The Noether symmetry approach in a ‘cosmic triad’ vector field scenario

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Abstract
To realize the accelerations in the early and late periods of our universe, we need to specify potentials for the dominant fields. In this paper, by using the Noether symmetry approach, we try to find suitable potentials in the ‘cosmic triad’ vector field scenario. Because the equation of state parameter of dark energy has been constrained in the range of $-1.21 \leq \omega \leq -0.89$ by observations, we derive the Noether conditions for the vector field in quintessence, phantom and quintom models, respectively. In the first two cases, constant potential solutions have been obtained. What is more, a fast decaying point-like solution with power-law potential is also found for the vector field in the quintessence model. For the quintom case, we find an interesting constraint $\tilde{C} V'_p = -C V'_q$ on the field potentials, where $C$ and $\tilde{C}$ are constants related to the Noether symmetry.

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1. Introduction

The inflationary paradigm was used as a resolution of the problems, such as horizon and flatness problems in standard cosmology [1]. The dark energy scenario was proposed to explain the current accelerating expansion of the universe found by the type Ia supernova observations [2]. Various candidates are suggested to explain the early and current accelerations. Assuming that a scalar field provides the driving force of the acceleration is the most popular and economic explanation. However, vector field scenarios are workable as well, which also have the advantage that their basic particle is present in nature. The vector field inflationary scenario was firstly proposed by Ford with the characteristic of a natural large-scale anisotropy [3]. Nevertheless, now, at least three methods [4] could solve the anisotropic problem: they are the time-like vector field scenario [5], the ‘cosmic triad’ vector field scenario [6] and the
‘N-flation’ vector field model [7]. As important applications in the early universe, the vector field can be used in inflation [8], used as a curvaton [9], and the stability is also discussed [10]. What is more, the vector field is one of the p-form fields [11] and can be identified as the electromagnetic field [12]. The vector field scenario is a worthy topic to work on.

However, these paradigms could not be satisfactorily established without considering their connection with a fundamental theory. Therefore, we have to face the problem of choosing suitable potentials from fundamental physics. The Noether symmetry approach has been revealed as a useful tool to find out exact solutions in cosmology. It is also an effective method to select models motivated at a more fundamental level. By choosing the constant of motion, the Noether symmetry reduces the dynamical system. In most cases, results are integrable because of the conserved quantities. Results of previous works, which concern the scalar field cosmology [13–15] and the \( f(R) \) cosmology [16], are encouraging. Then, it is natural to ask what potential will be obtained in the vector field scenario by applying the Noether symmetry approach. In the present paper, we try to give an answer. First of all, among the above three vector field scenarios, we choose the ‘cosmic triad’ model to discuss, which is easy to deal with.

Observations suggest that the dark energy equation of state (EoS) parameter is in the range of \(-1.21 \leq \omega \leq -0.89\) [17] which has a possibility of crossing the phantom divider \(\omega = -1\). Although the phantom type of matter with negative kinetic energy has well-known problems, it was implicitly suggested by astronomical observations and has also been widely studied as dark energy. It is phenomenologically significant and worthy of putting other theoretical difficulties aside temporarily. Therefore, for a complete description, corresponding to the classification of the scalar fields, we discuss three types of matter in the ‘cosmic triad’ vector field scenario, which are the quintessence type of vector fields with positive kinetic terms, the phantom type of vector fields with negative kinetic terms and the quintom type of vector fields with both positive and negative kinetic terms3.

The paper is organized as follows. In section 2, we introduce the ‘cosmic triad’ vector field model. In section 3, we introduce the Noether symmetry approach, and apply it to both quintessence and phantom cases. We perform the change of variables and obtain the solutions. In section 4, we discuss the application of the Noether symmetry approach in the vector field quintom case. In section 5, we draw our conclusions.

2. ‘Cosmic triad’ scenario

Based on observations, we assume that the geometry of spacetime is described by the flat Friedmann–Robertson–Walker (FRW) metric

\[
dx^2 = -dt^2 + a^2(t) \sum_{i=1}^{3} (dx^i)^2,
\]

where \(a\) is the scale factor. Meanwhile, as we consider both the vector fields and the dust matter in the system, the total action is

\[
S_{\text{tot}} = S_A + S_m.
\]

where \(S_A\) is the action for vector fields and \(S_m\) is the action for dust matter. The density of dust matter can be expressed as \(\rho_m = \rho_{m0}(a_0/a)^\gamma\), where \(\rho_{m0}\) is an initial constant and \(0 < \gamma \leq 2\).

3 The scalar field quintessence with a positive kinetic term is proposed by [18]; the scalar field phantom with a negative kinetic term is suggested by [19] and the quintom with both positive and negative kinetic terms is proposed by [20]. And with the help of modified gravity, a lot of models could cross \(\omega = -1\) as well, for example in [21].
Here, we limit our analysis to $\gamma = 1$ which corresponds to the pressureless dust matter with $P_m = 0$.

The ‘cosmic triad’ model [6], as a realistic vector field scenario, which can be derived from a gauge theory with the $SU(2)$ or $SO(3)$ gauge group, is proved to be compatible with the background metric, with the following action:

$$S_{A1} = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} + \sum_{a=1}^{3} \left( \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - V(A^{a2}) \right) \right],$$

(3)

where Latin indices label the gauge fields ($a, b = 1, 2, \ldots$), and Greek indices label the different spacetime components ($\mu, \nu = 1, 2, \ldots$). $F_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ is the field strength, $A^a_\mu$ is the vector field and $A^{a2} = g^{\mu\nu} A^a_\mu A^a_\nu$. For the purpose of describing both the positive and negative kinetic terms, we put the parameter $\epsilon$ in action. When $\epsilon = 1$, the kinetic term of the ‘cosmic triad’ is positive, and one has the vector field quintessence case. When $\epsilon = -1$, the kinetic term of the ‘cosmic triad’ is negative, and then one has the vector field phantom case. The action of the vector field quintom is

$$S_{A2} = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} + \sum_{a=1}^{3} \left( \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - V_q(A^{a2}) \right) + \sum_{b=1}^{3} \left( -\frac{1}{4} F_{\mu\nu}^b F^{b\mu\nu} - V_p(A^{b2}) \right) \right],$$

(4)

where the subscript $q$ denotes quintessence and the subscript $p$ denotes phantom.

Following the assumption in [6], the ‘cosmic triad’ scenario becomes compatible with spatial isotropy by using the assumptions below for the vector fields. For the quintessence and phantom cases,

$$A^b_\mu = \delta^b_\mu A(t) \cdot a,$$

(5)

and for action (4) which notifies the quintom, it is

$$A^{b\mu}_p = \delta^b_\mu A_p(t) \cdot a, \quad A^{b\mu}_q = \delta^b_\mu A_q(t) \cdot a,$$

(6)

where $A, A_p, A_q$ are scalars. The three kinds of vector fields with the same kinetic terms point in three mutually orthogonal spatial directions and share the same time-dependent length. It turns out that this scenario is able to drive a stage of accelerating expansion in the universe and exhibits tracking attractors that render cosmic evolution insensitive to initial conditions [6].

3. The Noether symmetry approach in quintessence and phantom cases

As discussed in the introduction, we are looking for solutions induced by symmetries. The Noether symmetry approach is such a powerful tool that it can find the solution for a given Lagrangian. This method looks for the related cyclic variables and consequently reduces the dynamics of the system to a manageable one. In vector field quintessence and phantom cases, we treat action (3) as a dynamical system in which the scale factor $a$ and the scalar field $A$ play the role of independent dynamical variables. Then, a configuration space $Q = (a, A)$ may be considered. To study the related symmetries, we need an effective point-like Lagrangian for the model whose variation with respect to the dynamical variables yields the correct equations of motion. Based on action (3), the point-like Lagrangian takes such a form:

$$L_1 = L_{A1} + L_m = 3a\dot{a}^2 - \frac{3}{m^2_{pl}} \left( \epsilon a^2 A^2 + a A^2 \dot{a}^2 + 2a^2 \ddot{A} \dot{A} - a^3 V(A^2) \right) + \frac{\rho_m}{m^2_{pl}},$$

(7)

where $m^2_{pl} = (8\pi G)^{-1}$ is the Planck mass.
The ‘energy function’ associated with $\mathcal{L}_1$ is
\[
E_{\mathcal{L}_1} = \left. \frac{\partial \mathcal{L}_1}{\partial q_i} \right|_{q_i = \alpha} - \mathcal{L}_1 = 3a^3 \left( \frac{\epsilon (A + HA)^2}{2} + V(A^2) - \frac{\rho_{m0}a^{-3}}{3} - m_{pl}^2 H^2 \right),
\]
where $q_i$ is the variable $a$ or $A$ in the configuration space separately. If we consider the vanishing of the ‘energy function’ as a constraint, we obtain the Friedmann equation
\[
H^2 = \frac{1}{m_{pl}^2} \left( \frac{\epsilon (A + HA)^2}{2} + V(A^2) + \frac{\rho_{m0}a^{-3}}{3} \right),
\]
and the Euler–Lagrange equations associated with $\mathcal{L}_1$ are
\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}_1}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}_1}{\partial q_i} = 0.
\]
When $q_i = a$, the explicit form of the equation can be written as
\[
\dot{H} = -\frac{[2\epsilon (A + HA)^2 + V(A)]}{2m_{pl}^2},
\]
which is called the Raychaudhuri equation, where $V' = dV/dA$. When $q_i = A$, the Euler–Lagrange equation is
\[
\dot{A} + 3H \dot{A} + (2H^2 + \dot{H})A + \epsilon V = 0.
\]

In the present paper, as both the positive and negative kinetic terms are discussed, it is natural and necessary to concern the EoS parameter which is related to the sign of the kinetic terms tightly. From action (3), we can give out the energy density and the pressure of the vector fields:
\[
\rho_A = \frac{3\epsilon}{2} (A + HA)^2 + 3V(A^2),
\]
\[
P_A = \frac{\epsilon}{2} (A + HA)^2 - 3V(A^2) + V'A,
\]
and then we can write the EoS parameter as
\[
\omega_A = \frac{P_A}{\rho_A} = \frac{\epsilon (A + HA)^2/2 - 3V(A^2) + V'A}{3\epsilon (A + HA)^2/2 + 3V(A^2)}.
\]
When $\epsilon = 1$, $\omega_A < -1$ requires $2(A + HA)^2 + V'A < 0$, and the potential must be ‘tachyonic’ which means a negative $V'$. When $\epsilon = -1$, $\omega_A < -1$ requires that $-2(A + HA)^2 + V'A < 0$, we do not need to make $V'$ smaller than 0. However, the vector field scenario is different from the scalar field case, which just needs $\phi^2 < 0$ if asking $\omega_{\phi} < -1$. Furthermore, the vector field quintessence and phantom allow that one crosses $\omega_A = -1$.

The Noether symmetry approach consists in considering the two equations (11) and (12) as a second-order dynamical system with the following vector field which is an infinitesimal generator of a point transformation on the configuration space $Q = (a, A)$
\[
X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial A} + \dot{a} \frac{\partial}{\partial \dot{a}} + \dot{A} \frac{\partial}{\partial \dot{A}},
\]
where $\alpha$ and $\beta$ are generic functions of $a$ and $A$. And the tangent space for the related bundle is $TQ = (a, A, \dot{a}, \dot{A})$. $X$ can be treated as a vector field on $T^2$, which is tangent bundle of $\mathbb{R}^2$, with natural coordinates $(a, A, \dot{a}, \dot{A})$. The Lagrangian is invariant under the transformation $X$ if
\[
L_X \mathcal{L}_1 = \alpha \frac{\partial \mathcal{L}_1}{\partial a} + \frac{\partial \mathcal{L}_1}{\partial t} \frac{\partial a}{\partial \dot{a}} + \beta \frac{\partial \mathcal{L}_1}{\partial A} + \frac{\partial \mathcal{L}_1}{\partial t} \frac{\partial A}{\partial \dot{A}} = 0,
\]
where \( L_X \) stands for the Lie derivative with respect to \( X \). We claim that the dynamical system has Noether symmetries when \( L_X \mathcal{L}_1 = 0 \). In this way, the transformation on the base space can preserve the second-order character of the dynamical field, with an explicit expression as

\[
\frac{a^2}{m_{\text{pl}}^2} (3\alpha V + a\beta V') - A^2 \epsilon a^2 \left( \frac{\partial \alpha}{\partial A} + \frac{\partial \beta}{\partial A} + \frac{3\alpha}{2} \right)
\]

\[-\dot{\alpha}^2 \left( \frac{\epsilon A^2}{2m_{\text{pl}}^2} a - \alpha + \frac{\epsilon a A^2}{2m_{\text{pl}}^2} \frac{\partial \alpha}{\partial a} - 2a \frac{\partial a}{\partial a} + \frac{\epsilon a A}{m_{\text{pl}}^2} \beta + \frac{\epsilon a^2 A}{m_{\text{pl}}^2} \frac{\partial \beta}{\partial a} \right)
\]

\[-a \dot{\alpha} \dot{A} \left( \frac{2\epsilon A}{2m_{\text{pl}}^2} \alpha - \frac{\epsilon a A}{2m_{\text{pl}}^2} \frac{\partial \alpha}{\partial a} + \frac{\epsilon A^2}{2m_{\text{pl}}^2} \frac{\partial \alpha}{\partial A} + \frac{\epsilon a^2 A}{2m_{\text{pl}}^2} \frac{\partial \beta}{\partial a} + \frac{\epsilon a A}{2m_{\text{pl}}^2} \frac{\partial \beta}{\partial A} \right) = 0. \tag{18}
\]

It gives a quadratic polynomial in terms of \( \dot{a} \) and \( \dot{A} \), whose coefficients are partial derivatives of \( \alpha \) and \( \beta \) with respect to the configuration variables \( a \) and \( A \). Thus the resulting expression is equal to zero if and only if these coefficients are zero:

\[
3\alpha V + a\beta V' = 0, \tag{19}
\]

\[
2A \frac{\partial \alpha}{\partial A} + 2a \frac{\partial \beta}{\partial A} + 3\alpha = 0, \tag{20}
\]

\[
\frac{\epsilon A^2}{2m_{\text{pl}}^2} \alpha - \alpha + \frac{\epsilon a A^2}{2m_{\text{pl}}^2} \frac{\partial \alpha}{\partial a} - 2a \frac{\partial a}{\partial a} + \frac{\epsilon a A}{m_{\text{pl}}^2} \beta + \frac{\epsilon a^2 A}{m_{\text{pl}}^2} \frac{\partial \beta}{\partial a} = 0, \tag{21}
\]

\[
2\epsilon A \frac{\partial A}{m_{\text{pl}}^2} + a \frac{\partial \beta}{m_{\text{pl}}^2} - 2a \frac{\partial a}{m_{\text{pl}}^2} + \frac{\epsilon a^2 A}{m_{\text{pl}}^2} \frac{\partial \beta}{\partial a} + \frac{\epsilon a A}{m_{\text{pl}}^2} \frac{\partial \beta}{\partial A} = 0. \tag{22}
\]

The last two equations can be simplified by using equation (20), which are reduced to

\[
\frac{a \beta}{m_{\text{pl}}^2} + \frac{a^2}{m_{\text{pl}}^2} \frac{\partial \beta}{\partial a} = \frac{\partial \alpha}{\partial A} \frac{\partial A}{\partial a}, \tag{23}
\]

\[-\alpha + a \frac{\partial \alpha}{m_{\text{pl}}^2} \frac{\partial a}{\partial a} + 2a \frac{\partial a}{\partial a} + 2A \frac{\partial a}{\partial A} = 0. \tag{24}
\]

In the present paper, we will use equations (19), (20), (23), (24) as the Noether conditions with implicit symmetries. Particularly speaking, equation (24) is a partial differential equation for \( \alpha \). We can find two solutions for this equation, which are \( \alpha = 0 \) and \( \alpha = C_2 A^{1/2} \) [22]. In the following subsections, combined with other Noether conditions, we will discuss the two corresponding full solutions in detail. Before that, we discuss the constants of motion firstly.

Following [13], the Noether conditions select constants of motion. The existence of a Noether symmetry in the model reduces the dynamics through cyclic variables. First, we have to define the conjugate momenta

\[
p_{\dot{q}_i} = \frac{\partial \mathcal{L}_1}{\partial \dot{q}_i}, \tag{25}
\]

whose explicit expressions with different variables are

\[
p_a = \frac{\partial \mathcal{L}_1}{\partial \dot{a}} = 3 \left( 2a \dot{a} - \epsilon a^2 \dot{a} + a^2 A \ddot{A} \right), \tag{26}
\]

\[
p_A = \frac{\partial \mathcal{L}_1}{\partial \dot{A}} = \frac{3 \epsilon a^3}{m_{\text{pl}}^2} (\dot{A} + H A). \tag{27}
\]
The equations of motion indicate \( \frac{\partial L_1}{\partial q_i} = \frac{dp_i}{dt} \), combined with equation (17), which gives

\[
L_X L_1 = \frac{d}{dt} (\alpha p_a + \beta p_A) = 0.
\]

(28)

Then, the required Noether constant of motion is deduced:

\[
\alpha p_a + \beta p_A = Q = \mu_0,
\]

(29)

where \( Q \) is the conserved charge with unclear physical meaning, and \( \mu_0 \) is the corresponding constant. In other words, a symmetry exists if at least one of the functions \( \alpha \) or \( \beta \) is different from zero. As byproducts, the constant of motion will be given out and the form of \( V(A^2) \) is determined in correspondence to such a symmetry.

3.1. Solution one: the constant potential

The simplest solution to equation (24) is \( \alpha = 0 \). And from the other Noether symmetry conditions, the complete solution is obtained:

\[
\alpha = 0, \quad \beta = C_1 a^{-1}, \quad V = V_0,
\]

(30)

where \( C_1 \) and \( V_0 \) are constants of integration. This solution is equivalent to the massless vector field plus a cosmological constant, and the solution is gauge invariant. Such a solution exists in the scalar field case as well. However, the difference between the scalar field and vector field is that the value of \( \beta \) is a constant in the scalar field scenario, while it varies as \( a^{-1} \) in the vector field scenario. The solution indicates that the cyclic coordinate is \( a \), and the constant of motion can be expressed as

\[
-\frac{3C_1}{m^2_{\text{pl}}} (a^3 \dot{A} + a^2 \dot{a} A) = Q = \mu_0,
\]

(31)

which can be simplified as

\[
(a A) = a (\dot{A} + HA) = \frac{\bar{\mu}_0}{a} = \frac{\mu_0 m^2_{\text{pl}}}{-3C_1 a},
\]

(32)

When \( \bar{\mu}_0 = 0 \), equation (32) shows that the kinetic energy of the vector field is zero, and the scalar field \( A(t) \) evolves as \( A \propto a^{-1} \). So the solution with \( \bar{\mu}_0 = 0 \) constrains the model to just a cosmological constant-type model.

When \( \bar{\mu}_0 \neq 0 \), according to equation (32), we obtain \( (\dot{A} + HA)^2 \propto 1/a^4 \). The kinetic energy of the vector field behaves as a radiation field, so the solution constrains the model to a radiation plus cosmological constant. For a viable dark energy or inflation model, the potential energy must dominate, and the model again behaves like the cosmological constant model.

Since \( V = V_0, V' = 0 \), the EoS parameter is

\[
\omega_A = \frac{\epsilon (\dot{A} + HA)^2/2 - 3V(A^2)}{3\epsilon (\dot{A} + HA)^2/2 + 3V(A^2)}.
\]

(33)

If \( \bar{\mu}_0 \neq 0 \), in the quintessence case, \( \epsilon = 1, \omega_A \geq -1 \), and in the phantom case, \( \epsilon = -1, \omega_A \leq -1 \). In both cases, the EoS parameter \( \omega_A \) is close to \(-1\) and it does not cross the phantom divider \( \omega_A = -1 \). That is the reason why we need to consider quintessence, phantom and quintom cases separately when the Noether symmetry is applied.
3.2. Solution two: the point-like solution

There is another solution which satisfies the Noether conditions:

\[ A^2 = 2\epsilon m^2_{pl}, \quad \alpha = C_2 A^{1/2}, \quad \beta = -\frac{4 C_2 A^{3/2}}{3a}, \quad V = V_0 \left( \frac{A^2}{m^2_{pl}} \right)^{9/8}, \tag{34} \]

where \( C_2 \) and \( V_0 \) are constants. For the phantom case where \( \epsilon = -1, A^2 = -2m^2_{pl}, \) the value of the fields is not physical, so the solution only applies to the quintessence case. In other words, the Noether symmetry may be used to select models. The value of the vector fields are constants; that is why we call it the point-like solution. In this solution, the potential has a power-law form.

Based on equation (34), the Noether constant of motion is

\[ 3C_2 A^{1/2} \left( 2\dot{a} - \frac{a A^2 \dot{\dot{a}} + a^2 A \ddot{A}}{m^2_{pl}} \right) + \frac{4C_2 A^{3/2}}{am^2_{pl}}(a^3 \dot{A} + a^2 A \ddot{a}) = Q = \mu_0. \tag{35} \]

When \( A^2 = 2m^2_{pl} \) and the value of the vector field keeps as constant \( \dot{A} = 0, \) the above equation can be simplified as

\[ 2A^{1/2} H = \frac{\bar{\mu}_0}{a^2}, \tag{36} \]

where \( \bar{\mu}_0 = \mu_0/3C_2. \) The solution is \( a(t) \propto t^{1/2}, \) which describes the radiation era. Therefore, this solution is unable to explain the current accelerating expansion.

In summary, for the quintessence and phantom cases, the ‘cosmic triad’ vector field scenario with Noether symmetry behaves effectively like the cosmological constant model.

4. The Noether symmetry approach in the quintom case

In this section, we are going to apply the Noether symmetry approach to the quintom case. We look for the solution which crosses the phantom divider \( \omega_A = -1. \) Based on action (4), we can obtain the point-like Lagrangian

\[ L_{A2} = 3a \dot{a}^2 - \frac{3}{m^2_{pl}} \left( \frac{a^3 A_q^2 + a A_q^2 \dot{a}^2 + 2a^2 A_q A_q \ddot{A}_q - a^3 V_q^2}{2} \right) \]

\[ + \frac{3}{m^2_{pl}} \left( \frac{a^3 A_p^2 + a A_p^2 \dot{a}^2 + 2a^2 A_p A_p \ddot{A}_p - a^3 V_p}{2} \right). \tag{37} \]

The energy density and pressure for the vector field are

\[ \rho_A = \frac{3}{2} \left( A_q + HA_q \right)^2 + 3V_q - \frac{3}{2} (A_p + HA_p)^2 + 3V_p, \tag{38} \]

\[ P_A = \frac{1}{2} \left( A_q + HA_q \right)^2 - 3V_q + \frac{dV_q}{dA_q} A_q - \frac{1}{2} (A_p + HA_p)^2 - 3V_p + \frac{dV_p}{dA_p} A_p, \tag{39} \]

and the equations of motion for \( A_q \) and \( A_p \) are

\[ \dot{A}_q + 3H \dot{A}_q + (2H^2 + \dot{H})A_q + V_q' = 0, \tag{40} \]

\[ \dot{A}_p + 3H \dot{A}_p + (2H^2 + \dot{H})A_p - V_p' = 0. \tag{41} \]
The configuration space is \( Q = (a, A_q, A_p) \), and the generator of the Noether symmetry is
\[
\tilde{X} = \tilde{\alpha} \frac{\partial}{\partial a} + \tilde{\beta} \frac{\partial}{\partial A_q} + \gamma \frac{\partial}{\partial A_p} + \dot{\tilde{\alpha}} \frac{\partial}{\partial \dot{a}} + \dot{\tilde{\beta}} \frac{\partial}{\partial \dot{A_q}} + \dot{\gamma} \frac{\partial}{\partial \dot{A_p}},
\]
(42)
where \( \tilde{\alpha}, \tilde{\beta}, \) and \( \gamma \) are the generic functions of \( a, A_p, A_q \). The Noether symmetry requires
\[
L_\tilde{X}(L_{\mathcal{A}_2} + L_M) = 0,
\]
so the Noether conditions are
\[
3\tilde{\alpha}(V + \dot{V}') + a\tilde{\beta}V_p' + a\gamma V_q' = 0,
\]
(43)
\[
2A_q \frac{\partial \tilde{\alpha}}{\partial A_q} + 2a \frac{\partial \tilde{\beta}}{\partial A_q} + 3\tilde{\alpha} = 0,
\]
(44)
\[
2A_p \frac{\partial \tilde{\alpha}}{\partial A_p} + 2a \frac{\partial \gamma}{\partial A_p} + 3\tilde{\alpha} = 0,
\]
(45)
\[
\frac{2A_q}{m_{pl}^2} \tilde{\alpha} - 2 \frac{\partial \tilde{\alpha}}{\partial A_q} + \frac{A_q^2}{m_{pl}^2} \frac{\partial \tilde{\alpha}}{\partial A_q} + \frac{a^2 \partial \tilde{\beta}}{m_{pl}^2} + aA_q \frac{\partial \tilde{\beta}}{\partial A_q} = 0,
\]
(46)
\[
\frac{2A_p}{m_{pl}^2} \tilde{\beta} + 2 \frac{\partial \tilde{\beta}}{\partial A_p} + \frac{A_p^2}{m_{pl}^2} \frac{\partial \tilde{\beta}}{\partial A_p} = 0,
\]
(47)
\[
\frac{A_q^2}{2m_{pl}^2} - \tilde{\alpha} - 2 \frac{\partial \tilde{\alpha}}{\partial a} + aA_q \frac{\partial \tilde{\beta}}{\partial A_q} = 0,
\]
(48)
The above equations have an obvious solution which is
\[
\tilde{\alpha} = 0, \quad \tilde{\beta} = C a^{-1}, \quad \gamma = \tilde{C} a^{-1},
\]
(49)
\[
\tilde{C} V_p' = -C V_q',
\]
(50)
where \( C \) and \( \tilde{C} \) are both constants of integration, and at least one of the three parameters \( \tilde{\alpha}, \tilde{\beta}, \gamma \) is not zero. In the following, according to the values of \( C, \tilde{C} \) and \( \mu_0 \), we will discuss the various solutions.

4.1. \( C = 0 \) and \( \tilde{C} \neq 0 \)

In the case of \( C = 0 \) and \( \tilde{C} \neq 0 \), we obtain \( V_p' = 0 \), and the conserved charge is
\[
Q = \tilde{\alpha} p_a + \tilde{\beta} p_{A_q} + \gamma p_{A_p} = \tilde{C} \frac{3a^2}{m_{pl}^2} (A_p + A_p H) = \mu_0;
\]
both \( a \) and \( A_q \) are cyclic variables. When \( \mu_0 \neq 0 \), the above equations show that the phantom-type field is the same as that from the constant potential solution, while the quintessence-type field is free of the constraint. This solution allows that one crosses the phantom divider \( \omega_A = -1 \). When \( \mu_0 = 0 \), the solution constrains the model to just a quintessence field plus the cosmological constant. This solution does not allow that one crosses the phantom divider \( \omega_A = -1 \). The case \( \tilde{C} = 0 \) and \( C \neq 0 \) can be treated exactly in the same way. And the results are similar, except for the case when the role of quintessence field is replaced by the phantom-type field.
4.2. $C \neq 0$ and $\tilde{C} \neq 0$

If both the parameters $C$ and $\tilde{C}$ are not zero, equation (50) gives a constraint on the form of potentials which is related to the first derivative of potentials. The conserved charge is

$$-C \frac{3a^2}{m_{pl}^2} (\dot{A}_q + A_q H) + \tilde{C} \frac{3a^2}{m_{pl}^2} (\dot{A}_p + A_p H) = Q = \mu_0.$$  (51)

Here, when $\mu_0 \neq 0$, we discuss a particular situation that both $\dot{A}_q + A_q H$ and $\dot{A}_p + A_p H$ are proportional to $a^{-2}$, which indicates that the kinetic energy of both fields behaves like a radiation field. Substituting the results into the equations of motion, we find that

$$V'_q = V'_p = 0.$$  (52)

Combining equations (50) and (52), we obtain

$$\tilde{\omega}_A = \frac{\tilde{P}_A}{\tilde{P}_A} = \frac{1}{2} (1 - \frac{\tilde{C}}{C}) (\dot{A}_q + A_q H)^2 + (1 - \frac{\tilde{C}}{C}) \frac{\partial V_q}{\partial A_q} A_q - 3V_q - 3V_p.$$  (53)

Obviously, when $C = \tilde{C}$, $\tilde{\omega}_A = -1$. If we require $\tilde{\omega}_A < (>) -1$, the following condition must be satisfied:

$$0 < (>) \left( \frac{\tilde{C}}{C} - 1 \right) \left( 2 \left( \frac{\tilde{C}}{C} + 1 \right) (\dot{A}_q + A_q H)^2 + V'_q A_q \right).$$  (54)

Therefore this model allows that one crosses the phantom divider $\tilde{\omega}_A = -1$ as expected. We take the following two examples to illustrate this point:

Example 1: $V_{q1} = V_0 - \frac{1}{2} m_1^2 A_q^2$, $V_{p1} = V_0 + \frac{1}{2} \tilde{m}_1^2 A_q^2$, $\frac{m_1^2}{\tilde{m}_1^2} = \frac{\tilde{C}}{C}$;  (55)

Example 2: $V_{q2} = V_0 + \frac{1}{2} m_2^2 A_q^2$, $V_{p1} = V_0 + \frac{1}{2} \tilde{m}_2^2 A_q^2$, $\frac{m_2^2}{\tilde{m}_2^2} = -\frac{\tilde{C}}{C}$.  (56)

In the first example, without loss of generality, assuming $\tilde{C}/C > 1$, when $2(\tilde{C}/C + 1)(\dot{A}_q + H A_q)^2 + V'_q A_q$ crosses zero, $\tilde{\omega}_A$ crosses over $\tilde{\omega}_A = -1$. It is not surprising that the quintessence-like field takes the dominant role. In the second example, if $\tilde{C}/C \ll -1$, or $m_2^2 \ll \tilde{m}_2^2$, it is easy to obtain $\tilde{\omega}_A < -1$.

The most interesting point is that the solution does not require a particular form for the potentials, so we have the freedom of choosing potentials by observations. Furthermore, this kind of constraint is coming from symmetries of the system.

5. Conclusion

In this paper, we have studied the choice of potentials in the ‘cosmic triad’ vector field model by using the Noether symmetry approach. The existence of Noether symmetry implies that the Lie derivative of the Lagrangian with respect to the related infinitesimal generator vanishes. The phase space is constructed by taking the scale factor $a$ and the field $A$ as independent dynamical variables. In the configuration space which is spanned by $a$ and $A$, the point-like Lagrangian of the model is constructed such that its variations with respect to these dynamical
variables yield correct field equations. Then, the dynamical system is simplified with Noether symmetry. The Noether symmetry is used to select a class of potentials. We have derived the Noether conditions for three different vector field models. In the quintessence and phantom cases, solutions with constant potential have been obtained. And a point-like solution with power-law potential exists for the quintessence case only. This suggests that we may use the Noether symmetry to select theoretical models. In the quintom case, we find that the Noether symmetry requires $\tilde{C}V'_p = -CV'_q$. This result gives a useful constraint on the quintom potentials, and the solution has the desired crossing over $\omega_A = -1$ behavior.

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