REGULAR COVARIANT REPRESENTATIONS AND THEIR WOLD-TYPE DECOMPOSITION

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Abstract. Olofsson introduced a growth condition regarding elements of an orbit for an expansive operator and generalized Richter’s wandering subspace theorem. Later on, using the Moore-Penrose inverse, Ezzahraoui, Mbekhta, and Zerouali extended the growth condition and obtained a Shimorin-Wold-type decomposition. Shimorin-Wold-type decomposition for completely bounded covariant representations, which are close to isometric representations, is obtained in [25]. This paper extends this decomposition for regular, completely bounded covariant representation having reduced minimum modulus \( \geq 1 \) that satisfies the growth condition. To prove the decomposition, we introduce the terms regular, algebraic core, and reduced minimum modulus in the completely bounded covariant representation setting and work out several fundamental results. Consequently, we shall analyze the weighted unilateral shift introduced by Muhly and Solel and introduce and explore a non-commutative weighted bilateral shift.

1. Introduction

The classical result of Wold [27] asserts that given isometry on a Hilbert space is either a unitary, a shift, or uniquely breaks as a direct summand of them. Beurling [3] proved that every \( M_z \)-invariant closed subspace of the Hardy space \( H^2(\mathbb{D}) \) is a copy of an inner function. One of the well-known implications of the Wold decomposition is that when the unitary part is zero, the Wold-decomposition gives uniqueness of the wandering subspace for a shift. Halmos [10] proved a wandering subspace theorem, which is an abstraction of the Beurling’s theorem, that characterized all the invariant subspaces of a shift. Richter [21] proved a wandering subspace theorem for an analytic concave operator which satisfies the growth condition that was explicitly mentioned and generalized by Olofsson in [17]. After that, Shimorin [24] provided an elementary proof of Richter’s theorem by giving a Wold-type decomposition for concave operators that can be considered close to an isometry. Olofsson [17] extended the Richter’s wandering subspace theorem as follows:

**Theorem 1.1. (Olofsson)** Suppose \( V \) is an analytic bounded linear map defined on Hilbert space \( \mathcal{H} \) such that

(i) \( V \) is an expansive operator,
(ii) there exist some positive numbers $d_m, d$ such that $\sum_{m \geq 2} \frac{1}{d_k} = \infty$ and
\[ \|V^m h\| \leq d\|h\|^2 + d_m(\|Vh\| - \|h\|^2), \quad h \in \mathcal{H}. \]

Then $V$ has the wandering subspace property.

Ezzahraoui, Mbekhta, and Zerouali in [7] using the reduced minimum modulus $\geq 1$, and a general condition of the expansive operator, extended Theorem 1.1 for the broader class of regular operators [13] and proved the following Wold-type decomposition:

**Theorem 1.2.** (Ezzahraoui-Mbekhta-Zerouali) If $V$ is a regular bounded linear operator defined on a Hilbert space $\mathcal{H}$ such that the reduced minimum modulus of $V$ is greater than or equal to 1, and satisfies the growth condition
\[ \|V^m h\| \leq \|V^\dagger Vh\|^2 + d_m(\|Vh\|^2 - \|V^\dagger Vh\|^2), \quad h \in \mathcal{H}\]
such that $\sum_{m \geq 2} \frac{1}{d_m} = \infty$, then
\[ \mathcal{H} = \bigcap_{n=1}^\infty V^n(\mathcal{H}) + [\mathcal{H} \ominus V(\mathcal{H})]_V. \]

The study of Wold decomposition has begun in Non-commutative Multivariate Operator Theory by Frazho [9] and by Popescu [20]. They explored Wold-decomposition for row-isometries considered by Cuntz [5]. Pimsner [19] introduced the notion of Cuntz-Pimsner algebra for faithful $C^*$-correspondence. Muhly and Solel [15] provided the Wold-type decomposition for representations of tensor algebras of $C^*$-correspondences, and they explored the invariant subspaces of particular subalgebras of Cuntz-Krieger algebras.

Let us discuss the structure of this article: Section 2 begins with the introduction of generalized range, the algebraic core, and regular covariant representation. Here, we explain the connection between generalized range and algebraic core. In Section 3 we analyze the regularity condition of the generalized inverse of a covariant representation, discuss its properties, and characterize the element of the generalized range. In Section 4 we study the reduced minimum modulus, Moore-Penrose inverse for the covariant representation $(\sigma, V)$ and its various properties. Assume $(\sigma, V)$ to be regular, completely bounded, covariant representation of a $C^*$-correspondence $E$ on a Hilbert space $\mathcal{H}$ with the reduced minimum modulus $\gamma(V) \geq 1$ and satisfying the growth condition
\[ \|\tilde{V}k(\xi_k)\|^2 \leq d_k(\|I_{E^\otimes k-1} \otimes \tilde{V})(\xi_k\|^2 - \|(I_{E^\otimes k-1} \otimes \tilde{V})\|\|\xi_k\|^2) \]
for every $k \geq 0; \xi_k \in E^\otimes k \otimes \mathcal{H}$ with $\sum_{k \geq 2} \frac{1}{d_k} = \infty$. Then $(\sigma, V)$ admits a Shimorin-Wold-type decomposition. It is the main content of [5]. In Section 5 we extend the commutant result of Bercovici, Douglas, and Foias [4] for a shift to left invertible completely bounded representation based on recent work of S. Sarkar [23]. In Section 6 we derived a sufficient condition on weight sequence $(Z_k)_{k \in \mathbb{N}_0}$ such that weighted shift $(\rho, S)$ on a $F(E) \otimes \mathcal{H}$ with weight sequence $(Z_k)$ given by Muhly and Solel in [16] admits the generating wandering subspace property. We also study the multivariable analog of the bilateral shift discussed in [7].
1.1. Preliminaries and Notations. Now we recall some well-known definitions and properties of Hilbert $C^*$-modules, $C^*$-correspondences (cf. [18, 14]) and covariant representations of $C^*$-correspondences (cf. [19, 14]).

Let $\mathcal{B}$ be a $C^*$-algebra and $E$ be a Hilbert $\mathcal{B}$-module. Assume $\mathcal{L}(E)$ to be $C^*$-algebra of all adjointable maps on $E$. Now $E$ is said to be a $C^*$-correspondence over $\mathcal{B}$ if $E$ is left $\mathcal{B}$ a module where the left action is through a non-zero $*$-homomorphism $\phi : \mathcal{B} \to \mathcal{L}(E)$ such that

$$a\xi := \phi(a)\xi \quad \text{for all} \quad (a \in \mathcal{B}, \xi \in E).$$

Here, we assume that every $*$-homomorphism is always essential, which means that the closure of the linear span of $\phi(\mathcal{B})E$ is $E$. The module $E$ has an operator space structure inherited as a subspace of the so called linking algebra (see [14]). Suppose $F$ and $E$ are two $C^*$-correspondences over $\mathcal{B}$. Then the tensor product, denoted by $F \otimes E$, satisfies the following properties

$$(\xi_1 a) \otimes \xi_2 = \xi_1 \otimes \phi(a)\xi_2,$$

for every $\xi_1, \xi_2 \in F$, $\xi_1, \xi_2 \in E$ and $a \in \mathcal{B}$.

Unless necessary, after this section, we simply write $F \otimes E$ instead of $F \otimes_\phi E$. Throughout this paper, we assume that $\mathcal{B}$ is a $C^*$-algebra, $\mathcal{H}$ is a Hilbert space and $E$ is a $C^*$-correspondence over $\mathcal{B}$ with left module action given by a $*$-homomorphism $\phi : \mathcal{B} \to \mathcal{L}(E)$.

**Definition 1.3.** Assume $\sigma : \mathcal{B} \to B(\mathcal{H})$ to be a representation and $V : E \to B(\mathcal{H})$ to be a linear function. If

$$V(cyd) = \sigma(c)V(\eta)\sigma(d), \quad \text{for all} \quad c, d \in \mathcal{B}, \eta \in E$$

then we say that the pair $(\sigma, V)$ is a covariant representation (cf. [14]) of $E$ on the Hilbert space $\mathcal{H}$. The covariant representation $(\sigma, V)$ is completely bounded (CB-representation) if $V$ is completely bounded. Additionally, if $V(\xi)^*V(\zeta) = \sigma(\langle \xi, \zeta \rangle), \zeta, \xi \in E$ then we say $(\sigma, V)$ is isometric.

Consider a CB-representation $(\sigma, V)$ of $E$ on $\mathcal{H}$, then there corresponds a bounded operator $\widetilde{V} : E \otimes \mathcal{H} \to \mathcal{H}$ defined by $\widetilde{V}(\xi \otimes h) := V(\xi)h$ where $\xi \in E$ and $h \in \mathcal{H}$. Note that $\widetilde{V}$ satisfies $\widetilde{V}(\phi(a) \otimes I_{\mathcal{H}}) = \sigma(a)\widetilde{V}, a \in \mathcal{B}$ and $\phi$ is a left action on $E$.

To categorise the covariant representation of $C^*$-correspondences, the following lemma from [14, Lemma 3.5] is useful.

**Lemma 1.4.** There is a map $(\sigma, V) \mapsto \widetilde{V}$ which gives a one-to-one correspondence between the collection of all bounded linear maps $\widetilde{V} : E \otimes \mathcal{H} \to \mathcal{H}$ which satisfies $\widetilde{V}(\phi(a) \otimes I_{\mathcal{H}}) = \sigma(a)\widetilde{V}, a \in \mathcal{B}$ and the collection of all CB-representations $(\sigma, V)$ of $E$ on $\mathcal{H}$. Additionally, $(\sigma, V)$ is isometric if and only $\widetilde{V}$ is an isometry. A CB-representation $(\sigma, V)$ is fully co-isometric if $\widetilde{V}$ is co-isometry, that is, $\widetilde{V}^*V = I_{\mathcal{H}}$.

For each $n \in \mathbb{N}$, define $E^{\otimes n} := E \otimes \cdots \otimes E = \bigotimes_{i=1}^{n} E$ (n fold tensor product) and $E^{\otimes 0} := \mathcal{B}$.

Then $E^{\otimes n}$ is a $C^*$-correspondence over $\mathcal{B}$ in a natural way, where the left action of $\mathcal{B}$ on $E^{\otimes n}$ is denoted by $\phi^n$ and, is given by $\phi^n(a)(\bigotimes_{i=1}^{n} \xi_i) = \phi(a)\xi_1 \otimes \bigotimes_{i=2}^{n} \xi_i$, $\xi_i \in E, 1 \leq i \leq n$. 


For each \( n \in \mathbb{N} \), define \( \widetilde{V}_n : E^\otimes n \otimes \mathcal{H} \to \mathcal{H} \) by
\[
\widetilde{V}_n(\bigotimes_{i=1}^n \xi_i \otimes h) = V(\xi_1)V(\xi_2) \cdots V(\xi_n)h,
\]
for \( \xi_i \in E, h \in \mathcal{H}, 1 \leq i \leq n \). It is trivial to see that for each \( n \in \mathbb{N} \),
\[
\widetilde{V}_n = \widetilde{V}(I_E \otimes \widetilde{V}_{n-1}) = \widetilde{V}_{n-1}(I_{E^\otimes (n-1)} \otimes \widetilde{V}).
\]

The following theorem from [26, Theorem 3.13] is an abstraction of Shimoren-Wold-type decomposition [24, Theorem 3.6] and of Wold decomposition for isometric representation due to Muhly and Solel [15].

**Theorem 1.5.** Consider a CB-representation \((\sigma, V)\) of \( E \) on \( \mathcal{H} \). If it satisfies one of the following conditions:

1. for every \( \xi \in E^\otimes 2 \otimes \mathcal{H}, \kappa \in E \otimes \mathcal{H} \)
\[
\|(I_E \otimes \widetilde{V})(\xi) + \kappa\|^2 \leq 2\|(\xi\|^2 + \|\widetilde{V}(\kappa)\|^2);
\]

2. \((\sigma, V)\) is concave, i.e.,
\[
\|\widetilde{V}(\eta \otimes h)\|^2 + \|\eta \otimes h\|^2 \leq 2\|(I_E \otimes \widetilde{V})(\eta \otimes h)\|^2, \eta \in E^\otimes 2, h \in \mathcal{H}.
\]

Then \((\sigma, V)\) admits Wold-type decomposition. To put it another way, there is a wandering subspace \( W \) for \((\sigma, V)\) that divides \( \mathcal{H} \) into the direct sum of \((\sigma, V)\)-reducing subspaces.

\[
(1.1) \quad \mathcal{H} = \bigvee_{n \in \mathbb{N}_0} \widetilde{V}_n(E^\otimes n \otimes W) \bigoplus \bigcap_{n \in \mathbb{N}_0} \widetilde{V}_n(E^\otimes n \otimes \mathcal{H})
\]

that both an isometric and a fully co-isometric representation of the restriction of \((\sigma, V)\) on \( \bigcap_{n \in \mathbb{N}_0} \widetilde{V}_n(E^\otimes n \otimes \mathcal{H}) \) is achieved. Moreover, the above decomposition is unique.

### 2. Regular, completely bounded, covariant representations

This section is based on [11, Chapter 1]. Also, we introduce regular covariant representations of \( E \) on \( \mathcal{H} \) and prove that its generalized range (hyper-range) is equal to the notion of the algebraic core (cf. [11, Chapter 1]).

Suppose \((\sigma, V)\) is a CB-representation of \( E \) on \( \mathcal{H} \). Define \( R^\infty(V) := \bigcap_{n \in \mathbb{N}_0} R(\widetilde{V}_n) \) will be used to denote by the generalized range of \((\sigma, V)\), where \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). Since \((R(\widetilde{V}_n))_{n \in \mathbb{N}_0}\) is a decreasing sequence, \( R^\infty(V) = \bigcap_{n \in \mathbb{N}_0} R(\widetilde{V}_n) = \bigcap_{n \in \mathbb{N}} R(\widetilde{V}_n) \).

**Lemma 2.1.** Consider a CB-representation \((\sigma, V)\) of \( E \) on \( \mathcal{H} \). Then
\[
(I_{E^\otimes n} \otimes \widetilde{V}_m)N(\widetilde{V}_{m+n}) = N(\widetilde{V}_{n}) \cap (I_{E^\otimes n} \otimes \widetilde{V}_m)(E^\otimes (m+n) \otimes \mathcal{H}), \quad m, n \in \mathbb{N}.
\]

**Proof.** Let \( \xi \in N(\widetilde{V}_{m+n}) \subseteq E^\otimes (m+n) \otimes \mathcal{H} \), then \( \widetilde{V}_n(I_{E^\otimes n} \otimes \widetilde{V}_m)(\xi) = \widetilde{V}_{m+n}(\xi) = 0 \), so that
\[
(I_{E^\otimes n} \otimes \widetilde{V}_m)N(\widetilde{V}_{m+n}) \subseteq N(\widetilde{V}_{n}) \cap (I_{E^\otimes n} \otimes \widetilde{V}_m)(E^\otimes (m+n) \otimes \mathcal{H}).
\]

On another hand, if \( \eta \in N(\widetilde{V}_n) \cap (I_{E^\otimes n} \otimes \widetilde{V}_m)(E^\otimes (m+n) \otimes \mathcal{H}) \) then \( \widetilde{V}_n(\eta) = 0 \) and \( \eta = (I_{E^\otimes n} \otimes \widetilde{V}_m)(\xi) \) for some \( \xi \in E^\otimes (m+n) \otimes \mathcal{H} \), and we get \( \widetilde{V}_{n+m}(\xi) = \widetilde{V}_n(I_{E^\otimes n} \otimes \widetilde{V}_m)(\xi) = 0 \). Therefore \( \xi \in N(\widetilde{V}_{m+n}) \), we obtain \( \eta \in (I_{E^\otimes n} \otimes \widetilde{V}_m)N(\widetilde{V}_{m+n}) \), so the opposite inclusion is verified. \( \square \)
Proof. 

(1) \(N(V_n) \subseteq (I_E \otimes \tilde{V}_n)(E^{\otimes(n+1)} \otimes \mathcal{H})\) for every \(n \in \mathbb{N}\); 

(2) \(N(V_n) \subseteq (I_E \otimes \tilde{V})(E^{\otimes(n+1)} \otimes \mathcal{H})\) for every \(n \in \mathbb{N}\); 

(3) \(N(V_n) \subseteq (I_E \otimes \tilde{V}_m)(E^{\otimes(n+m)} \otimes \mathcal{H})\) for every \(n, m \in \mathbb{N}\); 

(4) \(N(V_n) = (I_E \otimes \tilde{V}_m)N(V_{m+n})\) for every \(m, n \in \mathbb{N}\).

Proof. 

(1) \(\implies\) (2): We will prove it by Mathematical induction. For \(n = 1\) in (1), we have

\[N(V) \subseteq (I_E \otimes \tilde{V})(E^{\otimes2} \otimes \mathcal{H}),\]

which means (2) is true for \(n = 1\). Let us assume (2) is true for \(n = k\). To show that it is true for \(n = k + 1\), i.e.,

\[N(V_{k+1}) \subseteq (I_E \otimes \tilde{V}_k)(E^{\otimes(k+2)} \otimes \mathcal{H}).\]

For this purpose, consider \(\xi \in N(V_{k+1})\) then

\[(I_E \otimes \tilde{V}_k)(\xi) \in N(V) \subseteq (I_E \otimes \tilde{V}_k)(E^{\otimes(k+2)} \otimes \mathcal{H}),\]

here last inequality follows from the hypothesis (1). By Lemma 2.1 there exists \(\eta \in N(\tilde{V}_{k+2})\) such that \((I_E \otimes \tilde{V}_k)(\xi) = (I_E \otimes \tilde{V}_{k+1})(\eta)\) which implies that \(\xi - (I_E \otimes \tilde{V}_k)(\eta) \in N(I_E \otimes \tilde{V}_k) = E \otimes N(\tilde{V}_k)\). Now using the induction hypothesis, i.e., \(N(\tilde{V}_k) \subseteq (I_E \otimes \tilde{V})(E^{\otimes(k+1)} \otimes \mathcal{H})\), we get

\[\xi \in N(V_{k+1}) \subseteq (I_E \otimes \tilde{V}_k)(E^{\otimes(k+2)} \otimes \mathcal{H}),\]

which proves the desired inequality.

(2) \(\implies\) (3): We will prove inequality (3) by Mathematical induction. For \(m = 1\), nothing to prove. Suppose (3) is true for \(m = k\). Now we need to show that it is correct for \(m = k + 1\), that is,

\[N(V_n) \subseteq (I_E \otimes \tilde{V}_k)(E^{\otimes(n+k+1)} \otimes \mathcal{H}).\]

Let \(\xi \in N(V_n)\), using (2) there exists \(\eta \in N(V_{n+1})\) such that \(\xi = (I_E \otimes \tilde{V})(\eta)\) and by induction hypothesis, we have

\[\xi = (I_E \otimes \tilde{V})(\eta) \in (I_E \otimes \tilde{V}_k)(I_E \otimes \tilde{V}_{k+1})(E^{\otimes(n+k+1)} \otimes \mathcal{H}) = (I_E \otimes \tilde{V}_k)(E^{\otimes(n+k+1)} \otimes \mathcal{H}).\]

Thus we get (3).

(3) \(\implies\) (4): Assume \(N(V_n) \subseteq (I_E \otimes \tilde{V}_m)(E^{\otimes(n+m)} \otimes \mathcal{H})\), for \(m, n \in \mathbb{N}\). By Lemma 2.1 and (3),

\[(I_E \otimes \tilde{V}_m)N(V_{m+n}) = N(V_n) \cap (I_E \otimes \tilde{V}_m)(E^{\otimes(m+n)} \otimes \mathcal{H}) = N(V_n)\]

which proves (4).

(4) \(\implies\) (1): Obvious. \[\square\]
Note that if \( \tilde{V} \) satisfy (3) of the above theorem, we get

\[ N(\tilde{V}) \subseteq E^\otimes n \otimes R^\infty(V), \quad n \in \mathbb{N}. \]

In particular \( N(\tilde{V}) \subseteq E \otimes R^\infty(V) \). On another hand, if \( N(\tilde{V}) \subseteq E \otimes R^\infty(V) \), then \( \tilde{V} \) satisfies all conditions of above theorem.

**Definition 2.3.** Consider a CB-representation \((\sigma, V)\) of \( E \) on \( \mathcal{H} \). We call \((\sigma, V)\) is regular if range of \( \tilde{V} \) is closed and it satisfy any one of the equivalent conditions of Theorem 2.2.

**Definition 2.4 (Algebraic Core of \((\sigma, V)\)).** Consider a CB-representation \((\sigma, V)\) of \( E \) on \( \mathcal{H} \). The algebraic core of \((\sigma, V)\), denote by \( C(\tilde{V}) \), is the greatest invariant subspace \( K \) of \( H \) such that \( \tilde{V}(E \otimes K) = K \).

Clearly for all CB-representation \((\sigma, V)\) of \( E \) on \( \mathcal{H} \), we have

\[ C(\tilde{V}) = \tilde{V}(E \otimes C(\tilde{V})) = \tilde{V}(E \otimes \tilde{V}(E \otimes C(\tilde{V}))) = \tilde{V}(I_E \otimes \tilde{V})(E \otimes E \otimes C(\tilde{V})) = \tilde{V}_2(E^\otimes 2 \otimes C(\tilde{V})) = \cdots = \tilde{V}_n(E^\otimes n \otimes C(\tilde{V})) \subseteq \tilde{V}_n(E^\otimes n \otimes H), \quad n \in \mathbb{N}. \]

It follows that \( C(\tilde{V}) \subseteq R^\infty(V) \).

The following theorem gives us an analytic approach to the algebraic core of \((\sigma, V)\).

**Theorem 2.5.** Consider a CB-representation \((\sigma, V)\) of \( E \) on \( \mathcal{H} \). If \( h \in C(\tilde{V}) \) then there is a sequence \((\xi_n)\) with \( h = \xi_0 \) and \( \xi_n \in E^\otimes n \otimes C(\tilde{V}) \), such that \( (I_E \otimes \tilde{V})(\xi_{n+1}) = \xi_n \).

**Proof.** Let \( h \in C(\tilde{V}) \), by definition of \( C(\tilde{V}) \) there is an element \( \xi_1 \in E \otimes C(\tilde{V}) \) such that \( h = \tilde{V}(\xi_1) \). Since \( \xi_1 \in E \otimes C(\tilde{V}) = (I_E \otimes \tilde{V})(E^\otimes 2 \otimes C(\tilde{V})) \), there is \( \xi_2 \in E^\otimes 2 \otimes C(\tilde{V}) \) such that \( (I_E \otimes \tilde{V})(\xi_2) = \xi_1 \), and so \( h = \tilde{V}(\xi_1) = \tilde{V}(I_E \otimes \tilde{V})(\xi_2) = \tilde{V}_2(\xi_2) \). More generally, there exists a sequence \((\xi_n)\), with \( \xi_n \in E^\otimes n \otimes C(\tilde{V}) \), for which \( h = \xi_0 \) and \( (I_E \otimes \tilde{V})(\xi_{n+1}) = \xi_n \). \( \square \)

The following theorem establishes the relationship between the generalized range and the algebraic core of \((\sigma, V)\).

**Theorem 2.6.** Consider a regular CB-representation \((\sigma, V)\) of \( E \) on \( \mathcal{H} \). Then

\[ \tilde{V}(E \otimes R^\infty(V)) = R^\infty(V) = C(\tilde{V}). \]

**Proof.** For \( h \in R^\infty(V) \) \((= \bigcap_{n \in \mathbb{N}_0} R(\tilde{V}_n))\), there is \( \eta_n \in E^\otimes n \otimes \mathcal{H} \) such that \( h = \tilde{V}_n(\eta_n) \). Now apply an operator \( V(\xi) \) both sides

\[ V(\xi)h = V(\xi)\tilde{V}_n(\eta_n) = \tilde{V}_{n+1}(\xi \otimes \eta_n) \in R(\tilde{V}_{n+1}), \quad n \in \mathbb{N}, \]

where \( \xi \in E \). Since \( \tilde{V}(\xi \otimes h) \in R(\tilde{V}_n) \) for every \( n \in \mathbb{N} \), \( \tilde{V}(\xi \otimes h) \in R^\infty(\tilde{V}) \). Thus, we obtain

\[ \tilde{V}(E \otimes R^\infty(V)) \subseteq R^\infty(V). \]

For the converse part, let \( h \in R^\infty(V) \), then there is a sequence \((\eta_n)\), where \( \eta_n \in E^\otimes n \otimes \mathcal{H} \), so that

\[ h = \tilde{V}(\eta_1) = \tilde{V}_2(\eta_2) = \cdots = \tilde{V}_{n+1}(\eta_{n+1}) = \tilde{V}(I_E \otimes \tilde{V}_n)(\eta_{n+1}) \]
and hence \( \eta_1 - (I_E \otimes \tilde{V}_n)(\eta_{n+1}) \in N(\tilde{V}) \). Since \((\sigma, V)\) is regular,

\[
\eta_1 - (I_E \otimes \tilde{V}_n)(\eta_{n+1}) \in E \otimes R^\infty(V).
\]

Since \(E \otimes R^\infty(V)\) is a subspace, we get \(\eta_1 \in E \otimes R^\infty(V)\). Thus, \(h \in \tilde{V}(E \otimes R^\infty(V))\) which gives \(R^\infty(V) \subseteq \tilde{V}(E \otimes R^\infty(V))\), then \(\tilde{V}(E \otimes R^\infty(V)) = R^\infty(V)\) and by the definition of \(C(\tilde{V})\), we deduce that \(R^\infty(V) \subseteq C(\tilde{V})\). Since \(C(\tilde{V}) \subseteq R^\infty(V)\), we get \(R^\infty(V) = C(\tilde{V})\) which completes the proof of the theorem. \(\square\)

3. Regularity condition of the generalized inverse of a covariant representation

Definitions of the generalized inverse and bi-regularity of the covariant representation are provided in this section and derive some basic properties. Further, we prove the generalized range of a covariant representation is invariant under the generalized inverse.

**Definition 3.1.** Consider a CB-representation \((\sigma, V)\) of \(E\) on \(\mathcal{H}\). A bounded operator \(S : \mathcal{H} \to E \otimes \mathcal{H}\) is said to be a generalized inverse of \(\tilde{V}\) if \(\tilde{V}S\tilde{V} = \tilde{V}\) and \(S\tilde{V}S = S\).

Assume \((\sigma, V)\) to be a CB-representation of \(E\) on \(\mathcal{H}\) and let \(S\) be a generalized inverse of \(\tilde{V}\). For \(n \in \mathbb{N}\), define \(S^{(n)} : H \to E_{\otimes n} \otimes \mathcal{H}\) by

\[
S^{(n)} := (I_{E_{\otimes n-1}} \otimes S)(I_{E_{\otimes n-2}} \otimes S) \cdots (I_E \otimes S)S.
\]

Observe that

\[
(I_{E_{\otimes m}} \otimes S^{(n)})S^{(m)} = S^{(m+n)}, \quad m, n \in \mathbb{N}.
\]

Note that if \((\sigma, V)\) is isometric, then \(S = \tilde{V}^*\).

The next lemma will take a critical role in defining the bi-regularity condition of the CB-representation.

**Lemma 3.2.** Consider a CB-representation \((\sigma, V)\) of \(E\) on \(\mathcal{H}\). Suppose \(S\) is a generalized inverse of \(\tilde{V}\). Then for every \(m, n \in \mathbb{N}\),

\[
S^{(m)}N(S^{(m+n)}) = N(I_{E_{\otimes m}} \otimes S^{(n)}) \cap S^{(m)}(\mathcal{H}).
\]

**Proof.** Let \(n, m \in \mathbb{N}\) and \(h \in N(S^{(m+n)})\), then \((I_{E_{\otimes m}} \otimes S^{(n)})S^{(m)}h = S^{(m+n)}h = 0\) and thus \(S^{(m)}h \in N(I_{E_{\otimes m}} \otimes S^{(n)})\). Therefore, we obtain

\[
S^{(m)}N(S^{(m+n)}) \subseteq N(I_{E_{\otimes m}} \otimes S^{(n)}) \cap S^{(m)}(\mathcal{H}).
\]

On the other hand, let \(\xi \in N(I_{E_{\otimes m}} \otimes S^{(n)}) \cap S^{(m)}(\mathcal{H})\), then \((I_{E_{\otimes m}} \otimes S^{(n)})\xi = 0\) and \(\xi = S^{(m)}h\) for some \(h \in \mathcal{H}\). Consequently

\[
S^{(m+n)}h = (I_{E_{\otimes m}} \otimes S^{(n)})S^{(m)}h = (I_{E_{\otimes m}} \otimes S^{(n)})\xi = 0
\]

which implies \(h \in N(S^{(m+n)})\). Hence \(\xi = S^{(m)}h \in S^{(m)}(N(S^{(m+n)}))\), so the opposite inclusion is verified. \(\square\)

The following result presents some valuable connections between the kernel of generalized inverse and the generalized range, which leads to defining the bi-regularity condition.
Theorem 3.3. Consider a CB-representation $(\sigma, V)$ of $E$ on $H$ and let $S$ be a generalized inverse of $\tilde{V}$. Then the following statements are equivalent:

1. $N(I_{E^\otimes m} \otimes S) \subseteq R(S^{(m)})$ for each $m \in \mathbb{N}$;
2. $N(I_{E} \otimes S^{(n)}) \subseteq R(S)$ for each $n \in \mathbb{N}$;
3. $N(I_{E^\otimes m} \otimes S^{(n)}) \subseteq R(S^{(m)})$ for each $n, m \in \mathbb{N}$;
4. $N(I_{E^\otimes m} \otimes S^{(n)}) = S^{(m)}(N(S^{(m+n)}))$ for each $n, m \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2): We will prove it by Mathematical induction. For $n = 1$, we need to show that

$$N(I_{E} \otimes S) \subseteq R(S),$$

which is true when we substitute $\sim \xi$ to prove that (2) holds for $\sigma$. Therefore, there is $h$ such that

$$\sim \xi \in N(I_{E} \otimes S^{(k+1)}),$$

then

$$I_{E^\otimes k+1} \otimes S(I_{E} \otimes S^{(k)})(\xi) = (I_{E} \otimes (I_{E^\otimes k} \otimes S^{(k)}))(\xi) = (I_{E} \otimes S^{(k+1)})(\xi) = 0.$$}

Therefore $(I_{E} \otimes S^{(n)})(\xi) \in N(I_{E^\otimes k+1} \otimes S) \subseteq R(S^{(k+1)})$, here last inequality follows from (1). Therefore, there is $h \in N(S^{(k+2)})$ so that $(I_{E} \otimes S^{(k)})(\xi) = S^{(k+1)}(h) = (I_{E} \otimes S^{(k)})S(h).$ Thus we get $\xi - S(h) \in N(I_{E} \otimes S^{(k)}) \subseteq R(S).$ Hence $\xi \in R(S).

(2) $\Rightarrow$ (3): We will prove inequality (3) by Mathematical induction. For $m = 1$, nothing to prove. Suppose (3) is valid for $m = k$. Now we need to show that it is valid for $m = k + 1$, that is,

$$N(I_{E^\otimes k+1} \otimes S^{(n)}) \subseteq R(S^{(k+1)}).$$

Let $\xi \in N(I_{E^\otimes k+1} \otimes S^{(n)})$, then by (2) $N(I_{E^\otimes k+1} \otimes S^{(n)}) \subseteq (I_{E^\otimes k} \otimes S^{(n+1)})$, therefore there exists $\eta \in N(I_{E^\otimes k} \otimes S^{(n+1)})$ such that $\xi = (I_{E^\otimes k} \otimes S^{(n+1)})(\eta)$. Thus $\xi \in R(S^{(k+1)})$. If it follows that $\xi = (I_{E^\otimes k} \otimes S^{(n+1)})(\eta) \in R(S^{(k+1)})$ and hence $N(I_{E^\otimes k+1} \otimes S^{(n)}) \subseteq R(S^{(k+1)}).$

(3) $\Rightarrow$ (4): Suppose $N(I_{E^\otimes m} \otimes S^{(n)}) \subseteq R(S^{(m)})$, for $m, n \in \mathbb{N}$. By Lemma 3.2 and (3), we get

$$S^{(m)}N(S^{(m+n)}) = N(I_{E^\otimes m} \otimes S^{(n)}) \cap S^{(m)}(H) \cap N(I_{E^\otimes m} \otimes S^{(n)}),$$

which proves (4).

(4) $\Rightarrow$ (1): Trivial. □

The following definition draws inspiration from a recent article by Ezzahraoui, Mbekhta, and Zerouali in [S].

Definition 3.4. Consider a regular CB-representation $(\sigma, V)$ of $E$ on $H$ and let $S$ be a generalized inverse of $\tilde{V}$. We say that $(\sigma, V)$ is bi-regular if its generalized inverse $S$ satisfies any one of conditions of Theorem 3.3.

Theorem 3.5. Consider a regular CB-representation $(\sigma, V)$ of $E$ on $H$. If $S$ is the generalized inverse of $\tilde{V}$, then $\tilde{V}_{n}S^{(n)}\tilde{V}_{n} = \tilde{V}_{n}$, $n \in \mathbb{N}$. 


Proof. For every \( k \geq 1 \), we begin by demonstrating that \((I_E \otimes S^{(k)})N(\tilde{V}) \subseteq N(\tilde{V}_{k+1})\). For that, we need to prove the following inequality

\[(3.2)\quad (I_{E^{\otimes n}} \otimes S)N(\tilde{V}_n) \subseteq N(\tilde{V}_{n+1}), \quad \text{where } n \in \mathbb{N}.
\]

Let \( \xi \in N(\tilde{V}_n) \subseteq (I_{E^{\otimes n}} \otimes \tilde{V})(E^{\otimes n+1} \otimes \mathcal{H}) \) (using by Theorem 2.2), there is \( \eta \in E^{\otimes n+1} \otimes \mathcal{H} \) such that \( \xi = (I_{E^{\otimes n}} \otimes \tilde{V})(\eta) \). Observe that

\[
\tilde{V}_{n+1}(I_{E^{\otimes n}} \otimes S)(\xi) = \tilde{V}_n(I_{E^{\otimes n}} \otimes \tilde{V}S\tilde{V})(\eta) = \tilde{V}_n(I_{E^{\otimes n}} \otimes \tilde{V})(\eta) = 0
\]

and hence the Inequality \( (3.2) \). Using the Inequality \( (3.2) \), it is easy to see that

\[(3.3)\quad (I_E \otimes S^{(k)})N(\tilde{V}) \subseteq N(\tilde{V}_{k+1})\]

Now, we show that \( \tilde{V}_nS^{(n)}\tilde{V}_n = \tilde{V}_n \) for every \( n \in \mathbb{N} \). Consider

\[
\tilde{V}_nS^{(n)}\tilde{V}_n - \tilde{V}_n = \tilde{V}_n\left(\sum_{k=1}^{n}(I_{E^{\otimes k}} \otimes S^{(n-k+1)}\tilde{V}_{n-k+1}) - (I_{E^{\otimes n}} \otimes S^{(n-k)}\tilde{V}_{n-k})\right)
\]

\[
= \sum_{k=1}^{n} \tilde{V}_n(I_{E^{\otimes k}} \otimes S^{(n-k)})((I_{E^{\otimes k-1}} \otimes S\tilde{V}) - I_{E^{\otimes k} \otimes H})(I_{E^{\otimes k}} \otimes \tilde{V}_{n-k}).
\]

Note that \((I_{E^{\otimes k-1}} \otimes S\tilde{V}) - I_{E^{\otimes k} \otimes H}\) is bi-regular for every \( \xi_k \in E^{\otimes k} \otimes \mathcal{H} \) and Equations \( (3.2) \) and \( (3.3) \) follows that

\[
(I_{E^{\otimes k}} \otimes S^{(n-k)})(I_{E^{\otimes k-1}} \otimes S\tilde{V}) - I_{E^{\otimes k} \otimes H}\) \( \in \mathcal{N}(\tilde{V}_n), \)

for \( 1 \leq k \leq n \). Therefore,

\[
\tilde{V}_n(I_{E^{\otimes k}} \otimes S^{(n-k)})(I_{E^{\otimes k-1}} \otimes S\tilde{V}) - I_{E^{\otimes k} \otimes H} = 0.
\]

So, we get \( \tilde{V}_nS^{(n)}\tilde{V}_n = \tilde{V}_n \), for \( n \in \mathbb{N} \). \( \square \)

**Corollary 3.6.** Consider a bi-regular CB-representation \((\sigma, V)\) of \( E \) on \( \mathcal{H} \) and let \( S \) be a generalized inverse of \( \tilde{V} \). Then for each \( n \in \mathbb{N} \), \( S^{(n)} \) is a generalized inverse of \( \tilde{V}_n \), that is,

\[
S^{(n)}\tilde{V}_nS^{(n)} = S^{(n)} \quad \text{and} \quad \tilde{V}_nS^{(n)}\tilde{V}_n = \tilde{V}_n.
\]

**Proof.** Since \( S \) is a generalized inverse of \( \tilde{V} \), \( \tilde{V}S\tilde{V} = \tilde{V} \) and \( S\tilde{V}S = S \). Now \((\sigma, V)\) is bi-regular therefore by Theorem 3.5, we can see that \( \tilde{V}_nS^{(n)}\tilde{V}_n = \tilde{V}_n \) for \( n \in \mathbb{N} \). So it is sufficient to prove \( S^{(n)}\tilde{V}_nS^{(n)} = S^{(n)} \). For every \( k \in \mathbb{N} \), we begin by demonstrating that

\[(3.4)\quad \tilde{V}_k(N(I_{E^{\otimes k}} \otimes S)) \subseteq N(S^{(k+1)}).
\]

Let \( \xi \in N(I_E \otimes S^{(n)}) \subseteq R(S) \) for every \( n \in \mathbb{N} \) (using the fact that \((\sigma, V)\) is bi-regular), there is \( h \in \mathcal{H} \) such that \( \xi = Sh \). Note that

\[
S^{(n+1)}\tilde{V}\xi = S^{(n+1)}\tilde{V}Sh = (I_E \otimes S^{(n)})S\tilde{V}Sh = (I_E \otimes S^{(n)})Sh = (I_E \otimes S^{(n)})\xi = 0.
\]

It implies

\[(3.5)\quad \tilde{V}(N(I_E \otimes S^{(n)})) \subseteq N(S^{(n+1)}), \quad n \in \mathbb{N}.
\]
Using Equation (3.3) and by Mathematical induction, we get (3.4).
Now we show that \( S^{(n)} \tilde{V}_n S^{(n)} = S^{(n)} \) for \( n \in \mathbb{N} \). Consider
\[
S^{(n)} \tilde{V}_n S^{(n)} - S^{(n)} = \sum_{k=1}^{n} \tilde{V}_{n-k}(I_{E^\otimes n-k} \otimes \tilde{V})(I_{E^\otimes n-k} \otimes S)S^{(n-k)} - \tilde{V}_{n-k}S^{(n-k)} \]
\[
= \sum_{k=1}^{n} S^{(n)} \tilde{V}_{n-k}((I_{E^\otimes n-k} \otimes \tilde{V}S) - I_{E^\otimes n-k} \otimes H)S^{(n-k)},
\]
Note that \((I_{E^\otimes n-k} \otimes \tilde{V}S) - I_{E^\otimes n-k} \otimes H)\xi_{n-k} = (I_{E^\otimes n-k} \otimes (\tilde{V}S - I_{H}))\xi_{n-k} \in N(I_{E^\otimes n-k} \otimes S)\) for every \( \xi_{n-k} \in E^\otimes n-k \otimes H \), \( 1 \leq k \leq n \) and Equations (3.3) and (3.4) follows that
\[
\tilde{V}_{n-k}((I_{E^\otimes n-k} \otimes \tilde{V}S) - I_{E^\otimes n-k} \otimes H)\xi_{n-k} \in N(S^{(n-k+1)}) \subseteq N(S^{(n)}),
\]
for every \( 1 \leq k \leq n \). Therefore,
\[
S^{(n)} \tilde{V}_{n-k}((I_{E^\otimes n-k} \otimes \tilde{V}S) - I_{E^\otimes n-k} \otimes H) = 0.
\]
So we deduce from the above fact that \( S^{(n)} \tilde{V}_n S^{(n)} = S^{(n)} \), \( n \in \mathbb{N} \).

**Remark 3.7.** Consider a regular CB-representation \((\sigma, V)\) of \( E \) on \( H \) and let \( S \) be a generalized inverse of \( \tilde{V} \). Then
\[
R^\infty(V) = \{ h \in H : \tilde{V}_n S^{(n)} h = h, \text{ for all } n \in \mathbb{N} \}.
\]

Indeed, let \( h \in R^\infty(V) \), then for each \( n \in \mathbb{N} \) there is \( \xi_n \in E^\otimes n \otimes H \) such that \( \tilde{V}_n(\xi_n) = h \), for every \( n \in \mathbb{N} \). Now, apply \( \tilde{V}_n S^{(n)} \) in both sides, we get
\[
\tilde{V}_n S^{(n)} \tilde{V}_n \xi_n = \tilde{V}_n S^{(n)} h.
\]
Now by using Theorem 3.5 we have \( \tilde{V}_n \xi_n = \tilde{V}_n S^{(n)} h \). This implies \( h = \tilde{V}_n S^{(n)} h \), for all \( n \in \mathbb{N} \).

On the other hand, let \( h \in H \) be such that \( \tilde{V}_n S^{(n)} h = h \), for all \( n \in \mathbb{N} \). Then \( h \in R(\tilde{V}_n) \) and hence \( h \in R^\infty(V) \). Now we recall the definition of invariant subspace (cf. [25]) for the covariant representation \((\sigma, V)\).

**Definition 3.8.**
(1) Consider a CB-representation \((\sigma, V)\) of \( E \) on a Hilbert space \( H \) and suppose \( K \) is a closed subspace of \( H \). Then we say \( K \) is \((\sigma, V)\)-invariant if it is \( \sigma(B) \)-invariant and, is invariant by each operator \( V(\xi), \xi \in E \). In addition, if \( K^\perp \) is invariant by \( V(\xi) \) for \( \xi \in E \), then we say \( K \) is \((\sigma, V)\)-reducing. Restricting naturally this representation we get another representation of \( E \) on \( K \) which will be denoted as \((\sigma, V)|_K\).
(2) A closed subspace \( W \) of \( H \) is called wandering subspace for \((\sigma, V)\), if it is \( \sigma(B) \)-invariant and \( W \perp \tilde{V}_n( E^\otimes n \otimes W) \) for every \( n \in \mathbb{N} \). The representation \((\sigma, V)\) has
generating wandering subspace property (GWS-property) if there is a wandering subspace $W$ of $H$ satisfying

$$ \mathcal{H} = \bigvee_{n \in \mathbb{N}_0} \tilde{V}_n(E^\otimes n \otimes W) $$

and the corresponding wandering subspace $W$ is called generating wandering subspace (GWS).

**Corollary 3.9.** Consider a regular CB-representation $(\sigma, V)$ of $E$ on $H$ and let $S$ be a generalized inverse of $\tilde{V}$. Then

$$ S(R^\infty(V)) \subseteq E \otimes R^\infty(V). $$

**Proof.** Let $h \in R^\infty(V)$, then by Remark 3.7, $\tilde{V}Sh = h$ and thus there exist $\xi_n \in E^\otimes n \otimes \mathcal{H}$, $n \in \mathbb{N}$, such that

$$ \tilde{V}Sh = \tilde{V}_{n+1}(\xi_{n+1}) = \tilde{V}(I_E \otimes \tilde{V}_n)(\xi_{n+1}), $$

which implies that

$$ Sh - (I_E \otimes \tilde{V}_n)(\xi_{n+1}) \in N(\tilde{V}) \subseteq (I_E \otimes \tilde{V}_n)(E^\otimes (n+1) \otimes \mathcal{H}), $$

here the last inequality follows from Theorem 2.2. Let $\eta := Sh - (I_E \otimes \tilde{V}_n)(\xi_{n+1})$. Therefore,

$$ Sh = \eta + (I_E \otimes \tilde{V}_n)(\xi_{n+1}) \in (I_E \otimes \tilde{V}_n)(E^\otimes (n+1) \otimes \mathcal{H}), \quad n \in \mathbb{N}. $$

Hence $Sh \in E \otimes R(\tilde{V}_n)$ for every $n \in \mathbb{N}$, $Sh \in E \otimes R^\infty(V)$. Thus $S(R^\infty(V)) \subseteq E \otimes R^\infty(V)$. □

## 4. The Moore-Penrose Inverse and the Reduced Minimum Modulus

We begin this section by defining the Moore-Penrose inverse for $\tilde{V}$ and brief details of the Moore-Penrose inverse (for more details see [6]).

Consider a CB-representation $(\sigma, V)$ of $E$ on $H$ such that $\tilde{V}$ has closed range. If there is a unique operator $\tilde{V}^\dagger \in \mathcal{B}(\mathcal{H}, E \otimes \mathcal{H})$ so that

1. $N(\tilde{V}^\dagger) = R(\tilde{V})^\perp = N(\tilde{V}^*)$ and
2. $\tilde{V}^\dagger \tilde{V} \xi = \xi$, for $\xi \in N(\tilde{V})^\perp$,

then the operator $\tilde{V}^\dagger$ is called the Moore-Penrose inverse (MPI) of $\tilde{V}$ and it is satisfies the following:

$$ \tilde{V} = \tilde{V} \tilde{V}^\dagger \tilde{V}, \quad \tilde{V}^\dagger = \tilde{V}^\dagger \tilde{V} \tilde{V}^\dagger, \quad \tilde{V} \tilde{V}^\dagger = (\tilde{V} \tilde{V}^\dagger)^*, \quad \tilde{V}^\dagger \tilde{V} = (\tilde{V}^\dagger \tilde{V})^*. $$

**Remark 4.1.** Observe that $\tilde{V}^\dagger = (\tilde{V}^* \tilde{V})^{-1} \tilde{V}^*$ when $\tilde{V}$ has left inverse and if $\tilde{V}$ has right inverse, then $\tilde{V}^\dagger$ will be right inverse and equal to $\tilde{V}^*(\tilde{V} \tilde{V}^*)^{-1}$.

In the following definition, we introduce a notion of reduced minimum modulus (cf. [1], [2]) for the CB-representations of $E$ on $\mathcal{H}$:
Proposition 4.3. Consider a regular CB-representation \((\sigma,V)\) of \(E\) on \(\mathcal{H}\). Then for each \(n \in \mathbb{N}\),
\[
\gamma(\tilde{V}_n) \geq \gamma(\tilde{V})\gamma(I_E \otimes \tilde{V}) \cdots \gamma(I_{E^\otimes n-1} \otimes \tilde{V}).
\]

Proof. We prove it by Mathematical induction. The case \(n = 1\) is trivial. Assume that
\[
\gamma(\tilde{V}_n) \geq \gamma(\tilde{V})\gamma(I_E \otimes \tilde{V}) \cdots \gamma(I_{E^\otimes n-1} \otimes \tilde{V}).
\]

For \(\xi \in E^\otimes(n+1) \otimes \mathcal{H}\) and \(\eta \in N(\tilde{V}_{n+1})\), we have
\[
\text{dist}(\xi, N(\tilde{V}_{n+1})) = \text{dist}(\xi - \eta, N(\tilde{V}_{n+1})) \leq \text{dist}(\xi - \eta, N(I_{E^\otimes n} \otimes \tilde{V})),
\]
as \(N(I_{E^\otimes n} \otimes \tilde{V}) \subseteq N(\tilde{V}_{n+1})\), for every \(n \in \mathbb{N}\). By assumption \((\sigma,V)\) is regular and by (4) of Theorem 2.2,
\[
N(\tilde{V}_n) = (I_{E^\otimes n} \otimes \tilde{V})(N(\tilde{V}_{n+1}))
\]
and therefore
\[
\text{dist}((I_{E^\otimes n} \otimes \tilde{V})(\xi), N(\tilde{V}_n)) = \text{dist}((I_{E^\otimes n} \otimes \tilde{V})(\xi), (I_{E^\otimes n} \otimes \tilde{V})(N(\tilde{V}_{n+1})))
\]
\[
= \inf_{\eta \in N(\tilde{V}_{n+1})} \| (I_{E^\otimes n} \otimes \tilde{V})(\xi) - (I_{E^\otimes n} \otimes \tilde{V})(\eta) \|
\]
\[
= \inf_{\eta \in N(\tilde{V}_{n+1})} \| (I_{E^\otimes n} \otimes \tilde{V})(\xi - \eta) \|
\]
\[
\geq \gamma(I_{E^\otimes n} \otimes \tilde{V}) \inf_{\eta \in N(\tilde{V}_{n+1})} \text{dist}((\xi - \eta), N(I_{E^\otimes n} \otimes \tilde{V}))
\]
\[
\geq \gamma(I_{E^\otimes n} \otimes \tilde{V}) \text{dist}(\xi, N(\tilde{V}_{n+1})).
\]

From the above observation, we get
\[
\| \tilde{V}_{n+1}(\xi) \| = \| \tilde{V}_n(I_{E^\otimes n} \otimes \tilde{V})(\xi) \| \geq \gamma(\tilde{V}_n) \text{dist}((I_{E^\otimes n} \otimes \tilde{V})(\xi), N(\tilde{V}_n))
\]\[
\geq \gamma(\tilde{V}_n)\gamma(I_{E^\otimes n} \otimes \tilde{V}) \text{dist}(\xi, N(\tilde{V}_{n+1})).
\]

Consequently, from our induction hypothesis
\[
\gamma(\tilde{V}_{n+1}) \geq \gamma(\tilde{V}_n)\gamma(I_{E^\otimes n} \otimes \tilde{V}) \geq \gamma(\tilde{V})\gamma(I_E \otimes \tilde{V})\gamma(I_{E^\otimes 2} \otimes \tilde{V}) \cdots \gamma(I_{E^\otimes n} \otimes \tilde{V}),
\]
which completes the proof. \(\square\)

Corollary 4.4. Consider a regular CB-representation \((\sigma,V)\) of \(E\) on \(\mathcal{H}\). Then
\begin{enumerate}
  \item \(R^\infty(V)\) is closed.
  \item \(\tilde{V}\) is left invertible if \(R^\infty(V) = 0\).
\end{enumerate}
Proof. (1) Since \((\sigma, V)\) is a regular CB-representation of \(E\) on \(\mathcal{H}\), \(R(V)\) is closed. Thus for each \(n \in \mathbb{N}\), we have \(R(I_{E^\otimes n} \otimes \tilde{V})\) is closed and hence using Proposition 4.3 \(\gamma(V_n) > 0\). From this, we conclude that \(R(V_n)\) is closed for every \(n \in \mathbb{N}\), and therefore \(R(V)\) is closed.

(2) Since \(R(V) = 0\) and \((\sigma, V)\) is regular, \(\tilde{V}\) is injective and \(R(\tilde{V})\) is closed. Therefore \(\tilde{V}\) is left invertible. \(\square\)

The following proposition establishes the relationship between the reduced minimum modulus and MPI (see [12, Corollary 2.3]). The proof of the following proposition is similar to the operator theory case, so we can omit it.

**Proposition 4.5.** Consider a CB-representation \((\sigma, V)\) of \(E\) on \(\mathcal{H}\) such that \(\tilde{V}\) has closed range. Then

\[
\|\tilde{V}^\dagger\| = \frac{1}{\gamma(V)}.
\]

The following proposition explains numerous essential facts about MPI of \(\tilde{V}\), which will come in handy throughout the sequel. We omit the proof of this proposition, which is straightforward.

**Proposition 4.6.** Suppose \((\sigma, V)\) is a CB-representation of \(E\) on \(\mathcal{H}\) such that \(\tilde{V}\) has closed range. Then

1. \(\tilde{V}V^\dagger = P_{R(\tilde{V})}\) and \(V^\dagger \tilde{V} = P_{N(\tilde{V})}^\perp\),
2. \(R(\tilde{V}^\dagger) = R(\tilde{V}^*) = N(\tilde{V})^\perp\),
3. \(N(\tilde{V}^\dagger) = N(\tilde{V}V^\dagger) = N(\tilde{V}^*) = R(\tilde{V})^\perp\),
4. \(R(\tilde{V}) = R(\tilde{V}V^\dagger) = R(\tilde{V}^*)\),
5. \(N(\tilde{V}) = N(\tilde{V}^* V) = N(\tilde{V}^*)\),
6. \((\tilde{V}^*)^\dagger = (\tilde{V}^\dagger)^*\),
7. \((\tilde{V}^\dagger)^\dagger = \tilde{V}\),
8. \(\tilde{V}^* \tilde{V}^\dagger = \tilde{V}^\dagger \tilde{V}^* = \tilde{V}^*\).

Consider a CB-representation \((\sigma, V)\) of \(E\) on \(\mathcal{H}\) with \(\tilde{V}\) has closed range. For \(n \in \mathbb{N}\), define

\[
\tilde{V}^\dagger(n) := (I_{E^\otimes n-1} \otimes \tilde{V}^\dagger)(I_{E^\otimes n-2} \otimes \tilde{V}^\dagger) \cdots (I_E \otimes \tilde{V}^\dagger)\tilde{V}^\dagger.
\]

Note that the equality \(\tilde{V}^\dagger(n) = \tilde{V}^\dagger_n, n \geq 2\), is not true even if \(\tilde{V}\) is left invertible (cf. Example 4), where \(\tilde{V}^\dagger_n\) is MPI of \(V_n\). Moreover, if \(\tilde{V}\) is left invertible, so is \(\tilde{V}_n\). Thus

\[
\tilde{V}^\dagger(n) = (I_{E^\otimes n-1} \otimes (\tilde{V}^* \tilde{V})^{-1} \tilde{V}^*) (I_{E^\otimes n-2} \otimes (\tilde{V}^* \tilde{V})^{-1} \tilde{V}^*) \cdots (I_E \otimes (\tilde{V}^* \tilde{V})^{-1} \tilde{V}^*) (\tilde{V}^* \tilde{V})^{-1} \tilde{V}^*,
\]

but \(\tilde{V}^\dagger_n = (\tilde{V}^\dagger_n \tilde{V}_n)^{-1} \tilde{V}^*_n\). For this purpose, we define the following: For \(n \in \mathbb{N}\), the covariant representation \((\sigma, V)\) is said to be \(n\)-\textit{dagger} if \(\tilde{V}^\dagger(n) = \tilde{V}^\dagger_n\), and is said to be \(n\)-\textit{hyper-dagger} if it is \(n\)-dagger for all \(n \in \mathbb{N}\).
Proposition 4.7. Consider a regular CB-representation $(\sigma, V)$ of $E$ on $\mathcal{H}$. If $R(\tilde{V}^\dagger(n)) \subseteq N(\tilde{V}_n)^\perp$, $n \geq 2$, then the restriction map

$$\tilde{V}_n|_{(E^\otimes_n \odot R^\infty(V)) \cap R(\tilde{V}^\dagger(n))} : (E^\otimes_n \odot R^\infty(V)) \cap R(\tilde{V}^\dagger(n)) \to R^\infty(V)$$

is bijective. In particular,

$$\tilde{V}_n|_{(E \otimes R^\infty(V)) \cap N(\tilde{V})^\perp} : (E \otimes R^\infty(V)) \cap N(\tilde{V})^\perp \to R^\infty(V)$$

is bijective.

Proof. Since $(\sigma, V)$ is regular, by Theorem 2.6, $\tilde{V}(E \otimes R^\infty(V)) = R^\infty(V)$ and thus $\tilde{V}_n(E^\otimes_n \odot R^\infty(V)) = R^\infty(\tilde{V})$, for $n \in \mathbb{N}$. Also,

$$\tilde{V}_n((E^\otimes_n \odot R^\infty(V)) \cap R(\tilde{V}^\dagger(n))) \subseteq \tilde{V}_n(E^\otimes_n \odot R^\infty(V)) = R^\infty(V).$$

For $h \in R^\infty(V)$ and using Theorem 3.7, $h = \tilde{V}_n\tilde{V}^\dagger(n)h$, $n \in \mathbb{N}$. By Corollary 3.9, $\tilde{V}^\dagger(R^\infty(\tilde{V})) \subseteq E \otimes R^\infty(\tilde{V})$ and hence $\tilde{V}^\dagger(n)(h) \in (E^\otimes_n \odot R^\infty(V)) \cap R(\tilde{V}^\dagger(n))$. Therefore

$$h = \tilde{V}_n(\tilde{V}^\dagger(n)h) \in \tilde{V}_n((E^\otimes_n \odot R^\infty(V)) \cap R(\tilde{V}^\dagger(n))),$$

which proves $\tilde{V}_n|_{(E^\otimes_n \odot R^\infty(V)) \cap R(\tilde{V}^\dagger(n))}$ is onto. Now, we need to show that $\tilde{V}_n|_{(E^\otimes_n \odot R^\infty(V)) \cap R(\tilde{V}^\dagger(n))}$ is injective. Suppose $\xi \in (E^\otimes_n \odot R^\infty(V)) \cap R(\tilde{V}^\dagger(n))$ is such that $\tilde{V}_n|_{(E^\otimes_n \odot R^\infty(V)) \cap R(\tilde{V}^\dagger(n))}\xi = 0$. Then

$$\xi \in (E^\otimes_n \odot R^\infty(V)) \cap R(\tilde{V}^\dagger(n)) \cap N(\tilde{V}_n).$$

By hypothesis $R(\tilde{V}^\dagger(n)) \subseteq N(\tilde{V}_n)^\perp$,

$$\xi \in (E^\otimes_n \odot R^\infty(V)) \cap R(\tilde{V}^\dagger(n)) \cap N(\tilde{V}_n) = R(\tilde{V}^\dagger(n)) \cap N(\tilde{V}_n) = \{0\}.$$ 

This shows that $\tilde{V}_n|_{(E^\otimes_n \odot R^\infty(V)) \cap R(\tilde{V}^\dagger(n))}$ is injective. Note that, for $n = 1$, $R(\tilde{V}^\dagger) = N(\tilde{V})^\perp$. Hence $\tilde{V}_n|_{(E^\otimes_n \odot R^\infty(V)) \cap R(\tilde{V}^\dagger(n))}$ is bijective, for $n \in \mathbb{N}$. \qed

Proposition 4.8. Suppose $(\sigma, V)$ is a regular CB-representation of $E$ on $\mathcal{H}$. Then

$$(\tilde{V}|_{E \otimes R^\infty(V)})^\dagger = \tilde{V}^\dagger|_{R^\infty(V)}.$$ 

Proof. Since $(\sigma, V)$ is a regular CB-representation of $E$ on $\mathcal{H}$, $N(\tilde{V}|_{E \otimes R^\infty(V)}) = N(\tilde{V})$. Let $A = \tilde{V}|_{E \otimes R^\infty(V)}$, then $N(A) = N(\tilde{V})$ and by Theorem 2.6, $R(A) = R^\infty(V)$. An operator $A_0 = A|_{N(A)}^\perp : N(A) \supseteq R(A)$ is bijection and MPI of $A$, $A^\dagger = A_0^{-1}P_{R^\infty(V)}$. Since $\tilde{V}^\dagger = \tilde{V}_0^{-1}P_{R(\tilde{V})}$ where $\tilde{V}_0 = \tilde{V}|_{N(\tilde{V})^\perp} : N(\tilde{V})^\perp \to R(\tilde{V})$, thus $\tilde{V}^\dagger|_{R^\infty(V)} = \tilde{V}_0^{-1}P_{R^\infty(V)}$.

Let $h \in R^\infty(V)$ then $\tilde{V}^\dagger(h) = \tilde{V}_0^{-1}h$ and there exists $\xi \in E \otimes R^\infty(\tilde{V}) \cap N(\tilde{V})^\perp$ such that $\tilde{V}_0^{-1}h = \xi$. Now, consider $\eta = A^\dagger h = A_0^{-1}h$ then $\tilde{V}_0\xi = A_0\eta$, since $\tilde{V}|_{E \otimes R^\infty(V)} = A_0$ and $\tilde{V}_0$ is bijective, we deduce that $\xi = \eta$. Therefore $\tilde{V}^\dagger|_{R^\infty(V)} = A^\dagger$, $(\tilde{V}|_{E \otimes R^\infty(V)})^\dagger = \tilde{V}^\dagger|_{R^\infty(V)}$. \qed

Theorem 4.9. Consider a CB-representation $(\sigma, V)$ of $E$ on $\mathcal{H}$ such that $\tilde{V}$ has closed range. The following are equivalent:
(1) \((\sigma, V)\) is regular.

(2) The map \(\hat{V}_n : E^{\otimes n} \otimes R(\hat{V})^\perp \to R(\hat{V}_n) \cap R(\hat{V}_{n+1})^\perp\) defined by

\[
\hat{V}_n \xi = P_{R(\hat{V}_n) \cap R(\hat{V}_{n+1})^\perp} \tilde{V}_n \xi, \quad \xi \in E^{\otimes n} \otimes R(\hat{V})^\perp, \ n \in \mathbb{N}
\]

is an invertible map.

**Proof.** (1) \(\implies\) (2): Since \((\sigma, V)\) is regular, \(R(\hat{V}_n)\) is closed and hence \(R(\hat{V}_n) \cap R(\hat{V}_{n+1})^\perp\) is closed. Therefore \(P_{R(\hat{V}_n) \cap R(\hat{V}_{n+1})^\perp}\) is continuous, so we conclude that \(\hat{V}_n\) is well-defined bounded operator for every \(n \in \mathbb{N}\).

First we will show that \(\hat{V}_n\) is injective. Let \(\xi \in E^{\otimes n} \otimes R(\hat{V})^\perp\) be such that \(\hat{V}_n(\xi) = 0\), i.e., \(P_{R(\hat{V}_n) \cap R(\hat{V}_{n+1})^\perp} \tilde{V}_n(\xi) = 0\). Since \(R(\hat{V}_n) = R(\hat{V}_{n+1}) \oplus (R(\hat{V}_n) \cap R(\hat{V}_{n+1})^\perp)\), \(\tilde{V}_n(\xi) \in R(\hat{V}_{n+1})\) and there exists \(\eta \in E^{\otimes n+1} \otimes H\) such that \(\tilde{V}_n(\xi) = \tilde{V}_{n+1}(\eta)\). Using \((\sigma, V)\) is regular, \(\xi - (I_{E^{\otimes n}} \otimes \tilde{V})\eta \in E^{\otimes n} \otimes R(\tilde{V})\) but \(\xi \in E^{\otimes n} \otimes R(\tilde{V})^\perp\), thus \(\xi = 0\) and hence \(\hat{V}_n\) is injective.

Now we need to show that \(\hat{V}_n\) is surjective. For \(h \in R(\hat{V}_n) \cap R(\hat{V}_{n+1})^\perp\) is non-zero, i.e., \(h = P_{R(\hat{V}_n) \cap R(\hat{V}_{n+1})^\perp} \tilde{V}_n (I_{E^{\otimes n}} \otimes \tilde{V})(\xi)\) for some \(\xi \in E^{\otimes (n+1)} \otimes H\). Since \(\tilde{V}_n V_n^\dagger = 0\), the orthogonal projection on \(R(\tilde{V}_n)\), \(h = \tilde{V}_n V_n^\dagger h = \tilde{V}_n (I_{E^{\otimes n}} \otimes \tilde{V})(\xi) = \tilde{V}_{n+1}(\xi) \in R(\tilde{V}_{n+1})\). Thus \(h \in R(\hat{V}_{n+1})^\perp \cap R(\tilde{V}_{n+1})^\perp\) and hence \(h = 0\), which is absurd due to \(h \neq 0\). Therefore \(\hat{V}_n^\dagger h \notin E^{\otimes n} \otimes R(\tilde{V})\) and \(\hat{V}_n^\dagger h = \eta + (I_{E^{\otimes n}} \otimes \tilde{V})(\xi)\) for some \(\xi \in E^{\otimes (n+1)} \otimes H\), \(\eta \in E^{\otimes n} \otimes R(\tilde{V})^\perp\). Now \(h = \tilde{V}_n \hat{V}_n V_n^\dagger h = \tilde{V}_n(\eta) + \tilde{V}_{n+1}(\xi)\) and \(h = P_{R(\hat{V}_n) \cap R(\hat{V}_{n+1})^\perp} \tilde{V}_n(\eta)\), we conclude that \(h = \hat{V}_n(\eta)\) for some \(\eta \in E^{\otimes n} \otimes R(\tilde{V})^\perp\). Therefore \(\hat{V}_n\) is onto.

(2) \(\implies\) (1): Suppose \(\hat{V}_n : E^{\otimes n} \otimes R(\tilde{V})^\perp \to R(\hat{V}_n) \cap R(\hat{V}_{n+1})^\perp\) is an invertible operator. For \(\xi \in N(\hat{V}_n)\), then we have to show that \(\xi \in E^{\otimes n} \otimes R(\tilde{V})\). Since \(R(\tilde{V})\) is closed, \(\xi = \eta + (I_{E^{\otimes n}} \otimes \tilde{V})(\kappa)\) for some \(\kappa \in E^{\otimes (n+1)} \otimes H\) and \(\eta \in E^{\otimes n} \otimes R(\tilde{V})^\perp\). Then \(\tilde{V}_n(\eta) + \tilde{V}_{n+1}(\kappa) = \tilde{V}_n(\eta + (I_{E^{\otimes n}} \otimes \tilde{V})(\kappa)) = \tilde{V}_n \xi = 0\) and hence \(0 = P_{R(\hat{V}_n) \cap R(\hat{V}_{n+1})^\perp} \tilde{V}_{n+1}(\kappa) = -P_{R(\hat{V}_n) \cap R(\hat{V}_{n+1})^\perp} \tilde{V}_n(\eta)\), therefore \(\eta \in N(\hat{V}_n)\). But by hypothesis, \(\hat{V}_n\) is injective, \(\eta = 0\). Thus \(\xi = (I_{E^{\otimes n}} \otimes \tilde{V})(\kappa) \in E^{\otimes n} \otimes R(\tilde{V})\), which deduces that \((\sigma, V)\) is regular.

**Corollary 4.10.** Consider a regular CB-representation \((\sigma, V)\) of \(E\) on \(H\). Then

\[
\dim R(\hat{V}_n) \cap R(\hat{V}_{n+1})^\perp = \dim(E^{\otimes n} \otimes R(\tilde{V})^\perp), \ n \in \mathbb{N}.
\]

5. Shimorin-Wold-type decomposition for regular CB-representation of \(E\) on \(H\)

Suppose \((\sigma, V)\) is a CB-representation of \(E\) on a Hilbert space \(H\) such that \(\hat{V}\) has closed range. We use the notations \(W^\dagger\) and \(W\) throughout this section for

\[
W^\dagger := (E \otimes H) \ominus \hat{V}^\dagger(H) = R(\hat{V}^\dagger)^\perp = N(\hat{V}^*) = N(\tilde{V}),
\]

\[
W := H \ominus \hat{V}(E \otimes H) = R(\hat{V})^\perp = N(\tilde{V}^*).\]
Suppose that $\gamma(\tilde{V}) \geq 1$, then by Proposition \[1.5\] $\tilde{V}^\dagger$ is contraction and hence

\begin{equation}
\|\tilde{V}^\dagger \tilde{V}\xi\|^2 \leq \|\tilde{V}\|^2 \|\tilde{V}\xi\|^2 \leq \|\tilde{V}\xi\|^2, \quad \xi \in E \otimes \mathcal{H}.
\end{equation}

As $\tilde{V}^\dagger \tilde{V}$ is the orthogonal projection onto $N(\tilde{V})^\perp$, $\tilde{V}^\dagger \tilde{V} - \tilde{V}^\dagger \tilde{V} \leq 0$. We denote $D_{\tilde{V}}$ by $(\tilde{V}^\dagger \tilde{V} - \tilde{V}^\dagger \tilde{V})^\frac{1}{2}$, i.e. $D_{\tilde{V}} := (\tilde{V}^\dagger \tilde{V} - \tilde{V}^\dagger \tilde{V})^\frac{1}{2}$. Clearly for every $\xi \in E \otimes \mathcal{H},$

$$
\|D_{\tilde{V}}\xi\|^2 = \|\tilde{V}\xi\|^2 - \|\tilde{V}^\dagger \tilde{V}\xi\|^2.
$$

Consider a CB-representation $(\sigma, V)$ of $E$ on $\mathcal{H}$. We call it is concave (cf. \[26\]) if the following inequality

$$
\|\tilde{V}_2(\xi)\|^2 + \|\xi\|^2 \leq 2\| (I_E \otimes \tilde{V})\xi \|
$$

holds for all $\xi \in E^\otimes 2 \otimes \mathcal{H}$. If $(\sigma, V)$ is isometric then it is concave.

The following lemma in the article \[26\], Lemma 2.2] provides a crucial aspect of the concave CB-representation of $E$ on $\mathcal{H}$.

**Lemma 5.1.** Consider a concave CB-representation $(\sigma, V)$ of $E$ on $\mathcal{H}$. Then

\begin{equation}
\|\tilde{V}_k(\xi_k)\|^2 \leq \|\xi_k\|^2 + k((I_{E^\otimes (k-1)} \otimes \tilde{V})(\xi_k))\|^2 - \|\xi_k\|^2, \quad \xi_k \in E^\otimes k \otimes \mathcal{H}, k \in \mathbb{N}.
\end{equation}

Note that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. Now we can take general sequence, $(d_k)_k, k \geq 2$, of non-negative terms such that $\sum_{m \geq 2} \frac{1}{d_m} = \infty$, in place of the sequence $\frac{1}{n}$, which generalize the Inequality \[5.2\] in more general form. At the beginning of this section, we discuss the results related to this generalization of Inequality \[5.2\].

The following proposition is a abstraction of \[21\], Lemma 1], \[17\], Proposition 2.1], \[7\], Proposition 7] and \[26\], Lemma 2.2]:

**Proposition 5.2.** Consider a CB-representation $(\sigma, V)$ of $E$ on $\mathcal{H}$ so that $\gamma(\tilde{V}) \geq 1$. Suppose $d>0$ and $(d_k)_k, (k \geq 2)$ is some non-negative sequence. Then the following two inequalities are equivalent:

\begin{equation}
\|\tilde{V}_k(\xi_k)\|^2 \leq d_k((I_{E^\otimes (k-1)} \otimes D_{\tilde{V}})(\xi_k))\|^2 + d((I_{E^\otimes (k-1)} \otimes \tilde{V}^\dagger \tilde{V})(\xi_k))\|^2
\end{equation}

for every $\xi_k \in E^\otimes k \otimes \mathcal{H}$, and

\begin{equation}
\|\tilde{V}_k(\eta_k)\|^2 \leq d_k((I_{E^\otimes (k-1)} \otimes \tilde{V})\eta_k)^2 - \|\eta_k\|^2 + d\|\eta_k\|^2,
\end{equation}

for every $\eta_k \in E^\otimes (k-1) \otimes N(\tilde{V})^\perp$.

**Proof.** Due to the fact that $\gamma(\tilde{V})$ is strictly positive, the operator $\tilde{V}^\dagger$ exists. First, assume that the Inequality \[5.3\] holds, i.e.,

$$
\|\tilde{V}_k(\xi_k)\|^2 \leq d_k((I_{E^\otimes (k-1)} \otimes D_{\tilde{V}})(\xi_k))\|^2 + d((I_{E^\otimes (k-1)} \otimes \tilde{V}^\dagger \tilde{V})(\xi_k))\|^2
$$

for every $\xi_k \in E^\otimes k \otimes \mathcal{H}$. Therefore it also holds for every $\eta_k \in E^\otimes (k-1) \otimes N(\tilde{V})^\perp \subseteq E^\otimes (k-1) \otimes \mathcal{H}$, hence

$$
\|\tilde{V}_k(\eta_k)\|^2 \leq d_k((I_{E^\otimes (k-1)} \otimes D_{\tilde{V}})(\eta_k))\|^2 + d((I_{E^\otimes (k-1)} \otimes \tilde{V}^\dagger \tilde{V})(\eta_k))\|^2.
$$

Since $\tilde{V}^\dagger \tilde{V} = P_{N(\tilde{V})}^\perp$, $I_{E^\otimes (k-1)} \otimes \tilde{V}^\dagger \tilde{V} = P_{E^\otimes (k-1) \otimes N(\tilde{V})^\perp}$, and $(I_{E^\otimes (k-1)} \otimes \tilde{V}^\dagger \tilde{V})(\eta_k) = \eta_k$. Thus

$$
\|\tilde{V}_k(\eta_k)\|^2 \leq d_k((I_{E^\otimes (k-1)} \otimes (\tilde{V}^\dagger \tilde{V} - \tilde{V}^\dagger \tilde{V})^\frac{1}{2})(\eta_k))\|^2 + d\|\eta_k\|^2.
$$
which is the required inequality. 

On the other hand, suppose that the Inequality (5.4) holds. Let \( \xi_k \in E^{\otimes k} \otimes \mathcal{H} \), then 
\[
(I_{E^{\otimes k-1}} \otimes \widetilde{V}^\dagger)(\xi_k) \in E^{\otimes k-1} \otimes N(\widetilde{V})^\perp.
\]

By substituting \((I_{E^{\otimes k-1}} \otimes \widetilde{V}^\dagger)(\xi_k) \in E^{\otimes k-1} \otimes N(\widetilde{V})^\perp\) for \( \eta_k \) in Inequality (5.4) and using \( \widetilde{V}\widetilde{V}^\dagger = \widetilde{V} \), we can easily obtain Inequality (5.3). \( \square \)

Lemma 5.3. Consider a CB-representation \((\sigma, V)\) of \( E \) on \( \mathcal{H} \) so that \( \gamma(\widetilde{V}) \geq 1 \). Then for \( n \in \mathbb{N} \),
\[
\|h\|^2 = \sum_{i=0}^{n-1} \|(I_{E^{\otimes i}} \otimes P_W)\widetilde{V}^\dagger(i)h\|^2 + \|(\widetilde{V}^\dagger)^{n-1}h\|^2 + \sum_{i=1}^{n} \|(I_{E^{\otimes(i-1)}} \otimes D_V)\widetilde{V}^\dagger(i)h\|^2, \quad h \in \mathcal{H}
\]
where \( P_W := I_{H} - \widetilde{V}\widetilde{V}^\dagger \).

Proof. We use the Mathematical induction. Let \( h \in \mathcal{H} \), then
\[
\|(I_{H} - \widetilde{V}\widetilde{V}^\dagger)h\|^2 = \|h\|^2 - \|\widetilde{V}\widetilde{V}^\dagger h\|^2,
\]
which is equivalent to
\[
\|h\|^2 = \|(I_{H} - \widetilde{V}\widetilde{V}^\dagger)h\|^2 + \|\widetilde{V}\widetilde{V}^\dagger h\|^2. \tag{5.5}
\]
Since \( \|D_V(\xi)\|^2 = \|\widetilde{V}(\xi)\|^2 - \|\widetilde{V}^\dagger(\xi)\|^2 \) for every \( \xi \in E \otimes \mathcal{H} \), replace \( \widetilde{V}^\dagger h \) by \( \xi \), we get
\[
\|D_V(\widetilde{V}^\dagger h)\|^2 = \|\widetilde{V}(\widetilde{V}^\dagger h)\|^2 - \|\widetilde{V}^\dagger(\widetilde{V}^\dagger h)\|^2 = \|\widetilde{V}(\widetilde{V}^\dagger h)\|^2 - \|\widetilde{V}^\dagger h\|^2.
\]
Therefore
\[
\|h\|^2 = \|(I_{H} - \widetilde{V}\widetilde{V}^\dagger)h\|^2 + \|D_V(\widetilde{V}^\dagger h)\|^2 + \|\widetilde{V}^\dagger h\|^2,
\]
and thus
\[
\|h\|^2 = \|P_W h\|^2 + \|D_V(\widetilde{V}^\dagger h)\|^2 + \|\widetilde{V}^\dagger h\|^2
\]
which establishes the equality for \( n = 1 \). Now suppose that the equality true for \( n \). We observe that
\[
\|\widetilde{V}^\dagger(n) h\|^2 = \|(I_{E^{\otimes n}} \otimes P_W)\widetilde{V}^\dagger(n) h\|^2 + \|(I_{E^{\otimes n}} \otimes D_V)(\widetilde{V}^\dagger(n) h)\|^2 + \|\widetilde{V}^\dagger(n+1) h\|^2
\]
for every \( n \in \mathbb{N} \). Using the induction hypothesis and the previous equation, we obtain
\[
\|h\|^2 = \sum_{i=0}^{n-1} \|(I_{E^{\otimes i}} \otimes P_W)\widetilde{V}^\dagger(i) h\|^2 + \|(\widetilde{V}^\dagger)^{n-1} h\|^2 + \sum_{i=1}^{n} \|(I_{E^{\otimes(i-1)}} \otimes D_V)\widetilde{V}^\dagger(i) h\|^2
\]
\[
= \sum_{i=0}^{n-1} \|(I_{E^{\otimes i}} \otimes P_W)\widetilde{V}^\dagger(i) h\|^2 + \|(I_{E^{\otimes n}} \otimes P_W)\widetilde{V}^\dagger(n) h\|^2 + \|(I_{E^{\otimes(n-1)}} \otimes D_V)(\widetilde{V}^\dagger n) h\|^2
\]
\[
+ \|\widetilde{V}^\dagger(n+1) h\|^2 + \sum_{i=1}^{n} \|(I_{E^{\otimes(i-1)}} \otimes D_V)\widetilde{V}^\dagger(i) h\|^2
\]
Lemma 5.4. Consider \( n \in \mathbb{N} \) and a CB-representation \((\sigma, V)\) of \( E \) on \( \mathcal{H} \) such that \( \tilde{V} \) has closed range. Then

(1) For every \( h \in \mathcal{H} \), we have

\[
(I_{\mathcal{H}} - \tilde{V}_n \tilde{V}^{\dagger(n)})h = \sum_{i=0}^{n-1} \tilde{V}_i (I_{E^{\otimes i}} \otimes P_W) \tilde{V}^{\dagger(i)}h;
\]

(2) For \( 0 \leq i \leq n \) and \( \xi_n \in E^{\otimes n} \otimes \mathcal{H} \), we get

\[
(I_{E^{\otimes n} \otimes \mathcal{H}} - \tilde{V}^{\dagger(n)} \tilde{V}_n)(\xi_n) = \sum_{i=0}^{n-1} (I_{E^{\otimes n-i}} \otimes \tilde{V}^{\dagger(i)})(I_{E^{\otimes n-(i+1)}} \otimes P_W)(I_{E^{\otimes n-i}} \otimes \tilde{V}_i)(\xi_n),
\]

where \( P_W = I_{\mathcal{H}} - \tilde{V} \tilde{V}^{\dagger} \), \( P_W := I_{E^{\otimes H}} - \tilde{V}^{\dagger} \tilde{V} \).

Proof. (1) Consider

\[
I_{\mathcal{H}} - \tilde{V}_n \tilde{V}^{\dagger(n)} = \sum_{i=0}^{n-1} \tilde{V}_i \tilde{V}^{\dagger(i)} - \tilde{V}_{i+1} \tilde{V}^{\dagger(i+1)} = \sum_{i=0}^{n-1} \tilde{V}_i \tilde{V}^{\dagger(i)} - \tilde{V}_i (I_{E^{\otimes i}} \otimes \tilde{V})(I_{E^{\otimes i}} \otimes \tilde{V}^{\dagger}) \tilde{V}^{\dagger(i)}
\]

\[
= \sum_{i=0}^{n-1} \tilde{V}_i (I_{E^{\otimes i} \otimes \mathcal{H}} - (I_{E^{\otimes i}} \otimes \tilde{V})(I_{E^{\otimes i}} \otimes \tilde{V}^{\dagger})) \tilde{V}^{\dagger(i)} = \sum_{i=0}^{n-1} \tilde{V}_i (I_{E^{\otimes i} \otimes \mathcal{H}} - (I_{E^{\otimes i}} \otimes \tilde{V} \tilde{V}^{\dagger})) \tilde{V}^{\dagger(i)}
\]

\[
= \sum_{i=0}^{n-1} \tilde{V}_i (I_{E^{\otimes i}} \otimes P_W) \tilde{V}^{\dagger(i)}.
\]

(2) For second equality

\[
I_{E^{\otimes n} \otimes \mathcal{H}} - \tilde{V}^{\dagger(n)} \tilde{V}_n = \sum_{i=0}^{n-1} I_{E^{\otimes n-(i+1)}} \otimes (I_E \otimes \tilde{V}^{\dagger(i)} \tilde{V}_i - \tilde{V}^{\dagger(i+1)} \tilde{V}_{i+1})
\]

\[
= \sum_{i=0}^{n-1} I_{E^{\otimes n-(i+1)}} \otimes (I_E \otimes \tilde{V}^{\dagger(i)} \tilde{V}_i - (I_E \otimes \tilde{V}^{\dagger(i)}) \tilde{V}^{\dagger}(I_E \otimes \tilde{V}_i))
\]

\[
= \sum_{i=0}^{n-1} (I_{E^{\otimes n-i}} \otimes \tilde{V}^{\dagger(i)})(I_{E^{\otimes n-(i+1)}} \otimes (I_{E^{\otimes \mathcal{H}}} - \tilde{V}^{\dagger}(I_{E^{\otimes \mathcal{H}}} - \tilde{V}^{\dagger}))(I_{E^{\otimes n-i}} \otimes \tilde{V}_i)
\]

\[
= \sum_{i=0}^{n-1} I_{E^{\otimes n-(i+1)}} \otimes ((I_E \otimes \tilde{V}^{\dagger(i)}) P_{W_i}(I_E \otimes \tilde{V}_i)).
\]
Theorem 5.5. Consider \( n \in \mathbb{N} \) and a CB-representation \((\sigma, V)\) of \( E \) on \( \mathcal{H} \) such that \( \tilde{V} \) has closed range. Then

\[
(1) \quad N(\tilde{V}^{(n)}) \subseteq \bigvee_{i=0}^{n-1} \{ \tilde{V}_i(\xi_i) : \xi_i \in E^{\otimes_i} \otimes \mathcal{W} \}.
\]

\[
(2) \quad \text{If} \ (\sigma, V) \ \text{is regular, we have} \ N(\tilde{V}_n) = \bigvee_{i=0}^{n-1} \{ (I_E^{\otimes_{n-i}} \otimes \tilde{V}^{(i)}) (\xi_{n-i}) : \xi_{n-i} \in E^{\otimes_{n-(i+1)}} \otimes \mathcal{W}^\dagger \}.
\]

Proof. (1) Let \( h \in N(\tilde{V}^{(n)}) \), then \( \tilde{V}_n \tilde{V}^{(n)} h = 0 \). Therefore, by Lemma \([5,4]\) we conclude

\[
h = \sum_{i=0}^{n-1} \tilde{V}_i (I_E^{\otimes_i} \otimes P_W) \tilde{V}^{(i)} h,
\]

and thus \( h \in \bigvee_{i=0}^{n-1} \{ \tilde{V}_i(\xi_i) : \xi_i \in E^{\otimes_i} \otimes \mathcal{W} \} \).

(2) Let \( \xi_n \in N(\tilde{V}_n) \), then \( \tilde{V}^{(n)} \tilde{V}_n(\xi_n) = 0 \). Therefore using Lemma \([5,4]\) we have

\[
\xi_n = \sum_{i=0}^{n-1} (I_E^{\otimes_{n-i}} \otimes \tilde{V}^{(i)}) (I_E^{\otimes_{n-(i+1)}} \otimes P_W) (I_E^{\otimes_{n-i}} \otimes \tilde{V}_i) \xi_n
\]

and then \( \xi_n \in \bigvee_{i=0}^{n-1} \{ (I_E^{\otimes_{n-i}} \otimes \tilde{V}^{(i)}) (\xi_{n-i}) : \xi_{n-i} \in E^{\otimes_{n-(i+1)}} \otimes \mathcal{W}^\dagger \} \). On the other hand, let \( \xi_{n-i} \in E^{\otimes_{n-(i+1)}} \otimes \mathcal{W}^\dagger \), for \( 0 \leq i \leq n-1 \). Since \((\sigma, V)\) is regular, \( \mathcal{W}^\dagger = N(\tilde{V}) \subseteq E \otimes R^\infty(\tilde{V}) \) and by using Remark \([3,7]\) we have \( (I_E^{\otimes_{n-i}} \otimes \tilde{V}_i \tilde{V}^{(i)}) (\xi_{n-i}) = \xi_{n-i} \). Therefore

\[
\tilde{V}_n (I_E^{\otimes_{n-i}} \otimes \tilde{V}^{(i)}) (\xi_{n-i}) = \tilde{V}_{n-i} (I_E^{\otimes_{n-i}} \otimes \tilde{V}_i) (I_E^{\otimes_{n-i}} \otimes \tilde{V}^{(i)}) (\xi_{n-i})
\]

\[
= \tilde{V}_{n-i} (I_E^{\otimes_{n-i}} \otimes \tilde{V}_i \tilde{V}^{(i)}) (\xi_{n-i})
\]

\[
= \tilde{V}_{n-i} (\xi_{n-i}) = 0,
\]

that is, \( (I_E^{\otimes_{n-i}} \otimes \tilde{V}^{(i)}) (\xi_{n-i}) \in N(\tilde{V}_n) \) for \( \xi_{n-i} \in E^{\otimes_{n-(i+1)}} \otimes \mathcal{W}^\dagger, 0 \leq i \leq n-1 \). Hence

\[
\bigvee_{i=0}^{n-1} \{ (I_E^{\otimes_{n-i}} \otimes \tilde{V}^{(i)}) (\xi_{n-i}) : \xi_{n-i} \in E^{\otimes_{n-(i+1)}} \otimes \mathcal{W}^\dagger \} \subseteq N(\tilde{V}_n).
\]

\( \square \)

Consider a CB-representation \((\sigma, V)\) of \( E \) on \( \mathcal{H} \) and let \( \mathcal{W} \) be a wandering subspace for \((\sigma, V)\). In this section, we refer to the smallest \((\sigma, V)\)-invariant subspace of \( \mathcal{H} \) containing \( \mathcal{W} \), by the notation \([\mathcal{W}]_{\tilde{V}}\), i.e.,

\[
[\mathcal{W}]_{\tilde{V}} := \bigvee_{n \in \mathbb{N}_0} \tilde{V}_n (E^{\otimes n} \otimes \mathcal{W}).
\]
Note that if $\mathcal{K}$ is $(\sigma, V)$-reducing and $\mathcal{W} \subseteq \mathcal{K}$, then $[\mathcal{W}]_{\tilde{V}} \subseteq \mathcal{K}$. If $R^\infty(V)$ reduces $(\sigma, V)$, then $[\mathcal{W}]_{\tilde{V}} \subseteq R^\infty(V)^\perp$, $\mathcal{W} = \mathcal{H} \ominus \tilde{V}(E \otimes \mathcal{H})$. Also using the equalities
\[
R^\infty(V)^\perp = \left( \bigcap_{n \in \mathbb{N}_0} R(\tilde{V}_n) \right)^\perp = \bigvee_{n \in \mathbb{N}_0} N(\tilde{V}_n^*)
\]
we get the following duality relations.

**Corollary 5.6.** Consider a CB-representation $(\sigma, V)$ of $E$ on $\mathcal{H}$ so that $\tilde{V}$ has closed range. Then

1. $R^\infty(V^\dagger)^\perp \subseteq [\mathcal{W}]_{\tilde{V}}$, 
2. if $(\sigma, V)$ is regular, then $R^\infty(V)^\perp = \bigvee_{i=0}^{\infty} \{ \tilde{V}^\dagger(n)(\xi) : \xi \in E^{\otimes i} \otimes \mathcal{W} \}$.

**Theorem 5.7.** Consider a regular CB-representation $(\sigma, V)$ of $E$ on $\mathcal{H}$ so that $R^\infty(V)$ reduces $(\sigma, V)$, then $(\sigma, V)$ is bi-regular.

**Proof.** Since $R(\tilde{V})$ is closed and $\gamma(\tilde{V}^*) = \gamma(\tilde{V})$, one can easily see that $R(\tilde{V}^\dagger)$ is closed. Now, let $n \in \mathbb{N}$ and $\xi \in E \otimes N(\tilde{V}^\dagger(n))$, then by Theorem 5.5 we get $\xi \in E \otimes R^\infty(V)^\perp$. By using regularity of $(\sigma, V)$, $E \otimes R^\infty(V)^\perp \subseteq N(\tilde{V}^\dagger)$ and hence $\xi \in N(\tilde{V}^\dagger) = R(\tilde{V}^\dagger)$, which follows the required condition for bi-regularity, i.e., $E \otimes N(\tilde{V}^\dagger(n)) \subseteq R(\tilde{V}^\dagger)$. \hfill $\square$

**Theorem 5.8.** Consider a regular CB-representation $(\sigma, V)$ of $E$ on $\mathcal{H}$ so that $\gamma(\tilde{V}) \geq 1$ and satisfy the following inequality
\[
\|\tilde{V}_m(\xi_m)\|^2 \leq d_m(\|\tilde{V} \otimes \xi_m\|^2 - \|\tilde{V} \otimes \xi_m\|^2) + \|\tilde{V} \otimes \xi_m\|^2,
\]
where $\xi_m \in E^{\otimes m} \otimes \mathcal{H}, m \in \mathbb{N}$ and along with $\sum_{m \geq 2} \frac{1}{d_m} = \infty$. Then
\[
\mathcal{H} = [\mathcal{W}]_{\tilde{V}} + R^\infty(V),
\]
where $\mathcal{W} = \mathcal{H} \ominus \tilde{V}(E \otimes \mathcal{H})$.

**Proof.** For $m \in \mathbb{N}$, Inequality (5.6) can be rewrite as
\[
\|\tilde{V}_m(\xi_m)\|^2 - \|\tilde{V} \otimes \xi_m\|^2 \leq d_m\|\tilde{V} \otimes \xi_m\|^2, \quad \xi_m \in E^{\otimes m} \otimes \mathcal{H}.
\]
Now replace $\xi_m$ by $\tilde{V}^\dagger(m)h$, $h \in \mathcal{H}$, in the above inequality, we obtain
\[
\|\tilde{V}_m \tilde{V}^\dagger(m)h\|^2 - \|\tilde{V} \otimes \tilde{V}^\dagger(m)h\|^2 \leq d_m\|\tilde{V} \otimes \tilde{V}^\dagger(m)h\|^2
\]
and by using the identity $\tilde{V} \otimes \tilde{V}^\dagger = \tilde{V}^\dagger$,
\[
\|\tilde{V}_m \tilde{V}^\dagger(m)h\|^2 - \|\tilde{V}^\dagger(m)h\|^2 \leq d_m\|\tilde{V} \otimes \tilde{V}^\dagger(m)h\|^2.
\]
Therefore

\begin{equation}
(5.7) \quad \sum_{m \geq 2} \frac{\| \tilde{V}_m \tilde{V}^+(m) h \|^2 - \| \tilde{V}^+(m) h \|^2}{d_m} \leq \sum_{m \geq 2} \| (I_{E^m \otimes m} \otimes D_{\tilde{V}}) \tilde{V}^+(m) h \|^2 \leq \| h \|^2,
\end{equation}

since \((d_m)_{m \geq 2}\) is positive sequence and the last inequality holds due to Lemma 5.3. Moreover, from the hypothesis \(\sum_{m \geq 2} \frac{1}{d_m} = \infty\), we conclude

\[ \lim \inf_{m} \left\{ \| \tilde{V}_m \tilde{V}^+(m) h \|^2 - \| \tilde{V}^+(m) h \|^2 \right\} \leq 0. \]

Indeed, if \(\lim \inf_{m} \left\{ \| \tilde{V}_m \tilde{V}^+(m) h \|^2 - \| \tilde{V}^+(m) h \|^2 \right\} \geq \beta > 0\), then there is a \(m_0 \in \mathbb{N}\) so that \(\| \tilde{V}_m \tilde{V}^+(m) h \|^2 - \| \tilde{V}^+(m) h \|^2 \geq \frac{\beta}{2} > 0\) for \(m \geq m_0\). Thus

\[ \sum_{m \geq 2} \frac{\| \tilde{V}_m \tilde{V}^+(m) h \|^2 - \| \tilde{V}^+(m) h \|^2}{d_m} \geq \sum_{m \geq 2} \frac{\beta}{2d_m} = \infty, \]

which contradicts the Inequality (5.7). Since \(\tilde{V}^+\) is contraction, there exists a weakly convergent subsequence \(\{ \tilde{V}_{m_j} \tilde{V}^+(m_j) h \} \) of \(\{ \tilde{V}_m \tilde{V}^+(m) h \}\) which converges weakly to \(y\) for some \(y \in \mathcal{H}\). In symbols: \(\tilde{V}_{m_j} \tilde{V}^+(m_j) h \to y\). By the identity

\[ \tilde{V}_m \tilde{V}^+(m) \tilde{V}^+(m_j) h = \tilde{V}_{m_j} \tilde{V}^+(m_j) h, \quad \text{where} \quad m_j \geq m. \]

Apply \(m_j \to \infty\) to the above equation, we get \(\tilde{V}_m \tilde{V}^+(m) y = y\) and hence \(y \in R(\tilde{V}_m)\), for all \(m \in \mathbb{N}\). Now by Lemma 5.4, \(h - \tilde{V}_{m_j} \tilde{V}^+(m_j) h \in [\mathcal{W}]_{\tilde{V}}\). Since \([\mathcal{W}]_{\tilde{V}}\) is weakly closed and \(h - \tilde{V}_{m_j} \tilde{V}^+(m_j) h \to h - y, h - y \in [\mathcal{W}]_{\tilde{V}}\). Consequently \(\mathcal{H} = [\mathcal{W}]_{\tilde{V}} + R^\infty(V)\). \(\square\)

**Remark 5.9.** Hereafter whenever we say that \((\sigma, V)\) satisfies the growth condition, it means it satisfies (5.6).

**Theorem 5.10.** Consider a regular CB-representation \((\sigma, V)\) of \(E\) on \(\mathcal{H}\) so that \(\gamma(\tilde{V}) \geq 1\), which satisfies the growth condition, then the following hold:

1. \(R^\infty(V)\) reduces \((\sigma, V)\),
2. \((\tilde{V}|_{E \otimes R^\infty(V)})^+ = \tilde{V}^+|_{R^\infty(V)} = \tilde{V}^*|_{R^\infty(V)} = (\tilde{V}|_{E \otimes R^\infty(V)})^*\),
3. \((\sigma, V)\) is bi-regular,
4. the restriction map \(\tilde{V}|_{(E \otimes R^\infty(V)) \cap N(\tilde{V})}^+ : (E \otimes R^\infty(V)) \cap N(\tilde{V}) \to R^\infty(V)\) is an unitary,
5. \(\mathcal{H}\) admits an orthogonal decomposition,

\[ \mathcal{H} = [\mathcal{W}]_{\tilde{V}} \oplus R^\infty(V). \]

**Proof.** (1) By using Theorem 2.6 and Corollary 3.9 \(R^\infty(V)\) is \((\sigma, V)\)-invariant and \(\tilde{V}^+ R^\infty(V) \subseteq E \otimes R^\infty(V)\), therefore we shall check that \(R^\infty(V)^+\) is \((\sigma, V)\)-invariant. Let \(h \in R^\infty(V)\) and
replace $\xi_m \in E \otimes m \otimes H$ by $\widetilde{V}^{(m)} h$ in the growth condition, we get

$$\|\widetilde{V}_m \widetilde{V}^{(m)} h\|^2 - \| (I_{E \otimes m-1} \otimes \widetilde{V}^{(m)} \widetilde{V}^{(m)}) h\|^2 \leq d_m \|(I_{E \otimes m-1} \otimes D_{\widetilde{V}}) \widetilde{V}^{(m)} h\|^2.$$

By using the identity $\widetilde{V}^{(m)} \widetilde{V}^{(m)} = \widetilde{V}^{(m)}$,

$$\|\widetilde{V}_m \widetilde{V}^{(m)} h\|^2 - \| \widetilde{V}^{(m)} h\|^2 \leq d_m \|(I_{E \otimes m-1} \otimes D_{\widetilde{V}}) \widetilde{V}^{(m)} h\|^2.$$

Since $h \in R^\infty(V)$ and Remark 3.7 follows that, $\widetilde{V}_m \widetilde{V}^{(m)} h = h$ for every $m \in \mathbb{N}$. Therefore

$$\|h\|^2 - \| \widetilde{V}^{(m)} h\|^2 \leq d_m \|(I_{E \otimes m-1} \otimes D_{\widetilde{V}}) \widetilde{V}^{(m)} h\|^2.$$

Thus

$$\sum_{m \geq 2} \frac{\|h\|^2 - \| \widetilde{V}^{(m)} h\|^2}{d_m} \leq \sum_{m \geq 2} \|(I_{E \otimes m-1} \otimes D_{\widetilde{V}}) \widetilde{V}^{(m)} h\|^2 \leq \|h\|^2,$$

where the last inequality follows from the Lemma 5.3. By using the hypothesis $\sum_{m \geq 2} \frac{1}{d_m} = \infty$, we conclude

$$\liminf_m \{|h|^2 - \| \widetilde{V}^{(m)} h\|^2\} \leq 0.$$

Since $\widetilde{V}^{(m)}$ is contraction, $\lim_m \| \widetilde{V}^{(m)} h\| = \|h\|$ and it follows that

$$\|h\| = \lim_m \| \widetilde{V}^{(m)} h\| \leq \|\widetilde{V}^{(m)} h\| \leq \|h\|,$$

hence $\|\widetilde{V}^{(m)} h\| = \|h\|$, $h \in R^\infty(V)$. Therefore, for $\eta \in E \otimes R^\infty(V)$, $\widetilde{V}^{(m)} h \in R^\infty(V)$ and

$$\|D_{\widetilde{V}}(\eta)\|^2 = \| \widetilde{V}(\eta)\|^2 - \| \widetilde{V}^{(m)} \widetilde{V}(\eta)\|^2 = 0,$$

and thus

$$\widetilde{V}^{(m)} \widetilde{V}(\eta) = \widetilde{V}^{(m)} \widetilde{V}(\eta) \quad \text{for} \quad \eta \in E \otimes R^\infty(V) \quad \text{(5.8)}.$$

By Corollary 3.9, $\widetilde{V}^{(m)} h \in E \otimes R^\infty(V)$, for $h \in R^\infty(V)$ and replace $\eta$ by $\widetilde{V}^{(m)} h$ in Equation (5.8), we get

$$\widetilde{V}^{(m)} \widetilde{V}(\eta) = \widetilde{V}^{(m)} \widetilde{V}(\eta) \quad \text{for} \quad \eta \in E \otimes R^\infty(V).$$

But $\widetilde{V}^{(m)} \widetilde{V}^{(m)} \widetilde{V}^{(m)} = \widetilde{V}^{(m)}$ and $\widetilde{V}^{(m)} \widetilde{V}^{(m)} \widetilde{V}^{(m)} = \widetilde{V}^{(m)}$, therefore $\widetilde{V}^{(m)} h = \widetilde{V}^{(m)} h \in E \otimes R^\infty(V)$ for all $h \in R^\infty(V)$. This shows that $R^\infty(V)$ reduces $(\sigma, V)$.

(2) By the above observation, we conclude that $\widetilde{V}^{(m)} |_{R^\infty(V)} = \widetilde{V}^{(m)} |_{R^\infty(V)}$. Moreover,

$$(\widetilde{V} |_{E \otimes R^\infty(V)})^* = P_{E \otimes R^\infty(V)} \widetilde{V}^* |_{R^\infty(V)} = \widetilde{V}^* |_{R^\infty(V)}$$

and using Proposition 4.8, $(\widetilde{V} |_{E \otimes R^\infty(V)})^* = \widetilde{V}^* |_{R^\infty(V)}$. The statement (2) is proved.

(3) Since $(\sigma, V)$ is regular and $R^\infty(V)$ reduces $(\sigma, V)$, therefore it follows from Theorem 3.7 $(\sigma, V)$ is bi-regular.

(4) From Proposition 4.7, the restriction map $\widetilde{V} |_{(E \otimes R^\infty(V)) \cap N(\widetilde{V})}$ is bijective. So, we need to prove $\widetilde{V} |_{(E \otimes D_{\widetilde{V}} \cap N(\widetilde{V})}$ is an isometric. For this purpose, let $\eta \in E \otimes R^\infty(V) \cap N(\widetilde{V})$, $\widetilde{V}^{(m)} \widetilde{V} \eta = \eta$, and thus from Equation (5.8), $\|\widetilde{V} \eta\| = \|\eta\|$. Hence $\widetilde{V} |_{E \otimes R^\infty(V) \cap N(\widetilde{V})}$ is an isometric.
Remark 5.11. Observe that, from the above theorem, an orthogonal decomposition of \( H \) may not be unique. Because of this, if we take \((\sigma, V_n)\) as hyper-dagger, then the decomposition is unique.

Since \((\sigma, V_n)\) is CB-representation of \( E \otimes_n \) on \( H \), \( R^\infty(V) = R^\infty(V_n) \) and by Equation (5.8), we get

\[
\widetilde{V}_n \tilde{V}_n(\eta) = \tilde{V}_n \tilde{V}_n(\eta) \quad \text{and} \quad \tilde{V}_n \tilde{V}_n \eta = \eta, \quad \eta \in (E \otimes_n \otimes R^\infty(V)) \cap N(\tilde{V}_n) \perp, n \in \mathbb{N},
\]

thus \( \|\tilde{V}_n \eta\| = \|\eta\| \). Using Proposition 4.7 and \((\sigma, V_n)\) is n-dagger, \( R(\tilde{V}_n) = N(\tilde{V}_n) \perp \) and therefore the map \( \tilde{V}_n|_{(E \otimes_n \otimes R^\infty(V)) \cap N(\tilde{V}_n)} : (E \otimes_n \otimes R^\infty(V)) \cap N(\tilde{V}_n) \rightarrow R^\infty(V) \) is unitary, for all \( n \in \mathbb{N} \).

To prove the uniqueness, suppose \( H = H_1 \oplus H_2 \) is another decomposition of \( H \) into reducing subspaces such that \((\sigma, V)|_{H_1}\) has GWS-property and for \( n \in \mathbb{N} \),

\[
\tilde{V}_n|_{(E \otimes_n \otimes H_2) \cap N(\tilde{V}_n)} : (E \otimes_n \otimes H_2) \cap N(\tilde{V}_n) \rightarrow H_2
\]

is unitary. Then \( H_1 = [\tilde{W}]_{\tilde{V}_n} \), where \( \tilde{W} \) is wandering subspace for \((\sigma, V)|_{H_1}\). Note that \( \tilde{W} \) is uniquely determined by \( H_1, \tilde{W} = N(\tilde{V}_n|_{H_1}) \), therefore \( \tilde{W} \subseteq \tilde{W} \) and hence \([\tilde{W}]_{\tilde{V}_n} \subseteq [\tilde{W}]_{\tilde{V}_n} \).

For each \( n \in \mathbb{N} \),

\[
H_2 = \tilde{V}_n((E \otimes_n \otimes H_2) \cap N(\tilde{V}_n)) \subseteq \tilde{V}_n(E \otimes_n \otimes H)
\]

clearly \( H_2 \subseteq R^\infty(V) \). Moreover,

\[
H = H_1 \oplus H_2 \subseteq [\tilde{W}]_{\tilde{V}_n} \oplus R^\infty(V) = H,
\]

which proves uniqueness.

Let \((\sigma, V)\) be a CB-representation of \( E \) on \( H \) so that \( \tilde{V} \) is expansive, i.e., \( \|\xi\| \leq \|\tilde{V}(\xi)\|, \xi \in E \otimes H \). Then \( N(\tilde{V}) = 0 \) and \( \gamma(\tilde{V}) \geq 1 \), we get the following Wold-type decomposition.

Corollary 5.12. Consider a CB-representation \((\sigma, V)\) of \( E \) on \( H \) so that \( \tilde{V} \) is expansive and satisfies the growth condition. Then \( H \) has an unique orthogonal decomposition,

\[
H = [\tilde{W}]_{\tilde{V}_n} \oplus R^\infty(V)
\]

such that \((\sigma, V)|_{R^\infty(V)}\) is isometric as well as fully coisometric. That is, \((\sigma, V)\) admits Wold-type decomposition.

Corollary 5.13. Consider a regular CB-representation \((\sigma, V)\) of \( E \) on \( H \) so that \( \gamma(\tilde{V}) \geq 1 \), which satisfies the growth condition, then

\[
[\tilde{W}]_{\tilde{V}_n} = \bigvee_{n \in \mathbb{N}_0} N((\tilde{V}^{\dagger})^n) = R^\infty(V^{\dagger}) \perp.
\]
Proof. From statement (3) in Theorem 5.10, \((\sigma, V)\) is bi-regular, and thus \(\tilde{V}^{\dagger*}\) is regular. By using Corollary 5.6, we have

\[
R_\infty(V) = \bigvee_{i=0}^{n-1} \{ \tilde{V}^{\dagger*}(i)(x_i) : x_i \in E^{\otimes i} \otimes W \}.
\]

Replace \(\tilde{V}\) by \(\tilde{V}^{\dagger*}\) in the above equation, and we get

\[
R_\infty(V^{\dagger*}) = \bigvee_{i=0}^{n-1} \{ (\tilde{V}^{\dagger*})^{\dagger*}(i)(x_i) : x_i \in E^{\otimes i} \otimes W \} = [W]_{\tilde{V}}. \]

Let \((\sigma, V)\) be a CB-representation of \(E\) on \(H\) and let \(K\) be a \((\sigma, V)\)-invariant subspace of \(H\). Then we get a wandering subspace \(N(\tilde{V}^{\dagger*}|_K) = K \ominus \tilde{V}(E \otimes H)\) for \((\sigma, V)\). On the other hand, if we start with a wandering subspace \(W\) for \((\sigma, V)\), then \(K = \bigcup_{n \in \mathbb{N}_0} \tilde{V}_n(E^{\otimes n} \otimes W)\) is \((\sigma, V)\)-invariant. In fact \(K\) is the smallest \((\sigma, V)\)-invariant subspace which contains \(W\). In this case \(W = N(\tilde{V}^{\dagger*}|_K)\). Indeed,

\[
N(\tilde{V}^{\dagger*}|_K) = K \ominus \tilde{V}(E \otimes K) = \bigcup_{n \in \mathbb{N}_0} \tilde{V}_n(E^{\otimes n} \otimes W) \ominus \tilde{V}(E \otimes \bigcup_{n \in \mathbb{N}_0} \tilde{V}_n(E^{\otimes n} \otimes W))
\]

\[
= \bigcup_{n \in \mathbb{N}_0} \tilde{V}_n(E^{\otimes n} \otimes W) \ominus \bigcup_{n \in \mathbb{N}} \tilde{V}_n(E^{\otimes n} \otimes W) = W.
\]

Hence, an invariant subspace \(K\) determines the wandering subspace \(W\) in a unique way. This leads to the conclusion that there are one-to-one correspondences between the set of all \((\sigma, V)\)-invariant subspaces of \(H\) and the set of all wandering subspaces of \(H\). This conclusion is the refinement of [26, Theorem 2.4] and [17, Theorem 2.1].

Definition 5.14. Consider a CB-representation \((\sigma, V)\) of \(E\) on \(H\). If

\[
\bigcap_{n \in \mathbb{N}_0} \tilde{V}_n(E^{\otimes n} \otimes H) = \{0\},
\]

then we say \((\sigma, V)\) is analytic (pure) (cf. [26]).

Corollary 5.15. Consider an analytic, regular CB-representation \((\sigma, V)\) be of \(E\) on \(H\) such that \(\gamma(\tilde{V}) \geq 1\) and satisfies the growth condition. Let \(K\) be a \((\sigma, V)\)-invariant subspace. Then, a wandering subspace \(W\) is present in such a way that

\[
K = [W]_{\tilde{V}}.
\]

Proof. Since \(K\) is \((\sigma, V)\)-invariant subspace, \((\sigma, V)|_K\) is regular and satisfies the growth condition. Then by Theorem 5.10 \(K\) has an orthogonal decomposition

\[
K = [W]_{\tilde{V}} \oplus R_\infty(V|_K).
\]
for some wandering subspace $\mathcal{W}$ and $R^\infty(V|_K) = \bigcap_{n \in \mathbb{N}} \tilde{V}_n(E^\otimes n \otimes \mathcal{K})$, in fact $\mathcal{W} = N(\tilde{V}^*|_K)$.

Since $(\sigma, V)$ is analytic, $(\sigma, V)|_K$ is also analytic, that is, $R^\infty(V|_K) = 0$. Hence $\mathcal{K} = [\mathcal{W}]_V$. □

**Corollary 5.16.** Consider a concave CB-representation $(\sigma, V)$ of $E$ on $\mathcal{H}$. Then $(\sigma, V)$ admits Wold-type decomposition. That is,

$$\mathcal{H} = [\mathcal{W}]_V \oplus R^\infty(V).$$

such that $(\sigma, V)|_{R^\infty(V)}$ is isometric and co-isometric.

**Proof.** By using the Inequality (5.2), we obtain

$$\frac{n - 1}{n} \|\zeta\|^2 \leq \|(I_{E^\otimes(n-1)} \otimes \tilde{V})(\zeta)\|^2,$$

for $\zeta \in E^\otimes n \otimes \mathcal{H}$, $n \in \mathbb{N}$ and thus $\frac{n - 1}{n} \|\eta_{n-1} \otimes \zeta\|^2 \leq \|\eta_{n-1} \otimes \tilde{V}(\zeta)\|^2 \leq \|\tilde{V}(\zeta)\|^2$, $\zeta \in E^\otimes \mathcal{H}$, $\eta_{n-1} \in E^\otimes n - 1$ with $\|\eta_{n-1}\| \leq 1$. It follows that $\frac{n - 1}{n} \|\zeta\|^2 \leq \|\tilde{V}(\zeta)\|^2$ (See [26, Lemma 2.2]). This shows that $\|\zeta\| \leq \|\tilde{V}(\zeta)\|$, $\zeta \in E \otimes \mathcal{H}$. It follows that $\tilde{V}$ is left invertible, MPI $\tilde{V}^\dagger = \tilde{V}^{*\dagger} \tilde{V}^*$ is the left inverse of $\tilde{V}$, i.e., $\tilde{V}^\dagger \tilde{V} = I_{E \otimes \mathcal{H}}$. This makes it possible to write the Inequality (5.2) as

$$\|\tilde{V}_n(\zeta)\|^2 \leq \|(I_{E^\otimes(n-1)} \otimes \tilde{V}^\dagger)(\tilde{V})(\zeta)\|^2 + n(\|(I_{E^\otimes(n-1)} \otimes \tilde{V}^\dagger)(\zeta)\|^2 - \|(I_{E^\otimes(n-1)} \otimes \tilde{V}^\dagger)(\zeta)\|^2), \quad \zeta \in E^\otimes n \otimes \mathcal{H} \text{ and } n \in \mathbb{N}.$$

Note that $\|\xi\| \leq \|\tilde{V}(\xi)\|$, $\xi \in E \otimes \mathcal{H}$,

$$\frac{\|\tilde{V}(\xi)\|}{\text{dist}(\xi, N(\tilde{V}))} \geq \frac{\|\tilde{V}(\xi)\|}{\|\xi\|} \geq 1,$$

hence $\gamma(\tilde{V}) \geq 1$. Thus by Theorem 5.10, $(\sigma, V)$ admits Wold-type decomposition. □

6. **Contraction Intertwine With Left Invertible Covariant Representation**

Bercovici, Douglas, and Foias in [3] proved that if a contraction $A$, defined on a Hilbert valued Hardy space $H_\mathbb{D}(\mathbb{D})$, commute with shift operator $S$ on $H_\mathbb{D}^2(\mathbb{D})$ and $A^{*n}|_{P_S} \to 0$ strongly for $n \to \infty$ then so is $A^{*n} \to 0$ strongly for $n \to \infty$, where $P_S$ is projection on $N(S^*)$. After that, S.Sarkar in [23] proved the above result for the pure isometry case. At the beginning of this section, we define some notations and recall some definitions.

Consider a CB-representation $(\sigma, V)$ of $E$ on $\mathcal{H}$ and $\tilde{V}$ has left inverse. Define $\tilde{V}' : E \otimes \mathcal{H} \to \mathcal{H}$ by

$$\tilde{V}' := \tilde{V}(\tilde{V}^* \tilde{V})^{-1}.$$

It is easy to see that $\tilde{V}'(\phi(a) \otimes I_{\mathcal{H}}) = (\phi(a) \otimes I_{\mathcal{H}})\tilde{V}^* \tilde{V}$ for each $a \in \mathcal{B}$ and hence $\tilde{V}'(\phi(a) \otimes I_{\mathcal{H}}) = (\sigma(a) \otimes I_{\mathcal{H}})$, where $\phi$ is the left action on $E$. Then $(\sigma, V')$ is a CB-representation of $E$ on $\mathcal{H}$, where $V' : E \to B(\mathcal{H})$ is defined by $V'(\xi \otimes h) = V'(\xi)h, \xi \in E, h \in \mathcal{H}$, and it is called Cauchy dual (cf. (26) of $(\sigma, V)$).

**Notation 6.1.** $R^\infty(V') := \bigcap_{n \in \mathbb{N}} \tilde{V}_n'(E^\otimes n \otimes \mathcal{H})$ and $\mathcal{W} := N(\tilde{V}'^*)$.

Observe that $\mathcal{W}' = N(\tilde{V}^*) = \mathcal{W}$. 

Definition 6.2. Consider a CB-representation \((\sigma, V)\) of \(E\) on \(H\). A bounded linear operator \(A : H \to H\) is said to intertwine \((\sigma, V)\) if it satisfies the following

\[AV(\xi)h = V(\xi)Ah \quad \text{and} \quad A\sigma(a)h = \sigma(a)Ah,\]

where \(\xi \in E\), \(h \in H\) and \(a \in \mathcal{B}\).

To prove the primary conclusion of this section, we start with the lemma, which is quite helpful.

Lemma 6.3. Consider a CB-representation \((\sigma, V)\) of \(E\) on \(H\) and \(\tilde{V}\) has left inverse. Assume that \((\sigma, V)\) and its Cauchy dual \((\sigma, V')\) have GWS-property. Suppose \(K\) is a non-trivial proper \((\sigma, V')\)-invariant subspace of \(H\) and \(W \subseteq K^\perp\), where \(W = N(\tilde{V}^*)\). Then there is a non-zero \(h_1 \in K\) so that \(\tilde{V}^*h_1 \in E \otimes K^\perp\).

Proof. The subspace \(W \subseteq K^\perp\) satisfies the following conditions:

1. \(K\) is not \((\sigma, V')\)-reducing. Indeed, if \(K\) is \((\sigma, V')\)-reducing, then \(K \perp [W]_{\tilde{V}} = H\), which contradicts to the assumption that \(K\) is proper.

2. For any non-zero \(h \in K\), \(\tilde{V}^*h \neq 0\), otherwise \(h \in W \subseteq K^\perp\).

Note that \(N(\tilde{V}^*) = N(\tilde{V}'^*)\), it follows from our assumptions that \(H = [W]_{\tilde{V}}\). Since \(W \subseteq K^\perp\) and \(K\) is proper, there exists \(n \in \mathbb{N}\) such that \(\tilde{V}'_n(E^\otimes N \otimes W)\) is not orthogonal to \(K\). Then there exists a non-zero \(\eta_n \in E^\otimes n \otimes W\) such that \(\tilde{V}'_n(\eta_n)\) is not orthogonal to \(K\). Let \(N\) be the smallest positive integer so that \(\tilde{V}'_N(E^\otimes N \otimes W)\) is not orthogonal to \(K\) (such a \(N\) always exists because \(W \subseteq K^\perp\), that is, \(\tilde{V}'_l(E^\otimes l \otimes W)\) is orthogonal to \(K\) for all \(0 \leq l \leq N - 1\). Choose the non-zero elements \(h_1 \in K\) and \(h_2 \in K^\perp\) such that \(\tilde{V}'_N(\eta_N) = h_1 + h_2\). Then by the observation (2) and the minimality on \(N\), we obtain

\[\tilde{V}^*h_1 = \tilde{V}^*\tilde{V}'_N(\eta_N) - \tilde{V}^*h_2 = \tilde{V}^*\tilde{V}'(I_E \otimes \tilde{V}'_{N-1})(\eta_N) - \tilde{V}^*h_2 = (I_E \otimes \tilde{V}'_{N-1})(\eta_N) - \tilde{V}^*h_2 \in E \otimes K^\perp\]

This completes the proof. \(\square\)

Theorem 6.4. Consider a CB-representation \((\sigma, V)\) of \(E\) on \(H\) and \(\tilde{V}\) has left inverse. Suppose that \(A\) is a bounded linear operator on \(H\) which intertwine \((\sigma, V)\). Assume that \((\sigma, V)\) and its Cauchy dual \((\sigma, V')\) have the GWS-property. Then the following assertions are equivalent:

1. \(A\) is a pure contraction on \(H\).
2. \(P_W A|_W\) is a pure contraction, where \(W = N(\tilde{V}^*)\).

Proof. (1) \(\implies\) (2) : Note that \(W\) is a \(A^*\)-invariant subspace of \(H\), therefore for each \(n \in \mathbb{N}\), \((P_W A|_W)^* = A^*|_W\). Hence if \(A\) is pure contraction, then \((P_W A|_W)^*\) is also pure contraction.

(2) \(\implies\) (1) : Suppose \(P_W A|_W\) is pure contraction. Since \(A\) is contraction, \(\{A^n A'^n\}_n\) is a decreasing sequence of non-negative operators. Therefore there is a positive operator, denoted by \(D_A\), such that the sequence \(\{A^n A'^n\}_n\) converges to \(D_A^2\) in the strong operator topology. To prove \(A\) is a pure contraction, only we need to show that \(N(D_A) = H\). Since \(P_W A|_W\) is a pure contraction, \(W \subseteq N(D_A)\). Indeed, for \(h \in W\)

\[\|D_A h\| = \lim_{n \to \infty} \|A'^n h\| = \lim_{n \to \infty} \|(P_W A|_W)^* h\| = 0.\]
Case 1: If \( N(D_A) \) is \((\sigma, V)\)-reducing then
\[
\mathcal{H} = [W]_V \subseteq N(D_A) \subseteq \mathcal{H},
\]
that is, \( N(D_A) = \mathcal{H} \).

Case 2: If \( N(D_A) \) is not \((\sigma, V)\)-reducing then \( N(D_A) \) is a proper subspace of \( \mathcal{H} \). Note that \( N(D_A)^\perp \) is \((\sigma, V)\)-invariant. Indeed, for \( g \in N(D_A) \) and using intertwine property of \( A \), we obtain
\[
\|(I_E \otimes D_A)\tilde{V}^* g\|^2 = \lim_{n \to \infty} \langle \tilde{V}(I_E \otimes A^n A^{*n})\tilde{V}^* g, g \rangle = \lim_{n \to \infty} \langle A^n \tilde{V} \tilde{V}^* A^{*n} g, g \rangle \leq \|\tilde{V}\|^2 \|D_A g\|^2 = 0,
\]
thus \( \tilde{V}^* N(D_A) \subseteq E \otimes N(D_A) \) and hence \( N(D_A)^\perp \) is \((\sigma, V)\)-invariant. Therefore by Lemma \ref{lem}, there is an \( h_1 \in N(D_A)^\perp \) such that \( \tilde{V}^* h_1 \in E \otimes N(D_A) \) and \( h_1 \neq 0 \). For each \( m \in \mathbb{N} \), there exist \( \xi_{m} \in W \) and \( \xi_{m}^2 \in W^\perp \) such that
\[
A^{*m} h_1 = \xi_{m}^1 + \xi_{m}^2.
\]
Since \( A \) is contraction, \( \|\xi_{m}^2\| \leq \|h_1\| \) for all \( m \in \mathbb{N} \). Now apply \( \tilde{V}^* \) both the sides of the previous equation, we get
\[
\tilde{V}^* A^{*m} h_1 = \tilde{V}^* \xi_{m}^2.
\]
But \( \tilde{V}^* h_1 \in E \otimes N(D_A) \),
\[
\lim_{n \to \infty} \|\tilde{V}^* \xi_{m}^2\| = \lim_{n \to \infty} \|\tilde{V}^* A^{*m} h_1\| = \lim_{n \to \infty} \|(I_E \otimes A^{*m})\tilde{V}^* h_1\| = 0.
\]
Note that \( \xi_{m}^2 \in W^\perp = \tilde{V}(E \otimes \mathcal{H}) \), there is \( \eta_{m}^2 \in E \otimes \mathcal{H} \) such that \( \tilde{V}(\eta_{m}^2) = \xi_{m}^2 \) and also \( \tilde{V} \) has left inverse, hence \( \tilde{V} \) becomes bounded below. Then there exists \( \alpha > 0 \) such that
\[
\|\eta_{m}^2\| \leq \alpha \|\tilde{V}(\eta_{m}^2)\| = \alpha \|\xi_{m}^2\|.
\]
From the above inequality, we have
\[
\lim_{m \to \infty} \|\xi_{m}^2\|^2 = \lim_{m \to \infty} \|\tilde{V}(\eta_{m}^2)\|^2 \leq \lim_{m \to \infty} \|\tilde{V}^* \tilde{V}(\eta_{m}^2)\| \|\eta_{m}^2\| \leq \lim_{m \to \infty} \|\tilde{V}^* \xi_{m}^2\| \|\alpha \xi_{m}^2\| = 0,
\]
since the sequence \( \{\xi_{m}^2\} \) bounded by \( h_1 \). Thus for each \( n \in \mathbb{N} \),
\[
\|D_A h_1\| = \|D_A A^{*n} h_1\| = \|D_A (\xi_{m}^1 + \xi_{m}^2)\| \leq \|D_A\| \|\xi_{m}^2\|.
\]
If we let \( n \to \infty \) form the above inequality, \( D_A h_1 = 0 \), i.e., \( h_1 \in N(D_A) \). However, by assumptions \( h_1 \in N(D_A)^\perp \) and hence \( h_1 = 0 \). This contradicts to our assumptions \( N(D_A) \) is a proper subspace of \( \mathcal{H} \). Since \( \{0\} \neq W \subseteq N(D_A) \), the only possibility for \( N(D_A) \) is \( \mathcal{H} \) which is equivalent to showing that \( A \) is pure.

\[\square\]

**Corollary 6.5.** Consider an analytic, regular CB-representation \((\sigma, V)\) of \( E \) on \( \mathcal{H} \) so that \( \gamma(\tilde{V}) \geq 1 \) and satisfies the growth condition. Suppose \( A \) is a bounded linear operator on \( \mathcal{H} \) so that \( A \) intertwine \((\sigma, V)\). Then the following are equivalent:

1. \( A \) is a pure contraction on \( \mathcal{H} \).
2. \( P_W A |_W \) is a pure contraction, where \( W = N(\tilde{V}^*) \).
Proof. By Theorem 5.10

\[ \mathcal{H} = [\mathcal{W}] \tilde{\mathcal{V}} \oplus R^\infty(V). \]

Since \((\sigma, V)\) is pure and regular, \(R^\infty(V) = 0\) and hence \(\tilde{\mathcal{V}}\) is left invertible and \(\tilde{\mathcal{V}}^* = \tilde{\mathcal{V}}'\). Also note that \(\mathcal{H} = \bigvee_{n \in \mathbb{N}_0} \hat{\mathcal{V}}_n (E^\otimes n \otimes \mathcal{W}) = \bigvee_{n \in \mathbb{N}_0} \hat{\mathcal{V}}'_n (E^\otimes n \otimes \mathcal{W})\). Therefore by Theorem 6.4, \(A\) is pure contraction if and only if \(P_W A|_W\) is pure contraction.

7. Applications to unilateral and bilateral weighted shifts

In this section, we apply the Wold decomposition to unilateral shift defined by Muhly and Solel in [16] and an analog of bilateral shift used in [7]. Here we introduce an analog of unilateral weighted shift on the vector-valued Hardy space, and we discuss GWS-property.

7.1. Unilateral shift. In this subsection, we will consider \(\mathcal{B}\) to be \(W^*\)-algebra and \(E\) to be \(W^*\)-correspondence over \(\mathcal{B}\). Indeed, we shall use the setting of [16]. The Fock space of a \(W^*\)-correspondence \(E\), defined as \(\mathcal{F}(E) := \bigoplus_{n \in \mathbb{N}_0} E^\otimes n\), is a \(W^*\)-correspondence over \(\mathcal{B}\). The left module action of \(\mathcal{B}\) on \(\mathcal{F}(E)\) is denoted by \(\phi^\infty\) and, is defined by \(\phi^\infty(a) := \bigoplus_{n \in \mathbb{N}_0} \phi^n(a), a \in \mathcal{B}\). For \(\xi \in E\), we define the creation operator \(T_\xi\) on \(\mathcal{F}(E)\) by

\[ T_\xi(\eta_n) = \xi \otimes \eta_n, \quad \eta_n \in E^\otimes n, n \in \mathbb{N}_0. \]

A linear map \(T : E \to \mathcal{L}(\mathcal{F}(E))\) is defined by \(T(\xi) = T_\xi, \xi \in E\), then it satisfies \(T(a \xi b) = \phi^\infty(a)T(\xi)\phi^\infty(b)\), where \(a, b \in \mathcal{B}\) and \(\xi \in E\). Note that, the Fock module \(\mathcal{F}(E)\) is \(W^*\)-correspondence but not necessarily a Hilbert space in general, so that \((\phi^\infty, T)\) may not be covariant representation of \(E\) on \(\mathcal{L}(\mathcal{F}(E))\). But this can be overcome by composing the pair \((\phi^\infty, T)\) with a faithful representation of the \(W^*\)-algebra \(\mathcal{L}(\mathcal{F}(E))\) on a Hilbert space, this means that the pair \((\psi \circ \phi^\infty, \psi \circ T)\) is a covariant representation of \(E\) on \(B(\mathcal{H})\) where \(\psi : \mathcal{L}(\mathcal{F}(E)) \to B(\mathcal{H})\) is a faithful representation of the \(C^*\)-algebra \(\mathcal{L}(\mathcal{F}(E))\) on a Hilbert space \(\mathcal{H}\).

Notation 7.1. Let \(\phi^n(\mathcal{B})^c\) be the set of all the elements in \(\mathcal{L}(E^\otimes n)\) which commute with the image of \(\mathcal{B}\) under \(\phi^n, n \in \mathbb{N}_0\). Similarly, the commutant of \(\phi^\infty(\mathcal{B})\) in \(\mathcal{L}(\mathcal{F}(E))\) denoted as \(\phi^\infty(\mathcal{B})^c\). Observe that \(\phi^0(\mathcal{B})^c\) is simply the center of \(\mathcal{B}\).

Let \(E\) be \(W^*\)-correspondence over \(\mathcal{B}\). A sequence \(\{Z_k\}_{k \in \mathbb{N}_0}\), where \(Z_k \in \phi^k(\mathcal{B})^c\), is called weight sequence on \((\mathcal{B}, E)\) if \(Z_0 = I_B\) and \(\sup\|Z_k\| < \infty\). The following definition is due to Muhly-Solel [16]:

Definition 7.2. Let \(E\) be a \(W^*\)-correspondence over \(\mathcal{B}\) and \(Z = \{Z_k\}_{k \in \mathbb{N}_0}\) be a weight sequence on \((E, \mathcal{B})\). A linear map \(W : E \to \mathcal{L}(\mathcal{F}(E))\) is called weighted shift associated with the weight sequence \(Z = \{Z_k\}_{k \in \mathbb{N}_0}\), if \(W = DT\) where \(D\) is the diagonal operator on \(\mathcal{F}(E)\) corresponding to \(\{Z_k\}_{k \in \mathbb{N}_0}\), i.e., for \(\eta_n \in E^\otimes n\),

\[ W(\xi)\eta_n = DT(\xi)\eta_n = DT_\xi\eta_n = D(\xi \otimes \eta_n) = Z_{n+1}(\xi \otimes \eta_n), \quad \xi \in E. \]
Using the covariant condition of $T$ and $D$ lies in $\phi^c(\mathcal{B})$, one can easily see that $W$ satisfies
\[ W(a\xi b) = \phi_\infty(a) W(\xi) \phi_\infty(b), \]
where $a, b \in \mathcal{B}$ and $\xi \in E$. Suppose that $\pi$ is a representation of $\mathcal{B}$ on a Hilbert space $\mathcal{H}$. Define a covariant representaion $(\rho, S)$ of $E$ on a Hilbert space $\mathcal{F}(E) \otimes \mathcal{H}$ with weight $Z$ by
\[ \rho(a) = \phi_\infty(a) \otimes I_\mathcal{H} \text{ and } S(\xi) = W(\xi) \otimes I_\mathcal{H}, \quad a \in \mathcal{B}, \; \xi \in E. \]

Note that $S(\xi)(\eta) = (W(\xi) \otimes I_\mathcal{H})(\eta) = (DT_\xi \otimes I_\mathcal{H})\eta = \sum_{n \in \mathbb{N}_0} (Z_{n+1} \otimes I_\mathcal{H})(\xi \otimes \eta_n \otimes h_n)$, where $\eta = \oplus_{n \in \mathbb{N}_0} \eta_n \otimes h_n, \xi \in E, \eta_n \in E^{\otimes n}, h_n \in \mathcal{H}$ and $D = [Z_0, Z_1, Z_2, \ldots]$ is the diagonal operator on $\mathcal{F}(E)$. We refer to $(\rho, S)$ as the weighted induced representation induced by $\pi$ with weight $Z$.

Suppose that $Z_n$’s are surjective for each $n \in \mathbb{N}_0$, $\text{ran}(\tilde{S}) = \bigoplus_{n \in \mathbb{N}} (Z_n \otimes I_\mathcal{H})(E^{\otimes n} \otimes \mathcal{H}) = E \otimes \mathcal{F}(E) \otimes \mathcal{H}$ is closed and thus $N(\tilde{S}^*) = \mathcal{H}$, that is, $\mathcal{H}$ is a wandering subspace for $(\rho, S)$. In fact $\mathcal{H}$ is the generating wandering subspace for $(\rho, S)$, since $\tilde{S}(E^{\otimes n} \otimes \mathcal{H}) = E^{\otimes n} \otimes \mathcal{H}$.

**Theorem 7.3.** Let $(\rho, S)$ be a weighted shift on $\mathcal{F}(E) \otimes \mathcal{H}$ with weight $\{Z_k\}_{k \in \mathbb{N}_0}$ such that $Z_k$’s have closed range and injective for all $k \in \mathbb{N}$. Suppose that $\gamma(Z_k \otimes I_\mathcal{H}) \geq 1, k \in \mathbb{N}$ and satisfies the following inequality
\[
(Z_{k+n}(I_E \otimes Z_{k+n-1}) \cdots (I_{E^{\otimes k-1}} \otimes Z_{n+1}) \otimes I_\mathcal{H})^*(Z_{k+n}(I_E \otimes Z_{k+n-1}) \cdots (I_{E^{\otimes k-1}} \otimes Z_{n+1}) \otimes I_\mathcal{H})
\]
\[
- I_{E^{\otimes k+n} \otimes \mathcal{H}} \leq d_k(I_{E^{\otimes k-1}} \otimes (Z_{n+1} \otimes I_\mathcal{H})^*(Z_{n+1} \otimes I_\mathcal{H})) - I_{E^{\otimes k+n} \otimes \mathcal{H}}), \quad n \in \mathbb{N}
\]
along with $\sum_{k \geq 2} \frac{1}{d_k} = \infty$, then $(\rho, S)$ admits GWS-property.

**Proof.** Since $Z_k$’s are injective, $N(\tilde{S}) = \{0\}$ and thus the covariant representation $(\rho, S)$ is regular. The Moore-Penrose inverse of $\tilde{S}$, $\tilde{S}^\dagger : \mathcal{F}(E) \otimes \mathcal{H} \to E \otimes \mathcal{F}(E) \otimes \mathcal{H}$ is
\[
\tilde{S}^\dagger(\eta) = \bigoplus_{n \in \mathbb{N}} (Z_n \otimes I_\mathcal{H})^\dagger(\eta_n \otimes h_n), \quad \eta = \oplus_{n \in \mathbb{N}_0} \eta_n \otimes h_n \in \mathcal{F}(E) \otimes \mathcal{H}.
\]

Suppose if $\gamma(Z_n \otimes I_\mathcal{H}) \geq 1$, for $n \in \mathbb{N}$, then by Proposition 4.5 we get $\|\tilde{S}^\dagger(\eta)\|^2 = \bigoplus_{n \in \mathbb{N}} \|(Z_n \otimes I_\mathcal{H})^\dagger(\eta_n \otimes h_n)\|^2 \leq \sum_{n \in \mathbb{N}_0} \|\eta_n \otimes h_n\|^2$. This demonstrates that $\tilde{S}^\dagger$ is contraction and hence $\gamma(\tilde{S}) \geq 1$. Also, it is easy to see that $R^\infty(\tilde{S}) = \{0\}$. Now, we only need to show that $\tilde{S}$ satisfies the growth condition. For $\xi_1, \ldots, \xi_k$ are in $E, \eta_n \in E^{\otimes n}$ and $h \in \mathcal{H}$, a straightforward calculation follows that
\[ \tilde{S}_k(\bigotimes_{q=1}^k \xi_q \otimes \eta_n \otimes h) = (W_k \otimes I_\mathcal{H})(\bigotimes_{q=1}^n \xi_q \otimes \eta_n \otimes h_n) = (DT_{\xi_1} \otimes I_\mathcal{H}) \cdots (DT_{\xi_k} \otimes I_\mathcal{H})(\eta_n \otimes h_n)
\]
\[ = ((Z_{k+n}(I_E \otimes Z_{k+n-1}) \cdots (I_{E^{\otimes k-1}} \otimes Z_{n+1})) \otimes I_\mathcal{H})(\bigotimes_{q=1}^n \xi_q \otimes \eta_n \otimes h_n). \]
More generally, for $\zeta \in E^{\otimes k}$
\[ \tilde{S}_k(\zeta \otimes \oplus_{n \in \mathbb{N}_0} \eta_n \otimes h_n) = \sum_{n \in \mathbb{N}_0} ((Z_{k+n}(I_E \otimes Z_{k+n-1}) \cdots (I_{E^{\otimes k-1}} \otimes Z_{n+1})) \otimes I_\mathcal{H})(\zeta \otimes \eta_n \otimes h_n). \]
Since the weight sequence \( \{Z_k\}_{k \in \mathbb{N}_0} \) satisfies the Inequality (7.1) and by using previous equation, we obtain
\[
\| \widetilde{S}_k(\zeta \otimes \eta_n \otimes h_n) \| \leq d_k \left( \| (I_{E \otimes k-1} \otimes \widetilde{S})(\zeta \otimes \eta_n \otimes h_n) \| \right)^2 \leq d_0 \left( \| (I_{E \otimes k-1} \otimes \widetilde{S}_0)(\zeta \otimes \eta_n \otimes h_n) \| \right)^2
\]
along with \( \sum_{k \geq 2} \frac{1}{d_k} = \infty \), where \( \zeta \in E^\otimes k, \eta_n \in E^\otimes n \) and \( h_n \in \mathcal{H} \). Therefore \((\rho, S)\) satisfies the growth condition (5.6). Hence by Theorem 5.10 \((\rho, S)\) admits GWS-property. □

7.2. Bilateral Shift. Fix \( n \in \mathbb{N}, I_n = \{1, 2, \ldots, n\} \). Let \( \mathcal{H} \) be a separable Hilbert space with an orthonormal basis \( \{e_m : m \in \mathbb{Z}\} \). Consider a bounded set of complex numbers \( \{w_{i,m} \in \mathbb{C} : i \in I_n, m \in \mathbb{Z}\} \). Define an \( n \)-tuple of bounded operator \( V = (V_1, \ldots, V_n) \) on \( \mathcal{H} \) by (7.2)
\[
V_i(e_m) = w_{i,m} e_{i+n,m}, \ m \in \mathbb{Z}, \ i \in I_n.
\]

The operator \( V \) can be consider as a bounded operator from \( \bigoplus_{i=1}^n \mathcal{H} \) to \( \mathcal{H} \). Note that \( V = (V_1, V_2, \ldots, V_n) \) is non-commutative. Indeed, for distinct \( i, j \in I_n \), \( V_i V_j(e_m) = w_{i,m} w_{i,m} e_{i+n(j+nm)} \neq w_{i,m} w_{i,j+nm} e_{i+n(j+nm)} = V_j V_i(e_m) \). Suppose that for each \( i \in I_n, w_{i,0} = 0 \). Let \( h \in \mathcal{H} \) and \( h = \sum_{m \in \mathbb{Z}} a_m e_m, a_m \in \mathbb{C} \), we have
\[
V_i(h) = \sum_{m \in \mathbb{Z}} a_m V_i(e_m) = \sum_{m \in \mathbb{Z}\setminus\{0\}} a_m w_{i,m} e_{i+n,m},
\]
then \( \text{ran}(V_i) = \overline{\text{span}}\{e_{i+n,m} : m \in \mathbb{Z}\setminus\{0\}\} \). Since \( \{i + nm : m \in \mathbb{Z}\} \cap \{j + nm : m \in \mathbb{Z}\} = \emptyset \) for distinct \( i, j \in I_n \) and \( N(V_i) = \text{span}\{e_0\} \), \( \text{ran}(V_i) \perp \text{ran}(V_j) \) and it follows that
\[
\text{ran}(V) = \bigoplus_{i=1}^n \text{ran}(V_i) = \bigoplus_{i=1}^n \overline{\text{span}}\{e_{i+n,m} : m \in \mathbb{Z}\setminus\{0\}\} \quad \text{and} \quad N(V) = \bigoplus_{i=1}^n N(V_i) = \bigoplus_{i=1}^n \text{span}\{e_0\}.
\]

Now for \( l \geq 2 \) and \( i \in I_n \), we have for \( h = \sum_{m \in \mathbb{Z}} a_m e_m \in \mathcal{H} \),
\[
V_i^l(h) = \sum_{m \in \mathbb{Z}} a_m V_i^l(e_m) = \sum_{m \in \mathbb{Z}\setminus\{0\}} a_m w_{i,m} \left( \prod_{j=1}^{l-2} w_{i,(\sum_{k=0}^{l-2} in^{k+n+1})} \right) e_{(\sum_{k=0}^{l-1} in^{k+n+1})},
\]
where \( m \in \mathbb{Z}\setminus A_{i,l}, A_{i,l} = \{0, \sum_{k=0}^{l-2} in^k | 2 \leq j \leq l\} \). Therefore \( \text{ran}(V_i^l) = \overline{\text{span}}\{e_{(\sum_{k=0}^{l-1} in^{k+n+1})} : m \in \mathbb{Z}\setminus A_{i,l}\} \). Consider the set \( A_i = \bigcup_{l \in \mathbb{N}} A_{i,l} \), where \( A_{i,0} = \{0\}, i \in I_n \), then \( R^\infty(V_i) = \overline{\text{span}}\{e_{(\sum_{k=0}^{l-1} in^{k+n+1})} : m \in \mathbb{Z}\setminus A_i\} \) and
\[
R^\infty(V) = \bigoplus_{i=1}^n R^\infty(V_i) = \bigoplus_{i=1}^n \bigcap_{l=0}^\infty \overline{\text{span}}\{e_{(\sum_{k=0}^{l-1} in^{k+n+1})} : m \in \mathbb{Z}\setminus A_i\}.
\]

For each \( i \in I_n, e_0 \in \text{ran}(V_i^l) \) for all \( l \in \mathbb{N} \), then \( e_0 \in R^\infty(V_i) \) and it follows that \( N(V) = \bigoplus_{i=1}^n \overline{\text{span}}\{e_0\} \subseteq \bigoplus_{i=1}^n R^\infty(V_i) \), this shows that \( V \) is regular. Also, \( N(V_i) = \text{span}\{e_0\} : i \in I_n \) and \( \mathcal{H}_0 = \overline{\text{span}}\{e_m : m \in \mathbb{Z}\setminus\{0\}\} \).

Note that the map \( V_0 : \bigoplus_{i=1}^n \mathcal{H}_0 \to \bigoplus_{i=1}^n \text{ran}V_i \) defined by \( V_0|_{\mathcal{H}_0} = V_i|_{\mathcal{H}_0} : \mathcal{H}_0 \to \text{ran}V_i \) where \( \mathcal{H}_0 \) is at \( i \)th position, is invertible. Let \( e_{i+n,m} \in \text{ran}V_i, m \neq 0, V_i^{-1}(e_{i+n,m}) = \frac{1}{w_{i,m}} e_m \in \mathcal{H}_0 \), and also \( \text{ran}(V_i) \)'s are closed and orthogonal to each other. Therefore, the Moore-Penrose
inverse $V^\dagger : \mathcal{H} \to \bigoplus_{i=1}^n \mathcal{H}$, is of the form $V^\dagger = (V_1^\dagger, \ldots, V_n^\dagger)$, where $V_i^\dagger$ is Moore-Penrose inverse of $V_i$. More generally, for $h = \bigoplus_{i=1}^n h_i, h_i \in \text{ran}(V_i)$, then $h_i = \sum_{m \in \mathbb{Z}\setminus\{0\}} \alpha_{i+nm} e_{i+nm}$ for some $\alpha_{i+nm} \in \mathbb{C}$ and it yields that

$$V^\dagger h = V_0^{-1} h = \left( \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{\alpha_{1+nm}}{w_{1,m}} e_m, \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{\alpha_{2+nm}}{w_{2,m}} e_m, \ldots, \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{\alpha_{n+nm}}{w_{n,m}} e_m \right).$$

Suppose that $|w_{i,m}| \geq 1$ for all $i \in I_n, m \in \mathbb{Z}\setminus\{0\}$. Let $h' = h + h_0 \in \text{ran}(V) \oplus \text{ran}(V)^\perp$, using the previous equation, we get $\|V^\dagger h'\|^2 = \sum_{i=1}^n \sum_{m \in \mathbb{Z}\setminus\{0\}} |\alpha_{i+nm}|^2 \leq \sum_{i=1}^n \sum_{m \in \mathbb{Z}\setminus\{0\}} |\alpha_{i+nm}|^2 \leq \|h\|^2 \leq \|h'\|^2$. This shows that $V^\dagger$ is contraction, and thus Proposition 4.5 implies that $\gamma(V) \geq 1$.

Let $E$ be an $n$-dimensional Hilbert space with an orthonormal basis $\{\delta_i\}_{i \in I_n}$. Define a completely bounded covariant representation $(\rho, S^w)$ of $E$ on $\mathcal{H}$ by $\rho(a) = a I_{\mathcal{H}}$ and $S^w(\delta_i) = V_i, a \in \mathbb{C}, i \in I_n$, where $(V_1, \ldots, V_n)$ is in $(7.2)$. The representation $(\rho, S^w)$ is called bilateral weighted shift with weight $\{w_{i,m} : 1 \leq i \leq n, m \in \mathbb{Z}\}$.

Let $l \geq 2$ and $i \in I_n$, we can deduce that for $h = \sum_{m \in \mathbb{Z}} a_m e_m \in \mathcal{H}, a_m \in \mathbb{C}$,

$$S^w(\delta_i)^l h = \sum_{m \in \mathbb{Z}} a_m S^w(\delta_i)^l (e_m) = \sum_{m \in \mathbb{Z}} a_m w_{i,m} \left( \prod_{j=2}^l w_{i,(\sum_{k=0}^{j-2} i n^k + n^j m - 1)} \right) e_{\left( \sum_{k=0}^{j-1} i n^k + n^j m \right)}.$$

where $A_{i,l} = \{0, \sum_{k=0}^{j-2} i n^k | 2 \leq j \leq l \}$ and it follows that $\text{ran} S^w(\delta_i)^l = \overline{\text{span}} \{ e_{\sum_{k=0}^{j-1} i n^k + n^j m} | m \in \mathbb{Z} \setminus A_{i,l} \}$.

For $i \in I_n$, consider the set $A_i = \bigcup_{l \in \mathbb{N}} A_{i,l}$ then $R^\infty(S^w(\delta_i)) = \bigcap_{l=0}^\infty \overline{\text{span}} \{ e_{\sum_{k=0}^{j-1} i n^k + n^j m} : m \in \mathbb{Z} \setminus A_i \}$ and thus

$$R^\infty(S^w) = \bigoplus_{i=1}^n R^\infty(S^w_i) = \bigoplus_{i=1}^n \bigcap_{l=0}^\infty \overline{\text{span}} \{ e_{\sum_{k=0}^{j-1} i n^k + n^j m} : m \in \mathbb{Z} \setminus A_i \}.$$

Note that for $i \in I_n, e_0 \in \text{ran}(S^w(\delta_i)^l), \text{ and thus } e_0 \in R^\infty(S^w_n)$. It follows that $N(S^w) = \bigoplus_{i=1}^n \text{span} \{ \delta_i \otimes e_0 \} \subseteq E \otimes \bigoplus_{i=1}^n R^\infty(S^w_i)$, that is, $(\rho, S^w)$ is regular. Furthermore, based on the previous observation for $V_i$ $S^w$ is contraction, and thus $\gamma(S^w) \geq 1$. Next, we must determine under what condition on the set $\{w_{i,m} : 1 \leq i \leq n, m \in \mathbb{Z}\}$, $(\rho, S^w)$ satisfies the growth condition [3,6].

**Theorem 7.4.** Let $(\rho, S^w)$ be a bilateral weighted shift with weight $\{w_{i,m} : i \in I_n, m \in \mathbb{Z}\}$ on a separable Hilbert space $\mathcal{H}$ such that

(i) for each $i \in I_n, w_{i,m} = 1$ for all $m < 0, w_{i,0} = 0$ and $w_{i,m} \geq 1$ for every $m > 0$.

(ii) $\left( w_{i,m} \prod_{q=0}^{k-1} w_{i,q,n^k-q+1} \right) - 1 \leq d_k (w_{i,m}^2 - 1)$, where $(d_k)_{k \in \mathbb{N}}$ is positive sequence along with $\sum_{k=2}^\infty \frac{1}{d_k} = \infty$ and $n_{p,k} = \sum_{l=p}^\infty n^l p_{l} + n^{k-p+1} m$.

Then

1. $(\rho, S^w)$ satisfies the assumption of Theorem 5.10
2. $S^w$ is a regular contraction,
3. the subspace $\bigcup_{l=1}^n \{ e_{\sum_{k=0}^{j-1} i n^k + n^j m} : m \in \mathbb{Z} \setminus A_i, l \in \mathbb{N} \} \setminus \{ e_0 \}$ reduces $(\rho, S^w)$ and the restriction of $(\rho, S^w)$ on this subspace is isometric and fully co-isometric.
Proof. Since \( w_{i,m} = 1 \) for all \( m < 0 \), \( w^2_{i,k,m} \prod_{q=0}^{k-1} w^2_{i,k-q,n_{k-q+1,k}} \) is either 1 or 0 for all \( m < 0 \) and \( k \geq 2 \). Then, we get

\[
(7.3) \quad \left( w^2_{i,k,m} \prod_{q=0}^{k-1} w^2_{i,k-q,n_{k-q+1,k}} \right) - 1 \leq d_k \left( w^2_{i,k,m} - 1 \right), \quad k \geq 2, \ m \in \mathbb{Z} \setminus \{0\}.
\]

For \( k \in \mathbb{N} \), \( E^{\otimes k} \otimes \mathcal{H} = \operatorname{span}\{ \bigotimes_{q=1}^{k} \delta_q \otimes e_m : 1 \leq i_1, i_2, \ldots, i_k \leq n, \ m \in \mathbb{Z} \} \), by the previous discussion

\[
\tilde{S}^w_k(\eta_k) = \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n, \ m \in \mathbb{Z} \setminus \{0\}} \alpha_{i_1,i_2,\ldots,i_k,m} w_{i,k,m} \left( \prod_{q=0}^{k-1} w^2_{i,k-q,n_{k-q+1,k}} e_{n_1,k} \right),
\]

where \( \eta_k = \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n, \ m \in \mathbb{Z}} \alpha_{i_1,i_2,\ldots,i_k,m} \bigotimes_{q=1}^{k} \delta_q \otimes e_m \in E^{\otimes k} \otimes \mathcal{H} \) and \( n_{p,k} = \sum_{l=p}^{k} n^{l-p} i_l + n^{k-p+1} m \). Then

\[
\|\tilde{S}^w_k(\eta_k)\|^2 = \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n, \ m \in \mathbb{Z} \setminus \{0\}} |\alpha_{i_1,i_2,\ldots,i_k,m}|^2 w^2_{i,k,m} \prod_{q=0}^{k-1} w^2_{i,k-q,n_{k-q+1,k}}.
\]

Also, note that

\[
(I_{E^{\otimes k-1}} \otimes \tilde{S}^w)(\eta_k) = \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n, \ m \in \mathbb{Z}} \alpha_{i_1,i_2,\ldots,i_k,m} \bigotimes_{q=1}^{k-1} \delta_q \otimes \tilde{S}^w(\delta_{i_k} \otimes e_m),
\]

where \( m \in \mathbb{Z} \setminus \{0\} \) and \( \| (I_{E^{\otimes k-1}} \otimes \tilde{S}^w)(\eta_k) \|^2 = \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n, \ m \in \mathbb{Z} \setminus \{0\}} |\alpha_{i_1,i_2,\ldots,i_k,m}|^2 w^2_{i,k,m} \). Similarly

\[
(I_{E^{\otimes k-1}} \otimes \tilde{S}^w)^\dagger \tilde{S}^w(\eta_k) = \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n, \ m \in \mathbb{Z} \setminus \{0\}} \alpha_{i_1,i_2,\ldots,i_k,m} \bigotimes_{q=1}^{k-1} \delta_q \otimes \tilde{S}^w(\delta_{i_k} \otimes e_m),
\]

This implies that \( \| (I_{E^{\otimes k-1}} \otimes \tilde{S}^w)^\dagger \tilde{S}^w(\eta_k) \|^2 = \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n, \ m \in \mathbb{Z} \setminus \{0\}} |\alpha_{i_1,i_2,\ldots,i_k,m}|^2. \)
Finally, it follows from the above and by using the Equation (7.3), we obtain
\[ \|\tilde{S}_V^w(\eta_k)\|^2 \leq d_k\left(\|(I_{E\otimes k-1} \otimes \tilde{S}_V^w)(\eta_k)\|^2 - \|(I_{E\otimes k-1} \otimes \tilde{N}_{S}^{sw})\|^2\right) + \|(I_{E\otimes k-1} \otimes \tilde{S}_V^w)(\eta_k)\|^2, \]
which means \((\rho, S^w)\) satisfies the growth condition and it satisfies the hypothesis of Theorem 5.10 Since \(R^\infty(S^w)\) and \(\text{span}\{e_m : m \in \mathbb{Z} \setminus \{0\}\}\) are both \((\rho, S^w)\) reducing subspaces, \(R^\infty(S^w) \cap \text{span}\{e_m : m \in \mathbb{Z} \setminus \{0\}\} = V_i = 1\) is also \((\rho, S^w)\) reducing. Observe that \(E \otimes R^\infty(S^w) \cap N(S^w) = E \otimes V_i = 1\) and by using part 5 of Theorem 5.10, the restriction of \((\rho, S^w)\) on \(V_i = 1\) is isometric and fully co-isometric. This proves part 3 of the theorem.

**Remark 7.5.** Let \(k \in \mathbb{N}\), define a set \(\Gamma(k, I_n) = \{f : f_k \to I_n\}\) and \(\Gamma(0, I_n) = \{0\}\). Consider the Hilbert space
\[ \bigoplus_{f \in \Gamma(k, I_n)} \mathcal{H} = \bigoplus_{j=1}^{n^k} \bigoplus_{i=1}^{n^{k-1}} \left( \bigoplus_{j=1}^{n} \mathcal{H} \right) \]
with an orthonormal basis \(\{e_{m}^{(f,k)} : m \in \mathbb{Z}, f \in \Gamma(k, I_n)\}\), where \(e_{m}^{(f,k)} := e_m\) is the \(f\) position in the Hilbert space \(\bigoplus_{f \in \Gamma(k, I_n)} \mathcal{H}\). For \(k \in \mathbb{N}\) and \(f \in \Gamma(k, I_n)\), define \(V_f = V_{f(1)}V_{f(2)} \cdots V_{f(k)}\), where \(V_0 = I_{\mathcal{H}}\), and \(V^k = (V_f)_{f \in \Gamma(k, I_n)}\) be a \(|\Gamma(k, I_n)|\)-tuple of bounded operators on the Hilbert space \(\mathcal{H}\). The operator \(V^k\) can be consider as a bounded operator from \(\bigoplus_{f \in \Gamma(k, I_n)} \mathcal{H}\) to \(\mathcal{H}\).

Since \(\text{ran}V_i\) is orthogonal to each other, \(V^iV = (V^1V_1, V^2V_2, \ldots, V^nV_n)\). For \(k \in \mathbb{N}\), define \([V^iV]_k = (V^1V, V^2V, \ldots, V^nV) : \bigoplus_{j=1}^{n^{k-1}} \left( \bigoplus_{i=1}^{n} \mathcal{H} \right) \to \bigoplus_{j=1}^{n^{k-1}} \left( \bigoplus_{i=1}^{n} \mathcal{H} \right)\) and \([V]_k = (V, V, \ldots, V) : \bigoplus_{j=1}^{n^{k-1}} \left( \bigoplus_{i=1}^{n} \mathcal{H} \right) \to \bigoplus_{j=1}^{n^{k-1}} \mathcal{H}\) are both \(n^{k-1}\)-tuples of bounded operators on the Hilbert sapce \(\bigoplus_{i=1}^{n} \mathcal{H}\). Observe that for \(f \in \Gamma(k, I_n)\),
\[ V_f(e_{m}^{(f,k)}) = V_{f(1)}V_{f(2)} \cdots V_{f(k)}(e_{m}^{(f,k)}) = V_{f(1)}V_{f(2)} \cdots V_{f(k-1)}w_f(k),m e_{f(k)+nm} \]
\[ = V_{f(1)}V_{f(2)} \cdots V_{f(k-2)}w_f(k),m w_f(k-1),f(k)+nm e_{f(k-1)+nm} \]
\[ = \cdots \]
\[ = \prod_{i=1}^{k} w_f(k-i)\sum_{j=1}^{n} f(k-i+j)n^{j-1}+n^{i}m e_{\sum_{l=0}^{k-1} n^{l}f(k-l)+n^{k}m} \]
Assuming Equation (7.3) and by the previous equation, we can verify that
\[ ||V^k(h_f)||^2 \leq d_m(||[V]_k h_f||^2 - ||[V^iV]_k h_f||^2) + ||[V^iV]_k h_f||^2, \]
for all \(k \in \mathbb{N}\) and \(h_f \in \bigoplus_{f \in \Gamma(k, I_n)} \mathcal{H} = \bigoplus_{j=1}^{n^{k-1}} \left( \bigoplus_{i=1}^{n} \mathcal{H} \right)\), that is, the operator \(V\) satisfies the growth condition \([5,6]\). Hence \(V\) satisfies the hypothesis of Theorem 5.10. In particular, if \(n = 1\) then the subspace \(\text{span}\{e_m : m < 0\}\) is reducing for \(V\) and the restriction of \(V\) on the subspace \(\text{span}\{e_m : m < 0\}\) is unitary (see \([7, Proposition 9]\)). Note that, the number \(n\) may chosen to be \(\infty\) (infinity) also.
Acknowledgment. We are thankful to the referee for the valuable sugesstions and comments.
Azad Rohilla is supported by a UGC fellowship (File No: 16-6(DEC.2017) /2018(NET/CSIR)). Shankar Veerabathiran thanks ISI Bangalore for Visiting Scientist position. Harsh Trivedi is supported by MATRICS-SERB Research Grant, File No: MTR/2021/000286, by the Science and Engineering Research Board (SERB), Department of Science & Technology (DST), Government of India. We acknowledge the Centre for Mathematical & Financial Computing and the DST-FIST grant for the financial support for the computing lab facility under the scheme FIST (File No: SR/FST/MS-I/2018/24) at the LNMIIT, Jaipur.

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