Nature of the quantum insulator to superconductor and superconductor to normal state transitions in cuprate superconductors

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Abstract
Using the scaling theory of quantum critical phenomena we present a detailed analysis of various superconducting properties of La$_{2-x}$Sr$_x$CuO$_4$, Y$_{0.8}$Ca$_{0.2}$-123, Tl-1212 and Tl-2201. Our results include: the experimental data are fully consistent with a 2D insulator to superconductor (ISQ) transition in the underdoped limit. The quantum critical properties appear to be equivalent to those of a dirty 2D bosonic system with long range Coulomb interaction. Together with previous evidence, this transition appears to be a generic feature of cuprate superconductors. The 3D superconductor to normal state quantum transition (SNQ) turns out to be less unique. More extended experimental data is needed to settle the nature of this transition unambiguously.

1 Introduction
A characteristic property of cuprate superconductors is the existence of a chemical composition that gives a maximum transition temperature $T_c$, separating the so-called underdoped and overdoped regimes. This behavior appears to be a generic feature. In practice, however, there are only a few compounds for which the composition can be varied continuously throughout the entire doping range. La$_{2-x}$Sr$_x$CuO$_4$ is an example in which both, the underdoped and overdoped regimes, can be accessed by variation of the dopant concentration $x$. In the underdoped limit ($x = x_u \approx 0.05$) and $T = 0$ an insulator to superconductor quantum phase (ISQ) transition occurs, while in the overdoped limit ($x = x_o \approx 0.27$) a superconductor to normal state quantum (SNQ) transition takes place [1, 2, 3]. These quantum critical points are the endpoints of the phase transition line $T_c(x)$ which adopts its maximum around $x \approx 0.16$. Another generic feature of cuprate superconductors emerges from the doping dependence of the anisotropy parameter $\gamma = \sqrt{M_c/M_{ab}}$, where $M_c$ and $M_{ab}$ are the effective pair masses along the crystallographic c-axis and in the ab-plane, respectively. Magnetic torque, penetration depth and resistivity measurements
revealed, that whenever the underdoped regime can be accessed, \( \gamma (x) \) tends to diverge in the underdoped limit and decreases monotonously in the overdoped regime \([1, 2, 3, 4, 5]\). These behavior uncovers a dimensional crossover from 2D to 3D with increasing dopant concentration and implies a 2D-ISQ transition in the underdoped limit.

The main purpose of this work is to explore the nature of the SNQ transition in cuprate superconductors. Invoking the scaling theory of quantum critical phenomena we present a detailed analysis of various superconducting properties of La\(_{2-x}\)Sr\(_x\)CuO\(_4\), Y\(_{0.8}\)Ca\(_{0.2}\)-123, Tl-1212 and Tl-2201. In La\(_{2-x}\)Sr\(_x\)CuO\(_4\) these properties include the doping dependence of \( T_c \), zero temperature penetration depths \([5]\), linear \( T \) coefficient of specific heat coefficient \([6, 7]\) and in-plane penetration depth \( \lambda_{ab} \). In Y\(_{0.8}\)Ca\(_{0.2}\)-123, Tl-1212 and Tl-2201 we concentrate on the doping dependence of \( T_c \) and \( \lambda_{ab}^{-2} (T) \) \([8, 9]\). Our main results include: The experimental data is consistent with a 2D-ISQ transition in the underdoped limit. Together with the previous evidence, this transition appears to be a generic feature of cuprate superconductors \([1, 2, 3]\), although in practice, there are only a few compounds for which the dopant concentration can be varied continuously throughout the entire doping range. Its quantum critical properties fall into the universality class of a 2D bosonic system with long range Coulomb interaction \([10, 11]\). The 3D-SNQ transition turns out to be more complex. The experimental data for overdoped La\(_{2-x}\)Sr\(_x\)CuO\(_4\) reveals remarkable consistency with an intermediate mean-field clean d-wave superconductor to normal state quantum transition (CSNQ). This behavior is not confirmed by the data for overdoped thin films \([12]\) and the \( \mu \)SR data for bulk Y\(_{0.8}\)Ca\(_{0.2}\)-123, Tl-1212 and Tl-220, \([8, 9]\), where disorder appears to be relevant. We explore the applicability of a fluctuation dominated 3D-SNQ- and a mean-field d-wave disordered superconductor to the normal state quantum (DSNQ) transition. Unfortunately, more extended \( \mu \)SR data are needed to discriminate between these options unambiguously. When overdoped cuprate superconductors exhibit in the normal state Fermi liquid like properties and, doping and disorder are inseparable, the occurrence of the DSNQ transition turns out to be more likely. In this case, doping does not control the mobile carrier density only, but in terms of disorder and of the doping of the anisotropy parameter \( \gamma \), the nature and the dimensionality of the SNQ transition, as well. It is shown that the doping dependence of \( d/dT \left( 1/\lambda_{ab}^2 (T) \right) \) in overdoped cuprate superconductors allows to discriminate between the crossover from the 2D-ISQ transition to intermediate mean-field CSNQ or to 3D -SNQ critical behavior. It turns out, however, that more extended experimental data is needed to settle the nature of the 3D-SNQ transition unambiguously.

The rest of the paper is organized as follows. In Sec.2 we sketch the application of scaling theory of quantum critical phenomena to the 2D-ISQ and SNQ transitions in cuprate superconductors. Sec.2.1 treats the 2D-ISQ transition, Sec.2.2 is devoted to the fluctuation dominated 3D-SNQ transition, Sec.2.3 to the mean-field d-wave clean-superconductor to normal state quantum transition (CSNQ) and Sec.2.4 to the mean-field, d-wave, disordered superconductor to normal state quantum transition (DSNQ). In Sec.3 we present and discuss
the analysis of the experimental data.

2 Sketch of the scaling theory of quantum critical phenomena

2.1 Quantum insulator to superconductor transition (ISQ)

There is accumulating evidence that cuprate superconductors undergo in the underdoped limit and at \( T = 0 \) a quantum insulator to superconductor (ISQ) transition \[1, 2, 3\]. The transition is driven by variation of the dopant concentration \( x \) and occurs in the so called underdoped limit \( x < x_u \). Close to criticality \( (x \gtrsim x_u) \) the scaling theory of quantum critical phenomena predicts that in 2D transition temperature \( T_c \) and in-plane penetration depth \( \lambda_{ab}(T = 0) \) scale as \[1, 2, 14\]:

\[
T_c = a_{ISQ} (x - x_u)^{\nu \gamma}, \quad \frac{1}{\lambda_{ab}^2(T = 0)} = b_{ISQ} (x - x_u)^{-\gamma} \quad (1)
\]

\( z \) is the dynamic critical exponent of the ISQ-transition and \( \gamma \) the exponent of the zero temperature in-plane correlation length which diverges as \( \xi_{ab}(T = 0) \propto (x - x_u)^{-\nu} \). Thus in 2D \( T_c, 1/\lambda_{ab}^2(T = 0) \) and \( \rho_0 \) are related by \[1, 14\],

\[
k_B T_c = c \left( \frac{\rho_{oc} - \rho_0}{\rho_{oc}} \right)^z = \frac{\Phi_0^2}{16\pi^3 Q_2^2 \lambda_{ab}^2(T = 0)} \frac{d_s}{\gamma \lambda_{ab}^2(T = 0)} \quad (2)
\]

The critical residual resistivity \( \rho_{oc} \) separates the insulating from the superconducting phase. \( a_{ISQ} \) and \( b_{ISQ} \) are nonuniversal critical amplitudes related by Eq.(2). \( d_s \) denotes the thickness of the superconducting slab and \( Q_2 \) is a universal constant.

Along the phase transition line \( T_c(x) \), however, thermal fluctuations dominate the finite temperature critical behavior. There is accumulating evidence that it falls into the 3D – XY universality class \[\Box\]. In this case (see Appendix A), the transition temperature and finite temperature critical amplitudes of in-plane correlation length \( \xi_{ab,0}^+ \) and penetration depth \( \lambda_{ab,0} \) are related by

\[
k_B T_c = \left( \frac{\Phi_0^2}{16\pi^3} \right) R_\xi^{1/3} \xi_{ab,0}^+ / (\gamma \lambda_{ab,0}^2) \quad (Eql.(A)) \quad \gamma = \sqrt{M_e/M_{ab}} \quad \text{is a measure for the anisotropy.} \]

M_{ab} and M_e denote the in-plane and out-of-plane effective pair mass, entering the spatial gradient terms of the Ginzburg-Landau action. Supposing that there is a 2D-ISQ transition at a critical endpoint of the 3D – XY critical line, matching with the quantum behavior \( T_c \propto \delta^{2\gamma} \), \( \lambda_{ab,0}^2 \propto \lambda_{ab}^{-2}(0) \propto \delta^{-2\gamma} \) and \( \xi_{ab,0}^+ \propto \xi_{ab}(T = 0) \propto \delta^{-\gamma} \) requires that (Eq.(A))

\[
\gamma = \sqrt{M_e/M_{ab}} \propto (x - x_u)^{-\gamma} \quad (3)
\]
Noting that $\gamma = \infty$ corresponds to a 2D system of thickness $d_s$, this divergence is simply reflects the 3D-2D crossover. As underdoped cuprate superconductors are concerned, a rapid increase of $\gamma(x)$ is experimentally well established in terms of magnetic torque [4], penetration depth [3] and transport measurements. Thus, there is accumulating evidence that the experimental data, analyzed in terms of Eqs. (1), (2) and (3) are remarkably consistent with a generic 2D-ISQ transition with $\nu \approx 1$, $\nu \approx 1$.

These estimates are close to theoretical predictions [6, 11], from which $z = 1$, $\nu \approx 1$ would be expected for a dirty 2D bosonic system with long range Coulomb interactions independent of dimensionality and $\nu \geq 1 \approx 1.03$. In this transition, the loss of phase coherence is due to localization of Cooper pairs which is ultimately responsible for the ISQ transition. Accordingly, the 2D-ISQ transition in heavily underdoped cuprate superconductors appears to fall in the same universality class as the onset of superfluidity in $^4$He in disordered media, corrected for the long-range Coulomb interaction.

Noting that the singular part of the free energy density scales close to the ISQ transition as [1, 14]

$$f_s \propto (x - x_u)^{\nu(D+\nu)} F(T(x - x_u)^{-\nu}),$$

where $F(y)$ is a scaling function of its argument, we obtain for the specific heat coefficient in 2D the relations,

$$\gamma_c = \frac{C}{T} \propto \frac{d^2 f_s}{dT^2} \propto (x - x_u)^{\nu(D-\nu)}, \quad \frac{d^2 \gamma_c}{dT} \propto (x - x_u)^{\nu(D-2\nu)},$$

provided that the corresponding derivatives of the scaling function at $y = T(x - x_u) = 0$ exist. In the presence of a phase twist $k$ the free energy density reads [1, 14]

$$f_s \propto (x - x_u)^{\nu(D+\nu)} G(k(x - x_u)^{-\nu}).$$

Accordingly, the penetration depth scales as

$$\frac{1}{\lambda_{ab}^2(T = 0)} \propto \frac{d^2 f_s}{dk^2} \propto (x - x_u)^{\nu(D-2+\nu)}$$

while in the limit $T \to 0$,

$$\frac{1}{\lambda_{ab}^2(T)} \propto (x - x_u)^{\nu(D-2+\nu)} H(T(x - x_u)^{-\nu}).$$

holds. $H(x)$ is again a scaling function of its argument. The coefficient of the term linear in $T$ scale then as

$$\frac{d}{dT} \left( \frac{1}{\lambda_{ab}^2} \right) \propto (x - x_u)^{\nu(D-2)}$$
provided that \( dG(y) / dy \) and \( d^2G(y) / dy^2 \) exist at \( y = T(x - x_u) = 0 \). Combining Eqs. (3) and (10) we observe that in 2D for \( z = 1 \) and in the limits \( T \to 0 \) and \( x \to x_u \) the relation,

\[
\frac{d\gamma_c}{dT} \propto \frac{d}{dT} \left( \frac{1}{\lambda^2} \right) = \text{const.},
\]  

(11)

is a characteristic feature of a 2D-ISQ transition with \( z = 1 \).

In this context we note that \( \lambda_{ab}(0) \) and \( \lambda_c(0) \) are related by

\[
\left( \frac{\lambda_{ab}(0)}{\lambda_c(0)} \right)^2 = \frac{M_{ab}}{M_c} = \frac{1}{\gamma^2}
\]

(12)

so that with Eqs. (3) and (13)

\[
\frac{1}{\lambda^2_c(0)} \propto (x - x_u)^{-(z+2)} \propto (x - x_u)^{\gamma^2}
\]

(13)

and

\[
T_c \propto (x - x_u)^{1/3} \propto \left( \frac{1}{\lambda^2_c(0)} \right)^{z/(z+2)} \propto \left( \frac{1}{\lambda^2_{ab}(0)} \right)^{1/3}
\]

(14)

with \( z = 1 \), the marked difference to the relationship between \( T_c \) and \( 1/\lambda^2_{ab}(0) \) (Eq. (2)) simply reflects the 3D-2D crossover.

### 2.2 Fluctuation dominated quantum superconductor to normal state transition (3D-SNQ)

As the dopant concentration increases, \( \gamma \) is known to decrease drastically [1, 2, 3]. Accordingly, a 2D - 3D crossover takes place. For this reason, the SNQ occurs in 3D. As in 2D, \( T_c \) and dopant concentration \( x \) still scale as (Eq. (3))

\[
T_c = a_{SNQ} (x_o - x)^{\gamma^2}
\]

(15)

In a fluctuation dominated (nonmean-field) 3D-SNQ-transition, however, the scaling properties of the in-plane penetration depth follow from Eq. (8) so that

\[
\frac{1}{\lambda^2_{ab}(0)} = b_{SNQ} (x_o - x)^{\gamma^2(z+1)}
\]

(16)

Accordingly, transition temperature \( T_c \) and zero temperature in-plane penetration depth \( \lambda_{ab}(0) \) are related by

\[
T_c \propto \left( \frac{1}{\lambda^2_{ab}(0)} \right)^{1/(z+1)},
\]

(17)

where the factor of proportionality is nonuniversal. \( x_o \) denotes the dopant concentration in the overdoped limit.
From Eq. (5) we obtain in 3D for the specific heat coefficient the relations
\[ \gamma_c \propto \frac{d^2 f}{dT^2} \propto (x_o - x)^{\nu(3-z)}, \quad \frac{d\gamma_c}{dT} \propto (x_o - x)^{\nu(3-2z)}, \] (18)

Similarly, the scaling properties of \(1/\lambda^2 (T = 0)\) and \(d/dT \left(1/\lambda^2 (T)\right)\) are in 3D readily obtained from Eqs. (8), (10) and (16). The result is
\[ \frac{1}{\lambda^2_{ab}(T = 0)} = b_{SNQ} (x_o - x)^{\nu(z+1)}, \] (19)

and with Eq. (10) for the coefficients of the terms linear and quadratic in \(T\),
\[ \left( \frac{d}{dT} \left( \frac{1}{\lambda^2_{ab}} \right) \right)_{T=0} \propto (x_o - x)^{\nu} \propto \left( \frac{d\gamma_c}{dT} \right)_{T=0}^{1/(3-2z)}. \] (20)

Apparently, the term linear in \(T\), considered to be a characteristic feature of clean d-wave superconductors, vanishes here at criticality.

2.3 Mean-field clean-superconductor to normal state quantum transition (CSNQ)

It appears to be well documented, however, that overdoped cuprate superconductors exhibit in the normal state Fermi liquid like properties [13]. This suggests a mean-field SNQ transition in the overdoped limit.

In the clean limit \(T_c\) and the amplitude \(\Delta_0\) of the d-wave gap scale as
\[ T_c = a_{SNQ} (x_o - x)^{\nu} \propto \Delta_0, \] (21)
where [1]
\[ \nu = 1/2, \quad z = 1, \quad \nu = 1/2. \] (22)

While \(T_c\) tends to zero in the overdoped regime, the mobile carrier density in the normal state is known to remain finite [13]. Moreover, in a clean system there are no pair breaking mechanisms. Accordingly, the occurrence of a SNQ transition in the overdoped limit implies a vanishing pairing interaction. In this case the zero temperature in-plane penetration depth tends to a constant value[1][12]:
\[ \lim_{x \to x_o} \frac{1}{\lambda^2_{ab}(x, 0)} = const \] (23)

Note that in this transition, where fluctuations are absent, the scaling relation (17) does not hold. There is however the relation (16)
\[ T_c \left( \frac{d}{dT} \left( \frac{\lambda_{ab}(0)}{\lambda_{ab}(T)} \right) \right)^2_{T=0} = -0.645, \] (24)
between the leading temperature dependence of the in-plane penetration depth and transition temperature. Invoking Eqs. (21), (22) and (23) we find,

\[
\left( \frac{d}{dT} \left( \frac{1}{\lambda_{ab}(T)} \right)^2 \right)_{T=0} = -\frac{1}{a_{SNQ} \lambda_{ab}^2(x_o,0)} (x_o - x)^{-1/2}.
\]  

(25)

Moreover, in a mean-field d-wave superconductor the leading temperature dependence of the specific heat coefficient adopts the form \[17\]

\[
\left( T_c \frac{d\gamma}{dT} \right)_{T=0} = 1.53 \gamma_n, \quad \gamma_n = \frac{\pi^2}{3} k_B^2 N(0).
\]  

(26)

Invoking Eqs. (21) and (22) we obtain

\[
\left( \frac{d\gamma_c}{dT} \right)_{T=0} = -\frac{1.53 \gamma_n}{a_{SNQ}} (x_o - x)^{-1/2}.
\]  

(27)

Thus, close to a clean mean-field d-wave SNQ transition, the leading temperature dependence of \(1/\lambda_{ab}^2(T)\) and \(\gamma_c\) are according to Eqs. (25) and (27) linearly related by

\[
\left( \frac{d}{dT} \left( \frac{1}{\lambda_{ab}(T)} \right)^2 \right)_{T=0} = -\frac{1}{a_{SNQ} \lambda_{ab}^2(x_o,0)} (x_o - x)^{-1/2} \propto \left( \frac{d\gamma_c}{dT} \right)_{T=0}
\]  

(28)

This differs from the critical behavior at the fluctuation dominated 3D- SNQ transition (Eq. (21)), where \( (d/dT (1/\lambda_{ab}^2(T)))_{T=0} \) tends to zero.

### 2.4 Disordered tuned mean-field quantum superconductor to normal state transitions (DSNQ)

In most cuprate superconductors doping and disorder appear to be inseparable. Moreover, in a d-wave superconductor even nonmagnetic impurities are pair breaking and lead to a suppression of both, \(T_c\) and \(1/\lambda_{ab}^2(T=0)\) \[18\]. For an s-wave impurity potential of strength \(w\) and impurity concentration \(c\) the reduction of the mean-field \(T_c\) is given by \[18\]

\[
\ln \left( \frac{T_c}{T_{c0}} \right) = \Psi \left( \frac{1}{2} \right) - \Psi \left( \frac{1}{2} + \frac{1}{4\pi r T_c} \right),
\]  

(29)

where \(r\) is the normal-state relaxation rate

\[
\frac{1}{2r} = \frac{N(0) \pi c w^2}{1 + (N(0) \pi c w)^2}.
\]  

(30)

There is a critical relaxation rate \(r_c\) where \(T_c\) vanishes. Here an impurity tuned SNQ transition occurs. Close to the transition a mean-field treatment yields \[18\]

\[
T_c \propto (r - r_c)^{z\varphi}, \quad z\varphi = 1/2.
\]  

(31)
Note that for small $w$ the Born approximation holds, while large $w$, $w \rightarrow \infty$, corresponds to the unitary limit. In both cases, $T_c$ and $\lambda_{ab}(0)$ scale close to the mean-field SNQ transition as

$$T_c \sim \left( \frac{1}{\lambda^2(0)} \right)^{1/2}. \quad (32)$$

In the unitary limit this behavior appears to hold over a rather extended range. In contrast to the pure d-wave superconductor, the low temperature behavior of $1/\lambda^2(T)$ is no longer linear. Below the crossover temperature $T^*$ one enters the impurity dominated regime, where $1/\lambda^2(T) \propto T^2$ and at somewhat higher temperatures $1/\lambda^2(T)$ displays the temperature dependence of the pure state, $1/\lambda^2(T) \propto T$ (Eq.(28)). The interpolation formula

$$\frac{1}{\lambda^2_{ab}(T)} = \frac{1}{\lambda^2_{ab}(0)} + \frac{aT^2}{T^* + T} \quad (33)$$

provides a rough measure of the crossover temperature $T^*$. It is thus clear that in real systems where impurities cannot be avoided impurities will dominate below $T^*$. For this reason one expects that the nature of a mean-field SNQ transition, where $T_c$ tends to zero in the overdoped regime, is dominated by disorder and, on the mean-field level, a DSNQ transition is expected to occur.

3 Comparison with experiment

An essential requirement for the existence of the doping tuned 2D-ISQ and 3D-SNQ transitions is the 2D-3D crossover. Experimentally, this crossover will manifest itself in the doping dependence of the anisotropy parameter $\gamma$ (Eq.(3)). From magnetic torque and resistivity measurements it is known that $\gamma(x)$ tends to diverge in the underdoped limit and decreases monotonically with increasing dopant concentration $x$ [1, 2, 4]. Alternatively and even at $T = 0$, $\gamma$ can also be derived from penetration depth measurements in terms of $\gamma^2 = (\lambda_c(0)/\lambda_{ab}(0))^2 = M_c/M_{ab}$ (Eq.(12)). In Fig.?? we show $\gamma^2$ versus $x$ for La$_{2-x}$Sr$_x$CuO$_4$, derived from the penetration depth measurements of Panagopoulos et al. [3]. Although the data is rather sparse in the underdoped regime, comparison with the solid curve, confirms the doping tuned 2D-3D crossover in the ground state. In Fig.?? we displayed the data of Panagopoulos et al. [3], Uemura et al. [20] and Franck et al. [21] for La$_{2-x}$Sr$_x$CuO$_4$ in terms of $1/\lambda^2_{ab}(x,0)$ versus $x$. In the underdoped limit ($x \rightarrow x_u \approx 0.05$), the data reveal consistency with a 2D-ISQ transition, where for $z\tau = 1$ the scaling relation $1/\lambda^2_{ab}(x,0) \propto (x - x_u)$ (Eq.(9)) holds. According to Fig.??, this conclusion is also supported by the characteristic behavior of a 2D-ISQ transition, the linear relationship between $T_c$ and $1/\lambda^2_{ab}(x,0)$ (Eq.(2)) in the underdoped regime.

In the overdoped regime, matters are much less clear. Indeed, there is just one data point ($x = 0.24$ in Fig.??) favoring the suppression of $1/\lambda^2_{ab}(x,0)$ in the overdoped limit. Even from the plot $T_c$ versus $1/\lambda^2_{ab}(0)$ (Fig.??), which is not
affected by uncertainties in the dopant concentration, no clear-cut conclusion can be drawn. To indicate the drastically different behavior of the DSNQ and CSNQ transition, we sketched in Figs. ?? and ??, the crossover from the 2D-ISQ critical point to a 3D-SNQ or mean-field DSNQ - transition in terms of dashed and solid curves, while the dotted one mimics the crossover from the 2D-ISQ - to a CSNQ - transition, respectively.

Since in any real cuprate superconductor, impurities and imperfections are present, there is in a d-wave superconductor a temperature \( T^* \) (see Eq.(33)), below which disorder dominates. In the SNQ transition \( T_c \) tends to zero as

\[
T_c = a_{SNQ} (x_o - x)^{-\nu} \quad \text{(Eq. (15))}
\]

and with that, there is a crossover dopant concentration \( x^* \), given by

\[
a_{SNQ} (x_o - x^*)^{-\nu} = T^* \quad \text{for} \quad x < x^* \quad \text{clean behavior will dominate, while for} \quad x > x^* \quad \text{disorder sets the scale. It is thus evident that sufficiently close to the overdoped limit the critical behavior of the SNQ transition will be dominated by disorder. Experimentally, this conclusion is well confirmed by the in-plane penetration depth measurements on thin La\(_{2-x}\)Sr\(_x\)CuO\(_4\) films. The data clearly reveals that in overdoped films both, \( T_c \) and \( 1/\lambda_{ab}^2(0) \), are systematically reduced (0.16 < \( x \) ≤ 0.23). Thus, the strength of disorder has a strong influence on the crossover from the 3D-ISQ to the SNQ transition. For nearly clean samples it will be difficult to enter the regime where disorder controls the SNQ transition. To illustrate this point we consider the doping dependence of \( 1/\lambda_{ab}^2(T) \) at low temperatures. In Fig.?? we displayed the data of Panagopoulos et al. [5] for La\(_{2-x}\)Sr\(_x\)CuO\(_4\) at various dopant concentrations. In the doping and temperature range, 0.1 ≤ \( x \) ≤ 0.24 and 1 ≲ \( T \) ≲ 10 K respectively, the data is fully consistent with a linear temperature dependence and the magnitude of \( d/dT \left( 1/\lambda_{ab}^2(T) \right) \) increases with increasing dopant concentration. Although this systematic behavior clearly uncovers d-wave superconductivity and points to a clean mean-field CSNQ transition, a crossover to a disorder dominated SNQ transition, setting in around 1 K, cannot be ruled out. Indeed, fits to the interpolation formula ( Eq.(33) [19] yield for the crossover temperature the estimates \( T^* = 1.43 \pm 0.81, 0.44 \pm 0.39, 1.43 \pm 0.81, 0.61 \pm 0.16, 1.69 \pm 0.2 \) and 0.64 ± 0.39. for \( x = 0.1, 0.15, 0.2, 0.22 \) and 0.24, respectively. Noting that the data resulting from the linear fits (1/1/\( \lambda_{ab}^2(0) \)) are also included in Figs.(??) and (??). Consequently, the data shown in those figures stems from rather clean samples, where the disorder dominated SNQ regime is pushed rather close to the overdoped limit. To substantiate the occurrence of intermediate clean SNQ behavior further, we turn to the doping dependence of \( d/dT \left( 1/\lambda_{ab}^2(T) \right) \) displayed in Fig.??.

These properties scale linearly over the attained doping range, approach a constant value in the underdoped limit and their magnitude increases systematically with dopant concentration \( x \). This is just, of what one would expect, in a crossover from the 2D-ISQ transition with \( z = 1 \) to a clean d-wave mean-field SNQ transition. Indeed, close to the 2D-ISQ transition with \( z = 1 \), \( d\gamma_c/dT \) \( T=0 \propto (d(1/\lambda_{ab}^2)/dT)_{T=0} = \text{const} \) holds (Eq.(14)).
rewritten in form

$$\left( \frac{d}{dT} \left( \frac{1}{\lambda_{ab}^2(T)} \right) \right)_{T=0} = \frac{1}{a_{SNQ}\lambda_{ab}(x_0, 0)} (x_0 - x)^{-1/2}$$

$$= -0.21 (0.27 - x)^{-1/2} \times 10^{-8} \text{ cm}^{-2} \text{K}^{-1}$$ \hspace{1cm} (34)

applies. The solid curve in Fig. ?? corresponds to this asymptotic behavior with $a_{SNQ} = 120.5K$ taken from $T_c = a_{SNQ} (0.27 - x)^{1/2} (K)$ and $1/\lambda_{ab}^2(x_0, 0) \approx 25.3 \times 10^8 \text{ cm}^{-2}$ derived from the data shown in Fig. ??). In contrast, in the experimentally attained doping regime, the data is inconsistent with a fluctuation dominated 3D-SNQ transition, where $(d (1/\lambda_{ab}^2) /dT)_{T=0} \propto (x_0 - x)^{\nu}$ (Eq. (20)). It confirms, however, the evidence for intermediate mean-field CSNQ behavior. In addition the doping dependence of $(d (1/\lambda_{ab})^2 /dT)_{T=0}$ turns out to be a particular suitable property to discriminate between CSNQ and fluctuation dominated 3D-SNQ behavior.

Additional convincing evidence for a generic suppression of $T_c$ and $1/\lambda_{ab}^2(0)$ in overdoped cuprates emerges from the $\mu$SR data displayed in Fig. ?? for $Y_{0.8}Ca_{0.2}Ba_2(Cu_{1-z}Zn_z)O_{7-\delta}$ ($Y_{0.8}Ca_{0.2}Zn_{5}$), $Tl_{0.5-y}Pb_{0.5+y}Sr_{2}Ca_{1-x}Y_xCu_2O_7(Tl-1212)$ [5] and $TlBa_2CuO_{6+\delta}$ (Tl-2201) [6]. To document the reduction of $1/\lambda_{ab}^2(0)$ as a function of the dopant concentration as well, we displayed in Fig. ?? $\sigma_0(0) \propto 1/\lambda_{ab}^2(0)$ versus $x$ for $Tl_{0.5-y}Pb_{0.5+y}Sr_{2}Ca_{1-x}Y_xCu_2O_7(Tl-1212)$ [5]. Formally, this behavior points to a generic relationship of the form

$$T_c \propto (1/\lambda_{ab}^2(0))^x \propto \sigma_0(0)^x$$ \hspace{1cm} (35)

which is inconsistent with a mean-field CSNQ, but consistent with both, a fluctuation dominated 3D-SNQ, ( $x = z/(z + 1)$, Eq. (17)) or a mean-field DNQ transition ( $x = 1/2$, Eq. (32)). Unfortunately, the data shown in Figs. ?? and ?? are too sparse, to discriminate between these transitions.

Since Fermi liquid like behavior appears to be a generic feature of overdoped cuprates in the normal state, and in most cuprate superconductors doping and disorder appear to be inseparable, we pursue the scenario of a disorder tuned mean-field SNQ (DSNQ) transition. For this purpose, we need the relationship between the measure of disorder and dopant concentration. It is well documented [25] that the empirical relation [24],

$$T_c = T_c^{\text{max}} \left( 1 - 82.6 \times (x - 0.16)^2 \right),$$ \hspace{1cm} (36)

describes the doping dependence of most cuprate superconductors very well. $T_c$ adopts its maximum value at $x = 0.16$ and vanishes at $x_{o, u} = 0.16 \pm \sqrt{1/82.6}$ where the 2D-ISQ and 3D-SNQ transitions occur. Close to these quantum phase transitions the doping dependence is then given by

$$T_c = T_c^{\text{max}} \left\{ \frac{2\sqrt{82.6}(x - x_u)}{2\sqrt{82.6}(x_0 - x)} \right\}$$ \hspace{1cm} (37)
Note that $T_c \propto x - x_u$ is consistent with a 2D-ISQ transition (see Eqs. (4) and (4)). In the overdoped regime, where disorder is expected to be relevant, matching with the scaling relations for the disordered mean-field SNQ transition (Eqs. (31) and (32)) yields

$$T_c \propto (r - r_c)^{1/2} \propto (x_o - x)$$

and

$$T_c^2 \propto \frac{1}{\lambda^2_{ab}(0)} \propto (x_o - x)^2.$$  

(38)

(39)

To provide a comparison with the experimental data we included in Fig. ?? the critical behavior close to the 2D-ISQ (Eqs. (4) and (4)) and the disorder tuned DSNQ transition (Eq. (39)) in terms of the straight line and dashed curve, respectively. The straight line corresponds to

$$\frac{1}{\lambda^2_{ab}(0)} \propto \sigma_{0}(0) = 39.12 (x - 0.05),$$

(40)

and the dashed one to

$$\frac{1}{\lambda^2_{ab}(0)} \propto \sigma_{0}(0) = 5231.77 (0.27 - x)^2.$$

(41)

The solid curve sketches the crossover between these quantum phase transitions in terms of the interpolation formula

$$\sigma_{0}(0) = \left( \frac{1}{39.12 (x - 0.05)} + \frac{1}{5231.77 (0.27 - x)^2} \right)^{-1}.$$  

(42)

Note that the asymmetry in the doping dependence of $\sigma_{0}(0)$ simply reflects the crossover from $1/\lambda^2_{ab}(0) \propto x - x_u$ to $1/\lambda^2_{ab}(0) \propto (x_o - x)^2$. This differs from the symmetric doping dependence of $T_c$ (Eq. (4)). The solid and dashed curves in Fig. ??, follow from Eqs. (46) and (47) with $T^m_c = 106 K$ for Tl$_{0.5}$Pb$_{0.5+y}$Sr$_2$Ca$_{1-x}$Y$_x$Cu$_2$O$_{7}$ (Tl-1212). They resemble the outline of a fly’s wing and indicate the crossover from the 2D-ISQ transition to the mean-field DSNQ transition. It should be recognized, however, that the remarkable agreement between the experimental data and the interpolation scheme, points merely to a d-wave disorder tuned SNQ transition. Indeed, given the experimental data discussed here, it is not possible to discriminate between the fluctuation dominated 3D-SNQ and the mean-field DSNQ - transition. This conclusion is well documented in Fig. ??, in terms of the missing data allowing to estimate the critical behavior close to the SNQ transition. Here future experimental data is needed to discriminate between the critical exponents entering Eq. (19) and (39).

In summary, we have shown that the experimental data for the doping dependence of $T_c$, $\gamma$, $1/\lambda^2_{ab}$, $d/dT \left(1/\lambda^2_{ab}(T)\right)$ and $d\gamma_c/dT$, as well as for the relation
between $T_c$ and $1/\lambda_{ab}^2(0)$ are fully consistent with a 2D-ISQ transition in underdoped La$_{2-x}$Sr$_x$CuO$_4$. Together with the previous evidence [1, 2, 3], this transition appears to be a generic feature of cuprate superconductors, although in practice, there are only a few compounds for which the dopant concentration can be varied continuously throughout the entire doping range. Its quantum critical properties appear to be equivalent to those of a dirty 2D bosonic system with long range Coulomb interaction [10, 11]. This reveals that the loss of phase coherence due to localization of Cooper pairs is responsible for the 2D-ISQ transition in cuprate superconductors.

While the experimental data for overdoped La$_{2-x}$Sr$_x$CuO$_4$ revealed remarkable consistency with an intermediate mean-field clean d-wave superconductor to normal state quantum transition (CSNQ), this behavior is not confirmed by the data for overdoped thin films [12], where disorder is more pronounced. The behavior of these overdoped films and the $\mu$SR data for Y$_{0.8}$Ca$_{0.2}$-123, Tl-1212 and Tl-220 [8, 9] point to a rather different SNQ transition. We studied two potential candidates: the fluctuation dominated 3D-SNQ- and the mean-field, disordered, d-wave superconductor to the normal state quantum (DSNQ) transition. Unfortunately, more extended $\mu$SR data are needed to discriminate between these options unambiguously. When overdoped cuprate superconductors exhibit in the normal state Fermi liquid like properties and, doping and disorder are inseparable, the occurrence of the DSNQ transition turned out to be more likely. In this case, doping does not control the mobile carrier density only, but the nature of the SNQ transition and in terms of the anisotropy parameter $\gamma$, the dimensionality of the system as well. We identified the doping dependence of $d/dT \left(1/\lambda_{ab}^2(T)\right)$ in overdoped cuprate superconductors, as a property, allowing to discriminate between the crossover from the 2D-ISQ transition to intermediate mean-field CSNQ or to 3D-SNQ critical behavior. Moreover we have shown, that the doping dependence of $1/\lambda_{ab}^2(0)$ also offers an opportunity for future experimental studies to settle the nature of the 3D-SNQ transition unambiguously.

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Appendix A: Derivation of Eq.(3)

There is accumulating evidence that the observed finite temperature critical behavior of cuprate superconductors is consistent with 3D – XY universality [1, 2, 3, 4, 23, 27, 25, 23, 30]. This universality class is characterized by a set of critical exponents, describing the asymptotic behavior of the correlation length $\xi_i^\pm$, magnetic penetration depth $\lambda_i$, specific heat $A^\pm$, etc., in terms of

\[ \xi_i^\pm = \xi_{i,0}^\pm |t|^{-\nu}, \lambda_i = \lambda_{i,0}|t|^{-\nu/2}, C = \frac{A^\pm}{\alpha}|t|^{-\alpha}, \quad (A1) \]

where $3\nu = 2 - \alpha$. $i$ labels the crystallographic axes, $a, b$ and $c$. As usual, in the above expression $\pm$ refer to $t = T/T_c - 1 > 0$ and $t < 0$, respectively. The
critical amplitudes \(\xi_{i,0}^\pm, \lambda_{i,0}^2, A^\pm,\) etc., are nonuniversal, but there are universal critical amplitude relations, including \([1, 26, 27]\)

\[
(k_B T_c)^3 = \left( \frac{\Phi_0^2}{16\pi^3} \right)^3 \frac{\xi_{x,0}^+ \xi_{y,0}^- \xi_{z,0}^-}{\lambda_{x,0}^2 \lambda_{y,0}^2 \lambda_{z,0}^2}. \tag{A2}
\]

\[
(R^\pm)^3 = A^\pm \xi_{x,0}^\pm \xi_{y,0}^\pm \xi_{z,0}^\pm. \tag{A3}
\]

\[
\frac{A^+}{A^-} = R_A \tag{A4}
\]

and

\[
\frac{\xi_{x,0}^+ \xi_{y,0}^\pm \xi_{z,0}^-}{\xi_{x,0}^- \xi_{y,0}^\pm \xi_{z,0}^+} = R_\xi. \tag{A5}
\]

\(R^\pm, R_A\) and \(R_\xi\) are universal numbers \([31]\). These universal relations are extensions of the isotropic counterparts \([21]\) \((\xi_{x,0}^\pm = \xi_{y,0}^\pm = \xi_{z,0}^\pm,\) etc.) to extreme type II superconductors with effective pair mass anisotropy. This anisotropy enters the Ginzburg-Landau action in terms of the spatial gradient terms. In this case the correlation lengths and magnetic penetration depths scale as

\[
\frac{\xi_i^+}{\xi_j^-} = \sqrt{\frac{M_j}{M_i}}, \quad \frac{\lambda_i^2}{\lambda_j^2} = \frac{M_i}{M_j}. \tag{A6}
\]

Since in most cuprate superconductors, \(\xi_{x,0} = \xi_{y,0} = \xi_{ab,0}, \xi_{z,0} = \xi_{c,0}, \lambda_{x,0}^2 = \lambda_{y,0}^2 = \lambda_{c,0}^2,\) and \(\lambda_{x,0}^2 = \lambda_{c,0}^2,\) the universal relation \((A3)\) reduces with Eq.\((A4)\) to

\[
k_B T_c = \left( \frac{\Phi_0^2}{16\pi^3} \right)^{1/3} \frac{\xi_{ab,0}^+}{\lambda_{ab,0}^2} \frac{R_\xi^{1/3}}{\lambda_{ab,0}^2} \frac{R_\xi}{\lambda_{ab,0}^2 \gamma}, \quad \gamma = \sqrt{\frac{M_c}{M_{ab}}}. \tag{A7}
\]

Although \(T_c, \xi_{ab,0}^+, \xi_{ab,0}^-\) and \(\gamma\) are nonuniversal and depend on the dopant concentration, this relation holds irrespective of the doping level along the 3D \(-\) XY critical line, except at the critical endpoints, where quantum phase transitions occur. Close to the 2D-\(\text{ISQ}\) transition matching with the quantum behavior requires \(T_c \propto \delta^{-\gamma}, \lambda_{ab,0}^{-2} \propto \lambda_{ab,0}^{-2}(0) \propto \delta^{-\gamma}\) (Eq.\((1)\)) and \(\xi_{ab,0} \propto \xi_{ab}(T = 0) \propto \delta^{-\gamma}\) so that

\[
\gamma \propto \delta^{-\gamma}. \tag{A8}
\]

**References**

[1] T. Schneider and J. M. Singer, *Phase Transition Approach To High Temperature Superconductivity*, Imperial College Press, London, 2000
[2] T. Schneider and J. M. Singer, J. Superconductivity, **13**, 789 (2000)
[3] T. Schneider and J. M. Singer, Physica C **341-348**, 87 (2000)
[4] J. Hofer et al., Phys. Rev. B **62**, 631 (2000)
[5] C. Panagopoulos, J.R. Cooper and T. Xiang, Phys. Rev. B **57**, 13422 (1998)
[6] J. W. Loram et al., 10th Ann. HTS Workshop (World Scientific 1996) p.341
[7] M. Momono et al., Physica C **233**, 395 (1994)
[8] C. Niedermayer et al., Phys. Rev. Lett. **71**, 1764 (1993)
[9] C. Bernhard et al., Phys. Rev. Lett. **86**, 1614 (2001)
[10] M. P. A. Fisher et al., Phys. Rev. Lett. **64**, 587 (1990)
[11] I. F. Herbut, Phys. Rev. B **61**, 14723 (2000)
[12] J. P. Locquet et al., Phys. Rev. B **54**, 7481 (1996)
[13] Y. Yie, in Physical Properties of High Temperature Superconductors III, ed. D.M. Ginsberg, World Scientifique, Singapore, 1992, p.285
[14] K. Kim and P. W. Weichman, Phys. Rev. B **43**, 13583 (1999)
[15] J. M Singer et al., Eur. Phys. J. B **2**, 17 (1998)
[16] K. Maki and H. Won, J. Phys. I **6**, 1 (1996)
[17] K. Kübert and P.J. Hirschfeld, Solid State Commun. **76**, 55 (1988)
[18] H. Kim, G. Preosti and P. Muzikar, Phys. Rev. B **49**, 3544 (1994)
[19] P. J. Hirschfeld and N. Goldenfeld, Phys. Rev. B **48**, 4219 (1993)
[20] Y.J. Uemura et al., Phys. Rev. B **38**, 909 (1988), Phys.Rev.Lett. **62**, 2317 (1989)
[21] J. P. Franck, S. Harker and J.H. Brewer, Phys. Rev. Lett. **71**, 283 (1993)
[22] J. Perret, Thesis, University of Neuchatel (1999)
[23] Y. J. Uemura et al., Phys. Rev. Lett. **66**, 2605 (1991)
[24] M. Presland et al., Physica C **176**, 95 (1991)
[25] J. L. Tallon et al., Phys. Rev., B **51**, 12911 (1995)
[26] T. Schneider and D. Ariosa, Z. Phys. B **89**, 249 (1992)
[27] T. Schneider and H. Keller, Int. J. Mod. Phys. B **8**, 487 (1993)
[28] M. A. Hubbard et al, Physica C**259**, 309 (1996)
Fig. 1: $\gamma^2 = (\lambda_c(0)/\lambda_{ab}(0))^2 = M_c/M_{ab}$ versus $x$ for La$_{2-x}$Sr$_x$CuO$_4$ derived from the data of [5]. The solid curve mimics the critical behavior close to the 2D-ISQ transition according to Eqs. (3), (4) and (16) in terms of $\gamma^2 = 0.79 + 126(x - 0.05)^{-2}$.

Fig. 2: $\lambda_{ab}^{-2}(0)$ versus $x$ for La$_{2-x}$Sr$_x$CuO$_4$. ■: $\mu$SR - data [20]; ●: [3]; ▲: [21]. The dashed line mimics the crossover from the 2D-SIQ critical point with $z\nu \approx 1$ (Eqs. (3) and (4)) to 3D-SNQ or mean-field DSNQ - criticality, while the dotted one mimics the crossover from the 2D-SIQ - to the CSNQ - critical point (Eq. (23)).
Fig. 3: $T_c$ versus $\lambda_{ab}^{-2}(0)$ for La$_{2-x}$Sr$_x$CuO$_4$. ∆: [22]; ○: [20]; □: [5]. The solid and dashed curves mimic the crossover from the 2D-SIQ to the 3D-SNQ or mean-field DSNQ - critical point. The dotted curve indicates the crossover from the 2D-SIQ - to the CSNQ - critical point.

Fig. 4: $T_c$ versus $\lambda_{c}^{-2}(T=0)$ for La$_{2-x}$Sr$_x$CuO$_4$. ○: [5]. The solid line corresponds to $T_c$ versus $(1/\lambda_{c}^2(T=0))^{1/3}$ (Eq. (14)) with $z = 1$.
Fig. 5: $\lambda_{ab}^{-2}(T)$ versus $T$ for La$_{2-x}$Sr$_x$CuO$_4$ at various dopant concentrations $x$. Taken from [5]. The straight lines are linear fits yielding the estimates for $1/\lambda_{ab}^2(0)$ and $d/dT (1/\lambda_{ab}^2(0))$ at $T = 0$, plotted in Figs. ?? and ??, respectively.

Fig. 6: $(d/dT (\lambda_{ab}^{-2}(T)))_{T=0}$ and $(d\gamma_c/dT)_{T=0}$ versus $x$ for La$_{2-x}$Sr$_x$CuO$_4$. ■: Taken from [6] and •: from [6, 7]. The solid curve corresponds to Eq. (34) indicating the asymptotic behavior of $(d/dT (1/\lambda_{ab}^2(T)))_{T=0}$ close to CSNQ transition.
Fig.7: $T_c$ versus $\sigma_0 \propto \lambda_{ab}^{-2}(0)$ for $Y_{0.8}Ca_{0.2}Ba_2(Cu_{1-y}Zn_y)O_{7-\delta}$ ($Y_{0.8}Ca_{0.2}$-123), $Tl_{0.5-y}Pb_{0.5+y}Sr_2Ca_{1-y}Y_yCu_2O_7$ (Tl-1212) [9] and $TlBa_2CuO_{6+\delta}$ (Tl-2201) [8]. The solid and dashed lines sketch the crossover from the 2D-ISQ - to the mean-field DSNQ critical points in Tl-1212 according to Eqs.(36) and (42).

Fig.8: $\sigma_0 \propto \lambda_{ab}^{-2}(0)$ versus hole concentration $x$ for Tl-1212. Experimental data taken from [9]. The solid and dashed lines sketch the crossover from the 2D-QSI - to the mean-field DSNQ critical points in Tl-1212 according to Eqs. (36) and (42). The solid straight indicates the critical behavior close to the 2D-ISQ critical point (Eq.(42)) and the dashed the leading behavior close to the mean-field DSNQ critical point (Eq.(42)).
\[ \gamma^2 = \left( \frac{\lambda_c(0)}{\lambda_{ab}(0)} \right)^2 \]
