Global Stability of a Time-delayed Malaria Model with Standard Incidence Rate

Song-bai GUO¹,², Min HE¹, Jing-an CUI¹,†

¹School of Science, Beijing University of Civil Engineering and Architecture, Beijing 102616, China
(E-mail: guosongbai@bucea.edu.cn, †cuijingan@bucea.edu.cn)
²Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Abstract A four-dimensional delay differential equations (DDEs) model of malaria with standard incidence rate is proposed. By utilizing the limiting system of the model and Lyapunov direct method, the global stability of equilibria of the model is obtained with respect to the basic reproduction number $R_0$. Specifically, it shows that the disease-free equilibrium $E^0$ is globally asymptotically stable (GAS) for $R_0 < 1$, and globally attractive (GA) for $R_0 = 1$, while the endemic equilibrium $E^*$ is GAS and $E^0$ is unstable for $R_0 > 1$. Especially, to obtain the global stability of the equilibrium $E^*$ for $R_0 > 1$, the weak persistence of the model is proved by some analysis techniques.

Keywords malaria model; delay differential equations; Lyapunov functional; weak persistence; global stability

2020 MR Subject Classification 34K20; 37N25; 92D25

1 Introduction

Malaria is a disease caused by parasites and transmitted by the bites of infected female anopheles mosquitoes, which can be life-threatening[26]. The World Health Organization[27] estimated that there were 241 million malaria cases around the world in 2020, of which 627,000 deaths. Africa is the most affected region with a serious influence on local economic development[21]. Malaria is one of the most common infectious diseases in the world, bringing great pressure to the global control of infectious diseases[18]. The initial symptoms of malaria (fever, headache and chills) generally arise 10 to 15 days after the bite of an infected mosquito[26], which indicates that malaria has a certain incubation period.

Since the emergence of malaria, disease prevention and prediction have been of great importance. In this regard, mathematical modeling is a very important method. Many scholars have established human-mosquito malaria models to predict the development trend of the disease. In 1911, Ross[16] proposed a two-dimensional ordinary differential equations (ODEs) model of malaria and introduced the concept of threshold. In 1957, Macdonald[12] refined Ross’s model and first defined the concept of basic reproduction number. The refined model is called the Ross-Macdonald model. In recent years, many scholars further revised the classical Ross-Macdonald model (see, e.g., [1, 15, 23]). Subsequent studies have extended the revised models to higher dimensions (see, e.g., [4, 8, 13, 14, 19, 25]). For example, in 2015, Safan and Ghazi[19] proposed a four-dimensional ODEs model of malaria with standard incidence rates, and analyzed the local and global stability of equilibria of the model. In malaria transmission, time
delay can be considered due to the incubation period of the parasites, which means that individuals are not infectious until some time after they are infected\[9\]. In 2008, Ruan et al.\[17\] first proposed the Ross-Macdonald model with time delays and studied the effect of time delays on the basic reproduction number. Afterwards, lots of scholars had established some DDEs models of malaria (see, e.g., \[3, 9, 10, 20, 28\]). They mainly studied the local and global dynamics properties of these models. Consider that the standard incidence rate can be better to describe the transmission mechanism of malaria than the bilinear incidence rate (also see \[11\]). Thus, we propose a four-dimensional time-delayed malaria model with standard incidence rate. Next, we will focus on the local and global stability of equilibria of the model with respect to the basic reproduction number \(R_0\).

The remainder of this paper is organized as follows. In Section 2, we first give model formulation, and then obtain the well-posedness and dissipativeness of the system. In Section 3, we explore the existence conditions of equilibria of the system, and prove the local stability of equilibria with respect to \(R_0\). In Section 4, in order to get the global stability of the endemic equilibrium for \(R_0 > 1\), we verify the weak persistence of the system with a series of analysis techniques. In Section 5, we show that the global stability of the equilibria in terms of \(R_0\) by using the limiting system of the model and Lyapunov direct method.

2 The Model

We divide humans and mosquitoes into four compartments, such as \(S_h\): susceptible humans, \(I_h\): infected humans, \(S_v\): susceptible mosquitoes, \(I_v\): infected mosquitoes. We use the positive parameters \(\beta_h\), \(\beta_v\), \(\mu_h\) and \(\mu_v\) to represent the birth rates of humans and mosquitoes, and the death rates of humans and mosquitoes, respectively. The positive parameters \(C_{vh}\) and \(C_{hv}\) denote the infection rates of the infected mosquitoes biting susceptible humans and the susceptible mosquitoes biting the infected humans, respectively. The nonnegative parameter time delay \(\tau\) is the incubation period of malaria. Then, the following model of malaria transmission is proposed:

\[
\begin{align*}
\dot{S}_h(t) &= \beta_h - C_{vh} \frac{I_v(t)}{N_v(t)} S_h(t) - \mu_h S_h(t), \\
\dot{I}_h(t) &= C_{vh} \frac{I_v(t-\tau)}{N_v(t-\tau)} S_h(t-\tau) - \mu_h I_h(t), \\
\dot{S}_v(t) &= \beta_v - C_{hv} I_h(t) S_v(t) - \mu_v S_v(t), \\
\dot{I}_v(t) &= C_{hv} I_h(t) S_v(t) - \mu_v I_v(t),
\end{align*}
\]

(2.1)

where \(N_v(t) = S_v(t) + I_v(t)\).

The initial function of system (2.1) is

\[
\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T \in C_+ = \{\varphi \in C([-\tau, 0], \mathbb{R}_+^4) : \varphi_3(\theta) + \varphi_4(\theta) > 0, \forall \theta \in [-\tau, 0]\}
\]

with \(\mathbb{R}_+ = [0, \infty)\), where \(C\) represents the Banach space composed of continuous functions mapping from \([-\tau, 0]\) to \(\mathbb{R}_+^4\) with the supremum norm. Next, we will discuss the well-posedness and the dissipativeness of system (2.1) in \(C_+\).

**Theorem 2.1.** The solution \(x(t) = (S_h(t), I_h(t), S_v(t), I_v(t))^T\) of system (2.1) through any \(\varphi \in C_+\) exists, which is unique, nonnegative, and ultimately bounded on \([0, \infty)\).

**Proof.** By using the similar proof of [5, Proposition 2.1], we first can obtain the unique existence of the solution \(x(t)\) and \(x(t) \geq 0\) on \([0, \infty)\). Hence, for \(t \geq \tau\), it follows from system (2.1) that

\[
\begin{align*}
\dot{S}_h(t-\tau) + \dot{I}_h(t) &= \beta_h - \mu_h (S_h(t-\tau) + I_h(t)), \\
\dot{S}_v(t) + \dot{I}_v(t) &= \beta_v - \mu_v (S_v(t) + I_v(t)),
\end{align*}
\]
which yields
\[
\lim_{t \to \infty} (S_h(t - \tau) + I_h(t)) = \frac{\beta_h}{\mu_h}, \quad \lim_{t \to \infty} (S_v(t) + I_v(t)) = \frac{\beta_v}{\mu_v}.
\]

\[ \blacktriangleleft \]

**Remark 2.2.** It is not difficult to find that the solution \( x(t) \) of system (2.1) with any \( \varphi \in C_+ \) satisfies that \((S_h(t), S_v(t)) \to 0 \) for \( t > 0 \), and then \( C_+ \) is positively invariant for system (2.1).

### 3 Local Stability

The disease-free equilibrium \( E^0 = (S_h^0, 0, S_v^0, 0)^T = (\beta_h/\mu_h, 0, \beta_v/\mu_v, 0)^T \) of system (2.1) can be easily obtained. In order to get the existence of an endemic equilibrium (positive equilibrium) \( E^* = (S_h^*, I_h^*, S_v^*, I_v^*)^T \), the basic reproduction number \( R_0 = \sqrt{C_{vh}C_{hv}\beta_h/\mu_h^2\mu_v} \) is first given by using the similar method in [24].

**Lemma 3.1.** System (2.1) has a unique endemic equilibrium \( E^* \) if and only if \( R_0 > 1 \).

**Proof.** By system (2.1), the equilibrium equations can be given by

\[
\begin{align*}
\beta_h - C_{vh} \frac{I_v^*}{S_v^*} S_h^* - \mu_h S_h^* &= 0, \\
C_{vh} \frac{I_v^*}{S_v^* + I_v^*} S_h^* - \mu_h I_h^* &= 0, \\
(3.1) \\
\beta_v - C_{hv} I_h^* S_v^* - \mu_v S_v^* &= 0, \\
C_{hv} I_h^* S_v^* - \mu_v I_v^* &= 0.
\end{align*}
\]

Simplifying and sorting (3.1), we have

\[
\begin{align*}
S_h^* &= \frac{\beta_h}{C_{vh} C_{hv} I_v^*/\beta_v + \mu_h}, \\
I_h^* &= \frac{\beta_h C_{vh} C_{hv} I_v^*/\beta_v + \mu_h}{\beta_v}, \\
S_v^* &= \frac{\beta_v}{C_{hv} I_h^* + \mu_v}, \\
I_v^* &= \frac{\beta_v C_{hv} I_h^*}{\mu_v (C_{hv} I_h^* + \mu_v)}.
\end{align*}
\]

(3.2)

It follows from the second and the fourth equations in (3.2) that

\[
\left[ \mu_h \left( \frac{C_{vh} C_{hv} I_v^*}{C_{hv} I_h^* + \mu_v} + \mu_h \right) - \frac{C_{vh} C_{hv} \beta_h}{C_{hv} I_h^* + \mu_v} \right] I_h^* = 0.
\]

Solving the above equation for \( I_h^* \), we get that \( I_h^* = 0 \) or

\[
(3.3)
\]

Consequently, it follows from (3.3) that

\[
S_h^* = \frac{\beta_h (C_{hv} \beta_h / \mu_h + \mu_v)}{C_{hv} \beta_h + \mu_v \mu_h R_0^2}, \quad S_v^* = \frac{\beta_v (C_{hv} + \mu_h)}{C_{vh} \mu_v + \mu_v \mu_h R_0^2}, \quad I_v^* = \frac{\beta_v \mu_h (R_0^2 - 1)}{C_{vh} \mu_v + \mu_v \mu_h R_0^2}.
\]

Thus, the endemic equilibrium \( E^* \to 0 \) is uniquely obtained if and only if \( R_0 > 1 \). \( \square \)
For the local asymptotic stability of the disease-free equilibrium \( E^0 \), we have the following result.

**Theorem 3.2.** For any \( \tau \geq 0 \), the disease-free equilibrium \( E^0 \) is locally asymptotically stable (LAS) if \( R_0 < 1 \) and unstable if \( R_0 > 1 \).

**Proof.** By the direct calculation, the transcendental characteristic equation of the linearized system of system (2.1) at \( E^0 \) is taken by

\[
(\lambda + \mu_h)(\lambda + \mu_v)[(\lambda + \mu_h)(\lambda + \mu_v) - u_1u_2e^{-\lambda \tau}] = 0,
\]

where \( u_1 = C_{hv}\beta_v/\mu_v \), \( u_2 = C_{vh}\beta_h/\mu_v\beta_v\mu_h \). Obviously, equation (3.4) has two negative roots \( \lambda_1 = -\mu_v \) and \( \lambda_2 = -\mu_h \). Now, we consider the following transcendental equation

\[
G(\lambda) = \lambda^2 + q_1 \lambda + q_2 + q_3 e^{-\lambda \tau} = 0,
\]

where \( q_1 = \mu_h + \mu_v \), \( q_2 = \mu_v \mu_h \), \( q_3 = -u_1u_2 \). Note that

\[
q_1 > 0, \quad q_2 + q_3 = \mu_v \mu_h - \frac{C_{vh}C_{hv}\beta_h}{\mu_h} = \mu_v \mu_h (1 - R_0^2) > 0
\]

for \( R_0 < 1 \). Thus, it is known from the Routh-Hurwitz criterion that each root of equation (3.5) has a negative real part for \( R_0 < 1 \) and \( \tau = 0 \).

For \( R_0 < 1 \) and \( \tau > 0 \), assume that \( G(\lambda) \) has the complex root \( \lambda = wi \) (\( w \geq 0 \)). Then it follows from equation (3.5) that

\[
\begin{cases}
q_2 - w^2 = -q_3 \cos w\tau, \\
q_1 w = q_3 \sin w\tau.
\end{cases}
\]

Further, we have

\[
w^4 + (q_1^2 - 2q_2)w^2 + q_2^2 - q_3^2 = 0.
\]

(3.6)

Clearly,

\[
q_1^2 - 2q_2 = \mu_v^2 + \mu_h^2 > 0, \quad q_2^2 - q_3^2 = (q_2 - q_3)(q_2 + q_3) > 0
\]

for \( R_0 < 1 \) and \( \tau > 0 \). Thus, equation (3.6) is false, which means that equation (3.5) has no roots on the imaginary axis. Consequently, each root of equation (3.5) has a negative real part for \( R_0 < 1 \) and \( \tau > 0 \). Therefore, \( E^0 \) is LAS for \( R_0 < 1 \) and \( \tau \geq 0 \).

For \( R_0 > 1 \) and \( \tau \geq 0 \), it holds that

\[
G(0) = q_2 + q_3 = \mu_v \mu_h (1 - R_0^2) < 0, \quad \lim_{\lambda \to \infty} G(\lambda) = \infty.
\]

In consequence, equation (3.5) must admit a positive root, which implies that \( E^0 \) is unstable for \( R_0 > 1 \) and \( \tau \geq 0 \).

For the local asymptotic stability of the endemic equilibrium \( E^* \), we have the following conclusion.

**Theorem 3.3.** If \( R_0 > 1 \), then the endemic equilibrium \( E^* \) is LAS for any \( \tau \geq 0 \).

**Proof.** By the direct calculation, the transcendental characteristic equation of the linearized system of system (2.1) at \( E^* \) is given by

\[
\begin{vmatrix}
\lambda + C_{vh}I^*_vN^*_v + \mu_h & 0 & -C_{vh}I^*_vS^*_hN^*_v & C_{vh}S^*_vS^*_hN^*_v \\
-C_{vh}I^*_vN^*_ve^{-\lambda \tau} & \lambda + \mu_h & C_{vh}I^*_vS^*_hN^*_v e^{-\lambda \tau} & -C_{vh}S^*_vS^*_hN^*_v e^{-\lambda \tau} \\
0 & C_{hv}S^*_v & \lambda + C_{hv}I^*_v + \mu_v & 0 \\
0 & -C_{hv}S^*_v & -C_{hv}I^*_v & \lambda + \mu_v
\end{vmatrix} = 0.
\]

(3.7)
Let $m_1 = C_{vh}I^*_v/N_v^*$, $m_2 = C_{vh}I^*_h S_h^*/N_h^*$, $m_3 = C_{vh}S_v^* S_h^*/N_v^* N_h^*$, $m_4 = C_{hv} S_v^*$, $m_5 = C_{hv}I^*_h$. Then equation (3.7) becomes

$$
(\lambda + \mu_v)(\lambda + \mu_h + m_1)(\lambda + \mu_v + m_5) - m_4(m_3 + m_2)e^{-\lambda \tau} = 0. \tag{3.8}
$$

Obviously, equation (3.8) has characteristic roots $\lambda_1 = -\mu_v < 0$, $\lambda_2 = -\mu_h < 0$. Now, we consider the following transcendental equation

$$
G(\lambda) = \lambda^2 + p_1 \lambda + p_2 + p_3 e^{-\lambda \tau}, \tag{3.9}
$$

where $p_1 = \mu_h + m_1 + \mu_v + m_5$, $p_2 = (\mu_h + m_1)(\mu_v + m_5)$, $p_3 = -m_4(m_3 + m_2)$.

For $R_0 > 1$ and $\tau = 0$, we have

$$
p_1 = \mu_h + m_1 + \mu_v + m_5 > 0,
p_2 + p_3 = (\mu_h + m_1)(\mu_v + m_5) - m_4(m_3 + m_2)
= C_{hv}(\mu_h + C_{vh})I^*_h + C_{vh} \mu_v \mu_h \left( \mu_v + C_{hv} \frac{\beta_h}{\mu_h} \right) I^*_v + \mu_h \mu_v (1 - R_0^2)
= (R_0 + 1) \mu_v \mu_h (R_0 - 1) > 0,
$$

where $I^*_v = \beta_h \mu_v (R_0^2 - 1)/(C_{hv} \beta_h + \mu_v \mu_h R_0^2)$, $I^*_v = \beta_v \mu_h (R_0^2 - 1)/(C_{vh} \mu_v + \mu_v \mu_h R_0^2)$ and $R_0^2 = C_{vh} C_{hv} \beta_h \mu_h \mu_v$ are used. It follows from the Routh-Hurwitz criterion that each root of equation (3.9) has a negative real part.

For $R_0 > 1$ and $\tau > 0$, if $G(\lambda)$ has the root $\lambda = wi$ ($w \geq 0$), then by equation (3.9), we have

$$
\begin{cases}
p_2 - w^2 = -p_3 \cos w \tau, \\
p_1 w = p_3 \sin w \tau.
\end{cases}
$$

Consequently, it holds

$$
w^4 + (p_1^2 - 2p_2)w^2 + p_2^2 - p_3^2 = 0. \tag{3.10}
$$

Note that

$$
p_1^2 - 2p_2 = (\mu_h + m_1)^2 + (\mu_v + m_5)^2 + 2m_4(m_3 + m_2) > 0,
p_2^2 - p_3^2 = (p_2 - p_3)(p_2 + p_3) > 0.
$$

Accordingly, equation (3.10) is false, which hints that equation (3.9) has no roots on the imaginary axis. Hence, each root of equation (3.9) has a negative real part for $R_0 > 1$ and $\tau > 0$. Thus, $E^*$ is LAS for $R_0 > 1$ and $\tau \geq 0$.

4 Weak Persistence

To get the global stability of $E^*$, we need to study the weak persistence of system (2.1). Denote

$$D = \{ \varphi \in C_+: \varphi(0) > 0 \}.$$

Let $x(t) = (S_h(t), I_h(t), S_v(t), I_v(t))^T$ be the solution of system (2.1) through any $\varphi \in D$. It is not difficult to find that $D$ is a positive invariant set with respect to system (2.1) and $x(t) \not\to \emptyset$ for $t > 0$. Consequently, we investigate the weak persistence of system (2.1) in $D$.

Following the definition of weak persistence in [2], system (2.1) is called weakly persistent if

$$\limsup_{t \to \infty} U(t) > 0, \quad U = S_h, I_h, S_v, I_v \quad \text{for any } \varphi \in D.$$ 

Define $x_t = (S_{ht}, I_{ht}, S_{vt}, I_{vt})^T \in C_+$ for $t \geq 0$ as $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$. Then $x_t$ is the solution of system (2.1) through $\varphi$. Motivated by the persistence approach in [7], we have the following results.
Lemma 4.1. If $R_0 > 1$, $\theta \in (0, 1)$, and $\limsup_{t \to \infty} I_h(t) \leq \theta I_h^*$, then for any $\tau \geq 0$, there hold

$$
\liminf_{t \to \infty} S_v(t) \geq \bar{S}_v \equiv \frac{\beta_v}{\theta C_{hv} I_h^* + \mu_v} > S_v^*, \quad \liminf_{t \to \infty} S_h(t) \geq \bar{S}_h \equiv \frac{\beta_h}{C_{vh} (1 - S_v/S_v^0) + \mu_h} > S_h^*.
$$

Proof. According to the endemic equilibrium equations, we have $\bar{S}_v > S_v^*$, $\bar{S}_h > S_h^*$. For any $\varepsilon > 1$, there exists $\rho = \rho(\varphi, \varepsilon) > 0$ such that for $t > 0$, it follows

$$
I_h(t) < \varepsilon \theta I_h^*.
$$

Therefore, for $t > \rho$, we get

$$
\dot{S}_v(t) = \beta_v - C_{hv} I_h(t) S_v(t) - \mu_v S_v(t) > \beta_v - S_v(t) (\varepsilon \theta C_{hv} I_h^* + \mu_v).
$$

As a result, there holds

$$
\liminf_{t \to \infty} S_v(t) \geq \frac{\beta_v}{\theta C_{hv} I_h^* + \mu_v}.
$$

Letting $\varepsilon \to 1^+$, we obtain

$$
\liminf_{t \to \infty} S_v(t) \geq \bar{S}_v.
$$

It follows from system (2.1) that $\dot{N}_v(t) = \beta_v - \mu_v N_v(t)$, which yields

$$
\lim_{t \to \infty} N_v(t) = S_v^0, \quad \text{(4.1)}
$$

Hence, we have

$$
\limsup_{t \to \infty} \frac{I_v(t)}{N_v(t)} = \limsup_{t \to \infty} \left(1 - \frac{S_v(t)}{N_v(t)}\right) = 1 - \liminf_{t \to \infty} \frac{S_v(t)}{N_v(t)} \leq 1 - \frac{\bar{S}_v}{S_v^0}.
$$

For any $\varepsilon \in (0, 1)$, there is $\tilde{\rho} = \tilde{\rho}(\varphi, \varepsilon) > 0$, such that for $t > \tilde{\rho}$,

$$
\frac{I_v(t)}{N_v(t)} < 1 - \frac{\varepsilon \bar{S}_v}{S_v^0}.
$$

Hence, for $t > \tilde{\rho}$, we have

$$
\dot{S}_h(t) = \beta_h - C_{vh} \frac{I_v(t)}{N_v(t)} S_h(t) - \mu_h S_h(t) > \beta_h - S_h(t) \left[C_{vh} \left(1 - \frac{\varepsilon \bar{S}_v}{S_v^0}\right) + \mu_h\right].
$$

Consequently, there holds

$$
\liminf_{t \to \infty} S_h(t) \geq \frac{\beta_h}{C_{vh} \left(1 - \varepsilon \bar{S}_v/S_v^0\right) + \mu_h}.
$$

Letting $\varepsilon \to 1^-$, we obtain

$$
\liminf_{t \to \infty} S_h(t) \geq \bar{S}_h.
$$

Theorem 4.2. If $R_0 > 1$, then $\limsup_{t \to \infty} I_h(t) \geq I_h^*$ for any $\tau \geq 0$. \qed
Proof. We prove the conclusion by contradiction. Assume \( \limsup_{t \to \infty} I_h(t) < I^*_h \). Then there exists a \( \theta \in (0, 1) \) such that \( \limsup_{t \to \infty} I_h(t) \leq \theta I^*_h \). According to Lemma 4.1, for any \( \varepsilon \in (0, \varepsilon_0) \), there is an \( \varepsilon_0 > 0 \) such that

\[
\frac{\bar{S}_h}{S^0_v + \varepsilon} > \frac{S^*_h}{S^0_v}.
\]

By Lemma 4.1 and (4.1), we get that for any \( \varepsilon \in (0, \varepsilon_0) \), there is a \( T \equiv T(\varepsilon, \varphi) > 0 \) such that

\[
\frac{S_h(t)}{N_v(t)} > \frac{\bar{S}_h}{S^0_v + \varepsilon}, \quad S_v(t) > S^*_v
\]

for all \( t \geq T \). Next, we define the following functional

\[
L(\varphi) = \varphi_2(0) + \frac{C_{vh}\bar{S}_h}{(S^0_v + \varepsilon)\mu_v} \varphi_4(0) + C_{vh} \int_0^T \frac{\varphi_4(\theta)}{-\varphi_3(\theta) + \varphi_4(\theta)} \varphi_1(\theta) d\theta, \quad \varphi \in D.
\]

Clearly, \( L \) is continuous on \( D \) and then \( L(x_i) \) is bounded. Now, we calculate the derivative of \( L \) along the solution \( x_i \) for \( t \geq T \) as follows

\[
\dot{L}(x_i) \geq \left[ \frac{C_{vh}C_{hv}\bar{S}_h}{(S^0_v + \varepsilon)\mu_v} S_v(t) - \mu_h \right] I_h(t)
\]

\[
> \mu_h \left[ \frac{S^*_v\bar{S}_h}{(S^0_v + \varepsilon) S^0_v} R^2_0 - 1 \right] I_h(t).
\]

Set

\[
\bar{I}_h = \min_{\theta \in [-\tau, 0]} I_h(T + \tau + \theta), \quad \Lambda = \min \left\{ \bar{I}_h, \frac{\mu_v I_v(T)}{C_{hv} S^*_v} \right\}.
\]

Now, we start by verifying \( I_h(t) \geq \Lambda \) for \( t \geq T \). Otherwise, there is a \( T_0 \geq 0 \) such that \( I_h(t) \geq \Lambda \) for \( t \in [T, T + T_0 + \tau] \), \( I_h(T + T_0 + \tau) = \Lambda \) and \( \bar{I}_h(T + T_0 + \tau) \leq 0 \). For \( t \in [T, T + T_0 + \tau] \), we can obtain

\[
\dot{I}_v(t) = C_{hv} I_h(t) S_v(t) - \mu_v I_v(t) \geq C_{hv} \Lambda S^*_v - \mu_v I_v(t).
\]

By (4.2), for \( t \in [T, T + T_0 + \tau] \), we have

\[
I_v(t) \geq \frac{C_{hv} \Lambda S^*_v}{\mu_v} + \left( I_v(T) - \frac{C_{hv} \Lambda S^*_v}{\mu_v} \right) e^{-\mu_v t} \geq \frac{C_{hv} \Lambda S^*_v}{\mu_v}
\]

for \( t \in [T, T + T_0 + \tau] \). Note that \( R^2_0 = S^0_v S^0_v / S^*_v S^*_v \), we have

\[
\frac{S^*_v \bar{S}_h}{(S^0_v + \varepsilon) S^0_v} R^2_0 - 1 > \frac{S^*_v \bar{S}_h}{S^0_v \bar{S}_h} R^2_0 - 1 = 0.
\]

As a consequence, we conclude that

\[
\dot{I}_h(T + T_0 + \tau) = C_{vh} \frac{I_v(T + T_0)}{N_v(T + T_0)} S_h(T + T_0) - \mu_h I_h(T + T_0 + \tau)
\]

\[
> \Lambda \mu_h \left[ \frac{S^*_v \bar{S}_h}{(S^0_v + \varepsilon) S^0_v} R^2_0 - 1 \right] > 0,
\]

which contradicts \( \dot{I}_h(T + T_0 + \tau) \leq 0 \). Accordingly, \( I_h(t) \geq \Lambda \) for \( t \geq T \). Therefore, for \( t \geq T \), we have

\[
\dot{L}(x_i) > \Lambda \mu_h \left[ \frac{S^*_v \bar{S}_h}{(S^0_v + \varepsilon) S^0_v} R^2_0 - 1 \right] > 0,
\]

which yields \( L(x_i) \to \infty \) as \( t \to \infty \). As a consequence, this is a contradiction to the boundedness of \( L(x_i) \). \( \square \)
By Theorem 4.2, we can get the following corollary.

**Corollary 4.3.** If $R_0 > 1$, then system (2.1) is weakly persistent for any $\tau \geq 0$.

## 5 Global Stability

In this section, we will prove the global stability of $E^0$ and $E^*$ with respect to $R_0$. To do this, it follows from (4.1) that system (2.1) has the limiting system as follows:

\[
\begin{aligned}
\dot{S}_h(t) &= \beta_h - \frac{C_{vh}I_v(t)S_h(t)}{S_h^0} - \mu_h S_h(t), \\
\dot{I}_h(t) &= C_{vh}I_v(t - \tau)S_h(t - \tau) - \mu_h I_h(t), \\
\dot{S}_v(t) &= \beta_v - C_{hv}I_h(t)S_v(t) - \mu_v S_v(t), \\
\dot{I}_v(t) &= C_{hv}I_h(t)S_v(t) - \mu_v I_v(t).
\end{aligned}
\]

Using the similar proof as that of Theorem 2.1, we can get that the solution
\[
y(t) = (S_h(t), I_h(t), S_v(t), I_v(t))^T
\]
of system (5.1) with any $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T \in C_+$ exists, which is unique, nonnegative, and ultimately bounded on $[0, \infty)$. Obviously, $E^0$ and $E^*$ are the equilibria of system (5.1). In addition, $C_+$ is positively invariant for system (5.1) and $(S_h(t), S_v(t))^T \gg 0$ for $t > 0$. Let us define $y_t = (S_{ht}, I_{ht}, S_{vt}, I_{vt})^T \in C_+$ as $y_t(\theta) = y(t + \theta)$, $\theta \in [-\tau, 0]$ for $t \geq 0$. Then we can obtain the following the global stability result for the disease-free equilibrium $E^0$ of system (2.1).

**Theorem 5.1.** For any $\tau \geq 0$, the disease-free equilibrium $E^0$ is GAS for $R_0 < 1$ and GA for $R_0 = 1$ in $C_+$.

**Proof.** Firstly, it follows from Theorem 3.2 that $E^0$ is LAS for $R_0 < 1$. We thus just require to show that $E^0$ is GA for $R_0 \leq 1$. Let $x_t$ be the solution of system (2.1) through any $\varphi \in C_+$ and $y_t$ be the solution of system (5.1) with any $\psi \in C_+$. By Theorem 2.1, we have that $x_t$ is bounded. Hence, we can obtain that $\omega(\varphi) \subseteq C_+$ is compact, where $\omega(\varphi)$ is the $\omega$-limit set of $\varphi$ for system (2.1). To verify that $E^0$ is GA, we just require to prove $\omega(\varphi) = \{E^0\}$.

Next, we define a functional $V$ on $\Omega_1 = \{\psi \in C_+ : \psi_1(0) > 0, \psi_2(0) > 0\} \subseteq C_+$ as follows
\[
V(\psi) = V_1(\psi(0)) + \int_{-\tau}^0 \frac{\mu_v}{\beta_v} C_{vh} \psi_4(\theta) \psi_1(\theta) d\theta,
\]
where
\[
V_1(\psi(0)) = S_h^0 \left( \frac{\psi_1(0)}{S_h^0} - 1 - \ln \frac{\psi_1(0)}{S_h^0} \right) + \psi_2(0) + \frac{\mu_v \mu_h}{C_{hv} \beta_v} S_v^0 \left( \frac{\psi_3(0)}{S_v^0} - 1 - \ln \frac{\psi_3(0)}{S_v^0} \right) + \frac{\mu_v \mu_h}{C_{hv} \beta_v} \psi_4(0).
\]
Clearly, $V_1(\psi(0)) \leq V(\psi)$ and $V_1$ is positive definite with respect to $E^0$ on $\Omega_1$. We calculate the derivative of $V$ along the solution $y_t$ for $t \geq 1$ as follows
\[
\dot{V}(y_t) = \left( 1 - \frac{S_h^0}{S_h} \right) \dot{S}_h + \dot{I}_h + \frac{\mu_v \mu_h}{C_{hv} \beta_v} \left( 1 - \frac{S_v^0}{S_v} \right) \dot{S}_v + \frac{\mu_v \mu_h}{C_{hv} \beta_v} \dot{I}_v + \frac{\mu_v}{\beta_v} C_{vh} I_v S_h \left( \frac{S_{ht}}{S_h^0} \right) + \frac{\mu_v}{\beta_v} C_{vh} I_v S_h \left( \frac{S_{vt}}{S_v^0} \right)
\]

- $\frac{\mu_v}{\beta_v} C_{vh} I_v (t - \tau) S_h (t - \tau)$
For the endemic equilibrium $E^*$ of system (2.1), we have the following result.

**Theorem 5.2.** If $R_0 > 1$, then the endemic equilibrium $E^*$ is GAS for any $\tau \geq 0$ in $D$.

**Proof.** By Theorem 3.3, we have that $E^*$ is LAS for $R_0 > 1$. Hence, we only need to show that $E^*$ is GA for $R_0 > 1$. Let $x_t$ and $y_t$ be the solution of systems (2.1) and (5.1) with any $\phi \in D$ and any $\psi \in D$, respectively. Note that $D$ is a positive invariant set for system (5.1), and $x(t) \equiv 0$ for $t > 0$. Let $\Omega_2 = \{\psi \in C_+ : \psi(0) \gg 0\}$. Then $\Omega_2 \subseteq D$. To prove that $E^*$ is GA for $R_0 > 1$, we just require to verify $\omega(\phi) = \{E^*\}$.

Next, we define a functional $V$ on $\Omega_2$ as follows

$$V(\psi) = V_2(\psi(0)) + \mu_h I^*_h \int_{-\tau}^{0} \left( \frac{\mu_v C_v h \psi_1(\theta) \psi_1(\theta)}{\beta_v h \mu_h I^*_h} - 1 - \ln \frac{\mu_v C_v h \psi_1(\theta) \psi_1(\theta)}{\beta_v h \mu_h I^*_h} \right) d\theta,$$

where

$$V_2(\psi(0)) = \left( \psi_1(0) - S^*_h - S^*_h \ln \frac{\psi_1(0)}{S^*_h} \right) + \left( \psi_2(0) - I^*_v - I^*_v \ln \frac{\psi_2(0)}{I^*_v} \right) + \frac{\mu_h I^*_h}{\mu_v I^*_v} \left( \psi_3(0) - S^*_v - S^*_v \ln \frac{\psi_3(0)}{S^*_v} \right) + \frac{\mu_h I^*_h}{I^*_v} \left( \psi_4(0) - I^*_v - I^*_v \ln \frac{\psi_4(0)}{I^*_v} \right).$$

It easily follows that $V_2(\psi(0)) \leq V(\psi)$ and $V_2$ is positive definite with respect to $E^*$ on $\Omega_2$. We calculate the derivative of $V$ along the solution $y_t$ for $t \geq \tau + 1$ as follows

$$\dot{V}(y_t) = \left( 1 - \frac{S^*_h}{S^*_h} \right) S_h + \left( 1 - \frac{I^*_v}{I^*_v} \right) I_v + \frac{\mu_h I^*_h}{\mu_v I^*_v} \left( 1 - \frac{S^*_v}{S_v} \right) S_v + \frac{\mu_h I^*_h}{\mu_v I^*_v} \left( 1 - \frac{I^*_v}{I_v} \right) I_v$$

$$+ \frac{\mu_v}{\beta_v} C_v h I_v S_h - \frac{\mu_v}{\beta_v} C_v h I_v (t - \tau) S_h (t - \tau) + \mu_h I^*_h \ln \frac{I_v (t - \tau) S_h (t - \tau)}{I_v S_h}$$

$$= - \mu_h I^*_h S^*_v - \mu_v I^*_v \left( S_v - S^*_v \right)^2 + 4 \mu_h I^*_h S^*_h - \mu_h I^*_h S^*_h - \mu_h I^*_h \frac{I^*_v}{I_v} I_h S_v$$

$$- \mu_h I^*_h \frac{S^*_v}{S_v} - \mu_v I^*_h I_v (t - \tau) S_h (t - \tau) + \mu_h I^*_h \ln \frac{I_v (t - \tau) S_h (t - \tau)}{I_v S_h}.$$
Using the following equality
\[
\ln \frac{I_v(t - \tau)S_h(t - \tau)}{I_vS_h} = \ln \frac{S_h}{S_h} + \ln \frac{I_v(t - \tau)S_h(t - \tau)}{I_v I_v^* S_h} + \ln \frac{S_v}{S_v} + \ln \frac{I_v^* I_v S_v}{I_v I_v^* S_v},
\]
we can obtain that
\[
\dot{V}(y_t) = -\frac{\mu_h (S_h - S_h^*)^2}{S_h} - \frac{\mu_h I_v^* (S_h - S_h^*)^2}{S_v} + \mu_h I_v^* F\left(\frac{S_h}{S_h}\right) + \mu_h I_v^* F\left(\frac{S_v^*}{S_v}\right) + \mu_h I_v^* F\left(\frac{I_v^* I_v S_v}{I_v I_v^* S_v}\right),
\]
where \( F(x) = 1 - x + \ln x, x > 0 \). Since \( F(x) \) is a nonnegative definite function for \( x > 0 \), \( \dot{V}(y_t) \leq 0 \) for \( t \geq \tau + 1 \). Thus, by (5.4), (5.5) and [6, Corollary 3.3], we have that \( E^* \) is uniformly stable for system (5.1). Moreover, we can obtain \( \omega(\psi) \subseteq \Omega_2 \), which yields that \( \dot{V} \) is a Lyapunov functional on \( \{y_t : t \geq \tau + 1\} \subseteq \Omega_2 \). It follows from [6, Corollary 2.1] that \( \dot{V} = 0 \) on \( \omega(\psi) \).

Let \( y_t \) be the solution of system (5.1) with any \( \phi \in \omega(\psi) \). The invariance of \( \omega(\psi) \) hints that \( y_t \in \omega(\psi) \) for \( t \in \mathbb{R} \). It follows from (5.5) and [6, Theorem 3.2] that
\[
S_h(t) = S_h^*, \quad S_v(t) = S_v^*, \quad I_v^* I_v(t) = I_v^* I_v(t)
\]
for \( t \in \mathbb{R} \). Thus, by the fourth equation of system (5.1), we have
\[
I_v^* \dot{I}_v = I_v^* C_{hv} I_v S_v - I_v^* \mu_v I_v = I_v^* C_{hv} I_v S_v^* - \mu_v I_v I_v^* = (C_{hv} I_v^* S_v^* - \mu_v I_v^*) I_v = 0,
\]
where \( C_{hv} I_v^* S_v^* = \mu_v I_v^* \) is used. Consequently, \( I_v^* \) is a constant function and then \( I_v^* \) is also a constant function. In consequence, \( \eta(t) = 0 \) is an endemic equilibrium of system (5.1) for \( R_0 > 1 \). Considering that the endemic equilibrium is unique, we thus have \( \eta(t) = E^* \). Hence, it holds that \( \omega(\psi) = E^* \), which means that \( W^*(E^*) = D \), where \( W^*(E^*) \) is the stable set of \( E^* \) for system (5.1). From Theorem 4.2, we have \( \omega(\psi) \cap W^*(E^*) \neq \emptyset \). Thus, [22, Theorem 4.1] indicates that \( \omega(\psi) = \{ E^* \} \).

\[ \square \]

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