Avoiding order reduction when integrating diffusion-reaction boundary value problems with exponential splitting methods

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Abstract

In this paper, we suggest a technique to avoid order reduction in time when integrating reaction-diffusion boundary value problems under non-homogeneous boundary conditions with exponential splitting methods. More precisely, we consider Lie-Trotter and Strang splitting methods and Dirichlet, Neumann and Robin boundary conditions. Beginning from an abstract framework in Banach spaces, a thorough error analysis after full discretization is performed and some numerical results are shown which corroborate the theoretical results.

1 Introduction

Exponential splitting methods are very much used in the recent literature when integrating partial differential equations because they integrate the linear and stiff part of the problem in an exact way [17]. Due to the recent and high development of Krylov-type methods to calculate exponential-type functions over matrices which are applied

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over vectors [13], they constitute an effective tool to integrate such problems in a stable way.

In this paper, we will center on the first-order Lie-Trotter and second-order Strang methods. The order reduction which turns up with these methods when integrating linear problems with homogeneous boundary conditions was recently studied in [12]. In [1] we have suggested a technique to deal with non-homogeneous boundary conditions in linear problems. That technique has some similarities to that suggested in [3] for other exponential-type methods, which are Lawson ones. With that procedure, we managed to avoid order reduction completely in linear problems.

The aim of the present paper is to generalize that technique to nonlinear reaction-diffusion problems and to prove that order reduction can also be completely avoided.

There are other results in the literature concerning this problem or a more specific one. For example, in [7], a generalized Strang method is suggested for the specific nonlinear Schrödinger equation. However, in that paper, an abstract formulation of the problem is not given (as it is here), Neumann or Robin type boundary conditions are not considered, parabolic problems for which a summation-by-parts argument can be applied are not included and finally, Lie-Trotter method is not analyzed. On the other hand, in [10, 11], a completely different technique is suggested to avoid order reduction with the same methods and nonlinear problems than here, but the analysis for the local and global error is just performed in time. The error coming from the space discretization and the numerical approximation in time of the nonlinear and smooth part are not included and therefore, a practical implementation of the technique for the practitioners is not justified. However, in the present paper, the exact formulas to be implemented are described in (40)-(41) for Lie-Trotter and in (58), (60) and (62) for Strang. Moreover, the analysis is performed under quite general assumptions on the space discretization and time integration of the nonlinear part. For that, we use the maximum norm, which facilitates its applicability to quite general problems.

The paper is structured as follows. Section 2 gives some preliminaries on the abstract setting of the problem, on the assumptions of regularity which are required for the solution to be approximated and on Lie-Trotter and Strang methods. Section 3 describes the technique to avoid order reduction after time integration with Lie-Trotter method and explains how to deal with non-homogeneous Dirichlet, Robin and Neumann type boundary conditions. Moreover, a thorough local error analysis is given. In Section 4, the same is done for Strang method, for which just order 2 can be obtained in general for the local error. Section 5 states some hypotheses on the space discretization which include some finite-difference schemes, as the ones being used in the numerical experiments. (Similarly, collocation-type methods could be considered.) The detailed analysis of the local and global error after full discretization is performed in Section 6 for Lie-Trotter splitting and in Section 7 for Strang splitting. For the latter, no order reduction in time is observed for the global error if the bound (54) is satisfied by the discretization of the elliptic problem. Finally, Section 8 shows some numerical experiments which corroborate the previous results. Moreover, in two dimensions a double splitting is included considering the results in [1].
2 Preliminaries

Let $X$ and $Y$ be Banach spaces and let $A : D(A) \to X$ and $\partial : X \to Y$ be linear operators. Our goal is to study full discretizations, by using as time integrators Lie-Trotter and Strang exponential methods, of the nonlinear abstract non homogeneous initial boundary value problem

$$
\begin{align*}
    u'(t) &= Au(t) + f(t, u(t)), \quad 0 \leq t \leq T, \\
    u(0) &= u_0 \in X, \\
    \partial u(t) &= g(t) \in Y, \quad 0 \leq t \leq T,
\end{align*}
$$

(1)

where the functions $f : [0, T] \times X \to X$ (in general nonlinear) and $g : [0, T] \to Y$ are regular enough.

The abstract setting (1) permits to cover a wide range of nonlinear evolutionary problems governed by partial differential equations. We use the following hypotheses, which are closely related to the ones in [14], where the Strang splitting applied to a similar abstract problem with homogeneous boundary conditions is studied. In our case, we add suitable hypotheses in a such way that we are able to consider non homogeneous boundary values (cf. [4, 18]).

(A1) The boundary operator $\partial : D(A) \subset X \to Y$ is onto.

(A2) $\text{Ker}(\partial)$ is dense in $X$ and $A_0 : D(A_0) = \text{Ker}(\partial) \subset X \to X$, the restriction of $A$ to $\text{Ker}(\partial)$, is the infinitesimal generator of a $C_0$-semigroup $\{e^{tA_0}\}_{t \geq 0}$ in $X$, which type $\omega$ is assumed to be negative.

(A3) If $z \in \mathbb{C}$ satisfies $\Re(z) > 0$ and $v \in Y$, then the steady state problem

$$
\begin{align*}
    Ax &= zx, \\
    \partial x &= v,
\end{align*}
$$

(2) \quad (3)

possesses a unique solution denoted by $x = K(z)v$. Moreover, the linear operator $K(z) : Y \to D(A)$ satisfies

$$
\|K(z)v\| \leq C\|v\|,
$$

(4)

where the constant $C$ holds for any $z$ such that $Re(z) \geq \omega_0 > \omega$.

(A4) The nonlinear source $f$ belongs to $C^1([0, T] \times X, X)$.

(A5) The solution $u$ of (1) satisfies $u \in C^2([0, T], X)$, $u(t) \in D(A^2)$ for all $t \in [0, T]$ and $Au, A^2u \in C^1([0, T], X)$.

(A6) $f(t, u(t)) \in D(A)$ for all $t \in [0, T]$, and $Af(\cdot, u(\cdot)) \in C([0, T], X)$.

In the remaining of the paper, we always suppose that (A1)-(A6) are satisfied. However, we notice that we also assume more regularity in certain results which apply for Strang method.
Remark 1. From (A4), we deduce that $f : D(f) \subset [0, T] \times X \to X$ is locally Lipschitz continuous in $u$, uniformly in $t \in [0, T]$, with respect to the norm in $X$, that is,

$$\|f(t, v) - f(t, u)\| \leq L(c)\|v - u\|,$$

for $(t, u), (t, v) \in D(f)$ with $\|u\|, \|v\| \leq c$.

In order to define the Lie-Trotter and Strang splitting methods, we need to solve the nonlinear evolution equation

$$v'(t) = f(\tau + t, v(t)), \quad v(0) = v_0,$$

(6)

for several initial values $v_0$ and times $\tau > 0$. From (5), problem (6) has a unique solution, which is well defined for sufficiently small times (see Theorem 1.8.1 in [8]).

Remark 2. When problem (1) is linear, that is, when $f(t, \cdot) \equiv h(t)$, the results in [4, 18] show that, with the hypotheses (A1)-(A3), the problem

$$u'(t) = Au(t) + h(t), \quad 0 \leq t \leq T, \quad u(0) = u_0 \in X,$$

(7)

$$\partial u(t) = g(t) \in Y, \quad 0 \leq t \leq T,$$

is well posed and the solution depends continuously on data $u_0, h$, and $g$.

In order to define the time integrators which are used in this paper, we will consider initial boundary value problems which can be written as

$$u'(s) = Au(s), \quad u(0) = u_0, \quad \partial u(s) = v_0 + v_1s,$$

(8)

where $u_0 \in X$ and $v_0, v_1 \in Y$.

Assuming that $u_0 \in D(A)$ and $\partial u_0 = v_0$, the solution of (8) is given by (see e.g. [1])

$$u(t) = e^{tA_0} (u_0 - K(0)v_0) + K(0)(v_0 + v_1t) - \int_0^t e^{sA_0}K(0)v_1ds.$$

(9)

Notice that (8) is well defined for any $u_0 \in X$ and $v_0, v_1 \in Y$; therefore, it may be considered as a generalized solution of (8) even when $\partial u_0 \neq v_0$ or $u_0 \notin D(A)$. We will use this fact in order to establish the time integrator method in the following section.

Remark 3. From hypotheses (A1)-(A4), problem (1) with homogeneous boundary conditions has a unique classical solution for small enough time intervals (see Theorem 6.1.5 in [19]).

Regarding the nonhomogeneous case, we can assume that the boundary function $g : [0, T] \to Y$ satisfies $g \in C^1([0, T], Y)$ and we can look for a solution of (1) given by:

$$u(t) = v(t) + K(z)g(t), \quad t \geq 0,$$
for some fixed $\Re(z) > \omega$. Then, $v$ is solution of an IBVP with vanishing boundary values similar to the one in [12] and the well-posedness for the case of nonhomogeneous boundary values is a direct consequence if we take the abstract theory for initial boundary value problems in [4, 18] into account.

However, condition (A4) may be very strong. When $X$ is a function space with a norm $L^p$, $1 \leq p < +\infty$, and $f$ is an operator given by,

$$u \to f(u) = \phi \circ u,$$

with $\phi : \mathbb{C} \to \mathbb{C}$, [5] implies that $\phi$ is globally Lipschitz in $\mathbb{C}$. This objection disappears by considering the supremum norm, which is used in our numerical examples, where the nonlinear source is given by

$$u \to f(t, u) = \phi \circ u + h(t),$$

with $h : [0, T] \to X$, that is, $f$ is the sum of an operator like [10] and a linear term. In this way, problem [1] is well posed.

**Remark 4.** If we suppose that $A_0$ is the infinitesimal generator of an analytic semigroup, we can consider, for $\theta \in (0, 1)$ a new norm given by $\|u\|_\theta = \|(\omega I - A_0)^\theta u\|$, $\omega > 0$, when $u \in X_\theta = D((\omega I - A_0)^\theta)$. In this case, if $f$ satisfies a local Lipschitz condition in $u$ with this new norm, it is possible to obtain the well posedness of problem [1] even when $X$ is a function space with a norm $L^p$, $1 \leq p < +\infty$ (see [13]). However, this approach is not enough for our purposes since we also need to solve the nonlinear evolution equation [1].

**Example.** Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then, there exists a unique solution $h \in C(\overline{\Omega})$ of the problem

$$\begin{align*}
\Delta h &= 0 \text{ in } D(\Omega)', \\
h|_{\partial \Omega} &= \varphi \in C(\partial \Omega),
\end{align*}$$

where $D(\Omega)'$ denotes the space of distributions. We also remark that it can be proved that $h \in C^\infty(\Omega)$.

We take $X = C(\overline{\Omega})$ with the supremum norm and we consider the operator $A_0$ defined on $X$ by

$$\begin{align*}
D(A_0) &= \{u \in C_0(\Omega) : \Delta u \in X\} \\
A_0u &= \Delta u \in D(\Omega)',
\end{align*}$$

where $C_0(\Omega) = \{u \in X : u|_{\partial \Omega} = 0\}$. Then, the operator $A_0$ generates a bounded holomorphic semigroup $e^{tA_0}$ on $X$ [12, Section 2.4]. We denote $\omega < 0$ the type of this semigroup.

Now, we take $Y = C(\partial \Omega)$ and we define the linear operator

$$\begin{align*}
K : Y &\to K(Y) \subset X \\
\varphi &\to K(\varphi) = h,
\end{align*}$$
where \( h \) is the solution of (12). Then, we can define the (dense) subspace
\[
D(A) = D(A_0) \oplus K(Y),
\]
the extension of the operator \( A_0 \),
\[
A : D(A) \subset X \to X
\]
by means of \( Au = \Delta u \) for each \( u \in D(A) \), and the boundary operator
\[
\partial : D(A) \subset X \to Y,
\]
\[
u \to \partial u = u|_{\partial \Omega}.
\]
Finally, if \( z \in \mathbb{C} \) satisfies \( \Re(z) > \omega \), we define
\[
K(z) = (-A_0)(z - A_0)^{-1}K = K - z(z - A_0)^{-1}K,
\]
which satisfies \( AK(z) = AK - zA_0(z - A_0)^{-1}K = -zA_0(z - A_0)^{-1}K = zK(z) \).

Therefore, hypotheses (A1), (A2), and (A3) are satisfied.

We remark that the restriction to \( A = \Delta \) is only made for simplicity of presentation and more general elliptic operators can be considered (see [6]).

Because of hypothesis (A2), \( \{\varphi_j(tA_0)\}_{j=1}^3 \) are bounded operators for \( t > 0 \), where \( \{\varphi_j\} \) are the standard functions which are used in exponential methods [17] and which are defined by
\[
\varphi_j(tA_0) = \frac{1}{t^j} \int_0^t e^{(t-\tau)A_0} \frac{\tau^{j-1}}{(j-1)!} d\tau, \quad j \geq 1.
\]
(13)

It is well-known that they can be calculated in a recursive way through the formulas
\[
\varphi_{j+1}(z) = \frac{\varphi_j(z) - 1/j!}{z}, \quad z \neq 0, \quad \varphi_{j+1}(0) = \frac{1}{(j+1)!}, \quad \varphi_0(z) = e^z.
\]
(14)

For the time integration, we will center on exponential Lie-Trotter and Strang methods which, applied to a finite-dimensional nonlinear problem like
\[
U'(t) = MU(t) + F(t, U(t)),
\]
where \( M \) is a matrix, are described by the following formulas at each step
\[
U_{n+1} = \Psi_k F_{t_n} e^{kM} U_n,
\]
(16)
\[
U_{n+1} = \Psi_k F_{t_n} e^{kM} \Psi_{t_n}^k(U_n),
\]
(17)
where \( k > 0 \) is the time stepsize and \( \Psi_k F_{t_n} (U) \) and \( \Psi_k F_{t_n} e^{kM} \Psi_{t_n}^k(U) \) are the results of applying a certain \( p \)th-order numerical method \( (p \geq 1) \) to the following nonlinear differential problems:
\[
U'(s) = F(t_n + s, U(s)), \quad U'(s) = F(t_n + \frac{k}{2} + s, U(s)),
\]
with initial condition \( U(0) = U \) and \( t_n = nk \) for \( n \geq 0 \).
3 Time semidiscretization: exponential Lie-Trotter splitting

In this section, we give the technique to generalize Lie-Trotter exponential method, so that time order reduction is avoided even with non-vanishing and time-dependent boundary conditions. Besides, we prove the full-order of the local error of the time semidiscretization.

3.1 Description of the technique

Whenever $M$ is a matrix, $e^{sM}V$ is the solution at $t = s$ of

\[
\begin{align*}
\dot{U}(t) &= MU(t), \\
U(0) &= V.
\end{align*}
\]

More generally, matrix $M$ can be substituted by the infinitesimal generator $A_0$ of a $C_0$-semigroup in a certain Banach space $X$. Then, the corresponding semigroup is denoted by $e^{sA_0}$ and $e^{sA_0}v$, for $v \in D(A_0) \subset X$, is the solution of the corresponding abstract differential problem

\[
\begin{align*}
\dot{u}(t) &= A_0u(t), \\
u(0) &= v.
\end{align*}
\]

When $A_0$ is a linear (unbounded) operator associated to a differential operator defined on $\Omega \subset \mathbb{R}^n$, its domain $D(A_0)$ is formed by functions for which certain boundary operator vanishes on the boundary of $\Omega$ (see Example in Section 2).

Since we are interested in problems with nonvanishing boundary conditions, as those in (1), we replace the exponential matrices or semigroups with the solution of differential problems where the boundary values must be specified in a clever way. More precisely, we suggest to advance a stepsize from $u_n$ in the following way. Firstly, we consider the solution of

\[
\begin{align*}
v_n'(s) &= A v_n(s), \\
v_n(0) &= u_n, \\
\partial v_n(s) &= \partial \hat{v}_n(s),
\end{align*}
\]

where

\[
\hat{v}_n(s) = u(t_n) + sAu(t_n).
\]

Then, we consider the problem

\[
\begin{align*}
w_n'(s) &= f(t_n + s, w_n(s)), \\
w_n(0) &= v_n(k),
\end{align*}
\]

and $u^{n+1}$ is obtained advancing a time step $k \geq 0$ by means of a numerical integrator of order $p \geq 1$. That is,

\[
u^{n+1} = \Psi_k^{f,tn}(v_n(k)).
\]
Notice that we could have also started by integrating the nonlinear part of the equation and then the linear and stiff one. However, that would have led to a slightly more complicated expression for the boundary in the linear part.

**Remark 5.** In order to calculate \( \partial \hat{v}_n(s) \), apart from \( \partial u(t_n) = g(t_n) \), we also need \( \partial Au(t_n) \), for which we can use from (11),

\[
\partial Au(t_n) = \partial u'(t_n) - \partial f(t_n, u(t_n)) = g(t_n) - \partial f(t_n, u(t_n)).
\]

When the operator \( \partial \) corresponds to a Dirichlet boundary condition and the nonlinear term is given by (11),

\[
\partial f(t_n, u(t_n)) = \phi \circ g(t_n) + \partial h(t_n),
\]

and \( \partial \hat{v}_n(s) \) is exactly calculated from the given data. However, when \( \partial \) corresponds to a Robin or Neumann boundary condition, \( \partial Au(t_n) \) can only be calculated in an approximated way. For that, we write the boundary condition as

\[
\partial u = \alpha u|_{\partial \Omega} + \beta \partial_n u|_{\partial \Omega} = g, \quad \beta \neq 0,
\]

with \( \partial \Omega \) the boundary (or some part of it) of some domain \( \Omega \) and \( \partial_n \) the normal derivative to that boundary. Then, when \( f \) is again like in (11), it can be used that

\[
\partial f(t_n, u(t_n)) = \alpha[\phi(u(t_n)|_{\partial \Omega}) + h(t_n)|_{\partial \Omega}] + \beta[\phi'(u(t_n)|_{\partial \Omega})\partial_n u(t_n)|_{\partial \Omega} + \partial_n h(t_n)|_{\partial \Omega}].
\]

In this expression, \( u(t_n)|_{\partial \Omega} \) can be substituted by the numerical approximation at the previous step and \( \partial_n u(t_n)|_{\partial \Omega} \) by the result of applying the following formula which comes from (23)

\[
\partial_n u|_{\partial \Omega} = \frac{g(t_n) - \alpha u(t_n)|_{\partial \Omega}}{\beta}.
\]

### 3.2 Local error of the time semidiscretization

In order to study the local error, we consider the value which is obtained in (22) starting from \( u(t_n) \) in (19). Then, we obtain

\[
\bar{u}_{n+1} = \Psi_{k}^{t, t_n}(\bar{u}_n(k)),
\]

where \( \bar{u}_n(s) \) is the solution of

\[
\begin{align*}
\bar{v}_n'(s) & = A\bar{v}_n(s), \\
\bar{v}_n(0) & = u(t_n), \\
\partial \bar{v}_n(s) & = \partial \hat{v}_n(s).
\end{align*}
\]

with \( \hat{v}_n(s) \) that in (20).

Before bounding the local error \( \rho_{n+1} = \bar{u}_{n+1} - u(t_{n+1}) \), let us first study more thoroughly \( v_n(s) \).
Lemma 6. The solution of (24) is given by

\[ \n_n(s) = u(t_n) + sAu(t_n) + s^2 \varphi_2(sA_0)A^2u(t_n). \]

where \( \varphi_2(z) \) is defined in (13).

Proof. Notice that

\[ \n_n(s) - \hat{n}_n(s) = A\n_n(s) - Au(t_n) = A(\n_n(s) - \hat{n}_n(s)) + A(\hat{n}_n(s) - u(t_n)) \]
\[ = A(\n_n(s) - \hat{n}_n(s)) + sA^2u(t_n), \]
\[ \n_n(0) - \hat{n}_n(0) = 0, \]
\[ \partial(\n_n(s) - \hat{n}_n(s)) = 0. \]

Then,

\[ \n_n(s) = \hat{n}_n(s) + \int_0^s e^{(s-\tau)A_0}A^2u(t_n)d\tau = u(t_n) + sAu(t_n) + s^2 \varphi_2(sA_0)A^2u(t_n). \]

\[ \hat{n}_n(0) = \n_n(k). \]

\[ u(t_n) + kAu(t_n) + k^2u(t_n + u(t_n)) + O(k^2) = u(t_n + u(t_n)) + O(k^2). \]

Then,

\[ \rho_{n+1} = \n_{n+1} - u(t_{n+1}) = \Psi_{k}^{f,t_n}(\n_n(k)) - u(t_{n+1}) \]
\[ = [\Psi_{k}^{f,t_n}(\n_n(k)) - \n_n(k)] + [\n_n(k) - u(t_{n+1})] = O(k^{p+1}) + O(k^2) = O(k^2). \]

Theorem 7. Let us assume hypotheses (A1)-(A6) and that the numerical integrator \( \Psi_k \) integrates (21) with order \( p \geq 1 \) in \( X \). Then, when integrating (1) with Lie-Trotter method using the technique (19)-(22), the local error satisfies

\[ \rho_{n+1} = \n_{n+1} - u(t_{n+1}) = O(k^2). \]

Proof. Denoting by \( \n_n(s) \) the solution of

\[ \n_n'(s) = f(t_n + s, \n_n(s)) \]
\[ \n_n(0) = \n_n(k), \]

it happens that

\[ \n_n(k) = \n_n(k) + kf(t_n, \n_n(k)) + O(k^2) \]
\[ = u(t_n) + kAu(t_n) + k^2u(t_n, u(t_n)) + O(k^2) = u(t_{n+1}) + O(k^2). \]

Then,

\[ \rho_{n+1} = \n_{n+1} - u(t_{n+1}) = \Psi_{k}^{f,t_n}(\n_n(k)) - u(t_{n+1}) \]
\[ = [\Psi_{k}^{f,t_n}(\n_n(k)) - \n_n(k)] + [\n_n(k) - u(t_{n+1})] = O(k^{p+1}) + O(k^2) = O(k^2). \]
4 Time semidiscretization: exponential Strang splitting

With the same idea as in Section 3, we describe now how to generalize Strang exponential method in order to fight against order reduction in time. Instead of achieving order 3 for the local error (as when integrating non-stiff ODEs), we will just achieve order 2 for it. This is due to the fact that we want to guarantee that the boundary of the intermediate evolutionary partial differential equation problem can be calculated in terms of data. However, as we will see in Sections 7.3 and 8, that will mean in practice no order reduction for the global error because of a summation-by-parts argument.

Notice that, instead of starting with the integration of the linear part, as with Lie-Trotter method, we start with that of the nonlinear and smooth one. This is because, in such a way, just one stiff differential evolutionary problem per step arises for which we must suggest a boundary.

4.1 Description of the technique

For the time integration of (1), we firstly consider the problem

\[ v'_{n}(s) = f(t_{n} + s, v_{n}(s)), \]
\[ v_{n}(0) = u_{n}, \]

and denote by \( \Psi_{\frac{k}{2}}^{f,t_{n}}(u_{n}) \) the numerical approximation of this problem after time \( \frac{k}{2} \).

Secondly, we consider

\[ w'_{n}(s) = Aw_{n}(s), \]
\[ w_{n}(0) = \Psi_{\frac{k}{2}}^{f,t_{n}}(u_{n}), \]
\[ \partial w_{n}(s) = \partial \hat{w}_{n}(s), \]

where

\[ \hat{w}_{n}(s) = u(t_{n}) + \frac{k}{2}f(t_{n}, u(t_{n})) + sAu(t_{n}), \]

which comes from approximating \( v_{n}(\frac{k}{2}) + sAv_{n}(\frac{k}{2}) \). Thirdly, by considering

\[ z'_{n}(s) = f(t_{n} + \frac{k}{2} + s, z_{n}(s)), \]
\[ z_{n}(0) = w_{n}(k), \]

and advancing \( \frac{k}{2} \) with the numerical integrator, we obtain

\[ u_{n+1} = \Psi_{\frac{k}{2}}^{f,t_{n}+\frac{k}{2}}(w_{n}(k)). \]

Remark 8. Notice that the boundary values in (26) can be exactly or approximately calculated in terms of data under the same considerations of Remark 5.
4.2 Local error of the time semidiscretization

In order to study the local error, we consider the value \( \overline{w}_{n+1} \) which is obtained in (29) starting from \( u_n = u(t_n) \) in (25). Then, denoting by \( \overline{w}_n(s) \) the solution of (26) starting from \( \Psi_{\frac{k}{2}}^{f,t_n}(u(t_n)) \), we have the following result:

Lemma 9.

\[
\overline{w}_n(s) = u(t_n) + \frac{k}{2}f(t_n, u(t_n)) + sAu(t_n) + e^{sA_0}(\Psi_{\frac{k}{2}}^{f,t_n}(u(t_n)) - u(t_n) - \frac{k}{2}f(t_n, u(t_n))) \\
+ \frac{k}{2}s\varphi_1(sA_0)Af(t_n, u(t_n)) + s^2\varphi_2(sA_0)A^2u(t_n).
\]

Proof. It can be noticed that

\[
\overline{w}_n'(s) = A\overline{w}_n(s) - Au(t_n) = A(\overline{w}_n(s) - \hat{w}_n(s)) + A\hat{w}_n(s) - Au(t_n),
\]

\[
\overline{w}_n(0) - \hat{w}_n(0) = \Psi_{\frac{k}{2}}^{f,t_n}(u(t_n)) - u(t_n) - \frac{k}{2}f(t_n, u(t_n)),
\]

\[
\partial(\overline{w}_n(s) - \hat{w}_n(s)) = 0.
\]

Then,

\[
\begin{align*}
\overline{w}_n(s) - \hat{w}_n(s) &= e^{sA_0}(\overline{w}_n(0) - \hat{w}_n(0)) + \int_0^s e^{(s-\tau)A_0}\left[\frac{k}{2}Af(t_n, u(t_n)) + \tau A^2u(t_n)\right]d\tau \\
&= e^{sA_0}\left(\Psi_{\frac{k}{2}}^{f,t_n}(u(t_n)) - u(t_n) - \frac{k}{2}f(t_n, u(t_n))\right) \\
&+ \frac{k}{2}s\varphi_1(sA_0)Af(t_n, u(t_n)) + s^2\varphi_2(sA_0)A^2u(t_n).
\end{align*}
\]

\[\square\]

Theorem 10. Let us assume that hypotheses (A1)-(A6) are satisfied, and that the numerical integrator \( \Psi_k \) integrates (25) and (26) with order \( p \geq 1 \) in \( X \). Then, when integrating (4) with Strang method using the technique (25)-(28), the local error satisfies

\[
\rho_{n+1} = \overline{w}_{n+1} - u(t_{n+1}) = O(k^2).
\]

Proof. Notice that

\[
\begin{align*}
\rho_{n+1} &= \overline{u}_{n+1} - u(t_{n+1}) = [\Psi_{\frac{k}{2}}^{f,t_{n+\frac{1}{2}}}(\overline{w}_n(k)) - \overline{z}_n(\frac{k}{2})] + [\overline{z}_n(\frac{k}{2}) - u(t_{n+1})] \\
&= \overline{w}_n(k) + \frac{k}{2}f(t_n + k, \overline{w}_n(k)) - u(t_n) - kAu(t_n) - kf(t_n, u(t_n)) \\
&+ O(k^{p+1}) + O(k^2) \\
&= O(k^2),
\end{align*}
\]

(30)

where the last equality is deduced from Lemma 9 and the fact that \( p \geq 1 \). \[\square\]
Moreover, assuming a bit more regularity of the functions \( u \) and \( f \) and a bit more accuracy of the time numerical integrator for the nonlinear part, we have the following result:

**Theorem 11.** Whenever, apart from hypotheses (A1)-(A6), \( u \in C^3([0, T], X), f \in C^2([0, T] \times X, X) \) and \( f_u(\cdot, u(\cdot)) f(\cdot, u(\cdot)), f_u(\cdot, u(\cdot)) A u(\cdot) \in C([0, T], X) \), when integrating (7) with Strang method using the technique (25)-(29) with a numerical integrator \( \Psi_k \) which is of order \( p \geq 2 \) for problems (27) and (28), the local error satisfies

\[
A_0^{-1} \rho_{n+1} = O(k^3).
\]

**Proof.** Under the new assumptions, by explicitly writing the term in \( k^2 \) in (30), we have

\[
\rho_{n+1} = \frac{k^2}{4} f_t(t_n, u(t_n)) + \frac{k^2}{4} f_u(t_n, u(t_n)) f(t_n, u(t_n)) + \frac{k^2}{2} f_u(t_n, u(t_n)) A u(t_n)
\]

\[
+ \frac{k^2}{8} [f_t(t_n, u(t_n)) + f_u(t_n, u(t_n)) f(t_n, u(t_n))]
\]

\[
+ \frac{k^2}{8} e^{k A_0} [f_t(t_n, u(t_n)) + f_u(t_n, u(t_n)) f(t_n, u(t_n))]
\]

\[
+ \frac{k^2}{2} \varphi_1(k A_0) A f(t_n, u(t_n)) + k^2 \varphi_2(k A_0) A^2 u(t_n) - \frac{k^2}{2} u''(t_n) + O(k^3).
\]

By applying now \( A_0^{-1} \), considering (14) and simplifying terms and the notation for the sake of brevity,

\[
A_0^{-1} \rho_{n+1} = k^2 A_0^{-1} \left[ \frac{3}{8} f_t + \frac{3}{8} f_u f + \frac{1}{2} f_u A u - \frac{1}{2} u'' \right] + \frac{k^2}{8} (k \varphi_1(k A_0) + A_0^{-1})(f_t + f_u f)
\]

\[
+ \frac{k^2}{2} (k \varphi_2(k A_0) + A_0^{-1}) A f + k^2 (k \varphi_3(k A_0) + \frac{1}{2} A_0^{-1}) A^2 u + O(k^3)
\]

\[
= k^2 A_0^{-1} \left[ \frac{1}{2} (f_t + f_u f) + \frac{1}{2} f_u A u + \frac{1}{2} A f + \frac{1}{2} A^2 u - \frac{1}{2} u'' \right] + O(k^3),
\]

where, in order to see that the term in bracket vanishes, it suffices to differentiate (11) once with respect to time.

\[\square\]

### 5 Spatial discretization

Following the example in Section 2, we take \( X = C(\Omega) \) with the maximum norm and we consider a certain grid \( \Omega_h \) (of \( \Omega \)) over which the approximated numerical solution will be defined. In this way, this numerical approximation belongs to \( C^N \), where \( N \) is the number of nodes in the grid, endowed with the the maximum norm \( \| u_h \|_h = \|[u_1, \ldots, u_N]^T\|_h = \max_{1 \leq i \leq N} |u_i| \).

Notice that, usually, when considering Dirichlet boundary conditions, nodes on the boundary are not considered while, when using Neumann or Robin boundary conditions, the nodes on the boundary are taken into account.
In that sense, we consider the projection operator
\[ P_h : X \to \mathbb{C}^N, \] (32)
which takes a function to its values over the grid \( \Omega_h \). On the other hand, the operator \( A \), when applied over functions which satisfy a certain condition on the boundary \( \partial u = g \), is discretized by means of an operator
\[ A_{h,g} : \mathbb{C}^N \to \mathbb{C}^N, \]
which takes the boundary values into account. More precisely,
\[ A_{h,g}U_h = A_{h,0}U_h + C_hg, \]
where \( A_{h,0} \) is the matrix which discretizes \( A_0 \) and \( C_h : Y \to \mathbb{C}^N \) is another operator, which is the one which contains the information on the boundary.

We also assume that the source function \( f \) has also sense as function from \([0, T] \times \mathbb{C}^N \) on \( \mathbb{C}^N \) and, for each \( t \in [0, T] \) and \( u \in X \),
\[ P_h f(t, u) = f(t, P_h u). \] (33)

This fact is obvious when \( f \) is given by (11). By using this, the following semidiscrete problem arises after discretising (1) in space,
\[ U'_h(t) = A_{h,0}U_h(t) + C_h g(t) + f(t, U_h(t)), \]
\[ U_h(0) = P_h u(0), \] (34)

The subsequent analysis is carried out under the following hypotheses:

(H1) The matrix \( A_{h,0} \) satisfies
\begin{enumerate}
  \item \( \| e^{tA_{h,0}} \|_h \leq 1 \),
  \item \( A_{h,0} \) is invertible and \( \| A_{h,0}^{-1} \|_h \leq C \) for some constant \( C \) which does not depend on \( h \),
\end{enumerate}
where \( \| \cdot \|_h \) is the norm operator obtained from the maximum norm in \( \mathbb{C}^N \).

(H2) We define the elliptic projection \( R_h : D(A) \to \mathbb{C}^N \) as the solution of
\[ A_{h,0}R_h u + C_h \partial u = P_h Au. \] (35)

We assume that there exists a subspace \( Z \subset D(A) \), such that, for \( u \in Z \),
\begin{enumerate}
  \item \( A_0^{-1} u \in Z \) and \( e^{tA_0} u \in Z \), for \( t \geq 0 \),
  \item for some \( \varepsilon_h \) and \( \eta_h \) which are both small with \( h \),
\end{enumerate}
\[ \| A_{h,0}(P_h u - R_h u) \| \leq \varepsilon_h \| u \|_Z, \quad \| P_h u - R_h u \| \leq \eta_h \| u \|_Z. \] (36)

(Although obviously, because of (H1), \( \eta_h \) could be taken as \( C \varepsilon_h \), for some discretizations \( \eta_h \) can decrease more quickly with \( h \) than \( \varepsilon_h \) and that leads to better error bounds in the following section.)
(H3) The nonlinear source \( f \) belongs to \( C^1([0, T] \times X_h, X_h) \) and the derivative with respect to the variable \( u_h \) is uniformly bounded in a neighbourhood of the solution where the numerical approximation stays.

**Remark 12.** Hypothesis (H1a) can be deduced in our numerical experiments by using the logarithmic norm of matrix \( A_{h,0} \), which is given by [9]

\[
\mu(A_{h,0}) = \lim_{\tau \to 0^+} \frac{\|I + \tau A_{h,0}\| - 1}{\tau}.
\]

From the logarithmic norm, we obtain the bound

\[
\|e^{t A_{h,0}}\| \leq e^{\mu(A_{h,0})}.
\]

In particular, with the maximum norm (\( \|u\|_\infty = \max_i |u_i| \), which is the one used in our examples, we have

\[
\mu_\infty(A) = \max_i \left( \Re a_{ii} + \sum_{j \neq i} |a_{ji}| \right),
\]

which can be easily checked. For the matrices \( A_{h,0} \) in our numerical experiments, it is easily seen that \( \mu(A_{h,0})_\infty = 0 \) and (H1a) holds.

**Remark 13.** From (H3), the nonlinear term \( f \) satisfies, for some Lipschitz constant \( L \) independent of \( t \in [0, T] \) and the maximum norm,

\[
\|f(t, v_h) - f(t, u_h)\|_h \leq L\|v_h - u_h\|_h,
\]

when \( u_h, v_h \) belong to a compact set. In particular, we will be interested in considering as this set a neighborhood of the exact solution where the numerical approximation stays.

### 6 Full discretization: exponential Lie-Trotter splitting

#### 6.1 Final formula for the implementation

We apply the above space discretization to the evolutionary problems [19] and [21] and we obtain \( V_{h,n}(s), W_{h,n}(s) \) in \( \mathbb{C}^N \) as the solutions of

\[
\begin{align*}
V_{h,n}'(s) &= A_{h,0}V_{h,n}(s) + C_h \partial \hat{v}_n(s), \\
V_{h,n}(0) &= U_{h,n},
\end{align*}
\]

where \( \hat{v}_n(s) \) is that in [20], \( U_{h,n} \in \mathbb{C}^N \) is the numerical solution in the interior of the domain after full discretization at \( n \) steps, and

\[
\begin{align*}
W_{h,n}'(s) &= f(t_n + s, W_{h,n}(s)), \\
W_{h,n}(0) &= V_{h,n}(k).
\end{align*}
\]

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By using the variations of constants formula and the definition of the functions $\varphi_1$ and $\varphi_2$ in (13),

\[
V_{h,n}(k) = e^{kA_{h,0}}U_{h,n} + \int_0^k e^{(k-s)A_{h,0}}[C_h \partial[u(t_n) + sAu(t_n)]] ds
\]

\[
= e^{kA_{h,0}}U_{h,n} + k\varphi_1(kA_{h,0})C_h g(t_n) + k^2 \varphi_2(kA_{h,0})C_h (g'(t_n) - \partial f(t_n, u(t_n))),(40)
\]

and the numerical solution at step $n+1$ is therefore given by

\[
U_{h,n+1} = \Psi_{k}^{f,t_n}(V_{h,n}(k)), (41)
\]

where $\Psi_{k}^{f,t_n}$ stands for the previously mentioned numerical integrator applied to (39).

Moreover, we will take, as initial condition,

\[
U_{h,0} = P_h u(0). (42)
\]

**Remark 14.** Notice that, when

\[
\partial u(t_n) = \partial Au(t_n) = 0,
\]

it is also deduced from (1) that $\partial f(t_n, u(t_n)) = 0$. Therefore, formulas (40) - (41) just reduce to the standard time integration with Lie-Trotter method of the differential system which arises after discretizing (1) directly in space (see (34)):

\[
U'_h(t) = A_{h,0}U_h(t) + f(t, U_h(t)).
\]

Because of that, with the results which follow, we will be implicitly proving that there is no order reduction in the local error with the standard Lie-Trotter method under these assumptions.

**Remark 15.** We notice that, when $k$ is fixed, $e^{kA_{h,0}}$ and $\varphi_j(kA_{h,0})$ could be calculated once and for all at the very beginning. Besides, as better explained in the numerical experiments, $C_h$ will be represented by a matrix of dimension $O(\hat{N}^d) \times O(\hat{N}^{d-1})$ where $d$ is the dimension of the problem and $\hat{N}$ the number of grid points in each direction. Then, $\varphi_j(kA_{h,0})C_h$ ($j = 1, 2$) will be represented by matrices of the same order and therefore the computational cost of calculating the product of those matrices times the information on the boundary values is $O(\hat{N}^{2d-1})$, which is negligible compared with $O(\hat{N}^{2d})$, which corresponds to the calculation of the product of $e^{kA_{h,0}}$ times a vector of size $O(\hat{N}^d)$. On the other hand, for fixed and variable timestepsize $k$, Krylov techniques can also be used to calculate the terms in (40) without explicitly calculating $e^{kA_{h,0}}$, $\varphi_j(kA_{h,0})$ ($j = 1, 2$). As suggested in [13], it seems in principle cheaper to calculate the terms containing the $\varphi_j$-functions than those corresponding to the exponentials.
6.2 Local errors

In order to define the local error, we consider

\[ U_{n+1}^h = \Psi_k f_{t_n} (V_{h,n}(k)), \]  

(43)

where \( V_{h,n}(s) \) is the solution of

\[
\begin{align*}
\nabla_{h,n}(s) &= A_{h,0} V_{h,n}(s) + C_h \partial \hat{v}_n(s), \\
\nabla_{h,n}(0) &= P_h u(t_n). 
\end{align*}
\]

(44)

with \( \hat{v}_n(s) \) that in (20). We now define the local error at \( t = t_n \) as

\[ \rho_{h,n} = P_h u(t_n) - U_{h,n}, \]

and study its behaviour in the following theorem.

**Theorem 16.** Let us assume hypotheses (A1)-(A6), that \( \Psi_k \) integrates (39) with order \( p \geq 1 \) and (H1)-(H3). Then, when integrating (1) with Lie-Trotter method as described in (40)-(41), whenever \( u \) satisfies

\[ u, A u, A^2 u \in C([0, T], Z), \]

(45)

for the space \( Z \) in (H3), the local error after full discretization satisfies

\[
\begin{align*}
\rho_{h,n+1} &= O(k \varepsilon_h + k^2), \\
A_{h,0}^{-1} \rho_{h,n+1} &= O(k \eta_h + k^2).
\end{align*}
\]

(46)

where \( \varepsilon_h \) and \( \eta_h \) are those in (36).

**Proof.** Notice that

\[ U_{h,n+1} = \Psi_k f_{t_n} (V_{h,n}(k)) = V_{h,n}(k) + k f(t_n, V_{h,n}(k)) + O(k^2). \]

On the other hand, making the difference between (44) and (24) multiplied by \( P_h \),

\[
\begin{align*}
\nabla_{h,n}'(s) - P_h \nabla_n(s) &= A_{h,0} (\nabla_{h,n}(s) - P_h \nabla_n(s) + A_{h,0} (P_h - R_h) \nabla_n(s), \\
\nabla_{h,n}(0) - P_h \nabla_n(0) &= 0.
\end{align*}
\]

Then,

\[
\begin{align*}
\nabla_{h,n}(k) &= P_h \nabla_n(k) + \int_0^k e^{(k-s)A_{h,0}} A_{h,0} (P_h - R_h) \nabla_n(s) ds \\
&= P_h u(t_n) + kP_h Au(t_n) + O(k^2) + O(k \varepsilon_h),
\end{align*}
\]

(47)

where the last equality comes from Lemma 6 (H1) and (H2). From the definition of \( \rho_{h,n} \),

\[
\begin{align*}
\rho_{h,n+1} &= P_h u(t_{n+1}) - U_{h,n+1} = P_h (u(t_{n+1}) - \nabla_n + (P_h \nabla_n - U_{h,n+1}) \\
&= P_h \rho_{n+1} + P_h \nabla_n - \nabla_{h,n}(k) - k f(t_n, V_{h,n}(k)) + O(k^2).
\end{align*}
\]
Considering now Theorem 7, (32), (47), (H3) and the fact that, because of Lemma 6,
\[ u_{n+1} = \Psi_{f,t}^n(u_n(k)) = v_n(k) + kf(t_n, u_n(k)) + O(k^2), \]
the first part of the theorem is proved. To prove the second bound in (46), it suffices to apply the uniformly bounded matrix \( A_{h,0}^{-1} \) to the above formulas and to take the second part of (36) into account.

\[ \square \]

6.3 Global errors

We now study the global errors at \( t = t_n \), which are given by
\[ e_{h,n} = P_h u(t_n) - U_{h,n}. \]

**Theorem 17.** Under the same assumptions of Theorem 16, and assuming also that \( \partial f(t_n, u(t_n)) \) can be calculated exactly from data according to Remark 5, the global error which turns up when integrating (1) through formulas (40)-(42) satisfies
\[ e_{h,n} = O(k + \varepsilon_h), \]
where \( \varepsilon_h \) is that in (36).

**Proof.** It suffices to notice that
\[ e_{h,n+1} = [P_h u(t_{n+1}) - \overline{U}_{h_n}^{n+1}] + [\overline{U}_{h_n}^{n+1} - U_{h,n+1}] \]
\[ = \rho_{h,n+1} + \Psi_{f,t}^{n+1}(\overline{V}_{h,n}(k)) - \Psi_{f,t}^n(V_{h,n}(k)) \]
\[ = \rho_{h,n+1} + \overline{W}_{h,n}(k) - W_{h,n}(k) + O(k^{p+1}), \]
where \( \overline{W}_{h,n}(k) \) is the solution of (39) with initial condition \( \overline{V}_{h,n}(k) \), and the definition of \( \rho_{h,n+1} \), (41) and (43) have been used. Then, considering (39),
\[ \overline{W}_{h,n}^n(t) - W_{h,n}^n(s) = f(t_n + s, \overline{W}_{h,n}(s)) - f(t_n + s, W_{h,n}(s)), \]
\[ \overline{W}_{h,n}(0) - W_{h,n}(0) = \overline{V}_{h,n}(k) - V_{h,n}(k), \]
and
\[ \overline{W}_{h,n}(t) - W_{h,n}(t) = \overline{W}_{h,n}(0) - W_{h,n}(0) + \int_0^t [f(t_n + s, \overline{W}_{h,n}(s)) - f(t_n + s, W_{h,n}(s))] ds \]
\[ = \overline{V}_{h,n}(k) - V_{h,n}(k) + \int_0^t [f(t_n + s, \overline{W}_{h,n}(s)) - f(t_n + s, W_{h,n}(s))] ds. \]

Taking norms,
\[ \|\overline{W}_{h,n}(t) - W_{h,n}(t)\| \leq \|\overline{V}_{h,n}(k) - V_{h,n}(k)\| + \int_0^t \|f(t_n + s, \overline{W}_{h,n}(s)) - f(t_n + s, W_{h,n}(s))\| ds \]
\[ \leq \|\overline{V}_{h,n}(k) - V_{h,n}(k)\| + \int_0^t L\|\overline{W}_{h,n}(s) - W_{h,n}(s)\| ds, \quad (48) \]
where (H3) has been used. We can then apply Gronwall lemma and deduce that
\[
\| W_{h,n}(t) - W_{h,n}(t) \| \leq e^{Lt} \| V_{h,n}(k) - V_{h,n}(k) \|.
\]
Moreover,
\[
W_{h,n}(k) - W_{h,n}(k) = V_{h,n}(k) - V_{h,n}(k) + \int_0^k [f(t_n + s, W_{h,n}(s)) - f(t_n + s, W_{h,n}(s))] ds
\]
where
\[
\| E(V_{h,n}(k), V_{h,n}(k)) \| \leq L \int_0^k e^{sL} \| V_{h,n}(k) - V_{h,n}(k) \| ds \leq kC \| V_{h,n}(k) - V_{h,n}(k) \|,
\]
for a constant \( C \) which is independent of \( k \), when \( k \) is small enough.
On the other hand, by making the difference between (38) and (44),
\[
V_{h,n}'(s) - V_{h,n}'(s) = A_{h,0}(V_{h,n}(s) - V_{h,n}(s)),
\]
from what
\[
V_{h,n}(k) - V_{h,n}(k) = e^{kA_{h,0}}(P_h u(t_n) - U^n_h),
\]
and then
\[
e_{h,n+1} = e^{kA_{h,0}}e_{h,n} + \rho_{h,n+1} + k\tilde{E}_h(U^n_h, P_h u(t_n)),
\]
where, for some constant \( \bar{C} \),
\[
\| \tilde{E}_h(U^n_h, P_h u(t_n)) \| \leq \bar{C} \| e_{h,n} \|.
\]
From here,
\[
e_{h,n} = e^{nkA_{h,0}}e_{h,0} + \sum_{l=1}^{n} e^{(n-l)kA_{h,0}}\rho_{h,l} + k\sum_{l=0}^{n-1} e^{(n-l-1)kA_{h,0}}\tilde{E}_h(U^l_h, P_h u(t_l)).
\]
As \( e_{h,0} = P_h u(0) - U_{h,0} = 0 \), by taking norms,
\[
\| e_{h,n} \| \leq O(k + \varepsilon_h) + k\bar{C} \sum_{l=0}^{n-1} \| e_{h,l} \|,
\]
and using the discrete Gronwall lemma, the result follows.

Another finer result is the following, which will be very useful for non-Dirichlet boundary conditions. When \( \partial\hat{v}_n(s) \) is numerically approximated through \( U_{h,n} \), according to the comments made in Remark 5, we will denote by \( C_{h,n}^*(U_{h,n}) \) to the vector which approximates \( C_h \partial f(t_n, u(t_n)) \).
Theorem 18. Let us assume the same hypotheses of Theorem 16, that \( u \) belongs to \( C^3([0, T], X) \) and \( Af(\cdot, u(\cdot)), f_t(\cdot, u(\cdot)), f_u(\cdot, u(\cdot)) \) to \( C^1([0, T], X) \), and also that there exist constants \( C \) and \( C' \), independent of \( h \), such that

\[
\|A_{h,0}^{-1}[C_{h,n}^*(U^n_h) - C_h\partial f(t_n, u(t_n))]\| \leq C\|U^n_h - P_hu(t_n)\|, \quad (53)
\]

\[
\|kA_{h,0}\sum_{r=1}^{n-1}e^{rkA_{h,0}}\| \leq C', \quad 0 \leq nk \leq T. \quad (54)
\]

Then, the global error satisfies

\[
\varepsilon_{h,n} = O(k + \eta_h + k\varepsilon_h),
\]

where \( \eta_h \) and \( \varepsilon_h \) are those in (52).

Proof. The proof is very similar to that of Theorem 17 with the difference that now (50) must be substituted by

\[
V'_{h,n}(s) = A_{h,0}(V_{h,n}(s) - V_{h,n}(s)) - s[C_h\partial f(t_n, u(t_n)) - C_{h,n}^*(U_{h,n})].
\]

Therefore,

\[
V_{h,n}(k) - V_{h,n}(k) = e^{kA_{h,0}}(P_hu(t_n) - U^n_h) - k^2\varphi_2(kA_{h,0})[C_h\partial f(t_n, u(t_n)) - C_{h,n}^*(U_{h,n})]
\]

\[
= e^{kA_{h,0}}(P_hu(t_n) - U^n_h) - k[\varphi_1(kA_{h,0}) - I]A_{h,0}^{-1}[C_h\partial f(t_n, u(t_n)) - C_{h,n}^*(U_{h,n})]
\]

\[
= e^{kA_{h,0}}(P_hu(t_n) - U^n_h) + O(k\|\varepsilon_{h,n}\|),
\]

where the definition of \( \varphi_1 \) in (13) has been considered as well as (14) and (53). From this, (52) still applies for some other function \( \tilde{E}_h \) also satisfying (51). Then, we write one of the terms in (52) as

\[
\sum_{j=1}^{n} e^{(n-j)A_{h,0}}\rho_{h,j} = \left( \sum_{r=1}^{n-1}e^{rkA_{h,0}} \right)\rho_{h,1} + \sum_{j=2}^{n-1} \left( \sum_{r=1}^{j-1}e^{rkA_{h,0}} \right)(\rho_{h,n-j+1} - \rho_{h,n-j}) + \rho_{h,n}. \quad (55)
\]

As the first term in this decomposition can be written as

\[
\left( \sum_{r=1}^{n-1}e^{rkA_{h,0}} \right)\rho_{h,1} = (kA_{h,0}\sum_{r=1}^{n-1}e^{rkA_{h,0}})A_{h,0}^{-1}\rho_{h,1}, \quad (56)
\]

applying (51) and Theorem 16 this term is proved to be \( O(k + \eta_h) \). As for the second term in (55), because of (A4) and (A5), the term in \( k^2 \) in \( A_{0}^{-1}\rho_{n+1} \) is differentiable with respect to time \( t_n \) and therefore, \( A_{0}^{-1}(\rho_{n+1} - \rho_{n-j}) = O(k^3) \). When this is used in
the local error for the full discretization, \( A_{h,0}^{-1}(\rho_{h,n-j+1} - \rho_{h,n-j}) = O(k^3 + k^2\eta_h) \), from what
\[
\sum_{j=2}^{n-1} \left( \sum_{r=1}^{j-1} e^{r k A_{h,0}} \right) (\rho_{h,n-j+1} - \rho_{h,n-j}) = \sum_{j=2}^{n-1} \left( k A_{h,0} \sum_{r=1}^{j-1} e^{r k A_{h,0}} \right) \frac{1}{k} A_{h,0}^{-1}(\rho_{h,n-j+1} - \rho_{h,n-j}) = O(k + \eta_h).
\]
Finally, using Theorem [17] for the last term in (55), it is clear that
\[
\sum_{l=1}^{n} e^{k(n-l) A_{h,0}^\prime} \rho_{h,l} = O(k + \eta_h + k\varepsilon_h),
\]
and the proof of the theorem follows by applying discrete Gronwall lemma to
\[
\|e_{h,n}\|_h \leq O(k + \eta_h + k\varepsilon_h) + k\bar{C} \sum_{l=0}^{n-1} \|e_{h,l}\|_h.
\]

Remark 19. Bound (54) has been proved in [16] for analytic semigroups, covering the case in which the linear operator in (1) corresponds to that of a parabolic problem.

7 Full discretization: exponential Strang splitting

7.1 Final formula for the implementation

Firstly, we apply the space discretization in Section 5 to the evolutionary problem (25) and we obtain \( V_{h,n}(s) \in \mathbb{C}^N \) as the solution of
\[
V_{h,n}'(s) = f(t_n + s, V_{h,n}(s)),
V_{h,n}(0) = U_{h,n}.
(57)
\]
We will have to use the numerical integrator \( \Psi \) in order to approximate the solution of this problem. Then, we define
\[
V_h^n = \Psi_{\frac{t_n}{2}}(U_h^n).
(58)
\]
As a second step, discretizing (26), we consider \( W_{h,n}(s) \in \mathbb{C}^N \) as the solution of
\[
W_{h,n}'(s) = A_{h,0} W_{h,n}(s) + C_h \partial \hat{w}_n(s),
W_{h,n}(0) = V_h^n.
(59)
\]
where \( \hat{w}_n(s) \) is that in (27). By using the variations of constants formula and the definition of the functions \( \varphi_1 \) and \( \varphi_2 \) in (13), we can solve this problem exactly and we
\[ W_{h,n}(k) = e^{kA_{h,0}V_h^n} \int_0^k e^{(k-s)A_{h,0}}C_h \partial \tilde{w}_n(s) ds \]
\[ = e^{kA_{h,0}V_h^n} + kA_{h,0}C_h[g(t_n) + \frac{k}{2} \partial f(t_n, u(t_n))] \]
\[ + k^2 \varphi_2(kA_{h,0})C_h[g'(t_n) - \partial f(t_n, u(t_n))] \]  
(60)

Finally, from (28), we consider \( Z_{h,n}(s) \in \mathbb{C}^N \) as the solution of
\[ Z_{h,n}'(s) = f(t_n + \frac{k}{2} + s, Z_{h,n}(s)), \]  
(61)
\[ Z_{h,n}(0) = W_{h,n}(k), \]
and, numerically integrating this problem, we obtain
\[ U_{n+1}^h = \Psi_{\frac{k}{2}}^{f,t_n+\frac{k}{2}}(W_{h,n}(k)). \]  
(62)

**Remark 20.** Similar comments to those in Remark 14 apply here. Therefore, when \( \partial u(t) = \partial Au(t) = 0 \), with the results which follow we will be implicitly proving that the standard discretization with Strang method gives to rise to order 2 for the local error.

### 7.2 Local error

In order to define the local error, we consider
\[ \overline{U}_{h,n+1} = \Psi_{\frac{k}{2}}^{f,t_n+\frac{k}{2}}(W_{h,n}(k)), \]
where \( W_{h,n}(s) \) is the solution of
\[ W_{h,n}'(s) = A_{h,0}W_{h,n}(s) + C_h \partial \tilde{w}_n(s), \]  
(63)
\[ W_{h,n}(0) = \Psi_{\frac{k}{2}}^{P_h,f,t_n}(P_h u(t_n)), \]
and \( \tilde{w}_n(s) \) is that in (27).

Then, for the local error \( \rho_{h,n+1} = P_h u(t_{n+1}) - U_{h,n+1} \), we have the following results:

**Theorem 21.** Let us assume the same hypotheses of Theorem 11, also that \( f \in C^2([0,T] \times X_h, X_h) \), that \( \Psi_k \) integrates (67) with order \( p \geq 2 \) and (H1)-(H3). Then, when integrating (7) with Strang method as described in (58), (60) and (62), whenever \( u \) and \( f \) satisfy
\[ u(t_n), \quad Au(t_n), \quad A^2 u(t_n), \quad f(t_n, u(t_n)), \quad Af(t_n, u(t_n)) \in Z, \]  
(64)
for the space \( Z \) in (H2) and \( \Psi_{\frac{k}{2}}^{f,t_n} \) leaves this space invariant, the local error after full discretization satisfies
\[ \rho_{h,n+1} = O(k\varepsilon_h + k^2), \quad A_{h,0}^{-1}\rho_{h,n+1} = O(k\eta_h + k^2\varepsilon_h + k^3), \]
where \( \varepsilon_h \) and \( \eta_h \) are those in (36).
Proof. Notice that

\[ \rho_{h,n+1} = P_h u(t_n) - U_{h,n+1} = P_h \rho_{n+1} + P_h \bar{u}_{n+1} - U_{h,n+1} \]

\[ = P_h \rho_{n+1} + P_h \Psi_{\frac{f,t}{2}}^{f,t_{n+\frac{1}{2}}} (\bar{u}_{n}(k)) - \Psi_{\frac{f,t}{2}}^{f,t_{n+\frac{1}{2}}} (\bar{W}_{h,n}(k)) \]

\[ = P_h \rho_{n+1} + P_h \bar{u}_{n}(k) - \bar{W}_{h,n}(k) + k f(t_n + \frac{k}{2}, P_h \bar{u}_{n}(k)) - \frac{k}{2} f(t_n + \frac{k}{2}, \bar{W}_{h,n}(k)) \]

\[ + \frac{k^2}{4} [(f_t + f)u(t_n + \frac{k}{2}, P_h \bar{u}_{n}(k)) - (f_t + f)u(t_n + \frac{k}{2}, \bar{W}_{h,n}(k))] + O(k^3) \]

where we have used (33), the fact that \( \Psi_{\frac{f,t}{2}}^{f,t_{n+\frac{1}{2}}} \) integrates (63) with order \( p \geq 2 \) and that \( f \in C^2([0,T] \times X_h, X_h) \). Now, from (33) and the definition of \( \bar{w}_{n}(s) \) in Subsection 4.2

\[ P_h \bar{u}_{n}(s) - \bar{W}_{h,n}(s) = P_h A \bar{u}_{n}(s) - A_{h,0} \bar{W}_{h,n}(s) - C_h \partial \bar{w}_{n}(s) \]

\[ = A_{h,0}(P_h \bar{u}_{n}(s) - \bar{W}_{h,n}(s)) + A_{h,0}(R_h - P_h) \bar{w}_{n}(s), \]

where the last term is \( O(\varepsilon_h) \) according to Lemma 9 (63) and (H2). Moreover, again by (33),

\[ P_h \bar{u}_{n}(0) - \bar{W}_{h,n}(0) = P_h \Psi_{\frac{f,t}{2}}^{f,t_{n}} (u(t_n)) - \Psi_{\frac{f,t}{2}}^{f,t_{n}} (P_h u(t_n)) = 0. \]

Then,

\[ P_h \bar{u}_{n}(k) - \bar{W}_{h,n}(k) = \int_0^k e^{(k-s)A_{h,0}} A_{h,0}(R_h - P_h) \bar{w}_{n}(s) ds = O(k \varepsilon_h), \]

and the first result follows by using Theorem 10.

For the second result, applying \( A_{h,0}^{-1} \) to (33) and using the second formula in (H2b), it follows in the same way that

\[ A_{h,0}^{-1} \rho_{h,n+1} = A_{h,0}^{-1} P_h \rho_{n+1} + O(k \varepsilon_h) + O(k^2 \varepsilon_h) + O(k^3). \]

Now, the key of the proof is that, if \( \omega_{n+1} = A_{h,0}^{-1} \rho_{n+1}, R_h \omega_{n+1} = A_{h,0}^{-1} P_h \rho_{n+1}. \) This comes from the fact that \( \omega_{n+1} \) is the solution of

\[ A \omega_{n+1} = \rho_{n+1}, \quad \partial \omega_{n+1} = 0, \]

from what \( A_{h,0} R_h \omega_{n+1} = P_h \rho_{n+1}. \) Moreover, we will take into account that

\[ R_h \omega_{n+1} = P_h \omega_{n+1} + (R_h - P_h) \omega_{n+1} = O(k^3) + O(k \varepsilon_h), \]

where we have used Theorem 11 for the bound of the first term. As for the second, we have used the definition of \( \rho_{n+1}, (33), \) Lemma 9 hypothesis (H2a) and the fact that \( \Psi_{\frac{f,t}{2}}^{f,t_{n}} \) leaves \( Z \) invariant. Because of this, \( \omega_{n+1} \in Z \) and \( \| \omega_{n+1} \|_Z = O(k) \), and using (H2b) the result follows. \( \square \)
7.3 Global errors

From the first result for the local error $\rho_{h,n+1}$ in Theorem 21, a classical argument for the global error gives $e_{h,n} = O(k + \varepsilon_h)$. However, we also have this finer result, which will be very useful for Dirichlet and non-Dirichlet boundary conditions and parabolic problems:

**Theorem 22.** Let us assume the same hypotheses of Theorem 21, that the bound (34) is satisfied, and

(i) $u \in C^4([0, T], X)$, $f \in C^3([0, T] \times X, X), u(t) \in D(A^3)$ and $f(t, u(t)) \in D(A^2)$ for all $t \in [0, T]$ and $A^3 u, A^2 f(\cdot, u(\cdot)) \in C^1([0, T], X),$

(ii) $\partial f(t_n, u(t_n))$ is calculated exactly or just approximated from data according to Remark 8 and (53),

(iii) the term in $k^3$ for the local error when integrating (28) with $\Psi_k$ is differentiable with respect to $t_n$,

(iv) $u(\cdot), Au(\cdot), A^2 u(\cdot), f(\cdot, u(\cdot)) \in C^1([0, T], Z)$.

Then, the global error which turns up when integrating (1) through (58), (60) and (62), satisfies

$$e_{h,n} = O(k^2 + k\varepsilon_h + \eta_h),$$

where $\varepsilon_h$ and $\eta_h$ are those in (32).

**Proof.** Notice that

$$e_{h,n+1} = [P_h u(t_{n+1}) - \mathcal{U}_{h,n+1}] + [\mathcal{U}_{h,n+1} - U_{h,n+1}]$$

$$= \rho_{h,n+1} + \Psi^{t_{n+\frac{1}{2}}}_{\frac{1}{2}}(\mathcal{W}_{h,n}(k)) - \Psi^{t_{n+\frac{1}{2}}}_{\frac{1}{2}}(W_{h,n}(k))$$

$$= \rho_{h,n+1} + \mathcal{W}_{h,n}(k) - W_{h,n}(k) + kE(\mathcal{W}_{h,n}(k), W_{h,n}(k), k) + O(k^{p+1}), (66)$$

where, for some constant $C$,

$$\|E(\mathcal{W}_{h,n}(k), W_{h,n}(k), k)\| \leq C\|\mathcal{W}_{h,n}(k) - W_{h,n}(k)\|_h.$$ 

Now, notice that

$$\mathcal{W}_{h,n}(s) - W_{h,n}(s) = A_{h,0}(\mathcal{W}_{h,n}(s) - W_{h,n}(s)) + \left(\frac{k}{2} - s\right)[C_h \partial f(t_n, u(t_n)) - C^*_{h,n}(U_{h,n})],$$

$$\mathcal{W}_{h,n}(0) - W_{h,n}(0) = \Psi^{t_{n}}_{\frac{1}{2}}(P_h u(t_n)) - \Psi^{t_{n}}_{\frac{1}{2}}(U_{h,n}).$$
Then,
\[
\Psi_{h,n}(k) - \Psi_{h,n}(k) = e^{kA_{h,0}}(\Psi^{f,t_n}_h(P_h u(t_n)) - \Psi^{f,t_n}_h(U_{h,n}))
\]
\[
+ k^2 \left( \varphi_1(kA_{h,0}) - \varphi_2(kA_{h,0}) \right) [C_h \partial f(t_n, u(t_n)) - C^*_h(U_{h,n})]
\]
\[
= e^{kA_{h,0}}(\Psi_{h,n}(k) - V_{h,n}(k/2) + O(k^{p+1}))
\]
\[
+ k \left( e^{kA_{h,0}} \right) + \varphi_1(kA_{h,0})A^{-1}_{h,0} [C_h \partial f(t_n, u(t_n)) - C^*_h(U_{h,n})]
\]
\[
= e^{kA_{h,0}}(P_h u(t_n) - U_{h,n} + kE(P_h u(t_n), U_{h,n}, k) + O(k^{p+1}))
\]
\[
+ k \left( (e^{kA_{h,0}}) + \varphi_1(kA_{h,0})A^{-1}_{h,0} [C_h \partial f(t_n, u(t_n)) - C^*_h(U_{h,n})]
\]
\[
= e^{kA_{h,0}}e_{h,n} + k\bar{E}(P_h u(t_n), U_{h,n}, k) + O(k^{p+1}),
\]  
(67)

where $\Psi_{h,n}(s)$ is the solution of $\Psi_{h,n}(s)$ starting at $P_h u(t_n)$ and where the definition of $\varphi_1$ and $\varphi_2$ has been considered as well as $\Psi_k$ and the fact that $\Psi_k$ integrates $\vec{C}$ with order $p$. Moreover, for some constant $\bar{C}$,
\[
\|\bar{E}(P_h u(t_n), U_{h,n}, k)\|_h \leq \bar{C}\|e_{h,n}\|_h.
\]  
(68)

Then, inserting this in (66),
\[
e_{h,n+1} = e^{kA_{h,0}}e_{h,n} + k\bar{E}(P_h u(t_n), U_{h,n}, k) + \rho_{h,n+1} + O(k^{p+1}),
\]
where, for some constant $\bar{C}$,
\[
\|\bar{E}(P_h u(t_n), U_{h,n}, k)\|_h \leq \bar{C}\|e_{h,n}\|_h.
\]

Inductively, this means that
\[
e_{h,n} = e^{nkA_{h,0}}e_{h,0} + \sum_{l=1}^{n} e^{(n-l)kA_{h,0}}(\rho_{h,l} + O(k^{p+1}))
\]
\[
+ k\sum_{l=0}^{n-1} e^{(n-l-1)kA_{h,0}}\bar{E}(P_h u(t_n), U_{h,n}^l, k).
\]  
(69)

Then, we write one of the terms in (69) as in (53). As the first term in this decomposition can be written as (56), applying (54) and Theorem 21, this term is proved to be $O(k^2 + \eta_h)$. As for the second term in (53), hypothesis (i) makes that the term in $k^3$ in $A_0^{-1}n_{h+1}$ is differentiable with respect to time $t_n$ and therefore, $A_0^{-1}(\rho_{h,j+1} - \rho_{h,j}) = O(k^4)$. When this is used in the local error for the full discretization, $A_0^{-1}(\rho_{h,n-j+1} - \rho_{h,n-j}) = O(k^4 + k^2\eta_h)$, from what
\[
\sum_{j=2}^{n-1} (\sum_{r=1}^{j-1} e^{rkA_{h,0}})(\rho_{j,n-j+1} - \rho_{j,n-j}) = \sum_{j=2}^{n-1} (kA_{h,0})^{j-1} \sum_{r=1}^{j-1} e^{rkA_{h,0}} 1/k A_{h,0}(\rho_{j,n-j+1} - \rho_{j,n-j})
\]
\[
= O(k^3 + \eta_h).
\]

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Finally, using Theorem 21 for the last term in (55), it is clear that
\[ \sum_{l=1}^{n} e^{ik(n-l)A_{h,0}} \rho_{h,l} = O(k^2 + k\varepsilon_h + \eta_h). \]

By taking then norms and considering that \( e_{h,0} = 0 \), we have that
\[ \|e_{h,n}\|_h \leq O(k^2 + k\varepsilon_h + \eta_h) + k\bar{C} \sum_{l=0}^{n-1} \|e_{h,l}\|_h. \]

Applying a discrete Gronwall lemma, the result follows. \( \square \)

8 Numerical experiments

In this section we will show, through some examples, that order reduction is completely avoided with the technique suggested here for Lie-Trotter and Strang exponential splitting methods. For the sake of brevity, we have restricted here to finite differences for the space discretization, although collocation-type methods also satisfy hypotheses of Section 5.

8.1 One-dimensional problem

Firstly, we consider (1) where \( X = C([0,1]) \) and \( A \) is the second-order space derivative. Moreover, we take
\[ u_0(x) = e^{x^3}, \quad f(t, u) = u^2 - e^{t+x^3}(9x^4 + 6x + e^{t+x^3} - 1), \] (70)

and for the Dirichlet boundary conditions,
\[ g_0(t) = e^t, \quad g_1(t) = e^{t+1}, \] (71)

so that the exact solution of the problem is
\[ u(x, t) = e^{t+x^3}. \]

For the space discretization, we take \( h = 1/(N+1) \) and we consider the nodes \( x_j = jh, \ j = 0, \ldots, N + 1 \). Then, the discrete space is \( \mathbb{C}^N \), where \( N \) is the number of interior nodes, and the second derivative is approximated by means of the standard second-order difference scheme. Moreover, \( P_h \) is the projection on the interior nodal values, \( A_{h,0} = \text{tridiag}(1, -2, 1)/h^2 \) and \( C_h g(t) = [g_0(t), 0, \ldots, 0, g_1(t)]^T/h^2 \). Since \( \mu_{\infty}(A_{h,0}) = 0 \), hypothesis (H1a) is satisfied for the maximum norm; on the other hand, (H1b) can be verified directly from the formula of \( A_{h,0}^{-1} \). Moreover, in this case (H2b) is true with \( Z = C^4([0,1]) \) and \( \varepsilon_h, \eta_h = O(h^2) \), and \( f \) satisfies (H3).

Calculating \( \varphi_j(kA_{h,0})C_h g(t) \) just corresponds to making a linear combination of the first and last column of \( \varphi_j(kA_{h,0}) \), which can be both calculated once and for all at the
Table 1: Local and global error when integrating the one-dimensional problem corresponding to data (70) and (71) with Dirichlet boundary conditions with the suggested modification of Lie-Trotter method

|          | $k = 5 \times 10^{-4}$ | $k = 2.5 \times 10^{-4}$ | $k = 1.25 \times 10^{-4}$ |
|----------|------------------------|--------------------------|---------------------------|
| $L^\infty$-local error | 1.5838e-04          | 4.2830e-05           | 1.1390e-05                |
| Order    |                        | 1.8867                   | 1.9108                    |
| $L^\infty$-global error  | 6.8139e-03          | 3.4035e-03           | 1.7016e-03                |
| Order    |                        | 1.0015                   | 1.0001                    |

Table 2: Local and global error when integrating the one-dimensional problem corresponding to data (70) and (71) with Dirichlet boundary conditions with the suggested modification of Strang method

|          | $k = 1 \times 10^{-3}$ | $k = 5 \times 10^{-4}$ | $k = 2.5 \times 10^{-4}$ |
|----------|------------------------|------------------------|--------------------------|
| $L^\infty$-local error | 8.5559e-05          | 2.1777e-05           | 5.5000e-06                |
| Order    |                        | 1.9741                   | 1.9853                    |
| $L^\infty$-global error  | 1.6140e-04          | 4.2882e-05           | 1.1235e-05                |
| Order    |                        | 1.9122                   | 1.9324                    |

very beginning for fixed stepsize $k$. As integrator $\Psi_k$, we have considered the 4th-order classical Runge-Kutta method.

Considering the technique suggested in this paper, we have obtained the results in Table 1 when integrating till time $T = 0.2$ with Lie-Trotter method and $h = 10^{-3}$ and those in Table 2 with Strang method and $h = 2.5 \times 10^{-4}$. It is clear that orders 2 and 1 are obtained for the local and global errors respectively when integrating with Lie-Trotter and order 2 for the local and global errors when integrating with Strang method, as assured by Theorems 16, 17, 21 and 22 when the error in space is negligible. (This seems to be the case because decreasing $h$ does not practically change the errors.)

Let us now consider the same problem as for the previous experiment, but with a Neumann boundary condition at the right boundary. More precisely, the boundary conditions are

$$
\begin{align*}
  u(0, t) &= g_0(t), \\
  u_x(1, t) &= g_1(t),
\end{align*}
$$

(72)

with $g_0(t) = e^t$ and $g_1(t) = 3e^{1+t}$.

In this case, the values in the node $x = 1$ are included and the discrete space is $\mathbb{C}^{N+1}$, the matrix $A_{h,0}$ is the same as in the previous experiment except for the last row which is $[0, \ldots, 0, 2, -2]/h^2$ now, and $C_h \partial u(t) = [g_0(t)/h^2, 0, \ldots, 0, 2g_1(t)/h]^T$. Again, hypotheses (H1)-(H3) are satisfied with $Z = C^4([0, 1])$.

We notice that now, as it is can be proved by using Taylor expansions, all components of $A_{h,0}(R_hu - P_hu)$ are $O(h^2\|u_{xxxx}\|_{\infty})$ except for the last component which is
$k = 5 \times 10^{-4} \quad k = 2.5 \times 10^{-4} \quad k = 1.25 \times 10^{-4}$

|                  | $k = 5 \times 10^{-4}$ | $k = 2.5 \times 10^{-4}$ | $k = 1.25 \times 10^{-4}$ |
|------------------|-------------------------|---------------------------|---------------------------|
| $L^\infty$-local error | 2.0286e-04              | 5.1444e-05                | 1.2795e-05                |
| Order            | 1.9794                  | 2.0074                    |                           |
| $L^\infty$-global error | 3.9872e-02              | 1.9887e-02                | 9.9237e-03                |
| Order            | 1.0036                  | 1.0029                    |                           |

Table 3: Local and global error when integrating the one-dimensional problem corresponding to data (70), with Dirichlet and Neumann boundary conditions (72), with the suggested modification of Lie-Trotter method

just $O(h\|u_{xxxx}\|_\infty)$. Therefore, $\varepsilon_h$ is $O(h)$ and $\eta_h$ is, in principle, also $O(h)$ for every $u \in C^4([0,1])$. However,

$$R_h u - P_h u = A_{h,0}^{-1} \begin{bmatrix} O(h^2\|u_{xxxx}\|_\infty) \\ O(h^2\|u_{xxxx}\|_\infty) \end{bmatrix} + A_{h,0}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} O(h\|u_{xxxx}\|_\infty) \\ 0 \end{bmatrix}$$

$$= O(h^2\|u_{xxxx}\|_\infty + O(h^2\|u_{xxxx}\|_\infty A_{h,0}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 2/h \end{bmatrix})$$

(73)

where, for the last equality, we have used (H1b). Taking now into account that, when discretizing the problem $Av = 0, v(0) = 0, v_x(1) = 1$, it follows that $A_{h,0}R_h v + \begin{bmatrix} 0, \ldots, 0, 2/h \end{bmatrix}^T = \begin{bmatrix} 0, \ldots, 0, 0 \end{bmatrix}^T$, and it happens that

$$A_{h,0}^{-1} \begin{bmatrix} 0, \ldots, 0, 2/h \end{bmatrix}^T = -R_h v = -P_h v + O(h).$$

Therefore, the term above is bounded for small enough $h$ because $v(x) = x$ and, from (73), $\eta_h$ is in fact $O(h^2)$.

The results which are obtained with the technique proposed in this paper are shown in Table 3 for Lie-Trotter with $h = 10^{-3}$ and in Table 4 for Strang with $h = 2.5 \times 10^{-4}$. In both cases, the global errors are measured at time $T = 0.2$. We see that, for Lie-Trotter, orders 2 and 1 are observed for the local and global error respectively when $k$ decreases, while for Strang, order 2 is obtained for both the local and global errors. These results corroborate Theorems 16, 18, 21 and 22 (We also notice that hypotheses (53) and (54) apply here because of the use of Remark 5 to approximate $\partial f(t_n, u(t_n))$ and the parabolic character of the equation).
Table 4: Local and global error when integrating the one-dimensional problem corresponding to data (70), with Dirichlet and Neumann boundary conditions (72), with the suggested modification of Strang method

|                  | $k = 1 \times 10^{-3}$ | $k = 5 \times 10^{-4}$ | $k = 2.5 \times 10^{-4}$ |
|------------------|------------------------|------------------------|------------------------|
| $L^\infty$-local error | 2.6922e-05             | 5.0772e-06             | 9.1626e-07             |
| Order            |                        | 2.4067                 | 2.4702                 |
| $L^\infty$-global error | 1.8549e-04             | 4.6220e-05             | 1.0814e-05             |
| Order            |                        | 2.0048                 | 2.0957                 |

Table 5: Local and global error when integrating the two-dimensional problem corresponding to data (74) with the suggested modification of Lie-Trotter method

|                  | $k = 5 \times 10^{-3}$ | $k = 2.5 \times 10^{-3}$ | $k = 1.25 \times 10^{-3}$ |
|------------------|------------------------|------------------------|------------------------|
| $L^\infty$-local error | 6.1550e-02             | 1.9049e-02             | 5.7445e-03             |
| Order            |                        | 1.6921                 | 1.7295                 |
| $L^\infty$-global error | 6.1666e-01             | 2.9307e-01             | 1.4341e-01             |
| Order            |                        | 1.0732                 | 1.0311                 |

8.2 Two-dimensional problem

We have also considered the two-dimensional problem in the square $\Omega = [0, 1] \times [0, 1]$ corresponding to the Laplacian as operator $A$. Moreover, we have considered

$$u_0(x) = e^{x^3+y^3}, \quad x \in \Omega, \quad g(t, x) = e^{t+x^3+y^3}, \quad x \in \partial\Omega,$$

$$f(t, u) = u^2 - e^{t+x^3+y^3}(9(x^4 + y^4) + 6(x + y) + e^{t+x^3+y^3} - 1), \quad x \in \Omega, \quad (74)$$

which has $u(t, x) = e^{t+x^3+y^3}$ as exact solution.

Firstly, for the discretization of the Laplacian we have considered the standard five-point formula [20]. We notice that, in this case, the discrete space is $\mathbb{C}^{N^2}$, where $N$ is the number of interior nodes in each direction, and $A_{h,0}$ is a tridiagonal block-matrix of dimension $N^2$. Besides, the matrices in the diagonal are the same and are tridiagonal and the matrices at the subdiagonal and superdiagonal are the same and are diagonal. Notice also that $C_h g(t)$ would just have $4N - 4$ non-vanishing components, which is a number which is negligible compared with $N^2$, the total number of interior nodes. Again, (H1)-(H3) are satisfied for the infinity norm with $\varepsilon_h$, $\eta_h$ being $O(h^2)$.

Tables 5 and 6 show the orders which are observed in time for $h = 10^{-2}$ when integrating the problem till time $T = 1$ with the suggested modifications of Lie-Trotter and Strang method considering again $\Psi_k$ as the fourth-order classical Runge-Kutta method. Again, we see that the local and global order for Lie-Trotter are near 2 and 1 respectively and that the local and global order for Strang are near 2.

We also consider the use of a double splitting in order to obtain a very efficient time integrator. For that, we have followed the lines in [1], where the order reduction is avoided in the case of a dimension splitting of the Laplacian operator. We use similar
\[
\begin{array}{|c|c|c|}
\hline
k = 1 \times 10^{-2} & k = 5 \times 10^{-3} & k = 2.5 \times 10^{-3} \\
\hline
L^\infty\text{-local error} & 5.4180e-02 & 1.5992e-02 & 4.5641e-03 \\
Order & 1.7604 & 1.8090 & \\
\hline
L^\infty\text{-global error} & 3.0713e-01 & 7.7562e-02 & 2.1856e-02 \\
Order & 1.9854 & 1.8273 & \\
\hline
\end{array}
\]

Table 6: Local and global error when integrating the two-dimensional problem corresponding to data \((74)\) with the suggested modification of Strang method ideas for the discretization of \((19)\) in Lie-Trotter method and that of \((26)\) in Strang method.

More precisely, for the discretization of \((19)\) we firstly consider the problem

\[
\begin{align*}
    z'_n(s) &= A_1 z_n(s), \\
    z_n(0) &= u_n, \\
    \partial_t z_n(s) &= \partial_u(t_n) + s A_1 u(t_n)),
\end{align*}
\]

with \(A_1 u = \partial_{xx} u\), and \(\partial_t u = \{u(0, y) = u(1, y), y \in [0, 1]\}\). Then,

\[
\begin{align*}
    r'_n(s) &= A_2 r_n(s), \\
    r_n(0) &= z_n(k), \\
    \partial_r_n(s) &= \partial_{2u}(u(t_n) + k A_1 u(t_n) + s A_2 u(t_n)),
\end{align*}
\]

where \(A_2 u = \partial_{yy} u\) and and \(\partial_2 u = \{u(x, 0) = u(x, 1), x \in [0, 1]\}\). Finally, we make \(v_n(k) = r_n(k)\). We apply now the spatial discretization of problems \((75)\) and \((76)\) and we obtain

\[
\begin{align*}
    Z_{h,n}(k) &= e^{k A_{h,0.1}} U_{h,n} + k \varphi_1 (k A_{h,0.1}) C_h \partial_1 u(t_n) + k^2 \varphi_2 (k A_{h,0.1}) C_h \partial_2 A_1 u(t_n), \\
    R_{h,n}(k) &= e^{k A_{h,0.2}} U_{h,n} + k \varphi_1 (k A_{h,0.2}) C_h \partial_2 (u(t_n) + k A_1 u(t_n)) + k^2 \varphi_2 (k A_{h,0.2}) C_h \partial_2 A_2 u(t_n),
\end{align*}
\]

where \(A_{h,0.1}, A_{h,0.2}, \partial_1, \partial_2\) are the matrices and the boundaries associated to the spatial discretization of \(A_1\) and \(A_2\) respectively.

On the other hand, in order to integrate \((26)\) with Strang method, we firstly consider the problem,

\[
\begin{align*}
    r'_n(s) &= A_1 r_n(s), \\
    r_n(0) &= v_n \left( \frac{k}{2} \right), \\
    \partial_1 r_n(s) &= \partial_1 \left( u(t_n) + \frac{k}{2} f(t_n, u(t_n)) + s A_1 u(t_n) \right),
\end{align*}
\]
then,
\[ \phi'_n(s) = A_2 \phi_n(s), \]
\[ \phi_n(0) = r_n \left( \frac{k}{2} \right), \]
\[ \partial_2 \phi_n(s) = \partial_2 \left( u(t_n) + \frac{k}{2} f(t_n, u(t_n)) + \frac{k}{2} A_1 u(t_n) + s A_2 u(t_n) \right), \]

and finally
\[ \mu'_n(s) = A_3 \mu_n(s), \]
\[ \mu_n(0) = \phi_n(k), \]
\[ \partial_1 \mu_n(s) = \partial_1 \left( u(t_n) + \frac{k}{2} f(t_n, u(t_n)) + \frac{k}{2} A_1 u(t_n) + k A_2 u(t_n) + s A_1 u(t_n) \right), \]

and we make \( w_n(k) = \mu_n \left( \frac{k}{2} \right) \). Considering now the spatial discretization of the previous three problems, we obtain

\[ R_{h,n} \left( \frac{k}{2} \right) = e^{\frac{k}{2} A_{h,0,1}} V_{h,n} + \frac{k}{2} \varphi_1 \left( \frac{k}{2} A_{h,0,1} \right) C_h \partial_1 \left( u(t_n) + \frac{k}{2} f(t_n, u(t_n)) \right) \]
\[ + \frac{k^2}{4} \varphi_2 \left( \frac{k}{2} A_{h,0,1} \right) C_h \partial_1 A_1 u(t_n), \]
\[ \Phi_{h,n}(k) = e^{k A_{h,0,2}} R_{h,n} \left( \frac{k}{2} \right) + k \varphi_1 (k A_{h,0,2}) C_h \partial_2 \left( u(t_n) + \frac{k}{2} f(t_n, u(t_n)) + \frac{k}{2} A_1 u(t_n) \right) \]
\[ + k^2 \varphi_2 (k A_{h,0,2}) C_h \partial_2 A_2 u(t_n), \]
\[ \mu_{h,n} \left( \frac{k}{2} \right) = e^{\frac{k}{2} A_{h,0,1}} \Phi_{h,n}(k) \]
\[ + \frac{k}{2} \varphi_1 \left( \frac{k}{2} A_{h,0,1} \right) C_h \partial_1 \left( u(t_n) + \frac{k}{2} f(t_n, u(t_n)) + \frac{k}{2} A_1 u(t_n) + k A_2 u(t_n) \right) \]
\[ + \frac{k^2}{4} \varphi_2 \left( \frac{k}{2} A_{h,0,1} \right) C_h \partial_1 A_1 u(t_n), \]
\[ W_{h,n} = \mu_{h,n} \left( \frac{k}{2} \right). \]

We have considered the standard second order symmetric finite difference scheme for the discretization of \( A_1 \) and \( A_2 \). Notice that this procedure will be especially efficient since now the matrices \( A_{h,0,j} (j = 1, 2) \) for space discretization in one or another direction are block-diagonal matrices and, moreover, the blocks are tridiagonal. Therefore, multiplying \( e^{k A_{h,0,j}} \) or \( \varphi_1 (k A_{h,0,j}) \) times a vector of size \( N^2 \) just corresponds to \( N \) products of a matrix of dimension \( N \times N \) times a vector of size \( N \). Moreover, many components of \( C_h g(t) \) will vanish.

Although we do not make the analysis for this double splitting, it is natural to suspect that order reduction is also being completely avoided. Tables 7 and 8 corroborate that behavior with \( h = 10^{-2} \).
\[ k = 5 \times 10^{-3} \]
\[ k = 2.5 \times 10^{-3} \]
\[ k = 1.25 \times 10^{-3} \]

| \( L^\infty \)-local error | k = 10^{-2} | k = 5 \times 10^{-3} | k = 2.5 \times 10^{-3} |
|-------------------------|------------|----------------|----------------|
| Order                   | 6.5879e-02 | 1.8698e-02     | 5.2568e-03     |
| Order                   | 1.8169     | 1.8307         |                |

| \( L^\infty \)-global error | k = 10^{-2} | k = 5 \times 10^{-3} | k = 2.5 \times 10^{-3} |
|-----------------------------|------------|----------------|----------------|
| Order                       | 3.4096e-01 | 8.7881e-02     | 2.3635e-02     |
| Order                       | 1.9560     | 1.8946         |                |

Table 7: Local and global error when integrating the two-dimensional problem corresponding to data (74) with the double splitting of Lie-Trotter method and second-order difference scheme in space

Table 8: Local and global error when integrating the two-dimensional problem corresponding to data (74) with the double splitting of Strang method and second-order difference scheme in space

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