GLOBAL WEAK SOLUTION AND BOUNDEDNESS IN A
THREE-DIMENSIONAL COMPETING CHEMOTAXIS

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Abstract. We consider an initial-boundary value problem for a parabolic-
parabolic-elliptic attraction-repulsion chemotaxis model
\[
\begin{align*}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), \\
    v_t &= \Delta v - \beta v + \alpha u, \\
    0 &= \Delta w - \delta w + \gamma u,
\end{align*}
\]
in a bounded domain $\Omega \subset \mathbb{R}^3$ with positive parameters $\chi, \xi, \alpha, \beta, \gamma$ and $\delta$.

It is firstly proved that if the repulsion dominates in the sense that $\xi \gamma > \chi \alpha$, then for any choice of sufficiently smooth initial data $(u_0, v_0)$ the corresponding initial-boundary value problem is shown to possess a globally defined weak solution. To the best of our knowledge, this situation provides the first result on global existence of the above system in the three-dimensional setting when $\xi \gamma > \chi \alpha$, and extends the results in Lin et al. (2016) [19] and Jin and Xiang (2017) [14] to more general case.

Secondly, if the initial data is appropriately small or the repulsion is enough strong in the sense that $\xi \gamma$ is suitable large as related to $\chi \alpha$, then the classical solutions to the above system are uniformly-in-time bounded.

1. Introduction. This paper deals with the following initial-boundary competing system

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\[
\begin{aligned}
&\begin{cases}
  u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), & x \in \Omega, \quad t > 0, \\
  \tau v_t = \Delta v - \beta v + \alpha u, & x \in \Omega, \quad t > 0, \\
  0 = \Delta w - \delta w + \gamma u, & x \in \Omega, \quad t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, \quad t > 0, \\
  u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in \Omega,
\end{cases}
\end{aligned}
\]  

in an arbitrary bounded smooth domain \( \Omega \subset \mathbb{R}^3 \) with sufficiently smooth initial data \( u_0(x) \) and \( v_0(x) \). The above chemotaxis system describes the process of chemotactic interaction between one species (the density denoted by \( u(x, t) \)) and two different chemical signals (the concentration represented by \( v(x, t) \) and \( w(x, t) \), respectively).

More precisely, the positivity of the attractive chemotactic coefficient \( \chi \) implies the species move towards to the chemical signal \( v \), but gets away from the repulsive signal \( w \) because of the positivity of the repulsive chemotactic coefficient \( \xi \). Here the positive parameters \( \alpha, \beta, \gamma \) and \( \delta \) represent the production and degradation rates of the chemicals, respectively.

Some experimental examples for (1) can be found in [24] which models the aggregation of microglia in Alzheimer’s disease or in [31] to address the quorum sensing effect in the chemotactic movement. This model can be regard as the combination between the following one-species and one-single attractive Keller-Segel chemotaxis model (see [15])

\[
\begin{aligned}
&\begin{cases}
  u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega, \quad t > 0, \\
  \tau v_t = \Delta v - \beta v + \alpha u, & x \in \Omega, \quad t > 0,
\end{cases}
\end{aligned}
\]  

and the repulsive system (see [4])

\[
\begin{aligned}
&\begin{cases}
  u_t = \Delta u + \xi \nabla \cdot (u \nabla w), & x \in \Omega, \quad t > 0, \\
  \tau w_t = \Delta w - \delta w + \gamma u, & x \in \Omega, \quad t > 0,
\end{cases}
\end{aligned}
\]  

where \( \tau = 0, 1 \). From a mathematic point of view and the actual environment, for the attractive chemotaxis system (2), the solution may blow up in finite time in higher dimensions \( n \geq 2 \) if the initial data is suitable large [28, 9, 25, 38, 40, 3], although it remains bounded if \( n = 1 \) [30] or \( n = 2 \) with small initial data [3, 8, 29]. However, all the solutions of (3) would globally exist whenever \( n \geq 2 \) [4, 26, 27].

Compared with results in the classical attractive or repulsive chemotaxis system, the solution behavior of (1) could be essentially dominated by the competition of attraction and repulsion: namely, when the repulsion dominates (i.e., \( \xi \gamma > \chi \alpha \)) or the repulsion cancels attraction (i.e., \( \xi \gamma = \chi \alpha \)), the solution remains globally bounded in higher dimensions; In the case attraction dominates (i.e., \( \xi \gamma < \chi \alpha \)), unbounded solutions may exist. With more complicate consideration and for convenience, a general situation for (1) can be written as

\[
\begin{aligned}
&\begin{cases}
  u_t = \nabla \cdot (D(u) \nabla u) - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), & x \in \Omega, \quad t > 0, \\
  \tau_1 v_t = \Delta v - \beta v + \alpha u, & x \in \Omega, \quad t > 0, \\
  \tau_2 w_t = \Delta w - \delta w + \gamma u, & x \in \Omega, \quad t > 0,
\end{cases}
\end{aligned}
\]  

with \( \tau_1, \tau_2 \geq 0 \), and with nonlinear diffusion \( D \) satisfying

\[
D(s) \geq D_0 s^\theta \text{ for all } s > 0 \text{ with some } D_0 > 0, \text{ and } \theta \in \mathbb{R}.
\]  

For the simple case \( \tau_1 = \tau_2 = 0 \) and \( D \equiv 1 \), quite a comprehensive understanding has been achieved. Accordingly, global smooth and bounded solutions could be shown to exist if \( \xi \gamma \geq \chi \alpha \) and \( n \geq 2 \) [33, 17, 6], and the large time behavior of solution for any initial data was studied in [21], while suitable large data admits solutions blowing up in finite time if \( \xi \gamma < \chi \alpha \) and \( n = 2 \) [6, 16, 41].
The respective knowledge is much less complete in fully parabolic situations in the sense that $\tau_1 = \tau_2 = 1$ in higher dimensions. Even in the comparatively simple case $D \equiv 1$ and $\xi \gamma > \chi \alpha$ seems complex enough so as to allow for global boundedness results only in the two-dimensional setting or global existence of weak solutions in three-dimensional case up to now [11]. The balanced case $\xi \gamma = \chi \alpha$ is considered in [20] and the global existence of classical solutions for any $n \leq 3$ has been shown by semigroup techniques (see also [7, Theorem 1.1]). However, the blow-up phenomenon for $\xi \gamma < \chi \alpha$ is not clear, although the global classical solutions with uniform-in-time bound was established if $u_0$ is suitable small if $n = 2$ [18]. For more details about the global existence of classical solution, asymptotic behavior and pattern formation in one-dimensional space, we refer the readers to [12, 23, 22].

While for the case $\tau_1 = 1, \tau_2 = 0$, relying on the Lyapunov functional, the authors in [13] firstly show that the problem (1.4) with $D \equiv 1$ is globally solvable in the two-dimensional setting if either $\xi \gamma \geq \chi \alpha$, or $\xi \gamma < \chi \alpha$ and $\int_\Omega u_0 < \frac{4\pi}{\chi \alpha - \xi \gamma}$, whereas under the largeness condition that $\int_\Omega u_0 > \frac{4\pi}{\chi \alpha - \xi \gamma}$ and $\xi \gamma < \chi \alpha$, there exists solution which blows up in the finite time. With a nonlinear diffusion $D(u) = D_0 u^\theta$, the number $\theta = 1 - \frac{n}{2}$ has been uniquely detected to be the critical blow-up exponent in [19]: namely, if $\theta > 0$, the corresponding initial-boundary problem possesses a nonnegative globally bounded solution, whereas $\theta \leq 0$, blow up may occur in a ball $\Omega \subset \mathbb{R}^n (n \geq 3)$. However, taking into account that the effect of the repulsion plays an important role in boundedness, the range of $\theta$ has been extended in [14] to $\theta > 1 - \frac{4}{n+2}$ if the repulsion dominates or the repulsion cancels attraction in the sense that either $\xi \gamma > \chi \alpha$ or $\xi \gamma = \chi \alpha, \beta \geq \delta$.

However, these results are not available for (1) even if $n = 3$, since $\theta$ in (5) always remains positive for any $n \geq 2$. The first object in our paper is to develop an approach to make sure that the system (1) is globally solvable in the weak sense when $\xi \gamma > \chi \alpha$. As for the initial data, throughout the text we may assume that

$$\begin{cases} (u_0, v_0) \in C^0(\bar{\Omega}) \text{ are nonnegative,} \\ u_0 \neq 0 \text{ and } v_0 \neq 0. \end{cases}$$

(6)

**Theorem 1.1.** Assume that $\Omega \subset \mathbb{R}^3$ be an arbitrary smooth bounded domain, and suppose that the initial data $u_0$ and $v_0$ fulfill (6). Let $\chi > 0, \xi > 0, \alpha > 0, \beta > 0, \gamma > 0, \delta > 0$, and

$$\xi \gamma > \chi \alpha.$$  

(7)

Then there exists at least one pair $(u, v, w)$ of nonnegative functions

$$u \in L^2_{\text{loc}}(\Omega \times [0, \infty)) \cap L^4_{\text{loc}}([0, \infty); W^{1,4}(\Omega)) \text{ and}$$

$$(v, w) \in L^\infty([0, \infty); W^{1,2}(\Omega)) \cap L^2_{\text{loc}}([0, \infty); W^{2,2}(\Omega)),$$

(8)

which forms a global weak solution (in the sense of Definition 2.1 below) of (1).

A nature question connected to regularity is whether or not there exists globally bounded solution for (1). In the following theorem, we could see that whenever the initial data $u_0$ is suitable small or the repulsion is enough strong, and hence each of solutions becomes eventually smooth and bounded in the three-dimensional setting.

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain, and let $\chi, \xi, \alpha, \beta, \gamma$ and $\delta > 0$. Then for any $\lambda \in (0, 1)$ there exist $C = C(\lambda, \beta, \delta, \Omega) > 0$ with the following
property that whenever the initial data \((u_0, v_0)\) fulfill (6) and
\[
\int_{\Omega} u_0^2 \leq \frac{C}{\chi^2 \alpha^2 \gamma^2} \quad \text{and} \quad \int_{\Omega} v_0^2 \leq \frac{C \chi}{\chi^3 \alpha^3 \gamma^3},
\]
and
\[
\int_{\Omega} |\nabla v_0|^4 \leq \frac{C \chi^2 \gamma^2}{\chi^3 \alpha^2},
\]
then problem (1) possesses a global classical bounded solution.

The rest of this paper is organized as follows. In Section 2, we will introduce the definition of a weak solution and give an approximate problem, and some basic properties. In Section 3, some priori estimates are given to prove the existence of global weak solution. Section 4 is devoted to showing the global boundedness of solution under some smallness assumption on the initial data.

2. Preliminaries. The concept of (global) weak solution for (1) we shall pursue in this sequel will be given in the follows.

**Definition 2.1.** Suppose that \((u_0, v_0)\) satisfies (6), and \(T \in (0, \infty]\). Then a pair of nonnegative functions
\[
\begin{align*}
(u, v, w) &\in L^2_{loc}(\Omega \times [0, \infty)) \cap L^4_{loc}((0, \infty) ; W^{1,4}(\Omega)) \\
\end{align*}
\]
is called weak solution of (1) if
\[
-\int_0^T \int_{\Omega} u \phi_t - \int_{\Omega} u_0 \phi(\cdot, 0) = -\int_0^T \int_{\Omega} \nabla u \cdot \nabla \phi + \chi \int_{\Omega} u \nabla v \cdot \nabla \phi \\
- \xi \int_0^T \int_{\Omega} u \nabla \nabla \phi
\]
and
\[
-\int_0^T \int_{\Omega} v \phi_t - \int_{\Omega} v_0 \phi(\cdot, 0) = -\int_0^T \int_{\Omega} \nabla v \cdot \nabla \phi - \beta \int_{\Omega} v \phi + \alpha \int_0^T \int_{\Omega} u \phi
\]
as well as
\[
0 = -\int_0^T \int_{\Omega} \nabla w \cdot \nabla \phi - \delta \int_{\Omega} w \phi + \gamma \int_0^T \int_{\Omega} u \phi
\]
hold for any test functions \(\varphi \in C_0^\infty(\overline{\Omega} \times [0, \infty))\). Moreover, if \(T = \infty\), we call \((u, v, w)\) a global weak solution of (1).

In order to obtain the existence of global weak solution, we then need to consider the approximate system
\[
\begin{align*}
\begin{cases}
u_{tt} - \Delta v - \chi \nabla \cdot (u \nabla v) + \xi \Delta \nabla (u \nabla w) - \epsilon u_{\kappa} = 0, & x \in \Omega, \quad t > 0, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, \quad t > 0, \\
u_t = \Delta v - \beta v + \alpha u, & x \in \Omega, \quad t > 0, \\
0 = \Delta w - \delta w + \gamma u, & x \in \Omega, \quad t > 0, \\
u_e(x, 0) = u_0(x), v_e(x, 0) = v_0(x), & x \in \Omega,
\end{cases}
\end{align*}
\]
with some \(\kappa > 5\) and \(\epsilon \in (0, 1)\). The exponent \(\kappa > 5\) which ensures the global existence of (15) is not optimal and we pick \(\kappa > 5\) here for our convenience.
In contrast to (1), a fundamental difference between both systems consists in the circumstance that the term \(-cu^\varepsilon\) in (15) apparently destroys the construction of the Lyapunov functional, which plays an important role in establishing the behavior of solutions for (1) (see [13]). Therefore, motivated by the ideas in [13, Lemma 4.1], we explore similar appropriately energy structures (see Lemma 3.1) associated with (15) and achieve a priori estimates for solutions \((u_\varepsilon, v_\varepsilon, w_\varepsilon)\), \(\varepsilon \in (0, 1)\), which would facilitate a global existence of weak solutions. In deriving the assertion about boundedness, some estimates for the functional \(z(t) := \int_{\Omega} u^2(t) + \int_{\Omega} |\nabla v(t)|^4\) would be given if the initial data \(u_0\) is suitable small. Then, with these estimates at hand, we can further achieve eventual boundedness and regularity of \((u, v, w)\).

According to the well-known arguments for parabolic-parabolic or parabolic-elliptic logistic-type chemotaxis model (see [39, 37, 33]), a local existence theory of classical solution will be obtained in the following sense.

**Lemma 2.2.** Let \(\Omega \subset \mathbb{R}^n (n \geq 2)\) be a bounded domain, and let \(\chi, \xi, \alpha, \beta, \gamma\) and \(\delta > 0\). Suppose that \((u, v, w)\) is a nonnegative classical solution of

\[
\begin{align*}
u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) + f(u), \quad x \in \Omega, \quad t > 0, \\
v_t &= \Delta v - \beta v + \alpha u, \quad x \in \Omega, \quad t > 0, \\
0 &= \Delta w - \delta w + \gamma u, 
\end{align*}
\]

with homogeneous Neumann boundary conditions, for any initial data \(u_0\) and \(v_0\) fulfilling (6) and \(f \in W^{1,\infty}(\mathbb{R})\) satisfying \(f(0) \geq 0\). Then there exist \(T_{\max} \in (0, \infty]\) and nonnegative functions

\[
(u, w) \in C^0(\Omega \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times [0, T_{\max}]), \quad \text{and} \quad (v, w) \in C^0(\Omega \times [0, T_{\max}); W^{1,2}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times [0, T_{\max}]),
\]

which solve (16) classically in \(\Omega \times (0, T_{\max})\), and which are such that

\[
\text{if } T_{\max} < \infty, \text{ then } \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{W^1,\infty(\Omega)} \to \infty \quad (18)
\]

as \(t \to T_{\max}\).

Now, we let \((u_\varepsilon, v_\varepsilon, w_\varepsilon)\) denote the local solution to (15) with the maximal time \(T_{\max, \varepsilon}\). In view of above criterion for extensibility of local solutions, we can show that \(T_{\max, \varepsilon} = \infty\).

**Lemma 2.3.** Let \(\chi, \xi, \alpha, \beta, \gamma, \delta > 0\) and \(\kappa > 5\). For each \(\varepsilon \in (0, 1)\) the problem (15) has a (unique) nonnegative global classical solution in the sense that \(T_{\max, \varepsilon} = \infty\).

**Proof.** Given \(T \in (0, T_{\max, \varepsilon})\), we first integrate the first equation in (15) to see that

\[
\varepsilon \int_0^T \int_\Omega u_\varepsilon^\kappa \leq c_1 \quad \text{for all } t \in (0, T)
\]

with \(c_1 > 0\). Since \((-\frac{1}{2} - \frac{3}{2n}) \cdot \frac{\kappa}{\kappa - 1} > -1\) according to \(\kappa > 5\), the number

\[
c_2 := \left(\int_0^T (1 + (t - s)^{-\frac{1}{2} - \frac{3}{2n}})^{\frac{\kappa - 1}{\kappa}} ds\right)^{\frac{\kappa - 1}{\kappa}}
\]

is valid for all \(t \in (0, T)\). Making use of estimates for the heat semigroup in the variations-of-constants formula for \(v_\varepsilon\), we invoke Hölder’s inequality to obtain \(c_3, c_4\)
and $c_5 > 0$ such that

$$
\| \nabla v_\epsilon(\cdot, t) \|_{L^\infty(\Omega)} \leq \left\| \nabla e^{t(\Delta - \beta)} v_0 \right\|_{L^\infty(\Omega)} + \alpha \int_0^t \left\| \nabla e^{(t-s)(\Delta - \beta)} u_\epsilon(\cdot, s) \right\|_{L^\infty(\Omega)} \, ds
$$

$$
\leq c_3 \| \nabla v_0 \|_{L^\infty(\Omega)} + c_3 \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{d}{2p}}) \| u_\epsilon(\cdot, s) \|_{L^\infty(\Omega)} \, ds
$$

$$
\leq c_3 \left( \int_0^t (1 + (t-s)^{-\frac{1}{2} - \frac{d}{2p}}) \frac{s^{\frac{d}{2}}}{s^{\frac{d}{2p}}} \, ds \right) \left( \int_0^t \| u_\epsilon(\cdot, s) \|_{L^\infty(\Omega)} \, ds \right)^{\frac{d}{d-2}}
$$

$$
+ c_4
$$

$$
\leq c_4 + c_3 c_2 \left( \frac{c_1}{\epsilon} \right)^{\frac{d}{d-2}}
$$

$$
\leq c_5 \quad \text{for all } t \in (0, T).
$$

To estimate $\| \nabla w_\epsilon(\cdot, t) \|_{L^\infty(\Omega)}$ with $t \in (0, T)$, we need to multiply the first equation by $u_\epsilon^{-1}$ with $p > 3$ and use Young’s inequality to find that

$$
\frac{1}{p} \frac{d}{dt} \int_\Omega u_\epsilon^p = - (p-1) \int_\Omega u_\epsilon^{p-2} |\nabla u_\epsilon|^2 + \chi(p-1) \int_\Omega u_\epsilon^{p-1} \nabla u_\epsilon \cdot \nabla v_\epsilon
$$

$$
- \xi (p-1) \int_\Omega u_\epsilon^{p-1} \nabla u_\epsilon \cdot \nabla w_\epsilon - \epsilon \int_\Omega u_\epsilon^{p+1}
$$

$$
= - (p-1) \int_\Omega u_\epsilon^{p-2} |\nabla u_\epsilon|^2 + \chi(p-1) \int_\Omega u_\epsilon^{p-1} \nabla u_\epsilon \cdot \nabla v_\epsilon
$$

$$
+ \xi (p-1) \int_\Omega u_\epsilon^p (\delta w_\epsilon - \gamma u_\epsilon) - \epsilon \int_\Omega u_\epsilon^{p+1}
$$

$$
\leq c_6 \int_\Omega u_\epsilon^{p} |\nabla v_\epsilon|^2 + \frac{\xi \delta (p-1)}{p} \int_\Omega \int_\Omega u_\epsilon^{p} w_\epsilon - \frac{\xi \gamma (p-1)}{p} \int_\Omega u_\epsilon^{p+1} - \epsilon \int_\Omega u_\epsilon^{p+1}
$$

$$
\leq c_6^2 c_6 \int_\Omega u_\epsilon^{p} + c_6 \int_\Omega u_\epsilon^{p+1} - \epsilon \int_\Omega u_\epsilon^{p+1}
$$

$$
\leq c_6^2 c_6 \int_\Omega u_\epsilon^{p} + c_7 \int_\Omega u_\epsilon^{p+1} - \epsilon \int_\Omega u_\epsilon^{p+1}
$$

$$
\leq c_8 \quad \text{for all } t \in (0, T)
$$

with $c_6, c_7$ and $c_8 > 0$, where we have used (19) and the following inequality

$$
\| w_\epsilon(\cdot, t) \|_{L^{p+1}(\Omega)} \leq \frac{\gamma}{\delta} \| u_\epsilon(\cdot, t) \|_{L^{p+1}(\Omega)} \quad \text{for all } t \in (0, T)
$$

according to [42, Lemma 3.1]. Now integration over time yields $c_9 > 0$ such that

$$
\int_\Omega u_\epsilon^p \leq c_9 \quad \text{for all } t \in (0, T)
$$

with any $p > 3$ and thus, in conjunction with Agmon-Douglis-Nirenberg estimates (see [1, 2]) on linear elliptic equation, proves that

$$
\| \nabla w_\epsilon(\cdot, t) \|_{L^\infty(\Omega)} \leq c_{10} \quad \text{for all } t \in (0, T)
$$

with some $c_{10} > 0$. Then one may apply standard a priori estimates techniques to infer the existence of $c_{11} > 0$ such that

$$
\| u_\epsilon(\cdot, t) \|_{L^\infty(\Omega)} \leq c_{11} \quad \text{for all } t \in (0, T)
$$
and
\[ \|v_\epsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq c_{11} \] as well as \[ \|w_\epsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq c_{11} \] for all \( t \in (0,T) \).

As a consequence of these and Lemma 2.2, we actually have \( T_{\max, \epsilon} = \infty \).

Finally, let us briefly collect some elementary properties of the solution \((u_\epsilon, v_\epsilon, w_\epsilon)\) to (15).

**Lemma 2.4.** Let \( \kappa > 5 \). For any \( \epsilon \in (0,1) \) and \( T > 0 \), the solution \( u_\epsilon \) satisfies
\[ \|u_\epsilon(\cdot,t)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} \] for all \( t > 0 \) \hspace{1cm} (20)

and
\[ \epsilon \int_0^T \int_{\Omega} u_\epsilon^\kappa \leq C_1 \] for all \( t \in (0,T) \) \hspace{1cm} (21)

with \( C_1 > 0 \).

**Proof.** We need to integrate the first equation in (15) over \( \Omega \) and drop nonnegative term.

**Lemma 2.5.** Let \( n = 3 \), and \( \alpha > 0, \beta > 0 \). For any \( \epsilon \in (0,1) \), there exists \( C_2 = C_2(\alpha,\beta) > 0 \) such that the second component solution \( v_\epsilon \) satisfies
\[ \|v_\epsilon(\cdot,t)\|_{L^2(\Omega)} \leq C_2 \] for all \( t > 0 \). \hspace{1cm} (22)

Moreover, for any \( \eta > 0 \) we have
\[ \frac{d}{dt} \int_{\Omega} v_\epsilon^2 + \frac{1}{2} \int_{\Omega} |\nabla v_\epsilon|^2 + 2\beta \int_{\Omega} v_\epsilon^2 \leq \eta \int_{\Omega} u_\epsilon^2 + C_3 \] for all \( t > 0 \) \hspace{1cm} (23)

with \( C_3 = C_3(\eta,\alpha,\beta) > 0 \).

**Proof.** Since \( L^1 \)-norm of \( u_\epsilon \) has been given in Lemma 2.4, a straightforward semigroup technique implies (22).

Given \( \eta > 0 \), testing both side of the second equation, we use Young’s inequality to arrive at
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} v_\epsilon^2 + \int_{\Omega} |\nabla v_\epsilon|^2 + \beta \int_{\Omega} v_\epsilon^2 = \alpha \int_{\Omega} u_\epsilon v_\epsilon \]
\[ \leq \frac{\eta}{2} \int_{\Omega} u_\epsilon^2 + \frac{\alpha^2}{2\eta} \int_{\Omega} v_\epsilon^2 \] for all \( t > 0 \)

and obtain (23) according to (22).

**Lemma 2.6.** Let \( n = 3 \), and \( \gamma, \delta > 0 \). For any \( \epsilon \in (0,1) \), the third component solution \( w_\epsilon \) satisfies
\[ \|w_\epsilon(\cdot,t)\|_{L^1(\Omega)} \leq \frac{\gamma}{\delta} \|u_0\|_{L^1(\Omega)} \] for all \( t > 0 \). \hspace{1cm} (24)

Moreover, for any \( \eta > 0 \) one can find
\[ \|w_\epsilon(\cdot,t)\|_{L^2(\Omega)} \leq C_4 \] for all \( t > 0 \) \hspace{1cm} (25)

and
\[ \int_{\Omega} |\nabla w_\epsilon|^2 + \delta \int_{\Omega} w_\epsilon^2 \leq \eta \int_{\Omega} u_\epsilon^2 + C_4 \] for all \( t > 0 \) \hspace{1cm} (26)

with some \( C_4 = C_4(\eta,\gamma,\delta) > 0 \).
Lemma 3.1. Let the solution (3.1).

Proof. Integrating the third equation in (15) over $\Omega$ implies (24) immediately. Since $||u_\epsilon(\cdot,t)||_{L^1(\Omega)}$ is uniformly bounded for all $t > 0$, then applying $L^p - L^q$ estimates on linear elliptic equations with homogeneous Neumann boundary condition provides $c = c(\gamma, \delta) > 0$ such that

$$||w_\epsilon(\cdot,t)||_{L^2(\Omega)} \leq c \text{ for all } t > 0. \quad (27)$$

To prove (26), testing the third equation $0 = \Delta w_\epsilon - \delta w_\epsilon + \gamma u_\epsilon$ by $w_\epsilon$ and integrating by parts, we have

$$\int_\Omega |\nabla w_\epsilon|^2 + \delta \int_\Omega w_\epsilon^2 \leq \gamma \int_\Omega u_\epsilon w_\epsilon \text{ for all } t > 0, \quad (28)$$

where with $\eta > 0$ an application of Young’s inequality for (28) and using (27) would imply (26).

3. Global weak solutions if $\xi\gamma > \chi\alpha$.

3.1. A priori estimates. In this section we proceed to provide some estimates for the solution $(u_\epsilon, v_\epsilon, w_\epsilon)$ based on the analysis of the coupled functional

$$y_\epsilon(t) := \int_\Omega (u_\epsilon + 1) \ln(u_\epsilon + 1) + \frac{\chi}{2\alpha} \int_\Omega |\nabla v_\epsilon|^2 + \frac{\chi\beta + 2\chi^2\alpha}{2\alpha} \int_\Omega v_\epsilon^2 \text{ for all } t > 0 \quad (29)$$

under the assumption $\xi\gamma > \chi\alpha$. Here we note that since $u_\epsilon \geq 0$, it is reasonable for us to use the term $\int_\Omega (u_\epsilon + 1) \ln(u_\epsilon + 1)$ instead of $\int_\Omega u_\epsilon \ln u_\epsilon$ in (29).

Lemma 3.1. Let $n = 3$, and let $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$ and $\kappa > 5$. Given any $T > 0$, then there exists $C_5 = C_5(\chi, \xi, \alpha, \beta, \gamma, \delta) > 0$ such that for any $\epsilon \in (0, 1)$

$$\frac{d}{dt} \left[ \int_\Omega (u_\epsilon + 1) \ln(u_\epsilon + 1) + \frac{\chi}{2\alpha} \int_\Omega |\nabla v_\epsilon|^2 + \frac{\chi\beta + 2\chi^2\alpha}{2\alpha} \int_\Omega v_\epsilon^2 \right]
+ \int_\Omega (u_\epsilon + 1) \ln(u_\epsilon + 1) + \chi^2 \int_\Omega |\nabla v_\epsilon|^2 + 2\chi^2 \beta \int_\Omega v_\epsilon^2
\leq - \frac{1}{2} \int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon + 1} - \frac{\chi}{\alpha} \int_\Omega v_\epsilon^2 \frac{\xi\gamma - \chi\alpha}{8} \int_\Omega u_\epsilon^2
- \epsilon \int_\Omega u_\epsilon^5 \ln(u_\epsilon + 1) + C_5 \text{ for all } t \in (0, T). \quad (30)$$

Proof. First, observing the first equation in (15) can be rewritten as

$$(u_\epsilon + 1)_t = \Delta(u_\epsilon + 1) - \chi \nabla \cdot ((u_\epsilon + 1) \nabla v_\epsilon) + \xi \nabla \cdot ((u_\epsilon + 1) \nabla w_\epsilon)
+ \chi \Delta v_\epsilon - \xi \Delta w_\epsilon - \epsilon u_\epsilon^5$$

for all $t > 0$, we test this by $\ln(u_\epsilon + 1)$ and integrate by parts to see that

$$\frac{d}{dt} \int_\Omega (u_\epsilon + 1) \ln(u_\epsilon + 1) = \int_\Omega (u_\epsilon + 1)_t \ln(u_\epsilon + 1) + \int_\Omega u_\epsilon t_
u
= - \int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon + 1} + \chi \int_\Omega \nabla u_\epsilon \cdot \nabla v_\epsilon - \xi \int_\Omega \nabla u_\epsilon \cdot \nabla w_\epsilon
+ \chi \int_\Omega \Delta v_\epsilon \ln(u_\epsilon + 1) - \xi \int_\Omega \Delta w_\epsilon \ln(u_\epsilon + 1)
- \epsilon \int_\Omega u_\epsilon^5 \ln(u_\epsilon + 1) - \epsilon \int_\Omega u_\epsilon^5 \text{ for all } t > 0.$$
Here, in the second integral on the right side we again integrate by parts and make use of \( v_{\epsilon t} = \Delta v_{\epsilon} - \beta v_{\epsilon} + \alpha u_{\epsilon} \) to obtain
\[
\chi \int_\Omega \nabla u_{\epsilon} \cdot \nabla v_{\epsilon} = - \chi \int_\Omega u_{\epsilon} \Delta v_{\epsilon}
\]
\[
= - \chi \int_\Omega u_{\epsilon} v_{\epsilon t} - \chi \beta \int_\Omega u_{\epsilon} v_{\epsilon} + \chi \alpha \int_\Omega u_{\epsilon}^2
\]
\[
= - \frac{\chi}{\alpha} \int_\Omega v_{\epsilon t}^2 - \frac{\chi}{2\alpha} \frac{d}{dt} \int_\Omega |\nabla v_{\epsilon}|^2 - \frac{\chi \beta}{2\alpha} \frac{d}{dt} \int_\Omega v_{\epsilon}^2
\]
\[
- \chi \beta \int_\Omega u_{\epsilon} v_{\epsilon} + \alpha \int_\Omega u_{\epsilon}^2 \text{ for all } t > 0.
\]
Similarly, the third integral can be estimated as
\[
-\xi \int_\Omega \nabla u_{\epsilon} \cdot \nabla w_{\epsilon} = \xi \int_\Omega v_{\epsilon} \Delta w_{\epsilon} = \xi \int_\Omega u_{\epsilon} (\delta w_{\epsilon} - \gamma u_{\epsilon})
\]
\[
= \xi \delta \int_\Omega u_{\epsilon} w_{\epsilon} - \xi \gamma \int_\Omega u_{\epsilon}^2
\]
\[
\leq \frac{\xi \gamma - \chi \alpha}{4} \int_\Omega u_{\epsilon}^2 + \frac{\xi \gamma - \chi \alpha}{\xi \gamma - \chi \alpha} \int_\Omega u_{\epsilon}^2 - \xi \gamma \int_\Omega u_{\epsilon}^2 \text{ for all } t > 0,
\]
where we recall (25) to find \( c_1 = c_1(\chi, \xi, \alpha, \gamma, \delta) > 0 \) such that
\[
-\xi \int_\Omega \nabla u_{\epsilon} \cdot \nabla w_{\epsilon} \leq \frac{\xi \gamma - \chi \alpha}{4} \int_\Omega u_{\epsilon}^2 - \xi \gamma \int_\Omega u_{\epsilon}^2 + c_1 \text{ for all } t > 0.
\]
As for the term \( \chi \int_\Omega \Delta v_{\epsilon} \ln(u_{\epsilon} + 1) \), we use Young’s inequality to gain
\[
\chi \int_\Omega \Delta v_{\epsilon} \ln(u_{\epsilon} + 1) = - \chi \int_\Omega \frac{\nabla u_{\epsilon} \cdot \nabla v_{\epsilon}}{u_{\epsilon} + 1}
\]
\[
\leq \frac{1}{4} \int_\Omega \frac{|\nabla u_{\epsilon}|^2}{u_{\epsilon} + 1} + \chi \int_\Omega \frac{|\nabla v_{\epsilon}|^2}{u_{\epsilon} + 1}
\]
\[
\leq \frac{1}{4} \int_\Omega \frac{|\nabla u_{\epsilon}|^2}{u_{\epsilon} + 1} + \chi \int_\Omega |\nabla v_{\epsilon}|^2 \text{ for all } t > 0,
\]
and then use (23) with \( \eta = \frac{\xi \gamma - \chi \alpha}{4\chi} \) and with some \( c_2 = c_2(\chi, \xi, \alpha, \beta, \gamma) > 0 \)
\[
\chi^2 \frac{d}{dt} \int_\Omega v_{\epsilon}^2 + 2\chi^2 \int_\Omega |\nabla v_{\epsilon}|^2 + 2\chi^2 \beta \int_\Omega v_{\epsilon}^2 \leq \frac{\xi \gamma - \chi \alpha}{4} \int_\Omega u_{\epsilon}^2 + c_2 \text{ for all } t > 0
\]
to infer that
\[
\chi \int_\Omega \Delta v_{\epsilon} \ln(u_{\epsilon} + 1) \leq \frac{1}{4} \int_\Omega \frac{|\nabla u_{\epsilon}|^2}{u_{\epsilon} + 1} + \frac{\xi \gamma - \chi \alpha}{4} \int_\Omega u_{\epsilon}^2
\]
\[
- \chi^2 \frac{d}{dt} \int_\Omega v_{\epsilon}^2 - \chi^2 \int_\Omega |\nabla v_{\epsilon}|^2 - 2\chi^2 \beta \int_\Omega v_{\epsilon}^2 + c_2 \text{ for all } t > 0.
\]
Moreover, we take similar procedure to deal with
\[
-\xi \int_\Omega \Delta w_{\epsilon} \ln(u_{\epsilon} + 1) \leq \frac{1}{4} \int_\Omega \frac{|\nabla u_{\epsilon}|^2}{u_{\epsilon} + 1} + \frac{\xi \gamma - \chi \alpha}{4} \int_\Omega u_{\epsilon}^2 + c_3 \text{ for all } t > 0
\]
with some \( c_3 = c_3(\chi, \xi, \alpha, \gamma, \delta) > 0 \), thanks to (26). Finally, we may choose \( c_4 = c_4(\chi, \xi, \alpha, \gamma) > 0 \) such that
\[
\int_\Omega (u_{\epsilon} + 1) \ln(u_{\epsilon} + 1) \leq \frac{\xi \gamma - \chi \alpha}{8} \int_\Omega u_{\epsilon}^2 + c_4 \text{ for all } t > 0,
\]
where collecting above inequalities imply (30).

Upon integration in time, (30) first provides the following statements on regularity of solution \((u_\epsilon, v_\epsilon, w_\epsilon)\) under the assumption \(\xi\gamma > \chi\alpha\).

**Lemma 3.2.** Let \(n = 3\), and let \(\chi, \xi, \alpha, \beta, \gamma, \delta > 0\) and \(\kappa > 5\). Suppose that \(\xi\gamma > \chi\alpha\). Then for any \(\epsilon \in (0, 1)\) and \(T > 1\) there exists \(C_6 = C_6(\chi, \xi, \alpha, \beta, \gamma, \delta) > 0\) independent of \(T\) such that the solution satisfies

\[
\int_t^{t+1} \int_{\Omega} v_\epsilon^2 \leq C_6 \quad \text{for all } t \in (0, T-1) \tag{31}
\]

and

\[
\int_t^{t+1} \int_{\Omega} |\nabla u_\epsilon|^2 \leq C_6 \quad \text{for all } t \in (0, T-1) \tag{32}
\]

as well as

\[
\int_t^{t+1} \int_{\Omega} u_\epsilon^2 \leq C_6 \quad \text{for all } t \in (0, T-1) \tag{33}
\]

and

\[
\epsilon \int_t^{t+1} \int_{\Omega} u_\epsilon^\kappa \ln(u_\epsilon + 1) \leq C_6 \quad \text{for all } t \in (0, T-1) \tag{34}
\]

and

\[
\int_t^{t+1} \|u_\epsilon(\cdot, t)\|_{W^{1,4}(\Omega)}^{\frac{4}{3}} \leq C_6 \quad \text{for all } t \in (0, T-1). \tag{35}
\]

**Proof.** As Lemma 3.1 guarantees the existence of \(c_1 = c_1(\chi, \alpha, \beta) > 0\) and \(c_2 = c_2(\chi, \xi, \alpha, \beta, \gamma, \delta) > 0\) such that the function \(y_\epsilon\) defined by (29) fulfills

\[
y_\epsilon'(t) + c_1 y_\epsilon(t) + h_\epsilon(t) \leq c_2 \quad \text{for all } t > 0,
\]

where

\[
h_\epsilon(t) := \frac{1}{2} \int_{\Omega} |\nabla u_\epsilon|^2 + \frac{\chi}{\alpha} \int_{\Omega} v_\epsilon^2 + \frac{\xi\gamma - \chi\alpha}{8} \int_{\Omega} u_\epsilon^2 + \epsilon \int_{\Omega} u_\epsilon^\kappa \ln(u_\epsilon + 1) \quad \text{for all } t > 0.
\]

By a comparison argument, this in particular entails that

\[
y_\epsilon(t) \leq c_3 := \max \left\{ \frac{c_2}{c_1}, y_\epsilon(0) \right\} \quad \text{for all } t > 0.
\]

Then integration over time, we have

\[
\int_t^{t+1} h_\epsilon(s) ds \leq y_\epsilon(t) - y_\epsilon(t+1) - c_1 \int_t^{t+1} y_\epsilon(s) ds + c_2 \tag{36}
\]

\[
\leq c_4 := c_3 + c_2 \quad \text{for all } t \in (0, T-1).
\]

Therefore, (36) entails (31)-(34) hold if \(\xi\gamma > \chi\alpha\). Now an interpolation using Hölder’s and Young’s inequalities along with (32) and (33) provides \(c_5 > 0\) such
that
\[
\int_t^{t+1} \|u_\epsilon(\cdot, t)\|_{\dot{W}^{1, \frac{4}{3}}(\Omega)}^{\frac{4}{3}} = \int_t^{t+1} \left( \int_\Omega u_\epsilon^2 + \int_\Omega |\nabla u_\epsilon|^\frac{4}{3} \right) \leq \left( \int_t^{t+1} \int_\Omega u_\epsilon^2 \right)^{\frac{2}{3}} \cdot |\Omega|^{\frac{1}{3}} + \int_t^{t+1} \int_\Omega \frac{|\nabla u_\epsilon|^2}{u_\epsilon + 1} \cdot (u_\epsilon + 1)^\frac{2}{3} \\
+ \frac{1}{3} \int_t^{t+1} \int_\Omega (u_\epsilon + 1)^2 \\
\leq c_5 \text{ for all } t \in (0, T - 1),
\]
which finishes our proof.

The following estimates for \( u_\epsilon \) are immediate.

**Lemma 3.3.** Let \( \epsilon \in (0, 1) \), and let \( T > 0, \kappa > 5 \). If \( \xi \eta > \chi \alpha \), we have
\[
\int_0^T \int_\Omega u_\epsilon^2 \leq C_7(T) \text{ for all } t \in (0, T)
\]
and
\[
\epsilon \int_0^T \int_\Omega u_\epsilon^\kappa \ln(u_\epsilon + 1) \leq C_7(T) \text{ for all } t \in (0, T)
\]
as well as
\[
\int_0^T \|u_\epsilon(\cdot, t)\|_{\dot{W}^{1, \frac{4}{3}}(\Omega)}^{\frac{4}{3}} \leq C_7(T) \text{ for all } t \in (0, T)
\]
with some \( C_7(T) > 0 \).

The next lemma gives estimates on the derivatives of the second solution component \( v_\epsilon \).

**Lemma 3.4.** Let \( \epsilon \in (0, 1) \), and let \( T > 0 \). Then the solution \( v_\epsilon \) fulfills
\[
\int_\Omega |\nabla v_\epsilon|^2 \leq C_8 \text{ for all } t \in (0, T)
\]
and
\[
\int_0^T \int_\Omega |\Delta v_\epsilon|^2 \leq C_9(T) \text{ for all } t \in (0, T)
\]
as well as
\[
\int_0^T \int_\Omega |\nabla v_\epsilon|^q \leq C_9(T) \text{ for all } t \in (0, T)
\]
with some \( q \in [1, \frac{10}{3}) \), \( C_8 > 0 \) and \( C_9(T) > 0 \).

**Proof.** In view of Lemma 3.4 in [32], this from (33) shows that (40) holds. Since
\[
\int_0^T \int_\Omega u_\epsilon^2 \leq c_1(T) \text{ for all } t \in (0, T)
\]
with \( c_1(T) > 0 \) according to (31), then along with (22) and (37), there warrants the existence of \( c_2(T) > 0 \) such that

\[
\int_0^T \int_\Omega |\Delta v_\epsilon|^2 \leq \int_0^T \int_\Omega |v_{\epsilon t} + \beta v_\epsilon - \alpha u_\epsilon|^2 \\
\leq 3 \int_0^T \int_\Omega v_{\epsilon t}^2 + 3\beta^2 \int_0^T \int_\Omega v_\epsilon^2 + 3\alpha^2 \int_0^T \int_\Omega u_\epsilon^2 \\
\leq c_2(T) \text{ for all } t \in (0, T)
\]

by Young’s inequality. By means of the inequality \( \int_\Omega |D^2 v|^2 \leq c_3 \int_\Omega |\Delta v|^2 \) with \( c_3 > 0 \) (see [10, Lemma 4.3] for a detailed proof) and the Gagliardo-Nirenberg inequality and Young’s inequality, (40) and (41), we can find \( c_4, c_5 > 0 \) and \( c_6(T) > 0 \) such that

\[
||\nabla v_\epsilon||_{L^q(\Omega)}^q \leq c_4 ||D^2 v_\epsilon||_{L^2(\Omega)}^q + c_5 \text{ for all } t \in (0, T)
\]

and

\[
\int_0^T ||\nabla v_\epsilon||_{L^a(\Omega)}^a \leq c_5 \int_0^T ||D^2 v_\epsilon||_{L^2(\Omega)}^2 + c_5 T \\
\leq c_6(T) \text{ for all } t \in (0, T),
\]

where \( a = \frac{\frac{q}{2} - \frac{n}{2}}{1 + \frac{n}{2} - \frac{q}{2}} \in (0, 1) \) and

\[
q \cdot a = \frac{qn}{2} - n < 2,
\]

thanks to \( n = 3 \) and \( q < \frac{10}{3} \). Then we finish our proof. \( \square \)

Similarly, some time derivatives on the third solution component \( w_\epsilon \) can be given in the follow.

**Lemma 3.5.** Let \( \epsilon \in (0, 1) \), and let \( T > 0 \). Then the solution \( w_\epsilon \) fulfills

\[
\int_\Omega |\nabla w_\epsilon|^2 \leq C_{10} \text{ for all } t \in (0, T)
\]

and

\[
\int_0^T \int_\Omega |\Delta w_\epsilon|^2 \leq C_{11}(T) \text{ for all } t \in (0, T)
\]

as well as

\[
\int_0^T \int_\Omega |\nabla w_\epsilon|^q \leq C_{11}(T) \text{ for all } t \in (0, T)
\]

with some \( q \in [1, \frac{10}{3}) \), \( C_{10} > 0 \) and \( C_{11}(T) > 0 \).

**Proof.** By means of the boundedness of \( ||u \ln u||_{L^1(\Omega)} \) according to the proof of Lemma 3.2, we follow a similar proof in [35, Lemma A.4] to obtain (43). To see (44), we only need to multiply the third equation by \(-\Delta w_\epsilon\) and make use of Young’s
inequality to find that
\[
\int_0^T \int_\Omega |\Delta w_\epsilon|^2 + \delta \int_0^T \int_\Omega |\nabla w_\epsilon|^2 = -\gamma \int_0^T \int_\Omega u \Delta w_\epsilon + \frac{1}{2} \int_0^T \int_\Omega |\Delta w_\epsilon|^2 + \frac{\gamma^2}{2} \int_0^T \int_\Omega u_\epsilon^2 \quad \text{for all } t \in (0,T).
\]  
(46)
Therefore, (44) results from (46) and (37). Finally, (45) comes from (43) and (44) by the Gagliardo-Nirenberg inequality.

3.2. Passing to the limit. To prepare our subsequent compactness properties of \((u_\epsilon, v_\epsilon, w_\epsilon)\) by means of the Aubin-Lions lemma, we use Lemmas 3.3-3.5 to obtain the following regularity property with respect to the time variable.

Lemma 3.6. Let \(n = 3\), and let \(\epsilon \in (0,1), \kappa > 5\). Then for any \(T > 0\) one can find \(C_{12}(T) > 0\) with the property that
\[
\int_0^T \|u_\epsilon(t)\|_{W^{1,\infty}(\Omega)} \leq C_{12}(T) \quad \text{for all } t \in (0,T).
\]  
(47)
Proof. We fix \(\psi \in W^{1,\infty}(\Omega)\) such that \(\|\psi\|_{W^{1,\infty}(\Omega)} \leq 1\) and use Young’s inequality to find
\[
\int_\Omega u_\epsilon(t) \psi \leq \left| -\int_\Omega \nabla u_\epsilon \cdot \nabla \psi + \chi \int_\Omega u_\epsilon \nabla v_\epsilon \cdot \nabla \psi - \xi \int_\Omega u_\epsilon \nabla w_\epsilon \cdot \nabla \psi - \epsilon \int_\Omega u_\epsilon \psi \right|
\leq \int_\Omega |\nabla u_\epsilon| + \chi \int_\Omega u_\epsilon |\nabla v_\epsilon| + \xi \int_\Omega u_\epsilon |\nabla w_\epsilon| + \epsilon \int_\Omega u_\epsilon^2
\leq \int_\Omega |\nabla u_\epsilon|^2 + |\Omega| + 2 \int_\Omega u_\epsilon^2
+ \frac{\chi^2}{4} \int_\Omega |\nabla v_\epsilon|^2 + \frac{\xi^2}{4} \int_\Omega |\nabla w_\epsilon|^2 + \epsilon \int_\Omega u_\epsilon^2 \quad \text{for all } t \in (0,T)
\] with \(\epsilon \in (0,1)\) and \(\kappa > 5\). Then one can recall (39), (37), (40), (43) and (21) to obtain the boundedness result
\[
\int_0^T \sup_{\|\psi\|_{W^{1,\infty}(\Omega)} \leq 1} \left| \int_\Omega u_\epsilon(t) \psi \right| \leq c(T) \quad \text{for all } t \in (0,T)
\] with \(c(T) > 0\), independent of \(\epsilon\), which ends the proof.

Based on above lemmas and by extracting suitable subsequences in a standard way, we could see the solution of (1) is indeed globally solvable.

Lemma 3.7. Let \(n = 3\), and let \(T > 0\). Then there exist \((\epsilon_j)_{j \in \mathbb{N}} \subset (0,1)\) and nonnegative function \((u, v, w)\) fulfilling (8) such that \(\epsilon_j \to 0\) as \(j \to \infty\) and such that
\[
u u_\epsilon \to u \quad \text{a.e. in } \Omega \times (0,T),
\]  
(48)
\[
u u_\epsilon \to u \quad \text{in } L^r(\Omega \times (0,T)) \quad \text{for all } r \in [1, 2),
\]  
(49)
\[
\nabla u_\epsilon \to \nabla u \quad \text{in } L^2(\Omega \times (0,T)),
\]  
(50)
\[
\epsilon u_\epsilon^\kappa \to 0 \quad \text{in } L^1(\Omega \times (0,T)).
\]  
(51)
\[ v_\epsilon \to v \text{ \ a.e. in } \Omega \times (0,T), \quad (52) \]
\[ v_\epsilon \to v \text{ \ in } L^2(\Omega \times (0,T)), \quad (53) \]
\[ \nabla v_\epsilon \rightharpoonup \nabla v \text{ \ in } L^q(\Omega \times (0,T)) \text{ \ for all } q \in \left[ 1, \frac{10}{3} \right), \quad (54) \]
\[ \Delta v_\epsilon \rightharpoonup \Delta v \text{ \ in } L^2(\Omega \times (0,T)), \quad (55) \]
\[ w_\epsilon \to w \text{ \ a.e. in } \Omega \times (0,T), \quad (56) \]
\[ w_\epsilon \to w \text{ \ in } L^2(\Omega \times (0,T)), \quad (57) \]
\[ \nabla w_\epsilon \rightharpoonup \nabla w \text{ \ in } L^q(\Omega \times (0,T)) \text{ \ for all } q \in \left[ 1, \frac{10}{3} \right), \quad (58) \]
\[ u_\epsilon \nabla v_\epsilon \rightharpoonup u \nabla v \text{ \ in } L^1(\Omega \times (0,T)), \quad (59) \]
\[ u_\epsilon \nabla w_\epsilon \rightharpoonup u \nabla w \text{ \ in } L^1(\Omega \times (0,T)). \quad (60) \]

**Proof.** As Lemmas 3.3 and 3.6 warrant that for each \( T > 0, \)
\[ (u_\epsilon)_{\epsilon \in (0,1)} \text{ \ is bounded in } L^\frac{4}{3} \left( (0,T); W^{1,\frac{4}{3}}(\Omega) \right) \]
and
\[ (u_{\epsilon t})_{\epsilon \in (0,1)} \text{ \ is bounded in } L^1 \left( (0,T); (W^{1,\infty}(\Omega))^* \right), \]
then a straightforward application of the Aubin-Lions lemma allows us to pick some \( \epsilon = (\epsilon_j)_{j \in \mathbb{N}} \subset (0,1) \) and nonnegative \( u \in L^2_{\text{loc}}(\Omega \times [0,\infty)) \cap L^\frac{4}{3}_{\text{loc}}(\Omega) \) such that (48)-(50) hold. From Lemma 3.3, \( \{\epsilon u_\epsilon^c\}_\epsilon \) is equi-integrable, then according to [5, Thm. IV.8.9] and thus is weakly convergent along a subsequence, which implies (51).

Similarly, following some boundedness arguments from Lemmas 3.4 and 3.5, we also use the Aubin-Lions lemma to gain (52)-(58). Finally, we note that a combination of (49) and (54) entails (59), whereas (60) results from (49) and (58). \( \square \)

**Proof of Theorem 1.2.** Taking \( \varphi \in C^\infty_0(\bar{\Omega} \times [0,\infty)) \) and testing the first equation in (15), for \( T > 0 \) we obtain
\[ -\int_0^T \int_\Omega u_\epsilon \varphi_\epsilon t - \int_\Omega u_0 \varphi(\cdot,0) = -\int_0^T \int_\Omega \nabla u_\epsilon \cdot \nabla \varphi + \chi \int_0^T \int_\Omega u_\epsilon \nabla v_\epsilon \cdot \nabla \varphi - \xi \int_0^T \int_\Omega u_\epsilon \nabla w_\epsilon \cdot \nabla \varphi - \epsilon \int_0^T \int_\Omega u_\epsilon^c \varphi \]
valid for all \( \epsilon \in (0,1) \). Since, on picking \( \epsilon = \epsilon_j \to 0, \)
\[ -\int_0^T \int_\Omega u_\epsilon \varphi_\epsilon t \to -\int_0^T \int_\Omega w \varphi \]
due to (49), and
\[ -\int_0^T \int_\Omega \nabla u_\epsilon \cdot \nabla \varphi \to -\int_0^T \int_\Omega \nabla v \cdot \nabla \varphi \]
by (50), as well as
\[
\chi \int_0^T \int_\Omega u \nabla v \cdot \nabla \varphi \to \chi \int_0^T \int_\Omega u \nabla v \cdot \nabla \varphi \quad \text{and}
\]
\[-\xi \int_0^T \int_\Omega u \nabla w \cdot \nabla \varphi \to -\xi \int_0^T \int_\Omega u \nabla w \cdot \nabla \varphi.
\]
from (59) and (60), we obtain (12) from (51). Similarly, in
\[
-\int_0^T \int_\Omega v \varphi_t - \int_\Omega v_0 \varphi(\cdot,0) = -\int_0^T \int_\Omega \nabla v \cdot \nabla \varphi - \beta \int_0^T \int_\Omega v \varphi + \alpha \int_0^T \int_\Omega u \varphi
\]
we observe that (52) ensures
\[
-\int_0^T \int_\Omega v \varphi_t \to -\int_0^T \int_\Omega v \varphi_t \quad \text{and} \quad -\beta \int_0^T \int_\Omega v \varphi \to -\beta \int_0^T \int_\Omega v \varphi,
\]
whereupon in view of (54) and (48), the dominated convergence theorem warrants that as \(\epsilon = \epsilon_j \to 0\)
\[
-\int_0^T \int_\Omega \nabla v \cdot \nabla \varphi \to -\int_0^T \int_\Omega \nabla v \cdot \nabla \varphi
\]
and
\[
\alpha \int_0^T \int_\Omega u \varphi \to \alpha \int_0^T \int_\Omega u \varphi.
\]
Therefore, (13) holds. Moreover, (56)-(58) would ensure (14) holds, thereby we complete the proof of theorem. \(\square\)

4. Boundedness and eventual smoothness.

4.1. Preservation of smallness. We next address the question how the local solution of (1) obtained from Lemma 2.2 becomes smooth and bounded. A cornerstone for all our subsequence analysis is obtained by constructing ordinary differential equations for the functional
\[
z(t) := \int_\Omega u_0^2(\cdot,t) + \int_\Omega |\nabla v(\cdot,t)|^4
\]
with any \(t \in (0,T_{\text{max}})\), thus a smallness condition relating the physically relevant total mass may ensure the local solution in fact becomes globally bounded. The main idea in our proof comes from [36].

Lemma 4.1. Let \(\Omega \subset \mathbb{R}^3\) be a convex bounded domain, and let \(\chi, \xi, \alpha, \beta, \gamma\) and \(\delta > 0\). Then for any \(\lambda \in (0,1)\) there exist \(C_{13} = C_{13}(\delta, \Omega) > 0\), \(C_{14} = C_{14}(\Omega) > 0\), \(C_{15} = C_{15}(\beta, \Omega) > 0\), \(C_{16} = C_{16}(\beta, \Omega) > 0\) and \(C_{17} = C_{17}(\beta, \Omega) > 0\) with the following property: if the initial data \((u_0,v_0)\) fulfills (6) and
\[
\chi \int_\Omega u_0^3 + C_{14} \int_\Omega |\nabla v_0|^4 \leq \chi \alpha^3 \leq C_{15} \lambda \xi \gamma,
\]
and if for some \(t_0 \in [0, \infty)\),
\[
\int_\Omega u_0^2(\cdot,t_0) + \frac{2 \chi^3}{3 \lambda \xi \gamma \alpha} \int_\Omega |\nabla v(\cdot,t_0)|^4 \leq \frac{C_{16} \lambda \xi \gamma}{\chi^3 \alpha^3},
\]
then the local solution of problem (1) satisfies
\[
\int_\Omega u^2 \leq \frac{C_{17} \lambda \xi \gamma}{\chi^3 \alpha^3} \quad \text{and} \quad \int_\Omega |\nabla v|^4 \leq \frac{C_{17} \lambda \xi \gamma}{\chi^3 \alpha^3} \quad \text{for all} \ t \in (t_0,T_{\text{max}}),
\]
where \(\overline{u}_0 = \frac{1}{|\Omega|} \int_\Omega u_0\).
Proof. We let $c_1 := \frac{2x^3}{3\lambda^{5/6}}$ and invoke the Poincaré inequality, the Gagliardo-Nirenberg inequality and Young’s inequality with any $\zeta > 0$ to obtain $c_2 > 0$, $c_3 > 0$ and $c_4 > 0$, dependent only on $\Omega$, such that

$$\|\varphi\|_{W^{1,2}(\Omega)} \leq c_2 \|\nabla \varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in W^{1,2}(\Omega) \text{ with } \int_{\Omega} \varphi = 0$$

(64)

and

$$\|\varphi\|_{L^2(\Omega)} \leq c_3 \|\varphi\|_{W^{1,2}(\Omega)}^{\frac{2}{3}} \|\varphi\|_{L^2(\Omega)}^{\frac{1}{3}} \quad \text{for all } \varphi \in W^{1,2}(\Omega)$$

(65)

as well as

$$\|\varphi\|_{L^2(\Omega)}^{\frac{2}{3}} \leq c_4 \|\varphi\|_{W^{1,2}(\Omega)} \|\varphi\|_{L^2(\Omega)}^{\frac{1}{3}}$$

and

$$\leq c_4 \zeta \|\varphi\|_{W^{1,2}(\Omega)} + \frac{27c_4}{256\zeta^2} \|\varphi\|_{L^2(\Omega)}^2 \quad \text{for all } \varphi \in W^{1,2}(\Omega) \text{ with } \zeta > 0.$$ (66)

Then given $c_5 \in (0, \min\{2, 4\beta\})$ and by Young’s inequality, there provides $c_6 = \frac{48\zeta^2}{c_5^2} > 0$ such that

$$32c_1^2c_5^2a^2b^2 \leq c_5 c_1 a^2 + c_6 c_1 b^6 \quad \text{for all } a \geq 0 \text{ and } b \geq 0.$$ (67)

Then denoting $c_7 := c_7(\beta, \Omega) = \min \left\{ \frac{1}{c_5^2}, 4\beta - c_3 \right\}, c_8 := c_8(\Omega) = \frac{\|\Omega\|}{c_5^2}, c_9 := c_9(\delta, \Omega) = \max \left\{ \frac{3c_4^4|\Omega|^3}{32}, \frac{3c_4^4|\Omega|^3}{328} \right\}$ and $c_{10} := c_{10}(\beta, \Omega) = \frac{c_5 c_7}{576c_5},$ we choose $u_0$ small such that

$$\left[ \frac{2c_0 c_1}{(1-\lambda)^3} \xi \gamma \pi_0^3 + c_8 \pi_0^2 \right] \chi^3 \alpha^3 \leq c_{10} \lambda \xi \gamma,$$ (68)

and define

$$c_{11} := c_{11}(\lambda, \xi, \gamma, \delta, \Omega) = \frac{3c_4^2 \xi \gamma (\delta + 1 - \lambda)|\Omega|^3}{328(1-\lambda)^3}$$ (69)

and

$$c_{12} := c_{12}(\lambda, \chi, \xi, \alpha, \beta, \gamma, \Omega) = \frac{(c_5 c_7) \xi \gamma}{384c_5^2 \chi^3 \alpha^3}.$$ (70)

By means of straightforward computation and using the third equation in (1), one verifies the identity

$$\frac{d}{dt} \int_{\Omega} u^2 = -2 \int_{\Omega} |\nabla u|^2 + 2\chi \int_{\Omega} u \nabla u \cdot \nabla v$$

$$+ \xi \delta \int_{\Omega} u^2 w - \xi \gamma \int_{\Omega} u^3 \quad \text{for all } t \in (0, T_{\max}),$$ (71)

where

$$2\chi \int_{\Omega} u \nabla u \cdot \nabla v \leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 + 2\chi^2 \int_{\Omega} u^2 |\nabla v|^2$$

$$\leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{3} \int_{\Omega} u^3$$

$$+ \frac{32}{3\lambda^2 \xi^2 \gamma^2} \int_{\Omega} |\nabla v|^6 \quad \text{for all } t \in (0, T_{\max})$$ (72)
and
\[ \xi \delta \int u^2 w \leq \frac{2\xi}{3} \int u^3 + \frac{\xi \delta^3}{3\gamma^2} \int w^3 \quad \text{for all} \quad t \in (0, T_{\text{max}}) \] 
(73)

by Young’s inequality. Next, due to the fact that \( \frac{\partial^2 \nabla v^2}{\partial \nu} \leq 0 \) on \( \partial \Omega \times (0, \infty) \) thanks to the convexity of \( \Omega \) and making use of the identity \( 2\nabla v \cdot \nabla \Delta v = \Delta |\nabla v|^2 - 2|D^2 v| \), then a standard testing procedure (see [34]) applied to the second equation guarantees

\[
\frac{d}{dt} \int_\Omega |\nabla v|^4 + 2 \int_\Omega |\nabla |\nabla v|^2|^2 + 4 \int_\Omega |\nabla v|^2 |D^2 v|^2 + 4\beta \int_\Omega |\nabla v|^4 \\
\leq 4\alpha \int_\Omega |\nabla v|^2 \nabla u \cdot \nabla v \\
\leq \frac{1}{2c_1} \int_\Omega |\nabla u|^2 + 8c_1 \alpha^2 \int_\Omega |\nabla v|^6 \quad \text{for all} \quad t \in (0, T_{\text{max}}),
\] 
(74)

where in combination with (71)-(74), this shows that

\[
\frac{d}{dt} \left( \int_\Omega u^2 + c_1 \int_\Omega |\nabla v|^4 \right) + \int_\Omega |\nabla u|^2 + 2c_1 \int_\Omega |\nabla |\nabla v|^2|^2 + 4c_1 \beta \int_\Omega |\nabla v|^4 \\
\leq \left( \frac{32\chi}{3\lambda\xi^2 \gamma^2} + 8c_1 \alpha^2 \right) \int_\Omega |\nabla v|^6 + \frac{\xi \delta^3}{3\gamma^2} \int_\Omega w^3 \quad \text{for all} \quad t \in (0, T_{\text{max}}).
\] 
(75)

Now denote \( \pi = \frac{1}{|\Omega|} \int_\Omega u = \frac{1}{|\Omega|} \int_\Omega u_0 \), using (64), (65) and (67),

\[
\int_\Omega |\nabla u|^2 \geq \frac{1}{c_2^2} \int_\Omega |u - \pi|^2 \\
= \frac{1}{c_2^2} \int_\Omega u^2 - \frac{|\Omega|}{c_2^2} \pi^2 \quad \text{for all} \quad t \in (0, T_{\text{max}}),
\] 
(76)

and

\[
\left( \frac{32\chi}{3\lambda\xi^2 \gamma^2} + 8c_1 \alpha^2 \right) \int_\Omega |\nabla v|^6 = 32c_1^2 \alpha^2 \| |\nabla v|^2\|_{L^3(\Omega)}^3 \\
\leq 32c_1^2 \alpha^2 \| |\nabla v|^2\|_{W^{1,2}(\Omega)}^2 \| |\nabla v|^2\|_{L^2(\Omega)}^2 \\
\leq c_5 c_1 \| |\nabla v|^2\|_{W^{1,2}(\Omega)} + c_6 c_5 \| |\nabla v|^2\|_{L^2(\Omega)}^6 \\
\leq c_5 c_1 \int_\Omega |\nabla v|^2|^2 + c_5 c_1 \int_\Omega |\nabla v|^4 \\
+ c_6 c_5 \left( \int_\Omega |\nabla v|^4 \right)^3 \quad \text{for all} \quad t \in (0, T_{\text{max}}).
\] 
(77)

Moreover, to estimate the term \( \frac{\xi \delta^3}{3\gamma^2} \int_\Omega w^3 \) on the right side of (75), multiplying the third equation \( 0 = \Delta w - \delta w + \gamma u \) by \( w^2 \) and integrating by parts, thanks to Young’s inequality we have

\[
\frac{8}{9} \int_\Omega |\nabla w^2|^2 + \delta \int_\Omega w^3 = \gamma \int_\Omega uw^2 \\
\leq \frac{\xi \delta^3}{3\gamma^2} \int_\Omega w^3 + \frac{2\delta}{3} \int_\Omega w^3 \quad \text{for all} \quad t \in (0, T_{\text{max}}),
\]
where we can estimate

$$\frac{8}{3} \int_{\Omega} |\nabla w^2|^2 \leq \frac{\gamma^3}{\delta^2} \int_{\Omega} u^3 \quad \text{for all} \quad t \in (0, T_{\text{max}}).$$

(78)

With \( \zeta \in (0, \frac{1}{c_4}) \) an application of (66) would imply that

$$\int_{\Omega} u^3 = \|w^2\|_{L^2(\Omega)}^2 \leq c_4 \|w^3\|_{L^2(\Omega)}^2 + \frac{27c_4}{256c_3^3} \|w^\frac{3}{2}\|_{L^2(\Omega)}^2$$

$$= c_4 \left( \int_{\Omega} |\nabla w^2|^2 + \int_{\Omega} u^3 \right) + \frac{27c_4}{256c_3^3} \left( \int_{\Omega} w \right)^3$$

for all \( t \in (0, T_{\text{max}}) \),

whereupon this entails that

$$\int_{\Omega} u^3 \leq \frac{c_4 \zeta}{1 - c_4 \zeta} \int_{\Omega} |\nabla w^2|^2 + \frac{27c_4}{256(1 - c_4 \zeta)c_3^3} \left( \int_{\Omega} w \right)^3$$

for all \( t \in (0, T_{\text{max}}) \),

and together with (78) shows that

$$\int_{\Omega} u^3 \leq \frac{3c_4 \zeta \delta^3}{8(1 - c_4 \zeta)\delta^2} \int_{\Omega} u^3 + \frac{27c_4}{256(1 - c_4 \zeta)c_3^3} \left( \int_{\Omega} w \right)^3$$

for all \( t \in (0, T_{\text{max}}) \). (79)

Now taking \( \zeta = \frac{8(1 - \lambda)}{3c_4 \delta^2 + 8c_4(1 - \lambda)} \) in (79), we use (24) to obtain

$$\frac{\xi \delta^3}{3\gamma^2} \int_{\Omega} u^3 \leq \frac{c_4 \xi \delta \delta^3}{8(1 - c_4 \zeta)} \int_{\Omega} u^3 + \frac{9c_4 \xi \delta^3}{256\gamma^2(1 - c_4 \zeta)c_3^3} \left( \int_{\Omega} w \right)^3$$

$$\leq \frac{c_4 \xi \delta \delta^3}{8(1 - c_4 \zeta)} \int_{\Omega} u^3 + \frac{9c_4 \xi \gamma}{256(1 - c_4 \zeta)c_3} \left( \int_{\Omega} u_0 \right)^3$$

$$= (1 - \lambda) \frac{\xi \gamma}{3} \int_{\Omega} u^3 + \frac{3c_4 \xi \gamma(\delta + 1 - \lambda)}{32\delta(1 - \lambda)^3} \left( \int_{\Omega} u_0 \right)^3$$

for all \( t \in (0, T_{\text{max}}) \).

(80)

We now only need to collect (75)-(77), (80) and the definition of \( c_7, c_8 \) and \( c_{11} \) to conclude that

$$\frac{d}{dt} \left( \int_{\Omega} u^2 + c_1 \int_{\Omega} |\nabla v|^4 \right) + c_7 \left( \int_{\Omega} u^2 + c_1 \int_{\Omega} |\nabla v|^4 \right)$$

$$\leq c_6 \xi \delta^3 \left( \int_{\Omega} |\nabla v|^4 \right)^3 + \frac{3c_4 \xi \gamma(\delta + 1 - \lambda)}{32\delta(1 - \lambda)^3} \left( \int_{\Omega} u_0 \right)^3$$

$$+ \frac{1}{c_2 |\Omega|} \left( \int_{\Omega} u_0 \right)^2$$

$$\leq c_6 \xi \delta^3 \left( \int_{\Omega} |\nabla v|^4 \right)^3 + c_{11} \overline{m}_0^3 + c_8 \overline{m}_0^2$$

for all \( t \in (0, T_{\text{max}}) \),

and further consider

$$z(t) := \int_{\Omega} u^2 + c_1 \int_{\Omega} |\nabla v|^4$$

for all \( t \in (0, T_{\text{max}}) \)

fulfills

$$z'(t) + c_7 z(t) \leq c_6 \xi \delta^3 z^3(t) + c_{11} \overline{m}_0^3 + c_8 \overline{m}_0^2$$

for all \( t \in (0, T_{\text{max}}) \).
Therefore,
\[ z'(t) \leq p(z(t)) \quad \text{for all} \quad t \in (0, T_{\text{max}}) \]
with
\[ p(s) := -c_7 s + c_6 c_1^2 s^3 + c_{11} \bar{u}_0^3 + c_8 \bar{u}_0^2, \quad s \geq 0. \]
Observe that \( p \) attains a local minimum at
\[ s_0 := \sqrt{\frac{c_7}{3c_6c_1^3}}, \]
where the corresponding minimal value can be picked by
\[ p(s_0) = -c_7 \sqrt{\frac{c_7}{3c_6c_1^3}} + c_6 c_1^2 \sqrt{\frac{c_7}{3c_6c_1^3}}^3 + c_{11} \bar{u}_0^3 + c_8 \bar{u}_0^2 \]
\[ = -\sqrt{\frac{4c_7^2}{27c_6c_1^3}} + c_{11} \bar{u}_0^3 + c_8 \bar{u}_0^2, \]
and would be nonnegative, since
\[ c_{11} \bar{u}_0^3 + c_8 \bar{u}_0^2 \leq \frac{2c_9 \xi \gamma}{(1 - \lambda)^3} \bar{u}_0^3 + c_8 \bar{u}_0^2 \]
\[ \leq \frac{c_{10} \lambda \xi \gamma}{\chi^3 \alpha^3} = \sqrt{\frac{4c_7^2}{27c_6c_1^3}} \]
by (68) and the definition of \( c_1, c_6 \) and \( c_{10} \). Then, assume that
\[ z(t_0) \leq s_0 = \sqrt{\frac{c_7}{3c_6c_1^3}} = c_{12} \]
with some \( t_0 \in [0, T_{\text{max}}) \) and \( c_{12} \) given by (70), then the comparison principle for ordinary differential equations therefore shows that
\[ y(t) \leq c_{13} \quad \text{for all} \quad t \in [t_0, T_{\text{max}}) \]
with \( c_{13} := \min \{ s > s_0 | p(s) = 0 \} \). As
\[ p(\sqrt{\frac{c_7}{c_6c_1^3}}) = -c_7 \sqrt{\frac{c_7}{c_6c_1^3}} + c_6 c_1^2 \sqrt{\frac{c_7}{c_6c_1^3}}^3 + c_{11} \bar{u}_0^3 + c_8 \bar{u}_0^2 \]
\[ = c_{11} \bar{u}_0^3 + c_8 \bar{u}_0^2 \geq 0, \]
this shows that \( c_{13} \leq \sqrt{\frac{c_7}{c_6c_1^3}} \) and finally finishes our proof.

The assertions in Lemma 4.1 would guarantee the boundedness of solution in any of the spaces \( L^k(\Omega) \times W^{1,2k}(\Omega) \times W^{1,2k}(\Omega) \) for any \( k > 2 \) in the three-dimensional setting.

**Lemma 4.2.** Let \( \Omega \subset \mathbb{R}^3 \) be a convex bounded domain, and let \( \chi, \xi, \alpha, \beta, \gamma \) and \( \delta > 0 \). Suppose that \((u_0, v_0)\) satisfies (6). If for some \( t_0 \in [0, T_{\text{max}}) \) the solution of (1) has the following property
\[ \int_\Omega u^2 + \int_\Omega |\nabla v|^4 \leq C_{18} \quad \text{for all} \quad t \in [t_0, T_{\text{max}}) \] (81)
with some $C_{18} > 0$, then for any $k > 2$ there exists $C_{19} = C_{19}(k, \chi, \xi, \alpha, \beta, \gamma, \delta, C_{18}, 
abla, u_0) > 0$ such that the solution fulfills

$$\int_\Omega u^k + \int_\Omega |\nabla u|^{2k} \leq C_{19} \text{ for all } t \in [t_0, T_{\max}).$$

(82)

**Proof.** From the assumption, we find $c_1 > 0$ such that

$$\int_\Omega u^2 \leq c_1 \text{ and } \int_\Omega |\nabla u|^4 \leq c_1 \text{ for all } t \in [t_0, T_{\max}).$$

(83)

Upon testing the first equation by $u^{k-1}$ and picking $\zeta$ small enough in (79), we invoke Young’s inequality and (24) to obtain

$$\frac{d}{dt} \int_\Omega u^k = -k(k-1) \int_\Omega u^{k-2} |\nabla u|^2 + \chi k(k-1) \int_\Omega u^{k-1} \nabla u \cdot \nabla v$$

$$- \xi k(k-1) \int_\Omega u^{k-1} \nabla u \cdot \nabla w$$

$$\leq - \frac{k(k-1)}{2} \int_\Omega u^{k-2} |\nabla u|^2 + \frac{\chi^2 k(k-1)}{2} \int_\Omega u^k |\nabla v|^2 + \xi \int_\Omega u^k w$$

$$- \xi \int_\Omega u^k + 1$$

$$\leq - \frac{2(k-1)}{k} \int_\Omega |\nabla u|^2 + \frac{\chi^2 k(k-1)}{2} \int_\Omega u^k |\nabla v|^2$$

$$- \frac{\xi \gamma(k-1)}{4} \int_\Omega u^k + 1 \text{ for all } t \in (t_0, T_{\max})$$

with $c_2 = c_2(k, \xi, \gamma, \delta) > 0$ and $c_3 = c_3(k, \xi, \gamma, \delta, u_0) > 0$. Similarly,

$$\frac{d}{dt} \int_\Omega |\nabla v|^{2k} \leq 2k \alpha \int_\Omega |\nabla |^{2k-2} \nabla u \cdot \nabla v$$

$$- 2k \beta \int_\Omega |\nabla v|^{2k-2} |\nabla v|^2$$

$$\leq 4k(k-1) \alpha \int_\Omega |\nabla v|^{2k-2} |\nabla v|^2$$

$$- 4k \beta \int_\Omega |\nabla v|^{2k-2} |\nabla v|^2$$

$$\leq 4k(k-1)^2 \alpha^2 \int_\Omega |\nabla v|^{2k-2}$$

$$- \frac{k}{3} \int_\Omega |\nabla v|^{2k-2} |\nabla v|^2$$

$$+ 3k \alpha^2 \int_\Omega |\nabla v|^{2k-2}$$

$$\leq (4k(k-1)^2 + 3k) \alpha^2 \int_\Omega |\nabla v|^{2k-2} \text{ for all } t \in (t_0, T_{\max})$$
by Young’s inequality and the fact that $|\Delta v|^2 \leq 3|D^2 v|^2$ and the convexity of $\Omega$. Therefore,

$$\frac{d}{dt} \left\{ \int_\Omega u^k + \int_\Omega |\nabla v|^{2k} \right\} + c_4 \int_\Omega |\nabla u^2|^2 + c_4 \int_\Omega |\nabla v|^{2k} + c_4 \int_\Omega |\nabla|\nabla v|^{k}\right\}$$

$$\leq c_5 \int_\Omega u^k |\nabla v|^2 + c_5 \int_\Omega u^2 |\nabla v|^{2k-2}$$

$$- \frac{\xi \gamma (k-1)}{4} \int_\Omega u^{k+1} + c_3 \quad \text{for all} \quad t \in (t_0, T_{\max})$$

for all $t \in (t_0, T_{\max})$ with some $c_3 = c_3(k, \beta) > 0$ and $c_5 = c_5(k, \chi, \alpha, \gamma) > 0$, where by Young’s inequality again,

$$c_5 \int_\Omega u^k |\nabla v|^2 + c_5 \int_\Omega u^2 |\nabla v|^{2k-2}$$

$$\leq \frac{\xi \gamma (k-1)}{8} \int_\Omega u^{k+1} + c_6 \int_\Omega |\nabla v|^{2(k+1)}$$

(85)

whereas since $\|\nabla v\|_2^2 = \int_\Omega |v|^4 \leq c_1$, the Gagliardo-Nirenberg inequality implies that there exist $c_8 = c_8(k, \chi, \alpha, \gamma, \Omega) > 0$ and $c_9 = c_9(k, \chi, \alpha, \gamma, c_1, c_4, \Omega) > 0$ such that

$$c_6 \int_\Omega |\nabla v|^{2(k+1)} \geq c_6 \|\nabla v\|_2^{2(k+1)} L_2^{2(k+1)}(\Omega)$$

$$\leq c_8 \|\nabla |\nabla v|^k\|_2^{\frac{2(k+1)}{k+1}} L_2^{\frac{2(k+1)}{k+1}}(\Omega)$$

$$\leq c_8 \|\nabla v|\|_2^{k+1} L_2^{k+1}(\Omega) + c_9 \|\nabla v|\|_2^{k+1} L_2^{k+1}(\Omega)$$

(87)

$$\leq c_9 \int_\Omega |\nabla v|^{k+1} \quad \text{for all} \quad t \in (t_0, T_{\max}),$$

and because of $\vartheta = \frac{3k}{4} - \frac{3k^2}{2}$

$$\frac{2(k + 1)}{k} \cdot \vartheta = \frac{3(k+1)}{2} - \frac{3}{\frac{3k}{4} - \frac{1}{2}} < 2.$$

In consequence, (84)-(87) prove $h'(t) := \int_\Omega u^k + \int_\Omega |\nabla v|^{2k}$ satisfies

$$h'(t) + c_4 h(t) \leq c_3 + c_7 + c_9 \quad \text{for all} \quad t \in (t_0, T_{\max}),$$

(88)

this establishes (82) by solving (88).

Now in light of above lemmas, we can achieve the following boundedness result through a standard Moser-type iteration.

**Lemma 4.3.** Let $n = 3$, and let $\chi, \xi, \alpha, \beta, \gamma$ and $\delta > 0$. For any $\lambda \in (0, 1)$ if there exists $C_{20} = C_{20}(\lambda, \beta, \delta) > 0$ such that $(u_0, v_0)$ satisfies (6) and

$$\int_\Omega u_0^2 \leq \frac{C_{20}}{\chi^2 \alpha^2} \quad \text{and} \quad \int_\Omega u_0^2 \leq \frac{C_{20} \xi^2 \gamma}{\chi^3 \alpha^3}$$

(89)
and

$$\int_{\Omega} |\nabla v_0|^4 \leq \frac{C_{20} \xi^2 \gamma^2}{\chi^6 \alpha^2},$$  \hspace{1cm} (90)$$

then there exists $C_{21} > 0$ such that the solution $(u, v, w)$ fulfills

$$||u(\cdot,t)||_{L^\infty(\Omega)} + ||v(\cdot,t)||_{W^{1,\infty}(\Omega)} + ||w(\cdot,t)||_{W^{1,\infty}(\Omega)} \leq C_{21} \text{ for all } t \in (0, T_{\text{max}}).$$  \hspace{1cm} (91)$$

Proof. Fixing $\lambda \in (0,1)$, we take $C_{13}, C_{14}, C_{15}$ and $C_{16} > 0$ depending on $\beta, \delta$ and $\Omega$ from Lemma 4.1, and choose $C_{20} > 0$ small enough fulfilling

$$C_{20}^2 \leq \frac{C_{15} |\Omega|^{\frac{3}{2}}}{2C_{13}} \lambda (1 - \lambda)^3,$$  \hspace{1cm} (92)$$

and

$$C_{20} \leq \min \left\{ \frac{C_{15} |\Omega| \lambda}{2C_{14}}, \frac{C_{16} \lambda}{2}, \frac{3C_{16} \lambda^2}{4} \right\}.  \hspace{1cm} (93)$$

Now since the Cauchy-Schwarz inequality ensures $\int_{\Omega} u_0 \leq (\int_{\Omega} u_0^2)^{\frac{1}{2}} ||\Omega||\frac{1}{2}$, then we infer from (89), (92) and (93) that

$$\frac{C_{13}}{(1 - \lambda)^3} \xi \gamma u_0^3 \leq \frac{C_{13}}{(1 - \lambda)^3} \xi \gamma \left\{ \frac{1}{|\Omega|^\frac{1}{2}} \left( \int_{\Omega} u_0^2 \right)^{\frac{1}{2}} \right\}^3 \leq \frac{C_{15} \lambda \xi \gamma}{2 \chi^3 \alpha^3},$$

and

$$\frac{C_{14} u_0^2}{|\Omega|} \int_{\Omega} u_0^2 \leq \frac{C_{14} C_{20} \xi \gamma}{|\Omega|} \chi^{-1} \alpha^3 \leq \frac{C_{15} \lambda \xi \gamma}{2 \chi^3 \alpha^3},$$

where we directly obtain

$$\frac{C_{13}}{(1 - \lambda)^3} \xi \gamma u_0^3 + C_{14} u_0^2 \leq \frac{C_{15} \lambda \xi \gamma}{\chi^3 \alpha^3}.  \hspace{1cm} (94)$$

On the other hand, the second inequality in (89), (90), and (93) imply that

$$\int_{\Omega} u_0^2 + \frac{2 \chi^3}{3 \lambda \xi \gamma \alpha} \int_{\Omega} |\nabla v_0|^4 \leq \frac{C_{20} \xi \gamma}{\chi^3 \alpha^3} + \frac{2C_{20} \xi \gamma}{3 \lambda \chi^3 \alpha^3} \leq \frac{C_{20} \xi \gamma}{\chi^3 \alpha^3} + \frac{C_{16} \lambda \xi \gamma}{2 \chi^3 \alpha^3} \leq \frac{C_{15} \lambda \xi \gamma}{2 \chi^3 \alpha^3},  \hspace{1cm} (95)$$

whence we may employ Lemma 4.1 by considering (94) and (95) together to find $c_1 > 0$ such that

$$\int_{\Omega} u^2 + \int_{\Omega} |\nabla v|^4 \leq c_1 \text{ for all } t \in (0, T_{\text{max}}).$$
Then it follows from Lemma 4.2, applied to $t_0 = 0$ and arbitrary $k > 2$, yields $c_2 > 0$ such that
\[
\int_\Omega u^k + \int_\Omega |\nabla v|^{2k} \leq c_2 \text{ for all } t \in (0, T_{\text{max}}),
\] (96)
whereupon (91) finally can be derived from (96) by a Moser-type iteration in conjunction with standard parabolic and elliptic regularity arguments. \hfill \Box

**Proof of Theorem 1.2.** The conclusion in Theorem 1.2 follows from Lemmas 2.2 and 4.3. \hfill \Box

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