LIOUVILLE CORRESPONDENCES BETWEEN MULTICOMPONENT INTEGRABLE HIERARCHIES

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We establish Liouville correspondences for the integrable two-component Camassa–Holm hierarchy, the two-component Novikov (Geng–Xue) hierarchy, and the two-component dual dispersive water wave hierarchy using the related Liouville transformations. This extends previous results for scalar Camassa–Holm and KdV hierarchies and Novikov and Sawada–Kotera hierarchies to the multicomponent case.

Keywords: Liouville transformation, bi-Hamiltonian structure, two-component Camassa–Holm system, two-component Novikov system, two-component dual dispersive water wave system

DOI: 10.1134/S0040577920070028

1. Introduction

This paper is concerned with Liouville correspondences for integrable hierarchies of multicomponent systems with nonlinear dispersion and extends previous work on scalar hierarchies [1]–[4]. Specifically, we establish Liouville correspondences between several multicomponent integrable hierarchies generated by the two-component Camassa–Holm (CH) system, the two-component Novikov system (also known as the Geng–Xue (GX) system), and also the two-component dual dispersive water wave system.

A basic idea for investigating the integrability of a new system is to establish its relation to a known integrable system through some kind of transformations, which can include Bäcklund transformations, Miura transformations, gauge transformations, Darboux transformations, hodograph transformations, Liouville transformations, and the like.

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The research of J. Kang was supported by the National Science Foundation of China (Grant Nos. 11631007 and 11871395) and the National Science Basic Research Program of Shaanxi (No. 2019JC-28).
The research of X. Liu was supported by the National Science Foundation of China (Grant Nos. 11722111 and 11631007).
The research of C. Qu was supported by the National Science Foundation of China (Grant Nos. 11631007 and 11971251).

Prepared from an English manuscript submitted by the authors; for the Russian version, see Teoreticheskaya i Matematicheskaya Fizika, Vol. 204, No. 1, pp. 10–45, July, 2020. Received November 4, 2019. Revised February 26, 2020. Accepted March 6, 2020.
transformations, etc. [5]–[11]. Applying an appropriate transformation allows deriving solutions and analyzing the integrability properties of the considered system by adapting known solutions and integrable structures. For constructing solitons using the inverse scattering transform and also studying the long-time behavior of solutions based on the Riemann–Hilbert approach, the spatial isospectral problem in the Lax-pair formulation [12] plays a dominant role among other vital integrability properties. The transition from one isospectral problem to another via a change of variables can usually be identified as a form of Liouville transformation (also see [13] and [14] for this terminology). It is then expected that such a correspondence based on a Liouville transformation, which we call a Liouville correspondence for brevity, can be used to establish an inherent correspondence between associated integrability properties including symmetries, conserved quantities, soliton solutions, Hamiltonian structures, etc.

An intense interest in integrable systems of the CH type has been aroused in recent years as a result of discovering their novel properties including nonlinear dispersion, which (as a rule) supports nonsmooth soliton structures such as peakons, cuspons, and compactons. Moreover, the interest is related to the ability of such systems to model wave-breaking phenomena. Previous investigations established a variety of Liouville correspondences between integrable hierarchies of the CH type and certain classical integrable hierarchies. The best-studied example is the CH equation with a quadratic nonlinearity [15]–[18]

\[
m_t + 2u_x m + um_x = 0, \quad m = u - u_{xx}.
\]

(1.1)

The Liouville correspondence between the entire CH hierarchy induced by (1.1) and the usual Korteweg–de Vries (KdV) hierarchy was established in [3] and [4] and in addition provides a correspondence between the Hamiltonian functionals of the two hierarchies [3]. Moreover, the Liouville transformation relates their smooth traveling-wave solutions [19].

The modified CH (mCH) equation [20]

\[
m_t + ((u^2 - u_x^2)m)_x = 0, \quad m = u - u_{xx},
\]

(1.2)

is a prototypical integrable model with cubic nonlinearity and has several novel properties (described, e.g., in [1], [21]–[25]). In [1], we established a Liouville correspondence between the integrable mCH and modified KdV (mKdV) hierarchies, including explicit relations between their equations and Hamiltonian functionals. In contrast to the CH–KdV situation, the analysis in [1] was based on the interrelation between the respective recursion operators and the structure of conservation laws for flows of the mCH hierarchy. Moreover, we constructed a novel transformation mapping mCH equation (1.2) to CH equation (1.1) based on the associated Liouville correspondences between the CH–KdV hierarchies and between the mCH–mKdV hierarchies.

We note that all the equations in the CH, mCH, KdV, and mKdV hierarchies have a bi-Hamiltonian form. Moreover, the two Hamiltonians for the CH and mCH integrable hierarchies can be constructed from those of the respective KdV and mKdV hierarchies using the method of tri-Hamiltonian duality [26], [27], [20]. This approach is based on the observation that the majority of standard integrable equations that have a bi-Hamiltonian structure in fact admit a compatible triple of Hamiltonian structures if an appropriate scaling argument is used. Combinations of the members of the compatible Hamiltonian triple generate different types of bi-Hamiltonian integrable systems, which admit a duality relation. The tri-Hamiltonian duality relations help to establish the associated Liouville correspondences in the CH–KdV and mCH–mKdV cases [1], [3].

For the integrable hierarchies with generalized bi-Hamiltonian structures (i.e., compatible pairs of Dirac structures [28]) or without tri-Hamiltonian duality, establishing appropriate Liouville correspondences is a more challenging task. Nevertheless, the existence of such Liouville correspondences should be expected.
when there are Liouville transformations between the associated isospectral problems. In this case, the Novikov and Degasperis–Procesi (DP) [29], [30] integrable hierarchies are two representative examples. The Novikov integrable equation with cubic nonlinearity [31], [32]

\[ m_t = u^2 m_x + 3uu_x m, \quad m = u - u_{xx}, \]  

(1.3)
is associated with a third-order isospectral problem. In [31], a Liouville transformation was used to convert its isospectral problem into the isospectral problem of the Sawada–Kotera (SK) equation [33], [34]

\[ Q_\tau + Q_{yyyyy} + 5QQ_{yyy} + 5Q_y Q_{yy} + 5Q^2 Q_y = 0. \]  

(1.4)

Although the Novikov hierarchy is bi-Hamiltonian [31], its Hamiltonians do not support the tri-Hamiltonian duality construction, especially one that relates to the Hamiltonians of the SK equation. Moreover, SK equation (1.4) has a generalized bi-Hamiltonian formulation with the corresponding hierarchy generated by the recursion operator, a composition of symplectic and implicative operators, that does not satisfy the nondegeneracy and invertibility conditions [35]. Therefore, obtaining a Liouville correspondence for the Novikov integrable hierarchy requires a more refined analysis. In [2], using the Liouville transformation and several operator decomposition identities, we could establish a Liouville correspondence between the Novikov and SK integrable hierarchies. Also in [2], acting similarly and using a certain Liouville transformation proposed in [29] and [36], we established an analogous correspondence between the DP and Kaup–Kupershmidt [37] hierarchies.

We now consider multicomponent integrable systems of the CH type. An important example is the well-studied two-component CH (2CH) system [38]

\[ m_t + 2u_x m + um_x + \rho \rho_x = 0, \quad m = u - u_{xx}, \]  

(1.5)

which arises as an integrable shallow water model [39]. This system is of significant interest because it has nonlinear interactions between the free surface velocity components and the horizontal velocity components. Moreover, it can be used to model the phenomenon of wave breaking (see, e.g., [40]–[50]). System (1.5) is completely integrable and arises from the compatibility condition for the Lax pairs [39]

\[ \Psi_{xx} + \left(-\frac{1}{4} - \lambda m(t, x) + \lambda^2 \rho^2(t, x)\right) \Psi = 0, \]  

\[ \Psi_t = \left(\frac{1}{2\lambda} - u(t, x)\right) \Psi_x + \frac{1}{2} u_x(t, x) \Psi. \]  

(1.6)

In [51], Antonowicz and Fordy investigated the isospectral flows of the spectral problem

\[ \Psi_{xx} + \left(\lambda v_1(t, x) + \cdots + \lambda^N v_N(t, x)\right) \Psi = \alpha \Psi \]  

(1.7)

and the associated general \( N \)-component systems with \( N+1 \) compatible Hamiltonian structures. It was also proved in [51] that the Liouville transformation

\[ y = \int^x \sqrt{v_N(t, \xi)} \, d\xi, \quad \Phi = v_N^{1/4} \Psi, \]  

\[ u_0(t, y) = -\frac{\alpha}{v_N} - \frac{v_{N,xx}}{4v_N^2} + \frac{5v_N^2}{16v_N^4}, \quad u_k(t, y) = \frac{v_k}{v_N}, \quad k = 1, \ldots, N - 1, \]  

(1.8)
converts (1.7) into the energy-dependent Schrödinger isospectral problem [52], [53]

$$\Phi_{yy} + (u_0(t, y) + \lambda u_1(t, y) + \cdots + \lambda^{N-1} u_{N-1}(t, y))\Phi = -\lambda^N \Phi.$$  
(1.9)

We note that the $x$-part in (1.6) is a special case of (1.7). In [38], transformation (1.8) with $N = 2$ was used to convert Eq. (1.6) into

$$\Phi_{yy} + (Q(\tau, y) + \lambda P(\tau, y) + \lambda^2)\Phi = 0,$$

$$\Phi_{\tau} - \frac{1}{2\lambda} \rho(t, x)\Phi_y + \frac{1}{4\lambda} \rho_y(t, x)\Phi = 0,$$

which are a Lax pair for the integrable system

$$P_{\tau}(\tau, y) = \rho_y,$$

$$Q_{\tau}(\tau, y) = \frac{1}{2} \rho P_y(\tau, y) + \rho_y P(\tau, y),$$

$$\rho_{yyy} + 2 \rho Q(\tau, y) + 2(\rho Q(\tau, y))_y = 0.$$

In [38], this system was found to be the first negative flow of the Ablowitz–Kaup–Newell–Segur hierarchy [54] by using the spectral structure of Lax-pair formulation (1.10).

Moreover, 2CH system (1.5) is bi-Hamiltonian, and the compatible Hamiltonians therefore recursively generate the entire 2CH integrable hierarchy, where (1.5) forms the second flow in the positive direction. In addition, the bi-Hamiltonian structure can be derived from the bi-Hamiltonian structure of the Ito system [55] using the method of tri-Hamiltonian duality [20]. On the other hand, although the 2CH and Ito integrable systems are related by tri-Hamiltonian duality, in contrast to the CH-KdV and mCH-mKdV cases, the Liouville correspondence between these two hierarchies is unexpected because the transformation between the corresponding isospectral problems is not obvious. Nevertheless, we can expect to be able to establish a Liouville correspondence between the 2CH hierarchy and a second integrable hierarchy involving integrable system (1.11) as one of the flows in the negative direction. To achieve this goal, we must overcome several new difficulties.

First, the integrable structures including the recursion operator and Hamiltonians for the hierarchy that includes system (1.11) as a negative flow are not clear. The required integrability information was also not presented in [38]. On the other hand, as in the scalar case, verifying the Liouville correspondence relies on analyzing the underlying operators, which have a matrix form in the multicomponent case, and a more careful calculation of the nonlinear interplay among the various components is hence required. Here, we elucidate the entire integrable hierarchy, which we call the hierarchy associated with system (1.11). It forms the first negative flow in what we call the associated 2CH (a2CH) hierarchy. Below, we show that it has a bi-Hamiltonian structure and establish a Liouville correspondence between the 2CH and a2CH hierarchies. Moreover, we find that the second positive flow of the a2CH integrable hierarchy is the integrable system

$$Q_{\tau} = -\frac{1}{2} P_{yyy} - 2QP_y - Q_y P, \quad P_{\tau} = 2Q_y - 3PP_y,$$

which belongs to the integrable family studied in [56] and can be recognized as an integrable system of Kaup–Boussinesq type describing the motion of shallow water waves [57].

Novikov equation (1.3) has the two-component integrable generalization

$$m_t + 3uv_x m + umx = 0, \quad m = u - u_{xx},$$

$$n_t + 3uv_x n + unx = 0, \quad n = v - v_{xx},$$

(1.12)
which was introduced by Geng and Xue [58] and is therefore called the GX system (see [59] and the references therein). As a prototypical multicomponent integrable system with cubic nonlinearity, GX system (1.12) admits special peakon solutions and has recently attracted much attention [59]–[63]. It was shown in [60] that there exists a certain Liouville transformation converting the Lax pair of GX system (1.12) into the Lax pair of the integrable system
\begin{equation}
\frac{Q_t}{Q} = \frac{3}{2}(q_y + p_y) - (q - p)P, \quad \frac{P_t}{P} = \frac{3}{2}(q - p),
\end{equation}

\begin{align*}
p_{yy} + 2p_yP + pP_y + pP^2 - qP + 1 &= 0, \\
q_{yy} - 2q_yP - qP_y + qP^2 - qQ + 1 &= 0,
\end{align*}

where \( q = vm^{2/3}n^{-1/3} \) and \( p = um^{-1/3}n^{2/3} \). System (1.13) is bi-Hamiltonian; its bi-Hamiltonian structure was derived in [60]. We study the entire associated GX (aGX) integrable hierarchy, for which system (1.13) is the first negative flow. We also establish a Liouville correspondence between the integrable GX hierarchy generated by (1.12) and the aGX hierarchy.

Finally, we consider the dual dispersive water wave (DDWW) integrable system
\begin{align}
\rho_t &= ((\rho + v)u)_x, \quad \rho = v - v_x, \\
\gamma_t &= (\gamma u + 2v)_x, \quad \gamma = u + u_x,
\end{align}

which was recently derived in [64] from the dispersive water wave (DWW) integrable system using tri-Hamiltonian duality. The integrability of the DWW system was studied by Kaup and Kupershmidt [57], [65]. System (1.14) has a bi-Hamiltonian formulation and admits a variety of nonsmooth soliton solutions [64]. We find a transformation that establishes a Liouville correspondence between the DDWW integrable hierarchy generated by (1.14) with an associated DDWW integrable hierarchy, thus obtaining a bi-Hamiltonian description of the latter. We further investigate the explicit relation between the flows of these hierarchies and their Hamiltonian functionals.

We conclude this section by outlining the rest of the paper. In Sec. 2, we first present a Liouville transformation relating the isospectral problems of the 2CH and a2CH integrable hierarchies. Next, we use the Liouville transformation to establish a one-to-one correspondence between the flows of the 2CH and a2CH hierarchies. Moreover, we relate the Hamiltonian functionals appearing in the 2CH and a2CH hierarchies. In Secs. 3 and 4, we similarly investigate the Liouville correspondences for the respective GX and DDWW integrable hierarchies.

2. Liouville correspondence for the two-component Camassa–Holm hierarchy

2.1. Liouville transformation for the isospectral problem of the 2CH system. In this subsection, we present an explicit expression for the Liouville transformation for the isospectral problem of the 2CH system
\begin{align}
m_t + u_xm + (um)_x + \rho \rho_x &= 0, \quad m = u - u_{xx}, \\
\rho_t + (\rho u)_x &= 0,
\end{align}

which follows from the compatibility condition for the Lax pair
\begin{align}
\Psi_{xx} + \left(-\frac{1}{4} - \lambda m + \lambda^2 \rho^2\right)\Psi &= 0, \quad \Psi_t = \left(\frac{1}{2\lambda} - u\right)\Psi_x + \frac{u}{2}\Psi,
\end{align}
with the spectral parameter \( \lambda \). It was proved in [38] that the reciprocal transformation

\[
dy = \rho \, dx - \rho u \, dt, \\
d\tau = dt
\]

converts isospectral problem (2.2) into

\[
\Phi_{yy} + (Q + \lambda P + \lambda^2) \Phi = 0, \\
\Phi_\tau - \frac{1}{2\lambda} \rho \Phi_y + \frac{1}{4\lambda} \rho_y \Phi = 0,
\]

(2.4)

where

\[
\Phi = \sqrt{\rho} \Psi, \\
Q = -\frac{1}{4} \rho^{-2} + \frac{3}{4} \rho^{-4} \rho_x^2 - \frac{1}{2} \rho^{-3} \rho_{xx}, \\
P = -\frac{m}{\rho^2}
\]

(2.5)

Linear equations (2.4) have the form of a Lax pair, and the resulting compatibility condition \( \Phi_{yy\tau} = \Phi_{\tau yy} \) yields the integrable system

\[
P_{\tau} = \rho_y, \\
Q_{\tau} = \frac{1}{2} \rho P_y + \rho_y P, \\
\rho_{yy\tau} + 2\rho_y Q + 2(\rho Q)_y = 0.
\]

(2.6)

Therefore, the expected Liouville correspondence between 2CH system (2.1) and integrable system (2.6) is given by reciprocal transformation (2.3) and change of dependent variables (2.5). On the other hand, because of the spectral structure in (2.4), system (2.6) can be viewed as the first negative flow of a certain integrable hierarchy that satisfies isospectral problem (2.4), which we here call the a2CH hierarchy.

**2.2. Liouville correspondence between the 2CH and a2CH hierarchies.** Based on the obtained results, we generalize the Liouville correspondence between systems (2.1) and (2.6) to the entire integrable hierarchies. More precisely, we propose the Liouville transformation

\[
\tau = t, \\
y = \int x \rho(t, \xi) \, d\xi,
\]

\[
P(\tau, y) = -m(t, x) \rho^{-2}(t, x), \\
Q(\tau, y) = -\frac{1}{4} \rho^{-2}(t, x) + \frac{3}{4} \rho^{-4}(t, x) \rho_x^2(t, x) - \frac{1}{2} \rho^{-3}(t, x) \rho_{xx}(t, x).
\]

(2.7)

First, we write 2CH system (2.1) in the bi-Hamiltonian form [20]

\[
\begin{pmatrix} m \\ \rho \end{pmatrix}_t = \mathcal{K} \delta \mathcal{H}_1(m, \rho) = \mathcal{J} \delta \mathcal{H}_2(m, \rho), \\
\delta \mathcal{H}_n(m, \rho) = \left( \frac{\delta \mathcal{H}_n}{\delta m}, \frac{\delta \mathcal{H}_n}{\delta \rho} \right)^T, \quad n = 1, 2,
\]

(2.8)

with the compatible Hamiltonians

\[
\mathcal{K} = \begin{pmatrix} m \partial_x + \partial_x m & \rho \partial_x \\ \partial_x \rho & 0 \end{pmatrix}, \\
\mathcal{J} = \begin{pmatrix} \partial_x - \partial_x^3 & 0 \\ 0 & \partial_x \end{pmatrix}.
\]

(2.9)

The associated Hamiltonian functionals are

\[
\mathcal{H}_1(m, \rho) = -\frac{1}{2} \int (u^2 + u_x^2 + \rho^2) \, dx, \\
\mathcal{H}_2(m, \rho) = -\frac{1}{2} \int u(u^2 + u_x^2 + \rho^2) \, dx.
\]
According to the Magri theorem [66], the Hamiltonian pair induces the hierarchy of commutative bi-Hamiltonian systems

\[
\begin{align*}
(m, \rho)_t &= \mathbf{K}_n = \mathcal{K} \delta \mathcal{H}_{n-1}(m, \rho) = \mathcal{J} \delta \mathcal{H}_n(m, \rho), \\
\delta \mathcal{H}_n(m, \rho) &= \begin{pmatrix} \delta \mathcal{H}_n(m, \rho) \\ \delta \mathcal{H}_n(m, \rho) \end{pmatrix}^T, \quad n \in \mathbb{Z},
\end{align*}
\] (2.10)

based on the corresponding Hamiltonian functionals \( \mathcal{H}_n = \mathcal{H}_n(m, \rho) \). The members in hierarchy (2.10) are obtained by successively applying the recursion operator \( \mathcal{R} = \mathcal{K} \mathcal{J}^{-1} \) to a seed symmetry [67], [68]. The positive flows of (2.10) begin with the seed system

\[
(m, \rho)_t = \mathbf{K}_1 = -\begin{pmatrix} m \\ \rho \end{pmatrix}_x,
\]

and 2CH system (2.8) is the second member. We see that the Hamiltonian \( \mathcal{K} \) admits a Casimir functional

\[
\mathcal{H}_C(m, \rho) = \int \frac{m}{\rho} \, dx
\] (2.11)

with the variational derivative

\[
\delta \mathcal{H}_C(m, \rho) = \begin{pmatrix} \rho^{-1} \\ -m \rho^{-2} \end{pmatrix},
\]

which leads to the associated Casimir system

\[
(m, \rho)_t = \mathbf{K}_{-1} = \mathcal{J} \delta \mathcal{H}_C.
\]

This system is the first negative flow for hierarchy (2.10) and has the explicit form

\[
m_t = (\partial_x - \partial_x^3) \frac{1}{\rho}, \quad \rho_t = -\begin{pmatrix} m \\ \rho \end{pmatrix}_x, \quad m = u - u_{xx}.
\] (2.12)

Repeatedly applying the inverse recursion operator \( \mathcal{R}^{-1} = \mathcal{J} \mathcal{K}^{-1} \) to (2.12) produces members in the negative direction of (2.10), which have the form

\[
(m, \rho)_t = \mathbf{K}_{-n} = (\mathcal{J} \mathcal{K}^{-1})^{n-1} \mathcal{J} \begin{pmatrix} \rho^{-1} \\ -m \rho^{-2} \end{pmatrix}, \quad n = 1, 2, \ldots
\] (2.13)

In Lemma 2.2 below, we show that the a2CH integrable hierarchy involving system (2.6) is in fact generated by the recursion operator

\[
\mathcal{R} = \frac{1}{2} \begin{pmatrix} 0 & \partial_y^2 + 4Q + 2Q_y \partial_y^{-1} \\ -4 & 4P + 2P_y \partial_y^{-1} \end{pmatrix}.
\] (2.14)

Repeatedly applying \( \mathcal{R} \) to the usual seed symmetry \( \mathbf{K}_1 = (-Q_y, -P_y)^T \) gives the positive flows of the a2CH integrable hierarchy:

\[
\begin{pmatrix} Q \\ P \end{pmatrix}_\tau = \mathbf{K}_n = \mathcal{R}^{n-1} \mathbf{K}_1, \quad n = 1, 2, \ldots
\] (2.15)

On the other hand, because the trivial symmetry \( \mathbf{K}_0 = (0, 0)^T \) satisfies \( \mathcal{R} \mathbf{K}_0 = \mathbf{K}_1 \), the negative flow with \( n \in \mathbb{Z}^+ \) is given by

\[
\mathcal{R}^n \begin{pmatrix} Q \\ P \end{pmatrix}_\tau = \mathbf{K}_0, \quad n = 1, 2, \ldots
\] (2.16)
In particular, in the case \( n = 1 \), the first negative flow in (2.16) has the explicit form
\[
\left( \frac{1}{2} \partial_y^2 + 2Q + Q_y \partial_y^{-1} \right) P_\tau = 0, \quad Q_\tau = \left( P + \frac{1}{2} P_y \partial_y^{-1} \right) P_\tau.
\] (2.17)

More precisely, system (2.6) arising from the compatibility condition of Lax pair (2.4) is a reduction of first negative flow (2.17). Moreover, the second positive flow of the a2CH hierarchy in (2.15) is
\[
\left( \begin{array}{c} Q \\ P \end{array} \right)_\tau = \overline{K}_2 = \overline{K}_1 = \overline{K} (-Q_y - P_y) = \left( \begin{array}{c} -\frac{1}{2} P_{yy} - 2QP_y - Q_y P \\ 2Q_y - 3PP_y \end{array} \right),
\] (2.18)

which can be obtained from the \( y \)-component of Lax pair (2.4) together with
\[
\Phi_\tau + (2\lambda + P) \Phi_y - \frac{1}{2} P_y \Phi = 0.
\]

A Lax pair with the general form
\[
\Phi_{yy} = \left( -\lambda^2 + \lambda u(\tau, y) + \frac{\kappa}{2} u^2(\tau, y) + v(\tau, y) \right) \Phi,
\]
\[
\Phi_\tau = -\left( \lambda + \frac{1}{2} u(\tau, y) \right) \Phi_y + \frac{1}{4} u_y(\tau, y) \Phi,
\]
where \( \kappa \) is an arbitrary constant, was introduced in [56]. This Lax pair gives the integrable system
\[
u = \frac{3}{2} \kappa \]
\[
u - \frac{1}{4} u_{yy} + (uv)_y = \frac{1}{2} u^2 - \kappa \left( \frac{1}{2} + \kappa \right) u_y = 0.
\]

If \( \kappa = 0 \), then the resulting system is exactly (2.18) up to a change of variables \( u = P \), \( v = -Q \), and \( \tau = 2t \). Moreover, a change of the dependent variable
\[
Q(\tau, y) = N(\tau, y) + \frac{1}{4} P^2(\tau, y)
\]
converts (2.18) into
\[
N_\tau = -\frac{1}{2} P_{yy} - 2(PN)_y, \quad P_\tau = 2N_y - 2PP_y,
\]
which has the form of the Kaup–Boussinesq system considered in [57].

In what follows, we let \( 2\text{CH}_n \), \( 2\text{CH}_{-n} \) and \( \text{a2CH}_n \), \( \text{a2CH}_{-n} \) denote the \( n \)th equations in the positive and negative directions of 2CH hierarchy (2.10) and the a2CH hierarchy (here \( n \) is a positive integer). With this notation, we can formulate our main result for the Liouville correspondence between the 2CH and a2CH hierarchies.

**Theorem 2.1.** For each integer \( n \), the \( \text{CH}_{n+1} \) equation is mapped into the \( \text{a2CH}_{-n} \) equation under Liouville transformation (2.7).

The proof of this theorem relies on two preliminary lemmas.

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**Lemma 2.1.** Let \((m(t, x), \rho(t, x))\) and \((Q(\tau, y), P(\tau, y))\) be related by transformation (2.7). Then we have the operator identities

\[
\begin{align*}
\rho^{-3/2} \left( \frac{1}{4} - \partial_x^2 \right) \rho^{-1/2} &= -(Q + \partial_y^2), \\
\rho^{-2} (\partial_x - \partial_x^3) \rho^{-1} &= -(\partial_y^3 + 2Q\partial_y + 2\partial_y Q), \\
\rho^{-2}(\partial_x m + m\partial_x) \rho^{-1} &= -(P\partial_y + \partial_y P).
\end{align*}
\]

(2.19)

**Proof.** First, in view of transformation (2.7), we have

\[ \partial_x = \rho \partial_y. \]

It hence follows that

\[ \partial_x^2 \rho^{-1/2} = \frac{3}{4} \rho^{-5/2} \rho_x^2 - \frac{1}{2} \rho^{-3/2} \rho_{xx} + \rho^{3/2} \partial_y^2. \]

As a result, we obtain the relation

\[ \rho^{-3/2} \left( \frac{1}{4} - \partial_x^2 \right) \rho^{-1/2} = \rho^{-3/2} \left( \frac{1}{4} \rho^{-1/2} - \frac{3}{4} \rho^{-5/2} \rho_x^2 + \frac{1}{2} \rho^{-3/2} \rho_{xx} - \rho^{3/2} \partial_y^2 \right) = -(Q + \partial_y^2), \]

which proves the first identity in (2.19). Next, from (2.20), we obtain

\[
\begin{align*}
\partial_x \rho^{-1} &= -\rho^{-1} \rho_y + \partial_y, \\
\partial_x^3 \rho^{-1} &= -\rho^{-1} \rho_y^3 + 2\rho_y \rho_{yy} - \rho \rho_{yyy} + (\rho_y^2 - 2\rho \rho_{yy}) \partial_y + \rho^2 \partial_y^3.
\end{align*}
\]

Hence,

\[
\rho^{-2}(\partial_x - \partial_x^3) \rho^{-1} = -\rho^{-3} \rho_y + \rho^{-3} \rho_y^3 - 2\rho^{-2} \rho_y \rho_{yy} + \rho^{-1} \rho_{yyy} + (\rho^{-2} - \rho^{-2} \rho_y^2 + 2\rho^{-1} \rho_{yy}) \partial_y - \partial_y^3,
\]

and the second identity in (2.19) then follows. Finally, direct computation shows that

\[
\begin{align*}
\rho^{-2}(\partial_x m + m\partial_x) \rho^{-1} &= -\rho^{-1} \partial_y P \rho - P \rho \partial_y \rho^{-1} = \\
&= -\rho^{-1}(\partial_y P) \rho + P \rho \partial_y + P(\rho \partial_y - \partial_y) = -(P \partial_y + \partial_y P),
\end{align*}
\]

which proves the third identity in (2.19).

**Lemma 2.2.** Let \(\mathcal{K}\) and \(\mathcal{J}\) be the two compatible Hamiltonians given in (2.9) for the 2CH integrable hierarchy. Then as a result of transformations (2.7), for each positive integer \(n\), we have the equality

\[
A^{-1}(\mathcal{J}\mathcal{K}^{-1})^n A = \overline{\mathcal{K}},
\]

(2.21)

where the operator \(\overline{\mathcal{K}}\) is defined by (2.14) and

\[
A = \begin{pmatrix} -2\rho^2 & 0 \\ 0 & \rho \end{pmatrix}.
\]

(2.22)

Moreover, \(\overline{\mathcal{K}}\) is not only a hereditary operator itself but also the recursion operator for the a2CH hierarchy with positive flows given by (2.15) and negative flows by (2.16).
Before proving Lemma 2.2, we briefly discuss the notion of hereditary operators in the two-component setting. Let $\mathcal{A}$ denote the space of differential functions depending only on the indicated dependent variables and their spatial derivatives, and let $\mathcal{A}^n$ denote the corresponding space of $n$-component differential functions. An operator, for example, $\mathcal{R}$ as in (2.14), is called a hereditary operator if and only if it satisfies the condition

$$D_{\mathcal{R}[f]}g - D_{\mathcal{R}[g]}f = \mathcal{R}(D_{\mathcal{R}[f]}g - D_{\mathcal{R}[g]}f), \quad f, g \in \mathcal{A}^2.$$  \hfill (2.23)

The following proposition, proved in [18], [68], describes how a hereditary operator serves as the recursion operator of an integrable hierarchy.

**Proposition 2.1.** Let the hereditary operator $\mathcal{R}$ and the system

$$Q_\tau = G_1, \quad Q = (Q(\tau, y), P(\tau, y))^T, \quad G_1 = (G_1^1(Q), G_1^2(Q))^T \in \mathcal{A}^2,$$  \hfill (2.24)

satisfy the condition

$$\mathcal{R}_\tau \mid_{Q_\tau = G_1} = [D_{G_1}, \mathcal{R}],$$  \hfill (2.25)

where $\mathcal{R}_\tau$ is the time derivative of $\mathcal{R}$ and $D_{G_1}$ is the Fréchet derivative of $G_1$; $\mathcal{R}$ is hence the recursion operator for (2.24). Then $\mathcal{R}$ is also the recursion operator for each flow in the associated hierarchy

$$Q_\tau = G_n = \mathcal{R}_\tau^{n-1}G_1, \quad n \in \mathbb{Z}.$$  

**Proof of Lemma 2.2.** We prove (2.21) by induction. In the case $n = 1$, in view of the forms of $\mathcal{K}$ and $\mathcal{J}$, we obtain

$$A^{-1}\mathcal{J}\mathcal{K}^{-1}A = \frac{1}{2} \begin{pmatrix} 0 & -\rho^{-2}(\partial_x - \partial^3_x)\rho^{-1}\partial_x^{-1}\rho \\ -4 & -2\rho^{-2}(m\partial_x + \partial_x m)\rho^{-1}\partial_x^{-1}\rho \end{pmatrix},$$

which together with the second and third equations in (2.19) leads to (2.21) for $n = 1$. We next suppose that (2.21) holds for $n = k$ with some $k \in \mathbb{Z}^+$. Then for $n = k + 1$, we have

$$\mathcal{R}_\tau^{k+1} = \mathcal{R}_\tau^k \mathcal{R} = A^{-1}(\mathcal{J}\mathcal{K}^{-1})^k AA^{-1}\mathcal{J}\mathcal{K}^{-1}A = A^{-1}(\mathcal{J}\mathcal{K}^{-1})^{k+1}A,$$

which establishes the induction step and thus proves that (2.21) holds for any integer $n \geq 1$.

According to Proposition 2.1, to prove that the operator $\mathcal{R}$ is a recursion operator for the entire $a2CH$ hierarchy, we first prove that $\mathcal{R}$ is a recursion operator for the seed system

$$\begin{pmatrix} Q \\ P \end{pmatrix}_\tau = K_1 = -\begin{pmatrix} Q \\ P \end{pmatrix}_y.$$  \hfill (2.26)

To prove (2.25), we note that on one hand, the Fréchet derivative is

$$D_{\mathcal{R}_1} = -\begin{pmatrix} \partial_y & 0 \\ 0 & \partial_y \end{pmatrix}.$$

On the other hand, by virtue of (2.26), we have

$$\mathcal{R}_\tau = -\begin{pmatrix} 0 & Q_{yy}\partial_y^{-1} + 2Q_y \\ 0 & P_{yy}\partial_y^{-1} + 2P_y \end{pmatrix}.$$
Moreover, the commutator in (2.25) satisfies

\[
[D_{\mathcal{R}_1}, \mathcal{R}] = D_{\mathcal{R}_1} \cdot \mathcal{R} - \mathcal{R} \cdot D_{\mathcal{R}_1} = \frac{1}{2} \begin{pmatrix} 0 & -\partial_y (\partial_y^2 + 2Q_y \partial_y^{-1} + 4Q) \\ 4\partial_y & -2\partial_y (2P + P_y \partial_y^{-1}) \end{pmatrix} \cdot \mathcal{R} = \mathcal{R}_\tau,
\]

which proves (2.25) and demonstrates that \( \mathcal{R} \) is a recursion operator for (2.26).

Finally, a direct and tedious calculation shows that \( \mathcal{R} \) has hereditary property (2.23). This, together with the fact that \( \mathcal{R} \) is a recursion operator for seed system (2.26) suffices to prove that \( \mathcal{R} \) is a recursion operator for each flow in the \( a^2 \text{CH} \) integrable hierarchy, which completes the proof of the lemma.

**Proof of Theorem 2.1.** We first compute the \( \tau \)-derivatives of the functions \( Q(\tau, y) \) and \( P(\tau, y) \) appearing in Liouville transformation (2.7). More precisely,

\[
Q_\tau = Q + Q_y \int \rho_t(t, \xi) d\xi = Q + Q_y \partial_y^{-1} \rho^{-1} \rho_t. \tag{2.27}
\]

On the other hand, we note that \( Q \) given in transformation (2.7) can be rewritten in the compact form

\[
Q = -\rho^{-3/2} \left( \frac{1}{4} - \partial_x^2 \right) \rho^{-1/2}. \tag{2.28}
\]

Using operator identity (2.19), we hence obtain

\[
Q_\tau = -\left( \rho^{-3/2} \left( \frac{1}{4} - \partial_x^2 \right) \rho^{-1/2} \right)_\tau = -\left( 2Q + \frac{1}{2} \partial_y^2 \right) \rho^{-1} \rho_t. \tag{2.28}
\]

As a result, combining (2.27) and (2.28), we obtain

\[
Q_\tau = -\left( \frac{1}{2} \partial_y^2 + 2Q \partial_y + Q_y \right) \partial_y^{-1} \rho^{-1} \rho_t.
\]

Similarly,

\[
P_\tau = -\rho^{-2} m_t - (P \partial_y + \partial_y P) \partial_y^{-1} \rho^{-1} \rho_t,
\]

and hence

\[
\begin{pmatrix} Q \\ P \end{pmatrix}_\tau = -\mathcal{R} A^{-1} \begin{pmatrix} m \\ \rho \end{pmatrix}_t, \tag{2.29}
\]

where \( A \) is defined in (2.22).

We next consider \( \text{CH}_{-n} \) system (2.13) for \( n \geq 1 \). The second identity in (2.19) and transformation formulas (2.7) imply that first negative flow (2.12) of the \( 2 \text{CH} \) hierarchy satisfies

\[
\begin{pmatrix} m \\ \rho \end{pmatrix}_t = \mathcal{K}_{-1} = A \begin{pmatrix} Q \\ P \end{pmatrix}_y.
\]

Therefore, we can write \( n \)th negative flow (2.13) as

\[
\begin{pmatrix} m \\ \rho \end{pmatrix}_t = \mathcal{K}_{-n} = (\mathcal{J} \mathcal{K}^{-1})^{n-1} \mathcal{K}_{-1} = (\mathcal{J} \mathcal{K}^{-1})^{n-1} A \begin{pmatrix} Q \\ P \end{pmatrix}_y, \quad n = 1, 2, \ldots. \tag{2.30}
\]

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Substituting this expression in (2.29) and using formula (2.21), we obtain

$$\begin{align*}
\left( \frac{Q}{P} \right)_\tau &= -\mathcal{R}A^{-1}(\mathcal{J}\mathcal{K}^{-1})^{n-1}A\left( \frac{Q}{P} \right)_y = -A^{-1}(\mathcal{J}\mathcal{K}^{-1})^nA\left( \frac{Q}{P} \right)_y = \mathcal{R}^n\mathcal{K}_1 = \mathcal{K}_{n+1}.
\end{align*}$$

Hence, for each \(n \geq 1\), if \((m(t,x), \rho(t,x))\) is a solution of \(2\text{CH}_{-n}\) system (2.13), then the corresponding \((Q(\tau,y), P(\tau,y))\) is a solution of \(a2\text{CH}_{n+1}\) system (2.15).

Moreover, substituting the positive flow \(2\text{CH}_{n+1}\) with \(0 \leq n \in \mathbb{Z}\)

$$\begin{align*}
\left( \frac{m}{\rho} \right)_t &= \mathcal{K}_{n+1} = -(\mathcal{K}\mathcal{J}^{-1})^n\left( \frac{m}{\rho} \right)_x
\end{align*}$$

(2.31)

in (2.29) yields

$$\begin{align*}
\left( \frac{Q}{P} \right)_\tau &= \mathcal{R}A^{-1}(\mathcal{K}\mathcal{J}^{-1})^n\left( \frac{m}{\rho} \right)_x, \quad n = 0, 1, \ldots
\end{align*}$$

Acting with the operator \(\mathcal{R}^n\) on both sides of the obtained equation and again using formula (2.21), we obtain

$$\begin{align*}
\mathcal{R}^n\left( \frac{Q}{P} \right)_\tau &= \mathcal{R}^{n+1}A^{-1}(\mathcal{K}\mathcal{J}^{-1})^n\left( \frac{m}{\rho} \right)_x = A^{-1}\mathcal{J}\mathcal{K}^{-1}\left( \frac{m}{\rho} \right)_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{align*}$$

where we use \(\mathcal{K}^{-1}(m_x, \rho_x)^T = (1, 0)^T\). We conclude that for each \(n \geq 0\), if \((m(t,x), \rho(t,x))\) is a solution of \(2\text{CH}_{n+1}\) system (2.31), then the corresponding \((Q(\tau,y), P(\tau,y))\) is a solution of \(a2\text{CH}_{-n}\) system (2.16).

Finally, if \((m(t,x), \rho(t,x))\) is a solution of \(2\text{CH}_0\) system, then the corresponding \((Q(\tau,y), P(\tau,y))\) satisfies

$$\begin{align*}
\left( \frac{Q}{P} \right)_\tau &= -\mathcal{R}A^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -Q_y \\ -P_y \end{pmatrix},
\end{align*}$$

which implies that \((Q(\tau,y), P(\tau,y))\) is a solution of the \(a2\text{CH}_1\) system.

### 2.3. Correspondence between the Hamiltonian functionals of the 2CH and a2CH hierarchies.

In this subsection, we investigate the correspondence between the Hamiltonian functionals appearing in the 2CH and a2CH hierarchies. We first consider the effect of the Liouville transformation on the Hamiltonians. For this, we need some relevant results found in [69], [68]. In [69], a general transformation formula for scalar Hamiltonians under a change of variables was provided, which can be directly generalized to the multicomponent case.

We consider an \(n\)-component system of equations

$$\begin{align*}
m_i = K(m), \quad m = (m_1(t,x), \ldots, m_n(t,x))^T,
\end{align*}$$

(2.32)

where \(K(m) = (K_1(m), \ldots, K_n(m))^T \in \mathbb{A}^n\) is an \(n\)-component differential function depending on the vector function \(m\) and its \(x\)-derivatives up to a given order. We assume that system (2.32) and another \(n\)-component system involving the dependent variable \(Q\) and having the form

$$\begin{align*}
Q_i = K_i(Q), \quad Q = (Q_1(\tau,y), \ldots, Q_n(\tau,y))^T,
\end{align*}$$

(2.33)

where \(K(Q) = (K_1(Q), \ldots, K_n(Q))^T \in \mathbb{A}^n\), are related by the coordinate transformation

$$\begin{align*}
y = \int^{\tau} \Lambda(m) \, d\xi \equiv I(x,m), \quad \tau = t, \quad Q_i = F_i(m), \quad i = 1, \ldots, n,
\end{align*}$$

(2.34)

where \(\Lambda(m), F_i(m) \in \mathbb{A}\).
Theorem 2.2. Let $D(m)$ be a Hamiltonian of system (2.32). If $m(t,x)$ and $Q(\tau,y)$ respectively satisfy (2.32) and (2.33) and are related by coordinate transformation (2.34), then the corresponding Hamiltonian $\overline{D}(Q)$ of system (2.33) has the form

$$\overline{D}(Q) = \Lambda(m)^{-1}TD(m)T^*, \quad (2.35)$$

where $T$ is an $n \times n$ matrix operator with the elements

$$(T)_{i,j} = \Lambda(m)D_{F_i,m_j} - D_xQ_iD_{I,m_j}, \quad i, j = 1, \ldots, n, \quad (2.36)$$

and $T^*$ is (formally) adjoint to $T$ in the space $L^2$. Here, $D_x$ denotes the total derivative with respect to $x$, and $D_{I,m_j}$ and $D_{F_i,m_j}$ are the Fréchet derivatives of $I$ and $F_i$ with respect to $m_j$.

To prove Theorem 2.2, we use the following lemma [68] and adapt the proof in the scalar case [69].

Lemma 2.3. Let $H(m)$ and $\overline{H}(Q)$ be two functionals related by change of variables (2.34). Then their corresponding variational derivatives

$$\delta H(m) = \left( \delta H \left( \frac{\partial}{\partial m_1}, \ldots, \frac{\partial}{\partial m_n} \right) \right)^T, \quad \delta \overline{H}(Q) = \left( \delta \overline{H} \left( \frac{\partial}{\partial Q_1}, \ldots, \frac{\partial}{\partial Q_n} \right) \right)^T$$

satisfy

$$\delta H(m) = T^* \delta \overline{H}(Q), \quad (2.37)$$

where $T^*$ is the formal adjoint of the operator $T$ given in (2.36).

Proof of Theorem 2.2. If

$$m_t = K(m) = D\delta H(m) \quad (2.38)$$

is a Hamiltonian system in the variable $m(t,x)$, then the corresponding evolution equation in the variable $Q(\tau,y)$ is also Hamiltonian,

$$Q_\tau = \overline{K}(Q) = \overline{D}\delta \overline{H}(Q), \quad (2.39)$$

with the Hamiltonian $\overline{D}$ given in (2.35). To prove this, we first write transformation (2.34) in the implicit form

$$B(Q,m) = (B_1(Q,m), \ldots, B_n(Q,m))^T = 0, \quad (2.40)$$

where $B_i(Q,m) = Q_i - F_i(m)$ for $i = 1, \ldots, n$. Taking the $t$-derivative of each equation in system (2.40), we then obtain

$$\sum_{k=1}^{n} (DB_{i,m_k}m_{k,t} + DB_{i,Q_k}Q_{k,\tau}) = 0, \quad i = 1, \ldots, n,$$

where $DB_{i,m_k}$ and $DB_{i,Q_k}$ are the respective Fréchet derivatives of $B_i$ with respect to $m_k$ and $Q_k$. It is convenient to write the above expression in the vector form:

$$B_m m_t + B_Q Q_\tau = 0.$$

From transformation (2.40), we obtain $B_m = -\Lambda(m)^{-1}T$. This immediately yields

$$Q_\tau = \Lambda(m)^{-1}T m_t.$$

Finally, combining (2.38) and (2.39) together and also formula (2.37), we derive (2.35), which completes the proof of the theorem.

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A direct application of the above theorem is to derive a compatible pair of Hamiltonians for the a2CH integrable hierarchy, which is generated by the recursion operator $\mathcal{K}$ in (2.14), from the Hamiltonians $\mathcal{K}$ and $\mathcal{J}$ given by (2.9) of the 2CH integrable hierarchy. Indeed, because of transformation (2.7), the function $\Lambda(\cdot)$ given by (2.34) becomes $\Lambda(m, \rho) = \rho$. Moreover, the operator defined in (2.36) is given by

$$T = -\frac{1}{2} \begin{pmatrix} 0 & \rho \partial_y + 4Q + 2Q_y \partial_y^{-1} \rho^{-1} \\ 2\rho^{-1} & \rho(4P + 2P_y \partial_y^{-1}) \rho^{-1} \end{pmatrix},$$

and its adjoint is

$$T^* = -\frac{1}{2} \begin{pmatrix} 0 & 2\rho^{-1} \\ \partial_y^2 + 4Q - 2\partial_y^{-1}Q_y & 4P - 2\partial_y^{-1}P_y \end{pmatrix}. \quad (2.41)$$

We conclude that the resulting Hamiltonians for the a2CH hierarchy have the forms

$$\mathcal{K} = \Lambda^{-1} T J T^* = \frac{1}{4} \begin{pmatrix} \mathcal{L} \partial_y^{-1} \mathcal{L} & 2\mathcal{L} \partial_y^{-1}(P \partial_y + \partial_y P) \\ 2(P \partial_y + \partial_y P) \partial_y^{-1} \mathcal{L} & 4(P \partial_y + \partial_y P) \partial_y^{-1}(P \partial_y + \partial_y P) + 2\mathcal{L} \end{pmatrix}, \quad (2.42)$$

$$\mathcal{J} = \Lambda^{-1} T \mathcal{K} T^* = \frac{1}{2} \begin{pmatrix} 0 & \mathcal{L} \\ \mathcal{L} & 2(P \partial_y + \partial_y P) \end{pmatrix},$$

where $\mathcal{L} = \partial_y^2 + 2Q \partial_y + 2\partial_y Q$. We can hence write the a2CH hierarchy in bi-Hamiltonian form. In the positive direction,

$$\begin{pmatrix} Q \\ P \end{pmatrix}_\tau = \mathcal{K}_n = \mathcal{K} \delta \mathcal{H}_{n-1} = \mathcal{J} \delta \mathcal{H}_n, \quad \delta \mathcal{H}_n = \left( \frac{\delta \mathcal{H}_n}{\delta Q}, \frac{\delta \mathcal{H}_n}{\delta P} \right)^T, \quad n \geq 1,$$

where, for example,

$$\mathcal{H}_1 = -\int P \, dy, \quad \mathcal{H}_2 = -\int \left( \frac{1}{2} P^2 + 2Q \right) dy. \quad (2.43)$$

**Remark 2.1.** It follows from (2.14) and (2.42) that

$$\mathcal{K} = \mathcal{K} \mathcal{J}^{-1} = \frac{1}{2} \begin{pmatrix} 0 & \partial_y^2 + 2Q + Q_y \partial_y^{-1} \\ -4 & 4P + 2P_y \partial_y^{-1} \end{pmatrix} = \Lambda^{-1} T \mathcal{J} \mathcal{K}^{-1} T^{-1} \Lambda.$$

This formula provides an alternative decomposition of the recursion operator $\mathcal{K}$, differing from the one given in Lemma 2.2.

Having two pairs of Hamiltonians $\{\mathcal{K}, \mathcal{L}\}$ and $\{\mathcal{K}, \mathcal{J}\}$, based on the Magri theorem, we can now recursively construct an infinite hierarchy of Hamiltonian functionals. They are determined by the recurrence relations

$$\mathcal{K} \delta \mathcal{H}_{n-1} = \mathcal{J} \delta \mathcal{H}_n, \quad n \in \mathbb{Z}, \quad (2.44)$$

for the 2CH hierarchy and

$$\mathcal{K} \delta \mathcal{H}_{n-1} = \mathcal{J} \delta \mathcal{H}_n, \quad n \in \mathbb{Z}, \quad (2.45)$$

for the a2CH hierarchy. To establish a correspondence between the hierarchies of Hamiltonian functionals $\mathcal{H}_n$ and $\mathcal{H}_n$, we need a formula indicating how the correspondence between their variational derivatives is constructed.
Lemma 2.4. Let \( \{\mathcal{H}_n\} \) and \( \{\mathcal{H}_n\} \) be the respective hierarchies of Hamiltonian functionals determined by recurrence relations (2.44) and (2.45). Then their corresponding variational derivatives satisfy

\[
\delta \mathcal{H}_{-(n+1)}(m, \rho) = -\kappa^{-1} \mathcal{J} \delta \mathcal{H}_n(Q, P), \quad 0 \neq n \in \mathbb{Z}. \tag{2.46}
\]

Proof. We first prove (2.46) for \( n \geq 1 \) by induction. Because in the framework of the 2CH hierarchy, \( \mathcal{H}_1 = \mathcal{H}_C(m, \rho) \) with Casimir functional (2.11), we have \( \delta \mathcal{H}_1 = \delta \mathcal{H}_C = (\rho^{-1}, -m\rho^{-2})^T \). Using identities (2.19) and formula (2.44), we obtain

\[
\delta \mathcal{H}_2 = \kappa^{-1} \mathcal{J} \delta \mathcal{H}_1 = \kappa^{-1} \mathcal{A}(Q_y, P_y)^T = -\kappa^{-1} \mathcal{A} \mathcal{J} \delta \mathcal{H}_1,
\]

which proves that (2.46) holds for \( n = 1 \). We now suppose that (2.46) holds for \( n = k \) with some integer \( k \geq 1 \), i.e.,

\[
\delta \mathcal{H}_{-(k+1)} = -\kappa^{-1} \mathcal{A} \mathcal{J} \delta \mathcal{H}_k.
\]

Then for \( n = k + 1 \),

\[
\delta \mathcal{H}_{-(k+2)} = \kappa^{-1} \mathcal{J} \delta \mathcal{H}_{-(k+1)} = -\kappa^{-1} \mathcal{J} \kappa^{-1} \mathcal{A} \mathcal{J} \delta \mathcal{H}_k =
\]

\[
= -\kappa^{-1} \mathcal{J} \kappa^{-1} \mathcal{A} \mathcal{J} \mathcal{A} \mathcal{J} \delta \mathcal{H}_{k+1} = -\kappa^{-1} \mathcal{A} \mathcal{J} \delta \mathcal{H}_{k+1},
\]

where we use relation (2.21) with \( n = 1 \). This proves (2.46) for any integer \( n \geq 1 \).

Further, for \( n = -1 \), we first note that (2.46) is equivalent to

\[
\delta \mathcal{H}_{-1} = -\mathcal{J}^{-1} \mathcal{A}^{-1} \kappa \delta \mathcal{H}_0. \tag{2.47}
\]

Because \( \mathcal{H}_{-1} \) is a Casimir functional of the Hamiltonian \( \mathcal{K} \), to prove (2.47), it suffices to verify that \( h = -\mathcal{J}^{-1} \mathcal{A}^{-1} \kappa \delta \mathcal{H}_0 \) satisfies \( \mathcal{K} h = 0 \). Indeed, again using (2.21) with \( n = 1 \), we obtain

\[
\mathcal{K} h = \mathcal{A}^{-1} \mathcal{J} \mathcal{K}^{-1} \mathcal{A} \mathcal{A}^{-1} \kappa \delta \mathcal{H}_0 = \mathcal{A}^{-1} \mathcal{J} \delta \mathcal{H}_0 = 0,
\]

where we use the fact that \( \delta \mathcal{H}_0 \) is a constant vector. Now supposing that (2.46) holds for \( n = -k \leq -1 \), we conclude that

\[
\delta \mathcal{H}_{-(k-1+1)} = \delta \mathcal{H}_k = \mathcal{J}^{-1} \kappa \delta \mathcal{H}_{k-1} = -\mathcal{J}^{-1} \mathcal{A} \mathcal{J} \delta \mathcal{H}_{-k} =
\]

\[
= -\mathcal{J}^{-1} \mathcal{A} \mathcal{K} \delta \mathcal{H}_{-k-1} = -\kappa^{-1} \mathcal{A} \mathcal{J} \delta \mathcal{H}_{-k-1},
\]

which proves (2.46) for any \( n \leq -1 \) by induction and thus completes the proof of the entire lemma.

Finally, if it is given that \( (m(t, x), \rho(t, x)) \) and \( (Q(\tau, y), P(\tau, y)) \) are related by transformation (2.7), then we define the functional \( \mathcal{G}_n(Q, P) \equiv \mathcal{H}_n(m, \rho) \). A direct application of Lemma 2.3 yields

\[
\delta \mathcal{H}_n(m, \rho) = \mathcal{T}^* \delta \mathcal{G}_n(Q, P),
\]

where the operator \( \mathcal{T}^* \) is defined in (2.41). On the other hand, by Lemma 2.4, we have

\[
\delta \mathcal{H}_n(m, \rho) = -\kappa^{-1} \mathcal{J} \delta \mathcal{H}_{-(n+1)}(Q, P),
\]

which together with (2.44) and relation (2.21) yields

\[
\delta \mathcal{H}_n(m, \rho) = -\mathcal{J}^{-1} \mathcal{A} \mathcal{J} \delta \mathcal{H}_{-n}(Q, P).
\]

Moreover, a direct calculation shows that the operator \( \mathcal{T}^* \) admits the decomposition \( \mathcal{J}^{-1} \mathcal{A} \mathcal{J} = -\mathcal{T}^* \), which implies that

\[
\mathcal{H}_n(m, \rho) = \mathcal{G}_n(Q, P) = \mathcal{H}_{-n}(Q, P).
\]

We have thus proved the following theorem.
Theorem 2.3. Under Liouville transformation (2.7), for each nonzero integer \( n \), the Hamiltonian functionals \( H_n(m, \rho) \) of the 2CH hierarchy given by (2.44) are related to the Hamiltonian functionals \( \overline{H}_n(Q, P) \) of the a2CH hierarchy given by (2.45):

\[
H_n(m, \rho) = \overline{H}_{-n}(Q, P).
\]

For instance, in the case \( n = -1 \), using (2.7), we can verify that

\[
H_{-1}(m, \rho) = \int \frac{m}{\rho} \, dx = - \int P \, dy = \overline{H}_1(Q, P).
\]

In the case \( n = -2 \), it follows from (2.43) and (2.46) that

\[
\delta H_{-2} = \begin{pmatrix} \frac{m}{\rho^3} \\ -2Q + \frac{3}{2}P^2 \end{pmatrix}^T
\]

and hence

\[
H_{-2}(m, \rho) = - \int \rho \left( \frac{1}{2}P^2 + 2Q \right) \, dx = - \int \left( \frac{1}{2}P^2 + 2Q \right) \, dy = \overline{H}_2(Q, P).
\]

Both of these results agree with Theorem 2.3.

3. Liouville correspondence for the GX hierarchy

3.1. Liouville transformation for the isospectral problem of the GX system. The Lax-pair formulation for the GX system

\[
\begin{align*}
mt + 3vu_xm + uv m_x &= 0, \\
n_t + 3uv_xn + uvn_x &= 0,
\end{align*}
\]

has the form of a system for the vector function \( \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \) [58],

\[
\Psi_x = \begin{pmatrix} 0 & \lambda m & 1 \\ 0 & 0 & \lambda n \\ 1 & 0 & 0 \end{pmatrix} \Psi,
\]

\[
\Psi_t = \begin{pmatrix} -u_xv & \lambda^{-1}u_x - \lambda uv m & u_xv_x \\ \lambda^{-1}v & -\lambda^{-2} + u_xv - uv_x & -\lambda uvn - \lambda^{-1}v_x \\ -uv & \lambda^{-1}u & uv_x \end{pmatrix} \Psi.
\]

It was shown in [60] that the reciprocal transformation

\[
dy = \Delta^{1/3} \, dx - \Delta^{1/3} \, uv \, dt, \\
d\tau = dt
\]

(here and hereafter, \( \Delta = mn \)) converts isospectral problem (3.2) into a system for the vector function \( \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \),

\[
\Phi_y = \begin{pmatrix} 0 & \lambda & Q \\ 0 & P & \lambda \\ 1 & 0 & 0 \end{pmatrix} \Phi,
\]

\[
\Phi_x = \frac{1}{2} \begin{pmatrix} A & 2\lambda^{-1}(p_y + pP) & p + q \\ 2\lambda^{-1}q & A - 2\lambda^{-2} & 2\lambda^{-1}(Pq - q_y) \\ 0 & 2\lambda^{-1}p & A \end{pmatrix} \Phi,
\]

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where
\[ Q = \frac{1}{6} \Delta^{-5/3} \Delta_{xx} - \frac{7}{36} \Delta^{-8/3} \Delta_x^2 + \Delta^{-2/3}, \quad P = \frac{1}{2} m^{-4/3} n^{2/3} \left( \frac{m}{n} \right)_x, \] (3.5)
and
\[ A = q_y p - q_p y - 2 q P, \quad q = v m^{2/3} n^{-1/3}, \quad p = u m^{-1/3} n^{2/3}. \]

The compatibility condition \( \Phi_{yt} = \Phi_{ty} \) yields the integrable system
\[ Q_t = \frac{3}{2} (q_y + p_y) - (q - p) P, \quad P_t = \frac{3}{2} (q - p), \]
\[ p_{yy} + 2 p_y P + p P_y + p P^2 - p Q + 1 = 0, \]
\[ q_{yy} - 2 q_y P - q P_y + q P^2 - q Q + 1 = 0, \] (3.6)
which admits (3.4) as a Lax pair.

It turns out that there exists a Liouville correspondence between GX system (3.1) and system (3.6) in the sense that their respective isospectral problems (3.2) and (3.4) are related by transformations (3.3) and (3.5). In addition, in view of the spectral structure of the time evolution component of (3.4), reduced system (3.6) can be viewed as a negative flow of an integrable hierarchy, namely, the associated GX (aGX) integrable hierarchy.

### 3.2. Liouville correspondence between the GX and aGX integrable hierarchies.

We consider the transformation
\[ y = \int^x \Delta^{1/3} (t, \xi) d\xi, \quad \tau = t, \]
\[ Q = \Delta^{-2/3} + \frac{1}{6} \Delta^{-5/3} \Delta_{xx} - \frac{7}{36} \Delta^{-8/3} \Delta_x^2, \quad P = \frac{1}{2} m^{-4/3} n^{2/3} \left( \frac{m}{n} \right)_x. \] (3.7)

System (3.1) can be written in the bi-Hamiltonian form [61]
\[ \left( \frac{m}{n} \right)_t = \mathcal{K} \delta \mathcal{H}_1 (m, n) = \mathcal{J} \delta \mathcal{H}_2 (m, n), \]
where the compatible Hamiltonians are
\[ \mathcal{K} = \frac{3}{2} \left( \begin{array}{cc} 3 m^{1/3} \partial_x m^{2/3} \Omega^{-1} m^{2/3} \partial_x m^{1/3} + m \partial_x^{-1} m & 3 m^{1/3} \partial_x m^{2/3} \Omega^{-1} m^{2/3} \partial_x n^{1/3} - m \partial_x^{-1} n \\ 3 n^{1/3} \partial_x n^{2/3} \Omega^{-1} m^{2/3} \partial_x m^{1/3} - n \partial_x^{-1} m & 3 n^{1/3} \partial_x n^{2/3} \Omega^{-1} n^{2/3} \partial_x n^{1/3} + 3 n \partial_x^{-1} n \end{array} \right), \]
\[ \mathcal{J} = \left( \begin{array}{cc} 0 & \partial_x^2 - 1 \\ 1 - \partial_x^2 & 0 \end{array} \right), \] (3.8)
where \( \Omega = \partial_x^3 - 4 \partial_x \) and
\[ \mathcal{H}_1 (m, n) = \int un \, dx, \quad \mathcal{H}_2 (m, n) = \int (u_x v - u v_x) \, un \, dx \]
are the initial Hamiltonian functionals. Based on the bi-Hamiltonian structure of the GX system, we can construct the full GX integrable hierarchy by applying the resulting hereditary recursion operator \( \mathcal{R} = \mathcal{K} \mathcal{J}^{-1} \) to the particular seed system
\[ \left( \frac{m}{n} \right)_t = \mathbf{G}_1 (m, n) = \left( \frac{-m}{n} \right). \]
Hence, the \( \ell \)th member in the positive direction becomes
\[
\binom{m}{n}_t = G_\ell(m, n) = R^{\ell-1}G_1(m, n), \quad \ell = 1, 2, \ldots ,
\] (3.9)
and GX system (3.1) is exactly the second positive flow. In analogy with the 2CH hierarchy, the fact that the trivial symmetry \( G_0 = (0, 0)^T \) satisfies the equation \( RG_0 = G_1 \) implies that in the opposite direction, the negative flow is generated by the Casimir system. We note that the Hamiltonian \( K \) admits the Casimir functional
\[
\mathcal{H}_C(m, n) = 3\int \Delta^{1/3} dx
\] (3.10)
with the variational derivative \( \delta \mathcal{H}_C = (m - 2/3, n^{1/3} - 2/3)^T \). Therefore, the \( \ell \)th negative flow of the GX hierarchy is written as
\[
\binom{m}{n}_t = G_{-\ell}(m, n) = (JK_{\ell-1})J \delta \mathcal{H}_C, \quad \ell = 1, 2, \ldots .
\] (3.11)

Turning to the aGX integrable hierarchy, based on Theorem 2.2, from the Hamiltonian pair \( K \) and \( J \) introduced in (3.8) for the GX system, we can easily construct two Hamiltonians for transformed system (3.6) by applying Liouville transformation (3.7). The resulting Hamiltonians for system (3.6) were presented in [60]:
\[
\kappa = \Gamma \begin{pmatrix} 0 & \Theta \\ -\Theta^* & 0 \end{pmatrix} \Gamma^*, \quad \mathcal{J} = \frac{1}{2} \begin{pmatrix} \xi & 0 \\ 0 & -3\partial_y \end{pmatrix},
\] (3.12)
where the matrix operator \( \Gamma \) and the operators \( \Theta \) and \( \xi \) are defined by
\[
\Gamma = -\frac{1}{6} \begin{pmatrix} \xi \partial_y^{-1} & \xi \partial_y^{-1} \\ (3\partial_y^2 - 2\partial_y P)\partial_y^{-1} & -(3\partial_y^2 + 2\partial_y P)\partial_y^{-1} \end{pmatrix},
\] (3.13)
\[
\Theta = \partial_y^2 + P\partial_y + \partial_y P + P^2 - Q, \quad \xi = \partial_y^3 - 2Q\partial_y - 2\partial_y Q.
\]
Therefore, the aGX integrable hierarchy can be obtained by applying the resulting hereditary recursion operator \( \kappa = \kappa \mathcal{J}^{-1} \) to the seed system
\[
\begin{pmatrix} Q \\ P \end{pmatrix}_\tau = G_1 = -\begin{pmatrix} Q \\ P \end{pmatrix}_y.
\]
More precisely, the \( \ell \)th member in the positive direction has the form
\[
\begin{pmatrix} Q \\ P \end{pmatrix}_\tau = \kappa^\ell = R^{\ell-1}G_1, \quad \ell = 1, 2, \ldots .
\] (3.14)
For instance, the second positive flow of the aGX hierarchy (for \( \ell = 2 \)) is
\[
Q_\tau = \frac{1}{9} \left( P_{yyyy} - 2(QP)_{yyy} - 2(QP_{yy})_y - 2QP_{yy} + 6(Q^2 P)_y + 2Q^2 P_y - \frac{2}{9}((P^3)_{yyy} - 4(QP^3)_y + 2Q^3 P^3) \right),
\]
\[
P_\tau = \frac{2}{9} \left( PP_{yy} - \frac{3}{2} Q_{yy} + \frac{1}{2} P^2_y - \frac{5}{18} P^4 + QP^2 + \frac{3}{2} Q^2 \right)_y.
\]
The $\ell$th negative flow has the form
\[
\mathcal{R}_\ell \left( \begin{array}{c} Q \\ P \end{array} \right) = \mathcal{C}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \ell = 1, 2, \ldots.
\] (3.15)

Similarly to Theorem 2.1, we let $GX_\ell$, $GX_{-\ell}$ and $aGX_\ell$, $aGX_{-\ell}$ denote the $\ell$th positive and negative flows of the $GX$ and $aGX$ hierarchies. The scheme of the Liouville correspondence between the $GX$ and $aGX$ integrable hierarchies is described in the following theorem.

**Theorem 3.1.** Under Liouville transformation (3.7), for each integer $\ell \geq 1$,
1. If $(m(t, x), n(t, x))$ is a solution of $GX_\ell$ system (3.9), then $(Q(\tau, y), P(\tau, y))$ is a solution of $aGX_{-\ell}$ system (3.15), and
2. If $(m(t, x), n(t, x))$ is a solution of $GX_{-\ell}$ system (3.11), then $(Q(\tau, y), P(\tau, y))$ is a solution of $aGX_{\ell+1}$ system (3.14).

We need two preliminary lemmas.

**Lemma 3.1.** Let $(m(t, x), n(t, x))$ and $(Q(\tau, y), P(\tau, y))$ be related by transformation (3.7). Then we have the operator identities
\[
\Delta^{-1/2}(1 - \partial_x^2)\Delta^{-1/6} = Q - \partial_y^2, \quad \Delta^{-2/3} \Omega \Delta^{-1/3} = \mathcal{E},
\] (3.16)
where $\Omega = \partial_x^2 - 4\partial_x$ and $\mathcal{E}$ are given in (3.13).

**Proof.** In view of transformation (3.7), we have
\[
\partial_x = \Delta^{1/3}\partial_y,
\] (3.17)
whence it follows that
\[
\partial_x^2 \Delta^{-1/6} = \frac{7}{36} \Delta^{-13/6} \Delta^2_x - \frac{1}{6} \Delta^{-7/6} \Delta_{xx} - \frac{1}{3} \Delta^{-7/6} \Delta_x \partial_x + \Delta^{-1/6} \partial_x^2.
\]
Substituting this expression in $\Delta^{-1/2}(1 - \partial_x^2)\Delta^{-1/6}$ yields the first identity in (3.16). From (3.17), we obtain
\[
\partial_x^2 \Delta^{-1/3} = \frac{4}{9} \Delta^{-1/3} \Delta_x^2 - \frac{1}{3} \Delta^{-4/3} \Delta_{xx} - \frac{1}{3} \Delta^{-1} \Delta_x \partial_y + \Delta^{1/3} \partial_y^2.
\]
Straightforward computation then shows that
\[
\Delta^{-2/3} \Omega \Delta^{-1/3} = \Delta^{-2/3} \partial_x (\partial_x^2 - 4) \Delta^{-1/3} = \Delta^{-2/3} \partial_x \left( -4 \Delta^{-1/3} + \frac{4}{9} \Delta^{-7/3} \Delta_x^2 - \frac{1}{3} \Delta^{-4/3} \Delta_{xx} - \frac{1}{3} \Delta^{-1} \Delta_x \partial_y + \Delta^{1/3} \partial_y^2 \right) = 4 \Delta^{-2} \Delta_x - \frac{28}{27} \Delta^{-4} \Delta_x^3 + \frac{4}{3} \Delta^{-3} \Delta_x \Delta_{xx} + \frac{1}{3} \Delta^{-2} \Delta_{xx} + \left( -4 \Delta^{-2/3} + \frac{7}{9} \Delta^{-8/3} \Delta_x^2 - \frac{2}{3} \Delta^{-5/3} \Delta_{xx} \right) \partial_y + \partial_y^3.
\]
Using the formula for $Q$ in (3.7), we derive the second identity in (3.16). Again in view of (3.17), we further have
\[
m^{-4/3} n^{2/3} \partial_x mn^{-1} = \frac{n}{m} \partial_y m = \frac{n}{m} \left( \frac{m}{n} \right)_y + \partial_y.
\]
Hence, using the formula for $P$ in (3.7), we obtain the third identity in (3.16).
Lemma 3.2. We set
\[ E = \begin{pmatrix} \partial_y & 0 \\ 0 & \partial_y \end{pmatrix}, \quad B = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}. \] (3.18)

We then have the operator identity
\[ B^{-1}(JK^{-1})^\ell B = (-1)\ell B(U^\ell)B^{-1} \] (3.19)
for each positive integer \( \ell \), where
\[ U = \begin{pmatrix} \mathcal{E} & \mathcal{E} \\ \mathcal{F} + 3\partial_y^2 & \mathcal{F} - 3\partial_y^2 \end{pmatrix}, \quad \mathcal{F} = -(2P\partial_y + 2P_y), \]
(3.20)
\[ V = \frac{1}{54} \begin{pmatrix} 3\partial_y^{-1}\Theta\partial_y^{-1} & \partial_y^{-1}\Theta\partial_y^{-1}(2P - 3\partial_y) \\ -3\partial_y^{-1}\Theta^*\partial_y^{-1} & -\partial_y^{-1}\Theta^*\partial_y^{-1}(2P + 3\partial_y) \end{pmatrix}, \]
\( \mathcal{E} \) and \( \Theta \) are defined in (3.13), and \( K \) and \( J \) are the compatible Hamiltonians for the GX system given in (3.8).

Proof. The proof is by induction. We first prove (3.19) for \( \ell = 1 \). For any constants \( \alpha \) and \( \beta \), a direct calculation using transformation (3.7) yields the operator identities
\[ (\partial_y - 2\alpha P)(m/n)^\alpha = (m/n)^\alpha \partial_y, \quad (m/n)^{\beta/2} (\partial_y + \beta P)(m/n)^{\beta/2} = \partial_y. \] (3.21)

We note that (3.19) with \( \ell = 1 \) is equivalent to
\[ B^{-1}J = -EUVB^{-1} = \begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{pmatrix}. \]

Using the first operator identity in (3.16), we derive
\[ B^{-1}J = \begin{pmatrix} 0 & m^{-1}(\partial_y^2 - 1) \\ n^{-1}(1 - \partial_x^2) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \left(-\frac{m}{n}\right)^{-1/2}(Q - \partial_y^2)\Delta^{1/6} \\ \left(\frac{m}{n}\right)^{1/2}(Q - \partial_y^2)\Delta^{1/6} & 0 \end{pmatrix}. \]

On the other hand, a direct calculation shows that
\[ \mathcal{R}_{11} = \frac{1}{18} \Theta\partial_y^{-1} \left\{ \mathcal{E}\partial_y^{-1}(3\partial_x + m^{-1}m_x + n^{-1}n_x)\Omega^{-1}(3m\partial_x + m_x) - \\
- \left(\partial_y - \frac{2}{3}P\right) \left[ \left(\frac{1}{6}\mathcal{F}\partial_y^{-1} + \frac{1}{2}\partial_y\right) \left(\frac{3}{2}\partial_x + m^{-1}m_x\right)\Omega^{-1}(3m\partial_x + m_x) + \frac{3}{2}\partial_x^{-1}m \right] + \\
+ \left(\frac{1}{6}\mathcal{F}\partial_y^{-1} - \frac{1}{2}\partial_y\right) \left(\frac{3}{2}\partial_x + n^{-1}n_x\right)\Omega^{-1}(3m\partial_x + m_x) - \frac{3}{2}\partial_x^{-1}m \right\} = \\
= \frac{1}{2} \Theta\partial_y^{-1} \left\{ n^{-2/3}\partial_x m^{1/3} - \left(\partial_y - \frac{2}{3}P\right)m^{2/3}n^{-1/3} \right\}. \]
Using the first equation in (3.21), we then obtain \( R_{11} = 0 \). Next,
\[
R_{12} = \frac{1}{18} \Theta \partial_y^{-1} \left\{ \mathcal{E} \partial_y^{-1} (3 \partial_x + m^{-1} m_x + n^{-1} n_x) \Omega^{-1} (3 n \partial_x + n_x) - \right. \\
\left. \left( \partial_y - \frac{2}{3} P \right) \left[ \left( \frac{1}{6} \mathcal{F} \partial_y^{-1} + \frac{1}{2} \partial_y \right) \left( \frac{3}{2} \partial_x + m^{-1} m_x \right) \Omega^{-1} (3 n \partial_x + n_x) - \frac{3}{2} \partial_x^{-1} m \right] + \\
+ \left( \frac{1}{6} \mathcal{F} \partial_y^{-1} - \frac{1}{2} \partial_y \right) \left( \frac{3}{2} \partial_x + n^{-1} n_x \right) \Omega^{-1} (3 n \partial_x + n_x) - \frac{3}{2} \partial_x^{-1} m \right) \right\} = \\
\frac{1}{2} \Theta \partial_y^{-1} \left( \left( \partial_y - \frac{2}{3} P \right) m^{-1/3} n^{2/3} + m^{-2/3} \partial_x n^{1/3} \right) = \\
\frac{1}{2} \Theta \partial_y^{-1} \left( \left( \frac{m}{n} \right)^{-1/3} \partial_y \left( \frac{m}{n} \right)^{-1/6} + \left( \frac{m}{n} \right)^{1/3} \partial_y \left( \frac{m}{n} \right)^{-5/6} \right) \Delta^{1/6} = \\
\Theta \partial_y^{-1} \left( \frac{m}{n} \right)^{-1/2} (\partial_y - P) \Delta^{1/6}.
\]

We now claim that
\[
\Theta \partial_y^{-1} \left( \frac{m}{n} \right)^{-1/2} (\partial_y - P) \left( \frac{m}{n} \right)^{1/2} = - \left( \frac{m}{n} \right)^{-1/2} (Q - \partial_y^2) \left( \frac{m}{n} \right)^{1/2}.
\]

Indeed, based on the expression for \( Q \) given in (3.7), a direct calculation yields
\[
\left( \frac{m}{n} \right)^{-1/2} (Q - \partial_y^2) \left( \frac{m}{n} \right)^{1/2} = - \Theta,
\]
while it follows from the second equation in (3.21) that
\[
\Theta \partial_y^{-1} \left( \frac{m}{n} \right)^{-1/2} (\partial_y - P) \left( \frac{m}{n} \right)^{1/2} = \Theta,
\]
which proves the equality.

Further calculations yield
\[
R_{21} = - \frac{1}{18} \Theta^* \partial_y^{-1} \left\{ \mathcal{E} \partial_y^{-1} (3 \partial_x + m^{-1} m_x + n^{-1} n_x) \Omega^{-1} (3 m \partial_x + m_x) + \\
\left( \partial_y + \frac{2}{3} P \right) \left[ \left( \frac{1}{6} \mathcal{F} \partial_y^{-1} + \frac{1}{2} \partial_y \right) \left( \frac{3}{2} \partial_x + m^{-1} m_x \right) \Omega^{-1} (3 m \partial_x + m_x) + \frac{3}{2} \partial_x^{-1} m \right] + \\
+ \left( \frac{1}{6} \mathcal{F} \partial_y^{-1} - \frac{1}{2} \partial_y \right) \left( \frac{3}{2} \partial_x + n^{-1} n_x \right) \Omega^{-1} (3 m \partial_x + m_x) - \frac{3}{2} \partial_x^{-1} m \right) \right\} = \\
- \Theta^* \partial_y^{-1} \left( \frac{m}{n} \right)^{1/2} (\partial_y + P) \Delta^{1/6} = \left( \frac{m}{n} \right)^{1/2} (Q - \partial_y^2) \Delta^{1/6}.
\]
and
\[ R_{22} = -\frac{1}{18} \Theta^* \partial_y^{-1} \left\{ \mathcal{E} \partial_y^{-1} (3\partial_x + m^{-1} \partial_x + n^{-1} \partial_x) \Omega^{-1} (3n \partial_x + n_x) + \right. \]
\[ + \left. \left( \partial_y + \frac{2}{3} P \right) \left( \left( \frac{1}{6} \mathcal{F} \partial_y^{-1} + \frac{1}{2} \partial_y \right) \left( \left( \frac{3}{2} \partial_x + m^{-1} \partial_x \right) \Omega^{-1} (3n \partial_x + n_x) - \frac{3}{2} \partial_y^{-1} \right) + \right. \right. \]
\[ + \left. \left. \left( \frac{1}{6} \mathcal{F} \partial_y^{-1} - \frac{1}{2} \partial_y \right) \left( \left( \frac{3}{2} \partial_x + n^{-1} \partial_x \right) \Omega^{-1} (3n \partial_x + n_x) + \frac{3}{2} \partial_y^{-1} \right) \right) \right\} = \]
\[- \frac{1}{2} \Theta^* \partial_y^{-1} \left[ m^{-2/3} \partial_x n^{1/3} - \left( \frac{2}{3} P + \partial_y \right) \left( \frac{m}{n} \right)^{-1/3} n^{1/3} \right] = 0, \]
where we use formulas (3.21) and the relation
\[ \left( \frac{m}{n} \right)^{1/2} \left( Q - \partial_y^2 \right) \left( \frac{m}{n} \right)^{-1/2} = -\Theta^*. \]

Combining the obtained results, we conclude that (3.19) holds for \( \ell = 1 \). The induction procedure then shows that it holds in the general case. The lemma is thus proved.

**Remark 3.1.** It is remarkable that the matrix operators \( U \) and \( V \) defined by (3.20) satisfy the composition identity
\[ \mathcal{K} = UV. \quad (3.22) \]
Indeed, we define the matrix operator
\[ \Xi = \begin{pmatrix} 0 & \Theta \\ -\Theta^* & 0 \end{pmatrix} \Gamma^*. \quad (3.23) \]
It follows from definitions (3.13) and (3.20) of the operators \( \Gamma \) and \( U \) that
\[ \Gamma = -\frac{1}{6} U E^{-1}. \]
Hence, by virtue of (3.12) and (3.23), we obtain
\[ \mathcal{K} = -\frac{1}{6} U E^{-1} \Xi. \quad (3.24) \]
On the other hand, the identity
\[ \Xi \mathcal{F}^{-1} = \frac{1}{3} \begin{pmatrix} 0 & \Theta \\ -\Theta^* & 0 \end{pmatrix} \Gamma^* \begin{pmatrix} -6 \mathcal{E}^{-1} & 0 \\ 0 & -2 \partial_y^{-1} \end{pmatrix} = \]
\[ = -\frac{1}{9} \begin{pmatrix} 3 \Theta \partial_y^{-1} & \Theta \partial_y^{-1} (2P - 3 \partial_y) \\ -3 \Theta^* \partial_y^{-1} & -\Theta^* \partial_y^{-1} (2P + 3 \partial_y) \end{pmatrix} \]
yields \( E^{-1} \Xi = -6 V \mathcal{F} \). Substituting the result in (3.24), we obtain \( \mathcal{K} = \mathcal{K} \mathcal{F}^{-1} = U V \), which proves our claim.

Formula (3.22) can be viewed as a new operator factorization for the recursion operator \( \mathcal{K} \), which does not coincide with the decomposition \( \mathcal{K} = \mathcal{K} \mathcal{F}^{-1} \) using the pair of Hamiltonians. In the proof of Theorem 3.1 below, we see that this new factorization plays a key role in identifying the systems transformed from the negative and positive flows of the GX hierarchy into the respective positive and negative flows of the aGX hierarchy.
Proof of Theorem 3.1.} Directly calculating the $t$-derivatives of $Q(\tau, y)$ and $P(\tau, y)$ in accordance with Liouville transformation (3.7) yields

$$\left(\begin{array}{c} Q \\ P \end{array}\right)_\tau = \frac{1}{6} \mathcal{U} \mathbf{E}^{-1} \mathbf{B}^{-1} \begin{pmatrix} m \\ n \end{pmatrix},$$

(3.25)

where $\mathcal{U}$, $\mathbf{E}$, and $\mathbf{B}$ are the matrix operators defined in Lemma 3.2.

We suppose that $(m(t, x), n(t, x))$ is a solution of GX hierarchy (3.11) for some integer $\ell \geq 1$. Using formula (3.19) and (3.25), from (3.11), we then obtain

$$\left(\begin{array}{c} Q \\ P \end{array}\right)_\tau = \frac{1}{6} \mathcal{U} \mathbf{E}^{-1} \mathbf{B}^{-1} (\mathcal{K} \mathcal{J}^{-1})^\ell \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{1}{6} (-1)^{\ell} \mathcal{U} (\mathcal{V} \mathcal{U})^\ell \mathbf{E}^{-1} \mathbf{B}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{1}{6} (-1)^{\ell} (\mathcal{U} \mathcal{V})^\ell \mathbf{U} \begin{pmatrix} c \\ c \end{pmatrix} = \frac{2}{3} (-1)^{\ell} c (\mathcal{U} \mathcal{V})^\ell \mathbf{U},$$

where $c$ is the integration constant. If we choose $c = 3(-1)^{\ell}/2$, then the equality $\mathcal{R} = \mathcal{U} \mathcal{V}$ immediately shows that the corresponding $(Q(\tau, y), P(\tau, y))$ satisfies aGX hierarchy (3.14).

Further, we suppose that $(m(t, x), n(t, x))$ is a solution of GX system (3.9) for some integer $\ell \geq 1$. We see that after transformation (3.7), $(Q(\tau, y), P(\tau, y))$ satisfies

$$\mathcal{R}^\ell \left(\begin{array}{c} Q \\ P \end{array}\right)_\tau = \frac{1}{6} \mathcal{R}^{\ell-1} (\mathcal{K} \mathcal{J}^{-1})^\ell \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{1}{6} \mathcal{U} \mathcal{V}^\ell \mathbf{E}^{-1} \mathbf{B}^{-1} (\mathcal{K} \mathcal{J}^{-1})^\ell \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{1}{6} (-1)^{\ell} \mathcal{U} \mathcal{V}^\ell \mathbf{E}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where we use Lemma 3.2 and again the same factorization $\mathcal{R} = \mathcal{U} \mathcal{V}$. It hence follows that $(Q(\tau, y), P(\tau, y))$ is a solution of the aGX system, which completes the proof of the theorem.

### 3.3. Correspondence between the Hamiltonian functionals of the GX and aGX hierarchies.

We now study the correspondence between the Hamiltonian functionals involved in the GX and aGX hierarchies. Based on the pair of Hamiltonians $\mathcal{K}$ and $\mathcal{J}$ in (3.8) for the GX hierarchy and the pair of Hamiltonians $\mathcal{K}$ and $\mathcal{J}$ in (3.12) for the aGX hierarchy, we determine their respective infinite hierarchies of Hamiltonian functionals $\{\mathcal{H}_\ell\} = \{\mathcal{H}_\ell(m, n)\}$ and $\{\overline{\mathcal{H}}_\ell\} = \{\overline{\mathcal{H}}_\ell(Q, P)\}$ by the recurrence relations for $\ell \in \mathbb{Z}$

$$\mathcal{K} \delta \mathcal{H}_{\ell-1} = \mathcal{J} \delta \mathcal{H}_\ell,$$

(3.26)

$$\mathcal{K} \overline{\delta \mathcal{H}}_{\ell-1} = \mathcal{J} \overline{\delta \mathcal{H}}_\ell,$$

(3.27)

where $\delta \mathcal{H}_\ell = (\delta \mathcal{H}_\ell/\delta m, \delta \mathcal{H}_\ell/\delta n)^T$ and $\delta \overline{\mathcal{H}}_\ell = (\delta \overline{\mathcal{H}}_\ell/\delta Q, \delta \overline{\mathcal{H}}_\ell/\delta P)^T$. Establishing the correspondence between the two hierarchies of Hamiltonian functionals relies on two preliminary lemmas.

**Lemma 3.3.** Let $\{\mathcal{H}_\ell\}$ and $\{\overline{\mathcal{H}}_\ell\}$ be the respective hierarchies of Hamiltonian functionals determined by (3.26) and (3.27). Then for each $\ell \in \mathbb{Z}$, their corresponding variational derivatives are related according to the identity

$$\delta \mathcal{H}_\ell(m, n) = 6(-1)^{\ell} \mathcal{J}^{-1} \mathbf{B} \Xi \delta \overline{\mathcal{H}}_{\ell}(Q, P),$$

(3.28)

where the matrix operators $\mathbf{B}$ and $\Xi$ are defined in (3.18) and (3.23) and $\mathcal{J}$ is the first Hamiltonian of the GX hierarchy given by (3.8).
Proof. We first prove the recursive identity

\[ K^{-1} B \Xi \delta \overline{H}_{\ell-1} = -J^{-1} B \Xi \delta \overline{H}_{\ell} , \quad \ell \in \mathbb{Z}, \]

(3.29)

for the hierarchy of Hamiltonian functionals \( \{ \overline{H}_{\ell} \} \). Indeed, formula (3.19) with \( \ell = 1 \) yields

\[ K^{-1} B = -J^{-1} B E (\nabla U) E^{-1}. \]

Hence, the left-hand side of (3.29) becomes

\[ K^{-1} B \Xi \delta \overline{H}_{\ell-1} = -J^{-1} B E \nabla U E^{-1} \Xi K^{-1} J \delta \overline{H}_{\ell}. \]

Returning to expression (3.23) for the operator \( \Xi \), we immediately conclude that (3.29) holds for all \( \ell \in \mathbb{Z} \).

We next use induction to prove identity (3.28). Taking equality (3.29) with \( \ell = 1 \) into account, we must prove that for \( \ell = -2 \),

\[ \delta \mathcal{H}_{-2} = -6 K^{-1} B \Xi \delta \overline{H}_0. \]

(3.30)

Indeed, it follows from (3.10) and (3.26) that

\[ \delta \mathcal{H}_{-2} = K^{-1} J \delta \mathcal{H}_{-1} = K^{-1} J \delta \mathcal{H}_{C} = K^{-1} \left( (\partial_x^2 - 1) m^{1/3} n^{-2/3} \right). \]

(3.31)

Using transformation (3.7) and identities (3.16), we obtain

\[ (\partial_x^2 - 1) m^{1/3} n^{-2/3} = -\Delta^{1/2} (Q - \partial_y^2) \left( \frac{m}{n} \right)^{1/2} = m (P_y + P^2 - Q), \]

\[ (1 - \partial_x^2) m^{-2/3} n^{1/3} = \Delta^{1/2} (Q - \partial_y^2) \left( \frac{m}{n} \right)^{-1/2} = n (P_y - P^2 + Q). \]

Hence,

\[ \delta \mathcal{H}_{-2} = K^{-1} B \left( \frac{P_y + P^2 - Q}{P_y - P^2 + Q} \right). \]

(3.32)

On the other hand, because \( \overline{H}_0 = \int P \, dy \), taking the form of the operator \( \Xi \) into account, we obtain the equality

\[ 6 K^{-1} B \Xi \delta \overline{H}_0 = -K^{-1} B \left( \frac{\Theta \cdot 1}{-\Theta \cdot 1} \right), \]

which together with (3.31) and (3.32) proves identity (3.30) and implies that (3.28) holds for \( \ell = -2 \).

We now suppose that (3.28) holds for \( \ell = k \) with \( k \leq -2 \). From (3.29), we then deduce that for \( \ell = k - 1 \),

\[ \delta \mathcal{H}_{k-1} = K^{-1} J \delta \mathcal{H}_k = 6 (-1)^k K^{-1} B \Xi \delta \overline{H}_{(k+1)} = 6 (-1)^{k-1} J^{-1} B \Xi \delta \overline{H}_{-k}, \]

which implies that (3.28) holds for all \( \ell \leq -2 \).

In the opposite direction, we first prove that

\[ \delta \mathcal{H}_{-1} = 6 K^{-1} B \Xi \delta \overline{H}_{-1}. \]

(3.33)

It suffices to prove that \( \overline{K} \Xi^{-1} B^{-1} K \delta \mathcal{H}_{-1} = 0 \). Indeed, by (3.24), we have

\[ \overline{K} \Xi^{-1} B^{-1} K \delta \mathcal{H}_{-1} = -\frac{1}{6} \nabla E^{-1} B^{-1} K \delta \mathcal{H}_{-1} = \frac{1}{6} \nabla^{-1} E^{-1} B^{-1} J \delta \mathcal{H}_{-1}, \]

which completes the proof.
where
\[ B^{-1} J \delta \mathcal{H}_{-1} = \begin{pmatrix} P_y + P^2 - Q \\ P_y - P^2 + Q \end{pmatrix} = \mathbf{E} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]
whence (3.33) follows.

Further induction on \( \ell \) shows that if (3.28) holds for \( \ell = k \) with some \( k \geq 1 \), then for \( \ell = k + 1 \), again using (3.29), we conclude that
\[ \delta \mathcal{H}_{k+1} = J^{-1} \mathcal{K} \delta \mathcal{H}_k = 6(-1)^k J^{-1} \mathcal{K} B \widetilde{\mathcal{R}}_{-(k+1)} = 6(-1)^{k+1} J^{-1} B \widetilde{\mathcal{R}}_{-(k+2)}, \]
which completes the induction step and proves (3.28) in the general case.

The next lemma, which follows from Theorem 2.2, shows how Liouville transformation (3.7) acts on the variational derivatives.

**Lemma 3.4.** Let \((m(t, x), n(t, x))\) and \((Q(\tau, y), P(\tau, y))\) be related by Liouville transformation (3.7). If \( \mathcal{H}(m, n) = \overline{\mathcal{H}}(Q, P) \), then
\[ \delta \mathcal{H}(m, n) = -\frac{1}{6} \Delta^{1/3} B^{-1} \mathbf{E}^{-1} U^* \delta \overline{\mathcal{H}}(Q, P). \]

Finally, we claim that
\[ J^{-1} B \begin{pmatrix} 0 & \Theta \\ -\Theta^* & 0 \end{pmatrix} = \Delta^{1/3} B^{-1}, \]
which is equivalent to
\[ B \begin{pmatrix} 0 & \Theta \\ -\Theta^* & 0 \end{pmatrix} = J \begin{pmatrix} m^{-2/3} n^{1/3} & 0 \\ 0 & m^{1/3} n^{-2/3} \end{pmatrix}. \]

In fact,
\[ J \begin{pmatrix} m^{-2/3} n^{1/3} & 0 \\ 0 & m^{1/3} n^{-2/3} \end{pmatrix} = \begin{pmatrix} 0 & \Delta^{1/2} \left(Q - \partial_y^2\right) \left(\frac{m}{n}\right)^{1/2} \\ \Delta^{1/2} \left(Q - \partial_y^2\right) \left(\frac{m}{n}\right)^{-1/2} & 0 \end{pmatrix} = B \begin{pmatrix} 0 & \Theta \\ -\Theta^* & 0 \end{pmatrix}. \]

We now define the functional \( \mathcal{G}_\ell(Q, P) \equiv \mathcal{H}_\ell(m, n) \). On one hand, using Lemma 3.4, we obtain
\[ \delta \mathcal{H}_\ell(m, n) = -\frac{1}{6} \Delta^{1/3} B^{-1} \mathbf{E}^{-1} U^* \delta \overline{\mathcal{G}}_\ell(Q, P). \]

On the other hand, by virtue of Lemma 3.3,
\[ \delta \mathcal{H}_\ell(m, n) = 6(-1)^\ell J^{-1} B \Xi \delta \overline{\mathcal{R}}_{-(\ell+1)}(Q, P) = \]
\[ = (-1)^\ell J^{-1} B \begin{pmatrix} 0 & \Theta \\ -\Theta^* & 0 \end{pmatrix} \mathbf{E}^{-1} U^* \delta \overline{\mathcal{R}}_{-(\ell+1)}(Q, P) = \]
\[ = (-1)^\ell \Delta^{1/3} B^{-1} \mathbf{E}^{-1} U^* \delta \overline{\mathcal{R}}_{-(\ell+1)}(Q, P). \]

Combining the last two equations yields
\[ \mathcal{H}_\ell(m, n) = \mathcal{G}_\ell(Q, P) = 6(-1)^{\ell+1} \overline{\mathcal{R}}_{-(\ell+1)}(Q, P), \]
which establishes the correspondence between the sequences of Hamiltonian functionals admitted by the GX and aGX hierarchies. We have thus proved the following theorem.

**Theorem 3.2.** For any nonzero integer \( \ell \), each Hamiltonian conserved density \( \mathcal{H}_\ell(m, n) \) of the GX hierarchy is related by Liouville transformation (3.7) to the Hamiltonian conserved density \( \overline{\mathcal{H}}_\ell(Q, P) \) as
\[ \mathcal{H}_\ell(m, n) = 6(-1)^{\ell+1} \overline{\mathcal{H}}_{-(\ell+1)}(Q, P). \]
4. Liouville correspondence for the dual DWW hierarchy

The DWW system

\[ q_t = (-q_x + 2q_x + 2q)_x, \quad r_t = (r_x + r^2 + 2q)_x, \quad (4.1) \]

is an integrable physical system describing the propagation of shallow water waves [57], [65], [70], [71]. Its tri-Hamiltonian formulation was found by Kupershmidt [65] and can be used to construct a dual counterpart, the DDWW system proposed in [64]:

\[ \rho_t = ((\rho + v)u)_x, \quad \rho = v - v_x, \]
\[ \gamma_t = (\gamma u + 2v)_x, \quad \gamma = u + u_x. \quad (4.2) \]

The bi-Hamiltonian structure for the DDWW system

\[ \left( \begin{array}{c} \rho \\ \gamma \end{array} \right)_t = K \delta H_1 = J \delta H_2, \quad \delta H_n = \left( \begin{array}{c} \delta H_n \\ \delta \gamma \end{array} \delta \rho \right)^T, \quad n = 1, 2, \quad (4.3) \]

is governed by the pair of Hamiltonians [64]

\[ K = \left( \begin{array}{cc} \rho \partial_x + \partial_x \rho & \gamma \partial_x \\ \partial_x \gamma & 2 \partial_x \end{array} \right), \quad J = \left( \begin{array}{cc} 0 & \partial_x - \partial_x^2 \\ \partial_x + \partial_x^2 & 0 \end{array} \right), \]

and together with the associated Hamiltonian functionals

\[ H_1 = H_1(\rho, \gamma) = \int (u - u_x)v \, dx, \quad H_2 = H_2(\rho, \gamma) = \int (u + u_x)uv \, dx. \]

The members of the DDWW integrable hierarchy in the positive direction

\[ \left( \begin{array}{c} \rho \\ \gamma \end{array} \right)_t = M_n = K \delta H_{n-1}(\rho, \gamma) = J \delta H_n(\rho, \gamma), \quad n = 1, 2, \ldots, \quad (4.4) \]

are obtained by repeatedly applying the recursion operator \( R = KJ^{-1} \) to the seed system

\[ \left( \begin{array}{c} \rho \\ \gamma \end{array} \right)_t = M_1 = \left( \begin{array}{c} \rho \\ \gamma \end{array} \right)_x = J \delta H_1(\rho, \gamma). \]

We note that the Hamiltonian \( K \) in (4.3) admits the Casimir functional

\[ H_C(\rho, \gamma) = -\frac{1}{2} \int \sqrt{4\rho - \gamma^2} \, dx \]

with the variational derivative

\[ \delta H_C = \left( \begin{array}{c} \delta H_C \\ \delta \gamma \end{array} \delta \rho \right)^T = \left( \begin{array}{c} \frac{1}{\sqrt{4\rho - \gamma^2}} \frac{\gamma}{2\sqrt{4\rho - \gamma^2}} \end{array} \right)^T. \]

The functional \( H_C \) leads to the associated Casimir system

\[ \left( \begin{array}{c} \rho \\ \gamma \end{array} \right)_t = M_{-1} = J \delta H_C, \]
which is given explicitly by
\[
\rho_t = \frac{1}{2}(\partial_x - \partial_x^2)\frac{\gamma}{\sqrt{4 \rho - \gamma^2}}, \quad \gamma_t = -(\partial_x + \partial_x^2)\frac{1}{\sqrt{4 \rho - \gamma^2}},
\] (4.5)

and serves as the first negative flow in the DDWW integrable hierarchy. Starting from Casimir system (4.5), we can (formally) construct an infinite hierarchy of higher-order commuting bi-Hamiltonian systems and corresponding Hamiltonian functionals \( \{ \mathcal{H}_{-n} \} = \{ \mathcal{H}_{-n}(\rho, \gamma) \} \) in the negative direction:
\[
\left( \begin{array}{c} \rho \\ \gamma \end{array} \right)_t = M_{-n} = K \mathcal{H}_{-(n+1)}(\rho, \gamma) = J \delta \mathcal{H}_{-n}(\rho, \gamma), \quad n = 1, 2, \ldots,
\] (4.6)

where \( \mathcal{H}_{-1}(\rho, \gamma) = \mathcal{H}_C(\rho, \gamma) \) and \( \delta \mathcal{H}_{-n}(\rho, \gamma) = (\delta \mathcal{H}_{-n}/\delta \rho, \delta \mathcal{H}_{-n}/\delta \gamma)^T \).

The DDWW system can be expressed as the compatibility condition for the linear system for \( \Psi = (\psi_1 \psi_2) \) consisting of
\[
\Psi_x = \begin{pmatrix}
\frac{1}{2} \left( \frac{1}{\lambda} \gamma - 1 \right) & \frac{1}{2} \left( \frac{1}{\lambda} \right) \\
- \frac{1}{\lambda \sqrt{\lambda}} \rho & \frac{1}{2} \left( 1 - \frac{1}{\lambda} \gamma \right)
\end{pmatrix} \Psi,
\]
\[
\Psi_t = \begin{pmatrix}
\frac{1}{2} \left( \frac{1}{\lambda} \gamma u - \lambda \right) & \frac{1}{\sqrt{\lambda}} u + \sqrt{\lambda} \\
- \frac{1}{\sqrt{\lambda}} \left( \frac{1}{\lambda} \rho u + \rho + v_x \right) & \frac{1}{2} \left( \lambda - \frac{1}{\lambda} \gamma u \right)
\end{pmatrix} \Psi.
\] (4.7)

We note that Eq. (4.7) is reducible to the scalar equation
\[
\Psi_{xx} = \left( \frac{1}{2 \lambda} \gamma_x - \frac{1}{\lambda \gamma} \rho + \frac{1}{4} \left( \frac{1}{\lambda} \gamma - 1 \right)^2 \right) \Psi.
\] (4.8)

We now set \( \omega(\rho, \gamma) = \sqrt{4 \rho - \gamma^2} \) and introduce the coordinate transformation
\[
y = \int^\tau \omega(t, \xi) \, d\xi, \quad \tau = t,
\]
\[
Q(\tau, y) = \frac{1}{\omega(t, x)} \left( 1 - \frac{\omega_x(t, x)}{\omega(t, x)} \right), \quad P(\tau, y) = -\frac{\gamma(t, x)}{\omega(t, x)},
\]
\[
\Phi = \omega^{-1/2} \Psi.
\] (4.9)

A direct calculation shows that the Liouville transformations convert isospectral problem (4.8) into
\[
\Phi_{yy} = \frac{1}{4} \left( \frac{2}{\lambda} (PQ - P_y) - \frac{1}{\lambda^2} + Q^2 - 2Q_y \right) \Phi.
\]

In analogy with the 2CH and GX hierarchies, in this section, we aim to investigate how transformation (4.9) affects the underlying correspondence between the flows of the DDWW hierarchy and what we call the associated DDWW (aDDWW) integrable hierarchy.

We first investigate the integrable structure of the aDDWW hierarchy. Directly applying Theorem 2.2, we can easily construct its pair of Hamiltonians from the known pair of Hamiltonians \( \mathcal{K} \) and \( \mathcal{J} \) given by (4.3) of the DDWW system using transformation (4.9).
Theorem 4.1. Under coordinate transformation (4.9), the Hamiltonian pair \( \mathcal{K} \) and \( \mathcal{J} \) given by (4.3) admitted by DDWW system (4.2) is converted into the pair of Hamiltonians
\[
\mathcal{K} = 2 \begin{pmatrix}
\mathcal{X}(P \mathcal{X} + \mathcal{O} \partial_y^{-1}) \mathcal{O} \\
\mathcal{X}(P \mathcal{X} \partial_y + \mathcal{O}(1 + P \partial_y^{-1} P \partial_y)) \\
\mathcal{O} \mathcal{X} \mathcal{O} + (P_y \partial_y^{-1} + P) \mathcal{O} \partial_y^{-1} \mathcal{O} \\
\mathcal{X} \mathcal{X} \mathcal{O} \partial_y + (P_y \partial_y^{-1} + P) \mathcal{O}(1 + P \partial_y^{-1} P \partial_y)
\end{pmatrix},
\]
\[
\mathcal{J} = 2 \begin{pmatrix}
\mathcal{X} \mathcal{O} \\
\mathcal{X} \partial_y \\
\partial_y (1 + P \partial_y^{-1} P \partial_y)
\end{pmatrix},
\]
where the operators \( \mathcal{O}, \mathcal{X}, \) and \( \mathcal{Y} \) are defined by
\[
\mathcal{O} = Q \partial_y - \partial_y^2, \quad \mathcal{X} = \partial_y + Q + Q \partial_y^{-1}, \quad \mathcal{Y} = 1 + P^2 + P_y \partial_y^{-1} P.
\]

Proof. Let \( b \) and \( \nu \) be arbitrary nonzero constants. Similarly to Lemma 2.1, we can verify the operator identities
\[
\omega^{-(1+\nu)}(1 + b \partial_y^2) \omega^{\nu} = Q + b \partial_y, \quad \omega^{-2}(\partial_x - \partial_y^2) = Q \partial_y - \partial_y^2,
\]
\[
\omega^{-1}(\partial_x + \partial_y^2) \omega^{-1} = \partial_y Q + \partial_y^2, \quad \omega^{-2}(\rho \partial_x + \partial_x \rho) \omega^{-1} = \frac{1}{2}(\partial_y + P \partial_y P).
\]
The first and second identities imply that the matrix operators \( \mathbf{T} \) and \( \mathbf{T}^* \) in Theorem 2.2 are
\[
\mathbf{T} = -\omega \begin{pmatrix}
2 \mathcal{X} \omega^{-2} & \mathcal{X} P \omega^{-1} \\
2(P + P_y \partial_y^{-1}) \omega^{-2} & \mathcal{Y} \omega^{-1}
\end{pmatrix},
\]
\[
\mathbf{T}^* = -\begin{pmatrix}
2 \omega^{-1} \partial_y^{-1} \mathcal{O} & 2 \omega^{-1} \partial_y^{-1} P \partial_y \\
\partial_y^{-1} \mathcal{O} & 1 + P \partial_y^{-1} P \partial_y
\end{pmatrix}.
\]
Therefore, by virtue of (4.13), the formulas for Hamiltonians (4.11) follow directly from formula (2.35).

Having the Hamiltonian pair \( \mathcal{K} \) and \( \mathcal{J} \) given by (4.11), we obtain the positive flows in the aDDWW hierarchy by successively applying the recursion operator
\[
\mathcal{R} = \mathcal{K} \mathcal{J}^{-1} = \begin{pmatrix}
\mathcal{X} \mathcal{Y} \\
\mathcal{Y} (Q - \partial_y)
\end{pmatrix},
\]
to the seed symmetry \( \mathcal{M}_1 = (Q_y, P_y)^T \),
\[
\begin{pmatrix}
Q \\
P
\end{pmatrix}_n = \mathcal{M}_n = \mathcal{R}^{n-1} \mathcal{M}_1, \quad n = 1, 2, \ldots,
\]
and the negative flows are
\[
\mathcal{R}^n \begin{pmatrix}
Q \\
P
\end{pmatrix}_n = \mathcal{M}_0 = \begin{pmatrix}
0 \\
0
\end{pmatrix}, \quad n = 1, 2, \ldots.
\]

Lemma 4.1. Let \( \mathcal{K} \) and \( \mathcal{F} \), be the compatible Hamiltonians given by (4.3) for the DDWW hierarchy and \( \mathcal{R} \) be recursion operator (4.15) admitted by the aDDWW hierarchy. Let
\[
\mathbf{C} = \frac{1}{2} \begin{pmatrix}
0 & -\omega^2 (Q - \partial_y) \\
-2\omega & 0
\end{pmatrix}.
\]
Then applying transformation (4.9), we obtain the operator identity
\[
\mathbf{C}^{-1} (\mathcal{F} \mathcal{K}^{-1})^n \mathbf{C} = \mathcal{R}^n, \quad 0 < n \in \mathbb{Z}.
\]
Using this lemma and induction, we can obtain the following result for the correspondence between the DDWW and aDDWW hierarchies. Below, we use the notation DDWW\(_n\), DDWW\(_{-n}\) and aDDWW\(_n\), aDDWW\(_{-n}\) for the \(n\)th system in the positive and negative directions of the DDWW and aDDWW hierarchies.

**Theorem 4.2.** Under transformation (4.9), the DDWW\(_n\) system is related to the aDDWW\(_{-(n-1)}\) system for any \(n \in \mathbb{Z}\).

For brevity, we omit the proof of the theorem, which is based mainly on decomposition (4.18) of the recursion operator.

We can now establish the correspondence between the Hamiltonian functionals of the DDWW and the aDDWW hierarchies. In particular, for the DDWW hierarchy, bi-Hamiltonian structure (4.4), (4.6) produces a bi-infinite sequence of functionals \(\{H_n\}\) by

\[
\mathcal{K} \delta H_n = \mathcal{J} \delta H_{n+1}, \quad n \in \mathbb{Z}.
\]

(4.19)

On the other hand, the recurrence relation

\[
\overline{\mathcal{K}} \delta H_n = \overline{\mathcal{J}} \delta H_{n+1}, \quad n \in \mathbb{Z},
\]

(4.20)

with Hamiltonian pair (4.11) of the aDDWW hierarchy yields an infinite sequence of Hamiltonian functionals \(\{\overline{H}_n\}\) admitted by aDDWW flows (4.16) and (4.17).

The formula for the correspondence between the variational derivatives \(\delta H_n(\rho, \gamma)\) and \(\delta \overline{H}_n(Q, P)\) can be proved by a straightforward induction.

**Lemma 4.2.** Let \(\{H_n\}\) and \(\{\overline{H}_n\}\) be the hierarchies of Hamiltonian functionals determined by the respective recurrence relations (4.19) and (4.20). Then for each \(n \in \mathbb{Z}\), their respective variational derivatives satisfy the identity

\[
\delta H_n(\rho, \gamma) = \mathcal{K}^{-1} C \mathcal{J} \delta \overline{H}_{-(n+1)}(Q, P).
\]

Moreover, the following lemma (also a direct consequence of Lemma 2.3) gives a formula for the change of the variational derivative under transformation (4.9).

**Lemma 4.3.** Let \((\rho(t, x), \gamma(t, x))\) and \((Q(\tau, y), P(\tau, y))\) be related by Liouville transformation (4.9). If \(H(\rho, \gamma) = \overline{H}(Q, P)\), then

\[
\delta H(\rho, \gamma) = T^* \delta \overline{H}(Q, P),
\]

(4.21)

where \(T^*\) is the formal adjoint of \(T\) given in (4.14).

Finally, recalling the form of the Hamiltonians \(\mathcal{J}\) and using the identity \(\overline{\mathcal{K}} = \Delta^{-1} T C \mathcal{J}\) and (4.18) with \(n = 1\), we obtain

\[
T^* = \mathcal{J}^{-1} C \overline{\mathcal{K}}.
\]

(4.22)

Hence,

\[
\delta H_n(\rho, \gamma) = \mathcal{K}^{-1} C \mathcal{J} \delta \overline{H}_{-(n+1)}(Q, P) = \mathcal{J}^{-1} C \mathcal{J} \delta \overline{H}_{-n}(Q, P) = T^* \delta \overline{H}_{-n}(Q, P).
\]

Further, based on the condition in Lemma 4.3, we define the functional \(\mathcal{G}_n(Q, P) \equiv H_n(\rho, \gamma)\). Taking (4.21) and (4.22) into account, we then obtain

\[
T^* \delta \mathcal{G}_n(Q, P) = \delta H_n(\rho, \gamma) = T^* \delta \overline{H}_{-n}(Q, P),
\]

which yields \(\delta \mathcal{G}_n(Q, P) = \delta \overline{H}_{-n}(Q, P)\). Then \(H_n(\rho, \gamma) = \overline{H}_{-n}(Q, P)\). We have thus proved our main theorem on the Hamiltonian functionals of the two hierarchies.
Theorem 4.3. Each Hamiltonian functional $H_n(\rho, \gamma)$ of the DDWW hierarchy yields a Hamiltonian functional of the aDDWW hierarchy under transformation (4.9) according to the identity

$$\overline{H}_{-n}(Q, P) = H_n(\rho, \gamma), \quad n \in \mathbb{Z}.$$ 

Conflicts of interest. The authors declare no conflicts of interest.

REFERENCES

1. J. Kang, X. Liu, P. J. Olver, and C. Qu, “Liouville correspondence between the modified KdV hierarchy and its dual integrable hierarchy,” J. Nonlinear Sci., 26, 141–170 (2016).
2. J. Kang, X. Liu, P. J. Olver, and C. Qu, “Liouville correspondences between integrable hierarchies,” SIGMA, 13, 035 (2017).
3. J. Lenells, “The correspondence between KdV and Camassa–Holm,” Internat. Math. Res. Not., 2004, 3797–3811 (2004).
4. H. P. McKean, “The Liouville correspondence between the Korteweg–de Vries and the Camassa–Holm hierarchies,” Commun. Pure Appl. Math., 56, 998–1015 (2003).
5. D. Chen, Y. Li, and Y. Zeng, “Transformation operator between recursion operators of Bäcklund transformations: I,” Sci. Sinica Ser. A, 28, 907–922 (1985).
6. P. A. Clarkson, A. S. Fokas, and M. J. Ablowitz, “Hodograph transformations of linearizable partial differential equations,” SIAM J. Appl. Math., 49, 1188–1209 (1989).
7. A. Kundu, “Landau–Lifshitz and higher-order nonlinear systems gauge generated from nonlinear Schrödinger-type equations,” J. Math. Phys., 25, 3433–3438 (1984).
8. B. Matveev and M. A. Salle, Darboux Transformations and Solitons, Springer, Berlin (1991).
9. R. M. Miura, “Korteweg–de Vries equation and generalizations: I. A remarkable explicit nonlinear transformation,” J. Math. Phys., 9, 1202–1204 (1968).
10. C. Rogers and W. K. Schief, Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory (Cambridge Texts Appl. Math., Vol. 30), Cambridge Univ. Press, Cambridge (2002).
11. M. Wadati and K. Sogo, “Gauge transformations in soliton theory,” J. Phys. Soc. Japan, 53, 394–398 (1983).
12. R. P. Lax, “Integrals of nonlinear equations of evolution and solitary waves,” Commun. Pure Appl. Math., 21, 467–490 (1968).
13. R. Milson, “Liouville transformation and exactly solvable Schrödinger equations,” Internat. J. Theor. Phys., 37, 1735–1752 (1998).
14. F. W. J. Olver, Asymptotics and Special Functions, Acad. Press, New York (1974).
15. R. Camassa and D. D. Holm, “An integrable shallow water equation with peaked solitons,” Phys. Rev. Lett., 71, 1661–1664 (1993); arXiv:math-ph/9305000v1 (1993).
16. R. Camassa, D. D. Holm, and J. Hyman, “A new integrable shallow water equation,” in: Advances in Applied Mechanics (J. W. Hutchinson and T. Y. Wu, eds.), Vol. 31, Acad. Press, Boston, Mass. (1994), pp. 1–33.
17. A. Constantin and D. Lannes, “The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations,” Arch. Ration. Mech. Anal., 192, 165–186 (2009).
18. B. Fuchssteiner and A. S. Fokas, “Symplectic structures, their Bäcklund transformations and hereditary symmetries,” Phys. D, 4, 47–66 (1981).
19. J. Lenells, “Traveling wave solutions of the Camassa–Holm and Korteweg–de Vries equations,” J. Nonlinear Math. Phys., 11, 508–520 (2004).
20. P. J. Olver and P. Rosenau, “Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support,” Phys. Rev. E, 53, 1900–1906 (1996).
21. R. M. Chen, Y. Liu, C. Qu, and S. Zhang, “Oscillation-induced blow-up to the modified Camassa–Holm equation with linear dispersion,” Adv. Math., 272, 225–251 (2015).
22. G. Gui, Y. Liu, P. J. Olver, and C. Qu, “Wave breaking and peakons for a modified Camassa–Holm equation,” Commun. Math. Phys., 319, 731–759 (2013).
23. X. Liu, Y. Liu, and C. Qu, “Orbital stability of the train of peakons for an integrable modified Camassa–Holm equation,” Adv. Math., 255, 1–37 (2014).
24. Y. Liu, P. J. Olver, C. Qu, and S. Zhang, “On the blow-up of solutions to the integrable modified Camassa–Holm equation,” Anal. Appl., 12, 355–368 (2014).
25. Y. Matsuno, “Bäcklund transformation and smooth multisoliton solutions for a modified Camassa–Holm equation with cubic nonlinearity,” J. Math. Phys., 54, 051504 (2013); arXiv:1302.0107v2 [nlin.SI] (2013).
26. A. S. Fokas, P. J. Olver, and P. Rosenau, “A plethora of integrable bi-Hamiltonian equations,” in: Algebraic Aspects of Integrable Systems (Progr. Nonlin. Diff. Eq. Their Appl., Vol. 26, A. S. Fokas and I. M. Gelfand, eds.), Birkhäuser, Boston, Mass. (1997), pp. 93–101.
27. B. Fuchssteiner, “Some tricks from the symmetry-toolbox for nonlinear equations: Generalizations of the Camassa–Holm equation,” Phys. D, 95, 229–243 (1996).
28. I. Dorfman, Dirac Structures and Integrability of Nonlinear Evolution Equations, John Wiley and Sons, New York (1993).
29. A. Degasperis, D. D. Holm, and A. Hone, “A new integrable equation with peakon solutions,” Theor. Math. Phys., 133, 1463–1474 (2002).
30. A. Degasperis and M. Procesi, “Asymptotic integrability,” in: Symmetry and Perturbation Theory (Rome, Italy, 16–22 December 1998, A. Degasperis and G. Gaeta, eds.), World Scientific, Singapore (1999), pp. 23–37.
31. A. N. W. Hone, and J. P. Wang, “Integrable peakon equations with cubic nonlinearity,” J. Phys. A: Math. Theor., 41, 372002 (2008).
32. V. Novikov, “Generalizations of the Camassa–Holm equation,” J. Phys. A: Math. Theor., 42, 342002 (2009).
33. P. J. Caudrey, R. K. Dodd, and J. D. Gibbon, “A new hierarchy of Korteweg–de Vries equations,” Proc. Roy. Soc. London Ser. A, 351, 407–422 (1976).
34. K. Sawada and T. Kotera, “A method for finding N-soliton solutions of the K.d.V. equation and K.d.V.-like equation,” Prog. Theor. Phys., 51, 1355–1367 (1974).
35. B. Fuchssteiner and W. Oevel, “The bi-Hamiltonian structure of some nonlinear fifth- and seventh-order differential equations and recursion formulas for their symmetries and conserved covariants,” J. Math. Phys., 23, 358–363 (1982).
36. A. N. W. Hone and J. P. Wang, “Prolongation algebras and Hamiltonian operators for peakon equations,” Inverse Problems, 19, 129–145 (2003).
37. B. A. Kupershmidt, “A super Korteweg–de Vries equation: An integrable system,” Phys. Lett. A, 102, 213–215 (1984).
38. M. Chen, S.-Q. Liu, and Y. Zhang, “A two-component generalization of the Camassa–Holm equation and its solutions,” Lett. Math. Phys., 75, 1–15 (2006).
39. A. Constantin and R. I. Ivanov, “On an integrable two-component Camassa–Holm shallow water system,” Phys. Lett. A, 372, 7129–7132 (2008); arXiv:0806.0868v2 [nlin.SI] (2008).
40. J. Eckhardt, F. Gesztesy, H. Holden, A. Kostenko, and G. Teschl, “Real-valued algebro-geometric solutions of the two-component Camassa–Holm hierarchy,” Ann. Inst. Fourier, 67, 1185–1230 (2017).
41. J. Eckhardt and K. Grunert, “A Lagrangian view on complete integrability of the two-component Camassa–Holm system,” J. Integrable Syst., 2, xyy002 (2017).
42. J. Escher, J. Kohlmann, and J. Lenells, “The geometry of the two-component Camassa–Holm and Degasperis–Procesi equations,” J. Geom. Phys., 61, 436–452 (2011); arXiv:1009.0188v2 [math.AP] (2010).
43. P. Guha and P. J. Olver, “Geodesic flow and two(super)component analog of the Camassa–Holm equation,” SIGMA, 2, 054 (2006); arXiv:nlin/SI/0605041v1 (2006).
44. G. Gui and Y. Liu, “On the global existence and wave breaking criteria for the two-component Camassa–Holm system,” J. Funct. Anal., 258, 4251–4278 (2010).
45. D. D. Holm and R. I. Ivanov, “Two-component CH system: Inverse scattering, peakons, and geometry,” Inverse Problems, 27, 045013 (2011); arXiv:1009.5374v1 [nlin.SI] (2010).
46. A. N. W. Hone, V. Novikov, and J. P. Wang, “Two-component generalizations of the Camassa–Holm equation,” Nonlinearity, 30, 622–658 (2017); arXiv:1602.03431v1 [nlin.SI] (2016).
47. Y. Matsuno, “Multisoliton solutions of the two-component Camassa–Holm system and their reductions,” J. Phys. A: Math. Theor., 50, 345202 (2017).
48. C. Qu, J. Song, and R. Yao, “Multi-component integrable systems with peaked solitons and invariant curve flows in certain geometries,” SIGMA, 9, 001 (2013).
49. I. A. B. Strachan and B. M. Szablikowski, “Novikov algebras and a classification of multicomponent Camassa–Holm equations,” Stud. Appl. Math., 133, 84–117 (2014).
50. B. Xia, Z. Qiao, and R. Zhou, “A synthetical two-component model with peakon solutions,” Stud. Appl. Math., 135, 248–276 (2015).
51. M. Antonowicz and A. P. Fordy, “Coupled Harry Dym equations with multi-Hamiltonian structures,” J. Phys. A: Math. Gen., 21, 269–275 (1988).
52. M. Antonowicz and A. P. Fordy, “Coupled KdV equations with multi-Hamiltonian structures,” Phys. D, 28, 345–357 (1987).
53. M. Antonowicz and A. P. Fordy, “Factorisation of energy dependent Schrödinger operators: Miura maps and modified systems,” Commun. Math. Phys., 124, 465–486 (1989).
54. M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, “The inverse scattering transform–Fourier analysis for nonlinear problems,” Stud. Appl. Math., 53, 249–315 (1974).
55. M. Ito, “Symmetries and conservation laws of a coupled nonlinear wave equation,” Phys. Lett. A, 91, 335–338 (1982).
56. R. I. Ivanov and T. Lyons, “Integrable models for shallow water with energy dependent spectral problems,” J. Nonlinear Math. Phys., 19, supp. 1, 72–88 (2012).
57. D. J. Kaup, “A higher-order water-wave equation and the method for solving it,” Progr. Theoret. Phys., 54, 396–408 (1975).
58. X. Geng and B. Xue, “An extension of integrable peakon equations with cubic nonlinearity,” Nonlinearity, 22, 1847–1856 (2009).
59. H. Lundmark and J. Szmigielski, An Inverse Spectral Problem Related to the Geng–Xue Two-Component Peakon Equation (Memoirs Amer. Math. Soc., Vol. 244, No. 115), Amer. Math. Soc., Providence, R. I. (2016).
60. H. Li and W. Chai, “A new Liouville transformation for the Geng–Xue system,” Commun. Nonlinear Sci. Numer. Simul., 49, 93–101 (2017).
61. N. Li and Q. P. Liu, “On bi-Hamiltonian structure of two-component Novikov equation,” Phys. Lett. A, 377, 257–261 (2013).
62. N. Li and X. Niu, “A reciprocal transformation for the Geng–Xue equation,” J. Math. Phys., 55, 053505 (2014).
63. H. Lundmark and J. Szmigielski, “Dynamics of interlacing peakons and shockpeakons in the Geng–Xue equation,” J. Integrable Syst., 2, xyw014 (2017).
64. J. Kang, X. Liu, P. J. Olver, and C. Qu, “Bäcklund transformations for tri-Hamiltonian dual structures of multi-component integrable system,” J. Integrable Syst., 2, xyw016 (2017).
65. B. A. Kupershmidt, “Mathematics of dispersive water waves,” Commun. Math. Phys., 99, 51–73 (1985).
66. F. Magri, “A simple model of the integrable Hamiltonian equation,” J. Math. Phys., 19, 1156–1162 (1978).
67. P. J. Olver, “Evolution equations possessing infinitely many symmetries,” J. Math. Phys., 18, 1212–1215 (1977).
68. P. J. Olver, Applications of Lie Groups to Differential Equations (Grad. Texts Math., Vol. 107), Springer, New York (1993).
69. P. J. Olver, “Darboux’ theorem for Hamiltonian differential operators,” J. Differ. Equ., 71, 10–33 (1988).
70. L. J. F. Broer, “Approximate equations for long water waves,” Appl. Sci. Res., 31, 377–395 (1975).
71. G. B. Whitham, Linear and Nonlinear Waves, Wiley, New York (1974).