1. Introduction

In this work we prove a stability estimate from the Radon transform with limited angle-distance data to a local $L^p$-norm of the function. Our original motivation to study this problem was to obtain stability estimates for the inverse problem in electric impedance tomography (E.I.T.) proposed by Calderón. Nevertheless, we think that the results obtained on the Radon transform restricted to some partial data sets are interesting by themselves and are the main contribution of this work.

Calderón’s inverse problem deals with the recovery of a conductivity $\gamma$ in the interior of a smooth domain $\Omega$ from boundary measurements realized by the Dirichlet-to-Neumann map. Let $u$ be the solution of the Dirichlet boundary value problem

\begin{equation}
\begin{cases}
\text{div}(\gamma \nabla u) = 0 & \text{in } \Omega \\
u|_{\partial \Omega} = f \in H^{\frac{1}{2}}(\partial \Omega)
\end{cases}
\end{equation}

where $\gamma$ is a positive function of class $C^2$ on $\bar{\Omega}$. The Dirichlet-to-Neumann map assigns to a function $f \in H^{\frac{1}{2}}(\partial \Omega)$ on the boundary the corresponding Neumann data of (1.1)

$$\Lambda_{\gamma} f = \gamma \partial_{\nu} u|_{\partial \Omega}$$
where $\partial_\nu$ denotes the exterior normal derivative of $u$. This is a bounded operator $\Lambda_\gamma : H^{\frac{1}{2}}(\partial \Omega) \to H^{-\frac{1}{2}}(\partial \Omega)$ — in fact a pseudodifferential operator of order 1 when $\gamma$ is smooth. The inverse problem formulated by Calderón [12] is whether it is possible to determine $\gamma$ from $\Lambda_\gamma$. In fact in its initial formulation, the problem concerns only positive measurable conductivities bounded from above, and it was solved in dimension 2 in this degree of generality by Astala and Päivärinta [4] and remains so far open in higher dimensions.

This question is related to the inverse problem of determining a bounded potential $q \in L^\infty(\Omega)$ in the Schrödinger equation

\[
\begin{cases}
-\Delta u + qu = 0 & \text{in } \Omega \\
u|_{\partial \Omega} = f \in H^{\frac{1}{2}}(\partial \Omega),
\end{cases}
\] (1.2)

from boundary measurements. This reduction was exploited by Sylvester and Uhlmann in [43] and in combination with the boundary determination results on the conductivity obtained by Kohn and Vogelius [34] allowed them to solve the Calderón problem for smooth conductivities in dimension $n \geq 3$. When 0 is not a Dirichlet eigenvalue of the Schrödinger operator $-\Delta + q$, the measurements are implemented by the Dirichlet-to-Neumann map, which can be similarly defined as for the conductivity equation by

\[\Lambda_q f = \partial_\nu u|_{\partial \Omega}.\]

With a slight abuse of notations, we use the convention that whenever the subscript contains the letter $q$, the notation refers to the Dirichlet-to-Neumann map related to the Schrödinger equation (1.2), while if it contains the letter $\gamma$ it refers to the map related to the conductivity equation (1.1).

In the inverse problem with partial data one wonders whether one or the other of the Dirichlet-to-Neumann maps $\Lambda_\gamma, \Lambda_q$ measured only on a subset of the boundary, determines the conductivity $\gamma$ or the electric potential $q$ inside $\Omega$. In dimension two, this problem is settled by the articles [29, 30] of Imanuvilov, Uhlmann and Yamamoto using ideas from Bukhgeim [10] who dealt with the inverse problem for the Schrödinger equation with full data. See [23] for this problem on Riemann surfaces. In dimension higher than three, the first results were obtained by Bukhgeim and Uhlmann [11] but required measurements on roughly half of the boundary. The results obtained by Kenig, Sjöstrand and Uhlmann [33] are the most precise so far in dimension $n \geq 3$ since they require measurements on small subsets of the boundary for, say, strictly convex domains $\Omega$. This result has been extended to the Dirac system by Salo and Tzou in [39]. We should also mention the local inverse problem, in which all the measurement are restricted to input Dirichlet data supported on the same (the accessible boundary) subset as the output measurements. This problem was settled by Imanuvilov, Uhlmann and Yamamoto [29, 30] in dimension $n = 2$ and only for very special cases (the complement of the accessible boundary being a piece of a plane or a sphere) in dimension $n \geq 3$ by Isakov [31] and extended to Maxwell equation...
in [13] and [15]. The linearized inverse Calderón problem with partial data was studied in [20].

Let us describe Bukhgeim and Uhlmann result in more details. For this purpose, given a direction $\xi \in S^{n-1}$, we consider the $\xi$-illuminated face of $\partial \Omega$

$$\partial \Omega_-(\xi) = \{ x \in \partial \Omega : \langle \xi, \nu(x) \rangle \leq 0 \}$$

and the $\xi$-shadowed face

$$\partial \Omega_+(\xi) = \{ x \in \partial \Omega : \langle \xi, \nu(x) \rangle \geq 0 \},$$

where $\nu(x)$ is the exterior normal vector at $x$.

**Theorem 1.1** (Bukgheim and Uhlmann [11]). Let $\Omega$ be a bounded open set in $\mathbb{R}^n, n \geq 3$ with smooth boundary and let us consider $F \subset \partial \Omega$ an open neighborhood of the face $\partial \Omega_-(\xi)$. Let $q_1, q_2$ be two bounded potentials on $\Omega$, suppose that 0 is neither a Dirichlet eigenvalue of the Schrödinger operator $-\Delta + q_1$ nor of $-\Delta + q_2$, and that for all $f \in H^{1/2}(\partial \Omega)$ the two Dirichlet-to-Neumann maps coincide on $F$

$$\Lambda_{q_1} f |_{F} = \Lambda_{q_2} f |_{F},$$

then the two potentials agree $q_1 = q_2$.

To describe the uniqueness result of Kenig, Sjöstrand and Uhlmann we need to introduce the appropriate parts of $\partial \Omega$. Assume $y_0$ is not in the convex hull of $\Omega$, we define the $y_0$-illuminated face as

$$\partial \Omega_- (y_0) = \{ x \in \partial \Omega : \langle x - y_0, \nu(x) \rangle \leq 0 \}$$

and the $y_0$-shadowed face as

$$\partial \Omega_+ (y_0) = \{ x \in \partial \Omega : \langle x - y_0, \nu(x) \rangle \geq 0 \}.$$

Note the abuse of notation when writing $\partial \Omega_\pm (\xi)$ and $\partial \Omega_\pm (y_0)$, since the former one denotes the $\xi$-illuminated and $\xi$-shadowed faces from the direction $\xi$ while the latter one denotes $\xi$-illuminated and $\xi$-shadowed faces from the point $y_0$. Then

**Theorem 1.2** (Kenig, Sjöstrand and Uhlmann [33]). Let $\Omega$ be a bounded open set in $\mathbb{R}^n, n \geq 3$ with smooth boundary and let us consider $F, B \subset \partial \Omega$ two open neighborhoods respectively of the faces $\partial \Omega_- (y_0)$ and $\partial \Omega_+ (y_0)$. Let $q_1, q_2$ be two bounded potentials on $\Omega$, suppose that 0 is neither a Dirichlet eigenvalue of the Schrödinger operator $-\Delta + q_1$ nor of $-\Delta + q_2$, and that for all $f \in H^{1/2}(\partial \Omega)$ supported in $B$ the two Dirichlet-to-Neumann maps coincide on $F$

$$\Lambda_{q_1} f |_{F} = \Lambda_{q_2} f |_{F},$$

then $q_1 = q_2$.

The main goal of this article is to derive an estimate for the Radon transform which yields the corresponding stability estimates for the above uniqueness results (actually only a local estimate in the case of Theorem 1.2). This
estimate, which we call a quantitative version of Helgason-Holmgren theorem, will be obtained in section 2 (see Theorem 2.5). Stability estimates for the conductivity inverse problem in dimension higher than three go back to Alessandrini’s article [1]. This was followed by results in two dimensions by Liu [35], Barcelo, Barcelo and Ruiz [5], Barcelo, Faraco and Ruiz [6] and finally by Clop, Faraco and Ruiz [18] for discontinuous conductivities corresponding to the uniqueness results of Astala and Päivärinta [4] (see also [21]). Other stability results for the Calderón problem in dimension greater than two are [25] and [16]. In the case of Maxwell equations the stability was obtained in [14].

Concerning the inverse problem with partial data, stability estimates corresponding to the results of Bukhgeim and Uhlmann were derived by Heck and Wang [24], and in the presence of a magnetic field by Tzou [44]. We mention also the uniqueness results obtained by Ammari and Uhlmann [3] in the case where the potential is known close to the boundary, and the corresponding stability estimates obtained by Fathallah [22] and Ben Joud [7]. One single log stability estimate was obtained by Alessandrini and Kim [2] in the case of the conductivity equation when the conductivities coincide on a neighborhood of the boundary with a known one. The stability of the local problem under similar condition as in [31] was proved by Caro for Maxwell equations in [15].

Let $F$ and $B$ be boundary neighborhoods of the illuminated and shadowed faces respectively. The natural norm to consider on the partial Dirichlet-to-Neumann map is

$$\|\Lambda_q\|_{B\to F} = \sup \left\{ \left\langle \Lambda_q(\varphi) \right| \psi \right\} : \|\varphi\|_{H^{1/2}(\partial\Omega)} = \|\psi\|_{H^{1/2}(\partial\Omega)} = 1, \quad \text{supp} \varphi \subset B, \text{supp} \psi \subset F \right\},$$

where $\langle \cdot | \cdot \rangle$ denotes the duality between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. We’ll also have to consider a larger norm related to solutions of Schrödinger equation belonging to the space $H(\Omega, \Delta)$ (see section 3.1). This norm was considered by Nachman and Street in [37], where they prove the reconstruction of some 2-plane integrals of the potential from partial data. We will denote this norm as

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_{B\to F}$$

The class of allowable potentials under consideration will be in Besov spaces

$$\mathcal{K}(M, \lambda, p) = \left\{ q \in L^\infty(\Omega), q\mathbf{1}_\Omega \in W^{\lambda,p}(\mathbb{R}^n) : \|q\|_{L^\infty} + \|q\|_{W^{\lambda,p}} \leq M \right\},$$

where $\lambda > 0$. This class of potential has the advantage of allowing very rough functions if $\lambda$ is sufficiently small. Our stability results are as follows.

**Theorem 1.3.** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n \geq 3$ with smooth boundary. Given an open set $N$ in $S^{n-1}$ consider $F, B \subset \partial\Omega$ two open subsets of the boundary which are respective neighbourhoods of the faces $\partial\Omega_\pm(\xi)$ and
\[ \partial \Omega_+ (\xi) \text{ for all directions } \xi \in N. \] Given \( M > 0 \) there exists a constant \( C > 0 \) such that the following estimate holds true

\[ (1.3) \quad \| q_1 - q_2 \|_{L^p} \leq C \left( \log \| \Lambda q_1 - \Lambda q_2 \|_{B \to F} \right)^{-\lambda/2} \]

for all allowable potentials \( q_1, q_2 \in \mathcal{K}(M, \lambda, p) \) on \( \Omega \), with \( 1 \leq p < \infty \) and \( 0 < \lambda < 1/p \), for which 0 is neither a Dirichlet eigenvalue of the Schrödinger operator \( -\Delta + q_1 \) nor of \( -\Delta + q_2 \).

Next we consider the case of illumination from a point. Let \( N \) be an open set which does not cut the convex hull of \( \Omega \). We will define \( P \), the convex penumbra boundary from \( N \), as the set of points \( x \in \partial \Omega \) such that there exist a \( y \in N \) with \( \langle x - y, \nu(x) \rangle = 0 \) and the hyperplane through \( x \) normal to \( \nu(x) \) being a supporting hyperplane of \( \Omega \). In order to keep the exposition simple, and relate to the Radon transform (rather than the two-plane transform in high dimensions) we will restrict ourselves to the three dimensional case \( n = 3 \).

**Theorem 1.4.** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^3 \) with smooth boundary. Given an open set \( N \) in \( \mathbb{R}^3 \) which does not cut the convex hull of \( \Omega \), consider two open subsets \( F, B \) of the boundary which are respective neighbourhoods of the faces \( \partial \Omega_+(y) \) and \( \partial \Omega_-(y) \) for all \( y \in N \). Given \( M > 0 \), there exist an open neighborhood \( G \subset \mathbb{R}^3 \) of the convex penumbra \( P \) and a constant \( C > 0 \) such that the following estimate holds true

\[ (1.4) \quad \| q_1 - q_2 \|_{L^p(G)} \leq C \left( \log \| \Lambda q_1 - \Lambda q_2 \|_{B \to F} \right)^{-\lambda/2} \]

for all allowable potentials \( q_1, q_2 \in \mathcal{K}(M, \lambda, p) \) on \( \Omega \), with \( 1 \leq p < \infty \) and \( 0 < \lambda < 1/p \), for which 0 is neither a Dirichlet eigenvalue of the Schrödinger operator \( -\Delta + q_1 \) nor of \( -\Delta + q_2 \).

The proofs of these theorems will be carried out by using the approach of [19], which uses the Radon transform. One can see that the result on the Radon transform is general enough to be applied to get stability for partial data in the context of [33] in dimension three (the Dirichlet-to-Neumann map in this case controls the 2-plane transform, which is indeed the Radon transform in three dimensions). This can be achieved with the natural norm \( H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega) \) of the Dirichlet-to-Neumann map, by using the solutions constructed by Chung in [17].

Theorem 1.3 was proved in [21] by Heck and Wang, without the condition of the Dirichlet data being supported on \( B \) and the norm in the partial Dirichlet-to-Neumann map considered from \( H^{3/2}(\partial \Omega) \) to \( H^{1/2}(\partial \Omega) \) instead of the norm \( \| \cdot \|_{B \to F} \). They use the Fourier transform. The change to the Radon transform illustrates the use of Theorem 2.5.

The structure of this paper is as follows. In section 2 we will state and prove the theorem for the Radon transform which is the main result in this work. Section 3 and Section 4 are devoted to prove Theorem 1.3 and Theorem 1.4 applying the stability estimates for the Radon transform proven in Section 2.
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2. Stability for the local Radon transform

We recall that the Radon transform of a continuous compactly supported function $f$ is given by

$$Rf(s,\omega) = \int \delta((x,\omega) - s)f(x) \, dx, \quad s \in \mathbb{R}, \, \omega \in S^{n-1}.$$  

It is always possible to define, by duality, the Radon transform on compactly supported distributions. For the time being, we content ourselves with continuous functions with some kind of decay, but later on we will extend its definition to a wider class of functions. This transform is even. It is sometimes convenient to think of the Radon transform as a function on $\Pi^{n-1}$, the Grassmannian set of hyperplanes in $\mathbb{R}^n$. Let $H$ belong to $\Pi^{n-1}$, then

$$Rf(H) = \int_H f(x) \, d\mu_H(x)$$

where $d\mu_H$ is the Lebesgue measure on $H$. To relate both notations in a coherent way, we set

$$H_0(s,\omega) = \{x \in \mathbb{R}^n : \langle x,\omega \rangle = s\}.$$  

Note that $\omega$ is a unit normal to the hyperplane $H_0(s,\omega)$ and $s$ a signed distance to the origin. For later convenience we might change the origin of the affine reference for the Radon transform to the point $y_0 \in \mathbb{R}^n$. If one describes $H$ as

$$H = H_{y_0}(s,\omega) = \{x \in \mathbb{R}^n : \langle x - y_0,\omega \rangle = s\}$$  

for some $\omega \in S^{n-1}$ and $s \in \mathbb{R}$. Relating $\omega, s$ to $H$ as above, one can define

$$R_{y_0}f(s,\omega) = \int_H f \, d\mu_H.$$  

We will also make use of the following notation

$$H_{y_0}^\pm(s,\omega) = \{x \in \mathbb{R}^n : \pm(\langle x - y_0,\omega \rangle - s) < 0\}$$

to denote the half-spaces delimited by $H_{y_0}(s,\omega)$. 
We refer to Helgason’s book [26] for a general study of the Radon transform. Of particular importance is the issue of local inversion of the Radon transform: given a function \( f \) with some a priori regularity and some decay at infinity, such that \( Rf(H) = 0 \) for every \( H \in \Xi \subset \Pi^{n-1} \), does \( f \) vanish on \( E = \cup_{H \in \Xi} H \)? For instance, the celebrated Helgason’s support theorem reads as follows.

**Helgason’s support theorem.** Let \( f \) be a rapidly decreasing continuous function such that its Radon transform vanishes on all hyperplanes disjoint from a compact convex set \( K \)

\[ Rf(H) = 0, \quad H \cap K = \emptyset \]

then the support of \( f \) is contained in \( K \).

We are interested in the microlocal approach (which differs from Helgason’s original proof) to prove Helgason’s support theorem presented in [8, 9, 28]. This approach is somewhat flexible since it does not require the full family of hyperplanes used in Helgason’s theorem but can be adapted to provide weaker support results when the Radon transform only vanishes in a neighbourhood of a fixed hyperplane. A result that follows from this approach is:

**Microlocal Helgason-Holmgren Theorem.** Let \( f \) be a compactly supported continuous function such that its Radon transform vanishes in a neighborhood of \( H_0((x_0,\xi_0),\xi_0) \). If \( \text{supp} f \subset H_0^+(⟨x_0,\xi_0⟩,\xi_0) \) then \( x_0 \notin \text{supp} f \).

From the inverse problems point of view, the above result was used in [19] to prove the unique determination of the electric potential and the magnetic field in a magnetic Schrödinger equation from partial data. It served as a substitute to the original but somewhat more involved argument of Kenig, Sjöstrand and Uhlmann in [33] also based on analytic microlocal theory. Similar ideas were used in [20] to investigate a linearization of the Calderón problem with partial data.

The main result in this section, Theorem 2.5, is a quantitative version of the microlocal Helgason-Holmgren theorem. We want to relax the compact support and continuity assumptions on \( f \), in order to apply the corresponding results to the study of the stability of Calderón’s inverse problem. In the next paragraphs we review some concepts and results that will be basic in the microlocal approach and that will clarify the proof of Theorem 2.5.

### 2.1. Microlocal Helgason’s support and Kashiwara’s Watermelon theorems.

We will use the classical notation \( w^2 = w_1^2 + \cdots + w_n^2 \) to denote the holomorphic continuation of the Euclidean scalar product — particularly to avoid confusion with the norm \( |w|^2 = |w_1|^2 + \cdots + |w_n|^2 \) of complex vectors. The Segal-Bargmann transform of an \( L^\infty \) function is given by

\[ \mathcal{T}_h f(z) = \int e^{-\frac{i}{2} w^2} f(y) \, dy, \quad z \in \mathbb{C}^n. \]
Note that it has the following exponential growth

\[ |T_h f(z)| \leq (2\pi h)^{\frac{n}{2}} e^{\frac{1}{2h} (\text{Im } z)^2} \|f\|_{L^\infty}. \tag{2.3} \]

By duality, it is easy to extend this transform to tempered distributions. This transform has a wide range of applications in Analysis; amongst others, it provides a way of describing analytic singularities of a distribution on an open set \( \Omega \subset \mathbb{R}^n \).

**Definition 2.1.** A distribution \( f \in \mathcal{D}'(\Omega) \) is said to be microlocally exponentially small at \((x_0,\xi_0) \in T^*\Omega\) if there exist a cutoff function \( \chi \in C^\infty_0(\Omega)\) such that \( \chi(x_0) \neq 0 \), two constants \( c, C > 0 \) and a neighbourhood \( V_{z_0} \) of \( z_0 = x_0 - i\xi_0 \) in \( \mathbb{C}^n \) such that the following improved bound holds on the Segal-Bargmann transform

\[ |T_h (\chi f)(z)| \leq C e^{-c h + \frac{1}{4h} (\text{Im } z)^2}. \tag{2.4} \]

for all \( z \in V_{z_0} \) and all \( h \in (0, 1] \).

The analytic microsupport of a distribution \( f \) — which we denote by \( \mu\text{supp}_A f \) — is the complement of the set of covectors \((x_0,\xi_0) \in T^*\Omega\) at which \( f \) is microlocally exponentially small. The analytic wave front set \( \text{WF}_A f \) of \( f \) is the complement in \( T^*\Omega \setminus 0 \) of the set of covectors at which \( f \) is microlocally exponentially small.

The analytic microsupport is a closed conic set of the cotangent bundle and consists in two parts

\[ \mu\text{supp}_A f = \text{supp } f \times \{0\} \cup \text{WF}_A f. \]

The projection with respect to the space variable of the analytic wave front set is the analytic singular support

\[ \pi(\text{WF}_A f) = \text{supp}_A f, \quad \pi : T^*\Omega \to \Omega \]

i.e. the set of points \( x_0 \in \mathbb{R}^n \) which have no neighbourhood on which \( f \) is real analytic. A microlocal form of Helgason’s support theorem reads as follows.

**Microlocal Helgason’s theorem.** If the Radon transform \( \mathcal{R}f(s, \omega) \) of \( f \in C^0(\mathbb{R}^n) \) vanishes in the neighbourhood of \((s_0, \omega_0) \in \mathbb{R} \times S^{n-1}\) then \((x_0, \omega_0) \notin \text{WF}_A f\) where \( x_0 \in H_0(s_0, \omega_0) \).

A more invariant formulation would be that the conormal \( N^*(H_0(s_0, \omega_0)) \) to the hyperplane is contained in the complement of the analytic wave front set. As we will see it implies a local weaker (but quite flexible) form of Helgason’s support theorem.

In the situation where a distribution is supported on one side of a hyperplane, Kashiwara’s Watermelon theorem describes some of the covectors of

\[ ^1 \text{In fact, this analysis can be extended to hyperfunctions.} \]

\[ ^2 \text{That is, in the frequency variable: } (x, \xi) \in \mu\text{supp}_A f \Rightarrow (x, \lambda \xi) \in \mu\text{supp}_A f \text{ for positive } \lambda. \]
the analytic microsupport ([32], [11] Theorem 8.3.3, [27] Theorem 9.6.6, [42]).

**Kashiwara’s Watermelon theorem.** Let \( f \in \mathcal{D}'(\mathbb{R}^n) \) be a distribution supported on one side \( H^+ \) of a hyperplane \( H \). Let \( \nu_0 \) denote a unit normal to \( H \). If \( (x_0, \xi_0) \in \mu_{\text{supp}} A f \) then so does \( (x_0, \xi_0 + t\nu_0) \) for all \( t \in \mathbb{R} \).

Kashiwara’s Watermelon theorem is generally stated in terms of the analytic wave front set, we chose to use a formulation involving the analytic microsupport (for similar formulations, see also [36]) since it encompasses information about the support of the function. In particular, it immediately implies the following unique continuation property

\[
\text{supp } f \subset \{ \langle x - x_0, \nu_0 \rangle < 0 \} \quad \text{and} \quad (x_0, \nu_0) \notin \text{WF}_A(f) \Rightarrow x_0 \notin \text{supp } f.
\]

This is sometimes known as Holmgren’s microlocal uniqueness theorem (or the co-Holmgren theorem).

2.2. **Relating the Radon and the Segal-Bargman transforms.** In this paragraph, we want to connect the Radon and the Segal-Bargman transforms. We start from the identity

\[
\hat{f}(\sigma) = \int_{\mathbb{R}^n} e^{-is\sigma} \mathcal{R}f(s, \omega) \, ds
\]

and use Plancherel’s identity to compute the scalar product

\[
\int f \overline{g} \, dx = (2\pi)^{-n} \int_{S^{n-1}} \overline{\mathcal{R}f(\sigma, \omega)} \overline{\mathcal{R}g(\sigma, \omega)} \sigma^{n-1} \, d\sigma \, d\omega.
\]

Using the fact that the Radon transform is even, and once again Plancherel’s identity, we get

\[
(2.5) \quad \int f \overline{g} \, dx = \frac{1}{2} (2\pi)^{-n+1} \int_{-\infty}^{\infty} \int_{S^{n-1}} \mathcal{R}f(s, \omega) |D^{\lfloor n/2 \rfloor} \mathcal{R}g(\cdot, \omega)(s)| \, ds \, d\omega.
\]

We choose \( g \) to be the conjugate of the Gaussian kernel of the Segal-Bargman transform: we begin by computing its Radon transform

\[
\mathcal{R}(e^{-\frac{1}{2h} (z-x)^2})(s, \omega) = \int \delta(\langle x, \omega \rangle - s) e^{-\frac{1}{2h} (s-x)^2} \, dx
\]

\[
= (2\pi h)^{-\frac{n+1}{2}} e^{-\frac{1}{4h}(s-\langle \omega, \xi \rangle)^2}
\]
and plug $g = e^{-\frac{1}{2h}(z-x)^2}$ in the identity (2.5) to compute the Segal-Bargman transform of a function in terms of the Radon transform

$$\mathcal{T}_h f(z) = \frac{1}{2} (2\pi)^{-\frac{n-1}{2}} h^{-\frac{n-1}{2}} \int_0^\infty \int_{S^{n-1}} G_n(s, \langle \omega, z \rangle) \mathcal{R} f(s, \omega) \, ds \, d\omega$$

where the kernel $G_n$ is given by

$$G_n(s, w) = |D|^{n-1} \left(e^{-\frac{1}{2h}(s-w)^2}\right)(s), \quad s \in \mathbb{R}, \quad w \in \mathbb{C}.$$

We will use the following estimates of the kernel:

**Lemma 2.2.** The kernel $G_n$ satisfies the following bound

$$|G_n(s, w)| \leq B_n h^{-\frac{n-1}{2}} \left(1 + h^{-\frac{1}{2}} |s - w|\right)^n \left(1 + e^{\frac{1}{2h}((\text{Im } w)^2-(s-\text{Re } w)^2)}\right).$$

**Proof.** We need to distinguish two cases according to the parity of the dimension $n$.

Let us start with the case $n$ odd which is simpler. The kernel $G_n$ can explicitly be computed

$$G_n(s, w) = D_s^{n-1} \left(e^{-\frac{1}{2h}(s-w)^2}\right) = h^{-\frac{n-1}{2}} e^{-\frac{1}{2h}(s-w)^2} Q_n \left(\frac{s-w}{\sqrt{h}}\right)$$

where

$$Q_n(w) = e^{\frac{w^2}{2h}} D_w^{n-1} e^{-\frac{w^2}{2h}}$$

is a Hermite polynomial of degree $n - 1$, hence satisfies the bound

$$|Q_n(w)| \leq A_n (1 + |w|)^{n-1}.$$

The former estimate together with

$$\left|e^{-\frac{1}{2h}(s-w)^2}\right| = e^{-\frac{1}{2h}(s-\text{Re } w)^2 + \frac{1}{2h}((\text{Im } w)^2)}$$

imply the following bound on $G_n$

$$|G_n(s, w)| \leq A_n h^{-\frac{n-1}{2}} \left(1 + h^{-\frac{1}{2}} |s - w|\right)^{n-1} e^{\frac{1}{2h}((\text{Im } w)^2-(s-\text{Re } w)^2)}$$

when $n$ is odd.

Notice that identity (2.6) reads in odd dimensions

$$\mathcal{T}_h f(z) = \frac{1}{2} (2\pi)^{-\frac{n-1}{2}} h^{-\frac{n-1}{2}} \int_0^\infty \int_{S^{n-1}} e^{-\frac{1}{2h}(s-\langle \omega, z \rangle)^2} Q_n \left(\frac{s-\langle \omega, z \rangle}{\sqrt{h}}\right) \mathcal{R} f(s, \omega) \, d\omega \, ds.$$

The even dimensional case is a bit more involved: the kernel $G_n$ satisfies the following relations

$$G_n(s, w) = G_n(s - \text{Re } w, i \text{ Im } w), \quad G_n(s, w) = G_n(s, \bar{w}) = G_n(-s, -\bar{w})$$
and has the following expression

\[
G_n(s, w) = |D|^{n-1} \left( e^{-\frac{1}{2}(\omega^2)} \right)(s) \\
= h^{-\frac{n-1}{2}} e^{-\frac{\omega^2}{2}} \int_{-\infty}^{\infty} |\sigma|^{n-1} e^{-\frac{(\sigma+iw/\sqrt{h})^2}{2} + \frac{i}{\sqrt{h}} \sigma} d\sigma.
\]

By the relations (2.9), it suffices to prove the estimate when \(w = -i\tau\) is imaginary and \(s\) is non-negative, and by scaling, we might as well assume that \(h = 1\). For \(\tau \in \mathbb{R}, s \geq 0\) and \(n\) even, we may decompose the kernel as

\[
G_n(s, -i\tau) = e^{\frac{s^2}{2} - is\tau} \left( \int_{-\infty}^{\tau} + \int_{\tau}^{\infty} \right) |\sigma - \tau|^{n-1} e^{-\frac{s^2}{2} + is\sigma} d\sigma
\]

where the integral

\[
I_n(s, \tau) = \frac{e^{\frac{s^2}{2} - is\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\tau} (\tau - \sigma)^{n-1} e^{-\frac{s^2}{2} + is\sigma} d\sigma
\]

can be computed by an integration on the contour \((-\infty, \tau] \cup [\tau, \tau + is] \cup [\tau + is, -\infty + is)\)

\[
I_n(s, \tau) = \frac{e^{\frac{s^2}{2} - is\tau}}{\sqrt{2\pi}} \left( \int_{-\infty}^{\tau} (\tau - \sigma - is)^{n-1} e^{-\frac{(\sigma + is)^2}{2} + is(\sigma + is)} d\sigma \right.
\]

\[
+ (-i)^n \int_0^s \sigma^{n-1} e^{-\frac{s^2}{2} + is(\tau + is)} d\sigma.
\]

The first term is bounded by a constant times

\[e^{\frac{s^2}{2} - \frac{s^2}{2}} (1 + |s| + |\tau|)^{n-1}\]

while the second term\(^4\) is bounded by a constant times

\[(1 + |s|)^n.\]

This completes the proof of the lemma. \(\square\)

We restate (2.6) in term of the Radon transform centered at \(y_0\), see (2.2), as

\[
(2.10) \quad T_h f(\zeta) = \frac{1}{2} (2\pi)^{-\frac{n-1}{2}} h^{-\frac{n-1}{2}} \int_{-\infty}^{\infty} \int_{S^{n-1}} G_n(s, \langle \omega, \zeta - y_0 \rangle) \mathcal{R}_{y_0} f(s, \omega) ds d\omega.
\]

---

\(^4\)In the odd dimensional case, this term disappears when one computes \(I_n(s, \tau) + \bar{I}_n(s, -\tau)\).
Given \( \omega_0 \in S^{n-1} \) and \( \beta \in (0,1] \) we consider the following cap centered around \( \omega_0 \) on the hypersphere \( S^{n-1} \)

\[
\Gamma = \{ \omega \in S^{n-1} : (\omega, \omega_0)^2 > 1 - \beta^2 \}
\]

\[
= \{ \omega \in S^{n-1} : d_{S^{n-1}}(\omega_0, \omega) < \arcsin \beta \}
\]

\( d_{S^{n-1}} \) being the geodesic distance on \( S^{n-1} \).

Before proceeding to further computations, we also note that \( d_{\mu_{H_0(s,\omega)}} \wedge ds = dx \) and therefore

\[
\int_{-\infty}^{\infty} |Rf(s, \omega)| \, ds \leq \|f\|_{L^1}
\]

which leads to

\[
(2.12) \quad \int_{-\infty}^{\infty} \int_{S^{n-1}} |Rf(s, \omega)| \, d\omega \, ds \leq |S^{n-1}| \times \|f\|_{L^1}.
\]

We introduce the following set of functions: \( u \in X \) by definition if and only if \( u \in L^1(\mathbb{R}^n) \) and

\[
\|u\|_X = \int_{\mathbb{R}^n} (1 + |s|)^n \|R_0 u(s, \cdot)\|_{L^1(S^{n-1})} \, ds < \infty.
\]

Let us remark, see (2.12), that for functions in \( L^1(\mathbb{R}^n) \), the Radon transform is defined a.e. as a function in \( L^1(\mathbb{R} \times S^{n-1}) \). Our space \( X \) is more restrictive, a sufficient condition for a function to be in \( X \), is given by the estimate

\[
|u|_{S^{n-1}} \leq \int_{\mathbb{R}^n} (1 + |x|)^n |u(x)| \, dx.
\]

**Proposition 2.3** (Quantitative Microlocal Helgason’s theorem). Let \( f \) belong to \( X \). There exists a positive constant \( C \), only depending on \( n \), such that

\[
(2.15) \quad e^{-\frac{1}{2} \mu_{\text{Im}} |\zeta|^2} |\mathcal{H}_h f(\zeta)| \leq \frac{C}{h^{2n}} (1 + |\zeta| + |y_0|)^n
\]

\[
\times \left( \int_{|s|<\alpha} (1 + |s|)^n \|R_{y_0} f(s, \cdot)\|_{L^1(\Gamma)} \, ds + \|f\|_X \left( e^{-\frac{\gamma^2}{2}} + e^{-\frac{\beta^2}{2}} \right) \right),
\]

for all \( h \in (0,1] \), \( \gamma > 0 \) and \( \zeta \in \mathbb{C}^n \) such that \( |\text{Re} \, \zeta - y_0| < \alpha/2 \), \( |\text{Im} \, \zeta| \geq \gamma \) and \( \langle \omega_0, \frac{\text{Im} \, \zeta}{|\text{Im} \, \zeta|} \rangle^2 > 1 - \beta^2/4 \).

**Remark 2.4.** Note that when \( Rf \) vanishes on a neighborhood of \((s_0, \omega_0)\) the above estimate implies that \( (y_0, \pm \omega_0) \notin \text{WF}_A(f) \) when \( \langle y_0, \omega_0 \rangle = s_0 \). This is the microlocal version of Helgason’s support theorem as stated in the introduction of this section. Proposition 2.3 is therefore a quantitative version of this microlocal result.
Proof. It follows from Lemma 2.2 and (2.10) that
\[
|T_h f(\zeta)| \leq C \int_{-\infty}^{\infty} \int_{S^{n-1}} \left(1 + h^{-\frac{1}{2}} |s - \langle \omega, \zeta - y_0 \rangle| \right)^n \times \left(1 + e^{\frac{1}{2h}(\langle \omega, \text{Im} \zeta \rangle^2 - \langle s - \langle \omega, \text{Re} \zeta - y_0 \rangle \rangle^2)} \right) |R_{y_0} f(s, \omega)| \, d\omega \, ds.
\]

Let us split the integral into
\[
\int_{\mathbb{R} \times S^{n-1}} = \int_{\mathbb{R} \setminus (-\alpha, \alpha) \times S^{n-1}} + \int_{(-\alpha, \alpha) \times (S^{n-1} \setminus \Gamma)} + \int_{(-\alpha, \alpha) \times \Gamma} = I_1 + I_2 + I_3.
\]

To estimate $I_1$, notice that if $s \in \mathbb{R} \setminus (-\alpha, \alpha)$ and $\omega \in S^{n-1}$, then
\[
e^{-\frac{1}{2h}|\langle \omega, \text{Re} \zeta - y_0 \rangle - s|^2} \leq e^{-\frac{1}{2h}|\langle s - \langle \omega, \text{Re} \zeta - y_0 \rangle \rangle^2} \leq e^{-\frac{1}{2h} \frac{\alpha^2}{4}},
\]
and that
\[
1 \leq e^{-\frac{1}{2h} \gamma^2} e^{\frac{1}{2h} |\text{Im} \zeta|^2},
\]
for all $\zeta \in \mathbb{C}^n$ such that $|\text{Re} \zeta - y_0| < \alpha/2$ and $|\text{Im} \zeta| \geq \gamma$. Then it follows easily that
\[
I_1 \leq \frac{C}{h^\frac{n}{2}} e^{\frac{1}{2h} |\text{Im} \zeta|^2} (1 + |\zeta|)^n \|f\|_X \left(e^{-\frac{1}{2h} \frac{\alpha^2}{4}} + e^{-\frac{1}{2h} \gamma^2} \right).
\]

The integral $I_2$ can be bounded in a similar way. Notice that, for $s \in (-\alpha, \alpha)$ and $\omega \in S^{n-1} \setminus \Gamma$, it holds with $\theta = |\text{Im} \zeta|^{-1} |\text{Im} \zeta|$
\[
e^{\frac{1}{2h}\langle \omega, \text{Im} \zeta \rangle^2} \leq e^{\frac{1}{2h} |\text{Im} \zeta|^2} e^{-\frac{1}{2h} (|\text{Im} \zeta|^2 - \langle \omega, \text{Im} \zeta \rangle^2)} \leq e^{\frac{1}{2h} |\text{Im} \zeta|^2} e^{-\frac{1}{2h} \gamma^2 (1 - \langle \omega, \theta \rangle^2)},
\]
and again
\[
1 \leq e^{-\frac{1}{2h} \gamma^2} e^{\frac{1}{2h} |\text{Im} \zeta|^2},
\]
for all $\zeta \in \mathbb{C}^n$ such that $|\text{Im} \zeta| \geq \gamma$ and $\langle \omega, \theta \rangle^2 > 1 - \beta^2/4$. This completes the proof of the Proposition. \qed

2.3. A quantitative Helgason-Holmgren theorem. Now we state and prove the main result in this section, the quantitative version of microlocal Helgason-Holmgren theorem:

**Theorem 2.5.** Let $M \geq 1$ be constant. Given $y_0 \in \mathbb{R}^n$, $\omega_0 \in S^{n-1}$, $\alpha > 0$ and $\beta \in (0, 1]$, consider $(-\alpha, \alpha) \times \Gamma \subset \mathbb{R} \times S^{n-1}$ introduced in (2.11) above, and define the dependence domain of the Radon transform data
\[
E = \{ x \in \mathbb{R}^n : \langle \omega, x - y_0 \rangle = s, s \in (-\alpha, \alpha), \omega \in \Gamma \}.
\]

Assume that for some $p$, $1 \leq p < \infty$, and $\lambda$, $0 < \lambda < 1/p$, a function $q$ satisfies the following conditions:

(a) $1_E q \in X \cap L^\infty(\mathbb{R}^n)$, where $1_E$ stands for the characteristic function of the set $E$, furthermore
\[
\|q\|_{L^\infty(E)} + \|1_E q\|_X < M.
\]

(b) $y_0 \in \text{supp} q$ and $\text{supp} q \subset \{ x \in \mathbb{R}^n : \langle x - y_0, \omega_0 \rangle \leq 0 \}$. 


(c) $(\lambda,p,p)$-Besov regularity on the dependence domain

\[ \int_{\mathbb{R}^n} \left\| \frac{(1_E q - (1_E q)(\cdot - y))^p}{|y|^{n+\lambda p}} \right\|_{L^p(\mathbb{R}^n)} \, dy < M^p. \]

Then there exists a positive constant $C = C(M, |G|, \alpha, \beta, \lambda)$, such that

\[ \|q\|_{L^p(G)} \leq C \left| \log \int_{(-\alpha,\alpha)} (1 + |s|)^n \|R_{y_0} q(s, \cdot)\|_{L^1(\Gamma)} \, ds \right|^{-\frac{\lambda}{2}}, \]

where

\[ G = \left\{ x \in \mathbb{R}^n : |x - y_0| < \frac{\alpha}{8 \cosh(8\pi/\beta)} \right\}. \]

A precise value of $C$ in (2.16) can be given as

\[ C = C_n M \max \left(1, |G|^\frac{1}{p}\right) (1 + |y_0|) \left(\alpha^{-n} + \beta^{-n} + \alpha^\lambda\right). \]

Compared to Helgason support theorem, we relax the decay condition to the one given in (2.14) and the $L^\infty$ moduli of continuity are relaxed to integral moduli of continuity, which, under the condition $0 < \lambda < 1/p$, allow non continuous functions and are preserved (modulo constants) by multiplication by rough characteristic functions. These facts will be important in the applications.

It will be convenient to use the classical

\[ t_+ = \max(t, 0) \quad t_- = \min(t, 0) \]

to denote the positive and negative parts of a real number (or a function). The bounds on the Segal-Bargmann transform can be improved whenever the function $f$ is supported on one side of a hyperplane. Indeed if

\[ \text{supp} f \subset H_0^+(s, \omega_0) \]

then we have, for any $y_0 \in H_0(s, \omega_0)$,

\[ |T_{\partial} f(\zeta)| \leq (2\pi h)^\frac{n}{2} e^{\frac{\pi}{\hbar} (\text{Im} \zeta)^2 - \frac{\pi}{\hbar} (\text{Re} \zeta - y_0, \omega_0)^2} \|f\|_{L^\infty}. \]

Our first step will be to extrapolate estimate (2.15) to capture points $\zeta \in \mathbb{C}^n$ with $\text{Im} \zeta = 0$. Note that for those values of the parameter $\zeta$, the Segal-Bargmann transform is a gaussian transform. As in [11] (see also [27, Lemma 9.6.5]) the following maximum principle for subharmonic functions will be the keystone of the proof.

**Lemma 2.6.** Let $a, b$ and $\lambda$ be positive constants. Consider

\[ R = \{ z \in \mathbb{C} : |\text{Re} z| < a, |\text{Im} z| < b + \varepsilon \}, \]

for some $\varepsilon > 0$. Let $F$ be a subharmonic function on $R$ such that

\[ F(z) < (\text{Re} z_-)^2, \]

for all $z \in R$ and

\[ F(z) < -\lambda, \]
for \( z \in R \) such that \( |\text{Im} \, z| \geq b \). Then, for
\[
|\text{Im} \, z| < b, \quad |\text{Re} \, z| < \frac{\delta}{2},
\]
we have
\[
F(z) < -\frac{\lambda}{2a \cosh \left( \frac{b}{a} \right)} \delta,
\]
where
\[
\delta = \min \left( \frac{\lambda}{2a \cosh \left( \frac{b}{a} \right)}, \frac{a}{3} \right).
\]

**Proof.** The claim follows by comparison of the subharmonic function \( F(z) - \delta^2 \) with the harmonic function
\[
G(z) = -\lambda \frac{\cosh \left( \frac{a}{b} \right)}{\cosh \left( \frac{a}{b} \right)} \sin \left( \frac{\pi}{a} (x + \delta) \right),
\]
where \( z = x + iy \), is in the rectangle \( R_\delta = [-\delta, a - \delta] \times [-b, b] \). In fact, on the boundary of \( R_\delta \) we have
\[
F(x \pm ib) - \delta^2 < -\lambda \leq G(x \pm ib) \quad \text{for} \quad x \in [-\delta, a - \delta],
\]
\[
F(-\delta + iy) - \delta^2 < 0 = G(-\delta + iy)
\]
and
\[
F(a - \delta + iy) - \delta^2 < 0 = G(a - \delta + iy).
\]
From the maximum principle \( F(z) < G(z) + \delta^2 \) in \( R_\delta \), which means that
\[
F(x + iy) < \delta^2 - \lambda \frac{\cosh \left( \frac{a}{b} \right)}{\cosh \left( \frac{a}{b} \right)} \sin \left( \frac{\pi}{a} (x + \delta) \right).
\]
Since \( \sin t > 2t/\pi \) for \( 0 < t < \pi/2 \), one has that
\[
F(x + iy) < \delta^2 - \frac{\lambda}{a \cosh \left( \frac{a}{b} \right)} \frac{2}{a}(x + \delta),
\]
whenever \( 0 < x + \delta < a/2 \). So if \( x \) is restricted to \( |x| < \delta/2 \), then
\[
F(x + iy) < \delta^2 - \frac{\lambda}{a \cosh \left( \frac{a}{b} \right)} \delta
\]
\[
\leq -\frac{\lambda}{2a \cosh \left( \frac{a}{b} \right)} \delta.
\]
This completes the proof of the Lemma. \( \square \)

**Proposition 2.7.** Consider \( q \in X \cap L^\infty(\mathbb{R}^n) \) and let \( y_0 \in \mathbb{R}^n \) and \( \omega_0 \in S^{n-1} \) be such that \( y_0 \in \text{supp} \, q \) and \( \text{supp} \, q \subset \{ x \in \mathbb{R}^n : \langle x - y_0, \omega_0 \rangle \leq 0 \} \). Given \( \alpha > 0 \) and \( \beta \in (0, 1] \) consider the set
\[
\Gamma = \{ \omega \in S^{n-1} : \langle \omega, \omega_0 \rangle^2 > 1 - \beta^2 \}.\]
If one has
\[
\int_{(-\alpha, \alpha)} (1 + |s|)^n \| R_{y_0} q(s, \cdot) \|_{L^1(\Gamma)} \, ds \leq e^{-\frac{\alpha^2}{8}},
\]
there exists a positive constant $C$, only depending on $n$, such that
\[
(2.20) \quad e^{-\frac{1}{2\pi} |\text{Im} \zeta|^2} |T_h q(\zeta)| \\
\leq CM_q \left( 1 + |y_0| + \frac{\alpha}{\beta} \right)^n \left( \int_{-\alpha}^{\alpha} (1 + |s|)^n \| R_{y_0} q(s, \cdot) \|_{L^1(\Gamma)} \, ds \right)^\kappa,
\]
with
\[
\kappa < \frac{1}{8 \left( \cosh \left( \frac{8\pi}{n} \right) \right)^2}, \quad M_q := \max(1, \|q\|_{L^\infty(\mathbb{R}^n)} + \|q\|_X),
\]
\[
h = \frac{\alpha^2}{8 |\log \int_{-\alpha}^{\alpha} (1 + |s|)^n \| R_{y_0} q(s, \cdot) \|_{L^1(\Gamma)} \, ds|},
\]
for all $\zeta \in \mathbb{C}^n$ such that
\[
(2.21) \quad |\text{Re} \zeta - y_0| < \frac{\alpha}{8 \cosh(8\pi/\beta)}, \quad |\text{Im} \zeta| < \frac{2\alpha}{(4 - \beta^2)^{1/2}}.
\]

Proof. Let $\zeta \in \mathbb{C}^n$ and denote $z = (\omega_0, \zeta - y_0) \in \mathbb{C}$. We write $\zeta = (z + \langle \omega_0, y_0 \rangle)\omega_0 + w$ with $w \in \mathbb{C}^n$ such that $\langle \text{Re} \, w, \omega_0 \rangle = \langle \text{Im} \, w, \omega_0 \rangle = 0$. Let us denote
\[
\mathcal{I} = \int_{(-\alpha, \alpha)} (1 + |s|)^n \| R_{y_0} q(s, \cdot) \|_{L^1(\Gamma)} \, ds.
\]
Choose $\gamma = \frac{2\alpha}{\beta} > 0$ as in Proposition 2.3 then
\[
|T_h q((z + (\omega_0, y_0))\omega_0 + w)| \\
\leq CM_q (1 + \rho + |y_0|)^n e^{\frac{1}{2\pi} |\text{Im} \, z|^2 + \frac{1}{2\pi} |\text{Im} \, w|^2} \left( h^{-\frac{\alpha}{2}} \mathcal{I} + h^{-\frac{\alpha}{2}} e^{-\frac{\alpha^2}{32}} \right) \\
\leq CM_q (1 + \rho + |y_0|)^n e^{\frac{1}{2\pi} |\text{Im} \, z|^2 + \frac{1}{2\pi} |\text{Im} \, w|^2} e^{-\frac{1}{2\pi} \frac{\alpha^2}{16}} \\
\times \left( h^{-\frac{\alpha}{2}} e^{\frac{\alpha^2}{16}} \mathcal{I} + h^{-\frac{\alpha}{2}} e^{-\frac{3\alpha^2}{16}} \right)
\]
for all $z \in \mathbb{C}$ and $w \in \mathbb{C}^n$ such that
\[
(2.22) \quad |\text{Re} \, z|^2 + |\text{Re} \, w - y_0 + (\omega_0, y_0)\omega_0|^2 < \alpha^2/4,
\]
\[
(2.23) \quad |\text{Im} \, z| \geq 2\alpha/\beta,
\]
\[
(2.24) \quad |z + (\omega_0, y_0)|^2 + |w|^2 < \rho^2,
\]
and
\[
(2.25) \quad |\text{Im} \, z|^2/(|\text{Im} \, z|^2 + |\text{Im} \, w|^2) > 1 - \beta^2/4,
\]
where $\rho > 0$ is large enough. We next consider $w \in \mathbb{C}^n$ such that
\[
(2.26) \quad |\text{Re} \, w - y_0 + (\omega_0, y_0)\omega_0|^2 < 3\alpha^2/16.
\]
and
\begin{equation}
|\text{Im } w|^2 < \frac{4\alpha^2}{(4 - \beta^2)}.
\end{equation}
Then we have
\begin{equation}
|T_h q((z + \langle \omega_0, y_0 \rangle)\omega_0 + w)| \leq C M_q (1 + \rho + |y_0|^n) h^{-n/2}
\times e^{\frac{1}{2\pi} (|\text{Im } z|^2 - |\text{Re } z|^2) + \frac{1}{2\pi} |\text{Im } w|^2} \left( e^{\frac{1}{2\pi} \alpha^2 \tau} + e^{-\frac{1}{2\pi} \alpha^2 \tau} \right),
\end{equation}
and conditions (2.22)-(2.25) reduce to
\begin{equation}
|\text{Re } z|^2 < \frac{\alpha^2}{16}, \quad \frac{4\alpha^2}{\beta^2} \leq |\text{Im } z|^2 \leq \rho^2 - \frac{4\alpha^2}{4 - \beta^2},
\end{equation}
with \( \rho \) large enough.

Whenever \( I \leq e^{-\frac{\alpha^2}{8}} \), one can choose \( h = \frac{\alpha^2}{8|\log I|} \in (0, 1] \) so that
\[ e^{\frac{1}{2\pi} \alpha^2 \tau} = e^{-\frac{1}{2\pi} \alpha^2 \tau}. \]
From (2.19) we have
\begin{equation}
|T_h q((z + \langle \omega_0, y_0 \rangle)\omega_0 + w)| \leq C M_q h^{\frac{1}{2}} e^{\frac{1}{2\pi} |\text{Im } z|^2 + \frac{1}{2\pi} |\text{Im } w|^2 - \frac{1}{2\pi} (\text{Re } z)^2},
\end{equation}
for all \( w \in \mathbb{C}^n \) and \( z \in \mathbb{C} \). Recall that \( M_q = \max(1, \|q\|_{L_\infty(\mathbb{R}^n)} + \|q\|_{\mathcal{X}}) \).

Consider the sub-harmonic function \( \Phi \) defined as
\[ \Phi(z) = |\text{Re } z|^2 - |\text{Im } z|^2 + 2h \log |T_h q((z + \langle \omega_0, y_0 \rangle)\omega_0 + w)| + 2h \log \left( \frac{e^{-\frac{1}{2\pi} |\text{Im } w|^2}}{C h^{-n/2} M_q (1 + \rho + |y_0|^n)^n} \right), \]
where the variable \( w \) has been frozen. From (2.30) and (2.28) one derives that
\[ \Phi(z) < (\text{Re } z_-)^2, \]
for all \( z \in \mathbb{C} \); and
\[ \Phi(z) < h \log I = \frac{\alpha^2}{8}, \]
for all \( z \in \mathbb{C} \) satisfying (2.29).

It is clear that \( \Phi \) satisfies the conditions of Lemma 2.6 with parameters \( a = \frac{\alpha}{4}, b = \frac{2\alpha}{\beta} \) and \( \lambda = \frac{\alpha^2}{8} \), hence we might conclude that
\[ \Phi(z) < -\frac{h \log \int_{-\alpha}^{\alpha} (1 + |s|)^n \|R_{y_0 q(s, \cdot)}\|_{L^1(\Gamma)} \, ds)^{-1}}{2 (\cosh(8\pi/\beta))^2}, \]
for \( z \in \mathbb{C} \) such that
\[ |\text{Re } z| < \frac{\alpha}{8 \cosh(8\pi/\beta)}, \quad |\text{Im } z| < \frac{2\alpha}{\beta}. \]
Choosing \( \rho = 4\alpha/\beta + |y_0| \), one can translate this estimate into the statement of the proposition, notice that (2.26), (2.27) and (2.29) follow from (2.21) since
\[
|\text{Re} \zeta - y_0|^2 = |\text{Re} z|^2 + |\text{Re} w - y_0 + \langle \omega_0, y_0 \rangle \omega_0|^2
\]
and
\[
|\text{Im} \zeta|^2 = |\text{Im} z|^2 + |\text{Im} w|^2.
\]
This completes the proof of the proposition. \( \square \)

**Proof of Theorem 2.5.** The key point is the fact that the Segal-Bargmann transform restricted to real values is a convolution with the Gaussian. We exploit this by means of the following lemma. Since actually we need a backward estimate for the heat equation, which is an ill posed problem, we require the uniform Besov control of the potentials.

**Lemma 2.8.** Consider \( q \in L^p(\mathbb{R}^n) \) and \( G \) an open set in \( \mathbb{R}^n \). Assume that there exists \( \lambda \in (0, 1) \) such that
\[
L_q := \left( \int_{\mathbb{R}^n} \frac{\|q - q(\cdot - y)\|_{L^p(\mathbb{R}^n)}^p}{|y|^{n+\lambda p}} \, dy \right)^{1/p} < +\infty.
\]
Then, there exists a positive constant \( C \), only depending on \( n \), such that
\[
\|q\|_{L^p(G)} \leq C \left( h^{-\frac{n}{2}} \|T_h q\|_{L^p(G)} + L_q h^{\frac{\lambda}{2}} \right),
\]
for all \( h \in (0, 1] \).

**Proof.** Since
\[
q(x) = \frac{1}{(2\pi h)^{\frac{n}{2}}} T_h q(x) + \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}|y|^2} (q(x) - q(x - h^{\frac{1}{2}} y)) \, dy
\]
amost everywhere in \( G \),
\[
\|q\|_{L^p(G)} \leq \frac{1}{(2\pi h)^{\frac{n}{2}}} \|T_h q\|_{L^p(\mathbb{R}^n)}
\]
\[
+ \left\| \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}|y|^2} |q(\cdot - h^{\frac{1}{2}} y) - q(\cdot)| \, dy \right\|_{L^p(G)}.
\]
Minkowski’s inequality ensures that there exists a positive constant \( C \), only depending on \( n \), such that
\[
\left\| \int_{\mathbb{R}^n} |q - q(\cdot - h^{\frac{1}{2}} y)| e^{-\frac{1}{2}|y|^2} \, dy \right\|_{L^p(G)}^p \leq C^p h^{\frac{\lambda p}{2}} \int_{\mathbb{R}^n} \frac{\|q - q(\cdot - y)\|_{L^p(\mathbb{R}^n)}^p}{|y|^{n+\lambda p}} \, dy.
\]
This completes the proof of the Lemma. \( \square \)
Now we return to the proof of the Theorem. In the case
\[ \int_{(-\alpha,\alpha)} (1 + |s|)^n \| \mathcal{R}_{y_0}(q)(s, \cdot) \|_{L^1(\Gamma)} \, ds \leq e^{-\frac{2\alpha}{n}} \]
holds, then Theorem 2.5 is a consequence of Proposition 2.7 and Lemma 2.8 for the function $1_{E^q}$. On the other hand, if
\[ \int_{(-\alpha,\alpha)} (1 + |s|)^n \| \mathcal{R}_{y_0}(q)(s, \cdot) \|_{L^1(\Gamma)} \, ds \geq e^{-\frac{2\alpha}{n}}, \]
the conclusion of the statement of Theorem 2.5 is obvious. \qed

3. First application: illuminating $\Omega$ from infinity (BU)

We use the solutions of the Schrödinger equation in the maximal domain of the Laplace operator in $\Omega$. These solutions, satisfying the support condition, were constructed by Nachman and Street in [37] in the context of [33], but the construction in the case of illumination from infinity, [11], is easier and follows the same steps. We will collect some estimates for these solutions which are (some of them implicitly) contained in [11] and [37].

Let $H(\Omega; \Delta)$ denote the elements of $L^2(\Omega)$ such that their weak Laplacean also belong to $L^2(\Omega)$.

**Lemma 3.1** (Bukhgeim and Uhlmann[11]). Assume that $\partial \Omega \in C^2$. Then the trace maps
\[ \text{tr}_0 u = u|_{\partial \Omega} \]
and
\[ \text{tr}_1 u = \partial_u u|_{\partial \Omega}, \]
defined in $C^\infty$ have an extension, again denoted as $\text{tr}_j$, $j = 0, 1$ which is continuous from $H(\Omega; \Delta)$ to the Sobolev space $H^{-j+1/2}(\partial \Omega)$.

If we assume in addition that $\text{tr}_0 u \in H^{3/2}$, then $u \in H^2(\Omega)$ and
\[ \|u\|_{H^2(\Omega)} + \|\text{tr}_1 u\|_{H^{1/2}(\partial \Omega)} \leq C\left(\|u\|_{H(\Omega; \Delta)} + \|\text{tr}_0 u\|_{H^{3/2}(\partial \Omega)}\right), \]
for some constant $C > 0$.

The proof can be found in [11]. Let us remark that the definition of the extended trace maps is based on Green’s formulae for smooth functions. Consider $u \in H(\Omega; \Delta)$, then on one hand, for $\omega \in H^{1/2}(\partial \Omega)$, we have
\[ (3.1) \quad \text{tr}_0 u(\omega) = \int_{\Omega} (u \Delta \bar{v} - \Delta u \bar{v}) \, dx \]
where $v \in H^2(\Omega)$ is the extension
\[ (3.2) \quad v|_{\partial \Omega} = 0, \partial_v|_{\partial \Omega} = \omega. \]
On the other hand, for $\omega \in H^{3/2}(\partial \Omega)$, we have
\[ (3.3) \quad \text{tr}_1 u(\omega) = \int_{\Omega} (u \Delta \bar{v} - \Delta u \bar{v}) \, dx \]
where \( v \in H^2(\Omega) \) is the extension
\[
(3.4) \quad v|_{\partial \Omega} = \omega, \partial_{v}|_{\partial \Omega} = 0.
\]
The generalized Green’s formula reads as follows.

**Lemma 3.2.** For \( u \in H(\Omega; \Delta) \) and \( v \in H^2(\Omega) \), we have
\[
(3.5) \quad \int_{\Omega} (\Delta - q)u \bar{v} \, dx = \int_{\Omega} u(\Delta - q)v \, dx + \left< \text{tr}_1 u | \text{tr}_0 \bar{v} \right> - \left< \text{tr}_0 u | \text{tr}_1 \bar{v} \right>.
\]

### 3.1. The Dirichlet-to-Neuman map.

The next step is to define the Dirichlet-to-Neumann map associated to the Schrödinger equation \(-\Delta + q\). To achieve a definition in a extended domain that contains the traces of the solutions \( u \in H(\Omega; \Delta) \) of the equation \((-\Delta + q)u = 0\), we will need the following lemma (see [37]).

We will denote \( H(\partial \Omega) \) the range of the map
\[
\text{tr}_0 : H(\Omega; \Delta) \to H^{-1/2}(\partial \Omega),
\]
and also consider the space of solutions of Schrödinger equation
\[
b_q := \{ u \in L^2(\Omega) : (-\Delta + q)u = 0 \} \subset H(\Omega; \Delta).
\]

Then we have

**Lemma 3.3** (Nachman and Street [37]). If \( q \in L^\infty(\Omega) \) and 0 is not a Dirichlet eigenvalue of \(-\Delta + q\) in \( \Omega \) then the trace map
\[
\text{tr}_0 : b_q \to H(\partial \Omega)
\]
is one to one and onto.

We consider the inverse maps
\[
P_q = \text{tr}_0^{-1} : H(\partial \Omega) \to b_q
\]
and define the norm of \( H(\partial \Omega) \) as
\[
\| \varphi \|_{H(\partial \Omega)} := \| P_0 \varphi \|_{L^2(\Omega)}
\]
Then we have:

**Lemma 3.4** (Nachman and Street [37]). The map \( \text{tr}_0 : H(\Omega; \Delta) \to H(\partial \Omega) \) is continuous and, under the hypothesis of the previous lemma, the map
\[
\text{tr}_0 : b_q \to H(\partial \Omega)
\]
is a homeomorphism.

One can define the Dirichlet-to-Neumann map
\[
(3.6) \quad \Lambda_q : H(\partial \Omega) \to H^{-3/2}(\partial \Omega)
\]
as the map
\[
\Lambda_q(\varphi) = \text{tr}_1(P_q(\varphi)).
\]
To be more precise, by \([3.3]\) for \( \varphi \in H(\partial \Omega) \) and \( \psi \in H^{3/2}(\partial \Omega) \), we have
\[
(3.7) \quad \left< \Lambda_q \varphi | \psi \right> = \int_{\Omega} (P_q \varphi \Delta \bar{v} - qP_q \varphi \bar{v}) \, dx,
\]
where \( v \) is the extension in \( [34] \). It would be desirable to construct the Dirichlet-to-Neumann map as a selfdual operator, unfortunately this can not be achieved, instead we have:

**Lemma 3.5** (Nachman and Street [37]). Let \( q_j \), \( j = 1, 2 \) be \( L^\infty \) potentials so that 0 is not a Dirichlet eigenvalue of \(-\Delta + q_j\) in \( \Omega \). Then \( \Lambda_{q_2} - \Lambda_{q_1} \) extends to a continuous map \( \mathcal{H}(\partial\Omega) \rightarrow \mathcal{H}(\partial\Omega)^* \).

These lemmas can be found in [37]. Let us remark that we have, if \( \varphi, \psi \in \mathcal{H}(\partial\Omega) \),

\[
(\Lambda_{q_2} - \Lambda_{q_1}) (\varphi) | \psi \rangle = \int_{\Omega} P_{q_1}(\varphi)(q_1 - q_2) P_{q_2}(\psi) \, dx.
\]

This formula is the starting point of the recovery of values of the error in the interior of the domain. Notice that

\[
\| \Lambda_{q_1} - \Lambda_{q_2} \|_{\mathcal{H}(\partial\Omega) \rightarrow \mathcal{H}(\partial\Omega)^*} \geq \| \Lambda_{q_1} - \Lambda_{q_2} \|_{H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)}.
\]

### 3.2. The partial data map.

Given \( N \subset S^{n-1} \) open (which could be very small) and \( \xi \in N \), We assume \( F \subset \partial\Omega \) to be a neighborhood of \( \partial\Omega_- (\xi) \) for any \( \xi \in N \) and \( B \) a neighborhood of \( \partial\Omega_+ (\xi) \) for any \( \xi \in N \).

Given two potentials satisfying the hypothesis of Lemma [3.5] we will consider the difference of their partial data measurements as

\[
\| \Lambda_{q_1} - \Lambda_{q_2} \|_{B \rightarrow F} := \sup \left\{ \left\| (\Lambda_{q_1} - \Lambda_{q_2}) \varphi_B \right\|_{\mathcal{H}(\partial\Omega)} \right\},
\]

where sup is taken over the set

\[
\{ (\varphi_B, \psi_F) : \| \varphi_B \|_{\mathcal{H}(\partial\Omega)} = \| \psi_F \|_{\mathcal{H}(\partial\Omega)} = 1, \varphi_B \in \mathcal{E}'(B) \text{ and } \psi_F \in \mathcal{E}'(F) \}.
\]

### 3.3. Fadeev’s special solutions.

We collect and remark results in [11] and [37], concerning the existence and a priori bounds of \( H(\Omega, \Delta) \) solutions of the Schrödinger equation adapted to the support requirements of the partial data.

The modifications of [37] were done in the context of partial data considered in [33], but the construction can be adapted to the case of [11], the only point is to write the operator conjugated with the exponential weight in the appropriate coordinates so that it is given as perturbations of the laplacean with terms of the complex operators \( \partial_\xi \) and \( \partial_\zeta \). The final output of this construction is as follows.

We consider \( q \) as in Lemma [3.3], \( \xi \) and \( \zeta \) unit orthogonal vectors so that \( \xi \in N \). We write \( x = (x_1, x_2, x'') \) with respect to an orthonormal basis \( \{e_1, ..., e_n\} \) so that \( e_1 = \xi \) and \( e_2 = \zeta \) and \( x'' \in \mathbb{R}^{n-2} \).

**Theorem 3.6.** For \( \tau \geq 1 \) sufficiently large and \( q \in C^\infty(\mathbb{R}^{n-2}) \), and \( B \) and \( F \) as in the statement of Theorem [13] there exists a unique solution \( w_\tau \in H(\Omega; \Delta) \) of the equation \(-\Delta + q)w_\tau = 0 \) in \( \Omega \), such that \( \text{tr}_0 w_\tau \in \mathcal{H}(\partial\Omega) \cap \mathcal{E}'(B) \) and which can be written as

\[
w_\tau (x) = e^{\tau (\xi + i\zeta, x)} (g(x'') + R(\tau, x))
\]
where
\[ \|R(\tau,\cdot)\|_{L^2(\Omega)} \leq C \frac{1}{\tau} (\|qg\|_{L^2(\Omega)} + \tau^{1/2} \|\Delta g\|_{L^2(\Omega)}). \]

The same is true changing \( \tau \) by \(-\tau\) and \( B \) by \( F \).

The key ingredients in the proof are boundary Carleman estimates and some orthogonality properties of the reduced data solutions.

**Proposition 3.7.** Let \( q \in L^\infty(\Omega) \), there exist \( \tau_0 > 0 \) and \( C > 0 \) such that for all \( u \in C^\infty(\overline{\Omega}) \), \( u|_{\partial \Omega} = 0 \) and \( \tau > \tau_0 \)

\[
\begin{equation}
C C^2 \int_{\Omega} |e^{-\tau\langle x,\xi \rangle} u|^2 \, dx + \tau \int_{\partial \Omega^+} \langle x, \xi \rangle |e^{-\tau x \cdot \xi} \partial_v u|^2 \, dA \\
\leq \int_{\Omega} |e^{-\tau\langle x,\xi \rangle} (\Delta - q) u|^2 \, dx - \tau \int_{\partial \Omega^-} \langle x, \xi \rangle |e^{-\tau x \cdot \xi} \partial_v u|^2 \, dA.
\end{equation}
\]

3.4. Stability from the Dirichlet-to-Neumann map to the Radon transform. We will use identity (3.8) together with the special solutions of Theorem 3.6 to prove for \( q = (q_1 - q_2)1_\Omega \):

**Proposition 3.8.** For any \( g \in C^\infty(\mathbb{R}^{n-2}) \) there exists \( C \) which only depends on \( \Omega \) and the a priori bound of \( \|q\|_{L^\infty} \), such that

\[
\sup_{\xi \in N, \xi \in \xi^\perp} \left| \int_{[\xi,\xi]^\perp} g(x') \int_{\mathbb{R}^2} q(x'' + t\xi + s\zeta) \, dt \, ds \, dx'' \right| \\
\leq C (\tau^{-1/2} + e^{ct} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{B \rightarrow F}) \|g\|_{H^2(\Omega)},
\]

where \([\xi,\zeta]\) denotes the plane spanned by \( \xi \) and \( \zeta \).

**Corollary 3.9.** If we consider the open set in \( S^{n-1} \)

\[ M = \cup_{\xi \in N} [\xi] \perp, \]

then we have in natural coordinates of the Radon transform

\[
\begin{equation}
\sup_{\eta \in M} \left| \int_{\mathbb{R}} \tilde{g}(r) \mathcal{R}q(r,\eta) \, dr \right| \\
\leq C (\tau^{-1/2} + e^{ct} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{B \rightarrow F}) \|\tilde{g}\|_{H^2(\mathbb{R})}.
\end{equation}
\]

**Proof.** With the notation of the Proposition, fix \( \eta \in [\xi,\zeta] \) with \( |\eta| = 1 \). We write \( x'' \in [\xi,\zeta] \perp \) as \( x'' = x''' + r\eta \) where \( x''' \in \eta, [\xi,\zeta] \perp \) and choose appropriate \( g \) independent of \( x''' \), \( \tilde{g}(r) = g(r\eta) \) in the Proposition to get the control of the Radon transform on any hyperplane, obtained by translation of hyperplanes in the pencil through the origin that contains \( \xi \). We write \( x = r\eta + x''' + t\xi + s\zeta \) with \( x''' \in \eta, [\xi,\zeta] \perp \), so that \( dx'' = dx''' \, dr \). Hence

\[
\begin{equation}
\sup_{\xi \in N, \xi \in \xi^\perp} \left| \int_{\mathbb{R}} g(r\eta) \int_{\mathbb{R}^{n-1}} q(x'' + r\eta + t\xi + s\zeta) \, dt \, ds \, dx''' \, dr \right| \\
\leq C (\tau^{-1/2} + e^{ct} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{B \rightarrow F}) \|\tilde{g}\|_{H^2(\mathbb{R})}.
\end{equation}
\]
Translating the above in the natural variables \((r, \eta)\) of the Radon transform gives the Corollary.

**Proof of Proposition 3.8.** By Lemma 3.6 there exist \(v_1 \in B_{q_1}\) such that supp \(\operatorname{tr}_0 v_1 \subset B\) of the form

\[ v_1 = e^{-r(\xi+i\zeta,x)}(1 + R_1(\tau,x)) \]

and \(v_2 \in B_{q_2}\) such that supp \(\operatorname{tr}_0 v_2 \subset F\) and

\[ v_2 = e^{r(\xi-i\zeta,x)}(g(x'') + R_2(\tau,x)). \]

We might write (3.3), by using these solutions as

\[ \left< (\Lambda_{q_2} - \Lambda_{q_1}) (\operatorname{tr}_0 v_1) \right| \operatorname{tr}_0 v_2 \right> = \int_\Omega v_1(q_1 - q_2) v_2 \, dx. \]

We also have

\[ \left| \left< (\Lambda_{q_2} - \Lambda_{q_1}) (\operatorname{tr}_0 v_1) \right| \operatorname{tr}_0 v_2 \right| \leq \|\Lambda_{q_1} - \Lambda_{q_2}\|_{B \rightarrow F} \|\operatorname{tr}_0 v_1\|_{H(\partial \Omega)} \|\operatorname{tr}_0 v_2\|_{H(\partial \Omega)}, \]

which from Lemma 3.7 gives

\[ \left| \left< (\Lambda_{q_2} - \Lambda_{q_1}) (\operatorname{tr}_0 v_1) \right| \operatorname{tr}_0 v_2 \right| \leq C \|\Lambda_{q_1} - \Lambda_{q_2}\|_{B \rightarrow F} \|v_1\|_{H(\Omega;\Delta)} \|v_2\|_{H(\Omega;\Delta)}. \]

Since

\[ \|v_2\|_{H(\Omega;\Delta)} \leq C(\|q_2\|_{L^\infty} + 1) \|v_2\|_{L^2(\Omega)} \leq C(\|q_2\|_{L^\infty} + 1) \sup_{x \in \Omega} e^{r(\xi+i\zeta,x)} (\|g\|_{L^2(\Omega)} + \|R_2(\tau,\cdot)\|_{L^2(\Omega)}) \leq C(\|q_2\|_{L^\infty} + 1) e^{C \tau} \|g\|_{H^2(\Omega)}. \]

In a similar way,

\[ \|v_1\|_{H(\Omega;\Delta)} \leq C(\|q_1\|_{L^\infty} + 1) e^{C \tau}. \]

Hence

\[ \left| \left< (\Lambda_{q_2} - \Lambda_{q_1}) (\operatorname{tr}_0 v_1) \right| \operatorname{tr}_0 v_2 \right| \leq C e^{C \tau} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{B \rightarrow F} \|g\|_{H^2(\Omega)}, \]

with constants only depending on \(\Omega\) and the a priori bound assumed on \(\|q_j\|_{L^\infty}\).

We have

\[ \int_\Omega q v_1 v_2 \, dx = \int_\Omega q(1 + R_1(\tau,x))(g(x'') + R_2(\tau,x)) \, dx, \]

hence

\[ \left| \int_\Omega v_1 q v_2 \, dx \right| \geq \left| \int_\Omega q g(x'') \, dx \right| - C \tau^{-1/2} \|g\|_{H^2(\Omega)} \]

where \(C\) only depends on \(\Omega\) and the a priori bound of \(\|q_j\|_{L^\infty}\).

Putting all together, we get

\[ \left| \int_\Omega q g(x'') \, dx \right| \leq C e^{C \tau} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{B \rightarrow F} \|g\|_{H^2(\Omega)} + C \tau^{-1/2} \|g\|_{H^2(\Omega)}. \]

This ends the proof of Proposition 3.8. \(\square\)
Theorem 3.10 (Stability of Radon transform). We have the following estimate on \( q = (q_1 - q_2)1_\Omega \)

\[
\left( \int_M \left( \int_{\mathbb{R}} |\mathcal{R}q(r, \eta)|^2 \, dr \right)^{\frac{n+3}{4}} \, d\sigma(\eta) \right)^{\frac{2}{n+3}} 
\leq C(\tau^{-1/2} + e^{c\tau} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{B\rightarrow F}^{\ast})^{\frac{n-1}{n+3}}.
\]

Proof. The estimate can be obtained by interpolation of (3.11) and the following estimate for the Radon transform

\[
\int_{S^{n-1}} \int_{\mathbb{R}} (1 + \tau^2)^{\frac{n-1}{2}} |\hat{\mathcal{R}}q(\tau, \eta)|^2 \, d\tau \, d\sigma(\eta) \leq \|q\|_{L^2}^2.
\]

This estimate can be found in [38]. \(\square\)

3.5. End of proof of Theorem 1.3

Proof. Our aim is to use Theorem 2.5 together with estimate (3.12). We need the supporting condition (b) in Theorem 2.5. To achieve this condition let us take \( \eta \in M \), we know by translation that there exists \( y_0 \in \text{supp} \, q \) such that the hyperplane \( H_{y_0}(0, \eta) \), see (2.1), which contains \( y_0 \) satisfies condition (b).

Now \( M \) is a neighbourhood of \( \eta \) in the sphere and from the previous Theorem we can control the Radon transform for \( \omega \in M \) and \( s \in \mathbb{R} \).

We can take \( \alpha > 0 \) large enough, so that \( G \) in (2.17) contains \( \text{supp} \, q \) (the \( \beta \) depends on the size of \( M \)). Then

\[
\int_{(-\alpha,\alpha)} (1 + |s - (\omega, \zeta - y_0)|)^n \|\mathcal{R}_{y_0}(q_1 - q_2)(s, \cdot)\|_{L^1(M)} \, ds
\]

\[
\leq C(\text{supp} \, q, \beta) \left( \int_M \left( \int_{\mathbb{R}} |\mathcal{R}q(r, \eta)|^2 \, dr \right)^{\frac{n+3}{4}} \, d\sigma(\eta) \right)^{\frac{2}{n+3}}.
\]

The choice \( \tau = \frac{1}{2c} \log \|\Lambda_{q_1} - \Lambda_{q_2}\|_{B\rightarrow F}^{\ast} \) in the estimate (3.12) gives

\[
\int_{(-\alpha,\alpha)} (1 + |s - (\omega, \zeta - y_0)|)^n \|\mathcal{R}_{y_0}(q_1 - q_2)(s, \cdot)\|_{L^1(M)} \, ds \leq C(\text{supp} \, q, \beta) \times \left( (C|\log \|\Lambda_{q_1} - \Lambda_{q_2}\|_{B\rightarrow F}^{\ast}|-1/2 + (\|\Lambda_{q_1} - \Lambda_{q_2}\|_{B\rightarrow F}^{\ast})^{1/2})^{\frac{n+1}{n+3}}. \right)
\]

This together with (2.16) gives the desired estimate

\[
\|q_1 - q_2\|_{L^p(\Omega)} \leq C|\log |\log \|\Lambda_{q_1} - \Lambda_{q_2}\|_{B\rightarrow F}^{\ast}|^{-\frac{1}{2}},
\]

for small norms of the difference of Dirichlet-to-Neumann maps. This ends the proof of Theorem 1.3. \(\square\)
4. Second Application: illuminating $\Omega$ from a point (KSU)

To obtain stability in the case of $\overline{q}$, we could have followed the same approach, but in order to prove stability with the usual $H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)$-norm of the Dirichlet-to-Neumann map we will recall on the $H^1$-solutions of the Schrödinger equation constructed by Chung (see [17]), in his work on the magnetic case with partial data.

As above, we consider two potentials $q_1$ and $q_2$ being in $L^\infty(\Omega)$ and such that 0 is not an eigenvalue of $(-\Delta + q_j) : H^1_0(\Omega) \cap H(\Omega; \Delta) \to L^2(\Omega)$ for $j \in \{1, 2\}$. Let $\Lambda_{q_j}$ denote the Dirichlet-to-Neumann map corresponding to the coefficient $q_j$.

Let $\langle \cdot | \cdot \rangle$ denote the duality between $H^{1/2}(\partial \Omega)$ and $H^{-1/2}(\partial \Omega)$ and recall that $\Lambda_{q_j} : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)$ is defined as

$$\langle \Lambda_{q_j} f_j, g \rangle := \int_{\Omega} (\nabla u_j, \nabla v) \, dx + \int_{\Omega} q_j u_j v \, dx$$

for any $f_j, g \in H^{1/2}(\partial \Omega)$, where $u_j \in H^1(\Omega)$ is the weak solution of the Schrödinger equation $(-\Delta + q_j)u_j = 0$ in $\Omega$ with $u_j|_{\partial \Omega} = f_j$ and $v \in H^1(\Omega)$ with $v|_{\partial \Omega} = g$. Since 0 is not a Dirichlet eigenvalue, $\Lambda_{q_j}$ is a well-defined bounded linear operator.

An integration by parts shows that, for any $v_j \in H^1(\Omega)$ weak solution of the Schrödinger equation

$$(-\Delta + q_j)v_j = 0$$

in $\Omega$, one has

$$\langle (\Lambda_{q_1} - \Lambda_{q_2})(v_1|_{\partial \Omega}), v_2|_{\partial \Omega} \rangle = \int_{\Omega} (q_1 - q_2) v_1 v_2 \, dx.$$

Let $y_0$ be a point out of $\mathrm{ch}(\Omega)$, the convex hull of $\Omega$, and consider $N$ a neighbourhood of $y_0$ also away from $\mathrm{ch}(\Omega)$. Consider a hyperplane $H$ separating $N$ and $\mathrm{ch}(\Omega)$ and let $H^+$ denote the semi-space delimited by $H$ and containing $\overline{\Omega}$. Consider $R$ a positive constant such that, for all $y \in N$, $\Omega \subset B(y, R)$ and set $\Sigma$ a subset of $\mathbb{S}^{n-1}$ with non-empty interior and

$$\overline{\Sigma} \subset \{ \theta \in \mathbb{S}^{n-1} \text{ such that for all } y \in N, \ y + R \theta \notin H^+ \text{ and } y - R \theta \notin H^+ \}.$$

Let $F$ and $B$ denote open neighborhoods, for any $y \in N$, of the faces $\partial \Omega_- (y)$ and $\partial \Omega_+(y)$ respectively. Consider $\chi \in C^\infty(\partial \Omega)$ satisfying $\chi : \partial \Omega \to [0, 1]$, $\mathrm{supp} \chi \subset F$ and $\chi|_{F_\varepsilon} = 1$ with $\partial \Omega_- (y) \subset F_\varepsilon \subset \overline{F_\varepsilon} \subset F$ ($\chi$ and $F_\varepsilon$ may depend on $y$). Obviously, we can write

$$\langle (\Lambda_{q_1} - \Lambda_{q_2})(v_1|_{\partial \Omega}), v_2|_{\partial \Omega} \rangle = \langle (\Lambda_{q_1} - \Lambda_{q_2})(v_1|_{\partial \Omega}), (1 - \chi)v_2|_{\partial \Omega} \rangle$$

$$+ \langle (\Lambda_{q_1} - \Lambda_{q_2})(v_1|_{\partial \Omega}), \chi v_2|_{\partial \Omega} \rangle.$$

Let $w_2 \in H^1(\Omega)$ denote the weak solution of $(-\Delta + q_2)w_2 = 0$ in $\Omega$ with $w_2|_{\partial \Omega} = v_1|_{\partial \Omega}$. This implies $\partial_\nu (v_1 - w_2)|_{\partial \Omega} \in H^{1/2}(\partial \Omega)$ ($\nu$ stands for the
Proposition 4.1. by Chung [17]. More precisely, from [33] (see Lemma 3.4 in [19]) we have, and were constructed by Kenig, Sjöstrand and Uhlmann [33] and recently of the Schrödinger equation (4.1). These special solutions are a generaliza-

In order to obtain this information, we need to plug in (4.3) special solutions

generating point to control the 2-plane transform of

\[ \psi(\tau) = e^{(\varphi_2 + i\psi_2)}(a_2 + r_2(\tau)), \]
where \( \tau \) is a large parameter, \( \varphi_2 : (\mathbb{R}^n \setminus N) \times N \to \mathbb{R} \) defined by

\[ \varphi_2(x, y) = -\log |x - y|, \]

\( \psi_2 : \tilde{\Omega} \times N \times \Sigma \to \mathbb{R} \) defined by

\[ \psi_2(x, y, \theta) = d_{g_{n-1}} \left( \frac{x - y}{|x - y|}, \theta \right) \]

\( (H_0^{1/2}(F)) \) is the subspace of \( H^{1/2}(\partial \Omega) \) whose elements have their support in \( \mathcal{F} \), \( H^{-1/2}(F) \) is nothing but its dual –more about these spaces can be found in [15].

Assuming \( \text{supp } v_1|_{\partial \Omega} \subset \mathcal{B} \), one has

\[ \left| \left( \Lambda_1 - \Lambda_2 \right) (v_1|_{\partial \Omega}) \right| ||v_2||_{H^{1/2}(\partial \Omega)} \]

where \( \| \cdot \|_B \to F \) denotes the operator norm from \( H_0^{1/2}(B) \) to \( H^{-1/2}(F) \) and \( \epsilon \) is any positive constant (\( C^{0,1/2+\epsilon}(\partial \Omega) \) is the space of \( (1/2+\epsilon) \)-Hölder functions defined on the \( \partial \Omega \) –a close subset of \( \mathbb{R}^n \)).

For future references,

\[ \left| \left( \Lambda_1 - \Lambda_2 \right) (v_1|_{\partial \Omega}) \right| \leq \int _{\partial \Omega \setminus F} |\partial_x (v_1 - w_2)| ||v_2||_F + C \left\| \Lambda_1 - \Lambda_2 \right\| \left\| v_1 \right\|_{H_0^{1/2}(F)} \left\| v_2 \right\|_{H^{1/2}(\partial \Omega)}. \]

Since we assume that \( 0 \) is not an eigenvalue of \( (-\Delta + q) : H_0^1(\Omega) \cap H(\Omega; \Delta) \to L^2(\Omega) \) we have that \( \partial_x (v_1 - w_2)|_{\partial \Omega} \in L^2(\partial \Omega). \)

4.1. Controlling the Radon transform. Estimate [13] will be the starting point to control the 2-plane transform of \( q \in L^\infty(\mathbb{R}^n) \), for \( q = (q_1 - q_2)1_\Omega \). In order to obtain this information, we need to plug in [13] special solutions of the Schrödinger equation [11]. These special solutions are a generalization of the classical complex geometrical optic solutions (or Fadeev solutions) and were constructed by Kenig, Sjöstrand and Uhlmann [33] and recently by Chung [17]. More precisely, from [33] (see Lemma 3.4 in [19]) we have,

**Proposition 4.1.** There exists a solution \( v_2 \) of \( (-\Delta + q_2)v = 0 \) of the form

\[ v_2(\tau) = e^{i(\varphi_2 + i\psi_2)}(a_2 + r_2(\tau)), \]

where \( \tau \) is a large parameter, \( \varphi_2 : (\mathbb{R}^n \setminus N) \times N \to \mathbb{R} \) defined by

\[ \varphi_2(x, y) = -\log |x - y|, \]

\( \psi_2 : \tilde{\Omega} \times N \times \Sigma \to \mathbb{R} \) defined by

\[ \psi_2(x, y, \theta) = d_{g_{n-1}} \left( \frac{x - y}{|x - y|}, \theta \right) \]
with $\tilde{\Omega} = \cap_{y \in N} (H_+ \cap \{x \in \mathbb{R}^n : |x - y| < R\})$ and $d_{S^{n-1}}$ the geodesic distance on $S^{n-1}$ associated to the Euclidean metric restricted to $S^{n-1}$. Furthermore, $a_2 : \tilde{\Omega} \times N \times \Sigma \to \mathbb{C}$ as defined as

$$a_2(x, y, \theta) = (2|x - y - \theta \cdot (x - y)\theta|)^{-\frac{n-2}{2}} g \left( \frac{x - y - \theta \cdot (x - y)\theta}{|x - y - \theta \cdot (x - y)\theta|} \right),$$

with any $g : S^{n-2} \to \mathbb{C}$ smooth, $r_2(\tau) \in H^1(\Omega)$ and satisfies

$$\tau \|r_2(\tau)\|_{L^2(\Omega)} + \tau^{1/2} \|r_2(\tau)\|_{L^2(\partial \Omega)} + \|\nabla r_2(\tau)\|_{L^2(\Omega)^n} \leq C \left( \|g\|_{L^2(S^{n-2})} + \|\Delta_{S^{n-2}} g\|_{L^2(S^{n-2})} \right)$$

where $\Delta_{S^{n-2}}$ is the Laplace-Beltrami operator on $S^{n-2}$ for the canonical metric on $S^{n-2}$. Here $C$ depends on $\|q_2\|_{L^\infty(\Omega)}$, on $\Omega$, on the distance of $N$ to the hyperplane $H$.

Let us denote

$$Z(y) = \{x \in \partial \Omega : \langle x - y, \nu(x) \rangle = 0\},$$

and assume that $E$ is a compact subset of $\partial \Omega_-(y) \setminus Z(y)$ with $E \supset \partial \Omega \setminus \overline{B}$. Let us recall Proposition 7.2 in [17].

**Proposition 4.2.** There exists a solution $v_1$ of $v_1$ of $(-\Delta + q_1)v = 0$ vanishing on $E$ and having the form

$$v_1(\tau) = e^{\tau(\varphi_1 + i\psi_1)}(a_1 + r_1(\tau)) - e^{\tau} b,$$

where $\tau$ is the same as above, $\varphi_1 : (\mathbb{R}^n \setminus N) \times N \to \mathbb{R}$ defined by

$$\varphi_1(x, y) = -\varphi_2(x, y),$$

$\psi_1 : \tilde{\Omega} \times N \times \Sigma \to \mathbb{R}$ defined by

$$\psi_1(x, y, \theta) = -\psi_2(x, y, \theta),$$

$a_1 : \tilde{\Omega} \times N \times \Sigma \to \mathbb{C}$ as defined as

$$a_1(x, y, \theta) = (2|x - y - \theta \cdot (x - y)\theta|)^{-\frac{n-2}{2}},$$

additionally $r_1(\tau) \in H^1(\Omega)$ and satisfies

$$\tau \|r_1(\tau)\|_{L^2(\Omega)} + \|\nabla r_1(\tau)\|_{L^2(\Omega)^n} \leq C$$

where $C$ depends again on $\|q_1\|_{L^\infty(\Omega)}$. Finally, $l : \tilde{\Omega} \times N \times \Sigma \to \mathbb{C}$ is smooth and it satisfies $\Re l = \varphi_1 - k$ with $k(x) \simeq \text{dist}(x, E)$ in $G$, a neighbourhood of $E$ with non-empty interior in $\mathbb{R}^n$, and $b : \tilde{\Omega} \times N \times \Sigma \to \mathbb{C}$ is twice continuously differentiable in $\tilde{\Omega}$ and $\text{supp } b \subset G$.

Let us remark recall the definition of the 2-plane transform. Given $y \in \mathbb{R}^n$, and $\theta$ and $\eta$ unitary orthogonal vectors, we denote

$$Rq(y, \theta, \eta) := \int_{\mathbb{R} \times \mathbb{R}} q(y + t\theta + r\eta) \, dt \, dr.$$
This is just the integral of $q$ in the plane $\{y\} + [\theta, \eta]$, where $[\theta, \eta]$ denotes the plane spanned by $\theta$ and $\eta$. Notice that there is some redundancy on variables, since it is enough to define the above for $y \in [\theta, \eta]$, see [10].

We assume that $N$ contains the ball $B(0,\alpha)$ and that $\|\Lambda q_1 - \Lambda q_2\| \leq 1$.

**Theorem 4.3.** The following estimate holds

\begin{equation}
\sup_{y \in B(0,\alpha), \theta \in \Sigma} \|Rq(y, \theta, \eta)\|_{H^{-2}(S_\theta)} \leq C \left( \tau^{-1/4} \|q\|_{L^\infty(\mathbb{R}^n)}^{1/2} + e^{ct} \|\Lambda q_1 - \Lambda q_2\|_{B \to F}^{1/4} \right),
\end{equation}

where $S_\theta = \mathbb{S}^{n-1} \cap \theta^\perp$ and we consider the measure $d\sigma$, the volume form on $S_\theta$ associated to the canonical metric on $S_\theta$.

**Proof.** We next plug in (4.2) the solutions in the above Propositions and bound by below the absolute value of the term in the right hand side of this identity:

\begin{equation}
\left| \int_\Omega q v_1 v_2 \, dx \right| \geq \left| \int_\Omega q a_1 a_2 \, dx \right| - \|q\|_{L^\infty(\mathbb{R}^n)} \times \left( \|b\|_{L^\infty(\Omega \cap G)} \left( \|a_2\|_{L^2(\Omega)} + \|r_2\|_{L^2(\Omega)} \right) \left\| e^{-\tau k} \right\|_{L^2(\Omega \cap G)}^{1/2} + \|a_2\|_{L^2(\Omega)} \|r_1\|_{L^2(\Omega)} + \|a_1\|_{L^2(\Omega)} \|r_2\|_{L^2(\Omega)} + \|r_1\|_{L^2(\Omega)} \|r_2\|_{L^2(\Omega)} \right). \end{equation}

The last inequality, identity (4.2), the properties of the solutions $v_1$ and $v_2$ and (4.3) imply that

\begin{equation}
\int_\Omega q a_1 a_2 \, dx \leq \frac{C}{\tau^{1/2}} \|q\|_{L^\infty(\mathbb{R}^n)} \left( \|g\|_{L^2(S^{n-2})} + \|\Delta S^{n-2}g\|_{L^2(S^{n-2})} \right) + \int_{\partial\Omega \setminus F_\varepsilon} |\partial_{v_1} (v_1 - w_2)| |v_2| \, dA + C \|\Lambda q_1 - \Lambda q_2\|_{H^1(S^{n-2})} \cdot
\end{equation}

Mind that $\tau^{-1/2}$ comes from $\|e^{-\tau k}\|_{L^2(\Omega \cap G)}$, dependences of $C$ has not changed and $c$ only depends on $\Omega$.

We are next going to estimate the boundary integral term in (4.3). In order to do so, we are going to choose $F_\varepsilon$ in such a way that $\partial \Omega \setminus F_\varepsilon = \{ x \in \partial \Omega \}$.
\[ \partial \Omega : (x - y) \cdot \nu(x) \geq \varepsilon \} \text{ with } y \in N. \] Thus,

\[
\int_{\partial \Omega \setminus F_\varepsilon} |\partial_\nu (v_1 - w_2)||v_2| \, dA = \int_{\partial \Omega \setminus F_\varepsilon} e^{\tau \varphi^2} |\partial_\nu (v_1 - w_2)||a_2 + r_2| \, dA
\leq C \left( \int_{\partial \Omega \setminus F_\varepsilon} e^{2\tau \varphi^2} |\partial_\nu (v_1 - w_2)|^2 \, dA \right)^{\frac{1}{2}}
\times \left( \|g\|_{H^1(S^{n-2})} + \|\Delta S^{n-2}g\|_{L^2(S^{n-2})} \right)
\leq C \left( \frac{1}{\varepsilon} \int_{\partial \Omega \setminus F_\varepsilon} \langle \nu, (x - y) \rangle e^{2\tau \varphi^2} |\partial_\nu (v_1 - w_2)| \, dA \right)^{\frac{1}{2}}
\times \left( \|g\|_{H^1(S^{n-2})} + \|\Delta S^{n-2}g\|_{L^2(S^{n-2})} \right)
\]

We now focus on the integral boundary term on the last inequality. Note that

\[ (-\Delta + q^2)(v_1 - w_2) = -q|\Omega| v_1 \]

with \((v_1 - w_2)|_{\partial \Omega} = 0\). We are going to use the Carleman estimate with boundary terms proved in [33]

**Proposition 4.4.** Let \(q_2 \in L^\infty(\Omega)\), there exist \(\tau_0 > 0\) and \(C > 0\) such that for all \(u \in C^\infty(\Omega)\), \(u|_{\partial \Omega} = 0\) and \(\tau > \tau_0\)

\[
C \tau^2 \int_{\Omega} |e^{-\tau \varphi^2} u|^2 \, dx + \tau \int_{\partial \Omega} \langle x - y, \nu(x) \rangle e^{-\tau \varphi^2} \partial_\nu u|^2 \, dA
\leq \int_{\Omega} |e^{-\tau \varphi^2} (\Delta - q) u|^2 \, dx - \tau \int_{\partial \Omega} \langle x - y, \nu(x) \rangle |e^{-\tau \varphi^2} \partial_\nu u|^2 \, dA.
\]

Then we obtain,

\[
\left( \frac{1}{\varepsilon} \int_{\partial \Omega \setminus F_\varepsilon} \langle \nu, (x - y) \rangle e^{2\tau \varphi^2} |\partial_\nu (v_1 - w_2)| \, dA \right)^{1/2}
\leq C \left( \tau^{-1/2} \|e^{\tau \varphi^2} qv_1\|_{L^2(\Omega)} + e^{\tau \|\partial_\nu (v_1 - w_2)\|_{L^2(\partial \Omega \setminus (y))}} \right)
\leq C \left( \tau^{-1/2} \|q\|_{L^\infty(\mathbb{R}^n)} + e^{\tau \|\partial_\nu (v_1 - w_2)\|_{L^2(\partial \Omega \setminus (y))}} \right)
\]

Here the constant depends additionally on \(\varepsilon\) and the distance of \(N\) to \(\text{ch}(\Omega)\).

We finally look at the \(L^2\)-norm on the boundary. Let \(\tilde{w}_2 \in H^1(\Omega)\) denote the solution of (4.1) with \(j = 2\) and the following boundary condition \(\tilde{w}_2|_{\partial \Omega} = \)
\( \chi \partial_\nu (v_1 - w_2) |_{\partial \Omega} \in H^{1/2}(\partial \Omega) \). Then, one has

\[
\| \partial_\nu (v_1 - w_2) \|_{L^2(\partial \Omega_{\nu}(y))} = \left( \int_{\partial \Omega} \partial_\nu (v_1 - w_2) \tilde{w}_2 \, dA \right)^{1/2}
= \left( \int_{\Omega} \Delta (v_1 - w_2) \tilde{w}_2 \, dx + \int_{\Omega} \langle \nabla (v_1 - w_2), \nabla \tilde{w}_2 \rangle \, dx \right)^{1/2}
= \left( \int_{\Omega} q_v \tilde{w}_2 \, dx + \int_{\Omega} q_v (v_1 - w_2) \tilde{w}_2 + \langle \nabla (v_1 - w_2), \nabla \tilde{w}_2 \rangle \, dx \right)^{1/2}
= \left( \int_{\Omega} q_v \tilde{w}_2 \, dx \right)^{1/2}. 
\]

Notice that the square of the last term appears in (4.2) if we change \( v_2 \) by \( \tilde{w}_2 \). Thus,

\[
\| \partial_\nu (v_1 - w_2) \|^2_{L^2(\partial \Omega_{\nu}(y))} = \left( \langle \Lambda_{q_1} - \Lambda_{q_2} \rangle (v_1 |_{\partial \Omega}) \tilde{w}_2 |_{\partial \Omega} \right). 
\]

Taking into account that \( \text{supp} \, \tilde{w}_2 |_{\partial \Omega} \subset F \) one has

\[
\| \partial_\nu (v_1 - w_2) \|_{L^2(\partial \Omega_{\nu}(y))} \leq \left( C \| \Lambda_{q_1} - \Lambda_{q_2} \| \| v_1 \|_{H^1(\Omega)} \| \tilde{w}_2 \|_{H^1(\Omega)} \right)^{1/2}
\leq \left( C \| \Lambda_{q_1} - \Lambda_{q_2} \| \| v_1 \|_{H^1(\Omega)} \| \partial_\nu (v_1 - w_2) \|_{H^{1/2}(\partial \Omega)} \right)^{1/2}
\leq C \| \Lambda_{q_1} - \Lambda_{q_2} \|^{1/2} e^{c \tau} \left( \| v_1 - w_2 \|_{L^2(\Omega)} + \| \Delta (v_1 - w_2) \|_{L^2(\Omega)} \right)^{1/2}. 
\]

In order to bound \( \| \partial_\nu (v_1 - w_2) \|_{H^{1/2}(\partial \Omega)} \) in the last inequality, see [11]. Since \( v_1 - w_2 \) is solution of the problem (4.6) we can bound as follows

\[
\| \partial_\nu (v_1 - w_2) \|_{L^2(\partial \Omega_{\nu}(y))} \leq C \| \Lambda_{q_1} - \Lambda_{q_2} \|^{1/2} e^{c \tau} \| q \|_{L^\infty(\mathbb{R}^n)}, 
\]

where the constant \( C \) depends on \( \| q_j \|_{L^\infty(\Omega)} \) with \( j \in \{1, 2\} \). Summing up,

\[
\int_{\partial \Omega \setminus F} | \partial_\nu (v_1 - w_2) | v_2 | \, dA \leq C \left( \| g \|_{H^1(S^{n-2})} + \| \Delta g \|_{L^2(S^{n-2})} \right) 
\times \left( \tau^{-1/2} \| q \|_{L^\infty(\mathbb{R}^n)} + e^{c \tau} \| \Lambda_{q_1} - \Lambda_{q_2} \|^{1/2} \| q \|_{L^\infty(\mathbb{R}^n)} \right)^{1/2}. 
\]
Therefore, we finally get

\[
\int_{\Omega} q a_1 a_2 \, dx \leq C \frac{1}{\tau^{1/2}} \|q\|_{L^\infty(\mathbb{R}^n)} \left( \|g\|_{L^2(S^{n-2})} + \|\Delta S^{n-2} g\|_{L^2(S^{n-2})} \right)
\]

\[
+ C \left( \tau^{-1/2} \|q\|_{L^\infty(\mathbb{R}^n)} + e^{ct} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{1/2} \|\Lambda\|_{L^\infty(\mathbb{R}^n)} \right)^{1/2}
\]

\[
\times \left( \|g\|_{H^1(S^{n-2})} + \|\Delta S^{n-2} g\|_{L^2(S^{n-2})} \right)
\]

\[
+ C \|\Lambda_{q_1} - \Lambda_{q_2}\| e^{ct} \|g\|_{H^1(S^{n-2})}.
\]

In the following lines, we are going to relate the left hand side of the above estimate with the 2-plane transform of \(q\). Note that

\[
\int_{\Omega} q a_1 a_2 \, dx = 2^{-(n-2)} \int_{\mathbb{R} \times \mathbb{R} \times S_\theta} q(y + t\theta + r\eta) g(\eta) \, dt \, d\sigma(\eta),
\]

where \(S_\theta = \{\eta \in S^{n-1} : \langle \eta, \theta \rangle = 0\}\) and \(d\sigma\) is the volume form on \(S_\theta\) associated to euclidean metric restricted to \(S_\theta\). Note that given \(y \in N\) and \(\theta \in \Sigma\), one has

\[
\left| \int_{S_\theta} Rq(y, \theta, \eta) g(\eta) \, d\sigma(\eta) \right| \leq C \left( \tau^{-1/4} \|q\|_{L^\infty(\mathbb{R}^n)}^{1/2} + e^{ct} \|\Lambda_{q_1} - \Lambda_{q_2}\|^{1/4} \right)
\]

\[
\times \left( \|g\|_{H^1(S^{n-2})} + \|\Delta S^{n-2} g\|_{L^2(S^{n-2})} \right),
\]

for all \(y \in N\) and \(\theta \in \Sigma\). Obviously,

\[
\|Rq(y, \theta, \cdot)\|_{H^{-2}(S_\theta)} \leq C \left( \tau^{-1/4} \|q\|_{L^\infty(\mathbb{R}^n)}^{1/2} + e^{ct} \|\Lambda_{q_1} - \Lambda_{q_2}\|^{1/4} \right)
\]

for all \(y \in N\) and \(\theta \in \Sigma\). \(\square\)

We would like to have an expression similar to (4.4), with a norm of the Radon transform \(Rq\) in the left hand side, suitable to apply the stability result in section 2. This can be achieved in the three dimensional case, since the 2-plane transform \(Rq(y, \theta, \eta)\), the integral of \(q\) in the plane \(P = \{y\} + [\theta, \eta]\), is a reparametrisation of the Radon transform.

From now on, we will restrict our analysis to dimension \(n = 3\). As we pointed out before, the variable \(y\) in (4.4) is redundant, since the natural coordinates of the 2-plane transform are \((y, P)\) where \(P = [\theta, \eta]\) and \(y \in P^\perp\). The variable \(y\) in (4.4) can not be restricted to \(P^\perp\), due to the \(\eta\)-integral involved in the partial Sobolev norm.

We have to make sense of the integral in (4.4) in the appropriate coordinates of the Radon transform given by \((s, \omega)\) \(\in \mathbb{R} \times S^2\). We can write pointwise \(Rq(y, \theta, \eta) = \mathcal{R}q(s, \omega)\), where \(s = \langle y, \omega \rangle\) and \(\omega = \theta \times \eta\) (vector product). To simplify the notation, we will assume that \(\Sigma = \Sigma_{4\delta}\) is bounded by the two planes parallel to \(H\) at distance \(4\delta\) of the origin.

We introduce geodesic polar coordinates on \(S^2\) in the following way: given \(\omega_0 \in S^2\) we can find \(\theta_0 \in \Sigma_{\delta}\) and \(\eta_0\) so that they form an orthonormal frame.
We will denote \( \eta_0 = e_1, \theta_0 = e_2 \) and \( \omega_0 = e_3 \). Let \( \theta = \theta(\varphi) = \cos \varphi e_2 + \sin \varphi e_1 \) with \( |\varphi| < \delta \). Let \( \eta \) run the geodesic \( S_\theta \) according to \( \psi = \text{dist}(\eta, e_3) \), we have \( \eta = \eta(\varphi, \psi) = \cos \varphi \sin \psi e_1 - \sin \varphi \sin \psi e_2 + \cos \psi e_3 \). We will perform the following calculations on the time zone

\[
\Gamma = \Gamma_{\omega_0, \theta_0}(\delta) = \{ \eta(\varphi, \psi) : |\varphi| < \delta, 0 < |\psi| < \pi \}. 
\]

We will need to write \( Rq(y, \theta, \eta) \) in the standard coordinates \( Rq(s, \omega(\varphi, \psi)) \), where \( \omega(\varphi, \psi) = \theta(\varphi) \times \eta(\varphi, \psi) \). For a fix \( \varphi \) the map \( \eta \to \omega \) is just a rotation of angle \( \pi/2 \) on the geodesic \( S_\theta \), hence \( \omega(\varphi, \psi) = \eta(\varphi, \psi - \pi/2) \). As a map from \( \Gamma \) to \( \Gamma \) this rotation collapses the segments \( \psi = \pm \pi/2 \) to the points \( \pm \omega_0 \). We then restrict the time zone to

\[
(4.8) \quad \tilde{\Gamma} = \tilde{\Gamma}_{\omega_0, \theta_0}(\delta) = \{ \eta(\varphi, \psi) : |\varphi| < \delta, \psi \in J \},
\]

where \( J = \{ \delta/8 < |\psi| < \pi/2 - \delta/8 \} \cup \{ \pi/2 + \delta/8 < |\psi| < \pi - \delta/8 \} \).

In these coordinates we have:

**Lemma 4.5.** (Local estimate, \( n = 3 \)) Let \( B(0, \alpha) \subset N \). Then

\[
(4.9) \quad \int_{B(0, \alpha)} \int_{-\delta}^{\delta} \left( \int_{J} |Rq(y, \theta, \eta(\varphi, \cdot))|^{6/5} dy \right)^{5/6} d\varphi dy \leq C \left( \tau^{-1/4} \| q \|_{L^{\infty}(\mathbb{R}^n)}^{1/2} + e^{C\tau} \| \Lambda_{q_1} - \Lambda_{q_2} \|^{1/4} \right)^{1/3},
\]

where \( C \) only depends on \( \delta \) and \( \alpha \).

**Proof.** We will obtain (4.9) by interpolation of the following estimates.

\[
(4.10) \quad \int_{B(0, \alpha)} \int_{-\delta}^{\delta} \| Rq(y, \theta(\varphi), \eta(\varphi, \cdot)) \|_{H^{-2}(J)} d\varphi dy \leq C \left( \tau^{-1/4} \| q \|_{L^{\infty}(\mathbb{R}^n)}^{1/2} + e^{C\tau} \| \Lambda_{q_1} - \Lambda_{q_2} \|^{1/4} \right),
\]

and

\[
(4.11) \quad \int_{B(0, \alpha)} \int_{-\delta}^{\delta} \int_{J} (|\partial_\psi Rq(y, \theta(\varphi), \eta(\varphi, \psi))| + |Rq(y, \theta(\varphi), \eta(\varphi, \psi))|) d\psi d\varphi dy \leq C \| q \|_{L^2}.
\]

Estimate (4.10) follows easily from (4.11). To prove estimate (4.11) we start with the derivative term, we change the order of integration and write \( y = s\theta + \eta + y' \) with \( y' \in [\theta, \eta] \),

\[
\int_{B(0, \alpha)} \int_{-\delta}^{\delta} \int_{J} |\partial_\psi Rq(y, \theta(\varphi), \eta(\varphi, \psi))| d\psi d\varphi dy \leq \int_{-\delta}^{\delta} \int_{J} \int_{-\alpha}^{\alpha} \int_{B_\omega} |\partial_\psi Rq(s\omega + y', \theta(\varphi), \eta(\varphi, \psi))| dy' ds d\psi d\varphi,
\]

where \( \omega = \omega(\varphi, \psi) \) and \( B_\omega = B(0, \alpha) \cap [\theta, \eta] \).
Since \( y' \) does not change the 2-plane X-ray transform, we write the above as
\[
C\alpha^2 \int_{-\delta}^{\delta} \int_J \int_{\alpha}^{\alpha} |\partial_\psi Rq(s\omega, \theta(\varphi), \eta(\varphi, \psi))| \, ds \, d\psi \, d\varphi,
\]
This expression can be written in terms of the Radon transform as
\[
(4.12) \quad C\alpha^2 \int_{-\delta}^{\delta} \int_J \int_{\alpha}^{\alpha} |\partial_\psi Rq(s, \omega(\varphi, \psi))| \, ds \, d\psi \, d\varphi
\]
\[
= C\alpha^2 \int_{-\delta}^{\delta} \int_J \int_{\alpha}^{\alpha} |\partial_\psi Rq(s, \eta(\varphi, \psi - \pi/2))| \, ds \, d\psi \, d\varphi.
\]
To simplify the notation, following [38], we will use the homogeneous of degree \(-1\) extension of \( R(s, \eta) \) to \( \eta \in \mathbb{R}^3 \), to write
\[
\partial_\psi Rq(s, \eta(\varphi, \psi - \pi/2)) = \nabla_\eta Rq(s, \eta) \cdot \frac{\partial \eta}{\partial \psi} = \frac{\partial}{\partial s} R(xq)(s, \eta) \cdot \frac{\partial \eta}{\partial \psi},
\]
then (4.12) can be bounded by
\[
(4.13) \quad C\alpha^2 \int_{-\delta}^{\delta} \int_J \int_J \left| \frac{\partial}{\partial s} R(xq)(s, \eta(\varphi, \psi - \pi/2)) \right| \, ds \, d\psi \, d\varphi.
\]
From the fact that on \( \tilde{\Gamma} \) one has \( \delta/8 \leq |d\sigma(\eta)| = |\cos \psi| \leq 1 - \delta/8 \), we obtain that this can be bounded by
\[
\leq \frac{\alpha^2}{\delta} \int_{-\delta}^{\delta} \int_J \left| \frac{\partial}{\partial s} R(xq)(s, \eta) \right| \, d\sigma(\eta) \, ds.
\]
By using Cauchy-Schwarz inequality, this is majorized by
\[
\leq \frac{\alpha^{5/2}}{\delta^{1/2}} \left( \int_{-\alpha}^{\alpha} \int_{\tilde{\Gamma}} \left| \frac{\partial}{\partial s} R(xq)(s, \eta) \right|^2 \, d\sigma(\eta) \, ds \right)^{1/2}
\leq C\| R(xq) \|_{H^1(\mathbb{R} \times S^2)}.
\]
Since in dimension \( n = 3 \), we have
\[
\| R(xq) \|_{H^1(\mathbb{R} \times S^2)} \leq C\| (1 + |x|)q \|_{L^2},
\]
this fact, together with an easier estimate for the zero order term in (4.11) give,
\[
\int_{B(0, \alpha)} \int_{B(-\delta, \delta)} \int_{S_q} |\partial_\eta Rq(y \cdot \theta \times \eta, \theta \times \eta)| \, d\eta \, d\theta \, dy \leq C(\| |x| q \|_{L^2} + \| q \|_{L^2}),
\]
this ends the proof (4.11) and the Lemma.

Now we return to the standard coordinates.
Corollary 4.6. \((n=3)\) In the conditions of proposition 4.3, assume that \(B(0, \alpha) \subset N\). Then

\[
\int_{-\alpha/2}^{\alpha/2} \int_{S^2} |Rq(s, \omega)| \, d\sigma(\omega) \, ds 
\leq C \left( \tau^{-1/4} \|q\|_{L^\infty(R^n)}^{1/2} + e^{c\tau} \|\Lambda q_1 - \Lambda q_2\|^{1/4} \right)^{\frac{3}{4}},
\]

where \(R\) denotes the Radon transform.

Proof. Let us take the open covering of \(S^2\) given by \(\{\tilde{\Gamma}_a(\delta)\}_{a \in A}\), see (4.8), where \(A = \{(\omega_0, \theta_0) \in S^2 \times \Sigma_\delta : \langle \omega_0, \theta_0 \rangle = 0\}\). By taking a finite subcovering and a partition of unity subordinated to it, we might reduce to prove the statement locally for the sets \(\tilde{\Gamma}_a(\delta) \subset S^2\). Then, for the local coordinates given in the lemma, \(\omega = \eta(\varphi, \psi - \pi/2)\), we have

\[
\int_{\tilde{\Gamma}_a} |Rq(s, \eta(\varphi, \psi))| \, d\varphi \, d\psi \, ds.
\]

For any \(y \in R^3\) such that \(\langle y, \eta(\varphi, \psi) \rangle = s\) we have in terms of the two-plane transform

\[Rq(s, \eta(\varphi, \psi)) = Rq(y, \theta(\varphi), \eta(\varphi, \psi)).\]

By taking \(y \in B(0, \alpha/2)\), so that \(y = s\eta + y'\) with \(y' \in \eta^\perp\), and denoting \(B_\alpha = B(0, \alpha/2) \cap \eta^\perp\), we can write

\[Rq(s, \eta(\varphi, \psi)) = |B_\alpha|^{-1} \int_{B_\alpha} |Rq(s\eta + y', \theta, \eta)| \, dy'.\]

Inserting this in (4.15), we have

\[
\int_{-\alpha/2}^{\alpha/2} \int_{\tilde{\Gamma}_a} |Rq(s, \omega)| \, d\sigma(\omega) \, ds 
\leq C(\alpha, \delta) \int_{-\alpha/2}^{\alpha/2} \int_{J} \int_{B_\alpha} |Rq(s\eta + y', \theta, \eta)| \, dy' \, d\varphi \, d\psi \, ds 
\leq C(\alpha, \delta) \int_{B(0, \alpha)} \int_{J} \int_{\eta^\perp} |R(y, \theta, \eta)| \, dy \, d\varphi \, dy.
\]

The Corollary follows from the local estimate, together with Hölder’s inequality. \(\square\)

4.2. End of the proof of Theorem 1.4. We want to use the Corollary 4.6 together with Theorem 2.5. Assume \(x_0 \in P\), the convex penumbra from \(N\), and let \(y \in N\), so that \(\langle x_0 - y, \nu(x_0) \rangle = 0\). To simplify notation we assume \(y = 0\) and take \(\alpha > 0\), such that \(B(0, \alpha) \subset N\).
Assume \( x_0 \in \text{supp } q \) (otherwise there is nothing to estimate) then the plane \( H_0(0, \nu(x_0)) \), see (2.11), is a supporting plane of \( \text{supp } q \), as required by (b) of Theorem 2.5 and, from the Corollary, we have estimate (4.14). We need, as required by Theorem 2.5, to use the Radon transform \( R_{x_0} \) with respect to the affine reference with center at the point \( x_0 \in P \). Consider 
\[
\beta = \frac{\alpha}{4}, \quad \delta = \sup \{ |z - y| : z \in \Omega, y \in N \}.
\]
Then the set of planes 
\[
\{ H_{x_0}(s, \omega) : |s| < \alpha/8, \omega \in \Gamma \}
\]
contains the set 
\[
\{ H_{x_0}(s, \omega) : |s| < \alpha/2, \omega \in \Gamma \}
\]
where \( \Gamma = \Gamma(\nu(x_0), \beta) \) is defined in (2.11).

This allows us to write for \( I = (-\alpha/8, \alpha/8) \),
\[
\int_I (1 + |s - \langle \omega, \zeta - y_0 \rangle|) \int_\Gamma |R_{x_0} q(s, \omega)| \, d\sigma(\omega) \, ds 
\leq C \int_{-\alpha/2}^{\alpha/2} \int_{S^2} |R q(s, \omega)| \, d\sigma(\omega) \, ds 
\leq C \left( \tau^{-\frac{1}{3}} \| q \|_{L^\infty(\mathbb{R}^n)}^{1/2} + e^{c\tau} \| \Lambda_{q_1} - \Lambda_{q_2} \|_{B^0_{\infty, 1}} \right)^{\frac{1}{3}}.
\]
Finally, the choice \( \tau = \frac{1}{4c} \log \| \Lambda_{q_1} - \Lambda_{q_2} \|_{B^0_{\infty, 1}} \) together with Theorem 2.5 gives the estimate in Theorem 1.4 for \( G \) a neighborhood of \( x_0 \in P \). The claim for \( G \) a neighborhood of \( P \) follows by standard arguments.

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