Condensation of a Classical Scalar Field After Inflation and Dark Energy

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Abstract

In cosmological context, classical scalar fields are important ingredients for inflation models, many candidate models of dark energy, symmetry breaking and phase transition epochs, and their consequences such as baryo and lepto-genesis. We investigate the formation of these fields by studying the production of a light quantum scalar field during the decay of a heavy particle. For simplicity it is assumed to be a scalar too. We discuss the effects of the decay mode, the thermodynamical state of the decaying field, boundary conditions, and related physical parameters on the production and evolution of a condensate. For a simplified version of this model we calculate the asymptotic behaviour of the condensate and conditions for its contribution to the dark energy with an equation of state close to a cosmological constant. We also discuss the role of the back-reaction from interactions with other fields and expansion of the Universe on the evolution of the condensate.

1 Introduction

Symmetry breaking, phase transition, and related phenomena such as appearance of a dynamical mass (Higgs mechanism), superconductivity, inflation, leptogenesis, and quintessence field as a candidate for dark energy in early universe and cosmology are based on the existence of a classical scalar field. As the physics of the Universe and its content in its most elementary level is quantic, this scalar field, fundamental or composite, is always related to a quantum scalar field.

A classical field is more than just classical behaviour of a large number of scalar particles. In a quantum system, fields/particles are in superposition states and are quantum mechanically correlated with each others. Decoherence process removes the superposition/correlation between particles. However, this does not mean that after decoherence of scalar particles, they will behave collectively like a classical field. A simple example is the following:

Consider a closed system consisting of a macroscopic amount of unstable massive scalar particles that decay to a pair of light scalar particles with a global SU(2) symmetry and negligible interaction. If the unstable particle is a singlet of this symmetry, the remnant particles are entangled by their SU(2) state. After a time much larger than the lifetime of the massive particle, the system consists of a relativistic gas of pair entangled particles. If a detector measures this SU(2) charge without significant modification of their kinetic energy, the entanglement of pairs will break i.e. the system decoheres and becomes a relativistic gas. The equation of state of a relativistic ideal gas is \( w_{\text{rel}} = P/\rho \approx 1/3 \).

In contrast to an operator in quantum field theory, a classical scalar field \( \varphi(x) \) is a \( C \)-number. Its density \( \rho_{\varphi} \), pressure \( P_{\varphi} \) and kinetic energy are defined as:

\[
\rho_{\varphi} \equiv K_{\varphi} + V(\varphi) \\
P_{\varphi} \equiv K_{\varphi} - V(\varphi) \\
K_{\varphi} = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi
\]

\( V(\varphi) \) is the potential presenting the self-interaction of the field \( \varphi(x) \). When it is much smaller than kinetic energy \( K_{\varphi} \), one obtains \( P_{\varphi} \approx \rho_{\varphi} \). On the other hand, if \( V(\varphi) \gg K_{\varphi} \), \( P_{\varphi} \approx -\rho_{\varphi} \). Therefore in general, a relativistic gas and a scalar field don’t share the same equation of state. Thus, the proof
of decoherence in a system is not enough when a classical scalar field is needed to explain physical phenomena.

According to canonical quantization procedure, classical observables are replaced by operators acting on a Hilbert or Fock space of states respectively for a single particle and for a multi-particle quantum system. The expectation value of these operators are the outcome of measurements. Therefore, it is natural to define the classical observable related to a quantum scalar field as its expectation value:

$$\varphi(x) \equiv \langle \Psi | \Phi(x) | \Psi \rangle$$  \hspace{1cm} (4)

where $| \Psi \rangle$ is the state of the quantum system - an element of the Fock space. In analogy with particles in the ground state in quantum mechanics, the classical field $\varphi(x)$ is also called a condensate. Using canonical representation it is easy to see that for a free quantum scalar field $\langle \Psi | \Phi | \Psi \rangle = 0$. Therefore, a necessary condition for the buildup of a classical scalar field is an interaction (see also Appendix A).

In quintessence models a classical field is the basic content of the model and the source of the dark energy. Although in the framework of popular particle physics models such as supersymmetry, supergravity, and string theory many works have been concentrated on finding candidate scalar field to play the role of quintessence [1], little effort has been devoted to understand the necessary conditions for a quantum scalar fields to condense in a manner satisfying very special characteristics needed for a quintessence field. For instance, such a condensate must have a very small density, much smaller than other content of the Universe (smallness problem). Present observations show that dark energy has a behaviour close to a cosmological constant i.e. with expansion of the Universe either its energy density does not change or varies very slowly. Such a behaviour is not trivial. In the classical quintessence models usually the potential or other characteristics of the models are designed such that a tracking solution is obtained. However, apriori it is not trivial to prove that in the early Universe a quantum field could produce a condensate with properties similar to dark energy.

The purpose of the present work is to fill the gap between quantum processes producing various species of particles/fields in the early Universe - presumably during and after reheating - their classical component as defined in (4). In another word we want to see how the microscopic properties of matter is related to macro-physics and vis versa. As the quantum physics of that epoch is not well known, we consider the simple case of a scalar field - quintessence field - in interaction with two other scalar fields as a prototype process, and study the evolution of the classical component (condensate) of the quintessence field. Between many possible types of quantum scalar field and interaction models, we specially concentrate on a class of models is which the scalar field is one of the remnants of the decay of a heavy particle. Motivations for such a model are the results of studying the effects of a decaying dark matter on the the equation of states of the Universe [2]. It has been shown that a FLRW cosmology with a decaying dark matter and a cosmological constant at late times behaves similar to a cosmology with a stable dark matter and a dark energy component with $w = P/\rho \lesssim -1$. This is effectively what is concluded at least from some of present supernovae observations [3]. Recently, the same effect has been proved for the general case of interaction between dark matter and dark energy [4]. It has been also shown [5] that if a decaying dark matter has a small branching factor to a light scalar field, this can explain both observed density and equation of state of the dark energy without extreme fine-tuning of the potential or coupling constants. In other words, such a model solves both the smallness and the coincidence problems of the dark energy. These studies however are based on the assumption of a classical scalar field (a condensate). The present work should complete this investigation by studying the formation and evolution of classical component from quantum processes.

In Sec. 2 we construct the Lagrangian of a decaying dark matter model. We consider two decay modes for the heavy particle and use the closed time path integral method to calculate the contribution of interactions to the condensate. The same methodology has been used for studying inflation models [6], late-time warm inflation [7], the effects of renormalization and initial conditions on the physics of inflation [8], baryogenesis [9], and coarse-grained formulation of decoherence [10]. In Sec. 3 we obtain an analytical expression for the asymptotic behaviour of the condensate and discuss the importance
of the back-reaction of the quantum state of the Universe on the evolution of the condensate and properties of the dark energy. We summarize the results in Sec. 4. In Appendix A we make a remark about relation between decoherence and formation of a condensate. In Appendixes B we show that the effect of a non-vacuum state on the Green’s function can be included in the boundary conditions. Appendix C presents the solution of the evolution equation of the field in matter dominated era.

2 Decay in an Expanding Universe

We consider a simple decay mode for a heavy particle $X$ with only 2 types of particles/fields in the remnants: a light scalar $\Phi$ - light with respect to the decaying particle - and another field $A$ of an arbitrary type. In fact, in a realistic particle physics model, most probably $A$ will not be a final stable state and decays/fragments to other particles. Therefore, it should be considered as an intermediate state or a collective notation for other fields. In the simplest case studied here all the particles are assumed to be scalar. Extension to the case where the decaying particle $X$ and one of the remnants are spinors is straightforward. The quintessence field $\Phi$ must be a scalar to condensate. Nonetheless, in the extreme densities of the Universe after reheating, apriori the formation of Cooper-pair like composite scalars is possible if the interaction between spinors is enough strong. Thus, $\Phi$ can be such a field, but for the simple model studied here we ignore such complexities.

The simplest decaying modes are the followings:

\[
\begin{align*}
(a) & \quad X \to \Phi + A \\
(b) & \quad X \to \Phi + A'
\end{align*}
\]

Diagram (5-a) is a prototype decay mode when $X$ and $\Phi$ shares a conserved quantum number. For instance, one of the favorite candidates for $X$ is a sneutrino decaying to a much lighter scalar field (e.g. another sneutrino) carrying the same leptonic number [13] [14]. With seesaw mechanism in the superpartner sector (or even without it [14]) if SUSY breaking scale is lower than seesaw scale, a mass split between right and left neutrinos and sneutrinos will occur. As the right-hand neutrino super-field is assumed to be a singlet of the GUT gauge symmetry, it has only Yukawa-type of interaction. In such a setup $X$ can be a heavy right sneutrino decaying to a light sneutrino with the same leptonic number and a pair of Higgs or Higgsino [15]. In place of assuming two $A$ particles in the final state we could consider them as being different $A$ and $A'$. But this adds a bit to the complexity of the model and does not change its general behaviour. For this reason we simply consider the same field. Diagram (5-b) is representative of a case where $X$ and $A$ are fermions, or $\Phi$ carries a conserved charge [16].

It was easier to consider a simple 3-vertex $X \to \Phi + A$ similar to what is considered in Ref. [7]. However, such a vertex does not always allow simultaneous conservation of energy. Moreover, as we will see later, more complex diagrams considered here will show how the decay mode affects the evolution of the condensate component.
The corresponding Lagrangians of these effective interactions are the followings:

\[
\mathcal{L}_\Phi = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} m_\Phi^2 \Phi^2 - \frac{\lambda}{n} \Phi^n \right] (6)
\]

\[
\mathcal{L}_X = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu X \partial_\nu X - \frac{1}{2} m_X^2 X^2 \right] (7)
\]

\[
\mathcal{L}_A = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu A \partial_\nu A - \frac{1}{2} m_A^2 A^2 - \frac{\lambda'}{n'} A^{n'} \right] (8)
\]

\[
\mathcal{L}_{\text{int}} = \int d^4x \sqrt{-g} \begin{cases} 
  g\Phi X A^2, & \text{For } (5)\text{-a} \\
  g\Phi^2 X A, & \text{For } (5)\text{-b} 
\end{cases} (9)
\]

In addition to the interaction between \(X, \Phi\) and \(A\) we have assumed a power-law self-interaction for \(\Phi\) and \(A\). If \(A\) is a collective notation for other fields in the actual model, its self-interaction corresponds to the interaction between these unspecified fields. Again for the sake of simplicity in the rest of this work we consider \(\lambda' = 0\). The unstable particle \(X\) is assumed to have no self-interaction.

Although the model presented here is quite general, for physical and observational reasons we concentrate on the case of a heavy \(X\) particle as a candidate for the dark matter, \(\Phi\) as a quintessence field, and interactions in \((5)\) as candidate interactions for the decay of the dark matter and production of what is observed as dark energy. It is therefore necessary that \(X\) and \(\Phi\) have only a very weak interaction. Therefore couplings \(\lambda\) and \(g\) must be very small.

In a realistic particle physics model, renormalization as well as non-perturbative effects can lead to complicated potentials for scalar fields. An example relevant for dark energy is a pseudo-Nambu-Goldston boson as \(\Phi\) and potentials with a shift symmetry \([17]\). These models are interesting for the fact that the mass of the quintessence field does not receive quantum corrections and can be very small. Moreover, they can be easily implemented in SUSY theories in relation with right-neutrinos and sneutrinos (as candidate for \(X\)). The power-law potential considered here can be interpreted as the dominant term in the polynomial expansion of the potential. In any case, general aspects of the analysis presented here do not depend on the details of the particle physics and self-interaction, and can be applied to any model. The solution of the field equation and numerical quantities are however sensitive to the particle physics. The purpose of the present work is to investigate the behaviour of this model and to find what is most important for the formation and evolution of a condensate with characteristics similar to the observed dark energy. We leave the application of this analysis to realistic particle physics models to a future work and consider only simplest cases which are analytically tractable and permit exact or approximate analytical solutions.

We decompose \(\Phi(x)\) to a classical (condensate) and a quantum component:

\[
\Phi(x) = \varphi(x) + \phi(x) \quad \langle \Phi \rangle \equiv \langle \Psi | \Phi | \Psi \rangle = \varphi(x) \quad \langle \phi \rangle \equiv \langle \Psi | \phi | \Psi \rangle = 0 (10)
\]

Note that in \((10)\) both classical and quantum components depend on the spacetime \(x\). In studying inflation it is usually assumed that very fast expansion of the Universe washes out all the inhomogeneities and the condensed component is homogeneous. As we are studying the evolution after inflation, the distribution of unstable \(X\) can have non-negligible inhomogeneities, specially if the decay is slow and perturbations have time to grow.

We assume \(\langle X \rangle = 0\) and \(\langle A \rangle = 0\). Justification for these assumptions is the large mass and small coupling of \(X\) which should reduce their number and their quantum correlation. In other words, when mass is large, the minimum of the effective potential for the classical component is pushed to zero (see \([12]\) and \([13]\) below). We find a quantitative justification for negligible condensation of massive fields in Sec. 4.
The Lagrangian of $\Phi$ is decomposed to:

\[
\mathcal{L}_\Phi = \mathcal{L}_\varphi + \mathcal{L}_\phi + \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \left( \partial_\mu \varphi \partial_\nu \phi + \partial_\mu \phi \partial_\nu \varphi \right) - m_\varphi^2 \varphi^2 - \frac{\lambda}{n} \sum_{i=0}^{n} \left( \frac{n}{i} \right) \varphi^i \phi^{n-i} \right] + \int d^4x \sqrt{-g} \begin{cases} g\varphi A^2 + g\phi X A^2 & \text{For (5)-a} \\ g^2 X A^2 + 2g\varphi X A + g\phi^2 X A & \text{For (5)-b} \end{cases}
\]

(11)

Lagrangians $\mathcal{L}_\varphi$ and $\mathcal{L}_\phi$ are the same as (6) with respectively $\Phi \rightarrow \varphi$ and $\Phi \rightarrow \phi$. After replacing quantum terms by their expectation values, this Lagrangian leads to the following evolution equation for the condensate component:

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi) + m_\varphi^2 \varphi + \frac{\lambda}{n} \sum_{i=0}^{n-1} (i+1) \left( \frac{n}{i+1} \right) \varphi^i \phi^{n-i-1} - g \langle X A^2 \rangle = 0
\]

For (5)-a \hspace{1cm} (12)

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi) + m_\varphi^2 \varphi + \frac{\lambda}{n} \sum_{i=0}^{n-1} (i+1) \left( \frac{n}{i+1} \right) \varphi^i \phi^{n-i-1} - 2g \varphi \langle X A \rangle - 2g \langle \phi X A \rangle = 0
\]

For (5)-b \hspace{1cm} (13)

In the case of the interaction mode (5-a), the interaction Lagrangian depends linearly on $\Phi$ and appears as an external source in the field equation (12). For both decay modes, if $n \geq 2$, the term $i = 0$ in the sum of the self-interaction terms also contributes to the non-homogeneous component, and the term $i = 1$ contributes to the effective mass of the classical field $\varphi$. Note that what we call non-homogeneous component or external source terms have in fact implicit dependence on $\varphi$. The reason is the coupling between quantum interactions and the evolution of the classical component. We will show later that these terms play the role of a feedback between production and evolution of the condensate. In fact, (5-b) has a reach structure and various evolution histories are possible depending on the value and sign of $g$, the coupling to $\chi$, self-coupling $\lambda$, and the order of the self-interaction potential $n$. For instance, the mass can become imaginary (tachyonic) even without self-interaction leading to symmetry breaking. Tachyonic scalar fields have been suggested as quintessence field specially in the framework of models with $w < -1$. The decay mode (5-a) by contrast has a field equation very similar to the classical model studied in Ref. [5]. We discuss in detail the differences of these decay modes in the next sections.

In addition to interaction terms in the Lagrangian of the total field $\Phi$, the Lagrangian of the purely quantic component $\phi$ includes terms depending on the classical component $\varphi$. Derivative term $\frac{1}{2} g^{\mu\nu} \left( \partial_\mu \varphi \partial_\nu \phi + \partial_\mu \phi \partial_\nu \varphi \right)$, mass term $m^2 \varphi^2$, $i = n - 1$ term in the self-interaction, and $2g \varphi \phi X A$ are linear in $\phi$ and only affect the renormalization of the propagator [8]. For $n \geq 2$ the term $i = n - 2$ in the self-interaction sum contributes in the effective mass of $\phi$ and makes it time dependent.

We use Schwinger closed time path (also called in-in) formalism to calculate expectation values. Recent reviews of this formalism are available [11] and here we only present the results. Zero-order (tree) diagrams for the expectation values (12) and (13) are shown in (14), (15) and (16). The next relevant diagrams are of order $g^3$ and negligible for the dark energy model. One example of higher order diagrams is shown in (17). These types of diagrams are specially important for studying renormalization in the context of a realistic particle physics model. Thus for the phenomenological models considered here, we can ignore them.

\[
g \varphi \langle X A^2 \rangle = \text{Diagram (14)} + \ldots
\]
\[ g\phi^2(XA) = g\phi(x) \int \sqrt{-g} d^4 y \left[ G^\phi_A(x,y)G^\phi_A(x,y)G^\phi_X(x,y) - G^\phi_A(x,y)G^\phi_A(x,y)G^\phi_X(x,y) \right] \] (17)

\[ g\phi^2(XA) = g\phi^2 \int \sqrt{-g} d^4 y \left[ G^\phi_A(x,y)G^\phi_A(x,y) - G^\phi_A(x,y)G^\phi_A(x,y) \right] \] (18)

\[ g\phi(\phi XA) = g\phi(x) \int \sqrt{-g} d^4 y \left[ G^\phi_A(x,y)G^\phi_A(x,y)G^\phi_X(x,y) - G^\phi_A(x,y)G^\phi_A(x,y)G^\phi_X(x,y) \right] \] (19)

Future and past propagators \( G^> \) and \( G^< \) are defined as:

\[ G^>(x,y) \equiv -i\langle \psi(x)\psi^\dagger(y) \rangle = -i\text{tr}(\psi(x)\psi^\dagger(y)\rho) \] (20)

\[ G^<(x,y) \equiv i\langle \psi^\dagger(y)\psi(x) \rangle = i\text{tr}(\psi^\dagger(y)\psi(x)\rho) \] (21)

where \( \psi(x) \) presents one of \( \phi, X \) or \( A \) fields and \( \rho = |\Psi\rangle\langle\Psi| \) is the density (projection) operator for the state \( |\Psi\rangle \). The upper and lower signs in (21) are respectively for bosons and fermions. Definitions (20) and (21) correspond to the general case of a complex field. Here we only consider real fields and therefore \( \psi(x) = \psi^\dagger(x) \). Feynman propagators are related to \( G^>(x,y) \) and \( G^<(x,y) \):

\[ G_F(x,y) \equiv -i(T\psi(x)\psi^\dagger(y)) = G^>(x,y)\Theta(x^0 - y^0) + G^<(x,y)\Theta(y^0 - x^0) \] (22)

\[ \tilde{G}_F(x,y) \equiv -i(T\psi^\dagger(y)\psi(x)) = G^>(x,y)\Theta(y^0 - x^0) + G^<(x,y)\Theta(x^0 - y^0) \] (23)

The next step is the calculation of propagators.

### 2.1 Propagators in an expanding universe

Feynman propagators \( G^i_F(x,y), i = \phi, X, A \) can be determined using field equations from Lagrangians (4)-(9). As the concept of Green’s functions is only applicable to the linear differential equations, we have to linearize the field equations and treat interactions perturbatively, i.e. as quantum corrections. Free propagators of \( \phi, X \) and \( A \) are:

\[ \frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g}g^{\mu\nu}\partial_\nu G^\phi_F(x-y)) + (m^2 + (n-1)\lambda\phi^{-2})G^\phi_F(x-y) = -i\frac{\delta^4(x-y)}{\sqrt{-g}} \] (24)

\[ \frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g}g^{\mu\nu}\partial_\nu G^i_F(x-y)) + m_i^2 G^i_F(x-y) = -i\frac{\delta^4(x-y)}{\sqrt{-g}} \quad i = X, A \] (25)

The free propagator of \( \phi \) is independent of the type of interaction with \( X \) and \( A \) and therefore, equation (24) is valid for both interaction models presented in (5). Note also that \( G^\phi_F(x-y) \) is coupled to the classical field \( \phi \) even at the lowest quantum perturbation order. On the other hand,
evolution equations (12) and (13), depend on the interaction between quantum fields $\phi$, $X$ and $A$. Therefore, this model is coupled at all orders.

To proceed, we neglect the effect of spatial anisotropy and consider a flat homogeneous metric with synchronous or conformal time:

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^idx^j = a^2(\eta)(d\eta^2 - \delta_{ij}dx^idx^j) = dt^2 + a(d\eta)$$

(26)

where $t$ and $\eta$ are respectively comoving and conformal time. It is more convenient to write evolution and propagator equations with respect to conformal time $\eta$. After a variable change:

$$\chi \equiv a\phi \quad \Upsilon \equiv a\phi \quad X \equiv aX \quad A \equiv aA$$

(27)

and by using the metric (26), the evolution equation of the classical field $\chi$ takes the following form:

$$\chi'' - \delta_{ij}\partial_i\partial_j\chi + \left(a^2m^2_\phi - \frac{a''}{a}\right)\chi + \frac{\lambda a^{n-4}}{n}\sum_{i=0}^{n-1}(i+1)\left(\frac{n}{i+1}\right)\chi^i\langle \Upsilon^{n-i-1}\rangle - 2g\langle X\Upsilon\rangle - 2g\langle AX\rangle = 0$$

(28)

For (27)-a (28)

$$\chi'' - \delta_{ij}\partial_i\partial_j\chi + \left(a^2m^2_\phi - \frac{a''}{a}\right)\chi + \frac{\lambda a^{n-4}}{n}\sum_{i=0}^{n-1}(i+1)\left(\frac{n}{i+1}\right)\chi^i\langle \Upsilon^{n-i-1}\rangle - 2g\langle X\Upsilon\rangle - 2g\langle AX\rangle = 0$$

(29)

For (27)-b (29)

Propagator of quantum fields $\Upsilon$, $X$ and $A$ are:

$$\frac{d^2}{d\eta^2}G^\Upsilon_F(x,y) - \delta_{ij}\partial_i\partial_jG^\Upsilon_F(x,y) + \left(a^2m^2_\phi - \frac{a''}{a} + (n-1)\lambda a^2\varphi^{n-2}\right)G^\Upsilon_F(x,y) = -\frac{\delta^4(x-y)}{a}$$

(30)

$$\frac{d^2}{d\eta^2}G^X_F(x,y) - \delta_{ij}\partial_i\partial_jG^X_F(x,y) + \left(a^2m^2_\phi - \frac{a''}{a}\right)G^X_F(x,y) = -\frac{\delta^4(x-y)}{a}$$

(31)

$$G^\Upsilon_F = a(\eta)G^\phi_F \quad G^X_F = a(\eta)G^X_F \quad G^A_F = a(\eta)G^A_F$$

(32)

with $f' \equiv df/d\eta$. Note that the classical component of $\Phi$ appears as a spacetime dependent mass term for its quantum component $\phi$ (or equivalently $\Upsilon$). Fourier transform can be applied to (31), but the spacetime dependence of coefficients in (28), (29) and (30) makes this method useless for solving these equations. Nonetheless, if these terms are small and/or vary slowly, one can first ignore them and solve the equations. Then, by applying the WKB method, a more precise solution can be obtained. Equations (30)-31 are second order partial differential equations, and therefore propagators are linear combination of two independent solutions of the associated homogeneous equations. The delta function on the right hand side however leads to a discontinuity which appears as a consistency condition for the solutions and fixes the ambiguities in the propagator solution. This will be described in details in the rest of this section.

$X$ particles are presumably produced during reheating epoch [12] and begin their decay afterward. In this epoch relativistic particles dominate the density of the Universe. Thus, we first consider this epoch. Fortunately, for this epoch the homogeneous field equation has an exact solution. In matter dominated epoch only for special cases an analytical solution exists. They are discussed in the Appendix C. The expansion in the radiation domination epoch has the following time dependence:

$$a = a_0\left(\frac{t}{t_0}\right)^{\frac{1}{2}} = a_0\eta/\eta_0$$

(33)

And thus $a'' = 0$. After taking the Fourier transform of the spatial coordinates and neglecting the $\varphi$-dependent term, the solutions of the associated homogeneous equation of (30) and (31) are well.
known \[19\; 20\]:

\[
\left(\frac{d^2}{d\eta^2} + k^2 + a^2 m^2_i\right)U_k^i(\eta) = 0 \tag{34}
\]

\[
U_k = \int d^3x U^i(x) e^{ik\cdot x} = \Phi \cdot X, A \tag{35}
\]

\[
U_k(\eta) = c^a_k D_{q_i}(\alpha_i \eta) + d^a_k D_{q_i}(-\alpha_i \eta) \tag{36}
\]

\[
\alpha_i \equiv (1 + i) \sqrt{B_i}, \quad q_i \equiv \frac{1 + \frac{ik^2}{2\eta}}{2}, \quad B_i \equiv \frac{a_0 m_i}{H_0 \eta_0} = \frac{m_i}{\eta_0} = a^2_0 H_0 m_i \tag{37}
\]

where \(a_0\) and \(H_0\) are respectively expansion factor and Hubble constant at the initial conformal time \(\eta_0\). The function \(D_{q_i}(z)\) is the parabolic cylindrical function. From now on for simplicity we drop the species index \(i\) unless when its presence is necessary. We call two independent solutions of \(34\) in a general basis \(U_k\) and \(V_k\). If we want these solutions correspond to the coefficients of the canonical decomposition of \(\phi\) \(36\) in Appendix \(B\), we must choose a basis such that \(V_k = U_k^*\). In the rest of this work we only consider this basis. The corresponding equation for the free Feynman propagator (free 2-point Green’s function) is:

\[
\left(\frac{d^2}{d\eta^2} + k^2 + a^2 m^2_i\right)G_k(\eta, \eta') = -\frac{i \delta(\eta - \eta')}{a} \tag{38}
\]

When \(\eta \neq \eta'\), \(38\) is the same as \(34\) and therefore solutions of the former is a linear combination of two independent solutions of the latter. According to the definition of Feynman propagators \(20\) and \(21\) it can be divided to past and future propagating components \(G^-\) and \(G^+\). With \(\eta \leftrightarrow \eta'\) these propagators change their role: \(G^- \leftrightarrow G^+\). Therefore, \(G(\eta, \eta')\) has the following expansion:

\[
iG(\eta, \eta') = \left[ A^>_k U_k(\eta) U^<_k(\eta') + [B^>_k U^*_k(\eta) U_k(\eta')] \Theta(\eta-\eta') + \left[ A^<_k U^*_k(\eta) U^<_k(\eta') + [B^<_k U^*_k(\eta) U^*_k(\eta')] \Theta(\eta'-\eta) \right. \right. \tag{39}
\]

where \(A^>_k, B^>_k, A^<_k\) and \(B^<_k\) are integration constants. In the Appendix \(B\) we show that for the free propagators i.e. at the lowest perturbation order, if the state \(|\Psi\rangle\) is not vacuum, it is possible to include its effect in the boundary conditions imposed on the propagator. Comparing \(39\) with \(70\) in the Appendix \(B\) the relation between these constants and the initial state can be concluded:

\[
A^>_k = 1 + B^>_k, \quad B^>_k = 1 + A^<_k \tag{40}
\]

\[
A^<_k = B^<_k = \sum_i \sum_{k_1 k_2...k_n} \delta_{kk_i} |\Psi_{k_1 k_2...k_n}|^2 \tag{41}
\]

It is easy to see that with these relations the consistency condition defined as:

\[
G(\eta, \eta') \bigg|_{\eta = \eta'} = G^< (\eta, \eta') \bigg|_{\eta = \eta'} \tag{42}
\]

is automatically satisfied. Therefore propagators over a non-vacuum state \(\Psi\) only depends on this state and the solutions of the field equation considered as the free particle states. On the other hand, these solutions depend on two arbitrary constants \(c^a_k\) and \(d^a_k\) which should be fixed by initial conditions. We have used decomposition \(39\) along with canonical decomposition of the quantum field \(\Phi (\eta)\) to fix the integration constants \(c^a_k\) and \(d^a_k\). The advantage of this method is that it explicitly relate dynamical constants to the physical properties of environment in which the quantum field is living.

There is one more consistency condition that propagators should satisfy. By integrating two sides of equation \(58\) with respect to \(\eta\) in an infinitesimally region around \(\eta'\) we find the following constraint:

\[
\left. U^*_k(\eta) U_k(\eta') - U_k(\eta) U^*_k(\eta) \right|_{\eta = \eta'} = \frac{-i}{a(\eta)} \tag{43}
\]
This relation fixes one of the dynamical constants in \( (36) \). It can be chosen to be the normalization of the propagator. It is however interesting to note that multiplying both sides of \( (33) \) with an arbitrary constant rescales \( a(\eta) \) which is equivalent to redefinition of \( a_0 \). Rescaling of \( a_0 \) is equivalent to redefinition of coordinates and therefore is not an observable. In Minkovskyan spacetime the scale factor \( a \) is fixed to 1, and therefore there is no place for rescaling. Thus in a Minkovskyan space the normalization of the propagators is an observable and affecting final results. The scaling properties in FLRW or De-Sitter metric is a consequence of diffeomorphism invariance in the framework of curved spacetimes and general relativity.

2.2 Initial Conditions

Field equations are second order differential equations and need the initial value of the field and its derivative or a combination of them to be totally described. The general initial conditions for a bounded system, including both Neumann and Dirichlet conditions as special cases, are the followings \cite{21}:

\[
\eta^\nu \partial_\nu U = iKU, \quad \eta^\nu \partial_\nu U^* = -iK^*U^*, \quad g_{\mu\nu}n^\mu n^\nu = 1
\]  

(44)

The 4-vector \( n^\mu \) is the normal to the boundary surface. If the boundary is space-like, the normal \( n^\mu \) can be normalized to \( a^{-1}(1,0,0,0) \), then:

\[
\eta^\nu \partial_\nu U = iKU, \quad \eta^\nu \partial_\nu U^* = -iK^*U^*
\]  

(45)

Constants \( K_i \) and \( K_f \) depends on \( k \). In a general boundary problem the boundary conditions must be defined for all the boundaries. Thus, in a cosmological setup the initial conditions \( (45) \) must be applied to a past (initial) and future (final) boundary surfaces. This is the strategy suggested in Ref. \cite{21}. The past and future boundary conditions are respectively applicable only to past and future propagators. Assuming different values for \( K \) on the past and future boundary, one finds:

\[
K_j = -i\frac{U_k(\eta_j)}{a_j U_k(\eta_j)}, \quad j = i, f
\]  

(46)

Indexes \( i \) and \( f \) refer to the value of quantities at initial and final boundary conditions. These boundary conditions relate \( K_i \) and \( K_f \) to \( c_k \) and \( d_k \) in \( (46) \). In a cosmological context although \( K_f \) a priori can be decided based on observations, the value of \( K_i \) is unknown and leaves one arbitrary constant in the solution or it should be selected according to a special model considered for the physics of early universe. This arbitrariness of the general solution, or in other words the vacuum of the theory, is well known \cite{23}. In the case of inflation, this leads to a class of possible vacuum solutions called \( \alpha \)-vacuum. For instance if:

\[
K_i = K_f = \sqrt{k^2/a_0^2 + m^2}
\]  

(47)

one obtains the well known Bunch-Davies solution \cite{21}.

Another way of proceeding is using the consistency condition \( (43) \) to fix one of the arbitrary constants and applying the boundary condition \( (45) \) only to one of the initial or final 3-surfaces. Although this does not solve the problem of arbitrariness of \( K \) and its \( k \) dependence, it reduces it to only one of the boundary surfaces, for instance to the final 3-surface, and make the choice of \( (47) \) physically motivated. Besides, in fixing only one of the boundary conditions, the causality of the solution is transparent - the state of the second boundary is directly related on the choice of the first one through the evolution equation.

Another and somehow hidden arbitrariness in this formalism is the fact that a priori \( k \) dependence of the boundary constant \( K \) does not need to be the same for all the fields of the model. However, different \( k \) dependence for the boundary conditions breaks the Equivalence Principal. Similarly, a value different from \( (47) \) for \( K \) will lead to the breaking of the translation symmetry \cite{21} \cite{8}. In the context of quantum gravity the violence of both of these laws are expected and therefore, in a general framework they should be considered.
2.3 WKB approximation and back reactions

Finally after finding the solution of linearized field equation (34) and corresponding propagator (38) for the field $\Upsilon$, we should add the effect of spacetime dependence of the mass term. According to the WKB prescription we must replace $\alpha \Phi \eta$ in $\mathcal{U}$ with:

$$\alpha \Phi \eta \rightarrow \alpha \Phi \int d\eta \left( 1 + (n - 1) \lambda \frac{\varphi_k^{n-2}(\eta)}{m_{\phi}^2} \right)^{1/4}$$

where $\varphi_k$ is the Fourier transform of $\varphi(x)$ and $\alpha \Phi$ is defined in (37). With this correction apparently we have the solution for all the propagators at lowest order. However, $\varphi(x)$ (or equivalently $\chi$) evolves according to equation (28) or (29) that depend on the expectation values of the quantum fields, specially $\phi$. Therefore the propagators and the condensate are coupled even at lowest order, and only through a numerical calculation a final solution for each of them can be obtained. In fact the reason for coupling of classical and quantum fields is the self-coupling of $\Phi$. If the self-interaction of $\Phi$ is negligible, the solutions of evolution equation of the classical component (28) or (29) are similar to the field equation of the quantum component. In presence of a self-interaction however this equation is non-linear and must be solved numerically.

In addition to the self-coupling, the state $|\Psi\rangle$ for which we calculate propagators and expectation values is a source of back-reaction (classical effects of back-reaction are studied in Ref. [22]). It defines the quantum state of the Universe at the time when the interactions or more exactly the decay of $X$ is studied. However, $|\Psi\rangle$ evolves due to interactions between species and the expansion of the Universe. They are responsible for the variation of the number of particles and their momentum distribution and therefore evolution of $|\Psi\rangle$. In fact, in the path-integral formulation of closed path integral, a non-vacuum state $|\Psi\rangle$ adds a functional integral to the path-integral that presents the projection of the state on a predefined basis. When the Fock space is evolving, as it is the case in the cosmological context, the projection of $|\Psi\rangle$ evolves and path integrals become unfactorizable.

As an example we consider an initial non-zero distribution for the $X$ particles and no $A$ or $\Phi$ particles. At a later time, the number of $X$ particles in $|\Psi\rangle$ is reduced due to their decay and the expansion of the Universe. Their temperature or mean kinetic energy if they don’t have a thermal distribution also decreases. On the other hand, a non-zero number of $\Phi$ and $A$ particles are created. The latter at their production are relativistic and non-thermal. But if they have self-interaction and/or interaction with other fields which we ignored here, their momentum distribution will change both by interactions and due to the expansion of the Universe. This means that $|\Psi\rangle$ will change and its variation is reflected on the evolution of the classical component $\varphi$, the quantum component $\phi$, the expansion factor $a(\eta)$, and the thermalization of $A$ which we assume to have interactions with other particles (see also Appendix B).

3 Evolution of the classical field without self-interaction

To get an insight into the evolution of the classical component $\varphi$, in this section we neglect self-interaction of $\Phi$ which is the main source of the non-linearity and coupling of the equations. We find an analytical solution for the evolution of condensate and discuss the difference between two decay modes, as well as the effect of the other parameters. The evolution of $|\Psi\rangle$ is introduced by a simple parametrization. We discuss its effect and determine the range in which the condensate can have a behaviour similar to the dark energy.
3.1 Expectation values

Neglecting the self-interaction of $\Phi$, expression (36) is the exact solution of the field equation and (39) is the exact propagator. Therefore, we can calculate the expectation values (17) to (19):

$$g\langle XA \rangle(x) = \frac{-ig}{(2\pi)^6} \int d^3k_1 d^3k_2 d^3k_3 e^{-ix.(\vec{k}_1+\vec{k}_2+\vec{k}_3)} \delta(3)(\vec{k}_1+\vec{k}_2+\vec{k}_3) \int d\eta' \sqrt{-g}$$

$$g\langle XA \rangle(x) = \frac{-ig}{(2\pi)^6} \int d^3k_1 d^3k_2 d^3k_3 e^{-ix.(\vec{k}_1+\vec{k}_2+\vec{k}_3)} \delta(3)(\vec{k}_1+\vec{k}_2+\vec{k}_3) \int d\eta' \sqrt{-g}$$

$$g\langle YXA \rangle(x) = \frac{-ig}{(2\pi)^6} \int d^3k_1 d^3k_2 d^3k_3 e^{-ix.(\vec{k}_1+\vec{k}_2+\vec{k}_3)} \delta(3)(\vec{k}_1+\vec{k}_2+\vec{k}_3) \int d\eta' \sqrt{-g}$$

The most important aspect of these integrals for us is their time dependence because these expectation values contribute to the build-up and the time evolution of the condensate. Although the spatial spectrum is important specially for observational verification of the model, its role is secondary and comes later.

Regarding (36) and (39), it is easy to conclude that time integrals in (49) and (50) are similar to the following:

$$I(\eta) \equiv \prod_{i=1}^{N} D_{q_i}^{*} (\gamma_i | z) \int_{\eta_0}^{\eta} d\eta' \sqrt{-g} \prod_{i=1}^{N} D_{q_i} (\beta_i | z')$$

$$q_i \equiv -\frac{1 + i\beta_i^2}{2}, \quad z' \equiv \alpha_i \eta'$$

where $\alpha_i$ and $B_i$ are defined in (37) and $N$ is the number of fields in the expectation value brackets. For simplicity we use the product index in (52) for distinguishing the field species too. The constant coefficients of these integrals are of the form $\prod_{i} A_{k_i}^{A_i} C_{k_i}^{C_i}$ with $A_{k_i}^{A_i} \in \{A^{k_1}_k, B^{k_1}_k, A^{k_2}_k, B^{k_2}_k\}$, $C_{k_i} \in \{c_k^k, d_k^k\}$ and $C_{k_i}^{C_i} \in \{c_k^k, d_k^k\}$ for the corresponding species. These integrals don’t have analytical solutions. Using the asymptotic properties of the parabolic cylindrical functions, we estimate the late time behaviour of these integrals:

$$D_q(z) \sim e^{-\frac{z^2}{2}} z^q \left[ 1 - \frac{q(q-1)}{2z^2} + \ldots \right], \quad |z| \gg 1, \quad |z| \gg |q|$$

$$D_q(z) \sim 2^{q} e^{-\frac{z^2}{2}} \left[ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-q}{2}\right)} - \frac{\sqrt{2\pi z}}{\Gamma\left(\frac{1-q}{2}\right)} + \ldots \right], \quad |z| \ll 1$$

The validity of these regimes depends on the mass of the corresponding field and on the cosmological parameters at the initial time $\eta_0$:

$$|\beta_i| \eta_0 = \sqrt{\frac{2m_i}{H_0}}$$

For a light $\Phi$, its mass at the initial time $\eta_0$ can be comparable or lighter than the Hubble constant $H_0$ and therefore approximation (54) is applicable. As for $X$ and $A$, it seems unlikely that in any relevant particle physics model for these fields $\frac{2m_i}{H_0}$ be small and therefore approximation (53) must be applied.

Validity of both approximations depends on the spatial scale $k_i$ and the constant $B_i$ for each species. The initial time $\eta_0$ and Hubble constant $H_0$ also play important roles in these approximations and in the general behaviour of the model. Here we take these conditions for granted.
Applying these approximations to (52), at lowest order in $\eta$ this integral has the following time dependence:

$$I(\eta) \sim \left(\frac{a_0}{\eta_0}\right)^4 F(k) \eta^{5-N'} e^{\frac{4}{3} \sum_i \beta_i^2 B_i \eta^2}$$

(56)

$$F(k) \propto \frac{\prod_j \sinh \left(\frac{k^2}{m_{j}} \right)}{5-N' + \frac{4}{3} \sum_{i'} \frac{k^2_{i'}^2}{\Pi_{i'}}}$$

(57)

where $N'$ is the number of fields to which approximation (53) can be applied. Index $i'$ refers to all the fields. Indexes $i'$ refers to fields for which approximation (53) is applicable and $j$ for ones with approximation (54) approximation. If $j = 0$, the nominator in (57) is 1. In this case the effect of the fields for which $|\beta z|$ in the integration interval is small contributes only in the oscillating term of (56) and in the higher-order terms of the polynomial expansion.

For estimating the expectation values (49)-(51) in addition to time dependence included in integral (52) one has to take into account the time variation of the state $|\Psi\rangle$ both due to the expansion of the Universe and due to the back-reaction of the decay on the density of particles. Neglecting the effect of the decay on the density for a short duration after the massive production of $X$ particles during reheating, the main source of the time evolution of $|\Psi\rangle$ and $|\Psi_{k_1k_2...k_n}\rangle$ is the expansion. Assuming a very heavy $X$, it can be considered as non-relativistic at the time of its production and its density decreases by $a^3 \propto \eta^3$. The other two fields $A$ and specially $\Phi$ are relativistic and their density decreases by a factor of $a^4 \propto \eta^4$.

As the lifetime of $X$ particles is very long and thus $g$ is very small, the contribution of $A$ and $\phi$ particles to the total number, energy density, and entropy of the relativistic matter at this epoch is very small. Therefore, one expects that after an initial fast increasing trend of the expectation values in (49)-(51), they will slow down and probably approach a constant or begin to decrease. The turning point depends on the expansion rate, lifetime of $X$ particles or equivalently the coupling constant $g$, the density of $X$ particles at the end of reheating through its effect on $|\Psi_{k_1k_2...k_n}\rangle$, and finally the mass of $\Phi$ and $A$ particles that can be estimated from (56). Larger the number of fields with $m/H_0 \gg 1$, slower the time evolution of (56) and sooner the turning point of the expectation values. This observation is consistent with our initial assumption of no condensation for $X$ and $A$. We expect that only for the field $\Phi$ there is a time interval in which $m_{\Phi}/H_0 < 1$. Thus, at late times $\langle XA^2 \rangle(x)$ decreases faster than $\langle X A \rangle(x)$ and $\langle Y X A \rangle(x)$. In addition to the time dependent mass term due to $\langle XA^2 \rangle(x)$ in decay mode $b$, the time evolution of expectation values in the two decay modes $a$ and $b$ are different, and therefore the condensation evolution for these modes are different too. Nonetheless there can be an exception to this argument. If we consider $A$ as a single quantum field and not a collective notation, it is conceivable that at very early time symmetries keep it massless and it receives a dynamical mass after an epoch of symmetry breaking and phase transition. In this case the classical field $\chi$ will have a faster gross and at later times, a sudden change in the time evolution of the expectation values (49)-(51) and $\chi$ is expected. Note that this argument is based on the back-reaction of the $X$ particles decay on $|\Psi\rangle$.

In the next subsection we show that the most important term in the late time evolution of $\chi$ is the non-homogeneous term in (28) and (29). The coupling $g$ appears as a time independent constant and changes the amplitude of $\chi$. This means that the density of the condensate is proportional to $g^2 \propto \Gamma$. This confirms the results of the classical treatment of this model in Ref. [5]. Regarding the coincidence and smallness problem of the dark energy, the close relations between mass, number density and lifetime of $X$ and the mass and amplitude of the condensate is an evidence for an intrinsic feedback between dark matter and dark energy in this model. Moreover, this shows that the density of the dark energy is not more fine-tuned than the mass difference between left and right neutrinos.
in seesaw mechanism. In fact this similarity hints to a seesaw-like mechanism for the large difference between the mass of $X$ and $\Phi$ and supports the idea of right neutrino/sneutrino for $X$.

Although here we are mostly interested in the decay of a meta-stable heavy particle, it is also interesting to see how a light scalar produced during the fast decay of another field, e.g. inflaton, can condensate and whether this condensate can last for long time and contribute to the build up of the dark energy. As before, in this setup the expectation values (49)-(51) will have an initial rise. But, depending on the and whether this condensate can last for long time and contri bute to the build up of the dark energy.

To see how a light scalar produced during the fast decay of ano ther field, e.g. inflaton, can condensate and make classical density perturbations. This is another evidence for close relation between these processes as suggested in Appendix A.

3.2 Evolution of the condensate

Now we use the result of the previous sections to estimate the evolution of the classical field $\chi$ with cosmic time. Neglecting the self-interaction term in (28) and (29), the general solutions of these equations at the WKB approximation level are:

$$\chi(\eta) = \chi_1 D_\eta (\alpha \Phi \eta) + \chi_2 D_\eta (-\alpha \Phi \eta) + g \int_{\eta_0}^{\eta} d\eta' \langle X A^2 \rangle G^X (\eta, \eta')$$  \quad \text{For decay mode (a)} \quad (59)

$$\chi(\eta) = \chi_1' D_\eta (\alpha \Phi \int d\eta (1 - \frac{2g (X A)}{a^2 m^2_\Phi})^\frac{1}{2}) + \chi_2' D_\eta (-\alpha \Phi \int d\eta (1 - \frac{2g (X A)}{a^2 m^2_\Phi})^\frac{1}{2})$$

$$2g \int_{m_0}^{\eta} d\eta' \langle T X A \rangle G^X (\eta, \eta')$$  \quad \text{For decay mode (b)} \quad (60)

where $\chi_1$, $\chi_2$, $\chi_1'$ and $\chi_2'$ are integration constants. We assume the following physically motivated initial conditions for these solutions:

$$\chi(\eta_0) = \chi'(\eta_0) = 0$$  \quad (61)

The associated homogeneous equation of the differential equation (28) is the same as equation (44). Therefore the Green’s function $G^X (\eta, \eta')$ is the same as the Feynman propagator (39). In the case of mode (b) we have added the WKB approximation to the homogeneous solutions. This correction is first order in the coupling $g$, thus we can use the Green’s function (39) for this mode too. A
more precise solution can be obtained by replacing \( D_q(\alpha_\Phi \eta) \) terms in (39) with the WKB corrected homogeneous solution \( D_q(-\alpha_\Phi \int d\eta(1 - \frac{2g(XA)}{a^2 m_\Phi^2})^\frac{1}{2}) \).

Using the asymptotic behaviour of \( D_q(\eta) \) at large \( \eta \), we find that for both decay modes (a) and (b) the homogeneous part of the solutions (59) and (60) is proportional to \( \eta^{-1/2} \) and therefore decreases with time. In the same way we can find time dependence of the special solution. For large \( \eta \) it is proportional to \( \eta^{2+\epsilon} \) for both decay modes. Here \( \epsilon \) is added by hand to parametrize the unknown time variation of the cosmological state \(|\Psi\rangle\). If the mass of \( \Phi \) is very small such that \( B_\Phi \approx 0 \), the solution of the evolution equations is approximately a plane wave in \( \eta \) coordinate and \( \chi \propto \eta^{3+\epsilon} \) for the mode (b). For mode (a) this approximation does not modify the asymptotic behaviour of \( \chi \).

Finally the classical field \( \varphi \) in comoving coordinates is \( \chi = a\varphi \) and varies as:

\[
\varphi \propto t^{-1/2+2+\epsilon} \quad \text{Assuming } m \neq 0 \text{ for all fields, both decay modes.} \tag{62}
\]

\[
\varphi \propto t^{-1/2+3+\epsilon} \quad \text{For } m_\Phi \approx 0 \text{ and mode (b).} \tag{63}
\]

For all cases, the late time behaviour of \( \varphi \) depends on \( \epsilon \) i.e. the back reaction. Based on qualitative arguments, we expect that \( \epsilon \) depends on the density and lifetime of \( X \) field, and therefore on the expansion of the Universe. If we want that at late times \( \varphi \sim \text{cte} \), the only consistent model for the quintessence field is a model in which \( m_\Phi \neq 0 \) but small. In such a model the condensate grows during the radiation dominated era but stops growing when matter becomes dominant. However, self-interaction should somehow modify this result. None the less, the strong limit on the clustering of the dark energy shows that self-interaction of the quintessence field can not be very strong and therefore its over all effect must be small. This simple model considered here favors particle physics models with a PNGB scalar field and a cyclical potential to protect the small mass of \( \Phi \) [24].

At late times when the energy density of the dark matter \( X \) becomes comparable to the density of the dark energy \( \sim \frac{1}{2}m_\Phi^2 \varphi^2 \), there is a strong feedback between expansion rate and the density of dark matter. Faster expansion rates bring down the density of the dark matter and therefore the rate of \( \varphi \) production decreases. This reduces the density of the dark energy and the expansion. We have seen the same feedback in classical treatment of this process in Ref. [5].

## 4 Outline

Although we have not yet observed any elementary scalar field, we believe they play important roles in the foundation of fundamental forces in the nature, in Standard Model and in all suggested extension of it. The detailed studies of their behaviour and their consequences for other phenomena are however hampered by the fact that they can have complex non-linear self-interaction and interaction with other fields which make analytical calculations very difficult.

We used quantum field theory techniques to determine the evolution of the classical component - the condensate - of a scalar field produced during the decay of a much heavier particle. Such a process had necessarily happened during the reheating of the Universe, and similar phenomena can happen - both as decay and as interaction - after the reheating if a scalar has interaction with other fields. We showed that a significant amount of condensate can only be obtained for light fields with masses comparable or smaller than Hubble constant at the initial time. By considering two decay modes, we also showed that the type of interactions has very important role in the cosmological evolution of the condensate and its contribution to dark energy. Therefore, the details of such a model depends on the particle physics of the fields. The general behaviour is nonetheless universal. Due to the coupling between quantum phenomena, i.e the micro-physics of the Universe, and its classical macroscopic content, obtaining an exact analytical solution is impossible. We left this task for future and by using simplifying assumptions we deduced the general aspects of the asymptotic time evolution of the condensate. We showed that one of the most important ingredient of this model is the unknown back-reaction of the decay and other physical processes on the Fock space of the Universe. We parametrized
the time dependence of the Fock space with just one exponent $\propto \eta^\epsilon$. If during matter dominated epoch $-4 \lesssim \epsilon \lesssim -2$, the condensate density evolves very slowly with time and has a behaviour similar to the observed dark energy. This range of epsilon is for minimal model without self-interaction for the scalar field. In a realistic context, both self-interaction and interaction with other field, specially Standard Model fields must be considered and they can change this range.

The main goal of this work has been studying the contribution of the classical component of a gradually built-up field in the dark energy, specially in the context of a metastable heavy particle. Nonetheless, most of the results obtained here are applicable to other contexts and other models. Specifically, extensions of the Standard Model such as supersymmetric and supergravity models and string theory contain a large number of scalar fields. Condensation of these fields and their evolution and thereby their parameter space and symmetries can be constrained by the observation of the equation of state of the Universe. As we have shown here, the condensate evolution depends on a number of parameters including mass, decay rate of the producing particle, interactions and couplings. This is an additional and more detailed information with respect to simple density constraint for the scalar fields of particle physics models.

Appendixes

A A note on the relation between decoherence and condensation

Using canonical representation and Bogolubov transformation for a free scalar field, one can find the relation between creation and annihilation operators at two different cosmic times:

$$\begin{pmatrix}
    a_k(\eta) \\
    a_k^+(\eta)
\end{pmatrix} = \begin{pmatrix}
    u_k(\eta, \eta_0) & v_k(\eta, \eta_0) \\
    u_k^+(\eta, \eta_0) & v_k^+(\eta, \eta_0)
\end{pmatrix} \begin{pmatrix}
    a_k(\eta_0) \\
    a_k^+(\eta_0)
\end{pmatrix}$$

$$\Phi(\eta) = \frac{-ia}{\sqrt{2|k^2 + a^2m^2 - a^2|^\epsilon}}(a_k^+(\eta) + a_k(\eta))$$

where $a_k^+(\eta_0)$ and $a_k(\eta_0)$ are respectively creation and annihilation operator at an initial conformal time $\eta_0$, and $u_k(\eta, \eta_0), v_k(\eta, \eta_0)$ and their conjugates are proportional to the solutions of the field equation at $\eta$ and $\eta_0$. Using the definition of number operator $\hat{N} \equiv a_k^+a_k$, it is easy to show that even without any self-interaction the number of particles at $\eta$ and $\eta_0$ are not equal and cosmic expansion leads to particle production. The decoherence of these particles however needs an interaction to couple modes or fields. Although in a curved spacetime, specially during inflation, squeezed states can be achieved, the decoherence is not complete unless an interaction breaks the entanglement between degenerate quantum states by coupling causally unrelated modes - modes inside particle horizon to ones outside.

We observe that for a free field in which modes are independent, the linearity of (64) and the properties of the creation/annihilation operators leads to $\langle 0|\Phi(\eta)|0 \rangle = 0$, where $|0 \rangle$ is the vacuum at $\eta_0$, i.e. there is no condensation. Therefore decoherence is a necessary but not sufficient condition for the formation of a condensate.
B Free field Green’s function on non-vacuum states

In canonical representation, a free scalar field $\phi$ can be decomposed to creation and annihilation operators of an orthogonal basis of the Fock space:

$$\phi(x) = \sum_k U_k(x)a_k + U_k^*(x)a_k^\dagger, \quad [a_k, a_{k'}^\dagger] = \delta_{kk'} \quad [a_k, a_{k'}] = 0 \quad [a_k^\dagger, a_{k'}^\dagger] = 0$$ \hspace{1cm} (66)

where $U_k(x) \equiv U_k(\eta)e^{-ik\cdot x}$ is a solution of the free field equation \( \Box \phi \). Quantization of $\phi$ imposes the following relation:

$$U_k(\eta, x)U_k^a(\eta, y) - U_k^a(\eta, x)U_k^a(\eta, y) = i\delta^{(3)}(x - y)$$ \hspace{1cm} (67)

A Fock state $|\Psi\rangle$ is constructed by multiple applications of the creation operator $a_k^\dagger$ on the vacuum state $|0\rangle$ defined by:

$$a_k|0\rangle = 0, \quad |k_1k_2\ldots k_n\rangle \equiv a_{k_1}^\dagger a_{k_2}^\dagger\ldots a_{k_n}^\dagger|0\rangle$$ \hspace{1cm} (68)

$$|\Psi\rangle = \sum_{k_1k_2\ldots k_n} \Psi_{k_1k_2\ldots k_n}|k_1k_2\ldots k_n\rangle$$ \hspace{1cm} (69)

Applying these decompositions to 2-point free Green’s function of $\phi$, it can be written as:

$$iG_F(x, y) \equiv \langle \Psi|T\phi(x)\phi(y)|\Psi\rangle = \sum_k \sum_i \sum_{k_1k_2\ldots k_n} \delta_{kk_i} |\Psi_{k_1k_2\ldots k_n}|^2 \left[ U_k^*(x)U_k(y)\Theta(x_0 - y_0) + U_k(x)U_k^*(y)\Theta(y_0 - x_0) \right] + \sum_k \left[ U_k(x)U_k^*(y)\Theta(x_0 - y_0) + U_k^*(x)U_k(y)\Theta(y_0 - x_0) \right]$$ \hspace{1cm} (70)

From (70) we can extract the expression for future and past propagators:

$$iG^>(x, y) \equiv \langle \Psi|\phi(x)\phi(y)|\Psi\rangle = \sum_k \sum_i \sum_{k_1k_2\ldots k_n} \delta_{kk_i} |\Psi_{k_1k_2\ldots k_n}|^2 U_k^*(x)U_k(y) + \sum_k \left[ 1 + \sum_i \sum_{k_1k_2\ldots k_n} \delta_{kk_i} |\Psi_{k_1k_2\ldots k_n}|^2 \right] U_k(x)U_k^*(y)$$ \hspace{1cm} (71)

$$iG^<(x, y) \equiv \langle \Psi|\phi(x)\phi(y)|\Psi\rangle = \sum_k \sum_i \sum_{k_1k_2\ldots k_n} \delta_{kk_i} |\Psi_{k_1k_2\ldots k_n}|^2 U_k(x)U_k^*(y) + \sum_k \left[ 1 + \sum_i \sum_{k_1k_2\ldots k_n} \delta_{kk_i} |\Psi_{k_1k_2\ldots k_n}|^2 \right] U_k^*(x)U_k(y)$$ \hspace{1cm} (72)

Therefore, for free fields, $G_F(x, y)$ on any density operator $\rho$ can be written as a linear expansion with respect to $U_k(x)U_k^*(y)$ and $U_k^*(x)U_k(y)$, the independent solutions of the free field equation. The contribution from a non-vacuum state $|\Psi\rangle$ appears in the coefficients of the expansion, just as the initial/boundary condition effects appear in the expansion coefficients of the propagator \( \Box \phi \). Projection coefficients $|\Psi_{k_1k_2\ldots k_n}|^2$ determine the momentum distribution of the states in the environment. In the simplest case where quantum correlation between particles is negligible, it is proportional to one particle momentum distribution $f(k)$ that in the case of a thermal environment is Boltzmann, Fermi or Bose-Einstein distribution.

Here a general comment about expectation values including 2-point Green’s function is in order. An expectation value i.e. an inner product in the Fock space of a quantum system, this includes operators at different spacetime points is meaningful only if these points share the same or isomorphic Fock spaces. The state $|\Psi\rangle$ in (70) for which the expectation value is calculated must be a member of the Fock space at $x$ and $y$. However, in the fast changing early Universe, it is not evident that such
a condition exists unless $x$ and $y$ are enough close to each other. This reflects the back-reaction of interactions during and after reheating on the Fock space and coupling between operators and states as mentioned in Sec 2.3. Therefore, $|\Psi\rangle$ has an implicit spacetime dependence and one-particle distribution function gets the familiar form $f(x,k)$ used in the construction of classical Boltzmann equation [26].

C Propagators and evolution in matter dominated epoch

In the matter dominated epoch the relation between comoving and conformal time is defined as:

$$\eta = \int \frac{dt}{a} = \eta_0 \left( \frac{t}{t_0} \right)^{\frac{2}{3}} , \quad \eta_0 = \frac{3t_0}{a_0}$$  \hspace{1cm} (73)

$$\frac{a}{a_0} = \left( \frac{t}{t_0} \right)^{\frac{2}{3}} = \left( \frac{\eta}{\eta_0} \right)^2 , \quad \frac{a''}{a} = \frac{2}{\eta^2}$$  \hspace{1cm} (74)

By applying (74) to the field equations [28] and [29] and neglecting interactions, the field equation for both modes is the following:

$$\chi'' + (k^2 + \frac{m^2 a_0^2 \eta^4}{\eta_0^4} - \frac{2}{\eta^2})\chi = 0$$  \hspace{1cm} (75)

where $\chi$ presents one of the fields $\chi, \Upsilon, \chi$ or $A$. For two special cases of $m = 0$ and $k^2 = 0$ this equation has exact analytical solutions:

$$\chi(\eta) = \begin{cases} \sqrt{\eta} J_{\pm \frac{1}{2}}(\beta \eta^2) , & \beta = \frac{\sqrt{3} m a_0}{3 \eta_0} \quad \text{for } k^2 = 0 \\ \sqrt{\eta} J_{\pm \frac{1}{2}}(k \eta) , & \text{for } m = 0 \end{cases}$$  \hspace{1cm} (76)

As usual, in each case the WKB approximation can be used to find an approximate solution when the neglected terms are not zero. Another possible approximation is a linear interpolation between two cases in (76).

Bessel functions $J_{\pm 1/2}$ and $J_{\pm 3/2}$ have close analytical expressions:

$$J_{\pm \frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$ \hspace{1cm} and \hspace{1cm} $$J_{\pm \frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$  \hspace{1cm} (77)

$$J_{\pm \frac{1}{2}}(x) = \frac{1}{x} J_{\frac{1}{2}}(x) - J_{\frac{1}{2}}(x), \quad J_{\pm \frac{3}{2}}(x) = -J_{\frac{3}{2}}(x) - \frac{1}{x} J_{\frac{5}{2}}(x)$$  \hspace{1cm} (78)

and

$$\chi(\eta) = \begin{cases} \sqrt{\frac{\eta}{\eta_0}} \left( \chi_1 \sin(\beta \eta^2) + \chi_2 \cos(\beta \eta^2) \right) , & \text{for } k^2 = 0 \\ \sqrt{\frac{2}{\pi k}} \left( \chi_1' \frac{\sin(k \eta)}{k \eta} - \cos(k \eta) \right) + \chi_2' \left( -\sin(k \eta) - \frac{\cos(k \eta)}{k \eta} \right) , & \text{for } m = 0 \end{cases}$$  \hspace{1cm} (79)

where $\chi_1, \chi_2, \chi'_1$ and $\chi'_2$ are integration constants. At late times and for large scales (small $k$) the mass term in (75) is dominant and $k^2 = 0$ approximation can be applied. Using (79), we find that integrals analog to (52) (replacing $D_q$ with $\chi$ from (76)) have a late time behaviour $\propto \eta^{9-2N}$ where $N$ is the number of fields in the expectation value. If we assume that the mass of $\Phi$ is very small and $m = 0$, this approximation can be applied to this field. Under these conditions the expectation values containing one $\chi$ field are $\propto \eta^{9-2(N-1)}$ where here $N$ is the number of other fields. Finally from these results and the general solutions (59) and (60) (again replacing $U_k$ with $\chi$) we conclude that:

$$\varphi \propto t^{\frac{2}{3}} \quad , \quad \text{For } m \neq 0, \text{ both modes} \hspace{1cm} (80)$$

$$\varphi \propto t^{\frac{4}{3}} \quad , \quad \text{For } m = 0 \text{ and mode (b)} \hspace{1cm} (81)$$
Comparing (80) and (81) with (62), and (63) and assuming the same $\dot{\epsilon}$, we find that in the matter-dominated epoch the time evolution is slower than radiation-dominated epoch. In the matter-dominated epoch the expansion of the Universe is faster, the decay of $X$ particles is slower because for the same average density the Universe is younger, and therefore less $X$ particles have decayed. This compensates the density reduction due to a faster expansion. Similar to the radiation-dominated epoch, the final time evolution rate depends on the unknown parameter $\dot{\epsilon}$.

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