Long Distance Effects in Mixed Electromagnetic-Gravitational Scattering

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Abstract

Using the methods of effective field theory we examine long range effects in mixed electromagnetic-gravitational scattering. Recent calculations which have yielded differing results for such effects are examined and corrected. We consider various spin configurations of the scattered particles and find that universality with respect to spin-dependence is obtained in agreement with expectations.

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1 Introduction

Recently there have been a number of calculations of long distance effects in mixed electromagnetic-gravitational scattering (i.e., $\mathcal{O}(G\alpha)$ effects), using the methods of effective field theory [1, 2, 3]. The basic idea here is that such long distance corrections arise from pieces of the scattering amplitude which are nonanalytic in the momentum transfer. Such nonanalytic components can be isolated from one loop scattering amplitudes that involve both photons and gravitons using effective field theory methods, a procedure pioneered more than a decade ago by Donoghue in the case of the gravitational interaction [5]. There are two distinct forms of such corrections—a classical piece (independent of $\hbar$) which arises from square root components $\sim 1/\sqrt{-q^2}$ and a quantum ($\hbar$-dependent) component associated with logarithmic nonanalyticity $\sim \log -q^2$ where $q$ is the four-momentum transfer in the scattering reaction.

The first such calculation was done by N.E.J. Bjerrum-Bohr in the case of the scattering of two nonidentical spinless particles, particle $a$ of mass $m_a$ and charge $Z_a|e|$ and particle $b$ of mass $m_b$ and charge $Z_b|e|$, who quoted the one loop potential [1]

$$V_{CG}^{(2)}(\vec{r}) = G\alpha \left[ \frac{1}{2} \frac{m_a Z_b^2 + m_b Z_a^2}{r^2} + 3 \frac{Z_a Z_b (m_a + m_b)}{r^2} - \frac{4\hbar}{3\pi r^3} \left( \frac{Z_b^2 m_a}{m_b} + \frac{Z_a^2 m_b}{m_a} \right) - \frac{8Z_a Z_b \hbar}{\pi r^3} \right]. \tag{1}$$

A followup calculation by Butt involving the spin-averaged spin-1/2 – spin-1/2 scattering amplitude determined the form [2]

$$\left\{ \frac{1}{2} V_{CG}^{(2)}(\vec{r}) \right\}_{\text{spin-av.}} = G\alpha \left[ \frac{1}{2} \frac{m_a Z_b^2 + m_b Z_a^2}{r^2} + 3 \frac{Z_a Z_b (m_a + m_b)}{r^2} - \frac{4\hbar}{3\pi r^3} \left( \frac{Z_b^2 m_a}{m_b} + \frac{Z_a^2 m_b}{m_a} \right) - \frac{15}{6} \frac{Z_a Z_b \hbar}{\pi r^3} \right]. \tag{2}$$

What is surprising here is that while the classical ($\sim 1/r^2$) pieces of the spin-0 – spin-0 and of the spin-averaged spin-1/2 – spin-1/2 potentials agree, the quantum-mechanical ($\sim \hbar/r^3$) components have differing forms. These results then constitute a deviation from the universality of similar second

\[^1\text{Note, however, this result has now been corrected by Bjerrum-Bohr [4].}\]
order potentials found in the case of purely electromagnetic and purely gravitational scattering in recent calculations by Holstein and Ross [6, 7]. In the most recent calculation of this $O(G\alpha)$ potential in the case of spinless scattering by Faller, it is claimed to have still a third form [3]

$$0V^{(2)}_{CG}(r) = G\alpha \left[ \frac{1}{2} \frac{m_a Z_b^2 + m_b Z_a^2}{r^2} + 3 \frac{Z_a Z_b (m_a + m_b)}{r^2} \right] - \frac{4\hbar}{3\pi r^3} \left( \frac{Z_b^2 m_a}{m_b} + \frac{Z_a^2 m_b}{m_a} \right) - \frac{2 Z_a Z_b \hbar}{3\pi r^3}. \quad (3)$$

The surprising disagreement between these results and the possible breakdown of universality indicated by the discrepancy between them clearly calls for a new evaluation which resolves previous disagreements and clearly answers the question of universality. This is the purpose of the work below. We shall evaluate mixed gravitational-electromagnetic scattering at the one loop level in the case of spin-0 – spin-0, spin-0 – spin-1/2, spin-0 – spin-1, and spin-1/2 – spin-1/2 scattering, looking both at the spin-independent and spin-dependent pieces of the scattering amplitude. We quote the contributions of each diagram in order to assess and resolve the problems with previous evaluations.

In the next section we perform the calculation for the case of spinless scattering and compare with the previous evaluations by Bjerrum-Bohr [1] and Faller [3]. Then in the following sections we generalize this calculation to the case wherein one or both particles carry spin, connecting with the calculation by Butt [2]. Our results, which clearly resolve the questions raised above, are summarized in a concluding section. While some of our calculational methods are found in the appendix, we refer to our companion papers on purely electromagnetic scattering [6] and on purely gravitational scattering [7] for much of the calculational ingredients such as the loop integrals and parts of the Feynman rules needed.

## 2 Spin-0 – Spin-0 Scattering

We begin our discussion by considering the case of the scattering of two spinless particles, particle $a$ of mass $m_a$ and charge $Z_a|e|$ and particle $b$ of mass $m_b$ and charge $Z_b|e|$. We choose the initial (final) four-momentum of particle $a$ to be $p_1$ ($p_2 = p_1 - q$) and that of particle $b$ to be $p_3$ ($p_4 = p_3 + q$). This is the case originally studied by Bjerrum-Bohr [1].
The corresponding one loop diagrams that give rise to long distance contributions are drawn in Fig. 1 where the blobs are explained in Fig. 2. Since we will be comparing different calculations, it is useful to present the amplitudes for each diagram separately. The calculational techniques including the Feynman rules and the loop integrals needed are outlined in [6], [7] and in Appendix A. Defining the nonanalytic structures

\[ L = \log -q^2 \quad \text{and} \quad S = \frac{\pi^2}{\sqrt{-q^2}} \]

we find for the case of spinless scattering the contributions
Figure 2: Loop corrections subsumed in vertex and in vacuum polarization functions for mixed electromagnetic-gravitational scattering.

\[ 0 \mathcal{M}_{11}^{(2)}(q) = G\alpha Z_a Z_b(16L) \]
\[ 0 \mathcal{M}_{12}^{(2)}(q) = G\alpha Z_a Z_b(-16L - 8m_a S) \]
\[ 0 \mathcal{M}_{13}^{(2)}(q) = G\alpha Z_a Z_b(-16L - 8m_b S) \]
\[ 0 \mathcal{M}_{14}^{(2)}(q) = G\alpha Z_a Z_b \left[ \left( -\frac{8m_a m_b}{q^2} - \frac{7(m_a^2 + m_b^2)}{m_a m_b} + 4 \right) L - 5(m_a + m_b)S \right] \]
\[ + i8\pi Gm_a m_b \alpha Z_a Z_b \frac{L}{q^2} \sqrt{\frac{m_a m_b}{s - s_0}} \]
\[ 0 \mathcal{M}_{15}^{(2)}(q) = G\alpha Z_a Z_b \left[ \left( \frac{8m_a m_b}{q^2} + \frac{7(m_a^2 + m_b^2)}{m_a m_b} + \frac{68}{3} \right) L + 5(m_a + m_b)S \right] \]
\[ 0 \mathcal{M}_{16}^{(2)}(q) = G\alpha Z_a Z_b \left[ 2(m_a + m_b)S \right] \]
\[ 0 \mathcal{M}_{17}^{(2)}(q) = G\alpha \left[ -\frac{8(Z_a^2 m_b^4 + Z_b^2 m_a^4)}{3m_a m_b} L - (Z_a^2 m_b+ Z_b^2 m_a)S \right] \]
\[ M_1(q) = G\alpha Z_a Z_b \left[ \frac{4L}{3} \right]. \]  

(4)

from each diagram—(a)-(g) in Fig. 1—where \( s = (p_1 + p_3)^2 \) is the square of the center of mass energy and \( s_0 = (m_a + m_b)^2 \) is its threshold value.

Comparing with previous work, we agree diagram by diagram with the calculation of Bjerrum-Bohr. (In the case of the paper by Faller, we are also in agreement—once certain typos are corrected—except for the box and cross-box diagrams, which are too small in his calculation by a factor of two.) When all contributions are added together, we find the total amplitude

\[ M_{\text{tot}}^{(2)}(q) = G\alpha \left[ Z_a Z_b (-6(m_a + m_b)S + 12L) \right. \]
\[ \left. - (Z_a^2 m_b + Z_b^2 m_a)S - \frac{8(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} L \right] \]
\[ + i8\pi Gm_a m_b \alpha Z_a Z_b \frac{L m_r}{q^2 s - s_0}. \]  

(5)

For later use we shall work in the nonrelativistic limit and in the center of mass frame—\( \vec{p}_1 + \vec{p}_3 = 0 \). We have then

\[ s - s_0 = 2\sqrt{m_a^2 + \vec{p}_1^2} \sqrt{m_b^2 + \vec{p}_1^2} + 2\vec{p}_1^2 - 2m_a m_b \]  

(6)

and

\[ \sqrt{\frac{m_a m_b}{s - s_0}} \approx \frac{m_r}{p_0} \]  

(7)

where \( m_r = m_a m_b/(m_a + m_b) \) is the reduced mass and \( p_0 \equiv |\vec{p}_i|, \ i = 1, 2, 3, 4. \) The total amplitude then becomes

\[ M_{\text{tot}}^{(2)}(q) \approx G\alpha \left[ Z_a Z_b (-6(m_a + m_b)S + 12L) \right. \]
\[ \left. - (Z_a^2 m_b + Z_b^2 m_a)S - \frac{8(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} L \right] \]
\[ + i8\pi Gm_a m_b \alpha Z_a Z_b \frac{L m_r}{q^2 p_0}. \]  

(8)

\footnote{Our normalization of the amplitudes is a nonrelativistic one such that after applying all Feynman rules we divide the amplitude by a factor of \( \sqrt{2E_1 2E_2 2E_3 2E_4}. \)}

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At this point we encounter a difficulty in defining a proper second order potential in that the inclusion of loop effects has produced an imaginary piece of the transition amplitude, which clearly cannot be part of a properly defined (hermitian) potential. Of course, there is no mystery as to why this term is present, since unitarity requires the presence of such imaginary components. The solution is also clear. We must subtract from the second order transition amplitude the piece which arises from the iterated lowest order potential before attempting to identify a proper second order potential. Indeed, in lowest order spinless scattering, we use the Newtonian potential to describe the gravitational potential

\[ 0V_{G}^{(1)}(\vec{r}) = -G\frac{m_a m_b}{r} \]  

while for its electromagnetic analog we utilize the Coulomb potential

\[ 0V_{C}^{(1)}(\vec{r}) = \frac{Z_a Z_b \alpha}{r}. \]  

Working in momentum space we have

\[ 0V_{G}^{(1)}(\vec{q}) = \langle \vec{p}_f | 0\hat{V}_{G}^{(1)} | \vec{p}_i \rangle = -\frac{4\pi G m_a m_b}{q^2} = -\frac{4\pi G m_a m_b}{(\vec{p}_i - \vec{p}_f)^2} \]  

\[ 0V_{C}^{(1)}(\vec{q}) = \langle \vec{p}_f | 0\hat{V}_{C}^{(1)} | \vec{p}_i \rangle = \frac{4\pi \alpha Z_a Z_b}{q^2} = \frac{4\pi \alpha Z_a Z_b}{(\vec{p}_i - \vec{p}_f)^2} \]

respectively. The relevant second Born term of \( \mathcal{O}(G\alpha) \) is then

\[ 0\text{Amp}_{CG}^{(2)}(\vec{q}) = - \int \frac{d^3\ell}{(2\pi)^3} \frac{\langle \vec{\ell} | 0\hat{V}_{G}^{(1)} | \vec{p}_f \rangle \langle \vec{\ell} | 0\hat{V}_{C}^{(1)} | \vec{p}_i \rangle}{E(\vec{p}_0) - E(\vec{\ell}) + i\epsilon} \]

\[ - \int \frac{d^3\ell}{(2\pi)^3} \frac{\langle \vec{\ell} | 0\hat{V}_{C}^{(1)} | \vec{p}_f \rangle \langle \vec{\ell} | 0\hat{V}_{G}^{(1)} | \vec{p}_i \rangle}{E(\vec{p}_0) - E(\vec{\ell}) + i\epsilon} \]

\[ = i \int \frac{d^3\ell}{(2\pi)^3} 0V_{G}^{(1)}(\vec{\ell} - \vec{p}_f) G^{(0)}(\vec{\ell}) 0V_{C}^{(1)}(\vec{p}_i - \vec{\ell}) \]  

\[ + i \int \frac{d^3\ell}{(2\pi)^3} 0V_{C}^{(1)}(\vec{\ell} - \vec{p}_f) G^{(0)}(\vec{\ell}) 0V_{G}^{(1)}(\vec{p}_i - \vec{\ell}) \]  

\[ ^{3}\text{Note that we elect to use here the simple nonrelativistic forms for both the potentials and the propagator. We shall comment later on this choice.} \]
where
\[ G^{(0)}(\ell) = \frac{i}{p_0^2/2m_r - \ell^2/2m_r + i\epsilon} \] (15)
is the free propagator. The remaining integration can be performed exactly, as discussed in Appendix A, by including a photon "mass" term \( \lambda^2 \) as a regulator. Identifying \( \vec{p}_i \) with \( \vec{p}_1 \) and \( \vec{p}_f \) with \( \vec{p}_2 \), the iteration amplitude reads
\[
0 \text{Amp}_{CG}^{(2)}(\vec{q}) = i \int \frac{d^3 \ell}{(2\pi)^3} \left[ \frac{-4\pi G m_a m_b}{|\ell - \vec{p}_2|^2 + \lambda^2} \frac{Z_a Z_b \epsilon^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon |\vec{p}_1 - \ell|^2 + \lambda^2 \right. \\
+ \left. \frac{Z_a Z_b \epsilon^2}{|\ell - \vec{p}_2|^2 + \lambda^2} \frac{i}{2m_r} - \frac{4\pi G m_a m_b}{|\ell - \vec{p}_2|^2 + \lambda^2} \right]
\]
\[ \frac{\lambda^2}{2} 2H = i8\pi G m_a m_b \alpha Z_a Z_b \frac{L m_r}{q^2 p_0} \] (16)
which precisely reproduces the imaginary component of \( 0 \mathcal{M}_{tot}^{(2)}(\vec{q}) \), as expected. In order to generate a properly defined second order potential \( 0 V_{CG}^{(2)}(\vec{r}) \) we must subtract this second order Born term from the second order amplitude, yielding the result
\[
0 V_{CG}^{(2)}(\vec{r}) = -i \int \frac{d^3 q}{(2\pi)^3} e^{-i \vec{q} \cdot \vec{r}} \left[ \mathcal{M}_{tot}^{(2)}(\vec{q}) - 0 \text{Amp}_{CG}^{(2)}(\vec{q}) \right] \\
= G\alpha \left[ \frac{1}{2} m_a Z_b^2 + m_b Z_a^2 + \frac{Z_a Z_b (m_a + m_b)}{r^2} \right. \\
- \left. \frac{4\hbar}{3\pi r^3} \left( Z_b^2 \frac{m_a}{m_b} + Z_a^2 \frac{m_b}{m_a} \right) + \frac{6Z_a Z_b \hbar}{\pi r^3} \right] 
\] (17)
which disagrees with that quoted by both Bjerrum-Bohr and Faller [1, 3]. In the case of the Bjerrum-Bohr this is due to a typo which occurred in his total result. This mistake has been corrected in an erratum [4], so that our results are now in agreement. In the case of Faller the disagreement is due to his use of box plus cross-box contributions which are a factor of two smaller than the correct form. We notice that the resulting \( O(G\alpha) \) potential in Eq. (17) displays two different dependences on the charges of the scattered particles: The terms proportional to \( Z_a Z_b \) which vanish once one of the scattered particles has zero charge and the terms proportional to
$Z^2_a$ and $Z^2_b$ which can persist if one of the scattered particles is uncharged. Thus we avoid the temptation to call our results gravitational corrections to electromagnetic scattering since the $Z^2_a$ and $Z^2_b$ terms cannot be viewed as such corrections.

It should be noted here that the form of the classical ($\hbar$-independent) component of the second order potential is ambiguous, as pointed out by Sucher [8]. For a detailed discussion of these ambiguities in the case of purely electromagnetic scattering, see [6]. The point is that the form of the iterated lowest order potentials depends upon the precise form of the lowest potentials and whether relativistic effects are included in it and in the propagator $G^{(0)}(\ell)$. The existence of such ambiguity is not a problem, since the potential is not an observable. What is an observable and is invariant is the second order on-shell transition amplitude, which is given by

$$0M^{(2)}_{tot}(\vec{q}) = 0\text{Amp}^{(2)}(\vec{q}) - \int d^3r \ e^{i\vec{q} \cdot \vec{r}} 0V^{(2)}_{CG}(\vec{r}).$$

Because of this invariance, we shall for simplicity utilize the simple nonrelativistic potentials and propagator in obtaining the iterated second order amplitude.

3 Spin-0 – Spin-1/2 Scattering

Before proceeding to the spin-1/2 – spin-1/2 calculation of Butt, we first examine the simpler case of a spin-1/2 particle $b$ scattering from a spinless particle $a$, in order to check the hypothesis of universality. The calculation proceeds as in the previous section except that for the vertices for particle $b$ we use the spin-1/2 forms. One other subtlety that arises in the calculation once spin is involved is that it contains two independent kinematic variables: the momentum transfer $q^2$ and $s - s_0$, which is to leading order proportional to $p_0^2$ (where $p_0^2 \equiv \vec{p}_i^2, \ i = 1, 2, 3, 4$) in the center of mass frame. We find that our results differ if we perform an expansion first in $s - s_0$ and then in $q^2$ or vice versa. This ordering issue only occurs for the box diagram, diagram (d) of Fig. [1] where it stems from the reduction of vector and tensor box integrals. Their reduction in terms of scalar integrals involves the inversion of a matrix whose Gram determinant vanishes in the nonrelativistic threshold limit $q^2, s - s_0 \to 0$. More precisely, the denominators or the vector and tensor box integrals involve a factor of $(4p_0^2 - q^2)$ when expanded in the
nonrelativistic limit. Since \( q^2 = 4p_0^2 \sin^2 \frac{\theta}{2} \) with \( \theta \) the scattering angle, we notice that \( 4p_0^2 > q^2 \) unless we consider backward scattering where \( \theta = \pi \) and where the scattering amplitude diverges. And since \( p_0^2 \) originates from the relativistic structure \( s - s_0 \), we therefore must first expand our vector and tensor box integrals in \( q^2 \) and then in \( s - s_0 \). In this way we find

\[
\begin{align*}
\frac{1}{2} \mathcal{M}_k^{(2)}(q) &= G\alpha Z_a Z_b \left[ \frac{m_a}{1} \bar{u}(p_4) \bar{p}_1 u(p_3) 6 L \right] \\
\frac{1}{2} \mathcal{M}_k^{(2)}(q) &= G\alpha Z_a Z_b \left[ \frac{1}{m_a} \bar{u}(p_4) \bar{p}_1 u(p_3) (-6m_aS - 9L) \right] \\
\frac{1}{2} \mathcal{M}_k^{(2)}(q) &= G\alpha Z_a Z_b \left[ \bar{u}(p_4) u(p_3) (-6m_bS - 4L) \right. \\
&\quad \left. + \frac{1}{m_a} \bar{u}(p_4) \bar{p}_1 u(p_3) (-2m_bS - 2L) \right] \\
\frac{1}{2} \mathcal{M}_k^{(2)}(q) &= G\alpha Z_a Z_b \left[ \bar{u}(p_4) u(p_3) \left( \frac{4m_a m_b}{q^2} L + \frac{2m_a m_b (2m_a + 3m_b)}{s - s_0} S \\+ (3m_a + 8m_b)S + \frac{10m_a^2 + 10m_a m_b - 3m_b^2}{3m_a m_b} L \right) \right. \\
&\quad \left. + \frac{1}{m_a} \bar{u}(p_4) \bar{p}_1 u(p_3) \left( -12m_a m_b \frac{L}{q^2} - \frac{2m_a m_b (2m_a + 3m_b)}{s - s_0} S \\- (7m_a + 10m_b)S \\
&\quad \quad \quad \quad - \frac{44m_a^2 - m_a m_b + 36m_b^2}{6m_a m_b} L \right) \right] \\
&+ i8\pi Gm_a m_b \alpha Z_a Z_b \frac{L}{q^2} \sqrt{\frac{m_a m_b}{s - s_0}} \left( \frac{3}{2} m_a \bar{u}(p_4) \bar{p}_1 u(p_3) - \frac{1}{2} \bar{u}(p_4) u(p_3) \right) \\
\frac{1}{2} \mathcal{M}_k^{(2)}(q) &= G\alpha Z_a Z_b \left[ \bar{u}(p_4) u(p_3) \left( -\frac{4m_a m_b}{q^2} L \right. \\
&\quad \left. - \frac{4m_a - 3m_b}{2} S - \frac{10m_a^2 - 6m_a m_b - 3m_b^2}{3m_a m_b} L \right) \right. \\
&\quad \left. + \frac{1}{m_a} \bar{u}(p_4) \bar{p}_1 u(p_3) \left( \frac{12m_a m_b}{q^2} L + \frac{12m_a + 7m_b}{2} S \right) \right]
\end{align*}
\]
\[ \frac{1}{2} \mathcal{M}^{(2)}_{\gamma} (q) = G \alpha Z_a Z_b \left[ \bar{u}(p_4) u(p_3) m_b S + \frac{1}{m_a} \bar{u}(p_4) \not\! \not \! \! u(p_3) (2m_a + m_b) S \right] \]

\[ \frac{1}{2} \mathcal{M}^{(2)}_{\gamma} (q) = G \alpha \left[ \bar{u}(p_4) u(p_3) \left( \frac{Z_a^2 m_b}{2} S + \frac{4(Z_a^2 m_b^2 - Z_b^2 m_a^2)}{3m_a m_b} \right) + \frac{1}{m_a} \bar{u}(p_4) \not\! \not \! \! u(p_3) \left( - \frac{3Z_a^2 m_b + 2Z_b^2 m_a S}{2} \right) \right. \]

\[ \left. - \frac{4(3Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} L \right] \]

\[ \frac{1}{2} \mathcal{M}^{(2)}_{\text{tot}} (q) = G \alpha Z_a Z_b \left[ \frac{1}{m_a} \bar{u}(p_4) \not\! \not \! \! u(p_3) \left( \frac{4}{3} L \right) \right]. \] (19)

where our spinors are normalized as \( \bar{u}(p) u(p) = 1 \). Summing, we find the total transition amplitude

\[ \frac{1}{2} \mathcal{M}^{(2)}_{\text{tot}} (q) = G \alpha Z_a Z_b \left[ S \left( (m_a + 5m_b) \bar{u}(p_4) u(p_3) - (5m_a + 9m_b) \frac{1}{m_a} \bar{u}(p_4) \not\! \not \! \! u(p_3) \right) ight. \]

\[ + L \left( \frac{4}{3} \bar{u}(p_4) u(p_3) + \frac{32}{3} \frac{1}{m_a} \bar{u}(p_4) \not\! \not \! \! u(p_3) \right) \]

\[ + \frac{2m_a m_b (2m_a + 3m_b) S}{s - s_0} \left( \bar{u}(p_4) u(p_3) - \frac{1}{m_a} \bar{u}(p_4) \not\! \not \! \! u(p_3) \right) \]

\[ + G \alpha \left[ \bar{u}(p_4) u(p_3) \left( \frac{Z_a^2 m_b}{2} S + \frac{4(Z_a^2 m_b^2 - Z_b^2 m_a^2)}{3m_a m_b} \right) \right. \]

\[ + \frac{1}{m_a} \bar{u}(p_4) \not\! \not \! \! u(p_3) \left( - \frac{3Z_a^2 m_b + 2Z_b^2 m_a S}{2} \right) \]

\[ \left. - \frac{4(3Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} L \right] \]

\[ + i8\pi G m_a m_b \alpha Z_a Z_b \frac{L}{q^2} \sqrt{\frac{m_a m_b}{s - s_0}} \left( \frac{3}{2} \frac{1}{m_a} \bar{u}(p_4) \not\! \not \! \! u(p_3) - \frac{1}{2} \bar{u}(p_4) u(p_3) \right). \] (20)
Defining the spin four-vector as

\[ S_b^\mu = \frac{1}{2} \bar{u}(p_4) \gamma_5 \gamma^\mu u(p_3) \]  

(21)

where \( \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 \) and taking \( \epsilon^{0123} = +1 \), we find the identity

\[ \bar{u}(p_4) \gamma_\mu u(p_3) = \left( \frac{1}{1 - \frac{q^2}{4m_b^2}} \right) \left[ (p_3 + p_4)_\mu \bar{u}(p_4)u(p_3) - \frac{i}{m_b^2} \epsilon_{\mu\beta\gamma\delta} q^\beta p_3^\gamma S_b^\delta \right] \]  

(22)

whereby the transition amplitude can be written in the form

\[ \frac{1}{2} M_{\text{tot}}^{(2)}(q) = G\alpha Z_a Z_b \left[ \mathcal{U}_b \left( -6(m_a + m_b)S + 12L \right) \right. \]

\[ + i \frac{\mathcal{E}_b}{m_a m_b^2} \left( \frac{-2m_a m_b (2m_a + 3m_b)}{s - s_0} - 5m_a - \frac{15}{2} m_b \right) S \]

\[ \left. + \frac{32}{3} L \right] \]

\[ + G\alpha \left[ \mathcal{U}_b \left( -(Z_a^2 m_b + Z_b^2 m_a)S - \frac{8(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} L \right) \right. \]

\[ + i \frac{\mathcal{E}_b}{m_a m_b^2} \left( - \frac{3}{2} Z_a^2 m_b + Z_b^2 m_a \right) S - \frac{4(3Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} L \]

\[ \left. + \quad \frac{i8\pi G m_a m_b \alpha Z_a Z_b L}{q^2} \sqrt{\\frac{m_a m_b}{s - s_0}} \left( \mathcal{U}_b + \frac{3}{2} i \frac{\mathcal{E}_b}{m_a m_b^2} \right) \right] \]  

(23)

using the shorthand notation

\[ \mathcal{U}_b \equiv \bar{u}(p_4)u(p_3) \quad \text{and} \quad \mathcal{E}_b \equiv \epsilon_{\alpha\beta\gamma\delta} p_4^\alpha p_3^\beta q^\gamma S_b^\delta. \]  

(24)

This form has the advantage that one can easily read off the spin-independent component of the amplitude (which is the same as the spin-averaged amplitude) by setting \( \mathcal{U}_b \rightarrow 1 \) and \( \mathcal{E}_b \rightarrow 0 \) (see Eqs. (26) and (27) below), and we observe that the spin-independent part of Eq. (23) and the amplitude for spin-0 – spin-0 scattering in Eq. (5) have identical forms.

Before we can define an appropriate second order potential, we must, of course, remove the imaginary parts as done in the spinless scattering case.
above. However, we observe that when one or more particles carries spin a
ew from appears, proportional to \(1/(s-s_0)\), which also must be removed
before we can produce a proper higher order potential. Both forms must be
eliminated by subtraction of the iterated lowest order potential as before.
Before seeing how this is done, we first write the nonrelativistic amplitude in
the symmetric center of mass frame \((\vec{p}_1 = -\vec{p}_3 = \vec{p} + q/2)\) where
\[
S_b^\alpha \xrightarrow{NR} (0, \vec{S}_b) \quad \text{with} \quad \vec{S}_b = \frac{1}{2} \chi^b_i \vec{\sigma} \chi^i_b, \tag{25}
\]
\[
\bar{u}(p_4) u(p_3) \xrightarrow{NR} \chi^b_i \chi^i_b - \frac{i}{2m_b^2} \vec{S}_b \cdot \vec{p} \times \vec{q}, \tag{26}
\]
\[
\epsilon_{\alpha \beta \gamma \delta} p_1^\alpha p_3^\beta q^\gamma S^\delta_b \xrightarrow{NR} (m_a + m_b) \left( 1 + \frac{\vec{p}^2}{2m_a m_b} \right) \vec{S}_b \cdot \vec{p} \times \vec{q} \tag{27}
\]
and
\[
\frac{1}{s-s_0} \xrightarrow{NR} \frac{m_a m_b}{(m_a + m_b)^2} \frac{1}{p_0^2} + \frac{(m_a - m_b)^2}{4m_a m_b (m_a + m_b)^2}. \tag{28}
\]
We find then
\[
\frac{1}{2} \mathcal{M}_{\text{tot}}^{(2)}(q) \simeq \left[ G\alpha Z_a Z_b \left( -6(m_a + m_b)S + 12L \right) + i8\pi Gm_a m_b \alpha Z_a Z_b \frac{L m_r}{q^2 p_0} \right. \\
+ G\alpha \left( -(Z_a^2 m_b + Z_b^2 m_a)S - \frac{8(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} \right) \left. \chi^b_i \chi^i_b \right] \\
+ \left[ G\alpha Z_a Z_b \left( - \frac{3m_a^2 + 13m_b^2 m_a + 18m_a m_b^2 + 9m_b^3}{m_a + m_b} \right) S + \frac{14m_a + 32m_b}{3} \right] \left. \chi^b_i \chi^i_b \right] \\
- \frac{2G m_a m_b \alpha Z_a Z_b (2m_a + 3m_b)}{(m_a + m_b)} \left( -\frac{2\pi L}{p_0 q^2} + \frac{S}{p_0^2} \right) \\
+ \frac{G\alpha}{m_a m_b} \left( - \frac{1}{2} \left( \frac{Z_a^2 (2m_a m_b + 3m_b^2) + Z_b^2 (m_a^2 + 2m_a m_b)}{3m_a} \right) S \right. \\
- \left. \frac{4(Z_a^2 (2m_a m_b + 3m_b^2) + Z_b^2 m_b^3)}{3m_a \alpha Z_a Z_b (2m_a + 3m_b)} \right) \left. \vec{S}_b \cdot \vec{p} \times \vec{q}. \right]
\tag{29}
\]
Again, the spin-independent component—the piece proportional to \(\chi^b_i \chi^i_b\)—agrees with the corresponding expression for spinless scattering in Eq. \(\text{[5]}\).
The spin-dependent component is new and has the form of a spin-orbit coupling.

In order to remove the imaginary—\(iL/(q^2p_0)\)—and real—\(S/p_0^2\)—pieces, we iterate the lowest order potential, as done in the spinless case, but we note that in the case of a system with spin these lowest order potentials have, in addition to the usual spin-independent Newton/Coulomb pieces, a spin-dependent component which is of spin-orbit character. We refer the reader to [6] and [7] for the explicit derivation of the leading order potentials for spin-0 – spin-1/2 scattering. These lowest order potentials split into spin-independent \((S - I)\) and spin-dependent \((S - O)\) components—

\[
\begin{align*}
\langle \vec{p}_f | \frac{1}{2} \hat{V}^{(1)}_{C,S-I} | \vec{p}_i \rangle &= \langle \vec{p}_f | \frac{1}{2} \hat{V}^{(1)}_{G,S-I} | \vec{p}_i \rangle + \langle \vec{p}_f | \frac{1}{2} \hat{V}^{(1)}_{C,S-O} | \vec{p}_i \rangle \\
\langle \vec{p}_f | \frac{1}{2} \hat{V}^{(1)}_{G,S-O} | \vec{p}_i \rangle &= \langle \vec{p}_f | \frac{1}{2} \hat{V}^{(1)}_{G,S-I} | \vec{p}_i \rangle + \langle \vec{p}_f | \frac{1}{2} \hat{V}^{(1)}_{G,S-O} | \vec{p}_i \rangle 
\end{align*}
\]

where

\[
\begin{align*}
\langle \vec{p}_f | \frac{1}{2} \hat{V}^{(1)}_{C,S-I} | \vec{p}_i \rangle &= \frac{c_2^2}{q^2} \chi_f \chi_i = \frac{c_2^2}{(\vec{p}_i - \vec{p}_f)^2} \chi_f \chi_i \\
\langle \vec{p}_f | \frac{1}{2} \hat{V}^{(1)}_{C,S-O} | \vec{p}_i \rangle &= \frac{c_2^2}{q^2} \chi_f \chi_i = \frac{c_2^2}{(\vec{p}_i - \vec{p}_f)^2} \chi_f \chi_i \\
\langle \vec{p}_f | \frac{1}{2} \hat{V}^{(1)}_{G,S-I} | \vec{p}_i \rangle &= \frac{c_2^2}{q^2} \chi_f \chi_i = \frac{c_2^2}{(\vec{p}_i - \vec{p}_f)^2} \chi_f \chi_i \\
\langle \vec{p}_f | \frac{1}{2} \hat{V}^{(1)}_{G,S-O} | \vec{p}_i \rangle &= \frac{c_2^2}{q^2} \chi_f \chi_i = \frac{c_2^2}{(\vec{p}_i - \vec{p}_f)^2} \chi_f \chi_i \\
\end{align*}
\]

with \(c_2^2 \equiv 4\pi\alpha Z_a Z_b\) and \(c_2^2 \equiv -4\pi G m_a m_b\). We find then that the iterated amplitude splits also into spin-independent and spin-dependent pieces. The leading spin-independent iteration amplitude is

\[
\frac{1}{2} \text{Amp}^{(2)}_{S-I}(\vec{q}) = -\int \frac{d^3\ell}{(2\pi)^3} \left( \langle \vec{p}_f | \frac{1}{2} \hat{V}^{(1)}_{G,S-I} | \ell \rangle \langle \ell | \frac{1}{2} \hat{V}^{(1)}_{C,S-I} | \vec{p}_i \rangle \right) \frac{\vec{p}_f^2 - \ell^2 + i\epsilon}{2m_r}
\]

13
while the leading spin-dependent term is
\[ \lambda \rightarrow 0 \quad \chi_f \chi_i^b 2H = i 8 \pi G m_a m_b \alpha Z_a Z_b \frac{L}{q^2} \frac{m_r}{p_0} \chi_f \chi_i \quad (32) \]

while the leading spin-dependent term is
\[ \lambda \rightarrow 0 \quad \chi_f \chi_i^b 2H = i 8 \pi G m_a m_b \alpha Z_a Z_b \frac{L}{q^2} \frac{m_r}{p_0} \chi_f \chi_i \quad (32) \]
\[
\int \frac{d^3 \ell}{(2\pi)^3} \frac{c_G}{2} \, \frac{1}{2} (\vec{\ell} + \vec{p}_f) \times (\vec{\ell} - \vec{p}_f) \quad \frac{i}{\ell^2} - \frac{i}{2m_c} + i\epsilon |\vec{p}_i - \vec{\ell}|^2 + \lambda^2 \left( \frac{c_G}{2} \right)^2 (\vec{\ell} + \vec{p}_f) \times (\vec{\ell} - \vec{p}_f) \quad |\vec{\ell} - \vec{p}_f| \right)
\]

\[
\lambda \to 0 \quad \lambda \to 0 \quad \frac{i(4m_a + 6m_b)}{2m_a m_b^2} \vec{S}_b \cdot \vec{H} \times \vec{q}
\]

\[
= -2Gm_{a}m_{b}\alpha Z_{a}Z_{b}(2m_{a} + 3m_{b}) \left( \frac{i}{2\pi \rho_0 q^2 + \frac{S^2}{p_0^2}} \frac{i}{m_b} \vec{S}_b \cdot \vec{p} \times \vec{q} \right)
\]

\[
(33)
\]

(In principle one has to also include the iteration with two spin-orbit components of the leading potentials. However, this procedure but yields terms of higher order in \( p_0^2 \).)

We observe then that the second order Born term for spin-0 – spin-1/2 scattering is precisely of the form required to remove the offending imaginary \( iL/(q^2 p_0) \) and real \( S/p_0^2 \) pieces from the second order scattering amplitude in Eq. (29). What remains can be Fourier-transformed to yield a well-defined second order potential of the form

\[
\frac{1}{2} V^{(2)}_{CG}(\vec{r}) = -\int \frac{d^3q}{(2\pi)^3} \rho_0^{-1} \left[ \frac{1}{2} M^{(2)}_{\text{tot}}(\vec{q}) - \frac{1}{2} \text{Amp}^{(2)}_{CG}(\vec{q}) \right] \]

\[
\approx G\alpha \left[ \frac{1}{2} \left( Z_a^2 m_b + Z_b^2 m_a \right) \right. \quad \frac{3Z_a Z_b (m_a + m_b)}{r^2} \]

\[
- \frac{4\hbar}{3\pi r^3} \left( Z_b^2 \frac{m_a}{m_b} + Z_a^2 \frac{m_b}{m_a} \right) + \frac{6Z_a Z_b \hbar}{\pi r^3} \chi_i \chi_i^b^b
\]

\[
+ \left[ \frac{Z_a Z_b G\alpha (3m_a^3 + 13m_a^2 m_b + 18m_a m_b^2 + 9m_b^3)}{m_a m_b^2 (m_a + m_b) r^4} \right.
\]

\[
- \frac{G\alpha (Z_a^2 (2m_a m_b + 3m_b^2) + Z_b^2 (m_a^2 + 2m_a m_b))}{2m_a m_b^2 r^4}
\]

\[
- \frac{Z_a Z_b G\alpha \hbar (7m_a + 16m_b)}{\pi m_a m_b^2 r^5}
\]

\[
+ \left. \frac{2G\alpha \hbar (Z_a^2 (2m_a m_b + 3m_b^2) + Z_b^2 m_b^2)}{\pi m_a^2 m_b^5 r^5} \right] \vec{L} \cdot \vec{S}_b.
\]

\[
(34)
\]

where \( \vec{L} \equiv \vec{r} \times \vec{p} \) is the angular momentum and \( \vec{r} \equiv \vec{r}_a - \vec{r}_b \). Comparing with the second order potential found in the spinless case—Eq. (17)—we observe
that the \textit{spin-independent} component of the potential $\frac{1}{2}V_{CG}^{(2)}(\vec{r})$ (i.e., the piece proportional to $\chi_j^b \chi_i^b$) is \textit{identical} to the potential $0V_{CG}^{(2)}(\vec{r})$ found in the case of spinless scattering. This is the universality property that we had expected and verifies the correctness of our calculation. However, there also exists now a spin-orbit component of the potential having a new kinematic form, which we suspect is also universal. In order to check this assumption, we evaluate the case of a spin-1/2 particle $a$ scattering from a spin-1/2 particle $b$, the spin-independent piece of which was calculated by Butt [2]. Before leaving this section we note that all results were deriving using a $g$-factor of $g = 2$ at tree level throughout for the photon couplings, which is the natural value for spin-1/2 particle arising from the Dirac Lagrangian [10]. In particular, we observe that the form of the spin-orbit potential does depend on this choice. See [6] for a more detailed discussion on the $g$-dependence of spin-dependent components in the case of purely electromagnetic scattering.

\section{Spin-1/2 – Spin-1/2 Scattering}

As our next calculation, we evaluate the scattering of a pair of spin-1/2 particles. The spin-averaged version of this calculation was performed by Butt and was claimed not to obey universality [2]. This is one reason that we wish to examine this system in detail.

As before we list the result of the calculation and refer to the appendices for the Feynman rules and the relevant loop integrals. Using the spin-1/2 identity Eq. (22) and its pendant for particle $a$

\begin{equation}
\bar{u}(p_2)\gamma_\mu u(p_1) = \left(\frac{1}{1 - \frac{q^2}{4m_a^2}}\right) \left[\frac{(p_1 + p_2)_\mu}{2m_a} \bar{u}(p_2)u(p_1) + \frac{i}{m_a^2} \epsilon_{\mu\beta\gamma\delta} q^\beta p_1^\gamma S_a^\delta\right]
\end{equation}

where we have defined the spin vector

$S_a^\mu = \frac{1}{2} \bar{u}(p_2)\gamma_5 \gamma^\mu u(p_1)$

we present our results, as before, diagram by diagram:

$\frac{1}{2^2} M_{\Pi}^{(2)}(q) = 0$

$\frac{1}{2^2} M_{\Pi}^{(2)}(q) = G\alpha Z_a Z_b \left[U_a U_b (-3L - 6m_a S)\right]$
\[ \frac{1}{\pi} \mathcal{M}_{a}^{(2)}(q) = G \alpha Z_a Z_b \left[ U_a U_b \left( -3L - 6m_b S \right) + i \frac{\mathcal{E}_a U_b}{m_a m_b} \left( -3L - 6m_a S \right) + i \frac{\mathcal{E}_b U_a}{m_a m_b} \left( -3L - 6m_b S \right) \right] \]

\[ \frac{1}{\pi} \mathcal{M}_{b}^{(2)}(q) = G \alpha Z_a Z_b \left[ U_a U_b \left( -3L - 6m_b S \right) + i \frac{\mathcal{E}_a U_b}{m_a m_b} \left( -3L - 6m_a S \right) + i \frac{\mathcal{E}_b U_a}{m_a m_b} \left( -3L - 6m_b S \right) \right] \]

\[
\frac{1}{\pi} \mathcal{M}_{c}^{(2)}(q) = G \alpha Z_a Z_b \left[ U_a U_b \left( \frac{-8m_a m_b}{q^2} - \frac{21m_a^2 + 3m_a m_b + 21m_b^2}{3m_a m_b} \right) \\
+ S \left( -6(m_a + m_b) \right) \right] \\
+ i \frac{\mathcal{E}_a U_b}{m_a m_b} \left( \frac{-8m_a m_b}{q^2} - \frac{21m_a^2 + 3m_a m_b + 21m_b^2}{3m_a m_b} \right) \\
+ S \left( \frac{4m_a + 4m_b}{s - s_0} - \left( \frac{10m_a + 7m_b}{s - s_0} \right) \right) \\
+ i \frac{\mathcal{E}_b U_a}{m_a m_b} \left( \frac{-8m_a m_b}{q^2} - \frac{21m_a^2 + 3m_a m_b + 21m_b^2}{3m_a m_b} \right) \\
+ S \left( \frac{-2m_a m_b(2m_a + 3m_b)}{s - s_0} - \left( \frac{7m_a + 10m_b}{s - s_0} \right) \right) \\
+ S \left( \frac{2m_a m_b(2m_a + 3m_b)}{s - s_0} - \left( \frac{7m_a + 10m_b}{s - s_0} \right) \right) \\
+ S \left( \frac{2m_a m_b(2m_a + 3m_b)}{s - s_0} - \left( \frac{7m_a + 10m_b}{s - s_0} \right) \right) \\
+ S \left( \frac{-4m_a m_b(m_a + m_b)}{s - s_0} - \frac{43}{4} (m_a + m_b) \right) \\
+ i \frac{\mathcal{E}_a U_b}{m_a m_b} \left( \frac{-8m_a m_b}{q^2} - \frac{21m_a^2 + 3m_a m_b + 21m_b^2}{3m_a m_b} \right) \\
+ S \left( \frac{-2m_a m_b(2m_a + 3m_b)}{s - s_0} - \left( \frac{7m_a + 10m_b}{s - s_0} \right) \right) \\
+ S \left( \frac{-2m_a m_b(2m_a + 3m_b)}{s - s_0} - \left( \frac{7m_a + 10m_b}{s - s_0} \right) \right) \\
+ S \left( \frac{-4m_a m_b(m_a + m_b)}{s - s_0} - \frac{43}{4} (m_a + m_b) \right) \\
+ \frac{q^2 S_a \cdot S_b}{m_a m_b} \left( \frac{-4m_a m_b}{q^2} - \frac{37m_a^2 + 62m_a m_b + 37m_b^2}{6m_a m_b} \right) \\
\]
\[ +S \left( -\frac{2m_a m_b (m_a + m_b)}{s - s_0} - \frac{41}{4} (m_a + m_b) \right) \]

\[ + \left( 2S_a \cdot p_3 S_b \cdot p_1 + S_a \cdot q S_b \cdot p_1 - S_a \cdot p_3 S_b \cdot q \right) \left( -\frac{26L}{3m_a m_b} \right) \]

\[ + i 8 \pi G m_a m_b \alpha Z_a Z_b \frac{L}{q^2} \sqrt{\frac{m_a m_b}{s - s_0}} \left( U_a U_b + \frac{3i}{2} \frac{\mathcal{E}_a U_b}{m_a^2 m_b} + \frac{3i}{2} \frac{U_a \mathcal{E}_b}{m_a m_b^2} \right) \]

\[ + \frac{S_a \cdot q S_b \cdot q - \frac{1}{2} q^2 S_a \cdot S_b}{m_a m_b} \]

\[ \frac{1}{2} \mathcal{M}_1^{(2)}(q) = G \alpha Z_a Z_b \left[ \sum \left( \frac{8m_a m_b}{q^2} + \frac{21m_a^2 + 53m_a m_b + 21m_b^2}{3m_a m_b} \right) \right. \]

\[ + S 4(m_a + m_b) \]

\[ + i \frac{\mathcal{E}_a U_b}{m_a^2 m_b} \left( L \left( \frac{12m_a m_b}{q^2} + \frac{27m_a^2 + 38m_a m_b + 31m_b^2}{3m_a m_b} \right) \right. \]

\[ + S \left( \frac{7}{2} m_a + 6m_b \right) \]

\[ + i \frac{U_a \mathcal{E}_b}{m_a m_b^2} \left( L \left( \frac{12m_a m_b}{q^2} + \frac{31m_a^2 + 38m_a m_b + 27m_b^2}{3m_a m_b} \right) \right. \]

\[ + S \left( 6m_a + \frac{7}{2} m_b \right) \]

\[ + \frac{S_a \cdot q S_b \cdot q}{m_a m_b} \left( L \left( \frac{8m_a m_b}{q^2} + \frac{20m_a^2 + 12m_a m_b + 20m_b^2}{3m_a m_b} \right) \right. \]

\[ + S \left( \frac{9}{4} (m_a + m_b) \right) \]

\[ \left. - \frac{q^2 S_a \cdot S_b}{m_a m_b} \left( L \left( \frac{4m_a m_b}{q^2} + \frac{37m_a^2 + 42m_a m_b + 37m_b^2}{6m_a m_b} \right) \right) \right] \]
\[ i \mathcal{M}^{(2)}(p) = G \alpha Z_a Z_b \left[ U_a U_b \left( 2(m_a + m_b)S \right) \right. \\
+ \left. \frac{i \mathcal{E}_a U_b}{m_a^2 m_b} \left( (m_a + 2m_b)S \right) + \frac{i \mathcal{E}_b U_a}{m_a m_b^2} \left( (2m_a + m_b)S \right) \right. \\
+ \left. \frac{S_a \cdot q S_b \cdot q}{m_a m_b} (m_a + m_b) S - \frac{q^2 S_a \cdot S_b}{m_a m_b} (m_a + m_b) S \right] \]

\[ i \mathcal{M}^{(2)}(p) = G \alpha \left[ U_a U_b \left( -\frac{8(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} L - \left( Z_a^2 m_b + Z_b^2 m_a \right) S \right) \right. \\
+ \left. \frac{i \mathcal{E}_a U_b}{m_a^2 m_b} \left( -\frac{4(Z_a^2 m_b^2 + 3Z_b^2 m_a^2)}{3m_a m_b} - \left( Z_a^2 m_b + \frac{3}{2} Z_b^2 m_a \right) S \right) \right. \\
+ \left. \frac{i \mathcal{E}_b U_a}{m_a m_b^2} \left( -\frac{4(3Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} - \left( \frac{3}{2} Z_a^2 m_b + \frac{3}{2} Z_b^2 m_a \right) S \right) \right. \\
+ \left. \frac{S_a \cdot q S_b \cdot q}{m_a m_b} \left( -\frac{2(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} L - \frac{1}{2} \left( Z_a^2 m_b + Z_b^2 m_a \right) S \right) \right. \\
- \left. \frac{3m_a m_b}{3m_a m_b} \left( -\frac{2(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} L - \frac{1}{2} \left( Z_a^2 m_b + Z_b^2 m_a \right) S \right) \right] \]

\[ i \mathcal{M}^{(2)}(p) = G \alpha \left[ U_a U_b \left( \frac{4}{3} L \right) + \frac{i \mathcal{E}_a U_b}{m_a^2 m_b} \left( \frac{4}{3} L \right) + \frac{i \mathcal{E}_b U_a}{m_a m_b^2} \left( \frac{4}{3} L \right) \right. \\
+ \left. \frac{S_a \cdot q S_b \cdot q}{m_a m_b} \left( \frac{4}{3} L \right) - \frac{q^2 S_a \cdot S_b}{m_a m_b} \left( \frac{4}{3} L \right) \right] \]  

(36)

where we have defined

\[ U_a = \bar{u}(p_2) u(p_1) \quad U_b = \bar{u}(p_4) u(p_3) \]  

(37)

and

\[ \mathcal{E}_i = \epsilon_{\alpha \beta \gamma \delta} p^\alpha_1 p^\beta_3 q^\gamma S^\delta \]  

(38)

with \( i = a, b \) to keep our notation compact.
At this point we can compare with Butt’s spin-averaged result \[2\] by examining the spin-independent terms, which can be read off easily from our results by taking the limit \( \mathcal{U}_{a,b} \rightarrow 1, \mathcal{E}_{a,b} \rightarrow 0 \) and \( S_a \cdot q S_b \cdot q, S_a \cdot S_b \rightarrow 0 \). We do not concur with many of Butt’s diagrams and do not understand the reason for this disagreement. Undaunted, we sum the results of the above individual contributions and determine the total amplitude

\[
\frac{1}{2} \mathcal{M}_{tot}^{(2)}(q) = G\alpha Z_a Z_b \left[ \mathcal{U}_a \mathcal{U}_b \left( -6(m_a + m_b)S + 12L \right) + i \frac{\mathcal{E}_a \mathcal{U}_b}{m_a^2 m_b} \left( \frac{2m_a m_b (3m_a + 2m_b)}{s - s_0} - \frac{15}{2} \frac{m_a}{m_b} S + \frac{32}{3} \frac{L}{m_a m_b} \right) + \frac{\mathcal{U}_a \mathcal{E}_b}{m_a m_b} \left( \frac{2m_a m_b (2m_a + 3m_b)}{s - s_0} - 5m_a - \frac{15}{2} \frac{m_b}{m_a} S + \frac{32}{3} \frac{L}{m_a m_b} \right) + \frac{S_a \cdot q S_b \cdot q - q^2 S_a \cdot S_b}{m_a m_b} \left( -\frac{2m_a m_b}{s - s_0} - \frac{19}{2} \right) (m_a + m_b) S + \frac{S_a \cdot q S_b \cdot q}{m_a m_b} \frac{4}{3} \frac{L}{m_a m_b} \right] + G\alpha \left[ \mathcal{U}_a \mathcal{U}_b \left( - (Z_a^2 m_b + Z_b^2 m_a) S - \frac{8(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} S - \frac{4(Z_a^2 m_b^2 + 3Z_b^2 m_a^2)}{3m_a m_b} L \right) + i \frac{\mathcal{E}_a \mathcal{U}_b}{m_a m_b} \left( - \left( Z_a^2 m_b + \frac{3}{2} Z_b^2 m_a \right) S - \frac{4(Z_a^2 m_b^2 + 3Z_b^2 m_a^2)}{3m_a m_b} \right) + \frac{\mathcal{U}_a \mathcal{E}_b}{m_a m_b} \left( - \left( \frac{3}{2} Z_a^2 m_b + Z_b^2 m_a \right) S - \frac{4(3Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} \right) + \frac{S_a \cdot q S_b \cdot q - q^2 S_a \cdot S_b}{m_a m_b} \left( \frac{1}{2}(Z_a^2 m_b + Z_b^2 m_a) S - \frac{2(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} \right) \right] + i 8\pi G m_a m_b \alpha Z_a Z_b \frac{L \sqrt{m_a m_b}}{s - s_0} \left( \mathcal{U}_a \mathcal{U}_b + \frac{3}{2} \frac{\mathcal{U}_a \mathcal{E}_b}{m_a m_b} + \frac{3}{2} i \frac{\mathcal{E}_a \mathcal{U}_b}{m_a^2 m_b} \right) + \frac{S_a \cdot q S_b \cdot q - \frac{1}{2} q^2 S_a \cdot S_b}{m_a m_b} \right]. \tag{39}
\]

Finally, we take the nonrelativistic limit as before, yielding

\[
\frac{1}{2} \mathcal{M}_{tot}^{(2)}(\vec{q}) \approx \left[ G\alpha Z_a Z_b \left( -6(m_a + m_b)S + 12L \right) + i 8\pi G m_a m_b \alpha Z_a Z_b \frac{L}{q^2} \frac{m_r}{m_0} \right].
\]
\[\begin{align*}
+ G\alpha \left( -(Z_a^2 m_b + Z_b^2 m_a)S - \frac{8(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3 m_a m_b} L \right) \chi_f^{a+} \chi^{a \dagger} \chi_f^{b+} \\
+ \frac{G\alpha Z_a Z_b}{m_a m_b} \left( - \frac{9 m_a^2 + 18 m_a^2 m_b + 13 m_a^2 m_b^2 + 3 m_b^2}{m_a + m_b} S + \frac{32 m_a + 14 m_b}{3} L \right) \frac{2 G m_a m_b \alpha Z_a Z_b (3 m_a + 2 m_b)}{(m_a + m_b)} \left( - \frac{2 \pi L}{p_0 q^2} + \frac{S}{p_0^2} \right) \\
+ \frac{G\alpha m_a m_b}{m_a m_b} \left( - \frac{1}{2} \left( Z_a^2 (2 m_a m_b + m_b^2) + Z_b^2 (3 m_a^2 + 2 m_a m_b) \right) S \\
- \frac{4(Z_a^2 m_b^2 + Z_b^2 (3 m_a^2 + 2 m_a m_b))}{3 m_b} L \right) \frac{i}{m_a} \vec{S_a} \cdot \vec{p} \times \vec{q} \chi_f^{b+} \\
+ \frac{G\alpha Z_a Z_b}{m_a m_b} \left( - \frac{3 m_a^2 + 13 m_a^2 m_b + 18 m_a^2 m_b^2 + 9 m_b^2}{m_a + m_b} S + \frac{14 m_a + 32 m_b}{3} L \right) \frac{2 G m_a m_b \alpha Z_a Z_b (2 m_a + 3 m_b)}{(m_a + m_b)} \left( - \frac{2 \pi L}{p_0 q^2} + \frac{S}{p_0^2} \right) \\
+ \frac{G\alpha m_a m_b}{m_a m_b} \left( - \frac{1}{2} \left( Z_a^2 (2 m_a m_b + 3 m_b^2) + Z_b^2 (m_a^2 + 2 m_a m_b) \right) S \\
- \frac{4(Z_a^2 (2 m_a m_b + 3 m_b^2) + Z_b^2 m_a^2)}{3 m_a} L \right) \chi_f^{a+} \chi^{a \dagger} \frac{i}{m_b} \vec{S_b} \cdot \vec{p} \times \vec{q} \\
- G\alpha Z_a Z_b \frac{2(5 m_a^2 + 9 m_a m_b + 5 m_b^2)}{m_a + m_b} S \frac{\vec{S_a} \cdot \vec{q} \vec{S_b} \cdot \vec{q} - \vec{q}^2 \vec{S_a} \cdot \vec{S_b}}{m_a m_b} \\
+ G\alpha Z_a Z_b \frac{4}{3} L \frac{\vec{S_a} \cdot \vec{q} \vec{S_b} \cdot \vec{q}}{m_a m_b} \\
- \frac{2 G m_a^2 m_b^2 \alpha Z_a Z_b}{m_a + m_b} S \frac{\vec{S_a} \cdot \vec{q} \vec{S_b} \cdot \vec{q} - \vec{q}^2 \vec{S_a} \cdot \vec{S_b}}{p_0} \\
- \frac{2 G m_a^2 m_b^2 \alpha Z_a Z_b}{m_a + m_b} \left( - \frac{4 \pi L}{p_0 q^2} \right) \frac{\vec{S_a} \cdot \vec{q} \vec{S_b} \cdot \vec{q} - \frac{1}{2} \vec{q}^2 \vec{S_a} \cdot \vec{S_b}}{m_a m_b} \\
+ G\alpha \left( - \frac{1}{2} (Z_a^2 m_b + Z_b^2 m_a) S \\
- \frac{2(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3 m_a m_b} L \right) \frac{\vec{S_a} \cdot \vec{q} \vec{S_b} \cdot \vec{q} - \vec{q}^2 \vec{S_a} \cdot \vec{S_b}}{m_a m_b}.\end{align*}\]
The amplitude for spin-1/2 – spin-1/2 scattering now involves a spin independent component, two spin-orbit coupling pieces (one for each spin) which are symmetric under $a \leftrightarrow b$ and new spin-spin coupling terms. Comparing with the corresponding expression for the spin-0 – spin-1/2 scattering amplitude—Eq. (29)—we observe that the kinematic forms multiplying both the spin-independent and spin-orbit terms are identical. This is the universality which we expected and confirms the correctness of the individual diagram calculations.

Of course, before we can generate a proper potential we must, as before, subtract off the imaginary $iL/(q^2p_0)$ and the real $S/p_0^2$ pieces by iterating the lowest order electromagnetic and gravitational potentials. In the electromagnetic case the leading order potential for spin-1/2 – spin-1/2 scattering reads

$$\langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{C}^{(1)} \mid \tilde{p}_i \rangle = \langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{C,S-I}^{(1)} \mid \tilde{p}_i \rangle + \langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{C,S-O}^{(1)} \mid \tilde{p}_i \rangle + \langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{C,S-S}^{(1)} \mid \tilde{p}_i \rangle$$

(41)

with the components

$$\langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{C,S-I}^{(1)} \mid \tilde{p}_i \rangle = \frac{c^2}{q^2} \chi_f^{a} \chi_i^{b}$$

$$\langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{C,S-O}^{(1)} \mid \tilde{p}_i \rangle = \frac{c^2}{q^2} \frac{2m_a + m_b}{2m_a m_b} \frac{i}{m_a} \vec{s}_a \cdot \vec{p} \times \vec{q} \chi_f^{b} \chi_i^{b}$$

$$+ \frac{c^2}{q^2} \frac{m_a + 2m_b}{2m_a m_b} \chi_f^{a} \chi_i^{a} \frac{i}{m_b} \vec{s}_b \cdot \vec{p} \times \vec{q}$$

$$\langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{C,S-S}^{(1)} \mid \tilde{p}_i \rangle = \frac{c^2}{q^2} \frac{1}{m_a m_b} \vec{s}_a \cdot \vec{q} \vec{s}_b \cdot \vec{q}$$

(42)

and where $q = p_i - p_f$ and $p = \frac{1}{2}(p_i + p_f)$ when identifying $\tilde{p}_i$ with $\tilde{p}_1$ and $\tilde{p}_f$ with $\tilde{p}_2$. The leading order gravitational potential obtained in [7] is

$$\langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{G}^{(1)} \mid \tilde{p}_i \rangle = \langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{G,S-I}^{(1)} \mid \tilde{p}_i \rangle + \langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{G,S-O}^{(1)} \mid \tilde{p}_i \rangle + \langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{G,S-S}^{(1)} \mid \tilde{p}_i \rangle$$

(43)

with the components

$$\langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{G,S-I}^{(1)} \mid \tilde{p}_i \rangle = \frac{c^2}{q^2} \chi_f^{a} \chi_i^{b}$$

$$\langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{G,S-O}^{(1)} \mid \tilde{p}_i \rangle = \frac{c^2}{q^2} \chi_f^{a} \chi_i^{b}$$

$$\langle \tilde{p}_f \mid \frac{1}{2} \hat{V}_{G,S-S}^{(1)} \mid \tilde{p}_i \rangle = \frac{c^2}{q^2} \chi_f^{a} \chi_i^{b}$$
relevant for our calculation.) The leading spin-spin second Born term is then from the product of spin-orbit components of the lowest order gravitational and electromagnetic potentials, but this piece is higher order in \( iL/\epsilon \). evaulated the calculation of these pieces will go through as before and that by universality. This means that when the second order Born amplitude is to the results found in the case of spin-0 – spin-1/2 scattering, as required independent and spin-orbit pieces of the lowest order potential are identical in the second Born amplitude when iterated together with the leading spin-spin component and it is this piece which will generate a spin-spin term in the second Born amplitude when iterated together with the leading spin-independent piece of \( V_{G,S}^{(1)}(\vec{r}) \). (Note that a spin-spin component will also arise from the product of spin-orbit components of the lowest order gravitational and electromagnetic potentials, but this piece is higher order in \( p_0^2 \) and is not relevant for our calculation.) The leading spin-spin second Born term is then

\[
\left\langle \vec{p}_f \left| \frac{1}{2} \hat{V}_{G,S-S}^{(1)} \right| \vec{p}_i \right\rangle = c_G^2 \frac{4m_a + 3m_b}{2m_a m_b} \frac{i}{m_a} \hat{S}_a \cdot \vec{p} \times \vec{q} \chi_f \chi_i^b \\
+ c_G^2 \frac{3m_a + 4m_b}{2m_a m_b} \chi_f^a \chi_i \frac{i}{m_b} \hat{S}_b \cdot \vec{p} \times \vec{q}
\]

\[
\left\langle \vec{p}_f \left| \frac{1}{2} \hat{V}_{G,S-S}^{(1)} \right| \vec{p}_i \right\rangle = \frac{c_G^2}{2^6 m_a m_b} \hat{S}_a \cdot \vec{q} \hat{S}_b \cdot \vec{q}.
\]

Note that in both the electromagnetic and gravitational cases the spin-independent and spin-orbit pieces of the lowest order potential are identical to the results found in the case of spin-0 – spin-1/2 scattering, as required by universality. This means that when the second order Born amplitude is evaluated the calculation of these pieces will go through as before and that the offending \( iL/(q^2 p_0) \) and \( S/p_0^2 \) pieces will be removed just as in the spin-0 – spin-1/2 case. However, both lowest order potentials also contain a new spin-spin component and it is this piece which will generate a spin-spin term in the second Born amplitude when iterated together with the leading spin-independent piece of \( V_{G,S}^{(1)}(\vec{r}) \). (Note that a spin-spin component will also arise from the product of spin-orbit components of the lowest order gravitational and electromagnetic potentials, but this piece is higher order in \( p_0^2 \) and is not relevant for our calculation.) The leading spin-spin second Born term is then

\[
\frac{1}{2} \text{Amp}^{(2)}_{S-S}(\vec{q}) = - \int \frac{d^3 \ell}{(2\pi)^3} \left\langle \vec{p}_f \left| \frac{1}{2} \hat{V}_{G,S-I}^{(1)} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| \frac{1}{2} \hat{V}_{C,S-S}^{(1)} \right| \vec{p}_i \right\rangle \\
- \int \frac{d^3 \ell}{(2\pi)^3} \left\langle \vec{p}_f \left| \frac{1}{2} \hat{V}_{C,S-S}^{(1)} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| \frac{1}{2} \hat{V}_{G,S-I}^{(1)} \right| \vec{p}_i \right\rangle \\
- \int \frac{d^3 \ell}{(2\pi)^3} \left\langle \vec{p}_f \left| \frac{1}{2} \hat{V}_{G,S-I}^{(1)} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| \frac{1}{2} \hat{V}_{C,S-S}^{(1)} \right| \vec{p}_i \right\rangle \\
- \int \frac{d^3 \ell}{(2\pi)^3} \left\langle \vec{p}_f \left| \frac{1}{2} \hat{V}_{C,S-S}^{(1)} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| \frac{1}{2} \hat{V}_{G,S-I}^{(1)} \right| \vec{p}_i \right\rangle \\
= \frac{1}{m_a m_b} \hat{S}_a \cdot \hat{S}_b
\]

\[
\left( i \int \frac{d^3 \ell}{(2\pi)^3} \frac{c_G^2}{\ell^2 + \lambda^2} \frac{i}{\ell^2 + \lambda^2} \hat{p}_i \cdot \ell + \hat{p}_i \cdot \ell \right)^s
\]
\[
\begin{align*}
\lambda - 0 & \frac{2}{m_a m_b} \left[ \left( \vec{S}_a \cdot \vec{p}_i \vec{S}_b \cdot \vec{p}_f + \vec{S}_a \cdot \vec{p}_f \vec{S}_b \cdot \vec{p}_i \right) H \\
& \quad - \vec{S}_a \cdot (\vec{p}_i + p_f) \vec{S}_b \cdot \vec{H} - \vec{S}_a \cdot \vec{H} \vec{S}_b \cdot (\vec{p}_i + p_f) \\
& \quad + 2 S^a_s S^b_s H^{rs} \right] \\
& = - \frac{2Gm_a^2 m_b^2 \alpha Z_a Z_b}{m_a + m_b} \frac{S}{p_0^2} \frac{m_a m_b}{m_a m_b} \\
& \quad - \frac{2Gm_a^2 m_b^2 \alpha Z_a Z_b}{m_a + m_b} \left( -i \frac{4\pi L}{p_0 q^2} \right) \frac{S}{m_a} \cdot \frac{q}{q} \vec{S}_b \cdot \vec{q} - \frac{1}{2} q^2 \vec{S}_a \cdot \vec{S}_b \\
& \quad \left( 1 - \frac{Z_a Z_b}{3\pi r^3} \left( \frac{Z_a^2 m_a}{m_b} + \frac{Z_b^2 m_b}{m_a} \right) + \frac{6Z_a Z_b h}{\pi r^3} \right) \chi_f^a \chi_i^a \chi_f^b \chi_i^b \\
& \quad \left( - \frac{Z_a Z_b G \alpha (9m_a^3 + 18m_a^2 m_b + 13m_a m_b^2 + 3m_b^3)}{m_a^2 m_b (m_a + m_b) r^4} \\
& \quad - \frac{G \alpha (Z_a^3 (2m_a m_b + m_b^2) + Z_b^3 (3m_a^2 + 2m_a m_b))}{2m_a^2 m_b r^4} \right).
\end{align*}
\]

Comparing with the second order scattering amplitude Eq. (40) we see that the iterated result Eq. (45) is of precisely the right form to eliminate the \(iL/(q^2 p_0)\) and \(S/p_0^2\) pieces of the spin-spin correlation. The resulting second order potential is then well-defined and is of the form

\[\frac{1}{2} V^{(2)}_{CG}(\vec{r}) = - \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q} \cdot \vec{r}} \left[ \frac{1}{2} M^{(2)}_{tot}(\vec{q}) - \frac{1}{2} \text{Amp}_{CG}^{(2)}(\vec{q}) \right] \]

\[\approx G \alpha \left[ \frac{1}{2} \left( Z_a^2 m_b^2 + Z_b^2 m_a^2 + 3Z_a Z_b (m_a + m_b) \right) \right. \]

\[\left. - \frac{4\hbar}{3\pi r^3} \left( Z_a^2 m_a + Z_b^2 m_b \right) + \frac{6Z_a Z_b h}{\pi r^3} \right] \chi_f^a \chi_i^a \chi_f^b \chi_i^b \\
\left[ - \frac{Z_a Z_b G \alpha (9m_a^3 + 18m_a^2 m_b + 13m_a m_b^2 + 3m_b^3)}{m_a^2 m_b (m_a + m_b) r^4} \\
- \frac{G \alpha (Z_a^3 (2m_a m_b + m_b^2) + Z_b^3 (3m_a^2 + 2m_a m_b))}{2m_a^2 m_b r^4} \right].\]
Comparing with the earlier result found in Eq. (34) for spin-0 – spin-1/2 scattering, we confirm universality for the spin-independent and spin-orbit components together with a new spin-spin piece which itself is presumably universal.

5 Spin-0 – Spin-1 Scattering

Before closing we note that we have also evaluated the case of scattering of a spinless particle $a$ from a particle $b$ of unit spin as a further check of universality and of the complications which arise from the existence of quadrupole effects. Using the vertices given in [6], [7] and in Appendix A—note that the electromagnetic vertices for the unit spin system use the
"natural" value $g = 2$ [10]—we find the results

\[ 1 \mathcal{M}^{(2)}_a(q) = G a Z_a Z_{b}\left[ -12 L e_f^b \cdot e_i^b - \frac{9L}{m_a m_b}(e_f^{b*} \cdot q e_i^b \cdot p_1 - e_f^{b*} \cdot p_1 e_i^b \cdot q) \right] \]

\[ 1 \mathcal{M}^{(2)}_a(q) = G a Z_a Z_{b}\left[ (10L + 4m_a S)e_f^b \cdot e_i^b \right. \]
\[ \left. + \frac{8L + 3m_a S}{m_a m_b}(e_f^{b*} \cdot q e_i^b \cdot p_1 - e_f^{b*} \cdot p_1 e_i^b \cdot q) \right. \]
\[ \left. + \left( \frac{m_b^2}{2m_a L} + \frac{m_b^2}{4m_a} \right) \frac{1}{m_b^2} e_f^{b*} \cdot q e_i^b \cdot q \right] \]

\[ 1 \mathcal{M}^{(2)}_a(q) = G a Z_a Z_{b}\left[ (12L + 8m_b S)e_f^b \cdot e_i^b \right. \]
\[ \left. + \frac{5L + 2m_b S}{m_a m_b}(e_f^{b*} \cdot q e_i^b \cdot p_1 - e_f^{b*} \cdot p_1 e_i^b \cdot q) \right. \]
\[ \left. + (3L + 3m_b S) \frac{1}{m_b^2} e_f^{b*} \cdot q e_i^b \cdot q \right] \]

\[ 1 \mathcal{M}^{(2)}_a(q) = G a Z_a Z_{b}\left[ \frac{4m_a m_b}{q^2 L}\left(2e_f^b \cdot e_i^b + \frac{3}{m_a m_b}(e_f^{b*} \cdot q e_i^b \cdot p_1 - e_f^{b*} \cdot p_1 e_i^b \cdot q) \right) \right. \]
\[ \left. + \frac{S}{s - s_0}\left(4m_a + 6m_b\right)(e_f^{b*} \cdot q e_i^b \cdot p_1 - e_f^{b*} \cdot p_1 e_i^b \cdot q) \right. \]
\[ \left. - \frac{m_a(3m_a + 5m_b)}{m_b} e_f^{b*} \cdot q e_i^b \cdot q \right) \]
\[ + \left( \frac{12m_a^2}{3m_a m_b} - \frac{1}{m_a m_b} \right) \frac{1}{3m_a m_b} \left(7m_a + 5m_b S\right)e_f^b \cdot e_i^b \]
\[ - \frac{20}{3} \frac{L}{m_a m_b} e_f^{b*} \cdot p_1 e_i^b \cdot p_1 \]
\[ + \left( \frac{78m_a^2 + 4m_a m_b + 99m_b^2}{12m_a m_b} \right) L + \left( \frac{17}{2} m_a + 11m_b \right) S \]
\[ \times \frac{1}{m_a m_b}(e_f^{b*} \cdot q e_i^b \cdot p_1 - e_f^{b*} \cdot p_1 e_i^b \cdot q) \]
\[
+ \frac{10}{3} L \frac{1}{m_a m_b} (\epsilon_f^b \cdot q \epsilon_i^b \cdot p_1 + \epsilon_f^b \cdot p_1 \epsilon_i^b \cdot q) \\
- \left( \left( \frac{7m_a}{3m_b} + \frac{1}{6} + \frac{16m_b}{3m_a} + \frac{m_b^2}{4m_a^2} \right) L \\
+ \left( 2m_\epsilon + \frac{11}{2} m_b + \frac{m_b^2}{8m_a} \right) S \right) \times \frac{1}{m_b^2} \epsilon_f^b \cdot q \epsilon_i^b \cdot q \\
+ i8\pi G m_a m_b a Z_a Z_b L \frac{L}{q^2 \sqrt{s - s_0}} \left( -\epsilon_f^b \cdot \epsilon_i^b \\
- \frac{3}{2} \frac{1}{m_a m_b} (\epsilon_f^b \cdot q \epsilon_i^b \cdot p_1 - \epsilon_f^b \cdot p_1 \epsilon_i^b \cdot q) \\
+ \frac{1}{2} \frac{1}{m_b} \epsilon_f^b \cdot q \epsilon_i^b \cdot q \right) \\
\right)
\]

\[1 \mathcal{M}_1^{(2)}(q) = G a Z_a Z_b \left[ \frac{4m_a m_b}{q^2 L} \left( -2 \epsilon_f^b \cdot \epsilon_i^b - \frac{3}{m_a m_b} (\epsilon_f^b \cdot q \epsilon_i^b \cdot p_1 - \epsilon_f^b \cdot p_1 \epsilon_i^b \cdot q) \\
+ \frac{1}{m_b^2} \epsilon_f^b \cdot q \epsilon_i^b \cdot q \right) \\
+ \left( - \frac{12m_a^2 + 59m_a m_b + 21m_b^2}{3m_a m_b} L - \frac{1}{2} (m_a + m_b) S \right) \epsilon_f^b \cdot \epsilon_i^b \\
+ \frac{20}{3} L \frac{1}{m_a m_b} \epsilon_f^b \cdot p_1 \epsilon_i^b \cdot p_1 \\
+ \left( - \frac{78m_a^2 + 164m_a m_b + 99m_b^2}{12m_a m_b} L - \frac{9}{2} (m_a + m_b) S \right) \epsilon_f^b \cdot \epsilon_i^b \\
\times \frac{1}{m_a m_b} (\epsilon_f^b \cdot q \epsilon_i^b \cdot p_1 - \epsilon_f^b \cdot p_1 \epsilon_i^b \cdot q) \\
- \frac{10}{3} L \frac{1}{m_a m_b} (\epsilon_f^b \cdot q \epsilon_i^b \cdot p_1 + \epsilon_f^b \cdot p_1 \epsilon_i^b \cdot q) \\
+ \left( \frac{1}{4} + \frac{7}{6} \frac{16m_b}{3m_a} - \frac{m_b^2}{4m_a^2} \right) L \\
+ \left( \frac{5}{4} m_a + \frac{9}{4} m_b - \frac{m_b^2}{8m_a} \right) S \right) \times \frac{1}{m_b} \epsilon_f^b \cdot q \epsilon_i^b \cdot q \\
\right)
\]

\[1 \mathcal{M}_1^{(2)}(q) = G a Z_a Z_b \left[ - 2 (m_a + m_b) S \epsilon_f^b \cdot \epsilon_i^b \right] \]
\[ -(2m_a + m_b) S \frac{1}{m_a m_b} (\epsilon_f^b \cdot q \epsilon_i^b \cdot p_1 - \epsilon_f^b \cdot p_1 \epsilon_i^b \cdot q) \]

\[ + \left( -\frac{8}{3} L - \frac{3}{2} m_b S \right) \frac{1}{m_b^2} \epsilon_f^b \cdot q \epsilon_i^b \cdot q \]

\[ 1 \mathcal{M}_{U}^{(2)}(q) = G\alpha \left[ \left( \frac{8(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3 m_a m_b} L + (Z_a^2 m_b + Z_b^2 m_a) S \right) \epsilon_f^b \cdot \epsilon_i^b \right. \]

\[ + \left( \frac{4(3Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3 m_a m_b} L + \left( \frac{3}{2} Z_a^2 m_b + Z_b^2 m_a \right) S \right) \]

\[ \times \frac{1}{m_a m_b} \left( \epsilon_f^b \cdot q \epsilon_i^b \cdot p_1 - \epsilon_f^b \cdot p_1 \epsilon_i^b \cdot q \right) \]

\[ + \left( \frac{2(-3Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3 m_a m_b} L + \left( -\frac{3}{4} Z_a^2 m_b - \frac{1}{4} Z_b^2 m_a \right) S \right) \]

\[ \times \frac{1}{m_b^2} \epsilon_f^b \cdot q \epsilon_i^b \cdot q \]}

\[ 1 \mathcal{M}_{U}^{(2)}(q) = G\alpha Z_a Z_b \left[ -\frac{4}{3} L \epsilon_f^b \cdot \epsilon_i^b \right. \]

\[ - \frac{4}{3} L \frac{1}{m_a m_b} \left( \epsilon_f^b \cdot q \epsilon_i^b \cdot p_1 - \epsilon_f^b \cdot p_1 \epsilon_i^b \cdot q \right) \] \tag{47}

where \( \epsilon_{i\mu}^{b} \) is the polarization vector for the incoming spin 1 particle and \( \epsilon_{f\mu}^{b} \) is its outgoing polarization vector which satisfy \( \epsilon_f^{b} \cdot p_4 = 0 \) and \( \epsilon_i^{b} \cdot p_3 = 0 \) respectively.

It is useful at this point to define the spin four-vector

\[ S_{b\mu} = \frac{i}{2m_b} \epsilon_{\mu\beta\gamma\delta}^{b} \epsilon_f^{b \gamma} (p_3 + p_4)^\delta \] \tag{48}

which satisfies the identity

\[ \epsilon_f^{b \mu} \epsilon_i^{b} \cdot q - \epsilon_i^{b \mu} \epsilon_f^{b} \cdot q = \frac{1}{1 - \frac{q^2}{4m_b^2}} \left[ \frac{i}{m_b} \epsilon_{\mu\beta\gamma\delta}^{b} \epsilon_f^{b \gamma} (p_3 + p_4)^\delta - \frac{(p_3 + p_4)_{\mu}}{2m_b^2} \epsilon_f^{b \mu} \cdot q \epsilon_i^{b} \cdot q \right] \tag{49} \]

Using this definition of the spin vector and the identity Eq. \( \text{[49]} \) we find that the sum of the diagrams Eq. \( \text{[47]} \) has the form

\[ 1 \mathcal{M}_{\text{tot}}^{(2)}(q) = G\alpha Z_a Z_b \left[ - \epsilon_f^{b} \cdot \epsilon_i^{b} \left( -6(m_a + m_b) S + 12 L \right) \right] \]
$$+ i \frac{\mathcal{E}_b}{m_a m_b^2} \left( \left( \frac{2m_a m_b (2m_a + 3m_b)}{s - s_0} - 5m_a - \frac{15}{2} m_b \right) S + \frac{32}{3} L \right)$$

$$+ \epsilon_f \beta \cdot \epsilon_i \beta \left( - (Z_a^2 m_b + Z_b^2 m_a) S - \frac{8(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} L \right)$$

$$+ \frac{\mathcal{E}_b}{m_a m_b} \left( - \left( \frac{3}{2} Z_a^2 m_b + Z_b^2 m_a \right) S - \frac{4(3 Z_a^2 m_b^2 + Z_b^2 m_a^2)}{3m_a m_b} L \right)$$

$$+ \frac{\epsilon_f \beta \cdot \epsilon_i}{m_b^2} \left( \frac{3}{4} (Z_a^2 m_b + Z_b^2 m_a) S + \frac{2(Z_a^2 m_b^2 + Z_b^2 m_a^2)}{m_a m_b} L \right)$$

$$+ i 8\pi G m_a m_b \alpha Z_b \frac{L}{q^2} \sqrt{\frac{m_a m_b}{s - s_0}} \left( - \epsilon_f \beta \cdot \epsilon_i \beta + \frac{3}{4} \mathcal{E}_b \epsilon_i \beta \frac{\epsilon_f \beta \cdot \epsilon_i}{m_b^2} - \frac{\epsilon_f \beta \cdot \epsilon_i}{m_b^2} \right).$$

(50)

In order to generate a potential, we must first perform the nonrelativistic limit, wherein

$$\epsilon_i^{b0} \approx \frac{1}{m_b} \epsilon_i^{b} \cdot \vec{p}_3, \quad \epsilon_f^{b0} \approx \frac{1}{m_b} \epsilon_f^{b} \cdot \vec{p}_4$$

(51)

so that

$$\epsilon_f \beta \cdot \epsilon_i \beta \approx - \epsilon_f \beta \cdot \epsilon_i \beta + \frac{1}{m_b^2} \epsilon_f \beta \cdot \vec{p}_4 \epsilon_i^{b} \cdot \vec{p}_3$$

$$\approx - \epsilon_f \beta \cdot \epsilon_i \beta + \frac{1}{2m_b^2} \epsilon_f \beta \cdot \vec{p}_4 \times \epsilon_i^{b} \cdot \vec{p}_3$$

$$+ \frac{1}{2m_b^2} \left( \epsilon_f \beta \cdot \vec{p}_4 \epsilon_i^{b} \cdot \vec{p}_3 + \epsilon_f \beta \cdot \vec{p}_3 \epsilon_i^{b} \cdot \vec{p}_4 \right).$$

(52)

Since

$$- i \epsilon_f \beta \times \epsilon_i \beta = \left( 1, m_f \left| \vec{s}_b \right| 1, m_i \right),$$

(53)

Eq. (52) becomes

$$\epsilon_f \beta \cdot \epsilon_i \beta \approx - \epsilon_f \beta \cdot \epsilon_i \beta - \frac{i}{2m_b^2} \vec{s}_b \cdot \vec{p}_3 \times \vec{p}_4 + \frac{1}{2m_b^2} \left( \epsilon_f \beta \cdot \vec{p}_4 \epsilon_i^{b} \cdot \vec{p}_3 + \epsilon_f \beta \cdot \vec{p}_3 \epsilon_i^{b} \cdot \vec{p}_4 \right)$$

$$\approx - \epsilon_f \beta \cdot \epsilon_i \beta + \frac{1}{m_b^2} \epsilon_f \beta \cdot \vec{p} \epsilon_i^{b} \cdot \vec{p} + \frac{i}{2m_b^2} \vec{s}_b \cdot \vec{p} \times \vec{q} - \frac{1}{4m_b^2} \epsilon_f \beta \cdot \vec{q} \epsilon_i^{b} \cdot \vec{q}. $$

29
Comparing with the corresponding nonrelativistic reduction for the spin-1/2 particle in Eq. (26) we see that the structure of the spin-independent and spin-orbit pieces of $-\epsilon_f^b \epsilon_i^b$ is identical to that of $\bar{u}(p_4)u(p_3)$. However, in the unit spin case these terms are accompanied by new terms which are quadrupole in nature, as can see from the identity

$$T_{cd}^b \equiv \frac{1}{2} \left( \epsilon_{fc}^b \epsilon_{sd}^b + \epsilon_{ic}^b \epsilon_{fd}^b \right) - \frac{1}{3} \delta_{cd} \epsilon_f^b \epsilon_i^b$$

$$= - \left\langle 1, m_f \left| \frac{1}{2} (S_c S_d + S_d S_c) - \frac{2}{3} \delta_{cd} \right| 1, m_i \right\rangle$$

(55)

Finally, comparing the total amplitude for spin-0 – spin-1 scattering Eq. (50) and for spin-0 – spin-1/2 scattering Eq. (23) we observe that the spin-independent and spin-orbit pieces are identical once we replace the spin-1/2 structure $\bar{u}(p_4)u(p_3)$ by $-\epsilon_f^b \epsilon_i^b$ in the spin-1 case. Since the nonrelativistic reductions of these structures are also the same up to quadrupole corrections—

$$\bar{u}(p_4)u(p_3) \xrightarrow{\text{NR}} \chi_f^b \chi_i^b - \frac{i}{2m_b^2} \tilde{S}_b \cdot \vec{p} \times \vec{q}$$

$$-\epsilon_f^b \epsilon_i^b \xrightarrow{\text{NR}} \tilde{\epsilon}_f^b \tilde{\epsilon}_i^b - \frac{i}{2m_b^2} \tilde{S}_b \cdot \vec{p} \times \vec{q} + \ldots$$

(56)

where $\chi_f^b \chi_i^b = \delta_{m_f^b,m_i^b}$ and $\tilde{\epsilon}_f^b \tilde{\epsilon}_i^b = \delta_{m_f^b,m_i^b}$— we observe that the spin-independent and spin-orbit pieces of the nonrelativistic reductions of the spin-0 – spin-1/2 and spin-0 – spin-1 amplitudes are also identical. Finally, since the lowest order electromagnetic and gravitational amplitudes/potentials for spin-0 – spin-1 scattering have the same forms for the $S-I$ and $S-O$ components as their spin-0 – spin-1/2 counterparts [6, 7], it is clear that the second order potential for spin-0 – spin-1 scattering is identical to its spin-0 – spin-0 (for $S-I$) and both spin-0 – spin-1/2 and spin-1/2 – spin-1/2 (for both $S-I$ and $S-O$) counterparts. However, clearly there are new terms in the potential which are quadrupole in nature and have no counterpart in lower spin systems. The form of the quadrupole potentials is more complex and is currently under study.
6 Conclusions

Above we have presented a series of calculations of mixed electromagnetic-
gravitational (i.e., $O(G\alpha)$) effects in the long distance piece of the scattering
amplitude/potential of two massive particles for various spin combinations.

The basic idea is to use the methods of effective field theory for such a process,
as outlined by Donoghue for the related process of second order gravitational
scattering [5]. The pieces of the scattering amplitude which lead to power law
behavior of the coordinate space potential are nonanalytic in character and
are of two forms. One involves terms behaving as $1/\sqrt{-q^2}$ and the second in-
volves nonanalytic pieces having the form $\log(-q^2)$. When Fourier transformed
the first form leads to terms which are classical ($\hbar$-independent) and which
lead to long distance spin-independent behavior $V_{\text{class}}(r) \sim G\alpha m/r^2$; while
the second one is quantum mechanical ($\hbar$-dependent) and involves pieces
of the potential which have the form $V_{\text{qm}}(r) \sim \hbar G\alpha/r^3$. The project was
undertaken both because of the intrinsic interest of such a calculation but
also in order to resolve the disagreement between various papers on this
subject which have recently appeared [1, 2, 3]. Specifically, the existing publica-
tions have not satisfied the universality property, which asserts that the
kinematic forms for various pieces of the scattering amplitude should be iden-
tical, regardless of the spin-content of the scattering particles. By carefully
evaluating the scattering of particles with spins 0-0, 0-1/2, 0-1, and 1/2-1/2
we demonstrated that the results of all existing publications were incorrect
(note, however, that the simple typo in the work of Bjerrum-Bohr has now
been corrected [4]), and that universality is satisfied.

Our results are presented in the form of a second order potential, which
is defined in terms of the Fourier transform

$$V^{(2)}_{CG}(\vec{r}) = - \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q} \cdot \vec{r}} \left[ \mathcal{M}_{\text{tot}}^{(2)}(\vec{q}) - \text{Amp}^{(2)}_{CG}(\vec{q}) \right]$$

(57)

where $\mathcal{M}_{\text{tot}}^{(2)}(\vec{q})$ is the full second order scattering amplitude, which includes
all one loop amplitudes involving both graviton and photon exchange, and
$\text{Amp}^{(2)}(\vec{q})$ is the second Born approximation amplitude to the same process.
The subtraction of the iterated Born piece is necessary in order to eliminate
pieces of the second order scattering amplitude which diverge as the center
of mass momentum $p_0$ approaches zero. These terms are of two sorts:

i) imaginary components which behave as $i \log(-q^2)/(q^2 p_0)$
ii) real pieces which behave as \(1/(\sqrt{-q^2} p_0^2)\)

Both such terms were shown to disappear when the subtraction shown in Eq. (57) is performed. However, it should be noted that while the quantum mechanical component of the second order potential is unique, the classical component of the potential contains an ambiguity in that, as pointed out by Sucher [8], the iterated Born amplitude depends the choice for the lowest order potential (see also [6]). Moreover, due to general covariance the classical component also depends on the choice of coordinates [1, 9]. The subtraction of the offending forms i) and ii) above is independent of these choices, but what remains behind are finite pieces proportional to \(1/\sqrt{-q^2}\) whose coefficient depends upon this choice so that the resulting \(O(G\alpha)\) classical potential is not unique. Of course, this ambiguity is not of any physical significance, since the potential itself is not an observable. Rather the only observable is the second order on-shell scattering amplitude which is invariant and well defined.

Our results for the second order potential can be summarized succinctly—for arbitrary spin scattering there exists a spin-independent component which has the form

\[
V_{S-1}^{(2)}(\vec{r}) \simeq G\alpha \left[ \frac{1}{2} \left( Z_a^2 m_b + Z_b^2 m_a \right) \frac{1}{r^2} + 3 Z_a Z_b (m_a + m_b) \frac{1}{r^2} \right. \\
- \left. \frac{4\hbar}{3\pi r^3} \left( Z_b \frac{m_a}{m_b} + Z_a \frac{m_b}{m_a} \right) + \frac{6Z_a Z_b \hbar}{\pi r^3} \right] \delta_{m_a^i, m_b^i} \tag{58}
\]

where \(m_a^i, m_b^i\) represents the projection on the quantization axis of the spin of the indicated particle. In addition, if either particle carries spin there exists an additional shorter-range spin-orbit contribution to the potential which is seen in Eqs. (34) and (46) and whose universality is shown via 0-1/2, 0-1 and 1/2-1/2 scattering. One subtlety here is that the spin-orbit coupling does depend on the g-factor of the scattered particles with spin, and we have used the "natural" value \(g = 2\) throughout. See [6] for a discussion on the dependence of the spin-dependent terms on \(g\) in purely electromagnetic scattering. If both particles carry spin a new spin-spin coupling arises as seen in Eq. (46) which is even shorter ranged than the spin-orbit coupling and whose universality can also be presumed. Finally, if one of the spins is one or greater there exist also quadrupole components of the potential which are
even shorter range and (presumably) universal. These forms are still under study.

In any case, we have demonstrated that universality is valid for the long distance components in the case of mixed electromagnetic-gravitational scattering and that previous indications to the contrary were due to calculational errors.

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A Feynman Rules

In order to carry out the calculations described in the text we require the appropriate electromagnetic, gravitational, and mixed vertices. The purely electromagnetic vertices which involve photons but no gravitons are found in [6], and the purely gravitational vertices involving gravitons and no photons are found in [7]. Here we only list the vertices with couplings to both photons and gravitons which are specific to this work and which were not needed in [6] and [7].

For all couplings involving photons we use a g-factor of \( g = 2 \) at tree level. The g-factor does not affect the spin-independent components of our results, but the spin-dependent ones do depend on it. In [6] we offer a more detailed discussion and calculate purely electromagnetic scattering for arbitrary g-factors in an appendix.

A.1 Photon-Graviton Coupling

When minimally coupling the Maxwell Lagrangian

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}
\]  

(59)

with \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) to gravity, a vertex arises which couples a graviton to two photons. Additional contributions to this vertex arise from the
electromagnetic gauge fixing Lagrangian

\[ \mathcal{L}_{GF} = -\frac{1}{2}(\partial_{\mu}A^\mu)^2 \]  

(60)

coupled to gravity which have to be included in order to be consistent. Our vertex reads

\[
\tau_{\alpha,\beta,\mu\nu}(k, q) = \frac{i\kappa}{2} \left[ 2P_{\mu\nu,\alpha\beta} k \cdot (k + q) + \eta_{\mu\nu}k_\alpha(k + q)_\beta 
+ \eta_{\alpha\beta}(k_\mu(k + q)_\nu + (k + q)_\mu k_\nu) 
- \eta_{\mu\alpha}k_\nu(k + q)_\beta - \eta_{\mu\beta}k_\alpha(k + q)_\nu 
- \eta_{\nu\alpha}k_\mu(k + q)_\beta - \eta_{\nu\beta}k_\alpha(k + q)_\mu \right]
+ \frac{i\kappa}{2} \left[ \eta_{\mu\alpha}k_\nu k_\beta + \eta_{\mu\beta}(k + q)_\alpha(k + q)_\nu 
+ \eta_{\nu\alpha}k_\mu k_\beta + \eta_{\nu\beta}(k + q)_\alpha(k + q)_\mu 
+ \eta_{\mu\nu}(k + q)_\alpha k_\beta - k_\alpha k_\beta - (k + q)_\alpha(k + q)_\beta \right]
\]

(61)

with \( P_{\alpha\beta,\gamma\delta} \equiv \frac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\beta}\eta_{\gamma\delta}) \) where the first term stems from the Maxwell Lagrangian coupled to gravity and the second term is derived from the electromagnetic gauge fixing Lagrangian coupled to gravity. The gauge fixing contributions to the vertex were omitted in previous publications [11, 1, 12, 3]. Even though they change individual diagrams that contribute to form factors, they do not seem to affect any physical quantities so that the total results of the work in [11, 1, 12] are correct (up to the typo in [1] pointed out above which is also found in [12]).
A.2  Couplings to Spin-0 Particles

The Feynman rules for a scalar particle with both electromagnetic and gravitational interactions can be derived by coupling both interactions minimally to the Klein-Gordon Lagrangian. The relevant vertex with one photon and one graviton coupled is found to be

\[ \tau^{(1,1)}_{\mu\nu,\rho} = iZe\kappa P_{\mu\nu,\rho\lambda}(p_1 + p_2)^\lambda \]  \hspace{1cm} (62)

where the charge of the spin-0 particle is \( Ze \).

A.3  Couplings to Spin-1/2 Particles

Similarly, the mixed vertex for a spin-1/2 particle with g-factor \( g = 2 \) coupled to both a graviton and a photon is

\[ \frac{1}{2}\tau^{(1,1)}_{\mu\nu,\rho} = -\frac{iZe\kappa}{4}(2\eta_{\mu\nu}\gamma_\rho - \eta_{\mu\rho}\gamma_\nu - \eta_{\nu\rho}\gamma_\mu). \]  \hspace{1cm} (63)
A.4 Couplings to Spin-1 Particles

The mixed vertex for a spin-1 particle with g-factor $g = 2$ at tree level coupled to both a graviton and a photon has the expression

$$
1_{\beta,\alpha,\mu\nu,\rho}^{(1,1)} = -i Ze\kappa \left( (p_1 + p_2)^\lambda \left( I_{\alpha\beta,\mu\nu} \eta_{\rho\lambda} + \eta_{\alpha\beta} P_{\mu\nu,\rho\lambda} \right) \\
- (p_1 - k)^\lambda \left( I_{\alpha\rho,\mu\nu} \eta_{\beta\lambda} + \eta_{\alpha\rho} P_{\mu\nu,\beta\lambda} \right) \\
- (p_2 + k)^\lambda \left( I_{\alpha\lambda,\mu\nu} \eta_{\beta\rho} + \eta_{\alpha\lambda} P_{\mu\nu,\beta\rho} \right) \right).
$$

(64)

B Iteration Integrals

In this appendix we give the integrals

$$
[H; H_r; H_{rs}] = i \int \frac{d^3\ell}{(2\pi)^3} \frac{-4\pi G m_a m_b}{|\vec{\ell} - \vec{\ell}_f|^2 + \lambda^2} \frac{i[1; \ell, \ell, \ell, \ell, \ell]}{2m_r} \frac{4\pi \alpha Z_a Z_b}{L \frac{m_r}{q^2 p_0}}
$$

which are needed in order to perform the iteration of the lowest order Newton potentials. Here we list only the results; for a more detailed derivation, albeit with a different prefactor, see [6]. The leading expressions for the iteration integrals read

$$
H \simeq i 4\pi G m_a m_b \alpha Z_a Z_b \frac{L m_r}{q^2 p_0}
$$
\[ H_r \simeq (p_i + p_f)_r G m_a m_b \alpha Z_a Z_b \left( i 2 \pi \frac{L}{q^2} \frac{m_r}{p_0} - S \frac{m_r}{p_0^2} + \ldots \right) \]

\[ H_{rs} \simeq \delta_{rs} q^2 G m_a m_b \alpha Z_a Z_b \left[ -i \pi \frac{L}{q^2} \frac{m_r}{p_0} + \frac{1}{2} S \frac{m_r}{p_0^2} + \ldots \right] + (p_i + p_f)_r (p_i + p_f)_s G m_a m_b \alpha Z_a Z_b \left( i \pi \frac{L}{q^2} \frac{m_r}{p_0} - S \frac{m_r}{p_0^2} + \ldots \right) \]

\[ + (p_i - p_f)_r (p_i - p_f)_s G m_a m_b \alpha Z_a Z_b \left( i \pi \frac{L}{q^2} \frac{m_r}{p_0} - \frac{1}{2} S \frac{m_r}{p_0^2} + \ldots \right). \]

(66)

References

[1] N. E. J. Bjerrum-Bohr, Phys. Rev. D 66, 084023 (2002) [arXiv:hep-th/0206236].

[2] M. S. Butt, Phys. Rev. D 74, 125007 (2006) [arXiv:gr-qc/0605137].

[3] S. Faller, arXiv:0708.1701 [hep-th].

[4] N. E. J. Bjerrum-Bohr, arXiv:hep-th/0206236 - v3.

[5] J. F. Donoghue, Phys. Rev. Lett. 72, 2996 (1994) [arXiv:gr-qc/9310024].

J. F. Donoghue, Phys. Rev. D 50, 3874 (1994) [arXiv:gr-qc/9405057].

J. F. Donoghue, arXiv:gr-qc/9512024

[6] B.R. Holstein, A. Ross, arXiv:0802.0715 [hep-ph].

[7] B.R. Holstein, A. Ross, arXiv:0802.0716 [hep-ph].

[8] J. Sucher, Phys. Rev. D 49, 4284 (1994).

[9] N. E. J. Bjerrum-Bohr, J. F. Donoghue and B. R. Holstein, Phys. Rev. D 67, 084033 (2003) [Erratum-ibid. D 71, 069903 (2005)] [arXiv:hep-th/0211072].

[10] B. R. Holstein, Am. J. Phys. 74, 1104 (2006) [arXiv:hep-ph/0607187].

[11] J. F. Donoghue, B. R. Holstein, B. Garbrecht and T. Konstandin, Phys. Lett. B 529, 132 (2002) [arXiv:hep-th/0112237].

[12] N. E. J. Bjerrum-Bohr, arXiv:hep-th/0410097.