On the arithmetic product of combinatorial species

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Abstract

We introduce two new binary operations with combinatorial species; the arithmetic product and the modified arithmetic product. The arithmetic product gives combinatorial meaning to the product of Dirichlet series and to the Lambert series in the context of species. It allows us to introduce the notion of multiplicative species, a lifting to the combinatorial level of the classical notion of multiplicative arithmetic function. Interesting combinatorial constructions are introduced; cloned assemblies of structures, hyper-cloned trees, enriched rectangles, etc. Recent research of Cameron, Gewurz and Merola, about the product action in the context of oligomorphic groups, motivated the introduction of the modified arithmetic product. By using the modified arithmetic product we obtain new enumerative results. We also generalize and simplify some results of Canfield, and Pittel, related to the enumerations of tuples of partitions with restricted meet.

1 Introduction

Informally, a combinatorial species $F$ (see [4, 17]) is a class of labelled combinatorial structures that is closed by change of labels. Being more formal, $F$ is a rule assigning to each finite set $U$, a finite set $F[U]$. The elements of $F[U]$ are called $F$-structures on the set $U$. The rule $F$ not only acts on finite sets but also on bijections between finite sets. To each bijection $\sigma : U \rightarrow V$, the rule $F$ associates a bijection $F[\sigma] : F[U] \rightarrow F[V]$ that is called the transport of $F$-structures along $\sigma$. In other words, $F$ is an endofunctor of the category $\mathcal{B}$ of finite sets and bijections.

For two species of structures $F$ and $G$, other species can be constructed throughout combinatorial operations; addition $F + G$, product $F \cdot G$, cartesian product $F \times G$, substitution $F \circ G$ and derivative $F'$. See [1] for details.

To each species $F$ are associated three main series expansions. The exponential generating series,

$$F(x) = \sum_{n \geq 0} |F[n]| \frac{x^n}{n!},$$

(1)

where $|F[n]|$ is the number of $F$-structures on the set $[n] = \{1, 2, \ldots \}$. The isomorphism types generating series,

$$\bar{F}(x) = \sum_{n \geq 0} |F[n]| / \sim |x^n|,$$

(2)
where \( F[n]/\sim \) denotes the set of isomorphism types of \( F \)-structures on \([n]\). The cycle index series,

\[
Z_F(x_1, x_2, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in S_n} \text{fix} F[\sigma] x_1^{\sigma(1)} x_2^{\sigma(2)} \cdots, \tag{3}
\]

Here \( S_n \) denotes the symmetric group, \( \text{fix} F[\sigma] := |\text{Fix} F[\sigma]| \), where \( \text{Fix} F[\sigma] \) is the set of \( F \)-structures on \([n]\) left fixed by the permutation \( F[\sigma] \), and \( \sigma_k \) is the number of cycles of length \( k \) of \( \sigma \).

Let \( F \) be a species of structures and \( n \geq 0 \) any integer. Unless otherwise be explicitly stated, we will denote by \( F_n \) the species \( F \) concentrated in cardinality \( n \),

\[
F_n[U] = \begin{cases} F[\{U\}], & \text{if } |U| = n, \\ \emptyset, & \text{if } |U| \neq n, \end{cases} \tag{4}
\]

where \( U \) is a finite set. We will also use the notation \( F_+ \) for the species of nonempty \( F \)-structures,

\[
F_+[U] = \begin{cases} F[\{U\}], & \text{if } |U| \geq 1, \\ \emptyset, & \text{if } |U| = 0. \end{cases} \tag{5}
\]

Yeh \[25, 26\] established the relationship between operations with actions of finite permutation groups and operations with species (product, substitution, cartesian product and derivative), via the decomposition of a species as a sum of molecular species. For example, consider the product of two species

\[
M \cdot N[U] := \sum_{U_1 + U_2 = U} M[U_1] \times N[U_2]. \tag{6}
\]

When \( M \) and \( N \) are molecular, there are permutation groups \( H \leq S_m \) and \( K \leq S_n \), such that \( M = X^m_H \) and \( N = X^n_K \). We have that

\[
\frac{X^m}{H} \cdot \frac{X^n}{K} = \frac{X^{m+n}}{H \times K}, \tag{7}
\]

where the direct product \( H \times K \) acts naturally over the disjoint union \([m] + [n] \equiv [m + n]\), in what is called the intransitive action. There is another natural action \( H \times K : [m] \times [n] \), the product action, without a species counterpart. Some enumerative problems have been solved by Harary \[13\] and by Harrison and High \[15\] using the cycle index polynomial of the product action.

We can define the arithmetic product of two molecular species by the formula

\[
\frac{X^m}{H} \boxplus \frac{X^n}{K} = \frac{X^{mn}}{H \times K}, \tag{8}
\]

where the action of \( H \times K \) over \([mn] \equiv [m] \times [n]\) is the product action. Then we can extend this product by linearity. But, in order to have set theoretical definition for the arithmetic product like formula (8) for the ordinary product, we need a notion of decomposition of a set into factors. In other words, a set-theoretical analogous of the factoring of a positive integer as a product of two positive integers. In this way we arrived to the concept of rectangle on a finite set. This concept was previously introduced in other context with the name of cartesian decomposition \[2\], and is a particular kind of what is called in \[21\] a small transversal of a partition.
The most interesting combinatorial construction associated to the arithmetic product is the assembly of cloned structures. Informally, an assembly of cloned $N$-structures is an assembly of $N$-structures in the ordinary sense, where in addition, all structures in the assembly are isomorphic replicas of the same structure. Moreover, information about 'homologous vertices' or 'genetic similarity' between each pair in the assembly is also provided. The structures of $M \sqdiagonal N$ have some resemblance with the structures of the substitution $M(N)$. An element of $M \sqdiagonal N$ can be represented as a cloned assembly of $N$-structures together with an external $M$-structure (an $M$-assembly of cloned $N$-structures). Because of the symmetry $M \sqdiagonal N = N \sqdiagonal M$ it also can be represented as an $N$-assembly of cloned $M$-structures. For example, for $M$ an arbitrary species and $L_+$ the species of non-empty lists, the structures of $M \sqdiagonal L_+$ could be thought of either as $M$-assemblies of cloned lists, or as lists of cloned $M$-structures.

There is a link between oligomorphic groups [5] and combinatorial species, implicit in the work of Cameron, and which we hope to have made explicit here. To each oligomorphic group $G$ we can associate a combinatorial species $F_G$. There is a correspondence between operations with oligomorphic groups and operations with species that is very similar to that established by Yeh between finite permutation groups and molecular species. For example, the intransitive product action of two oligomorphic groups translates to the ordinary product of the respective species and the wreath product to the operation of substitution. Recently, Cameron, Gewurz, and Merola have studied the product action of oligomorphic groups (see [7, 10, 12]). This has motivated us to introduce the modified arithmetic product in order to have the appropriate correspondence between operations. We have made use of this operation to obtain many new enumerative results. Using a simple manipulation of generating series (the shift trick) we greatly simplified and generalized some results of Canfield [8], and Pittel [22].

2 The arithmetic product

Definition 2.1 For a finite set $U$, we say that an ordered pair $(\pi, \tau)$ of partitions of $U$ is a partial rectangle on $U$ when $\pi \wedge \tau = 0$. If moreover $\pi$ and $\tau$ are independent partitions (every block of $\pi$ meets every block of $\tau$) we call it a rectangle. More generally, a partial rectangle of dimension $k$, or a $k$-partial rectangle is a tuple $(\pi_1, \pi_2, \ldots, \pi_k)$ of partitions such that $\pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_k = 0$. It is called a $k$-rectangle if,

$$|B_1 \cap \cdots \cap B_k| = 1, \text{ for all } B_1 \in \pi_1, \ldots, B_k \in \pi_k. \quad (9)$$

This definition of rectangle is equivalent to the “cartesian decomposition” of Baddeley, Praeger and Schneider [2]. If $(\pi, \tau)$ is a rectangle on $U$, we can arrange the elements of $U$ in a matrix whose rows are the blocks of $\pi$ and whose columns are the blocks of $\tau$. Two matrices represent the same rectangle if we can obtain one from the other by interchanges of rows or columns. The same can be say about the partial rectangles except for the fact that some of the entries of the matrix could be empty. Figure 1 shows an example of partial rectangle and rectangle on a set with 12 elements (the symbol * stands by a empty intersection).

For a rectangle $(\pi, \tau)$ on $U$ obviously holds $|U| = |\pi||\tau|$. The height of a rectangle $(\pi, \tau)$ is $|\pi|$. Naturally height$(\pi, \tau)$ divides $|U|$. For $|U| = n$ represent the number of
rectangles of height $d$ with the symbol $\{n\}_d$. It is no difficult to see that

$$\left\{ \frac{n}{d} \right\} = \frac{n!}{d!(n/d)!}. \quad (10)$$

Consider $\mathcal{R}$ the species of rectangles, that is, for $U$ a finite set,

$$\mathcal{R}[U] = \{ (\pi, \tau) \mid (\pi, \tau) \text{ is a rectangle on } U \}. \quad (11)$$

If $n \geq 1$, we have

$$|\mathcal{R}[n]| = \sum_{d|n} \left\{ \frac{n}{d} \right\}. \quad (12)$$

In an analogous way, for the species $\mathcal{R}^{(k)}$, of $k$-rectangles, we have

$$|\mathcal{R}^{(k)}[n]| = \sum_{d_1d_2\ldots d_k=n} \left\{ \frac{n}{d_1, d_2, \ldots, d_k} \right\}, \quad (13)$$

where

$$\left\{ \frac{n}{d_1, d_2, \ldots, d_k} \right\} = \frac{n!}{d_1!d_2!\cdots d_k!}. \quad (14)$$

**Definition 2.2** (Arithmetic Product of Species) Let $M$ and $N$ be species of structures such that $M[\emptyset] = N[\emptyset] = \emptyset$. The arithmetic product of $M$ and $N$, is defined as follows

$$(M \circ N)[U] := \sum_{(\pi, \tau) \in \mathcal{R}[U]} M[\pi] \times N[\tau], \quad (15)$$

where the sum represents the disjoint union and $U$ is a finite set. In words, the elements of $(M \circ N)[U]$ are tuples of the form $(\pi, \tau, m, n)$, where $m \in M[\pi]$ and $n \in N[\tau]$. Recall that given a bijection $\sigma : U \rightarrow V$ and a partition $\pi$ of $U$, $\sigma$ induces the partition $\pi' = \sigma(\pi) = \{ \sigma(A) \mid A \in \pi \}$ of $V$ and another bijection $\sigma^\pi : \pi \rightarrow \pi'$, sending $A \mapsto \sigma(A)$, for every $A \in \pi$. Similarly for the partition $\tau$.

The transport along a bijection $\sigma : U \rightarrow V$ is carried out by setting

$$(M \circ N)[\sigma](\pi, \tau, m, n)) = (\pi', \tau', M[\sigma^\pi](m), N[\sigma^\tau](n)). \quad (16)$$

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Figure 2: Graphical representation of arithmetic product.

Figure 2 illustrates an \((M \boxtimes N)\)-structure on a set with 12 elements. Here the capital letters (except \(M\) and \(N\)) are the labels for the blocks of two partitions forming the rectangle.

**Example 2.1** Figure 3 shows that the species \(R\) of rectangles satisfies the combinatorial equation \(R = E_+ \boxtimes E_+\), where \(E_+\) is the species of non empty sets.

**Proposition 2.1** Let \(M\) and \(N\) be species of structures such that \(M[\emptyset] = N[\emptyset] = \emptyset\). Then the exponential generating series of species \(M \boxtimes N\) is

\[
(M \boxtimes N)(x) = \sum_{n \geq 1} \sum_{d \mid n} \frac{n!}{d!} |M[d]| |N[n/d]| \frac{x^n}{n!}.
\]  

(17)
Proof.

\[(M \sqcap N)[n] = \sum_{(\pi, \tau) \in \mathcal{R}[n]} |M[\pi]| |N[\tau]|\]

\[= \sum_{d|n} \sum_{(\pi, \tau) \in \mathcal{R}[n]} |M[d]| |N[n/d]| \]

\[= \sum_{d|n} |\{(\pi, \tau) \in \mathcal{R}[n] \mid \text{height}(\pi, \tau) = d\}| |M[d]| |N[n/d]| \]

\[= \sum_{d|n} \binom{n}{d} |M[d]| |N[n/d]|.\]

Example 2.2 (Regular octopuses [4, p. 56]) Consider the species \(C\) of oriented cycles and \(L_+\) of non empty linear orders. Figure 4 represents an \((C \sqcap L_+)-structure\) on a set with 8 elements. Since \(|C[n]| = (n-1)!\) and \(|L_+[n]| = n!\), we obtain

\[|(C \sqcap L_+)[n]| = \sum_{d|n} \binom{n}{d} |C[d]| |L_+[n/d]| \]

\[= \sigma(n)(n-1)!,\]

where \(\sigma(n)\) is the sum of non negative divisors of \(n\). Then, the exponential generating series is

\[(C \sqcap L_+)(x) = \sum_{n \geq 1} \sigma(n)(n-1)! \frac{x^n}{n!} \]

\[= \sum_{n \geq 1} \sigma(n) \frac{x^n}{n}.\]
Example 2.3 (Ordered lists of equal size) Figure 5 illustrates an \((L_+ \boxtimes L_+)\)-structure on a set with 6 elements. Since \(|L_+[n]| = n!\), the number of \((L_+ \boxtimes L_+)\)-structures on a set of \(n\) elements is

\[
|L_+[n]| = \sum_{d|n} \binom{n}{d} |L_+[d]| |L_+[n/d]|
\]

\[
= d(n)n!,
\]

where \(d(n)\) is the number of non negative divisors of \(n\). Then, we obtain the generating series

\[
(L_+ \boxtimes L_+)(x) = \sum_{n \geq 1} n!d(n) \frac{x^n}{n!}
\]

\[
= \sum_{n \geq 1} d(n)x^n.
\]

Example 2.4 Let \(S_+\) be the species of nonempty permutations. It is clear that

\[
(S_+ \boxtimes S_+)(x) = (L_+ \boxtimes L_+)(x) = \sum_{n \geq 1} d(n)x^n,
\]

The structures of \((S_+ \boxtimes S_+)[U]\) are rectangles enriched with permutations on each side. Formally, they are tuples of the form \((\pi, \tau, \sigma_1, \sigma_2)\), where \((\pi, \tau) \in \mathcal{R}[U]\), \(\sigma_1 \in S_+[\pi]\), and \(\sigma_2 \in S_+[\tau]\). By the definition of rectangle, for each element \(b \in U\) there exists a unique pair of sets \((A_b, B_b)\) such that \(b \in A_b \cap B_b\). The pair \((\sigma_1, \sigma_2)\) induces the permutation \(\sigma_1 \boxtimes \sigma_2 \in S_+[U]\), which sends the element \(b \in A_b \cap B_b\) to the unique element in \(\sigma_1(A_b) \cap \sigma_2(B_b)\) (see Figure 5). Let \(\mathcal{R}\) be the species defined as follows

\[
\mathcal{R}[U] = \{(\pi, \tau, \sigma) : (\pi, \tau) \in \text{Fix}\mathcal{R}[\sigma], \sigma \in S[U]\}.
\]

The function

\[
\boxtimes_U : (S_+ \boxtimes S_+)[U] \rightarrow \mathcal{R}[U]
\]

\[
(\pi, \tau, \sigma_1, \sigma_2) \rightarrow (\pi, \tau, \sigma_1 \boxtimes \sigma_2)
\]

is a natural bijection with inverse \((\pi, \tau, \sigma) \mapsto (\pi, \tau, \sigma_\pi^\tau, \sigma^\tau)\). The family \(\{\boxtimes_U\}_{U \in \mathbb{B}}\), defines a species isomorphism \(\boxtimes : S_+ \boxtimes S_+ \rightarrow \mathcal{R}\).
Proposition 2.2 Let $M$, $N$ and $R$ be species of structures such that $M[\emptyset] = N[\emptyset] = R[\emptyset] = \emptyset$, and $X$ the singular species. The product $\boxtimes$ has the following properties:

\[ M \boxtimes N = N \boxtimes M, \quad (19) \]
\[ M \boxtimes (N \boxtimes R) = (M \boxtimes N) \boxtimes R, \quad (20) \]
\[ M \boxtimes (N + R) = M \boxtimes N + M \boxtimes R, \quad (21) \]
\[ M \boxtimes X = X \boxtimes M = M, \quad (22) \]
\[ (M \boxtimes N)^* = M^* \boxtimes N^*, \quad (23) \]
\[ M \boxtimes X^n = M(X^n), \quad (24) \]
\[ M \boxtimes L_+ = \sum_{n \geq 1} M(X^n). \quad (25) \]

All the properties are not difficult to prove. In particular, the reader may verify that both sides of equation (20) evaluated at a set $U$, are naturally equivalent to the set

\[ \sum_{(\pi_1,\pi_2,\pi_3) \in R^{(3)}} M[\pi_1] \times N[\pi_2] \times R[\pi_3]. \quad (26) \]

In general, for a family $\{M_i\}_{i=1}^k$ of species with $M_i[\emptyset] = \emptyset$, we have

\[ (\boxtimes_{i=1}^k M_i)[U] = \sum_{(\pi_1,\pi_2,\ldots,\pi_k) \in R^{(k)}[U]} \prod_{i=1}^k M_i[\pi_i], \quad (27) \]

for every $i = 1, 2, \ldots, k$.

From equation (25), the structures of $M \boxtimes L_+$ may be thought of as $M$-assemblies of lists of equal size (see Figure 7).
2.1 The arithmetic product and generating series

Definition 2.3 For any two monomials $x^n$ and $x^m$ we define the arithmetic product $x^n \square x^m := x^{nm}$. Extend this product by linearity to exponential formal power series with zero constant term.

We easily obtain that

$$
\left( \sum_{n \geq 1} a_n \frac{x^n}{n!} \right) \square \left( \sum_{n \geq 1} b_n \frac{x^n}{n!} \right) = \sum_{n \geq 1} c_n \frac{x^n}{n!},
$$

where

$$
c_n = \sum_{d|n} \left( \frac{n}{d} \right) a_d b_{n/d}. \tag{29}\n$$

Observe that the exponential formal power series with the arithmetic product form a ring with identity $x$. The substitution $x^n \leftarrow \frac{1}{n^s}$, makes it isomorphic to the ring of modified formal Dirichlet series,

$$
\sum_{n \geq 1} \frac{a_n}{n! n^s}.
$$

This motivates the following definition.

Definition 2.4 Let $M$ be a species of structures satisfying the condition $M[\varnothing] = \varnothing$. Then the modified Dirichlet generating series of $M$ is

$$
D_M(s) = \sum_{n \geq 1} \frac{|M[n]|}{n! n^s}. \tag{30}\n$$
Thus, for the species $E_+, L_+, C$ and $S_+$, we have:

\[
\mathcal{D}_{E_+}(s) = \sum_{n \geq 1} \frac{1}{n!} s^n, \quad (31)
\]

\[
\mathcal{D}_{L_+}(s) = \sum_{n \geq 1} \frac{1}{n^n} = \zeta(s), \quad (32)
\]

\[
\mathcal{D}_C(s) = \sum_{n \geq 1} \frac{1}{n^{s+1}} = \zeta(s+1), \quad (33)
\]

\[
\mathcal{D}_{S_+}(s) = \sum_{n \geq 1} \frac{1}{n^n} = \zeta(s). \quad (34)
\]

From Proposition 2.1, we obtain the following.

**Proposition 2.3** For species of structures $M$ and $N$ with the condition $M[\emptyset] = \emptyset = N[\emptyset]$, we have:

\[
(M \boxdot N)(x) = M(x) \Box N(x), \quad (35)
\]

and

\[
\mathcal{D}_{M \boxdot N}(s) = \mathcal{D}_M(s) \cdot \mathcal{D}_N(s). \quad (36)
\]

For a formal power series $R(x)$ we have

\[
x^n \boxdot R(x) = R(x^n). \quad (37)
\]

Thus we have the generating series identity

\[
(M \boxdot N)(x) = \sum_{n \geq 1} \frac{|M[n]|}{n!} N(x^n). \quad (38)
\]

In particular

\[
x^n \boxdot \frac{x}{1-x} = \frac{x^n}{1-x^n}, \quad (39)
\]

and we obtain that the generating series of $M \boxdot L_+$ is the Lambert series

\[
(M \boxdot L_+)(x) = \sum_{n \geq 1} \frac{|M[n]|}{n!} \frac{x^n}{1-x^n}. \quad (40)
\]

By (25) we also have

\[
(M \boxdot L_+)(x) = \sum_{n \geq 1} M(x^n) \quad (41)
\]

Using the previous two equations we get

\[
(C \boxdot L_+)(x) = \sum_{n \geq 1} \frac{x^n}{n(1-x^n)} = \sum_{n \geq 1} \ln \left( \frac{1}{1-x^n} \right) = \sum_{n \geq 1} \frac{\sigma(n)}{n} x^n, \quad (42)
\]

\[
(L_+ \Box L_+)(x) = \sum_{n \geq 1} \frac{x^n}{1-x^n} = \sum_{n \geq 1} d(n)x^n, \quad (43)
\]

\[
(L_+^* \Box L_+)(x) = \sum_{n \geq 1} \frac{x^n}{1-x^n} = \sum_{n \geq 1} \frac{x^n}{n(1-x^n)^2} = \sum_{n \geq 1} \frac{\sigma(n)}{n} x^n. \quad (44)
\]
(see [9] and [27] for more properties of Lambert series). Those identities translate to Dirichlet generating series as

\[ D_{L \boxtimes L}(s) = \zeta(s+1)\zeta(s) = \sum_{n \geq 1} \frac{\sigma(n)}{n} n^{-s}, \quad (45) \]

\[ D_{L \boxtimes L}(s) = \zeta^2(s) = \sum_{n \geq 1} d(n)n^{-s}, \quad (46) \]

\[ D_{L \bullet L}(s) = \zeta(s-1)\zeta(s) = \sum_{n \geq 1} \sigma(n)n^{-s}. \quad (47) \]

By equation (38) we also obtain:

\[ (C \boxtimes M)(x) = \ln \prod_{n \geq 1} \left( \frac{1}{1 - x^n} \right)^{|M[n]|}, \quad (48) \]

\[ E(C \boxtimes M)(x) = \prod_{n \geq 1} \left( \frac{1}{1 - x^n} \right)^{|M[n]|}. \quad (49) \]

Let \( M \) be a species of structures. To describe the compatibility of the product \( \boxtimes \) with the transformation \( M \to Z_M \), it is necessary to define a product \( \boxtimes \) for two index series (see [13]). First we have the following Lemma.

**Lemma 2.1** Let \((\pi, \tau, \sigma_1, \sigma_2)\) be an element of \((S_+ \boxtimes S_+) [U]\) and \(\sigma = \sigma_1 \boxtimes \sigma_2 \in S[U]\). If the cycle type of \(\sigma, \sigma_1,\) and \(\sigma_2\), are respectively \(\alpha, \beta\) and \(\gamma\), then we have

\[ \alpha_k = \sum_{[i, l] = k} (i, l)\beta_1\gamma_l, \quad k = 1, 2, \ldots, [d, n/d], \quad (50) \]

where \(d = \text{height}(\pi, \tau)\), \([i, l]\) denotes the least common multiple of \(i\) and \(l\), and \((i, l)\) the greatest common divisor.

**Proof.**

Analogous to the proof of proposition 7(b) in ([4] p. 74). \(\square\)

We will say that \(\alpha = \beta \boxtimes \gamma\) when they satisfy the equation (51). Define now the operation \(\boxtimes\) on monomials by

\[ \left( \prod_{i=1}^{m} x_i^{\beta_i} \right) \boxtimes \left( \prod_{l=1}^{n} x_l^{\gamma_l} \right) := \prod_{i=1}^{m} \prod_{l=1}^{n} x_{\beta_i\gamma_l(i, l)} = \prod_{k=1}^{nm} x_k^{\alpha_k}, \quad (51) \]

where \(\alpha_k = \sum_{[i, l] = k} (i, l)\beta_1\gamma_l = (\beta \boxtimes \gamma)_k\). Equivalently

\[ x^\beta \boxtimes x^\gamma := x^{\beta \boxtimes \gamma}. \quad (52) \]

Finally extend linearly this operation to polynomials and formal power series. Note that \(x_1^i \boxtimes x_1^j = x_1^{ij}\) as in Definition 2.3.

**Definition 2.5** For \(\alpha, \beta,\) and \(\gamma\) as above, define the coefficient

\[ \left\{ \begin{array}{ll} \alpha & \text{if } \beta \boxtimes \gamma = \alpha, \\ \beta, \gamma & \text{otherwise}. \end{array} \right\} \quad (53) \]
Lemma 2.2 Let σ be a permutation on a finite set U with cycle type α. For β and γ as above, let \( S^\sigma_{\beta, \gamma} \) be the set of tuples \((\pi, \tau, \sigma_1, \sigma_2) \in (S_+ \boxtimes S_+)[U]\), such that \( \sigma_1 \boxtimes \sigma_2 = \sigma \), and the cycle type of \( \sigma_1 \) and \( \sigma_2 \) are respectively β and γ. Then

\[
|S^\sigma_{\beta, \gamma}| = \left\{ \alpha \beta, \gamma \right\}.
\]

Proof. The group Aut(σ) acts transitively on \( S^\sigma_{\beta, \gamma} \) in the following manner: for \( \eta \in \text{Aut}(\sigma) \),

\[
\eta \cdot (\pi, \tau, \sigma_1, \sigma_2) := (\eta(\pi), \eta(\tau), \eta^\pi \sigma_1(\eta^\tau)^{-1}, \eta^\tau \sigma_2(\eta^\tau)^{-1}).
\]

The order of the group fixing any element of \( S^\sigma_{\beta, \gamma} \) is \( \text{aut}(\beta) \text{aut}(\gamma) \). \( \square \)

Proposition 2.4 Let M and N be two species of structures. Then, the cycle index series and the type generating series associated to the species \( M \boxtimes N \) satisfy the identities

\[
Z_{M \boxtimes N}(x_1, x_2, \ldots) = Z_M(x_1, x_2, \ldots) \boxtimes Z_N(x_1, x_2, \ldots),
\]

\[
\tilde{M} \boxtimes \tilde{N}(x) = \tilde{M}(x) \boxtimes \tilde{N}(x).
\]

Proof. It is not difficult to deduce the second identity from the first. Using equation (52) we obtain

\[
Z_M(x) \boxtimes Z_N(x) = \sum_{\alpha} \left( \sum_{\beta \boxtimes \gamma = \alpha} \left\{ \alpha \beta, \gamma \right\} \text{fixM}[\beta] \text{fixN}[\gamma] \right) \frac{x^{\alpha}}{\text{aut}(\alpha)}.
\]

Then, all we have to prove is that

\[
\text{fix}(M \boxtimes N)[\alpha] = |\text{Fix}(M \boxtimes N)[\sigma]| = \sum_{\beta \boxtimes \gamma = \alpha} \left\{ \alpha \beta, \gamma \right\} \text{fixM}[\beta] \text{fixN}[\gamma],
\]

where \( \sigma \) is any permutation on a finite set U, with cycle type \( \alpha \). Since

\[
\text{Fix}(M \boxtimes N)[\sigma] = \{(\pi, \tau, m, n) \mid (\pi, \tau) \in \text{FixR}[\sigma], m \in \text{FixM}[\sigma^\pi], n \in \text{FixN}[\sigma^\tau]\},
\]

we have

\[
\text{fix}(M \boxtimes N)[\alpha] = \sum_{(\pi, \tau) \in \text{FixR}[\sigma]} |\text{FixM}[\sigma^\pi]| |\text{FixN}[\sigma^\tau]|\]

(61)

\[
= \sum_{(\pi, \tau, \sigma_1, \sigma_2) \in (S_+ \boxtimes S_+)[U]} |\text{FixM}[\sigma_1]| |\text{FixN}[\sigma_2]|.
\]

(62)

The last identity is obtained from bijection (19) in example (2.4). Classifying the permutations \( \sigma_1 \) and \( \sigma_2 \) according with their cycle type, we get

\[
\text{fix}(M \boxtimes N)[\alpha] = \sum_{\beta, \gamma} \sum_{(\pi, \tau, \sigma_1, \sigma_2) \in S^\sigma_{\beta, \gamma}} \text{fixM}[\beta] \text{fixN}[\gamma].
\]

(63)

By lemma (2.2) we obtain the result. \( \square \)
2.1.1 The cyclotomic identity

There are various bijective proofs of the cyclotomic identity

\[
\frac{1}{1 - \alpha x} = \prod_{n \geq 1} \left( \frac{1}{1 - x^n} \right)^{\lambda_n(\alpha)}.
\]

See for example: Metropolis-Rota [20], Taylor [24] and Bergeron [3]. We propose here a very simple one, as an application of the combinatorics of the arithmetic product.

Let \( C(\alpha) \) be the species \( \alpha \)-colored cycles, or necklaces, following the terminology of Metropolis and Rota (see [20]). The elements of \( C(\alpha) \) are pairs of the form \((\sigma, f)\), where \( \sigma \in C[U] \) and \( f : U \to A \) is an arbitrary function assigning colors (letters) in a totally ordered set \( A \) (alphabet) with \( |A| = \alpha \), to the labelled beads of the cycle \( \sigma \).

Denote by \( S(\alpha) \) the species of assemblies of necklaces. It is clear that

\[
C(\alpha)(x) = \ln \left( \frac{1}{1 - \alpha x} \right), \quad \text{(64)}
\]

\[
S(\alpha)(x) = \frac{1}{1 - \alpha x}. \quad \text{(65)}
\]

Let \((\sigma, f)\) be a necklace in \( C(\alpha)[U] \), where \( |U| = n \). The integer

\[
d = \min\{1 \leq k \leq n \mid f \circ \sigma^k = f\}
\]

is called the period of \((\sigma, f)\). When \( d = n \), the necklace is called aperiodic. The flat part \( \overline{C(\alpha)} \) of \( C(\alpha) \) (see [18]) is the species of aperiodic necklaces.

Let \((\sigma, f)\) be an aperiodic necklace. To each of the \( n \) possible presentations of the cycle \( \sigma \) as an ordered tuple \( \sigma = (a_1, a_2, \ldots, a_n) \) corresponds a different word \( f(a_1)f(a_2)\ldots f(a_n) \) in the alphabet \( A \). The lowest of them in the lexicographic order is called a Lyndon word. The ordering \( \sigma = (a_1, a_2, \ldots, a_n) \) such that the corresponding word \( w \) is Lyndon will be called the standard presentation of \( \sigma \). Thus, the necklace \((\sigma, f)\) can be identified with the pair \((w, l)\), where \( w \) is a Lyndon word and \( l \), a linear order on \( U \), is the standard presentation of the cycle \( \sigma \). The number of Lyndon words on \( A \) is well known to be \( \lambda_n(\alpha) := \frac{1}{n} \sum_{d|n} \mu(d) \alpha^{n/d} \), where \( \mu \) is the classical Möbius function. Then, the decomposition of \( \overline{C(\alpha)} \) as a sum of molecular species is

\[
\overline{C(\alpha)} = \sum_{n \geq 1} \lambda_n(\alpha)X^n. \quad \text{(66)}
\]

**Proposition 2.5** We have the equalities:

\[
C(\alpha) = C \boxdot \overline{C(\alpha)}, \quad \text{(67)}
\]

\[
S(\alpha) = E(C \boxdot \overline{C(\alpha)}). \quad \text{(68)}
\]

**Proof.** Equation (68) is immediate from (67). To prove (67) observe that the structures of \( C \boxdot \overline{C(\alpha)} \) are regular octopuses where each tentacle (linear order) is decorated with the same Lyndon word. Join the decorated tentacles following the external cycle of the octopus to obtain a necklace whose period is the common length of the tentacles. Conversely, given a necklace of period \( d \), there is a unique way of cutting it into pieces...
of length $d$ such that the word on each piece is Lyndon. It is easy to see how to get an element of $C \square C^{(\alpha)}$ out of this sliced necklace. See for example [23, pages 4-5], where a similar bijection is used to count ordinary octopuses.

Equation (68) can be interpreted as the cyclotomic identity lifted at a combinatorial level. By equations (48) and (49), we get:

$$C^{(\alpha)}(x) = \ln \prod_{n \geq 1} \left( \frac{1}{1 - x^n} \right)^{\lambda_n(\alpha)}, \quad (69)$$

$$S^{(\alpha)}(x) = \frac{1}{1 - \alpha x} = \prod_{n \geq 1} \left( \frac{1}{1 - x^n} \right)^{\lambda_n(\alpha)}. \quad (70)$$

3 Assemblies of cloned structures

In this section we will see that an $(M \square N)$-structure can be interpreted as an “$M$-assembly of cloned $N$-structures”. The intuition behind this is the following: an element of $(M \square N)[U]$ consist of a rectangle $(\pi, \tau)$ on $U$ enriched with an $M$-structure $m$ on one side $(\pi)$ and an $N$-structure $n$ on the other side $(\tau)$. Because $\tau$ have the same number of elements than any block of $\pi$, we could laid an isomorphic copy (clone) of $n$ on each block $B$ of $\pi$. Those copies of $n$ together with the “external structure” $m \in M[\pi]$ form an $M$-assembly of cloned $N$-structures.

To make this definition precise we need some formalism. Two elements of $U$ belonging to the same block of $\tau$ will be called homologous. For example let $A$ be the species of rooted trees. In Figure 9 we represent in two ways a structure of $C \square A$ as a $C$-assembly of cloned rooted trees. In both of them homologous elements are represented with the same color (pattern). Roots of cloned trees in the right hand side are connected like the original cycle on $\pi$ in the left hand side, the rest of homologous elements are connected with closed segmented curves.

We now express conveniently the relation among homologous elements. Let $B \in \pi$ and $b \in B$. It is clear that there is only one block $C \in \tau$ such that $\{b\} = B \cap C$. For
(B, B') ∈ π × π, we define the bijection

\[ \Phi_{B, B'}^\tau : B \rightarrow B' \]
\[ b \mapsto b' \]

where \( b' \) is the unique element of \( B' \cap C \). In other words, \( \Phi_{B, B'}^\tau \) sends each element of \( B \) to its homologous in \( B' \). It is easy to verify that

(i) \( \Phi_{B, B} = \text{Id}_B \)

(ii) \( \Phi_{B', B''}^\tau \circ \Phi_{B, B'}^\tau = \Phi_{B, B''}^\tau \), for all \( B, B', B'' \in \pi \).

**Definition 3.1** Let \( U \) be a finite set and \( M, N \) two species of structures such that \( M[\emptyset] = N[\emptyset] = \emptyset \). An \( M \)-assembly of cloned \( N \)-structures is a triple \((\{n_B\}_{B \in \pi}, \tau, m)\), where:

(i) \( (\pi, \tau) \in \mathcal{R}[U] \),

(ii) \( \{n_B\}_{B \in \pi} \) is an assembly of \( N \)-structures \((n_B \in N[B], \text{for each } B \in \pi)\), along with the condition

\[ N[\Phi_{B, B'}]n_B = n_{B'} \quad (71) \]

for every pair \((B, B') \in \pi \times \pi\),

(iii) \( m \in M[\pi]\).

![Figure 9: A C-assembly of cloned rooted trees.](image)

**Proposition 3.1** Let be \( M \) and \( N \) be two species of structures. Then the species \( M \Join N \) and the species of \( M \)-assemblies of cloned \( N \)-structures are isomorphic.

**Proof.** Let \( U \) be a finite set and assume that \((\pi, \tau, m, n) \in (M \Join N)[U]\). For each \( B \in \pi \), let \( \Psi_{\tau, B} : \tau \rightarrow B \) be the bijection that sends each block \( C \in \tau \) to the unique element \( b \) in \( C \cap B \). For \((B, B') \in \pi \times \pi\), we have

\[ \Phi_{B, B'} = \Psi_{\tau, B'} \circ \Psi_{\tau, B}^{-1} \quad (72) \]
Let \( \Upsilon_U \) be the function
\[
\Upsilon_U : (M \boxtimes N)[U] \rightarrow M\text{-assemblies of cloned } N\text{-structures on } U,
\]
that sends \((\pi, \tau, m, n)\) to \((\{n_B\}_{B \in \pi}, \tau, m)\), where \(n_B = N[\Psi_{\tau,B}](n)\), for each \(B \in \pi\).
From equation (72) condition (71) is satisfied.

Let now \((\{n_B\}_{B \in \pi}, \tau, m)\) be an \(M\)-assemblies of cloned \(N\)-structures on \(U\). From condition (71) and equation (72), the \(N\)-structure \(n \in N[\tau]\), where \(n := N[\Psi_{\tau,B}^{-1}](n_B)\), remains the same independently of the block \(B \in \pi\) that we choose. It is easy to check that \(\Upsilon_U\) has as inverse the function that sends \((\{n_B\}_{B \in \pi}, \tau, m)\) to \((\pi, \tau, m, n)\).

The family of bijections \(\{\Upsilon_U\}_{U \in \mathbb{B}}\) is the desired isomorphism. \(\Box\)

### 3.1 Hyper-cloned rooted trees

The species of \(R\)-enriched rooted trees could be defined by the implicit combinatorial equation
\[
A_R = X \cdot R(A_R).
\]
(74)

When \(|R[\varnothing]| = 1\) equation (74) becomes
\[
A_R = X + X \cdot R_+(A_R).
\]
(75)

By changing the operation of substitution of species by the arithmetic product in equation (75) we obtain the combinatorial implicit equation for a new kind of structures, the \(R\)-enriched hyper-cloned rooted trees (\(R\)-enriched HRT’s)
\[
H_R = X \cdot (R_+ \boxtimes H_R).
\]
(76)

This equation leads to the following recursive definition: an \(H_R\)-structure on a set \(U\) is either a singleton vertex (when \(|U| = 1\)), or is obtained by choosing a vertex in \(a_0 \in U\) (the root) and attaching to it an \(R_+\)-assembly of cloned \(H_R\)-structures on \(U \setminus \{a_0\}\). To give an explicit description of this kind of structures we need some previous notation.

Let \(t_U\) be an \(R\)-enriched rooted tree on \(U\). The subset of \(U\) formed by the non-leave elements will be denoted as \(U^+\). For \(a \in U^+\), the set \(U_a\) will be the set of vertices in \(U\) that precede \(a\) when the edges are oriented towards the root. The partition \(\pi_a\) will be the partition of \(U_a\) induced by the forest of \(R\)-enriched trees that are attached to \(a\), and \(\{t^a_B\}_{B \in \pi_a}\) such forest.

**Definition 3.2** An \(R\)-enriched HRT on a finite set \(U\) is an \(R\)-enriched tree \(t_U\) together with a family of partitions, \(\{\tau_a\}_{a \in U^+}, \tau_a \in \text{Par}[U_a]\), satisfying the conditions:

(i) For every \(a \in U^+\), \((\pi_a, \tau_a)\) is a rectangle on \(U\),

(ii) For every pair \(B, B' \in \pi_a\),
   \begin{itemize}
   
   (a) \(\Phi^\tau_{a, B', B} : t^a_B \rightarrow t^a_{B'}\) is an isomorphism of \(R\)-enriched trees,
   (b) If \(\Phi^\tau_{a, B, B'}(a_1) = a_2\) with \(a_1 \in B \cap U^+\), then \(\Phi^\tau_{a, B, B'}(\tau_{a_1}) = \tau_{a_2}\).
   \end{itemize}
Figure 10: Element of \( H_E \). Homologous elements according to \( \tau_{a_1}, \tau_{a_2}, \tau_{a_3} \) are linked with different kinds of lines.

By condition (ii)(b), the family \( \{\tau_a\}_{a \in U^+} \) is completely determined by the partitions \( \{\tau_{a_0}, \tau_{a_1}, \ldots, \tau_{a_k}\} \) on any branch \( \{a_0, a_1, \ldots, a_k\} \subseteq U^+ \) of \( t_U \).

From equation (76) we get the recursion:

\[
|H_R[1]| = 1, \tag{77}
\]

\[
|H_R[n + 1]| = \sum_{d|n} \left\{ \begin{array}{c} n \\ d \end{array} \right\} |R[d]| |H_R[n/d]| n \geq 1. \tag{78}
\]

In particular, \( H_L \) is the species of rooted achiral trees (see [14]). The ten first coefficients of the sequence \( |H_E[n]| \) are shown in Table 1.

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( |H_E[n]| \) | 1 | 1 | 2 | 3 | 10 | 11 | 192 | 193 | 3554 | 10080 |

Table 1: The ten first coefficients \( |H_E[n]| \).

4 Multiplicative species

We have a notion of multiplicative species, the “categorified” analogous of the multiplicative arithmetic function in analytic number theory (see [11]).

**Definition 4.1** Let \( M \) be a species of structures satisfying the condition \( M[\emptyset] = \emptyset \). We say that \( M \) is multiplicative if,

\[
M_{rs} = M_r \Box M_s, \tag{79}
\]

whenever \( (r, s) = 1 \).
For example, using the isomorphisms of groups \( \{1\} \simeq \{1\} \times \{1\} \) and \( C_{rs} \simeq C_r \times C_s \), when \((r,s)=1\), we have the species \( L_+ \) and \( C \) are multiplicative, respectively. The following three propositions are proved straightforwardly.

**Proposition 4.1** Let \( M \) and \( N \) be two multiplicative species of structures. Then the species \( M^\bullet \) and \( M \boxtimes N \) are multiplicative.

Combining the above examples and the above proposition, we find that the species of regular octopuses \( C \boxtimes L_+ \) is multiplicative.

**Proposition 4.2** Let \( M \) be a species of structures. Then \( M \) is multiplicative, if and only if, \( M_1 = X \), and, for \( n \geq 2 \),

\[ M_n = M_{p_1^\alpha_1} \boxtimes M_{p_2^\alpha_2} \boxtimes \cdots \boxtimes M_{p_k^\alpha_k}, \quad (80) \]

with \( n = \prod_{i=1}^k p_i^{\alpha_i} \) the canonical prime factorization of the integer \( n \).

**Proposition 4.3** (Euler product formula) Let \( M \) be a multiplicative species of structures. Then

\[ M = \bigsqcup_{p \in \mathbb{P}} (X + M_p + M_p^2 + \cdots), \quad (81) \]

where \( \mathbb{P} \) denotes the set of prime numbers.

The following corollary follows immediately by taking generating series in the previous proposition.

**Corollary 4.1** Let \( M \) be a multiplicative species of structures. Then:

\[ M(x) = \bigsqcup_{p \in \mathbb{P}} \left( x + M_p(x) + M_p^2(x) + \cdots \right), \quad (82) \]

\[ D_M(s) = \prod_{p \in \mathbb{P}} \left( 1 + \sum_{k \geq 1} \frac{|M[p^k]|}{p^k p^{-k s}} \right), \quad (83) \]

\[ Z_M(x_1, x_2, \ldots) = \bigsqcup_{p \in \mathbb{P}} \left( x_1 + Z_{M_p}(x_1, x_2, \ldots) + Z_{M_p^2}(x_1, x_2, \ldots) + \cdots \right). \quad (84) \]

**Example 4.1** For the multiplicative species \( C \) of cyclic permutations we have the identities:

\[ C(x) = \bigsqcup_{p \in \mathbb{P}} \left( x + \sum_{k \geq 1} \frac{x^{p^k}}{p^k} \right) = \bigsqcup_{p \in \mathbb{P}} \left( x - \frac{x^p}{p} \right)^\square(-1), \quad (85) \]

\[ D_C(s) = \zeta(s+1) = \prod_{p \in \mathbb{P}} \left( 1 - p^{-(s+1)} \right)^{-1}, \quad (86) \]

\[ Z_C(x_1, x_2, \ldots) = \bigsqcup_{p \in \mathbb{P}} \left( (1 - p^{-1}) \sum_{j \geq 0} \sum_{k \geq j} \frac{x^{p^k-j}}{p^{k-j}} \right). \quad (87) \]
5 Oligomorphic groups and species

Let $A$ be an at most countable set, and $U$ a finite set. Denote by $A^U$ and by $(A)_U$ the set of functions and injective functions from $U$ to $A$ respectively. A permutation group $G$ on the set $A$ is called oligomorphic if $G$ has only finitely many orbits on $A^U$ for every finite set $U$.

For $h \in A^U$ denote by $\ker(h)$ the partition of $U$ whose blocks are the non-empty pre-images of elements of $A$ by $h$. Recall that each function $f \in A^U$ can be identify with a pair $(\pi, \hat{h})$, where $\pi = \ker(h)$ and $\hat{h} \in (A)_\pi$ is the injective function $\hat{h}(B) := h(b)$, for every $B \in \pi, b$ being an arbitrary element of $B$.

**Definition 5.1** Let $G, A$ and $U$ be as above. Define the species of structures $F^*_G$ by $F^*_G[U] = A^U/G$, the (finite) set of orbits of $A^U$ under the action of $G$. For a bijection $\sigma : U \rightarrow V$, define the bijection $F^*_G[\sigma] : F^*_G[U] \rightarrow F^*_G[V] \quad f \mapsto f \circ \sigma^{-1}$. (88)

This bijection is well defined since $\sigma$ commutes with the action of $G$ over $A^U$. In an analogous way we define the species $F_G$ of $G$-orbits of injective functions.

Recall that for a finite set $A$ and a species $M$, the species of $M$-enriched functions is denoted by $M^A$ (see [17]). Observe that when $A$ is finite and $G$ is the identity subgroup of $S_A$, $F^*_G[U] = A^U$ is isomorphic to $(1 + X)^A$ and $F^*_G[U] = A^U$ is isomorphic to $E^A$.

**Remark.** Cameron [5] has studied the three following counting problems: how many elements in (a) $F_G[n]$, (b) $\tilde{F}_G[n]$, (c) $F^*_G[n]$? Equivalently, how many $G$-orbits in (a) $n$-tuples of distinct elements, (b) $n$-sets, (c) all $n$-tuples?

**Proposition 5.1** We have the following combinatorial identity

$$F^*_G = F_G(E^+).$$ (89)

**Proof.** Let $h \in A^U$, since the action of $G$ does not affect the kernel of $h$, the orbit $\overline{h}$ of $h$ can be identified with the pair $(\pi, \hat{h})$, where $\pi = \ker(h)$. Obviously $\hat{h} \in F_G[\pi]$. This defines a natural bijection from $F^*_G[U]$ to $F_G(E^+)[U]$. \hfill \Box

5.1 The modified arithmetic product

Let $G$ and $H$ be two oligomorphic groups of permutations on the sets $A$ and $B$ respectively. In [7] Cameron et al. deal with the enumerative problem in the remark above for the product group $G \times H$ acting over $A \times B$. We now introduce the analogous problem in the more general context of species of structures.

**Definition 5.2** (MODIFIED ARITHMETIC PRODUCT OF SPECIES) Let $M$ and $N$ be two species of structures. Denote by $P_R$ the species of partial rectangles. We define the modified arithmetic product of $M$ and $N$ by

$$(M \Box N)[U] := \sum_{(\pi, \tau) \in P_R[U]} M[\pi] \times N[\tau],$$ (90)

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Figure 11: An \((M \boxplus N)\)-structure on a 10-set.

where the sum represents the disjoint union and \(U\) is a finite set. For a bijection \(\sigma : U \to V\), the transport \((M \boxplus N)[\sigma]\) is as in Definition 2.2.

Some of the properties of the arithmetic product in Proposition 2.2 have their analogous in this context.

**Proposition 5.2** Let \(M, N\) and \(R\) be species of structures. The product \(\boxplus\) has the following properties:

\[
\begin{align*}
M \boxplus N &= N \boxplus M, \\
M \boxplus (N \boxplus R) &= (M \boxplus N) \boxplus R, \\
M \boxplus (N + R) &= M \boxplus N + M \boxplus R, \\
X \boxplus M &= M_+, \\
M \boxplus (1 + X) &= (X + 1) \boxplus M = M, \\
M \boxplus (1 + X)^{[n]} &= M((1 + X)^{[n]}_+). 
\end{align*}
\]

**Proof.** We will only prove identity (96). An element of \((M \boxplus (1 + X)^{[n]}_+) [U]\) is of the form \((\pi, \tau, m, f)\), where \((\pi, \tau)\) is a partial rectangle on \(U\), \(m \in M[\pi]\), and \(f: \tau \to [n]\) is an injective function. Recall that the pair \((\tau, f)\) can be identified with a function \(f : U \to [n]\) whose kernel is \(\tau\). Since \(\pi \land \tau = 0\), the restriction \(f_B\) of \(f\) to each block \(B\) of \(\pi\) is injective. Conversely, if all the functions in a family \(\{f_B\}_{B \in \pi}\) are injective, then \(\pi \land \tau = 0\), \(\tau\) being the kernel of \(f := \cup_{B \in \pi} f_B\). Then, the correspondence

\[
\Omega_U : (M \boxplus (1 + X)^{[n]}_+) [U] \to M((1 + X)^{[n]}_+) [U]
\]

\[
(\pi, \tau, m, f) \mapsto (\{f_B\}_{B \in \pi}, m),
\]

is a natural bijection. \(\Box\)
Like in equation (27) the product of a family \( \{ M_i \}_{i=1}^k \) of species of structures is given by
\[
(\prod_{i=1}^k M_i)[U] = \sum_{(\pi_1, \pi_2, \ldots, \pi_k) \in P_R^k[U]} \prod_{i=1}^k M_i[\pi_i],
\]
where \( P_R^k \) is the species of \( k \)-partial rectangles.

We have the following Theorem.

**Theorem 5.1** Let \( G \) and \( H \) be two oligomorphic groups acting on sets \( A \) and \( B \) respectively. Then
\[
F_{G \times H} = F_G \bigsqcup F_H.
\]

**Proof.** Let \( h : U \rightarrow A \times B \) be an injective function. Let \( h_1 : U \rightarrow A \) and \( h_2 : U \rightarrow B \) be its components, i.e. \( h(u) = (h_1(u), h_2(u)) \) for \( u \in U \). Let \( \pi = \ker(h_1) \) and \( \tau = \ker(h_2) \). It is clear that \( \ker(h) = \ker(h_1) \land \ker(h_2) = \pi \land \tau \), and since \( h \) is injective, \( \pi \land \tau = 0 \). Then \( h \) can be identify with the tuple \((\pi, \tau, h_1, h_2)\), and its orbit under the action of \( G \times H \) with \((\pi, \tau, h_1, h_2)\), where \( h_1 \in F_G[\pi] \) and \( h_2 \in F_H[\tau] \). This defines a natural bijection between \( F_{G \times H}[U] \) and \((F_G \bigsqcup F_H)[U]\).

Take \( A = [m], B = [n], \) and \( G, H \) being the identity subgroups of \( S_m \) and \( S_n \) respectively. We obtain the isomorphism
\[
(1 + X)^{|m|} \bigsqcup (1 + X)^{|n|} = (1 + X)^{|m|\times |n|}.
\]

The exponential generating series of the modified arithmetic product of species of structures is not as straightforward to compute as in the arithmetic product case. However, the identity
\[
F_{G \times H}^*(x) = F_G^*(x) \times F_H^*(x),
\]
proved in [7], provides a device to compute this series,
\[
(F_G \bigsqcup F_H)(x) = F_{G \times H}(x).
\]
It has motivated the following general combinatorial identity.

**Theorem 5.2** Let \( \{ M_i \}_{i=1}^k \) be a family of species of structures. Then we have
\[
\prod_{i=1}^k M_i(\text{E}_+^+) = \times_{i=1}^k M_i(\text{E}_+^+),
\]
where \( \times \) is the operation of cartesian product of species,
\[
(M \times N)[U] = M[U] \times N[U].
\]

**Proof.** It is enough to prove the identity for \( k = 2 \). Consider the species \( \text{Par} \) of set partitions. For a finite set \( U \), let \( \leq_U \) be the refinement order on \( \text{Par}[U] \). For \( \eta \in \text{Par}[U] \), let \( C_\eta \) denote the order coideal of \((\text{Par}[U], \leq_U)\) of the elements greater than or equal to \( \eta \). For \( \pi \in C_\eta \) and \( B \in \pi \), let \( \hat{B} = \{ C \in \eta \mid C \subseteq B \} \) and \( \hat{\pi} = \{ \hat{B} \mid B \in \pi \} \). Clearly, \( \hat{\pi} \) is a partition of \( \eta \), and it is easy to see that the correspondence
\[
\frac{C_\eta}{\pi} \rightarrow (\text{Par}[\eta], \leq_\eta)
\]
\[
\frac{\hat{\pi}}{} \rightarrow \hat{\pi}
\]
is an order isomorphism. Then the partition \( \pi \land \tau = \eta \) is an element of \( C_\eta \) if and only if \((\hat{\pi}, \hat{\tau})\) is a partial rectangle on \( \eta \). The right hand side of (102), for \( k = 2 \), evaluated in a set \( U \) is equal to

\[
(M_1(E_+) \times M_2(E_+))[U] = \left( \sum_{\pi \in \text{Par}[U]} M_1[\pi] \right) \times \left( \sum_{\tau \in \text{Par}[U]} M_2[\tau] \right)
\]

(104)

\[
= \sum_{(\pi, \tau) \in \text{Par}[U] \times \text{Par}[U]} M_1[\pi] \times M_2[\tau]
\]

(105)

\[
= \sum_{\eta \in \text{Par}[U]} \sum_{\pi \land \tau = \eta} M_1[\pi] \times M_2[\tau].
\]

(106)

For any partition \( \varrho \) in \( C_\eta \), let \( f_{\varrho, \eta} : \varrho \rightarrow \hat{\eta} \) be the bijection sending each block \( B \) of \( \varrho \) to \( \hat{B} \). The family of bijections

\[
\alpha_U : \sum_{\eta \in \text{Par}[U]} \sum_{\pi \land \tau = \eta} M_1[\pi] \times M_2[\tau] \rightarrow \sum_{\eta \in \text{Par}[U]} \sum_{(\hat{\pi}, \hat{\tau}) \in P_K[\eta]} M_1[\hat{\pi}] \times M_2[\hat{\tau}]
\]

(107)

\[
\alpha_U := \sum_{\eta \in \text{Par}[U]} \sum_{\pi \land \tau = \eta} M_1[f_{\pi, \eta}] \times M_2[f_{\tau, \eta}]
\]

(108)

defines a natural transformation

\[
\alpha : M_1(E_+) \times M_2(E_+) \rightarrow (M_1 \square M_2)(E_+).
\]

(109)

Taking exponential generating series and cycle index series in identity (102), we obtain the following

**Corollary 5.1** Let \( \{M_i\}_{i=1}^k \) be as above. Then the following generating function identities hold

\[
(\square_i=1^k M_i)(e^x - 1) = \times_{i=1}^k M_i(e^x - 1),
\]

(110)

\[
Z_{\square_i=1^k M_i}(x) \ast Z_{E_+}(x) = \times_{i=1}^k (Z_{M_i}(x) \ast Z_{E_+}(x)),
\]

(111)

where \( \times \) means coefficient-wise product or Hadamard product as in [4], \( \ast \) means plethystic substitution, and

\[
Z_{E_+}(x) = \exp \left( \sum_{n \geq 1} \frac{x^n}{n} \right) - 1.
\]

(112)

Using equation (110) with \( M_i = E \), for \( i = 1, \ldots, k \), we recover the first identity of Theorem 1 in [3],

\[
P_R^{(k)}(e^x - 1) = (e^{e^x-1})^{x^k} = \sum_{n \geq 0} (B_n)^k \frac{x^n}{n!}.
\]

(113)

where \( B_n \) is the \( n \)-th Bell number, the number of partitions of the set \([n]\).

In order to have the identity \((M \square N)(x) = M(x) \square N(x)\) for two species of structures \( M \) and \( N \), following (110) we make the following Definition.
Definition 5.3 For two formal power series $F(x)$ and $G(x)$ define the product $\boxtimes$ by
\[
F(x) \boxtimes G(x) = (F(e^x - 1) \times G(e^x - 1)) \circ (\ln(1 + x)).
\] (114)

It is easy to see that this product is commutative and distributive with respect to the sum,
\[
(F(x) + H(x)) \boxtimes G(x) = F(x) \boxtimes G(x) + H(x) \boxtimes G(x).
\] (115)

5.2 The shift trick

Sometimes the equation (114) is too clumsy to make computations. We will provide a more efficient method. Previous to that we need the following Lemma.

Lemma 5.1 Let $F(x)$ be a formal power series. For $m$ and $n$ nonnegative integers we have the identities:
\[
F(x) \boxtimes (1 + x)^n = F((1 + x)^n - 1),\] (116)
\[
(1 + x)^m \boxtimes (1 + x)^n = (1 + x)^{mn}.
\] (117)

Proof. Equation (116) follows from identity (96). Equation (117) follows from (116) or by taking generating functions in (99). \(\blacksquare\)

From this lemma we recover the following result of Pittel [22].

Proposition 5.3 Let $k$ be a fixed positive integer. The exponential generating series $P_{R,k}(x)$, of the number of partial rectangles $(\pi, \tau)$, with $|\pi| = k$, is
\[
P_{R,k}(x) = \frac{1}{k!e} \sum_{l \geq 0} \frac{1}{l!} \left( (x + 1)^l - 1 \right)^k.
\] (118)

Proof. The required species is $P_{R,k} := E_k \boxtimes E$. Its exponential generating series is
\[
(E_k \boxtimes E)(x) = \frac{x^k}{k!e} e^x = \frac{1}{k!e} \sum_{l \geq 0} \frac{1}{l!} x^k \boxtimes (x + 1)^l.
\] (119)

Use equation (116) to finish the proof. \(\blacksquare\)

The algorithm to compute the product $F(x) \boxtimes G(x)$, of two generating series $F(x)$ and $G(x)$, runs as follows:
1. Express $F(x) = F_1(x + 1)$ and $G(x) = G_1(x + 1)$ as power series of $(x + 1)$,
2. use the distributive property and equation (117) to compute $H(x + 1) = F_1(x + 1) \boxtimes G_1(x + 1)$,
3. express back $H(x + 1)$ as a power series of $x$.

Now we solve some enumerative problems.

Theorem 5.3 The number $M(m, n, r)$ of $m \times n$ $(0,1)$-matrices with exactly $r$ entries equal to 1 and no zero row or columns, is given by
\[
M(m, n, r) = \sum_{l \geq r} \sum_{d \mid l} (-1)^{n + m - (d + l/d)} \binom{m}{d} \binom{n}{l/d} \binom{l}{r}.
\] (120)
Proof. The structures of the species $X^n \square X^m$ are the linearly ordered $m \times n$ partial rectangles. A structure of $(X^n \square X^m)[r]$ can be thought of as a $m \times n$ matrix with entries $1, 2, \ldots, r$, without repetitions, zero elsewhere, and no zero row or columns. Then, $M(m,n,r)$ is the coefficient of $x^r$ in the generating series

$$(X^m \square X^n)(x) = x^m \square x^n.$$  \hfill (121)

By shifting we get

$$x^m \square x^n = (x + 1 - 1)^m \square (x + 1 - 1)^n$$  \hfill (122)

$$= \sum_{j,k} \binom{m}{j} \binom{n}{k} (-1)^{m+n-(j+k)} (x + 1)^{j} \square (x + 1)^{k}$$  \hfill (123)

$$= \sum_{j,k} \binom{m}{j} \binom{n}{k} (-1)^{m+n-(j+k)} (x + 1)^{j} \square (x + 1)^{k}$$  \hfill (124)

$$= \sum_{j,k} \binom{m}{j} \binom{n}{k} (jk) (-1)^{m+n-(j+k)} x^r.$$  \hfill (125)

Making the change $l = jk$, we obtain the result. \hfill \Box

Corollary 5.2 The number $|P_{\mathcal{R},m,n}[r]|$ of $m \times n$ partial rectangles on $r$ elements, is given by

$$|P_{\mathcal{R},m,n}[r]| = \sum_{l \geq r} \sum_{d \mid l} \left\{ \frac{l}{d} \right\} \frac{(-1)^{m+n-(d+1/l)}}{(m-d)!(n-l/d)!(l-r)!}.$$  \hfill (126)

Proof. $|P_{\mathcal{R},m,n}[r]|$ is equal to $|(E_m \square E_n)[r]|$, which is the coefficient of $\frac{x^r}{r!}$ in the generating series

$$(E_m \square E_n)(x) = \frac{x^m}{m!} \square \frac{x^n}{n!}.$$  \hfill (127)

Then,

$$|P_{\mathcal{R},m,n}[r]| = r! \frac{M(m,n,r)}{m!n!} = \sum_{l \geq r} \sum_{d \mid l} \left\{ \frac{l}{d} \right\} \frac{(-1)^{m+n-(d+1/l)}}{(m-d)!(n-l/d)!(l-r)!}.$$  \hfill \Box

We now give a very short direct proof of the beautiful formula obtained by Pittel \cite{22}.

Theorem 5.4 The number $|P_{\mathcal{R}}^{(k)}[n]|$, of $k$-tuples of partitions $(\pi_1, \pi_2, \ldots, \pi_k)$ on $[n]$ satisfying $\pi_1 \land \pi_2 \land \cdots \land \pi_k = \emptyset$, is given by

$$|P_{\mathcal{R}}^{(k)}[n]| = e^{-k} \sum_{i_1, i_2, \ldots, i_k \geq 1} \frac{(i_1 \cdots i_k)_n}{i_1! \cdots i_k!},$$  \hfill (128)

where $(m)_n = m(m-1) \cdots (m-n+1)$.  

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Proof. By the definition of $\square$-product,
\[ P^{(k)}_{\mathcal{R}} = E^{\square k} = \underbrace{E \square \cdots \square E}_{k \text{ factors}}. \]

The exponential generating series of this species is $(e^x)^{\square k}$. Following the algorithm, we have $e^x = e^{-1}e^{(x+1)}$. Then
\[
E^{\square k}(x) = \left( e^{-1}e^{(x+1)} \right)^{\square k}
= e^{-k} \sum_{i_1, i_2, \ldots, i_k \geq 0} \frac{(x + 1)^{i_1} \cdots \cdots (x + 1)^{i_k}}{i_1! \cdots i_k!}
= e^{-k} \sum_{i_1, i_2, \ldots, i_k \geq 0} \frac{(x + 1)^{i_1} \cdots i_k!}{i_1! i_2! \cdots i_k!}
= \sum_{n \geq 0} \left( e^{-k} \sum_{i_1, i_2, \ldots, i_k \geq 0} \frac{(i_1 \cdots i_k)_n}{i_1! i_2! \cdots i_k!} \right) \frac{x^n}{n!}.
\]

As a corollary, we obtain an remarkable identity

**Corollary 5.3** For $n \geq 1$,
\[ |P^{(k)}_{\mathcal{R}}[n]| = e^{-k} \sum_{l \geq n} \frac{|\mathcal{R}^{(k)}[l]|}{(l - n)!}. \]

Proof. Making the change $l = i_1 i_2 \cdots i_k$ in equation (128) we obtain
\[
|P^{(k)}_{\mathcal{R}}[n]| = e^{-k} \sum_{l \geq n} \sum_{i_1, i_2, \ldots, i_k \geq l} \frac{(l)_n}{i_1! \cdots i_k!}
= e^{-k} \sum_{l \geq n} \frac{1}{(l - n)!} \sum_{i_1, i_2, \ldots, i_k \geq l} \frac{l!}{i_1! \cdots i_k!}. \]

To finish the proof we recall equation (13).

Theorem 5.4 is a particular case of the following general result, that can be proved without much extra effort.

**Theorem 5.5** Let $\{M_i\}_{i=1}^k$ be a family of species of structures whose exponential generating series, $M_i(x) = F_i(x+1)$, expressed as power series of $(x+1)$, are given,
\[ M_i(x) = F_i(x+1) = \sum_{n \geq 0} b_i^{(j)} \frac{(x + 1)^n}{n!}, \quad i = 1, \ldots, k. \]

Then,
\[
\langle \square \rangle_{i=1}^k M_i(x) = \sum_{n \geq 0} \left( \sum_{i_1, i_2, \ldots, i_k \geq 0} \frac{1}{i_1! i_2! \cdots i_k!} \frac{(i_1 \cdots i_k)_n}{i_1! i_2! \cdots i_k!} \right) \frac{x^n}{n!}
= a_0 + \sum_{n \geq 1} \left( \sum_{l \geq n} \frac{1}{(l - n)!} \left[ \frac{x^n}{n!} \right] (F_1 \square \cdots \square F_k)(x) \right) \frac{x^n}{n!},
\]

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where \( a_0 = |(\square^k_{i=1} M_i)[0]| = \prod_{i=1}^k |M_i[0]|. \)

### Example 5.1

Given the expansions:

\[
L(x) = \frac{1}{2 \left(1 - \frac{x+1}{2}\right)} = \sum_{n \geq 0} \frac{(x+1)^n}{2^{n+1}}, \quad (139)
\]

\[
C(x) = \ln \left(\frac{1}{2 \left(1 - \frac{x+1}{2}\right)}\right) = \ln(2^{-1}) + \sum_{n \geq 1} \frac{(x+1)^n}{n 2^n}. \quad (140)
\]

We obtain formulas for:

- The number of \((0,1)\)-matrices of any size, with \( n \) ones, and with no zero row or column,

\[
\frac{|(L \square L)[n]|}{n!} = \frac{1}{4n!} \sum_{r,s \geq 0} (rs)_n 2^{r+s}. \quad (141)
\]

- The number of matrices of any size up to column permutations, with \( n \) different elements, zero elsewhere and with no zero row or column,

\[
\frac{|(L \square E)[n]|}{n!} = \frac{1}{2e} \sum_{r,s \geq 0} (rs)_n 2^{r+s}. \quad (142)
\]

- The number of linearly ordered \( k \)-partial rectangles on \([n]\),

\[
|L^{\square^k}[n]| = \frac{1}{2^k} \sum_{i_1 \cdots i_k \geq 0} \frac{(i_1 \cdots i_k)_n}{2^{i_1 + \cdots + i_k}}. \quad (143)
\]

- The number of cyclic \( k \)-partial rectangles on \([n]\),

\[
|C^{\square^k}[n]| = \sum_{i_1 \cdots i_k \geq 1} \frac{(i_1 \cdots i_k - 1)_{n-1}}{2^{i_1 + \cdots + i_k}}. \quad (144)
\]

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