DUAL $\pi$-RICKART MODULES

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Abstract. Let $R$ be an arbitrary ring with identity and $M$ a right $R$-module with $S = \text{End}_R(M)$. In this paper we introduce dual $\pi$-Rickart modules as a generalization of $\pi$-regular rings as well as that of dual Rickart modules. The module $M$ is said to be dual $\pi$-Rickart if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer $n$ such that $\text{Im} f^n = eM$. We prove that some results of dual Rickart modules can be extended to dual $\pi$-Rickart modules for this general settings. We investigate relations between a dual $\pi$-Rickart module and its endomorphism ring.

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1. Introduction

Throughout this paper $R$ denotes an associative ring with identity, and modules are unitary right $R$-modules. For a module $M$, $S = \text{End}_R(M)$ is the ring of all right $R$-module endomorphisms of $M$. In this work, for the $(S, R)$-bimodule $M$, $l_S(\cdot)$ and $r_M(\cdot)$ are the left annihilator of a subset of $M$ in $S$ and the right annihilator of a subset of $S$ in $M$, respectively. A ring is reduced if it has no nonzero nilpotent elements. Baer rings [8] are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. Principally projective rings were introduced by Hattori [3] to study the torsion theory, that is, a ring $R$ is called left (right) principally projective if every principal left (right) ideal is projective. The concept of left (right) principally projective rings (or left (right) Rickart rings) has been comprehensively studied in the literature. Regarding a generalization of Baer rings as well as principally projective rings, recall that a ring $R$ is called generalized left (right) principally projective if for any $x \in R$, the left (right) annihilator of $x^n$ is generated by an idempotent for some positive integer $n$. A number of papers have been written on generalized principally
A ring \( R \) is (von Neumann) regular if for any \( a \in R \) there exists \( b \in R \) with \( a = aba \). The ring \( R \) is called \( \pi \)-regular if for each \( a \in R \) there exist a positive integer \( n \) and an element \( x \) in \( R \) such that \( a^n = a^nxn \). Similarly, call a ring \( R \) strongly \( \pi \)-regular if for every element \( a \in R \) there exist a positive integer \( n \) (depending on \( a \)) and an element \( x \in R \) such that \( a^n = a^{n+1}x \), equivalently, there exists \( y \in R \) such that \( a^n = ya^{n+1} \).

Every regular ring is \( \pi \)-regular and every strongly \( \pi \)-regular ring is \( \pi \)-regular. There are regular or \( \pi \)-regular rings which are not strongly \( \pi \)-regular.

According to Rizvi and Roman, a module \( M \) is said to be Rickart \([10]\) if for any \( f \in S \), \( r_M(f) = eM \) for some \( e^2 = e \in S \). The class of Rickart modules is studied extensively by different authors (see \([1]\) and \([11]\)). Recently the concept of a Rickart module is generalized in \([16]\) by the present authors. The module \( M \) is called \( \pi \)-Rickart if for any \( f \in S \), there exist \( e^2 = e \in S \) and a positive integer \( n \) such that \( r_M(f^n) = eM \). Dual Rickart modules are defined by Lee, Rizvi and Roman in \([12]\). The module \( M \) is called dual Rickart if for any \( f \in S \), \( \text{Im} f = eM \) for some \( e^2 = e \in S \).

In the second section, we investigate general properties of dual \( \pi \)-Rickart modules and Section 3 contains the results on the structure of endomorphism ring of a dual \( \pi \)-Rickart module. In what follows, we denote by \( \mathbb{Z} \), \( \mathbb{Q} \), \( \mathbb{R} \) and \( \mathbb{Z}_n \) integers, rational numbers, real numbers and the ring of integers modulo \( n \), respectively, and \( J(R) \) denotes the Jacobson radical of a ring \( R \).

### 2. Dual \( \pi \)-Rickart Modules

In this section, we introduce the concept of a dual \( \pi \)-Rickart module that generalizes the notion of a dual Rickart module as well as that of a \( \pi \)-regular ring. We prove that some properties of dual Rickart modules hold for this general setting. Although every direct summand of a dual \( \pi \)-Rickart module is dual \( \pi \)-Rickart, a direct sum of dual \( \pi \)-Rickart modules is not dual \( \pi \)-Rickart. We give an example to show that a direct sum of dual \( \pi \)-Rickart modules may not be dual \( \pi \)-Rickart. It is shown that the class of some abelian dual \( \pi \)-Rickart modules is closed under direct sums.

We start with our main definition.

**Definition 2.1.** Let \( M \) be an \( R \)-module with \( S = \text{End}_R(M) \). The module \( M \) is called dual \( \pi \)-Rickart if for any \( f \in S \), there exist \( e^2 = e \in S \) and a positive integer \( n \) such that \( \text{Im} f^n = eM \).
For the sake of brevity, in the sequel, \( S \) will stand for the endomorphism ring of the module \( M \) considered. Dual \( \pi \)-Rickart modules are abundant around. Every semisimple module, every injective module over a right hereditary ring and every module of finite length are dual \( \pi \)-Rickart. Also every quasi-projective strongly co-Hopfian module, every quasi-injective strongly Hopfian module, every Artinian and Noetherian module is dual \( \pi \)-Rickart (see Corollary 2.19). Every finitely generated module over a right Artinian ring is a dual \( \pi \)-Rickart module (see Proposition 2.20).

**Proposition 2.2.** Let \( R \) be a ring. Then the right \( R \)-module \( R \) is a dual \( \pi \)-Rickart module if and only if \( R \) is a \( \pi \)-regular ring.

**Proof.** If the right \( R \)-module \( R \) is a dual \( \pi \)-Rickart module and \( f \in R \), then there exist \( e^2 = e \in R \) and a positive integer \( n \) such that \( \text{Im} f^n = eR \). There exist \( x, y \in R \) such that \( e = f^n x \) and \( f^n = ey \). Multiplying the first equation from the right by \( f^n \), we have \( f^n x f^n = ey = f^n \). Conversely, assume that \( R \) is a \( \pi \)-regular ring. Let \( g \in R \). Then there exist a positive integer \( n \) and \( x \in R \) such that \( g^n = g^n x g^n \). Hence \( e = g^n x \) is an idempotent of \( R \). Since \( e \in g^n R \) and \( g^n = g^n x g^n = eg^n \in eR \), we have \( \text{Im} g^n = eR \). Therefore the right \( R \)-module \( R \) is dual \( \pi \)-Rickart. \( \square \)

It is clear that every dual Rickart module is dual \( \pi \)-Rickart. The following example shows that every dual \( \pi \)-Rickart module need not be dual Rickart.

**Example 2.3.** Let \( R \) denote the ring \( \left( \begin{array}{cc} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{array} \right) \) and \( M \) the right \( R \)-module \( \left( \begin{array}{cc} 0 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{array} \right) \) with usual matrix operations. If \( f \in S = \text{End}_R(M) \), then there exist \( a, b, c \in \mathbb{Z}_2 \) such that

\[
\begin{pmatrix} 0 & x \\ y & z \end{pmatrix} \quad \begin{pmatrix} 0 & ax \\ by & cx + bz \end{pmatrix}
\]

By using this image of \( f \), we prove that there exists a positive integer \( n \) such that \( \text{Im} f^n \) is a direct summand of \( M \). Consider the following cases for \( a, b, c \in \mathbb{Z}_2 \).

**Case 1.** If \( a = b = c = 1 \), then \( f \) is an epimorphism.

**Case 2.** If \( a = 0, b = 0, c = 1 \), then \( f^2 = 0 \).

**Case 3.** If \( a = 0, b = 1, c = 1 \) or \( a = 0, b = 1, c = 0 \), then in either case \( \text{Im} f = \left\{ \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \mid x, y \in \mathbb{Z}_2 \right\} \) is a direct summand of \( M \).
Case 4. If $a = 1$, $b = 0$, $c = 1$, then $\text{Im} f = \left\{ \begin{pmatrix} 0 & x \\ 0 & x \end{pmatrix} \mid x \in \mathbb{Z}_2 \right\}$ is a direct summand of $M$.

Case 5. If $a = 1$, $b = 0$, $c = 0$, then $\text{Im} f = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mid x \in \mathbb{Z}_2 \right\}$ is a direct summand of $M$.

Case 6. If $a = 1$, $b = 1$, $c = 0$, then $f$ is an identity map.

Case 7. If $a = 0$, $b = 0$, $c = 0$, then $f$ is a zero map.

In all cases there exists a positive integer $n$ such that $\text{Im} f^n$ is a direct summand of $M$ and so $M$ is a dual $\pi$-Rickart module. The module $M$ is not dual Rickart by the second case, since $\text{Im} f = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \mid x \in \mathbb{Z}_2 \right\}$.

Our next aim is to find conditions under which a dual $\pi$-Rickart module is dual Rickart.

**Proposition 2.4.** Let $M$ be a dual Rickart module. Then $M$ is dual $\pi$-Rickart. The converse holds if $S$ is a reduced ring.

**Proof.** The first statement is clear. Suppose that $S$ is a reduced ring and $M$ is a dual $\pi$-Rickart module. Let $f \in S$. There exist a positive integer $n$ and an idempotent $e \in S$ such that $\text{Im} f^n = eM$. If $n = 1$, there is nothing to do. Assume that $n > 1$. Then $(1 - e)f^n M = 0$ and so $(1 - e)f^n = 0$. Since $S$ is a reduced ring, $e$ is central and $((1 - e)f)^n = 0$. Also it implies $(1 - e)f = 0$ or $f = ef$. Thus $\text{Im} f \leq eM$. The reverse inclusion $eM \leq \text{Im} f$ follows from $eM = f^n M \leq f(f^{n-1})M \leq fM$. Therefore $eM = \text{Im} f$ and $M$ is a dual Rickart module. □

By using a different condition on an endomorphism ring of a module we show that a dual $\pi$-Rickart module is dual Rickart. To do this we need the following lemma.

**Lemma 2.5.** Let $M$ be a module. Then $M$ is dual $\pi$-Rickart and $S$ is a domain if and only if every nonzero element of $S$ is an epimorphism.

**Proof.** The sufficiency is clear. For the necessity, let $M$ be a dual $\pi$-Rickart module and $0 \neq f \in S$. Then there exist a positive integer $n$ and an idempotent $e \in S$ such that $\text{Im} f^n = eM$. Hence $f^n = ef^n$. Since $S$ is a domain and $f^n$ is nonzero, we have $e = 1$ and so $\text{Im} f^n = M$. This implies that $\text{Im} f = M$. Thus $f$ is an epimorphism. □
Recall that a module $M$ has $C_2$ condition if any submodule $N$ of $M$ which is isomorphic to a direct summand of $M$ is a direct summand, while a module $M$ is said to have $D_2$ condition if any submodule $N$ of $M$ with $M/N$ isomorphic to a direct summand of $M$, then $N$ is a direct summand of $M$. In the next result we obtain relations between $\pi$-Rickart and dual $\pi$-Rickart modules by using $C_2$ and $D_2$ conditions. An endomorphism $f$ of a module $M$ is called morphic if $M/fM \cong \text{Ker} f$. The module $M$ is called morphic if every endomorphism of $M$ is morphic.

**Theorem 2.6.** Let $M$ be a module. Then we have the following.

1. If $M$ is a dual $\pi$-Rickart module with $D_2$ condition, then it is $\pi$-Rickart.
2. If $M$ is a $\pi$-Rickart module with $C_2$ condition, then it is dual $\pi$-Rickart.
3. If $M$ is projective morphic, then it is $\pi$-Rickart if and only if it is dual $\pi$-Rickart.

**Proof.** Since $M/\text{Ker} f^n \cong \text{Im} f^n$, $D_2$ and $C_2$ conditions complete the proof of (1) and (2). The proof of (3) is clear. \hfill \Box

The next result is an immediate consequence of Theorem 2.6.

**Corollary 2.7.** Let $M$ be a module with $C_2$ and $D_2$ conditions. Then $M$ is a dual $\pi$-Rickart module if and only if it is $\pi$-Rickart.

In [12, Proposition 2.6], it is shown that $M$ is a dual Rickart module if and only if the short exact sequence $0 \rightarrow \text{Im} f \rightarrow M \rightarrow M/\text{Im} f \rightarrow 0$ splits for any $f \in S$. In this direction we can give a similar characterization for dual $\pi$-Rickart modules.

**Lemma 2.8.** The following are equivalent for a module $M$.

1. $M$ is a dual $\pi$-Rickart module.
2. For every $f \in S$ there exists a positive integer $n$ such that the short exact sequence $0 \rightarrow \text{Im} f^n \rightarrow M \rightarrow M/\text{Im} f^n \rightarrow 0$ splits.

**Proof.** For any $f \in S$ and any positive integer $n$ consider the short exact sequence $0 \rightarrow \text{Im} f^n \rightarrow M \rightarrow M/\text{Im} f^n \rightarrow 0$. The short exact sequence splits in $M$ if and only if $\text{Im} f^n$ is a direct summand of $M$ if and only if $M$ is a dual $\pi$-Rickart module. \hfill \Box
One may suspect that every submodule of a dual $\pi$-Rickart module is dual $\pi$-Rickart. The following example shows that this is not the case.

**Example 2.9.** Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Then $S = \text{End}_\mathbb{Z}(\mathbb{Q})$ is isomorphic to $\mathbb{Q}$. Since every element of $S$ is an isomorphism or zero, $\mathbb{Q}$ is dual $\pi$-Rickart.

Now consider the submodule $\mathbb{Z}$ and $f \in \text{End}_\mathbb{Z}(\mathbb{Z})$ defined by $f(x) = 2x$, where $x \in \mathbb{Z}$. Since the image of any power of $f$ can not be a direct summand of $\mathbb{Z}$, the submodule $\mathbb{Z}$ is not dual $\pi$-Rickart.

Although every submodule of a dual $\pi$-Rickart module need not be dual $\pi$-Rickart by Example 2.9, we now prove that every direct summand of dual $\pi$-Rickart modules is also dual $\pi$-Rickart.

**Proposition 2.10.** Let $M$ be a dual $\pi$-Rickart module. Then every direct summand of $M$ is also dual $\pi$-Rickart.

**Proof.** Let $M = N \oplus P$ with $S_N = \text{End}_R(N)$. Define $g = f \oplus 0|_P$, for any $f \in S_N$ and so $g \in S$. By hypothesis, there exist a positive integer $n$ and $e^2 = e \in S$ such that $\text{Im}g^n = eM$ and $g^n = f^n \oplus 0|_P$. Hence $eM = \text{Im}g^n = f^n N \leq N$. Let $M = eM \oplus Q$ for some submodule $Q$. Thus $N = eM \oplus (N \cap Q) = f^n N \oplus (N \cap Q)$. Therefore $N$ is dual $\pi$-Rickart.

**Corollary 2.11.** Let $R$ be a $\pi$-regular ring with $e = e^2 \in R$. Then $M = eR$ is a dual $\pi$-Rickart $R$-module.

Here we give the following result for $\pi$-regular rings.

**Corollary 2.12.** Let $R = R_1 \oplus R_2$ be a $\pi$-regular ring with direct sum of the rings $R_1$ and $R_2$. Then the rings $R_1$ and $R_2$ are also $\pi$-regular.

We now characterize $\pi$-regular rings in terms of dual $\pi$-Rickart modules.

**Theorem 2.13.** Let $R$ be a ring. Then $R$ is $\pi$-regular if and only if every cyclic projective $R$-module is dual $\pi$-Rickart.

**Proof.** The sufficiency is clear. For the necessity, let $M = mR$ be a projective module. Then $R = r_R(m) \oplus I$ for some right ideal $I$ of $R$. Let $I \varphi \to M$ denote the isomorphism and $f \in S$. By Proposition 2.2 and Proposition 2.10, $(\varphi^{-1} f \varphi)^n I = (\varphi^{-1} f^n \varphi) I$ is a direct summand of $I$ for some positive integer $n$. Hence $I = (\varphi^{-1} f^n \varphi) I \oplus K$ for some right ideal $K$ of $I$. Thus $\varphi I = (f^n \varphi) I \oplus \varphi K$, and so $M = f^n M \oplus \varphi K$. Therefore $M$ is dual $\pi$-Rickart.
Theorem 2.14. Let $R$ be a ring and consider the following conditions.

1. Every free $R$-module is dual $\pi$-Rickart.
2. Every projective $R$-module is dual $\pi$-Rickart.
3. Every flat $R$-module is dual $\pi$-Rickart.

Then (3) $\Rightarrow$ (2) $\Leftrightarrow$ (1). Moreover (2) $\Rightarrow$ (3) holds for finitely presented modules.

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) Clear. (1) $\Rightarrow$ (2) Let $M$ be a projective $R$-module. Then $M$ is a direct summand of a free $R$-module $F$. By (1), $F$ is dual $\pi$-Rickart, and so is $M$ due to Proposition 2.10.

(2) $\Rightarrow$ (3) is clear from the fact that finitely presented flat modules are projective. □

The next example reveals that a direct sum of dual $\pi$-Rickart modules need not be dual $\pi$-Rickart.

Example 2.15. Let $R$ denote the ring $\begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ and $M$ the $R$-module $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$. Let $f \in S$. Then there exist $a, c, u, t \in R$ such that $f \begin{pmatrix} x & y \\ r & s \end{pmatrix} = \begin{pmatrix} ax + ur & ay + us \\ cx + tr & cy + ts \end{pmatrix}$ where $\begin{pmatrix} x & y \\ r & s \end{pmatrix} \in M$. Consider $f \in S$ defined by $a = c = 0$, $u = 3$ and $t = 2$. This implies that $f \begin{pmatrix} x & y \\ r & s \end{pmatrix} = \begin{pmatrix} 3r & 3s \\ 2r & 2s \end{pmatrix}$ and for any positive integer $n$ we obtain $f^n \begin{pmatrix} x & y \\ r & s \end{pmatrix} = \begin{pmatrix} 3(2^n-1)r & 3(2^n-1)s \\ 2^n r & 2^n s \end{pmatrix}$

It follows that $f^n M$ can not be a direct summand. On the other hand, consider the submodules $N = \begin{pmatrix} R & R \\ 0 & 0 \end{pmatrix}$ and $K = \begin{pmatrix} 0 & 0 \\ R & R \end{pmatrix}$ of $M$. Then End$_R(N)$ and End$_R(K)$ are isomorphic to $R$. Hence $N$ and $K$ are dual $\pi$-Rickart modules but $M$ is not dual $\pi$-Rickart.

The following lemma is useful to show that a direct sum of some dual $\pi$-Rickart modules is dual $\pi$-Rickart.

Lemma 2.16. Let $M$ be a module and $f \in S$. If $Im f^n = eM$ for some central idempotent $e \in S$ and a positive integer $n$, then $Im f^{n+1} = eM$. 

Proof. Let \( f \in S \) and \( \text{Im} f^n = eM \) for some central idempotent \( e \in S \) and a positive integer \( n \). It is clear that \( \text{Im} f^{n+1} \subseteq \text{Im} f^n \). Let \( f^n(x) \in \text{Im} f^n \), then \( f^n(x) = ef^n(x) = f^n e(x) \). Since \( e(x) \in \text{Im} f^n \), \( e(x) = f^n(y) \) for some \( y \in M \). So \( f^n(x) = f^n(f^n(y)) = f^{n+1}(f^{n-1}(y)) \in \text{Im} f^{n-1} \). This completes the proof. \( \square \)

A ring \( R \) is called abelian if every idempotent is central, that is, \( ae = ea \) for any \( a, e^2 = e \in R \). A module \( M \) is called abelian \([4]\) if \( fem = efm \) for any \( f \in S \), \( e^2 = e \in S \), \( m \in M \). Note that \( M \) is an abelian module if and only if \( S \) is an abelian ring. We now prove that a direct sum of dual \( \pi \)-Rickart modules is dual \( \pi \)-Rickart for some abelian modules.

**Proposition 2.17.** Let \( M_1 \) and \( M_2 \) be abelian \( R \)-modules. If \( M_1 \) and \( M_2 \) are dual \( \pi \)-Rickart with \( \text{Hom}_R(M_i, M_j) = 0 \) for \( i \neq j \), then \( M_1 \oplus M_2 \) is a dual \( \pi \)-Rickart module.

**Proof.** Let \( S_1 = \text{End}_R(M_1) \), \( S_2 = \text{End}_R(M_2) \) and \( M = M_1 \oplus M_2 \). We may describe \( S \) as \( \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \). Let \( \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \in S \), where \( f_1 \in S_1 \) and \( f_2 \in S_2 \).

Then there exist positive integers \( n, m \) and \( e_1^2 = e_1 \in S_1 \) and \( e_2^2 = e_2 \in S_2 \) such that \( \text{Im} f_1^n = e_1 M_1 \) and \( \text{Im} f_2^m = e_2 M_2 \). Consider the following cases:

(i) Let \( n = m \). Obviously, \( \text{Im} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^n = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M \).

(ii) Let \( n < m \). By Lemma 2.16 we have \( \text{Im} f_1^n = \text{Im} f_1^m = e_1 M_1 \). Clearly, \( \begin{pmatrix} e_1 \\ 0 \\ e_2 \end{pmatrix} M \leq \text{Im} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^m \). Now let \( \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \text{Im} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^m \).

Then \( m_1 \in \text{Im} f_1^m = e_1 M_1 \) and \( m_2 \in \text{Im} f_2^m = e_2 M_2 \). Hence \( \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \). Thus \( \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M \). Therefore \( \text{Im} \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^m \leq \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M \).

(iii) Let \( m < n \). Since \( M_2 \) is abelian, the proof is similar to case (ii). \( \square \)

We close this section with the relations among strongly co-Hopfian modules, Fitting modules and dual \( \pi \)-Rickart modules.

Recall that a module \( M \) is called co-Hopfian if every injective endomorphism of \( M \) is an automorphism, while \( M \) is called strongly co-Hopfian \([5]\), if for any endomorphism \( f \) of \( M \) the descending chain

\[
\text{Im} f \supseteq \text{Im} f^2 \supseteq \cdots \supseteq \text{Im} f^n \supseteq \cdots
\]
stabilizes.

We now give a relation between abelian and strongly co-Hopfian modules by using dual $\pi$-Rickart modules.

**Corollary 2.18.** Let $M$ be a dual $\pi$-Rickart module and $S$ an abelian ring. Then $M$ is strongly co-Hopfian.

**Proof.** It follows from Lemma 2.16 and [5, Proposition 2.6]. □

A module $M$ is said to be a Fitting module [5] if for any $f \in S$, there exists an integer $n \geq 1$ such that $M = \text{Ker} f^n \oplus \text{Im} f^n$. Due to Armendariz, Fisher and Snider [2] or [15, Proposition 5.7], the module $M$ is Fitting if and only if $S$ is strongly $\pi$-regular.

We now give the following relation between Fitting modules and dual $\pi$-Rickart modules.

**Corollary 2.19.** Every Fitting module is a dual $\pi$-Rickart module.

Then we have the following result.

**Proposition 2.20.** Let $R$ be an Artinian ring. Then every finitely generated $R$-module is dual $\pi$-Rickart.

**Proof.** Let $M$ be a finitely generated $R$-module. Then $M$ is an Artinian and Noetherian module. Hence $M$ is a Fitting module and so it is dual $\pi$-Rickart. □

**Proposition 2.21.** Let $R$ be a ring and $n$ a positive integer. If the matrix ring $M_n(R)$ is strongly $\pi$-regular, then $R^n$ is a dual $\pi$-Rickart $R$-module.

**Proof.** Let $M_n(R)$ be a strongly $\pi$-regular ring. Then by [5, Corollary 3.6], $R^n$ is a Fitting $R$-module and so it is dual $\pi$-Rickart. □

### 3. The Endomorphism Ring of a Dual $\pi$-Rickart Module

In this section we study relations between a dual $\pi$-Rickart module and its endomorphism ring. We prove that the endomorphism ring of a dual $\pi$-Rickart module is always a generalized left principally projective ring, the converse holds if the module is self-cogenerator. The modules whose endomorphism rings are $\pi$-regular are characterized. It is shown that if the module satisfies $D_2$ condition, then it is dual $\pi$-Rickart if and only if the endomorphism ring of the module is a $\pi$-regular ring.
Lemma 3.1. If $M$ is a dual $\pi$-Rickart module, then $S$ is a generalized left principally projective ring.

Proof. Let $f \in S$. By assumption, there exist $e^2 = e \in S$ and a positive integer $n$ such that $\text{Im}f^n = eM$. Hence $l_S(f^nM) = S(1-e) = l_S(f^n)$. Thus $S$ is a generalized left principally projective ring. □

The next result is a consequence of Theorem 2.10 and Lemma 3.1.

Corollary 3.2. If $R$ is a $\pi$-regular ring, then $eRe$ is a generalized left principally projective ring for any $e^2 = e \in R$.

Corollary 3.3. Let $M$ be a dual $\pi$-Rickart module and $f \in S$. Then $Sf^n$ is a projective left $S$-module for some positive integer $n$.

Proof. Clear from Lemma 3.1 since $Sf^n \cong S/l_S(f^n)$. □

Recall that a module is called self-cogenerator if it cogenerates all its factor modules. The following result shows that the converse of Lemma 3.1 is true for self-cogenerator modules. On the other hand, Theorem 3.4 generalizes the result [17, 39.11].

Theorem 3.4. Let $M$ be a module and $f \in S$.

(1) If $Sf^n$ is a projective left $S$-module for some positive integer $n$, then the submodule $N = \bigcap \{ \text{Ker} g \mid g \in S, \text{Im}f^n \leq \text{Ker} g \}$ is a direct summand of $M$.

(2) If $M$ is self-cogenerator and $S$ is a generalized left principally projective ring, then $M$ is a dual $\pi$-Rickart module.

Proof. (1) Let $Sf^n$ be a projective left $S$-module for some positive integer $n$. Since $Sf^n \cong S/l_S(f^n)$, $l_S(f^n) = Se$ for some $e^2 = e \in S$. We prove $(1-e)M = N$. Due to $ef^nM = 0$, we have $f^nM \leq (1-e)M$. By definition of $N$ we have $N \leq (1-e)M$. Let $g \in S$ with $\text{Im}f^n \leq \text{Ker} g$. Then $gf^nM = 0$ or $gf^n = 0$. Hence $g \in l_S(f^n) = Se$ and $ge = g$. So $g(1-e)M = 0$ from which we have $(1-e)M \leq \text{Ker} g$ for all $g$ with $\text{Im}f^n \leq \text{Ker} g$. Thus $(1-e)M \leq N$. Therefore $(1-e)M = N$.

(2) Assume that $M$ is self-cogenerator and $S$ is generalized left principally projective. There exist $e^2 = e \in S$, a positive integer $n$ such that $l_S(f^n) = Se$ and $M/\text{Im}f^n$ is cogenerated by $M$. By [17] 14.5, 

$$\bigcap \{ \text{Ker} g \mid g \in \text{Hom}(M/\text{Im}f^n, M) \} = 0.$$
Hence
\[ \text{Im} f^n = \bigcap \{ \text{Ker} g \mid g \in S, \text{Im} f^n \subseteq \text{Ker} g \}. \]
Thus conditions of (1) are satisfied and so \( \text{Im} f^n \) is a direct summand. \( \square \)

For an \( R \)-module \( M \), it is shown that, if \( S \) is a von Neumann regular ring, then \( M \) is a dual Rickart module (see [12, Proposition 3.8]). We obtain a similar result for dual \( \pi \)-Rickart modules.

**Lemma 3.5.** Let \( M \) be a module. If \( S \) is a \( \pi \)-regular ring, then \( M \) is dual \( \pi \)-Rickart.

**Proof.** Let \( f \in S \). Since \( S \) is \( \pi \)-regular, there exist a positive integer \( n \) and \( g \in S \) such that \( f^n = f^n g f^n \). Then \( e = f^n g \) is an idempotent of \( S \).

Now we show that \( \text{Im} f^n = f^n g M \). It is clear that \( f^n M = e f^n M \leq e M \).

For the other inclusion, let \( m \in M \). Hence \( em = f^n g m \in f^n M \). Thus \( \text{Im} f^n = e M \). \( \square \)

Since every strongly \( \pi \)-regular ring is \( \pi \)-regular, we have the next result.

**Corollary 3.6.** Let \( M \) be a module. If \( S \) is a strongly \( \pi \)-regular ring, then \( M \) is dual \( \pi \)-Rickart.

The converse statement of Corollary 3.6 does not hold in general, that is, there exists a dual \( \pi \)-Rickart module having not a strongly \( \pi \)-regular endomorphism ring.

**Example 3.7.** Let \( D \) be a division ring, \( M \) a vector space over \( D \) with an infinite basis \( \{ e_i \in M \mid i = 1, 2, ... \} \) and \( S = \text{End}_D(M) \). As a semisimple right \( D \)-module, \( M \) is dual \( \pi \)-Rickart, and by [17, 3.9] \( S \) is a regular and so \( \pi \)-regular ring. Assume that \( S \) is a strongly \( \pi \)-regular ring and we reach a contradiction. Let \( f \in S \) defined by \( f(e_i) = e_{i+1} \) for all \( i = 1, 2, 3, ... \).

By assumption, there is a positive integer \( n \) such that \( f^n = f^n+1 g \) for some \( g \in S \). Then \( f^n = f^n+1 g \) implies \( f^n S = f^{n+1} S \) and so \( f^n M = f^{n+1} M \).

Since \( f^n(e_i) = e_{i+n} \) for all \( i \), we have \( f^n M = \sum_{i>n} e_i D \neq f^{n+1} M \). This is a contradiction. Hence \( S \) is not a strongly \( \pi \)-regular ring (see also [15, 5.5]).

The proof of Lemma 3.8 may be in the context.

**Lemma 3.8.** Let \( M \) be a module. Then \( S \) is a \( \pi \)-regular ring if and only if there exists a positive integer \( n \) such that \( \text{Ker} f^n \) and \( \text{Im} f^n \) are direct summands of \( M \) for any \( f \in S \).
Now we recall some known facts that will be needed about \( \pi \)-regular rings.

**Lemma 3.9.** Let \( R \) be a ring. Then

1. If \( R \) is \( \pi \)-regular, then \( eRe \) is also \( \pi \)-regular for any \( e^2 = e \in R \).
2. If \( M_n(R) \) is \( \pi \)-regular for any positive integer \( n \), then so is \( R \).
3. If \( R \) is a commutative ring, then \( R \) is \( \pi \)-regular if and only if \( M_n(R) \) is \( \pi \)-regular for any positive integer \( n \).

**Proposition 3.10.** Let \( R \) be a commutative \( \pi \)-regular ring. Then every finitely generated projective \( R \)-module is dual \( \pi \)-Rickart.

*Proof.* Let \( M \) be a finitely generated projective \( R \)-module. So the endomorphism ring of \( M \) is \( eM_n(R)e \) with some positive integer \( n \) and an idempotent \( e \) in \( M_n(R) \). Since \( R \) is commutative \( \pi \)-regular, \( M_n(R) \) is also \( \pi \)-regular, and so is \( eM_n(R)e \) by Lemma 3.9. Hence \( M \) is dual \( \pi \)-Rickart by Lemma 3.5. \( \square \)

**Theorem 3.11.** Let \( M \) be a module with \( D_2 \) condition. Then \( M \) is dual \( \pi \)-Rickart if and only if \( S \) is \( \pi \)-regular.

*Proof.* The necessity holds by Lemma 3.5. For the sufficiency, let \( 0 \neq f \in S \). Since \( M \) is dual \( \pi \)-Rickart, \( \text{Im} f^n \) is a direct summand of \( M \) for some positive integer \( n \). Because of \( M/\text{Ker} f^n \cong \text{Im} f^n \), \( D_2 \) condition implies that \( \text{Ker} f^n \) is a direct summand of \( M \). The rest is obvious from Lemma 3.8. \( \square \)

The following is a consequence of Proposition 3.10 and Theorem 3.11.

**Corollary 3.12.** Let \( R \) be a commutative ring and satisfy \( D_2 \) condition. Then the following are equivalent.

1. \( R \) is a \( \pi \)-regular ring.
2. Every finitely generated projective \( R \)-module is dual \( \pi \)-Rickart.

Recall that a module \( M \) is called quasi-projective if it is \( M \)-projective. Since every quasi-projective module has \( D_2 \) condition, we have the following.

**Corollary 3.13.** If \( M \) is a quasi-projective dual \( \pi \)-Rickart module, then the endomorphism ring of \( M \) is a \( \pi \)-regular ring.

**Theorem 3.14.** The following are equivalent for a ring \( R \).

1. \( M_n(R) \) is \( \pi \)-regular for every positive integer \( n \).
2. Every finitely generated projective \( R \)-module is dual \( \pi \)-Rickart.
Proof. (1) ⇒ (2) Let $M$ be a finitely generated projective $R$-module. Then $M \cong eR^n$ for some positive integer $n$ and $e^2 = e \in M_n(R)$. Hence $S$ is isomorphic to $eM_n(R)e$. By (1), $S$ is $\pi$-regular. Thus $M$ is $\pi$-Rickart due to Lemma 3.5.

(2) ⇒ (1) $M_n(R)$ can be viewed as the endomorphism ring of a projective $R$-module $R^n$ for any positive integer $n$. By (2), $R^n$ is dual $\pi$-Rickart. Then $M_n(R)$ is $\pi$-regular by Corollary 3.13. □

Recall that an $R$-module $M$ is called duo if every submodule of $M$ is fully invariant, i.e., for any submodule $N$ of $M$, $fN \leq N$ for each $f \in S$, equivalently, every right $R$-submodule of $M$ is also left $S$-submodule. Our next aim is to determine to find conditions under which any factor module of a dual $\pi$-Rickart module is also dual $\pi$-Rickart.

**Corollary 3.15.** Let $M$ be a quasi-projective module and $N$ a fully invariant submodule of $M$. If $M$ is dual $\pi$-Rickart, then so is $M/N$.

**Proof.** Let $f \in S$ and $\pi$ denote the natural epimorphism from $M$ to $M/N$. Consider the following diagram.

$$
\begin{array}{ccc}
M & \xrightarrow{\pi} & M/N \\
\downarrow{f} & & \downarrow{f^*} \\
M & \xrightarrow{\pi} & M/N \\
\end{array}
$$

Since $N$ is fully invariant, we have $\text{Ker} \pi \subseteq \text{Ker} f$. By the Factor Theorem, there exists a unique homomorphism $f^*$ such that $f^* \pi = \pi f$. Hence we define a homomorphism $\varphi : S \to \text{End}_R(M/N)$ with $\varphi(f) = f^*$ for any $f \in S$. As $M$ is quasi-projective, $\varphi$ is an epimorphism. Thus $\text{End}_R(M/N) \cong S/\text{Ker} \varphi$. By Corollary 3.13, $S$ is $\pi$-regular, and so is $S/\text{Ker} \varphi$. Therefore $M/N$ is dual $\pi$-Rickart due to Lemma 3.5. □

**Corollary 3.16.** Let $M$ be a quasi-projective duo module. If $M$ is dual $\pi$-Rickart, then $M/N$ is also dual $\pi$-Rickart for every submodule $N$ of $M$.

**Corollary 3.17.** If $M$ be a quasi-projective dual $\pi$-Rickart module, then so is $M/\text{Rad}(M)$ and $M/\text{Soc}(M)$.

**Proposition 3.18.** Let $M$ be a dual $\pi$-Rickart module. Then every endomorphism of $M$ with a small image in $M$ is nilpotent.
Proof. Let \( f \in S \) with \( \text{Im} f \) small in \( M \). Then \( \text{Im} f^n \) is a direct summand of \( M \) for some positive integer \( n \). Also \( \text{Im} f^n \) is small in \( M \). Hence \( f^n = 0 \). \( \square \)

Corollary 3.19. Let \( M \) be a dual \( \pi \)-Rickart discrete module. Then \( J(S) \) is nil and \( S/J(S) \) is von Neumann regular.

Proof. Since \( M \) is discrete, by \cite[Theorem 5.4]{B}, \( J(S) \) consists of endomorphisms with small image. By Proposition 3.18, \( J(S) \) is nil and again by \cite[Theorem 5.4]{B}, \( S/J(S) \) is von Neumann regular. \( \square \)

Theorem 3.20. The following are equivalent for a module \( M \).

(1) \( M \) is a dual \( \pi \)-Rickart module.

(2) \( S \) is a generalized left principally projective ring and \( f^n M = r_M(l_S(f^n M)) \) for all \( f \in S \) and a positive integer \( n \).

Proof. (1) \(\Rightarrow\) (2) By Lemma 3.1, we only need to show that \( f^n M = r_M(l_S(f^n M)) \) for all \( f \in S \). Since \( M \) is dual \( \pi \)-Rickart, for any \( f \in S \), \( f^n M = eM \) for some \( e^2 = e \in S \) and a positive integer \( n \). Thus \( r_M(l_S(f^n M)) = r_M(l_S(eM)) = eM = f^n M \).

(2) \(\Rightarrow\) (1) Let \( f \in S \). Since \( S \) is a generalized left principally projective ring, \( l_S(f^n M) = Se \) for some \( e^2 = e \in S \) and a positive integer \( n \). By hypothesis, \( f^n M = r_M(l_S(f^n M)) = r_M(Se) = (1-e)M \). Thus \( M \) is dual \( \pi \)-Rickart. \( \square \)

Corollary 3.21. Let \( M \) be a module. Then \( M \) is dual \( \pi \)-Rickart if and only if \( f^n M = r_M(l_S(f^n M)) \) and \( r_M(l_S(f^n M)) \) is a direct summand of \( M \).

Theorem 3.22. Let \( M \) be a dual \( \pi \)-Rickart module. Then the left singular ideal \( Z_l(S) \) of \( S \) is nil and \( Z_l(S) \subseteq J(S) \).

Proof. Let \( f \in Z_l(S) \). Since \( M \) is dual \( \pi \)-Rickart, \( \text{Im}(f^n) = eM \) for some positive integer \( n \) and \( e = e^2 \in S \). Then, by Lemma 3.1, \( l_S(f^n) = S(1-e) \). Since \( l_S(f^n) \) is essential in \( S \) as a left ideal, we have \( l_S(f^n) = S \). This implies that \( f^n = 0 \) and so \( Z_l(S) \) is nil. On the other hand, for any \( g \in S \) and \( f \in Z_l(S) \), according to previous discussion, \( (gf)^n = 0 \) for some positive integer \( n \). Hence \( 1 - gf \) is invertible. Thus \( f \in J(S) \). Therefore \( Z_l(S) \subseteq J(S) \). \( \square \)

Proposition 3.23. The following are equivalent for a module \( M \).

(1) \( M \) is an indecomposable dual \( \pi \)-Rickart module.

(2) Each element of \( S \) is either an epimorphism or nilpotent.
Proof. (1) ⇒ (2) Let \( f \in S \). Then \( f^n M \) is a direct summand of \( M \) for some positive integer \( n \). As \( M \) is indecomposable, we see that \( f^n M = 0 \) or \( f^n M = M \). This implies that \( f \) is an epimorphism or nilpotent.

(2) ⇒ (1) Let \( e = e^2 \in S \). If \( e \) is nilpotent, then \( e = 0 \). If \( e \) is an epimorphism, then \( e = 1 \). Hence \( M \) is indecomposable. Also for any \( f \in S \), \( f M = M \) or \( f^n M = 0 \) for some positive integer \( n \). Therefore \( M \) is dual \( \pi \)-Rickart.

\[ \square \]

**Theorem 3.24.** Consider the following conditions for a module \( M \).

(1) \( S \) is a local ring with nil Jacobson radical.

(2) \( M \) is an indecomposable dual \( \pi \)-Rickart module.

Then (1) ⇒ (2). If \( M \) is a morphic module, then (2) ⇒ (1).

Proof. (1) ⇒ (2) Clearly, each element of \( S \) is either an epimorphism or nilpotent. Then \( M \) is indecomposable dual \( \pi \)-Rickart due to Proposition 3.23.

(2) ⇒ (1) Let \( f \in S \). Then \( f^n M = eM \) for some positive integer \( n \) and an idempotent \( e \) in \( S \). If \( e = 1 \), then \( f \) is an epimorphism. Since \( M \) is morphic, \( f \) is invertible by [9, Corollary 2]. If \( e = 0 \), then \( f^n = 0 \). Hence \( 1 - f \) is invertible. This implies that \( S \) is a local ring. Now let \( 0 \neq f \in J(S) \). Since \( f \) is not invertible and \( M \) is morphic, \( f \) is nilpotent by Proposition 3.23. Therefore \( J(S) \) is nil.

\[ \square \]

The next result can be obtained from Theorem 3.24 and [6, Lemma 2.11].

**Corollary 3.25.** Let \( M \) be an indecomposable dual \( \pi \)-Rickart module. If \( M \) is morphic, then \( S \) is a left and right \( \pi \)-morphic ring.

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