Fall of a Particle to the Center of a Singular Potential: Exact results

Michael I. Tribelsky

1 M. V. Lomonosov Moscow State University, Moscow, 119991, Russia
2 National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), Moscow, 115409, Russia

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A fall of a particle to the center of a singular potential is one of a few fundamental problems of quantum mechanics. Nonetheless, its solution is not complete yet. The known results just indicate that if the singularity of the potential is strong enough, the spectrum of the Schrödinger equation is not bounded from below. However, the wave functions of the problem do not admit the limiting transition to the ground state. Therefore, the unboundedness of the spectrum is only a necessary condition. To prove that a quantum particle indeed can fall to the center, a wave function describing the fall should be obtained explicitly. This is done in the present paper. Firstly, classical collapse is inspected. Then, the quantum problem is analyzed. A family of exact self-similar solutions of the time-dependent Schrödinger equation corresponding to the fall is obtained and discussed. A law for the collapse of the region of the wave function localization to a single point is obtained explicitly. It is shown that the known necessary conditions for the particle to fall simultaneously are sufficient. Comparison of the classical and quantum collapse exhibits striking similarity of the two cases.

I. INTRODUCTION

It is hardly possible to overrate the importance of the Schrödinger equation (SE). Being the keystone of entire non-relativistic quantum mechanics, it also arises in many quite classical problems, see, e.g., [1–5]. Among the vast diversity of problems related to the SE and its applications, there are several revealing its fundamental properties. They have paramount importance and enter into all main textbooks on quantum mechanics. A fall of a particle to the center of a spherically symmetric potential singular at \( r = 0 \), also known as quantum collapse, is one of them [6, 7]. Here \( r \) stands for the coordinate in the spherical coordinate frame. The collapse is a non-relativistic analog of the big-bang since, in the latter, the extension of the conformal time below the big-bang moment gives rise to the collapsing Universe preceding the expanding one observed now [8].

Nonetheless, despite the long-lasting history of the quantum collapse the problem has not been solved completely yet. In particular, even the key question if complete collapse may take place does not have a convincing answer (for more details see below). The reason for that is the fact that the quantum collapse problem is ill-posed: solutions of the SE with a potential which may exhibit the collapse have at the origin of the coordinate frame a singularity of such a kind that they do not have any definite value at this point [6, 7]. This makes impossible to employ the boundary condition imposed at this point and hence to determine the spectrum of the Hamiltonian.

Various regularization procedures proposed to solve the problem, such as a singular potential cutoff at the vicinity of \( r = 0 \), a shift of the boundary conditions from the origin to a small but finite value of \( r \), the renormalization group approach implementation, and the incorporation into the problem nonlinear terms [7, 9–17] to fix the problem with the wave function behavior at \( r = 0 \). However, they do not give the law describing the contraction of the wave function localization region in the course of collapse. Moreover, since the regularizations stabilize the collapse, the fundamental question wether the complete collapse, i.e., a contraction of the wave function localization region into a single point indeed may be exhibited by a solution of the SE, remains open.

Note that implementation of any regularization procedure explicitly or implicitly implies the convergence of the regularized problem solution to that of the initial one when the corresponding regularization parameter (such as the radius of the cutoff) vanishes. This is not the case for the problem in question — due to the non-existence of the wave functions’ limits at \( r \to 0 \) smooth transition from a regularized problem to the initial one cannot be performed. Therefore, the properties of the former and latter may be qualitatively different. Below we will see that the solution to the quantum collapse problem does exhibit very unusual properties, for example, the norm of the wave function occurs to be explicitly time-dependent. It raises the questions about conservation of the probability to find the particle anywhere in the entire space, the correct calculation of mean values, etc. Discussion of these questions sheds new light on fundamental principles of quantum mechanics and its significance extends far beyond the framework of the collapse problem.

We stress that while in actual physical systems the singularity in the potential compatible with the collapse usually does not hold up to \( r = 0 \) being replaced by another behavior at small enough \( r \), it does not remove the highlighted above open questions. The point is that in the most common cases, when the initial size of the wave function localization region is much larger than the radius corresponding to the crossover to another potential, an overwhelming part of the wave function dynamic is described by the SE with the collapse-compatible potential.
For this reason, in the present paper any regularization procedure intentionally is not applied. Answers to the above questions are given by obtaining a family of explicit exact solutions of the full time-dependent the SE with a singular potential.

Last but not least, all solutions discussed in this paper are exact and therefore have the same validity as the underlying dynamical equations themselves. However, one thing is to obtain a result, another is to interpret it. The obtained results are so unusual that their interpretation is not a straightforward matter. The author did his best in this way. However, he does not claim that the given interpretation is the ultimate truth. Perhaps other (even better) interpretations are possible. Thus, the main goal of this publication is to present new exact solutions. As for their interpretation,... maybe somebody else does it better.

The paper has the following structure. In Sec. II the collapse is discussed in the framework of classical mechanics. Sec. III is the key part of the paper. It is devoted to conclusions. Cumbersome calculations are moved to Appendix. Preliminary results of this study were reported in Ref. [18].

II. CLASSICAL COLLAPSE

To understand the main features of the collapse let us first consider its realization in classical mechanics. In this case the equations of motion are exactly integrable for any spherically-symmetric potential \( U(r) \). The implicit dependence \( r(t) \) is given by the expression

\[
t = \pm \int \frac{dr}{\sqrt{\frac{2}{m} [E - U(r)] - \frac{M^2}{m^2 r^2}}},
\]

where \( M \) is the angular momentum, \( m \) stands for the particle’s mass and \( E \) is its energy. In Eq. (1) we have dropped the constant of integration, since it always may be included in \( t \) by the corresponding shift of the origin of the \( t \)-axis. Note that in contrast to the conventional description [19] we keep two signs of the square root, so that \( t \) may be both positive and negative.

It is implied that \( U(r) \) is singular at \( r \to 0 \). Then, in the vicinity of the coordinate origin \( E \) may be neglected relative to \( U \). Next, since the expression under the square root should be non-negative we arrive at the conclusion that the collapse is possible, provided

\[
[r^2 U(r)]_{r \to 0} < -M^2/(2m).
\]

In what follows the potential

\[
U(r) = -\beta/r^2; \quad (\beta > 0)
\]

is of a special interest for us. It should be stressed that potential (3) does exist in nature. For example, the potential singular as \( 1/r^2 \) describes the interaction of a point electric charge with a particle with zero total electric charge but a fixed finite dipole moment [15], etc. Thus, in addition to purely academic interest, the collapse in this potential may have a pretty practical meaning.

The application of condition (2) to potential (3) gives rise to the following restriction:

\[
\beta > \frac{M^2}{2m}.
\]

To begin with, we consider Eq. (1) at \( E = 0 \). Then, Eq. (1) results in two following solutions:

\[
r = \sqrt{-\chi t} \quad \text{at} \quad t < 0 \quad \text{and} \quad r = \sqrt{\chi t}, \quad \text{at} \quad t > 0,
\]

where \( \chi = \frac{2\sqrt{2m\beta - M^2}}{m} > 0 \), see above.

It is also relevant to give the expressions for the radial component of the particle’s momentum \( p_r = m\text{ }d\text{ }r/d\text{ }t \). They are as follows

\[
p_r = \frac{-\chi m}{2\sqrt{-\chi t}} \quad \text{at} \quad t < 0 \quad \text{and} \quad p_r = \frac{\chi m}{2\sqrt{\chi t}} \quad \text{at} \quad t > 0.
\]

The first solution is valid at \( -\infty < t \leq 0 \) and describes the collapse. The second one corresponds to the escape of the particle from the origin of the coordinate frame and valid at \( 0 \leq t < \infty \).

What happens at \( t = 0 \)? Does the collapse smoothly transfer to the escape? To answer these questions we should consider a more general case of a finite \( E \).

Let us begin with the case \( E < 0 \). In this case, once again, Eq. (1) is readily integrated and gives rise to the following result:

\[
r = \sqrt{\frac{2m\beta - M^2 - 4E^2t^2}{2m|E|}}
\]

A drastic difference between Eq. (5) and Eq. (7) is that for the former the solution exists only during the finite period of time: \(-t_0 \leq t \leq t_0\), where

\[
t_0 = \frac{\sqrt{2m\beta - M^2}}{2|E|}.
\]

At \( t = t_0 \) the particle escapes from the singular point of the potential, departures from it to the maximal distance

\[
r_{\text{max}} = \sqrt{\frac{2m\beta - M^2}{2m|E|}}
\]

achieved at \( t = 0 \), turns back, and returns to the coordinate origin at \( t = t_0 \). That is all; there is not any
continuation of the solution beyond the point \( t = t_0 \). The dynamic is over — the particle remains trapped at the singularity forever.

The solution at \( E > 0 \) may be obtained from Eq. \( \text{[7]} \) by the formal replacement \( |E| \to -E \). It exists at \(-\infty < t \leq -t_0 \) and at \( t_0 \leq t < \infty \). The branch at negative \( t \) describes the collapse to the origin, while the one with \( t > 0 \) corresponds to the escape from it. It is important that the two branches are not connected to each other and, therefore describe two independent solutions.

The collapsing stage is over at \( t = -t_0 \). At this moment the particle becomes trapped at the singular point and remains there. At \( t = t_0 \) the escape of the particle begins. However, the escape is completely independent from the collapse — they are two different solutions to the same problem.

One should not be confused with the mutual symmetry with respect to \( t = 0 \) of the moments when the collapse finishes and the moment when the escape begins. The symmetry is a side-effect caused by the specific choice of the integration constant in Eq. \( \text{[1]} \), which is assumed to be equal to zero. Any other choice of the its value breaks the symmetry.

Finally note that at \( E \neq 0 \) the leading approximation to the expansion of the solution \( \text{[1]} \) in small departures from the point \( t = \pm t_0 \) reproduces Eq. \( \text{[5]} \) with exactly the same value of \( \chi \). However, it is important that each time only one of the two branches of Eq. \( \text{[5]} \) is reproduced: either the collapse or the escape, depending of the choice of the signs of \( \pm t_0 \) and \( E \). The other branch does not exist at the given vicinity.

Bearing this in mind, we may conclude that the two branches in Eq. \( \text{[5]} \) are not connected to each other. The collapsing branch is terminated at \( t = 0 \). Then, the particle remains trapped at \( r = 0 \) and may stay there for any period of time. The escape is an independent process, which may start any time (remember the time-shift option in Eq. \( \text{[1]} \) related to a specific choice of the constant of integration). The question “what does initiate the escape?” lies beyond the scope of the present paper and, for the time being, remains open.

Another conclusion, we can draw, is that at the final stages of the collapse (initial stages of the escape) the dynamic at \( E = 0 \) is an attractor for the dynamic at any finite \( E \). Formally, it follows from the mentioned convergence at any finite \( E \) of the r.h.s. of Eq. \( \text{[7]} \) to that of Eq. \( \text{[5]} \) at \( t \to \pm t_0 \). Regarding the convergence itself, it is explained by the fact that due to the singularity of \( U(r) \) in the r.h.s. of Eq. \( \text{[1]} \), \( E \) always may be neglected relative to \( U \) at \( r \to 0 \).

Secondly, a particle’s localization in a given space point must not contradict the uncertainty relations. The latter implies a divergence of the uncertainty of the particle momentum if the localization occurs. Nonetheless, the quantum collapse exhibits remarkable similarity with its classical analog, see below.

Estimates of the spectrum of the SE with the potential

\[
U(r) = -\beta/r^s, \quad \beta > 0, \quad s > 0
\]  

reveal that at \( s > 2 \), there are bound states with energy \( E < 0 \), which are not limited from below. This indicates a possibility of the collapse. At \( s < 2 \), the spectrum is bounded from below, and the collapse cannot happen. The case \( s = 2 \) requires more accurate consideration \([3, 7]\).

Note now that while the spectrum bounded from below does mean that the fall to the center is not possible, the unboundedness of the spectrum from below is just the necessary condition for the fall to happen. To make the fall sure, one has to build the corresponding wave function explicitly. The latter is hardly possible with the help of the stationary-state eigenfunctions since for the given problem, they do not have any limit at \( r \to 0 \). This behavior prevents one from implementing a direct limiting transition to the ground state with \( E \to -\infty \).

On top of that, even if the expression for the stationary-state eigenfunction with \( E \to -\infty \) were obtained, it would not mean the fall can indeed occur. To prove the occurrence of the fall, one has to show that there is at least a single initial state whose wave function has a non-zero projection to this eigenfunction, i.e., that the state corresponding to the localization of the particle at \( r = 0 \) may be indeed excited.

Last but not least is the law describing the contraction with the time of the region of the wave function spatial localization if the fall does happen. For the time being, none of these issues has been resolved. Thus, strictly speaking, the question about a fall of a quantum particle to the origin of a centrally symmetric potential remains open.

The difficulties mentioned above have arisen owing to the attempt to describe an essentially time-dependent process of the fall with the help of stationary-state eigenfunctions. In a sense, it looks like trying to force a square peg into a round hole. To save the day, one has to consider a spatiotemporal evolution of a wave package, and this consideration should not be based on the projection of the corresponding wave function onto the stationary-state eigenmodes \([20]\).

III. QUANTUM COLLAPSE

A. Preliminary

In quantum mechanics, the case is more tricky. Firstly, just a few potentials make the SE exactly integrable. For the time being, none of these issues has been resolved. Thus, strictly speaking, the question about a fall of a quantum particle to the origin of a centrally symmetric potential remains open.

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B. Problem formulation

Thus, we have to start from the full time-dependent version of the SE:

\[
\frac{i\hbar}{\partial t} \Psi = \hat{H} \Psi; \quad \hat{H} \equiv -\frac{\hbar^2}{2m} \Delta + U(r), \quad (11)
\]
where $\Delta$ stands for the Laplasian. Eq. (11) should be supplemented by the initial condition $\Psi = \Psi_0(r)$ and the standard boundary conditions stipulating convergence of $\int |\Psi|^2 d^3r$ for the states corresponding to the bounded in space motion.

Now, let us recall the main concepts of dimensional analysis [21, 22]. Any mathematical function must be a function of dimensionless variables. To make the dimensional $r$ dimensionless one must rescale it dividing by a constant with the same dimension. If such a constant is unique, it determines the characteristic spatial scale of the problem. For the problem in question, the role of this constant may play the initial radius of the wave function spatial region, $r_0$, provided $r_0 < \infty$.

However, if the fall does take place, in the course of it the region of the wave function localization contracts to a point. Then, at final stages of the collapse $r_0$, regardless of its specific numerical value, becomes much larger than the characteristic scale of the wave function spatial variations and ceases to play the role of the scale. This is a prerequisite for a self-similar solution to come into being. In this solution $r$ is rescaled by a certain power of $t$ so that the characteristic spatial scale varies in time having nothing to do with $r_0$. In other words, at the final stages of the collapse, the dynamic of the wave function may become self-similar and hence universal, i.e., independent of the initial condition, provided the latter leads to the collapse, cf. Eq. (5).

Then, we have to check whether Eq. (11) has self-similar solutions and, if so, to find them explicitly. To prove the possibility of the fall, obtaining just a single solution of such a kind is enough. However, in what follows a certain family of such solutions will be found.

C. Self-Similar Problem

As well as it was in Eq. (5), we count the time down from the moment of the complete collapse so that the collapse occurs at $t = 0$, and its dynamic corresponds to the negative values of $t$. Once again, we consider the potential [10]. Let us look for a self-similar solution in the form

$$\Psi = \sum_{\ell} \sum_{m=-\ell}^{\ell} C_{\ell m} Y_{\ell}^m(\theta, \varphi) \Phi_\ell(-\chi t) R_\ell(\xi).$$

Here $C_{\ell m}$ are constants $Y_{\ell}^m(\theta, \varphi)$ stand for the spherical harmonic functions, $\xi = r/(\chi t)^\mu$, where $\chi$ and $\nu$ are unknown yet constants; $\Phi_\ell(-\chi t)$ and $R_\ell(\xi)$ are unknown functions, which, however, do not depend on $m$ owing to the degeneracy of the $Y_{\ell}^m(\theta, \varphi)$ with respect to $m$ in Eq. (11).

Substituting Eq. (12) into Eq. (11) and employing the orthogonality of $Y_{\ell}^m(\theta, \varphi)$ we obtain a detached equation for $\Phi_\ell R_\ell$, where $\ell$ may be any nonnegative integer. Then, without loss of generality, we may drop the sums in Eq. (12), while $C_{\ell m}$ may be dropped owing to the linearity of the problem. Eventually, the SE is reduced to an equation for $\Phi_\ell(-\chi t) R_\ell(\xi)$. The analysis of this equation presented in Appendix indicates that it is consistent with the stipulated self-similarity if and only if

$$\Phi = (-\chi t)^\mu, \quad \mu = \text{const}, \quad \nu = 1/2, \quad s = 1/\nu = 2,$$  
(13)

where the value of $\mu$ (generally speaking, complex) does not obey any constraints yet. Regarding $\chi$, it is convenient to suppose that $\chi = \hbar/m$. Not that according to Eq. (13) $\Phi$ occurs $\ell$-independent.

In this case the SE results in the following equation for $R_\ell(\xi)$:

$$R'_\ell + \left( \frac{2}{\xi} + i\xi \right) R_\ell + \left( \frac{\gamma R_\ell}{\xi^2} - 2i\mu \right) R_\ell = 0.$$  
(14)

Here $\gamma_\ell \equiv 2m\beta/\hbar^2 - \ell(1 + 1)$ and prime denotes $d/d\xi$.

The necessary collapse conditions include the inequality $\gamma_\ell > 1/4$ [6, 7]. It restricts the value of $\ell$ in sum, Eq. (12) by the constraint

$$\ell(\ell + 1) < \frac{2m\beta}{\hbar^2} - \frac{1}{4}.$$  
(15)

Note that the eigenvalue of the square of the angular momentum is $\hbar^2 \ell(\ell + 1)$. Then, designating it as $M^2$ we may rewrite Eq. (15) as

$$\beta > \frac{M^2}{2m} + \frac{\hbar^2}{4}.$$  
(16)

In the classical limit ($\hbar \to 0$) Eq. (16) coincides with the discussed above condition for the classical collapse, see Eq. (4).

Next, according to Eq. (14), each $R_\ell$ is independent of others. Then, finding an explicit expression for $\Psi$ in Eq. (12) suffices to solve Eq. (14) at an arbitrary given value of $\ell$ satisfying inequality (13). Then, from now on, $\ell$ may be regarded as a fixed quantity, and $\Psi$ is understood just as a single term in the r.h.s. of Eq. (12). It makes possible to simplify the notations dropping the subscript $\ell$ in $R_\ell$ and $\gamma_\ell$.

Eq. (14) is exactly integrable. Its general solution is

$$R(\xi) =$$

$$\frac{1}{\sqrt{\xi}} \left[ C_1 \xi^{-i\frac{\alpha}{4}} F_1 \left( \frac{-1 + i\alpha}{4} - \mu; 1 - \frac{i\alpha}{2}; -i\xi^2 \right) \right. + C_2 \xi^{i\frac{\alpha}{4}} F_1 \left( \frac{-1 - i\alpha}{4} - \mu; 1 + \frac{i\alpha}{2}; -i\xi^2 \right) \right].$$  
(17)

Here $C_{1,2}$ are constants of integration, $1F_1(a; b; z)$ designates the Kummer confluent hypergeometric function of the first kind [23], and $\alpha \equiv \sqrt{4\gamma - 1} > 0$. Positiveness of the expression under the square root follows from Eq. (15).
Eq. (17) requires a comment: It includes the real positive quantity $\xi$ in the imaginary powers $\pm i\alpha/2$. Such an expression can be transformed as follows:

$$z^{\pm i\frac{\alpha}{2}} = e^{(2\pi n + \log z) \pm i\frac{\alpha}{2} \log z} = e^{2\pi n} e^{i\frac{\alpha}{2} \log z} = e^{(\pm \pi n + \log z) \pm i \sin \left(\frac{\alpha}{2} \log z\right)},$$

(18)

where $n$ is an arbitrary integer and $z > 0$.

Eq. (18) has an infinite number of branches corresponding to different values of $n$. At any $n$, all these branches have the same phase equal to $(\alpha/2) \log z$, while their modulus vary with $n$ as $\exp(\mp \pi n)$. Therefore, the real and imaginary parts of every branch have the number of zeros which in the same manner increases without limit as $z \to 0$. For the sake of simplicity, in Eq. (17) and in what follows, only the single branch with $n = 0$ is selected.

D. Norm and Mean values

Eq. (17) is a solution to the problem. However, we must normalize the obtained wave function to use it for calculations of the probability density and the mean values of operators. The norm is given by the following expression

$$\langle \Psi | \Psi \rangle = C_r (-\chi t)^{2\mu'} \int_0^\infty R \left(\frac{r}{\sqrt{-\chi t}}\right)^2 r^2 dr$$

$$= C_r (-\chi t)^{2\frac{3}{2} + 2\mu'} \int_0^\infty |R(\xi)|^2 \xi^2 d\xi$$

$$= C_\xi (-\chi t)^{2\frac{3}{2} + 2\mu'}.$$  

(19)

Here $\mu' = \Re \mu$; and $C_r$, $C_\xi$ are certain constants. Note, that $C_\xi$ is meaningful, provided $\int_0^\infty |R(\xi)|^2 \xi^2 d\xi$ converges. The analysis presented in Appendix shows that by the proper choice of the ratio $C_2/C_1$ in Eq. (17) the convergence may be granted at any values of $\mu$ but those which have $\mu' = -3/4$. In the latter case the integral exhibits the logarithmic divergence at the upper limit. On the other hand, as it follows from Eq. (19), $\mu' = -3/4$ is the only value of $\mu'$ when the norm may be time-independent. Thus, the norm of the obtained wave function is always time-dependent.

Then, we face a very unusual in quantum mechanics case of a wave function with a time-dependent norm, and the question is “how to use this norm to calculate the probability density and mean values of operators?” There are two options: (i) to equalize the probability density that at a given moment of time the particle is at a given point of space to $|\Psi|^2/\text{const}$ or (ii) to suppose that this quantity equals $|\Psi|^2/\langle \Psi | \Psi \rangle$.

In case (i) $\langle \Psi | \Psi \rangle$ varies in time as $(-\chi t)^{2\frac{3}{2} + 2\mu'}$. At $\frac{3}{2} + 2\mu' > 0$ it corresponds to the “leakage” of the probability through the singularity in the origin. In other words, the singularity acts as a non-relativistic analog of a black hole, and in the course of the collapse the particle gradually leaves our world, concentrating at the singular point. In case (ii) the probability to find the particle somewhere in the 3D coordinate space is fixed and equals unity.

To understand which one is correct, note that case (i) has a certain internal discrepancy in the choice of the specific value of the normalizing constant. Indeed, if the “leakage” does take place, the value of this constant should be selected so that at the beginning of the collapse its value equal to $\langle \Psi | \Psi \rangle$. Then, during the collapse $\langle \Psi | \Psi \rangle/\text{const}$ gradually decreases from unity to zero.

However, the point is that the collapse begins at $t = -\infty$ when, according to Eq. (19), $\langle \Psi | \Psi \rangle$ turns to infinity. On the other hand, if the normalizing constant is selected so that the equality $\langle \Psi | \Psi \rangle = \text{const}$ holds at a certain finite value of $t$, the probability to find the particle somewhere in the 3D coordinate space at any preceding moment of time becomes larger than unity.

However, the final argument in favor of case (ii) is given by the uncertainty relations. To apply them to the obtained solution we have to calculate the mean value of $\hat{r}$ and $\hat{p}_r \equiv -i\hbar \partial/\partial r$, where the latter stands for the radial component of the momentum. At any given but fixed $t$ from the domain $-\infty < t < 0$ the calculations according to case (i) give rise to the expressions

$$\langle \hat{r} \rangle = c_r^{(i)} (-\chi t)^{2(1+\mu')}; \quad \langle \hat{p}_r \rangle = \hbar c_r^{(i)} (-\chi t)^{1+2\mu'},$$

(20)

where $c_r^{(i)}$ are dimensionless constants. Then, $\Delta r \Delta p_r \sim \hbar (-\chi t)^{3+4\mu'}$ and, obviously, do not satisfy the uncertainty relations (remember that $\mu' \neq -3/4$).

In case (ii) the corresponding expressions read

$$\langle \hat{r} \rangle = c_r^{(ii)} \sqrt{-\chi t}; \quad \langle \hat{p}_r \rangle = c_r^{(ii)} \frac{\hbar}{\sqrt{-\chi t}},$$

(21)

where $c_r^{(ii)}$ are dimensionless constants of the order of unity. Then, in this case $\Delta r \Delta p_r \sim \hbar$, in full compliance with the uncertainty relations.

Based on the discussed above, we select case (ii). We stress that the temporal dependence (21) agrees with that obtained for the classical self-similar solution, cf. Eqs. (3), (6).

Next, we have to calculate the energy of the particle. Note that since the obtained wave function does not belong to the set of the stationary states, only the mean energy, $E = \langle \hat{H} \rangle$ makes sense. According to Eq. (11),

$$E = \langle \Psi | \hat{H} | \Psi \rangle / \langle \Psi | \Psi \rangle = i\hbar \langle \Psi | \partial \Psi / \partial t \rangle / \langle \Psi | \Psi \rangle.$$  

For the self-similar $\Psi$, given by Eqs. (12), (13)

$$\frac{\partial}{\partial t} \left( (-\chi t)^{\mu} R \left( \frac{r}{\sqrt{-\chi t}} \right) \right) = \frac{(\chi t)^{\mu}}{2t} \left[ 2\mu R(\xi) - \xi R'(\xi) \right],$$

(22)

Convergence of the integrals in $\langle \Psi | \partial \Psi / \partial t \rangle$ with $\partial \Psi / \partial t$ following from Eq. (A2) is the same as that in $\langle \Psi | \Psi \rangle$, see Appendix.
Seemingly, this gives rise to a paradoxical conclusion that in the closed system under consideration \( E \) is not conserved and varies in time as \( 1/t \). To avoid the paradox one must conclude that \( \langle \Psi \delta \Psi/\delta t \rangle = 0 \). The same results may be obtained by a tedious integration in a complex plane of the explicit expression for \( \langle \Psi \delta \Psi/\delta t \rangle \). However, actually, it is not required — the above “physical” proof that \( \langle E \rangle = 0 \) is quite rigorous.

Note, the vanishing of \( \langle E \rangle \) also agrees with the physical properties of the obtained solution. Indeed, if \( \langle E \rangle \) were finite and negative, the problem would have had the classically inaccessible region at \( r > r_σ \), where \( r_σ \) satisfies the condition \( U(r_σ) = \langle E \rangle \). On the other hand at \( t \to -\infty \) the radius of the localization region for \( R(\xi) \) defined by Eq. (17) diverges. It means that in the beginning of the collapse an overwhelming part of the wave function of the particle would have been localized in the classically inaccessible region, which is physically meaningless.

If \( \langle E \rangle \) were finite and positive, at \( r \to \infty \) the wave function would have been transformed into a convergent spherical wave, which disagrees with the asymptotics of solution Eq. (17) at \( r \to \infty \), see Appendix.

Finally, we emphasize that the condition \( \langle E \rangle = 0 \) follows just from the fact of the existence of a self-similar solution of the SE. At any \( \langle E \rangle \neq 0 \) the problem has a constant with the physical dimension of length. It plays the role of the problem’s characteristic spatial scale and destroys the self-similarity, cf. above the corresponding discussion of the classical collapse, where the exact self-similar solution \( r = \sqrt{|\xi|} \) exists only at \( E = 0 \).

What about the escape? Since the SE is invariant for time reversal accompanied with complex conjugation, these transformations applied to Eqs. (12), (17) generate the wave function describing the escape of the particle from the center at \( t > 0 \).

### E. Specific Example

To give an impression about the behavior of the obtained solution we consider its specific example at \( \mu = 0 \). In this case, \( R(\xi) \) is given by Eq. (B11). Dropping constant \( C \) and assigning \( \mu \) the zero value we obtain

\[
R(\xi) = \frac{1}{\sqrt{\xi}} \left[ \left( \frac{i \xi^2}{2} \right)^{-\frac{\nu}{2}} \Gamma \left( 1 + \frac{i \alpha}{2} \right) \Gamma \left( 5 - i \alpha \right) \right] \\
\times f_1 \left( 1 - i \frac{\alpha}{4} ; 1 - i \frac{\alpha}{2} - \frac{i \xi^2}{2} \right) - \left( \frac{i \xi^2}{2} \right)^{\frac{\nu}{2}} \Gamma \left( 1 - i \frac{\alpha}{2} \right) \Gamma \left( 5 + i \alpha \right) \\
\times f_1 \left( 1 - i \frac{\alpha}{4} ; 1 + i \frac{\alpha}{2} - \frac{i \xi^2}{2} \right).
\]

Plots of \(|R(\xi)|\), \(\text{Re} R(\xi)\) and \(\text{Im} R(\xi)\) at \( \mu = 0 \) and \( \alpha = 30 \) are presented in Fig. 1 in two substantially different ranges of variations of \( \xi : 1 \leq \xi \leq 10 \) and \( 10^{-16} \leq \xi \leq 10^{-15} \). It is seen that \( \text{Re} R(\xi) \) and \( \text{Im} R(\xi) \) both have self-similar profiles whose characteristic scale monotonically tends to zero at \( \xi \to 0 \). In particular, the phase shift in oscillations of \( \text{Re} R(\xi) \) and \( \text{Im} R(\xi) \) remains fixed at any \( \xi \) so that zero of one function corresponds to a local extremum of the other and vice versa. As a result the oscillations of the real and imaginary parts of \( R(\xi) \) do not affect its modulus, which is a smooth monotonic function of \( \xi \).

This feature is generic for the problem in question and does not depend on the value of \( \alpha \), see Fig. 2 where \(|R(\xi)|^2 \) corresponding to Eq. (23) is presented at several values of \( \alpha \) in the log-log scale. All curves in this figure obey two simple power laws, following from asymptotics (B1) and (B6) with a narrow crossover region from one asymptotic to the other.

### IV. Conclusions

Thus, we have obtained a family of exact solutions (17), describing the quantum collapse (escape) and owing to that constructively proved the existence of the
phenomenon. We also have shown that the necessary conditions for the quantum collapse \([6, 7]\) simultaneously are sufficient.

In addition we have compared the problems of classical and quantum collapse and found out remarkable similarity between them. Specifically, the obtained family of the exact self-similar solutions of the SE describing either the collapse of a particle to the center of the singular potential \([9]\) at \(t < 0\) or escape from it at \(t > 0\) corresponds to the dynamics of a particle with \(\langle E \rangle = 0\) solely. The same restriction is valid for the self-similar solution in the classical case, see Eq. \([5]\). Though, it could be expected since at \(\langle E \rangle \neq 0\) the self-similarity is destroyed owing to the existence in the problem of the characteristic spatial scale \([9]\), what could not be expected is the same dynamic law: \(r = \sqrt{\chi |t|}\) valid in both classical and quantum cases. In the classical case it describes the variation of the distance from the particle to the center of the potential. In the quantum case the dynamic of the spatial localization characteristic scale of the wave function obeys the same law, see Eq. \([13]\).

In the classical case the collapse begins at \(t \to -\infty\) and the dynamic is over at \(t = 0\). Though at \(0 < t < \infty\) Eq. \([5]\) describes the escape from the center, actually, the end of the collapse and the beginning of the escape may be separated by an arbitrary time-gap, which is removed in solution Eq. \([5]\) with the help of the shift of the origin of the \(t\)-axis. In other words the collapse and escape are completely independent of each other and the end of the former does not mean the beginning of the latter. The same may be made in the quantum case since the solution \([17]\) does not admit the analytical continuation through the point \(t = 0\).

Regarding the dynamics at \(\langle E \rangle \neq 0\), though in the classical case the overall dynamic is not self-similar owing to the existence of a constant characteristic scale, \(r_{\text{max}}\) and lasts for the finite time, \(2t_0\), the final stages of the collapse and the initial stages of the escape are self-similar, since at these stages \(r \ll r_{\text{max}}\) and the latter ceases to play the role of the characteristic scale. In other words, the self-similar solution is an attractor for non-self-similar ones. The same claim and for the same reasons is valid in the quantum case too. The characteristic time for the dynamics of quantum processes at \(\langle E \rangle \neq 0\) may be estimated as \(nr_{\text{max}}^2/\hbar\).

Hopefully, this study sheds new light on the old problem of non-relativistic quantum mechanics and provides a better understanding of its fundamental issues.

Appendix A: Self-Similar Version of the SE

Let us find the conditions for the reduction of the SE to a self-similar form which \textit{may} have a solution of the type of

\[
\Psi = \Phi(-\chi t)R(\xi)Y_\nu^{\ell m}(\theta, \varphi). \tag{A1}
\]

with \(\xi = r/(-\chi t)^\nu\). For this type of \(\Psi(r, t)\) we have \(\partial^n \Psi / \partial r^n = (-\chi t)^{-n\nu} \partial^n \Phi / \partial \xi^n\), while

\[
\frac{\partial \Psi}{\partial t} = \frac{\nu \xi \chi \Phi(z) R'(\xi)}{z} - \chi R(\xi) \Phi'(z). \tag{A2}
\]

Here \(z \equiv -\chi t\). Our goal is to reduce the SE to an equation depending solely on \(\xi\) and independent of \(t\). The necessary condition for that is \(\Phi(z)/z\) and \(\Phi'(z)\) in the l.h.s of the SE have the same dependence on \(t\), so that it makes a common for them factor, which may be canceled with the same factor in the r.h.s. of the SE. It means that

\[
\frac{d \Phi}{dz} = \mu \frac{\Phi}{z}, \tag{A3}
\]

where \(\mu\) is an arbitrary constant. Integrating this equation we readily obtain

\[
\Phi = \text{const} \cdot z^\nu \tag{A4}
\]

Substituting Eq. \([A1]\) with this form of \(\Phi(z)\) into Eq. \([11]\) with \(U(r)\) in the form of Eq. \([10]\) we arrive at the following equation:

\[
\begin{align*}
z^{1-2\nu} R''(\xi) + 2 \left[ \frac{i \nu \xi \chi}{\hbar} + \frac{z^{1-2\nu}}{\xi} \right] R'(\xi) \\
+ \left[ \frac{2 \beta m z^{1-\nu s}}{\hbar^2 \xi^s} - \frac{2 \mu \chi}{\hbar} - \frac{\ell (\ell + 1) z^{1-2\nu}}{\xi^2} \right] R(\xi) = 0.
\end{align*} \tag{A5}
\]

It is seen straightforwardly that Eq. \([A5]\) does not depend on \(z\) if and only if \(\nu = 1/2\) and \(s = 1/\nu\), cf. Eq. \([13]\). Regarding \(\chi\), the choice \(\chi = \hbar/m\) is just a matter of convenience in order to turn the corresponding coefficient to unity. Then, Eq. \([A5]\) transforms into Eq. \([14]\).
Appendix B: Norm Convergence

Here we discuss the convergence of \( \int_0^\infty |R(\xi)|^2 \xi^2 d\xi \), where \( R(\xi) \) is given by Eq. (17). Since \( _1F_1(a; b; z) \) is a holomorphic function of \( z \) on the whole complex plane \([23]\) only the convergence at the lower and upper limits should be examined.

The case of the lower limit is simple. Taking into account that \( _1F_1(a; b; z) = 1 + O(z) \) at \( z \to 0 \) \([23]\), we readily obtain that in the proximity of \( \xi = 0 \) the most singular terms in the solution give rise to the expression

\[
|R(\xi)| \approx \frac{1}{\sqrt{\xi}} \left| C_1 \xi^{-3/2} + C_2 \xi^{1/2} \right| \leq \frac{|C_1| + |C_2|}{\sqrt{\xi}}. \tag{B1}
\]

This means the singularity of \( |R(\xi)|^2 \xi^2 \) at \( \xi = 0 \) is integrable. It is remarkable that estimate \( \text{(B1)} \) does not depend on \( \mu \). Thus, the integral in Eq. \( \text{(19)} \) converges at the lower limit at any \( \mu \).

The case \( \xi \to \infty \) is more tricky. The asymptotic expansion of \( _1F_1(a; b; z) \) at \( |z| \to \infty \) reads \([23]\)

\[
_1F_1(a; b; z) = \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a} G(a, a+1-b, -z) + \frac{\Gamma(b)}{\Gamma(a)} e^z a^{-b} G(b, a-1, -a, z), \tag{B2}
\]

where \( G(x) \) stands for the Euler gamma function and

\[
G(a, b, z) = 1 + \frac{ab}{1! z} + \frac{a(a+1)b(b+1)}{2! z^2} + \ldots + \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! z^n}. \tag{B3}
\]

In Eq. \( \text{/B3} \), \( (x)_n \) designates the rising factorial (the Pochhammer function) defined as follows:

\[
(x)_0 = 1, \quad (x)_n = x(x+1)(x+2) \cdots (x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}.
\]

Then, according to Eqs. \( \text{/B2}, \text{/B3} \), the asymptotic expansion for \( _1F_1(a; b; z) \) has two groups of terms originated in the ones proportional to \(-z)^{-a} G\) and \(e^z a^{-b} G\), respectively. Following Eq. \( \text{/B3} \) at \( z \to \infty \) to select the most singular terms in every group we have to replace \( G \) by \( 1 \). It results in the following asymptotic expression for \( R(\xi) \approx 1R(0)(\xi) + 2R(0)(\xi) \), where

\[
1R(0)(\xi) = \left| C_1(0)^2 \xi^{2\mu} \right| \times \left[ \frac{\left( \frac{i}{2} \right)^{\frac{3}{2}} \Gamma(1 - \frac{i\alpha}{4} - \mu)}{\Gamma(\frac{2-i\alpha}{4} + \mu)} C_1 + \frac{\left( \frac{i}{2} \right)^{\frac{3}{2}} \Gamma(1 + \frac{i\alpha}{2})}{\Gamma(\frac{2+i\alpha}{4} + \mu)} C_2 \right], \tag{B5}
\]

\[
2R(0)(\xi) = xC(0)e^{-\frac{i\pi}{2}} \xi^{-3-2\mu} \times \left[ \frac{\left( \frac{-i}{2} \right)^{\frac{3}{2}} \Gamma(1 - \frac{i\alpha}{4} - \mu)}{\Gamma(\frac{-1-i\alpha}{4} - \mu)} C_1 + \frac{\left( \frac{-i}{2} \right)^{\frac{3}{2}} \Gamma(1 + \frac{i\alpha}{2})}{\Gamma(\frac{-1+i\alpha}{4} - \mu)} C_2 \right], \tag{B6}
\]

Here \( 1, 2 C^{(0)} \) are constants. The cumbersome explicit expressions for them may be obtained upon the step-by-step implementation of the above procedure. We do not need these expressions for the further analysis.

With the proper choice of the ratio \( C_1/C_2 \) either \( 1R(0)(\xi) \) or \( 2R(0)(\xi) \) may be nullified. If instead of the leading terms solely we employ the entire infinite series \[B3\], the expression for \( R(\xi) \) becomes the following:

\[
R(\xi) = \sum_{n=0}^{\infty} \left[ 1R^{(n)}(\xi) + 2R^{(n)}(\xi) \right]. \tag{B7}
\]

A remarkable thing, however, is that the expressions for \( 2R^{(n)}(\xi) \) preserves the same structure as that for Eqs. \[B5, B9\]. The only difference is in the change of the prefactor: \( C^{(0)}(0)^{2\mu} \to C^{(n)}(n)^{2\mu-n} \) for \( 1R^{(n)}(\xi) \) and \( 2C^{(0)}e^{-\frac{i\pi}{2}} \xi^{-3-2\mu} \to 2C^{(n)}e^{-\frac{i\pi}{2}} \xi^{-3-2(\mu+n)} \) for \( 2R^{(n)}(\xi) \); the expression in the square delimiters remains the same, common for all \( n \). It means that nullifying the expression in the delimiters we nullify the entire infinite series of the corresponding terms. Specifically, at

\[
C_2 = -\frac{\left( \frac{i}{2} \right)^{\frac{3}{2}} \Gamma(1 - \frac{i\alpha}{4} - \mu)}{\Gamma(\frac{2-i\alpha}{4} + \mu)} C_1 \tag{B8}
\]

all \( 2R^{(n)} \) vanish, and \( R(\xi) = \sum_{n=0}^{\infty} 1R^{(n)}(\xi) \).

At

\[
C_2 = -\frac{\left( \frac{i}{2} \right)^{\frac{3}{2}} \Gamma(1 + \frac{i\alpha}{2})}{\Gamma(\frac{2+i\alpha}{4} + \mu)} C_1 \tag{B9}
\]

all \( 1R^{(n)} \) vanish, and \( R(\xi) = \sum_{n=0}^{\infty} 2R^{(n)}(\xi) \).

In both cases the most singular term is the one with \( n = 0 \). Let us inspect the contribution of these terms to the integral of the norm for the obtained wave function. In case \[B8\] \( |R(\xi)|^2 \sim \xi^{4\mu} \). Then, at \( \xi \to \infty \) the integral in Eq. \[19\] \( \sim \xi^{4\mu+3} \). Its convergence requires \( \mu' < -3/4 \).

Similarly, in case \[B9\] \( |R(\xi)|^2 \sim \xi^{-6-4\mu'} \). Then, the norm integral converges at the upper limit, provided \( \mu' > 3/4 \).

At \( \mu' = -3/4 \) the integral in Eq. \[19\] diverges at the upper limit as \( \log \xi \) owing to the contribution of \( |1, 2 R^{(0)}(\xi)|^2 \). Since the sole value of \( \mu' \) when the norm of the obtained wave function does not depend on time is just \( \mu' = -3/4 \), only time-dependent norms are physically meaningful for the given wave function.

It is convenient to present the explicit form of the obtained solution admitting the normalization. It is as follows:
At \( \mu' < -3/4 \)

\[
R(\xi) = \frac{C}{\sqrt{\xi}} \left[ \left( -\frac{i\xi^2}{2} \right)^{-\frac{i\alpha}{2}} \Gamma \left( 1 + \frac{i\alpha}{2} \right) \Gamma \left( -\frac{1 + i\alpha}{4} - \mu \right) \right]_1 F_1 \left( -\frac{1 + i\alpha}{4} - \mu; 1 - \frac{i\alpha}{2}; -\frac{i\xi^2}{2} \right) - \left( -\frac{i\xi^2}{2} \right)^{\frac{i\alpha}{2}} \Gamma \left( 1 - \frac{i\alpha}{2} \right) \Gamma \left( \frac{1 - i\alpha}{4} - \mu \right) \right]
\]

At \( \mu' > -3/4 \)

\[
R(\xi) = \frac{C}{\sqrt{\xi}} \left[ \left( \frac{i\xi^2}{2} \right)^{-\frac{i\alpha}{2}} \Gamma \left( 1 + \frac{i\alpha}{2} \right) \Gamma \left( \frac{5 - i\alpha}{4} + \mu \right) \right]_1 F_1 \left( -\frac{1 + i\alpha}{4} - \mu; 1 - \frac{i\alpha}{2}; -\frac{i\xi^2}{2} \right) - \left( \frac{i\xi^2}{2} \right)^{\frac{i\alpha}{2}} \Gamma \left( 1 - \frac{i\alpha}{2} \right) \Gamma \left( \frac{5 + i\alpha}{4} + \mu \right) \right]
\]

For the sake of symmetry we have rescaled the constant of integration so that in Eq. (B10)

\[
C_1 = \left( -\frac{i}{2} \right)^{-\frac{i\alpha}{2}} \Gamma \left( 1 + \frac{i\alpha}{2} \right) \Gamma \left( -\frac{1 + i\alpha}{4} - \mu \right) C
\]

while in Eq. (B11)

\[
C_1 = \left( \frac{i}{2} \right)^{-\frac{i\alpha}{2}} \Gamma \left( 1 + \frac{i\alpha}{2} \right) \Gamma \left( \frac{5 - i\alpha}{4} + \mu \right) C
\]

Note that r.h.s.' of both Eq. (B10) and Eq. (B11) vanish at \( \alpha = 0 \). Since the necessary collapse condition reads \( \alpha > 0 \), see Eq. (15), it means that the obtained solution smoothly vanishes at the continuous transformation of the potential from the collapsing to the non-collapsing type at \( \alpha \to 0 \).

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[20] I should confess that the idea is not mine. When I was a junior scientist, my adviser Dr. Sergei I. Anisimov from Landau Institute, told me about this approach to the problem. Moreover, he said that together with Dr. Igor E. Dzyaloshinskii, they had found a solution describing the collapse of the wave package. Many years later, in connection with light scattering by nanoparticles, I came across the problem mathematically analogous to quantum collapse. I remembered this conversation with Dr. Anisimov and asked him for details and references. The reply was that these results had never been published and details he did not remember. Now, when both Dr. Anisimov and Dr. Dzyaloshinskii have passed away, I decide to reobtain these results and make them available to a broad readership as a small token of my great respect to these distinguished scholars.
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