Universal spin squeezing from the tower of states of $U(1)$-symmetric spin Hamiltonians

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Spin squeezing – a central resource for quantum metrology – results from the non-linear, entangling evolution of an initially factorized spin state. Here we show that universal squeezing dynamics is generated by a very large class of $S = 1/2$ spin Hamiltonians with axial symmetry, in relationship with the existence of a peculiar structure of the low-lying Hamiltonian eigenstates – the so-called Anderson’s tower of states. Such states are fundamentally related to the appearance of spontaneous symmetry breaking in quantum systems, and they are parametrically close to the eigenstates of a planar rotor (Dicke states), in that they feature an anomalously large value of the total angular momentum. We show that, starting from a coherent spin state, a generic $U(1)$-symmetric Hamiltonian featuring the Anderson’s tower of states generates the same squeezing evolution at short times as the one governed by the paradigmatic one-axis-twisting (or planar rotor) model of squeezing dynamics. The full squeezing evolution is seemingly reproduced for interactions decaying with distance $r$ as $r^{-\alpha}$ when $\alpha < 5d/3$ in $d$ dimensions. Our results connect quantum simulation with quantum metrology by unveiling the squeezing power of a large variety of Hamiltonian dynamics that are currently implemented by different quantum simulation platforms.

Introduction. The controlled generation and manipulation of massively entangled quantum states is one of the central tasks of modern quantum technology platforms [1]. In the context of $S = 1/2$ (or qubit) ensembles, a fundamental class of entangled quantum states is represented by spin-squeezed states [2], namely states which feature a net polarization of the collective spin, along with suppressed fluctuations of a spin component transverse to the polarization axis. Introducing the collective spin of $N$ qubits, $J = \sum_{i=1}^{N} \hat{S}_i$ – where $\hat{S}_i$ is a $S = 1/2$ spin operator – and assuming that $\langle J^{y,z} \rangle = 0$ while $\langle J^{x} \rangle \neq 0$, a state is spin-squeezed if [3]

$$\xi^2_R = \frac{N \min_{\perp} [\text{Var}(\hat{J}^z)]}{\langle J^x \rangle^2} < 1 \quad (1)$$

where $\min_{\perp}$ indicates the minimization over the spin components transverse to the polarization axis $x$. A state exhibiting squeezing is entangled [4]; specifically, if $\xi^2_R < 1/k$ with integer $k$, the state cannot be represented as a separable state among clusters of less than $k+1$ spins [5,7], and, most importantly, it offers a fundamental metrological advantage over separable states when used as the input for Ramsey interferometry [3]. Identifying different many-body dynamics leading to spin squeezing is therefore of paramount importance: the entanglement of the resulting states can be certified, and their metrological potential exploited by accessing their most basic property, namely the collective spin.

The paradigmatic spin-squeezing dynamics is the one generated by the so-called one-axis-twisting (OAT) Hamiltonian [8]

$$\hat{H}_{OAT} = \frac{(\hat{j}^x)^2}{2I}, \quad (2)$$

namely, the Hamiltonian of a planar rotor with moment of inertia $I$. For the energy to be extensive, one must assume that $I \sim N$. Under this assumption, starting from the initial state $|CSS\rangle = \otimes_{i=1}^{N} |\downarrow_2\rangle_i$ (the coherent spin state - CSS - polarized along the x axis) the strongest squeezing is achieved at a time $t_{\text{min}} \sim N^{-2/3}I \sim N^{1/3}$; and it scales as $\xi^2_{R,\text{min}} \sim N^{-2/3}$ [8]. This Hamiltonian (as well as related ones) and the corresponding squeezing dynamics have been realized in seminal experiments exploiting interactions in Bose-Einstein condensates [9–11] and in trapped ions [12], to cite a few relevant examples.

Interestingly, the OAT Hamiltonian of Eq. (2) plays also a special role in condensed matter, to explain the mechanism of spontaneous breaking of a continuous symmetry in quantum spin models [13–15]. Hereafter, we consider models with an axial rotational ($U(1)$) symmetry, and we shall choose the $z$ axis as the symmetry axis. Such models have a ground state which is also $U(1)$ symmetric for any finite size. As suggested by Anderson in pioneering works [16,17], a quantum spin model in the thermodynamic limit can break a continuous symmetry (such as $U(1)$) by developing a finite order parameter due to the existence of a set of low-energy Hamiltonian eigenstates – the so-called Anderson’s tower of states (ToS). These states are approximately eigenstates of a planar rotor Hamiltonian of the kind of Eq. (2); and their energy decreases as $O(1/N)$ in the thermodynamic limit (due to the scaling of the moment of inertia $I$), making them nearly degenerate with the $U(1)$ symmetric ground state. Spontaneous symmetry breaking (SSB) is therefore the result of the collapse of the ToS.

The central insight of the present work is that generic quantum spin Hamiltonians can produce a spin-squeezing dynamics thanks to the emergent ToS structure of their low-energy eigenstates – in other words, several models with $U(1)$ symmetry generate the same dynamics as
that of the OAT Hamiltonian Eq. (2) at sufficiently short times, irrespective of the nature (long-range vs. short-range) of their interactions. This is strictly true when the dynamics is initialized in a CSS polarized in the symmetry plane. Such a state, maximizing the total spin \( \langle \hat{J}^2 \rangle \) with \( \alpha = \infty \), has a very strong overlap with the ToS, because the latter states have in turn an anomalously large average value of \( \hat{J}^2 \) among all the states in the same energy range – namely they behave as “quantum many-body scars” [18, 19], whose presence alters profoundly the dynamics of the system with respect to a generic initial state with the same initial energy. We underpin this insight with a combination of exact diagonalization (ED) and time-dependent variational calculations of the dynamics of various \( S = 1/2 \) XX models, fully corroborating the universal picture of spin-squeezing dynamics. Our results imply that a large variety of current quantum simulation setups implementing \( U(1) \) symmetric models of quantum magnetism [20, 21] can be viewed as generators of spin-squeezed states, of potential immediate interest for quantum metrology tasks.

Models and method. We specialize our attention to the case of XX models with power-law decaying interactions (\( \alpha \)-XX model), whose Hamiltonian reads

\[
\hat{H} = -\sum_{i < j} J_{ij}^{(\alpha)} \left( \hat{S}_i^x \hat{S}_j^x + \hat{S}_i^y \hat{S}_j^y + \hat{S}_i^z \hat{S}_j^z \right),
\]

where \( J_{ij}^{(\alpha)} = J |\mathbf{r}_i - \mathbf{r}_j|^{-\alpha} \) with \( \alpha \geq 0 \). The limit \( \alpha = 0 \) corresponds to infinite-range interactions (reproducing the OAT Hamiltonian), while the opposite limit \( \alpha \to \infty \) corresponds to nearest-neighbor interactions [25]. The above Hamiltonian occupies a prominent role in quantum simulation, as it is currently implemented on various lattice geometries by a rich variety of very different quantum simulation platforms, including trapped ions (\( 0 < \alpha < 3 \)) [20]; Rydberg atoms (\( \alpha = 3 \)) [21]; spinor gases (\( \alpha = \infty \)) [22]; and superconducting circuits (\( \alpha = \infty \)) [23, 24].

Throughout the paper, we will consider the quench dynamics generated by XX Hamiltonians starting from the CSS \( x \) state, \( |\Psi(t)\rangle = \exp(-i\hat{H}t)|\text{CSS}_x\rangle \). Under this assumption, the sign of the \( J \) coupling is irrelevant as long as one follows the expectation value of operators which are real matrices (e.g. on the computational basis of the eigenstates of the \( \hat{S}_i^z \) operators) [26]. Hereafter we will assume \( J > 0 \) for definiteness. We shall study this model on lattices with \( N = L^d \) sites (in \( d = 1 \) and 2) with periodic boundary conditions. With \( J > 0 \), SSB in the ground state is expected for \( \alpha < 3 \) in \( d = 1 \) [27] and for all values of \( \alpha \) in \( d = 2 \) [28].

Studying the real-time dynamics generated by the Hamiltonian with \( \alpha > 0 \) is generically a challenging prob-
lem. Recent results based on a semi-classical approach \cite{31} already indicate the robustness of squeezing when moving away from the $\alpha = 0$ limit. Here we adopt a different strategy, going beyond any semi-classical framework. Specifically, we use exact diagonalization for small systems \cite{30,31}, and a time-dependent variational Monte Carlo (tVMC) approach to tackle large system sizes. The latter are both relevant to current experimental realizations and instrumental to our scaling analysis of the spin-squeezing dynamics discussed below. Our tVMC calculations are based on the pair-product Ansatz (or two-spin long-range entangled-plaquette state, 2LR-EPS \cite{32})

$$|\Psi(t)\rangle = \sum_{\sigma} \prod_{i<j} \psi_{ij}(\sigma_i, \sigma_j; t)|\sigma\rangle$$ \hspace{1cm} (4)

where $|\sigma\rangle = \{|\sigma_i\rangle\}$ is the joint eigenstate of all $\hat{S}_z$ operators. The evolution of the variational parameters $\psi_{ij}(\sigma_i, \sigma_j; t)$ is dictated by the time-dependent variational principle, and it requires Monte Carlo sampling of the probability distribution $|\langle \sigma| \Psi(t) \rangle|^2$ \cite{33,34}. The chosen Ansatz has the advantage of reproducing the exact dynamics in the $\alpha = 0$ limit \cite{35}; and, as we shall see, it remains very accurate for $\alpha > 0$.

Tower of states as quantum many-body scars. The existence of an Anderson’s ToS in the low-energy spectrum of the $\alpha$-XX model can be directly verified using ED on small system sizes \cite{13}. Fig. 1 shows the low-energy spectrum of a $N = 16$ chain with $\alpha = 1, 3$ and $\infty$; the ToS is clearly visible when plotting the energies as a function of the quantum number $J_z$, since the ground states in each sector have an energy which grows almost exactly as $(J_z)^2$, as expected for the eigenstates of $\hat{H}_{\text{OAT}}$. Remarkably, this behavior is also present in the case $\alpha \geq 3$, for which SSB should not be present in the thermodynamic limit. Yet the ground state, which is strictly speaking a gapless spin liquid with algebraically decaying correlations for $N \to \infty$ \cite{27,48}, has similar features as those of a long-range ordered state for finite $N$.

The states of the ToS, being ground states of different $J_z$ sectors, can be naturally expected to exhibit non-typical features among all the states in the corresponding energy range. In particular, when mapping the $\alpha$-XX model onto hardcore bosons \cite{36}, $J_z$ parametrizes the particle number, such that the ToS is readily understood as the set of the (quasi-)condensate ground states for different particle numbers \cite{37}. In particular, the ToS represent the states featuring the most slowly decaying correlations $C_{ij}^{(y)} = \langle \hat{S}_i^{+}(y)\hat{S}_j^{+}(y) \rangle$ (corresponding to the one-body density matrix for bosons) among all the states at the same $J_z$: therefore they are the states maximizing the total angular momentum $\langle \hat{J}_z^2 \rangle = \langle \hat{J}_z \rangle^2 + \sum_{ij} C_{ij}^{(y)}$. Fig. 1 shows that they are also the states with the largest $\langle \hat{J}_z^2 \rangle$ throughout their energy range. As such, they represent a paradigmatic example of quantum many-body scars \cite{18}. Due to their anomalously large value of $\langle \hat{J}_z^2 \rangle$, the states of the ToS are naturally expected to play a significant role in the dynamics of the system when starting from the $\{\text{CSS}_x\}$ state, given that for this state $\langle \hat{J}_z \rangle$ is maximal. Indeed, as shown in \cite{35}, the CSS has maximal overlap with the ToS.

\begin{figure}
\centering
\includegraphics[width=1\textwidth]{fig2.png}
\caption{Squeezing dynamics. (a-c) Dynamics of the squeezing parameter for the 1d $\alpha$-XX Hamiltonian with $\alpha = 1$ (a), $\alpha = 3$ (b) and $\alpha = \infty$ (c). Time is rescaled by the Kac prefactor $K_N^{(\alpha)}$ – see text. The dashed line is the exact result for $N = 16$, while the other data are obtained via tVMC (solid lines); (d-e) Scaling exponents for the dynamics of squeezing: (d) $\mu$ exponent for the scaling of the optimal time ($t_{\text{min}}^{(\alpha)} \sim N^{\nu}$); (e) $\nu$ exponent for the scaling of the minimum squeezing parameter $\xi_{R,\text{min}}^2 \sim N^{-\nu}$. In (d-e), the horizontal dotted line shows the OAT exponents, and the vertical dotted line marks $\alpha = 5d/3$ (see text).}
\end{figure}

Squeezing dynamics and its scaling. The existence of a set of quantum-scar states with an energy dependence on $J_z$ akin to that of the OAT Hamiltonian in Eq. (2), and which are strongly overlapping with the CSS, suggests that the Hamiltonian evolution starting from the CSS will strongly resemble that of the OAT model, namely it will feature squeezing at short times. This is indeed what is shown in Fig. 2 for different $\alpha$ values in $d = 1$ and for variable system sizes up to $N = 128$ spins. Our tVMC results are indistinguishable from the exact ones (which we obtain for $N = 16$) up to $\alpha \approx 1$, remaining accurate for all values of $\alpha$. For each $\alpha$, we identify the minimal value of the squeezing parameter $\xi_{R,\text{min}}^2$ and the corresponding optimal time $t_{\text{min}}$, and we follow their scaling with the system size – see also \cite{35} for Supplemental Data.

The results of the scaling analysis of the spin-squeezing
dynamics in the 1d $\alpha$-XX model are summarized in Fig. 2(d-e). Following the example of the OAT model, we postulate \( 29, 38 \) the power-law scalings \( \xi_{R,\text{min}}^2 \sim N^{-\nu} \) and \( t_{\text{min}} K_N^{(\alpha)} \sim N^{\mu} \). Here we have introduced the Kac normalization \( K_N^{(\alpha)} = N^{-1} \sum_i \sum_{j \neq i} |r_i - r_j|^{-\alpha} \) of the power-law couplings in order to properly scale time, by considering evolutions with Hamiltonians with extensive energies \( 39 \) – this is appropriate for all the quantum-simulation platforms cited above. The system sizes we have considered (up to \( N = 128 \)) do not necessarily capture the asymptotic scaling limit, but they are comparable with the typical sizes achieved by state-of-the-art quantum simulators for the $\alpha$-XX model. In Fig. 2(d-e) we observe that the scaling properties of the OAT limit $\alpha = 0$ (namely $\mu = 1/3$ and $\nu = 2/3$) are essentially maintained throughout the range $0 < \alpha \lesssim 1.5$. This observation reveals (far beyond what is accessible to ED) the dominant role that the ToS has on the dynamics of the system in this range of $\alpha$, exceeding the regime of long-range interactions $\alpha \lesssim 1$. On the other hand for $\alpha \gtrsim 2$ both $\mu$ and $\nu$ are found to sharply drop to zero; this observation signals that the dynamics leaks significantly out of ToS manifold, and that further Hamiltonian eigenstates entering in the dynamics have the effect of curbing the growth of squeezing and suppressing its scaling behavior. In fact, within spin-wave (SW) theory \( 20, 25 \), we can estimate that a time $t_{\text{SW}} \sim N^{z/d}$ sets a lower bound to the retardation effects for propagation of correlations across the system, where $z$ is the dynamical exponent dictating the spin-wave dispersion relation $\omega \sim k^z$ \( 39 \). A necessary condition for the squeezing dynamics of the OAT to be reproduced at $\alpha > 0$ is that $t_{\text{SW}} < t_{\text{min}}$, which leads to the condition $z < d/3$, translating into $\alpha < 5d/3$ for $d \leq 3$, namely $\alpha < 1.666\ldots$ in $d = 1$. Our numerical results strongly suggest that this necessary condition may also be sufficient.

**Universality of the squeezing dynamics.** The connection between the OAT dynamics and the short-time dynamics of generic $\alpha$-XX models can be made fully quantitative by identifying the moment of inertia $I^{(\alpha)}$ of the effective OAT Hamiltonian emerging from the $\alpha$-XX model. The quantitatively correct way to do this is to assume that the short-time dynamics of the $\alpha$-XX model prepared in the CSS remains confined in the sector with maximum total angular momentum $J_{\text{max}}$ – this assumption will be justified below. If this is true, then the effective Hamiltonian governing the dynamics is given by the restriction of the $\alpha$-XX model onto the subspace of Dicke states $|J_{\text{max}}, J^z\rangle$ which are eigenstates of $J^z$ with $J = N/2$, as well as of $J^z$; given that $\mathcal{H}$ commutes with $J^z$ and it is invariant under inversion of the spins along $z$, the resulting projected Hamiltonian is necessarily an even function of $J^z$, $\langle J_{\text{max}}, J^z | \mathcal{H} | J_{\text{max}}, (J^z)' \rangle = \delta_{J^z, (J^z)'} \text{const.}$, and $J^z = \langle J^z \rangle [\text{const.} + (J^z)^2/(2I^{(\alpha)}) + \mathcal{O}(J^z)^4]$. The eigenvalues of the Hamiltonian projected onto the Dicke states turn out to be almost perfect quadratic functions of $J^z$ \( 39 \), which allow us to systematically extract the corresponding moment of inertia $I^{(\alpha)}$. The picture of the effective dynamics projected onto the Dicke-state manifold would then predict that all $\alpha$-XX models squeeze the fluctuations of the CSS in the same way as the OAT Hamiltonian does – so that universal dynamics should be manifested when properly rescaling time by $t_\alpha = I^{(\alpha)}$. This is indeed observed in Fig. 3(a-b), exhibiting a perfect collapse for the short-time evolution of the squeezing parameter at all $\alpha$ values for $d = 1$; for $d = 2$ the collapse holds up to longer times, as a result of the increased connectivity of the lattice.

To connect this observation with the existence of the ToS in the spectrum one can adopt a Fourier decomposition of the $\alpha$-XX model \( 13 \) as $\mathcal{H} = \sum_q \hat{\mathcal{H}}_q + \text{const.}$, where

$$\hat{\mathcal{H}}_q = -\frac{J}{2N} \gamma_{q,N}^{(\alpha)} \left( \hat{j}_q \hat{j}^*_q + \hat{j}_q^* \hat{j}^*_q \right). \quad (5)$$

Here $\hat{j}_q^{(x,y)} = \sum_{\gamma} e^{iq \cdot \gamma} \hat{S}_\gamma^{(x,y)}$ and $\gamma_{q,N}^{(\alpha)} = \sum_i e^{iq \cdot \gamma_i} |r_i|^{-\alpha}$ is the Fourier transform of the couplings calculated on an $N$-site system. The $\hat{\mathcal{H}}_{q=0}$ Hamiltonian corresponds to a OAT model with $[I^{(\alpha)}]^{-1} = \mathcal{J}(\gamma_{0,N} + 1)/N$. The emergence of a ToS in the spectrum of the $\alpha$-XX model can be understood as the fact that the $\hat{\mathcal{H}}_{q \neq 0}$ terms in the Hamiltonian perturb only weakly

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**FIG. 3. Universal squeezing dynamics.** (a-b) Squeezing dynamics for the $\alpha$-XX model in (a) $d = 1$ ($N = 16$) and (b) $d = 2$ ($N = 4 \times 4$), for various values of $\alpha$. Collapse of the curves at short times is obtained by rescaling time with the $I^{(\alpha)}$ parameter; (c-d) Comparison between the three estimates of the moment of inertia: from the Hamiltonian projection onto Dicke states ($I^{(\alpha)}$); from the Fourier transform of the coupling constant ($I^{(\alpha)}_{\text{Ft}}$); and from the ToS spectrum ($I^{(\alpha)}_{\text{ToS}}$). System Hamiltonian and sizes as in panels (a-b).
the eigenstates of the $H_{q=0}$ part with maximal $\langle \hat{J}^2 \rangle$; as a consequence there exist special Hamiltonian eigenstates which are parametrically closer to Dicke states with $J = J_{\text{max}}$; the smaller $\alpha$ is. These states have energies $E \approx (J^2)^2/(2I_{\text{Tos}}^{(0)}) + \text{const.}$, with a moment of inertia $I_{\text{Tos}}^{(0)}$ that can be extracted by fitting the energy spectrum as in Fig. 1. The $I_{\text{Tos}}^{(0)}$ parameter differs from the $I_\gamma^{(0)}$ parameter in that it is renormalized by the $H_{q\neq0}$ perturbations; and it differs from the $I^{(0)}$ parameter, because it includes the mixing of the $J_{\text{max}}$ sector of Hilbert space with all the other $J$ sectors. Yet these three definitions of the effective moment of inertia – a dynamical one ($I^{(0)}$), an Hamiltonian one ($I_\gamma^{(0)}$) and a spectral one ($I_{\text{Tos}}^{(0)}$) – coincide for $\alpha = 0$, and they turn out to be very close to each other for all $\alpha$ values, as shown in Fig. 3(c-d) for both $d = 1$ and $d = 2$. This result corroborates the picture in which the $\alpha$-XX Hamiltonians couple weakly the $J_{\text{max}}$ sector with all the other sectors of Hilbert space, resulting simultaneously in 1) the existence of quantum scars (the ToS) in the spectrum, predominantly overlapping with the $J_{\text{max}}$ sector; and 2) the existence of a universal spin-squeezing dynamics at short times, which remains temporarily confined in the $J_{\text{max}}$ sector (see [35] for the dynamics of $\langle \hat{J}^2 \rangle$), reproducing the behavior of the OAT model.

Conclusions. Focusing on $U(1)$ symmetric models, in this work we have unveiled a dynamical manifestation of the Anderson’s tower of states – the fundamental spectral mechanism of spontaneous breaking of continuous symmetries in quantum mechanics – in the form of an effective squeezing dynamics of coherent spin states, fully equivalent at short times to that generated by a planar-rotor Hamiltonian. In particular we find that the one-dimensional XX model with power-law interactions generates squeezing that scales with system size when the decay exponent takes values $\alpha \lesssim 2$. These results establish an important link between the simulation of nonequilibrium dynamics of $U(1)$ quantum spin Hamiltonians – a common task of nearly all quantum simulation platforms – and quantum metrology.

Data and additional details about the tVMC numerical simulations are made publicly available [40].

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The $\alpha$-XX Hamiltonian features extensive eigenvalues only for $\alpha > d$, $d$ being the number of dimensions; later, when appropriate, we will adopt a Kac normalization of the coupling constant in order to reinstate the extensive nature of the energy.

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See Supplemental Material for: 1) the overlap of the initial state with states of the spectrum; 2) the proof of the exactness of the pair-product Ansatz for infinite-range interactions; 3) supplemental data for the scaling of squeezing in the $\alpha$-XX model; 4) the necessary condition for the scaling of squeezing of the OAT model to persist at $\alpha > 0$; 5) the diagonal elements of the $\alpha$-XX Hamiltonian on the $|J_{\text{max}}, J_z\rangle$ basis; 6) the time evolution of $\langle J^2 \rangle$. 

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The states of the ToS, being ground states of hardcore bosons with long-range hopping, are expected to exhibit area-law scaling of entanglement entropy, at least for $\alpha > d$, while logarithmic scaling of the entropy with subsystem size can be exactly proven for $\alpha = 0$. This is in stark contrast with typical states at finite energy density.

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In particular, in the limit $N \to \infty$, we have that $K_N^{(\alpha)} \sim N^0$ for $\alpha > 1$, $K_N^{(\alpha)} \sim \log N$ for $\alpha = 1$ and $K_N^{(\alpha)} \sim N^{1-\alpha}$ for $\alpha < 1$.

T. Comparin, F. Mezzacapo, and T. Roscilde, Supporting data (link to be added at a later stage).
Supplemental Material

Universal spin squeezing from the tower of states of $U(1)$-symmetric spin Hamiltonians

OVERLAP OF COHERENT SPIN STATE WITH THE TOWER OF STATES

Through exact diagonalization, we obtain the $n_{\text{max}}$ lowest-energy eigenstates $|n\rangle$ of the $\alpha$-XX Hamiltonian $\hat{H}$ (in $d = 1$ and with $N = 16$). Fig. S1 shows that for $\alpha = 0$ the eigenstates forming the ToS are the only ones with a finite overlap $|\langle n|\text{CSS}\rangle|$ with the CSS. As $\alpha$ increases, also other states in the spectrum acquire a finite overlap.

FIG. S1. Overlap of the low-energy eigenstates of the $\alpha$-XX model (with $d = 1$ and $N = 16$) with $|\text{CSS}\rangle$, as a function of either $(J^z)^2$ (top panels) or the total angular momentum (lower panel). Blue (orange) dots corresponds to states in (outside) the ToS.

PAIR-PRODUCT ANSATZ IS EXACT FOR THE ONE-AXIS TWISTING Hamiltonian

The exact evolution of the CSS under the OAT Hamiltonian is restricted to the $J = J_{\text{max}}$ sector, and it reads

$$e^{-i\mathcal{H}_{\text{OAT}} t}|\text{CSS}\rangle = \sum_{J^z = -J_{\text{max}}}^{J_{\text{max}}} (J_{\text{max}}, J^z)|\text{CSS}\rangle \exp \left[ -i t \frac{(J^z)^2}{2I} \right] |J_{\text{max}}, J^z\rangle,$$

where we set $\hbar = 1$ and where

$$\langle J_{\text{max}}, J^z|\text{CSS}\rangle = 2^{-J_{\text{max}}} \sqrt{\frac{2J_{\text{max}}}{J_{\text{max}} - J^z}}.$$  

Provided that a certain variational Ansatz correctly represents the initial state, then it can also exactly reproduce the time-evolved state if the coefficients $\langle \sigma|\Psi(t) \rangle$ can take values proportional to $\exp[-i t (J^z)^2/(2I)]$, with $J^z = \sum_{i=1}^{N} \sigma_i$ and where $\sigma_i = \pm 1/2$ is the eigenvalue of $\hat{S}_i^z$. The pair-product Ansatz used in this work has this property, as we show by construction. If we set its coefficients to be

$$\psi_{jk}(\sigma_j, \sigma_k; t) = \exp [w_{jk} \sigma_j \sigma_k], \quad w_{jk} = -\frac{i t}{I},$$

then the value of $|\Psi(t)\rangle$ on a given basis state $|\sigma\rangle$ reads

$$\langle \sigma|\Psi(t) \rangle = \prod_{j < k} \psi_{jk}(\sigma_j, \sigma_k; t) = \exp \left[ -i t \sum_{j < k} \sigma_j \sigma_k \right] = \exp \left[ -i t \frac{(J^z)^2}{2I} \right] \exp \left[ i t \frac{N^2}{8I} \right].$$
This corresponds to the required form of $\langle \sigma | \Psi(t) \rangle$, up to an irrelevant global phase factor. Therefore the pair-product Ansatz can describe the exact time evolution of $|CSS_x\rangle$. Note that the initial state is trivially represented with this Ansatz by setting all coefficients $\psi_{jk}(\sigma_j, \sigma_k; t = 0)$ equal to each other.

![Graph](image)

**FIG. S2.** Spin squeezing generated by the $\alpha$-XX Hamiltonian, with $d = 1$ and $\alpha = 0$. For each system size $N$, the result of tVMC with the pair-product Ansatz (solid lines) perfectly matches with the exact solution for the OAT Hamiltonian [8].

The Hamiltonian of the $\alpha$-XX model with $\alpha = 0$,

$$\hat{H} = \frac{J}{2} \left[ (\hat{J}^2)^2 - \hat{J}^2 \right] + \text{const}, \quad (10)$$

corresponds to the OAT Hamiltonian with moment of inertia $I = 1/J$, up to a constant shift and to an additional $\hat{J}^2$ term (which only adds an irrelevant global phase factor to the time-evolved state). Thus the pair-product Ansatz is also exact for the $\alpha$-XX model with $\alpha = 0$. We explicitly verify this fact by comparing the tVMC dynamics of spin squeezing to the exact expression for the OAT squeezing dynamics [8]. As shown in Fig. S2, the tVMC calculation for the $\alpha$-XX model with $\alpha = 0$ is exact. Note that the dynamics of variational parameters $\var{w_{jk}}$ is fully obtained through the tVMC scheme [34], that is, without postulating the expected expression in Eq. (8).

**SCALING OF SQUEEZING FOR DIFFERENT INTERACTION RANGES**

Here we show additional results for the spin-squeezing dynamics, with several values of $\alpha$ – see Fig. S3. For $N = 16$, we compare the squeezing dynamics obtained through tVMC with the exact one, and we observe that the accuracy of the pair-product Ansatz improves when $\alpha$ decreases (becoming exact at $\alpha = 0$). We also show the same curves for spin squeezing for the Kac-normalized $\alpha$-XX model, which corresponds to multiplying time by $K_N^{(\alpha)}$ – see Fig. S4. For any value of $\alpha$, results for different system sizes collapse onto the same curve at short times. The dependence on system size appears at larger time, where we observe the presence or absence of scaling of the optimal squeezing with $N$. Fig. S5 shows the scaling of the optimal squeezing $\xi_{R,\text{min}}^2$ and of the corresponding time $t_{\text{min}}$. The power-law fits yield the exponents $\mu$ and $\nu$ shown in Fig. 2(d-e) of the main text.
**NECESSARY CONDITION ON THE PERSISTENCE OF $\alpha = 0$ SCALING OF SQUEEZING FROM SPIN-WAVE THEORY**

In a (Kac-normalized) one-axis-twisting (OAT) Hamiltonian, with all-to-all interactions, the optimal squeezing time scales as

$$t_{\text{min}} \sim N^{1/3} = L^{d/3}.$$  \hfill (11)

Taking the OAT Hamiltonian literally, its dynamics is characterized by the total absence of retardation effects: any signal propagates instantaneously from end to end of the system. As a consequence, the dynamics of the spin system is

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**FIG. S3.** Time evolution of spin squeezing generated by the $\alpha$-XX Hamiltonian, with $d = 1$ and $\alpha \in \{0, 1/2, 1, 3/2, 2, 5/2, 3, \infty\}$. Results for system sizes $N = 16, 32, 64, 128$ are obtained via tVMC with the pair-product Ansatz (solid lines), and for each $\alpha$ we also show the exact result for $N = 16$ (dashed black line). Statistical error bars on the tVMC curves are of the order of the line width or smaller.

**FIG. S4.** Same data as in Fig. S3, with time rescaled by the Kac prefactor.
FIG. S5. Finite-size scaling of the optimal time $t_{\text{min}}^{K_N^{(\alpha)}}$ (left panel) and of the minimal squeezing $\xi^2_{R,\text{min}}$ (right panel), for several values of $\alpha$. Dots are tVMC results, extracted from curves in Fig. S3. Straight lines are power-law fits with $t_{\text{min}}^{K_N^{(\alpha)}} \sim N^\mu$ and $\xi^2_{R,\text{min}} \sim N^{-\nu}$. The fit range is $32 \leq N \leq 128$.

completely captured by that of the collective spin variables, and there is no relative dynamics between spins. Even in the picture of the OAT model, there is still a finite time required for the establishment of maximal squeezing, scaling with system size. This scaling also contains the Kac normalization, namely the interactions have no retardation, but their strength is decreasing as $N^{-1}$ to keep the energy extensive. This has the fundamental effect of making the optimal time increase with system size.

When considering instead $\alpha > 0$, the relevant excitations involved in the dynamics starting from the CSS state are not only the states from Anderson’s ToS, but also spin-wave (SW) excitations. These excitations may have in general a finite group velocity, or a group velocity diverging with system size sufficiently slowly, so that they could lead to retardation effects in the dynamics of the collective spin, thereby altering substantially the picture of the OAT model.

SW theory for the $\alpha$-XX model [26, 28] predicts that SW excitations have a dispersion relation $\omega \sim k^z$, with dynamical exponents $z$ taking values

$$ z = \begin{cases} \frac{\alpha - d}{2} & \alpha \geq d + 2 \\ d & d \leq \alpha \leq d + 2 \\ 0 & \alpha \leq d \end{cases} . $$

(12)

The associated group velocity is therefore $v_g \sim k^{z-1}$. On a finite system of linear size $L$ the maximum group velocity is therefore scaling as $v_{g,\text{max}} \sim k_{\text{min}}^{z-1} \sim L^{1-z}$ where $k_{\text{min}} = 2\pi/L$. Associated with this maximal group velocity there is an intrinsic minimal retardation time

$$ t_{\text{SW}} = \frac{L}{v_{g,\text{max}}} \sim L^z $$

(13)

which is the time needed for the fastest spin-wave excitations to traverse the entire system of linear size $L$.

When $\alpha \leq d$, the vanishing of the $z$ exponent implies that the retardation time does not scale with system size, so that the spatial decay of interactions is not expected to affect the collective spin dynamics at the time scale of $t_{\text{min}}$ (which instead grows with $L$). On the other hand, when $\alpha > d$ one has that $z > 0$, so that retardation effects can play a role. Given that $t_{\text{SW}}$ is the minimal retardation time in the collective spin dynamics, one can argue that a necessary condition for retardation effects not to affect the collective spin dynamics up to the optimal squeezing time $t_{\text{min}}$ is that

$$ t_{\text{SW}} < t_{\text{min}} . $$

(14)

For this condition to remain valid for large system sizes, one should then require that $L^z < L^{d/3}$, namely

$$ z < \frac{d}{3} \implies \alpha < \frac{5}{3}d . $$

(15)

The above necessary condition is valid if $\frac{5}{3}d \leq d + 2$, namely if $d \leq 3$, covering all situations of interest.
DIAGONAL PART OF THE $\alpha$-XX HAMILTONIAN ON THE $|J_{\text{max}}, J_z\rangle$ REDUCED BASIS

Fig. S6 shows the diagonal elements of the 1d $\alpha$-XX Hamiltonian with $N = 16$ on the reduced basis $|J_{\text{max}}, J_z\rangle$. The Hamiltonian, commuting with $\hat{J}_z$, is diagonal on this basis, and the diagonal matrix elements show a clear quadratic dependence on $J_z$ of the kind $\langle J_{\text{max}}, J_z | \hat{H} | J_{\text{max}}, J_z \rangle = \text{const.} + (J_z)^2/(2I^{(\alpha)})$, from which we can extract systematically the moment of inertia $I^{(\alpha)}$ shown in the main text.

FIG. S6. Diagonal matrix elements of the $\alpha$-XX Hamiltonian $\hat{H}$ (for $d = 1$ and $N = 16$) on the $|J_{\text{max}}, J_z\rangle$ states, as a function of $(J_z)^2$. For each $\alpha$, straight lines are linear fits.

TIME EVOLUTION OF THE TOTAL SPIN

For any $\alpha > 0$, the total spin $\hat{J}^2$ has a non-trivial dynamics, since it does not commute with the Hamiltonian $\hat{H}$. In the initial state $|\text{CSS}_x\rangle$, $\langle \hat{J}^2 \rangle$ is maximum and equal to $\langle \hat{J}^2 \rangle(0) = J_{\text{max}}(J_{\text{max}} + 1)$. During the dynamics, for $\alpha > 0$, the $J = J_{\text{max}}$ sector is mixed with other sectors, and $\langle \hat{J}^2 \rangle(t)$ slowly departs from its initial value – see Fig. S7. The decay is stronger the larger $\alpha$, due to the stronger $\hat{H}_{\alpha \neq 0}$ terms in the Hamiltonian.

FIG. S7. Dynamics of the total spin $\langle \hat{J}^2 \rangle$ in the one-dimensional $\alpha$-XX model, for several values of $\alpha$. For each $\alpha$, tVMC results are shown up to $N = 128$ (solid lines), and compared with the exact curve for $N = 16$ (dashed black line).