MONOTONICITY FOR FRACTIONAL LAPLACIAN SYSTEMS IN UNBOUNDED LIPSCHITZ DOMAINS

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Abstract. In this paper, we first establish a narrow region principle for systems involving the fractional Laplacian in unbounded domains, which plays an important role in carrying on the direct method of moving planes. Then combining this direct method with the sliding method, we derive the monotonicity of bounded positive solutions to the following fractional Laplacian systems in unbounded Lipschitz domains Ω

\[
\begin{cases}
(-\Delta)^s u = f(u,v), & \text{in } \Omega, \\
(-\Delta)^t v = g(u,v), & \text{in } \Omega, \\
u, v \equiv 0, & \text{on } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

without any decay assumptions on the solution pair \((u, v)\) at infinity.

1. Introduction. During the last decade the monotonicity results for bounded positive solutions to the second order elliptic equations and systems have been studied extensively in various unbounded domains since the pioneering works of Berestycki, Caffarelli and Nirenberg (cf. [3, 4, 5, 6]). Among them, we are interested in [16] by Dancer, who generalized the monotonicity results for scalar equations in [6] to the following elliptic systems

\[
\begin{cases}
-\Delta u = f(u,v), & \text{in } \mathbb{R}^n_+, \\
-\Delta v = g(u,v), & \text{in } \mathbb{R}^n_+, \\
u, v \equiv 0, & \text{on } \partial \mathbb{R}^n_+,
\end{cases}
\]

(1)

where the half space \(\mathbb{R}^n_+ := \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_n > 0\}\). He proved that any bounded nonnegative solution pair \((u, v)\) of (1) is strictly increasing with respect to \(x_n\) in \(\mathbb{R}^n_+\) under the assumptions on \(f\) and \(g\) that

\[f, g \in C^1, f(0,0) \geq 0, g(0,0) \geq 0, \frac{\partial f}{\partial v} \geq 0 \text{ and } \frac{\partial g}{\partial u} \geq 0 \text{ for } u, v \geq 0.\]

In addition, if \(n > 3\), it also need to assume the hypothesis

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either (1) has no bounded semi-trivial solutions or any bounded semi-trivial solutions is a function of $x_n$ only.

Afterwards, Chen, Lin and Zou [14] extended the above monotonicity results by removing the hypothesis (H).

In recent years considerable attentions have been paid to the study of elliptic equations and systems with the fractional Laplacian $(-\Delta)^s$. Here, $(-\Delta)^s$ is a nonlocal pseudo-differential operator defined by

$$(-\Delta)^s u(x) := C_{n,s} PV \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy,$$

where $0 < s < 1$, $PV$ stands for the Cauchy principal value and $C_{n,s}$ is a normalization positive constant. In order to guarantee the integral in (2) is well defined, we need to assume that $u \in C^{1,1}_{\text{loc}}(\mathbb{R}^n) \cap L^2_s$ with

$$L^2_s := \{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} dx < \infty \}.$$

Apart from its theoretical interest, this nonlocal operator arises in a variety of physical phenomena such as anomalous diffusion, quasi-geostrophic dynamics, phase transition models, image reconstruction problems and so on (cf. [7, 9, 1, 18]). It can also be used to model American options in mathematical finance (cf. [2]). Thus, it is particularly important to investigate the fractional Laplacian.

Due to the non-locality of the fractional Laplacian, the situation is however different from considering the usual Laplace operator. To overcome this difficulty, Caffarelli and Silvestre [8] first introduced the extension method that reduced the nonlocal problem into a local one in higher dimensions. Another effective approach to deal with the higher order fractional Laplacian is the method of moving planes in integral forms, which turns given pseudo differential equations into their equivalent integral equations, we refer [13] for details. However, sometimes one need to assume $\frac{1}{2} \leq s < 1$ or impose additional integrability conditions on the solutions by using the aforementioned two methods. Recently, Chen et al. [11] developed a direct method of moving planes based on various maximum principles which can circumvent these drawbacks. This direct method was then widely used to establish the symmetry, monotonicity and nonexistence of positive solutions for all kinds of equations and systems involving the fractional Laplacian (cf. [15, 19, 20] and the references therein).

Nevertheless, when studying the fractional equations in unbounded domains, in the case that the Kelvin transform is no longer valid, one needs to impose some decay conditions on the solutions at infinity to carry out the method of moving planes. Very recently, some new progress has been made on this aspect. Dipierro et al. [17] first generalized the symmetry and monotonicity results of [5] to non-decaying bounded positive solutions for the fractional elliptic equations in unbounded Lipschitz domains by a general version of the sliding method. Lately, Chen and Hu [10] obtained the monotonicity of bounded positive solutions for the equations involving the fractional Laplacian in unbounded Lipschitz domains based on the method of moving planes. In contrast, there are few papers concerned with the monotonicity results for non-decaying bounded positive solutions to the fractional elliptic systems in unbounded domains, we are aware solely the reference [21], which studied
fractional Laplacian systems of the type
\[
\begin{cases}
(-\Delta)^s u = f(v), & \text{in } \mathbb{R}^n_+,
\cr
(-\Delta)^s v = g(u), & \text{in } \mathbb{R}^n_+,
\cr
u, v \equiv 0, & \text{on } \mathbb{R}^n \setminus \mathbb{R}^n_+.
\end{cases}
\]

They derived the bounded positive solution \((u, v)\) is strictly increasing in \(x_n\)-direction, under the conditions on the locally Lipschitz continuous functions \(f\) and \(g\) that \(f(0) \geq 0\), \(g(0) \geq 0\) and strictly nondecreasing in \(v\) and \(u\) respectively.

Inspired by the above literature, in this paper, we investigate the monotonicity of bounded positive solutions for a class of fractional elliptic systems in the unbounded domain \(\Omega\):
\[
\begin{cases}
(-\Delta)^s u = f(u, v), & \text{in } \Omega,
\cr
(-\Delta)^t v = g(u, v), & \text{in } \Omega,
\cr
u, v \equiv 0, & \text{on } \mathbb{R}^n \setminus \Omega,
\end{cases}
\tag{3}
\]
where \(0 < s, t < 1\), \(n \geq 2\) and \(\Omega\) is the region above the graph of a globally Lipschitz continuous function \(\phi : \mathbb{R}^{n-1} \to \mathbb{R}\), i.e.,
\[
\Omega := \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > \phi(x')\}
\tag{4}
\]
with \(x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\) and
\[
m := \inf_{\mathbb{R}^{n-1}} \phi(x') > -\infty.
\]
In particular, \(\Omega\) becomes the upper half space when \(\phi \equiv 0\).

Before illuminating the difficulties of our problem, we first introduce the following notation. Let \(T_\lambda := \{x \in \mathbb{R}^n \mid x_n = \lambda \text{ for } \lambda \in \mathbb{R}\}\) be the moving planes,
\[
\Sigma_\lambda := \{x \in \mathbb{R}^n \mid x_n < \lambda\}
\]
be the region below the plane \(T_\lambda\) and
\[
x^\lambda := (x_1, x_2, \ldots, 2\lambda - x_n)
\]
be the reflection of \(x\) with respect to \(T_\lambda\). Let \((u, v) \in (C_\text{loc}^{1,1}(\Omega) \cap \mathcal{L}_{2s}) \times (C_\text{loc}^{1,1}(\Omega) \cap \mathcal{L}_{2t})\) be a solution pair of system (3), we denote the reflected functions by \(u_\lambda(x) := u(x^\lambda)\) and \(v_\lambda(x) := v(x^\lambda)\). Let
\[
\begin{cases}
U_\lambda(x) := u(x) - u_\lambda(x),
\cr
V_\lambda(x) := v(x) - v_\lambda(x).
\end{cases}
\]
It is obvious that \(U_\lambda\) and \(V_\lambda\) are anti-symmetric functions, i.e., \(U_\lambda(x^\lambda) = -U_\lambda(x)\) and \(V_\lambda(x^\lambda) = -V_\lambda(x)\).

At a glance, the main difficulty lies in establishing a narrow region principle, a main ingredient in the method of moving planes, for the fractional systems in unbounded domains without imposing any asymptotic conditions on the solution \((u, v)\) at infinity. In this case, the maximum values of \(U_\lambda(x)\) and \(V_\lambda(x)\) may not be attained, and the maximizing sequences may tend to infinity. To conquer such difficulties, we will develop a new method by estimating the singular integral defined in (2) along a sequence of approximate maximum points and combining the iterative method to derive the narrow region principle.

Combining the method of moving planes with the sliding method, we obtain the monotonicity of bounded positive solution \((u, v)\) for (3) in the unbounded Lipschitz domain \(\Omega\).
Theorem 1.1. Let \( u \in C^{1,1}_{\text{loc}}(\Omega) \cap \mathcal{L}_{2s} \) and \( v \in C^{1,1}_{\text{loc}}(\Omega) \cap \mathcal{L}_{2t} \) be a bounded positive solution pair of (3) with \( f(\cdot, \cdot), g(\cdot, \cdot) \) being locally Lipschitz continuous and satisfying

\((H_1)\) \( f(u, v) \) is strictly increasing in \( v \) and \( g(u, v) \) is strictly increasing in \( u \).

Meanwhile, suppose that at least one of \( f \) and \( g \) satisfies

(i) \( f(u, 0) \geq C_1 u \) on \([0, u_0]\) for some positive constant \( C_1 \) and small \( u_0 \),
(ii) \( g(0, v) \geq C_2 v \) on \([0, v_0]\) for some positive constant \( C_2 \) and small \( v_0 \).

Then \( u \) and \( v \) are strictly increasing with respect to \( x_n \) in \( \Omega \).

Remark 1. Theorem 1.1 generalizes results that the monotonicity of bounded positive solutions in the above mentioned reference [5, 17, 14, 10, 21] to the fractional Laplacian systems in unbounded Lipschitz domains. Furthermore, since we use a different method, our conditions on nonhomogeneous terms are weaker than that in [5, 17, 21].

In the sequel, we deal with the case that \( \Omega \) is a coercive epigraph, i.e., we assume the function \( \phi \) in (4) is continuous and satisfies

\[
\lim_{|x'| \to \infty} \phi(x') = +\infty. \tag{5}
\]

A simple example of such coercive epigraph \( \Omega \) is

\[
\Omega = \{ x = (x', x_n) \in \mathbb{R}^n \mid x_n > |x'|^p \},
\]

where \( p > 0 \). Now we have the following result:

Theorem 1.2. Let \( \Omega \) be a coercive epigraph, and \( u \in C^{1,1}_{\text{loc}}(\Omega) \cap \mathcal{L}_{2s}, v \in C^{1,1}_{\text{loc}}(\Omega) \cap \mathcal{L}_{2t} \) be a positive solution pair (not necessarily bounded) of (3) with \( f(\cdot, \cdot), g(\cdot, \cdot) \) being locally Lipschitz continuous and satisfying \((H_1)\). Then \( u \) and \( v \) are strictly increasing with respect to \( x_n \) in \( \Omega \).

Note that the conditions in Theorem 1.2 are simpler than that in Theorem 1.1, the crucial reason is that the assumption (5) makes the region

\[
\Omega_\lambda := \{ x \in \Omega \mid m < x_n < \lambda \}
\]

is bounded for any \( \lambda \in \mathbb{R} \) with \( \lambda > m \), even if \( \Omega \) is unbounded. Thus, the direct method of moving planes as in the bounded domain case [11] can be applied directly to prove Theorem 1.2. This proof is well known, so we omit it here.

A major prototype of our model (3) is the following fractional Gross-Pitaevskii systems

\[
\begin{aligned}
(-\Delta)^s u &= u - u^3 + uv^2, \quad \text{in } \Omega, \\
(-\Delta)^t v &= v - v^3 + vu^2, \quad \text{in } \Omega, \\
u, v &= 0, \quad \text{on } \mathbb{R}^n \setminus \Omega.
\end{aligned} \tag{6}
\]

Obviously, the functions on the right side satisfy all the conditions in Theorem 1.1 and 1.2, then our main results conclude that the bounded positive solutions \( u \in C^{1,1}_{\text{loc}}(\Omega) \cap \mathcal{L}_{2s} \) and \( v \in C^{1,1}_{\text{loc}}(\Omega) \cap \mathcal{L}_{2t} \) of (6) are strictly increasing with respect to \( x_n \) in \( \Omega \). It is worth mentioning that the above model is very significant in view of the physical interest.

Our paper is organized as follows. In section 2, we establish the narrow region principle in unbounded domains. Finally, the proof of Theorem 1.1 is completed in Section 3.
2. **Narrow region principle in unbounded domains.** In this section, we construct the narrow region principle for anti-symmetric functions \( (U_\lambda, V_\lambda) \) in unbounded domains, which is the key ingredient to carry on the method of moving planes for the fractional Laplacian systems. In what follows, \( C \) denotes a general positive constant whose value may be different from line to line.

**Theorem 2.1.** (Narrow region principle) Let \( \tilde{\Omega} \) be a narrow region in \( \Sigma_\lambda \), possibly unbounded, such that it is contained in \( \{ x \mid \lambda - l < x_n < \lambda \} \) with a small \( l \). Assume that \( U_\lambda \in C^{1,1}_{loc}(\tilde{\Omega}) \cap L_{2s}, V_\lambda \in C^{1,1}_{loc}(\tilde{\Omega}) \cap L_{2t} \) are bounded and uniformly Hölder continuous on \( \Sigma_\lambda \), and satisfy

\[
\begin{align*}
(-\Delta)^s U_\lambda(x) + C_1(x) U_\lambda(x) + C_2(x) V_\lambda(x) &\leq 0, \quad x \in \tilde{\Omega}, \\
(-\Delta)^t V_\lambda(x) + C_3(x) U_\lambda(x) + C_4(x) V_\lambda(x) &\leq 0, \quad x \in \tilde{\Omega}, \\
U_\lambda(x), V_\lambda(x) &\leq 0, \quad x \in \Sigma_\lambda \setminus \tilde{\Omega}, \\
U_\lambda(x^k) = -U_\lambda(x), V_\lambda(x^k) = -V_\lambda(x), &\quad x \in \Sigma_\lambda,
\end{align*}
\]

where \( C_1(x), C_2(x), C_3(x) \) and \( C_4(x) \) are bounded and \( C_2(x), C_3(x) < 0 \) in \( \tilde{\Omega} \). Then

\[ U_\lambda(x), V_\lambda(x) \leq 0, \quad x \in \tilde{\Omega} \tag{8} \]

for sufficiently small \( l \).

Furthermore, if \( U_\lambda(x) \) or \( V_\lambda(x) \) attains zero somewhere in \( \tilde{\Omega} \), then

\[ U_\lambda(x) = V_\lambda(x) \equiv 0, \quad x \in \mathbb{R}^n. \tag{9} \]

**Remark 2.** We believe that this narrow region principle and the idea behind the proof will be useful in studying other types of nonlocal systems in various unbounded domains.

**Proof of Theorem 2.1.** We argue by contradiction. Without loss of generality, we assume that there exists \( x^0 \in \tilde{\Omega} \) such that \( U_\lambda(x^0) > 0 \). Otherwise, the same arguments as follows can also yield a contradiction for the case that there exists \( x^1 \in \tilde{\Omega} \) such that \( V_\lambda(x^1) > 0 \).

By virtue of the boundedness of \( U_\lambda(x) \), we have

\[ A := \sup_{\Sigma_\lambda} U_\lambda(x) \in (0, +\infty). \tag{10} \]

It implies that there exists a sequence \( \{x^k\} \subset \tilde{\Omega} \) such that

\[ U_\lambda(x^k) \to A \text{ as } k \to \infty. \tag{11} \]

From now on, let \( \varphi(x) \in C^\infty_0(\mathbb{R}^n) \) satisfy

\[ \varphi(x) := \begin{cases} c_1 e^{\frac{1}{|x|^2-1}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \tag{12} \]

where the positive constant \( c_1 = e \) such that \( \varphi(0) = \max_{\mathbb{R}^n} \varphi(x) = 1 \). We denote

\[ \varphi_{r_k}(x) := \varphi \left( \frac{x-x^k}{r_k} \right) \text{ and } \varphi_{r_k}^\lambda(x) := \varphi \left( \frac{x^\lambda-x^k}{r_k} \right), \]

where \( r_k := \text{dist} \{x^k, T_\lambda\} \). Then it is not difficult to verify that

\[ \Phi_{r_k}(x) := \varphi_{r_k}(x) - \varphi_{r_k}^\lambda(x) \]
is an anti-symmetric function with respect to $T_\lambda$. We can choose $\varepsilon_k > 0$ and $\varepsilon_k \to 0$ as $k \to \infty$ such that

$$U_\lambda(x^k) + \varepsilon_k \Phi_{r_k}(x^k) = U_\lambda(x^k) + \varepsilon_k > A.$$ 

While if $x \in \Sigma_\lambda \setminus B_{r_k}(x^k)$, then it follows from (10) that

$$U_\lambda(x) + \varepsilon_k \Phi_{r_k}(x) = U_\lambda(x) \leq A.$$ 

Hence, there exists $\bar{x}^k \in B_{r_k}(x^k)$ such that

$$U_\lambda(\bar{x}^k) + \varepsilon_k \Phi_{r_k}(\bar{x}^k) = \max_{\Sigma_\lambda} (U_\lambda(x) + \varepsilon_k \Phi_{r_k}(x)) > A. \quad (13)$$

Moreover, since $\max_{\Sigma_\lambda} \Phi_{r_k}(x) = \Phi_{r_k}(x_k)$, then in terms of (13) and (11), we obtain

$$U_\lambda(\bar{x}^k) \to A > 0 \text{ as } k \to \infty. \quad (14)$$

Since $U_\lambda(x) \leq 0$ in $\Sigma_\lambda \setminus \hat{\Omega}$, then (14) implies that $\bar{x}^k$ must be in $\hat{\Omega}$ for sufficiently large $k$.

Now a combination of (13) and $|\bar{x}^k - y| < |\bar{x}^k - y^\lambda|$ for $y \in \Sigma_\lambda$ yields that

$$(-\Delta)^s (U_\lambda(\bar{x}^k) + \varepsilon_k \Phi_{r_k}(\bar{x}^k)) \geq C U_\lambda(\bar{x}^k) r_k^{2s}.$$ \quad (16)
On the other side, applying the first inequality in (7) and \( \varphi(x) \in C_0^\infty(\mathbb{R}^n) \), we obtain
\[
(-\Delta)^s \left(U_\lambda(\tilde{x}^k) + \varepsilon_k \Phi_{r_k}(\tilde{x}^k)\right) \\
= (-\Delta)^s U_\lambda(\tilde{x}^k) + \varepsilon_k (-\Delta)^s \varphi_{r_k}(\tilde{x}^k) - \varepsilon_k (-\Delta)^s \varphi_\lambda(\tilde{x}^k) \\
\leq -C_1(\tilde{x}^k)U_\lambda(\tilde{x}^k) - C_2(\tilde{x}^k)V_\lambda(\tilde{x}^k) + \frac{\varepsilon_k}{r_k^{2s}} \left[(-\Delta)^s \varphi\right]\left(\frac{\tilde{x}^k - x^k}{r_k}\right) \\
- \frac{\varepsilon_k}{r_k^{2s}} \left[(-\Delta)^s \varphi\right]\left(\frac{x^k - \tilde{x}^k}{r_k}\right) \\
\leq -C_1(\tilde{x}^k)U_\lambda(\tilde{x}^k) - C_2(\tilde{x}^k)V_\lambda(\tilde{x}^k) + \frac{C\varepsilon_k}{r_k^{2s}} \\
\leq C U_\lambda(\tilde{x}^k) + \frac{C\varepsilon_k}{r_k^{2s}} - \frac{CA}{r_k^{2s}}.
\]
(17)

Then combining (15) with (17) and utilizing the boundedness of \( C_1(x) \), we have
\[
C_2(\tilde{x}^k)V_\lambda(\tilde{x}^k) \leq -C_1(\tilde{x}^k)U_\lambda(\tilde{x}^k) + \frac{C\varepsilon_k}{r_k^{2s}} - \frac{CA}{r_k^{2s}} \\
\leq C U_\lambda(\tilde{x}^k) + \frac{C\varepsilon_k}{r_k^{2s}} - \frac{CA}{r_k^{2s}}.
\]
(18)

For each fixed \( l > 0 \), due to \( U_\lambda(x) \leq 0 \) in \( \Sigma_\lambda \setminus \bar{\Omega} \) and \( U_\lambda(x^k) \to A > 0 \) as \( k \to \infty \), we can deduce that \( r_k \) is bounded away from zero for sufficiently large \( k \) by the uniform Hölder continuity of \( U_\lambda \). Let \( r_0 \) be the limit of \( r_k \) as \( k \to \infty \), then we have
\[
CU_\lambda(\tilde{x}^k) + \frac{C\varepsilon_k}{r_k^{2s}} - \frac{CA}{r_k^{2s}} \\
\to CA - \frac{CA}{r_0^{2s}} < 0,
\]
(19)

as \( k \to \infty \) for small enough \( l \), which is ensured by (14) and \( \varepsilon_k \to 0 \) as \( k \to \infty \).

Inserting (19) into (18) and utilizing \( C_2(x) < 0 \), we conclude that
\[
V_\lambda(\tilde{x}^k) > 0
\]
for sufficiently large \( k \).

It means that
\[
B := \sup_{\Sigma_\lambda} V_\lambda(x) \in (0, +\infty).
\]
(20)

Thus, there exists a sequence \( \{y^k\} \subset \bar{\Omega} \) such that
\[
V_\lambda(y^k) \to B \text{ as } k \to \infty.
\]
(21)

Setting \( R_k := \text{dist} \{y^k, T_\lambda\} \) and denoting
\[
\varphi_{R_k}(x) := \varphi\left(\frac{x - y^k}{R_k}\right), \quad \varphi_\lambda_{R_k}(x) := \varphi\left(\frac{x^k - y^k}{R_k}\right),
\]
then it is obvious that
\[
\Phi_{R_k}(x) := \varphi_{R_k}(x) - \varphi_{\lambda_{R_k}}(x)
\]
is also an anti-symmetric function with respect to \( T_\lambda \). We can select \( \gamma_k > 0 \) and \( \gamma_k \to 0 \) as \( k \to \infty \) such that
\[
V_\lambda(y^k) + \gamma_k \Phi_{R_k}(y^k) = V_\lambda(y^k) + \gamma_k > B.
\]

As a consequence, there exists \( \bar{y}^k \in B_{R_k}(y^k) \) such that
\[
V_\lambda(\bar{y}^k) + \gamma_k \Phi_{R_k}(\bar{y}^k) = \max_{\Sigma_\lambda} (V_\lambda(x) + \gamma_k \Phi_{R_k}(x))) > B
\]
(22)
and then
\[ V_\lambda(\tilde{y}^k) \to B \text{ as } k \to \infty. \] (23)

In the sequel, by proceeding similarly as (15), we can compute
\[ (-\Delta)^l (V_\lambda(\tilde{y}^k) + \gamma_k \Phi_{R_k}(\tilde{y}^k)) \geq \frac{CB}{R_k^{2l}}. \] (24)

Besides, it follows from (16) and (17) that
\[ \left( \frac{C}{r_k} + C_1(\tilde{x}^k) \right) U_\lambda(\tilde{x}^k) \leq -C_2(\tilde{x}^k) V_\lambda(\tilde{x}^k) + \frac{C\varepsilon_k}{r_k}. \]

The boundedness of \( C_1(x) \) guarantees \( \frac{C}{r_k} + C_1(\tilde{x}^k) > \frac{C}{r_k} \) with \( 0 < C_0 < C \) for sufficiently large \( k \), then we have
\[ U_\lambda(\tilde{x}^k) \leq -C r_k^{2s} C_2(\tilde{x}^k) V_\lambda(\tilde{x}^k) + C\varepsilon_k. \] (25)

Next, applying the second inequality in (7) and combining \( \varphi(x) \in C^0_\infty(\mathbb{R}^n) \), \( C_2, C_3(x) < 0 \), the boundedness of \( C_i(x) \), \( i = 2, 3, 4 \) with (13), (25) and (22), we derive
\begin{align*}
(-\Delta)^l (V_\lambda(\tilde{y}^k) + \gamma_k \Phi_{R_k}(\tilde{y}^k)) & \leq -C_3(\tilde{y}^k) U_\lambda(\tilde{y}^k) - C_4(\tilde{y}^k) V_\lambda(\tilde{y}^k) + \frac{C\gamma_k}{R_k^{2l}} \\
& \leq -C_3(\tilde{y}^k) \left( U_\lambda(\tilde{x}^k) + \varepsilon_k \Phi_{R_k}(\tilde{x}^k) \right) + \varepsilon_k C_3(\tilde{y}^k) \Phi_{R_k}(\tilde{y}^k) - C_4(\tilde{y}^k) V_\lambda(\tilde{y}^k) + \frac{C\gamma_k}{R_k^{2l}} \\
& \leq -C_3(\tilde{y}^k) \left( -C r_k^{2s} C_2(\tilde{x}^k) V_\lambda(\tilde{x}^k) + C\varepsilon_k + \varepsilon_k C_3(\tilde{y}^k) \Phi_{R_k}(\tilde{y}^k) \right) - C_4(\tilde{y}^k) V_\lambda(\tilde{y}^k) + \frac{C\gamma_k}{R_k^{2l}} \\
& \leq C r_k^{2s} (V_\lambda(\tilde{y}^k) + \gamma_k \Phi_{R_k}(\tilde{y}^k)) - C r_k^{2s} C_2(\tilde{x}^k) V_\lambda(\tilde{x}^k) + C\varepsilon_k + C V_\lambda(\tilde{y}^k) + \frac{C\gamma_k}{R_k^{2l}} \\
& \leq C \left( r_k^{2s} + 1 \right) V_\lambda(\tilde{y}^k) + C\gamma_k r_k^{2s} + C\varepsilon_k + \frac{C\gamma_k}{R_k^{2l}}. \quad (26)
\end{align*}

In fact, for each fixed \( l > 0 \), it follows from \( V_\lambda(x) \leq 0 \) in \( \Sigma_\lambda \setminus \tilde{\Omega} \), \( V_\lambda(\tilde{y}^k) \to B > 0 \) as \( k \to \infty \) and the uniform Hölder continuity of \( V_\lambda \) that \( R_k \) is also bounded away from zero for sufficiently large \( k \). Hence, by virtue of (26), \( \varepsilon_k, \gamma_k \to 0 \) as \( k \to \infty \) and (23), we derive
\begin{align*}
(-\Delta)^l (V_\lambda(\tilde{y}^k) + \gamma_k \Phi_{R_k}(\tilde{y}^k)) & \leq C \left( r_k^{2s} + 1 \right) V_\lambda(\tilde{y}^k) + C\gamma_k r_k^{2s} + C\varepsilon_k + \frac{C\gamma_k}{R_k^{2l}} \to CB,
\end{align*}
as \( k \to \infty \). Therefore, we deduce a contradiction with (24) for sufficiently small \( l \), which means that (8) is valid.

Next, in order to prove (9), we assume that there exists \( \hat{x} \in \tilde{\Omega} \) such that \( U_\lambda(\hat{x}) = 0 \), then
\[ U_\lambda(\hat{x}) = \max_{x \in \Sigma_\lambda} U_\lambda(x). \]

On one hand, we compute
\[ (-\Delta)^s U_\lambda(\hat{x}) = C_{n,s} PV \int_{\mathbb{R}^n} \frac{-U_\lambda(y)}{|\hat{x} - y|^{n+2s}} \, dy \]
\[ = C_{n,s} PV \int_{\Sigma_\lambda} \left( \frac{1}{|\hat{x} - y|^n} - \frac{1}{|\hat{x} - y|^{n+2s}} \right) U_\lambda(y) \, dy. \]
If $U_\lambda(x) \neq 0$ in $\Sigma_\lambda$, then due to $|\hat{x} - y^\lambda| > |\hat{x} - y|$, (7) and (8), we have

$$(-\Delta)^s U_\lambda(\hat{x}) > 0.$$ 

On the other hand, a combination of (7), (8) and $C_2(x) < 0$ yields that

$$(-\Delta)^s U_\lambda(\hat{x}) \leq -C_2(\hat{x}) V_\lambda(\hat{x}) \leq 0.$$ 

This contradiction implies that

$$U_\lambda(x) \equiv 0, \quad x \in \Sigma_\lambda.$$ 

Hence, it follows from the anti-symmetry of $U_\lambda$ that

$$U_\lambda(x) \equiv 0, \quad x \in \mathbb{R}^n.$$ 

(27)

Afterwards, using (7), (8) and $C_2(x) < 0$ again, we derive

$$V_\lambda(x) \equiv 0, \quad x \in \bar{\Omega}.$$ 

Then in analogy to the proof of (27), we can deduce

$$V_\lambda(x) \equiv 0, \quad x \in \mathbb{R}^n.$$ 

Similarly, one can show that if $V_\lambda(x)$ attains zero at one point in $\bar{\Omega}$, then both $U_\lambda(x)$ and $V_\lambda(x)$ are identically zero in $\mathbb{R}^n$. This completes the proof of Theorem 2.1.

3. Monotonicity of positive solutions. Based on the narrow region principle established in the previous section, this section is devoted to deriving the monotonicity of bounded positive solutions for (3) in the unbounded Lipschitz domain $\Omega$ by virtue of the method of moving planes and the sliding method.

We start by proving that a pair of bounded positive solutions for (3) is bounded away from zero at any points away from the boundary under the assumptions in Theorem 1.1.

Lemma 3.1. Assume that the conditions in Theorem 1.1 are true, then for any $R > 0$, there exist $\tau_0, \tau_1 > 0$ with $\tau_0$ and $\tau_1$ depending on $R$ such that at least one of $u$ and $v$ satisfies

$$u(x) > \tau_0 \text{ if } \text{dist}\{x, \partial \Omega\} > R \text{ for } x \in \Omega, \quad (28)$$

$$v(x) > \tau_1 \text{ if } \text{dist}\{x, \partial \Omega\} > R \text{ for } x \in \Omega. \quad (29)$$

Proof. Without loss of generality, we assume that the condition (i) on $f$ is valid. Let $\lambda_1$ be the principle eigenvalue of fractional Laplacian in the unit ball $B_1(0)$ and $\psi$ be the corresponding eigenfunction such that

$$
\begin{cases}
(-\Delta)^s \psi(x) = \lambda_1 \psi(x), & x \in B_1(0), \\
\psi(x) \equiv 0, & x \in \mathbb{R}^n \setminus B_1(0).
\end{cases}
$$

It is well known that $\psi(0) = \max_{x \in B_1(0)} \psi(x)$. For any $R > 0$, let $x^0 \in \Omega$ with $\text{dist}\{x^0, \partial \Omega\} > R$, we denote

$$\psi_{R, \tau, 0}(x) := \tau \psi \left( x - \frac{x^0}{R} \right),$$

where the positive constant $\tau$ will be determined later. By a direct calculation, we have

$$
\begin{cases}
(-\Delta)^s \psi_{R, \tau, 0}(x) = \frac{\lambda_1}{R^n} \psi_{R, \tau, 0}(x), & x \in B_R(x^0), \\
\psi_{R, \tau, 0}(x) \equiv 0, & x \in \mathbb{R}^n \setminus B_R(x^0),
\end{cases}
$$
and \( \psi_{R,\tau,0}(x^0) = \max_{x \in B_R(x^0)} \psi_{R,\tau,0}(x) = \tau \psi(0) \). Now choosing \( \tau > 0 \) sufficiently small such that \( \frac{\lambda_1}{R^{2s}} \tau^2 \leq C_1 \) and \( \tau^{-\frac{1}{2}} \psi_{R,\tau,0}(x^0) = \tau^\frac{1}{2} \psi(0) \leq u_0 \). Thus, a combination of (i) and \((H_1)\) yields that

\[
(-\Delta)^s \psi_{R,\tau,0}(x) \leq C_1 \tau^{-\frac{1}{2}} \psi_{R,\tau,0}(x) \\
\leq f(\tau^{-\frac{1}{2}} \psi_{R,\tau,0}(x), 0) \\
\leq f(\tau^{-\frac{1}{2}} \psi_{R,\tau,0}(x), v(x)), \text{ in } B_R(x^0).
\]

In conclusion, we have

\[
\begin{cases}
(-\Delta)^s u(x) = f(u(x), v(x)), & x \in B_R(x^0), \\
(-\Delta)^s \psi_{R,\tau,0}(x) \leq f(\tau^{-\frac{1}{2}} \psi_{R,\tau,0}(x), v(x)), & x \in B_R(x^0), \\
\tau^{-\frac{1}{2}} \psi_{R,\tau,0}(x) \leq (\neq)u(x), & x \in \mathbb{R}^n \setminus B_R(x^0).
\end{cases}
\tag{30}
\]

We also take \( \tau \) sufficiently small such that \( \tau^{-\frac{1}{2}} \psi_{R,\tau,0}(x) \leq \tau^{\frac{1}{2}} \psi(0) < u(x) \) in \( B_R(x^0) \), which provides a starting point to slide the centre of ball continuously. Let \( x^1 \) be any point in \( \{x \in \Omega \mid \text{dist}\{x, \partial\Omega\} > R\} \). Connecting \( x^0 \) and \( x^1 \) by a continuous path \( \gamma = \{x^t \in \Omega \mid 0 \leq t \leq 1, \text{dist}\{x, \partial\Omega\} > R\} \).

Now we utilize the sliding method to show that

\[
u(x) > \tau^{-\frac{1}{2}} \psi_{R,\tau,t}(x), \quad x \in B_R(x^t)
\tag{31}
\]

for all \( 0 \leq t \leq 1 \), where

\[
\psi_{R,\tau,t}(x) := \tau \psi \left( \frac{x-x^t}{R} \right).
\]

If not, then there exists a first \( t \) such that the graph of \( \tau^{-\frac{1}{2}} \psi_{R,\tau,t} \) touches that of \( u \) at some point \( y^0 \in B_R(x^t) \), i.e. \( u(y^0) = \tau^{-\frac{1}{2}} \psi_{R,\tau,t}(y^0) \). Hence, \( y^0 \) is the minimum point of \( u - \tau^{-\frac{1}{2}} \psi_{R,\tau,t} \) in \( B_R(x^t) \). On one hand, by proceeding similarly as (30), we have

\[
(-\Delta)^s u(y^0) - (-\Delta)^s \psi_{R,\tau,t}(y^0) \geq f(u(y^0), v(y^0)) - f(\tau^{-\frac{1}{2}} \psi_{R,\tau,t}(y^0), v(y^0)) = 0.
\]

On the other hand, we can compute

\[
(-\Delta)^s u(y^0) - (-\Delta)^s \psi_{R,\tau,t}(y^0) \\
= (-\Delta)^s u(y^0) - (-\Delta)^s \tau^{-\frac{1}{2}} \psi_{R,\tau,t}(y^0) + (\tau^{-\frac{1}{2}} - 1)(-\Delta)^s \psi_{R,\tau,t}(y^0) \\
= C_{n,s} \text{PV} \int_{\mathbb{R}^n} \frac{u(y^0) - \tau^{-\frac{1}{2}} \psi_{R,\tau,t}(y^0) - (u(y) - \tau^{-\frac{1}{2}} \psi_{R,\tau,t}(y))}{|y^0 - y|^{n+2s}} dy \\
+ (\tau^{-\frac{1}{2}} - 1) \frac{\lambda_1}{R^{2s}} \psi_{R,\tau,t}(y^0) \\
\leq C_{n,s} \int_{\mathbb{R}^n \setminus B_R(x^t)} \frac{-u(y)}{|y^0 - y|^{n+2s}} dy + \tau^\frac{1}{2} \frac{\lambda_1}{R^{2s}} \psi(0) < 0
\]

for small enough \( \tau > 0 \). This contradiction guarantees that the assertion (31) is true for all \( 0 \leq t \leq 1 \). Let \( t = 1 \), then it follows from (31) that

\[
u(x^1) > \tau^{-\frac{1}{2}} \psi_{R,\tau,1}(x^1) = \tau^{\frac{1}{2}} \psi(0) > 0,
\]

for any \( x^1 \in \{x \in \Omega \mid \text{dist}\{x, \partial\Omega\} > R\} \). Let \( \tau_0 = \tau^{\frac{1}{2}} \psi(0) \) and we arrive at (28).
By proceeding similarly as the proof of (28), we can deduce from the assumption (ii) on \( g \) that there exists a positive constant \( r_1 \) that depends on \( R \) such that (29) is valid for \( R > 0 \). Hence, the proof of Lemma 3.1 is completed.

Now we turn our attention to prove Theorem 1.1.

Proof of Theorem 1.1. In terms of \( m = \inf_{\mathbb{R}^{n-1}} \phi(x') > -\infty \), let

\[
\Omega_\lambda := \{ x \in \Omega \mid m < x_m < \lambda \}.
\]

Meanwhile, a direct calculation shows that \( U_\lambda(x) \) and \( V_\lambda(x) \) satisfy the following fractional Laplacian system

\[
\begin{aligned}
(-\Delta)^s U_\lambda(x) + C_1(x)U_\lambda(x) + C_2(x)V_\lambda(x) &= 0, \quad x \in \Omega_\lambda, \\
(-\Delta)^t V_\lambda(x) + C_3(x)U_\lambda(x) + C_4(x)V_\lambda(x) &= 0, \quad x \in \Omega_\lambda, \\
U_\lambda(x), V_\lambda(x) &\leq 0, \quad x \in \Sigma_\lambda \setminus \Omega_\lambda,
\end{aligned}
\]

(32)

where

\[
C_1(x) = -\frac{f(u(x), v(x)) - f(u_\lambda(x), v(x))}{u(x) - u_\lambda(x)},
\]

\[
C_2(x) = -\frac{f(u_\lambda(x), v(x)) - f(u_\lambda(x), v_\lambda(x))}{v(x) - v_\lambda(x)},
\]

\[
C_3(x) = -\frac{g(u(x), v(x)) - g(u_\lambda(x), v(x))}{u(x) - u_\lambda(x)},
\]

\[
C_4(x) = -\frac{g(u_\lambda(x), v(x)) - g(u_\lambda(x), v_\lambda(x))}{v(x) - v_\lambda(x)},
\]

for \( U_\lambda(x), V_\lambda(x) \neq 0 \). The locally Lipschitz continuity of \( f(\cdot, \cdot), g(\cdot, \cdot) \) and the assumption \( (H_1) \) in Theorem 1.1 guarantee that \( C_1(x), C_2(x), C_3(x) \) and \( C_4(x) \) are bounded and \( C_2(x), C_4(x) < 0 \) for \( U_\lambda(x), V_\lambda(x) \neq 0 \). Besides, the boundedness of \( u \) and \( v \) imply the nonhomogeneous terms \( f \) and \( g \) of system (3) are bounded. Then given that \( \Omega \) is a globally Lipschitz epigraph, one can derive that \( u \) and \( v \) are uniformly Hölder continuous (cf. [12, 17]). Hence, by virtue of Theorem 2.1, we deduce that

\[
U_\lambda(x), V_\lambda(x) \leq 0 \text{ in } \Omega_\lambda
\]

(33)

for sufficiently small \( \lambda > m \), where \( \Omega_\lambda \) is the narrow region.

In fact, (33) provides a starting point to move the plane \( T_\lambda \). Next, we continue to move the plane \( T_\lambda \) upwards along the \( x_n \)-axis as long as (33) holds to its limiting position. To be more precise, let

\[
\lambda_0 := \sup \{ \lambda > m \mid U_{\mu}(x) \leq 0, V_{\mu}(x) \leq 0, \quad x \in \Omega_{\mu}, \quad m < \mu \leq \lambda \}.
\]

We now argue the assertion that

\[
\lambda_0 = +\infty.
\]

(34)

Otherwise, if \( \lambda_0 < +\infty \), we claim that the plane can move up a little more such that (33) is still valid. That is to say, there exists a small \( \epsilon > 0 \) such that (33) holds for any \( \lambda \in (\lambda_0, \lambda_0 + \epsilon) \), which is a contradiction with the definition of \( \lambda_0 \). It follows from the continuity that

\[
U_{\lambda_0}(x), V_{\lambda_0}(x) \leq 0 \text{ in } \Omega_{\lambda_0}.
\]

(35)
Then by proceeding similarly as the proof of (9) in Theorem 2.1, we can show that
\[ U_{\lambda_0}(x), V_{\lambda_0}(x) < 0 \] in \( \Omega_{\lambda_0} \).

In the following, we prove that
\[ \sup_{\Omega_{\lambda_0 - \delta}} U_{\lambda_0}(x), \sup_{\Omega_{\lambda_0 - \delta}} V_{\lambda_0}(x) \leq -C < 0 \] for any \( 0 < \delta < \lambda_0 - m \), where \( C \) is a positive constant.

If not, it may well be assumed that \( \sup_{\Omega_{\lambda_0}} U_{\lambda_0}(x) = 0 \) for some \( \delta > 0 \). Then there exists a sequence \( \{x^k\} \subset \Omega_{\lambda_0 - \delta} \) such that
\[ U_{\lambda_0}(x^k) \to 0 \text{ as } k \to \infty. \] (38)

We first claim that dist\( \{x^k, \partial \Omega\} \) is bounded away from zero for sufficiently large \( k \). Without loss of generality, assuming the condition (i) on \( f \) holds in what follows. In order to prove this assertion, we need to use the global Lipschitz continuity of \( \phi \).

We denote the Lipschitz constant of \( \phi \) by \( c_0 \), then
\[ |\phi(x') - \phi(y')| \leq c_0 |x' - y'| \leq \max\{c_0, 1\} |x' - y'| \] for any \( x', y' \in \mathbb{R}^{n-1} \). If \( x \in \partial \Omega \cap \Sigma_{\lambda_0 - \delta} \), then the distance between \( x \) and \( T_{\lambda_0} \) is at least \( \delta \). Let \( x^{\lambda_0} \) be the reflection of \( x \) with respect to \( T_{\lambda_0} \) and \( c_1 = \max\{c_0, 1\} \), we show that
\[ \text{dist}(x^{\lambda_0}, \partial \Omega) \geq \frac{\delta}{c_1} \] (40)

In fact, for any \( y \in \partial \Omega \), we apply (39) to derive
\[ \left( \text{dist}(x^{\lambda_0}, y) \right)^2 = |x' - y'|^2 + [(2\lambda_0 - x_n) - y_n]^2 \geq \frac{|x_n - y_n|^2}{c_1^2} + [(2\lambda_0 - x_n) - y_n]^2. \]

We consider two possible cases.

i) If \( y_n \geq \lambda_0 \), then
\[ \left( \text{dist}(x^{\lambda_0}, y) \right)^2 \geq \frac{|x_n - y_n|^2}{c_1^2} \geq \frac{\delta^2}{c_1^2}. \]

ii) If \( y_n < \lambda_0 \), then
\[ \left( \text{dist}(x^{\lambda_0}, y) \right)^2 \geq [(2\lambda_0 - x_n) - y_n]^2 \geq \delta^2 \geq \frac{\delta^2}{c_1^2}. \]

Hence, this implies that (40) is true.

Now we can conclude that dist\( \{x^k, \partial \Omega\} \) must be bounded away from zero for sufficiently large \( k \). Otherwise, passing to a subsequence, we have
\[ \text{dist}(x^k, \partial \Omega) \geq \frac{\delta}{2c_1} \] for sufficiently large \( k \). Then the uniform Hölder continuity of \( u \) and Lemma 3.1 implies that
\[ U_{\lambda_0}(x^k) = u(x^k) - u((x^k)^{\lambda_0}) \leq -\tau_0 < 0 \] for \( k \) sufficiently large. Here \( \tau_0 \) depends on \( \delta \) and \( c_1 \), while it is independent of \( k \). This contradicts the assumption (38). Hence, we deduce that dist\( \{x^k, \partial \Omega\} \) is bounded away from zero for sufficiently large \( k \).
where \( \delta_0 := \min\{\delta, \text{dist}\{x^k, \partial \Omega\}\} \) and \( \varphi(x) \in C^\infty_0(\mathbb{R}^n) \) is defined in (12). We denote

\[
\eta_{\delta_0}(x) := \varphi_{\delta_0}(x) - \varphi_{\delta_0}^\lambda(x),
\]

which is an anti-symmetric function with respect to \( T_{\lambda_0} \). Choosing \( \varepsilon_k > 0 \) and \( \varepsilon_k \to 0 \) as \( k \to \infty \) such that

\[
U_{\lambda_0}(x^k) + \varepsilon_k \eta_{\delta_0}(x^k) = U_{\lambda_0}(x^k) + \varepsilon_k > 0.
\]

Hence, there exists \( \bar{x}^k \in B_{\delta_0}(x^k) \) such that

\[
U_{\lambda_0}(\bar{x}^k) + \varepsilon_k \eta_{\delta_0}(\bar{x}^k) = \max_{\Sigma_{\lambda_0}}(U_{\lambda_0}(x) + \varepsilon_k \eta_{\delta_0}(x)) > 0 \tag{41}
\]

and then

\[
U_{\lambda_0}(\bar{x}^k) \to 0 \text{ as } k \to \infty. \tag{42}
\]

Similarly, we can also show that \( \text{dist}\{\bar{x}^k, \partial \Omega\} \) is bounded away from zero for sufficiently large \( k \).

Now we calculate directly to obtain

\[
\begin{align*}
(-\Delta)^s \left(U_{\lambda_0}(\bar{x}^k) + \varepsilon_k \eta_{\delta_0}(\bar{x}^k)\right) & = C_{n,s} PV \int_{\Sigma_{\lambda_0}} \frac{U_{\lambda_0}(\bar{x}^k) + \varepsilon_k \eta_{\delta_0}(\bar{x}^k) - (U_{\lambda_0}(y) + \varepsilon_k \eta_{\delta_0}(y))}{|\bar{x}^k - y|^{n+2s}} dy \\
& \quad + C_{n,s} \int_{\Sigma_{\lambda_0}} \frac{U_{\lambda_0}(\bar{x}^k) + \varepsilon_k \eta_{\delta_0}(\bar{x}^k) + U_{\lambda_0}(y) + \varepsilon_k \eta_{\delta_0}(y)}{|\bar{x}^k - y_{\lambda_0}|^{n+2s}} dy \\
& = C_{n,s} PV \int_{\Sigma_{\lambda_0}} \left( \frac{1}{|\bar{x}^k - y|^{n+2s}} - \frac{1}{|\bar{x}^k - y_{\lambda_0}|^{n+2s}} \right) \\
& \quad \times (U_{\lambda_0}(\bar{x}^k) + \varepsilon_k \eta_{\delta_0}(\bar{x}^k) - U_{\lambda_0}(y) - \varepsilon_k \eta_{\delta_0}(y)) dy \\
& \quad + 2C_{n,s} \int_{\Sigma_{\lambda_0}} \frac{U_{\lambda_0}(\bar{x}^k) + \varepsilon_k \eta_{\delta_0}(\bar{x}^k)}{|\bar{x}^k - y_{\lambda_0}|^{n+2s}} dy \\
& \geq C_{n,s} \int_{\Sigma_{\lambda_0} \setminus B_{\frac{3}{4}}(\bar{x}^k)} \left( \frac{1}{|\bar{x}^k - y|^{n+2s}} - \frac{1}{|\bar{x}^k - y_{\lambda_0}|^{n+2s}} \right) \\
& \quad \times (U_{\lambda_0}(\bar{x}^k) + \varepsilon_k \eta_{\delta_0}(\bar{x}^k) - U_{\lambda_0}(y)) dy \\
& \geq C_{n,s} \int_{\Sigma_{\lambda_0} \setminus B_{\frac{3}{4}}(\bar{x}^k)} \left( \frac{1}{|\bar{x}^k - y|^{n+2s}} - \frac{1}{|\bar{x}^k - y_{\lambda_0}|^{n+2s}} \right) (-U_{\lambda_0}(y)) dy \\
& \geq C_{n,s} \int_{\Omega \setminus B_{\frac{3}{4}}(\bar{x}^k)} \left( \frac{1}{|z|^{n+2s}} - \frac{1}{(|z|^2 + |2\lambda_0 - 2\bar{x}^k - z_n|^{2s})^{\frac{n+2s}{2}}} \right) \\
& \quad \times (-U_{\lambda_0}(\bar{x}^k + z)) dz, \tag{43}
\end{align*}
\]

which is ensured by \( B_{\frac{3}{4}}(x^k) \subset B_{\frac{3}{4}}(\bar{x}^k) \), (41) and \( \lambda_0 - \bar{x}^k_n > \frac{3}{4} \delta \). Meanwhile, starting from the system (32) and combining \( C_2(x) < 0 \), the boundedness of \( C_1(x) \), (36),

Let

\[
\varphi_{\delta_0}(x) := \varphi \left( \frac{x - x^k}{\delta_0} \right) \quad \text{and} \quad \varphi_{\delta_0}^\lambda(x) := \varphi \left( \frac{x^\lambda_{\lambda_0} - x^k}{\delta_0} \right),
\]
Note that such that \( U \) due to the boundedness and uniform Hölder continuity of \( k \rightarrow \infty \) for fixed \( \delta > 0 \). A combination of (43) and (44) yields that
\[
\lim_{k \to \infty} \int_{\Sigma_{\frac{1}{2}} \setminus B_{\frac{1}{2}}(0)} \left( \frac{1}{|z|^{n+2s}} - \frac{1}{(|x|^2 + |2\lambda_0 - 2x_n - z_n|^2)^{\frac{s}{2}} + \varepsilon} \right) (-U_{\lambda_0}(x_k + z)) \, dz = 0.
\]
Let
\[
U_{\lambda_0}^k(z) := u^k(z) - w_{\lambda_0}^k(z) := u(z + x^k) - w_{\lambda_0}(z + x^k).
\]
Due to the boundedness and uniform Hölder continuity of \( u \), we can use Arzelà-Ascoli theorem to deduce there exist \( U_{\lambda_0}^\infty \) and a subsequence still denoted by \( \{U_{\lambda_0}^k\} \) such that
\[
U_{\lambda_0}^k(z) \to U_{\lambda_0}^\infty(z), \text{ as } k \to \infty \text{ uniformly in } \mathbb{R}^n.
\]
Note that
\[
\lim_{k \to \infty} \left( \frac{1}{|z|^{n+2s}} - \frac{1}{(|x|^2 + |2\lambda_0 - 2x_n - z_n|^2)^{\frac{s}{2}} + \varepsilon} \right) > 0
\]
for fixed \( \delta > 0 \), then by virtue of (45) and (46), we get
\[
U_{\lambda_0}^\infty(x) = 0, \forall x \in \Sigma_{\frac{1}{2}} \setminus B_{\frac{1}{2}}(0).
\]
Furthermore, the first equation in system (32), (42) and the similar arguments as in the estimates of (43) and (44) indicate that
\[
-\nabla \cdot (\nabla V_{\lambda_0}(x^k)) = - (\nabla \cdot (U_{\lambda_0}(x^k) + \varepsilon_k \eta_{\delta_0}(x^k))) + C_1(x^k)U_{\lambda_0}(x^k) - \varepsilon_k (-\nabla \cdot \eta_{\delta_0}(x^k))
\]
\[
\geq 2C_{n,s} (U_{\lambda_0}(x^k) + \varepsilon_k \eta_{\delta_0}(x^k)) \int_{\Sigma_{\lambda_0}} \frac{1}{|x|^2 - |y|^2} \, dy + C U_{\lambda_0}(x^k) - \frac{C \varepsilon_k}{\delta_0^2}
\]
\[
\to 0,
\]
as \( k \to \infty \) for fixed \( \delta > 0 \). Thus, \( C_2(x) < 0 \) and (36) imply that \( V_{\lambda_0}(x^k) \to 0 \) as \( k \to \infty \). That is to say,
\[
\sup_{\Omega_{\lambda_0} - \frac{\varepsilon}{2}} V_{\lambda_0}(x) = 0.
\]
Let
\[
\varphi_{\delta_1}(x) := \varphi \left( \frac{x - x^k}{\delta_1} \right), \quad \varphi_{\delta_0}^\lambda(x) := \varphi \left( \frac{x\lambda_0 - x^k}{\delta_0^\lambda} \right),
\]
and
\[
\eta_{\delta_1}(x) := \varphi_{\delta_1}(x) - \varphi_{\delta_0}^\lambda(x)
\]
be an anti-symmetric function, where \( \delta_1 := \min\{\delta, \text{dist}\{x^k, \partial \Omega\}\} \). Selecting \( \gamma_k > 0 \) and \( \gamma_k \to 0 \) as \( k \to \infty \) such that
\[
V_{\lambda_0}(x^k) + \gamma_k \eta_{\delta_1}(x^k) = V_{\lambda_0}(x^k) + \gamma_k > 0.
\]
Hence, there exists \( \tilde{x}^k \in B_{\frac{1}{2}}(x^k) \) such that
\[
V_{\lambda_0}(\tilde{x}^k) + \gamma_k \eta_{\delta_1}(\tilde{x}^k) = \max_{\Omega_{\lambda_0}} (V_{\lambda_0}(x) + \gamma_k \eta_{\delta_1}(x)) > 0.
\]
Let
\[ V^k_{\lambda_0}(z) := v^k(z) - v^k_{\lambda_0}(z) := u(z + \bar{x}^k) - v_{\lambda_0}(z + \bar{x}^k). \]
Since \( v \) is bounded and uniformly Hölder continuous, we can apply Arzelà-Ascoli theorem again to deduce there exist \( V_{\lambda_0}^\infty \) and a subsequence still denoted by \( \{ V_{\lambda_0}^k \} \) such that
\[ V^k_{\lambda_0}(z) \to V_{\lambda_0}^\infty(z), \quad \text{as } k \to \infty \text{ uniformly in } \mathbb{R}^n. \]
Through a similar argument for (47), we can conclude
\[ V_{\lambda_0}^\infty(x) \equiv 0, \quad \forall x \in \Sigma_{\frac{1}{2}} - B_{\frac{1}{2}}(0). \quad (50) \]
Therefore, if \( \sup \Omega_{\lambda_0}(x) = 0 \) for some \( \delta > 0 \), then a combination of (47) and (50) yields that
\[ U_{\lambda_0}^\infty(x) = V_{\lambda_0}^\infty(x) \equiv 0, \quad \forall x \in \Sigma_{\frac{1}{2}} - B_{\frac{1}{2}}(0). \quad (51) \]
Next we devote ourselves to deriving a contradiction to (51). By the assumption \( (i) \) on \( f \), for any \( R > 0 \), we can choose \( x + \bar{x}^k \in \left( \Sigma_{\frac{1}{2}} - B_{\frac{1}{2}}(0) \right) \cap \Omega' \) with \( \text{dist}(x + \bar{x}^k, \partial\Omega') > R \), where
\[ \Omega' = \{ x = (x', x_n) \in \mathbb{R}^n \mid x_n < 2m - \phi(x') \}. \]
Thus, it follows from (28) in Lemma 3.1 that
\[ U_{\lambda_0}(x) = u(x + \bar{x}^k) - u_{\lambda_0}(x + \bar{x}^k) = -u_{\lambda_0}(x + \bar{x}^k) < -\tau_0 < 0, \]
which is contradictive with (51). Also, by the assumption \( (ii) \) on \( g \), we can also deduce a contradiction to (51) by (29) in Lemma 3.1. Either one of the two contradictions above establishes (37).
Next, applying (37) and combining the continuity of \( U_\lambda(x) \) and \( V_\lambda(x) \) with respect to \( \lambda \), we can derive there exist \( \epsilon > 0 \) such that
\[ U_\lambda(x), V_\lambda(x) \leq 0 \text{ in } \Omega_{\lambda_0 - \delta} \]
for any \( \lambda \in (\lambda_0, \lambda_0 + \epsilon) \). From Theorem 2.1 with the narrow region being \( \Omega = \Omega_\lambda \setminus \Omega_{\lambda_0 - \delta} \) for sufficiently small \( \delta \) and \( \epsilon \), we deduce
\[ U_\lambda(x), V_\lambda(x) \leq 0 \text{ in } \Omega_\lambda \]
for any \( \lambda \in (\lambda_0, \lambda_0 + \epsilon) \). It contradicts with the definition of \( \lambda_0 \), then we must have \( \lambda_0 = +\infty \).
Together with (36), we conclude that \( u \) and \( v \) are strictly increasing with respect to \( x_n \) in \( \Omega \). The proof of Theorem 1.1 is completed.

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