A canonical form for Gaussian periodic processes

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Abstract: This article provides a representation theorem for a set of Gaussian processes; this theorem allows to build Gaussian processes with arbitrary regularity and to write them as limit of random trigonometric series. We show via Karhunen-Loève theorem that this set is isometrically equivalent to \( \ell^2 \). We then prove that regularity of trajectory path of anyone of such processes can be detected just by looking at decrease rate of \( \ell^2 \) sequence associated to him via isometry.

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1. Introduction

The aim of this article is to provide a simple and general method for constructing continuous and periodic Gaussian processes of arbitrary regularity. Periodic real processes arise as natural tool in analyzing continuous processes on the circle (e.g., in image analysis, when processing noises of closed lines). More recently, they found a great development in the theory of Random Fields on the sphere (see [8] and the reference therein).

The Brownian Bridge \( B_t \) is a well known example of a continuous and periodic Gaussian process on \([0, 1]\). Although it is not stationary, it becomes stationary if we remove the path-integral \( \int_0^1 B_s ds \), see [4]. If we model a noise with such a process, its supremum may be used to test the null hypothesis. Moreover, this supremum is the limiting distribution of an optimal test statistic for the uniformity of the distribution on a circle, [4]; in Section 2.1, we show that this process is strongly related to one generated by \( B_t \) by randomly choosing the starting point on \([0, 1]\). To generalize this example, we propose an approach which is linked to Karhunen-Loève’s expansion that gives uncorrelated coefficient. This expansion is optimal in regression functional studies, as shown in [3]. In fact, periodic processes are easily decomposed with Fourier basis, and stationarity will cause the coefficients of the sin and cos of the same frequency to be equal. Thus, we underline a natural isometry between this representation of Gaussian
processes and $\ell^2$. The asymptotic decay of the Karhunen-Loève’s coefficients will be related to the regularity of the paths, as a consequence of Fourier analysis. Thus, it will be possible to define a “periodic fractional Brownian motion” by choosing an appropriate asymptotic decay (see also [1, 5]) of the coefficients.

For what concerns notations, $s, t, \ldots$ relates to time variables, and will often belong to $[0, 1]$. We denote by $\{x_t\}_{t \in [0,1]}, \{y_t\}_{t \in [0,1]}, \ldots$ stochastic adapted processes defined on a given filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,1]}, \mathbb{P})$, while $(X_n)_n, (Y_n)_n, (Z_n)_n, \ldots$ are sequences of random variables. $\mathcal{C}(s, t)$ is a positive semidefinite function (it will be the correlation function of a stochastic process). When a process is stationary, its covariance function will often be replaced by the associated covariogram function $\tilde{\mathcal{C}}(t)$. The sequence $(e_k(t))_{k \in \mathbb{N}}$ denotes a sequence of orthogonal functions on $L^2([0,1])$. Finally, we denote by $|t|_1$ the fractional part sawtooth function of the real number $t$, which is defined by the formula $|t|_1 = t - \text{floor}(t)$.

2. A canonical form for Gaussian periodical processes

The first example of signal theory usage in the description of stochastic processes can be found in [6], where is exposed a theorem that allows to represent Gaussian processes as limits of stochastic Fourier series. The classical statement of Karhunen-Love’s theorem is the following, as described in [2].

**Theorem 2.1** (Karhunen-Love). Let $\{x_t\}_{t \in [a,b]}, a, b < \infty$, such that $E[x_t] = 0$, $\forall t \in [a, b]$, and $\text{Cov}(x_t, x_s) = \mathcal{C}(t, s)$, continuous in both variables. Then

$$x_t = \sum_{k=1}^{\infty} Z_k e_k(t), \quad a \leq t \leq b,$$

where $e_k$ are the eigenfunction of following integral operator from $L^2[a, b]$ in itself

$$f \in L^2[a, b] \rightarrow g(t) = \int_a^b \mathcal{C}(t, \tau)f(\tau)d\tau, \quad a \leq t \leq b,$$

and $e_k$ form an orthonormal bases for the space spanned by eigenfunctions corresponding to nonzero eigenvalues. The $Z_k$ are given by

$$Z_k = \int_a^b x_t e_k(t)dt$$

and are orthogonal random variables ($E[Z_k Z_j] = 0$ for $k \neq j$), with zero mean and variance $\lambda_k^2$, where $\lambda_k^2$ is the eigenvalue corresponding to $e_k$. The series $\sum_{k=1}^{\infty} Z_k e_k(t)$ converges in mean square to $x_t$, uniformly in $t$, that is

$$E\left( (x_t - \sum_{k=1}^{\infty} Z_k e_k(t))^2 \right) \xrightarrow{n \to \infty} 0$$

uniformly for $t \in [a, b]$. Moreover if $x_t$ is Gaussian, the $Z_k$ in expansion are real independent Gaussian random variables.
In this paper we provide a result that allows to build Gaussian processes of arbitrary regularity, proceeding on the way tracked by Adler (see [1]). The result will be based on Karhunen-Love’s decomposition theorem, and will deal with following set of processes.

**Definition 2.1.** \( \mathcal{H} \) is the set of real Gaussian stochastic processes \( \{x_t\}_{t \in [0,1]} \) such that they are continuously stationary (so if \( C(s,t) \) is the covariance function then there exist a real continuous function \( \tilde{C}(s-t) = C(s,t) \forall s,t, \mathbb{R} \)), periodical (i.e. \( x_0 = x_1, \text{a.s.} \)) with \( \mathbb{E}(x_t) = 0, \forall t \in \mathbb{R} \).

The set \( \mathcal{H} \) is a Banach space, when it is equipped with the inner product given by

\[
\langle \{x_t\}_{t \in [0,1]}, \{y_t\}_{t \in [0,1]} \rangle = \int_0^1 \mathbb{E}(x_t y_t) dt \in \mathbb{R}_+.
\]

**Remark.** We remark that if \( \{x_t\}_{t \in [0,1]} \in \mathcal{H} \) and if \( \tilde{C}(s-t) = C(s,t) \) is its covariogram function, then \( \tilde{C}(t_0) = \tilde{C}(t_0 + 1) \).

We are going to specialize Karhunen-Love’s decomposition theorem to \( \mathcal{H} \), showing that a process is in \( \mathcal{H} \) if and only if it can be written as limit of a canonical trigonometric random series. From this result we will show that \( \mathcal{H} \) may be seen as a Hilbert space, isometrically equivalent to the space of the coefficients \( \ell^2 \); via this isometry it will be easy to create Gaussian stationary processes with arbitrary regularity, by looking at decreasing speed rate of canonical series coefficients. Moreover, this result allows also to detect information about regularity of a process, since it relates it with the regularity of its covariogram function.

**Theorem 2.2.** Let \( \{x_t\}_{t \in [0,1]} \in \mathcal{H} \) with covariance \( C(s,t) = \tilde{C}(t-s) \); then in mean square, uniformly in \( t \),

\[
x_t = c_0 Y'_0 + \sum_{k=1}^{\infty} c_k (Y_k \sqrt{2} \sin(2k\pi t) + Y'_k \sqrt{2} \cos(2k\pi t))
\]

where \( (Y_n)_n, (Y'_n)_n \) are two independent sequence of independent standard Gaussian variables, and \( (c_k) \in \ell^2 \) is such that

\[
c_n^2 = \int_0^1 \tilde{C}(s) \cos(2n\pi s) ds, \quad n = 0, 1, 2, \ldots
\]

**Proof.** By Mercer Theorem (see, e.g., [2]) we know that if \( (e_n)_n \) is an orthonormal bases for the space spanned by the eigenfunctions corresponding to nonzero eigenvalues of integral operator

\[
x \in L^2[0,1] \rightarrow y(t) = \int_0^1 C(t,\tau)x(\tau)d\tau, \quad a \leq t \leq b,
\]

then, uniformly, absolutely and in \( L^2[0,1] \times [0,1], C(s,t) = \sum_{k=0}^{\infty} \epsilon_k(t)e_k(s)\lambda_k \), where \( \lambda_k \) is the eigenvalue corresponding to \( \epsilon_k \). By Remark 2 we are going to
see that \((\cos(2n\pi s), \sin(2n\pi s))_n\) are eigenfunctions relative to operator whose kernel is \(C(s, t)\). In fact, let \(a_n = \int_0^1 \tilde{C}(s) \cos(2n\pi s)ds\), then
\[
\int_0^1 \cos(2n\pi t) \tilde{C}(t - \tau) dt = a_n \cos(2n\pi \tau),
\]
the same relation holding when \(\cos\) is replaced by \(\sin\). It follows from Mercer Theorem that
\[
C(s, t) = a_0 + \sum_{k=1}^{\infty} 2a_k \cos(2k\pi(s - t))
\]
uniformly, absolutely and in \(L^2[0, 1] \times [0, 1]\) and that \((a_n)_n\) forms \((c_n)_n\) such that \(\tilde{C}(s, t) = a_0 + \sum_{k=1}^{\infty} 2a_k \cos(2k\pi(s - t))\).

**Theorem 2.3.** Let \((Y_n)_n, (Y'_n)_n\) be two independent sequence of independent standard Gaussian variables, and \((c_k)_k \in \ell^2\). Then the sequence
\[
y_t^{(n)} = c_0 Y'_0 + \sum_{k=1}^{n} c_k (Y_k \sqrt{2} \sin(2k\pi t) + Y'_k \sqrt{2} \cos(2k\pi t)).
\]
converges in mean square, uniformly in \(t\) to \(\{y_t\}_{t \in [0, 1]} \in \mathcal{H}\). Moreover if \(C(s, t)\) is the \(y_t\) covariance function, then uniformly, absolutely and in \(L^2[0, 1] \times [0, 1]\),
\[
C(s, t) = c_0^2 + \sum_{k=1}^{\infty} 2c_k^2 \cos(2k\pi(s - t)).
\]

**Proof.** First of all we notice that Gaussian process \(y_t^{(n)}\) converges to a periodical \(\{y_t\}_{t \in [0, 1]}\) in mean square uniformly in \(t\), because
\[
\sup_{t \in [0, 1]} E[|y_t^{(n)} - y_t^{(m)}|^2] = 2 \sum_{k=1}^{\infty} c_k^2 \xrightarrow{m,n} 0.
\]
Let’s look at \(\{y_t\}_{t \in [0, 1]}\) properties: it is a calculation to show that \(E[y_t] = 0\) for all \(t\), and that
\[
\text{Cov}(y_t, y_s) = c_0^2 + 2 \sum_{k=1}^{\infty} c_k^2 \cos(2k\pi(s - t)),
\]
which is a continuous function, and that
\[
E[y_t^2] = c_0^2 + 2 \sum_{k=1}^{\infty} c_k^2 = 2\|c_n\|^2 - c_0^2.
\]
Moreover \( \{y_t\}_{t \in [0,1]} \) is a Gaussian process, because the two sequences \((Y_n)_n\) and \((Y'_n)_n\) are Gaussians.

\[ \{y_t\}_{t \in [0,1]} \text{ is a Gaussian process, because the two sequences } (Y_n)_n \text{ and } (Y'_n)_n \text{ are Gaussians.} \]

**Corollary 2.1.** Let us consider a couple \( Z = ((\bar{Y}_n)_n,(\bar{Y}'_n)_n) \) of independent sequence of independent standard Gaussian variables. For each \( \{z_t\}_{t \in [0,1]} \in \mathcal{H} \), there exists an \( \{x_t\}_{t \in [0,1]} \in \mathcal{H}_Z \) having the same law, where

\[
\mathcal{H}_Z = \left\{ \{x_t\}_{t \in [0,1]} \in \mathcal{H} : x_t = a_0 \bar{Y}'_0 + \sqrt{2} \sum_{k=1}^{\infty} a_k (\bar{Y}_k \sin(2k \pi t) + \bar{Y}'_k \cos(2k \pi t)), (a_n)_n \in \ell^2 \right\}
\]

and the limit is in mean square and uniformly in \( t \).

**Geometry of \( \mathcal{H}_Z \): isometry with \( \ell^2 \)**

For \( (a_n)_n \in \ell^2 \), define

\[
x_t = a_0 \bar{Y}'_0 + \sqrt{2} \sum_{k=1}^{\infty} a_k (\bar{Y}_k \sin(2k \pi t) + \bar{Y}'_k \cos(2k \pi t)).
\]

From Theorem 2.2 and Theorem 2.3 it follows that \( \|x_t\|_{\mathcal{H}} = \sqrt{a_0^2 + 2 \sum_{n} a_n^2} \), and hence it is naturally defined an isometry between the representative space \( \mathcal{H}_Z \) and \( \ell^2 \).

**Regularity of the paths**

We have seen that to each stochastic process in \( \mathcal{H} \) can be associated a sequence in \( \ell^2 \). We are now showing how are related the decrease rate of such sequence with the regularity of the process trajectory path. We first recall a classic regularity theorem.

**Theorem 2.4 (see [10]).** Let \( \{x_t\}_{t \in [0,1]} \) a real stochastic process such that there exist three positive constants \( \gamma, c \) and \( \epsilon \) so that

\[
E\left(|x_t - x_s|^{\gamma}\right) \leq c|t-s|^{1+\epsilon};
\]

so there exists a modification \( \{\tilde{x}_t\}_{t \in [0,1]} \) of \( \{x_t\}_{t \in [0,1]} \), such that

\[
E((\sup_{s \neq t} |\tilde{x}_t - \tilde{x}_s|^{\gamma}) < \infty
\]

for all \( \alpha \in [0,\frac{\gamma}{2}] \); in particular the trajectories of \( \{\tilde{x}_t\}_{t \in [0,1]} \) are Holder continuous of order \( \alpha \).
It is simple to apply this last theorem to processes staying in \( H \). It is well known that if \( Y \approx N(0, \sigma^2) \), then \( E(|Y|^p) = \sigma^p \frac{2^p \Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}} \), (see, e.g., [9]). From this fact we deduce the following result.

**Theorem 2.5.** Assume that \( \{x_t\}_{t \in [0,1]} \in H \) and let \( C(s, t) = \tilde{C}(s - t) \) be its covariance function. If \( \tilde{C} \) is Holder continuous of order \( \alpha \), then almost all trajectories of \( \{x_t\}_{t \in [0,1]} \) are Holder continuous of order \( \beta < \frac{\alpha}{2} \).

**Proof.** Since

\[
E(|x_{t+h} - x_t|^2) = E(x_t^2 + x_{t+h}^2 - 2x_{t+h}x_t) = 2(\tilde{C}(0) - \tilde{C}(h)) \leq M|h|^\alpha,
\]

we deduce that

\[
E(|x_{t+h} - x_t|^{2p}) = 2^p C_p (\tilde{C}(0) - \tilde{C}(h))^p \leq M|h|^{p\alpha}
\]

so, by Theorem 2.4, if \( p\alpha = 1 + \epsilon \), for all \( p > \frac{1}{\alpha} \), almost all trajectories of \( \{x_t\}_{t \in [0,1]} \) are Holder continuous of order \( \beta \), with \( \beta < \frac{p\alpha - 1}{2p} \); but this is true for each \( p > \frac{1}{\alpha} \), we conclude that almost all trajectory path of \( \{x_t\}_{t \in [0,1]} \) are Holder continuous of order \( \beta < \frac{\alpha}{2} \).

A very useful result for our analysis will be the following one, whose proof may be found in [7].

**Theorem 2.6 (Boas’ Theorem).** Let \( f \in L^1[0,1] \) be a function whose Fourier expansion has only nonnegative cosine terms, and let \( (a_n) \) be the sequence of its cosine coefficient. Then

\[
\text{f is Holder continuous of order } \alpha \iff a_k = O\left(\frac{1}{k^{\alpha+1}}\right).
\]

Boas’ Theorem may be used in connection with Theorem 2.2 and Theorem 2.3 to deduce more regularity properties of the processes in \( H \). In fact, take \( (c_n) \) as in Theorem 2.2 and Theorem 2.3. From Boas’ Theorem we have that if \( k^2 c_k^2 = O\left(\frac{1}{k^{2\alpha+1}}\right) \) for \( 0 < \alpha \leq 1 \), then \( \partial^2 \tilde{C} \) is Holder continuous of order \( \alpha \). This link between the regularity of \( \tilde{C} \) and the paths of \( \{x_t\}_{t \in [0,1]} \) is underlined in the following theorem.

**Theorem 2.7.** With the notations of Theorem 2.3, if \( c_k^2 = O\left(\frac{1}{k^{3+\alpha}}\right) \), then almost all trajectories of \( \{x_t\}_{t \in [0,1]} \) are Lipshitz continuous, and, as function of \( t \), \( \{x_t\}_{t \in [0,1]} \) have a continuous derivative \( \{x'_t\}_{t \in [0,1]} \) Holder continuous of order \( \beta < \frac{\alpha}{2} \).

**Proof.** It is clear that

\[
\partial^2 \tilde{C}(\delta) = 2\partial^2 \sum_{k=1}^{\infty} c_k^2 \cos(2k\pi(\delta)) = -2 \sum_{k=1}^{\infty} (2\pi)^2 k^2 c_k^2 \cos(2k\pi(\delta))
\]
and that $\partial^2 \tilde{C}$ is Holder continuous of order $\alpha$, for some $0 < \alpha \leq 1$. Moreover we have that uniformly in $t$ and in mean square

$$x_t = c_0 Y'_0 + \sqrt{2} \sum_{k=1}^{\infty} c_k (Y_k \sin(2k\pi t) + Y'_k \cos(2k\pi t)).$$

and, from Theorem 2.2, there also exist a stochastic process in $\mathcal{H}$ such that uniformly in $t$ and in mean square

$$\tilde{x}_t = 2\sqrt{2\pi} \sum_{k=1}^{\infty} kc_k (Y_k \cos(2k\pi t) - Y'_k \sin(2k\pi t)),$$

which has covariogram function Holder continuous of order $\alpha$ given by

$$\tilde{C}(\delta) = 2 \sum_{k=1}^{\infty} (2\pi)^2 k^2 c_k^2 \cos(2k\pi \delta).$$

If we define

$$y^{(n)}(t) := c_0 Y'_0 + \sqrt{2} \sum_{k=1}^{n} c_k (Y_k \sin(2k\pi t) + Y'_k \cos(2k\pi t))$$

$$\tilde{y}^{(n)}(t) := 2\sqrt{2\pi} \sum_{k=1}^{n} kc_k (Y_k \cos(2k\pi t) - Y'_k \sin(2k\pi t)),$$

than $y^{(n)}(t) = y^{(n)}_0 + \int_0^t \tilde{y}^{(n)}(\tau) d\tau$, a.s. for any $n$, while for each fixed $t$, in mean square we have $\int_0^t \tilde{y}^{(n)}(\tau) d\tau \to \int_0^t \tilde{x}_t d\tau$. Since

$$\sqrt{E((x_t - x_0 - \int_0^t \tilde{x}_t d\tau)^2) \leq \sqrt{E((y^{(n)}(t))}^2 + \sqrt{E((y^{(n)}_0 + \int_0^t \tilde{y}^{(n)}(\tau) d\tau - x_0 - \int_0^t \tilde{x}_t d\tau)^2)^2 \to 0,}$$

it follows that a.s. $x_t = x_0 + \int_0^t \tilde{x}_t d\tau$. By Theorem 2.5 we know that almost all trajectory path of $\tilde{x}_t$ are Holder continuous of order $\beta < \frac{\alpha}{2}$, and thesis follows.

A natural generalization of this result is the following:

**Corollary 2.2.** If, in previous notation, $c_k^2 = O(\frac{1}{k^{1+2m+\delta}})$ then almost all trajectory path of $\{\partial^k x_t\}_{t \in [0,1]}$ with $k < m$, are Lipschitz continuous, and admit, as function of $t$, a continuous derivative $\{\partial^m x_t\}_{t \in [0,1]}$ Holder continuous of order $\beta$, for all $\beta < \frac{\alpha}{2}$. \qed
2.1. The centered Brownian bridge

Take a Brownian bridge \( \{ x_t \}_{t \in [0,1]} \). This process is Gaussian, periodic but not stationary, since \( x_0 = x_1 \equiv 0 \). If we randomize the starting point of the process, by shifting the \( t \)-axis of a \([0,1]\)-uniform random variable \( U \), we obtain the process

\[
\hat{x}_t = x_{1-U^1},
\]

which is expected to belong to \( \mathcal{H} \). A process with the law of \( \{ \hat{x}_t \}_{t \in [0,1]} \) is called centered Brownian bridge. Let us recall that a Brownian bridge may also be represented as

\[
x_t = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} W_k \frac{\sin(k\pi t)}{k}.
\]

One may expect that the periodic process

\[
y_t = aY_0 + \sqrt{2} \sum_{k=1}^{\infty} \frac{1}{2k\pi} (Y_k \sin(2k\pi t) + Y'_k \cos(2k\pi t)),
\]

which shares the same asymptotic behavior of the coefficients of \( \{ x_t \}_{t \in [0,1]} \) is closely related to \( \{ \hat{x}_t \}_{t \in [0,1]} \). The next theorem shows this facts.

**Theorem 2.8.** A centered Brownian bridge \( \{ \hat{x}_t \}_{t \in [0,1]} \) given in (2.1) belongs to \( \mathcal{H} \), with covariogram function \( \tilde{C}(\delta) = \left( |\delta| - \frac{1}{2} \right) + \frac{1}{24} \). It may be represented as

\[
\hat{x}_t = \sqrt{\frac{1}{12}} Y'_0 + \sqrt{2} \sum_{k=1}^{\infty} \frac{1}{2k\pi} (Y_k \sin(2k\pi t) + Y'_k \cos(2k\pi t)).
\]

**Proof.** By conditioning on \( U \), it is simple to prove that \( E(\hat{x}_t) = 0 \) and \( E(\hat{x}_s \hat{x}_t) = \frac{(s-t-1/2)^2}{2} + \frac{1}{\pi} = \tilde{C}(|t - s|) \). A straightforward calculation gives

\[
c^2_0 = \int_0^1 \tilde{C}(s) \, ds = \frac{1}{12}, \quad c^2_n = \int_0^1 \tilde{C}(s) \cos(2n\pi s) \, ds = \frac{1}{(2\pi n)^2}, \quad n \geq 1,
\]

and hence the Karhunen-Loève theorem gives

\[
\hat{x}_t = \sqrt{\frac{1}{12}} Y'_0 + \sum_{k=1}^{\infty} \frac{1}{2k\pi} (Y_k \sqrt{2} \sin(2k\pi t) + Y'_k \sqrt{2} \cos(2k\pi t)).
\]

What remains to prove is that all the \( (Y_k, Y'_k) \)s are Gaussian. To sketch the proof for \( Y_k \) (the same arguments apply to \( Y'_k \)), we first recall that, conditioned on \( U = s \),

\[
Z_k := \frac{Y_k}{2\pi k} = \int_0^1 \hat{x}_t \sqrt{2} \sin(2k\pi t) \, dt = \int_0^1 \sqrt{2} \sum_{n=1}^{\infty} W_n \frac{\sin(n\pi(t-s)_1)}{n\pi} \sqrt{2} \sin(2k\pi t) \, dt
\]

\[
= \cos(2k\pi t) \frac{W_{2k}}{2k\pi} + \frac{2 \sin(2k\pi s)}{\pi^2} \sum_{n=1,n\neq 2k}^{\infty} W_n (-1)^n \frac{1}{4k^2 - n^2}.
\]
Therefore, the characteristic function \( \Phi_{s,Z_k}(t) \) of \( Z_k \) conditioned on \( U = s \) is
\[
\Phi_{s,Z_k}(t) = e^{-\frac{t^2}{2} \frac{4\sin^2(2k\pi s)}{\pi^2}} \sum_{n=1, n \neq 2k}^{\infty} \left( \frac{(-1)^{n-1}}{4k^2 - n^2} \right)^2
\]
and hence the characteristic function \( \Phi_{Z_k}(t) \) of \( Z_k \) is
\[
\Phi_{Z_k}(t) = \int_0^1 e^{-\frac{t^2}{2} \frac{4\sin^2(2k\pi s)}{\pi^2}} \sum_{n=1, n \neq 2k}^{\infty} \left( \frac{(-1)^{n-1}}{4k^2 - n^2} \right)^2 ds.
\]
Now, since
\[
\frac{1}{\pi} \sum_{n=1, n \neq 2k}^{\infty} \left( \frac{(-1)^n - 1}{4k^2 - n^2} \right)^2 = \frac{1}{k^2 \pi^4} \sum_{m=0}^{\infty} \left( \frac{1}{2k + 2m + 1} + \frac{1}{2k - 2m - 1} \right)^2 = \frac{1}{4k^2 \pi^2}
\]
we get \( \Phi_{Z_k}(t) = e^{-\frac{t^2}{2} \frac{4\sin^2(2k\pi s)}{\pi^2}}, \) which concludes the proof.

The word “centered” in the definition of \( \{\hat{x}_t\}_{t \in [0,1]} \) is clearly related to the randomization of the starting point of the underlying Brownian bridge.

In fact, we can say more: \( \{\hat{x}_t\}_{t \in [0,1]} \) is strongly related to the \( y \)-centralization of the Brownian bridge, i.e. to the following process
\[
\hat{x}_t = x_t - \int_0^1 x_t \, dt, \tag{2.2}
\]
where \( \{x_t\}_{t \in [0,1]} \) is a Brownian bridge. This last process stays in \( \mathcal{H} \), and his sup is the limiting distribution of an optimal test statistic for the uniformity of the distribution on a circle, see [11, 4].

**Corollary 2.3.** If \( \{\hat{x}_t\}_{t \in [0,1]} \) is a centered Brownian Bridge, then it holds
\[
\hat{x}_t = \hat{\hat{x}}_t + Z
\]
where \( Z \) is an independent random variable with null expectation and variance \( 1/12 \), and \( \{\hat{\hat{x}}_t\}_{t \in [0,1]} \) is defined as in (2.2). Furthermore
\[
\hat{x}_t = \sqrt{2} \sum_{k=1}^{\infty} \frac{1}{2k \pi} (Y_k \sin(2k\pi t) + Y'_k \cos(2k\pi t)). \tag{2.3}
\]

**Proof.** The covariogram function of \( \{\hat{x}_t\}_{t \in [0,1]} \) is \( \hat{C}(\delta) = \frac{(\delta^2 - 1/2)^2}{2} - \frac{1}{\pi^2} \) (see [4]). It is sufficient to calculate the covariance function of \( \hat{x}_t + Z \) and to use the Corollary 2.1 to complete the proof.

### 2.2. A computational parametric model for smoothing

Results provided in this paper allows to create a Gaussian parametric family of stationary and periodic processes of arbitrary regularity. In fact, let us consider the following family of processes in \( \mathcal{H} \) that extends (2.3)
\[
x_t = \sum_{k=1}^{\infty} \frac{a}{k^6} (Y_k \sin(2k\pi t) + Y'_k \cos(2k\pi t)). \tag{2.4}
\]
Theorem 2.5 states that the paths become more regular as $p$ increases. This property is shown in Figure 1, which suggests how to smooth a process by changing $p$.

Summing up, model (2.4) gives a family of Gaussian processes whose trajectories are arbitrarily regular. In application, maximum likelihood estimates of $a$ and $p$ is a straightforward consequence of a FFT of the observed discretized process $\{x_t\}_{t \in [0,1]}$.

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