) explorers to cover a sphere of volume \( V \) as shown in [8], and we believe that we need a number of explorers of order \( |\mathcal{C}| \) to cover a cell \( \mathcal{C} \).

Feature \( P_c \) is essential for having the following coupling between flashing and internal DLA processes.

\textbf{Theorem 1.3} There is a coupling between the two processes such that, for all \( k, N \geq 1 \), and for a sequence \( \{r_k, k \in \mathbb{N}\} \) going to infinity (that we describe in Section 3), and \( h_k = r_k^{1/(d+1)} \)

\begin{itemize}
  \item if \( A^*(N) \subset \mathbb{B}(0, r_k + h_k) \), then \( A(N) \subset \mathbb{B}(0, r_k + h_k) \),
\end{itemize}
• if $\mathbb{B}(0, r_k + h_k) \subset A^*(N)$, then $\mathbb{B}(0, r_k + h_k) \subset A(N)$.

For the flashing process, we control easily the inner error. Then, to control the outer error we follow the approach of [7], though with a simpler proof (allowed by the cell structure used to build our growth model).

**Theorem 1.4** There is a positive constant $A_d$ such that, (1.6) hold for the flashing process.

Also, we know that flashing internal DLA does exhibit power-law fluctuations.

**Theorem 1.5** With probability 1,

$$\lim_{n \to \infty} \frac{n^{\frac{1}{d+1}}}{\delta_f(n)} = 0.$$  \hfill (1.10)

Theorem 1.3 and Theorem 1.4 imply Theorem 1.2.

Let us now describe the heuristics behind Theorem 1.5. It is useful to organize the flow of explorers in the flashing process into *exploration waves*, in the way of Section 3 of [7]. That is, in the $k$-th exploration wave, the explorers either stop as they reach the *bulk* of $S_k$, or settle before reaching $S_k$. Consider now the exploration wave associated with the last shell making $\mathbb{B}(0, n)$, say $S^*$. Assume that at this time, the cluster fills $\mathbb{B}(0, n) \setminus S^*$. From our definition, $S^*$ has width of order $n^{\frac{1}{d+1}}$, and the last shell receives a number of stopped explorers equal to its volume. There is necessarily one cell in $S^*$ which receives of the order of its own volume, and for a coupon-collector process, it is very unlikely that the explorers stopped in this very cell can cover the bulk of this cell before escaping $S^*$. By feature $P_c$, if a hole is left in $S^*$ after the explorers leave $S^*$, this hole remains uncovered forever. These heuristics pose two questions concerning internal DLA, we are unable to answer at the moment.

• How many explorers stopped in the *bulk* of a cell are needed to cover the whole cell?

• What is the correct order of fluctuations in internal DLA?

The rest of the paper is organized as follows. Section 2 introduces the main notation, and recall well known useful facts. In Section 3 we build the flashing process, give an alternative construction, and prove Theorem 1.3. In Section 4 we obtain a sharp estimate on the expected number of explorers crossing a given cell, and prove $P$-b. Both proofs are based on classical potential theory estimates. In Section 5 we prove Theorems 1.4 and 1.5. Finally, in the Appendix, Sébastien Blachère gives a sharp estimate on the expected time spent in an annulus by a random walk.

## 2 Notation and useful tools

### 2.1 Notation

We say that $z, z' \in \mathbb{Z}^d$ are nearest neighbors when $\|z - z'\| = 1$, and we write $z \sim z'$. For any subset $\Lambda \subset \mathbb{Z}^d$, we define

$$\partial \Lambda = \{ z \in \mathbb{Z}^d \setminus \Lambda : \exists z' \in \Lambda, z' \sim z \}.$$  \hfill (2.1)
For any \( r \leq R \) we define the annulus
\[
A(r, R) = B(0, R) \setminus B(0, r) \quad \text{and} \quad A_r(R) = A(r, R) \cap \mathbb{Z}^d
\] (2.2)
A trajectory \( \gamma \) is a discrete nearest-neighbor path on \( \mathbb{Z}^d \). That is \( \gamma : \mathbb{N} \to \mathbb{Z}^d \) with \( \gamma(t) \sim \gamma(t + 1) \) for all \( t \). The law of the simple random walk started in \( z \), is denoted with \( \mathbb{P}_z \). For a subset \( \Lambda \) in \( \mathbb{Z}^d \), and a trajectory \( \gamma \), we define the hitting time of \( \Lambda \) as
\[
H(\Lambda; \gamma) = \min \{ t \geq 0 : \gamma(t) \in \Lambda \}.
\]
We often omit \( \gamma \) in the notation when no confusion is possible. We use the shorthand notation
\[
B_n = B(0, n), \quad \mathbb{B}_n = \mathbb{B}(0, n), \quad H_R = H(B_R^c), \quad \text{and} \quad H_z = H(\{z\}).
\]
For any \( a, b \) in \( \mathbb{R} \) we write \( a \wedge b = \min\{a, b\} \), and \( a \vee b = \max\{a, b\} \). Let \( \Gamma \) be a finite collection of trajectories on \( \mathbb{Z}^d \). For \( R > 0 \), \( z \) in \( \mathbb{Z}^d \) and \( \Lambda \) a subset of \( \mathbb{Z}^d \), we call \( M(\Gamma, R, z) \) (resp. \( M(\Gamma, R, \Lambda) \)) the number of trajectories which exit \( B(0, R) \) on \( z \) (resp. in \( \Lambda \)):
\[
M(\Gamma, R, z) = \sum_{\gamma \in \Gamma} 1_{\{\gamma(H_R) = z\}}, \quad \text{and} \quad M(\Gamma, R, \Lambda) = \sum_{z \in \Lambda} M(\Gamma, R, z).
\] (2.3)
When we deal with a collection of independent random trajectories, we rather specify its initial configuration \( \eta \in \mathbb{N}^{\mathbb{Z}^d} \), so that \( M(\eta, R, z) \) is the number of random walks starting from \( \eta \) and hitting \( B(0, R)^c \) on \( z \). Two types of initial configurations are important here: (i) the configuration \( n\mathbf{1}_{z^*} \) formed by \( n \) walkers starting on a given site \( z^* \), (ii) for \( \Lambda \subset \mathbb{Z}^d \), the configuration \( \mathbf{1}_{\Lambda} \) that we simply identify with \( \Lambda \). For any configuration \( \eta \in \mathbb{N}^{\mathbb{Z}^d} \) we write
\[
|\eta| = \sum_{z \in \mathbb{Z}^d} \eta(z).
\] (2.4)

We are in dimension 3 or more, and Green’s function of the simple random walk is well defined and denoted \( G \). That is, for any \( x, y \in \mathbb{Z}^d \)
\[
G(x, y) = \mathbb{E}_x \left[ \sum_{n \geq 0} 1_{\{S(n) = y\}} \right].
\] (2.5)
For any \( \Lambda \subset \mathbb{Z}^d \), we define Green’s function restricted to \( \Lambda \), \( G_{\Lambda} \), as follows. For \( x, y \in \Lambda \)
\[
G_{\Lambda}(x, y) = \mathbb{E}_x \left[ \sum_{0 \leq n < H(\Lambda^c)} 1_{\{S(n) = y\}} \right].
\] (2.6)

### 2.2 Some useful tools

We recall here some well known facts. Some of them are proved for the reader’s convenience. This section can be skipped at a first reading.
In [3], the authors emphasized the fact that the spherical limiting shape of internal DLA was intimately linked to strong isotropy properties of Green’s function. This isotropy is expressed by the following asymptotics (Theorem 4.3.1 of [9]). In \( d \geq 3 \), there is a constant \( K_g \), such that for any \( z \neq 0 \),
\[
\left| G(0, z) - \frac{C_d}{\|z\|^{d-2}} \right| \leq \frac{K_g}{\|z\|^{d}} \quad \text{with} \quad C_d = \frac{2}{v_d(d-2)},
\]  
where \( v_d \) stands for the volume of the euclidean unit ball in \( \mathbb{R}^d \). The first order expansion
\[
(2.7)
\]  
is proved in [9] for general symmetric walks with finite \( d + 3 \) moments and vanishing third moment. All the estimates we use are eventually based on (2.7) and we emphasize the fact that the estimate is uniform in \( \|z\| \).

The following lemma is also used in the Appendix.

**Lemma 2.1** Each \( z^* \in \mathbb{Z}^d \setminus \{0\} \) has a nearest-neighbor \( z \) (i.e. \( z^* \sim z \)) such that
\[
\|z\| \leq \|z^*\| - \frac{1}{2\sqrt{d}}.
\]  

**Proof.** Without loss of generality we can assume that all the coordinates of \( z^* \) are non-negative. Let us denote by \( b \) the maximum of these coordinates and note that
\[
\|z^*\|^2 \leq db^2, \quad \text{and} \quad b \geq 1.
\]  

Denote by \( z \) the nearest-neighbor obtained from \( z^* \) by decreasing by one unit a maximum coordinate. Using (2.9)
\[
\|z^*\|^2 - \|z\|^2 = b^2 - (b - 1)^2 = 2b - 1 \geq b \geq \frac{\|z^*\|}{\sqrt{d}}.
\]  

Note that (2.8) follows from \( 2\|z^*\| (\|z^*\| - \|z\|) \geq \|z^*\|^2 - \|z\|^2 \), and (2.10).

We recall a roughly but useful result about the exit site distribution from a sphere. This is Lemma 1.7.4 of [6].

**Lemma 2.2** There are two positive constants \( c_1, c_2 \) such that for any \( z \in \partial B(0, n) \), and \( n > 0 \)
\[
\frac{c_1}{n^{d-1}} \leq \mathbb{P}_0(S(H_n) = z) \leq \frac{c_2}{n^{d-1}}.
\]  

Finally, we recall a well known large deviations estimate for independent Bernoulli variables (see for instance Lemma 4.3 of [1]).

**Lemma 2.3** For any positive integer \( n \), and \( \{X_1, \ldots, X_n\} \) independent Bernoulli variables, we have for any \( x > 0 \), and with \( X = X_1 + \cdots + X_n \)
\[
\max \left( P(X - E[X] \geq x), P(X - E[X] \leq -x) \right) \leq \exp \left( -\left( \min \left( \frac{x^2}{4\text{var}[X]}, \frac{x}{2} \right) \right) \right).
\]  

(2.12)
Remark 2.4 Note that for a sum of Bernoulli, $\text{var}[X] \leq E[X]$, and the following inequality is useful
\[
\max (P(X - E[X] \geq x), P(X - E[X] \leq -x)) \leq \exp \left(-\min \left(\frac{x^2}{4E[X]}, \frac{x}{2}\right)\right). \tag{2.13}
\]
Also, note that if $E[X] \leq E[Y]$, where $Y$ is a sum of $m$ independent Bernoulli variables, then
\[
\max (P(X - E[X] \geq x), P(Y - E[Y] \leq -x)) \leq \exp \left(-\min \left(\frac{x^2}{4E[Y]}, \frac{x}{2}\right)\right). \tag{2.14}
\]

3 The flashing process

In this section, we construct the flashing process. We then present a useful alternative construction of the same process. Finally, we prove Theorem 1.3 which couples the two processes.

3.1 Construction of the process

Partitioning the lattice. We partition the lattice into shells ($\mathcal{S}_j : j \geq 0$). For a given parameter $h_0 > 0$ the first shell $\mathcal{S}_0$ is the ball $\mathbb{B}(0, h_0)$. The next shells are the annuli
\[
\mathcal{S}_j = A(r_j - h_j, r_j + h_j), \quad j \geq 1, \tag{3.1}
\]
where $r_j$ and $h_j$ are defined inductively by $r_1 - h_1 = h_0$, and for $j \geq 1$
\[
r_{j+1} - h_{j+1} = r_j + h_j, \quad \text{and} \quad h_j = r_j^{d+1}. \tag{3.2}
\]
We omit the easy check that (3.2) yields
\[
r_j \sim \left(\frac{2d}{d+1}j\right)^{\frac{d+1}{d-1}}. \tag{3.3}
\]
We also define
\[
\Sigma_0 = \{0\} \quad \text{and} \quad \Sigma_j = \partial \mathbb{B}(0, r_j), \quad j \geq 1. \tag{3.4}
\]

Flashing times. Consider $\{X_j, Y_j, j \geq 0\}$ a sequence of independent Bernoulli variables such that
\[
P(X_j = 1) = 1 - P(X_j = 0) = \frac{1}{h_j}, \tag{3.5}
\]
\[
P(Y_j = 1) = 1 - P(Y_j = 0) = \begin{cases} \frac{1}{2} & \text{if } j = 0, \\ 1 & \text{if } j \geq 1. \end{cases} \tag{3.6}
\]
Consider also a sequence of continuous independent variables $\{R_j, j \geq 0\}$ each of which has density $g_j : [0, h_j] \to \mathbb{R}^+$ with
\[
g_j(h) = \frac{dh^{d-1}}{h_j^d}. \tag{3.7}
\]
For $j \geq 0$, and $z_j$ in $\Sigma_j$, let $S$ be a random walk starting in $z_j$, and define a stopping time $\sigma$ as follows. If $R_j = h$ for some $h \leq h_j$ then

$$\sigma = \begin{cases} 
0 & \text{if } X_j = 1, \\
H(\mathbb{B}(z, h) \cap (r_j + h_j - \|z_j\|)) & \text{if } X_j = 0 \text{ and } Y_j = 1, \\
H(A(r_j - h, r_j + h)) & \text{if } X_j = 0 \text{ and } Y_j = 0.
\end{cases} \quad (3.8)$$

We set $H_j = H(\Sigma_j)$, and we define the stopping times $(\sigma_j : j \geq 0)$ as

$$\sigma_j = H_j + \sigma(S \circ \theta_{H_j}), \quad (3.9)$$

where $\theta$ stands for the usual time-shift operator. For $j \geq 0$ we note that, by construction, $S(t) \in S_j$ for all $t$ such that $H_j \leq t < \sigma$ and we say that $\sigma_j$ is a flashing time when $S(\sigma_j)$ is contained in the intersection between $S_j$ and the cone with base $B(S(H_j), h_j/2)$. We call such an intersection a cell centered at $S(H_j)$, that we denote $C(S(H_j))$. In other words, for any $z \in \Sigma_j$

$$C(z) = S_j \cap \{ x \in \mathbb{R}^d : \exists \lambda \geq 0, \exists y \in B(z, h_j/2), x = \lambda y \}. \quad (3.10)$$

The uniform hitting property. The main property of the hitting time $\sigma$ constructed above is the following proposition, which yields property $Pb$ of the flashing process to be defined soon.

**Proposition 3.1** There are two positive constants $\alpha_1 < \alpha_2$, such that, for $j \geq 0$, $z_j \in \Sigma_j$, and $z^* \in C(z_j)$,

$$\frac{\alpha_1}{h_j^d} \leq \mathbb{P}_{z_j}(S(\sigma) = z^*) \leq \frac{\alpha_2}{h_j^d}. \quad (3.11)$$

The proof of Proposition 3.1 is given in Section 4.

The flashing process. Consider a family of $N$ independent random walks $(S^*_i : 1 \leq i \leq N)$ with their hitting times, and stopping times $(H_{i,j}, z_{i,j}, \sigma_{i,j} : j \geq 0)$. Let also $z_{i,j} = S_i(H_{i,j})$ be the first hitting position of $E_i$ on $\Sigma_i$.

We define the cluster inductively. Set $A^*(0) = \emptyset$. For $i \geq 1$, we define $\tau_i^*$ as the first flashing time associated with $S^*_i$ when the explorer stands outside $A^*(i-1)$. In other words,

$$\tau_i^* = \min \{ \sigma_{i,j} : j \geq 0, S_i^*(\sigma_{i,j}) \in C(z_{i,j}) \cap A^*(i-1)^c \}, \quad (3.12)$$

and

$$A^*(i) = A^*(i-1) \cup \{ S_i^*(\tau_i^*) \}. \quad (3.13)$$

### 3.2 Exploration Waves

Rather than building $A^*(N)$ following the whole journey of one explorer after another, we can build $A^*(N)$ as an increasing union of clusters formed by stopping explorers on successive shells. Similar wave constructions are introduced in [8] and [7], with an equality in law between alternative constructions. However, the features of the flashing process are
such that in our case the two constructions are strictly equivalent. We use this alternative construction in the proof of Theorem 1.3.

We denote by \( \xi_k \in (\mathbb{Z}^d)^N \) the explorers positions after the \( k \)-th wave. We denote by \( A_k^*(N) \) and the set of sites where settled explorers are after the \( k \)-th wave. Our construction will be such that

\[
\xi_k(i) \not\in \Sigma_k \iff \xi_k(i) \in \bigcup_{j<k} S_j \iff \xi_k(i) \in A_k^*(N).
\]

(3.14)

For \( k = 0 \) we set \( \xi_0(i) = 0 \), and \( A_0^*(i) = \emptyset \), for \( 1 \leq i \leq N \). Then, for all \( k \geq 0 \), we set \( A_{k+1}(0) = A_k^*(N) \). For \( i \) in \( \{1, \ldots, N\} \), we set the following.

- If \( \xi_k(i) \not\in \Sigma_k \), then
  \[
  \xi_{k+1}(i) = \xi_k(i) \in \bigcup_{j<k} S_j, \quad \text{and} \quad A_{k+1}(i) = A_k^*(i-1).
  \]

- If \( \xi_k(i) \in \Sigma_k \) and \( S_i(\sigma_{i,k}) \in C(z_{i,k}) \cap A_k^*(i-1)^c \), then
  \[
  \xi_{k+1}(i) = S_i(\sigma_{i,k}) \in \Sigma_k, \quad \text{and} \quad A_{k+1}^*(i) = A_k^*(i-1) \cup \{S_i(\sigma_{i,k})\}.
  \]

- If \( \xi_k(i) \in \Sigma_k \) and \( S_i(\sigma_{i,k}) \not\in C(z_{i,k}) \cap A_k^*(i-1)^c \), then
  \[
  \xi_{k+1}(i) = S_i(H_{i,k+1}) \in \Sigma_{k+1}, \quad \text{and} \quad A_{k+1}^*(i) = A_k^*(i-1)\setminus \{ \).

In words, for each \( k \geq 1 \), during the \( k \)-th wave of exploration, the unsettled explorers move one after the other in the order of their labels until either settling in \( S_{k-1} \), or reaching \( \Sigma_k \) where they stop. We then define \( A^*(N) \) by

\[
A^*(N) = \bigcup_{k \geq 1} A_k^*(N).
\]

(3.15)

We explain now why this construction yields the same cluster as our previous definition. An explorer cannot settle inside a shell it has left, and thus cannot settle in any shell \( S_j \) with \( j < k \) if it reaches \( \Sigma_k \). Now, since each wave of exploration is organized according to the label ordering, the fact that an explorer has to wait for the following explorers before proceeding its journey beyond \( \Sigma_k \) does not interfere with the site where it eventually settles.

### 3.3 Coupling internal DLA and flashing processes

We use here the first definition of the flashing process, and realize the internal DLA process using the same randomness.

**Proposition 3.2** There is a coupling between the flashing and original internal DLA processes such that, for all \( N \geq 1 \),

\[
\bigcup_{i=1}^N \{S_i(t) : 0 \leq t \leq \tau_i\} \subset \bigcup_{i=1}^N \{S_i^*(t) : 0 \leq t \leq \tau_i^*\}
\]

(3.16)

and there is a one to one mapping \( \psi_N : A(N) \rightarrow A^*(N) \) such that for \( z \in A(N) \)

\[
\text{if for } k \geq 1, \quad z \not\in \bigcup_{j<k} S_j, \quad \text{then} \quad \psi_N(z) \not\in \bigcup_{j<k} S_j.
\]

(3.17)
Fluctuations for internal DLA

Theorem 1.3 is a simple consequence of Proposition 3.2. On the one hand, if \( A^*(N) \subset \cup_{j<k} S_j \) for some \( k \geq 1 \), then, recalling that a flashing explorer cannot settle a shell it has left, the orbits of the \( N \) flashing explorers are all contained in \( \cup_{j<k} S_j \) and, by (3.16), so is \( A(N) \). On the other hand, if \( \cup_{j<k} S_j \subset A^*(N) \), then, (3.17) implying that with such a coupling

\[
|A(N) \cap \cup_{j<k} S_j| \geq |A^*(N) \cap \cup_{j<k} S_j| = |\cup_{j<k} S_j|, \tag{3.18}
\]

which implies that \( \cup_{j<k} S_j \subset A(N) \).

**Proof of Proposition 3.2.** We build the coupling together with the map \( \psi_N \) by induction on \( N \). We use the trajectories of the flashing explorers to drive the internal DLA trajectories. We need a little more notation to do so. For each \( i \leq N \), set for simplicity \( g^*(i) = S^*_i(\tau^*_i) \), and denote by \( t_{i,N} \) the length of the flashing trajectory \( (S^*_i(t) : 0 \leq t \leq t_{i,N}) \) used to form the trajectories of the original explorers. Necessarily, we need \( t_{i,N} \leq \tau^*_i \) to have (3.16). For convenience, we partition \( A(N) \) into blue sites, say \( B(N) \), and red sites, say \( R(N) \).

We build a one to one map \( f_N : A(N) \to \{1, \ldots, N\} \), together with the blue-red partition as follows. For each \( z \in A(N) \), there are two possibilities. Either there is \( i \leq N \) such that \( S^*_i(\tau^*_i) = z \) and \( t_{i,N} = \tau^*_i \), and we say that \( z \in B(N) \) and we set \( f_N(z) = i \). Otherwise \( z \in R(N) \). We then define \( f_N(z) \) as the label \( i \) of the flashing explorer that was driving the random walk \( S_j \) when the \( j \)-th explorer settled in \( z \), and this will imply, by induction, that \( t_{i,N} < \tau^*_i \).

Finally the one to one map \( \psi_N : A(N) \to A^*(N) \) is the composition \( g^* \circ f_N \). Note, first, that for all \( z \in B(N) \), \( \psi_N(z) = z \); second, \( B(N) \subset A(N) \cap A^*(N) \) and last,

\[
z \in B(N) \iff t_{f_N(z),N} = \tau^*_{f_N(z)}, \quad z \in R(N) \iff t_{f_N(z),N} < \tau^*_{f_N(z)}. \]

We now start our induction with \( N = 1 \). The first trajectory \( (S^*_1(t) : 0 \leq t \leq \tau^*_1) \) ends in \( g^*(1) \). We use this trajectory to build that of the first internal DLA-explorer. We set \( S_1(0) = S^*_1(0) \) and immediately stop here since the origin \( 0 = S_1(0) \) was initially unoccupied. As a consequence \( t_{1,1} = 0 \), \( \tau_1 = 0 \) and there are two possibilities: either \( \tau^*_1 = 0 \), \( B(1) = \{0\} \), \( R(1) = \emptyset \) and \( f_1(0) = 1 \), or, \( \tau^*_1 > 0 \), \( R(1) = \{0\} \), \( B(1) = \emptyset \) and \( f_1(0) = 1 \).

Assume now that we have built from the trajectories \( \{S^*_i(t) : 0 \leq t \leq \tau^*_i\}, \ i \leq N \), the clusters \( A^*(N) \), and the sets \( B(N) \) and \( R(N) \), together with the times \( \{t_{i,N} \ i \leq N\} \), and the one to one map \( f_N : A(N) \to \{1, \ldots, N\} \). We launch a new flashing explorer with trajectory \( (S^*_N(t) : 0 \leq t \leq \tau^*_N) \) that ends in \( g^*(N+1) \), and we start to define the \( (N+1) \)-th trajectory for the original internal DLA process by following \( S^*_N+1 \):

\[
S_{N+1}(0) = S^*_N(0), \quad S_{N+1}(1) = S^*_N(1), \ldots \tag{3.19}
\]

- If \( \{S^*_N(t) : 0 \leq t \leq \tau^*_N \} \) is not contained in \( A(N) \) then \( S_{N+1} \) settles the first time \( S^*_N+1 \) exits \( A(N) \), that is at time \( (\text{resp. on a site } z) \)

\[
t_{N+1,N+1} = \inf\{k \geq 0 : S^*_N+1 \not\in A(N)\}, \quad (\text{resp. } z = S^*_N+1(t_{N+1,N+1}) \in A(N) \}
\]

We then set

\[
\forall i \leq N, \quad t_{i,N+1} = t_{i,N}, \quad f_{N+1}|_{A(N)} = f_N, \quad \text{and } f_{N+1}(z) = N + 1,
\]
and, if $t_{N+1,N+1} = \tau_{N+1}^*$ (resp. $t_{N+1,N+1} < \tau_{N+1}^*$),
\[
\mathcal{B}(N+1) = \{z\} \cup \mathcal{B}(N) \quad \text{(resp. } \mathcal{B}(N)) ,
\]
\[
\mathcal{R}(N+1) = \mathcal{R}(N) \quad \text{(resp. } \{z\} \cup \mathcal{R}(N)) ,
\]

- If $\{S_i^*(t) : 0 \leq t \leq \tau_{N+1}^*\} \subseteq \mathcal{A}(N) = \mathcal{B}(N) \cup \mathcal{R}(N)$ then $S_{N+1}^*$ settles necessarily in a red site $z$ since $\mathcal{B}(N) \subseteq \mathcal{A}^*(N)$. This red site is occupied by an explorer that was driven by a flashing explorer $i = f_N(z)$ when it settled, and we have $t_i,N < \tau_i^*$. After reaching $S_{N+1}(\tau_{N+1}^*) = S_{N+1}(\tau_{N+1}^*)$ in our definition of $S_{N+1}$, we set
\[
S_{N+1}(1+\tau_{N+1}^*) = S_i^*(1+t_i,N), \quad S_{N+1}(2+\tau_{N+1}^*) = S_i^*(2+t_i,N), \ldots \tag{3.20}
\]
we set $t_{N+1,N+1} = \tau_{N+1}^*$, we turn blue the site $z$, we set $f_{N+1}(z) = N + 1$ and we proceed as previously: If $\{S_i^*(t) : t_i,N < t \leq \tau_i^*\} \not\subseteq \mathcal{A}(N)$ then $S_{N+1}$ settles at the first exit from $\mathcal{A}(N)$ in some $z' = S_i^*\{t_i,N+1\} \in \mathcal{A}(N)^c$ with $t_i,N+1 \leq \tau_i^*$ and we set
\[
\forall j \leq N, \quad j \neq i, \quad t_i,N+1 = t_j,N, \quad f_{N+1}|_{\mathcal{A}(N)\{z\}} = f_N|_{\mathcal{A}(N)\{z\}}, \quad \text{and} \quad f_{N+1}(z') = i,
\]
and if $t_{i,N+1} = \tau_i^*$ (resp. $t_{i,N+1} < \tau_i^*$)
\[
\mathcal{B}(N+1) = \{z'\} \cup \mathcal{B}(N) \cup \{z\} \quad \text{(resp. } \mathcal{B}(N) \cup \{z\},
\]
\[
\mathcal{R}(N+1) = \mathcal{R}(N) \setminus \{z\} \quad \text{(resp. } \{z'\} \cup \mathcal{R}(N) \setminus \{z\}).
\]
Otherwise, $\{S_i^*(t) : t_i,N < t \leq \tau_i^*\} \subseteq \mathcal{A}(N)$ and $S_i^*$ settles on a red site $z'$. With $i' = f_N(z')$ we have $t_{i',N} < \tau_i^*$ and after $S_{N+1}(\tau_i^* - t_{i',N} + \tau_{N+1}^*) = S_i^*(\tau_i^*)$ we can set
\[
S_{N+1}(1+\tau_i^* - t_{i',N} + \tau_{N+1}^*) = S_i^*(1+t_i,N), \ldots \tag{3.21}
\]
we set $t_{i,N+1} = \tau_i^*$, we turn blue the site $z'$, we set $f_{N+1}(z') = i$ and so on.

Since the number of red sites is finite this procedure necessarily reaches an end and one immediately checks that we define in this way $\mathcal{A}_N = \mathcal{B}_{N+1} \cup \mathcal{R}_{N+1}$ together with the times $\{t_{i,N+1}, \quad i \leq N + 1\}$, and a function $f_{N+1} : \mathcal{A}(N+1) \rightarrow \{1, \ldots, N\}$ with all the required properties ($f_{N+1}$ is one to one and $t_{f_{N+1}(z),N+1} < \tau_{f_{N+1}(z)}^*$ for all $z$ in $\mathcal{R}_{N+1}$).

Note first that we have the orbits inclusion by construction. Now, the law of $(\mathcal{A}(N) : N \geq 1)$ is that of the internal DLA process. Indeed, the part of the flashing trajectories $\{(S_i^*(t) : t_i,N < t \leq \tau_i^*, i \leq N\}$, that can be used together with $(S_{N+1}^*(t) : 0 \leq t \leq \tau_{N+1}^*)$ to build $\mathcal{A}(N+1)$ have increments that are independent from $\mathcal{A}(N)$.

Finally, for any $k \geq 1$, $\psi_N : \mathcal{A}(N) \rightarrow \mathcal{A}^*(N)$ does associate with any site outside $\cup_{j<k} \mathcal{S}_j$ a site outside $\cup_{j<k} \mathcal{S}_j$. Indeed, with any blue site, $\psi_N = g^* o f_N$ associates that site itself, while with each red site $\psi_N$ associates the end point of a flashing trajectory that visits that site. And a flashing trajectory that exits $\cup_{j<k} \mathcal{S}_j$ necessarily settles outside $\cup_{j<k} \mathcal{S}_j$.

### 4 Estimates on the Harmonic measure

We gather in this section two results which deal with the hitting probability of sets. The first one relies on a discrete mean value theorem for the Green’s function. This latter theorem relies on Green’s function estimates in [7], and Proposition [A1] given in the Appendix. The second result is Proposition [3.1] which we prove in Section 4.2. The set we wish to hit is not a sphere, and the proof is inspired by Lemma 5 of [8], which only gives an upper bound.
4.1 A discrete mean value theorem

Our main result in this section is the following.

**Theorem 4.1** Let \( \{\Delta_n, n \in \mathbb{N}\} \) be a positive sequence with \( \Delta_n \leq Kn^{1/3} \) for some constant \( K \), and set \( r_n = n - \Delta_n \). There is a constant \( K_a \), such that for any \( \Lambda \subset \partial B_n \)

\[
|E[M(|B_{r_n}|1_0, n, \Lambda)] - E[M(B_{r_n}, n, \Lambda)]| \leq K_a|\Lambda|.
\] (4.1)

Written explicitly, (4.1) reads

\[
||B_{r_n}| \times P_0(S(H_n) = z^*) - \sum_{y \in B_{r_n}} P_y(S(H_n) = z^*)| \leq K_a.
\] (4.2)

We now recall a classical decomposition (Lemma 6.3.6 of [9]). For a finite subset \( \Lambda \), and set \( z^* \in \partial \Lambda \)

\[
P_y(S(H(\partial \Lambda) = z^*) = \frac{1}{2d} \sum_{z \in \Lambda, z \sim z^*} G_\Lambda(y, z).
\] (4.3)

By (4.2) and (4.3) with \( \Lambda = B_n \), we have reduced Theorem 4.1 to proving a discrete mean value theorem which we formulate next. We keep the same notation as Theorem 4.1.

**Proposition 4.2** For \( z \in B_n \), and \( n - ||z|| \leq 1 \),

\[
||B_{r_n}| \times G_n(0, z) - \sum_{y \in B_{r_n}} G_n(y, z)| \leq K_a.
\] (4.4)

**Remark 4.3** Note that a related (but distinct) property was also at the heart of [8]. Namely, for \( \epsilon > 0 \), and \( n \) large enough, if \( z \in B_n \), and \( n - ||z|| \geq \epsilon n \),

\[
|B_{r_n}| \times G_n(0, z) \geq \sum_{y \in B_{r_n}} G_n(y, z).
\] (4.5)

**Proof.** We use an improved version of Lemma 2 of [7]. Using \( G_n(0, z) = G(0, z) - \mathbb{E}_z[G(0, S(H_n))] \) (Proposition 1.5.8 of [6]), and (2.7), one obtains by a Taylor expansion that for a constant \( K_1 \) (independent on \( n \))

\[
|\omega d G_n(0, z) - 2z\frac{\alpha(z)}{n^{d-1}}| \leq \frac{K_1}{n^d}, \quad \text{where} \quad \alpha(z) = \mathbb{E}_z[||S(H_n)|| - ||z||].
\] (4.6)

Now, \( r_n^d = n^d - d\Delta_n n^{d-1} + O(\Delta_n^2 n^{d-2}) \), so that using (4.6), and the hypothesis \( \Delta_n = O(n^{1/3}) \), and \( 0 \leq n - ||z|| \leq 1 \)

\[
|\mathbb{E}_{r_n} G_n(0, z)| = (r_n^d + O(r_n^{d-1})) \left( 2z \frac{\alpha(z)}{n^{d-1}} + O\left(\frac{1}{n^d}\right) \right)
\]

\[
= (n^d - d\Delta_n n^{d-1} + O(\Delta_n^2 n^{d-2}) + O(n^{d-1})) \left( 2z \frac{\alpha(z)}{n^{d-1}} + O\left(\frac{1}{n^d}\right) \right)
\]

\[
= 2\alpha(z)(n - d\Delta_n) + O(1)
\] (4.7)
A martingale argument (Lemma 3 of [7]) yields for a constant $K_l$
\[
\left| \sum_{y \in B_n} G_n(y, z) - 2\alpha(z)n \right| \leq K_l. \tag{4.8}
\]

Proposition A.1 of the Appendix reads here as follows. There is $K_b$ such that for $z \in B_n$ with $n - \|z\| \leq 1$
\[
\left| \sum_{y \in B_n(G_n(z, \Delta_n))} G_n(y, z) - 2\alpha(z) \right| \leq K_b, \tag{4.9}
\]

Now, combining (4.8) and (4.9) we obtain (when $0 \leq n - \|z\| \leq 1$)
\[
\left| \sum_{y \in B_n} G_n(y, z) - 2n \left( \alpha(z) - \alpha_0(z) \right) \right| \leq K_l + K_b. \tag{4.10}
\]

Now, we combine (4.6) and (4.10) we obtain for a constant $K_2$,
\[
\left| \left( |B_n|G_n(0, z) - \sum_{y \in B_n} G_n(y, z) \right) + 2 \left( \alpha_0(z) - \alpha(z) \right) \right| \leq K_2. \tag{4.11}
\]

We now bound $|\alpha_0(z) - \alpha(z)|$ by the following expression
\[
\mathbb{P}_z \left( H(B_{r_n}) < H_n \right) \times \left( \alpha_0(z) + \mathbb{E}_z \left[ \| S(H_n) \| - \| z \| \right] H_n > H(B_{r_n}) \right). \tag{4.12}
\]

Now, it is a classical estimate (see (A.6)) that there is $K_0$ such that for any $z \in A(n - 1, n)$,
\[
\mathbb{P}_z \left( H(B_{r_n}) < H_n \right) \leq \frac{K_0}{\Delta_n}. \tag{4.13}
\]

Thus,
\[
\Delta_n |\alpha_0(z) - \alpha(z)| \leq 2\Delta_n \mathbb{P}_z (H(B_{r_n}) < H_n) \leq 2K_0. \tag{4.14}
\]

The desired result follows at once.

\section*{4.2 Proof of Proposition 3.1}

For $j \geq 0$, consider $z_j$ in $\Sigma_j$. We show that for all $z^*$ in $\mathcal{C}(z_j)$ and for suitable positive constants $\alpha_1, \alpha_2$,
\[
\frac{\alpha_1}{h^d_j} \leq \mathbb{P}_{z_j}(S(\sigma_j) = z^*) \leq \frac{\alpha_2}{h^d_j}. \tag{4.15}
\]

First, $z^* = z_j$ is a flashing position when $X_j = 1$. This happens with probability $1/h^d_j$, and gives the result. Now, consider $z^* \in \mathcal{C}(z_j) \setminus \{z_j\}$. We recall that the unbiased Bernoulli $Y_j$ decides whether we flash on $\partial B(z_j, R_j)$ or on $\partial A(r_j - R_j, r_j + R_j)$, where $R_j$ has density $g_j$ given in (3.7).
Step 1: Proof of the upper bound in (4.15). The following obvious facts follow from Lemma 2.8:

(i) \( z^* \in \partial B(z_j, \| z^* - z_j \|) \), and \( z^* \in \partial A(r_j - \| z^* \| - r_j, r_j + \| z^* \| - r_j) \).

(ii) \( z^* \notin \partial B(z_j, \| z^* - z_j \| - 1) \), and \( z^* \notin \partial A(r_j - \| z^* \| - r_j + 1, r_j + \| z^* \| - r_j - 1) \).

This means that if \( Y_j = 1 \), then \( R_j \in \{ \| z^* - z_j \| - 1, \| z^* - z_j \| \} \), whereas if \( Y_j = 0 \), then \( R_j \in \{ \| z^* \| - r_j | - 1, \| z^* \| - r_j | \} \). Thus, there is a constant \( C \) such that

\[
(i) \quad P(Y_j = 1, R_j \in [\| z^* - z_j \| - 1, \| z^* - z_j \|]) \leq C \frac{\| z^* - z_j \|^{d-1}}{h_j^d},
\]

and

\[
(ii) \quad P(Y_j = 0, R_j \in [\| z^* \| - r_j | - 1, \| z^* \| - r_j |]) \leq C \frac{\| z^* \| - r_j |^{d-1}}{h_j^d}.
\]

In the case \( z^* \in \partial B(z_j, \partial R_j) \), the upper bound (4.18) then follows from (i) of (4.16), and (2.11) of Section 2.2. We consider now \( Y_j = 0 \). To simplify the notation we set for \( h > 0 \),

\[ D_h = A(r_i - h, r_i + h), \quad \text{and} \quad \tilde{D}_h = A(r_i - \frac{h}{2}, r_i + \frac{h}{2}), \]

and define two stopping times

\[ \tau = \inf \{ n \geq 0 : S(n) \in \partial D_h \cup \{z_j\} \}, \quad \text{and} \quad \tau^+ = \inf \{ n \geq 1 : S(n) \in \partial D^c_h \cup \{z_j\} \}. \]

It is enough to prove that for some constant \( c \), and for \( h \) such that \( z^* \in \partial D_h \), (and \( h \in [\| z^* \| - r_j | - 1, \| z^* \| - r_j |] \))

\[
\mathbb{P}_{z_j}(S(H(D^c_h)) = z^*) \leq \frac{c}{h^{d-1}}.
\]

We use a last exit decomposition, and the strong Markov property to get

\[
\mathbb{P}_{z_j}(S(H(D^c_h)) = z^*) \leq \mathbb{E}_{z_j}(z_j)\mathbb{P}_{z^*}(H(\tilde{D}_h) < \tau^+) \max_{x \in \partial D^c_h} \mathbb{P}_x(S(\tau) = z_j)
\]

\[
= \mathbb{P}_{z^*}(H(\tilde{D}_h) < \tau^+) \max_{x \in \partial D^c_h} G_{D_h}(x, z_j)
\]

\[
\leq \mathbb{P}_{z^*}(H(\tilde{D}_h) < \tau^+) \max_{x \in \partial D^c_h} G(x, z_j).
\]

It follows, from a Gambler’s ruin estimate, that for a constant \( K_0 \)

\[
\mathbb{P}_{z^*}(H(\tilde{D}_h) < \tau^+) \leq \frac{K_0}{h}.
\]

Now, from the Greens’ function asymptotics (2.7)

\[
\sup_{x \in \partial D^c_h} G(x, z_j) \leq \sup_{x \in \partial D^c_h} \left( \frac{C_d}{\| x - z_j \|^{d-2}} + \frac{K_0}{\| x - z_j \|^{d}} \right).
\]

Note that the distance between \( z_j \) and \( \tilde{D}_h \) is of order \( h \). We use (4.20) and (4.21) in (4.19) to obtain (4.18).

Step 2: Proof of the lower bound in (4.15). Note the following two facts.
Fluctuations for internal DLA

Thus, for some constant $K$.

Let $z^*$ be the closest site of $\partial \mathbb{B}(z_j, h)$, such a way that, applying Harnack’s inequality 20 times (see Theorem 6.3.9 in [9]) to the harmonic functions $G_{D_h}(x, \cdot)$, we can estimate from below the last factor in (4.24). There is a constant $K_2$ such that

$$\min_{x \in \Gamma \cap \partial \mathbb{B}_h} G_{D_h}(x, z_j) \geq c_H^{20} \min_{x \in \Gamma \cap \partial \mathbb{B}_h} G_{D_h}(x, y^*)$$

$$\geq c_H^{20} \min_{x \in \Gamma \cap \partial \mathbb{B}_h} G(x, y^*) - \mathbb{E}_x [G(S(H(D^c_h)), y^*)]$$

$$\geq c_H^{20} \frac{C_d}{h^{d-2}} \left( \frac{1}{(\sqrt{2}/2)^{d-2}} - 1 \right) \geq \frac{K_2}{h^{d-2}}.$$
As a consequence of (4.25), we just need to prove that the first factor in (4.24) is of order $1/h_j$ at least. We realize the event \( \{ H(\Gamma) < \tau^+ \} \) in two moves: we first hit the sphere \( B(x^*, R^*/2) \), and then we exit from the cap \( \partial B(x^*, R^*) \) which lies in \( \bar{D}_h \).

\[
\mathbb{P}_{x^*} (H(\Gamma) < \tau^+) \geq \mathbb{P}_{x^*} (H(B(x^*, R^*/2)) < H(B^c(x^*, R^*))) \geq \inf_{y \in \partial B(x^*, R^*/2)} \mathbb{P}_y (H(B^c(x^*, R^*) \cap \bar{D}_h) = H(B^c(x^*, R^*)))
\]

We invoke again Harnack’s inequality to have for \( y \in \partial B(x^*, R^*/2) \)

\[
\mathbb{P}_y (H(B^c(x^*, R^*) \cap \bar{D}_h) = H(B^c(x^*, R^*))) \geq c_H \mathbb{P}_0 (H(E \cap B^c(x^*, R^*)) = H(B^c(x^*, R^*))).
\]

We invoke now (2.11) to obtain for some constant \( K_3 \)

\[
\mathbb{P}_0 (H(B^c(x^*, R^*) \cap \bar{D}_h) = H(B^c(x^*, R^*))) \geq c_1 \frac{|\partial B(x^*, R^*) \cap \bar{D}_h|}{|\partial B(x^*, R^*)|} \geq K_3.
\]

We gather now (4.26), (4.27) and (4.28) to obtain the desired lower bound.

5 The flashing process fluctuations

In this section we prove Theorems 1.4 and 1.5. To do so we use the construction in terms of exploration waves of Section 3.2.

5.1 Tiles

We recall that we have defined a cell of \( S_j \) in (3.10), as the intersection of a cone with \( S_j \). We need also a smaller structure. We define, for any \( z_j \) in \( \Sigma_j \),

\[
\tilde{C}(z_j) = S_j \cap \left\{ x \in \mathbb{R}^d : \exists \lambda \geq 0, \exists y \in B(z_j, h_j/5), x = \lambda y \right\}.
\]

As in Lemma 12 in [7], concerning locally finite coverings, we claim that, for \( h_o \) large enough, there exist a positive constants \( c_1 \), and, for each \( j \geq 0 \), a subset \( \tilde{\Sigma}_j \) of \( \Sigma_j \) such that

\[
|\tilde{\Sigma}_j| \leq c_1 \frac{|\Sigma_j|}{h_j^{d-1}} \quad \text{and} \quad S_j = \bigcup_{z_j \in \tilde{\Sigma}_j} \tilde{C}(z_j).
\]

For any \( z_j \in \Sigma_j \), we call tile centered at \( z_j \), the intersections of \( \tilde{C}(z_j) \) with \( \Sigma_j \). We denote by \( T(z_j) \) a tile centered at \( z_j \), and by \( T_j \) the set of tiles associated with the shell \( S_j \):

\[
T_j = \left\{ T(z_j) : \ z_j \in \tilde{\Sigma}_j \right\}.
\]

Let us explain the reason for \( h_j/5 \) in the definition of a tile. It implies a fundamental feature of the flashing process. For any \( z \in S_j \), there is \( \bar{z}_j \in \tilde{\Sigma}_j \) such that

\[
z \in \bigcap \left\{ C(y) : \ y \in T(\bar{z}_j) \right\}.
\]
Indeed, let $z_j \in \Sigma_j$ be the site realizing the minimum of $\{\|z - y\| : y \in \Sigma_j\}$. There is $\lambda > 0$ and $u \in B(z_j, 1)$, such that $z = \lambda u$. Now, there is $\tilde{z}_j \in \Sigma_j$ such that $\|\tilde{z}_j - z_j\| < h_j/5$, and for any $y \in \mathcal{T}(\tilde{z}_j)$, we have $\|y - z_j\| < 2h_j/5$. Thus

$$u \in \bigcap \left\{ B(y, \frac{h_j}{2}) : y \in \mathcal{T}(\tilde{z}_j) \right\} \implies z \in \bigcap \{ C(y) : y \in \mathcal{T}(\tilde{z}_j) \}.$$ 

### 5.2 The inner ball

For $n \geq 0$, we take $N = |B_n|$, we recall that $\mathcal{A}^*(N) = \bigcup_{k \geq 1} \mathcal{A}^*_k(N)$, and write $\mathcal{A}^*$ instead of $\mathcal{A}^*(N)$. We consider

$$T^* = \min \{ k \geq 1 : \cup_{j<k} S_j \not\subset \mathcal{A}^*_k \}.$$ 

(5.5)

We have, for $l$ with $r_l < n$,

$$P(\mathcal{B}(0, r_l) \not\subset \mathcal{A}^*) \leq \sum_{k \leq l} P(T^* = k + 1)$$ 

(5.6)

and we estimate from above the probability $P(T^* = k + 1)$ assuming $r_k < n$. For $k \geq 1$ and $\Lambda \subset \Sigma_k$, we call $W_k(\Lambda)$ the number of unsettled explorers that stand in $\Lambda$ after the $k$-th wave, that is

$$W_k(\Lambda) = \sum_{i=1}^N 1_{\Lambda}(\xi_k(i)).$$ 

(5.7)

We now look at the crossings of tiles of $\mathcal{T}_k$. On the one hand, we will use that if $W_k(\mathcal{T})$ is large, then it is unlikely that a hole appears in the cell containing $\mathcal{T}$. On the other hand, if $r_k$ is small it is unlikely that $W_k(\mathcal{T})$ is small. We make precise what we intend by small and large. For this purpose, we will show in (5.16) that for some constant $\kappa_1 > 0$, and any tile $\mathcal{T} \in \mathcal{T}_k$

$$E[W(\mathcal{T})] \geq \kappa_1 (n - r_k) h_k^{d-1}, \quad \text{and we define} \quad h = n^{\frac{d}{d+1}} \geq \sup_{k : r_k \leq n} h_k.$$ 

(5.8)

For any positive constant $A$, we write

$$P(T^* = k + 1) = P\left( T^* = k + 1, \exists \mathcal{T} \in \mathcal{T}_k, W_k(\mathcal{T}) < Ah_k^d \log n \right) + P\left( T^* = k + 1, \forall \mathcal{T} \in \mathcal{T}_k, W_k(\mathcal{T}) \geq Ah_k^d \log n \right),$$

and we estimate separately each term in the right hand side of (5.9).

**Estimating the first term.** We show here that (for $\kappa_1$ and $h$ appearing in (5.8) for $k$ such that

$$r_k \leq n - \frac{2A}{\kappa_1} h \log n,$$ 

(5.9)

there is a constant $\kappa_2 > 0$, and $n$ large enough, such that

$$P\left( T^* = k + 1, \exists \mathcal{T} \in \mathcal{T}_k, W_k(\mathcal{T}) < Ah_k^d \log n \right) \leq |S_k| \exp \left( -\kappa_2 A^2 \log^2 n \right).$$ 

(5.10)
On \( \{T^* = k + 1\} \), we have \( A_k^* = \mathbb{B}(0, r_k - h_k) \). On \( \{T^* = k + 1\} \), and for any \( T \subset T_k \), we consider a variable \( L_k(T) = M(\mathbb{B}(0, r_k - h_k), r_k, T) \) independent of \( W_k(T) \), and define \( M_k(T) = W_k(T) + L_k(T) \). We have the equality in law, on \( \{T^* = k + 1\} \),

\[
M_k(T) \overset{\text{law}}{=} M(N1_{\{0\}}, r_k, T), \quad \text{and} \quad W_k(T) = M_k(T) - L_k(T). \quad (5.11)
\]

As a consequence

\[
P(T^* = k + 1, \exists T \in T_k, W_k(T) < Ah_k^d \log n) \leq |\Sigma_k| \max_{T \in T_k} P(M_k(T) - L_k(T) < Ah_k^d \log n).
\]

We first estimate the number of explorers stopped on a tile \( T \) of \( T_k \).

Now, \( M_k(T) \) and \( L_k(T) \) are dependent random variables, but both are sums of independent Bernoulli variables, for which Lemma 2.3 is designed. We introduce two notations. For a variable \( X \), let \( \bar{X} = X - E[X] \), and let

\[
2\bar{x}_k = E[M_k(T) - L_k(T)] - Ah_k^d \log(n).
\]

Since we need \( \bar{x}_k \) of (5.13) to be positive, we will choose \( A \) and \( k \) such that

\[
E[M_k(T) - L_k(T)] \geq 2Ah_k^d \log(n), \quad \text{which implies} \quad \bar{x}_k \geq \frac{1}{2}Ah_k^d \log(n).
\]

Then,

\[
P(M_k(T) - L_k(T) < Ah_k^d \log n) = P(M_k(T) - \bar{L}_k(T) < -2\bar{x}_k).
\]

In order to estimate \( E[M_k(T) - L_k(T)] \), we invoke Theorem 4.1 with \( n = r_k \), and \( \Delta_n = h_k \) (the hypothesis \( h_k = O(r_k^{1/3}) \) holds here). We have for some positive constants \( \kappa', \kappa_1 \), and for \( n \) large enough

\[
E[M_k(T) - L_k(T)] = E[M(|\mathbb{B}_n| - |\mathbb{B}_{r_k-h_k}|)1_0, r_k, T] + E[M(|\mathbb{B}_{r_k-h_k}|1_0, r_k, T) - E[M(\mathbb{B}_{r_k-h_k}, r_k, T)] \\
\geq (|\mathbb{B}_n| - |\mathbb{B}_{r_k-h_k}|) P_0(S(H_k) \in T) - O(h_k^{d-1}) \\
\geq \kappa'(n^d - (r_k - h_k)h_k^{d-1})h_k^{d-1} - O(h_k^{d-1}) \\
\geq \kappa_1(n - r_k)h_k^{d-1}. \quad \text{(recall that \( n - r_k > h \)).}
\]

Note that the ultimate inequality in (5.16) is the estimate in (5.8). In view of (5.16), condition (5.14) is ensured if \( A \) and \( k \) satisfy (5.9).

Note that (5.16) implies that \( E[M_k(T)] \geq E[L_k(T)] \), so that (2.14) of Remark 2.4 requires only an upper bound on \( E[M_k(T)] \). Thus, we only treat the latter quantity.

We distinguish two cases: (i) when \( r_k \) is close to \( n \), (ii) when \( r_k \) is small compared to \( n \).

**Step 1:** We assume \( n - h \geq r_k \geq n/2 \).

We set here \( 2\bar{x}_k = Ah_k^d \log n \). (5.14) and (5.15) imply that

\[
P(M_k(T) - L_k(T) < Ah_k^d \log n) \leq P(M_k(T) < -x_k) + P(\bar{L}_k(T) > x_k).
\]

(5.17)
To be in the CLT regime of Lemma 2.3 when dealing with the right hand side of (5.17), we need
\[ 0 < x_k < E[M_k(T)]. \]  
(5.18)

Let us now estimate \( E[M_k(T)] \). Note that by using (2.11) and \( r_k \geq n/2 \), we have for positive constants \( K_1, K'_1 \)
\[ K'_1 n^d \left( \frac{h_k}{r_k} \right)^{d-1} \geq E[M_k(T)] = |\mathbb{B}_n| \mathbb{P}_0(S(H_k) \in \mathcal{T}) \geq K_1 n^d \left( \frac{h_k}{r_k} \right)^{d-1} \geq K_2 n^d h_k^{d-1}. \]  
(5.19)

Thus, \( E[M_k(T)] > x_k \) and we are in the Gaussian regime for (2.12). Similarly, as in (5.19), we have a lower bound
\[ E[M_k(T)] \leq K'_2 n h_k^{d-1}. \]  
(5.20)

Thus, there is \( \kappa_2 > 0 \) such that for a large enough \( n \),
\[ P(\bar{M}_k(T) \leq -x_k) \leq e^{-\frac{x_k^2}{4K_2^2 n h_k^{d-1}}} \leq \exp \left(-\frac{A^2 h_k^2 \log^2 n}{16 K_2^2 n h_k^{d-1}}\right) = \exp \left(-\kappa_2 A^2 \log^2 n\right). \]  
(5.21)

As already noted, (2.14) yields
\[ P(\bar{L}_k(T) \geq x_k) \leq \exp(-\kappa_2 A^2 \log^2 n). \]  
(5.22)

**Step 2:** We assume \( r_k < n/2 \).

We have, using (5.16), for \( n \) large enough
\[ 2\bar{x}_k \geq \kappa'(n^d - r_k^d) \left( \frac{h_k}{r_k} \right)^{d-1} - Ah_k^d \log n \geq \frac{\kappa'}{2} n^d \left( \frac{h_k}{r_k} \right)^{d-1} - Ah_k^d \log n \geq \frac{\kappa'}{4} n^d \left( \frac{h_k}{r_k} \right)^{d-1}. \]  
(5.23)

We define here
\[ x_k = \frac{\kappa'}{16} n^d \left( \frac{h_k}{r_k} \right)^{d-1}, \quad \text{(and note that)} \quad x_k \geq \frac{\kappa'}{16} n h_k^{d-1}. \]  
(5.24)

As previously, we have (5.17). From (5.19), we have for some positive \( K'_2 \)
\[ E[M_k(T)] \leq K'_2 n^d \left( \frac{h_k}{r_k} \right)^{d-1}. \]  
(5.25)

Now, using Lemma 2.3 for \( n \) large enough
\[ P(M_k(T) - L_k(T) < Ah_k^d \log n) \leq 2 \exp \left(-\frac{x_k}{4} \min \left( 1, \frac{\kappa'}{8 K'_2} \right) \right) \leq \exp \left(-\kappa_2 A^2 \log^2 n\right). \]  
(5.26)

Collecting (5.21), (5.22) and (5.26) together with (5.12) and (5.2), we conclude that (5.10) holds.
Estimating the last term. The last term in the right hand side of (5.9) is bounded using a simple coupon-collector argument. Indeed, the event \( \{ T^* = k + 1 \} \) means that there is one uncovered site in \( S_k \). By (5.4), there is \( z_k \in \tilde{\Sigma}_k \), such that this site is a possible settling position of all explorers stopped in \( \mathcal{T}(z_k) \). Now, if \( \{ W_k(\mathcal{T}(z_k)) \geq Ah_k^d \log n \} \), Proposition 3.1 tells us that the probability of not covering this site is less than \( (1 - \alpha^2/h_k^d) \) to the power \( Ah_k^d \log(n) \). In other words,

\[
P \left( T^* = k + 1, \forall T \in \mathcal{T}_k, W_k(T) \geq Ah_k^d \log n \right) \leq |S_k| \left( 1 - \frac{\alpha^2}{h_k^d} \right)^{Ah_k^d \log n}
\]

(5.27)

Conclusion. First, choose \( A \) large enough so that

\[
|B_n| \exp (-\alpha^2 A \log n) \leq \frac{1}{n^2}.
\]

(5.28)

Recall the decomposition (5.6 and (5.9), and assume that \( r_l \) satisfies (5.9). Then, (5.10) and (5.27) yield that for \( n \) large enough

\[
P(B(0, r_l + h_l \not\subset A^* (|B_n|))) \leq |B_n| \left( \exp (-\kappa^2 A^2 \log^2 n) + \exp (-\alpha^2 A \log n) \right) \leq \frac{2}{n^2}.
\]

(5.29)

The right-hand side in (5.29) is summable, and Borel-Cantelli lemma yields the inner control of Theorem 1.4.

5.3 The outer ball

This section follows closely [7]. The features of the flashing process allow for some simplification. We keep the notation of the previous section. There, we proved that for some positive constant \( \delta \)

\[
P \left( \{ B(0, n - \delta h \log n) \subset A^* \} \right) = 1 - e^\delta(n), \quad \text{with} \quad \sum_{n \geq 1} e^\delta(n) < +\infty.
\]

As consequence, the following conditional law can be seen as a slight modification of \( P \).

\[
P^\delta(\cdot) = P (\cdot | \{ B(0, n - \delta h \log n) \subset A^* \}).
\]

(5.30)

We begin by proving that, under \( P^\delta \), the probability to find some \( k \) with \( r_k < 2n \) and some tile \( \mathcal{T} \) in \( \mathcal{T}_k \) with \( W_k(\mathcal{T}) \) larger than or equal to \( 2Ah^d \log n \) for a large enough \( A \) decreases faster than any power of \( n \). First, note that, under \( P^\delta \), we have

\[
W_k(\mathcal{T}) \leq M_k(\mathcal{T}) - L_k^\delta(\mathcal{T}), \quad \text{with} \quad L_k^\delta = M (B(0, n - \delta h \log n), r_k, \mathcal{T}).
\]

(5.31)

Now,

\[
P^\delta \left( W_k(\mathcal{T}) \geq 2Ah^d \log n \right) \leq P^\delta \left( M_k(\mathcal{T}) - L_k^\delta(\mathcal{T}) \geq 2Ah^d \log n \right).
\]

(5.32)
By Theorem 4.1 for some positive constants $K'$, $K$ and for $n$ large enough

$$
E \left[ M_k(T) - L^\delta_k(T) \right] \leq K' \left( n^d - (n - \delta h \log n)^d \right) \frac{h^{d-1}_k}{r^{d-1}_k} + O(h^{d-1}_k)
$$

$$
\leq K'n^d d \frac{\delta h \log n h^{d-1}_k}{r^{d-1}_k} + O(h^{d-1}_k)
$$

$$
\leq K'd\delta hh^{d-1}_k \log n + O(h^{d-1}_k) \leq Kh^d \log n.
$$

(5.33)

Choosing $A \geq K$, we get for $n$ large enough so that $P(\mathbb{B}(0, n - \delta h \log n) \subset A^*) \geq 1/2$

$$
P^\delta \left( W_k(T) \geq 2Ah^d \log n \right) \leq 2P \left( M_k(T) - E[M_k(T)] \geq \frac{A}{2} h^d \log n \right)
$$

$$
+ 2P \left( L^\delta_k(T) - E[L^\delta_k(T)] \leq -\frac{A}{2} h^d \log n \right)
$$

(5.34)

As in the previous section $E[M_k(T)]$ is of order $n^d h^{d-1}_k / r^{d-1}_k$, i.e., of order $nh^{d-1}$. In addition $E[L^\delta_k(T)]$ is smaller than $E[M_k(T)]$. We conclude once again by invoking Lemma 2.3.

Now, let $F_k$ denote the event that no tile $T$ in $\Sigma_k$ contains more than $2Ah^d \log n$ unsettled explorers after the $k$-th exploration wave. We denote by $F_k = \sigma(\xi_0, \ldots, \xi_k)$, and note that $F_k$ and $\{\mathbb{B}(0, n - \delta h^d \log(n)) \subset A^*\}$ are $\mathcal{F}_k$-measurable.

For any tile $T \in \mathcal{T}_k$, let $z_k \in \Sigma_k$ be such that $T = T(z_k)$ and denote by $\tilde{C} = \tilde{C}(z_k)$. We are entitled, by Proposition 3.1, to use a coupon-collector estimate on the number of settled explorers during the $k + 1$-th exploration wave. On $F_k \cap \{\mathbb{B}(0, n - \delta h^d \log(n)) \subset A^*\}$, and for some positive constant $K_1$

$$
E \left[ A^*_k \cap \tilde{C} \mid F_k \right] \geq |\tilde{C}| \left( 1 - \left( 1 - \frac{\alpha_1}{h^d_k} \right)^{W_k(T)} \right)
$$

$$
\geq |\tilde{C}| \left( 1 - \exp \left\{ -\alpha_1 \frac{W_k(T)}{h^d_k} \right\} \right)
$$

$$
\geq \frac{|\tilde{C}|}{h^d_k} W_k(T) \frac{h^d_k}{W_k(T)} \left( 1 - \exp \left\{ -\alpha_1 \frac{W_k(T)}{h^d_k} \right\} \right)
$$

$$
\geq K_1 W_k(T) \inf_{x \leq 2A \log n} \frac{1 - e^{-\alpha_1 x}}{x}.
$$

(5.35)

We now write for some positive constant $K_2$

$$
\inf_{x \leq 2A \log n} \frac{1 - e^{-\alpha_1 x}}{x} \geq \frac{1}{2A \log n} \inf_{x \leq 2A \log n} \frac{1 - e^{-\alpha_1 x / 2A \log n}}{x / 2A \log n}
$$

$$
\geq \frac{1}{2A \log n} \inf_{x \leq 1} \frac{1 - e^{-\alpha_1 x}}{x} \geq K_2 \frac{1}{\log n}.
$$

We conclude that on $F_k \cap \{\mathbb{B}(0, n - \delta h^d \log(n)) \subset A^*\}$,

$$
E \left[ A^*_{k+1} \cap \tilde{C} \mid F_k \right] \geq K_1 K_2 \frac{W_k(T)}{\log n}.
$$

(5.36)
Summing over all tiles we get, for a different constant $K$, (because of the finite, $k$-independent overlapping between tiles), we obtain on $F_k \cap \{ B(0, n - \delta h^d \log(n)) \subset \mathcal{A}_k \}$

$$E \left[ A_{k+1}^* \cap \mathcal{S}_k \left| \mathcal{F}_k \right. \right] \geq K \frac{W_k(S_k)}{\log n}.$$ 

Also, since $W_k(S_k) \leq |B(0, n)|$,

$$E \left[ 1_{F_k \cap \{ B(0, n - \delta h^d \log(n)) \subset \mathcal{A}_k \}} A_{k+1}^* \cap \mathcal{S}_k \right] \geq K \frac{E[1_{\{ B(0, n - \delta h^d \log(n)) \subset \mathcal{A}_k \}} W_k(S_k)]}{\log n} - n^d P(F_k^c).$$

Since $P(\{ B(0, n - \delta h^d \log(n)) \subset \mathcal{A}_k \}) \geq 1/2$,

$$E^\delta \left[ A_{k+1}^* \cap \mathcal{S}_k \right] \geq K \frac{E^\delta [W_k(S_k)]}{\log n} - 2n^d P(F_k^c). \quad (5.37)$$

In other words, noting that $A_{k+1}^* \cap \mathcal{S}_k = W_k(S_k) - W_{k+1}(S_{k+1})$

$$E^\delta [W_{k+1}(S_{k+1})] \leq \left( 1 - \frac{K}{\log n} \right) E^\delta [W_k(S_k)] + 2n^d P(F_k^c). \quad (5.38)$$

By iterating $(5.38)$, we obtain that for any $\epsilon$, $E^\delta[W_l + \epsilon \log^2 n (S_{l+\epsilon \log^2 n})]$, decreases faster than any power of $n$, when $l$ the lowest index for which $r_l \geq n$. Also, the probability (under $P$!) of seeing at least one explorer reaching the shell $S_{l+\epsilon \log^2 n}$ is summable. Using Borel-Cantelli lemma, this yields the proof of Theorem 1.4.

### 5.4 Optimality of the fluctuation exponent

Let time $k$ be such that $r_k = n - Ah$, for a large arbitrary constant $A$. We show that $P(T^* = k + 1)$ decays faster than any polynomial in $n$.

On the event $\{ T^* = k + 1 \}$, we have, after the $k$-th wave and for some constant $K$,

$$|B_{r_k}| = M(B_{r_k}, r_k, \mathcal{S}_k) = M(A_k^*, r_k, \mathcal{S}_k) \implies W_k(S_k) = |B_n| - |B_{r_k}| \leq AKn^{d-1} \times h. \quad (5.39)$$

This means that there exists $z_k \in \mathcal{S}_k$ such that, for some positive constant $K'$,

$$W_k(B(z_k, 3h) \cap \mathcal{S}_k) \leq K'n^{d-1} \times h \times \frac{h^{d-1}}{n^{d-1}} \leq K'h^d. \quad (5.40)$$

By construction, only the explorers stopped inside $B(z_k, 3h)$ can cover $\mathcal{C}(z_k)$ (for $n$ large enough). As a consequence, we can think of a coupon-collector problem, where an album of size $|\mathcal{C}(z_k)|$ has to be filled when we collect no more than $K'h^d$ coupons. Thus, the probability of $\{ T^* = k + 1 \}$ is bounded from above by the probability of filling such an album, which is less than $\exp(-c(A)h^{d/2})$, for some explicit constant $c(A)$ dependent on $A$. This result is based on the following simple coupon-collector lemma (together with Proposition 3.1), which we did not find in the vast literature on such problems.
Lemma 5.1 Consider an album of $L$ items for which are bought independent random coupons, each of them covering one (or possibly none) of the possible $L$ items. If $Y_i$ is the item associated with the $i$-th coupons, we assume that for positive constants $\alpha_1, \alpha_2$, such that for any $j = 1, \ldots, L$,
\[
\frac{\alpha_1}{L} \leq P(Y_i = j) \leq \frac{\alpha_2}{L}.
\] (5.41)

Let $\tau_L$ be the number of coupons needed to complete the album. Then, for any $A > 0$,
\[
P(\tau_L < A) \leq \exp \left(-\frac{\alpha_2^2 A^2 e^{-2\alpha_2 A}}{4} \sqrt{L}\right).
\] (5.42)

Proof. We denote by $\sigma_i$ the time needed to collect the $i$-th distinct item after having collected $i - 1$ distinct items. The sequence $\{\sigma_1, \sigma_2, \ldots, \sigma_L\}$ is not independent, but if $G_k = \sigma(\{Y_1, \ldots, Y_k\})$, and $\tau(k) = \sigma_1 + \cdots + \sigma_k$, then for $i = 1, \ldots, L$,
\[
\left(1 - \frac{\alpha_1 (L - i + 1)}{L}\right)^k \geq P(\sigma_i > k | G_{\tau(i-1)}) \geq \left(1 - \frac{\alpha_2 (L - i + 1)}{L}\right)^k.
\] (5.43)

Indeed, calling $E(i-1)$ the set of the first $i-1$ collected items,
\[
P(\sigma_i > k | G_{\tau(i-1)}) = P(\{Y_{\tau(i-1)+1}, \ldots, Y_{\tau(i-1)+k}\} \subset E(i-1) | G_{\tau(i-1)})
\] (5.44)
\[
= (P(Y \in E(i-1) | G_{\tau(i-1)}))^k
\] (5.44)
\[
= (1 - P(Y \notin E(i-1) | G_{\tau(i-1)}))^k.
\] (5.44)

Using (5.41) we deduce (5.43) from (5.44). Now, (5.43) gives that
\[
\frac{L}{\alpha_1 (L - i + 1)} \geq E[\sigma_i | G_{\tau(i-1)}] \geq \frac{L}{\alpha_2 (L - i + 1)};
\] (5.45)
as well as
\[
E[\sigma_i^2 | G_{\tau(i-1)}] \leq \frac{L^2}{\alpha_1^2 (L - i + 1)^2}.
\] (5.46)

Now, we look for $B$ such that
\[
\sum_{i=\sqrt{L}}^{B \sqrt{L}} E[\sigma_{L-i}] \geq 2AL.
\] (5.47)

Note that
\[
\sum_{i=\sqrt{L}}^{B \sqrt{L}} E[\sigma_{L-i}] \geq \frac{L}{\alpha_2} \sum_{i=\sqrt{L}}^{B \sqrt{L}} \frac{1}{i+1} \geq \frac{L}{\alpha_2} \log(B).
\]
Thus, condition (5.47) holds for $B \geq \exp(2\alpha_2 A)$. Finally, note that
\[
\max \left\{ E[\sigma_{L-i} | G_{\tau(L-i-1)}], \ i = \sqrt{L}, \ldots, B \sqrt{L} \right\} \leq \frac{\sqrt{L}}{\alpha_1},
\]
and set
\[
X_i = \frac{E[\sigma_{L-i} | G_{\tau(L-i-1)}] - \sigma_{L-i}}{\left(\sqrt{L}/\alpha_1\right)} \leq 1.
\]
For $x \leq 1$, note that $e^x \leq 1 + x + x^2$ to obtain for $0 \leq \lambda \leq 1$, by successive conditioning

$$P \left( \sum_{i=\sqrt{L}}^{B\sqrt{T}} \sigma_{L-i} \leq AL \right) \leq P \left( \sum_{i=\sqrt{L}}^{B\sqrt{T}} X_i \geq \alpha_1 A\sqrt{T} \right)$$

$$\leq e^{-\lambda \alpha_1 A\sqrt{T}} \prod_{i=\sqrt{L}}^{B\sqrt{T}} \left( 1 + \lambda^2 \sup_{i=\sqrt{L}} E[X_i^2 \mid \mathcal{G}_{\tau(L-i-1)}} \right) \tag{5.48}$$

$$\leq \exp \left( -\lambda \alpha_1 A\sqrt{T} + \lambda^2 \sum_{i=\sqrt{L}}^{B\sqrt{T}} \sup_{i=\sqrt{L}} E[X_i^2 \mid \mathcal{G}_{\tau(L-i-1)}} \right).$$

Finally, we have, using (5.46),

$$\sum_{i=\sqrt{L}}^{B\sqrt{T}} \sup_{i=\sqrt{L}} E[X_i^2 \mid \mathcal{G}_{\tau(L-i-1)}} \leq \sum_{i=\sqrt{L}}^{B\sqrt{T}} \alpha_1^2 \sup_{i=\sqrt{L}} E[\sigma_{L-i}^2 \mid \mathcal{G}_{\tau(L-i-1)}}] / L \leq 2B\sqrt{T}. \tag{5.49}$$

The results follows as we optimize on $\lambda \leq 1$ in the upper bound in (5.48).

## A Time spent in an annulus (By S.Blachère)

This appendix is devoted to an asymptotic expansion of the expected time spent in an annulus $A(r_n, n)$ for $r_n < n$, when the random walk is started at some point $z$ within the annulus, and before it exits the outer shell.

**Proposition A.1** Consider a sequence $\{\Delta_n, n \in \mathbb{N}\}$ with $\Delta_n \leq Kn^{1/3}$ for some constant $K$. Let $r_n = n - \Delta_n$, and $z \in A(r_n, n)$. There is a constant $K_b$, independent on $z$ and $n$, such that

$$\left| \sum_{y \in A(r_n, n)} G_n(z, y) - (2d \Delta_n \alpha_0(z) - 2d(n - \|z\|)^2) \right| \leq K_b ((n - \|z\|) \lor 1), \tag{A.1}$$

with

$$\alpha_0(z) = E_z \left[ \|S(H_n)\| - \|z\| \mid H(B^c(0, n)) < H(B(0, r_n)) \right].$$

**Remark A.2** The statement is true in dimension 2, when Green’s function is replaced by the potential kernel

$$a(x, y) = E_x \left[ \sum_{l=0}^{\infty} 1 \{S(l) = x\} - 1 \{S(l) = y\} \right]. \tag{A.2}$$

**Proof.** Our strategy is to decompose a path into successive strands lying entirely in the annulus. The first strand is special since the starting point is any $z \in A(r_n, n)$. The other strands, if any, start all on $\partial B(0, r_n)$. We estimate the time spent inside the annulus for
Secondly, we show that for \( z \in \mathcal{A}(r_n, n) \). We define the following stopping times \((D_i, U_i, i \geq 0)\), corresponding to the \(i\)th downward and upward crossings of the sphere of radius \( r_n \). Let \( \theta(n) \) act on trajectories by time-translation of \( n\)-units. Let \( \tau = H(B_{r_n}) \wedge H_n, D_0 = U_0 = 0, \) and

\[
D_1 = \tau \mathbf{1}_{H(B_{r_n}) < H_n} + \infty \mathbf{1}_{H_n < H(B_{r_n})}.
\]

If \( D_1 < \infty \), then \( U_1 = H_{r_n} \circ \theta(D_1) + D_1 \), whereas if \( D_1 = \infty \), then we set \( U_1 = \infty \). We now proceed by induction, and assume \( D_i, U_i \) are defined. If \( D_i = \infty \), then \( D_{i+1} = \infty \), whereas if \( D_i < \infty \), (and necessarily \( U_i < \infty \)) then

\[
D_{i+1} = U_i + (\tau \mathbf{1}_{\tau = H(B_{r_n})} + \infty \mathbf{1}_{\tau = H_n}) \circ \theta(U_i), \quad \text{and} \quad U_{i+1} = D_{i+1} + H_{r_n} \circ \theta(D_{i+1}).
\]

With this notation, we can write

\[
\sum_{y \in \mathcal{A}(r_n, n)} G_n(z, y) = E_z[\tau] + \sum_{i=1}^{\infty} E_z[\tau \circ \theta(U_i) \mathbf{1}_{D_i < \infty}] = E_z[\tau] + \mathbb{P}_z(D_1 < \infty) \times I(z), \tag{A.3}
\]

where

\[
I(z) = \sum_{i=1}^{\infty} E_z[\tau \circ \theta(U_i) | D_i < \infty] \prod_{j=1}^{i-1}(1 - P_z(D_{j+1} = \infty | D_j < \infty)). \tag{A.4}
\]

Now, we compute each term of the right hand side of (A.3).

We have divided the proof in three steps.

**Step 1:** First, we show that there is a positive constant \( K \), (independent of \( z \) and \( n \)) such that when \( z \in \mathcal{A}(r_n, n) \), then

\[
|\mathbb{P}_z(D_1 < \infty) - \frac{\alpha_0(z)}{\Delta_n}| \leq \frac{K}{\Delta_n^2} \left((n - \|z\|) \lor 1 \right). \tag{A.5}
\]

Note that when \( z \in B(0, n) \), and \( n - \|z\| \leq 1 \), (A.5) yields

\[
|\mathbb{P}_z(D_1 < \infty) - \frac{E_z[\|S(\tau)\| - \|z\| | D_1 = \infty]}{\Delta_n}| \leq \frac{K}{\Delta_n^2}. \tag{A.6}
\]

Secondly, we show that for \( z \in \mathcal{A}(r_n, n) \), and \( i \geq 1 \)

\[
|\mathbb{P}_z(D_{i+1} = \infty | D_i < \infty) - \frac{E_z[(\|S(U_i)\| - \|S(D_{i+1})\| | D_{i+1} = \infty) \mathbf{1}_{D_{i+1} < \infty}] - \mathbb{E}_z[G(0, S(\tau)) | D_1 = \infty]}{\Delta_n}| \leq \frac{K}{\Delta_n^2}. \tag{A.7}
\]

Our starting point is the classical Gambler’s ruin estimate

\[
\mathbb{P}_z(D_1 < \infty) = \frac{G(0, z) - E_z[G(0, S(\tau)) | D_1 = \infty]}{E_z[G(0, S(\tau)) | D_1 < \infty] - E_z[G(0, S(\tau)) | D_1 = \infty]}. \tag{A.8}
\]
We now expand Green’s function using asymptotics (2.7). For this purpose, it is convenient to define a random variable

\[ X(z) = \frac{1}{\|z\|} (\|S(\tau)\|^2 - \|z\|^2), \quad \text{and to set } \eta = \frac{d - 2}{2}. \]

By expressing \( S(\tau) \) in terms of \( X(z) \), we have

\[ \frac{1}{\|S(\tau)\|^{d-2}} = \frac{1}{\|z\|^{d-2}} \left( 1 + \frac{X(z)}{\|z\|} \right)^{-\eta}. \]  \hspace{1cm} (A.9)

Note that for any \( z \in A(r_n, n) \), \( X(z)/\|z\| \) is small. Indeed,

\[ \frac{X(z)}{\|z\|} = \frac{\|S(\tau)\| - \|z\|}{\|z\|} \frac{\|S(\tau)\| + \|z\|}{\|z\|} \]  \hspace{1cm} (A.10)

Since \( \Delta_n = n - r_n = O(n^{1/3}) \), we have for \( n \) large enough

\[ \frac{X(z)}{\|z\|} \leq \frac{2(n + 1)\Delta_n}{(n - \Delta_n)^2} \leq \frac{8\Delta_n}{n}, \quad \text{and} \quad \sup_{z \in A(r_n, n)} \left( \frac{1}{\|z\|} \right)^3 \left( \frac{X(z)}{\|z\|} \right) \leq \frac{8^3\Delta_n^3}{n} \times \frac{1}{n^2}. \]  \hspace{1cm} (A.11)

More precisely, \( X(z) \) is of order \( 2(\|S(\tau)\| - \|z\|) \). Indeed, \( \Delta_n^3 \leq K'n \) for some \( K' > 0 \), and (A.10) yields

\[ X(z) = 2(\|S(\tau)\| - \|z\|) + \left( \frac{\|S(\tau)\| - \|z\|}{\|z\|} \right)^2 \rightarrow |X(z) - 2(\|S(\tau)\| - \|z\|)| \leq \frac{K'}{\Delta_n}. \]  \hspace{1cm} (A.12)

Finally, we have a constant \( K \) such that

\[ \left| \left( 1 + \frac{X(z)}{\|z\|} \right)^{-\eta} - \left( 1 - \eta \frac{X(z)}{\|z\|} + \frac{\eta}{2} \left( \frac{X(z)}{\|z\|} \right)^2 \right) \right| \leq \frac{K}{n^2}. \]  \hspace{1cm} (A.13)

For any \( z \neq 0 \), Green’s function asymptotics (2.7) and (A.13) yields

\[ |G(0, S(\tau)) - G(0, z) - \eta C_d \left( \frac{X(z)}{\|z\|^{d-1}} + \frac{\eta + 1}{2} \frac{X(z)^2}{\|z\|^{d+1}} \right)| \leq \frac{K}{n^d}. \]  \hspace{1cm} (A.14)

Using (A.8) and (A.14), we obtain

\[ \mathbb{P}_z(D_1 < \infty) = \frac{E_z[X(z)|D_1 = \infty] - \hat{C}(z) + O(\frac{1}{n})}{E_z[X(z)|D_1 = \infty] - E_z[X(z)|D_1 < \infty] + \hat{C}(z) - C(z) + O(\frac{1}{n})}. \]  \hspace{1cm} (A.15)

where

\[ \hat{C}(z) = \frac{\eta + 1}{2} E_z \left[ \frac{X^2(z)}{\|z\|^2} \right] D_1 = \infty, \quad \text{and} \quad C(z) = \frac{\eta + 1}{2} E_z \left[ \frac{X^2(z)}{\|z\|^2} \right] D_1 < \infty. \]  \hspace{1cm} (A.16)

Using (A.11), we have some rough estimates on \( \hat{C} \) and \( C \). For any \( z \in A(r_n, n) \),

\[ \hat{C}(z) = O \left( \frac{\Delta_n^2}{n} \right) = O \left( \frac{1}{\Delta_n} \right), \quad \text{and} \quad C(z) = O \left( \frac{\Delta_n^2}{n} \right) = O \left( \frac{1}{\Delta_n} \right). \]  \hspace{1cm} (A.17)
Using (A.12), we have better estimates for $\bar{C}$ and $C$:

$$C(z) = d\frac{(n - \|z\|)^2}{\|z\|} + O\left(\frac{\Delta_n}{n}\right), \quad \text{and} \quad \bar{C}(z) = \frac{d(\|z\| - r_n)^2}{\|z\|} + O\left(\frac{\Delta_n}{n}\right). \quad (A.18)$$

The rough estimates (A.17) allow us to derive from (A.15) an estimate for $\mathbb{P}_z(D_1 < \infty)$, for any $z \in A(r_n, n)$.

$$\mathbb{P}_z(D_1 < \infty) = \frac{E_z[\|S(\tau)\| - \|z\||D_1 = \infty] + O\left(\frac{1}{\Delta_n}\right)}{E_z[\|S(\tau)\| - \|z\||D_1 = \infty] - E_z[\|S(\tau)\| - \|z\||D_1 < \infty] + O\left(\frac{1}{\Delta_n}\right)} \quad (A.19)$$

$$= \frac{\alpha_0(z) + O\left(\frac{1}{\Delta_n}\right)}{\Delta_n(1 + O\left(\frac{1}{\Delta_n}\right))}.$$}

This yields (A.5) since $\alpha_0(z) \leq 1 + (n - \|z\|) \vee 1 \leq 2(n - \|z\|) \vee 1$.

**Case where** $z \in \partial B(0, r_n)$.

On $\{D_1 = \infty\}$, we have

$$X(z) = 2(\|S(\tau)\| - \|z\|) + O\left(\frac{1}{\Delta_n}\right). \quad (A.20)$$

On $\{D_1 < \infty\}$, we have

$$X(z) = 2(\|S(\tau)\| - \|z\|) + O\left(\frac{1}{n}\right). \quad (A.21)$$

This implies that using (A.18)

$$\bar{C}(z) = d\frac{\Delta_n^2}{\|z\|} + O\left(\frac{\Delta_n}{n}\right), \quad \text{and} \quad C(z) = O\left(\frac{1}{n}\right). \quad (A.22)$$

Thus,

$$\mathbb{P}_z(D_1 = \infty) = \frac{2E_z[\|z\| - \|S(\tau)\||D_1 < \infty] + \bar{C}(z) + O\left(\frac{1}{\Delta_n}\right)}{E_z[X(z)|D_1 = \infty] - E_z[X(z)|D_1 < \infty] + \bar{C} - \bar{C} + O\left(\frac{1}{\Delta_n}\right)} \quad (A.23)$$

$$= \frac{E_z[\|z\| - \|S(\tau)\||D_1 < \infty] + O\left(\frac{1}{\Delta_n}\right)}{\Delta_n + O(1)}$$

$$= \frac{\Delta_n}{\Delta_n + O(1)} + O\left(\frac{1}{\Delta_n}\right).$$

In order to obtain (A.7), we write (A.23) on $\{D_i < \infty\}$, and $z = S(U_i)$ as follows. There is a constant $K$ such that on the event $\{D_i < \infty\}$,

$$\left| E_{S(U_i)}[1_{D_{i+1} = \infty} - \frac{E_{S(U_i)}[\|S(U_i)\| - \|S(\tau)\|]}{\Delta_n \times P_{S(U_i)}(D_1 < \infty)}] \right| \leq \frac{K}{\Delta_n^2}. \quad (A.24)$$
Note that (A.23) implies that $P_{S(U_i)}(D_1 < \infty) = 1 + O(1/\Delta_n)$, so that (A.24) reads as we integrate over $\{D_i < \infty\}$ with respect to $E_z$

$$
\mathbb{P}_z (D_{i+1} = \infty, D_i < \infty) = \frac{\mathbb{E}_z \left[ \mathbf{1}_{D_i < \infty} (\|S(U_i)\| - \|S(\tau)\|) \mathbf{1}_{D_{i+1} < \infty} \right]}{\Delta_n} \leq \frac{K \mathbb{P}_z (D_i < \infty)}{\Delta_n^2}.
$$

We obtain (A.7) as we divide both sides of (A.25) by $\mathbb{P}_z (D_i < \infty)$.

**Step 2:** We show now that for any $z \in A(r_n, n)$ we have

$$
|E_z [\tau] - (d\Delta_n \alpha_0(z) - 2d(n - \|z\|)^2) | \leq K ((n - \|z\|) \vee 1).
$$

When $z \in B_n$ and $n - \|z\| \leq 1$, (A.26) reads

$$
|E_z [\tau] - (d\Delta_n \alpha_0(z) - 2d(n - \|z\|)^2) | \leq K.
$$

When $z \in A(r_n, n)$, and $i \geq 1$, we show that

$$
\left| \frac{\mathbb{E}_z [\tau \circ \theta(U_{i-1}) | D_{i} < \infty]}{d\Delta_n^2} - \frac{\mathbb{E}_z [(\|S(U_i)\| - \|S(\tau)\|) \mathbf{1}_{D_{i+1} < \infty} | D_{i} < \infty]}{\Delta_n} \right| \leq \frac{K \Delta_n^2}{\Delta_n^2}.
$$

Using that $\{\|S(n)\|^2 - n, n \in \mathbb{N}\}$ is a martingale, and the optional sampling theorem (see Lemma 2 of [8])

$$
E_z [\tau] = E_z [(\|S(\tau)\|^2) - \|z\|^2 = \|z\| \times E_z [X(z)] = \|z\| \times (E_z [X(z) | D_1 = \infty] \mathbb{P}_z (D_1 = \infty) + E_z [X(z) | D_1 < \infty] \mathbb{P}_z (D_1 < \infty)).
$$

Thus, using (A.15), simple algebra yields

$$
E_z [\tau] = \|z\| \times ((C(z) - \bar{C}(z)) \mathbb{P}_z (D_1 < \infty) + \bar{C}(z)) + O(1).
$$

By recalling (A.18) and (A.5)

$$
E_z [\tau] = d \left( (\|z\| - r_n)^2 - (n - \|z\|)^2 + O(\Delta_n) \right) \left( \frac{\alpha_0(z)}{\Delta_n} + O \left( \frac{\Delta_n}{\Delta_n^2} \right) \right) + O(1)
$$

$$
= d \left( (\|z\| - r_n)^2 - (n - \|z\|)^2 + O \left( \frac{\Delta_n}{\Delta_n^2} \right) \right) + O(1)
$$

$$
= d \Delta_n \alpha_0(z) - 2d(n - \|z\|)^2 + O \left( \frac{\Delta_n}{\Delta_n^2} \right) + O(1)
$$

Note that in the case where $n - \|z\| \leq 1$, (A.31) yields (A.27).

Assume now that $z \in \partial B(0, r_n)$. From (A.30), we have

$$
E_z [\tau] = \|z\| \times ((\bar{C}(z) - \overline{C}(z)) \mathbb{P}_z (D_1 = \infty) + \overline{C}(z)) + O(1).
$$

We use (A.7), (A.20) and (A.21) to obtain

$$
E_z [\tau] = d \Delta_n \left( (d \Delta_n^2 + O(\Delta_n)) \left( \frac{\mathbb{E}_z [\|z\| - \|S(\tau)\| | D_1 < \infty]}{\Delta_n} + O \left( \frac{1}{\Delta_n^2} \right) \right) \right) + O(1)
$$

$$
= d \Delta_n \mathbb{E}_z [\|z\| - \|S(\tau)\| | D_1 < \infty] + O(1).
$$

Now, write (A.33) as follows. There is a constant \( K \) such that for any \( z \in \partial \mathcal{B}(0, r_n) \)

\[
\left| \frac{\mathbb{E}_z[\tau]}{d \Delta_n^2} - \frac{\mathbb{E}_z[\|z\| - \|S(\tau)\|I_{D_1 < \infty}]}{\Delta_n \mathbb{P}_z(D_1 < \infty)} \right| \leq \frac{K}{\Delta_n^2}.
\] (A.34)

Note that by (A.23), we have that \( \Delta_n \mathbb{P}_z(D_1 < \infty) = \Delta_n + O(1) \), and \( \|z\| - \|S(\tau)\|I_{D_1 < \infty} \leq 1 \), we have

\[
\left| \frac{\mathbb{E}_z[\tau]}{d \Delta_n^2} - \frac{\mathbb{E}_z[\|z\| - \|S(\tau)\|I_{D_1 < \infty}]}{\Delta_n} \right| \leq \frac{K}{\Delta_n^2}.
\] (A.35)

We replace \( z \) by \( S(U_i) \) in (A.35) under the event \( \{D_i < \infty\} \) to obtain

\[
\left| \frac{\mathbb{E}_{S(U_i)}[\tau]}{d \Delta_n^2} - \frac{\mathbb{E}_{S(U_i)}[\|S(U_i)\| - \|S(D_1 \circ \theta(U_i))\|I_{D_1 \circ \theta(U_i) < \infty}]}{\Delta_n} \right| \leq \frac{K}{\Delta_n^2}.
\] (A.36)

We multiply both sides of (A.36) by \( I_{D_i < \infty} \), take the expectation on both side of (A.36), and divide by \( \mathbb{P}_z(D_i < \infty) \) to obtain (A.28).

**Step 3:** For \( i \geq 1 \), we show the following bounds

\[
2 \geq \gamma_i \geq \frac{1}{4d \sqrt{d}}, \quad \text{where} \quad \gamma_i = \mathbb{E}_z \left[ (\|S(U_i)\| - \|S(D_{i+1})\|) 1_{D_{i+1} < \infty} | D_i < \infty \right].
\] (A.37)

The upper bound is obvious. For the lower bound, first we restrict to \( \{D_i < \infty\} \), so that \( U_i < \infty \). By Lemma 2.1, \( S(U_i) \) has a nearest neighbor \( x \), within \( \mathbb{B}(0, r_n) \) such that \( \|S(U_i)\| - \|x\| \geq 1/(2 \sqrt{d}) \), and (A.37) is immediate.

**Step 4:** We show (A.1) using (A.3). For \( p \) such that \( 1 \leq p \leq \infty \), let

\[
\sigma_p = \sum_{i=1}^{p} \mathbb{E}_z [\tau \circ \theta(U_i) | D_i < \infty] \prod_{j=1}^{i-1} (1 - \mathbb{P}_z(D_{j+1} = \infty | D_j < \infty)).
\] (A.38)

Now, (A.4) reads \( l(z) = \lim_{p \to \infty} \sigma_p \). We establish in this step that, for some constant \( \tilde{K} \), any integer \( n \)

\[
\lim_{p \to \infty} |1 - \frac{\sigma_p}{d \Delta_n^2}| \leq \frac{\tilde{K}}{\Delta_n}.
\] (A.39)

Once we prove (A.39), we have all the bounds to estimate the right hand side of (A.3). Indeed, using (A.26), (A.5) and (A.39), we have

\[
\mathbb{E}_z[\tau] + \mathbb{P}_z(D_1 < \infty) \times l(z) = d \Delta_n \alpha_0(z) - 2d (n - \|z\|)^2 + O ((n - \|z\|) \wedge 1) + \left( \frac{\alpha_0(z)}{\Delta_n} + O \left( \frac{(n - \|z\|) \wedge 1}{\Delta_n^2} \right) \right) \times (d \Delta_n^2 + O(\Delta_n))
\]

\[
= 2d \Delta_n \alpha_0(z) - 2d (n - \|z\|)^2 + O ((n - \|z\|) \wedge 1).
\] (A.40)

In order now to prove (A.39), we introduce first some shorthand notation. For \( p \) and \( j \) positive integers

\[
a_p = 1 - \frac{\sigma_p}{d \Delta_n^2}, \quad \alpha_j = P_z(D_{j+1} = \infty | D_j < \infty), \quad \text{and} \quad \beta_j = \frac{\mathbb{E}_z [\tau \circ \theta(U_j) | D_j < \infty]}{d \Delta_n^2}.
\] (A.41)
With this notation (A.7) and (A.28) read as follows.

\[ |\alpha_j - \frac{\gamma_j}{\Delta_n}| \leq \frac{K}{\Delta_n^2}, \text{ and } |\beta_j - \frac{\gamma_j}{\Delta_n}| \leq \frac{K}{\Delta_n^2}, \] so that \[ |\alpha_j - \beta_j| \leq \frac{2K}{\Delta_n^2}. \] (A.42)

Let us rewrite (A.38) as

\[ a_p = a_{p-1} - \beta_p \prod_{j=1}^{p-1} (1 - \alpha_j). \] (A.43)

In order to establish (A.39), we show by induction that

\[ |a_p - \prod_{j=1}^{p} (1 - \alpha_j)| \leq \epsilon_p, \] (A.44)

with for \( p > 1 \)

\[ \epsilon_p = \epsilon_{p-1} + \frac{2K}{\Delta_n^2} \prod_{j=1}^{p-1} (1 - \alpha_j) \text{ and } \epsilon_1 = \frac{2K}{\Delta_n^2}. \] (A.45)

Note that it is easy to estimate \( \epsilon_p \) from (A.45). There is a constant \( \kappa_S \) such that

\[ \epsilon_d \leq \frac{2K}{\Delta_n^2} \left( 1 + \sum_{k=1}^{p} \exp \left( - \sum_{j=1}^{k} \alpha_j \right) \right) \leq \frac{2K}{\Delta_n^2} \left( 1 + \sum_{k=1}^{p} \exp \left( - \sum_{j=1}^{k} \frac{\gamma_j}{2\Delta_n} \right) \right) \leq \frac{2K}{\Delta_n^2} \kappa_S \Delta_n = \frac{2K \kappa_S}{\Delta_n}. \]

Now, (A.44) holds for \( p = 1 \), and assume it holds for \( p - 1 \). Then

\[ (1 - \beta_p) \prod_{j=1}^{p-1} (1 - \alpha_j) - \epsilon_{p-1} \leq a_p \leq (1 - \beta_p) \prod_{j=1}^{p-1} (1 - \alpha_j) + \epsilon_{p-1}. \] (A.46)

Then by (A.42), we have (A.44) with \( \epsilon_p \) satisfying (A.45).

Now (A.39) follows as we notice that Step 3 implies that

\[ \lim_{p \to \infty} \prod_{j=1}^{p} (1 - \alpha_j) = 0. \]

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