Conformal Kaehler Euclidean submanifolds

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Abstract

Let \( f : M^{2n} \to \mathbb{R}^{2n+\ell}, n \geq 5 \), denote a conformal immersion into Euclidean space with codimension \( \ell \) of a Kaehler manifold of complex dimension \( n \) and free of flat points. For codimensions \( \ell = 1, 2 \) we show that such a submanifold can always be locally obtained in a rather simple way, namely, from an isometric immersion of the Kaehler manifold \( M^{2n} \) into either \( \mathbb{R}^{2n+1} \) or \( \mathbb{R}^{2n+2} \), the latter being a class of submanifolds already extensively studied.

Throughout the paper \( f : M^{2n} \to \mathbb{R}^{2n+\ell} \) denotes a conformal Kaehler submanifold, that is, \( (M^{2n}, J) \) is a connected Kaehler manifold of complex dimension \( n \geq 2 \) and \( f \) a conformal immersion into Euclidean space with codimension \( \ell \). That the immersion is conformal means that there is a positive function \( \lambda \in C^\infty(M) \) such that the metric induced by \( f \) is related to the original Kaehler metric by \( \langle \cdot, \cdot \rangle_f = \lambda^2 \langle \cdot, \cdot \rangle_{M^{2n}} \). The immersion is called a real Kaehler submanifold if \( \lambda \equiv 1 \). Our goal is to describe, up to a conformal congruence of the ambient space, the local situation of the conformal Kaehler submanifold submanifolds if \( \ell \) is at most two. Recall that two immersions \( f, g : M^n \to \mathbb{R}^N \) are said to be conformally congruent if \( g = \tau \circ f \) for some conformal (Moebius) transformation \( \tau \) of \( \mathbb{R}^N \).

In our first result, by a real Kaehler hypersurface we mean a real Kaehler submanifold with codimension one of a manifold free of flat points. These submanifolds have been locally classified by Dajczer and Gromoll \[5\] by means of the Gauss parametrization in terms of a pseudoholomorphic surface in a sphere and a smooth function on the surface. Florit and Zheng \[9\] showed that metrically complete real Kaehler hypersurfaces are just cylinders over a surface in \( \mathbb{R}^3 \).

**Theorem 1.** Any conformal immersion \( f : M^{2n} \to \mathbb{R}^{2n+1}, n \geq 4 \), of a simply connected Kaehler manifold free of flat points is conformally congruent to a real Kaehler hypersurface.

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For codimension two, simple examples of conformal Kaehler submanifolds are obtained by composing a holomorphic hypersurface $M^2 \rightarrow \mathbb{C}^{n+1}$ or the extrinsic product of a pair of real Kaehler hypersurfaces with a conformal transformation of the ambient space. But there are many other examples of real Kaehler submanifolds that can be composed with a conformal ambient transformation; for instance see Dajczer and Gromoll [6] for the class of complex ruled submanifolds, including the metrically complete that are among the ones produced by a Weierstrass type representation. See also Dajczer and Florit [4] for the case of submanifolds of rank two.

Theorem 2. Let $f : M^{2n} \rightarrow \mathbb{R}^{2n+2}$, $n \geq 5$, be a conformal Kaehler submanifold where $M^{2n}$ is free of flat points. Then there is an open dense subset $M_0$ of $M^{2n}$ such that along any connected component $N^{2n}$ of $M_0$ one of the following holds:

(i) $f|_N$ is conformally congruent to a real Kaehler submanifold $g : N^{2n} \rightarrow \mathbb{R}^{2n+2}$.

(ii) $f|_N = h \circ g$ is a composition of a real Kaehler hypersurface $g : N^{2n} \rightarrow \mathbb{R}^{2n+1}$ and a conformal immersion $h : V \rightarrow \mathbb{R}^{2n+2}$ where $V \subset \mathbb{R}^{2n+1}$ is open and $g(N) \subset V$.

Notice that $h$ in part (ii) is just a conformally flat hypersurface. The submanifolds in this class have been parametrically described by do Carmo, Dajczer and Mercuri [2].

1 Preliminaries

1.1 The isometric light-cone representative

The light-cone $V^{m+1}$ of the standard flat Lorentzian space $L^{m+2}$ is one of the two connected components of the set of all light-like vectors, that is,

$$\{v \in L^{m+2} : \langle v, v \rangle = 0, \ v \neq 0\}$$

endowed with the degenerate metric inherited from $L^{m+2}$.

The Euclidean space $\mathbb{R}^m$ can be realized as an umbilic hypersurface of $V^{m+1}$ as follows: Given light-like vectors $v, w \in L^{m+2}$ such that $\langle v, w \rangle = 1$ and a linear isometry $C : \mathbb{R}^m \rightarrow \{v, w\}^\perp$, define $\Psi : \mathbb{R}^m \rightarrow V^{m+1} \subset L^{m+2}$ by

$$\Psi(x) = v + Cx - \frac{1}{2}||x||^2w.$$ 

Then $\Psi$ is an isometric embedding of $\mathbb{R}^m$ as an umbilical hypersurface in the light cone given as an intersection of $V^{m+1}$ with an affine hyperplane, namely,

$$\Psi(\mathbb{R}^m) = \{y \in V^{m+1} : \langle y, w \rangle = 1\}.$$ 

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The normal bundle of $\Psi$ is $N_\Psi \mathbb{R}^m = \text{span}\{\Psi, w\}$ and the second fundamental form is

$$\alpha^\Psi(X, Y) = -\langle X, Y \rangle_{\mathbb{R}^m} w.$$ 

We observe that $\Psi(\mathbb{R}^m)$ is independent of the triple $v, w, C$ in the sense that different triples produce submanifolds congruent by an isometry of $\mathbb{L}^{m+2}$.

If $f: M^n \to \mathbb{R}^m$ is a conformal immersion with conformal factor $\lambda \in C^\infty(M)$, then the isometric immersion

$$F = \frac{1}{\lambda} \Psi \circ f: M^n \to \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2},$$

is called the isometric light-cone representative of $f$. The normal bundle of $F$ decomposed orthogonally as

$$N_F M = \Psi_* N_f M \oplus L^2$$

such that $F \in \Gamma(L^2)$ and the second fundamental form of $F$ satisfies

$$\langle \alpha^F(X, Y), F \rangle = -\langle X, Y \rangle$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$. The full expression of the second fundamental form of $F$ as well as additional information on the isometric light-cone representatives can be found in [7] and [10].

**Proposition 3.** Two conformal immersions $f, g: M^n \to \mathbb{R}^m$ are conformally congruent if and only if their isometric light-cone representatives $F, G: M^n \to \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ are isometrically congruent.

**Proof:** See Proposition 9.18 in [7].

**Proposition 4.** Let $F: M^n \to \mathbb{V}^{n+p+1} \subset \mathbb{L}^{n+p+2}$ be an isometric immersion that carries a normal light-like vector field $\delta$ that is constant in $\mathbb{L}^{n+p+2}$ and satisfies $\langle F, \delta \rangle = 1$. If $M^n$ is simply connected there exists an isometric immersion $f: M^n \to \mathbb{R}^{n+p}$ that has $F$ as its isometric light-cone representative.

**Proof:** With respect to the orthogonal splitting $N_F M = \text{span}\{\delta, F\} \oplus L$ we have

$$\alpha^F(X, Y) = -\langle X, Y \rangle \delta + \alpha_L(X, Y)$$

where $\alpha_L = \pi_L \circ \alpha^F$. Clearly $\alpha_L: TM \times TM \to L$ satisfies the Gauss equation. In fact, being $F$ a normal vector field parallel in the normal connection and $\delta$ constant, it is easy to see that $\alpha_L$ also satisfies the Codazzi and Ricci equations when $L$ is taken with the induced connection $(\nabla^\perp)_L$ from $N_F M$. Hence, there are an isometric immersion $f: M^n \to \mathbb{R}^{n+p}$ and a vector bundle isometry $\phi: L \to N_f M$ such that

$$f \nabla^\perp \circ \phi = \phi \circ (\nabla^\perp)_L$$

and $\alpha^f = \phi \circ \alpha_L$. [3]
Let $G = \Psi \circ f : M^n \to \mathbb{V}^{n+p+1} \subset \mathbb{L}^{n+p+2}$ be the isometric light-cone representative of $f$. Then $N_G M = \text{span} \{G, w\} \oplus \Psi_* N_f M$ and

\[ \alpha^G(X, Y) = -\langle X, Y \rangle w + \Psi_* \alpha^f(X, Y). \] (4)

Let $T : N_G M \to N_F M$ be the vector bundle isometry defined by $T w = \delta$, $T \circ G = F$ and $T \Psi_* \xi = \phi^{-1} \xi$ for $\xi \in N_f M$. Using (2), (3) and (4) we obtain that $T \circ \alpha^G = \alpha^F$ and $F \nabla^\perp \circ T = T \circ G \nabla^\perp$. Hence $F$ and $G$ are isometrically congruent.

1.2 Flat bilinear forms

Let $V^n$ and $W^{p,p}$ be real vector spaces of dimensions $n$ and $2p$, respectively, where the latter is endowed with an inner product of signature $(p, p)$. This means that $p$ is the dimension of the subspaces of maximal dimension where the inner product is either positive or negative definite. A vector subspace $L \subset W^{p,p}$ is called degenerate if $L \cap L^\perp \neq \{0\}$ and nondegenerate otherwise.

A bilinear form $\beta : V^n \times V^n \to W^{p,p}$ (maybe not symmetric) is called flat if

\[ \langle \beta(X, Y), \beta(Z, T) \rangle - \langle \beta(X, T), \beta(Z, Y) \rangle = 0 \]

for all $X, Y, Z, T \in V^n$. It is said that $\beta$ is null when

\[ \langle \beta(X, Y), \beta(Z, T) \rangle = 0 \]

for all $X, Y, Z, T \in V^n$. Thus null bilinear forms are trivially flat. We denote

\[ S(\beta) = \text{span} \{\beta(X, Y) : X, Y \in V^n\} \]

and

\[ N(\beta) = \{Y \in V^n : \beta(X, Y) = 0 \text{ for all } X \in V^n\}. \]

**Proposition 5.** Let $V^n$ and $U^p$, $2p < n$ and $1 \leq p \leq 5$, be real vector spaces such that there is $J \in \text{End}(V)$ satisfying $J^2 = -I$ and $U^p$ has an inner product of any signature. Let $\alpha : V^n \times V^n \to U^p$ be a symmetric bilinear form and let $\beta : V^n \times V^n \to U^p \oplus U^p$ be the bilinear form given by

\[ \beta(X, Y) = (\alpha(X, Y), \alpha(X, JY)). \] (5)

Assume that $\beta$ is flat when $W^{p,p} = U^p \oplus U^p$ is endowed with the inner product given by

\[ \langle \langle (\xi_1, \xi_2), (\eta_1, \eta_2) \rangle \rangle = \langle \xi_1, \eta_1 \rangle_{U^p} - \langle \xi_2, \eta_2 \rangle_{U^p}. \] (6)

If the subspace $S(\beta)$ is nondegenerate then $\dim N(\beta) \geq n - 2p$. 
Proof: For $p \leq 5$ the proof of Proposition 10 in [1], where the inner product on $U^p$ is positive definite and $\alpha$ satisfies a certain condition, can be adapted to this case. With the notations in there and using the same type of arguments used there it is easy to conclude that the only cases one needs to consider are $2 \leq \kappa < \tau \leq p - 1$ where $\kappa$ and $\tau$ are even. Thus, we only have to deal with the case $\tau = 4$ and $\kappa = 2$.

By Fact 11 in [1] there exist $Y_1, Y_2 \in RE^p(\beta) \cap RE(\hat{\beta})$ such that

$$\hat{U}(X) = \hat{B}Y_1(V) + \hat{B}Y_2(V).$$

We cannot have $B_{Y_j}(N) = U(X), j = 1, 2$, since otherwise dim $U(Y_j) \leq 3 < \tau$. Therefore, by Fact 12 in [1] it remains to consider the case dim $B_{Y_j}(N) \leq 2, j = 1, 2$. Set $B_1 = B_{Y_1}|_{N}: N \to U(X), N_1 = \ker B_1, B_2 = B_{Y_2}|_{N_1}: N_1 \to U(X)$ and $N_2 = \ker B_2$. Then $N_2 \subset N(\beta)$ and

$$\dim N(\beta) \geq \dim N_2 \geq \dim N_1 - 2 \geq \dim N - 4 \geq n - 2p,$$

and this concludes the proof.

2 The proofs

The following application of Proposition 5 is the main ingredient in the proofs of the theorems in this paper.

Proposition 6. Let $V^n$ and $U^p$, $2p < n$ and $1 \leq p \leq 5$, be real vector spaces such that there is $J \in \text{End}(V)$ satisfying $J^2 = -I$ and $U^p$ carries an either positive definite or Lorentzian inner product. Assume that the bilinear form $\beta: V^n \times V^n \to W^{p,p} = U^p \oplus U^p$ defined by (5) is flat with respect to the inner product (6). If dim $N(\beta) \leq n - 2p - 1$ then $U = S(\beta) \cap S(\beta)^\perp$ satisfies dim $U = s > 0$ is even. Moreover, let $L \subset U^p$ denote the projection of $U$ on the first factor of $W^{p,p}$. Then, we have:

(i) If the subspace $L$ is nondegenerate then dim $L = s$ and $L$ inherits a positive definite inner product. With respect to the orthogonal splitting $U^p = L \oplus L^\perp$ we denote $\alpha_1 = \pi_L \circ \alpha$ and $\alpha_2 = \pi_{L^\perp} \circ \alpha$. Then

$$\alpha_1(X, JY) = \alpha_1(JX, Y) \text{ for all } X, Y \in V^n$$

and

$$\dim N(\alpha_2) \cap JN(\alpha_2) \geq n - 2(p - s).$$

(ii) If the subspace $L$ is degenerate let $0 \neq \delta \in L \cap L^\perp$. Then there is an orthogonal splitting $U^p = U_0 \oplus U_1^{s-2} \oplus U_2^{p-s}$, $s = 2$ or $4$, with $U_0 = \text{span} \{\delta, \zeta\}$, where $\zeta \in U^p$.
is a light-like vector satisfying \( \langle \delta, \zeta \rangle = 1 \), and \( L = \text{span} \{ \delta \} \oplus U^{s-2} \) such that \( \alpha_j = \pi_{U_j} \circ \alpha \), \( j = 0, 1, 2 \), satisfy

\[
\langle \alpha(X, Y), \delta \rangle = 0 \quad \text{and} \quad \alpha_1(X, JY) = \alpha_1(JX, Y) \quad \text{for all} \quad X, Y \in V^n
\]

and

\[
\dim N(\alpha_2) \cap JN(\alpha_2) \geq n - 2(p - s).
\]

Proof: By Proposition 3, we have \( s > 0 \). If \( 0 \neq (\xi, \bar{\xi}) \in U \), then

\[
(\xi, \bar{\xi}) = \sum_i \beta(X_i, Y_i) = \sum_i (\alpha(X_i, Y_i), \alpha(X_i, JY_i))
\]

and

\[
0 = \left\langle \langle \beta(X, Y), (\xi, \bar{\xi}) \rangle \right\rangle = \langle \alpha(X, Y), \xi \rangle - \langle \alpha(X, JY), \bar{\xi} \rangle
\]

for any \( X, Y \in V^n \). Then \( (\xi, -\xi) = \sum_i \beta(X_i, JY_i) \in S(\beta) \) and

\[
\left\langle \langle (\beta(X, Y), (\xi, -\xi) \rangle \right\rangle = \langle \alpha(X, Y), \xi \rangle + \langle \alpha(X, JY), \xi \rangle = 0
\]

for any \( X, Y \in V^n \). Thus also \( (\xi, -\xi) \in U \). It follows that \( s \) is even and

\[
\pi_1(U) = L = \pi_2(U),
\]

where \( \pi_j : W^{p,p} \to U^p, j = 1, 2 \), denote the projections onto the factors.

Case (i): We have that the inner product induced on \( L \) is positive definite. In fact, if otherwise there are vectors \( \delta, \bar{\delta} \in L \) such that \( \delta \) is time-like and \( \langle \delta, \bar{\delta} \rangle, (\bar{\delta}, -\delta) \in U \). But then also \( \bar{\delta} \) would be a time-like vector orthogonal to \( \delta \) in contradiction with the signature of \( U^p \).

We have \( \beta = \beta_1 + \beta_2 \) where

\[
\beta_j(X, Y) = (\alpha_j(X, Y), \alpha_j(X, JY)), \quad j = 1, 2.
\]

Since \( \beta_1 \) is null, then

\[
0 = \left\langle \langle \beta_1(X, Y), \beta_1(Z, W) \rangle \right\rangle = \langle \alpha_1(X, Y), \alpha_1(Z, W) \rangle - \langle \alpha_1(X, JY), \alpha_1(Z, JW) \rangle.
\]

Then \( T : S(\alpha_1) \to S(\alpha_1) \) defined by

\[
T\alpha_1(X, Y) = \alpha_1(X, JY)
\]

is a linear isometry and

\[
\alpha_1(JX, Y) = \alpha_1(Y, JX) = T\alpha_1(Y, X) = T\alpha_1(X, Y) = \alpha_1(X, JY).
\]
Being $\beta$ flat and $\beta_1$ null, then also $\beta_2$ is flat. Since the subspace $S(\beta_2)$ is nondegenerate we have from Proposition 5 that

$$\dim N(\beta_2) \geq n - 2(p - s).$$

To conclude the proof of this case observe that $N(\beta_2) = N(\alpha_2) \cap JN(\alpha_2)$.

Case (ii): Let $\tilde{\delta} \in L$ be such that $(\delta, \tilde{\delta}), (\tilde{\delta}, -\delta) \in U$. Since the inner product on $U^p$ has Lorentzian signature, then the vectors $\delta, \tilde{\delta}$ must be linearly dependent. Thus $(\delta, 0), (0, \delta) \in U$, and hence

$$0 = \langle \beta(X, Y), (\delta, 0) \rangle = \langle \alpha(X, Y), \delta \rangle$$

for all $X, Y \in V^n$.

Assume $s = 2$, in which case $U = \text{span} \{ (\delta, 0), (0, \delta) \}$. We have $\alpha = \alpha_0 + \alpha_2$, where

$$\alpha_0(X, Y) = \langle \alpha(X, Y), \zeta \rangle \delta.$$ 

Hence $\beta = \beta_0 + \beta_2$, where

$$\beta_0(X, Y) = \langle \alpha(X, Y), \zeta \rangle (\delta, 0) + \langle \alpha(X, JY), \zeta \rangle (0, \delta)$$

and

$$\beta_2(X, Y) = (\alpha_2(X, Y), \alpha_2(X, JY)).$$

Because $\beta$ is flat and $\beta_0$ is null, then $\beta_2$ is flat. Since the subspace $S(\beta_2)$ is nondegenerate, then Proposition 5 gives

$$\dim N(\beta_2) \geq n - 2p + 4.$$ 

Assume $s = 4$. Then there are space-like vectors $\xi, \bar{\xi} \in L$ such that

$$U = \text{span} \{ (\delta, 0), (0, -\delta), (\xi, \bar{\xi}), (\bar{\xi}, -\xi) \}.$$ 

Set $U_1 = \text{span} \{ \xi, \bar{\xi} \}$ and choose $\zeta \perp U_1$. Let $\beta_j : V^n \times V^n \rightarrow U_j \oplus U_j$ be given by

$$\beta_j(X, Y) = (\alpha_j(X, Y), \alpha_j(X, JY)), \ j = 0, 1, 2.$$ 

Then $\beta = \beta_0 + \beta_1 + \beta_2$ where $\beta_0$ and $\beta_1$ are null.

If $T : U_1 \rightarrow U_1$ be the linear isometry defined by

$$T \alpha_1(X, Y) = \alpha_1(X, JY),$$

then

$$\alpha_1(JX, Y) = \alpha_1(Y, JX) = T \alpha_1(Y, X) = T \alpha_1(X, Y) = \alpha_1(X, JY).$$

Since $\beta_2$ is flat and $S(\beta_2)$ is a nondegenerate subspace, then Proposition 5 gives

$$\dim N(\beta_2) \geq n - 2p + 8,$$
We have that $U$ is free of flat points and let $\alpha^F: TM \times TM \to N_F M$ be the second fundamental form of its isometric light-cone representative $F: M^{2n} \to V^{2n+p+1} \subset \mathbb{L}^{2n+p+2}$. At any point $x \in M^{2n}$, let $\beta: T_x M \times T_x M \to W^{p+2,p+2} = N_F M(x) \oplus N_F M(x)$ be the bilinear form given by

$$\beta(X,Y) = (\alpha^F(X,Y), \alpha^F(X, JY))$$

(7)

where the inner product in $W^{p+2,p+2}$ is as in (3). Using the Gauss equation and that the curvature tensor of $M^{2n}$ satisfies $J \circ R(X,Y) = R(X,Y) \circ J$ it is easy to verify that $\beta$ is flat. Moreover, since

$$\langle \alpha^F(X,Y), F \rangle = -\langle X,Y \rangle$$

we have $N(\beta) \subset N(\alpha^F) = \{0\}$.

**Proof of Theorem 1.** From Proposition 6 applied at $x \in M^{2n}$ to $\beta: T_x M \times T_x M \to W^{3,3}$ defined by (7) in terms of $\alpha^F$ satisfying (11) we have $s(x) = 2$. We also have that $L$ is a degenerate subspace. In fact, if otherwise, by part (i) there exists an orthogonal splitting $N_F M(x) = L \oplus L^\perp$ such that $L$ inherits a positive definite inner product and $L^\perp = \text{span} \{\eta\}$ where $\eta \in N_F M(x)$ is a unit time-like vector. Moreover, the $J$-invariant subspace $\Delta = N(\beta_2)$ satisfies dim $\Delta \geq 2n - 2$ and, since $L = \text{span} \{\xi, \bar{\xi}\}$ where $(\xi, \bar{\xi}) \in \mathcal{U}$, then the shape operators of $F$ satisfy

$$A^F_\xi = -J A^F_\bar{\xi} \quad \text{and} \quad \Delta \subset \ker A^F_\eta.$$

Since $F \in N_F M(x)$ then $F = a\xi + b\bar{\xi} + c\eta$. Hence

$$JZ = -J A^F_\eta Z = -J(a A^F_\xi Z + b A^F_{\bar{\xi}} Z) = A^F_{a \xi - b \bar{\xi}} Z$$

for any $Z \in \Delta$. Therefore $J|_\Delta = A^F_{a \xi - b \bar{\xi}}|_\Delta$, and this is a contradiction.

Since the subspace $L$ is degenerate, by part (ii) at any point there is a splitting

$$N_F M = \text{span} \{\delta, \zeta\} \oplus U_2$$

such that $A^F_\delta = 0$ and the $J$-invariant subspace $\Delta = N(\beta_2)$ satisfies dim $\Delta \geq 2n - 2$. Moreover, since $M^{2n}$ is free of flat points then dim $\Delta = 2n - 2$.

Because $F, \delta \in N_F M(x)$ are linearly independent we may take $\zeta = F$. Hence, we have a normal basis $\{\delta, F, \xi\}$ with $\xi \perp \text{span} \{\delta, F\}$ of unit length such that

$$A^F_\delta = 0, \quad A^F_F = -I \quad \text{and} \quad \Delta \subset \ker A^F_\xi.$$

We have that $\mathcal{U} = S(\beta) \cap S(\beta)^+ \cap \text{span} \{\delta, F, \xi\}$ is a common choice for the isometric light-cone representative $F$. Hence, we have a normal basis $\{\delta, F, \xi\}$ of unit length such that $A^F_\delta = 0, A^F_F = -I$ and $\Delta \subset \ker A^F_\xi$. We have that $\mathcal{U} = S(\beta) \cap S(\beta)^+ \subset \mathcal{Z}$ has constant dimension and hence is smooth. It follows easily that also the frame $\{\delta, F, \xi\}$ can be taken to be smooth.
The Codazzi equation for $A^F_\delta$ is

$$A^F_{\nabla^\perp X} Y = A^F_{\nabla^\perp Y} X$$

for all $X, Y \in TM$. Using that $M^{2n}$ does not have flat points, it is not difficult to conclude that $\delta$ is parallel in the normal connection, and hence constant in the ambient space. Thus, as in the proof of Proposition 4, there exists an isometric immersion $g: M^{2n} \to \mathbb{R}^{2n+1}$ that has $A^\gamma = A^F_\xi$ as shape operator and its isometric light-cone representative $G = \Psi \circ g$ is isometrically congruent to $F$. Therefore, by Proposition 3, $f$ and $g$ are conformal.

For the proof of Theorem 2 we need the following two technical results.

**Lemma 7.** Let $g: M^n \to \mathbb{R}^{n+1}$ be an isometric immersion and let $f: M^n \to \mathbb{R}^{n+p}$ be a conformal immersion. Then let $G: M^n \to \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3}$ and $F: M^n \to \mathbb{V}^{n+p+1} \subset \mathbb{L}^{n+p+2}$ be the isometric light-cone representatives of $g$ and $f$, respectively. Given an open subset $U \subset M^n$, there exists a conformal immersion $h: \mathbb{V} \to \mathbb{R}^{n+p}$ of an open subset $V \supset g(U)$ of $\mathbb{R}^{n+1}$ such that $f|_U = h \circ g|_U$ if and only if there exists an isometric immersion $H: W \to \mathbb{V}^{n+p+1}$ of an open subset $W \subset \mathbb{V}^{n+2}$ with $G(U) \subset W$ such that $F|_U = H \circ G|_U$.

**Proof:** See Proposition 2 in [8].

**Lemma 8.** Let $F: M^n \to \mathbb{V}^{n+3} \subset \mathbb{L}^{n+4}$ be an isometric immersion and let $\xi$ be a normal vector field of unit length that satisfies $\langle \xi, F \rangle = 0$, rank $A^F_\xi = 1$ and is parallel along $\ker A^F_\xi$. Then, there exist open subsets $V \subset M^n$ and $W \subset \mathbb{V}^{n+2}$ and local isometric immersions $G: V \to \mathbb{V}^{n+2}$ and $H: W \to \mathbb{V}^{n+3}$ with $G(V) \subset W$ such that $F|_V = H \circ G$.

**Proof:** See Lemma 2 in [3].

**Proof of Theorem 2.** We proceed making use of the definitions and notations in the proof of Theorem 1. Proposition 6 applied to $\beta: T_x M \times T_x M \to W^{4,4}$ at $x \in M^{2n}$ gives $s(x) = 2$ or $4$. In what follows we work on an open dense subset $M_*$ of $M^{2n}$ where $\dim S(\beta)$ is locally constant. Let $M_2$ be the open subset of $M_*$ where $s(x) = 2$. A similar argument as in the proof of Theorem 1 gives that the subspace $L$ is degenerate at each point of $M_2$. Then along $M_2$ there is a smooth orthogonal splitting of the normal bundle of $F$ as

$$N_F M = \text{span} \{ \delta, F \} \oplus P$$

with $\langle \delta, F \rangle = 1$ such that $A^F_\delta = 0$, $A^F_\xi = -I$ and $\Delta = N(\beta_2)$ satisfies $\dim \Delta \geq 2n - 4$.

Let $P = \text{span} \{ \xi_1, \xi_2 \}$ where the smooth frame is orthonormal. In the sequel, we work on a connected component $M'_2$ of the open subset of $M_2$ where $\dim \Delta$ and the
ranks of the $A^F_{\xi_j}$’s are locally constant. The corresponding Codazzi equation are

$$
\nabla_X A^F_{\xi_i} Y - A^F_{\xi_i} \nabla_X Y + \langle \nabla^\perp_X \xi_i, \delta \rangle Y - \langle \nabla^\perp_X \xi_i, \xi_j \rangle A^F_{\xi_j} Y = \nabla_Y A^F_{\xi_i} X - A^F_{\xi_i} \nabla_Y X + \langle \nabla^\perp_Y \xi_i, \delta \rangle X - \langle \nabla^\perp_Y \xi_i, \xi_j \rangle A^F_{\xi_j} X, \quad 1 \leq i \neq j \leq 2,
$$

(8)

for any $X, Y \in \mathfrak{X}(M)$. It follows that

$$
A^F_{\xi_i} [S, T] = \langle \nabla^\perp_S \xi_j, \delta \rangle T - \langle \nabla^\perp_T \xi_j, \delta \rangle S
$$

for any $S, T \in \Gamma(\Delta)$. Hence $\langle \nabla^\perp_S \xi_j, \delta \rangle = 0$, $j = 1, 2$ and $S \in \Gamma(\Delta)$. Then (8) yields

$$
-A^F_{\xi_i} \nabla_X S + \langle \nabla^\perp_X \xi_i, \delta \rangle S = \nabla_S A^F_{\xi_i} X - A^F_{\xi_i} \nabla_X S - \langle \nabla^\perp_S \xi_i, \xi_j \rangle A^F_{\xi_j} X, \quad 1 \leq i \neq j \leq 2,
$$

for any $S \in \Gamma(\Delta)$ and $X \in \mathfrak{X}(M)$. In particular,

$$
\langle \nabla^\perp_X \delta, \xi_j \rangle \langle S, T \rangle = \langle \nabla_S T, A^F_{\xi_j} X \rangle, \quad j = 1, 2,
$$

(9)

for any $X \in \mathfrak{X}(M)$ and $S, T \in \Gamma(\Delta)$. Thus

$$
\langle \nabla_S T \rangle \text{Im} A^F_{\xi_j} = \langle S, T \rangle Z_j, \quad j = 1, 2,
$$

(10)

where $Z_j \in \Gamma(\text{Im} A^F_{\xi_j})$ and $S, T \in \Gamma(\Delta)$. Now (9) reads as

$$
\langle \nabla^\perp_X \delta, \xi_j \rangle = \langle A^F_{\xi_j} Z_j, X \rangle, \quad j = 1, 2,
$$

(11)

for any $X \in \mathfrak{X}(M)$.

We denote

$$
R(x) = \text{span} \{ A^F_{\xi_j} Z_j(x) : x \in M'_2, \quad j = 1, 2 \}.
$$

We claim that the open subset $N_2 \subset M'_2$ defined by

$$
N_2 = \{ x \in M'_2 : \text{dim} \ R(x) = 2 \}
$$

is empty. In fact, the Codazzi equation for $A^F_{\xi}$ is

$$
\langle \nabla^\perp_X \delta, \xi_1 \rangle A^F_{\xi_1} Y + \langle \nabla^\perp_X \delta, \xi_2 \rangle A^F_{\xi_2} Y = \langle \nabla^\perp_Y \delta, \xi_1 \rangle A^F_{\xi_1} X + \langle \nabla^\perp_Y \delta, \xi_2 \rangle A^F_{\xi_2} X
$$

(12)

for any $X, Y \in \mathfrak{X}(M)$. We have from (11) that $\delta$ is parallel along $R^\perp$. Hence

$$
\langle \nabla^\perp_X \delta, \xi_1 \rangle A^F_{\xi_1} X + \langle \nabla^\perp_X \delta, \xi_2 \rangle A^F_{\xi_2} X = 0
$$

(13)

for any $X \in \Gamma(R^\perp)$ and $Y \in \mathfrak{X}(M)$. In particular, the vectors $A^F_{\xi_1} X, A^F_{\xi_2} X$ cannot be linearly independent for any $X \in R^\perp$. If otherwise (13) yields that $\delta$ is parallel, and then (11) gives $R = 0$, a contradiction.
We argue that

$$R^\perp \subset \ker A^F_{\xi_1} \cap \ker A^F_{\xi_2}. \quad (14)$$

Suppose that $X \notin \ker A^F_{\xi_1} \cap \ker A^F_{\xi_2}$ for $X \in \Gamma(R^\perp)$. By the above $A^F_{\xi_1}X = \gamma A^F_{\xi_2}X$ where $A^F_{\xi_2}X \neq 0$ but $\gamma \in C^\infty(N_2)$ may vanish. We obtain from (13) that

$$\gamma \langle \nabla^\perp Y, \xi_1 \rangle + \langle \nabla^\perp Y, \xi_2 \rangle = 0$$

for any $Y \in \mathfrak{X}(M)$. Then (11) gives $\gamma A^F_{\xi_1}Z_1 + A^F_{\xi_2}Z_2 = 0$, which is a contradiction.

Since $M^{2n}$ is free of flat points, we have from (14) that we may choose the frame $\xi_1, \xi_2$ such that $\ker A^F_{\xi_1} = R^\perp = \ker A^F_{\xi_2}$. From (10) we obtain $Z_1 = Z_2 = Z \in R$. Then (11) and (12) give

$$\langle A^F_{\xi_1}Z, X \rangle A^F_{\xi_1}Y + \langle A^F_{\xi_2}Z, X \rangle A^F_{\xi_2}Y = \langle A^F_{\xi_1}Z, Y \rangle A^F_{\xi_1}X + \langle A^F_{\xi_2}Z, Y \rangle A^F_{\xi_2}X \quad (15)$$

for any $X, Y \in \mathfrak{X}(M)$. By the Gauss equation (15) is equivalent to $R(X, Y)Z = 0$, and this is a contradiction since $M^{2n}$ is free of flat points. Thus the claim that $N_2$ is empty has been proved.

Let $N_1 \subset M'_2$ be the open subset defined as

$$N_1 = \{ x \in M'_2 : \dim R(x) = 1 \}.$$

Similarly as above, we obtain that $\delta$ is parallel along the hyperplane $R^\perp$ and that the vectors $A^F_{\xi_1}X, A^F_{\xi_2}X$ cannot be linearly independent for any $X \in R^\perp$. And from (11) we have that $\delta$ is not a parallel vector field. Hence, we can choose the frame $\{ \xi_1, \xi_2 \}$ for $P$ such that

$$\langle \nabla^\perp X, \xi_2 \rangle = 0 \quad (16)$$

for any $X \in \mathfrak{X}(M)$. It now follows from (12) that $R^\perp \subset \ker A^F_{\xi_1}$.

We have $R^\perp = \ker A^F_{\xi_1}$. In fact, otherwise $A^F_{\xi_1} = 0$ and the Codazzi equation gives

$$\langle \nabla^\perp X, \delta \rangle S = -\langle \nabla^\perp S, \xi_2 \rangle A^F_{\xi_2}X$$

for any $S \in \Gamma(\Delta)$ and $X \in \mathfrak{X}(M)$. It follows that $\delta$ is a parallel vector field, and this is a contradiction.

We obtain from (11) and (16) that $A^F_{\xi_2}Z_2 = 0$. Since $A^F_{\xi_2}|_{\text{Im } A^F_{\xi_2}}$ is an isomorphism, then $Z_2 = 0$. If $\text{Im } A^F_{\xi_1} \subset \text{Im } A^F_{\xi_2}$, it follows from (10) that $Z_1 = 0$. Thus $\text{Im } A^F_{\xi_1} \notin \text{Im } A^F_{\xi_2}$ and since $M^{2n}$ has no flat points we have that $2 \leq \dim \text{Im } A^F_{\xi_2} \leq 3$. In fact, it holds that $\dim \text{Im } A^F_{\xi_2} = 2$ since, otherwise, $\Delta^\perp = \text{Im } A^F_{\xi_1} \oplus \text{Im } A^F_{\xi_2}$. Then (10) gives

$$\langle \nabla S T \rangle_{\Delta^\perp} = \langle S, T \rangle Z_1$$

for any $S, T \in \Delta$, and hence

$$\langle S, T \rangle JZ_1 = J(\nabla S T)_{\Delta^\perp} = (J \nabla S T)_{\Delta^\perp} = (\nabla S J T)_{\Delta^\perp} = \langle S, J T \rangle Z_1.$$
Thus \( Z_1 = 0 \), and this is a contradiction.

If \( Y, Z \in \Gamma(\ker A_{\xi_1}^F) \) are linearly independent, then the Codazzi equation for \( A_{\xi_1}^F \) is

\[
A_{\xi_1}^F[Z,Y] = A_{\xi_2}^F((\nabla^\bot_Y \xi_1, \xi_2)Z - (\nabla^\bot_Z \xi_1, \xi_2)Y).
\]

Since \( \text{Im} \ A_{\xi_1}^F \not\subset \text{Im} \ A_{\xi_2}^F \) and \( A_{\xi_2}^F|_{\text{Im} A_{\xi_2}^F} \) is an isomorphism, then \( \langle \nabla^\bot_Y \xi_1, \xi_2 \rangle = 0 \) for any \( Y \in \Gamma(\ker A_{\xi_1}^F) \). Thus \( \xi_1 \) is parallel along \( \ker A_{\xi_1}^F \).

By Lemma 8 there exist an simply connected open neighborhood \( V \subset N_1 \) of any \( x \in N_1 \), an open subset \( W \subset V \) and local isometric immersions \( G: V \to \mathbb{R}^{2n+2} \) and \( H: W \to \mathbb{R}^{2n+3} \) with \( G(V) \subset W \) such that \( F|_V = H \circ G \). An elementary argument gives that there exists a conformal immersion \( g: V \to \mathbb{R}^{2n+1} \) that has \( G \) as isometric light-cone representative; see pg. 7 of [10] or Proposition 9.9 of [7]. By Lemma 7 there exist a conformal immersion \( h: U \to \mathbb{R}^{2n+2} \) such that \( f|_V = h \circ g|_V \) with \( g(V) \subset U \). Finally, by Theorem 1 applied to \( g \) we are as in part (ii) of Theorem 2.

Let \( N'_0 \) be an open simply connected subset of the set

\[
N_0 = \text{int}\{x \in M_2 : R(x) = 0\}.
\]

Then \( \delta \) is constant on \( N'_0 \) from (11). By Proposition 4 there is an isometric immersion \( g_0: N'_0 \to \mathbb{R}^{2n+2} \) whose isometric light-cone representative is \( F|_{N'_0} \). From Proposition 3 we are in part (i) of Theorem 2.

To conclude, let \( M_4 \subset M^{2n} \) be the interior of the set \( \{x \in M^{2n} : s(x) = 4\} \). Then \( L(x) \) for \( x \in M_4 \) is a degenerate subspace since, otherwise, \( N_F M_4(x) = L(x) \) which contradicts the fact that \( L(x) \) has a positive definite inner product. By Proposition 6 there is a smooth orthogonal vector bundle decomposition

\[
N_F M_4 = \text{span} \{\delta, F\} \oplus U^2_1
\]

such that \( A_{\delta}^F = 0, A_{F}^F = -I \) and \( \langle \delta, F \rangle = 1 \). Moreover, we have \( U^2_1 \) = span \( \{\xi_1, \xi_2\} \) and

\[
A_{\xi_1}^F = JA_{\xi_2}^F.
\]

Comparing the Codazzi equations for \( A_{\xi_1}^F \) and \( A_{\xi_2}^F \) by means of (17), it follows easily that \( \delta \in \Gamma(N_F M_4) \) is parallel, hence constant in \( \mathbb{L}^{n+4} \). Along any simply connected open subset of \( M_4 \) we now combine Proposition 4 and Lemma 7 to conclude that we are again in part (i) of Theorem 2. Notice that in this case \( f \) is minimal.

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