EXAMPLE OF QUASIANALYTIC CONTRACTION WHOSE SPECTRUM IS A PROPER SUBARC OF THE UNIT CIRCLE

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Abstract. A partial answer on [KS2, Question 2] is given. Namely, an operator $R$ similar to a quasianalytic contraction whose quasianalytic spectral set is equal to its spectrum and is a proper subarc of the unit circle is constructed, but no estimates of $\|R^{-1}\|$ is given.

1. Introduction

Let $H$ be a (complex, separable) Hilbert space, and let $T$ be a (linear, bounded) operator on $H$. The lattice of all (closed) subspaces $E$ of $H$ such that $TE \subset E$ is called the invariant subspace lattice of $T$ and is denoted by $\text{Lat} T$. The commutant $\{T\}'$ is the set of all operators $A$ on $H$ such that $AT = TA$. Recall that $\{T\}'$ is an algebra closed in the weak operator topology. The lattice of all subspaces $E$ of $H$ such that $AE \subset E$ for every $A \in \{T\}'$ is called the hyperinvariant subspace lattice of $T$ and is denoted by $\text{Hlat} T$; the subspaces $E$ are called hyperinvariant subspaces of $T$.

Let $H$ and $K$ be two Hilbert space. Denote by $L(H, K)$ the space of all (linear, bounded) transformations acting from $H$ to $K$. Set $L(H) = L(H, H)$, then $L(H)$ is the algebra of all (linear, bounded) operators acting on $H$. Let $T \in L(H), R \in L(K), X \in L(H, K)$ be such that $XT = RX$. If $X$ is invertible, that is, $X^{-1} \in L(K, H)$, then $T$ and $R$ are called similar. If, in addition, $X$ is a unitary transformation, then $T$ and $R$ are called unitarily equivalent.

An operator $T \in L(H)$ is called power bounded, if $\sup_{n \in \mathbb{N}} \|T^n\| < \infty$. A power bounded operator $T$ is of class $C_{1\cdot}$, if $\inf_{n \in \mathbb{N}} \|T^n\| > 0$ for every $0 \neq x \in H$, and is of class $C_{0\cdot}$, if $\lim_{n} \|T^n x\| = 0$ for every $x \in H$. $T$ is of class $C_{a\cdot}$, if $T^*$ is of class $C_{a\cdot}$, and $T$ is of class $C_{ab}$, if $T$ is of classes $C_{a\cdot}$ and $C_{b\cdot}, a, b = 0, 1$.

An operator $T \in L(H)$ is called polynomially bounded, if there exists a constant $M$ such that

$$\|p(T)\| \leq M \max\{|p(z)| : z \in \text{clos} \mathbb{D}\}$$

for every (analytic) polynomial $p$,

where $\mathbb{D}$ is the open unit disc. For a natural number $n$ a $n \times n$ matrix can be regarded as an operator on $\ell_2^n$, its norm is denoted by the symbol $\| \cdot \|_{L(\ell_2^n)}$.

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For a family of polynomials \([p_{ij}]_{i,j=1}^n\) put
\[
\|p_{ij}(\alpha)\|_{\ell^p_{\infty}} = \sup\{\|p_{ij}(\alpha)\|_{\ell^p_{\infty}}, \alpha \in \text{clos } \mathbb{D}\}.
\]
For \(T \in \mathcal{L}(\mathcal{H})\) and a family of polynomials \([p_{ij}]_{i,j=1}^n\) the operator
\[
[p_{ij}(T)]_{i,j=1}^n \in \mathcal{L}(\ell^n_{\infty})
\]
is defined. \(T\) is called completely polynomially bounded, if there exists a constant \(M\) such that
\[
\|p_{ij}(T)\|_{i,j=1}^n \leq M \|p_{ij}\|_{i,j=1}^n \quad \text{for every family of polynomials } [p_{ij}]_{i,j=1}^n.
\]
(1.1)

An operator \(T\) is called a contraction if \(\|T\| \leq 1\). The following criterion for an operator to be similar to a contraction is proved in [P]:

An operator \(T\) is similar to a contraction if and only if \(T\) is completely polynomially bounded.

We recall some definitions and results on unitary asymptotes and quasianalytic operators. For references on unitary asymptotes see [NFBK, Ch. IX.1], [Kér1], [Kér3], [Kér4], for quasianalytic operators see also [E], [KS1], [KS2], [Gam1], [Gam2]; see also references therein.

A pair \((X,U)\), where \(U\) is a unitary operator, and \(X\) is a transformation such that \(XT = UX\), is called a unitary asymptote of an operator \(T\), if for any other pair \((Y,V)\), where \(V\) is a unitary operator, and \(Y\) is a transformation such that \(YT = VY\), there exists a unique transformation \(Z\) such that \(ZU = VZ\) and \(Y = ZX\). Two pairs \((X,U)\) and \((X',U')\), where \(U\) and \(U'\) are unitary operators, and \(X\) and \(X'\) are transformations such that \(XT = UX\) and \(X'T = U'X'\), are equivalent, if there exists an invertible transformation \(Z\) such that \(ZU = U'Z\) and \(X' = ZX\). It follows from the definition that a unitary asymptote of \(T\) is defined up to equivalence. If two operators are similar and one of them has a unitary asymptote, then there is also a unitary asymptote, too, and their unitary asymptotes are equivalent. If \(ZU = U'Z\) for an invertible transformation \(Z\) and unitary operators \(U\) and \(U'\), then \(U\) and \(U'\) are unitarily equivalent ([NFBK, Proposition II.3.4], [Co, Proposition II.10.6], [RR, Proposition 1.5]).

Let \(T \in \mathcal{L}(\mathcal{H})\) have a unitary asymptote \((X,U)\), where \(U \in \mathcal{L}(\mathcal{K})\). Then there exists the mapping
\[
\gamma_T : \{T\}' \to \{U\}', \quad \gamma_T(A) = D,
\]
where \(D \in \{U\}'\) is a unique operator such that \(XA = DX\), and \(\gamma_T\) is a unital algebra-homomorphism. Furthermore,
\[
\sigma(\gamma_T(A)) \subset \sigma(A) \quad \text{for every } A \in \{T\}'.
\]
For \(\mathcal{E} \subset \mathcal{K}\) set
\[
X^{-1}\mathcal{E} = \{x \in \mathcal{H} : Xx \in \mathcal{E}\}.
\]
Then \(X^{-1}\mathcal{E} \in \text{Hlat} T\) for every \(\mathcal{E} \in \text{Hlat} U\). We will assume that \(\mathcal{K} \neq \{0\}\).

It is well known that \(\text{Hlat} U \neq \{0\}, \mathcal{K}\), if \(U\) is not the multiplication by a unimodular constant on \(\mathcal{K}\). But it is possible that
\[
X^{-1}\mathcal{E} = \{0\} \quad \text{for every } \mathcal{E} \in \text{Hlat} U \text{ such that } \mathcal{E} \neq \mathcal{K}.
\]
Such an operator \(T\) is called quasianalytic.
Denote by $m$ the normalized linear measure on the unit circle $\mathbb{T}$. For a measurable (with respect to $m$) set $\sigma \subset \mathbb{T}$ denote by $U_\sigma$ the operator of multiplication by the independent variable on $L^2(\sigma) := L^2(\sigma, m)$. It is well known that $U_\sigma$ is cyclic, 
\[
\{U_\sigma\}' = \{\eta(U_\sigma) : \eta \in L^\infty(\sigma, m)\},
\]
where $\eta(U_\sigma)$ is the operator of multiplication by $\eta$, and
\[
\text{Hlat} U_\sigma = \{L^2(\tau) : \tau \subset \sigma\},
\]
(where $\tau$ are measurable with respect to $m$).

Let $\sigma \subset \mathbb{T}$ be a measurable (with respect to $m$) set, and let an operator $T$ have a unitary asymptote $(X, U_\sigma)$. Then for every $A \in \{T\}'$ there exists a function $\eta \in L^\infty(\sigma, m) =: L^\infty(\sigma)$ such that $\gamma_T(A) = \eta(U_\sigma)$. The mapping
\[
\tilde{\gamma}_T : \{T\}' \to L^\infty(\sigma), \quad \tilde{\gamma}_T(A) = \eta,
\]
is a unital algebra-homomorphism, and $\tilde{\gamma}_T$ does not depend on the choice of $X$. Furthermore, $\tilde{\gamma}_T(T) = \chi$, where $\chi(z) = z, z \in \mathbb{T}$. The range $\tilde{\gamma}_T(\{T\}')$ is called the functional commutant of $T$, see [KS1] and references therein.

Every power bounded operator $T$ has a unitary asymptote, and if $T$ of class $C_1$, then $\gamma_T$ is injective. If, in addition, a unitary operator from the unitary asymptote (which also will be called the unitary asymptote) of $T$ is $U_\sigma$ for some $\sigma \subset \mathbb{T}$, then $\tilde{\gamma}_T$ is injective, too. Therefore, $\{T\}'$ is an abelian algebra, because $L^\infty(\sigma)$ is an abelian algebra. Moreover, if $R \in \{T\}'$ and $\{R\}'$ is abelian, then $\{T\}' = \{R\}'$ (see [KS1, Proposition 11], quasianalyticity is not used in the proof here).

Let $T$ be a polynomially bounded operator. Then, clearly, $T$ is power bounded, therefore, $T$ has a unitary asymptote. If, in addition, $T$ is of class $C_0$, then the spectral measure of the unitary asymptote of $T$ is absolutely continuous with respect to $m$ [Kér3, Theorem 13 and Proposition 15]. If $T$ is quasianalytic, then $T$ is of class $C_1$ [Kér3, Proposition 33]. For definition of the quasianalytic spectral set of $T$ we refer to [Kér3], [Kér4] and [KS1]. We recall only that for quasianalytic polynomially bounded operator $T$ the quasianalytic spectral set coincides with the measurable (with respect to $m$) set on which the spectral measure of the unitary asymptote of $T$ is concentrated.

Let $T$ be a polynomially bounded quasianalytic operator. Then the spectrum $\sigma(T)$ of $T$ is a connected set, and $m(\sigma(T) \cap \mathbb{T}) > 0$. Therefore, if $\sigma(T) \subset \mathbb{T}$, then $\sigma(T)$ is a subarc of $\mathbb{T}$. Examples of quasianalytic contractions $T$ such that $\sigma(T) = \mathbb{T}$ or $\sigma(T) \cap \mathbb{T} \neq \mathbb{T}$ are known. But in all known (to the author) examples the interior of the polynomially convex hull of $\sigma(T)$ is non-empty. (Recall that the polynomially convex hull of a compact set $\sigma \subset \mathbb{C}$ is the union of $\sigma$ and all the bounded components of $\mathbb{C} \setminus \sigma$; for example, the polynomially convex hull of $\mathbb{T}$ is close to $D$.) In this paper, a quasianalytic operator $R$ similar to a contraction is constructed such that $\sigma(R) = \{e^{it} : t \in [0, \pi]\}$ and the unitary asymptote of $R$ is $U_{\sigma(R)}$. Therefore, the quasianalytic spectral set of $R$ is $\sigma(R)$. For this purpose, an appropriate quasianalytic operator $T$ with $\sigma(T) = \mathbb{T}$ is constructed, and it is proved that there exists $R \in \{T\}'$ such that $R^2 = T$. The existence of non-trivial hyperinvariant subspaces of $T$ and $R$ is based on result from [E].
The following notation will be used. By $H^p$ the Hardy space is denoted (on a some domain of $\mathbb{C}$, which will be mentioned). By $I_H$ and $P_E$ the identity operator on a Hilbert space $H$ and the orthogonal projection on the subspace $E$ are denoted. For two positive functions $w(t)$ and $\phi(t)$, the notation $w \asymp \phi$ means that $0 < \inf_t w(t)/\phi(t) \leq \sup_t w(t)/\phi(t) < \infty$. By $1$ and $\chi$ the unit constant function and the identity function are denoted: $1(z) = 1$ and $\chi(z) = z$, $z \in \text{clos } \mathbb{D}$.

2. Square root of operator

**Theorem 2.1.** Let $T$ and $R$ be two operators on a Hilbert space such that $R^2 = T$. Then $R$ is power bounded if and only if $T$ is power bounded, and then $R$ is of class $C_1$. if and only if $T$ is of class $C_1$, and $R$ is of class $C_0$ if and only if $T$ is of class $C_0$. Furthermore, $R$ is polynomially bounded if and only if $T$ is polynomially bounded, and $R$ is similar to a contraction if and only if $T$ is similar to a contraction.

**Proof.** The proofs of statements concerning power boundedness are very simple, therefore, they are omitted. The proof of “only if” part of statements concerning polynomial boundedness and similarity of contractions are very simple, too. Suppose that $T$ is similar to a contraction. Then $T$ is completely polynomially bounded [P]. We will to prove that $R$ is completely polynomially bounded. Then it will be proved that $R$ is similar to a contraction [P].

Let $p$ be a polynomial. Then $p(z) = \sum_{n=0}^{N} c_n z^n$, $z \in \mathbb{C}$, for some $N \in \mathbb{N}$. For convenience, set $c_n = 0$ for $n \in \mathbb{N}$, $n \geq N + 1$. Set

$$p_0(z) = \sum_{n \geq 0} c_{2n} z^n \quad \text{and} \quad p_1(z) = \sum_{n \geq 0} c_{2n+1} z^n, \quad z \in \mathbb{C}.$$  

Clearly,

$$p_0(z^2) = \frac{p(z) + p(-z)}{2} \quad \text{and} \quad z p_1(z^2) = \frac{p(z) - p(-z)}{2}, \quad z \in \mathbb{C}.$$  

For a family of polynomials $[p_{ij}]_{i,j=1}^{n}$ we have

$$[(p_{ij})_0(z^2)]_{i,j=1}^{n} = \frac{1}{2} \left( [p_{ij}(z)]_{i,j=1}^{n} + [p_{ij}(-z)]_{i,j=1}^{n} \right) \quad \text{and} \quad z [(p_{ij})_1(z^2)]_{i,j=1}^{n} = \frac{1}{2} \left( [p_{ij}(z)]_{i,j=1}^{n} - [p_{ij}(-z)]_{i,j=1}^{n} \right), \quad z \in \mathbb{C}.$$  

Therefore,

$$\|[(p_{ij})_0(z^2)]_{i,j=1}^{n}\|_{L(\ell^{2})} \leq \frac{1}{2} \left( \| [p_{ij}(z)]_{i,j=1}^{n} \|_{L(\ell^{2})} + \| [p_{ij}(-z)]_{i,j=1}^{n} \|_{L(\ell^{2})} \right) \quad \text{and} \quad z \|[(p_{ij})_1(z^2)]_{i,j=1}^{n}\|_{L(\ell^{2})} \leq \frac{1}{2} \left( \| [p_{ij}(z)]_{i,j=1}^{n} \|_{L(\ell^{2})} + \| [p_{ij}(-z)]_{i,j=1}^{n} \|_{L(\ell^{2})} \right) \quad \text{and} \quad z \| [(p_{ij})_1(z^2)]_{i,j=1}^{n}\|_{L(\ell^{2})}, \quad z \in \mathbb{C}.$$  

EXAMPLE OF QUASIANALYTIC CONTRACTION 4

$\|$
Clearly, for every $\zeta \in \text{clos } \mathbb{D}$ there exists $z \in \text{clos } \mathbb{D}$ such that $z^2 = \zeta$. Therefore,

$$
||[(p_{ij})_0]^n_{i,j=1}||_{H^\infty(\ell_2^2)} \leq ||[p_{ij}]^n_{i,j=1}||_{H^\infty(\ell_2^3)} \quad \text{and}
$$

$$
||[(p_{ij})_1]^n_{i,j=1}||_{H^\infty(\ell_2^2)} \leq ||[p_{ij}]^n_{i,j=1}||_{H^\infty(\ell_2^3)}.
$$

We have

$$
[p_{ij}(R)]^n_{i,j=1} = [(p_{ij})_0(T)]^n_{i,j=1} + \left(\oplus_{j=1}^n R\right) \cdot [(p_{ij})_1(T)]^n_{i,j=1}.
$$

Since $T$ is completely polynomially bounded, $(1.1)$ is fulfilled for $T$ with some constant $M$. Therefore,

$$
||[p_{ij}(R)]^n_{i,j=1}|| \leq ||[(p_{ij})_0(T)]^n_{i,j=1}|| + ||R|| ||[(p_{ij})_1(T)]^n_{i,j=1}||
$$

$$
\leq M ||[(p_{ij})_0]^n_{i,j=1}||_{H^\infty(\ell_2^3)} + ||R|| M ||[(p_{ij})_1]^n_{i,j=1}||_{H^\infty(\ell_2^3)}
$$

$$
= (1 + ||R||) M ||[p_{ij}]^n_{i,j=1}||_{H^\infty(\ell_2^3)}.
$$

Thus, $R$ is completely polynomially bounded.

If we suppose only that $T$ is polynomially bounded, then the proof of polynomial boundedness of $R$ is similar.  \hfill \Box

**Lemma 2.2.** Set $q(e^{it}) = e^{it}, t \in (0, 2\pi)$. Suppose that $T$, $R \in \mathcal{L}(\mathcal{H})$ are such that $T$ is a polynomially bounded operator of class $C_0$, $(X, U_T)$ is a unitary asymptote of $T$, $R^2 = T$ and $XR = q(U_T)X$. Then $(X, q(U_T))$ is a unitary asymptote of $R$.

**Proof.** By Theorem 2.1, $R$ is a polynomially bounded operator of class $C_0$. Therefore, $R$ has a unitary asymptote $(Y, V)$, and the spectral measure of $V$ is absolutely continuous with respect to $m$ [Kér3, Theorem 13 and Proposition 15]. By [Kér2, Theorem 6], $(Y, V^2)$ is a unitary asymptote of $T$. Therefore, there exists an invertible transformation $Z_1$ such that $Y = Z_1X$ and $Z_1U_T = V^2Z_1$.

Since $q(U_T)$ is unitary, there exists a transformation $Z_2$ such that $X = Z_2Y$ and $Z_2V = q(U_T)Z_2$. Therefore, $X = Z_2Z_1X$ and $Z_2Z_1U_T = U_TZ_2Z_1$. It follows from the definition of a unitary asymptote that

$$
\bigvee_{n \geq 0} U_T^{-n}X\mathcal{H} = L^2(\mathbb{T}).
$$

Therefore, $Z_2Z_1 = I_{L^2(\mathbb{T})}$. Thus, $Z_2 = Z_1^{-1}$. It is proved that the pairs $(X, q(U_T))$ and $(Y, V)$ are equivalent. \hfill \Box

**Corollary 2.3.** In assumption of Lemma 2.2, $T$ is quasianalytic if and only if $R$ is quasianalytic.

**Proof.** It is well known and easy to see that $q(U_T)$ is unitarily equivalent to $U_\sigma$ with $\sigma = \{e^{it} : t \in (0, \pi)\}$. Therefore, $\{q(U_T)\}'$ is an abelian algebra. Since $q(U_T) \in \{U_T\}'$ and $\{U_T\}'$ is abelian, we conclude that

$$
\{q(U_T)\}' = \{U_T\}'
$$

by [KS1, Proposition 11] (quasianalyticity is not used in the proof there). Thus,

$$
(2.1) \quad \text{Hlat } q(U_T) = \text{Hlat } U_T = \{L^2(\tau) : \tau \subset \mathbb{T}\}.
$$
Suppose that $T$ is quasianalytic. It follows from the definition (1.3) of quasianalyticity and (2.1) that
\[(2.2)\quad X^{-1}L^2(\tau) = \{0\}\]
for every measurable set $\tau \subset \mathbb{T}$ such that $m(\tau) < 1$. Since $(X, \varphi(U_T))$ is a unitary asymptote of $R$, we obtain that $R$ is quasianalytic by (2.2) and (2.1).

Conversely, if $R$ is quasianalytic, then $T$ is quasianalytic by the same reasoning.

Note that “if” part is a particular case of [Kér2, Corollary 13] (applied to $R$).

\[\square\]

3. Construction of $T$

Recall that $1(z) = 1$ and $\chi(z) = z$, $z \in \text{clos} \, \mathbb{D}$, and $H^2(\mathbb{D})$ denotes the Hardy space in $\mathbb{D}$. Set $S = U_T|_{H^2(\mathbb{D})}$ and $H^2(\mathbb{D}) = L^2(\mathbb{T}) \ominus H^2(\mathbb{D})$.

In the following proposition [Gam1, Proposition 3.1] (based on [Ca]), [Kér1, Theorem 3] (applied to $T^*$) and [KS1, Sec. 5] are combined, therefore, its proof is omitted.

**Proposition 3.1.** Suppose that $\mathcal{H}_0$ is a Hilbert space, $T_0 \in \mathcal{L}(\mathcal{H}_0)$ is a contraction of class $C_{00}$, $X_0 \in \mathcal{L}(\mathcal{H}_0, H^2(\mathbb{D}))$ is such that $\ker X_0 = \{0\}$, $\text{clos} \, X_0 \mathcal{H}_0 = H^2(\mathbb{D})$ and $X_0 T_0 = P_{H^2(\mathbb{D})} U_T |_{H^2(\mathbb{D})} X_0$. Put
\[
T = \left( S \oplus \begin{pmatrix} \cdot & X_0^* \mathcal{N} \end{pmatrix} \right)_{T_0}.
\]

Then $T$ is similar to a contraction and $T$ is of class $C_{10}$. Therefore, $T$ admits an $H^\infty$-functional calculus. Furthermore, $(I_{H^2(\mathbb{D})} \oplus X_0, U_T)$ is a unitary asymptote of $T$. Therefore, $T \subset \sigma(T) \subset \text{clos} \mathbb{D}$, and $T$ is quasianalytic if and only if
\[(3.1)\quad P_{H^2(\mathbb{D})} L^2(\tau) \cap X_0 \mathcal{H}_0 = \{0\}\]
for every measurable set $\tau \subset \mathbb{T}$ such that $m(\tau) < 1$.

Let $\eta \in L^\infty(\mathbb{T})$. Then $\tilde{\gamma}_T(\{\gamma \}^1)$ (where $\tilde{\gamma}_T$ is defined in (1.4)) if and only if the mapping
\[(3.2)\quad (I_{H^2(\mathbb{D})} \oplus X_0)^{-1} \eta(U_T) (I_{H^2(\mathbb{D})} \oplus X_0)\]
is defined and bounded, and then $\tilde{\gamma}_T^{-1}(\eta)$ is equal to the operator in (3.2).

Let $\nu$ be a positive finite Borel measure on $\mathbb{D}$. Clearly, the operator of multiplication by the independent variable on $L^2(\nu)$ is a contraction of class $C_{00}$. Denote by $P^2(\nu)$ the closure of (analytic) polynomials in $L^2(\nu)$, and by $S_\nu$ the operator of multiplication by the independent variable in $P^2(\nu)$, i.e.
\[S_\nu \in \mathcal{L}(P^2(\nu)), \quad (S_\nu f)(z) = zf(z), \quad f \in P^2(\nu), \quad z \in \mathbb{D}.
\]
Since $S_\nu$ is the restriction on an invariant subspace of a contraction of class $C_{00}$, $S_\nu$ is a contraction of class $C_{00}$, too. Furthermore, if $H^2(\mathbb{D}) \subset L^2(\nu)$, then the natural embedding of $H^2(\mathbb{D})$ into $L^2(\nu)$ is bounded and
\[P^2(\nu) = \text{clos}_{L^2(\nu)} H^2(\mathbb{D}).\]
Set
\begin{equation}
(Wh)(z) = \overline{\vartheta(h)}, \quad h \in L^2(T), \quad z \in \mathbb{T}.
\end{equation}
Clearly, \( W \in \mathcal{L}(L^2(\mathbb{T})) \) is unitary, \( W = W^{-1} \), and \( WH^2(\mathbb{D}) = H^2(\mathbb{D}) \).

**Proposition 3.2.** Let \( \nu \) be a positive finite Borel measure on \( \mathbb{D} \) such that every \( f \in P^2(\nu) \) is analytic in \( \mathbb{D} \), for every \( \lambda \in \mathbb{D} \) the mapping \( f \mapsto f(\lambda) \), \( P^2(\nu) \to \mathbb{C} \) is bounded, and \( H^2(\mathbb{D}) \subset P^2(\nu) \). Let \( J_\nu \in \mathcal{L}(H^2(\mathbb{D}), P^2(\nu)) \) be the natural imbedding. Set \( H_0 = P^2(\nu), T_0 = S^*_\nu, X_0 = WH_0^*, \) and define \( T \) as in Proposition 3.1. Then \( \sigma(T) = \mathbb{T} \).

*Proof.* Let \( \lambda \in \mathbb{D} \). There exists \( k_\lambda \in P^2(\nu) \) such that \( (f, k_\lambda) = f(\lambda) \) for every \( f \in P^2(\nu) \). Clearly, \( S^*_\nu k_\lambda = \overline{\lambda} k_\lambda \). For every \( \lambda \in \mathbb{D} \) and every \( f \in P^2(\nu) \) define
\[
 f_\lambda(z) = \frac{f(z) - f(\lambda)}{z - \lambda}, \quad z \in \mathbb{D}.
\]
Then \( f_\lambda \in P^2(\nu) \). Therefore, \( P^2(\nu) = (S_\nu - \lambda I) P^2(\nu) + \mathbb{C}1 \) for every \( \lambda \in \mathbb{D} \). (For proof, see, for example, [ARS, Lemma 4.5].) Thus, \( S_\nu - \lambda I \) is left-invertible, therefore, \((S_\nu - \lambda I)^* P^2(\nu) = P^2(\nu)\). Furthermore,
\[
\dim \ker(S_\nu - \lambda I)^* = 1.
\]

Let \( \lambda \in \mathbb{D}, h_\circ \in H^2(\mathbb{D}), \) and let \( f_0 \in P^2(\nu) \). There exists \( f \in P^2(\nu) \) such that \((S_\nu - \lambda I)^* f = f_0 \). Set \( h(z) = \frac{h_\circ(z) - h_\circ(\overline{\lambda})}{z - \overline{\lambda}}, \ z \in \mathbb{D} \). Then \( h \in H^2(\mathbb{D}) \). Taking into account that
\[
X_0^* \overline{\nu} = J_\nu W^{-1} = J_\nu 1 = 1,
\]
we obtain that
\[
(T - \overline{\lambda}I)(h + (h_\circ(\overline{\lambda}) - (f, 1)))k_\lambda = h_\circ + f_0.
\]
If \( \lambda \in \mathbb{D}, h \in H^2(\mathbb{D}), \) and \( f \in P^2(\nu) \) are such that \((T - \overline{\lambda}I)(h + f) = 0\), then there exists \( c \in \mathbb{C} \) such that \( f = ck_\lambda \) and \((z - \overline{\lambda})h(z) = -c \) for every \( z \in \mathbb{D} \). Therefore, \( c = 0 \). Thus, \( \mathbb{D} \cap \sigma(T) = \emptyset \). By Proposition 3.1, \( T \subset \sigma(T) \subset \text{clos} \mathbb{D} \). \( \square \)

For \( 0 \leq r < 1 \) set
\begin{equation}
D_r = \{ z \in \mathbb{C} : |z - r| < 1 - r \}, \quad \Gamma_r = \partial D_r = \{ z \in \mathbb{C} : |z - r| = 1 - r \},
\end{equation}
denote by \( \nu_r \) the arc length measure on \( \Gamma_r \). (Of course, \( D_0 = \mathbb{D}, \Gamma_0 = \mathbb{T}, \) and \( \nu_0 = 2\pi \mathbb{m} \).) Using a linear change of variable and well-known properties of \( H^2(\mathbb{D}) \), it is easily seen that every \( f \in P^2(\nu_r) \) is analytic in \( D_r \), and for every \( \lambda \in D_r \) the mapping \( f \mapsto f(\lambda), P^2(\nu_r) \to \mathbb{C} \) is bounded.

**Lemma 3.3.** Let \( \{a_n\}_{n=1}^\infty \) and \( \{r_n\}_{n=1}^\infty \) be families of numbers such that \( a_n > 0, 0 < r_{n+1} < r_n \) for every \( n, \sum_{n=1}^\infty a_n < \infty \), and \( r_n \to 0 \). Set
\[
\nu = \frac{1}{2\pi} \sum_{n=1}^\infty a_n \nu_{r_n}.
\]
Then \( H^2(\mathbb{D}) \subset L^2(\nu) \), if \( f \in P^2(\nu) \), then \( f \) is analytic in \( \mathbb{D} \), and for every \( \lambda \in \mathbb{D} \) the mapping \( f \mapsto f(\lambda), P^2(\nu) \to \mathbb{C} \) is bounded.
Proof. Let $h \in H^2(D)$. Since \( \frac{1}{2\pi} \int_{\Gamma_r} |h|^2 d\nu_r \leq 2 \|h\|_{H^2(D)}^2 \) (see, for example, [N, Lemma I.A.6.3.3]), we conclude that $H^2(D) \subset L^2(\nu)$. On the other hand, $P^2(\nu) \subset P^2(\nu_r)$ for every $n$, $D_{r_n} \subset D_{r_{n+1}}$ and $\cup_{n=1}^{\infty} D_{r_n} = D$. Therefore, every $f \in P^2(\nu)$ is analytic in $D$, and for every $\lambda \in \mathbb{D}$ the mapping $f \mapsto f(\lambda)$, $P^2(\nu) \to \mathbb{C}$ is bounded. □

**Remark 3.4.** The construction of the measure $\nu$ from Lemma 3.3 is close to [KT1], [KT2].

### 4. Transfer to the half-plane and Fourier transform

Set $C_+ = \{z \in \mathbb{C} : \text{Im} z > 0\}$, $C_- = \{z \in \mathbb{C} : \text{Im} z < 0\}$ and

\[
\varpi(z) = \frac{z - i}{z + i}, \quad z \in \mathbb{C}.
\]

It is well known and easy to see that $\varpi|_{C_+}$ is a conformal mapping of $C_+$ onto $\mathbb{D}$, and for every $0 \leq r < 1$ and every $f$ for which the integrals below are defined we have

\[
\frac{1}{2\pi} \int_{\Gamma_r} f d\nu_r = \frac{1}{\pi} \int_{-\infty}^{+\infty} (f \circ \varpi)(t + i \frac{r}{1 - r}) \frac{dt}{t^2 + (\frac{r}{1 - r})^2},
\]

where $\Gamma_r$ and $\nu_r$ are defined by (3.4) and just after (3.4), respectively. Set

\[
(Jf)(z) = \frac{1}{\sqrt{\pi}} \frac{1}{z + i}(f \circ \varpi)(z)
\]

for all functions $f$ and $z \in \mathbb{C}$ for which the definition (4.3) has sense. Then $J$ is a unitary transformation from $L^2(T)$ onto $L^2(\mathbb{R})$, and for $\eta \in L^\infty(T)$ the operator $J\eta(U_T)J^{-1}$ is the multiplication by $\eta \circ \varpi$ acting on $L^2(\mathbb{R})$. Furthermore, $JH^2(D) = H^2(C_+)$ (see, for example, [N, Sec. I.A.6.3.1]). Since $f \in H^2(D)$ if and only if $f_*(z) := \frac{1}{z} f(\frac{z}{2})$, $|z| > 1$, is from $H^2(D)$, we have

\[
(Jf_*)(z) = -(Jf)(-z), \quad z \in C_-,
\]

and

\[
(JWJ^{-1}h)(z) = -h(-z) \quad \text{for } h \in H^2(C_+) \text{ and } z \in C_-
\]

For a measure $\mu$ defined as in Lemma 3.3 set

\[
d\mu = \sum_{n=1}^{\infty} a_n dt|_{\mathbb{R} + iv_n} \text{ with } v_n = \frac{r_n}{1 - r_n}, \quad n \geq 1.
\]

A straightforward calculation based on (4.2) shows that

\[
J \text{ is a unitary transformation from } L^2(D, \nu) \text{ onto } L^2(C_+, \mu).
\]

Since $P^2(\nu) = \text{clos}_{L^2(\nu)} H^2(D)$ and $JH^2(D) = H^2(C_+)$, we conclude that $H^2(C_+) \subset L^2(C_+, \mu)$ and

\[
J P^2(\nu) = \text{clos}_{L^2(\mu)} H^2(C_+).
\]

Denote by $J_\mu$ the natural imbedding of $H^2(C_+)$ into $\text{clos}_{L^2(\mu)} H^2(C_+)$.  

Let $\mathcal{D}(\mathbb{R})$ be the space of test functions, that is, the space of functions from $C^\infty(\mathbb{R})$ with compact support. Let $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ be the spaces of rapidly decreasing functions and of tempered distributions, respectively.
Recall that $S'(\mathbb{R})$ is the dual space of $S(\mathbb{R})$, $D(\mathbb{R})$ is contained and dense in $S(\mathbb{R})$, and $L^p(\mathbb{R}) \subset S'(\mathbb{R})$, $1 \leq p \leq \infty$. The Fourier transform $\mathcal{F}$ of a function $f$ defined on $\mathbb{R}$ and its inverse $\mathcal{F}^{-1}$ act by the formulas
\begin{equation}
(\mathcal{F}f)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ists}f(s)ds, \quad (\mathcal{F}^{-1}f)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ists}f(s)ds, \quad t \in \mathbb{R}.
\end{equation}

It is well known that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are linear continuous mutually inverse bijections on $S(\mathbb{R})$ and on $S'(\mathbb{R})$, and $\mathcal{F}$ is unitary on $L^2(\mathbb{R})$. It follows from (4.4) that
\begin{equation}
\mathcal{F}\mathcal{J}W\mathcal{J}^{-1} = \mathcal{J}W\mathcal{J}^{-1}\mathcal{F}.
\end{equation}

For $\Psi \in S'(\mathbb{R})$ the multiplication $\mathcal{M}_\Psi$ by $\Psi$ and the convolution $\mathcal{C}_\Psi$ with $\Psi$ are linear continuous mappings from $S(\mathbb{R})$ to $S'(\mathbb{R})$. If $\Psi \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, then $\mathcal{M}_\Psi$ and $\mathcal{C}_\Psi$ act in a usual way: $(\mathcal{M}_\Psi f)(t) = \Psi(t)f(t)$ and
\begin{equation}
(\mathcal{C}_\Psi f)(t) = \int_{\mathbb{R}} f(t-s)\Psi(s)ds, \quad t \in \mathbb{R}, \quad f \in S(\mathbb{R}).
\end{equation}

It is well known that
\begin{equation}
\mathcal{F}\mathcal{M}_\Psi \mathcal{F}^{-1} f = \frac{1}{\sqrt{2\pi}} \mathcal{C}_\Psi f, \quad f \in S(\mathbb{R}).
\end{equation}

If $\eta \in L^\infty(\mathbb{T})$, then $\mathcal{C}_{\mathcal{F}(\eta \circ \omega)}$ has an extension from $S(\mathbb{R})$ onto $L^2(\mathbb{R})$ defined by (4.11), which is a (linear, bounded) operator on $L^2(\mathbb{R})$ (that is, $\mathcal{C}_{\mathcal{F}(\eta \circ \omega)} \in \mathcal{L}(L^2(\mathbb{R}))$).

For $\alpha > 0$ set
\begin{equation}
\theta_\alpha(z) = e^{inz}, \quad z \in \mathbb{C}.
\end{equation}

Then
\begin{equation}
(\mathcal{F}(\theta_\alpha^n f))(t) = (\mathcal{F}f)(t - n\alpha), \quad t \in \mathbb{R}, \quad n \in \mathbb{Z}, \quad f \in L^2(\mathbb{R})
\end{equation}
and
\begin{equation}
\frac{1}{\sqrt{2\pi}} (\mathcal{C}_{\theta_\alpha^n} f)(t) = f(t - \alpha), \quad t \in \mathbb{R}, \quad f \in L^2(\mathbb{R}).
\end{equation}

Set $\mathcal{K}_\alpha = H^2(\mathbb{C}_+) \ominus \theta_\alpha H^2(\mathbb{C}_+)$. By the Paley–Wiener theorem,
\begin{equation}
H^2(\mathbb{C}_+) = \mathcal{F}^{-1}L^2(0, +\infty)
\end{equation}
and
\begin{equation}
\theta_\alpha^n \mathcal{K}_\alpha = \theta_\alpha^n \mathcal{F}^{-1}L^2(0, \alpha) = \mathcal{F}^{-1}L^2(n\alpha, (n+1)\alpha) \text{ for every } n \in \mathbb{Z}.
\end{equation}

For references see, for example, [Ka, Ch. VI] or [R, Ch. 7].

For $\infty \leq b_1 < b_2 \leq +\infty$ and $w: (b_1, b_2) \to (0, +\infty)$ set
\begin{equation}
L^2((b_1, b_2), w) = \{ f: (b_1, b_2) \to \mathbb{C} : \int_{b_1}^{b_2} |f(t)|^2 w(t)dt < \infty \}.
\end{equation}

**Proposition 4.1.** Let $\nu$ be as in Lemma 3.3, and let $\mu$ be defined by $\nu$ as in (4.5). For $\alpha > 0$ set
\begin{equation}
\frac{1}{\omega_\alpha^n(-n-1)} = \sum_{k=1}^{\infty} a_k e^{-2\alpha \nu w_k}, \quad n \geq 0,
\end{equation}
\[ \phi_{\alpha,n}(t) = \sum_{k=1}^{\infty} a_k e^{-2\alpha v_k t} e^{-2v_k t}, \quad t \in (0, \alpha), \quad \text{and} \]
\[ \phi_\alpha(t) = \phi_{\alpha,n}(t - n\alpha), \quad t \in (n\alpha, (n + 1)\alpha), \quad n \geq 0. \]

Then

(4.18) \[ \mathcal{F} \text{ is a unitary transformation} \]

from \( \text{clos}_{L^2(\mu)} H^2(\mathbb{C}_+) \) onto \( L^2((0, +\infty), \phi_\alpha) \),

(4.19) \[ J^*_{\mu} \text{clos}_{L^2(\mu)} H^2(\mathbb{C}_+) = \{ \oplus_{n=0}^{\infty} \theta_n^\alpha \mathcal{F}^{-1} f_n : f_n \in L^2(0, \alpha), \]
\[ \sum_{n=0}^{\infty} \|f_n\|_{L^2(0, \alpha)}^2 \omega^2_\alpha (-n - 1) < \infty \}, \]

(4.20) \[ (\mathcal{F} JW J^*_{\mu} \mathcal{F}^{-1} f)(t) = -\phi_\alpha(-t) f(-t), \]
\[ t \in (-\infty, 0), \quad f \in L^2((0, +\infty), \phi_\alpha). \]

Proof. Set \( \mathcal{L}_{\alpha,n} = \theta_n^\alpha \mathcal{F}^{-1} L^2(0, \alpha), n \geq 0 \). By (4.16),

\[ H^2(\mathbb{C}_+) = \oplus_{n=0}^{\infty} \mathcal{L}_{\alpha,n} \quad \text{and} \quad \|\theta_n^\alpha \mathcal{F}^{-1} f\|_{H^2(\mathbb{C}_+)} = \|f\|_{L^2(0, \alpha)}, \quad f \in L^2(0, \alpha). \]

For \( v > 0 \) define \( A_{\alpha,v} \in \mathcal{L}(L^2(0, \alpha)) \) by the formula \( (A_{\alpha,v} f)(t) = e^{-vt} f(t), \)
\( f \in L^2(0, \alpha), \quad t \in (0, \alpha) \). Then \( (\mathcal{F}^{-1} f)(t + iv) = (\mathcal{F}^{-1} A_{\alpha,v} f)(t), t \in (0, \alpha). \)

Let \( n, m \geq 0 \), and let \( f, g \in L^2(0, \alpha) \). We have

\[ (\theta_n^\alpha \mathcal{F}^{-1} f, \theta_m^\alpha \mathcal{F}^{-1} g)_{L^2(\mu)} \]
\[ = \sum_{k=1}^{\infty} a_k \int_{\mathbb{R}} (\theta_n^\alpha \mathcal{F}^{-1} f)(t + iv_k) \overline{(\theta_m^\alpha \mathcal{F}^{-1} g)(t + iv_k)} dt \\
= \sum_{k=1}^{\infty} a_k \int_{\mathbb{R}} e^{i\alpha(t + iv_k)u} e^{-\alpha u} e^{-iv_k m}(\mathcal{F}^{-1} A_{\alpha,v_k} f)(t) \overline{(\mathcal{F}^{-1} A_{\alpha,v_k} g)(t)} dt \\
= \sum_{k=1}^{\infty} a_k e^{-\alpha v_k (n + m)} \int_{\mathbb{R}} e^{i(n - m)at} (\mathcal{F}^{-1} A_{\alpha,v_k} f)(t) \overline{(\mathcal{F}^{-1} A_{\alpha,v_k} g)(t)} dt \\
= \sum_{k=1}^{\infty} a_k e^{-\alpha v_k (n + m)} (\theta_n^\alpha \mathcal{F}^{-1} A_{\alpha,v_k} f, \theta_m^\alpha \mathcal{F}^{-1} A_{\alpha,v_k} g)_{L^2(\mathbb{R})}. \]

If \( n \neq m \), then

\[ (\theta_n^\alpha \mathcal{F}^{-1} A_{\alpha,v_k} f, \theta_m^\alpha \mathcal{F}^{-1} A_{\alpha,v_k} g)_{L^2(\mathbb{R})} = 0, \]
because $\mathcal{F}^{-1}L^2(0, \alpha) = \mathcal{K}_n$ by (4.16). If $n = m$, then
\[
(\theta^n \mathcal{F}^{-1}f, \theta^n \mathcal{F}^{-1}g)_{L^2(\mu)} = \sum_{k=1}^{\infty} a_k e^{-2\alpha \nu_k} (\theta^n \mathcal{F}^{-1} A_{\alpha, \nu_k} f, \theta^n \mathcal{F}^{-1} A_{\alpha, \nu_k} g)_{L^2(\mathbb{R})}
\]
(4.21)
\[
= \sum_{k=1}^{\infty} a_k e^{-2\alpha \nu_k} (A_{\alpha, \nu_k} f, A_{\alpha, \nu_k} g)_{L^2(0, \alpha)}
\]
\[
= \int_0^\alpha \sum_{k=1}^{\infty} a_k e^{-2\alpha \nu_k} e^{-2\nu_k t} f(t) g(t) dt
\]
\[
= \int_0^\alpha f(t) g(t) \phi_{\alpha, n}(t) dt.
\]
Clearly,
\[
e^{-2\alpha t} \leq \frac{1}{\omega^2_{\alpha}(-n-1)} \leq \phi_{\alpha, n}(t) \leq \frac{1}{\omega^2_{\alpha}(-n-2)}, \quad t \in (0, \alpha), \quad n \geq 0.
\]
It is proved that $L_{\alpha, n}$ is orthogonal to $L_{\alpha, m}$ for $n, m \geq 0$ and $n \neq m$, and $L_{\alpha, n}$ is closed in $L^2(\mu)$ for every $n \geq 0$. Consequently,
\[
clos_{L^2(\mu)} H^2(\mathbb{C}_+) = \{ \oplus_{n=0}^{\infty} h_n : h_n \in L_{\alpha, n}, \quad \sum_{n=0}^{\infty} \|h_n\|^2_{L^2(\mu)} < \infty \}.
\]
Let $\{f_n\}_{n=0}^{\infty} \subset L^2(0, \alpha)$. Set $h_n = \theta^n \mathcal{F}^{-1} f_n$, $n \geq 0$, and
\[
f(t) = f_n(t - n\alpha), \quad t \in (n\alpha, (n+1)\alpha), \quad n \geq 0.
\]
By (4.13), $\mathcal{F}(\oplus_{n=0}^{\infty} h_n) = f$. The relation (4.18) follows from the latest equality, (4.21), (4.23) and the definition of $\phi_{\alpha}$.

Let $J_{\mu, \alpha, n}$ be the natural imbedding of $L_{\alpha, n}$ as a subspace of $H^2(\mathbb{C}_+)$ into $L_{\alpha, n}$ as a subspace of $L^2(\mu)$. Since the spaces $L_{\alpha, n}$, $n \geq 0$, are orthogonal and dense in both spaces $H^2(\mathbb{C}_+)$ and $\clos_{L^2(\mu)} H^2(\mathbb{C}_+)$, we conclude that $J_{\mu} = \oplus_{n=0}^{\infty} J_{\mu, \alpha, n}$. Consequently,
\[
J_{\mu}^* = \oplus_{n=0}^{\infty} J_{\mu, \alpha, n}^*.
\]
Let $f, g \in L^2(0, \alpha)$. By (4.21),
\[
(J_{\mu, \alpha, n}^* \theta^n \mathcal{F}^{-1} f, \theta^n \mathcal{F}^{-1} g)_{H^2(\mathbb{C}_+)} = (\theta^n \mathcal{F}^{-1} f, \theta^n \mathcal{F}^{-1} g)_{L^2(\mu)}
\]
\[
= \int_0^\alpha f(t) g(t) \phi_{\alpha, n}(t) dt = (\phi_{\alpha, n} f, g)_{L^2(0, \alpha)}
\]
\[
= (\theta^n \mathcal{F}^{-1} \phi_{\alpha, n} f, \theta^n \mathcal{F}^{-1} g)_{H^2(\mathbb{C}_+)}.
\]
Thus,
\[
J_{\mu, \alpha, n}^* \theta^n \mathcal{F}^{-1} f = \theta^n \mathcal{F}^{-1} \phi_{\alpha, n} f; \quad f \in L^2(0, \alpha).
\]
By (4.24) and (4.23),
\[
J_{\mu}^* \clos_{L^2(\mu)} H^2(\mathbb{C}_+) = \{ \oplus_{n=0}^{\infty} \theta^n \mathcal{F}^{-1} \phi_{\alpha, n} f_n : f_n \in L^2(0, \alpha), \quad \sum_{n=0}^{\infty} \|\theta^n \mathcal{F}^{-1} f_n\|^2_{L^2(\mu)} < \infty \}.
\]
EXAMPLE OF QUASIANALYTIC CONTRACTION

Let $g_n = \phi_{a,n} f_n$, then

$$J^*_\mu \text{clos}_{L^2(\mu)} H^2(\mathbb{C}_+) = \{ \sum_{n=0}^{\infty} \theta^n_{\alpha} F^{-1} g_n : g_n \in L^2(0, \alpha), \sum_{n=0}^{\infty} \left\| \theta^n_{\alpha} F^{-1} \frac{g_n}{\phi_{\alpha,n}} \right\|^2_{L^2(\mu)} < \infty \}. \tag{4.22}$$

By (4.21),

$$\left\| \theta^n_{\alpha} F^{-1} \frac{g_n}{\phi_{\alpha,n}} \right\|^2_{L^2(\mu)} = \int_0^\alpha \frac{|g_n(t)|^2}{\phi_{\alpha,n}(t)^2} dt = \int_0^\alpha \frac{|g_n(t)|^2}{\phi_{\alpha,n}(t)} dt.$$

It follows from the latest equality and (4.22) that

$$\sum_{n=0}^{\infty} \left\| \theta^n_{\alpha} F^{-1} \frac{g_n}{\phi_{\alpha,n}} \right\|^2_{L^2(\mu)} < \infty \text{ if and only if } \sum_{n=0}^{\infty} \|g_n\|^2_{L^2(0, \alpha)} \omega^2_n(-n - 1) < \infty.$$

The equality (4.19) is proved.

Let $f \in L^2((0, +\infty), \phi_\alpha)$. Then $f = \oplus_{n=0}^{\infty} f|_{(na,(n+1)a)}$. Set

$$f_n(t) = f(t + na), \quad t \in (0, \alpha), \quad n \geq 0.$$

By (4.13) and (4.24),

$$J^*_\mu F^{-1} f|_{(na,(n+1)a)} = J^*_\mu \theta^n_{\alpha} F^{-1} f_n = \theta^n_{\alpha} F^{-1}(\phi_{\alpha,n} f_n) = F^{-1}(\phi_{\alpha} f)|_{(na,(n+1)a)}.$$

The equality (4.20) follows from (4.9) and (4.4). \hfill \Box

**Remark 4.2.** The idea of Proposition 4.1 is from [FR1], [FR2].

**Theorem 4.3.** Let $T$ be defined as in Proposition 3.2 with $\nu$ as in Lemma 3.3. Define $\mu$ as in (4.5). For $\alpha > 0$ set

$$\bar{\phi}_\alpha: \mathbb{R} \to (0, +\infty), \quad \bar{\phi}_\alpha(t) = \frac{1}{\phi_\alpha(-t)}, \quad t \in (-\infty, 0), \quad \bar{\phi}_\alpha(t) = 1, \quad t \in (0, +\infty),$$

where $\phi_\alpha$ is defined as in Proposition 4.1.

Let $\eta \in L^\infty$. Then $\eta \in \hat{\gamma}_T(T')$ (where $\hat{\gamma}_T$ is defined in (1.4)) if and only if $\mathcal{C}_{\mathcal{F}(\eta, \phi_\alpha)} \in \mathcal{L}(L^2(\mathbb{R}, \bar{\phi}_\alpha))$, and then $\hat{\gamma}_T^{-1}(\eta)$ is unitarily equivalent to $\frac{1}{\sqrt{2\pi}} \mathcal{C}_{\mathcal{F}(\eta, \omega_\alpha)} \in \mathcal{L}(L^2(\mathbb{R}, \bar{\phi}_\alpha))$.

**Proof.** By Proposition 3.1, $\eta \in \hat{\gamma}_T(T')$ if and only if the mapping from (3.2) is defined and bounded. By (4.6), (4.7), and (4.18),

$$\mathcal{F} \mathcal{J} : H^2(\mathbb{D}) \oplus P^2(\nu) \to L^2(0, +\infty) \oplus L^2((0, +\infty), \phi_\alpha) \text{ is unitary.}$$

Set

$$Y_\alpha = (\mathcal{F} \mathcal{J})_{H^2(\mathbb{D}) \to L^2(-\infty, 0)} X_0(\mathcal{F} \mathcal{J})^{-1}_{L^2((0, +\infty), \phi_\alpha) \to P^2(\nu)},$$

where lower index of $\mathcal{F} \mathcal{J}$ and $(\mathcal{F} \mathcal{J})^{-1}$ show spaces between they act. Taking into account the definition of $X_0$, the equality

$$J^*_\nu = J^{-1} J^*_\mu J,$$

applying equalities (4.20) and (4.7), we obtain that $Y_\alpha$ acts by the formula

$$Y_\alpha : L^2((0, +\infty), \phi_\alpha) \to L^2(-\infty, 0), \quad \{Y_\alpha f\}(t) = -\phi_\alpha(t) f(-t), \quad t \in (-\infty, 0), \quad f \in L^2((0, +\infty), \phi_\alpha).$$
By (4.11), the mapping from (3.2) is defined and bounded if and only if the mapping

\[(I_{L^2(0,\infty)} \oplus Y_\alpha^{-1}) \frac{1}{\sqrt{2\pi}} C_{F(p\omega)}(I_{L^2(0,\infty)} \oplus Y_\alpha)\]

is defined and bounded.

Define \(V_\alpha : L^2((-\infty, 0), \tilde{\phi}_\alpha) \to L^2((0, +\infty), \phi_\alpha)\) by the formula

\[(V_\alpha f)(t) = -\frac{1}{\phi_\alpha(t)} f(-t), \quad t \in (0, +\infty), \quad f \in L^2((-\infty, 0), \tilde{\phi}_\alpha).\]

Then \(V_\alpha\) is unitary and \(Y_\alpha V_\alpha\) is the natural imbedding of \(L^2((-\infty, 0), \tilde{\phi}_\alpha)\) into \(L^2(0, +\infty)\). Thus, \(I_{L^2(0,\infty)} \oplus Y_\alpha V_\alpha\) is the natural imbedding of \(L^2(\mathbb{R}, \tilde{\phi}_\alpha)\) into \(L^2(\mathbb{R})\). Multiplying the mapping from (4.26) by \(I_{L^2(0,\infty)} \oplus V_\alpha^{-1}\) from the left side and by \(I_{L^2(0,\infty)} \oplus V_\alpha\) from the right side, we obtain the conclusion of the theorem. \(\square\)

5. Properties of constructed weights

Let \(w, \phi : \mathbb{R} \to (0, +\infty)\) be two measurable functions. If \(w \asymp \phi\), then the natural imbedding

\(L^2(\mathbb{R}, w) \to L^2(\mathbb{R}, \phi)\)

is a (bounded) transformation with the bounded inverse. Let \(\alpha > 0\). Let \(\tilde{\phi}_\alpha\) be defined in Theorem 4.3, and let \(\omega_\alpha^2(-n - 1), n \geq 0\), be defined by (4.17). We may assume that \(\sum_{k=1}^{\infty} a_k \leq 1\). Then \(\omega_\alpha^2(-n - 1) \geq 1\) for \(n \geq 0\).

Set

\[(5.1) \quad w_\alpha(t) = \omega_\alpha^2(-n - 1), \quad t \in (-n + 1)\alpha, -n\alpha), \quad n \geq 0,
\]

\[w_\alpha(t) = 1, \quad t \in (0, +\infty).\]

By (4.22),

\[(5.2) \quad w_\alpha \asymp \tilde{\phi}_\alpha.\]

Therefore, we can consider \(L^2(\mathbb{R}, w_\alpha)\) instead of \(L^2(\mathbb{R}, \tilde{\phi}_\alpha)\).

**Lemma 5.1.** Let \(\{a_k\}_{k=1}^{\infty}\) and \(\{v_k\}_{k=1}^{\infty}\) be families of numbers such that \(0 < a_k \leq 1, 0 < v_{k+1} < v_k\) for every \(k\), and \(\sum_{k=1}^{\infty} a_k < \infty\). For \(\alpha > 0\) define \(\omega_\alpha^2(-n - 1), n \geq 0\), by (4.17). Then

\[\omega_\alpha^2(-n - m - 1) \leq \left(1 + 2\sum_{k=1}^{\infty} a_k\right)\omega_\alpha^2(-n - 1)\omega_\alpha^2(-m - 1), \quad m, n \geq 0.\]
Lemma 5.3. Let \( a \) such that \( a > 0 \). First, consider the case where \( w(0) \leq C \omega \). We have
\[
\sum_{k=0}^{\infty} a_k e^{-2\alpha n v_k} \sum_{l=1}^{\infty} a_l e^{-2\alpha m v_l} = \sum_{k=1}^{\infty} a_k e^{-2\alpha n v_k} \left( \sum_{l=1}^{k-1} a_l e^{-2\alpha m v_l} + a_k e^{-2\alpha n v_k} + \sum_{l=k+1}^{\infty} a_l e^{-2\alpha m v_l} \right)
\]
\[
= \sum_{k=2}^{\infty} a_k e^{-2\alpha n v_k} \sum_{l=1}^{k-1} a_l e^{-2\alpha m v_l} + \sum_{k=1}^{\infty} a_k e^{-2\alpha (n+m) v_k} \sum_{l=1}^{k-1} a_l e^{-2\alpha m v_l} + \sum_{k=1}^{\infty} a_k e^{-2\alpha (n+m) v_k} \sum_{l=1}^{\infty} a_l e^{-2\alpha m v_l}
\]
\[
\leq \sum_{k=2}^{\infty} a_k e^{-2\alpha n v_k} e^{-2\alpha m v_k} \sum_{l=1}^{k-1} a_l + \sum_{k=1}^{\infty} a_k e^{-2\alpha (n+m) v_k} \sum_{l=1}^{k-1} a_l + \sum_{k=1}^{\infty} a_k e^{-2\alpha (n+m) v_k} \sum_{l=1}^{\infty} a_l
\]
\[
= \left( 1 + 2 \sum_{k=1}^{\infty} a_k \right) \frac{1}{\omega_\alpha^{2}(-n - m - 1)}.
\]
\[
\square
\]

Corollary 5.2. Let \( \{a_k\}_{k=1}^{\infty} \) and \( \{v_k\}_{k=1}^{\infty} \) be families of numbers such that \( a_k > 0, 0 < v_{k+1} < v_k \) for every \( k \), and \( \sum_{k=1}^{\infty} a_k \leq 1 \). Set
\[
C = 1 + 2 \sum_{k=1}^{\infty} a_k.
\]

For \( \alpha > 0 \) define \( w_\alpha \) by (5.1). Then
\[
w_\alpha(t + s) \leq C^2 \omega_\alpha(-2)^2 w_\alpha(t) w_\alpha(s), \quad t, s \in \mathbb{R}.
\]

Proof. First, consider the case where \( t, s < 0 \). Then there exists \( m, n \geq 0 \) such that \( t \in \left( -(n+1)\alpha, -na \right) \) and \( s \in \left( -(m+1)\alpha, -ma \right) \). Then \( t + s \in \left( -(n+m+2)\alpha, -(n+m)\alpha \right) \). Therefore,
\[
w_\alpha(t + s) \leq \omega_\alpha^2(-n - m - 2) \leq C \omega_\alpha^2(-n - 1) \omega_\alpha^2(-m - 2) \leq C \omega_\alpha^2(-n - 1) C \omega_\alpha^2(-m - 1) \omega_\alpha^2(-2) = C^2 \omega_\alpha^2(-2) w_\alpha(t) w_\alpha(s).
\]

In remaining cases the conclusion of the lemma follows from the fact that \( w_\alpha \) is nonincreasing and \( w_\alpha \equiv 1 \) on \( (0, +\infty) \). \( \square \)

Lemma 5.3. Let \( \{a_k\}_{k=1}^{\infty} \) and \( \{v_k\}_{k=1}^{\infty} \) be families of numbers such that \( a_k > 0, 0 < v_{k+1} < v_k \) for every \( k \), \( v_k \to 0 \) and \( \sum_{k=1}^{\infty} a_k < \infty \). For \( \alpha > 0 \) define \( \omega_\alpha^2(-n - 1), n \geq 0 \), by (4.17). Then for every \( \varepsilon > 0 \) there exists a finite constant \( C_\varepsilon \) (which also depends on \( \alpha \)) such that \( \omega_\alpha^2(-n - 1) \leq C_\varepsilon e^{\varepsilon n} \) for all \( n \geq 0 \).
Then there exists $\|C\| = \frac{1}{\sum_{k=k_k} a_k}$. □

Recall that $C_\varphi$ denote the convolution with a function $\varphi$, see (4.10).

**Lemma 5.4.** Let $C > 0$, and let $w: \mathbb{R} \to (0, +\infty)$ and $\psi: \mathbb{R} \to \mathbb{C}$ be measurable functions such that $w(t + s) \leq Cw(t)w(s)$ for all $s \in \mathbb{R}$ and $\psi \sqrt{w} \in L^1(\mathbb{R})$. Then $C_\varphi \in L(L^2(\mathbb{R}, w))$ and $\|C_\varphi\| \leq \sqrt{C}\|\psi \sqrt{w}\|_{L^1(\mathbb{R})}$.

**Proof.** Define $B \in L(L^2(\mathbb{R}, w), L^2(\mathbb{R}))$ by the formula $Bf = \sqrt{w}f$, $f \in L^2(\mathbb{R}, w)$. Then $B$ is unitary, and

$$(BC_\varphi B^{-1}f)(t) = \int_{\mathbb{R}} \psi(t-s)f(s)\frac{\sqrt{w(t)}}{\sqrt{w(s)}}ds, \ t \in \mathbb{R}, \ f \in L^2(\mathbb{R}).$$

Therefore,

$$|(BC_\varphi B^{-1}f)(t)| \leq \int_{\mathbb{R}} |\psi(t-s)f(s)|\frac{\sqrt{w(t)}}{\sqrt{w(s)}}ds$$

$$\leq \int_{\mathbb{R}} |\psi(t-s)f(s)|\frac{\sqrt{Cw(t-s)w(s)}}{\sqrt{w(s)}}ds$$

$$= \sqrt{C} \int_{\mathbb{R}} |\psi(t-s)|\sqrt{w(t-s)}|f(s)|ds$$

$$= \sqrt{C}(C_{\psi \sqrt{w}}f)(t), \ t \in \mathbb{R}, \ f \in L^2(\mathbb{R}).$$

It is well known (and can be deduced from (4.11)) that if $\varphi \in L^1(\mathbb{R})$, then $C_\varphi \in L(L^2(\mathbb{R}))$ and $\|C_\varphi\| \leq \|\varphi\|_{L^1(\mathbb{R})}$. Setting $\varphi = |\psi|\sqrt{w}$, we obtain

$$\|BC_\varphi B^{-1}f\|_{L^2(\mathbb{R})} \leq \sqrt{C}\|\psi \sqrt{w}\|_{L^1(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}.$$ 

Clearly, $\|\psi \sqrt{w}\|_{L^1(\mathbb{R})} = \|\psi \sqrt{w}\|_{L^1(\mathbb{R})}$ and $\|f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$. The conclusion of the lemma follows from the unitarity of $B$. □

Recall that $\varpi$ is defined by (4.1) and $F$ is the Fourier transform (see (4.8)).

**Lemma 5.5.** Let $C > 0$, and let $w: \mathbb{R} \to [1, +\infty)$ be a nonincreasing function such that $w(t + s) \leq Cw(t)w(s)$ for all $s \in \mathbb{R}$ and

(5.3) \[ \int_{\mathbb{R}} \frac{\log w(t)}{1 + t^2} < \infty. \]

Then there exists $\eta \in L^\infty(\mathbb{T})$ such that $\eta(e^{it}) = 0$ for $t \in (\pi, 2\pi)$, $\eta \not\equiv 0$, and $C_{F(\eta \varpi w)} \in L(L^2(\mathbb{R}, w))$.

**Proof.** There exists a function $h: \mathbb{R} \to (0, +\infty)$ such that $h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $h\sqrt{w} \in L^1(\mathbb{R})$ and

(5.4) \[ \int_{\mathbb{R}} \frac{\log h(t)}{1 + t^2} > -\infty. \]
Indeed, take \( c > 0, \varepsilon > 0 \) and set

\[
h(t) = 1, \quad t \in (-c, c) \quad \text{and} \quad h(t) = \frac{1}{|t|^{1+\varepsilon}\sqrt{w(t)}}, \quad t \in (\infty, -c) \cup (c, +\infty).
\]

Since \( w \) is nonincreasing, \( w \) is bounded on \((-c, c)\), therefore, \( h\sqrt{w} \in L^1(\mathbb{R}) \). Since \( w \geq 1 \) on \( \mathbb{R} \), \( h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). Furthermore,

\[
\int_{\mathbb{R}} \frac{\log h(t)}{1 + t^2} = \left( \int_{-\infty}^{-c} + \int_{c}^{+\infty} \right) \left( -\frac{(1 + \varepsilon) \log |t| + \frac{1}{2} \log w(t)}{1 + t^2} \right) > -\infty
\]

by (5.3).

By (5.4), there exists \( \psi \in H^1(\mathbb{C}_+) \cap H^2(\mathbb{C}_+) \) such that \( |\psi| = h \) a.e. on \( \mathbb{R} \) (see, for example, [Gar, Theorem II.4.4]). By Lemma 5.4, \( C_\psi \in \mathcal{L}(L^2(\mathbb{R}, w)) \).

Set \( \eta = (\mathcal{F}^{-1} \psi) \circ \omega^{-1} \). Since \( \psi \in H^2(\mathbb{C}_+) \) and \( (\mathcal{F}^{-1} \psi)(t) = (\mathcal{F} \psi)(t) \) for \( t \in \mathbb{R} \), we have \( (\mathcal{F}^{-1} \psi) \in L^2(-\infty, 0) \) by (4.15). It remains to note that \( \omega((0, +\infty)) = \{ e^{it} : t \in (\pi, 2\pi) \} \). \qed

The following lemma will be applied in Sec. 9.

**Lemma 5.6.** Suppose that \( \delta > 0 \), and \( B \in \mathcal{L}(L^2(\mathbb{R})) \) is such that for every \( -\infty < b_1 < b_2 < +\infty \)

\[
BL^2(b_1, b_2) \subset L^2(b_1 - \delta, b_2 + \delta).
\]

Suppose that a sequence \( \{\omega(n)\}_{n \in \mathbb{Z}} \) of positive numbers is such that

\[
\frac{\omega(n)}{\omega(n + 1)} \asymp 1, \quad n \in \mathbb{Z}.
\]

For \( \alpha > \delta \) set \( w_\alpha(t) = \omega^2(n), \quad t \in (n\alpha, (n + 1)\alpha), \quad n \in \mathbb{Z} \). Then \( B \in \mathcal{L}(L^2(\mathbb{R}, w_\alpha)) \), and

\[
\|B\|_{\mathcal{L}(L^2(\mathbb{R}, w_\alpha))}^2 \leq 3 \|B\|_{\mathcal{L}(L^2(\mathbb{R}))}^2 \left( 1 + \sup_{n \in \mathbb{Z}} \frac{\omega^2(n)}{\omega^2(n + 1)} + \sup_{n \in \mathbb{Z}} \frac{\omega^2(n)}{\omega^2(n - 1)} \right).
\]

**Proof.** Let \( f \in L^2(\mathbb{R}, w_\alpha) \). Set \( f_n = f|_{(n\alpha, (n + 1)\alpha)}, \quad n \in \mathbb{Z} \). By assumption, \( Bf_n \) is well-defined, and \( Bf_n \in L^2((n - 1)\alpha, (n + 2)\alpha) \). We have

\[
Bf = \sum_{n \in \mathbb{Z}} (Bf_n|_{((n-1)\alpha,n\alpha)} + Bf_n|_{(n\alpha,(n+1)\alpha)} + Bf_n|_{((n+1)\alpha,(n+2)\alpha)})
= \oplus_{n \in \mathbb{Z}} (Bf_{n-1} + Bf_n + Bf_{n+1})|_{(n\alpha,(n+1)\alpha)}.
\]
Furthermore,
\[
\|(Bf_{n-1} + Bf_n + Bf_{n+1})\|_{(n\alpha,(n+1)\alpha)}^2 \leq 3\|Bf_{n-1}\|_{(n\alpha,(n+1)\alpha)}^2 + \|Bf_n\|_{(n\alpha,(n+1)\alpha)}^2 + \|Bf_{n+1}\|_{(n\alpha,(n+1)\alpha)}^2 + \|Bf_{n+1}\|_{(n\alpha,(n+1)\alpha)}^2
\]
\[
\leq 3\omega^2(n)\|f_{n-1}\|_{L^2(\mathbb{R})}^2 + \|f_n\|_{L^2(\mathbb{R})}^2 + \|f_{n+1}\|_{L^2(\mathbb{R})}^2
\]
\[
= 3\omega^2(n)\|B\|_{L^2(\mathbb{R})}^2 \left( \frac{1}{\omega^2(n-1)}\|f_{n-1}\|_{L^2(\mathbb{R},\omega_\alpha)}^2 + \frac{1}{\omega^2(n)}\|f_n\|_{L^2(\mathbb{R},\omega_\alpha)}^2 + \frac{1}{\omega^2(n+1)}\|f_{n+1}\|_{L^2(\mathbb{R},\omega_\alpha)}^2 \right)
\]
Therefore,
\[
\|Bf\|_{L^2(\mathbb{R},\omega_\alpha)}^2 = \sum_{n \in \mathbb{Z}} \|(Bf_{n-1} + Bf_n + Bf_{n+1})\|_{(n\alpha,(n+1)\alpha)}^2 \leq 3\|B\|_{L^2(\mathbb{R})}^2 \sum_{n \in \mathbb{Z}} \left( \frac{\omega^2(n)}{\omega^2(n-1)}\|f_{n-1}\|_{L^2(\mathbb{R},\omega_\alpha)}^2 + \frac{\omega^2(n)}{\omega^2(n+1)}\|f_{n+1}\|_{L^2(\mathbb{R},\omega_\alpha)}^2 \right)
\]
\[
= 3\|B\|_{L^2(\mathbb{R})}^2 \sum_{n \in \mathbb{Z}} \left( \frac{\omega^2(n+1)}{\omega^2(n)} + \frac{\omega^2(n-1)}{\omega^2(n)} \right)\|f_n\|_{L^2(\mathbb{R},\omega_\alpha)}^2
\]
\[
\leq 3\|B\|_{L^2(\mathbb{R})}^2 \left( 1 + \sup_{n \in \mathbb{Z}} \frac{\omega^2(n+1)}{\omega^2(n)} + \sup_{n \in \mathbb{Z}} \frac{\omega^2(n-1)}{\omega^2(n)} \right)\|f\|_{L^2(\mathbb{R},\omega_\alpha)}^2
\]
\[
= 3\|B\|_{L^2(\mathbb{R})}^2 \left( 1 + \sup_{n \in \mathbb{Z}} \frac{\omega^2(n+1)}{\omega^2(n)} + \sup_{n \in \mathbb{Z}} \frac{\omega^2(n-1)}{\omega^2(n)} \right)\|f\|_{L^2(\mathbb{R},\omega_\alpha)}^2
\]

6. **Quasianalyticity**

We will apply Beurling’s quasianalyticity theorem, see [Ko, Ch. VII.B.5].

**Theorem 6.1** (Beurling). Let \(-\infty < b_1 < b_2 < +\infty\). For \(c > 0\) set
\[
\mathcal{G}(c) = \{t + iy : b_1 < t < b_2, 0 < y < c\}.
\]
For a function \(f\) analytic in \(\mathcal{G}(c)\) set
\[
\mathcal{Q}_f(c) = \sup_{0 < y < c} \left( \int_{b_1}^{b_2} |f(t + iy)|^2 \, dt \right)^{1/2}.
\]
For \( \varphi \in L^2(b_1, b_2) \) and \( u \in [1, +\infty) \) define \( M(u) \) by the relation
\[
e^{-M(u)} = \inf \left\{ \left( \int_{b_1}^{b_2} |\varphi(t) - f(t)|^2 dt \right)^{1/2} : f \text{ is analytic in } G(c) \right\}
\]
and \( \varsigma_{G(c)}(f) \leq e^u \).

If
\[
\int_1^{+\infty} \frac{M(u)}{u^2} du = \infty
\]
and \( |\{ t \in \mathbb{R} : b_1 < t < b_2, \varphi(t) = 0 \}| > 0 \) (where \( | \cdot | \) is the linear measure of a subset of \( \mathbb{R} \)), then \( \varphi \equiv 0 \).

To apply Theorem 6.1 we need the following simple lemma.

**Lemma 6.2.** Let \( M : (0, +\infty) \to (0, +\infty) \) be a nondecreasing function. Let \( \alpha > 0 \). Then
\[
\sum_{n=1}^{\infty} \frac{M(n\alpha)}{n^2} = \infty \quad \text{if and only if} \quad \int_1^{+\infty} \frac{M(u)}{u^2} du = \infty.
\]

**Proof.** We have
\[
\int_1^{+\infty} \frac{M(u\alpha)}{u^2} du = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{M(u\alpha)}{u^2} du
\]
and
\[
\frac{M(n\alpha)}{(n+1)^2} \leq \frac{M(u\alpha)}{u^2} \leq \frac{M((n+1)\alpha)}{n^2}
\]
for \( u \in [n, n+1] \).

Since \( \frac{n}{n+1} \to 1 \) when \( n \to \infty \), we conclude that
\[
\sum_{n=1}^{\infty} \frac{M(n\alpha)}{n^2} = \infty \quad \text{if and only if} \quad \int_1^{+\infty} \frac{M(u\alpha)}{u^2} du = \infty.
\]

The lemma follows from the equality
\[
\int_1^{+\infty} \frac{M(u\alpha)}{u^2} du = \alpha \int_1^{+\infty} \frac{M(u)}{u^2} du.
\]

**Theorem 6.3.** Let \( T \) be defined as in Proposition 3.2 with \( \nu \) as in Lemma 3.3. For \( \alpha > 0 \) define \( \omega_\alpha^2(-n-1), n \geq 0, \) by (4.17). Then \( T \) is quasianalytic if and only if
\[
\sum_{n=0}^{\infty} \frac{\log \omega_\alpha(-n-1)}{(n+1)^2} = \infty. \tag{6.1}
\]

**Proof.** “If” part. By Proposition 3.1, \( T \) is quasianalytic if and only if (3.1) is fulfilled. By the construction of \( T, H_0 = P^2(\nu) \) and \( X_0 = W J^\nu \), where \( W \) is defined by (3.3). Recall that \( J \) is defined by (4.3) and \( J_\mu \) is defined after (4.7). Applying \( J \) and taking into account (4.4), (4.7), and (4.25), the relation (3.1) can be rewritten as follows.

Let \( h \in H^2(\mathbb{C}_+), \) let \( g \in J^\nu_\mu \) clos \( L^2(\mu) H^2(\mathbb{C}_+), \)
and let \( |\{ t \in \mathbb{R} : h(t) = g(-t) \}| > 0. \) Then \( g \equiv 0. \)
We have $g \in H^2(\mathbb{C}_+)$. Set $g_\alpha(z) = g(-z)$, $z \in \mathbb{C}_-$. Then $g_\alpha \in L^2(\mathbb{R}) \oplus H^2(\mathbb{C}_+)$. Therefore, it sufficient to prove that $h(t) - g(-t) = 0$ for a.e. $t \in \mathbb{R}$.

By (4.19),

$$g = \oplus_{n=0}^{\infty} \theta^n \mathcal{F}^{-1} g_n, \text{ where } g_n \in L^2(0, \alpha) \quad \text{and}$$

$$C_{1g} := \sum_{n=0}^{\infty} \| g_n \|^2_{L^2(0, \alpha)} \omega^2(\alpha)(-n - 1) < \infty.$$ 

Since $\omega^2(\alpha)(-n - 1) \rightarrow \infty$ when $n \rightarrow \infty$,

$$C_{2g} := \sum_{n=0}^{\infty} \| g_n \|^2_{L^2(0, \alpha)} < \infty.$$ 

Let $-\infty < b_1 < b_2 < +\infty$ be such that

$$|\{t \in (b_1, b_2) : h(t) = g(-t)\}| > 0. \quad (6.2)$$

For $c > 0$ define $\mathcal{G}(c)$ as in Theorem 6.1.

Set $\varphi(t) = h(t) - g(-t)$, $t \in (b_1, b_2)$,

$$f_{1n}(z) = (\theta^n \mathcal{F}^{-1} g_n)(-z) \quad \text{and}$$

$$f_n(z) = h(z) - \oplus_{k=0}^{n} f_{1k}(z), \quad z \in \mathbb{C}_+ \cup \mathbb{R}, \quad n \geq 0.$$ 

Clearly, $f_{1n}$ and $f_n$ are analytic in $\mathbb{C}_+$,

$$f_{1n}(z) = \frac{1}{\sqrt{2\pi}} e^{-i\pi z} \int_{0}^{\alpha} e^{-izs} g_n(s)ds, \quad z \in \mathbb{C}_+,$$ 

and

$$2\pi(\mathcal{S}(c)(f_{1n}))^2 = \sup_{0 < y < c} \int_{b_1}^{b_2} |e^{-i(t + iy)n\alpha}|^2 \int_{0}^{\alpha} e^{-i(t + iy)s} g_n(s)ds^2 dt$$

$$\leq \sup_{0 < y < c} \int_{b_1}^{b_2} e^{2y\alpha} \left( \int_{0}^{\alpha} |e^{-i(t + iy)s} g_n(s)|ds \right)^2 dt$$

$$\leq e^{2\alpha n\alpha} \sup_{0 < y < c} \int_{b_1}^{b_2} \left( \int_{0}^{\alpha} |g_n(s)|ds \right)^2 dt$$

$$\leq e^{2\alpha (n+1)\alpha} \sup_{0 < y < c} \int_{b_1}^{b_2} \left( \int_{0}^{\alpha} |g_n(s)|ds \right)^2 dt$$

$$\leq e^{2\alpha (n+1)\alpha} \int_{b_1}^{b_2} \alpha \left( \int_{0}^{\alpha} |g_n(s)|^2 ds \right) dt$$

$$= e^{2\alpha (n+1)\alpha} (b_2 - b_1) \alpha \| g_n \|^2_{L^2(0, \alpha)}.$$
Therefore,
\[
\varsigma \mathcal{G}(c)(f_n) \leq \varsigma \mathcal{G}(c)(h) + \sum_{k=0}^{n} \varsigma \mathcal{G}(c)(f_{1k})
\]
\[
\leq \varsigma \mathcal{G}(c)(h) + (b_2 - b_1)^{1/2} \frac{\sqrt{\alpha}}{2\pi} \sum_{k=0}^{n} e^{c(k+1)\alpha} \|g_k\|_{L^2(0,\alpha)}
\]
\[
\leq \varsigma \mathcal{G}(c)(h) + (b_2 - b_1)^{1/2} \frac{\sqrt{\alpha}}{2\pi} \left( \sum_{k=0}^{n} e^{2c(k+1)\alpha} \right)^{1/2} \left( \sum_{k=0}^{n} \|g_k\|_{L^2(0,\alpha)}^2 \right)^{1/2}
\]
\[
\leq \varsigma \mathcal{G}(c)(h) + (b_2 - b_1)^{1/2} \frac{\sqrt{\alpha}}{2\pi} (n + 1)^{1/2} e^{c(n+1)\alpha} C_{2g}^{1/2}.
\]
Set \( C_1 = (b_2 - b_1)^{1/2} \frac{\sqrt{\alpha}}{2\pi} \). Since \( h \in H^2(C_+) \), we have \( \varsigma \mathcal{G}(c)(h) \leq \|h\|_{H^2(C_+)} \).

Thus,
\[
\varsigma \mathcal{G}(c)(f_n) \leq \|h\|_{H^2(C_+)} + C_1 C_{2g}^{1/2} (n + 1)^{1/2} e^{c(n+1)\alpha}, \quad n \geq 0.
\]
Take \( 0 < c < 1/\alpha \). Then there exists \( C_2 \) (which depends on \( c \)) such that
\[
\|h\|_{H^2(C_+)} + C_1 C_{2g}^{1/2} (n + 1)^{1/2} e^{c(n+1)\alpha} \leq C_2 e^a
\]
for all \( n \in \mathbb{N} \).

We obtain that
\[
\varsigma \mathcal{G}(c)\left( \frac{1}{C_2} f_n \right) \leq e^n \text{ for all } n \in \mathbb{N}.
\]

We have
\[
\left( \int_{b_1}^{b_2} |\varphi(t) - f_n(t)|^2 \, dt \right)^{1/2} \leq \|\varphi - f_n\|_{L^2(\mathbb{R})} = \| \oplus_{k=n+1}^{\infty} f_{1k} \|_{L^2(\mathbb{R})}
\]
\[
= \left( \sum_{k=n+1}^{\infty} \|g_k\|_{L^2(0,\alpha)}^2 \right)^{1/2} \leq \frac{1}{\omega_{\alpha}(-n - 2)} \left( \sum_{k=n+1}^{\infty} \omega_{\alpha}^2(-k - 1) \|g_k\|_{L^2(0,\alpha)}^2 \right)^{1/2}
\]
\[
\leq \frac{1}{\omega_{\alpha}(-n - 2)} C_{1g}
\]
(because \( \omega_{\alpha}(-k - 1) \geq \omega_{\alpha}(-n - 2) \) for \( k \geq n + 1 \)).

Define \( M(n) \) as in Theorem 6.1 applying to \( \frac{1}{C_2} \varphi \). Then
\[
e^{-M(n)} \leq \left( \int_{b_1}^{b_2} \left| \frac{1}{C_2} \varphi(t) - \frac{1}{C_2} f_n(t) \right|^2 \, dt \right)^{1/2} \leq \frac{1}{\omega_{\alpha}(-n - 2)} C_{1g}.
\]
Therefore, \( M(n) \geq \log \omega_{\alpha}(-n - 2) - \log C_{1g} \). By assumption (6.1),
\[
\sum_{n=1}^{\infty} \frac{M(n)}{n^2} = \infty.
\]

By Lemma 6.2, \( \frac{1}{C_2} \varphi \) satisfies the conclusion of Theorem 6.1, that is, \( \frac{1}{C_2} \varphi \equiv 0 \).

Therefore, \( h(t) - g(-t) = 0 \) for a.e. \( t \in (b_1, b_2) \).

Since \((b_1, b_2)\) is an arbitrary interval satisfying (6.2), we conclude that \( h(t) - g(-t) = 0 \) for a.e. \( t \in \mathbb{R} \).

"Only if" part. Take \( \alpha > 0 \). Define \( w_{\alpha} \) by (5.1). If the sum in (6.1) is finite, then (5.3) is fulfilled for \( w_{\alpha} \). By Lemma 5.5, there exists \( \eta \in L^\infty(T) \) such that \( \eta(e^{it}) = 0 \) for \( t \in (\pi, 2\pi) \), \( \eta \not\equiv 0 \), and \( \mathcal{C}_F(\eta w_{\alpha}) \in \mathcal{L}(L^2(\mathbb{R}, w_{\alpha})) \). By (5.2), \( \mathcal{C}_F(\eta w_{\alpha}) \in \mathcal{L}(L^2(\mathbb{R}, \phi_{\alpha})) \), too. By Theorem 4.3, \( \eta \in \hat{\gamma}_T(\{T\}') \) (where
\( \tilde{\gamma}_T \) is defined in (1.4)). Since \( \eta \neq 0 \) and \( \eta = 0 \) on the set of positive measure, \( T \) is not quasianalytic by [KS1, Proposition 21].

**Lemma 6.4.** Let \( \{a_k\}_{k=1}^\infty \) and \( \{v_k\}_{k=1}^\infty \) be families of numbers such that \( a_k > 0, 0 < v_{k+1} < v_k \) for every \( k \), and \( \sum_{k=1}^\infty a_k < \infty \). Let \( \{c_n\}_{n=1}^\infty \) be a subsequence such that
\[
\sum_{n=1}^\infty \frac{v_k}{n} = \infty,
\]
and let \( \{c_n\}_{n=1}^\infty \) be defined by the equality
\[
\sum_{k=c_n}^\infty a_k = e^{-nc_n}, \quad n \geq 1.
\]
For \( \alpha > 0 \) define \( \omega^2_\alpha \) \((-n-1), n \geq 0\), by (4.17). If \( 2\alpha v_k \leq c_n \) for sufficiently large \( n \), then (6.1) is fulfilled.

**Proof.** Set \( a = \sum_{k=1}^\infty a_k \). We have
\[
\frac{1}{\omega^2_\alpha (n+1)^2} = \sum_{k=1}^{\infty} a_k e^{-2\alpha v_k} = \sum_{k=1}^{k_n} a_k e^{-2\alpha v_k} + \sum_{k=k_n+1}^{\infty} a_k e^{-2\alpha v_k} \leq \sum_{k=1}^{k_n} a_k e^{-2\alpha v_k} + \sum_{k=k_n+1}^{\infty} a_k \leq a e^{-2\alpha v_k} + e^{-nc_n} = \frac{1}{a} e^{-n(c_n - 2\alpha v_k)}.
\]
Therefore,
\[
\log \frac{1}{\omega^2_\alpha (n+1)^2} \leq \log a - 2\alpha v_k n + \frac{1}{a} e^{-n(c_n - 2\alpha v_k)}.
\]
Consequently,
\[
2 \sum_{n=0}^\infty \frac{\log \omega (n+1)^2}{(n+1)^2} \geq -2 \sum_{n=0}^\infty \frac{\log a + 2\alpha \sum_{n=0}^\infty \frac{n v_k}{(n+1)^2} - \frac{1}{a} \sum_{n=0}^\infty \frac{e^{-n(c_n - 2\alpha v_k)}}{(n+1)^2}}. \quad \square
\]

**Example 6.5.** Let \( 0 < a < 1 \). Set \( a_n = (1-a)^{n-1} \) and \( v_n = \frac{1}{\log(n+1)}, n \geq 1 \). Then \( \{a_n\}_{n=1}^\infty \) and \( \{v_n\}_{n=1}^\infty \) satisfy the assumption of Lemma 6.4 with \( k_n = n \) and \( c_n = -\log a, n \geq 1 \).

7. **Existence of hyperinvariant subspaces**

Recall the definition of a bilateral weighted shift, see [E]. Let \( \omega : \mathbb{Z} \to (0,\infty) \) be a nonincreasing function. Set
\[
\ell^2_\omega = \{u = \{u(n)\}_{n \in \mathbb{Z}} : \|u\|^2_\omega = \sum_{n \in \mathbb{Z}} |u(n)|^2 \omega^2(n) < \infty\}.
\]
The *bilateral weighted shift* \( S_\omega \in \mathcal{L}(\ell^2_\omega) \) acts by the formula
\[
(S_\omega u)(n) = u(n-1), \quad n \in \mathbb{Z}, \quad u \in \ell^2_\omega.
\]
Theorem 7.1. Let $T$ be defined as in Proposition 3.2 with $\nu$ as in Lemma 3.3. For $\alpha > 0$ set
\[ \vartheta_\alpha(z) = e^{\alpha z + \frac{1}{\alpha}}, \quad z \in \mathbb{D}. \]
Set $\omega_\alpha(n) = 1$, $n \geq 0$, and define $\omega_\alpha(n)$ for $n \leq -1$ by (4.17). Then $\vartheta_\alpha(T)$ is similar to $\oplus_{j \in \mathbb{N}} S_{\omega_\alpha}$.

Proof. Recall that $\varpi$ and $\theta_\alpha$ are defined by (4.1) and (4.12), respectively. Clearly, $\vartheta_\alpha \circ \varpi = \theta_\alpha$. By Theorem 4.3, $\vartheta_\alpha(T)$ is unitarily equivalent to $\frac{1}{\sqrt{2\pi}} C_{F\theta_\alpha}$ acting on $L^2(\mathbb{R}, \delta_\alpha)$. Define $w_\alpha$ by (5.1). By (5.2), $\frac{1}{\sqrt{2\pi}} C_{F\theta_\alpha}$ is similar to the same operator acting on $L^2(\mathbb{R}, w_\alpha)$. We have
\[ L^2(\mathbb{R}, w_\alpha) = \{ \oplus_{n \in \mathbb{Z}} f_n : f_n \in L^2(\alpha, (n+1)\alpha), \sum_{n \in \mathbb{Z}} \| f_n \|_{L^2(\alpha, (n+1)\alpha)}^2 \omega_\alpha^2(n) < \infty \}. \]
By (4.14), $\frac{1}{\sqrt{2\pi}} (C_{F\theta_\alpha} f)(t) = f(t - \alpha)$, $t \in \mathbb{R}$.

Therefore, $\frac{1}{\sqrt{2\pi}} C_{F\theta_\alpha}$ on $L^2(\mathbb{R}, w_\alpha)$ is unitarily equivalent to the bilateral shift on the weighted space of sequences $\{ f_n \}_{n \in \mathbb{Z}}$, where $f_n \in L^2(0, \alpha)$. Since dim $L^2(0, \alpha) = \infty$, we conclude that $\frac{1}{\sqrt{2\pi}} C_{F\theta_\alpha}$ on $L^2(\mathbb{R}, w_\alpha)$ is unitarily equivalent to $\oplus_{j \in \mathbb{N}} S_{\omega_\alpha}$. \qed

Corollary 7.2. Let $T$ be defined as in Proposition 3.2 with $\nu$ as in Lemma 3.3. Then for every $\alpha > 0$ there exists a singular inner function $\eta_\alpha \in H^\infty$ such that the range of $(\eta_\alpha \circ \vartheta_\alpha)(T)$ is not dense.

Proof. Without loss of generality, we may assume that $\sum_{k=1}^{\infty} a_k \leq 1$. (Else, the weight $\{ \omega_\alpha(n) \}_{n \in \mathbb{Z}}$ constructed in (4.17) can be replaced by $\omega_\alpha^2(n) = \sum_{k=1}^{\infty} a_k \omega_\alpha^2(n)$ for $n \leq -1$ and $\omega_\alpha(0) = 1$ for $n \geq 0$.)

Define the weight $\omega_\alpha = \{ \omega_\alpha(n) \}_{n \in \mathbb{Z}}$ by (4.17). It follows from Lemmas 5.1 and 5.3 that $\omega_\alpha$ is a dissymmetric weight (see [E] for definition). By [E, Theorem 5.7], there exists a singular inner function $\eta_\alpha \in H^\infty$ (which depends on $\omega_\alpha$) such that the range of $\eta_\alpha(S_{\omega_\alpha})$ is not dense. Therefore, the range of $\eta_\alpha(\oplus_{j \in \mathbb{N}} S_{\omega_\alpha})$ is not dense. Taking into account that
\[ (\eta_\alpha \circ \vartheta_\alpha)(T) = (\eta_\alpha(\vartheta_\alpha(T)) \]
and applying Theorem 7.1 we obtain the conclusion of the corollary. \qed

Remark 7.3. Let $T$ be an operator which admits an $H^\infty$-functional calculus (see [Kér3, Sec. 5]). Let $\vartheta \in H^\infty$ be a singular inner function with at least two singularities. If $\vartheta(T)$ is invertible and $\sigma(T) = T$, then $T$ cannot be quasianalytic by [Gam2, Theorems 2.5 and 2.6].

For $\alpha > 0$, let $\vartheta_\alpha$ be defined in Theorem 7.1. Then $\vartheta_\alpha$ has the only singularity at a point $1 \in T$. Let $T$ be defined as in Proposition 3.2 with $\nu$ as in Lemma 3.3. By Proposition 3.2, $\sigma(T) = T$. By Theorem 7.1, $\vartheta_\alpha(T)$ is similar to $\oplus_{j \in \mathbb{N}} S_{\omega_\alpha}$. By [E], $S_{\omega_\alpha}$ is invertible. Thus, $\vartheta_\alpha(T)$ is invertible. By results of Sec. 6, $T$ can be quasianalytic.

8. Convolution and Fourier transform

The results of this section will be applied in Sec. 9. Recall that $\mathcal{F}$ and $\mathcal{C}_\varphi$ denote the Fourier transform and convolution with a function $\varphi$, see (4.8) and (4.10).
Lemma 8.1. Suppose that $\delta > 0$, $\psi \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, and

\begin{equation}
\int_{-\delta}^{\delta} \frac{e^{its} - 1}{s} \psi(s) ds < \infty.
\end{equation}

For $t \in \mathbb{R}$ set

$$
\psi_1(t) = \int_{-\delta}^{\delta} \frac{e^{its} - 1}{s} \psi(s) ds.
$$

Then $\psi_1 \in L^\infty(\mathbb{R})$. For $f \in D(\mathbb{R})$ and $t \in \mathbb{R}$ set

$$
(A_1 \psi f)(t) = \int_{-\delta}^{\delta} \psi(s) \frac{f(t-s) - f(t)}{s} ds.
$$

Then $A_1 \psi f \in L^\infty(\mathbb{R})$, and if $-\infty < b_1 < b_2 < +\infty$ are such that $f(t) = 0$ for $t \in (-\infty, b_1] \cup [b_2, +\infty)$, then $(A_1 \psi f)(t) = 0$ for $t \in (-\infty, b_1 - \delta] \cup [b_2 + \delta, +\infty)$. Furthermore,

$$
F^{-1} A_1 \psi f = \psi_1 F^{-1} f.
$$

Consequently, $A_1 \psi$ can be extended from $D(\mathbb{R})$ onto $L^2(\mathbb{R})$ and

$$
A_1 \psi \in L(L^2(\mathbb{R})).
$$

Proof. We have

$$
|\psi_1(t)| \leq \int_{-\delta}^{\delta} \left| \frac{\psi(s) - \psi(0)}{s} \right| \left| e^{its} - 1 \right| ds + |\psi(0)| \left| \int_{-\delta}^{\delta} \frac{e^{its} - 1}{s} ds \right|,
$$

and

$$
\int_{-\delta}^{\delta} \frac{e^{its} - 1}{s} ds = i \int_{-\delta}^{\delta} \frac{\sin ts}{s} ds = i \int_{-\delta t}^{\delta t} \frac{\sin s}{s} ds.
$$

Since

$$
\sup_{c > 0} \left| \int_{-c}^{c} \frac{\sin s}{s} ds \right| < \infty,
$$

we conclude that $\psi_1 \in L^\infty(\mathbb{R})$.

Let $f \in D(\mathbb{R})$ and $t \in \mathbb{R}$. Then

$$
\max_{s \in [-\delta, \delta]} \left| \frac{f(t-s) - f(t)}{s} \right| \leq \max_{s \in [-\delta, \delta]} |f'(s)| \leq \max_{s \in \mathbb{R}} |f'| < \infty.
$$

Since

$$
|(A_1 \psi f)(t)| \leq \max_{s \in [-\delta, \delta]} \left| \frac{f(t-s) - f(t)}{s} \right| \int_{-\delta}^{\delta} |\psi(s)| ds,
$$

we conclude that $A_1 \psi f \in L^\infty(\mathbb{R})$. Let $-\infty < b_1 < b_2 < +\infty$ be such that $f(t) = 0$ for $t \in (-\infty, b_1] \cup [b_2, +\infty)$. Then $f(t-s) = f(t) = 0$ for $t \in (-\infty, b_1 - \delta] \cup [b_2 + \delta, +\infty)$ and $s \in [-\delta, \delta]$. Let $x \in \mathbb{R}$. By Fubini's
Theorem 8.2. Let $\Psi \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R})$ be such that $\Psi' \in L^1(\mathbb{R})$. Set $\psi = \mathcal{F}\Psi'$. Suppose that $\psi \in L^1(\mathbb{R})$ and $\psi$ satisfies (8.1) for some $\delta > 0$ (and, consequently, for arbitrary finite $\delta$). For $f \in D(\mathbb{R})$ and $t \in \mathbb{R}$ set

$$(A_\psi f)(t) = \int_\mathbb{R} \psi(s) \frac{f(t-s) - f(t)}{s} ds.$$ 

Then

$$A_\psi f = iC\mathcal{F}\Psi f - i\sqrt{2\pi} \Psi(0)f, \quad f \in D(\mathbb{R}).$$

Proof. Fix $\delta > 0$. Set

$$(8.2) \quad \psi_2(t) = \chi_{(-\infty,-\delta] \cup [\delta,\infty)}(t) \frac{\psi(t)}{t}, \quad t \in \mathbb{R}, \quad \text{and} \quad c_\psi = \int_\mathbb{R} \psi_2(t) dt.$$ 

(Of course, $\psi_2$ and $c_\psi$ depend on $\delta$.) Clearly, $\psi_2 \in L^1(\mathbb{R})$. Therefore, $C\psi_2 f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for every $f \in D(\mathbb{R})$. We have

$$(8.3) \quad A_\psi f = A_1\psi f + C\psi_2 f - c_\psi f, \quad f \in D(\mathbb{R}).$$

Set

$$\Phi(t) = \int_\mathbb{R} \frac{e^{its} - 1}{s} \psi(s) ds, \quad t \in \mathbb{R}.$$ 

By Lemma 8.1, $\Phi(t) \in L^\infty(\mathbb{R})$ and $\mathcal{F}^{-1}A_1\psi f = \psi_1 \mathcal{F}^{-1} f$ for $f \in D(\mathbb{R})$. Since $\psi_2 \in L^1(\mathbb{R})$, we have

$$\mathcal{F}^{-1}C\psi_2 f = \sqrt{2\pi}(\mathcal{F}^{-1}\psi_2) \cdot (\mathcal{F}^{-1} f), \quad f \in D(\mathbb{R})$$

(see, for example, [Ka, Theorem VI.1.3] or [R, Theorem 7.2]). Thus,

$$(8.4) \quad \mathcal{F}^{-1}A_\psi f = \Phi \mathcal{F}^{-1} f, \quad f \in D(\mathbb{R}).$$

We will show that $\Phi' = i\sqrt{2\pi}\Psi'$. We have

$$\Phi'(t) = \int_\mathbb{R} \left(\frac{e^{its} - 1}{s} \right)' \psi(s) ds = i \int_\mathbb{R} e^{its} \psi(s) ds, \quad t \in \mathbb{R},$$

because $\psi \in L^1(\mathbb{R})$. Since $L^1(\mathbb{R}) \subset S'(\mathbb{R})$ (see, for example, [Ka, Sec. VI.4.1] or [R, Example 7.12(d)]), we have $\psi, \Psi' \in S'(\mathbb{R})$. Taking into account that $\psi = \mathcal{F}\Psi'$, we conclude that

$$\int_\mathbb{R} e^{its} \psi(s) ds = \sqrt{2\pi}(\mathcal{F}^{-1}\psi)(t) = \sqrt{2\pi}\Psi'(t).$$
9. Square root again

Denote by $\varrho$ the branch of square root defined in $\mathbb{C} \setminus [0, +\infty)$. Set

\[
\Psi = \varrho \circ \varpi,
\]

where $\varpi$ is defined by (4.1). Then $\Psi$ is analytic in $\mathbb{C} \setminus \{iy : |y| \geq 1\}$, $\Psi(\mathbb{C} \setminus \{iy : |y| \geq 1\}) = \mathbb{C}_+$, and

\[
\Psi'(z) = \frac{i}{\Psi(z)} \frac{1}{(z+1)^2}, \quad z \in \mathbb{C} \setminus \{iy : |y| \geq 1\}.
\]

For $\lambda \notin \mathbb{C}_+ \cup \mathbb{R}$ set

\[
\Psi_{\lambda} = \frac{1}{\Psi - \lambda},
\]

then $\Psi$ is analytic in $\mathbb{C} \setminus \{iy : |y| \geq 1\}$ and $\Psi_{\lambda}' = -\frac{1}{(\Psi - \lambda)^2} \Psi'$.

The proof of the following theorem can be found in [Ka, Sec. VI.7.1].

**Theorem 9.1** (Paley–Wiener). Let $c > 0$, and let a function $f$ be analytic in $\{z \in \mathbb{C} : z = t + iy, \quad t \in \mathbb{R}, \quad y \in (-c, c)\}$ and such that

\[
\sup_{y \in (-c, c)} \int_{\mathbb{R}} |f(t + iy)|^2 dt < \infty.
\]

Then $\int_{\mathbb{R}} e^{2c|t|}|(\mathcal{F}(f|_{\mathbb{R}}))(t)|^2 dt < \infty$.

**Lemma 9.2.** Let $\lambda \notin \mathbb{C}_+ \cup \mathbb{R}$, and let $\Psi_{\lambda}$ be defined by (9.2). Then for every $0 < c < 1$ $\Psi_{\lambda}'$ satisfies (9.3).

**Proof.** We have $|\Psi_{\lambda}'| \leq \frac{1}{\text{dist}(\lambda, \mathbb{C}_+)^2} |\Psi'|$. Therefore, it sufficient to proof that $\Psi'$ satisfies (9.3), which follows from the equality

\[
|\Psi'(t + iy)|^2 = \frac{1}{(t^2 + (1 - y)^2)^{1/2}(t^2 + (1 + y)^2)^{3/2}}, \quad t \in \mathbb{R}, \quad |y| < 1. \quad \square
\]

**Lemma 9.3.** Let $\lambda \notin \mathbb{C}_+ \cup \mathbb{R}$, and let $\Psi_{\lambda}$ be defined by (9.2). Set $\psi_{\lambda} = \mathcal{F}\Psi_{\lambda}'$.

Let $\delta > 0$. Then $\psi_{\lambda}$ satisfies (8.1).

**Proof.** Since $\Psi_{\lambda}' \in L^1(\mathbb{R})$, we have

\[
\frac{\psi_{\lambda}(t) - \psi_{\lambda}(0)}{t} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-its} - 1}{t} \Psi_{\lambda}'(s) ds.
\]

Therefore,

\[
\int_0^{\delta} \left| \frac{\psi_{\lambda}(t) - \psi_{\lambda}(0)}{t} \right| dt \leq \frac{1}{\sqrt{2\pi}} \int_0^{\delta} \int_{\mathbb{R}} \left| \frac{e^{-its} - 1}{t} \right| \left| \Psi_{\lambda}'(s) \right| ds dt \\
\leq \frac{1}{\sqrt{2\pi}} \text{dist}(\lambda, \mathbb{C}_+)^2 \int_0^{\delta} \int_{\mathbb{R}} \left| \frac{e^{-its} - 1}{t} \right| \left| \Psi'(s) \right| ds dt.
\]
We have \(|\Psi'(s)| = \frac{1}{1+s^2}, \ s \in \mathbb{R}\), and
\[
\int_0^\delta \int_\mathbb{R} \frac{e^{-its} - 1}{t} |\Psi'(s)| ds dt = 2 \int_0^\delta \int_\mathbb{R} \frac{\sin \frac{t \phi}{2}}{1 + s^2} ds dt
\]
\[
= 2 \int_0^\delta \int_\mathbb{R} \frac{\sin \frac{t}{2}}{1 + \left(\frac{t}{2}\right)^2} \frac{ds}{|t|} dt = 2 \int_\mathbb{R} \frac{\sin \frac{s}{2}}{1 + s^2} \frac{ds}{2} \int_0^\delta \frac{1}{t^2 + s^2} dt ds
\]
\[
= 2 \int_\mathbb{R} \frac{\sin \frac{s}{2}}{|s|} \arctan \frac{\delta}{|s|} ds < \infty.
\]
Thus,
\[
\int_0^\delta \left| \frac{\psi_\lambda(t) - \psi_\lambda(0)}{t} \right| dt < \infty.
\]
The estimate for \(\int_{0-\delta}^0 \left| \frac{\psi_\lambda(t) - \psi_\lambda(0)}{t} \right| dt\) is obtained similarly. \(\square\)

The following two lemma are proved exactly as Lemmas 9.2 and 9.3, therefore, their proofs are omitted.

**Lemma 9.4.** Let \(\Psi\) be defined by (9.1). Then for every \(0 < c < 1\) \(\Psi'\) satisfies (9.3).

**Lemma 9.5.** Let \(\Psi\) be defined by (9.1). Set \(\psi = \mathcal{F}\Psi\). Let \(\delta > 0\). Then \(\psi\) satisfies (8.1).

Recall that \(\hat{\gamma}_T\) is defined in (1.4).

**Theorem 9.6.** Let \(T\) be defined as in Proposition 3.2 with \(v\) as in Lemma 3.3. Let \(\varrho\) be the branch of square root defined in \(\mathbb{C} \setminus [0, +\infty)\). Then there exists \(R \in \{T\}'\) such that \(\hat{\gamma}_T(R) = \varrho|_T\), and \(\sigma(R) \subset \mathbb{C}_+ \cup \mathbb{R}\).

**Proof.** Since \(\hat{\gamma}_T\) is a unital algebra-homomorphism, it sufficient to prove that \(\varrho|_T \in \hat{\gamma}_T({\{T\}'})\) and \(1 - \frac{1}{\varrho|_T}\) \(\in \hat{\gamma}_T({\{T\}'})\) for every \(\lambda \notin \mathbb{C}_+ \cup \mathbb{R}\) (see [KS1]).

Let \(\Psi\) be defined by (9.1), and for \(\lambda \notin \mathbb{C}_+ \cup \mathbb{R}\) let \(\Psi_\lambda\) be defined by (9.2). Take \(\alpha > 0\). Let \(\hat{\lambda}_\alpha\) be defined in Theorem 4.3. By Theorem 4.3, it is sufficient to prove that \(\mathcal{C}_\varrho\Psi \in L(L^2(\mathbb{R}, \hat{\lambda}_\alpha))\), and \(\mathcal{C}_\varrho\Psi_\lambda \in L(L^2(\mathbb{R}, \hat{\lambda}_\alpha))\) for every \(\lambda \notin \mathbb{C}_+ \cup \mathbb{R}\). We may assume that \(\sum_{k=1}^\infty a_k \leq 1\) (where \(\{a_k\}_{k=1}^\infty\) are from the construction of \(\nu\)). Let \(w_\alpha\) be defined by (5.1). By (5.2), \(w_\alpha \simeq \hat{\lambda}_\alpha\). Therefore, it sufficient to prove that \(\mathcal{C}_\varrho\Psi \in L(L^2(\mathbb{R}, w_\alpha))\), and \(\mathcal{C}_\varrho\Psi_\lambda \in L(L^2(\mathbb{R}, w_\alpha))\) for every \(\lambda \notin \mathbb{C}_+ \cup \mathbb{R}\). Since \(\sup_{(b_1, b_2)} w_\alpha < \infty\) for every \(-\infty < b_1 < b_2 < +\infty\), we have \(D(\mathbb{R}) \subset L^2(\mathbb{R}, w_\alpha)\).

Take \(0 < \delta < \alpha\). Set \(\psi = \mathcal{F}\Psi\). By Theorem 9.1 and Lemmas 9.4 and 9.5, \(\Psi\) satisfies Theorem 8.2. By Theorem 8.2, it sufficient to prove that the mapping \(A_\psi\) which is defined on \(D(\mathbb{R})\) can be extended as a (linear, bounded) operator onto \(L^2(\mathbb{R}, w_\alpha)\).

By (8.3), it sufficient to prove that \(A_{1\psi} \) and \(C_{\psi_2}\), where \(\psi_2\) is defined by (8.2), can be extended as (linear, bounded) operators onto \(L^2(\mathbb{R}, w_\alpha)\). By Lemma 5.1, \(w_\alpha\) satisfies the assumptions of Lemma 5.6. By Lemma 8.1, \(A_{1\psi}\) satisfies the assumptions of Lemma 5.6. Thus, \(A_{1\psi} \in L(L^2(\mathbb{R}, w_\alpha))\) by Lemma 5.6. By Theorem 9.1 and Lemma 5.3, \(\psi_2\sqrt{w_\alpha} \in L^1(\mathbb{R})\). By Corollary 5.2, \(w_\alpha\) satisfies the assumptions of Lemma 5.4. By Lemma 5.4, \(C_{\psi_2} \in L(L^2(\mathbb{R}, w_\alpha))\).
Thus, $C_{K^*} \in \mathcal{L}(L^2(\mathbb{R}, \tilde{\varphi}(\alpha)))$. For $\Psi_\lambda$ with $\lambda \notin \mathbb{C}_+ \cup \mathbb{R}$, the proof is the same. □

**Corollary 9.7.** There exists a quasianalytic contraction $R$ with $\sigma(R) = \{e^{it} : t \in [0, \pi]\}$ and with a unitary asymptote $U_\sigma(R)$. Consequently, $\sigma(R)$ coincides with the quasianalytic spectral set of $R$ and $\sigma(R) \neq \mathbb{T}$.

**Proof.** Let $T$ and $R$ be from Theorem 9.6. Clearly, $(\varphi|_T)^2 = \chi$ (where $\chi(z) = z$, $z \in \mathbb{T}$). Since $\hat{\chi}_T(R) = \varphi|_T$ and $\hat{\chi}_T(T) = \chi$, we conclude that $R^2 = T$, because $\hat{\chi}_T$ is a unital algebra-homomorphism.

Set $\sigma = \{e^{it} : t \in [0, \pi]\}$. By Theorem 9.6, $\sigma(R) \subset \mathbb{C}_+ \cup \mathbb{R}$. By Proposition 3.2, $\sigma(T) = \mathbb{T}$. Since $R^2 = T$, we have $\sigma(T) = \{\lambda^2 : \lambda \in \sigma(R)\}$. Therefore, $\sigma(R) \subset \mathbb{T}$. Consequently,

$$\sigma(R) \subset \mathbb{T} \cap (\mathbb{C}_+ \cup \mathbb{R}) = \sigma.$$

By Proposition 3.1, $T$ is similar to a contraction. By Theorem 2.1, $R$ is similar to a contraction, too. By Lemma 2.2, $\varphi(U_T)$ is a unitary asymptote of $R$. Since $\varphi(U_T)$ is unitarily equivalent to $U_\sigma$, we have that $U_\sigma$ is a unitary asymptote of $R$. By (1.2) applied to $R$, $\sigma \subset \sigma(R)$. Thus, $\sigma(R) = \sigma$.

Let $\nu$ be chosen such that $T$ is quasianalytic. (It is possible by results of Sec. 6.) By Corollary 2.3, $R$ is quasianalytic. □

**Remark 9.8.** Let $T$ and $R$ be operators constructed in the proof of Corollary 9.7. By Proposition 3.1 and Lemma 2.2, unitary asymptotes of $T$ and $R$ are cyclic unitary operators. Therefore, $\{T\}'$ and $\{R\}'$ are abelian algebras. Since $R \in \{T\}'$, we conclude that $\{R\}' = \{T\}'$ by [KS1, Proposition 11]. Consequently, $\text{Hlat } R = \text{Hlat } T$. By Corollary 7.2, $\text{Hlat } T$ is nontrivial. Therefore, $\text{Hlat } R$ is nontrivial, too.

**Remark 9.9.** Let $T$ be defined as in Proposition 3.2 with $\nu$ as in Lemma 3.3. For $\alpha > 0$, let $\vartheta_\alpha$ be defined in Theorem 7.1. By Remark 7.3, $\vartheta_\alpha(T)$ is invertible.

For every $0 < r < 1$ set $G_r = \mathbb{D} \setminus D_r$, where $D_r$ is defined in (3.4). Then $\inf_{G_r} |\vartheta_\alpha| > 0$. Denote by $\kappa_{\nu}$ a conformal mapping of $\mathbb{D}$ onto $G_r$. Let $Q$ be an operator which admits an $H^\infty$-functional calculus (see [Kér3, Theorem 23]). By [Kér2] and [Gam2], $\vartheta_\alpha(\kappa_{\nu}(Q))$ is invertible.

A question appears: whether exists an operator $Q_\nu$ such that $T = \kappa_{\nu}(Q_\nu)$? It is possible to prove that there exists $Q_1r \in \{T\}'$ such that $\hat{\chi}_T(Q_1r) = \kappa_{\nu}^{-1}$ and $\sigma(Q_1r)$ is a proper subarc of $\mathbb{T}$. But the estimate obtained by the author is $\|Q_1^r\| \leq Cn(\log n)^2$ for sufficiently large $n \in \mathbb{N}$, which does not allow to define $\kappa_{\nu}(Q_1r)$.

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EXAMPLE OF QUASIANALYTIC CONTRACTION

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