Yang-Baxter $\sigma$ model: Quantum aspects

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Abstract

We study the quantum properties at one-loop of the Yang-Baxter $\sigma$-models introduced by Klimeš$k^{1,2}$. The proof of the one-loop renormalizability is given, the one-loop renormalization flow is investigated and the quantum equivalence is studied.
1 Introduction

The Yang-Baxter $\sigma$-models were first introduced by Klimčík \cite{1,2} as a special case, at the classical level, of a non-linear $\sigma$-model with Poisson-Lie symmetry \cite{3,4}. Recall that the Poisson-Lie symmetry appears to be the natural generalization of the so-called Abelian $T$-duality \cite{5} and non-Abelian $T$-duality \cite{6,7,8} of non-linear $\sigma$-models. In particular, two dynamically equivalent $\sigma$-models can be obtained at the classical level providing that Poisson-Lie symmetry condition holds. That condition takes a very elegant formulation in the case where the target space is a compact semi-simple Lie group which naturally leads to the concept of the Drinfeld double \cite{9}. The Drinfeld double is the $2n$-dimensional linear space where both dynamically equivalent theories live. For the Poisson-Lie $\sigma$-models, a proof of the one-loop renormalizability and quantum equivalence was given in \cite{10,11,12,13}. We are interested by a special class of classical Poisson-Lie $\sigma$-models, the Yang-Baxter $\sigma$-models. Those classical models exhibit the special feature to be both Poisson-Lie symmetric with respect to the right action of the group on itself and left invariant. Thus, using the right Poisson-Lie symmetry or the left group action leads to two different dual theories. Those two dynamically equivalent dual pair of models live in two non-isomorphic Drinfeld doubles, the cotangent bundle of the Lie group for the left action, and the complexified of the Lie group for the right Poisson-Lie symmetry. Classical properties were investigated in the past and it has been showed that Yang-Baxter $\sigma$-models are integrable \cite{1}. More recently, based on the previous work of Refs.\cite{14,15,16}, authors of Ref.\cite{17} proved that they belong to a more general class of integrable $\sigma$-models. In particular, they showed that the $\varepsilon$-deformation parameter of the Poisson-Lie symmetry can be re-interpreted as a classical $q$-deformation of the Poisson-Hopf algebra.

If classical properties are well investigated, very little is known about the quantum version of the Yang-Baxter $\sigma$-models. In the case where the Lie group is $SU(2)$, the Yang-Baxter $\sigma$-model coincides with the anisotropic principal model which is known to be one-loop renormalizable. This low dimensional result can let us hope a generalization for any Yang-Baxter $\sigma$-models. However, contrary to the anisotropic principal model, the Yang-Baxter $\sigma$-models contain a non-vanishing torsion which could potentially give rise to some difficulties. On the other hand, another generalization of the anisotropic chiral model, the squashed group models are one-loop renormalizable for a special choice of torsion \cite{22}. Furthermore, the one-loop renormalizability of the Poisson-Lie $\sigma$-model cannot provide any help here since the proof was established for a theory containing $n^2$ parameters when the Yang-Baxter $\sigma$-models contain only two: the $\varepsilon$ deformation and the coupling constant $t$. At the quantum level, the Yang-Baxter $\sigma$-models are no more a special case of the Poisson-Lie $\sigma$ models. The
main result of this article consists in proving the one-loop renormalizability
of Yang-Baxter $\sigma$-models.

The plan of the article is as follows. In Section 2 we introduced the
Yang-Baxter $\sigma$-models on a Lie group and all algebraic tools needed. In
section 3, the counter-term of the Yang-Baxter $\sigma$-models, i.e. the Ricci
tensor, is calculated. Section 4 is dedicated to the proof of the one-loop
renormalizability, and the computation of the renormalization flow is done
in Section 5. In Section 6, we study the quantum equivalence and we ex-
press the Yang-Baxter $\sigma$-action in terms of the usual one of the Poisson-Lie
$\sigma$-models. Outlooks take place in Section 7.

2 Yang-Baxter $\sigma$-models

2.1 The complexified double

We considered the case of the Yang-Baxter models studied in Ref.[1]. In that
case the Drinfeld double $D$ can be the complexification of a simple compact
and simply-connected Lie group $G$, i.e. $D = G^C$, or the cotangent bundle
$T^* G$. Let us consider the case of the complexified Drinfeld double, it turns
out that $D = G^C$ admits the so-called Iwasawa decomposition

$$G^C = G AN = AN G.$$ 

(1)

In particular, if $D = SL(n, C)$, then the group $AN$ can be identified with
the group of upper triangular matrices of determinant 1 and with positive
numbers on the diagonal and $G = SU(n)$. 
Furthermore, the Lie algebra $D$ turns out to be the complex Lie algebra $G^C$,
which suggests to use the roots space decomposition of $G^C$:

$$G^C = H^C \bigoplus (\oplus_{\alpha \in \Delta} C E_\alpha),$$

(2)

where $\Delta$ is the space of all roots. Consider the Killing-Cartan form $\kappa$ on
$G^C$, and let us take an orthonormal basis $H_i$ in the $r$-dimensional Cartan
sub-algebra $H^C$ of $G^C$ with respect to the bilinear form $\kappa$ on $G^C$, i.e:

$$\kappa(H_i, H_j) \equiv \delta_{ij}$$

(3)

This permits to define a canonical bilinear form on $H^*$, and more specifically
endows the roots space $\Delta \subset H^*$ with an Euclidean metric, i.e.

$$(\alpha, \beta) = \delta^{ij} \alpha_i \beta_i, \quad \alpha_i = \alpha(H_i).$$

Moreover, the inner product on the roots space part of $G^C$ is chosen such as:

$$\kappa(E_\alpha, E_{-\alpha}) \equiv 1,$$

(4)
and to fix the normalization, we impose the following non-linear condition $E_\alpha = E_{\alpha}^\dagger$. With all those conventions, the generators of $G^C$ verify:

\[
[H_i, E_\alpha] = \alpha_i E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \alpha_i^* H_i,
\]
\[
[H_i, H_j] = 0, \quad [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \quad \alpha + \beta \in \Delta.
\] (5)

The structure constants $N_{\alpha,\beta}$ vanish if $\alpha + \beta$ is not a root.

Since $G^C$ is a Lie algebra, the structure constants verify the Jacobi identity which leads on one hand to:

\[
N_{\alpha,\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha},
\] (6)

and on the other hand to:

\[
N_{\alpha,\beta+(k-1)\alpha} N_{\beta+k\alpha,-\alpha} + N_{-\alpha,\beta+(k+1)\alpha} N_{\alpha,\beta+k\alpha} = -(\alpha,\beta + k\alpha).
\] (7)

In the non-vanishing case, the structure constants $N_{\alpha,\beta}$ can be calculated from the last relation

\[
N_{\alpha,\beta}^2 = n(m+1)(\alpha,\alpha),
\] (8)

with $(n, m) \in \mathbb{N}$ such that $\beta + n\alpha$ and $\beta - m\alpha$ are the last roots of the chain containing $\beta$ (see Ref. [21] for more details).

Since $H_i$ is an orthonormal basis in $\mathcal{H}^C$, we obtain the relations:

\[
\sum_{\alpha \in \Delta} \alpha_i \alpha_j = \delta_{ij}, \quad \text{and} \quad \sum_{\alpha \in \Delta} (\alpha, \alpha) = r.
\] (9)

A basis of the compact Lie real form $G$ of $G^C$ can be obtained by the following transformations:

\[
T_i = i H_i, \quad B_\alpha = \frac{i}{\sqrt{2}} (E_\alpha + E_{-\alpha}), \quad C_\bar{\alpha} = \frac{1}{\sqrt{2}} (E_\alpha - E_{-\alpha}),
\] (10)

with $\alpha \in \Delta^+$ (positive roots). With our choice of normalization, the vectors of the basis verify $\kappa(T_i, T_j) \equiv \kappa_{ij} = -\delta_{ij}$, $\kappa(B_\alpha, B_\beta) \equiv \kappa_{\alpha\beta} = -\delta_{\alpha\beta}$, $\kappa(C_\alpha, C_\beta) \equiv \kappa_{\bar{\alpha}\bar{\beta}} = -\delta_{\alpha\beta}$ and all others are zero.

Let us define now a $\mathbb{R}$-linear operator $R : G \to G$ such that:

\[
RT_i = 0, \quad RB_\alpha = C_\bar{\alpha}, \quad RC_\bar{\alpha} = -B_\alpha,
\] (11)

this operator $R$ is the so-called the Yang-Baxter operator [2] which satisfies the following modified Yang-Baxter equation:

\[
[RA, RB] = R([RA, B] + [A, RB]) + [A, B], \quad (A, B) \in G.
\] (12)

Let us define the skew-symmetric bracket:

\[
[A, B]_R = [RA, B] + [A, RB], \quad (A, B) \in G,
\] (13)

which fulfills the Jacobi identity, and defines a new Lie algebra $(G, [\cdot, \cdot]_R)$. It turns out that this new algebra is nothing but the Lie algebra of the $AN$ group of the Iwasawa decomposition of $G^C$ and will be denoted $G^R$ the dual algebra.
2.2 The Yang-Baxter action

We shall now consider the action of the Yang-Baxter $\sigma$-models expressed on the Lie group $G$, which takes the expression:

$$S(g) = -\frac{1}{2t} \int \kappa(g^{-1}\partial_+ g, (1 - \varepsilon R)^{-1} g^{-1}\partial_- g) d\xi^+ d\xi^-, \quad g \in G$$  \hspace{1cm} (14)

where $\partial_+ = \partial_\tau + \partial_\sigma$ and $\partial_- = \partial_\tau - \partial_\sigma$, $\mathbb{1}$ is the identity map on $G$, $t$ is the coupling constant, and $\varepsilon$ is the deformation parameter.

We can immediately check that the Yang-Baxter models (14) are left action invariant, $G$ acting on himself. Concerning the right Poisson-Lie symmetry, it is well known that such $\sigma$-models have to fulfill a zero curvature condition to be Poisson-Lie invariant. Indeed, if we take the following $G^*$-valued Noether current 1-form $J(g)$:

$$J(g) = -(1 + \varepsilon R)^{-1} g^{-1}\partial_+ g d\xi^+ + (1 - \varepsilon R)^{-1} g^{-1}\partial_- g d\xi^-,$$  \hspace{1cm} (15)

we can easily verify that the fields equations of (14) are equivalent to the following zero curvature condition:

$$\partial_+ J_-(g) - \partial_- J_+(g) + \varepsilon [J_-(g), J_+(g)]_R = 0.$$  \hspace{1cm} (16)

We remark that if the deformation $\varepsilon$ vanishes then the action of the group $G$ is an isometry, since the Noether current are closed 1-forms on the worldsheets and the action (14) coincides with that of the principal chiral $\sigma$-model. The operator $(1 - \varepsilon R)^{-1}$ on $G$ can be decomposed in a symmetric part interpreted as a metric $g$ on $G$ and a skew-symmetric part interpreted as a torsion potential $h$ on $G$. An attentive study of the action (14) gives the following expressions for $g$ and $h$:

$$g = \kappa_{ij}(g^{-1}dg)^i(g^{-1}dg)^j + \frac{1}{1 + \varepsilon^2} \left( \kappa_{\alpha\beta}(g^{-1}dg)^\alpha(g^{-1}dg)^\beta + \kappa_{\bar{\alpha}\bar{\beta}}(g^{-1}dg)^{\bar{\alpha}}(g^{-1}dg)^{\bar{\beta}} \right),$$  \hspace{1cm} (17)

$$h = -\frac{\varepsilon}{1 + \varepsilon^2} (g^{-1}dg)^\alpha \wedge (g^{-1}dg)^\beta \kappa_{\alpha\alpha}.$$  \hspace{1cm} (18)

In order to prove the one-loop renormalizability, we need to calculate the Ricci tensor associated to the manifold $(G, g, h)$.

3 Counter-term of the Yang-Baxter $\sigma$-models

In this paper, for the calculus of the counter-term, we choose the standard approach based on the Ricci tensor. This choice provides a clear and an elegant expression of the counter-term in terms of the roots of $G^C$. However the calculus could have been done by using our formula of [12] for the counter-term in an equivalent way.
3.1 Geometry with torsion on a Lie group $G$

Let us consider a pseudo-Riemannian manifold $(G, g)$ as the base of its frame bundle, where $G$ is a compact semi-simple Lie group and $g$ a non-degenerated metric. Moreover, we choose the left Maurer-Cartan form $g^{-1}dg$, $g \in G$ as the basis of 1-forms on $G$, and in that basis the metric coefficients $g_{ab}$ and the torsion components $T_{abc}$ are all constant. On that frame bundle we define a metric connection $\Omega$ with its covariant derivative $D$ such that $Dg = 0$. Furthermore, if we define by $dD$ the exterior covariant derivative, the torsion can be written $T = dD(g^{-1}dg)$. From these definitions we will obtain the expression of the connection $\Omega$.

**Metric connection.**

By using the relation $Dg = 0$ we obtain:

$$\Omega^s_{ac}g_{sb} + \Omega^s_{bc}g_{as} = 0. \quad (19)$$

With $g_{ab}$ constant and if we denote $\Omega_{abc} = g_{as}\Omega^s_{bc}$, the previous relation becomes:

$$\Omega_{abc} = -\Omega_{bac} \quad (20)$$

Thus the two first indices of the connection $\Omega$ are skew-symmetric.

**The torsion.**

We said that the torsion verifies $T = dD(g^{-1}dg)$ or in terms of components:

$$T^a = \Omega^a_b \wedge (g^{-1}dg)^b + d(g^{-1}dg)^a. \quad (21)$$

Since $g^{-1}dg$ is the left Maurer-Cartan form on $G$, we get:

$$d(g^{-1}dg) = -(g^{-1}dg) \wedge (g^{-1}dg),$$

on the other hand $T^a$ is the 2-form torsion, i.e. we can write it as:

$$T^a = \frac{1}{2}T^a_{bc}(g^{-1}dg)^b \wedge (g^{-1}dg)^c.$$

Consequently, the components of the torsion are related to the skew-symmetric part of the connection as:

$$T^a_{bc} = \Omega^a_{cb} - \Omega^a_{bc} - f^a_{bc}, \quad (22)$$

with $f^a_{bc}$ the structure constants of the Lie algebra $G$.

Note that in the case of the non-linear $\sigma$-models the torsion is defined by $T = dh$ where $h$ is the 2-form potential torsion, we will exploit that a little further to express the connection for the Yang-Baxter $\sigma$-models.

**The connection.**

From the relations $(20)$ $(22)$, we can find the components of the connection:

$$2\Omega_{abc} = (-T_{abc} - T_{bca} + T_{cab}) + (f_{abc} - f_{cab} - f_{bca}), \quad (23)$$

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with the conventions $\Omega_{abc} = g_{as} \Omega^s_{bc}$, $T_{abc} = g_{as} T^s_{bc}$ and $f_{bc} = f_{bc}^s g_{sa}$.

Let us introduce the Levi-Civita connection $L$ which is in fact the second term of the r.h.s in Eq.(23), and rewrite the connection $\Omega$ for a totally skew-symmetric torsion:

$$\Omega_{abc} = L_{abc} - \frac{1}{2} T_{abc}. \quad (24)$$

The curvature and the Ricci.

By definition the 2-form curvature $F$ fulfills $F = dD\Omega$, i.e.

$$F^a_b = d\Omega^a_b + \Omega^a_s \wedge \Omega^s_b. \quad (25)$$

Moreover, since $\Omega^a_b$ is a 1-form of $G$, $\Omega^a_b = \Omega^a_{bc}(g^{-1}dg)^c$, we obtain the general expression for the curvature:

$$F^a_{bcd} = \Omega^a_{sc} \Omega^s_{bd} - \Omega^a_{sd} \Omega^s_{bc} - \Omega^a_{bs} f^s_{cd}. \quad (26)$$

The Ricci tensor is such that $Ric_{ab} = F^s_{asb}$ and can be written as:

$$Ric_{ab} = -\Omega^r_{ar}(\Omega^r_{bs} + f^r_{bs}). \quad (27)$$

We are now able to decompose the symmetric and skew-symmetric parts of the Ricci tensor in terms of the torsion-less Ricci tensor $Ric^L$ and the torsion $T$ as:

$$Ric_{(ab)} = Ric^L_{(ab)} + \frac{1}{4} T^r_{as} T^s_{br} \quad (28)$$

$$Ric_{[ab]} = \frac{1}{2} f^r_{at} T^t_{bs} - \frac{1}{2} g_{at} f^r_{sr} T^s_{bu} + \frac{1}{2} g_{at} f^r_{ar} T^r_{bu} - (a \leftrightarrow b). \quad (29)$$

3.2 Application to Yang-Baxter

Ricci symmetric part:

Recall that in the case of the Yang-Baxter $\sigma$-models and with our normalization choice, the metric is given by:

$$g_{ij} = -\delta_{ij}, \quad g_{a\beta} = -\frac{1}{1 + \epsilon^2} \delta_{a\beta}, \quad g_{\bar{a}\beta} = -\frac{1}{1 + \epsilon^2} \delta_{\bar{a}\beta}. \quad (30)$$

Let us introduce the bi-invariant connection $\Gamma$ on the Lie group $G$, it corresponds to the Levi-Civita connection in the case of a vanishing deformation, i.e. $\Gamma = L(\epsilon = 0)$. From the equations (23) we can obtain the Levi-Civita coefficients:

$$L^\alpha_{\bar{a}i} = -L^\alpha_{\bar{a}i} = (1 - \epsilon^2) \Gamma^\alpha_{\bar{a}i}, \quad (31)$$

$$L^\alpha_{i\bar{a}} = -L^\alpha_{i\bar{a}} = (1 + \epsilon^2) \Gamma^\alpha_{i\bar{a}}, \quad (32)$$
where we keep the convention for the indices $i \in \mathcal{H}$ and $\alpha \in \Delta^+$. All others
Levi-Civita coefficients are equal to those of the bi-invariant connection $\Gamma$.

We can now express the torsion-less Ricci tensor $Ric^L$ as a deformation of
the usual Ricci tensor $Ric^\Gamma$ of the bi-invariant connection on Lie group, i.e.

$$Ric^L_{\alpha\beta} = Ric^\Gamma_{\alpha\beta} - \frac{\varepsilon^2}{2} (\alpha, \alpha) \delta_{\alpha\beta}$$ (33)

$$Ric^L_{\bar{\alpha}\bar{\beta}} = Ric^\Gamma_{\bar{\alpha}\bar{\beta}} - \frac{\varepsilon^2}{2} (\alpha, \alpha) \delta_{\alpha\beta}$$ (34)

$$Ric^L_{ij} = (1 + \varepsilon^2)^2 Ric^\Gamma_{ij}$$ (35)

It is well-known that for the Riemannian bi-invariant structure the Ricci
tensor takes the expression:

$$Ric^\Gamma_{ab} = -\frac{1}{4} \kappa_{ab}, \ (a, b) \in G,$$ (36)

therefore, the components of $Ric^L$ are the following:

$$Ric^L_{\alpha\beta} = Ric^L_{\bar{\alpha}\bar{\beta}} = -\frac{1}{4} \kappa_{\alpha\beta} - \frac{\varepsilon^2}{2} (\alpha, \alpha) \delta_{\alpha\beta}$$ (37)

$$Ric^L_{ij} = -\frac{1}{4} \kappa_{ij} (1 + \varepsilon^2)^2$$ (38)

Concerning the contribution of the Torsion to the symmetric part of the Ricci
tensor, we have to express the Torsion in terms of the constant structures
of $G$. For a non-linear $\sigma$-model the Torsion 3-form is calculated from the
potential torsion 2-form such $T = dh$, which implies that:

$$T_{abc} = -3 f_{[ab} h_{c]} s, \ (a, b, c, s) \in G.$$ (39)

Moreover, since the torsion potential involves only root indices

$$h = -\frac{\varepsilon}{1 + \varepsilon^2} (g^{-1}dg)^\alpha \wedge (g^{-1}dg)^\bar{\alpha} \kappa_{\alpha\alpha},$$

the torsion components vanish for the Cartan sub-algebra indices ($T_{i\bar{c}} = 0$).

We can now calculate the torsion contribution, and we obtain for the non-
vanishing coefficients:

$$\frac{1}{4} T^r_{\alpha s} T^s_{\alpha r} = \frac{1}{4} T^r_{\bar{\alpha} s} T^s_{\bar{\alpha} r} = \frac{\varepsilon^2}{2} (\frac{1}{2} \kappa_{\bar{\alpha}\alpha} + (\alpha, \alpha)).$$ (40)

In the calculus we used the fact that the Killing $\kappa$ can be expressed in terms
of the root $\alpha$ and the constant structures $N_{\alpha,\beta}$ such as:

$$\frac{1}{2} \kappa_{\alpha\alpha} = \alpha^i \alpha_i + \frac{1}{2} \sum_{\beta \in \Delta^+} (N_{\alpha,\beta})^2 + (N_{\alpha,-\beta})^2.$$ (41)
Adding both contributions to the Ricci tensor and using our normalization, we obtain the final expression of the symmetric part:

\[
Ric_{\alpha\beta} = Ric_{\bar{\alpha}\bar{\beta}} = -\frac{\kappa_{\alpha\beta}}{4}(1 - \varepsilon^2) = \frac{1}{4}(1 - \varepsilon^2)\delta_{\alpha\beta}
\] (42)

\[
Ric_{ij} = -\frac{\kappa_{ij}}{4}(1 + \varepsilon^2)^2 = \frac{\delta_{ij}}{4}(1 + \varepsilon^2)^2.
\] (43)

We observe that, in the case of the Yang-Baxter model, the torsion induced by the Poisson-Lie symmetry is precisely that which avoids the dependence of the Ricci tensor in the root length \((\alpha, \alpha)\).

**Ricci skew-symmetric part**

Using the fact the \(T_{ab} = 0\), the only non-vanishing non-diagonal components of the Ricci tensor can be written:

\[
Ric_{\alpha\bar{\alpha}} = 2f_{\alpha\beta\gamma}T_{\bar{\alpha}\beta\gamma}\kappa^{\gamma\gamma}\kappa^{\beta\bar{\beta}} - f_{\bar{\alpha}\beta\gamma}T_{\alpha\beta\gamma}\kappa^{\gamma\gamma}\kappa^{\beta\bar{\beta}} - f_{\bar{\alpha}\bar{\beta}\gamma}T_{\alpha\bar{\beta}\gamma}\kappa^{\gamma\gamma}\kappa^{\bar{\beta}\bar{\beta}}.
\] (44)

The first r.h.s term can be expressed as a function of the structure constants, \(N_{\alpha,\beta}\) such as:

\[
2f_{\alpha\beta\gamma}T_{\bar{\alpha}\beta\gamma}\kappa^{\gamma\gamma}\kappa^{\beta\bar{\beta}} = 2\varepsilon\sum_{\beta\in\Delta^+} (N_{\alpha,\beta})^2 - (N_{\alpha,-\beta})^2.
\] (45)

The two other terms are nothing but the contribution of the roots space (see Eq.(41)) to the component \(\kappa_{\bar{a}\bar{a}}\) of the Killing form, i.e.:

\[
- f_{\bar{\alpha}\beta\gamma}T_{\alpha\beta\gamma}\kappa^{\gamma\gamma}\kappa^{\beta\bar{\beta}} - f_{\bar{\alpha}\bar{\beta}\gamma}T_{\alpha\bar{\beta}\gamma}\kappa^{\gamma\gamma}\kappa^{\bar{\beta}\bar{\beta}} = -\frac{\varepsilon}{2}\left(\kappa_{\bar{a}\bar{a}} + 2(\alpha, \alpha)\right).
\] (46)

By summing the Bianchi relations (7) on positive roots, we obtain that:

\[
\sum_{\beta\in\Delta^+} (N_{\alpha,\beta})^2 - (N_{\alpha,-\beta})^2 = -2(\rho, \alpha) + (\alpha, \alpha),
\] (47)

with

\[
\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha
\]

the Weyl vector.

Finally, the skew-symmetric part of the Ricci tensor is given by:

\[
Ric_{\alpha\bar{a}} = -Ric_{\bar{a}\alpha} = -\varepsilon\left(2(\alpha, \rho) + \frac{1}{2}\kappa_{\alpha\alpha}\right).
\] (48)
4 One-loop renormalizability

At one-loop the counter-terms for a non-linear $\sigma$-model on $G$ are given by:

$$\frac{1}{4\pi\epsilon} \int \text{Ric}_{ab}(g^{-1}\partial_-g)^a(g^{-1}\partial_+g)^b, \quad \epsilon = 2 - d. \quad (49)$$

We require, for the renormalizability, that all divergences have to be absorbed by fields-independent deformations of the parameters $(t, \epsilon)$ and a possible non-linear fields renormalization of the fields $(g^{-1}\partial_{\pm}g)^a$. Thus, if we suppose that all parameters are the independent coupling constants of the theory, the Ricci tensor in our frame has to verify the relations:

$$\text{Ric}_{ab} = -\chi_0 (1 - \epsilon R)^{-1}_{ab} + \chi_{\epsilon} \frac{\partial}{\partial \epsilon} (1 - \epsilon R)^{-1}_{ab} + D_b u_a, \quad (50)$$

with $u$ a vector that contributes to the fields renormalization, $\chi_0$ and $\chi_{\epsilon}$ are coordinates-independent. Decomposing into symmetric and skew-symmetric parts, the previous relation for the Yang-Baxter $\sigma$-models becomes:

$$\text{Ric}_{ij} = -\chi_0 g_{ij}, \quad (51)$$

$$\text{Ric}_{\alpha\alpha} = -\chi_0 g_{\alpha\alpha} - \chi_{\epsilon} \frac{2\epsilon}{1 + \epsilon^2} g_{\alpha\alpha}, \quad (52)$$

$$\text{Ric}_{\alpha\bar{\alpha}} = -\chi_0 h_{\alpha\bar{\alpha}} + \chi_{\epsilon} \frac{1 - \epsilon^2}{\epsilon(1 + \epsilon^2)} h_{\alpha\bar{\alpha}} + D_{\bar{\alpha}} u_\alpha. \quad (53)$$

From the equations (51) and (52), we extract immediately:

$$\chi_0 = \frac{1}{4}(1 + \epsilon^2)^2, \quad \text{and} \quad \chi_{\epsilon} = -\frac{1}{4}\epsilon(1 + \epsilon^2)^2. \quad (54)$$

Since $\chi_0$ and $\chi_{\epsilon}$ are now fixed, they have to fulfill in the same time the relation (53), which gives the following constraint:

$$\epsilon \left( -\frac{1}{2} + 2(\rho, \alpha) \right) = -\frac{1}{2}\epsilon + D_{\bar{\alpha}} u_\alpha. \quad (55)$$

Furthermore, the covariant derivative of $u$ can be easily calculated:

$$Du = -\frac{1}{2} \sum_{\alpha \in \Delta^+} (u, \alpha)(g^{-1}dg)^\alpha \wedge (g^{-1}dg)_{\bar{\alpha}}. \quad (56)$$

Let us define the vector $\varepsilon\bar{u} = u$, and insert (56) in the constraint (55) we obtain:

$$(4\rho - \bar{u}, \alpha) = 0. \quad (57)$$

Then, if we impose $\bar{u} = 4\rho$ the constraint is fulfilled for any root $\alpha$ since $(\cdot, \cdot)$ is the canonical scalar product on $\mathbb{R}^r$. We can conclude that the Yang-Baxter $\sigma$-models are one-loop renormalizable.

We note that it is quite elegant to find a field renormalization given by the Weyl vector.
5 Renormalization flow

Let us introduce the $\beta$-functions of the two parameters $(t, \varepsilon)$, they satisfy:

$$\beta_t = \frac{dt}{d\lambda} = -t^2 \chi_0, \quad \beta_{\varepsilon} = \frac{d\varepsilon}{d\lambda} = t\chi_{\varepsilon},$$

(58)

where $\lambda = \frac{1}{\pi} \ln \mu$, with $\mu$ the mass energy scale. We obtain the following system of differential equations:

$$\frac{dt}{d\lambda} + \frac{1}{4}(1 + \varepsilon^2)^2 t^2 = 0$$

(59)

$$\frac{d\varepsilon}{d\lambda} + \frac{1}{4}\varepsilon(1 + \varepsilon^2)^2 t = 0.$$ 

(60)

The set of differential equations can be exactly solved, and solutions take the following general expressions:

$$t(\varepsilon) = A\varepsilon, \quad \hat{\lambda}(\varepsilon) = B\lambda(\varepsilon) = \frac{3}{2} \arctan \varepsilon + \frac{1 + \frac{3}{2} \varepsilon^2}{\varepsilon(1 + \varepsilon^2)^2} + C,$$

(61)

with $(A, B, C) \in \mathbb{R}$ three integrative constants. We note that divergences occur for $\varepsilon$ and $t$ when the energy scale $\hat{\lambda}$ goes to $\pm \frac{3\pi}{4} + C$. On the other hand, for $\hat{\lambda} \rightarrow \infty$ the parameters $\varepsilon$ and $t$ are vanishing, leading to an asymptotic freedom. We can illustrate the situation with the following plot (Fig.1) of $\lambda$ as a function of $\varepsilon$ where we choose $B = 1$ and $C = 0$.

Figure 1: Energy scale $\lambda$ as a function of the deformation parameter $\varepsilon$
6 Poisson-Lie models and duality

Now we will express the Yang-Baxter $\sigma$-models in terms of the usual Poisson-Lie $\sigma$-models’ expression. Recall that general Right symmetric Poisson-Lie $\sigma$-models can be written:

$$S(g) = \frac{1}{2t} \int (\partial_+ gg^{-1})^a (M + \Pi_R(g))_{ab}^{-1} (\partial_+ gg^{-1})^b. \quad (62)$$

Here $\Pi_R(g)$ is the so-called Right Poisson-Lie bi-vector and $M$ an $n^2$ real matrix.

Using the adjoint action of an element $g \in G$ we can rewrite the action (14) such as the previous (62), with

$$\Pi_R(g) = \text{Ad}_g R \text{Ad}_g^{-1} - R$$

Let us focus on the dual models, as evoked earlier there exists two non-isomorphic Drinfeld doubles for the action (62). Consequently, we have two different dual theories for one single initial theory on $G$, and all three are classically equivalent. We will consider each case and argue that they are all quantum-equivalent at one-loop.

We start by considering the Drinfeld double $D = G^C$, in that case we saw that the dual group is the factor $AN$ in the Iwasawa decomposition. The corresponding algebra is the Lie algebra $G_R$ generated by the $\mathbb{R}$-linear operator $(R - i)$ on $G$, whose its group is a non-compact real form of $G^C$ (see [2, 20] for details). The dual action can be expressed as:

$$S(\hat{g}) = \frac{1}{2t} \int (\partial_+ \hat{g} \hat{g}^{-1})_a \left[(M^{-1} + \hat{\Pi}_R(\hat{g}))^{-1} \right]^{ab} (\partial_+ \hat{g} \hat{g}^{-1})_b. \quad (63)$$

K.Sfetsos and K.Siampos proved in [10] that for Right Poisson-Lie symmetric $\sigma$-models the quantum equivalence holds providing that the matrix $M$ is invertible. In the Yang-Baxter $\sigma$-models this condition is always satisfied and the inverse of $M$ is given by:

$$M^{-1} = \frac{\varepsilon^2}{1 + \varepsilon^2} \left( \frac{1}{\varepsilon} \mathbb{1} + R \right)$$

When we consider the dual model associated to the left action of $G$, the Drinfeld double is the cotangent bundle $T^*G = G \rtimes G^*$. Then the dual group is the dual linear space $G^*$ of $G$, which is an Abelian group with the addition of vectors as the group law. The corresponding action is that of the non-Abelian $T$-dual $\sigma$-models [6, 7, 8] and has the well-known expression:

$$S(\hat{g} = e^{s\chi}) = \frac{1}{2\varepsilon t} \int d\xi^+ d\xi^- \partial_+ \chi_a ((M^{-1})_{ab} + f^{abc}_{ab}\chi_c)^{-1} \partial_+ \chi_b, \chi \in G^*, s \in \mathbb{R}. \quad (64)$$
It has been showed in [19] that those models are one-loop renormalizable. Since the action (64) is Left Poisson-Lie symmetric, Sfetsos-Siampos condition [10] still holds (in their Left formulation) and implies again the quantum equivalence at one-loop.

7 Outlooks

Yang-Baxter $\sigma$-models are one case of non-trivial Poisson-Lie symmetric $\sigma$-models which keep the renormalizability and the quantum equivalence at the one-loop level, and are known to be classically integrable. Those models appear to be a semi-classical $q$-deformation of Poisson algebra, and can be a starting point in the quest for a quantum $q$-deformation fully renormalizable thanks to the relative simplicity of these models containing only two parameters.

Furthermore, for low dimensional compact Lie groups $G$ the geometry associated to the Yang-Baxter $\sigma$-models can be viewed as a torsionless Einstein-Weyl geometry. We plan in the future to study the Weyl connections with torsion on Einstein manifolds, with the hope to learn more about the geometric aspects of the Poisson-Lie $\sigma$-models.

I thank G. Valent for discussions and C. Carbone for proofreading.

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