Critical velocities in two-component superfluid Bose gases

Keywords  multi-component Bose-Einstein condensate, critical velocity

Abstract  On the ground of the Landau criterion we study the behavior of critical velocities in a superfluid two-component Bose gas. It is found that under motion of the components with different velocities the velocity of each component should not be lower than a minimum phase velocity of elementary excitations ($s_-$). The Landau criterion yields a relation between the critical velocities of the components ($v_{c1}$, $v_{c2}$). The velocity of one or even both components may exceed $s_-$. The maximum value of the critical velocity of a given component can be reached when the other component does not move. The approach is generalized for a two-component condensate confined in a cylindrical harmonic potential.

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1 Introduction

At present considerable attention is given to the study of two-component Bose-Einstein condensates. The progress in cooling and trapping of atomic rarefied gases allows to obtain such condensates experimentally. From the fundamental point of view, they are considered as objects in which some cosmological and astrophysical processes can be modelled. In two component systems the superfluid components may flow with different velocities. In such a situation an unusual (nondissipative) kind of the drag effect takes place. It makes possible to create a controlled phase difference between two Bose-condensates placed in a two-well potential, and to observe effects, similar to ones that occur in superconductive systems with Josephson contacts in magnetic fields. In this article...
we study a related problem, namely, the critical velocities in a two component superfluid system with components flowing with different velocities.

2 The Landau criterium for the two-component system

According to the Landau criterion, the critical velocity in a single-component condensate is determined by the expression

$$v_c = \min \left( \frac{E_0(k)}{\hbar k} \right),$$

where $E_0(k)$ is the excitation spectrum in an immovable condensate and $k$ is the wave number. The velocity is given in the frame of reference connected with walls or obstacles. Eq. (1) can be applied to a two-component condensate only in the case when both components move with the same velocity.

The Landau criterium can be reformulated as the requirement of positivity of energies of elementary excitations in the frame of reference, connected with walls or obstacles. These energies depend on the velocities of the components $v_1$ and $v_2$ (in the same frame of reference), and the Landau criterium should yield some joint condition on $v_1$ and $v_2$.

To find the energy of elementary excitations we use the Gross-Pitaevskii equation for the two-component system

$$i\hbar \frac{\partial \psi_j}{\partial t} = -\frac{\hbar^2}{2m_j} \nabla^2 \psi_j + \gamma_j |\psi_j|^2 \psi_j + \gamma_{12} |\psi_{(3-j)}|^2 \psi_j, \quad (j = 1, 2),$$

where $\psi_j$ are the wave functions of the components, $m_j$ are the masses of the particles, $\gamma_j, \gamma_{12}$ are the interaction constants ($\gamma_j = 4\pi\hbar^2 a_{jj}/m_j, \gamma_{12} = 2\pi\hbar^2 (m_1 + m_2)a_{12}/(m_1 m_2)$ where $a_{ik}$ are the scattering lengths).

The wave function of the component can be represented as the sum of the stationary part and the fluctuating part $\psi_j = \psi_{0j} + \delta \psi_j$, where $\delta \psi_j \ll \psi_{0j}$. The stationary part of the condensate wave function can be presented in the form:

$$\psi_{0j}(r, t) = \sqrt{n_j} e^{i\phi_j(r)} e^{-i\mu_j t/\hbar},$$

where $\mu_j = \frac{m_j v_j^2}{2} + \gamma_j n_j + \gamma_{12} n_{3-j}$ are the chemical potentials of the components. The gradients of the phases $\phi_j$ are connected with the superfluid velocities by the relation $v_j = \frac{\hbar}{m_j} \nabla \phi_j$.

The fluctuating part can be written as

$$\delta \psi_j(r, t) = e^{-i\mu_j t/\hbar} e^{i\phi_j(r)} \left[ u_j e^{i(k \cdot r - \omega t)} + v_j^* e^{-i(k \cdot r - \omega t)} \right].$$

The substitution of (4) into the linearized version of (2) leads to a system of equations for $u - v$ coefficients, whose determinant gives the dispersion equation for the spectrum of elementary excitations

$$[E_1^2 - (E - \hbar v_1 \cdot k)] [E_2^2 - (E - \hbar v_2 \cdot k)] - 4\varepsilon_1 \varepsilon_2 \gamma_{12}^2 n_1 n_2 = 0,$$
where $E_j = \sqrt{\varepsilon_j (\varepsilon_j + 2\gamma_j n_j)}$ is the Bogolyubov spectrum for the $j$-component (in the absence of interaction between the components), and $\varepsilon_j = \hbar^2 k^2 / 2m_j$.

For $v_1 = v_2 = v$ the equation (5) yields

$$E_{\pm} = \sqrt{\frac{E_1^2 + E_2^2}{2} \pm \sqrt{\left(\frac{E_1^2 - E_2^2}{2}\right)^2 + 4\gamma_1\gamma_2 \varepsilon_1 \varepsilon_2 n_1 n_2 + \hbar^2 v \cdot v}}. \quad (6)$$

In this study we assume that the condition of stability of a two-component condensate relative to phase separation is fulfilled ($\gamma_1 \gamma_2 - \gamma_1^2 > 0$).

The requirement of positivity of (6) at all $\mathbf{k}$ is equivalent to the condition (1). This condition yields the following expression for the critical velocity

$$v_c = s_- = \frac{1}{\sqrt{2}} \sqrt{s_1^2 + s_2^2 - \sqrt{(s_1^2 - s_2^2)^2 + 4s_1^2 s_2^2 \frac{\gamma_2^2}{\gamma_1 \gamma_2}}}, \quad (7)$$

where $s_j = \sqrt{\gamma_j n_j / m_j}$ are the bare velocities of the sound modes for the components (in the absence of interaction between the components).

In a general case $v_1 \neq v_2$ the Landau criterion requires the existence of two positive solutions of Eq. (5) at all $\mathbf{k}$. This requirement is equivalent to the following two inequalities:

$$[E_1^2 - (\hbar v_1 \cdot \mathbf{k})^2][E_2^2 - (\hbar v_2 \cdot \mathbf{k})^2] - 4\varepsilon_1 \varepsilon_2 \gamma_1 \gamma_2 n_1 n_2 > 0, \quad (8)$$

$$E_1^2 > (\hbar v_1 \cdot \mathbf{k})^2 \quad \text{(or} \quad E_2^2 > (\hbar v_2 \cdot \mathbf{k})^2) \quad . \quad (9)$$

The critical values of $v_1$ and $v_2$ correspond to the case, when the inequality (8) turns into the equality at least for one $\mathbf{k}$. In the case considered the sufficient condition for fulfilling the inequalities (8), (9) for all $\mathbf{k}$ is their fulfillment at $k \to 0$ for all directions of $\mathbf{k}$, complanar to $v_1$ and $v_2$. Therefore, the inequalities (8), (9) can be replaced with the system of inequalities

$$(s_1^2 - v_1^2 \cos^2 \alpha) (s_2^2 - v_2^2 \cos^2 (\theta - \alpha)) - \frac{\gamma_2^2}{\gamma_1 \gamma_2} s_1 s_2^2 > 0, \quad s_1^2 > v_1^2 \cos^2 \alpha \quad (10)$$

(where $\theta$ is an angle between $v_1$ and $v_2$, and $\alpha$ is an angle between $\mathbf{k}$ and $v_1$), that should be fulfilled for all $\alpha$. The results of the analysis of (10) is given in Fig.1

If the components move in the same direction their critical velocities are related by the equation

$$(s_1^2 - v_{c1}^2)(s_2^2 - v_{c2}^2) = \frac{\gamma_2^2}{\gamma_1 \gamma_2} s_1 s_2^2 \quad (11)$$

with the additional condition $v_{c1} < s_1$. According to (11) only for $v_1 = v_2$ the critical velocity coincides with the velocity of the lowest hydrodynamic mode $s_-$. In a general case one of the velocities may exceed $s_-$. (the second velocity should be less than $s_-$. If one of the components does not move the velocity of the other component may reach the maximum value

$$v_{c,j,\text{max}} = s_j \sqrt{1 - \frac{\gamma_2^2}{\gamma_1 \gamma_2}} \quad (12)$$
Fig. 1 Connection between the critical velocities at different $\theta$ (for the parameters $\gamma_{12} = \sqrt{\gamma_1 \gamma_2}/2$). The shown value $s_-$ corresponds to the case $s_1 = s_2$.

One can see from Fig. 1 that at $\theta \neq 0, \pi$ both components may move with velocities that exceed $s_-$. Under the motion of the components in mutually perpendicular directions ($\theta = \pi/2$) the velocities can simultaneously reach the maximum critical values (12).

Here we do not consider the possibility of excitation of vortices. Therefore, strictly speaking, our analysis yields only the upper bound for the critical velocities. Nevertheless, in a number of situations the estimation for the critical velocities presented in this paper is justified completely. For example, this occurs when the superfluid flows past an obstacle with a small (less than the healing length) linear size(11,12,13).

3 Critical velocities in the two-component condensate, confined in a cylindrical harmonic potential

In Bose gases confined in optical or magnetic traps surface excitations have the minimum phase velocity(14). In such systems the process of generating of vortices is connected with the excitation of surface modes, and the critical velocity coincides with the phase velocity of the lowest surface mode(15).

Let us study the critical velocities for a two-component Bose gas, confined in a harmonic cylindrical potential $V(r) = m \omega_0^2 (x^2 + y^2)/2$. We will consider the case when superfluid flows are directed along $z$. For simplicity we assume $n_1(r) = n_2(r) = n(r)$ ($r$ - radial coordinate), $m_1 = m_2 = m$, $\gamma_1 = \gamma_2 = \gamma$ and $0 < \gamma_{12} < \gamma$. Let us consider the system whose Thomas-Fermi radius $R_{TF} = [2(\gamma + \gamma_{12})n_0/m \omega_0^2]^{1/2}$ is much large than the oscillator length of the trap. To find the spectrum of elementary excitations we pass from the Gross-Pitaevskii equation to the linearized system of hydrodynamic equations for the densities $n_j(r,t)$ and the velocities $v_j(r,t)$.
of the components:

$$\frac{\partial \delta n_j}{\partial t} + \nabla (n \delta v_j + v_{0j} \delta n_j) = 0,$$

$$m \frac{\partial \delta v_j}{\partial t} + \nabla ((\gamma \delta n_j + \gamma_{12} \delta n_{3-j} + m v_{0j} \cdot \delta v_j) = 0,$$

where \( n = n_0(1 - r^2/R_T^2) \) is the equilibrium density of the components \( n_0 \) is the density in the center of trap), \( v_{0j} = (0,0,v_j) \) are their superfluid velocities and \( \delta n_j \) and \( \delta v_j \) are the fluctuations of these values.

The modes we are interested in are localized near the surface and the problem considered can be reduced to the problem for the spectrum of excitations in a Bose gas in a linear potential\textsuperscript{16,17,18}. With this simplification we obtain the following dispersion equation

$$\left[ (\omega - kv_1)^2 - \frac{2\gamma n_0 k}{m R_T} \right] \left[ (\omega - kv_2)^2 - \frac{2\gamma n_0 k}{m R_T} \right] - \frac{4\gamma_{12} n_0^2 k^2}{m^2 R_T^2} = 0.$$ \( (14) \)

The region of the applicability of \( (14) \) is bounded from above by the condition \( k \lesssim k_m \) where \( k_m \) is the wave vector for which the contributions of kinetic and potential energies to the excitation energy become comparable. The value of \( k_m \) can be estimated by equating the kinetic energy of the particles \( \hbar^2 k^2/2m \) to the energy of the lowest mode (in the hydrodynamic limit) \( E_{0,-} = \hbar [2(\gamma - \gamma_{12}) n_0 k/m R_T]^1/2 \) what gives \( k_m = 2\sqrt{\frac{mn_0(\gamma - \gamma_{12})}{\hbar^2 R_T^2}} \). At the point \( k \approx k_m \) the dependence of the excitation energy on \( k \) has a bend and the critical velocity can be estimated by substituting of \( k = k_m \) into the equation \( (14) \).

As it follows from \( (14) \) for \( v_1 = v_2 \) the critical velocity is the minimum phase velocity of the lowest surface mode

$$v_c = s_{sf} = \left( \frac{2\gamma n}{m R_T k_m} \right)^{1/2} \sqrt{\frac{1 - \gamma_{12}}{\gamma}}$$ \( (15) \)

If only one component moves, the critical velocity reaches the value

$$v_{c,\text{max}} = \left( \frac{2\gamma n}{m R_T k_m} \right)^{1/2} \sqrt{1 - \frac{\gamma_{12}}{\gamma}} = s_{sf} \sqrt{1 + \frac{\gamma_{12}}{\gamma}}.$$ \( (16) \)

Thus, the highest velocity can be reached when only one component flows.

4 Conclusions

In conclusion, let us discuss a number of possibilities to observe the predicted behavior. One of them is to create barriers not penentable for one of the components. If in the absence of the barriers the components move with equal velocities the appearance of such barriers may increase of the critical velocity. The motion of the
components with different velocities (and in different directions) can also be realized in systems, in which the components are separated spatially, for example, in a bilayer geometry (in that case a long-range interaction between the components is required for observing the effect predicted). The effect is also may be observed in multilayer condensates of electron-hole pairs which can appear in semiconducting heterostructures with an even ($> 2$) number of two-dimensional electron layers. In electron-hole condensates the critical velocities can be easily measured since they are proportional to the value of critical currents.

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