The correction to the nonlinear inviscid model of the Rayleigh–Taylor instability suppressed by a phase transition

V V Konovalov, T P Lyubimova
Institute of Continuous Media Mechanics of the Ural Branch, RAS, Perm 614013, Russia
E-mail: lubimova@psu.ru

Abstract. The Rayleigh–Taylor instability, complicated by heat and mass transfer, is studied for a thin vapor film covering the horizontal surface of a flat heater, which is immersed in a subcooled volume of liquid. A correction to the nonlinear inviscid model of the instability suppression by a phase transition is proposed. It is based on the fact that, at finite Prandtl numbers, the strength of the phase transition effect can have the limitation described before for the linear problem. It is shown that the nonlinearity alone, without taking the viscosity into account, is able to stabilize the thin vapor film, but only for the situations when the convective heat exchange in the liquid obeys the Newton–Richman law.

1. Introduction
For the case of saturated film boiling, the development of the Rayleigh–Taylor instability at the boundary of a vapor film, covering a horizontal flat heater in a volume of liquid, leads to permanent separation of vapor bubbles from it, whose loss is compensated by evaporation. However, as observed in experiment [1], the presence of a strong subcooling in the liquid promotes the instability suppression and complete stabilization of the interface. In a stable base state, there is no phase transition and the entire heat flux from the heater is removed in the processes of thermal conductivity at vapor and natural convection in the liquid. Although such a regime has no practical interest, its stability can be fairly simply studied by the methods of the linear or weakly nonlinear analysis. This allows the most complete representation of interaction of various physical factors, which determine also the process of saturated film boiling important for technical applications.

A number of approaches to the calculation of the heat and mass transfer effect on the Rayleigh–Taylor instability are known. Not all of them can at least qualitatively explain the appearance of a stable vapor film during film boiling. Thus, the linear stability analysis based on an inviscid approximation [2]-[4] does not reveal narrowing of the instability region. As a result, its boundary with respect to wavenumbers is given by the classical relation of gravitational and capillary forces [5], and the action of the phase transition is presented by a decrease in the growth rate of perturbations of the base state.

A linear approach taking into account viscosities of the phases seems to be more successful. Although for the case of their thick layers, the cut-off wavenumber shifts toward the long-wave region [6], [7], the complete disappearance of the Rayleigh–Taylor instability, complicated by heat and mass transfer, is possible only for the configuration when the thickness of liquid or its vapor layer is finite. This follows from [8], where an estimate for the heat flux of the instability stabilization was obtained.
within the framework of the "lubrication" approximation for a thin vapor film. The results of rigorous linear stability analysis [9]-[12] serve to refine it.

Earlier, some attempts were made to consider the nonlinearity factor. The easiest way to do this is the above-mentioned "lubrication" approximation. Thus, in [13], stable nonlinear regimes were found for the case of subcooled film boiling, in particular a spatially periodic solution with a smaller subcooling followed by the regime of "traveling waves".

Generally speaking, the weakly nonlinear analysis is not limited to the case of small thicknesses of the phase layers. Thus, in the framework of the inviscid approximation, the method of multiple scales was applied in [14] to the Rayleigh–Taylor instability in a system with their arbitrary thicknesses. In this case, the phase transition rate was expanded in a series in terms of the amplitude of interface perturbations. The first-order phenomenological coefficient, like similar coefficients in the following orders of the expansion, was found from the problem of one-dimensional stationary thermal conductivity. Such a quasi-equilibrium approach was originally proposed for the linear stability problem [3], [6], [7], [10]. The fact that an asymptotic solution was obtained in [14] only near the cutoff wavenumber should obviously limit its applicability. Nevertheless, it is clear that the stability region expands if a certain parameter, directly proportional to the square of the phenomenological coefficient and inversely proportional to the thickness of the vapor film, exceeds its critical value.

In general, as noted in [15], [16], the phenomenological coefficient is a function of all processes occurring in a two-layer, two-phase system. Thus, under the condition of homogeneous liquid and its vapor, which corresponds, strictly speaking, to weak or moderate heating, the strength of the phase transition effect has its limit [16]. In [11], this was expressed in the existence of a threshold thickness for the stable vapor layer on the heater surface. Moreover, this threshold disappears with allowance for a thermal inhomogeneity for thermophysical properties of the vapor [12].

In the present paper, the nonlinear inviscid model from [14] has been revised. In the main order, the thermal problem was solved within the rigorous viscous approach. Thus, a real, non-estimated value of the phenomenological coefficient has been obtained, which is included in the Ginzburg–Landau equation from [14], which describes the effect of nonlinearity.

2. The linear problem
Let us consider a semi-infinite volume of liquid, whose lower boundary is separated from the solid horizontal surface of a flat heater by a layer of generated vapor with thickness \( h \) determined by thermal conditions. The existence of the two-layer, two-phase system is provided by heat flux \( q \) removed from the heater by thermal conductivity in the subcooled from above liquid and its vapor. A gravity force with acceleration vector \( \bar{g} \) acts in the direction from the heavy to light phase. Below, \( \rho_j, \nu_j, \kappa_j, \) and \( \chi_j \) are the densities of the media and their coefficients of kinematic viscosity, thermal conductivity and diffusivity, respectively. In what follows, index \( j \) denotes either the liquid (\( j = 1 \)) or its vapor (\( j = 2 \)). Its interface is characterized by the coefficient of surface tension, \( \gamma \), and latent heat of phase transition, \( L \).

We represent our problem in a dimensionless form, choosing gravity-capillary length 
\[ d_{rg} = \sqrt{\gamma (\rho_1 - \rho_2) g} \] as a length scale, gravity-capillary time 
\[ t_{rg} = \sqrt{\frac{\rho_1 - \rho_2}{d_{rg}^2 \gamma}} \] as a time scale, 
\[ d_{rg} L \] is flow velocity, 
\[ \frac{\rho_1 + \rho_2}{d_{rg}^2} \] is pressure, 
\[ q d_{rg} \left( \kappa_1 + \kappa_2 \right) \] is temperature, and 
\[ \frac{\rho_1 + \rho_2}{d_{rg}^2} L \] is phase transition rate.

As a result, the linear stability problem is characterized by the following dimensionless parameters. These are the dimensionless densities and coefficients of thermal conductivity of liquid and its vapor (related to each other by relations \( \tilde{\rho}_1 + \tilde{\rho}_2 = 1 \) and \( \tilde{\kappa}_1 + \tilde{\kappa}_2 = 1 \)),

\[ \tilde{\rho}_j = \frac{\rho_j}{\rho_1 + \rho_2}, \quad \tilde{\kappa}_j = \frac{\kappa_j}{\kappa_1 + \kappa_2}, \]
the Reynolds and Peclét numbers for each of the phases,
\[ \text{Re}_j = \frac{1}{\text{v}_j} \frac{d^2}{t_{\text{fr}}} \quad \text{Pe}_j = \frac{1}{\chi_j} \frac{d^2}{t_{\text{fr}}} \]
the Bond number, determined by the vapor film thickness in the base state, \( h \),
\[ \text{Bo} = \frac{h}{d_{\text{fr}}} \]
as well as the dimensionless specific heat of evaporation, \( \Lambda \), and pressure effect parameter \( \Pi \),
\[ \Lambda = \frac{L (\rho_1 + \rho_2) d_{\text{fr}}}{q} \quad \Pi = \frac{(\kappa_1 + \kappa_2) T_1}{(\rho_1 + \rho_2) L^2 t_{\text{fr}}} \]

From the Navier–Stokes equation, continuity equation for an incompressible fluid, and energy equation, we obtain the following relations for the perturbation amplitudes of pressure, \( P_j \), vertical velocity, \( U_{zj} \), and temperature, \( \Theta_j \) [11]:
\[ D P_j = 0, \quad \left( \omega - \frac{1}{\text{Re}_j} D \right) U_{zj} = -\frac{1}{\rho_j} \frac{d P_j}{d z}, \quad \left( \omega - \frac{1}{\text{Pe}_j} \right) D \Theta_j = \frac{U_{zj}}{k_j}. \tag{1} \]

Here, \( k \) and \( \omega \) are wavenumber and growth rate of perturbations, respectively, and
\[ D = \frac{d^2}{d z^2} - k^2. \]

In the volume of liquid at \( z \to \infty \), the constant temperature condition is satisfied, and the assumed flow velocity vanishes,
\[ \Theta_1 = 0, \quad U_{z1} = 0. \tag{2} \]

On a rigid, ideally heat-conducting surface of the heater at \( z = -\text{Bo} \), the conditions of temperature constancy and no slip are established,
\[ \Theta_2 = 0, \quad U_{z2} = 0, \quad \frac{d U_{2z}}{d z} = 0. \tag{3} \]

Onto the unperturbed liquid-vapor interface at \( z = 0 \), we transfer the kinematic condition, modified by taking into account a phase transition, and also the conditions of mass balance, tangential velocity and stress continuity, normal stress balance, temperature continuity, and heat flux balance. The last condition relates deviations of the interface temperature from the equilibrium saturation temperature \( T_s \) to pressure perturbations [9],
\[ \omega F = U_{z1} + \frac{\Xi}{\tilde{\rho}_1}, \quad U_z + \frac{\Xi}{\tilde{\rho}} = 0, \quad \left[ \frac{d U_z}{d z} \right] = 0, \]
\[ \left[ \frac{\tilde{\rho}}{\text{Re}} \left( D + 2k^2 \right) U_z \right] = 0, \quad \left[ -P + \frac{2\tilde{\rho}}{\text{Re}} \frac{d U_z}{d z} \right] + \left( 1 - k^2 \right) F = 0, \]
\[ \left[ \Theta - \frac{E}{k} \right] = 0, \quad \left[ k \frac{d \Theta}{d z} \right] = \Lambda \Xi, \quad \Theta_2 - \frac{E}{k_2} = \Lambda \Pi \left( \frac{P_2}{\tilde{\rho}_2} - \frac{P_1}{\tilde{\rho}_1} \right). \tag{4} \]

3. The nonlinearity factor

In order to evaluate the role of nonlinearity, we turn to [14]. There, by excluding secular terms in the third order of an expansion in an amplitude \( A \) for an inviscid solution near the cut-off wavenumber \( k_c \), the Ginzburg–Landau equation is written,
\[ \frac{\partial A}{\partial t_2} + \left( s_1 + s_2 |A|^2 \right) A = 0, \tag{5} \]
which determines dependence of amplitude \( A \) on "slow" time \( t_2 \). For the configuration of a thick layer of heavy liquid, coefficients \( s_1 \) and \( s_2 \) get the following, dimensionless form:

\[
    s_1 = \frac{2(k - k_c)k_c}{\alpha (\coth k Bo + 1)},
\]

\[
    s_2 = \frac{\alpha^2 k^2 \left( \frac{\alpha^2}{6k_c^2} \coth^2 k Bo + 1 \right) \coth(k Bo) \coth(2k Bo) - \frac{3}{2} k^2}{\tilde{\rho}_2} \frac{\coth(k Bo) - \coth(2k Bo) - \frac{3}{2} k^2}{\tilde{\rho}_2/k^2}. \tag{7}
\]

Here, it can be assumed that \( k_c = 1 \), which corresponds to the inviscid solution.

Now, we propose a modification of inviscid model (5)-(7). First, coefficient \( s_1 \), which controls evolution of small perturbations, can be replaced by their growth rate \( \omega \) defined above in the framework of the linear viscous problem. Then,

\[
    \frac{\partial A}{\partial t_2} + \left( s_2 |A|^2 - \omega \right) A = 0. \tag{8}
\]

Our evaluation approach does not take into account the effect of viscosity on the nonlinear term in relationship (8) with coefficient \( s_2 \). Hence, the model is accurate only for weak dissipation conditions at \( \text{Re}_{j} \to \infty \).

Following [14], we neglect the knowingly small contribution related to the second-order phenomenological coefficient \( \alpha_2 \) from the introduced there expansion of the phase transition rate \( \Xi \) in a series in amplitude \( F \) for the interface perturbations,

\[
    \Xi = -\alpha F - \alpha_2 F^2 + \ldots.
\]

As for the coefficient of the first order, \( \alpha \), it is found as a solution of boundary value problem (1)-(4) from the previous section.

Equation (8) admits the existence of the linear stability region under the condition that \( \text{Re}_0 \omega < 0 \), and amplitude \( A \) is small enough. If \( \omega > 0 \), which corresponds to wavenumbers \( k < k_c \), where \( k_c \) is the cut-off wavenumber calculated from the linear viscous problem, then the main order of the solution indicates the Rayleigh–Taylor instability. According to [14], its nonlinear suppression is possible at \( s_2 > 0 \), at least in a left vicinity of value \( k_c \). In the limit of a thin vapor layer, when \( \coth^2 k Bo \approx 1/(k Bo) \), this reduces to the estimate of [14] with \( k \approx 1 \),

\[
    \frac{\alpha^2}{\tilde{\rho}_2^2} > 12k^2.
\]

In contrast to [14], where the quasi-stationary approximation was used, which according to [16] is correct in the limits \( \text{Pr}_j \to 0 \) and/or \( \text{Bo} \to 0 \), our model corresponds to finite values of the Prandtl numbers \( \text{Pr}_j = \text{Pe}_e \text{Re}_j^{1/2} \). A consequence of this must be the limitation on the strength of the phase transition effect, described before for the linear problem [16].

4. Results

The calculations are performed for the water–water vapor system at normal atmospheric pressure and corresponding saturation temperature of 100 °C. Its dimensional parameters are listed in table 1. The dimensionless values determined by them are \( \text{Re}_1 = 1307.1, \text{Re}_2 = 18.6, \text{Pe}_i = 2306.7, \text{Pe}_e = 19.6, \tilde{\rho}_2 = 6.2 \cdot 10^{-4}, \tilde{k}_2 = 3.4 \cdot 10^{-2}, \) and \( \Pi = 3.2 \cdot 10^{12} \). In the expressions for the dimensionless specific
heat of evaporation, $\Lambda = 3.4 \cdot 10^8 / q \left[ \text{W m}^{-2} \right]$, and Bond number $\text{Bo} = 399.4 \ h \left[ \text{m} \right]$, heat flux $q$ and thickness of the vapor film, $h$, are measured in the units of $\text{W m}^{-2}$ and $\text{m}$, respectively.

Table 1. Dimensional parameters for the water–water vapor system.

| Parameter | Value 1 | Value 2 |
|-----------|---------|---------|
| $g$       | 9.81 m s$^{-2}$ | 1.7 $\cdot 10^{-7}$ m$^{2}$ s$^{-1}$ |
| $T$       | 373.15 K | 2.0 $\cdot 10^{-5}$ m$^{2}$ s$^{-1}$ |
| $\rho_1$  | 9.6 $\cdot 10^{2}$ kg m$^{-3}$ | 6.8 $\cdot 10^{-1}$ W m$^{-1}$ K$^{-1}$ |
| $\rho_2$  | 6.0 $\cdot 10^{-1}$ kg m$^{-3}$ | 2.4 $\cdot 10^{-2}$ W m$^{-1}$ K$^{-1}$ |
| $v_1$     | 3.0 $\cdot 10^{-2}$ m$^{2}$ s$^{-1}$ | 5.9 $\cdot 10^{-2}$ N m$^{-1}$ |
| $v_2$     | 2.1 $\cdot 10^{-2}$ m$^{2}$ s$^{-1}$ | $L = 2.3 \cdot 10^{6}$ J kg$^{-1}$ |

Some results have been given in the framework of a convective model introduced by the following modification of the heat balance condition:

$$-	ilde{A} \left( \Theta_2 - \frac{F}{\kappa_2} \right) - \tilde{\kappa}_2 \frac{d\Theta_2}{dz} = \Lambda \Xi .$$

This corresponds to the convective heat transfer in the liquid obeying the Newton–Richman law. Further, for our calculations, the dimensionless coefficient of convective heat transfer was taken as $\tilde{A} = 100$.

**Figure 1.** Critical values of parameter $\Lambda$ as a function of Bond number for the vapor film. The following models are presented: the linear viscous, thermal conductive model (solid line), linear viscous model with the convective heat exchange in liquid (dashed line), quasi-equilibrium, nonlinear, thermal conductive model (long-dash line), and non-equilibrium nonlinear model with natural convection (dotted and dashed line).
Figure 1 shows the dependencies of the critical value of parameter $\Lambda$ on Bond number $Bo$, determined in four different approaches. As already known from [11], the factor of natural convection in liquid enhances the phase transition effect greatly (compare the solid and dashed lines in the figure), increasing the threshold thickness of the vapor film, above which the instability stabilization is impossible.

It can be seen that the factor of nonlinearity is two to three orders of magnitude weaker than the linear viscous mechanism. When the deviation from the quasi-equilibrium model, presented by the long-dash line in figure 1, is taken into account, the corresponding stabilization curve disappears for the case of thermal conductivity in liquid. If natural convection is allowed, the threshold thickness of the vapor layer appears on the curve (see the dotted and dashed lines in the figure).

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