Anomalous Hall Effect in non-commutative mechanics

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Abstract

The anomalous velocity term in the semiclassical model of a Bloch electron deviates the trajectory from the conventional one. When the Berry curvature (alias noncommutative parameter) is a monopole in momentum space, as found recently in some ferromagnetic crystals while observing the anomalous Hall effect, we get a transverse shift, similar to that in the optical Hall effect.

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1 Introduction

The Anomalous Hall Effect (AHE), characterized by the absence of a magnetic field, is observed in some ferromagnetic crystals. While this has been well established experimentally, its explanation is still controversial. One, put forward by Karplus and Luttinger \cite{Karplus} fifty years ago, suggests that the effect is due to an anomalous current.

Many years later, it has been argued \cite{Luttinger} that the semiclassical dynamics of a Bloch electron in a crystal should involve a Berry curvature term, $\Theta$.

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In the \( n^{th} \) band the equations of motion read, in an electromagnetic field,

\[
\dot{\mathbf{r}} = \frac{\partial \epsilon_n(k)}{\partial k} - \dot{k} \times \Theta, \tag{1}
\]
\[
\dot{k} = eE + e\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}), \tag{2}
\]

where \( \mathbf{r} = (x^i) \) and \( k = (k_j) \) denote the electron’s intracell position and quasimomentum, respectively; \( \epsilon_n(k) \) is the band energy. The relation (1) exhibits an anomalous velocity term, \( \dot{k} \times \Theta \), which is the mechanical counterpart of the anomalous current.

The model is distinguished by the non-commutativity of the position coordinates: in the absence of a magnetic field, \( \{x^i, x^j\} = \epsilon^{ijn} \Theta_n \) \[3, 4\]. In the free noncommutative model in 3 space dimensions, \( \Theta \) can only be momentum-dependent such that \( \partial_{k^i} \Theta^i = 0 \) \[3\].

A remarkable discovery concerns the AHE in the metallic ferromagnet \( \text{SrRuO}_3 \). Fang et al. \[5\] found in fact that the experimental data are consistent with \( \Theta \) taking the form of a monopole in momentum space,

\[
\Theta = \frac{\mathbf{k}}{k^3}, \tag{3}
\]

\( k \neq 0 \). \[3\] is, furthermore, the only possibility consistent with rotational symmetry \[3\].

Here we propose to study the AHE in the semiclassical framework (as advocated in \[3\]), with non-commutative parameter \[3\]. For \( \mathbf{B} = 0 \) and a constant electric field, \( \mathbf{E} = \text{const.} \), and assuming a parabolic profile \( \epsilon_n(k) = k^2/2 \), eqn. (2), \( \mathbf{k} = e\mathbf{E} \), is integrated as \( \mathbf{k}(t) = e\mathbf{E} t + \mathbf{k}_0 \). The velocity relation (1) becomes in turn

\[
\dot{\mathbf{r}} = \mathbf{k}_0 + e\mathbf{E}t + \frac{e\theta E \mathbf{k}_0}{k^3} \mathbf{n}, \tag{4}
\]

where \( \mathbf{n} = \mathbf{k}_0 \times \hat{\mathbf{E}} \) [“hats” denote vectors normalized to unit length]. The component of \( \mathbf{k}_0 \) parallel to \( \mathbf{E} \) has no interest; we can assume therefore that \( \mathbf{k}_0 \) is perpendicular to the electric field. Writing \( \mathbf{r}(t) = x(t)\mathbf{k}_0 + y(t)\hat{\mathbf{E}} + z(t)\mathbf{n} \), eqn. (4) yields that the component parallel to \( \mathbf{k}_0 \) moves uniformly, \( x(t) = \mathbf{k}_0 t \), and its component parallel to the electric field is uniformly accelerating, \( y(t) = \frac{1}{2}eEt^2 \). (Our choices correspond to choosing time so that the turning point is at \( t = 0 \).) However, owing to the anomalous term in (1), the particle is also deviated perpendicularly to \( \mathbf{k}_0 \) and \( \mathbf{E} \), namely by

\[
z(t) = \frac{\theta}{k_0} \frac{eEt}{\sqrt{k_0^2 + e^2E^2t^2}}. \tag{5}
\]
Figure 1: The anomalous velocity term deviates the trajectory from the plane.

It follows that the trajectory leaves its initial plane and suffers, between \( t = -\infty \) to \( t = \infty \), a finite transverse shift, namely

\[
\Delta z = \frac{2\theta}{k_0}.
\]  

(6)

Most contribution to the shift comes when the momentum is small, i.e., “near the \( k \)-monopole.”

\( \theta \) becomes a half-integer upon quantization, \( \theta = N/2 \), and hence (6) is indeed \( N/k_0 \). The constant \( k_0 \neq 0 \), the minimal possible value of momentum, plays the role of an impact parameter. Let us observe that while (6) does not depend on the field \( E \) or the electric charge \( e \), the limit \( eE \to 0 \) is singular. For \( eE = 0 \), the motion is uniform along a straight line.

The transverse shift is reminiscent of the recently discovered optical Hall effect [7] and can also be derived, just like in the optical case, using the angular momentum. The free expression \[3\], \( J = r \times k - \theta \hat{k} \), is plainly broken by the electric field to its component parallel to \( E \),

\[
J = J_y = z(t)k_0 - \theta \frac{eEt}{\sqrt{k_0^2 + e^2E^2t^2}},
\]  

(7)
whose conservation yields once again the shift \( \theta \).

How can the same argument work for a Bloch electron and for light? The answer relies, for both problems, on having the same “k-monopole” contribution, \(-\theta \hat{\mathbf{k}}\) in the angular momentum.

Our model is plainly not realistic: what we described is, rather, the deviation of a freely falling non-commutative particle from the classical parabola found by Galileo. Particles in a metal are not free, though, and their uniform acceleration in the direction of \( \mathbf{E} \) should be damped by some mechanism. It is nevertheless remarkable that we obtain qualitative information from such a toy model.

Note added. I am indebted to Dr. S. Murakami for calling my attention to similar work in the context of the spin Hall effect in semiconductors \[8\]. I would also like to thank Dr. Y. Kats for informing me about the experimental status of the AHE, see \[9\].

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