WHEN IS THE ALBANESE MORPHISM AN ALGEBRAIC FIBER SPACE IN POSITIVE CHARACTERISTIC?

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Abstract. In this paper, we study the Albanese morphisms in positive characteristic. We prove that the Albanese morphism of a variety with nef anti-canonical divisor is an algebraic fiber space, under the assumption that the general fiber is $F$-pure. Furthermore, we consider a notion of $F$-splitting for morphisms, and investigate it of the Albanese morphisms. We show that an $F$-split variety has $F$-split Albanese morphism, and that the $F$-split Albanese morphism is an algebraic fiber space. As an application, we provide a new characterization of abelian varieties.

1. Introduction

The Albanese morphism is an important tool in the study of varieties with non-positive Kodaira dimension. In characteristic zero, Kawamata proved that the Albanese morphism of a smooth projective variety with Kodaira dimension zero is an algebraic fiber space, that is, a separable surjective morphism with connected fibers [19, Theorem 1]. Zhang showed the same statement for a smooth projective variety with nef anti-canonical divisor [34, Corollary 2]. Under the same assumption, Cao recently proved that the Albanese morphism is locally isotrivial [4, 1.2. Theorem]. In positive characteristic, Hacon and Patakfalvi proved that the Albanese morphism of a smooth projective variety $X$ is surjective if the $S$-Kodaira dimension $\kappa_S(X)$ of $X$ is zero [12, Theorem 1.1.1]. Here, $S$-Kodaira dimension is an analog of usual Kodaira dimension defined by using the trace maps of Frobenius morphisms. Recently Wang showed that the Albanese morphism of a threefold with semi-ample anti-canonical divisor is surjective if the general fiber is $F$-pure [33, Theorem B]. In this paper, we generalize his result to varieties of arbitrary dimension, which can be viewed as a positive characteristic counterpart of the above result of Zhang.

Theorem 1.1. Let $X$ be a normal projective variety over an algebraically closed field of characteristic $p > 0$, and $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that $-m(K_X + \Delta)$ is a nef Cartier divisor for an integer $m > 0$ not divisible by $p$. Let $a : X \to A$ be the Albanese morphism of $X$, and $X_\eta$ be the geometric generic fiber over the image of $a$. If $(X_\eta, \Delta|_{X_\eta})$ is $F$-pure, then $a$ is a separable surjective morphism with connected fibers.

We also study the relation between the Albanese morphisms and Frobenius splitting. The notion of an $F$-split variety was introduced by Mehta and Ramanathan as a variety with splitting of the Frobenius morphism [23], which are considered to be related to varieties of Calabi-Yau type [10] [11] [27] [31]. As a generalization of $F$-splitting of varieties, we consider a notion of the $F$-splitting of a pair $(f, \Gamma)$ consists
of a morphism \( f : V \to W \) and an effective \( \mathbb{Q} \)-Weil divisor \( \Gamma \) on \( V \) (Definition 5.1). In this paper, we focus on the \( F \)-splitting of the Albanese morphism. Let \( X \) be a normal projective variety over an algebraically closed field \( k \) of characteristic \( p > 0 \), \( \Delta \) be an effective \( \mathbb{Q} \)-Weil divisor on \( X \), and \( a : X \to A \) be the Albanese morphism of \( X \). Then there is the following relationship between the \( F \)-splitting of \( a \) and that of \( X \).

**Theorem 1.2.** \((X, \Delta)\) is \( F \)-split if and only if \((a, \Delta)\) is \( F \)-split and \( A \) is ordinary.

We study the Albanese morphism \( a \) under the assumption that \((a, \Delta)\) is locally \( F \)-split (Definition 5.1), which is weaker than the assumption that it is \( F \)-split. For instance, a flat morphism with normal \( F \)-split fibers is locally \( F \)-split, but not necessarily \( F \)-split. The next theorem shows that the local \( F \)-splitting of \((a, \Delta)\) requires that \( a \) is an algebraic fiber space and that \( \Delta \) and fibers satisfy some geometric properties.

**Theorem 1.3.** Assume that \((a, \Delta)\) is locally \( F \)-split. Then \( a \) is a separable surjective morphism with connected fibers. Furthermore, if \( m \Delta \) is Cartier for an integer \( m > 0 \) not divisible by \( p \), then the following holds:

1. The support of \( \Delta \) does not contain any irreducible component of any fiber.
2. For every scheme-theoretic point \( z \in A \), \((X_z, \Delta_z)\) is \( F \)-split, where \( z \) is the algebraic closure of \( z \). In particular, \( X_z \) is reduced.
3. \( a \) is smooth in codimension one. In other words, there exists an open subset \( U \) of \( X \) such that \( \text{codim}(X \setminus U) \geq 2 \) and \( a|_U : U \to A \) is a smooth morphism. In particular, the general geometric fiber of \( a \) is normal.

This theorem recovers the result of Hacon and Patakfalvi when \( K_X \) is numerically trivial, because the condition \( \kappa_S(X) = 0 \) is equivalent to the \( F \)-splitting of \( X \) in that case. As a corollary of this theorem, we provide a new characterization of abelian varieties. Before stating the precise statement, we recall that the first Betti number \( b_1(X) \) of \( X \) is defined as a dimension of the \( \mathbb{Q}_l \)-vector space \( H^1_{\text{ét}}(X, \mathbb{Q}_l) \) for a prime \( l \neq p \) and is equal to \( 2 \dim A \).

**Theorem 1.4.** Assume that \((a, \Delta)\) is locally \( F \)-split (resp. \((X, \Delta)\) is \( F \)-split). Then \( b_1(X) \leq 2 \dim X \). Furthermore, the equality holds if and only if \( X \) is an abelian variety (resp. ordinary abelian variety) and \( \Delta = 0 \).

As an application of Theorem 1.4, we give a necessary and sufficient condition for a normal projective variety to have \( F \)-split Albanese morphism (Theorem 6.6). We conclude this paper with a classification of minimal surfaces with \( F \)-split or locally \( F \)-split Albanese morphisms (Theorem 7.1).

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2. Notations

Throughout this paper, we fix an algebraically closed field $k$ of characteristic $p > 0$. By $k$-scheme we mean a separated scheme of finite type over Spec $k$, and by variety over $k$ we mean an integral $k$-scheme. A curve and surface mean respectively a variety of dimension one and two. We denote by $\text{CDiv}(X)$ the group of Cartier divisors for a scheme $X$. Set $\mathbb{K} := \mathbb{Q}$ or $\mathbb{K} := \mathbb{Z}_p$. Here $\mathbb{Z}_p$ denotes the localization of $\mathbb{Z}$ at $(p) = p\mathbb{Z}$. $\mathbb{K}$-Cartier divisors on $X$ are elements of $\text{CDiv}(X) \otimes \mathbb{K}$. When $X$ is normal, we define $\mathbb{K}$-Weil divisors similarly to the above. We denote by $\sim_\mathbb{K}$ the equivalent relation of $\mathbb{K}$-Weil divisors induced by the usual linear equivalence of Weil divisors. Let $\varphi : S \to T$ be a morphism of schemes and let $T'$ be a $T$-scheme. Then we denote by $S_{T'}$ and $\varphi_{T'} : S_{T'} \to T'$ respectively the fiber product $S \times_T T'$ and its second projection. For a Cartier, $\mathbb{Z}_p$-Cartier or $\mathbb{Q}$-Cartier divisor $D$ on $S$ (resp. an $\mathcal{O}_S$-module $\mathcal{G}$), the pullback of $D$ (resp. $\mathcal{G}$) to $S_{T'}$ is denoted by $D_{T'}$ (resp. $\mathcal{G}_{T'}$) if it is well-defined. Let $f : X \to Z$ be a morphism between $k$-schemes. We denote by $F_X : X \to X$ the absolute Frobenius morphism of $X$. We often denote by $X^e$ the source of the $e$-time iteration $F_X^e$ of $F_X$, and denote $f : X \to Z$ by $f^{(e)} : X^e \to Z^e$ when we regard $X$ and $Z$ as $X^e$ as $Z^e$, respectively. The induced morphism $(F_X^e, f^{(e)}) : X^e \to X \times_Z Z^e =: X_{Z^e}$ is denoted by $F_{X/Z}^{(e)}$.

3. Trace maps of relative Frobenius morphisms

In this section, we define and study the trace maps of relative Frobenius morphisms.

3.1. Base change by Frobenius morphisms. Let $f : X \to Z$ be a morphism (not necessarily surjective) between $k$-schemes. For each integer $e > 0$, we have the following diagram:

\[
\begin{array}{ccc}
X^e & \xrightarrow{F_X^{(e)}} & F_X^e \\
\downarrow{F_{X/Z}^{(e)}} & & \downarrow{(F_Z^e)_X} \\
X_{Z^e} & \xrightarrow{(F_Z^e)_X} & X \\
\downarrow{f_{Z^e}} & & \downarrow{f} \\
Z^e & \xrightarrow{F_Z^e} & Z
\end{array}
\]

We first consider the property of the $k$-scheme $X_{Z^e}$ when $Z$ is a smooth variety.

**Lemma 3.1.** With the notation as above, assume that $Z$ is a smooth variety.

1. If $X$ is a Gorenstein $k$-scheme of pure dimension, then so is $X_{Z^e}$. Furthermore, the dualizing sheaf $\omega_{X_{Z^e}}$ of $X_{Z^e}$ is isomorphic to $f_{Z^e}^* \omega_{Z^e}^{1-p^e} \otimes (\omega_X)_{Z^e}$, where $\omega_X$ is the dualizing sheaf of $X$.

2. Suppose that $f$ is dominant and separable. If $X$ is a variety, then so is $X_{Z^e}$.

**Proof.** We first note that since $(F_Z^e)_X$ is homeomorphic, if $X$ is of pure dimension (resp. irreducible), then so is $X_{Z^e}$. We show (1). Since $F_Z^e$ is flat as shown by Kunz, $F_Z^e$ is Gorenstein morphism [14 V,§9], and so is $(F_Z^e)_X$. Hence $X_{Z^e}$ is a Gorenstein $k$-scheme. Furthermore, by [14] Theorems 3.6.1 and 4.3.3, we have

\[
\omega_{X_{Z^e}} \cong \omega_{(F_Z^e)_X} \otimes (\omega_X)_{Z^e} \cong f_{Z^e}^* \omega_{Z^e} \otimes (\omega_X)_{Z^e} \cong f_{Z^e}^* \omega_{Z^e}^{1-p^e} \otimes (\omega_X)_{Z^e}.
\]
Next we show (2). We may assume that $X = \text{Spec } A$ and $Z = \text{Spec } B$. Let $K$ be
the function field of $X$. Since $f$ is separable, $F^e_B \otimes_B K$ is reduced. Since $F^e_Z$ is
flat, the morphism $F^e_*B \otimes_B A \to F^e_*B \otimes_B K$ is injective, and hence $F^e_*B \otimes_B A$ is
reduced. \hfill \square

3.2. Trace maps of relative Frobenius morphisms.

(3.2.1) Let $\pi : X \to Z$ be a finite surjective morphism between Gorenstein $k$-schemes
of pure dimension, and let $\omega_X$ and $\omega_Z$ be dualizing sheaves of $X$ and $Z$, respectively.
We denote by $\text{Tr}_e : \pi_*\omega_X \to \omega_Z$ the morphism obtained by applying the functor
$\mathcal{H}om_Z(\_\_\_, \omega_Z)$ to the natural morphism $\pi^\# : \mathcal{O}_Z \to \pi_*\mathcal{O}_X$. This is called the trace map
of $\pi$.

(3.2.2) Let $f : X \to Z$ be a morphism (not necessarily surjective) from a Gorenstein
variety $X$ to a smooth variety $Z$. Then by Lemma 3.1 we see that,

$$\omega_{X^e} \otimes F^{(e)}_{X/Z} \omega^{-1}_{X^eZ^e} \cong \omega_{X^e} \otimes f^{(e)}_*\omega_{Z^e}^{-1} \otimes \omega_{X^e} \cong \omega_{X^e/Z^e}^{-1}.$$ 

Hence we have $(F^{(e)}_{X/Z}, \omega_{X^e}) \cong (F^{(e)}_{X/Z^e}, \omega_{X^e/Z^e})$ by the projection formula. We define

$$\phi^{(e)}_{X/Z} := \text{Tr}_{X/Z} \otimes \omega_{X^eZ^e}^{-1} : F^{(e)}_{X/Z^e} \omega_{X^e/Z^e} \to \mathcal{O}_{X^eZ^e}.$$ 

Additionally, we have the following isomorphisms:

$$F^{(e)}_{X/Z^e} \omega_{X^e/Z^e} \cong F^{(e)}_{X/Z^e} \mathcal{H}om(F^{(e)}_{X/Z}, \omega_{X^eZ^e}, \omega_{X^e}) \cong \mathcal{H}om(F^{(e)}_{X/Z^e} \mathcal{O}_{X^e}, \omega_{X^eZ^e})$$
$$\cong \mathcal{H}om(F^{(e)}_{X/Z^e} \mathcal{O}_{X^e}, \mathcal{O}_{X^eZ^e})$$

Here, the second isomorphism follows from the Grothendieck duality and the projection formula.

(3.2.3) Let $f : X \to Z$ be a morphism from a normal variety $X$ to a smooth variety $Z$.
Let $\iota : U \to X$ be an open immersion from the smooth locus $U$ of $X$. Then we have

$$\iota_{Z^e}F^{(e)}_{U/Z^e} \omega_{U/Z} \cong F^{(e)}_{X/Z^e} \omega_{X/Z}, \text{ and } \iota_{Z^e} \mathcal{O}_{UZ^e} \cong \mathcal{O}_{XZ^e}.$$ 

Hence we can define

$$\phi^{(e)}_{X/Z} := \iota_{*}\phi^{(e)}_{U/Z} : F^{(e)}_{X/Z^e} \omega_{X/Z} \to \mathcal{O}_{XZ^e}.$$ 

Let $K_{X/Z}$ be an Weil divisor on $X$ such that $\mathcal{O}_X(K_{X/Z}) \cong \omega_{X/Z}$. Let $\Delta$ be an
effective $\mathbb{Q}$-Weil divisor on $X$. For every $e > 0$, we define

$$L^{(e)}_{(X/Z, \Delta)} := \mathcal{O}_X([1 - p^e](K_{X/Z} + \Delta))] \subseteq \mathcal{O}_X([1 - p^e]K_{X/Z}),$$
and

$$\phi^{(e)}_{(X/Z, \Delta)} : F^{(e)}_{X/Z^e} L^{(e)}_{X/Z} \hookrightarrow F^{(e)}_{X/Z^e} \mathcal{O}_X([1 - p^e]K_{X/Z}) \xrightarrow{\phi^{(e)}_{X/Z}} \mathcal{O}_{XZ^e}.$$ 

It is easily seen that the above morphism is obtained by the application of the functor
$\mathcal{H}om_{XZ^e}(\_\_\_, \mathcal{O}_{XZ^e})$ to the natural morphism

$$\mathcal{O}_{XZ^e} \to F^{(e)}_{X/Z^e} \mathcal{O}_{X^e} \hookrightarrow F^{(e)}_{X/Z^e} \mathcal{O}_{X^e}([p^e - 1]\Delta]).$$

When $Z = \text{Spec } k$, we may identify $Z^e$, $X^e$ and $F^{(e)}_{X/Z}$ with $Z$, $X$ and $F^e_x$, respectively. In this case we denote $\phi^{(e)}_{(X,Z,\Delta)}$ by $\phi^{(e)}_{(X,\Delta)}$. 

4. VARIETIES WITH NEF ANTI-CANONICAL DIVISORS

In this section, we prove Theorem 1.1 which states that the Albanese morphism of a normal projective variety with nef anti-canonical divisor is an algebraic fiber space if the geometric generic fiber is $F$-pure.

We first recall the notion of weak positivity which was introduced by Viehweg.

**Definition 4.1** ([20 Notation]). Let $\mathcal{G}$ be a coherent sheaf on a normal projective variety $Y$.

(i) $\mathcal{G}$ is said to be generically globally generated if the natural morphism $H^0(Y, \mathcal{G}) \otimes \mathcal{O}_Y \to \mathcal{G}$ is generically surjective.

(ii) $\mathcal{G}$ is said to be weakly positive if for every ample Cartier divisor $H$ on $Y$ and every integer $a > 0$, there exists an integer $b > 0$ such that $(S^a \mathcal{G})^\ast (bH)$ is generically globally generated. Here $\mathcal{G}^\ast$ and $S^m \mathcal{G}$ denote the double dual and the $m$-th symmetric product of $\mathcal{G}$, respectively.

Theorem 1.1 is proved as an application of Theorems 4.2 and 4.4 below.

**Theorem 4.2** ([7]). Let $f : X \to Y$ be a surjective morphism between normal varieties. Let $\Delta$ be an effective divisor on $X$ such that $m(K_X + \Delta)$ is an integral Cartier divisor for an integer $m > 0$ not divisible by $p$. Assume that $(X_\pi, \Delta|_{X_\pi})$ is $F$-pure, where $X_\pi$ is the geometric generic fiber. If $-(K_X + \Delta + f^*D)$ is a nef $\mathbb{Q}$-Cartier divisor on $X$ for some $\mathbb{Q}$-Cartier divisor $D$ on $Y$, then $\mathcal{O}_Y(-n(K_Y + D))$ is weakly positive for an integer $n > 0$ such that $nD$ is integral.

**Remark 4.3.** (1) The above theorem is a special case of [7, Theorem 4.5] (see [7, Remark 4.7]).

(2) The $F$-purity of the geometric generic fiber is equivalent to the $F$-purity of the geometric general fiber (see [28, Corollary 3.31] or [7, Lemma 2.3]).

**Theorem 4.4** ([13 Theorem 0.2]). Let $X$ be a normal projective variety with $\kappa(X, K_X) = 0$. Let $a : X \to A$ be the Albanese morphism of $X$. If $a : X \to \text{Im}(a)$ is generically finite and separable, then $a$ is surjective.

The following lemma is also used in the proof of Theorem 1.1.

**Lemma 4.5.** Let $D$ be an effective Weil divisor on a normal projective variety $Y$. If $\mathcal{O}_Y(-D)$ is weakly positive, then $D = 0$.

**Proof.** Let $\pi : Y' \to Y$ be the blowing-up of $Y$ along $D$. Then we have the natural surjection $\pi^* \mathcal{O}_Y(-D) \to \mathcal{O}_{Y'}(-D')$, where $D'$ is the exceptional divisor of $\pi$. Since $\mathcal{O}_Y(-D)$ is weakly positive, $\pi^* \mathcal{O}_Y(-D)$ is again weakly positive. Then by the above surjection, we see that $\mathcal{O}_{Y'}(-D')$ is also weakly positive. Since the weak positivity of a line bundle is equivalent to the pseudo-effectivity, we see that $-D'$ is pseudo-effective. Hence $D' = 0$, and so $D = 0$. \qed

**Proof of Theorem 4.4** Let $Z$ be the normalization of $\text{Im}(a)$ and $f : X \to Z$ be the induced morphism. Now we have the natural morphism $\Omega_{A|Z}^1 \to \Omega_Z^1$ which is generically surjective. Hence $H^0(Z, \omega_Z) \neq 0$. Furthermore, by Theorem 4.2 we obtain that $\omega_Z^{-1}$ is weakly positive. Therefore we have $\omega_Z^{-1} \cong \Omega_Z$ by Lemma 4.5. By Theorem 4.4 we see that $a$ is surjective, or equivalently $Z = A$. Let $a : X \xrightarrow{g} Y \xrightarrow{h} A$
be the Stein factorization of $a$. Since the geometric generic fiber of $a$ is $F$-pure, it is reduced, and hence $a$ is separable. This implies that $h$ is also separable, and therefore we have an injection $\mathcal{O}_Y \cong h^*\omega_A \to \omega_Y$. By the same argument as before we see that $\omega_Y \cong \mathcal{O}_Y$, and by the Zariski-Nagata purity we obtain that $a$ is an étale morphism. Hence we see that $Y$ is an abelian variety by [25, Section 18, Theorem] and $h$ is an isomorphism. Consequently we obtain $a_*\mathcal{O}_X \cong \mathcal{O}_A$, which is our assertion. 

\section{Splittings of Relative Frobenius}

In this section, we introduce and study the notion of $F$-split morphisms.

\textbf{Definition 5.1.} Let $X$ be a normal variety and $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$. Let $f : X \to Z$ be a projective morphism to a smooth variety $Z$. $f$ is said to be \textit{sharply $F$-split} ($F$-split for short) with respect to $\Delta$ if there exists an $e > 0$ such that the composite

\begin{equation}
\begin{array}{c}
\mathcal{O}_{X_{\Delta}} \xrightarrow{F_{X/Z}^e} F_{X/Z_{\Delta}}^e \mathcal{O}_{X_{\Delta}} \hookrightarrow F_{X/Z_{\Delta}}^e \mathcal{O}_{X_{\Delta}}(\lceil (p^e - 1)\Delta \rceil)
\end{array}
\end{equation}

of the natural homomorphism $F_{X/Y}(e)$ and the natural inclusion $F_{X/Z_{\Delta}}^e \mathcal{O}_{X_{\Delta}} \hookrightarrow F_{X/Z_{\Delta}}^e \mathcal{O}_{X_{\Delta}}(\lceil (p^e - 1)\Delta \rceil)$ is injective and splits as an $\mathcal{O}_{X_{\Delta}}$-module homomorphism. $f$ is said to be \textit{locally sharply $F$-split} (locally $F$-split for short) with respect to $\Delta$ if there exists an open covering $\{V_i\}$ of $Z$ such that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is $F$-split with respect to $\Delta|_{f^{-1}(V_i)}$ for every $i$.

We often say that the pair $(f, \Delta)$ is $F$-split (resp. locally $F$-split) if $f$ is $F$-split (resp. locally $F$-split) with respect to $\Delta$. $f$ is simply said to be $F$-split (resp. locally $F$-split) if $(f, 0)$.

\textbf{Remark 5.2.} (1) If the morphism $(5.1)_{e}$ splits, then $(5.1)_{ne}$ also splits for every integer $n > 0$.

(2) When $Z = \text{Spec } k$, it is easily seen that $(f, \Delta)$ is $F$-split if and only if $(X, \Delta)$ is $F$-split. Note that we now assume that $k$ is algebraically closed.

(3) Let $\Delta'$ be an effective $\mathbb{Q}$-divisor on $X$ with $\Delta' \leq \Delta$. If $(f, \Delta)$ is $F$-split (resp. locally $F$-split), then so is $(f, \Delta')$.

(4) Hashimoto has dealt with morphisms with local splittings of $(5.1)_{e}$ in [15].

\textbf{Example 5.3.} Let $X$, $\Delta$, $Z$ and $f$ be as in Definition 5.1. Assume that $X$ is the projective space bundle $\mathbb{P}(E)$ associated with a locally free coherent sheaf $E$ and that $f : X \to Z$ is its projection. Then $f$ is locally $F$-split. Furthermore, if $E$ is the direct sum of line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on $Z$, then $f$ is $F$-split. The first statement follows from the second. We assume that $E = \bigoplus_{i=1}^n \mathcal{L}_i$. For every $m \geq 0$, there exists the natural injective morphism

$$
\psi_m : \bigoplus_{m_1 + \cdots + m_n = m} \mathcal{L}_i^{m_i} \cong F^n_\ast S^m E \to S^{mp} E.
$$
Then obviously the image of $\psi_m$ is $\bigoplus_{m_1+\ldots+m_n=m} S^{m_i}F^{(l)}_{\mathbb{Z}} \subseteq S^{mp}\mathcal{E}$, and hence $\psi_m$ splits. The morphism $\mathcal{O}_{X_{Z}} \to F^{(l)}_{X/Z_{*}}\mathcal{O}_{X_{Z}}$ corresponds to the morphism

$$
\psi := \bigoplus_{m \geq 0} \psi_m : \bigoplus_{m \geq 0} S^{m}F^{1}_{\mathbb{Z}} \to \bigoplus_{m \geq 0} S^{mp}\mathcal{E} \subseteq \bigoplus_{m \geq 0} S^{m}\mathcal{E}.
$$

Since $\psi_m$ splits for every $m \geq 0$, $\psi$ also splits, and hence $\mathcal{O}_{X_{Z}} \to F^{(l)}_{X/Z_{*}}\mathcal{O}_{X_{Z}}$ splits. Note that as we see in Theorem 7.1, there exists an indecomposable vector bundle $\mathcal{E}$ on an elliptic curve $\mathcal{F}$ such that $\mathbb{P}(\mathcal{E}) \to \mathcal{F}$ is not $F$-split.

We first prove that $F$-split morphisms are surjective.

**Lemma 5.4.** Let $X, \Delta, Z$ and $f$ be as in Definition 5.1. Assume that $f$ is locally $F$-split. Then there exists an $e > 0$ such that for each $i \geq 0$, $\mathcal{G}^i := R^i f_* \mathcal{O}_X$ is a vector bundle satisfying $F^e_{\mathbb{Z}} \mathcal{G}^i \cong \mathcal{G}^i$. In particular, $f$ is surjective.

**Proof.** Applying the functor $R^i f_* \mathcal{O}_Z$ to $\mathcal{O}_{X/Z_{*}} \to F^{(e)}_{X/Z_{*}} \mathcal{O}_{X_{Z}}$, we obtain the morphism $R^i f_* \mathcal{O}_{X/Z_{*}} \to R^i f_* \mathcal{O}_X = \mathcal{G}^i$ which is injective and splits locally. Since $f_* \mathcal{O}_Z$ is flat, we have $R^i f_* \mathcal{O}_{X/Z_{*}} \cong F^e_{\mathbb{Z}} R^i f_* \mathcal{O}_X = F^e_{\mathbb{Z}} \mathcal{G}^i$. Hence we obtain the morphism $F^e_{\mathbb{Z}} \mathcal{G}^i \to \mathcal{G}^i$ which is injective and splits locally. It is easily seen that this morphism is an isomorphism. By the lemma below, we see that $\mathcal{G}^i$ is locally free. □

**Lemma 5.5 ([22] Lemma 1.4).** Let $M$ be a finitely generated module over a regular local ring $R$ of positive characteristic. If $F^e_{R} M \cong M$ for some $e > 0$, then $M$ is free.

The following proposition shows that locally $F$-splitting requires some conditions on boundaries and fibers.

**Proposition 5.6.** Let $X, \Delta, Z$ and $f$ be as in Definition 5.1. Assume that $\Delta$ is $\mathbb{Z}_{(p)}$-Cartier and $(f, \Delta)$ is locally $F$-split. Then the following holds:

1. The support of $\Delta$ does not contain any irreducible component of any fiber.
2. For every $z \in Z$, $(X_{\overline{z}}, \Delta_{\overline{z}})$ is $F$-split, where $\overline{z}$ is the algebraic closure of $z$. In particular, $X_{\overline{z}}$ is reduced.
3. There exists an open subset $U \subseteq X$ such that $\text{codim}(X \setminus U) \geq 2$ and $f|_U : U \to Y$ is a smooth morphism. In particular, general geometric fibers of $f$ are normal.

Note that $f$ is surjective as shown by Lemma 5.4.

**Proof.** Let $z \in Z$. Restricting the homomorphism $(5.11)_e$ to $X_{\overline{z}}$, we obtain the homomorphism of $\mathcal{O}_{X_{\overline{z}}}$-modules

$$
\mathcal{O}_{X_{\overline{z}}} \xrightarrow{F^{(e)}_{X_{\overline{z}}}} F^{(e)}_{X_{\overline{z}}} \mathcal{O}_{(X_{\overline{z}})^e} \to F^{(e)}_{X_{\overline{z}}} \mathcal{O}_{(X_{\overline{z}})^e}((p^e - 1)\Delta))|_{(X_{\overline{z}})^e}
$$

which is injective and splits for some $e > 0$. This implies that the homomorphism $\mathcal{O}_{X_{\overline{z}}} \to (\mathcal{O}_X(p^e - 1)\Delta)|_{X_{\overline{z}}}$ is not zero over each irreducible component. Hence the support of $\Delta$ does not contain any component of $X_{\overline{z}}$, and $(X_{\overline{z}}, \Delta_{\overline{z}})$ is $F$-split. Thus
(1) and (2) hold. We show (3). Let \( \pi : Y \to X_{Z^e} \) be the normalization of \( X_{Z^e} \). Then \( F_{X/Y}^{(e)} : X^{(e)} \to Y^{(e)} \) factors through \( Y \), and we have morphisms

\[
\mathcal{O}_{X_{Z^e}} \to \pi_* \mathcal{O}_Y \to F_{X/Y}^{(e)} \mathcal{O}_{X^{(e)}}
\]

of \( \mathcal{O}_{X_{Z^e}} \)-modules. Therefore the morphism \( \mathcal{O}_{X_{Z^e}} \to \pi_* \mathcal{O}_Y \) splits. Since \( \pi_* \mathcal{O}_Y / \mathcal{O}_{X_{Z^e}} \) is a torsion module and \( \pi_* \mathcal{O}_Y \) is torsion free, we see that \( \pi_* \mathcal{O}_Y / \mathcal{O}_{X_{Z^e}} = 0 \). Hence \( X_{Z^e} \) is normal. Since \( F_{X/Z}^{(e)} \mathcal{O}_{X^{(e)}} \) is torsion free, there exists an open subset \( U \subseteq X \) such that \( F_{X/Z}^{(e)} \mathcal{O}_{X^{(e)}}|_{U_{Z^e}} \cong F_{U/Z}^{(e)} \mathcal{O}_{U^{(e)}} \) is locally free over \( U_{Z^e} \). From this, we see that \( F_{U/Z}^{(e)} \mathcal{O}_{U^{(e)}} \mathcal{O}_{U^{(e)}}|_{U_{Z^e}} \) is locally free for every \( z \in Z \). Consequently, we deduce that \( U_z \) is regular by Kunz’s theorem, and thus \( f|_U : U \to Z \) is smooth. \( \square \)

On the contrary to the above, \( f \) is not necessarily \( F \)-split even if every fiber is \( F \)-split (see Theorem \[7.1\] for example). However, if \( K_X \) is \( \mathbb{Z}(p) \)-linearly trivial relative to \( f \), then the converse holds as seen in the next theorem. This is used in the proofs of Proposition \[6.9\] and Theorem \[7.1\].

**Theorem 5.7 ([6 Theorem 3.17]).** Let \( f : X \to Z \) be a surjective projective morphism from a normal variety \( X \) to a smooth variety \( Z \) satisfying \( f_* \mathcal{O}_X \cong \mathcal{O}_Z \). Let \( \Delta \) be an effective \( \mathbb{Z}(p) \)-Weil divisor on \( X \) such that \( K_X + \Delta \sim_{\mathbb{Z}(p)} f^* C \) for some Cartier divisor \( C \) on \( Z \). Let \( \overline{\eta} \) be the geometric generic point of \( Y \).

(i) If \( (X_{\overline{\eta}}, \Delta_{\overline{\eta}}) \) is not \( F \)-split, then so is \( (X_{\overline{\eta}}, \Delta_{\overline{\eta}}) \) for general \( z \in Z \).

(ii) If \( (X_{\overline{\eta}}, \Delta_{\overline{\eta}}) \) is \( F \)-split, then there exists an effective \( \mathbb{Z}(p) \)-Weil divisor \( \Delta_Z \) on \( Z \) such that the following holds:

1. \((K_Z + \Delta_Z) \sim_{\mathbb{Z}(p)} C \).
2. \((X, \Delta) \) is \( F \)-split if and only if so is \((Z, \Delta_Z) \).
3. The followings are equivalent:
   1. \((f, \Delta) \) is \( F \)-split.
   2. \((f, \Delta) \) is locally \( F \)-split.
   3. \((X_{\overline{\eta}}, \Delta|_{X_{\overline{\eta}}} \) is \( F \)-split for every codimension one point \( z \in Z \), where \( \overline{z} \) is the algebraic closure of \( z \).
4. \( \Delta_Z = 0 \).

**Proof.** (i) follows from \([6\) Observation 3.19]. (1) of (ii) follows directly from \([6\) Theorem 3.17 (1)]. \([6\) Theorem 3.17 (2)] shows that \( S^0(X, \Delta, \mathcal{O}_X) \cong S^0(Z, \Delta_Z, \mathcal{O}_Z) \) (see \([30\) §3] or \([6\) Definition 3.2] for the definition of \( S^0 \)). Hence (2) of the Theorem follows from the fact that \((X, \Delta) \) is \( F \)-split if and only if \( S^0(X, \Delta, \mathcal{O}_X) = H^0(X, \mathcal{O}_X) \). To prove (3), we recall the construction on \( \Delta_Z \). Replacing \( X \) and \( Z \) by its smooth locus respectively, we may assume that \( X \) and \( Z \) are smooth. For an \( e > 0 \) with \( e | (p^e - 1) \), we have

\[
f^{(e)}_{(*)} \mathcal{O}_{X_{Z^e}} = f^{(e)}_* \mathcal{O}_{X^e} ((1 - p^e)(K_{X^e/Z^e} + \Delta)) \cong \mathcal{O}_{Z^e} ((1 - p^e)(C - K_{Z^e}))
\]

by the projection formula. Then we set

\[
\theta^{(e)} : \mathcal{O}_{Z^e} ((1 - p^e)(C - K_{Z^e})) \rightarrow f^{(e)}_* \mathcal{O}_{X^e} \mathcal{L}_{(X/Z, \Delta)}^{(e)} \mathcal{O}_{X_{Z^e}}
\]
Since \( (f_{Z*}\phi_{X/Z,\Delta}) \otimes k(\pi) \cong H^0(X_{\pi^*}, \phi_{(X_{\pi}\pi,\Delta^{\pi})}) \) is surjective because of the assumption, \( \theta^{(e)} \) is nonzero. Hence there exists an effective divisor \( E \) on \( Z \) such that \( O_Z(-E) \) is equal to the image of \( \theta^{(e)} \). We define \( \Delta_Z := (p^e - 1)^{-1}E \). By the definition, \( \Delta_Z = 0 \) if and only if \( \theta^{(e)} \) is surjective. Furthermore, by the argument similar to the above, we see that for a codimension one point \( z \in Z \), \( (X_{\pi}, \Delta|_{X_{\pi}}) \) is \( F \)-split if and only if \( \theta^{(e)} \otimes k(\pi) \) is non-zero, or equivalently, \( \Delta \) is zero around \( z \). Now we prove (3). (3-1)\( \Rightarrow \) (3-2) is obvious. (3-2)\( \Rightarrow \) (3-3) follows from Proposition 5.6. (3-3)\( \Rightarrow \) (3-4) follows from the above argument. If \( \theta^{(e)} \) is surjective, or equivalently is an isomorphism, then \( H^0(X_{\pi^*}, \phi_{(X_{\pi},\Delta^{\pi})}) \cong H^0(Z^e, \theta^{(e)}) \) is also surjective, and hence \( \phi_{(X_{\pi^{(e)}},\Delta^{\pi})} \) splits. This proves (3-4)\( \Rightarrow \) (3-1). \( \square \)

When \( f : X \to Z \) is \( F \)-split with respect to \( \Delta \), there exists a \( \mathbb{Z}_{(p)} \)-Weil divisor \( \Delta' \geq \Delta \) on \( X \) such that \( K_{X/Z} + \Delta' \sim_{\mathbb{Z}_{(p)}} 0 \) as seen in the below.

**Observation 5.8.** Let \( X, \Delta, Z \) and \( f \) be as in Definition 5.1. Assume that \( (f, \Delta) \) is \( F \)-split. Then there exists an \( e > 0 \) such that \( \phi_{(X,\Delta)}^{(e)} : F_{X/Z}^{(e)} L_{X/Z,\Delta}^{(e)} \to \mathcal{O}_{X^e} \) splits as a homomorphism of \( \mathcal{O}_{X^e} \)-module. Here, we recall that

\[
L_{X/Z,\Delta}^{(e)} := \mathcal{O}_{X^e}(\lfloor (1 - p^e)(K_{X/Z} + \Delta) \rfloor).
\]

Then there exists an element \( s \in H^0(X^e, [1 - p^e](K_{X/Z} + \Delta)) \) such that \( \phi_{(X/Z,\Delta)}^{(e)} \) sends \( s \) to \( 1 \). Let \( E \) be an effective Weil divisor on \( X^e \) defined by \( s \). Set \( \Delta^{e'} := (p^e - 1)^{-1}[(p^e - 1)\Delta + E] \geq \Delta \). Then by the choice of \( E \) we have

\[
L_{(X,\Delta')}^{(e)} := \mathcal{O}_{X^e}(1 - p^e)(K_{X/Z} + \Delta') = \mathcal{O}_{X^e}(\lfloor (1 - p^e)(K_{X/Z} + \Delta) - E \rfloor) \cong \mathcal{O}_{X^e},
\]

and \( \phi_{(X/Z,\Delta')}^{(e)} : F_{X/Z}^{(e)} L_{(X/Z,\Delta')}^{(e)} \to \mathcal{O}_{X^e} \) splits.

Next we consider the case of finite morphisms.

**Proposition 5.9.** Let \( X, \Delta, Z \) and \( f \) be as in Definition 5.1. Assume that \( \dim X = \dim Z \). Then the following conditions are equivalent:

1. \( (f, \Delta) \) is \( F \)-split.
2. \( (f, \Delta) \) is locally \( F \)-split.
3. \( f \) is étale and \( \Delta = 0 \).

In the case when \( \Delta = 0 \), the proposition has been shown in [15, 2.19 Theorem].

**Proof.** (1)\( \Rightarrow \) (2) is obvious. Let \( f \) be étale and \( \Delta = 0 \). Then \( F_{X/Z}^{(e)} : X^e \to X_{Z^e} \) is a finite morphism of degree one between normal varieties, and hence it is an isomorphism, which implies (3)\( \Rightarrow \) (1). We show (2)\( \Rightarrow \) (3). By Lemma 5.4 and Proposition 5.6 \( f \) is a separable surjective morphism, and hence we obtain that \( f \) is generically finite by the assumption. Let \( e > 0 \) be an integer such that the morphism

\[
\mathcal{O}_{X^e} \to F_{X/Z}^{(e)} \mathcal{O}_{X^e}(\lfloor (p^e - 1)\Delta \rfloor)
\]

splits. Since \( F_{X/Z}^{(e)} \) is a finite morphism of degree zero, \( F_{X/Z}^{(e)} \mathcal{O}_{X^e}(\lfloor (p^e - 1)\Delta \rfloor) \) is a torsion free sheaf of rank one. Note that \( X_{Z^e} \) is a variety as seen in Lemma 3.4. Therefore the cokernel of the above morphism is zero, or equivalently, the above
morphism is an isomorphism. Hence \( \Delta = 0 \) and \( F_X^{(e)} \) is an isomorphism. Then for every \( z \in Z \), \( F_{X/Z}^{(e)} \) is also an isomorphism, where \( \overline{z} \) is the algebraic closure of \( z \in Z \). This implies that \( X_{\overline{z}} \) is isomorphic to a disjoint union of copies of the spectrum of \( k(\overline{z}) \), and thus \( f \) is finite. Since \( f_* \mathcal{O}_X \) is locally free as shown by Lemma 5.4, \( f \) is flat. Hence the smoothness of \( X_{\overline{z}} \) implies that \( f \) is étale. \( \square \)

The following lemma is used in proofs of Proposition 6.9 and Theorem 7.1.

**Lemma 5.10.** Let \( X, \Delta, Z \) and \( f \) be as in Definition 5.1. Assume that \( \Delta \) is a \( \mathbb{Z}(p) \)-Weil divisor and that \((f, \Delta)\) is locally \( F \)-split. Then the Iitaka-Kodaira dimension \( \kappa(X, K_X/Z + \Delta) \) of \( K_X/Z + \Delta \) is non-positive. Furthermore, if \((f, \Delta)\) is \( F \)-split, then \( \kappa(X, -(K_X/Z + \Delta)) \geq 0 \).

**Proof.** The second statement follows from Observation 5.8. By Lemma 5.4, \( f \) is surjective. Assume that \( \kappa(X, K_X/Z + \Delta) \geq 0 \). Then \( \kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}}/\overline{\eta} + \Delta_{\overline{\eta}}) \geq 0 \), where \( \overline{\eta} \) is the geometric generic point of \( Z \). Since \( X_{\overline{\eta}} \) is \( F \)-split, we have \( H^0(X_{\overline{\eta}}, (1-p^e)(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}})) \neq 0 \) for some \( e > 0 \), and hence \((1-p^e)(K_{X_{\overline{\eta}}} + \Delta) \sim 0 \). Then the morphism

\[
\phi_{(X/Z, \Delta)}^{(e)}: f_{Z_*}^*(1-p^e)(K_{X/Z} + \Delta) \rightarrow f_{Z_*}^*(1-p^e)(K_{X/Z} + \Delta)
\]

is a surjective morphism between torsion free coherent sheaves of the same rank, and thus it is an isomorphism. Hence \( H^0(X, (1-p^e)(K_X/Z + \Delta)) \neq 0 \), which implies that \( \kappa(X, K_X/Z) = 0 \). This is our assertion. \( \square \)

In the rest of this section, we consider the composition of morphisms in the next proposition, which is used frequently in Section 6.

**Proposition 5.11.** Let \( X, \Delta, Z \) and \( f \) be as in Definition 5.1 and \( Y \) be a normal variety. Assume that \( f: X \rightarrow Z \) can be factored into projective morphisms \( g: X \rightarrow Y \) with \( g_* \mathcal{O}_X \cong \mathcal{O}_Y \) and \( h: Y \rightarrow Z \).

1. If \((f, \Delta)\) is \( F \)-split, then so is \( h \).
2. Assume that \( Y \) is smooth. If \((g, \Delta)\) and \( h \) are \( F \)-split, then so is \((f, \Delta)\).
3. The converse of (2) holds if \( K_Y \sim_{\mathbb{Z}(p)} h^* K_Z \).

**Proof.** Let \( e > 0 \) be an integer. Now we have the following commutative diagram:
Here \( \pi^{(e)} := (F_{Y/Z}^{(e)})_X \). We first show (1). The above diagram induces a commutative diagram of \( \mathcal{O}_{YZ^e} \)-modules

\[
\begin{array}{ccc}
\mathcal{O}_{YZ^e} & \xrightarrow{\cong} & F_{Y/Z}^{(e)} \mathcal{O}_{Y^e} \\
\cong & \quad & \cong \\
g_{Z^e*} \mathcal{O}_{XZ^e} & \xrightarrow{\cong} & g_{Z^e*} F_{X/Z}^{(e)} \mathcal{O}_{X^e}
\end{array}
\]

Here the left vertical morphism is an isomorphism because of the flatness of \( (F_{Z}^{e})_Y \). Since the lower horizontal morphisms splits, so is the upper one.

Next we show (2) and (3). As explained in Observation 5.3, if \((g, \Delta)\) (resp. \((f, \Delta)\)) is \( F \)-split, then there exists an effective \( \mathbb{Z}_p \)-Weil divisor \( \Delta' \geq \Delta \) on \( X \) such that \( K_{X/Y} + \Delta' \) (resp. \( K_{X/Z} + \Delta' \)) is \( \mathbb{Z}_p \)-linearly trivial and that \((g, \Delta')\) (resp. \((f, \Delta')\)) is also \( F \)-split. Thus we may assume that \( \Delta \) is a \( \mathbb{Z}_p \)-Weil divisor and that \( \langle p^e - 1 \rangle (K_{X/Y} + \Delta) \sim 0 \) (resp. \( \sim (p^e - 1)(f^* K_Z - g^* K_Y) \)) for every \( e > 0 \) divisible enough. In particular, \( L^{(e)}_{(X/Y, \Delta)} \) (resp. \( L^{(e)}_{(X/Z, \Delta)} \)) is isomorphic to the pullback by \( g^{(e)} \) of a line bundle on \( Y^{(e)} \).

Let \( V \subseteq Y \) be an open subset such that \( g \) is flat at every point in \( X_V := g^{-1}(V) \) and \( \text{codim}(Y \setminus V) \geq 2 \). Let \( u : U \rightarrow X_V \) be the open immersion of the regular locus of \( X_V \). Set \( g' := g \circ u : U \rightarrow Y \). Then we have \( g'_* \mathcal{O}_U \cong g_* \mathcal{O}_X \cong \mathcal{O}_Y \) because of the assumptions. Additionally, by the flatness of \( F_{Z}^{e} \), we see that \( g'_{Z^e*} \mathcal{O}_{UZ^e} \cong \mathcal{O}_{Y^{Z^e}} \).

Thus by the projection formula, we see that

\[
H^0(U^{Z^e}, (gZ^e* L)|_{U^{Z^e}}) \cong H^0(Y^{Z^e}, g'_{Z^e* (g'Z^e* L)}) \cong H^0(Y^{Z^e}, L) \cong H^0(X^{Z^e}, gZ^e* L)
\]

for every line bundle \( L \) on \( Y^{Z^e} \), and hence there exists the following commutative diagram:

\[
\begin{array}{ccc}
H^0(U^e, L^{(e)}_{(X/Z, \Delta)}) & \xrightarrow{H^0(U_{Z^e}, \phi^{(e)}_{(U/Z, \Delta|U)})} & H^0(U_{Z^e}, \mathcal{O}_{UZ^e}) \\
\cong & \quad & \cong \\
H^0(X^e, L^{(e)}_{(X/Z, \Delta)}) & \xrightarrow{H^0(X_{Z^e}, \phi^{(e)}_{(X/Z, \Delta)})} & H^0(X_{Z^e}, \mathcal{O}_{XZ^e})
\end{array}
\]

Note that in particular, \( H^0(U_{Y^e}, \mathcal{O}_{U_{Y^e}}) \cong H^0(Y^e, \mathcal{O}_{Y^e}) \cong k \). Clearly, the splitting of \( \phi^{(e)}_{(X/Z, \Delta)} \) is equivalent to the surjectivity of \( H^0(X^{Z^e}, \phi^{(e)}_{(X/Z, \Delta)}) \). From this we see that the \( F \)-splitting of \( (f, \Delta) \) is equivalent to the \( F \)-splitting of \( (f|_U : U \rightarrow Z, \Delta|_U) \). By an argument similar to the above, we also see that the \( F \)-splitting of \( (g, \Delta) \) is equivalent to the \( F \)-splitting of \( (g|_U, \Delta|_U) \).

Assume that we can choose \( V = Y \) and \( U = X \), in other words, \( X \) and \( Y \) are regular and \( g \) is flat. Let \( e > 0 \) be an integer. By the flatness of \( g \), we have the...
Proof of Theorem 1.3. Let \( f : X \to Y \) be a morphism, and thus so is \( \pi(f) \) following commutative diagram:

\[
\begin{array}{c}
g_{Z*}O_{YZ} \\
\cong \\
O_{XZ} \\
\downarrow \\
\pi(e)^*O_{XY} \\
\end{array} \xrightarrow{g_{Z*}(F^{(e)}_{Y/Z})} \xrightarrow{\pi(e)^*F^{(e)}_{Y/Z*}} \xrightarrow{\pi(e)^*g_{Y*}O_{Y*}} \xrightarrow{\not\sim} \xrightarrow{\not\sim}
\]

This implies that

\[
\text{Hom}(\pi(e)^*, O_{XZ}) \cong g_{Z*}^* \text{Hom}(F^{(e)}_{Y/Z}, O_{YZ}) = g_{Z*}^* \phi_{Y/Z}^{(e)}.
\]

Applying the functor \( \text{Hom}(\_ , O_{XZ}) \) and the Grothendieck duality to the natural morphism

\[
O_{XZ} \xrightarrow{\pi(e)^*} \pi(e)^*O_{XY} \to F^{(e)}_{X/Z*} O_{X*}((p^e - 1)\Delta)),
\]

we obtain the morphism

\[
\phi^{(e)}_{(X/Z, \Delta)} : F^{(e)}_{X/Z*} L^{(e)}_{(X/Z, \Delta)} \to \pi(e)^* \phi^{(e)}_{X/Y, \Delta} \cong \omega^{(e)}_{\pi(e)^*} \to g_{Z*}^* F^{(e)}_{Y/Z*} L^{(e)}_{Y/Z} \xrightarrow{g_{Z*}^* \phi_{Y/Z}^{(e)}} O_{XZ}.
\]

Note that \( \omega^{(e)}_{\pi(e)^*} \cong \omega_{X_Y} \otimes \pi(e)^* \omega_{X_Z} \cong g_{Z*}^* \omega_{Y_{EZ}}^{1-p^e} \).

Now we prove the assertion. If \((g, \Delta)\) is \(F\)-split and \(h\) is \(F\)-split, then both of \(\phi^{(e)}_{(X/Y, \Delta)} \) and \(\phi^{(e)}_{Y/Z} \) split for every \(e > 0\) divisible enough. Hence \(\phi^{(e)}_{(X/Z, \Delta)} \) also splits, or equivalently, \((f, \Delta)\) is \(F\)-split. Conversely, assume that \((f, \Delta)\) is \(F\)-split and that \((p^e - 1)K_{Y/Z} \sim 0\) for an \(e > 0\). Then \(\omega_{\pi(e)^*}^{1-p^e} \cong O_{X_Y} \). Since for every \(e > 0\) divisible enough \(H^0(X_{Z^*}, \phi^{(e)}_{X/Z, \Delta})\) is surjective, \(H^0(X_{Z^*}, \pi(e)^* \phi^{(e)}_{X/Y, \Delta})\) is a nonzero morphism, and thus so is \(H^0(X_{Y^*}, \phi^{(e)}_{X/Y, \Delta})\). This is surjective because its target space \(H^0(X_{Y^*}, O_{X_{Y^*}}) \cong H^0(Y^*, O_{Y*}) \cong k\). Hence \(\phi^{(e)}_{(X/Y, \Delta)} \) splits, and so \((g, \Delta)\) is \(F\)-split. Note that the \(F\)-splitting of \(h\) follows directly from (1).

6. Varieties with \(F\)-split Albanese morphisms

In this section, we prove Theorems 1.2, 1.3, 1.4 and 6.6. Throughout this section, we denote by \(X\) and \(\Delta\) respectively a normal projective variety and an effective \(\Q\)-Weil divisor on \(X\).

We first recall that the Albanese morphism of \(X\) is defined as a morphism \(a : X \to A\) to an Abelian variety \(A\) (called the Albanese variety) such that every morphism \(b : X \to B\) to an abelian variety \(B\) factors through \(a\). The existence of the Albanese morphism for a normal projective variety is proved for instance in [9] [§9].

**Proof of Theorem 1.3** Assume that \((a, \Delta)\) is locally \(F\)-split. The surjectivity of \(a\) follows from Lemma 5.4. Let \(X \xrightarrow{f} Z \xrightarrow{g} A\) be the Stein factorization of \(a\). As seen in Proposition 5.11 (1), \(g\) is \(F\)-split, and hence we see that \(g\) is étale by Proposition 5.9. Therefore Section 18, Theorem] shows that \(Z\) is an abelian variety, and hence \(g\) is an isomorphism and \(a_* O_X \cong g_* O_Z \cong O_A\). (1)-(3) follows directly from Proposition 5.6.

The next lemma is used to prove Theorems 1.2 and 1.4.
**Lemma 6.1.** Let $F$ be a coherent sheaf or rank $r$ on a normal variety $Y$. Let $F'$ be an indecomposable direct summand of $F$ with rank $r'$. Set $I := \{ L \in \text{Pic}(Y) | F \otimes L \cong F \}$ and $I' := \{ L \in I | F' \otimes L \cong F' \}$. Then $\bigoplus_{[L] \in I/I'} F' \otimes L$ is a direct summand of $F$. In particular, $\#(I/I') \leq r/r'$.

**Proof.** For every $L \in I$, $F' \otimes L$ is again a direct summand of $F$. Furthermore, $F \otimes L \cong F \otimes L'$ if and only if $L' \otimes L^{-1} \in I$. Hence by Krull-Schmit theorem [1], we see that $\bigoplus_{[L] \in I/I'} F' \otimes L$ is a direct summand of $F$. This implies $r' \#(I/I') \leq r$, which is our claim. \hfill \square

To prove Theorem 1.2 we recall a characterization of ordinary abelian varieties due to Sannai and Tanaka.

**Theorem 6.2** ([29 Theorem 1.1]). Let $Y$ be a normal projective variety over an algebraically closed field $k$ of characteristic $p > 0$. $Y$ is an ordinary abelian variety if and only if $K_Y$ is pseudo-effective and $F_{Y*}\mathcal{O}_Y$ is isomorphic to a direct sum of line bundles for infinitely many $e > 0$.

**Remark 6.3.** In [3], it is proved that we only need to check $F_{Y*}\mathcal{O}_X$ for $e = 1, 2$ in the above theorem.

For convenience, we use the following notation.

**Notation 6.4.** Let $\varphi : S \to T$ be a morphism of schemes. We denote by $\text{Pic}(S)[\varphi]$ (resp. $\text{Pic}^0(S)[\varphi]$) the kernel of the induced homomorphism $\varphi^* : \text{Pic}(T) \to \text{Pic}(S)$ (resp. $\varphi^* : \text{Pic}^0(T) \to \text{Pic}^0(S)$). Then for every $e > 0$, $\text{Pic}(X)[p^e]$ is the set of $p^e$-torsion line bundles. We denote it by $\text{Pic}(X)[p^e]$.

**Proof of Theorem 1.2.** We first prove that if $(X, \Delta)$ is $F$-split, then $(a, \Delta)$ is $F$-split and $A$ is ordinary. We have the following commutative diagram

$$
\begin{array}{ccc}
H^1(X, \mathcal{O}_X) & \xrightarrow{F_{X*}} & H^1(X, \mathcal{O}_X) \\
\uparrow{\alpha^*} & & \uparrow{\alpha^*} \\
H^1(A, \mathcal{O}_A) & \xrightarrow{F_{A*}} & H^1(A, \mathcal{O}_A)
\end{array}
$$

Since $X$ is $F$-split, the upper horizontal arrow is bijective. Furthermore, by [24 Lemma(1.3)] we see that the vertical arrows are injective. (Note that although $X$ is assumed to be smooth in [24 Lemma(1.3)], the proof does not use smoothness of $X$.) Hence the lower horizontal arrow is injective, and thus $A$ is ordinary. Let $X \xrightarrow{f} Z \xrightarrow{g} A$ be the Stein factorization of $a$. Then as shown by Proposition 5.11 (1), $Z$ is $F$-split, or equivalently, $\mathcal{O}_Z$ is a direct summand of $\mathcal{F}(e) := F_{Z*}\mathcal{O}_{Z^e}$ for every $e > 0$. Since $a^* : \text{Pic}^0(A) \to \text{Pic}^0(X)$ is bijective, $g^* : \text{Pic}^0(A) \to \text{Pic}^0(Z)$ is injective. Hence

$$p^{e-\dim A} = \#\text{Pic}^0(A)[F_A^e] \leq \#\text{Pic}^0(Z)[F_Z^e].$$

Then by the projection formula and Lemma 6.1 (set $\mathcal{F} := \mathcal{F}(e)$ and $\mathcal{F}' := \mathcal{O}_Z$), we obtain

$$p^{e-\dim A} \leq \# \{ L \in \text{Pic}(Z) | \mathcal{F}(e) \otimes L \cong \mathcal{F}(e) \} \leq \text{rank } \mathcal{F}(e) = p^{e-\dim Z}. $$
This implies that dim $Z = \dim A$ and that $\bigoplus_{L \in \text{Pic}(Z)[p^e]} L \subseteq \mathcal{F}(e)$ is a direct summand of maximum rank. Since $\mathcal{F}(e)$ is torsion free, the inclusion is an isomorphism. Therefore $F_Z^e$ is flat, or equivalently, $Z$ is smooth. Now it is enough to show that $\omega_Z$ is pseudo-effective. Indeed, if it holds, then Theorem 6.2 shows that $Z$ is an ordinary abelian variety, since $\mathcal{F}(e)$ is a direct sum of line bundles for every $e > 0$. Then $g : Z \to A$ is an isomorphism, and by Proposition 5.11(3), we see that $(a, \Delta)$ is $F$-split, which is our assertion. We show the pseudo-effectivity of $\omega_Z$. Fix an $e > 0$. Now we have $(\mathcal{F}(e))^* \cong \mathcal{F}(e)$ and $F_Z^e \mathcal{F}(e) \cong \bigoplus \mathcal{O}_{Z^e}$. Furthermore, by (3.2.2) of Subsection 3.2 we obtain that

$$F_Z^e \omega_{Z^e}^{1-p^e} \cong \text{Hom}(F_Z^e \mathcal{O}_{Z^e}, \mathcal{O}_Z) = (\mathcal{F}(e))^* \cong \mathcal{F}(e).$$

Hence there exists a surjection $F_Z^e \mathcal{F}(e) \cong F_Z^e \omega_{Z^e}^{1-p^e} \to \omega_{Z^e}^{1-p^e}$, which implies that $\omega_{Z^e}^{1-p^e}$ is globally generated. Since $H^0(Z^e, \omega_{Z^e}^{1-p^e}) \cong H^0(Z, \mathcal{F}(e)) \cong k$, we get $\omega_{Z^e}^{1-p^e} \cong \mathcal{O}_Z$, or equivalently $\omega_Z^{p^e-1} \cong \mathcal{O}_Z$, and thus $\omega_Z$ is pseudo-effective.

The converse follows directly from Proposition 5.11. □

**Proof of Theorem 1.4.** Assume that $(a, \Delta)$ is locally $F$-split. By Theorem 1.3 $a$ is surjective with $a_* \mathcal{O}_X \cong \mathcal{O}_X$, and hence the first statement follows. We show the second statement. The if part is obvious. For the only if part, we assume $\dim A = \dim X$. Then by Proposition 5.9 we see that $a$ is an isomorphism and $\Delta = 0$. □

**Remark 6.5.** For a smooth projective variety $V$ over an algebraically closed field of characteristic zero, we have $b_1(V)/2 = h^{1,0}(V) := \dim H^0(V, \Omega^1_V)$. However, in positive characteristic, we only have the inequality $b_1(V)/2 \leq h^{1,0}(V)$. Igusa constructed a smooth projective surface $S$ with $b_1(S) = h^{1,0}(S) = 2$ [1]. In [8], ordinary abelian varieties of odd characteristic are characterized as smooth projective $F$-split varieties $V$ with $h^{1,0}(V) = \dim V$.

The purpose of the remainder of this section is to prove the next theorem.

**Theorem 6.6.** Let $\gamma_A$ be the $p$-rank of $A$. Assume that there exists a morphism $f : X \to B$ to an abelian variety $B$ of $p$-rank $\gamma_B$ such that $(f, \Delta)$ is $F$-split. Then $(a, \Delta)$ is $F$-split and $\gamma_A = \gamma_B + \dim A - \dim B$. In particular, if $B$ is ordinary, then $(X, \Delta)$ is $F$-split.

To prove this, we need to prove Proposition 5.9 which is an application of Theorem 1.4. We first observe line bundles whose pullbacks by relative Frobenius morphisms are trivial.

**Observation 6.7.** Let $f : X \to Z$ be a separable surjective morphism to a smooth projective variety $Z$ such that $f_* \mathcal{O}_X \cong \mathcal{O}_Z$. 
(1) We consider the following commutative diagram of Picard groups:

\[
\begin{array}{ccc}
\text{Pic}(X^e) & \overset{F^e_{X/Z}}{\longrightarrow} & \text{Pic}(X) \\
\downarrow_{F^e_{X/Z}} & & \downarrow_{f^*} \\
\text{Pic}(X_{Z^e}) & \overset{(F^e_{Z})_{X}}{\longrightarrow} & \text{Pic}(X) \\
\downarrow_{f_{Z^e}^*} & & \downarrow_{f^*} \\
\text{Pic}(Z^e) & \underset{F^e_{Z}}{\longleftarrow} & \text{Pic}(X) \\
\end{array}
\]

Clearly, \( f^* \) induces an injective morphism \( \text{Pic}(Z)[p^e] \overset{f^*}{\longrightarrow} \text{Pic}(X)[(F^e_{Z})_{X}] \). We show that this is an isomorphism. Let \( L \in \text{Pic}(X)[(F^e_{Z})_{X}] \). Then by the flatness of \( F^e_{Z} \), we have

\[
F^e_{Z} f_* L \cong f_{Z^e}^* L \cong f_{Z^e}^* O_{X_{Z^e}} \cong F^e_{Z} f_* O_X \cong O_{Z^e}.
\]

Hence \( f_* L \) is a \( p^e \)-torsion line bundle on \( Z \), and the natural morphism \( f^* f_* L \to L \) is an isomorphism. Therefore we deduce that the above homomorphism is surjective. Note that a non-zero homomorphism between numerically trivial line bundles on a projective variety is an isomorphism.

By the above argument, we have the following exact sequence

\[
0 \to \text{Pic}(Z)[p^e] \overset{f^*}{\longrightarrow} \text{Pic}(X)[p^e] \to \text{Pic}(X_{Z^e})[F^e_{X/Z}]\]

(2) Set \( \mathcal{F} := F^e_{X/Z} O_{X^e} \) and \( I := \{ L \in \text{Pic}(X_{Z^e}) | \mathcal{F} \otimes L \cong \mathcal{F} \} \). Then we have \( \text{Pic}(X_{Z^e})[F^e_{X/Z}] \subseteq I \) by the projection formula. Let \( \mathcal{F}' \) be an indecomposable direct summand of \( \mathcal{F} \) and let \( I' := \{ L \in \text{Pic}(X_{Z^e}) | \mathcal{F}' \otimes L \cong \mathcal{F}' \} \). Then by Lemma 6.7 we obtain that \( \bigoplus_{[L] \in I/I'} \mathcal{F}' \otimes L \) is a direct summand of \( \mathcal{F} \). In particular,

\[
\text{rank } \mathcal{F}' \cdot \#(I/I') \leq \text{rank } \mathcal{F} = p^{e(\text{dim } X - \text{dim } Z)}.
\]

The following lemma is used to prove Proposition 6.9.

**Lemma 6.8.** Let \( f : X \to Z \) be an \( F \)-split morphism to a smooth projective variety \( Z \). Let \( X_z \) be the general closed fiber of \( f \). Then \( h^1(X, O_X) \leq h^1(X_z, O_{X_z}) + h^1(Z, O_Z) \).

**Proof.** Set \( \mathcal{G}^i := R^i f_* O_X \). Then we have \( \text{rank } \mathcal{G}^i = h^i(X_z, O_{X_z}) \) and \( F^e_{Z} \mathcal{G}^i \cong \mathcal{G}^i \) for some \( e > 0 \) by Lemma 6.4. As shown by [211, 1.4. Satz], there exists an étale cover \( \pi : Z' \to Z \) such that \( \pi^* \mathcal{G}^i \cong \bigoplus O_{Z'} \) for each \( i \), and hence

\[
\dim H^0(Z, \mathcal{G}^i) \leq \dim H^0(Z', \pi^* \mathcal{G}^i) = \text{rank } \mathcal{G}^i = h^i(X_z, O_{X_z}).
\]

Therefore by the Leray spectral sequence, we have

\[
h^1(X, O_X) \leq h^0(Z, \mathcal{G}^1) + h^1(Z, O_Z) \leq h^1(X_z, O_{X_z}) + h^1(Z, O_Z).
\]

\[ \square \]

**Proposition 6.9.** Let \( f : X \to Z \) be an \( F \)-split morphism to an abelian variety \( Z \). Suppose that the Albanese morphism \( a : X \to A \) of \( X \) is a finite morphism. Then \( a \) is an isomorphism, or equivalently, \( X \) is an abelian variety.
Proof. Let \( f : X \to Z' \xrightarrow{\pi} Z \) be the Stein factorization. As shown by Proposition 5.11, \( \pi \) is \( F \)-split. Hence we see that \( \pi \) is étale by Proposition 5.9. This implies that \( Z' \) is also an abelian variety by [25, Section 18, Theorem] and that \( (f', \Delta) \) is \( F \)-split by Proposition 5.11. Replacing \( Z \) by \( Z' \), we may assume that \( f_*\mathcal{O}_X \cong \mathcal{O}_Z \). We can factor \( f \) into \( f : X \xrightarrow{a} A \xrightarrow{g} Z \). Let \( z \in Z \) be a general closed point. Then as shown by Proposition 5.6, \( X_z \) is integral, normal and \( F \)-split. We recall that \( a \) is a finite morphism by the assumption. Then the induced morphism \( X_z \to (A_z)_{\text{red}} \) is a finite morphism to an abelian variety, and therefore \( X_z \) is an ordinary abelian variety by Theorem 1.4. Hence by Lemma 6.8, we have

\[
\dim A \leq h^1(X, \mathcal{O}_X) \leq h^1(X_z, \mathcal{O}_{X_z}) + h^1(Z, \mathcal{O}_Z) = \dim X_z + \dim Z = \dim X.
\]

This means that \( a \) is surjective. Since \( f \) is \( F \)-split, it is separable, and hence so is \( g \), which implies that \( A_z \) is reduced. We may assume \( X_z \to A_z \) is an isogeny of abelian varieties. Considering \( p \)-torsion points, we see that \( A_z \) is also ordinary. Therefore the \( p \)-rank \( \gamma_A \) of \( A \) is equal to

\[
\gamma_{A_z} + \gamma_Z = \dim A_z + \gamma_Z = \dim A - \dim Z + \gamma_Z,
\]

and hence \( g \) is \( F \)-split because of Theorem 5.7 (ii).

Claim 6.10. \( a : X \to A \) is separable.

If the claim holds, then \( 0 \sim a^*K_A \leq K_X \). Since \( f \) is \( F \)-split, we have \( \kappa(X, K_{X/Z}) = \kappa(X, K_X) \leq 0 \) by Lemma 5.10, and hence \( K_X = 0 \). Applying the Zariski-Nagata purity, \( a \) is an étale morphism. Hence we obtain that \( X \) is an abelian variety by [25, Section 18, Theorem]. This is our assertion. \( \square \)

Proof of Claim 6.10. We factor \( a : X \to A \) into two finite morphisms \( i : X \to Y \) and \( s : Y \to A \) such that \( i \) is purely inseparable and \( s \) is separable. We show that \( i \) is an isomorphism. We fix an \( e > 0 \) such that there exists a morphism \( b : Y^e \to X \) such that the following diagram commutes:

\[
\begin{array}{ccc}
X^e & \xrightarrow{F_X^e} & X \\
\downarrow i^e \downarrow & & \downarrow i \\
Y^e & \xrightarrow{F_Y^e} & Y.
\end{array}
\]

This induces the following commutative diagram:

\[
\begin{array}{ccc}
X^e & \xrightarrow{F_{X/Z}^e} & X_{Z^e} & \xrightarrow{(F_{Z}^e)_X} & X \\
\downarrow i^e \downarrow & & \downarrow i_{Z^e} \downarrow & & \downarrow i \\
Y^e & \xrightarrow{F_{Y/Z}^e} & Y_{Z^e} & \xrightarrow{(F_Z^e)_Y} & Y.
\end{array}
\]

Note that since \( f : X \to Z \) and \( g \circ s : Y \to Z \) are separable, \( X_{Z^e} \) and \( Y_{Z^e} \) are varieties as shown by Lemma 3.1. Since \( \mathcal{O}_{X_{Z^e}} \to F_{X/Z}^e \mathcal{O}_{X^e} \) splits, \( \mathcal{O}_{X_{Z^e}} \to b_{Z^e}^* \mathcal{O}_{Y^e} \) also splits. From this, the coherent sheaf \( F := F_{Y/Z}^e \mathcal{O}_{Y^e} \) on \( Y_{Z^e} \) has \( i_{Z^e}^* \mathcal{O}_{X_{Z^e}} \) as a
Since the finite morphism

Let \( H \) be a \( p^r \)-torsion line bundle on \( A \). Now we have the morphisms \( Y \stackrel{g}{\rightarrow} A \). Set \( M \) to be the \( p^r \)-torsion line bundle \((s^*L)^e \) on \( Y_{Z^e} \). We show that \( M \in I \) and that \( M \in I' \) if and only if \( L \in g^*\text{Pic}(Z)[p^e] \). By the projection formula, we have

\[
F \otimes M = (F_{Y/Z_e} \mathcal{O}_Y) \otimes (s^*L)^e \cong F_{Y/Z_e} \mathcal{O}_Y \cong F^e_{Y/Z_e} (s^*L)^e \cong F \cong F_{Y/Z_e} \mathcal{O}_Y \cong F_{Y/Z_e} \mathcal{O}_Y,
\]

and hence \( M \in I \). If \( L \in g^*N \) for an \( N \in \text{Pic}(Z)[p^e] \), then \( M \cong s_{Z^e} F_{Z^e}^e N \cong \mathcal{O}_{Y_{Z^e}} \in I' \). Conversely, if \( M \in I' \), then again by the projection formula, we have

\[
0 \neq H^0(Y_{Z^e}, F') \cong H^0(Y_{Z^e}, F' \otimes M) \cong H^0(Y_{Z^e}, (i_{Z^e*} \mathcal{O}_{X^e}) \otimes M) \cong H^0(X_{Z^e}, i_{Z^e*} M) \cong H^0(X_{Z^e}, (a^*L)^e).
\]

Therefore \((a^*L)^e \cong \mathcal{O}_{X^e}\). By Observation 6.7 (1), we get \( a^*L \in \mathfrak{I}^*(\text{Pic}(Z)[p^e]) = a^*g^*(\text{Pic}(Z)[p^e]) \). Since \( a^* : \text{Pic}^0(A) \rightarrow \text{Pic}^0(X) \) is an isomorphism, we have \( L \in g^*(\text{Pic}(Z)[p^e]) \). From the argument above, we have the following injective morphism

\[
G := \text{Pic}(A)[p^e]/g^*\text{Pic}(Z)[p^e] \xrightarrow{(a^*(L))^e} I/I'.
\]

Let \( r' \) be the rank of \( F' \). Since the number of \( p^e \)-torsion line bundles on \( A \) (resp. \( Z \)) is equal to \( p^{e-\gamma_A} \) (resp. \( p^{e-\gamma_Z} \)), we have

\[
p^{e-(\dim A - \dim Z)} r' = p^{e-(\gamma_A-\gamma_Z)} r' \leq \# G \cdot r' \leq \#(I/I') \cdot r' \leq p^{e-(\dim A - \dim Z)}
\]

by Observation 6.7 (2). This implies that \( r' = 1 \) and that \( \bigoplus_{[\mathcal{L}] \in G} F' \otimes (s^*L)^e \subseteq F \) is a direct summand of maximal rank. Since \( F \) is torsion free, this inclusion is an isomorphism. Since \( i_{Z^e*} \mathcal{O}_{X^e} \) is a direct summand of \( F \), there exists a subset \( H \subseteq G \) such that \( \bigoplus_{[\mathcal{L}] \in H} F' \otimes (s^*L)^e \cong i_{Z^e*} \mathcal{O}_{X^e} \). Then the rank \( i_{Z^e*} \mathcal{O}_{X^e} = \# H \cdot r' = \# H. \) We show \( \# H = 1 \). For \( \mathcal{L} \in \text{Pic}(A)[p^e] \) with \([\mathcal{L}] \in H \), we have

\[
0 \neq H^0(Y_{Z^e}, F') = H^0(Y_{Z^e}, F' \otimes (s^*L)^e \otimes (s^*L^{-1})^e) \subseteq H^0(Y_{Z^e}, (i_{Z^e*} \mathcal{O}_{X^e}) \otimes (s^*L^{-1})^e) = H^0(X_{Z^e}, (a^*L)^{-1}).
\]

By an argument similar to the above, we see that \( L^e \in g^*\text{Pic}(Z)[p^e] \), and thus \( [\mathcal{L}] = [\mathcal{O}_A] \in G \). Hence \( H = \{[\mathcal{O}_A]\} \). Since \( \deg i = \deg i_{Z^e} = \text{rank} i_{Z^e*} \mathcal{O}_{X^e} = 1 \), we see that \( i \) is an isomorphism, which is our assertion. \( \square \)

**Proof of Theorem 6.6.** Let \( X \xrightarrow{g} X' \) be the Stein factorization of \( a \). Then we can factors \( f \) into \( X \xrightarrow{h} X' \xrightarrow{g'} A \xrightarrow{\Delta} B \). By Proposition 5.11 (1), \( h \circ g' \) is \( F \)-split. Since the finite morphism \( g' : X' \rightarrow A \) is the Albanese morphism of \( X' \), we see that \( g' \) is an isomorphism by Proposition 6.9. Therefore Proposition 5.11 (3) shows that \((a, \Delta)\) is \( F \)-split. Since \( h : A \rightarrow B \) is an \( F \)-split morphism whose closed fibers \( A_z \) are ordinary abelian varieties, we obtain

\[
\gamma_A = \gamma_A + \gamma_B = \dim A_z + \gamma_B = \dim A - \dim B + \gamma_B.
\]

\( \square \)
7. Minimal surfaces with $F$-split Albanese morphisms

The aim of this section is to specify minimal surfaces with $F$-split or locally $F$-split Albanese morphisms. Note that if a smooth projective surface has $F$-split (resp. locally $F$-split) Albanese morphism, then so are its minimal surfaces. Indeed, let $S_1$ be a smooth projective surface and $\pi : S_1 \to S_2$ be the contraction of a $(-1)$-curve. Then it is easily seen that the induced morphism $\text{Alb}(\pi) : \text{Alb}(S_1) \to \text{Alb}(S_2)$ is an isomorphism. Hence if $S_1$ has $F$-split (resp. locally $F$-split) Albanese morphism, then so does $S_2$ by Proposition 5.11 (1).

Throughout this section, we denote by $X$ a smooth projective minimal surface and by $a : X \to A$ the Albanese morphism of $X$.

**Theorem 7.1.** If $a$ is locally $F$-split, then one of the following holds:

0) $b_1(X) = 0$ and $X$ is $F$-split.

(1-1) $b_1(X) = 2$, $\kappa(X) = -\infty$ and $X$ is the projective space bundle $\mathbb{P}(E)$ associated with a rank two vector bundle $E$ on $A$. Furthermore, $a$ is $F$-split if and only if either

(a) $E$ is decomposable,

(b) $E$ is indecomposable, $p \geq 3$ and $\deg E$ is odd, or

(c) $E$ is indecomposable, $p = 2$ and $A$ is ordinary.

(1-2) $b_1(X) = 2$, $\kappa(X) = 0$ and $X$ is a hyperelliptic surface such that every closed fiber of $a$ is an ordinary elliptic curve. In this case, $a$ is $F$-split.

(2) $X$ is an abelian surface.

Note that the first Betti number $b_1(X)$ is equal to $2 \dim A$. By Theorem 1.4 we see that $b_1(X) = 0, 2$ or $4$.

• If $b_1(X) = 0$, then the $F$-splitting of $a$ is equivalent to the $F$-splitting of $X$.

• If $b_1(X) = 4$, then $X$ is an abelian surface as shown by Theorem 1.4.

• The case when $b_1(X) = 2$ is dealt with in the remainder of this section. As shown by Lemma 5.10 we have $\kappa(X) \leq 0$. We consider the cases $\kappa(X) = -\infty$ and $\kappa(X) = 0$ respectively in Subsections 7.1 and 7.2.

7.1. The case $b_1(X) = 2$ and $\kappa(X) = 0$. In this case, by Bombieri and Mumford’s classification of minimal surfaces with Kodaira dimension zero [3], we see that $X$ is a hyperelliptic or quasi-hyperelliptic surface. If $a$ is locally $F$-split, then $a$ has normal geometric generic fiber as shown by Proposition 5.6 and hence $X$ is hyperelliptic. In particular, there exist two elliptic curves $E_0$ and $E_1$ such that $X \cong E_1 \times E_0/B$, where $B$ is a finite subgroupscheme of $E_1$ [3, Theorem 4]. Furthermore, every closed fiber of $a$ is isomorphic to $E_0$, and $A \cong E_1/B$.

**Proposition 7.2.** The followings are equivalent:

1) $a$ is $F$-split.

2) $a$ is locally $F$-split.

3) $E_0$ is ordinary.

**Proof.** (1)$\Rightarrow$(2) is obvious. If $a$ is locally $F$-split, then the general fibers are $F$-split by Proposition 5.6 and hence $E_0$ is $F$-split. Thus (2)$\Rightarrow$(3) holds. We prove (3)$\Rightarrow$(1).

Assume that $E_0$ is $F$-split. Since $a$ is flat and every fiber has the trivial canonical
bundle, $K_X \sim a^*C$ for a Cartier divisor $C$ on $A$. Then by Theorem \ref{t:rank} (ii), we obtain an effective $\mathbb{Q}$-divisor $\Delta_A$ on $A$ such that $C \sim_{\mathbb{Z}(p)} K_A + \Delta_A \sim \Delta_A$. Since $K_X \sim_{\mathbb{Q}} 0$, we have $\Delta_A = 0$, and hence $a$ is $F$-split as shown by Theorem \ref{t:rank} (ii)-(3). This is our claim.

7.2. The case $b_1(X) = 2$ and $\kappa(X) = -\infty$. In this case, $X$ is a ruled surface over an elliptic curve. We start with recalling some facts on elliptic curves. In the theorem and lemmas, we denote by $C$ an elliptic curve.

**Theorem 7.3.** Let $E_C(r, d)$ be the isomorphism class of indecomposable vector bundles of rank $r$ and of degree $d$.

1. \cite[Theorem 10]{not1} For every $E, E' \in E_C(r, d)$, there exists an $L \in \text{Pic}^0(C)$ of degree zero such that $E \otimes L \cong E'$. For $L_1, L_2 \in \text{Pic}^0(C)$, $E \otimes L_1 \cong E \otimes L_2$ if and only if $L_1' \cong L_2'$, where $r' := r/(r, d)$. Furthermore, when $d = 0$, there exists a unique element $E_{r, 0}$ in $E_C(r, 0)$ such that $H^0(C, E_{r, 0}) \not\cong 0$.

2. \cite[Proposition 2.1]{not1} Let $\pi : C' \to C$ be an isogeny of degree $r$ and $L$ be a line bundle of degree $d$ on $C'$. If $r$ and $d$ are coprime, then $\pi_*L \in E_C(r, d)$.

3. \cite[Theorem 2.16]{not1} Let $r > 0$ and $d$ be coprime integers and $E$ be an element of $E_C(rh, dh)$ for some $h > 0$. When $C$ is ordinary, $F_C^*E$ is indecomposable.

When $C$ is supersingular, $F_C^*E$ is indecomposable if and only if either $h = 1$, or $h \neq 1$ and $p \nmid r$.

In the following lemmas, we denote by $C$ an elliptic curve.

**Lemma 7.4.** Let $\pi : C' \to C$ be a finite morphism of degree $d$ from an elliptic curve $C'$. Let $L$ be a line bundle on $C$ such that $\pi^*L \cong O_{C'}$. Then $L^d \cong O_C$.

**Proof.** Set $F := \pi_*O_{C'}$. Then by the projection formula, we have $F \otimes L \cong F$. We consider the action of $\{L^i \mid i \in \mathbb{Z}\}$ on the set $D_F$ of indecomposable direct summands of $F$. We take an $F^i \in D_F$. Let $r := rF$ and $r' := r/(r, \text{deg } F)$. Let $n > 0$ be the order of $L$. Then by Theorem \ref{t:rank} (1), we see that $F^i \otimes L^i \cong F$ if and only if $n|r't$. Let $t$ be the minimum positive integer such that $n|r't$. Then $\bigoplus_{0 \leq i < r} F^i \otimes L^i$ is a direct summand of $F$ of rank $rt$. Note that $n|r't$. From the above argument we obtain $n|\text{rank } F = d$, or equivalently, $L^d \cong O_A$. \hfill $\Box$

**Lemma 7.5.** Then there exists a finite morphism $\pi : C' \to C$ of from an elliptic curve $C'$ such that $\pi^*E_{2, 0} \cong O_C^{\oplus 2}$ and $\pi_*O_{C'} \cong E_{p, 0}$.

**Proof.** Recall that $E_{2, 0}$ is obtained as a nontrivial extension $0 \to O_C \to E_{2, 0} \to O_C \to 0$. Let $\xi$ be the element of $\text{Ext}^1(O_C, O_C) \cong H^1(C, O_C)$ corresponding to this extension. If $C$ is ordinary, or equivalently, if $F_C^* : H^1(C, O_C) \to H^1(C, O_C)$ is an isomorphism, then we may assume that $F_C^*\xi = \xi$. In this case, $\xi$ defines an étale cover $\pi : C' \to C$ of degree $p$ such that $\pi^*\xi = 0$, and thus $\pi^*E_{2, 0} \cong O_C^{\oplus 2}$. If $C$ is supersingular, then we set $\pi := F_C$. Since $\pi^*\xi = 0$, we have $\pi^*E_{2, 0} \cong O_C^{\oplus 2}$.

Next we prove the second statement. Set $F := \pi_*O_{C'}$. By the Grothendieck duality, we have $F^* \cong F$. Let $F'$ be an indecomposable direct summand of $F$. Note that $\deg F' > 0$ if and only if $F'$ is ample. Since $\text{Hom}(F', F) \cong F \otimes F^* \cong \pi_*(\pi^*F^*)$ has non-zero global section, we see that $\deg F' \leq 0$. From this we get that $\deg F' = 0$. Then by Theorem \ref{t:rank} (1), we have an $L \in \text{Pic}^0(C)$ such that $H^0(C, F' \otimes L) \cong k$. 

Then by the projection formula, we see that $F' \otimes L \subseteq F \otimes L \cong \pi_*\pi^*L$ has non-zero global section, and hence $\pi^*L \cong O_{C'}$. By Lemma 7.4, we have $L^p \cong O_C$. If $L \cong O_C$, then $F$ has $p$ direct summands of rank one, and one of them is isomorphic to $O_C$. This means that the natural homomorphism $O_C \rightarrow \pi_*O_{C'}$ splits, and hence $\pi^*: H^1(C, O_C) \rightarrow H^1(C', O_{C'})$ is an isomorphism, which is a contradiction. Hence $L \cong O_C$, or equivalently, $H^0(C, F') \cong k$. Since $H^0(C, F) \cong k$, we conclude that $F$ is indecomposable, and therefore $F \in \mathcal{E}_{p,0}$. 

Lemma 7.6. The $m$-th symmetric product $S^m \mathcal{E}_{2,0}$ of $\mathcal{E}_{2,0}$ is a direct sum of vector bundles of the form $\mathcal{E}_{r,0}$. 

Proof. Let $\mathcal{F}$ be an indecomposable direct summand of $S^m \mathcal{E}_{2,0}$ of rank $r$. By Theorem 7.3, we may write $\mathcal{F} \cong \mathcal{E}_{r,0} \otimes L$ for an $L \in \text{Pic}^0(C)$. Let $\pi: C' \rightarrow C$ be as in Lemma 7.5. Since $\pi^*S^m \mathcal{E}_{2,0}$ is trivial, we have $\pi^*L \cong O_{C'}$. By Lemma 7.4, we see that $L^p \cong O_C$. Since supersingular elliptic curves have no non-trivial $p$-torsion line bundle, we may assume that $C$ is ordinary. Then since $F_{C'}^* \mathcal{E}_{r,0} \cong \mathcal{E}_{r,0}$, we get that $F_{C'}^* \mathcal{F} \cong \mathcal{E}_{r,0} \otimes L^p \cong \mathcal{E}_{r,0}$ and $F_{C'}^*S^m \mathcal{E}_{2,0} \cong S^m \mathcal{E}_{2,0}$. Hence we conclude that $L \cong O_C$. 

Now we return to study the $F$-splitting of the Albanese morphism $a: X \rightarrow A$ of $X$. We may regard $X$ and $a$ respectively as $\mathbb{P}(\mathcal{E})$ for a vector bundle on $A$ of rank two and its projection. If $\mathcal{E}$ is decomposable, then $a$ is $F$-split as seen in Example 5.3. Assume that $\mathcal{E}$ is indecomposable. We only need to consider the two cases: $\text{deg} \mathcal{E} = 0$ and $\text{deg} \mathcal{E} = 1$. 

7.2.1. The case $\text{deg} \mathcal{E} = 0$. In this case, we may assume that $\mathcal{E} = \mathcal{E}_{2,0}$ by Theorem 7.3 (1). Then we have a finite morphism $\pi: A' \rightarrow A$ from an elliptic curve $A'$ such that $\pi^* \mathcal{E}_{2,0} \cong O_{A'}^{\oplus 2}$, as seen in Lemma 7.5. In particular, $X_{A'} \cong \mathbb{P}(\pi^* \mathcal{E}_{2,0}) \cong \mathbb{P}^1 \times A^1$. We show the following: 

Proposition 7.7. $a: X \rightarrow A$ is $F$-split if and only if $A$ is ordinary and $p = 2$. 

To prove Proposition 7.7, we prepare the claims below. 

Claim 7.8. There exists an algebraic fiber space $g: X \rightarrow Y \cong \mathbb{P}^1$ such that $g^*O_Y(1) \cong O_X(p)$. 

Claim 7.9. $S^p \mathcal{E}_{2,0} \cong \mathcal{E}_{p,0} \oplus O_A$. 

Proof of Claims 7.8 and 7.9. Since the Iitaka-Kodaira dimensions of line bundles are preserved under the pullback by any surjective projective morphism [17, Theorem 10.5], we have 

$$\kappa(X, O_X(1)) = \kappa(X_{A'}, O_{X_{A'}}(1)) = \kappa(\mathbb{P}^1, O_{\mathbb{P}^1}(1)) = 1.$$ 

Since $\nu(X, O_X(1))$ is also equal to one, we deduce that $O_X(1)$ is semi-ample. Let $g: X \rightarrow Y$ be the Iitaka fibration associated to $O_X(1)$. Then $Y \cong \mathbb{P}^1$ obviously. Let $B$ be the general fiber of $g$. Then $B$ is an elliptic curve. Now we have the following
commutative diagram:

\[
\begin{array}{ccc}
B_{A'} & \longrightarrow & B \\
\downarrow & & \downarrow \\
X_{A'} & \stackrel{\pi_X}{\longrightarrow} & X \\
\downarrow a_{A'} & & \downarrow a \\
A' & \stackrel{\pi}{\longrightarrow} & A
\end{array}
\]

By the construction, we have \(O_X(B) \cong O_X(m) \otimes a^*L\) for an \(m > 0\) and a torsion line bundle \(L\) on \(A\). We consider the exact sequence

\[
0 \to O_X(l) \otimes O_X(-B) \to O_X(l) \to O_B \to 0
\]

for \(l \in \mathbb{Z}\). Taking the direct image, we obtain exact sequences

\[
0 \to L^* \to S^m E_{2,0} \to a_\ast O_B \to 0 \quad \text{and} \quad 0 \to S^{m-1} E_{2,0} \to a_\ast O_B \to 0
\]

when \(l = m\) and \(l = m - 1\), respectively. By the first exact sequence and by Lemma \(7.6\) we see that \(L \cong O_A\), or equivalently \(O_X(B) \cong O_X(m)\). By the second one and Lemma \(7.6\) we obtain that \(a_\ast O_B \cong E_{m,0}\). Hence \(\pi^*a_\ast O_B \cong \oplus O_{A'}^m\). Since \(\pi : A' \to A\) and \((a | B) : B \to A\) are flat, we obtain

\[
\dim H^0(B, (a | B)^* E_{p,0}) = \dim H^0(B, (a | B)^* \pi_\ast O_{A'})
\]

\[
= \dim H^0(B_{A'}, O_{B_{A'}}) = \dim H^0(A', \pi^* a_\ast O_B) = m.
\]

Since \(a : X \to A\) is a non-trivial projective space bundle, we have \(m \geq 2\). From this, we deduce that \((a | B)^* : H^1(A, O_A) \to H^1(B, O_B)\) is zero and \((a | B)^* E_{p,0} \cong O_{B}^p\). Consequently, we obtain \(m = p\). Since \(H^0(A, S^p E_{2,0}) \cong H^0(X, O_X(p)) \cong H^0(Y, O_Y(1)) = \mathbb{P}^2\), we see that the first exact sequence splits, which implies that \(S^p E_{2,0} \cong E_{p,0} \oplus O_A\).

Now we start the proof of Proposition \(7.7\).

**Proof of Proposition \(7.7\).** We use the same notation as the proof of Claim \(7.8\). First we prove the if part. We show that \((a, B)\) is \(F\)-split. Now we have \(\omega_X \otimes O_X(B) \cong O_X(-2) \otimes O_X(p) \cong O_X\). Hence by Theorem \(5.7\) (ii)-(3), it is enough to show that \((X_z, B|_{X_z})\) is \(F\)-split for a fiber \(X_z \cong \mathbb{P}^1\) of \(a\). Since \(A\) is ordinary, \(\pi : A' \to A\) is étale. Since \(\pi^* a_\ast O_B\) is a trivial vector bundle of rank two on \(A'\), we see that \(B_{A'}\) is a disjoint union of sections of \(a_{A'} : X_{A'} \to A'\). This implies that the divisor \(B|_{X_z}\) is a sum of two distinct points, and therefore we conclude that \((X_z, B|_{X_z})\) is \(F\)-split.

Next we prove the only if part. We first show that \(A\) is ordinary by contradiction. Assume that \(A\) is supersingular. Then \(\pi = F_A\). In this case we see that \(B_{A'} = pS\) as divisors, where \(S\) is a section of \(a_{A'} : X_{A'} \to A'\). Set

\[
\psi^{(e)} := H^0(S, \phi_{X_{A'/A'}}^{(1)} \otimes \omega_X^{1-p^{e-1}}) \cdot H^0((X_{A'})^1, \omega_X^{1-p^{e}}) \to H^0(S, \phi_{X_{A'/A'}}^{(1)} \otimes \omega_X^{1-p^{e-1}}).
\]

Then by Claim \(7.8\) we have

\[
H^0(X, \omega_X^{1-p^{e}}) = H^0(X, O_X(2p^e - 2))
\]

\[
= (g \cdot H^0(Y, O_Y(2p^{e-1} - 1))) \cdot (H^0(X, O_X(p - 2))).
\]
Since every fiber of \( g \circ \pi_X \) is equal to \( pS \) as divisors for a section \( S \) of \( a_{A'} \), we see that for every \( s \in \pi_X^* X^e \) there exists a \( t \in H^0(\mathcal{O}_{X, A'}(2p^{e-1} - 1)) \) such that \( s = t^p \). Hence we have \( \psi(e)(s \cdot r) = \psi(e)(r) \cdot t = 0 \) for every \( r \in \pi_X^* H^0(X, \mathcal{O}_X(p - 2)) \). Since \( \phi_{X, A'/A'}^e \) factors through \( \phi^{(1)} \otimes \omega_{X, A'/A'} \), we deduce that \( \phi_{X, A'/A'}^e \) sends \( s \cdot r \) to \( 0 \). Since \( \phi_{X, A'/A'}^e \) is obtained as a pullback of \( \phi_{X, A'}^e \), we conclude that \( H^0(X, \mathcal{O}_{X, A'}) \) is the zero map. Therefore \( a_{A} \) is not \( F \)-split, which is a contradiction. Thus \( A \) is ordinary. We show \( p = 2 \). Since \( a_{A} \) is \( F \)-split, we see that \( X \) is \( F \)-split, or equivalently, \( \phi^{(1)}_{X, A'} : F_{X/A'} \omega_{X}^{1-p} \rightarrow \mathcal{O}_X \) splits. Then by an argument similar to the proof of Proposition 5.11 (3), we obtain the splitting \( \phi^{(1)}_{X, A'} : F_{X/A'}^{(1)} \omega_{X}^{1-p} \rightarrow \mathcal{O}_{X, A} \), which is equivalent to the splitting of \( \mathcal{O}_{X, A} \rightarrow F_{X/A}^{(1)} \mathcal{O}_{X} \). Applying the functor \( a_{A} \otimes (\mathcal{O}_{X, A}(1)) \), we obtain the morphism
\[
\mathcal{E}_{2,0} \cong F_{A}^* \mathcal{E}_{2,0} \cong a_{A}^* a_{A}^* \mathcal{O}_{X, A}(1) \rightarrow a_{A}^* a_{A}^* \mathcal{O}_{X} \cong \mathcal{O}_{A} \oplus \mathcal{O}_{A}^e
\]
which splits as \( \mathcal{O}_{A} \)-modules. Since \( \mathcal{S}_{p} \mathcal{E}_{2,0} \cong \mathcal{E}_{p,0} \oplus \mathcal{O}_{A} \) as shown by Claim 6.9, we get that \( \mathcal{E}_{p,0} \cong \mathcal{E}_{2,0} \) and thus \( p = 2 \). \[ \square \]

7.2.2. The case \( \deg \mathcal{E} = 1 \). The following proposition is the conclusion of this case.

**Proposition 7.10.** If \( \deg \mathcal{E} = 1 \), then \( a_{A} \) is \( F \)-split if and only if \( A \) is ordinary or \( p > 2 \).

**Proof.** We first prove the if part. When \( p > 2 \), we take the étale cover of degree two corresponding to a torsion line bundle \( \mathcal{L} \) of order two. Then \( \pi_{*} \mathcal{O}_{A} \cong \mathcal{O}_{A} \oplus \mathcal{L} \) and
\[
\pi_{A_{1}}^{*} \mathcal{O}_{A_{1} \times A_{1}'} \cong \pi_{*}^{*} \pi_{*} \mathcal{O}_{A} \cong \mathcal{O}_{A} \oplus \mathcal{O}_{A}.
\]
Here the first isomorphism follows from the flatness of \( \pi \). Hence \( A_{1} \times A_{1}' \) is a disjoint union of two copies of \( A_{1} \). By Theorem 7.3 (1) and (2), there exists a line bundle \( \mathcal{M} \) of degree one such that \( \pi_{*} \mathcal{M} \cong \mathcal{E} \). Then
\[
\pi_{*} \mathcal{E} \cong \pi_{*} \pi_{*} \mathcal{M} \cong \pi_{A_{1}}^{*} \mathcal{M}_{A_{1}} \cong \mathcal{M} \oplus \mathcal{M}.
\]
Therefore \( X_{1} := X_{A_{1}} \cong \mathbb{P}(\mathcal{M} \oplus \mathcal{M}) \) is \( F \)-split over \( A_{1} \). Now we have followings:

\[
\begin{array}{ccc}
X_{1} & \xrightarrow{\phi(e)} & X_{e} \\
\xrightarrow{F_{X_{A_{1}}}^{(e)}} & & \xrightarrow{F_{X/A}^{(e)}} \\
X_{1} & \xrightarrow{(\phi(e))_{\mathcal{X}}} & X_{A} \\
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{O}_{X_{1}} & \xrightarrow{(\pi(e))_{\mathcal{X}}} & \mathcal{O}_{X_{1} A_{1}} \\
\xrightarrow{F_{X_{A_{1}}}^{(e)}} & & \xrightarrow{F_{X_{A}}^{(e)}} \\
\mathcal{O}_{X_{1}} & \xrightarrow{(\pi(e))_{\mathcal{X}}} & \mathcal{O}_{X_{1} A_{1}} \\
\end{array}
\]

Since \( \pi \) is a finite étale morphism of degree not divisible by \( p \), the upper horizontal morphism in the above splits. Then the diagonal morphism also splits, and hence so is the left morphism. Consequently, we see that \( X \) is \( F \)-split over \( A \). When \( p = 2 \) and \( A \) is ordinary, \( F_{X}^* \mathcal{E} \in \mathcal{E}_{A}(2, 2) \) as shown by Theorem 7.3 (3). Then by
Proposition 7.7, we see that \( a_{A^1} : X_{A^1} \to A^1 \) is \( F \)-split. Replacing \( \pi \) by \( F_A \), we can prove the assertion by the same argument as the above.

Next we prove the only if part by contraposition. Assume that \( p = 2 \) and \( A \) is supersingular. Then Theorem 7.3 (3) shows that \( F^*A^*E \in E_A(2, 2) \). Hence as seen in Proposition 7.7, \( a_{A^1} : X_{A^1} \to A^1 \) is not \( F \)-split. This requires that \( a : X \to A \) is also not \( F \)-split, which completes the proof.  

\[ \square \]

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