INCOMPRESSIBLE BOUSSINESQ EQUATIONS AND BORDERLINE BESOV SPACES

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Abstract. We prove local-in-time existence and uniqueness of an inviscid Boussinesq-type system. We assume the density equation contains nonzero diffusion and that our initial vorticity and density belong to a space of borderline Besov type.

1. Introduction

Consider the two dimensional Boussinesq system given by

\[
\begin{aligned}
&\partial_t u + (u, \nabla) u - \nu \Delta u + \nabla P = \begin{pmatrix} 0 \\ \rho \end{pmatrix} \\
&\partial_t \rho + (u, \nabla) \rho = \kappa \Delta \rho \\
&\text{div } u = 0 \\
&u(x, 0) = u_0(x), \quad \rho(x, 0) = \rho_0(x).
\end{aligned}
\]

Where \( u = (u_1, u_2) \) is the velocity field, \( \rho \) is the scalar density and \( P \) is the pressure. This system is used to model, among other things, the influence of gravitational forces on the motion of a lighter or denser fluid. As such, it has relevance in the study of atmospheric and oceanographic dynamics and turbulence where rotation or stratification occurs (cf. [Maj03]). Of mathematical interest, note that for \( \rho \equiv 0 \) we recover the incompressible Navier-Stokes and Euler Equations (for \( \nu > 0 \) and \( \nu \equiv 0 \), respectively). Furthermore, the system \((B_0,0)\) has vortex stretching similar to that of the 3D axisymmetric flow, thus providing a model under which to approach the formation of finite time singularities (see [MB02] for discussion of this relationship). While there has been some numerical study of finite time singularity for \((B_0,0)\), the results remain inconclusive [ES94, PS92].

From an existence and regularity perspective, much progress has been made recently. When both \( \kappa \) and \( \nu \) are strictly positive, [CD80, Guo89] give global existence of smooth solutions using standard energy method arguments. In the case of \( \kappa \equiv \nu \equiv 0 \), local well-posedness as well as a blow-up criterion similar to well-known result of Beale, Kato and Majda [BKM84] has been shown in a number of function spaces [CKN99, CN97, LWZ10]. For \( \kappa \equiv 0, \nu > 0 \), [Tan06] shows global existence of a solution to \((B_{0,\nu})\) for nondecaying initial data: \((u_0, \rho_0) \in L^\infty(\mathbb{R}^2) \times B^0_{\infty,1}(\mathbb{R}^2)\).

In this paper, we study the case of \( \nu \equiv 0, \kappa > 0 \). Recall that in the celebrated paper [Yud63], V. Yudovich proves the existence and uniqueness of a weak solution to the 2D incompressible Euler Equations,

\[
\begin{aligned}
&\partial_t u + (u, \nabla) u + \nabla P = 0 \\
&\text{div } u = 0 \\
&u(x, 0) = u_0(x),
\end{aligned}
\]

for initial vorticity, \( \omega_0 = \text{curl } u_0 \in L^\infty(\mathbb{R}^2) \). This result has been extended for initial vorticity in more general function spaces including, among others, Besov spaces. Recall that the (inhomogeneous) Besov Space, \( B^{s}_{p,q} \), can be characterized as the set of all \( f \in \mathcal{S}' \) such that

\[
\left( \sum_{j=-1}^{\infty} 2^{jq_1} \| \Delta_j f \|_{L^p}^q \right)^{\frac{1}{q}} < \infty,
\]

1991 Mathematics Subject Classification. 35Q35, 76B03.
where $s \in \mathbb{R}$, $p, q \in [1, \infty]$ and $\Delta_j$ is the Littlewood-Paley operator defined below. In [Vil99], M. Vishik proves the existence and uniqueness of a local-in-time solution to the incompressible 2D Euler Equations with initial data in a critical Besov-type space (critical in the sense that $\text{supp} \hat{\phi} \subset \{ \xi \in \mathbb{R}^n \mid |\xi| \leq 1 \}$). He introduces the $B_{\infty,1}$ spaces, based on the $B_{\infty,1}$-norm, such that

$$B_{\Gamma} = \left\{ f \in S'((\mathbb{R}^n)) \mid \sum_{j=-1}^{N} \| \Delta_j f \|_\infty = O(\Gamma(N)) \right\},$$

and shows that for initial data in such a space, sufficient control on the growth of $\Gamma$ as $n \to \infty$ gives existence in a slightly weaker $B_{\Gamma}$-type space for positive time. More recently, R. Danchin and M. Paicu in [DP09] prove a Yudovich-type result in $\mathbb{R}$ for a thorough discussion of critical, sub-critical and super-critical Besov Spaces). He introduces the $B_{\Gamma}$ spaces, based on the $B_{\infty,1}$-norm, such that

$$B_{\Gamma} = \left\{ f \in S'((\mathbb{R}^n)) \mid \sum_{j=-1}^{N} \| \Delta_j f \|_\infty = O(\Gamma(N)) \right\},$$

Remark 1.1. Note that for $\Gamma$ such that $\Gamma(\alpha) = C\alpha$ when $\alpha \geq 1$, $\Gamma(\alpha) = 1$ otherwise, we have $L^\infty \subset B_{\Gamma}$, thus the two spaces are comparable for such a choice of $\Gamma$.

The structure of the paper is as follows: In the remainder of this section we define more precisely some concepts we will need throughout the paper. In section 2, we prove a priori estimates for vorticity and density, generalizing the results found in [Vis99] and [DP09] as well as a technical commutator estimate needed in the remainder of the paper. Finally, in sections 3 and 4 we prove the uniqueness and existence of these solutions, respectively.

In order to define the $B_{\Gamma}$ spaces discussed in this paper, we first need to define the Littlewood-Paley decomposition. Let $\Phi \in S(\mathbb{R}^n)$ be a function such that $\text{supp} \Phi \subset \{ \xi \in \mathbb{R}^n \mid |\xi| \leq 1 \}$ and $|\hat{\Phi}(\xi)| \geq C > 0$ on $\{ |\xi| \leq \frac{5}{6} \}$. Let $\varphi \in S(\mathbb{R}^n)$ be a radial function such that $\text{supp} \varphi \subset \{ \xi \in \mathbb{R}^n \mid \frac{1}{2} \leq |\xi| \leq 2 \}$, $|\hat{\varphi}(\xi)| \geq C > 0$ on $\{ \frac{1}{3} \leq |\xi| \leq \frac{2}{3} \}$. Set $\varphi_j(x) = 2^{jn} \varphi(2^j x)$ for $j \in \mathbb{Z}$ (i.e., $\varphi_j(\xi) = \hat{\varphi}(2^{-j} \xi)$). Based on this choice of $\Phi$ and $\varphi$, the Littlewood-Paley operators are then given by:

**Definition 1.2.** For $f \in S'((\mathbb{R}^n))$, we have

$$\Delta_{-1} f = \hat{\Phi} \left( \frac{1}{i} \frac{\partial}{\partial x} \right) f = \mathcal{F}^{-1}(\hat{\Phi} \cdot \hat{f}) = \Phi * f,$$

$$\Delta_j f = \varphi_j \left( \frac{1}{i} \frac{\partial}{\partial x} \right) f = \mathcal{F}^{-1}(\hat{\varphi} \cdot \hat{f}) = \varphi_j * f \text{ for } j \geq 0,$$

$$\Delta_j f = 0 \text{ for } j \leq -2,$$

$$S_j f = \sum_{k=-1}^{j} \Delta_k f.$$

From the above, we define the Littlewood-Paley decomposition of $f \in S'$ as

$$f = \sum_{j=-1}^{\infty} \Delta_j f.$$

**Definition 1.3.** The space $B_{\Gamma}$ is the set of all $f \in S'$ such that for any $N \geq -1$:

$$\sum_{j=-1}^{N} \| \Delta_j f \|_\infty \leq C \Gamma(N), \text{ where } \| f \|_{\Gamma} = \sup_{N \geq -1} (\Gamma(N))^{-1} \sum_{j=-1}^{N} \| \Delta_j f \|_\infty.$$

For the purposes of this paper, let $\Gamma : \mathbb{R} \to [1, \infty)$ satisfy the following conditions:

(i) $\Gamma(\alpha) = 1$ for $\alpha \in (-\infty, -1]$, $\lim_{\alpha \to -\infty} \Gamma(\alpha) = \infty$

(ii) There is a constant $C > 0$ such that $C^{-1} \Gamma(\beta) \leq \Gamma(\alpha) \leq C \Gamma(\beta)$ for $\alpha, \beta \in [-1, \infty)$, $|\alpha - \beta| \leq 1$. 

2 JACOB GLENN-LEVIN
function. Then we have the following estimates:

\[ C 2^{-\alpha \Gamma(\alpha)} \geq \int_{0}^{\alpha} 2^{-\xi} \Gamma(\xi) d\xi. \]

Define \( \Gamma_1(\alpha) = (\alpha + 2)\Gamma(\alpha) \) for \( \alpha \geq -1, \Gamma_1(\alpha) = 1 \) otherwise and assume:

(i) \( \Gamma_1 \) satisfies (iii),

(ii) \( \Gamma_1 \) is convex,

(iii) \( \int_{0}^{\alpha} (\Gamma_1(\alpha))^{-1} d\alpha = \infty. \)

2. A Priori and Commutator Estimates

Let \( K \) be the Biot-Savart kernel. Denote by \( X_u(x, t; \tau) := X(x, t; \tau) \) the flow mapping generated by \( u. \) \( X \) solves

\[ \frac{\partial}{\partial t} X(x, t; \tau) = u(X(x, t; \tau), t); \quad X(x, 0; \tau) = x(t) \]

and for any \( 0 \leq \tau \leq t \leq T, X(\cdot, t; \tau) \) is a volume preserving homeomorphism \( \mathbb{R}^2 \rightarrow \mathbb{R}^2 \). The goal for this section is to prove the following a priori bound for \( \omega: \)

**Theorem 2.1.** Let \( 1 < p_0 < 2 < p_1 < \infty. \) Let \( f \in B_{p_1} \cap L^{p_0} \cap L^{p_1} \), and let \( g \in W^{1, p_0} \cap W^{1, p_1} \) such that \( \nabla g \in B_{p_1} \). Let \( u \in \mathcal{K} \ast \omega, \omega_0 = f \) and \( \rho_0 = g. \) Finally, let \( (u, \rho) \) solve \( (B_{k, 0}). \) then we have the following a priori estimates:

1. There exists \( T > 0 \) (depending on \( \Gamma, f \) and \( g) \) such that \( \omega, \nabla \rho \in L^\infty([0, T]; L^{p_0} \cap L^{p_1}), \nabla \rho \in L^\infty([0, T]; B_{p_1}), \) and \( \omega \in L^\infty([0, T]; B_{p_1}) \) when

\[ (\alpha + 2)\Gamma'(\alpha) \leq C \quad \text{for a.e.} \quad \alpha \in [-1, \infty). \]

2. Furthermore, \( \omega, \nabla \rho \in L^\infty_{\text{loc}}([0, \infty); L^{p_0} \cap L^{p_1}), \nabla \rho \in L^\infty_{\text{loc}}([0, \infty); B_{p_1}), \) and \( \omega \in L^\infty_{\text{loc}}([0, \infty); B_{p_1}) \) under the stronger assumption that

\[ \Gamma'(\alpha) \Gamma_1(\alpha) \leq C \quad \text{for a.e.} \quad \alpha \in [-1, \infty). \]

**Remark 2.2.** Note that while we do not explicitly use assumptions (2.1) and (2.2) in the main body of this paper, they are an essential piece of Proposition 2.3.

Observe that if we apply the curl operator to the equation satisfied by the velocity field \( u \) then curl \( \nabla P = 0, \) and we have the vorticity equation \( \partial_t \omega + (u, \nabla) \omega = \partial_1 \rho. \) Integrating along characteristic curves, we have that for flow map \( X(x, t; \tau) = X_u(x, t; \tau), \)

\[ \omega(x(t), t) = \omega_0(X^{-1}(t; 0)x(t)) + \int_{0}^{t} \partial_1 \rho(X^{-1}(t; \tau)x(t), \tau) d\tau. \]

For any \( p \in [1, \infty), \) this implies that

\[ \|\omega(t)\|_p \leq \|\omega_0\|_p + \int_{0}^{t} \|\nabla \rho\|_p d\tau, \]

since \( X \) is a volume-preserving homeomorphism. In regards to the \( B_{p_1} \) norm, an initial estimate gives us:

\[ \|\omega(t)\|_{B_{p_1}} \leq \|\omega_0(X^{-1}(t))\|_{B_{p_1}} + \int_{0}^{t} \|\nabla \rho(X^{-1}(t; \tau))\|_{B_{p_1}} d\tau. \]

With (2.4) in mind, we must address the following two concerns: First, does the gradient of density remain in the initial \( B_{p_1} \) space for positive time? Assuming the first question, how does the inverse flow map \( X^{-1} \) act on the \( B_{p_1} \) spaces? In order to address this second question, we make use of a key proposition of Vishik in [Vis99]:

**Proposition 2.3 (Vishik).** Let \( \omega_0 \in B_{p_1} \cap L^{p_0} \cap L^{p_1}, \) and let \( \rho_0 \in W^{1, p_0} \cap W^{1, p_1} \) such that \( \nabla \rho_0 \in B_{p_1}. \) Let \( (u, \rho) \) be a regular solution to \( (B_{k, 0}) \) such that \( u \in \mathcal{K} \ast C([0, T]; B_{p_1} \cap L^{p_0} \cap L^{p_1}), \) where \( T \) is defined for each case below. Let \( X_u(t; \tau) \) be the flow map given by \( u, \) and let \( f \in B_{p_1} \cap L^{p_0} \cap L^{p_1} \) be an arbitrary function. Then we have the following estimates:
1. If (2.1) holds, then there exists a \( T > 0 \) and a \( C > 0 \) (both depending on \( \Gamma \) and the initial data), such that for \( 0 \leq \tau \leq t \leq T \),
\[
\| f \circ X^{-1}_u(t; \tau) \|_{L^1_t} \leq C \| f \|_{\Gamma}.
\]

2. Under assumption (2.2), let \( T > 0 \) be arbitrary. Then there exists \( \lambda(\cdot) \in L^\infty_{loc}(0, \infty) \) (depending on \( \Gamma \) and the initial data), such that:
\[
\| f \circ X^{-1}_u(t; \tau) \|_{L^1_t} \leq C \| f \|_{\Gamma} \lambda(t)
\]
for all \( 0 \leq \tau \leq t \leq T \).

Remark 2.4. For ease of exposition, we fix \( T > 0 \) for the remainder of this section. In the case of assumption (2.1), this \( T \) depends on our choice of \( \Gamma \), while under assumption (2.2), this choice of \( T \) is arbitrary.

The second tool we need to prove Theorem 2.5 is the following a priori control on the density \( \rho \):

**Theorem 2.5.** Assume \( \omega_0 \) and \( \rho_0 \) are defined as above. Set \( \alpha = \left( \frac{1 + \kappa t}{\kappa} \right) \). Then there is a constant \( C > 0 \) (depending only on \( \Gamma, f, g \) and \( T \)) such that for all \( t > 0 \),
\[
\int_0^t \| \nabla \rho(t) \|_{L^2(\omega_0 \cap B_{\rho_0})} \, dt \leq \Upsilon(t),
\]
where
\[
\Upsilon(t) = \left[ \alpha^2 \left( \| \rho_0 \|_{B_{\rho_0}^{-1} \cap B_\infty^{-1}}^2 + \alpha t \| \rho_0 \|_{L^2(\omega_0)} \right) \right]^{\frac{\tau}{2}} \exp \left[ C\alpha^3 \| \rho_0 \|_{L^2} \right].
\]

Using Proposition 2.3 together with Theorem 2.5, we conclude that for \( t \in [0, T] \), the terms on the right hand side of (2.2) are bounded by a constant multiple of \( \| \omega_0 \|_{L^1} \) and \( \int_0^t \| \nabla \rho(t) \|_{L^1} \, dt \), respectively, hence proving Theorem 2.1.

It remains to prove Theorem 2.5. First, note that \( \rho \) solves
\[
\partial_t \rho - \kappa \Delta \rho = h,
\]
where \( h = -(u, \nabla) \rho \). Written in this form, it becomes evident that we may treat the right hand side as the nonhomogeneous part of a heat equation and make use of the smoothing properties of the Laplacian. Let \( \{e^{t\Delta}\}_{t>0} \) stand for the heat semi-group. Applying the \( \Delta_j \) operator (for \( j \geq 0 \)) to the above equation, we have
\[
\partial_t \Delta_j \rho - \kappa \Delta \Delta_j \rho = \Delta_j h,
\]
which implies that
\[
\Delta_j \rho(t) = e^{\kappa t} \Delta_j \rho_0 + \int_0^t e^{\kappa (t-\tau)} \Delta_j h(\tau) \, d\tau.
\]
As in [DP09], we make use of the following result of Chemin (see [Che99] for proof):

**Proposition 2.6** (Chemin). There exists positive constants \((C, c)\) such that for any \( 1 \leq p \leq \infty \) and \( \nu > 0 \),
\[
\| e^{\nu \Delta} \Delta_j h \|_p \leq C e^{-c \nu 2^j} \| \Delta_j h \|_p.
\]

Chemin’s proposition applied to (2.6) gives us
\[
\| \Delta_j \rho(t) \|_p \leq C \left( e^{-C \kappa 2^j t} \| \Delta_j \rho_0 \|_p + \int_0^t e^{-C \kappa 2^j (t-\tau)} \| \Delta_j h(\tau) \|_p \, d\tau \right),
\]
and integrating both sides in \( t \) leads to the following inequality for \( j \geq 0 \):
\[
\kappa 2^j \int_0^t \| \Delta_j \rho(\tau) \|_p \, d\tau \leq C 2^{-j} \left( \| \Delta_j \rho_0 \|_p + \int_0^t \| \Delta_j h(\tau) \|_p \, d\tau \right).
\]

Let \( 1 \leq p \leq \infty \). From an application of the maximum principle to (2.5), we have
\[
\| \Delta_{-1} \rho(t) \|_p \leq \| \Delta_{-1} \rho_0 \|_p + \int_0^t \| \Delta_{-1} h(\tau) \|_p \, d\tau
\]
\[
\Rightarrow \int_0^t \| \Delta_{-1} \rho(\tau) \|_p \, d\tau \leq C t \left( \| \Delta_{-1} \rho_0 \|_p + \int_0^t \| \Delta_{-1} h(\tau) \|_p \, d\tau \right).
\]
Summing $j \geq -1$, we conclude that for any $1 \leq p \leq \infty$,  
\begin{equation}
\kappa \int_0^t \| \rho \|_{B^{1}_{p,1}} \, d\tau \leq C(1 + \kappa t) \left( \| \rho_0 \|_{B^{1}_{p,1}} + \int_0^t \| (u, \nabla) \rho \|_{B^{1}_{p,1}} \, d\tau \right).
\end{equation}

Observe that for $p = \infty$, the norm on the left hand side of (2.9) is equivalent to the $B^{0}_{\infty,1}$-norm of $\nabla \rho$, which is itself an upper bound for $\| \nabla \rho \|_p$. Similarly, for $p = p_0$, the $B^{1}_{p_0,1}$-norm of $\rho$ on the left hand side is a bound for $\| \nabla \rho \|_{p_0}$. Therefore, in order to prove the a priori bound on the $B^1_T \cap L^{p_0}$-norm of $\nabla \rho$, it suffices to control the right hand side of (2.9) by suitable bounds and then utilize a Gronwall-type estimate.

\textbf{Remark 2.7.} For the space $B^{1}_{p_0,1}$, we use the embedding $W^{1,p_0} \hookrightarrow B^{1}_{p_0,\infty} \hookrightarrow B^{1}_{p_0,1}$ to conclude that $\rho_0 \in B^{1}_{p_0,1}$. For details, see [Pee76].

Since the cases $p = \infty$ and $p = p_0$ are nearly identical, we will address only the case $p = \infty$. Use Bony’s paraproduct decomposition to write

\[ \rho = R(u, \nabla) + \sum_{m=1}^{2} T_{\partial_m} \rho u_m + T_{u_m} \partial_m \rho, \]

where $R(f, g) = \sum_{|j-k| \leq 1} \Delta_j f \Delta_k g$, and $T_{fg} = \sum_{j=0}^{\infty} S_j f \Delta_j g$. Since $\text{div} \, u = 0$, we have $R(u, \nabla) = \text{div} \, R(u, \rho)$. Because div maps $B^{0}_{\infty,1} \to B^{1}_{\infty,1}$, it suffices to bound $\| R(u, \rho) \|_{B^{0}_{\infty,1}}$. To do so, we write

\[ \| R(u, \rho) \|_{B^{0}_{\infty,1}} = \sum_{j=-1}^{\infty} \left\| \sum_{|k-l| \leq 1} \Delta_j \Delta_k u \Delta_l \rho \right\|_{\infty} \]

\[ \leq \sum_{j=-1}^{\infty} \sum_{|k-l| \leq 1} \left\| \Delta_j \Delta_k u \Delta_l \rho \right\|_{\infty} \]

\[ \leq C \sum_{j=-1}^{\infty} \left\| \Delta_j u \right\|_{\infty} \left\| \Delta_l \rho \right\|_{\infty} \]

\[ \leq C \rho \| u \|_{B^{0}_{\infty,1}}, \]

where $\| \rho \|_{B^{0}_{\infty,\infty}} = \sup_{j \geq -1} \| \Delta_j \rho \|_{\infty}$. Since $\| u \|_{B^{0}_{\infty,1}}$ is bounded by a constant multiple of $\| \omega \|_{p_0} + \| \omega \|_{\Gamma_1}$, we conclude that

\begin{equation}
\| R(u, \nabla \rho) \|_{B^{1}_{\infty,1}} \leq C \| \rho \|_{B^{0}_{\infty,\infty}} (\| \omega \|_{p_0} + \| \omega \|_{\Gamma_1}).
\end{equation}

For the second term in the paraproduct decomposition, we use the following estimate:

\[ \| T_{\partial_m} \rho u_m \|_{B^{1}_{\infty,1}} = \sum_{j=-1}^{\infty} 2^{-j} \left\| \sum_{k=0}^{\infty} S_{k-2} \partial_m \rho \Delta_k u_m \right\|_{\infty} \]

\[ \leq C \sum_{j=-1}^{\infty} \sum_{|k-l| \leq M_0} 2^{-j} \left\| \Delta_j \rho \partial_m \right\|_{\infty} \left\| \Delta_k u_m \right\|_{\infty} \]

\[ \leq C \| \rho \|_{B^{0}_{\infty,\infty}} \sum_{k=-1}^{\infty} \| \Delta_k u \|_{\infty} \]

\[ \leq C \| \rho \|_{B^{0}_{\infty,\infty}} \| u \|_{B^{0}_{\infty,1}}. \]

The above is true regardless of our choice of $m$, and a near identical argument shows the same expression bounds the $B^{1}_{\infty,1}$-norm of $T_{u_m} \partial_m \rho$. Thus

\begin{equation}
\left\| \sum_{m=1}^{2} T_{\partial_m} \rho u_m + T_{u_m} \partial_m \rho \right\|_{B^{1}_{\infty,1}} \leq C \| \rho \|_{B^{0}_{\infty,\infty}} (\| \omega \|_{p_0} + \| \omega \|_{\Gamma_1}).
\end{equation}
By Hölder’s inequality, we can write
\begin{equation}
\int_0^t \|\rho\|_{L^p_{B_{\infty,\infty}}} \left(\|\omega\|_{p_0} + \|\omega\|_{\Gamma_1}\right) d\tau \leq \\tag{2.12}
\end{equation}
\begin{equation}
\left(\int_0^t \|\rho\|_{L^p_{B_{\infty,\infty}}}^2 d\tau\right)^{\frac{1}{2}} \left(\int_0^t \left(\|\omega\|_{p_0} + \|\omega\|_{\Gamma_1}\right)^2 d\tau\right)^{\frac{1}{2}}, \\tag{2.13}
\end{equation}
and bound each integral individually. To handle the first integral, observe that if we take the $L^2$ inner product of $\rho$ with the equation satisfied by $\rho$, we have
\[\langle \rho, \partial_t \rho \rangle + \langle \rho, (u, \nabla) \rho \rangle + \langle \rho, \kappa \Delta \rho \rangle = 0,\]
and following an integration by parts in the space variable and a time integration over $[0,t]$,
\begin{equation}
\|\rho(t)\|_2^2 + 2\kappa \int_0^t \|\nabla \rho(\tau)\|_2^2 d\tau = \|\rho_0\|_2^2 \\tag{2.14}
\end{equation}
for all $t \in \mathbb{R}_+$. By the definition of $\|\rho\|_{L^p_{B_{\infty,\infty}}}$ and Bernstein’s inequality, we have
\begin{equation}
\|\rho\|_{L^p_{B_{\infty,\infty}}} \leq \sup_{j \geq -1} \|\Delta_j \rho\|_\infty \leq \sup_{j \geq -1} 2^j \|\Delta_j \rho\|_2 \leq C \|\Delta_{-1} \rho\|_2 + \sup_{j \geq 0} 2^j \|\Delta_j \rho\|_2 \leq C \|\rho\|_2 + \|\nabla \rho\|_2. \\tag{2.15}
\end{equation}

Squaring both sides and integrating over time, we have by (2.14)
\[
\int_0^t \|\rho\|_{L^p_{B_{\infty,\infty}}}^2 \, d\tau \leq C \int_0^t \|\rho\|_2^2 \, d\tau + \int_0^t \|\nabla \rho\|_2^2 \, d\tau \leq C \|\rho_0\|_2^2 t + \frac{1}{\kappa} \|\rho_0\|_2^2 \leq C \|\rho_0\|_2^2.
\]
Combining the above with the bound given by (2.9), we conclude that
\begin{equation}
\int_0^t \|\nabla \rho(\tau)\|_{\Gamma} \, d\tau \leq C \alpha \left[\|\rho_0\|_{B_{\infty,-1}} + C \int_0^t \|\rho\|_{B_{\infty,\infty}} \left(\|\omega\|_{p_0} + \|\omega\|_{\Gamma_1}\right) \, d\tau\right] \\tag{2.16}
\end{equation}
To achieve the desired a priori bound on the gradient of the density, it then suffices to control the $L^2$ (in time) integral of the $L^p$ and $B_{\Gamma_1}$ norms (in space) of vorticity. By (2.3) and Proposition 2.3, we have for all $t \in [0, T]$:
\begin{equation}
\|\omega(t)\|_{p_0} \leq \|\omega_0\|_{p_0} + \int_0^t \|\nabla \rho\|_{p_0} \, d\tau, \\tag{2.17}
\end{equation}
\begin{equation}
\|\omega(t)\|_{\Gamma_1} \leq C \left(\|\omega_0\|_{\Gamma} + \int_0^t \|\nabla \rho(\tau)\|_{\Gamma} \, d\tau\right). \\tag{2.18}
\end{equation}
Define $\Theta(t) = \int_0^t \|\nabla \rho(\tau)\|_{B_{\Gamma_1} \cap L^p} \, d\tau$, where $\|\cdot\|_{B_{\Gamma_1} \cap L^p} = \max \left\{\|\cdot\|_{p_0}, \|\cdot\|_{\Gamma}\right\}$. Then inserting the above into (2.16) gives
\[
\int_0^t \|\nabla \rho(\tau)\|_{\Gamma} \, d\tau \leq C \alpha \left[\|\rho_0\|_{B_{\infty,-1}} + \right] \tag{2.19}
\]
Remark 2.8. The argument with respect to $B^0_{p_0,1}$ yields an identical estimate, with $\|\nabla p\|_{B^0_{p_0}}$ and $\|p_0\|_{B^{-1}_{p_0}}$ replacing the first two norms in (2.19), respectively.

Combining (2.19) and the equivalent $B^0_{p_0,1}$ estimate and squaring both sides, we have
\[
\Theta^2(t) \leq C \alpha^2 \left[ \|\rho_0\|^2_{B^{-1}_{p_0,1} \cap B^{1}_{\infty,1}} + C \alpha \|\rho_0\|^2_2 \left( \int_0^t \Theta^2(\tau) d\tau \right) \right]
\]

An application of Gronwall's inequality to $\Theta^2(t)$ gives
\[
\Theta^2(t) \leq C \alpha^2 \left( \|\rho_0\|^2_{B^{-1}_{p_0,1} \cap B^{1}_{\infty,1}} + C \alpha \|\rho_0\|^2_2 \|\omega_0\|^2_{L_{p_0} \cap B^1_2} \right) \exp \left[ C \alpha^3 \|\rho_0\|^2_2 t \right]
\]

and taking a square root of both sides yields the desired bound.

Finally, in order to apply the a priori estimate to the proofs of uniqueness and existence, we first must understand what happens when the $\Delta_j$ operator is applied to the nonlinear term $(u, \nabla)\rho$. We follow the general approach introduced in [BC94] and write
\[
R_j(u, \rho) = \Delta_j(u, \nabla)\rho - (S_{j-2}u, \nabla)\Delta_j\rho.
\]

Let $M_0$ be the constant such that $\Delta_j \Delta_k f = 0$ if $|j - k| > M_0$. Note that $M_0$ depends strictly on our choice of $\varphi$ and $\Phi$ defining the $\Delta_j$ operators.

Theorem 2.9. For $u, \rho$ defined above:
\[
\|R_j(u, \rho)\|_\infty \leq C \sum_{|j-l| \leq M_0} \|S_{l-2} \rho\|_\infty \|\Delta_l \nabla u\|_\infty + \|S_{l-2} \nabla u\|_\infty \|\Delta_l \rho\|_\infty
\]
\[
+ C 2^j \sum_{|l-m| \leq 1, i \geq 1} 2^{-l} \|\Delta_l \nabla u\|_\infty \|\Delta_m \rho\|_\infty
\]
where the $\sum'$ implies that for $l = -1$, the factor $\|\Delta_l \nabla u\|_\infty$ should be replaced by $\|\Delta_{-1} u\|_\infty$.

Proof. The proof roughly follows that of Theorem 6.1 in [Vis99] and is therefore omitted. \qed

3. Uniqueness of the Flow

Let $\Pi : \mathbb{R} \to [1, \infty)$ be a function such that (i)-(iii) of Section (1) are satisfied. In addition, assume the following holds for $\Pi$:
\[
\int_1^\infty [\Pi(\xi)]^{-1} d\xi = \infty
\]
\[
\Pi(\xi)^{2^{-\xi}} \text{ is nonincreasing for } \alpha \geq C, \lim_{\xi \to \infty} \Pi(\xi)^{2^{-\xi}} = 0.
\]

Remark 3.1. We use $\Pi$ in place of $\Gamma$ and $\Gamma_1$ in this section since the uniqueness result utilizes weaker assumptions on $\Pi$ than those needed in Section 2.

Theorem 3.2. For $t \in [0, T]$, let $(u_1, \rho_1), (u_2, \rho_2)$ be two solutions to $(B_{u,0})$, and let $\omega_{1,2} = \text{curl } u_{1,2}$. We assume that for $1 < p_0 < 2$:
\[
\omega_{1,2}, \nabla \rho_{1,2} \in L^\infty([0, T]; L^{p_0}), \|\omega_{1,2}(\cdot)\|_{\Pi}, \|\nabla \rho_{1,2}(\cdot)\|_{\Pi} \in L^\infty([0, T]),
\]
\[
\|\nabla \rho_{1,2}(\cdot)\|_{\Pi} + \|\nabla u_{1,2}(\cdot)\|_{\Pi} \leq \|f(\cdot)\|_{L^{p_0}} + \|g(\cdot)\|_{L^{p_0}}.
\]
Then $(u_1, \rho_1) = (u_2, \rho_2)$ for $t \in [0, T]$. 

Proof. Define $v = u_1 - u_2$, $\omega = \omega_1 - \omega_2$, $\rho = \rho_1 - \rho_2$ and $P = P_1 - P_2$. We then have (for $\dot{j} = \frac{\partial}{\partial f}$):

$$
\begin{align*}
\dot{v} &= -(u_1, \nabla)v - (v, \nabla)u_2 - \nabla P + \begin{pmatrix} 0 \\ \rho \end{pmatrix} \\
\dot{\rho} - \kappa \Delta \rho &= -(u_1, \nabla)\rho + (v, \nabla)\rho_2 \\
\text{div } v &= 0 \\
\left| v \right|_{t=0} = \rho |_{t=0} = 0.
\end{align*}
$$

We handle the equation for $\dot{\rho}$ first. Applying the $\Delta_j$ operator, we have

$$
\Delta_j \dot{\rho} - \kappa \Delta_j \Delta_j \rho = -((S_{j-2}u_1, \nabla)\Delta_j \rho + R_j(u_1, \rho) + (S_{j-2}v, \nabla)\Delta_j \rho_2 + R_j(v, \rho_2).
$$

Using (2.7) and the fact that $\rho(0) = 0$ gives

$$

\begin{align*}
\| \Delta_j \rho(t) \|_{\infty} &\leq \int_0^t e^{-C\kappa 2^j(t-\tau)} \left( \| R_j(u_1, \rho) \|_{\infty} \\
&\quad + \| (S_{j-2}u_1, \nabla) \Delta_j \rho \|_{\infty} + \| R_j(v, \rho_2) \|_{\infty} \\
&\quad + \| (S_{j-2}v, \nabla) \Delta_j \rho_2 \|_{\infty} \right) d\tau.
\end{align*}
$$

It therefore suffices to bound the four terms on the right hand side. By Theorem 2.9, we have for fixed $j$

$$
\| R_j(u_1, \rho) \|_{\infty} \leq C \sum_{|j-l| \leq M_0} \{ \| S_{l-2} \rho \|_{\infty} \| \Delta_l \nabla u_1 \|_{\infty} + \| S_{l-2} \nabla u_1 \|_{\infty} \| \Delta_l \rho \|_{\infty} \}
$$

$$
+ C 2^j \sum_{l \geq j-M_0} \sum_{l-m \leq 1} 2^{-l} \| \Delta_l \nabla u_1 \|_{\infty} \| \Delta_m \rho \|_{\infty}.
$$

Let $A_1, A_2$ be the sum of the first and second lines above from $j = -1$ to $N$, respectively. (We will determine $N \geq 1$ later.) Then we have

$$
A_1 = C \sum_{j=-1}^N \sum_{|j-l| \leq M_0} \{ \| S_{l-2} \rho \|_{\infty} \| \Delta_l \nabla u_1 \|_{\infty} + \| S_{l-2} \nabla u_1 \|_{\infty} \| \Delta_l \rho \|_{\infty} \}
$$

$$
\leq C \left( \sup_{-1 \leq l \leq N+M_0} \| S_{l-2} \rho \|_{\infty} \right) \sum_{l=-1}^{N+M_0} \| \Delta_l \nabla u_1 \|_{\infty}
$$

$$
+ C \left( \sup_{-1 \leq l \leq N+M_0} \| S_{l-2} \nabla u_1 \|_{\infty} \right) \sum_{l=-1}^{N+M_0} \| \Delta_l \rho \|_{\infty}
$$

$$
\leq C \left( \sum_{l=-1}^{N+M_0} \| \Delta_l \nabla u_1 \|_{\infty} \right) \left( \sum_{l=-1}^{N+M_0} \| \Delta_l \rho \|_{\infty} \right).
$$

Using Bernstein’s inequality, $\omega_1 \in B\Omega$ and the fact that $\nabla u \mapsto \omega$ is a bounded operator on $L^{p_0}$, we have

$$
\sum_{l=-1}^{N+M_0} \| \Delta_l \nabla u_1 \|_{\infty} \leq \| \Delta_{-1} \nabla u_1 \|_{\infty} + C \sum_{l=0}^{N+M_0} \| \Delta_l \omega_1 \|_{\infty}
$$

$$
\leq C \| \Delta_{-1} \nabla u_1 \|_{p_0} + C \Pi(N + M_0) \| \omega_1 \|_{\Pi}
$$

$$
\leq C \| \Delta_{-1} \omega_1 \|_{p_0} + C \Pi(N + M_0) \| \omega_1 \|_{\Pi}
$$

$$
\leq C \| \omega_1 \|_{p_0} + C \Pi(N + M_0) \| \omega_1 \|_{\Pi}
$$

$$
\leq C \| \omega_1 \|_{\Pi(N + M_0)}
$$

where $\| \cdot \| = \| \cdot \|_{\Pi \cap L^{p_0}}$ and we make use of the fact that $\Pi(\alpha) \geq 1$ for all $\alpha \in \mathbb{R}$. Combined with the previous estimate, we conclude

$$
A_1 \leq C \| \omega_1 \|_{\Pi(N + M_0)} \sum_{l=-1}^{N+M_0} \| \Delta_l \rho \|_{\infty}.
$$
For the second term, we write:

\[
A_2 = C \sum_{j=1}^{N} \sum_{j \geq j-M_0 } \sum_{|l-m| \leq 1} 2^{j-l} \| \Delta_l \nabla u_1 \|_\infty \| \Delta_m \rho \|_\infty \\
\leq C \sum_{m=-1}^{\infty} \| \Delta_m \rho \|_\infty \left( \sum_{j=1}^{\min(N,m+M_0+1)} 2^{j-m} \right) \sum_{|l-m| \leq 1}^\prime \| \Delta_l \nabla u_1 \|_\infty \\
\leq C \sum_{m=-1}^{\infty} \left( 2^{\min(N,m)-m} \sum_{|l-m| \leq 1}^\prime \| \Delta_l \nabla u_1 \|_\infty \right) \| \Delta_m \rho \|_\infty \\
= \sum_{m=-1}^{N} \left( \sum_{|l-m| \leq 1}^\prime \| \Delta_l \nabla u_1 \|_\infty \right) \| \Delta_m \rho \|_\infty \\
+ \sum_{m=N+1}^{\infty} \left( 2^{N-m} \sum_{|l-m| \leq 1}^\prime \| \Delta_l \nabla u_1 \|_\infty \right) \| \Delta_m \rho \|_\infty \\
\leq \text{CII}(N) \| \omega_1 \| \sum_{m=-1}^{N} \| \Delta_m \rho \|_\infty + C \sum_{m=N+1}^{\infty} \left( 2^{N-m} \Pi(m) \| \omega_1 \| \right) \| \Delta_m \rho \|_\infty \\
\leq \text{CII}(N) \| \omega_1 \| \sum_{m=-1}^{\infty} \| \Delta_m \rho \|_\infty \\
+ C \sum_{m=N+1}^{\infty} \left( \int_N^\infty 2^{N-\xi} \Pi(\xi) d\xi \| \omega_1 \| \right) \| \Delta_m \rho \|_\infty \\
\leq \text{CII}(N) \| \omega_1 \| \sum_{m=-1}^{\infty} \| \Delta_m \rho \|_\infty . \quad \text{(by (iii))}
\]

Combined with the estimate for \( A_1 \), we conclude that

\[
(3.11) \sum_{j=-1}^{N} \| R_j(u_1, \rho) \|_\infty \leq C \| \omega_1 \| \Pi(N) \sum_{j=-1}^{\infty} \| \Delta_j \rho \|_\infty .
\]

Next, we estimate \( \| R_j(v, \rho_2) \|_\infty \). Similar to \( R_j(u_1, \rho) \), we split the estimate into two terms:

\[
\| R_j(v, \rho_2) \|_\infty \leq C \sum_{|j-l| \leq M_0} \left\{ \| S_l \nabla v \|_\infty \| \Delta_l \nabla \rho_2 \|_\infty + \| S_l \nabla \rho_2 \|_\infty \| \Delta_l \nabla v \|_\infty \right\} \\
+ C 2^j \sum_{|l-m| \leq 1}^{\min(j-M_0)} 2^{-l} \| \Delta_l \nabla v \|_\infty \| \Delta_m \rho_2 \|_\infty
\]

and conclude that

\[
(3.12) \sum_{j=-1}^{N} \| R_j(v, \rho_2) \|_\infty \leq C \| \nabla \rho_2 \| \Pi(N) \sum_{j=-1}^{\infty} \| \Delta_j v \|_\infty .
\]

Finally, we estimate \( \| (S_{j-2} v, \nabla) \Delta_j \rho_2 \|_\infty \) and \( \| (S_{j-2} u_1, \nabla) \Delta_j \rho \|_\infty \). We have

\[
\| (S_{j-2} v, \nabla) \Delta_j \rho_2 \|_\infty \leq \| S_{j-2} v \|_\infty \| \Delta_j \nabla \rho_2 \|_\infty ,
\]

and
from which the estimate

\begin{equation}
\sum_{j=-1}^{N} \| (S_{j-2}v, \nabla) \Delta_j \rho \|_\infty \leq C \left( \sum_{j=-1}^{N} \| S_{j-2}v \|_\infty \right) \sum_{j=-1}^{N} \| \Delta_j \nabla \rho \|_\infty \leq C \| \nabla \rho \| \Pi(N) \sum_{j=-1}^{N} \| \Delta_j \|_\infty
\end{equation}

easily follows. Similarly, we can write

\begin{equation}
\sum_{j=-1}^{N} \| (S_{j-2}u_1, \nabla) \Delta_j \rho \|_\infty \leq C \| \omega_1 \Pi(N) \sum_{j=-1}^{N} \| \Delta_j \rho \|_\infty.
\end{equation}

Combining (3.11) and (3.12)-(3.14), we sum (3.8) from \( j = -1 \) to \( N \) and estimate it as:

\begin{equation}
\sum_{j=-1}^{N} \| \Delta_j \rho(t) \|_\infty \leq C \Pi(N) \int_{0}^{t} \sum_{j=-1}^{N} \left( \| \Delta_j \rho(\tau) \|_\infty + \| \Delta_j v(\tau) \|_\infty \right) \mathrm{d}\tau,
\end{equation}

where we have used 3.13 to bound \( \| \omega_1(\tau) \|, \| \nabla \rho_2(\tau) \| \) uniformly on \([0, T]\).

Next, we apply the \( \Delta_j \) operator to \( \dot{v} \) and find

\begin{equation}
\Delta_j \dot{v} + (S_{j-2}u_1, \nabla) \Delta_j v = -R_j(u_1, v) - (S_{j-2}v, \nabla)u_2 - R_j(v, u_2) - \Delta_j \nabla P + \left( \begin{array}{c} 0 \\ \Delta_j \rho \end{array} \right).
\end{equation}

Define the flow mappings \( \{ X_j(x, t; \tau) \} \) given by:

\begin{align*}
& X_j(x, t; \tau) = S_{j-2}u_1(X_j(x, t; \tau), t) \\
& X_j(x, 0; \tau) = x(\tau)
\end{align*}

(\text{where } S_{j-2} \text{ is } S_{-1} \text{ when } j = -1, 0). Integrating (3.16) along \( \{ X_j(\alpha, t; \tau) \} \) and taking the \( L^\infty \)-norm of both sides yields

\begin{equation}
\| \Delta_j v(t) \|_\infty \leq \int_{0}^{t} \| R_j(u_1, v) \|_\infty + \| (S_{j-2}v, \nabla)u_2 \|_\infty \\
+ \| R_j(v, u_2) \|_\infty + \| \Delta_j \nabla P \|_\infty + \| \Delta_j \rho \|_\infty \mathrm{d}\tau.
\end{equation}

The estimates for the first three terms hew closely to estimates 3.11, 3.13, and 3.12, respectively, and will therefore not be repeated. For the pressure term, we take the divergence of (3.16) and use (3.3) to find:

\begin{equation}
\Delta_j \Delta P = -\text{div} R_j(u_1, v) - \text{div} R_j(v, u_2) - \text{tr}(\nabla \Delta_j v \cdot \nabla S_{j-2}u_1) \\
- \text{tr}(\nabla \Delta_j u_2 \cdot \nabla S_{j-2}v) + \Delta_j \partial_2 \rho.
\end{equation}

We consider two cases, \( j \geq 0 \) and \( j = -1 \). For \( j \geq 0 \), observe first that

\(- \nabla \Delta_j P(x) = \mathcal{F}_{x+z}(i\xi |\xi|^{-2}(\Delta_j \Delta P)(\xi)).\)

Using the estimate for \( \Delta_j \Delta P \) above, as well as a Littlewood-Paley argument and Bernstein’s inequality gives:

\begin{equation}
\| \Delta_j \nabla P \|_\infty \leq C \left( \| R_j(u_1, v) \|_\infty + \| R_j(v, u_2) \|_\infty \\
+ 2^{-j} \| \nabla \Delta_j v \|_\infty \| S_{j-2} \nabla u_1 \|_\infty \\
+ 2^{-j} \| \nabla \Delta_j u_2 \|_\infty \| S_{j-2} \nabla v \|_\infty + \| \Delta_j \rho \|_\infty \right)
\leq C \left( \| R_j(u_1, v) \|_\infty + \| R_j(v, u_2) \|_\infty + \| \Delta_j v \|_\infty \| S_{j-2} \nabla u_1 \|_\infty \\
+ \| \Delta_j u_2 \|_\infty \| S_{j-2} \nabla v \|_\infty + \| \Delta_j \rho \|_\infty \right).
\end{equation}
For $j = -1$, we use
\begin{equation}
\|\Delta_{-1} \nabla P\|_\infty \leq C \|\Delta_{-1} \nabla P\|_{p_2}
\end{equation}
\begin{equation}
\leq C \left\| \sum_{k=1}^{n} \partial_k \Delta_{-1} \{ u_1^{(k)} v + v^{(k)} u_2 \} \right\|_{p_2} + \|\Delta_{-1} \rho\|_{p_2}
\end{equation}
\begin{equation}
\leq C \|\Delta_{-1} (u_1 \otimes v)\|_{p_2} + C \|\Delta_{-1} (v \otimes u_2)\|_{p_2} + \|\Delta_{-1} \rho\|_{p_2}
\end{equation}
\begin{equation}
\leq C \sum_{|\tau-m| \leq M_0} \left( \|\Delta t u_1\|_{p_2} + \|\Delta t u_2\|_{p_2} \right) \|\Delta m v\|_\infty + \|\Delta_{-1} \rho\|_{p_2},
\end{equation}

where $p_2 \in \left[ \frac{n \rho_0}{n-\rho_0}, \infty \right)$. Sobolev embedding and (3.3) then gives the desired control over pressure in terms of the other members of the right hand side of (3.17).

Combining this estimate with those for the previous three terms and using (3.3), we have:
\begin{equation}
\sum_{j=-1}^{N} \|\Delta_j v(t)\|_\infty \leq C \Pi(N) \int_{0}^{t} \sum_{j=-1}^{\infty} \left( \|\Delta_j v(\tau)\|_\infty + \|\Delta_j \rho(\tau)\|_\infty \right) d\tau,
\end{equation}

and therefore:
\begin{equation}
\sum_{j=-1}^{N} \|\Delta_j v(t)\|_\infty + \|\Delta_j \rho(t)\|_\infty \leq C \Pi(N) \int_{0}^{t} \sum_{j=-1}^{\infty} \|\Delta_j v(\tau)\|_\infty + \|\Delta_j \rho(\tau)\|_\infty d\tau.
\end{equation}

We wish to use a Gronwall-type estimate, so we need to control $\sum_{j=N+1}^{\infty} \left( \|\Delta_j v(t)\|_\infty + \|\Delta_j \rho(t)\|_\infty \right)$ as well. Control of these two quantities is nearly identical, so we address only $\rho(t)$. Suppressing time, we use Bernstein’s inequality to write $\sum_{j=N+1}^{\infty} \|\Delta_j \rho\|_\infty \leq C \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \nabla \rho\|_\infty$. Next, we set $d_k = \sum_{j=-1}^{k} \|\Delta_j \nabla \rho\|_\infty$, and use Abel’s lemma to write
\begin{equation}
\sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \nabla \rho\|_\infty \leq \sum_{j=N+1}^{\infty} 2^{-j} (d_j - d_{j-1})
\end{equation}
\begin{equation}
\leq -2^{(N-1)} d_N + \sum_{j=N+1}^{\infty} d_j (2^{-j} - 2^{-(j+1)})
\end{equation}
\begin{equation}
\leq -2^{(N-1)} d_N + \|\nabla \rho\|_\Pi \sum_{j=N+1}^{\infty} 2^{-j} \Pi(j)
\end{equation}
\begin{equation}
\leq -2^{(N-1)} d_N + \|\nabla \rho\|_\Pi \int_{j=N+1}^{\infty} 2^{-j} \Pi(j) dj
\end{equation}
\begin{equation}
\leq C 2^{-N} \Pi(N),
\end{equation}

due to conditions (ii) and (iii) on $\Pi$ and (3.3). This allows us to write:
\begin{equation}
\sum_{j=-1}^{\infty} \left( \|\Delta_j v\|_\infty + \|\Delta_j \rho\|_\infty \right) \leq \sum_{j=-1}^{N} \left( \|\Delta_j v\|_\infty + \|\Delta_j \rho\|_\infty \right) + C 2^{-N} \Pi(N).
\end{equation}

Set $F(t) = \int_{0}^{t} \sum_{j=-1}^{\infty} \left( \|\Delta_j v(\tau)\|_\infty + \|\Delta_j \rho(\tau)\|_\infty \right) d\tau$, and use (3.22) and (3.23) to achieve the estimate
\begin{equation}
F'(t) \leq C \Pi(N) F(t) + C 2^{-N} \Pi(N).
\end{equation}

$F(t)$ is a monotonically nondecreasing, absolutely continuous function, and given (3.3), we have $\|F'(t)\|_{L^\infty([0,T])} \leq C$. Since $F(0) = 0$ by construction, this implies that there exists some $t_0$ such that $F(t) \equiv 0$ on $[0,t_0]$, $F(t) > 0$ on $(t_0,T]$. 
If \( t_0 = T \), then we have uniqueness. Therefore we assume that \( t_0 < T \). Fix \( \varepsilon > 0 \) sufficiently small that \( t_0 + \varepsilon < T \) and \( F(t) < 2^{-M_1 - 1} \) on \( (t_0, t_0 + \varepsilon) \) (where \( M_1 \) will be determined later). Choose \( t \in (t_0, t_0 + \varepsilon) \) and let \( N = \max\{1, \left[-\log_2 F(t) \right] \} \). This gives
\[
F'(t) \leq C\Pi(-\log_2 F(t)) F(t), \quad F(0) = 0.
\]
By (3.25), we have \( \int_0^{1/2} F^{-1}(\Pi(-\log_2 F))^{-1} dF = C \int_1^\infty (\Pi(\alpha))^{-1} d\alpha = \infty \), and the Osgood Uniqueness Theorem (see \cite{Che98} for statement) applies to (3.25).

By (3.26), we have
\[
\Pi(-\log_2 \eta(\tau, \delta)) \eta(\tau, \delta) \leq \Pi(-\log_2 \eta(\tau, \delta)) \eta(\tau, \delta) \leq \delta + \int_0^1 \Pi(-\log_2 \eta(\tau, \delta)) \eta(\tau, \delta) d\tau = \eta(t_1, \delta),
\]
contradicting the definition of \( t_1 \). Finally, choose \( M_1 \) such that (3.2) holds for \( \alpha \geq M_1 \). As \( \delta \to 0^+ \), we must have that \( F \equiv 0 \) on \( [t_0, t_0 + \varepsilon] \), contradicting the definition of \( t_0 \). This implies that \( t_0 = T \), and uniqueness is proven. \( \square \)

4. Construction of the Flow

Let \( \Gamma, \Gamma' \) satisfy (i)-(vi). This section is dedicated to proving the following theorems:

**Theorem 4.1.** For \( 1 < p_0 < 2 < p_1 < \infty \), let \( f \in B_{\Gamma} \cap L^{p_0} \cap L^{p_1} \) and \( g \in W^{1,p_0} \cap W^{1,p_1} \) such that \( \nabla g \in B_{\Gamma} \). Assume that \( (\alpha + 2)\Gamma'(\alpha) \leq C \) for a.e. \( \alpha \in [-1, \infty) \). Then there exists a \( T > 0 \) (depending on \( \Gamma, f \) and \( g \)) and a solution \( (u, \rho) \) to the system of equations \((B_{\kappa,0})\) with \( u = K \ast \omega \), such that
\[
\omega(\cdot) \in L^\infty([0, T]; L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0, T]; B_{\Gamma}),
\]
\[
\nabla \omega(\cdot) \in L^\infty([0, T]; L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0, T]; B_{\Gamma}).
\]

**Theorem 4.2.** Let \( f \) and \( g \) be as in Theorem 4.1. Assume that \( \Gamma'(\alpha) \Gamma_1(\alpha) \leq C \) for a.e. \( \alpha \in [-1, \infty) \). Then there exist \( (u, \rho) \) solving \((B_{\kappa,0})\) such that
\[
\omega(\cdot) \in L^\infty_{loc}([0, \infty); L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0, \infty); B_{\Gamma}),
\]
\[
\nabla \rho(\cdot) \in L^\infty_{loc}([0, \infty); L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0, \infty); B_{\Gamma}).
\]

It should be noted that \( C_{w^*}([0, T]; B_{\Gamma}) \) is the space of weak-* continuous functions with values in \( B_{\Gamma} \) in the sense of duality, \( H_{\Gamma}^* = B_{\Gamma} \). We define the predual \( H_{\Gamma_1} \) of \( B_{\Gamma} \) as:
\[
H_{\Gamma_1} = \left\{ f \in S' \mid \exists \{d_j\}_{j=1}^\infty, d_j \geq 0, \sum_{j=-1}^\infty d_j < \infty \right\}
\]
and state without proof that the dual of \( H_{\Gamma_1} \) is isomorphic to \( B_{\Gamma} \). (For related discussion, see \cite{Pee76}, Chapter 3).

**Proof of Theorem 4.1.** We use approximation by Sobolev-regular solutions to prove the existence theorem, for which we will need the following result from \cite{Cha06}:
Proposition 4.3 (Chae). Let $\kappa > 0$ be fixed, and $\text{div} u_0 = 0$. Let $r > 2$ be an integer, and $(u_0, \rho_0) \in H^r(\mathbb{R}^2)$. Then there exists a unique solution $(u, \rho)$ to $(B_{\kappa,0})$ with $u \in C([0,\infty);H^r(\mathbb{R}^2))$ and $\rho \in C([0,\infty);H^r(\mathbb{R}^2)) \cap L^1([0,T];H^{r+1}(\mathbb{R}^2))$.

Following [Tri01], we have that for $f \in S'$, the $H^r$ Sobolev norm is equivalent to the following Littlewood-Paley decomposition:

\[
\|\Delta_j f\|_2 + \left( \sum_{k=0}^{\infty} 2^{2kr} \|\Delta_k f\|_2^2 \right)^{\frac{1}{2}}
\]

A simple application of the Cauchy-Schwarz inequality gives:

Proposition 4.4. For $r > 1$, $H^r(\mathbb{R}^2) \subset B_r(\mathbb{R}^2)$.

For any $m \geq 1$, we construct the solution $(u_m, \rho_m)$ given by Proposition 4.3 such that
\[
\omega_m(0) = S_m f \in \cap_{r > 2} H^r \quad \text{and} \quad \nabla \rho_m(0) = S_m \nabla g \in \cap_{r > 2} H^r.
\]

Since $\|S_m h\|_p \leq \|h\|_p$ for $p \in [1,\infty]$ and any $h \in S'$ by definition of the $S_m$ operator, we have
\[
\|\omega_m(0)\|_{p_0}, \quad \|\nabla \rho_m(0)\|_{p_0} \leq C \quad \text{and} \quad \|\omega_m(0)\|_{p_1}, \quad \|\nabla \rho_m(0)\|_{p_1} \leq C.
\]

Furthermore, the definition of $\Delta_j$ and $S_m$ give us
\[
\|\omega_m(0)\|_\Gamma \leq C \|f\|_\Gamma, \quad \|\nabla \rho_m(0)\|_\Gamma \leq C \|g\|_\Gamma.
\]

Combined with Propositions 4.3 and 4.4 we conclude that
\[
\omega_m(\cdot) \in L^\infty_{loc}((0,\infty);B_{\Gamma_1}), \quad \nabla \rho_m(\cdot) \in L^\infty_{loc}((0,\infty);B_{\Gamma_1}).
\]

Furthermore, using Theorem 2.7 along with (4.6), (4.7) and (4.8) we conclude that there is a $T > 0$ such that
\[
\omega_m(\cdot), \nabla \rho_m(\cdot) \in L^\infty([0,T]; L^{p_0} \cap L^{p_1}),
\]
\[
\omega_0(\cdot) \in L^\infty([0,T]; B_{\Gamma_1}),
\]
\[
\nabla \rho_0(\cdot) \in L^\infty([0,T]; B_{\Gamma_1}).
\]

Fix two indices, $m$ and $l$. Then we set
\[
\begin{align*}
\omega_{m,l} &= K \ast \omega_{m,l}, \\
v &= u_m - u_l, \\
\rho &= \rho_m - \rho_l.
\end{align*}
\]

We use the same estimate as in the uniqueness proof of Section 3 only in this case we cannot assume that $\Delta_j v(0)$ and $\Delta_j \rho(0)$ are zero. To wit, we have for any $N \geq 1$,
\[
\begin{align*}
\sum_{j=-1}^{N} \left( \|\Delta_j v(t)\|_\infty + \|\Delta_j \rho(t)\|_\infty \right)
\leq & \sum_{j=-1}^{N} \left( \|\Delta_j v(0)\|_\infty + \|\Delta_j \rho(0)\|_\infty \right) \\
+ & CT_1(N) \int_0^t \sum_{j=-1}^{\infty} \left( \|\Delta_j v(\tau)\|_\infty + \|\Delta_j \rho(\tau)\|_\infty \right) d\tau \\
+ & C2^{-N}T_1(N).
\end{align*}
\]

As in the previous section, we define
\[
F(t) = \int_0^t \sum_{j=-1}^{\infty} \left( \|\Delta_j v(\tau)\|_\infty + \|\Delta_j \rho(\tau)\|_\infty \right) d\tau,
\]
which, for fixed \( t \in [0, T] \) and \( N = \max \{1, \lfloor \log_2 F(t) \rfloor \} \) allows us to write (4.11) as

\[
F'(t) \leq \sum_{j=1}^{N} \left( \| \Delta_j v(0) \|_\infty + \| \Delta_j \rho(0) \|_\infty \right) + C \Gamma_1(- \log_2 F(t)) F(t).
\]  

Denote the first term on the right hand side by

\[
\kappa_{m,l} + \iota_{m,l} := \sum_{j=1}^{N} \| \Delta_j v(0) \|_\infty + \sum_{j=1}^{N} \| \Delta_j \rho(0) \|_\infty.
\]

The bounds on \( \kappa_{m,l} \) and \( \iota_{m,l} \) are similar so we demonstrate only the latter. To bound \( \iota_{m,l} \), we first write

\[
\iota_{m,l} = \sum_{j=1}^{N} \| \Delta_j \rho(0) \|_\infty
\]

\[
\leq \sum_{j=1}^{N} 2^{-j} \| \Delta_j (S_m - S_l)g \|_\infty
\]

\[
= \sum_{j=1}^{N} 2^{-j} \left\| \Delta_j \left( \sum_{k=l+1}^{m} \Delta_k g \right) \right\|_\infty
\]

\[
\leq \sum_{k=l+1}^{m} \sum_{k-j \leq M_0} 2^{-k} \| \Delta_j \Delta_k g \|_\infty
\]

\[
\leq C \sum_{k=l+1}^{\infty} 2^{-k} \| \Delta_k g \|_\infty.
\]

Using an Abel’s Lemma argument identical to (3.23), we conclude that

\[
\iota_{m,l} \leq C 2^{-l} \Gamma(l).
\]  

After integrating (4.12) in time, we use (4.13) to write

\[
F(t) \leq C 2^{-l} \Gamma(l) + C \int_{0}^{t} \Gamma_1(- \log_2 F(\tau)) F(\tau) d\tau.
\]

As in the proof of uniqueness, we have that \( \Gamma_1(- \log_2 F) \) is monotonically nondecreasing for small \( F \geq 0 \). If we let \( \eta \) solve the ODE:

\[
\begin{align*}
\dot{\eta} &= C \Gamma_1(- \log_2 \eta) \\
\eta(0) &= C 2^{-l} \Gamma(l)
\end{align*}
\]

Then a simple Gronwall argument gives \( F(t) \leq \eta(t, C 2^{-l} \Gamma(l)) \) for \( t \in [0, T] \). Combined with (4.12), we have

\[
F'(t) \leq C 2^{-l} \Gamma(l) + C \Gamma_1[- \log_2 \eta(t, C 2^{-l} \Gamma(l))] \eta(t, C 2^{-l} \Gamma(l))
\]

for all \( t \in [0, T] \). This implies that \( \{u_m\} \) and \( \{\rho_m\} \) are Cauchy sequences in the Banach space \( L^\infty([0, T]; B^0_{\infty, 1}) \). Therefore, there exists \( u, \rho \) such that:

\[
u_m \to u, \quad \rho_m \to \rho \in L^\infty([0, T]; B^0_{\infty, 1}).
\]

As we show next, this in fact implies that for \( \omega = \text{curl} \ u \),

\[
\|
\| \omega \|_{\Gamma_1}, \| \nabla \rho \|_{\Gamma} \in L^\infty([0, T]).
\]

Define the seminorm \( \nu_N \) on \( L^\infty([0, T]; B^0_{\infty, 1}) \) given by

\[
\nu_N(f) = \left\| \sum_{j=-1}^{N} \| \Delta_j f(\cdot) \|_\infty \right\|_{L^\infty([0, T])}.
\]
By (4.15), it is clear that $\nu_N(u_m - u)$ and $\nu_N(\rho_m - \rho)$ tend to zero as $m \to \infty$. Using Bernstein's inequality, we have
\[
\left\| \sum_{j=-1}^{N} \| \Delta_j (\omega_m - \omega) \|_\infty \right\|_{L^\infty([0,T])} \leq C 2^N \nu_N(u_m - u),
\]
and similarly for $\nabla \rho_m$. Working with $\nabla \rho$, this yields
\[
(4.17) \quad \left\| \sum_{j=-1}^{N} \| \Delta_j \nabla \rho_m \|_\infty - \| \Delta_j \nabla \rho \|_\infty \right\|_{L^\infty([0,T])} \leq C 2^N \nu_N(u_m - u).
\]

By (4.9), we have that $\sum_{j=-1}^{N} \| \Delta_j \nabla \rho_m(t) \|_\infty \leq C \Gamma(N)$, where $C$ is independent of our choice of $m$. Using (4.17) and the above, we have
\[
\left\| \sum_{j=-1}^{N} \| \Delta_j \nabla \rho \|_\infty \right\|_{L^\infty([0,T])} \leq \left\| \sum_{j=-1}^{N} \| \Delta_j \nabla \rho_m \|_\infty \right\|_{L^\infty([0,T])} \leq C 2^N \nu_N(u_m - u) + C \Gamma(N),
\]
and passing to the limit as $m \to \infty$ gives the second inclusion of (4.10). The bound on $\omega$ is similar (with $\Gamma_1$ in place of $\Gamma$), and is therefore omitted.

It remains to show that $(u, \rho)$ satisfy the Boussinesq equations. Note that since $\{u_m\}$ and $\{\rho_m\}$ are in fact Cauchy sequences in $C([0,T]; B^0_{\infty,1})$ (in addition to $L^\infty([0,T]; B^0_{\infty,1})$), we can conclude that
\[
(4.18) \quad u_m \to u, \rho_m \to \rho \text{ in } L^\infty(\mathbb{R}^2 \times [0,T]) \cap C(\mathbb{R}^2 \times [0,T]),
\]
and after choosing a subsequence, we have:
\[
(4.19) \quad \omega_m \to \omega, \quad \nabla \rho_m \to \nabla \rho \text{ in } L^\infty([0,T]; L^p) \cap L^\infty([0,T]; L^p),
\]
\[
(4.20) \quad \dot{u}_m \to \dot{u}, \quad \dot{\rho}_m \to \dot{\rho} \text{ in } L^\infty([0,T]; L^p) \cap L^\infty([0,T]; L^p).
\]

Let $\beta \in S$, $\text{div} \beta = 0$ be a test function, and let $\theta \in D([0,T])$. By definition of $(u_m, \rho_m)$, we have
\[
\langle u_m(0), \beta \rangle \theta(0) + \int_0^T \langle u_m(\tau), \beta \rangle \dot{\theta}(\tau) d\tau + \langle u_m(\tau), (u_m(\tau), \nabla) \beta \rangle \theta(\tau) - \langle \rho_m(\tau), \beta \rangle \theta(\tau) d\tau = 0,
\]
\[
\langle \rho_m(0), \beta \rangle \theta(0) + \int_0^T \langle \rho_m(\tau), \beta \rangle \dot{\theta}(\tau) d\tau + \langle \rho_m(\tau), (u_m(\tau), \nabla) \beta \rangle \theta(\tau) + \kappa \langle \rho_m(\tau), \Delta \beta \rangle \theta(\tau) d\tau = 0.
\]

From (4.5) and the definition of $\omega_m, \rho_m$, we have
\[
\langle u_m(0), \beta \rangle \to \langle K * f, \beta \rangle,
\]
\[
\langle \rho_m(0), \beta \rangle \to \langle g, \beta \rangle.
\]

Sending $m \to \infty$ and utilizing (4.18)-(4.20), we conclude that $(u, \rho)$ solve $(B_{\kappa, \theta})$. 
It remains to show that $\omega(\cdot), \nabla \rho(\cdot)$ are weak-* continuous with values in $B_{r_1}$ and $B_{r_2}$, respectively. Since the proofs are nearly identical up to our choice of target space, we consider only the case of $\nabla \rho$. 

$\{\rho_m\}$ is a Cauchy sequence in $C([0,T]; B_{\infty,1}^{0})$, therefore we have that

$$||\nabla \rho - \nabla \rho_m||_{C([0,T]; B_{\infty,1}^{-1})} \to 0 \text{ as } m \to \infty.$$ 

Fix $h \in H_R$. Consider $\pi(t) := (\nabla \rho(t), h)$ for $t \in [0,T]$, and define $\pi_m(t)$ similarly for $\rho_m(t)$. For any $t_0 \in [0,T]$, we have

$$\pi(t) - \pi(t_0) = (\pi - \pi_m)(t) - (\pi - \pi_m)(t_0) + (\pi_m(t) - \pi_m(t_0)).$$

By (4.22), we have that for fixed $m$, $\pi_m(t) - \pi_m(t_0) \to 0$ as $t \to t_0$. For any $\hat{h} \in H_R$, we have

$$|((\pi - \pi_m)|t)\| \leq \left(\left(\left(\nabla \rho - \nabla \rho_m\right)(t), h - \hat{h}\right)\right) + \left(\left(\nabla \rho - \nabla \rho_m\right)(t), \hat{h}\right).$$

Let $\|\cdot\|_{\Gamma}$ be the norm of $H_{\Gamma}$, given by $\|f\|_{\Gamma} = \inf(d_j) \sum_{j=-1}^{\infty} d_j$. Next, for arbitrary $\delta > 0$, consider the space $B_{1,1+\delta,1}^{-1}$. Note that any Besov Space based on the $L^1$-norm contains every $L^1$-function with bounded Fourier spectrum (since the Littlewood-Paley decomposition of such a function has only finitely many nonzero terms), and that these functions are dense in $H_R$. Therefore $B_{1,1+\delta,1}^{-1}$ is dense in $H_R$, and we can choose $\hat{h} \in B_{1,1+\delta,1}^{-1}$ such that $\|h - \hat{h}\|_{\Gamma} < \varepsilon$ for any $\varepsilon > 0$. Because $\|\nabla \rho\|_{\Gamma}$ and $\|\nabla \rho_m\|_{\Gamma}$ are uniformly bounded on $[0,T]$, we have that

$$\left(\left(\nabla \rho - \nabla \rho_m\right)(t), h - \hat{h}\right) \leq C \varepsilon.$$ 

Finally, we use the duality $(B_{1,1+\delta,1}^{-1})' = B_{1+\delta,1}^{-1}$ and the embedding $B_{1,1+\delta,1}^{-1} \hookrightarrow B_{1+\delta,1}^{-1}$, to write

$$\left(\left(\nabla \rho - \nabla \rho_m\right)(t), \hat{h}\right) \leq C \|\nabla \rho - \nabla \rho_m\|_{C([0,T]; B_{\infty,1}^{-1})} \|\hat{h}\|_{B_{1+\delta,1}^{-1}}.$$ 

By choosing $m$ sufficiently large, we can make the right hand side of (4.23) less than $\varepsilon$. Combined with (4.20), this gives $\limsup_{t \to t_0} |\pi(t) - \pi(t_0)| \leq C \varepsilon$, which yields the desired result for $\nabla \rho$. \qed

**Proof of Theorem 4.2.** The proof follows that of Theorem A.1 except that our choice of $T > 0$ is arbitrary and no longer depends on $\Gamma$. Since the proof is identical to the above except in that respect, it is omitted. \qed

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