A Scalable Max-Consensus Protocol For Noisy Ultra-Dense Networks

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Abstract—We introduce ScalableMax, a novel communication scheme for achieving max-consensus in a wireless network that harnesses the broadcast and superposition properties of the wireless channel. In a sufficiently dense network, the amount of communication resources necessary grows logarithmically with the number of nodes, while in state-of-the-art alternatives, this growth is at least linear. ScalableMax can handle additive noise and works well in a high SNR regime. For medium and low SNR, we propose the ScalableMax-EC scheme, which expands upon the ideas of ScalableMax introducing a novel error correction scheme. It achieves lower error rates at the cost of using more channel resources. However, it retains the logarithmic growth with the number of agents in the system.

I. INTRODUCTION AND PRIOR WORK

The problem of achieving max-consensus in a network of agents arises in many current and envisioned practical applications, particularly in regard to distributed and cooperative control. Examples most notably include task assignment [11], leader election [2], rendezvous [3], clock synchronization [4], spectrum sensing [5], distributed decision-making [6] and formation control [7]. Future generations of mobile networks are anticipated to be several orders of magnitude denser than today because of expected infrastructure densification [8]. Distributed and cooperative control of multiple agents in various ultra-dense networks will be a major challenge. Therefore the growth in complexity of consensus algorithms with the number of agents in the network could become much more important than it is today. In this work, we present max-consensus protocols that are practical to implement in wireless communication systems and exhibit a more favorable asymptotic complexity behavior than state-of-the-art alternatives.

Historically, max-consensus algorithms are analyzed based on the properties of the communication network graph [9, 6, 10]. The exchange of information between all neighboring agents is assumed to happen simultaneously, with complexity independent of the number of agents. Hence, these algorithms are designed to minimize the total number of information exchanges required to reach consensus. However, in wireless networks, these assumptions are often unrealistic due to the presence of interference and noise. On the other hand, the specific characteristics of the wireless channel can be exploited by making use of its broadcast and superposition properties. Iutzeler et al. proposed and analyzed three communication strategies: Random-Pairwise, Random-Walk and Random-Broadcast [11, 12]. They leverage the broadcast property of the wireless channel, reduce interference using random scheduling of agents sharing the same channel, and protect the transmitted messages using forward error correction. This leads to a linear growth of communication resources necessary with the number of agents. Alternatively, the maximum can be approximated with linear functions and thus, linear consensus protocols can be applied, e.g., [13], [14], [10]. In [15], [16], the superposition property of the wireless channel is harnessed to achieve constant complexity in the number of agents in networks with bounded diameter. But these works neither consider noise introduced by the approximation of the maximum function nor by the wireless channel. [17] proposes to use a stochastic approximation based algorithm to tackle the residual additive noise, but the convergence rate is much slower than that of standard consensus algorithms.

II. NOTATION

We denote the sets of finite and infinite binary sequences with \( \{0,1\}^\leq \) and \( \{0,1\}^{\infty} \), respectively. Given \( S_1, S_2 \in \{0,1\}^\leq \cup \{0,1\}^{\infty} \), they are compatible, or \( S_1 \parallel S_2 \), if they coincide on the intersection of their domains. \( S_1 \) is lexicographically greater than \( S_2 \), or \( S_1 > S_2 \), if there is \( k \) such that \( S_1(k) > S_2(k) \), while for all \( k' < k \), \( S_1(k') = S_2(k') \). We write \( S_1 \geq S_2 \) if \( S_1 \parallel S_2 \) or \( S_1(k) > S_2(k) \). \( \emptyset \) denotes the empty sequence. Given \( S \in \{0,1\}^\leq \) and \( b \in \{0,1\} \), \( S^{-}b \) is the sequence that results from appending \( b \) at the end of \( S \). Finally, \( \mathbb{1} \) denotes the indicator function and \( \lvert \cdot \rvert \) the cardinality of a set.

III. SYSTEM MODEL AND PROBLEM STATEMENT

A. Preliminaries

A Wireless Multiple Access Channel (WMAC) is a system with inputs \( \alpha_1, \ldots, \alpha_n \) and output \( \gamma := \sum_{k=1}^{n} h_k \alpha_k + N \), where \( \alpha_k \in \mathbb{C} \) is the signal transmitted by transmitter \( k \), the complex random variable \( h_k \) is the channel fading coefficient of transmitter \( k \), and the complex random variable \( N \) is the additive noise at the receiver. A multicast channel takes input \( \beta \in \mathbb{C} \) from a single transmitting node, and produces outputs \( \Gamma_1, \ldots, \Gamma_n \) defined as \( \Gamma_k = h_k \beta + N_k \) for \( k = 1, \ldots, n \), where the complex random variables \( N_k \) and \( h_k \) represent the additive noise and the channel fading coefficient at receiver \( k \), respectively. One channel use is defined as a realization of either a WMAC, a multicast or a point-to-point channel.
Assumption 1 (Channel Assumptions). We assume that the inputs and outputs of WMAC channels are real valued. Moreover, the fading coefficients \( h_1, \ldots, h_n \) are assumed to be deterministically equal to 1. The only assumption on the additive noise distribution is that it is symmetric around 0. White Gaussian noise is one example of such a noise distribution.

We remark that the proposed schemes can be extended to sufficiently large but otherwise arbitrary real fading coefficients, and the assumption is made for clarity and brevity of the exposition. We refer the reader to [13] and [15] for further discussion on the practical applicability of Assumption 1. Multicast communication satisfying arbitrarily low errors can be realized employing state-of-the-art coding schemes with forward error correction. In the following, we therefore neglect this residual error and assume that multicast transmission of binary sequences is possible without error. Consider a wireless network defined by an undirected connected graph \( G = (A, E) \). The nodes in \( A \) can communicate with each other through channels represented by the edges in \( E \). Besides point-to-point communication along individual edges, we also harness the broadcast and superposition properties of the wireless channel.

Assumption 2. Given a network graph \( G = (A, E) \), a WMAC with receiver \( C \in A \) and transmitters \( A' \subseteq A \) or a multicast channel with receivers \( A' \) and transmitter \( C \) can be realized iff for all \( A \in A' \), we have \( \{C, A\} \in E \).

For simplicity, we start with considering a special network topology where Assumption 2 is particularly useful.

Assumption 3. We assume that there exists a designated node \( C \in A \), the coordinator, with links to all other nodes in \( A \).

In Section VI, we consider the case in which such a network coordinator is not necessary, and show how to extend the proposed solutions to general wireless network graphs.

B. Problem Statement

In this section, we define the general max-consensus problem and simplify it to a relaxed version which can be solved more efficiently.

Problem 1 (Max-consensus). Each agent \( A_k \in A \) holds an input \( S_k \in S \), where \( S \) is a finite totally ordered set. We say that the system has achieved max-consensus if all agents agree on a common output sequence \( S \) that is equal to the maximum of the inputs from all agents i.e. \( S = \max_{A_k \in A} S_k \). The objective is to design protocols that can achieve max-consensus with a minimum number of channel uses.

We can assume without loss of generality that \( S \) is a set of binary sequences of a certain fixed length, equipped with lexicographic ordering which coincides with the usual ordering on dyadic rationals. For example, consider a Wireless Sensor Network where sensor nodes are sensing a physical phenomenon described by a real number. The sensors, due to their limited sensitivity, can only read the value up to a quantized number, represented by a finite sequence of binary digits. In the following, we assume that each agent holds an infinite-length binary sequence, and that no two agents hold the same sequence. In practice, we concatenate uniform random bits as necessary, which also makes the input sequences of agents distinct. In the relaxed version of the max-consensus problem, we seek to narrow down the set of all agents to a smaller set which still contains the agent holding the maximum input.

Definition 1 (Weak \( m \)-max-consensus). Each agent in \( A \) holds an input sequence \( S_k \in \{0, 1\}^\infty \), where no two inputs are the same. At any point in time, the coordinator can terminate the scheme with a termination condition \( \varphi = \varphi(x) \) either of the form \( \varphi(x) = x \geq S \) or of the form \( \varphi(x) = x > S \), where \( x \) is a free variable and \( S \in \{0, 1\}^{\leq \infty} \) is called the coordinator’s output estimate. We say that the termination is successful iff \( 1 \leq |M| \leq m \), where \( M := \{A_k : \varphi(S_k)\} \) is the set of agents satisfying the termination condition.

Remark 1 (From weak \( m \)-max-consensus to max-consensus). Further steps are required after reaching a weak \( m \)-max-consensus to find the true maximum among the remaining set \( M \) of agents. We remark without giving details that as long as \( m \) does not grow with \( n \), this reduction can be achieved with a constant number of channel uses through a series of point-to-point and multicast communications.

Remark 2 (Designing \( m \)). The agent holding the true maximum input sequence is guaranteed to be an element of \( M \) as long as \( M \neq \emptyset \). \( m \) is a designable parameter which does not need to grow with the number of agents in the system. The higher it is, the more we can harness the combined signal strength of multiple transmitters to combat noise, but the more communication resources are necessary to simplify the max-consensus problem to weak \( m \)-max-consensus.

IV. ScalableMaxScheme

In this section, we propose a scheme that achieves weak \( m \)-max-consensus and scales logarithmically with the number of agents. The max-consensus problem can be simplified to weak \( m \)-max-consensus as per Remark 1. The coordinator starts the scheme and generates an output estimate \( S \in \{0, 1\}^{\leq \infty} \) based on information received from the agents. In the following, we detail the communication protocols and information shared between agents and the coordinator. For every possible coordinator output estimate \( S \), we define the set \( P_S := \{k : S_k > S\} \) of protesting agents, the set \( A_S := \{k : S_k \geq S\} \) of active agents and the set \( R_S := \{k : S_k \geq S - 1\} \) of raising agents. The coordinator uses noisy estimates of the cardinalities of these sets in order to refine its output estimate.

We use an iteration counter \( t \), where each iteration consists of a transmission of digital information through the multicast channel and three uses of the WMAC, and thereby corresponds to a constant number of channel uses. Conceptually, we thus
split every iteration \(t\) into four time instants \(4t, \ldots, 4t+3\). The coordinator starts with \(S(0) := \emptyset\). At every time instant of the form \(4t\), it transmits \(S(t)\) through the multicast channel. We remark that since \(S(t)\) differs from \(S(t-1)\) in at most one digit, it is sufficient to transmit only the change, and hence, the length of the transmitted sequence can be considered constant.

At time instants not divisible by 4, the agents transmit through the WMAC, the signal of each being either 1 or 0, according to the following scheme:

- \(4t + 1\): protest
- \(4t + 2\): activity
- \(4t + 3\): raising

We denote the signal transmitted by agent \(A_k\) at time instant \(t\) with \(\alpha_k(t)\) and the corresponding received signal \(\gamma(t) = \sum_{k=1}^n \alpha_k(t) + N(t)\). These values are not defined if \(t\) is divisible by 4, since the agents do not transmit in these steps.

After step \(4t + 3\), the coordinator either determines a new output estimate \(S(t+1)\) or it makes a termination decision according to Fig. 1. In Fig. 2, we show a graphical representation of part of the decision process.

\[
\text{if } \gamma(4t + 1) > m/4 \text{ then} \\
\quad \text{Terminate with } \varphi(x) = x > S(t); \\
\text{end} \\
\text{if } \gamma(4t + 2) < 3m/4 \text{ then} \\
\quad \text{Terminate with } \varphi(x) = x \geq S(t); \\
\text{end} \\
\text{if } \gamma(4t + 3) < m/4 \text{ then} \\
\quad S(t + 1) \leftarrow S(t) \wedge 0; \\
\text{else} \\
\quad S(t + 1) \leftarrow S(t) \wedge 1; \\
\quad \text{if } \gamma(4t + 3) < 3m/4 \text{ then} \\
\quad \quad \text{Terminate with } \varphi(x) = x \geq S(t + 1); \\
\text{end} \\
\text{end}
\]

Fig. 1. Post-processing of received signals in ScalableMax.

\[
\begin{align*}
\text{Fig. 2. Visualization of post-processing at the coordinator of received signals } &\gamma(4t + 1) \text{ and } \gamma(4t + 2). \\
\text{Dashed lines show the possible noiseless combined signals that are possible and solid lines delimit the numbered decision regions.} \\
\text{Only in decision region } 1 \text{ is } &\gamma(4t + 3) \text{ taken into account, and the decision can be to append } 0 \text{ or } 1 \text{ to the output estimate, or to append } 1 \text{ and terminate with } \varphi \geq. \\
\text{Received signals in region } &2 \text{ and } 3 \text{ lead to termination with conditions } \geq \text{ and } >, \text{ respectively, with the current unmodified output estimate.} \\
\text{In ScalableMax-EC, the regions marked } &4 \text{ are part of regions } 2 \text{ and } 3, \text{ respectively.} \\
\text{In ScalableMax-EC, they correspond to the coordinator removing the last digit from its current output estimate, thus correcting an error that may have been made in previous steps due to high noise.}
\end{align*}
\]

where the inequality is due to the union bound. Elementary transformations yield that \(\mathbb{P}(d \geq d_0) \leq \varepsilon\) as long as

\[
d_0 \geq \log_2 n + \log_2 (n - 1) + \log_2 (1/\varepsilon) - 1,
\]

so in the case that the agents’ inputs are uniformly distributed, the description length depends logarithmically on \(n\).

In general, according to our assumptions in Section [11] each of the agents’ inputs consists of finitely many, say \(p\), arbitrary bits and infinitely many uniform bits. Then \(\mathbb{P}(d \geq d_0) \leq \varepsilon\) if

\[
d_0 \geq p + \log_2 n + \log_2 (n - 1) + \log_2 (1/\varepsilon) - 1.
\]

We omit the proof of the theorem due to lack of space and instead present only a brief sketch. With an appropriate case distinction, it is not hard to show that the scheme succeeds if the noise realizations throughout the scheme never exceed \(m/4\). We can therefore derive an error bound if we know through how many iterations the scheme has to go until termination. Observe that the coordinator’s output estimate cannot reach a length of more than \(d\) if the noise is always less than \(m/4\). Moreover, each digit added to the coordinator’s output estimate as well as the termination decision in the end corresponds to one iteration and thus entails three analog multicast steps. For these reasons, we can argue that in case the noise samples are always less than \(m/4\), the scheme terminates after at most \(d + 1\) iterations and we observe at most \(3(d + 1)\) noise samples, from which the theorem follows.

V. SCALABLEMAX-EC SCHEME

In this section, we introduce the ScalableMax-EC scheme which expands upon the ideas of the previous section, intro-
producing error correction. We achieve this with two main modifications to the ScalableMax scheme. First, the coordinator can now additionally make correction decisions, i.e., remove a digit from its current output estimate $S(t)$. Second, in the cases in which the above scheme would terminate, the coordinator does not do so immediately, but rather raises a termination counter and only terminates when this counter reaches a termination threshold $\tau$, which is a parameter of the scheme. For each condition $\text{cond} \in \{ \text{"\textgreater"}, \text{"\textgeq"}, \text{"append"} \}$ and each possible output estimate $S \in \{0, 1\}^{\infty}$, the coordinator keeps a termination counter $T(S, \text{cond})$, which is initially 0. Coordinator and agents communicate as they do in the above scheme, but the post-processing in the coordinator after step $4t + 3$ differs and is conducted according to Fig. 3. We visualize a part of this decision process in Fig. 2.

```plaintext
if $\gamma(4t + 1) > 3m/4$ then
    $S(t + 1) \leftarrow S(t)$ with last digit removed (if any);
else if $\gamma(4t + 1) > m/4$ then
    $T(S(t), \text{"\textgreater"}) \leftarrow T(S(t), \text{"\textgreater"}) + 1$;
    if $T(S(t), \text{"\textgreater"}) = \tau$ then
        Terminate with $\varphi(x) = x\geq S(t)$;
    end
else if $\gamma(4t + 2) < m/4$ then
    $S(t + 1) \leftarrow S(t)$ with last digit removed (if any);
else if $\gamma(4t + 2) < 3m/4$ then
    $T(S(t), \text{"\textgeq"}) \leftarrow T(S(t), \text{"\textgeq"}) + 1$;
    if $T(S(t), \text{"\textgeq"}) = \tau$ then
        Terminate with $\varphi(x) = x\geq S(t)$;
    end
else if $\gamma(4t + 3) < m/4$ then
    $S(t + 1) \leftarrow S(t)\sim 0$;
else if $\gamma(4t + 3) < 3m/4$ then
    $T(S(t), \text{"append"}) \leftarrow T(S(t), \text{"append"}) + 1$;
    if $T(S(t), \text{"append"}) = \tau$ then
        Terminate with $\varphi(x) = x\geq S(t)\sim 1$;
    end
else
    $S(t + 1) \leftarrow S(t)\sim 1$;
end
```

Fig. 3. Post-processing of received signals in ScalableMax-EC.

VI. SIMULATION RESULTS

We model the problem as described in Section III and run the ScalableMax and ScalableMax-EC algorithms as described in sections IV and V. Uniform random bits are used as the agents’ input sequences $(S_k)_{k=1}^n$. We assume unit transmission power and white Gaussian noise. Each plotted data point is an average over $10^5$ identical and independent simulation runs.

In Fig. 4, we compare the performance of the proposed schemes in terms of error rate, i.e., the rate of unsuccessful termination of the scheme, and in Fig. 4b the average number of iterations required in successfully terminated runs of the schemes. As the noise power increases, so does the chance of unfavorable decisions by the coordinator. ScalableMax-EC has a mechanism to correct such bad decisions, and thus exhibits lower error rates, but at the cost of needing more iterations than ScalableMax.

In Fig. 5 we compare the scalability of the ScalableMax-EC scheme with the state-of-the-art Random-Broadcast (RB) and Random-Pairwise (RP) schemes described in [11]. To this end, we extend our scheme with a RB step to determine the maximum among the $m = 8$ agents that remain after a ScalableMax-EC run. We choose $\tau$ such that this combination achieves an overall error rate of at most 0.005. For comparison, we plot the number of iterations necessary in RB and RP to achieve an error rate of 0.005, given that all digital
transmissions arrive error free. The ScalableMax-EC scheme scales logarithmically with the total number of agents, while RB and RP scale at least linearly (see also [11]).

We conclude with two observations made during the simulations which are not shown in the plots. First, if the agents' input sequences are quantized versions of Gaussian random numbers, the number of iterations needed increases slightly depending on the variance of the random numbers and the granularity of the quantization. Second, the performance in terms of iterations can be improved significantly by choosing a suitable $S(0)$ other than $\emptyset$. One example that performs well is the coordinator output estimate at which the scheme was terminated successfully in an identically distributed but independent earlier simulation run. Finding other ways to choose suitable $S(0)$ in practical scenarios remains an open point for future research.

VII. EXTENSION TO NON-COMPLETE NETWORK GRAPHS

We consider the case in which Assumption 3 does not apply. Instead, we make the weaker assumption that we have a set of designated coordinators $\{C_1, \ldots, C_c\}$ such that every agent in the network has a link to at least one of the coordinators and that we have a way of scheduling between them. Note that because of the connectivity requirement on $G$, some agents necessarily have links with two or more coordinators. We denote the subgraph of $G$ induced by $C_\ell$ and its neighbors with $G_\ell$ and achieve max-consensus with these steps:

1) For each $\ell \in \{1, \ldots, c\}$, find a max-consensus in $G_\ell$ and update the inputs of all agents (to be used in all future max-consensus steps) to be the consensus value.

2) Repeat step 1 a total of $c$ times.

After the initial execution of step 1 at least one subgraph of agents will have the true maximum as the input for future consensus schemes. The connectivity requirement on $G$ ensures that after each further execution of step 1 this property is propagated to at least one additional subgraph, so after $c$ repetitions, the whole network has achieved max-consensus.

Note that the ScalableMax or ScalableMax-EC scheme is executed a total of $c^2$ times, so our scheme can be advantageous compared to the random-pairwise or random-broadcast scheme only as long as the network can be partitioned into subgraphs of very large size with a very small number of coordinators, which can e.g. be the case in ultra-dense networks of not overly large diameter.

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