Floating Bodies of Equilibrium in Three Dimensions.
The central symmetric case

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Abstract Three-dimensional central symmetric bodies different from spheres that can float in all orientations are considered. For relative density $\rho = \frac{1}{2}$ there are solutions, if holes in the body are allowed. For $\rho \neq \frac{1}{2}$ the body is deformed from a sphere. A set of nonlinear shape-equations determines the shape in lowest order in the deformation. It is shown that a large number of solutions exists. An expansion scheme is given, which allows a formal expansion in the deformation to arbitrary order under the assumption that apart from $x = 0, \pm 1$ there is no $x$, which obeys $P_{p,2}(x) = 0$ for two different integer $p$s, where $P$ are Legendre functions.

1 Introduction and Summary

A long standing problem asked by Stanislaw Ulam in the Scottish Book [1] (problem 19) is, whether a sphere is the only solid of uniform density which will float in water in any position. Such a solid is called a floating body of equilibrium. It will be in indifferent equilibrium in all orientations.

The simpler, two-dimensional, problem to find non circular cross-sections of a long cylindrical log which floats without tending to rotate (the axis of the log is assumed to be parallel to the water surface.) was solved for relative density $\rho_d = 1/2$ in 1938 by Auerbach [2] and for densities $\rho_d \neq 1/2$ by the present author [3, 4, 5].

Here we start to investigate the problem for three dimensional systems. In section 2 the basic equations for a partly immersed body in indifferent equilibrium are derived. In the following section the theorem that a star-shaped inversion-symmetric body in arbitrary dimension $d$ and density $1/2$ is a sphere, is reconsidered. One easily sees that a floating body of equilibrium, which is not star-shaped can have a shape different from a ball. A class of such bodies is explicitly given.

The following sections, which are concerned with star-shaped floating bodies of equilibrium with arbitrary relative densities, can be read without having
read section 3. In section 4 and 5 we start considering the expansion of the surface of a floating body of equilibrium around the sphere. To perform the expansion the distance of the surface is measured from the center of gravity. It is expanded in spherical harmonics, which are defined as the eigenfunctions of the Laplacean operator on the unit sphere with eigenvalues \(-l(l+1)\). For given \(l\) the eigenfunctions are linear combinations of \(2l+1\) linearly independent functions.

In section 6 the equations are considered for the deformation in first order. If the angle \(\theta_0\) is a zero of the associated Legendre function \(P_{p,-2}(\cos \theta_0) = 0\), then the spherical harmonics with \(l = p\) contribute to the deformation. This equation is fulfilled at \(\theta_0 = \pi/2\) for all odd \(p\), which corresponds to \(\rho_d = 1/2\). Thus one has to expect that at this special density there is a large set of solutions. In the following we assume that with the exception of \(\cos \theta_0 = 0, \pm 1\) there is no \(\theta_0\), which solves \(P_{p,-2}(\cos \theta_0) = 0\) for two different integer \(p\).

In section 7 the equations in second order in the deformation are derived. One finds that only spherical harmonics for the deformation are allowed whose square projected onto the space of harmonics with the same \(p\) is proportional to the original deformation. This is the contents of the shape equations (111). This projection vanishes for odd \(p\). Thus one has to distinguish between the problem for odd and even \(p\). In the following only the case of even \(p\), that is for central symmetric bodies, is pursued. A large number of solutions for the shape equations is found in section 8 assuming invariance under various subgroups of the orthogonal group \(O(3)\). By considering only these special groups only solutions with mirror symmetry are found. It is not clear whether there are shapes with inversion symmetry, but without mirror symmetry.

In section 9 it is shown how contributions in higher order can be obtained, which lead to a formal expansion for the shape.

## 2 General Considerations

**Potential energy** Denoting the volume of the body by \(V\), the volume above the water by \(V_a\), that below the water by \(V_b\), one obtains (Archimedes’ law)

\[
V_a = (1 - \rho_d)V, \quad V_b = \rho_dV,
\]

where \(\rho_d\) is the relative density of the body with respect to the liquid. Denote the total mass of the body by \(m\), the masses above/below the water-line by \(m_{a,b}\), the center of mass above/below the water-line by \(C_{a,b}\), and the distance of \(C_{a,b}\) from the water-line by \(d_{a,b}\). Then the potential energy \(V\) of the system is given by

\[
V = m_agd_a + (m - m_b)gd_b = m(1 - \rho_d)g(d_a + d_b).
\]

Thus the difference in height between the two centers of mass, \(d = d_a + d_b\), is constant, since it has to be independent of the orientation. This does not imply,
that $d_a$ and $d_b$ are separately constant. Moreover the line $C_aC_b$ connecting the two centers of mass has to be perpendicular to the water-level. Placing the center of mass of the body in the origin and denoting the coordinates of $C_a, b$ by $(0, 0, z_a, z_b)$ one obtains
\[ z_a = \rho_d d, \quad z_b = -(1 - \rho_d)d. \tag{3} \]
Thus the loci of the centers of gravity lie on spheres.

The reverse is also true: If $z_a$ is independent of the orientation, then also $z_b$ has this property. Then $z_a - z_b = d_a + d_b$ is constant and thus its potential energy. We will use this property to determine bodies which can float in all orientations.

One can conclude: If a body has the property that a plane in arbitrary orientation cutting through it, so that the two volumina $V_a, b$ are constant and the centers of gravity of these volumina lie on spheres, then it is a floating body of equilibrium, since the body assumes in all orientations the same potential energy.

These properties and another one we will derive below were derived by Pierre Bouguer and Charles Dupin a long time ago. Auerbach writes in his 1938 paper: *Il résulte aisément des théorèmes classiques de Bouguer et Dupin que la condition nécessaire et suffisante pour qu’un corps soit une solution de ce problème est que la surface de centres de carène soit une sphère. De plus, on peut affirmer que dans ce cas l’ellipse centrale d’inertie de la flotaison est un cercle dont le rayon est le même pour toute position d’équilibre.* Indeed Pierre Bouguer (1698-1758) and Charles Dupin (1784-1873) wrote books on the hydrostatics of ships. These conditions are also found in the textbooks by Appell and Webster. Compare also the article by Gilbert.

**Moment of inertia** Rotate the body by an infinitesimal angle. Since the volume above and below the water has to be conserved the rotation is around the center of gravity $M$ of the water-plane area $F$ (intersection of the plane of the water-surface with the body). The center of gravity $M$ of the water-plane area is given by
\[ x_M = \frac{1}{F} \int dF \; x, \quad y_M = \frac{1}{F} \int dF \; y. \tag{4} \]
An infinitesimal rotation $\delta \phi_{x,y}$ will bring a wedge of thickness
\[ \zeta(x, y) = \delta \phi_x (x - x_M) + \delta \phi_y (y - y_M) \tag{5} \]
above the waterline, if positive; if negative, its modulus describes the thickness of a wedge disappearing below the waterline. There are two contributions to the shift of the centers of gravity in $x$ and $y$ direction,
\[ \delta x_a = -z_a \delta \phi_x + \frac{1}{V_a} \int dF \; x \zeta(x, y), \quad \delta y_a = -z_a \delta \phi_y + \frac{1}{V_a} \int dF \; y \zeta(x, y), \tag{6} \]
\[ \delta x_b = -z_b \delta \phi_x - \frac{1}{V_b} \int dF \; x \zeta(x, y), \quad \delta y_b = -z_b \delta \phi_y - \frac{1}{V_b} \int dF \; y \zeta(x, y). \tag{7} \]
where the first term is due to the rotation of the centers of gravity, and the second one comes from the appearance and disappearance of the wedges. The requirement $\delta x_a = \delta x_b$, $\delta y_a = \delta y_b$ yields for arbitrary $\delta \phi_{x,y}$ the conditions
\[ I_{ij} = \delta_{ij} I, \quad I = \rho_d (1 - \rho_d) V d \tag{8} \]
for the moments of inertia of the water-plane area

\[ I_{xx} = \int dF (x - x_M)^2, \]  
\[ I_{xy} = \int dF (x - x_M)(y - y_M), \]  
\[ I_{yy} = \int dF (y - y_M)^2. \]  

The moment \( I \) is independent of the orientation.

If (8) is not fulfilled, but both eigenvalues of the matrix

\[ \hat{I} = \begin{pmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{pmatrix} \]  

are larger than \( I \), then the body is in stable equilibrium. If both are less, then the body is in unstable equilibrium, and if one is larger, one less than \( I \), then it is in saddle-point equilibrium. Here we are interested in an indifferent equilibrium, for which equality (8) holds. The corresponding condition yields in two dimensions that the length of the waterline has to be constant.\[3\]

3 Central Symmetric Case \( \rho_d = 1/2 \)

In this section central symmetric bodies with relative density \( \rho_d = 1/2 \) are considered. Central symmetry, which is also called inversion symmetry, means: If the point at \( r \) belongs to the body, then also the point \(-r\) belongs to it, where the center has been placed at the origin. The plane of the water-plane area goes through the origin in all orientations, since it cuts the body into two equal halves.

The following theorem due to Schneider\[11, 12\] and to Falconer\[13\], also referred to by Hensley in the Scottish book\[4\] holds: For arbitrary dimension \( d \) and density \( 1/2 \), if the body is star-shaped, symmetric, bounded and measurable, then it differs from a ball by a set of measure 0. It follows from theorem 1.4 of Schneider \[12\] (similarly corollary 3.1 in \[11\]): 'Let \( \Omega_d \) be the unit sphere \(|u| = 1\), and \cdot the inner product. If \( \Phi \) is an even real-valued, countably additive set function on \( \Omega_d \) satisfying \( \int_{\Omega_d} |u| \cdot v|d\Phi(u) = 0 \) for each \( v \in \Omega_d \), then \( \Phi = 0 \).'

The theorem will be reproved in the following assuming that the distance of the surface from the center of the body is a continuous function of \( u \). This is done first for \( d = 3 \) dimensions and then generalized to dimensions \( d > 3 \). The premise that the body is star-shaped is important. It is shown in subsection \[5.3\] that there are non-spherical floating bodies of equilibrium, if one allows holes to be drilled into the body.

3.1 Star-shaped body in three dimensions

If the body is star-shaped, i.e., there exists a point \( A \) such that for each point \( P \) in the body the segment \( AP \) lies in the body. Since the set of these points \( A \) (called kernel) forms a convex set and since it is central symmetric, too, the origin is such a point \( A \). We denote the extension of the body from the origin
in direction of the unit vector $\mathbf{u}$ by $r(\mathbf{u})$. Let $\Omega_u$ be the unit sphere and $\mathbf{v}$ the normal on the water surface, then
\[
\frac{1}{2} V_d = \int d^3r d\Omega_u r|\mathbf{u} \cdot \mathbf{v}| = \frac{1}{4} \int d\Omega_u r^4(\mathbf{u})|\mathbf{u} \cdot \mathbf{v}|
\] (13)
has to be independent of $\mathbf{v}$. One expands $|\mathbf{u} \cdot \mathbf{v}|$ in Legendre polynomials
\[
|\mathbf{u} \cdot \mathbf{v}| = \sum_{n=0}^{\infty} c_n P_{2n}(\mathbf{u} \cdot \mathbf{v}), \quad c_n = (-)^{n+1} \frac{(2n+1/2)\Gamma(n-1/2)}{2\sqrt{\pi} (n+1)!}.
\] (14)
Note that all coefficients $c_n$ differ from 0, $c_n \neq 0$. Using the addition theorem
\[
P_{2n}(\mathbf{u} \cdot \mathbf{v}) = \frac{4\pi}{2n+1} \sum_m Y_{2n,m}(\mathbf{u}) Y_{2n,m}(\mathbf{v}),
\] (15)
where the $Y_{2n,m}$ are $4n+1$ orthonormalized (real) spherical harmonics, one obtains
\[
V_d = \sum_{n,m} \frac{2\pi c_n}{2n+1} Y_{2n,m}(\mathbf{v}) \int d\Omega_u r^4(\mathbf{u}) Y_{2n,m}(\mathbf{u}).
\] (16)
$V_d$ is only independent of $\mathbf{v}$, if all coefficients $c_n$ except $c_0$ vanish. Since the functions $Y_{2n,m}$ form a complete orthogonal set of central symmetric functions, this implies that $r^4(\mathbf{u})$ has to be independent of $\mathbf{u}$.

### 3.2 Star-shaped body in higher dimensions

The proof is similar to that in $d$ dimensions. $r^3$ and $r^4/4$ have to be replaced by $r^d$ and $r^{d+1}/(d+1)$, resp. Instead of Legendre polynomials one has to expand in ultraspherical (Gegenbauer) polynomials $G_n^{(d/2-1)}(\mathbf{u} \cdot \mathbf{v})$,
\[
|\mathbf{u} \cdot \mathbf{v}| = \sum_{n=0}^{\infty} c_n G_n^{(d/2-1)}(\mathbf{u} \cdot \mathbf{v}),
\]
\[
c_n = (-)^{n+1} \frac{(2n+d/2-1)\Gamma(n-1/2)\Gamma(d/2-1)}{2\pi \Gamma(n+d/2+1/2)}.
\] (17)
Repeated application of Gegenbauer’s addition theorem (see appendix A) yields
\[
G_n^{(d/2-1)}(\mathbf{u} \cdot \mathbf{v}) = \frac{2\pi^{d/2}}{(n+d/2-1)\Gamma(d/2-1)} \sum_m Y_{n,m}^{(d)}(\mathbf{u}) Y_{n,m}^{(d)}(\mathbf{v}),
\] (18)
where the functions $Y_{n,m}^{(d)}$ form a complete set of orthonormal functions on $\Omega_d$ obeying
\[
\triangle_u Y_{n,m}^{(d)}(\mathbf{u}) = -n(n+d-2)Y_{n,m}^{(d)}(\mathbf{u}),
\] (19)
with the Laplacean operator $\triangle_u$ on the sphere $\Omega_d$. Since all $c_n \neq 0$ the same argument applies that $r^{d+1}u$ has to be independent of $\mathbf{u}$. 

5
3.3 Bodies with holes

If the requirement that the body is star-shaped is omitted, but holes in the body are allowed, then the condition that the body K can float in all orientations requires that the integral \( \int_0^\infty dr \chi(r, \mathbf{u}) r^d \) does not depend on the direction \( \mathbf{u} \), where the indicator function

\[
\chi(r, \mathbf{u}) = \begin{cases} 
1 & \mathbf{ru} \in K \\
0 & \mathbf{ru} \notin K.
\end{cases}
\]  

(20)

is introduced. Thus one may drill holes into the body and add the corresponding amount of material at the outside creating a bulge.

Figure 2: Non-spherical central symmetric floating body of equilibrium with two holes described in this section at \( \rho_d = 1/2 \) projected onto the (x,y)-, (x,z)-, and (y,z)-plane. The inner part of the cylindrical holes end at the sphere, the outer part at the bulge.

As an example consider two circular cylindrical holes of radius \( r_h \) around the axes \( x = \pm b, \ y = 0 \) drilled into a sphere of radius \( R_0 \). Using spherical coordinates

\[
x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta
\]  

(21)

the bounds of the hole is given by

\[
r = R_\pm = \frac{b \cos \phi \pm \sqrt{r_h^2 - b^2 \sin^2 \phi}}{\sin \theta}.
\]  

(22)

One obtains the indicator function

\[
\chi(r, \theta, \phi) = \begin{cases} 
\hat{\theta}(R_- - r) + \hat{\theta}(R' - r)\hat{\theta}(r - R_+) & \text{if } |\sin \phi| < \frac{b}{r_h} \text{ and } R_- < R_0, \\
\hat{\theta}(R_0 - r) & \text{otherwise}
\end{cases}
\]  

(23)

with the Heaviside step function \( \hat{\theta} \), which equals 1 for positive argument and 0 for negative argument. \( \hat{\theta}(0) \) may be chosen 1 (0) for a closed (an open) body. \( R' \) is given by

\[
R' = \sqrt{R_0^4 + R_+^4 - R_-^4}.
\]  

(24)

Obviously there are bodies with several holes and the holes need not be circular cylinders. Thus there is a large variety of bodies with central symmetry at
density $\rho_0 = 1/2$, if one does not require star-shape. The body described here has the property that any fluid above the center can flow off and any air below the center can climb up up. In two dimensions a body with this property is not possible, since two holes through the body would cut it into three pieces. Otherwise, however, fluid above the center cannot always outflow nor air below the center cannot always bubble up.

### 4 Parametrization of the Surface

In the following bodies of general densities are considered, however, star-shape is assumed. Thus in spherical coordinates the radius $r$ of the surface of the body is a unique function of the angles $\theta_K$, $\phi_K$ in the body-fixed system.

It is expanded in the set of spherical harmonics $C_{l,m}(\theta_K) e^{im\phi_K}$ (Racah uses this normalization for $C$; however here $C$ contains only the $\theta$-dependence),

$$r = \sum_{l,m} a_{l,m} C_{l,m}(\theta_K) e^{im\phi_K},$$  \hspace{1cm} (25)

which are related to the conventional spherical harmonics by

$$Y_{l,m}(\theta_K, \phi_K) = \sqrt{\frac{2l+1}{4\pi}} C_{l,m}(\theta_K) e^{im\phi_K}.$$  \hspace{1cm} (26)

In contrast to section 3 and appendix A, where the spherical harmonics were assumed to be real, here the spherical harmonics conventionally applied in physics are used$\textsuperscript{[14,15,16]}$. In particular one has

$$C_{00} = 1, \quad C_{10}(\theta) = \cos(\theta), \quad C_{1,\pm1}(\theta) = \mp \frac{1}{\sqrt{2}} \sin \theta.$$  \hspace{1cm} (27)

and in general

$$C_{lm}(\theta) = (-)^m \sqrt{\frac{(l-m)!}{(l+m)!}} P_{lm}(\cos \theta)$$  \hspace{1cm} (28)

holds with the Legendre functions defined by

$$P_{lm}(x) = (1-x^2)^{m/2} \frac{1}{2l!} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l.$$  \hspace{1cm} (29)

The following relation holds

$$C_{l-m}(\theta) = (-)^m C_{lm}(\theta).$$  \hspace{1cm} (30)

The body is rotated by means of Euler angles ($\alpha$, $\beta$) into a position where the water-surface is parallel to the $x$, $y$-plane. A third rotation around the axis perpendicular to the water-surface is not necessary. The functions $C_{lm'}(\theta_K) e^{im'\phi_K}$ transform according to

$$C_{lm'}(\theta_K) e^{im'\phi_K} = \sum_m D_{mm'}^l(\alpha, \beta, 0) C_{lm}(\theta) e^{im\phi},$$  \hspace{1cm} (31)

from which we obtain the radius

$$r(\theta, \phi; \beta, \alpha) = \sum_{l,m,m'} C_{lm}(\theta) D_{mm'}^{l}(\alpha, \beta, \phi) a_{l,m},$$  \hspace{1cm} (32)
with the Wigner rotation matrix elements\([16, 14, 15]\)

\[ D_{l_{m m'}}^l(\alpha, \beta, \phi) = e^{im\phi} d_{m m'}^l(\beta)e^{im'\alpha}. \]  

(33)

The height of the water level above the center of mass of the body be \( h(\beta, \alpha) \). Then the intersection of the water level with the surface of the body is given by

\[ h(\beta, \alpha) = \cos(\Theta)r(\Theta, \phi; \beta, \alpha). \]  

(34)

This determines \( \Theta(\phi, \beta, \alpha) \) of the waterline on the surface of the body for given orientation \((\alpha, \beta)\). At water-level the waterline is given in polar coordinates \((\rho, \phi)\) by

\[ \rho(\phi, \beta, \alpha) = \sin(\Theta)r(\Theta, \phi; \beta, \alpha). \]  

(35)

The total volume \( V \) and the volume \( V_a \) above the water surface are given by

\[ V = \frac{1}{3} \int d\phi \int_0^\pi d\theta r^3 \sin \theta, \]  

(36)

\[ V_a = \frac{1}{3} \int d\phi \int_0^{\Theta(\phi)} d\theta r^3 \sin \theta - \frac{h}{6} \int d\phi \rho^2. \]  

(37)

The volume \( V_a \), which is a segment, is divided into a sector (first integral) and a cone (second integral). Obviously one has \( V_b = V - V_a \). Similarly one obtains the coordinates \( z \) of the centers of gravity by means of \( Z_c = V z_c \) and \( Z_{a,b} = V_{a,b} z_{a,b} \) from

\[ Z_c = \frac{1}{4} \int d\phi \int_0^\pi d\theta r^4 \sin \theta \cos \theta, \]  

(38)

\[ Z_a = \frac{1}{4} \int d\phi \int_0^{\Theta(\phi)} d\theta r^4 \sin \theta \cos \theta - \frac{h^2}{8} \int d\phi \rho^2. \]  

(39)

The integrals \([36, 39]\) can often be expressed in the form

\[ \int d\phi \sum_{lm m'} D_{l_{m m'}}^l(\alpha, \beta, \phi) \hat{f}_{lm'} = 2\pi \sum_{lm'} D_{0 m'}^{0 l}(\alpha, \beta, 0) \hat{f}_{lm'} \]

\[ = 2\pi \sum_{lm'} C_{lm'}(\beta)e^{im'\alpha} \hat{f}_{lm'}. \]  

(40)

Since the integrals must not depend on of \( \alpha, \beta \) all \( \hat{f}_{lm'} \) have to vanish with the exception of \( \hat{f}_{00} \).

4.1 The Sphere

For a sphere of radius \( r_0 \) with the waterline at \( \Theta = \theta_0 \) one obtains (see figure 1)

\[ h = r_0 \cos \theta_0, \quad \rho = r_0 \sin \theta_0, \]  

(41)

independent of the orientation. Since the cosine and sine of \( \theta_0 \) is often needed, the abbreviations

\[ c_0 = \cos \theta_0, \quad s_0 = \sin \theta_0 \]  

(42)
are used. The volumes are
\[ V_a = \frac{\pi}{3} r_0^3 (2 - 3c_0 + c_0^3) = \frac{\pi}{3} r_0^3 (1 - c_0)^2 (2 + c_0) \]
\[ = \frac{\pi}{3} (2r_0^3 - 3r_0^2 h + h^3), \quad (43) \]
\[ V_b = \frac{\pi}{3} r_0^3 (2 + 3c_0 - c_0^3) = \frac{\pi}{3} r_0^3 (1 + c_0)^2 (2 - c_0) \]
\[ = \frac{\pi}{3} (2r_0^3 + 3r_0^2 h - h^3), \quad (44) \]
and the z-coordinates
\[ z_a = \frac{3r_0}{4} \frac{(1 + c_0)^2}{2 + c_0}, \quad z_b = \frac{3r_0}{4} \frac{(1 - c_0)^2}{2 - c_0} \quad (45) \]
One easily calculates
\[ d = z_a - z_b = \frac{3r_0}{(2 + c_0)(2 - c_0)}, \quad I = \frac{\pi}{4} \beta^4 \quad (46) \]
and verifies eq. (8).

5 Expansion

Starting out from the sphere the body will be deformed and an expansion in the deformation parameter \( \epsilon \) will be performed. That is, \( a \) and similarly the other quantities are expanded in powers of \( \epsilon \),
\[ a_{l,m} = \sum_n \epsilon^n a_{n;l,m}. \quad (47) \]
Starting point of the expansion is the sphere of radius \( r_0 \),
\[ a_{0;l,m} = r_0 \delta_{l,0} \delta_{m,0}. \quad (48) \]
Similarly \( r \) is expanded,
\[ r(\theta, \phi; \beta, \alpha) = \sum_n \epsilon^n r_n(\theta, \phi; \beta, \alpha) \quad (49) \]
with
\[ r_n(\theta, \phi; \beta, \alpha) = \sum_{l,m,m'} C_{lm}(\theta) D_{mm'}^l(\alpha, \beta, \phi) a_{n;l,m'}. \quad (50) \]
Since \( r_n \) has to be evaluated around \( \theta = \theta_0 \), the expansion
\[ r_n(\theta, \phi; \beta, \alpha) = \sum_k (\theta - \theta_0)^k r_{nk}(\phi, \beta, \alpha) \quad (51) \]
with
\[ r_{nk}(\phi, \beta, \alpha) = \frac{1}{k!} \sum_{l,m,m'} \left. \frac{d^k C_{lm}(\theta)}{d\theta^k} \right|_{\theta = \theta_0} D_{mm'}^l(\alpha, \beta, \phi) a_{n;l,m'}. \quad (52) \]
is used. \( r_0 \) does not depend on \( \theta, \phi, \beta, \alpha \). Similarly \( \Theta(\phi) \), \( h \) and \( \rho \) are expanded in powers of \( \epsilon \),

\[
\Theta(\phi; \beta, \alpha) = \sum_n \epsilon^n \Theta_n(\phi; \beta, \alpha),
\]

\[
h(\beta, \alpha) = \sum_n \epsilon^n h_n(\beta, \alpha),
\]

\[
\rho(\phi, \beta, \alpha) = \sum_n \epsilon^n \rho_n(\phi, \beta, \alpha).
\]

\( \Theta_0 = \theta_0 \), \( h_0 = r_0 c_0 \), \( \rho_0 = r_0 s_0 \) do not depend on the angles, since the body is a sphere in lowest (zeroth) order.

### 5.1 Evaluation of the Integrals

In order to evaluate the integrals for the part of the body above the water-level I divide the integral over \( \theta \) into one which runs up to \( \theta_0 \) and the one from \( \theta = \theta_0 \) to \( \Theta(\phi) \). Let us first consider the integral up to \( \theta_0 \). For the volume we expand in powers of \( \epsilon \) and in spherical harmonics

\[
r^3/3 = \sum_n \epsilon^n f_n = \sum_{nlm} \epsilon^n C_{lm}(\theta) e^{im\phi} f_{n;lm}.
\]

Due to the \( \phi \)-integration only terms with \( m = 0 \) will contribute. With \( \zeta = \cos \theta \) one obtains

\[
\int_0^{\theta_0} d\theta C_{l0}(\theta) \sin \theta = \int_{c_0}^1 d\zeta P_l(\zeta)
\]

\[
= \sqrt{1 - \zeta^2} P_{l,-1}(\zeta) \bigg|_{c_0}^1 = -s_0 P_{l,-1}(c_0).
\]

Thus the first contribution \( V_{an}^{(1)} \) reads

\[
V_{an}^{(1)} = \int d\phi \int_0^{\theta_0} \sin \theta f_n = 2\pi \left( (1 - c_0) f_{n;00} - s_0 \sum_{l \geq 1} f_{n;0l} P_{l,-1}(c_0) \right).
\]

The term for \( l = 0 \) is written separately, since \( P_{0,-1} \) is not defined. The total volume yields

\[
V_n = 4\pi f_{n;00}.
\]

Next the corresponding integral for \( Z \) is considered. The expansion

\[
r^4/4 = \sum_n \epsilon^n f^z_n = \sum_{nlm} \epsilon^n C_{lm}(\theta) e^{im\phi} f^z_{n;lm}.
\]

yields

\[
\int_0^{\theta_0} d\theta C_{l0}(\theta) \sin \theta \cos \theta = \int_{c_0}^1 d\zeta \zeta P_l(\zeta)
\]

\[
= \zeta \sqrt{1 - \zeta^2} P_{l,-1}(\zeta) \bigg|_{c_0}^1 - \int_{c_0}^1 \sqrt{1 - \zeta^2} P_{l,-1}(\zeta)
\]

\[
= (\zeta \sqrt{1 - \zeta^2} P_{l,-1}(\zeta) - (1 - \zeta^2) P_{l,-2}(\zeta)) \bigg|_{c_0}^1
\]

\[
= -s_0 c_0 P_{l,-1}(c_0) + s_0^2 P_{l,-2}(c_0).
\]
Thus the first contribution to $Z_{n}^{(1)}$ reads

$$Z_{n}^{(1)} = \int d\phi \int_{0}^{\theta_0} d\theta f_{n}^{z} \sin \theta \cos \theta = 2\pi \left( \frac{1}{2} (1 - c_0^2) f_{n:00}^z + \frac{1}{3} (1 - c_0^3) f_{n:10}^z \right)$$

$$+ s_0 \sum_{l \geq 2} f_{n:10}^z (s_0 P_{l,-1}(c_0) - c_0 P_{l,-1}(c_0)) .$$  \hspace{1cm} (65)

For the complete volume one obtains

$$Z_{cn} = \frac{4\pi}{3} f_{n:10}^z .$$  \hspace{1cm} (66)

We require that the origin coincides with the center of mass of the complete body. This yields $f_{n:1,0} = 0$. This can always be done by an appropriate choice of the $a_{n:1,m}$.

Since $\Theta(\phi) - \theta_0 = O(\epsilon)$ the integral over the interval $\theta = \theta_0 \ldots \Theta(\phi)$ can be expanded in powers of $\epsilon$. Thus the second contributions to $V_{n}$ and $Z_{n}$ yield

$$V_{n}^{(2)} = \int d\phi g_{n},$$

with

$$\sum_{n} \epsilon^{n} g_{n} = \frac{1}{3} \int_{\theta_0}^{\Theta(\phi)} d\theta r^3 \sin \theta - \frac{h}{6} \rho^2 .$$  \hspace{1cm} (68)

Similarly the second contribution to $Z_{n}$ is obtained from

$$Z_{n}^{(2)} = \int d\phi g_{n}^z ,$$

with

$$\sum_{n} \epsilon^{n} g_{n}^z = \frac{1}{4} \int_{\theta_0}^{\Theta(\phi)} d\theta r^4 \sin \theta \cos \theta - \frac{h^2}{8} \rho^2 .$$  \hspace{1cm} (70)

It is useful to perform the expansion procedure explicitly in zeroth, first and second order. Then it becomes apparent how to continue to higher orders.

5.2 Zeroth Order: The Sphere again

In zeroth order one obtains

$$f_{0} = \frac{1}{3} r_{0}^3 , \quad f_{0}^z = \frac{1}{4} r_{0}^4 ,$$

$$g_{0} = -\frac{1}{6} r_{0}^3 c_{0} s_{0}^2 , \quad g_{0}^z = -\frac{1}{8} r_{0}^4 c_{0}^2 s_{0}^2 .$$  \hspace{1cm} (72)

One obtains from these expressions

$$V_{n}^{(1)} = 2\pi (1 - c_0) f_{n:00} = \frac{2\pi}{3} r_{0}^3 (1 - c_0) ,$$

$$V_{n}^{(2)} = -\frac{\pi}{3} r_{0}^3 c_{0} s_{0}^2 .$$  \hspace{1cm} (74)
which yields the volume in agreement with (43), and

\[
Z^{(1)}_{a0} = \pi (1 - c_0^2) f_{0,00}^z = \frac{\pi}{4} (1 - c_0^2) r_0^4, \quad (75)
\]

\[
Z^{(2)}_{a0} = -\frac{\pi}{4} (1 - c_0^2) c_0^2 r_0^4, \quad (76)
\]

which yields \( z_a \) in eq. (45).

6 First order in the deformation

The deformation of the sphere in first order in \( \epsilon \) is considered in this section. One obtains from eqs. (34,35)

\[
\Theta_1 = r_{10} c_0 - h_1 \frac{r_0 s_0}{s_0}, \quad \rho_1 = \frac{r_{10} - h_1 c_0}{s_0}. \quad (77)
\]

Following the procedure in subsection 5.1 one obtains \( f_1 = r_0^2 r_1 \) or more explicitly

\[
f_1 = \sum_{lm} f_{1;lm} C_{lm}(\theta) e^{im \phi}, \quad f_{1;lm} = r_0^2 \sum_{m'} D_{lm,m'}^{(l)}(\alpha, \beta, 0) a_{1;lm'}, \quad (78)
\]

and thus the volume

\[
V_1 = 4 \pi r_{1;00} r_0^2. \quad (79)
\]

We choose to keep the volume unchanged in this order, which implies

\[
r_{1;00} = a_{1;00} = 0. \quad (80)
\]

Next one obtains \( g_1 = -\frac{1}{2} h_1 r_0^2 s_0^2 \) and thus

\[
V_{a1} = -2 \pi s_0 r_0^2 \sum_{l \geq 1} r_{1;0} P_{l,-1}(c_0) - \pi h_1 r_0^2 s_0^2. \quad (81)
\]

Since \( V_{a1} \) is constant, one obtains

\[
h_1 = h_{1;00} - \frac{2}{s_0} \sum_{l \geq 1} r_{1;0} P_{l,-1}(c_0) \quad (82)
\]

with the \( \alpha, \beta \) independent contribution \( h_{1;00} \).

Next one determines \( Z \). With \( f_{1}^z = r_0^2 r_1 \) one obtains

\[
Z_{c1} = \frac{4 \pi}{3} r_0^3 r_{1;10}. \quad (83)
\]

Since the center of gravity stays at the origin, one requires \( r_{1;10} = 0 \), which is equivalent to \( a_{1;1m} = 0 \). With \( g_{1}^z = -\frac{1}{2} h_1 r_0^3 c_0 s_0^2 \) one obtains

\[
Z_{a1} = \pi s_0 r_0^3 \left( \sum_{l \geq 2} r_{1;0} (s_0 P_{l,-2}(c_0) - c_0 P_{l,-1}(c_0)) - h_1 c_0 s_0 \right). \quad (84)
\]

Inserting eq. (82) into the expression for \( Z_{a1} \) yields

\[
Z_{a1} = \pi s_0^2 r_0^3 \left( \sum_{l \geq 2} r_{1;0} P_{l,-2}(c_0) - h_{1;00} c_0 \right) \quad (85)
\]
$Z_{a1}$ must be constant. We note that

$$r_{1;0} = \sum_{m} D_{0m}^{l} (\alpha, \beta, 0) a_{1;lm}. \quad (86)$$

In order that $Z_{a1}$ is constant all coefficients $r_{1;0} P_{l,-2}(c_0)$ except for $l = 0$ have to vanish. Therefore for any given $l$ either all coefficients $a_{1;lm}$ or $P_{l,-2}(c_0)$ have to vanish. We note that $P_{l,-2}(\pm 1) = 0$ for all $l$. However this corresponds to the limits $\rho_4 = 0$ and $\rho_4 = 1$, in which case only a sphere is possible\[17\]. Further one has $P_{l,-2}(0) = 0$ for all odd $l$. This corresponds to $\rho_4 = 1/2$. It may well be that similarly to the two-dimensional case one has a large variety of solutions for this special density. This case will not be considered further in this paper.

In the following only cases are considered, for which $c_0$ is different from 0 and $\pm 1$. We will assume a $\cos \theta_0 = c_0$ for which $P_{p,-2}(c_0) = 0$ and assume that for this $c_0$ there is only one solution $p$. (Numerical calculation up to $l = 100$ shows that apart from $c_0 = 0, \pm 1$ there is no $c_0$, which for two different $l$ yields $P_{l,-2}(c_0) = 0$.) Thus all $a_{1;lm} = 0$ with the exception of $l = p$ and one obtains

$$r_1 = \sum_{m} C_{pm}(\theta) D_{mn}^{p}(\alpha, \beta, \phi) a_{1;pm'}, \quad (87)$$

where the amplitudes $a_{1;pm'}$ are as yet undetermined. Similarly $h_{1;00}$ is undetermined. Note that the zeroes of $P_{p,2}(x)$ and $P_{p,-2}$ are identical.

7 Second Order in the deformation

One obtains from eqs. (34, 35) in second order in $\epsilon$

$$\Theta_2 = \frac{1}{2r_0^2} (-2s^2 h_2 r_0 + 2c h_0 r_2 - 2c h_0 r_1 h_1$$

$$+ 2c h_0 r_1 r_0 - c_0 h_1^2 - c_0 (2 - c_0^2) r_1^2 + 2 h_1 r_1), \quad (88)$$

$$\rho_2 = \frac{1}{2r_0^2} (-2c h_0^2 h_2 r_0 + 2s^2 h_2 r_0 - 2s h_0 r_1 h_1$$

$$+ 2s h_0 r_1 r_0 - h_1^2 - c_0^2 r_1^2 + 2 c h_1 r_1). \quad (89)$$

These quantities depend on $\phi$, $\alpha$ and $\beta$. Quantities on the right hand side depending on $\theta$ have to be evaluated at $\theta = \theta_0$. The deviation of $\theta$ from $\theta_0$ is taken into account by the derivatives $r_{11}$ with respect to $\theta$, compare eqs. (51, 52). For products $b^{(1)} b^{(2)}$ of

$$b^{(i)} = \sum_{lm} b_{lm}^{(i)} C_{lm}(\theta)e^{ilm\phi} \quad (90)$$

we use the notation

$$b^{(1)} b^{(2)} = \sum_{lm} (b^{(1)} b^{(2)})_{lm} C_{lm}(\theta)e^{ilm\phi} \quad (91)$$

with

$$(b^{(1)} b^{(2)})_{lm} = \sum_{l_1,m_1,l_2,m_2} \langle l_{1}m_{1}l_{2}m_{2} \rangle (l_{1}m_{1}l_{2}m_{2}) b_{l_{1}m_{1}}^{(1)} b_{l_{2}m_{2}}^{(2)}. \quad (92)$$
where the \( \langle lm; l_1m_1, l_2m_2 \rangle \) are Clebsch-Gordan coefficients. Use is made of

\[
C_{l_1, m_1}(\theta)C_{l_2, m_2}(\theta) = \sum_{lm} \langle lm; l_1m_1, l_2m_2 \rangle \langle 00; l_10, l_20 \rangle C_{l_1, m_1}(\theta). \tag{93}
\]

Then one obtains from

\[
f_2 = r_0 r_1^2 + r_0^2 r_2, \tag{94}
\]

\[
f_2^z = r_0^2 r_2 + \frac{3}{2} r_1^2 r_1^2 \tag{95}
\]

the expressions

\[
f_{2,lm} = r_0 \sum_{m'} D^{(l)}_{m,m'}(\alpha, \beta, 0)(a_1^2)_{lm'} + r_0 a_{2,lm'}. \tag{96}
\]

\[
f_{2z,lm} = r_0^2 \sum_{m'} D^{(l)}_{m,m'}(\alpha, \beta, 0) \frac{3}{2} (a_1^2)_{lm'} + r_0 a_{2,lm'}. \tag{97}
\]

For the complete volume one obtains

\[
V_2 = 4\pi f_{2,00} = 4\pi (r_0^2 a_{2,00} + \frac{r_0}{2p + 1} \sum_m (-)^m a_{1,pm} a_{1,pm}), \tag{98}
\]

\[
Z_{c2} = \frac{4\pi}{3} f_{2,1,0}. \tag{99}
\]

Since the center of mass should stay at the origin and since \((a_1^2)_{1m} \) vanishes for odd \( l \) one has \((a_1^2)_{1m} = 0 \) and requires \( a_{2,1m} = 0 \).

Then the first contributions to \( V_{a2} \) and \( Z_{a2} \) yield

\[
V^{(1)}_{a2} = 2\pi \left( (1 - c_0) f_{2,00} - s_0 \sum_{l \geq 2} f_{2,10} P_{l,-1}(c_0) \right), \tag{100}
\]

\[
Z^{(1)}_{a2} = 2\pi \left( \frac{1}{2} (1 - c_0^2) f_{2z,00} + \frac{1}{3} (1 - c_0^3) f_{2z,10}^z \right.
\]

\[
+ s_0 \sum_{l \geq 2} f_{2z,10}^z (s_0 P_{l,-2}(c_0) - c_0 P_{l,-1}(c_0)) \right). \tag{101}
\]

with eqs. \[59, 65, 67\].

The second contributions are obtained from

\[
g_2 = \frac{r_0}{2} (-s_0^2 h_2 r_0 + c_0 r_1^2 - 2 c_0 h_1 + c_0 h_1^2), \tag{102}
\]

\[
g_2^z = \frac{1}{4} r_0^2 (-2 c_0 s_0^2 h_2 r_0 + 2 c_0^2 r_1^2 - 4 c_0 r_1 h_1 + (3 c_0^2 - 1) h_1^2). \tag{103}
\]

Together with eqs. \[67, 68\]. The condition that \( V_{a2} \) is independent of \( \alpha, \beta \) yields an equation for \( h_2 \). Thus \( h_2 \) is determined up to some additive \( \alpha, \beta \)-independent contribution \( h_{2,00} \). Instead of considering \( Z_{a2} \) it is more practical to consider the difference \( Z_{a2} - c_0 r_0 V_{a2} \)

\[
Z^{(2)}_{a2} - c_0 r_0 V^{(2)}_{a2} = \text{const} - 2\pi r_0^2 P_{p,-1}(c_0) \sum_{lm} D^{l}_{0m}(a_1^2)_{lm}
\]

\[
+ 2\pi r_0^2 h_{1,00} s_0 P_{p,-1}(c_0) \sum_{m} D^{p}_{0m} a_{1,pm}. \tag{104}
\]
Then one obtains

\[ Z_{a2} - c_0 r_0 V_{a2} - \text{const} = 2 \pi r_0^2 \sum_{l \geq 2, m} D'_{lm}(\alpha, \beta, 0) \left( A_{1l} r_0 a_{2;lm} + A_{2l} h_{1;00} a_{1,lm} + A_{3l} (a_1^2)_{;lm} \right) \]  

with

\[ A_{1l} = s_0^2 P_{l,-2}(c_0), \]  
\[ A_{2l} = s_0 \delta_{l,p} P_{p,-1}(c_0), \]  
\[ A_{3l} = \frac{3}{2} s_0^2 P_{l,-2}(c_0) - \frac{1}{2} s_0 c_0 P_{l,-1}(c_0) - P_{p,-1}^2(c_0). \]

From these expressions \( a_{2;lm} \) is determined for \( l \neq 0, 1, p \)

\[ r_0 s_0^2 P_{l,-2}(c_0) a_{2;lm} = \left( -\frac{3}{2} s_0^2 P_{l,-2}(c_0) + \frac{1}{2} s_0 c_0 P_{l,-1}(c_0) + P_{p,-1}^2(c_0) \right)(a_1^2)_{;lm}. \]  

For \( l = p \) one obtains the equation

\[ s_0 h_{1;00} a_{1;pm} = \left( \frac{1}{2} c_0 s_0 + P_{p,-1}(c_0) \right)(a_1^2)_{;pm}. \]  

For odd \( p \) the only non-trivial solution can be obtained for \( h_{1;00} = 0 \), since the right-hand side vanishes for odd \( p \). For even \( p \) a set of quadratic equations

\[ \gamma a_{1;pm} = \sum_{m_1, m_2} \langle pm; pm_1, pm_2 \rangle a_{1;pm_1, a_{1;pm_2}} \]  

is obtained with the constant

\[ \gamma = \frac{s_0 h_{1;00}}{\left( \frac{1}{2} c_0 s_0 + P_{p,-1}(c_0) \right) \langle p0; p0, p0 \rangle}. \]

I call the equations (111) shape-equations. They determine the possible shapes of the body in first order in \( \epsilon \). Descriptively it means that the projection of the square of the deformation \( a_1 \) onto the harmonics with \( l = p \) is proportional to the deformation. I expect that there are cubic shape-equations for odd \( p \).

The denominator \( \frac{1}{2} c_0 s_0 + P_{p,-1}(c_0) \) can be expressed as \( s_0^2 c_0 K_p(c_0) \) with \( K_p \) an even polynomial of order \( p - 4 \) in \( c_0 \). A recurrence relation for \( K_p \) is derived in appendix C. Apparently \( K_p(c_0) \) is positive for real argument \( c_0 \) and even \( p \geq 4 \). I do not have a proof for general \( p \). However, numerical calculations show that up to \( p = 36 \) the functions \( K_p(c_0) \) do not have any real zero.

The solutions of these equations are invariant under rotations. If \( a_{1;pm} \) is a solution, then also

\[ a'_{1;pm} = \sum_{m} D^p_{mm'}(\alpha, \beta, \gamma) a_{1;pm'} \]  

is solution for arbitrary Euler angles \( \alpha, \beta, \gamma \).

8 Solution of the Shape Equations

The complete solutions of the shape-equations (111) is not known to me. After all they constitute a set of \( 2p + 1 \) quadratic equations. However, a number of
special solutions can be given by restricting to shapes invariant under subgroups of the rotation group. Under a subgroup of O(3) a certain set of functions indexed by $i$

$$\sum_m v_{im} C_{pm}(\theta_K) e^{im\phi_K}$$  \hspace{1cm} (114)

is invariant. Thus the amplitude $a_{1;pm}$ is a linear combination of the $v_{i,m}$s,

$$a_{1;pm} = \gamma \sum_i \alpha_i v_{i,m}.$$  \hspace{1cm} (115)

The $v$s should be orthonormal, $\sum_m v_{jm}^* v_{im} = \delta_{ji}$. Then the shape equations read

$$\sum_i \alpha_i v_{i,m} = \sum_{i_1 i_2} \alpha_{i_1} \alpha_{i_2}$$  \hspace{1cm} (116)

$$\times \sum_{m_1 m_2} \sqrt{2p+1} (-)^{m_1+m_2} \left( \begin{array}{ccc} p & p & p \\ m_1 & m_2 & -m \end{array} \right) v_{i_1,m_1} v_{i_2,m_2}.$$  

Multiplication by $v_{j,m}^*$ summing over $m$ and using orthonormality yields

$$\alpha_j = \sum_{j_1 j_2} w_{j_1, i_1, i_2} \alpha_{i_1} \alpha_{i_2},$$  \hspace{1cm} (117)

$$w_{j_1, i_1, i_2} = \sqrt{2p+1} \sum_{m_1 m_2} \left( \begin{array}{ccc} p & p & p \\ m_1 & m_2 & -m \end{array} \right) (-)^{m_1+m_2} v_{j,m}^* v_{i_1,m_1} v_{i_2,m_2}.$$  \hspace{1cm} (118)

If we require $v_{j,m}^* = (-)^m v_{j,-m}$, which implies real $\alpha$, then we obtain after replacing $-m$ by $m$ under the sum

$$w_{j_1, i_1, i_2} = \sqrt{2p+1} \sum_{m_1 m_2} \left( \begin{array}{ccc} p & p & p \\ m_1 & m_2 & m \end{array} \right) v_{j,m} v_{i_1,m_1} v_{i_2,m_2}.$$  \hspace{1cm} (119)

Since the 3j Wigner symbol is invariant under change of the sign of all $m$s and the product of the $v$s changes into its conjugate complex under this change of sign, $w$ is real. Moreover it is invariant under permutation of the indices $j, i_1, i_2$.

We consider two classes of subgroups of the O(3). The first class contains shapes with an $n$ fold rotation axis, the second one shapes with tetrahedral, octahedral and icosahedral symmetry. Obviously the shapes of the last class are special shapes of the first class of groups. Without loss of generality I put $\gamma = 1$ in this section. Otherwise one should read $a/\gamma$ instead of $a$.

**8.1 Solutions with rotational symmetry around the z-axis**

If the z-axis is an $n$ fold rotation axis, then the shapes are invariant under the cyclic group $C_n$. Since $p$ is even, the shapes have inversion symmetry. Thus they are invariant under the groups $C_{nh}$ for even $n$ and under $S_{2n}$ for odd $n$.

Only amplitudes $a_{1;pn}$ with integer $k$ running from $-[p/n]$ to $[p/n]$ contribute. If in particular $n > p$, then only $a_{1;p,0}$ contributes and since $(p0; p0, p0)$
Table 1 Solutions of the shape equations for the dihedral groups $D_{nh}$ even

| (p,n)   | $\nu_e, \nu_o$ | $a_{1;p,0}$ | $a_{1;p,n}$ | $a_{1;p,2n}$ | $a_{1;p,3n}$ |
|---------|----------------|-------------|-------------|-------------|-------------|
| (4,\infty) | (1,0)         | 2.48576     |             |             |             |
| (4,4)   | (1,1)         | 1.59799     | -0.95498    |             |             |
| (4,2)   | (2,1)         | -0.39950    | -1.26332    | 0.71624     |             |
| (4,2)   | (2,1)         | 0.93216     | -0.98258    | 1.29984     |             |
| (6,\infty) | (1,0)         | -2.98035    |             |             |             |
| (6,6)   | (1,1)         | 2.70941     | -3.74359    |             |             |
| (6,4)   | (1,1)         | 7.45088     | -13.9393    |             |             |
| (6,2)   | (2,2)         | 2.70941     | -3.47040    | 0.00000     | -1.40385    |
| (6,2)   | (2,2)         | -12.1077    | -1.48526    | 0.99999     |             |
| (6,2)   | (2,2)         | 0.93136     | -0.95436    | 1.04545     | -1.41554    |
| (6,2)   | (2,2)         | 0.38868     | -0.77601    | -0.63361    | 0.52318     |
| (8,\infty) | (1,0)         | 3.40498     |             |             |             |
| (8,6)   | (1,1)         | 1.83345     | -1.24558    |             |             |
| (8,4)   | (2,1)         | -0.22918    | -1.48526    | 0.99999     |             |
| (8,4)   | (2,1)         | 2.97936     | -1.12039    | 1.70705     |             |
| (10,\infty) | (1,0)        | -3.78269    |             |             |             |
| (10,10) | (1,1)         | 7.74691     | -13.5249    |             |             |
| (10,8)  | (1,1)         | 1.37562     | -1.60639    |             |             |
| (10,6)  | (1,1)         | -1.82513    | -1.31296    |             |             |
| (10,4)  | (2,1)         | -139.718    | -123.111    | -48.3395    |             |
| (10,4)  | (2,1)         | 0.99612     | -1.00375    | -1.19470    |             |
| (12,\infty) | (1,0)        | 4.12619     |             |             |             |
| (12,10) | (1,1)         | 1.34073     | -1.10158    |             |             |
| (12,8)  | (1,1)         | -1.74997    | -2.08835    |             |             |
| (12,6)  | (2,1)         | 0.91262     | -1.15888    | 0.62268     |             |
| (12,4)  | (2,2)         | -5.08382    | -1.21363    | -1.98403    | 5.50958     |
| (12,4)  | (2,2)         | 5.05473     | -1.95363    | 1.30127     | -2.91433    |
| (12,6)  | (2,2)         | -0.55198    | -0.93569    | 0.89981     | -0.56841    |
| (12,4)  | (2,2)         | -1.64037    | -0.45582    | -1.99424    | 0.56834     |

does not vanish for even $p$, there exist for all $p > 2$ shapes invariant under any rotations around the z-axis, that is under $C_{\infty h}$. If $n \leq p$, then several amplitudes contribute. Real $r$ implies the condition $a_{1,p,-m} = (-)^ma_{1,p,m}^\ast$. Distinguishing between real and imaginary part of the amplitudes one obtains $2[p/n] + 1$ equations for $2[p/n] + 1$ unknowns. By an appropriate rotation around the z-axis $a_{1,p,n}$ can be made real. Then only $2[p/n]$ unknowns are left. I did not find solutions for real $a_{1,p,n} \neq 0$ and complex $a_{1,p,kn}$ for $|k| \neq 0, 1$, but I cannot exclude the existence of such solutions.

If one requires that all $a_{1,p,kn}$ are real, then it is easy to find solutions. They are invariant under dihedral groups $D_{nh}$ for even $n$, and under $D_{nd}$ for odd $n$. Shapes were determined up to $p = 12$ and $\nu := [p/n] + 1 \leq 4$.

The number $\nu$ of independent amplitudes is separated into $\nu_e$ amplitudes $a_{1,p,kn}$ with even $k$ and $\nu_o$ amplitudes with odd $k$. The shape eq. expresses the amplitude for even $k$ on the r.h.s. by products of two amplitudes with even
ks and products of two amplitudes with odd ks, but no mixed contributions, whereas the eqs. for the amplitudes with odd ks contain on the r.h.s. mixed products of one amplitude with even and one with odd k. As a consequence to any solution with non-vanishing odd amplitudes there exists a second one with reversed sign of the odd amplitudes. In the tables only one of these solutions is given.

### 8.2 Tetrahedral, octahedral and icosahedral groups

Other subgroups of the O(3) are the tetrahedral, the octahedral and the icosahedral groups. Due to inversion symmetry the corresponding groups are $T_h$, $O_h$, and $I_h$. Only floating bodies of equilibrium with mirror symmetry have been found, since all these groups have mirror planes. I do not know whether there exist any shapes without this symmetry.

Solutions of the shape equations are listed in the following tables. For the tetrahedral group one can again distinguish between even and odd contributions. The numbers $\nu(p)$ of linearly independent functions $r_K$ obey

\[
\nu_{\text{oct}}(p) = \nu_{\text{tet}}(p) = \nu_{\text{tet}}(p + 6), \quad (120)
\]

\[
\nu_{\text{oct}}(p + 12) = \nu_{\text{oct}}(p) + 1, \quad \nu_{\text{ico}}(p + 30) = \nu_{\text{ico}}(p) + 1. \quad (121)
\]

The spherical harmonics invariant under these groups are constructed by requiring that they are invariant under two elements generating the group. The first group element is a rotation about the z-axis by $\pi$ for $T_h$, $\pi/2$ for $O_h$, and $2\pi/5$ for $I_h$. The second is a rotation about another axis by $2\pi/3$, $\pi/2$, and $2\pi/5$ for $T_h$, $O_h$, and $I_h$, resp., described by the rotation matrix $D(\alpha, \beta, \gamma)$.
Table 3 Solutions of the shape equations for the tetrahedral group $T_h$

| $p$ \ $(\nu_e, \nu_o)$ | $a_{1; p, 0}$ | $a_{1; p, 2}$ | $a_{1; p, 4}$ | $a_{1; p, 6}$ | $a_{1; p, 8}$ | $a_{1; p, 10}$ | $a_{1; p, 12}$ | $a_{1; p, 14}$ | $a_{1; p, 16}$ | $a_{1; p, 18}$ |
|---------------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| 6 \ (1,1)           | 0.33868      | -0.77601     | -0.63361     | 0.52318      |              |              |              |              |              |              |
| 10 \ (1,1)          | 0.82311      | -0.66870     | -0.82942     | -0.13114     | -0.98720     |              |              |              |              |              |
| 12 \ (2,1)          | -6.12511     | -4.20630     | 9.82449      | 16.5288      | -19.1622     | -10.4288     | 0.69076      |              |              |              |
| 12 \ (2,1)          | 0.83701      | -0.23817     | 0.36961      | 0.93590      | 0.14896      | -0.59051     | 0.49855      |              |              |              |
| 16 \ (2,1)          | -0.56063     | -0.48811     | -2.41712     | 1.26394      | 0.99991      |              |              |              |              |              |
| 16 \ (2,1)          | 0.98186      | -0.40466     | 0.07028      | 1.04785      | 0.40520      | 0.49200      | 2.10476      | -1.00183     | -1.20704     |              |
| 18 \ (2,2)          | -0.00998     | -2.11050     | -0.15926     | -1.76012     | 3.00022      |              |              |              |              |              |
| 18 \ (2,2)          | 0.45850      | 0.72989      | -0.13223     | -0.01543     | -0.89094     |              |              |              |              |              |
| 18 \ (2,2)          | 0.68875      | 0.37789      | -0.39434     | -0.58803     | -0.50005     |              |              |              |              |              |
| 18 \ (2,2)          | -4.73748     | 0.38021      | 2.98822      | -4.42878     | 2.25081      |              |              |              |              |              |
| 20 \ (2,2)          | 1.38678      | -0.45717     | -0.35519     | 0.89628      | 0.49923      | -0.00998     | -2.11050     | -0.15926     | -1.76012     | 3.00022      |
| 22 \ (2,2)          | 0.92948      | 0.86536      | 0.08246      | 1.02995      | -0.82637     | 0.40742      |              |              |              |              |
| 22 \ (2,2)          | -246.049     | -310.255     | 115.358      | -273.782     | 161.314      |              |              |              |              |              |
| 22 \ (2,2)          | 104.246      | 128.285      | 83.0346      | 118.399      | -135.985     |              |              |              |              |              |
| 22 \ (2,2)          | 197.709      | 272.876      |              |              |              |              |              |              |              |              |
| 22 \ (2,2)          | -0.16701     | -1.08554     | -0.64291     | -0.95793     | 2.35878      |              |              |              |              |              |
| 22 \ (2,2)          | 0.36474      | -0.42041     | 0.29053      | -2.02467     | -0.47579     |              |              |              |              |              |
| 22 \ (2,2)          | 1.12764      | 0.95476      |              |              |              |              |              |              |              |              |
| 22 \ (2,2)          | 2.63933      | -9.21867     | -1.96687     | -4.21038     | 0.54457      |              |              |              |              |              |
| 22 \ (2,2)          | -1.31708     | -1.88933     | -0.28461     | -3.39911     | 0.30053      |              |              |              |              |              |
| 22 \ (2,2)          | 0.61160      | 7.22339      |              |              |              |              |              |              |              |              |
| 22 \ (2,2)          | 2.63933      | -3.50688     | -1.96687     | -7.01919     | 0.54457      |              |              |              |              |              |
| 22 \ (2,2)          | 5.59288      | -1.88933     | 3.69040      | -3.39911     | -5.87815     |              |              |              |              |              |
| 22 \ (2,2)          | -1.11601     | 3.97000      |              |              |              |              |              |              |              |              |
| 22 \ (2,2)          | 0.67761      | 0.69275      | -0.19515     | -0.34288     | -0.82641     |              |              |              |              |              |
| 22 \ (2,2)          | 0.84351      | -0.26707     | 0.48523      | 0.03159      | -0.75424     |              |              |              |              |              |
| 22 \ (2,2)          | -0.71327     | -0.39702     |              |              |              |              |              |              |              |              |
| 22 \ (2,2)          | -1.68119     | 2.41122      | 1.93518      | -1.18573     | -2.47898     |              |              |              |              |              |
| 22 \ (2,2)          | 2.91702      | 1.68361      | 1.67804      | 4.15670      | -2.60831     |              |              |              |              |              |
| 22 \ (2,2)          | -0.22901     | -1.37298     |              |              |              |              |              |              |              |              |

tetrahedral $D(\pi_2, \pi_2, 0)$,
octahedral $D(0, \pi_2, 0)$,
icosahedral $D(0, \beta, \pi_2)$, \( \cos \beta = \frac{1}{\sqrt{5}}, \sin \beta = \frac{2}{\sqrt{5}} \) \tag{122}
shapes are denoted by $S$ tables in different orientations. A list of equal shapes is given in table 4. The table for the tetrahedral group. Some solutions appear several times in the tables for the dihedral groups. The solutions invariant under $I_h$ wishes to insert them, then one should multiply the amplitudes $T$ since different orientations of the equivalent mirror-planes have been chosen for amplitudes. $O$ to $\nu$ hint apart from the case $(\nu, \nu) = (2,2)$ for the tetrahedral group. Therefore in

| $p$ | $(\nu_1, \nu_2)$ | $a_{1:p,0}$ | $a_{2:p,2}$ | $a_{1:p,4}$ | $a_{1:p,6}$ | $a_{1:p,8}$ | $a_{1:p,10}$ | $a_{1:p,12}$ | $a_{1:p,14}$ | $a_{1:p,16}$ | $a_{1:p,18}$ | $a_{1:p,20}$ | $a_{1:p,22}$ | $a_{1:p,24}$ | $a_{1:p,26}$ |
|-----|-----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 26  | (2,2)           | -12.9853    | -17.3122    | 4.24577     | -16.3533    | 9.95326     | 1.94686     | 6.90372     | 5.42601     | 4.24205     | 1.16469     | 4.03855     | -6.97531    | 11.0403     | 15.8632     |
| 26  | (2,2)           | 0.44976     | 3.76044     | -3.19985    | 1.10830     | 7.15029     | 1.21251     | 1.12982     | 0.86551     | -4.45935    | 0.16510     | -6.08814    | -0.94012    | 3.61602     | -2.81306    |
| 26  | (2,2)           | 0.88059     | -5.99324    | -2.81215    | -2.56502    | 5.52238     | -1.39801    | 0.66375     | -0.71139    | -3.85344    | -0.12650    | -5.19225    | 0.69595     | 2.55744     | 4.69009     |
| 26  | (2,2)           | 0.89299     | 0.89158     | -0.39047    | -0.59796    | -0.44156    | 0.86347     | -0.43012    | 0.92514     | -0.43095    | 0.18639     | -0.46987    | -1.08764    | -0.62933    | -0.44414    |

If for $T_h$ the amplitudes $a_{1:p,2k}$ for odd $k$ vanish, then the solutions belong to $O_h$. I did not tabulate them in the table for the tetrahedral group. If one wishes to insert them, then one should multiply the amplitudes $a_{4k}$ by $(-)^k$, since different orientations of the equivalent mirror-planes have been chosen for $T_h$ and $O_h$. For $O_h$ and $I_h$ one can no longer distinguish between even and odd amplitudes.

The solutions which correspond to the groups $O_h$ and $I_h$ are indicated in the tables for the dihedral groups. The solutions invariant under $I_h$ are indicated in the table for the tetrahedral group. Some solutions appear several times in the tables in different orientations. A list of equal shapes is given in table 4. The shapes are denoted by $S_{G,p,i}$, where $G$ denotes one of the groups $D_n$, $T$, $O$ and $I$, resp. and $i$ numbers the solutions for given $G$ and $p$ ($i$ is omitted, if only one solution for $G$ and $p$ is listed).

Maple was able to solve the set of shape equations without giving any further hint apart from the case $(\nu_1, \nu_2) = (2,2)$ for the tetrahedral group. Therefore in
Table 5 Solutions of the shape equations for the octahedral group $O_h$

| $p$ | $\nu$ | $a_{1;p,0}$ | $a_{1;p,4}$ | $a_{1;p,8}$ | $a_{1;p,12}$ | $a_{1;p,16}$ |
|-----|-------|-------------|-------------|-------------|-------------|-------------|
| 4   | 1     | 1.59799     | 0.95498     |             |             |             |
| 6   | 1     | 7.45088     | -13.9393    |             |             |             |
| 8   | 1     | 2.97936     | 1.12039     | 1.70705     |             |             |
| 10  | 1     | 0.99612     | -1.00375    | -1.19470    |             |             |
| 12  | 2     | -0.55198    | -0.93569    | 0.89981     | -0.56841    |             |
| 12  | 2     | 5.05473     | 1.95363     | 1.30127     | 2.91433     |             |
| 12  | 2     | -1.64037    | 0.45582     | -1.99424    | -0.56834    |             |
| 14  | 1     | 1.27193     | -0.93532    | -1.00368    | -1.21871    |             |
| 16  | 2     | -1.25843    | -2.46818    | 0.71252     | 1.71627     | -1.52220    |
| 16  | 2     | 6.23060     | 1.98796     | 1.77254     | 1.91038     | 3.42907     |
| 16  | 2     | -7.22633    | 3.60347     | -5.11674    | -8.22622    | -1.60500    |
| 18  | 2     | -20.7093    | 32.0293     | -71.4902    | 109.660     | -11.7988    |
| 18  | 2     | -3.80429    | 2.89479     | -0.31593    | 5.60740     | 2.23271     |
| 18  | 2     | 1.04507     | -0.44079    | -1.43304    | 0.18341     | -1.13512    |
| 20  | 2     | 1.59394     | -0.85772    | 0.65038     | 1.28437     | 1.54183     |
|     |       | 0.26933     |             |             |             |             |
| 22  | 2     | -1.20555    | 0.85156     | -0.10269    | 0.83004     | 1.41591     |
|     |       | 0.57425     |             |             |             |             |
| 22  | 2     | -0.22188    | -0.74186    | 2.78358     | -0.47953    | -2.36217    |
|     |       | 1.34346     |             |             |             |             |
| 22  | 2     | 2.68380     | -0.84569    | -3.04632    | -1.10895    | -0.08719    |
|     |       | -2.72485    |             |             |             |             |
| 24  | 3     | 1.18937     | 0.54116     | -0.11816    | 1.28275     | -0.82343    |
|     |       | 0.61481     | 0.68382     |             |             |             |
| 24  | 3     | 1.67166     | 0.04356     | 0.12474     | 1.32901     | 0.00398     |
|     |       | 1.05199     | 0.65127     |             |             |             |
| 24  | 3     | 1.35890     | -0.55989    | 0.33293     | 0.71232     | 0.94372     |
|     |       | 1.02056     | 0.27116     |             |             |             |
| 24  | 3     | 1.85881     | 0.00554     | 0.86643     | -0.36989    | 1.76993     |
|     |       | 0.50703     | 0.77945     |             |             |             |
| 24  | 3     | 0.75146     | 0.64693     | 0.61258     | -1.06502    | 0.92200     |
|     |       | -0.46093    | 0.64810     |             |             |             |
| 26  | 2     | -1.87265    | 1.19575     | 0.00293     | 0.73397     | 1.43595     |
|     |       | 1.71925     | 0.82798     |             |             |             |
| 26  | 2     | -0.08100    | -3.16660    | 7.90156     | 1.47491     | -4.48413    |
|     |       | -6.19642    | 4.25103     |             |             |             |
| 26  | 2     | 4.86364     | -0.57937    | -5.85327    | -2.88496    | -0.19160    |
|     |       | 0.38293     | -5.16865    |             |             |             |

In this case the equations were rearranged as shown in appendix B. Apparently the case $(2,2)$ is easier solved for the dihedral groups, since some of the coefficients $w_{j,i_1,i_2}$ vanish in this later case.

In general $2^n$ solutions were obtained including the trivial one, for which all amplitudes vanish. Only real solutions are listed, since in the calculation $a_{1;p,-m}$ was expressed by $(-)^m a_{1;p,m}$, whereas it should be $(-)^m a_{1;p,m}^*$. 

21
Table 5 cont. Solutions of the shape equations for the octahedral group $O_h$

| $p$ | $\nu$ | $a_{1;p,0}$ | $a_{1;p,4}$ | $a_{1;p,8}$ | $a_{1;p,12}$ | $a_{1;p,16}$ |
|-----|-------|----------|----------|----------|----------|----------|
| 28  | 3     | -11.2790 | -15.3654 | -2.10025 | -3.08484 | 3.01089  |
|     |       | 7.52651  | 1.63005  | -11.7977 |          |          |
| 28  | 3     | -8.19362 | -77.0755 | -6.62562 | 9.30006  | 32.7654  |
|     |       | 48.2929  | 35.2893  | -41.3066 |          |          |
| 28  | 3     | 1.31496  | -0.12505 | -0.16182 | 1.13151  | 0.54722  |
|     |       | -0.02126 | 1.04212  | 0.39059  |          |          |
| 28  | 3     | 4.96143  | 2.93182  | 3.87509  | -3.45602 | 0.33561  |
|     |       | 4.06348  | -1.77305 | 3.70403  |          |          |
| 28  | 3     | 8.08355  | 2.16990  | 4.11477  | -1.80224 | 1.82715  |
|     |       | 5.36033  | 0.32544  | 4.84849  |          |          |
| 30  | 3     | -10.3305 | 3.75338  | 5.18714  | 4.88790  | 4.09221  |
|     |       | 4.37127  | 4.62163  | 7.22909  |          |          |
| 30  | 3     | -4.1266  | 1.90227  | 1.61647  | -0.22958 | 5.29145  |
|     |       | -0.7024  | 3.63614  | 2.20732  |          |          |
| 30  | 3     | -0.3293  | -0.47577 | 1.23931  | 1.34217  | -1.44945 |
|     |       | 0.41590  | -1.13186 | 1.04467  |          |          |
| 30  | 3     | 0.80723  | 0.36595  | -1.83491 | -0.45764 | -0.88932 |
|     |       | 1.37893  | 0.22661  | -1.34800 |          |          |
| 30  | 3     | 73.3784  | -105.971 | 206.495  | -395.192 | 731.230  |
|     |       | -842.698 | 143.693  | 7.16578  |          |          |
| 32  | 3     | -0.94505 | -1.62873 | 1.80496  | -2.69679 | -1.24200 |
|     |       | 1.76241  | 2.78195  | -1.70814 | -0.86640 |          |
| 32  | 3     | 2.02108  | -0.08638 | -0.29673 | 1.28426  | 1.07854  |
|     |       | 0.37970  | 0.19005  | 1.46548  | 0.62050  |          |
| 32  | 3     | -1.16350 | 1.20893  | -0.44735 | 0.08591  | -0.46335 |
|     |       | -1.25548 | -1.57057 | -0.62832 | 0.12415  |          |
| 32  | 3     | 7.53901  | 12.2486  | 8.93730  | -5.85501 | -5.29201 |
|     |       | -0.18349 | 1.41254  | -7.58121 | 10.0563  |          |
| 32  | 3     | 47.6029  | 30.0645  | 24.1029  | -3.30451 | -1.96647 |
|     |       | 7.99337  | 11.6529  | -3.80089 | 35.2026  |          |
| 34  | 3     | -11.4914 | 9.42515  | 25.0654  | 30.3572  | -9.41248 |
|     |       | 9.23599  | 1.92303  | -17.1336 | 25.2407  |          |
| 34  | 3     | -11.5774 | 6.11285  | -3.55783 | -2.00760 |          |
|     |       | -5.79444 | 0.80954  | 4.16799  | 5.11033  | 0.56450  |
| 34  | 3     | 0.94328  | 2.59973  | 0.35243  | 5.36668  |          |
| 34  | 3     | 0.51337  | 0.24287  | -1.08014 | -0.06906 | -0.82773 |
|     |       | 0.02861  | 0.97644  | -0.20870 | -0.77065 |          |
| 34  | 3     | 1.09993  | -0.90039 | 0.44903  | -0.02780 | -0.34114 |
|     |       | -0.84004 | -1.20752 | -0.96928 | -0.08435 |          |

**Case** $\nu = 2$  As an example the set of equations (117) for the case $\nu = 2$ is considered,

$$a_1 = w_{111}a_1^2 + 2w_{112}a_1a_2 + w_{122}a_2^2,$$

(123)
Table 6 Solutions of the shape equations for the icosahedral group $I_h$

| p | $\nu$ | $a_{1;p,0}$ | $a_{1;p,5}$ | $a_{1;p,10}$ | $a_{1;p,15}$ | $a_{1;p,20}$ | $a_{1;p,25}$ | $a_{1;p,30}$ | $a_{1;p,35}$ |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 1 | -1.08376 | 0.86454 |
| 10 | 1 | -0.84286 | -1.34290 | -0.73338 |
| 12 | 1 | 1.18220 | -0.61091 | 0.98335 |
| 16 | 1 | 2.53390 | 1.87102 | -2.40430 | -1.86451 |
| 18 | 1 | -1.24551 | 0.70244 | -0.56268 | 1.11003 |
| 20 | 1 | 0.35816 | 1.06413 | 1.72730 | 0.53368 | 0.94937 |
| 22 | 1 | -9.42141 | -3.84903 | 6.02057 | -6.60823 | -6.45637 |
| 24 | 1 | 1.28346 | -0.84415 | 0.57424 | -0.59346 | 1.25220 |
| 26 | 1 | -3.35369 | -6.46073 | -4.17676 | 7.32138 | 3.70065 |
| 28 | 1 | 5.20342 |
| 30 | 2 | -21.2790 | -4.33343 | 10.4040 | -11.4419 | 12.2013 |
| 30 | 2 | 14.1643 |
| 30 | 2 | -1.29745 | 1.00742 | -0.67877 | 0.53443 | -0.66788 |
| 30 | 2 | 1.41535 | -0.00751 |
| 30 | 2 | 0.43597 | -0.78987 | -0.90073 | -1.69849 | -0.33963 |
| 30 | 2 | -0.64517 | -0.90325 |
| 30 | 2 | 3.94227 | -1.15076 | 6.83983 | 4.80458 | 4.41657 |
| 30 | 2 | -3.58280 | 3.85631 |
| 32 | 1 | -25.3074 | -34.3654 | -6.83507 | 30.6595 | -30.5265 |
| 34 | 1 | 7.10336 | 0.32668 | -2.79493 | 3.20114 | -3.38581 |
| 36 | 2 | 1.28515 | -1.18256 | 0.83404 | -0.59249 | 0.54385 |
| 36 | 2 | -0.77680 | 1.60342 | -0.01678 |

\[
\alpha_2 = w_{112} \alpha_1^2 + 2 w_{122} \alpha_1 \alpha_2 + w_{222} \alpha_2^2. \tag{124}
\]

The ratio
\[
r = \frac{\alpha_2}{\alpha_1}. \tag{125}
\]
is introduced for the non-trivial solutions. Then the equation
\[
r = \frac{w_{112} + 2 w_{122} r + w_{222} r^2}{w_{111} + 2 w_{112} r + w_{122} r^2} \tag{126}
\]
is obtained, which rewrites as an equation of third order in $r$,
\[
w_{122} r^3 + (2w_{112} - w_{222}) r^2 + (w_{111} - 2w_{112}) r - w_{112} = 0. \tag{127}
\]
Thus there are three non-trivial solutions. One or three of them are real.
9 Higher Orders in the deformation

As in the preceding section only central symmetric bodies are considered. Then the expansion of \( r \) and \( h \) in spherical harmonics contains only those with even \( l \). In higher orders the equation for \( V_{\alpha \beta} \) can be used to determine \( h_n(\beta, \alpha) \) up to an \( \beta, \alpha \)-independent contribution \( h_{n:00} \). The difference \( Z_{\alpha \beta} - c_0 r_0 V_{\alpha \beta} \) yields the condition

\[
A_1 r_0 a_{n,lm} + A_2 (h_{1:00} a_{n-1,lm} + h_{n-1:00} a_{1,lm}) + 2 A_3 (a_1 a_{n-1};lm) = I_{n;lm} \tag{128}
\]

for \( l > 0 \), where \( I_{n;lm} \) contains only terms \( a_m \) with \( m < n - 1 \). For \( l \neq p \) it yields

\[
a_{n,lm} = -\frac{1}{A_1} (I_{n;lm} - 2 A_3 (a_1 a_{n-1};lm)). \tag{129}
\]

For \( l = p \) one obtains a set of linear eqs. for the \( a_{n-1,pm} \)

\[
a_{n-1,pm} - 2 \sum_{m'} M_{mm'} a_{n-1,pm'} = I_{n;m} \tag{130}
\]

with

\[
M_{mm'} = (pm; pm - m', pm') \frac{a_{1,pm - m'}}{\gamma}, \tag{131}
\]

\[
I_{n;m} = -\frac{h_{n-1:00}}{h_{1:00}} + \frac{I_{n;pm}}{s_0 h_{3:00} P_{p-1}(c_0)}
+ \frac{c_0 s_0 + 2 P_{p-1}(c_0)}{s_0 h_{1:00}} \sum_{l \neq p, m_1 m_2} \langle p; p0, l0 \rangle \langle p; pm_1, lm_2 \rangle a_{1,pm_1} a_{n-1;lm_2}.
\tag{132}
\]

This set of equations has a unique solution, if the matrix \( M \) does not have an eigen-value 1/2. An eigen-value 1/2 appears, if the shape-eqs. have a double-solution. This means that besides \( a_{1,p} \), also \( a_{1,p} + \delta a_{1,p} \) is solution to first order in \( \delta a_{1,p} \), which yields the equations

\[
\delta a_{1,pm} - 2 \sum_{m'} M_{mm'} \delta a_{1,pm'} = 0. \tag{133}
\]

This is the case, if one does not fix the orientation of the body, since then the change of \( a_{1,p} \) by infinitesimal rotations yields such solutions \( a_{1,p} + \delta a_{1,p} \). Here we have fixed the symmetry axes in the calculations. Thus this case does not appear. We cannot exclude that in special cases a double-solution appears. We have determined the eigen-values of \( M \) for all shapes listed in the tables. No eigenvalue 1/2 appeared. The closest one were 0.4541 for \( S_{D_{4h,12.5}} \) and 0.5979 for \( S_{T_{26.2}} \). Always one of the eigenvalues equals 1 for the eigen-vector \( a_{1,p} \).

9.1 Example for double-solution

We give an example for a double-solution by further considering the case \( \nu = 2 \). Starting from eq. \((127)\) and denoting the solutions by \( r_1, r_2, r_3 \) one obtains

\[
w_{111} = w_{122} (2 + r_1 r_2 + r_1 r_3 + r_2 r_3), \tag{134}
\]

\[
w_{112} = w_{122} r_1 r_2 r_3, \tag{135}
\]

\[
w_{222} = w_{222} (r_1 + r_2 + r_3 + 2 r_1 r_2 r_3). \tag{136}
\]
Considering the solution \( r = r_1 \) one obtains
\[
\alpha_1 = \frac{1}{w_{122}N}, \quad \alpha_2 = \frac{r_1}{w_{122}N},
\]
\[
N = 2 + r_1 r_2 + r_1 r_3 + r_2 r_3 + 2 r_1^2 r_2 r_3 + r_1^2. \tag{137}
\]
This yields the matrix \( M \) with the matrix-elements
\[
M_{ij} = w_{ij1\alpha_1 + w_{ij2\alpha_2}}, \tag{138}
\]
\[
M_{11} = \frac{2 + r_1 r_2 + r_1 r_3 + r_2 r_3 + r_1^2 r_2 r_3}{N}, \tag{139}
\]
\[
M_{12} = M_{21} = \frac{r_1 (1 + r_2 r_3)}{N}, \tag{140}
\]
\[
M_{22} = \frac{1 + r_1 (r_1 + r_2 + r_3) + 2 r_1^2 r_2 r_3}{N}. \tag{141}
\]
The eigenvalues of this matrix are
\[
\mu_1 = 1, \quad \mu_2 = \frac{1 + r_1 r_2 + r_1 r_3 + r_1^2 r_2 r_3}{N}. \tag{142}
\]
One finds
\[
\mu_2 - \frac{1}{2} = \frac{(r_1 - r_2)(r_1 - r_3)}{2N}. \tag{143}
\]
Thus one obtains an eigenvalue \( \mu = 1/2 \) if and only if \( r_1 \) equals one of the other solutions \( r_i \).

### 9.2 Reparametrization

By iterating the expansion in \( \epsilon \) it became apparent that the coefficients \( r_{n;00} \) and \( h_{n;00} \) were undetermined, whereas all other coefficients \( r_{n;im} \) and \( h_{n;im} \) were determined.

The reason is the following: One is free to reparametrize \( r \) by introducing a new \( \tilde{r} = r s_r(\epsilon) \) with a function \( s_r(\epsilon) \) that has a Taylor expansion in \( \epsilon \). Simultaneously \( h \) has to be changed to \( \tilde{h} = h s_h(\epsilon) \). Thus one is free to choose \( r_{n;00} \) arbitrarily. One may for example choose \( r_{n;00} = 0 \) for \( n > 0 \) or require a volume independent of \( \epsilon \).

If the \( r_{n;00} \)s are fixed one is still free to reparametrize \( h \) to \( \tilde{h} = h s_h(\epsilon) \), where again \( s_h(\epsilon) \) must be Taylor expandable. A possible choice is \( h_{n;00} = 0 \) for \( n > 1 \).

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### A Supplement to section 3

#### A.1 Ultraspherical (Gegenbauer) polynomials

For more information on ultraspherical polynomials see [18, 19].

Introducing coordinates
\[
u_i = \cos \theta_i \prod_{k=i+1}^{d} \sin \theta_k, \tag{144}
\]
The Legendre polynomials are ultraspherical polynomials, \( P_n \) (148) yields thus it is sufficient to consider the expansion eq. (17). Orthogonality of the \( G_n \) is represented by Rodriguez' formula

\[
Y^{(d)}_{m_d,m_{d-1},...,m_2,m_1}(\theta_d, \ldots \theta_1) = \sin^{m_d-1} \theta_d \frac{C^{(m_d-1,d/2-1)}_{m_d-m_{d-1}}(\cos \theta_d)}{\sqrt{N^{(m_d-1,d/2-1)}_{m_d-m_{d-1}}}} Y^{(d-1)}_{m_d-1,...,m_2,m_1}(\theta_d-1, \ldots \theta_1), \quad (145)
\]

\[
Y^{(1)}_{m_1}(\theta_1) = \frac{1}{\sqrt{2}} \cos^{m_1} \theta_1 \quad (146)
\]

with the restriction \( m_k \geq m_{k-1} \) and \( m_1 = 0, 1 \). For given \( d \) and \( m_d \) these are \((2m_d + d - 2)(m_d + d - 3)!/((d - 2)!m_d!)\) ultraspherical harmonics. They form a complete orthonormal set of functions and obey eq. (19), where \( n = m_d \) and \( m \) stands for the set \( \{m_{d-1}, \ldots, m_1\} \).

Gegenbauer’s addition theorem [20] reads

\[
(n + \alpha)\Gamma(\alpha)G_n^{(\alpha)}(x') \cdot \sqrt{(1 - x'^2)(1 - x^2)} \cos \phi = \sum_{l=0}^{n} \frac{(1 - x^2)^{l/2}(1 - x'^2)^{l/2}}{N^{(\alpha+l)}_{n-l}} C^{(\alpha+l)}_{n-l}(x) G^{(\alpha+l)}_{n-l}(x') \]

\[
\times (l + \alpha - 1/2)\Gamma(\alpha - 1/2)G_l^{(\alpha-1/2)}(\cos \phi), \quad (147)
\]

with the norm

\[
\int_{-1}^{1} dx (1 - x^2)^{n-1/2} C_n^{(\alpha)}(x) C_n^{(\alpha)}(x) = N_n^{(\alpha)} \delta_{n,m}, \quad (148)
\]

\[
N_n^{(\alpha)} = \frac{2^{1-2\alpha} \pi^{\alpha} \Gamma(n + 2\alpha)}{n!(n + \alpha)\Gamma^{2}(\alpha)}. \quad (149)
\]

Repeated use of this addition theorem with \( u = x e_d + \sqrt{1 - x^2} u' \), \( e_d \) unit vector, \( x = \cos \theta_d(u) \), similarly for \( v \), and \( u' \cdot v' = \cos \phi \) yields eq. (18).

### A.2 Determination of the expansion coefficients \( c_n \)

The Legendre polynomials are ultraspherical polynomials, \( P_n(x) = G_n^{(1/2)}(x) \). Thus it is sufficient to consider the expansion eq. (17). Orthogonality of the polynomials (145) yields

\[
c_n N_n^{(d/2-1)} = 2 \int_{0}^{1} dx x(1 - x^2)^{d/2-3/2} G_n^{(d/2-1)}(x). \quad (150)
\]

\( G \) is represented by Rodriguez' formula

\[
G_n^{(d/2-1)}(x) = \kappa_n^{(d/2-1)}(1 - x^2)^{3/2-d/2} \frac{Q^{2n}}{dx^{2n}}(1 - x^2)^{d/2-3/2+2n}, \quad (151)
\]

\[
\kappa_n^{(d/2-1)} = \frac{\Gamma(d/2)}{2^{2n}\Gamma(d/2)^2}, \quad (152)
\]

Thus

\[
c_n N_n^{(d/2-1)} = 2 \kappa_n^{(d/2-1)} \int_{0}^{1} dx x \frac{Q^{2n}}{dx^{2n}}(1 - x^2)^{d/2-3/2+2n}. \quad (153)
\]
Partial integration yields [13]
\[ 2^{\frac{(d/2-1)}{2}} \kappa^{\frac{(d/2+1)}{2}} G_{2n-2}^{(d/2+1)}(0). \] (154)

With
\[ G_{2n-2}^{(d/2+1)}(0) = \binom{n + d/2 - 1}{n - 1} \] (155)

one obtains the expression for \( c_n \) in eq. (17).

**B The case \((\nu_e, \nu_o) = (2, 2)\)**

The shape-equations can be written
\[ 0 = c_x - c'_xuv, \] (156)
\[ 0 = c_y - c'_yuv, \] (157)
\[ 0 = c_{uv}u + c_{uv}v, \] (158)
\[ 0 = c_{uv}u + c_{uv}v. \] (159)

with
\[ c_x = x - w_{000}x^2 - 2w_{002}xy - w_{022}y^2, \] (160)
\[ c_y = y - w_{002}x^2 - 2w_{022}xy - w_{222}y^2, \] (161)
\[ c'_{xuv} = w_{011}u^2 + 2w_{013}uv + w_{033}v^2, \] (162)
\[ c'_{yuv} = w_{211}u^2 + 2w_{213}uv + w_{233}v^2, \] (163)
\[ c_{uu} = 1 - 2w_{011}x - 2w_{211}y, \] (164)
\[ c_{uv} = -2w_{013}x - 2w_{213}y, \] (165)
\[ c_{vv} = 1 - 2w_{033}x - 2w_{233}y. \] (166)

Eqs. (158, 159) yield
\[ \frac{c_{uu}c_{vv} - c_{uv}^2}{u} = 0, \] (167)
\[ \frac{u}{v} = \frac{c_{uu}}{c_{uv}} = \frac{c_{uv}}{c_{uv}}. \] (168)

Insertion in (162, 163) yields
\[ c'_{xuv} = -\frac{uv}{c_{uv}} c_{xuv}, \quad c_{xuv} = w_{011}c_{uv} - 2w_{013}c_{uv} + w_{033}c_{uv}, \] (169)
\[ c'_{yuv} = -\frac{uv}{c_{uv}} c_{yuv}, \quad c_{yuv} = w_{211}c_{uv} - 2w_{213}c_{uv} + w_{233}c_{uv}. \] (170)

and substitution in eqs. (156, 157) and elimination of \( \frac{uv}{c_{uv}} \) yields
\[ c_x c_{yuv} - c_y c_{xuv} = 0. \] (171)

The eqs. (167, 171) allow the determination of \( x \) and \( y \).

From (156, 169, 168) one obtains
\[ c_x c_{yuv} - c_y c_{xuv} u^2 = 0 \] (172)

which allows the calculation of \( u \). Finally eq. (158) yields \( v \). One notes that the solutions with \( u = v = 0 \) have been eliminated by use of (167, 168). They can be obtained from \( c_x = c_y = 0 \).
C Denominator of the coefficient $\gamma$

The Legendre functions obeys the recursion relation

$$(p + 2)P_{p+1,-1}(x) - (2p + 1)xP_{p,-1}(x) + (p - 1)P_{p-1,-1}(x) = 0. \quad (173)$$

Using it also for $p \to p \pm 1$, one can eliminate $P_{p\pm 1,-1}(x)$ and obtain the recursion relation

$$(p + 2)(p + 3)(2p - 1)K_{p+2}(x) + (2p + 1)(2p^2 + 2p - 3 - (2p - 1)(2p + 3)x^2)K_p(x) + (p - 1)(p - 2)(2p + 3)K_{p-2}(x) = 0. \quad (174)$$

Substituting

$$\frac{1}{2}s_0c_0 + P_{p,-1}(c_0) = s_0c_0(1 - c_0^2)K_p(c_0) \quad (175)$$

one obtains with $c_0 = x$

$$(p + 2)(p + 3)(2p - 1)K_{p+2}(x) + (2p + 1)(2p^2 + 2p - 3 - (2p - 1)(2p + 3)x^2)K_p(x) + (p - 1)(p - 2)(2p + 3)K_{p-2}(x) = \frac{1}{2}(2p - 1)(2p + 1)(2p + 3). \quad (176)$$

One easily calculates

$$K_2 = 0, \quad K_4 = \frac{7}{8} \quad (177)$$

The recursion relation (176) yields

$$K_6 = \frac{3}{16}(1 + 11x^2),$$
$$K_8 = \frac{11}{128}(9 - 26x^2 + 65x^4),$$
$$K_{10} = \frac{13}{256}(5 + 89x^2 - 289x^4 + 323x^6),$$
$$K_{12} = \frac{1}{1024}(743 - 5032x^2 + 34238x^4 - 72353x^6 + 52003x^8). \quad (178)$$

Numerical calculations yield no real zeroes of $K_p(x)$ up to $p = 36$. Probably all $K_p(x)$ are strictly positive for real $x$.

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