STRONG-CORRELATIONS VERSUS U-CENTER PAIRING
and FRACTIONAL AHARONOV-BOHM EFFECT

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The influence of the interaction between electrons on the Aharonov-Bohm effect is investigated in the framework of the Hubbard model. The repulsion between electrons associated with strong correlation is compared with the case of attraction such as $U$-center pairing. We apply the Bethe ansatz method and exact numerical diagonalization to the Hubbard hamiltonian. It is shown that the quasi half quantum flux periodicity occurs for any nonzero values of $U$ for two electrons. For large number of sites, or strong $U$, the quasi periodicity becomes an exact half quantum flux periodicity. However the character of the state created on the ring is different in both cases. In the case of $U$-center pairing the electrons are bound in pairs located on the same site. For strong correlations (large positive $U$) the electrons tend to be far from each other as possible. We show by numerical solution of Bethe ansatz equations that for three electrons the flux periodicity of the ground state energy is equal to $1/3$. The one third periodicity may occur even for small values of the ratio $U/t$, for the very dilute system. It is shown analytically using the Bethe ansaz equations for $N$ electrons, that for dilute systems with arbitrary value of the Hubbard repulsion $U$, the fractional Aharonov-Bohm effect occurs with period $f_T = 1/N$ in units of elementary flux quantum. Such period occurs when the value $Nt/LU$ is small, where $L$ is the number of sites. Parity effects dissapear in this fractional regime.
I. INTRODUCTION

The importance of strong correlations in condensed matter physics became obvious after the discovery of the quantum Hall effect and high temperature superconductivity [1]. In fact the correlations are created by Coulomb forces which are especially important in low-dimensional systems [2]-[8]. In spite of rapid progress in the understanding of strongly-correlated states such as the fractional quantum Hall effect [2], or the one-dimensional Luttinger liquid[3], the microscopic detail of these correlations still needs to be further developed [7], [8]. In order to shed light on their nature we investigate using Bethe ansatz as well as by direct diagonalisation an exactly solvable Hubbard model, describing a finite number of electrons in magnetic field, located on a ring. The filling factor is kept as a free parameter, by varying the number of sites. Lieb recently pointed out, that for small chains, the general “Bethe ansatz” solution, while correct, is too complicated for numerical calculations [10]. Examining this problem, we first investigate the simplest case, when two electrons are located on the ring. In this case the solution may be given analytically for certain cases. We use the Aharonov-Bohm effect as a tool to study the pairing of the correlated state, and compare it with the $U$-center pairing. In the limit of large repulsion, when $U/t \to \infty$ Kusmartsev [11] found that the Aharonov-Bohm effect might be fractional. The Aharonov-Bohm period is be changed from the conventional one, to $1/N$, where the $N$ is the number of electrons on the ring. Kusmartsev’s result has since been confirmed in other investigations [12], [13], [14]. In the present work we show that the fractional $1/N$ Aharonov-Bohm effect may occur for an arbitrary $N$ and for an arbitrary (even very small) ratio $U/t$ for the very dilute system, when filling factor drops to zero.

The numerical calculations for a given number of sites are based on the exact diagonal-
ization of the two electron Hubbard hamiltonian. With this method we can then analyze all the finite size effects, which may give many instances of both level crossings and of permanent degeneracies (as a function of $U$), as have been found by Heilmann and Lieb [16] in their studies of the energy levels of the benzene molecule having 6 sites and 6 electrons.

We obtain the same results from this direct diagonalization as from the Bethe-ansatz method. [20]- [29]. This contrasts with solutions of Bethe equations obtained in the thermodynamic limit [29], [29] [25] or calculations by Woynarovich and Eckle [24], who evaluated the asymptotics of finite size effects on the ground state energy. In fact, finite size objects such as rings may have complicated non-abelian dynamical symmetry groups which were not accounted for on the basis of the known invariance groups (spin, pseudospin and rotational symmetries of the ring) [10]. In fact we found that finite scaling finite size effects were very important. There is a scaling behavior of ground state energy which does not depend on size $L$ or on $U$ but which depends only on $UL/N = const = \alpha$, where $U$ is measured in units of $t$ and $L$ is measured in units of lattice constant. Such scaling occurs only at small values of $\alpha$. Due to this scaling symmetry the fractional Aharonov-Bohm effect may arise even for small values of $U$, for very dilute electron systems.

II. MODEL AND BETHE EQUATIONS

To describe the fractional Aharonov-Bohm effect for dilute systems we study the Hubbard hamiltonian

$$H = -t \sum_{<i,j>,\sigma} a_{i\sigma}^+ a_{j\sigma} + U \sum_{i=1}^{L} n_{i+} n_{i-}$$  \hspace{1cm} (2.1)

involving as parameters the electron hopping integral $t$ and the on-site repulsive Coulomb potential $U$ and the number of sites $L$. The operator $a_{i\sigma}^+ (a_{i\sigma})$ creates (destroys) an electron
with spin projection $\sigma$ ($\sigma = +$ or $-$) at a ring site $i$, and $n_{i\sigma}$ is the occupation number operator $a_{i\sigma}^+ a_{i\sigma}$. The summations in Eq. (1) extend over the ring sites $i$ or $-$ as indicated by $<i,j>$, $\sigma$ – over all distinct pairs of nearest-neighbor sites, along the ring with the spin projection $\sigma$. The effect of the transverse magnetic field is included via twisted boundary conditions, with the Bethe ansatz substitution for the wave function.

For the case of the magnetic field we will use the same form of the wave function as has been proposed in the Refs [23], [26]

$$
\psi(x_1, \ldots, x_N) = \sum_P [Q, P] \exp[i \sum_{j=1}^N k_{Pj} x_{Qj}] 
$$

(2.2)

where $P = (P_1, \ldots, P_N)$ and $Q = (Q_1, \ldots, Q_N)$ are two permutations of $(1, 2, \ldots, N)$, and $N$ is the number of electrons.

The coefficients $[Q, P]$ as well as $(k_1, \ldots, k_N)$ are determined from the Bethe equations which in a magnetic field are changed by the addition of the flux phase $2\pi f$ [26], [11]

$$
e^{i(k_j L - 2\pi f)} = \prod_{\beta=1}^M \left( \frac{it \sin k_j - i\lambda_\beta - U/4}{it \sin k_j - i\lambda_\beta + U/4} \right) 
$$

(2.3)

and

$$
- \prod_{j=1}^N \left( \frac{it \sin k_j - i\lambda_\alpha - U/4}{it \sin k_j - i\lambda_\alpha + U/4} \right) = \prod_{\beta=1}^M \left( \frac{i\lambda_\alpha - i\lambda_\beta + U/2}{i\lambda_\alpha - i\lambda_\beta - U/2} \right) 
$$

(2.4)

The $f$ is the flux in units of elementary fundamental quantum flux $\phi_0$. The explicit form of Bethe equations in magnetic field is [29], [17], [11], [12], [14]

$$
L k_j = 2\pi I_j + 2\pi f - \sum_{\beta=1}^M \theta(4(t \sin k_j - \lambda_\beta)/U) 
$$

(2.5)

$$
- \sum_{j=1}^N \theta(4(t \sin k_j - \lambda_\beta)/U) = 2\pi J_\beta + \sum_{\lambda_\alpha=1}^M \theta(2(\lambda_\beta - \lambda_\alpha)/U) 
$$

(2.6)
where $\theta(x) = 2\arctan(x)$ and the quantum numbers $I_j$ and $J_\beta$, which are associated with charge and spin degrees of freedom, respectively, are either integers or half odd integers, depending on the parities of the numbers of down and up-spin electrons, respectively:

$$I_j = \frac{M}{2} \pmod{1} \quad \text{and} \quad J_\beta = \frac{N - M + 1}{2} \pmod{1}. \quad (2.7)$$

The actual values (sets) of these numbers must be chosen to minimize the total energy for the given value of the flux $f$.

**III. HALF-FLUX PERIODICITY FOR TWO ELECTRONS**

For the case of interest, for two electrons on the ring the system of equations (3 and 4) simplify to two decoupled equations

$$Lx = 2\arctan\left(\frac{\epsilon}{\sin x}\right) + \pi n \quad (3.8)$$

and

$$Ly = 2\pi f + \pi m \quad (3.9)$$

where $x = (k_1 - k_2)/2$, $y = (k_1 + k_2)/2$ and $\epsilon = U/(4t \cos((2\pi f + \pi m)/L))$. The numbers $n, m$ may have both positive and negative values.

When $L = 2$, $(U > 0)$, the first of equations (5 and 6) may be solved immediately and the solution is valid for the range of flux $|f| < 1/2$, where we must put $m = n = 0$. For other values of the flux the solution must be periodically continued. The result is

$$k_{1,2} = \pm\arccos(-\epsilon/2 + \sqrt{\epsilon^2/4 + 1}) + \pi f \quad (3.10)$$

which shows that with the increase of $U$ there is an increase of the difference between the $k$ vectors of the first and second electron. The increase of this difference improves the quasi
half quantum flux periodicity of the Aharonov-Bohm effect. The ground state energy is described by simple expression:

\[ E_{\text{ground}} = -2t\left(-\frac{\epsilon}{2} + \sqrt{\frac{\epsilon^2}{4} + 1}\right) \cos(\pi f) \]  

(3.11)

\[ j_2 = -\frac{\partial E_{\text{ground}}}{\partial f} = -\frac{8\pi t^2 \sin(2f\pi)}{\sqrt{(U^2 + 64t^2 \cos(f\pi))}} \]  

(3.12)

where \( f \) is limited to the region \( |f| \leq 1/2 \). One sees that the value of the persistent current at the fixed value of the flux \( f \) monotonically decreases as \( U \) increases. One sees that interactions change the current-flux dependence in a way which is similar to that of disorder or temperature, as shown in our recent paper \[30]. That is, the interactions cause the jumps in current-flux dependence, or the cuspidal points in the energy-flux curves disappear. With stronger interactions these curves become gradually smoother. The energy dependence is a single-flux periodical function at any value of \( U \). The reason for the single flux periodicity is that the two site ring with two electrons is a very special case in that it is half-full with two electrons. In the limit \( U \to \infty \) the current and the energy vanishes, which coincides with the result obtained in Ref \[11]. That is, for the half-filled cases in that limit, the persistent current equals zero.

However, this single flux periodicity is broken, when we go away from half-filling and immediately obtain quasi-half flux periodicity. For example, for the Aharonov-Bohm effect on a ring having four sites (1/4-filling) the explicit formulas may also be written. In this case equation (5) simplifies to the cubic equation

\[ z^3 + \epsilon z^2 - z - \epsilon/2 = 0 \]  

(3.13)

where \( z = \cos x \). With the aid of the Cardano formula the solution can be given explicitly. In the limit of large value of \( \epsilon \) it has the form:
\[ k_1(2) = (+1)\arccos\left(\frac{1}{\sqrt{2}} + 1/(4\epsilon)\right) + (2\pi f + \pi m)/4, \quad \text{when } \epsilon \gg 1 \]  
\[ (3.14) \]

with the flux energy dependence:

\[ E_{\text{ground}} = -4\cos\left(\frac{\pi f}{2}\right)\left(\frac{1}{\sqrt{2}} + \frac{1}{4\epsilon}\right) \]  
\[ (3.15) \]

A nontrivial fact here is that there is another solution, associated with singular values of \( \lambda_\alpha \), which for an arbitrary value of \( L \) has the form

\[ E_{\text{ground}} = -4\cos\left(\frac{2\pi(f - \frac{1}{2})}{L}\right)\cos\left(\frac{\pi}{L}\right) \]  
\[ (3.16) \]

This formula coincides that describing the flux-energy dependence of two noninteracting spinless fermions on a ring with \( L \) sites. For the problem under consideration this dependence also corresponds to a triplet state. This means that on the two sites ring there in a region of flux values near half-odd integer numbers in which a very surprising degeneracy between triplet and singlet states occurs. This means that at this value of flux the matrix element for interaction vanishes for the singlet state. For the singlet state there already occurs a trapping of the flux quantum, which is shares between two electrons. The trapping of the quantum flux occurs at each cuspoidal point of the energy-flux curve. On the other hand the trapping does not occur for the triplet state. This degeneracy, reflecting some hidden symmetry, may be schematically expressed with the aid of the formula: singlet + flux quantum == triplet.

For the two sites ring discussed above, this solution (3.16) exactly equals zero, giving a single flux periodicity in the flux energy dependence for the half-filled ring. Thus, for a non-half-filled ring the ground state flux-energy dependence is determined by two solutions (3.15) and (3.16), which give the needed quasi-half periodicity.

From the equations (3.14) one sees that the phase increases with \( U \to \infty \) from zero to \( \pi/4 \). The general result, which is valid for any value of \( L \), is as follows: With the increase
of $U$ the difference between $k_1$ and $k_2$ increases, and the solution is given by the following formula

$$k_{1,2} = (\pm \theta + 2\pi f + \pi n)/L \quad (3.17)$$

where the function $\theta$ depends on $\epsilon$ and increases with $U$ and $L$. For the case $L = 2$ the explicit dependence $\theta(U, f)$ is given by equation (7).

The ground state energy is determined from the formula

$$E = -2t \sum_{j=1}^{N} \cos k_j \quad (3.18)$$

where the Beyers-Yang theorem [27], [28] (see also Refs [26], [15]) is used to remove flux. From equations (9) and (10) one may conclude that for the ground state energy the momenta $k_1$ and $k_2$ must be single quantum flux periodical functions. Because of the difference between $k_1$ and $k_2$, due to the function $\theta$, the ground state energy $E$ as a function of flux becomes a quasi half flux periodic function.

With the increase of $U$ or $L$ ($L > 2$) the shift $\theta$ increases; and consequently the half quantum flux periodicity improves. This effect exists in two cases. Good half flux quantum periodicity appears for a small ring with a large $U$ or for a large ring with small $U$. Numerical results indicating these effects are presented in Figs. 1 and 2.

For the electrons on the Hubbard ring the repulsive potential $U$ causes the particles to locate on opposite sides of the ring. Because of finite size effects (or alternatively the kinetic energy of electrons) this localization is not complete. The criteria for strong $U$ to have good quasi half quantum flux periodicity is then that $U$ is much larger than the kinetic energy. This is why the half flux periodicity improves both with larger $L$ and larger $U$.

It is interesting to compare the results with the Aharonov-Bohm effect in the case when there exists a pairing of the electrons induced by a negative $U$-potential (Pairing due to a
negative $U$ center). In a one dimensional system with negative $U$ pairing one expects that the two electrons will tend to pair together on the same site. On the ring these electrons will also have the tendency to move in pairs. The kinetic energy, due to the finite size, will try to destroy the pairs. Therefore we again have in this case the approximate half quantum flux periodicity. For the same reason as in the correlated state discussed above, the half quantum flux periodicity is improved with the increase of $|U|$ and $L$. However the character of this state is different from the correlated one. For large negative $U$ one needs an activation energy (spectral gap) to destroy the localized pair.

For illustration we show the ground state energy dependence on the flux in two cases: constant value of $U$ with increase in the number of sites; and at constant value of $L$ with the decrease of the negative value of $U$. We see(Fig. 3) that in both cases half quantum flux periodicity improves as both $|U|$ and $L$ increase. The change in slope of the ground state energy as a function of flux will correspond to change in the direction of persistent current. In ring superconductors this current keeps flux quantized in units of half quantum flux, for which the negative $U$ case is a plausible model. The magnetization also behaves similarly to the current.

**IV. 1/3 FLUX PERIODICITY FOR 3 ELECTRONS**

In contrast with the previous section, the three body problem does not allow an explicit solution. As discussed by Lieb [10] the direct numerical solution of Bethe ansatz equations for small chains is a much difficult problem than the case of the thermodynamic limit. However, the problem for finite chains may be solved if we give the Bethe ansatz equations in a form convenient for numerical iteration procedure. The problem is to solve these equations...
that is to find numerically the values of the variables \( k_j \) and \( \lambda_\alpha \). In order to find such solutions we must represent the Bethe equations in a form convenient for iteration procedure, which is usually used in numerical calculations. The first equation (2.5) is already in the required form if we divide both sides by \( L \). In the second equation (2.6) we add to both sides the function \( N\theta(4\lambda_\beta/U) \). With the aid of these tricks one reduces the second equation (2.6) to the form:

\[
\lambda_\beta = \frac{U}{4} \tan \left[ \frac{N\theta(\frac{4\lambda_\beta}{U}) + 2\pi J_\beta + \sum_{\alpha=1}^{M} \theta(\frac{2\lambda_\beta+2\lambda_\alpha}{U}) - \sum_{j=1}^{N} \theta(\frac{4t\sin k_j}{U} - \frac{4\lambda_\beta}{U})}{2N} \right] \tag{4.19}
\]

With the substitution \( \lambda_\beta = t_\beta U/4 \) this equation may be simplified and we arrive at a couple of equations, convenient for the iteration procedure:

\[
k_j = \frac{2\pi I_j + \sum_{\beta=1}^{M} \theta(4t\sin k_j/U - t_\beta)}{L} \tag{4.20}
\]

and

\[
t_\beta = \tan \left[ \frac{N\theta(t_\beta) + 2\pi J_\beta + \sum_{\alpha=1}^{M} \theta((t_\beta - t_\alpha)/2) + \sum_{j=1}^{N} \theta(4t\sin k_j/U - t_\beta)}{2N} \right] \tag{4.21}
\]

In what follows below we use \( U \) to represent the ratio \( U/t \). We have solved these equations iteratively, for the case of 3 or 4 electrons on the ring, for different values of \( F, U \) and \( L \). The convergence depends on the value of the parameters \( U \) and \( L \) and is fast if the value \( U \) or \( L/N \) are large. The results may be classified as follows. For small number of sites, and small value of \( U \), the ground state energy dependence remains that of the free particle case. The ground state energy dependence on flux is a single flux periodical function. The ground state energy corresponds to the case when two particles have down-spin and particle has an up-spin, or alternatively, two particles have up-spins and one particle down-spin. In that case the energy decreases monotonically when flux increases from zero upto \( f = 0.295167 \) and
then energy increases when flux increases upto 1/2. This behavior must be symmetrically reflected on the second half of the elementary flux unit and then periodically continued for an arbitrary values of the flux. If the value of $U$ becomes larger than the critical value, which, for the ring of 4 sites is equal to $U_c \sim 20$, there appear new minima at integer values of the flux in the energy dependence. In Fig.5 this dependence is calculated for $U = 50$. The shape of that curve is very different from the free fermion case. Each of these parabolic curves is associated with the state characterized by a definite set of quantum numbers $I_j$ and $J_\alpha$. To show the validity of the Bethe ansatz equations for the value $M > N/2$ we have solved these equations for two cases, associated with the values of $M = 2$ and $M = 1$. For these cases the Bethe equations have different forms although, physically, the states are equivalent to each other. In fact, the flux-energy dependency for these cases are identical, although the quantum numbers are distinct. Each state associated with the parabolic curve on Fig.5 is represented by a vertical column in the Table I. One sees that for the transition of one state to the other at the value $M = 2$, the set of quantum numbers is changed drastically.

**TABLE I.** The sets of quantum numbers $I_j$ and $J_\alpha$ corresponding to the parabolic curve with the lowest energy and the value of the flux at the minimum values are given in the vertical column.

| State’s Number | 1                     | 2                     | 3                     | 4                     |
|----------------|-----------------------|-----------------------|-----------------------|-----------------------|
| $M = 2 I_j$   | (-1,0,1)              | (-1,0,1)              | (-2,-1,0)             | (-2,-1,0)             |
| $M = 2 J_\beta$ | (1,2)                 | (-1,0)                | (0,1)                 | (1,2)                 |
| $M = 1 I_j$   | (-3/2,-1/2,1/2)       | (-3/2,-1/2,1/2)       | (-3/2,-1/2,1/2)       | (-3/2,-1/2,1/2)       |
| $M = 1 J_\beta$ | (3/2)                 | (1/2)                 | (-1/2)                | (-3/2)                |
| Flux $f_{min}$ | 0                     | 1/3                   | 2/3                   | 1                     |
With further increase of $U$ the minima become more profound, transforming gradually to the curve consisting of equidistant parabolas, which is the $1/3$ flux periodical function. However one gets the same effect with the increase of the number of sites $L$. For example, for the value $U = 50$ of Fig.6 the flux-energy dependence is shown for the 6 site ring. With the increase of the sites number from $L = 4$ to $L = 6$ the $1/3$ flux quasi-periodicity has been gradually increased. If we take a smaller value of $U$, for example $U = 8$, for the same number of sites $L = 6$, two parabolic curves disappear from the ground state energy curve, which now consists of only two parabolas (see, Fig.7). For a larger number of sites, $L = 12$ for example, one has again four parabolas for the ground state energy and $1/3$-flux-periodical dependence also appears (Fig. 8). It is clear from these calculations that the general tendency of the appearance $1/3$ flux periodicity of the ground state energy is either with the increase of $U$, at the fixed $L$ or with the increase of the value of $L$, for fixed value of $U$.

The appearance of the $1/3$ flux periodicity may arise also at small values of $U$, provided that the value $L$ is large. This is illustrated in Fig.9, where the value of $U$ is equal to 1 and the number of sites is $L = 128$. One can claim even more, that the shape of the ground state energy-flux dependence does not depend on the particular values of $U$ and $L$, but depends on their product. Fig.10 shows the flux-energy dependence for the value $U = 0.5$ and the value $L = 256$ with the same product $UL=128$ as in the former case, given in the Fig.9. The comparison these two Figures, which are identical if the energy scale is neglected, allows one to conclude that there is a scaling symmetry, whereby the product $UL$ is constant and the shape of the flux-energy dependence is not changed. Note that to have $1/3$ flux periodicity this product must have a large value. The results of calculations we made for the case of four electrons ($N = 4$) is to a large extent the same, but with the difference that $1/4$ flux periodicity appears.
The general conclusion can be drawn that the fractional Aharonov-Bohm effect appears when the parameter $\alpha = tN/UL$ is small, but not exclusively in the case when $t/U$ is small, as discussed in previous work by Kusmartsev [15] and by Yu and Fowler [14].

Encouraged by these numerical investigations of very small rings we investigate the general case of arbitrary number of electrons on the ring, when $\alpha$ is small. These parameter values correspond to realistic situations when $U/t$ has a some fixed value but the system has a very dilute density.

V. FRACTIONAL 1/N FLUX PERIODICITY FOR N ELECTRONS

Let us show that the fractional Aharonov-Bohm effect is created for an arbitrary number of electrons $N$ on the ring and arbitrary values of $U$. From equation (4.20) one sees that the numerator of the right-hand side cannot be large than $2\pi N$. This holds since for values of quantum numbers satisfy $I_j \leq N/2$, the flux $f < 1$ and $\sum_{j=1}^{M} \theta(x_j) \leq \pi M$. If $2\pi N/L \ll 1$ one has values of $k_j \ll 1$. Hence, on the right side of the equation (4.20) the value $4 \sin k_j/U \sim 4k_j/U \sim N/UL = \alpha$, is a small parameter. Therefore in zeroth approximation, for small $\alpha$, the expression $4 \sin k_j/U$ may be neglected, and one gets an expression for $k_j$ in the form:

$$k_j = \frac{2\pi I_j + 2\pi f + \sum_{\beta=1}^{M} \theta(t_\beta)}{L} \quad (5.22)$$

In an analogous way, from equation (4.21) one gets an equation, which does not depend on $k_j$:

$$N\theta(t_\beta) = 2\pi J_\beta + \sum_{\alpha=1}^{M} \theta((t_\beta - t_\alpha)/2), \quad (5.23)$$
This coincides with the equation obtained by Yu and Fowler [14], in the limit of large $U/t$. From the equation (5.23) we calculate the sum needed for the right hand side of equation (5.22). After the substitution into that equation one gets

$$k_j = \frac{2\pi}{L} (I_j + f + \sum_{\beta=1}^{M} J_\beta)$$

(5.24)

This expression was first obtained in the limit $U/t \to \infty$ by Kusmartsev [11], and then rederived by Yu and Fowler [14].

In our new derivation we have not made any assumption about the value of $U$. The ground state energy is here $1/N$ flux periodical function, where the ground state energy-flux dependence consists of parabolic curves with minima at the flux value $f_{\text{min}} = p/N$, where $p$ is the number of parabolic curves.

$$E_{\text{ground}} = -D \cos \left( \frac{2\pi}{L} (f - \frac{p}{N}) \right),$$

(5.25)

where flux value $f$ is changed in the region $(2p - 1)/2N < f < (2p + 1)/N$ and $D = 2\sin(\pi N/L)/\sin(\pi/L)$. Also from the above derivation it is clear that the parameter of our expansion is equal to

$$\alpha = \frac{Nt}{LU} = \rho t/U$$

(5.26)

where $\rho = N/L$ is the filling factor. Summarizing, for any fixed value of the ratio $t/U$, there exists a dilute density limit associated with $\alpha \ll 1$. In this dilute limit the conventional Aharonov-Bohm effect dissapears, and the fractional effect takes over. Let us note that for the almost completely polarized system, for example, when number of up-spins is much larger than the number of down-spins $M/N << 1$ the equation (5.23) may be also solved analytically. In that case one assumes that the value of $t_\beta$ in the equation (5.23) is small. This gives that
This looks like a spectrum of free spinless fermions on the chain with \(2N\) sites.

With the aid of the parameter defined above, one may find the first correction. Making an expansion in powers of the parameter \(\alpha\), and using the second Bethe equation (2.6), we get the form:

\[
N \theta(t_\beta) - \frac{8}{U} \frac{1}{(1 + t_\beta^2)} \sum_{j=1}^{N} \sin(k_j) = 2 \pi J_\beta + \sum_{\lambda=1}^{M} \theta((t_\beta - t_\alpha)/2),
\]

which may be reduced to the equation:

\[
N \theta(t_\beta) - \frac{4}{NU} \frac{1}{(1 + t_\beta^2)} \sum_{j=1}^{N} \sin(k_j)) = 2 \pi J_\beta + \sum_{\lambda=1}^{M} \theta((t_\beta - t_\alpha)/2),
\]

With the substitution \(x_\beta = t_\beta - \frac{4}{NU} \sum_{j=1}^{N} \sin(k_j)\) this equation is reduced to eq. (5.23), with unknown variables \(x_\beta\). The equation derived for the variables \(x_\alpha\) is independent of the flux \(f\) and the value of \(U\). It is just the equation for an isotropic Heisenberg antiferromagnet on the ring having \(2N\) sites and \(M\) down spins. The solution for \(x_\alpha\) is independent of the flux \(f\) or the value \(U\). However, the variable \(t_\beta\), expressed via \(x_\alpha\) with the aid of the formula:

\[
t_\beta = x_\beta + \frac{4}{NU} \sum_{j=1}^{N} \sin(k_j),
\]

does depend on both parameters: \(U\) or \(\alpha\) and \(f\). The first correction to \(t_\beta\) does not depend on the index \(\beta\). The dependence of \(t_\beta\) on the flux \(f\) comes about through its explicit dependence on the momenta \(k_j\) via the second term of the right hand side of the eq. (5.30).

Substituting the equation for variables \(t_\beta\) into the equation (5.22) for the momenta \(k_j\) and making an expansion using the parameter \(\alpha\), we get the form:

\[
Lk_j = 2\pi I_j + 2\pi f + \sum_{\beta=1}^{M} \theta(x_\beta) + \frac{8}{NU} \sum_{l=1}^{N} (1 - N\delta_{lj}) \sin k_l \sum_{\beta=1}^{M} \frac{1}{1 + t_\beta^2}
\]

\[
(5.31)
\]
For small parameter \( \alpha \) this system of linear equations may be solved, with the result:

\[
k_j = \frac{2\pi I_j}{L(1 + \frac{8B}{UL})} + \frac{2\pi f}{L} + \frac{2\pi}{NL} \sum_{\beta=1}^{M} J_\beta + \frac{2\pi}{L} \frac{8B}{(UL + 8B)} \sum_{l=1}^{N} I_l
\]

(5.32)

where \( B = \sum_{\beta=1}^{M} \frac{1}{1+x_\beta} \) is a real number. Taking this solution into account, the formula for the ground state energy takes on the form:

\[
E_{\text{ground}} = -\tilde{D} \cos \left( \frac{2\pi}{L} \left( f - \frac{p}{N} + \frac{8B}{(UL + 8B)} \sum_{l=1}^{N} I_l \right) \right)
\]

(5.33)

where \( p = -\sum_{\beta=1}^{M} J_\beta \), \( \tilde{D} = 2 \sin(\pi N/\tilde{L}) / \sin(\pi/\tilde{L}) \) and \( \tilde{L} = L(1 + \frac{8B}{UL}) \).

Here the values of quantum numbers \( x_\beta \) do not depend on the magnetic field. Precisely speaking, they do not change their values when the flux changes within a single parabola. The ground state energy will be associated with a new set of the quantum numbers \( J_\beta \). This means that with the first correction taken into account, these parabolic curves, which the ground state energy consists of, change their position mostly along the vertical axis but also slightly along the horizontal axis, but do not change in form. Thus, it is in this case, the quasi \( 1/N \) periodicity is preserved.

To conclude, our investigation sheds light on the case in which the ring contains many electrons in the limit of very dilute electron density. In the correlated state an effective phase shift appears between the momenta of the different electrons, a shift which is associated with the repulsive interaction. Because of the shift the periodicity of the Aharonov-Bohm flux may have a fractional value. In contrast this effect is not expected to occur for the negative \( U \) center model, where at best one will have only the half quantum flux periodicity.
VI. LOW VERSUS HIGH DENSITY LIMIT

There is, however, a correspondence between the states associated with the positive and negative $U$ values. The dilute density limit of electrons on the ring described by the Hubbard Hamiltonian with positive values of $U$ correspond to the high electron density limit, described by the same Hubbard model with negative values of $U$, with the aid of the transformation

$$c_{i,+} \rightarrow a_{i,+} \quad (6.34)$$

and

$$c_{i,-} \rightarrow a_{i,-} + i_{-} \quad (6.35)$$

the Hamiltonian (1) transforms into

$$H = -t \sum_i (a_{i,+}^\dagger a_{i+1,+} - a_{i,-}^\dagger a_{i+1,-}) + U \sum_i n_{i,+} - U \sum_i n_{i,+} n_{i,-} \quad (6.36)$$

Introducing an auxiliary field acting only on the spin-down electrons with a flux through the ring of $\Phi_- = \pi L$ the Hamiltonian (6.36) may be transformed to the form:

$$H = -t \sum_{i,\sigma} a_{i,+}^\dagger a_{i+1,+} + U \sum_i n_{i,+} - U \sum_i n_{i,+} n_{i,-}, \quad (6.37)$$

This is a negative $U$-center model in a magnetic field of strength $U$, creating the Zeeman term. Note that with this transformation the number of spin-up particles is not changed $N_{+,\text{new}} = N_+$, but the number of spin-down particles is equal to $N_{-,\text{new}} = L - N_-$. In other words, spin-down particles are equivalent to holes in the original Hamiltonian. Thus the spin-up particles are moving in a field of flux $\Phi_+ = f$ and spin-down particles are moving in a field of flux $\Phi_- = f + L\pi$. One sees also that the system with odd and even number
of sites will have different energy-flux dependence. One of them will be transformed into
the other one with a shift of a half flux quantum. Therefore, there is a parity effect here.
Let us now discuss the ring with an even number of sites. We have shown that electrons
on the ring, described by the Hubbard Hamiltonian with positive values of $U$ in low density
limit, behave in the same way as the high density electron case, described by the Hubbard
Hamiltonian with the negative values of $U$, provided that the system is highly polarized.

This comparison shows that the Zeeman energy term:

$$U = -\mu_B H \sum_{i=1}^L (n_{i+} - n_{i-})$$

(6.38)

where $H$ is the magnetic field associated with the flux quantum threading the loop, cannot be
simply dropped, and in some cases may give very interesting new physics, as, for example,
fractional Aharonov-Bohm periodicity for the model of negative $U$-centers. In the latter
case, one again has the fractional $1/N$ flux periodicity of the ground state energy and the
persistent current. This the number $N$ is equal to the sum of the number of spin-up particles
and the number of holes. Physically it is not clear why the fractional $1/N$ periodicity would
occur in that model. The detail investigation of the influence of the Zeeman energy on the
Aharonov-Bohm effect we postpone for a forthcoming paper, assuming for now that this
energy is small (i.e. the ring has a very large radius).

VII. PARITY EFFECTS ON A HUBBARD RING

For spinless fermions there is a difference in responses to a magnetic field for the cases
of even and odd number of particles on a ring [15]. This is the so-called parity effect. The
effect is practically unchanged if there is an interaction between these spinless fermions.
When the number of spinless fermions on the ring changes from odd to even, there is a
statistical half-flux quantum which shifts the energy-flux dependence by exactly half of the fundamental flux quantum. Therefore, for small values of the flux and at odd number of spinless fermions, the ring behaves as diamagnet. When there is an even number of particles it behaves a paramagnet. Kusmartsev obtained this result by exact solution with the aid of Bethe-ansatz, in the model of interacting spinless fermions on the ring \([31]\) \([15]\). This was also independently qualitatively discussed by Legett for general case (called as Legett conjecture) \([32]\) and was proven by Daniel Loss \([18]\), with the aid of bosonisation method in the framework of the same model \([31]\), \([15]\) but for arbitrary coupling. However taking spins into account, the situation is drastically changed.

Taking spins into account for noninteracting electrons gives the diamagnetic response only when there is \(N = 4n + 2\) particles on the ring, where \(n\) is an arbitrary integer. For all other cases the response has a paramagnetic character. With finite temperature and disorder there occurs a double parity effect, in which, for \(N = 4n + 1\), and \(N = 4n\), the response is paramagnetic; and for \(N = 4n + 1\) and \(N = 4n + 2\) the response is diamagnetic \([30]\).

With the inclusion of the Hubbard interaction there appears an additional phase shift due to scattering of a given particle on the other particles, via two-particles interactions. Each scattering event gives a phase shift \(\theta(x)\) in the Bethe equations. For the case of spinless fermions, the parity effect is conserved, in spite of the appearance of the new phases \([15]\).

However, with the Hubbard interaction, one has a different picture. The analytical solution shows that the phase shift \(\theta(x)\) creates a quasi-half flux periodicity, which improves when the parameter \(\alpha\) becomes smaller and smaller. The interaction creates an additional to the statistical flux which appears between the two electrons.

It is interesting that in the limit \(U \to \infty\) this phase shift is exactly equal to half-flux
quantum. Therefore if the flux of the external magnetic field is equal to a half-flux quantum, then, with that additional statistical half-flux quantum, the total flux is equal to the unit of fundamental flux quantum.

In comparison with the case of noninteracting electrons, where the periodicity is in units of flux quantum, here we already have a periodicity at half a flux quantum. For the three electron case the phase shift is different. The additional statistical flux arises on counting one permutation with each of other electrons. The value of that phase can be estimated in the limit of $\alpha \to 0$ and equals

$$2\pi f_{\text{stat}} = \sum_{\beta=1}^{M} \theta(t_{\beta}) = 2\pi \sum_{\alpha} J_{\alpha}$$

where we get a fraction $f_{\text{stat}} = \frac{1}{N}$. In this case one may think that this flux is attached to each electron, that is all $N$ electron share 1 unit of the flux quantum. Putting a new electron on the ring creates a new system, where $N + 1$ electrons will now share a unit of quantum flux. In this system, the response has a purely diamagnetic character, for any number of electrons.

In general terms, the appearance of the parity effect and conventional Aharonov-Bohm effect may be described as follows:

At small values of $U$, or more precisely speaking large $\alpha$, we have the conventional parity effect for free electrons (see, Table II). With an increase of the interaction there exists the critical value of $U = U_{\text{cr1}}$ or $\alpha = \alpha_{\text{cr1}}$ where, for values $U > U_{\text{cr1}}$ or ($\alpha < \alpha_{\text{cr1}}$) the parity effect looks similar to the parity effect for spinless electrons. That is, for even number of electrons the magnetic response has a diamagnetic character and for an odd number of electrons the response is paramagnetic. Note that for spinless fermions the response is diamagnetic for an odd number of electrons. With further increase of the coupling constant
$U$ there is a second critical value, at $U = U_2$ ($\alpha = \alpha_2$), with the new type of parity effect. Thus for $U > U_2$ ($\alpha < \alpha_2$) the paramagnetic response occurs only for $N = 4n + 1$ electrons. Finally, for $U > U_3$, so that $U_3 > U_2$, the parity effect disappears and the ring behave as diamagnet for any number of electrons. The value of $U_3$ depends on the electron density. As discussed above, the parameter $\alpha$ is what matters. From our investigation we may conclude that the critical value for the disappearance of the parity effect equals $\alpha_3 \sim 0.02$ With the disappearance of the parity effect the quasi-$1/N$– fractional Aharonov-Bohm periodicity will appear. The classification of the different parity regimes are shown in the Table II.

TABLE II. The table shows the classification of different regimes of the parity effect, with change of the Hubbard interaction. The notations $\text{dia-}$ means that the ring behaves as diamagnet, the notations $\text{para-}$ means that the ring behaves as a paramagnet. The number of electrons on the ring is equal to $4n + 2, 4n + 1, 4n, 4n - 1$, respectively.

| Particle’s Number | $U < U_{c1}$ | $U_{c1} < U < U_{c2}$ | $U_{c2} < U < U_{c3}$ | $U_{c3} < U$ |
|-------------------|---------------|-------------------------|-------------------------|--------------|
| $4n + 2$          | $\text{dia-}$ | $\text{dia-}$           | $\text{dia-}$           | $\text{dia-}$ |
| $4n + 1$          | $\text{para-}$ | $\text{para-}$          | $\text{para-}$          | $\text{dia-}$ |
| $4n$              | $\text{para-}$ | $\text{dia-}$           | $\text{dia-}$           | $\text{dia-}$ |
| $4n - 1$          | $\text{para-}$ | $\text{para-}$          | $\text{dia-}$           | $\text{dia-}$ |
The parity effect is preserved with disorder or finite temperature. However the change of this classification with the change of the temperature is nontrivial and will be discussed in a forthcoming paper.

VIII. POSSIBLE EXPERIMENTS

It is worth noting that there is a possible practical realization, where the above model may be applicable; that is the case of a ring consisting of quantum dots, placed in succession. These structures may be built and investigated using modern technology \cite{33}. The single quantum dot will act as a potential well for the electrons. If the radius $R$ of the quantum dot decreases, the charging energy $e^2/R$ increases and there occurs a case in which no more than two electrons with opposite spin can be accommodated on a single dot. The system of quantum dots is to be described with a Hubbard hamiltonian with $U = e^2/R$. Therefore in a ring consisting of quantum dots, with high charging energy, one may observe the destruction of the simple Aharonov-Bohm effect and the appearance of the fractional period.

The effect of the fractional or $1/N$ periodicity is directly related to the phenomena of Coulomb Blockade. The novel feature of both these phenomena is due to many body effects associated with the interacting current carriers. That is, the motion or a change of state of a single electron, changes the states of all other electrons.

If we put the ring of quantum dots in a transverse magnetic field single electrons will have the tendency to move along the ring. However hops onto a dot already containing two electrons is not allowed, since this will cost the energy equal to the charging energy. One may have incoherent or independent hops. However, there is another possibility. This is a process which will not cost the charging energy, in which all electrons on the ring make
coherent hops (cotunneling). In other words this is the simultaneous motion of all electrons on the ring. It is clear that if we move all electrons together the charging energy is not important ($U$ may be arbitrarily large) and the total change in the phase of the many-body wave function will be equal to $2\pi fN$.

From the gauge invariance of the ground state of the ring we conclude that the equivalent state to $f = 0$ is $Nf = 1$. Hence the period of interacting electrons on the ring is $f = 1/N$, in agreement with the results obtained in this paper.

**IX. CONCLUSIONS**

In the present work we have studied the effects of the electron-electron correlations on the Aharonov-Bohm effect in a quantum ring. We have shown the correlations result in a fractional Aharonov-Bohm effect, which appears when the parameter $\alpha = Nt/LU$ is small. This case may occur when $U/t$ is large or in low density limit when the filling $N/L$ is small. The conclusion that the low density limit of the Hubbard model is equivalent to a strong coupling limit $U/t >> 1$ coincides with one obtained by Shulz [34], a work which describes Luttinger liquid properties in the framework of the bosonisation approach. Using the Aharonov-Bohm effect we proved this theorem far beyond the low-frequency limit of that theory.

We have found also a very interesting scaling symmetry, hold for the low density limit, namely, that the shape of the ground and excited flux energy dependencies depends only on the parameter $\alpha = \frac{tN}{UL}$. In other words when $\alpha << 1$ the flux-energy dependencies obtained for the different values $U$ and $L$, provided that the parameter $\alpha = const$ is fixed, can be transformed one into another by a scaling transformation of the energy scale. This confirms
the Lieb suggestion \cite{10} that the Hubbard Hamiltonian describing a system of the finite size has some very nontrivial internal symmetries.

The density of electrons may be well controlled in many experimental situations, for example, by doping, or by applying the gate voltage to change of the position of the chemical potential as it is used in quantum wells. Therefore the predicted fractional Aharonov-Bohm effect is a good challenge for experimentalists.

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Figure Captions

**Fig.1** The ground state energy dependence on the flux of the external field for the Hubbard ring at the fixed value of $U$ at the different values of $L$. Here $U = 50t$ and $L$ is 3, 4 or 6 sites. ◇ is for $L = 3$, + is for $L = 4$ and □ is for $L = 6$. Ground state energy at zero flux is subtracted off so as to normalize the figures. Energies are in units of $t$ and flux is in units of quantum flux.

**Fig.2** The ground state energy versus external field flux for the Hubbard ring at the fixed value of $L$ and different values of $U$. Here $L$ is 5 sites and $U = 5t, 20t, 50t$. ◇ is for $U = 5t$, + is for $U = 20t$ and □ is for $U = 50t$.

**Fig.3.** The same as in Fig1., but for the case of $U$–center pairing. Energies are in units of $t$. Here $U = -10t$ and $L$ takes on values of 3, 5 and 8 sites. ◇ is for 3 sites, + is for 5 sites and □ is for 8 sites. Ground state energy at zero flux has been subtracted off to normalize the curves. Flux is in units of the quantum flux.

**Fig.4** The same as in Fig2., but for the case of $U$–center pairing. Here $L$ is kept fixed at 5 sites but $U = -1t, -5t or -20t$. ◇ is for $U = -t$, + is for $U = -5t$ and □ is for $U = -20t$.

**Fig.5** The behavior of the ground state energy and the first excited levels as a function of flux for three electrons at the values $L = 4, N = 3$ and $U = 50$ in the region of flux within the single fundamental flux quantum. Two particles have up-spins and one particle has down-spin.

**Fig.6** The same as in Fig.5 but at the value $L = 6$. At the cuspidal point there appears the flux absorption.

**Fig.7** The same as in Fig.5 but at the value $L = 6$ and $U = 8$. 

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Fig. 8  The same as in Fig.5 but at the value $L = 12$ and $U = 8$.

Fig. 9  The same as in Fig.5 but at the value $L = 128$ and $U = 1$. The energy is expressed in the units $t10^3$. The zero energy corresponds to $-2.99t$.

Fig. 10  The same as in Fig.5 but at the value $L = 256$ and $U = 0.5$. One sees that the shape of this dependence is scaled to the one shown in Fig.9. The energy is expressed in the units $t10^4$. The zero energy corresponds to $-2.999t$. 