QUANTUM THEORY
AND
LOCAL HIDDEN VARIABLE THEORY:
GENERAL FEATURES AND TESTS
FOR
EPR STEERING

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Abstract

Quantum states for bipartite composite systems are categorised as either separable or entangled, but the states can also be divided differently into Bell local or Bell non-local states. The latter categorisation is based on whether or not the probability \( P(a,b|A,B,c) \) for measured outcomes \( a,b \) on sub-system observables \( A,B \) for state preparation process \( c \), is given by a local hidden variable theory (LHVT) form

\[
P(a,b|A,B,c) = \sum_\lambda P(\lambda|c)P(a|A,c,\lambda)P(b|B,c,\lambda)
\]

(where preparation \( c \) results in a probability distribution \( P(\lambda|c) \) for hidden variables \( \lambda \), \( P(a|A,c,\lambda) \) is the probability for measured outcome \( a \) on sub-system observable \( A \) when the hidden variables are \( \lambda \) with \( P(b|B,c,\lambda) \) the analogous observable \( B \) probability). Quantum states where \( P(a,b|A,B,c) \) is given by a LHVT form are Bell local, if not they are Bell non-local and associated with Bell inequality violation experiments. This paper presents a detailed classification of quantum states for bipartite systems and describes the inter-relationships between the various types. For the Bell local states there are three cases depending on whether both, one of or neither of the LHVT probabilities \( P(a|A,c,\lambda) \) and \( P(b|B,c,\lambda) \) are also given by a quantum probability involving sub-system density operators. Cases where one or both are given by a quantum probability are known as local hidden states (LHS) and such states are non-steerable. The steerable states are the Bell local states where there is no LHS, or the Bell non-local states.

In a previous paper tests for entanglement for two mode systems involving identical massive bosons were obtained. In the present paper we consider tests for EPR steering in such systems. We find that spin squeezing in the any spin component, a Bloch vector test, the Hillery-Zubairy spin variance test and a two mode quadrature squeezing test all show that the LHS model fails, and hence the quantum state is EPR steerable. We also find a generalisation of the Hillery-Zubairy spin variance test which shows that EPR steering occurs. Correlation tests are also presented.

Keywords

- Bell locality
- Quantum entanglement
- EPR steering
- Spin squeezing test
- Two mode quadrature squeezing tests
- Spin variance tests
1 Introduction

Recent papers by Dalton et al [1], [2], [3] have dealt with the topic of bipartite quantum entanglement and experimental tests for demonstrating it in the context of systems of identical massive bosons. However, although the quantum states of composite systems can just be classified into disjoint sets of separable or entangled states, it is also possible to classify them into distinct categories based on local hidden variable theory [4], where the two basic disjoint sub-sets of quantum states are the Bell local states and the Bell non-local states. Based on the work of Wiseman et al [5], [6], [7] the Bell local states for bipartite systems can then be divided into three disjoint sub-categories. The categories of states associated with local hidden variable theory have differing features regarding entanglement, EPR steering and Bell non-locality - as will be explained below. This paper is one of a series aimed at developing tests based on experimentally measureable quantities for demonstrating which category applies for quantum states of bipartite composite systems of identical massive bosons. The focus of the present paper is on tests for demonstrating EPR steering in these systems.

The local hidden variable theory treatment has origins in papers by Einstein, Schrodinger, Bell and Werner ([8], [9], [10], [4], [11]). Einstein suggested that quantum theory, though correct was incomplete - in that it did not deal satisfactorily with the issue of whether the possible measured outcomes for observable quantities (such as position and momentum) could be regarded as elements of reality irrespective of whether an actual measurement has taken place. This issue was raised for the EPR paradox in the context of entangled states, in which the entangled state for two well-separated and no longer interacting subsystems had well-defined values for the position difference and the momentum sum. Because of these correlations the choice of measuring the position (or the momentum) for the first sub-system would instantaneously affect the outcome for measuring the position (or the momentum) of the second sub-system - a feature we now refer to as steering - but which Einstein called "spooky action at a distance" and regarded as being in conflict with causality. The paradox is that by measuring the position for the first sub-system we then know the position for the second sub-system without doing a measurement, so by then measuring the momentum for the second sub-system a joint precise measurement of both the position and momentum for the second sub-system would have occurred - which is apparently incompatible with the Heisenberg uncertainty principle. The paradox is clearly based on the assumption that the positions and momenta always have actual values (reality) irrespective of whether a measurement has taken place. The Schrodinger cat paradox [10] is another example, but now involving a macroscopic sub-system (the cat).in an entangled state with a microscopic sub-system (the two state radioactive atom). Is the cat always either alive or dead before the box is opened to see which is the case? Bohm [12] described a similar paradox to EPR, but now involving a system consisting of two spin 1/2 particles in a singlet state, and where observables were spin components with quantised measured outcomes - instead of the continuous outcomes that applied to EPR.
Einstein suggested that quantum theory could be the statistical outcome of an underlying deterministic theory - essentially what we now refer to as a hidden variable theory - where in its simplest form the possible measured outcomes for all observables always have specific values, and measurement merely reveals what these values are. This is in direct contradiction to the Copenhagen interpretation of quantum theory, in which the values for observables do not have a presence in reality until measurement takes place. However, it was not until 1965 before a quantitative general form for hidden variable theory which could be tested in experiments was proposed by Bell [4]. Basically, the key idea is that hidden variables are determined probabilistically when the state for the composite system is prepared, and these would determine the values for all the sub-system observables even after the sub-systems have separated - and even if the observables were incompatible (such as two different spin components). In the Schrodinger cat experiment the hidden variables would specify whether the cat is dead or alive, even before the box is opened. In the EPR experiment they would specify both the position and momentum for a localised sub-system even though in quantum theory these pairs of observables are incompatible with simultaneous precise measurements. Quantum states for composite systems that could be described by hidden variable theory (the Bell local states) were such that certain inequalities would apply involving the mean values of products for the results of measuring pairs of observables for both sub-systems - the Bell inequalities [4], [13]. Based on the entangled singlet state of two spin $\frac{1}{2}$ particles Clauser et al [14] proposed an experiment that could demonstrate a violation of a Bell inequality. This would show that hidden variable theory could not account for experiments that can be explained by quantum theory. In more elaborate versions of hidden variable theory (see [2] for a description), the hidden variables would merely determine the probabilities of measurement outcomes for each of the sub-systems, whilst the overall expressions for the joint sub-system measurement outcomes are still obtained via classical probability theory. Bell inequalities of the same form still apply in this more elaborate version of hidden variable theory. Subsequent experimental work violating Bell inequalities confirmed that there are some quantum states for which a hidden variable theory does not apply and where quantum theory was needed to explain the results (see Brunner et al [15] for a recent review). States for which a hidden variable theory does not apply (and hence violate Bell inequalities) are the Bell non-local states.

It was recognised [11] that all separable states could be described by hidden variable theory (and hence are Bell local) and hence a state had to be entangled to be Bell non-local. However, Werner [11] showed that some entangled states could also be described by hidden variable theory - and hence not violate a Bell inequality. The relationship between the classification of states into separable or entangled on one hand, and a classification into Bell local and Bell non-local states on the other hand is therefore not a simple one. In addition to Bell locality or non-locality, there is the question of which categories of states demonstrate the feature of steering [8], [9], [10], in which a choice of measurement on one sub-system can be used to instantly affect the outcomes for possible measurements.
on the other sub-system - even if it are well separated. As we will see, steerability requires the absence of the so-called local hidden states - which are sub-system quantum states whose density operator is specified by the hidden variables.

In the work by Wiseman et al. [5], [6], [7] states for bipartite systems defined in terms of local hidden variable theory were first categorised by whether they are Bell local or Bell non-local. Within the states that are Bell local a more detailed categorisation was made based on a hierarchy of non-disjoint sub-sets - firstly by whether they are EPR steerable or not, and then secondly for EPR non-steerable states by whether they are separable or not. In the present paper we find it convenient to categorise the Bell local states into three sub-sets which are disjoint, though still related to the hierarchy of non-disjoint sub-sets introduced by Wiseman et al. The disjoint sub-sets of states are defined by whether two, one or none of the sub-system hidden variable probabilities is associated with a local hidden state determined from the hidden variables. Category 1 states involve two hidden states, and this Bell local sub-set is the same as the separable states, These are non-steerable. Category 2 states involve only one hidden state and for this Bell local sub-set the states are entangled, though non-steerable. Category 3 states do not involve any hidden state, and these Bell local states are both entangled and steerable. The physical reason why the various categories of states are steerable or non-steerable is described in [5], [6], [7], but for completeness this is set out in Appendix 8. We will also designate the states that are Bell non-local as Category 4 states, and these states are both entangled and steerable.

It is of some interest to devise tests for which specific category a quantum state falls into in the context of systems of identical massive bosons, such as occur in Bose-Einstein condensates for cold bosonic atomic gases. The focus of this paper is on whether the quantum state is EPR steerable - which means showing that it is not a Category 1 or a Category 2 state. In previous work tests have been obtained (see [3] for details of a range of tests found by various authors) for showing that a state is entangled, which therefore rules them out from being in Category 1. Hence we only need to consider tests for showing that the state is also not in Category 2. In a later paper we will consider tests for Bell non-locality that can be applied when the measurable quantities for the two sub-systems have a range of outcomes other than the more limited +1, −1 outcomes considered by Clauser et al. [14].

In Section 2 we begin with a brief review of measurement probabilities in bipartite systems, focusing both on quantum expressions for joint measurement probabilities and local hidden variable theory expressions, and discuss the question of how to interconvert between quantum and local hidden variable theory quantities. Important relationships between the mean values for measurements given by quantum theory and by local hidden variable theory are highlighted. We then consider in Section 3 the detailed description of how the quantum states for composite systems may be categorised. We relate our categories of states to the hierarchy of sub-sets discussed in Refs. [5], [6], [7], [16].

We then consider in Section 4 various tests for EPR steering, taking into
account that the local hidden states must comply with the local particle number super-selection rule, since they must be possible quantum states for the particular sub-system considered on its own. Furthermore, in accordance with Einstein’s basic idea that quantum theory predictions for measurement probabilities are correct, but for Bell local states can also be interpreted in terms of an underlying reality represented by a hidden variable theory, we can link certain hidden variable theory probabilities to quantum theory expressions. This linkage does not of course apply for Bell non-local states. Also, since annihilation and creation operators do not have counterpart classical observables we replace these by quadrature amplitudes when treating Bell local states, including in expressions for spin operators. It turns out that previous tests (see Refs. [1], [2], [3] for details) for quantum entanglement (Bloch vector test, spin squeezing in any spin component $S_x$, $S_y$ or $S_z$, the Hillery-Zubairy spin variance test [17], a strong correlation test, a two mode quadrature squeezing test) can also be applied as tests for EPR steering in systems of identical massive bosons. However, in addition a new spin variance test for EPR steering involving the sum of the variances for spin operators $S_x$, $S_y$ and the mean boson number has been obtained which also involves the mean value for $S_z$, generalising a result in He et al [18]. This test is a generalisation of the Hillery-Zubairy spin variance test. In addition weak and strong correlation tests for EPR steering are also derived, though the latter have been previously obtained by Cavalcanti et al [23], and the correlation tests are equivalent to other tests. Section 5 provides a summary of the main results. Experiments demonstrating EPR steering in two mode Bose-Einstein condensates based on these tests, such as in Refs. [19], [20], [21], are discussed elsewhere [22].

Details are set out in Appendices. In Appendix 6 general properties of mean values and variances are reviewed. The Werner states are described in Appendix 7 since in various parameter regimes they provide examples of the four categories of states in the local hidden variable theory model. The idea behind EPR steering is discussed in Appendix 8. Details for the derivation of the variance for two mode quadratures, spin operator forms of correlation inequalities and EPR steering tests are presented in Appendices 9, 10 and 11.
2 Measurement Probabilities in Bipartite Systems

2.1 Basic Measurement Probabilities in Physics

This paper deals with measurements on bipartite composite quantum systems, where we have two distinguishable sub-systems $A$ and $B$ which are each associated with measurable physical observables $\Omega_A$ and $\Omega_B$ for which possible outcomes are denoted $\alpha$ and $\beta$. The composite system exists in various quantum states, whose preparation is symbolised by $c$. Quantum theory has the key feature that such measurements the occurrence of particular outcomes are specified by probabilities rather than being deterministic, and the basic quantity of interest is the joint probability $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ for measurement of any pair of sub-system observables $\Omega_A$ and $\Omega_B$ to obtain any of their possible outcomes $\alpha$ and $\beta$ when the preparation process is $c$. As the sub-systems are distinct simultaneous precise measurement outcomes apply for the pairs of observables $\Omega_A$ and $\Omega_B$ in both quantum and hidden variable theory (in the latter case the observables are classical variables and not Hermitian operators). The probability $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ is of course real and positive and its sum for all outcomes for both $\Omega_A$ and $\Omega_B$ is equal to unity. The sum of the joint probability over the possible outcomes $\alpha$ for measuring $\Omega_A$ defines the single probability $P(\beta | \Omega_B, c)$ for measuring $\Omega_B$ with outcome $\beta$, irrespective of the outcome for measuring $\Omega_A$. A similar definition applies for the single probability $P(\alpha | \Omega_A, c)$ for measuring $\Omega_A$ with outcome $\alpha$, irrespective of the outcome for measuring $\Omega_B$.

Thus:

$$\sum_{\alpha, \beta} P(\alpha, \beta | \Omega_A, \Omega_B, c) = 1 \quad (1)$$

$$P(\beta | \Omega_B, c) = \sum_{\alpha} P(\alpha, \beta | \Omega_A, \Omega_B, c) \quad (2)$$

$$P(\alpha | \Omega_A, c) = \sum_{\beta} P(\alpha, \beta | \Omega_A, \Omega_B, c) \quad (3)$$

The single probabilities also satisfy the expected probability sum rules

$$\sum_{\beta} P(\beta | \Omega_B, c) = 1 \quad \sum_{\alpha} P(\alpha | \Omega_A, c) = 1 \quad (4)$$

which follow from (1).

From the joint measurement probability $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ and the single measurement probabilities $P(\alpha | \Omega_A, c)$, $P(\beta | \Omega_B, c)$ we can introduce conditional probabilities $P(\beta | \Omega_B | \alpha, \Omega_A, c)$ and $P(\alpha | \Omega_A | \beta, \Omega_B, c)$. Here $P(\beta | \Omega_B | \alpha, \Omega_A, c)$ is the probability that measurement of the observable $\Omega_B$ yields the outcome $\beta$ given that measurement of the observable $\Omega_A$ yields the outcome $\alpha$. This (and the corresponding expression for $P(\alpha | \Omega_A | \beta, \Omega_B, c)$) is given by Bayes’ theorem as

$$P(\beta | \Omega_B | \alpha, \Omega_A, c) = \frac{P(\alpha, \beta | \Omega_A, \Omega_B, c)}{P(\alpha | \Omega_A, c)} \quad (5)$$

$$P(\alpha | \Omega_A | \beta, \Omega_B, c) = \frac{P(\alpha, \beta | \Omega_A, \Omega_B, c)}{P(\beta | \Omega_B, c)}$$
All these expressions apply irrespective of whether the joint and single measurement probabilities are obtained from quantum theory or local hidden variable theory formulae.

Discussions of hidden variable theory (HVT) are usually based on Einstein’s viewpoint in which he accepted that quantum theory provided correct expressions for joint and single measurement probabilities on composite quantum systems, but felt that such probabilities reflected quantum theory being incomplete. Einstein believed that an underlying realist theory could be found, based on what are now referred to as hidden variables - which would specify the real or underlying state of the system. Thus, quantum theory is not wrong, it is merely incomplete.

Accordingly, we begin by first presenting the quantum theory expressions for joint and single measurement probabilities for composite quantum systems, and then the possible underlying local hidden variable theory (LHVT) expressions. These quantum expressions [6], [7] and [8] will be regarded as always applying - irrespective of additional local hidden variable theory formulae that may apply as well. For quantum theory, the preparation process is reflected in the density operator for the system $c \rightarrow \hat{\rho}(c)$. Only von Neumann measurements will be considered. Furthermore, in accordance with the viewpoint that quantum theory is correct, the measurement outcomes $\alpha$ and $\beta$ will be the same as the possible quantum theory outcomes, determined as the eigenvalues of the corresponding quantum Hermitian operators $\hat{\Omega}_A$ and $\hat{\Omega}_B$. For simplicity we treat these as quantized - the generalisation for continuous eigenvalues is straightforward.

2.2 Quantum Theory - Measurement Probabilities

In quantum theory the joint probability $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ for measurement of any pair of sub-system observables $\Omega_A$ and $\Omega_B$ to obtain any of their possible outcomes $\alpha$ and $\beta$ when the preparation process is $c$ is given by an expression based on the sub-system observables $\Omega_A$ and $\Omega_B$ being represented by quantum Hermitian operators $\hat{\Omega}_A$ and $\hat{\Omega}_B$. Here simultaneous precise measurement applies because the system operators involved, $\hat{\Omega}_A \otimes \hat{I}_B$ and $\hat{I}_A \otimes \hat{\Omega}_B$ commute and therefore have complete sets of simultaneous eigen vectors.

We have for the joint measurement probability (see [5], Eq. (2))

$$P(\alpha, \beta|\Omega_A, \Omega_B, c) = Tr((\hat{\Pi}_A^\alpha \otimes \hat{\Pi}_B^\beta)\hat{\rho}) \quad (6)$$

where $\hat{\Pi}_A^\alpha$ and $\hat{\Pi}_B^\beta$ are projectors onto the eigenvector spaces for $\hat{\Omega}_A$ and $\hat{\Omega}_B$ associated with the real eigenvalues $\alpha$ and $\beta$ that in quantum theory are the possible measurement outcomes. We have $\hat{\Omega}_A \hat{\Pi}_A^\alpha = \alpha \hat{\Pi}_A^\alpha = \hat{\Pi}_A^\alpha \hat{\Omega}_A$, and similar expressions for $\hat{\Pi}_B^\beta$. Clearly the quantum expression for the joint probability satisfies the general probability requirement (1) - the sum rules over $\alpha$ and $\beta$ being based on the projector properties $\sum_\alpha \hat{\Pi}_A^\alpha = \hat{1}^A$ and $\sum_\beta \hat{\Pi}_B^\beta = \hat{1}^B$ involving the sub-system unit operators and $Tr\hat{\rho} = 1$. 

8
The quantum theory expressions for the single measurement probabilities

\[
P(\alpha|\Omega_A, c) = Tr((\hat{\Pi}_A^\alpha \otimes \hat{1}_B)\hat{\rho})
\]

\[
P(\beta|\Omega_B, c) = Tr((\hat{1}_A \otimes \hat{\Pi}_B^\beta)\hat{\rho})
\]

(7)

for (respectively) measuring \(\Omega_A\) to have outcome \(\alpha\) irrespective of \(\Omega_B\) and \(\beta\) or for measuring \(\Omega_B\) to have outcome \(\beta\) irrespective of \(\Omega_A\) and \(\alpha\) both follow from [2] or [3] and the projector properties. The single measurement probabilities can be expressed in terms of reduced density operators \(\hat{\rho}^A\) and \(\hat{\rho}^B\) for the subsystems

\[
\hat{\rho}^A = Tr_B(\hat{\rho}) \quad \hat{\rho}^B = Tr_A(\hat{\rho})
\]

\[
P(\alpha|\Omega_A, c) = Tr_A(\hat{\Pi}_A^\alpha \hat{\rho}^A) \quad P(\beta|\Omega_B, c) = Tr_B(\hat{\Pi}_B^\beta \hat{\rho}^B)
\]

(8)

The proof of the results (8) for \(\rho_{system}\) density operator overall abilities are related via (3) and (2), as easily shown using \(\sum \hat{\Pi}_\alpha^A = \hat{1}_A\) and \(\sum \hat{\Pi}_\beta^B = \hat{1}_B\). Using similar considerations and \(Tr\hat{\rho} = 1\), the single probabilities also satisfy the sum rules (4).

The mean value for joint measurement outcomes of the observables \(\hat{\Omega}_A\) and \(\hat{\Omega}_B\) will be given by

\[
\langle \hat{\Omega}_A \otimes \hat{\Omega}_B \rangle = \sum_{\alpha, \beta} \alpha \beta P(\alpha, \beta|\Omega_A, \Omega_B, c)
\]

\[
= Tr(\hat{\Omega}_A \otimes \hat{\Omega}_B)\hat{\rho}
\]

(9)

where the results \(\sum_\alpha \hat{\Pi}_\alpha^A = \hat{\Omega}_A\) and \(\sum_\beta \hat{\Pi}_\beta^B = \hat{\Omega}_B\) and (6) have been used.

The mean value for the measurement of a single observable \(\hat{\Omega}_A\) is

\[
\langle \hat{\Omega}_A \rangle = \sum_\alpha \alpha P(\alpha|\Omega_A, c) = Tr(\hat{\Omega}_A \otimes \hat{1}_B)\hat{\rho} = Tr_A(\hat{\Omega}_A \hat{\rho}^A)
\]

(10)

as can be derived from (6) and (8).

It is worth noting that for systems of identical massive bosons super-selection rules (SSR) require the overall density operator \(\hat{\rho}\) to commute with the total number operator \(N\) (global particle number SSR - see for example Refs. [2], [3] and references therein for discussions on SSR). Consequently the density operator for a two mode system

\[
\hat{\rho} = \sum_{n_A, n_B} \sum_{m_A, m_B} \rho(n_A, n_B; m_A, m_B)(|n_A\rangle \otimes |n_B\rangle)(\langle m_A| \otimes \langle m_B|)
\]

(11)

is such that \(\rho(n_A, n_B; m_A, m_B) = 0\) unless \(n_A + n_B = m_A + m_B\). It is then straightforward to show that the reduced density operator \(\hat{\rho}^A\) for mode \(A\) is given by

\[
\hat{\rho}^A = \sum_{n_A} \left(\sum_{n_B} \rho(n_A, n_B; n_A, n_B)\right)(|n_A\rangle \langle n_A|)
\]

(12)
which is SSR compliant for the sub-system particle number $N_A$ (local particle number SSR). This feature will turn out to be relevant for evaluating terms associated with the EPR steering tests.

2.3 Local Hidden Variable Theory - Measurement Probabilities

A hidden variable theory is based on hidden variables $\lambda$ which describe the real or underlying state of the system, and which are determined with a probability $P(\lambda|c)$ for a preparation process $c$. The probability $P(\lambda|c)$ is real, positive and its sum over all possible hidden variables is also unity. Thus

$$\sum \lambda P(\lambda|c) = 1. \quad (13)$$

The preparation process is thus reflected in the probability function for the hidden variables $c \rightarrow P(\lambda|c)$. For composite systems the overall picture for a local hidden variable theory is that although the hidden variables are global and associated with the original preparation process, they act locally in determining the probabilities of separate measurements on the sub-systems - even in the situation where the sub-systems are localised in well-separated spatial regions and the two sub-system measurements occur simultaneously.

In local hidden variable theory the joint probability $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ for measurement of any pair of sub-system observables $\Omega_A$ and $\Omega_B$ to obtain any of their possible outcomes $\alpha$ and $\beta$ when the preparation process is $c$ is given by an expression involving measurement probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$ for the separate sub-systems which depend on the hidden variables $\lambda$. Here the sub-system observables $\Omega_A$ and $\Omega_B$ are represented by c-numbers rather than Hermitian operators. Here $P(\alpha|\Omega_A, c, \lambda)$ is the probability that measurement of the observable $\Omega_A$ of sub-system $A$ results in outcome $\alpha$ when the hidden variable are $\lambda$, with a similar definition for $P(\beta|\Omega_B, c, \lambda)$. It is important to note that each sub-system observable $\Omega_C$ has its own single measurement outcome probability $P(\gamma|\Omega_C, c, \lambda)$, not necessarily related to those for a different observable for the same sub-system.

For a LHVT the joint probability $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ for measurement of any pair of sub-system observables $\Omega_A$ and $\Omega_B$ to obtain any of their possible outcomes $\alpha$ and $\beta$ when the preparation process is $c$ is given by (see [5], Eq. (3), [7], Eq. (15))

$$P(\alpha, \beta|\Omega_A, \Omega_B, c) = \sum \lambda P(\alpha|\Omega_A, c, \lambda) P(\beta|\Omega_B, c, \lambda) P(\lambda|c) \quad (14)$$

In LHVT the hidden variables $\lambda$ are first determined (probabilistically) via the preparation process, these then act locally to determine the sub-system measurement probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$. The probabilities are then finally combined in accordance with classical probability theory to determine the
joint measurement probability. We can also write (14) as

\[
P(\alpha, \beta | \Omega_A, \Omega_B, c) = \sum_{\lambda} P(\alpha, \beta | \Omega_A, \Omega_B, c, \lambda) P(\lambda | c) P(\alpha, \beta | \Omega_A, \Omega_B, c, \lambda) = P(\alpha | \Omega_A, c) P(\beta | \Omega_B, c, \lambda)
\]  

(15)

where the last equation expresses the requirement that the joint probability \( P(\alpha, \beta | \Omega_A, \Omega_B, c, \lambda) \) for a preparation leading to hidden variables \( \lambda \) is given by the product of sub-system probabilities \( P(\alpha | \Omega_A, c, \lambda) \) and \( P(\beta | \Omega_B, c, \lambda) \) for that global value of \( \lambda \). States for which the joint probability is given by the local hidden variable theory Eq. (14) are referred to as Bell local. States where this does not apply are the Bell non-local states.

In a non-local hidden variable theory the decomposition in (14) does not occur and we would just have

\[
P(\alpha, \beta | \Omega_A, \Omega_B, c) = \sum_{\lambda} P(\alpha, \beta | \Omega_A, \Omega_B, c, \lambda) P(\lambda | c)
\]  

(16)

with no local sub-system probabilities involved. Here \( P(\alpha, \beta | \Omega_A, \Omega_B, c, \lambda) \) is the joint probability that measurement of the observables \( \Omega_A, \Omega_B \), of sub-systems \( A, B \) results in outcomes \( \alpha, \beta \) when the hidden variable are \( \lambda \), but is now not factorisable.

The single measurement probabilities \( P(\alpha | \Omega_A, c, \lambda) \) and \( P(\beta | \Omega_B, c, \lambda) \) must of course satisfy the general requirements of being real, positive and such that their sum over all possible outcomes is unity for each value of the LHV in accordance with the general requirements (4). Thus

\[
\sum_{\alpha} P(\alpha | \Omega_A, c, \lambda) = 1 \quad \sum_{\beta} P(\beta | \Omega_B, c, \lambda) = 1
\]  

(17)

By combining (13) and (17) it is straightforward to show that the joint probability \( P(\alpha, \beta | \Omega_A, \Omega_B, c) \) satisfies the standard probability sum rule (1).

The overall probability \( P(\alpha | \Omega_A, c) \) of measurement of the observable \( \Omega_A \) of sub-system \( A \) results in outcome \( \alpha \) when the preparation process is \( c \) irrespective of the outcome for measurement of the observable \( \Omega_B \) of sub-system \( B \) is defined as in (3), so it is given by the sum over the possible values \( \lambda \) of the hidden variables of the \( P(\alpha | \Omega_A, c, \lambda) \) times the preparation probability \( P(\lambda | c) \), with a similar expression for \( P(\beta | \Omega_B, c) \). Thus using (17)

\[
P(\alpha | \Omega_A, c) = \sum_{\lambda} P(\alpha | \Omega_A, c, \lambda) P(\lambda | c)
\]

(18)

The results (18) show that in a LHVT the measurement probability for an observable \( \Omega_A \) of sub-system \( A \) is independent of the results for measuring an observable \( \Omega_B \) of sub-system \( B \), and do not even depend on which observable \( \Omega_B \) is being measured. The same applies if the sub-systems are reversed. This important result for LHVT is called the no-signaling theorem and shows that a
choice of observable to be measured in one sub-system cannot affect the result of measurements in the other sub-system.

Also, we have using (17) and (15)

\[
P(\alpha|\Omega_A, c, \lambda) = \sum_{\beta} P(\alpha, \beta|\Omega_A, \Omega_B, c, \lambda)
\]

\[
P(\beta|\Omega_B, c, \lambda) = \sum_{\alpha} P(\alpha, \beta|\Omega_A, \Omega_B, c, \lambda)
\]

relating the joint probability \(P(\alpha, \beta|\Omega_A, \Omega_B, c, \lambda)\) for a preparation leading to hidden variables \(\lambda\) to the corresponding single probabilities for the same \(\lambda\).

Here we have followed the approach of Refs. [5], [6] (but not the notation) where the sub-system probabilities \(P(\alpha|\Omega_A, c, \lambda)\) and \(P(\beta|\Omega_B, c, \lambda)\) are not necessarily given by quantum expressions such as (7) though they may be. Later we will introduce a more specific notation (subscript \(Q\)) to distinguish cases where \(P(\alpha|\Omega_A, c, \lambda)\) and/or \(P(\beta|\Omega_B, c, \lambda)\) are given by quantum expressions from those where they are not. When the \(P_Q(\gamma|\Omega_C, c, \lambda)\) for sub-system \(C\) are determined from a quantum expression which involves a density operator \(\rho^C(c, \lambda)\) for sub-system \(C\) determined from the hidden variables \(\lambda\), then \(\rho^C(c, \lambda)\) specifies a so-called local hidden state (LHS).

We can use (14) to obtain an expression for the mean value of the joint measurement of observables \(\Omega_A\) and \(\Omega_B\) when the preparation process is \(c\). This will be given by

\[
\langle \Omega_A \otimes \Omega_B \rangle = \sum_{\alpha, \beta} \alpha \beta P(\alpha, \beta|\Omega_A, \Omega_B, c)
\]

\[
= \sum_{\lambda} \langle \Omega_A(c, \lambda) \rangle \langle \Omega_B(c, \lambda) \rangle P(\lambda|c)
\]

(20)

where \(\langle \Omega_A(c, \lambda) \rangle \equiv \langle \Omega_A(\lambda) \rangle\) is the expectation value of observable \(\Omega_A\) when the preparation process \(c\) leads to hidden variables \(\lambda\), with \(\langle \Omega_B(c, \lambda) \rangle \equiv \langle \Omega_B(\lambda) \rangle\) the corresponding expectation value for observable \(\Omega_B\). These are given by

\[
\langle \Omega_A(c, \lambda) \rangle = \sum_{\alpha} \alpha P(\alpha|\Omega_A, c, \lambda)
\]

\[
\langle \Omega_B(c, \lambda) \rangle = \sum_{\beta} \beta P(\beta|\Omega_B, c, \lambda)
\]

(21)

The mean value for the measurement of a single observable \(\Omega_A\) is

\[
\langle \Omega_A \rangle = \sum_{\alpha} \alpha P(\alpha|\Omega_A, c) = \sum_{\lambda} \langle \Omega_A(c, \lambda) \rangle P(\lambda|c)
\]

(22)

as can be derived from (18) and (21). A similar result applies for \(\langle \Omega_B \rangle\).

In a non-fuzzy version of LHVT \(\langle \Omega_A(c, \lambda) \rangle = \alpha(c, \lambda)\) and \(\langle \Omega_B(c, \lambda) \rangle = \beta(c, \lambda)\), where \(\alpha(c, \lambda)\) and \(\beta(c, \lambda)\) are specific allowed outcomes for measurement of the observables when the preparation process \(c\) leads to hidden variables \(\lambda\). Here the hidden variables \(\lambda\) determine unique measurement outcomes \(\alpha(c, \lambda)\) and \(\beta(c, \lambda)\). In the non-fuzzy case

\[
\langle \Omega_A \otimes \Omega_B \rangle = \sum_{\lambda} \alpha(c, \lambda) \beta(c, \lambda) P(\lambda|c)
\]

(23)
which is a form originally used for $\langle \Omega_A \otimes \Omega_B \rangle$ by Bell (see Ref. [4]). Thus in a non-fuzzy version of LHVT the hidden variables uniquely specify the measurement outcomes, and it is only because the hidden variables are not known that they must be averaged over.

2.4 Links between Quantum and Local Hidden Variable Theory

In accordance with Einstein’s basic idea that quantum theory predictions for $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ and $P(\alpha|\Omega_A, c)$, $P(\beta|\Omega_B, c)$ are correct, but can be interpreted in terms of an underlying reality represented by a hidden variable theory, it follows that the same joint probability in (14) can also be determined from the quantum theory expression (6). Similarly for the single measurement probabilities $P(\alpha|\Omega_A, c)$, $P(\beta|\Omega_B, c)$. Note that this assumes that the particular quantum state for the composite system can be interpreted via local hidden variable theory, which therefore excludes the Bell non-local states. As we have already noted, there are actual Bell non-local states where the quantum results are not accountable via LHVT - either theoretically or experimentally. So it is only when we are considering Bell local states that these inter-relationships can be applied.

The conditional probabilities are given by the general expressions (5) that apply for both quantum and LHVT cases. Again, using (17) and (18) the general relationships (3) and (2) between the joint and single measurement probabilities occur. Also, using (17) and (13) we then see that the general relationship (11) applies in LHVT.

Hence for Bell local states, equating the LHVT (18) and quantum theory (8) expressions for the single measurement probability $P(\alpha|\Omega_A, c)$ we obtain a LHVT - quantum theory relationship

$$P(\alpha|\Omega_A, c) = \sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P(\lambda|c) = Tr((\hat{\Pi}_A^\alpha \otimes \hat{1}_B) \hat{\rho}) = Tr_A(\hat{\Pi}_A^\alpha \hat{\rho}_A) = P(\alpha|\Omega_A, c)$$

(24)

This shows that the hidden variable theory probability $P(\alpha|\Omega_A, c, \lambda)$ associated with single sub-system A measurements and the reduced density operator $\hat{\rho}_A^A$ for sub-system A are inter-related. A similar result applies for $P(\beta|\Omega_B, c)$. However, this relationship does not mean that $P(\alpha|\Omega_A, c, \lambda)$ can always be determined from a sub-system density operator which is not dependent on the overall quantum state $\hat{\rho}$ describing both sub-systems together - in general the reduced density operator for each sub-system is determined from the full density operator $\hat{\rho}$. However, when there is a local hidden state, the reduced density operator $\hat{\rho}_A^A$ may be replaced by the form $\hat{\rho}_A^A(c, \lambda)$ - which is determined specifically for sub-system A for preparation process $c$ via the hidden variables $\lambda$.

Similar considerations apply for Bell local states to the joint measurement probability $P(\alpha, \beta|\Omega_A, \Omega_B, c)$. We have a second LHVT - quantum theory rela-


development

\begin{align}
P(\alpha, \beta | \Omega_A, \Omega_B, c) &= \sum_\lambda P(\alpha, \beta | \Omega_A, \Omega_B, c, \lambda) P(\lambda | c) = Tr((\hat{\Pi}^A_\alpha \otimes \hat{\Pi}^B_\beta)\hat{\rho}) \tag{25}
\end{align}

Also, for *Bell local* states we can inter-relate the quantum and LHVT *mean values* of the joint measurement of observables \(\Omega_A\) and \(\Omega_B\) when the preparation process is \(c\). Using (9) and (20) we have

\begin{align}
\langle \hat{\Omega}_A \otimes \hat{\Omega}_B \rangle &= Tr(\hat{\Omega}_A \otimes \hat{\Omega}_B)\hat{\rho} = \sum_\lambda \langle \Omega_A(c, \lambda) \rangle \langle \Omega_B(c, \lambda) \rangle P(\lambda | c) = \langle \Omega_A \otimes \Omega_B \rangle \tag{26}
\end{align}

in cases where the LHVT can be applied.

In the case of *mean values* for a *single* observable, we have similarly

\begin{align}
\langle \hat{\Omega}_A \rangle &= \langle \hat{\Omega}_A \otimes \hat{1}_B \rangle = Tr(\hat{\Omega}_A \otimes \hat{1}_B)\hat{\rho} = Tr_A(\hat{\Omega}_A \hat{\rho}^A) \\
&= \sum_\lambda \langle \Omega_A(c, \lambda) \rangle P(\lambda | c) = \langle \Omega_A \otimes \hat{1}_B \rangle = \langle \Omega_A \rangle \tag{27}
\end{align}

for *Bell local* states. A similar result applies for \(\langle \hat{\Omega}_B \rangle\). These results are all useful for *inter-converting* LHVT and quantum theory expressions, for the *Bell local* states.

We will also need to consider the mean values for observables which in quantum theory are given by the *sum* of *products* of sub-system Hermitian operators, where the operators for each sub-system do not necessarily commute - \([\hat{\Omega}_A, \hat{\Omega}_A] \neq 0\) etc. Thus for

\begin{align}
\hat{\Omega} &= \hat{\Omega}_A \otimes \hat{\Omega}_B + \hat{\Omega}_A^2 \otimes \hat{\Omega}_B^2 \tag{28}
\end{align}

the mean value will be given in *quantum theory* by

\begin{align}
\langle \hat{\Omega} \rangle &= \langle \hat{\Omega}_A \otimes \hat{\Omega}_B \rangle + \langle \hat{\Omega}_A^2 \otimes \hat{\Omega}_B^2 \rangle \\
&= Tr(\hat{\Omega}_A \otimes \hat{\Omega}_B)\hat{\rho} + Tr(\hat{\Omega}_A^2 \otimes \hat{\Omega}_B^2)\hat{\rho} \\
&= \sum_{\alpha_1, \beta_1} \alpha_1 \beta_1 P(\alpha_1, \beta_1 | \Omega_A, \Omega_B, c) + \sum_{\alpha_2, \beta_2} \alpha_2 \beta_2 P(\alpha_2, \beta_2 | \Omega_A^2, \Omega_B, c) \tag{29}
\end{align}

where

\begin{align}
P(\alpha_1, \beta_1 | \Omega_A, \Omega_B, c) &= Tr(\hat{\Pi}_{\alpha_1} \otimes \hat{\Pi}_{\beta_1})\hat{\rho} \quad P(\alpha_2, \beta_2 | \Omega_A^2, \Omega_B, c) = Tr(\hat{\Pi}_{\alpha_2} \otimes \hat{\Pi}_{\beta_2})\hat{\rho} \tag{30}
\end{align}

In *LHVT* the corresponding observable is

\begin{align}
\Omega &= \Omega_A \otimes \Omega_B + \Omega_A^2 \otimes \Omega_B^2 \tag{31}
\end{align}
and for *Bell local* states, the mean value of $\Omega$ is given by

$$
\langle \Omega \rangle = \langle \Omega_{A1} \otimes \Omega_{B1} \rangle + \langle \Omega_{A2} \otimes \Omega_{B2} \rangle
$$

$$
= \sum_{\lambda} P(\lambda|c) \langle \Omega_{A1}(\lambda) \rangle \langle \Omega_{B1}(\lambda) \rangle + \sum_{\lambda} P(\lambda|c) \langle \Omega_{A2}(\lambda) \rangle \langle \Omega_{B2}(\lambda) \rangle
$$

$$
= \sum_{\alpha_1\beta_1} \alpha_1\beta_1 P(\alpha_1,\beta_1|\Omega_{A1,B1},c) + \sum_{\alpha_2\beta_2} \alpha_2\beta_2 P(\alpha_2,\beta_2|\Omega_{A2,B2},c)
$$

(32)

where in LHVT

$$
P(\alpha_1,\beta_1|\Omega_{A1,B1},c) = \sum_{\lambda} P(\lambda|c) P(\alpha_1|\Omega_{A1},c,\lambda) P(\beta_1|\Omega_{B1},c,\lambda)
$$

$$
P(\alpha_2,\beta_2|\Omega_{A2,B2},c) = \sum_{\lambda} P(\lambda|c) P(\alpha_2|\Omega_{A2},c,\lambda) P(\beta_2|\Omega_{B2},c,\lambda)
$$

(33)

We will use these expressions (29) and (32) to interconvert between quantum theory and LHVT when the latter applies.

To determine these mean values *experimentally*, two sets of joint measurements for $\hat{\Omega}_{A1}, \hat{\Omega}_{B1}$ and then $\hat{\Omega}_{A2}, \hat{\Omega}_{B2}$ (or the classical observables $\Omega_{A1}, \Omega_{B1}$ and then $\Omega_{A2}, \Omega_{B2}$) would be required, unless a technique exists for measuring the outcomes for $\hat{\Omega}$ (or $\Omega$) directly.

### 3 Classes of Quantum States for Bipartite Systems

#### 3.1 Two Hierarchies of Bipartite Quantum States

There is various ways the quantum states for bipartite systems can be *categorised* and quantum states falling into a particular category in one scheme *may not* all end up in the same category in a different scheme. Jones et al [6] (as elaborated by Cavalcanti et al [7]), established a heirarchy of *bipartite quantum states* can be established based on *LHV models* for the joint probability $P(\alpha,\beta|\Omega_{A},\Omega_{B},c)$ for measurement of *any* pair of sub-system observables $\Omega_{A}$ and $\Omega_{B}$ to obtain *any* of their possible outcomes $\alpha$ and $\beta$ when the *preparation* process is $c$. However before considering this heirarchy we first identify a *classification* based purely on *quantum state models*.

#### 3.2 Separable and Entangled States

The quantum states for bipartite composite systems may be divided into *two classes* - the *separable* and the *entangled* states. We will refer to this scheme as the *Quantum Theory Classification Scheme* (QTCS).

The *separable* states are those whose preparation is described by the density operator

$$
\hat{\rho}_{sep} = \sum_{R} P_{R} \hat{\rho}_{R}^{A} \otimes \hat{\rho}_{R}^{B}
$$

(34)
where $\hat{\rho}_R^A$ and $\hat{\rho}_R^B$ are possible quantum states for sub-systems $A$ and $B$ respectively and $P_R$ is the probability that this particular pair of sub-system states is prepared. Each distinct pair is listed by $R$. This follows the preparation process for separable states described by Werner [11]. Such quantum states are of the same form as what Werner [11] referred to as uncorrelated states, but which nowadays would be referred to as separable or non-entangled states. The entangled states are simply the quantum states that are not separable.

A detailed discussion of the significance of separable and entangled states, and tests for distinguishing these is given in many articles and textbooks (see for example [2], [3]). Clearly a given quantum state is either separable or entangled - it cannot be both. Werner [11] referred to the entangled states as EPR correlated states, but nowadays these would be referred to as entangled states.

For the present we note that if the quantum state is separable then from [6] and [34] the joint probability $P(\alpha, \beta | \Omega_A, \Omega_B, c)$ is given by

$$P(\alpha, \beta | \Omega_A, \Omega_B, c) = \sum_R P_R Tr_A(\hat{\Pi}_\alpha^A \hat{\rho}_R^A) Tr_B(\hat{\Pi}_\beta^B \hat{\rho}_R^B)$$  \hspace{1cm} (35)

$$= \sum_R P_R P(\alpha | \Omega_A, c(A, R)) P(\beta | \Omega_B, c(B, R))$$  \hspace{1cm} (36)

where

$$P(\alpha | \Omega_A, c(A, R)) = Tr_A(\hat{\Pi}_\alpha^A \hat{\rho}_R^A)$$

$$P(\beta | \Omega_B, c(B, R)) = Tr_B(\hat{\Pi}_\beta^B \hat{\rho}_R^B)$$  \hspace{1cm} (37)

are the sub-system probabilities for outcomes $\alpha$, $\beta$ for measurements of observables $\Omega_A$, $\Omega_B$ when the sub-system preparations specify density operators as $c(A, R) \rightarrow \hat{\rho}_R^A$, $c(B, R) \rightarrow \hat{\rho}_R^B$.

Alternatively, if the joint probability is given by (35) for all observables and outcomes then

$$P(\alpha, \beta | \Omega_A, \Omega_B, c) = \sum_R P_R Tr(\hat{\Pi}_\alpha^A \otimes \hat{\Pi}_\beta^B)(\hat{\bar{\rho}}_R^A \otimes \hat{\bar{\rho}}_R^B) = Tr((\hat{\Pi}_\alpha^A \otimes \hat{\Pi}_\beta^B)\hat{\bar{\rho}})$$  \hspace{1cm} (38)

where $\hat{\bar{\rho}} = \sum_R P_R \hat{\rho}_R^A \otimes \hat{\rho}_R^B$ - so the state is separable. Thus the density operator definition and the joint probability expression for a separable state are equivalent.

### 3.3 Bell Local and Non-Local States

Based on LHVT the quantum states for bipartite composite systems may also be differently divided into two other classes - the Bell local and the Bell-non-local states. We will refer to this scheme as the Local Hidden Variable Theory Classification Scheme (LHVTCS). As we will see, there is no simple relationship between the entangled states on the one hand and the Bell non-local states on
The Bell local states are those for which the joint probability $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ is given by the LHV theory expression (14) as well as the quantum theory expression (6). In contrast, the Bell non-local states are those for which there is no LHV theory expression (14) for the joint probability - this is only given by the quantum theory expression (6). As explained in many textbooks and articles (see for example [2]) the Bell local states obey the Bell inequalities [4], and the existence of some quantum states (such as the two qubit Bell states [5]) for which the Bell inequalities are not obeyed and which was confirmed experimentally shows that Einstein’s hope that an underlying reality represented by a local hidden variable theory could underpin quantum theory is not realised.

Before looking at a further classes of quantum states defined in terms of LHV theory we first present an important result, namely that all separable states are Bell local. The formal similarity between the hidden variable theory expression for the joint probability (14) and the quantum expression (36) for a separable state is noticeable. We can then identify the probabilistic choice $R$ for the preparation of the particular pair of sub-system states $\hat{\rho}_R^A$ and $\hat{\rho}_R^B$ with a particular choice of hidden variables $\lambda$, thus $R \rightarrow \lambda$. Then the probability $P_R$ for this particular pair of sub-system states $\hat{\rho}_R^A$ and $\hat{\rho}_R^B$ can be identified with the hidden variable creation probability $P(\lambda|c)$, thus $P_R \rightarrow P(\lambda|c)$. Next, the probabilities $P(\alpha|\Omega_A, c(A, R))$ and $P(\beta|\Omega_B, c(B, R))$ for the single sub-system probabilities can be identified with the hidden variable probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$, thus $P(\alpha|\Omega_A, c(A, R)) \rightarrow P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda) \rightarrow P(\beta|\Omega_B, c, \lambda)$. With these identifications the joint probability $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ for a separable state (36) is of the general form for the joint probability $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ for a Bell local state (14). Hence the separable states are Bell local.

An immediate corollary of this result that all quantum separable states are Bell local is that all Bell non-local states are quantum entangled. After all, if the quantum state is Bell non-local then the LHV theory expression (14) for the joint probability does not apply - whereas it does apply if the quantum state is separable. Hence Bell non-local states cannot be separable and thus must be quantum entangled. Thus, these last results therefore provide a general relationship between the classes of quantum states based just on quantum theory and the classes based on local hidden variable theory. This is that all quantum separable states are Bell local and all Bell non-local states are quantum entangled. Note however that the converses are not true. As we will see, some Bell local states are not quantum separable, that is they are quantum entangled. Similarly, some quantum entangled states are not Bell non-local, that is they are Bell local. This last result was established by Werner [11].
3.4 Categories of Bell Local States

In regard to the quantum separable states we have from before that the single probabilities are given by quantum theory expressions

\[ P(\alpha | \Omega_A, c, \lambda) = \text{Tr}_A(\hat{\Pi}_A^A \hat{\rho}_R^A) = P_Q(\alpha | \Omega_A, c, \lambda) \]
\[ P(\beta | \Omega_B, c, \lambda) = \text{Tr}_B(\hat{\Pi}_B^B \hat{\rho}_R^B) = P_Q(\beta | \Omega_B, c, \lambda) \]  

(39)

where the subscript \( Q \) indicates that a quantum theory expression applies. Hence for separable states we have

\[ P(\alpha, \beta | \Omega_A, \Omega_B, c) = \sum_{\lambda} P_Q(\alpha | \Omega_A, c, \lambda) P_Q(\beta | \Omega_B, c, \lambda) P(\lambda | c) \]  

(40)

where the single probabilities are given by quantum theory expressions. The \( \hat{\rho}_R^A \) and \( \hat{\rho}_R^B \) thus specify local hidden states.

This situation for separable states suggests that the Bell local states for bipartite systems may be divided up into three classes depending on the number of single sub-system probabilities that are definitely described by quantum expressions involving the density operator \( \hat{\rho}^C(c, \lambda) \) for a local hidden state and a projector \( \hat{\Pi}_C^C \) associated with measurement outcome \( \omega \) for observable \( \hat{\Omega}_C \).

There are three possibilities - (1) Category 1 states where both \( P(\alpha | \Omega_A, c, \lambda) \) and \( P(\beta | \Omega_B, c, \lambda) \) are given by quantum expressions as in (39), (2) Category 2 states where only one is given by a quantum expression and (3) Category 3 states where neither is given by a quantum expression. The three classes or categories are mutually exclusive - a given Bell local state can only be in one of the three classes. We now introduce a different notation in which (as in (39)) the presence of the sub-script \( Q \) on a sub-system LHV probability indicates that it can be obtained from a quantum expression involving a sub-system density operator, and the absence of the sub-script \( Q \) indicates that it cannot be obtained from a quantum expression. Note that our notation differs from that in Refs. [5], [6], [7] where the \( P(\alpha | \Omega_A, c, \lambda) \) could be either \( P(\alpha | \Omega_A, c, \lambda) \) (non-quantum) or \( P_Q(\alpha | \Omega_A, c, \lambda) \) (quantum) in our notation. Hence in the present notation the joint probabilities for the Bell local states in Categories 1, 2 and 3 are given by

\[ P(\alpha, \beta | \Omega_A, \Omega_B, c) = \sum_{\lambda} P_Q(\alpha | \Omega_A, c, \lambda) P_Q(\beta | \Omega_B, c, \lambda) P(\lambda | c) \quad \text{Cat} \ 1 \]  

(41)

\[ P(\alpha, \beta | \Omega_A, \Omega_B, c) = \sum_{\lambda} P(\alpha | \Omega_A, c, \lambda) P_Q(\beta | \Omega_B, c, \lambda) P(\lambda | c) \quad \text{Cat} \ 2 \]  

(42)

\[ P(\alpha, \beta | \Omega_A, \Omega_B, c) = \sum_{\lambda} P(\alpha | \Omega_A, c, \lambda) P(\beta | \Omega_B, c, \lambda) P(\lambda | c) \quad \text{Cat} \ 3 \]  

(43)

When a quantum expression applies

\[ P_Q(\alpha | \Omega_A, c, \lambda) = \text{Tr}_A(\hat{\Pi}_A^A \hat{\rho}^A(c, \lambda)) \]
\[ P_Q(\beta | \Omega_B, c, \lambda) = \text{Tr}_B(\hat{\Pi}_B^B \hat{\rho}^B(c, \lambda)) \]  

(44)

where \( \hat{\rho}^A(c, \lambda) \) and \( \hat{\rho}^B(c, \lambda) \) are the sub-system density operators for the local hidden states associated with hidden variables \( \lambda \) for preparation \( c \). By convention in Category 2 we choose \( B \) to be the sub-system where the single probability is given by a quantum expression.
We also list as Category 4 states those for which the joint probability is not given by any of Eqs. (41), (42) and (43).

\[ P(\alpha, \beta | \Omega_A, \Omega_B, c) \neq \text{Eqs. (41), (42) or (43)} \]

Category 4 (45)

For these states the joint probability is only given by the quantum theory expression (6). The Category 4 states are of course the Bell non-local states, and such states do occur. If Einstein’s realist approach applied there would be no Category 4 states.

Clearly, all separable states are Category 1 states, and all Category 1 states are separable, as may be shown by substituting (6) for \( P(\alpha, \beta | \Omega_A, \Omega_B, c) \) for the single sub-system probabilities and \( P_B \) for \( P(\lambda | c) \). The Category 1 states may also be just referred to as separable states. However, Category 2, Category 3 and Category 4 states must be quantum entangled states.

In Refs. [5], [6] and [7] the definition of a local hidden state (LHS) model for sub-system B is introduced. These are states in which the sub-system measurement probability \( p(\beta | \Omega_B, c, \lambda) \) for sub-system B is given by a quantum expression - \( P_Q(\beta | \Omega_B, c, \lambda) \) in our notation, but the sub-system measurement \( p(\alpha | \Omega_A, c, \lambda) \) for sub-system A may either be given by a quantum expression or it may not be. Thus we see that the LHS model for sub-system B applies to both Category 1 and Category 2 states, since in these cases there is a local hidden state \( \hat{\rho}^B(c, \lambda) \) involved. However, the LHS model for sub-system B does not apply to either Category 3 or Category 4 states. In Refs. [5], [6] and [7] the relevant expressions (in their notation) that are used to define LHS states for sub-system B are Eqs. (6), (3.6) and (18) respectively.

The feature of EPR steering of sub-system B from sub-system A is fully discussed in these three papers, and requires the failure of the LHS model for sub-system B. This means that there must be no local hidden state \( \hat{\rho}^B(c, \lambda) \) for sub-system B. For such states the sub-system B said to be non-steerable from sub-system A. For completeness, a brief presentation of the physical argument involved based on a consideration of states that are conditional on the outcomes of measurements on sub-system A, is set out in Appendix 8. Hence Category 1 and Category 2 states are non-steerable, whereas Category 3 and Category 4 states are steerable since no local hidden state for sub-system B is involved. The Category 3 states, which are Bell local, entangled, non LHS and steerable are sometimes referred to as EPR entangled states. Thus, based on their distinction via the number of sub-systems associated with a local hidden state, the four different categories of bipartite states have differing features in regard to EPR steering.

As we have now seen, the Bell local states for bipartite systems can be divided up into three non-overlapping subsets, each of which has different features for the sub-system LHV probabilities \( P(\alpha | \Omega_A, c, \lambda) \) and \( P(\beta | \Omega_B, c, \lambda) \). This distinctiveness between the sub-sets is of particular convenience when we consider tests for various categories of states. However, it should again be emphasised that other researchers ([5], [6] and [7]) have used a hierarchy of non disjoint
sub-sets. This is because in certain of their definitions the sub-system probabilities can be either given by quantum or non-quantum expressions. In their scheme the sub-sets overlap, with each set being a sub-set of a larger set. In their scheme Category 1 states (the separable states) would be a sub-set of a set (the LHS states) consisting of Category 1 and Category 2 states. In their scheme the Category 1 and Category 2 states would be combined and be a sub-set of a combined set (the Bell local states) consisting of Category 1, Category 2 and Category 3 states. It is important to note that our scheme and that in Refs. [5], [6] and [7] are not the same though they are related, and this needs to be taken into account when discussing tests. The overall scheme used here is shown in Figure 1, where the features for all the different sets of states for bipartite composite systems are set out.

The states introduced by Werner [11] provide examples of the three categories of Bell local states and of the Bell non-local states. These are certain $U \otimes U$ invariant states $((\hat{U} \otimes \hat{U}) \hat{\rho}_W (\hat{U}^\dagger \otimes \hat{U}^\dagger)) = \hat{\rho}_W$, where $\hat{U}$ is any unitary
operator) for two $d$ dimensional sub-systems. Depending on the parameter $\eta$ (or $\phi$) the Werner states (see Eq. (146)) may be separable or entangled. They may also be Bell local and in one of the three categories described above, or they may be Bell non-local. For completeness the Werner states are described in Appendix 7.
4 Bloch Vector, Spin and Two Mode Quadrature Squeezing, Spin Variance Sum and Correlation Tests for EPR Steering

4.1 General Considerations

In a number of papers (see the review papers [2], [3] and references therein) various tests for quantum entanglement have been formulated, recently in the particular context of bipartite systems of identical massive bosons [1]. These include spin and two mode quadrature squeezing, Bloch vector and correlation tests. An important issue then is: Are these tests also valid for detecting EPR steering or do some of them fail? Of course any test that detects EPR steering must of necessity also detect entanglement, but a test that demonstrates entanglement does not necessarily demonstrate EPR steering. In this situation we are looking for conditions where LHS model for sub-system $B$ also fails - or in other words, the quantum state does not have a joint measurement probability as in Eqs. (41) and (42) for Category 1 or Category 2 states. As the tests for quantum entanglement previously obtained have already found the conditions under which Category 1 probabilities fail, we then know that the quantum state must be in Category 2, Category 3 or Category 4. If we can then show that it is not in Category 2 because the joint measurement probability (42) also fails, then the state must be in Category 3 or Category 4 - in other words it is an EPR steerable state. We would then have found a test for EPR steering. Note that for the Category 2 states the sub-system $A$ probabilities $P(\alpha|\Omega_A,c,\lambda)$ in LHVT are not given by a quantum expression involving a sub-system density operator. This feature must be taken into account when considering the tests for EPR steering. However, the issue of how to treat mean values and variances in the context of LHV in general and the LHS model in particular requires some consideration, so we have set this out in Appendix 6.

Note however that a test that demonstrates EPR steering only shows that the quantum state is either Category 3 or Category 4, both of which are entangled states. To demonstrate Bell non-locality (Category 4 states) will require different tests - notably those involving violations of a Bell inequality. This will be the subject of a later paper.

In the present paper, as in previous work in Refs. [1], [2], [3], we focus on tests for bipartite systems involving identical massive bosons. Consequently, when quantum states either for the overall system or for a sub-system are involved these must comply with the symmetrization principle and super-selection rules involving the total boson number for either the overall system or for the sub-system. In particular, for Category 2 states (as well as Category 1 states) the local hidden state $\hat{\rho}^B(c,\lambda)$ for the sub-system $B$ that is treated quantum mechanically must have zero coherences between Fock states with differing sub-system boson number $N_B$. The LHS is still a possible quantum state for sub-system $B$. The issue of super-selection rules is discussed fully in [2].

Also, as in these papers both the overall system and the two sub-systems
will be specified in terms of modes (or single particle states that the particles may occupy) based on a second quantization treatment, rather than in terms of labeled identical particles - as might be thought appropriate in a first quantization method. Cases with differing numbers of particles are just different states of the (multi) modal system, not different systems, as in first quantization.

In addition, since the mean values of various observables are involved in the tests for showing the state is not Category 2, we can use Eqs. (27) and (26) to replace LHVT theory expressions by quantum theory expressions at suitable stages in the derivations - both when a sub-system B LHS $\hat{\rho}_B(c,\lambda)$ occurs or when we wish to evaluate the mean value of a sub-system A observable $\Omega_A$ allowing for all values of the hidden variables $\lambda$. However, there will be situations for Category 2 states where we need to consider the mean value of a sub-system A observable $\Omega_A$ when the hidden variables have particular values. In this case some general properties of classical probabilities $P(\alpha|\Omega_A,c,\lambda)$ are useful. One is that the mean of the square of a real observable is never less than the square of the mean for the observable.

$$\langle \Omega_A^2(c,\lambda) \rangle \geq (\langle \Omega_A(c,\lambda) \rangle)^2$$

(46)

Another is a Cauchy inequality

$$\sum_\lambda C(\lambda) P(\lambda|c) \geq \left( \sum_\lambda \sqrt{C(\lambda)} P(\lambda|c) \right)^2.$$  

(47)

for $C(\lambda) \geq 0$, such as the case $C(\lambda) = \langle \Omega_A^2(c,\lambda) \rangle$. The proof of the first is elementary, the second is proved in [2].

Finally, since LHVT deals with physical quantities that are classical observables we need to express various non-Hermitian quantum mechanical operators that we need to consider - such as mode annihilation and creation operators - in terms of quantum operators that are Hermitian. Any non-Hermitian operator $\hat{\Omega}$ can always be expressed in terms of Hermitian operators $\hat{\Omega}_1$ and $\hat{\Omega}_2$ as $\hat{\Omega} = \hat{\Omega}_1 + i \hat{\Omega}_2$ and the latter operators would be equivalent to classical observables $\Omega_1$ and $\Omega_2$, so the corresponding classical observable will be $\Omega = \Omega_1 + i \Omega_2$. The mean value $\langle \hat{\Omega} \rangle$ will then be equal to $\langle \hat{\Omega}_1 \rangle + i \langle \hat{\Omega}_2 \rangle$. Note that two independent sets of measurements for the generally incompatible $\hat{\Omega}_1$ and $\hat{\Omega}_2$ would be needed to separately determine $\langle \hat{\Omega}_1 \rangle$ and $\langle \hat{\Omega}_2 \rangle$. For the corresponding classical observable we take $\langle \Omega \rangle = \langle \Omega_1 \rangle + i \langle \Omega_2 \rangle$ - see Eq. (143) in Appendix 6.

In the next SubSection we introduce pairs of quadrature operators to replace the annihilation and creation operators for each mode. As we will see, we also need new auxiliary Hermitian operators as well, which are sums of products of quadrature operators and these will also be associated with classical observables in the LHVT. All the physical observables that we need to consider have quantum operators that can be written as linear combinations of products $\hat{\Omega}_A \otimes \hat{\Omega}_B$, where both $\hat{\Omega}_A$ and $\hat{\Omega}_B$ are Hermitian - including cases where $\hat{\Omega}_A = \hat{1}_A$ or $\hat{\Omega}_B = \hat{1}_B$. Such products can then be replaced by $\Omega_A \otimes \Omega_B$, where $\Omega_A$ and $\Omega_B$
are the corresponding classical observables. Using this procedure both quantum and hidden variable theory expressions can be used for the joint measurement probabilities and mean values.

4.2 Quadrature Amplitudes

In the case of quantum mode annihilation or creation operators the corresponding Hermitian components are the quadrature operators for each mode. In quantum theory these are given by the Hermitian operators

\[
\hat{x}_A = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger) \quad \hat{p}_A = \frac{1}{\sqrt{2}i}(\hat{a} - \hat{a}^\dagger)
\]

\[
\hat{x}_B = \frac{1}{\sqrt{2}}(\hat{b} + \hat{b}^\dagger) \quad \hat{p}_B = \frac{1}{\sqrt{2}i}(\hat{b} - \hat{b}^\dagger)
\]

(48)

which have the same commutation rules as the position and momentum operators for distinguishable particles in units where \(\hbar = 1\). Thus \([\hat{x}_A, \hat{p}_A] = [\hat{x}_B, \hat{p}_B] = i\) as for cases where \(A, B\) were distinguishable particles. By scaling these operators with a length scale given by the Compton wavelength \(\lambda_C = \hbar/mc\) and a momentum scale given by the Compton momentum \(p_C = mc\), so that with \(\hat{X}_{A,B} = \lambda_C\hat{x}_{A,B}\) and \(\hat{P}_{A,B} = p_C\hat{p}_{A,B}\) we then have \([\hat{X}_A, \hat{P}_A] = [\hat{X}_B, \hat{P}_B] = i\hbar\), just as for normal quantum position and momentum operators. It is then reasonable to assume that there are equivalent classical observables \(x_A, p_A, x_B, p_B\) and that their measurement outcomes would be real numbers, and further more for sub-systems not being treated quantum mechanically (such as sub-system \(A\) in the context of the LHS model) these outcomes can actually be measured in experiment and probabilities and mean values such as \(P(\alpha|\Omega_A, c, \lambda)\) and \(\langle \Omega_A(\lambda) \rangle\) can be assigned as in a hidden variable treatment of sub-system \(A\). In considering Category 2 states the probabilities and mean values such as \(P(\beta|\Omega_B, c, \lambda)\) and \(\langle \Omega_B(\lambda) \rangle\) for the sub-system \(B\) are also given by quantum expressions involving sub-system density operators \(\rho^B(\lambda)\).

We can write the mode annihilation and creation operators in terms of the quadrature operators via

\[
\hat{a} = \frac{1}{\sqrt{2}}(\hat{x}_A + i\hat{p}_A) \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x}_A - i\hat{p}_A)
\]

\[
\hat{b} = \frac{1}{\sqrt{2}}(\hat{x}_B + i\hat{p}_B) \quad \hat{b}^\dagger = \frac{1}{\sqrt{2}}(\hat{x}_B - i\hat{p}_B)
\]

(49)

which can also be written in terms of the scaled quadrature amplitudes.

We then find that the spin operators (defined as \(\hat{S}_x = (\hat{b}^\dagger\hat{a} + \hat{a}^\dagger\hat{b})/2, \hat{S}_y = (\hat{b}^\dagger\hat{a} - \hat{a}^\dagger\hat{b})/2i, \hat{S}_z = (\hat{b}\hat{b} - \hat{a}\hat{a})/2\)) and the number operators (defined as \(\hat{N} = \hat{N}_A + \hat{N}_B\) with \(\hat{N}_A = \hat{a}^\dagger\hat{a}, \hat{N}_B = \hat{b}^\dagger\hat{b}\), the separate mode number operators - note that \(\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \hat{N}/2\) (for \(\hat{N}/2 + 1\)), can be expressed in terms of the quadrature
operators as

\[ \hat{S}_x = \frac{1}{2}(\hat{x}_A\hat{x}_B + \hat{p}_A\hat{p}_B) \quad \hat{S}_y = \frac{1}{2}(\hat{p}_A\hat{x}_B - \hat{x}_A\hat{p}_B) \]
\[ \hat{S}_z = \frac{1}{4}(\hat{x}_B^2 - \hat{x}_A^2 + \hat{p}_B^2 - \hat{p}_A^2) - \frac{1}{2}\hat{V}_B + \frac{1}{2}\hat{V}_A \]
\[ \hat{N} = \frac{1}{2}(\hat{x}_B^2 + \hat{x}_A^2 + \hat{p}_B^2 + \hat{p}_A^2) - \hat{V}_B - \hat{V}_A \]

(50)

which are all linear combinations of products of two quadrature operators. Here we have introduced the auxiliary Hermitian operators

\[ \hat{V}_A = \frac{1}{2i}(\hat{x}_A\hat{p}_A - \hat{p}_A\hat{x}_A) = \frac{1}{2}\hat{1}_A \]
\[ \hat{V}_B = \frac{1}{2i}(\hat{x}_B\hat{p}_B - \hat{p}_B\hat{x}_B) = \frac{1}{2}\hat{1}_B \]

(51)

In terms of the quadrature and auxiliary operators the mode number and mode number difference operators (defined as \( \hat{N}_- = \hat{N}_B - \hat{N}_A \)) are

\[ \hat{N}_A = \frac{1}{2}(\hat{x}_A^2 + \hat{p}_A^2) - \hat{V}_A \quad \hat{N}_B = \frac{1}{2}(\hat{x}_B^2 + \hat{p}_B^2) - \hat{V}_B \]
\[ \hat{N}_- = \frac{1}{2}(\hat{x}_B^2 + \hat{p}_B^2 - \hat{x}_A^2 - \hat{p}_A^2) - \hat{V}_B + \hat{V}_A \]

(52)

(53)

so that \( \hat{N}_- = 2\hat{S}_z \) as expected.

We also introduce two further distinct auxiliary Hermitian combinations of the quadrature operators for each mode

\[ \hat{U}_A = \frac{1}{2}(\hat{x}_A\hat{p}_A + \hat{p}_A\hat{x}_A) = \frac{1}{2i}((\hat{a})^2 - (\hat{a}^\dagger)^2) \]
\[ \hat{U}_B = \frac{1}{2}(\hat{x}_B\hat{p}_B + \hat{p}_B\hat{x}_B) = \frac{1}{2i}((\hat{b})^2 - (\hat{b}^\dagger)^2) \]

(54)

where using the commutation rules the operators \( \hat{U}_A \) and \( \hat{U}_B \) can also be expressed in terms of mode annihilation and creation operators.

The \( \hat{U}_A \) and \( \hat{U}_B \) operators appear in the expressions for \( \hat{S}_x^2 \), \( \hat{S}_y^2 \) and \( \hat{S}_z^2 \). We find that for \( \hat{S}_x^2 \)

\[ \hat{S}_x^2 = \frac{1}{4}(\hat{x}_A^2\hat{x}_B^2 + \hat{p}_A^2\hat{p}_B^2) + \frac{1}{2}(\hat{U}_A\hat{U}_B - \hat{V}_A\hat{V}_B) \]

(55)

and in the case of \( \hat{S}_y^2 \) we get

\[ \hat{S}_y^2 = \frac{1}{4}(\hat{p}_A^2\hat{x}_B^2 + \hat{x}_A^2\hat{p}_B^2) - \frac{1}{2}(\hat{U}_A\hat{U}_B + \hat{V}_A\hat{V}_B) \]

(56)

The spin operators thus involve the quadrature operators for both modes.
In addition to the spin operators we can also define two mode quadrature operators in terms of the quadrature operators for both modes [3]. These depend on a phase parameter $\theta$. There are two sets given by

\[
\hat{X}_\theta(\pm) = \frac{1}{2} \left( \hat{a} e^{-i\theta} \pm \hat{b} e^{+i\theta} + \hat{a}^\dagger e^{+i\theta} \pm \hat{b}^\dagger e^{-i\theta} \right)
\]

\[
\hat{P}_\theta(\pm) = \frac{1}{2i} \left( \hat{a} e^{-i\theta} \mp \hat{b} e^{+i\theta} - \hat{a}^\dagger e^{+i\theta} \mp \hat{b}^\dagger e^{-i\theta} \right)
\]

It is easy to see that $\hat{P}_\theta(\pm) = \hat{X}_{\theta+\pi/2}(\pm)$ and that $[\hat{X}_\theta(\pm), \hat{P}_\theta(\pm)] = [\hat{X}_\theta(-\pm), \hat{P}_\theta(-\pm)] = i$. The Heisenberg uncertainty principle is given by $\langle \Delta X^2_\theta(\pm) \rangle \langle \Delta P^2_\theta(\pm) \rangle \geq 1/4$ and a state is two mode quadrature squeezed if one of $\langle \Delta X^2_\theta(\pm) \rangle$ or $\langle \Delta P^2_\theta(\pm) \rangle$ is less than $1/2$. In Reference [3] we showed that two mode quadrature squeezing was a sufficiency test for entanglement. We can also write the two mode quadrature operators in terms of the single mode quadrature operators as

\[
\hat{X}_\theta(\pm) = \frac{1}{\sqrt{2}} \left( \hat{x}_A \cos \theta + \hat{\rho}_A \sin \theta \pm \hat{x}_B \cos \theta \pm \hat{\rho}_B \sin \theta \right)
\]

\[
\hat{P}_\theta(\pm) = \frac{1}{\sqrt{2}i} \left( -\hat{x}_A \sin \theta + \hat{\rho}_A \cos \theta \mp \hat{x}_B \sin \theta \pm \hat{\rho}_B \cos \theta \right)
\]

The square of the two mode quadrature operators $\hat{X}_\theta(\pm)$ are given by

\[
\hat{X}_\theta(\pm)^2 = \frac{1}{2} \left\{ \hat{x}^2_A \cos^2 \theta + \hat{\rho}^2_A \sin^2 \theta + 2\hat{\rho}_A \sin \theta \cos \theta \right\}
\]

\[
+ \frac{1}{2} \left\{ \hat{x}^2_B \cos^2 \theta + \hat{\rho}^2_B \sin^2 \theta + 2\hat{\rho}_B \sin \theta \cos \theta \right\}
\]

\[
\pm \left\{ \hat{x}_A \hat{x}_B \cos^2 \theta + \hat{\rho}_A \hat{\rho}_B \sin^2 \theta + \hat{\rho}_A \hat{\rho}_B \sin \theta \cos \theta \pm \hat{\rho}_A \hat{x}_B \sin \theta \cos \theta \right\}
\]

The expression for $\hat{P}_\theta(\pm)^2$ can be obtained using $\hat{P}_\theta(\pm) = \hat{X}_{\theta+\pi/2}(\pm)$. The fundamental quantum Hermitian operators $\hat{x}_A, \hat{\rho}_A, \hat{x}_B, \hat{\rho}_B$ for the two mode system plus the auxiliary Hermitian operators $\hat{U}_A, \hat{V}_A, \hat{U}_B, \hat{V}_B$ all correspond to physical quantities that could be measured, with real eigenvalues as the outcomes. In the local hidden variable theory these quantities correspond to classical observables $x_A, p_A, x_B, p_B$ and $U_A, V_A, U_B, V_B$, for which single observable hidden variable probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\beta|\Omega_B, c, \lambda)$ apply - from which joint probabilities $P(\alpha, \beta|\Omega_A, \Omega_B, c)$ can be obtained via [14]. The physical observables involved in the tests such as the spin operators, their squares and the number operators can all be expressed in terms of the quadrature and auxiliary operators as sums of products of the form $\hat{\Omega}_A \otimes \hat{\Omega}_B$. For the local hidden variable theory treatment the corresponding classical observable will be the same as the quantum expressions, but now with the quantum Hermitian operators replaced by the corresponding classical observable. Thus we now have
for the classical spin components $S_x$, $S_y$ and $S_z$ and the number observable $N$

$$
S_x = \frac{1}{2}(x_A x_B + p_A p_B) \quad S_y = \frac{1}{2}(p_A x_B - x_A p_B)
$$

$$
S_z = \frac{1}{4}(x_B^2 - x_A^2 + p_B^2 - p_A^2) - \frac{1}{2}V_B + \frac{1}{2}V_A
$$

$$
N = \frac{1}{2}(x_B^2 + x_A^2 + p_B^2 + p_A^2) - V_B - V_A
$$

(60)

Also for the sub-system particle numbers and their difference

$$
N_A = \frac{1}{2}(x_A^2 + p_A^2) - V_A \quad N_B = \frac{1}{2}(x_B^2 + p_B^2) - V_B
$$

$$
N_+ = \frac{1}{2}(x_B^2 + p_B^2 - x_A^2 - p_A^2) - V_B + V_A
$$

(61)

The two mode quadrature observables are

$$
X_\theta(\pm) = \frac{1}{\sqrt{2}}(x_A \cos \theta + p_A \sin \theta \pm x_B \cos \theta \pm p_B \sin \theta)
$$

$$
P_\theta(\pm) = \frac{1}{\sqrt{2}}(-x_A \sin \theta + p_A \cos \theta \mp x_B \sin \theta \mp p_B \cos \theta)
$$

(62)

The related LHVT expressions for $S_x^2$ and $S_y^2$ are obvious. The reverse process for the replacement of the classical observables $x_A, x_B, p_A, p_B$ by $\hat{x}_A, \hat{x}_B, \hat{p}_A, \hat{p}_B$ and $U_A, U_B, V_A, V_B$ by $\hat{U}_A, \hat{U}_B, \hat{V}_A, \hat{V}_B$ requires using (48), (54) and (51) to give the correct quantum Hermitian operators. Carrying out this replacement in the classical spin components $S_x$, $S_y$ and $S_z$ and the number observable $N$ also gives the correct quantum operators, as also occurs for the squares of these observables as well. Once again we emphasise that we only need single measurement LHVT probabilities $P(\alpha|\Omega_A, c, \lambda)$ with $\Omega_A = x_A, p_A, U_A$ or $V_A$ and $P(\beta|\Omega_B, c, \lambda)$ with $\Omega_B = x_B, p_B, U_B$ or $V_B$ to treat the classical observables such as $S_x$, $S_y$ and $S_z$ and $N$ or $X_\theta(\pm), P_\theta(\pm)$ via hidden variable theory.

The local hidden variable theory for these new observables is defined by several independent single measurement probability functions. For $x_A, p_A, U_A$ and $V_A$ these are $P(\alpha_A|x_A, c, \lambda), P(\beta_A|p_A, c, \lambda), P(\xi_A|U_A, c, \lambda)$ and $P(\eta_A|V_A, c, \lambda)$, with analogous probabilities for $x_B, p_B, U_B$ and $V_B$.

### 4.3 Spin Operators: Means and Variances - Category 2 States

#### 4.3.1 Mean Values of Spin Components $S_x$ and $S_y$ - Category 2 States

We now consider the mean value for spin components for the Category 2 states. For example in the case of the spin component $S_z$
\[
\langle S_x \rangle = \sum_\lambda P(\lambda|c) \langle S_x(\lambda) \rangle \\
= \frac{1}{2} \left( \sum_\lambda P(\lambda|c)((x_A(\lambda)) Q + \langle p_A(\lambda) \rangle Q + \langle p_B(\lambda) \rangle Q) \right)
\]

(63)

using (60) and (20). This expression involves the hidden variable mean values for the (classical) observables \(x_A\) and \(p_A\) of sub-system \(A\) and the local hidden state mean values for the quantum quadrature operators \(\hat{x}_B\) and \(\hat{p}_B\). The latter must also correspond to quantum mean values, for a physically realisable quantum state for sub-system \(B\). Thus \(\langle x_B(\lambda) \rangle_Q = Tr(\hat{x}_B \hat{\rho}_B(\lambda))\) and \(\langle p_B(\lambda) \rangle_Q = Tr(\hat{p}_B \hat{\rho}_B(\lambda))\). Since sub-system \(B\) is to be treated quantum mechanically then the density operator \(\hat{\rho}_B(\lambda)\) would be required to both satisfy the symmetrisation principle and be local particle number SSR compliant. Hence there is a constraint based on the local hidden state \(\hat{\rho}_B(\lambda)\) being a possible state for sub-system \(B\) that requires the state to be local particle number SSR compliant.

In this case then since both \(\hat{x}_B\) and \(\hat{p}_B\) are just linear combinations of \(\hat{b}\) and \(\hat{b}^\dagger\) we have

\[
\langle x_B(\lambda) \rangle_Q = Tr\left( \frac{1}{\sqrt{2}} (\hat{b} + \hat{b}^\dagger) \hat{\rho}_B(\lambda) \right) = 0
\]

(64)

\[
\langle p_B(\lambda) \rangle_Q = Tr\left( \frac{1}{\sqrt{2}i} (\hat{b} - \hat{b}^\dagger) \hat{\rho}_B(\lambda) \right) = 0
\]

(65)

and thus

\[
\langle S_x(\lambda) \rangle = 0 \quad \langle S_y(\lambda) \rangle = 0
\]

(66)

Hence reverting to quantum operators using (26) we have

\[
\langle \hat{S}_x \rangle = 0 \quad \langle \hat{S}_y \rangle = 0
\]

(67)

These two results are the same as for a quantum separable (Category 1) state. We do not need to know the outcome for \(\langle x_A(\lambda) \rangle\) or \(\langle p_A(\lambda) \rangle\).

### 4.3.2 Mean Values of Spin Component \(S_z\) and Number \(N\) - Category 2 States

For the other spin component \(S_z\) we find using (60) and (20) that for the Category 2 states

\[
\langle S_z \rangle = \frac{1}{4} \sum_\lambda P(\lambda|c)((x_B^2(\lambda)) Q + \langle p_B^2(\lambda) \rangle_Q - \langle x_A^2(\lambda) \rangle - \langle p_A^2(\lambda) \rangle)
\]

\[- \frac{1}{2} \sum_\lambda P(\lambda|c)((V_B(\lambda)) Q - (V_A(\lambda)))
\]

(68)
Note the presence of terms involving $V_B$ and $V_A$. As in the quantum separable state case $\langle S_z \rangle$ is not necessarily zero.

For the number observable $N$ we find using (60) and (20) that for the Category 2 states

\[ \langle N \rangle = \frac{1}{2} \sum_\lambda P(\lambda|c)\left(\langle x_B^2(\lambda) \rangle_Q + \langle p_B^2(\lambda) \rangle_Q + \langle x_A^2(\lambda) \rangle + \langle p_A^2(\lambda) \rangle \right) \]
\[ - \sum_\lambda P(\lambda|c)(\langle V_B(\lambda) \rangle_Q + \langle V_A(\lambda) \rangle) \]  

(69)

Note the presence of terms involving $V_B$ and $V_A$. We will return to these results when we consider the spin squeezing and spin variance tests.

4.3.3 Variances of Spin Components $S_x$ and $S_y$ - Category 2 States

As $\langle S_x(\lambda) \rangle = \langle S_y(\lambda) \rangle = 0$ from (65) we see that $\langle \Delta S_x^2(\lambda) \rangle = \langle S_x^2(\lambda) \rangle$ and $\langle \Delta S_y^2(\lambda) \rangle = \langle S_y^2(\lambda) \rangle$. Using (26), the LHVT expression for $S_x^2$ obtained from the classical form of (55) and after applying the inequality (142) we then have the following inequalities for Category 2 states

\[ \langle \Delta S_x^2 \rangle \geq \sum_\lambda P(\lambda|c) \left( \frac{1}{4} \langle x_B^2(\lambda) \rangle_Q + \langle p_B^2(\lambda) \rangle_Q \right) \]
\[ \langle \Delta S_y^2 \rangle \geq \sum_\lambda P(\lambda|c) \left( \frac{1}{4} \langle p_A^2(\lambda) \rangle_Q + \langle x_A^2(\lambda) \rangle_Q \right) \]  

(70)

4.3.4 Evaluation of Expressions Needed - Category 2 States

To consider spin squeezing, spin variance and correlation tests for EPR steering based on the Category 2 states we will need to consider the following additional quantum theory based expressions: $\langle x_B^2(\lambda) \rangle_Q$, $\langle p_B^2(\lambda) \rangle_Q$, $\langle V_B(\lambda) \rangle_Q$, $\langle U_B(\lambda) \rangle_Q$ and the following non-quantum expressions $\langle x_A^2(\lambda) \rangle$, $\langle p_A^2(\lambda) \rangle$, $\langle V_A(\lambda) \rangle$.

Starting with the quantum theory expressions (48) we find that

\[ \langle x_B^2(\lambda) \rangle_Q = \text{Tr}(\hat{b}\hat{b}^\dagger)\rho_B(\lambda) + \frac{1}{2} \]
\[ = \langle N_B(\lambda) \rangle_Q + \frac{1}{2} \]  

(71)

\[ \langle p_B^2(\lambda) \rangle_Q = \langle N_B(\lambda) \rangle_Q + \frac{1}{2} \]  

(72)
where the commutation rules have been used and the SSR constraints eliminate the $Tr((\hat{b})^2\hat{\rho}^B(\lambda))$ and $Tr((\hat{b})^2\hat{\rho}^B(\lambda))$ terms. Note that $\langle N_B(\lambda) \rangle_Q \geq 0$.

Then using (54) we find that

$$
\langle U_B(\lambda) \rangle_Q = \frac{1}{2i}Tr((\hat{b})^2 - (\hat{b})^2)\hat{\rho}^B(\lambda)
= 0
$$

again due to the SSR constraints on the hidden state $\hat{\rho}^B(\lambda)$.

Also, using (51)

$$
\langle V_B(\lambda) \rangle_Q = \frac{1}{2}Tr_B(\hat{1}_B\hat{\rho}^B(\lambda))
= \frac{1}{2}
$$

since the trace of a density operator is unity. Using (71), (72) and (74) we confirm the result that $\langle N_B(\lambda) \rangle_Q \geq 0$ (73)

Note the analogous result for sub-system $A$.

Using the results (71), (72), (73) and (74) and (75) we have using (61)

$$
\langle x^2_A(\lambda) \rangle + \langle p^2_A(\lambda) \rangle = 2 \langle N_A(\lambda) \rangle + 2 \langle V_A(\lambda) \rangle
$$

Note the analogous result for sub-system $B$.

Using the results (71), (72), (73) and (74) and (75) we now have for Category 2 states

$$
\langle \Delta S^2 \rangle \\
\geq \sum_\lambda P(\lambda|c) \left( \frac{1}{2}(\langle N_B(\lambda) \rangle_Q + \frac{1}{2}(\langle N_A(\lambda) \rangle + \langle V_A(\lambda) \rangle) - \frac{1}{4} \langle V_A(\lambda) \rangle \right)
\geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{4} \langle \hat{V}_A \otimes \hat{N}_B \rangle + \frac{1}{4} \langle \hat{V}_A \otimes \hat{I}_B \rangle + \frac{1}{4} \langle \hat{V}_A \otimes \hat{I}_B \rangle
\geq \frac{1}{2} \langle \hat{V}_A \otimes \hat{I}_B \rangle + \frac{1}{4} \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{4} \langle \hat{N}_A \otimes \hat{I}_B \rangle
\geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{4} \langle \hat{I}_A \otimes \hat{N}_B \rangle + \frac{1}{4} \langle \hat{N}_A \otimes \hat{I}_B \rangle \\
$$

where we have used (51) and the LHVT and quantum theory interconversions (26) and (27) for the Bell local Category 2 states. These inequalities are the same as those for Category 1 states (see (3)). Note that the SSR have been used in deriving these last results.
Also from \[68\]

\[
\langle S_z \rangle = \frac{1}{4} \sum_\lambda P(\lambda|c) \left( 2 \langle N_B(\lambda) \rangle_Q + 1 - 2 \langle N_A(\lambda) \rangle - 2 \langle V_A(\lambda) \rangle \right)
\]

\[\frac{1}{2} \sum_\lambda P(\lambda|c) \left( \frac{1}{2} - \langle V_A(\lambda) \rangle \right)\]

\[
= \frac{1}{2} \langle \hat{T}_A \otimes \hat{N}_B \rangle - \frac{1}{2} \langle \hat{N}_A \otimes \hat{T}_B \rangle
\]

\[\frac{1}{2} \langle S_z \rangle \leq \frac{1}{4} \langle \hat{T}_A \otimes \hat{N}_B \rangle + \frac{1}{4} \langle \hat{N}_A \otimes \hat{T}_B \rangle
\]

The last line follows from both \(\langle \hat{T}_A \otimes \hat{N}_B \rangle = \sum_{n_A n_B} n_B \rho_{n_A n_B : n_A n_B}\) and \(\langle \hat{N}_A \otimes \hat{T}_B \rangle = \sum_{n_A n_B} n_A \rho_{n_A n_B : n_A n_B}\) never being negative. This result is the same as that for Category 1 states (see \[3\]).

Combining \[76\] and \[78\] we find that for Category 2 states

\[
\langle \Delta S^2_z \rangle - \frac{1}{2} \langle S_z \rangle \geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{N}_B \rangle
\]

Similarly

\[
\langle \Delta S^2_y \rangle - \frac{1}{2} \langle S_y \rangle \geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{N}_B \rangle
\]

The right side \(\frac{1}{2} \langle \hat{N}_A \otimes \hat{N}_B \rangle = \frac{1}{2} \sum_{n_A n_B} n_A n_B \rho_{n_A n_B : n_A n_B}\) is never negative.

Finally, from \[69\] and \[26\]

\[
\langle N \rangle = \sum_\lambda P(\lambda|c) \left( \langle N_B(\lambda) \rangle_Q + \frac{1}{2} + \langle N_A(\lambda) \rangle + \langle V_A(\lambda) \rangle \right)
\]

\[\sum_\lambda P(\lambda|c) \left( \frac{1}{2} + \langle V_A(\lambda) \rangle \right)\]

\[= \sum_\lambda P(\lambda|c) \left( \langle N_B(\lambda) \rangle_Q + \langle N_A(\lambda) \rangle \right)\]

\[= \langle \hat{T}_A \otimes \hat{N}_B \rangle + \langle \hat{N}_A \otimes \hat{T}_B \rangle
\]

This result is the same as that for Category 1 states (see \[3\]).

### 4.4 Two Mode Quadrature Operators: Means and Variances - Category 2 States

#### 4.4.1 Mean Values for Two Mode Quadratures \(X_\theta(\pm)\) and \(P_\theta(\pm)\) - Category 2 States

We now consider the mean value for two mode quadrature observables for the Category 2 states. For example in the case of the quadratures \(X_\theta(\pm)\)

\[
\langle X_\theta(\pm) \rangle = \frac{1}{\sqrt{2}} \left( \sum_\lambda P(\lambda|c) \langle x_A(\lambda) \rangle \cos \theta + \langle p_A(\lambda) \rangle \sin \theta \pm \langle x_B(\lambda) \rangle_Q \cos \theta \pm \langle p_B(\lambda) \rangle_Q \sin \theta \right)
\]

(82)
using Eq. (62). A similar result is found for \( P_\theta(\pm) \). We then use the previous results \(64\) for sub-system \( B \) to find

\[
\langle X_\theta(\pm) \rangle = \frac{1}{\sqrt{2}} \sum_\lambda P(\lambda|c) \left( \langle x_A(\lambda) \rangle \cos \theta + \langle p_A(\lambda) \rangle \sin \theta \right)
\]

\[
\langle P_\theta(\pm) \rangle = \frac{1}{\sqrt{2}i} \sum_\lambda P(\lambda|c) \left( -\langle x_A(\lambda) \rangle \sin \theta + \langle p_A(\lambda) \rangle \cos \theta \right)
\]

(83)

4.4.2 Variances for Two Mode Quadratures - Category 2 States

Using \(26\) and the LHVT expression for \( X_\theta(\pm)^2 \) obtained from the equivalent of Eq. \(59\) for classical observables we have for Category 2 states,

\[
\langle X_\theta(\pm)^2 \rangle = \frac{1}{2} \sum_\lambda P(\lambda|c) \left( \langle x_A(\lambda) \rangle^2 \cos^2 \theta + \langle p_A(\lambda) \rangle^2 \sin^2 \theta + \langle U_A(\lambda) \rangle \right)
\]

\[
+ \frac{1}{2} \sum_\lambda P(\lambda|c) \left( \langle N_B(\lambda) \rangle_Q + \frac{1}{2} \right)
\]

(84)

where we have used the previous results \(64\) and \(73\) for sub-system \( B \) to eliminate terms involving \( \langle x_B(\lambda) \rangle_Q \), \( \langle p_B(\lambda) \rangle_Q \), \( \langle U_B(\lambda) \rangle_Q \) and the results \(71\) and \(72\) for \( \langle x_B^2(\lambda) \rangle_Q \) and \( \langle p_B^2(\lambda) \rangle_Q \) to simplify the last term.

We next use the LHVT - quantum theory equivalences \(27\) to replace \(83\) and \(84\) by their quantum forms. Quantum forms for the variances are then obtained. Finally we use the result from SubSection \(2.2\) the reduced density operator for sub-system \( A \) satisfies the local particle number SSR to obtain expressions for \( \langle x_A \rangle \), \( \langle p_A \rangle \), \( \langle x_A^2 \rangle \), \( \langle p_A^2 \rangle \) and \( \langle U_A \rangle \) to give the following results for the variances \( \langle \Delta X_\theta(\pm)^2 \rangle \) and \( \langle \Delta P_\theta(\pm)^2 \rangle \) for Category 2 states.

\[
\langle \Delta X_\theta(\pm)^2 \rangle = \frac{1}{2} \langle \Delta \hat{X}_\theta(\pm)^2 \rangle = \frac{1}{2} \langle \hat{N} \rangle + \frac{1}{2} \geq \frac{1}{2}
\]

\[
\langle \Delta P_\theta(\pm)^2 \rangle = \frac{1}{2} \langle \Delta \hat{P}_\theta(\pm)^2 \rangle = \frac{1}{2} \langle \hat{N} \rangle + \frac{1}{2} \geq \frac{1}{2}
\]

(85)

Details are given in Appendix \(9\). The same results apply for Category 1 (separable) states (see Appendix L in Ref. \[3\]).

4.5 Two Mode Correlations: Inequalities for Bell Local States

The paper by Cavalcanti et al \[23\] derives certain inequalities for \( |\langle \hat{a}^\dagger \hat{b} \rangle|^2 \) for Category 1 and Category 2 states which lead to strong correlation tests for EPR steering. We will show here that these inequalities lead to more useful tests in
terms of spin operators for quantum entanglement and EPR steering. These inequalities are set out here in Eqs. (93) and (95) for Category 1 and Category 2 states respectively. The inequality in Eq. (93) has also been previously obtained for separable states by Hillery and Zubairy [17]. They two inequalities correspond to Eqs. (15) and (14) in Ref. [23] where there are \( N = 2 \) sub-systems ("sites"), with Eq. (15) applying when both sub-systems are associated with a LHS \( (T = 2 \) - two "trusted sites") and Eq. (14) when only one sub-system has a LHS \( (T = 1 \) - one "trusted site"). The inequalities obtained by Cavalcanti et al [23] were based on their general expression in Eq. (4) for the LHV theory joint measurement probability, for which Eqs. (41) and (42) for Category 1 and Category 2 states are special cases. Hence these inequalities would apply for the present paper. For completeness however, rather than just quoting the inequalities in Ref. [23] we will also derive them here using the approach set out in the present paper. A further inequality for \( |\langle \hat{a}^\dagger \hat{b} \rangle| \) will also be derived that would apply to Category 3 states.

For Category 1 states the result gives a strong correlation test and the Hillery-Zubairy [17] test for quantum entanglement, whilst for Category 2 states this result gives a strong correlation test plus a generalised Hillery-Zubairy test for EPR steering, originally set out in He et al [18] for the case where \( \langle \tilde{S}_z \rangle \neq 0 \). For Category 3 states no useful test for Bell non-locality occurs.

### 4.5.1 General Correlation Inequality for \( |\langle \hat{a}^\dagger \hat{b} \rangle| \) - Bell Local States

We have using (49) to introduce quadrature operators

\[
\langle \hat{a}^\dagger \hat{b} \rangle = \frac{1}{2} (\langle \hat{x}_A \hat{x}_B \rangle + \langle \hat{p}_A \hat{p}_B \rangle + i (\langle \hat{x}_A \hat{p}_B \rangle - \langle \hat{p}_A \hat{x}_B \rangle)) \tag{86}
\]

so introducing LHVT expressions \( \langle \hat{a}^\dagger \hat{b} \rangle = \frac{1}{2} \sum \lambda P(\lambda|c) (\langle x_A(\lambda) \rangle - i \langle p_A(\lambda) \rangle) (\langle x_B(\lambda) \rangle + i \langle p_B(\lambda) \rangle) \)

and then as \( |\langle \hat{a}^\dagger \hat{b} \rangle| \leq \frac{1}{2} \sum \lambda P(\lambda|c) |(\langle x_A(\lambda) \rangle - i \langle p_A(\lambda) \rangle) |(\langle x_B(\lambda) \rangle + i \langle p_B(\lambda) \rangle) | \)

and \( |(\langle x_A(\lambda) \rangle - i \langle p_A(\lambda) \rangle) | = \sqrt{(\langle x_A(\lambda) \rangle)^2 + (\langle p_A(\lambda) \rangle)^2} \)

e tc, we then find that

\[
|\langle \hat{a}^\dagger \hat{b} \rangle| \leq \frac{1}{4} \left( \sum \lambda P(\lambda|c) \sqrt{(\langle x_A(\lambda) \rangle)^2 + (\langle p_A(\lambda) \rangle)^2} \right) \sum \lambda P(\lambda|c) \langle x_A(\lambda) \rangle + \langle p_A(\lambda) \rangle \right)^2 \tag{87}
\]

Using the inequality \( \sum_{R} P_R C_R \geq \left( \sum_{R} P_R \sqrt{C_R} \right)^2 \) with \( \sum_{R} P_R = 1 \) and \( C_R \geq 0 \) we then have the key inequality

\[
|\langle \hat{a}^\dagger \hat{b} \rangle| \leq \frac{1}{4} \sum \lambda P(\lambda|c) \left( (\langle x_A(\lambda) \rangle)^2 + (\langle p_A(\lambda) \rangle)^2 \right) \left( (\langle x_B(\lambda) \rangle)^2 + (\langle p_B(\lambda) \rangle)^2 \right) \tag{88}
\]
use Schwarz’ inequality to give
leads to some outcomes different to (89).

4.5.2 Stronger Correlation Inequalities for Bell Local States

Stronger inequalities can now be derived for the quantities \( (x_A(\lambda))^2 + \langle p_A(\lambda) \rangle^2 \) and \( (x_B(\lambda))^2 + \langle p_B(\lambda) \rangle^2 \) in the cases of Categories 1, 2 and 3 states. This leads to some outcomes different to (89).

If the sub-system \( C \) does not involve a local hidden state \( \hat{\rho}_C \) then we can use Schwarz’ inequality to give \( (x_C(\lambda))^2 \leq \langle x_C^2(\lambda) \rangle \) and \( \langle p_C(\lambda) \rangle^2 \leq \langle p_C^2(\lambda) \rangle \). This is equivalent to the variances of \( x_C \) and \( p_C \) being non-negative. Thus

\[
(x_C(\lambda))^2 + \langle p_C(\lambda) \rangle^2 \leq \langle x_C^2(\lambda) \rangle + \langle p_C^2(\lambda) \rangle \quad (90)
\]

On the other hand, if the sub-system \( C \) does involve a local hidden state \( \hat{\rho}_C \) then we can obtain a stronger inequality via quantum theory. For any real \( \eta \) the quantity \( \langle (\Delta x_C - i\eta \Delta \hat{\rho}_C)(\Delta x_C + i\eta \Delta \hat{\rho}_C) \rangle_\lambda = Tr(\Delta \hat{x}_C - \langle \hat{x}_C \rangle_\lambda)(\Delta \hat{x}_C + \langle \hat{x}_C \rangle_\lambda) \hat{\rho}_C \geq 0 \), where \( \Delta \hat{x}_C = \hat{x}_C - \langle \hat{x}_C \rangle_\lambda \). Thus for all \( \eta \) we have \( \langle \Delta x_C^2 \rangle_\lambda - \eta^2 \langle \Delta \hat{\rho}_C^2 \rangle_\lambda \geq 0 \) using \( [\hat{x}_C, \hat{\rho}_C] = i \). Putting \( \eta = 1 \) gives the inequality \( \langle \Delta x_C^2 \rangle_\lambda + \langle \Delta \hat{\rho}_C^2 \rangle_\lambda - 1 \geq 0 \), which can be written as \( \langle x_C^2 \rangle_\lambda + \langle p_C^2 \rangle_\lambda \leq \langle x_C^2 \rangle_\lambda + \langle p_C^2 \rangle_\lambda - 1 \). In terms of LHVT notation this inequality is

\[
(x_C(\lambda))^2 + \langle p_C(\lambda) \rangle^2 \leq \langle x_C^2(\lambda) \rangle + \langle p_C^2(\lambda) \rangle - 1 \quad (91)
\]

For Category 1 states both sub-systems involve a local hidden state, so the key inequality [85] gives

\[
\left| \langle a^\dagger b \rangle \right|^2 \leq \frac{1}{4} \sum_\lambda P(\lambda|c) \left( \langle x_A^2(\lambda) \rangle + \langle p_A^2(\lambda) \rangle - 1 \right) \left( \langle x_B^2(\lambda) \rangle + \langle p_B^2(\lambda) \rangle - 1 \right) \quad (92)
\]

Using (26), (27), (52) and (51) we can then convert these inequalities to expressions involving number operators, \( \hat{N}_C = \hat{c}^\dagger \hat{c} \) \( (C = A, B) \)

\[
\left| \langle a^\dagger b \rangle \right|^2 \leq \left( \hat{N}_A + \hat{V}_A - \frac{1}{2} 1_A \right) \otimes \left( \hat{N}_B + \hat{V}_B - \frac{1}{2} 1_B \right) \quad (93)
\]

\[
= \left( \hat{N}_A \otimes \hat{N}_B \right) \quad (94)
\]
For Category 2 states with sub-system B involving a local hidden state \( \hat{\rho}_B^L \), the key inequality \( (88) \) gives

\[
| \langle \hat{a}^\dagger \hat{b} \rangle |^2 \leq \frac{1}{4} \sum_{\lambda} P(\lambda|c) \left( \langle x_A^2(\lambda) \rangle + \langle p_A^2(\lambda) \rangle \right) \left( \langle x_B^2(\lambda) \rangle + \langle p_B^2(\lambda) \rangle \right) - 1 \tag{94}
\]

Similarly to the Category 1 case we then find that for Category 2 states (with B involving the local hidden state)

\[
| \langle \hat{a}^\dagger \hat{b} \rangle |^2 \leq \left( \hat{N}_A^2 + 1 \right) \otimes \left( \hat{N}_B - \frac{1}{2} \right) \tag{95}
\]

For Category 3 states with neither sub-system involving a local hidden state, the key inequality \( (88) \) gives

\[
| \langle \hat{a}^\dagger \hat{b} \rangle |^2 \leq \frac{1}{4} \sum_{\lambda} P(\lambda|c) \left( \langle x_A^2(\lambda) \rangle + \langle p_A^2(\lambda) \rangle \right) \left( \langle x_B^2(\lambda) \rangle + \langle p_B^2(\lambda) \rangle \right) - 1 \tag{96}
\]

In the case of the Category 3 states we then have

\[
| \langle \hat{a}^\dagger \hat{b} \rangle |^2 \leq \left( \hat{N}_A + \frac{1}{2} \right) \otimes \left( \hat{N}_B + \frac{1}{2} \right) \tag{97}
\]

where we note that \( \hat{N}_A + \frac{1}{2} \hat{1}_A = \hat{a}^\dagger \hat{a} + \frac{1}{2} = (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) / 2 \). This result is the same as the general result \( (89) \) found for all Bell local states. Note also that this derivation of Eqs. \((93), (95)\) and \((97)\) did not make use of the SSR.

As will be seen in the next Section, all these inequalities \((93), (95)\) and \((97)\) can be expressed in terms of spin operator variances.

### 4.5.3 Correlations as Spin Operator Inequalities - Bell Local States

The inequalities \((93), (95)\) and \((97)\) derived above can be put into a more useful form involving spin operators - whose mean values and variances can be measured. We use the definitions of the spin operators in SubSection 4.2 (see also Ref. [3])

\[
| \langle \hat{a}^\dagger \hat{b} \rangle |^2 = \left( \hat{S}_x \right)^2 + \left( \hat{S}_y \right)^2
\]

\[
\hat{N}_A = \frac{1}{2} \hat{N} - \hat{S}_z \quad \hat{N}_B = \frac{1}{2} \hat{N} + \hat{S}_z
\]

\[
\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{\hat{N}}{2} \left( \frac{\hat{N}}{2} + 1 \right)
\]

(98)

to find after some straightforward calculations and introducing the variances \( \langle \Delta \hat{S}_x^2 \rangle = \langle \hat{S}_x^2 \rangle - \langle \hat{S}_x \rangle^2 \) etc the following results.
For Category 1, 2 and 3 states

\[ \langle \Delta S_x^2 \rangle + \langle \Delta S_y^2 \rangle - \frac{1}{2} \langle \hat{N} \rangle \geq 0 \]  
\[ \text{Cat 1 States} \quad (99) \]

\[ \langle \Delta S_x^2 \rangle + \langle \Delta S_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle \geq 0 \]  
\[ \text{Cat 2 States} \quad (100) \]

\[ \langle \Delta S_x^2 \rangle + \langle \Delta S_y^2 \rangle + \frac{1}{4} \geq 0 \]  
\[ \text{Cat 3 States} \quad (101) \]

Details are given in Appendix 10. For Category 2 states with \( A \) involving the LHS then the left side would have involved \( -\frac{1}{2} \langle \hat{S}_z \rangle \).

The inequality (100) for Category 2 states can also be obtained more directly \textit{without} using the strong correlation inequalities from the results in Section 4.3 set out in Eqs. (76), (81) and (77). Details are given in Appendix 11. The inequality (99) for Category 1 states was also derived in Refs. [17] and [3].

We note in passing that Eq. (101) does not lead to a test for Bell non-locality. From the Heisenberg Uncertainty Principle this inequality applies for all quantum states. Hence the inequalities (97) or (101) do not provide a test for Bell non-locality.

4.6 Tests for EPR Steering

Having obtained in Sections 4.3, 4.4 and 4.5 a number of inequalities for spin and quadrature observables that apply for Category 2 (and Category 1) states, we now apply these to obtain tests for EPR steering. First, we consider whether tests that have been shown to be sufficient to demonstrate quantum entanglement (violation of Category 1) (see Ref. [3] for details) are also valid for demonstrating EPR steering. Obviously a test that demonstrates EPR steering must also demonstrate quantum entanglement, but a test that demonstrates entanglement does not necessarily demonstrate EPR steering. We first consider the Bloch vector tests, then spin squeezing tests for \( S_z \) and for the other spin components, followed by the Hillery-Zubairy spin variance test and finally two mode quadrature squeezing tests. Of these possible tests, the Bloch vector test, spin squeezing in any spin component, the Hillery-Zubairy spin variance test and squeezing in any two mode quadrature are valid for demonstrating EPR steering. Second, we consider a modified version of the Hillery-Zubairy spin variance test, which also shows that EPR steering occurs. Third, we consider for completeness weak and strong correlation tests, though more useful tests equivalent to these but involving spin operators have already been obtained.

4.6.1 Bloch Vector Test

From (67) for Category 2 (or Category 1) states we immediately see that if

\[ \langle S_x \rangle \neq 0 \quad \text{or} \quad \langle S_y \rangle \neq 0 \]  
\[ (102) \]
then the quantum state cannot be in Category 2 (or Category 1). Hence reverting to quantum operators the Bloch vector test $\langle \hat{S}_x \rangle \neq 0$ or $\langle \hat{S}_y \rangle \neq 0$ now also shows that the state is EPR steered as well as just being entangled.

4.6.2 Spin Squeezing Tests

From Eq. (67) we immediately see that if the observable $S_z$ is squeezed with respect to $S_x$ or with respect to $S_y$, then the LHS model fails, because spin squeezing in $S_z$ requires $\langle \Delta S_z^2 \rangle$ to be less than either $|\langle S_x \rangle|/2$ or $|\langle S_y \rangle|/2$ and this is impossible for a LHS model - where $\langle S_x \rangle = \langle S_y \rangle = 0$ for both Category 1 (see Ref. [3]) and Category 2 states. This condition also rules out $S_x$ or $S_y$ being squeezed with respect to $S_z$ or $S_x$ being squeezed with respect to $S_x$ or $S_y$. In Ref. [3] it was shown that spin squeezing involving $S_z$ provided a test for entanglement. Here we see that spin squeezing involving the observable $S_z$ shows the state is EPR steered as well as merely being entangled.

From Eqs. (79) and (80) we see that $\langle \Delta S_x^2 \rangle - \frac{1}{2} |\langle S_z \rangle| \geq 0$ and $\langle \Delta S_y^2 \rangle - \frac{1}{2} |\langle S_z \rangle| \geq 0$ for Category 2 states. Hence we find that for Category 2 states there is no spin squeezing in $S_x$ compared to $S_y$ (or vice versa). For Category 1 states we also find that $\langle \Delta S_x^2 \rangle - \frac{1}{2} |\langle S_z \rangle| \geq \frac{1}{2} \left\langle \hat{N}_A \otimes \hat{N}_B \right\rangle \geq 0$ and $\langle \Delta S_y^2 \rangle - \frac{1}{2} |\langle S_z \rangle| \geq \frac{1}{2} \left\langle \hat{N}_A \otimes \hat{N}_B \right\rangle \geq 0$ (see Eq. (31) in Ref. [3]). Hence spin squeezing in $S_x$ versus $S_y$ (or vice versa) is a test for entanglement, so the state is not in Category 1. Thus spin squeezing in $S_x$ versus $S_y$ (or vice versa) is therefore also a test for EPR steering.

Overall then we now see that spin squeezing in any spin component $S_\alpha$ with respect to another component $S_\beta$

$$\langle \Delta S_\alpha^2 \rangle < \frac{1}{2} |\langle S_\gamma \rangle| \quad \text{and} \quad \langle \Delta S_\beta^2 \rangle > \frac{1}{2} |\langle S_\gamma \rangle|$$

(103)

(where $\alpha, \beta, \gamma$ are $x, y, z$ in cyclic order) is a sufficiency test for EPR steering.

4.6.3 Two Mode Quadrature Squeezing Test

For Category 1 states it has been shown (see Eqs. (155), (157) in Ref. [3]) that if there is squeezing in any of the two mode quadrature operators

$$\langle \Delta X_\theta(\pm) \rangle^2 < \frac{1}{2} \quad \text{or} \quad \langle \Delta P_\theta(\pm) \rangle^2 < \frac{1}{2}$$

(104)

then the state is entangled - that is, it is not in Category 1. Due to the Heisenberg uncertainty principle $\langle \Delta X_\theta(\pm) \rangle \langle \Delta X_\theta(\pm) \rangle \geq 1/4$ only one of the pair of quadrature operators is squeezed.

However we have shown for Category 2 states (see Eq. (85)) that

$$\langle \Delta X_\theta(\pm) \rangle = \langle \Delta P_\theta(\pm) \rangle = \frac{1}{2} \langle N \rangle + \frac{1}{2}$$

(105)
and as the right side is never less than one half, it follows that two mode quadrature squeezing in either $X_\theta(\pm)$ or $P_\theta(\pm)$ shows that the state cannot be in Category 2. Thus two mode quadrature squeezing as in (104) provides a test for EPR steering.

4.6.4 Hillary-Zubairy Spin Variance Test

The Hillery-Zubairy spin variance test [17] for quantum entanglement is

\[ \Delta \hat{S}_x^2 + \Delta \hat{S}_y^2 - \frac{1}{2} \langle \hat{N} \rangle < 0. \]

We now consider the quantity $\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{2} \langle \hat{N} \rangle$ for Category 2 states using (76) and (81). We find that

\[ \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{2} \langle \hat{N} \rangle \geq \langle \hat{N}_A \otimes \hat{N}_B \rangle \geq 0 \]  

This result also applies for Category 1 states (see Eqs. (82), (83) in Ref. [3] for details, or directly from Eq. (99)).

Thus we can say that if

\[ \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{2} \langle \hat{N} \rangle < 0 \]  

then the Category 2 model fails. It also shows that Category 1 (separable states) fails, this being the Hillery-Zubairy spin variance test [17] for entanglement. This condition can also be written as

\[ E_{HZ} = \frac{\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle}{\frac{1}{2} \langle \hat{N} \rangle} < 1 \]

which is the form given in Ref. [18].

Hence there is a Hillary-Zubairy spin variance inequality test for EPR steering.

4.6.5 Generalised Hillery-Zubairy Spin Variance Test

The results (76), (81) and (77) provide a generalisation of the Hillery-Zubairy spin variance test [17] for EPR steering

\[ \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle < 0 \]

in the case where the LHS occurs in sub-system $B$. The details are set out in Appendix 11. The test also follows from the spin operator form (100) of the strong correlation condition (95) obtained by Cavalcanti et al [23]. For
\[ \langle \hat{S}_z \rangle = 0 \] this test was previously obtained by He et al. [18]. If sub-system A involves the LHS then \( \frac{1}{2} \langle \hat{S}_z \rangle \) is replaced by \( -\frac{1}{2} \langle \hat{N} \rangle \). Since \( \frac{1}{2} \langle \hat{N} \rangle \geq \langle \hat{S}_z \rangle \geq -\frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle \geq 0 \), so as \( \langle \Delta \hat{S}_z^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle = \langle \Delta \hat{S}_z^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle \) and we have just shown that \( \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle \) is never negative, then if (109) is satisfied then (107) will also apply, ruling out the state being in Category 1. Since \( 0 \leq \frac{1}{4} \langle \hat{N} \rangle - \frac{1}{2} \langle \hat{S}_z \rangle \leq \frac{1}{2} \langle \hat{N} \rangle \) it is of course harder to find states where \( \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle < \frac{1}{4} \langle \hat{N} \rangle - \frac{1}{2} \langle \hat{S}_z \rangle \) to show EPR steering than merely being \( < \frac{1}{2} \langle \hat{N} \rangle \), as would show entanglement. The generalised Hillery-Zubairy spin variance test (109) for EPR steering is a more difficult test to satisfy than the Hillery-Zubairy test for entanglement. In the generalised form (109) the EPR steering test now allows for asymmetry \( \langle \hat{S}_z \rangle \neq 0 \).

The generalised Hillery-Zubairy EPR steering test can also be written as

\[
E_{HZ} = \frac{\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle}{\frac{1}{2} \langle \hat{N} \rangle} < \langle \hat{N}_A \rangle
\]

after substituting \( \langle \hat{N} \rangle = \langle \hat{N}_A \rangle + \langle \hat{N}_B \rangle \) and \( \langle \hat{S}_z \rangle = (\langle \hat{N}_B \rangle - \langle \hat{N}_A \rangle) / 2 \), which is consistent with the result \( E_{HZ} < 1/2 \) given in Ref. [18] for \( \langle \hat{S}_z \rangle = 0 \). This form of the test may be compared to the EPR steering test in (108), which is consistent since the right side is always less than unity because \( \langle \hat{N}_A \rangle \leq \langle \hat{N} \rangle \).

4.6.6 Weak Correlation Test

The quantum operator \( \hat{a}^\dagger \hat{b} \) is not an observable, but from the definitions for the spin operator we can write \( \hat{a}^\dagger \hat{b} = \hat{S}_x - i \hat{S}_y \). We can then interpret \( a^b \) to be \( S_x - i S_y \), where now \( S_x \) and \( S_y \) are observables whose mean values are definable in a LHV theory.

From (143) and (67) we see that for Category 2 (and Category 1) states

\[
\langle a^b \rangle = \langle S_x \rangle - i \langle S_y \rangle
\]

\[
= 0
\]

so that

\[
| \langle a^b \rangle |^2 = (\langle S_x \rangle)^2 + (\langle S_y \rangle)^2 = 0
\]

for quantum states in Category 2 (or Category 1). This means that if

\[
| \langle a^b \rangle |^2 > 0
\]
the state cannot be either Category 1 or Category 2. This constitutes a so-called weak correlation test for EPR steering. However because \( |\langle a^\dagger b \rangle|^2 = \langle S_x \rangle^2 + \langle S_y \rangle^2 \) this test is really just equivalent to the Bloch vector test. So no useful test for either quantum entanglement or EPR steering involving \( \langle \hat{a}^\dagger \hat{a} \rangle^2 + \langle \hat{N}_A \otimes \hat{N}_B \rangle \) is established at this point. Later however (see Section 4.6.7) it will be shown that related tests can be obtained both for quantum entanglement and EPR steering.

4.6.7 Strong Correlation Test

Hillery and Zubairy \[17\] showed that for separable states (Category 1 states) that \( |\langle a^\dagger b \rangle|^2 \leq \langle \hat{a}^\dagger \hat{a} \rangle^2 \) and \( \langle \hat{S}_x \rangle^2 + \langle \hat{S}_y \rangle^2 \) is established here in Eq. (93). The proof of this result was valid irrespective of whether the sub-system states \( \hat{\rho}_R^A \) and \( \hat{\rho}_R^B \) were local particle number SSR compliant or not (see Ref. \[3\] for details). The quantum result

\[
|\langle a^\dagger b \rangle|^2 = \langle \hat{S}_x \rangle^2 + \langle \hat{S}_y \rangle^2
\]

is a strong correlation test for quantum entanglement. Hence as the numbers of bosons \( N_A \) and \( N_B \) are observables in the LHV model (and therefore the mean \( \langle N_A \otimes N_B \rangle \) can be defined) we see that for Category 1 states the LHV result

\[
|\langle a^\dagger b \rangle|^2 \leq \langle N_A \otimes N_B \rangle
\]

applies. Thus if

\[
|\langle a^\dagger b \rangle|^2 > \langle N_A \otimes N_B \rangle
\]

we have a strong correlation test for entanglement. However, there is a different strong correlation test for EPR steering that applies - and which is harder to satisfy.

In the case of Category 2 states from the inequality in Eq. (95) we see that if

\[
|\langle a^\dagger b \rangle|^2 > \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{1}_B \rangle
\]

the state cannot be in Category 2 (nor in Category 1) so it must be EPR steerable. Thus the inequality \(117\) is a strong correlation test for EPR steering.

Note that the condition is harder to satisfy than the strong correlation test \(114\) for entanglement since \( \langle \hat{1}_A \otimes \hat{N}_B \rangle \) is positive, but obviously if \(117\) is satisfied the state is entangled as well as being EPR steerable. If \( \hat{A} \) involved the LHS then the right side would have been \( \langle \hat{N}_A \otimes \left( \hat{N}_B + \frac{1}{2} \hat{1}_B \right) \rangle \).

However, as these tests are just equivalent to the Hillery-Zubairy spin variance test and the generalised Hillery-Zubairy spin variance test, so no useful additional test has been obtained.
5 Summary and Conclusion

We have reviewed two possible classification schemes for the quantum states of bipartite composite systems. In the first (Quantum Theory Classification Scheme) the states are classified as being either quantum separable or quantum entangled. In the second (Local Hidden Variable Theory Classification Scheme) the states are initially classified as being Bell local or Bell non-local. The Bell non-local states are quantum entangled and EPR steerable - these are listed as Category 4 states. However, the Bell local states can be divided up into three categories depending on whether both, one or neither of the sub-system single measurement probability is given by a quantum theory expression involving a sub-system density operator. The Category 1 states (both) are the same as the quantum separable states and are non-entangled, LHS states and non-steered. The Category 2 states (one) are quantum entangled LHS states (LHS) and are non-steerable. The Category 3 (neither) states are quantum entangled and EPR steerable.

A detailed study of how observables are treated in terms of quantum theory and local hidden variable theories has been carried out, including how the two approaches are related and how to replace quantum operators for observables with classical entities. For systems involving identical bosons the mode annihilation, creation operators are replaced by quadrature amplitudes.

Tests for EPR steering (EPR entanglement) based on violation of the LHS model have been examined. Such tests were obtained based on whether the Bloch vector is in the $xy$ plane (Bloch vector test) and on whether there is spin squeezing in any of the spin components $S_x$, $S_y$ or $S_z$ (spin squeezing test). The Hillery spin variance test based on the sum of variances in $S_x$ and $S_y$ also demonstrates EPR steering. In addition, two mode quadrature squeezing also provides a test for EPR steering. A new generalised Hillery-Zubairy spin variance test for EPR steering was found, involving the sum of variances in $S_x$ and $S_y$, but now containing a different multiple of the mean value for $N$ along with a term involving the mean value for $S_z$. This allows for asymmetry and is a stronger version of the Hillery spin variance test. Correlation tests based on the mean value of $\langle a^\dagger b \rangle$ have also been obtained, but these are just equivalent to previous tests based on the spin operators. No EPR steering test based on the difference between the variances of the number difference and number sum was found. We note that some of the tests (Bloch vector, spin squeezing, two mode quadrature squeezing) were based on applying the super-selection rules for the total particle number as well as that for the local particle number for the sub-system LHS. However, since the stronger correlation inequalities from which they can be derived do not depend on the SSR (see Section 4.5.2) the Hillery-Zubairy spin variance test and its generalisation involving the mean value for $S_z$ do not depend on these rules.
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6 Appendix A - Mean Values and Variances - General Features

6.1 Mean Values and Variances - Quantum Models

In a fully quantum treatment, any observable represented by a hermitian operator \( \hat{\Omega} \) - whose measured outcomes are its eigenvalues \( \theta \), can be written as \( \hat{\Omega} = \sum \theta \hat{\Pi}_\theta \) in terms of its projectors \( \hat{\Pi}_\theta \) and we can determine the probability \( P(\hat{\Omega}, \theta) \) for the outcome \( \theta \) via \( P(\hat{\Omega}, \theta) = \text{Tr}(\hat{\Pi}_\theta \hat{\rho}) \) - where \( \hat{\rho} \) is the density operator that specifies the quantum state. Hence the mean value of the measured outcomes can be defined and then determined as follows

\[
\langle \hat{\Omega} \rangle_Q = \sum \theta P(\hat{\Omega}, \theta) = \text{Tr}(\hat{\Omega} \hat{\rho})
\]

We can also extend the concept of the mean value for measured outcomes to the case of a non-Hermitian operator \( \hat{\Omega} \) - which although it does not correspond to an observable can be written in the form \( \hat{\Omega} = \hat{\Omega}_1 + i \hat{\Omega}_2 \), where both \( \hat{\Omega}_1 \) and \( \hat{\Omega}_2 \) are each observable Hermitian operators, not necessarily commuting. We simply define the mean for \( \hat{\Omega} \) via

\[
\langle \hat{\Omega} \rangle \equiv \langle \hat{\Omega}_1 \rangle + i \langle \hat{\Omega}_2 \rangle = \text{Tr}(\hat{\Omega}_1 + i \hat{\Omega}_2) \hat{\rho}
\]

where \( \langle \hat{\Omega}_1 \rangle \) and \( \langle \hat{\Omega}_2 \rangle \) are defined as in (118), and we see that the result is given by the trace process. This definition and result can be applied to provide a meaning for the quantum mean values of operators such as an annihilation operator \( \hat{a} = \frac{1}{\sqrt{2}}(\hat{x}_A + i \hat{p}_A) \) - which can be written in terms of quadrature operators or a transition operator \( \hat{b} \hat{\beta} = \hat{S}_x + i \hat{S}_y \) - which can be expressed in terms of spin operators. The latter case applies for considering correlation tests. If \( \hat{\Omega} \) can be written as the sum of products of Hermitian sub-system operators \( \hat{\Omega}_A \) and \( \hat{\Omega}_B \) the last expression can be used to evaluate the mean value based on the quantum probability distributions for measurements of each \( \hat{\Omega}_A \) and \( \hat{\Omega}_B \).

Note that in expressing \( \langle \hat{\Omega} \rangle \) in terms of \( \langle \hat{\Omega}_1 \rangle \) and \( \langle \hat{\Omega}_2 \rangle \) we are considering the results of two independent sets of measurements, one set for \( \hat{\Omega}_1 \) and the other for \( \hat{\Omega}_2 \). We do not imply that there is a joint probability \( P(\omega_1, \omega_2|\Omega_1, \Omega_2, c) \) for simultaneous outcomes \( \omega_1, \omega_2 \) of a combined measurement of \( \Omega_1, \Omega_2 \) following preparation \( c \). We only require single measurement probabilities \( P(\omega_1|\Omega_1, c) \) and \( P(\omega_2|\Omega_2, c) \) to exist in order to define the mean values via \( \langle \hat{\Omega}_1 \rangle = \sum \omega_1 P(\omega_1|\Omega_1, c) \), which corresponds to the set of measurements on \( \hat{\Omega}_1 \) alone. In von-Neumann’s proof that hidden variable theories were inconsistent with quantum theory, he
had evidently used the equivalent of $\langle \hat{\Omega} \rangle = \sum_{\omega_1} \sum_{\omega_2} (\omega_1 + i\omega_2) P(\omega_1, \omega_2|\Omega_1, \Omega_2, c)$ based on one set of measurements, whereas we just use $\langle \hat{\Omega} \rangle = \sum_{\omega_1} (\omega_1) P(\omega_1|\Omega_1, c) + i \sum_{\omega_2} (\omega_2) P(\omega_2|\Omega_2, c)$ - which rests on two independent sets of measurements.

In the case of quantum separable states the mean values for jointly measuring $\Omega_A$ in sub-system $A$ and $\Omega_B$ in sub-system $B$ for preparation $\rho$ would be given by

$$\langle \Omega_A \Omega_B \rangle = \sum_R P_R \langle \Omega_A \rangle_R \langle \Omega_B \rangle_R$$

where $\langle \Omega_A \rangle_R = \sum_\alpha \alpha P_Q(\alpha|\Omega_A, \rho, R) = Tr(\hat{\Omega}_A \hat{\rho}_R^A)$ and $\langle \Omega_B(\lambda) \rangle_Q = \sum_\beta \beta P_Q(\beta|\Omega_B, \rho, R) = Tr(\hat{\Omega}_B \hat{\rho}_R^B)$ are the mean values for measurement outcomes for $\Omega_A$ and $\Omega_B$. For the quantum separable state the mean value for any sum of products of sub-system operators which is Hermitian overall would be given by

$$\langle \sum_i \hat{\Omega}_{A_i} \hat{\Omega}_{B_i} \rangle = \sum_R P_R \sum_i \langle \hat{\Omega}_{A_i} \rangle_R \langle \hat{\Omega}_{B_i} \rangle_R$$

where $\langle \hat{\Omega}_{A_i} \rangle_R = Tr(\hat{\Omega}_{A_i} \hat{\rho}_R)$ and $\langle \hat{\Omega}_{B_i} \rangle_R = Tr(\hat{\Omega}_{B_i} \hat{\rho}_R)$ are quantum mean values, since we can always write $\hat{\Omega}_{A_i} = \hat{\Omega}_{A_i}^{(1)} + i \hat{\Omega}_{A_i}^{(2)}$ where both $\hat{\Omega}_{A_i}^{(1)}$ and $\hat{\Omega}_{A_i}^{(2)}$ are Hermitian and can be regarded as observables so with $\hat{\Omega}_{A_i} \hat{\Omega}_{B_i} = \hat{\Omega}_{A_i}^{(1)} \hat{\Omega}_{B_i}^{(1)} - \hat{\Omega}_{A_i}^{(2)} \hat{\Omega}_{B_i}^{(2)} + i(\hat{\Omega}_{A_i}^{(1)} \hat{\Omega}_{B_i}^{(2)} - \hat{\Omega}_{A_i}^{(2)} \hat{\Omega}_{B_i}^{(1)})$ which is of the form $\hat{\Omega}_1 + i\hat{\Omega}_2$, where both $\hat{\Omega}_1$ and $\hat{\Omega}_2$ are each observable Hermitian operators (the $A$ and $B$ operators commute). We can then invoke the probability distributions for the $\hat{\Omega}_{A_i}^{(1)}$, $\hat{\Omega}_{B_i}^{(1)}$, $\hat{\Omega}_{A_i}^{(2)}$ and $\hat{\Omega}_{B_i}^{(2)}$ to derive the expression for the mean value of $\hat{\Omega}_{A_i} \hat{\Omega}_{B_i}$ by also using (120). So (122) applies even if quantum operators $\hat{\Omega}_{A_i}$ and $\hat{\Omega}_{B_i}$ do not represent observables.

Variances can be obtained based on considering the mean values of the square of $\hat{\Omega}$. For observable represented by a hermitian operator $\hat{\Omega}$ the variance is defined by the mean of the squared variation of outcomes from the mean and equal to the difference between the mean of $\hat{\Omega}^2$ and the square of the mean of $\hat{\Omega}$.

$$\langle \Delta \hat{\Omega}^2 \rangle_Q = \sum_\theta (\theta - \langle \hat{\Omega} \rangle_Q)^2 P(\hat{\Omega}, \theta) = \langle \hat{\Omega}^2 \rangle_Q - \langle \hat{\Omega} \rangle_Q^2$$

In the case of a mixed state (such as the QSS)

$$\hat{\rho} = \sum_R P_R \hat{\rho}_R$$
the mean for a Hermitian operator $\hat{\Omega}$ in a mixed state is the average of means for separate components
\[
\langle \hat{\Omega} \rangle = \sum_R P_R \langle \hat{\Omega}_R \rangle \tag{125}
\]
where $\langle \hat{\Omega}_R \rangle = Tr(\hat{\rho}_R \hat{\Omega})$. The variance for a Hermitian operator $\hat{\Omega}$ in a mixed state is always never less than the the average of the variances for the separate components (see [24])
\[
\langle \Delta \hat{\Omega}^2 \rangle \geq \sum_R P_R \langle \Delta \hat{\Omega}_R^2 \rangle \tag{126}
\]
where $\langle \Delta \hat{\Omega}^2 \rangle = Tr(\hat{\rho} \Delta \hat{\Omega}^2)$ with $\Delta \hat{\Omega} = \hat{\Omega} - \langle \hat{\Omega} \rangle$ and $\langle \Delta \hat{\Omega}_R^2 \rangle = Tr(\hat{\rho}_R \Delta \hat{\Omega}_R^2)$ with $\Delta \hat{\Omega}_R = \hat{\Omega} - \langle \hat{\Omega}_R \rangle$. To prove this result we have using (125) both for $\hat{\Omega}$ and $\hat{\Omega}^2$
\[
\langle \Delta \hat{\Omega}^2 \rangle = \langle \hat{\Omega}^2 \rangle - \langle \hat{\Omega} \rangle^2
= \sum_R P_R \left( \langle \hat{\Omega}_R^2 \rangle - \langle \hat{\Omega}_R \rangle^2 \right) + \sum_R P_R \langle \hat{\Omega}_R \rangle^2 - \left( \sum_R P_R \langle \hat{\Omega}_R \rangle \right)^2
= \sum_R P_R \langle \Delta \hat{\Omega}_R^2 \rangle + \sum_R P_R \langle \hat{\Omega}_R \rangle^2 - \left( \sum_R P_R |\langle \hat{\Omega}_R \rangle| \right)^2 \tag{127}
\]
The variance result (126) follows because the sum of the last two terms is always $\geq 0$ using the result (135) in Appendix E of Ref [2], with $C_R = \langle \hat{\Omega}_R^2 \rangle$, $\sqrt{C_R} = |\langle \hat{\Omega}_R \rangle|$ which are real and positive.

In considering the means and variances in the context of LHV several difficult issues need to be dealt with. Firstly, in a LHV the observables are basically considered as classical c-numbers, but given that the predictions from quantum theory are accepted as being correct these classical observables must correspond to underlying quantum Hermitian operators - especially as in the LHS model where the probabilities $P_Q(\beta|\Omega_B, \alpha, \lambda)$ for sub-system $B$ are also to be given by quantum formulae. Also, there are several entanglement tests involving spin components, these are represented by the spin operators $\hat{S}_x = (\hat{b}^\dagger \hat{a} + \hat{a}^\dagger \hat{b})/2$, $\hat{S}_y = (\hat{b}^\dagger \hat{a} - \hat{a}^\dagger \hat{b})/2i$ and $\hat{S}_z = (\hat{b}^\dagger \hat{b} - \hat{a}^\dagger \hat{a})/2$, where $\hat{a}$ and $\hat{b}$ are mode annihilation operators. The tests also involve the total number operator $\hat{N} = (\hat{b}^\dagger \hat{b} + \hat{a}^\dagger \hat{a})$. All these operators are Hermitian and represent observable quantities applying for the overall two mode system. We may also consider number operators for the two modal sub-systems defined by $\hat{N}_A = \hat{a}^\dagger \hat{a}$ and $\hat{N}_B = \hat{b}^\dagger \hat{b}$, which again are Hermitian and represent observable quantities for each sub-system. The question then arises: How do you define the spin components and the boson number when the observables are supposed to be non-quantum? Secondly, when considering
entanglement tests involving spin components, both sub-system $A$ and $B$ involve mode annihilation operators - which are non-Hermitian and not themselves associated with measurable observables. What meaning can we give to LHV probabilities $P(\alpha | \Omega_A, c, \lambda)$ and associated mean values $\langle \Omega_A(\lambda) \rangle = \sum_{\alpha} \alpha P(\alpha | \Omega_A, c, \lambda)$ for sub-system $A$ when during the discussion of spin squeezing tests in the LHS model we consider situations where $\Omega_A$ corresponds to a mode annihilation or creation operator? To give meaning to the LHS model do we need to consider LHV probabilities $P(\alpha_1, \alpha_2 | \Omega_{A_1}, \Omega_{A_2}, c, \lambda)$ associated with the outcomes of measuring two observables $\Omega_{A_1}, \Omega_{A_2}$ for sub-system $A$ when the hidden variables are $\lambda$ and which may correspond to quantum operators that do not commute? What happens when we need to consider a product such as $\Omega_{A_1} \Omega_{A_2} \Omega_{B_1} \Omega_{B_2}$ such as may occur when we are considering expressions for variances? Would this mean that for products of sub-system observables we should determine the mean values via 

$$\langle \Omega_{A_1} \Omega_{A_2} \Omega_{B_1} \Omega_{B_2} \rangle = \sum_{\lambda} P(\lambda | c) \langle \Omega_{A_1} \Omega_{A_2}(\lambda) \rangle \langle \Omega_{B_1} \Omega_{B_2}(\lambda) \rangle_Q$$

where $\langle \Omega_{A_1} \Omega_{A_2}(\lambda) \rangle = \sum_{\alpha_1, \alpha_2} \alpha_1 \alpha_2 P(\alpha_1, \alpha_2 | \Omega_{A_1}, \Omega_{A_2}, c, \lambda)$ and $\langle \Omega_{B_1} \Omega_{B_2}(\lambda) \rangle_Q = \sum_{\beta_1, \beta_2} \beta_1 \beta_2 P_Q(\beta_1, \beta_2 | \Omega_{B_1}, \Omega_{B_2}, c, \lambda)$? But what meaning is there to the quantum expression when the corresponding operators $\hat{\Omega}_{B_1}, \hat{\Omega}_{B_2}$ do not commute?

None of these questions arise in considering whether spin squeezing is a test for standard quantum entanglement, since no hidden variables are involved nor are issues of the existence of probabilities for measurement of individual sub-system operators that may become involved in the evaluation. However, when non-quantum LHV expressions for measurement probabilities are involved, the analogous results to those for quantum mean values need further consideration. Until these issues are resolved we cannot begin to modify the operator based proof regarding the consequences for spin variances and means for the LHS state. The proof would involve expressions giving meaningful interpretations to the mean values of what would appear to be non-physical quantities such as mode annihilation and creation operators for sub-system $A$.

6.2 General Results for Mean and Variance in LHV Theory

Before dealing with the above issues it is useful to prove some results for mean values and variances in LHV that are analogous to similar results in quantum theory. We now consider the measurement of an observable $\Omega$ with outcomes $\omega$ for a preparation process $c$. The probability $P(\omega|\Omega, c)$ for this outcome can be
written in LHV as

\[ P(\omega|\Omega, c) = \sum_{\lambda} P(\lambda|c) P(\omega|\Omega, c, \lambda) \]  

(128)

where \( \lambda \) are the hidden variables and \( P(\lambda|c) \) is the probability for preparation process \( c \) that the hidden variables are \( \lambda \) and \( P(\omega|\Omega, c, \lambda) \) is the probability of outcome \( \omega \) for measurement of \( \Omega \) when the hidden variables are \( \lambda \).

The mean value for measurement outcomes for observable \( \Omega \) will then be given by

\[ \langle \Omega \rangle = \sum_{\omega} \omega P(\omega|\Omega, c) \]  

(129)

\[ = \sum_{\lambda} P(\lambda|c) \langle \Omega(\lambda) \rangle \]  

(130)

\[ \langle \Omega(\lambda) \rangle = \sum_{\omega} \omega P(\omega|\Omega, c, \lambda) \]  

(131)

where the first equation is the definition and the second equation shows that the mean value is given by weighting the mean value \( \langle \Omega(\lambda) \rangle \) that would apply if the hidden variables are \( \lambda \) by the probability \( P(\lambda|c) \) for these hidden variables when the preparation is \( c \). The result (130) is similar to the quantum result for the mixed state \( \hat{\rho} = \sum_{R} P_{R} \hat{\rho}_{R} \) where \( \langle \hat{\Omega} \rangle = \sum_{R} P_{R} \langle \hat{\Omega}_{R} \rangle \) where \( \langle \hat{\Omega}_{R} \rangle = Tr(\hat{\Omega} \hat{\rho}_{R}) \).

The result for the mean value of a function \( F(\Omega) \) would be

\[ \langle F(\Omega) \rangle = \sum_{\lambda} P(\lambda|c) \langle F(\Omega)_{\lambda} \rangle \]  

(132)

\[ \langle F(\Omega)_{\lambda} \rangle = \sum_{\omega} F(\omega) P(\omega|\Omega, c, \lambda) \]

In the case where the outcomes for two observables \( \Omega \) and \( \Lambda \) with outcomes \( \omega \) and \( \mu \) for a preparation process \( c \), the mean value for a function \( F(\Omega, \Lambda) \) would be

\[ \langle F(\Omega, \Lambda) \rangle = \sum_{\lambda} P(\lambda|c) \langle F(\Omega, \Lambda)_{\lambda} \rangle \]  

(133)

\[ \langle F(\Omega, \Lambda)_{\lambda} \rangle = \sum_{\omega\mu} F(\omega, \mu) P(\omega, \mu|\Omega, \Lambda, c, \lambda) \]

This result will be useful when we consider steering tests.

The variance for measurement outcomes for observable \( \Omega \) will then be given by

\[ \langle \Delta \Omega^2 \rangle = \sum_{\omega} (\omega - \langle \Omega \rangle)^2 P(\omega|\Omega, c) \]  

(134)

\[ = \sum_{\omega} (\omega^2 - 2\omega \langle \Omega \rangle + \langle \Omega \rangle^2) P(\omega|\Omega, c) \]

\[ = \langle \Omega^2 \rangle - \langle \Omega \rangle \]  

(135)

\[ \langle \Omega^2 \rangle = \sum_{\omega} \omega^2 P(\omega|\Omega, c) \]  

(136)
where the first equation is the definition and the third equation shows that the variance is given by the difference between the mean of the squared observable and the square of the mean, as in standard statistics. Here we have used $\sum P(\Omega|\omega, c) = 1$ and (129). We can then write

$$\langle \Omega^2 \rangle = \sum_{\lambda} P(\lambda|c) \langle \Omega^2(\lambda) \rangle$$  \hspace{1cm} (137)

$$\langle \Omega^2(\lambda) \rangle = \sum_{\omega} \omega^2 P(\omega|\Omega, \lambda, c)$$  \hspace{1cm} (138)

where the second line gives the definition for the mean of the square of the observable when the hidden variables are $\lambda$ and the first line expresses the mean of the square of the observable in terms of an average over this quantity.

We then have

$$\langle \Delta \Omega^2 \rangle = \sum_{\lambda} P(\lambda|c) \langle \Delta \Omega^2(\lambda) \rangle - \langle \sum_{\lambda} P(\lambda|c) \langle \Omega(\lambda) \rangle \rangle^2$$

$$\geq \sum_{\lambda} P(\lambda|c) \langle \langle \Omega^2(\lambda) \rangle - \langle \Omega(\lambda) \rangle^2 \rangle + \sum_{\lambda} P(\lambda|c) \langle \Omega(\lambda) \rangle^2 - \langle \sum_{\lambda} P(\lambda|c) \langle \Omega(\lambda) \rangle \rangle^2$$

$$\geq \sum_{\lambda} P(\lambda|c) \langle \langle \Omega^2(\lambda) \rangle - \langle \Omega(\lambda) \rangle^2 \rangle$$  \hspace{1cm} (139)

which establishes an important inequality. The second line follows from the modulus of a sum being less than the sum of the moduli and the last line follows from the Cauchy inequality $\sum P_R C_R \geq \langle \sum P_R \sqrt{C_R} \rangle^2$ with $\sqrt{C_R} = |\langle \Omega(\lambda) \rangle|$. But we also have

$$\langle \Delta \Omega^2(\lambda) \rangle = \sum_{\omega} (\omega - \langle \Omega(\lambda) \rangle)^2 P(\omega|\Omega, c, \lambda)$$  \hspace{1cm} (140)

$$= \sum_{\omega} \omega^2 P(\omega|\Omega, c, \lambda) - \langle \Omega(\lambda) \rangle^2$$

$$= \langle \Omega^2(\lambda) \rangle - \langle \Omega(\lambda) \rangle^2$$  \hspace{1cm} (141)

showing that when the hidden variable is $\lambda$ the variance for measured outcomes of observable $\Omega$ is equal to the difference between the mean value for measured outcomes of the square of the observable and the square of the mean value (as expected).

We finally have the inequality

$$\langle \Delta \Omega^2 \rangle \geq \sum_{\lambda} P(\lambda|c) \langle \Delta \Omega^2(\lambda) \rangle$$  \hspace{1cm} (142)

This result may be compared to the quantum theory result $\langle \Delta \hat{\Omega}^2 \rangle \geq \sum_R P_R \langle \Delta \hat{\Omega}^2 \rangle_R$.

Finally, we consider mean values in LHV for complex combinations of observables $\Omega_1$ and $\Omega_2$, which have measured outcomes $\omega_1$ and $\omega_2$. We simply define

$$\langle (\Omega_1 + i\Omega_2) \rangle = \langle \Omega_1 \rangle + i \langle \Omega_2 \rangle$$  \hspace{1cm} (143)
where in HVT we have

\[
\langle \Omega_1 \rangle = \sum_\lambda P(\lambda|c) \sum_{\omega_1} \omega_1 P(\Omega_1|\omega_1, c, \lambda)
\]

\[
\langle \Omega_2 \rangle = \sum_\lambda P(\lambda|c) \sum_{\omega_2} \omega_2 P(\Omega_2|\omega_2, c, \lambda)
\]

(144)

since the two fundamental probabilities \(P(\omega_1|\Omega_1, c, \lambda)\) and \(P(\omega_2|\Omega_2, c, \lambda)\) always exist in a LHV, even if in quantum theory the corresponding operators \(\hat{\Omega}_1\) and \(\hat{\Omega}_2\) do not commute. This is an important feature to recognise about LHV. The result (143) may be compared to the quantum result (120). Thus, we see that many results in LHV are analogous to the results in quantum theory.

With these results now established we can see that for the LHS model the mean values for jointly measuring \(\Omega_A\) in sub-system \(A\) and \(\Omega_B\) in sub-system \(B\) for preparation \(c\) would be given by

\[
\langle \Omega_A \otimes \Omega_B \rangle = \sum_\lambda P(\lambda|c) \langle \Omega_A(\lambda) \rangle \langle \Omega_B(\lambda) \rangle_Q
\]

(145)

where \(\langle \Omega_A(\lambda) \rangle = \sum_\alpha \alpha P(\alpha|\Omega_A, c, \lambda)\) and \(\langle \Omega_B(\lambda) \rangle_Q = \sum_\beta \beta P_Q(\beta|\Omega_B, c, \lambda) = Tr(\hat{\Omega}_B \hat{\rho}_B^c)\) are the definitions of the mean values for measurement outcomes for \(\Omega_A\) and \(\Omega_B\). The latter is also determined from quantum theory, the former is not. Variances can be obtained based on considering the mean values of the squares of \(\Omega_A\) and \(\Omega_B\). The similarities and differences between the LHS and the QSS expressions (145) and (121) should be noted.
7 Appendix B - Werner States

As examples of the three categories of Bell local states we may consider the states introduced by Werner [11] as $U \otimes U$ invariant states ($\left( \hat{U} \otimes \hat{U} \right) \hat{\rho}_W \left( \hat{U}^\dagger \otimes \hat{U}^\dagger \right) = \hat{\rho}_W$, where $\hat{U}$ is any unitary operator) for two $d$ dimensional sub-systems. Depending on the parameter $\eta$ (or $\phi$) the Werner states may be separable or entangled. They may also be Bell local in one of the three categories described above, or they may be Bell non-local. The density operator for the Werner states is given by

$$\hat{\rho}_W = (d^3 - d)^{-1} \left[ (d - \phi) \hat{1} + (d \phi - 1) \hat{V} \right]$$

where $\hat{1}$ is the unit operator and $\hat{V}$ is the flip operator ($\hat{V} \left( |\psi\rangle \otimes |\chi\rangle \right) = \left( |\chi\rangle \otimes |\psi\rangle \right)$). The two expressions are interconvertable with $\phi = (1 - (d + 1)\eta)/d$. For a positive density operator we have $-1 \leq \phi \leq +1$. Werner had showed that if $\eta < 1/(d + 1)$ (or $\phi > 0$) the state $\hat{\rho}_W$ is separable, for $\eta > 1/(d + 1)$ (or $\phi < 0$) the state was entangled. Thus Werner states with $\eta < 1/(d + 1)$ or $\phi > 0$ are separable. Wiseman et al [5] considered the above categories for such Werner states and determined the parameter boundaries for the various categories. These results are shown in Figure 2 (taken from Figure 1 in Ref [5]), where the parameter regimes for the various categories of quantum states are explained.
Figure 2: Parameter $\eta$ (see text) boundaries for Werner States. The blue line corresponds to $\eta = 1/(d+1)$, the red line to $\eta = (1 - d^{-1})$ and the green line to $\eta = 1$ for $d \geq 3$. For $\eta$ below blue line the states are Category 1 - separable states. These states are also Bell local, LHS and non-steerable. For $\eta$ between blue line and red line the states are Category 2. These states are also Bell local, non-steerable and entangled. For $\eta$ between red line and green line the states are Category 3 - Bell local, steerable and entangled (EPR entangled). For $\eta$ above green line the states are Category 4 - Bell non-local, steerable and entangled. This is only possible for $d = 2$. Figure taken from Wiseman et al Ref 5.
8 Appendix C - Idea of EPR Steering

In this Appendix we consider for reasons of completeness the physical idea behind EPR steering, as presented in the papers [5], [6] and [7].

We can derive expressions within LHV theory for the conditional probabilities defined in [5]. These expressions apply for all three Bell local categories considered here. We will focus on LHS states, which in terms of our LHVCS may be either in Category 1 or Category 2. We will initially consider the latter.

In the case of Category 2 states (which are LHS states) we obtain from (42) and (5)

\[ P(\beta|\Omega_B|\alpha, \Omega_A, c) = \sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) \text{Tr}_B((\hat{\Pi}_B^{\beta})\rho_B^{\beta}(\lambda)) \sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P(\lambda|c) \] (147)

using (18) and (44).

It is also important to realise that these LHS model states are still related to an overall quantum state, but one which is non-separable since we cannot derive the density operator (34) for separable states from Category 2 expression (42) for the joint probability. For Category 2 LHS states, \( P(\alpha|\Omega_A, c, \lambda) \) is not given by a quantum expression. However, as in [6], [7] we can relate the quantities in the LHS model (42) to a density operator for sub-system B that is conditional on the results for measurements on sub-system A.

From (7) the quantum theory result for the probability that measurement of observable \( \Omega_A \) results in outcome \( \alpha \) is given by

\[ P(\alpha|\Omega_A, \rho) = \text{Tr}((\hat{\Pi}_A^\alpha \otimes \hat{1}_B) \hat{\rho}) \] (148)

where \( \hat{\rho} \) is the density operator for the overall quantum state (the preparation symbol \( c \) is left out for simplicity). In the Copenhagen interpretation of quantum theory the normalised state that is produced as a result of this measurement is the conditional state

\[ \hat{\rho}_{\text{cond}}(\alpha|\Omega_A, \rho) = (\hat{\Pi}_A^\alpha \otimes \hat{1}_B) \hat{\rho} (\hat{\Pi}_A^\alpha \otimes \hat{1}_B) / P(\alpha|\Omega_A, \rho) \] (149)

This state has a trace of unity, as required. To confirm that \( \hat{\rho}_{\text{cond}}(\alpha|\Omega_A, \rho) \) does lead to the correct quantum expression for the conditional probability \( P(\beta|\Omega_B|\alpha|\Omega_A, \rho) \) that measurement of \( \Omega_B \) in sub-system B will result outcome \( \beta \) given that measurement of \( \Omega_A \) resulted in outcome \( \alpha \) based on the quantum state \( \hat{\rho} \), we calculate the probability of that measurement of \( \Omega_B \) in
Bayes’ theorem (5) following from sub-system B state describing sub-system post-measurement $\hat{\rho}$ to a new conditioned state $\rho_{\Omega}$. The key idea is that when a measurement of $\hat{A}$ is made on sub-system $A$ resulting in outcome $\alpha$ (the bipartite quantum state prepared being $\rho$) this results in both the overall quantum state changing, and hence the quantum state for both the overall system and its sub-system $B$ changing to a new conditioned state $\hat{\rho}_{\text{cond}}(\alpha|\Omega_A, \rho)$ (given in Eq. (149)) and hence the post-measurement state describing sub-system $B$ changing to

$$\hat{\rho}_{\text{cond}}(\alpha|\Omega_A, \rho)^B = Tr_A(\hat{\rho}_{\text{cond}}(\alpha|\Omega_A, \rho))$$

from its pre-measurement state $\hat{\rho}^B = Tr_A(\hat{\rho})$ given by the reduced density operator $[8]$. This strange quantum effect allows for an experiment carried out on sub-system $A$ to instantly change (or "steer") the quantum state for sub-system $B$ into a new quantum state, even when the two sub-systems are localised in well-separated spatial regions and the experimenter on $A$ may have no direct access to sub-system $B$. For those who accept the Copenhagen interpretation of quantum theory there is nothing really strange involved. Quantum states merely specify all that can be known about the physical state (and no distinction between "physical state" and "quantum state" is made), so as the measurement of $\Omega_A$ has led to a particular outcome $\alpha$ our knowledge about the state has changed, and hence the quantum state for both the overall system and its sub-systems should change accordingly. Using quantum theory we can obtain an explicit formula for $\hat{\rho}_{\text{cond}}(\alpha|\Omega_A, \rho)^B$ and this is

$$\hat{\rho}_{\text{cond}}(\alpha|\Omega_A, \rho)^B = \sum_{\beta_i, \gamma_n} |B\beta_i\rangle \langle B\gamma_n| \left( \sum_i \hat{\rho}_{A\alpha i, B\beta l} : A\alpha i, B\gamma n \right)$$

where the original density operator $\rho$ is expressed in terms of orthonormal basis states $|A\alpha i\rangle \otimes |B\beta n\rangle$ that are eigenstates for $\hat{A}_A$ and $\hat{A}_B$, with $i = 1, 2, ..., d_A$ and $n = 1, 2, ..., d_B$ allowing for degeneracy.

We can also show that the sum of the conditional density operators $\hat{\rho}_{\text{cond}}(\alpha|\Omega_A, \rho)^B$ each weighted by the probability $P(\alpha|\Omega_A, \rho)$ for the measurement outcome $\alpha$ for $\Omega_A$ gives the reduced density operator $\hat{\rho}^B$ associated with the original state $\rho$. This result is not surprising, since carrying out the measurement of any choice of $\Omega_A$ and then discarding the results would be described by reduced density
operator.

\[ \sum_{\alpha} P(\alpha|\Omega_A, \rho) \hat{\rho}_{cond}(\alpha|\Omega_A, \rho)^B = \hat{\rho}^B = Tr_A \hat{\rho} \] (153)

The proofs of (152) and (153) are straightforward.

Thus, we have seen how according to quantum theory the quantum state describing sub-system B changes as a result of measuring \( \Omega_A \) on sub-system A and obtaining outcome \( \alpha \). Furthermore, we have obtained quantum theory expressions (150) for the conditional probability \( P(\beta|\Omega_B, \rho_{cond}) \) for measurement of \( \Omega_B \) on sub-system B and obtaining outcome \( \beta \) when measurement of \( \Omega_A \) on sub-system A resulted in outcome \( \alpha \) and (152) for the quantum state describing sub-system B. The question then is: Although quantum theory gives the correct results for the conditional probability \( P(\beta|\Omega_B, \rho_{cond}) \), can the same results also be explained in a local hidden variable theory?

Following the operational definition for steering in Refs. [5], [6] and [7], the quantum state \( \rho \) is only considered to be EPR steerable when the conditional probability \( P(\beta|\Omega_B|\alpha, \Omega_A, c) \) can not be explained via a local hidden variable theory. For the LHS cases of Category 1 and Category 2 states we will see that a LHV theory explanation applies. We consider what expression for a density operator for sub-system B would give the LHS result for the conditional probability \( P(\beta|\Omega_B|\alpha, \Omega_A, c) \) for measurement of \( \Omega_B \) to have outcome \( \beta \), given that measurement of \( \Omega_A \) has outcome \( \alpha \) and the preparation process is \( c \).

In the case of Category 2 states we use Eqs. (42) and (44) in conjunction with (5) and (18) to find

\[ P(\beta|\Omega_B|\alpha, \Omega_A, c) = \frac{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) Tr_B((\hat{\Pi}_B^\beta)^B(\lambda)) \hat{\rho}^B(\lambda) P(\lambda|c)}{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P(\lambda|c) P(\lambda|c)} \] (154)

We then define a new normalised quantum state \( \hat{\rho}_{cond}(\alpha|\Omega_A, c) \) for sub-system B by the expression

\[
\hat{\rho}_{cond}(\alpha|\Omega_A, c) = \frac{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) \hat{\rho}^B(\lambda) P(\lambda|c)}{Tr_B \left( \sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) \hat{\rho}^B(\lambda) P(\lambda|c) \right)} = \frac{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) \hat{\rho}^B(\lambda) P(\lambda|c)}{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P(\lambda|c)} \] (155)

It is to be noted that this state for sub-system B involves local HVT and not quantum expressions for the measurement probabilities \( P(\alpha|\Omega_A, c, \lambda) \) for sub-system A. We then see from [7] that for this state the probability for measure-
ment of $\Omega_B$ to have outcome $\beta$ is given by

$$
Tr_B(\hat{\Pi}_\beta \hat{\rho}_{\text{cond}}^B(\alpha|\Omega_A, c)) = \frac{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) Tr_B((\hat{\Pi}_\beta^B)\hat{\rho}^B(\lambda)) P(\lambda|c)}{\sum_{\lambda} P(\alpha|\Omega_A, c, \lambda) P(\lambda|c)}
$$

$$
= P(\beta|\Omega_B||\alpha, \Omega_A, c)
$$

which is the same as (147) obtained for the Category 2 states (which are LHS states). Thus the sub-system $B$ quantum state (155) has been constructed purely from the Category 2 LHS model probabilities $P(\alpha|\Omega_A, c, \lambda)$ and $P(\lambda|c)$, together with the LHS model quantum state $\hat{\rho}^B(\lambda)$ - which is a possible quantum state for sub-system $B$ based on hidden variables $\lambda$. The sub-system $B$ quantum state $\hat{\rho}_{\text{cond}}^B(\alpha|\Omega_A, c)$ in (155) determines the correct probability for measurement of $\Omega_B$ to have outcome $\beta$. The same analysis would apply to the LHS states in Category 1, the only difference being that $P(\alpha|\Omega_A, c, \lambda)$ would be replaced by $P_Q(\alpha|\Omega_A, c, \lambda)$ in terms of our notation. So in both of these cases there could be a hidden state $\hat{\rho}^B(\lambda)$ associated with hidden variables that could explain (along with suitable choices for $P(\alpha|\Omega_A, c, \lambda)$ and $P(\lambda|c)$) the measurements on sub-system $B$. The treatment however does not apply to the quantum states in Category 3, where the LHV model in Eq.(43) does not include a quantum state $\hat{\rho}^B(\lambda)$ for sub-system $B$. Hence, the conditional probability $P(\beta|\Omega_B||\alpha, \Omega_A, c)$ can be explained via the LHS model for both Category 1 and Category 2 states, showing that the Category 1 and Category 2 quantum states are non-steerable. However, the Category 3 states are EPR steerable.
9 Appendix D - Variances of Two Mode Quadratures - Category 2 States

We use the LHVT - quantum theory equivalences (27) to replace (83) and (84) by their quantum forms. Quantum forms for the variances are then obtained. After reverting back to LHVT expressions results for the variances \( \langle \Delta X_\theta(\pm)^2 \rangle \) and \( \langle \Delta P_\theta(\pm)^2 \rangle \) for Category 2 states are obtained, as follows.

For the mean values of the two mode quadratures we have

\[
\langle X_\theta(\pm) \rangle = \langle \hat{X}_\theta(\pm) \rangle = \frac{1}{\sqrt{2}} (\langle \hat{x}_A \rangle \cos \theta + \langle \hat{p}_A \rangle \sin \theta)
\]

\[
\langle X_\theta(\pm)^2 \rangle = \langle \hat{X}_\theta(\pm)^2 \rangle
= \frac{1}{2} \left( (\langle \hat{x}_A^2 \rangle \cos^2 \theta + (\langle \hat{x}_A \hat{p}_A + \hat{p}_A \hat{x}_A \rangle) \sin \theta \cos \theta + (\langle \hat{p}_A^2 \rangle \sin^2 \theta) \right)
+ \frac{1}{2} \left( \langle \hat{N}_B \rangle + 1 \right)
\]

(157)

(158)

on substituting for \( \hat{U}_A \) from (54).

The variance is then given by

\[
\langle \Delta X_\theta(\pm)^2 \rangle = \langle \Delta \hat{X}_\theta(\pm)^2 \rangle
= \frac{1}{2} \left( (\langle \Delta \hat{x}_A \cos \theta + \Delta \hat{p}_A \sin \theta \rangle \langle \Delta \hat{x}_A \cos \theta + \Delta \hat{p}_A \sin \theta \rangle) \right) + \frac{1}{2} \left( \langle \hat{N}_B \rangle + \frac{1}{2} \right)
\]

(159)

where \( \Delta \hat{x}_A = \hat{x}_A - \langle \hat{x}_A \rangle, \Delta \hat{p}_A = \hat{p}_A - \langle \hat{p}_A \rangle \). The expression for \( \langle \Delta P_\theta(\pm)^2 \rangle \) can be obtained using \( \hat{P}_\theta(\pm) = \hat{X}_{\theta+\pi/2}(\pm) \).

However, we can make use of the SSR to simplify these expressions further. As shown in SubSection 2.2 the reduced density operator for sub-system A satisfies the local particle number SSR. Consequently

\[
\langle \hat{x}_A \rangle = Tr_A(\hat{x}_A \hat{\rho}^A) = \langle x_A \rangle = 0 \quad \langle \hat{p}_A \rangle = Tr_A(\hat{p}_A \hat{\rho}^A) = \langle p_A \rangle = 0
\]

(160)

using the same arguments as for \( \langle x_B(\lambda) \rangle_Q \) and \( \langle p_B(\lambda) \rangle_Q \) in Eq. (64). Furthermore, the same steps as for \( \langle x_B^2(\lambda) \rangle_Q \), \( \langle p_B^2(\lambda) \rangle_Q \) and \( \langle U_B(\lambda) \rangle_Q \) lead to

\[
\langle \hat{x}_A^2 \rangle = \langle x_A^2 \rangle = \langle \hat{N}_A \rangle + \frac{1}{2} \quad \langle \hat{p}_A^2 \rangle = \langle p_A^2 \rangle = \langle \hat{N}_A \rangle + \frac{1}{2}
\]

\[
\langle \hat{U}_A \rangle = \langle U_A \rangle = 0
\]

(161)
Using these results we then find that

\[
\langle \Delta X_\theta(\pm)^2 \rangle = \langle \Delta \hat{X}_\theta(\pm)^2 \rangle = \frac{1}{2} \left( \langle \hat{N}_A \rangle + \frac{1}{2} \right) + \frac{1}{2} \left( \langle \hat{N}_B \rangle + \frac{1}{2} \right)
\]

\[
= \frac{1}{2} \langle \hat{N} \rangle + \frac{1}{2}
\]

\[
\langle \Delta P_\theta(\pm)^2 \rangle = \langle \Delta \hat{P}_\theta(\pm)^2 \rangle = \frac{1}{2} \langle \hat{N} \rangle + \frac{1}{2}
\]

Exactly the same results apply for Category 1 (separable) states (see Appendix L in Ref. [3]).
10 Appendix E - Correlation Inequalities and Spin Operators

The inequalities (93), (95) and (97) derived above can be put into a more useful form involving spin operators - whose mean values and variances can be measured. We use the definitions of the spin operators in SubSection 4.2 (see also Ref. [3])

\[ \left| \langle \hat{a} \hat{b} \rangle \right|^2 = \langle \hat{S}_x \rangle^2 + \langle \hat{S}_y \rangle^2 \]
\[ \hat{N}_A = \frac{1}{2} \hat{N} - \hat{S}_z \quad \hat{N}_B = \frac{1}{2} \hat{N} + \hat{S}_z \]
\[ \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{\hat{N}}{2} \left( \frac{\hat{N}}{2} + 1 \right) \] (163)

We see that

\[ \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle \geq \frac{1}{4} \left( \langle \hat{N}_A + \hat{N}_B \rangle^2 \right) + \frac{1}{2} \langle \hat{N}_A + \hat{N}_B \rangle - \left| \langle \hat{a} \hat{b} \rangle \right|^2 - \frac{1}{4} \langle \hat{N}_B - \hat{N}_A \rangle^2 \]
\[ \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle \geq \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{N}_A \otimes \hat{B}_B \rangle + \frac{1}{2} \langle \hat{I}_A \otimes \hat{N}_B \rangle \]
\[ - \langle \hat{N}_A \otimes \hat{N}_B \rangle \quad \text{Cat 1 States} \]
\[ \geq \frac{1}{2} \left( \langle \hat{N}_A \otimes \hat{B}_B \rangle + \frac{1}{2} \langle \hat{I}_A \otimes \hat{N}_B \rangle \right) \quad \text{Cat 1 States} \]
\[ \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle \geq \langle \hat{N}_A \otimes \hat{B}_B \rangle + \frac{1}{2} \langle \hat{N}_A \otimes \hat{B}_B \rangle + \frac{1}{2} \langle \hat{I}_A \otimes \hat{N}_B \rangle \]
\[ - \left( \langle \hat{N}_A + \frac{1}{2} \hat{I}_A \rangle \otimes \langle \hat{N}_B \rangle \right) \quad \text{Cat 2 States} \]
\[ \geq \frac{1}{2} \langle \hat{N}_A \otimes \hat{B}_B \rangle \quad \text{Cat 2 States} \]
\[ \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle \geq \langle \hat{N}_A \otimes \hat{B}_B \rangle + \frac{1}{2} \langle \hat{N}_A \otimes \hat{B}_B \rangle + \frac{1}{2} \langle \hat{I}_A \otimes \hat{N}_B \rangle \]
\[ - \left( \langle \hat{N}_A + \frac{1}{2} \hat{I}_A \rangle \otimes \langle \hat{B}_B + \frac{1}{2} \hat{I}_B \rangle \right) \quad \text{Cat 3 States} \]
\[ \geq - \frac{1}{4} \quad \text{Cat 3 States} \] (164)

So we have

\[ \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{2} \langle \hat{N} \rangle \geq 0 \quad \text{Cat 1 States} \]
\[ \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{2} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle \geq 0 \quad \text{Cat 2 States} \]
\[ \langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle + \frac{1}{4} \geq 0 \quad \text{Cat 3 States} \] (165)
11 Appendix F - Spin Variances - EPR Steering Test

The EPR steering test in (109) can also be obtained more directly without using the strong correlation inequalities from the results in Section 4.3. Using (76), (81) and (77). We find for Category 2 states

$$\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle \geq \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle$$

$$\geq 0$$

(166)

This result was also obtained in Appendix 10. Details are:

$$\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle \geq \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle$$

$$- \frac{1}{4} \langle \hat{1}_A \otimes \hat{N}_B \rangle - \frac{1}{4} \langle \hat{N}_A \otimes \hat{1}_B \rangle$$

$$+ \frac{1}{4} \langle \hat{1}_A \otimes \hat{1}_B \rangle - \frac{1}{4} \langle \hat{N}_A \otimes \hat{1}_B \rangle$$

$$\geq \langle \hat{N}_A \otimes \hat{N}_B \rangle + \frac{1}{2} \langle \hat{1}_A \otimes \hat{N}_B \rangle$$

$$\geq 0$$

since both $\langle \hat{N}_A \otimes \hat{N}_B \rangle$ and $\langle \hat{1}_A \otimes \hat{N}_B \rangle$ are positive quantities. In this form it shows that if $\langle \Delta \hat{S}_x^2 \rangle + \langle \Delta \hat{S}_y^2 \rangle - \frac{1}{4} \langle \hat{N} \rangle + \frac{1}{2} \langle \hat{S}_z \rangle < 0$ then the state cannot be Category 2.