\textbf{\textit{ p-ADIC HURWITZ NUMBERS}}

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\textbf{Abstract.} We introduce stable tropical curves, and use these to count covers of the \( p \)-adic projective line of fixed degree and ramification types by Mumford curves in terms of tropical Hurwitz numbers. Our counts depend on the branch loci of the covers.

1. Introduction

The recent success of tropical geometry lies in its ability to provide elementary methods in the enumerative geometry of algebraic curves. For example, plane Gromov-Witten invariants can be defined in the tropical setting in order to obtain the classical recursion formulæ [9, 8].

By its own nature, it seems straightforward to relate tropical geometry to non-archimedean geometry and obtain an interpretation of e.g. the \( j \)-invariant of elliptic curves as the cycle length of their tropicalisations [13]. This interpretation becomes strongly supported by the tropicalisation of \( p \)-adic analytic spaces in [11]. The latter, in fact, has its precedent in the use of tropical tori in order to study the Jacobian of \( p \)-adic Mumford curves in [10].

Important objects in the enumerative geometry of curves are moduli spaces and their compactifications. These spaces exist also in tropical geometry, although they are defined mostly in an ad-hoc fashion and not always are compact. Hence, a foundational approach to moduli spaces via something like “tropical stacks” would be desirable. A first step in this direction is found in the upcoming book [16], where tropical varieties are defined. This should lead into proving theorems about tropicalised spaces (or stacks) by “tropicalising” existing proofs for their classical or \( p \)-adic counterparts.

The aim of this article is to relate Hurwitz numbers for covers of the \( p \)-adic projective line by Mumford curves to tropical Hurwitz numbers. The fact whether or not the upper curve is a Mumford curve depends on the branch locus in a not so obvious way. First results have been found for hyperelliptic covers in [10]. The results were extended to cyclic covers [19, 4] and to general finite Galois covers [3]. In the latter case, the motivation was inverse Galois theory, and Hurwitz spaces for Galois covers were used. However, the situation for covers which are not Galois remained unclear. It is the use of tropical geometry here which allows in principle to count degree \( d \) covers of \( \mathbb{P}^1 \) by genus \( g \) Mumford curves and fixed ramification types \( \eta_0, \eta_1, \ldots \), depending on the branch locus. The most important ingredient for actually deciding whether a given tropical cover comes from a Mumford curve is a tropical version of the Riemann-Hurwitz formula (Theorem [1.9]) which yields the characterising criterion that the ramification divisor be effective.

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Our main enumerative result is that the classical Hurwitz number $H^d_{g}(\eta_0, \ldots, \eta_{n+1})$ counts covers by Mumford curves, if the branch tree is binary (Theorem 5.6). In particular, the double Hurwitz numbers studied by Cavalieri et al. [5] count Mumford curves, as their tropical covers correspond to tropical maps with effective ramification above comb-shaped binary trees (Corollary 5.5).

We do not refrain from using ad-hoc constructions for the moduli spaces of stable tropical curves and maps. However, a more foundational approach is the content of ongoing work. We benefit from the definition of projective tropical variety in [16] which is in fact a straightforward imitation of the corresponding classical notion. In this sense, our $n$-pointed tropical curves are tropically quasi-projective, and the moduli spaces we introduce here are most likely projective.

It turns out that the philosophy of tropical curves being generically tropicalisations of Mumford curves can be proven also in the case of Hurwitz covers.

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2. Generalities

Let $p$ be a prime number. It will hardly ever be referred to. Our ground field will be $K = \mathbb{C}_p$, the completion of the algebraic closure of the field $\mathbb{Q}_p$ of $p$-adic numbers. The valuation map will be denoted by $v: K \rightarrow \mathbb{R} \cup \{\infty\}$.

3. Tropicalised stable maps to $\mathbb{P}^1$

Here, we introduce the notion of stable tropical curve and stable tropical map to $\mathbb{P}^1$. These notions generalise the existing definitions from [9, 5].

3.1. Tropicalisation of quasi-projective curves. Let $\iota: C \rightarrow \mathbb{P}^N$ be the embedding of a $p$-adic quasi-projective curve into the projective space $\mathbb{P}^N$. We will define a tropicalisation map for $\iota$ generalising the approach of [17].

There is the extended valuation map on affine space $\mathbb{A}^N$:

$$v: \mathbb{A}^N \rightarrow \mathbb{T}\mathbb{A}^N := \left(\mathbb{T}\mathbb{A}^1\right)^N, \ (x_1, \ldots, x_N) \mapsto (v(x_1), \ldots, v(x_N)),$$

which has been used in [17] to study Berkovich-analytifications of affine curves.

The next step is to glue the maps $v_i = v: U_i \rightarrow \mathbb{T}\mathbb{A}^N$ on the standard affine charts $U_i \cong \mathbb{A}^N$ of projective $N$-space $\mathbb{P}^N$. This yields our tropicalisation map

$$\text{Val} : \mathbb{P}^N \rightarrow \mathbb{T}\mathbb{P}^N$$

to the tropical projective space from [16]. The space $\mathbb{T}\mathbb{P}^N$ is obtained by gluing the tropical affine pieces $\mathbb{T}\mathbb{A}^N$ in the “tropically” analogous way as usual projective space $\mathbb{P}^N$ is constructed from $\mathbb{A}^N$.

Let $(C, p_1, \ldots, p_n)$ be a smooth irreducible $n$-punctured projective curve of genus $g$ defined over $K$ with $n$ $K$-rational punctures $p_1, \ldots, p_n$. Let $\iota: C \rightarrow \mathbb{P}^N$ be the pluricanonical embedding of the curve given by $\omega_C(p_1 + \cdots + p_n)^{\otimes e}$, a sufficiently high power of the ample line bundle $\omega_C(p_1 + \cdots + p_n)$. It has the property

$$O(1)\vert_C = \omega_C(p_1 + \cdots + p_n)^{\otimes e}.$$  

The closure $\Gamma := \text{trop}_t(\iota(C))$ of $\text{Val}(\iota(C))$ is known to be an abstract tropical curve with $n$ punctures and first Betti number at most $g$, and the punctures are represented by unbounded edges of $\Gamma$ (cf. the following subsection). From the point
of view of Berkovich [2, §4], the tropical curve \( \Gamma \) coincides with the skeleton of \( C^\text{an} \), and the unbounded edges of \( \Gamma \) are completed by the punctures \( p_1, \ldots, p_n \in \hat{C}^\text{an} \), the completion of \( C \). The induced tropicalisation map \( C^\text{an} \to \Gamma \) is the retraction map to the skeleton.

3.2. Deligne-Mumford compactification of \( M^\text{trop}_{g,n} \). Let \( 2g - 2 + n > 0 \) and \( \mathcal{M}_{g,n} \) the moduli space of \( n \)-pointed stable curves defined over \( K^0 \), and \( \mathcal{M}_{g,n} := \mathcal{M}_{g,n} \times_{\Spec K} \Spec K \).

For \( S = \Spec K_0 \) and an \( n \)-pointed stable curve \((C \to S, s_1, \ldots, s_n : S \to C)\) fix \( e \) as in the previous subsection, and let \( \Phi : C \to \mathbb{P}(V^*) \cong \mathbb{P}^N_S \) be the induced embedding, where \( V = H^0(C, \omega_{C/S}(s_1 + \cdots + s_n)^{\otimes e}) \) and \( N = e(2g - 2 + n) - g \).

By [11, Lemma 7.4], the line bundle \( \mathcal{L} := \omega_{C/S}(s_1 + \cdots + s_n)^{\otimes e} \) has a canonical formal metric \( \| \|_{\mathcal{L}} \). This formal metric can be applied to the projective embedding as follows. Namely, take a basis \( \sigma_0, \ldots, \sigma_N \) of \( V^* \) such that \( \sigma_i \) restricted to the generic fibre \( C_0 \) of \( C \) has a pole in the \( i \)-th puncture, and let \( L := \mathcal{L}|_{C_0} \). The restricted projective embedding \( \Phi_L : C_0 \to \mathbb{P}^N \) is defined on the chart \( U_i = \{ x \in C_0 \mid \sigma_i(x) \neq 0 \} \) as

\[
\begin{align*}
\sigma(x) = (\frac{\sigma_1(x)}{\sigma_i(x)}, \ldots, 1 : \ldots : \frac{\sigma_N(x)}{\sigma_i(x)})
\end{align*}
\]

By definition of the canonical formal metric, the restriction \( \| \|_\mathcal{L} \) to \( L \) satisfies \(-\log \| \sigma(x) \| = v(\sigma(x)) \) for \( \sigma \in H^0(C_0, L) \). Hence, on \( U_i \),

\[
\text{Val}(\Phi_L(x)) = (v(\sigma_0(x)) - v(\sigma_1(x)), \ldots, v(\sigma_i(x)) - v(\sigma_N(x))) \in \mathbb{T}^N_k,
\]

and these maps glue to a map \( \text{Val} \circ \Phi_L : C_0 \to \mathbb{T}^N \). Note that we have used the property (1) and that the formal canonical metric is the restriction of the formal canonical metric on \( O(1) \).

Define \( \text{trop}(C) := \text{Val}(\Phi_L(C))^{\text{cl}} \subseteq T \mathbb{P}^N \). For \( [C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n} \), we will label in the tropical curve \( \Gamma = \text{trop}(C) \) the punctures \( \text{Val}(\Phi_L(p_i)) \) by their original names \( p_i \). This is reminiscent of the fact that the skeleton of a Berkovich-analytic space \( X \) lies inside the space \( X \).

**Remark 3.1.** If \( (C, p_1, \ldots, p_n) \) is an \( n \)-pointed projective line defined over a finite extension \( k \) of \( \mathbb{Q}_p \), then one can check that \( \text{trop}(C \times_K k) \) is isometric to the subtree \( \mathcal{T}^* \{p_1, \ldots, p_n\} \) of the Bruhat-Tits tree \( \mathcal{T}_k \) of \( \text{PGL}_2(k) \), defined in [12]. That tree is the smallest subtree of \( \mathcal{T}_k \) containing all geodesic lines between the points \( \{p_1, \ldots, p_n\} \subseteq \mathbb{P}^1(k) \).

We propose a modified definition of tropical curve in order to have a compact moduli space containing the tropicalisations of \( p \)-adic curves.

**Definition 3.2.** Let \( I := I_1 \coprod \cdots \coprod I_n \) be a finite disjoint union of copies of the closed or half-open unit interval \([0, 1]\) or \([0, 1)\). Let the boundary \( \partial I = (I_1 \setminus I_i) \coprod \cdots \coprod (I_n \cup I_p) \) be partitioned into the disjoint union of non-empty sets \( P_1 \cup \cdots \cup P_r \). The topological quotient space \( \Gamma = I/\sim \) obtained by the equivalence relation \( \sim \) which identifies the points in each \( P_i \) is called a semi-graph. The equivalence classes of the \( P_i \) are called vertices, and the open intervals \( I^e_i \) edges of \( \Gamma \). An edge \( e \) is called bounded, if its closure is the image of a closed interval under the canonical projection \( I \to \Gamma \). An edge which is not bounded is called a puncture.
Definition 3.3. A tropical curve is a finite connected semi-graph $\Gamma$ whose vertices $v$ are labeled with natural numbers $g_v$, and whose bounded edges are assigned lengths in $\mathbb{R}_{> 0} \cup \{\infty\}$. A tropical curve is stable if the vertices with label 0 have at least three edges or punctures attached to them, and those with label 1 have at least one edge or puncture emanating. A tropical curve is smooth, if it is stable and all bounded edges have finite length. The subsets $\Gamma^0, \Gamma^1, \Gamma_0^1, \Gamma_\infty^1$ of $\Gamma$ consist of the vertices, edges, bounded edges and punctures.

The abstract tropical curves from [9] are smooth tropical curves according to Definition 3.3.

Definition 3.4. The arithmetic genus of a tropical curve $\Gamma$ is the number

$$g(\Gamma) := 1 + \sum_{v \in \Gamma^0} (g_v - 1) + \#\Gamma^1_0.$$  

The moduli space of $n$-pointed stable tropical curves of arithmetic genus $g$ is

$$\bar{M}_{g,n}^{trop} := \{\Gamma \mid g(\Gamma) = g \text{ and } \#\Gamma^1_\infty = n\} / \sim,$$

where $\Gamma$ is a stable tropical curve, and $\Gamma \sim \Gamma'$ means that there is a homeomorphism $\Gamma \to \Gamma'$ which takes vertices to vertices, bounded edges to bounded edges, punctures to punctures and respects the labellings.

We will in the following often suppress the labelling of vertices. However, it should be born in mind that the interpretation is meant to be that of the labelling by “genus of component of special fibre in stable model of a $p$-adic curve”. The following theorem makes apparent the meaningfulness of this interpretation. Later, Corollary 4.12 could be used to recover the suppressed labelling of vertices in finite covers of tropical curves.

Theorem 3.5. The moduli space $\bar{M}_{g,n}^{trop}$ is a compact polyhedral complex of pure dimension $3g - 3 + n$, and the map

$$\text{trop}: M_{g,n} \to M_{g,n}^{trop}, [C, p_1, \ldots, p_n] \mapsto [\text{trop}(C), p_1, \ldots, p_n]$$

is well-defined and has dense image in $\bar{M}_{g,n}^{trop}$.

Proof. The space $\bar{M}_{g,n}^{trop}$ is clearly a compactification of the space $M_{g,n}^{trop}$ of $n$-pointed smooth tropical curves by adding extra cells. That the space $M_{g,n}$ is a polyhedral complex of the named pure dimension was seen in [9, Ex. 2.13] for $g = 0$. The general case can be proved in a similar way.

Let $X \to \text{Spec } K^0$ be a stable $n$-pointed curve. The canonical formal metric $\|d_L\|$ on the line bundle $L$ from the beginning of this subsection restricts to a metric $\|\|$ on the line bundle $L = L|_X$ on the generic fibre $X$. This endows the stable reduction graph $\Gamma$ of $X$ with a metric. Namely, $(L, \|\|)$ induces a commuting diagram

$$\begin{array}{ccc}
X & \overset{\Phi_L}{\longrightarrow} & \mathbb{P}^N \\
\downarrow & & \downarrow \\
\Gamma & \overset{\text{Val}}{\longrightarrow} & \text{T}^*_\mathbb{P}^N \\
\end{array}$$

where $\Gamma$ is the combinatorial graph underlying the closure of $\text{Val}(\Phi_L(X))$ in $\text{T}^*_\mathbb{P}^N$.

If $X$ is smooth, then $\Gamma$ is given the structure of a tropical curve, as follows easily with [6] Thm. 2.1.1].
Then for any constant sheaf \( \mathcal{A} \) the tropicalisation map \( \text{trop} : \overline{\mathcal{M}} \to \mathcal{M} \) is well known to equal the arithmetic genus of the tropicalisation of \( \mathcal{A} \) as the fibre in \( t = 0 \) of a family \( X_t \) of curves flat over \( K \) with constant reduction graph \( \Gamma \), and such that \( X_t \) is smooth for \( t \neq 0 \). Let \( X_t \) be the stable \( K^0 \)-model of \( X_t \), and let \( x_t \in X_t \) converge to \( x_0 \). In other words, there is a section \( s : T \to X \), where \( X \to T \) is the family \( X_t \), such that \( s(t) = x_t \). Let \( L := \omega_{X/T}(t_1, \ldots, t_n)^{\otimes e} \), where \( t_1, \ldots, t_n : T \to X \) are the n punctures of \( X \to T \), and let \( L_t := L|_{X_t} \). It is well known that the reduction map \( \pi_t : X^n \to X_{t, \sigma} \) to the special fibre \( X_{t, \sigma} \) of \( X_t \) has the property that the pre-image of a double point \( \xi_t \in X_{t, \sigma} \) is an open annulus \( A_t \). The edge \( e \) in \( \Gamma \) corresponding to \( \xi_t \) corresponds for \( t \neq 0 \) to an edge \( e_t \) in \( \Gamma_t := \text{Val}(\Phi_L(X_t)) \) and has length equal to the thickness of the annulus \( A_t \). Now assume that \( A_t \) contains \( x_t \). Because \( x_t \) is not a puncture, it defines a point \( v \in e \) corresponding to a point \( v_t \in e_t \) for \( t \neq 0 \). Let \( S_t \subseteq A_t \) be a circle containing \( x_t \) such that \( \pi_t(S_t) = v_t \). We can realise the family \( X \) in such a way that \( S_t \) shrinks for \( t \to 0 \) to the double point \( x_0 \), while the outer boundary circle of \( A_t \) remains of constant radius 1. In this case, the length of \( e_t \) increases unboundedly to \( \infty \), the length of \( e_0 \). By construction, the latter is the edge corresponding to \( e \) in the limit \( \Gamma_0 \) of the family of tropical curves \( \Gamma_t \). Hence, \( \Gamma_0 \) is a stable tropical curve.

The arithmetic genus of the tropicalisation of \( x \in \overline{\mathcal{M}}_{g,n} \) is well-known to equal the arithmetic genus of any model of \( x \) over \( \text{Spec} K^0 \).

The space \( \overline{\mathcal{M}}_{g,n} \) is also called the Deligne-Mumford compactification of \( \mathcal{M}_{g,n} \). Important for us is a stratification of \( \overline{\mathcal{M}}_{g,n} \) according to Betti number.

**Lemma 3.6.** Let \( C \) be an irreducible, non-singular projective curve of genus \( g \). Then for any constant sheaf \( A \) on \( C \), it holds true that \( H^1(C, A) \cong \mathbb{A}^0 \), where \( b \) is the first Betti number of \( \text{trop}(C) \).

**Proof.** [7 Prop. 7.4.3].

Lemma 3.6 implies that

\[
\overline{\mathcal{M}}_{g,n} = \prod_{b=0}^{g} \overline{\mathcal{M}}_{g,n}^b,
\]

where \( \overline{\mathcal{M}}_{g,n}^b \) consists of the isomorphism classes of punctured curves \( (C, p_1, \ldots, p_n) \) with \( h^1(C) = b \).

**Corollary 3.7.** The tropicalisation map \( \text{trop} : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}^{\text{trop}} \) satisfies

\[
\text{trop}(\overline{\mathcal{M}}_{g,n}^b) \subseteq \overline{\mathcal{M}}_{b,n}^{\text{trop}},
\]

where the image is dense.

**Proof.** This is immediate from Proposition 3.5 and Lemma 3.6. Alternatively, a more careful proof Proposition 3.5 would yield the same result.

It is known that \( \overline{\mathcal{M}}_{g,n} \) is a projective variety [13 Thm. 6.1]. It would be desirable to construct a tropical ample line bundle on \( \overline{\mathcal{M}}_{g,n}^{\text{trop}} \) in a similar way as can be done for its classical counterpart.

### 3.3. Stable maps of degree d to \( \mathbb{P}^1 \)

Let \( f : \overline{\mathcal{C}} \to \mathbb{P}^1 \) be a stable map of degree \( d \) of \( p \)-adic curves which is defined over \( K \). We assume for \( \mathcal{C} := \overline{\mathcal{C}} \setminus \{ p_1, \ldots, p_n \} \) that \( [C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n} \). Let \( B_f \subseteq \mathbb{P}^1(K) \) be the set of branch points of \( f \). Then
there exist \( K^0 \)-models \( C \) and \( C' \) of \( C \) and \( C' := \mathbb{P}^1 \setminus B_f \) such that \( f \) extends to a map \( F: C \to C' \). This yields an induced map between tropical curves
\[
trop(f): \Gamma := \trop(C \setminus f^{-1}(B_f)) \to \Gamma' := \trop(\mathbb{P}^1 \setminus B_f)
\]
called the tropicalisation of \( f \).

4. A tropical Riemann-Hurwitz formula

Let \( \Gamma \) be a tropical curve. For reasons of convenience, we assume \( \Gamma \) to be smooth in the remainder of this article. It is often convenient for us to deal with the tropical analogon of complete or projective curve. For this reason, we adjoin to \( \Gamma \) an additional vertex at the end of each unbounded edge of \( \Gamma \). This yields the completion \( \hat{\Gamma} \) of \( \Gamma \). However, for us it is important to keep track of the set of ends \( \hat{\Gamma} \setminus \Gamma \) as they can be interpreted as \( K \)-rational points on non-Archimedean curves for a suitable valued field \( K \).

The degree \( \deg(v) \) of a vertex \( v \in \Gamma^0 \) is defined as
\[
\deg(v) := \# \{ e \mid e \vdash v \},
\]
where \( e \vdash v \) means an edge \( e \) emanating from a vertex \( v \). A tropical curve will be called binary, if the degree of each vertex is three. A binary tree is a binary tropical curve whose first Betti number vanishes.

**Definition 4.1.** Let \( \Gamma \) be a tropical curve. Then a divisor on \( \Gamma \) is a formal finite sum
\[
D = \sum_{v \in \Gamma^0} D(v)[v]
\]
with \( D(v) \in \mathbb{Z} \). The divisor is expressed as \( D = D_0 + D_\infty \), where
\[
D_\infty := \sum_{v \in \hat{\Gamma} \setminus \Gamma^0} D(v)[v]
\]
is called the part at infinity of \( D \). The divisor
\[
K_\Gamma := \sum_{v \in \Gamma^0} (\deg(v) - 2)[v]
\]
is called the canonical divisor of \( \Gamma \).

If \( \Gamma \) is the tropicalisation of a smooth \( p \)-adic curve \( X \), then the canonical divisor
\[
K_\Gamma = \sum_{v \in \Gamma^0} (\deg(v) - 2)[v] - \sum_{e \in \Gamma^1_\infty} [e]
\]
is the sum of a divisor on \( X \) and a sum of vertices of \( \Gamma \). However, the part consisting of points of the completion \( \hat{X} \) of \( X \) is different from the canonical divisor of \( \hat{X} \). In fact, it is merely minus the sum of the punctures on \( X \) defined by \( X \). On the other hand, any divisor on \( \Gamma \) can be interpreted as an element of the free abelian group on the points of the Berkovich analytification \( X^\text{an} \) of the \( p \)-adic curve \( \hat{X} \).

**Remark 4.2.** Clearly, for a tropical curve \( \Gamma \), it holds true that
\[
\deg K_\Gamma = 2b_1(\Gamma) - 2,
\]
where \( b_1(\Gamma) = b_1(\hat{\Gamma}) = \# \Gamma^1 - \# \Gamma^0 + 1 \) is the first Betti number of \( \Gamma \).
A morphism \( \phi : \Gamma \to \Gamma' \) of tropical curves will mean for us in this article a continuous map between the underlying topological spaces which takes edges to edges and vertices to vertices.

**Definition 4.3.** A weighted tropical curve is a tropical curve \( \Gamma \) together with a map \( w : \Gamma^1 \to \mathbb{N} \setminus \{0\} \), called the weights. A morphism of tropical curves \( \phi : \Gamma \to \Gamma' \) is said to be \( w \)-harmonic, if the weights \( w \) on \( \Gamma \) are such that for all \( v \in \Gamma^0 \) the quantity

\[
m_{\phi, w}(v) := \sum_{e \in \phi^{-1}(e')} w(e)
\]

is independent of the choice of edge \( e' \parallel \phi(v) \in \Gamma^1 \). If \( w \) is constant and equal to 1, then a \( w \)-harmonic morphism will be called harmonic. The quantity \( m_{\phi, w} \) is called the multiplicity of \( v \) in \( \phi \) with respect to \( w \).

We will often suppress the weights \( w \) in the notation of multiplicity, when the weights are clear from the context.

**Lemma 4.4.** The following inequality holds true:

\[
\deg(v) \leq m_{\phi}(v) \deg(\phi(v)) =: \deg_w(v),
\]

and is in general not an equality.

*Proof.* The statement follows immediately from the obvious equality

\[
\sum_{e \parallel v} w(e) = m_{\phi}(v) \deg(\phi(v))
\]

and the fact that the values of the weights are at least one. \( \square \)

The degree of a \( w \)-harmonic morphism generalises the corresponding notion for graph morphisms from \([1]\):

**Definition 4.5.** The degree of a \( w \)-harmonic morphism \( \phi : \Gamma \to \Gamma' \) of tropical curves is defined as

\[
\deg \phi := \sum_{e \in \phi^{-1}(e')} w(e),
\]

where \( e' \in \Gamma^1 \) is arbitrary.

**Remark 4.6.** Clearly, a morphism of degree \( d \) is surjective. We must, however, prove that the degree of \( \phi \) is well-defined.

*Proof.* The corresponding modification of the proof of \([1]\) Lemma 2.4] proves the statement. \( \square \)

When we speak of a morphism of degree \( d \), we usually mean that the weights of \( \Gamma \) are fixed. There is some freedom of choice due to the deck group of \( \phi \), i.e. the automorphisms of \( \Gamma \) leaving the fibres of \( \phi \) invariant.

**Lemma 4.7.** For any vertex \( v' \in \Gamma' \), we have

\[
\deg(\phi) = \sum_{v \in \phi^{-1}(v')} m_{\phi}(v).
\]

*Proof.* The proof is as simple as that of \([1]\) Lemma 2.6]. \( \square \)
Theorem 4.9 (Tropical Riemann-Hurwitz formula). Let $\phi: \Gamma \to \Gamma'$ be a morphism of tropical curves of degree $d$. Then

$$\phi: \text{Div} \Gamma \to \text{Div} \Gamma', \quad D \mapsto \sum_{v \in \Gamma^0} D(v)[\phi(v)]$$

$$\phi^*: \text{Div} \Gamma' \to \text{Div} \Gamma, \quad D' \mapsto \sum_{v' \in \Gamma'^0} \sum_{v \in \phi^{-1}(v')} m_{\phi}(v)D(v')[v]$$

are the push-forward and the pull-back homomorphisms, respectively.

Let $\phi: \Gamma \to \Gamma'$ be a morphism of tropical curves of degree $d$. The divisor $R_\phi := K_\Gamma - \phi^*K_{\Gamma'}$ is called the ramification divisor of $\phi$. The branch divisor of $\phi$ is $\text{br}(\phi) := \phi_*R_\phi$. A tip of a tropical curve $\Gamma$ is a vertex of degree one. The set of tips of $\Gamma$ will be denoted by $\Gamma^0_\infty$. A vertex $v \in \Gamma^0$ such that $R_\phi(v) = 0$ will be called unramified, just like in the classical case.

**Theorem 4.9 (Tropical Riemann-Hurwitz formula).** Let $\phi: \Gamma \to \Gamma'$ be a morphism of tropical curves of degree $d$. Then it holds true that

1. $2b_1(\Gamma) - 2 = (2b_1(\Gamma') - 2) \cdot d + \deg R_\phi$
2. $(R_{\phi})_0 = \sum_{v \in \Gamma^0} (2m_{\phi}(v) - 2 - (\deg_{\omega}(v) - \deg(v))) [v]$.  
3. $(R_{\phi})_\infty = \sum_{v \in \Gamma^0_\infty} (\deg_{\phi}(v) - 1)[v]$
4. $b_1(\Gamma) \leq b_1(\Gamma')$.  

In particular, the degree of $R_\phi$ is non-negative and even.

**Proof.** Formula (3) follows easily from the formula

$$\deg \phi^* D' = \deg \phi \cdot \deg D',$$

whose proof is an adaptation of the unweighted version \[1, \text{Lemma 2.13}]. This implies that $\deg R_\phi$ is of even degree. The formulae (4) and (5) are straightforward.

Let for any graph $G$ denote $C(G)$ the chain complex associated to $G$, after possibly assigning orientations to the edges. Then $\phi$ induces a morphism of chain complexes $C(\Gamma) \to C(\Gamma')$ which, by assumption, admits a well-defined section $\sigma: C(\Gamma') \to C(\Gamma)$ satisfying

$$[e'] \mapsto \sum_{e \in \phi^{-1}(e')} w(e)[e].$$

The existence of the section $\sigma$ implies that there is an injective homomorphism $H_1(\Gamma', \mathbb{Z}) \to H_1(\Gamma, \mathbb{Z})$ from which (6) follows. \(\square\)

Let $f: C \to C'$ be a morphism of smooth projective $p$-adic curves of degree $d$, and $\phi: \Gamma \to \Gamma'$ its tropicalisation according to Section 3. Assume that $\phi$ is a morphism of tropical curves in our sense. Let $v \in \Gamma^0$ and $v' := \phi(v)$. The star of $v$ denotes the tropical curve consisting of $v$ labelled with $g_v$ and the edges in $\Gamma$ adjacent to $v$. The restricted morphism $\phi_v: \Gamma_v \to \Gamma_{v'}$ is the tropicalisation of a morphism $f_v: C_v \to C_{v'}$ constructed as follows. Via tropicalisation, the star $\Gamma_v$ corresponds to an affinoid subset $U_v$ of $C$. It can be completed with affinoid disks to a projective curve $C_v$ of genus $g_v$. \[1, \text{Thm. 7.5.16}\], and one obtains by continuation of $f|_{U_v}: U_v \to U_{v'}$ a morphism $f_v: C_v \to C_{v'}$ of degree $m_v \leq d$. The branch locus $B_{f_v}$ is contained in the set of punctures $\Gamma_{v'}$. Let $e' \models v'$. Then we
assign to \( e \in f^{-1}_v(e') \) as weight \( w(e) \) the ramification degree of the corresponding point in the cover \( f_v \). Note that \( w(e) \) does not depend on the vertex \( v \) adjacent to \( e \). Hence, we obtain weights \( w \) on \( \Gamma \) and call these the weights induced by \( f \).

**Lemma 4.10.** The tropical morphism as above is \( w \)-harmonic for the weights \( w \) on \( \Gamma \) induced by the morphism \( f \).

**Proof.** This follows from the local equality

\[
\sum_{e \in f^{-1}_v(e')} w(e) = m_v
\]

for each vertex \( v \in \Gamma \).

Whenever \( \Gamma \to \Gamma' \) is the tropicalisation of a finite map of smooth \( p \)-adic curves, we assume that the weights of \( \Gamma \) are induced by \( f \).

**Theorem 4.11** (RHM-criterion). Assume that \( \phi: \Gamma \to \Gamma' \) is a morphism of tropical curves which is the tropicalisation of a map \( f: X \to X' \) of smooth projective \( p \)-adic curves of degree \( d \), where \( X' \) is a Mumford curve. Then the following statements are equivalent:

1. \( X \) is a Mumford curve.
2. \( R_\phi \) is effective.
3. \( R_\phi = R_f \), where \( R_f \) is the ramification divisor of \( f \).

**Proof.**

3 \( \Rightarrow \) 1. This follows from Theorem 4.9 and the classical Riemann-Hurwitz theorem for curves.

1 \( \Rightarrow \) 3. Now assume that \( X \) is a Mumford curve. First, consider the case that \( \Gamma \) has precisely one vertex \( v \). Then \( R_\phi(v) = 0 \), as otherwise \( \deg R_\phi \neq \deg R_f \) yields the contradiction \( g(X) \neq b_1(\Gamma) \) (where the latter is 0). In the general case, the \( \Gamma_v \) of a vertex \( v \in \Gamma \) yields as above a morphism of tropical curves associated to the cover \( f_v: X_v \to X'_v \) of projective smooth \( p \)-adic curves. From the assumption, it follows that \( X_v \) and \( X'_v \) are Mumford curves. By the first case, it holds true that \( R_\phi(v) = R_{\phi_v}(v) = 0 \). It follows that \( R_\phi = R_f \), as claimed.

3 \( \Rightarrow \) 2. This is obvious.

2 \( \Rightarrow \) 3. Note that for \( v \in \Gamma^0 \) it holds true that

\[
2m_\phi(v) - 2 + 2g(X_v) = \deg R_{f_v} = \sum_{x \in \Gamma_v^{\infty}} (w(e) - 1) = \deg w(v) - \deg(v)
\]

where the first equality holds true by the classical Riemann-Hurwitz theorem. Hence, \( R_\phi(v) > 0 \) implies (by Theorem 4.11) \( \deg R_{f_v} < 2m_\phi(v) - 2 \), from which it follows by 7 that \( g(X_v) < 0 \). This is a contradiction. So, if \( R_\phi \) is effective, then necessarily \( R_\phi = R_f \). 

We remark that, due to Theorem 4.11 we cannot define a straightforward tropical version of the Fantechi-Pandharipande map \( \bar{M}_{g,0}(\mathbb{P}^1, d) \to \mathbb{P}^r \), where \( r \) is determined by the classical Riemann-Hurwitz theorem. What we can get for general tropical curves is a map from \( \bar{M}_{g,0}(\mathbb{P}^1, d) \) into the monoid of effective divisors on \( (\mathbb{P}^1)^{an} \).

**Corollary 4.12.** In the situation as in Theorem 4.11 it holds true that

\[
R_\phi(v) = -2g(C_v) = -2g_v,
\]

where \( v \in \Gamma^0 \) and \( C_v \) is defined after the proof of Theorem 4.9.
Proof. This follows easily from (7).

Remark 4.13. The result of Theorem 4.11 is that the degree of the ramification divisor $R_\phi$ depends on the relative position of the branch points of the covering $f: X \to X'$.

5. Tropical Hurwitz numbers

In this section, we develop tropical methods for counting Mumford curves covering the $p$-adic projective line depending on the position of the branch points. This can be viewed as an extension of [3, 4], where the focus was on Galois covers of Mumford curves.

5.1. Binary branch trees. After dealing with comb-shaped targets, we prove that tropical Hurwitz numbers above binary trees count solely Mumford curves.

Definition 5.1. By a tropical comb we mean a tropical curve $\Gamma$ of the form:

```
  ←  ←  ↓  ↓  ...
  •          •
  ↓  ↓
```

The horizontal geodesic line will be called the backbone, the two horizontal ends the handles, and the other ends the teeth of $\Gamma$.

Let a set $\{\eta_0 = (\eta_0^1, \ldots, \eta_0^{\ell_0}), \ldots, \eta_{n+1} = (\eta_{n+1}^1, \ldots, \eta_{n+1}^{\ell_{n+1}})\}$ of partitions of a natural number $d$, and a tropical comb $\Gamma'$ with $n$ vertices $v_1, \ldots, v_n$, be given, where

- the number $n + 2 \geq 3$ is the number of branch points defined by the classical Riemann-Hurwitz formula. We will construct covers $\Gamma \to \Gamma'$ according to the following rules:

1. Start with $\ell_0$ ends above the left handle, and weight these with the entries of $\eta_0$. These ends will be called strands.
2. Above $v_1$ create vertices by joining and splitting or continuing strands above the next edge of the backbone. Weight the new outgoing strands such that for each vertex $v$ the sum of incoming weights equals the sum of outgoing weights. This number is $m_v$.
3. From all vertices $v$ above $v_1$ let strands emanate above the tooth $t_1$ attached to $v_1$ such that the total number of strands above $t_1$ is $\ell_1$, their weights are the entries of $\eta_1$, and the sum of the weights of the new strands emanating from $v$ equals again $m_v$.
4. In (2) and (3) make sure that all vertices $v$ above $v_1$ satisfy $\deg(v) = m_v + 2$.
5. Repeat the procedure successively above the vertices $v_2, \ldots, v_n$, observing rule (4) each time.
6. Above the right handle make sure there are $\ell_{n+1}$ strands weighted with the entries of $\eta_{n+1}$.

Definition 5.2. The number $H^g_d(\eta_0, \ldots, \eta_{n+1})$ is defined to be the number of isomorphism classes of covers $\Gamma \to \Gamma'$ satisfying the above rules (1) to (6) with $b_1(\Gamma) = g$. It is called tropical Hurwitz number.

Notice, that if we set $\eta_0 = \eta$, $\eta_{n+1} = \nu$ and $\ell_i = d - 1$ for $i = 1, \ldots, n$, then $H^g_d(\eta_0, \ldots, \eta_{n+1})$ specialises to the double Hurwitz number $H^g_d(\eta, \nu)$ from [5]. In fact, the authors of [5] consider covers of the tropical projective line, which can be obtained from our covers by contracting all teeth. The rules (2) to (4) applied
to the case of simple ramification enforce the vertices of the upper tropical curve to have precisely three adjoint strands above the backbone of the comb.

Let \( H_d^g(\eta_0, \ldots, \eta_{n+1}) \) denote the Hurwitz number of smooth connected covers of \( \mathbb{P}^1 \) of degree \( d \) and genus \( g \) ramified above \( n + 2 \) points with ramification profiles \( \eta_0, \ldots, \eta_{n+1} \). If we fix the branch locus \( \mathbf{B} \subseteq \mathbb{P}^1(K) \), then we can associate to the lower curve the moduli point \( x = [\mathbb{P}^1 \setminus \mathbf{B}, \mathbf{B}] \in M_{0,n-1} \). The quantity \( H_d^g(\eta_0, \ldots, \eta_{n+1})^{\text{Mumf}}(x) \) then denotes the number of Mumford curves of genus \( g \) covering any representative of \( x \) in degree \( d \) with the assigned ramification profiles. Taking into consideration Remark 5.3 below, this restricted Hurwitz number depends only on the combinatorial type of the tropicalisation of the lower punctured curve. The inequality \( H_d^g(\eta_0, \ldots, \eta_{n+1})^{\text{Mumf}}(x) \leq H_d^g(\eta_0, \ldots, \eta_{n+1}) \) trivially holds true.

**Remark 5.3.** When we speak in the following of certain topological types of tropical curves, we require that all bounded edges have some minimal length \( \geq 0 \), depending on whether or not the prime \( p \) divides the ramification index of an adjacent vertex. The reason is that only edges of sufficient length can possibly come from Mumford curves; in the contrary case, there fails to be a discrete action on the corresponding Bruhat-Tits tree. The precise minimal edge lengths can be calculated with the methods from [4].

**Theorem 5.4.** It holds true that

\[
H_d^g(\eta_0, \ldots, \eta_{n+1})^\text{trop} = H_d^g(\eta_0, \ldots, \eta_{n+1})^{\text{Mumf}}(x) = H_d^g(\eta_0, \ldots, \eta_{n+1})
\]

where \( x \in M_{0,n-1} \) is in comb position.

**Proof.** We need to prove that \( H_d^g(\eta_0, \ldots, \eta_{n+1})^\text{trop} = H_d^g(\eta_0, \ldots, \eta_{n+1}) \). Then the first equality is a consequence of Theorem 4.11, as for any \( \phi : \Gamma \to \Gamma' \) according to the rules (1) to (6), the ramification divisor \( R_\phi \) is effective by construction. Clearly, the branch locus of any cover of \( p \)-adic curves with tropicalisation equal to \( \phi \) is in comb position.

Let \( \phi : \Gamma \to \Gamma' \) be a tropical cover counted by \( H_d^g(\eta_0, \ldots, \eta_{n+1}) \). It can be obtained by glueing local covers \( \phi_v : \Gamma_v \to \Gamma'_v \) defined after the proof of Theorem 4.11. The trees \( \Gamma'_v \) consist of the vertex \( v_i = \phi(v) \) and three ends, one of which is the tooth \( t_i \). For fixed \( v_i \), the cover

\[
\phi_i : \Gamma_i := \prod_{v \in \phi^{-1}(v_i)} \Gamma_v \to \Gamma'_v
\]

is of ramification type \( (\gamma_i^-, \eta_i, \gamma_i^+) \), and the glueing condition is \( \gamma_i^+ = \gamma_{i+1}^- \). Of course, \( \gamma_0^- = \eta_0 \) and \( \gamma_{n+1}^+ = \eta_{n+1} \). The covers \( \phi_v \) correspond to covers \( f_v : \mathbb{P}^1 \to \mathbb{P}^1 \) of the same degrees and ramification types. Hence, \( H_d^g(\eta_0, \ldots, \eta_{n+1})^\text{trop} \) counts all possible genus \( g \) covers of the projective line obtainable by glueing genus 0 covers of \( \mathbb{P}^1 \) according to the same glueing conditions. By specialising the Degeneration Theorem (or join-cut recursion) [15] Thm. 3.15, it follows that this method counts all the way up to \( H_d^g(\eta_0, \ldots, \eta_{n+1}) \).

As a corollary, we obtain that double tropical Hurwitz numbers count Mumford curves:

**Corollary 5.5.** It holds true that

\[
H_d^g(\eta, \nu)_{\text{CJM}}^{\text{CJM}} = H_d^g(\eta, \nu) = H_d^g(\eta, \nu)^{\text{Mumf}}(x),
\]
Proof. Let \( h : \Gamma \to \mathbb{R} \cup \{ \pm \infty \} \) be a tropical map to \( \mathbb{P}^1 \) of degree \( \Delta \) in the sense of \([5]\), where the multiset \( \Delta \) is given as 
\[
\Delta = \{-\eta_1, \ldots, -\eta_{\ell(h)}, \nu_1, \ldots, \nu_{\ell(h)}\}.
\]

Let \( v' \in \mathbb{R} \) be a branch point of \( h \). We may assume that \( h^{-1}(v') \) is a discrete subset of \( \Gamma \). By subdivision of edges, we may further assume that every pre-image of \( v' \) is a vertex. Now, attach to every branch point in \( \mathbb{R} \) a half-line in order to obtain a tropical comb. Attach also to every vertex \( v \) of \( \Gamma \) a set of \( m(v) - 1 \) half-lines, where \( m(v) \) is the sum of the weights on edges coming into \( v \) from the left. The resulting map is an extension of \( h \) to a morphism \( \phi : \Gamma \to \Gamma' \) of degree \( d \) simply as 
\[
\phi \to \phi + \nu \to \phi \lor \nu \to \phi \lor \nu + \infty \to \phi \lor \nu + \infty + \nu.
\]

Clearly, \((R_\phi)_0 = 0\). Hence, \( H^0_d(\eta, \nu)^{\text{CJM}} \) counts covers of \( p \)-adic \( \mathbb{P}^1 \) of degree \( d \) by curves \( C \) such that \( g(C) = b_1(\text{trop}(C)) \), i.e. Mumford curves. Hence, \( H^0_d(\eta, \nu)^{\text{CJM}} = H^0_d(\eta, \nu)^{\text{trop}} \) which, by Theorem \([5,4]\) counts Mumford curves ramified over points in comb position. \(\square\)

For \( x = [C, p_1, \ldots, p_n] \in M_{0,n-1} \), denote \([\text{trop}(C), p_1, \ldots, p_n]\) simply as \( \mathcal{T}(x) \).

**Theorem 5.6.** Let \( x \in M_{0,n-1} \) such that \( \mathcal{T}(x) \) is binary. Assume that \( \eta_0, \ldots, \eta_{n+1} \) are \( n+2 \) partitions of \( d \). Then 
\[
H^0_d(\eta_0, \ldots, \eta_{n+1})^{\text{Mumf}}(x) = H^0_d(\eta_0, \ldots, \eta_{n+1}).
\]

Proof. The binary tree \( T = \mathcal{T}(x) \) can be obtained by glueing combs \( T_i = \mathcal{T}(x_i) \) along an extra end \( e_i \) for each glueing morphism. The consequence is a construction of tropical degree \( d \) maps \( f : \Gamma \to \Gamma' \) from genus \( g \) curves via local pieces \( f_i \) from genus \( g_i \) curves and the end \( e_i \) corresponds to a branch point of \( f_i \). Clearly, the genera sum up to at most \( b_1(\Gamma) \leq g \). Since the vertices in \( \Gamma_i \) above the origin of \( e_i \) are unramified by Lemma \([5,3]\) and Theorem \([4,11]\) the same holds true for the vertices of \( \Gamma \). Putting together \( b_1(\Gamma_i) = g_i \) and Theorem \([4,11]\) again, it follows that the Hurwitz number does indeed count Mumford curves above \( x \).

Using again the classical join-cut recursion as in the proof of Theorem \([5,4]\) it follows that the tropical count equals the classical count. \(\square\)

An immediate consequence is:

**Corollary 5.7.** The number \( H^0_d(\eta_0, \ldots, \eta_{n+1})^{\text{trop}} \) does not depend on the ordering of the partitions \( \eta_i \).

### 5.2. Edge Contractions

Let \( x \in M_{0,n-1} \) and \( T = \mathcal{T}(x) \) the associated tropical curve. Assume that \( T \) is a binary curve, and contract a bounded edge \( e' \in T^2 \) in order to obtain the tropical curve \( \Gamma' \). Let \( x' \in M_{0,n} \) be a punctured projective line such that \( \mathcal{T}(x') = \Gamma' \). Then \( H^0_d(\eta, \nu) \) does not count anymore only Mumford curves covering \( x' \). Namely, any tropical curve \( \Gamma \) with \( b_1(\Gamma) = g \) above \( \Gamma' \) having the interior of a graph shaped like

\[\text{wiener}^1\]

\[1\text{This is called wiener in [5].}\]
above an edge $e'$ will be contracted to the tropicalisation of a curve $C$ having the property $b_1(\text{trop}(C)) < g$. Hence, for the double Hurwitz numbers we obtain the result:

\[
H^g_d(\eta, \eta_1, \ldots, \eta_n, \nu)_{\text{Mumf}}(x') = H^g_d(\eta, \nu)^{CIM} - W(e')
\]

where $W(e')$ is the number of double Hurwitz covers of degree $d$ and genus $g$ such that $\phi^{-1}(e')$ contains a wiener. Here, $e'$ is the segment in $T$ consisting of $e'$ and its adjacent vertices.

In the case of general ramification types above combs, we must let $W(e')$ count multi-wieners above $e'$ in order to obtain the correct count in (8) for Mumford curves. By a multi-wiener, we mean a graph of the following kind:

\[\text{Multi-wiener diagram}\]

This gives an algorithm for computing Hurwitz numbers for Mumford curves covering $\mathbb{P}^1$ with given branch locus in $M_{0,n+2}$.

5.2.1. An algorithm. Let $\Gamma' = \mathcal{J}(x)$, where $x \in M_{0,n-1}$ is the branch locus. Let $\gamma$ be a shortest path in $M_{0,n-1}^{\text{trop}}$ to the nearest cell of maximal dimension (it consists of binary trees) successively passing through cells of dimensions increasing by one. At the other end of the path $\gamma$ lies a point $y \in M_{0,n-1}$ such that $T = \mathcal{J}(y)$ is a binary tree. The path induces a choice of edges $E_\gamma = \{e_1, \ldots, e_{\ell(\gamma)}\}$ in $T$ which get contracted in order to obtain $\Gamma'$. Let $W(E_\gamma)$ be the number of Hurwitz covers $\phi: \Gamma \to T$ counted by $H^g_d(\eta_0, \ldots, \eta_{n+1})^{\text{trop}}$ such that $b_1(\phi^{-1}(E_\gamma)) > 0$, where $E_\gamma$ is the closure of $E_\gamma$ in $T$.

Proposition 5.8. The quantity $W(E_\gamma)$ does not depend on the choice of path the $\gamma$. It holds true that

\[
H^g_d(\eta_0, \ldots, \eta_{n+1})_{\text{Mumf}}(x) = H^g_d(\eta_0, \ldots, \eta_{n+1})^{\text{trop}} - W(E_\gamma).
\]

Proof. This follows immediately from Theorem 5.6 and the fact that the classical Hurwitz number is independent of the positions of the branch points. □

5.3. Cyclic covers: the Harbater-Mumford condition. Assume that $f : C \to \mathbb{P}^1$ is a cyclic cover of degree $n$. The methods we have developed allow to recover the well-known Harbater-Mumford condition which are necessary for $C$ to be a Mumford curve. Namely, if $C$ is a Mumford curve, then the local tropical pieces of the cover are themselves cyclic covers $\Gamma_v \to \Gamma_v'$ of degree $m_v$. By tropical Riemann-Hurwitz, the degree of the vertex $v$ must be $m_v + 2$. Hence, there can be ramification only above two of the three ends, and each ramification degree is $m_v$. It follows that the corresponding cover of $p$-adic curves $f_v : \mathbb{P}^1 \to \mathbb{P}^1$ is cyclic of degree $m_v$, and the two branch cycles $\sigma$ and $\tau$ are inverse to each other.

However, the Harbater-Mumford condition is not sufficient. In fact, the previous section shows that the branch points must form a tree with sufficiently many edges in order for the upper curve to have the correct first Betti number. In [4], we have calculated the precise branch trees (depending also on the prime number $p$) for which cyclic covers have Mumford curves on the top.
Example 5.9. Let $E$ be an elliptic curve, and $f: E \rightarrow \mathbb{P}^1$ the cover of degree 2 which we may assume to be ramified in the points $0, 1, \lambda, \infty$ with $v(\lambda) = 0$. It is known that $E$ is a Tate curve, if and only if $v(\lambda - 1) > 2 \cdot v(2)$ (cf. [18, Thm. 5] for $p \neq 2$ and [4, Ex. 3.8] for $p$ arbitrary). In fact, after making the edge in the branch tree long enough, the upper tropical curve trop($E \setminus f^{-1}\{0, 1, \lambda, \infty\}$) obtains a wiener, the length of which can be viewed as the tropical $j$-invariant of the tropical elliptic curve by the well-known formulae relating $j$, $\lambda$ and the Tate-parameter $q$ for which $E$ has the representation as Tate curve $E \cong \mathbb{G}_m/q^2$.

In the case $v(j) \leq 4 \cdot v(2)$, there is still a corresponding moduli point in $\overline{M}_{1,1}^{\text{trop}}$, namely the point representing the tree $\bullet$.

The other extreme point in $\overline{M}_{1,1}$ corresponds to the rational curve with one self-intersection point. Its tropicalisation has a loop of infinite length. Hence, we have exhibited $\overline{M}_{1,1}^{\text{trop}}$ as a cw-complex isomorphic to the segment $\bullet \overline{\bullet}$ and whose edge contains densely the tropicalisations of Tate curves.

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