Non-Parametric Inference of Relational Dependence (Supplementary File)

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A PROOFS

In this section we present the proofs of consistency for HSIC and relational HSIC under weak dependence. The approach here is to extend the results of Chwialkowski et al. [2014] and Leucht and Neumann [2013], who analyze degenerate $U$ and $V$-statistics (which includes HSIC as a specific instantiation) under weak dependence in spaces that admit euclidean distances to the more general setting of graph structured spaces. Much of the results carry through after modifications to accommodate the fact that the number of reachable instances at a specific distance is irregular. We present a modification of the relevant proof which shows the convergence of the distribution of degenerate $V$-statistics, which may be of independent interest, and then describe the application to our setting and the extension to relational variables.

A.1 $V$-STATISTICS UNDER RELATIONAL WEAK DEPENDENCE

Let $X = \{X_1, \ldots, X_n\}$ be the set of given observations. Define $h$ to be a symmetric function, taking $m$ arguments. A $V$-statistic is a function defined with respect to $h$ taking the form

$$V(h, X)_n = \frac{1}{n^m} \sum_{i_1, \ldots, i_m \in N^m} h(X_{i_1}, \ldots, X_{i_m})$$

where $N^m$ is defined as the Cartesian product of the set $1, \ldots, n$ and $n$ is the total number of observations. In the sequel, we will write $V(h, X)$ as $V(X)$ to reduce notational clutter. We will refer to $h$ as the core.

We say that a core $h$ is $j$-degenerate if for every $x_1, \ldots, x_j$,

$$E[h(X_1, \ldots, X_j, X_{j+1}^*, \ldots, X_m^*)] = 0$$

where $X_{j+1}^*, \ldots, X_m^*$ are independent samples drawn from the same distribution as $X_1$. A core is called canonical if for all $j \leq m - 1$ it is $j$-degenerate. Finally, we call a $V$-statistic with a 1-degenerate core a degenerate $V$-statistic.

We now provide a proof of consistency of degenerate $V$-statistics for relational data under weak dependence. The strategy of this proof is to first approximate the $V$-statistic with weighted sums of squares, and then apply the central limit theorem to this approximation. The approximation used is the spectral decomposition of the core

$$h(x, y) = \sum_k \lambda_k \Phi(x) \Phi(y)$$

where $\lambda_k$ are the nonzero eigenvalues of $E[h(x, X_0) \Phi(X_0)] = \lambda \Phi(x)$, and $\Phi(x)$ are the associated eigenvectors. This strategy largely mirrors what is found by Leucht and Neumann [2013]. However, in that case, the approximations are

\[\text{In order to prevent confusion, we follow Chwialkowski et al. [2014] and do not follow the canonical convention of calling } h \text{ the kernel.}\]
constructed as a function of distance in time. Our contribution is a generalization of the approximations to network domains that follow the aforementioned assumptions. This is done by considering sets of instances separated by shortest path distance of $k$, rather than assuming that there is always a single instance at distance $k$, and adapting results accordingly.

**Theorem 2.** Let $(Z_k)_k$ be centered, jointly normal random variables with $\text{Cov}(Z_j, Z_k) = \sum_{r=-\infty}^{\infty} \text{Cov}(\Phi_j(X_0), \Phi_k(X_r))$, and $(\lambda_k, (\Phi_k)_k)$ be the sequence of non-zero eigenvalues and corresponding eigenfunctions of $\text{E}[h(x, X_0)\Phi(X_0)] = \lambda \Phi(x)$. Under the aforementioned assumptions, $V_n \xrightarrow{d} Z := \sum_k \lambda_k Z_k^2$, as $n \to \infty$, and $\text{E}Z = \sum_{r \in \mathbb{Z}} \text{E}h(X_0, X_r) < \infty$ i.e., the infinite series that defines $Z$ converges in $L_1$.

**Proof.** Let $(\lambda_k)_k$ be an enumeration of the positive eigenvalues of $\text{E}h(x, X_0)\Phi(X_0) = \lambda \Phi(x)$ sorted in decreasing order, and $(\Phi_k)_k$ be the corresponding eigenfunctions. Following [Leucht and Neumann 2013], we set $\lambda_k := 0, \Phi_k \equiv 0, \forall k > L$, when the number $L$ of non-zeros eigenvalues is finite. We are given a version of Mercer’s theorem (given by Theorem 2 of Sun SUN [2005]) that\[ h^{(k)}(x, y) = \sum_{k=1}^{K} \lambda_k \Phi_k(x)\Phi_k(y) \xrightarrow{K \to \infty} h(x, y), \forall x, h \in \text{supp}(P^{X_0}) \]

[Leucht and Neumann 2013] provide the prerequisites necessary for the equation to converge absolutely and uniform on compact subsets of $\text{supp}(P^{X_0})$, which apply directly in our setting as well. We will consider an approximation of $V_n$ by a $V$-statistic with a kernel with finite spectral decomposition given by $V_n^{(K)} = \frac{1}{n} \sum_{i=1}^{n} h^{(K)}(X,s,t)$. Because $h$ is positive semi-definite by definition, all eigenvalues are non-negative, implying $V_n - V_n^{(K)} \geq 0$. This implies

\[ E |V_n - V_n^{(K)}| = E |V_n - V_n^{(K)}| = E \left[ h(X_0, X_0) - h^{(K)}(X_0, X_0) \right] + \sum_{r=1}^{n-1} 2(1 - r/n)E \left[ h(X_0, X_r) - h^{(K)}(X_0, X_r) \right] \]

By majorized convergence the first term converges to zero as $K \to \infty$. For the second term, repeated application of Cauchy-Schwarz gives

\[ \sum_{r=1}^{n-1} 2(1 - r/n)E \left[ h(X_0, X_r) - h^{(K)}(X_0, X_r) \right] \leq 2 \sum_{r=1}^{\infty} \sum_{j \in \Delta_r} E \left[ \sum_{k=K+1}^{\infty} \lambda_k \Phi_k(X_0)\Phi_k(X_j) \right] \leq 2 \sum_{r=1}^{\infty} \sum_{j \in \Delta_r} \lambda_k \Phi_k^2(X_0) \right] \leq 2 \sum_{k=K+1}^{\infty} \lambda_k \sum_{r=1}^{\infty} \left[ E \left[ \sum_{j \in \Delta_r} \lambda_k \left( \Phi_k(X_j) - \Phi_k(X_j) \right)^2 \right] \leq 2 \sum_{k=K+1}^{\infty} \lambda_k \sum_{r=1}^{\infty} \sqrt{2 \text{max}(\deg) \cdot \text{Lip}(h)} \sqrt{r} \]
We refer to the attribute class of an item class $I$. Theorem 1. The relational dependence is specifically defined between two item classes $I$. Moreover, let $G(I)$ denote a set of items of an item class $I$. Here, we point out two major differences in a multi-relational system:

1. The relational dependence is specifically defined between two item classes $I$ and $I$.
2. The path predicate $\rho$ is likely to be defined with relational queries rather than random walks over a neighborhood.

Now, we revisit definition 1 from the main text with the new notation as follows:

**Definition 1** (Relational Variable). Given a relational schema $\mathcal{S} = (\mathcal{E}, \mathcal{R}, \mathcal{A})$, its instantiation $G$, two item classes $I, J \in \mathcal{I}$ and a path predicate $\rho$, a relational variable $\sigma(v_i, X, G, \rho)$ is the set of attributes $v_j.X$ selected by $\rho$ of items $v_j \in G(J)$ reachable from items $v_i \in G(I)$ such that $X \subset \mathcal{A}(J)$, where the path predicate $\rho$ is a function given by:

$$\rho(v_i, G) : G(I) \mapsto \mathcal{P}(G(J))$$

The necessary assumptions and relational dependence definitions still hold. The major difference arises in the compact representation of the relational kernel. Equation 1 stays valid with an updated notion of path predicate. However, the compact representation in equation 2 is no longer trivial since the adjacency matrix $A$ is no longer directly applicable. There are two potential workarounds. First, since the compact representation is not mandatory for our method to work, we can still work with equation 1 for multi-relational systems. Second, we can essentially consider the bipartite graph between sets of items between item classes $I, J \in \mathcal{I}$ and use the adjacency matrix $A_{I,J}$ of this bipartite graph instead of $A$. Similarly a corresponding degree matrix $D_{I,J}$ can be constructed from $A_{I,J}$.

## C EXPERIMENTS

### C.1 SYNTHETIC ATTRIBUTE GENERATION

Here, we describe the synthetic attribute generation procedure for the three cases mentioned in the main text. Note that, only the generation of $v_i.Y$ differs in null and alternate hypothesis while others stay the same. We consider polynomial dependency model for most of our experiments. $v_i.X$ for case 1 and $v_i.Z$ for cases 2,3 is drawn from a uniform distribution $U(0, 1)$ while $v_i.X$ is always binarized to resemble the effect of treatment assignment. The outcome $v_i.Y$ is generated according to the following equation for marginal dependence (case 1):

$$v_i.Y \sim \begin{cases} U(0, 1) & \text{null} \\ \beta_d \cdot (g(\sigma_x(v_i)))^2 + \epsilon & \text{alternate} \end{cases}$$

(3)

Conditional dependence (case 2) is reflected by the following equation:

$$\begin{align*}
    v_i.X & \sim \beta_c \cdot (v_i.Z)^2 + \epsilon \\
    v_i.Y & \sim \begin{cases} 
        \beta_c \cdot (v_i.Z)^2 + \epsilon & \text{null} \\
        \beta_d \cdot (g(\sigma_x(v_i)))^2 + \beta_c \cdot (v_i.Z)^2 + \epsilon & \text{alternate} \end{cases}
\end{align*}$$

(4)

Here, $\beta_d$ and $\beta_c$ are dependence and confounding coefficients respectively. $\beta_c$ is considered 1.0 in our experiments. $\epsilon$ is noise drawn from standard normal ($N(0, 1)$) distribution. $g$ refers to the mean aggregate function. We can get the generating function for case 3 by replacing $g(\sigma_x(v_i))$ and $v_i.Z$ with $v_i.X$ and $g(\sigma_x(v_i))$ respectively in equation 4. Next, we consider the following procedure to simulate linear threshold model for the diffusion experiment which falls under case 1:

$$\begin{align*}
    T_i & \sim U(0, 1) \\
    v_i.x_{i+1} & = 1(mean(\sigma_x(v_i)) > T_i) \\
    v_i.y_{i+1} & = 1(g(\sigma_x(v_i)) > T_i)
\end{align*}$$

(5)

where we reassign $v_i.x$ values to simulate each diffusion step based on its value in previous step. The $v_i.y$ values are assigned based on $v_i.x$ values in the last diffusion step.

### C.2 IMPACT OF VARIED NOISE VARIANCE

We conducted an experiment where we draw noise variance from a normal distribution $\sigma^2 \sim N(1, 0.2)$ over different trials. From figure we can see a slight change of type-II errors compared to Figure 1 in the main paper. However, the trend seems to be very similar.
Figure 5: Relational dependence impact on Type I/II errors while variance of noise varied \( \sim \mathcal{N}(1, 0.2) \) over multiple trials.

C.3 IMPACT OF ACTIVATION PROBABILITY ON DIFFUSION

Figure 6: Type II error for the Linear Threshold Model on Twitter ego-network.

In order to showcase the applicability of the proposed method on large scale real world relational data, we show an extended version of the diffusion experiment from the main paper. We consider a similar semi-synthetic setup with Twitter ego-network which is a larger real world network consisting 11,176 nodes and 1,44,653 edges [Leskovec and Mcauley, 2012]. We consider a sample of 10,000 nodes and vary the initial activation probabilities. Figure 6 shows the Type-II errors (y-axis) for different diffusion step sizes (x-axis). The lines correspond to the initial activation probabilities (AP) for the diffusion process. We see the general trend of decreasing Type-II error with higher step sizes. It seems to be almost saturated with step 10. Moreover, the result indicates that the test is sensitive to activation probabilities and with higher activation probability, it shows higher type II error.

C.4 COMPARISON TO SOBOLEV INDEPENDENCE CRITERION (SIC)

To show the effectiveness of relational CI methods vs. CI methods developed for i.i.d. data, we compare both RCI methods (NIRD, KRCIT) to a recent i.i.d. CI test, the Sobolov Independence Criterion [Mroueh et al., 2019]. SIC is an interpretable dependency measure between multivariate random variables characterized by integral probability metric between the joint distribution and the product of the marginals. We perform the SIC test on the flattened representation of the relational data, similar to KRCIT. Figure 7 extends the results shown in Figure 1 in the main text. In all three cases, the i.i.d. baseline SIC exhibits high Type I error which shows its poor calibration to reasoning over the relational data.
Figure 7: Type I/II errors with polynomial dependency model on synthetic networks for all three cases.

D REAL-WORLD DEMONSTRATION

One of the main challenges in social studies is to identify the effect of friends on their peers and the strength of such effects in different domains, e.g., health and violence. Studies show that patterns of interactions among adolescents can reveal possible reasons for changes in their behavior over time. The central question in such studies is how to identify and measure the existence of such effects. The proposed independence test can facilitate reasoning over the existence of dependence between peers’ behaviors in social networks by providing a mechanism for falsifying statistical hypotheses.

As a demonstration, we examine the 50 Women dataset [Michell and Amos, 1997]. This dataset has the smoking, sport, drug, alcohol consumption habits of 50 female students, along with their friendship information, over the course of three years. Each of the behavioral variables are coded as categorical variables indicating how regularly women engage in each of the behaviors. Assuming independence between the behavior peers as the null hypothesis, the goal of this analysis is to explore whether the habits of a student’s friends are associated with her habits in subsequent years.

Table 1 shows p-values estimated by our kernel test method considering four attributes in 50 Women dataset. We use column Period to indicate the years we consider for the test, e.g., in Period 1 → 2, 3, we explore students’ behavior change from first year to the second and third year. We consider both the original categorical coding and a binarization of the categorical attributes, which is 1 if the student uses a substance at least once during the year and 0 otherwise. The number of students who did not engage in the behavior during the first time point is shown in column t0, e.g., in the first row of the table, 4 students did not drink alcohol in the first year. We exclude t0 for categorical data (indicated by NA) because the frequency of the habit is intrinsic to the hypothesis of interest in these cases. The last two columns (NIDR_all and NIDR_t0) show p-values measured by NIRD. In NIDR_all and NIDR_t0 we consider all women (whether they have the habit in a year or not), and women who do not have the habit in the first time point, respectively. Overall we find:

- Sports activity of peers is not associated with whether a student plays a sport or not. High values of NIDR_t0 and NIDR_all are enough evidences to accept the null hypothesis of independence.
- Peer smoking habits are associated with students’ frequency of smoking: NIDR_all = 0 and NIDR_t0 < 0.022 for all time periods, except period 2 → 3 where NIDR_t0 ≈ 0.28.
- Peer drug use is not associated with subsequent drug use in previously non-drug using students (NIDR_t0 > 0.05). However, when we consider the effect of drug users on non-drug users and vice versa, it becomes associated in both the
Table 1: Real-world demonstration: exploration of the dependence between the habits of students and their first-hop neighbors in 50 Women dataset

| period | attribute | attribute type | t0 | NIRD_all | NIRD_t0 |
|--------|-----------|----------------|----|----------|---------|
| 1 → 2  | alcohol   | binary         | 4  | 0.425532 | 0.000000|
| 1 → 2  | alcohol   | categorical    | NA | 0.000000 | NA      |
| 1 → 2  | drug      | binary         | 35 | 0.000000 | 0.138298|
| 1 → 2  | drug      | categorical    | NA | 0.000000 | NA      |
| 1 → 2  | smoke     | binary         | 35 | 0.000000 | 0.021277|
| 1 → 2  | smoke     | categorical    | NA | 0.000000 | NA      |
| 1 → 2  | sport     | binary         | 12 | 0.925532 | 0.978723|
| 1 → 2  | sport     | categorical    | NA | 0.925532 | NA      |
| 1 → 2,3| alcohol   | binary         | 5  | 0.114583 | 0.197917|
| 1 → 2,3| alcohol   | categorical    | NA | 0.000000 | NA      |
| 1 → 2,3| drug      | binary         | 35 | 0.000000 | 0.583333|
| 1 → 2,3| drug      | categorical    | NA | 0.000000 | NA      |
| 1 → 2,3| smoke     | binary         | 36 | 0.000000 | 0.000000|
| 1 → 2,3| smoke     | categorical    | NA | 0.000000 | NA      |
| 1 → 2,3| sport     | binary         | 12 | 1.000000 | 0.166667|
| 1 → 2,3| sport     | categorical    | NA | 1.000000 | NA      |
| 2 → 3  | alcohol   | binary         | 3  | 0.125000 | 0.666667|
| 2 → 3  | alcohol   | categorical    | NA | 0.281250 | NA      |
| 2 → 3  | drug      | binary         | 32 | 0.000000 | 0.125000|
| 2 → 3  | drug      | categorical    | NA | 0.000000 | NA      |
| 2 → 3  | smoke     | binary         | 31 | 0.000000 | 0.281250|
| 2 → 3  | smoke     | categorical    | NA | 0.000000 | NA      |
| 2 → 3  | sport     | binary         | 20 | 0.864583 | 0.479167|
| 2 → 3  | sport     | categorical    | NA | 0.864583 | NA      |

use and rate of consumption ($NIDR_{all} = 0$).

- Peer alcohol consumption is associated with the level subsequent alcohol use ($NIDR_{all} = 0$, except in period $2 \rightarrow 3$ where $NIDR_{all} > 0.1$), but not with the decision for a non-drinking student to begin drinking.

Different studies [Michell and Amos, 1997; Pearson and Michell, 2000] deploy 50 women data to explore the association between gender, risk-taking or social position and smoking or drug usage in groups of youngsters. In particular our results comport with Pearson et al. [Pearson and Michell, 2000] who show that drug usage and smoking are contagious among group of friends who are highly connected and people who are loosely connected to a friendship group.

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