On the Iwasawa asymptotic class number formula for
\[ \mathbb{Z}_p^r \rtimes \mathbb{Z}_p \]-extensions

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Abstract
Let \( p \) be an odd prime and \( F_{\infty, \infty} \) a \( p \)-adic Lie extension of a number field \( F \) with Galois group isomorphic to \( \mathbb{Z}_p^r \rtimes \mathbb{Z}_p, r \geq 1 \). Under certain assumptions, we prove an asymptotic formula for the growth of \( p \)-exponents of the class groups in the said \( p \)-adic Lie extension. This generalizes a previous result of Lei, where he establishes such a formula in the case \( r = 1 \). An important and new ingredient towards extending Lei’s result rests on an asymptotic formula for a finitely generated (not necessarily torsion) \( \mathbb{Z}_p[\mathbb{Z}_p^r] \)-module which we will also establish in this paper. We then continue studying the growth of \( p \)-exponents of the class groups under more restrictive assumptions and show that there is an asymptotic formula in our noncommutative \( p \)-adic Lie extension analogous to a refined formula of Monsky (which is for the commutative extension) in a special case.

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1 Introduction
The starting point of this paper is the following Iwasawa’s celebrated asymptotic class number formula (cf. \cite[Theorem 11]{11}).

Theorem (Iwasawa). Let \( F_{\infty} \) be a \( \mathbb{Z}_p \)-extension of a number field \( F \). Denote by \( F_n \) the intermediate subfield of \( F_{\infty} \) with index \( |F_n : F| = p^n \). Write \( e_n \) for the \( p \)-exponent of the \( p \)-class group of \( F_n \). Then there exist \( \mu, \lambda \) and \( \nu \) (independent of \( n \)) such that

\[ e_n = \mu p^n + \lambda n + \nu \]

for \( n \gg 0 \).

This result is the first of its kind which describes the precise growth of the \( p \)-exponent of the \( p \)-class group in an infinite tower of number fields. Subsequently, it is natural to search for such a formula,
The main goal of this paper is to extend Lei’s result. Namely, we obtain an asymptotic class number formula for a \( \mathbb{Z}_p \times \mathbb{Z}_p \)-extension under an analogous hypothesis to that of Lei (see Theorem 3.2.1) which we denote by \( \Gamma := G/H \simeq \mathbb{Z}_p \). (Therefore, we have \( d = r+1 \).) Write \( F^c \) for the subextension of \( F_{\infty, \infty} \) fixed by \( H \). The following ramification conditions (R1) – (R4) are always assumed to be satisfied for our extension \( F_{\infty, \infty}/F \).

(R1) The number field \( F \) has only one prime above \( p \) which we denote by \( \mathfrak{p} \).

(R2) The prime \( \mathfrak{p} \) is totally ramified in \( F_{\infty, \infty}/F \).

(R3) The set of primes of \( F \) ramified in \( F_{\infty, \infty} \) is finite and we shall denote this set by \( \Sigma \).

(R4) Every prime of \( \Sigma \) is finitely decomposed in \( F^c/F \).

For \( 0 \leq m, n \leq \infty \), we let \( H_m = H^{p^m} \), \( \Gamma_n = \Gamma^{p^n} \) and \( G_{n,m} = H_m \times \Gamma_n \). Set \( F_{n,m} \) to be the fixed field of \( G_{n,m} \). The \( p \)-exponent of the \( p \)-class group of \( F_{n,m} \) is then denoted by \( e_{n,m} \). Finally, we write \( \mathcal{X} = \text{Gal}(\mathcal{M}/F_{\infty, \infty}) \), where \( \mathcal{M} \) is the maximal unramified abelian pro-\( p \)-extension of \( F_{\infty, \infty} \). The following is the main theorem of this paper noting that our \( r \) here is \( d-1 \) of the previous authors.
Theorem (Theorem 3.2.1). Retain the setting of the preceding paragraphs. Suppose further that $\mathcal{X}$ is finitely generated over $\mathbb{Z}_p[H]$. Then we have

$$e_{n,n} = \text{rank}_{\mathbb{Z}_p[H]}(\mathcal{X})np^{rn} + O(p^{rn}).$$

We now say a little on the ideas of the proof. In [12], Lei established this formula for the case $r = 1$. There he made use of the analysis of torsion $\mathbb{Z}_p[H]$-modules in Cucuo and Monsky’s work [3]. Note that the hypothesis of the theorem only assumes that $\mathcal{X}$ is finitely generated over $\mathbb{Z}_p[H]$. In the situation of $r = 1$, there is a nice enough structure theory for $\mathbb{Z}_p[H]$-modules, and combined with the fact that $H$ is pro-cyclic (since $r = 1$), Lei was able to reduce the problem to considering torsion $\mathbb{Z}_p[H]$-modules which in turn allowed him to obtain a crucial estimate (see [12, Proposition 5.2]) required for his eventual proof. Unfortunately, when $r \geq 2$, the structure theory for $\mathbb{Z}_p[H]$-modules is less refined, and more importantly, the group $H$ is no longer pro-cyclic. Therefore, Lei’s approach does not seem to be able to carry over. Hence this suggests the necessitation to work directly with finitely generated $\mathbb{Z}_p[H]$-modules which are not necessarily torsion. This is precisely the core of Section 2, where we obtain the following asymptotic formula for a finitely generated (not necessarily torsion) $\mathbb{Z}_p[H]$-modules analogous to those in [3].

Theorem (Theorem 2.4.1). Let $H = \mathbb{Z}_p^r$, $r \geq 1$. Let $M$ be a finitely generated (not necessarily torsion) $\mathbb{Z}_p[H]$-module. Then we have

$$e(M_{H_m}) = \mu_H(M)p^{rm} + O(mp^{(r-1)m}).$$

Here $e(M_{H_m})$ denotes the $p$-exponent of the torsion subgroup of $M_{H_m}$ and $\mu_H(M)$ denotes the $\mu_H$-invariant of the module $M$ (see Section 2 for the definition). Our estimate is cruder than those in [3, 16] which is somewhat expected, since we are now working with possibly nontorsion modules. Fortunately, this crude estimate is enough for us to establish the analogue of the crucial estimate of Lei for $r \geq 2$ (see Proposition 3.2.2) which in turn allows us to prove our main theorem.

The remainder of the paper is concerned with (raising) questions on what we can say about the asymptotic growth under more stringent hypothesis on the structure of the Galois group $\mathcal{X}$. The motivation behind this study stems from the refined asymptotic formula of Monsky [16, Theorem 3.13] (or see above), where he obtained an asymptotic formula up to an error term of $O(np^{(d-2)n})$. It is then natural to ask if our estimates can be improved in this direction for the class of noncommutative $p$-adic Lie extensions considered in this paper. Unfortunately, we do not see any way of doing this in general at this point of writing which is why we took the following more restricted approach of study. Namely, we would like to ask for a possible shape of the asymptotic formula when $\mathcal{X}$ is a finitely generated torsion $\mathbb{Z}_p[H]$-module? We remark that this line of approach is also inspired by Lei’s and our approach in obtaining the asymptotic formulas, where we investigate the growth under an extra assumption on $\mathcal{X}$ which essentially eliminates the leading term $p^{(r+1)n}$ (or $p^{dn}$ in Cucuo-Monsky’s notation). Therefore, in this more restrictive context that we adopted, our question is as follows.

Suppose that $\mathcal{X}$ is a finitely generated torsion $\mathbb{Z}_p[H]$-module, does there exist some $\alpha^*$ such that

$$e_{n,n} = \alpha^* p^{rn} + O(np^{(r-1)n})?$$
Note that it follows from our Theorem 3.2.1 and the assumption in the question that we have $e_{n,n} = O(p^n)$ (see Corollary 3.2.3) and hence the above question makes sense. Had $G_{\infty,\infty}$ been abelian, the result of Monsky ([16, Theorem 3.13]; also see beginning of this section) will confirm this speculation. (Again, we note that our $r$ is Monsky’s $d - 1$). At present, the best we can do in our noncommutative setting is the following estimate.

**Theorem (Theorem 3.2.4).** Suppose that $X$ is a finitely generated torsion $\mathbb{Z}_p[H]$-module. Then we have

$$e_{n,n} \leq \beta^* p^n + O(np^{(r-1)n})$$

for some nonnegative integer $\beta^*$.

In fact, the $\beta^*$ appearing in our upper bound has a description of the sum of certain $\mu_H$-invariants (see the proof of Theorem 3.2.4). However, we have to confess that, at this point of writing, we do not know whether there is any relation between our $\beta^*$ and Monsky’s $\alpha^*$. Despite so, our description of $\beta^*$ enables us to obtain the following estimate by imposing an even stronger assumption on $X$, thus answering our question in a special case.

**Theorem (Theorem 3.2.5).** Suppose that $X$ is a finitely generated torsion $\mathbb{Z}_p[H]$-module with trivial $\mu_H$-invariant. Then we have

$$e_{n,n} = O(np^{(r-1)n}).$$

We then also discuss some interesting examples (see Subsection 3.3). In particular, in one of the examples, we specialize to the extension $F_{\infty,\infty} = \mathbb{Q}(\mu_{p^{\infty}}, p^{-p^{\infty}})$ and show that $e_{n,n} = O(n)$ for an irregular prime $p < 1000$ by combining the above theorem with results of Sharifi [19]. This estimate, together with another result of Cucuo-Monsky [3, Theorem II] in the commutative situation, naturally leads us to ask whether there exist $\lambda$ and $\nu$ such that $e_{n,n} = \lambda n + \nu$ for sufficiently large $n$? Unfortunately, we (again!) do not have an answer to this at this point of writing.

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2 Algebraic preliminaries

As before, $p$ will denote a fixed odd prime. Throughout this section, $H$ will always denote a multiplicative group which is isomorphic to a direct sum of $r$ copies of the additive group of the $p$-adic ring of integers, where $r \geq 1$. For each $m$, write $H_m = H^{p^m}$. If $N$ is a $Z_p$-module, denote by $N(p)$ the submodule of $N$ consisting of elements of $N$ which are annihilated by some power of $p$. In the event that $N$ is finitely generated over $Z_p$, we write $e(N)$ for the $p$-exponent of $N(p)$, i.e., $|N(p)| = p^{e(N)}$.

2.1 Some technical lemmas

In this subsection, we collect several technical results required for subsequent discussion. We first make a remark that if $M$ is a finitely generated $Z_p[H]/llbracket H/rrbracket$-module, then $H_i(H_m, M)$ is finitely generated over $Z_p$ (for instance, see [15, Lemma 3.2.3]). Thus, it makes sense to speak of rank $Z_p H_i(H_m, M)$ and $e(H_i(H_m, M))$ which we do without further comment. We begin by showing that the exponent of the $Z_p$-torsion subgroup of $H_i(H_m, M)$ is bounded by a linear function.

Lemma 2.1.1. Let $M$ be a finitely generated $Z_p[H]$-module. Then there exists $c$ (independent of $m$) such that $p^m+c$ annihilates $H_i(H_m, M)(p)$ for every $m$ and $i$.

Proof. Since $Z_p[H]$ has finite global dimension $r+1$ (cf. [17, Page 288, Exercise 5]), we have a finite free resolution

$$0 \rightarrow P_{r+1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

of $M$. Set $K_i = \ker(P_i \rightarrow P_{i-1})$, where $P_{-1}$ is understood to be $M$. Since there are only finitely many $K_i$’s, we may apply [3, Theorem 2.8] to find a $c$ (independent of $m$) such that $p^m+c$ annihilates $K_i(p)$ for every $i$. A straightforward exercise in homological algebra tells us that $H_i(H_m, M) \rightarrow (K_{i-1}H_m)$ which in turn implies that $p^m+c$ annihilates $H_i(H_m, M)(p)$ for every $m$ and $i$. \qed

Before stating the next result, we recall the definition of a structure in the sense of Cucuo-Monsky [3]. Let $M$ be a finitely generated $Z_p[H]$-module. A structure $S$ on $M$ consists of an integer $m_0$ and a finite set of pairs $(\tau_i, M_i)$ where $\tau \in H - H_1$ and the $M_i$ are submodules of $M$ (cf. [3, Definition 4.1]). For $m \geq m_0$ and for each $i$, set

$$\alpha_{i,m,m_0} = \frac{\tau_i^{p^m} - 1}{\tau_i^{p^{m_0}} - 1}.$$

We then define $A_m(S, M)$ to be the submodule

$$I_{H_n}M + \sum_{i} \alpha_{i,m,m_0} M_i$$

of $M$, where here $I_{H_n}$ denotes the augmentation ideal of $H_n$ in $Z_p[H]$. We can now state the following analogue of Lemma 2.1.1 for structures.
Lemma 2.1.2. Let $M$ be a finitely generated $\mathbb{Z}_p[H]$-module and $S$ a structure on $M$. Then there exists $c$ (independent of $m$) such that $p^{(r+1)m+c}$ annihilates the torsion subgroup of $M/A_m(S,M)$ for every $m \geq m_0$.

Proof. This is proven in [3, Theorem 4.5] for a torsion $\mathbb{Z}_p[H]$-module but it can be checked that the same proof carries over for general modules. \hfill \square

We record another technical lemma which has been utilized in [3] without mention. For clarity of presentation, we have stated this and its corollary which will be frequently inferred in subsequent discussion of this paper.

Lemma 2.1.3. Let $0 \rightarrow A \xrightarrow{i} B \xrightarrow{g} C \rightarrow 0$ be a short exact sequence of finitely generated $\mathbb{Z}_p$-modules. Suppose that $p^j$ annihilates $A(p), B(p)$ and $C(p)$. Then we have an exact sequence

$$0 \rightarrow A(p) \rightarrow B(p) \rightarrow C(p) \rightarrow A_f/p^j$$

of finite abelian groups, where here $A_f = A/A(p)$. Consequently, we have the following inequalities

$$|e(A) - e(B)| \leq e(C),$$

$$|e(B) - e(C)| \leq e(A) + j \text{ rank}_{\mathbb{Z}_p}(A),$$

$$e(C) \leq e(B) + j \text{ rank}_{\mathbb{Z}_p}(A).$$

Proof. For a finitely generated $\mathbb{Z}_p$-module $N$, we have an identification $N = N(p) \oplus N_f$, where $N_f = N/N(p)$. Let $x \in N$. We then write $x = (x_t, x_f)$ which corresponds to such an identification, where $x_t \in N(p)$ and $x_f \in N_f$. An application of the snake lemma to the diagram

$$
\begin{array}{c}
0 \rightarrow A \xrightarrow{i} B \xrightarrow{g} C \xrightarrow{0} \\
\quad \downarrow p^j \quad \downarrow p^j \quad \downarrow p^j \\
0 \rightarrow A \xrightarrow{p^j} B \xrightarrow{p^j} C \xrightarrow{0}
\end{array}
$$

yields an exact sequence

$$0 \rightarrow A(p) \rightarrow B(p) \rightarrow C(p) \rightarrow A/p^j.$$

It therefore remains to show that $\text{im}(C(p) \rightarrow A/p^j) \subseteq A_f/p^j$ which essentially amounts to a finer analysis of the snake lemma argument. Let $c = (c_t, 0) \in C(p)$. Then there exists $b = (b_t, b_f) \in B$ such that $g(b) = c$. It follows from our hypothesis on $p^j$ that $p^jb = (0, p^fb_f)$. From the snake lemma argument, there is a unique $a = (a_t, a_f) \in A$ being sent to $p^jb$ under the map $i$. Since the torsion component of $p^j b$ is trivial and $i$ is injective, we must have $a_t = 0$. This in turn tells us that $c$ is sent to the class of $(0, a_f)$ in $A/p^j$ and hence lies in $A_f/p^j$. \hfill \square

Corollary 2.1.4. Let $A \rightarrow B \rightarrow C \rightarrow D$ be an exact sequence of finitely generated $\mathbb{Z}_p$-modules. Suppose that $p^j$ annihilates $A(p), B(p), C(p)$ and $D(p)$. Then we have

$$|e(B) - e(C)| \leq e(A) + e(D) + j \text{ rank}_{\mathbb{Z}_p}(A).$$
Proof. Write $U = \ker(A \to B)$, $V = \ker(B \to C)$ and $W = \ker(C \to D)$ and $D' = \text{im}(C \to D)$. It can be readily verified that $U(p), V(p), W(p)$ and $D'(p)$ are also annihilated by $p^j$. Therefore, we have

$$|e(B) - e(C)| \leq |e(B) - e(W)| + |e(W) - e(C)|$$

$$\leq e(V) + j \text{ rank}_{Z_p}(V) + e(D')$$

$$\leq e(A) + j \text{ rank}_{Z_p}(U) + j \text{ rank}_{Z_p}(V) + e(D)$$

$$= e(A) + j \text{ rank}_{Z_p}(A) + e(D),$$

where the second and third inequalities follow from repeated applications of Lemma 2.1.3 on the appropriate short exact sequences, and the final equality is a consequence of the simple observation that $\text{rank}_{Z_p}(U) + \text{rank}_{Z_p}(V) = \text{rank}_{Z_p}(A)$. \hfill \Box

2.2 Estimates for pseudo-null modules

In this subsection, we estimate the $Z_p$-rank and $p$-exponent of $H_i(H_m, M)$ for a pseudo-null $Z_p[H]$-module $M$, where we recall that a finitely generated $Z_p[H]$-module is said to be pseudo-null if its localization at every prime ideal of height one is trivial. For $M_{H_m}$, such estimates are worked out in [3, Theorem 3.2]. For the $p$-exponent of $H_i(H_m, M)$ when $M$ is a $p$-torsion module, this is done in [13, Theorem 2.5.1] for general $p$-adic Lie groups. The following proposition can thus be viewed as a generalization of these previous results when $H \cong Z^r_p$.

Proposition 2.2.1. Let $M$ be a pseudo-null $Z_p[H]$-module. Then we have

$$\text{rank}_{Z_p}(M_{H_m}) = O(p^{(r-2)m}), \quad e(M_{H_m}) = O(p^{(r-1)m})$$

and

$$\text{rank}_{Z_p}(H_i(H_m, M)) = O(p^{(r-1)m}), \quad e(H_i(H_m, M)) = O(mp^{(r-1)m})$$

for $i \geq 1$.

Proof. As mentioned above, the estimates for $M_{H_m}$ are contained in [3, Theorem 3.2]. It therefore remains to establish the estimates for the higher homology groups. We first make the following remark. For a finitely generated $Z_p[H]$-module $M$, it is an easy exercise to verify that $M(p)$ is a $Z_p[H]$-submodule of $M$. Since the ring $Z_p[G]$ is Noetherian, the module $M(p)$ is therefore also finitely generated over $Z_p[H]$. As a consequence, one can find an integer $t$ such that $p^t$ annihilates $M(p)$. It then follows that for every $i$, $H_i(H_m, M(p))$ is a finitely generated $Z_p$ module annihilated by $p^t$. This in turn implies that $H_i(H_m, M(p))$ is finite.

Now consider the short exact sequence

$$0 \to M(p) \to M \to M_f \to 0,$$

where we write $M_f := M/M(p)$. From this short exact sequence, we have an exact sequence

$$H_i(H_m, M(p)) \to H_i(H_m, M) \to H_i(H_m, M_f) \to H_{i-1}(H_m, M(p)).$$
Since \( H_i(H_m, M(p)) \) and \( H_{i-1}(H_m, M(p)) \) are finite by the remark in the preceding paragraph, we have \( \text{rank}_{Z_p}(H_i(H_m, M)) = \text{rank}_{Z_p}(H_i(H_m, M_f)) \) and
\[
e(H_i(H_m, M)) \leq e(H_i(H_m, M(p))) + e(H_i(H_m, M_f)),
\]
where here we emphasize that the latter inequality follows from the finiteness of \( H_i(H_m, M(p)) \). (Warning: note that for a sequence \( A \to B \to C \) exact at \( B \) but not necessarily short exact, we do not have \( e(B) \leq e(A) + e(C) \) in general.) Now by [18, Theorem 2.5.1], we have \( e(H_i(H_m, M(p))) = O(p^{(r-1)m}) \). Therefore, we are essentially reduced to proving the estimates under the assumption that \( M(p) = 0 \) which we will do for the remainder of the proof. We proceed first by establishing the estimate for \( \text{rank}_{Z_p}(H_i(H_m, M)) \). The \( H_m \)-homology of the short exact sequence \( 0 \to M \to M \to M/p \to 0 \) yields an exact sequence
\[
H_{i+1}(H_m, M/p) \to H_i(H_m, M) \to H_i(H_m, M) \to H_i(H_m, M/p).
\]
Note that
\[
\text{rank}_{Z_p} H_i(H_m, M) = e(H_i(H_m, M)/p) - e(H_i(H_m, M)[p]).
\]
From the above exact sequence, we see that \( e(H_i(H_m, M)/p) \leq e(H_i(H_m, M/p)) \) and \( e(H_i(H_m, M)[p]) \leq e(H_{i+1}(H_m, M/p)) \), and these latter two quantities are \( O(p^{(r-1)m}) \) by another application of [18, Theorem 2.5.1]. Hence this gives the required estimate for \( \text{rank}_{Z_p}(H_i(H_m, M)) \).

We now turn to estimating \( e(H_i(H_m, M)) \). By Lemma [2.1.1] there exists \( c \) (independent of \( m \)) such that \( p^{rm+c} \) annihilates \( H_1(H_m, M)(p) \) and \( H_2(H_m, M)(p) \) for every \( m \). From the short exact sequence
\[
0 \to M \to M \to M/p^{rm+c} \to 0,
\]
we obtain the following exact sequence
\[
H_2(H_m, M) \to H_2(H_m, M/p^{rm+c}) \to H_1(H_m, M)(p) \to 0,
\]
where the surjectivity follows from the fact that since \( p^{rm+c} \) annihilates \( H_1(H_m, M)(p) \), one has the equality \( H_1(H_m, M)[p^{rm+c}] = H_1(H_m, M)(p) \). Note that \( p^{rm+c} \) clearly annihilates \( H_1(H_m, M/p^{rm+c}) \). We may then apply Corollary [2.1.3] (taking \( D = 0 \)) to obtain the equality
\[
e(H_1(H_m, M)) \leq e(H_2(H_m, M/p^{rm+c})) + (rm + c) \text{rank}_{Z_p}(H_2(H_m, M)).
\]
By a similar argument to that in [18, Corollary 2.4], one can show that there exists a constant \( C \) independent of \( m \) such that
\[
e(H_2(H_m, M/p^{rm+c})) \leq C(rm + c)p^{(r-1)m} = O(mp^{(r-1)m}).
\]
On the other hand, the rank estimate established in the previous paragraph yields
\[
(rm + c) \text{rank}_{Z_p}(H_2(H_m, M)) = O(mp^{(r-1)m}).
\]
Putting these estimates together, we obtain \( e(H_1(H_m, M)) = O(mp^{(r-1)m}) \) which gives the required bound for the first homology group. The estimates for higher homology groups can be proved similarly. □
We end the subsection with the following result which compares the variation of the $p$-exponents of the $H_m$-invariants of two pseudo-isomorphic $\mathbb{Z}_p[H]$-modules. Recall that two finitely generated $\mathbb{Z}_p[H]$-modules are said to be pseudo-isomorphic if there is a $\mathbb{Z}_p[H]$-homomorphism $\varphi : M \to N$ whose kernel and cokernel are pseudo-null $\mathbb{Z}_p[H]$-modules. Note that the property of being pseudo-isomorphic is not a symmetric relation for nontorsion modules.

**Proposition 2.2.2.** Let $M$ and $N$ be two finitely generated (not necessarily torsion) $\mathbb{Z}_p[H]$-modules. Suppose that there is a $\mathbb{Z}_p[H]$-homomorphism $\varphi : M \to N$ whose kernel and cokernel are pseudo-null $\mathbb{Z}_p[H]$-modules. Then for each $i \geq 0$, we have

$$|e(H_i(H_m, M)) - e(H_i(H_m, N))| = O(mp^{(r-1)m}).$$

**Proof.** The statement will follow if it holds in the two special cases of exact sequences

$$0 \to P \to M \to N \to 0,$$

$$0 \to M \to N \to P \to 0,$$

where $P$ is a finitely generated pseudo-null $\mathbb{Z}_p[H]$-module. We will prove the second case, the first case has a similar argument. By Lemma 2.1.1, there exists $c$ which is independent of $m$ and such that $p^{rm+c}$ annihilates the torsion subgroups of $H_i(H_m, M), H_i(H_m, N)$ and $H_i(H_m, P)$ for every $i$ and $m$ (noting that $H_i(H_m, -) = 0$ for $i \geq r + 1$ and so there are only finite number of these groups, thus enabling one to find such a $c$). Applying $H_m$-invariant, we obtain an exact sequence

$$H_{i+1}(H_m, P) \to H_i(H_m, M) \to H_i(H_m, N) \to H_i(H_m, P).$$

It then follows from an application of Corollary 2.1.4 that

$$|e(H_i(H_m, M)) - e(H_i(H_m, N))| \leq e(H_{i+1}(H_m, P)) + e(H_i(H_m, P)) + (rm + c) \text{ rank}_{\mathbb{Z}_p} H_{i+1}(H_m, P).$$

The required estimates now follow from Proposition 2.2.1. 

### 2.3 Estimates for elementary modules

In this subsection, we prove the following estimate for modules of the form $M = \mathbb{Z}_p[H]/f^*$, where $f \in \mathbb{Z}_p[H]$ is a generator of a prime ideal of $\mathbb{Z}_p[H]$ of height one. For the next proposition, we write $\delta_{f,p}$ as 1 or 0 accordingly as $f\mathbb{Z}_p[H] = p\mathbb{Z}_p[H]$ or $f\mathbb{Z}_p[H] \neq p\mathbb{Z}_p[H]$.

**Proposition 2.3.1.** Let $M = \mathbb{Z}_p[H]/f^*$, where $f \in \mathbb{Z}_p[H]$ is a generator of a prime ideal of $\mathbb{Z}_p[H]$ of height one. Then $\text{ rank}_{\mathbb{Z}_p}(M_{H_m}) = \text{ rank}_{\mathbb{Z}_p}(H_1(H_m, M)) = O(p^{(r-1)m})$ and $e(M_{H_m}) = \delta_{f,p}sp^m + O(mp^{(r-1)m})$. Furthermore, we have $H_i(H_m, M) = 0$ for $i \geq 2$ and $e(H_i(H_m, M)) = 0$ for $i \geq 1$.

**Proof.** Since $M$ is torsion over $\mathbb{Z}_p[H]$, it follows from a formula of Harris (cf. [8, Theorem 1.10]) that $\text{ rank}_{\mathbb{Z}_p}(M_{H_m}) = O(p^{(r-1)m})$. Now as $\mathbb{Z}_p[H]$ has no zero divisors, we have an exact sequence

$$0 \to \mathbb{Z}_p[H] \overset{f}{\to} \mathbb{Z}_p[H] \to M \to 0.$$
Now note that since $\mathbb{Z}_p[[H]]$ is a free $\mathbb{Z}_p[[H_m]]$-module, we have $H_i(H_m, \mathbb{Z}_p[[H]]) = 0$ for $i \geq 1$. Therefore, by considering the $H_m$-homology, we obtain an exact sequence

$$0 \to H_i(H_m, M) \to \mathbb{Z}_p[H/H_m] \to \mathbb{Z}_p[H/H_m] \to M_{H_m} \to 0$$

and $H_i(H_m, M) = 0$ for $i \geq 2$. The latter implies that $H_i(H_m, M) = 0$ and $e(H_i(H_m, M)) = 0$ for $i \geq 2$.

On the other hand, it follows from the four terms exact sequence that $\text{rank}_{\mathbb{Z}_p}(M_{H_m}) = \text{rank}_{\mathbb{Z}_p}(H_1(H_m, M))$. Thus, this completes all the required estimates for $\mathbb{Z}_p$-rank. Also, since $\mathbb{Z}_p[H/H_m]$ has no $\mathbb{Z}_p$-torsion, we have $e(H_1(H_m, M)) = 0$.

It therefore remains to establish the estimate for $e(M_{H_m})$. The proof proceeds as in [3] Theorem 2.5 with some slight difference which we describe. Write $\zeta = (\zeta_1, ..., \zeta_r) \in \mu_{p^\infty}$. Recall that two such $\zeta$ is said to be conjugate if there is an automorphism of $\mathbb{Q}_p$ sending one to the other. By identifying $\mathbb{Z}_p[[H]]$ with $\mathbb{Z}_p[[T_1, ..., T_r]]$, we may view every element $g$ of $\mathbb{Z}_p[[H]]$ as a power series in $r$ variables and it makes sense to speak of $g(\zeta - 1) := g(\zeta_1 - 1, ..., \zeta_r - 1)$ which is an element of $\mathbb{Z}_p[\zeta] := \mathbb{Z}_p[\zeta_1, ..., \zeta_r]$. Such an assignment gives rise to a homomorphism $\mathbb{Z}_p[H/H_m] \to \oplus \mathbb{Z}_p[\zeta]$. Summing these homomorphisms over the conjugacy class of $\zeta = (\zeta_1, ..., \zeta_r) \in \mu_{p^\infty}$, we obtain a homomorphism $\varphi: \mathbb{Z}_p[H/H_m] \to \oplus \mathbb{Z}_p[\zeta]$ which is called the cyclotomic embedding in [3] Section 2. By [3] Theorem 2.2, this is injective with a finite cokernel annihilated by $p^m$. One can easily check that there is a commutative diagram

$$\begin{CD}
Z_p[H/H_m] @>>> \oplus \mathbb{Z}_p[\zeta] \\
@V{f}VV @V{f(\zeta - 1)}VV \\
Z_p[H/H_m] @>>> \oplus \mathbb{Z}_p[\zeta]
\end{CD}$$

where the vertical map are given by multiplication by the corresponding element labelled on the map.

Note that the cokernel of the vertical map on the left is precisely $M_{H_m}$. Denote by $N_m$ the cokernel of the vertical map on the right. By [3] Theorem 2.4 and Lemma 2.4, we have

$$|e(M_{H_m}) - e(N_m)| \leq rm \text{rank}_{\mathbb{Z}_p}(\ker f)$$

Since $\mathbb{Z}_p[H/H_m]$ is finitely generated over $\mathbb{Z}_p$, the latter is equal to $rm \text{rank}_{\mathbb{Z}_p}(\text{coker} f) = rm \text{rank}_{\mathbb{Z}_p}(M_{H_m})$ which is precisely $O(mp^{r-1}m)$ by our rank estimate (this is the essential difference from that in [3] Theorem 2.5). It therefore remains to estimate $e(N_m)$. For this, one proceeds similarly to that in [3] Theorem 2.5 and obtains $e(N_m) = \delta_{f,p}sp^{rm} + O(mp^{r-1}m)$. The conclusion of the proposition is now immediate from these estimates.

**2.4 Estimates for general modules**

We now prove the estimates for a finitely generated (not necessarily torsion) $\mathbb{Z}_p[[H]]$-module. Before that, we recall the notion of the $\mu$-invariant which we do with slightly more generality. Let $\mathcal{G}$ be a uniform pro-$p$ group in the sense of [4]. It then follows from [9] Proposition 1.11 (see also [20] Theorem 3.40)]
that there is a $\mathbb{Z}_p[\mathcal{G}]$-homomorphism

$$\varphi : M(p) \longrightarrow \bigoplus_{i=1}^{t} \mathbb{Z}_p[\mathcal{G}] / p^{\alpha_i},$$

whose kernel and cokernel are pseudo-null $\mathbb{Z}_p[\mathcal{G}]$-modules, and where the integers $t$ and $\alpha_i$ are uniquely determined. The $\mu_G$-invariant of $M$ is defined to be $\mu_G(M) = \sum_{i=1}^{t} \alpha_i$. We can state the following main result of this section.

**Theorem 2.4.1.** Let $M$ be a finitely generated $\mathbb{Z}_p[H]$-module. Then $e(M_{H,m}) = \mu_H(M)p^m + O(mp^{(r-1)m})$ and $e(H_i(H_m, M)) = O(mp^{(r-1)m})$ for every $i \geq 1$.

The proof of the theorem will take up the remainder of this subsection. We begin by establishing the estimates for torsion $\mathbb{Z}_p[H]$-modules. We mention that even for torsion modules, our estimate is less precise than that in [3, Theorem 3.4] due to the fact that we do not work under the stronger hypothesis on the rank growth as assumed there.

**Lemma 2.4.2.** Let $M$ be a finitely generated torsion $\mathbb{Z}_p[\mathcal{H}]$-module. Then $e(M_{H,m}) = \mu_H(M)p^m + O(mp^{(r-1)m})$ and $e(H_i(H_m, M)) = O(mp^{(r-1)m})$ for every $i \geq 1$.

**Proof.** By Proposition 2.2.2, we may assume that $M$ is a direct sum of modules of the form $\mathbb{Z}_p[H]/f^s$, where $f$ is a generator of a prime ideal of height one. It then suffices to prove the result for each $\mathbb{Z}_p[\mathcal{H}]/f^s$ which is precisely Proposition 2.3.1.

We turn to the situation of a torsionfree $\mathbb{Z}_p[H]$-module. The following preparatory lemma is perhaps well-known to experts but for a lack of proper reference, we shall state this formally here.

**Lemma 2.4.3.** Let $M$ be a finitely generated torsion $\mathbb{Z}_p[\mathcal{H}]$-module. Then there exists a map $M \longrightarrow \mathbb{Z}_p[\mathcal{H}]^s$ with $\mathbb{Z}_p[\mathcal{H}]$-torsion kernel and cokernel.

In particular, if $M$ is a torsionfree $\mathbb{Z}_p[\mathcal{H}]$-module, then $M$ injects into $\mathbb{Z}_p[\mathcal{H}]^s$ with $\mathbb{Z}_p[\mathcal{H}]$-torsion cokernel.

**Proof.** The proof is similar to that in the proof of [13, Lemma 4.9]. For the convenience of the readers, we repeat the argument here. Let $K(H)$ denote the field of fractions of $\mathbb{Z}_p[H]$. Write $M^+ = \text{Hom}_{\mathbb{Z}_p[H]}(M, \mathbb{Z}_p[H])$. Then by [20, Proposition 2.5], there is a canonical map $M \longrightarrow M^{++}$ with $\mathbb{Z}_p[\mathcal{H}]$-torsion kernel and cokernel. Choose $f_1, \ldots, f_s \in M^+$ such that they form a basis for $K(H) \otimes_{\mathbb{Z}_p[H]} M^+$. Then these elements give rise to a map $\mathbb{Z}_p[\mathcal{H}]^s \longrightarrow M^+$ which clearly has $\mathbb{Z}_p[\mathcal{H}]$-torsion kernel and cokernel. Taking $\mathbb{Z}_p[\mathcal{H}]$-dual, we obtain a map $M^{++} \longrightarrow \mathbb{Z}_p[\mathcal{H}]^s$ with $\mathbb{Z}_p[\mathcal{H}]$-torsion kernel and cokernel. Combining this with the above canonical map, we obtain the required map. The second assertion is immediate from the first.

We can now prove our estimate for a torsionfree $\mathbb{Z}_p[\mathcal{H}]$-module.

**Lemma 2.4.4.** Let $M$ be a finitely generated torsionfree $\mathbb{Z}_p[\mathcal{H}]$-module with rank$_{\mathbb{Z}_p[H]}(M) = s$. Then for each $i \geq 0$, we have $e(H_i(H_m, M)) = O(mp^{(r-1)m})$.  

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Proof. By Lemma 2.4.3 we have a short exact sequence

\[ 0 \to M \to \mathbb{Z}_p[H]^s \to N \to 0, \]

for some torsion \( \mathbb{Z}_p[H]\)-module \( N \). This short exact sequence in turn yields an exact sequence

\[ 0 \to H_1(H_m, N) \to M_{H_m} \to \mathbb{Z}_p[H/H_m]^s \to N_{H_m} \to 0 \]

and isomorphisms

\[ H_{i+1}(H_m, N) \cong H_i(H_m, M) \]

for \( i \geq 1 \). Hence it follows that \( e(H_i(H_m, M)) = e(H_{i+1}(H_m, N)) \) for \( i \geq 0 \), where the equality for the case \( i = 0 \) follows from the fact that \( \mathbb{Z}_p[H/H_m]^s \) has no \( \mathbb{Z}_p \)-torsion. But since \( N \) is torsion, the required estimate of the lemma is now a consequence of Lemma 2.4.2.

We finally come to the proof of Theorem 2.4.1.

Proof of Theorem 2.4.1 By [17, Proposition 5.1.7], there is a pseudo-isomorphism \( M \sim T(M) \oplus M_{tf} \), where \( T(M) \) is the \( \mathbb{Z}_p[H]\)-torsion submodule of \( M \) and \( M_{tf} = M/T(M) \). By Proposition 2.2.2 we are thus reduced to proving the estimates for \( T(M) \) and \( M_{tf} \) which are precisely the contents of Lemmas 2.4.2 and 2.4.4. Hence we have proven our theorem.

2.5 An estimate for certain \( \mathbb{Z}_p[\Gamma] \)-modules

In this subsection, we quote the following useful result which will be used in the proof of our main theorem.

Lemma 2.5.1. Let \( M \) be a finitely generated \( \mathbb{Z}_p[\Gamma] \)-module with the properties that \( M \) is finitely generated over \( \mathbb{Z}_p \) and that \( M_{\Gamma_n} \) is finite for every \( n \geq 1 \). Let \( f \) be a generator of the characteristic ideal of \( M \). Let \( n_0 \) be an integer such that every irreducible distinguished polynomial that divides \( f \) has degree \( < p^{n_0-1}(p-1) \). Then

\[ e(M_{\Gamma_n}) - e(M_{\Gamma_0}) = \text{rank}_{\mathbb{Z}_p}(M)(n - n_0) + e(M(p)_{\Gamma_n}) - e(M(p)_{\Gamma_n}). \]

for all \( n \geq n_0 \).

Proof. This is a special case of [12, Proposition 4.6] noting that \( \mu_T(M) = 0 \) as \( M \) is finitely generated over \( \mathbb{Z}_p \).

3 Arithmetic

3.1 Setup

We recall the arithmetic setup following that in [12]. As before, \( p \) always denotes an odd prime and \( r \) an integer \( \geq 1 \). Let \( F \) be a number field. Denote by \( F_{\infty, \infty} \) a \( p \)-adic Lie extension of \( F \) whose Galois group
$G = \text{Gal}(F_{\infty,\infty})$ is isomorphic to $\mathbb{Z}_p^r \times \mathbb{Z}_p$. Let $H$ be a closed normal subgroup of $G$ which is isomorphic to $\mathbb{Z}_p^r$ and has the property that $\Gamma := G/H \cong \mathbb{Z}_p$. We write $F^c$ for the fixed field of $H$. The ramification assumptions $(R1) - (R4)$ in the introductory section are always assumed to be valid.

For $0 \leq m, n \leq \infty$, write $H_m = H^{p^m}$, $\Gamma_n = \Gamma^{p^n}$ and $G_{n,m} = H_m \times \Gamma_n$. The fixed field of $G_{n,m}$ is then denoted to be $F_{n,m}$. Let $M$ be the maximal unramified abelian pro-$p$ extension of $F_{\infty,\infty}$. We write $X = \text{Gal}(M/F_{\infty,\infty})$, $Y = \text{Gal}(M/F)$ and $Y_{n,m} = \text{Gal}(M/F_{n,m})$. Denote by $C_{n,m}$ the subgroup of $Y_{n,m}$ generated by $[Y_{n,m}, Y_{n,m}]$ and all the inertia groups in $Y_{n,m}$ and write $B_{n,m} = C_{n,m} \cap X$. The relation between $B_{n,m}$ and the class group of $F_{n,m}$ is established in [12, Lemma 2.2 and Formula (2)] which we record here.

**Proposition 3.1.1.** We have $X/B_{n,m} \cong \text{Cl}(F_{n,m})(p)$ and a short exact sequence

$$0 \rightarrow B_{n,m}/I_{G_{n,m}}X \rightarrow X_{G_{n,m}} \rightarrow \text{Cl}(F_{n,m})(p) \rightarrow 0,$$

where $I_{G_{n,m}}$ is the augmentation ideal of $G_{n,m}$ in $\mathbb{Z}_p[G]$.

We also record the useful estimate of Lei [12 Corollary 2.10].

**Proposition 3.1.2 (Lei).** $B_{n,m}/I_{G_{n,m}}X$ is finitely generated over $\mathbb{Z}_p$ and there exists $C$ independent of $1 \leq n, m \leq \infty$ such that

$$\text{rank}_{\mathbb{Z}_p}(X/B_{n,m}) \leq C p^{(r-1)m},$$

for every $m$ and $n$.

From now on, we always write $e_{n,m}$ for $p$-exponent of $\text{Cl}(F_{n,m})(p)$. For each $n$, we also denote by $X(\Gamma_n)$ the $\mathbb{Z}_p[G]$-submodule of $X$ which is generated by elements of the form $(\gamma - 1)x$, where $\gamma \in \Gamma_n$ and $x \in X$.

**Proposition 3.1.3 (Lei).** For a fixed $n$, we have

$$e_{n,m} = \mu_H(X/\chi(\Gamma_n))p^{r^m} + \text{rank}_{\mathbb{Z}_p[H]}(X/X(\Gamma_n))mp^{(r-1)m} + O(p^{(r-1)m}).$$

**Proof.** This is [12 Corollary 3.4] noting that our $r$ here is $d - 1$ there.

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**3.2 Main results**

We can now state and prove the main theorem of this paper which is a generalization of [12 Corollary 5.3].

**Theorem 3.2.1.** Retain the setting in the previous subsection. Suppose that $X$ is finitely generated over $\mathbb{Z}_p[H]$. Then we have

$$e_{n,n} = \text{rank}_{\mathbb{Z}_p[H]}(X)p^{r^m} + O(p^{r^n}).$$

Before proving our theorem, we first show the following crucial estimate which is established by Lei in the case $r = 1$ (see [12 Proposition 5.2]). This estimate is now possible in general thanks to our analysis in Section 2.
Proposition 3.2.2. Retain all the assumptions of Theorem 3.2.1 Then

\[ e(\mathcal{X}/B_{\infty,m}) \leq \mu_H(\mathcal{X})p^{rm} + O(mp^{(r-1)m}). \]

Proof. Since \( B_{\infty,m} = A_m(S, \mathcal{X}) \) for some structure \( S \) on \( \mathcal{X} \) (cf. [12] Proofs of Lemma 3.3 and Proposition 5.2), we may apply Lemmas 2.1.1 and 2.1.2 to obtain an \( e \) which is independent of \( m \) and has the property that \( p^{(r+1)m+c} \) annihilates the torsion subgroups of \( \mathcal{X}_{H,m} \) and \( \mathcal{X}/B_{\infty,m} \) for every \( m \). This in turn allows us to apply Lemma 2.1.3 to the exact sequence in Proposition 3.1.1 which yields the following inequality

\[ e(\mathcal{X}/B_{\infty,m}) \leq \mu_H(\mathcal{X})p^{rm} + (r+1)m + c \].

Thanks to Theorem 2.4.1 we have that \( e(\mathcal{X}_{H,m}) = \mu_H(\mathcal{X})p^{rm} + O(mp^{(r-1)m}) \). On the other hand, by virtue of Proposition 3.1.2, one has \( (r+1)m + c \) \( \text{rank}_{\mathbb{Z}}(\mathcal{X}/B_{\infty,m}/I_{H,m} \mathcal{X}) = O(mp^{(r-1)m}) \). The required bound of the proposition is now a consequence of these estimates.

We are in position to prove our theorem.

Proof of Theorem 3.2.1 By [1] Proposition 2.3, there exists a finite collection of \( g_i \in \mathbb{Z}_p\Gamma[G] \) such that each \( \mathbb{Z}_p\Gamma[G]/\mathbb{Z}_p[G]g_i \) is finitely generated over \( \mathbb{Z}_p\Gamma[H] \) and such that there is a surjection

\[ \bigoplus \mathbb{Z}_p\Gamma[G]/\mathbb{Z}_p[G]g_i \to \mathcal{X}. \]

By [12] Corollary 5.4, there exists an integer \( t \) (independent of \( m \)) such that the \( \mathbb{Z}_p\Gamma[G] \)-characteristic ideal of \( \bigoplus \mathbb{Z}_p[G]/\mathbb{Z}_p[G]g_i \), and hence \( \mathcal{X}_{H,m} \), factorizes into polynomials of degree \( \leq t \). Since \( \mathcal{X}/B_{\infty,m} \) is a quotient of \( \mathcal{X}_{H,m} \), the same conclusion holds for \( \mathcal{X}/B_{\infty,m} \). Now fix a choice of \( n_0 \) such that \( t < p^{n_0} - 1 \).

For now, fix an arbitrary positive integer \( m \). As \( (\mathcal{X}/B_{\infty,m})_{\Gamma_n} \cong Cl(F_{n,m})(p) \) (cf. [12] Lemma 2.2) is finite, we may apply Lemma 2.5.1 to conclude that whenever \( n \geq n_0 \), one has

\[ |e_{n,m} - e_{n_0,m} - (n - n_0) \text{rank}_{\mathbb{Z}_p}(\mathcal{X}/B_{\infty,m})| = |e((\mathcal{X}/B_{\infty,m})(p)_{\Gamma_n}) - e((\mathcal{X}/B_{\infty,m})(p)_{\Gamma_{n_0}})\]

\[ \leq e((\mathcal{X}/B_{\infty,m})(p)_{\Gamma_n}) + e((\mathcal{X}/B_{\infty,m})(p)_{\Gamma_{n_0}}) \leq 2e(\mathcal{X}/B_{\infty,m}). \]

Since \( m \) is arbitrary, the above formula, in particular, holds when \( m = n \geq n_0 \). In other words, we have

\[ |e_{n,n} - e_{n_0,n} - (n - n_0) \text{rank}_{\mathbb{Z}_p}(\mathcal{X}/B_{\infty,n})| \leq 2e(\mathcal{X}/B_{\infty,n}) \]

for \( n \geq n_0 \). Therefore, in order to estimate \( e_{n,n} \), it remains to estimate every other terms in this inequality.

As a start, Proposition 3.2.2 tells us that \( e(\mathcal{X}/B_{\infty,n}) = O(p^{rn}) \). By Proposition 3.1.3 we have

\[ e_{n_0,n} = \mu_H(\mathcal{X}/\mathcal{X}(\Gamma_{n_0}))p^{rn} + O(np^{(r-1)n}) = O(p^{rn}) \]

since \( n_0 \) is fixed. On the other hand, it follows from [8] Theorem 1.10 that \( \text{rank}_{\mathbb{Z}_p}(\mathcal{X}_{H,n}) = \text{rank}_{\mathbb{Z}_p}[H](\mathcal{X})p^{rn} + O(p^{(r-1)n}) \). Combining this with Propositions 3.1.1 and 3.1.2, we obtain

\[ \text{rank}_{\mathbb{Z}_p}(\mathcal{X}/B_{\infty,n}) = \text{rank}_{\mathbb{Z}_p}[H](\mathcal{X})p^{rn} + O(p^{(r-1)n}) \]
which in turn implies that

\[
(n - n_0) \text{rank}_{\mathbb{Z}_p}(\mathcal{X} / B_{\infty,n}) = \text{rank}_{\mathbb{Z}_p}[H](\mathcal{X})np^r n - \text{rank}_{\mathbb{Z}_p}[H](\mathcal{X})n_0 p^r n + O(np^{(r-1)n})
\]

\[
= \text{rank}_{\mathbb{Z}_p}[H](\mathcal{X})np^r n + O(p^r n).
\]

The conclusion of the theorem now follows by combining these estimates.

As an immediate corollary, we have the following.

**Corollary 3.2.3.** Suppose that \( \mathcal{X} \) is a finitely generated torsion \( \mathbb{Z}_p[H] \)-module. Then we have

\[
e_{n,n} = O(p^n).
\]

As mentioned in the introductory section, we like to ask if there exists some \( \alpha^* \) such that

\[
e_{n,n} = \alpha^* p^r n + O(np^{(r-1)n})
\]

when \( \mathcal{X} \) is a finitely generated torsion \( \mathbb{Z}_p[H] \)-module. In the event that \( G_{\infty,\infty} \) is commutative, this will follow from the result of Monsky (cf. [16, Theorem 3.13]). In our noncommutative situation, the best we can do at present is the following.

**Theorem 3.2.4.** Suppose that \( \mathcal{X} \) is a finitely generated torsion \( \mathbb{Z}_p[H] \)-module. Then we have

\[
e_{n,n} \leq \beta^* p^r n + O(np^{(r-1)n})
\]

for some \( \beta^* \).

**Proof.** By a similar argument to that in Theorem 3.2.1, there exists an \( n_0 \) such that

\[
e_{n,n} \leq e_{n_0,n} + \text{rank}_{\mathbb{Z}_p}((\mathcal{X}/B_{\infty,n})(n - n_0) + 2e(\mathcal{X}/B_{\infty,n}).
\]

Performing the same calculations as before, we obtain the required estimate noting that this time round our calculation gives

\[
\beta^* = \mu_H(\mathcal{X}) + \mu_H(\mathcal{X}/\mathcal{X}(\Gamma_{n_0})).
\]

As mentioned in the introduction, we are not able to relate our \( \beta^* \) with Monsky’s \( \alpha^* \). Despite so, the calculations in the preceding proof gives us the following result which answers our question in a special case.

**Theorem 3.2.5.** Suppose that \( \mathcal{X} \) is a finitely generated torsion \( \mathbb{Z}_p[H] \)-module with \( \mu_H(\mathcal{X}) = 0 \). Then we have

\[
e_{n,n} = O(np^{(r-1)n}).
\]

**Proof.** As seen from the calculation of Theorem 3.2.4, we have

\[
e_{n,n} \leq (\mu_H(\mathcal{X}) + \mu_H(\mathcal{X}/\mathcal{X}(\Gamma_{n_0}))) p^r n + O(np^{(r-1)n})
\]

for some \( n_0 \). By our hypothesis, we have \( \mu_H(\mathcal{X}) = 0 \) which in turn implies that \( \mu_H(\mathcal{X}/\mathcal{X}(\Gamma_{n_0})) = 0 \). Hence the conclusion of corollary now follows.
3.3 Examples

We conclude the paper with some examples and further questions.

(1) A nonzero integer \(\alpha\) is said to be amenable for the prime \(p\) if either \(p|\alpha\) or \(p \parallel \alpha^{p-1} - 1\) holds. Denote by \(\mu_p\) (resp., \(\mu_p^\infty\)) the multiplicative group of \(p\)-root of unity (resp., \(p\)-power root of unity). Let \(F = \mathbb{Q}(\mu_p), F^c = \mathbb{Q}(\mu_p^\infty)\) and \(F_{\infty,\infty} = \mathbb{Q}(\mu_p^\infty, \alpha_1^{-p^-\infty}, \ldots, \alpha_r^{-p^-\infty})\), where \(\alpha_i\) are nonzero integers whose image in \(\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^p\) are linearly independent over \(\mathbb{F}_p\) and the products \(\alpha_1^{n_1} \cdots \alpha_r^{n_r}\) are all amenable for \(p\) for every \(0 \leq n_i \leq p-1\). It is well-known that the ramification properties (\(R1\)), (\(R2\)) and (\(R4\)) are satisfied. Property (\(R3\)) is satisfied by [22] Theorem 5.2 and Lemma 6.1. By the theorem of Ferrero-Washington [9] and a descent argument (for instance, see [19] Proposition 2.1), we have that \(\mathcal{X}\) is finitely generated over \(\mathbb{Z}_p[[H]]\). Hence Theorem 3.2.1 applies in this situation.

In view of [18] Question 1.3 and the two paragraphs after, one would expect \(\mathcal{X}\) to be a finitely generated torsion \(\mathbb{Z}_p[[H]]\)-module. Granted this, Corollary 3.2.3 applies. However, it remains currently an open problem to prove this torsionness condition in general (but see below discussion).

(2) Let \(F = \mathbb{Q}(\mu_p), F^c = \mathbb{Q}(\mu_p^\infty)\) and \(F_{\infty,\infty} = \mathbb{Q}(\mu_p^\infty, p^{-p^-\infty})\). Now if \(p\) is a regular prime, then the \(p\)-class group of any intermediate field in this tower is trivial by a classical result of Iwasawa [10]. We therefore assume that the prime \(p\) is irregular in our discussion here. This was the extension, where Venjakob [21] first made his guess on the form of the asymptotic class number formula and which we now know is correct thanks to the work of Lei [12].

Now if \(p\) is an irregular prime \(<1000\), a result of Sharifi [19] Propositions 3.3 and 2.1a] asserts that \(\mathcal{X}\) is finitely generated over \(\mathbb{Z}_p\). In view of the results of Cucuo-Monsky for a commutative \(p\)-adic Lie extension (see [3] Theorem II), one is led to the following speculation:

Does there exist \(\lambda\) and \(\nu\) such that \(e_{n,n} = \lambda n + \nu\) for sufficiently large \(n\)?

At this point of writing, we do not have an answer to this. We do however remark that it follows from Sharifi’s result and our Theorem 3.2.3 that one at least has the suggestive estimate \(e_{n,n} = O(n)\).

For irregular primes \(>1000\) and beyond, our current state of knowledge seems even less and we have nothing to say on this.

References

[1] J. Coates, T. Fukaya, K. Kato, R. Sujatha and O. Venjakob, The \(GL_2\) main conjecture for elliptic curves without complex multiplication, Publ. Math. IHES 101 (2005) 163-208.

[2] J. Coates, P. Schneider and R. Sujatha, Modules over Iwasawa algebra, J. Inst. Math. Jussieu 2(1) (2003) 73-108.

[3] A. Cucuo and P. Monsky, Class numbers in \(\mathbb{Z}_p^d\)-extensions, Math. Ann. 255(2) (1981) 235-258.

[4] D. Delbourgo and A. Lei, Estimating the growth in Mordell-Weil ranks and Shaferevich-Tate groups over Lie extensions, Ramanujan J. 43 (2017) 29-68.

[5] J. Dixon, M. P. F. Du Sautoy, A. Mann and D. Segal, Analytic Pro-\(p\) Groups, 2nd edn, Cambridge Stud. Adv. Math. 38, Cambridge Univ. Press, Cambridge, UK, 1999.
[6] B. Ferrero and L. C. Washington, The Iwasawa invariant $\mu_p$ vanishes for abelian number fields, Ann. of Math. 109 (1979) 377-395.

[7] Y. Hachimori and R. Sharifi, On the failure of pseudo-nullity of Iwasawa modules, J. Algebraic Geom. 14(3) (2005) 567-591.

[8] M. Harris, Correction to $p$-adic representations arising from descent on abelian varieties, Comp. Math. 121 (2000) 105-108.

[9] S. Howson, Structure of central torsion Iwasawa modules, Bull. Soc. Math. France 130(4) (2002) 507-535.

[10] K. Iwasawa, A note on class numbers of algebraic number fields, Abh. Math. Sem. Univ. Hamburg 20 (1956) 257-258.

[11] K. Iwasawa, On $\Gamma$-extensions of algebraic number fields, Bull. Amer. Math. Soc. 65 (1959) 183-226.

[12] A. Lei, Estimating class numbers over metabelian extensions, Acta Arith. 180(4) (2017) 347-364.

[13] M. F. Lim, Notes on the fine Selmer groups, Asian J. Math. 21(2) (2017) 337-362.

[14] M. F. Lim, Comparing the $\pi$-primary submodules of the dual Selmer groups, Asian J. Math. 21(6) (2017) 1153-1182.

[15] M. F. Lim and R. Sharifi, Nekovář duality over $p$-adic Lie extensions of global fields, Doc. Math. 18 (2013) 621-678.

[16] P. Monsky, Fine estimate for the growth of $e_n$ in $\mathbb{Z}_p$-extensions, in Algebraic Number Theory—in honor of K. Iwasawa, ed. J. Coates, R. Greenberg, B. Mazur and I. Satake, Adv. Stud. in Pure Math. 17, 1989, pp. 309-330.

[17] J. Neukirch, A. Schmidt and K. Wingberg, Cohomology of Number Fields, 2nd Ed., Grundlehren Math. Wiss. 323, Springer 2008.

[18] G. Perbet, Sur les invariants d’Iwasawa dans les extensions de Lie $p$-adiques, Algebra Number Theory 5 (2011) 819-848.

[19] R. Sharifi, On Galois groups of unramified pro-$p$ extensions, Math. Ann. 342(2) (2008) 297-308.

[20] O. Venjakob, On the structure theory of the Iwasawa algebra of a $p$-adic Lie group, J. Eur. Math. Soc. 4(3) (2002) 271-311.

[21] O. Venjakob, A non-commutative Weierstrass preparation theorem and applications to Iwasawa theory, J. reine angew. Math. 559 (2003) 153-191.

[22] F. Viviani, Ramification groups and Artin conductors of radical extensions of $\mathbb{Q}$, J. Théor. Nombres Bordeaux 16(3) (2004) 779-816.