POLAR ACTIONS ON COMPACT RANK ONE SYMMETRIC SPACES ARE TAUّT

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Abstract. We prove that the orbits of a polar action of a compact Lie group on a compact rank one symmetric space are tautly embedded with respect to $\mathbb{Z}_2$-coefficients.

1. Introduction

The main result of this paper is the following theorem.

Theorem 1. A polar action of a compact Lie group on a compact rank one symmetric space is taut.

A proper isometric action of a Lie group $G$ on a complete Riemannian manifold $X$ is called polar if there exists a connected, complete submanifold $\Sigma$ that meets all orbits of $G$ in such a way that the intersections between $\Sigma$ and the orbits of $G$ are all orthogonal. Such a submanifold is called a section. It is easily seen that a section $\Sigma$ is totally geodesic in $X$. An action admitting a section that is flat in the induced metric is called hyperpolar. See the monographs [PT88, BCO03], and the survey [Tho05] for recent results and a list of references.

A properly embedded submanifold $M$ of a complete Riemannian manifold $X$ is called $F$-taut, where $F$ is a coefficient field, if the energy functional $E_q : \mathcal{P}(X, M \times q) \to \mathbb{R}$ is a $F$-perfect Morse function for every $q \in X$ that is not a focal point of $M$, where $\mathcal{P}(X, M \times q)$ denotes the space of $H^1$-paths $\gamma : [0, 1] \to X$ such that $\gamma(0) \in M$ and $\gamma(1) = q$ [GH91, TT97]. Recall that $E_q$ is a Morse function if and only if $q$ is not a focal point of $M$, and $\gamma$ is a critical point of $E_q$ if and only if $\gamma$ is a geodesic perpendicular to $M$ at $\gamma(0)$ that is parametrized proportionally to arc length, in which case the index of $\gamma$ as a critical point of $E_q$ is equal to the sum of the multiplicities of the focal points of $M$ along $\gamma$ by the Morse index theorem. Write $\mathcal{M} = \mathcal{P}(X, M \times q)$ and let $\mathcal{M}^c$ denote the sublevel set $\{E_q \leq c\}$. Since $E_q$ is bounded below and satisfies the Palais-Smale condition [PS64] (see e.g. Theorem A in [GH91]), the number $\mu_k(E_q|\mathcal{M}^c)$ of critical points of index $k$ of a Morse function $E_q$ restricted to $\mathcal{M}^c$ is finite and the weak Morse inequalities say that $\mu_k(E_q|\mathcal{M}^c) \geq \beta_k(\mathcal{M}^c, F)$ for all $k$ and all $c$ and all $F$, where $\beta_k(\mathcal{M}^c, F)$ denotes the $k$-th $F$-Betti number of $\mathcal{M}^c$; $E_q$ is called $F$-perfect if $\mu_k(E_q|\mathcal{M}^c) = \beta_k(\mathcal{M}^c, F)$ for all $k$ and all $c$.

A proper isometric action of a Lie group $G$ on a complete Riemannian manifold $X$ is called taut if all the orbits of $G$ in $X$ are taut submanifolds. Herein we will always consider $F = \mathbb{Z}_2$ and usually drop the reference to the field.

A proper isometric action of a Lie group is taut (resp. polar) if and only if the same is true for the restriction of that action to the connected component of the group. This is clear in the case of taut actions and is easy in the case of polar actions, see Proposition 2.4

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in [HPTT94]. We can thus assume that we have a connected group in the statement of Theorem 1.

It follows from results of Bott-Samelson and Conlon [BS58, Con71] that a hyperpolar action of a compact Lie group on a complete Riemannian manifold is taut. (Hyperpolar actions on compact irreducible symmetric spaces were classified by Kollross [Kol02].) However, there are many examples of polar actions on compact rank one symmetric spaces that are not hyperpolar as shown by the classification of polar actions on these spaces due to Podestà and Thorbergsson [PT99]. On the other hand, it has been conjectured for some time that the classes of polar and hyperpolar actions on compact irreducible symmetric spaces of higher rank coincide. This has been verified for actions on these spaces with a fixed point [Bri98], real Grassmannians of rank two [PT02], complex Grassmannians [BG05], and, more recently, this result has also been claimed for compact irreducible Hermitian symmetric spaces [Bil04], and compact irreducible type I symmetric spaces [Kol05].

The compact rank one symmetric spaces are the spheres and the projective spaces over the four normed division algebras. Regarding the proof of Theorem 1, the cases of spheres and real projective spaces are very simple. In the cases of complex and quaternionic projective spaces, we use a lifting argument based on results of [GH91, IT97, HLO00], see Proposition 3. It is only in the case of the Cayley projective plane $\mathbb{C}P^2$ that we resort to the classification in [PT99]. According to that result, a polar action on $\mathbb{C}P^2$ has either cohomogeneity one so that it is hyperpolar and then it is taut by the results of Bott-Samelson and Conlon referred to above, or it has cohomogeneity two and is given by one of the following subgroups of $F_4$, the isometry group of $\mathbb{C}P^2$:

$$\begin{align*}
\text{Spin}(8), & \quad S^1 \cdot \text{Spin}(7), \quad \text{SU}(2) \cdot \text{SU}(4), \quad \text{SU}(3) \cdot \text{SU}(3).
\end{align*}$$

We prove that each one of these actions is taut by generalizing an argument based on the reduction principle that was first used in [GT00] to prove that certain finite dimensional orthogonal representations are taut, see Proposition 6.

As is easy to see and we will explain in a moment, the classes of taut actions on $S^n$, $\mathbb{R}P^n$ and $\mathbb{R}^{n+1}$ all coincide. (A linear action on an Euclidean space is usually called a representation.) On the other hand, Proposition 3 implies that the class of taut actions on $CP^n$ (resp. $HP^n$) corresponds to a certain subclass of taut representations. In particular, it is worth noting that there are taut actions on $CP^n$ and $HP^n$ which are not polar. In fact, Proposition 3 implies that the representations of $SO(2) \times \text{Spin}(9)$ (resp. $U(2) \times \text{Sp}(n)$, $\text{Sp}(1) \times \text{Sp}(n)$, where $n \geq 2$) that were shown to be taut in [GT00, GT02] induce taut actions on $CP^{15}$ (resp. $CP^{4n-1}$, $HP^{2n-1}$), and these are not polar by the classification in [PT99].

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2. A LIFTING ARGUMENT

We begin now the proof of Theorem 1. Consider the unit sphere $S^n$ in Euclidean space $\mathbb{R}^{n+1}$. A polar action action of $G$ on $S^n$ is the restriction of a polar representation of $G$ on $\mathbb{R}^{n+1}$. On Euclidean space, the classes of polar and hyperpolar representations coincide since the sections are linear subspaces. It follows from the results of Bott-Samelson and Conlon quoted in the introduction that a polar representation is taut. One easily sees that a compact submanifold of $S^n$ is taut in $S^n$ if and only if it is taut in $\mathbb{R}^{n+1}$ (this can be seen by recalling that in the cases in which $X = S^n$ or $X = \mathbb{R}^{n+1}$, a properly embedded submanifold...
$M$ of $X$ is taut if and only the squared distance function $L_q : M \to \mathbb{R}$, $L_q(x) = d(x, q)^2$ is perfect for generic $q \in X$, see section 2 in [T97]. It follows that the action of $G$ on $S^n$ is taut.

Next we consider the cases of polar actions on real, complex or quaternionic projective spaces. By making use of the principal bundles
\[
\mathbb{Z}_2 \to S^n \to \mathbb{R}P^n, \quad S^1 \to S^{2n+1} \to \mathbb{C}P^n, \quad S^3 \to S^{4n+3} \to \mathbb{H}P^n,
\]
we reduce the problem to the already solved case of spheres via the following lifting argument (see Lemma 6.1 in [HLO00]; see also Proposition 4.1 in [GH91] and Theorem 6.12 in [T97]; the case of $\mathbb{R}P^n$ is of course simpler and can be dealt with directly, compare Proposition 6.3, p. 45, in [Ber99]).

**Lemma 2.** (Heintze, Liu, Olmos [HLO00]) Let $\hat{X}$ and $X$ be Riemannian manifolds, $\pi : \hat{X} \to X$ a Riemannian submersion, $M$ a properly embedded submanifold of $X$, and $\hat{M} = \pi^{-1}(M)$. Then a normal vector $\xi \in \nu\hat{X}$ is a multiplicity-$m$ focal direction of $\hat{M}$ if and only if $\pi_*\xi$ is a multiplicity-$m$ focal direction of $M$.

**Proposition 3.** Let $\hat{X}$ and $X$ be Riemannian manifolds, and suppose there is a proper isometric action of a Lie group $G$ on $\hat{X}$ such that $\pi : \hat{X} \to X$ is a principal $G$-fiber bundle and a Riemannian submersion. Let $M$ be a properly embedded submanifold of $X$ and $\hat{M} = \pi^{-1}(M)$. Then $\hat{M}$ is taut in $\hat{X}$ if and only if $M$ is taut in $X$.

**Proof.** Let $\hat{q} \in \hat{X}$ and $q = \pi(\hat{q})$. It follows from Lemma 2 that $\hat{q}$ is not a focal point of $\hat{M}$ if and only if $q$ is not a focal point of $M$. Consider the energy functionals $E_q : \mathcal{P}(\hat{X}, \hat{M} \times \hat{q}) \to \mathbb{R}$ and $E_q : \mathcal{P}(X, M \times q) \to \mathbb{R}$. A critical point of $E_q$ is a geodesic joining $q$ to a point in $\hat{M}$ that is perpendicular to $\hat{M}$, and similarly for the critical points of $E_{\hat{q}}$. Since a geodesic in $\hat{X}$ is horizontal with respect to $\pi$ if and only if it is perpendicular to one fiber of $\pi$, it follows that the critical points of $E_q$ are horizontal geodesics and there is a bijective correspondence between critical points of $E_q$ and critical points of $E_{\hat{q}}$ given by the projection. By the Morse index theorem, the index of a geodesic as a critical point of the energy functional is the sum of the multiplicities of the focal points along the geodesic. It follows from this and Lemma 2 that the projection maps critical points of $E_{\hat{q}}$ to critical points of $E_q$ with the same value of the energy and the same indices. In order to finish the proof of the lemma, we only need to show that $\mathcal{P}(\hat{X}, \hat{M} \times \hat{q})^c$ and $\mathcal{P}(X, M \times q)^c$ have the same homotopy type for all $c$. In fact, the map
\[
\Phi : \mathcal{P}(X, M \times q) \times \mathcal{P}(G, G \times 1) \to \mathcal{P}(\hat{X}, \hat{M} \times \hat{q}),
\]
given by $\Phi(x, g)(t) = \hat{x}(t)g(t)$, where $\hat{x}(t)$ is the horizontal lift of $x(t)$ with $\hat{x}(1) = \hat{q}$, is a diffeomorphism. Since $\mathcal{P}(G, G \times 1)$ is contractible, the result follows. □

It follows from Proposition 3 that polar actions on $\mathbb{R}P^n$, $\mathbb{C}P^n$, and $\mathbb{H}P^n$ are taut. In fact, the case of $\mathbb{R}P^n$ is clear, and if $G$ acts polarly on $\mathbb{C}P^n$ (resp. $\mathbb{H}P^n$), then $T^1 \times G$ (resp. $\text{Sp}(1) \times G$) acts polarly (and hence tautly) on $S^{2n+1} \subset \mathbb{C}^{n+1}$ (resp. $S^{4n+3} \subset \mathbb{H}^{n+1}$) by [PT99]. Hence $G$ acts tautly on $\mathbb{C}P^n$ (resp. $\mathbb{H}P^n$) by Proposition 3.

It remains to consider the case of the Cayley projective plane $\mathbb{Ca}P^2$. In this case, there is no convenient Riemannian submersion from a sphere, so we need to use a different argument. According to [PT99] and the discussion in the introduction, we need to prove that the four polar actions on $\mathbb{Ca}P^2$ given by the groups $G$ listed in [4] are taut.
3. The reduction principle

We now discuss a reduction principle that will be used to prove that the remaining four polar actions on $\mathbb{C}aP^2$ are taut. Let $G$ be a compact Lie group acting isometrically on a connected, complete Riemannian manifold $X$. It is not necessary to assume that $G$ is connected. Denote by $H$ a fixed principal isotropy subgroup of the $G$-action on $X$, let $X^H$ be the totally geodesic submanifold of $X$ that is left point-wise fixed by the action of $H$, and let $cX$ denote the closure in $X$ of the subset of regular points of $X^H$. Then it is known that $cX$ consists of those components of $X^H$ which contain regular points of $X$, and that any component of $cX$ intersects all the orbits. Let $N$ be the normalizer of $H$ in $G$. Then the group $\bar{N} = N/H$ acts on $cX$ with trivial principal isotropy subgroup in such a way that the inclusion $cX \to X$ induces a homeomorphism between orbit spaces $cX/\bar{N} \to X/G$. The pair $(\bar{N}, cX)$ is called the reduction of $(G, X)$. In case $cX$ is not connected, the pair $(\bar{N}, cX)$ can be further reduced as follows. Let $\bar{X}$ be a connected component of $cX$, and let $\bar{G}$ denote the subgroup of $\bar{N}$ that leaves $\bar{X}$ invariant. Then there is a homeomorphism between orbit spaces $\bar{X}/\bar{G} \to cX/\bar{N}$. We refer the reader to [Str94, SS95, GS00] for the proof and a discussion of these assertions.

From now on, we will work with a fixed connected component $\bar{X}$ of $X^H$. Let $p \in \bar{X}$. It is clear that $Gp \cap \bar{X} = \bar{G}p$. If $M = Gp$, we will denote $M \cap \bar{X}$ by $\bar{M}$. If, in addition, $p$ is a $G$-regular point, then the normal space to the principal orbit $M$ at $p$ is contained in $T_p\bar{X}$, because the slice representation at $p$ is trivial.

There is a natural action of $G$ on the space $\mathcal{P}(X)$ of $H^1$-paths $\gamma : [0, 1] \to X$ given by $(g; \gamma)(t) = g\gamma(t)$, where $g \in G$ and $t \in [0, 1]$, since the Sobolev class $H^1$ is invariant under diffeomorphisms. In particular, if $q \in \bar{X}$ and $M$ is a $G$-orbit in $X$, this action restricts to an action of $H$ on the space $\mathcal{P}(X, M \times q)$, and it is clear that $\mathcal{P}(X, M \times q)^H = \mathcal{P}(X^H, M^H \times q) = \mathcal{P}(\bar{X}, \bar{M} \times q)$.

**Lemma 4.** Let $M$ be a $G$-orbit and $q \in \bar{X}$ be a regular point for $G$. Consider the energy functional $E_q : \mathcal{P}(X, M \times q) \to \mathbb{R}$. Then the critical set of $E_q$ coincides with the critical set of the restriction of $E_q$ to $\mathcal{P}(\bar{X}, \bar{M} \times q)$.

**Proof.** Let $\gamma$ be a critical point of $E_q : \mathcal{P}(X, M \times q) \to \mathbb{R}$. Then $\gamma$ is a geodesic of $X$ that is perpendicular to $M$ at $p = \gamma(0)$ and such that $\gamma(1) = q$. Since $\gamma$ is perpendicular to $M$ at $p$, it is also perpendicular to $Gq$ at $q$. Since $Gq$ is a principal orbit, $Gq = H$ fixes $\gamma$ point-wise. It follows that $\gamma$ is a geodesic of $X^H$, and thus of $\bar{X}$, that is perpendicular to $Gq$, from which we deduce that $\gamma$ is perpendicular to $G\bar{X} = M$. Thus $\gamma$ is a critical point of $E_q : \mathcal{P}(\bar{X}, \bar{M} \times q) \to \mathbb{R}$.

On the other hand, let $\gamma$ be a critical point of $E_q : \mathcal{P}(\bar{X}, \bar{M} \times q) \to \mathbb{R}$. Then $\gamma$ is a geodesic of $\bar{X}$ that is perpendicular to $\bar{M} = G\bar{X}$ at $p = \gamma(0)$ and such that $\gamma(1) = q$. It follows that $\gamma$ is perpendicular to $G\bar{X}$. Since $G\bar{X}$ is a principal orbit, the normal spaces of $G\bar{X}$ in $T_q\bar{X}$ and of $Gq$ in $T_q\bar{X}$ coincide. Therefore $\gamma$ is perpendicular to $Gq$, and hence, to $G\bar{X}$. Since $\bar{X}$ is totally geodesic in $X$, it follows that $\gamma$ is a critical point of $E_q : \mathcal{P}(X, M \times q) \to \mathbb{R}$. $\square$

Let $Q$ be any subset of $\mathcal{P}(X)$. For $c > 0$, denote by $Q^c$ the subset $Q \cap E^{-1}([0, c])$, where $E : \mathcal{P}(X) \to \mathbb{R}$ is the energy functional. Also, denote by $\beta(Z)$ the (possibly infinite) sum of the $\mathbb{Z}_2$-Betti numbers of a topological space $Z$ (say, with respect to singular homology).

**Lemma 5.** Let $M$ be an arbitrary $G$-orbit, $q \in X$ and $c > 0$. Suppose there is a subgroup $L \subset H$ which is a finitely iterated $\mathbb{Z}_2$-extension of the identity and such that the connected
component of the fixed point set $X^L$ containing $q$ coincides with $X$. Then the numbers
\[ \beta(\mathcal{P}(X, M \times q)^c) \text{ and } \beta(\mathcal{P}(X, M \times q)) \] are finite and
\[ \beta(\mathcal{P}(X, M \times q)^c) \leq \beta(\mathcal{P}(X, M \times q)) . \]

Proof. We will work with the finite dimensional approximations of the spaces $\mathcal{P}(X, M \times q)^c$ and $\mathcal{P}(X, M \times q)^c$ [Mi73]. Let $\rho$ denote the injectivity radius of $X$; note that the injectivity radius of $\bar{X}$ cannot be smaller than $\rho$. Fix a positive integer $n$ such that $n > c/\rho^2$. Denote by $\Omega$ the space of continuous, piecewise smooth curves $\gamma : [0, 1] \to X$ of energy less than or equal to $c$ such that $\gamma(0) \in M$, $\gamma(1) = q$ and $\gamma|_{[\frac{2(k-1)}{n}, \frac{2k}{n}]}$ is a geodesic of length less than $\rho$ for $k = 1, \ldots, n$. It is known that $\Omega$ is an open finite dimensional manifold, and there is a deformation retract of $\mathcal{P}(X, M \times q)^c$ onto $\Omega$. It is clear that $\Omega$ is $H$-invariant; moreover, $\mathcal{P}(X, \bar{M} \times q)^c$ contains $\Omega^H$, and the definition of the deformation retract shows that it restricts to a deformation retract of $\mathcal{P}(X, \bar{M} \times q)^c$ onto $\Omega^H$.

We can write $L = \langle h_1, \ldots, h_n \rangle \subset H$ such that $h_{i+1}$ normalizes $H_i$ and $h_{i+1} \in H_i$ for $i = 0, \ldots, n - 1$, where $H_0$ is the identity group and $H_i$ is the subgroup of $H$ generated by $h_1, \ldots, h_i$. Now each transformation $h_{i+1}$ has order 2 in the fixed point set $\Omega^{H_i}$, so by a theorem of Floyd [Flo52],
\[ \beta(\Omega^L) \leq \beta(\Omega^{H_{n-1}}) \leq \ldots \leq \beta(\Omega^{H_0}) . \]
But $\Omega^L = \Omega^H$ and $\Omega^{H_0} = \Omega$, so this completes the proof. \hfill \Box

Proposition 6. Suppose there is a subgroup $L \subset H$ which is a finitely iterated $\mathbb{Z}_2$-extension of the identity and such that a connected component of the fixed point set $X^L$ coincides with $\bar{X}$. Suppose also that the reduced action of $\bar{G}$ on $\bar{X}$ is taut. Then the action of $G$ on $X$ is taut.

Proof. Let $M$ be an arbitrary $G$-orbit. We will prove that $M$ is taut. Suppose $q \in X$ is not a focal point of $M$, and consider the energy functional $E_q : \mathcal{P}(X, M \times q) \to \mathbb{R}$; this is a Morse function. We must show that $E_q$ is perfect, and in fact it is enough to do that for $q$ in a dense subset of $X$, so we may take $q$ to be a regular point. Since $G$ acts on $X$ by isometries, we may moreover assume that $q \in \bar{X}$. To simplify the notation, denote $\mathcal{P}(X, M \times q)$ by $\mathcal{M}$ and $\mathcal{P}(\bar{X}, \bar{M} \times q)$ by $\bar{\mathcal{M}}$. If the action of $\bar{G}$ on $\bar{X}$ is taut, then each $G$-orbit in $\bar{X}$ is taut, hence $M$ is taut. Note that $q$ is not a focal point of $\bar{M}$, so the restriction $E_q | \bar{\mathcal{M}}$ is a perfect Morse function. Therefore $\mu_k(E_q | (\bar{\mathcal{M}})^c) = \beta_k((\bar{\mathcal{M}})^c)$ for all $k$ and all $c$. Let $\mu$ denote $\sum_k \mu_k$. Then
\[ \mu(E_q | (\bar{\mathcal{M}})^c) = \beta((\bar{\mathcal{M}})^c) , \]
for all $c$. Now Lemmas [4] and [5] and the Morse inequalities imply that
\[ \beta((\bar{\mathcal{M}})^c) \leq \beta(\mathcal{M}^c) \leq \mu(E_q | \mathcal{M}^c) = \mu(E_q | (\bar{\mathcal{M}})^c) . \]
It follows that $\mu(E_q | \mathcal{M}^c) = \beta(\mathcal{M}^c)$ for all $c$. This implies that $\mu_k(E_q | \mathcal{M}^c) = \beta_k(\mathcal{M}^c)$ for all $k$ and all $c$, and thus $E_q$ is perfect. \hfill \Box

We finish the proof of Theorem [11] by using Proposition [6] to prove that the four polar actions on $X = \mathcal{C}a^P^2$ listed in [11] are taut. For each action, it is enough to show that the reduced action is taut, and to exhibit the group $L$ with the required properties. Some facts about the involved actions that are used below can be found in [PT99]. In particular, it is shown there that the sections for each one of these actions are isometric to $\mathbb{R}P^2$. 

5
\[
G = \text{Spin}(8). \text{ Since this group is contained in Spin}(9), \text{ it admits a fixed point } p \in X. \text{ The slice representation at } p \text{ is the sum of two inequivalent 8-dimensional representations of Spin}(8). \text{ A principal isotropy subgroup } H \text{ of } G \text{ on } X \text{ can be taken to be a principal isotropy subgroup of the slice representation, so } H \cong G_2. \text{ The homogeneous space Spin}(8)/G_2 \text{ is diffeomorphic to } S^7 \times S^7, \text{ so it follows that the normalizer } N \text{ of } G_2 \text{ in Spin}(8) \text{ is isomorphic to } \mathbb{Z}_2^7 \cdot G_2. \text{ Since } \tilde{N} = N/H \text{ is discrete, } \tilde{G} \text{ is also discrete, so } \tilde{X} \text{ is two-dimensional and thus coincides with a section } \Sigma \cong \mathbb{R}P^2. \text{ Now the reduced action of } G \text{ on } X \text{ is taut, because its orbits are points and points are taut in a symmetric space } [\text{Tate}, 1997]. \text{ The connected component containing } p \text{ of the fixed point set of a subgroup } L \text{ of } H \text{ is completely determined by the fixed point set of its linearized action on } T_pX. \text{ The action of } H \text{ on } T_p\Sigma \text{ is of course trivial, and its action on the orthogonal complement of } T_p\Sigma \text{ in } T_pX \text{ is equivalent to the sum of two copies of the 7-dimensional representation of } G_2. \text{ Hence the required } L \text{ can be constructed as follows. Consider the standard embedding of } SU(3) \text{ in } G_2. \text{ The normalizer of } SU(3) \text{ in } G_2 \text{ is isomorphic to } \mathbb{Z}_2 \cdot SU(3); \text{ let } h \text{ be a generator of the } \mathbb{Z}_2 \text{-factor. We take } L \text{ to be generated by the diagonal elements of } SU(3) \text{ with } \pm 1 \text{ entries and by } h.
\]

\[
G = SO(2) \cdot \text{Spin}(7). \text{ There is also a fixed point } p \text{ in this case. The slice representation at } p \text{ is } R^2 \otimes R^8, \text{ where } R^2 \text{ is the standard representation of } SO(2) \text{ and } R^8 \text{ is the spin representation of Spin}(7). \text{ The principal isotropy subgroup of } G \text{ on } X \text{ is } H = \mathbb{Z}_2 \cdot SU(3). \text{ The normalizer } N \text{ of } H \text{ in } G \text{ is isomorphic to } SO(2) \times SO(2) \cdot SU(3). \text{ Since } \tilde{N} = N/H \text{ is two-dimensional, } \tilde{X} \text{ is a four-dimensional connected complete totally geodesic submanifold of } X \text{ that contains a section } \Sigma \cong \mathbb{R}P^2. \text{ It follows that } \tilde{X} \text{ must be isometric to } CP^2 \text{ (compare Theorem 1 in [Wolf, 1963]), and then the reduced action of } \tilde{G} \text{ on } \tilde{X} \text{ is equivalent to the action of a maximal torus } T^2 \text{ of } SU(3) \text{ on } CP^2 = SU(3)/SU(1) \times SU(2)). \text{ This action is known to be polar, and hence it is taut by the argument in section 2. The subgroup } L \text{ of } H \text{ can be taken to be generated by the diagonal elements of } SU(3) \text{ with } \pm 1 \text{ entries.}
\]

\[
G = SU(2) \cdot SU(4). \text{ There is also a fixed point } p \text{ in this case. The slice representation at } p \text{ is } C^2 \otimes C^4, \text{ where } C^2 \text{ is the standard representation of } SU(2) \text{ and } C^4 \text{ is the standard representation of } SU(4). \text{ Therefore the principal isotropy subgroup } H \text{ of } G \text{ on } X \text{ is } T^1SU(2). \text{ The normalizer of } H \text{ in } G \text{ is } T^3 \cdot SU(2). \text{ It follows that } \tilde{X} \text{ is isometric to } CP^2, \text{ and the reduced action is taut as in the preceding case. The subgroup } L \text{ can be taken to be generated by the diagonal elements of } H \text{ with } \pm 1 \text{ entries.}
\]

\[
G = SU(3) \cdot SU(3). \text{ There is an orbit of type } CP^2. \text{ The corresponding isotropy is } U(2) \cdot SU(3), \text{ and the slice representation is } C^2 \otimes C^3. \text{ Therefore the principal isotropy subgroup } H \text{ of } G \text{ on } X \text{ is } T^2, \text{ the normalizer of } H \text{ in } G \text{ is } T^1, \tilde{X} \text{ is isometric to } CP^2, \text{ and the reduced action is as taut in the preceding case. The subgroup } L \text{ can be taken to be generated by the diagonal elements of } H \text{ with } \pm 1 \text{ entries.}
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References

[BCO03] J. Berndt, S. Console, and C. Olmos, Submanifolds and holonomy, Research Notes in Mathematics, no. 434, Chapman & Hall/CRC, Boca Raton, 2003.

[Ber99] I. Bergmann, Polar actions, Ph.D. thesis, Univ. Augsburg, 1999.

[BG05] L. Biliotti and A. Gori, Coisotropic and polar actions on complex Grassmannians, Trans. Amer. Math. Soc. 357 (2005), no. 5, 1731–1751.

[Bil04] L. Biliotti, Coisotropic and polar actions on compact irreducible Hermitian symmetric spaces, E-print math. DG/0408409, 2004.
[Brü98] M. Brück, Äquifokale Familien in symmetrischen Räumen, Ph.D. thesis, Universität zu Köln, 1998.

[BS58] R. Bott and H. Samelson, Applications of the theory of Morse to symmetric spaces, Amer. J. Math. 80 (1958), 964–1029, Correction in Amer. J. Math. 83 (1961), 207–208.

[Con71] L. Conlon, Variational completeness and K-transversal domains, J. Differential Geom. 5 (1971), 135–147.

[Flo52] E.E. Floyd, On periodic maps and the Euler characteristics of associated spaces, Trans. Amer. Math. Soc. 72 (1952), 138–147.

[GH91] K. Grove and S. Halperin, Elliptic isometries, condition (C) and proper maps, Arch. Math. (Basel) 56 (1991), 288–299.

[GS00] K. Grove and C. Searle, Global G-manifold reductions and resolutions, Arch. Math. (Basel) 56 (1991), 288–299.

[GT00] C. Gorodski and G. Thorbergsson, Representations of compact Lie groups and the osculating spaces of their orbits, Preprint, Univ. of Cologne, (also E-print math. DG/0203196), 2000.

[GT02] C. Gorodski and G. Thorbergsson, Cycles of Bott-Samelson type for taut representations, Ann. Global Anal. Geom. 21 (2002), 287–302.

[HLO00] E. Heintze, X. Liu, and C. Olmos, Isoparametric submanifolds and a Chevalley-type restriction theorem, E-print math. DG/0004028, 2000.

[HPTT94] E. Heintze, R. S. Palais, C.-L. Terng, and G. Thorbergsson, Hyperpolar actions and k-flat homogeneous spaces, J. Reine Angew. Math. 454 (1994), 163–179.

[Kol02] A. Kollross, A classification of hyperpolar and cohomogeneity one actions, Trans. Amer. Math. Soc. 354 (2002), 571–612.

[Kol05] A. Kollross, Polar actions on symmetric spaces, E-print math. DG/0506312, 2005.

[Mil73] J. Milnor, Morse theory, Annals of Mathematical Studies, vol. 51, Princeton University Press, 1973.

[PS64] R. S. Palais and S. Smale, A generalized Morse theory, Bull. Amer. Math. Soc. 70 (1964), 165–172.

[PT88] R. S. Palais and C.-L. Terng, Critical point theory and submanifold geometry, Lect. Notes in Math., no. 1353, Springer-Verlag, 1988.

[PT99] F. Podestà and G. Thorbergsson, Polar actions on rank one symmetric spaces, J. Differential Geom. 53 (1999), 131–175.

[PT02] F. Podestà and G. Thorbergsson, Polar and coisotropic actions on Kähler manifolds, Trans. Amer. Math. Soc. 354 (2002), 1759–1781.

[SS95] T. Skjelbred and E. Straume, A note on the reduction principle for compact transformation groups, preprint, 1995.

[Str94] E. Straume, On the invariant theory and geometry of compact linear groups of cohomogeneity ≤ 3, Diff. Geom. and its Appl. 4 (1994), 1–23.

[Tho05] G. Thorbergsson, Transformation groups, Rend. Mat. Appl. (7) 25 (2005), no. 1, 1–16.

[TT97] C.-L. Terng and G. Thorbergsson, Taut immersions into complete Riemannian manifolds, Tight and Taut Submanifolds (T. E. Ryan and S.-S. Chern, eds.), Math. Sci. Res. Inst. Publ. 32, Cambridge University Press, 1997, pp. 181–228.

[Wol63] J. A. Wolf, Elliptic spaces in Grassmann manifolds, Illinois J. Math. 7 (1963), 447–462.