On commensurability of fibrations on a hyperbolic 3-manifold

Hidetoshi Masai

Tokyo institute of technology, DC2

13th, January, 2013
Contents

1 Introduction
   - Fibered Manifolds
   - Thurston norm
   - Fibered Commensurability

2 Sketch of Proof
   - Construction
Notations

- surface = compact orientable surface of negative Euler characteristic possibly with boundary.
- hyperbolic manifold = orientable manifold whose interior admits complete hyperbolic metric of finite volume.
- $F$ : surface
- $\phi : F \rightarrow F$, automorphism (isotopy class of self-homeomorphisms) which may permute components of $\partial F$.
- $(F, \phi)$: pair of surface $F$ and automorphism $\phi$. 
Fibered Manifolds

Definition

- \([F, \phi] = F \times [0, 1]/((\phi(x), 0) \sim (x, 1))\) is called the mapping torus associated to \((F, \phi)\).
- A 3-manifold \(M\) is called fibered if we can find \((F, \phi)\) s.t. \([F, \phi] \cong M\).

Mapping tori and classification of automorphisms

- \(\phi\) is periodic \(\iff [F, \phi]\) is a Seifert fibered space.
- \(\phi\) is reducible \(\iff [F, \phi]\) is a toroidal manifold.
- \(\phi\) is pseudo Anosov \(\iff [F, \phi]\) is a hyperbolic manifold.
Thurston norm

- $M$ : fibered hyperbolic 3-manifold.
- $F = F_1 \sqcup F_2 \sqcup \cdots F_n$ : (possibly disconnected) compact surface.
- $\chi_-(F) = \sum |\chi(F_i)|$ 
  ($F_i$ : components with negative Euler characteristic).

**Definition (Thurston)**

$\omega \in H^1(M; \mathbb{Z}) \subset H^1(M; \mathbb{R})$.

We define $\|\omega\|$ to be

$$\min\{\chi_-(F) \mid (F, \partial F) \subset (M, \partial M) \text{ embedded, and} \ [F] \in H_2(M, \partial M; \mathbb{Z}) \text{ is the Poincare dual of } \omega. \}$$
Definition

$F$ is called a minimal representative of $\omega \iff F$ realize the minimum $\chi_-(F)$.

We can extend this norm to $H^1(M; \mathbb{Q})$ by $\|\omega\| = \|r\omega\|/r$.

Theorem (Thurston)

- $\|\cdot\|$ extends continuously to $H^1(M; \mathbb{R})$,
- $\|\cdot\|$ turns out to be semi-norm on $H^1(M; \mathbb{R})$, and
- The unit ball $U = \{\omega \in H^1(M; \mathbb{R}) \mid \|\omega\| \leq 1\}$ is a compact convex polygon

Definition

$\|\cdot\|$ is called the Thurston norm on $H^1(M; \mathbb{R})$. 
Fibered cone

Figure: $H^1(M, \mathbb{R})$
Fibered cone

Figure: $H^1(M, \mathbb{R})$


Fibered cone

Figure: $H^1(M, \mathbb{R})$
Question.
What is "a relationship" among fibrations on a hyperbolic manifold (or, on the same fibered cone)?

Example.
(Fried) Mapping tori of (un)stable laminations with respect to the pseudo Anosov monodromies on the same fibered cone are isotopic.
Commensurability of Automorphisms

**Definition (Calegari-Sun-Wang (2011))**

A pair $\left(\tilde{F}, \tilde{\phi}\right)$ covers $(F, \phi)$ if there is a finite cover $\pi : \tilde{F} \to F$ and representative homeomorphisms $\tilde{f}$ of $\tilde{\phi}$ and $f$ of $\phi$ so that $\pi \tilde{f} = f \pi$ as maps $\tilde{F} \to F$.

**Definition (CSW)**

Two pairs $(F_1, \phi_1)$ and $(F_2, \phi_2)$ are said to be commensurable if $\exists \left(\tilde{F}, \tilde{\phi}_i\right), k_i \in \mathbb{Z} \setminus \{0\} \ (i = 1, 2)$ such that $\tilde{\phi}_1^{k_1} = \tilde{\phi}_2^{k_2}$.

This commensurability generates an equivalence relation.
Sketch of Proof

Fibered Commensurability

Commensurability

\[(\tilde{F}_1, \tilde{\phi}_1^{k_1}) \quad = \quad (\tilde{F}_2, \tilde{\phi}_2^{k_2})\]

\[
\begin{array}{cc}
(F_1, \phi_1) & \quad (F_2, \phi_2) \\
\downarrow & \downarrow \\
(F_1, \phi_1) & \quad (F_2, \phi_2)
\end{array}
\]

Remark. The above is different from the below.

\[(\tilde{F}_1, \tilde{\phi}_1^{k_1}) \quad = \quad (\tilde{F}_2, \tilde{\phi}_2^{k_2})\]

\[
\begin{array}{cc}
(F_1, \phi_1) & \quad (F_2, \phi_2) \\
\downarrow & \downarrow \\
(F_1, \phi_1) & \quad (F_2, \phi_2)
\end{array}
\]
Fibered Commensurability

**Definition (CSW)**

A fibered pair is a pair \((M, \mathcal{F})\) where
- \(M\) is a compact 3-manifold with boundary a union of tori and Klein bottles,
- \(\mathcal{F}\) is a foliation by compact surfaces.

**Remark.** Since \([F, \phi]\) has a foliation whose leaves are homeomorphic to \(F\), fibered pair is a generalization of the pair of type \((F, \phi)\).
Fibered Commensurability 2

**Definition (CSW)**

A fibered pair \((\tilde{M}, \tilde{F})\) covers \((M, F)\) if there is a finite covering of manifolds \(\pi : \tilde{M} \to M\) such that \(\pi^{-1}(F)\) is isotopic to \(\tilde{F}\).

**Definition (CSW)**

Two fibered pairs \((M_1, F_1)\) and \((M_2, F_2)\) are commensurable if there is a third fibered pair \((\tilde{M}, \tilde{F})\) that covers both.
Proposition. [CSW]
The covering relation on pairs of type \((F, \phi)\) is transitive.

Definition
An element \((F, \phi)\) (or \((M,F)\)) is called \textit{minimal} if it does not cover any other elements.
∃ exactly 2 commensurability classes; with or without boundaries.

each commensurability class contains $\infty$-many minimal elements.

(hint: consider elements with maximal period)
Reducible Case

**Theorem (CSW)**

- ∃ manifold with infinitely many incommensurable fibrations.
- ∃ manifold with infinitely many fibrations in the same commensurable class.

**Remark.** The manifolds in this theorem are graph manifolds.
Pseudo Anosov Case

Theorem (CSW)

Suppose \( \partial M = \emptyset \). Then every hyperbolic fibered commensurability class \( [(M, F)] \) contains a unique minimal element.

Remark. The assumption \( \partial M = \emptyset \) is not explicitly written in their paper.

Result 1

Every hyperbolic fibered commensurability class \( [(M, F)] \) contains a unique minimal element.
Corollary (CSW)

$M$: hyperbolic fibered 3-manifold.  
Then number of fibrations on $M$ commensurable to a fibration on $M$ is finite.

Recall that if (the first Betti number of $M) > 1$, then $M$ admits infinitely many distinct fibrations (Thurston).

Question [CWS].

- When two fibrations on $M$ are commensurable?
- Are there any example of two commensurable fibrations on $M$ with non homeomorphic fiber?
Invariants, pseudo-Anosov case (CSW)

- Commensurability class of dilatations.
- Commensurability class of the vectors of the numbers of $n$-pronged singular points on $\text{Int}(F)$.

**Example.** $(0,0,1,1,1,0,...)$ means it has one 3 (4, and 5)-pronged singularity.

**Remark.** Let $\{p_i\}_{i \in I}$ be the set of singular points and $\{n_i\}_{i \in I}$ their prong number, then

$$\sum_i \frac{2 - n_i}{2} = \chi(F).$$
Definition

Two fibrations $\omega_1 \neq \omega_2 \in H^1(M; \mathbb{Z})$ are symmetric if

$\exists$ homeomorphism $\varphi : M \rightarrow M$ such that $\varphi^*(\omega_1) = \omega_2$ or $\varphi^*(\omega_1) = -\omega_2$. 
Definition

Two fibrations $\omega_1 \neq \omega_2 \in H^1(M; \mathbb{Z})$ are symmetric if $\exists$ homeomorphism $\varphi : M \rightarrow M$ such that $\varphi^*(\omega_1) = \omega_2$ or $\varphi^*(\omega_1) = -\omega_2$.

Result 2

Two fibrations on $S^3 \setminus 6_2^2$ or the Magic 3-manifold are either symmetric or non-commensurable.

Figure: the fibered link associated to a braid $\sigma \in B_3$
Fibrations on a manifold

**Result 3**

$M$: fibered hyperbolic 3-manifold which does not have hidden symmetry.

Then, any two non-symmetric fibrations of $M$ are not fibered commensurable.
Fibrations on a manifold

**Result 3**

$M$ : fibered hyperbolic 3-manifold which does not have hidden symmetry.

Then, any two non-symmetric fibrations of $M$ are not fibered commensurable.

**Remark**

- ”Most” hyperbolic 3-manifolds do not have hidden symmetry.
- $S^3 \setminus 6_2$ and the Magic 3-manifold have lots of hidden symmetries.
Hidden Symmetries

$M$: hyperbolic 3-manifold

$\rho : \pi_1(M) \to \text{PSL}(2, \mathbb{C})$: a holonomy representation.

$\Gamma := \rho(\pi_1(M))$
**Hidden Symmetries**

Let $M$ be a hyperbolic 3-manifold.

Define $\rho : \pi_1(M) \to \text{PSL}(2, \mathbb{C})$ as a holonomy representation, and let $\Gamma := \rho(\pi_1(M))$.

**Definition**

\[ N(\Gamma) := \{ \gamma \in \text{PSL}(2, \mathbb{C}) \mid \gamma \Gamma \gamma^{-1} = \Gamma \} \]
\[ C(\Gamma) := \{ \gamma \in \text{PSL}(2, \mathbb{C}) \mid \gamma \Gamma \gamma^{-1} \text{ and } \Gamma \text{ are weakly commensurable} \} \]

$N(\Gamma)$ and $C(\Gamma)$ are called the normalizer and commensurator, respectively.

Two groups $\Gamma_i < \text{PSL}(2, \mathbb{C})$ ($i = 1, 2$) are said to be weakly commensurable if $[\Gamma_i : \Gamma_1 \cap \Gamma_2] < \infty$ for both $i = 1, 2$. 
**Manifold with (no) Hidden Symmetry**

**Definition**

An elements in $C(\Gamma) \setminus N(\Gamma)$ is called a hidden symmetry.

**Definition**

A hyperbolic 3-manifold $M$ said to have no hidden symmetry $\iff$ the image $\Gamma := \rho(\pi_1(M))$ of a holonomy representation $\rho$ does not have hidden symmetry.

**Remark.** By Mostow-Prasad rigidity theorem, this definition does not depend on the choice of a holonomy representation.
Commensurable fibrations on the same fibered cone

**Result 4**

One can construct an infinite sequence of manifolds \( \{ M_i \} \) with

- non-symmetric (fibers are of different topology), and
- commensurable fibrations whose corresponding elements in \( H^1(M_i; \mathbb{Z}) \) are on the same fibered cone.
Construction

Lemma

$M$ : fibered hyperbolic 3-manifold.

$\omega_1 \neq \pm \omega_2 \in H^1(M; \mathbb{Z}) :$ primitive elements correspond to symmetric fibrations.

Then, for all $n >> 1 (n \in \mathbb{N})$, there exists a finite cover $p_n : M_n \to M$ of degree $n$ such that $p_n^*(\omega_1)$ and $p_n^*(\omega_2)$ correspond to commensurable but non-symmetric fibrations.
Idea

Let \((F_1, \phi_1)\) and \((F_2, \phi_2)\) be corresponding pair of \(\omega_1\) and \(\omega_2\), respectively.

Then let \(p_n : M_n \rightarrow M\) be the covering that corresponds to \((F_1, \phi_1^n)\) (dynamical cover).

Then for large enough \(n\), we see that \(p_n^{-1}(F_2)\) is not homeomorphic to \(F_1\).

**Figure:** Schematic picture of the dynamical covering
To be precise

( $b$ is the first Betti number of $M$)

- $H^1(M; \mathbb{Z}) \cong \text{Hom}(H_1(M)/\text{Tor}, \mathbb{Z}) \cong \mathbb{Z}^b$
- $0 \to \pi_1(F_i) \to \pi_1(M) \xrightarrow{\rho_i} \pi_1(S^1) \cong \mathbb{Z} \to 0$. 


To be precise

( $b$ is the first Betti number of $M$)

- $H^1(M; \mathbb{Z}) \cong \text{Hom}(H_1(M)/\text{Tor}, \mathbb{Z}) \cong \mathbb{Z}^b$
- $0 \to \pi_1(F_i) \to \pi_1(M) \xrightarrow{\rho_i} \pi_1(S^1) \cong \mathbb{Z} \to 0.$
- $A_i = \text{ab}(\pi_1(F_i))/\text{Tor} \subset H_1(M)$ ($\text{ab} : \text{abelianization}$)
- $A_i = \text{Ker}(\omega_i) \cong \mathbb{Z}^{b-1}$
To be precise

\((b \text{ is the first Betti number of } M)\)

- \(H^1(M; \mathbb{Z}) \cong \text{Hom}(H_1(M)/\text{Tor}, \mathbb{Z}) \cong \mathbb{Z}^b\)
- \(0 \rightarrow \pi_1(F_i) \rightarrow \pi_1(M) \xrightarrow{\rho_i} \pi_1(S^1) \cong \mathbb{Z} \rightarrow 0.\)
- \(A_i = \text{ab}(\pi_1(F_i))/\text{Tor} \subset H_1(M) \ (\text{ab} : \text{abelianization})\)
- \(A_i = \text{Ker}(\omega_i) \cong \mathbb{Z}^{b-1}\)
- \(\rho_1 : \pi_1(M) \xrightarrow{\text{ab}} H_1(M) \xrightarrow{\omega_1} \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}.\)
- for sufficiently large \(n\), \(\exists b \in A_2 \text{ s.t. } \rho_1(b) \neq 0\)
$S^3 \setminus 6^2_2$ has a symmetry that permutes the components of cusps.

**Figure:** $6^2_2$ (Generated by "Kirby Calculator")
Thurston Norm on $H^1(S^3 \setminus 6_2^2, \mathbb{R})$[Hironaka]

Figure: $H^1(S^3 \setminus 6_2^2, \mathbb{R})$
By taking conjugate we can prove that \( \exists h_1 : M \to M \) s.t. 
\[
h_1^*(\omega) = -\omega.
\]
By taking conjugate we can prove that $\exists h_1 : M \to M$ s.t.

$$h_1^*(\omega) = -\omega.$$ 

+ the fact that $6_2^2$ is amphicheiral, we can find a symmetry $h_2 : M \to M$ s.t.

- $h_2^*(U) = -U$, and
- $h_2^*(T) = T$.

$\Rightarrow$ Fibrations $aU + bT$ and $-aU + bT$ are symmetric.
⇒ By taking covers of $S^3 \setminus 6^2_2$, we prove

**Result 4**

One can construct an infinite sequence of manifolds with non-symmetric but commensurable fibrations (whose corresponding elements in $H^1(M; \mathbb{Z})$ are in the same fibered cone).
Questions

Question 1.
When a manifold has non-symmetric but commensurable fibrations.

- $S^3 \setminus 6_2^2$ and the Magic 3-manifold have many hidden symmetry.

Question 2.
How many commensurable fibrations can a manifold have up to symmetry?
Thank you for your attention

Figure: Marseille