Note on the residual finiteness of Artin groups

Rubén Blasco-García, Arye Juhász and Luis Paris

Communicated by Pierre-Emmanuel Caprace

Abstract. Let $A$ be an Artin group. A partition $\mathcal{P}$ of the set of standard generators of $A$ is called admissible if, for all $X, Y \in \mathcal{P}, X \neq Y$, there is at most one pair $(s, t) \in X \times Y$ which has a relation. An admissible partition $\mathcal{P}$ determines a quotient Coxeter graph $\Gamma / \mathcal{P}$.

We prove that, if $\Gamma / \mathcal{P}$ is either a forest or an even triangle free Coxeter graph and $A_X$ is residually finite for all $X \in \mathcal{P}$, then $A$ is residually finite.

1 Introduction and statements

Let $S$ be a finite set. A Coxeter matrix over $S$ is a square matrix $M = (m_{s,t})_{s,t \in S}$, indexed by the elements of $S$, with coefficients in $\mathbb{N} \cup \{\infty\}$, such that $m_{s,s} = 1$ for all $s \in S$ and $m_{s,t} = m_{t,s} \geq 2$ for all $s, t \in S, s \neq t$. We represent $M$ by a labelled graph $\Gamma$, called Coxeter graph, defined as follows. The set of vertices of $\Gamma$ is $S$ and two vertices $s, t$ are connected by an edge labelled with $m_{s,t}$ if $m_{s,t} \neq \infty$.

If $a, b$ are two letters and $m$ is an integer $\geq 2$, we set

$$\Pi(a, b : m) = \begin{cases} (ab)^{m^2/m} & \text{if } m \text{ is even}, \\ (ab)^{m-1}a & \text{if } m \text{ is odd}. \end{cases}$$

In other words, $\Pi(a, b : m)$ denotes the word $aba \cdots$ of length $m$. The Artin group $A = A_\Gamma$ of $\Gamma$ is defined by the presentation

$$A = \langle S \mid \Pi(s, t : m_{s,t}) = \Pi(t, s : m_{s,t}) \text{ for all } s, t \in S, s \neq t \text{ and } m_{s,t} \neq \infty \rangle.$$

Recall that a group $G$ is residually finite if for each $g \in G \setminus \{1\}$ there exists a group homomorphism $\varphi : G \to K$ such that $K$ is finite and $\varphi(g) \neq 1$. Curiously the list of Artin groups known to be residually finite is quite short. It contains the Artin groups of spherical type (because they are linear), the right-angled Artin groups (because they are also linear), the even Artin groups of FC type

The first named author was partially supported by Gobierno de Aragón, European Regional Development Funds, MTM2015-67781-P (MINECO/ FEDER) and by the Departamento de Industria e Innovación del Gobierno de Aragón and Fondo Social Europeo Phd grant.
R. Blasco-García, A. Juhász and L. Paris

(see Blasco-García, Martinez-Perez and Paris [1]) and groups that can be constructed from these families using standard operations that preserve the property of being residually finite. Within the framework of Artin groups, these operations are essentially of three kinds: taking a subgroup, taking a direct product, taking a free product (see Gruenberg [11]). The application of the second and third operations is quite obvious. The application of the first operation is less obvious, but examples of Artin groups embedded into Artin groups exist in the literature. Here are three examples (to our knowledge, these are the main ones). If \( X \) is a subset of \( S \), then, by Van der Lek [14], the subgroup of \( A \) generated by \( X \) is an Artin group. We also know that the Artin group of type \( A_n \) is a subgroup of the Artin group of type \( A_n \) (see Charney and Peifer [5] for instance). Finally, there is a general construction of homomorphisms between Artin groups, called \textit{LCM homomorphisms}, that are known to be injective in many cases, for example when the groups are of FC type (see Crisp [7] and Godelle [9]). Our purpose here is to increase this list of Artin groups known to be residually finite with an extra operation.

For \( X \subset S \) we denote by \( \Gamma_X \) the full subgraph of \( \Gamma \) spanned by \( X \) and by \( A_X \) the subgroup of \( A \) generated by \( X \). By Van der Lek [14], \( A_X \) is the Artin group of \( \Gamma_X \). We say that \( \Gamma \) is \textit{even} if \( m_{s,t} \) is either even or \( \infty \) for all \( s, t \in S, s \neq t \). We say that \( \Gamma \) is \textit{triangle free} if \( \Gamma \) has no full subgraph which is a triangle. Here by a \textit{partition} of \( S \) we mean a set \( \mathcal{P} \) of pairwise disjoint subsets of \( S \) satisfying \( \bigcup_{X \in \mathcal{P}} X = S \). We say that a partition \( \mathcal{P} \) is \textit{admissible} if, for all \( X, Y \in \mathcal{P}, X \neq Y \), there is at most one edge in \( \Gamma \) connecting an element of \( X \) with an element of \( Y \). In particular, if \( s \in X \) and \( t \in Y \) are connected in \( \Gamma \) by an edge and \( s' \in X \), \( s' \neq s \), then \( s' \) is not connected in \( \Gamma \) by an edge to any vertex of \( Y \). An admissible partition \( \mathcal{P} \) of \( \Gamma \) determines a new Coxeter graph \( \Gamma / \mathcal{P} \) defined as follows. The set of vertices of \( \Gamma / \mathcal{P} \) is \( \mathcal{P} \). Two distinct elements \( X, Y \in \mathcal{P} \) are connected by an edge labelled with \( m \) if there exist \( s \in X \) and \( t \in Y \) such that \( m_{s,t} = m \). Our main result is the following.

**Theorem 1.1.** Let \( \Gamma \) be a Coxeter graph, let \( A = A_\Gamma \), and let \( \mathcal{P} \) be an admissible partition of \( S \) such that

(a) the group \( A_X \) is residually finite for all \( X \in \mathcal{P} \),

(b) the Coxeter graph \( \Gamma / \mathcal{P} \) is either even and triangle free, or a forest.

Then \( A \) is residually finite.

**Corollary 1.2.** The following statements hold.

(1) If \( \Gamma \) is even and triangle free, then \( A \) is residually finite.

(2) If \( \Gamma \) is a forest, then \( A \) is residually finite.
Remark. When $\Gamma$ is a forest, the Artin group $A$ is the fundamental group of a graph manifold with boundary by Gordon [10], and is thus virtually special by Przytycky and Wise [13], hence is linear and residually finite.

Remark. Recall that the Coxeter group $W$ of $\Gamma$ is the quotient of $A$ by the relations $s^2 = 1, s \in S$. We say that $\Gamma$ is of spherical type if $W$ is finite. We say that a subset $X$ of $S$ is free of infinity if $m_{s,t} \neq \infty$ for all $s, t \in X$. We say that $\Gamma$ is of FC type if for each free of infinity subset $X$ of $S$ the Coxeter graph $\Gamma_X$ is of spherical type. Artin groups of FC type were introduced by Charney and Davis [4] in their study of the $K(\pi, 1)$ problem for Artin groups and there is an extensive literature on them. It is easily checked that any triangle free Coxeter graph (in particular any forest) is of FC type. So, all the groups that appear in Corollary 1.2 are of FC type. On the other hand, we know by Blasco-Garcia, Martinez-Perez and Paris [1] that all even Artin groups of FC type are residually finite (which gives an alternative proof to Corollary 1.2 (1)). The next challenge would be to prove that all Artin groups of FC type are residually finite. Another interesting challenge would be to prove that all three generators Artin groups are residually finite.

2 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the following.

Theorem 2.1 (Boler and Evans [2]). Let $G_1$ and $G_2$ be two residually finite groups and let $L$ be a common subgroup of $G_1$ and $G_2$. Assume that both $G_1$ and $G_2$ split as semi-direct products $G_1 = H_1 \rtimes L$ and $G_2 = H_2 \rtimes L$. Then $G = G_1 \ast_L G_2$ is residually finite.

The rest of the section forms the proof of Theorem 1.1.

Lemma 2.2. If a Coxeter graph $\Gamma$ has one or two vertices, then $A = A_\Gamma$ is residually finite.

Proof. If $\Gamma$ has only one vertex, then $A \simeq \mathbb{Z}$ which is residually finite. Suppose that $\Gamma$ has two vertices $s, t$. If $m_{s,t} = \infty$, then $A$ is a free group of rank 2 which is residually finite. If $m_{s,t} \neq \infty$, then $\Gamma$ is of spherical type, hence, by Digne [8] and Cohen and Wales [6], $A$ is linear, and therefore $A$ is residually finite. \qed

Lemma 2.3. Let $\Gamma$ be a Coxeter graph and let $A = A_\Gamma$. Let $s \in S$. Set $Y = S \setminus \{s\}$, denote by $\Gamma_1, \ldots, \Gamma_\ell$ the connected components of $\Gamma_Y$ and, for $i \in \{1, \ldots, \ell\}$, denote by $Y_i$ the set of vertices of $\Gamma_i$. If $A_{Y_i \cup \{s\}}$ is residually finite for all $i \in \{1, \ldots, \ell\}$, then $A$ is residually finite.
Proof. We argue by induction on \( \ell \). If \( \ell = 1 \), then \( Y_1 \cup \{ s \} = S \) and \( A_{Y_1 \cup \{ s \}} = A \), hence \( A \) is obviously residually finite. Suppose that \( \ell \geq 2 \) and that the inductive hypothesis holds. We set \( X_1 = Y_1 \cup \cdots \cup Y_{\ell-1} \cup \{ s \} \), \( X_2 = Y_\ell \cup \{ s \} \) and \( X_0 = \{ s \} \). Let \( G_1 = A_{X_1} \), \( G_2 = A_{X_2} \), and \( L = A_{X_0} \cong \mathbb{Z} \). The groups \( G_1 \) and \( G_2 \) are residually finite by induction. It is easily seen in the presentation of \( A \) that \( A = G_1 \ast_L G_2 \). Furthermore, the homomorphism \( \varphi_1 : G_1 \to L \) which sends \( t \) to \( s \) for all \( t \in X_1 \) is a retraction of the inclusion map \( L \hookrightarrow G_1 \), hence \( G_1 \) splits as a semi-direct product \( G_1 = H_1 \rtimes L \). Similarly, \( G_2 \) splits as a semi-direct product \( G_2 = H_2 \rtimes L \). We conclude by Theorem 2.1 that \( A \) is residually finite. \( \square \)

Lemma 2.4. Let \( \Gamma \) be a Coxeter graph, let \( A = A_\Gamma \), and let \( \mathcal{P} \) be an admissible partition of \( S \) such that

(a) the group \( A_X \) is residually finite for all \( X \in \mathcal{P} \),

(b) the Coxeter graph \( \Gamma/\mathcal{P} \) has at most two vertices.

Then \( A \) is residually finite.

Proof. If \( |\mathcal{P}| = 1 \), there is nothing to prove. Suppose that \( |\mathcal{P}| = 2 \) and one of the elements of \( \mathcal{P} \) is a singleton. We set \( \mathcal{P} = \{ X, Y \} \) where \( X = S \setminus \{ t \} \) and \( Y = \{ t \} \) for some \( t \in S \). If there is no edge in \( \Gamma \) connecting \( t \) to an element of \( X \), then \( A = A_X \ast A_Y \), hence \( A \) is residually finite. So, we can assume that there is an edge connecting \( t \) to an element \( s \in X \). Note that this edge is unique by the definition of admissibility. We denote by \( \Gamma_1, \ldots, \Gamma_\ell \) the connected components of \( \Gamma_{X \setminus \{ s \}} \) and, for \( i \in \{ 1, \ldots, \ell \} \), we denote by \( X_i \) the set of vertices of \( \Gamma_i \). For all \( i \in \{ 1, \ldots, \ell \} \) the group \( A_{X_i \cup \{ s \}} \) is residually finite since \( A_{X_i \cup \{ s \}} \subset A_X \). On the other hand, \( A_{\{ s, t \}} \) is residually finite by Lemma 2.2. Noticing that the connected components of \( \Gamma_{S \setminus \{ s \}} \) are precisely \( \Gamma_1, \ldots, \Gamma_\ell \) and \( \{ t \} \), we deduce from Lemma 2.3 that \( A \) is residually finite.

Now assume that \( |\mathcal{P}| = 2 \) and both elements of \( \mathcal{P} \) are of cardinality \( \geq 2 \). Set \( \mathcal{P} = \{ X, Y \} \). If there is no edge in \( \Gamma \) connecting an element of \( X \) with an element of \( Y \), then \( A = A_X \ast A_Y \), hence \( A \) is residually finite. So, we can assume that there is an edge connecting an element \( s \in X \) to an element \( t \in Y \). Again, this edge is unique. Let \( \Omega_1, \ldots, \Omega_p \) be the connected components of \( \Gamma_{X \setminus \{ s \}} \) and let \( \Gamma_1, \ldots, \Gamma_q \) be the connected components of \( \Gamma_Y \). We denote by \( X_i \) the set of vertices of \( \Omega_i \) for all \( i \in \{ 1, \ldots, p \} \) and by \( Y_j \) the set of vertices of \( \Gamma_j \) for all \( j \in \{ 1, \ldots, q \} \). The group \( A_{X_i \cup \{ s \}} \) is residually finite since \( X_i \cup \{ s \} \subset X \) for all \( i \in \{ 1, \ldots, p \} \), and, by the above, the group \( A_{Y_j \cup \{ s \}} \) is residually finite for all \( j \in \{ 1, \ldots, q \} \). It follows by Lemma 2.3 that \( A \) is residually finite. \( \square \)

Remark. Alternative arguments from Pride [12] and/or from Burillo and Martino [3] can be used to prove partially or completely Lemma 2.4.
Lemma 2.5. Let $\Gamma$ be a Coxeter graph, let $A = A_\Gamma$, and let $\mathcal{P}$ be an admissible partition of $S$ such that

(a) the group $A_X$ is residually finite for all $X \in \mathcal{P}$,

(b) the Coxeter graph $\Gamma / \mathcal{P}$ is even and triangle free.

Then $A$ is residually finite.

Proof. We argue by induction on the cardinality $|\mathcal{P}|$ of $\mathcal{P}$. The case $|\mathcal{P}| \leq 2$ is covered by Lemma 2.4. So, we can suppose that $|\mathcal{P}| \geq 3$ and that the inductive hypothesis holds. Since $\Gamma / \mathcal{P}$ is triangle free, there exist $X, Y \in \mathcal{P}$ such that none of the elements of $X$ is connected to an element of $Y$. We set $U_1 = S \setminus X$, $U_2 = S \setminus Y$, and $U_0 = S \setminus (X \cup Y)$. We have $A = A_{U_1} \ast_{A_{U_0}} A_{U_2}$ and, by the inductive hypothesis, $A_{U_1}$ and $A_{U_2}$ are residually finite. Since $\Gamma / \mathcal{P}$ is even, the inclusion map $A_{U_0} \hookrightarrow A_{U_1}$ admits a retraction $\rho_1 : A_{U_1} \rightarrow A_{U_0}$ which sends $t$ to $1$ if $t \in Y$ and sends $t$ to $t$ if $t \in U_0$. Similarly, the inclusion map $A_{U_0} \hookrightarrow A_{U_2}$ admits a retraction $\rho_2 : A_{U_2} \rightarrow A_{U_0}$. By Theorem 2.1 it follows that $A$ is residually finite. □

The following lemma ends the proof of Theorem 1.1.

Lemma 2.6. Let $\Gamma$ be a Coxeter graph, let $A = A_\Gamma$, and let $\mathcal{P}$ be an admissible partition of $S$ such that

(a) the group $A_X$ is residually finite for all $X \in \mathcal{P}$,

(b) the Coxeter graph $\Gamma / \mathcal{P}$ is a forest.

Then $A$ is residually finite.

Proof. We argue by induction on $|\mathcal{P}|$. The case $|\mathcal{P}| \leq 2$ being proved in Lemma 2.4, we can assume that $|\mathcal{P}| \geq 3$ plus the inductive hypothesis. Set $\Omega = \Gamma / \mathcal{P}$. Let $\Omega_1, \ldots, \Omega_\ell$ be the connected components of $\Omega$. For $i \in \{1, \ldots, \ell\}$ we denote by $\mathcal{P}_i$ the set of vertices of $\Omega_i$ and we set $Y_i = \bigcup_{X \in \mathcal{P}_i} X$ and $\Gamma_i = \Gamma Y_i$. The set $\mathcal{P}_i$ is an admissible partition of $Y_i$ and $\Gamma_i / \mathcal{P}_i = \Omega_i$ is a tree for all $i \in \{1, \ldots, \ell\}$. Moreover, we have $A = A_{Y_1} \ast \cdots \ast A_{Y_\ell}$, hence $A$ is residually finite if and only if $A_{Y_i}$ is residually finite for all $i \in \{1, \ldots, \ell\}$. So, we can assume that $\Omega = \Gamma / \mathcal{P}$ is a tree.

Since $|\mathcal{P}| \geq 3$, $\Omega$ has a vertex $X$ of valence $ \geq 2$. Choose $Y \in \mathcal{P}$ connected to $X$ by an edge of $\Omega$. Let $s \in X$ and $t \in Y$ such that $s$ and $t$ are connected by an edge of $\Gamma$. Recall that by definition $s$ and $t$ are unique. Let $Q'$ be the connected component of $\Omega_{\mathcal{P}\setminus\{X\}}$ containing $Y$, let $\mathcal{P}'_Q$ be the set of vertices of $Q'$, let $U' = \bigcup_{Z \in \mathcal{P}'_Q} Z$, let $U = U' \cup \{s\}$, and let $\mathcal{P}_Q = \mathcal{P}'_Q \cup \{s\}$. Observe that $\mathcal{P}_Q$
is an admissible partition of $U$, that $A_Z$ is residually finite for all $Z \in \mathcal{P}_Q$, that $\Gamma_U/\mathcal{P}_Q$ is a tree, and that $|\mathcal{P}_Q| < |\mathcal{P}|$. By the inductive hypothesis it follows that $A_U$ is residually finite. Let $R$ be the connected component of $\Omega_{\mathcal{P}\setminus\{Y\}}$ containing $X$, let $\mathcal{P}_R$ be the set of vertices of $R$, and let $V = \bigcup_{Z \in \mathcal{P}_R} Z$. Observe that $\mathcal{P}_R$ is an admissible partition of $V$, that $A_Z$ is residually finite for all $Z \in \mathcal{P}_R$, that $\Gamma_V/\mathcal{P}_R$ is a tree, and that $|\mathcal{P}_R| < |\mathcal{P}|$. By the inductive hypothesis it follows that $A_V$ is residually finite. Let $\Delta_1, \ldots, \Delta_q$ be the connected components of $\Gamma_{\mathcal{S}\setminus\{s\}}$. Let $i \in \{1, \ldots, q\}$. Let $Z_i$ be the set of vertices of $\Delta_i$. It is easily seen that either $Z_i \cup \{s\} \subset U$, or $Z_i \cup \{s\} \subset V$, hence, by the above, $A_{Z_i \cup \{s\}}$ is residually finite. We conclude by Lemma 2.3 that $A$ is residually finite.

\textbf{Acknowledgments.} The authors thank the referee for pointing out to them that the fact that $A$ is residually finite when $\Gamma$ is a forest follows from Gordon [10] and Przytycki and Wise [13] (see the remark after Corollary 1.2).

\textbf{Bibliography}

[1] R. Blasco-Garcia, C. Martinez-Perez and L. Paris, Poly-freeness of even Artin groups of FC type, \textit{Groups Geom. Dyn.}, to appear.

[2] J. Boler and B. Evans, The free product of residually finite groups amalgamated along retracts is residually finite, \textit{Proc. Amer. Math. Soc.} \textbf{37} (1973), 50–52.

[3] J. Burillo and A. Martino, Quasi-potency and cyclic subgroup separability, \textit{J. Algebra} \textbf{298} (2006), no. 1, 188–207.

[4] R. Charney and M. W. Davis, The $K(\pi, 1)$-problem for hyperplane complements associated to infinite reflection groups, \textit{J. Amer. Math. Soc.} \textbf{8} (1995), no. 3, 597–627.

[5] R. Charney and D. Peifer, The $K(\pi, 1)$-conjecture for the affine braid groups, \textit{Comment. Math. Helv.} \textbf{78} (2003), no. 3, 584–600.

[6] A. M. Cohen and D. B. Wales, Linearity of Artin groups of finite type, \textit{Israel J. Math.} \textbf{131} (2002), 101–123.

[7] J. Crisp, Injective maps between Artin groups, in: \textit{Geometric Group Theory Down Under} (Canberra 1996), de Gruyter, Berlin (1999), 119–137.

[8] F. Digne, On the linearity of Artin braid groups, \textit{J. Algebra} \textbf{268} (2003), no. 1, 39–57.

[9] E. Godelle, Morphismes injectifs entre groupes d’Artin–Tits, \textit{Algebr. Geom. Topol.} \textbf{2} (2002), 519–536.

[10] C. M. Gordon, Artin groups, 3-manifolds and coherence, \textit{Bol. Soc. Mat. Mexicana (3)} \textbf{10} (2004), 193–198.

[11] K. W. Gruenberg, Residual properties of infinite soluble groups, \textit{Proc. London Math. Soc. (3)} \textbf{7} (1957), 29–62.
[12] S. J. Pride, On the residual finiteness and other properties of (relative) one-relator groups, *Proc. Amer. Math. Soc.* **136** (2008), no. 2, 377–386.

[13] P. Przytycki and D. T. Wise, Graph manifolds with boundary are virtually special, *J. Topol.* **7** (2014), no. 2, 419–435.

[14] H. Van der Lek, *The homotopy type of complex hyperplane complements*, Ph.D. thesis, Nijmegen, 1983.

Received October 10, 2017; revised December 22, 2017.

**Author information**

Rubén Blasco-García, Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain.  
E-mail: rubenb@unizar.es

Arye Juhász, Department of Mathematics, Technion, Israel Institute of Technology, Haifa 32000, Israel.  
E-mail: arju@technion.ac.il

Luis Paris, IMB, UMR 5584, CNRS, Université Bourgogne Franche-Comté, 21000 Dijon, France.  
E-mail: lparis@u-bourgogne.fr