Large Isolating Cuts Shrink the Multiway Cut

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Abstract. We propose a preprocessing algorithm for the multiway cut problem that establishes its polynomial kernelizability when the difference between the parameter $k$ and the size of the smallest isolating cut is at most $\log(k)$. To the best of our knowledge, this is the first progress towards kernelization of the multiway cut problem. We pose two open questions that, if answered affirmatively, would imply, combined with the proposed result, unconditional polynomial kernelizability of the multiway cut problem.

1 Introduction

1.1. Overview of the proposed results. Given a pair $(G, T)$ where $G$ is a graph and $T$ is a specified set of vertices, called terminals, a (vertex) Multiway Cut (mwc) of $(G, T)$ is a set of non-terminal vertices whose removal from $G$ separates all the terminals. The mwc problem asks to compute the smallest mwc of $G$. It is NP-hard for $|T| \geq 3$.

In this paper we concentrate on the parameterized version $(G, T, k)$ of the mwc problem where we are given a parameter $k$ and asked whether there is an mwc of $(G, T)$ of size at most $k$. The goal we work towards is understanding the kernelizability of the mwc problem. In other words, we want to understand, whether there is a polynomial time algorithm that transforms $(G, T, k)$ into an equivalent instance $(G', T', k')$ (equivalent in the sense that the former is the ‘YES’ instance iff the latter is) such that $|V(G')|$ is upper-bounded by a polynomial of $k'$ and $k'$ itself is upper bounded by a polynomial of $k$. Informally speaking we want to shrink the instance of the mwc problem to a size polynomially dependent on the parameter.

The kernelizability of the mwc is considered by the parameterized complexity community as an interesting and challenging question. In this paper we propose a partial result and pose two open questions that, if resolved affirmatively, will imply, together with this result, that the mwc problem is kernelizable. An informal overview is given below.

Let $(G, T, k)$ be an instance of the mwc problem. An isolating cut [5] is a set of non-terminal vertices separating a terminal $t$ from the rest of terminals. Let $r$ be the smallest size of an isolating cut. Clearly we can assume $r \leq k$ otherwise, $(G, T, k)$ is a 'NO' instance. In this paper we propose an algorithm transforming the initial instance into an equivalent one whose size is $O(2^{k-r}r^2k^2)$. The runtime of this algorithm is $O(A(n) + 2^{k-r}n^3r^2k^4)$ where $A(n)$ is the runtime of the
constant ratio approximation algorithm for the vertex MWC problem proposed in \cite{6}. Thus we demonstrate that for every fixed constant $c$ the subclass of MWC problem consisting of instances with $k-r \leq c \times \log k$ is polynomially kernelizable. To the best of our knowledge, this is the first progress towards kernelization of the MWC problem. Two more merits of the proposed results are that it might be a building block in an unconditional kernelization of the MWC problem and that it gives a new insight into the structure of important separators \cite{8}. To justify these merits, we provide below a more detailed overview of the proposed result.

The main ingredient of the proposed algorithm is computing for each $t \in T$ the union $U_t$ of all important isolating cuts of $t$ of size at most $k$. An almost immediate consequence of Lemma 3.6. of \cite{8} shows that the union of all $U_t$ contains a solution of $(G,T,k)$ if such exists. Therefore, 'contracting' the rest of non-terminal vertices results in an instance equivalent to $(G,T,k)$. Prior to computing the sets $U_t$ we ensure that the size of $|T|$ is at most $2k(k+1)$. This is done in Section 3 by running the approximation algorithm of \cite{6} and processing the output in the flavour of a simple quadratic kernelization algorithm for the Vertex Cover problem (i.e. noticing that the vertices of the given MWC adjacent to a large number of terminal components must be present in any solution and, after removal of these vertices and the already separated terminals, the number of remaining terminals is small).

But what is the size of $U_t$ and what is the time needed for its computation? To understand this, we study (in Section 2) important $X-Y$ separators of $G$ \cite{8} where $X$ and $Y$ are two arbitrary subsets of vertices. As a result, we obtain a combinatorial theorem saying that if $r$ is the smallest size of an $X-Y$ separator then for an arbitrary $x$ the size of the union of all $X-Y$ important separators of size at most $r+x$ is at most $2^x+1 \times r$ and these vertices can be computed in time $O(n^32^x r^2 (r+x)^2)$. The exponential part of the runtime follows from the need to enumerate so-called principal important separators whose union includes all the needed vertices. We argue that the principal important separators constitute a generally small subset of the whole set of important separators and pose the first open question asking whether the number of principal separators can be bounded by a polynomial of $n$. The affirmative answer to this question implies the polynomial runtime of the algorithm proposed in this paper. In this case the algorithm can be a first step of a kernelization method of the MWC. However, it cannot be the only step. We demonstrate the upper bound on the number of vertices is tight and hence generally cannot polynomially depend on $k$. Therefore, a natural question is whether the output of this algorithm can be further processed to obtain an unconditional kernelization. We pose this as our second open question.

1.2. Related work. There are many publications related to the topics considered in the paper. We overview only those that are of a direct relevance for the proposed results.

\footnote{The results are obtained without any regard to the MWC problem, hence they might be of an independent interest.}
The fixed-parameter tractability of the mwc problem has been established in [8] and the runtime has been improved to $O^*(4^k)$ in [4]. A parameterization of the mwc problem above a guaranteed value has been recently proposed in [9], where we show that the problem is in XP under this parameterization leaving open the fixed-parameter tractability status.

The notion of important separator has been introduced in [8]. As noticed in [7], the recent algorithms for a number of challenging graph separation problems, including the one of [4], are based on enumeration of important separators. Further on, [7] proves an upper bound $4^k$ on the number of important separators of size at most $k$ and notices that the algorithm of [4] in fact implicitly establishes this upper bound. An alternative upper bound, suitable for the case where the smallest important separator is large, is established in [9].

Constant ratio approximation algorithms for the mwc problem have been first proposed in [5] for the edge version and in [6] for the vertex version. The research on kernelization has been given its current shape by the landmark paper [1], which allowed to classify fixed-parameter tractable problems into kernelizable ones and those that are probably not. Among the many known kernelizability and non kernelizability results, let us mention the kernelization methods for multicut for trees [3] and Feedback Vertex Set [10] and non-kernelizability proof for the Disjoint Cycles problem [2]. Although far from being analogous to the mwc problem, all these problems are related to the flow maximization/cut minimization tasks and hence might be a source of ideas useful for the final settling of the kernelizability of mwc problem.

2 Bounding the union of important separators

Let $X$ and $Y$ be two disjoint sets of vertices of the given graph $G$. A set $K \subseteq V(G) \setminus (X \cup Y)$ is an $X - Y$ separator if in $G \setminus K$ there is no path from $X$ to $Y$. Let $A, B$ be two disjoint subsets of $V(G)$. We denote by $NR(G, A, B)$ the set of vertices that are not reachable from $A$ in $G \setminus B$. Let $K_1$ and $K_2$ be two $X - Y$ separators. We say that $K_1 \prec_* K_2$ if $NR(G, Y, K_1) \subset NR(G, Y, K_2)$.

A minimal $X - Y$ separator $K$ is called important if there is no $X - Y$ separator $K'$ such that $K \prec_* K'$ and $|K| \geq |K'|$. This notion was first introduced in [8] in a slightly different although equivalent way (see Proposition 3 of [9]). Let $r$ be the size of a smallest important $X - Y$ separator and let $S$ be an arbitrary important separator. We call $|S| - r$ the excess of $S$. Then the following theorem holds.

**Theorem 1.** Let $U$ be the union of important $X - Y$ separators of excess at most $x$. Then $|U| \leq 2^{x+1}r$. Moreover, $U$ can be computed in time $O(n^3 2^{2x} r^2 (r + x)^2)$.

In this section we prove Theorem 1 and show the tightness of the upper bound of $|U|$. The proof of Theorem 1 in divided into two stages. On the first stage we introduce a partially ordered family of subsets of the given set satisfying a number of certain properties. We call such family of sets an IS-family. We prove Theorem 1 in terms of the IS family. Then we show that the family of
all important separators with the ≺∗ relation is in fact an IS family from where Theorem 1 immediately follows.

The advantage of such 'axiomatic' way of proof is the possibility to clearly specify the properties of the family of important separators (viewed as a partially ordered family of sets) that imply the above upper bound. An additional potential advantage is that some deep algebraic techniques might become applicable for further investigation of the kernelization of multiway cut.

2.1 From Important Separators to Partially Ordered Families of Sets

Let \( V \) be a finite set. Let \((F, ≺)\) be a pair where \( F \) is a family of subsets of \( V \) and ≺ is an order relation on the elements of \( F \). Let \( S \in F \) and \( v \in V \). We say that \( S \) covers \( v \) if there is \( S' \in F \) such that \( S' ≺ S \) and \( v \in S' \setminus S \). We define \( \text{Pred}(S) \) to be the set of all \( S' \) such that \( S' ≺ S \) and there is no \( S'' \in F \) such that \( S' ≺ S'' ≺ S \). Symmetrically, we define \( \text{Succ}(S) \) to be the set of all \( S' \in F \) such that \( S ≺ S' \) and there is no \( S'' \in F \) such that \( S ≺ S'' ≺ S' \). We define the visible set of \( S \) denoted by \( \text{Vis}(S) \) to be the set of all \( v \in V \) satisfying the following two conditions:

- there is \( S' \in \text{Pred}(S) \) such that \( v \in S' \);
- \( v \) is not covered by any element of \( \text{Pred}(S) \).

\((F, ≺)\) is called an IS-family if the following conditions are true.

- **Smallest element (SE) condition.** There is a unique element of \( F \) denoted by \( \text{sm}(F) \) such that for any other \( S \in F \), \( \text{sm}(F) ≺ S \).
- **Strict monotonicity (SM) condition.** Let \( S_1, S_2 \in F \). If \( S_1 ≺ S_2 \) then \( |S_1| < |S_2| \).
- **Single witness (SW) condition.** Let \( S \in F \) and let \( v \in S \). Let \( S' \) be a minimal element such that \( S ≺ S' \) and \( v \in S \setminus S' \). We call \( S' \) a witness of \( v \) w.r.t. \( S \). The condition requires that there is at most one witness of \( v \) w.r.t. \( S \).
- **Transitive Elimination (TE) condition** Let \( S_1 ≺ S_2 ≺ S_3 \) be three elements of \( F \) and let \( v \in S_1 \setminus S_2 \). Then \( v \in S_1 \setminus S_3 \).
- **Large visible set (LVS) condition** Let \( S \in F \) and let \( S' \in \text{Pred}(S) \). Then \( |S'| ≤ |\text{Vis}(S)| \). For the subsequent proofs we will use the extended LVS condition stating that for each \( S'' ≺ S \), \( |S''| ≤ |\text{Vis}(S)| \), which immediately follows from the combination of LVS and SM conditions.
- **Distinct visible set (DVS) condition** For each \( S \in F \) such that \( S \neq \text{sm}(F) \). Then \( \text{Vis}(S) \not\subset S \).
- **Efficient Computability (EC) condition** Let \( n = |V| \). In \( O(n^3) \) we can compute \( \text{sm}(F) \) as well as the witness of \( v \) w.r.t. \( S \) for the given \( S \in F \) and \( v \in S \) (or return 'NO' in case such witness does not exist). The relation \( S_1 ≺ S_2 \) can be tested in \( O(|S_1|) \).
In the rest of this subsection we assume that $(F, \prec)$ is an IS-family. Our reasoning consists of three stages. On the first stage, we prove 3 propositions stating simple properties of an IS family. On the second stage we prove Theorem 2 our main counting result. The main body of the proof is provided in the 3 preceding lemmas. On the last stage we prove an analogue of Theorem 1 for IS families: Corollary 1 proves the upper bound on the size of the union of the respective sets and Theorem 3 establishes an algorithm for computing of these sets.

For $S \in F$ let us denote $S \setminus \bigcup_{S' \prec S} S'$ by $\hat{S}$.

**Proposition 1.** $\hat{S} = S \setminus Vis(S)$.

**Proof.** It is clear from the definition that $\hat{S} \subseteq S \setminus Vis(S)$. Conversely, consider $v \in (S \setminus Vis(S)) \setminus \hat{S}$. What can we say about such $v$? First, that $v \in S$. Then, since $v \in \bigcup_{S' \prec S} S' \setminus Vis(S)$, there may be two possibilities. According to one of them, $v \in S' \prec S$ such that $S' \notin Pred(S)$ and $v$ does not belong to any $S'' \in Pred(S)$. It follows that there is $S'' \in Pred(S)$ such that $S' \prec S''$ and $v \in S' \setminus S''$. Since $S'' \prec v$, $v \notin S$ by the TE condition in contradiction to our assumption. The other possibility may be that $v \in S' \in Pred(S)$ and $v$ is covered by another $S'' \in Pred(S)$. Then analogous reasoning applies. By definition of a covered vertex, there is $S^* \prec S''$ such that $v \in S^* \setminus S''$ and again $v \notin S$ by the TE condition, yielding an analogous contradiction.

**Proposition 2.** Let $S_1, S_2, v$ be such that $S_1 \prec S_2$ and $v \in S_1 \setminus S_2$. Let $S^*$ be the witness of $v$ w.r.t. $S_1$. Then $S^* \preceq S_2$.

**Proof.** Let $S''$ be a minimal element of $F$ such that $S_1 \prec S'' \preceq S_2$ and $v \in S_1 \setminus S''$. Then $S''$ is a witness of $v$ w.r.t. $S_1$. By the SW condition, $S'' = S^*$.

**Proposition 3.** Let $S \in F$ and let $v \in Vis(S) \setminus S$. Then there is $S^* \prec S$ such that $v \in \hat{S}^*$ and $S$ is the witness of $v$ w.r.t. $S^*$.

**Proof.** Let $S^*$ be a minimal element of $F$ preceding $S$ such that $v \in S^*$. Then $v \in \hat{S}^*$. Indeed, otherwise, there is $S''$ such that $v \in S' \prec S'' \prec S$ in contradiction to the choice of $S^*$. Assume by contradiction that $S$ is not the witness of $v$ w.r.t. $S^*$ and let $S''$ be this witness. According to Proposition 2 $S'' \preceq S$. Let $S_2 \in Pred(S)$ be such that $S'' \preceq S_2$. Clearly $S^* \preceq S_2$. If $S'' = S_2$, then $v \in S^* \setminus S_2$ by definition of $S''$. Otherwise, $v \in S^* \setminus S_2$ by the TE condition. It follows that $S_2$ covers $v$. Consequently, $v \notin Vis(S)$, a contradiction proving that $S$ is indeed the witness of $v$ w.r.t. $S^*$.

For $S \in F$, let's call $|S| - |sm(F)|$, the excess of $S$ and denote it $ex(S)$.

**Lemma 1.** Let $S \in F$ such that $S \neq sm(F)$. For $v \in Vis(S) \setminus S$, let $S(v)$ be such that $v \in \hat{S(v)}$ and $S$ is the witness of $v$ w.r.t. $S(v)$ (the existence of such $S(v)$ follows from Proposition 3). Then $|\hat{S}| \leq \sum_{v \in Vis(S) \setminus S} 2^{ex(S) - ex(S(v))}$. 
Proof. If $Vis(S) = S$ then by Proposition $\text{h}(S) = \emptyset$ and we are done. Otherwise, the DVS condition allows us to fix a $v^* \in Vis(S) \setminus S$. Let us define a function $f$ on $V$ as follows: $f(v^*) = 2^{ex(S) - ex(S(v^*))}$ and for $w \neq v^*$, $f(w) = 1$. For $S \subseteq V$, the function naturally extends to $f(S) = \sum_{v \in S} f(v)$.

Claim. $|\text{h}(S)| \leq f(Vis(S) \setminus S)$

Observe that $f(Vis(S)) = |Vis(S) \setminus \{v^*\}| + f(v^*) = |Vis(S)| + f(v^*) - 1$. By the extended LVS condition, the rightmost part of the above equality does not increase if we replace $Vis(S)$ by $S(v)$, i.e. $f(Vis(S)) \geq |S(v)| + f(v^*) - 1$. Since $ex(S) - ex(S(v)) \geq 1$, by the SM condition, $f(v^*) \geq ex(S) - ex(S(v)) + 1$. That is, $f(Vis(S)) \geq |S(v)| + ex(S) - ex(S(v)) = |S|$. Furthermore $f(Vis(S)) = f(Vis(S) \setminus S) + f(Vis(S) \cap S) = f(Vis(S) \setminus S) + (Vis(S) \cap S)$. On the other hand, $|S| = |S \setminus Vis(S)| + |Vis(S) \cap S| = |\text{h}(S)| + |Vis(S) \cap S|$, the last equality follows from Proposition $\text{h}$ Thus the desired claim follows by removal $|Vis(S) \cap S|$ from the both sides of the inequality $f(Vis(S)) \geq S$. □

Observe that due to the SM condition, for each $v \in Vis(S) \setminus S$, $2^{ex(S(v)) - ex(S(v))} \geq f(v)$, hence $f(Vis(S) \setminus S) \leq \sum_{v \in Vis(S) \setminus S} 2^{ex(S(v)) - ex(S(v))}$. Therefore the lemma follows from the above claim. ■

For $x \geq 0$, let $E_x$ be the subset of $F$ consisting of all the elements of excess at most $x$. Let $S \subseteq E_x$. The $x$-hat of $S$ denoted by $\text{hat}_x(S)$ is a subset of $\text{h}(S)$ consisting of all elements $v$ such that there is no $S' \subseteq E_x$ such that $S \prec S'$ and $v \in S \setminus S'$.

Lemma 2. For any $x \geq 0$

\[
\sum_{S \subseteq E_x} 2^{x - ex(S)} |\text{hat}_x(S) \setminus \text{hat}_{x+1}(S)| \geq \sum_{S' \subseteq E_{x+1} \setminus E_x} |\text{h}(S')|.
\]

Proof. Denote the elements of $E_x$ by $S_1, \ldots, S_m$. Denote $\{(v, i) | 1 \leq i \leq m, v \in \text{hat}_x(S_i) \setminus \text{hat}_{x+1}(S_i)\}$ by OS. For each $(v, i) \in OS$, let $aw(v, i) = 2^{x - ex(S_i)} + 1$. For $OS' \subseteq OS$, let $aw((OS')) = \sum_{(v, i) \in OS'} aw(v, i)$. It is not hard to see that the left part of the desired inequality is $aw(OS)$. Indeed, for the given $i$, if we sum up $aw(v, i)$ for all $(v, i) \in OS$ then the total amount will be exactly $2^{x - ex(S_i)} + 1 |\text{hat}_x(S)|$.

Consider $(v, i) \in OS$. Then, since $v \notin \text{hat}_{x+1}(S_i)$, there is $S' \subseteq E_{x+1} \setminus E_x$ such that $S_i \prec S'$ and $v \in S_i \setminus S'$. We claim that $S'$ is in fact the witness of $v$ w.r.t. $S_i$. Indeed, otherwise, according to Proposition $\text{h}$ $S'$ succeeds the witness of $v$ w.r.t. $v$ hence, by the SM condition, the size of the latter is at most $x$. However, this contradicts $v \in \text{hat}_x(S_i)$. By the SW condition, the above $S'$ is unique for $(v, i)$. So, we can say that $(v, i)$ is witnessed by $S'$.

Denote the elements of $E_{x+1} \setminus E_x$ by $S'_1, \ldots, S'_q$. Partition $OS$ into $OS_1, \ldots, OS_q$ such that the elements of $OS_q$ are witnessed by $S'_q$. To confirm the lemma, it remains to prove that, for the given $i$, $aw(OS_i) \geq |\text{h}(S_i)|$.

Let $v \in Vis(S'_i)$. According to Proposition $\text{h}$ there is $S' \prec S'_i$ such that $v \in \text{h}(S')$ and $S'_i$ is the witness of $v$ w.r.t. $S'$. By the SM condition $ex(S') \leq x$, that is $S' \subseteq E_x$. Observe that in fact $v \in \text{hat}_x(S') \setminus \text{hat}_{x+1}(S')$. Indeed, otherwise there is an element $S''$ of $E_x$ such that $S' \prec S''$ and $v \in S' \setminus S''$. But then $S \prec S''$ by Proposition $\text{h}$ in contradiction to the SM condition. Let
rem 2, the rightmost item of the above inequality is clearly upperbounded by $2\varepsilon$ by definition, the left set is clearly a superset of the right one, so let $\varepsilon$ vertex of the left set. If $\varepsilon > 0$ be the excess of $S$. Otherwise, let $\varepsilon$ be the excess of $S$. \[\text{Corollary 1.} \]

For $x \geq 0$, denote $\sum_{S \in E_x} 2^{\varepsilon - ex(S)} |\hat{x}(S)|$ by $M(x)$. Then the following statement takes place.

Lemma 3. For each $x \geq 0$, $M(x + 1) \leq 2M(x)$.

Proof. First of all, observe that for each $S \in E_{x+1} \setminus E_x$, $\hat{x}(S) = \hat{x}(S)$ just because, by the SM condition, there is no $S' \in E_{x+1}$ such that $S \approx S'$. Furthermore, by definition, the excess of $S$ is $x + 1$. Therefore $|\hat{x}(S)| = 2^{\varepsilon - ex(S)} |\hat{x}_{x+1}(S)|$. That is, we can rewrite the inequality of Lemma 2 as

\[\sum_{S \in E_x} 2^{\varepsilon - ex(S)} |\hat{x}(S)\setminus \hat{x}_{x+1}(S)| \geq \sum_{S \in E_{x+1} \setminus E_x} 2^{\varepsilon - ex(S)} |\hat{x}_{x+1}(S')|\]

Furthermore, observe that for each $S \in E_x$, $\hat{x}_{x+1}(S) \subseteq \hat{x}(S)$, therefore $\hat{x}_{x+1}(S) = \hat{x}_{x+1}(S) \cap \hat{x}(S)$. Then we can safely add $\sum_{S \in E_x} 2^{\varepsilon - ex(S)} |\hat{x}(S)\setminus \hat{x}_{x+1}(S)|$ to the left part of the inequality of the previous paragraph and $\sum_{S \in E_x} 2^{\varepsilon - ex(S)} |\hat{x}_{x+1}(S)|$ to the right part of this inequality. Then after noticing that for each $S \in E_x$, $|\hat{x}(S)\setminus \hat{x}_{x+1}(S)| + |\hat{x}(S)\cap \hat{x}_{x+1}(S)| = |\hat{x}(S)|$ and that the right part in fact explores $|\hat{x}_{x+1}(S')|$ for all elements $S' \in E_{x+1}$, the resulting inequality is transformed into:

\[\sum_{S \in E_x} 2^{\varepsilon - ex(S)} |\hat{x}(S)| \geq \sum_{S \in E_{x+1}} 2^{\varepsilon - ex(S') + 1} |\hat{x}_{x+1}(S')|\]

It remains to notice that the left part of this inequality is $2M(x)$ and the right part is $M(x + 1)$.

Now we are ready to state the main counting result.

Theorem 2. For each $x \geq 0$, $M(x) \leq 2^x |\text{sm}(F)|$.

Proof. Applying inductively Lemma 3 it is easy to see that $M(x) \leq 2^x M(0)$. By definition, $M(0) = \sum_{S \in E_0} 2^{\varepsilon - ex(S)} |\hat{0}(S)|$. Since the only element of $E_0$ is $\text{sm}(F)$ whose excess is 0 and $\hat{0}(\text{sm}(F)) = \text{sm}(F)$, the theorem follows.

The following corollary is the first statement of Theorem 1 in terms of an IS family

Corollary 1. $|\bigcup_{S \in E_x} S| \leq 2^{x+1} |\text{sm}(F)|$.

Proof. Observe that $\bigcup_{S \in E_x} S = \text{sm}(F) \bigcup \bigcup_{j=1}^x \bigcup_{S' \in E_{x-j+1}} \hat{0}(S')$. Indeed, by definition, the left set is clearly a superset of the right one, so let $v$ be a vertex of the left set. If $v \in \text{sm}(F)$ then the containment in the right set is clear. Otherwise, let $S^* \in E_x$ be a minimal set containing $v$ and let $j > 0$ be the excess of $S^*$. Then, by definition of sets $E_x$, $S^* \in E_j \setminus E_{j-1}$. From the minimality of $S^*$ subject to the containment of $v$, it follows that $v \in \hat{0}(S^*)$. Furthermore, by the SM condition, there is no $S'' \in E_j$ such that $S^* \approx S''$. This implies that $v \in \hat{0}(S^*)$, confirming the observation.

It follows from this equality that $|\bigcup_{S \in E_x} S|$ is upper-bounded by $|\text{sm}(F)| + \sum_{j=1}^x \sum_{S' \in E_{x-j+1}} |\hat{0}(S')| \leq M_0 + \sum_{j=1}^x M_i \leq \sum_{j=0}^x M_i$. According to Theorem 2 the rightmost item of the above inequality is clearly upperbounded by $2^{x+1} |\text{sm}(F)|$, hence the corollary follows. ■
To prove the second statement of Theorem 1, we need to compute $\bigcup_{S \in E_x} S$. We obtain the required algorithm in four simple steps. First we introduce the notion of principal sets of $F$, then we show that the union of principal sets of excess at most $x$ in fact includes all the vertices of $\bigcup_{S \in E_x} S$. Furthermore, we show that the number of principal sets can be upper bounded by $2^{x+1}|sm(F)|$. Finally, we show that subject to EC condition, these principal sets can be computed in time polynomial in their bound and in $n = |V|$. (Recall that $V$ is the universe of for the sets of $F$).

We say that a set $S \in F$ is principal if $hat(S) \neq \emptyset$. Denote by $Pr_x$ the family of all principal sets of excess at most $x$. By definition, $\bigcup_{S \in Pr_x} \subseteq \bigcup_{S \in E_x}$. For the other direction, let $v \in \bigcup_{S \in Pr_x}$. Then, arguing as in the proof of Corollary 1, we observe the existence of $S^*$ of excess at most $x$ such that $v \in hat(S^*)$. Clearly $S^* \in Pr_x$. Thus we have established the following proposition.

**Proposition 4.** $\bigcup_{S \in Pr_x} S = \bigcup_{S \in E_x} S$

**Proposition 5.** For each $x \geq 0$, $|Pr_x| \leq 2^{x+1}|sm(F)|$.

**Proof.** By definition, the number of elements of $|Pr_x|$ is upper-bounded by the sum of the sizes of their hats, which in turn, is bounded by the sum of sizes of hats of all elements of $E_x$. Taking into account that for each $1 \leq i \leq x$ and for each $S \in E_i \setminus E_{i-1}$, $hat(S) = hat_i(S)$ (argue as in the proof of Corollary 1), our upper bound can be represented as $|sm(F)| + \sum_{i \geq 1} \sum_{S \in E_i \setminus E_{i-1}} |hat_i(S)|$. Now, apply the second paragraph of the proof of Corollary 1. ■

**Theorem 3.** $Pr_x$ can be computed in time $O(n^3 2^{2x} r^2 (r+x)^2)$ where $r = |sm(F)|$.

**Proof sketch.** The algorithm works iteratively. First it computes $Pr_0$. For each $i > 0$, it computes $Pr_i$ based on $Pr_{i-1}$. Since $Pr_0 = \{sm(F)\}$, for $i = 0$, the result directly follows from the EC condition. Now consider computing of $Pr_i$ for $i > 0$ assuming that $Pr_{i-1}$ have been computed.

The algorithm explores all the elements of $Pr_{i-1}$ and for each such element $S$ and for each $v \in S$, applies the witness computation algorithm of the EC condition. If the witness $S'$ of $S$ has been returned, $S'$ joins $Pr_i$ if $ex(S') = i$, $S'$ has not been already generated and the union of elements of $Pr_i$ preceding $S'$ is not a superset of $S$. In the rest of the proof, postponed to the appendix, we prove correctness and the runtime of this algorithm. ■

### 2.2 Back to Important Separators.

**Lemma 4.** The family of all important $X - Y$ separators of graph $G$ partially ordered by the $\prec$ relation is an IS-family.

**Proof sketch.** The SE condition is established by Lemma 3.3. of [8]. The SM condition immediately follows from the definition of an important separator. For the SW condition, let $K$ be an important $X - Y$ separator and let $v \in K$. Assume that a witness of $v$ w.r.t. $K$ exists. Replace $NR(G, Y, K)$ by a single vertex
x and split v into n + 1 copies. Let G* be the resulting graph. We prove that there is a bijection between the witnesses of v w.r.t. K and smallest important x – Y separators of G* and apply to G* the SE condition. For the TE condition, we observe (e.g. Proposition 1 of [9]), that if K1≺∗K2 then K1 \ K2 ⊆ NR(G, Y, K2). Thus if K2≺∗K3, K1 \ K2 ⊆ NR(G, Y, K2) ⊆ NR(G, Y, K3), the last inclusion is obtained by definition of the ≺* relation. Thus, no vertex of K1 \ K2 can belong to K3. For the visible set conditions, we first prove that if K is an important X – Y separator different from the smallest one then for each K′ ∈ Pred(K), K* = Vis(K) \ NR(G, Y, K′) is also an X – Y separator such that K′ ≺* K*. The LVS and DVS conditions will immediately follow from this claim combined with the definition of an important separator. The O(n^3) algorithm for computing sm(F), as required by the EC condition follows from Lemma 1 in [9]. As shown in the proof of the SW condition, computing of a witness is essentially equivalent to computing of an important separator. Finally the fast testing of K1≺∗K2 is easy to establish by maintaining an important separator in an appropriate data structure.

Proof of Theorem 1 The theorem immediately follows from combination of Corollary [1], Theorem [3] and Lemma [4].

2.3 Lower bounds and possibilities for further improvement

We start with showing that the obtained upper bound on the number of vertices involved in important separators of size at most x is quite tight.

Theorem 4. For each x and r there is a graph H with two specified terminals s and t such that the size of the smallest s – t separator is r and the size of the union of all important separators of excess at most x is 2^{x+1}r – r.

Proof. Take r complete rooted binary trees of height x with 2^x leaves (of course, replace arcs by undirected edges). Add two new vertices s and t. Connect s to the roots of the trees and t to all the leaves. This is the resulting graph H for the given x and r. It is not hard to see that any minimal s – t separator of this graph is an important one. It only remains to show that each non-terminal vertex participates in a s – t separator of excess at most x. In fact, we can show that any vertex v whose depth in the respective binary tree is i participates in a separator of excess i. We compute such separator by obtaining a sequence S1, . . . , Si of separators, where S1 is the desired separator. Si is just the set of neighbours of s. To obtain Sj+1 from Sj, we specify the unique u ∈ Sj such that u is the ancestor of v (the uniqueness easily follows by induction) and replace it by its children. The correctness of this construction can be easily established by induction on the constructed sequence of separators, we omit the tedious details.

In the previous subsection we introduced the notion of a principal set of an IS-family. The corresponding notion of a principal important separator K means that K \ ⋃_{K′ ≺∗K} K′ = ∅. Proposition 5 along with Lemma 4 implies that the number of principal important X – Y separators of excess x is at most 2^{x+1}r.
where \( r \) is the size of the smallest important \( X - Y \) separator and the class of graphs considered in Theorem 4 shows that this bound is tight. On the other hand, the number of principal important separators in this class of graphs is linear in the overall number \( n \) of vertices. This leads us to the following question

**Open Question 1** Is the number of principal important \( X - Y \) separators of the given graph \( G \) bounded by a polynomial of \( |V(G)| \)?

First of all observe that this question is reasonable because the number of principal separators is generally much smaller than the overall number of important separators. Indeed, in the class of instances considered in Theorem 4 the overall number of important separators is exponential in \( n \) (consider the important separators including leaves of the binary trees).

To see the significance of this open question, suppose that the answer is yes. Then the algorithm claimed in Theorem 1 runs in a polynomial time. Indeed, its exponential runtime is caused by the fact that the algorithm explores all pairs of principal important separators, so, replacing the upper bound has an immediate effect on the runtime. Such poly-time algorithm would mean that it is possible to test in a polynomial time whether the given vertex belongs to an important separator, which is itself quite an interesting achievement. Moreover, the whole preprocessing algorithm for the MWC problem proposed in this paper will have a polynomial time. This means that the output of this algorithm can be used for the further preprocessing, potentially making easier the unconditional kernelization of the MWC problem.

3 Preprocessing of multiway cut

Let \((G, T)\) be an instance of the MWC problem. An isolating cut of \( t \in T \) is a \( t - T \setminus \{t\} \) separator. If such separator is important, we call it important isolating cut of \( t \).

We start from a proposition that allows us to harness the machinery of important separators for the preprocessing of the MWC problem. The proposition is easily established by iterative application the argument of Lemma 3.6 of [8].

**Lemma 5.** Let \((G, T)\) be an instance of the MWC problem. Then there is a smallest MWC \( S \) of \((G, T)\) such that each \( v \in S \) belongs to an important isolating cut of some \( t \in T \).

With Lemma 5 in mind, we can use the algorithm claimed in Theorem 1 for the preprocessing. In particular, for each \( t \in T \), let \( r_t \) be the size of the smallest isolating cut. Compute the set of all vertices participating in the important isolating cuts of \( t \). Let \( V^* \) be the set of all the computed vertices together with the terminals. Let \( G^* \) be the graph obtained from \( G[V^*] \) by making adjacent all non-adjacent \( u, v \) such that \( G \) has a \( u - v \) path with all intermediate vertices lying outside \( V^* \). It is not hard to infer from Lemma 5 that the size of the
optimal solution of \((G^*, T)\) is the same as of \((G, T)\). According to Theorem 1, the number of vertices of \(G^*\) is at most \(|T|(2^{k-r}r + 1)\) where \(r = \min_t \in T_r\) and 1 is added on the account of terminals. This bound is not good in the sense that \(|T|\) may be not bounded by \(k\) at all. Therefore prior to computing the union of important separators, we reduce the number of terminals. This is possible due to the following theorem.

**Theorem 5.** There is a polynomial-time algorithm that transforms the instance \((G, T, k)\) of the mwc problem into an equivalent instance \((G', T', k')\) such that \(k' \leq k\) and \(|T'| \leq 2k(k'+1)\). Then runtime of this algorithm is the same as the runtime of the fixed-ratio approximation algorithm for the mwc problem \([6]\). \(\square\)

**Proof.** We start from observation that if \(u\) is a non-terminal vertex such that there are \(k+2\) terminals connected to \(u\) by paths intersecting only at \(u\) then \(u\) participates in any mwc of \((G, T)\) of size at most \(k\). Indeed, removal of a set of at most \(k\) vertices not containing \(u\) would leave at least 2 of these paths undestroyed and hence the corresponding terminals would be connected. An immediate consequence of this observation is that if \(S\) is a mwc of \(G\) and there is \(v \in S\) adjacent to at least \(k+2\) components of \(G \setminus S\) containing terminals then this vertex participates in any mwc of \(G\) of size at most \(k\).

Having the above in mind, we apply the ratio 2 approximation algorithm for the mwc problem proposed in \([6]\). \(^3\) If the resulting mwc is of size greater than \(2k\), the algorithm simply returns "NO". Otherwise, let \(S\) be the resulting mwc. If \(|T| > |S|(k+1)\) then, taking into account that each component is adjacent to at least one vertex of \(S\), it follows from the pigeonhole principle that at least one vertex of \(S\) is adjacent to at least \(k+2\) components of \(G \setminus S\) containing terminals. Remove \(v\) and remove isolated components of \(G \setminus \{v\}\) (i.e. those that contain at most one terminal), decrease the parameter by 1 and recursively apply the same operation to the new data. Eventually, one of three possible situations occur. First, after removal of \(k\) or less vertices, the resulting graph has no terminals. In this case we have just found the desired mwc of \((G, T)\) in a polynomial time. Second, after removal of \(k\) vertices, there are still terminals, not separated by the removed vertices. In this case, again in a polynomial time, we have found that \((G, T)\) has no mwc of size at most \(k\). Finally, it may happen that after removal of some \(S' \subseteq S\) of size at most \(k\), the number of terminals in the remaining graph is at most \(|S' \setminus S'|(k - |S'| + 1)\). Then the resulting graph is returned as the output of the preprocessing. \(\blacksquare\)

Thus, Theorem 5 together with Theorem 1 and Lemma 5 lead to the following result.

**Corollary 2.** There is an algorithm that for an instance \((G, T, k)\) of the mwc problem finds an equivalent instance of \(O(k^2r2^{k-r})\) vertices in time \(O(A(n) + 2^{k-r}n^{3/2}k^4)\) where \(r\) is the smallest isolating cut and \(A(n)\) is the time complexity

\(^2\) This algorithm is based on solving a linear program.

\(^3\) In fact, the approximation ratio of this algorithm is \(2 - 2/|T|\), but ratio 2 is sufficient for our purpose.
of the approximation algorithm proposed in [6]. In particular, if $k - r = c \times \log(k)$ for any fixed $c$ then the MWC problem is polynomially kernelizable.

The output of the above algorithm is much richer than just another instance of the MWC problem. Indeed, for each terminal, the algorithm in fact computes all principal important isolating cuts. This leads to the follows interesting question.

**Open Question 2** Is there an algorithm that gets the above output as input and, in time polynomial in $n$ and the number of the principal isolating cuts, produces an equivalent instance of the MWC problem of size polynomial in $k$?

Observe that if Open Questions 1 and 2 are answered affirmatively then, together with Proposition[4], Theorem[3] and Lemma[1] they imply an unconditional polynomial kernelization of the MWC problem. Moreover, we believe that investigation of Open Question 2 would give a significant insight into the structure of the MWC problem. Indeed it would reveal whether or not we can 'filter' in a reasonable time some principal isolating cuts, which in turn would require proof of some interesting structural dependencies related to the MWC problem.

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A Proofs omitted from the main body

The rest of proof of Theorem 3 For the correctness, we need to show that the set of new added elements is precisely $\mathbf{Pr}_1 \setminus \mathbf{Pr}_{i-1}$. This is established in the following three paragraphs.

Every element of $\mathbf{Pr}_1 \setminus \mathbf{Pr}_{i-1}$ is collected during the gathering stage. Let $S \in \mathbf{Pr}_1 \setminus \mathbf{Pr}_{i-1}$. Since $\text{hat}(S) \neq \emptyset$, by Proposition 1, $S \neq \text{Vis}(S)$. It follows from the DVS condition that $\text{Vis}(S) \nsubseteq S$. Let $v \in \text{Vis}(S) \setminus S$. By Proposition 4 there is $S^*$ such that $v \in \text{hat}(S^*)$ and $S$ is the witness of $v$ w.r.t. $S^*$. Taking into account the SM condition, we conclude that $S^* \in \mathbf{Pr}_{i-1}$, that is $S$ will be generated by the above algorithm.

An element of $\mathbf{Pr}_1 \setminus \mathbf{Pr}_{i-1}$ will not be filtered out. $\text{hat}(S)$ consists of elements that are not contained in any element of $\mathbf{F}$ preceding $S$. The algorithm checks $S$ only against the union of a subset of such elements.

An element that is not in $\mathbf{Pr}_1 \setminus \mathbf{Pr}_{i-1}$ will be filtered. Let $S \in \mathbf{F}$ be such that $\text{ex}(S) = i$ and $\text{hat}(S) = \emptyset$. Let $v \in S$. It is sufficient to show that there is $S^* \in \mathbf{Pr}_{i-1}$ such that $v \in S^*$ and $S^* \prec S$. Choose $S^*$ to be a minimal element preceding $S$ and containing $v$. Due to the minimality of $S^*$, $v$ does not belong to any element preceding $S^*$ hence $v \in \text{hat}(S^*)$, i.e. $S^*$ is a principal set. Due to the SM condition, $\text{ex}(S^*) \leq i - 1$. It follows that $S^* \in \mathbf{Pr}_{i-1}$.

Let us compute the runtime. The main cycle goes through all elements of $\mathbf{Pr}_{i-1}$, the number of such elements is at most $2^{x+1}r$ according to Proposition 5. Let us compute the time spent per element. Since each element of $\mathbf{Pr}_{i-1}$ is of excess at most $x$, i.e. of size at most $r + x$, the algorithm explores at most $r + x$ vertices and for each vertex either computes the respective witness or concludes its absence. It follows from the EC condition that the overall time spent for computation of witnesses per element of $\mathbf{Pr}_{i-1}$ is $O(n^3(r + x))$. Let us compute time spent per witness. Denote the considered witness by $S_1$. Then $S_1$ is compared against all elements of $\mathbf{Pr}_{i-1}$ where the number of such elements is at most $2^{x+1}r$ as noticed above. For each $S_2 \in \mathbf{Pr}_{i-1}$, it is checked whether $S_2 \prec S_1$ which can be done in $O(r + x)$ according to the EC condition. If the test returns a positive answer then $S_2$ is added to the union of elements preceding $S_1$ which, using appropriate data structures, can be done in $O(|S_2|)$, i.e. again in $O(r + x)$. Thus the total runtime of this operation is $O(2^{x+1}r(r + x))$. After finishing the comparison against $\mathbf{Pr}_{i-1}$, the algorithm checks whether or not all the elements are in the resulting union of predecessors. This can be done in $O(|S_1|)$ i.e. in $O(r + x)$, clearly this runtime can be ignored in the light of the already spent $O(2^{x+1}r(r + x))$. Multiplying the number of considered witnesses by the runtime spent per witness, the desired runtime of $O(n^32^{x+1}r^2(r + x)^2)$ follows.

Proof of Lemma 4 We show that the set of important separators partially ordered by the $\prec^*$ meets all the conditions of the IS-family.

the union can be stored as a binary vector of size $n$ indexed by the elements of the universe and adding a set to the union just means ticking the respective entries $|S_1|$ times
**SE Condition** See Lemma 3.3 of [9].

**SM Condition** Immediately follows from the definition of an important separator.

**SW Condition** Let $K$ be an important separator. Let $G^*$ be the graph obtained from $G$ by contraction of all the vertices of $NR(G, Y, K) \setminus X$. In other words, to obtain $G^*$ from $G$, remove all vertices of $NR(G, Y, K)$ and add an edge between each vertex $u \in X$ and $v \in K$ such that there is a $u - v$ path all intermediate vertices of which belong to $NR(G, Y, K)$. It is not hard to see that the definition of an important separator implies that $K$ is the smallest $X - Y$ separator of $G^*$. Let $v \in K$. Assume that $v$ is not adjacent to $Y$ in $G$ (otherwise there is no witness of $v$ w.r.t. $K$). Let $G''$ be a graph obtained from $G^*$ by splitting $v$ into $n + 1$ copies. It is not hard to observe that the set of important $X - Y$ separators of $G''$ is the set of important $X - Y$ separators of $G$ that do not contain $v$, moreover the partial order relation is preserved. Then a witness of $K$ w.r.t. $v$ in $G$ is a smallest important $X - Y$ separator of $G''$. By the SE condition, this separator is unique.

**TE condition.** Observe (e.g. Proposition 1 of [9]), that if $K_1 \prec^* K_2$ then $K_1 \setminus K_2 \subseteq NR(G, Y, K_2)$. Thus if $K_2 \prec^* K_3$, $K_1 \setminus K_2 \subseteq NR(G, Y, K_3) \subseteq NR(G, Y, K_3)$, the last inclusion is obtained by definition of the $\prec^*$ relation. Thus, no vertex of $K_1 \setminus K_2$ can belong to $K_3$.

In order to establish the visible set conditions, we prove an intermediate claim.

**Claim.** Let $K$ be an important $X - Y$ separator, which is not the smallest one and let $K' \in Pred(K)$. Then $K^* = Vis(K) \setminus NR(G, Y, K')$ is a $X - Y$ separator such that $K' \preceq^* K^*$.

**Proof.** Let $v \in NR(G, Y, K')$ and let $p$ be a $v - Y$ path of $G$. Let $u$ be the last vertex of $p$ that belongs to $\bigcup_{K'' \in Pred(K)} K''$. Clearly, $u$ is not covered by any element of $Pred(K)$ because otherwise it would not be the last vertex of $p$ that belongs to an element of $Pred(K)$. Hence by definition $u \in Vis(K)$. Clearly, $u$ cannot belong to $NR(G, Y, K')$ because otherwise it will be followed in $p$ by an element of $K'$. Consequently, $u \in Vis(K) \setminus NR(G, Y, K')$, confirming the claim. □

**LVS condition.** According to the above claim $|K'| \leq |K^*|$ because otherwise we get a contradicition to being $K'$ an important separator. Taking into account that $K^* \subseteq Vis(K)$, the condition follows.

**DVS condition.** $K^* \subseteq Vis(K)$, hence the latter is an $X - Y$ separator. Therefore, if $Vis(K) \subset K$ then $K$ is not a minimal $X - Y$ separator in contradiction to its importance.

**EC condition** The $O(n^3)$ algorithm for computing a smallest important separator follows from Lemma 1 in [9]. This immediately implies existence of such algorithm for the witness computation. Indeed, the single witness condition proof of Lemma 4 shows that witness computation can be reduced to computing the smallest important separator and such the reduction can be clearly performed in $O(n^3)$. Finally, observe that it is possible to maintain an important separator $K$ in a way that for each vertex $v$ testing whether $v \in NR(G, Y, K)$ can be
performed in $O(1)$: associate $K$ with a binary vector of size $n$ indexed by $V(G)$ where 1-s correspond to the elements of $NR(G, Y, K)$. In the light of Proposition 1 in [9], this immediately implies that $K_1 \prec^* K_2$ can be tested in $O(K_1)$.

**Proof of Lemma 5** Let $S_1$ be an arbitrary smallest mwc of $(G, T)$. If all vertices of $S_1$ belong to important isolating cuts, we are done. Otherwise, let $v \in S_1$ be a vertex that does not belong to any smallest isolating cut. Due to the minimality of $S_1$, there is $t \in T$ such that $v$ belongs to a minimal isolating cut $S' \subseteq S_1$ of $t$. It follows that there is an important isolating cut $S'' \succ^* S'$ of $t$ such that $|S''| \leq |S'|$. The proof of Lemma 3.6. of [8] shows that $S_2 = (S_1 \setminus S') \cup S''$ is a mwc of $(G, T)$ of size not exceeding $S_1$. In other words $S_2$ is an optimal solution of $(G, T)$ and the number of vertices of $S_2$ not involved in any important isolating cuts is smaller than that of $S_1$. Applying such modification iteratively, we eventually obtain a smallest multiway cut without such 'undesired' vertices.

■