DESINGULARIZATION OF REGULAR ALGEBRAS

MOHSEN ASGHARZADEH

Abstract. The goals of this paper are two-fold. First, motivated by the uniformization theorem of Zariski, we investigate the rings that can be written as a direct limit of noetherian regular rings. Second, as reverse to the first one, we study properties of a direct limit of noetherian regular rings. We establish the regularity of a direct limit of noetherian regular rings in several cases. By using an extended version of a result of Hartshorne, we remark that a coherent super-regular ring dominates a noetherian regular local ring via a flat extension. Many examples are presented to illustrate the idea and to sharpening the results. We give several applications of the results.

1. Introduction

This paper is dedicated to an essay on the notion of desingularization. In non-technical terms, this notion represents an object as a direct limit of a family of objects that behave better. Concerning this, we recall the following examples from the literature. The first writes a flat module as a direct limit of free modules (Lazard’s theorem). The second one is a beautiful theorem of Popescu [34]. It says that any regular homomorphism is a direct limit of smooth homomorphisms. The third one restate the notion of purity by a direct limit of certain splitting direct systems.

The historical reason for desingularization comes from the fundamental paper of Zariski [42]. He proved that any zero-characteristic valuation ring of a function field can be written by a direct limit of essentially of finite type smooth algebras over the base field and so a direct limit of noetherian regular rings. A direct system \( \{ R_i : i \in I \} \) of noetherian regular rings is called a desingularization of a ring \( R \) if \( R = \lim_{\rightarrow} R_i \).

In this paper, we are interested in:

Question 1.1. When can we represent a ring by a direct limit of noetherian regular rings?

Suppose a ring \( A \) has a desingularization \( \{ A_i : i \in I \} \), that is, it satisfies Question 1.1. In absence of many classical results of commutative and homological algebra for the ring \( A \), the direct-limit-argument provides a bridge to transfer information from the fruitful land of noetherian rings \( \{ A_i : i \in I \} \) to widely unknown realm of non-noetherian ring \( A \), that is, desingularization play a role with things that behave well with direct limit. Here, are some samples:

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Application 1.2. Desingularization helps to

(i) compute modules of differential forms, and so De Rham cohomology, see e.g. Example 9.9 and Example 9.11;
(ii) compute flat dimension of modules, see e.g. Example 9.15;
(iii) show a certain ring is Cohen-Macaulay, see e.g. subsection 9.E;
(iv) show a module is projective, see e.g. Proposition 9.13;
(v) prove the ring under consideration is stably coherent, when we present examples of desingularization with flat morphisms, see e.g. Theorem 9.3(iii);
(vi) define non-noetherian closure operations, see e.g. Proposition 9.20.
(vii) compute Čech cohomology modules and determines certain Castelnuovo-Mumford regularity, see Corollary 9.7.

It may be worth to note that Application 1.2(iv) provides a simple proof of [26, Theorem B] in a zero characteristic case; see Corollary 9.14.

We answer Question 1.1 via certain purity assumption; see Proposition 3.4 and in a low dimensional case; see Theorem 4.7. We give desingularization of many examples in the sequel, by applying different approaches due to different situations.

The gluing analogue of Question 1.1 studies the behavior of regularity under taking the direct limit. Under the assuming $R := \varinjlim R_i$ is coherent we prove, in Proposition 5.4, that $R$ is regular, where $\{R_i : i \in I\}$ is a directed system of coherent regular local rings of bounded Krull dimension. This is a reason to have an essential attention to the case of coherent regular rings in the sequel. We remark that Proposition 5.3 implies [36, Theorem 4.1] by a new and simple argument.

Section 6 is about of desingularization of supper-regular rings, see Proposition 6.3. Recall from [41] that a quasi-local ring $R$ is called super-regular, if $\text{gl} \dim R = \text{w} \dim R < \infty$. As an application of the notion of supper-regularity, we compute global dimension of certain perfect algebras in Proposition 6.4 and we give their desingularizations in Proposition 6.5. The final result of this section is Example 6.7. This shows that the assumptions of the results is needed. The example inspired by some ideas of Nagata [30] and Kabele [24].

To show that the local assumption of Theorem 4.7 is needed, we give a coherent regular ring which is not a filter limit of its noetherian regular subrings; see Example 7.2.

Let $R$ be a ring containing a field $k$ and $\underline{x} := x_1, \ldots, x_n$ a regular sequence in the Jacobson radical of a coherent ring $R$. In Proposition 8.3 we show that $R$ is a flat extension of a polynomial ring with $n$ variables over $k$. This extends a result of Hartshorne to the coherent case; see [19]. It follows by a result of Dieudonné that the mentioned result is not true for general rings. Our proof applies the concept of balanced big Cohen-Macaulay modules. We conclude, by the mentioned result, that a coherent super-regular ring dominates a noetherian regular local ring via a flat extension.

*this may answers a question asked in: [http://mathoverflow.net/questions/70017/is-the-direct-limit-of-a-direct-system-of-regular-rings-regular](http://mathoverflow.net/questions/70017/is-the-direct-limit-of-a-direct-system-of-regular-rings-regular)
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In Section 9 we present the proof of Application 1.2. Here we state an example with details of presentation. For a field $F$ of prime characteristic, set $R_F := \prod_{N}(F[X])$. We show it is a direct limit of polynomial rings, and by applying a direct-limit-argument, we transfer the following claims from polynomial rings to the ring $R_F$:

(i) If $F$ is finite, then $R_F$ is stably coherent.

(ii) Suppose $X := X_1, \ldots, X_\ell$. Then the Čech cohomology modules are

$$
H^i_X(R_F) = \begin{cases} 
X_1^{-1} \cdots X_\ell^{-1}([F][X_1^{-1}, \ldots, X_\ell^{-1}] & \text{for } i = \ell \\
0 & \text{for } i \neq \ell.
\end{cases}
$$

(iii) The Castelnuovo-Mumford regularity of $R_F$ is zero.

(iv) Suppose $F$ is finite. The De Rham cohomology modules are computable as follows

$$
H^i_{dR}(R_F/F) \simeq \begin{cases} 
\prod F & \text{for } i = 0 \\
0 & \text{for } i \neq 0.
\end{cases}
$$

Concerning the above display enumerate items, it may be worth to note that we have no data when the base field is of zero characteristic.

Our concluding remarks and questions are presented in Section 10.

The Cohen-Macaulay analog of desingularization is the subject of [3]. Throughout this paper, rings are commutative but not necessarily noetherian. We refer the reader to the books [15] and [12] for all unexplained definitions in the sequel.

2. Basic concepts

Let $a$ be a finitely generated ideal of a ring $R$ by a generating set $x := x_1, \ldots, x_r$. We denote the Koszul complex of $R$ with respect to $x$ by $K_\bullet(x)$. The Koszul grade of $a$ on an $R$-module $M$ is defined by

$$
K_{\text{grade}}_R(a, M) := \inf\{i \in \mathbb{N} \cup \{0\} | H^i(\text{Hom}_R(K_\bullet(x), M)) \neq 0\}.
$$

Notation 2.1. Let $R$ be a ring.

(i) By $p. \dim_R(\sim)$ (resp. $\text{fl. dim}_R(\sim)$), we mean the projective dimension (resp. flat dimension) of a module over $R$.

(ii) By $\text{gl. dim}(R)$ (resp. $\text{w. dim}(R)$), we mean the global dimension (resp. weak dimension) of $R$.

(iii) The notation $\text{Sym}_R(\sim)$ stands for the symmetric algebra of a module over $R$.

Definition 2.2. (For more details, see [17])

(i) A ring is said to be regular, if each of its finitely generated ideal has finite projective dimension.

(ii) A ring is called coherent, if each of its finitely generated ideal is finitely presented.

(iii) A ring $R$ is called stably coherent, if the polynomial ring $R[T]$ is coherent.

We cite the following as our main key-word in this paper.
Definition 2.3. We say a ring $R$ has a desingularization if there is a direct system \( \{ R_i : i \in I \} \) of noetherian regular rings such that their direct limit is $R$. The direct system \( \{ R_i : i \in I \} \) is called a desingularization of $R$.

Clearly, rings with a desingularization are reduced.

3. Desingularization; via purity

Definition 3.1. Let $M \subset N$ be two modules over a ring $R$. Recall that $M$ is pure in $N$ if $M \otimes L \to N \otimes L$ is monomorphism for all $R$-modules $L$.

The following extends [30, Page 206] where it is shown $R_0 \subset \hat{R}$ is cyclically pure.

Example 3.2. Let $K$ be a field of characteristic $p$ and $x_1, \ldots, x_n$ be indeterminates. Set $\hat{R} = F[[x_1, \ldots, x_n]]$, $R_0 = Kp[[x_1, \ldots, x_n]][K]$. Then $R_0 \subset \hat{R}$ is pure. Indeed, it is proved in [30] Page 206 that $R_0$ is regular local and $R_0 \subset \hat{R}$ is integral. Let $\{ R_i \}$ be filtered direct family of module finite extensions of $R_0$ with direct union $R$. By direct summand theorem for rings of prime characteristic [12, Theorem 9.2.3], the extension $R_0 \subset R_i$ splits, and so pure. As purity is preserved by taking direct limit, $R_0 \to \varprojlim R_i = \hat{R}$ is pure.

We will use the following result several times in the sequel.

Lemma 3.3. Let $\varphi$ be a map from a noetherian regular local ring $(R, m)$ to a noetherian Cohen-Macaulay local ring $(S, n)$. Suppose $\dim R + \dim S/mS = \dim S$. Then $\varphi$ is flat.

Proof. See [29, Theorem 23.1].

Any ring is a direct system of noetherian rings. In this subsection, we are interested on direct systems with pure morphisms. This condition is strong as the next result says.

Proposition 3.4. Let $\{ R_i : i \in I \}$ be a pure direct system of noetherian local rings and suppose that the maximal ideal of $(R, m) := \varprojlim_{i \in I} R_i$ has a finite free resolution. Then the following assertions are true:

(i) there exists $i \in I$ such that $R_j$ is regular for all $i \leq j$.

(ii) there exists $i \in I$ such that $R_j \to R_k$ is flat for all $i \leq j \leq k$.

(iii) $R$ is noetherian and regular.

Proof. (i): Look at the following finite free resolution of $m$:

$$
0 \to F_N \to \cdots \to F_{j+1} \to F_j \to \cdots \to F_0 \to \text{m} \to 0
$$

where $F_j$ is finite free and $f_j$ is given by a finite row and colon matrix. By $I_t(f_j)$, we mean the ideal generated by $t \times t$ minors of $f_j$. Let $r_i$ be the expected rank of $f_j$; see [12]...
Section 9.1] for its definition. By [12, Theorem 9.1.6] which is a theorem of Buchsbaum-Eisenbud-Northcott, one has

\[ \text{K. grade}_R(I_{r_i}(f_j), R) \geq i. \]

There is an index \( i \in I \) such that all of components of \( \{f_j\} \) are in \( R_i \). Let \( F_j(i) \) be the free \( R_i \)-module with the same rank as \( F_j \). Consider \( f_j \) as a matrix over \( R_i \), and denote it by \( f_j(i) \). Recall that \( m \) is finitely generated. Choosing \( i \) sufficiently large, we may assume that \( m = m_i R \). Look at the following complex of finite free modules:

\[
0 \longrightarrow F_N(i) \longrightarrow \cdots \longrightarrow F_{j+1}(i) \xrightarrow{f_j(i)} F_j(i) \longrightarrow \cdots \longrightarrow F_0(i) \longrightarrow m_i \longrightarrow 0
\]

We are going to show it is exact. Recall that \( I_t(f_j(i)) \) is the ideal generated by \( t \times t \) minors of \( f_j(i) \). Clearly, \( r_j \) is the expected rank of \( f_j(i) \). Let \( z := z_1, \ldots, z_s \) be a generating set for \( I_t(f_j(i)) \). In view of the purity, there are monomorphisms

\[
0 \longrightarrow H_i(\mathbb{K}_*(z)) \longrightarrow H_i(\mathbb{K}_*(z) \otimes_{R_i} R)
\]

for all \( i \); see [12, Exercise 10.3.31]. Then,

\[ \text{K. grade}_R(I_{r_i}(f_j), R) \leq \text{K. grade}_{R_i}(I_{r_i}(f_j(i)), R_i). \]

Thus,

\[ \text{K. grade}_{R_i}(I_{r_i}(f_j), R_i) \geq j. \]

Again, due to [12, Theorem 9.1.6],

\[ 0 \longrightarrow F_N(i) \longrightarrow \cdots \longrightarrow F_0(i) \]

is acyclic. Thus, \( p \text{. dim}(R_i/m_i) < \infty \). By Local-Global-Principle, \( R_i \) is regular.

(ii): By purity, \( \dim R_m \geq \dim R_n \) for all \( n \leq m \), see [10, Remark 4 and Corollary 5].

Again, in the light of purity,

\[ m_m = (m_m R) \cap R_m = (m_n R) \cap R_m = (m_n R_m) R \cap R_m = m_n R_m, \]

for all \( n \leq m \). Thus \( m_n R_m = m_m \) when \( n \leq m \). Denote the minimal number of elements of \( R_m \) that need to generate \( m_m \) by \( \mu(m_m) \). Consequently, \( \mu(m_m) \leq \mu(m_n) \). By part (i), \( (R_i, m_i) \) is regular. Hence

\[ \dim R_m = \mu(m_m) \leq \mu(m_n) = \dim R_n \]

and so \( \dim R_m = \dim R_n \). In view of Lemma 3.3, we observe that \( A_n \rightarrow A_m \) is flat when \( n \leq m \).

(iii): Recall from (ii) that \( m_m = m_n R_m \) for all \( n < m \). In view of [32], \( R \) is noetherian. It is regular, because \( p \text{. dim}_R(R/m) < \infty \).
Remark 3.5. Denote algebraic closure of \(\mathbb{Q}\) by \(\overline{\mathbb{Q}}\). Look at \(V := \overline{\mathbb{Q}}[x](x)\) and let \(\{F_i\}\) be a filter family of finitely generated subfields of \(\overline{\mathbb{Q}}\) with direct union \(\overline{\mathbb{Q}}\). Set \(R_i := F_i[x](x)\). Then \(V\) is a directed union of \(\{R_i\}\). Note that \(R_i\) is torsion-free and finite over \(R_j\) for \(j < i\). Due to our 1-dimensional assumption, \(R_i\) is flat over \(R_j\). By the localness, \(R_i\) is free over \(R_j\). Note that \(\{1\}\) extends to a minimal generating set of \(R_i\). This forces that \(R_j\) is a direct summand over \(R_i\) and so pure. In particular, \(\{R_i\}\) is a nontrivial pure direct system of noetherian local rings and that the maximal ideal of \(\lim_{\rightarrow i\in I} R_i\) has a finite free resolution. For a more general case, see [20].

Lemma 3.6. Let \(R\), \(S\) and \(T\) be commutative rings. Let \(\varphi : R \to S\) and \(\theta : S \to T\) be ring homomorphisms. If \(\theta \varphi\) is pure, then \(\varphi\) is pure.

Proof. This is trivial. \(\square\)

Recall from [21] that a quasilocal ring \((R, \mathfrak{m})\) with finitely generated maximal ideal \(\mathfrak{m}\) is called Koszul-regular, if the Koszul complex of \(R\) with respect to a generating set of \(\mathfrak{m}\) is exact.

Proposition 3.7. Let \((R, \mathfrak{m})\) be Koszul-regular. Then \(R\) is a pure direct limit of a direct system of countable Koszul-regular rings \(\{R_i\}\). In general, \(\{R_i\}\) are not necessarily noetherian.

Proof. Let \(p\) be the prime subring of \(R\). It is countable. Let \(x := x_1, \ldots, x_n\) be a generating set for \(\mathfrak{m}\). Look at \(R_0 := p[x](x)\). Let \(r \in R\). We will find a countable Koszul-regular ring \(R_1\) such that \(r \in R_1\) and both of \(R_0 \subseteq R_1\) and \(R_1 \subseteq R\) are pure.

Look at the family of all linear equations \(\sum r_{ij} X_j = a_i\) in \(R_0[r]\). Let \(X_1\) be the family of all its solutions in \(R\). First, we claim that \(X_1\) is countable. To show this, let \(\mathcal{Y}_1\) be the family of all solutions of \(\sum r_{ij} X_j = 0\) in \(R\). If we fix one solution \(t \in X_1\) (supposing that at least one solution exists), then there exists a bijection

\[ X_1 \longrightarrow \mathcal{Y}_1 \]

by sending \(x \mapsto x + t\). But \(\mathcal{Y}_1\) is countably generated module over \(R_0[r]\). In order to show \(X_1\) is countable it remains to remark that \(R_0[r]\) is countable.

Define \(A_1 := R_0[r][X_1]\). Repeat this argument to construct the following chain of countable rings

\[ A_1 \subseteq A_2 \subseteq \cdots \]

Set \(R_1 := \bigcup_{n \in \mathbb{N}} A_n\). By a characterization of purity in terms of equations [20] Theorem 7.13], \(R_1 \subseteq R\) is a pure extension. By applying the same argument for each \(a \in R \setminus R_1\), we find a pure map \(R_2 \subseteq R\) such that \(a \in R_2\) and \(R_2\) is countable. By the above lemma, \(R_1 \subseteq R_2\) is pure. Thus, there is a directed family of countable subrings \(\{R_i : i \in I\}\) of \(R\) such that \(R_i \subseteq R_j\) is pure and \(\bigcup R_i = R\).

\[ \text{It may be worth to note by direct summand conjecture that any module-finite extension of regular rings splits. This conjecture is true in equal-characteristic case, see [12] Theorem 9.2.3].} \]
Let $m_i := m \cap R_i$. Recall that $\bar{x} = x_1, \ldots, x_n$ is a generating set for $m$. Choose $j \in I$ sufficiently large, such that $x_m \in R_i$ for all $m$ and $i > j$. Due to the purity,

$$\bar{x}R_i = (\bar{x}R_i)R \cap R_i = m \cap R_i = m_i,$$

This shows that $m_i$ is finitely generated for all $i > j$. Set $S_i := R_i \setminus m_i$. Then

$$S_i^{-1}R_i \subset S_i^{-1}R \cong R$$

is pure, since $R_i \subset R$ is pure. Replace $(R_i)_m$ with $R_i$, we can assume that $(R_i, m_i)$ is quasilocal. Denote the Koszul complex of $R$ (resp. $R_i$) with respect to $m$ (resp. $m_i$) by $K$ (resp. $K_i$). The exactness of $K$ implies the exactness of $K_i$ because of the purity; see [12 Exercise 10.3.31]. Thus $(R_i, m_i)$ is Koszul-regular and $\bigcup R_i = R$.

For the last claim, in the light of [24 Example 3], there is a Koszul-regular ring $(R, m)$ such that $m$ can not be generated by a regular sequence. In particular, $m$ has a finite free resolution. In view of Proposition [34.3] the rings $\{R_i\}$ are not noetherian. \hfill $\square$

4. Desingularization; A low dimensional case

Our main result in this section is Theorem [4.7]. We start by the following lemmas.

**Lemma 4.1.** (uniformization theorem of Zariski) Let $V$ be a valuation domain containing a field $k$ of zero characteristic. Then $V = \varprojlim_{i \in I} A_i$ where $A_i$ is a localization of a finite type smooth algebra over $k$.

**Idea of the proof.** By the usual direct-limit-argument, we reduce things to the case of function field. Second step constructs a noetherian regular ring $A_0 := S^{-1}B_0$, where $B_0 := k[x_1, \ldots, x_d]$ is a polynomial ring and that $V$ birationally dominates $A_0$. In the third step, for each positive integer $i$, define the $i$-th quadratic transform $A_i$ of $A_0$ along $V$. We remark that $A_i$ is a localization of a smooth algebra $B_i$. Let us recall the process for $i = 1$, e.g., $B_1 := k[x_1][x_i/x_1 : 2 \leq i \leq d]$ and $A_1$ is a localization of $B_1$ at the maximal ideal. The final step shows that $V = \bigcup A_i$. In particular, $A_i \subset V$. For more details see [34 (1.2) Theorem] and [42].

Our reference for unexplain notions from tight closure is [21].

**Example 4.2.** In the prime characteristic case, we present a nontrivial desingularization of a valuation ring. Let $R$ be a noetherian local domain of prime characteristic $p$ such that $R^\infty$, its perfect closure, is coherent. Recall that the perfect closure of a reduced ring $A$ of prime characteristic $p$ is defined by adjoining to $A$ all higher $p$-power roots of all elements of $A$ and denote it by $A^\infty$. The following holds:

(i) If $R$ is complete, then $R^\infty$ is the perfect closure of a normal $F$-finite domain. Indeed, by [17 Corollary 6.2.10], coherent regular rings are greatest common divisor and so integrally closed. Then, by [5 Theorem 1.2], $R^\infty$ is integrally closed. Let $F$ be the fraction field of $R$ and set $A := F \cap R^\infty$. Take $x \in A^\infty$. Then $x^p^n \in A$ for some $n$. Due to definition of $A$, $x^p \in R^\infty$. Thus $x^{p^{n+m}} \in R$ for some $m$. So $x \in R^\infty$, i.e., $A^\infty = R^\infty$. We
know that \( A \) is noetherian, because \( R \) is complete. This yields that \( A \) is a complete local normal domain. By replacing \( R \) with \( A \), we may assume that \( R \) is complete and normal. It remains to apply the Gamma construction \( R^\Gamma \). Then \( R \to R^\Gamma \) is purely inseparable extension and \( R^\Gamma \) is \( F \)-finite. Also, \( R^\Gamma \) is domain and is flat over \( R \). Since the perfect closure of \( R \) and \( R^\Gamma \) are the same, we need to show \( R^\Gamma \) is normal. In view of \( [12, \text{Theorem } 2.2.22] \), \( R \) satisfies Serre’s condition \((R_1)\) and \((S_2)\). Fibers of \( R \to R^\Gamma \) are Gorenstein. Thus, in the light of \( [12, \text{Theorem } 2.2.21] \) \( R^\Gamma \) satisfies \((S_2)\). If \( I \) defines the singular locus of \( R \), then \( IR^\Gamma \) defines the singular locus of \( IR^\Gamma \). Since \( R \) satisfies \((R_1)\), \( R^\Gamma \) is as well. Again, by applying Serre’s characterization of normality, \( R^\Gamma \) is normal.

(ii) If \( R \) is 1-dimensional, we claim that \( R^\infty \) is a valuation domain and has a desingularization. Indeed, first note that normalization of \( R \) is noetherian, since \( R \) is 1-dimensional. By applying the proof of part (i), we can assume that \( R \) is normal. Hence, \( R \) is a discrete valuation domain. For each positive integer \( n \), set \( R_n := \{ x \in R^\infty | x^{p^n} \in R \} \), where \( p \) is characteristic of \( R \). One may find easily that \( R_n \) is a discrete valuation domain. Direct union of tower of valuation domains is a valuation domain. This along with \( R^\infty = \bigcup R_n \), yields the claim.

**Lemma 4.3.** (see \([16, (1.9) \text{Proposition}]\)) Let

\[
\begin{array}{ccc}
D & \overset{i}{\longrightarrow} & A \\
\downarrow \pi_1 & & \downarrow \pi \\
B & \overset{j}{\longrightarrow} & C,
\end{array}
\]

be a cartesian diagram and let \( S_A \) (resp. \( S_B \)) be a multiplicative closed subset of \( A \) (resp. \( B \)) such that \( \pi(S_B) = i(S_A) =: S_C \). Then \( (S_B \times_{S_C} S_A)^{-1} D = (S_B)^{-1} B \times (S_C)^{-1} C (S_A)^{-1} A \).

**Lemma 4.4.** Any localization of a ring with desingularization has a desingularization. Any direct limit of a direct system of rings with desingularization is a ring with desingularization.

**Proof.** Let \( R \) be a ring with the desingularization \( \{ R_i : i \in I \} \) and let \( S \subset R \) be a multiplicative closed set. For each \( i \), define \( S_i := S \cap R_i \). Then \( \{ S_i^{-1} R_i : i \in I \} \) is a desingularization of \( S^{-1} R \). \( \square \)

Let \( \mathcal{D} \) be a category with direct limit and let \( \mathcal{C} \subset \mathcal{D} \) be a subclass. Denote the class of all objects \( M \in \mathcal{D} \) with \( M = \varprojlim M_i \) and \( M_i \in \mathcal{C} \) by \( \varprojlim \mathcal{C} \). It is not true in general that \( \varprojlim \mathcal{C} \) is closed under taking direct limit. To find an example see \([23, \text{Example } 1.1]\).

**Lemma 4.5.** Let \( D \) be a noetherian regular ring an let \( \{ R_i : i \in I \} \) be a direct system and suppose for each \( i \in I \) that \( R_i \) has a desingularization of finite type algebras over \( D \). If \( R_i \to R_{i'} \) preserves \( D \) for all \( i \leq i' \), then \( \varprojlim R_i \) has a desingularization.
Proof. Let \((R^i_j)_{j_i \in J}i\) be a desingularization of \(R_i\) consisting of finite-type regular algebras over \(D\). Suppose \(i \leq i'\) and denote the natural map \(R_i \to R_i'\) by \(f_{i'i}\). Also, \(g_{i'i'}^{j'} : R_{i'}^{j'} \to R_{i'}\) are the natural maps. As \(R^i_j\) is of finite-type, 
\[ f_{ii'}(g_{ji}^j(R^j_i)) \subseteq g_{ji'}^{j'}(R^{j'}_{i'}) \]
for some \(j' \geq j_i\). Then one can defines a map 
\[ \varphi^{(i',j')_{(i,j)}} : R^i_j \rightarrow R^{i'}_{j'} \]
These imply that \(\{R^i_j; \varphi^{(i',j')_{(i,j)}}\}\) is a desingularization for \(\lim_{\rightarrow} R_i\). \(\square\)

Discussion 4.6. (Vasconcelos’ umbrella ring) (i): Let \((V, p)\) be a valuation ring and \(D\) be a subring of \(V/p\) which is not a field. Let \(A\) be the inverse image of \(D\) under the canonical map, \(\pi\), taking \(V\) on to \(V/p\). That is 
\[ i : A \rightarrow V \]
\[ D \rightarrow V/p, \]
where \(i, j\) are the inclusion maps. If \(V = A_p\), we say we have the basic construction. In this case one has (see [40, 4.15]): (1) \(V = A_p\), (2) \(D = A/p\), (3) \(V/p = A_p/p\), and (4) \(pA_p = p\). Also, recall from [40, 4.14 Proposition] that \(V = A_p\) if and only if \(V/p\) is the fraction field of \(D\).

(ii): A ring \(R\) is called umbrella, if there is \(p \in \text{spec}(R)\) such that 
(1) \(p = pR_p\), (2) \(R/p\) is a 2-dimensional noetherian regular ring, (3) \(R_p\) is a valuation domain, (4) \(R\) has countably many principal prime ideals.

Theorem 4.7. Let \((R, m)\) be a characteristic zero quasilocal ring of global dimension less than three and locally has a coefficient field. Then \(R\) is a direct limit of noetherian regular rings.

Proof. If \(\text{gl.dim}(R) = 0\), then \(R\) is a field. If \(\text{gl.dim}(R) = 1\), then \(R\) is a valuation ring. In this case the claim follows by Lemma 4.1. Suppose that \(\text{gl.dim}(R) = 2\). In the light of [40, Theorem 2.2], \(R\) is one of the following rings: 1) a noetherian regular ring; 2) a valuation ring; 3) an umbrella ring. Hence, in view of Lemma 4.1, we may assume that \(R\) is an umbrella ring. Then by applying [40, 4.19 Corollary], \(R\) is the pullback of the following basic diagram 
\[ R \xrightarrow{i} V \]
\[ D \xrightarrow{j} V/p, \]
in which
(a) $V$ has a global dimension one or two.
(b) $D$ is a noetherian regular local ring of global dimension two.
(c) $\dim_D(V/p) = 1$.

Keep in mind that $V = R_p$ and $pR_p = p$. Thus $Q := V/p$ is a coefficient field of $V$. In view of Lemma 4.1, $V = \lim_{\gamma \in \Gamma} A_\gamma$ where $A_\gamma = S_{\gamma}^{-1}B_\gamma$ and $B_\gamma$ is a smooth algebra of finite type over $Q$.

Claim: We can reduce the situation to the case that $A_\gamma = B_\gamma$. Indeed, look at $S_\gamma \subseteq B_\gamma \hookrightarrow A_\gamma \hookrightarrow V \simeq R_p \twoheadrightarrow R_p/pR_p \simeq Q$.

Denote the composite map $\beta_1 : B_\gamma \rightarrow Q$. This implies that $\beta_1$ is the restriction of the projection $V \rightarrow Q$, i.e., $\beta_1(f) = f(0)$. But $S_\gamma \subseteq R_p \setminus pR_p$, so $0 \notin S := \beta_1(S_\gamma)$ and $S$ is multiplicative closed. Conclude that $f(0) \neq 0$ for all $f \in S_\gamma$. Now, define $\tilde{S}_\gamma := \{ f \in S_\gamma \}$.

Then $\tilde{S}_\gamma$ is a multiplicative closed subset of $B_\gamma$ and that $S_\gamma^{-1}B_\gamma \simeq \tilde{S}_\gamma^{-1}B_\gamma$. After replacing $\tilde{S}_\gamma$ with $S_\gamma$ we may assume that $f(0) = 1 \ \forall f \in S_\gamma$ (*).

We denote the pullback of the following diagram by $R_\gamma$:

$$
\begin{array}{ccc}
R_\gamma & \stackrel{\psi_2}{\longrightarrow} & A_\gamma \\
\downarrow{\psi_1} & & \downarrow{\phi_1} \\
D & \stackrel{\phi_2}{\longrightarrow} & Q
\end{array}
$$

Note that $\phi_1$ is surjective. In view of [40, Page 28], $\psi_1$ is surjective. Thus, by [37, 1 Theorem], $R_\gamma$ has a finite global dimension. However, it is not necessarily noetherian.

The strategy in the ext part is to find a desingularization for $R_\gamma$. Now, look at the following pullback diagram:

$$
\begin{array}{ccc}
C_\gamma & \longrightarrow & B_\gamma \\
\downarrow{\alpha_2} & & \downarrow{\beta_1} \\
D & \longrightarrow & Q
\end{array}
$$

Let $x \in R$. By definition of fiber product, $x = (d, v)$ where $d \in D$ and $v \in V$ with the property $j(v) = \pi(v)$. As $V = \lim_{\gamma \in \Gamma} A_\gamma$, we observe that $v \in A_\gamma$. The pair $x_\gamma = (d, v)$ has the property $\phi_2(d) = \phi_1(v)$ and so $x_\gamma \in R_\gamma$. Let $f_\gamma : R_\gamma \rightarrow \lim_{\gamma \in \Gamma} R_\gamma$ be the natural map. The assignment $x \rightarrow f_\gamma(x_\gamma)$ defines the following isomorphism

$$
R \simeq \lim_{\gamma \in \Gamma} R_\gamma \quad (\dagger)
$$

*The claim follows with more directed argument. We prefer the stated argument, because of its future possible applications.
Set \( S_D = \{1\} \). Then, by (\#), \( \alpha_2(S_D) = \beta_1(S_\gamma) = 1 \). Thus, by Lemma 4.3, \( R_\gamma \) is a localization of \( C_\gamma \). By Lemma 4.3 and (\#), we can assume that \( A_\gamma = B_\gamma \), as claimed.

After changing of the variables, we may assume that \( A_\gamma \) is of polynomial type over \( Q \) of dimension say \( n_\gamma \). Hence,

\[
R_\gamma = \{ f \in Q[X_1, \ldots, X_{n_\gamma}] : f(0, \ldots, 0) \in D \} \simeq \otimes_{1 \leq i \leq n_\gamma} \{ f \in Q[X_i] : f(0) \in D \}.
\]

Note that \( Q \) is the fraction field of \( D \) and so a flat \( D \)-module. Let \( F_\gamma \) be the flat module \( \oplus_{n_\gamma} Q \). Under the identification \( \text{Sym}_Q(Q) = Q[X] \), the image of \( \text{Sym}_D(Q) \) in the natural map

\[
\text{Sym}_D(Q) \longrightarrow \text{Sym}_Q(Q)
\]

is \( D + XQ[X] \), so

\[
\text{Sym}_D(F_\gamma) \simeq \otimes_{n_\gamma} \text{Sym}_D(Q) \\
\simeq \otimes_{n_\gamma} \{ f \in Q[X_i] : f(0) \in D \} \\
\simeq R_\gamma.
\]

Let \( \{ F_i : i \in I \} \) be a direct system of free modules with direct limit \( F_\gamma \). Such a thing exists, because of the Lazard’s theorem. Then, in view of [17, 8.3.3],

\[
R_\gamma = \text{Sym}_D(F_\gamma) = \varinjlim_{i \in I} \text{Sym}_D(F_i).
\]

Remark that \( \text{Sym}_D(F_i) \) is the polynomial ring of finite dimension over \( D \) and so a noetherian regular ring. Now, Lemma 4.3 combined with (\#), completes the proof. \( \square \)

In the next result, as another example, we give a desingularization of a certain low dimensional ring.

**Example 4.8.** Let \( R: = \{ n + \sum_{i=1}^m n_i t^i : n \in \mathbb{Z}, n_i \in \mathbb{Z}[1/2] \} \). The following assertions hold.

(i) \( R \) is a direct limit of a direct system of 2-dimensional noetherian regular rings with non-pure homomorphisms.

(ii) Any finitely generated prime ideal of \( R \) is generated by a regular sequence and there is a finitely generated maximal ideal.

(iii) \( R \) has a maximal ideal which is not finitely generated.

**Proof.** (i): For each \( i \in \mathbb{N} \), set \( R_i := \mathbb{Z}[t/2^i] \). Note that \( R_i \) is a noetherian regular ring. Direct \( \{ R_i : i \in I \} \) by means of inclusion. Let \( f \in R \). Then \( f = n + \sum_{i=1}^m n_i t^i \) where \( n_i \in \mathbb{Z}[1/2] \). There is \( k \in \mathbb{Z} \) such that \( n_i = m_i/2^k \) for some \( m_i \in \mathbb{Z} \) and all \( i \). Hence, \( f \in R_k \) and so \( R = \varinjlim_{i \in I} R_i \). For the second claim, consider the equation \( 2X = t/2^i \). It has the solution \( t/2^{i+1} \in R_{i+1} \). Note that the equation has no solution in \( R_i \). In the light of [29, Theorem 7.13], the map \( R_i \to R_{i+1} \) is not pure.

(ii): Let us first recall that prime ideals of \( \mathbb{Z}[X] \) are of the following tree forms:

(1): (p) where \( p \) is a prime integer.

(2): (f) where \( f \) is irreducible in \( \mathbb{Z}[X] \).
(3): \((p, f)\) where \(p\) is a prime integer and \(f\) is irreducible in \(\mathbb{Z}/p\mathbb{Z}[X]\).

Now, let \(p\) be a finitely prime ideal of \(R\) and let \(\underline{x} := x_1, \ldots, x_n\) be a generating set for it. There is \(i\) such that \(x_k \in R_j\) for all \(1 \leq k \leq n\) and \(j \geq i\). Set \(p_j := p \cap R_j\). Then, \(p_j\) is generated by at most two elements. We look at \(\underline{x}R \subset p_jR \subset p\) to conclude \(p_jR = p\). This implies that \(p\) is generated by at most two elements. Without loss of generality, we may assume that \(p = (p, f)\) where \(p\) is a prime integer and \(f\) is irreducible in \(R_j/pR_j\) for \(j \geq i\). Thus \((p, f)R_k\) is a ideal of height two is generated by two elements in a noetherian regular ring \(R_k\) for all \(k \geq j\). It turns out that \(p, f\) is a regular sequence in \(R_k\) for all \(k \geq j\). So, \(p, f\) is a regular sequence in \(R\).

For the last claim, set \(m := (2, t + 1)\). Note that \(t + 1\) is irreducible in \(R_j\) for all \(j\). Thus \(m_j := m \cap R_j\) is a maximal ideal of \(R_j\) for all \(j\) and \(m = \bigcup m_j\). Conclude from this that \(m\) is a maximal ideal of \(R\).

(iii): Look at \(m := (2, t/2^j)_{j=0}^{\infty}\). Clearly, \(R/m \simeq \mathbb{Z}/2\mathbb{Z}\) and
\[
t/2^{j+1} \notin (2, t, t/2, \ldots, t/2^j).
\]
Thus, \(m\) is maximal and is not finitely generated.

\[\square\]

5. DIRECT LIMIT OF REGULAR RINGS; HOMOLOGICAL PROPERTIES

In this subsection we study homological properties of direct limit of noetherian regular rings. Let \(\{A_\gamma : \gamma \in \Gamma\}\) be a filtered direct system of noetherian regular rings. By a result of McDowell, \(A := \varprojlim_{\gamma \in \Gamma} A_\gamma\) is regular when the morphisms are flat. Also, under assuming \(A\) is noetherian, one may shows that \(A\) is regular. We need the following extended version of it to failure the coherence property of certain algebras.

**Lemma 5.1.** Let \(\{R_i : i \in I\}\) be a directed system of noetherian regular rings and \(p\) a finitely generated prime ideal of \(R := \varinjlim R_i\). If \(R_p\) is coherent, then \(R_p\) is regular.

**Proof.** To simplify the notation, we replace \(R_p\) with \((R, m, k)\) and \((R_i)_{p \cap R_i}\) with \((R_i, m_i, k_i)\). Let \(\underline{x} := x_1, \ldots, x_n\) be a generating set for \(m\). We assume that \(x_i \in R_j\) for all \(i\) and \(j\). Set \(A_i := R_i/(\underline{x})\). By \([27]\) Lemma 2.5.1]
\[
0 \rightarrow H_2(R_i, A_i, A_i) \rightarrow H_1(\mathbb{K}_* (\underline{x})) \otimes A_i, A_i \rightarrow R_i^n/(\underline{x}) R_i^n \otimes A_i, A_i \rightarrow (\underline{x})/(\underline{x})^2 \otimes A_i, A_i \rightarrow 0,
\]
where \(H_2(-, - , - )\) is the second Andre-Quillen homology. Note that Koszul homology behaves well with direct limit. Also, \([27]\) Proposition 1.4.8] says that Andre-Quillen homology behaves well with direct limit. Keep in mind that direct limit is exact. These induce the following exact sequence
\[
0 \longrightarrow H_2(R, k, k) \longrightarrow H_1(\mathbb{K}_* (\underline{x})) \longrightarrow R^n/mR^n \xrightarrow{\pi} m/m^2 \longrightarrow 0.
\]
In view of \([27]\) Proposition 1.4.8, Corollary 2.5.3], \(H_2(R, k, k) = \varinjlim H_2(R_i, k_i, k_i) = 0\). Thus \(H_1(\mathbb{K}_* (\underline{x})) = 0\), because \(\pi\) is an isomorphism. Since \(R\) is coherent, \(H_i(\mathbb{K}_* (\underline{x}))\) are
finitely generated. Also, by induction and by the following exact sequence
\[ H_{j+1}(\mathbb{K}_*(x)) \xrightarrow{\partial_n} H_{j+1}(\mathbb{K}_*(\mathfrak{x})) \rightarrow H_j(\mathbb{K}_*(\mathfrak{x})) \]
we get the acyclicity of \( \mathbb{K}_*(\mathfrak{x}) \). So, \( \text{fl.dim}(k) < \infty \). Again, as \( R \) is coherent and in view of [17, Corollary 2.5.10], we see that any finitely generated ideal of \( R \) has finite projective dimension, that is \( R \) is regular.

Second part of the next result extends [36, Theorem 4.1].

**Lemma 5.2.** Let \( \{R_i : i \in I\} \) be a directed system of rings such that their weak dimensions bounded by an integer \( n \). Set \( R := \varinjlim R_i \). The following assertions hold:

(i) any finitely presented \( R \)-module has finite flat dimension bounded by \( n \).

(ii) If \( R \) is coherent, then any finitely presented \( R \)-module has finite projective dimension bounded by \( n \).

**Proof.** (i): Let \( M \) and \( N \) be two \( R \)-modules. The desired claim follows, by recalling from [13, VI, Exercise 17] that
\[ \text{Tor}_j^R(M,N) \cong \varinjlim_i \text{Tor}_j^{R_i}(M,N), \]
which is zero for all \( j > n \).

(ii): Let \( M \) be a finitely presented \( R \)-module and of finite flat dimension. In view of part i), we need to show that \( p\text{-dim}(M) = \text{fl.dim}(M) \). Indeed, we prove the claim by induction on \( n := \text{fl.dim}(M) \). In the case \( n = 0 \), there is nothing to prove, since by [17, Theorem 2.1.4(3)], finitely presented flat modules are projective. Look at the exact sequence
\[ 0 \rightarrow \Omega(M) \rightarrow R^n \rightarrow M \rightarrow 0. \]
Due to the coherent assumption, \( \Omega(M) \) is finitely presented. Adopting the induction assumption,
\[ p\text{-dim}(M) = p\text{-dim}(\Omega(M)) + 1 = \text{fl.dim}(\Omega(M)) + 1 = \text{fl.dim}(M), \]
as claimed. □

**Corollary 5.3.** Let \( \{R_n : n \in \mathbb{N}\} \) be a directed system of noetherian regular local rings such that their Krull dimension bounded by an integer. Then \( R := \varinjlim R_i \) is regular.

**Proof.** Any ideal of \( R \) is countably generated. It follows by the proof of [33, Corollary 2.47], that \( p\text{-dim}(<) \leq \text{fl.dim}(<) + 1 \). Since \( R_i \) is noetherian, \( \text{w.dim}(R_i) = \text{dim}(R_i) \). Thus, Lemma 5.2 yields the claim. □

**Proposition 5.4.** Let \( \{R_i : i \in I\} \) be a directed system of coherent regular quasilocal rings such that their Krull dimension bounded by an integer. If \( R := \varinjlim R_i \) is coherent, then \( R \) is regular.
Proof. Let $I$ be a finitely generated ideal of $R$ that generated by $x := x_1, \ldots, x_n$. There is an index set $i$ such that $x \subseteq R_j$ for all $j \geq i$. Denote $\not\in R_i$ by $I_i$ and define $m := m \cap R_i$.

In view of [7, Lemma 3.2], $K = \text{grade}(R_i(m, R_i)) \leq \dim R_i$. Note that $R_i/I_i$ has finite free resolution. By Auslander-Buchsbaum-Northcott [31, Chap. 6, Theorem 2],

$$\text{fl. dim}(R_i/I_i) \leq p. \dim(R_i/I_i) = \text{K. grade}(m_i, R_i) - \text{K. grade}(m_i, R_i/I_i) \leq \text{K. grade}(m_i, R_i) \leq \dim R_i.$$ 

Thus

$$\{w. \dim R_i : i \in I\} \leq \sup \{\dim R_i : i \in I\} \leq n.$$ 

So, Lemma 5.2 completes the proof. □

As an easy example of regularity of a direct system with unbounded Krull dimension, we state the following.

Example 5.5. Let $\{(R_i, \varphi_{ij}) : i \in I\}$ be a direct system of polynomial rings over a field $F$. Suppose $\varphi_{ij}$ are homogeneous and of degree zero. Then $R = \varprojlim R_i$ is regular.

Proof. Let $V_i$ be a $F$-vector space such that $R_i = \text{Sym}_F(V_i)$. Our assumptions guarantee that $\varphi_{ij}$ induces by $\phi_{ij} : V_i \to V_j$. Look at $\{V_i : i \in I\}$ and set $V := \varinjlim V_i$. Then $V$ is a vector space with a base $B$. Hence, in view of [17, 8.3.3],

$$R = \varinjlim \text{Sym}_F(V_i) \simeq \text{Sym}_F(V) = F[X_b : b \in B]$$

is a polynomial ring and so regular. □

Lemma 5.3 implies the following (extended version of a) folklore result of Kunz.

Proposition 5.6. Let $R$ be a quasilocal ring of prime characteristic $p$ with a desingularization. Then the Frobenious map is flat.

Proof. Let $\{R_i \in I\}$ be a desingularization of $R$. We map assume that $R_i$ is local and of prime characteristic $p$. Denote the Frobenious map $R_i \to R_i$ by $\varphi_i$. It is clear by Lemma 5.3 that $\varphi_i$ is flat. Also, $\varprojlim \varphi_i$ is the Frobenious map of $R$. It remains to recall that direct limit of flat morphisms is flat. □

Direct limit of weakly $F$-regular rings is the subject of Corollary 9.21.

6. DESINGULARIZATION; VIA SUPER-REGULARITY

We start by recalling the following important class of regular rings which were introduced by Vasconcelos [41].

Definition 6.1. A quasilocal ring $R$ is called super-regular, if

$$\text{gl. dim } R = w. \dim R < \infty.$$
We will use the following result several times in the sequel.

**Lemma 6.2.** (see [41]) Let \((R, \mathfrak{m})\) be a coherent super-regular ring. Then \(\mathfrak{m}\) can be generated by a regular sequence.

The following result provides the desingularization of a class of super-regular rings.

**Proposition 6.3.** Let \(\{(R_i, \mathfrak{m}_i, \kappa_i) : i \in J\}\) be a direct system of noetherian local rings with the property that \(\mathfrak{m}_i^2 = \mathfrak{m}_i \cap \mathfrak{m}_{i+1}^2\). If \(R := \lim_{\rightarrow} R_i\) is coherent and super-regular, then \(\{R_i\}\) is a desingularization for \(R\).

**Proof.** Denote the maximal ideal of \(R\) by \(\mathfrak{m}\) and denote the residue field of \(R\) by \(\kappa\). In view of Lemma 6.2, \(\mathfrak{m}\) is generated by a regular sequence. Thus the ring equipped with the following natural ring-isomorphism

\[
\theta : \text{Sym}_\kappa(\bigoplus_{\mu(m)} \kappa) \longrightarrow \text{Gr}_R(\mathfrak{m}) := \bigoplus_{i=0}^{\infty} \mathfrak{m}_i / \mathfrak{m}_i + 1.
\]

Here \(\text{Gr}_R(\sim)\) is the associated graded ring. Look at the natural epimorphism

\[
\theta_i : \text{Sym}_{\kappa_i}((\bigoplus_{\mu(m_i)} \kappa_i)) \twoheadrightarrow \text{Gr}_{R_i}(\mathfrak{m}_i).
\]

Keep in mind the following map

\[
\varphi : V_i := \mathfrak{m}_i / \mathfrak{m}_i^2 \rightarrow V_{i+1} := \mathfrak{m}_{i+1} / \mathfrak{m}_{i+1}^2.
\]

Now we show the symmetric extension of \(\varphi\) is monomorphism. Looking at

\[
\begin{array}{ccc}
\text{Sym}_{\kappa_i}(V_i) & \xrightarrow{f} & \text{Sym}_{\kappa_i}(V_{i+1}) \\
\xrightarrow{\varphi} & \xrightarrow{g} & \xrightarrow{\varphi} \\
\text{Sym}_{\kappa_i+1}(V_{i+1}) & \xrightarrow{h} & \text{Sym}_{\kappa_i+1}(\bigoplus_{\dim_{\kappa_i}(\kappa_{i+1})} V_{i+1})
\end{array}
\]

- Since \(V_i\) is a direct summand of \(V_{i+1}\) as a \(\kappa_i\)-vector space, \(f\) is monomorphism.
- The map \(g\) is monomorphism, because \(\kappa_i\) is a field.
- The horizontal isomorphism follows by [17 8.3.2].
- The vertical isomorphism follows by \(\bigoplus V_{i+1} \simeq V_{i+1} \otimes_{\kappa_i} \kappa_{i+1}\).
- Since \(V_{i+1}\) is direct summand of \(\bigoplus V_{i+1}\) as \(\kappa_{i+1}\)-vector space, \(i\) is monomorphism.

By these, \(h\) is monomorphism. So

\[
\varphi_{i,i+1} : \text{Sym}_{\kappa_i}(V_i) \longrightarrow \text{Sym}_{\kappa_{i+1}}(V_{i+1})
\]

is monomorphism, as claimed.

Set \(K_i := \ker \varphi_i\). Remark that \(\varphi_{i,i+1}(K_i) \subseteq K_{i+1}\) and again denote it by

\[
\varphi_{i,i+1} : K_i \hookrightarrow K_{i+1}.
\]
Note that \( \lim_{\to} \kappa_i = \kappa \), \( \lim_{\to} \left( m^n_i / m_i^{n+1} \right) = m^n / m^{n+1} \). Also, Sym and Gr behaved nicely with direct limit. Finally remark that \( \theta \) is injective. Put all of these together to observe that

\[ K_i \hookrightarrow \lim_{\to} K_i = \ker \theta = 0. \]

So, \( \theta_i \) is an isomorphism. This means that \( m_i \) is generated by a regular sequence and the regularity of \( R_i \) follows by this, because \( R_i \) is noetherian. The proof is now complete. \( \square \)

Here is a nice application of the notion of supper-regularity.

**Proposition 6.4.** Let \( R \) be a noetherian local domain which is either excellent or homomorphic image of a Gorenstein local ring and suppose that its perfect closure is coherent. If \( R \) is not a field, then \( \text{gl.dim}(R^\infty) = \dim R + 1 \).

**Proof.** Let \( x := x_1, \ldots, x_d \) be a system of parameters for \( R \). By [4, Lemma 3.1], \( x \) is a regular sequence on \( R^\infty \). It turns out that \( \text{fl.dim}(R^\infty / xR^\infty) = d \). Combining this with [5, Theorem 1.2],

\[ w. \dim(R^\infty) = \dim R < \infty. \]

The same citation says that \( \text{gl.dim}(R^\infty) \leq \dim R + 1 \). Now suppose on the contrary that

\[ \text{gl.dim}(R^\infty) \neq \dim R + 1. \]

This says that \( \text{gl.dim}(R^\infty) = w. \dim R \). Note that \( R^\infty \) is quasilocal. In sum, \( R^\infty \) is a coherent super-regular quasilocal ring. Call its maximal ideal by \( m_{R^\infty} \). In the light of Lemma 6.2, we see that \( m_{R^\infty} \) is finitely generated. One may find easily that \( m_{R^\infty} = m_{R^\infty}^2 \).

By Nakayama’s Lemma, \( m_{R^\infty} = 0 \). This is the case if and only if \( R \) is a field, as desired. \( \square \)

The following provides a desingularization of the above result in a special case.

**Proposition 6.5.** Let \( R \) be a quasilocal containing a field of prime characteristic which is purely inseparable extension a noetherian regular ring \( R_0 \). If \( R \) contains all roots of \( R_0 \), then \( R \) has a desingularization.

**Proof.** Let \( R_0 \) be the noetherian regular ring such that \( R \) is integral over it. Without loss of generality we can assume that \( R_0 \) contains a field. Write \( R \) as a direct union of a filter system \( \{ R_i \} \) of its subrings which are module finite over \( R_0 \). Again, without loss of generality we can assume that \( (R_i, m_i) \) is local. Let \( x \) be a regular system of parameters on \( R_0 \). Remark that if \( y \in R_1 \), there is \( n_1 \in \mathbb{N} \) such that \( y^{p^{n_1}} \in R_1 \). We may assume that \( R_0^{1/p} \subseteq R_1 \). Use these to show that \( x \) is a regular sequence on \( R_1 \). Since \( R_0 \to R_1 \) is integral, \( \dim R_0 = \dim R_1 \). Hence,

\[ \dim R_1 \geq \text{depth} R_1 \geq m = \dim R_0 = \dim R_1, \]

i.e., \( R_1 \) is Cohen-Macaulay. The closed fiber \( m_0 R_1 \) is \( m_1 \)-primary. In view of Lemma 3.3, \( R_0 \to R_1 \) is flat and by [12, Theorem 2.2.12], \( R_1 \) is regular. Applying the same method, one can observe that \( R \) has the desingularization \( \{ R_i \} \) equipped with flat morphisms. \( \square \)
**Corollary 6.6.** Let $R$ be as Proposition 6.5. Then $R$ is stably coherent.

**Proof.** This follows by Proposition 6.5, where we showed that $R$ is a direct limit of a flat direct system of noetherian rings, see [17, Theorem 2.3.3]. □

The next says that the assumption of Proposition 6.5 and 6.3 is essential.

**Example 6.7.** Let $F$ be a field of characteristic 2 with $[F : F^2] = \infty$. Let $R_0 = F[[x, y]]$ be the formal power series on variables $\{x, y\}$ and look at $R := F[[x, y]][F]$. Let $\{b_i : i \in \mathbb{N}\} \subset F$ be an infinite set of 2-independent elements. Set $E_n := \sum_{i=n}^{\infty} (xy)^i b_i$, $e_n := E_n/y^n$ and $f_n := E_n/x^n$. Define $R := R_0[e_i, f_i : i \in \mathbb{N}]$. This ring is quasilocal, denote its unique maximal ideal by $m$.

(i) Recall from [30, Page 206] that $R_0 \to R$ is integral and purely inseparable. By [24, Example 1], $\dim_R(m) = \infty$ and that the maximal ideal of $R$ is finitely generated. Thus, $R$ is not regular. We conclude, immediately, from Lemma 5.1 that $R$ is not coherent. Now, Corollary 6.6 implies that the assumption of Proposition 6.5 is needed, i.e., $R_0^\infty \not\subseteq R$.

(ii) For the mentioned property of Proposition 6.3, set $\kappa := R/m$ and recall from [24, Example 1] that $\theta : \text{Sym}_\kappa(\oplus_{\mu(\mathfrak{m})}\kappa) \to \text{Gr}_R(m)$ is an isomorphism. Look at filtered direct system $\{(R_n, \mathfrak{m}_n) : n \in \mathbb{N}\}$ of noetherian local subrings of $R$ that containing $R_0$ with direct limit $R$. Suppose on the contrary that $R_n$ is regular. Keep in mind that $R_n \to R_{n+1}$ is integral. It follows by Lemma 3.3 that the extension is flat. Note that flat colimit of noetherian rings is coherent, see [17, Theorem 2.3.3]. On the other hand, and in view of Lemma 5.1, $R$ is not coherent. This is a contradiction that we search for it. Finally, we remark that $m_i^2 \neq m_i \cap m_{i+1}^2$.

### 7. Desingularization; Non-examples

In what follows we will use the following result several times.

**Lemma 7.1.** Suppose $F$ is a field. The following holds.

(i) If $F$ is a finite field, then $\prod_n F \simeq_{\text{ring}} \lim_{\gamma \in \Gamma} (\bigoplus_{1 \leq i \leq n_\gamma} F_i)$ where $F_i \subset F$ is a field.

(ii) If $F$ is a finite prime field, then $\prod_n F \simeq_{\text{ring}} \lim_{\gamma \in \Gamma} (\bigoplus_{n_\gamma} F)$.

(iii) If $F$ is an infinite field, then $\prod_n F \not\simeq_{\text{ring}} \lim_{\gamma \in \Gamma} (\bigoplus_{1 \leq i \leq n_\gamma} F_i)$ where $F_i \subset F$ is a field.

**Proof.** Let $S \subset \prod_n F$ be the subring consisting of all elements that have only finitely many distinct coordinates. By [18, Proposition 5.2], $S$ expressible as the directed union of artinian regular subrings $\{A_j\}$ of $\prod_n F$. Any artinian regular ring is isomorphic with a finite direct product of fields. Replace $A_j$ with one of its isomorphic versions, without loss of the generality, we may assume that $A_j \subset \bigoplus_{n_j} F$. Then

$$
S = \lim_{\gamma \in \Gamma} \bigoplus_{1 \leq i \leq n_\gamma} F \subset \prod_n F.
$$
Suppose now that $F$ is finite. Then $S = \prod_N F$ and this proves (i) and (ii).

(iii): The claim follows by [28, Introduction]. Note that [28, Introduction] states the assertion for $\mathbb{Q}$. It works for all infinite fields by the same argument.

The local assumption of Theorem 4.7 is needed.

**Example 7.2.** The ring $R := \prod_N \mathbb{Q}$ is coherent and regular. But $R$ is not a filter limit of its noetherian regular subrings.

**Proof.** By [17, Theorem 6.1.20], $\mathbb{Q}$ is a uniformly coherent ring. This means that $\prod_N \mathbb{Q}$ is coherent; see [17, Theorem 6.1.2]. It is well-known that $R$ is von Neumann regular, i.e., $R$ is of zero weak dimension. This implies that $R$ is regular (and of global dimension less than 3).

Suppose now on the contrary that $R$ is a filter limit of its noetherian regular subrings \{${R_i : i \in I}$\}. Denote the invertible elements of a ring $A$ by $U(A)$. For each $i$, define $S_i := \{r \in R_i : r \in U(R)\}$. It is a multiplicative closed subset of $R_i$ and $S_i \subset S_j$ if $i \leq j$. Thus

$$(S_i)^{-1}R_i \hookrightarrow (S_i)^{-1}R_j \hookrightarrow (S_j)^{-1}R_j \hookrightarrow (S_j)^{-1}R \simeq R.$$

By replacing, $(S_i)^{-1}R_i$ with $R_i$, without loss of generality we can assume that

$$U(R) \cap R_i = U(R_i).$$

Take $r \in R_i \setminus U(R_i)$. Then $r \in R \setminus U(R)$. Hence $r$ is a zero-divisor in $R$ and so in $R_i$. Thus $\dim R_i = 0$, because $R_i$ is Cohen-Macaulay. Keep in mind that zero dimensional noetherian regular rings are von Neumann regular (finite direct product of fields). By using Lemma 7.1 we get a contradiction. □

**Remark 7.3.** A filter system \{${R_i : i \in I}$\} of noetherian Cohen-Macaulay rings is called a deMacaulayfication of $\lim_{\rightarrow} R_i$.

(i) As an example, for any normal semigroup $H \subseteq \mathbb{Z}^n$ (not necessarily affine), the semigroup ring $\mathbb{Q}[H]$ has a deMacaulayfication. To see this, look at the set $\Gamma := \{X \subseteq H | X \text{ is finite}\}$. We direct $\Gamma$ by means of inclusion. Set $C_X := \mathbb{Z}_{\geq 0}X$ and let $C'_X$ be its normalization. Then $C'_X$ is normal and finitely generated. It is now easy to show $H = \lim_{\rightarrow X \in \Gamma} C'_X$. By a theorem of Hochster, $\mathbb{Q}[C'_X]$ is Cohen-Macaulay. So, the deMacaulayfication of $\mathbb{Q}[H]$ is $\{\mathbb{Q}[C'_X]\}$, see [3].

(ii) Non Cohen-Macaulay rings may have deMacaulayfication. For more details, see [3].

(iii) We cite [9], to see a ring $R$ without any deMacaulayfication, but satisfying in the Cohen-Macaulay property $ht(a) = K grading $R(a, R)$. Its proof uses a theorem of Bloch-Deligne-Illusie on identifying the slope $< 1$ part of rigid cohomology and Witt vector cohomology.
8. Regularity and Regular Sequences

The following extends the main result of [19] to the coherent case by a new argument. Our proof uses the concept of balanced big Cohen-Macaulay modules. Recall that a module $M$ over a noetherian local ring $R$ is balanced big Cohen-Macaulay if every system of parameters of $R$ is a regular sequence over $M$.

**Notation 8.1.** We denote the $i$-th Čech cohomology modules with respect to $x \subset R$ by $H^i_\varpi(-)$.

**Remark 8.2.** Here we remark that Čech cohomology modules does not change if we replace a sequence with any sequence with the same radical as the first one. In the case of nonnoetherian rings, we note that Čech cohomology does not necessarily coincide with derived local cohomology $\lim_{n \to \infty} \text{Ext}^\bullet_R(R/(x)^n, \sim)$.

**Proposition 8.3.** Let $R$ be a ring containing a field $k$ and $\underline{x} := x_1, \ldots, x_n$ a finite sequence in the Jacobson radical of $R$. The following holds:

- (i) If $H^n_\varpi(R) \neq 0$, then $R$ contains the polynomial ring with $n$ variables over $k$.
- (ii) If $R$ is coherent and $\underline{x}$ a regular sequence, then $R$ is a flat extension of a polynomial ring with $n$ variables over $k$. The converse holds without any coherent assumptions on $R$.

**Proof.** Let $A_0 := k[\underline{x}]$. Then it is noetherian and $(\underline{x})$ is its maximal ideal. First, we claim that $\underline{x}$ is algebraically independent over $k$. Let $a$ be the set of all polynomials $f \in k[\underline{X}]$ such that $f(\underline{x}) = 0$. It is an ideal and $k[\underline{X}]/a \simeq k[\underline{x}]$.

(i): Look at the top local cohomology isomorphism

$$0 \neq H^n_\varpi(R) \cong H^n_\varpi(k[\underline{x}]) \otimes_{k[\underline{x}]} R.$$ 

Hence $0 \neq H^n_\varpi(k[\underline{x}])$. Conclude by this that $\dim k[\underline{x}] = n$. Thus

$$n = \dim k[\underline{x}] = \dim k[\underline{X}]/a = \dim k[\underline{X}] - \mathrm{ht}(a) = n - \mathrm{ht}(a).$$

Clearly, $\mathrm{ht}(a) = 0$. As $k[\underline{X}]$ is domain, $a = 0$ and so $\underline{x}$ is algebraically independent over $k$, as claimed.

(ii): Since $\underline{x}$ is a regular sequence on $R$, $H^n_\varpi(R) \neq 0$. By part (i), $R_0 := k[\underline{x}]/(\underline{x})$ is a noetherian regular local ring. Clearly, $R$ is a big Cohen-Macaulay $R_0$-algebra.

Claim: $R$ is a balanced big Cohen-Macaulay $R_0$-algebra.

Indeed, let $\underline{y} := y_1, \ldots, y_n$ be a system of parameter for $R_0$. By induction on $n$, we show that $\underline{y}$ is a regular sequence on $R$. If $n = 1$, the claim is clear, because $H^0_\varpi(y_1; R) = (0 :_R y_1)$. Recall that $\text{K.grade}_R(\underline{y}, R) = \text{K.grade}_R(\underline{x}, R) = n$. Koszul cohomology modules are finitely presented, because $R$ is coherent. Then by applying an easy induction,

$$\text{K.grade}_R((y_1, \ldots, y_i), R) = i.$$
Set $\overline{R} := R/(y_1, \ldots, y_{n-1})$. By the induction hypothesis, $y_1, \ldots, y_{n-1}$ is a regular sequence on $R$. Thus $\text{K.grade}_R(y, \overline{R}) = 1$. Note that $H^0(y, \overline{R}) = (0 :_R y_n)$. Therefore, $y_n$ is a regular sequence over $\overline{R}$. So, $y$ is a regular sequence over $R$, as claimed.

In view of [22] 7.6, Flatness, $R$ is flat over $R_0$. Since localization is flat, $R_0$ is flat over $k[x]$. Due to the transversality of flatness, $R$ is flat over $k[x]$.

The proof of the converse part is trivial. □

The coherent condition of Proposition 8.3 (ii) is needed.

Example 8.4. Let $A$ be a non-Noetherian quasilocal ring containing a field $k$ with two elements $x$ and $y$ of the maximal ideal of $A$, such that the sequence $x, y$ is regular, while the sequence $y, x$ is not. Such a thing exists; see [14]. We claim that $R$ is not a flat extension of the polynomial ring $k[X, Y]$. Suppose on the contrary that $k[X, Y] \to R$ is flat. As $Y, X$ is a regular sequence on $k[X, Y]$ and in view of flatness, $y, x$ is a regular sequence, a contradiction.

Corollary 8.5. Adopt the assumption of Proposition 8.3 (ii). The following holds:

(i) Any permutation of $\underline{x}$ is a regular sequence.

(ii) For each $m_i \in \mathbb{N}$, $\underline{x}^{m_i} := x_1^{m_1}, \ldots, x_n^{m_n}$ is a regular sequence.

Proof. Let $k[X] \to R$ be the natural flat extension provided by Proposition 8.3 (ii). As any permutation of $X$ (resp. $X^{m_i}$) is a regular sequence on $k[X]$ and in view of flatness, any permutation of $\underline{x}$ (resp. $\underline{x}^{m_i}$) is a regular sequence. □

A ring extension $(A, m_A, k_A) \to (B, m_B, k_B)$ is called unramified (at the closed point) if it is flat, $m_A B = m_B$ and $k_B/k_A$ is separable.

Corollary 8.6. Let $(R, m)$ be a coherent super-regular ring containing a field. The following holds:

(i) $R$ dominates a noetherian regular local ring via a flat extension.

(ii) $R$ is an unramified extension of a noetherian regular local ring if $R$ has a coefficient field.

Proof. By Lemma 6.2, $m$ is generated by a regular $\underline{x} := x_1, \ldots, x_n$. Set $(R_0, m_0) := k[\underline{x}](\underline{x})$. By Proposition 8.3 $R_0 \to R$ is flat.

(i) : It remains to recall that $m \cap R_0 = m_0$.

(ii) : Note that $m_0 R = m$ and $R_0/m_0 \cong k = R/m$ which is trivially separable. □

9. Applications

Desingularization play a role with things that are behaved well with direct limit. In this section we collect some of its applications, mainly by a computational point of view. We do
this task by using several examples. The results divide into six subsections. We left their more properties to the reader. The first two subsections involved on desingularization of certain products.

**9.A: Stably coherent rings.** In this section we show how a ring which involved in a product can represent by a direct limit of noetherian regular rings. We show the ring under consideration is stably coherent by constructing a desingularization with flat morphisms.

**Notation 9.1.** For each ring $k$ and any finite family $\underline{X} = \{X_1, \ldots, X_n\}$ of variables, set $R_k := \prod_N (k[\underline{X}])$.

**Example 9.2.** Let $F$ be a finite field of characteristic two. Denote the prime subfield of $F$ by $F_2$. The following holds.

(i) $R_F = R_{F_2} \otimes_{F_2} F$.

(ii) $R_F$ is a flat direct limit of noetherian regular rings.

**Proof.** We prove both of them at the same time. First note that

\[
\prod_F F \simeq \prod_N (F \otimes_{F_2} F_2) \simeq (\prod_F F_2) \otimes_{F_2} F, \quad (9.2.1)
\]

because $F$ is finitely presented as an $F_2$-module.

(9.2.2): The ring $\prod F_2$ is boolean. Clearly, subrings of a boolean ring are boolean. Applying these, $\prod F_2$ is a direct limit of its boolean subalgebras $\{P(I_\gamma) : \gamma \in \Gamma\}$ that are finitely generated over $F_2$. Remark that $P(I_\gamma) \simeq \bigoplus I_\gamma F_2$ for some finite index set $I_\gamma$. That is $\prod F_2$ is a direct limit of $\bigoplus I_\gamma F_2$. One can deduce this directly from Lemma [7.1].

By $L$ we mean $(i_1, \ldots, i_n)$. Also, the notation $X^L$ stands for $X_1^{i_1} \cdots X_n^{i_n}$. Suppose $(\sum_{L(1)} a_{L(1)} X^{L(1)}, \ldots) \in \prod F_2[\underline{X}]$. The assignment

\[
(\sum_{L} a_{L(1)} X^{L(1)}, \ldots, \sum_{L} a_{L(\ell)} X^{L(\ell)}, \ldots) \mapsto \sum_{L} (a_{L(1)}, \ldots, a_{L(\ell)}, \ldots, X^{L}),
\]

gives an isomorphism

\[
\prod F_2[\underline{X}] \simeq (\prod F_2)[\underline{X}] \quad (9.2.3)
\]

Note that

\[
\lim_{\gamma \in \Gamma} R_{\gamma}[\underline{X}] \simeq (\lim_{\gamma \in \Gamma} R_{\gamma})[\underline{X}] \quad (9.2.4)
\]

for any filtered system $\{R_\gamma\}$ of rings. Thus

\[
\prod F_2[\underline{X}] \simeq (\prod F_2)[\underline{X}] \simeq (\lim_{\gamma \in \Gamma} P(I_\gamma))[\underline{X}] \simeq \lim_{\gamma \in \Gamma} (P(I_\gamma))[\underline{X}] \quad (9.2.5)
\]

*This is not true for formal power series. In fact, one can find a ring $A$ with a desingularization such that $A[[X]]$ does not have any desingularization.
Also, $\prod F[X] \simeq (\prod F)[X]$. Now,

\[
\prod F[X] \overset{9.2.3}{\simeq} (\prod F)[X] = (\prod F_2) \otimes_{\mathbb{F}_2} F[X]
\overset{9.2.1}{\simeq} \lim_{\gamma \in \Gamma} (\bigoplus_{\gamma} F)[X]
\overset{9.2.2}{\simeq} \lim_{\gamma \in \Gamma} (\bigoplus_{\gamma} \mathbb{F}_2)[X] \otimes_{\mathbb{F}_2} F
\overset{9.2.4}{\simeq} \lim_{\gamma \in \Gamma} (\bigoplus_{\gamma} \mathbb{F}_2[X]) \otimes_{\mathbb{F}_2} F,
\]

where 9.2.* follows by a natural isomorphism.

The ring $\bigoplus_{\gamma} F[X]$ is a finite direct product of $F[X]$. Hence $\bigoplus_{\gamma} F[X]$ is a noetherian regular ring. Suppose $I_\gamma \subset I_\delta$. Then $\bigoplus_{\gamma} F \to \bigoplus_{\delta} F$ is flat, since $\bigoplus_{\gamma} F$ is of zero weak dimension. Also,

\[
(\bigoplus_{\gamma} F)[X] \to (\bigoplus_{\delta} F)[X]
\]

is flat. This immediately implies that $R_F$ is a flat direct limit of noetherian regular rings.

**Lemma 9.3.** Adopt the above notation. The following assertions hold.

(i) Let $L/F$ be a finitely presented ring extension and $\prod_n F \simeq \lim_{\gamma \in \Gamma} (\bigoplus_{n} F)$. Then

\[
\begin{align*}
(1) & : \prod_N L \simeq \lim_{\gamma \in \Gamma} (\bigoplus_{n} L) \\
(2) & : R_L \simeq R_F \otimes_F L.
\end{align*}
\]

(ii) If $F$ is a finite field, then $R_F$ is a flat direct limit of noetherian regular rings.

**Proof.** (i): This follows by the proof of Example 9.2. (ii): In view of Lemma 7.1, this follows by the proof of Example 9.2.

**Lemma 9.4.** Let $\{R_i\}$ be a direct family of rings and $\underline{x} \subset \lim_{\gamma \in \Gamma} R_i$ be a finite set. Take $i$ be such that $\underline{x} \subset R_i$. Then

\[
H^\ell_i(\lim_{j \geq i} R_j) = \lim_{j \geq i} H^\ell_j(R_j).
\]

**Proof.** This is easy and we leave it to the reader.

**Corollary 9.5.** Let $F$ be a field of positive characteristic. The following holds.

(i) If $F$ is a finite field, then $R_F$ is stably coherent.

(ii) Suppose $X := X_1, \ldots, X_\ell$. Then

\[
H^i_X(R_F) = \begin{cases} 
X_1^{-1} \cdots X_\ell^{-1}(\prod F)[X_1^{-1}, \ldots, X_\ell^{-1}] & \text{for } i = \ell \\
0 & \text{for } i \neq \ell.
\end{cases}
\]

(iii) $\text{reg}(R_F) = 0$. 

Thus is an isomorphism. Now we use the data that we have proved in the case of finite fields. By Lemma 7.1, 
\[ H^i_{\Delta}(R_F) = H^i_{\Delta}(\lim_{\rightarrow} (\bigoplus_{1 \leq i \leq n} F_i)[X]) = \lim_{\rightarrow} H^i_{\Delta}(\bigoplus_{1 \leq i \leq n} F_i)[X], \]  
where \( F_i \subset F \) is a field. Set \( X^{-1} := X_1^{-1}, \ldots, X_\ell^{-1} \). This is well known that 
\[ H^i_{\Delta}(\bigoplus_{1 \leq i \leq n} F_i)[X]) = \begin{cases} 
X_1^{-1} \cdots X_\ell^{-1}(\bigoplus_{1 \leq i \leq n} F_i)[X^{-1}] & \text{for } i=\ell \\
0 & \text{for } i \neq \ell. 
\end{cases} \]
By Lemma 7.1,
\[ \lim_{\rightarrow} X_1^{-1} \cdots X_\ell^{-1}(\bigoplus_{1 \leq i \leq n} F_i)[X^{-1}] \simeq X_1^{-1} \cdots X_\ell^{-1}(\lim_{\rightarrow} (\bigoplus_{1 \leq i \leq n} F_i))[X^{-1}] \simeq X_1^{-1} \cdots X_\ell^{-1}(\prod F)[X^{-1}], \]
put this along with (*) yields the claim.

Suppose now that \( F \) is a general field of positive characteristic \( p \) and \( F_p \) is a prime subfield. Note that \( \prod F_p \) is of zero weak dimension. Hence, \( \prod F_p \to \prod F \) is flat. It implies that \( \psi : (\prod F_p)[X] \to (\prod F)[X] \) is flat. Look at 
\[ (\prod F_p)[X] \xrightarrow{\psi} (\prod F)[X] \]
\[ \simeq \prod (F_p[X]) \xrightarrow{\phi} (\prod (F[X])), \]
where the isomorphisms provide by (9.2.3). So, \( \phi : R_{F_p} \to R_F \) is flat. We apply flat base change for Čech cohomology modules, that is, the following map
\[ H^i_{\Delta}(R_F) \to H^i_{\Delta}(R_{F_p}) \otimes_{R_{F_p}} R_F \]
is an isomorphism. Now we use the data that we have proved in the case of finite fields. Thus 
\[ H^i_{\Delta}(R_F) \simeq (X_1^{-1} \cdots X_\ell^{-1}(\prod F_p)[X^{-1}]) \otimes_{R_{F_p}} R_F \simeq X_1^{-1} \cdots X_\ell^{-1}(\prod F_p \otimes_{R_{F_p}} R_F)[X^{-1}] \simeq X_1^{-1} \cdots X_\ell^{-1}(\prod F)[X^{-1}] \simeq X_1^{-1} \cdots X_\ell^{-1}(\prod F)[X^{-1}] \]
Therefore, 
\[ H^i_{\Delta}(R_F) = \begin{cases} 
X_1^{-1} \cdots X_\ell^{-1}(\prod F)[X_1^{-1}, \ldots, X_\ell^{-1}] & \text{for } i=\ell \\
0 & \text{for } i \neq \ell, 
\end{cases} \]
as claimed.

(iii): Let \( \{R_i\} \) be any nonempty family of \( \mathbb{N} \)-graded rings. For each \( n \in \mathbb{N} \) define \( R(n) = \prod_i (R_i)_n \subseteq R := \prod_i R_i \). Then \( R(n) \) is an abelian group, \( R(n).R(m) \subseteq R(n + m) \) and \( \bigoplus R(n) = R \), i.e., \( \{R(n)\} \) defines structure of a graded-ring on \( R \). We note that direct
limit of a direct system of \( \mathbb{N} \)-graded (resp. \( \mathbb{Z} \)-graded) rings (resp. modules) is \( \mathbb{N} \)-graded (resp. \( \mathbb{Z} \)-graded).

Suppose now that \( F \) is a general field of positive characteristic \( p \) and \( \mathbb{F}_p \) is a prime subfield. Then

\[
(R_F)_0 = \prod F, \quad \text{and} \quad (R_{\mathbb{F}_p})_0 = \prod \mathbb{F}_p.
\]

Also, in view of the above diagram,

\[
R_{\mathbb{F}_p} \otimes (R_{\mathbb{F}_p})_0 (R_F)_0 := \prod (\mathbb{F}_p[X]) \otimes \prod \mathbb{F}_p \prod F
\]

\[
\cong (\prod \mathbb{F}_p)[X] \otimes \prod \mathbb{F}_p \prod F
\]

\[
\cong (\prod F)[X]
\]

\[
\cong R_F,
\]

where (9.5.*) is a natural isomorphism. Note that \( (R_F)_+ := \bigoplus_{n > 0} (R_F)_n \) is finitely generated. Hence we can apply the Čech cohomology modules with respect to a generating set of it. Thus by graded flat base change theorem,

\[
H^i_{(R_F)_+}(R_F) \longrightarrow H^i_{(R_{\mathbb{F}_p})_+}(R_{\mathbb{F}_p}) \otimes R_{\mathbb{F}_p} R_F
\]

is a graded-preserving isomorphism. One may read this as follows

\[
H^i_{(R_F)_+}(R_F)_n \cong H^i_{(R_{\mathbb{F}_p})_+}(R_{\mathbb{F}_p})_n \otimes (R_{\mathbb{F}_p})_0 (R_F)_0 \quad (\ast)
\]

We indexed the last nonzero graded component of a graded module by \( \text{end}(\sim) \). By definition, Castelnuovo-Mumford regularity is equal with

\[
\text{reg}(R_F) := \sup \{ \text{end}(H^i_{(R_F)_+}(R_F)) + i \mid i \in \mathbb{N} \cup \{0\} \}.
\]

Let \( (x_i) \in \prod \mathbb{F}_p \) be such that \( (x_i)(y_i) = (1) \) for some \( (y_i) \in \prod F \). Then \( y_i = x_i^{-1} \). As \( \mathbb{F}_p \) is a field, \( y_i \in \mathbb{F}_p \). So \( (x_i) \prod \mathbb{F}_p = \prod \mathbb{F}_p \). This implies that \( (R_{\mathbb{F}_p})_0 \rightarrow (R_F)_0 \) is faithfully flat. Thus, in view of (\ast), we get that

\[
\text{reg}(R_F) = \text{reg}(R_{\mathbb{F}_p}),
\]

and so we may assume that \( F = \mathbb{F}_p \) is a finite field.

Claim: The map \( \varphi : R_F \rightarrow \lim (\bigoplus_{1 \leq i \leq n, F_i}[X]) \) of Lemma 9.3 respects the graded structures, that is, \( \varphi(R_F(n)) = (\lim (\bigoplus_{1 \leq i \leq n, F_i}[X]))(n) \).

Indeed, let \( f \in R_F(n) \). Then, by the definition, \( f := (\sum I(a_{I(1)}, \ldots)) \) where \( |I| = n \). Under the identification (9.2.3), \( f = \sum I(a_{I(1), \ldots})X^I \) which is graded in \( (\prod F)[X] \) of degree \( n \). Without loss of the generality, we identify \( (a_{I(1), \ldots}) \) with \( (b_{I(1), \ldots}) \otimes y \) where \( (b_{I(1), \ldots}) \in \prod \mathbb{F}_p \) and \( y \in F \). In view of Lemma 7.1 and up to an isomorphism, one can assume that \( (b_{I(1), \ldots}) \in \bigoplus \mathbb{F}_p \) for some \( \ell \). Look \( f \) at \( (\bigoplus \mathbb{F}_p)[X] \otimes F \). Then it is of degree \( n \). Clearly,

\[
(\bigoplus \mathbb{F}_p)[X] \otimes F \rightarrow (\bigoplus \mathbb{F}_p)[X] \rightarrow \lim (\bigoplus \mathbb{F}_p)[X]
\]

is degree-preserving. This yields the claim.
The map $\varphi$ induces a graded structure on $R_F$. This two graded structures on $R_F$ are the same, because $\varphi$ is a degree-zero graded isomorphism. We work with the second graded structure on $R_F$ and remark that its irrelevant ideal $(R_F)_+$ is the direct limit of the irrelevant ideals of the direct system. By definition,

$$\text{reg}(R_F) = \sup\{\text{end}(H^i_{(R_F)_+}(R_F)) + i \mid i \in \mathbb{N} \cup \{0\}\} = \sup\{\text{end}(H^i_X(R_F)) + i \mid i \in \mathbb{N} \cup \{0\}\},$$

which is zero by (ii).

**Remark 9.6.** It is shown by Soublin that $\prod \mathbb{N} \mathbb{Q}[\llbracket X, Y \rrbracket]$ is not stably coherent; see [38]. We have no data on desingularization of $\prod \mathbb{N} \mathbb{Q}[\llbracket X, Y \rrbracket]$.

**9.B: Differential forms and De Rham cohomology.** Our aim in this subsection is to compute modules of differential forms and De Rham cohomology modules, by the help of desingularization. Our reference for these topics is [15, Chapter 16].

**Lemma 9.7.** Let $F$ be a field and let $R$ be a ring with a desingularization $\{F[\mathbb{X}_i] : i \in I\}$ of polynomial rings. Then the $R$-module $\Omega^1_{R/F}$ is flat.

**Proof.** In view of [15, Theorem 16.8],

$$\Omega^1_{R/F} \simeq \lim_{\to i}(\Omega^1_{F[\mathbb{X}_i]/F} \otimes_{F[\mathbb{X}_i]} R).$$

But $\Omega^1_{F[\mathbb{X}_i]/F}$ is free as an $F[\mathbb{X}_i]$-module. This implies that $\Omega^1_{R/F}$ is flat as an $R$-module.

To give an application of the above Lemma, we present the following definition.

**Definition 9.8.** (see [2, Section 3]) Let $I \subseteq \mathbb{N}$ be an infinite index set. Denote the i-th component of $\alpha \in \prod \mathbb{N} \mathbb{Q}$ by $\alpha_i$. Then $\alpha$ is called $I$-supported if $\alpha_i \neq 0$ for all $i \in I$. Also, an element $\alpha \in \prod \mathbb{N} \mathbb{Q}$ is called almost positive, if there exists only finitely many $i \in \mathbb{N}$ such that $\alpha_i$ is negative. The notation $M \subseteq \prod_{i \in \mathbb{N}} \mathbb{Q}$ stands for the set of all almost positive and $I$-supported elements. We now define $\tilde{H} := M \cup \{0\}$.

**Example 9.9.** Let $F$ be a field and $\tilde{H}$ be as above. Let $R$ be the semigroup ring $F[\tilde{H}]$. The following assertions hold.

(i) $R$ has a desingularization consisting of polynomial rings over $F$.

(ii) The $R$-module $\Omega^1_{R/F}$ is flat.

**Proof.** First recall that for a semigroup $H$, $k[H]$ is the $k$-vector space $\bigoplus_{h \in H} kX^h$. It equipped with a multiplication structure whose table is given by $X^hX^{h'} := X^{h+h'}$.

(i): This is proved in [2, Theorem 4.9]. In fact $R$ is a direct limit of polynomial rings over $F$ with toric maps, i.e., monomials go to the monomials. It may be worth to note that this follows by $\tilde{H} \simeq_{\text{semigroup}} \lim_{\to i} Q_{\geq 0}^{\mathbb{N}_i}$; see [2, Section 3].

(ii): This is an immediate corollary of Lemma 9.7. □
Recall that $H^i_{dR}(-)$ is the De Rham cohomology, see [15, Exercise 16.15] for its definition.

**Lemma 9.10.** Let $F$ be a field, and $A$ a ring with a desingularization $\{R_j : j \in I\}$ consisting of polynomial rings equipped with flat morphisms. There is the following isomorphism of $F$-vector spaces

$$H^i_{dR}(A/F) \simeq \lim_{j \in I} \lim_{k(j)} H^i_{dR}(R_j/F)^{n_{k(j)}},$$

where $n_{k(j)}$ are positive integers.

**Proof.** Denote the exterior algebra by $\bigwedge(\sim)$. We collect the following facts.

1. Note that for an $R$-algebra $S$ and an $S$-module $M$, one has

   $$\bigwedge(M \otimes_R S) \simeq \bigwedge(M) \otimes_R S$$

   as graded rings.

2. Let $\{M_i\}$ be a flat direct family of modules and let $x \in (\varinjlim M_i)^{\otimes n}$. Then, for some $i_1 \leq \ldots \leq i_n$, one has $x \in M_{i_1} \otimes \ldots \otimes M_{i_n} \subseteq (M_n)^{\otimes n}$, because of the flatness. It turns out that $\bigoplus (\varinjlim M_i)^{\otimes n} \simeq \varinjlim (\bigoplus (M_i)^{\otimes n})$. Denote the ideal of $\bigoplus (\varinjlim M_i)^{\otimes n}$ (resp. $\bigoplus (M_i)^{\otimes n}$) generated by $x \otimes x$ where $x \in \varinjlim M_i$ (resp. $x \in M_i$) by $I$ (resp. $I_i$). Then $I = \varinjlim I_i$ and so

   $$\bigwedge(\varinjlim M_i) = \bigoplus (\varinjlim M_i)^{\otimes n} / I \simeq \varinjlim \bigoplus (M_i)^{\otimes n} / I_i \simeq \varprojlim \bigwedge(M_i).$$

3. Recall that $\Omega^1_{R_j/F} \simeq \varinjlim \Omega^1_{R_j/F} \otimes R_j$. $R_j$.

4. For each $j$, there are positive integers $n_{k(j)}$ such that $R \simeq \varinjlim R_j^{n_{k(j)}}$, because $R$ is flat as an $R_j$-module and the claim follows due to the Lazard’s theorem. 

5. Direct limit commutes with cohomology.

Then, in the light of definition,

$$H^i_{dR}(R) := H^i(\ldots \to \bigwedge^{i-1} \Omega^1_{R_j/F} \to \bigwedge^i \Omega^1_{R_j/F} \to \bigwedge^{i+1} \Omega^1_{R_j/F} \to \ldots)$$

$$\simeq \lim_{j \in \mathbb{I}} \lim_{k(j)} H^i(\bigoplus^{n_{k(j)}}(\ldots \to \bigwedge^{i-1} \Omega^1_{R_j/F} \otimes R_j \to \bigwedge^i \Omega^1_{R_j/F} \otimes R_j \to \ldots))$$

$$\simeq \lim_{j \in \mathbb{I}} \lim_{k(j)} H^i(\bigoplus^{n_{k(j)}}(\ldots \to \bigwedge^{i-1} \Omega^1_{R_j/F} \otimes R_j \to \bigwedge^i \Omega^1_{R_j/F} \otimes R_j \to \ldots))$$

$$\simeq \lim_{j \in \mathbb{I}} \lim_{k(j)} H^i(\bigoplus^{n_{k(j)}}(\ldots \to \bigwedge^{i-1} \Omega^1_{R_j/F} \otimes R_j \to \bigwedge^i \Omega^1_{R_j/F} \otimes R_j \to \ldots))$$

$$\simeq \lim_{j \in \mathbb{I}} \lim_{k(j)} H^i(\bigoplus^{n_{k(j)}}(\ldots \to \bigwedge^{i-1} \Omega^1_{R_j/F} \otimes R_j \to \bigwedge^i \Omega^1_{R_j/F} \otimes R_j \to \ldots))$$

$$\simeq \lim_{j \in \mathbb{I}} \lim_{k(j)} H^i(\bigoplus^{n_{k(j)}}(\ldots \to \bigwedge^{i-1} \Omega^1_{R_j/F} \otimes R_j \to \bigwedge^i \Omega^1_{R_j/F} \otimes R_j \to \ldots))$$

$$\simeq \lim_{j \in \mathbb{I}} \lim_{k(j)} H^i_{dR}(R_j)^{n_{k(j)}}.$$ 

\[ \square \]

*The morphisms of the De Rham complex associated to $R_j/F$ are not $R_j$-morphisms, that is, one can not commutes a tensor product with a flat $R_j$-module by cohomology modules via a direct method.
Here we give an application of the above lemma.

Example 9.11. Let $F$ be a finite field and $R_F := \prod_{\mathbb{N}}(F\lbrack X\rbrack)$ be as Notation 9.4. There is the following isomorphism of $F$-vector spaces

$$H^1_{dR}(R_F/F) \cong \left\{ \begin{array}{ll} \prod_{\mathbb{N}} F & \text{for } i=0 \\ 0 & \text{for } i>0. \end{array} \right.$$  

Proof. In the light of Lemma 9.3, $R_F$ is a flat direct limit of the noetherian regular rings $\{ (\bigoplus_{1 \leq i \leq n_\gamma} F_i)\lbrack X\rbrack | j \in I \}$, where $F_i \subset F$ is a field. Set $R_j := (\bigoplus_{1 \leq i \leq n_\gamma} F_i)\lbrack X\rbrack$. By Lemma 9.10, there is the following isomorphism of $F$-vector spaces

$$H^1_{dR}(A/F) \cong \lim_{\longleftarrow} \bigoplus_{j \in I} H^1_{dR}(R_j)^{n_k(j)}$$

where $n_k(j)$ are positive integers. In view of the Poincaré Lemma [15, Exercise 16.15(c)],

$$H^1_{dR}((\bigoplus_{1 \leq i \leq n_\gamma} F_i)\lbrack X\rbrack) = \left\{ \begin{array}{ll} \bigoplus_{1 \leq i \leq n_\gamma} F_i & \text{for } i=0 \\ 0 & \text{for } i>0. \end{array} \right.$$  

Note that $I$ is an index set with cardinality $\aleph_1$. Also, $\{ n_k(j) \}$ is a set of cardinality at most $\aleph_1$. Remark that $\prod_{\mathbb{N}} F \cong \ker(f) \oplus f(F)$ and so $H^1_{dR}(R_F/F)$ is an $F$-vector space with a base of cardinality bounded from $\aleph_1$ to $\aleph_1 \times \aleph_1 \times \aleph_1 = \aleph_1$, because $F$ is finite. It remains to mention that any $F$-vector space of cardinality $\aleph_1$ is isomorphic with $\prod_{\mathbb{N}} F$. □

9.C: Projective modules. Our reference for projective modules is [25]. The results inspired by [39].

Fact 9.12. Let $R$ be a ring and $P$ a finitely generated $R$-module. Then $P$ is projective if and only if there exists a free $R$-module $F$ of finite rank and an $R$-linear endomorphism $f$ of $F$ such that $P$ is a submodule of $F$, $f^2 = f$ and $f(F) = P$.

Proof. Due to definition, there exists a free $R$-module $F$ of finite rank such that $P$ is a direct summand of $F$. Using this, construct an $R$-linear endomorphism $f$ of $F$ such that $f^2 = f$ and $f(F) = P$. Conversely, given a free $R$-module $F$ of finite rank and $R$-linear endomorphism $f$ of $F$ with $f^2 = f$ and $f(F) = P$. It implies that

$$F \cong \ker(f) \oplus f(F) \cong \ker(f) \oplus P.$$  

Hence, $P$ is a finitely generated projective $R$-module. □

Proposition 9.13. Let $A$ be a ring containing a field and suppose that $A$ has a desingularization. Then any finitely generated projective $A[X_n : n \in \mathbb{N}]$-module is free.
Proof. Let $P$ be a finitely generated projective module over $A[X_{\infty}] := A[X_n : n \in \mathbb{N}]$. Let $f$ and $F$ be as Fact 9.12. Using a basis of $F$, we regard $f$ as an idempotent matrix of size $n$. Remark that $f$ as a matrix, has only finitely many entries in $A$. For an index set $I$, let $\{A_i : i \in I\}$ be a desingularization for $A$. Without loss of generality we assume that each $A_i$ contains a field. Take the integer $m$ and $i \in I$ such that all entries of the matrix $f$ are in $A_i[X_m] := A_i[X_1, \ldots, X_m]$. By Fact 9.12, $Q = f(A_i[X_m]^n)$ is a projective $A_i[X_m]$-module. Note by the celebrated Bass-Quillen theorem that every finitely generated projective $A_i[X_m]$-module is free. Thus, $Q$ is a free $A_i[X_m]$-module. One has
\[
P = f(F) 
\cong f(A_i[X_m]^n) \otimes_{A_i[X_m]} A[X_{\infty}] 
\cong Q \otimes_{A_i[X_m]} A[X_{\infty}].
\]
Therefore, $P$ is free as an $A[X_{\infty}]$-module. \hfill \Box

The next result provides a simple proof of [26, Theorem B], in the following zero characteristic case.

**Corollary 9.14.** Let $V$ be a Bezout domain containing a field $k$ of zero characteristic. Then any finitely generated projective $V[X_1, \ldots, X_n]$-module is free.

**Proof.** By [35, Theorem 1'], we assume that $V$ is valuation. In view of Lemma 4.1, $V$ has a desingularization. The claim is now clear by Proposition 9.13. \hfill \Box

**9.D: Projective dimension.** Lemma 5.2 implies regularity of many examples presented in above except Example 9.9. It implies the following:

**Example 9.15.** Let $R$ be as Example 9.9 and $I = (f_1, \ldots, f_\ell)$ a finitely generated homogeneous ideal of $R$. Then $I$ has finite projective dimension.

**Proof.** For a each semigroup $G$, recall that $k[G]$ is the $k$-vector space $\bigoplus_{g \in G} kX^g$. It carries a natural multiplication whose table is given by $X^gX^{g'} := X^{g+g'}$. Note that $k[G]$ is equipped with a structure of $G$-graded ring defined by $G$. By homogeneous, we mean homogenous elements of $k[G]$ with respect to this graded structure.

By part (i) of Example 9.9 there are polynomial rings $R_\gamma$ such that $R = \lim_{\gamma \in \Gamma} (R_\gamma, \varphi_{\gamma\delta})$ and $f \in R_\gamma$ for all $\gamma$. Set $I_\gamma := fR_\gamma$. Since $\varphi_{\gamma\delta}$ sends monomials to monomials, we can assume that $I_\gamma$ is generated by monomials. By using the Taylor resolution [15, Ex. 17.11], $p.\dim(R_\gamma/I_\gamma) \leq \ell$. Similar as Lemma 5.2
\[
\Tor^R_j(R/I, -) \cong \lim_{\gamma \in \Gamma} \Tor^R_j(R_\gamma/I_\gamma, -) = 0,
\]
for all $j > \ell$. This results that $\text{fl.\dim}(R/I) \leq \ell$. Note that any ideal of $R$ can be generated by a set of cardinality $\aleph_1$. Thus the proof of [33, Corollary 2.47], implies that $p.\dim(R/I) \leq \ell + 2$. \hfill \Box
9.E: Cohen-Macaulay rings. It may be worth to recall an application of desingularization to Cohen-Macaulay rings from our joint work with Dorreh [2].

Remark 9.16. (i) We note that there are many characterizations of noetherian Cohen-Macaulay rings. In the non-noetherian case, these are not necessarily equivalent. For more details see [7].

(ii) Recall from [2] that the equality of height and grade of ideals does not hold in the associated semigroup ring over a field $k$ of a normal and non-affine subsemigroup of $\mathbb{Z}^2$.

(iii) The concept of strong parameter sequence that we use here is a non-noetherian version of system of parameters in the local algebra.

Let $k$ be a field and $H \subseteq \mathbb{Z}^n$ be a normal semigroup but not necessarily finitely generated. Theorem 1.1 in [2] shows that any monomial strong parameter sequence of $k[H]$ is a regular sequence. One of the main ideas it was to reduce things to the following desingularization situation.

Lemma 9.17. (see [2]) Let $k$ be a field and $H \subseteq \mathbb{Z}^n$ be a positive and normal semigroup. Then there is a direct system $\{A_n : n \in \mathbb{N}\}$ with the following properties:

(i) $A_n$ is a noetherian polynomial ring over $k$ for all $n \in \mathbb{N}$.

(ii) $A_n \to A_m$ is toric for all $n \leq m$.

(iii) $k[H]$ is a direct summand of $\lim_{n \to \infty} A_n$.

9.F: non-noetherian closure operations. In this subsection we show how a direct limit argument helps to extend a closure operation from noetherian situation to non-noetherian case. We work with tight closure and left the other operations (and properties) to the reader. Let $A$ be a ring of prime characteristic $p$. By $A^0$ we mean the complement of the set of all minimal primes of $A$. Let $I$ be an ideal of $A$. The notation $I^q$ stands for the ideal of $A$ generated by $q = p^e$-th powers of all elements of $I$. Recall that the tight closure $I^*$ of $I$ is the set of $x \in A$ such that there exists $c \in A^0$ with the property $cx^q \in I^q$ for $q \gg 0$. For more details see [21].

Definition 9.18. Adopt the above notation and let $x \in A$. Suppose there is a direct family $\{A_j : j \in J\}$ of rings with $A = \varprojlim_{j \in J} A_j$ for which $x \in (I \cap A_j)^*$ when $x \in A_j$ for some $j \in J$. Then, we say $x \in I^{*\lim}$, if $A_j$ is of finite type over $\mathbb{F}_p$ as a subalgebras of $A$ for all $j \in J$.

Lemma 9.19. Let $I$ be an ideal of a ring $A$. The following holds.

(i) $I^{*\lim} \subseteq I^*$ when $A$ is reduced.

(ii) $I^* \subseteq I^{*\lim}$ if $A$ is of finite type over $\mathbb{F}_p$.

(iii) $(-)^{*\lim}$ is a closure operation that is: 1) $I \subseteq I^{*\lim}$, 2) $(I^{*\lim})^{*\lim} = I^{*\lim}$, 3) If $J \subseteq I$, then $J^{*\lim} \subseteq I^{*\lim}$.

Proof. (i): Let $x \in I^{*\lim}$ and let $\{A_j : j \in J\}$ be as definition. Then, there is $j \in J$ and $c \in (A_j)^0 \subseteq A^0$ such that $x \in A_j$ and $cx^q \in (I \cap A_j)^q \subseteq I^{[q]}$. So $x \in I^*$.
(ii): Suppose $A$ is of finite type over $\mathbb{F}_p$ and $x \in I^*$. Look at the singleton direct system \{A\}. By definition $x \in I^{*\text{tim}}$ and so $I^* \subseteq I^{*\text{tim}}$.

(iii): The first and the third are trivial. Let $x \in (I^{*\text{tim}})^{\ast\text{tim}}$ and let $\{A_j : j \in J\}$ be the direct family of finite type $\mathbb{F}_p$-subalgebras of $A$ with direct limit $A$. Suppose that $x \in A_i$. Then $x \in (I^{*\text{tim}} \cap A_i)^{\ast} \subseteq A_i$. Let $c \in (A_i)^0$ be such that
\[ cx^q = a_{1q}y_1^q + \ldots + a_{bq}y_b^q \quad \forall q \gg 0 \quad (\dagger) \]
where $a_{kj} \in A_i$ and $y_k \in I^{*\text{tim}} \cap A_i$. By definition, $y_k \in (I \cap A_i)^{\ast}$. Suppose $\{b_1, \ldots, b_\ell\}$ is a generating set for $(I \cap A_i)^{\ast} \subseteq A_i$. Take $d_j \in (A_i)^0$ be such that $d_jy_j^q \in (I \cap A_i)^{[q]}$. Set $d := \prod d_j \in (A_i)^0$. Then
\[ d(I \cap A_i)^{\ast[q]} \subseteq (I \cap A_i)^{[q]} . \]
In particular, $dy_j \in (I \cap A_i)^{[q]}$. Multiply $(\dagger)$ with $d$ we have
\[ (cd)x^q = a_{1q}(dy_1^q) + \ldots + a_{\ell q}(dy_\ell^q) \in (I \cap A_i)^{[q]} , \]
i.e., $x \in (I \cap A_i)^{\ast}$. So $x \in I^{*\text{tim}}$. The reverse inclusion follows by the first and the third items. \hfill \Box

One of advantage of $I^{*\text{tim}}$ to $I^*$ is the tightness property:

**Proposition 9.20.** Let $A$ be a quasilocal ring of prime characteristic $p$ with a desingularization $\{R_i : i \in I\}$. The following holds:

(i) For each ideal $I$, one has $I^{*\text{tim}} = I$.

(ii) There is a coherent quasilocal regular ring with a desingularization and a (non-finitely generated) ideal $I$ such that $I \neq I^*$. In particular, $I^{*\text{tim}} \neq I^*$.

**Proof.** (i): In view of Lemma 9.18 $I \subseteq I^{*\text{tim}}$. To see the reverse inclusion, let $x \in I^{*\text{tim}}$ and suppose $\{A_j : j \in J\}$ is as definition. Take $j \in J$ be such that $x \in A_j$. As $A_j$ is of finite type over $\mathbb{F}_p$, one has $A_j \subseteq R_{i(j)}$ for some $i(j) \in I$. By definition, $x \in (I \cap A_j)^{\ast}$. In view of persistence property, we have
\[ x \in (I \cap A_j)^{\ast}R_{i(j)} \subseteq ((I \cap A_j)R_{i(j)})^{\ast} . \]
Note that the persistence of tight closure holds for the map $A_j \to R_{i(j)}$. By [12, Theorem 10.1.7], the tightness property, $x \in ((I \cap A_j)R_{i(j)}) \subseteq I$. So $I^{*\text{tim}} \subseteq I$.

(ii): Take $R$ be a 1-dimensional noetherian integral domain of prime characteristic $p$ such that $A := R^\infty$ is coherent. Such a thing exists. In view of Example 4.2, $A$ has a desingularization. Let $(V, v)$ be a discreet valuation ring birationally dominate $R$. Let $v$ be a value map of $V$ and take $y \in R$ such that $v(y) = \ell \in \mathbb{N} \setminus \{0\}$. Let $r \in R^\infty$. Then $r^\ell p^\ell \in R$ for some $n$. The assignment $r \mapsto v(r^\ell p^n)$ defines a value map on $R^\infty$. Set $I := \{x \in R^\infty : v(x) > \ell/p\}$. To finish the argument, it remains to recall from [4, Example 2.2] that $I \neq I^*$.

Recall that a ring $A$ is *weakly $F$-regular* if every ideal of $A$ is tightly closed.
Corollary 9.21. The following assertions hold.

(i) Let \( \{ R_i \} \) be a direct family of finite type algebras over \( \mathbb{F}_p \) such that \( I^* = I \) for all \( I \leq R_i \) and all \( i \). Then, for each ideal \( I \leq \lim_{j \in J} R_j \), one has \( I^{* \lim} = I \).

(ii) Direct limit of weakly \( F \)-regular rings may not be weakly \( F \)-regular.

(iii) Direct limit of weakly \( F \)-regular rings is weakly \( F \)-regular, if the direct limit is reduced and of finite type over \( \mathbb{F}_p \).

Proof. (i): This is immediate by definition.

(ii): Recall that noetherian regular rings satisfy in the tightness property. So they are weakly \( F \)-regular. The claim is now clear by Proposition 9.20(ii).

(iii): In view of the first two items of Lemma 9.19, one has \( I^* \subseteq I^{* \lim} \subseteq I^* \). To deduce the claim it remains to recall from part (i) that \( I^{* \lim} = I \). \( \square \)

10. Concluding remarks and questions

In this section we state our concluding remarks and questions.

Remark 10.1. Quasilocal rings with a desingularization are normal domain, because direct limit of normal domains is normal. Now, having Question 1.1 in mind, one difficulty is that quasilocal regular rings may have a zero-divisor. Also, there are quasilocal regular domains such that they are not normal. These suggest to work with coherent rings.

One source for producing regular rings is the ring of continuous functions from certain topological spaces to certain fields.

Question 10.2. Under what conditions a ring of continuous functions has a desingularization?

We cite from [28] the desingularization theory of rings of continuous functions.

Remark 10.3. Denote the ring of continuous function from a topological space \( X \) to topological field \( F \) by \( C(X,F) \).

(i) Let \( X \) be a profinite topological space and \( F \) a discrete field. Then \( C(X,F) \) is a direct limit of finite product of fields.

(ii) Adopt the assumption of part (i). Set \( X := \{1/n : n \in \mathbb{N}\} \cup \{0\} \) and look at the subring \( R := \{f \in C(X,C) : f(0) \in \mathbb{R}\} \). Then \( R \) is a direct limit of \( \{C^n \times \mathbb{R} : n \in \mathbb{N}\} \), and so \( R \) has a desingularization.

(iii) Let \( X \) be the Stone-Čech compactification of a discrete space and \( F \) a field. Then \( C(X,F) \) is a direct limit of finite product of fields.

Due to the results of Section 5, we ask the following.

Question 10.4. Let \( \{ R_i : i \in I \} \) be a direct family of noetherian regular rings. Suppose \( R := \lim_{i} R_i \) is coherent. Is \( R \) regular?
Having Notation 9.1 in mind we ask:

**Question** 10.5. Suppose $F$, in Example 9.11, is a general field. What is $H^i_{dR}(R_F/F)$?

**Question** 10.6. Suppose $F$, in Corollary 9.5, is a general field. What can say about $\text{reg}(R_F)$?

The next question is open for noetherian rings and has connection in the counterexample to the localization problem, see [11]. There is a notherian ring $R = K[x, y, z, t]/(G)$ and the multiplicative system $S = K[t]\setminus 0$. Now $R_S = \lim_{P \in S} R_P$, and there is an element $f$ and an ideal $I$ in $R$ such that $f \not\in I^*$ in $R$, but in $R_S$. It is not known whether $f \in I^*$ in $R_P$ for some $P$.

**Question** 10.7. Is $* = *_{\lim}$?

Witt vector of a prime-characteristic perfect filed is a discreet valuation domain. It may be natural to search a nonnoetherian version of this result. We remark that the ring of Witt vectors (set theoretically) defined by product. Having the results of subsection 9.A, we ask the following:

**Question** 10.8. Is there a desingularization theory for the ring of Witt vectors of certain coherent perfect algebras?

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*E-mail address: mohenasgharzadeh@gmail.com*