The \( p \)-Laplacian in thin channels with locally periodic rough boundaries

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Abstract

In this work we analyze the asymptotic behavior of the solutions of the \( p \)-Laplacian equation with homogeneous Neumann boundary conditions set in bounded thin domains as

\[ R^\varepsilon = \{ (x, y) \in \mathbb{R}^2 : x \in (0, 1) \text{ and } 0 < y < \varepsilon G(x, x/\varepsilon) \} \]

We take a smooth function \( G : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R} \), \( L \)-periodic in the second variable, which allows us to consider locally periodic oscillations at the upper boundary. The thin domain situation is established passing to the limit in the solutions as the positive parameter \( \varepsilon \) goes to zero.

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1 Introduction

Let \( G \) be a function

\((H)\)

\[ G : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R} \]

satisfying that there exist a finite number of points

\[ 0 = \xi_0 < \xi_1 < \cdots < \xi_{N-1} < \xi_N = 1 \]

such that \( G : (\xi_{i-1}, \xi_i) \times \mathbb{R} \rightarrow (0, \infty) \) is \( C^1 \) and such that \( G, \partial_x G \) and \( \partial_y G \) are uniformly bounded in \( (\xi_{i-1}, \xi_i) \times \mathbb{R} \) getting limits when we approach \( \xi_{i-1} \) and \( \xi_i \). Further, we assume there exist two constants \( G_0 \) and \( G_1 \) such that

\[ 0 < G_0 \leq G(x, y) \leq G_1, \quad \forall (x, y) \in (0, 1) \times \mathbb{R}, \]

and a real number \( L > 0 \) such that \( G(x, y + L) = G(x, y) \) for all \( (x, y) \in (0, 1) \times \mathbb{R} \)

\( \ast G(x, \cdot) \) is a \( L \)-periodic function for each \( x \in (0, 1) \).

We denote by \( R^\varepsilon \) the following family of open bounded sets

\[ R^\varepsilon = \{ (x, y) \in \mathbb{R}^2 : x \in (0, 1) \text{ and } 0 < y < \varepsilon G_\varepsilon(x) \} \quad \text{for} \; \varepsilon > 0, \]

where

\[ G_\varepsilon(x) = G \left( x, \frac{x}{\varepsilon} \right). \]
In this note, we are interested in analyzing the asymptotic behavior of the solutions of a \( p \)-Laplacian equation posed in the thin domain \( \mathbb{R}^\varepsilon \) with rough boundary. We consider

\[
\begin{aligned}
-\Delta_p u_\varepsilon + |u_\varepsilon|^{p-2}u_\varepsilon &= f_\varepsilon \text{ in } \mathbb{R}^\varepsilon \\
|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \eta_\varepsilon &= 0 \text{ on } \partial \mathbb{R}^\varepsilon
\end{aligned}
\]  

where \( \eta_\varepsilon \) is the unit outward normal vector to the boundary \( \partial \mathbb{R}^\varepsilon \), \( 1 < p < \infty \) with \( p^{-1} + p'^{-1} = 1 \), and \( \Delta_p \) denotes the \( p \)-Laplacian differential operator. We also assume \( f_\varepsilon \in L^{p'}(\mathbb{R}^\varepsilon) \) uniformly bounded.

The variational formulation of (1.2) is given by

\[
\int_{\mathbb{R}^\varepsilon} \left( |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla \varphi + |u_\varepsilon|^{p-2} u_\varepsilon \varphi \right) \, dx \, dy = \int_{\mathbb{R}^\varepsilon} f_\varepsilon \varphi \, dx \, dy, \quad \varphi \in W^{1,p}(\mathbb{R}^\varepsilon). 
\]  

Existence and uniqueness of the solutions are guaranteed by Minty-Browder’s Theorem setting a family of solutions \( u_\varepsilon \). We study the asymptotic behavior of \( u_\varepsilon \) as \( \varepsilon \to 0 \), as the domain \( \mathbb{R}^\varepsilon \) becomes thinner and thinner, also exhibiting a high oscillating boundary at the top due to \( L \)-periodicity of \( G(x, \cdot) \) as illustrated at Figure 1.

According to [1] and references there in, it is expected that the sequence \( u_\varepsilon \) will converge to a function of just one variable \( x \in (0, 1) \) satisfying a one-dimensional equation of the same type. Combining boundary perturbation techniques [3, 4, 5] and monotone operator analysis [10], we identify the effective limit model of (1.3) at \( \varepsilon = 0 \).

As we will see, the homogenized limit equation is a one-dimensional \( p \)-Laplacian equation with variable coefficients \( q(x) \) and \( r(x) \). It assumes the following form

\[
\begin{aligned}
\left\{ 
- (q(x)|u'|^{p-2}u')' + r(x)|u|^{p-2}u &= \tilde{f} \quad \text{in } (0, 1), \\
 u'(0) = u'(1) &= 0,
\end{aligned}
\]

where

\[
q(x) = \frac{1}{L} \int_{Y^\ast(x)} |\nabla v|^{p-2}\partial_{y_1} v \, dy_1 dy_2 \\
r(x) = \frac{|Y^\ast(x)|}{L}
\]  

and \( |Y^\ast(x)| \) denotes the Lebesgue measure of the representative cell

\[
Y^\ast(x) = \{ (y_1, y_2) : 0 < y_1 < L, 0 < y_2 < G(x, y_1) \}
\]

which also depends on variable \( x \in (0, 1) \). The function \( v \) used to set the homogenized coefficient \( q(x) \) in (1.4) is an auxiliar function which is the unique solution of the problem

\[
\int_{Y^\ast(x)} |\nabla v|^{p-2} \nabla v \nabla \varphi \, dy_1 dy_2 = 0 \quad \forall \varphi \in W^{1,p}_{\#}(Y^\ast(x)), \quad \langle \varphi \rangle_{Y^\ast(x)} = 0, \\
(v - y_1) \in W^{1,p}_{\#}(Y^\ast(x)) \quad \text{with} \quad \langle (v - y_1) \rangle_{Y^\ast(x)} = 0.
\]  

Figure 1: A locally periodic thin channel with rough boundary.
where
\[ W_{\#}^{1,p}(Y^*(x)) = \{ \varphi \in W^{1,p}(Y^*(x)) : \varphi|_{\partial_{cft}Y^*(x)} = \varphi|_{\partial_{right}Y^*(x)} \} \]
is the space of periodic functions on the horizontal variable \( y_1 \), and \( \langle \varphi \rangle_\mathcal{O} \) denotes the average of any function \( \varphi \in L^1_{\text{loc}}(\mathbb{R}^N) \) on measurable sets \( \mathcal{O} \subset \mathbb{R}^N \).

It is worth noticing that problem (1.3) is well posed for each \( x \in (0, 1) \), due to Minty-Browder’s Theorem, and then, the coefficient \( q(x) \) is also well defined. Further, \( q(x) \) is a positive function setting a well posed homogenized equation. Indeed, since \( v \) is the solution of (1.5), there exists \( \psi \in W_{\#}^{1,p}(Y^*(x)) \) with \( \langle \psi \rangle_{Y^*(x)} = 0 \) for each \( x \in (0, 1) \) such that \( v = y_1 + \psi \) implying
\[
0 < \int_{Y^*(x)} |\nabla v|^p \, dy_1 \, dy_2 = \int_{Y^*(x)} |\nabla v|^{p-2} \nabla v (y_1 + \psi) \, dy_1 \, dy_2 = \int_{Y^*(x)} |\nabla v|^{p-2} \partial_{y_1} v \, dy_1 \, dy_2 = L q(x).
\]

Several are the works in the literature dealing with issues related to the effect of thickness and rough on the feature of the solutions of partial differential equations. Indeed, thin structures with oscillating boundaries appear in many fields of science: fluid dynamics (lubrication), solid mechanics (thin rods, plates or shells) or even physiology (blood circulation). Therefore, analyzing the asymptotic behavior of models set on thin structures understanding how the geometry and the roughness affect the problem is a very relevant issue in applied science. In these directions, see for instance \([6, 7, 9, 11, 12]\) and references therein.

Here, we are improving results from \([3, 4]\) where the Laplacian operator in locally periodic thin domains was considered dealing with the \( p \)-Laplacian equation for any \( p \in (1, \infty) \). It is worth noticing that the techniques developed in \([3, 4]\) can not be directly applied in or case. On one side, the results obtained in \([4]\) do not guarantee strong convergence in \( L^p \) for the unfolding operator applied on the solutions of the quasilinear operators. On the other side, the analysis performed in \([3]\) just work on \( L^2 \)-spaces. Our goal here is to overcome this situation. We discretize the oscillating region passing to the limit using uniform estimates on two parameters: one associated to the roughness, and other given by the variable profile of the thin domain. In this way, a continuous dependence property for the solutions with respect to \( G \) in \( L^p \)-norms is crucial.

The main result of the paper is the following.

**Theorem 1.1.** Let \( u_\varepsilon \) be the solution of (1.2) with \( f^\varepsilon \in L^p(R^\varepsilon) \) uniformly bounded. Suppose that
\[
\hat{f}^\varepsilon(x) = \frac{1}{\varepsilon} \int_0^{\varepsilon \xi} f^\varepsilon(x, y) \, dy
\]
satisfies \( \hat{f}^\varepsilon \rightharpoonup \hat{f} \) weakly in \( L^p(0, 1) \).

Then, there exists \( u \in W^{1,p}(0, 1) \) such that
\[
\frac{L}{Y^*(x)} \int_0^{\varepsilon \xi} u_\varepsilon(x, y) \, dy \rightharpoonup u \text{ weakly in } L^p(0, 1), \quad \text{as } \varepsilon \to 0,
\]
with \( u \) satisfying the homogenized equation
\[
\int_0^1 \{ q(x)|u|^p-2u\varphi' + r(x)|u|^p-2w\varphi \} \, dx = \int_0^1 f \varphi \, dx, \quad \forall \varphi \in W^{1,p}(0, 1),
\]
where the homogenized coefficients \( q(x) \) and \( r(x) \) are given by (1.4).

Notice that our paper also goes a step beyond \([13]\) where the \( p \)-Laplacian operator is studied in standard thin domains with no oscillatory boundary (as those ones introduced in \([8]\)), and the recent one \([2]\) where purely periodically thin domains in oscillating boundary has been considered.

The paper is organized as follows. In Section 2 we introduce some notations and state some basic results which will be needed in the sequel. In Section 3 we prove the continuous dependence of the
solutions in $L^p$ spaces with respect to the function $G$ uniformly in $\varepsilon > 0$. In Section 4, we perform the asymptotic analysis of (1.2) in piecewise periodic thin domains (that is, in thin domains set by functions $G$ which are piecewise constants in the first variable $x$, and $L$-periodic in the second one). See Figure 2 below which illustrates piecewise periodic open sets. Finally, we provide a proof of the main result in Section 5, namely Theorem 1.1, as a consequence of the analysis performed in the previous sections. Furthermore, we include an Appendix where the dependence of the auxiliary solution $v$ on admissible functions $G$ is analysed.

2 Preliminaries

In this section, we introduce some basic facts, definitions and results concerning to the unfolding method making some straightforward adaptations to our propose. First, let us just recall some basic properties to the $p$-Laplacian which can be found for instance in [10].

**Proposition 2.1.** Let $x,y \in \mathbb{R}^n$.

- If $p \geq 2$, then
  $$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq c_p |x - y|^p.$$

- If $1 < p < 2$, then
  $$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq c_p |x - y|^2 (|x| + |y|)^{p-2}.$$

**Corollary 2.1.** Let $a_p : \mathbb{R}^n \to \mathbb{R}^n$ such that $a_p(s) = |s|^{p-2}s$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then, $a_p$ is the inverse of $a_{p'}$. Moreover,

- If $1 < p' < 2$ (i.e, $p \geq 2$), then
  $$|u|^{p'-2}u - |v|^{p'-2}v \leq c|u - v|^{p'-1}.$$

- If $p' \geq 2$ (i.e, $1 < p < 2$), then
  $$|u|^{p'-2}u - |v|^{p'-2}v \leq c|u - v|(|u| + |v|)^{p'-2} \leq c|u - v|(1 + |u| + |v|)^{p'-2}.$$

**Proposition 2.2.** Let $x,y \in \mathbb{R}^n$ and $p \geq 1$. Then,

$$|y|^p \geq |x|^p + p|x|^{p-2}x \cdot (y - x)$$

Moreover,

$$|y|^p \geq |x|^p + p|x|^{p-2}x \cdot (y - x) + c_p y - x|^p$$

if $p \geq 2,$

$$|y|^p \geq |x|^p + p|x|^{p-2}x \cdot (y - x) + c_p |x - y|^2(1 + |x| + |y|)^{p-2}$$

if $1 < p < 2.$
From now on, we use the following rescaled norms
\[
\|\varphi\|_{L^p(R^d)} = \varepsilon^{-1/p} \|\varphi\|_{L^p(R^d)} \quad \forall \varphi \in L^p(R^d), 1 \leq p < \infty,
\]
\[
\|\varphi\|_{W^{1,p}(R^d)} = \varepsilon^{-1/p} \|\varphi\|_{W^{1,p}(R^d)} \quad \forall \varphi \in W^{1,p}(R^d), 1 \leq p < \infty.
\]
For completeness we may denote \(\|\varphi\|_{L^\infty(R^d)} = \|\varphi\|_{L^\infty(R^d)}\).

Next, we get the following uniform bound for the solutions of (1.2):

**Proposition 2.3.** Consider the variational formulation of our problem:

\[
\int_{R^d} \left\{ |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \varphi + |u_\varepsilon|^{p-2} u_\varepsilon \varphi \right\} dx dy = \int_{R^d} f^\varepsilon \varphi dx dy, \quad \varphi \in W^{1,p}(R^d),
\]

where \(f^\varepsilon\) satisfies
\[
\|f^\varepsilon\|_{L^{p'}(R^d)} \leq c
\]
for some positive constant \(c\) independent of \(\varepsilon > 0\). Then,
\[
\|u_\varepsilon\|_{L^p(R^d)} \leq c, \quad \|u_\varepsilon\|_{W^{1,p}(R^d)} \leq c,
\]
\[
\|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon\|_{L^{p'}(R^d)} \leq c.
\]

**Proof.** Take \(\varphi = u_\varepsilon\) in (2.1). Then,
\[
\|u_\varepsilon\|_{W^{1,p}(R^d)} = \int_{R^d} \left\{ |\nabla u_\varepsilon|^p + |u_\varepsilon|^p \right\} dx dy \leq \|f^\varepsilon\|_{L^{p'}(R^d)} \|u_\varepsilon\|_{L^p(R^d)}.
\]
Hence,
\[
\|u_\varepsilon\|_{W^{1,p}(R^d)} \leq c.
\]
Therefore, the sequence \(u_\varepsilon\) and \(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon|\), are respectively bounded in \(L^p(R^d)\) and \((L^{p'}(R^d))^2\) under the norm \(||\cdot||\). \(\square\)

### 2.1 Unfolding operator

Here, we present the unfolding operator for thin domains in the purely and locally periodic setting. They have been introduced in [3, 5] where details an proofs can be found.

#### 2.1.1 Purely periodic

Let \(G_i : \mathbb{R} \to \mathbb{R}\) be a \(L\)-periodic function, lower semicontinuous satisfying \(0 < g_{0,i} \leq G_i(x) \leq g_{1,i}\) with \(g_{0,i} = \min_{x \in \mathbb{R}} G_i(x)\) and \(g_{1,i} = \sup_{x \in \mathbb{R}} G_i(x)\) for any \(i = 1, ..., N\). Now consider the thin region
\[
R^\varepsilon_i = \left\{(x, y) \in \mathbb{R} : \xi_{i-1} < x < \xi_i, 0 < y < \varepsilon G_i(x/\varepsilon)\right\}.
\]
The basic cell associated to \(R^\varepsilon_i\) is
\[
Y^*_i = \left\{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L \text{ and } 0 < y_2 < G_i(y_1)\right\}.
\]
By
\[
\langle \varphi \rangle_\mathcal{O} := \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} \varphi(x) \, dx,
\]
we denote the average of \(\varphi \in L^1_{\text{loc}}(\mathbb{R}^2)\) for any open measurable set \(\mathcal{O} \subset \mathbb{R}^2\). We also set functional spaces which are defined by periodic functions in the variable \(y_1 \in (0, L)\). Namely
\[
L^p(Y^*_i) = \left\{ \varphi \in L^p(Y^*_i) : \varphi(y_1, y_2) \text{ is } L\text{-periodic in } y_1 \right\},
\]
\[
L^p((0,1) \times Y^*_i) = \left\{ \varphi \in L^p((0,1) \times Y^*_i) : \varphi(x, y_1, y_2) \text{ is } L\text{-periodic in } y_1 \right\},
\]
\[
W^{1,p}(Y^*_i) = \left\{ \varphi \in W^{1,p}(Y^*_i) : \varphi|_{\partial_{x,l} Y^*_i} = \varphi|_{\partial_{x,r} Y^*_i} \right\}.
\]
For each $\varepsilon > 0$ and any $x \in (\xi_{i-1}, \xi_i)$, there exists an integer denoted by $\left[ \frac{x}{\varepsilon} \right]_L$ such that
\[
x = \varepsilon \left[ \frac{x}{\varepsilon} \right]_L + \varepsilon \left\{ \frac{x}{\varepsilon} \right\}_L \text{ where } \left\{ \frac{x}{\varepsilon} \right\}_L \in [0, L).
\]
We still set
\[
I^i_\varepsilon = \text{Int} \left( \bigcup_{k=0}^{N^i_\varepsilon} [k\varepsilon, (k + 1)\varepsilon] \right),
\]
where $N^i_\varepsilon$ is largest integer such that $\varepsilon L(N^i_\varepsilon + 1) \leq \xi_i$, as well
\[
\Lambda^i_\varepsilon = (\xi_{i-1}, \xi_i) \backslash I^i_\varepsilon = [\varepsilon L(N^i_\varepsilon + 1), \xi_i),
\]
\[
R^0_{i\varepsilon} = \left\{ (x, y) \in \mathbb{R}^2 : x \in I^i_\varepsilon, \quad \varepsilon < y < \varepsilon G_i \left( \frac{x}{\varepsilon} \right) \right\},
\]
\[
R^1_{i\varepsilon} = \left\{ (x, y) \in \mathbb{R}^2 : x \in \Lambda^i_\varepsilon, \quad 0 < y < \varepsilon G_i \left( \frac{x}{\varepsilon} \right) \right\}.
\]

**Definition 2.4.** Let $\varphi$ be a Lebesgue-measurable function in $R^i_\varepsilon$. The unfolding operator $T^i_\varepsilon$ acting on $\varphi$ is defined as the following function in $((\xi_{i-1}, \xi_i) \times Y^*_i,
\]
\[
T^i_\varepsilon \varphi(x, y_1, y_2) = \left\{ \begin{array}{ll}
\varphi \left( \varepsilon \left[ \frac{x}{\varepsilon} \right]_L + \varepsilon y_1, \varepsilon y_2 \right), & \text{for } (x, y_1, y_2) \in I^i_\varepsilon \times Y^*_i, \\
0, & \text{for } (x, y_1, y_2) \in \Lambda^i_\varepsilon \times Y^*_i.
\end{array} \right.
\]

**Proposition 2.5.** The unfolding operator satisfies the following properties:

1. $T^i_\varepsilon$ is linear;
2. $T^i_\varepsilon(\varphi \psi) = T^i_\varepsilon(\varphi)T^i_\varepsilon(\psi)$, for all $\varphi, \psi$ Lebesgue measure in $R^i_\varepsilon$;
3. $\forall \varphi \in L^p(R^i_\varepsilon)$, $1 \leq p \leq \infty$,
\[
T^i_\varepsilon(\varphi) \left( x, \left\{ \frac{x}{\varepsilon} \right\}_L, \frac{y}{\varepsilon} \right) = \varphi(x, y),
\]
for $(x, y) \in R^0_{i\varepsilon}$.
4. Let $\varphi$ a Lebesgue measurable function in $Y^*_i$ extended periodically in the first variable. Then, $\varphi^\ast(x, y) = \varphi \left( \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right)$ is measurable in $R^i_\varepsilon$ and
\[
T^i_\varepsilon(\varphi^\ast)(x, y_1, y_2) = \varphi(y_1, y_2), \forall (x, y_1, y_2) \in I^i_\varepsilon \times Y^*_i.
\]
Moreover, if $\varphi \in L^p(Y^*_i)$, then $\varphi^\ast \in L^p(R^i_\varepsilon)$;
5. If $f : Y^*_i \times \mathbb{R}^2 \to \mathbb{R}^2$ is $L$-periodic in $y_1$ and $\varphi : R^i_\varepsilon \to \mathbb{R}^2$ a measurable function, then
\[
T^i_\varepsilon \left[ f \left( \frac{\cdot}{\varepsilon}, \frac{\cdot}{\varepsilon}, \varphi \right) \right] = f(y_1, y_2, T^i_\varepsilon \varphi)
\]
for $(x, y_1, y_2) \in I^i_\varepsilon \times Y^*_i$;
6. Let $\varphi^\ast \in L^1(R^i_\varepsilon)$. Then,
\[
\frac{1}{L} \int_{((\xi_{i-1}, \xi_i) \times Y^*_i)} T^i_\varepsilon(\varphi)(x, y_1, y_2) dx dy_1 dy_2 = \frac{1}{\varepsilon} \int_{R^0_{i\varepsilon}} \varphi(x, y) dx dy
\]
\[
= \frac{1}{\varepsilon} \int_{R^1_{i\varepsilon}} \varphi(x, y) dx dy - \frac{1}{\varepsilon} \int_{R^1_{i\varepsilon}} \varphi(x, y) dx dy;
\]
7. $\forall \varphi \in L^p(R^i_\varepsilon)$, $T^i_\varepsilon(\varphi) \in L^p((\xi_{i-1}, \xi_i) \times Y^*_i)$, $1 \leq p \leq \infty$. Moreover
\[
\| T^i_\varepsilon(\varphi) \|_{L^p((\xi_{i-1}, \xi_i) \times Y^*_i)} = \left( \frac{L}{\varepsilon} \right)^\frac{1}{p} \| \varphi \|_{L^p(R^i_\varepsilon)} \leq \left( \frac{L}{\varepsilon} \right)^\frac{1}{p} \| \varphi \|_{L^p(R^i_\varepsilon)}.
\]
If $p = \infty$,
\[
\| T^i_\varepsilon(\varphi) \|_{L^\infty((\xi_{i-1}, \xi_i) \times Y^*_i)} = \| \varphi \|_{L^\infty(R^i_\varepsilon)} \leq \| \varphi \|_{L^\infty(R^i_\varepsilon)};
\]
8. \( \forall \varphi \in W^{1,p}(R^n_1), 1 \leq p \leq \infty, \)
\[ \partial_{y_1} T^i_\varepsilon(\varphi) = \varepsilon T^i_\varepsilon(\partial_x \varphi) \quad \text{and} \quad \partial_{y_2} T^i_\varepsilon(\varphi) = \varepsilon T^i_\varepsilon(\partial_y \varphi) \quad \text{a.e. in} \ (\xi_{i-1}, \xi_i) \times Y^*_i; \]

9. If \( \varphi \in W^{1,p}(R^n_1) \), then \( T^i_\varepsilon(\varphi) \in L^p ((\xi_{i-1}, \xi_i); W^{1,p}(Y^*_i)) \), \( 1 \leq p \leq \infty \). Besides, for \( 1 \leq p < \infty \), we have
\[
\left\| \partial_{y_1} T^i_\varepsilon(\varphi) \right\|_{L^p((\xi_{i-1}, \xi_i) \times Y^*_i)} = \varepsilon \left( \frac{L}{\varepsilon} \right)^{\frac{1}{p}} \left\| \partial_x \varphi \right\|_{L^p(R^n_0)} \leq \varepsilon \left( \frac{L}{\varepsilon} \right)^{\frac{1}{p}} \left\| \partial_x \varphi \right\|_{L^p(R^n_1)}
\]
\[
\left\| \partial_{y_2} T^i_\varepsilon(\varphi) \right\|_{L^p((\xi_{i-1}, \xi_i) \times Y^*_i)} = \varepsilon \left( \frac{L}{\varepsilon} \right)^{\frac{1}{p}} \left\| \partial_y \varphi \right\|_{L^p(R^n_0)} \leq \varepsilon \left( \frac{L}{\varepsilon} \right)^{\frac{1}{p}} \left\| \partial_y \varphi \right\|_{L^p(R^n_1)}.
\]
If \( p = \infty, \)
\[
\left\| \partial_{y_1} T^i_\varepsilon(\varphi) \right\|_{L^\infty((\xi_{i-1}, \xi_i) \times Y^*_i)} = \varepsilon \left\| \partial_x \varphi \right\|_{L^\infty(R^n_0)} \leq \varepsilon \left\| \partial_x \varphi \right\|_{L^\infty(R^n_1)}
\]
\[
\left\| \partial_{y_2} T^i_\varepsilon(\varphi) \right\|_{L^\infty((\xi_{i-1}, \xi_i) \times Y^*_i)} = \varepsilon \left\| \partial_y \varphi \right\|_{L^\infty(R^n_0)} \leq \varepsilon \left\| \partial_y \varphi \right\|_{L^\infty(R^n_1)}.
\]

10. Let \( (\varphi_\varepsilon) \) be a sequence in \( L^p(R^n_1), 1 < p \leq \infty \) with the norm \( \| \varphi_\varepsilon \|_{L^p(R^n_1)} \) uniformly bounded.
    Then,
\[
\frac{1}{\varepsilon} \int_{R^n_1} |\varphi_\varepsilon| \, dx dy \rightarrow 0.
\]

11. Let \( (\varphi_\varepsilon) \) be a sequence in \( L^p(\xi_{i-1}, \xi_i), 1 \leq p < \infty \), such that
\[ \varphi_\varepsilon \rightharpoonup \varphi \text{ strongly in } L^p(\xi_{i-1}, \xi_i). \]
    Then,
\[ T^i_\varepsilon \varphi_\varepsilon \rightharpoonup \varphi \text{ strongly in } L^p((\xi_{i-1}, \xi_i) \times Y^*_i). \]

The above result sets several basic and somehow immediate properties of the unfolding operator. Property 6 of Proposition 2.5 will be essential to pass to the limit when dealing with solutions of differential equations since it allows us to transform any integral over the thin sets depending on the parameter \( \varepsilon \) and function \( G_i \) into an integral over the fixed set \( (\xi_{i-1}, \xi_i) \times Y^*_i \).

**Remark 2.1.** Since \( |\cdot|^{p-2} \) is monotone, we have that \( T^i_\varepsilon f_\varepsilon \rightarrow f \text{ strongly in } L^p((\xi_{i-1}, \xi_i) \times Y^*_i) \) implies
\[ T^i_\varepsilon (|f|^{p-2} f_\varepsilon) \rightarrow |f|^{p-2} f \text{ strongly in } L^{p'}((\xi_{i-1}, \xi_i) \times Y^*_i). \]

**Proposition 2.6.** Let \( f \in L^p((0,1); L^p_\#(Y^*)) \) and extend it periodically in \( y_1 \)-direction defining
\[ f_\varepsilon(x,y) := f\left(x, \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) \in L^p(R^n). \]
Then,
\[ T^i_\varepsilon f_\varepsilon \rightarrow f \text{ strongly in } L^p((\xi_{i-1}, \xi_i) \times Y^*_i). \]

**Proof.** See [2]. □

**Theorem 2.7.** Let \( (\varphi_\varepsilon) \subset W^{1,p}(R^n_1), 1 < p < \infty, \) with \( \| \varphi_\varepsilon \|_{W^{1,p}(R^n_1)} \) uniformly bounded. Then, there exists \( \varphi^i \in W^{1,p}(\xi_{i-1}, \xi_i) \) and \( \varphi^i_\varepsilon \in L^p((\xi_{i-1}, \xi_i); W^{1,p}(Y^*_i)) \) such that \( (\text{up to a subsequence}) \)
\[ T^i_\varepsilon \varphi_\varepsilon \rightharpoonup \varphi^i \text{ strongly in } L^p((\xi_{i-1}, \xi_i); W^{1,p}(Y^*_i)), \]
\[ T^i_\varepsilon \partial_{x_1} \varphi_\varepsilon \rightharpoonup \partial_{x_1} \varphi^i + \partial_{y_1} \varphi^i_\varepsilon \text{ weakly in } L^p((\xi_{i-1}, \xi_i) \times Y^*_i), \]
\[ T^i_\varepsilon \partial_{y_1} \varphi_\varepsilon \rightharpoonup \partial_{y_1} \varphi^i_\varepsilon \text{ weakly in } L^p((\xi_{i-1}, \xi_i) \times Y^*_i). \]

**Proof.** See [5] Theorem 3.1 and 4.1 respectively. □
2.1.2 Locally Periodic Unfolding

Now, let us set the locally periodic unfolding operator seeing some properties that will be needed in the sequel.

**Definition 2.8.** We define the locally periodic unfolding operator $T_{x}^{\varepsilon} \varphi$ acting on $\varphi$, as the function $T_{x}^{\varepsilon} \varphi$ defined in $(0,1) \times (0,L) \times (0,G_{1})$ by expression

$$T_{x}^{\varepsilon} \varphi(x,y_{1},y_{2}) = \tilde{\varphi} \left( \frac{x}{\varepsilon} L + \varepsilon y_{1}, \varepsilon y_{2} \right) \text{ for } (x,y_{1},y_{2}) \in (0,1) \times (0,L) \times (0,G_{1}),$$

where $\tilde{\varphi}$ denotes the extension by zero to the whole space.

As in classical periodic homogenization, we have the unfolding operator reflecting two scales. The macroscopic one, denoted by $x$ which gives the position in the interval $(0,1)$, and the microscopic scale given by $(y_{1},y_{2})$ which sets the position in the cell $(0,L) \times (0,G_{1})$. However, due to the locally periodic oscillations of the domain $R^{c}$, the definition given here differs from the usual ones. In this case, we do not have a fixed cell that describes the domain $R^{c}$ which makes the extension by zero needed.

**Theorem 2.9.** Let $\varphi_{\varepsilon} \in W^{1,p}(R^{c})$ for $1 < p < \infty$ such that $|||\varphi_{\varepsilon}|||_{W^{1,p}(R^{c})}$ is uniformly bounded. Then, there exists $\varphi \in W^{1,p}(0,1)$ such that, up to subsequences,

$$T_{x}^{\varepsilon} \varphi_{\varepsilon} \rightharpoonup \varphi \chi_{(0,1) \times Y^{*}(x)},$$

weakly in $L^{p}((0,1) \times (0,L) \times (0,G_{1}))$, where $\chi_{(0,1) \times Y^{*}(x)}$ is the characteristic function of $(0,1) \times Y^{*}(x)$.

**Proof.** See [14], Theorem 2.3.9. \hfill $\square$

**Remark 2.2.** We point out that the convergence above can not be improved because of the definition of locally periodic unfolding operator.

**Proposition 2.10.** 1. Let $\varphi \in L^{1}(R^{c})$. Then,

$$\frac{1}{L} \int_{(0,1) \times (0,L) \times (0,G_{1})} T_{x}^{\varepsilon} \varphi(x,y_{1},y_{2}) dx dy_{1} dy_{2} = \frac{1}{\varepsilon} \int_{R^{c}} \varphi(x,y) dx dy.$$

2. Let $\varphi \in L^{p}(0,1)$. Then,

$$T_{x}^{\varepsilon} \varphi \rightarrow \chi_{(0,1) \times Y^{*}(x)} \varphi \text{ strongly in } L^{p}((0,1) \times (0,L) \times (0,G_{1})).$$

**Proof.** See [14] Propositions 2.2.5 and 2.3.6. \hfill $\square$

**Proposition 2.11.** Let $\varphi_{\varepsilon} \in L^{p}(R^{c})$ such that

$$T_{x}^{\varepsilon} \varphi_{\varepsilon} \rightharpoonup \chi_{(0,1) \times Y^{*}(x)} \varphi \text{ weakly in } L^{p}((0,1) \times (0,L) \times (0,G_{1})),$$

where $\varphi(x,y_{1},y_{2}) = \varphi(x)$. Then,

$$\frac{L}{\varepsilon} \int_{0}^{\varepsilon G_{c}(-)} \varphi_{\varepsilon}(\cdot,y) dy \rightharpoonup |Y^{*}(\cdot)| \varphi \text{ weakly in } L^{p}(0,1).$$

**Proof.** Notice that

$$\frac{1}{L} \int_{(0,1) \times (0,L) \times (0,G_{1})} T_{x}^{\varepsilon} \varphi_{\varepsilon} T_{x}^{\varepsilon} \psi(x) dx dy_{1} dy_{2} \rightarrow \frac{1}{L} \int_{(0,1) \times (0,L) \times (0,G_{1})} \varphi(x) \psi(x) \chi_{(0,1) \times Y^{*}(x)} dx dy_{1} dy_{2},$$

for all $\psi \in L^{p'}(0,1)$. By the Proposition 2.10 we have

$$\frac{1}{L} \int_{(0,1) \times (0,L) \times (0,G_{1})} T_{x}^{\varepsilon} \varphi_{\varepsilon} T_{x}^{\varepsilon} \psi(x) dx dy_{1} dy_{2} = \frac{1}{\varepsilon} \int_{R^{c}} \varphi_{\varepsilon}(x,y) \psi(x) dx dy$$

$$= \int_{0}^{1} \left( \frac{1}{\varepsilon} \int_{0}^{\varepsilon G_{c}(x)} \varphi_{\varepsilon}(x,y) dy \right) \psi(x) dx.$$
and
\[ \frac{1}{L} \int_{(0,1) \times (0,L) \times (0,G)} \varphi(x) \psi(x) x(0,1) \times Y^*(x) dx dy = \frac{1}{L} \int_0^1 |Y^*(x)| \varphi(x) \psi(x) dx, \]
for all \( \psi \in L^p(0,1) \). Thus,
\[ \frac{1}{\varepsilon} \int_0^{\varepsilon G(x)} \varphi(x,y) dy \rightarrow \frac{1}{L} |Y^*(x)| \varphi(x) \]
weakly in \( L^p(0,1) \).

\[ \square \]

### 3 A domain dependence result

In the next we analyze how the solutions of (1.2) depends on the function \( G_\varepsilon \). Let us take
\[ G_\varepsilon(x) = G \left( x, \frac{x}{\varepsilon} \right) \quad \text{and} \quad \hat{G}_\varepsilon(x) = \hat{G} \left( x, \frac{x}{\varepsilon} \right) \]
satisfying hypothesis (H) and considering the associated thin domains \( R_\varepsilon \) and \( \hat{R}_\varepsilon \) by
\[ R_\varepsilon = \left\{ (x,y) \in \mathbb{R}^2 : x \in (0,1), 0 < y < \varepsilon G_\varepsilon(x) \right\} \quad \text{and} \quad \hat{R}_\varepsilon = \left\{ (x,y) \in \mathbb{R}^2 : x \in (0,1), 0 < y < \varepsilon \hat{G}_\varepsilon(x) \right\}. \]

Now, let \( u_\varepsilon \) and \( \hat{u}_\varepsilon \) be the solutions of (1.2) for the domains \( R_\varepsilon \) and \( \hat{R}_\varepsilon \) respectively with \( f_\varepsilon \in L^p(\mathbb{R}^2) \). We have the following result.

**Theorem 3.1.** Let \( G_\varepsilon \) and \( \hat{G}_\varepsilon \) be piecewise \( C^1 \) functions satisfying (H) with
\[ ||G_\varepsilon - \hat{G}_\varepsilon||_{L^\infty(0,1)} \leq \delta. \]
Assume also \( f_\varepsilon \in L^p(\mathbb{R}^2) \) satisfying \( ||f_\varepsilon||_{L^p(\mathbb{R}^2)} \leq 1 \).

Then, there exists a positive real function \( \rho : [0, \infty) \rightarrow [0, \infty) \) such that
\[ |||u_\varepsilon - \hat{u}_\varepsilon|||_{W^{1,p}(R_\varepsilon \cap \hat{R}_\varepsilon)} + |||u_\varepsilon|||_{W^{1,p}(R_\varepsilon \setminus \hat{R}_\varepsilon)} + |||\hat{u}_\varepsilon|||_{W^{1,p}(\hat{R}_\varepsilon \setminus R_\varepsilon)} \leq \rho(\delta), \quad (3.1) \]
with \( \rho(\delta) \to 0 \) as \( \delta \to 0 \) uniformly for all \( \varepsilon > 0 \).

**Remark 3.1.** The important part of this result is that the function \( \rho(\delta) \) does not depend on \( \varepsilon \). As we will see, it only depends on the positive constants \( G_0 \) and \( G_1 \).

In order to prove Theorem 3.1, we use the fact that \( u_\varepsilon \) and \( \hat{u}_\varepsilon \) are minimizer of the the functionals
\[ V_\varepsilon(\varphi) = \frac{1}{p\varepsilon} \int_{R_\varepsilon} (|\nabla \varphi|^p + |\varphi|^p) dx dy - \frac{1}{\varepsilon} \int_{R_\varepsilon} f_\varepsilon \varphi dx dy \]
\[ \hat{V}_\varepsilon(\hat{\varphi}) = \frac{1}{p\varepsilon} \int_{R_\varepsilon} (|\nabla \hat{\varphi}|^p + |\hat{\varphi}|^p) dx dy - \frac{1}{\varepsilon} \int_{R_\varepsilon} f_\varepsilon \hat{\varphi} dx dy \]
that is,
\[ V_\varepsilon(u_\varepsilon) = \min_{\varphi \in W^{1,p}(R_\varepsilon)} V_\varepsilon(\varphi) \quad \text{and} \quad \hat{V}_\varepsilon(\hat{u}_\varepsilon) = \min_{\hat{\varphi} \in W^{1,p}(\hat{R}_\varepsilon)} \hat{V}_\varepsilon(\hat{\varphi}). \]

We will need to evaluate the minimizers plugging them into different functionals. For this, we set the following operators introduced in [3]:
\[ P_{1+\eta} : W^{1,p}(U) \rightarrow W^{1,p}(U(1+\eta)) \]
\[ (P_{1+\eta}\varphi)(x,y) = \varphi \left( x, \frac{y}{1+\eta} \right), \quad (x,y) \in U(1+\eta), \quad (3.3) \]
where
\[ U(1+\eta) = \left\{ (x, (1+\eta)y) \in \mathbb{R}^2 : (x,y) \in U \right\} \quad (3.4) \]
and $U \subset \mathbb{R}^2$ is an arbitrary open set. We also consider the following norm in $W^{1,p}(U)$

$$||w||_{W^{1,p}(U)}^p = \frac{1}{1 + \eta} \left[ ||w||_{L^p(U)}^p + ||K_{1+\eta} \nabla w||_{L^p(U)}^p \right] \quad (3.5)$$

where

$$K_{1+\eta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 + \eta \end{pmatrix}.$$

We can easily see that

$$||w||_{W^{1,p}(U)}^p = ||P_{1+\eta} w||_{W^{1,p}(U(1+\eta))}^p \quad (3.6)$$

and

$$\frac{1}{1 + \eta} ||w||_{W^{1,p}(U)}^p \leq ||w||_{W^{1,p}(U)} \leq (1 + \eta) ||w||_{W^{1,p}(U)}^p \quad \text{as } \eta \geq 0.$$

Also, we need the following result about the behavior of the solutions near of the oscillating boundary.

**Lemma 3.2.** Let $u_\varepsilon$ be the solution of problem (1.2) and let $P_{1+\eta}$ be the operator given by (3.3). Then, there exists a positive function $\rho = \rho(p, \eta)$ satisfying $\rho(p, \eta) \to 0$ as $\eta \to 0$, such that

$$||u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p + ||u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p + ||P_{1+\eta} u_\varepsilon - u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p \leq \rho(p, \eta),$$

for $1 < p < \infty$.

**Proof.** Since $\eta > 0$, we have that $R^2 \left( \frac{1}{1+\eta} \right) \subset R^2$. Then,

$$V(u_\varepsilon) = \frac{1}{p} ||u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p - \frac{1}{\varepsilon} \int_{R^2} f^\varepsilon u_\varepsilon \, dx \, dy$$

$$= \frac{1}{p} ||u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p + \frac{1}{p} ||u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p - \frac{1}{\varepsilon} \int_{R^2} f^\varepsilon u_\varepsilon \, dx \, dy$$

$$= \frac{1}{p} ||u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p + \frac{1}{p} ||P_{1+\eta} u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p - \frac{1}{\varepsilon} \int_{R^2} f^\varepsilon u_\varepsilon \, dx \, dy$$

$$\geq \frac{1}{p} ||u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p + \frac{1}{p(1 + \eta)} ||P_{1+\eta} u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p - \frac{1}{\varepsilon} \int_{R^2} f^\varepsilon u_\varepsilon \, dx \, dy. \quad (3.7)$$

Now, let us first assume $p \geq 2$. We use the notations of Corollary 2.1.1 to simplify proofs. By Proposition 2.2, 3.2 and 1.3 for $\varphi = P_{1+\eta} u_\varepsilon - u_\varepsilon$, we get

$$||P_{1+\eta} u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p \geq ||u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p + \frac{p}{\varepsilon} \int_{R^2} [ap(\nabla u_\varepsilon) \nabla (P_{1+\eta} u_\varepsilon - u_\varepsilon)] \, dx \, dy + c_p ||P_{1+\eta} u_\varepsilon - u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p$$

$$= \rho V(u_\varepsilon) + \frac{p}{\varepsilon} \int_{R^2} f^\varepsilon u_\varepsilon \, dx \, dy + \frac{p}{\varepsilon} \int_{R^2} f^\varepsilon (P_{1+\eta} u_\varepsilon - u_\varepsilon) \, dx \, dy + c_p ||P_{1+\eta} u_\varepsilon - u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p \quad (3.8)$$

Putting together (3.7) and (3.8), we obtain

$$V(u_\varepsilon) \geq \frac{1}{p} ||u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p + \frac{1}{p(1 + \eta)} ||P_{1+\eta} u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p - \frac{1}{\varepsilon} \int_{R^2} f^\varepsilon u_\varepsilon \, dx \, dy$$

$$\geq \frac{1}{p} ||u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p + \frac{1}{p(1 + \eta)} V(u_\varepsilon)$$

$$+ \frac{1}{\varepsilon(1 + \eta)} \int_{R^2} f^\varepsilon P_{1+\eta} u_\varepsilon \, dx \, dy + \frac{c_p}{1 + \eta} ||P_{1+\eta} u_\varepsilon - u_\varepsilon||_{W^{1,p}(R^2 \setminus R^2 \left( \frac{1}{1+\eta} \right))}^p - \frac{1}{\varepsilon} \int_{R^2} f^\varepsilon u_\varepsilon \, dx \, dy,$$
Consequently
\[ \frac{\eta}{1 + \eta} V(u_\varepsilon) \geq \frac{1}{p} \| u_\varepsilon \|^p \int_{W^{1,p}(R^\varepsilon \setminus R^\varepsilon(\varepsilon \eta^\varepsilon\frac{1}{1 + \eta}))}^p \]
\[ + \frac{1}{\varepsilon} \int_{R^\varepsilon} f^e \left( \frac{P_1 + \eta u_\varepsilon}{1 + \eta} - u_\varepsilon \right) \, dx \, dy + \frac{c_p}{1 + \eta} \| P_1 + \eta u_\varepsilon - u_\varepsilon \|^p \int_{W^{1,p}(R^\varepsilon)}^p \]
which implies
\[ \frac{1}{p} \| u_\varepsilon \|^p \int_{W^{1,p}(R^\varepsilon \setminus R^\varepsilon(\varepsilon \eta^\varepsilon\frac{1}{1 + \eta}))}^p + \frac{c_p}{1 + \eta} \| P_1 + \eta u_\varepsilon - u_\varepsilon \|^p \int_{W^{1,p}(R^\varepsilon)}^p \]
\[ \leq \frac{\eta}{1 + \eta} V(u_\varepsilon) + \frac{1}{\varepsilon} \int_{R^\varepsilon} f^e \left[ u_\varepsilon - \frac{P_1 + \eta u_\varepsilon}{1 + \eta} \right] \, dx \, dy. \]
(3.9)

Now, let us analyze the integral:
\[ \frac{1}{\varepsilon} \int_{R^\varepsilon} f^e \left[ u_\varepsilon - \frac{P_1 + \eta u_\varepsilon}{1 + \eta} \right] \, dx \, dy. \]

To do this, notice that
\[ u_\varepsilon(x, y) - (P_1 + \eta u_\varepsilon)(x, y) = u_\varepsilon(x, y) - u_\varepsilon \left( x, \frac{y}{1 + \eta} \right) = \int_{\frac{y}{1 + \eta}}^y \partial_y u_\varepsilon(x, s) \, ds, \]
which implies
\[ |u_\varepsilon(x, y) - (P_1 + \eta u_\varepsilon)(x, y)| \leq \left[ \int_{\frac{y}{1 + \eta}}^y |\partial_y u_\varepsilon(x, s)|^p \, ds \right]^{1/p} \left( \frac{\eta y}{1 + \eta} \right)^{1/p}. \]

Putting the power p, multiplying by 1/\varepsilon, integrating between 0 and \varepsilon G_\varepsilon(x) and using that (y/(1 + \eta), y) \subset (\varepsilon G_\varepsilon(x)), we get
\[ \frac{1}{\varepsilon} \int_{\varepsilon G_\varepsilon(x)} \, dy \leq \left[ \frac{1}{\varepsilon} \int_{\varepsilon G_\varepsilon(x)} |\partial_y u_\varepsilon(x, s)|^p \, ds \right]^{1/p} \left( \frac{\eta}{1 + \eta} \right)^{1/p} \]
(3.9)

Thus, we have
\[ \| u_\varepsilon - P_1 + \eta u_\varepsilon \|_{L^p(R^\varepsilon)} \leq \| \partial_y u_\varepsilon \|_{L^p(R^\varepsilon)} \left( \frac{\eta}{1 + \eta} \right)^{1/p} \frac{G_1}{\varepsilon^{1/p'}}, \]
for \varepsilon < 1. Consequently, we get
\[ \frac{1}{\varepsilon} \int_{R^\varepsilon} f^e \left[ u_\varepsilon - \frac{P_1 + \eta u_\varepsilon}{1 + \eta} \right] \, dx \, dy \]
\[ \leq \frac{\eta}{1 + \eta} \int_{R^\varepsilon} |f^e u_\varepsilon| \, dx \, dy + \frac{\eta}{1 + \eta} \int_{R^\varepsilon} \| f^e u_\varepsilon - f^e P_1 + \eta u_\varepsilon \| \, dx \, dy \]
\[ \leq \frac{\eta}{1 + \eta} \| f^e \|_{L^{p'}(R^\varepsilon)} \| u_\varepsilon \|_{L^p(R^\varepsilon)} + \| f^e \|_{L^{p'}(R^\varepsilon)} \| \partial_y u_\varepsilon \|_{L^p(R^\varepsilon)} \frac{\eta^{1/p'}}{(1 + \eta)^{1/p'}} \frac{G_1}{\varepsilon^{1/p'}}. \]
(3.10)

Hence, due Proposition 2.3, (3.9) and (3.10), one gets
\[ \frac{1}{p} \| u_\varepsilon \|_{W^{1,p}(R^\varepsilon \setminus R^\varepsilon(\varepsilon \eta^\varepsilon\frac{1}{1 + \eta}))}^p + c_p \| P_1 + \eta u_\varepsilon - u_\varepsilon \|^p \int_{W^{1,p}(R^\varepsilon)}^p \]
\[ \leq \frac{\eta}{1 + \eta} c + \frac{\eta}{1 + \eta} c + \frac{\eta^{1/p'}}{(1 + \eta)^{1/p'}} c \]
\[ \leq c\eta + c\eta^{1/p'}. \]
(3.11)
we can argue as in (3.11) and (3.12), to get, for $p < 1$

Thus, due Proposition 2.3 and (3.10), we get for $p > 2$

Hence, due to (3.8), we get

Now, notice that

On the other hand, we have

Thus, due Proposition 2.3 and (3.10), we get for $p > 2$

and then,

Thus, due Proposition 2.3 and (3.10), we get for $p > 2$ that

Notice that to the case $p > 2$, we have mainly estimated the term $|x - y|^p$. Now, for the case $1 < p < 2$, we have to estimate $(1 + |x| + |y|)^{p-2}|x - y|^2$ in view of Propositions 2.1 and 2.2. Indeed, we can argue as in (3.11) and (3.12), to get, for $1 < p < 2$ that

and

Now, notice that

\[
\begin{align*}
\|P_{1+\eta}u_{\varepsilon} - u_{\varepsilon}\|_{W^{1,p}(R^d)}^p & \leq \left( \frac{1}{\varepsilon} \int_{R^d} |\nabla P_{1+\eta} u_{\varepsilon} - \nabla u_{\varepsilon}|^2 (1 + |\nabla P_{1+\eta} u_{\varepsilon}| + |\nabla u_{\varepsilon}|)^{p-2} \, dx \right)^{p/2} \\
& \quad \cdot \left( \frac{1}{\varepsilon} \int_{R^d} (1 + |\nabla P_{1+\eta} u_{\varepsilon}| + |\nabla u_{\varepsilon}|)^p \, dx \right)^{(2-p)/2} \\
& \quad + \left( \frac{1}{\varepsilon} \int_{R^d} |P_{1+\eta} u_{\varepsilon} - u_{\varepsilon}|^2 (1 + |P_{1+\eta} u_{\varepsilon}| + |u_{\varepsilon}|)^{p-2} \, dx \right)^{p/2} \\
& \quad \cdot \left( \frac{1}{\varepsilon} \int_{R^d} (1 + |P_{1+\eta} u_{\varepsilon}| + |u_{\varepsilon}|)^p \, dx \right)^{(2-p)/2} 
\end{align*}
\]
Finally, putting together the last inequality and (3.13), we also obtain
\[
\frac{1}{p} \|u_\varepsilon\|^p _{W^{1,p}(\mathbb{R}^n \setminus \beta \mathbb{R}^n(1 + \eta))} + \|P_{1+\eta} u_\varepsilon - u_\varepsilon\|^p _{W^{1,p}(\mathbb{R}^n)} \leq c\eta + c\eta^{1/p} + c\eta^{1/p}
\]
for \(1 < p < 2\) finishing the proof.

Now, we are in condition to show Theorem 3.1.

**Proof of Theorem 3.1**  Taking \(\eta = \delta/G_0\), we get under condition \(|G_\varepsilon - \hat{G}_\varepsilon| \leq \delta\) that
\[
R_\varepsilon \left(\frac{1}{1 + \eta}\right) \subset \hat{R}_\varepsilon \subset R_\varepsilon (1 + \eta) \quad \text{and} \quad \hat{R}_\varepsilon \left(\frac{1}{1 + \eta}\right) \subset \hat{R}_\varepsilon (1 + \eta).
\]

(3.14) Applying Lemma 3.2 we get
\[
\|u_\varepsilon\|^p _{W^{1,p}(\mathbb{R}^n \setminus \beta \mathbb{R}^n(1 + \eta))} \leq \|u_\varepsilon\|^p _{W^{1,p}(\mathbb{R}^n \setminus \beta \mathbb{R}^n(1 + \eta))} \leq c\eta \quad \text{and}
\]
\[
\|u_\varepsilon\|^p _{W^{1,p}(\beta \mathbb{R}^n \setminus \beta \mathbb{R}^n(1 + \eta))} \leq \|u_\varepsilon\|^p _{W^{1,p}(\beta \mathbb{R}^n \setminus \beta \mathbb{R}^n(1 + \eta))} \leq c\eta.
\]

(3.15) Now, let us focus to the first term of (3.1). We have
\[
V_\varepsilon(u_\varepsilon) \leq V_\varepsilon ((P_{1+\eta} \hat{u}_\varepsilon) | R_\varepsilon)
\]
\[
= \frac{1}{p} \| (P_{1+\eta} \hat{u}_\varepsilon) | R_\varepsilon \| ^p _{W^{1,p}(R_\varepsilon)} - \frac{1}{\varepsilon} \int _{R_\varepsilon} f^\varepsilon (P_{1+\eta} \hat{u}_\varepsilon) | R_\varepsilon | dxdy
\]
\[
\leq \frac{1}{p} \| P_{1+\eta} \hat{u}_\varepsilon \| ^p _{W^{1,p}(\hat{R}_\varepsilon (1 + \eta))} - \frac{1}{\varepsilon} \int _{\hat{R}_\varepsilon} f^\varepsilon P_{1+\eta} \hat{u}_\varepsilon | dxdy + \frac{1}{\varepsilon} \int _{\hat{R}_\varepsilon \setminus R_\varepsilon} f^\varepsilon P_{1+\eta} \hat{u}_\varepsilon | dxdy.
\]

(3.16) But from the definition of \(P_{1+\eta}\) (see (3.3)) and a change of variables, we get
\[
\|P_{1+\eta} \hat{u}_\varepsilon\|^p _{W^{1,p}(\hat{R}_\varepsilon (1 + \eta))} \leq (1 + \eta) ||\hat{u}_\varepsilon||^p _{W^{1,p}(\hat{R}_\varepsilon)}.
\]

(3.17) From Lemma 3.2 we get
\[
\frac{1}{\varepsilon} \int _{\hat{R}_\varepsilon} f^\varepsilon P_{1+\eta} \hat{u}_\varepsilon | dxdy \leq \|f^\varepsilon\|^p _{L^p(\varepsilon \hat{R}_\varepsilon)} \|P_{1+\eta} \hat{u}_\varepsilon - \hat{u}_\varepsilon\|^p _{L^p(\varepsilon \hat{R}_\varepsilon)} \leq c\eta^{1/p}.
\]

(3.18) Also, by (3.14), (3.15) and Lemma 3.2 we obtain
\[
\frac{1}{\varepsilon} \int _{\hat{R}_\varepsilon \setminus R_\varepsilon} f^\varepsilon P_{1+\eta} \hat{u}_\varepsilon | dxdy \leq \|f^\varepsilon\|^p _{L^p(\varepsilon \hat{R}_\varepsilon \setminus R_\varepsilon)} \|P_{1+\eta} \hat{u}_\varepsilon\|^p _{L^p(\varepsilon \hat{R}_\varepsilon \setminus R_\varepsilon)} \leq c\eta^{1/p}.
\]

(3.19) Hence, using (3.2), (3.16), (3.17), Proposition 2.3, (3.18), (3.19), we get
\[
V_\varepsilon(u_\varepsilon) \leq \frac{(1 + \eta)}{p} \|\hat{u}_\varepsilon\| ^p _{W^{1,p}(\hat{R}_\varepsilon)} - \frac{1}{\varepsilon} \int _{\hat{R}_\varepsilon} f^\varepsilon P_{1+\eta} \hat{u}_\varepsilon | dxdy + \frac{1}{\varepsilon} \int _{\hat{R}_\varepsilon \setminus R_\varepsilon} f^\varepsilon P_{1+\eta} \hat{u}_\varepsilon | dxdy
\]
\[
= (1 + \eta) \hat{V}_\varepsilon(\hat{u}_\varepsilon) + \frac{(1 + \eta)}{\varepsilon} \int _{R_\varepsilon} f^\varepsilon \hat{u}_\varepsilon | dxdy - \frac{1}{\varepsilon} \int _{\hat{R}_\varepsilon} f^\varepsilon P_{1+\eta} \hat{u}_\varepsilon | dxdy + \frac{1}{\varepsilon} \int _{\hat{R}_\varepsilon \setminus R_\varepsilon} f^\varepsilon P_{1+\eta} \hat{u}_\varepsilon | dxdy
\]
\[
= (1 + \eta) \hat{V}_\varepsilon(\hat{u}_\varepsilon) + \frac{\eta}{\varepsilon} \int _{R_\varepsilon} f^\varepsilon \hat{u}_\varepsilon | dxdy + \frac{1}{\varepsilon} \int _{R_\varepsilon} f^\varepsilon (\hat{u}_\varepsilon - P_{1+\eta} \hat{u}_\varepsilon) | dxdy + \frac{1}{\varepsilon} \int _{\hat{R}_\varepsilon \setminus R_\varepsilon} f^\varepsilon P_{1+\eta} \hat{u}_\varepsilon | dxdy
\]
\[
\leq (1 + \eta) \hat{V}_\varepsilon(\hat{u}_\varepsilon) + \eta \|f^\varepsilon\|^p _{L^p(\varepsilon \hat{R}_\varepsilon)} \|\hat{u}_\varepsilon\|^p _{L^p(\varepsilon \hat{R}_\varepsilon)} + c\eta^{1/p}
\]
\[
= (1 + \eta) \hat{V}_\varepsilon(\hat{u}_\varepsilon) + \tilde{\rho}(\eta),
\]

where \(\tilde{\rho}\) denotes a function such that \(\tilde{\rho}(\eta) \to 0\) as \(\eta \to 0\).
On the other hand, by (3.2), (3.5), (3.6), (3.14) and Proposition 2.2, we get, for $p \geq 2$,

$$V_{\varepsilon}(u_{\varepsilon}) = \frac{1}{p} ||u_{\varepsilon}||_{W^{1,p}(R^p)}^p - \frac{1}{\varepsilon} \int_{R^p} f^\varepsilon u_{\varepsilon} dxdy$$

$$= \frac{1}{p} ||P_{1+\eta}u_{\varepsilon}||_{W^{1,p}(R^p(1+\eta))^p}^p - \frac{1}{\varepsilon} \int_{R^p} f^\varepsilon u_{\varepsilon} dxdy$$

$$\geq \frac{1}{p(1+\eta)} ||P_{1+\eta}u_{\varepsilon}||_{W^{1,p}(R^p)}^p - \frac{1}{\varepsilon} \int_{R^p} f^\varepsilon u_{\varepsilon} dxdy$$

$$\geq \frac{1}{p(1+\eta)} \left[ ||\tilde{u}_{\varepsilon}||_{W^{1,p}(\tilde{R}^p)}^p + \frac{p}{\varepsilon} \int_{\tilde{R}^p} (ap(\nabla \tilde{u}_{\varepsilon})\nabla(P_{1+\eta}u_{\varepsilon} - \tilde{u}_{\varepsilon}) + a_p(\tilde{u}_{\varepsilon})(P_{1+\eta}u_{\varepsilon} - \tilde{u}_{\varepsilon})) dxdy + c_p||P_{1+\eta}u_{\varepsilon} - \tilde{u}_{\varepsilon}||_{W^{1,p}(\tilde{R}^p)}^p \right] - \frac{1}{\varepsilon} \int_{R^p} f^\varepsilon u_{\varepsilon} dxdy$$

$$= \frac{1}{p(1+\eta)} \left[ p\tilde{V}(\tilde{u}_{\varepsilon}) + \frac{p}{\varepsilon} \int_{\tilde{R}^p} f^\varepsilon \tilde{u}_{\varepsilon} dxdy + \frac{p}{\varepsilon} \int_{\tilde{R}^p} f^\varepsilon (P_{1+\eta}u_{\varepsilon} - \tilde{u}_{\varepsilon}) dxdy + c_p||P_{1+\eta}u_{\varepsilon} - \tilde{u}_{\varepsilon}||_{W^{1,p}(\tilde{R}^p)}^p \right] - \frac{1}{\varepsilon} \int_{R^p} f^\varepsilon u_{\varepsilon} dxdy$$

$$= \frac{1}{1+\eta} \tilde{V}(\tilde{u}_{\varepsilon}) + \frac{1}{\varepsilon} \int_{\tilde{R}^p} f^\varepsilon \frac{1}{1+\eta} P_{1+\eta}u_{\varepsilon} dxdy - \frac{1}{\varepsilon} \int_{\tilde{R}^p} f^\varepsilon u_{\varepsilon} dxdy + \frac{1}{p(1+\eta)} ||P_{1+\eta}u_{\varepsilon} - \tilde{u}_{\varepsilon}||_{W^{1,p}(\tilde{R}^p)}^p. \quad (3.21)$$

Now, due (3.10), a Hölder’s inequality and Lemma 3.2 we obtain

$$\left| \frac{1}{\varepsilon} \int_{\tilde{R}^p} f^\varepsilon \frac{1}{1+\eta} P_{1+\eta}u_{\varepsilon} dxdy - \frac{1}{\varepsilon} \int_{\tilde{R}^p} f^\varepsilon u_{\varepsilon} dxdy \right| \leq \left| \frac{1}{\varepsilon} \int_{\tilde{R}^p \setminus \tilde{R}^p} f^\varepsilon P_{1+\eta}u_{\varepsilon} dxdy \right| + \left| \frac{1}{\varepsilon} \int_{\tilde{R}^p \setminus \tilde{R}^p} f^\varepsilon P_{1+\eta}u_{\varepsilon} dxdy \right| \quad (3.22)$$

$$+ \left| \frac{1}{(1+\eta)\varepsilon} \int_{\tilde{R}^p \setminus \tilde{R}^p} f^\varepsilon P_{1+\eta}u_{\varepsilon} dxdy - \frac{1}{\varepsilon} \int_{\tilde{R}^p} f^\varepsilon u_{\varepsilon} dxdy \right| \leq cp(\delta)^{1/p}.$$

First, one can put together (3.20) and (3.21), and then use (3.22) to lead us to

$$\frac{c_p}{p(1+\eta)} ||P_{1+\eta}u_{\varepsilon} - \tilde{u}_{\varepsilon}||_{W^{1,p}(\tilde{R}^p)}^p \leq \eta^2 + 2\eta \tilde{V}(\tilde{u}_{\varepsilon}) + \rho(\delta)^{1/p} + \tilde{\rho}(\delta),$$

which implies that

$$||P_{1+\eta}u_{\varepsilon} - \tilde{u}_{\varepsilon}||_{W^{1,p}(\tilde{R}^p)}^p \leq \tilde{\rho}(\delta), \quad (3.23)$$

for $p \geq 2$, where $\tilde{\rho}(\eta)$ is a nonnegative function that tends to zero as $\eta \to 0$.

From Lemma 3.2 we have $||u_{\varepsilon} - P_{1+\eta}u_{\varepsilon}||_{W^{1,p}(\tilde{R}^p)}^p \leq cp(\delta)$. It follows from (3.23) that

$$||u_{\varepsilon} - \tilde{u}_{\varepsilon}||_{W^{1,p}(\tilde{R}^p \cap \tilde{R}^p)}^p \leq \tilde{\rho}(\delta),$$

for $p \geq 2$, where $\tilde{\rho}(\eta)$ is a nonnegative function that tends to zero as $\eta \to 0$.

For $1 < p < 2$, we can perform analogous argument to obtain

$$\frac{c_p}{p(1+\eta)} \left[ \frac{1}{\varepsilon} \int_{\tilde{R}^p} |\nabla P_{1+\eta}u_{\varepsilon} - \nabla u_{\varepsilon}|^2 (1 + |\nabla P_{1+\eta}u_{\varepsilon}| + |\nabla u_{\varepsilon}|)^{p-2} dxdy \right. \right.$$

$$+ \left. \frac{1}{\varepsilon} \int_{\tilde{R}^p} |P_{1+\eta}u_{\varepsilon} - u_{\varepsilon}|^2 (1 + |P_{1+\eta}u_{\varepsilon}| + |u_{\varepsilon}|)^{p-2} dxdy \right] \leq \frac{\eta}{1+\eta} \tilde{V}(\tilde{u}_{\varepsilon}) + \rho(\delta)^{1/p}$$

which gives us

$$||u_{\varepsilon} - \tilde{u}_{\varepsilon}||_{W^{1,p}(\tilde{R}^p \cap \tilde{R}^p)}^p \leq \tilde{\rho}(\delta),$$

where $\tilde{\rho}(\eta)$ is a nonnegative function that tends to zero as $\eta \to 0$. 

\qed
Remark 3.2. It follows from \[3.23\] that there exists \( \rho : [0, \infty) \mapsto [0, \infty) \) such that

\[
||| P_{1+\delta/G_0} u_\varepsilon - \hat{u}_\varepsilon |||^p_{W^{1,p}(R)} \leq \rho(\delta)
\]

with \( \rho(\delta) \to 0 \) as \( \delta \to 0 \) uniformly in \( \varepsilon \) and any piecewise \( C^1 \) functions \( G_\varepsilon \) and \( \hat{G}_\varepsilon \) uniformly bounded with \( ||G_\varepsilon - \hat{G}_\varepsilon||_{L^\infty(0,1)} \leq \delta \) and \( f^\varepsilon \in L^p(\mathbb{R}^2) \) satisfying \( ||f^\varepsilon||_{L^p(\mathbb{R}^2)} \leq 1 \).

4 The piecewise periodic case

Now, we analyze the limit of \( \{u_\varepsilon\}_{\varepsilon > 0} \) assuming the upper boundary of \( R^\varepsilon \) is piecewise periodic.

More precisely, we assume \( G \) satisfies (H) being independent on the first variable in each interval \( (\xi_{i-1}, \xi_i) \times \mathbb{R} \). We suppose \( G \) satisfies \( G(x,y) = G_i(y) \) in \( x \in I_i = (\xi_{i-1}, \xi_i) \) with \( G_i(y + L) = G_i(y) \) for all \( y \in \mathbb{R} \). Moreover, we assume the function \( G_i(\cdot) \) is \( C^1 \) for all \( i = 1, \ldots, N \) and there exist \( 0 < G_0 < G_1 \) such that \( G_0 \leq G_i(\cdot) \leq G_1 \) for all \( i = 1, \ldots, N \).

Notice that the domain \( R^\varepsilon \) can now be rewritten as

\[
R^\varepsilon = \left( \bigcup_{i=1}^N \hat{R}_i^\varepsilon \right) \cup \left( \bigcup_{i=1}^{N-1} \{(\xi, y) : 0 < y < \varepsilon \min\{G_{i-1}(\xi_i/\varepsilon), G_i(\xi_i/\varepsilon)\}\} \right)
\]

with

\[
\hat{R}_i^\varepsilon = \{(x, y) \in \mathbb{R} : \xi_{i-1} < x < \xi_i, 0 < y < \varepsilon G_i(x/\varepsilon)\}.
\]

See Figure 2 which illustrates this piecewise periodic thin domain.

We have the following result.

Theorem 4.1. Let \( u_\varepsilon \) be the solution of problem \[1.2\] with \( f^\varepsilon \in L^p(R^\varepsilon) \) and \( ||f^\varepsilon||_{L^p(R^\varepsilon)} \leq c \), for some \( c > 0 \) independent of \( \varepsilon > 0 \). Suppose the function

\[
\hat{f}^\varepsilon(x) = \frac{1}{\varepsilon} \int_0^{\varepsilon G(x, \varepsilon)} f(x, y) dy
\]

satisfies

\[
\hat{f}^\varepsilon \rightharpoonup \hat{f} \text{ weakly in } L^p(0,1).
\]

Then, there exist \( u \in W^{1,p}(0,1) \) and \( u_i^1 \in L^p((\xi_{i-1}, \xi_i); W^{1,p}(Y_i^*)) \) such that

\[
\begin{aligned}
T^\varepsilon u_\varepsilon &\rightharpoonup u \text{ weakly in } L^p((\xi_{i-1}, \xi_i); W^{1,p}(Y_i^*)), \\
T^\varepsilon(\delta_x u_\varepsilon) &\rightharpoonup \partial_x u + \partial_y u_1^1(x, y_1, y_2) \text{ weakly in } L^p((\xi_{i-1}, \xi_i); W^{1,p}(Y_i^*)), \\
T^\varepsilon(\delta_y u_\varepsilon) &\rightharpoonup \partial_y u_1^1(x, y_1, y_2) \text{ weakly in } L^p((\xi_{i-1}, \xi_i); W^{1,p}(Y_i^*))
\end{aligned}
\]

and \( u \) is the solution of the problem

\[
\int_0^1 \{ q(x)|u'|^{p-2}u' \varphi' + r(x)|u|^{p-2}u \varphi \} \; dx = \int_0^1 \hat{f}\varphi \; dx, \quad \varphi \in W^{1,p}(0,1),
\]

where \( q, r : (0,1) \to \mathbb{R} \) are piecewise constant functions given by

\[
q(x) = q_i, \quad \text{if } x \in (\xi_{i-1}, \xi_i),
\]

\[
r(x) = r_i, \quad \text{if } x \in (\xi_{i-1}, \xi_i),
\]

with \( r_i \) and \( q_i \) are given by

\[
q_i = \int_{Y_i^*} |\nabla v_i|^p \, dy_1 \, dy_2,
\]

\[
r_i = \frac{|Y_i^*|}{L},
\]

and \( v_i \) is the solution of the auxiliary problem

\[
\int_{Y_i^*} |\nabla v_i|^{p-2} \nabla v_i \nabla \psi dy_1 dy_2 = 0, \quad \forall \psi \in W^{1,p}_{#}(Y_i^*), \quad \langle \psi \rangle_{Y_i^*} = 0
\]

\[
(v_i - y_1) \in W^{1,p}_{#}(Y_i^*), \quad \langle v - y_1 \rangle_{Y_i^*} = 0.
\]
Proof. First, by (4.1), we can rewrite (4.3) taking account the partition \(\{\xi_i\}_{i=1}^N\) as
\[
\sum_{i=1}^N \int_{R_i^\varepsilon} \left\{ |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \varphi + |u_\varepsilon|^{p-2} u_\varepsilon \varphi \right\} \, dxdy = \int_{R^\varepsilon} f^\varepsilon \varphi \, dxdy, \quad \varphi \in W^{1,p}(R^\varepsilon). \tag{4.5}
\]
Hence, using (4.5) and Proposition 2.5, we obtain from (1.3) with test functions \(\varphi(x, y) = \varphi_1(x) \in C_0^\infty(\xi_{i-1}, \xi_i)\) that
\[
\int_{(\xi_{i-1}, \xi_i) \times Y_i^*} T_\varepsilon^i \left( |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \right) T_\varepsilon^i \varphi \, dxdy + L \int_{R_i^\varepsilon} |u_\varepsilon|^{p-2} u_\varepsilon \varphi \, dxdy = L \int_{R_i^\varepsilon} f^\varepsilon \varphi \, dxdy. \tag{4.6}
\]
By Proposition 2.3 and Theorem 2.7, there exist
\[ u_i^1 \in W^{1,p}(\xi_{i-1}, \xi_i) \quad \text{with} \quad a_0^i \in L^p((\xi_{i-1}, \xi_i) \times Y_i^*). \]
such that, up to subsequences,
\[
\begin{align*}
T_\varepsilon^i u_\varepsilon & \rightarrow u_i \ \text{strongly in} \ L^p((\xi_{i-1}, \xi_i) \times Y_i^*), \\
T_\varepsilon^i (\partial_x u_\varepsilon) & \rightarrow \partial_x u_i + \partial_y u_1^1(x, y_1, y_2) \ \text{weakly in} \ L^p((\xi_{i-1}, \xi_i) \times W^{1,p}(Y_i^*)), \\
T_\varepsilon^i (\partial_y u_\varepsilon) & \rightarrow \partial_y u_i + \partial_y u_1^1(x, y_1, y_2) \ \text{weakly in} \ L^p((\xi_{i-1}, \xi_i) \times W^{1,p}(Y_i^*)). 
\end{align*}
\tag{4.7}
\]
We still have from Remark 2.1 that
\[
\int_{(\xi_{i-1}, \xi_i) \times Y_i^*} a_0^i \nabla \varphi_i + |u_i|^{p-2} u_i \varphi_i \, dxdy = \int_0^1 f \varphi_i \, dx \quad \text{for each} \ i = 1, \ldots, N. \tag{4.8}
\]
Now take \(\phi \in C_0^\infty(\xi_{i-1}, \xi_i)\) and \(\psi \in W^{1,p}_\#(Y_i^*)\). Extend \(\psi\) periodically in the variable \(y_1\) and define the sequence
\[
v_\varepsilon(x, y) = \varepsilon \phi(x) \psi\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right). \tag{4.9}
\]
We have
\[
\begin{align*}
T_\varepsilon^i v_\varepsilon & \rightarrow 0 \ \text{strongly in} \ L^p((\xi_{i-1}, \xi_i) \times Y_i^*), \\
T_\varepsilon^i (\partial_x v_\varepsilon) & \rightarrow \phi \partial_x \psi \ \text{strongly in} \ L^p((\xi_{i-1}, \xi_i) \times Y_i^*), \\
T_\varepsilon^i (\partial_y v_\varepsilon) & \rightarrow \phi \partial_y \psi \ \text{strongly in} \ L^p((\xi_{i-1}, \xi_i) \times Y_i^*). 
\end{align*}
\]
Thus, taking \(v_\varepsilon\) as a test function in (4.6), we obtain at \(\varepsilon = 0\) that
\[
\int_{(\xi_{i-1}, \xi_i) \times Y_i^*} a_0^i \phi(x) \nabla_y \psi \, dxdy = 0. \tag{4.10}
\]
Hence, we get from (4.9) and the density of the tensor product \(C_0^\infty(\xi_{i-1}, \xi_i) \otimes W^{1,p}_\#(Y_i^*)\) in \(L^p((\xi_{i-1}, \xi_i) \times W^{1,p}_\#(Y_i^*))\) that
\[
\int_{(\xi_{i-1}, \xi_i) \times Y_i^*} a_0^i \nabla_y \psi \, dxdy = 0, \quad \forall \psi \in L^p((\xi_{i-1}, \xi_i}; W^{1,p}_\#(Y_i^*)). \tag{4.10}
\]
Now, let us identify \(a_0^i\) given by (4.7) for each \(i\). For this sake, take \(u_1 \in L^p((\xi_{i-1}, \xi_i); W^{1,p}_\#(Y_i^*))\) and \(u_1^1 \in W^{1,p}(\xi_{i-1}, \xi_i)\) given by (4.7). Extend \(\nabla_y u_1^1\) periodically in the \(y_1\)-direction, and then define
\[
W_\varepsilon(x, y) = (\partial_x u_1^1(x, 0), \nabla_y u_1^1(x, y_1 \varepsilon)), \quad (x, y) \in R_i^\varepsilon.
\]
Notice that $W_\varepsilon \in L^p(R_\varepsilon^i) \times L^p(R_\varepsilon^i)$, and due to Proposition 2.6 we have

$$T^i_\varepsilon W_\varepsilon \to (\partial_x u^i, 0) + \nabla_y u^i_1,$$

and

$$T^i_\varepsilon (|W_\varepsilon|^{p-2} W_\varepsilon) \to |(\partial_x u^i, 0) + \nabla_y u^i_1|^{p-2} [(\partial_x u^i, 0) + \nabla_y u^i_1]$$

(4.11)

strongly in $L^p((\xi_{i-1}, \xi_i) \times Y_i^*)^2$ and $L^p((\xi_{i-1}, \xi_i) \times Y_i^*)^2$ respectively. Moreover, we can see that the right hand side of the inequality

$$0 \leq \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} T^i_\varepsilon \left( |\nabla u^i|^{p-2} \nabla u^i - |W_\varepsilon|^{p-2} W_\varepsilon \right) T^i_\varepsilon (\nabla u^i - W_\varepsilon) \, dx dy_1 dy_2$$

(4.12)

converges to zero as $\varepsilon \to 0$. Notice that by the monotonicity of $|\cdot|^{p-2}$ (see Proposition 2.1) inequality (4.12) is obtained. To pass the limit in (4.12), we evaluate each term of the integral. Using (4.6), (4.7) and denoting $dY = dy_1 dy_2$, we get that

$$\lim_{\varepsilon \to 0} \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} T^i_\varepsilon \left( |\nabla u^i|^{p-2} \nabla u^i \right) T^i_\varepsilon (\nabla u^i) \, dx dy = \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} a_0 \nabla u^i \, dx dy. \quad (4.13)$$

On the other hand, due to (4.7), (4.11) and (4.10), we get

$$\lim_{\varepsilon \to 0} \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} T^i_\varepsilon \left( |\nabla u^i|^{p-2} \nabla u^i \right) T^i_\varepsilon (W_\varepsilon) \, dx dy = \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} a_0 \left( \nabla u^i + \nabla_y u^i_1 \right) \, dx dy = \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} a_0 \nabla u^i \, dx dy, \quad (4.14)$$

since $u^i_1 \in L^p((\xi_{i-1}, \xi_i) \times Y_i^*)$.

Finally, we have

$$\lim_{\varepsilon \to 0} \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} T^i_\varepsilon \left( |W_\varepsilon|^{p-2} W_\varepsilon \right) T^i_\varepsilon (\nabla u^i - W_\varepsilon) \, dx dy = 0, \quad (4.15)$$

by (4.11) and (4.7). Indeed, we have $T^i_\varepsilon (\nabla u^i - W_\varepsilon) \rightharpoonup 0$ weakly in $L^p((\xi_{i-1}, \xi_i) \times Y_i^*)$. Thus, from (4.13), (4.14) and (4.15), we can pass to the limit in (4.12) to get

$$\int_{(\xi_{i-1}, \xi_i) \times Y_i^*} T^i_\varepsilon \left( |\nabla u^i|^{p-2} \nabla u^i - |W_\varepsilon|^{p-2} W_\varepsilon \right) T^i_\varepsilon (\nabla u^i - W_\varepsilon) \, dx dy \to 0. \quad (4.16)$$

Next, suppose $p \geq 2$. By Proposition 2.1 and (4.16), we have

$$\int_{(\xi_{i-1}, \xi_i) \times Y_i^*} |T^i_\varepsilon \nabla u^i - T^i_\varepsilon W_\varepsilon|^p \, dx dy \leq c \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} T^i_\varepsilon \left( |\nabla u^i|^{p-2} \nabla u^i - |W_\varepsilon|^{p-2} W_\varepsilon \right) \left( T^i_\varepsilon \nabla u^i - T^i_\varepsilon W_\varepsilon \right) \, dx dy \to 0 \quad \text{as } \varepsilon \to 0.$$
Now, suppose \( 1 < p \leq 2 \). Then,

\[
\int_{(\xi_{i-1}, \xi_i) \times Y_i^*} \left| T_{i}^t \nabla u_\varepsilon - T_{i}^{t}W_\varepsilon \right|^p \, dx dy
\]

\[
= \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} \left| T_{i}^t \nabla u_\varepsilon - T_{i}^{t}W_\varepsilon \right|^p \frac{\left(1 + |T_{i}^t \nabla u_\varepsilon| + |T_{i}^{t}W_\varepsilon|\right)^{(p-2)p/2}}{(1 + |T_{i}^t \nabla u_\varepsilon| + |T_{i}^{t}W_\varepsilon|)^{(p-2)p/2}} \, dx dy.
\]

Hence, using a Hölder’s inequality for the exponent \( \frac{2}{p} \) (and its conjugate \( \frac{2}{2-p} \)) and Proposition 2.1,

\[
\int_{(\xi_{i-1}, \xi_i) \times Y_i^*} \left| T_{i}^t \nabla u_\varepsilon - T_{i}^{t}W_\varepsilon \right|^p \, dx dy
\]

\[
\leq \left[ \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} \left| T_{i}^t \nabla u_\varepsilon - T_{i}^{t}W_\varepsilon \right|^2 \left(1 + |T_{i}^t \nabla u_\varepsilon| + |T_{i}^{t}W_\varepsilon|\right)^{p-2} \, dx dy \right]^{p/2}
\]

\[
\cdot \left[ \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} \left(1 + |T_{i}^t \nabla u_\varepsilon| + |T_{i}^{t}W_\varepsilon|\right)^{(2-p)/2} \, dx dy \right]^{(2-p)/2}
\]

\[
\leq c \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} T_{i}^t \left(|\nabla u_\varepsilon|^p - |\nabla u_\varepsilon|^p - 2|W_\varepsilon|^p - 2W_\varepsilon\right) (T_{i}^t \nabla u_\varepsilon - T_{i}^{t}W_\varepsilon) \, dx dy.
\]

Consequently, as \( \varepsilon \to 0 \), one gets for any \( p > 1 \) that

\[
\int_{(\xi_{i-1}, \xi_i) \times Y_i^*} \left| T_{i}^t \nabla u_\varepsilon - T_{i}^{t}W_\varepsilon \right|^p \, dx dy \to 0.
\]

(4.17)

Now, we prove

\[
\int_{(\xi_{i-1}, \xi_i) \times Y_i^*} T_{i}^t \left(|\nabla u_\varepsilon|^p - |\nabla u_\varepsilon|^p - 2|W_\varepsilon|^p - 2W_\varepsilon\right) \varphi \, dx dy
\]

for any test function \( \varphi \in C^\infty_0((\xi_{i-1}, \xi_i) \times Y_i^*) \times C^\infty_0((\xi_{i-1}, \xi_i) \times Y_i^*) \).

Let \( p \geq 2 \). Therefore, from Corollary (2.1.1) and Hölder’s inequality, we get

\[
\int_{(\xi_{i-1}, \xi_i) \times Y_i^*} T_{i}^t \left(|\nabla u_\varepsilon|^p - |\nabla u_\varepsilon|^p - 2|W_\varepsilon|^p - 2W_\varepsilon\right) \varphi \, dx dy
\]

\[
\leq c \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} (1 + |T_{i}^t \nabla u_\varepsilon| + |T_{i}^{t}W_\varepsilon|)^{p-2} \left| T_{i}^t \nabla u_\varepsilon - W_\varepsilon\right| \varphi \, dx dy
\]

\[
\leq \left[ \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} (1 + |T_{i}^t \nabla u_\varepsilon| + |T_{i}^{t}W_\varepsilon|)^p \, dx dy \right]^{1/p'} \cdot \left[ \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} \left| T_{i}^t \nabla u_\varepsilon - T_{i}^{t}W_\varepsilon \right|^p \, dx dy \right]^{1/p}.
\]

For \( 1 < p < 2 \), we perform analogous arguments. Using Corollary (2.1.1) and Hölder’s inequality, one gets

\[
\int_{(\xi_{i-1}, \xi_i) \times Y_i^*} T_{i}^t \left(|\nabla u_\varepsilon|^p - |\nabla u_\varepsilon|^p - 2|W_\varepsilon|^p - 2W_\varepsilon\right) \varphi \, dx dy
\]

\[
\leq c \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} |T_{i}^t \nabla u_\varepsilon - T_{i}^{t}W_\varepsilon|^{p-1} \, dx dy = c \left[ \int_{(\xi_{i-1}, \xi_i) \times Y_i^*} \left| T_{i}^t \nabla u_\varepsilon - T_{i}^{t}W_\varepsilon \right|^p \, dx dy \right]^{1/p'}.
\]

Therefore, for any \( p > 1 \), we get from (4.17), (4.18) and (4.18) that

\[
\int_{(\xi_{i-1}, \xi_i) \times Y_i^*} \left[ a_0^i - a_p \left((\partial_x u_i^t, 0) + \nabla_y u_i^t\right)\right] \varphi \, dx dy = 0,
\]

(4.18)

for any \( \varphi \in C^\infty_0((\xi_{i-1}, \xi_i) \times Y_i^*) \times C^\infty_0((\xi_{i-1}, \xi_i) \times Y_i^*) \) and \( i = 1, \cdots, N \).
Now, let us associate $a^i_0$ with the auxiliary problem (4.14). We first rewrite (4.10) as
\[
\int_{(\xi_{i-1}, \xi_i) \times Y^*_i} \left| \frac{\partial x^i u^i}{y^i} \right|^2 \left( \frac{\partial x^i u^i}{y^i} \right) \nabla_y \psi \, dx \, dY = 0, \tag{4.19}
\]
for any $\psi \in L^p((\xi_{i-1}, \xi_i); W^{1,p}_{#}(Y^*_i))$.

From Minty-Browder Theorem, one can prove that (4.19) sets a well posed problem in the following sense: for each $u \in W^{1,p}_{#}(\xi_{i-1}, \xi_i)$, (4.19) possesses an unique solution $u_1 \in L^p((\xi_{i-1}, \xi_i); W^{1,p}_{#}(Y^*_i)/\mathbb{R})$.

Notice that $W^{1,p}_{#}(Y^*_i)/\mathbb{R}$ is identified with the closed subspace of $W^{1,p}_{#}(Y^*_i)$ consisting of all its functions with zero average.

Multiplying the solution $v^i$ of the equation (4.4) by $(u^i)'$ we obtain a function $(u^i)'v$ which depends on $(\xi_{i-1}, \xi_i)$, and belongs to the space $L^p((\xi_{i-1}, \xi_i); W^{1,p}_{#}(Y^*_i)/\mathbb{R})$. Next, multiplying (4.4) by $\left| \frac{\partial x^i u^i}{y^i} \right|^2 \frac{\partial x^i u^i}{y^i}$ and $\phi \in C_0^\infty(\xi_{i-1}, \xi_i)$, and integrating in $(\xi_{i-1}, \xi_i)$, we get
\[
\int_{(\xi_{i-1}, \xi_i) \times Y^*_i} \phi \left| \frac{\partial x^i u^i}{y^i} \right|^2 \frac{\partial x^i u^i}{y^i} \nabla_y \phi \, dx \, dY = 0, \quad \forall \phi \in W^{1,p}_{#}(Y^*_i)/\mathbb{R}.
\]
Thus, from the density of tensor product $C_0^\infty(\xi_{i-1}, \xi_i) \otimes (W^{1,p}_{#}(Y^*_i)/\mathbb{R})$, we get
\[
\int_{(\xi_{i-1}, \xi_i) \times Y^*_i} \left| \frac{\partial x^i u^i}{y^i} \right|^2 \frac{\partial x^i u^i}{y^i} \nabla_y v^i \nabla_y \phi \, dx \, dY = 0, \quad \forall \psi \in L^p((\xi_{i-1}, \xi_i); W^{1,p}_{#}(Y^*_i)/\mathbb{R}). \tag{4.20}
\]

Hence, from equations (4.19) and (4.20), we get, by uniqueness, that
\[
\frac{\partial x^i u^i}{y^i}(\xi_{i-1}, y_2) = \left( \frac{\partial x^i u^i}{y^i}(\xi_{i-1}, 0) \right) + \left( \frac{\partial x^i u^i}{y^i}(\xi_{i-1}, y_2) \right) a.e. \text{ in } (\xi_{i-1}, \xi_i) \times Y^*_i.
\]

Moreover, taking test functions $\phi \in C_0^\infty (U^N_{i-1}(\xi_{i-1}, \xi_i))$ in (4.5), we can use Proposition 2.5 and 4.8 in each interval $(\xi_{i-1}, \xi_i)$ to obtain
\[
\sum_{i=1}^N \frac{1}{L} \int_{(\xi_{i-1}, \xi_i) \times Y^*_i} \left| \frac{\partial x^i u^i}{y^i} \right|^2 \frac{\partial x^i u^i}{y^i} \nabla_y \phi + \left| u^i \right|^2 u^i \phi \, dx = \int_0^1 \hat{f} \phi \, dx,
\]
which is equivalent to
\[
\sum_{i=1}^N \int_{\xi_{i-1}}^{\xi_i} \left[ q^i \frac{\partial x^i u^i}{y^i} \frac{\partial x^i u^i}{y^i} \frac{\partial x^i u^i}{y^i} \phi + \frac{|Y^*_i|}{L} \left| u^i \right|^2 u^i \phi \right] \, dx = \int_0^1 \hat{f} \phi \, dx. \tag{4.21}
\]

Then, using $q^i$ and $r^i$ are given by (4.3), one can obtain the following limit problem
\[
\int_0^1 \left[ q(x) |u'|^2 u' \phi' + r(x) |u|^2 u \phi \right] \, dx = \int_0^1 \hat{f} \phi \, dx, \quad \forall \phi \in W^{1,p}(0, 1),
\]
which has a unique solution $u \in W^{1,p}(0, 1)$, by Minty-Browder’s Theorem. Thus, from (4.21), we get
\[
u(x) = u^i(x) \text{ a.e. in } (\xi_{i-1}, \xi_i)
\]
concluding the proof since $q_i > 0$ for each $i$. Indeed, by (4.4), we can take $(v^i - y_1) \in W^{1,p}_{#,0}(Y^*_i)$ as a test function, obtaining
\[
q^i = \frac{1}{L} \int_{Y^*_i} |\nabla v^i|^2 |\nabla v^i| \left( (1, 0) + |\nabla v^i| (0, 1) \right) \, dy_1 \, dy_2 = \frac{1}{L} \int_{Y^*_i} |\nabla v^i|^2 \, dy_1 \, dy_2 > 0.
\]
\[\square\]
5 The locally periodic case

In this section, we provide the proof of our main result, Theorem 1.1.

Proof of Theorem 1.1. Using Proposition 2.3 and Theorem 2.9, there is \( u_0 \in W^{1,p}(0,1) \) such that, up to subsequences,

\[
T_{\varepsilon}^p u_\varepsilon \rightarrow \chi u_0 \text{ weakly in } L^p((0,1) \times (0,L) \times (0,G_1)),
\]

where \( \chi \) is the characteristic function of \((0,1) \times Y^*(x)\).

We show that \( u_0 \) satisfies the Neumann problem (4.2). To do this, we use a kind of discretization argument on the oscillating thin domains. We first proceed as in [3] Theorem 2.3 fixing a parameter \( \delta > 0 \) in order to set a function \( G^\delta(x,y) \) with the property \( 0 \leq G^\delta(x,y) - G(x,y) \leq \delta \) in \((0,1) \times \mathbb{R}\) and such that the function \( G^\delta \) satisfies (H) and is piecewise periodic.

Let us construct this function. Recall that \( G \) is uniformly \( C^1 \) in each of the domains \((\xi_{i-1}, \xi_i) \times \mathbb{R}\).

Also, it is periodic in the second variable. In particular, for \( \delta > 0 \) small enough and for a fixed \( z \in (\xi_{i-1}, \xi_i) \) we have that there exists a small interval \((z - \eta, z + \eta) \) with \( \eta \) depending only on \( \delta \) such that \( |G(x,y) - G(z,y)| + |\partial_y G(x,y) - \partial_y G(z,y)| < \delta/2 \) for all \( x \in (z - \eta, z + \eta) \cap (\xi_{i-1}, \xi_i) \) and for all \( y \in \mathbb{R} \).

This allows us to select a finite number of points: \( \xi_{i-1} = \xi_{i-1}^0 < \xi_{i-1}^0 < \cdots < \xi_{i-1}^n = \xi_i \) with \( \xi_{i-1}^0 - \xi_{i-1}^n < \eta \) in such a way that \( G^\delta(x,y) = G(\xi_{i-1},y) + \delta/2 \) defined for \( x \in (\xi_{i-1}^0, \xi_{i-1}^0) \) and \( y \in \mathbb{R} \) satisfies \( |\partial_y G^\delta(x,y) - \partial_y G(\xi_{i-1},y)| \leq \delta \) in \((\xi_{i-1}^0, \xi_{i-1}^0) \times \mathbb{R}\). Notice that this construction can be done for all \( i = 1, \ldots, N \). In particular, if we rename all the constructed points \( \xi_i^0 \) by \( 0 = z_0 < z_1 < \cdots < z_m = 1 \), for some \( m = m(\delta) \), we get that \( G^\delta(x,y) = G^\delta(\xi_{i-1},y) \) for \( x \in (z_{i-1}, z_i) \times \mathbb{R} \) and \( i = 1, \ldots, m \) is a piecewise \( C^1 \)-function which is \( L \)-periodic in the second variable \( y \).

Finally, we set \( C^\delta(x) = G^\delta(x,x/\varepsilon) \) considering the following domains

\[
R_i^\varepsilon = \{(x,y) : x \in (0,1), 0 < y < \varepsilon C^\delta(x)\}.
\]

It follows from Theorem 4.1 that, for each \( \delta > 0 \) fixed, there exist \( u^\delta \in W^{1,p}(0,1) \) and \( u_1^\delta \in L^p((\xi_{i-1}, \xi_i); W^{1,p}(Y_i^*)) \) in such way that the solutions \( u_{\varepsilon, \delta} \) of (1.2) in \( R_i^\varepsilon \) satisfy

\[
\begin{aligned}
T_{\varepsilon}^\delta u_{\varepsilon, \delta} &\rightarrow u^\delta \text{ strongly in } L^p((z_{i-1}, z_i); W^{1,p}(Y_i^*)) ,

T_{\varepsilon}^\delta (\partial_y u_{\varepsilon, \delta}) &\rightarrow \partial_y u^\delta + \partial_y u_1^\delta (x, y_1, y_2) \text{ weakly in } L^p((z_{i-1}, z_i); W^{1,p}(Y_i^*)) ,

T_{\varepsilon}^\delta (\partial_y u_{\varepsilon, \delta}) &\rightarrow \partial_y u_1^\delta (x, y_1, y_2) \text{ weakly in } L^p((z_{i-1}, z_i); W^{1,p}(Y_i^*)) ,

T_{\varepsilon}^\delta (|\nabla u_{\varepsilon, \delta}|^p-2\nabla u_{\varepsilon, \delta}) &\rightarrow q^\delta A_p(\partial_y u^\delta) \text{ weakly in } L^p((z_{i-1}, z_i) \times Y_i^*) .
\end{aligned}
\]

Also, we have that \( u^\delta \) is the solution of the Neumann problem

\[
\int_0^1 \left\{ q^\delta(x)(u^\delta)'|^{p-2}(u^\delta)' \phi' + r^\delta(x)|u^\delta|^{p-2}u^\delta \phi \right\} dx = \int_0^1 f(x) dx, \quad \forall \phi \in W^{1,p}(0,1),
\]

with

\[
q^\delta(x) = \frac{1}{L} \sum_{i=1}^{N-1} \chi_{I_i}(x) \int_{Y_i^*} |\nabla v|^p-2 \partial_y v^i dy_1 dy_2 \quad \text{and} \quad r^\delta(x) = \sum_{i=1}^{N-1} \chi_{I_i}(x) \frac{|Y_i^*|}{L} .
\]

\( \chi_{I_i} \) is the characteristic function of \((\xi_{i-1}, \xi_i) \) and \( v^i \) is the solution of (4.4) in \( Y_i^* \) which is given by

\[
Y_i^* = \{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L \text{ and } 0 < y_2 < G_i(y_1)\} .
\]

Now, we pass to the limit in (5.4) as \( \delta \rightarrow 0 \). From Lemma 6.1, we have the uniform convergence of \( q^\delta \) and \( r^\delta \) to \( q \) and \( r \) where

\[
q(x) = \frac{1}{L} \int_{Y^*(x)} |\nabla v|^p-2 \partial_y v^i dy_1 dy_2 \quad \text{and} \quad r(x) = \frac{|Y^*(x)|}{L} .
\]

Notice that \( q(x) > 0 \). Furthermore, we have the solutions \( u^\delta \in W^{1,p}(0,1) \) of (5.4) are uniformly bounded in \( \delta \). Thus, there exists \( u^* \in W^{1,p}(0,1) \) such that \( u^\delta \rightharpoonup u^* \rightharpoonup u^* \) weakly in \( W^{1,p}(0,1) \) and strongly in \( L^p(0,1) \).

Indeed, we have the strong convergence

\[
u^\delta \rightarrow u^* \in W^{1,p}(0,1) .
\]
To prove this, we use the following norm

\[ \| \cdot \|_{L^p_f(0,1)} = \int_0^1 q^\delta \cdot |p| \, dx. \]

By Proposition 2.1 and (5.4), we get for \( \varphi = u^\delta - u^* \) and \( p > 2 \)

\[ \|(u^\delta)' - (u^*)'\|_{L^p_f(0,1)} \leq c \int_0^1 q^\delta \left[ a_p \left( (u^\delta)' - (u^*)' \right) - a_p \left( (u^*)' \right) \right] \, dx \]

\[ = c \int_0^1 (\hat{f} - a_p(u^\delta)) (u^\delta - u^*) \, dx - c \int_0^1 q^\delta a_p \left( (u^*)' \right) \, dx \]

\[ \to 0. \]

Hence, using the equivalence of norms, we get

\[ \|(u^\delta)' - (u^*)'\|_{L^p_f(0,1)} \leq \|(u^\delta)' - (u^*)'\|_{L^p_f(0,1)} \to 0, \]

as \( \delta \to 0 \), which implies (5.7). Thus, we have that \( u^* \in W^{1,p}(0,1) \) satisfies

\[ \int_0^1 \left\{ q(x) |(u^*)'|^{p-2} (u^*)' \varphi' + r(x) |u^*|^{p-2} u^* \varphi \right\} \, dx = \int_0^1 \hat{f} \varphi \, dx, \tag{5.8} \]

for all \( \varphi \in W^{1,p}(0,1) \) and for \( p \geq 2 \). For \( 1 < p < 2 \), we use similar arguments as in the proof of Theorem 4.1 when we obtained (4.17).

To finish the proof, we need to show that \( u^* = u_0 \) in \((0,1)\) where \( u_0 \) is given in (5.1).

Let \( \eta \) be a positive small number and let \( \varphi \in C_0^\infty(0,1) \). Notice that

\[ \int_0^1 (u_0 - u^*) \varphi \, dx = \int_0^1 \left( u_0 - \frac{L}{|Y^*(x)|} \int_0^{\varepsilon G_1(x)} u_\varepsilon(x,y) \, dy \right) \varphi(x) \, dx \]

\[ + \int_0^1 \left( \frac{L}{|Y^*(x)|} \int_0^{\varepsilon G_1(x)} u_\varepsilon(x,y) - P_{1+\delta/G_0} u_{\varepsilon,\delta}(x,y) \, dy \right) \varphi(x) \, dx \]

\[ + \int_0^1 \left( \frac{L}{|Y^*(x)|} \int_0^{\varepsilon G_1(x)} P_{1+\delta/G_0} u_{\varepsilon,\delta}(x,y) - u^\delta(x) \, dy \right) \varphi(x) \, dx \]

\[ + \int_0^1 \left( \frac{L}{|Y^*(x)|} \int_0^{\varepsilon G_1(x)} u^\delta(x) - u^*(x) \, dy \right) \varphi(x) \, dx, \tag{5.9} \]

where \( P_{1+\delta/G_0} \) is the operator defined in (3.3).

Now, due to definition (3.3), notation (3.4) and an appropriated change of variables, we get

\[ \int_0^1 \left( \frac{L}{\varepsilon} \int_0^{\varepsilon G_1(x)} P_{1+\delta/G_0} u_{\varepsilon,\delta}(x,y) - u^\delta(x) \, dy \right) \varphi(x) \, dx \leq c \| P_{1+\delta/G_0} u_{\varepsilon,\delta} - u^\delta \|_{L^p_f(0,1)} \]

\[ \leq c \| P_{1+\delta/G_0} u_{\varepsilon,\delta} - u^\delta \|_{L^p_f(0,1)} = c \| u_{\varepsilon,\delta} - u^\delta \|_{L^p_f(0,1)}. \]

Thus, we can rewrite (5.9) as

\[ \left| \int_0^1 (u_0 - u^*) \varphi \, dx \right| \leq \left| \int_0^1 \left( u_0 - \frac{L}{\varepsilon} \int_0^{\varepsilon G_1(x)} u_\varepsilon(x,y) \, dy \right) \varphi(x) \, dx \right| \]

\[ + c \| u_{\varepsilon,\delta} - P_{1+\delta/G_0} u_{\varepsilon,\delta} \|_{L^p_f(0,1)} + c \| u_{\varepsilon,\delta} - u^\delta \|_{L^p_f(0,1)} + c \| u^\delta - u^* \|_{L^p_f(0,1)}. \]

From (5.3) and Remark 3.2, we can take \( \delta > 0 \) small enough such that \( \| u_{\varepsilon,\delta} - P_{1+\delta/G_0} u_{\varepsilon,\delta} \|_{L^p_f(0,1)} \leq \eta \) and \( \| u_{\varepsilon,\delta} - u^\delta \|_{L^p_f(0,1)} \leq \eta \) uniformly in \( \varepsilon > 0 \). Also, from (5.7), we can choose \( \varepsilon_1 > 0 \) such that \( \| u^* - u^\delta \|_{L^p_f(0,1)} \leq \eta \) for \( 0 < \varepsilon < \varepsilon_1 \).
Moreover, from (5.1) and Proposition 2.11, we have
\[
\int_0^1 \left(u_0 - \frac{L}{|Y^*(x)|}\right) \int_0^{\varepsilon G(x)} u_\varepsilon(x,y)dy \varphi(x)dx \to 0, \quad \varepsilon \to 0.
\]

Therefore, there exists \(\varepsilon_2 > 0\) such that
\[
\left| \int_0^1 \left(u_0 - \frac{L}{|Y^*(x)|}\right) \int_0^{\varepsilon G(x)} u_\varepsilon(x,y)dy \varphi(x)dx \right| \leq \eta
\]
whenever \(0 < \varepsilon < \varepsilon_2\). Hence, setting \(\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}\) we get
\[
\left| \int_0^1 (u_0 - u^*)\varphi dx \right| \leq 4\eta
\]
Since \(\varphi\) and \(\eta\) are arbitrarily, we conclude that \(u^* = u_0\). \hfill \Box

\section{Appendix}

In the proof of the main result, we used \(q^\delta \to q\) uniformly to obtain (5.8). Recall that \(q^\delta\) and \(q\) are given by (5.5) and (5.6) respectively. Here we prove such convergence. For this sake, let us first set
\[
A(M) = \{ G \in C^1(\mathbb{R}) : G \text{ is } L \text{ - periodic, } 0 < G_0 \leq G(\cdot) \leq G_1 \text{ with } |G'(s)| \leq M \}.
\]

Hence, for any \(G \in A(M)\), we can consider the problem
\[
\int_{Y_G^*} |\nabla \bar{v}|^{p-2} \nabla \bar{v} \nabla \varphi dy_1 dy_2 = 0, \quad \forall \varphi \in W_{\#}^{1,p}(Y_G^*)
\]
where \(W_{\#}^{1,p}(Y_G^*)\) is the space of functions \(W_{\#}^{1,p}(Y_G^*)\) with zero average,
\[
Y_G^* = \{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L, 0 < y_2 < G(y_2)\}
\]
and we are looking for solutions \(\bar{v}\) such that \((\bar{v} - y_1) \in W_{\#}^{1,p}(Y_G^*)\).

Now, for any \(G, \bar{G} \in A(M)\), let us consider the following transformation
\[
L : Y_G^* \to Y_{\bar{G}}^*
\]
\[
(z_1, z_2) \to (z_1, F(z_1)z_2) = (y_1, y_2)
\]
where
\[
F = \frac{\bar{G}}{G}.
\]

The Jacobian matrix for \(L\) is
\[
JL(z_1, z_2) = \begin{pmatrix}
1 & 0 \\
F'(z_1)z_2 & F(z_1)
\end{pmatrix}
\]
with \(\det(JL) = F\). Also, we can consider
\[
\mathcal{L} \nabla U = \begin{pmatrix}
1 & -F'z_2 \\
0 & \frac{1}{F}
\end{pmatrix} \nabla U = \begin{pmatrix}
\partial_{z_1} U - \frac{F'}{F} z_2 \partial_{z_2} U, \\
1 \partial_{z_2} U
\end{pmatrix}
\]
and
\[
\mathcal{B} \nabla U = \begin{pmatrix}
\partial_{z_1} U + \frac{F'}{F} z_2 \partial_{z_2} U, \\
\frac{F'}{F} z_2 \partial_{z_1} U + \frac{1}{F^2} \left[1 + (z_2 F')^2\right] \partial_{z_2} U
\end{pmatrix}.
\]

It is not difficult to see that \(\mathcal{B} = \mathcal{L}^T \mathcal{L}\).
Then, we can use the change of variables given by \( L \) to rewrite (6.2) in the region \( Y_G^* \) as
\[
\int_{Y_G^*} |\mathcal{L}\nabla\bar{v}|^{p-2} \mathcal{L}\nabla\bar{v} \mathcal{L} \nabla \left( \frac{\varphi}{F} \right) F \, dz_1 dz_2 = 0, \forall \varphi \in W^{1,p}_{\#,0}(Y_G^*). \tag{6.3}
\]
Notice that this problem still has unique solution \( \bar{v} \in W^{1,p}(Y_G^*) \) with \( (\bar{v} - z_1) \in W^{1,p}_{\#,0}(Y_G^*) \) by Minty-Browder’s Theorem.

By the coercivity of (6.3), we get
\[
\|\nabla \bar{v}\|^p_{L^p(Y_G^*)} \leq \int_{Y_G^*} |\mathcal{L}\nabla\bar{v}|^{p-2} \mathcal{L}\nabla\bar{v} \mathcal{L} \nabla \left( \frac{\bar{v}}{F} \right) F \, dz_1 dz_2 \leq c \|\nabla \bar{v}\|^p_{L^p(Y_G^*)},
\]
which means that the solutions are uniformly bounded by a constant independent on \( G \) and \( G \).

Now, let us compare the solutions of (6.2) for \( G = G \) and (6.3). We need to analyze
\[
\int_{Y_G^*} |\mathcal{L}\nabla\bar{v}|^{p-2} \mathcal{L}\nabla\bar{v} - |\nabla v|^{p-2} \nabla v \right) (\mathcal{L}\nabla\bar{v} - \nabla v) dz_1 dz_2.
\tag{6.4}
\]

Notice that \( \mathcal{L}(1,0) = (1,0) \). We will distribute the terms finding estimative for each one.

First, observe that for any test function \( \varphi \in W^{1,p}_{\#,0}(Y_G^*) \) in (6.3), we have
\[
\int_{Y_G^*} |\mathcal{L}\nabla\bar{v}|^{p-2} \mathcal{L}\nabla\bar{v} \mathcal{L} \nabla \varphi dz_1 dz_2 = \int_{Y_G^*} |\mathcal{L}\nabla\bar{v}|^{p-2} \mathcal{L}\nabla\bar{v} \left( \frac{F'}{F}, 0 \right) dz_1 dz_2. \tag{6.5}
\]

Now, take \( \varphi = (\bar{v} - z_1) \) in (6.5). Then,
\[
\int_{Y_G^*} |\mathcal{L}\nabla\bar{v}|^{p-2} \mathcal{L}\nabla\bar{v} \mathcal{L} \nabla (\bar{v} - z_1) dz_1 dz_2 = \int_{Y_G^*} |\mathcal{L}\nabla\bar{v}|^{p-2} \mathcal{L}\nabla\bar{v} \left( \frac{F'}{F}, 0 \right) dz_1 dz_2. \tag{6.6}
\]

On the other side, we can compute
\[
\int_{Y_G^*} \left| \mathcal{L}\nabla\bar{v} \right|^{p-2} \mathcal{L}\nabla\bar{v} \left( (1,0) - \nabla v \right) dz_1 dz_2 = \int_{Y_G^*} \left| \mathcal{L}\nabla\bar{v} \right|^{p-2} \mathcal{L}\nabla\bar{v} ((1,0) - \nabla v + \mathcal{L}\nabla v - (1,0) + (1,0) - \mathcal{L}\nabla v) dz_1 dz_2 \tag{6.7}
\]

by (6.5) with \( \varphi = (z_1 - v) \).

Next, take \( (\bar{v} - z_1) \in W^{1,p}_{\#,0}(Y_G^*) \) as a test function in (6.2). Then,
\[
\int_{Y_G^*} |\nabla v|^{p-2} \nabla v (\nabla \bar{v} - (1,0)) dz_1 dz_2 = 0. \tag{6.8}
\]

Finally, due to (6.8), we have
\[
\int_{Y_G^*} |\nabla v|^{p-2} \nabla v (\mathcal{L}\nabla \bar{v} - (1,0)) dz_1 dz_2
\]
\[
= \int_{Y_G^*} |\nabla v|^{p-2} \nabla v (\mathcal{L}\nabla \bar{v} - (1,0)) dz_1 dz_2 - \int_{Y_G^*} |\nabla v|^{p-2} \nabla v (\nabla \bar{v} - (1,0)) dz_1 dz_2
\]
\[
= \int_{Y_G^*} |\nabla v|^{p-2} \nabla v (\mathcal{L} - I) \nabla \bar{v} dz_1 dz_2. \tag{6.9}
\]
Hence, putting together (6.4), (6.6), (6.7), (6.8) and (6.9), we obtain

\[ \int_{Y_G} \left[ (\mathcal{L} \nabla \tilde{v}) |^{p-2} \mathcal{L} \nabla \tilde{v} - |\nabla v| |^{p-2} \nabla v \right] (\mathcal{L} \nabla \tilde{v} - \nabla v) dz_1 dz_2 \]
\[ = \int_{Y_G} |\mathcal{L} \nabla \tilde{v}| |^{p-2} \mathcal{L} \nabla \tilde{v}(\tilde{v} - z_1) \left( \frac{F'}{F}, 0 \right) dz_1 dz_2 \]
\[ - \int_{Y_G} |\mathcal{L} \nabla \tilde{v}| |^{p-2} \mathcal{L} \nabla \tilde{v}(\mathcal{L} - I) \nabla v dz_1 dz_2 + \int_{Y_G} |\mathcal{L} \nabla \tilde{v}| |^{p-2} \mathcal{L} \nabla \tilde{v}(z_1 - v) \left( \frac{F'}{F}, 0 \right) dz_1 dz_2 \]
\[ - \int_{Y_G} |\nabla v| |^{p-2} \nabla v(\mathcal{L} - I) \nabla \tilde{v} dz_1 dz_2. \] (6.10)

Now, one can apply Hölder and Poincaré-Wirtinger’s inequalities in (6.10) to obtain

\[ \int_{Y_G} |\mathcal{L} \nabla \tilde{v}| |^{p-2} \mathcal{L} \nabla \tilde{v} - |\nabla v| |^{p-2} \nabla v \right] (\mathcal{L} \nabla \tilde{v} - \nabla v) dz_1 dz_2 \]
\[ \leq \| \mathcal{L} \nabla \tilde{v} \|_{L^p(Y_G)} \| \nabla \tilde{v} \|_{L^p(Y_G)} \left\| \frac{F'}{F} \right\|_{L^\infty} + \| \mathcal{L} \nabla \tilde{v} \|_{L^p(Y_G)} \| \mathcal{L} - I \|_{L^\infty} \| \nabla v \|_{L^p(Y_G)} \] (6.11)
\[ + \| \mathcal{L} \nabla \tilde{v} \|_{L^p(Y_G)} \| \nabla v \|_{L^p(Y_G)} \left\| \frac{F'}{F} \right\|_{L^\infty} + \| \nabla v \|_{L^p(Y_G)} \| \mathcal{L} - I \|_{L^\infty} \| \nabla \tilde{v} \|_{L^p(Y_G)}. \]

Note that

\[ \left\| \frac{F'}{F} \right\|_{L^\infty} \leq c \| \tilde{G} - G \|_{C^1} \text{ and } \| \mathcal{L} - I \|_{L^\infty} \leq c \| \tilde{G} - G \|_{C^1}. \] (6.12)

Also, \( \| \nabla v \|_{L^p(Y_G)} \), \( \| \nabla \tilde{v} \|_{L^p(Y_G)} \), \( \| \mathcal{L} \nabla \tilde{v} \|_{L^p(Y_G)} \) and \( \| \mathcal{L} \nabla v \|_{L^p(Y_G)} \) are uniformly bounded. Thus, by (6.11)

\[ \int_{Y_G} \left[ |\mathcal{L} \nabla \tilde{v}| |^{p-2} \mathcal{L} \nabla \tilde{v} - |\nabla v| |^{p-2} \nabla v \right] (\mathcal{L} \nabla \tilde{v} - \nabla v) dz_1 dz_2 \leq c \| \tilde{G} - G \|_{C^1}. \] (6.13)

If \( p \geq 2 \), we get from Proposition 2.1 and (6.13) that

\[ \| \mathcal{L} \nabla \tilde{v} - \nabla v \|_{L^p(Y_G)} \leq c \int_{Y_G} \left[ |\mathcal{L} \nabla \tilde{v}| |^{p-2} \mathcal{L} \nabla \tilde{v} - |\nabla v| |^{p-2} \nabla v \right] (\mathcal{L} \nabla \tilde{v} - \nabla v) dz_1 dz_2 \]
\[ \leq c \| \tilde{G} - G \|_{C^1}. \]

On the other side, if \( 1 < p < 2 \), we get from Hölder’s inequality, Proposition 2.1 and (6.13), that

\[ \| \mathcal{L} \nabla \tilde{v} - \nabla v \|_{L^p(Y_G)} \leq c \left\{ \int_{Y_G} \left[ |\mathcal{L} \nabla \tilde{v}| |^{p-2} \mathcal{L} \nabla \tilde{v} - |\nabla v| |^{p-2} \nabla v \right] (\mathcal{L} \nabla \tilde{v} - \nabla v) dz_1 dz_2 \right\}^{p/2} \]
\[ \leq c \| \tilde{G} - G \|_{C^1}^{p/2}, \]

Therefore, for \( 1 < p < \infty \), we have

\[ \| \mathcal{L} \nabla \tilde{v} - \nabla v \|_{L^p(Y_G)} \leq c \| \tilde{G} - G \|_{C^1}^{\alpha}, \] (6.14)

where \( \alpha = 1/2 \) if \( 1 < p < 2 \) and \( \alpha = 1/p \) if \( p \geq 2 \).

Finally, since

\[ \| \nabla \tilde{v} - \nabla v \|_{L^p(Y_G)} \leq \| \mathcal{L} \nabla \tilde{v} - \nabla \tilde{v} \|_{L^p(Y_G)} + \| \mathcal{L} \nabla \tilde{v} - \nabla v \|_{L^p(Y_G)}, \]

we conclude by (6.14) and (6.12) that

\[ \| \nabla \tilde{v} - \nabla v \|_{L^p(Y_G)} \leq c \| \tilde{G} - G \|_{C^1} + c \| \tilde{G} - G \|_{C^1}^{\alpha}. \]

We have the following lemma:
Lemma 6.1. Let us consider the family of admissible functions $G \in A(M)$ for some constant $M > 0$ where $A(M)$ is defined by (6.1).

Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $G, \bar{G} \in A(M)$ with $\|\bar{G} - G\| \leq \delta$, then

$$\|\nabla \bar{v} - \nabla v\|_{L^p(Y^*_G)} \leq c(\varepsilon + \varepsilon^\alpha),$$

where $\alpha = 1/2$ if $1 < p < 2$ and $\alpha = 1/p$ if $p \geq 2$ and $c$ is a constant which depends only on $p, G_0, G_1$.

In particular, we have that

$$|q(\bar{G}) - q(G)| \leq c(\varepsilon + \varepsilon^\alpha),$$

where

$$q(\bar{G}) = \int_{Y^*_G} |\nabla \bar{v}|^{p-2}\partial_{y_1}\bar{v} dy_1 dy_2$$

and $\bar{v}$ is the solution of (6.2) in the region $Y^*_G$ set by $\bar{G}$.

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