HIGH-ORDER S-LEMMA WITH APPLICATION TO STABILITY OF A CLASS OF SWITCHED NONLINEAR SYSTEMS*

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Abstract. This paper extends some results on the S-Lemma proposed by Yakubovich and uses the improved results to investigate the asymptotic stability of a class of switched nonlinear systems.

Firstly, the strict S-Lemma is extended from quadratic forms to homogeneous functions with respect to any dilation, where the improved S-Lemma is named the strict homogeneous S-Lemma (the SHS-Lemma for short). In detail, this paper indicates that the strict S-Lemma does not necessarily hold for homogeneous functions that are not quadratic forms, and proposes a necessary and sufficient condition under which the SHS-Lemma holds.

It is well known that a switched linear system with two sub-systems admits a Lyapunov function with homogeneous derivative (LFHD for short), if and only if it has a convex combination of the vector fields of its two sub-systems that admits a LFHD. In this paper, it is shown that this conclusion does not necessarily hold for a general switched nonlinear system with two sub-systems, and gives a necessary and sufficient condition under which the conclusion holds for a general switched nonlinear system with two sub-systems. It is also shown that for a switched nonlinear system with three or more sub-systems, the “if” part holds, but the “only if” part may not.

At last, the S-Lemma is extended from quadratic polynomials to polynomials of degree more than 2 under some mild conditions, and the improved results are called the homogeneous S-Lemma (the HS-Lemma for short) and the non-homogeneous S-Lemma (the NHS-Lemma for short), respectively.

Besides, some examples and counterexamples are given to illustrate the main results.

Key words. strict homogeneous S-Lemma, switched nonlinear system, Lyapunov function with homogeneous derivative, convex combination, homogeneous S-Lemma, non-homogeneous S-Lemma

AMS subject classifications. 93C10, 70K20, 90C26

1. Introduction and Preliminaries.

1.1. S-Lemma. The S-Lemma, firstly proposed by Yakubovich [1], characterizes when a quadratic function is copositive with another quadratic function. The basic idea of this widely used method comes from control theory but it has important consequences in quadratic and semi-definite optimization, convex geometry, and linear algebra as well [3, 8].

A real-valued function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be copositive with a real-valued function $g: \mathbb{R}^n \to \mathbb{R}$ if $g(x) \geq 0$ implies $f(x) \geq 0$. Furthermore, $f$ is said to be strictly copositive with $g$ if $f$ is copositive with $g$, and $g(x) \geq 0$ and $x \neq 0$ imply $f(x) > 0$.

**THEOREM 1.1 (S-Lemma, [1]).** Let $f, g: \mathbb{R}^n \to \mathbb{R}$ be quadratic functions such that $g(\bar{x}) > 0$ for some $\bar{x} \in \mathbb{R}^n$. Then $f$ is copositive with $g$ if and only if there exists $\xi \geq 0$ such that $f(x) - \xi g(x) \geq 0$ for all $x \in \mathbb{R}^n$.

**THEOREM 1.2 (strict S-Lemma).** Let $f, g: \mathbb{R}^n \to \mathbb{R}$ be quadratic forms. Then $f$ is strictly copositive with $g$ if and only if there exists $\xi > 0$ such that $f(x) - \xi g(x) > 0$ for all nonzero $x \in \mathbb{R}^n$.

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* This work is supported by National Natural Science Foundation of China (No. 61174047), Program for New Century Excellent Talents in University of Ministry of Education of China and Basic Research Foundation of Northwestern Polytechnical University (No. JC201230). The first author is supported by the China Scholarship Council. A short version of this paper was presented in the 31st Chinese Control Conference, Hefei, China, July 25-27, 2012.

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1In the original version, $\xi \geq 0$. However, $f$ is strictly copositive with $g$ implies $f$ and $g$ have no common zero point except $0 \in \mathbb{R}^n$. Then by Theorem 1.4 a positive real number $\xi$ can be found.
Theorem 1.1 and Theorem 1.2 were firstly obtained based on the following Theorem 1.3 given in [2] via the separation theorem for convex sets.

**Theorem 1.3** ([2]). Let \( f, g : \mathbb{R}^n \rightarrow \mathbb{R} \) be quadratic forms. Then the set \( \{(f(x), g(x)) : x \in \mathbb{R}^n\} \) is convex. Particularly, if \( f \) and \( g \) have no common zero point except \( 0 \in \mathbb{R}^n \), then the set \( \{(f(x), g(x)) : x \in \mathbb{R}^n\} \) is closed as well as convex, and is either the entire \( xy \)-plane or an angular sector of angle less than \( \pi \).

Yakubovich [1] gave an example indicating the set \( \{(f(x), g(x)) : x \in \mathbb{R}^n\} \) is not convex, which indicates neither Theorem 1.1 nor Theorem 1.2 holds for three or more quadratic functions. We shall also give an example to support it (see Example 2.4) later. Despite the general non-convexity of the set \( \{(f(x), g(x)) : x \in \mathbb{R}^n\} \), one can impose additional conditions on quadratic functions \( f(x), g_1(x), \ldots, g_m(x) \) to make the set \( \{(f(x), g_1(x), \ldots, g_m(x)) : x \in \mathbb{R}^n\} \) be convex. There are many such extensions with applications to control theory (linear systems) [3, 8, 11, 12, 13, 14, 24]. However, the case that these functions are (homogeneous) polynomials that have degree more than 2 or even general homogeneous functions has not been studied yet, which can be used to deal with nonlinear systems. In this paper, we focus on the latter case.

1.2. Homogeneous Function and Even (Odd) Function. In this subsection we introduce some preliminaries related to homogeneous functions and even (odd) functions.

Any given \( n \)-tuple \( (r_1, \ldots, r_n) \) with each \( r_i \) positive is called a dilation; the set \( \{x \in \mathbb{R}^n : (x_1^{1/r_1} + \cdots + x_n^{1/r_n})^{1/l} = 1\} \) denotes the generalized unit sphere, where \( l > 0 \). Specially, the set \( \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\} \) denotes the unit sphere. Based on the concept of dilations, the concept of homogeneous functions is introduced as follows [16, 19]:

**Definition 1.4.** A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be homogeneous of degree \( k \in \mathbb{R} \) with respect to the dilation \( (r_1, \ldots, r_n) \), if

\[
f(\epsilon^{r_1} x_1, \ldots, \epsilon^{r_n} x_n) = \epsilon^k f(x_1, \ldots, x_n)
\]

for all \( \epsilon > 0, \) and \( x_1, \ldots, x_n \in \mathbb{R} \).

It can be easily seen that \( f \) is homogeneous of degree \( k \) with respect to the dilation \( (r_1, \ldots, r_n) \) if and only if \( f \) is homogeneous of degree \( k/r \) with respect to the dilation \( (r_1, \ldots, r_n)/r \), where \( r = \min\{r_1, \ldots, r_n\} \). Without loss of generality, we assume that \( r_i \geq 1, i = 1, \ldots, n \) hereinafter. By Definition 1.4, homogeneous polynomials are analytic and homogeneous functions of degree a nonnegative integer with respect to the trivial dilation (1, \ldots, 1).

A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is called even (odd) if \( f(-x) = f(x)(-f(x)) \) for all \( x \in \mathbb{R}^n \). For example, a homogeneous polynomial of even (odd) degree is an even (odd) function. However, a homogeneous function is not necessarily a polynomial or not necessarily an even (odd) function. For example, the odd and homogeneous function \( |x|^2 \) is not a polynomial, where \( \text{sgn}(\cdot) \) denotes the sign function; the polynomial \( x + y^2 \) that is homogeneous of degree 2 with respect to the dilation (2, 1) is neither an even (odd) function nor a homogeneous polynomial; the homogeneous function \( x^3 + |x|^3 \) is neither a polynomial nor an even (odd) function.

1.3. Applications of the Strict S-Lemma to Stability of Switched Linear Systems. Wicks and Peleties [9] showed that if a switched linear system with two sub-systems has an asymptotically stable convex combination of its sub-systems, there exists a quadratic Lyapunov function and a computable stabilizing switching law. Feron [10] proved the converse is also true by constructing a quadratically stable convex combination of the two sub-systems based on two total derivatives (two quadratic forms) of the existing quadratic Lyapunov function and using the strict S-Lemma. These results reveal the difference degree between linear
systems and switched linear systems from the perspective of stability. Due to their substantial contributions, these results were quoted widely and embodied in the monograph [15] on switched systems. However, these results have not been extended to nonlinear cases. This is for reason that it is difficult for switched nonlinear systems to construct stable convex combinations of the sub-systems, and the strict S-Lemma can not be used to deal with derivatives of Lyapunov functions of higher degrees. Then interesting issues arise: May the strict S-Lemma be extended to nonlinear functions of higher degrees? May the above necessary and sufficient condition for switched linear systems be extended to switched nonlinear systems?

Homogeneous nonlinear systems are a class of nonlinear systems that have properties similar to linear systems, and many interesting results of linear systems were extended to homogeneous nonlinear systems (cf. [16, 17, 18, 19, 20, 23]). Cheng and Martin [21] proposed the concept of the Lyapunov function with homogeneous derivative (LFHD for short) and applied it to testify the stability of a class of nonlinear polynomial systems. A nonlinear system admitting a LFHD is not necessarily homogenous, but still have some properties of homogeneous systems. For example, if a nonlinear component-wise homogeneous polynomial system admits a LFHD (cf. [21]), its global stability is easily guaranteed. A nonlinear system admitting a LFHD can be regarded as an approximation of the center manifolds of a large class of nonlinear systems. Hence to study such systems is theoretically significant and interesting. Cheng and Martin [21] also gave methods to construct a LFHD for a component-wise homogeneous polynomial systems.

In this paper, we use the concept of LFHD to characterize a class of switched nonlinear systems.

1.4. Model. In order to describe this problem clearly, the system considered in this paper is formulated as

\[
\dot{x} = f_{\sigma(t)}(x), \quad x = x(t) \in \mathbb{R}^n, \tag{1.2}
\]

where \( \sigma : [0, +\infty) \to \Lambda = \{1, 2, \cdots, N\} \) is a piece-wise constant, right continuous function, called the switching signal. \( N \) is an integer no less than 2, and each \( f_i \) is a continuous function of the state \( x \). A convex combination of the sub-systems of system (1.2) denotes the system

\[
\dot{x} = \sum_{i=1}^{N} \lambda_i f_i(x), \quad 0 \leq \lambda_1, \lambda_2, \cdots, \lambda_N \leq 1, \quad \text{and} \quad \sum_{i=1}^{N} \lambda_i = 1.
\]

Throughout this paper, it is assumed that system (1.2) admits a LFHD. That is to say, there exists a positive definite and continuously differentiable function \( V : \mathbb{R}^n \to \mathbb{R} \), such that each of \( \dot{V}(x) |_{S_i} \) is a continuous, even and homogeneous function of the same degree with respect to the same dilation, and

\[
\bigcup_{i=1}^{N} \left\{ x \in \mathbb{R}^n : \dot{V}(x) |_{S_i} < 0 \right\} \supset \mathbb{R}^n \setminus \{0\}, \tag{1.3}
\]

where \( S_i \) denotes the \( i \)-th sub-system, \( \dot{V}(x) |_{S_i} \) denotes the derivative of \( V(x) \) along the solution trajectory of \( S_i, i \in \Lambda \).

It can be proved that if system (1.2) admits a LFHD, then for any given initial state, there exists a switching law driving the initial state to the equilibrium point as \( t \to \infty \) [26].

Based on the concept of LFHD, the necessary and sufficient conditions given in [9, 10] can be restated as: If for system (1.2), each \( f_i \) is linear and \( N = 2 \), then system (1.2) admits a (quadratic) LFHD if and only if there exits a convex combination of its two sub-systems that admits a (quadratic) LFHD. In this paper, we will extend these results to nonlinear system (1.2) with \( N = 2 \), and show that the necessary one does not hold when \( N > 2 \).

The contributions of the paper include:
Theorem 2.1. We extend the strict S-Lemma to the strict homogeneous S-Lemma (the SHS-Lemma for short, from the case \( f, g \) are quadratic forms to homogeneous functions with respect to any dilation). In detail, we indicate that the strict S-Lemma does not necessarily hold for homogeneous functions that are not quadratic forms, and give a necessary and sufficient condition under which the SHS-Lemma holds.

We use the SHS-Lemma to give a necessary and sufficient condition under which system \((1.2)\) when \( N = 2 \) admits a LFHD if and only if there exists a convex combination of its sub-systems that admits a LFHD, and show the “if” part still holds when \( N > 2 \).

A counterexample is given to show that even though system \((1.2)\) when \( N > 2 \) admits a LFHD, there may exist no convex combination of its sub-systems that admits a LFHD.

The S-Lemma is extended to polynomials of degree more than 2 under some mild conditions, and the extended results are called the homogeneous S-Lemma (the HS-Lemma) and the non-homogeneous S-Lemma (the NHS-Lemma), respectively.

The remaining part of this paper is organized as follows: Section 2 gives the main results and some examples supporting the main results. The SHS-Lemma is first shown, then based on it, the asymptotic stability of switched nonlinear systems with two sub-systems is analyzed; a counterexample about switched linear systems with more than two sub-systems is given; at last, some non-strict S-Lemmas are shown. Section 3 is a brief conclusion.

2. Main Results. Until now, there have been four approaches to proving the S-Lemma (cf. [2, 1], [4, 5], [6] and [7], respectively). It turns out that the two approaches given in [4, 5] and [6] cannot be generalized to prove the SHS-Lemma, since for homogeneous polynomials of degree more than 2, the positive definiteness cannot only be determined by their coefficient matrices or the eigenvalues of their coefficient matrices; the approach given in [7] cannot either, since unlike quadratic polynomials, graphs of polynomials of degree greater than 2 are not necessarily spherically convex (The concept of spherical convexity is referred to [7]). The most fundamental approach, the approach given in [2, 1] can be generalized to deal with the case that the homogeneous functions are odd functions. However for the case that the homogeneous functions are even, it does not work either. In this paper, we propose a new approach that can be used to deal with both the two cases and to prove the SHS-Lemma.

2.1. Strict Homogeneous S-Lemma with Application to Stability of Switched Nonlinear Systems with Two Sub-systems. We first prove Theorem 2.1 that is an extension of Theorem 1.3 to some extent, and then prove the SHS-Lemma (Theorem 2.2) based on Theorem 2.1.

**Theorem 2.1.** Let \( f, g : \mathbb{R}^n \to \mathbb{R} \) be continuous, homogeneous functions of degree \( k \leq n \) with respect to the same dilation \((r_1, \ldots, r_n)\), and assume \( f \) and \( g \) have no common zero point except \( 0 \in \mathbb{R}^n \) when \( k > 0 \). Then the set \( \{(f(x), g(x)) : x \in \mathbb{R}^n \} := U \) is closed. If \( k = 0 \), the set \( U \) is a singleton. Next assume \( k > 0 \). If \( f \) and \( g \) are both odd functions, the set \( U \) is convex. In detail, the set \( U \) either equals \( \mathbb{R}^2 \), or is a straight line passing through the origin. If \( f \) and \( g \) are both even functions, the set \( U \) is an angular sector.

**Remark 2.1.** Note that in Theorem 2.1 the assumption that \( f \) and \( g \) have no common zero point except \( 0 \in \mathbb{R}^n \) is crucial. It is because if \( f \) and \( g \) do have a common nonzero zero point, the set \( \{(f(x), g(x)) : x \in \mathbb{R}^n \} \) may be neither convex nor an angular sector. For example, polynomials \( f(x, y) = -x^3 + y^3 \) and \( g(x, y) = y^3 - \frac{1}{2}x^3 - \frac{1}{2}xy^2 \) have the common nonzero zero point \((1, 1)\). And the set \( \{(-x^3 + y^3, y^3 - \frac{1}{2}x^3 - \frac{1}{2}xy^2) : x, y \in \mathbb{R} \} := U \) is neither convex nor an angular sector (see Fig. 2.7). This is because \((1, 1)\) and \((-1, -\frac{1}{2})\) are...
both in $U$, but $(0, \frac{1}{2}) = \frac{1}{2}[(1, 1) + (-1, -1)]$ is not in $U$; $(-1, -1)$ and $(1, \frac{1}{2})$ are both in $U$, but $(0, -\frac{1}{2}) = \frac{1}{2}[-1, -1] + (1, \frac{1}{2})$ is not in $U$.

On the other hand, if $f$ and $g$ have a common nonzero zero point, the set $\{(f(x), g(x)) : x \in R^n\}$ may be a convex set. For example, polynomials $x^2 - 2xy + y^2$ and $x^2 - y^2$ have the common nonzero zero point $(1, 1)$, but the set $\{(x^2 - 2xy + y^2, x^2 - y^2) : x, y \in R\}$ is still convex by Theorem 1.3.

Proof. [of Theorem 2.1] Let $U$ denote the set $\{(f(x), g(x)) : x \in R^n\}$ for short.

$k = 0$:

Let $\epsilon_m$ be $1/m$, $m = 1, 2, \cdots$. We have $\lim_{m \to \infty} (\epsilon_m x_1, \cdots, \epsilon_m x_n) = 0$ for all $x_1, \cdots, x_n \in R$. Further

$$f(x_1, \cdots, x_n) = \lim_{m \to \infty} \epsilon_m f(x_1, \cdots, x_n) = \lim_{m \to \infty} f(\epsilon_m x_1, \cdots, \epsilon_m x_n) = f(0)$$

for all $(x_1, \cdots, x_n) \in R^n$ by the continuity and homogeneity of $f$. Similarly $g$ is also constant. Hence the set $U$ is a singleton, which is closed.

$k > 0$:

In this case, $f(0) = g(0) = 0$.

Firstly we prove the set $U$ is closed.

Because $f$ and $g$ are continuous and they have no common zero point except $0 \in R^n$, $(f(x), g(x))/||f(x), g(x)||$ is a continuous function defined on $R^n \backslash \{0\}$ and maps the unit sphere of $R^n$ onto a compact subset of the unit sphere of $R^2$, where $||\cdot||$ is the Euclidean norm. The compact subset is also compact in $R^2$, and then closed. Further by the homogeneity of $f$ and $g$, the set $U$ is closed.

Secondly we prove if $u \in U$, then $\lambda u \in U$ for all $\lambda > 0$.

For any given $u \in U$, there exists $z_1 = (z_{11}, \cdots, z_{1n}) \in R^n$ such that

$$u = (u_f, u_g) = (f(z_1), g(z_1)). \quad (2.1)$$

For any given $\lambda > 0$, there exists $\epsilon > 0$ such that $\lambda = \epsilon^k$. Then

$$\lambda u = (\epsilon^k f(z_1), \epsilon^k g(z_1)) = (f(\epsilon^k z_{11}, \cdots, \epsilon^k z_{1n}), g(\epsilon^k z_{11}, \cdots, \epsilon^k z_{1n})) \in U.$$
Thirdly we define a closed curve that plays a central role in the following proof. We use \( f(\theta) \) and \( g(\theta) \) to denote the functions

\[
\begin{align*}
  f(z) &= |\cos \theta|^r \text{sgn}(\cos \theta) + z_2 |\sin \theta|^r \text{sgn}(\sin \theta) + \cdots + z_n |\sin \theta|^r \text{sgn}(\sin \theta) \\
  g(z) &= |\cos \theta|^r \text{sgn}(\cos \theta) + z_2 |\sin \theta|^r \text{sgn}(\sin \theta) + \cdots + z_n |\sin \theta|^r \text{sgn}(\sin \theta)
\end{align*}
\]

respectively for short hereinafter, where \( \text{sgn}(\cdot) \) denotes the sign function.

The function \( (f(\theta), g(\theta)) \) can be seen as a continuous function defined over the closed interval \([0, 2\pi]\), and \( f(\theta) \) and \( g(\theta) \) both have period \( 2\pi \), then the curve \( \{(f(\theta), g(\theta)) : \theta \in [0, 2\pi]\} := \ell \) is a path-connected, bounded and closed set. And \( \{tv \in [0, 2\pi] : t \geq 0, v \in \ell\} \subset U \).

Since \( f(x) \) and \( g(x) \) have no common zero point except \( 0 \in R^n \), \( f(\theta) = g(\theta) = 0 \) implies \( z_1 |\cos \theta|^r \text{sgn}(\cos \theta) + z_2 |\sin \theta|^r \text{sgn}(\sin \theta) = 0 \), then \( z_1 = -z_2 |\tan \theta|^r \text{sgn}(\tan \theta) \) or \( z_1 |\cot \theta|^r \text{sgn}(\cot \theta) = -z_2 \) for all \( i = 1, \ldots, n \). Then \( u \) and \( v \) are linearly dependent. Hence the curve \( \ell \) does not pass through the origin if \( u \) and \( v \) are linearly independent. Similarly, if \( f \) and \( g \) are both even functions, \( u \) and \( v \) are linearly dependent, and either \( u^j v^j < 0 \) or \( u^j v^j < 0 \), the curve \( \ell \) is also path-connected, bounded, closed and does not pass through the origin either.

At last, we give the conclusion.

Next assume that \( f(x) \) and \( g(x) \) are both odd functions.

Assume that the set \( U \) is not a line passing through the origin, then there exist linearly independent vectors \( u, v \in U \). It is easy to get \( f(\theta) = -f(\theta + \pi) \) and \( g(\theta) = -g(\theta + \pi) \) for all \( \theta \in R \). That is, the curve \( \ell \) is central symmetric. Then \( \ell \) is homeomorphic to the unit sphere of \( R^2 \). Hence \( \{tv \in [0, 2\pi] : t \geq 0, v \in \ell\} \subset R^2 \subset U \subset R^2 \). That is, \( U = R^2 \), and \( U \) is convex.

Next assume that \( f \) and \( g \) are both even functions.

Assume \( U \neq R^2 \), that is to say, there exists a vector \( u' \in R^2 \) such that \( u' \notin U \), then the set \( U \) is contained in an angular sector of angle less than \( 2\pi \) whose boundary is in \( U \) since \( U \) is closed. The boundary of the angular sector is the union of two half lines. Choose two points \( u, v \) in different half lines. Then the corresponding curve \( \ell \) is path-connected, closed and does not pass through the origin. And furthermore, \( \{tv \in [0, 2\pi] : t \geq 0, v \in \ell\} = U \) equals the angular sector.

**Example 2.1.** We give some examples to illustrate Theorem 2.1

\( k \) is odd:
1. \( \{(f(x) = x^3, g(x) = x^3) : x \in R\} \) is a straight line passing through the origin.
2. \( \{(f(x, y) = x^3, g(x, y) = x^3) : x, y \in R\} = R^2 \).

\( k \) is even:

In this case, we give some examples to show the angle, denoted by \( \Phi \), of the set \( U \) (see the proof of Theorem 2.1) satisfies \( \Phi = \pi, \pi < \Phi < \frac{3}{2}\pi, \frac{3}{2}\pi < \Phi < 2\pi \) and \( \Phi = 2\pi \), respectively. The case \( \Phi < \pi \) is seen in Example 2.2 (see Fig. 2.3). In each of the following four examples, \( f \) and \( g \) have no common zero point except \( 0 \in R^2 \).

1. \( \{(f(x, y) = x^4 - y^4 - x^2y^2, g(x, y) = -x^4 + y^4) : x, y \in R\} (\Phi = \pi) \): 
   \( f(1, 0), g(1, 0) = (1, -1), (f(0, 1), g(0, 1) = (-1, 1) \) and \( f(x, y) + g(x, y) < 0 \)
   for all \( (x, y) \in R^2 \) imply the angle of \( U \) equals \( \pi \).
2. \( \{(f(x, y) = -x^4 + y^4 - x^2y^2, g(x, y) = x^4 - y^4 + x^2y) : x, y \in R\} (\pi < \Phi < \frac{3}{2}\pi) \):
   \( f(1, -2), g(1, -2) = (23, -17) \) and \( (f(2, 1), g(2, 1)) = (-17, 23) \) and
   \( (f(3, 4), g(3, 4)) = (-17, -67) \) imply \((23, -17), (-17, 23), (-17, -67) \in U \). The
three points show that the angle of $U$ is greater than $\pi$. The inequalities $f(x, y) \geq 0$ and $g(x, y) \geq 0$ have no common solution shows that the angle of $U$ is less than $\frac{3\pi}{2}$.

3. $\{(f(x, y) = x^6 - y^6 + 20x^5y - 20x^3y^3, g(x, y) = -x^6 + y^6 - 10xy^5) : x, y \in R\}$ ($\frac{3\pi}{2} < \Phi < 2\pi$):

$f(0, 1), g(0, 1) = (−1, 1)$, $(f(2, 3), g(2, 3)) = (−3065, −4195),$
$(f(2, 1), g(2, 1)) = (543, −83) and $(f(−5, −6), g(−5, −6)) = (133969, 419831)$

imply $−1, 1, (−3065, −4195), (543, −83), (133969, 419831) \in U.$

$\{(543, −83), (133969, 419831)\} = 37899194 > 0$, where $\langle \cdot, \cdot \rangle$ denotes the inner product.

$f(x, y) = 1$ and $g(x, y) = 0$ have no common solution.

Hence $\frac{3\pi}{2} < \Phi < 2\pi$.

4. $\{(f(x, y) = x^6 − y^6, g(x, y) = −x^6 + y^6 − x^3y^3) : x, y \in R\}$ ($\Phi = 2\pi$):

$f(x, y) = a$ and $g(x, y) = b$ have a common solution for all $a, b \in R$.

Based on Theorem 2.1 we give the following Theorem 2.2. We still call it the strict homogeneous S-Lemma.

**Theorem 2.2 (SHS-Lemma).** Let $f, g : R^n \to R$ both be continuous, even and homogenous functions of degree $0 \leq k \in R$ with respect to the same dilation $(r_1, \ldots, r_n)$.

If and only if there exist $a, b \in R$ such that $a^2 + b^2 > 0$ and neither $\begin{cases} f(x) = a & \text{nor} \\ g(x) = b & \end{cases}$

(i) $f$ is strictly copositive with $g$;

(ii) there exists $\xi > 0$ such that $f(x) - \xi g(x) > 0$ for all $0 \neq x \in R^n$.

**Remark 2.2.** Theorem 2.1 shows that if $f$ and $g$ are both homogeneous of odd degree and $f$ and $g$ have no nonzero common zero point, the set $\{(f(x), g(x)) : x \in R^n\}$ is either the whole $R^2$ or a straight line passing through the origin. In the former case, (i) of Theorem 2.2 cannot hold. In the latter case, if (i) of Theorem 2.2 holds, there exist $\alpha_1, \alpha_2 \in R$ such that $\alpha_1\alpha_2 < 0$ and $\alpha_1 f(x) + \alpha_2 g(x) = 0$ for all $x \in R^n$, which indicates (ii) of Theorem 2.2 cannot hold (For example, $f(x) = g(x) = x^3 : R \to R$). Hence in Theorem 2.2 we assume that $k$ is even.

**Proof.** [of Theorem 2.2] If $k = 0$, $f$ and $g$ are both constant functions by Theorem 2.1. Then (i) is obviously equivalent to (ii).

Next we assume that $k > 0$.

(iii) $\Rightarrow$ (i) holds naturally.

By Theorem 2.1 (i) implies the set $\{(f(x), g(x)) : x \in R^n\}$, denoted by $U$, is an angular sector of angle less than $\frac{3\pi}{2}$ and

$$U \cap \{(r_1, r_2) : r_1 \leq 0, r_2 \geq 0\} = \emptyset.$$ 

Next we assume that there exist $a, b \in R$ such that $a^2 + b^2 > 0$ and neither $\begin{cases} f(x) = a & \text{nor} \\ g(x) = b & \end{cases}$

The foregoing assumption and (i) imply the angle of $U$ is less than $\pi$. Then there exist $\xi_1 < 0$ and $\xi_2 > 0$ such that

$$\xi_1 f(x) + \xi_2 g(x) < 0$$

for all $0 \neq x \in R^n$. Set $\xi = −\xi_2, \xi_1 > 0$, then $f(x) − \xi g(x) > 0$ for all $x \in R^n$.

In particular, when $k = 2$, (i) implies the above assumption (see Theorem 1.3).
Next we assume for all \(a, b \in R\) such that \(a^2 + b^2 > 0\), either \(\begin{cases} f(x) = a \\ g(x) = b \end{cases}\) or \(\begin{cases} f(x) = -a \\ g(x) = -b \end{cases}\) have a solution, which together with (i) implies the angle of \(U\) is no less than \(\pi\). Hence (ii) does not hold.

Based on Theorem 2.2 we give the following Theorem 2.3.

**Theorem 2.3.** System (1.2) admits a LFHD \(V : R^n \rightarrow R\), and there exist \(a, b \in R\) such that \(a^2 + b^2 > 0\) and neither \(\begin{cases} \dot{V}(x) |_{s_1} = a \\ \dot{V}(x) |_{s_2} = b \end{cases}\) nor \(\begin{cases} \dot{V}(x) |_{s_1} = -a \\ \dot{V}(x) |_{s_2} = -b \end{cases}\) have a solution, if and only if there exists a convex combination of its two sub-systems that admits a LFHD when \(N > 2\) (The “if part” still holds when \(N > 2\)).

**Proof.** “if”: This part is trivial just like the triviality of the “if” part of the SHS-Lemma.

“only if”: This part is proved by the SHS-Lemma.

Since \(V\) is a LFHD of system (1.2) when \(N = 2\), that is to say,

\[
\bigcup_{i=1}^{2} \left\{ x \in R^n : \dot{V}(x) |_{s_i} < 0 \right\} \supset R^n \setminus \{0\},
\]

then \(-\dot{V}(x) |_{s_1}\) is strictly copositive with \(\dot{V}(x) |_{s_2}\).

By Theorem 2.2 and the assumption related to \(V\) in Theorem 2.3 there exists \(\xi > 0\) such that

\[
\dot{V}(x) |_{s_1} + \xi \dot{V}(x) |_{s_2} < 0 \quad \text{for all} \quad 0 \neq x \in R^n.
\]

Take \(\lambda_1 = \frac{1}{1 + \xi}, \lambda_2 = \frac{\xi}{1 + \xi}\), then \(V\) is a LFHD of system \(\dot{x} = \lambda_1 f_1(x) + \lambda_2 f_2(x)\).

**Remark 2.3.** Theorem 2.3 indicates the existence of an asymptotically stable convex combination of the two sub-systems, but it does not show how to find the convex combination. Luckily, there are only two sub-systems, so we can use Young’s inequality to construct the convex combination. Example 2.2 illustrates the procedure and the case that \(\Phi < \pi\) in Theorem 2.7 by showing a switched polynomial system and Example 2.3 illustrates the procedure by showing a switched non-polynomial system.

In fact, Theorem 2.3 supplies a method to find a LFHD for a switched polynomial system with two sub-systems: (i) Construct its convex combination of its sub-systems with coefficients variable parameters; (ii) construct a LFHD by using the methods proposed in [21].

**Example 2.2.** Consider the switched polynomial system \(S\) with two sub-systems as follows:

\[
S_1 : \begin{cases} \dot{x}_1 = 7x_1^3 - 3x_2^3 + 2x_1x_2^2 \\ \dot{x}_2 = 5x_1^3 - 5x_2^2 \end{cases},
\]

\[
S_2 : \begin{cases} \dot{x}_1 = -5x_1^3 - x_1x_2^3 \\ \dot{x}_2 = -x_1^3 + x_2^3 \end{cases}.
\]

It is obvious that the origin is the unique equilibrium point for both sub-systems \(S_1\) and \(S_2\).

Firstly, we prove the origin is unstable both for sub-system \(S_1\) and for sub-system \(S_2\).

For sub-system \(S_1\), choose \(V_1(x) = \frac{1}{4} (5x_1^4 - x_2^4)\). On the line \(x_2 = 0\), \(V_1(x) > 0\) at points arbitrarily close to the origin, and \(V_1(x) = 15x_1^6 + 5 (2x_1^3 - x_2^3)^2 + 10x_1^4x_2^2\) is positive definite. Then by Chetaev’s theorem ([21] 4.3 of [23]), the origin is unstable.
For sub-system $S_2$, choosing $V_2(x) = \frac{1}{4} \left( -x_1^4 + x_2^4 \right)$, similarly we have the origin is unstable.

Secondly, we prove switched system $S$ admits a LFHD. Choosing $V(x) = \frac{1}{4} \left( x_1^4 + x_2^4 \right)$ that is positive definite, then

$$
\dot{V}(x) \bigg|_{S_1} = 7x_1^6 + 2x_1^3x_2^3 - 5x_2^6 + 2x_1^4x_2^2,
$$

$$
\dot{V}(x) \bigg|_{S_2} = -5x_1^6 - x_1^3x_2^3 + x_2^6 - x_1^4x_2^2.
$$

By Young's inequality, we have (see Fig. 2.2)

\[
\left\{ (x_1, x_2) : \dot{V}(x) \bigg|_{S_1} < 0 \right\} \cup \left\{ (x_1, x_2) : \dot{V}(x) \bigg|_{S_2} < 0 \right\} \supset \mathbb{R}^2 \setminus \{ (0, 0) \}.
\]

The procedure is as follows: By Young's inequality, we have

\[
\dot{V}(x) \bigg|_{S_1} \leq \frac{25}{3} x_1^6 + 2x_1^3x_2^3 - \frac{13}{3} x_2^6
= -\frac{13}{3} \left( x_1^3 - \frac{3 - \sqrt{334}}{13} x_2^3 \right) \left( x_2^3 - \frac{3 + \sqrt{334}}{13} x_1^3 \right),
\]

\[
\dot{V}(x) \bigg|_{S_2} \leq -5x_1^6 - x_1^3x_2^3 + x_2^6
= -\left( x_2^3 - \frac{1 - \sqrt{21}}{2} x_1^3 \right) \left( x_2^3 - \frac{1 + \sqrt{21}}{2} x_1^3 \right),
\]

and then

\[
\left\{ (x_1, x_2) : \dot{V}(x) \bigg|_{S_1} < 0 \right\} \supset \left\{ (x_1, x_2) : \frac{25}{3} x_1^6 + 2x_1^3x_2^3 - \frac{13}{3} x_2^6 < 0 \right\}
= \left\{ (x_1, x_2) : x_2 \left( 3 - \frac{\sqrt{334}}{13} \right)^{1/3} x_1 > 0, x_2 - \left( 3 + \frac{\sqrt{334}}{13} \right)^{1/3} x_1 > 0 \right\}
\cup \left\{ (x_1, x_2) : x_2 \left( 3 - \frac{\sqrt{334}}{13} \right)^{1/3} x_1 < 0, x_2 - \left( 3 + \frac{\sqrt{334}}{13} \right)^{1/3} x_1 < 0 \right\},
\]

\[
\left\{ (x_1, x_2) : \dot{V}(x) \bigg|_{S_2} < 0 \right\} \supset \left\{ (x_1, x_2) : -5x_1^6 - x_1^3x_2^3 + x_2^6 < 0 \right\}
= \left\{ (x_1, x_2) : x_2 \left( 1 - \frac{\sqrt{21}}{2} \right)^{1/3} x_1 > 0, x_2 - \left( 1 + \frac{\sqrt{21}}{2} \right)^{1/3} x_1 > 0 \right\}
\cup \left\{ (x_1, x_2) : x_2 \left( 1 - \frac{\sqrt{21}}{2} \right)^{1/3} x_1 < 0, x_2 - \left( 1 + \frac{\sqrt{21}}{2} \right)^{1/3} x_1 < 0 \right\}.
\]
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The Stable Region of Sub-system 1:
\[ \{ (x_1, x_2) : (x_1^2 + x_2^2 - 3x_1 + 2x_2 + 3x_1^2 + 3x_2^2 / 2 | 0) \} \]

The Stable Region of Sub-system 2:
\[ \{ (x_1, x_2) : (x_1^2 - x_2^2 - 3x_1 + 2x_2 - 3x_1^2 + 3x_2^2 / 2 | 0) \} \]

The Intersection of the Stable Regions of the Two Sub-systems
\[ \{ (x_1, x_2) : (x_1^2 - x_2^2 - 3x_1 + 2x_2 - 3x_1^2 + 3x_2^2 / 2 | 0) \} \]

The Union of the Stable Regions of the Two Sub-systems
\[ \{ (x_1, x_2) : (x_1^2 + x_2^2 - 3x_1 + 2x_2 + 3x_1^2 + 3x_2^2 / 2 | 0) \} \]

FIG. 2.2. The stable regions of Example 2.2 on the unit disk

As \( \frac{1 + \sqrt{13}}{2} > \frac{3 + \sqrt{13}}{2} > 0 > \frac{3 - \sqrt{13}}{2} > \frac{1 - \sqrt{13}}{2} \), we have

\[
\left\{ (x_1, x_2) : \dot{V}(x)|_{S_1} < 0 \right\} \cup \left\{ (x_1, x_2) : \dot{V}(x)|_{S_2} < 0 \right\}
\]

\[ \supset \left\{ (x_1, x_2) : \frac{25}{3} x_1^6 + 2 x_1^3 x_2^3 - \frac{13}{3} x_2^6 < 0 \right\} \cup \left\{ (x_1, x_2) : -5x_1^6 - x_1^3 x_2^3 + x_2^6 < 0 \right\}
\]

\[ \supset \mathbb{R}^2 \setminus \{(0,0)\}.
\]

Thirdly, we prove the LFHD \( \dot{V} \) satisfies the assumption in Theorem 2.3.

\( \dot{V}(x)|_{S_1} = 2 \) and \( \dot{V}(x)|_{S_2} = -1 \) imply \( x_1^6 + x_2^6 = 0 \), then \( x_1 = x_2 = 0 \). That is to say, they have no common solution.

\( \dot{V}(x)|_{S_1} = -2 \) and \( \dot{V}(x)|_{S_2} = 1 \) also imply \( x_1^6 + x_2^6 = 0 \), then \( x_1 = x_2 = 0 \), which also means they have no common solution.

To illustrate Theorem 2.7, the best we can do is to picture the set

\[
\left\{ \left( \dot{V}(x)|_{S_1}, \dot{V}(x)|_{S_2} \right) : x_1, x_2 \in \mathbb{R} \right\} := U \text{ (see Fig. 2.3)}.
\]

From Fig. 2.3 we see that \( U \) is an angular sector of angle less than \( \pi \).

At last, we construct a convex combination of sub-system \( S_1 \) and sub-system \( S_2 \) that admits a LFHD.
Let $0 < \lambda < 1$, then a convex combination of sub-system $S_1$ and sub-system $S_2$, $\lambda S_1 + (1-\lambda)S_2$, is formulated as follows:

$$
\begin{aligned}
\dot{x}_1 &= (12\lambda - 5)x_1^3 - 3\lambda x_2^3 + (3\lambda - 1)x_1x_2^2, \\
\dot{x}_2 &= (6\lambda - 1)x_1^3 + (1 - 6\lambda)x_2^3.
\end{aligned}
$$

(2.6)

We might as well take $V(x) = \frac{1}{4}(x_1^4 + x_2^4)$, a positive definite function, then we have

$$
\dot{V}(x) = (12\lambda - 5)x_1^6 + (3\lambda - 1)x_1^3x_2^3 + (1 - 6\lambda)x_2^6 + (3\lambda - 1)x_1^4x_2^2.
$$

(2.7)

Now we try to find a $\lambda \in (0, 1)$ such that (2.7) is negative definite. If $3\lambda - 1 \geq 0$, by Young’s inequality, we get

$$
\dot{V}(x) \leq \left(\frac{31}{2} \lambda - \frac{37}{6}\right)x_1^6 + \left(\frac{1}{6} - \frac{7}{2}\lambda\right)x_2^6.
$$

Let $\frac{31}{2} \lambda - \frac{37}{6} < 0$ and $\frac{1}{6} - \frac{7}{2}\lambda < 0$, together with $3\lambda - 1 \geq 0$, we get $\frac{1}{3} \leq \lambda < \frac{37}{93}$.

If $3\lambda - 1 \leq 0$, by Young’s inequality, we get

$$
\dot{V}(x) \leq \left(9\lambda - \frac{9}{2}\right)x_1^6 + \left(\frac{3}{2} - \frac{15}{2}\lambda\right)x_2^6 + (3\lambda - 1)x_1^4x_2^2.
$$

Let $9\lambda - \frac{9}{2} < 0$, $\frac{3}{2} - \frac{15}{2}\lambda < 0$ and $3\lambda - 1 < 0$, we get $\frac{1}{3} < \lambda < \frac{1}{3}$. Hence, if $\frac{1}{3} < \lambda < \frac{37}{93}$, system (2.6) admits a LFHD.

**Example 2.3.** Consider the following switched system $S$ with two sub-systems as follows:

$$
S_1 : \begin{cases}
\dot{x}_1 = -4x_1, \\
\dot{x}_2 = 4x_1^4x_2 + 4x_2^3
\end{cases}, \quad S_2 : \begin{cases}
\dot{x}_1 = 2x_1 + x_1^\frac{1}{2}x_2^2, \\
\dot{x}_2 = -8x_2^3.
\end{cases}
$$

It is obvious that the origin is the unique equilibrium point for both sub-system $S_1$ and sub-system $S_2$.

Firstly, we prove the origin is unstable both for sub-system $S_1$ and for sub-system $S_2$. 

For sub-system $S_1$, choose $V_1(x) = -3x_1^{\frac{4}{3}} + x_2^2$. On the line $x_1 = 0$, $V_1(x) > 0$ at points arbitrarily close to the origin, and $\dot{V}_1(x) = (4x_1^{\frac{2}{3}} + x_2^2)^2 + 7x_2^4$ is positive definite. Then by Chetaev’s theorem (Theorem 4.3 of [22]), the origin is unstable.

For sub-system $S_2$, choosing $V_2(x) = 3x_1^{\frac{4}{3}} - x_2^2$, then $\dot{V}_2(x) = 4x_1^{\frac{2}{3}} + (2x_1^{\frac{2}{3}} + x_2^2)^2 + 15x_2^4$, similarly we have the origin is unstable.

Secondly, we prove switched system $S$ admits a LFHD.

Choosing $V(x) = 3x_1^{\frac{4}{3}} + x_2^2$ that is positive definite, then
\[
\dot{V}(x)\bigg|_{S_1} = -16x_1^{\frac{4}{3}} + 8x_1^{\frac{2}{3}}x_2^2 + 8x_2^4,
\]
\[
\dot{V}(x)\bigg|_{S_2} = 8x_1^{\frac{4}{3}} + 4x_1^{\frac{2}{3}}x_2^2 - 16x_2^4,
\]
which are both homogeneous functions of degree 4 with respect to the dilation $(3, 1)$.

By Young’s inequality, we have
\[
\{(x_1, x_2) : \dot{V}(x)\bigg|_{S_1} < 0\} \cup \{(x_1, x_2) : \dot{V}(x)\bigg|_{S_2} < 0\} \supset R^2 \setminus \{(0, 0)\} \text{ (see Fig. 2.4).}
\]

Thirdly, we prove the LFHD $V$ satisfies the assumption in Theorem 2.3
\[
\dot{V}(x)\bigg|_{S_1} = 1 \text{ and } \dot{V}(x)\bigg|_{S_2} = -1 \implies 2x_1^{\frac{4}{3}} - 3x_1^{\frac{2}{3}}x_2^2 + 2x_2^4 = 0, \text{ which has no solution.}
\]

That is to say, they have no common solution.
\[
\dot{V}(x)\bigg|_{S_1} = -1 \text{ and } \dot{V}(x)\bigg|_{S_2} = 1 \implies 2x_1^{\frac{4}{3}} - 3x_1^{\frac{2}{3}}x_2^2 + 2x_2^4 = 0, \text{ then they have no common solution.}
\]

At last, we construct a convex combination of sub-system $S_1$ and sub-system $S_2$ that admits a LFHD.

Let $0 < \lambda < 1$, then a convex combination of sub-system $S_1$ and sub-system $S_2$, $\lambda S_1 + (1-\lambda) S_2$, is formulated as follows:
\[
\begin{cases}
\dot{x}_1 = (2 - 6\lambda)x_1 + (1 - \lambda)x_1^{\frac{4}{3}}x_2^2, \\
\dot{x}_2 = 4\lambda x_1^{\frac{2}{3}}x_2 + (12\lambda - 8)x_2^3.
\end{cases}
\]

We might as well take $V(x) = 3x_1^{\frac{4}{3}} + x_2^2$, a positive definite function, then we have
\[
\dot{V}(x) = 8(1 - 3\lambda)x_1^{\frac{4}{3}} + 4(1 + \lambda)x_1^{\frac{2}{3}}x_2^2 + 8(3\lambda - 2)x_2^4,
\]
\[
\leq (10 - 22\lambda)x_1^{\frac{4}{3}} + (26\lambda - 14)x_2^4
\]
by Young’s inequality.

Now we try to find a $\lambda \in (0, 1)$ such that \eqref{2.10} is negative definite.

Let $10 - 22\lambda < 0$ and $26\lambda - 14 < 0$, we get $\frac{5}{11} < \lambda < \frac{7}{11}$.

Hence, if $\frac{5}{11} < \lambda < \frac{7}{11}$, system \eqref{2.9} admits a LFHD.

Next we give a direct corollary of Theorem 2.1 We use a generalization of the basic idea in [2] to give an interesting proof that is only suitable for homogeneous polynomials of odd degree.

**Corollary 2.4.** Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be homogeneous polynomials of degree $k \geq 1$, and assume that $f$ and $g$ have no common zero point except $0 \in \mathbb{R}^n$. Then the set $\{(f(x), g(x)) : x \in \mathbb{R}^n\}$, denoted by $U$, is closed. If $k$ is odd, the set $U$ is convex. In detail, the set $U$ either
The Stable Region of Sub-system 1: \[
\{ (x_1, x_2) : x_1^2 + x_2^2 \leq 1 \}
\]
The Stable Region of Sub-system 2: \[
\{ (x_1, x_2) : (x_1^2 + x_2^2)_{-0.5} \}
\]
The Intersection of the Stable Regions of the Two Sub-systems:
\[
\{ (x_1, x_2) : (x_1^2 + x_2^2)_{-0.5} \}
\]
The Union of the Stable Regions of the Two Sub-systems:
\[
\{ (x_1, x_2) : (x_1^2 + x_2^2)_{-0.5} \}
\]

**FIG. 2.4.** The stable regions of Example 2.3 on the generalized unit disk \{ \((x_1, x_2) : x_1^2 + x_2^2 \leq 1\) \}

equals \(R^2\), or is a straight line passing through the origin. If \(k\) is even, the set \(U\) is an angular sector.

**Proof.**
Let \(U\) denote the set \(\{ (f(x), g(x)) : x \in R^n \}\) for short.
If \(A \in U\), then each point in the ray starting at the origin and passing through \(A\) is in the set \(U\) by the homogeneity of \(f\) and \(g\). Hereinafter, we assume that \(f\) and \(g\) have no common zero point except \(0 \in R^n\) and \(k\) is odd.

Next we prove the set \(U\) is a convex set.
To this end, we only need to prove that for any \(u, v \in U\), \(\lambda u + (1 - \lambda) v \in U\) for all \(\lambda \in [0, 1]\).

There exist \(z_1, z_2 \in R^n\) such that
\[
uf = f(z_1), \ ug = g(z_1), \ vf = f(z_2) \text{ and } vg = g(z_2),
\]
where \(u = (uf, ug)\) and \(v = (vf, vg)\).
By the homogeneity of \(f\) and \(g\), if \(u\) and \(v\) are linearly dependent, then \(\lambda u + (1 - \lambda) v \in U\) for all \(\lambda \in [0, 1]\).
Without loss of generality, we assume that \(u\) and \(v\) are linearly independent and
\[
u_g v_f - u_f v_g := d > 0.
\]
Below we try to find a vector \(z \in R^n\) such that
\[
(f(z), g(z)) = \lambda u + (1 - \lambda)v
\]
for some $\lambda \in (0, 1)$.

We make the following ansatz $z = \rho (z_1 \cos \theta + z_2 \sin \theta)$, where $\rho$ and $\theta$ are real variables.

Substitute $z = \rho (z_1 \cos \theta + z_2 \sin \theta)$ into (2.12), we get
\[
\begin{align*}
\rho^k f(z_1 \cos \theta + z_2 \sin \theta) &= \lambda u_f + (1 - \lambda)v_f, \\
\rho^k g(z_1 \cos \theta + z_2 \sin \theta) &= \lambda u_g + (1 - \lambda)v_g.
\end{align*}
\tag{2.13}
\]

Hereinafter, we use $f(\theta)$ and $g(\theta)$ to denote $f(z_1 \cos \theta + z_2 \sin \theta)$ and $g(z_1 \cos \theta + z_2 \sin \theta)$, respectively for short. Then there exists no $\theta'$ such that $f(\theta') = g(\theta') = 0$. This is because if there does exist $\theta'$ such that $f(\theta') = g(\theta') = 0$, then $z_1 \cos \theta' + z_2 \sin \theta' = 0$, that is to say, $z_1$ and $z_2$ are linearly dependent; furthermore, $u$ and $v$ are linearly dependent, which is a contradiction. (2.13) shows that $\rho^k = d/T(\theta)$ and $\lambda = S(\theta)/T(\theta)$, where
\[
\begin{align*}
T(\theta) &= f(\theta)(u_g - v_g) - g(\theta)(u_f - v_f), \\
S(\theta) &= g(\theta)u_f - f(\theta)v_g.
\end{align*}
\tag{2.14}
\]

Denote $S(\theta)/T(\theta) := \Lambda(\theta)$, a function of $\theta$ having period $\pi$. Then we need to prove
\[
\Lambda ([0, 2\pi] \cap \{\theta : T(\theta) > 0\}) \supset [0, 1].
\tag{2.15}
\]

It is easy to get $S(0) = T(0) = T(\frac{\pi}{2}) = d$ and $S(\frac{\pi}{2}) = 0$. So, $\Lambda(0) = 1$ and $\Lambda(\frac{\pi}{2}) = 0$. Since $f$ and $g$ are homogeneous polynomials of $\cos \theta$ and $\sin \theta$, $T$ and $S$ can be expressed as
\[
\begin{align*}
T(\theta) &= \sum_{i=0}^{k} \alpha_i \cos^i \theta \sin^{k-i} \theta, \\
S(\theta) &= \sum_{i=0}^{k} \beta_i \cos^i \theta \sin^{k-i} \theta.
\end{align*}
\]

It is obvious that $S(0) = T(0) = T(\frac{\pi}{2}) = d$ and $S(\frac{\pi}{2}) = 0$, then $\alpha_0 = \alpha_k = \beta_k = d$ and $\beta_0 = 0$. Hence,
\[
\begin{align*}
T(\theta) &= d (\cos^k \theta + \sin^k \theta) + \sum_{i=1}^{k-1} \alpha_i \cos^i \theta \sin^{k-i} \theta, \\
S(\theta) &= d \cos^k \theta + \sum_{i=1}^{k-1} \beta_i \cos^i \theta \sin^{k-i} \theta.
\end{align*}
\]

Notice that $T$ and $S$ are both continuous functions of $\theta$, if $T(\theta) \neq 0$ for all $\theta \in [0, \frac{\pi}{2}]$ or $[\frac{\pi}{2}, \pi]$, then $\Lambda(\theta)$ is also a continuous function defined on the interval $[0, \frac{\pi}{2}]$ or $[\frac{\pi}{2}, \pi]$. So
\[
\Lambda ([0, \pi]) \supset [0, 1].
\tag{2.16}
\]

Next we assume that $T$ has a zero point.

We claim that $T$ and $S$ have no common zero point. If there exists $\hat{\theta} \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ such that $T(\hat{\theta}) = S(\hat{\theta}) = 0$, then
\[
\begin{bmatrix}
-v_g & v_f \\
u_g & -u_f
\end{bmatrix}
\begin{bmatrix}
f(z_1 \cos \hat{\theta} + z_2 \sin \hat{\theta}) \\
g(z_1 \cos \hat{\theta} + z_2 \sin \hat{\theta})
\end{bmatrix} = 0.
\tag{2.17}
\]
Premultiplying both sides of (2.17) by $\begin{bmatrix}
-v_g & v_f \\
u_g & -u_f
\end{bmatrix}^{-1}$, we get
\[
\begin{bmatrix}
f(z_1 \cos \hat{\theta} + z_2 \sin \hat{\theta}) \\
g(z_1 \cos \hat{\theta} + z_2 \sin \hat{\theta})
\end{bmatrix} = 0.
\]
Then $z_1 \cos \hat{\theta} + z_2 \sin \hat{\theta} = 0$, that is, $z_1$ and $z_2$ are linearly dependent, and furthermore, $u$ and $v$ are linearly dependent.
Here consider the interval $[0, 2\pi]$. By (2.14), we have $T(2\pi) = S(2\pi) = d > 0$ and $T(\pi) = S(\pi) = -d$ where $d$ is shown in (2.11).

Recall the linearly independent vectors $u = (u_f, u_g), v = (v_f, v_g) \in U$. (2.14) together with that $f$ and $g$ have no common zero point except $0 \in R^n$ shows that

1. $S(\theta) = 0$ implies $f(\theta) = v_f t, g(\theta) = v_g t$ and $T(\theta) = td$ for some nonzero real number $t$.
2. $T(\theta) = 0$ implies $f(\theta) = (u_f - v_f) t, g(\theta) = (u_g - v_g) t$ and $S(\theta) = td$ for some nonzero real number $t$.
3. $S(\theta) = T(\theta)$ implies $f(\theta) = u_f t, g(\theta) = u_g t$ and $T(\theta) = T(\theta) = td$ for some nonzero real number $t$.

where $t$ cannot be 0, because there exists no $\theta'$ such that $f(\theta') = g(\theta') = 0$.

Denote the set of zero points of $S$ that are not minima or maxima in the interval $[0, 2\pi]$ by $0$. It is to get $S(\theta) = -S(\theta + \pi)$ for all $\theta \in R$. Then $|0 \cap (0, \pi)| = |0 \cap (\pi, 2\pi)| := l$ is an odd number. We also have if $S(\theta) = 0$, then $T(\theta)T(\theta + \pi) < 0$. Hence $\{|w : w \in U, \pi, T(w) > 0\} = |\{w : w \in U \cap (0, 2\pi), T(w) < 0\}| = l$. Denote $0$ by $\{0, 0, \cdots, 0, 2\}$, where $0 < 0 < \cdots < 0 < 2 < 2\pi$.

1. Assume $l = 1$. Based on the foregoing discussion, it holds that $T(0_1)T(0_2) < 0$.
   - Then either $\lambda(0, 0_1) \cap \{\theta : T(\theta) > 0\} \supset [0, 1]$ or $\Lambda(0_2, 2\pi) \cap \{\theta : T(\theta) > 0\} \supset [0, 1]$.
2. Assume $l > 1$. If $T(0_1) > 0$, $\Lambda(0, 0_1) \cap \{\theta : T(\theta) > 0\} \supset [0, 1]$. If $T(0_2) > 0$, $\Lambda(0_2, 2\pi) \cap \{\theta : T(\theta) > 0\} \supset [0, 1]$. If $T(0_1) < 0$ and $T(0_2) < 0$, there exists $1 \leq i < l$ such that $T(0_1)T(0_{2+i+1}) < 0$. Suppose the contrary: If for each $1 \leq j < l, T(0_j)T(0_{2j+1}) > 0$, then $\{|w : w \in U \cap (0, 2\pi), T(w) > 0\}$ is an even number, which is a contradiction. Since $S(\theta) > 0$ for all $\theta \in (0, 2\pi)$, and $\Lambda(0_2, 2\pi) \cap \{\theta : T(\theta) > 0\} \supset [0, 1]$.

Hence (2.15) holds.

Based on the above discussion, the set $U$ is a convex set if $k$ is odd and $f$ and $g$ have no common zero point except $0 \in R^n$.

Given nonzero $(a_1, a_2) \in U$, then $-a_1, -a_2 \in U$, and then $\{(a_1 t, a_2 t) : t \in R\} \subset U$. Thus $(a_1 t, a_2 t) = U$ may hold (see Example 2.1). $\Lambda(0_1, 0_2) \cap \{\theta : T(\theta) > 0\} \supset [0, 1]$. If $\{(a_1 t, a_2 t) : t \in R\} \neq U$, there exists nonzero $(a_3, a_4) \in U$ such that $a_1 a_4 - a_2 a_3 \neq 0$, then $(-a_3, -a_4) \in U$, and then $U = R^2$ by the convexity of the set $U$ and the homogeneity of $f$ and $g$. \qed

2.2. A Counterexample for Switched Polynomial Systems with Three Sub-systems.

In this subsection, we give an example showing that even if a switched system with three sub-systems admits a LFHD, there may exist no convex combination of its sub-systems admitting a LFHD.

**Example 2.4.** Let $S$ be a switched linear system with three sub-systems, and the three sub-system matrices are

\[
S_1 : A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S_2 : A_2 = \begin{bmatrix} -\sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}, \quad S_3 : A_3 = \begin{bmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix}.
\]

Firstly, it is easy to obtain that $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ is a LFHD (see Fig. 2.5).

Secondly, we prove none of the linear combinations of the three sub-systems is asymptotically stable.

Denote $A = \sum_{i=1}^{3} \lambda_i A_i = \begin{bmatrix} \lambda_1 - \sqrt{3} \lambda_2 - \sqrt{3} \lambda_3 & -\lambda_2 + \lambda_3 \\ -\lambda_2 + \lambda_3 & -\lambda_1 + \sqrt{3} \lambda_2 + \sqrt{3} \lambda_3 \end{bmatrix}$.
where $\lambda_1, \lambda_2, \lambda_3$ are real variables. Notice that $A$ is real symmetric, then we have $A$ either is a zero matrix, or has a positive eigenvalue and a negative eigenvalue. Hence, system $\dot{x} = Ax$ is either stable or unstable, but cannot be asymptotically stable.

We can easily get the unique stable convex combination is $A^* = \lambda_1^* A_1 + \lambda_2^* A_2 + \lambda_3^* A_3 = 0$, where $\lambda_1^* = \frac{2\sqrt{3}}{2\sqrt{3}+2}, \lambda_2^* = \lambda_3^* = \frac{1}{2\sqrt{3}+2}$.

Taking $f(x) = -V(x)_{S_1}, g_i(x) = V(x)_{S_i+1}, i = 1, 2$, we have the set $\{(f(x), g_1(x), g_2(x)) : x \in R^n\}$ is not convex.

2.3. Extended S-Lemma. In this subsection, based on Theorem 2.4, by borrowing the idea of Yakubovich, we give some extended versions of the S-Lemma under some mild conditions.

In the SHS-Lemma, if $f$ and $g$ have a nonzero common zero point, the set $\{(f(x), g(x)) : x \in R^n\}$ implies there exists $\xi \geq 0$ such that $f - \xi g$ is nonnegative (see Example 2.5). Later, under the assumption that the two polynomials considered have no nonzero common zero point and some extra mild assumptions, we extend the S-Lemma to the case of non-homogeneous polynomials of the same degree greater than 2 (Theorem 2.5). And based on Theorem 2.5, we extend the S-Lemma to the case of non-homogeneous polynomials of the same degree greater than 2.
(Theorem 2.7) and of the same even degree greater than 2 (Theorem 2.8).

Example 2.5. Recall Remark 2.7. Choose \( f(x, y) = -x^3 + y^3 \) and \( g(x, y) = y^3 - \frac{1}{2}x^3 - \frac{1}{2}xy^2 \). \( f \) and \( g \) have the common nonzero zero point \((1, 1)\). It can be calculated that the two boundaries (see Fig. 2.1) of the set \( \{(-x^3 + y^3, y^3 - \frac{1}{2}x^3 - \frac{1}{2}xy^2) : x, y \in R \} \) are \( y = \frac{1}{2}x \) and \( y = \frac{2}{3}x \). Then \( f \) is copositive with \( g \), but not strictly copositive with \( g \). Further we have there exists no \( \xi \geq 0 \) such that \( f - \xi g \) is nonnegative for all \( x, y \in R \).

2.3.1. Homogeneous S-Lemma. Theorem 2.5 (HS-Lemma). Let \( f, g : R^n \to R \) be homogeneous polynomials of degree \( k \geq 1 \). If \( f \) and \( g \) have no common zero point except \( 0 \in R^n \) and there exists at most one vector \((a, b) \in R^2 \) such that \( a^2 + b^2 = 1, a + \delta(a)b > 0 \) and both \( \begin{cases} f(x) = a \\ g(x) = b \end{cases} \) and \( \begin{cases} f(x) = -a \\ g(x) = -b \end{cases} \) have a solution, where \( \delta(t) = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{if } t \neq 0, \end{cases} \) that is to say, there exists at most one straight line passing through the origin that is contained in the set \( U \), then the following two items are equivalent:

\( i) \) \( f \) is copositive with \( g \);

\( (ii) \) there exists \( \xi \geq 0 \) such that \( f(x) - \xi g(x) \geq 0 \) for all \( x \in R^n \). In particular, if there exists a vector \( z \in R^n \) such that \( f(z) < 0 \), then there exists \( \xi > 0 \) such that \( f(x) - \xi g(x) \geq 0 \) for all \( x \in R^n \).

Proof. \((ii) \Rightarrow (i)\) holds obviously. We only prove \((i) \Rightarrow (ii)\).

Next we assume \( f \) and \( g \) have no common zero point except \( 0 \in R^n \) and there exists at most one vector \((a, b) \in R^2 \) such that \( a^2 + b^2 = 1, a + \delta(a)b > 0 \) and both \( \begin{cases} f(x) = a \\ g(x) = b \end{cases} \) and \( \begin{cases} f(x) = -a \\ g(x) = -b \end{cases} \) have a solution. Then by Theorem 2.4 the set \( \{(f(x), g(x)) : x \in R^n \} := U \) is a straight line passing through the origin if \( k \) is odd, and is an angular sector of angle no greater than \( \pi \) if \( k \) is even.

Since \( f \) is copositive with \( g \),

\[ U \cap \{(u, v) : u < 0, v \geq 0 \} = \emptyset. \]

Then there exist \( \xi_1 < 0, \xi_2 \geq 0 \) such that

\[ \xi_1 f(x) + \xi_2 g(x) \leq 0 \]

for all \( x \in R^n \).

Setting \( \xi = -\xi_2/\xi_1 \geq 0 \), we have

\[ f(x) - \xi g(x) \geq 0 \]

for all \( x \in R^n \).

In particular, if there exists \( z \in R^n \) such that \( f(z) < 0 \), there exist \( \xi_1 < 0, \xi_2 > 0 \) such that

\[ \xi_1 f(x) + \xi_2 g(x) \leq 0 \]

for all \( x \in R^n \). Hence there exists \( \xi > 0 \) such that

\[ f(x) - \xi g(x) \geq 0 \]

for all \( x \in R^n \).
2.3.2. **Non-homogeneous S-Lemma.** For convenience, we use the semi-tensor product of matrices to represent a polynomial hereinafter. The concept of the semi-tensor product of matrices is referred to [25] and the references therein. Here we only introduce the semi-tensor product of two column vectors.

**Definition 2.6 ([25]).** Let \( u, v \in \mathbb{R}^n \) be two column vectors, set \( u = (u_1, u_2, \cdots, u_n)^T \), the semi-tensor product of \( u \) and \( v \) is defined as

\[
u \Join v = (u_1 v^T, u_2 v^T, \cdots, u_n v^T)^T;
\]

since the semi-tensor product preserves the associative law, \( u \Join v \Join \cdots \Join u \)
inductively.

By **Definition 2.6** a polynomial \( f(x) : \mathbb{R}^n \to \mathbb{R} \) of degree \( k \) can be represented as

\[
f(x) = f_0 + f_1 x + \cdots + f_k x^k,
\]

where each \( f_i \in \mathbb{R}^{n \times n} \) is a constant row vector, \( i = 0, 1, \cdots, k \), called coefficient vector. Note that \( f_0 \) and \( f_1 \) are unique, but other coefficient vectors may be not.

Based on **Theorem 2.5** we have the following Theorems 2.7 and 2.8.

**Theorem 2.7** (NHS-Lemma). Let \( f, g : \mathbb{R}^n \to \mathbb{R} \) be polynomials of degree \( k \geq 1 \) in the form of

\[
f(x) = f_0 + f_1 x + \cdots + f_k x^k \text{ and } g(x) = g_0 + g_1 x + \cdots + g_k x^k,
\]

where \( f_i, g_i \in \mathbb{R}^{n \times n} \) are constant row vectors, \( i = 0, 1, \cdots, k \).

Let us introduce homogeneous functions:

\[
\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R}, \tilde{f}(x, t) = f_0 t^k + f_1 t x^{k-1} + \cdots + f_k x^k;
\]

\[
\tilde{g} : \mathbb{R}^{n+1} \to \mathbb{R}, \tilde{g}(x, t) = g_0 t^k + g_1 t x^{k-1} + \cdots + g_k x^k.
\]

Assume \( \tilde{f} \) is copositive with \( \tilde{g} \), \( \tilde{f} \) and \( \tilde{g} \) have no common zero point except \( 0 \in \mathbb{R}^{n+1} \), and there exists at most one vector \((a, b) \in \mathbb{R}^2 \) such that \( a^2 + b^2 = 1 \), \( a + \delta(a)b > 0 \) and both

\[
\begin{cases}
\tilde{f}(x, t) = a \\
\tilde{g}(x, t) = b
\end{cases}
\]

and

\[
\begin{cases}
\tilde{f}(x, t) = -a \\
\tilde{g}(x, t) = -b
\end{cases}
\]

have a solution, where \( \delta(\cdot) \) is seen in **Theorem 2.5**.

Then there exists \( \xi \geq 0 \) such that \( f(x) - \xi g(x) \geq 0 \) for all \( x \in \mathbb{R}^n \).

**Proof.** Note that \( \tilde{f} \) is copositive with \( \tilde{g} \) implies \( f \) is copositive with \( g \) by taking \( t = 1 \), and \( \tilde{f} \) and \( \tilde{g} \) have no common nonzero zero point implies \( f \) and \( g \) have no common zero point. But the converse is not true.

Then by **Theorem 2.5** there exists \( \xi \geq 0 \) such that

\[
\tilde{f}(x, t) - \xi \tilde{g}(x, t) \geq 0 \text{ for all } (x, t) \in \mathbb{R}^{n+1}.
\]

Choosing \( t \equiv 1 \), we have

\[
f(x) - \xi g(x) \geq 0 \text{ for all } x \in \mathbb{R}^n.
\]

\[\Box\]

**Theorem 2.8.** Let \( f, g : \mathbb{R}^n \to \mathbb{R} \) be polynomials of even degree \( k \). Assume \( f \) is copositive with \( g \), \( f \) and \( g \) have no common zero point. Denote

\[
f(x) := f_0 + f_1 x + \cdots + f_k x^k,
\]

\[
g(x) := g_0 + g_1 x + \cdots + g_k x^k,
\]
We assume that there exists nonzero $f_0$ and $	ilde{g}$.

Assume that homogeneous polynomials $f_ky^k$ and $g_ky^k$ have no common zero point except $0 \in \mathbb{R}^n$, $f_ky^k$ is copositive with $g_ky^k$, and assume

1. there exist no nonzero vector $(a, b) \in \mathbb{R}^2$ such that both
   \[
   \begin{cases}
   f_kx^k = -a \\
g_kx^k = -b
   \end{cases}
   \]
   have a solution,

2. there exist no $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 0$, either $ac < 0$ or $bd < 0$, and
   both
   \[
   \begin{cases}
   f(x) = a \\
g(x) = b
   \end{cases}
   \]
   have a solution,

3. there exist no $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 0$, either $ac < 0$ or $bd < 0$, and
   both
   \[
   \begin{cases}
   f_kx^k = c \\
g_kx^k = d
   \end{cases}
   \]
   have a solution,

4. and there exist no $a, b, c, d \in \mathbb{R}$ such that $ad - bc = 0$, either $ac < 0$ or $bd < 0$, and
   both
   \[
   \begin{cases}
   f_kx^k = a \\
g_kx^k = b
   \end{cases}
   \]
   have a solution.

Then there exists $x \geq 0$ such that $f(x) - xg(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Proof. Let us introduce homogeneous functions:

\[ \tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R}, \tilde{f}(x, t) = f_0t^k + f_1xt^{k-1} + \cdots + f_kx^k \]

and

\[ \tilde{g} : \mathbb{R}^{n+1} \to \mathbb{R}, \tilde{g}(x, t) = g_0t^k + g_1xt^{k-1} + \cdots + g_kx^k. \]

Firstly we prove $\tilde{f}$ is copositive with $\tilde{g}$. That is to say, we prove $\tilde{f}(x, t) < 0$ and $\tilde{g}(x, t) \geq 0$ have no common solution. Suppose the contrary: Assume that there exists $(x_1, t_1)$ such that $f(x_1, t_1) < 0$ and $\tilde{g}(x_1, t_1) \geq 0$.

1. If $t_1 \neq 0$, then
   \[
   f(x_1/t_1) = \tilde{f}(x_1, t_1)/t_1^k < 0,
   \]
   \[
   g(x_1/t_1) = \tilde{g}(x_1, t_1)/t_1^k \geq 0,
   \]
   which contradicts that $f$ is copositive with $g$.

2. If $t_1 = 0$, then
   \[ f_kx_1^k < 0 \text{ and } g_kx_1^k \geq 0, \]
   (2.18)
   that is to say, $f_ky^k$ is not copositive with $g_ky^k$, which is a contradiction.

Secondly we prove $\tilde{f}$ and $\tilde{g}$ have no common nonzero zero point. Suppose the contrary: We assume that there exists nonzero $(x_2, t_2) \in \mathbb{R}^{n+1}$ such that $f(x_2, t_2) = \tilde{g}(x_2, t_2) = 0$.

1. If $t_2 \neq 0$,
   \[
   f(x_2/t_2) = \tilde{f}(x_2, t_2)/t_2^k = 0,
   \]
   \[
   g(x_2/t_2) = \tilde{g}(x_2, t_2)/t_2^k = 0,
   \]
   which contradicts that $f$ and $g$ have no common zero point.

2. If $t_2 = 0$, then $x_2 \neq 0$ and $f_kx_2^k = g_kx_2^k = 0$, which contradicts $f_ky^k$ and $g_ky^k$ have no common zero point except $0 \in \mathbb{R}^n$. 

where $f_i, g_i \in \mathbb{R}^n$ are constant row vectors, $i = 0, 1, \cdots, k$. 

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Thirdly we prove there exist no nonzero vector \((a, b) \in \mathbb{R}^2\) such that both \(\begin{cases} \tilde{f}(x, t) = a \\ \tilde{g}(x, t) = b \end{cases}\) and \(\begin{cases} \tilde{f}(x, t) = -a \\ \tilde{g}(x, t) = -b \end{cases}\) have a solution. Suppose the contrary: If there exist \(a, b \in \mathbb{R}\), \((x_1, t_1), (x_2, t_2) \in \mathbb{R}^{n+1}\) such that \(a^2 + b^2 \neq 0\), then \((x_1, t_1) \neq 0\) and \((x_2, t_2) \neq 0\).

1. If \(t_1 \neq 0\) and \(t_2 \neq 0\), then
   \[
   f(x_1/t_1) = \frac{\tilde{f}(x_1, t_1)}{t_1^k} = a/t_1^k, \quad g(x_1/t_1) = \frac{\tilde{g}(x_1, t_1)}{t_1^k} = b/t_1^k,
   \]
   \[
   f(x_2/t_2) = \frac{\tilde{f}(x_2, t_2)}{t_2^k} = -a/t_2^k, \quad g(x_2/t_2) = \frac{\tilde{g}(x_2, t_2)}{t_2^k} = -b/t_2^k,
   \]
   which contradicts Item 2 in Theorem 2.8.

2. If \(t_1 = 0\) and \(t_2 = 0\), then
   \[
   f_k x_1^k = a, \quad g_k x_1^k = b, \quad f_k x_2^k = -a, \quad g_k x_2^k = -b,
   \]
   which contradicts Item 1 in Theorem 2.8.

3. If \(t_1 \neq 0\) and \(t_2 = 0\), then
   \[
   f(x_1/t_1) = \frac{\tilde{f}(x_1, t_1)}{t_1^k} = a/t_1^k, \quad g(x_1/t_1) = \frac{\tilde{g}(x_1, t_1)}{t_1^k} = b/t_1^k,
   \]
   \[
   f_k x_2^k = -a, \quad g_k x_2^k = -b,
   \]
   which contradicts Item 3 in Theorem 2.8.

4. If \(t_1 = 0\) and \(t_2 \neq 0\), then
   \[
   f(x_2/t_2) = \frac{\tilde{f}(x_2, t_2)}{t_2^k} = -a/t_2^k, \quad g(x_2/t_2) = \frac{\tilde{g}(x_2, t_2)}{t_2^k} = -b/t_2^k,
   \]
   which contradicts Item 4 in Theorem 2.8.

Based on the above discussion, by Theorem 2.8 there exists \(\xi \geq 0\) such that \(\tilde{f}(x, t) - \xi \tilde{g}(x, t) \geq 0\) for all \((x, t) \in \mathbb{R}^{n+1}\). Taking \(t = 1\), we have \(f(x) - \xi g(x) \geq 0\) for all \(x \in \mathbb{R}^n\).

In order to illustrate Theorem 2.8 we give the following Example 2.6.

**Example 2.6.** Consider \(f(x_1, x_2) = -7x_1^6 - 2x_1^3x_2^3 + 5x_2^6 - 2x_1^4x_2^2 - 2\) and \(g(x_1, x_2) = -5x_1^6 - x_1^3x_2^3 + x_2^6 - x_1^4x_2^2 - 1\) both from \(\mathbb{R}^2\) to \(\mathbb{R}\).

\(f\) and \(g\) are both polynomials of even degree 6. Easily we have
\[
\begin{cases} 
-7x_1^6 - 2x_1^3x_2^3 + 5x_2^6 - 2x_1^4x_2^2 - 2 < 0 \\
-5x_1^6 - x_1^3x_2^3 + x_2^6 - x_1^4x_2^2 - 1 \geq 0
\end{cases}
\Rightarrow -3(x_1^6 + x_2^6) > 0.
\]

Then \(f\) is copositive with \(g\). We easily get that \(f\) and \(g\) have no common zero point, since
\[
\begin{cases} 
-7x_1^6 - 2x_1^3x_2^3 + 5x_2^6 - 2x_1^4x_2^2 - 2 = 0 \\
-5x_1^6 - x_1^3x_2^3 + x_2^6 - x_1^4x_2^2 - 1 = 0
\end{cases}
\Rightarrow x_1^6 + x_2^6 = 0 \Rightarrow x_1 = x_2 = 0.
\]

By (2.5) we have \(-7x_1^6 - 2x_1^3x_2^3 + 5x_2^6 - 2x_1^4x_2^2 - 2\) and \(-5x_1^6 - x_1^3x_2^3 + x_2^6 - x_1^4x_2^2 - 1\) have no common zero point except \((0, 0)\), the former is copositive with the latter and Item 7 in Theorem 2.8 holds. It is easy to get Items 2, 3, and 4 in Theorem 2.8 hold.

Based on the above discussion, by Theorem 2.8 there exists \(\xi \geq 0\) such that \(f(x_1, x_2) - \xi g(x_1, x_2) \geq 0\) for all \(x_1, x_2 \in \mathbb{R}\).

Since each \(\frac{1}{3} \leq \lambda \leq \frac{17}{93}\) makes system (2.6) asymptotically stable, here we might as well choose \(\lambda = \frac{1}{3}\), i.e., choose \(\xi = 2\). We have \(f(x_1, x_2) - 2g(x_1, x_2) = 3(x_1^6 + x_2^6) \geq 0\) for all \(x_1, x_2 \in \mathbb{R}\).
3. Conclusions. This paper studied the relationship between (i) a switched nonlinear system admits a LFHD and (ii) the system has a convex combination of its sub-systems that admits a LFHD. By using the strict homogeneous S-Lemma presented and proved in this paper, a necessary and sufficient condition was given under which (i) is equivalent to (ii) when the system has two sub-systems, and a counterexample was given to show that (i) does not imply (ii) when the system has more than two sub-systems.

Besides, the S-Lemma was extended from quadratic polynomials to polynomials of degree more than 2 under some mild conditions.

4. Acknowledgments. The authors thank Dr. Ragnar Wallin for supplying some of the references, and thank the anonymous reviewers and the AE for their valuable comments that led to an improvement for the readability and an increase of the range of applications of the manuscript.

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