The Prolongation Problem for the Heavenly Equation

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Abstract
We provide an exact regular solution of an operator system arising
as the prolongation structure associated with the heavenly equation.
This solution is expressed in terms of operator Bessel coefficients.

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1 Introduction

It is well known that the so–called heavenly equation

\begin{equation}
    u_{xx} + u_{yy} + (e^u)_{zz} = 0,
\end{equation}

where \( u = u(x, y, z) \) and subscripts mean partial derivatives, occurs in the study of
heavenly spaces (Einstein spaces with one rotational Killing vector).

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and in the context of extended conformal symmetries [9]. Reduced versions of Eq. (1) have been found in [1] via the symmetry approach, providing instanton and meron-like configurations. Furthermore, an algebra of the Virasoro type without central charge was associated with the heavenly equation by resorting to its invariance under conformal transformations [1].

To investigate the algebraic aspects of the integrability properties of Eq. (1), we shall apply the prolongation technique, which could provide the relative linear spectral problem [4, 8, 11, 12]. In more than two independent variables, the extension of the prolongation procedure is generally nontrivial and some aspects remain to be explored (see, for example, [8, 11]). However, the main result of this note is represented by the exact solution of the prolongation system associated with Eq. (1). This system consists of three operator equations in the form of commutator relations which can be written as second order (operator) differential equations resembling formally equations of the Bessel type. Our solution is achieved through a series expansion defining operator Bessel coefficients.

2 The prolongation structure

Let us introduce on a manifold with local coordinates \((x, y, z, u, p, q, r)\) the closed differential ideal defined by the set of 3–forms:

\[
\begin{align*}
\theta_1 &= du \wedge dx \wedge dy - rdx \wedge dy \wedge dz, \\
\theta_2 &= du \wedge dy \wedge dz - pdx \wedge dy \wedge dz, \\
\theta_3 &= du \wedge dx \wedge dz + qdx \wedge dy \wedge dz, \\
\theta_4 &= dp \wedge dy \wedge dz - dq \wedge dx \wedge dz + e^u dr \wedge dx \wedge dy + e^u r^2 dx \wedge dy \wedge dz,
\end{align*}
\]

where \(\wedge\) stands for the exterior product.

It is easy to verify the following

**Proposition 2.1** On every integral submanifold defined by \(u = u(x, y, z), p = u_x, q = u_y, r = u_z\), with \(dx \wedge dy \wedge dz \neq 0\), the ideal (2)–(5) is equivalent to Eq. (1).

Now let us consider the 2–forms:

\[
\Omega^k = H^k(u, u_x, u_y, u_z; \xi^m)dx \wedge dy + F^k(u, u_x, u_y, u_z; \xi^m)dx \wedge dz + G^k(u, u_x, u_y, u_z; \xi^m)dy \wedge dz + A_m d\xi^m \wedge dx + B_m d\xi^m \wedge dz + d\xi^k \wedge dy,
\]
where \( \xi = \{\xi^m\} \), \( k, m = 1, 2, \ldots, N \) (\( N \) arbitrary), and \( H^k \), \( F^k \) and \( G^k \) are, respectively, the pseudopotential and functions to be determined. Furthermore, the quantities \( A_{km}^k \) and \( B_{km}^k \) denote the elements of two \( N \times N \) constant regular matrices, and the summation convention over repeated indices is understood.

**Definition 2.1** The forms \( \Omega^k \) are called the *prolongation forms* associated with Eq. (1). Let \( I \) be the prolonged ideal generated by the forms \( \theta_j \) and \( \Omega^k \). We say that \( I \) is closed if \( d\Omega^k \in I(\theta_j, \Omega^k) \).

**Lemma 2.1** The closure condition for \( I \) yields

\[
H^k = e^u u_z L^k(\xi^m) + P^k(u, \xi^m),
\]
\[
F^k = -u_y L^k(\xi^m) + N^k(\xi^m),
\]
\[
G^k = u_z L^k(\xi^m) + M^k(u, \xi^m),
\]

where \( L^k, P^k, N^k, M^k \) are functions of integration.

**Proof.** The closure condition is equivalent to the following constraints:

\[
H^k_{ux} - e^u G^k_{ux} = 0, \quad F^k_{uy} + G^k_{ux} = 0,
\]
\[
H^k_{ux} = H^k_{uy} = F^k_{ux} = F^k_{uy} = G^k_{uy} = G^k_{ux} = 0,
\]
\[
u u_z H^k_u - u_y F^k_u + u_x G^k_u - e^u u_z^2 G^k_{ux} + [G, H]^k = 0,
\]
\[
F^k_{\xi^l} - G^k_{\xi^m} A^m_{l^l} - H^k_{\xi^m} B^m_{l^l} = 0,
\]
\[
F^k_{\xi^m}(B^{-1})^m_{l^l} G^m - G^k_{\xi^m}(B^{-1})^m_{l^l} F^m = 0,
\]
\[
[A, B] = 0,
\]

where \([G, H]^k = G^j H^k_{\xi^l} - H^j G^k_{\xi^l} \) (Lie bracket), \( H^k_{\xi^l} = \frac{\partial H^k}{\partial \xi^l} \), \( H^k_u = \frac{\partial H^k}{\partial u} \), and so on. Equations (10)–(11) provide the result.

In the following we shall omit the indices \( k, m \) for simplicity.

**Proposition 2.2** The following prolongation equations hold

\[
P_u = e^u [L, M],
\]
\[
M_u = -[L, P],
\]
\[
[M, P] = 0.
\]
Proof. It is a straightforward consequence of Lemma 2.1 (Eq. (12)). QED

Hereafter, $L$, $P$, $M$ will be regarded as (regular) operators, in the sense that

$$L \to L^j \frac{\partial}{\partial \xi^j}, \quad M \to M^j \frac{\partial}{\partial \xi^j}, \quad P \to P^j \frac{\partial}{\partial \xi^j},$$

while the Lie brackets become commutators.

3 Exact solution of the prolongation equations

Now, let us look for an exact solution to Eqs. (16)–(18).

Definition 3.1 For any $A = A^j \frac{\partial}{\partial \xi^j}$, we define $\mathcal{L}[A]$ by

$$\mathcal{L}[A] = [L, A].$$

(19)

Definition 3.2 We define the $n$–th power of the operator $\mathcal{L}$ by setting

$$\mathcal{L}^n[A] = [L, [L, \ldots, [L, A] \ldots]],$$

(20)

where $L$ appears $n$–times, and $\mathcal{L}^0[A] = A$.

Remark 3.1 The prolongation Eqs. (16) and (17) can be written as the second order operator equations

$$P_{tt} - \frac{1}{t} P_t + \mathcal{L}^2[P] = 0,$$

(21)

$$M_{tt} + \frac{1}{t} M_t + \frac{1}{t} \mathcal{L}^2[M] = 0,$$

(22)

where $t = 2e^{\frac{u}{2}}$. QED
Remark 3.2 Notice that the above equations resemble formally conventional Bessel equations of the type
\[
\kappa_{tt} - \frac{1}{t} \kappa_t + \omega^2 \kappa = 0, \quad \chi_{tt} + \frac{1}{t} \chi_t + \frac{1}{t} \omega^2 \chi = 0,
\]
with \( \omega \) a constant, whose regular solutions at \( t = 0 \) are given by
\[
\kappa(t) = p_0 \frac{t}{2} J_1(t \omega), \quad \chi(t) = m_0 J_0(t \omega),
\]
with \( p_0, m_0 \) constants of integration.

We are interested in solutions of Eqs. (21) and (22) which are regular at \( t = 0 \).
We proceed in a heuristic way generalizing the scheme working out for the solution of the operator equations:

\[
\tilde{P}_{tt} + \mathcal{L}^2[\tilde{P}] = 0, \quad \tilde{M}_{tt} + \mathcal{L}^2[\tilde{M}] = 0,
\]
whose formal solution is
\[
\tilde{P}(t, \xi) = e^{it\mathcal{L}}[A_0(\xi)] + e^{-it\mathcal{L}}[B_0(\xi)],
\]
\[
\tilde{M}(t, \xi) = e^{it\mathcal{L}}[C_0(\xi)] + e^{-it\mathcal{L}}[D_0(\xi)],
\]
where \( A_0, B_0, C_0, D_0 \) can be determined from the initial conditions.

Definition 3.3 Let \( J_0(\cdot) \) and \( J_1(\cdot) \) be formally defined by
\[
J_0(t\mathcal{L}[M_0]) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!^2} \left( \frac{t}{2} \right)^{2m} \mathcal{L}^{2m}[M_0],
\]
\[
J_1(t\mathcal{L}[P_0]) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \left( \frac{t}{2} \right)^{1+2m} \mathcal{L}^{1+2m}[P_0].
\]

Remark 3.3 Here \( J_0 \) and \( J_1 \) are borrowed by the series expansion of the Bessel functions \( J_\nu \), with \( \nu \) an integer:
\[
J_\nu(\tau) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+\nu)!} \left( \frac{\tau}{2} \right)^{\nu+2m} \mathcal{L}^{\nu+2m}[P_0],
\]
for \( \nu = 0 \) and \( \nu = 1 \), respectively (see [13]).
The following result holds, whose proof is straightforward.

**Proposition 3.1** A solution of equations (21) and (22) regular at \( t = 0 \) is given by

\[
P = \frac{t}{2} J_1(tL[P_0]), \quad M = J_0(tL[M_0]),
\]

where \( M_0 \equiv M_0(\xi) = M(t; \xi) \mid_{t=0} \) and \( P_0 \equiv P_0(\xi) \) is such that \([L, P_0] = [L, M_0] \).

**Remark 3.4** Inserting the operators (29) into Eqs. (21) and (22) yields the condition \([L, M_0], M_0] = 0\).

**Remark 3.5** We point out that the search of solutions to Eqs. (21) and (22), regular at \( t = 0 \), is equivalent to tackle the corresponding Cauchy problem with the initial conditions \( P(0; \xi) = 0, \ P_t(t, \xi) \mid_{t=0} = 0, \ M(0; \xi) = M_0, \ M_t(t, \xi) \mid_{t=0} = 0 \).

For practical purposes, e.g. to determine the spectral problem and Bäcklund transformations associated with Eq. (1), we shall rewrite the solutions (29) in a form which contains the operator \( L \) and not the operator \( \mathcal{L} \). We shall give a regular solution of Eqs. (21) and (22) following a scheme similar to that working out for the case (23).

By induction it is easy to prove the following Lemma.

**Lemma 3.1** For \( n > 0 \), (20) takes the form

\[
L^n[A] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} L^{n-k} AL^k.
\]

As a straightforward consequence we have

\[
e^{itL}[A_0] = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} L^{n-k}[A_0] L^k
\]

\[
= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} L^j[A_0] \left( \sum_{k=0}^{\infty} \frac{(it)^k}{k!} L^k \right) \equiv e^{itL}[A_0]e^{-itL},
\]

which is just the *Baker–Campbell–Hausdorff expansion* (see e.g. [2]).

Thus, we can rewrite the solutions of the operator equations (23) in a more suitable form.
3.1 Operator Bessel coefficients

In order to express the solutions (29) of Eqs. (21) and (22) in terms of the operator \( L \), we shall introduce operator Bessel coefficients by means of a formal expansion analogous to that used in the case of Bessel functions.

**Definition 3.4** Let \( X \) be a regular operator. We define the operator Bessel coefficients \( \mathbf{J}_m(tX) \), as the coefficients of the formal expansion:

\[
e^{\frac{1}{2}X(z-1/z)} = \sum_{m=-\infty}^{\infty} z^m \mathbf{J}_m(tX).
\]

(32)

**Remark 3.6** We stress that the Laurent series on the right side is uniformly convergent.

First we prove a technical Lemma.

**Lemma 3.2** Operator Bessel coefficients satisfy the following recurrence and derivation formulae:

\[
J_{-k}(tL) = (-1)^k J_k(tL),
\]

(33)

\[
2kJ_k(tL) = tL[J_{k-1}(tL) + J_{k+1}(tL)],
\]

(34)

\[
2 \frac{d}{dt}[J_k(tL)] = L[J_{k-1}(tL) - J_{k+1}(tL)].
\]

(35)

\[
\frac{d}{dt}[t^kJ_k(tL)] = Lt^{k-1}J_k(tL),
\]

(36)

\[
\frac{d}{dt}[t^{-k}J_k(tL)] = -Lt^{-k}J_{k+1}(tL),
\]

(37)

**Proof.** By differentiating with respect to \( z \) the formal expansion (32), with \( X = L \), we obtain

\[
\frac{1}{2}tL(1 + \frac{1}{z^2}) \sum_{k=-\infty}^{\infty} z^k J_k(tL) = \sum_{k=-\infty}^{\infty} kz^{k-1}J_k(tL).
\]

Then, by equating coefficients of \( z^{k-1} \) in the above identity we obtain formula (34). Furthermore, if we differentiate the formal expansion with respect to \( t \) we have

\[
\frac{1}{2}L(z - 1/z) \sum_{k=-\infty}^{\infty} z^k J_k(tL) = \sum_{k=-\infty}^{\infty} z^k \frac{d}{dt} J_k(tL).
\]

By equating coefficients of \( z^k \) on either side of this identity we obtain formula (33). Formulae (34) and (37) can be determined by adding and subtracting (34) and (35), while Eq. (33) follows directly from (32) and from the hypothesis that \( L \) is a regular operator.
3.2 A form of the solution of the prolongation equations in terms of $L$

In the following we shall provide an equivalent solution to Eqs. (21) and (22) in terms of $L$ which is in some sense an analogue of formula (31).

**Proposition 3.2** We can rewrite the solution (29) in terms of $L$ as follows:

\begin{align*}
\mathcal{P} &= \frac{t}{2} \sum_{k=-\infty}^{\infty} J_{k+1}(tL)P_0 J_k(tL), \\
M &= \sum_{k=-\infty}^{\infty} J_k(tL)M_0 J_k(tL).
\end{align*}

**Proof.** To verify that the operators (38) obey Eqs. (21) and (22), we refer to Lemma 3.2. In fact, by resorting to (36), (37) and (33) we have

\begin{align*}
P_{tt} &= \frac{1}{2} \sum_{k=-\infty}^{\infty} \left[ J_k(tL)\mathcal{L}[P_0]J_k(tL) \right] + \frac{t}{2} \sum_{k=-\infty}^{\infty} \left[ J_k(tL)\mathcal{L}^2[P_0]J_{k+1}(tL) \right].
\end{align*}

On the other hand, since

\begin{align*}
\mathcal{L}^2[P] &= -\frac{t}{2} \sum_{k=-\infty}^{\infty} (-1)^k [J_k(tL)L^2P_0J_{1-k}(tL)] - 2J_k(tL)L_0J_{1-k}(tL) \\
&\quad + J_k(tL)P_0L^2J_{1-k}(tL) = -\frac{t}{2} \sum_{k=-\infty}^{\infty} \left[ J_k(tL)\mathcal{L}^2[P_0]J_{k+1}(tL) \right],
\end{align*}

we obtain the result.

In a similar way, by virtue of (34), we can write

\begin{align*}
LJ_{k+2}(tL) &= \frac{2(k+1)}{t}J_{k+1}(tL) - LJ_k(tL).
\end{align*}

This, with the help of Lemma 3.2, gives the following expression for $M_{tt}$:

\begin{align*}
- \sum_{k=-\infty}^{\infty} \left[ J_k(tL)\mathcal{L}^2[M_0]J_k(tL) \right] + \sum_{k=-\infty}^{\infty} (-1)^{k-1} \left[ J_{k-1}(tL)LM_0J_{-k}(tL) - 2J_{k-1}(tL)M_0LJ_{-k}(tL) + J_{-k}(tL)M_0LJ_{k-1}(tL) \right].
\end{align*}

Then, using the expression

\begin{align*}
\mathcal{L}^2[M] &= \sum_{k=-\infty}^{\infty} \left[ J_k(tL)\mathcal{L}^2[M_0]J_{-k}(tL) \right],
\end{align*}

we conclude the proof.
and taking into account (33) and (34), we have
\[ tM_{tt} + M_t + \mathcal{L}^2[M] = \sum_{k=-\infty}^{\infty} \left[ 2k \mathbf{J}_{k-1}(tL)LM_0 \mathbf{J}_k(tL) + -2(k - 1) \mathbf{J}_{k-1}(tL)M_0L \mathbf{J}_k(tL) \right] = 0. \]

Hence Eq. (22) is satisfied. This completes the proof.

4 Conclusions

We have solved the prolongation problem for Eq. (1) in terms of a series expansion of operators which can be interpreted as generalized Bessel coefficients. The operators (38) have been derived under the hypothesis that \( P \) and \( M \) are regular at \( t = 0 \), i.e., at \( u \to -\infty \). The “key” for our result is based on Eqs. (21) and (22), which constitute an extended form (of operator Bessel type) of the operator equations (23). We point out that the introduction of the operator (30) has strongly facilitated our task. To this regard, a useful step is represented by Eq. (31), where a correspondence is established between the operators \( \mathcal{L} \) and \( L \) via the Baker–Campbell–Hausdorff formula.

In theory, the knowledge of \( H, F \) and \( G \) (see (3)–(5)) may be exploited to find the spectral problem related to Eq. (1). However, in opposition to what happens in other cases, the determination of the spectral problem of Eq. (1) within the prolongation scheme offers notable difficulties, mainly owing to the fact that we have been able to get only a solution of the prolongation equations (16)–(18) which is regular at \( t = 0 \). At present the construction of a prolongation algebra whose elements depend uniquely on the pseudopotential variables remain an open problem.

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