ON GENERALIZED KUMMER SURFACES AND THE
ORBIFOLD BOGOMOLOV-MIYAOKA-YAU INEQUALITY

XAVIER ROLLEAU

Abstract. A generalized Kummer surface $X = \text{Km}(T, G)$ is the resolution of a quotient of a torus $T$ by a finite group of symplectic automorphisms $G$. We complete the classification of generalized Kummer surfaces by studying the two last groups which have not been yet studied. For these surfaces we compute the associated Kummer lattice $K_G$, which is the minimal primitive sub-lattice containing the exceptional curves of the resolution $X \to T/G$. We then prove that a K3 surface is a generalized Kummer surface of type $\text{Km}(T, G)$ if and only if its Néron-Severi group contains $K_G$. For smooth-orbifold surfaces $X$ of Kodaira dimension $\geq 0$, Kobayashi proved the orbifold Bogomolov Miyaoka Yau inequality $c_1^2(X) \leq 3c_2(X)$. For Kodaira dimension 2, the case of equality is characterized as $X$ being uniformized by the complex 2-ball $B_2$. For smooth-orbifold K3 and Enriques surfaces we characterize the case of equality as being uniformized by $\mathbb{C}^2$.

1. Introduction

A K3 surface $X$ is called a generalized Kummer surface and we write $X = \text{Km}(T, G)$ if it is the resolution of a quotient $T/G$ where $T$ is a torus and $G$ is a finite group of automorphisms of $T$. Let $X = \text{Km}(T, G)$ be a generalized Kummer surface and let $F_G$ be the sub-lattice of the Néron Severi group $\text{NS}(X)$ generated by the exceptional divisors $C_G$ of the resolution $X \to T/G$. The minimal primitive sub-lattice $K_G$ of $\text{NS}(X)$ containing the lattice $F_G$ is called the Kummer lattice of $G$.

In [17] Nikulin computes the Kummer lattice $K_{\mathbb{Z}/2\mathbb{Z}}$ and obtains the famous result that for a K3 surface $X$, it is equivalent to be a Kummer or to contain 16 disjoint $(-2)$-curves or that there exists a primitive embedding of the lattice $K_{\mathbb{Z}/2\mathbb{Z}}$ in the Néron-Severi group of $X$.

That result linking the primitive embedding of a lattice $K_G$ contained in $\text{NS}(X)$ to a geometric description of $X$ has then been extended by Bertin [2] and Garbagnati [4] to the 7 other symplectic automorphism groups $G$ acting on some torus $T$ such that the action of $G$ preserves the origin of $T$.

It turns out that there are symplectic groups $G$ which has not been yet studied: when $G$ has no global fix points on the torus $T$. Up to taking quotient of $G$ by its translation subgroup, one can suppose that $G$ contains no translations. Fujiki has described and classified such pairs $(T, G)$: then $G$ is isomorphic to $\hat{Q}_8$ (the quaternion group) or $\hat{T}_{24}$ (the binary tetrahedral group of order 24). We compute that the singularities of the quotient $T/\hat{Q}_8$ are $C_{\hat{Q}_8} = 6A_3 + A_1$ and the ones of

2000 Mathematics Subject Classification. Primary 14J28, 14L30, 32J25, Secondary 14J50.

Key words and phrases. Generalized K3 surfaces, Kummer Lattices, Quotient surfaces, orbifold Bogomolov-Miyaoka-Yau inequality.
Theorem 1. Let $X$ be a K3 surface and $G$ be the group $\hat{Q}_8$ or $\hat{T}_{24}$. The following conditions are equivalent:

i) $X$ is a generalized Kummer surface $X = \text{Km}(T, G)$

ii) the Kummer lattice $K_G$ is primitively embedded in $\text{NS}(X)$.

iii) $X$ contains a configuration of ADE curves $C_G$.

Then we turn our attention to a related question, which was our initial motivation. Let $C$ be a configuration of disjoint ADE curves on a smooth surface $X$ and let $X \rightarrow \mathcal{X}$ be the contraction of the connected components of $C$. To the singular surface $\mathcal{X}$ one can associate its orbifold Chern numbers, denoted by $c_1^2(\mathcal{X}), c_2(\mathcal{X}) \in \mathbb{Q}$, which depends on the Chern numbers $c_1^2(X), c_2(X)$ of $X$ and on the number and type of the ADE singularities of $\mathcal{X}$. These orbifold Chern numbers have the following property:

Theorem 2. (Orbifold Bogomolov-Miyaoka-Yau inequality, [13, 14, 10, 16]) Suppose that $X$ is a minimal algebraic surface of Kodaira dimension $\kappa \geq 0$. Then:

A) One has

$$c_1^2(\mathcal{X}) \leq 3c_2(\mathcal{X}).$$

B) Suppose $X$ has general type. Equality holds in (1.1) if and only if there exists a discrete cocompact lattice $\Gamma$ in $\text{PU}(2, 1)$ such that $\mathcal{X} = \mathbb{B}_2/\Gamma$. In other words, one has equality if and only if $\mathcal{X}$ is uniformisable by the unit ball $\mathbb{B}_2$.

Here a discrete cocompact lattice means a subgroup which is discrete in $\text{PU}(2, 1)$, such that the points with non-trivial isotropy are isolated, these isotropy groups are finite, and the quotient $\mathbb{B}_2/\Gamma$ is compact. A consequence of Theorem 2 is that in case of equality in (1.1), there always exists a finite uniformisation of $\mathcal{X}$ i.e. a smooth ball quotient surface $Z$ having a finite group of automorphisms $G$ such that $\mathcal{X} = Z/G$.

It is now natural to ask if there is an analog of part B) of Theorem 2 for surfaces of Kodaira dimension 0 and 1. In this paper we study that problem for surfaces having Kodaira dimension $\kappa = 0$, for which equality $c_1^2(\mathcal{X}) = 3c_2(\mathcal{X})$ is in fact equivalent to $c_2(\mathcal{X}) = 0$. Let $X$ be a K3 surface, let $C$ be a configuration of ADE curves on $X$ and let $X \rightarrow \mathcal{X}$ be the contraction of the curves in $C$. We obtain the following result:

Theorem 3. The equality $c_1^2(\mathcal{X}) = 3c_2(\mathcal{X})$ holds if and only if there exists a discrete cocompact lattice $\Gamma$ in the affine linear group $\mathbb{C}^2 \times \text{GL}_2(\mathbb{C})$ such that $\mathcal{X} = \mathbb{C}^2/\Gamma$.

Using the now complete classification of generalized Kummer surfaces, we will in fact see that in case of equality $c_1^2(\mathcal{X}) = 3c_2(\mathcal{X})$, the K3 $X$ is a generalized Kummer surface, which result implies Theorem 3.

We then obtain the same result as Theorem 3 for Enriques surfaces: for an Enriques surface $X$ and $X \rightarrow \mathcal{X}$ the contraction of a configuration $C$ of ADE curves, 1

Note that there exist stronger versions of Theorem 3 in particular with other quotient singularities, but for surfaces of Kodaira dimension 0 which is the case of interest for us, the only quotient singularities one can obtain are ADE.
one has $c_2^2(\mathcal{X}) = 3c_2(\mathcal{X})$ if and only if $C$ is the union of 8 disjoint $(-2)$-curves. We moreover construct the Enriques surfaces containing such a configuration.

Among algebraic surfaces with Kodaira dimension 0, it remains the Abelian and bi-elliptic surfaces, which satisfy $c_2^2 = 3c_2 = 0$. The universal cover of these surfaces is $\mathbb{C}^2$, and they do not contain rational curves; therefore the question is closed for surfaces with $\kappa = 0$.

The paper is organized as follows: in Section 2 we recall the notations and the main results we will need, which are mainly the results of Garbagnati [7]. In section 3 we recall Fjuki’s beautiful classification of automorphism groups of 2-dimensional tori and we give a more detailed account of the previous work on generalized Kummer surfaces. Then we describe the Kummer lattices $K_M$ of a lattice is at most the rank of the lattice, $l_M$ of the manuscript and useful remarks.

Acknowledgements. The author wishes to thank the referee for careful reading of the manuscript and useful remarks.

2. Preliminaries

2.1. Notations. $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the cyclic group of order $n$.

$Q_8$ is the quaternion group (order 8), has a unique involution $\iota$, $Q_8/\iota \cong (\mathbb{Z}_2)^2$.

$D_{12}$ is the binary dihedral group (order 12).

$T_{24}$ is the binary tetrahedral group (order 24, isomorphic to $SL_2(F_3)$; $Q_8$ is a normal subgroup of it).

For $n \in \mathbb{Z}$, $[n]: T \rightarrow T$ is the multiplication by $n$ map on a torus $T$.

For more on the problematic of generalized Kummer surfaces, we recommend the paper of Garbagnati [7], from which we tried to follow the notations.

2.2. Lattices, divisible sets. For a lattice $L$, we denote by $L^\vee$ its dual. The length of $L$ is the minimal number of generators of its discriminant group $L^\vee/L$. A sublattice $M$ of $L$ is said primitive if $L/M$ is torsion free.

Proposition 4. ([18 Proposition 1.6.1]) Let $L$ be a unimodular lattice, $M$ be a primitive sublattice of $L$ and $M^\perp$ the orthogonal to $M$ in $L$. The discriminant group of $M$ is isomorphic to the discriminant group of $M^\perp$. In particular, since the length of a lattice is at most the rank of the lattice, $l(M) = l(M^\perp) \leq \min(\text{rk}(M), \text{rk}(M^\perp))$.

A set of disjoint smooth rational curves $(C_i)_{i \in I}$ on a surface $X$ is called even if there exist an invertible sheaf $\mathcal{L}$ such that $\mathcal{O}(\sum_i C_i) = \mathcal{L}^{\otimes 2}$. On a K3 surface, an even set contains 8 or 16 disjoint curves.

Let $C_i^1, 1 \leq i \leq n, j \in \{1, 2\}$ be a set of $n$ disjoint $A_2$ configurations (so that $C_i^1C_j^2 = 1$) on a surface. The divisor $D = \sum_{i=1}^n C_i^1 + 2C_j^2$ is called 3-divisible if $\mathcal{O}(D) = \mathcal{L}^{\otimes 3}$ where $\mathcal{L}$ is an invertible sheaf. On a K3 surface, the support of a 3-divisible divisor contains 6 or 9 disjoint $A_2$ configurations.

We will use repeatedly the following consequence of Proposition 4.

Lemma 5. Let $X$ be a K3 surface containing 12 disjoint $(-2)$-curves. Then there exists an even set supported on 8 of these curves.

Let $X$ be a K3 surface containing 13 disjoint $(-2)$-curves. Then there exist two linearly independent even sets of curves, supported on 12 of these curves.
Proof: This is well known, see eg [3] Remark 8.10. The discriminant group of the lattice generated by the 12 curves is \((\mathbb{Z}_2)^2\), it has length \(12 > \min(12, 22 - 12)\), thus there exist at least one non-trivial divisible class.

The second part follows e.g. from the first: there is an even set, then remove one curve from that even set, there exists still a set of 12 disjoint curves, thus another even set, with a different support. □

In the present paper an \(ADE\) configuration \(C\) on a surface will have a polysemic meaning. It could mean a set of \(ADE\) singularities on a surface \(X\), or the set of the exceptional curves of its minimal resolution \(X \to \tilde{X}\).

For numbers \(\alpha_n, \delta_n, \varepsilon_n \in \mathbb{N}^*\) with \(\delta_i = 0\) for \(i \leq 3\) and \(\varepsilon_i = 0\) for \(i \notin \{6, 7, 8\}\), we write symbolically \(C = \sum_{n \geq 1} (n \alpha_n + \delta_n D_n + \varepsilon_n E_n)\) if for any \(n \geq 1\), \(C\) contains \(\alpha_n\) (resp. \(\delta_n, \varepsilon_n\)) configurations of type \(A_n\) (resp. \(D_n, E_n\)).

Let \(C\) be an \(ADE\) configuration. We are looking for obstructions or criteria for some sub-configurations of \(C\) to be part of an even set or a 3-divisible set. In Remark 6 below, when we speak of a configuration \(A_n\) or \(D_n\), we implicitly assume it is maximal in \(C\), i.e. it is not contained in a \(A_m\) or \(D_m\) contained in \(C\) for some \(m > n\).

Remark 6. A) An irreducible component \(C\) of a configuration \(A_n\) in \(C\) can be part of an even set \(E\) if and only if \(n\) is odd, the \(\frac{n+1}{2}\) disjoint curves in that \(A_n\) configuration are in \(E\) and \(C\) is among these curves, since otherwise there always exists a curve \(C'\) supported on \(A_n\) such that \(C'E = 1\), and therefore \(E\) cannot be even.

B) The discriminant group of \(D_n\) is \((\mathbb{Z}_2)^2\) if \(n \geq 4\) is even and is \(\mathbb{Z}_4\) if \(n\) is odd (see [3] Theorem 2.3.5). Accordingly, \(k\) disjoint curves and the 2 extremal disjoint closest curves on a \(D_{2k}\) can be possibly part of an even set; the two closest extremal disjoint curves on a \(D_{2k+1}\) can be possibly part of an even set.

C) The discriminant group of \(A_n\) is \(\mathbb{Z}_{n+1}\). Since \(\mathbb{Z}_4\) does not contain the group \(\mathbb{Z}_{4}\), a sub-configuration \(A_2\) of a configuration \(A_3\) in \(C\) cannot be part of a 3-divisible set. There is no such an obstruction for the two disjoint \(A_2\) in a configuration \(A_5\).

2.3. Double, bi-double and triple covers, lifts of automorphisms. To an even set \(E\) (resp. a 3-divisible divisor \(E = \sum_{i=1}^n C_1^i + 2C_2^2\)) on a K3 surface \(X\), one can associate a double (resp. triple) cyclic cover of \(X\) branched on the support of \(E\). The minimal desingularisation \(Y\) of that cyclic cover has an involution (resp. an automorphism of order 3) \(\tau\) such that \(Y/\tau\) is (isomorphic to) \(\tilde{X}\), the surface obtained by contracting the curves on the support of \(E\). We call \(Y\) the \(surface\) associated to \(E\).

Lemma 7. Let \(E\) be an even set on a K3 surface \(X\) and let \(Y\) be the surface associated to \(E\).

A) An automorphism \(\sigma\) of order \(n\) of the K3 surface \(X\) lifts to \(Y\) if and only if \(E = \sigma^n E\).

B) Suppose that \(\sigma\) lifts to an automorphism \(\sigma' \in \text{Aut}(Y)\). Let \(\tau\) be an element of the transformation group of the cover \(Y\) (thus \(Y/\tau\) is birational to \(X\)). There is an exact sequence

\[0 \to \langle \tau \rangle \to \langle \tau, \sigma' \rangle \to \langle \sigma \rangle \to 1.\]

Proof. A K3 surface satisfies \(\text{NS}(X) = \text{Pic}(X)\), then part A) is [20] Proposition 4.2. Part B) follows from the fact \(\sigma' \tau \sigma'^{-1}\) is a lift of the identity, thus a power of
A bi-double cover $Y \to X$ of a surface $X$ is a Galois cover of group $(\mathbb{Z}_2)^2$. It is determined by divisors $D_1, D_2, D_3$ and invertible sheaves $L_1, L_2, L_3$ such that for \{i, j, k\} = \{1, 2, 3\}, one has

\begin{align}
2L_i \equiv D_j + D_k,
\end{align}

(see [3]). The surface $Y$ is embedded in the total space of the vector bundle $L = L_1 \oplus L_2 \oplus L_3$ as the variety with equation

$$
\begin{bmatrix}
x_1 & w_3 & w_2 \\
w_3 & x_2 & w_1 \\
w_2 & w_1 & x_3
\end{bmatrix} = 1,
$$

where $D_i = \text{div}(x_i)$ and $w_1, w_2, w_3$ are coordinates of the $L_i$.

**Example 8.** Let $E = \sum_{i=1}^{12} C_i$ be a $12A_1$ configuration on a K3 surface, such that $E$ has 2 linearly independent even sets $\ell_1, \ell_2$. Up to reordering, one can suppose

$$
\ell_1 = \sum_{i=5}^{12} C_i, \quad \ell_2 = \sum_{i=1}^{4} C_i + \sum_{i=9}^{12} C_i.
$$

Then $\ell_3 = \sum_{i=1}^{8} C_i$ is also even. Let be $L_i := \frac{1}{4}\ell_i$ and $D_j = E - \ell_j$. The datas $D_i, L_j$ satisfy the relations 2.1 and determine a bi-double cover $Y \to X$; $Y$ is a smooth K3.

Let $\sigma$ be an automorphism of a smooth surface $X$ admitting a bi-double cover determined by divisors $D_i, i \in \{1, 2, 3\}$ and invertible sheaves $L_i, i \in \{1, 2, 3\}$ as above. Suppose that there is an action of $\sigma$ on $\{1, 2, 3\}$ such that $\sigma^*L_i = L_{\sigma i}$ and $\sigma^*D_i = D_{\sigma i}$. Then

**Lemma 9.** The automorphism $\sigma$ lifts to an automorphism of $Y$.

**Proof.** One can choose coordinates $w_i$ so that $\sigma^*w_i = w_{\sigma i}$ and equations $x_i$ of $D_i$ such that $\sigma^*x_i = x_{\sigma i}$. Then the automorphism $\sigma$ lifts to an automorphism of $L$, and the equations of $Y$ are preserved, thus it restricts to an automorphism of $Y$. \(\square\)

### 2.4. Roots of a lattice and $(-2)$-curves

Let $X$ be an algebraic K3 surface. Let $h$ be a pseudoample divisor on $X$ (i.e. $h^2 > 0$ and $hD \geq 0$ for all effective divisor $D$) and let

$$
L = h^\perp := \{l \in \text{NS}(X) \text{ such that } lh = 0\}
$$

be the orthogonal of $h$ in $\text{NS}(X)$. We will use the following result proved by Garbagnati [17, Proposition 3.2] (see also [2] Lemma 3.1 of Bertin):

**Proposition 10.** Let us assume that there exists a root lattice $R$ such that:

1. $L$ is an overlattice of finite index of $R$
2. the roots of $R$ and of $L$ coincide.

Then there exists a basis of $R$ which is supported on smooth irreducible rational curves.

**Remark 11.** According to a recent preprint of Schütt [24], hypothesis (2) is always satisfied.
Let $X$ be a K3 surface and $F \subset \text{NS}(X)$ be a sub-lattice. A minimal primitive sub-lattice of $H^2(X, \mathbb{Z})$ containing $F$ is a lattice $K_F$ containing $F$ such that $K_F/F$ is finite and $H^2(X, \mathbb{Z})/K_F$ is free. That lattice $K_F$ is unique and is equal to the lattice $\text{NS}(X) \cap F \otimes \mathbb{Q}$.

Let $X$ be a non-algebraic K3 surface. By Grauert’s ampleness criterion for complex surfaces, since $X$ is not algebraic any divisor on $X$ has self intersection $\leq 0$. Therefore the irreducible curves are $-2$-curves or of arithmetic genus 1. By Riemann-Roch, if there is a curve of arithmetic genus 1, then it is a fiber of a fibration $X \to P^1$. That fibration is then unique and contracts every $(-2)$-curves on $X$ (this is still because the divisors have self-intersection $\leq 0$). The class of a fiber generates the kernel of the natural map $\text{NS}(X) \to \text{Num}(X)$, in particular $\text{NS}(X)$ is degenerate. Therefore since the signature of the intersection form on $H^{1,1}$ is $(1, 19)$, if $\text{NS}(X)$ contains a negative definite sub-group of rank 19 then there is no curves of arithmetic genus 1 on the non algebraic K3 surface $X$.

Remark 12. Let $X$ be a non algebraic K3 surface containing no curves of arithmetic genus 1. The negative definite lattice $\text{NS}(X)$ has rank $\rho \leq 19$ and $\rho$ is equal to the number of $(-2)$-curves on $X$. In particular the minimal primitive sub-lattice containing the $(-2)$-curves on $X$ is $\text{NS}(X)$ itself.

Let $\delta$ be a $(-2)$-class on $X$ i.e. an element of $\text{NS}(X)$ such that $\delta \in \text{NS}(X)$ satisfies $\delta^2 = -2$. By Riemann-Roch $-\delta$ or $\delta$ is effective; say $\delta \geq 0$ : there exists $(-2)$-curves $C_i$ such that $\delta = \sum n_i C_i$, with $n_i \geq 1$. Therefore the $(-2)$-classes on $X$ form a root system of the lattice $F$ generated by the $(-2)$-curves on $X$ and the $(-2)$-curves form a simple base $B$ of $F$. One can then apply [7, Lemma 3.2] (see also [17, Remark 4.5]) and conclude that if one has a direct sum decomposition as root lattice $F = \sum_{n \geq 1} A_n^{\alpha_n}$, then for each of the factors $A_n$ there is a simple base constituted of $(-2)$-curves.

2.5. Orbifold settings. Let $C = \sum_{n \geq 1} \alpha_n A_n + \delta_n D_n + \varepsilon_n E_n$ be a ADE configuration of curves on a smooth surface $X$. Let us define the quantity

$$m(C) := \sum_{n \geq 1} (\alpha_n + \delta_n + \varepsilon_n)(n+1) - \sum_{n \geq 1} \frac{\alpha_n}{n+1} - \sum_{n \geq 4} \frac{\delta_n}{4(n-2)} - \frac{\varepsilon_6}{24} - \frac{\varepsilon_7}{48} - \frac{\varepsilon_8}{120}.$$ 

Let $X \to \mathcal{X}$ be the contraction map of the curves contained in $C$. Since $\mathcal{X}$ contains only ADE singularities, the orbifold Chern numbers of $\mathcal{X}$ are

$$c_1^2(\mathcal{X}) = K_X^2, \quad \text{and} \quad c_2(\mathcal{X}) = c_2(X) - m(C),$$

(see e.g. [21]). Let $X$ be a K3 surface. The orbifold Miyaoka-Yau inequality [13] tells us that

$$m(C) \leq 24.$$ 

Moreover, since each configuration $A_n, D_n$ or $E_n$ contributes for a n-dimensional subspace in the negative definite part (of rank at most 19) of the Néron-Severi group, one has the restriction

$$\sum_{n \geq 1} n(\alpha_n + \delta_n + \varepsilon_n) \leq 19.$$ 

Suppose that the K3 orbifold $\mathcal{X}$ has a finite uniformization $Y \to \mathcal{X}$, i.e. $Y$ is a smooth surface with an action by a finite group $G$ of order $n$ such that $\mathcal{X} = Y/G$ and $Y \to Y/G$ is ramified in codimension 2. Then
Lemma 13. The Chern numbers of Y are $K_Y^2 = nc_1^2(X)$ and $c_2(Y) = nc_2(X)$. The surface Y is Abelian or a K3.

Proof. The first part follows by the definition of the orbifold Chern numbers, see e.g. [13, 21]. Since $Y \to X$ is ramified in codimension 2, the canonical divisor $K_Y$ is the pull-back of $K_X$, which is trivial, thus $K_Y$ is trivial and Y is a K3 or is an Abelian surface. □

The double cover of an even set of 8 (resp. 16) $A_1$ is a K3 (resp. a torus). The triple cover of a 3-divisible set of 6 (resp. 9) $A_2$ is a K3 (resp. a torus).

3. Classification of symplectic groups and generalized Kummer surfaces

3.1. Fujiki’s constructions of Abelian tori with symplectic action of a group. In [6], Fujiki constructs and classifies pairs $(T, G)$ of complex tori $T$ with a faithful action by a group $G$ containing no translations. Let us describe his results when $G$ acts symplectically and is not cyclic.

Let $H = \mathbb{R}[1, i, j, k]$ be the quaternion field, so that $i^2 = j^2 = k^2 = -1, ij = -ji = k$.

Let

$\mathfrak{a} = \mathbb{Z}[1, i, j, t]$ be the ring of Hurwitz quaternions, where $t = \frac{1}{2}(1 + i + j + k)$.

This is a maximal order of $F = \mathbb{Q}[i, j, k]$ and its group of invertible elements is

$\mathfrak{a}^\times = \{1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$,

which is the binary tetrahedral group $T_{24}$. Let be

$\mathfrak{a}_0 = \mathbb{Z}[1, i, j, k]$;

the subgroup $\mathfrak{a}_0^\times = \{1, \pm i, \pm j, \pm k\}$ is the quaternion group $Q_8$. Let be

$F' = \mathbb{Q}[1, i, \sqrt{3}j, \sqrt{3}k]$,

and $\mathfrak{b} = \mathbb{Z}[1, i, h, l]$, where

$h = \frac{1}{2}(i + \sqrt{3}j), \ l = \frac{1}{2}(1 + \sqrt{3}k)$.

The subgroup

$\mathfrak{b}^\times = \{\pm 1, \pm i, \pm h, \pm l, \pm ih, \pm il\}$

is the binary dihedral group $D_{12}$ of order 12.

Let us define the following lattices in $\mathbb{H}$:

$\Lambda_{Q_8} = \mathfrak{a}_0, \ \Lambda_{D_{12}} = \mathfrak{b}, \ \Lambda_{T_{24}} = \mathfrak{a}$.

The set $X$ of pure quaternions:

$X = \{q \in \mathbb{H} | q^2 = -1\} = \{ai + bj + ck | a^2 + b^2 + c^2 = 1\}$,

is isomorphic to $\mathbb{P}^1_\mathbb{C}$. For $q \in X$, one can identify $\mathbb{R} + q\mathbb{R}$ with $\mathbb{C}$ by sending $q$ to $\sqrt{-1}$. By multiplication on the right, $X$ parametrizes complex structures of $\mathbb{H} = F \otimes \mathbb{R} = \mathbb{R}^4$.

For $G = Q_8, D_{12}$ or $T_{24}$, such a complex structure induces a complex structure on the real torus $T_q := \mathbb{H}/\Lambda_G$. The left multiplication on $\mathbb{H}$ induces a left action of
\( G = \Lambda_G^X \) on \( T_q = \mathbb{H}/\Lambda_G \), which is compatible with the complex structure induced by \( q \), in other words that action is holomorphic. In that way we get a holomorphic family of pairs

\[(T_q, G)_{q \in \mathcal{X}}\]

of a complex tori \( T_q \) with an action of the group automorphism \( G \) (preserving \( 0 \in T_q \)), parametrized by \( q \in \mathcal{X}_G = \mathcal{X} \cong \mathbb{P}^1 \).

We say that a group acts symplectically on a torus (or is symplectic) if its analytic representation is in \( SL_2(\mathbb{C}) \subset GL_2(\mathbb{C}) \). According to [5], the three groups \( G = \text{Q}_8, D_{12} \) and \( T_{24} \) acts symplectically on the torus \( T_q = \mathbb{H}/\Lambda_G \).

**Definition.** We say that two pairs \((T_1, G_1)\) and \((T_2, G_2)\) of tori \( T_1, T_2 \) with action by groups \( G_1, G_2 \) are isomorphic if there is an isomorphism of \( T_1 \) with \( T_2 \) such that the action of \( G_1 \) on \( T_2 \) (induced by transport of structure) is \( G_2 \) (in particular \( G_1 \cong G_2 \)). We say that a symplectic group \( G \) acting on a torus \( T \) is reduced if it contains no translations.

Let \( G \) be a symplectic group of automorphism of a torus \( T \) and let \( G_0 \) be its subgroup of translations.

**Lemma 14.** The group \( G_0 \) is normal in \( G \) and \( G/G_0 \) is a reduced symplectic group of automorphisms of the torus \( T/G_0 \).

**Proof.** It is easy to check that \( G_0 \) is normal (the translation subgroup of a torus is normal). The quotient \( T/G_0 \) is of course a torus ; the group \( G/G_0 \) acts on \( T/G_0 \) symplectically since the analytic representations of an element in \( G \) or its image in \( G/G_0 \) are the same. \( \square \)

We say that a finite reduced group \( G \) is maximal if \( G \) is not a strict sub-group of another reduced finite symplectic group. Let \( G \) be a non-cyclic group of symplectic automorphisms of a torus \( T \), fixing one point globally (which we can suppose to be the origin; that hypothesis implies that \( G \) is reduced). We have

**Theorem 15.** (Fujiki [5] Proposition 3.5 and Theorem 3.11) The group \( G \) is isomorphic to one of the groups \( \text{Q}_8, D_{12} \) or \( T_{24} \).

If \( G \) is maximal, then there exists \( q \in \mathcal{X} \) such that \((T, G)\) is isomorphic to \((T_q, G)\), where \( T_q = \mathbb{H}/\Lambda_G \) with complex structure given by \( q \).

If \( G \) is not maximal, then \( G = \text{Q}_8 \) and there exists \( q \in \mathcal{X} \) such that \((T, G)\) is isomorphic to \((T_q, \text{Q}_8)\), where \( T_q = \mathbb{H}/\Lambda_{T_{24}} \) and \( \text{Q}_8 \subset T_{24} \) is the unique quaternion group of order 8 contained in \( T_{24} \).

For \( q \in \mathcal{X} \) and \( T_q = \mathbb{H}/\Lambda_{T_{24}} \), let us now denote by \( \hat{\Lambda}(T_q) \) and \( \hat{\Lambda}_0(T_q) \) respectively the group of real affine automorphisms and the group of translations of \( T_q \). Then \( \hat{\Lambda}(T_q) \) is naturally a semi-direct product \( \hat{\Lambda}(T_q) = \text{Aut}_\mathbb{H}\Lambda_{T_{24}} \rtimes \hat{\Lambda}_0(T_q) \). Let \( \lambda \in \Lambda_{T_{24}}^\times \) (acting by left multiplication) and \( r \in T_q \), then the action \( (\lambda; r) x \to \lambda x + r \) is biholomorphic on \( T_q \) so that we have the natural embedding \( \Lambda_{T_{24}}^\times \rtimes \hat{\Lambda}_0(T_q) \subset \hat{\Lambda}(T_q) \).

Let us define the sub-groups \( \hat{\text{Q}}_8 \) and \( \hat{T}_{24} \) of \( \hat{\Lambda}(T_q) \) as follows:

\[ \hat{\text{Q}}_8 = \{1, \pm i, \pm j', \pm k'\} \]

for \( j' = (j; \alpha), k' = (k; \alpha) \) where \( \alpha = \frac{1}{2} (1 + i) \) and

\[ \hat{T}_{24} = \langle \hat{\text{Q}}_8, (t; \frac{1}{2} s) \rangle, \text{ for } s = \frac{1}{2} (1 + i - j + k); \]
(we recall that $t = \frac{1}{2}(1 + i + j + k)$) thus by definition $\hat{Q}_8 \subset \hat{T}_{24}$. For $q \in \mathcal{X}$, the group $\hat{T}_{24}$ acts symplectically on the torus $T_q = \mathbb{H}/\Lambda_{T_{24}}$. That action is without global fixed points and so is the action of the sub-group $Q_8 \subset \hat{T}_{24}$. One has $\hat{Q}_8 \simeq Q_8$ and $\hat{T}_{24} \simeq T_{24}$ as abstract groups.

**Theorem 16.** (Fujiki [6] Theorem 3.17) Let $G$ be a reduced finite group acting symplectically on a torus $T$, such that there is no global fixed point. The group $G$ is isomorphic to $Q_8$ or $T_{24}$. If $G \simeq Q_8$ (resp. $T_{24}$), then there exists $q \in X$ such that $(T, G)$ is isomorphic to $(T_q, \hat{Q}_8)$ (resp. $(T_q, \hat{T}_{24})$), where in both cases $T_q = \mathbb{H}/\Lambda_{G}$.

**Remark 17.**

A) Any action of $Q_8$ on the torus $\mathbb{H}/\Lambda_{Q_8}$ has a global fixed point.

B) By [6, Proposition 5.7 p 62], the complex torus $T_q$ is algebraic if and only if $\exists \mu \in \mathbb{R}$, $\mu q \in \Lambda_G$. There are an infinite number of such $q \in X$. Moreover if $T_q$ is algebraic, it has maximal Picard number.

On the following table are summarized the 10 $ADE$ configurations on generalized Kummer surfaces:

| Configuration | Groups | References for $K_G$ | $\rho$ |
|--------------|--------|----------------------|-------|
| $16A_1$      | $\mathbb{Z}/2\mathbb{Z}$ | [17, 19] | 16     |
| $9A_2$       | $\mathbb{Z}/3\mathbb{Z}$ | [2]       | 18     |
| $6A_1 + 4A_3$| $\mathbb{Z}/4\mathbb{Z}$ | [2]       | 18     |
| $5A_1 + 4A_2 + A_5$ | $\mathbb{Z}/6\mathbb{Z}$ | [2]       | 18     |
| $2A_1 + 3A_2 + 2D_4$ | $Q_8$ | [2, §4.2.2, 23 Prop. 2.1] | 19     |
| $3A_1 + 4D_4$ | $Q_8 \subset \hat{T}_{24}$ | [2, §4.2.3] | 19     |
| $A_1 + 6A_3$ | $Q_8$ | [2]       | 19     |
| $A_1 + 2A_2 + 3A_3 + D_5$ | $Q_{12}$ | [2, §4.2.5] | 19     |
| $A_1 + 4A_2 + D_4 + E_6$ | $\hat{T}_{24}$ | [2, §4.2.4, 23 Prop. 2.1] | 19     |
| $4A_2 + 2A_3 + A_5$ | $\hat{T}_{24}$ | [2]       | 19     |

The column $\rho$ gives the contribution of the given configuration of $(-2)$-curves to the Picard number of the K3 surface.

About generalized Kummer surfaces, one must cite the work of Enriques and Severi [5] more than one century ago, who were the first to study generalized Kummer surfaces obtained as quotients of Jacobians of curves. They saw the 10 cases of the above table. They also described the resulting singularities (with errors for some non-cyclic groups).

In [4] Çinkir and Önsiper study generalized Kummer surfaces and describe the quotient singularities (but some cases are missing). In [19] Önsiper and Sertöz give a generalization of Shioda-Inose structures to these generalized Kummer surfaces. In [2], Bertin describes the primitive sub-lattices containing the configurations for cyclic groups $\mathbb{Z}_n$, $n \in \{3, 4, 6\}$, after the work of Nikulin [17] (and Morrison [15]) for $n = 2$. In [23], Wendland studies that problem for some non-cyclic groups preserving globally a point, a work which was later corrected and completed by Garbagnati in [7].

To be more exhaustive, one must also mention that Fujiki studied the possible $ADE$ singularities in [6], and Katsura [11] worked out the possible symplectic groups in characteristic $> 0$, illustrating each cases by examples.
3.2. The configuration $\hat{Q}_8$ : $A_1 + 6A_3$. Let $X$ be the K3 surface obtained as the desingularization of the quotient $T_q/\hat{Q}_8$ of a torus $T_q = \mathbb{H}/A_{24}$ by the action of $\hat{Q}_8 \subset \hat{T}_{24}$ described in Section 3.1

Lemma 18. The singularities of the quotient surface $T_q/\hat{Q}_8$ are $A_1 + 6A_3$.

Proof. Since the square of any order 4 elements in $Q_8$ is the multiplication by $-1$ map $[-1]_7$, the fixed point sets of these elements are included into the fixed point set of $[-1]_7$, i.e. the set of 2-torsion points of $T_q$. For $a, b, c, d \in \{0, 1\}$, let us denote by $abcd$ the 2-torsion point $\frac{a}{2} + \frac{b}{2}i + \frac{c}{2}j + \frac{d}{2}k \in T_q$. One has $i(0011) = 0101, i(1001) = 1111, i(0001) = 1011, i(1011) = 1101, j'(0001) = 1100, j'(1010) = 0110$ etc, and we obtain that the fixed point set of the order 4 elements $i, j', k'$ is $(k; \alpha)$ (where $\alpha = \frac{1}{2}(1 + i)$) of $Q_8$ are

$$\begin{align*}
\text{Fix}(i) &= \{0000, 1100, 1010, 0110\} \\
\text{Fix}(j') &= \{0011, 0101, 1001, 1111\} \\
\text{Fix}(k') &= \{0001, 1011, 0111, 1101\}.
\end{align*}$$

Using that $k' = ij', j' = -ik'$ etc, we compute that on the quotient surface there are $2A_3$ which are the images of $\text{Fix}(i)$, $2A_3$ images of $\text{Fix}(j')$ and $2A_3$ images of $\text{Fix}(k')$; the image of the 4 remaining 2-torsion points in $T_q$ (the orbit of 1000) is a $A_1$.

Let now $X$ be any K3 surface containing a configuration $A_1 + 6A_3$. For $1 \leq r \leq 6$, we denote by

$$C_r^s, 1 \leq s \leq 3$$

the resolution of the $6A_3$, where $C_1^2C_2^2 = C_3^2C_4^2 = 1$ and the other intersection numbers among the curves $C_r^s$ are $0$ or $-2$. Let $C_0$ be the resolution of the $A_1$. The discriminant group of the lattice $F_{Q_8}$ generated by the curves $C_r^s, 1 \leq r \leq 6, s \in \{1, 2, 3\}$ and $C_0$ is $\mathbb{Z}_2 \times (\mathbb{Z}_4)^6$, it is generated by $t_0 = \frac{1}{2}C_0$ and

$$t_r = \frac{1}{4}(C_r^1 + 2C_r^2 + 3C_r^3), r \in \{1, \ldots, 6\}.$$

Let $K_{Q_8}$ be the Kummer lattice of $\hat{Q}_8$: the minimal primitive sub-lattice of $\text{NS}(X)$ containing the lattice $F_{Q_8}$.

Proposition 19. The lattice $K_{Q_8}$ is generated by $F_{Q_8}$ and by the divisors

$$\delta_1 = (1, 1, 1, 1, 2, 0), \delta_2 = (1, 3, 2, 0, 1, 3)$$

in the base $t_1, \ldots, t_6$ (up to reorder the $t_r$ and $C_r^s$).

The lattice $K_{Q_8}$ has discriminant group $\mathbb{Z}_2 \times (\mathbb{Z}_4)^2$; the index of $F_{Q_8}$ in $K_{Q_8}$ equals 16.

Proof. The curves $C_1^r, C_2^3, r \in \{1, \ldots, 6\}$ and $C_0$ form a configuration of 13 disjoint $A_1$. Therefore there exists two linearly independent even sets supported on 12 of these curves. The curve $C_0$ cannot be part of such an even set (see Remark 9). Therefore, up to permuting the indices, the three even sets are

$$\begin{align*}
v_1 &= C_1^1 + C_3^3 + C_3^1 + C_4^3 + C_4^1 + C_1^3 + C_4^1 + C_4^3 + C_4^1 + C_4^3 \\
v_2 &= C_3^1 + C_3^3 + C_4^2 + C_4^1 + C_5^3 + C_5^1 + C_6^3 + C_6^1 + C_6^3 \\
v_3 &= C_1^1 + C_1^3 + C_2^2 + C_2^3 + C_3^3 + C_4^3 + C_6^3 + C_6^1
\end{align*}$$
and \( \frac{1}{2} v_1, \frac{1}{2} v_2, \frac{1}{2} v_3 \) are in fact element of \( \text{NS}(X) \). In the discriminant group of \( F_{\hat{Q}_8} \), one has \( \frac{1}{2} v_1 + \frac{1}{2} v_2 = \frac{1}{2} v_3 \). Let us denote by \( L_8 \) the lattice spanned by \( F_{\hat{Q}_8} \) and \( \frac{1}{2} v_1, \frac{1}{2} v_2, \frac{1}{2} v_3 \). The discriminant group of \( L_8 \) is \( (\mathbb{Z}/2\mathbb{Z})^5 \times (\mathbb{Z}/4\mathbb{Z})^2 \), of length \( 7 > \text{rk}(L_8^3) = 3 \), thus there exists other divisibilities.

Since a set of 12 disjoint \( A_1 \) contains at most two linearly independent even sets, these are divisibilities by 4. Comparing the length, one obtain that there are two linearly independent 4-divisible classes. In \( A_1 + 6A_3 \) there is no sub-configuration \( 4A_3 + 6A_1 \) that could come from the quotient of a torus by an order 4 automorphism. A quotient of a K3 by an order 4 automorphism has singularities \( 4A_3 + 2A_1 \) and the \( 4 \times 2 \) disjoint configurations \( A_1 \) supported on the sub-configuration \( 4A_3 \) of \( 4A_3 + 2A_1 \) must be divisible by 2. Thus in our configuration \( 6A_3 + A_1 \), these \( 4A_3 \) are (supported on 4 times the following elements):

\[
t_1, t_2, t_3, t_4 \text{ or } t_1, t_2, t_5, t_6 \text{ or } t_3, t_4, t_5, t_6.
\]

Once the \( 4A_3 \) are chosen, there are two choices for the \( 2A_1 \) such that \( 4A_3 + 2A_1 \) becomes 4-divisible: one can take two disjoint curves in the resolution of the \( 5^{th} \) or of the \( 6^{th} \) \( A_3 \)’s. Up to permuting the \( t_j \), and also since one has some freeness to permute \( C_i^1 \) with \( C_i^3 \), one can suppose that

\[
\delta_1 = (1, 1, 1, 1, 2, 0)
\]

(written in the canonical base of the subgroup \( \mathbb{Z}^6 \subset F_{\hat{Q}_8}^\vee \) generated by the \( t_i, i \in \{1, \ldots, 6\} \)) is integral. The relations \( \delta_1 \delta_2 \in \mathbb{Z}, \delta_2 \in 2\mathbb{Z} \) forces the other generator \( \delta_2 \) supported on \( t_1, t_2, t_5, t_6 \) to be \( \delta_2 = (1, 3, 2, 0, 1, 3) \) (or \( (3, 1, 2, 0, 3, 1) \), both generates the same group in the discriminant group). Then \( \delta_3 = (2, 0, 3, 1, 3, 3) \) is supported on \( t_3, t_4, t_5, t_6 \) and equals \( \delta_1 + \delta_2 \) in the discriminant group.

The lattice generated by \( F_{\hat{Q}_8} \) and the \( \delta_i, i \in \{1, 2, 3\} \) has discriminant group \( \mathbb{Z}_2 \times (\mathbb{Z}_4)^2 \) (of length 3). Another divisibility by 2 is not possible because a set of 13 disjoint \( A_1 \) support at most two linearly independent even sets. If there was another independent 4-divisible set, it would create other even sets. Therefore that lattice is primitive and equals \( K_{\hat{Q}_8} \).

**Remark 20.** Let \( Y \) be the K3 associated to the \( \mathbb{Z}_4 \)-cover defined by \( \delta_1 \). There exists on \( Y \) a configuration \( 4A_3 + 6A_1 \), therefore \( Y = \text{Km}(T', \mathbb{Z}_4) \) for some torus \( T' \). The order 4 automorphism \( \tau \) such that \( Y/\tau \) is birational to \( X \) lifts to an automorphism of \( T' \). The group generated by the lifts and the automorphism \( \sigma \in \text{Aut}(T') \) such that \( Y \) is birational to \( T'/\sigma \) has order 16. Thus by the classification of Fujiki, it contains a translation.

Let us now prove the following result:

**Proposition 21.** Let \( X \) be a K3 surface containing a configuration \( A_1 + 6A_3 \). Then there exists \( q \in X \) such that \( X = \text{Km}(T_\eta, \hat{Q}_8) \) where \( T_\eta = \mathbb{H}/\mathfrak{a} \).

**Proof.** By the proof of Proposition 19 there exist two linearly independent even sets which are supported on the 12 disjoint rational curves of the sub-configuration \( 6A_3 \) in \( A_1 + 6A_3 \).

Taking the associated bidouble cover and its minimal model, the pull-back of the \( A_1 \) and central curves in the six \( A_3 \) is a set of 16 disjoint \( A_1 \) curves \( C_i \), this is therefore a Kummer surface \( \text{Km}(T) \). Since the automorphisms in the group \( (\mathbb{Z}/2\mathbb{Z})^2 \) preserves the branch locus \( \sum_1^{16} C_i \), these automorphisms lift to automorphisms of \( T \). By the
classification of Fujiwki that group must be isomorphic to $Q_8$ and the result follows from Fujiwki’s classification Theorem 16.

Let now $X$ be a K3 surface such that there exists a primitive embedding of $K_{Q_8}$ into $\text{NS}(X)$.

**Theorem 22.** There exists a complex torus $T$ and a group of automorphism $G \simeq Q_8$ such that $X = \text{Km}(T, G)$.

*Proof.* Using Magma, one computes that the number of roots in $K_{Q_8}$ (37 of such) equals the number of roots of $F_{Q_8}$. Thus by Lemma 10 there exists a configuration $A_1 + 6A_3$ of smooth irreducible rational curves. We then apply Proposition 21 and Remark 12.

3.3. **The configuration $\hat{T}_{24}$:** $4A_2 + 2A_3 + A_5$. Let $X = \text{Km}(T, \hat{T}_{24})$ be a K3 surface obtained as the desingularization of the quotient of a complex torus $T = T_q$ by the action of $\hat{T}_{24}$.

One computes that the order 3 automorphism $w = (t; \frac{1}{12})^2$ fixes a unique 2-torsion point on the torus $T_q = \mathbb{H}/\Lambda_{T_{24}}$; that point is not in the fixed point sets of the automorphisms $i, j' = (j, \alpha), k' = (k, \alpha)$. The K3 surface $T_q/\hat{T}_{24}$ is a quotient of $T_q/\hat{Q}_8$ (where $\hat{Q}_8 \subset \hat{T}_{24}$ is the unique normal subgroup of order 8) by the order 3 automorphism $w'$ induced by $w$.

An order 3 automorphism on a smooth K3 has 6 isolated fixed points. In our situation, two of these fixed points are on the isolated $A_1$ in $6A_3 + A_1$, thus taking the resolution one gets an $A_5$. The configurations $6A_3$ on $T_q/\hat{Q}_8$ are permuted by 3, creating $2A_3$ on the quotient surface, there are moreover $4A_2$ coming from the 4 other fixed points of $w'$. We thus obtain:

**Lemma 23.** The K3 surface $X = \text{Km}(T, \hat{T}_{24})$ contains a configuration $4A_2 + 2A_3 + A_5$.

Let now $X$ be any K3 surface containing a configuration $4A_2 + 2A_3 + A_5$.

**Proposition 24.** There exists a torus $T$ with an action of the group $\hat{T}_{24}$ such that $X = \text{Km}(T, \hat{T}_{24})$.

*Proof.* The configuration $4A_2 + 2A_3 + A_5$ contains 8 disjoint $A_2$ sub-configurations. The discriminant group of $8A_2$ is $(\mathbb{Z}_3)^8$. It has length $8 > \min(16, 22 - 16) = 6$, therefore there exists a non-trivial 3-divisible class $D$ with support on 6 of the 8A_2. By Remark 6 the support of $D$ is the sub-configuration $6A_2$ contained in $4A_2 + A_5$. The surface associated to the triple cover branched on the support of $D$ is a K3 surface $Y$ with a configuration $6A_3 + A_1$ and having an order 3 automorphism $\sigma$.

We proved in Proposition 21 that the surface $Y$ is of type $Y = \text{Km}(T, \hat{Q}_8)$. The automorphism $\sigma$ must preserves the 2 linearly independent even sets on $Y$ supported on the $6A_3$, otherwise there would be other divisibilities relations. Therefore by Lemma 11 the automorphism $\sigma$ lifts to the $(\mathbb{Z}_2)^2$-cover of $Y$, which contains a $16A_1$ configuration. These $16A_1$ are pull-back of curves in $X$, thus $\sigma$ lifts to an automorphism $\hat{\sigma}$ of $T$, and $X$ is the Kummer surface associated to the group generated by $\hat{Q}_8$ and $\hat{\sigma}$, which has order divisible by 3. By Theorem 16 that group is $\hat{T}_{24}$. □

Let again $X$ be a K3 surface containing a configuration $4A_2 + 2A_3 + A_5$. The discriminant group of the lattice $F_{\hat{T}_{24}}$ generated by the curves in $4A_2 + 2A_3 + A_5$ is $(\mathbb{Z}_3)^4 \times (\mathbb{Z}_4)^2 \times \mathbb{Z}_6$. 
It has length 5. There exists an integral class \( \gamma = \frac{1}{3} D \), where \( D \) is supported on the 6 disjoint \( A_2 \) on the sub-configuration \( 4A_2 + A_5 \) (see proof of Proposition 24). The discriminant group of the lattice generated by \( \gamma \) and \( F_{\hat{T}_{24}} \) is
\[
(Z_4)^2 \times (Z_4)^2 \times Z_6 \cong (Z_{12})^2 \times Z_6,
\]
which has length \( 3 = 22 - 19 \). By Remark 0 there are no other set of 6 disjoint \( A_2 \) which is 3-divisible, nor there are even sets, therefore we get the following result:

**Proposition 25.** The lattice generated by \( F_{\hat{T}_{24}} \) and \( \delta \) is the minimal primitive sub-lattice \( K_{\hat{T}_{24}} \subset \text{NS}(X) \) containing \( F_{\hat{T}_{24}} \). The discriminant group of \( K_{\hat{T}_{24}} \) is \( (Z_{12})^2 \times Z_6 \).

Thus if \( X = \text{Km}(\hat{T}, \hat{T}_{24}) \), then there is a primitive embedding of \( K_{\hat{T}_{24}} \) into \( \text{NS}(X) \). Conversely, let \( X \) be any K3 surface:

**Theorem 26.** Suppose that there is a primitive embedding of \( K_{\hat{T}_{24}} \) into \( \text{NS}(X) \), then \( X = \text{Km}(\hat{T}, \hat{T}_{24}) \).

*Proof.* Using MAGMA, it turns out that \( K_{\hat{T}_{24}} \) has the same roots as \( F_{\hat{T}_{24}} \). We then apply Lemma 10 and Remark 12. \( \square \)

4. **The case of equality in the orbifold Bogomolov Miyaoka Yau inequality**

4.1. **K3 surfaces.** For an orbifold \( \mathcal{X} \) with only ADE singularities, such that \( \mathcal{X} \) has Kodaira dimension 0 or 1, one has \( c_{\text{II}}(\mathcal{X}) = 0 \) and the second orbifold Chern number is defined by \( c_2(\mathcal{X}) = c_2(X) - m(C) \), where the rational number \( m(C) \geq 0 \) depends only on the type and number of the singularities of \( \mathcal{X} \) (see Section 2). For a K3 surface, part A) of Theorem 2 is thus equivalent to \( m(C) \leq c_2(X) = 24 \). Our aim is to characterize configurations \( C \) for which equality
\[
c_2(\mathcal{X}) = 0
\]
holds, i.e. when \( m(C) = c_2(X) \). For any configuration \( C \) among the following 10 configurations
\[
16A_1, \ 9A_2, \ 6A_1 + 4A_3, \ 5A_1 + 4A_2 + A_5, \\
3A_1 + 4D_4, \ 2A_1 + 3A_3 + 2D_4, \ A_1 + 2A_2 + 3A_3 + D_5, \\
A_1 + 4A_2 + D_4 + E_6, \ 4A_2 + 2A_3 + A_5, \ A_1 + 6A_3,
\]
one has \( m(C) = 24 \), moreover:

**Theorem 27.** Suppose that a K3 \( X \) contains the configuration \( C \) and let \( X \to \mathcal{X} \) be the contraction of the curves in \( C \). There exists a finite group of automorphisms \( G \) acting on a torus \( T \) such that \( X = \text{Km}(T, G) \) and \( \mathcal{X} = T/G \).

This is a result of Nikulin [17] for 16A1, Bertin [2] for the cases 9A2, 6A1 + 4A3, 5A1 + 4A2 + A5, of Propositions 21 and 24 for the two last cases and Garbagnati [12] for the remaining cases. A direct consequence is:

**Theorem 28.** For each of the 10 above cases, there exists a lattice \( \Gamma \) in the affine automorphism group of \( \mathbb{C}^2 \) such that \( \mathcal{X} = \mathbb{C}^2/\Gamma \).
In other words, each of the orbifold surfaces $X$ is uniformisable by $\mathbb{C}^2$.

It is easy to compute that there are 8 other possible configurations $C$ with Milnor number $\rho \leq 19$ (since $H^2(X, \mathbb{Z})$ has signature $(3, 19)$) and $m(C) = 24$. These configurations are:

- $C_1 = 11A_1 + 2A_3$
- $C_2 = 7A_1 + A_3 + 2D_4$
- $C_3 = 5A_1 + A_3 + A_7 + D_4$
- $C_4 = 6A_1 + 2A_2 + A_3 + D_5$
- $C_5 = 5A_1 + A_2 + D_4 + D_8$
- $C_6 = 5A_1 + A_3 + A_4 + D_7$
- $C_7 = 2A_1 + 2A_2 + 2D_4 + D_5$
- $C_8 = A_1 + 4A_2 + 2D_5$.

The aim of this section is to prove the following result, which with Theorem 28 implies Theorem 6.

**Proposition 29.** For any $i \in \{1, \ldots, 8\}$ there is no complex K3 surface containing a configuration $C_i$.

**Remark 30.** Some of these configurations $C_i$ may exist in characteristic $p > 0$. Indeed by [11] Corollary 3.17 and Remark 7.3, the cyclic groups $\mathbb{Z}_5, \mathbb{Z}_8, \mathbb{Z}_{10}, \mathbb{Z}_{12}$, the binary dihedral groups $\mathbb{D}_{n-2}$ (of order $4n - 8$, creating singularity $D_n$) with $n \in \{4, \ldots, 8\}$ and the binary octahedral and icosahedral groups act symplectically on some Abelian surfaces in characteristic $p > 0$.

**4.1.1. Configuration C_1 = 11A_1 + 2A_3.** The $11A_1$ plus one curve from each $A_3$ form a set of 13 disjoint curves. By Remark 6 there are two linearly independent even sets supported on 12 curves. This is impossible since a unique curve on a $A_3$ cannot be part of an even set. Such a configuration $C_1 = 11A_1 + 2A_3$ does not exist on a complex K3.

**4.1.2. Configuration C_2 = 7A_1 + A_3 + 2D_4.** There are 14 disjoint rational curves: $7A_1$, one curve in $A_3$ plus 3 curves for each $D_4$. If there are 14 disjoint rational curves on a K3 surface, there are three independent even sets, supported on all the curves. But one curve in $A_3$ can not be in the support of an even set. Therefore that configuration does not exist on a complex K3 surface.

**4.1.3. Configuration C_3 = 5A_1 + A_3 + A_7 + D_4.** Let us consider the following set of 12 disjoint rational curves supported on $C_3$: $5A_1$ plus the two disjoint curves in $A_3$, plus two disjoint curves in $A_7$ (at the extrema) and three disjoint curves in $D_4$. It contains an even set of curves $E$. The two curves on the $A_7$ cannot be on the support of $E$. One must take 0 or 2 curves in the $D_4$, thus the even set is made of $4A_1$ plus the two disjoint curves on the $A_3$ and two disjoint curves on the $D_4$. The K3 double cover will have a configuration

$$2A_1 + A_1 + 2A_7 + A_3,$$

but it would have Picard number > 20, contradiction.

**4.1.4. Configuration C_4 = 6A_1 + 2A_2 + A_3 + D_5.** There is a set of 13 disjoint rational curves on $C$. There must be two linearly independent even set supported on 12 of these curves. But an even set cannot contain the curves in a $A_2$.

**4.1.5. Configuration C_5 = 5A_1 + A_2 + D_4 + D_8.** Let us consider the following 14 curves: $5A_1$ plus one curve in $A_2$, plus the 3 disjoint curves in $D_4$ and the 5 disjoint curves in $D_8$. As for configuration $C_2$, there are three independent even sets, supported on all the curves. But the curve in $A_2$ cannot be in the support of a 2-divisible even set.
4.1.6. Configuration $C_6 = 5A_1 + A_3 + A_4 + D_7$. The sub-configuration $4A_1 + A_3 + A_4 + D_7$ contains 13 disjoint rational curves (the $5A_1$ plus 2 disjoint curves in $A_3$, 2 in $A_4$ and 4 curves in $D_7$), thus there exist two independents even sets supported on 12 curves. However the 2 disjoint curves in $A_4$ cannot be part of an even set.

4.1.7. Configuration $C_7 = 2A_1 + 2A_2 + 2D_4 + D_5$. There are 13 disjoint rational curves on $C_7$, thus there exists two linearly independent even sets and we obtain a contradiction as before by looking at the possible supports for these two even sets.

4.1.8. Configuration $C_8 = A_1 + 4A_2 + 2D_5$. The discriminant group of $C_8$ is
\[ \mathbb{Z}_2 \times (\mathbb{Z}_3)^2 \times (\mathbb{Z}_{12})^2 \]
it has length 5, but the lattice has rank 19 and a minimal primitive sub-lattice of rank 19 has a discriminant with length at most 3. Thus there exists some divisibilities by 2 or 3. But it is easy to check using Remark 6 that no such an even set can exist, nor there exists a 3-divisible set of $6A_2$.

4.2. Enriques surfaces. An Enriques surface $Z$ has invariants $K_Z^2 = 0$, $c_2 = 12$ with $2K_Z = 0$. It is the quotient of a K3 by a fix-point free involution. Let $C$ be a configuration of $ADE$ curves on an Enriques surface $Z$ such that the associated orbifold $Z$ has Chern numbers $c_1^2(Z) = 3c_2(Z)$.

**Proposition 31.** The configuration $C$ is $C = 8A_1$. There exists an Abelian surface $A$ isogeneous to the product of two elliptic curves, a group of automorphisms $G \simeq (\mathbb{Z}_2)^2$ of the surface $A$ generated by the involution $[-1]$ and a fix-point free involution such that $Z$ is the minimal resolution of $A/G$.

**Proof.** For an Enriques surface the condition $c_1^2(Z) = 3c_2(Z)$ is equivalent to $c_2(Z) = 0$ i.e. $m(C) = 12$.

Let $X \rightarrow Z$ be the étale double cover of $Z$. The K3 surface $X$ contains the configuration $2C$, which verifies $m(2C) = 24$, thus the only possibilities are $C = 8A_1$ and $C = 3A_1 + 2A_2$.

Let $\sigma$ be the Enriques involution on $X$ so that $Z = X/\sigma$. The involution $\sigma$ preserves the $16A_1$ (resp $6A_1 + 4A_2$) on $X$, thus it lifts to an automorphism $\sigma'$ on the Abelian surface $A$ such that $X = \text{Km}(A)$ (resp $X = \text{Km}(A, \mathbb{Z}_4)$). Since $\sigma$ has no fix points on $X$, $\sigma'$ has no fix points either on $A$.

Let us study the case $C = 8A_1$. Suppose that a lift $\sigma'$ of $\sigma$ has order 4; then $\sigma'^2$ is the transformation of the double cover $A \rightarrow X$, i.e. $\sigma'^2 = [-1]$. Since $H^0(Z, K_Z) = 0$, $\sigma'$ must not preserve the space $H^0(A, K_A)$, thus (up to replacing $\sigma'$ by $\sigma''$) the eigenvalues of the analytic representation of $\sigma'$ are $i, i$ and $A$ is the surface $(\mathbb{C}/\mathbb{Z}[i])^2$, $\sigma'$ is the multiplication by $i$ map composed by some translation. But such morphism has always fixed points. Therefore $\sigma'$ has order 2, commutes with $[-1]$ and the eigenvalues of its analytic representation are $(1, -1)$. Then there exists coordinates of $T_A \simeq \mathbb{C}^2$ such that $\sigma': A \rightarrow A$ is given by
\[ \sigma'(z_1, z_2) = (-z_1, z_2) + v \]
where $v \in A$. Thus there exist a product $E_1 \times E_2$ of elliptic curves and an isogeny $E_1 \times E_2 \rightarrow A$; moreover since $\sigma'$ commutes with $[-1]$, one must have $v = -v$ i.e. $v$ is a 2 torsion point. Since $\sigma'$ has no fix points $v$ is non-trivial.

Let us study the case $3A_1 + 2A_3$. Suppose there exists an Abelian surface $A$ and a group $G$ of order 8, such that $A/G$ is an Enriques surface with a configuration
2A_4 + 3A_1. For an automorphism \( \tau \) let \( \tau_0 \) be the linear part of \( \tau \), and \( G_0 \) the group \( \{ \tau_0 \mid \tau \in G \} \). An element \( \tau \) in the kernel \( K \) of \( G \to G_0 \) is a translation, but then \( A/G \) is the surface \( A'/G_0 \) where \( A' = A/K \) and \( G_0 \) has order 4, which lead to a contradiction. The group \( G \) is therefore isomorphic to \( G_0 \) and since it contains an order 4 element, it is among the following groups

\[ \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{D}_4, \text{ or } Q_8. \]

There are no order 8 automorphisms acting on an Abelian surface \([6]\) thus \( G \neq \mathbb{Z}_8 \). The group \( G \) is generated by a fix-point free involution \( \sigma \) and an automorphism \( \mu \) of order 4 such that \( A/\mu \) is a K3 with \( 4A_3 + 6A_1 \) (in particular \( \mu^2 = [-1] \)), moreover the involution \( \sigma \) induces a fix-point free non-symplectic involution on \( A/\mu \). The group \( G \) is not \( Q_8 \) since that groups has a unique involution.

Suppose that this is \( \mathbb{Z}_4 \times \mathbb{Z}_2 = \langle \mu \rangle \times \langle \sigma \rangle \), then \( \sigma_0 = \sigma_0(z_1, z_2) = (-z_1, z_2) \)

\[ \sigma(z_1, z_2) = (-z_1, z_2) + v \]

where \( v \) is a non-trivial 2-torsion point. Moreover since \( \mu_0 \sigma_0 = \sigma_0 \mu_0 \), the element \( \mu \) must act diagonally, thus

\[ \mu(z_1, z_2) = (iz_1, -iz_2). \]

Therefore \( A = C \times C \), where \( C \) is the elliptic curve \( \mathbb{C}/\mathbb{Z}[i] \). One has

\[ \sigma \mu(z_1, z_2) = (-iz_1, -iz_2) + v, \]

which has always some fix points, creating \( 1/4(1, 1) \) singularities, but there is no such singularities on Enriques surfaces.

It remains the dihedral group \( \mathbb{D}_4 \) of order 8. There is only one faithful 2-dimensional representation of \( \mathbb{D}_4 \), which is generated by

\[ \sigma_0(z_1, z_2) = (-z_1, z_2), \mu(z_1, z_2) = (-z_2, z_1). \]

Taking \( \sigma(z_1, z_2) = (-z_1, z_2) + v \) where \( v \) is a 2-torsion point, the involution \( \sigma \mu \) has a one dimensional fixed point set, thus the quotient of \( A \) by \( \mathbb{D}_4 \) is a rational surface (see e.g. \([11]\)). We thus proved that there is no Enriques surface containing a configuration \( 3A_1 + 2A_3 \).

**Example 32.** (Lieberman, see \([12]\)). Let \( A \) be the product of two elliptic curves \( A = E_1 \times E_2 \) and let \((e_1, e_2)\) be a 2-torsion point on \( A \), with \( e_1 \neq 0, e_2 \neq 0 \). Then the endomorphism \( \tau : A \to A \) given by

\[ \tau(z_1, z_2) = (-z_1 + e_1, z_2 + e_2) \]

induces a fix-point free involution on the Kummer surface \( \text{Km}(A) \). The associated Enriques surface contains a \( 8A_1 \) configuration.

Let \( T_0 \) be a sub-group of torsion points on \( A = E_1 \times E_2 \) as above, such that \( \tau(T_0) = T_0 \) and \((e_1, 0), (0, e_2)\) are not element of \( T_0 \). Then \( \tau \) induces a fix-point free involution \( \tau' \) on the quotient \( A/T_0 \) and \( A/(T_0, [-1], \tau) \) is an Enriques surface containing \( 8A_1 \). Reciprocally, from the proof of Proposition \([24]\) every Enriques surface containing \( 8A_1 \) is obtained by that construction.
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Xavier Rouleau,
Aix-Marseille Université, CNRS, Centrale Marseille,
UMR 7373, 13453 Marseille, France
Xavier.Rouleau@univ-amu.fr