On a Special Transformation to a Non-Inertial, Radially Rigid Reference Frame

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Abstract

We discuss the conditions under which a body, moving non-inertially in Minkowski space, can preserve its size. Under these conditions, using a series expansion of the generalized Lorentz transformation, we find a coordinate transformation connecting the laboratory inertial reference frame \( S \) and the rigid non-inertial reference frame \( s \) which moves without its own rotation with respect to \( S \). Direct consequences of this transformation are: (a) desynchronization, in system \( s \), of the coordinate clocks of \( s \) which were previously synchronized in \( S \), and (b) a kinematic contraction of a ruler of system \( s \) observed in \( S \). We also consider the dependence of the transformation vector parameter on the proper coordinates of \( s \).

1. INTRODUCTION

In special relativity (SR), there exists a special transformation of the 4-coordinates which has the meaning of a transition from a laboratory inertial reference frame \( S : (T, R) \) to a radially rigid, arbitrarily accelerated non-inertial reference frame \( s : (t, r) \) (see Section 2 below). It has initially received the name of the generalized Lorentz transformation [1], but further on it will be called the special Lorentz–Moller–Nelson (LMN) transformation.

The reference frame obtained with this transformation possesses the following properties. Its time coordinate is the proper time of the origin, the spatial coordinate axes are Cartesian, and the square root of the sum of squares of the coordinates of a point is the distance between this point and the origin. In such

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2The 3-space metric of an arbitrary radially rigid nonstationary reference frame is still variable and non-Euclidean. In other words, the distance between two fixed points in this reference frame can change, therefore the rigidity of such a reference frame is only radial.
4-coordinates, the metric has an especially simple form \([2]\) (see also \([3]\), p. 404, Eq.(13.71)), with the correction that there is no space-time curvature in SR.

An important role of the LMN transformation in gravitational field theory is contained in the existence of a kinematic approach to gravity. By this approach, an observer at rest in a gravitational field can locally consider him/herself as if moving with acceleration in a space without gravitational field. Meanwhile, an observer falling in a gravitational field can be considered as if being at rest in a laboratory inertial reference frame. By the equivalence principle (EP), all observational results will be independent of whether there is a local gravitational field “as a matter of fact”. Due to this circumstance, to connect the two reference frames, the falling one and the one at rest, one can locally ignore the gravitational field and consider motions in flat space-time. Another important meaning of this transformation in physics is that the explicit coefficients of its differential form represent a connection between tensors in the laboratory frame and the moving, radially rigid frame.

The reference frames considered in general relativity (GR) are usually non-rigid since perfectly rigid bodies do not exist. \(^3\) It can therefore seem that this transformation is useless, or is of methodological interest only. However, under certain conditions, to be discussed in Section 3, the radial rigidity of a reference frame will be a good approximation. In this case, the LMN transformation is meaningful in GR.

Thus it makes sense to find out the physical consequences of the LMN transformation and its characteristic features that distinguish it from the conventional Lorentz transformation. Let us recall that if \(s\) is an inertial reference frame, then, apart from time slowing-down, there are two other basic kinematic effects. These are the Fitzgerald-Lorentz length contraction and Einstein’s relativity of simultaneity. A non-inertial nature of a reference frame, in general, exerts an influence on these effects, slightly changing their magnitudes \([4]\), \([5]\). Besides, there emerges the effect of inhomogeneity of motion of points belonging to an accelerated reference frame \([4]\). It is also necessary to take into account that each effect should be considered in both reference frames \(S\) and \(s\), since the results in the relativistic case are different, and, in particular, for time it is necessary to distinguish the coordinate desynchronization of clocks from the physical one.

In \([6]\), in Eqs. (4) and (13) for an arbitrary reference frame \(s\), two effects were considered: (a) in a laboratory inertial frame, the desynchronization effect for coordinate clocks at rest, synchronized in frame \(s\), and (b) the Lorentz length contraction effect for an \(S\)-frame ruler with respect to \(s\).

\(^3\)In relativity, speaking of a perfectly rigid body, one usually means a fictitious body with conserved distances between a selected point which moves in a certain way and any other point, left to itself.
In [4], [5], for a rectilinear motion of $s$, four effects were considered: (a) Lorentz contraction of an $s$-ruler in $S$; (b) desynchronization of coordinate clocks of $s$ in $s$ if they had been synchronized in $S$; (c) inhomogeneity of the point motion velocities of the reference frame $s$, and (d) desynchronization in $S$ of the clocks measuring the physical time in the accelerated frame. The inhomogeneity of point motion velocities of $s$ in the case of a rectilinear motion was also considered in [7, p. 65, Eq. (6)].

The purpose of the present paper is to determine the first two effects as functions of the proper acceleration, but now for arbitrary motions of frame $s$, and to find out the approximate dependence of the transformation parameter $v$ on the proper coordinates. The physical consequences of the special LMN transformation manifest themselves most clearly if one finds, beforehand, its inverse transformation. The form of the special LMN transformation, unlike that of the Lorentz transformation, is nonlinear, therefore a search for its inverse transformation is not a simple task. This transformation will be sought for by consecutive approximations, being restricted to second powers of proper coordinates of frame $s$.

Further on, in Sections 3 to 5, we consider step by step each relativistic effect, and in Section 6 our resulting relations are compared with those already known.

2. THE SPECIAL TRANSFORMATION TO A RADIIALLY RIGID REFERENCE FRAME

The transformation from the laboratory inertial reference frame $S: (T, R)$ to the rigid non-inertial frame $s: (t, r)$, moving with respect to $S$ without rotation, has the form [1] (here and henceforth $c = 1$)

$$T = \frac{vr}{\sqrt{1-v^2}} + \int_{0}^{t} \frac{dt}{\sqrt{1-v^2}}$$

(1)

$$R = r + \frac{1-\sqrt{1-v^2}}{v^2\sqrt{1-v^2}}(vr)v + \int_{0}^{t} \frac{vdtd}{\sqrt{1-v^2}}.$$

(2)

Here $T$ and $R$ are the time and spatial coordinates of the inertial frame $S$; $t$ and $r$ are the same for the non-inertial frame $s$, respectively. This transformation is parametrized by only one quantity $v(t)$ (the time-dependent velocity of frame $s$), therefore such an orbital motion of $s$ can be called translational. It turns out, however, that the reference frame $s$ also has a certain proper rotation, connected in a certain way with its orbital motion. This rotation is the Thomas proper precession.

Substituting the differentials of (1) and (2) into the expression for the interval
of the inertial frame in rectilinear coordinates,

\[ ds^2 = dT^2 - dR^2, \]  

we obtain the well-known interval

\[ ds^2 = \left[ (1 + W r)^2 - (\Omega \times r)^2 \right] dt^2 - 2(\Omega \times r) dr dt - dr^2, \]  

for a rigid reference frame with its proper acceleration \( W \) and rotating with the Thomas proper precession rate \( \Omega \), where

\[ W = \frac{\dot{v}}{\sqrt{1 - v^2}} + \frac{1 - \sqrt{1 - v^2}}{v^2(1 - v^2)} (\dot{v}v)v, \]  

\[ \Omega = \Omega_T = \frac{1 - \sqrt{1 - v^2}}{v^2\sqrt{1 - v^2}} v \times \dot{v}. \]

This angular frequency depends on the nature of the orbital motion of the reference frame. These relations can be rewritten in terms of the laboratory acceleration \( \dot{V} = dV/dT \)

\[ W = \frac{\dot{V}}{1 - V^2} + \frac{1 - \sqrt{1 - V^2}}{V^2(1 - V^2)} (\dot{V}V)V, \]  

\[ \Omega_T = \frac{1 - \sqrt{1 - V^2}}{V^2(1 - V^2)} V \times \dot{V}. \]

From (7) it follows that

\[ \dot{V} = (1 - V^2)W - \frac{(1 - V^2)(1 - \sqrt{1 - V^2})}{V^2}(VW)V, \]  

\[ V\dot{V} = \sqrt{1 - V^2}^3 VW. \]

3. THE RIGID BODY APPROXIMATION

The LMN transformation is only correct for a perfectly rigid body, whereas a real, arbitrarily moving reference frame does not preserve its proper rulers. For example, the proper size of an elastic body, whose certain point is subject to a jump of velocity or acceleration, will inevitably change. Let us make it clear, in which case a moving elastic body can be treated as an almost rigid one in its proper reference frame. This question is answered by elasticity theory. According to it, the self-energy of an elastic deformation of a ruler is a functional of the specific law of motion of the ruler’s certain point, taken for the origin. A sudden impact, applied to the origin of a ruler fabricated from a homogeneous elastic material,
launches deformation waves propagating with a characteristic period equal to the ratio of the ruler’s proper length $r$ to the velocity of sound in its material, $v_s$. If the proper acceleration $W$ is changing slowly enough during this period, i.e., if

$$\frac{dW}{dt} \ll \frac{v_s W}{r}$$

(11)

the proper acceleration of the ruler’s origin can be considered as an adiabatically slowly changing parameter. In this case, both the ruler’s free energy and its proper deformation, averaged over the period, will depend (apart from the choice of the material) only parametrically on the proper acceleration value at a given time instant.

Such an impact on the elastic ruler’s scale, averaged over the period, since it depends on the material, can be easily taken into account by simply redenoting the ruler’s graduations. Apart from such a systematic action, the ruler’s coordinate will possess an error $\delta r$ due to oscillations caused by the inhomogeneous acceleration. Its value will have the order of magnitude of a product of the scale graduation’s velocity $v_r$ due to a jerk $dW/dt$ by the period of the elastic wave. This velocity will in turn be of the order

$$v_r \sim \frac{r^2}{v_s^2} \frac{dW}{dt}.$$  

(12)

Consequently,

$$\delta r \sim \frac{r^3}{v_s^3} \frac{dW}{dt} \ll r.$$  

(13)

Hence,

$$\frac{dW}{dt} \ll \frac{v_s^3}{r^2}.$$  

(14)

Under the conditions (11), (14), the elastic ruler can be considered to preserve its size. These requirements are certainly classical because the ruler of the reference frame has been assumed to be a classical body, and the whole reasoning concerned only its proper reference frame.

The rigid body approximation constrains the required precision in calculating the quantities characterizing the non-inertial reference frame. Such quantities are represented by power series in $r$. If we restore $c$, then the first-order term will be proportional to $W r/c^2$. The next terms of the expansion of the parameter $v$ in powers of $r$ will be proportional to $(W r)^2/c^4$ and $r^2 dW/c^3 dt$. For a real rather than perfect body, there will emerge one more term connected with an error in determining the coordinate (13) due the ruler’s oscillations. This term will be of the order $r^3 W dW/c^2 v_s^3 dt$. The strictness of the conditions (11) and (14) does not in any way prevent one to choose the quantity $dW/dt$ to be of the order

$$\frac{dW}{dt} \sim \frac{W^2}{c}.$$  

(15)
or

\[
\frac{dW}{dt} \sim \frac{v_s^3}{c^2 r} W .
\]  

(16)

In this case the terms with \( r^2 dW/c^3 dt \) and with \( r^3 W dW/c^2 v_s^2 dt \), respectively, will be of the same order as \((W r)^2/c^4\). Therefore, to neglect the corrections caused by the elastic oscillations (i.e., by the non-rigidity of the reference frame), it is necessary to restrict oneself in the expansions to terms containing the acceleration (but not its derivatives) only linearly.

A separate problem is to calculate the velocities of points of an almost rigid reference frame. Then, for an elastic body, the velocity expansion contains an additional periodic term of the order \( r^2 dW/cv_s^2 dt \) (12), for which nothing prevents one from choosing

\[
\frac{dW}{dt} \sim \frac{v_s^2}{c r} W .
\]  

(17)

In this case such a correction will be of the same order as \( W r/c^2 \). However, if one averages the velocity expression over the elastic wave period, this averaged correction vanishes. Thus, in the expansion for the velocities of points of the reference frame, all quantities should be understood as being time-averaged. In what follows, the quantity \( W r/c^2 \) will be considered to be very small but non-zero. Its higher powers will be neglected

\[(W r)^2/c^4 \cong 0 .\]  

(18)

Thus, let a point of a rigid body, taken for the origin, move sufficiently smoothly, without abrupt jerks in its proper acceleration. Then the elastic properties of this body forming the reference frame can be ignored, and one can consider its proper size to preserve in the process of motion. In this case, the transformation (1), (2) to a radially rigid non-inertial reference frame moving with its proper Thomas precession, is physically meaningful.

4. THE TRANSFORMATION PARAMETER AS A FUNCTION OF LABORATORY TIME AND PROPER COORDINATES

The inverse transformation of (1), (2) can be obtained in the general case in an analytic form only approximately, as a power series in the components of \( r \). In what follows, the dependence of a function on a certain quantity is shown by a subscript. Let us denote the time integral as

\[
\int_0^t \frac{dt}{\sqrt{1 - v^2}} = \theta_t .
\]  

(19)
Then, expressing $t$ from (19) and substituting it into $v(t)$, one can present it as a function of $\theta$

$$v = V_{\theta}. \quad (20)$$

So the first equation of the transformation (1),(2) takes the form

$$T = \theta + \frac{V_{\theta}r}{\sqrt{1 - V_{\theta}^2}}. \quad (21)$$

The function $V_{\theta}$ is the velocity of a point $r$ of the frame $s$ at time $\theta$ with respect to the inertial reference frame $S$. This instant $\theta$ is different for different points of frame $s$. Let us make clear how it is related to $T$ and $r$. To solve this equation, we shall use a power expansion in $r$. In the first approximation, $\theta$ is

$$\theta = T - \frac{Vr}{\sqrt{1 - V^2}} \quad (22)$$

where $V = V_T$ is the velocity of the origin of frame $s$ as a function of the laboratory time $T$.

Therefore, in the first approximation in powers of $r$, the parameter $v$ is

$$v = V_{\theta} = V - \frac{(Vr)\dot{V}}{\sqrt{1 - V^2}}, \quad \dot{V} = \frac{dV}{dT}. \quad (23)$$

Substituting here Eq. (9), we obtain another representation of this equation:

$$v = V - \sqrt{1 - V^2}(Vr)W + \frac{\sqrt{1 - V^2}(1 - \sqrt{1 - V^2})}{V^2}(Vr)(VW)V. \quad (24)$$

Thus the parameter $v$ depends, in addition to laboratory time $T$, on the point position $r$ in the coordinate frame, i.e., it is inhomogeneous. From Eq. (23) one can see that to obtain an invariable position of some point of the coordinate system in a reference frame (that is, to obtain a rigid reference frame), it is necessary that this point move in the laboratory inertial frame in a certain agreement, or correlation, with the motion of the origin.

Let us now find the function $\theta(T, r)$ in the second approximation. Expanding $V_{\theta}/\sqrt{1 - V_{\theta}^2}$ in a Taylor series and leaving only the first term, we obtain

$$\frac{v}{\sqrt{1 - v^2}} = \frac{V_{\theta}}{\sqrt{1 - V_{\theta}^2}} = \frac{V}{\sqrt{1 - V^2}} - \frac{(Vr)\dot{V}}{1 - V^2} - \frac{(\dot{V}V)(Vr)V}{(1 - V^2)^2}. \quad (25)$$

Substituting the expansion (25) into (21), we obtain, omitting the subscript showing the $T$ dependence of quantities, in the second-order approximation showing the $T$ dependence of quantities:

$$\theta = T - \frac{Vr}{\sqrt{1 - V^2}} + \frac{(Vr)\dot{V}r}{1 - V^2} + \frac{(\dot{V}V)(Vr)^2}{(1 - V^2)^2}. \quad (26)$$
5. NONLINEAR CONTRACTION OF THE COORDINATE RULER

Let us denote the time integral in Eq. (2) as

$$\int_0^t \frac{v dt}{\sqrt{1 - v^2}} = \lambda_t.$$  (27)

Then, substituting into $\lambda_t$ the dependence $t_\theta$ found from (19), one can, similarly to (20), present it as a function of $\theta$

$$\lambda_t = \Lambda_\theta.$$  (28)

Then Eq. (2) takes the form

$$R = \Lambda_\theta + r + \frac{1 - \sqrt{1 - V_\theta^2}}{V_\theta^2 \sqrt{1 - V_\theta^2}} (r V_\theta) V_\theta,$$  (29)

where $\Lambda_\theta$ is

$$\Lambda_\theta = \int_0^\theta V_\theta d\theta.$$  (30)

From Eq. (26) it is evident that an arbitrary vector $x_\theta$ is equal to

$$x_\theta = x - \dot{x} \frac{V_r}{\sqrt{1 - V^2}} + \ddot{x} \frac{(V_r)(\dot{V}r)}{1 - V^2} + \dddot{x} \frac{(\ddot{V}V)(V_r)}{(1 - V^2)^2} + \frac{1}{2} \dddot{x} \frac{(V_r)^2}{1 - V^2},$$  (31)

where

$$\dot{x} = \frac{dx}{dT}, \quad \ddot{x} = \frac{d^2x}{dT^2},$$

and all quantities in the r.h.s. of the equalities are taken at time $T$. As an arbitrary vector $x_\theta$ in (31) one can take, for instance, the vector $\Lambda_\theta$ from (30). Hence one can write, omitting the subscript showing the $T$ dependence $T (\Lambda = \int_0^T V dT)$

$$\lambda_t = \Lambda_\theta = \Lambda - \frac{(V_r)V}{\sqrt{1 - V^2}} + \frac{(V_r)(\dot{V}r)V}{1 - V^2} + \frac{(V_r)^2(\ddot{V}V)V}{(1 - V^2)^2} + \frac{(V_r)^2\dot{V}}{2(1 - V^2)}.$$  (32)

Substituting (32) into (29) and using that

$$\frac{1 - \sqrt{1 - v^2}}{v^2 \sqrt{1 - v^2}} v_\alpha v_\beta = \frac{1 - \sqrt{1 - V^2}}{V^2 \sqrt{1 - V^2}} V_\alpha V_\beta - \frac{1 - \sqrt{1 - V^2}}{V^2(1 - V^2)} (V_r)(V_\alpha V_\beta + V_\alpha \ddot{V}_\beta)

- \frac{(1 - \sqrt{1 - V^2})^2(1 + 2\sqrt{1 - V^2})}{V^4(1 - V^2)^2} \cdot (\dot{V}V)(V_r)V_\alpha V_\beta$$  (33)
after some transformations we obtain that

\[
L = r - \frac{1 - \sqrt{1 - V^2}}{V^2} (Vr) V + \frac{1 - \sqrt{1 - V^2}}{V^2 \sqrt{1 - V^2}} (Vr) (Vr) V
\]

\[
- \frac{(1 - \sqrt{1 - V^2})^2}{2V^2 (1 - V^2)} (Vr)^2 \dot{V} + \frac{(1 - \sqrt{1 - V^2})^2}{V^4 (1 - V^2)} (Vr)^2 (\dot{V} V) V,
\]

where

\[
L = R - \int_0^T V dT.
\]

Here \(L\) is the length of a perfectly rigid rod at time \(T\) in the laboratory inertial reference frame \(S\), whose proper length is \(r\) and whose origin moves relative to \(S\) with the velocity \(V\) and acceleration \(\dot{V}\). According to this equation, a sufficiently long straight rod of length \(r\) looks curved in \(S\). Let us solve the resulting vector equation (34) with respect to \(r\). To this end, we multiply both parts of (34) scalarly by \(V\) and by \(\dot{V}\). We obtain:

\[
L V = \sqrt{1 - V^2} Vr + \frac{1 - \sqrt{1 - V^2}}{\sqrt{1 - V^2}} (Vr) (Vr) V + \frac{(1 - \sqrt{1 - V^2})^2}{2V^2 (1 - V^2)} (Vr)^2 (\dot{V} V),
\]

(35)

\[
L \dot{V} = \dot{V} r - \frac{1 - \sqrt{1 - V^2}}{V^2} (Vr) (\dot{V} V) + \frac{1 - \sqrt{1 - V^2}}{V^2 \sqrt{1 - V^2}} (Vr) (Vr) (\dot{V} V)
\]

\[
- \frac{(1 - \sqrt{1 - V^2})^2}{2V^2 (1 - V^2)} (Vr)^2 \dot{V}^2 + \frac{(1 - \sqrt{1 - V^2})^2}{V^4 (1 - V^2)} (Vr)^2 (\dot{V} V)^2.
\]

(36)

Furthermore, we note that it is sufficient to calculate the factors \(\dot{V} r\) and \(V r\) in the third, fourth and fifth terms of the r.h.s. of Eq. (34) by leaving only the first power of \(L\). Therefore (35) implies

\[
V r = \frac{L V}{\sqrt{1 - V^2}}.
\]

(37)

With the same accuracy, from (36) with (37) it follows

\[
\dot{V} r = L \dot{V} + \frac{1 - \sqrt{1 - V^2}}{V^2 \sqrt{1 - V^2}} (L V) \dot{V} V.
\]

(38)

The factor \(V r\) in the second term of (34) should be calculated up to the second power of \(L\). Due to the equalities (37) (38), we obtain from (35), that

\[
V r = \frac{L V}{\sqrt{1 - V^2}} - \frac{1 - \sqrt{1 - V^2}}{V^4 (1 - V^2)} (L V) (L \dot{V})
\]

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\[ - \frac{(1 - \sqrt{1 - V^2})^2(1 + 2\sqrt{1 - V^2})}{2V^2\sqrt{1 - V^2}} (LV)^2(\dot{V}V). \] 

Now we substitute (39) into (34), then, using (37), (38) we obtain up to \(O(L^2)\) inclusive that

\[ r = L + \frac{1 - \sqrt{1 - V^2}}{V^2\sqrt{1 - V^2}} (LV)V - \frac{(1 - \sqrt{1 - V^2})^2(1 + 3\sqrt{1 - V^2})}{2V^4\sqrt{1 - V^2}} (\dot{V}V)(LV)^2V \]

\[ + \frac{(1 - \sqrt{1 - V^2})^2}{2V^2(1 - V^2)^2} (LV)^2\ddot{V} - \frac{1 - \sqrt{1 - V^2}}{V^2\sqrt{1 - V^2}} (LV)(L\dot{V})V = \text{invariant.} \] 

Eqs. (34) and (40) can be expressed in terms of the proper acceleration. Substituting (39), we obtain

\[ L = r - \frac{1 - \sqrt{1 - V^2}}{V^2} (Vr)V + \frac{(1 - \sqrt{1 - V^2})\sqrt{1 - V^2}}{V^2} (Vr)(Wr)V \]

\[ - \frac{(1 - \sqrt{1 - V^2})^2}{2V^2}(Vr)^2W + \frac{(1 - \sqrt{1 - V^2})^3}{2V^4}(Vr)^2(VW)V, \] 

\[ r = L + \frac{1 - \sqrt{1 - V^2}}{V^2\sqrt{1 - V^2}} (LV)V - \frac{1 - \sqrt{1 - V^2}}{V^2\sqrt{1 - V^2}} (LV)(LW)V + \]

\[ + \frac{(1 - \sqrt{1 - V^2})^2}{2V^2(1 - V^2)} (LV)^2W - \frac{(1 - \sqrt{1 - V^2})^2}{V^4(1 - V^2)^2}(VW)(LV)^2V. \] 

The nonlinearity of the Lorentz contraction means that, in general, it is impossible to simulate an arbitrary reference frame \(s\), sufficiently large in size from the viewpoint of a laboratory observer, by a sequence of instantaneously comoving inertial reference frames. Such simulation is only possible for an observer of frame \(s\). This nonlinearity is well explained by the fact that the points of a rigid reference frame move in different ways. The region of the coordinate axis of frame \(s\), adjacent to its front end moves, at a rectilinear acceleration of \(s\) with a smaller velocity than the velocity \(V\) of the origin. Therefore, the region of the coordinate axis close to its front end with respect to the lab reference frame \(S\) is less contracted that the neighborhood of the origin. Thus the whole length of the accelerated rod in the process of acceleration along its own direction should be slightly larger than the length of a similar but inertial rod, whose all points move with the same velocity \(V\) at a given time instant [4].

Eqs. (34) or (40) are different forms of one of the equations of the sought-for inverse transformation, up to the second order in \(r\) (or \(L\)) inclusive. It is evident that the direct and inverse LMN transformations are different from each other. This feature is connected with a radical difference between inertial and non-inertial reference frames.
6. DESYNCHRONIZATION OF COORDINATE CLOCKS

Similarly to (31), expanding an arbitrary scalar function \( y_\theta \) in Taylor series up to \( O(r^2) \) inclusive, we obtain

\[
y_\theta = y - \left( \frac{Vr}{\sqrt{1-V^2}} - \frac{(Vr)(\ddot{V}r)}{1-V^2} \right) \dot{y} + \frac{(Vr)^2(\ddot{VV})}{(1-V^2)^2} \ddot{y} + \frac{(Vr)^2}{2(1-V^2)} \dddot{y},
\]

where \( \dot{y} = dy/dT \) and \( \dddot{y} = \frac{d^3y}{dT^3} \). The equation inverse to (19) is

\[
t = \int_0^\theta \sqrt{1-V_\theta^2} d\theta,
\]

where \( \theta \) is a root of Eq. (21). Substituting into (43) instead of the scalar function \( y_\theta \) the function \( t_\theta \) from (44), we obtain

\[
t = \int_0^T \sqrt{1-V^2} dT - \left\{ Vr - \frac{(Vr)(\ddot{V}r)}{\sqrt{1-V^2}} - \frac{(Vr)^2(\ddot{VV})}{2\sqrt{1-V^2}} \right\}.
\]

It is the second equation of the inverse transformation. The first term in the r.h.s. of this expression is the proper time of the origin, \( \tau (r = 0) \). Evidently, the world time \( t \) evidently, the world time \( t \) cannot coincide with the proper time of the origin due to relativity of simultaneity. The second term in curly brackets in (45) determines the desynchronization of two coordinate clocks in the non-inertial frame \( s \), if these clocks were initially synchronized in the lab frame \( S (dT = 0) \). From (46) it is clear that the desynchronization magnitude, in the present approximation, does not depend on the law of motion of frame \( s \), but is determined by the instantaneous values of its velocity and acceleration.

Substituting into (45) the values \( \ddot{V}r \) and \( Vr \) from (38) and (39), we finally obtain another form of the second equation,

\[
t = \int_0^T \sqrt{1-V^2} dT - \left\{ \frac{LV}{\sqrt{1-V^2}} - \frac{(LV)(L\dot{V})}{\sqrt{1-V^2}} - \frac{(LV)^2(L\ddot{V})}{\sqrt{1-V^2}} \right\}.
\]

Substituting the equality (9) into (45) and (46), we obtain

\[
t = \int_0^T \sqrt{1-V^2} dT - Vr + \sqrt{1-V^2}(Vr)(Wr) + \frac{(1-\sqrt{1-V^2})^2}{2V^2}(Vr)^2(VW),
\]

\[
t = \int_0^T \sqrt{1-V^2} dT - \frac{LV}{\sqrt{1-V^2}} + \frac{(LV)(LW)}{\sqrt{1-V^2}} + \frac{(1-\sqrt{1-V^2})}{V^2(1-V^2)}(LV)^2(VW).
\]

Lastly, substituting into (23) the relation (37), we obtain with the same accuracy

\[
v = V - \frac{(VL)\dot{V}}{1-V^2}.
\]
7. COMPARISON OF THE OBTAINED FORMULAS WITH KNOWN RESULTS

First of all, it is necessary to make clear whether Eqs. (7) and (8) are true. Eq. (7) is known for long time [8, p. 109, Eq. (194)]. Eq. (8) also well agrees with quite a number of other sources, e.g., [9, Eq. (33)] and [10, Eq. (34)].

It is easy to notice that in the case of a constant velocity \( V \) the inverse LMN transformation (34), (45) and (40), (46) turn into the conventional Lorentz transformation.

Let us now consider the easily verifiable consequences of the above inverse transformation in the case of a rectilinear motion of frame \( s \) without rotation in the direction of the \( X \) and \( x \) axes. In this case, the direction of the laboratory acceleration coincides with that of the velocity. Then one can verify that Eqs. (23), (24) and (49) are in agreement with the already known formulas for the velocities of points of the coordinate system of \( s \) [7, p. 65, Eq. (6)], [4, Eq. (19)].

Suppose now that two events have occurred at the same time according to the clocks of the lab inertial reference frame \( S \), but at different points \( x_1 \) and \( x_2 \) of the accelerated reference frame \( s \). Let us find the world time interval between events 1 and 2 in the accelerated frame \( s \). From Eq. (47) it follows that the time instants \( t_1 \) and \( t_2 \) by the clocks of frame \( s \) are

\[
t_\alpha = \int_0^T \sqrt{1 - V^2} dT - \left\{ V x_\alpha - V \left(1 - \frac{V^2}{2}\right) W x^2 \right\}, (\alpha = 1, 2). \tag{50}
\]

Consequently,

\[
\Delta t = t_2 - t_1 = - \left( V x - V \left(1 - \frac{V^2}{2}\right) W x^2 \right)_{x=x_2}^{x=x_1}. \tag{51}
\]

Similarly, from Eq. (41) it follows that the length of a rod of proper length \( x \), situated along the direction of its rectilinear motion, is

\[
L = \sqrt{1 - V^2} x + \frac{V^2 \sqrt{1 - V^2}}{2} W x^2. \tag{52}
\]

Eqs. (51) and (52) are already known from a direct calculation using the equation of rectilinear uniformly accelerated motion ( [5, Eq. (18)] and [11, Eq. (26)], [4, Eq. (30)], respectively). Eqs. (34) and (40), though contradicting to the conventional Lorentz contraction, still well agree with [6].

8. CONCLUSION

Summarizing, one can make the general conclusion that the special LMN transformation and the inverse transformation suggested here are useful for a description
of real, sufficiently rigid bodies. The physical consequences of the LMN transformation are nontrivial, they agree with each other and with the results obtained by other authors.

9. DISCUSSION

The LMN transformation with the arbitrary parameter \( v(t) \) describes an arbitrary non-inertial rigid motion of a perfectly rigid body which does not exist. However, under the constraints (11), (14) and for a reference frame not too large in size, (18), the idea of rigid motion in relativity theory is quite meaningful.

The parameter \( v(t) \) in the LMN transformation is connected with the velocity of the origin by the general approximate formula (23). This means that the points of a radially rigid non-inertial frame move with respect to \( S \) (on the average over the oscillation period) inhomogeneously, with different velocities.

The equations that represent the inverse special LMN transformation (in two forms, (34), (45) and (40), (46), respectively) possess an essential nonlinearity depending on the acceleration of the origin. The direct and explicit kinematic consequences of this transformation are: (a) desynchronization (in Eqs. (45), (46) it is shown in curly brackets) of coordinate clocks of the non-inertial frame \( s \) if they were previously synchronized in the lab frame \( S \), and (b) a nonlinear contraction in \( S \) of a ruler of frame \( s \) (34), (40). In the second order with respect to the proper coordinates, these effects depend on the velocity and proper acceleration only. This transformation may be used for calculating other effects which are possible at rigid non-inertial motion.

In conclusion, let us briefly dwell on the perspectives of future studies. The present paper was devoted to a remarkable transformation from a laboratory inertial frame to a radially rigid reference frame which moves arbitrarily but rotates with a strictly determined Thomas frequency. This proper rotation is a special case of an arbitrary rotation, therefore the above formulas of the inverse special LMN transformation can be generalized. Another direction of research is contained in the following. The above transformation is applicable to curved spacetime only locally, in quite a restricted region of space. This circumstance is a shortcoming. It is therefore required to extend the special LMN transformation to curved spacetime. One should expect that the transformation to be found will be similar to the LMN transformation but will contain the curvature tensor. It is quite probable that this more general transformation will have a wider applicability area.

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