From Conformal Haag-Kastler Nets to Wightman Functions∗

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Abstract
Starting from a chiral conformal Haag-Kastler net on 2 dimensional Minkowski space we present a canonical construction that leads to a complete set of conformally covariant N-point-functions fulfilling the Wightman axioms.

Our method consists of an explicit use of the representation theory of the universal covering group of $SL(2,\mathbb{R})$ combined with a generalization of the conformal cluster theorem to N-point-functions [FrJ].

This paper continues work done in [FrJ] and [Jör3].

1 Introduction
The formulation of quantum field theory in terms of Haag Kastler nets of local observable algebras ("local quantum physics" [Haag]) has turned out to be well suited for the investigation of general structures. Discussion of concrete models, however, is mostly done in terms of pointlike localized fields.

In order to be in a precise mathematical framework, these fields might be assumed to obey the Wightman axioms [StW]. Even then, the interrelation between both concepts is not yet completely understood (see [BaW], [BoY] for the present stage).

Heuristically, Wightman fields are constructed out of Haag-Kastler nets by some scaling limit which, however, is difficult to formulate in an intrinsic way [Buc2]. In a dilation invariant theory scaling is well defined, and in the presence of massless particles the construction of a pointlike field was performed in [BuF].

Here, we study the possibly simplest situation: Haag-Kastler nets in 2 dimensional Minkowski space with trivial translations in one light cone direction ("chirality") and covariant under the real Möbius group which acts on the other lightlike direction.

In [FrJ], it has been shown that in the vacuum representation pointlike localized fields can be constructed. Their smeared linear combinations are affiliated to the original net and generate it. We do not know at the moment whether they satisfy all Wightman axioms, since we have not yet found an invariant domain of definition.

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In [Jor3], we have generalized this to the charged sectors of a theory. We have constructed pointlike localized fields carrying arbitrary charge with finite statistics and therefore intertwining between the different superselection sectors of the theory. (In Conformal Field Theory these objects are known as “Vertex Operators”.) We have obtained the unbounded field operators as limits of elements of the reduced field bundle [FRS1, FRS2] associated to the net of observables of the theory.

In this paper, we start again from chiral conformal Haag-Kastler nets and present an canonical construction of N-point-functions that can be shown to fulfill the Wightman axioms. We proceed by generalizing the conformal cluster theorem [FrJ] to higher N-point-functions and by examining the momentum space limit of the algebraic N-point-functions at \( p = 0 \).

We are not able to prove that these Wightman fields can be identified with the pointlike localized fields constructed in [FrJ] and [Jor3].

2 First Steps

In this section, we give an explicit formulation of the setting from which this work starts. We then present the proof of the conformal cluster theorem and the results on the construction of pointlike localized fields in [FrJ] and [Jor3].

2.1 Assumptions

Let \( \mathcal{A} = (\mathcal{A}(I))_{I \in \mathcal{K}_0} \) be a family of von Neumann algebras on some separable Hilbert space \( H \). \( \mathcal{K}_0 \) denotes the set of nonempty bounded open intervals on \( \mathbb{R} \). \( \mathcal{A} \) is assumed to satisfy the following conditions.

i) Isotony:
\[
\mathcal{A}(I_1) \subset \mathcal{A}(I_2) \quad \text{for} \quad I_1 \subset I_2, \quad I_1, I_2 \in \mathcal{K}_0. \tag{1}
\]

ii) Locality:
\[
\mathcal{A}(I_1) \subset \mathcal{A}(I_2)' \quad \text{for} \quad I_1 \cap I_2 = \emptyset, \quad I_1, I_2 \in \mathcal{K}_0 \tag{2}
\]
\((\mathcal{A}(I_2)')'\) is the commutant of \( \mathcal{A}(I_2) \)).

iii) There exists a strongly continuous unitary representation \( U \) of \( G = SL(2, \mathbb{R}) \) in \( H \) with \( U(-1) = 1 \) and
\[
U(g) \mathcal{A}(I) U(g)^{-1} = \mathcal{A}(gI), \quad I, gI \in \mathcal{K}_0 \tag{3}
\]
\((SL(2, \mathbb{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) acts on \( \mathbb{R} \cup \{\infty\} \) by \( x \mapsto \frac{ax + b}{cx + d} \) with the appropriate interpretation for \( x, gx = \infty \).

iv) The conformal Hamiltonian \( H \), which generates the restriction of \( U \) to \( SO(2) \), has a nonnegative spectrum.

v) There is a unique (up to a phase) \( U \)-invariant unit vector \( \Omega \in H \).

vi) \( H \) is the smallest closed subspace containing the vacuum \( \Omega \) which is invariant under \( U(g), \quad g \in SL(2, \mathbb{R}), \quad \text{and} \quad A \in \mathcal{A}(I), I \in \mathcal{K}_0 \) (“cyclicity”). \footnote{This assumption is seemingly weaker than cyclicity of \( \Omega \) w.r.t. the algebra of local observables on \( \mathbb{R} \).}
It is convenient to extend the net to intervals $I$ on the circle $S^1 = R \cup \{\infty\}$ by setting
\[ A(I) = U(g) A(g^{-1}I) U(g)^{-1}, \quad g^{-1}I \in K_0, \quad g \in SL(2, R). \] (4)

The covariance property guarantees that $A(I)$ is well defined for all intervals $I$ of the form $I = gI_0, \quad I_0 \in K_0, \quad g \in SL(2, R)$, i.e. for all nonempty nondense open intervals on $S^1$ (we denote the set of these intervals by $K$).

## 2.2 Conformal Cluster Theorem

In this subsection, we derive a bound on conformal two-point-functions in algebraic quantum field theory (see [FrJ]). This bound specifies the decrease properties of conformal two-point-functions in the algebraic framework to be exactly those known from theories with pointlike localization. The Conformal Cluster Theorem plays a central role in this work.

**Conformal Cluster Theorem (see [FrJ]):** Let $(A(I))_{I \in K_0}$ be a conformally covariant local net on $R$. Let $a, b, c, d \in R$ and $a < b < c < d$. Let $A \in \mathcal{A}( (a,b) ), \quad B \in \mathcal{A}( (c,d) ), \quad n \in N$ and $\Omega = \Omega_k = P_k \Omega = P_k A^* \Omega = 0, \quad k < n$. $P_k$ here denotes the projection on the subrepresentation of $U(G)$ with conformal dimension $k$. We then have
\[ ||(\Omega, B \Omega) || \leq \frac{(b-a)(d-c)}{(c-a)(d-b)} ||A|| ||B||. \] (5)

**Proof:** Choose $R > 0$. We consider the following 1-parameter subgroup of $G = SL(2, R)$:
\[ g_t : x \mapsto \frac{x \cos \frac{t}{2} + R \sin \frac{t}{2}}{-\frac{R}{2} \sin \frac{t}{2} + \cos \frac{t}{2}}. \] (6)

Its generator $H_R$ is within each subrepresentation of $U(G)$ unitarily equivalent to the conformal Hamiltonian $H$. Therefore, the spectrum of $A \Omega$ and $A^* \Omega$ w.r.t. $H_R$ is bounded below by $n$. Let $0 < t_0 < t_1 < 2\pi$ such that $g_{t_0}(b) = c$ and $g_{t_1}(a) = d$. We now define
\[ F(z) = \begin{cases} \begin{array}{l} (\Omega, B z^{-H_R} A \Omega) \quad |z| > 1 \\ (\Omega, A z^{H_R} B \Omega) \quad |z| < 1 \\ (\Omega, A \alpha_{g_t} (B) \Omega) \quad z = e^{it}, \quad t \in [t_0, t_1] \end{array} \end{cases}, \] (7)
a function analytic in its domain of definition, and then
\[ G(z) = (z - z_0)^n (z^{-1} - z_0^{-1})^n F(z), \quad z_0 = e^{rac{i}{2}(t_0 + t_1)}. \] (8)

(Confer the idea in [FrJ].) At $z = 0$ and $z = \infty$ the function $G(\cdot)$ is bounded because of the bound on the spectrum of $H_R$ and can therefore be analytically continued. As an analytic function it reaches its maximum at the boundary of its domain of definition, which is the interval $[e^{it_0}, e^{it_1}]$ on the unit circle:
\[ \sup |G(z)| \leq ||A|| ||B|| |e^{it_0} - e^{i\frac{1}{2}(t_0 + t_1)}|^{2n} = ||A|| ||B|| (2 \sin \frac{t_0 - t_1}{2})^{2n}. \] (9)

This leads to
\[ ||(\Omega, B \Omega) || = |G(1)| = |G(1)| |1 - e^{i\frac{1}{2}(t_0 + t_1)}|^{-2n} = |G(1)| (2 \sin \frac{t_0 + t_1}{4})^{-2n} \leq \sup |G| (2 \sin \frac{t_0 + t_1}{4})^{-2n} \leq ||A|| ||B|| \left( \frac{\sin \frac{t_0 - t_1}{4}}{\sin \frac{t_0 + t_1}{4}} \right)^{2n}. \] (10)
Determining $t_0$ and $t_1$ we obtain
\[
\lim_{R \to \infty} R t_0 = 2(c - b) \quad \text{and} \quad \lim_{R \to \infty} R t_1 = 2(d - a). \tag{11}
\]

We now assume $a - b = c - d$ and find
\[
(t_0 - t_1)(t_0 + t_1)^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)} =: x. \quad \text{Since the bound on } |(\Omega, BA\Omega)| \text{ can only depend on the conformal cross ratio } x, \text{ we can drop the assumption and the theorem is proven.} \quad \square
\]

2.3 The Construction of Pointlike Localized Fields from Conformal Haag-Kastler Nets

This subsection presents the argumentation and results of [FrJ] and [Jo3]:

The idea for the definition of conformal fields is the following: Let $A$ be a local observable,
\[
A \in \bigcup_{I \in K_0} A(I), \tag{12}
\]
and $P_\tau$ the projection onto an irreducible subrepresentation $\tau$ of $U$. The vector $P_\tau A\Omega$ may then be thought of as $\varphi_\tau(h) \Omega$ where $\varphi_\tau$ is a conformal field of dimension $n_\tau =: n$ and $h$ is an appropriate function on $\mathbb{R}$. The relation between $A$ and $h$, however, is unknown at the moment, up to the known transformation properties under $G$, $U(g) P_\tau A\Omega = \varphi_\tau(h_g) \Omega$ \tag{13}

with $h_g^n(x) = (cx - a)^{2n-2} h\left(\frac{dx-b}{cx+a}\right)$, $g = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in G$. We may now scale the vector $P_\tau A\Omega$ by dilations $D(\lambda) = U\left(\begin{array}{cc} \lambda^{\frac{2}{n}} & 0 \\ 0 & \lambda^{-\frac{2}{n}} \end{array}\right)$ and find
\[
D(\lambda) P_\tau A\Omega = \lambda^n \varphi_\tau(h_\lambda) \Omega \tag{14}
\]
where $h_\lambda(x) = \lambda^{-1} h\left(\frac{x}{\lambda}\right)$. Hence, we obtain formally for $\lambda \downarrow 0$
\[
\lambda^{-n} D(\lambda) P_\tau A\Omega \longrightarrow \int dx h(x) \varphi_\tau(0) \Omega. \tag{15}
\]

In order to obtain a Hilbert space vector in the limit, we smear over the group of translations $T(a) = U\left(\begin{array}{cc} 1 & a \\ 0 & 1 \end{array}\right)$ with some test function $f$ and obtain formally
\[
\lim_{\lambda \downarrow 0} \lambda^{-n} \int da f(a) T(a) D(\lambda) P_\tau A\Omega = \int dx h(x) \varphi_\tau(f) \Omega. \tag{16}
\]

We now interpret the left-hand side as a definition of a conformal field $\varphi_\tau$ on the vacuum, and try to obtain densely defined operators with the correct localization by defining
\[
\varphi_\tau^I(f) A'\Omega = A' \varphi_\tau^I(f) \Omega, \quad f \in D(I), \ A' \in A(I)', \ I \in K. \tag{17}
\]

In order to make this formal construction meaningful, there are two problems to overcome.

The first one is the fact that the limit on the left-hand side of (16) does not exist in general if $A\Omega$ is replaced by an arbitrary vector in $H$. This corresponds to the possibility that the
function $h$ on the right-hand side might not be integrable. We will show that after smearing the operator $A$ with a smooth function on $G$, the limit is well defined. Such operators will be called regularized. The second problem is to show that the smeared field operators $\varphi^I_\tau(f)$ are closable, in spite of the nonlocal nature of the projections $P_\tau$.

We omit the technical parts of [FTJ] and [J"or3] and summarize the results in a compact form and as general as possible.

Due to the positivity condition, the representation $U(\tilde{G})$ is completely reducible into irreducible subrepresentations and the irreducible components $\tau$ are up to equivalence uniquely characterized by the conformal dimension $n_\tau \in \mathbb{R}_+$ ($n_\tau$ is the lower bound of the spectrum of the conformal Hamiltonian $H$ in the representation $\tau$).

Associated with each irreducible subrepresentation $\tau$ of $U$ we find for each $I \in K_0$ a densely defined operator-valued distribution $\varphi^I_\tau$ on the space $D(I)$ of Schwartz functions with support in $I$ such that the following statements hold for all $f \in D(I)$.

i) The domain of definition of $\varphi^I_\tau(f)$ is given by $\mathcal{A}(I') \Omega$.

\begin{equation}
\varphi^I_\tau(f) \Omega \in P_\tau H_{red}
\end{equation}

with $P_\tau$ denoting the projector on the module of $\tau$.

\begin{equation}
U(\tilde{g}) \varphi^I_\tau(x) U(\tilde{g})^{-1} = (cx + d)^{-2n_\tau} \varphi^g_\tau(\tilde{g}x)
\end{equation}

with the covering projection $\tilde{g} \mapsto g$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, $I, gI \in K_0$.

iv) $\varphi^I_\tau(f)$ is closable.

v) The closure of $\varphi^I_\tau(f), f \in D(I)$, is affiliated to $\mathcal{A}(I)$.

vi) $\mathcal{A}(I)$ is the smallest von Neumann algebra to which all operators $\varphi^I_\tau(f), f \in D(I)$, are affiliated.

vii) The exchange algebra of the reduced field bundle [FRS2] and the existence of the closed field operators $\varphi^I_\tau(f)$, mapping a dense set of the vacuum Hilbert space into some charged sector with finite statistics, suffice to construct closed field operators $\varphi^I_{\tau,\alpha}(f)$, mapping a dense set of an arbitrary charged sector $\alpha$ with finite statistics into some (other) charged sector with finite statistics. Here, the irreducible module $\tau$ of $U(\tilde{G})$ labels orthogonal irreducible fields defined in the same sector $\alpha$.

viii) The closure of any $\varphi^I_{\tau,\alpha}(f), f \in D(I)$, is affiliated to $\mathcal{F}_{red}(I)$.

ix) $\mathcal{F}_{red}(I)$ is the smallest von Neumann algebra to which all operators $\varphi^I_{\tau,\alpha}(f), f \in D(I)$, are affiliated.

With the existence of pointlike localized fields we are able to give a proof of a generalized Bisognano-Wichmann property. We can identify the conformal group and the reflections as generalized modular structures in the reduced field bundle. Especially, we obtain a PCT operator on $H_{red}$ proving the PCT theorem for the full theory.

Moreover, the existence of pointlike localized fields gives a proof of the hitherto unproven Spin-Statistics theorem for conformal Haag-Kastler nets in 1+1 dimensions.
It was also possible to prove an operator product expansions for arbitrary local observables: For each $I \in \mathcal{K}_0$ and each $A \in \mathcal{A}(I)$ there is a local expansion

$$A = \sum_{\tau} \phi^I_\tau(f_{\tau,A})$$

(20)

into a sum over all irreducible modules $\tau$ of $U(G)$ with

$$\text{supp} f_{\tau,A} \subset I,$$

(21)

which converges on $\mathcal{A}(I')\Omega$ $\ast$-strongly (cf. the definition in [BrR]). Here, $I'$ denotes the complement of $I$ in $\mathcal{K}_0$.

3 Canonical Construction of Wightman Fields

Starting from a chiral conformal Haag-Kastler net, pointlike localized fields have been constructed in [FrJ, Jör3]. Their smeared linear combinations are affiliated to the original net and generate it. We do not know at the moment whether these fields satisfy all Wightman axioms, since we have not found an invariant domain of definition.

In this section, we construct in a canonical manner a complete set of pointlike localized correlation functions out of the net of algebras we have been starting from. We proceed by generalizing the conformal cluster theorem to higher N-point-functions and by examining the momentum space limit of the algebraic N-point-functions at $p = 0$. This canonically constructed set of correlation functions can be shown to fulfill the conditions for Wightman functions (cf. [StW] and [Jos]). Hence, we can construct an associated field theory fulfilling the Wightman axioms.

We are not able to prove that these Wightman fields can be identified with the pointlike localized fields constructed in [FrJ] and [Jör3]. We do not know either how the Haag-Kastler theory, we have been starting from, can be reconstructed from the Wightman theory.

Such phenomena have been investigated by Borchers and Yngvason [BoY]. Starting from a Wightman theory, they could not rule out in general the possibility that the associated local net has to be defined in an enlarged Hilbert space.

3.1 Conformal Two-Point-Functions

First, we will determine the general form of conformal two-point-functions of local observables: It has been shown (cf. e.g. [Jör1]) that a two-point-function $(\Omega, BU(x)A\Omega)$ of a chiral local net with translation covariance is of Lebesgue class $L^p$ for any $p > 1$. The Fourier transform of this two-point-function is a measure concentrated on the positive half line. Therefore, it is - with the possible exception of a trivial delta function at zero - fully determined by the Fourier transform of the commutator function $(\Omega, [B, U(x)AU(x)^{-1}]\Omega)$. Since $A$ and $B$ are local observables, the commutator function has compact support and an analytic Fourier transform $G(p)$. The restriction $\Theta(p)G(p)$ of this analytic function to the positive half line is then the Fourier transform of $(\Omega, BU(x)A\Omega)$.

In the conformally covariant case with $P_k A\Omega = P_k A^*\Omega = 0$, $k < n$, the conformal cluster theorem implies that the two-point-function $(\Omega, BU(x)A\Omega)$ decreases as $x^{-2n}$. Therefore, its Fourier transform is $2n - 2$ times continuously differentiable and can be written as $\Theta(p) p^{2n-1} H(p)$ with an appropriate analytic function $H(p)$.
Using this result, we are able to present a sequence of canonically scaled two-point-functions of local observables converging as distributions to the two-point-function known from conventional conformal field theory (cf. [Jör1, Reh]):

\[
\lim_{\lambda \to 0} \lambda^{-2n} (\Omega, B U(\lambda^{-1} x) A \Omega) = \lim_{\lambda \to 0} \lambda^{-2n} \mathcal{F}_{p \to x} \Theta(p) (\lambda p)^{2n-1} H(\lambda p) \lambda dp = H(0) (x+i\varepsilon)^{-2n}.
\]  

(22)

### 3.2 Conformal Three-Point Functions

We consider the properties of chiral algebraic three-point functions

\[
(\Omega, A_1 U(x_1 - x_2) A_2 U(x_2 - x_3) A_3 \Omega)
\]

of local observables \(A_i, i = 1, 2, 3\). The general form of a (truncated) chiral three-point function of local observables is restricted by locality and by the condition of positive energy. The Fourier transform of an algebraic three-point function can be shown to be the sum of the restrictions of analytic functions to disjoint open wedges in the domain of positive energy:

If \(F\) now denotes the Fourier transform of \((\Omega, A_1 U(\cdot) A_2 U(\cdot) A_3 \Omega)\), we get by straightforward calculations as a first result (cf. [Jör4])

\[
F(p, q) = \Theta(p) \Theta(q - p) G^+(p, q) + \Theta(q) \Theta(p - q) G^-(p, q)
\]

with appropriate analytic functions \(G^+\) and \(G^-.\)

In the case of conformal covariance the general form of these algebraic three-point functions is even more restricted by the following generalization of the conformal cluster theorem [Fr.3].

**Theorem:** Let \((\mathcal{A}(I))_{I \in \mathbb{K}_0}\) be a conformally covariant local net on \(\mathbb{R}\). Let \(a_i, b_i \in \mathbb{R}, i = 1, 2, 3\), and \(a_1 < b_1 < a_2 < b_2 < a_3 < b_3\). Let \(A_i \in \mathcal{A}( (a_i, b_i) ) , n_i \in \mathbb{N}, i = 1, 2, 3\), and

\[
P_k A_i \Omega = P_k A_i^* \Omega = 0, \ k < n_i.
\]

(25)

\(P_k\) here denotes the projection on the subrepresentation of \(U(SL(2, R))\) with conformal dimension \(k\). We then have the following bound:

\[
| (\Omega, A_1 A_2 A_3 \Omega) | \leq \frac{(a_1 - b_1) + (a_2 - b_2)}{(a_2 - a_1) + (b_2 - b_1)}^{(n_1 + n_2 - n_3)}
\]

(26)

\[
| (a_1 - b_1) + (a_3 - b_3) |^{(n_1 + n_3 - n_2)}
\]

\[
| (a_3 - a_1) + (b_3 - b_1) |^{(n_2 + n_3 - n_1)}
\]

\[
| (a_2 - b_2) + (a_3 - b_3) |^{(n_2 + n_3 - n_1)} \parallel A_1 \parallel \parallel A_2 \parallel \parallel A_3 \parallel.
\]

If we additionally assume

\[
a_1 - b_1 = a_2 - b_2 = a_3 - b_3,
\]

(27)

we get

\[
| (\Omega, A_1 A_2 A_3 \Omega) | \leq r_{12}^{(n_1 + n_2 - n_3)/2} r_{23}^{(n_2 + n_3 - n_1)/2} r_{13}^{(n_1 + n_3 - n_2)/2} \parallel A_1 \parallel \parallel A_2 \parallel \parallel A_3 \parallel,
\]

(28)

with the conformal cross ratios

\[
\frac{(a_i - b_i)(a_j - b_j)}{(a_i - a_j)(b_i - b_j)} = r_{ij}, i, j = 1, 2, 3.
\]

(29)
Proof: This proof follows, wherever possible, the line of argument in the proof of the conformal cluster theorem for two-point functions (cf. [FrJ]). Choose \( R > 0 \). Let us consider the following one-parameter subgroup of \( SL(2, \mathbb{R}) \):

\[
g_t : x \mapsto \frac{x \cos \frac{t}{2} + R \sin \frac{t}{2}}{-x \sin \frac{t}{2} + \cos \frac{t}{2}}.
\] (30)

Its generator \( H_R \) is within each subrepresentation of \( U(SL(2, \mathbb{R})) \) unitarily equivalent to the conformal Hamiltonian \( H \). Therefore, the spectrum of \( A_i \Omega \) and \( A_i^\dagger \Omega \) with respect to \( H_R \) is bounded from below by \( n_i, i = 1, 2, 3 \). Let \( 0 < t_{ij}^- < t_{ij}^+ < 2\pi \) such that

\[
g_{t_{ij}^-}(b_i) = a_j
\] (31)

and

\[
g_{t_{ij}^+}(a_i) = b_j
\] (32)

for \( i, j = 1, 2, 3, \ i < j \). We now define

\[
F(z_1, z_2, z_3) := ( \Omega, A_{i_1} \left( \frac{z_{i_1}}{z_{i_2}} \right)^H_R A_{i_2} \left( \frac{z_{i_2}}{z_{i_3}} \right)^H_R A_{i_3} \Omega )
\] (33)

in a domain of definition given by

\[
|z_{i_1}| < |z_{i_2}| < |z_{i_3}|
\] (34)

with permutations \((i_1, i_2, i_3)\) of \((1, 2, 3)\). This definition can uniquely be extended to certain boundary values with \(|z_j| = |z_k|, j, k = 1, 2, 3, j \neq k\):

\[
F
\]

shall be continued to this boundary of its domain of definition if

\[
t_{jk} := -i \log \frac{z_j}{z_k} \notin \lfloor t_{jk}^- , t_{jk}^+ \rfloor + 2\pi \mathbb{Z}
\] (35)

or equivalently if

\[
g_{t_k}(\{a_k, b_k\}) \cap g_{t_j}(\{a_j, b_j\}) \neq \emptyset ,
\] (36)

using the notation

\[
t_i := -i \log z_i, \ i = 1, 2, 3 .
\] (37)

Thereby, boundary points with coinciding absolute values are included in the domain of definition. The definition of \( F \) is chosen in analogy to the analytic continuation of general Wightman functions (cf., e.g., [StW, Jos]) such that the edge-of-the-wedge theorem for distributions with several variables [StW] proves \( F \) to be an analytic function:

Permuting the local observables \( A_i, i = 1, 2, 3 \), we have six three-point functions

\[
( \Omega, A_{i_1} U(x_{i_1} - x_{i_2}) A_{i_2} U(x_{i_2} - x_{i_3}) A_{i_3} \Omega )
\] (38)

These six functions have by locality identical values on a domain

\[
E := \{ (y_1, y_2) \in \mathbb{R}^2 | |y_1| > c_1, |y_2| > c_2, |y_1 + y_2| > c_3 \}
\] (39)

with appropriate \( c_1, c_2, c_3 \in \mathbb{R}_+ \). Each single function can be continued analytically by the condition of positive energy to one of the six disjoint subsets in

\[
U := \mathbb{R}^2 + iV := \{ (z_1, z_2) \in \mathbb{C}^2 | \text{Im}z_1 \neq 0 \neq \text{Im}z_2, \text{Im}z_1 + \text{Im}z_2 \neq 0 \}.
\] (40)
In this geometrical situation, the edge-of-the-wedge theorem (cf. [StW], theorem 2.14) proves the assumed analyticity of $F$.

With the abbreviation

$$z_{ij}^0 := e^{(t_{ij}^- + t_{ij}^+)/2}, \ i, j = 1, 2, 3,$$

we then define

$$G(z_1, z_2, z_3) := F(z_1, z_2, z_3) \prod_{(i,j,k) \in T(1,2,3)} (\frac{z_i}{z_j} - z_{ij}^0)^{(n_i + n_j - n_k)/2}(\frac{z_j}{z_i} - z_{ji}^0)^{(n_j + n_k - n_i)/2},$$

where $T(1,2,3)$ denotes the set $\{(1,2,3), (1,3,2), (2,3,1)\}$. The added polynomial in $z_i$, $i = 1, 2, 3$, is constructed such that the degree of the leading terms are restricted by the assumption on the conformal dimensions of the three-point function $F$. Also, using the binomial formula, it can be controlled by straightforward calculations that no half odd integer exponents appear after multiplication of the product. Hence, at $z_i = 0$ and $z_i = \infty$, $i = 1, 2, 3$, the function $G$ is bounded because of the bound on the spectrum of $H_R$ and can therefore be analytically continued. We can find estimates on $G$ by the maximum principle for analytic functions. In order to get the estimate needed in this proof, we do not use the maximum principle for several complex variables [BoM]. Instead, we present an iteration of the maximum principle argument used in the proof of the conformal cluster theorem [FrJ] for the single variables $z_i$, $i = 1, 2, 3$, of $G(\cdot, \cdot, \cdot)$ and derive a bound on $G(1,1,1)$:

Applying the line of argument known from the case of the two-point functions now to $G(\cdot, 1, 1)$, we get the estimate

$$|G(1,1,1)| \leq \sup_{z_1} |G(z_1, 1, 1)| = \sup_{z_1 \in B_{1,1,1}} |G(z_1, 1, 1)|.$$  

The boundary of the domain of definition of the maximal analytical continuation of $G(\cdot, 1, 1)$ is here denoted by

$$B_{1,1,1} := \{e^{it} | t \notin [t_{12}^-, t_{12}^+] \cup [t_{13}^-, t_{13}^+] + 2\pi \mathbb{Z}\}.$$  

Applying this argument to $G(z_1, \cdot, 1)$, we analogously get the estimate

$$|G(z_1, 1, 1)| \leq \sup_{z_2} |G(z_1, z_2, 1)| = \sup_{z_2 \in B_{z_1,1}} |G(z_1, z_2, 1)|$$

with $B_{z_1,1}$ denoting the boundary of the domain of definition of the maximal analytical continuation of $G(z_1, \cdot, 1)$. Applying this argument finally to $G(z_1, z_2, \cdot)$, we analogously get the estimate

$$|G(z_1, z_2, 1)| \leq \sup_{z_3} |G(z_1, z_2, z_3)| = \sup_{z_3 \in B_{z_1,z_2}} |G(z_1, z_2, z_3)|$$

with $B_{z_1,z_2}$ denoting the boundary of the domain of definition of the maximal analytical continuation of $G(z_1, z_2, \cdot)$.

Having iterated this maximum principle argument for the single variables $z_i$, $i = 1, 2, 3$, we can combine the derived estimates and get

$$|G(1,1,1)| \leq \sup_{t_{jk} = -i \log \frac{z_j}{z_k} \notin [t_{jk}^-, t_{jk}^+ + 2\pi \mathbb{Z}], j \neq k} |G(z_1, z_2, z_3)|.$$
Hence, the boundary values of \( G \) have to be evaluated on the domain described by

\[
g_{t_k}([a_k, b_k]) \cap g_{t_j}([a_j, b_j]) \neq \emptyset
\]  

(48)

with \( t_i = -i \log z_i, \ i = 1, 2, 3 \). We find the supremum with the same calculation as in the proof of the conformal cluster theorem above (cf. [FrJ]):

\[
|G(1, 1, 1)| \leq \|A_1\| \|A_2\| \|A_3\| \prod_{(i,j,k) \in T(1,2,3)} |e^{it_{ij}^e} - e^{i(t_{ij}^+ + t_{ij}^+)/2}|_{n_i + n_j - n_k}
\]

(49)

This leads to another estimate:

\[
|(\Omega, A_1 A_2 A_3 \Omega)| = |F(1, 1, 1)|
\]

\[
= |G(1, 1, 1)| \prod_{(i,j,k) \in T(1,2,3)} |1 - e^{i(t_{ij}^+ + t_{ij}^+)/2}|_{n_i + n_j - n_k}
\]

(50)

\[
= |G(1, 1, 1)| \prod_{(i,j,k) \in T(1,2,3)} |2 \sin \frac{t_{ij}^- + t_{ij}^+}{4}|_{n_i + n_j - n_k}
\]

(51)

\[
\lim_{R \to \infty} R t_{ij}^- = 2(a_j - b_i)
\]

(52)

and

\[
\lim_{R \to \infty} R t_{ij}^+ = 2(b_j - a_i)
\]

(53)

and the first bound in the theorem is proven. If we now assume

\[
a_1 - b_1 = a_2 - b_2 = a_3 - b_3,
\]

(54)

we find

\[
\left( \frac{t_{ij}^- - t_{ij}^+}{t_{ij}^- + t_{ij}^+} \right)^2 = \frac{(a_i - b_i) (a_j - b_j)}{(a_i - a_j) (b_i - b_j)} = r_{ij}, \quad i, j = 1, 2, 3,
\]

and the theorem is proven.

This theorem can be used to get deeper insight in the form of the Fourier transforms of algebraic three-point functions. As in the case of the two-point functions, we proceed by transferring the decrease properties of the function in position space into regularity properties of the Fourier transform in momentum space.

In conventional conformal field theory, the three-point function with conformal dimensions \( n_i, \ i = 1, 2, 3 \), is known up to multiplicities as

\[
f_{n_1 n_2 n_3}(x_1, x_2, x_3) = (x_1 - x_2 + i\varepsilon)^{-(n_1 + n_2 - n_3)} (x_2 - x_3 + i\varepsilon)^{-(n_2 + n_3 - n_1)} (x_1 - x_3 + i\varepsilon)^{-(n_1 + n_3 - n_2)}
\]

(55)
Its Fourier transform

\[ \tilde{f}_{n_1n_2n_3}(p, q) =: \Theta(p) \Theta(q) Q_{n_1n_2n_3}(p, q) \]  

(56)
can be calculated to be a sum of the restrictions of homogeneous polynomials \(Q_{n_1n_2n_3}^+\) and \(Q_{n_1n_2n_3}^-\) of degree \(n_1 + n_2 + n_3 - 2\) to disjoint open wedges \(W_+\) and \(W_-\) in the domain of positive energy (cf. \[Reh\]).

By the bound in the cluster theorem above, we know that a conformally covariant algebraic three-point function \( (\Omega, A_1 U(x_1 - x_2) A_2 U(x_2 - x_3) A_3 \Omega) \) of local observables \(A_i\) with minimal conformal dimensions \(n_i, i = 1, 2, 3\), decreases in position space at least as fast as the associated pointlike three-point function \(f_{n_1n_2n_3}(x_1, x_2, x_3)\) known from conventional conformal field theory. Hence, the Fourier transform \(F_{A_1A_2A_3}(p, q)\) of this algebraic three-point function has to be at least as regular in momentum space as the Fourier transform \(\tilde{f}_{n_1n_2n_3}(p, q)\) of the associated pointlike three-point function known from conventional conformal field theory.

Technically, we use a well-known formula from the theory of Fourier transforms,

\[ \mathcal{F} (\text{Pol}(X) S) = \text{Pol}(\frac{\partial}{\partial Y}) \mathcal{F} S, \]  

(57)
for arbitrary temperate distributions \(S\) and polynomials \(\text{Pol}(\cdot)\) with a (multi-dimensional) variable \(X\) in position space and an appropriate associated differential operator \(\frac{\partial}{\partial Y}\) in momentum space. \(\mathcal{F}\) denotes the Fourier transformation from position space to momentum space.

Let now \(S\) be the conformally covariant algebraic three-point function of local observables \(A_i\) with minimal conformal dimensions \(n_i, i = 1, 2, 3\):

\[ S := (\Omega, A_1 U(x_1 - x_2) A_2 U(x_2 - x_3) A_3 \Omega) \]  

(58)
and \(X\) be a pair of two difference variables out of \(x_i - x_j, i, j = 1, 2, 3\). By the cluster theorem proved above, we can now choose an appropriate homogeneous polynomial \(\text{Pol}(X)\) of degree \(n_1 + n_2 + n_3 - 4\) such that the product \(\text{Pol}(X) S\) is still absolutely integrable in position space. Using the formula given above, we see that \(\text{Pol}(\frac{\partial}{\partial Y}) \mathcal{F} S\) is continuous and bounded in momentum space. Furthermore, we have already derived the form of the Fourier transform \(F\) of an arbitrary (truncated) algebraic three-point function in a chiral theory to be

\[ F(p, q) = \Theta(p) \Theta(q - p) G^+(p, q) + \Theta(q) \Theta(p - q) G^-(p, q) \]  

(59)
with appropriate analytic functions \(G^+\) and \(G^-\). Thereby, we see that in the case of conformal covariance with minimal conformal dimensions \(n_i, i = 1, 2, 3\), the analytic function \(G^+\) \((G^-)\) can be expressed as the product of an appropriate homogeneous polynomial \(P^+\) \((P^-)\) of degree \(n_1 + n_2 + n_3 - 2\) restricted to the wedge \(W_+\) \((W_-)\) and an appropriate analytic function \(H^+\) \((H^-)\). Hence, we have proved that the Fourier transform \(F_{A_1A_2A_3}\) of the algebraic three-point function \( (\Omega, A_1 U(x_1 - x_2) A_2 U(x_2 - x_3) A_3 \Omega) \) can be written as

\[ F_{A_1A_2A_3}(p, q) = \Theta(p) \Theta(q) P_{A_1A_2A_3}(p, q) H_{A_1A_2A_3}(p, q) \]  

(60)
with an appropriate homogeneous function \(P_{A_1A_2A_3}(p, q)\) of degree \(n_1 + n_2 + n_3 - 2\) and an appropriate continuous and bounded function \(H_{A_1A_2A_3}(p, q)\).
These results suffice to control the pointlike limit of the considered correlation functions. Scaling an algebraic three-point function in a canonical manner, we construct a sequence of distributions that converges to the three-point function of conventional conformal field theory:

\[ \lim_{\lambda \downarrow 0} \lambda^{-(n_1+n_2+n_3)} \left( \Omega, A_1 U \left( \frac{x_1 - x_2}{\lambda} \right) A_2 U \left( \frac{x_2 - x_3}{\lambda} \right) A_3 \Omega \right) = \lim_{\lambda \downarrow 0} \lambda^{-(n_1+n_2+n_3)} \mathcal{F}_{p \rightarrow x_1-x_2} \mathcal{F}_{q \rightarrow x_2-x_3} F_{A_1A_2A_3}(\lambda p, \lambda q) \lambda^2 \, dp \, dq \]

\[ = \lim_{\lambda \downarrow 0} \lambda^{-(n_1+n_2+n_3)} \mathcal{F}_{p \rightarrow x_1-x_2} \mathcal{F}_{q \rightarrow x_2-x_3} \Theta(p) \Theta(q) \lambda^{n_1+n_2+n_3-2} P_{A_1A_2A_3}(p, q) H_{A_1A_2A_3}(\lambda p, \lambda q) \lambda^2 \, dp \, dq \]

\[ = (x_1 - x_2 + i\varepsilon)^{-(n_1+n_2-n_3)} (x_2 - x_3 + i\varepsilon)^{-(n_2+n_3-n_1)} (x_1 - x_3 + i\varepsilon)^{-(n_1+n_3-n_2)} H_{A_1A_2A_3}(0, 0). \]  

### 3.3 Conformal N-Point Functions

Since the notational expenditure increases strongly as we come to the construction of higher N-point functions, we concentrate on qualitatively new aspects not occurring in the case of two-point functions and three-point functions. These qualitatively new aspects in the construction of higher N-point functions are related to the fact that in conventional field theory the form of higher N-point functions is not fully determined by conformal covariance. In conventional conformal field theory conformal covariance restricts the form of correlation functions of field operators \( \varphi_i(x_i), \ i = 1, 2, \ldots, N \), with conformal dimension \( n_i \) in the following manner (cf. [ChH, Reh]):

\[ (\Omega, \prod_{1 \leq i \leq N} \varphi_i(x_i)) \Omega = \left( \prod_{1 \leq i < j \leq N} \frac{1}{(x_j - x_i + i\varepsilon)^{c_{ij}}} \right) f(r_{v_1u_1}^{v_3u_3}, \ldots, r_{v_{N-3}u_{N-3}}^{v_{N-3}u_{N-3}}). \]  

(62)

Here, \( f(\cdot, \ldots, \cdot) \) denotes an appropriate function depending on \( N-3 \) algebraically independent conformal cross ratios

\[ r_{tu}^{vs} = \frac{(x_v - x_s) (x_t - x_u)}{(x_v - x_t) (x_s - x_u)}. \]  

(63)

The exponents \( c_{ij} \) must fulfill the consistency conditions

\[ \sum_{j=1}^{N} c_{ij} = 2n_i, \quad c_{ij} = c_{ji}, \quad 1 \leq i \leq N. \]  

(64)

These conditions do not fully determine the exponents \( c_{ij} \) in the case of \( N \geq 4 \). Hence, in conventional conformal field theory four-point functions and higher N-point functions are not fully determined by conformal covariance.

In the case of conformal two-point functions and conformal three-point functions, our strategy to construct pointlike localized correlation functions was the following: First, we proved...
that the algebraic correlation functions decrease in position space as fast as the associated correlation functions in conventional field theory, which are uniquely determined by conformal covariance. Then, we transferred this property by Fourier transformation into regularity properties in momentum space. Finally, we were able to prove that the limit $\lambda \downarrow 0$ of canonically scaled algebraic correlation functions converges to (a multiple of) the associated pointlike localized correlation functions in conventional conformal field theory.

In the case of four-point functions and higher N-point functions, the situation has changed and we cannot expect to be able to fully determine the form of the pointlike localized limit in this construction, since for $N > 4$ the correlation functions in conventional conventional field theory are not any longer uniquely determined by conformal covariance.

Beginning with the discussion of the general case with $N \geq 4$, we consider algebraic N-point functions

$$\left( \Omega, \left( \prod_{1 \leq i \leq N} U(-x_i) A_i U(x_i) \right) \Omega \right) \tag{65}$$

of local observables $A_i$ with minimal conformal dimensions $n_i$, $i = 1, 2, ..., N$, in a chiral theory with conformal covariance. We want to examine the pointlike limit of canonically scaled correlation functions

$$\lim_{\lambda \downarrow 0} \lambda^{-(\sum_{1 \leq i \leq N} n_i)} \left( \Omega, \left( \prod_{1 \leq i \leq N} U(-\frac{x_i}{\lambda}) A_i U(\frac{x_i}{\lambda}) \right) \Omega \right). \tag{66}$$

Our procedure in the construction of pointlike localized N-point functions for $N \geq 4$ will be the following: We consider all possibilities to form a set of exponents $c_{ij}$ fulfilling the consistency conditions

$$\sum_{j=1}^{N} c_{ij} = 2n_i, \ c_{ij} = c_{ji}, \ i = 1, 2, 3, ..., N. \tag{67}$$

For each consistent set of exponents a bound on algebraic N-point functions in position space can be proved. Each single bound on algebraic N-point functions in position space can be transferred into a regularity property of algebraic N-point functions in momentum space. We can use the same techniques as in the case of three-point functions. Finally, we will control the canonical scaling limit in (66) and construct pointlike localized conformal N-point functions.

We present the following generalization of the conformal cluster theorem proved above (cf. [Fr.J]) to algebraic N-point functions of local observables:

**Theorem:** Let $(\mathcal{A}(I))_{I \in \mathcal{K}_0}$ be a conformally covariant local net on $\mathbb{R}$. Let $a_i, b_i \in \mathbb{R}, \ i = 1, 2, 3, ..., N$, and $a_i < b_i < a_{i+1} < b_{i+1}$ for $i = 1, 2, 3, ..., N-1$. Let $A_i \in \mathcal{A}(\ (a_i, b_i) \ )$, $n_i \in \mathbb{N}$, and

$$P_k A_i \Omega = P_k A_i^* \Omega = 0, \ k < n_i, \ i = 1, 2, 3, ..., N. \tag{68}$$

$P_k$ here denotes the projection on the subrepresentation of $U(SL(2, \mathbb{R}))$ with conformal dimension $k$. We then have for each set of exponents $c_{ij}$ fulfilling the consistency conditions

$$\sum_{j=1}^{N} c_{ij} = 2n_i, \ c_{ij} = c_{ji}, \ i = 1, 2, 3, ..., N, \tag{69}$$

the following bound:
\[ \leq \left( \prod_{1 \leq i < j \leq N} \frac{|(a_i - b_i) + (a_j - b_j)|^{e_{ij}}}{|a_j - a_i + (b_j - b_i)|} \right) \prod_{1 \leq i \leq N} \|A_i\|. \] (70)

If we additionally assume
\[ a_1 - b_1 = a_2 - b_2 = ... = a_N - b_N, \] (71)
we can introduce conformal cross ratios and get
\[ |(\Omega, \left( \prod_{1 \leq i \leq N} A_i \right) \Omega)| \]
\[ \leq \left( \prod_{1 \leq i < j \leq N} \frac{(a_i - b_i)(a_j - b_j)}{(a_i - a_j)(b_j - b_j)}^{c_{ij}/2} \right) \prod_{1 \leq i \leq N} \|A_i\|. \] (72)

**Proof:** If we pay attention to the obvious modifications needed for the additional variables, we can use in this proof the assumptions, the notation, and the line of argument introduced in the proof of the cluster theorem in the case of three-point functions.

We choose an arbitrary set of exponents \( c_{ij} \) fulfilling the consistency conditions
\[ \sum_{j=1, j \neq i}^{N} c_{ij} = 2n_i, \quad c_{ij} = c_{ji}, \quad i = 1, 2, 3, ..., N. \] (73)

Let \( R > 0 \). We consider the generator \( H_R \) of the following one-parameter subgroup of \( SL(2, \mathbb{R}) \):
\[ g_t : x \mapsto x \cos \frac{t}{2} + R \sin \frac{t}{2}, \quad -\frac{R}{2} \sin \frac{t}{2} + \cos \frac{t}{2}. \] (74)

We know that \( H_R \) is within each subrepresentation of \( U(SL(2, R)) \) unitarily equivalent to the conformal Hamiltonian \( H \). Therefore, the spectrum of \( A_i \Omega \) and \( A^*_i \Omega \) with respect to \( H_R \) is bounded from below by \( n_i \), \( i = 1, 2, ..., N \). Let \( 0 < t^-_{ij} < t^+_{ij} < 2\pi \) such that
\[ g_{t^-_{ij}}(b_i) = a_j, \] (75)
and
\[ g_{t^+_{ij}}(a_i) = b_j, \] (76)
for \( i, j = 1, 2, ..., N, \ i < j \). We introduce
\[ F(z_1, ..., z_N) := (\Omega, \left( \prod_{i=1}^{N} z^p_{p(i)}^{H_R} A_{p(i)} z^{H_R}_{p(i)} \right) \Omega) \] (77)
in a domain of definition given by
\[ |z_{p(1)}| < |z_{p(2)}| < ... < |z_{p(N)}| \] (78)
with permutations \((p(1), p(2), ..., p(N))\) of \((1, 2, ..., N)\). This definition can uniquely be extended in analogy to the case of three-point functions to boundary points with \(|z_j| = |z_k|\), \( j, k = 1, 2, ..., N, \ j \neq k \), if
\[ g_{t_k}([a_k, b_k]) \cap g_{t_j}([a_j, b_j]) \neq \emptyset. \] (79)
thereby introducing
\[ t_i := -i \log z_i, \quad i = 1, 2, \ldots, N. \] (80)

The line of argument presented above in the case of three-point functions and developed for general Wightman functions in [StW, Jos] proves that this continuation is still an analytic function. We then define
\[ G(z_1, \ldots, z_N) := F(z_1, \ldots, z_N) \prod_{1 \leq i < j \leq N} \frac{z_i^0 - z_j^0}{z_i - z_j^0} \frac{c_{ij}/2}{c_{ji}/2}, \] (81)

using the abbreviation
\[ z_{ij}^0 := e^{(t_{ij}^- + t_{ij}^+)/2}, \quad i, j = 1, 2, \ldots, N. \] (82)

This function is constructed such that with the consistency conditions for \( c_{ij} \) and with the bound on the spectrum of \( H_R \) we get the following result in analogy to the cluster theorem for three-point functions: At the boundary points \( z_i = 0 \) and \( z_i = \infty, \quad i = 1, 2, \ldots, N, \) the function \( G \) is bounded and can therefore be analytically continued. As in the case of three-point functions, we get with the maximum principle for analytic functions further estimates on \( G \): Iterating the well-known maximum principle argument for the single variables, one obtains
\[ |G(1, \ldots, 1)| \leq \sup_B |G(z_1, \ldots, z_N)|, \] (83)

where \( B \) denotes the set of boundary points
\[ B := \{ |z_j| = |z_k| \mid g_{h_k}([a_k, b_k]) \cap g_{t_j}([a_j, b_j]) \neq \emptyset, \quad j \neq k \}, \] (84)

with \( t_i = -i \log z_i, \quad i = 1, 2, \ldots, N \). The supremum of the boundary values of \( G \) can be calculated in full analogy to the case of the three-point functions and to the proof of the conformal cluster theorem (cf. [FrO]). We obtain straightforward:
\[ \left| (\Omega, \left( \prod_{1 \leq i \leq N} A_i \right) \Omega) \right| \leq \left( \prod_{1 \leq i \leq N} \|A_i\| \right) \prod_{1 \leq i < j \leq N} \frac{\sin \frac{t_{ij}^- - t_{ij}^+}{4} |c_{ij}|}{\sin \frac{t_{ij}^- + t_{ij}^+}{4}}. \] (85)

This estimate converges in the limit \( R \downarrow 0 \) with
\[ \lim_{R \to \infty} R t_{ij}^- = 2(a_j - b_i) \] (86)
and
\[ \lim_{R \to \infty} R t_{ij}^+ = 2(b_j - a_i) \] (87)
for \( i, j = 1, 2, \ldots, N \) to the first bound asserted in the theorem. If we assume
\[ a_1 - b_1 = a_2 - b_2 = \ldots = a_N - b_N, \] (88)
we find
\[ \left( \frac{t_{ij}^- - t_{ij}^+}{t_{ij}^- + t_{ij}^+} \right)^2 = \frac{(a_i - b_j) (a_j - b_j)}{\|a_i - a_j\| (b_i - b_j)} = r_{ij}, \quad i, j = 1, 2, \ldots, N, \] (89)
and get the second bound. Hence, the theorem is proven. \( \square \)
For each consistent set of exponents $c_{ij}, i, j = 1, 2, 3, \ldots, N$, we have proved a different bound on conformal four-point functions of chiral local observables. Hence, we know that the algebraic N-point function

$$ ( \Omega, \left( \prod_{1 \leq i \leq N} U(-x_i) A_i U(x_i) \right) \Omega ) \tag{90} $$

decreases in position space at least as fast as the set of associated pointlike N-point functions known from conventional conformal field theory. Therefore, the Fourier transform of the algebraic N-point function has to be at least as regular in momentum space as the Fourier transforms of the associated pointlike N-point functions known from conventional conformal field theory.

Technically, we follow the line of argument in the case of three-point functions and use the formula

$$ \mathcal{F}(\text{Pol}(X)S) = \text{Pol} \left( \frac{\partial}{\partial Y} \right) \mathcal{F}S \tag{91} $$

for arbitrary temperate distributions $S$ and polynomials $\text{Pol}(\cdot)$ with a (multi-dimensional) variable $X$ in position space and an appropriate associated differential operator $\frac{\partial}{\partial Y}$ in momentum space. $\mathcal{F}$ denotes the Fourier transformation from position space to momentum space.

Now, we choose $S$ to be an algebraic N-point function

$$ ( \Omega, \left( \prod_{1 \leq i \leq N} U(-x_i) A_i U(x_i) \right) \Omega ) \tag{92} $$

doing local observables $A_i$ with minimal conformal dimensions $n_i$, $i = 1, 2, \ldots, N$, and $X$ to be a tuple of $N - 1$ algebraically independent difference variables out of $x_i - x_j$, $i, j = 1, 2, \ldots, N$.

The estimates in the cluster theorem proved above imply, that appropriate homogeneous polynomials $\text{Pol}(X)$ of degree

$$ \text{deg}(\text{Pol}) = \left( \sum_{i=1}^{N} n_i \right) - 2N + 2 \tag{93} $$

can be found such that the product $\text{Pol}(X)S$ is still absolutely integrable in position space. We then see that $\text{Pol}(\frac{\partial}{\partial Y})\mathcal{F}S$ is continuous and bounded in momentum space. By locality and the condition of positive energy, the Fourier transform $F$ of an arbitrary (truncated) algebraic N-point function is known to be of the form

$$ F(p_1, \ldots, p_{N-1}) = G(p_1, \ldots, p_{N-1}) \prod_{i=1}^{N-1} \Theta(p_i) \tag{94} $$

where $G$ denotes a sum of restrictions of appropriate analytic functions to subsets of momentum space (cf. the case of three-point functions in the section above). One can now proceed in analogy to the argumentation in the case of three-point functions: In a situation with conformal covariance and minimal conformal dimensions $n_i$, $i = 1, 2, \ldots, N$, the function $G$ can be expressed as the product of an appropriate homogeneous polynomial $P$ of degree

$$ \text{deg}(P) = \left( \sum_{i=1}^{N} n_i \right) - N + 1 \tag{95} $$
and an appropriate function \( H \), where \( H \) denotes another sum of restrictions of analytic functions to subsets of momentum space. Hence, we have proved that the Fourier transform of the algebraic N-point function

\[
(\Omega, \left( \prod_{1 \leq i \leq N} U(-x_i) A_i U(x_i) \right) \Omega)
\]

can be written as

\[
F(p_1, \ldots, p_{N-1}) = P(p_1, \ldots, p_{N-1}) \prod_{i=1}^{N-1} \Theta(p_i)
\]

with an appropriate homogeneous function \( P \) of degree

\[
\deg(P) = \left( \sum_{i=1}^{N} n_i \right) - N + 1
\]

and an appropriate continuous and bounded function \( H \).

Using this result, we can now show in full analogy to the procedure in the last section that by canonically scaling an algebraic N-point function we construct a sequence of distributions that converges to an appropriate pointlike localized N-point function of conventional conformal field theory:

\[
\lim_{\lambda \downarrow 0} \lambda^{-\left( \sum_{i \leq N} n_i \right)} \left( \Omega, \left( \prod_{1 \leq i \leq N} U\left( \frac{x_i}{\lambda} \right) A_i U\left( \frac{x_i}{\lambda} \right) \right) \Omega \right)
\]

\[
= \lim_{\lambda \downarrow 0} \lambda^{-\left( \sum_{i \leq N} n_i \right)} F_{p_i \to x_i - x_{i+1}} F(\lambda p_1, \ldots, \lambda p_{N-1}) \lambda^{N-1} \prod_{1 \leq i \leq N-1} dp_i
\]

\[
= \lim_{\lambda \downarrow 0} F_{p_i \to x_i - x_{i+1}} P(p_1, \ldots, p_{N-1}) H(\lambda p_1, \ldots, \lambda p_{N-1}) \prod_{1 \leq i \leq N-1} \Theta(p_i) dp_i
\]

\[
= \left( \prod_{1 \leq i < j \leq N} \left( \frac{1}{(x_j - x_i + i\varepsilon)^{c_{ij}}} \right) \right) f(t_{u_{1u_1}}^{v_{u_1}}, \ldots, t_{u_{N-3u_{N-3}}}^{v_{N-3u_{N-3}}}).
\]

Again, \( f(\cdot, \ldots, \cdot) \) denotes an appropriate function depending on \( N - 3 \) algebraically independent conformal cross ratios

\[
t_{u_{1u_1}}^{v_{u_1}} := \frac{(x_v - x_s)(x_t - x_u)}{(x_v - x_t)(x_s - x_u)}.
\]

The exponents \( c_{ij} \) must fulfill the consistency conditions

\[
\sum_{j=1}^{N} c_{ij} = 2n_i, \quad c_{ij} = c_{ji}, \quad 1 \leq i \leq N,
\]

which do not fully determine the exponents. Hence, the general form of the pointlike localized conformal correlation functions constructed from algebraic quantum field theory has been determined to be exactly the general form of the N-point functions known from conventional conformal field theory. In both approaches conformal covariance does not fully determine the form of N-point functions for \( N > 4 \).
3.4 Wightman Axioms and Reconstruction Theorem

The most common axiomatic system for pointlike localized quantum fields is the formulation of Wightman axioms given in [StW] and [Jos]. (If braid group statistics has to be considered and the Bose-Fermi alternative does not hold in general, the classical formulation of [StW] and [Jos] has to be modified for the charged case by introducing the axiom of weak locality instead of locality [FRS1, FRS2].)

The construction of pointlike localized correlation functions in this paper uses sequences of algebraic correlation functions of local observables. The algebraic correlation functions obviously fulfill positive definiteness, conformal covariance, locality, and the spectrum condition. Hence, if the sequences converge, the set of pointlike limits of algebraic correlation functions fulfills the Wightman axioms (see [StW]) by construction. By the reconstruction theorem in [StW] and [Jos], the existence of Wightman fields associated with the Wightman functions is guaranteed and this Wightman field theory is unique up to unitary equivalence.

We do not know at the moment whether the Wightman fields can be identified with the pointlike localized field operators constructed in [FrJ] from the Haag-Kastler theory. We do not know either whether the Wightman fields are affiliated to the associated von Neumann algebras of local observables and how the Haag-Kastler net we have been starting from can be reconstructed from the Wightman fields. Possibly, the Wightman fields cannot even be realized in the same Hilbert space as the Haag-Kastler net of local observables.

We do know, however, that the Wightman theory associated with the Haag-Kastler theory is non-trivial: The two-point functions of this Wightman fields are, by construction, identical with the two-point functions of the pointlike localized field operators constructed in [FrJ]. And we have already proved that those pointlike field vectors can be chosen to be non-vanishing and that the vacuum vector is cyclic for a set of all field operators localized in an arbitrary interval.

It shall be pointed out again that those pointlike fields constructed in [FrJ, Jor3] could not be proved to fulfill the Wightman axioms, since we were not able to find a domain of definition that is stable under the action of the field operators.

To summarize this paper, we state that starting from a chiral conformal Haag-Kastler theory we have found a canonical construction of non-trivial Wightman fields. The reconstruction of the original net of von Neumann algebras of local observables from the Wightman fields could not explicitly be presented, since we do not know whether the Wightman fields can be realized in the same Hilbert space as the Haag-Kastler net.

Actually, Borchers and Yngvason [BoY] have investigated similar situations and have shown that such problems can occur in quantum field theory. In [BoY] the question is discussed under which conditions a Haag-Kastler net can be associated with a Wightman theory. The condition for the locality of the associated algebra net turned out to be a property of the Wightman fields called “central positivity”. Central positivity is fulfilled for Haag-Kastler nets and is stable under pointlike limits [BoY]. Hence, the Wightman fields constructed in this thesis fulfill central positivity. The possibility, however, that the local net has to be defined in an enlarged Hilbert space could not be ruled out in general by [BoY].

Furthermore, it has been proved in [BoY] that Wightman fields fulfilling generalized H-bounds (cf. [DSW]) have associated local nets of von Neumann algebras that can be defined in the same Hilbert space. The closures of the Wightman field operators are then affiliated to the associated local algebras. We could not prove generalized H-bounds for the Wightman fields constructed in this thesis. Actually, we suppose that the criterion of generalized H-bounds is too strict for general conformal – and therefore massless – quantum field theories. (Generalized) H-bounds have been proved, however, for massive theories, i.e. for models in quantum field
theory with massive particles (cf. also [DrF, FrH, Sum, Buc1]).

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