NON-TRIVIAL $m$-QUASI-EINSTEIN METRICS ON QUADRATIC LIE GROUPS

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Abstract. We call a metric $m$-quasi-Einstein if $\text{Ric}_m^X$, which replaces a gradient of a smooth function $f$ by a vector field $X$ in $m$-Bakry-Emery Ricci tensor, is a constant multiple of the metric tensor. It is a generalization of Einstein metrics which contains Ricci solitons. In this paper, we focus on left-invariant metrics on quadratic Lie groups whose Lie algebras are quadratic Lie algebras. First, we prove that $X$ is a left-invariant Killing field if the left-invariant metric on a quadratic Lie group is $m$-quasi-Einstein for $m$ finite. Then we prove by constructing that solvable quadratic Lie groups $G(n)$ admit infinitely many non-trivial $m$-quasi-Einstein metrics on $G(n)$ for $m$ finite. Finally we obtain a Ricci soliton on $G(n)$ which implies that the theorem in the first step is invalid for $m$ infinite.

1. Introduction

A natural extension of the Ricci tensor is the $m$-Bakry-Emery Ricci tensor

(1.1) \[ \text{Ric}_m^f = \text{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df \]

where $0 < m \leq \infty$, $f$ is a smooth function on $M^n$, and $\nabla^2 f$ stands for the Hessian form. Instead of a gradient of a smooth function $f$ by a vector field $X$, $m$-Bakry-Emery Ricci tensor was extended by Barros and Ribeiro Jr in [BR12] and Limoncu in [Lim10] for an arbitrary vector field $X$ on $M^n$ as follows:

(1.2) \[ \text{Ric}_m^X = \text{Ric} + \frac{1}{2} L_X g - \frac{1}{m} X^* \otimes X^* \]

where $L_X g$ denotes the Lie derivative on $M^n$ and $X^*$ denotes the canonical 1-form associated to $X$. With this setting $(M^n, g)$ is called an $m$-quasi-Einstein metric, if there exist a vector field $X \in \mathfrak{X}(M^n)$ and constants $m$ and $\lambda$ such that

(1.3) \[ \text{Ric}_m^X = \lambda g. \]

An $m$-quasi-Einstein metric is called trivial when $X \equiv 0$. The triviality definition is equivalent to say that $M^n$ is an Einstein manifold. When $m = \infty$, the equation (1.3) reduces to a Ricci soliton, for more details see [Cao09] and the references therein. When $m$ is a positive integer and $X$ is a gradient vector field, it corresponds to a warped product Einstein metric, for more details see [HPW12]. Classically the study on $m$-quasi-Einstein are considered when $X$ is a gradient of a smooth function $f$ on $M^n$, see [And99, AK09, CSW11, Cor00, ELM08, KK03].

In this paper, we focus on left-invariant metrics on quadratic Lie groups which include compact Lie groups and semisimple Lie groups as a special class. First, we prove the following theorem.
Theorem 1.1. Let $G$ be a quadratic Lie group with a left-invariant metric $\langle \cdot, \cdot \rangle$. If $X$ is a vector field on $G$ such that $\langle \cdot, \cdot \rangle$ is $m$-quasi-Einstein for $m$ finite, i.e. $\text{Ric}_m^G = \lambda \langle \cdot, \cdot \rangle$, then $X$ is a left-invariant Killing field.

Theorem 1.1 for compact Lie groups is proved in [CL13], and there are non-trivial $m$-quasi-Einstein metrics on homogeneous manifolds [BRJ12, CL13]. Furthermore, we will prove that Theorem 1.1 for $m$ infinite fails for solvable quadratic Lie groups. It is different from the compact case. In fact, compact homogeneous Ricci solitons are Einstein, which is equivalent with that the vector field is a Killing field, by the work of Petersen-Wylie [PW09] and Perelman [Per02]. In [Jab11], Jablonski gives a new proof. In essential, Theorem 1.1 for $m$ infinite holds for semisimple Lie groups by Jablonski’s proof.

Here we pay attention to solvable quadratic Lie groups. In particular, we study a class of simply connected solvable quadratic Lie groups $G(n)$ for $n \geq 1$, the derived algebras of whose Lie algebras are Heinsenberg Lie algebras of dimension $2n + 1$, and prove

Theorem 1.2. Solvable quadratic Lie groups $G(n)$ admit infinitely many non-equivalent non-trivial $m$-quasi-Einstein metrics on $G(n)$ for $m$ finite.

In order to prove Theorem 1.2, we first obtain a formula of the Ricci curvature with respect to a left-invariant metric which holds for any quadratic Lie group. It is a natural extension of the formula on compact semisimple Lie groups, which is given by Sagle [Sag70] and simpler proved by D’Atri and Ziller in [DZ79]. Based on it, we get a computable formula of the Ricci curvature on quadratic Lie groups, i.e. Theorem 3.2, which is the fundament of the proof for Theorem 1.2.

But the proof of Theorem 1.1 doesn’t work for Ricci solitons, i.e. $\infty$-quasi-Einstein metrics. Up to now, we know that Theorem 1.1 for $m$ infinite holds for compact Lie groups and semisimple Lie groups. Finally by our formula of Ricci curvature, we obtain a non-trivial Ricci soliton on $G(n)$ for every $n \geq 1$, which shows that Theorem 1.1 for $m$ infinite fails for solvable quadratic Lie groups.

2. The proof of Theorem 1.1

A quadratic Lie algebra is a Lie algebra $\mathfrak{g}$ together with a pseudo-Riemannian metric $\langle \cdot, \cdot \rangle: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$ that is invariant under the adjoint action, i.e.

$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0$ for any $X, Y, Z \in \mathfrak{g}$.

A Lie group $G$ is called a quadratic Lie group if and only if the Lie algebra $\mathfrak{g}$ is a quadratic Lie algebra. It is easy to see that $\text{trad} X = 0$ for any $X \in \mathfrak{g}$, i.e. a quadratic Lie group is unimodular.

Let $\langle \cdot, \cdot \rangle$ be a left-invariant metric on $G$, i.e.

$\langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle = 0$ for any $X, Y, Z \in \mathfrak{g}$.

In fact, the Levi-Civita connection corresponding to the left-invariant metric $\langle \cdot, \cdot \rangle$ is determined by the following equation

(2.1) $\langle \nabla_X Y, Z \rangle = \frac{1}{2} \left\{ \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle \right\}$.

There is a linear map $D$ of $\mathfrak{g}$ satisfying

$\langle X, Y \rangle = \langle X, D(Y) \rangle$ for any $X, Y \in \mathfrak{g}$.

It follows that $D$ is invertible, and symmetric with respect to $\langle \cdot, \cdot \rangle$. Since $\langle \cdot, \cdot \rangle$ is positive definite and $\langle \cdot, \cdot \rangle$ is symmetric, by a result in linear algebra, we can choose an orthonormal basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$ such that

$\langle X_i, X_j \rangle = \lambda_i \delta_{ij}$,
for some $\lambda_i \in \mathbb{R}$. Then $D(X_i) = \lambda_i(X_i)$.

For any vector field $X$ on $G$, $X = \sum_{i=1}^{n} f_i X_i$, where every $f_i$ is a smooth function on $G$. With respect to the left-invariant $\langle \cdot, \cdot \rangle$, 

$$\text{Ric}^m_X(X_i, X_i) = \text{Ric}(X_i, X_i) + \langle \nabla_X X, X_i \rangle - \frac{1}{m} (X, X_i)^2$$

$$= \text{Ric}(X_i, X_i) + X_i f_i + \sum_{j=1}^{n} f_j \langle \nabla_{X_j} X, X_i \rangle - \frac{1}{m} f_i^2$$

$$= \text{Ric}(X_i, X_i) + X_i f_i + \sum_{j=1}^{n} f_j \langle [X_i, X_j], X_i \rangle - \frac{1}{m} f_i^2$$

$$= \text{Ric}(X_i, X_i) + X_i f_i + \frac{1}{\lambda_i} \sum_{j=1}^{n} f_j \langle [X_i, X_j], X_i \rangle - \frac{1}{m} f_i^2$$

$$= \text{Ric}(X_i, X_i) + X_i f_i - \frac{1}{m} f_i^2.$$

Furthermore in addition $\text{Ric}^m_X = \lambda \langle \cdot, \cdot \rangle$ for $m$ finite, i.e. $\langle \cdot, \cdot \rangle$ is $m$-quasi-Einstein. Then for any $i$, there exists some constant $a_i$ such that 

$$X_i f_i - \frac{1}{m} f_i^2 = a_i.$$

It follows that for any $g \in G$ and $t \in \mathbb{R}$,

$$\left(2.2\right) \quad \frac{df_i(g \exp t X_i)}{dt} = \frac{1}{m} f_i^2(g \exp t X_i) + a_i$$

The equation (2.2) satisfies the local Lipschitz condition. Then it has a unique solution if the initial condition is given.

Furthermore, we have the following lemma.

**Lemma 2.1.** Let $G$ be a quadratic Lie group with a left-invariant metric $\langle \cdot, \cdot \rangle$. If $X$ is a vector field on $G$ such that $\text{Ric}^m_X = \lambda \langle \cdot, \cdot \rangle$ for $m$ finite, i.e. $\langle \cdot, \cdot \rangle$ is $m$-quasi-Einstein, then $X$ is left-invariant.

**Proof.** It is enough to prove that the global smooth solutions of the equation (2.1) are constant. We will discuss it in three cases.

If $a_i < 0$, then it is clear that $f_i = \pm \sqrt{-ma_i}$ are the solutions of the equation (2.2). We claim that the equation (2.2) has no other global smooth solutions. Otherwise, assume that $f_i \neq \pm \sqrt{-ma_i}$ is a global smooth solution of the equation (2.2), i.e. there exists $g \in G$ such that 

$$f_i(g) \neq \pm \sqrt{-ma_i}.$$ 

By the equation (2.2), we have 

$$\left(2.4\right) \quad \frac{m df_i}{f_i^2 + ma_i} = dt.$$

By the integral from 0 to $t$ on the identity (2.4), we have 

$$\frac{m}{2 \sqrt{-ma_i}} \left\{ \ln \left( \frac{f_i(g \exp t X_i)}{f_i(g \exp t X_i) + \sqrt{-ma_i}} \right) - \ln \left( \frac{f_i(g) - \sqrt{-ma_i}}{f_i(g) + \sqrt{-ma_i}} \right) \right\} = t.$$ 

Let $t_0 = -\frac{m}{2 \sqrt{-ma_i}} \ln \left( \frac{f_i(g) - \sqrt{-ma_i}}{f_i(g) + \sqrt{-ma_i}} \right)$, which is finite by (2.3). Then by the above equation, we have 

$$\ln \left( \frac{f_i(g \exp t_0 X_i)}{f_i(g \exp t_0 X_i) + \sqrt{-ma_i}} \right) = 0,$$

which implies that $f_i(g \exp t_0 X_i) = \infty$. It is impossible since $f_i$ is smooth.
If \( a_i = 0 \), then it is clear that \( f_i = 0 \) is a solution of the equation (2.2). We claim that the equation (2.2) has no other global smooth solutions. Otherwise, assume that \( f_i \neq 0 \) is a global smooth solution of the equation (2.2), i.e. there exists \( g \in G \) such that

\[
(2.5) \quad f_i(g) \neq 0.
\]

By the equation (2.2), we have

\[
(2.6) \quad \frac{m df_i}{f_i^2} = dt.
\]

By the integral from 0 to \( t \) on the identity (2.6), we have

\[
-m \left( \frac{1}{f_i(g \exp t X_i)} - \frac{1}{f_i(g)} \right) = t.
\]

Let \( t_0 = \frac{m}{f_i(g)} \), which is finite by (2.5). Then by the above equation, we have

\[
\frac{1}{f_i(g \exp t_0 X_i)} = 0,
\]

which implies that \( f_i(g \exp t_0 X_i) = \infty \). It is impossible since \( f_i \) is smooth.

If \( a_i > 0 \), then we claim that the equation (2.2) has no global smooth solutions. In fact, by the equation (2.2), we have

\[
(2.7) \quad \frac{m df_i}{f_i^2 + ma_i} = dt.
\]

By the integral from 0 to \( t \) on the identity (2.7), we have

\[
-m \left\{ \arctan \frac{f_i(g \exp t X_i)}{\sqrt{ma_i}} - \arctan \frac{f_i(g)}{\sqrt{ma_i}} \right\} = t.
\]

Let \( t_0 = -\frac{m}{\sqrt{ma_i}} \left( \frac{\pi}{2} - \arctan \frac{f_i(g)}{\sqrt{ma_i}} \right) \), which is finite for any given \( g \in G \). Then by the above equation, we have

\[
\arctan \frac{f_i(g \exp t_0 X_i)}{\sqrt{ma_i}} = \frac{\pi}{2},
\]

which implies that \( f_i(g \exp t_0 X_i) = +\infty \). It is impossible since \( f_i \) is smooth.

Hence for any case, the global smooth solutions of the equation (2.2) are constant. Then we have the lemma. \( \square \)

**Remark 2.2.** For the quadratic Lie groups case, \( \text{Ric}(X_i, X_i) \geq \lambda \) for any \( i \) by the proof of Lemma 2.1. Furthermore “ = ” holds for any \( i \), if and only if \( X = 0 \) by the proof of Lemma 2.1 for \( a_i = 0 \), if and only if the \( m \)-quasi-Einstein metric \( \langle \cdot, \cdot \rangle \) is Einstein.

Let \( M \) denote the set of left-invariant metrics on a unimodular Lie group \( G \). For any left-invariant metric \( Q \) on \( G \), the tangent space \( T_Q M \) at \( Q \) is left-invariant symmetric, bilinear forms on \( g \). Define a Riemannian metric on \( M \) by

\[
(v, w)_Q = \text{tr} \, vw = \sum_i v(e_i, e_i)w(e_i, e_i),
\]

where \( v, w \in T_Q M \) and \( \{ e_i \} \) is a \( Q \)-orthonormal basis of \( g \). Let \( G \) be a unimodular Lie group with a left-invariant metric \( \langle \cdot, \cdot \rangle \). Given \( Q \in M \), denote by \( \text{ric}_Q \) and \( \text{sc}_Q \) the Ricci and scalar curvatures of \((G, Q)\), respectively. The gradient of the function \( \text{sc} : M \to \mathbb{R} \) is

\[
(2.8) \quad (\text{grad sc})_Q = -\text{ric}_Q
\]

relative to the above Riemannian metric on \( M \), see [Heb98] or [Nik98].
Assume that $\langle \cdot, \cdot \rangle$ is $m$-quasi-Einstein. That is, there exists a vector field $X$ on $G$ such that $\text{Ric} + \frac{m}{2} \mathcal{L}_X \langle \cdot, \cdot \rangle - \frac{1}{m} X^* \otimes X^* = \lambda \langle \cdot, \cdot \rangle$. In addition, assume that $X$ is a left-invariant vector field, i.e. $X \in \mathfrak{g}$. Since $\langle \cdot, \cdot \rangle$ is a left-invariant metric, for a orthonormal basis relative to $\langle \cdot, \cdot \rangle$, we have

$$
(2.9) \quad \text{Ric} = \lambda \text{Id} - \frac{1}{2} [(\text{ad} X) + (\text{ad} X)^t] + \frac{|X|^2}{m} \text{Pr}|_X.
$$

Here $\text{Pr}|_X$ is the projection of a vector to $X$.

**Lemma 2.3** ([CL13]). Let $G$ be a unimodular Lie group with a left-invariant metric $\langle \cdot, \cdot \rangle$. If $X$ is a left-invariant vector field on $G$ such that $\langle \cdot, \cdot \rangle$ is $m$-quasi-Einstein, i.e. $\text{Ric}|_X^m = \lambda \langle \cdot, \cdot \rangle$, then $X$ is a Killing field.

**Proof.** Since scalar curvature is a Riemannian invariant, $sc(\langle \cdot, \cdot \rangle) = sc(\phi_t^* \langle \cdot, \cdot \rangle)$, where $\phi_t = \exp t \text{ad} X$ is a 1-parameter subgroup of $\text{Aut}(G)$. By the equations (2.8) and (2.9), and the fact that $G$ is unimodular, we have

$$
0 = \frac{d}{dt} |_{t=0} sc(\phi_t^* \langle \cdot, \cdot \rangle) = (\text{grad } sc, \text{ad} X)_{\langle \cdot, \cdot \rangle}
$$

$$
= -\lambda tr (\text{ad} X) + tr \left( \frac{1}{2} [(\text{ad} X) + (\text{ad} X)^t] | (\text{ad} X) - \frac{|X|^2}{m} \text{tr} (\text{Pr}|_X)(\text{ad} X) \right)
$$

$$
= -\lambda tr (\text{ad} X) + tr \left( \frac{1}{2} [(\text{ad} X) + (\text{ad} X)^t]^2 - \frac{|X|^2}{m} \text{tr} (\text{ad} X)(\text{Pr}|_X),
$$

$$
= \frac{1}{4} [ (\text{ad} X) + (\text{ad} X)^t ]^2.
$$

Here $\langle \text{ad} X(Y), Z \rangle = \langle Y, (\text{ad} X)^t(Z) \rangle$ for any $X, Y, Z \in \mathfrak{g}$. Thus we have $\frac{1}{2} [(\text{ad} X) + (\text{ad} X)^t] = 0$ and hence $X$ is a Killing field.

Since a quadratic Lie group is unimodular, we know that Theorem 1.1 follows from Lemmas 2.1 and 2.3.

### 3. Ricci curvature on quadratic Lie groups

Let notations be as those in above section. By the equation (2.1) and the invariance of $\langle \cdot, \cdot \rangle$, we have

$$
\langle \nabla_X Y, Z \rangle = \frac{1}{2} \{ \langle [X, Y], Z \rangle - \langle Y, [X, Z] \rangle + \langle [Z, X], Y \rangle \}
$$

$$
= \frac{1}{2} \{ \langle [X, Y], D^{-1} Z \rangle - \langle [Y, Z], D^{-1} X \rangle + \langle [Z, X], D^{-1} Y \rangle \}
$$

$$
= \frac{1}{2} \{ \langle D^{-1} [X, Y], Z \rangle - \langle D^{-1} Y, [X, D^{-1} Y] \rangle + \langle [X, D^{-1} Y], Z \rangle \}
$$

$$
= \frac{1}{2} \langle D^{-1} X, Y \rangle - [D^{-1} X, Y] + [X, D^{-1} Y], DZ \rangle.
$$

It follows that

$$
(3.1) \quad \nabla_X Y = \frac{1}{2} \{ [X, Y] - D[D^{-1} X, Y] + D[X, D^{-1} Y] \},
$$

that is,

$$
\nabla_X = \frac{1}{2} (\text{ad} X - D \text{ad} (D^{-1} X) + D \text{ad} XD^{-1}).
$$

One has a simple formula of the Ricci curvature on a compact semisimple Lie group with respect to a left-invariant metric which was derived in [Sag70], and a simpler proof is given in [DZ79]. The proof given in [DZ79] is easily extended to a quadratic Lie group. That is,

**Lemma 3.1.** For any $X, Y \in \mathfrak{g}$, $\text{Ric}(X, Y) = -\text{tr}(\nabla_X - \text{ad} X)(\nabla_Y - \text{ad} Y)$. 
Proof. For any \( X_i \in \mathfrak{g} \), we have
\[
\text{Ric}(X,Y) = \text{tr}(X \mapsto \nabla_X Y - \nabla_Y X - \nabla_{[X,Y]} Y)
= \text{tr}(\nabla_X Y - \text{ad}(\nabla_Y X) - \nabla_X Y + \nabla_Y \text{ad}X + \nabla_Y \text{ad}X - \text{ad}Y \text{ad}X)
= \text{tr}(\nabla_X Y) - \text{tr}(\text{ad}(\nabla_X Y)) - \text{tr}(\text{ad}(\nabla_Y X))(\nabla_Y - \text{ad}Y).
\]

By the left-invariancy of \((\cdot,\cdot)\), we know that \(\nabla_Z\) is skew-symmetric with respect to \((\cdot,\cdot)\). It follows that \(\text{tr}\nabla_Z = 0\) for any \(Z \in \mathfrak{g}\). In particular, \(\text{tr}(\text{ad}(\nabla_X Y)) = 0\).

Since \(G\) is quadratic, we have \(\text{tr}\text{ad}X = 0\) for any \(X \in \mathfrak{g}\). In particular, \(\text{tr}(\text{ad}(\nabla_X Y)) = 0\). Then the theorem follows. \(\Box\)

Assume that \(\{X_1,\ldots,X_n\}\) is the orthonormal basis of \(\mathfrak{g}\) with respect to \((\cdot,\cdot)\) given in Section 2, and \(C_{ij}^k\) are the structure constants with respect to the basis. That is,
\[
[X_i, X_j] = \sum_{l=1}^n C_{ij}^l X_l.
\]

By the invariance of \((\cdot,\cdot)\), we have
\[
(3.2) \quad C_{ij}^l \lambda_l = C_{ji}^l \lambda_i = C_{il}^j \lambda_j.
\]

Let \(\mu_i = \frac{1}{X_i}\). Then we have
\[
\nabla_{X_i} X_j = \frac{1}{2}[X_i, X_j] - D[D^{-1}X_i, X_j] + D[X_i, D^{-1}X_j]
= \frac{1}{2}(\text{Id} - \mu_i D + \mu_j D)([X_i, X_j])
= \frac{1}{2} \sum_{l=1}^n \frac{\mu_i - \mu_j + \mu_l}{\mu_l} C_{ij}^l X_l.
\]

Then by Lemma 3.1 and the equation (3.2), we have
\[
\text{Ric}(X_j, X_k) = -\text{tr}(\nabla_{X_j} - \text{ad}X_j)(\nabla_{X_k} - \text{ad}X_k)
= -\sum_{i=1}^n \langle(\nabla_{X_j} - \text{ad}X_j)(\nabla_{X_k} - \text{ad}X_k), X_i \rangle
= -\sum_{i=1}^n \langle\left\{\sum_{l=1}^n \frac{\mu_i - \mu_k + \mu_l}{\mu_l} - 1\right\} C_{ki}^l \rangle(\nabla_{X_j} - \text{ad}X_j)X_i, X_j\rangle
= -\frac{1}{4} \sum_{i=1}^n \sum_{l=1}^n \frac{-\mu_i - \mu_k + \mu_i - \mu_j + \mu_j}{\mu_l} C_{ki}^l C_{jl}^l
= \frac{1}{4} \sum_{i=1}^n \sum_{l=1}^n \frac{-\mu_i - \mu_k + \mu_i - \mu_j + \mu_j}{\mu_l} C_{ki}^l C_{jl}^l
\]

Furthermore, by the equation (3.2), we know
\[
\frac{1}{4} \sum_{i>l} \frac{-\mu_i - \mu_k + \mu_i - \mu_j - \mu_j}{\mu_l} C_{ki}^l C_{jl}^l
= \frac{1}{4} \sum_{i>l} (-\mu_i - \mu_k + \mu_i - \mu_j - \mu_j) \frac{C_{ki}^l C_{jl}^l}{\mu_i \mu_l}
= \frac{1}{4} \sum_{i>l} (-\mu_i - \mu_k + \mu_i - \mu_j - \mu_j) \frac{C_{ki}^l C_{jl}^l}{\mu_i \mu_l}.
\]

Thus we have the following theorem.
Theorem 3.2. Let \( \{X_1, \ldots, X_n\} \) be as above. Then
\[
\text{Ric}(X_j, X_k) = -\frac{1}{2} \sum_{i<l} ((\mu_l - \mu_i)^2 - \mu_k \mu_j) \frac{C^j_{kl}}{\mu_l - \mu_i}
\]

4. \( m \)-quasi-Einstein metrics on \( G(n) \) for \( m \) finite

Let \( G \) be a simply connected Lie group with the Lie algebra \( g \), where \( \{D, X, Y, Z\} \) is a basis of \( g \) such that the non-zero brackets are given by
\[
[D, X] = X, [D, Y] = -Y, [X, Y] = Z.
\]
It is easy to check that the symmetric bilinear form on \( g \) satisfying
\[
(D, Z) = (X, Y)
\]
is invariant. Thus \( G \) is a quadratic Lie group if we take
\[
(D, Z) = (X, Y) = \frac{1}{2} \text{ and others zero.}
\]
Let \( e_1 = D + Z, e_2 = D - Z, e_3 = X + Y \) and \( e_4 = X - Y \). Consider the left-invariant metric \( \langle \cdot, \cdot \rangle \) on \( G \) defined by
\[
\langle e_i, e_j \rangle = \delta_{ij} \lambda_i^2, \lambda_i \neq 0, \text{ for any } i, j = 1, 2, 3, 4.
\]
Let \( f_i = \frac{e_i}{\lambda_i} \) for any \( 1 \leq i \leq 4 \). Then we have
\[
\langle f_i, f_j \rangle = \delta_{ij}, \langle f_1, f_1 \rangle = \frac{1}{\lambda_1^2}, \langle f_2, f_2 \rangle = -\frac{1}{\lambda_2^2}, \langle f_3, f_3 \rangle = \frac{1}{\lambda_3^2}, \langle f_4, f_4 \rangle = -\frac{1}{\lambda_4^2};
\]
and the non-zero structure constants corresponding to the basis \( \{f_i\}_{i=1,2,3,4} \) are
\[
C^1_{23} = \frac{\lambda_4}{\lambda_1 \lambda_3}, C^4_{23} = \frac{\lambda_4}{\lambda_2 \lambda_3}, C^3_{14} = \frac{\lambda_3}{\lambda_1 \lambda_4}, C^2_{34} = \frac{\lambda_2}{\lambda_3 \lambda_4}, C^1_{34} = -\frac{\lambda_1}{\lambda_3 \lambda_4}.
\]
The left-invariant Killing vector field with respect to \( \langle \cdot, \cdot \rangle \) is of the form
\[
a(\lambda_1 f_1 - \lambda_2 f_2).
\]
By Theorem 3.2, the Ricci curvatures are given by
\[
\begin{align*}
\text{Ric}(f_1, f_3) &= \text{Ric}(f_1, f_4) = \text{Ric}(f_2, f_3) = \text{Ric}(f_2, f_4) = \text{Ric}(f_3, f_4) = 0, \\
\text{Ric}(f_1, f_2) &= -\frac{1}{2} ((\lambda_3^2 + \lambda_4^2)^2 + \lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_4^2), \\
\text{Ric}(f_1, f_1) &= \frac{1}{2} ((\lambda_3^2 + \lambda_4^2)^2 - \lambda_1^4) \frac{1}{\lambda_1^2 \lambda_3^2 \lambda_4^2}, \\
\text{Ric}(f_2, f_2) &= -\frac{1}{2} ((\lambda_3^2 + \lambda_4^2)^2 - \lambda_2^4) \frac{1}{\lambda_2^2 \lambda_3^2 \lambda_4^2}, \\
\text{Ric}(f_3, f_3) &= \frac{1}{2} ((\lambda_1^2 + \lambda_2^2)^2 - \lambda_3^4) \frac{1}{\lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_4^2}, \\
\text{Ric}(f_4, f_4) &= -\frac{1}{2} ((\lambda_1^2 + \lambda_2^2)^2 - \lambda_4^4) \frac{1}{\lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_4^2},
\end{align*}
\]
Assume that \( X \) is a vector field on \( G \) such that \( \langle \cdot, \cdot \rangle \) is \( m \)-quasi-Einstein. Then by Theorem 3.2 \( X \) is a left-invariant Killing vector field with respect to \( \langle \cdot, \cdot \rangle \). That is,
\[
X = a(\lambda_1 f_1 - \lambda_2 f_2) \text{ for some } a.
\]
Thus $\langle \cdot, \cdot \rangle$ is $m$-quasi-Einstein with the constant $\lambda$ and $a = 1$ if and only if the following equations holds, i.e.

\[
\begin{align*}
-\frac{1}{2}((\lambda_3^2 + \lambda_2^2) - \lambda_1^2) + \frac{1}{\lambda_1^2 \lambda_2^2 \lambda_3^2} \frac{1}{m} \lambda_1 \lambda_2 = 0, \\
-\frac{1}{2}((\lambda_3^2 + \lambda_2^2) - \lambda_1^2) + \frac{1}{\lambda_1^2 \lambda_2^2 \lambda_3^2} \frac{1}{m} = \lambda, \\
-\frac{1}{2}((\lambda_1^2 + \lambda_2^2) - \lambda_3^2) + \frac{1}{\lambda_1^2 \lambda_2^2 \lambda_3^2} \frac{1}{m} = \lambda, \\
-\frac{1}{2}((\lambda_1^2 + \lambda_2^2) - \lambda_3^2) + \frac{1}{\lambda_1^2 \lambda_2^2 \lambda_3^2} = \frac{1}{2}((\lambda_1^2 + \lambda_2^2 - \lambda_3^2) + \frac{1}{\lambda_1^2 \lambda_2^2 \lambda_3^2} = \lambda.
\end{align*}
\]

By the first equation, we can represent $m$ by $\{\lambda_i\}_{i=1,2,3,4}$. Put the representation of $m$ into the second and third equations, and then eliminate $\lambda$. Finally we have

\[\lambda_3^2 = \lambda_2^2 \text{ and } \lambda_1^2 \lambda_2^2 = 4 \lambda_4^2.\]

It is easy to check that $\{\lambda_i\}_{i=1,2,3,4}$ satisfying the equations (4.9) are the solutions of the above equations. On the other hand, non-zero $\{\lambda_i\}_{i=1,2,3,4}$ satisfying the equations (4.9) give an $m$-quasi-Einstein metric on $G$.

For the above case, $[g, g]$ is the Heinsenberg Lie algebra of dimension 3. Motivated by the above example, we try to give examples when $\{\lambda_i\}_{i=1,2,3,4}$.

Let $G(n)$ be a simply connected Lie group with the Lie algebra $g(n)$, where $\{D, X_s, Y_s, Z\}$ is a basis of $g(n)$ such that the non-zero brackets are given by

\[
[D, X_s] = a_s X_s, [D, Y_s] = -a_s Y_s, [X_s, Y_s] = Z.
\]

We can assume that $0 < a_1 \leq a_2 \leq \ldots \leq a_n$ by adjusting the order and the sign of the basis if necessary. It is easy to check that the symmetric bilinear form on $g(n)$ satisfying

\[\langle D, Z \rangle = a_s (X_s, Y_s) \text{ for any } s = 1, 2, \ldots, n\]

is invariant. Thus $G$ is a quadratic Lie group if we take

\[\langle D, Z \rangle = a_s (X_s, Y_s) = \frac{1}{2} \text{ for any } s \text{ and others zero.}\]

Let $e_1 = D + Z, e_2 = D - Z, e_{2s+1} = \sqrt{a_s} (X_s + Y_s)$ and $e_{2s+2} = \sqrt{a_s} (X_s - Y_s)$ for any $1 \leq s \leq n$. Consider the left-invariant metric $\langle \cdot, \cdot \rangle$ on $G(n)$ defined by

\[\langle e_i, e_j \rangle = \delta_{ij} \lambda_i^2, \lambda_i \neq 0, \text{ for any } i, j = 1, 2, \ldots, 2n + 2.\]

Let $f_i = \frac{e_i}{\lambda_i}$ for any $1 \leq i \leq 2n + 2$. Then we have

\[\langle f_i, f_j \rangle = \delta_{ij}, \langle f_{2s+1}, f_{2s+1} \rangle = \frac{1}{\lambda_{2s+1}^2}, \langle f_{2s+2}, f_{2s+2} \rangle = -\frac{1}{\lambda_{2s+2}^2}, \forall 0 \leq s \leq n\]

and the non-zero structure constants corresponding to the basis $\{f_i\}_{i=1,2n+2}$ are

\[
\begin{align*}
C^{2s+2}_{1(2s+1)} &= a_s \lambda_{2s+2} \lambda_{2s+1}^2, \\
C^{2s+2}_{2(2s+1)} &= a_s \lambda_{2s+2} \lambda_{2s+1} \lambda_{2s+2}, \\
C^{2s+1}_{1(2s+2)} &= a_s \lambda_{2s+1} \lambda_{2s+2}, \\
C^{2s+1}_{2(2s+2)} &= a_s \lambda_{2s+1} \lambda_{2s+2}, \\
C^{2}_{(2s+1)(2s+2)} &= a_s \lambda_{2s+1} \lambda_{2s+2}, \\
C^{1}_{(2s+1)(2s+2)} &= \frac{a_s \lambda_1}{\lambda_{2s+1} \lambda_{2s+2}}.
\end{align*}
\]
The left-invariant Killing vector field with respect to $\langle \cdot, \cdot \rangle$ is of the form
\begin{equation}
(4.18) \quad a(\lambda_1 f_1 - \lambda_2 f_2).
\end{equation}
By Theorem 3.2, Ric$(f_1, f_2) = 0$ except $i = j$ or $(i, j) = (1, 2)$. Furthermore,
\[
\begin{cases}
\text{Ric}(f_1, f_2) = \sum_{s=1}^{n} \frac{1}{2} \left( (\lambda_s^2 + \lambda_{s+2}^2) - \lambda_s^2 \lambda_{s+2}^2 \right), \\
\text{Ric}(f_1, f_1) = \sum_{s=1}^{n} \frac{1}{2} \left( (\lambda_s^2 + \lambda_{s+2}^2) - \lambda_s^2 \lambda_{s+2}^2 \right), \\
\text{Ric}(f_2, f_2) = \sum_{s=1}^{n} \frac{1}{2} \left( (\lambda_s^2 + \lambda_{s+2}^2) - \lambda_s^2 \lambda_{s+2}^2 \right),
\end{cases}
\]
and for any $1 \leq s \leq n$,
\[
\begin{cases}
\text{Ric}(f_{2s+1}, f_{2s+1}) = -\frac{1}{2} \left( (\lambda_s^2 + \lambda_{s+2}^2) - \lambda_{s+1}^2 \lambda_{s+2}^2 \right), \\
\text{Ric}(f_{2s+2}, f_{2s+2}) = -\frac{1}{2} \left( (\lambda_s^2 + \lambda_{s+2}^2) - \lambda_{s+1}^2 \lambda_{s+2}^2 \right),
\end{cases}
\]
Assume that $X$ is a vector field on $G(n)$ such that $\langle \cdot, \cdot \rangle$ is m-quasi-Einstein. By Theorem 1.1 $X$ is a left-invariant Killing vector field with respect to $\langle \cdot, \cdot \rangle$. That is,
\begin{equation}
(4.19) \quad X = a(\lambda_1 f_1 - \lambda_2 f_2) \text{ for some } a.
\end{equation}
Assume that $\langle \cdot, \cdot \rangle$ is m-quasi-Einstein with the constant $\lambda$ and $a = 1$. Then for any $1 \leq s \leq n$, Ric$(f_{2s+1}, f_{2s+1}) = \text{Ric}(f_{2s+2}, f_{2s+2})$. It follows that
\begin{equation}
(4.20) \quad \lambda_{2s+1}^2 = \lambda_{2s+2}^2.
\end{equation}
For any $1 \leq i \neq j \leq n$, since Ric$(f_{2i+1}, f_{2i+1}) = \text{Ric}(f_{2j+1}, f_{2j+1})$, we have
\begin{equation}
(4.21) \quad \lambda_{2i+1}^2 \lambda_{2j+1} = \lambda_{2j+2}^2 \lambda_{2i+2}.
\end{equation}
By the first equation, we can represent $m$ by $\{\lambda_i\}_{i=1, \ldots, 2n+2}$, and then put the representation of $m$ into Ric$_X^m(f_1, f_1)$ and Ric$_X^m(f_2, f_2)$. It is easy to check that
\[
\text{Ric}_{X}^m(f_1, f_1) = \text{Ric}_{X}^m(f_2, f_2).
\]
Since Ric$_X^m(f_1, f_1) = \text{Ric}_{X}^m(f_3, f_3)$, we have
\begin{equation}
(4.22) \quad \lambda_1^2 \lambda_2^2 = 4 \sum_{s=1}^{n} \frac{a^2}{a_1^2} \lambda_s^2.
\end{equation}
It is easy to check that $\{\lambda_i\}_{i=1, \ldots, 2n+2}$ satisfying the equations $(4.20)$, $(4.21)$ and $(4.22)$ are the solutions. On the other hand, non-zero $\{\lambda_i\}_{i=1, \ldots, 2n+2}$ satisfying the equations $(4.20)$, $(4.21)$ and $(4.22)$ give an m-quasi-Einstein metric on $G(n)$. It is easy to check that
\begin{equation}
(4.23) \quad S = \sum_i \text{Ric}(f_i, f_i) = -n(\lambda_1^2 + \lambda_2^2)\left(\frac{2}{\lambda_1^2 \lambda_2^2} + \frac{a^2}{2\lambda_3^2}\right).
\end{equation}
With respect to the orthonormal basis $\{f_i\}_{i=1, \ldots, 2n+2}$, the determinant of the metric matrix with respect to $\langle \cdot, \cdot \rangle$ is 1. Take $\lambda_3 = c \neq 0$. Then by the equation $(4.22)$, $\lambda_1^2 \lambda_2^2$ is a constant. It follows that $S$ is based on the choice of $\lambda_1^2$ and $\lambda_2^2$. That
is, $G(n)$ admits infinitely many non-equivalent non-trivial $m$-quasi-Einstein metrics for $m$ finite, i.e. Theorem 1.2 holds.

5. Ricci solitons on $G(n)$

The proof of Theorem 1.1 is invalid for a Ricci soliton, i.e. an $\infty$-quasi-Einstein metric. In essential, the proof of Lemma 2.1 is invalid for a Ricci soliton. A natural problem is whether quadratic Lie groups admit non-trivial Ricci solitons. As we know, left-invariant Ricci solitons on compact Lie groups are trivial, i.e. Einstein. This section is to construct non-trivial Ricci solitons on $G(n)$.

Since $G(n)$ is simply connected, we know that the left-invariant metric $\langle \cdot, \cdot \rangle$ on $G(n)$ satisfying

\[ \text{Ric} = \lambda \langle \cdot, \cdot \rangle + \frac{1}{2}(d + d^\ast) \]

is in fact a Ricci soliton for any $d \in \text{Der} g(n)$.

Let notations be as those in section 4. It is easy to see that the linear map $\phi$ on $g(n)$ defined by

\[ \phi(D) = 0, \phi(Z) = 2Z, \phi(X_i) = X_i, \phi(Y_i) = Y_i \]

is a derivation on $g(n)$. Let $d = a\phi$ for some constant $a$ which is also a derivation. With respect to the basis $\{f_i\}_{i=1,\ldots,2n+2}$, the matrix of $d$ is

\[ \begin{pmatrix} a & -\frac{a\lambda_1}{\lambda_2} & 0 \\ -\frac{a\lambda_2}{\lambda_1} & a & 0 \\ 0 & 0 & aI_{2n-2} \end{pmatrix}. \]

Since $\{f_i\}_{i=1,\ldots,2n+2}$ is an orthonormal basis on $g(n)$, it follows that the matrix of $d^\ast$ is the transport of $d$. Then the matrix of $\frac{1}{2}(d + d^\ast)$ is

\[ \begin{pmatrix} a & -\frac{a(\lambda_1^2 + \lambda_2^2)}{2\lambda_1\lambda_2} & 0 \\ -\frac{a(\lambda_1^2 + \lambda_2^2)}{2\lambda_1\lambda_2} & a & 0 \\ 0 & 0 & aI_{2n} \end{pmatrix}. \]

Assume that the equation (5.1) holds. Then for any $1 \leq s \leq n$, $\text{Ric}(f_{2s+1}, f_{2s+1}) = \text{Ric}(f_{2s+2}, f_{2s+2})$. It follows that

\[ \lambda_{2s+1}^2 = \lambda_{2s+2}^2. \]

For any $1 \leq i \neq j \leq n$, since $\text{Ric}(f_{2i+1}, f_{2i+1}) = \text{Ric}(f_{2j+1}, f_{2j+1})$, we have

\[ \lambda_{2i+1}^2 = \lambda_{2j+1}^2. \]

By $\text{Ric}(f_1, f_1) = \text{Ric}(f_2, f_2)$, we have

\[ \lambda_1^2 = \lambda_2^2. \]

Since $\text{Ric}(f_1, f_1) = \text{Ric}(f_3, f_3)$, we have $(n + 2)\lambda_1^4a_1^2 = \sum_{s=1}^n (4a_s^2)\lambda_s^4$. That is,

\[ \lambda_1^2 = \sqrt{\frac{\sum_{s=1}^n (4a_s^2)\lambda_s^4}{(n + 2)a_1^2}} \lambda_1^2. \]

Since $\text{Ric}(f_1, f_2) = \lambda(f_1, f_2) - \frac{a(\lambda_1^2 + \lambda_2^2)}{2\lambda_1\lambda_2}$, we have

\[ a(\lambda_1^2 + \lambda_2^2) = \sum_{s=1}^n (4a_s^2) + \frac{na_1^2\lambda_1^2\lambda_2^2}{\lambda_3^2}. \]
By the equations (5.5) and (5.6), we have

\begin{equation}
\alpha = \frac{(n+1)a_1 \sqrt{\sum_{s=1}^{n} (4\alpha^2_s)}}{\sqrt{n+2}\lambda^2_3}.
\end{equation}

Thus if we take non-zero \( \{\lambda_i\}_{i=1,2,...,2n+2} \) satisfying the equations (5.2), (5.3), (5.4) and (5.5), then the derivation \( d = a\phi \) where \( a \) satisfies the equation (5.7) determines a non-trivial Ricci soliton on \( G(n) \) for every \( n \geq 1 \).

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