LARGE SETS AVOIDING AFFINE COPIES OF INFINITE SEQUENCES

ANGEL CRUZ, CHUN-KIT LAI, AND MALABIKA PRAMANIK

Abstract. A conjecture of Erdős states that for any infinite set $A \subseteq \mathbb{R}$, there exists $E \subseteq \mathbb{R}$ of positive Lebesgue measure that does not contain any nontrivial affine copy of $A$. The conjecture remains open for most fast-decaying sequences, including the geometric sequence $A = \{2^{-k} : k \geq 1\}$. In this article, we consider infinite decreasing sequences $A = \{a_k : k \geq 1\}$ in $\mathbb{R}$ that converge to zero at a prescribed rate; namely $\log(a_n/a_{n+1}) = e^{\varphi(n)}$, where $\varphi(n)/n \to 0$ as $n \to \infty$. This condition is satisfied by sequences whose logarithm has polynomial decay, and in particular by the geometric sequence. For any such sequence $A$, we construct a Borel set $O \subseteq \mathbb{R}$ of Hausdorff dimension 1, but Lebesgue measure zero, that avoids all nontrivial affine copies of $A \cup \{0\}$.

1. Introduction

Following the terminologies in [10], let us call a set $A \subseteq \mathbb{R}$ universal if every set of positive Lebesgue measure in $\mathbb{R}$ contains a non-trivial affine copy of the set $A$. In other words, $A$ is universal if for every $E \subseteq \mathbb{R}$ with $m(E) > 0$, there exist $x \in \mathbb{R}$ and $\delta \neq 0$ such that $x + \delta A \subseteq E$. A classical result of Steinhaus [16] shows, using the Lebesgue density theorem, that finite sets must be universal. In 1974, Erdős proposed a conjecture, now known as the Erdős similarity conjecture [2, 3, Chapter 4]:

Conjecture. There are no infinite universal sets.

We will call an infinite sequence $A = \{a_k\}$ a null sequence if $a_k > 0$ and strictly decreases to 0. It is easy to see that the conjecture will be resolved in full generality if all null sequences are shown to be non-universal. The conjecture is currently open for sequences with exponential decay, and in particular for $A = \{2^{-k} : k \geq 1\}$.

In this paper, we study, for a given a compact set $K \subseteq \mathbb{R}$, the following set:

\[(1.1) \quad \mathcal{E} = \mathcal{E}_K := \{x \in K : \forall \delta \neq 0, \exists k \in \mathbb{N} \text{ s.t. } x + \delta a_k \notin K\},\]

which is the set of translates $x$ such that $x + \delta A \nsubseteq K$ for every $\delta \neq 0$. We call $\mathcal{E}_K$ the set of Erdős points of $K$. In the appendix, we will show that

(a) A set $A$ is universal if and only if every compact set $K \subseteq \mathbb{R}$ with $m(K) > 0$ contains a nontrivial affine copy of $A$ (Lemma [A.1]), and

(b) For every compact set $K \subseteq \mathbb{R}$, the set $\mathcal{E}_K$ is Borel measurable (Proposition [A.2]).

Our main result is the following:

Date: April 28, 2022.

2010 Mathematics Subject Classification. 28A80 (primary), 11B30 (secondary).
Theorem 1.1. Let $A = \{a_k : k \geq 1\}$ be a null sequence. Suppose that

\[
\log \left( \frac{a_n}{a_{n+1}} \right) = e^{\varphi(n)}
\]

where $\varphi(n)$ is strictly increasing and $\frac{\varphi(n)}{n} \to 0$ as $n \to \infty$. Then there exists a compact set $K$ of Lebesgue measure arbitrarily close to 1 such that the set of Erdős points has Hausdorff dimension 1.

The set $K$ mentioned in Theorem 1.1 will be of Cantor type, whose construction is described in Section 2.1. By restricting the set $K$ to $E$, we establish the following corollary.

Corollary 1.2. Let $A = \{a_k : k \geq 1\}$ be a null sequence obeying (1.2). Then there exists a Borel set $E$ of Hausdorff dimension 1 that does not contain any nontrivial affine copy of $A \cup \{0\}$.

Remarks:
1. The condition in (1.2) is satisfied for all exponential decay sequences $a_n = 2^{-np}$ where $p \geq 1$. We will prove the theorem by exhibiting a subset of $E$ that has Hausdorff dimension 1 but Lebesgue measure zero (see Proposition 4.4).
2. Finding a compact set $K$ with $m(E_K) > 0$ would prove the non-universality of $A$ satisfying (1.2). This is because the set $E_K$ avoids all nontrivial affine copies of $A \cup \{0\}$ by definition. This would show that $A \cup \{0\}$ is not universal. By a result of Svetic [17, Lemma 2.1], $A$ would not be universal either.
3. For certain sequences $\{a_k : k \geq 1\}$ with faster decay than (1.2) e.g. $a_k = 2^{-2^k}$, our method provides partial information; specifically, the construction of $K$ leads to a lower bound on the Hausdorff dimension of $E_K$ that is positive but strictly smaller than 1.

The Erdős similarity conjecture has long been a focal point of research. We refer the interested reader to [17] for a comprehensive survey of the conjecture. Let us summarize some significant progress here. Given an infinite set $A \subseteq \mathbb{R}$, Komjáth [7] proved the existence of a set $E \subseteq \mathbb{R}$ of positive Lebesgue measure that does not contain any translate of $A$. This result leaves open the possibility that $E$ might contain a scaled copy of $A$. Falconer [5] proved non-universality of slowly decaying sequences. Specifically, he showed that sets that contain an infinite sequence $\{x_n : n \in \mathbb{N}\}$ with $x_{n+1}/x_n \to 1$ is non-universal. Bourgain [11] showed that for any three infinite sets $\{S_i : i = 1, 2, 3\}$ in $\mathbb{R}$, the sumset $S_1 + S_2 + S_3$ cannot be universal (see also a recent survey by Tao [18] about this result). Bourgain remarked that variants of his method can be used to establish non-universality of certain double sums as well, such as $\{2^{-n}\} + \{2^{-n}\}$. Using a probabilistic construction, Kolountzakis [9] showed that for any infinite set $A$, one can find a set $E \subseteq [0, 1]$ with Lebesgue measure arbitrarily close to 1 such that the exceptional set of dilates

$$\{\delta \in \mathbb{R} : \exists x \in \mathbb{R} \text{ such that } x + \delta A \subseteq E\}$$

has Lebesgue measure zero. His work also established non-universality of certain infinite structures with large gaps, for instance sets of the form $\{2^{-\alpha n}\} + \{2^{-\alpha n}\}$, where $0 < \alpha < 2$. Despite all the efforts above, determining whether the set $\{2^{-n} : n \in \mathbb{N}\}$ is universal is still an open problem. Our Theorem 1.1 and Corollary 1.2 show that it is possible to construct a
Borel set avoiding all non-trivial affine copies of \( \{2^{-n}\} \cup \{0\} \), but which is large in the sense of Hausdorff dimension.

The conjecture has also led to many related questions of interest concerning existence or avoidance of patterns in sets. For instance, Steinhaus’s theorem fails for Lebesgue-null sets of large Hausdorff dimension; given any countable collection of three tuples of points, Keleti \[8\] constructed a compact set in \( \mathbb{R} \) of Hausdorff dimension one not containing any nontrivial affine copy of any of the given triplets. In contrast, Laba and Pramanik \[10\] obtained certain sufficient conditions, involving ball growth and Fourier decay of measures, under which a set of dimension strictly less than one contains a three-term non-trivial arithmetic progression. See \[11, 15\] for subsequent refined investigation between Fourier dimension and existence of configurations.

On the other extreme, and though apparently a contradiction in terms, small sets can also contain many patterns. Erdős and Kakutani \[4\] constructed a perfect set of measure zero but Hausdorff dimension one which contains an affine copy of all finite sets. Recently, Máthé \[12\] constructed such a perfect set with Hausdorff dimension zero. Molter and Yavicoli \[13\] constructed an \( F_\sigma \)-set of Hausdorff dimension zero containing affine copies of large families of infinite sets. Yang \[19\] studied the topological properties of sets containing affine copies of many infinite sequences.

We now briefly describe the strategy of our proof. Given a fast decaying sequence obeying certain decay conditions, we will describe in Section 2 the construction of a Cantor set \( K \) of positive Lebesgue measure that permits an explicit description of \( O \), a subset of its Erdős points. The relevant statement is given in Theorem 2.1 and it proof appears in Section 4. In Section 4 we estimate the Hausdorff dimension of \( O \). The proof of Theorem 1.1 is completed here.

2. Setup of the construction

Let us set up the notation used in this paper. The Lebesgue measure of a measurable set \( K \subseteq \mathbb{R} \) will be denoted by \( m(K) \). The notation \( a_k \searrow 0 \) (or \( a_k \nearrow \infty \)) will mean that the sequence \( \{a_k : k \geq 1\} \) is strictly decreasing to zero (or strictly increasing to infinity). The notation \( a_n \asymp b_n \) means that there exist absolute constants \( C, c > 0 \) such that \( Cb_n \geq a_n \geq cb_n \) for all sufficiently large \( n \).

2.1. A Cantor construction. Let \( N_1, N_2, \ldots \) be a sequence of positive integers greater than 3. For each \( n \in \mathbb{N} \), let us choose a subset

\[
B_n \subset \mathbb{Z}_{N_n}, \quad \text{where} \quad \mathbb{Z}_N := \{0, 1, \ldots, N - 1\}.
\]

The set \( B_n \) will represent the \( n^{\text{th}} \) digit set of our Cantor construction.

Given the sequence of tuples \( \mathcal{N} := \{(N_n, B_n) : n \in \mathbb{N}\} \), we define

\[
\delta_n := \frac{1}{N_1 \cdots N_n}, \quad \text{and} \quad \Sigma_n := B_1 \times \ldots \times B_n = \{(b_1, \ldots, b_n) : b_j \in B_j \forall j = 1, \ldots, n\}.
\]
Each ordered list of integers $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_n}$ corresponds to a closed interval $I_\mathbf{b} \subseteq [0, 1]$ given by

$$I_\mathbf{b} = \sum_{j=1}^{n} b_j \delta_j + [0, \delta_n].$$

Among them, the intervals of the form $\{I_\mathbf{b} : \mathbf{b} \in \Sigma_n\}$ are called the $n^{th}$-level basic intervals of the Cantor construction associated to $\mathcal{N}$. Their union leads to the set

$$(2.1)\quad K_n := \bigcup_{\mathbf{b} \in \Sigma_n} I_\mathbf{b},$$

often called the $n^{th}$ Cantor iterate. The iterates $\{K_n : n \in \mathbb{N}\}$ form a nested sequence of sets that is decreasing in $n$. Taking their intersection over all the levels $n$, we arrive at the following set generated by the set of tuples $\mathcal{N}$:

$$K = K(\mathcal{N}) := \bigcap_{n=1}^{\infty} K_n = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{b} \in \Sigma_n} I_\mathbf{b}.$$ 

A set $K$ obtained through the prescription above is sometimes called a Cantor-Moran set (as it was first studied by Moran [14]). Such a set should be viewed as a natural generalization of the standard middle-third Cantor set in which $N_n = 3$ and $B_n = \{0, 2\}$ for all $n$. Similar to the middle-third Cantor set, it is readily seen that elements of $K$ are identified by their digit expansion:

$$(2.2)\quad K = \left\{ \sum_{n=1}^{\infty} b_n \delta_n : b_n \in B_n \right\}.$$ 

The above construction is quite general. For our choice of $K$ we will fix a positive integer $M_n < N_n$ and choose our digit sets $B_n$ to be

$$(2.3)\quad B_n := \mathbb{Z}_{N_n} \setminus \left( \{1, 2, \ldots, M_n\} \cup \{N_n - 1 - M_n, \ldots, N_n - 2\} \right).$$

In other words, $\{1, 2, \ldots, M_n\}$ and $\{N_n - 1 - M_n, \ldots, N_n - 2\}$ are two forbidden bands for the $n^{th}$ digit set. A consequence of this is the following. Suppose that $y = \sum_{j=1}^{m} b_j \delta_j \in K$. Then

$$(2.4)\quad b_m = 0 \text{ implies } \left[ y + (\delta_m, (M_m + 1)\delta_m) \right] \cap K = \emptyset,$$

$$b_m = N_m - 1 \text{ implies } \left[ y + [-(M_m + 1)\delta_m, 0) \right] \cap K = \emptyset.$$ 

We note that this Cantor set satisfies

$$(2.5)\quad K = 1 - K.$$ 

Indeed, the relation $(2.5)$ follows from the identity $1 = \sum_{j=1}^{\infty} (N_j - 1)\delta_j$; we observe that

$$(2.6)\quad \text{for all } x = \sum_{j=1}^{\infty} b_j \delta_j, \quad 1 - x = \sum_{j=1}^{\infty} (N_j - 1 - b_j)\delta_j.$$
2.2. A fast decaying sequence. Suppose that $A = \{a_k : k \geq 1\}$ is a sequence of positive numbers such that
\begin{equation}
(2.7) \quad a_1 = 1, \quad a_k \searrow 0 \quad \text{and} \quad \frac{a_k}{a_{k+1}} \nearrow \infty \quad \text{as} \quad k \to \infty.
\end{equation}
For $n \geq 1$, and given positive integers $M_n \geq 1$ and $N_n \geq 3$, we set
\begin{equation}
(2.8) \quad k_n := \sup\{k \geq 1 : \frac{a_k}{a_{k+1}} \leq M_n\}.
\end{equation}
Fix $\varepsilon > 0$. The main assumptions on $M_n$ and $N_n$ are the following:
\begin{align}
(2.9) & \quad \frac{\delta_n}{a_{k_n+1}} \to \infty \text{ as } n \to \infty, \\
(2.10) & \quad M_n = \lfloor 2\varepsilon N_n n^{-2} \rfloor \text{ for all } n \geq 1.
\end{align}
where $\lfloor x \rfloor$ denotes the largest integer less than or equal to $x$.

**Theorem 2.1.** Let $A = \{a_k : k \geq 1\}$ be a sequence of positive numbers satisfying (2.7). Suppose there exist $\varepsilon > 0$ and sequences $M_n, N_n$ satisfying the assumptions (2.9) and (2.10). For these $M_n, N_n$ and the Cantor set $K$ in (2.2) with digits $B_n$ in (2.3), the following conclusions hold.

(a) $m(K) > 1 - \pi^2 \varepsilon / 3$; in other words, the Lebesgue measure of $K$ can be made arbitrarily close to one by choosing $\varepsilon$ sufficiently small.

(b) Define
\begin{equation}
(2.11) \quad \mathcal{O} = \mathcal{O}[K] := \left\{ x \in K : x = \sum_{j=1}^{\infty} b_j \delta_j \left| b_j = 0 \text{ for infinitely many indices } j, \right. \right. \\
\left. \left. b_j = N_j - 1 \text{ for infinitely many indices } j \right\} \right.
\end{equation}
Then $\mathcal{O} \subset \mathcal{E}$, where $\mathcal{E} = \mathcal{E}_K$ is the set of Erdős points of $K$ given by (1.1).

This theorem is proved in the next section.

3. Proof of Theorem 2.1

The proof is divided into several lemmas. The first lemma estimates the Lebesgue measure of the Cantor set.

**Lemma 3.1.** For the set $K$ defined in (2.2), the following conclusions hold.

(a) The Lebesgue measure of $[0,1] \setminus K$ is
\begin{equation}
(3.1) \quad m([0,1] \setminus K) = \frac{2M_1}{N_1} + (N_1 - 2M_1) \frac{2M_2}{N_1N_2} + (N_1 - 2M_1)(N_2 - 2M_2) \frac{2M_3}{N_1N_2N_3} + \cdots.
\end{equation}

(b) Fix $\varepsilon > 0$. Suppose that (2.10) holds. Then
\begin{equation}
(3.2) \quad m([0,1] \setminus K) \leq \sum_{j=1}^{\infty} \frac{2M_j}{N_j} \leq \sum_{n=1}^{\infty} \frac{2\varepsilon}{n^2} = \frac{\pi^2 \varepsilon}{3}.
\end{equation}

Thus we can make $m(K)$ arbitrarily close to 1 by choosing $\varepsilon$ small, provided we can find sequences $M_n$ and $N_n$ that obey (2.10) for that choice of $\varepsilon$. 
Proof. The set $K^c = [0, 1] \setminus K$ is an increasing union of the sets $K_n^c = [0, 1] \setminus K_n$, where $K_n$ is as in (2.1). This means that

$$m(K^c) = m(K_1^c) + \sum_{j=1}^{\infty} m(K_{j+1}^c \setminus K_j^c) = m(K_1^c) + \sum_{j=1}^{\infty} m(K_{j+1}^c \cap K_j).$$

According to our construction,

$$m(K_1^c) = \frac{2M_1}{N_1} \quad \text{and} \quad m(K_{j+1}^c \cap K_j) = \left[\prod_{\ell=1}^{j} \left(1 - \frac{2M_\ell}{N_\ell}\right)\right] \frac{2M_{j+1}}{N_{j+1}} \quad \text{for} \quad j \geq 1,$$

from which (3.1) follows. The $j$th summand in (3.1) is bounded above by $M_j/N_j$, which combined with (2.10) leads to the conclusion (3.2). □

We conclude this section with the proof of Theorem 2.1, using the two lemmas we just established.

**Proof of Theorem 2.1.** It is clear from Lemma 3.1 that the set $K$ has Lebesgue measure arbitrarily close to 1. This establishes part (a) of Theorem 2.1.

It remains to prove part (b), i.e., every $x \in O$ is an Erdős point. Equivalently, for every $\delta \neq 0$, we aim to establish that $x + \delta A \not\subseteq K$. Let us write $x \in O$ in terms of its digit expansion

$$x = \sum_{j=1}^{\infty} b_j \delta_j.$$

Suppose first $\delta > 0$. It follows from the definition of $O$ that $b_n = 0$ for infinitely many indices $n$. Let us choose $n_0$ for which the conclusion of Lemma 3.2 holds for all $n \geq n_0$, and then
pick a large enough $m \geq n_0$ so that $b_m = 0$. Lemma 3.2 ensures the existence of $k \leq k_m$ such that $\delta a_k \in [\delta_m, M_m \delta_m)$. We can then write

$$x = \sum_{j=1}^{m} b_j \delta_j + \sum_{j=m+1}^{\infty} b_j \delta_j := y + \varepsilon_m, \quad \text{where } 0 \leq \varepsilon_m \leq \delta_m.$$ 

Since $b_m = 0$, it follows from (2.4) that $[y + (\delta_m, (M_m + 1)\delta_m)] \cap K = \emptyset$. We consider two cases:

- If $\varepsilon_m > 0$, this implies that $x + \delta a_k \in y + \varepsilon_m + [\delta_m, M_m \delta_m) \subset y + (\delta_m, (M_m + 1)\delta_m]$. Hence $x + \delta a_k \not\in K$ by (2.4).

- If $\varepsilon_m = 0$, then $x = y$ and $x + \delta a_k \in y + [\delta_m, M_m \delta_m)$. If $x + \delta a_k \neq y + \delta_m$, the point $x + \delta a_k \not\in K$, again by (2.4). This leaves the subcase $\delta a_k = \delta_m$. For this we consider the index $k - 1 < k_m$, for which (2.8) yields

$$1 < \frac{a_{k-1}}{a_k} < M_m, \quad \text{hence } x + \delta a_{k-1} = x + \delta_m \frac{a_{k-1}}{a_k} \in x + (\delta_m, M_m \delta_m) \not\in K.$$ 

Combining the two cases, it follows that $x + \delta A \not\subseteq K$ for every $\delta > 0$.

It remains to investigate the situation where $\delta < 0$. In this case, we notice that if $x \in \mathcal{O}$, then $1 - x \in \mathcal{O}$ due to (2.6). From our previous paragraph, we can find $a_k \in A$ such that $(1-x) - \delta a_k \not\in K$, which implies that

$$x + \delta a_k \not\in 1-K.$$ 

But $K = 1-K$ by (2.5). This obtains the desired conclusion for $\delta < 0$, completing the proof. \hfill \square

4. Erdős points of Cantor-like sets with forbidden digits

4.1. Uncountability of Erdős points. We now turn our attention to proving Theorem 1.1. Let us start by showing that for any convergent sequence $A$ (not necessarily obeying (1.2)), the construction in Section 2 leads to a Cantor-like set $K$ whose set of Erdős points is uncountable.

**Theorem 4.1.** Let $A$ be any null sequence. Then it is possible to choose a null subsequence $\{a_k : k \geq 1\} \subseteq A$ and parameters $M_n, N_n$ such that conditions (2.9) and (2.10) hold. As a result, it follows from Theorem 2.1 that there exists a Cantor set $K$ of Lebesgue measure arbitrarily close to 1 whose set of Erdős points contains $\mathcal{O}$ and is uncountable.

**Proof.** We first note that $\mathcal{O}$ is uncountable by a standard diagonal argument. We only need to see that all points in $\mathcal{O}$ are Erdős points. Given a positive sequence decaying to 0, we can extract a fast-decaying subsequence, which we still denote as $\{a_k : k \geq 1\}$, consisting of positive numbers, such that $a_1 = 1$,

$$a_k \searrow 0, \quad R_k := \frac{a_k}{a_{k+1}} \nearrow \infty, \quad R_k > k, \quad R_{k+1} > R_k + 1.$$ 

Indeed any sequence of positive real numbers decreasing to zero admits a subsequence with the above properties. Therefore, our proof will follow from Theorem 2.1 if we can show the existence of the Cantor set satisfying (2.9) and (2.10).
Fix $\varepsilon > 0$ that is the reciprocal of a positive integer and let $C$ be a positive integer to be determined later. It remains to define the positive integers $M_n$ and $N_n$. Set $M_n := \lfloor R_{Cn} \rfloor + 1$. We also define

$$N_n := \frac{2n^2}{\varepsilon} M_n,$$

so that $\delta_n = \frac{1}{N_1 \cdots N_n} = \left(\frac{\varepsilon}{2}\right)^n \frac{1}{(n!)^2 M_1 \cdots M_n}$.

Since $1/\varepsilon$ is a positive integer, so is $N_n$. The definition (4.2) immediately implies (2.10). To verify (2.9), we first recall from (4.1) the requirement that $R_{k+1} > R_k + 1$, which implies that

$$R_{Cn} < M_n = \lfloor R_{Cn} \rfloor + 1 \leq R_{Cn} + 1 < R_{Cn+1}.$$ 

Hence it follows from the definition of $R_k$ in (2.8) that $k_n = \sup \{k : R_k \leq M_n\} = Cn$. The definition (4.1) and $a_1 = 1$ implies the relation $1/a_{k+1} = R_1 \cdots R_k$ for all $k \geq 1$. Combining this with (4.2), we obtain:

$$\frac{\delta_n}{a_{k+1}} = \frac{\delta_n}{a_{Cn+1}} = \delta_n \prod_{j=1}^{Cn} R_j = \left(\frac{\varepsilon}{2}\right)^n \frac{\prod_{j=1}^{Cn} R_j}{(n!)^2 M_1 \cdots M_n} \geq \left(\frac{\varepsilon}{2}\right)^n \frac{\prod_{j=1}^{Cn} R_j}{(n!)^2 \prod_{j=1}^{Cn} R_{Cj} + 1} \geq \left(\frac{\varepsilon}{4}\right)^n \frac{\prod_{j=1}^{Cn} R_j}{(n!)^2 \prod_{j=1}^{Cn} R_{Cj}} \quad \text{(using $R_{Cj} + 1 < 2R_{Cj}$)} \geq \left(\frac{\varepsilon}{4}\right)^n \frac{1}{(n!)^2} \prod_{1 \leq j \leq Cn, j \notin CZ} R_j.$$ 

The assumed bound $R_k > k$ lets us estimate the last quantity from below:

$$\frac{\delta_n}{a_{k+1}} \geq \left(\frac{\varepsilon}{4}\right)^n \frac{1}{(n!)^2} \prod_{1 \leq j \leq Cn, j \notin CZ} j = \left[\left(\frac{\varepsilon}{4}\right)^n \frac{1}{(n!)^2}\right] (Cn)! = \left(\frac{\varepsilon}{4C}\right)^n \frac{(Cn)!}{(n!)^3}.$$ 

By Stirling’s approximation, $n! \asymp \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$. Using this, we estimate the lower bound in (4.4):

$$\left(\frac{\varepsilon}{4C}\right)^n \frac{(Cn)!}{(n!)^3} \asymp \left(\frac{\varepsilon}{4C}\right)^n \frac{\sqrt{2\pi Cn} (Cn)^{Cn} e^{-Cn}}{(2\pi n)^3/2 n^{3n} e^{-3n}} \asymp \left(\frac{\varepsilon C}{4C}\right)^n \frac{n^{(C-3)n}}{n e^{(C-3)n}}.$$ 

If we take for example $C = 5$ and let $\kappa = \frac{20 n^2}{\varepsilon}$, the last quantity becomes

$$\frac{n^2}{n^{\kappa n}} = \frac{n^n}{n \kappa^n},$$

which diverges to infinity as $n \to \infty$. We have thus verified all the requirements, and hence completed the proof. \hfill \Box

4.2. Hausdorff dimension of a subset of Erdős points. We now need to estimate the Hausdorff dimension of the set $E = E_K$ consisting of the Erdős points of $K$. To do this, we will identify a subset $\mathcal{O}_s$ of Hausdorff dimension 1 contained in $\mathcal{O}$. Since $\mathcal{O} \subseteq \mathcal{E}$ by Theorem 2.1(a), we conclude that $\mathcal{E}$ must have Hausdorff dimension 1 as well. We now specify our desired subset.
Given a sequence of tuples $N = \{(N_n, B_n) | n \in \mathbb{N}\}$, we pick a subsequence indexed by $S = \{n_1 < n_2 < n_3 < n_4 < \ldots\}$. With this subset we define the set

$$O_S := \left\{ x = \sum_{j=1}^{\infty} \frac{b_j}{N_1 \ldots N_j} : b_j \in B_j, b_{n_{2j-1}} = 0, b_{n_{2j}} = N_{2j} - 1 \right\} \subseteq O.$$

We can define this set another way,

$$O_S = \bigcap_{k=1}^{\infty} E_k,$$

where

$$E_1 = \bigcup \left\{ I_b : b \in B_1 := \prod_{i=1}^{n_1-1} B_i \times \{0\} \right\},$$

$$E_2 = \bigcup \left\{ I_b : b \in B_2 := B_1 \times \prod_{i=1}^{n_2-1} B_i \times \{N_2 - 1\} =: B_2 \right\}$$

$$\vdots$$

$$E_k = \bigcup \left\{ I_b : b \in B_k := B_{k-1} \times \prod_{i=1}^{n_k-1} B_i \times \{\varrho_k\} \right\}, \text{ where}$$

$$\varrho_k = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ N_k - 1 & \text{if } k \text{ is even}. \end{cases}$$

For $b \in B_k$, the Lebesgue measure of the interval $I_b$ is given by

$$m(I_b) = \delta_{n_k} = \frac{1}{N_1 N_2 \ldots N_{n_k-1} N_{n_k}}.$$

We denote $|B|$ as the cardinality of a finite set $B$. We have the following proposition.

**Proposition 4.2.** Using the above notation, the Hausdorff dimension of $O_S$ is equal to

$$1 - \limsup_{j \to \infty} \frac{\log(\prod_{i=1}^{n_j-1} |B_{n_i}|)}{\log(\prod_{\ell=1}^{n_j-1} N_{n_\ell})}.$$

**Proof.** Using the well-known result in Falconer textbook [6, Example 4.6, Chapter 4], a Cantor set $O_S$ constructed as in Section 4.5 has Hausdorff dimension

$$\dim_H O_S \geq \liminf_{j \to \infty} \frac{\log(m_1 \ldots m_{j-1})}{-\log(m_j \varepsilon_j)}, \text{ where}$$

$$m_j = \text{number of } j^{th} \text{ level intervals in a } (j-1)^{th} \text{ level interval}$$

$$= |B_{n_j-1+1}| \ldots |B_{n_j-1}| \text{, and}$$

$$\varepsilon_j = \text{minimum gap length among } j^{th} \text{ level intervals}$$

$$= \delta_{n_j} - \delta_{n_{j-1}} = \delta_{n_j} - \left( 1 - \frac{1}{N_{n_j}} \right).$$
From (2.10) we have that

\[ N_n(1 - \frac{2\varepsilon}{n^2} + \frac{1}{N_n}) \geq |B_n| = N_n - 2M_n \geq N_n(1 - \frac{2\varepsilon}{n^2}). \]

Using this, the fraction in (4.6) can be written as

\[
\frac{\mathcal{N}}{\mathcal{D}} := \frac{\log(m_1 \cdots m_{j-1})}{-\log(m_j \varepsilon_j)} \quad \text{where} \quad \mathcal{N} := \log(\prod_{\ell=1}^{n_j-1} |B_{\ell}|) - \log(\prod_{\ell=1}^{j-1} |B_{n_{\ell}}|) \\
\geq \log(\prod_{\ell=1}^{n_j-1} N_{\ell}) + \log\left(\prod_{\ell=1}^{n_j-1} \left(1 - \frac{2\varepsilon}{n_{\ell}^2}\right)\right) - \log(\prod_{\ell=1}^{j-1} |B_{n_{\ell}}|), \quad \text{and} \\
\mathcal{D} = -\log\left(\frac{1}{N_1 \cdots N_{n_j-1}}\right) - \log\left(\frac{|B_{n_{j-1}+1}|}{N_{n_{j-1}+1}} \cdots \frac{|B_{n_{j-1}}|}{N_{n_{j-1}}}\right) - \log\left(1 - \frac{1}{N_{n_j}}\right) \\
\leq \log\left(\prod_{\ell=1}^{n_j-1} N_{\ell}\right) - \log\left(\prod_{\ell=n_{j-1}}^{n_j-1} \left(1 - \frac{2\varepsilon}{n_{\ell}^2} + \frac{1}{N_{\ell}}\right)\right) - \log\left(1 - \frac{1}{N_{n_j}}\right). 
\]

Simplifying the expressions above leads to

\[
\dim_H(O_S) = \lim_{j \to \infty} \frac{\mathcal{N}}{\mathcal{D}} \geq \lim_{j \to \infty} \left[ 1 + \frac{\log(\prod_{\ell=1}^{j-1} (1 - \frac{2\varepsilon}{n_{\ell}^2}))}{\log(\prod_{\ell=1}^{n_j-1} N_{\ell})} - \frac{\log(\prod_{\ell=1}^{j-1} |B_{n_{\ell}}|)}{\log(\prod_{\ell=1}^{n_j-1} N_{\ell})} \right] \\
\times \left[ 1 - \frac{\log(\prod_{\ell=n_{j-1}}^{n_j-1} (1 - \frac{2\varepsilon}{n_{\ell}^2} + \frac{1}{N_{\ell}}))}{\log(\prod_{\ell=1}^{n_j-1} N_{\ell})} - \frac{\log\left(1 - \frac{1}{N_{n_j}}\right)}{\log(\prod_{\ell=1}^{n_j-1} N_{\ell})}\right]^{-1}. 
\]

We note that the products \(\prod_{\ell=1}^{\infty}(1 - \frac{2\varepsilon}{n_{\ell}^2})\) and \(\prod_{\ell=1}^{\infty}(1 - \frac{2\varepsilon}{n_{\ell}^2} + \frac{1}{N_{\ell}})\) are finite and positive. Therefore, except the last term in the numerator, all the other terms in \(\mathcal{N}\) and \(\mathcal{D}\) tend to zero as \(j\) goes to infinity. We obtain that

\[
\dim_H(O_S) \geq 1 - \limsup_{j \to \infty} \frac{\log(\prod_{\ell=1}^{j-1} |B_{n_{\ell}}|)}{\log(\prod_{\ell=1}^{n_j-1} N_{\ell})}. 
\]

In remains to establish that \(\dim_H(O_S)\) is equal to the right hand side above. It follows from [6 Example 4.6] that this is a consequence of the condition \(m_j \varepsilon_j \geq c\delta_{n_j-1}\). We will verify this condition. Applying the right inequality in (4.7), we obtain

\[
m_j \varepsilon_j = |B_{n_{j-1}+1}| \cdots |B_{n_j-1}| \delta_{n_j-1} \left(1 - \frac{1}{N_{n_j}}\right) \\
\geq \delta_{n_j-1} \times \left[ \prod_{\ell=n_{j-1}+1}^{n_j-1} \left(1 - \frac{2\varepsilon}{n_{\ell}^2}\right) \right] \left(1 - \frac{1}{N_{n_j}}\right) \\
\geq \frac{1}{2} \left[ \prod_{\ell=1}^{\infty} \left(1 - \frac{2\varepsilon}{n_{\ell}^2}\right) \right] \delta_{n_j-1} \geq c\delta_{n_j-1}.
\]
The last step follows from the fact that the product is bounded below by a positive number. This completes the verification of the condition, and hence the proof. □

4.3. Proof of Theorem 1.1. We are now in a position to compute the Hausdorff dimension of the set of Erdős points of $K$, and complete the proof of Theorem 1.1.

Proof. Let us recall that $R_k := \frac{a_k}{a_{k+1}}$. Set

$$M_n := \lfloor R_C n \rfloor + 1, \quad N_n = \frac{2n^2}{\varepsilon} M_n, \quad B_n \text{ as in (2.3)},$$

and construct the Cantor set $K = K(N)$ as described in Section 2.1. In Theorem 4.1, we deduced that $\mathcal{O} = \mathcal{O}[K] \subset \mathcal{E}$ where any integer $C \geq 5$ will work. To complete the proof, we need to show $\dim_H(\mathcal{O}) = 1$. Using Proposition 4.2, it suffices to show for some subsequence $S$, $\dim_H(\mathcal{O}_S) = 1$. In other words, for $\mathcal{O}_S$ we need to establish that

$$\limsup_{j \to \infty} \frac{\log(\prod_{\ell=1}^{j} |B_{n_\ell}|)}{\log(\prod_{\ell=1}^{n_j} N_\ell)} = 0.$$

In view of (4.7), this is equivalent to showing that

$$\limsup_{j \to \infty} \frac{\log(\prod_{\ell=1}^{j} N_{n_\ell})}{\log(\prod_{\ell=1}^{n_j} N_\ell)} = 0.$$

Expressing $N_n$ in terms of $R_n$, we obtain

$$\frac{\log(\prod_{\ell=1}^{j} N_{n_\ell})}{\log(\prod_{\ell=1}^{n_j} N_\ell)} \leq \frac{\log(\prod_{\ell=1}^{j} \frac{2n_\ell^2}{\varepsilon} (2R_C n_\ell))}{\log(\prod_{\ell=1}^{n_j} \frac{2n_\ell^2}{\varepsilon} R_C \ell)} \leq \frac{j \log(4/\varepsilon) + 2 \log(n_1 \cdots n_j) + \log(\prod_{\ell=1}^{j} R_{n_\ell})}{n_j \log(2/\varepsilon) + 2 \log(n_j!) + \log(\prod_{\ell=1}^{n_j} R_C \ell)} \leq \frac{j}{\varepsilon} \log \left( \frac{\prod_{\ell=1}^{j} R_C n_\ell}{\prod_{\ell=1}^{n_j} R_C \ell} \right) = \left[ \sum_{\ell=1}^{j} \exp(\varphi(Cn_\ell)) \right]^{-1} \left[ \sum_{\ell=1}^{n_j} \exp(\varphi(C\ell)) \right]^{-1},$$

where the constant implicit in $\lesssim \varepsilon$ depends only on $\varepsilon$, and the last inequality follows from the assumption (4.1) that $R_{Ck} > R_k > k$, so that

$$j \log(4/\varepsilon) + 2 \log(n_1 \cdots n_j) \lesssim \varepsilon \log \left( \prod_{\ell=1}^{j} R_C n_\ell \right).$$

Let us choose the subsequence $n_\ell$ in the following way: $n_\ell$ is the largest integer $n$ such that $\varphi(Cn) \leq \ell + 1$. From our assumption, $\frac{\varphi(Cn)}{n} \to 0$ as $n \to \infty$. In Lemma 4.3 below, we will see that this condition is equivalent to

$$\omega(r) := \# \{ \ell : r < \varphi(C\ell) \leq r + 1 \} \to \infty \text{ as } r \to \infty.$$
We deduce

\[
\sum_{\ell=1}^{j} \exp(\phi(Cn_{\ell})) \leq \sum_{\ell=1}^{j} e^{\ell+1} \lesssim e^{j}, \quad \text{whereas}
\]

\[
\sum_{\ell=1}^{n_{j}} \exp(\phi(Cl)) \geq \sum_{r=1}^{j} e^{r} \#\{ \ell : r+1 > \phi(Cl) \geq r \} = \sum_{r=1}^{j} e^{r} \omega(r) \geq e^{j} \omega(j).
\]

Combining (4.11) and (4.12) with (4.9) and (4.10), it follows that

\[
\log\left(\prod_{\ell=1}^{j} N_{n_{\ell}}\right) \lesssim \left[ \sum_{\ell=1}^{j} \exp(\phi(Cn_{\ell})) \right] \left[ \sum_{\ell=1}^{n_{j}} \exp(\phi(Cl)) \right]^{-1} \lesssim \frac{1}{\omega(j)} \to 0 \text{ as } j \to \infty,
\]

as claimed in (4.8). \(\square\)

**Lemma 4.3.** Let \(x_{n}\) be a strictly increasing sequence of positive real numbers. Then \(\frac{x_{n}}{n} \to 0\) as \(n \to \infty\) if and only if

\[
\omega(k) = \#\{ \ell : k < x_{\ell} \leq k+1 \} \to \infty \text{ as } k \to \infty.
\]

**Proof.** (\(\implies\)) Suppose towards a contradiction, it is possible that \(\sup_{k} \omega(r) =: C_{0} < \infty\). For every \(n \geq 1\), let \(r = r(n)\) be the unique integer such that \(x_{n} \in (r, r+1]\). Since \(x_{n}\) is strictly increasing, we conclude that \(r(n)\) is monotone non-decreasing, with \(r(n) \to \infty\) as \(n \to \infty\). Thus

\[
n \leq \#\{ \ell : 1 < x_{\ell} \leq r(n) + 1 \} = \sum_{k=1}^{r(n)} \omega(k) \leq C_{0} r(n).
\]

But this means that

\[
\frac{x_{n}}{n} \geq \frac{r(n)}{C_{0} r(n)} = \frac{1}{C_{0}} > 0,
\]

contradicting the assumption that \(\varphi(n)/n \to 0\).

(\(\Longleftarrow\)) For every \(M \geq 1\), there exists \(k_{M} \geq 1\) such that \(\omega(k) \geq M\) for all \(k \geq k_{M}\). With the same definition of \(r(n)\), we know that

\[
n \geq \#\{ \ell : 1 < x_{\ell} \leq r(n) \} = \sum_{k=1}^{r(n)} \omega(k)
\]

Hence,

\[
\frac{x_{n}}{n} \leq \frac{r(n) + 1}{\sum_{k=1}^{r(n)} \omega(k)} \leq \frac{r(n) + 1}{\sum_{k=k_{M}}^{r(n)} \omega(k)} \leq \frac{r(n) + 1}{(r(n) - k_{M}) M},
\]

meaning that \(\limsup_{n \to \infty} \frac{x_{n}}{n} \leq \frac{1}{M}\) for all \(M \geq 1\). As \(M\) is arbitrary, the proof is complete. \(\square\)
4.4. Remark and open questions. Our main result found a subset $O$ of Hausdorff dimension 1 inside the set of Erdős points of $K$. The following proposition shows however that $O$ has Lebesgue measure zero.

**Proposition 4.4.** Under the assumption of Theorem 2.1, the set $O$ constructed in (2.11) has Lebesgue measure zero.

*Proof.* For $n \in \mathbb{N}$, let

$$J_n = \bigcup \left\{ I_b : b \in \prod_{i=1}^{n-1} B_i \times \{0, N_n - 1\} \right\}$$

This $J_n$ collects all the $n$th level intervals that have 0 or $N_n - 1$ at the $n$th digit. In particular, all the points in $K$ such that $b_n = 0$ or $N_n - 1$ are in $J_n$. By definition of $O$,

$$O \subset \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} J_n. \hspace{1cm} (4.13)$$

This implies that

$$m(J_n) = \frac{1}{N_1 \ldots N_n} \cdot |B_1| \ldots |B_{n-1}| \cdot 2 \leq \frac{2}{N_n}.$$ 

From (2.1), the Lebesgue measure of $K$ is equal to

$$\lim_{n \to \infty} m(K_n) = \prod_{n=1}^{\infty} \left( 1 - \frac{2M_n}{N_n} \right).$$

This number is positive if and only if $\sum_{n=1}^{\infty} M_n / N_n < \infty$. But $M_n > 1$, this implies that

$$\sum_{n=1}^{\infty} m(J_n) \leq 2 \sum_{n=1}^{\infty} \frac{1}{N_n} < \infty.$$

Hence, by the Borel-Cantelli Lemma and (4.13), $m(O) = 0$. □

We do not know if there are more points in $E_K$ other than points in $O$ and we do not know how the decay rate condition in Theorem 1.1 be removed. In view of this, we conclude the paper with two open problems.

1. Can condition (1.2) about the decay rate of the sequence in Theorem 1.1 be removed?
2. Can one strengthen the arguments in this paper to verify whether $E_K$ has positive Lebesgue measure? If true, that would resolve the Erdős similarity conjecture for sequences $\{a_k\}$ with (1.2).

**Appendix A. Measurability of Erdős Points**

Here we prove the statements (a) and (b) in page 1.

**Lemma A.1.** A set $A$ is universal if and only if every compact set $K$ of positive Lebesgue measure contains a nontrivial affine copy of $A$. 
Proof. We only need to prove the “if” part of the statement. Suppose that \( A \) is universal for all compact sets. Given a set of positive Lebesgue measure \( E \), the inner regularity of Lebesgue measure allows us to find a compact subset \( K \subset E \) with positive Lebesgue measure. By our assumption, \( K \), and hence \( E \), contains a nontrivial affine copy of \( A \). The proof is complete. \( \square \)

Let \( K \) be any compact subset of \( \mathbb{R} \) and let \( A = \{a_k\} \) be a bounded sequence of real numbers. Define

\[
\mathcal{F} = \{(x, \delta) \in \mathbb{R}^2 : x + \delta A \subset K \} \subseteq \mathbb{R}^2,
\]

and

\[
F = \{x \in K : \exists \delta \neq 0, x + \delta A \subset K \}.
\]

We further write

\[
\mathcal{F} = \bigcup_{j=-\infty}^{\infty} \mathcal{F}_j,
\]

where \( \mathcal{F}_j = \{(x, \delta) \in \mathcal{F} : \delta \in [-2^{j+1}, -2^j] \cup [2^j, 2^{j+1}] \} \).

We note that \( E = K \setminus F = K \cap F^c \). In order to show that the set \( E \) of Erdős points is Borel measurable, it suffices to prove the same for \( F \). We do this below.

**Proposition A.2.** For each \( j \), the set \( \mathcal{F}_j \) is closed in \( \mathbb{R}^2 \) and \( F = K \setminus \mathcal{E} \) is Borel measurable on \( \mathbb{R} \).

**Proof.** Let \( (x_k, \delta_k) \in \mathcal{F}_j \) and \( (x_k, \delta_k) \to (x, \delta) \in \mathbb{R}^2 \). Then for all \( n \in \mathbb{N} \),

\[
x_k + \delta_k a_n \in K.
\]

As \( K \) is closed, \( x + \delta a_n \in K \) also and \( \delta \) is still in the range of interest. Hence, \( \mathcal{F}_j \) is closed and thus compact since the set is bounded. To see that \( F \) is measurable, we note that

\[
F = \pi_x(\mathcal{F}) = \bigcup_{j=-\infty}^{\infty} \pi_x(\mathcal{F}_j),
\]

where \( \pi_x \) is the projection onto the \( x \)-axis. As \( \mathcal{F}_j \) is closed, \( F \) is a countable union of compact sets and \( F \) is a countable union of the image of the compact sets under \( \pi_x \), which are compact. Hence, \( F \) is a \( F_\sigma \) set. \( \square \)

That gives us our desired result that \( \mathcal{E} \) is in fact a measurable set. We end this appendix by the following interesting proposition which says that if \( A \) is universal for all sets of positive Lebesgue measure, then almost all points of \( K \) are not Erdős points. In other words, we should be able to find affine copies almost everywhere.

**Proposition A.3.** Suppose that \( A \) is universal for all sets of positive Lebesgue measure. Then for all closed set \( K \), \( m(K) = m(F) \).

**Proof.** Suppose that \( A \) is universal. Using Svetic’s paper Lemma 2.1, we know also that \( \overline{A} \) is also universal. Assume for the sake of contradiction \( m(K \setminus F) > 0 \). Then \( K \setminus F \) will also contain an affine copy of \( \overline{A} \) since \( K \setminus F \) is measurable. However,

\[
K \setminus F = \{x \in K : \forall \delta \neq 0, \exists n \text{ s.t. } x + \delta a_n \notin K \}.
\]

This means that for all \( x \in K \setminus F \) and for all \( \delta \neq 0 \), we can find \( n \) such that \( x + \delta a_n \notin K \setminus F \). This means there is no affine copy of \( \overline{A} \) in \( K \setminus F \). Hence, \( \overline{A} \) is not universal, a contradiction. Thus \( m(K \setminus F) = 0 \) and hence \( m(K) = m(F) \). \( \square \)
References

[1] J. Bourgain, Construction of sets of positive measure not containing an affine image of a given infinite structure, Israel J. of Math. 60 (1987), 333-344.
[2] P. Erdős, Problems, Math. Balkanica (Papers presented at The Fifth Balkan Mathematical Congress), 4(1974), 203-204.
[3] P. Erdős, My Scottish Book ‘Problems’, The Scottish Book, second edition, R. Daniel Mauldin, Birkhauser, 2015.
[4] P. Erdős and S. Kakutani, On a Perfect Set, Colloq. Math., 4 (1957), 195-196.
[5] K. J. Falconer, On a problem of Erdős on sequences and measurable sets, Proc. Amer. Math. Soc., 90(1984), 77-78.
[6] K. J. Falconer, Fractal Geometry, Mathematical Foundations and Applications, Third Edition, Wiley.
[7] P. Komjáth, Large sets not containing images of a given sequence, Canad. Math. Bull., 26(1983), 41-43.
[8] T. Keleti, Construction of one-dimensional subsets of the reals not containing similar copies of given patterns, Anal. PDE 1 (2008), no. 1, 29-33.
[9] M. N. Kolountzakis, Infinite patterns that can be avoided by measure, Bull. London Math. Soc., 29(1997), 415-424.
[10] I. Laba and M. Pramanik, Arithmetic progressions in sets of fractional dimension, Geom. Funct. Anal. 19 (2009), no. 2, 429-456.
[11] C.-K. Lai, Perfect fractal sets with zero Fourier dimension and arbitrarily long arithmetic progressions, Annales Academiæ Scientiarum Fennicae, 42 (2017), 1009-1017.
[12] A. Máthé, Covering the real line with translates of a zero-dimensional compact set, Fund. Math. 213 (2011), 213-219.
[13] U. Molter and A. Yavicoli, Small Sets containing any patterns, Math. Proc. Camb. Philo. Soc. (2018), 1-17.
[14] P.A. Moran, Additive functions of intervals and Hausdorff measure. Math. Proc. Camb. Philo. Soc., 42 (1946). 15-23.
[15] P. Shmerkin, Salem Sets with no arithmetic progression, Int. Math. Res. Notices, 7 (2017), 1929-1941.
[16] H. Steinhaus, Sur les distances des points dans les ensembles de mesure positive, Fund. Math., 1(1920), 93-104.
[17] R. E. Svetica, The Erdős similarity problem: A Survey Real Analysis Exchange, Real Analysis Exchange 25(2000), 181-184.
[18] T. Tao, Exploring the toolkit of Jean Bourgain, Bull. Amer. Math. Soc. 58 (2021), 155-171.
[19] T. Yang, On sets containing an affine copy of bounded decreasing sequences. J. Fourier Anal. Appl. 26 (2020), no. 5, Paper No 72.

Angel Cruz, San Francisco State University, Department of Mathematics, 1600 Holloway Avenue, CA 94132, US
Email address: angelc@mail.sfsu.edu

Chun-Kit Lai, San Francisco State University, Department of Mathematics, 1600 Holloway Avenue, CA 94132, US
Email address: cklai@sfsu.edu

Malabika Pramanik, University of British Columbia, Department of Mathematics, 1984 Mathematics Road, Vancouver, BC V6T 1Z2
Email address: malabika@math.ubc.ca