Detecting multipartite entanglement with untrusted measurements in asymmetric quantum networks

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The future of quantum communication relies on quantum networks composed by observers sharing multipartite quantum states. The certification of multipartite entanglement will be crucial to the usefulness of these networks. In many real situations it is natural to assume that some observers are more trusted than others in the sense that they have more knowledge of their measurement apparatuses. Here we propose a general method to certify all kinds of multipartite entanglement in this scenario and experimentally demonstrate it in an optical experiment. Our work fills a gap in the characterization of quantum correlations and provides a basis for semi-device-independent cryptographic applications in quantum networks.

The most widely used techniques to detect entanglement rely either on having knowledge of the quantum state, obtained through quantum state tomography, or on the use of measurements that constitute an entanglement witness [1]. A frequently disregarded assumption behind these methods is that the measurements and devices used are well characterized. However, a mismatch between the theoretical description of the measurements and their actual implementation may lead to erroneous conclusions about the presence of entanglement [2]. A way of avoiding this assumption is to use device-independent techniques [3], where the measuring devices are not trusted to behave as expected, and no specific description of the experimental observables is assumed. In this approach the measurement devices are considered as black boxes that the parties can access with classical inputs (corresponding to the measurement choices) that provide classical outputs (considered as the measurement results). The presence of entanglement is then verified analyzing the correlation statistics between the data lists corresponding to the measurement results. The violation of Bell inequalities [4] certify the presence of entanglement in this scenario, which can be thought of as a device-independent entanglement witness. The device-independent approach is especially important in adversarial scenarios, such as device-independent quantum key distribution [5], where an adversary can use a mismatch between the real implementation of the protocol and its description to fake its performance [6–8]. However, the violation of a Bell inequality requires a high degree of correlation between the parties tolerating then very low levels of noise and demanding highly efficient detectors and high-quality entangled states [3].

An intermediate scenario between the standard and the device-independent cases is that of quantum steering [9, 10]. This is the situation where, in the bipartite case, one of the parties uses a trusted measuring device but the other does not. As such, we refer to this approach as the semi-device-independent one. Apart from the fundamental importance of characterizing separability in different scenarios, quantum steering appears as a practical situation that is less demanding experimentally than the device-independent approach. It requires fewer assumptions than the standard case and lower strength for the quantum correlations to be witnessed or certified. For these reasons the study of quantum steering, including its applications [11, 12] and experimental demonstrations [13–17], have increased rapidly over recent years.

In the multipartite case, much knowledge has been acquired concerning standard entanglement detection [1] and the device-independent case [18–22]. However, only few results were found in the semi-device-independent case. For instance, Ref. [23] provides inequalities to rule out fully separable states, Ref. [24] developed a probabilistic protocol to detect the presence of a particular multipartite entangled state, and Ref. [25] discussed a hybrid model where each party is sometimes trusted and sometimes untrusted.

Here we propose a general method to detect all kinds of entanglement that can be present in a quantum network, where some of the parties use untrusted measurements and must use data lists. We show how the different types of entanglement constrain the corresponding observed experimental data and present an efficient method to obtain semi-device-independent entanglement witnesses. We furthermore implement this method in an optical experiment and demonstrate the presence of genuine tripartite entanglement in both scenarios where either one or two parties perform untrusted measurements. Finally, we also quantify the advantage that the present approach provides over the device-independent one in terms of tolerance to noise.

For simplicity, in the main text we will explain our idea for the case of detecting genuine multipartite entanglement in a tripartite system. This case contains all the basic ingredients needed to understand both how to detect other types of entanglement and how to treat systems composed of more parties. These procedures are

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The probability distributions of Alice and Bob’s measurements is encoded in $p(ab|x)=\text{tr}\sigma^C_{a|x}p_B$.

If the initial state $\rho^{ABC}$ contains no genuine multipartite entanglement, i.e. it is biseparable, then it has the form

$$\rho^{ABC} = \sum_\lambda \rho^{A:BC}_\lambda \rho^{A}_\lambda \otimes \rho^{BC}_\lambda + \sum_\mu \rho^{B:AC}_\mu \rho^{B}_\mu \otimes \rho^{AC}_\mu + \sum_\nu \rho^{AB:C}_\nu \rho^{AB}_\nu \otimes \rho^{C}_\nu,$$

(3)

where $\rho^{A:BC}_\lambda$, $\rho^{B:AC}_\mu$ and $\rho^{AB:C}_\nu$ are probability distributions. Then the assemblages (1) and (2) have the form

$$\sigma^C_{a|x} = \text{tr}(M_{a|x} \otimes \mathbb{I}_B \otimes \mathbb{I}_C \rho^{ABC})$$

$$= \sum_\lambda \rho^{A:BC}_\lambda p_A(a|x) \rho^{A}_\lambda$$

$$+ \sum_\mu \rho^{B:AC}_\mu \rho^{B}_\mu \otimes \sigma^C_{a|x,\mu}$$

$$+ \sum_\nu \rho^{AB:C}_\nu \sigma^C_{a|x,\nu} \otimes \rho^{C}_\nu$$

(4)

and

$$\sigma^C_{ab|xy} = \text{tr}_{AB}(M_{a|x} \otimes M_{b|y} \otimes \mathbb{I}_C \rho^{ABC})$$

$$= \sum_\lambda \rho^{A:BC}_\lambda p_A(a|x) \rho^{C}_{a,\lambda,y}$$

$$+ \sum_\mu \rho^{B:AC}_\mu \rho^{C}_{b,y} \sigma^C_{a|x,\mu}$$

$$+ \sum_\nu \rho^{AB:C}_\nu \rho^{C}_{a,b|xy}$$

(5)

respectively. Thus, the fact that the original state is biseparable imposes constraints on the observed assemblages. For instance, in the second line of (4) the dependence on the variables $a$ and $x$ is only through the distribution $p_A(a|x)$ and not through the quantum states, while the third and fourth lines contain only separable states. These constraints allow the parties holding the trusted devices to determine if the distributed state must have been genuine multipartite entangled: if the observed data admits no decomposition of the form (4) or (5) then there exists no biseparable state that could explain it. Therefore, even not knowing the initial state or what type of measurements the untrusted parties performed, it is possible to discriminate the assemblages that were produced by genuinely multipartite entangled initial states from those produced by biseparable ones.

Checking for the existence of decompositions of the form (4) or (5) is a computational hard task, since deciding if a state is separable, or if a probability distribution has a quantum realization, are both computational demanding [26, 27]. However, we show, in the Supplementary Material, how to combine techniques developed in the framework of entanglement [28, 29] and nonlocality [27] to relax the problem so that checking for an approximate decomposition become a certain type of optimization problem known as a semidefinite program (SDP),
for which efficient numerical methods exist. Crucial for experimental usefulness (see Methods), this SDP methods also provides a witness, similar to an entanglement witness or a Bell inequality, whose violation certifies entanglement for any experimental assemblage. As examples, we use our method to produce the following inequality that is satisfied by all biseparable states when Alice holds the untrusted measurements:

\[
1 + 0.1547 \langle Z_B Z_C \rangle - \frac{1}{2} \left( \langle A_2 Z_B \rangle + \langle A_2 Z_C \rangle + \langle A_0 X_B X_C \rangle - \langle A_0 Y_B Y_C \rangle - \langle A_1 X_B Y_C \rangle - \langle A_1 Y_B X_C \rangle \right) \geq 0,
\]

with \( A_i \) being observables in Alice’s system with outcomes labeled ±1 and X, Y and Z representing the Pauli operators. The pure GHZ state violates this inequality by \(-0.8453 \not\geq 0\). Similar inequalities for two untrusted parties and for the W state are presented in the appendix.

In order to illustrate the utility and efficiency of our approach, we use this technique to demonstrate the presence of genuine tripartite entanglement in a real laboratory setting where one or two parties perform untrusted measurements. The experimental setup is shown in Fig 2 and is set to produce a GHZ state encoded in the polarization and path degree of freedom of two photons [30, 31] with high fidelity. The experimental procedure starts by preparing photons in a state close to

\[
|\Psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \otimes |0\rangle_{B_1},
\]

where \( A_p \) and \( B_p \) represent the polarization qubit of photons A and B respectively, where 0 and 1 stand for horizontal and vertical polarization states, and \( B_s \) represents the spatial degree of freedom of photon B. To obtain a GHZ state, we couple the spatial degree of freedom with the polarization using Beam Displacer (BD1) which transforms \(|0\rangle_{B_1} \rightarrow |0\rangle_{B_2} |0\rangle_{B_s} \) and \(|1\rangle_{B_1} \rightarrow |1\rangle_{B_2} |1\rangle_{B_s} \). Once we obtain the desired state, every qubit is measured in the eigenstates of the three Pauli operators. For the polarization degrees of freedom this is carried out using a quarter-wave-plate (QWP), a half-wave plate (HWP) and a polarizing beam splitter (PBS) or BD2, depending on the photon. For the spatial degrees of freedom this is carried out using the interferometer described in Fig. 2 [30, 31].

Although this experiment is tailored to produce a GHZ state and perform Pauli measurements, the analysis we perform on the experimental data makes no assumption about the state nor the untrusted measurements. We consider two cases, one where part \( A_p \) is untrusted and parts \( B_p \) and \( B_s \) hold the trusted devices, and another when parts \( B_p \) and \( B_s \) hold the untrusted devices and part \( A_p \) the trusted one. We experimentally determine...
the assemblages obtained in these two cases and use the SDP method described in the Supplementary Material to derive inequalities of the form $S \geq 0$ whose violation certify that they cannot be written in the biseparable form (4) or (5), respectively. Experimental violation of these inequalities are shown in Fig. 2 b), for 215 experimental runs, giving an average violation of $S = -0.82 \pm 0.05$ for one untrusted part and $S = -0.56 \pm 0.04$ for two untrusted parties. This proves that there exists no biseparable tripartite state and measurements performed by the untrusted parties that could have generated the observed assemblages. We thus guarantee in a semi-device-independent manner the presence of genuine multipartite entanglement in the experimental system.

Finally, an advantage of the semi-device-independent approach is able to certify weaker quantum correlations as compared to the fully device-independent scenario [21]. For example, let us consider how much white noise can be added to tripartite GHZ and W states, until we are unable to detect genuine multipartite entanglement. Specifically, we quantify the minimum $w$ for which our method guarantees that following states are genuinely multipartite entangled:

$$\rho_\psi = w|\psi\rangle\langle\psi| + (1-w)\mathbb{1}/8,$$

where $|\psi\rangle$ can be either the GHZ or the W state. The results are summarized in Table III, together with the known bounds for the device-independent case [21]. One can see that trusting some of the parties offers a significant advantage in terms of noise tolerance.

| no. untr. meas. | 1    | 2    | 3    |
|-----------------|------|------|------|
| GHZ             | $\approx 0.54$ | $\approx 0.65$ | $2/3 \approx 0.70$ |
| W               | $\approx 0.57$ | $\approx 0.67$ | $\approx 0.72$ |

TABLE I: Critical robustness to white noise $w$. We provide a comparison between the critical robustness to white noise of the GHZ and W states above which genuine multipartite entanglement can be detected in 3 different scenarios: when 1 and 2 parties hold untrusted devices, for which we used the semi-device-independent method developed here and when all devices are untrusted, i.e. the device-independent case developed in [21].

In conclusion, we have derived a method to detect multipartite entanglement when some of the apparatuses used in a quantum network are untrusted or uncharacterized. The method allows the detection of multipartite entanglement in quantum networks where some of the observers use their measurement apparatuses simply as data lists. This scenario is experimentally less demanding than the nonlocality scenario, as it tolerates more noise, for instance. We have performed an experiment demonstrating the existence of genuine tripartite entanglement in a setup, without any assumption on the source or the measurements being performed in some of the subsystems. The results provide a feasible test for multipartite entanglement in quantum networks and bridges the two well known cases of multipartite entanglement and multipartite Bell nonlocality. Moreover the scenario considered is a natural generalization of bipartite quantum steering [10]. Since steering has found applications in cryptographic protocols [11, 33], we believe that our results can be used as a starting point to define semi-device-independent cryptographic applications in future quantum networks.

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Note added: After finishing this work we became aware of arXiv:1412.7212, which reported an experiment demonstrating multipartite entanglement in an asymmetric network composed by continuous-variable entangled states.

### Appendix A: Data analysis

Due to experimental errors and finite statistics the observed assemblage is not compatible with any physical state and local measurements. In particular it does not satisfy the no-signaling property. Since our method is tailored to detect entanglement of physical states we can not use the observed data directly. We then use the following steps: (i) we first find a physical assemblage that approximates the observed data. This is done through a least squares optimisation to find the physical assemblage which will have all probabilities as close as possible to those experimentally observed. Crucially, this can be done through semi-definite programming. (ii) Second we use the SDP method discussed in the Appendix to generate an inequality that is satisfied by assemblages coming from biseparable states. (iii) We finally show that the observed data violates this inequality, thus proving the existence of tripartite entanglement.

### Appendix B: Characterising multipartite assemblages

Here we determine the constraints that different types of entanglement in the initial state $\rho^{ABC}$ impose on the assemblages produced by untrusted measurements and define the corresponding sets that they characterise. We will first consider the question of whether or not there is any entanglement in the state, before moving on to entanglement in a given bipartition, for which there are a number of different cases, given the asymmetry of the scenario, and finish with the question of detecting genuine multipartite entanglement.
Case 1. Multipartite entanglement

Let us then start by considering a state $\rho_{A:B:C}$ that is fully separable i.e.

$$\rho_{A:B:C} = \sum_\lambda p_\lambda \rho_\lambda^A \otimes \rho_\lambda^B \otimes \rho_\lambda^C,$$  

(B1)

where $p_\lambda$ defines a probability distribution.

Case 1A. Multipartite entanglement with one untrusted party

We treat first the case where a single party, taken to be $A$, performs a set of untrusted measurements $\{M_a|x\}_{a,x}$ on her share of the state, providing the parties $B$ and $C$ with the assemblage

$$\sigma_{a|x}^{BC} = \text{tr}_A(M_a|x \otimes \mathbb{1}_B \otimes \mathbb{1}_C \rho_{A:B:C}^{A:B:C}) = \sum_\lambda p(a|x, \lambda) \rho_\lambda^B \otimes \rho_\lambda^C,$$  

(B2)

where $p(a|x, \lambda) = p_\lambda \text{tr}(M_{a|x} \rho_\lambda^A)$. Notice first that the dependence of Bob and Charlie’s assemblage on $a$ and $x$ comes only from the common pre-shared variable $\lambda$. This is a typical instance of an unsteerable assemblage, or, in other words, this is a local hidden state (LHS) model for the assemblage $\sigma_{a|x}^{BC}$ [10]. Notice further that it is composed only by separable (unnormalised) states for $B$ and $C$. Thus, in this case, testing for the presence of multipartite entanglement reduces to testing if the assemblage $\sigma_{a|x}^{BC}$ is steerable and separable at the same time. This type of assemblages forms a set $\Sigma_{BC}^{A:B:C}$, given by

$$\Sigma_{BC}^{A:B:C} = \{ \sigma^{BC}_{a|x} | \sigma^{BC}_{a|x} = \sum_\mu D(a|x, \mu) \rho_\mu^{BC} : \rho_\mu^{BC} \in \text{SEP} \}$$  

(B3)

where we have used the fact that any probability distribution $p(a|x, \lambda)$ can be written as a convex combination of deterministic ones $D(a|x, \mu)$, i.e. $p_\lambda(a|x) = \sum_\mu q(\mu|\lambda) D(a|x, \mu)$, and denote the set of (unnormalised) separable quantum states by SEP.

Case 1B. Multipartite entanglement with two untrusted parties

Now we consider that two of the parties, $A$ and $B$, have untrusted measuring devices. In this case they prepare an assemblage for Charlie, which is given by

$$\sigma_{ab|xy}^C = \text{tr}_{AB}(M_{a|x} \otimes M_{b|y} \otimes \mathbb{1}_C \rho_{A:B:C}^{A:B:C}) = \sum_\lambda p_\lambda (ab|xy) \rho_\lambda^C.$$  

(B4)

Once again, since the only dependence of the assemblage on $a$, $b$, $x$ and $y$ is through $\lambda$ this assemblage is also unsteerable. Moreover, because the set of probability distributions (also called a behaviour) $p_\lambda(ab|xy)$ arises from local measurements on a separable state it must be local, i.e. it can be written as $p_\lambda(ab|xy) = \sum_{\mu\nu} q(\mu|\nu|\lambda) D(a|x, \mu) D(b|y, \nu)$ [3]. Therefore, the relevant set of assemblages $\Sigma_C^{A:B:C}$ is now given by

$$\Sigma_C^{A:B:C} = \{ \sigma_{ab|xy}^C | \sigma_{ab|xy}^C = \sum_{\mu\nu} D(a|x, \mu) D(a|x, \nu) \rho_\mu^{BC}, \rho_\nu^{BC} \geq 0 \}.$$  

(B5)

Case 2. Entanglement in a bipartition

Let us now consider the case where the state $\rho_{A:B:C}$ is separable with respect to a single given bipartition. Choosing this partition to be $A : BC$ we now consider states of the form

$$\rho_{A:B:C} = \sum_\lambda p_\lambda \rho_\lambda^A \otimes \rho_\lambda^{BC}.$$  

(B6)

Crucially, given the asymmetry of picking a bipartition as well as the asymmetry of picking the trusted party (or parties), we will see below that we have two inequivalent situations to consider, for both the cases of one or of two untrusted parties.

Case 2A. Entanglement in a bipartition with one untrusted party

When only one of the parties performs uncharacterised measurements the asymmetry of (B6) leads to two different situations: (i) when the lone party $A$ has the untrusted devices and (ii) when $B$ (or equivalently $C$) does. In the first case Bob and Charlie’s assemblage is given by

$$\sigma_{a|x}^{BC} = \text{tr}_A(M_{a|x} \otimes \mathbb{1}_B \otimes \mathbb{1}_C \rho_{A:B:C}^{A:B:C}) = \sum_\lambda p(a|x, \lambda) \rho_\lambda^{BC},$$  

(B7)

Once again the dependence of Bob and Charlie’s assemblage on $a$ and $x$ comes only from the common variable $\lambda$, and so this assemblage is unsteerable. In comparison to previously, the states distributed to Bob and Charlie are now arbitrary entangled states, and hence there is no additional structure that the decomposition imposes. The set $\Sigma_{BC}^{A:B:C}$ defined by assemblages of the form (B7) is therefore given by

$$\Sigma_{BC}^{A:B:C} = \{ \sigma_{a|x}^{BC} | \sigma_{a|x}^{BC} = \sum_\mu D(a|x, \mu) \rho_\mu^{BC}, \rho_\mu^{BC} \geq 0 \}.$$  

(B8)

In the second case, where Bob is the one not trusting his measurements, Alice and Charlie are left with the following assemblage:

$$\sigma_{ab|y}^C = \text{tr}_B(\mathbb{1}_A \otimes M_{b|y} \otimes \mathbb{1}_C \rho_{A:B:C}^{A:B:C}) = \sum_\lambda p_\lambda \rho_\lambda^A \otimes \sigma_{\mu|y,\lambda}^C.$$  

(B9)
which is a fundamentally different situation. This assemblage now has two main features: (i) the only dependence on the variables \( b \) and \( y \) are due to Charlie’s states and the variable \( \lambda \). In other words, \( \sigma_{b|y}^{AC} \) is unsteerable from Bob to Alice but not necessarily from Bob to Charlie. This implies that if we trace out system \( C \) (or apply any quantum-to-classical map to it) the resulting assemblage for Alice alone will be unsteerable; (ii) it is composed by separable (unnormalised) states. The relevant set, \( \Sigma_{AC}^{A:BC} \), is now given by

\[
\Sigma_{AC}^{A:BC} = \left\{ \sigma_{b|y}^{AC} \middle| \text{tr}_C \sigma_{b|y}^{AC} = \sum_\mu D(b|y,\mu)\rho_\mu^A, \right. \\
\left. \rho_\mu^A \geq 0, \sigma_{b|y}^{AC} \in \text{SEP} \right\} \quad (B10)
\]

**Case 2B. Entanglement in a bipartition with two untrusted parties**

Again the asymmetry of the decomposition (B6) leads to two different situations: In one the untrusted measurements are at \( A \) and \( B \) (or similarly \( A \) and \( C \)), whilst in the other they are at \( B \) and \( C \). In the first case the assemblage obtained is given by

\[
\sigma_{ab|xy}^C = \text{tr}_{AB}(M_{a|x} \otimes M_{b|y} \otimes I_C \rho_{A:BC}) \\
= \sum_\lambda \rho(a|x,\lambda)\sigma_{b|y,\lambda}^C. \quad (B11)
\]

This assemblage has only one main feature, that it may contain only steering from Bob to Charlie, and not from Alice to Charlie. It then defines the set \( \Sigma_{C}^{A:BC} \) as

\[
\Sigma_{C}^{A:BC} = \left\{ \sigma_{ab|xy}^C \middle| \sigma_{ab|xy}^C = \sum_\mu D(a|x,\mu)\sigma_{b|y,\mu}^C, \sigma_{b|y,\mu}^C \geq 0 \right\} \quad (B12)
\]

In the second case the resulting assemblage is given by

\[
\sigma_{bc|yz}^A = \text{tr}_C(\mathbb{I}_A \otimes M_{b|y} \otimes M_{c|x} \rho_{A:BC}) \\
= \sum_\lambda \rho(b|y,z,\lambda)\rho_\lambda^A. \quad (B13)
\]

Here, there are two main features: (i) this assemblage is unsteerable; (ii) The behaviour \( \rho(bc|yz,\lambda) \) arises from local measurements on a possibly entangled state \( \rho_\lambda^{ALC} \), it may contain nonlocal quantum correlations \([3]\). The final set we define is therefore \( \Sigma_{A}^{A:BC} \), given by

\[
\Sigma_{A}^{A:BC} = \left\{ \sigma_{bc|yz}^A \middle| \sigma_{bc|yz}^A = \sum_\lambda \rho(b|y,z,\lambda)\sigma_\lambda^A, \right. \\
\left. \sigma_\lambda^A \geq 0, \rho(b|y,z,\lambda) \in Q \right\} \quad (B14)
\]

where we have denoted by \( Q \) the set of probability distributions which can arise from local measurements on quantum states.

**Case 3. Genuine multipartite entanglement**

Let us now turn to the question of genuine multipartite entanglement (GME) detection. Genuine tripartite entangled states are the ones that can not be written as tight entanglement (GME) detection. Genuine multipartite entanglement with one untrusted party

\[
\rho_{\text{bisp}} = \sum_\lambda \rho_{A:BC}^{\lambda} \rho_{A}^{\lambda} \otimes \rho_{BC}^{\lambda} + \sum_\lambda \rho_{AB:C}^{\lambda} \rho_{A}^{\lambda} \otimes \rho_{AB}^{\lambda}, \quad (B15)
\]

where \( \rho_{A:BC}^{\lambda}, \rho_{A}^{\lambda} \) and \( \rho_{AB:C}^{\lambda} \) are probability distributions. Our goal once again is to determine what constraints the form (B15) imposes on the obtained assemblages, which will now follow straightforwardly given the analysis made before.

**Case 3A. Genuine multipartite entanglement with one untrusted party**

When Alice is the one holding the untrusted devices, Bob and Charlie’s assemblage is given by

\[
\sigma_{a|x}^{BC} = \text{tr}(M_{a|x} \otimes I_B \otimes I_C \rho_{\text{bisp}}) \\
= \sum_\lambda \rho_{A:BC}^{\lambda} \rho(a|x,\lambda)\rho_\lambda^A + \sum_\lambda \rho_{AB:C}^{\lambda} \rho_{a|x,\lambda}^B \otimes \rho_\lambda^C. \quad (B16)
\]

The terms \( \Gamma_{a|x}^{A:BC} \) and \( \Gamma_{a|x}^{B:AC} \) can be seen as assemblages having the same structure as the assemblages (B7) and (B9) respectively, while the assemblage \( \Gamma_{a|x}^{C:AB} \) is identical to \( \Gamma_{a|x}^{B:AC} \), except that the role of Bob and Charlie is interchanged.
Case 3B. Genuine multipartite entanglement with two untrusted parties

Consider now that Alice and Bob perform untrusted measurements, leading to:

\[ \sigma^C_{ab|xy} = \text{tr}_{AB}(M_{a|x} \otimes M_{b|y} \otimes \mathbb{1}_C \rho_{\text{bisep}}) \]

\[ = \sum_{\lambda} \rho^{A:BC}_{\lambda|a} p(a|x, \lambda) \sigma^C_{\beta|y,\lambda} + \sum_{\lambda} \rho^{B:AC}_{\lambda|b} p(b|y, \lambda) \sigma^C_{\alpha|x,\lambda} + \sum_{\lambda} \rho^{AB:C}_{\lambda|c} p(ab|xy, \lambda) \rho^C_{\lambda} . \]  

(B17)

Again, the assemblages \( \Pi^{A:BC}_{a|b|xy} \) and \( \Pi^{B:AC}_{b|a|xy} \) are seen to have the same structure as (B11), while the assemblage \( \Pi^{C:AB}_{a|b|xy} \) has the same structure as (B13).

Appendix C: SDP tests and semi-device-independent entanglement witnesses

We have previously determined the constraints that each kind of entanglement imposes, and defined the corresponding sets of assemblages these constraints define. We now turn to the following practical question: given that we have observed a specific assemblage, can we test for a certain type of entanglement by checking whether or not the assemblage belongs to one of the previously defined sets?

Crucially, it turns out that all of the sets defined above are either specified solely in terms of positive semi-definite (PSD) constraints and linear matrix inequalities (LMIs), or can be approximated from the outside by a set with such a specification. Testing for membership inside such a set is an optimisation problem known as a semi-definite program (SDP), for which efficient numerical methods exist for the case of small systems, allowing for an answer to this question [34]. Moreover, due to the theory of duality, the dual SDP provides us with a semi-device-independent witness that allows us to certify the presence of the different types of entanglement solely from the knowledge of the assemblage. This is similar to the ideas of entanglement witnesses and Bell inequalities in the standard and fully device-independent scenarios respectively.

1. Deriving the SDP tests

In those cases where the sets defined above are not specified solely in terms of PSD constraints and LMIs our strategy is to show that there exist suitable relaxations which are, that is to define bigger sets which are specified solely in terms of such constraints.

Working through in the order that they appeared, we shall consider each set in turn. The first set is \( \Sigma^{A:B:C} \), given in equation (B3). It is the final constraint, \( \rho^{BC}_\mu \in \text{SEP} \), that does not have the desired form, as the set of (unnormalised) separable states has in general a complicated structure. The one case where the set in fact has a simple characterisation is if the dimensions match \( d_Bd_C \leq 6 \), in which case the set of separable states is exactly the set of states positive under partial transposition (PPT) [35]. In this simple case we can rewrite \( \rho^{BC}_\mu \in \text{SEP} \) as \( (\rho^{BC}_\mu)^{T_B} \geq 0 \), where \( T_B \) denotes the partial transposition with respect to system B. Since this operation is a linear map (on the state), this is now a PSD constraint, and the set is in fact in the desired form.

In all other dimensions, we can use the relaxation of the separable states to those that have a \( k \)-symmetric PPT extension [28]. That is, we define the set \( \text{SYM}^{(k)}_{BC} \)

\[ \text{SYM}^{(k)}_{BC} = \left\{ \rho^{BC}_\mu \mid \rho^{BC}_\mu = \text{tr}_{B_2 \cdots B_k} \rho^{B_1 \cdots B_k C}, \right. \]

\[ \left. S_{ij} \rho^{B_1 \cdots B_k C} S_{ij}^\dagger = \rho^{B_1 \cdots B_i, B_{i+1} \cdots B_k} \quad \forall i \neq j, \quad (\rho^{B_1 \cdots B_k C})^{T_C} \geq 0, \rho^{B_1 \cdots B_k C} \geq 0 \right\} \]  

(C1)

where \( S_{ij} \) is the swap operator between \( B_i \) and \( B_j \). This demands that \( \rho^{BC}_\mu \) can be extended to a state with \( k \) Bobs, which is symmetric under interchange and PPT, such that the reduced state of a single Bob and Charlie is the original state. Such sets are all specified in terms of PSD constraints and LMIs, and converge to the set of separable states as \( k \to \infty \) [28, 29]. For the case \( k = 1 \) it also reduces to the set of PPT state. We thus define the sequence of relaxations

\[ \Sigma^{A:B:C(k)}_{BC} = \left\{ \sigma^{BC}_{a|x} \mid \sigma^{BC}_{a|x} = \sum_{\mu} D(a|x, \mu) \rho^{BC}_\mu, \quad \rho^{BC}_\mu \in \text{SYM}^{(k)}_{BC} \right\} \]  

(C2)

which now have the desired structure for each \( k \).

Moving on, the set \( \Sigma^{A:B:C} \) given in (B5) already has the desired structure. This is also true for the set \( \Sigma^{A:BC} \).
defined in (B8). The set $\Sigma_{AC}^{A:BC}$ (10) contains the requirement that $\sigma_{by}^{AC} \in SEP$, which is dealt with in exactly the same way as above. Thus we define the relaxed set

$$
\Sigma_{AC}^{A:BC(k)} = \left\{ \sigma_{by}^{AC} \mid \text{tr}_C \sigma_{by}^{AC} = \sum_{h} D(b|y, \mu) \rho_{A}^{A}, \right. \\
\left. \rho_{A}^{A} \geq 0, \sigma_{by}^{AC} \in \text{SYM}_{AC}^{(k)} \right\}  \quad (C3)
$$

The set $\Sigma_{AC}^{A:BC}$ given in (B12) has the desired structure.

Finally, the set $\Sigma_{C:AB}^{C:AB}$ (which is has the same structure as (B14)) is less straightforward because of the constraint $p(ab|xy, \lambda) \in Q$, i.e. that the behaviours $p(ab|xy, \lambda)$ should have a quantum realisation. As in the nonlocality scenario of deciding if a behaviour has a quantum realisation, the exact answer to this problem is in general intractable. However, we can use the idea introduced in [36] (see also [37]) to obtain a semi-definite relaxation.

The basic idea is to apply the method of the NPA hierarchy [27] only to the untrusted devices, whilst leaving Charlie quantum. We thus relax to $\Pi_{ab|xy}^{C:AB} \in Q^{(k)}$, where $Q^{(k)}$ is defined by

$$
Q^{(k)}_{ab|xy} = \left\{ \sigma_{ab|xy}^{C} \mid \Gamma_{C}^{(k)} \geq 0, \text{tr} (G_{j} \Gamma_{C}^{(k)}) = \text{tr} (h_{j} \sigma_{ab|xy}^{C}) \forall j \right\},  \quad (C4)
$$

for some sets of operators $\{G_{j}^{(k)}\}$ and $\{h_{j}^{(k)}\}$, which encode the constraints that arise in the original NPA hierarchy [27], coming from (i) orthogonality of measurement outcomes, and (ii) commutativity of Alice and Bob measurements. The main difference with the NPA approach is that whereas previously the elements of the matrix $\Gamma_{C}^{(k)}$ were complex numbers with certain ones equal to the nonlocal behaviour, now one should think of the elements as matrices, with certain ones equal to members of the assemblage.

Last, in order to be able to impose semidefinite constraints to the set $\Sigma_{C:AB}^{C:AB}$ we need to constrain the number of terms in the summation in $\lambda$. We do this by noticing that any quantum behaviour can be written as a convex combination of extremal non-signalling behaviours $D^{NS}(bc|yz, \nu)$ [3].

Given the above, set $\Sigma_{C:AB}^{C:AB}$ is relaxed to

$$
\Sigma_{C:AB}^{C:AB(k)} = \left\{ \sigma_{ab|xy}^{C} \mid \sigma_{ab|xy}^{C} = \sum_{\nu} D^{NS}(ab|xy, \nu) \sigma_{\nu}^{C}, \right. \\
\left. \sigma_{ab|xy}^{C} \in Q^{(k)}_{C} \right\}.  \quad (C5)
$$

Having found appropriate relaxations of all of the sets which we wish to consider, we can now straightforwardly write down an approximate optimisation problem in the form of an SDP that needs to be solved to check for the desired type of entanglement in each given scenario. Let us describe explicitly the approximate test that checks for the existence of a decomposition of the form (B17), i.e. that checks for genuine multipartite entanglement with two untrusted parties. We provide all the other SDP tests for the other decompositions described above in Table II.

In (B17) we see that we have to find 3 assemblages, each contained in a different set, with each set either in a form directly usable, or for which we just gave an outer approximation above. Thus, by introducing the maximally mixed assemblage $id^{C}_{ab|xy} = \frac{1}{m_{A} m_{B}} \mathbb{1}/d_{C}$ we arrive at the following SDP test for genuine multipartite entanglement with 2 untrusted parties

$$
\begin{align*}
\max \quad & p \\
\text{s.t.} \quad & \Pi_{ab|xy}^{A:BC} + \Pi_{ab|xy}^{B:AC} + \Pi_{ab|xy}^{C:AB} = \sigma_{ab|xy}^{obs} - p id^{C}_{ab|xy} \\
& \Pi_{ab|xy}^{A:BC} \in \Sigma_{C}^{A:BC}, \quad \Pi_{ab|xy}^{B:AC} \in \Sigma_{C}^{B:AC}, \quad \Pi_{ab|xy}^{C:AB} \in \Sigma_{C}^{C:AB(k)}  \quad (C6)
\end{align*}
$$

where $\sigma_{ab|xy}^{obs}$ is the observed assemblage of Charlie. Since $id^{C}_{ab|xy}$ is clearly contained in all 3 sets, being producible from the maximally mixed state, a sufficiently large negative $p$ will always be a solution, hence the SDP is strictly feasible. A strictly negative optimal solution $p^{*} < 0$ certifies that $\sigma_{ab|xy}^{obs}$ being measured does not have the desired decomposition, i.e. that the state is genuinely multipartite entangled. On the other hand an optimal value $p^{*} = 0$ indicates that a decomposition can be found. Note however that in this case, given the relaxation of the problem, one is not able to conclude anything regarding the separability of the state. One can take the parameter $k$ larger to obtain a better approximation to the original problem.

2. Sem device-independent entanglement witnesses

The dual of the SDP (C6) is also readily written down [34], and is given by

$$
\begin{align*}
\min \quad & \text{tr} \sum_{ab|xy} F_{ab|xy} \sigma_{ab|xy}^{obs} \\
\text{s.t.} \quad & \text{tr} \sum_{ab|xy} F_{ab|xy} \sigma_{ab|xy}^{C} \geq 0 \\
& \forall \sigma_{ab|xy}^{C} \in \Sigma_{C}^{A:BC} \cup \Sigma_{C}^{B:AC} \cup \Sigma_{C}^{C:AB(k)} \\
& \text{tr} \sum_{ab|xy} F_{ab|xy} id^{C}_{ab|xy} = 1
\end{align*}
$$

which is seen to constitute a witness for genuine multipartite entanglement. That is, the dual provides a set of operators $\{F_{ab|xy}\}_{ab|xy}$ such that the linear functional $\beta = \sum_{ab|xy} F_{ab|xy} \sigma_{ab|xy}$ is greater than zero for all assemblages which arise from measurements on a bi-separable state. An observed value $\beta^{obs} < 0$ thus provides a witness which certifies the genuine multipartite entanglement of the state in a semi-device-independent manner. The final condition, $\text{tr} \sum_{ab|xy} F_{ab|xy} id^{C}_{ab|xy} = 1$ is a convention, which simply defines an overall scale for the witness.
TABLE II: Collection of all SDP tests for the tripartite case. All expressions with indices should be understood to hold for each value of the index. Various SDPs depend upon a parameter $k$, such that for larger values of $k$ we obtain a better approximate characterisation of the set, and therefore a more stringent SDP test. All programs are strictly feasible, such that a negative optimal value $p^* < 0$ certifies that the assemblage has the corresponding type of entanglement. In this case the dual provides a certificate in the form of a semi-device-independent witness. An optimal solution $p^* > 0$ indicates that the assemblage is inside the corresponding set, i.e. that one cannot conclude that the state contains the desired type of entanglement.

| Form of state | Untrusted parties | Known objects | SDP |
|---------------|-------------------|---------------|-----|
| $\sum_{\lambda} p_{\lambda} \rho_A^\lambda \otimes \rho_B^\lambda \otimes \rho_C^\lambda$ | A | $\sigma_{AB}^C$ | max $p$ s.t. $\sum_{\mu} D_{\mu}(a|x) \sigma_{AB}^C = \sigma_{AB}^{C|A} - p \pi_{AB}^{C|A}$, (C7) |
| $\sum_{\lambda} p_{\lambda} \rho_A^\lambda \otimes \rho_B^{BC} \otimes \rho_A^\lambda$ | A | $\sigma_{AB}^{C|BC}$ | max $p$ s.t. $\sum_{\mu} D_{\mu}(a|x) \sigma_{AB}^{C|BC} = \sigma_{AB}^{C|BC} - p \pi_{AB}^{C|BC}$, (C9) |
| $\sum_{\lambda} p_{\lambda} \rho_A^\lambda \otimes \rho_{BC} \otimes \rho_A^\lambda$ | B | $\sigma_{BC}^A$ | max $p$ s.t. $\sum_{\mu} D_{\mu}(a|x) \sigma_{BC}^A = \sigma_{BC}^{A|BC} - p \pi_{BC}^{A|BC}$, (C11) |
| $\sum_{\lambda} p_{\lambda} \rho_{A} \otimes \rho_{AB} \otimes \rho_{AC} \otimes \rho_{BC} \otimes \rho_{A} \otimes \rho_{AC} \otimes \rho_{BC}$ | A | $\sigma_{AB}^{C|BC}$ | max $p$ s.t. $\sum_{\mu} D_{\mu}(a|x) \sigma_{AB}^{C|BC} = \sigma_{AB}^{C|BC} - p \pi_{AB}^{C|BC}$, (C13) |
| $\sum_{\lambda} p_{\lambda} \rho_{A} \otimes \rho_{BC} \otimes \rho_{A} \otimes \rho_{BC}$ | A and B | $\sigma_{AB}^{C|BC}$ | max $p$ s.t. $\sum_{\mu} D_{\mu}(a|x) \sigma_{AB}^{C|BC} = \sigma_{AB}^{C|BC} - p \pi_{AB}^{C|BC}$, (C14) |
More generally, the dual of each SDP in Table II provides witness operators \( \{ F_{ax} \} \), for the case of Alice untrusted, or \( \{ F_{abxy} \} \), for the case of Alice and Bob untrusted (or a permutation of the parties) which constitute a semi-device-independent entanglement witnesses of the form

\[
\text{tr} \sum_{ax} F_{ax} \sigma_{ax} \geq 0 \quad \forall \sigma_{ax} \in \Sigma
\]

\[
\text{tr} \sum_{abxy} F_{abxy} \sigma_{abxy} \geq 0 \quad \forall \sigma_{abxy} \in \Sigma'
\]

(C16)

with corresponding violations \( \beta_{\text{obs}} = \text{tr} \sum_{ax} F_{ax} \sigma_{ax}^{\text{obs}} < 0 \) or \( \beta_{\text{obs}} = \text{tr} \sum_{abxy} F_{abxy} \sigma_{abxy}^{\text{obs}} < 0 \) respectively, where \( \Sigma \) and \( \Sigma' \) are sets, or union of sets (depending upon the type of entanglement one is checking for), as defined above.

Finally, we note that it is possible to put these witnesses into a more friendly form in the case of binary measurement outcomes. To do so we use the definition of the observed assemblage, and introduce the observables \( A_x = M_0|x - M_1|x \) to arrive at

\[
\text{tr} \sum_{ax} F_{ax} \sigma_{ax}^{\text{obs}} = \text{tr} \sum_{ax} M_{ax} \otimes F_{ax} \rho
\]

\[
= \frac{1}{2} \text{tr} \sum_{ax} (\mathbb{1}_A + (-1)^a A_x) \otimes F_{ax} \rho
\]

\[
= \text{tr} (\mathbb{1}_A \otimes \frac{1}{2} \sum_{ax} F_{ax} + \sum_x A_x \otimes \frac{1}{2} \sum_a (-1)^a F_{ax}) \rho
\]

\[
= \text{tr} (\mathbb{1}_A \otimes J_0 + \sum_x A_x \otimes J_x) \rho
\]

(C17)

for the case of one untrusted party, where we have defined the observables \( J_0 = \frac{1}{2} \sum_{ax} F_{ax} \) and \( J_x = \frac{1}{2} \sum_a (-1)^a F_{abxy} \) for Bob and Charlie. For the case of two untrusted parties, an analogous but longer calculation gives

\[
\text{tr} \sum_{abxy} F_{abxy} \sigma_{abxy}^{\text{obs}} = \text{tr} (\mathbb{1}_A \otimes \mathbb{1}_B \otimes K_0 + \sum_x A_x \otimes \mathbb{1}_B \otimes K_x
\]

\[
+ \sum_y \mathbb{1}_A \otimes B_y \otimes K_y + \sum_{xy} A_x \otimes B_y \otimes K_{xy}) \rho
\]

(C18)

where we have introduced the observables \( B_y \) for Bob, as well as the observables \( K_0 = \frac{1}{4} \sum_{abxy} F_{abxy} \), \( K_j = \frac{1}{4} \sum_{abxy} (-1)^a F_{abxy} \), \( K_y = \frac{1}{4} \sum_{ab} (-1)^b F_{abxy} \), and \( K_{xy} = \frac{1}{4} \sum_{ab} (-1)^{a+b} F_{abxy} \) for the trusted party Charlie, that need to be measured in the corresponding configurations, given above.

Appendix D: Generalisation to more parties

We have presented our main results in the tripartite case. Notice however that the same procedure can readily be followed to derive SDPs to test the presence of different kinds of entanglement for general \( N \)-partite systems.

First of all one specifies the scenarios by fixing (i) a particular type of entanglement and (ii) the pattern of trusted and untrusted parties. The entanglement can be chosen arbitrarily, for example one may ask that the state is not fully separable, be separable across a given number of fixed bipartitions, or be a convex combination of states separable over a given number of partitions (but not necessarily fixed). The pattern may also be chosen arbitrarily, ranging from all but one party trusted, to all but one untrusted.

Given the specification, one then enumerates the list of properties which the corresponding assemblages have. These properties will fall into two classes - those which impose constraints which are directly applicable, i.e. are in the form of PSD constraints and LMI, and those which are not. As in the tripartite case, the objective is then to relax the non-directly applicable constraints to find an approximate SDP test.

The main difficulty in our approach is that as the number of parties increases, and the local dimension of the Hilbert space, we expect that the difficulty of the problem will grow to the point where current numerical techniques are unable to solve efficiently the tests. For example, one class of constraints that will arise is that multipartite assemblages will need to have quantum realisations. In principle such a constraint can still be imposed by applying the NPA hierarchy [27] to the untrusted devices, however in the multipartite setting this soon becomes intractable. Alternatively, one may have constraints that a multipartite quantum state is separable. One can again relax this using the generalisation of the k-shareability condition [29].

In summary, the approach presented here is most suitable to scenarios involving relatively small numbers of parties, where it provides powerful tests for multipartite entanglement (and explicitly provides witnesses in each case). This is however expected as this is also the case in standard entanglement detection techniques [1] (due to the increase of the Hilbert space dimension) and in the fully device-independent approach [3] (due to the number of the space of local probability distributions).

Appendix E: Examples: GHZ and W states

In order to demonstrate the usefulness of our previous characterisation we apply the above SDP to two exemplary genuine multipartite states, namely the GHZ and W states. More specifically we are interested in how much white noise can be added to these states until our method fails to detect either entanglement or GME, i.e. we want find the minimum w, denoted by \( w^* \), allowing us to detect either entanglement or GME in the
\[ \rho_{\text{GHZ}} = w|\text{GHZ}\rangle\langle\text{GHZ}| + (1 - w)\mathbb{I}/8; \]
\[ \rho_{W} = w|W\rangle\langle W| + (1 - w)\mathbb{I}/8, \]
where \(|\text{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2} \) and \(|W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3} \). Table III gives a summary of the results, in terms of the numbers provided by our methods and a comparison to what was known regarding entanglement witnesses and Bell inequalities. All results were obtained using CVX [34] for MATLAB to solve the SDP, and the optimisation toolbox to numerically search for the best choices of measurements for Alice (and Bob). Since such a search over measurements choices provides no guarantee that the global optimum is obtained, all results constitute upper bounds. However, all of our numerical evidence suggests that the values obtained cannot be improved.

As we can see the values of \(w^*\) lies in between the bound for entanglement, where the largest number of assumptions are made, and the bound from nonlocality, where no assumptions are made. Furthermore, as one would expect, stronger bounds are possible with 2 parties trust their devices compared to the case of only 1.

We end by presenting the steering witnesses we obtain in the above for the GHZ and W states that certify genuine tripartite entanglement in a semi-device-independent fashion.

Starting with the GHZ state and the case of two untrusted parties (and three measurements), the optimal witness is

\[ 1 - \alpha\langle A_2B_2\rangle - \alpha\langle A_2Z\rangle - \alpha\langle B_2Z\rangle - \beta\langle A_0B_0X\rangle + \beta\langle A_0B_1Y\rangle + \beta\langle A_1B_0Y\rangle + \beta\langle A_1B_1X\rangle \geq 0 \]

(E2) where \(\alpha = 0.1831\) and \(\beta = 0.2582\), and the pure GHZ state achieves a violation \(-0.5821 \not> 0\). For the case of the GHZ state and only a single untrusted party, the witness is

\[ 1 + 0.1547\langle Z_BZ_C\rangle - \frac{1}{2}(\langle A_2Z_B\rangle + \langle A_2Z_C\rangle + \langle A_0X_BX_C\rangle - \langle A_0Y_BY_C\rangle - \langle A_1XBY_C\rangle - \langle A_1YBX_C\rangle) \geq 0 \]

(E3) with the pure GHZ state now achieving a violation of \(-0.8453 \not> 0\). Interestingly, we note first that the structure of both witnesses is the same, the only difference being in the coefficients. Furthermore the only terms which appear are those which arise from the stabiliser relations of the GHZ state.

Moving on to the W state, for two untrusted parties the optimal witness is

\[ 1 + 0.2517(\langle A_2\rangle + \langle B_2\rangle) + 0.3520(\langle Z\rangle - 0.1112(\langle A_0X\rangle + \langle A_1Y\rangle + \langle B_0X\rangle + \langle B_1Y\rangle) + 0.1296(\langle A_2Z\rangle + \langle B_2Z\rangle) - 0.1943(\langle A_0B_0\rangle + \langle A_1B_1\rangle) + 0.2277(\langle A_2B_2\rangle - 0.1590(\langle A_0B_0Z\rangle + \langle A_1B_1Z\rangle) + 0.2228(\langle A_2B_2Z\rangle - 0.2298(\langle A_0B_2X\rangle + \langle A_1B_2Y\rangle + \langle A_2B_0X\rangle + \langle A_2B_1Y\rangle) \geq 0 \]

(E4) and the pure W state obtains the violation \(-0.4803 \not> 0\). For one untrusted party the witness is

\[ 1 + 0.4405(\langle Z_B\rangle + \langle Z_C\rangle) - 0.0037(\langle Z_BZ_C\rangle) - 0.1570(\langle X_BX_C\rangle + \langle Y_BY_C\rangle + \langle A_2X_BX_C\rangle + \langle A_2Y_BY_C\rangle) + 0.2424(\langle A_2 + A_2Z_BZ_C\rangle) + 0.1848(\langle A_2Z_B\rangle + \langle A_2Z_C\rangle) - 0.2533(\langle A_0X_B\rangle + \langle A_0X_C\rangle + \langle A_1Y_B\rangle + \langle A_1Y_C\rangle + \langle A_0X_BX_C\rangle + \langle A_1Y_BZ_C\rangle + \langle A_1Z_BY_C\rangle) \geq 0 \]

(E5) with he pure W state achieving the violation \(-0.7594 \not> 0\). Again, we note that structurally the witnesses are the same in the case of one and two untrusted parties.

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