On the embedding of left-symmetric algebras into differential Perm-algebras

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Abstract

Given an associative algebra satisfying the left commutativity identity $abc = bac$ (Perm-algebra) with a derivation $d$, the new operation $a \circ b = ad(b)$ is left-symmetric (pre-Lie). We derive necessary and sufficient conditions for a left-symmetric algebra to be embeddable into a differential Perm-algebra.

1. Introduction

The class of left-symmetric algebras (also known as pre-Lie algebras) initially appeared in deformation theory and geometry [7, 13, 19]. By definition, a left-symmetric algebra is a linear space with one bilinear multiplication $\circ$ satisfying the identity

$$(a, b, c) = (b, a, c),$$

where $(a, b, c) = (a \circ b) \circ c - a \circ (b \circ c)$ is the associator.

If, in addition, the right commutativity holds on a left-symmetric algebra $(A, \circ)$, i.e.,

$$(a \circ b) \circ c = (a \circ c) \circ b, \quad a, b, c \in A,$$

then $(A, \circ)$ is said to be a Novikov algebra. Novikov algebras appeared in formal variational calculus [6] and, independently, as a tool for studying Poisson brackets of hydrodynamic type [2]. The structure theory of Novikov algebras is well-developed, see, for example, [14] and references therein.

For example, if $A$ is an associative and commutative algebra with a derivation $d$ then the same space $A$ relative to the new product

$$a \circ b = ad(b), \quad a, b \in A,$$

is a Novikov algebra. As shown in [3], this example is generic: every Novikov algebra embeds into a differential commutative algebra. An alternative way to prove the latter is to apply the following observation: the free Novikov algebra $\text{Nov}(X)$ generated by a set $X$ embeds into the algebra $\text{ComDer}(X,d)$ of differential polynomials in $X$ [5], and it is easy to show [12] that every homomorphic image of $\text{Nov}(X)$ also embeds into a commutative differential algebra.
In this article, we study the class of left-symmetric algebras obtained from non-commutative differential algebras by means of the operation \((1.1)\). Namely, if \(A\) is an associative algebra satisfying the identity of left commutativity \(abc = bac\) (i.e. \(A\) is a Perm-algebra) then for every derivation \(d\) on \(A\) the operation \((1.1)\) turns \(A\) into a left-symmetric algebra which is denoted by \(A^{(d)}\).

Let us call a left-symmetric algebra \(special\) if it can be embedded into a differential Perm-algebra via \((1.1)\). Although Perm-algebras are very close to commutative ones, the description of the class of special left-symmetric algebras differs from that of Novikov algebras. The most important difference is that it is not a variety: a homomorphic image of a special left-symmetric algebra may not be special. We show the class of special left-symmetric algebras to be a quasi-variety. It is easy to see that every special left-symmetric algebra satisfies the following identities: \(((ax)y)b = (ay)x)b\) and \((x, ya, b) = (y, xa, b)\). The corresponding variety of \(SLS\)-algebras contains the class of special left-symmetric algebras and we find explicitly the set of quasi-identities defining special algebras within the class of \(SLS\)-algebras. Finally, we prove that \(SLS\)-algebras form the smallest variety containing special left-symmetric algebras. The latter follows from the observation that the free \(SLS\)-algebra is special.

We believe that the same approach as developed in this article will be useful for finding the complete list of special identities for Gelfand–Dorfman algebras. For the latter, special algebras are those embeddable into differential Poisson algebras [11]. The class of special Gelfand–Dorfman algebras does form a variety, and it is known that there are two independent identities of degree 4 [12]. This is an open problem if other special identities exist for Gelfand–Dorfman algebras.

Throughout the article, \(\mathbb{k}\) is an arbitrary base field. We will use the following notations. If \(Var\) is a variety of algebras then let \(Var(X)\) stand for the free algebra in \(Var\) generated by a set \(X\). The same symbol \(Var\) will be used to denote the operad governing the variety \(Var\).

For example, if \(Com\) denotes the variety of associative and commutative algebras then \(Com(X) = \mathbb{k}[X]\) is the ordinary polynomial algebra. By \(LSym\) we denote the variety of left-symmetric algebras, \(Nov \subset LSym\) stands for Novikov algebras, and so on.

The class of pairs \((A, d), A \in Var,\ d\) is a derivation on \(A\) is also a variety denoted \(VarDer\). The free algebra in \(VarDer\) generated by a set \(X\) is denoted \(VarDer(X, d)\). Here we include \(d\) into the notation for convenience. As a \(Var\)-algebra, \(VarDer(X, d)\) is isomorphic to \(Var(X^{(d)})\), where \(X^{(d)} = X \cup X' \cup X'' \cup \ldots \cup X^{(n)} \cup \ldots\), \(X^{(n)} = \{x^{(n)} | x \in X\}, n \geq 0\), are disjoint copies of \(X\). The derivation \(d\) acts on the generators of \(Var(X^{(d)})\) as \(d(x^{(n)}) = x^{(n+1)}\).

The interest to the study of differential Perm-algebras has one more motivation. The operad \(Perm\) plays a specific role in the theory of dialgebras [15] and corresponding operads (called replicated operads) [8, 17]: given an operad \(Var\), the Manin white product of operads \(Perm \circ Var\) is exactly the operad governing the class \(diVar\) of \(Var\)-dialgebras. Note that for \(Perm\) the Manin white product coincides with the Hadamard product of operads.

On the other hand, the operad \(Nov\) has its own distinguished role in the combinatorics of derivations on non-associative algebras [11]: given a binary operad \(Var\), the Manin white product \(Nov \circ Var\) is the operad governing the class of derived \(Var\)-algebras. The latter are obtained from \(Var\)-algebras with a derivation \(d\) relative to the new operations

\[
a \prec b = ad(b), \quad a \succ b = d(a)b
\]

(for each binary product in \(Var\)).

In the study of differential Perm-algebras, \(Perm\) and \(Nov\) meet each other. This is why the theory of dialgebras (in particular, Novikov dialgebras) provides us with effective tools for studying speciality of left-symmetric algebras.

The article is organized as follows.

In Section 2 we recall the definition of a \(Var\)-dialgebra and prove an analogue of the embedding theorem from [3] for Novikov dialgebras. This is done in a routine way by means of the
general methods described in [8] (see also [9]). Since every Novikov dialgebra is in particular a left-symmetric algebra, we obtain that an embedding of a left-symmetric algebra into a differential Perm-algebra is equivalent to the embedding into a Novikov dialgebra.

Section 3 is devoted to the study of the forgetful functor from diLSym to LSym. It turns out that the left adjoint functor has a very natural description: given an algebra $A \in LSym$, we may define the structure of its universal enveloping left-symmetric dialgebra on the tensor algebra $T(A)$.

In Section 4, we prove that the free SLS-algebra is special. Namely, the subalgebra of PermDer$(X,d)$ generated by the set $X$ relative to the operation $a \circ b = ad(b)$ is isomorphic to the free left-symmetric algebra satisfying two identities of SLS-algebras mentioned above. In contrast to [5], we do not find the monomial basis of SLS$(X)$, but still we can describe its linear basis explicitly as a subset of PermDer$(X,d)$.

Finally, in Section 5 we derive the necessary conditions of speciality for an SLS-algebra in the form of quasi-identities. Then we prove that these conditions are also sufficient. Given an SLS-algebra $A$, we construct an appropriate quotient of its universal enveloping left-symmetric dialgebra constructed in Section 3 to get a Novikov dialgebra envelope of $A$. The results of Section 2 then lead us to the desired conclusion.

### 2. Novikov dialgebras and differential Perm-algebras

Let Perm denote the class of associative algebras satisfying the identity $xyz - yzx = 0$.

The operad governing the variety of such algebras is also denoted by Perm. It is clear (see [4]) that $\dim \text{Perm}(n) = n$, and the composition rule is easy to describe [10].

Given a binary quadratic operad $O$, the Hadamard product of operads $O \otimes \text{Perm}$ coincides with the Manin white product [16], and the class of algebras governed by $O \otimes \text{Perm}$ is known as the variety of di $O$-algebras (replicated $O$-algebras [17], or $O$-dialgebras [10]). In particular, if Com and Lie denote the operads governing commutative and Lie algebras then diCom and diLie are isomorphic to Perm and Leib, respectively, where Leib is the operad of Leibniz algebras.

For an arbitrary binary operad $O$ with $\dim O(2) = 2$ (one binary product with no symmetry) the operad $\text{di} O = O \otimes \text{Perm}$ is generated by two binary operations $\vdash$ and $\dashv$. The defining relations of $\text{di} O$ are easy to derive from those of $O$ [10]. Namely, every $\text{di} O$-algebra satisfies so-called $0$-identities

\[(x_1 \vdash x_2) \dashv x_3 = (x_1 \vdash x_2) \dashv x_3,\]
\[x_1 \dashv (x_2 \vdash x_3) = x_1 \dashv (x_2 \vdash x_3).\] 

Moreover, for every multi-linear defining identity $f(x_1, \ldots, x_n) = 0$ of $O$ and for every $i = 1, \ldots, n$ one should claim

\[f_i(x_1, \ldots, x_n) = f(x_1, \ldots, x_i, \ldots, x_n) = 0\]

to hold on $\text{di} O$. Here $f(x_1, \ldots, x_i, \ldots, x_n)$ is derived from $f$ in the following way: in each nonassociative monomial $(x_{i_1} \cdots x_{i_k})$ the initial binary product is replaced with the products $\vdash$ and $\dashv$ so that the horizontal dashes point to the fixed variable $x_i$.

**Example 1.** Let Nov stand for the variety of Novikov algebras as well as for the corresponding operad. Then diNov is defined by (2.1) together with

\[(x_1 \vdash x_2) \vdash x_3 - x_1 \vdash (x_2 \vdash x_3) = (x_2 \vdash x_1) \vdash x_3 - x_2 \vdash (x_1 \vdash x_3),\]
\[(x_1 \vdash x_2) \dashv x_3 - x_1 \dashv (x_2 \vdash x_3) = (x_2 \vdash x_1) \dashv x_3 - x_2 \dashv (x_1 \vdash x_3),\]

(2.2)
\[(x_1 \downarrow x_2) \downarrow x_3 = (x_1 \downarrow x_3) \downarrow x_2,\]
\[(x_1 \uparrow x_2) \downarrow x_3 = (x_1 \downarrow x_3) \uparrow x_2.\]

(2.3)

A linear space \( N \) equipped with bilinear operations \( \uparrow \) and \( \downarrow \) is a Novikov dialgebra if and only if (2.1)–(2.3) hold for all \( x_1, x_2, x_3 \in N \).

Obviously, every Perm-algebra (diCom-algebra) with a derivation \( d \) turns into a Novikov dialgebra (diNov-algebra) relative to the operations \( x \downarrow y = xd(y), \quad x \uparrow y = d(y)x \). The embedding statement may be derived by means of the general construction from [8] that allows us to reduce the problem on a Novikov dialgebra to an “ordinary” Novikov algebra by means of conformal algebras (pseudo-algebras). One may refer to [1] for details on pseudo-algebras, but for our purposes the only essential instance is that every pseudo-algebra may be turned into a dialgebra [10].

**Theorem 1.** Every Novikov dialgebra \( (N, \uparrow, \downarrow) \) may be embedded into a differential Perm-algebra in such a way that

\[ x \uparrow y = xd(y), \quad x \downarrow y = d(y)x \]

for \( x, y \in N \).

**Proof.** If \( N \) is a Novikov dialgebra with operations \( \uparrow \) and \( \downarrow \) then

\[ N_0 = \text{span} \{a \uparrow b - a \downarrow b \mid a, b \in N\} \]

is an ideal in \( N \) and \( \bar{N} = N/N_0 \) is a Novikov algebra. The space \( N \) is a Novikov bimodule over \( \bar{N} \) relative to the action

\[ \bar{a} \circ b = a \uparrow b, \quad a \circ \bar{b} = a \downarrow b, \]

for \( a, b \in N \). The corresponding split null extension \( \hat{N} = \bar{N} \times N \) is a Novikov algebra [18] in which \( N \circ N = 0 \).

Let \( H \) be a cocommutative bialgebra with an element \( T \) such that \( \varepsilon(T) = 0 \), where \( \varepsilon \) is the counit in \( H \) (e.g., \( H = \mathbb{K}[T] \)). Consider the current pseudo-algebra \( \text{Cur}\hat{N} = H \otimes \hat{N} \). Then \( \text{Cur}\hat{N} \) is a Novikov dialgebra relative to the operations

\[ (f \otimes x) \uparrow (g \otimes y) = \varepsilon(f)g \otimes x \circ y, \quad (f \otimes x) \downarrow (g \otimes y) = \varepsilon(g)f \otimes x \circ y \]

for \( x, y \in \hat{N}, f, g \in H \).

The dialgebra \( N \) embeds into \( \text{Cur}\hat{N} \) via

\[ a \mapsto \hat{a} = 1 \otimes \bar{a} + T \otimes a, \quad a \in N. \]

Let \( (B, \cdot, d) \) be a commutative differential algebra which contains \( \hat{N} \) in such a way that

\[ x \cdot d(y) = x \circ y, \quad x, y \in \hat{N}. \]

Then \( C = \text{Cur}B \) may be considered as a commutative dialgebra (i.e. Perm-algebra) and it is easy to check that \( \hat{d} = \text{id} \otimes d \) is a derivation of \( C \). Thus we have

\[ N \subset \text{Cur}\hat{N} \subset \text{Cur}B = C, \]

and it is easy to check that the operations \( \uparrow, \downarrow \) on \( N \) are related with the operations \( \cdot, \hat{d} \) on \( C \) in the desired way. Indeed, if \( a, b \in N, \hat{a}, \hat{b} \in \text{Cur}\hat{N} \) then the product of \( \hat{d}(\hat{b}) \) and \( \hat{a} \) in the Perm-algebra \( C \) may be calculated as follows:

\[ \hat{d}(\hat{b})\hat{a} = (1 \otimes d(\hat{b}) + T \otimes d(b))(1 \otimes \bar{a} + T \otimes a) = 1 \otimes d(\hat{b}) \cdot \bar{a} + T \otimes d(b) \cdot a \]

\[ = 1 \otimes (\bar{a} \circ \hat{b}) + T \otimes (a \circ \hat{b}) = 1 \otimes a \downarrow \hat{b} + T \otimes (a \uparrow b) = a \downarrow \hat{b}. \]

In a similar way, \( a \uparrow b = \hat{a} \hat{d}(\hat{b}) \) in \( C \) for all \( a, b \in N \).
Lemma 1. A left-symmetric algebra \((A, \circ)\) embeds into a differential Perm algebra in such a way that \(a \circ b = ad(b)\) for \(a, b \in A\) if and only if \((A, \circ)\) embeds into a Novikov dialgebra in such a way that \(a \circ b = a \circ b\) for \(a, b \in A\).

3. Universal enveloping left-symmetric dialgebras

According to the general rule of constructing defining identities of dialgebras, every left-symmetric dialgebra \((A, \lhd, \rhd)\) is a left-symmetric algebra relative to the operation \(\lhd\). Thus we have a forgetful functor \(\text{diLSym} \to \text{LSym}\). In this section, we explicitly construct its left adjoint functor \(\mathcal{D} : \text{LSym} \to \text{diLSym}\) which will be applied in the sequel.

Let \((A, \circ)\) be a left-symmetric algebra. Consider the (associative) tensor algebra \(T(A)\) of the space \(A\) (without the identity):

\[
T(A) = A \oplus A^2 \oplus A^3 \oplus \ldots
\]

We will write \(a_1 \ldots a_n\) for the element \(a_1 \otimes \ldots \otimes a_n \in A^\otimes n\) for brevity.

Define the map \(M : T(A) \to A\) as follows:

\[
M(a_1 \ldots a_n) = (((a_1 \circ a_2) \circ a_3) \ldots) \circ a_n), \quad a_i \in A.
\]

Let us define a binary operation \(\lhd\) on the space \(T(A)\) by induction on the length of the second factor:

\[
\begin{align*}
u \lhd a &= M(u) \circ a, \quad u \in T(A), a \in A, \\
u \lhd (wa) &= (u \lhd w)a + w(M(u) \circ a) - wM(u)a, 
\end{align*}
\]

for \(u, w \in T(A), a \in A\). Finally, define a binary operation \(\rhd\) on \(T(A)\) as

\[
u \rhd w = uM(w), \quad u, w \in T(A).
\]

Lemma 1. For every \(u, w \in T(A)\) we have

\[
u \lhd w = M(u) \lhd w, \quad u \rhd w = u \lhd M(w), \\
M(u \lhd w) = M(u \rhd w) = M(u) \circ M(w).
\]

Proof. The desired equalities are obvious for the operation \(\rhd\), as well as for \(w = a \in A\). Proceed by induction on the length of \(w\):

\[
\begin{align*}
M(u \lhd wa) &= M(u \lhd w) \circ a + M(w) \circ (M(u) \circ a) - (M(w) \circ M(u)) \circ a \\
&= (M(u) \circ M(w)) \circ a - (M(w), M(u), a) = M(u) \circ (M(w) \circ a) \\
&= M(u) \circ M(wa)
\end{align*}
\]

by the left symmetry. \(\square\)

Proposition 1. For every left-symmetric algebra \((A, \circ)\), the space \(T(A)\) equipped with the operations (3.1), (3.2) is a left-symmetric dialgebra.

Proof. It is enough to check that (2.1) and (2.2) hold on \((T(A), \lhd, \rhd)\).

Let us start with (2.1). Consider \(u, v, w \in T(A)\) and note that

\[
(u \lhd v) \lhd w = M(u \lhd v) \lhd w = (M(u) \circ M(v)) \lhd w = M(u \lhd v) \lhd w = (u \lhd v) \lhd w
\]

by Lemma 1. The remaining 0-identity holds by similar reasons.
Next, proceed to (2.2). Let us state the calculation for the more complicated case: show that the “right” associator
\[(u \vdash v) \vdash w - u \vdash (v \vdash w)\]
is symmetric relative to the exchange of \(u\) and \(v\). Indeed, for \(w = c \in A\) it is enough to apply Lemma 1 and the left symmetry of \(A\). For the general case, apply induction on the length of \(w\). Since we may replace \(u\) and \(v\) with \(M(u)\) and \(M(v)\) by Lemma 1, consider \(u = a, v = b\) \((a, b \in A)\):
\[
(a \vdash b) \vdash wc - a \vdash (b \vdash wc) = ((a \circ b) \vdash w)c + w((a \circ b) \circ c) - w(a \circ b)c
- a \vdash ((b \vdash w)c + w(b \circ c) - wbc)
= ((a \circ b) \vdash w)c + w((a \circ b) \circ c) - w(a \circ b)c
- (a \vdash (b \vdash w)c - w((a \circ b) \circ c) + (b \vdash w)ac
- (a \vdash w)(b \circ c) - w(a \circ (b \circ c)) + wa(b \circ c)
+ (a \vdash wb)c + wb(a \circ c) - wbac.
\]
Let us remove those summands that already form symmetric expressions (relative to the exchange of \(a\) and \(b\)) and expand the last \((a \vdash wb)\):
\[-w(a \circ b)c - (b \vdash w)(a \circ c) + (b \vdash w)ac - (a \vdash w)(b \circ c) + wa(b \circ c)
+(a \vdash w)bc + w(a \circ b)c - wabc + wb(a \circ c) - wbac
= (a \vdash w)bc + (b \vdash w)ac - (a \vdash w)(b \circ c) - (b \vdash w)(a \circ c)
+ wa(b \circ c) + wb(a \circ c) - wbac - wabc.
\]
The expression obtained has the desired symmetry.

The remaining relation from (2.2) can be proved in a similar but simpler way. □

Denote the left-symmetric dialgebra \((T(A), \vdash, \dashv)\) by \(D(A)\). The injection \(\iota : A \rightarrow D(A), \iota(a) = a\), is a homomorphism from \(A\) to \(D(A)^o = (D(A), \vdash)\) by the definition.

**Proposition 2.** Let \((A, \circ)\) be a left-symmetric algebra. Then for every left-symmetric dialgebra \(B\) and for every homomorphism \(\tau : A \rightarrow B^o\) there exists unique homomorphism of dialgebras \(\varphi : D(A) \rightarrow B\) such that \(\varphi \iota = \tau\).

**Proof.** It is enough to check that the map
\[
\varphi(a_1a_2...a_{n-1}a_n) = (\ldots((x_1 \dashv x_2) \dashv \ldots \dashv x_{n-1}) \dashv x_n), \quad x_i = \tau(a_i), \quad (3.3)
\]
is a homomorphism of dialgebras preserving \(\vdash\) and \(\dashv\). The latter follows from the following observation: for all \(w \in T(A), x \in B\) we have
\[
\varphi(w) \vdash x = \tau(M(w)) \vdash x, \quad x \dashv \varphi(w) = x \dashv \tau(M(w))
\]
by (2.1). It remains to prove
\[
\varphi(a \vdash w) = \tau(a) \vdash \varphi(w), \quad a \in A,
\]
by induction on the length of \(w \in T(A)\), and apply Lemma 1. □

Thus \(D(A)\) is the universal enveloping left-symmetric dialgebra of of a left-symmetric algebra \(A\). As we can see, every \(A \in LSym\) embeds into its universal dialgebra envelope which is just \(T(A)\) as a linear space. The functor \(D(\cdot)\) is left adjoint to the forgetful functor \(diLSym \rightarrow LSym\) which is induced by the morphism of operads \(x_1 \circ x_2 \mapsto x_1 \vdash x_2\).

Note that the same sort of functor acts on \(diNov\): every Novikov dialgebra is in particular a left-symmetric algebra relative to the operation \(\vdash\). So one may construct a universal enveloping
Novikov dialgebra for a left-symmetric algebra \( A \), but the problem is to distinguish those algebras that are embedded into such envelopes. We will completely resolve this problem in Section 5.

### 4. A linear basis of the free SLS-algebra

Let \( P \) be a Perm-algebra with a derivation \( d \). As above, denote \( d(x) \) by \( x' \) for \( x \in P \). A new binary operation

\[
x \circ y = xy', \quad x, y \in P,
\]

(4.1)

turns \( P \) into a left-symmetric algebra. Moreover, the following identities hold on \( P \):

\[
((x \circ y) \circ z) \circ u = ((x \circ z) \circ y) \circ u, \quad (4.2)
\]

\[
(x, y \circ z, u) = (y, x \circ z, u), \quad (4.3)
\]

where \((a, b, c)\) stands for the associator \((a \circ b) \circ c - a \circ (b \circ c)\).

**Definition 1.** A left-symmetric algebra \( A \) satisfying (4.2) and (4.3) is said to be an SLS-algebra.

Let \( X \) be a nonempty set. In this section, we prove that the free SLS-algebra \( SLS\langle X \rangle \) embeds into the free differential Perm-algebra \( Perm\text{Der}\langle X, d \rangle \) relative to the operation (4.1). Namely, define a homomorphism

\[\tau : SLS\langle X \rangle \to Perm\text{Der}\langle X, d \rangle\]

such that

\[\tau(x) = x, \quad x \in X, \quad \tau(f \circ g) = \tau(f)\tau(g)'.\]

The main purpose of this section is to prove that \( \tau \) is injective.

A Novikov algebra is in particular an SLS-algebra. Hence, there exists a homomorphism

\[\phi : SLS\langle X \rangle \to Nov\langle X \rangle.\]

Choose a monomial basis \( \bar{N} \) in \( Nov\langle X \rangle \) and for every \( u \in \bar{N} \) choose its monomial pre-image in \( SLS\langle X \rangle \). Denote the set of such pre-images by \( N \). There exists a monomial basis \( B \) of \( SLS\langle X \rangle \) such that \( N \subset B \). Let us choose and fix such a set \( B \).

Every non-associative word \( u \) in \( X \) can be written as

\[u = L(u_1, \ldots, u_k, x) := u_1 \circ (u_2 \circ (\ldots \circ (u_k \circ x)\ldots)),\]

(4.4)

where \( u_j \) are non-associative words, \( x \in X \). Let \( k \) be the length of \( u \). For a linear combination of (4.4), the length is the maximal length of its summands.

**Lemma 2.** Every element of \( SLS\langle X \rangle \) may be written as a linear combination of words of the form (4.4) with \( u_j \in N, \ x \in X \).

**Proof.** It follows from (4.2) that if \( \phi(a) = \phi(b) \) then \( a \circ u = b \circ u \) for all \( a, b, u \in SLS\langle X \rangle \). This observation makes the lemma obvious. \( \Box \)

There exist various presentations of an element \( f \in SLS\langle X \rangle \) by linear combinations of (4.4) as in Lemma 2. However, for every nonzero \( f \) we may choose a presentation with minimal length. In this way, we obtain a well-defined length function

\[\ell : SLS\langle X \rangle \setminus \{0\} \to \mathbb{Z}_+\]

which depends only on the choice of \( B \).

Recall the notion of weight in a free commutative differential algebra [5]. Let \( X^{(\omega)} \) stand for the disjoint union \( X \cup X' \cup X'' \cup \ldots \), then \( Com\text{Der}\langle X, d \rangle \) as a commutative algebra coincides with
the polynomial algebra $\text{Com}(X^{(\omega)})$. Define

$$\text{wt}(x^j) = j - 1, \quad x \in X, \ j \geq 0,$$

and set $\text{wt}(uv) = \text{wt}(u) + \text{wt}(v)$ for all monomials $u, v \in \text{Com}(X^{(\omega)})$.

As shown in [5], the subspace $W$ spanned in $\text{Com}(X^{(\omega)})$ by all monomials of weight $-1$ is isomorphic to the free Novikov algebra $\text{Nov}(X)$ relative to the map

$$\text{Nov}(X) \rightarrow \text{Com}(X^{(\omega)})$$

defined in the same way as $\tau$. We will not distinguish notations for the maps presenting horizontal arrows in the commuting diagram below.

Hence, for every monomial $u \in W$ we may find its unique pre-image $\tau^{-1}(u) \in \text{Nov}(X)$, write $\tau^{-1}(u)$ as a linear combination of $N$, and consider the same combination as an element of $SLS(X)$ since $N \subseteq B$. Thus we obtain a well-defined map $\tau^{-1} : W \rightarrow kN \subseteq SLS(X)$ which depends only on the choice of $B$.

If $u \in \text{SLS}(X)$ is a monomial of the form (4.4) then

$$\tau(u) = \tau(u_1)\tau(u_2)\cdots\tau(u_k)x^{(k)} + g_u \in \text{Perm}(X^{(\omega)})$$

where all monomials in $g_u$ end with $x^{(j)}$ for $j < k$.

Recall that a linear basis of $\text{Perm}(X^{(\omega)})$ consists of all monomials of the form $ux^{(j)}$, where $j \geq 0$, $x \in X$, and $u$ is from a linear basis of the free associative commutative algebra $\text{Com}(X^{(\omega)})$ (see, e.g., [4]). In particular, associative words with different last letters are linearly independent in the free $\text{Perm}$-algebra.

Assume $\ker \tau \neq 0$ in $\text{SLS}(X)$. Choose a nonzero element $f \in \ker \tau$ of minimal length, $\ell(f) = k$. Without loss of generality we may suppose that all monomials from $f$ in a presentation given by Lemma 2 end with the same letter $x \in X$:

$$f = \sum_j \alpha_jL(u_{1j}, u_{2j}, \ldots, u_{kj}, x) + \tilde{f},$$

where $\ell(\tilde{f}) < k$ or $\tilde{f} = 0$. Then

$$\tau(f) = \sum_j \alpha_j\tau(u_{1j})\cdots\tau(u_{kj})x^{(k)} + h,$$

where all monomials in $h$ end with $x^{(j)}$ for $j < k$. Hence,

$$\sum_j \alpha_j\tau(u_{1j})\cdots\tau(u_{kj}) = 0$$

in $\text{ComDer}(X, d) = \text{Com}(X^{(\omega)})$.

The following statement leads us to a contradiction with the choice of $f$.

**Proposition 3.** Let

$$F = \sum_j \alpha_jw_{1j} \otimes \cdots \otimes w_{kj} \in W^{\otimes k}, \quad \alpha_j \in k,$$
Proof. Suppose \( \mu(F) = 0 \) for \( F \in W^{\otimes k} \). Then \( F \) may be presented as a linear combination of tensors of the form

\[
w_1 \otimes \ldots \otimes w_{j-1} \otimes (ab \otimes cd - cb \otimes ad) \otimes w_{j+1} \otimes \ldots \otimes w_k,
\]

where \( w_j, a, b, c, d \) are monomials in \( \text{Com}(X^{(a)}) \) and all tensor factors are of weight \(-1\).

It remains to prove that the associator identity

\[
(\tau^{-1}(ab), \tau^{-1}(cd), u) = (\tau^{-1}(cb), \tau^{-1}(ad), u)
\]

holds in \( \text{SLS}(X) \) for all appropriate \( a, b, c, d \in \text{Com}(X^{(a)}) \) and for all \( u \in \text{SLS}(X) \). Since \( \text{wt}(ab) = \text{wt}(cd) = -1 \) and due to left-symmetry of an SLS-algebra, it is enough to consider the case when \( \text{wt}(a) = \text{wt}(c) = -1 \).

Let \( x = \tau^{-1}(a) \). It follows from the definition of \( \tau \) that for every \( b \) such that \( \text{wt}(b) = 0 \) the element \( \tau^{-1}(ab) \) considered in \( \text{Nov}(X) \) is a linear combination of elements

\[
R(x, u_1, \ldots, u_m) = (\ldots((x \circ u_1) \circ u_2) \circ \ldots) \circ u_m, \quad u_i \in N.
\]

It follows from (4.2) that for every \( u \in B \) we may write

\[
\tau^{-1}(ab) \circ u = R(x, u_1, \ldots, u_m) \circ u \in \text{SLS}(X)
\]

In a similar way, one may present \( \tau^{-1}(cd) \) in \( \text{Nov}(X) \) as \( R(y, v_1, \ldots, v_l) \). Then

\[
(\tau^{-1}(ab), \tau^{-1}(cd), u) = (R(x, u_1, \ldots, u_m), R(y, v_1, \ldots, v_l), u),
\]

\[
(\tau^{-1}(cb), \tau^{-1}(ad), u) = (R(y, u_1, \ldots, u_m), R(x, v_1, \ldots, v_l), u).
\]

The right-hand sides of the last two relations are equal due to the identities of SLS-algebras. Indeed, let us prove the identity

\[
(R(x, u_1, \ldots, u_m), R(y, v_1, \ldots, v_l), u) = (R(y, u_1, \ldots, u_m), R(x, v_1, \ldots, v_l), u)
\]

by induction on \( m \) and \( l \).
If \( m = l = 0 \) then (4.6) coincides with the left symmetry identity. Assume \( m = 0, l > 0 \) and (4.6) is already proved for smaller \( \ell \)s. Then (4.3) together with left symmetry imply
\[
(x, R(y, v_1, \ldots, v_l), u) = (x, R(y, v_1, \ldots, v_{l-1}) \circ v_l, u)
\]
\[
= (R(y, v_1, \ldots, v_{l-1}), x \circ v_l, u) = (x \circ v_l, R(y, v_1, \ldots, v_{l-1}), u)
\]
\[
= (y, R(x \circ v_l, v_1, \ldots, v_{l-1}), u)
\]
by induction. It remains to note that \( R(x \circ v_l, v_1, \ldots, v_{l-1}) \) in the right-hand side may be replaced with \( R(x, v_1, \ldots, v_{l-1}, v_l) \) by (4.2).

For \( m > 0 \), proceed in a similar way:
\[
(R(x, u_1, \ldots, u_m), R(y, v_1, \ldots, v_l), u) = (y, R(R(x, u_1, \ldots, u_m), v_1, \ldots, v_l), u)
\]
\[
= (y, R(x, u_1, \ldots, u_m, v_1, \ldots, v_l), u) = (y, R(x, v_1, \ldots, v_l, u_1, \ldots, u_m), u)
\]
\[
= (y, R(x, v_1, \ldots, v_l), u_1, \ldots, u_m), u) = (R(x, v_1, \ldots, v_l), R(y, u_1, \ldots, u_m), u)
\]
\[
= (R(y, u_1, \ldots, u_m), R(x, v_1, \ldots, v_l), u).
\]

As a result,
\[
L(\tau^{-1}(w_1), \ldots, \tau^{-1}(w_{l-1}), \tau^{-1}(ab), \tau^{-1}(cd), \tau^{-1}(w_{l+1}), \ldots \tau^{-1}(w_k), x)
\]
\[
-L(\tau^{-1}(w_1), \ldots, \tau^{-1}(w_{l-1}), \tau^{-1}(cb), \tau^{-1}(ad), \tau^{-1}(w_{l+1}), \ldots \tau^{-1}(w_k), x)
\]
\[
\equiv L(\tau^{-1}(w_1), \ldots, \tau^{-1}(w_{l-1}), (\tau^{-1}(ab), \tau^{-1}(cd), u) - (\tau^{-1}(cb), \tau^{-1}(ad), u))
\]
modulo summands of smaller length, where \( u = L(\tau^{-1}(w_{l+1}), \ldots \tau^{-1}(w_k), x) \). The right-hand side of (4.7) is zero by (4.6).

Let us summarize the exposition of the section.

**Theorem 2.** For a nonempty set \( X \), the subalgebra \( F \) generated by \( X \) in the free differential \( \text{PermDer}(X, d) \) relative to the operation \( f \circ g = fd(g), f, g \in \text{PermDer}(X, d) \) is isomorphic to the free SLS-algebra generated by \( X \).

**Corollary 2.** A linear basis of the subalgebra \( F \) mentioned in Theorem 2 consists of all monomials \( u \in \text{Perm}(X^{(o)}) \) of weight \(-1\) such that \( u = vx^{(n)}, \ x \in X, \ n > 0 \).

The proof is completely analogous to that of [5] for \( \text{ComDer}(X, d) \).

### 5. A criterion of speciality

Let us say that an SLS-algebra \( (A, \circ) \) is nice if for every \( x \in A \) there exists a linear map \( \mu_x : A \circ A \to A \) such that \( \mu_x (a \circ b) = (a \circ x) \circ b \) for all \( a, b \in A \).

It is easy to see that \( (A, \circ) \) is nice if and only if for every \( a_i, b_i \in A \) \( (i = 1, \ldots, n) \) the equality
\[
\sum_i a_i \circ b_i = 0
\]
implies
\[
\sum_i (a_i \circ x) \circ b_i = 0
\]
for all \( x \in A \).

**Example 2.** A Novikov algebra is a nice SLS-algebra, the operator \( \mu_x \) coincides with the right multiplication by \( x \).
**Example 3.** Let $V$ be a linear space and let $A = T(V)/I$, where $I$ is the ideal spanned by

$$a_0a_1\ldots a_na_{n+1} - a_0a_{\sigma(1)}\ldots a_{\sigma(n)}a_{n+1}, \quad a_i \in V, \; n \geq 2, \; \sigma \in S_n.$$  

Obviously, $A$ is an associative (thus, left-symmetric) algebra satisfying the identities of SLS-algebras. This is not a Novikov algebra, but it is easy to see that if $\sum_i u_iv_i = 0$ in $A$ then $\sum_i u_ixv_i = 0$. Hence, $A$ is a nice SLS-algebra which is not Novikov.

**Example 4.** Suppose $V$ is a 3-dimensional space with a basis $x, y, z$. Let $A$ be the nice SLS-algebra from Example 3, and let $A' = A/(xz)$. Then $xyz \neq 0$ in $A'$, so $A'$ is an SLS-algebra which is not nice.

Example 4 shows us that the class of nice SLS-algebra is not a variety: it is not closed relative to homomorphic images.

The main purpose of this section is to prove the following criterion.

**Theorem 3.** A left-symmetric algebra $(A, \circ)$ embeds into a differential Perm-algebra in such a way that $a \circ b = ad(b)$ for $a, b \in A$ if and only if $A$ is a nice SLS-algebra.

The “only if” part is easy: the defining identities of an SLS-algebra hold in a differential Perm-algebra, and the map $\mu_x$ acts as a (left) multiplication by $d(x)$:

$$\mu_x(a \circ b) = d(x)ad(b) = ad(x)d(b) = (a \circ x) \circ b$$

for $a, b, x \in A$.

To prove the “if” part, we first construct the universal enveloping Novikov dialgebra for a nice SLS-algebra and then apply Corollary 1.

Suppose $(A, \circ)$ is a nice SLS-algebra. As above, let $T(A)$ stand for the associative tensor algebra of the space $A$ (without an identity), and let $D(A)$ be the left-symmetric dialgebra constructed on $T(A)$ relative to the operations defined by (3.1), (3.2).

Let $V$ stand for the right ideal of the algebra $T(A)$ generated by

$$g(a, x) = ax - \mu_x(a), \quad a \in A \circ A \subseteq A, \; x \in A.$$  

Choose a linear basis $X$ of $A$ in the form $X = X_0 \cup X_1$ where $X_0$ is a linear basis of $A \circ A$ and $X_1$ is a complement of $X_0$. It is easy to see that

$$V = \text{Span} \ \{ax_1\ldots x_n - \mu_{x_1}\ldots \mu_{x_i}(a) | a \in X_0, x_i \in X, i \geq 1\}.$$  

In particular, $V \cap A = 0$ since every generator of $V$ as of a linear space contains a unique principal term of degree $\geq 1$.

Note that

$$(a \circ b)f - M(abf) \in V \quad (5.1)$$

for all $a, b \in A, f \in T(A)$, by the definition of $M$.

Denote

$$K = \{f \in T(A) | M(abf) = 0 \text{ for all } a, b \in A\}$$

For example, $xy - yx \in K$. Moreover, if $f \in K$ then for all $u, v \in T(A) \cup \{1\}$ we have $ufv \in K$. Indeed,

$$M(ufvb) = M(M(au)fx_1\ldots x_n b) = M(M(M(au)fx_1)x_2\ldots x_nb) = 0$$

for $v = x_1\ldots x_n, \; n \geq 0$. Here we used the obvious corollary of the definition of $M$:

$$M(uv) = M(M(u)v), \quad u, v \in T(A).$$

In particular, if $f \in K$ and $u, v \in T(A)$ then $M(ufv) = 0$. 
Another series of elements in $K$ is given by

$$h(x,y,z) = xyz + (x \circ z) \circ y - (x \circ y)z - (x \circ z)y \in K$$

for all $x,y,z \in A$. Indeed, compute

$$M(ah(x,y,z)b) = M(axyz + a(x \circ z) \circ y - a(x \circ y)z - a(x \circ z)y) \circ b$$

$$= (((a \circ x) \circ y) \circ z + a \circ ((x \circ z) \circ y) - (a \circ (x \circ y)) \circ z - (a \circ (x \circ z)) \circ y) \circ b$$

$$= (((a \circ x) \circ z) \circ y + a \circ ((x \circ z) \circ y) - (a \circ z) \circ (x \circ y) - (a \circ (x \circ z) \circ y) \circ b = 0 \circ b = 0.$$

Let $I \subseteq T(A)$ stand for the linear span of all $ufv, u \in T(A), v \in T(A) \cup \{1\}$; this is a proper subset of $K$. In particular, $uxy - uyx \in I$ for all $u \in T(A), x,y \in A$. Hence, we may permute variables in $T(A)$ modulo $I$ leaving the first letter unchanged.

**Lemma 3.** The set $I$ is a two-sided ideal of $D(A)$.

**Proof.** For all $a \in A, u,v \in T(A)$ we have

$$a \vdash (uv) \equiv (a \vdash u)v + u(a \vdash v) - uav \pmod{I}. \quad (5.2)$$

It is easy to prove by induction on the length of $v$.

Let $f \in K, u \in T(A), v \in T(A) \cup \{1\}$. If $v \neq 1$ then $M(ufv) = 0$ and thus

$$ufv \vdash T(A) + T(A) \vdash ufv = 0.$$

Consider the case $v = 1$. For every $a,b \in A$, left symmetry implies

$$M(aM(uf)b) = (a \circ M(uf)) \circ b$$

$$= (a \circ M(uf)) \circ b + (M(uf) \circ a) \circ b - M(uf) \circ (a \circ b) = 0$$

since $M(uf) \circ x = M(ufx) = 0$ for all $x \in A$. Hence, $M(uf) \in K$.

Therefore, (5.2) leads to

$$uf \vdash w = M(uf) \vdash w \equiv -(n - 1)wM(uf) \equiv 0 \pmod{I}$$

for every $w \in T(A)$ of length $n$.

Similarly,

$$w \vdash uf = wM(uf) \in I$$

for every $w \in T(A)$.

As a straightforward corollary of the definition we get $I \vdash T(A) \subseteq I$. It remains to show $T(A) \vdash I \subseteq I$. Choose $ufv \in I$ for $f \in K, w \in T(A)$, and apply (5.2) for $x = M(w)$:

$$w \vdash ufv \equiv (x \vdash u)fv + u(x \vdash f)v + uf(x \vdash v) - 2uxfv \pmod{I}$$

(for $v \neq 1$; if $v = 1$ then the equation is analogous but simpler). Obviously, we need to prove $x \vdash f \in K$ since all other summands belong to $I$. Indeed, consider

$$x \vdash af \equiv (x \circ a)f + a(x \vdash f) - axf \pmod{I}.$$

Since $M(I) \circ b = 0$, we obtain

$$M(a(x \vdash f)b) = M((x \vdash af)b) - M((x \circ a)b) + M(afxb)$$

$$= M(x \vdash af) \circ b = (x \circ M(af)) \circ b = 0$$

due to left symmetry. Hence, $T(A) \vdash I \subseteq I$. \hfill \square

**Lemma 4.** The set $I = I + V$ is a two-sided ideal of the dialgebra $D(A)$.

**Proof.** Since $I$ is already known to be an ideal, it remains to consider the right ideal $V$ of $T(A)$. By definition, $V \vdash T(A) \subseteq V$. 

Denote
\[ g(a \circ b, x) = (a \circ b)x - (a \circ x) \circ b, \quad a, b, x \in A. \]
Then \( M(g(a \circ b, x)v) = 0 \) for all \( v \in T(A) \) by (4.2), and thus \( VA \vdash T(A) = 0 \). For \( v = 1 \), we may see \( M(g(a \circ b, x)) = (a \circ b)x - (a \circ x) \circ b \). Hence, \( uM(g(a \circ b, x)) = u((a \circ b)x - (a \circ x) \circ b) \equiv uh(a, x, b) - uh(a, b, x) \equiv 0 \pmod{I} \). It follows that
\[ V \vdash T(A) \subseteq I \]
by (5.2).
Similarly, \( u \vdash g(a \circ b, x)v = uM(g(a \circ b, x)v) \in I \). for all \( u \in T(A), v \in T(A) \cup \{1\} \).
To complete the proof it remains to calculate
\[
\begin{align*}
c \vdash g(a \circ b, x) &= c \vdash (a \circ b)x - c \circ ((a \circ x) \circ b) \\
&= (c \circ (a \circ b))x + (a \circ b)(c \circ x) - (a \circ b)c x - c \circ ((a \circ x) \circ b) \\
&= (c \circ x) \circ (a \circ b) + (a \circ (c \circ x)) \circ b - ((a \circ c) \circ x) \circ b - c \circ ((a \circ x) \circ b) \pmod{V}.
\end{align*}
\]
The latter relation is zero in every SLS-algebra. Then \( u \vdash g(a \circ b, x)v \in J \) for all \( u, v \in T(A) \) by (5.2).

\( \square \)

**Lemma 5.** For all \( u, w, v \in T(A) \) we have
\[
\begin{align*}
(u \vdash v) \vdash w - (u \vdash w) \vdash v & \in J, \\
(u \vdash v) \vdash w - (u \vdash w) \vdash v & \in V \subseteq J. \\
\end{align*}
\]
In particular, \( D(A)/J \) is a Novikov dialgebra.

**Proof.** The first relation follows from the definition of operations:
\[
(u \vdash v) \vdash w - (u \vdash w) \vdash v = uM(v)M(w) - uM(w)M(v) \in I.
\]
Let us prove the second one by induction on the length of \( w \). For \( w = a \in A \),
\[
\begin{align*}
(u \vdash v) \vdash a &= M(u \vdash v) \circ a = (M(u) \circ M(v)) \circ a, \\
(u \vdash a) \vdash v &= M(u) \circ a)M(v) \equiv \mu_{M(v)}(M(u) \circ a) \pmod{V}.
\end{align*}
\]
Hence, \( (u \vdash v) \vdash a - (u \vdash a) \vdash v \in V \subseteq J \). Next,
\[
\begin{align*}
(u \vdash v) \vdash wa &= ((u \vdash v) \vdash w)a + w(M(u \vdash v) \circ a) - wM(u \vdash v)a \\
&= ((u \vdash v) \vdash w)a + w(M(u) \circ M(v)) \circ a) - wM(u) \circ M(v)a, \\
(u \vdash wa) \vdash v &= (u \vdash w)M(v) \\
&= (u \vdash w)M(v) + w(M(u) \circ a)M(v) - wM(u)aM(v) \\
&= (u \vdash w)M(v)a + w(M(u) \circ a)M(v) - wM(u)aM(v) \pmod{I}.
\end{align*}
\]
Hence,
\[
\begin{align*}
(u \vdash v) \vdash wa - (u \vdash wa) \vdash v \\
&= ((u \vdash v) \vdash w)a - (u \vdash w)M(v)a + wh(M(u), a, M(v)) \\
&= (u \vdash v) \vdash w - (u \vdash w) \vdash v)a + wh(M(u), a, M(v)) \equiv 0 \pmod{f}.
\end{align*}
\]
\( \square \)

**Remark 1.** Suppose \( A \) is a Novikov algebra. If \( T \) is an ideal of the left-symmetric algebra \( D(A) \) such that \( D(A)/T \) is a Novikov algebra then \( T \) contains \( uh(x, y, z) \) for all \( u \in T(A), x, y, z \in A \) (it follows from the proof of Lemma 5). Therefore, the universal enveloping Novikov dialgebra
$\mathcal{N}(A)$ of $A$ is actually an image of $A \oplus A^2 \oplus A^3 \subset T(A)$. In particular, if $A$ is finite-dimensional then so is $\mathcal{N}(A)$.

**Lemma 6.** $A \cap J = \{0\}$.

**Proof.** Recall that a linear basis $X$ of $A$ is chosen in the form $X = X_0 \cup X_1$ where $X_0$ is a basis of $A \circ A$ and $X_1$ is its complement. Then $T(A)$ is the space of non-commutative polynomials in $X$.

By definition, the space $V$ is spanned by

$$g(a,x_1...x_n) = ax_1...x_n - \mu_{x_i}(a), \quad a \in X_0, x_1,...,x_n \in X. \quad (5.3)$$

Suppose $u \in T(A)$ is a word starting with a letter from $X_0$: $u = (a \circ b)w$. Then for every $f \in K, v \in T(A) \cup \{1\}$ we have

$$uv = (a \circ b)wfv \equiv M(awfvb) = 0 \pmod{V}$$

by (5.1).

Assume $I = I + V$ has a nonzero intersection with $A$. Then we may write the following equation in $T(A)$:

$$f_1 + f_0 + g = a \in A, \quad a \neq 0,$$

where $g \in V, f_i$ is a linear combination of the elements like $uv$ with $u$ starting with a letter from $X_0, i = 0, 1$. As we noted above, $f_0 \in V$. Every monomial in $f_1$ is of length at least 2 by the definition of $I$. All monomials that emerge in elements of $V$ start with a letter from $X_0$, so $f_1$ must be zero.

Therefore, we come to $a = f_0 + g \in V \cap A$ which is impossible for $a \neq 0$. \qed

**Corollary 3.** A nice SLS-algebra $(A, \circ)$ may be embedded into a Novikov dialgebra $(\mathcal{N}(A), \triangleright, \triangleleft)$ in such a way that $a \triangleright b = a \circ b$ for $a, b \in A$.

Indeed, $\mathcal{N}(A) = D(A)/J$ is the desired dialgebra.

**Remark 2.** Note that the generators of $I$ and $V$ represent the necessary conditions on a Novikov dialgebra which contains $(A, \circ)$ in such a way that $a \triangleright b = a \circ b$ for $a, b \in A$. Hence, $D(A)/J$ is the universal enveloping Novikov dialgebra of $(A, \circ)$.

Now we may finish the proof of **Theorem 3.** A nice SLS-algebra $(A, \circ)$ embeds into a Novikov dialgebra $(\mathcal{N}(A), \triangleright, \triangleleft)$, and the latter embeds into a differential Perm-algebra $(P, d)$ by **Theorem 1.** Since $a \circ b = a \triangleright b = ad(b) \in P$ for $a, b \in A$, this is a desired embedding.

**Corollary 4.** The free SLS-algebra SLS$(X)$ is nice.

**Corollary 5.** The variety SLS is the smallest one containing the class of all special left-symmetric algebras.

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