Monodromy zeta-functions of deformations and Newton diagrams

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Abstract

For a one-parameter deformation of an analytic complex function germ of several variables, there is defined its monodromy zeta-function. We give a Varchenko type formula for this zeta-function if the deformation is non-degenerate with respect to its Newton diagram.

1 Introduction

Let $F$ be the germ of an analytic function on $(\mathbb{C}^{n+1}, 0)$, where $\mathbb{C}^{n+1} = \mathbb{C}_\sigma \times \mathbb{C}_z^n$, $\sigma$ is the coordinate on $\mathbb{C}$, and $z = (z_1, z_2, \ldots, z_n)$ are the coordinates on $\mathbb{C}^n$. The germ $F$ provides a deformation $f_\sigma = F(\sigma, \cdot)$ of the function germ $f = f_0$ on $(\mathbb{C}^n, 0)$. We give formulae for the monodromy zeta-functions of the deformations of the hypersurface germs $\{f = 0\} \cap (\mathbb{C}^*)^n$ and $\{f = 0\}$ at the origin in terms of the Newton diagram of $F$. A reason to study deformations of hypersurface germs and their monodromy zeta-functions was inspired by their connection with zeta-functions of deformations of polynomials: [3].

Let $A$ be the complement to an arbitrary analytic hypersurface $Y$ in $\mathbb{C}^n$: $A = \mathbb{C}^n \setminus Y$. Let $V = \{F = 0\} \cap (\mathbb{C}_\sigma \times A) \cap B_\varepsilon$, where $B_\varepsilon \subset \mathbb{C}^{n+1}$ is the closed ball of radius $\varepsilon$ with the centre at the origin. Let $D_\delta^* \subset \mathbb{C}_\sigma$ be the punctured disk of radius $\delta$ with the centre at the origin. For $0 < \delta \ll \varepsilon$ small enough the restriction to $V$ of the projection $\mathbb{C}^{n+1} \to \mathbb{C}_\sigma$ onto the first factor provides a fibration over $D_\delta^*$ ([7]). Denote by $V_c$ the fibre over the point $c$. Consider the monodromy transformation $h_{F,A}: V_c \to V_c$ of the above fibration restricted to the loop $c \cdot \exp(2\pi it)$, $t \in [0,1]$, $|c|$ is small enough.

The zeta-function of an arbitrary transformation $h: X \to X$ of a topological space $X$ is the rational function $\zeta_h(t) = \prod_{i \geq 0}(\det(\Id - th_\ast|H_i^c(X;\mathbb{C}))^{(-1)^i}$, where $H_i^c(X;\mathbb{C})$ is the $i$-th homology group with closed support.

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Definition. The zeta-function of the monodromy transformation \( h_{F,A} \) will be called the monodromy zeta-function of the deformation \( f_{\sigma} \) on \( A \): \( \zeta_{f_{\sigma}|A}(t) = \zeta_{h_{F,A}}(t) \).

For a power series \( S = \sum c_k y^k \), \( y^k = y_1^{k_1} \cdots y_m^{k_m} \), one defines its Newton diagram as follows. Denote by \( \mathbb{R}_+ \subset \mathbb{R} \) the set of non-negative real numbers. Denote by \( \Gamma'(S) \) the convex hull of the union \( \cup_{\alpha \neq 0} (k+\mathbb{R}_+^m) \). The Newton diagram of the series \( S \) is the union of compact faces of \( \Gamma'(S) \). For a germ \( G \) on \( \mathbb{C}^m \) at the origin, its Newton diagram \( \Gamma(G) \) is the Newton diagram of its Taylor series at the origin.

For a generic germ \( F \) on \( (\mathbb{C}^{n+1},0) \) with fixed Newton diagram \( \Gamma \in \mathbb{R}_+^{n+1} \) the zeta-functions \( \zeta_{f_{\sigma}|(\mathbb{C}^+)^n}(t), \zeta_{f_{\sigma}|\mathbb{C}^n}(t) \) are also fixed. We provide explicit formulas for these zeta-functions in terms of the Newton diagram \( \Gamma \).

2 The main result (a Varchenko type formula)

Let \( F \) be a germ of a function on \( (\mathbb{C}^{n+1},0) \). Let \( k = (k_0, k_1, \ldots, k_n) \) be the coordinates on \( \mathbb{R}^{n+1} \) corresponding to the variables \( \sigma, z_1, \ldots, z_n \) respectively. For \( I \subset \{0,1,\ldots,n\} \) denote by \( \mathbb{R}^I, \Gamma^I(F) \) the sets \( \{k \mid k_i = 0, i \notin I \} \subset \mathbb{R}^{n+1} \) and \( \Gamma(F) \cap \mathbb{R}^I \) respectively.

An integer covector is called primitive if it is not a multiple of another integer covector. Let \( P^I \) be the set of primitive integer covectors in the dual space \( (\mathbb{R}^I)^* \) such that all their components are strictly positive. For \( \alpha \in P^I \), let \( \Gamma^I_\alpha(F) \) be the subset of the diagram \( \Gamma^I(F) \) where \( \alpha|_{\Gamma^I(F)} \) reaches its minimal value: \( \Gamma^I_\alpha(F) = \{x \in \Gamma^I(F) \mid \alpha(x) = \min(\alpha|_{\Gamma^I(F)})\} \) (for \( \Gamma^I(F) = \emptyset \) we assume \( \Gamma^I_\alpha(F) = \emptyset \)). Consider the Taylor series of the germ \( F \) at the origin: \( F = \sum_{k \in \Gamma^I_\alpha} F_k \sigma^{k_0} z_1^{k_1} \cdots z_n^{k_n} \). Denote: \( F_\alpha = \sum_{k \in \Gamma^I_\alpha} F_k \sigma^{k_0} z_1^{k_1} \cdots z_n^{k_n} \).

Definition. A germ \( F \) of a function on \( (\mathbb{C}^{n+1},0) \) is called non-degenerate with respect to its Newton diagram if for any \( \alpha \in P^I \) the 1-form \( dF_\alpha \) does not vanish on the germ \( \{F_\alpha = 0\} \cap (\mathbb{C}^*)^{n+1} \) at the origin (see [9]).

For \( I \in \{0,1,\ldots,n\} \) such that \( 0 \in I \), we denote:
\[
\zeta^I_F(t) = \prod_{\alpha \in P^I} (1 - t^{\alpha(\frac{\partial}{\partial \theta})^- l_{-1}} V_l(\Gamma^I_\alpha(F))),
\]
where \( l = |I| - 1 \), \( \frac{\partial}{\partial \theta} \) is the vector in \( \mathbb{R}^I \) with the single non-zero coordinate \( k_0 = 1 \), and \( V_l(\cdot) \) denotes the \( l \)-dimensional integer volume, i.e. the volume in a rational \( l \)-dimensional affine hyperplane of \( \mathbb{R}^I \) normalized in such way that the volume of the minimal parallelepiped with integer vertices is equal to 1. We assume that \( V_0(\emptyset) = 1 \) and for \( n \geq 0 \) one has \( V_n(\emptyset) = 0 \).

Theorem 1 Let \( F \) be non-degenerate with respect to its Newton diagram \( \Gamma(F) \). Then one has
\[
\zeta_{f_{\sigma}|(\mathbb{C}^+)^n}(t) = \zeta^{0,1,\ldots,n}_F(t),
\]
\[
\zeta_{f_{\sigma}|\mathbb{C}^n}(t) = (1 - t) \times \prod_{I : 0 \in I \subset \{0,1,\ldots,n\}} \zeta^I_F(t).
\]
Remarks. 1. The equation (1) implies the equation (2) because of the following multiplicative property of the zeta-function. Let $h: X \to X$ be a transformation of a CW-complex $X$. Let $Y \subset X$ be a subcomplex of $X$. Assume that $h(Y) \subset Y$, $h(X \setminus Y) \subset (X \setminus Y)$. Then $\zeta_{h|X}(t) = \zeta_{h|X \setminus Y}(t) \times \zeta_{h|Y}(t)$.

One can see that $\zeta_{f_0|\{0\}}(t) = (1 - t) \times \zeta_{\tilde{F}}(0)(t)$. In fact, in the case $\Gamma^{\{0\}} = \emptyset$ one has $\zeta_{f_0|\{0\}}(t) = (1 - t)$, $\zeta_{\tilde{F}}(0)(t) = 1$. Otherwise $\zeta_{f_0|\{0\}}(t) = 1$, $\zeta_{\tilde{F}}(0)(t) = (1 - t)^{-1}$.

2. The zeta-function $\zeta_{f_0|\alpha}(t)$ coincides with the monodromy zeta-function of the germ of the function $\sigma: \{ F = 0 \} \to C_\sigma$ at the origin. The main theorem of [8] provides a formula for the zeta-functions of germs of functions on complete intersections in non-degenerate cases. One can apply this formula to the germ $\sigma$ and verify that the formula (2) agrees with the one of M. Oka. But (2) cannot be deduced from the result of M. Oka because the function $\sigma$ doesn’t satisfy the condition of "convenience" ([8], p. 17).

Examples. 1. Let $F(\sigma, z) = f(z) - \sigma$. The monodromy zeta-function of the deformation $f_\sigma$ coincides with the (ordinary) monodromy zeta-function $\zeta_f(t)$ of the germ $f$ on $(\mathbb{C}^n, 0)$ (see, e.g., [9]). In this case the $l$-dimensional faces $\Gamma_{\alpha}(F)$ (where $l = |I| - 1 > 0$) are cones of integer height 1 over the corresponding $(l - 1)$-dimensional faces $\Gamma_{\alpha|\{0\}}(f)$. One has:

$$V_\alpha(\Gamma_{\alpha}^I(F)) = V_{\alpha - 1}(\Gamma_{\alpha|\{0\}}^I(f)) / l$$

with $\alpha(\partial/\partial k_0) = \min(\alpha|_{\Gamma_{\alpha|\{0\}}(f)})(f_\alpha)$. This means that in this case the equation (2) coincides with the Varchenko formula (9).

2. For a deformation $F(\sigma, z)$ of the form $f_0(z) - \sigma f_1(z)$ the fibre

$$\{(\sigma) \times \{ f_\sigma = 0 \} \} \cap B_\varepsilon$$

is the disjoint union of the sets

$$\{(\sigma) \times \{ f_0/f_1 = 1 \} \} \cap B_\varepsilon$$

and

$$\{(\sigma) \times \{ f_0 = f_1 = 0 \} \} \cap B_\varepsilon.$$

If $f_0(0) = f_1(0) = 0$, then $\zeta_{f_0|\alpha}(t) = (1 - t) \times \zeta_{f_0/f_1|\alpha}(t)$, otherwise $\zeta_{f_0|\alpha}(t) = \zeta_{f_0/f_1|\alpha}(t)$ (the zeta-function of the meromorphic function $f_0/f_1$: [2]).

For $I \subset \{0, 1, \ldots, n\}$ such that $0 \in I$, and for a covector $\alpha \in P^I$, assume that the face $\Gamma_{\alpha}^I(F)$ has dimension $l$, where $l = |I| - 1 > 1$. Then $\Gamma_{\alpha}^I(F)$ is the convex hull of the corresponding faces $\Delta_{\alpha,0} = \{0\} \times \Gamma^I_{\alpha|\{k_0=0\}}(f_0)$ and $\Delta_{\alpha, 1} = \{1\} \times \Gamma^I_{\alpha|\{k_0=0\}}(f_1)$, which lie in the hyperplanes $\{k_0 = 0\}$, $\{k_0 = 1\}$ respectively. It is not difficult to show (see, e.g., [4], Lemma 1) that $V_{\alpha}(\Gamma_{\alpha}^I(F)) = V_{\alpha}^I / l$, where

$$V_{\alpha}^I = V_{\alpha - 1}(\Delta_{\alpha,0}^I, \ldots, \Delta_{\alpha,0}^I) + V_{\alpha - 1}(\Delta_{\alpha,0}^I, \ldots, \Delta_{\alpha,0}^I, \Delta_{\alpha,1}^I) + \ldots + V_{\alpha - 1}(\Delta_{\alpha,0}^I, \ldots, \Delta_{\alpha,1}^I) + V_{\alpha - 1}(\Delta_{\alpha,1}^I, \ldots, \Delta_{\alpha,1}^I).$$

Here $V_{\alpha - 1}$ denotes the $(l - 1)$-dimensional Minkowski’s mixed volume; see, e.g., [8]. Moreover, $\alpha(\partial/\partial k_0) = \min(\alpha|_{\Gamma_{\alpha|\{0\}}(f_0)}) - \min(\alpha|_{\Gamma_{\alpha|\{0\}}(f_1)})$, thus (2) coincides with the main result of [2].
3 A’Campo type formula

Proof of Theorem 1 uses an A’Campo type formula (1) written in terms of the integration with respect to the Euler characteristic (2).

For a constructible function \( \Phi \) on a constructible set \( Z \) with values in a (multiplicative) Abelian group \( G \) its integral \( \int_Z \Phi \, d\chi \) with respect to the Euler characteristic \( \chi \) is defined as \( \prod_{g \in G} g^{\chi(\Phi^{-1}(g))} \) (see 10). Further we consider \( G = \mathbb{C}(t)^* \) to be the multiplicative group of non-zero rational functions in the variable \( t \).

Let \( F \) be a germ of an analytic function on \((\mathbb{C}^{n+1},0)\) defined on a neighbourhood \( U \) of the origin. Let \( Y \) be a hypersurface in \( \mathbb{C}^n \). Denote \( S = (\mathbb{C}_\sigma \times Y) \cup \{\sigma = 0\} \). Consider a resolution \( \pi: (X,D) \to (U,0) \) of the germ of the hypersurface \( \{F = 0\} \cup S \) at the origin, where \( D = \pi^{-1}(0) \) is the exceptional divisor.

**Theorem 2** Assume \( \pi \) to be an isomorphism outside of \( \pi^{-1}(U \cap S) \). Then

\[
\zeta_{f*|C^n \setminus Y}(t) = \int_{D \cap W} \zeta_{|W \setminus Z, x}(t) \, d\chi,
\]

where \( W \) is the proper preimage of \( \{F = 0\} \) (i.e. the closure of \( \pi^{-1}(V) \), \( V = (\{F = 0\} \cap U) \setminus S \)), \( \Sigma = \sigma \circ \pi, \ Z = \pi^{-1}(\mathbb{C}_\sigma \times Y) \) and \( \zeta_{|W \setminus Z, x}(t) \) is the monodromy zeta-function of the germ of the function \( \Sigma \) on the set \( W \setminus Z \) at the point \( x \in D \cap W \).

**Proof.** The map \( \pi \) provides an isomorphism \( W \setminus (Z \cup \{\Sigma = 0\}) \to V \), which is also an isomorphism of fibrations provided by the maps \( \Sigma \) and \( \sigma \) over sufficiently small punctured neighbourhood of zero \( \mathbb{D}_\delta^* \subset \mathbb{C}_\sigma \). Therefore the monodromy zeta-functions of this fibrations coincide, \( \zeta_{f*|C^n \setminus Y}(t) = \zeta_{|W \setminus Z}(t) \) (the monodromy zeta-function of the ”global” function \( \Sigma \) on \( W \setminus Z \)).

Applying the localization principle (3) to \( \Sigma \) we obtain:

\[
\zeta_{f*|C^n \setminus Y}(t) = \int_{W \cap \{\Sigma = 0\}} \zeta_{|W \setminus Z, x}(t) \, d\chi.
\]

The integration is multiplicative with respect to subdivision of its domain. One has \( W \cap \{\Sigma = 0\} = (D \cap W) \cup ((W \cap \{\Sigma = 0\}) \setminus D) \). Thus the right hand side of (3) is the product \( \left[ \int_{D \cap W} \zeta_{|W \setminus Z, x}(t) \, d\chi \right] \cdot \left[ \int_{W \cap \{\Sigma = 0\} \setminus D} \zeta_{|W \setminus Z, x}(t) \, d\chi \right] \). The first factor coincide with the right hand side of (3); we prove that the second factor equals 1.

For a point \( x \in D \), its neighbourhood \( U(x) \subset X \) with a coordinate system \( u_1, u_2, \ldots, u_{n+1} \) is called convenient if each of manifolds \( D, Z \) can be defined on \( U(x) \) by an equation of type \( u^k = 0 \) and each of functions \( \Sigma, F = \pi \) has the form \( a u^k \), where \( a(0) \neq 0 \). One can assume that \( X \) is covered by a finite number of convenient neighbourhoods.

For an arbitrary convenient neighbourhood \( U_0 \), choose an order of coordinates \( u_i \) on it such that \( D = \{u_1 u_2 \cdots u_l = 0\} \).

**Proposition 1** The zeta-function \( \zeta_{|W \setminus Z, x}(t) \) at a point \( x \in U_0 \setminus D \) is well-defined by the coordinates \( u_1, u_2, \ldots, u_{n+1} \) of \( x \).
Proof. The germ of the manifold \( Z \) at the point \( x \) is defined by an equation 
\[ u_{i+1} \cdots u_{n+1} = 0; \] 
in a neighbourhood of \( x \) one has 
\[ \tilde{F} = a u_{i+1}^{k_{2,i+1}} \cdots u_{n+1}^{k_{2,n+1}}; \] 
\( \Sigma = b u_{i+1}^{k_{3,i+1}} \cdots u_{n+1}^{k_{3,n+1}} \), where \( a(x) \neq 0, b(x) \neq 0 \), \( k_{1,j} \in \{0, 1\}; k_{2,j}, k_{3,j} \geq 0 \). The zeta-function 
\( \zeta_{|\omega \cap \Sigma}(t) \) is well-defined by the numbers \( k_{i,j}, i = 1, 2, 3, j = l+1, \ldots, n+1 \), which do not depend on \( u_1, \ldots, u_n \).

For a rational function \( Q(t) \), we define a set \( X_Q = \{ x \in W \cap (\{ \Sigma = 0 \} \setminus D) \} \) 
\( \zeta_{|\omega \cap \Sigma}(t) = Q(t) \). It follows from the proposition above that for any convenient 
neighbourhood \( U_0 \) we have \( \chi(U_0 \cap X_Q) = 0 \). Thus for all \( Q(t) \) we have \( \chi(X_Q) = 0 \) and 
\[ \int_{W \cap (\{ \Sigma = 0 \} \setminus D)} \zeta_{|\omega \cap \Sigma}(t) dx = \prod_Q Q^\chi(X_Q) = 1. \]

□

4 Proof of Theorem[1]

Using the Newton diagram \( \Gamma(F) \) of the germ \( F \) on \( (\mathbb{C}^{n+1}, 0) \) one can construct 
an unimodular simplicial subdivision \( \Lambda \) of the set of covectors with non-negative 
coordinates \( (R^{n+1})^n_+ \) (see, e.g., [9]). Consider the toroidal modification map \( p : \) 
\( (X_\Lambda, D) \to (\mathbb{C}^{n+1}, 0) \) corresponding to \( \Lambda \). Let \( U \subseteq \mathbb{C}^{n+1} \) be a small enough ball 
with the centre at the origin, \( X = p^{-1}(U), \pi = p|_X \). Let \( Y = \{ z_1 z_2 \cdots z_n = 0 \} \subset \mathbb{C}_n \). Then \( S = (Y \times \mathbb{C}_\sigma) \cup \{ \sigma = 0 \} \) is the union of the coordinate hyperplanes of 
\( \mathbb{C}^{n+1} \). Since \( F \) is non-degenerate with respect to its Newton diagram \( \Gamma(F) \), \( \pi \) is a resolution of the germ \( S \cup \{ F = 0 \} \) (see, e.g., [8]). Finally, \( \pi \) is an isomorphism 
outside of \( S \), so the resolution \((X, \pi)\) satisfies the assumptions of Theorem[2]

Compute the right hand side of (3). Let \( x \in D \cap W \) be a point of the \((n-l+1)\)- 
dimensional torus \( T_\lambda \) corresponding to an \( l \)-dimensional cone \( \lambda \in \Lambda \). Let \( \lambda \) be 
generated by integer covectors \( \alpha_1, \ldots, \alpha_l \) and let \( \lambda \) lie on the border of a cone 
\( \lambda' \in \Lambda \) generated by \( \alpha_1, \ldots, \alpha_l, \alpha_{l+1} \). Let \( (u_1, \ldots, u_{n+1}) \) be the coordinate 
system corresponding to the set \( \{ \alpha_1, \ldots, \alpha_{l+1} \} \). There exists a coordinate system 
\( (u_1, \ldots, u_i, u_{i+1}, \ldots, u_{n+1}) \) in a neighbourhood \( U' \) of the point \( x \) such that 
\( u_i(x) = 0, i = l+1, \ldots, n+1 \) and \( \tilde{F} = F \circ \pi = a u_{1}^{k_{1,1}} u_{2}^{k_{1,2}} \cdots u_{l}^{k_{1,l}} u_{n+1}^{k_{1,n+1}} \) (where \( a(0) \neq 0 \)). The zero level set 
\( \{ \Sigma = 0 \} \) is a normal crossing divisor contained in \( \{ u_1 u_2 \cdots u_l = 0 \} \). Therefore 
\( \Sigma = \sigma \circ \pi = u_{k_{2,1}}^{k_{2,1}} u_{k_{2,2}}^{k_{2,2}} \cdots u_{k_{2,l}}^{k_{2,l}} \). One has: \( W \cap U' = \{ u_{n+1} = 0 \}, \) 
\( (Z \cup \{ \Sigma = 0 \}) \cap U' = \{ u_1 u_2 \cdots u_l = 0 \} \), thus 
\( \zeta_{|\omega \cap \Sigma}(t) = \zeta_{g|_{\{ u_1 \neq 0, \cdots, l \leq i \}}}(t), \) where \( g \) is the germ of the following function of \( n \) variables: 
\( g(u_1, \ldots, u_i, u_{l+1}, \ldots, u_n) = u_{k_{2,1}}^{k_{2,1}} u_{k_{2,2}}^{k_{2,2}} \cdots u_{k_{2,l}}^{k_{2,l}} \).

Assume that one of the exponents \( k_{2,1}, k_{2,2}, \ldots, k_{2,l} \) (say, \( k_{2,1} \)) is equal to zero. 
Then \( g \) doesn’t depend on \( u_1 \). We may assume that the monodromy transformation 
of its Milnor fibre also doesn’t depend on \( u_1 \). Denote \( h = g|_{\{ u_1 = 0 \}} \). The monodromy 
transformations of the fibre of \( g|_{\{ u_2 u_3 \cdots u_l \neq 0 \}} \) and one of \( h|_{\{ u_2 u_3 \cdots u_l \neq 0 \}} \) 
are homotopy equivalent, so 
\( \zeta_{g|_{\{ u_2 u_3 \cdots u_l \neq 0 \}}}(t) = \zeta_{h|_{\{ u_2 u_3 \cdots u_l \neq 0 \}}}(t) \). On the other hand 
the multiplicative property of the zeta-function implies that 
\( \zeta_{g|_{\{ u_1 \neq 0, i \leq l \}}}(t) \times \zeta_{h|_{\{ u_2 u_3 \cdots u_l \neq 0 \}}}(t) = \zeta_{g|_{\{ u_2 u_3 \cdots u_l \neq 0 \}}}(t) \) and thus 
\( \zeta_{g|_{\{ u_1 \neq 0, i \leq l \}}}(t) = 1. \)
Now assume that all the exponents \( k_{2,1}, k_{2,2}, \ldots, k_{2,l} \) are positive. Then the non-zero fibre of the function \( g \) doesn’t intersect \( \{ u_1 u_2 \ldots u_l = 0 \} \), so \( \zeta_{\{ u_1 u_2 \ldots u_l = 0 \}}(t) = \zeta_g(t) \). In the case \( l > 1 \) one has \( \zeta_g(t) = 1 \). In the case \( l = 1 \) one has: \( g = u_1^{k_{2,1}}, \zeta_g(t) = 1 - t^{k_{2,1}} \).

We see that the integrand in (3) differs from 1 only at points \( x \) that lie in strata of dimension \( n \). From here on \( l = 1 \). If all the components of \( \alpha = \alpha_1 \) are positive, then \( T_\lambda \subset D \). Otherwise, \( T_\lambda \cap D = \emptyset \). From here on \( \alpha \in P^{[0,1,\ldots,n]} \) (see the definitions before Theorem 1).

Using the coordinates \((u_2, \ldots, u_{n+1})\) on the torus \( T_\lambda = \{ u_1 = 0 \} \) we obtain: \( T_\lambda \cap W = \{ Q_\alpha = 0 \} \), where for the power series \( F = \sum F_k \sigma^{k_0} z_1^{k_1} \ldots z_n^{k_n} \) we denote \( Q_\alpha = \sum_{k \in \Gamma_\alpha^{(0,\ldots,n)}} F_k \sigma^{k_0} z_1^{k_1} \ldots z_n^{k_n} \). So \( T_\lambda \cap W \) is the zero level set of the Laurent polynomial \( Q_\alpha \). Using results of [5], [6] we obtain: \( \chi(T_\lambda \cap W) = (-1)^{n-1} n! V_n(\Delta(Q_\alpha)) \), where \( \Delta(\cdot) \) denotes the Newton polyhedron. Since the polyhedra \( \Delta(Q_\alpha) \) and \( \Gamma_\alpha = \Gamma_\alpha^{(0,\ldots,n)}(F) \) are isomorphic as subsets of integer lattices, their volumes are equal: \( V_n(\Delta(Q_\alpha)) = V_n(\Gamma_\alpha) \). In a neighbourhood of a point \( x \in T_\lambda \cap W \) one has \( \Sigma = a u_1^{\alpha(\partial/\partial \kappa_\alpha)} \), where \( a(x) \neq 0 \). Therefore \( \zeta_{\Sigma|W \setminus x}(t) = 1 - t^{\alpha(\partial/\partial \kappa_\alpha)} \). Thus one has:

\[
\int_{T_\lambda \cap W} \zeta_{\Sigma|W \setminus x}(t)^{d\chi} = (1 - t^{\alpha(\partial/\partial \kappa_\alpha)})^{\chi(T_\lambda \cap W)} = (1 - t^{\alpha(\partial/\partial \kappa_\alpha)})^{(-1)^{n-1} n! V_n(\Gamma_\alpha)}. \tag{5}
\]

Multiplying (5) for all strata \( T_\lambda \subset D \) of dimension \( n \) we get (1).

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**References**

[1] A’Campo N., La fonction zeta d’une monodromie, Comm. Math. Helv., 50, p. 233–248, 1975.

[2] Gusein-Zade S. M., Luengo I., Melle-Hernandez A., Zeta-functions for germs of meromorphic functions and Newton diagrams, Funct. Anal. Appl., 1998.

[3] Gusein-Zade S. M., Siersma D., Deformations of polynomials and their zeta-functions, Journal of Mathematical Sciences, 144, no. 1, p. 3782–3788, 2007.

[4] Gusev G. G., The Euler characteristic of the bifurcation set for a polynomial of degree two, to appear in Russian Mathematical Surveys.

[5] Khovanskii A. G., Newton polyhedra and toroidal varieties, Funct. Anal. Appl., 11, no. 4, p. 56–64, 1977.

[6] Khovanskii A. G., Newton polyhedra and the genus of complete intersections, Funct. Anal. Appl., 12, no. 1, p. 51–61, 1978.

[7] Lê Dũng Tráng, Some remarks on relative monodromy, Real and complex singularities, Ed. by P. Holm, Nordhoff Publ., p. 397–403, 1977.
[8] Oka M., Principal zeta-function of non-degenerate complete intersection singularity, J. Fac. Sci. Univ. Tokyo. Sect. IA., 37, p. 11–32, 1990.

[9] Varchenko A. N., Zeta function of monodromy and Newton’s diagram, Inv. Math. 37, p. 253–262, 1976.

[10] Viro O. Y., Some integral calculus based on Euler characteristic, Lecture Notes in Math., vol. 1346, Springer-Verlag, p. 127–138, 1988.

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