Area (or Entropy) Products in Modified Gravity and Kerr-MOG/CFT Correspondence

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Abstract

We examine the thermodynamic features of inner and outer horizons of modified gravity (MOG) and its consequences on the holographic duality. We derive the thermodynamic product relations for this gravity. We consider both spherically symmetric solutions and axisymmetric solutions of MOG. We find that the area product formula for both cases is not mass-independent because they depend on the ADM mass parameter while in Einstein gravity this formula is mass-independent (universal). We also explicitly verify the first law which is fulfilled at the inner horizon (IH) as well as at the outer horizon (OH). We derive thermodynamic products and sums for this kind of gravity. We further derive the Smarr like mass formula for this kind of black hole (BH) in MOG. Moreover, we derive the area bound for both the horizons. Furthermore, we show that the central charges of the left and right moving sectors are the same via universal thermodynamic relations. We also discuss the most important result of the Kerr-MOG/CFT correspondence. We derive the central charges for Kerr-MOG BH which is $c_L = 12J$ and it is similar to Kerr BH. We also derive the dimensionless temperature of a extreme Kerr-MOG BH which is $T_L = \frac{\alpha + 2}{4\pi \sqrt{1 + \alpha}}$, where $\alpha$ is a MOG parameter. This is actually dual CFT temperature of the Frolov-Thorne thermal vacuum state. In the limit $\alpha = 0$, we find the dimensionless temperature of Kerr BH. Consequently, Cardy formula gives us microscopic entropy for extreme Kerr-MOG BH, $S_{\text{micro}} = \frac{\alpha + 2}{4\pi} \pi J$ for the CFT which is completely in agreement with macroscopic Bekenstein-Hawking entropy. Therefore we may conjecture that in the extremal limit the Kerr-MOG BH is holographically dual to a chiral 2D CFT with central charge $c_L = 12J$. Finally, we derive the mass-independent area (or entropy) product relations for regular MOG BH.

1 Introduction

Perhaps, a BH is the most fascinating as well as thermal object of the universe. A thermal object in the sense that it has both temperature and entropy. Again this entropy is proportional to the area of the event horizon (EH) and Cauchy horizon (CH). They are defined as

$$S_{\pm} = \frac{A_{\pm}}{4}.$$  \hspace{1cm} (1)

where, $S_{\pm}$ is the so-called the Bekenstein-Hawking entropy (in units in which $G = h = c = k_B = 1$) and $A_{\pm}$ is the area of both the horizons. Similarly, the temperature is proportional to the surface gravity of EH ($\mathcal{H}^+$) and CH ($\mathcal{H}^-$). They are defined as

$$T_{\pm} = \frac{\kappa_{\pm}}{2\pi}.$$ \hspace{1cm} (2)

where $T_{\pm}$ is the so-called the Hawking temperature computed on $\mathcal{H}^\pm$ and $\kappa_{\pm}$ is defined as the surface gravity of the BH computed on $\mathcal{H}^\pm$. 

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It is established that in terms of the above quantities, the first law of BH thermodynamics for both horizons ($\mathcal{H}^\pm$) becomes

$$dM = \pm \kappa_\pm dA_\pm + \Omega_\pm dJ + \Phi_\pm dQ.$$ (3)

where $\Omega_\pm = \frac{\partial M}{\partial J}$ and $\Phi_\pm = \frac{\partial M}{\partial Q}$.

In recent times, the area (or entropy) product formula of $\mathcal{H}^\pm$ has been a fascinating topic of research due to the seminal work of Ansorg and Hennig [3]. In their work, the authors derived that for a Kerr-Newman (KN) class of BH in the Einstein-Maxwell (EM) gravity the area (or entropy) product formula of $\mathcal{H}^\pm$ becomes

$$A_+ A_- = 64\pi^2 \left( J^2 + \frac{Q^4}{4} \right).$$ (4)

This is a universal product relation in the sense that it is independent of the Arnowitt-Deser-Misner (ADM) mass parameter [See also [6, 23, 22, 21, 24, 25, 26, 27, 28]].

On the other hand, Cvetić et al. [4] proposed that for BHs in $D = 4$ and $D \geq 4$ the area (or entropy) product formulas obey the quantization formula

$$A_+ A_- = \left( 8\pi \ell_{pl}^2 \right)^2 N, \quad N \in \mathbb{N}. $$ (5)

where $\ell_{pl}$ is the Planck length [5, 7, 8, 9, 10].

It should be emphasized that one of the major achievements in string theory is that of holographic duality which connects quantum theory of gravity to the quantum field theory without any nomenclature of Einstein’s general theory of gravity [12]. For the extreme class of BH in string theory, Strominger and Vafa [13] have successfully given the idea of the microscopic origin of the Bekenstein-Hawking entropy but up to date no idea has been found in the literature for the non-extreme class of BH. It may be noted that a Kerr BH is dual to a CFT for $AdS_3$ BH, which was introduced first by Brown and Henneaux [17]. Guica et al. [18] (See also [19]) first demonstrated the Kerr/CFT correspondence by using the near-horizon limiting procedure. They in fact computed the central charges for the Kerr BH by using the asymptotic symmetry group (ASG) method by imposing some boundary conditions. They also proved that by using some boundary conditions, the extreme Kerr BH has a feature which is dual to a chiral CFT. When one takes the extreme limit for Kerr BH, one obtains the Frolov-Thorne vacuum [41] state for extreme Kerr BH which produces a thermal state with a temperature $T_L = \frac{1}{2\pi r_c}$. They also microscopically computed the entropy for an extremal Kerr BH by using the Cardy formula and proved that it generates the macroscopic Bekenstein-Hawking entropy formula. We derive this result for the Kerr-MOG BH by using thermodynamic procedure.

However, in this work we wish to examine the area (or entropy) product relations, broadly speaking we would like to investigate the thermodynamic properties of inner and outer horizons of scalar-tensor-vector gravity (STVG) or MOG [29]. We have considered both the spherically symmetric solution and the axisymmetric solution of MOG. We show that the area (or entropy) product formula for both the situations is mass-dependent while in Einstein gravity (EG) this formula is mass-independent. We also explicitly verify the first law of BH mechanics which is completely in agreement with both the inner horizon (IH) as well as the outer horizon (OH). We further derive the other thermodynamic relations like thermodynamic products and sums. Moreover, we derive the Smarr type of mass formula for this class of BH in MOG. Also, we derive the area (or entropy) bound for both the horizons. Finally, we show that the central charges of the left and right moving sectors of the dual CFT in MOG/CFT correspondence are the same by using universal thermodynamic relations.

It should be noted the argument made in [10, 11], “the area product being mass-independent is equivalent to the relation $T_+ S_+ = T_- S_-$”, where $T_\pm$ and $S_\pm$ are the Hawking temperature and the entropies of $\mathcal{H}^\pm$. We show that this argument is violated in case of a Kerr-MOG BH. Because

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1 The ASG is a one type of conformal group.
for a Kerr-MOG BH, the relation \( T_+ S_+ = -T_- S_- \) is satisfied while the area product is not mass-independent. This is one of the key predictions for a Kerr-MOG BH.

Furthermore, we have derived the central charges of the left moving sectors and right moving sectors of dual CFT in Kerr-MOG/CFT correspondence. We found that the central charges are the same in both sectors i.e. \( c_L = c_R = 12J \). We also computed the dimensionless temperature of microscopic CFT, which is completely in agreement with the ones computed from hidden conformal symmetry. In the extremal limit, we determine the temperature of Frolov-Thorne vacuum state which is the so-called dimensionless temperature \( T_L = \frac{1}{4\pi} \frac{\sqrt{1+\alpha}}{\alpha} \). When one takes the limit \( \alpha = 0 \), one obtains the dimensionless temperature of the Kerr BH. Using the Cardy formula, we compute microscopic entropy for an extreme Kerr-MOG BH, \( S_{\text{micro}} = \frac{\alpha^{3/2}}{\sqrt{\alpha}} \pi J \) for the CFT, which is exactly in agreement with the macroscopic Bekenstein-Hawking entropy. Thus we can conjecture that an extreme Kerr-MOG BH is holographically dual to a chiral 2D CFT with the central charge \( c_L = 12J \).

It is well known that general relativity is the most successful and well examined theory of gravity. However, the STVG is a modification of laws of gravitation on a length scale where Newtonian gravity or general gravity have not been explicitly examined. One such type of gravity is called MOG. This framework correctly explains the observations of the solar system \[29\], the rotation curves of the cluster of galaxies and the dynamics of galaxies clusters. There has been no needs to the idea of dark matter \[30, 31, 32, 33\].

One characteristic is that the STVG is a formulation of MOG where the fields are scalar fields and massive vector fields, and it can be used to describe the growth of the structure and the power spectrum of matter and the acoustical power spectrum of the cosmic microwave background (CMB) data.

Now we shortly review the modified action for the STVG \[29, 36\] which is given by

\[
\mathcal{I} = \mathcal{I}_G + \mathcal{I}_V + \mathcal{I}_S + \mathcal{I}_M, \tag{6}
\]

where \( \mathcal{I}_G \) is the Einstein-Hilbert action for gravity, \( \mathcal{I}_V \) is the action for massive vector field \( \phi_a \), \( \mathcal{I}_S \) is the action for scalar fields and \( \mathcal{I}_M \) is the action for pressure less matter. They are defined as

\[
\mathcal{I}_G = \frac{1}{16\pi G} \int (R + 2\Lambda) \sqrt{-g} \, d^4x, \tag{7}
\]

\[
\mathcal{I}_V = -\frac{1}{4\pi} \int \left[ \mathcal{K} + V(\phi_a) \right] \sqrt{-g} \, d^4x, \tag{8}
\]

\[
\mathcal{I}_S = \int \frac{1}{G} \left[ \frac{1}{2} g^{\alpha\beta} \left( \nabla_a G \nabla_b G \frac{G}{G^2} + \nabla_a \mu \nabla_b \mu \frac{\mu}{\mu^2} \right) - \frac{V_G(G)}{G^2} - \frac{V_\mu(\mu)}{G^2} \right] \sqrt{-g} \, d^4x, \tag{9}
\]

and

\[
\mathcal{I}_M = -\int \left[ \rho \sqrt{u^a u_a} + Q u^a \phi_a \right] \sqrt{-g} \, d^4x + J^a \phi_a \tag{10}
\]

where \( R = g^{ab} R_{ab} \), \( g = \det(g_{ab}) \), \( \nabla_a \) is the covariant derivative corresponds to the metric \( g_{ab} \). The potential, \( V(\phi_a) \) indicates that the potentials are associated with the vector field \( \phi_a \) and \( V_G(G), V_\mu(\mu) \) denote the potentials with respect to the scalar fields, \( G \) and \( \mu \), respectively.

The MOG theory \[33\] admits the parameters \( G, \omega \) and \( \mu \). Where \( G \) is the Gravitational constant, \( \omega \) is the coupling constant and \( \mu \) is the mass of the massive vector field which is vary with space & time. The coupling constant \( \omega \) in the STVG action in \[30\] is the scalar field and it should be treated as a constant value thus hereafter we assume throughout the work \( \omega = 1 \).
We have also used the value $c = 1$. Finally, the kinetic term with respect to the field $\phi_a$ is defined by

$$K = \frac{1}{4} B^{ab} B_{ab}. \quad (11)$$

where $B_{ab} = \partial_a \phi_b - \partial_b \phi_a$.

Now the field equation for the STVG \[29\] is

$$G_{ab} - \Lambda g_{ab} + Q_{ab} = -8\pi G T_{ab}. \quad (12)$$

where $Q_{ab} = G(\nabla^a \nabla_b \Theta g_{ab} - \nabla_a \nabla_b)$ and $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$, $\Lambda$ is the cosmological constant and $\Theta = \frac{1}{G}$. We do not write the explicit field equations for $B^{ab}$ because it can be found in \[36\]. Now we should define the covariant current density as

$$J^a = \kappa T^{mab} u_b. \quad (13)$$

where $T^{mab}$ is the energy-momentum tensor for matter, $\kappa = \sqrt{\alpha G N}$, $\alpha = G - G_N G_N$ is the scalar field, $G_N$ is the Newtonian constant and $u^a$ is the four velocity.

The perfect fluid energy-momentum tensor for matter is defined as

$$\mathcal{T}^{mab} = (\rho_m + p_m) u^a u^b - p_m g^{ab}. \quad (14)$$

where $\rho_m$ and $p_m$ are correspond to the density and pressure of matter respectively. Using the normalization condition of the four velocity and with the help of Eq.(13), and Eq. (14) one obtains

$$J^a = \kappa \rho_m u^b. \quad (15)$$

Whereas the gravitational source charge is defined as

$$Q = \kappa \int J_0(x) d^3 x. \quad (16)$$

The values $Q = \sqrt{\alpha G_N} M$ and $G = G_N (1 + \alpha)$ are derived from the weak field approximation\[8\] of the STVG field equations.

One could study the geodesic motion by using the geodesic equation of a test particle which should read, for the time-like particle

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = \frac{q}{m} B^{ad} \frac{dx^d}{d\tau}. \quad (17)$$

and for the light-like particle

$$\frac{d^2 x^a}{d\lambda^2} + \Gamma^a_{bc} \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda} = 0. \quad (18)$$

where $\lambda$ is an affine parameter and $\Gamma^a_{bc}$ denotes the Christoffel symbols.

The parameters $m$ and $q = \sqrt{\alpha G_N} m$ denote test particle mass and gravitational charge respectively. Let us choose the potential for the massive vector field, $\phi_a$ of the form

$$V(\phi_a) = -\frac{1}{2} \mu^2 \phi^a \phi_a, \quad (19)$$

where $\mu$ is the mass of massive vector field and $\partial_b \phi^b = 0$, and $\phi_a = (\phi_0, \phi_i)(i = 1, 2, 3)$. Therefore the radial source free static, spherically symmetric solutions of $\phi_0$ could be obtained from the following equation

$$\frac{d^2 \phi_0}{dr^2} + \frac{2}{r} \frac{d\phi_0}{dr} - \mu^2 \phi_0 = 0. \quad (20)$$

\[The perturbation of the metric around the Minkowski metric $\eta_{ab}$ can be written in the form $g_{ab} = \eta_{ab} + \lambda h_{ab}$.
The solution of the above equation is $\phi_0(r) = -Q e^{\frac{\mu + \eta}{2}}$ and the gravitational charge $Q = \sqrt{\alpha G_N M}$, and $M$ is the mass of the source particle. For matter-free MOG, one has to set the energy momentum tensor equal to zero as well as the cosmological constant and then one obtains the energy momentum tensor for the vector field $\phi_a$ which is briefly discussed in the following section.

The manuscript is organized as follows. In Sec. 2, we study the thermodynamic properties of a static, Spherically symmetric MOG BH. In Sec. 3, we analyze the thermodynamic properties of the Kerr-MOG BH. In Sec. 4, we give most important results of the Kerr-MOG/CFT correspondence. Sec. 5 is devoted to studying the thermodynamic properties of a regular MOG BH. Finally, in Sec. 6, we summarize the results.

2 Area Product formula in a static, Spherically symmetric MOG BH

To derive the static, spherically symmetric solution of a MOG BH we shall use the modified Einstein’s field equations which can be written as \[ G_{ab} = -8\pi G T_{\phi ab} \] (21)

Since we are working with matter-free MOG field equations, the the energy-momentum tensor of the matter sector, $T_{mab} = 0$ and the energy-momentum tensor for the massive vector field, $\phi_a$ read

$$T_{\phi ab} = -\frac{1}{4\pi} \left( B^a_{\ b} B^b_{\ c} - \frac{1}{4} g_{ab} B^c_{\ d} B_{\ df} \right)$$ (22)

where $B_{ab}$ is previously defined and $\phi_a$ is the vector field with source charge, $Q = \sqrt{\alpha G_N M}$. We also require other vacuum field equations:

$$\nabla_b B^{ab} = \frac{1}{\sqrt{-g}} \partial_a \left( \sqrt{-g} B^{ab} \right) = 0 \ ,$$ (23)

and

$$\nabla_a B_{bc} + \nabla_b B_{ca} + \nabla_c B_{ab} = 0 \ .$$ (24)

where $\nabla_a$ denotes the covariant derivative with respect to the metric tensor $g_{ab}$. Now, we assume the static, spherically symmetric metric whose form is given by

$$ds^2 = e^{\eta} dt^2 - e^{\mu} dr^2 - r^2 d\Omega^2 \ .$$ (25)

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. For the static solution, $\phi_0 \neq 0$ and $\phi_1 = \phi_2 = \phi_3 = 0$. Also from Eq. 23 one obtains

$$\partial_r \left( \sqrt{-g} B^{0r} \right) = -\sin \theta \partial_r \left( e^{\frac{\mu + \eta}{2}} r^2 \phi_0' \right) = 0 \ ,$$ (26)

where $\phi_0' = \partial_r \phi_0$. After integration, one finds

$$\phi_0' = e^{\frac{\mu + \eta}{2}} \frac{Q}{r^2} \ ,$$ (27)

where $Q$ is the gravitational source charge of $B_{ab}$. The components of the energy-momentum tensor for the massive vector field are

$$T^0_{\phi 0} = T^1_{\phi 1} = -T^2_{\phi 2} = -T^3_{\phi 3} = \frac{1}{2} e^{(\mu - \eta)} (\phi_0')^2 = \frac{Q^2}{8\pi r^4} \ .$$ (28)

The main interesting feature of MOG is that the weak field acceleration law is attractive and repulsive. The Yukawa contribution due to a spin 1 (graviton) is a repulsive force, while the scalar spin 0 (graviton) described by the scalar field $G$ is an attractive one.
Now let us put $\lambda = e^\eta$ and solving Eqs. (21), one can obtain $\eta' = -\mu'$ and

$$
\lambda + \eta' = 1 - \frac{GQ^2}{r^2},
$$

$$
\eta' = -\mu' + \frac{GQ^2}{r} - 2GM.
$$

where $2GM$ is an integration constant. Substituting these values in Eq. (25), one obtains the gravitational field metric

$$
ds^2 = \left[1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right] dt^2 - \frac{dr^2}{\left[1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right]} - r^2 d\Omega^2.
$$

where $G = G_N(1 + \alpha)$. Interestingly, the form of this metric is similar to the static, spherically symmetric Reissner-Nordström solution of a charged body and the value of $Q > 0$.

Now by postulating that this independent charge $Q$ is proportional to the mass of the source particle i.e. $Q = \kappa M$, we also have $\kappa = \pm \sqrt{\alpha G_N}$, of which we have previously defined the value of $\alpha$. For physical stable stars, galaxies etc. and to maintain the repulsive gravitational Yukawa force we choose the value of independent charge to be positive i.e. $Q > 0$. Therefore the metric of Eq. (31) becomes

$$
ds^2 = \left[1 - \frac{2GM}{r} + \frac{\alpha G G_N M^2}{r^2}\right] dt^2 - \frac{dr^2}{\left[1 - \frac{2GM}{r} + \frac{\alpha G G_N M^2}{r^2}\right]} - r^2 d\Omega^2.
$$

After substituting the value of $G = G_N(1 + \alpha)$, one obtains the static, spherically symmetric MOG BH

$$
ds^2 = \left[1 - \frac{2G_N(1 + \alpha) M}{r} + \frac{G_N^2 M^2 \alpha (1 + \alpha)}{r^2}\right] dt^2 - \frac{dr^2}{\left[1 - \frac{2G_N(1 + \alpha) M}{r} + \frac{G_N^2 M^2 \alpha (1 + \alpha)}{r^2}\right]} - r^2 d\Omega^2.
$$

where $G_N$ is the modified Newtonian constant which is related to the Newtonian constant via the relation $G = G_N(1 + \alpha)$ and the modified charge parameter is $Q = \sqrt{\alpha G_N} M$, where $\alpha$ is a free parameter. The above metric can be obtained by putting these values in the usual Reissner-Nordström BH solution.

The BH has both EH and CH situated at

$$
r_\pm = G_N M \left(1 + \alpha \pm \sqrt{1 + \alpha}\right).
$$

$r_+$ is called EH and $r_-$ is called CH. The MOG BH does not possess a naked singularity. The area of both horizons ($A_\pm$) for the MOG BH is

$$
A_\pm = 4\pi r_\pm^2 = 4\pi (1 + \alpha) M^2 G_N^2 \left[\alpha + 2(1 + \alpha \pm \sqrt{1 + \alpha})\right].
$$

The Bekenstein-Hawking entropy of $A_\pm$ (in units in which $\hbar = c = 1$) should read

$$
S_\pm = \frac{A_\pm}{4G} = \pi G_N M^2 \left[\alpha + 2(1 + \alpha \pm \sqrt{1 + \alpha})\right].
$$

The Hawking temperature of $A_\pm$ reads

$$
T_\pm = \frac{\kappa_\pm}{2\pi} = \pm \frac{1}{2\pi G_N M \sqrt{1 + \alpha} \left(1 \pm \sqrt{1 + \alpha}\right)^2}.
$$
where $\kappa_{\pm}$ is called surface gravity of $\mathcal{H}^{\pm}$.

The Smarr formula is derived to be

$$M^2 = \frac{\mathcal{A}_{\pm}}{4\pi(1 + \alpha)G_N^2[\alpha + 2(1 + \alpha \pm \sqrt{1 + \alpha})]}.$$  \hspace{1cm} (38)

and the first law is satisfied to be

$$\pm dM = \sqrt{1 + \alpha}T_{\pm}dS_{\pm}.$$  \hspace{1cm} (39)

The specific heat for a MOG BH is given by

$$C_{\pm} = \frac{\partial M}{\partial T_{\pm}} = -2\pi G_N M^2 \sqrt{1 + \alpha} (1 \pm \sqrt{1 + \alpha})(\sqrt{1 + \alpha} \pm 1).$$  \hspace{1cm} (40)

The Komar energy is calculated to be

$$E_{\pm} = 2T_{\pm}S_{\pm} = \frac{\pi M^3 G_N^3 \alpha^2 (1 + \alpha)^{\frac{3}{2}}}{(1 \pm \sqrt{1 + \alpha})(\sqrt{1 + \alpha} \pm 1)}.$$  \hspace{1cm} (41)

Finally, the Gibbs free energy is given by

$$G_{\pm} = M - T_{\pm}S_{\pm} = M \left[1 - \frac{\pi M^2 G_N^3 \alpha^2 (1 + \alpha)^{\frac{3}{2}}}{2(1 \pm \sqrt{1 + \alpha})(\sqrt{1 + \alpha} \pm 1)}\right].$$  \hspace{1cm} (42)

The main interest here is to examine the area product formula of $\mathcal{H}^{\pm}$ whether it is mass independent or not. Thus, the product is computed to be

$$\mathcal{A}_{\pm} \mathcal{A}_{\mp} = 16\pi^2 \alpha^2 (1 + \alpha)^2 M^4 G_N^4.$$  \hspace{1cm} (43)

It implies that the area (or entropy ) product formula does depend on the mass parameter thus the area (or entropy) product formula for the MOG BH in spherically symmetric cases is not universal. It is also clearly evident that all the other thermodynamic products are mass dependent. In Fig. 1, we show how the area product varies with the free parameter in the case of a spherically symmetric MOG BH.

Now the irreducible mass product of $\mathcal{H}^{\pm}$ for the MOG BH is

$$M_{\text{irr},+} M_{\text{irr},-} = \frac{M^2 G_N^2 \alpha (1 + \alpha)}{4}. \hspace{1cm} (44)$$

where the irreducible mass $M_{\text{irr},\pm}$ of $\mathcal{H}^{\pm}$ is defined by

$$M_{\text{irr},\pm} = \frac{\sqrt{\mathcal{A}_{\pm}}}{16\pi}.$$  \hspace{1cm} (45)

The other relevant thermodynamic relations are

$$\frac{1}{\mathcal{A}_{\pm}} + \frac{1}{\mathcal{A}_{\mp}} = \frac{8\pi M^2 G_N^2 (1 + \alpha)(2 + \alpha)}{(2 + \alpha)}, \quad \mathcal{A}_{\pm} - \mathcal{A}_{\mp} = 8\pi (1 + \alpha)MG_N T_{\pm} \mathcal{A}_{\pm}$$

$$T_{\pm} S_{\pm} + T_{-\pm} S_{-\pm} = 0.$$  \hspace{1cm} (46)

The above thermodynamic relations of all the horizons may be used to determine the MOG BH entropy in terms of the Cardy formula which provides some evidence for a BH/CFT correspondence [7]. The most important relation among all the above relations is

$$T_{\pm} S_{\pm} + T_{-\pm} S_{-\pm} = 0$$  \hspace{1cm} (47)
Figure 1: The figure shows the variation of area product of $\mathcal{H}^\pm$ with free parameter $\alpha$ for spherically symmetric MOG BH with $M = G_N = 1$.

which we may called as a *universal* relation because it indicates that the central charges in left and right moving sectors of dual CFT in MOG/CFT correspondence are the same for two horizons BH in MOG.

Moreover, using the above relations we are now able to compute the area bound for this BH following the work of Xu et al. [20]. Since $r_+ \geq r_-$ thus $A_+ \geq A_- \geq 0$. Now the area product relation gives

$$A_+ \geq \sqrt{A_+ A_-} = 4\pi M^2 G_N^2 \alpha (1 + \alpha) \geq A_- \ .$$

(48)

and the area sum gives

$$8\pi M^2 G_N^2 (1 + \alpha)(2 + \alpha)$$

$$= A_+ + A_- \geq A_+ \geq \frac{A_+ + A_-}{2} = 4\pi M^2 G_N^2 (1 + \alpha)(2 + \alpha) \geq A_- \ .$$

(49)

Therefore, one obtains the area bound for $\mathcal{H}^+$

$$4\pi M^2 G_N^2 (1 + \alpha)(2 + \alpha) \leq A_+ \leq 8\pi M^2 G_N^2 (1 + \alpha)(2 + \alpha) \ .$$

(50)

and the area bound for $\mathcal{H}^-$ becomes

$$0 \leq A_- \leq 4\pi M^2 G_N^2 \alpha (1 + \alpha) \ .$$

(51)

From these relations one can easily derive the entropy bound for this BH. For completeness, we derive the irreducible mass bound for the MOG BH in spherically symmetric cases. For $\mathcal{H}^+$, it is given by

$$\frac{MG_N}{2\sqrt{(1 + \alpha)(2 + \alpha)}} \leq M_{\text{irr,}+} \leq \frac{MG_N}{\sqrt{2}} \sqrt{(1 + \alpha)(2 + \alpha)} \ .$$

(52)

and for $\mathcal{H}^-$, it is given by

$$0 \leq M_{\text{irr,}-} \leq \frac{MG_N}{2\sqrt{\alpha(1 + \alpha)}} \ .$$

(53)


3 Area Product formula in an axisymmetric MOG BH

We have seen that in the previous section, the Eq. (33) has the same form as the RN BH solution in the Einstein-Maxwell gravity when \( Q = \sqrt{\alpha G_N} M \) which is just a postulation mentioned in [36] thus one can easily obtain the metric for a Kerr-MOG BH by putting the above unique relation in the Kerr-Newman metric [37] and the metric becomes [36]

\[
ds^2 = \frac{\Delta}{\rho^2} [dt - a \sin^2 \theta d\phi]^2 - \frac{\sin^2 \theta}{\rho^2} \left[(r^2 + a^2) d\phi - a dt\right]^2 - \rho^2 \left[\frac{dr^2}{\Delta} + d\theta^2\right].
\]

where

\[
a \equiv \frac{J}{M}, \quad \rho^2 \equiv r^2 + a^2 \cos^2 \theta
\]

\[
\Delta \equiv r^2 - 2Mr + a^2 + G_N^2 \alpha (1 + \alpha) M^2 \equiv (r - r_+)(r - r_-)
\]

It describes the BH with horizon radii:

\[
r_{\pm} = G_N (1 + \alpha) M \left[1 \pm \sqrt{1 - \frac{a^2}{(1 + \alpha)^2 M^2 G_N^2} - \frac{\alpha}{1 + \alpha}}\right].
\]

It should be noted that when \( \alpha = 0 \), one obtains the usual Kerr BH. The above metric is equivalent to the Kerr-Newman BH provided that the gravitational charge \( Q = \sqrt{\alpha G_N} M \).

The area of \( H^\pm \) for the Kerr-MOG BH is

\[
A_{\pm} = 4\pi (1 + \alpha) M^2 G_N^2 \left[(2 + \alpha) \pm 2(1 + \alpha) \sqrt{1 - \frac{a^2}{(1 + \alpha)^2 M^2 G_N^2} - \frac{\alpha}{1 + \alpha}}\right].
\]

Similarly, the entropy of \( H^\pm \) for the Kerr-MOG BH is

\[
S_{\pm} = \pi M^2 G_N \left[(2 + \alpha) \pm 2(1 + \alpha) \sqrt{1 - \frac{a^2}{(1 + \alpha)^2 M^2 G_N^2} - \frac{\alpha}{1 + \alpha}}\right].
\]

The angular velocity of \( H^\pm \) computed on the horizon is

\[
\Omega_{\pm} = \frac{a}{(1 + \alpha) M^2 G_N^2 (2 + \alpha) \pm 2(1 + \alpha) \sqrt{1 - \frac{a^2}{(1 + \alpha)^2 M^2 G_N^2} - \frac{\alpha}{1 + \alpha}}}.
\]

The Hawking temperature of \( H^\pm \) should read

\[
T_{\pm} = \pm \frac{\sqrt{1 - \frac{a^2}{(1 + \alpha)^2 M^2 G_N^2} - \frac{\alpha}{1 + \alpha}}}{2\pi G_N M \left[(2 + \alpha) \pm 2(1 + \alpha) \sqrt{1 - \frac{a^2}{(1 + \alpha)^2 M^2 G_N^2} - \frac{\alpha}{1 + \alpha}}\right]}.
\]

One obtains the Smarr like formula by solving the following quartic equation of \( M \):

\[
\alpha^2 G_N^2 M^4 - \left(\frac{\alpha + 2}{\alpha + 1}\right) \left(\frac{A_\pm}{2\pi}\right) M^2 + 4J^2 + \left(\frac{A_\pm}{4\pi G}\right)^2 = 0.
\]

In the limit \( \alpha = 0 \), one finds the Smarr formula for the Kerr BH. By solving the above Eq. (61), one obtains the Smarr formula for the Kerr-MOG BH:

\[
M^2 = \frac{A_\pm}{4\pi \alpha^2 G_N^2} \left[\left(\frac{\alpha + 2}{\alpha + 1}\right) + \sqrt{\frac{2(\alpha + 2)}{(\alpha + 1)^2} - \frac{64\pi^2 \alpha^2 G_N^2 J^2}{A_\pm^2}}\right].
\]

\(^{3}\)It should be noted that the Eq. (33) and Eq. (54) are a new kind of solutions in MOG due to the special unique relation \( Q = \sqrt{\alpha G_N} M \) in this sense they may be considered as a unique hairy BH solutions in the STVG theory.
It can be now easily verified that Kerr-MOG BH satisfies the first law for both the OH and IH as

\[ dM = T_+dS_+ + \Omega_+dJ \] (63)

\[ dM = -T_-dS_- + \Omega_-dJ \] (64)

The Komar energy for the Kerr-MOG BH is

\[ E_\pm = \pm G_N(1+\alpha) M \sqrt{1 - \frac{a^2}{(1+\alpha)^2 M^2 G_N^2} - \frac{\alpha}{1+\alpha}} \] (65)

Finally, the Gibbs free energy should read

\[ G_\pm = M \left[ 1 \mp \frac{G_N(1+\alpha)}{2} \sqrt{1 - \frac{a^2}{(1+\alpha)^2 M^2 G_N^2} - \frac{\alpha}{1+\alpha}} \right] - \frac{\alpha^2 G_N^2 M^2}{4} \] (66)

Now the area product is evaluated to be

\[ A_+ A_- = 64\pi^2 (1+\alpha)^2 G_N^2 \left[ J^2 + \frac{\alpha^2 G_N^2 M^2}{4} \right] \] (67)

Again it is explicitly mass dependent. This means that the area (or entropy) product formula for the Kerr-MOG BH is not universal. It follows from the above formula that all other thermodynamic products are strictly mass dependent. Therefore we could conclude that all the products of thermodynamic parameters in MOG are always mass dependent. Thus they could not be treated as a universal quantity. It could be noted that when the parameter \( \alpha \) goes to the zero value, one obtains the above results for Kerr BH.

In Fig. 2, we show the axisymmetric MOG BH and how the area product of \( H_\pm \) varies with the free parameter. One could observe from the figure due to the presence of the spin parameter area product is quite different from that of the spherically symmetric MOG BH.

Proceeding analogously, the irreducible mass product of \( H_\pm \) for the Kerr-MOG BH are

\[ M_{irr,+} M_{irr,-} = \frac{G_N(1+\alpha)}{2} \sqrt{J^2 + \frac{\alpha^2 G_N^2 M^2}{4}} \] (68)

The other important thermodynamic relations for Kerr-MOG BH are

\[ 1 \frac{A_- + A_+}{A_+ - A_-} = 8\pi M^2 G_N^2 (1+\alpha)(2+\alpha), \quad A_+ - A_- = 8\pi (1+\alpha) M G_N T_+ A_+ \]

\[ 1 \frac{1}{A_+} + \frac{1}{A_-} = \frac{M^2 (2+\alpha)}{8\pi (1+\alpha) \left[ J^2 + \frac{\alpha^2 G_N^2 M^2}{4} \right]} \]

\[ 1 \frac{1}{A_+} - \frac{1}{A_-} = -\frac{M \sqrt{G_N^2 M^2 (1+\alpha) - a^2}}{4\pi G_N (1+\alpha) \left[ J^2 + \frac{\alpha^2 G_N^2 M^2}{4} \right]} \]

\[ T_+ S_+ + T_- S_- = 0, \quad \frac{\Omega_+}{T_+} + \frac{\Omega_-}{T_-} = 0 \] (69)

\(^6\)It might be plausible that the mass-dependent formulas that we have derived in Eq. 143 and in Eq. 67 do not seem to be generic in the STVG theory due to the special unique relation \( Q = \sqrt{\alpha G_N M} \). This is just a postulation that has been mentioned in Ref. [35]. Thus the main resulting mass-dependence seems to be an artifact by this postulation. Alternatively, we could say that the mass-dependence relation may be just the consequence of the unique postulation \( Q = \sqrt{\alpha G_N M} \). Due to this special relation, one could expect that it critically affects the area (or entropy) formula; that is why the product formula is not universal. This is the main difference between the Einstein’s gravity & the MOG.
Figure 2: The figure depicts the variation of $A_p$ with free parameter $\alpha$ for Kerr-MOG BH with $M = G_N = 1$. Where $A_p = \frac{A_+A_-}{64\pi^2}$.

These relations for all the horizons of the Kerr-MOG BH could be further used to determine the BH entropy in terms of the Cardy formula which gives some clue for a MOG BH/CFT correspondence in the axisymmetric spacetime. The following *universal* relation

$$T_+S_+ + T_-S_- = 0$$  \hspace{1cm} (70)

implies that the central charges in the left and right moving modes of the dual CFT in MOG/CFT correspondence are same for the MOG-Kerr BH.

It should be mentioned that in [10] [11], "the area product being mass-independent is equivalent to the relation $T_+S_+ = T_-S_-$." We proved that this argument is violated in the Kerr-MOG BH, since here, we see that the relation $T_+S_+ = -T_-S_-$ is satisfied but the area product is *not* mass-independent. This is an interesting observation for MOG.

Analogously, using the above relations one can easily calculate the area bound for this BH. Since $r_+ \geq r_-$ thus $A_+ \geq A_- \geq 0$. Now the area product relation is given by

$$A_+ \geq \sqrt{A_+A_-} = 8\pi G_N(1 + \alpha)\sqrt{J^2 + \frac{\alpha^2 G_N^2 M^2}{4}} \geq A_-.$$  \hspace{1cm} (71)

and the area sum is calculated to be

$$8\pi M^2 G_N^2(1 + \alpha)(2 + \alpha)$$

$$= A_+ + A_- \geq A_+ \geq \frac{A_+ + A_-}{2} = 4\pi M^2 G_N^2(1 + \alpha)(2 + \alpha) \geq A_-.$$  \hspace{1cm} (72)

Likewise, the area bound for $\mathcal{H}^+$ is

$$4\pi M^2 G_N^2(1 + \alpha)(2 + \alpha) \leq A_+ \leq 8\pi M^2 G_N^2(1 + \alpha)(2 + \alpha).$$  \hspace{1cm} (73)

and the area bound for $\mathcal{H}^-$ is

$$0 \leq A_- \leq 8\pi G_N(1 + \alpha)\sqrt{J^2 + \frac{\alpha^2 G_N^2 M^2}{4}}.$$  \hspace{1cm} (74)
From these relations, one could easily derive the entropy bound for this BH. It should be noted that, in the limit $\alpha = 0$, one obtains the result for the Kerr BH \cite{20}.

For our record, one could derive the irreducible mass bound for MOG-Kerr BH. Thus for $\mathcal{H}^+$, it is calculated to be

$$
\frac{MG_N}{2} \sqrt{(1 + \alpha)(2 + \alpha)} \leq M_{\text{irr},+} \leq \frac{MG_N}{\sqrt{2}} \sqrt{(1 + \alpha)(2 + \alpha)}.
$$

(75)

and for $\mathcal{H}^-$, it is

$$
0 \leq M_{\text{irr},-} \leq \sqrt{\frac{G_N(1 + \alpha)}{2}} \left( \frac{j^2 + \alpha^2 G_N^2 M^2}{4} \right)^\frac{1}{4}.
$$

(76)

Finally, it should be noted that using the symmetric properties of $r_\pm$, one obtains

$$
T_+ = -T_+|_{r_+\leftrightarrow r_-}, S_+ = S_+|_{r_+\leftrightarrow r_-}, \Omega_+ = \Omega_+|_{r_+\leftrightarrow r_-}, A_+ = A_+|_{r_+\leftrightarrow r_-}, M_{\text{irr},+} = -M_{\text{irr},+}|_{r_+\leftrightarrow r_-}, E_- = -E_+|_{r_+\leftrightarrow r_-}, T_- S_- = -T_+ S_+|_{r_+\leftrightarrow r_-}.
$$

(77)

Perhaps, most importantly, the mass-dependent formulas that we have derived in Eq. \ref{43} and in Eq. \ref{77} do not seem to be generic in MOG, since the metrics of Eq. \ref{33} and Eq. \ref{54} have the same form as the RN BH solution and the KN BH solution in the Einstein-Maxwell system \cite{3}. Due to the special unique relation $Q = \sqrt{\alpha G_N M}$ which is a postulation we have mentioned earlier, the main mass-dependence result seems to just an artifact due this postulation. In the modified theory of gravity, there are several examples of BHs where the charge associated with the new degrees of freedom (scalar or vector field) is related to the mass, whereas the charge is secondary. An example of this type is the very controversial Bocharova-Bronnikov-Melnikov-Bekenstein (BBMB) BH \cite{38} solution in Einstein-conformally coupled scalar theory, where electric and magnetic fields of Einstein-Maxwell systems relate $Q = M$ and consequently the metric is obtained as extreme RN BH solution. Also this is a single parameter solutions where the parameter is $M$, only the total mass parameter with the scalar field which is diverges at the horizon but the geometry is regular there. But in our case, the metrics of Eq. \ref{33} and Eq. \ref{54} do not seem to be this type, more importantly the charge parameter $Q$ is quite independent from the mass parameter $M$. Hence the mass-dependence relation might be just the consequences of the unique postulation $Q = \sqrt{\alpha G_N M}$.

\section{Kerr-MOG/CFT correspondence}

In this section, we should derive the central charges $c_R$ and $c_L$ of the right and left moving sectors of the dual CFT in Kerr-MOG/CFT correspondence. We should prove that the central charges of the right and left moving sectors are same i.e. $c_R = c_L$ for Kerr-MOG BH. Also we should determine the dimensionless temperature of microscopic CFT from the above thermodynamic relations. Furthermore using Cardy formula, we should explicitly compute the right and left moving entropies in 2D CFT. Moreover, in the extreme limit we find the Frolov-Thorne vacuum state temperature which is thermally populated with a Boltzmann distribution. Finally by using Cardy formula we should determine the microscopic entropy of extreme Kerr-MOG BH and it is exactly same to the macroscopic Bekenstein-Hawking entropy.

Now we could proceed as in terms of OH radius $r_+$ and IH radius $r_-$, we could write the ADM mass and spin parameter

$$
M = \frac{(r_+ + r_-)}{2G} \quad \text{and} \quad a = \sqrt{r_+ r_- - \left( \frac{\alpha}{1 + \alpha} \right) \left( r_+ + r_- \right)^2}. \quad (78)
$$

It should be mentioned that MOG is a purely gravitational theory depending on only mass and spin. The metric solutions are similar algebraically to RN metrics and KN metrics but this comparison should end there. Because astrophysical bodies and BHs are electrically neutral. The electric charge if present is negligible and a BH would instantly blow up if $Q_{\text{electric}} = M$ due to the ratio of Coulomb force to gravity $10^{40}$. Any electric charge would have very little effect on the spacetime metric.
Therefore the angular momentum is derived to be
\[ J = \frac{(r_+ + r_-)}{2G} \sqrt{r_+ r_- - \left( \frac{\alpha}{1 + \alpha} \right) \left( \frac{(r_+ + r_-)}{4} \right)^2}. \] (79)

Again in terms of \( r_+ \) and \( r_- \), we could derive the entropy, Hawking temperature and angular velocity of \( \mathcal{H}^+ \) for Kerr-MOG BH
\[ S_+ = \frac{\pi}{2G} \left[ 2r_+ - \left( \frac{\alpha}{1 + \alpha} \right) \left( \frac{(r_+ + r_-)}{2} \right) \right]. \] (80)
\[ T_+ = \frac{2\pi(r_+ + r_-)}{2r_+ - \left( \frac{\alpha}{1 + \alpha} \right) \left( \frac{(r_+ + r_-)}{2} \right)} \] (81)
\[ \Omega_+ = \frac{\sqrt{r_+ r_- - \left( \frac{\alpha}{1 + \alpha} \right) \left( \frac{(r_+ + r_-)}{2} \right)}}{2r_+ - \left( \frac{\alpha}{1 + \alpha} \right) \left( \frac{(r_+ + r_-)}{2} \right)}. \] (82)

Using the features of symmetry of \( r_\pm \), one obtains the following relation for \( \mathcal{H}^- \)
\[ T_+ \mid_{r_+ \leftrightarrow r_-}, S_+ = S_+ \mid_{r_+ \leftrightarrow r_-}, \Omega_+ \mid_{r_+ \leftrightarrow r_-}. \] (83)

Thus the first law of BH thermodynamics may be rewritten as in terms of right and left moving sectors of dual CFT.
\[ \frac{dM}{2} = T_R dS_R + \Omega_R dJ. \] (84)
\[ = T_L dS_L + \Omega_L dJ. \] (85)

and using the definitions of \( \beta_{R,L} = \beta_+ \pm \beta_- \), \( \beta_\pm = \frac{1}{T_\pm}, \Omega_{R,L} = \beta_+ \Omega_{R,L} = \frac{(S_+ + S_-)}{2} \) [14][15][16].

Now one could easily derive the temperature and entropy for left moving sectors and right moving sectors as
\[ T_L = \frac{1}{4\pi(r_+ + r_-)}, \quad T_R = \frac{\left( \frac{\alpha + 1}{\alpha + 2} \right) \left( \frac{(r_+ - r_-)}{2\pi(r_+ + r_-)} \right)}{2r_+ - \left( \frac{\alpha}{1 + \alpha} \right) \left( \frac{(r_+ + r_-)}{2} \right)} \]
\[ S_L = \frac{(\alpha + 2)}{(\alpha + 1)} \frac{\pi(r_+ + r_-)^2}{4G}, \quad S_R = \frac{\pi(r_+ - r_-)^2}{2G} \]
\[ \Omega_L = 0, \quad \Omega_R = 2 \left( \frac{\alpha + 1}{\alpha + 2} \right) \frac{\sqrt{r_+ r_- - \left( \frac{\alpha}{1 + \alpha} \right) \left( \frac{(r_+ + r_-)}{4} \right)^2}}{(r_+ + r_-)^2}. \] (86)

With the help of Eq. (84) & Eq. (85), one could obtain the first law of BH thermodynamics for left moving sectors and right moving sectors of dual CFT
\[ dJ = \frac{T_L}{\Omega_R - \Omega_L} dS_L - \frac{T_R}{\Omega_R - \Omega_L} dS_R. \] (87)

From the above Eq. (87), one could easily determine the dimensionless temperature of the left and right moving sectors of the dual CFT correspondence. They are defined as
\[ T_L^J = \frac{T_L}{\Omega_R - \Omega_L}, \quad T_R^J = \frac{T_R}{\Omega_R - \Omega_L}. \] (88)
For Kerr-MOG BH, one can derive that

\[ T^J_L = \left( \frac{\alpha + 2}{\alpha + 1} \right) \frac{(r_+ + r_-)}{8\pi \sqrt{r_+ r_- - \left( \frac{\alpha}{1 + \alpha} \right)^2 \frac{(r_+ + r_-)^2}{4}}} \]  

\[ T^J_R = \frac{(r_+ - r_-)}{4\pi \sqrt{r_+ r_- - \left( \frac{\alpha}{1 + \alpha} \right)^2 \frac{(r_+ + r_-)^2}{4}}} \]  

they are exactly the microscopic temperature of dual CFT.

Now we are ready to determine the central charges \[ \text{(10)} \] in left and right moving sectors of the Kerr-MOG/CFT correspondence via the Cardy formula

\[ S^J_L = \frac{\pi^2}{3} c^J_L T^J_L, \quad S^J_R = \frac{\pi^2}{3} c^J_R T^J_R. \]  

Thus the central charges of dual CFT are

\[ c^J_L = 12J, \quad c^J_R = 12J. \]  

This means that the central charges of left moving sectors and right moving sectors of dual CFT are same for Kerr-MOG BH. This is an interesting result for Kerr MOG BH. It is also more interesting because it is independent of free parameter \( \alpha \) & the result is exactly same as we have seen in case of Kerr BH \[ \text{(42)} \] and Kerr-Newman BH \[ \text{(10)}. \] This kind of observation indicates that Kerr-MOG BH is dual to \( c_L = c_R = 12J \) of 2D CFT at temperature \((T_L, T_R)\) for each value of \( M \) and \( J \).

Now we are going to see what happens in the extremal limit \( r_+ = r_- \)?

\[ T_L = \frac{1}{8\pi r_+}, \quad T_R = 0 \]

\[ S_L = \left( \frac{\alpha + 2}{\alpha + 1} \right) \frac{\pi r_+^2}{G}, \quad S_R = 0 \]

\[ \Omega_L = 0, \quad \Omega_R = \frac{\sqrt{1 + \alpha}}{2 + \alpha} \frac{1}{2r_+^2}. \]

\[ T^J_L = \frac{1}{4\pi \sqrt{1 + \alpha}} \frac{\alpha + 2}{2}, \quad T^J_R = 0. \]

\[ \text{This is the left moving temperature which is actually Frolov-Thorne vacuum quantum state temperature \[ \text{(41)}, \] and finally one obtains the central charge for extremal Kerr-MOG BH} \]

\[ c^J_L = 12J. \]  

Finally, one could obtain the microscopic entropy via Cardy formula in chiral dual CFT

\[ S_{\text{micro}} = \frac{\pi^2}{3} c^J_L T^J_L = \frac{\alpha + 2}{\sqrt{1 + \alpha}} \pi J. \]  

which is perfectly match with the macroscopic Bekenstein-Hawking entropy for extreme Kerr-MOG BH. In the limit \( \alpha = 0 \), one finds the macroscopic Bekenstein-Hawking entropy for extreme Kerr BH, \( S_{\text{micro}} = 2\pi J \) \[ \text{(18)}. \]

In the following section, we shall analyze the thermodynamics properties of regular BH in MOG which is so called singularity free solution of classical general theory of relativity. First, the idea of regular BH solution has been incorporated by Bardeen in 1980 \[ \text{(39)}. \] Subsequently, Ayón-Beato and García \[ \text{(40)} \] derived a singularity-free solution of the Einstein field equations which is coupled to a non-linear electrodynamics in 1998.
The metric of a regular MOG BH \cite{[36]} is given by

\[ ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) . \]  

(97)

where the function \( U(r) \) is defined by

\[ U(r) = 1 - \frac{2GMr^2}{(r^2 + \alpha GM^2)^2} + \frac{\alpha G_N GM^2 r^2}{(r^2 + \alpha G_N GM^2)^2} . \]  

(98)

This is a solution with an EH but without singularities. It should be noted that if the value of the free parameter \( \alpha \) is less than a critical value i.e. \( \alpha_c = 0.673 \) then there exists two physical horizons namely EH & CH. Otherwise for \( \alpha > \alpha_c \), there exists no horizon \cite{[36]}.

To obtain the BH horizon equation we have to set \( U(r) = 0 \) i.e.

\[ r^8 + (6\alpha G_N GM^2 - 4G^2 M^2)r^6 + (11\alpha^2 G_N^2 G^2 M^4 - 4\alpha G_N G^2 M^4)r^4 
+ 6(\alpha G_N GM^2)^3 r^2 + (\alpha G_N GM^2)^4 = 0 . \]  

(99)

Now putting \( r^2 = x \), one obtains the following polynomial equation

\[ x^4 + (6\alpha G_N GM^2 - 4G^2 M^2)x^3 + (11\alpha^2 G_N^2 G^2 M^4 - 4\alpha G_N G^2 M^4)x^2 
+ 6(\alpha G_N GM^2)^3 x + (\alpha G_N GM^2)^4 = 0 . \]  

(100)

This is a fourth order polynomial equation. To finding the roots one could apply the Vieta’s theorem and one obtains

\[ \begin{align*}
\sum_{i=1}^{4} x_i & = 4G^2 M^2 - 6\alpha G_N GM^2 , \\
\sum_{1 \leq i < j \leq 4} x_i x_j & = 11\alpha^2 G_N^2 G^2 M^4 - 4\alpha G_N G^2 M^4 , \\
\sum_{1 \leq i < j < k \leq 4} x_i x_j x_k & = -6(\alpha G_N GM^2)^3 , \\
\prod_{i=1}^{4} x_i & = (\alpha G_N GM^2)^4 .
\end{align*} \]  

(101)(102)(103)(104)

Eliminating the mass parameter, we should find the following mass-independent equation

\[ \begin{align*}
\sum_{1 \leq i < j \leq 4} x_i x_j & = \frac{(7\alpha^2 - 4\alpha)}{(4 - 2\alpha)^2} \left( \sum_{i=1}^{4} x_i \right)^2 , \\
\sum_{1 \leq i < j < k \leq 4} x_i x_j x_k & = \frac{6\alpha^3}{(2\alpha - 4)^3} \left( \sum_{i=1}^{4} x_i \right)^3 , \\
\prod_{i=1}^{4} x_i & = \frac{\alpha^4}{(4 - 2\alpha)^4} \left( \sum_{i=1}^{4} x_i \right)^4 .
\end{align*} \]  

(105)(106)(107)
In terms of area $A_i = 4\pi x_i$, the above mass-independent equation could be written as

$$
\sum_{1 \leq i < j \leq 4} A_i A_j = \frac{(7\alpha^2 - 4\alpha)}{(4 - 2\alpha)^2} \left( \sum_{i=1}^{4} A_i \right)^2,
$$

(108)

$$
\sum_{1 \leq i < j < k \leq 4} A_i A_j A_k = \frac{6\alpha^3}{(2\alpha - 4)^3} \left( \sum_{i=1}^{4} A_i \right)^3,
$$

(109)

$$
\prod_{i=1}^{4} A_i = \frac{\alpha^4}{(4 - 2\alpha)^4} \left( \sum_{i=1}^{4} A_i \right)^4.
$$

(110)

Eliminating further one obtains the following mass independent relation

$$
\sum_{1 \leq i < j < k \leq 4} A_i A_j A_k = \left( \frac{6\alpha}{2\alpha - 4} \right)^{\frac{3}{2}} \left( \sum_{1 \leq i < j \leq 4} A_i A_j \right)^{\frac{3}{2}},
$$

(111)

$$
\prod_{i=1}^{4} A_i = \left( \frac{\alpha}{7\alpha - 2} \right)^2 \left( \sum_{1 \leq i < j \leq 4} A_i A_j \right)^2.
$$

(112)

It must be noted that the above formulae for the horizon areas could be obtained by observing that the horizon coordinate positions $r_i$ are related to the zeros $x_i$ of a fourth order polynomial via the above relation $r_i^2 = x$. On the other hand any horizon must indeed be a zero of that polynomial, it is by no means clear that any zero also corresponds to a horizon. Indeed, for a physically relevant horizon, the zero must be real and positive. Therefore, equations that involve all four roots of the polynomial are not necessarily physically resonable. This is depends upon the value of free parameter $\alpha$ that has been discussed above. Hence, there are at most two actual horizons so that the above equations that we have derived which depend on all four roots, are all unphysical. Reasonable equations could be constructed if two roots are eliminated such that relations between only two horizon areas are obtained.

One could write the explicit expressions in terms of two horizons area by using the equations (101), (102), (103) and (104). Thus one obtains

$$
\frac{A_1 A_2}{16\pi^2} - \frac{(A_1 + A_2)^2}{16\pi^2} = (11\alpha^2 G_N^2 G^2 - 4\alpha G_N G^3) M^4 - \left( \frac{A_1 + A_2}{4\pi} \right) (4G^2 - 6\alpha G_N G) M^2 - \frac{16\pi^2 (\alpha G_N G)^4 M^8}{A_1 A_2}.
$$

(113)

and

$$
\frac{A_1 A_2}{64\pi^3} = \frac{(A_1 + A_2)}{4\pi} (4G^2 - 6\alpha G_N G) M^2 + 4\pi (\alpha G_N G)^4 \left( \frac{A_1 + A_2}{4\pi} \right) M^8 + 6(\alpha G_N G)^3 M^6.
$$

(114)

It could be easily seen that from the above two equations, there has been no way to eliminate the mass parameter from these equations. It indicates that even for regular MOG BH there has been no way to construct mass-independent formula in terms of two physical horizons area. It is indeed true that the mass-independent relations that we have constructed in terms of four horizons area which are unphysical. This is also an another new result for regular MOG BH.

The Hawking temperature of $\mathcal{H}_\pm$ for MOG regular BH should read

$$
T_\pm = \frac{GM}{2\pi} \left[ \frac{r_\pm (r_\pm^2 - 2\alpha G_N G M^2)}{(r_\pm^2 + \alpha G_N G M^2)^2} \right] - \frac{\alpha G_N G M^2}{2\pi} \left[ \frac{r_\pm (r_\pm^2 - \alpha G_N G M^2)}{(r_\pm^2 + \alpha G_N G M^2)^3} \right].
$$
Whereas the Komar energy of $\mathcal{H}^\pm$ is derived to be

$$E^\pm = GM \left[ \frac{r^3_\pm (r^2_\pm - 2\alpha G_N GM^2)}{(r^2_\pm + \alpha G_N GM^2)^{\frac{3}{2}}} \right] - \frac{\alpha G_N GM^2}{2} \left[ \frac{r^3_\pm (r^2_\pm - \alpha G_N GM^2)}{(r^2_\pm + \alpha G_N GM^2)^{\frac{3}{2}}} \right],$$

where $r^\pm$ is the root of Eq. (5). Finally, the Gibbs free energy is computed to be

$$G^\pm = M - \frac{GM}{2} \left[ \frac{r^3_\pm (r^2_\pm - 2\alpha G_N GM^2)}{(r^2_\pm + \alpha G_N GM^2)^{\frac{3}{2}}} \right] - \frac{\alpha G_N GM^2}{2} \left[ \frac{r^3_\pm (r^2_\pm - \alpha G_N GM^2)}{(r^2_\pm + \alpha G_N GM^2)^{\frac{3}{2}}} \right].$$

From the above thermodynamic relations, one can conclude that the product is strictly mass dependent.

## 6 Conclusion

We investigated the features of inner and outer horizon thermodynamics of MOG. We derived the thermodynamic product relations particularly emphasized on area (or entropy) products for this gravity. We considered both spherically symmetric solution and axisymmetric solution of MOG. We found that the area (or entropy) product formula for both cases is not mass-independent because they depend on ADM parameter while in EG this formula is universal. We also examined the first law which is fulfilled at the IH as well as OH. We also derived other thermodynamic relations like products and sums.

We further derived the Smarr mass formula and Christodoulou’s irreducible mass formula for this kind of BH in MOG. Moreover, we derived the area (or entropy) bound for all the horizons. Furthermore, we showed the central charges of the left and right moving modes of the dual CFT in MOG/CFT correspondence are same by using a universal thermodynamic relations. For regular MOG BH, we derived some complicated combinations of four horizon area (or entropy) product relations that are mass independent but it is unphysical. On the other hand, we derived explicit expressions in terms of two physical horizons area while it is mass dependent.

We proved that the statement could made in [10, 11], “the area product being mass-independent is equivalent to the relation $T_+ S_+ = T_- S_-$” breaks down in case of Kerr-MOG BH. Interestingly, we pointed out that the relation $T_+ S_+ = -T_- S_-$ is satisfied but the area product is not mass-independent. Moreover, we derived the central charges for Kerr-MOG BH, $c_L = 12J$ which is usually derived using asymptotic symmetry group analysis. We also computed the dimensionless temperature for extreme Kerr-MOG BH. Using famous Cardy formula, we derived the microscopic entropy for extreme Kerr-MOG BH which is precisely equal to the macroscopic Bekenstein-Hawking entropy. Thus we conjectured that extreme Kerr-MOG BH is holographically dual to a chiral 2D CFT with $c_L = 12J$.

To sum up, the Ansorg & Hennig’s “mass-independence conjecture” could break down in case of MOG due to the special unique relation $Q = \sqrt{\alpha G_N M}$ consequently it critically affects on the thermodynamic product relations.

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