On extendability by continuity of valuations on convex polytopes.

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Abstract

There is a well known construction of weakly continuous valuations on convex compact polytopes in $\mathbb{R}^n$. In this paper we investigate when a special case of this construction gives a valuation which extends by continuity in the Hausdorff metric to all convex compact subsets of $\mathbb{R}^n$. It is shown that there is a necessary condition on the initial data for such an extension. In the case of $\mathbb{R}^3$ more explicit results are obtained.

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0 Introduction.

The aim of this paper is to discuss a well known construction of valuations on convex compact polytopes in $\mathbb{R}^n$ and study the question when the constructed valuations can be extended by continuity in the Hausdorff metric from polytopes to the class of all convex compact sets. We show that in general there is a non-trivial obstruction to such an extension. The case of 3-dimensional space is studied in greater detail, and we obtain necessary and sufficient conditions on the initial data of the construction under which the valuations do extend by continuity to all convex compact sets. The main results of the paper are Theorems 0.5, 0.7, 0.10 below.

Let $V$ be a finite dimensional real vector space, $n = \dim V$. Let $\mathcal{P}(V)$ denote the family of all convex compact polytopes in $V$, and let $\mathcal{K}(V)$ denote the family of all non-empty convex compact subsets in $V$.

0.1 Definition. A valuation on $\mathcal{P}(V)$ (resp. $\mathcal{K}(V)$) is a functional

\[ \phi: \mathcal{P}(V) \rightarrow \mathbb{C} \]  

( resp. $\phi: \mathcal{K}(V) \rightarrow \mathbb{C}$)

which is additive in the following sense: for any $A, B \in \mathcal{P}(V)$ (resp. $\mathcal{K}(V)$) such that the union $A \cup B$ is also convex, one has

\[ \phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B). \]

As a generalization of the notion of measure, valuations on convex sets have long played an important role in geometry. Since the middle 1990s’ we have witnessed a breakthrough in the structure theory of valuations [28], [35], [11]-[15], [12]-[14], [30]-[33], [39]. This progress in turn has led to immense advances in the integral geometry of isotropic (in particular, Hermitian) spaces (see [2], [6], [15], [16]) and an understanding why certain classical notions from geometric analysis are indeed fundamental (see [23], [32], [30], [31], [38]).

0.2 Definition. A valuation $\phi$ on $\mathcal{K}(V)$ is called continuous if $\phi$ is continuous in the Hausdorff metric on $\mathcal{K}(V)$.

The notion of weak continuity of a valuation is often more appropriate for polytopes; it is due to Hadwiger [24]. Let us recall it. Fix $\xi := \{\xi_1, \ldots, \xi_s\} \subset$
Let \( V^* \) an \( s \)-tuple of linear functionals on \( V \). Let \( \mathcal{P}_\xi \) denote the family of polytopes in \( \mathcal{P}(V) \) of the form

\[
P_\xi(y) = \{ x \in V | \xi_i(x) \leq y_i \forall i = 1, \ldots, s \}
\]

where \( y = (y_1, \ldots, y_s) \in \mathbb{R}^s \) are such that \( P_\xi(y) \) are non-empty and compact.

**0.3 Definition.** A valuation \( \phi \) on \( \mathcal{P}(V) \) is called *weakly continuous* if for any \( \xi \) the function

\[
y \mapsto \phi(P_\xi(y))
\]

is continuous with respect to \( y \) in the set where \( P_\xi(y) \) is non-empty and compact.

Clearly the restriction to \( \mathcal{P}(V) \) of a continuous valuation on \( \mathcal{K}(V) \) is weakly continuous. The first question studied in this article is the converse one: when does a weakly continuous valuation on \( \mathcal{P}(V) \) extend to a continuous valuation on \( \mathcal{K}(V) \)? Clearly this extension is unique if it exists.

Weakly continuous translation invariant valuations on \( \mathcal{P}(V) \) were described completely by P. McMullen \[34\]. All such valuations are obtained by a general relatively explicit construction. Let us describe a particular case of it which will be studied in this article in greater detail.

Let us fix a Euclidean metric on \( V \). Let us denote by \( G_k(V) \) the manifold of pairs \( (E, l) \) where \( E \subset V \) is a \( k \)-dimensional linear subspace of \( V \), and \( l \) is a line orthogonal to \( E \).

Let \( f : G_k(V) \to \mathbb{C} \) be a continuous function. For a polytope \( P \in \mathcal{P}(V) \) let us denote by \( \mathcal{F}_k(P) \) the set of \( k \)-faces of \( P \). For any \( k \)-face \( F \in \mathcal{F}_k(P) \) let us denote by \( \bar{F} \) the (only) \( k \)-dimensional linear subspace of \( V \) containing a translate of \( F \). Let us denote by \( \gamma_F \) the exterior angle of \( P \) at \( F \), that is \( \gamma_F \) is the subset of the unit sphere \( S(F^\perp) \) of \( F^\perp \) consisting of all \( n \in S(F^\perp) \) such that for some (equivalently, any) \( x \) belonging to the relative interior of \( F \) the scalar product \( (n, p - x) \) is non-positive for any \( p \in P \).

Finally for \( P \in \mathcal{P}(V) \) define

\[
\phi_f(P) = \sum_{F \in \mathcal{F}_k(P)} \text{vol}(F) \int_{\gamma_F} f(F, l) dl,
\]

where \( dl \) denotes the Lebesgue measure on the sphere \( S(\bar{F}^\perp) \) such that the total measure of \( S(\bar{F}^\perp) \) is equal to 1 (here we consider \( f(\bar{F}, \cdot) \) as an even function on the unit sphere \( S(\bar{F}^\perp) \)).
By the (easy part of) P. McMullen’s theorem [34], \( \phi_f \) is a translation invariant even weakly continuous valuation on \( \mathcal{P}(V) \).

0.4 Example. Take \( f \equiv 1 \). Then \( \phi_f \) is proportional to the \( k \)-th intrinsic volume \( V_k \) (see the book [36] for this notion).

Let us mention that expressions similar to (0.1) have been studied in recent preprints [26], [37].

The goal of this paper is to study the following question: when does the valuation \( \phi_f \) extend by continuity in the Hausdorff metric to \( \mathcal{K}(V) \)? It turns out that such an extension does not always exist. Notice that by a result of Groemer [22] if \( \phi_f \) extends by continuity to \( \mathcal{K}(V) \) then this extension will be automatically a valuation on \( \mathcal{K}(V) \). In this case we will denote this extension by the same symbol \( \phi_f \).

Our first main result is as follows.

0.5 Theorem. Let \( n := \dim V \geq 3 \) and \( 1 \leq k \leq n - 2 \). There exists a real analytic function \( f : G_k(V) \rightarrow \mathbb{C} \) such that \( \phi_f \) does not extend to \( \mathcal{K}(V) \) by continuity.

0.6 Remark. (1) It is well known that for \( k = n - 1 \) the valuation \( \phi_f \) does extend by continuity to \( \mathcal{K}(V) \) for any continuous function \( f \).

(2) Obviously the simplest case when the assumptions of Theorem 0.5 are satisfied is \( n = 3, k = 1 \). It turns out that in this case one has a much more precise statement, see Theorem 0.10 below.

(3) We would like to emphasize the following phenomenon: the class of smoothness of the function \( f \) is not really important, it can be real analytic, and in fact even \( O(n) \)-finite\(^1\).

(4) By Proposition 0.9 below, the subset of the space \( C^\infty(G_k(V)) \) of infinitely smooth functions \( f \) such that \( \phi_f \) admits an extension to \( \mathcal{K}(V) \) by continuity is a closed linear subspace.

(5) In a very recent preprint [26] Theorem 0.5 was proved by a very different method.

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\(^1\)Recall the definition of \( O(n) \)-finite vectors in a representation of \( O(n) \) in a vector space \( \mathcal{U} \): a vector \( \xi \in \mathcal{U} \) is called \( O(n) \)-finite if it is contained in a finite dimensional \( O(n) \)-invariant linear subspace. It is well known that if \( \mathcal{U} \) is the space of (smooth, continuous, or \( L^2 \)) functions on a homogeneous space of \( O(n) \), then all \( O(n) \)-finite functions are real analytic.
Now we are going to explain some more precise forms of Theorem 0.5. In the rest of the paper we will always assume that \( f \) is infinitely smooth.

Let us denote by \( C^\infty(X) \) the space of infinitely smooth functions on a manifold \( X \). Let us denote by \( Gr_k(V) \) the Grassmannian of \( k \)-dimensional linear subspaces of \( V \). Let us define the linear map

\[
S: C^\infty(G_k(V)) \longrightarrow C^\infty(Gr_k(V))
\]

by

\[
S(g)(E) = \int_{l \in \mathbb{P}(E^\perp)} g(E, l) dl,
\]

where \( \mathbb{P}(E^\perp) \) denotes the set of lines in \( E^\perp \) (i.e. the projectivization of \( E^\perp \)), \( dl \) is the normalized Lebesgue measure on \( \mathbb{P}(E^\perp) \). Clearly the map \( S \) is onto.

Here is our second main result.

0.7 Theorem. Let \( n := \dim V \geq 4 \) and \( 2 \leq k \leq n - 2 \). There exists a closed proper subspace of \( C^\infty(Gr_k(V)) \) with the following property: if \( f \in C^\infty(G_k(V)) \) is such that \( \phi_f \) extends by continuity to \( K(V) \), then \( S(f) \) belongs to this closed proper subspace. This implies that the subset of \( f \in C^\infty(G_k(V)) \) such that \( \phi_f \) extends by continuity to \( K(V) \) is contained in a closed proper linear subspace.

0.8 Remark. This theorem already implies Theorem 0.5 for \( 2 \leq k \leq n - 2 \). Actually, as it will be seen from the proof, the closed proper subspace of \( C^\infty(Gr_k(V)) \) mentioned in the theorem is equal to the image of the so called cosine transform on the Grassmannian \( Gr_k(V) \). For \( 2 \leq k \leq n - 2 \) it is known to be a proper subspace by a result of Goodey, Howard, and Reeder [20]; see also [8] for a more precise description of this subspace. The proof of the proposition uses the Klain imbedding [29] of even valuations to functions on the Grassmannian and the result by J. Bernstein and the author [8] stating that the image of the Klain imbedding is equal to the range of the cosine transform.

The case \( k = 1 \) is not covered by Theorem 0.7; here the situation is more subtle since the range of the cosine transform on \( C^\infty(Gr_1(V)) \) is equal to the whole space. Here we will (easily) see that in order to prove Theorem 0.5 it suffices to do that for \( \dim V = 3 \). In this case we have a much more precise statement. In order to formulate it, it will be necessary to rewrite the construction of \( \phi_f \) in a more invariant way without using explicitly the
Euclidean metric on $V$. Then the group $GL_n(\mathbb{R})$ will act on all spaces. It will be necessary to consider $f$ not as a function on $G_k(V)$ but as a section of a certain $GL_n(\mathbb{R})$-equivariant line bundle over $G_k(V)$.

Let us describe this. We will do it for $\dim V \geq 3$ and any $1 \leq k \leq n - 2$. First in metric free terms we identify $G_k(V)$ with the set of pairs $(E, l)$ where $E \in Gr_k(V)$ is a $k$-dimensional linear subspace, and $l$ is a line contained in the annihilator $E^\perp \subset V^*.$

Let $p: G_k(V) \longrightarrow Gr_k(V)$ denote the projection $p(E, l) = E.$ Let $L \longrightarrow Gr_k(V)$ be the line bundle whose fiber over $E \in Gr_k(V)$ is equal to the space of Lebesgue measures on $E$. Note that a choice of a Euclidean metric on $V$ defines a trivialization of $L$.

Next let us denote by $\omega_{G_k/Gr_k}$ the line bundle over $G_k(V)$ of relative densities of $G_k(V)$ over $Gr_k(V)$. Let us recall its definition. Let $q: X \longrightarrow Y$ be a smooth map between smooth manifolds. Assume that $q$ is a submersion, i.e. its differential at every point is onto. (In our case $X = G_k(V), Y = Gr_k(V), q = p$.) The fiber over a point $x \in X$ of the line bundle of relative densities $|\omega_{X/Y}|$ is, by definition, equal to the (1-dimensional) space of Lebesgue measures on the tangent space at $x$ to the fiber $p^{-1}(p(x))$. If $q$ is proper, then we have a canonical map $C^\infty(X, |\omega_{X/Y}|) \longrightarrow C^\infty(Y)$ given by integration along the fibers of $p$.

Note also that a choice of a Euclidean metric on $V$ defines a trivialization of $|\omega_{G_k/Gr_k}|$ (indeed a Euclidean metric on $V$ defines a Haar measure on each fiber of $p$). Let

$$M := p^*L \otimes |\omega_{G_k/Gr_k}|.$$ 

Thus a choice of a Euclidean metric on $V$ defines a trivialization of $M$, and hence an isomorphism

$$C^\infty(G_k(V), M) \simeq C^\infty(G_k(V)).$$ (0.2)

In this notation and with the isomorphism (0.2), the above construction of $\phi_f$ can be rewritten as follows. Let $f \in C^\infty(G_k(V), M)$. Let $P \in \mathcal{P}(V), F \in \mathcal{F}_k(P)$. Then $\int_{\gamma_F} f$ is a Lebesgue measure on $F$. Hence $\int_F \left(\int_{\gamma_F} f\right)$ is a number. Then

$$\phi_f(P) = \sum_{F \in \mathcal{F}_k(P)} \int_F \left(\int_{\gamma_F} f\right).$$ (0.3)

\[\text{To be more precise, } \gamma_F \text{ is a subset of the manifold } \mathbb{P}_+(E^\perp) \text{ of oriented lines in } E^\perp, \text{ and here we integrate } f \text{ over the image of } \gamma_F \text{ in } \mathbb{P}(E^\perp) \text{ under the natural two sheeted covering } \mathbb{P}_+(E^\perp) \longrightarrow \mathbb{P}(E^\perp) \text{ given by forgetting the orientation.}\]
Set

\[ \Xi := \{ f \in C^\infty(G_k(V), M) | \phi_f \text{ extends by continuity to } \mathcal{K}(V) \} . \]

Thus under the identification (0.2) the subspace \( \Xi \) is isomorphic to the space of smooth functions \( f \) on \( G_k(V) \) such that \( \phi_f \) extends by continuity to \( \mathcal{K}(V) \).

In Proposition 1.1 we will show

0.9 Proposition. \( \Xi \) is a closed \( GL(V) \)-invariant linear subspace of \( C^\infty(G_k(V), M) \).

The next main result of this paper is the following one; later on it will be formulated more explicitly.

0.10 Theorem. Let \( n = 3 \), \( k = 1 \). The natural representation of \( GL_3(\mathbb{R}) \) in \( C^\infty(G_1(V), M) \) has length three. The (closed \( GL_3(\mathbb{R}) \)-invariant) subspace \( \Xi \) has length two. Moreover \( \Xi \) coincides with the image of an explicit \( GL_3(\mathbb{R}) \)-equivariant map closely related to the construction of valuations using integration with respect to the normal cycle (described below).

0.11 Remark. (1) As stated, Theorem 0.10 does not provide a unique representation theoretical characterization of the subspace \( \Xi \). However with a little bit of extra work it can be shown that \( \Xi \) is the only \( GL_3(\mathbb{R}) \)-equivariant closed subspace of \( C^\infty(G_1(V), M) \) of length two whose irreducible subquotients have the minimal (and equal to each other) Gelfand-Kirillov dimension among the three irreducible subquotients of the whole space (the third irreducible subquotient has strictly larger Gelfand-Kirillov dimension). We will not pursue this characterization in this paper, but people familiar with the Gelfand-Kirillov dimension can easily deduce the above characterization from our arguments.

(2) For any \( n \) and \( k \) the recent preprint [26] contains a sufficient condition for a section to belong to \( \Xi \) (see Theorem 4.1 there).

The proof of this theorem uses the Klain imbedding theorem, construction of valuations using integration with respect to the normal cycle, and a detailed study of the representation of \( GL_3(\mathbb{R}) \) in \( C^\infty(G_1(\mathbb{R}^3), M) \); the latter step uses the Beilinson-Bernstein localization theorem.

The paper is organized as follows. In Section 1 we prove the main results on valuations except the case \( n = 3, k = 1 \). In Section 2 we prove the case \( n = 3, k = 1 \) modulo some representation theoretical computations which
are postponed to Section 4. In Section 3 we remind some representation theoretical background; that section does not contain new results. In Section 4 we make the computations with representations of $GL_3(\mathbb{R})$; the main result of this section is Theorem 4.1.

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1 Proof of main results on valuations when $n > 3$.

We keep the notation of the introduction.

1.1 Proposition. The space $\Xi$ is a closed $GL(V)$-invariant linear subspace of $C^\infty(G_k(V), M)$.

The linearity and $GL(V)$-invariance of $\Xi$ are obvious. In order to prove that $\Xi$ us closed we will need some preparations.

Let us denote by $Val_{ev}^k(V)$ the space of translation invariant continuous even $k$-homogeneous valuations on $K(V)$. Equipped with the topology of uniform convergence on compact subsets of $K(V)$, $Val_{ev}^k(V)$ is a Banach space. Let us recall the construction of the Klain imbedding $K: Val_{ev}^k(V) \hookrightarrow C(Gr_k(V), L)$ which is a continuous linear map commuting with the natural action of $GL(V)$. Let $\phi \in Val_{ev}^k(V)$. For any subspace $E \in Gr_k(V)$ the restriction of $\phi|_E$ is a $k$-homogeneous valuation. By a result of Hadwiger [25], $\phi|_E$ is a Lebesgue measure on $E$. Let us denote by $K$ the map $\phi \mapsto [E \mapsto \phi|_E]$. Thus $K: Val_{ev}^k(V) \rightarrow C(Gr_k(V), L)$. A deep result due to D. Klain [29] (heavily based on [28]) says that $K$ is injective.

Let us denote by $Val_{ev, sm}^k(V)$ the subspace of smooth $k$-homogeneous even valuations (a valuation $\phi \in Val(V)$ is called smooth if the map $GL(V) \rightarrow Val(V)$ given by $g \mapsto g(\phi)$ is $C^\infty$-differentiable). $Val_{ev, sm}^k(V)$ has a natural structure of a Fréchet space. Moreover

$$K: Val_{ev, sm}^k(V) \hookrightarrow C^\infty(Gr_k(V), L).$$
The Casselman-Wallach theorem (see Theorem 3.9 below) implies the following claim.

1.2 Claim. The Klain imbedding

\[ K : Val_k^{ev,sm}(V) \hookrightarrow C^\infty(Gr_k(V), L) \]

has a closed image and induces an isomorphism of \( Val_k^{ev,sm}(V) \) onto its image as topological vector spaces.

Let us denote by \( T \) the map \( \Xi \rightarrow Val_k^{ev}(V) \) given by \( f \mapsto \phi_f \). Note also that we have the canonical map

\[ S : C^\infty(G_k(V), M) \rightarrow C^\infty(Gr_k(V), L) \quad (1.1) \]
given by the integration along the fibers of the projection \( p : G_k(V) \rightarrow Gr_k(V) \). The map \( S \) was defined in the introduction under a choice of Euclidean metric. To rewrite it in invariant terms let us recall that by definition \( M = p^* L \otimes \omega_{G_k/Gr_k} \). Then for any \( f \in C^\infty(G_k(V), M) \) and any \( E \in Gr_k(V) \), the restriction \( f|_{p^{-1}(E)} \) is a section over \( p^{-1}(E) \) of the line bundle \( |\omega_{p^{-1}(E)}| \otimes L|_E \). Hence one has \( \int_{p^{-1}(E)} f \in L|_E \). Now it is clear that \( S \) is a continuous linear map commuting with the natural action of \( GL(V) \).

The next lemma is obvious.

1.3 Lemma. The composition

\[ K \circ T : \Xi \rightarrow C(Gr_k, L) \]

is equal to \( S|_{\Xi} \).

1.4 Lemma. For any \( f \in \Xi \) the valuation \( \phi_f \) is smooth, i.e.

\[ T(\Xi) \subset Val_k^{ev,sm}(V). \]

Proof. By Lemma 1.3 for any \( f \in \Xi \), \( K(\phi_f) \in C^\infty(Gr_k(V), L) \). Let us show that this implies that \( \phi_f \) is smooth. By Claim 1.2 \( K(Val_k^{ev,sm}(V)) \) is a closed subspace of \( C^\infty(Gr_k(V), L) \). Its closure \( \overline{K(Val_k^{ev,sm}(V))} \) in \( C(Gr_k(V), L) \) contains \( K(Val_k^{ev}(V)) \). It is easy to see that

\[ \overline{K(Val_k^{ev,sm}(V))} \cap C^\infty(Gr_k(V), L) = K(Val_k^{ev,sm}(V)). \]
Hence there exists $\psi \in \text{Val}_k^{ev,sm}(V)$ such that $K(\psi) = K(\phi_f)$. Since $K$ is injective on $\text{Val}_k^{ev}(V)$ we get $\psi = \phi_f$. Q.E.D.

**Proof** of Proposition 1.1. It remains to show that $\Xi$ is a closed subspace of $C^\infty(G_k(V), M)$. First let us show that the map $T : \Xi \to \text{Val}_k^{ev,sm}(V)$ is continuous when $\Xi$ is equipped with the topology induced from $C^\infty(G_k(V), M)$.

By Claim 1.2 it is enough to show that $K \circ T$ is continuous. By Lemma 1.3 $K \circ T = S$ which is obviously continuous.

Now let us assume $f \in \Xi$. Let $\{f_N\} \subset \Xi$ be a sequence such that $f_N \to f$. The sequence $\{Tf_N\} \subset \text{Val}_k^{ev,sm}(V)$ is a Cauchy sequence due to the continuity of $T$. Hence there exists $\phi \in \text{Val}_k^{ev,sm}(V)$ such that $Tf_N = \phi_{f_N} \to \phi$. Hence $(K \circ T)(f_N) \to K(\phi)$. Also $Sf_N \to Sf$. Hence $Sf = K(\phi)$. By construction, it is also clear that for any polytope $P \in \mathcal{P}(V)$ one has $\phi_{f_N}(P) \to \phi_f(P)$. But also $\phi_{f_N}(P) \to \phi_f(P)$. Hence $\phi_f(P) = \phi_f(P)$ for any $P \in \mathcal{P}(V)$. This means that $\phi_f$ admits an extension by continuity to $K(V)$. Hence $f \in \Xi$. Q.E.D.

1.5 **Theorem.**

$$T(\Xi) = \text{Val}_k^{ev,sm}(V).$$

**Proof.** Recall that $T : \Xi \to \text{Val}_k^{ev,sm}(V)$ is a continuous $GL(V)$-equivariant map. Moreover it is non-zero by Example 0.4. By the Irreducibility Theorem [1] the space $\text{Val}_k^{ev,sm}(V)$ is $GL(V)$-irreducible. Hence $\text{Im}T$ is dense in $\text{Val}_k^{ev,sm}(V)$. By Proposition 1.1 $\Xi$ is a closed subspace of $C^\infty(G_k(V), M)$. Hence by the Casselman-Wallach theorem (see Theorem 3.9 below) $\text{Im}T$ is closed. Hence $\text{Im}T = \text{Val}_k^{ev,sm}(V)$. Q.E.D.

1.6 **Theorem.** The subspace $S(\Xi) \subset C^\infty(Gr_k(V), L)$ coincides with the range of the cosine transform.

**Proof.** Recall that by Lemma 1.3 $S|_\Xi = K \circ T$. Thus $S(\Xi) = (K \circ T)(\Xi) = K(\text{Val}_k^{ev,sm}(V))$ where the last equality follows from Theorem 1.5. But $K(\text{Val}_k^{ev,sm}(V))$ coincides with the range of the cosine transform by [8]. Q.E.D.

Theorem 1.6 immediately implies Theorem 0.7.

So far we have proven Theorem 0.7. It implies Theorem 0.3 in all cases except when $k = 1$ and $n \geq 3$. Let us deduce the case $k = 1, n \geq 3$ from the case $k = 1, n = 3$. Let $V$ be an $n$-dimensional vector space, $n > 3$. Let us
fix a 3-dimensional subspace $W \subset V$. Let $f \in C^\infty(G_1(V))$ be an arbitrary smooth function. Let $\phi_f$ be the corresponding valuation on polytopes $P(V)$. Let us describe explicitly the restriction of $\phi_f$ to $P(W) \subset P(V)$ in terms intrinsic to $W$. We will construct a function $g \in C^\infty(G_1(W))$ such that the restriction of $\phi_f$ to $P(W)$ is equal to $\phi_g$.

The function $g$ can be described as follows. Let $p_W : V \rightarrow W^\perp$ denote the orthogonal projection to $W^\perp$. Then

$$g(E, l) = \kappa_n \int_{m \in S(l \oplus W^\perp)} f(E, m) \sqrt{1 - |p_W(m)|^2} dm,$$

(1.2)

where the integration is over the unit sphere of the space $l \oplus W^\perp$, $dm$ is rotation invariant Haar measure on this sphere, $| \cdot |$ under the square root is the Euclidean norm, and $\kappa_n \neq 0$ is a normalizing constant depending on $n$ only. We leave it to the reader to verify this elementary formula.

Clearly the map $f \mapsto g$ defined by (1.2) is a continuous linear operator

$$U : C^\infty(G_1(V)) \rightarrow C^\infty(G_1(W)).$$

It is easy to see that this map is onto.

Let $g$ be an $O(3)$-finite function on $G_1(W)$ such that the valuation $\phi_g$ does not extend by continuity to all convex compact subsets of $W$; for the moment we just assume its existence. It suffices to show that there exists an $O(n)$-finite function $f$ on $G_1(V)$ such that $U(f) = g$. Then we would have that the restriction of $\phi_f$ to $P(W)$ is equal to $\phi_g$; this implies that $\phi_f$ does not extend by continuity to all convex compact subsets of $V$.

Notice that existence of such an infinitely smooth function $f$ follows from the surjectivity of $U$ on infinitely smooth functions. Next we have the obvious imbedding $O(3) \subset O(n)$. The map $U$ commutes with the action of $O(3)$ on both spaces. Clearly any $O(n)$-finite function is $O(3)$-finite. It follows that since $O(n)$-finite functions are dense in $C^\infty(G_1(V))$, their image under $T$ is dense in $C^\infty(G_1(W))$, and any function in this image is $O(3)$-finite. But any dense linear $O(3)$-invariant subspace of $C^\infty(G_1(W))$ consisting of $O(3)$-finite functions is equal to the space of all $O(3)$-finite functions on $C^\infty(G_1(V))$; this easily follows from the fact that the natural representation of $O(3)$ in functions on $G_1(W)$ has finite multiplicities due to transitivity of the action on $G_1(W)$. This finishes the proof of Theorem 0.5.

The following lemma will be needed in the next section.
1.7 Lemma. Let $n \geq 3$, $1 \leq k \leq n - 1$. Then
\[ \Xi \cap \text{Ker} S \neq \text{Ker} S. \]

Proof. Let us fix a Euclidean metric on $V$, and let us identify $C^\infty(G_k(V), M)$ with $C^\infty(G_k(V))$. In order to prove the lemma it is enough to construct a function $f \in \text{Ker} S$ and a simplex $\Delta$ such that $\phi_f(\Delta) \neq 0$. Let us fix an arbitrary $n$-simplex $\Delta$. Let $F_1, \ldots, F_t$ be all of its $k$-faces (then $t = \binom{n+1}{k+1}$). Let $A_i := p^{-1}(F_i) \subset G_k(V)$. Thus each $A_i$ is isomorphic to $\mathbb{R}P^{n-k-1}$.

Let us choose an arbitrary function $g \in C^\infty(G_k(V))$ such that $g|_{A_i} \equiv 0$ for $i > 1$, $\int_{\gamma F_i} g \neq 0$, and $\int_{A_i} g = 0$. Then clearly $\phi_g(\Delta) \neq 0$. Let $h := p^*(g) \in C^\infty(G_k(V))$. Let $f := g - p^*g$. Then clearly $f \in \text{Ker} S$, $f|_{A_i} \equiv g|_{A_i}$ for any $i = 1, 2, \ldots, t$, and hence $\phi_f(\Delta) = \phi_g(\Delta) \neq 0$. Hence $f \notin \Xi$. This proves the lemma. Q.E.D.

2 The case $n = 3$, $k = 1$.

The main result of this section, Theorem 2.5, is stated below. Let us start with some technical preparations.

Let us study the natural representation of the group $GL(V)$ in the space $C^\infty(G_k(V), M)$. Let us denote by $P$ the parabolic subgroup of $GL(V)$ with the Levi component $GL_k \times GL_{n-k-1} \times GL_1$. Let $\chi : P \to \mathbb{C}^*$ be the character of $P$ given by
\[
\chi \left( \begin{bmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & C \end{bmatrix} \right) = |\det A|^{-1} \cdot |\det B| \cdot |C|^{-(n-k-1)}
\]

where $A \in GL_k$, $B \in GL_{n-k-1}$, $C \in GL_1$.

The following proposition is straightforward.

2.1 Proposition. The natural representation of $GL(V)$ in the space $C^\infty(G_k(V), M)$ is equal to the induced representation $\text{Ind}^P_G \chi$ (the induction is not unitary).

Let us study in greater detail the case $n = 3$, $k = 1$. Then the corresponding parabolic subgroup $P$ is the subgroup of upper triangular matrices of size 3, and
\[
\chi \left( \begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix} \right) = |a|^{-1} |b||c|^{-1}. \quad (2.1)
\]
In Section 4 below we will prove the following representation theoretical result.

2.2 Proposition. Let $n = 3$, $k = 1$. The representation $\text{Ind}_{P}^{G} \chi$ is of length 3 as a $GL(V)$-representation. Moreover the irreducible subquotients are pair-wise non-isomorphic.

Recall from Section 1 that we have the map (integration along the fibers)

$$S : C^\infty(G_1, M) \longrightarrow C^\infty(Gr_1(V), L).$$

We will need a lemma.

2.3 Lemma. $\text{Ker} S$ has length 2 as $GL(V)$-module.

Proof. By Theorem 1.6 the subspace $S(\Xi) \subset C^\infty(Gr_1(V), L)$ coincides with the range of the cosine transform. But in this case the cosine transform is an isomorphism (see e.g. [10]). Moreover the source and the target spaces of the cosine transform are irreducible $GL(V)$-representations (this is well known; see e.g. [27]). Hence $\text{Im}(S) = C^\infty(Gr_1(V), L)$ is irreducible. This and Proposition 2.2 imply that the length of $\text{Ker}(S)$ is equal to 2. Q.E.D.

Let us remind an important construction of continuous valuations called integration with respect to the normal cycle; it will be necessary for the formulation of the main result of this section. For a convex compact subset $K \subset V$ and a point $x \in K$ let us define a tangent cone $T_x K$ to $K$ at $x$:

$$T_x K := \bigcup_{t > 0} t(K - x).$$

Clearly $T_x K$ is a convex cone in $V$. Define the dual cone

$$(T_x K)^\circ := \{ \xi \in V^* | < \xi, v > \leq 0 \text{ for all } v \in T_x K \}.$$ 

Let us denote by $\mathbb{P}_+(V^*)$ the oriented projectivization of $V^*$, namely the set of oriented one dimensional linear subspaces of $V^*$. Another way to view it is as the quotient space $(V^* \setminus \{0\})/\mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ acts of $V^*$ by usual multiplication.

Now let us define the normal cycle $N(K)$ as a subset of $\mathbb{P}_+(V^*) \times V$:

$$N(K) := \cup_{x \in K} (T_x K)^\circ / \mathbb{R}_{>0}.$$
It is well known that $N(K)$ is a compact set of Hausdorff dimension $n - 1$ \((n = \dim V)\). It is easy to see that an orientation of $V$ induces an orientation of $N(K)$.

To construct a continuous valuation, let us fix
\[
\omega \in C^\infty(\mathbb{P}_+^n(V^*), \Omega^{n-1}) \otimes or(V),
\]
where $\Omega^{n-1}$ denotes the vector bundle whose sections are $n - 1$ forms, and $or(V)$ is the orientation line of $V$. (Remind the definition of $or(V)$: this is the one dimensional space of $\mathbb{C}$-valued functions $H$ on the set of all bases of $V$ satisfying the property $H(gB) = sgn(g)H(B)$ for any $g \in GL(V)$ and any basis $B$). Then define a functional on $\mathcal{K}(V)$:
\[
K \mapsto \int_{N(K)} \omega.
\]
It is well known (see e.g. [9], Section 2.1) that this functional is a continuous valuation on $\mathcal{K}(V)$. It is clear that if $\omega$ is translation invariant with respect to $V$ then the corresponding valuation is translation invariant. The space of translation invariant $\omega$’s can be written
\[
\bigoplus_{k=0}^{n-1} C^\infty(\mathbb{P}_+^n(V^*), \Omega^{n-k-1}) \otimes \wedge^k V^* \otimes or(V).
\]
Elements of $C^\infty(\mathbb{P}_+^n(V^*), \Omega^{n-k-1}) \otimes \wedge^k V^* \otimes or(V))$ define $k$-homogeneous valuations. Even valuations are determined by even sections of the latter space, when the parity on them is induced by the involution on $\mathbb{P}_+^n(V^*)$ of the change of orientation. This space of even sections over $\mathbb{P}_+^n(V^*)$ can be viewed as the space of all sections of an appropriate vector bundle over the projective space $\mathbb{P}(V^*)$ of non-oriented lines in $V^*$. It is not hard to see that this vector bundle is
\[
C^\infty(\mathbb{P}(V^*), \Omega^{n-k-1} \otimes \varepsilon_{n-k}) \otimes \wedge^k V^* \otimes or(V),
\]
where $\varepsilon_{n-k}$ is the trivial line bundle for $n - k$ even, and for $n - k$ odd its fiber over $l \in \mathbb{P}(V^*)$ is equal to the orientation line $or(l)$ of $l$.

Let us introduce more notation. Let $\mathcal{P}Val^+_k$ denote the space of all valuations on convex compact polytopes which are $k$-homogeneous and even (without any continuity assumptions). Let
\[
\mathcal{F}: C^\infty(G_k(V), M) \rightarrow \mathcal{P}Val^+_k
\]
be given by \( f \mapsto \phi_f \) as in (0.3). Let 
\[ \mathcal{G} : C^\infty(\mathbb{P}(V^*), \Omega^{n-1} \otimes \varepsilon_{n-k}) \otimes \wedge^k V^* \otimes \text{or}(V) \longrightarrow \mathcal{P} \text{Val}_k^+ \]

be the map \( \omega \mapsto [K \mapsto \int_{N(K)} \omega] \). The image of \( \mathcal{G} \) is contained in continuous valuations \( \text{Val}_k^+ \). Clearly \( \mathcal{F}, \mathcal{G} \) are \( GL(V) \)-equivariant.

2.4 Proposition. There exists a \( GL(V) \)-equivariant map 
\[ \mathcal{H} : C^\infty(\mathbb{P}(V^*), \Omega^{n-1} \otimes \varepsilon_{n-k}) \otimes \wedge^k V^* \otimes \text{or}(V) \longrightarrow C^\infty(G_k(V), M) \]
such that \( \mathcal{G} = \mathcal{F} \circ \mathcal{H} \).

Proof. Let us construct \( \mathcal{H} \) explicitly. Fix 
\[ \omega \in C^\infty(\mathbb{P}(V^*), \Omega^{n-1} \otimes \varepsilon_{n-k}) \otimes \wedge^k V^* \otimes \text{or}(V). \]

Let \( E \in \text{Gr}_k(V) \), and let \( l \subset E^\perp \) be a line. Restriction of \( \omega \) to \( \mathbb{P}(E^\perp) \subset \mathbb{P}(V^*) \) is an element of \( C^\infty(\mathbb{P}(E^\perp), \Omega^{n-1} \otimes \varepsilon_{n-k}) \otimes \wedge^k V^* \otimes \text{or}(V) \). The canonical map \( V^* \longrightarrow E^* \) defines a linear map \( \wedge^k V^* \longrightarrow \wedge^k E^* \). Hence \( \omega \) defines an element in 
\[ C^\infty(\mathbb{P}(E^\perp), \Omega^{n-1} \otimes \varepsilon_{n-k}) \otimes \wedge^k E^* \otimes \text{or}(V). \]

Taking the value of this section over \( l \in \mathbb{P}(E^\perp) \) we get an element in 
\[ \wedge^{n-1-k} T_l^* \mathbb{P}(E^\perp) \otimes \wedge^k E^* \otimes \text{or}(V) \otimes \varepsilon_{n-k}|_l. \quad (2.2) \]

Let us show that the last space is canonically isomorphic to 
\[ |\omega_{G_k/\text{Gr}_k}|_l \otimes L|_E. \]

Clearly 
\[ |\omega_{G_k/\text{Gr}_k}|_l = \wedge^{n-1-k} T_l^* \mathbb{P}(E^\perp) \otimes \text{or}(T_l^* \mathbb{P}(E^\perp)), \quad (2.3) \]
\[ L|_E = \wedge^k E^* \otimes \text{or}(E). \quad (2.4) \]

Comparing (2.3)-(2.4) with (2.2) we see that we have to construct a canonical isomorphism 
\[ \text{or}(T_l^*(\mathbb{P}(E^\perp))) \otimes \text{or}(E) = \text{or}(V) \otimes \varepsilon_{n-k}|_l. \quad (2.5) \]
To prove this, we will use the following general canonical isomorphisms

\[ or(X^*) = or(X), \]
\[ or(X/Y) = or(X) \otimes or(Y), \]
\[ or(X \otimes Y) = or(X)^{\otimes \dim Y} \otimes or(Y)^{\otimes \dim X}. \]

Then the left hand side in (2.5) is equal to

\[ or(Hom(l, E^\perp/l)^*) \otimes or(E) = \]
\[ or(l \otimes (E^\perp/l)^*) \otimes or(E) = \]
\[ or(l)^{\otimes (n-k-1)} \otimes or(E^\perp)^* \otimes or(l) \otimes or(E) = \]
\[ or(l)^{\otimes (n-k)} \otimes or(V/E) \otimes or(E) = or(l)^{\otimes (n-k)} \otimes or(V) = \]
\[ or(V) \otimes \varepsilon_{n-k}|l|. \]

Thus the isomorphism (2.5) is constructed. Hence the map \( \mathcal{H} \) is constructed. It is easy to see that it satisfies

\[ \mathcal{G} = \mathcal{F} \circ \mathcal{H}. \]

Q.E.D.

In the rest of this section we will assume that \( n = 3, k = 1 \). The following theorem is the main result of this section.

**2.5 Theorem.** Let \( n = 3, k = 1 \).

1. The subspace \( \Xi \subset C^\infty(G_1(V), M) \) is equal to the image of \( \mathcal{H} \).
2. The length of \( \Xi \) as a \( GL(V) \)-module is equal to 2, while the length of the whole space \( C^\infty(G_1(V), M) \) is equal to 3, and all irreducible subquotients are pairwise non-isomorphic.

**2.6 Remark.** Part (1) of Theorem 2.5 provides some kind of analytic (or geometric) description of \( \Xi \). Part (2) gives some complimentary information of the subspace \( \Xi \): for example it may take some extra work to show that part (1) implies that \( \Xi \) does not coincide with the whole space. In fact, the proof of part (1) uses part (2).

Part (2), as it is stated, does not provide a unique characterization of the subspace \( \Xi \) inside \( C^\infty(G_1(V), M) \). However with a little bit of extra work one can obtain the following slightly more precise statement: \( \Xi \) is the only closed \( GL(V) \)-invariant subspace of length 2 whose irreducible subquotients
have minimal Gelfand-Kirillov dimensions among irreducible subquotients of $C^\infty(G_1(V), M)$. (In fact, one can show that the Gelfand-Kirillov dimensions of the three irreducible subquotients of the whole space are equal to 2, 2, and 3.)

**Proof of Theorem 2.5.** First it will be useful to compute the kernel of $\mathcal{H}$. Recall that for $n = 3, k = 1$ we have

$$\mathcal{H} : C^\infty(\mathbb{P}(V^*), \Omega^1) \otimes V^* \otimes or(V) \longrightarrow C^\infty(G_1(V), M).$$

Let us fix $\omega \in C^\infty(\mathbb{P}(V^*), \Omega^1) \otimes V^* \otimes or(V)$. Clearly $\omega \in Ker(\mathcal{H})$ if and only if for any 1-dimensional linear subspaces $E \subset V$ and $l \subset E^\perp (\subset V^*)$ the image of $\omega$ vanishes under the map

$$T^*_l(\mathbb{P}(V^*)) \otimes V^* \otimes or(V) \longrightarrow T^*_l(\mathbb{P}(E^\perp)) \otimes E^* \otimes or(V) \quad (2.6)$$

which is just the tensor product of the natural restriction maps

$$T^*_l(\mathbb{P}(V^*)) \longrightarrow T^*_l(\mathbb{P}(E^\perp)),$$

$$V^* \longrightarrow E^*,$$

and of the identity map on $or(V)$.

Let us fix $l \subset V^*$ and describe explicitly the intersection of kernels of the maps (2.6) over all lines $E \subset l^\perp$. Thus we have

$$T^*_l(\mathbb{P}(V^*)) = l \otimes (V^*/l)^* = l \otimes l^\perp,$$

$$T^*_l(\mathbb{P}(E^\perp)) = l \otimes (E^\perp/l)^* = l \otimes (l^\perp/E).$$

Then (2.6) becomes a map

$$l \otimes (l^\perp \otimes V^*) \otimes orV \longrightarrow l \otimes ((l^\perp/E) \otimes E^*) \otimes or(V), \quad (2.7)$$

where the map is tensor product of the identity map on the first copy of $l$ and on $or(V)$, of the canonical quotient map $l^\perp \longrightarrow l^\perp/E$, and of the obvious natural map $V^* \longrightarrow E^*$.

It suffices to describe the intersection of the kernels of maps

$$l^\perp \otimes V^* \longrightarrow (l^\perp/E) \otimes E^*, \quad (2.8)$$

when $l$ is fixed, and $E$ runs over all lines $E \subset l^\perp$. The map (2.8) can be identified with the map

$$Hom(V, l^\perp) \longrightarrow Hom(E, l^\perp/E) \quad (2.9)$$
given by the composition $\phi \mapsto p_E \circ \phi \circ i_E$, where $i_E : E \rightarrow V$ is the identity imbedding, and $p_E : l^\perp \rightarrow l^\perp / E$ is the quotient map. The kernel of (2.9) consists of maps $\phi : V \rightarrow l^\perp$ such that $\phi(E) \subset E$. Hence the intersection of kernels of (2.9) over all $E \subset l^\perp$, when $L$ is fixed, consists of $\phi : V \rightarrow l^\perp$ such that any line $E \subset l^\perp$ is mapped to $E$, or equivalently the restriction of $\phi$ to $l^\perp$ is proportional to the identity map. Let us denote by $X_l$ this space of maps tensored by $l \otimes or(V)$. Then $X_l$ is the intersection of the kernels of (2.7), which is the same as the intersection of kernels of (2.6) over all $E \subset l^\perp$ with fixed $l$.

Now let us describe explicitly the quotient of $T^*_l(\mathbb{P}(V^*)) \otimes V^* \otimes or(V)$ by $X_l$. Clearly

$$X_l \supset l \otimes Hom(V/l^\perp, l^\perp) \otimes or(V) = l \otimes (l^\perp \otimes l) \otimes or(V).$$

Then we have furthermore

$$T^*_l(\mathbb{P}(V^*)) \otimes V^* \otimes or(V) / (l \otimes (l^\perp \otimes l) \otimes or(V)) =$$

$$l \otimes (l^\perp \otimes V^*) \otimes or(V) / (l \otimes (l^\perp \otimes l) \otimes or(V)) =$$

$$l \otimes (l^\perp \otimes V^*/l) \otimes or(V) =$$

$$l \otimes End(V^*/l) \otimes or(V).$$

We have to quotient out the last space by $X_l / (l \otimes (l^\perp \otimes l) \otimes or(V))$. The result is

$$l \otimes End_0(V^*/l) \otimes or(V), \quad (2.10)$$

where $End_0$ denotes the trace zero endomorphisms.

Let us denote by $N$ the bundle over $\mathbb{P}(V^*)$ whose fiber over $l$ is equal to $X_l$ and (2.10). Clearly $N$ is $GL(V)$-equivariant. Thus so far we obtained that the map $\mathcal{H}$ uniquely factorizes via a linear map

$$\mathcal{H} : C^\infty(\mathbb{P}(V^*), N) \rightarrow C^\infty(G_1(V), M) \quad (2.11)$$

which is an injective $GL(V)$ equivariant map. As we have mentioned previously, $Im(\mathcal{H}) \subset \Xi$. Hence

$$Im(\mathcal{H}) \subset \Xi. \quad (2.12)$$

It is easy to see that the composition

$$S \circ \mathcal{H} : C^\infty(\mathbb{P}(V^*), N) \rightarrow C^\infty(Gr_1(V), L)$$

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is a non-zero $GL(V)$-equivariant map. Since the target is an irreducible representation, the image of $S \circ \mathcal{H}$ is everywhere dense. Furthermore the Casselmann-Wallach theorem (see Theorem 3.9 below) implies immediately that $S \circ \mathcal{H}$ is onto, but we will not need this fact.

Let us show that

$$\text{Im}(\mathcal{H}) \cap \text{Ker}S \neq \{0\}.$$  

(2.13)

A short proof of this claim was personally communicated to us by S. Sahi. We reproduce here his argument.

Let us assume on the contrary that $\text{Im} \mathcal{H} \cap \text{Ker}S = \{0\}$. Then

$$S \circ \mathcal{H} : C^\infty(\mathbb{P}(V^*), N) \longrightarrow C^\infty(\text{Gr}_1(V), L)$$

is an isomorphism of the underlying Harish-Chandra modules (and in fact of the actual spaces by the Casselmann-Wallach theorem, though we will not need this fact here). We will get a contradiction by comparing the $SO(3)$-actions of them.

Let us fix a Euclidean metric and an orientation on $V$. Then the space $C^\infty(\text{Gr}_1(V), L)$ can be $SO(3)$-equivariantly identified with the space of smooth functions on $\mathbb{RP}^2$. It is well known that this space is a direct sum of irreducible representations of $SO(3)$ with highest weights $2m$, $m \in \mathbb{Z}_{\geq 0}$, each entering with multiplicity 1 exactly.

Now let us show that $C^\infty(\mathbb{P}(V^*), N)$ contains some extra irreducible $SO(3)$-representations. In the following computation the symbol ”0” on the bottom denotes the traceless part. We have

$$N|_l = l \otimes \text{End}_0(V^*/l) \otimes \text{or}(V) =$$

$$l \otimes [(V^*/l)^* \otimes (V^*/l)]_0 \otimes \text{or}(V) \overset{SO(3)}{\simeq}$$

$$l \otimes [(V^*/L)^{\otimes 2}]_0 =$$

$$l \otimes (\wedge^2(V^*/l) \oplus \text{Sym}^2(V^*/l))_0).$$

But $l \otimes \wedge^2(V^*/l) = \wedge^3 V^* \overset{SO(3)}{\simeq} \mathbb{R}$. Thus we see that the $SO(3)$-module $C^\infty(\mathbb{P}(V^*), N)$ is a direct sum of the space of functions on $\mathbb{P}(V^*)$ (which coincides with the previous space) and another infinite dimensional space. This implies (2.13).

Since $S \circ \mathcal{H} \neq 0$ and $\text{Im} \mathcal{H} \cap \text{Ker}S \neq 0$, it follows that the length of $C^\infty(\mathbb{P}(V^*), N)$ as $GL(V)$-module is at least 2. The injectivity of $\mathcal{H}$ and (2.12)
imply
\[ 2 \leq \text{length}(C^\infty(\mathbb{P}(V^*), N)) \leq \text{length}(\Xi). \]  
(2.14)

But by Proposition 2.2 and Lemma 1.7 we obtain
\[ \text{length}(\Xi) < 3. \]

Consequently
\[ \mathcal{H}(C^\infty(\mathbb{P}(V^*), N)) = \Xi, \]
and both have length 2.

This implies Theorem 2.5.

3 Reminder on representation theory.

We recall the Beilinson-Bernstein theorem on localization of \( g \)-modules following [10], and the Casselman-Wallach theorem [19], [42]. Let \( G \) be a complex reductive algebraic group. Let \( T \) denote a Cartan subgroup of \( G \). Let \( B \) be a Borel subgroup of \( G \) containing \( T \). Let \( g \) denote the Lie algebra of \( G \), \( t \) the Lie algebra of \( T \), and \( b \) the Lie algebra of \( B \). Let \( n \) denote the nilpotent radical of \( b \).

3.1 Remark. In our applications we will take \( G = GL_3(\mathbb{C}) \), \( T \) will be the subgroup of diagonal invertible matrices, \( B \) will be the subgroup of upper triangular invertible matrices, \( n \) is the Lie algebra of upper triangular matrices with zeros on the diagonal.

Let \( R(t) \subset t^* \) be the set of roots of \( t \) in \( g \). The set \( R(t) \) is naturally divided into the set of roots whose root spaces are contained in \( n \) and its complement. The latter set is denoted by \( R^+(t) \); it is equal to the set of roots of \( t \) in \( g/b \). If \( \alpha \) is a root of \( t \) in \( g \) then the dimension of the corresponding root subspace \( g_\alpha \) is called the multiplicity of \( \alpha \). Let \( \rho_b \) be the half sum of the roots contained in \( R^+(t) \) counted with their multiplicities.

For a root \( \alpha \) let \( \alpha^V \in t \) denote the corresponding coroot (see e.g. [21], §2.4.2). We say that \( \lambda \in t^* \) is dominant if for any root \( \alpha \in R^+(t) \) we have \( \langle \lambda, \alpha^V \rangle \neq -1, -2, \ldots \). We shall say that \( \lambda \in t^* \) is regular if for any root \( \alpha \in R^+(t) \) we have \( \langle \lambda, \alpha^V \rangle \neq 0. \)
For the definitions and basic properties of the sheaves of twisted differential operators we refer to [17]. Here we will present only the explicit description of the sheaf \( D_\lambda \) in order to agree about the normalization.

Let \( X \) be the full flag variety of \( G \) (then \( X = G/B \)). \( X \) can be identified with the variety of all Borel subalgebras of \( g \). Let \( O_X \) denote the sheaf of regular functions on \( X \). Let \( U(g) \) denote the universal enveloping algebra of \( g \). Let \( U^o \) be the sheaf \( U(g) \otimes_C O_X \), and \( g^o := g \otimes_C O_X \). Let \( T_X \) be the tangent sheaf of \( X \). We have the canonical morphism \( \alpha : g^o \rightarrow T_X \).

Let also \( b^o := \text{Ker} \alpha = \{ \xi \in g^o | \xi_x \in b_x \forall x \in X \} \), where \( b_x \subset g \) denotes the Borel subalgebra corresponding to \( x \in X \). Let \( \lambda : b \rightarrow \mathbb{C} \) be a linear functional which is trivial on \( n \) (thus \( \lambda \in t^* \)). Since \( b_x/[b_x,b_x] \) are canonically isomorphic for different \( x \in X \), \( \lambda \) defines a morphism \( \lambda^o : b^o \rightarrow O_X \). We will denote by \( D_\lambda \) the sheaf of twisted differential operators corresponding to \( \lambda - \rho_b \), i.e. \( D_\lambda \) is isomorphic to \( U^o/I_\lambda \), where \( I_\lambda \) is the two sided ideal generated by the elements of the form

\[
\xi - (\lambda - \rho_b)^o(\xi),
\]

where \( \xi \) is a local section of \( b^o \). Let \( D_\lambda := \Gamma(X,D_\lambda) \) denote the ring of global sections of \( D_\lambda \). We have a canonical morphism \( U(g) \rightarrow D_\lambda \). For the full flag variety \( X \) this map is onto ([10]). Let us also denote by \( D_\lambda - \text{mod} \) (resp. \( D_\lambda - \text{mod} \)) the category of \( D_\lambda \)- (resp. \( D_\lambda \)-) modules.

In this notation one has the following result due to Beilinson and Bernstein [10].

**3.2 Theorem (Beilinson-Bernstein).** (1) If \( \lambda \in t^* \) is dominant then the functor of global sections \( \Gamma : D_\lambda - \text{mod} \rightarrow U(g) - \text{mod} \) is exact, namely \( \Gamma \) maps exact sequences to exact ones.

(2) If \( \lambda \in t^* \) is dominant and regular then the functor \( \Gamma \) is also fully faithful, namely \( \Gamma \) induces a bijection on the set of morphisms between any two objects of \( D_\lambda - \text{mod} \).

**3.3 Remark.** Let us notice that when \( \lambda = \rho_b \) we get the usual (untwisted) ring \( D \) and sheaf \( D \) of differential operators. This \( \lambda = \rho_b \) is regular and dominant, and hence the Beilinson-Bernstein theorem is applicable.

Note also that the functor \( \Gamma \) can be considered as taking values in the category \( D_\lambda - \text{mod} \), rather than \( U(g) - \text{mod} \). Considered in this way, \( \Gamma \) has a left adjoint functor, called the Beilinson-Bernstein localization functor, \( \Delta : D_\lambda - \text{mod} \rightarrow D_\lambda - \text{mod} \). It is defined as \( \Delta(M) = D_\lambda \otimes_{D_\lambda} M \).
Part (1) of the next lemma was proved in [10], parts (2), (3) can be found in [17], Proposition I.6.6.

3.4 Lemma. Suppose $\Gamma : D_\lambda - \text{mod} \rightarrow D_\lambda - \text{mod}$ is exact. Then

(1) the localization functor $\Delta : D_\lambda - \text{mod} \rightarrow D_\lambda - \text{mod}$ is the right inverse of $\Gamma$:

$$\Gamma \circ \Delta = \text{Id};$$

(2) $\Gamma$ sends simple objects to simple ones or to zero.

(3) $\Gamma$ sends distinct simple objects to distinct ones or to zero.

Now let us discuss the Casselman-Wallach theorem.

3.5 Definition. Let $\pi$ be a continuous representation of a Lie group $G_0$ in a Fréchet space $F$. A vector $\xi \in F$ is called $G_0$-smooth if the map $g \mapsto \pi(g)\xi$ is an infinitely differentiable map from $G_0$ to $F$.

It is well known (see e.g. [42], Section 1.6) that the subset $F^{sm}$ of smooth vectors is a $G_0$-invariant linear subspace dense in $F$. Moreover it has a natural topology of a Fréchet space (which is stronger than the topology induced from $F$), and the representation of $G_0$ in $F^{sm}$ is continuous. Moreover all vectors in $F^{sm}$ are $G_0$-smooth.

Let $G_0$ be a real reductive group. Assume that $G_0$ can be imbedded into the group $GL_N(\mathbb{R})$ for some $N$ as a closed subgroup invariant under the transposition. Let us fix such an imbedding $p : G_0 \hookrightarrow GL_N(\mathbb{R})$. (In our applications $G_0$ will be either $GL_n(\mathbb{R})$ or a direct product of several copies of $GL_n(\mathbb{R})$.) Let us introduce a norm $| \cdot |$ on $G_0$ as follows:

$$|g| := \max\{||p(g)||, ||p(g^{-1})||\}$$

where $|| \cdot ||$ denotes the usual operator norm in $\mathbb{R}^N$.

3.6 Definition. Let $\pi$ be a smooth representation of $G_0$ in a Fréchet space $F$ (namely $F^{sm} = F$). One says that this representation has moderate growth if for each continuous semi-norm $\lambda$ on $F$ there exists a continuous semi-norm $\nu_\lambda$ on $F$ and $d_\lambda \in \mathbb{R}$ such that

$$\lambda(\pi(g)v) \leq |g|^{d_\lambda} \nu_\lambda(v)$$

for all $g \in G$, $v \in F$. 
The proof of the next lemma can be found in [42], Lemmas 11.5.1 and 11.5.2.

3.7 Lemma. (i) If \((\pi, G_0, H)\) is a continuous representation of \(G_0\) in a Banach space \(H\), then \((\pi, G_0, H^{sm})\) has moderate growth.

(ii) Let \((\pi, G_0, V)\) be a representation of moderate growth. Let \(W\) be a closed \(G_0\)-invariant subspace of \(V\). Then the representations of \(G_0\) in \(W\) and \(V/W\) have moderate growth.

Recall that a continuous Fréchet representation \((\pi, G_0, F)\) is said to have finite length if there exists a finite filtration

\[0 = F_0 \subset F_1 \subset \cdots \subset F_m = F\]

by \(G_0\)-invariant closed subspaces such that \(F_i/F_{i-1}\) is irreducible for any \(i\), i.e. it does not have proper closed \(G_0\)-invariant subspaces. The sequence of all consecutive quotients

\[F_1, F_2/F_1, \ldots, F_m/F_{m-1}\]

is called the Jordan-Hölder series of the representation \(\pi\). It is well known (and easy to see) that the Jordan-Hölder series of a finite length representation is unique up to a permutation.

3.8 Definition. A Fréchet representation \((\rho, G_0, F)\) of a real reductive group \(G_0\) is called admissible if its restriction to a maximal compact subgroup \(K\) of \(G_0\) contains an isomorphism class of any irreducible representation of \(K\) with at most finite multiplicity. (Recall that a maximal compact subgroup of \(GL_n(\mathbb{R})\) is the orthogonal group \(O(n)\).)

3.9 Theorem (Casselman-Wallach [19], [42]). Let \(G_0\) be a real reductive group. Let \((\rho, G_0, F_1)\) and \((\pi, G_0, F_2)\) be smooth representations of moderate growth in Fréchet spaces \(F_1, F_2\). Assume in addition that \(F_2\) is admissible of finite length. Then any continuous morphism of \(G_0\)-modules \(f : F_1 \to F_2\) has closed image.

4 Representations of \(GL_3(\mathbb{R})\)

Let \(P\) be the Borel subgroup \(GL_3(\mathbb{R})\) consisting of upper triangular \(3 \times 3\) matrices. The positive roots are \((-1,1,0), (-1,0,1), (0,-1,1)\). Hence the half sum of the positive roots is \(\rho = (-1,0,1)\).
The main goal of this section is to prove Proposition 2.2. We will immediately see that Proposition 2.2 is equivalent to the following result.

4.1 Theorem. Let $\psi(x, y, z) = |x|^{-1}$. Then the representation $\text{Ind}^G_P\psi$ (the induction is not normalized!) has length three, and all the irreducible subquotients are pairwise non-isomorphic.

The equivalence of Proposition 2.2 and Theorem 4.1 follows from an easy observation that the two induced representations considered in these statements are associate, and the well known fact (see e.g. [41], Proposition 4.1.20) that associate representations have the same Jordan-Hölder series.

To prove Theorem 4.1, let us observe first that the character $\psi$ is dominant and regular. Let us denote by $D_\psi$ the corresponding sheaf of twisted differential operators on the flag manifold $X$.

Now let us construct an $O(3, \mathbb{C})$-equivariant $D_\psi$-module $M_\psi$ on $X$ such that the $U(g)$-module of global sections $\Gamma(X, M_\psi)$ coincides with the Harish-Chandra module of $\text{Ind}^G_P\psi$. Let $U$ be the open $O(3, \mathbb{C})$-orbit in $X$. Let $j: U \hookrightarrow X$ be the identity imbedding (which is an affine morphism). Fix a flag $(E \subset F) \in U$. The category of $O(3, \mathbb{C})$-equivariant coherent $D_\psi$-modules on $U$ is equivalent to the category of finite dimensional representations of the group of connected components of the stabilizer of $(E, F)$ in $O(3, \mathbb{C})$. This stabilizer is equal to

$$O(1) \times O(1) \times O(1) \simeq \{\pm 1\} \times \{\pm 1\} \times \{\pm 1\}.$$ 

Let us consider the representation of this group given by

$$(A, B, C) \mapsto A.$$

Let $M_{\psi 0}$ be the corresponding $O(3, \mathbb{C})$-equivariant $D_\psi$-module on $U$. (It is easy to see that $M_{\psi 0}$ is a locally free rank one $O_U$-module.) Define

$$M_\psi := j_* M_{\psi 0}.$$ 

It is not hard to see that $M_\psi$ is the required $D_\psi$-module.

By the Beilinson-Bernstein Theorem 3.2 and Lemma 3.4, Theorem 4.1 is equivalent to saying that $M_\psi$ has length three with non-isomorphic irreducible subquotients in the category of $O(3, \mathbb{C})$-equivariant $D_\psi$-modules.

The sheaf $D_\psi$ is the sheaf of differential operators acting on algebraic sections of the $GL_3(\mathbb{C})$-equivariant line bundle $L$ over $X$ which is defined as
follows: for a complex flag \((E \subset F)\) in \(\mathbb{C}^3\), the fiber of \(L\) over \((E \subset F)\) is equal to \(E^*\). Then we have an equivalence of categories of \(O(3, \mathbb{C})\)-equivariant \(D_\psi\)-modules and of \(O(3, \mathbb{C})\)-equivariant \(D\)-modules (here \(D\) denotes the sheaf of usual untwisted differential operators on \(X\)) which is given by \(\mathcal{F} \mapsto \mathcal{F} \otimes_{\mathcal{O}_X} L^{-1}\), where \(\mathcal{O}_X\) denotes the sheaf of regular functions on \(X\).

Thus we have to prove the following proposition.

**4.2 Proposition.** The \(O(3, \mathbb{C})\)-equivariant \(D\)-module
\[
\mathcal{M} := \mathcal{M}_\psi \otimes_{\mathcal{O}_X} L^{-1}
\] (4.1)
has length three and all the irreducible subquotients are pairwise non-isomorphic.

Now let us describe explicitly the \(O(3, \mathbb{C})\)-orbits on the variety of complete flags in \(\mathbb{C}^3\). Let us denote by \(X\) the variety of complete flags in \(\mathbb{C}^3\), namely the set of pairs \((E, F)\) where \(E\) is a complex line, \(F\) is a complex 2-plane, and \(E \subset F\). The following proposition is essentially well known and its proof is left to the reader as an (easy) exercise. We denote by \(\mathcal{B}\) a non-degenerate symmetric bilinear form on \(\mathbb{C}^3\) such that \(O(3, \mathbb{C})\) is the group of linear transformations of \(\mathbb{C}^3\) preserving this form.

**4.3 Proposition.** Two complete flags \((E, F)\) and \((E', F')\) in \(\mathbb{C}^3\) belong to the same \(O(3, \mathbb{C})\)-orbit if and only if the following two conditions are satisfied:
(i) ranks of the restrictions of \(\mathcal{B}\) to \(E\) and to \(E'\) are equal;
(ii) ranks of the restrictions of \(\mathcal{B}\) to \(F\) and to \(F'\) are equal.

**4.4 Remark.** It is easy to see that the rank of the restriction of \(\mathcal{B}\) to any line in \(\mathbb{C}^3\) is either 0 or 1, and the rank of the restriction of \(\mathcal{B}\) to any plane in \(\mathbb{C}^3\) is either 1 or 2.

Proposition 4.3 implies that the \(O(3, \mathbb{C})\)-orbits on \(X\) are described by the following diagram:

\[
\begin{array}{ccc}
(0,0) & \searrow & (0,1) \\
\nearrow & & \searrow \\
(1,0) & & (1,1)
\end{array}
\]
Let us explain the notation used in this diagram. Here the pair of numbers \((i, j)\) corresponds to the orbit of flags \((E \subset F)\) such that the restriction of the form \(B\) to \(E\) has a kernel of dimension \(i\), and the restriction of the form \(B\) to \(F\) has a kernel of dimension \(j\). The arrows point from an orbit to another orbit contained in its closure.

One can easily check the following lemma.

**4.5 Lemma.**
1. The orbit \((0, 0)\) is open.
2. The codimensions of the orbits \((0, 1)\) and \((1, 0)\) are equal to 1.
3. The codimension of the orbit \((1, 1)\) is equal to 2.

Let us start proving Proposition 4.2 by describing the module \(\mathcal{M}\) more explicitly. Let \(U\) be the open \(O(3, \mathbb{C})\)-orbit in \(X\), namely \(U\) corresponds to \((0, 0)\) in the above diagram. Let \(j : U \hookrightarrow X\) be the identity imbedding. Fix a flag \((E, F) \in U\). The category of \(O(3, \mathbb{C})\)-equivariant coherent \(\mathcal{D}\)-modules on \(U\) is equivalent to the category of finite dimensional representations of the group of connected components of the stabilizer of \((E, F)\) in \(O(3, \mathbb{C})\). This stabilizer is isomorphic to the group \(O(1) \times O(1) \times O(1)\). The representation of this group corresponding to the restriction \(M_0\) of \(M\) to \(U\) is equal to \((A, B, C) \mapsto A\) (recall that \(O(1) = \{+1, -1\}\)). Then one can easily see that

\[
\mathcal{M} = j_* M_0, \tag{4.2}
\]

where \(j_*\) denotes the push-forward in the category of \(\mathcal{D}\)-modules.

Let us now come back to our situation and let us describe the perverse sheaf corresponding to our \(\mathcal{D}\)-module \(\mathcal{M}\). Let us denote by \(M\) (resp. \(M_0\)) the \(O(3, \mathbb{C})\)-equivariant perverse sheaf on \(X\) (resp. \(U\)) corresponding to \(\mathcal{M}\) (resp. \(\mathcal{M}_0\)) via the Riemann-Hilbert correspondence (we refer to [13] for this notion). Clearly \(M = j_* M_0\) where \(j_*\) is the push-forward in the derived category of sheaves; since \(j : U \hookrightarrow X\) is an affine open imbedding, perverse sheaves go to perverse sheaves (see [11]). It is clear that \(M_0\) is a local system on \(U\) of rank one, shifted by 3 to the left. The following lemma describes the monodromies of \(M_0\) around the codimension one orbits \((0, 1)\) and \((1, 0)\).

**4.6 Lemma.** The monodromy of \(M_0\) around the orbit \((1, 0)\) is equal to \(-1\), and its monodromy around the orbit \((0, 1)\) is equal to 1.

The proof is by a direct computation; it is left to the reader. Let us now discuss the local system \(M_0\) near the minimal orbit \((1, 1)\). Let \(\mathcal{T}\) denote a
normal slice at some point on this orbit. It will be shown below that $T$ is isomorphic to $\mathbb{C}^2$ with the following stratification:

$$T = S \cup P \cup Q \cup \{0\}, \quad (4.3)$$

where $P$ and $Q$ are parabolas passing through the origin $0$ minus this origin and such that their closures $\bar{P}$ and $\bar{Q}$ are tangent to each other at $0$. $S$ is the open strata which is the complement of the two parabolas. $P$ is the intersection of $T$ with the orbit $(0, 1)$, and $Q$ is the intersection of $T$ with the orbit $(1, 0)$. Let us describe the stratification $(4.3)$ explicitly.

Let us choose a basis $e_1, e_2, e_3$ in $\mathbb{C}^3$ so that the matrix $B$ of our quadratic form in this basis is

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

Consider the flag $x_0 := (E_0 \subset F_0)$ defined by $E_0 = \text{span}\{e_1\}$, $F_0 = \text{span}\{e_1, e_2\}$. Clearly, this flag belongs to the minimal orbit. By a straightforward computation one easily checks that the Lie algebra of the stabilizer of $x_0$ in $O(3, \mathbb{C})$ consists of the matrices of the form

$$\text{Lie}(\text{St}(x_0)) = \left\{ \begin{bmatrix} a & b & 0 \\ 0 & 0 & -b \\ 0 & 0 & -a \end{bmatrix} \right\}.$$

The transversal in $\text{Lie}(O(3, \mathbb{C}))$ to this subalgebra can be chosen to be equal to

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ -c & 0 & 0 \\ 0 & c & 0 \end{bmatrix} \right\}.$$

It also can be identified with the tangent space at $x_0$ of the $O(3, \mathbb{C})$-orbit of $x_0$.

Consider now the set of matrices of the form $U = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & a & 0 \end{bmatrix}$. Consider the vectors

$$\xi_1(a, b) = e_1 + Ue_1 = \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}, \quad \xi_2(a, b) = e_2 + Ue_2 = \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix}.$$
Consider the flag \( X(a, b) := (E(a, b) \subset F(a, b)) \), where we denote \( E(a, b) = \text{span}\{\xi_1(a, b)\} \) and \( F(a, b) = \text{span}\{\xi_1(a, b), \xi_2(a, b)\} \). It is easy to see that the flags \( X(a, b) \) with \( a, b \in \mathbb{C} \), form a transversal to the minimal orbit; this transversal will be denoted by \( T \). Let us now compute the intersection of this transversal with the other orbits. We have:

\[
\begin{align*}
B(\xi_1(a, b), \xi_1(a, b)) &= \begin{bmatrix} 1 & a & b \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 2b + a^2, \\
B(\xi_1(a, b), \xi_2(a, b)) &= \begin{bmatrix} 1 & a & b \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix} = 2a, \\
B(\xi_2(a, b), \xi_2(a, b)) &= \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix} = 1.
\end{align*}
\]

Thus we obtain that the matrix of the restriction of the quadratic form \( B \) to \( F(a, b) \) is equal to \[
\begin{bmatrix}
2b + a^2 & 2a \\
2a & 1
\end{bmatrix}.
\]

Thus we see that the intersection of the transversal with the closure of the orbit \((0, 1)\) consists of pairs \( \tilde{P} := \{(a, b)|2b = 3a^2\} \), and the intersection of the transversal with the closure of the orbit \((1, 0)\) consists of the pairs \( \tilde{Q} := \{(a, b)|2b = -a^2\} \). In particular we see that \( \tilde{P} \) and \( \tilde{Q} \) are two different parabolas on \( \mathbb{C}^2 \) with a single common point \((0, 0)\) and which are tangent to each other at this point. To be consistent with the above notation, denote \( P := \tilde{P}\setminus\{0\}, Q := \tilde{Q}\setminus\{0\} \).

Let us agree on the following notation: for any \( O(3, \mathbb{C})\)-equivariant perverse sheaf \( T \) on \( X \) or on a locally closed subset of \( X \), we will denote by \( \tilde{T} \) the restriction of \( T \) to the intersection of the transversal \( T \) with this subset. Then Lemma 4.6 implies that in this notation \( \tilde{M}_0 \) is a rank one local system on \( S \) (up to a cohomological shift) with monodromy +1 around \( P \), and monodromy -1 around \( Q \). Let \( \tilde{j} : S \rightarrow T \) denote the identity imbedding. It is easy to see that the restriction \( \tilde{j}_*\tilde{M}_0 \) is equal to \( \tilde{j}_*\tilde{M}_0 \). We will need the following lemma.

**4.7 Lemma.** The perverse sheaf \( \tilde{j}_*\tilde{M}_0 \) has length 3. Moreover the smooth parts of all the irreducible subquotients are supported on different strata, and they are rank one local systems (up to a cohomological shift).
Before we prove Lemma 4.7 let us finish the proof of Theorem 4.1. It is easy to see that the functor of restriction to the transversal $T$ is an exact and faithful functor from the category of $O(3, \mathbb{C})$-equivariant perverse sheaves on $X$ to the category of perverse sheaves on $T$ (this is because $T$ intersects every orbit). This and Lemma 4.7 imply that $j_*M_0$ has length at most 3 as $O(3, \mathbb{C})$-equivariant perverse sheaf. Hence it remains to show that the restriction to $T$ of any irreducible subquotient of $j_*M_0$ (as $O(3, \mathbb{C})$-equivariant perverse sheaf) is an irreducible perverse sheaf; that clearly will imply that the length of $j_*M_0$ is equal to 3.

Let $N$ be an irreducible subquotient of $j_*M_0$ as an $O(3, \mathbb{C})$-equivariant perverse sheaf. Consider its restriction $\tilde{N}$ to $T$. We want to prove that $\tilde{N}$ is irreducible. Assume in the contrary that it is not irreducible. By Lemma 4.7 the smooth parts of different irreducible subquotients of $\tilde{N}$ are supported on different strata, and all of these smooth parts are rank one local systems.

Any irreducible $O(3, \mathbb{C})$-equivariant perverse sheaf, can be constructed as follows. There exists a unique $O(3, \mathbb{C})$-orbit $O \subset X$ and an irreducible $O(3, \mathbb{C})$-equivariant local system $L$ on $O$ such that $N$ is isomorphic to the Goresky-MacPherson extension of $L$. It is easy to see that $\tilde{N}$ is the Goresky-MacPherson extension of $\tilde{L}$.

Let us show that $rk(L) = 1$. Assume in the contrary that $rk(L) \geq 2$. Then $\tilde{N}$ is not irreducible by Lemma 4.7 moreover there are at least $rk(L)$ irreducible subquotients such that their smooth part is a rank one local system supported on $O \cap T$. This contradicts Lemma 4.7 according to which the smooth parts of all irreducible subquotients of $\tilde{N}$ must be supported on different orbits.

Thus $\tilde{L}$ is a rank one local system. Its Goresky-MacPherson extension must be irreducible. Thus $\tilde{N}$ is irreducible. Thus we have shown that the restriction to $T$ maps irreducible $O(3, \mathbb{C})$-equivariant subquotients of $j_*M_0$ to irreducible perverse sheaves. Hence length of $M = j_*M_0$ is equal to 3.

Now it remains to prove Lemma 4.7.

Proof of Lemma 4.7. It will be more convenient to make a change of variables on the transversal $T$ as follows:

$$z = a,$$

$$w = \frac{1}{2}(\frac{a^2}{2} + b).$$
In coordinates \((z, w)\)

\[ \bar{Q} = \{ w = 0 \}, \bar{P} = \{ w = z^2 \}. \]

Recall that on \(S = \mathcal{T}\backslash (\bar{P} \cup \bar{Q})\) we have a rank one local system \(\tilde{M}_0\) whose monodromy around \(\bar{P}\) is equal to +1, and around \(\bar{Q}\) to \(-1\). Let \(j: S \rightarrow \mathcal{T}\) be the identity imbedding. We want to show that \(j_*\tilde{M}_0\) is a length 3 perverse sheaf, and the smooth parts of different irreducible subquotients are (appropriately shifted) rank one local systems with different supports.

Let \(f: S \rightarrow \mathcal{T}\backslash \bar{Q}\) be the identity imbedding, which is an affine morphism. Since the monodromy of \(\tilde{M}_0\) around \(P\) is equal to +1, there is a short exact sequence of perverse sheaves:

\[ K[2] \rightarrow f_*\tilde{M}_0 \rightarrow N[1], \quad (4.4) \]

where \(K\) is a rank one local system on \(\mathcal{T}\backslash \bar{Q}\), and \(N\) is a rank one local system on \(P(= \bar{P}\backslash \{0\})\).

Let \(g: \mathcal{T}\backslash \bar{Q} \rightarrow \mathcal{T}\) be the identity imbedding. It is an affine imbedding, hence \(g_*\) is an exact functor between categories of perverse sheaves. Also \(j = g \circ f\). Hence applying \(g_*\) to (4.4) we get a short exact sequence of perverse sheaves on \(\mathcal{T}\)

\[ g_*K[2] \rightarrow j_*\tilde{M}_0[2] \rightarrow j_*N[1]. \quad (4.5) \]

It is easy to see that the monodromy of \(K\) around \(\bar{Q}\) is equal to \(-1\). Hence \(g_*K[2]\) is an irreducible perverse sheaf; clearly its smooth part is the rank one local system \(K[2]\) supported on \(\mathcal{T}\backslash \bar{Q}\).

It remains to show that \(j_*N[1]\) has length 2, and the smooth parts of the two irreducible subquotients are (shifted) rank one local systems on \(P\) and \(\{0\}\) respectively. Let us denote by \(h: P \rightarrow \bar{P}\) the identity map. It is equivalent to prove that \(h_*N[1]\) has length 2, and the irreducible subquotients have appropriate supports. It suffices to show that the monodromy of \(N\) around \(0 \in \bar{P}\) is equal to +1. To prove this, let us fix the loop around 0 in \(P\) given by

\[ \{ (e^{i\theta}, e^{2i\theta}) \mid 0 \leq \theta \leq 2\pi \}. \quad (4.6) \]

Around each point of this loop let us fix a small loop

\[ \{ (e^{i\theta}, e^{2i\theta} + \varepsilon e^{i\omega}) \mid 0 \leq \omega \leq 2\pi \}, \quad (4.7) \]
where $0 < \varepsilon < 1$ is fixed, $\theta \in [0, 2\pi]$ is arbitrary. It is easy to see that each point on the loops (4.7) belongs to the open stratum $S \subset T$, and all these points together form a 2-dimensional real torus $T^2$. For a fixed $\theta$, the loop (4.7) is a loop in $S$ around $P$.

Let us write the torus $T^2$ as a product of circles $T^2 = S_1 \times S_2$, where $S_1$ corresponds to the parameter $\theta$ in (4.7), and $S_2$ to $\omega$. Let $\pi_1: T^2 \to S_1$ denote the natural projection.

Let us denote by $\bar{M}_0$ the restriction of the (shifted by 2) local system $M_0$ to the torus $T^2$. Clearly the restriction of $N[1]$ to the circle (4.6) is equal to $R^1\pi_1\ast (\bar{M}_0)$ (where $R^1\pi_1\ast$ is the first right derived functor of the push-forward under $\pi_1$) which is a rank one local system on the circle $S_1$. We want to show that its monodromy is equal to +1.

Any local system on torus $T^2$ is characterized (up to isomorphism) by its monodromies along the meridians. The monodromy of $\bar{M}_0$ along the meridian $S_2$ is equal to +1. Then evidently the monodromy of $R^1\pi_1\ast \bar{M}_0$ is equal to the monodromy of $\bar{M}_0$ along the meridian $S_1$, which we can choose to be

$$\{(e^{i\theta}, e^{2i\theta} + \varepsilon) | \theta \in [0, 2\pi]\}. \quad (4.8)$$

Thus it remains to show that the monodromy of $\bar{M}_0$ along the circle (4.8) is equal to +1. To do that, let us go back from the torus to $\mathbb{C}^2 = T$. In $\mathbb{C}^2$ we can close the loop (4.8) by a disk $D$ defined by

$$D := \{(z, z^2 + \varepsilon) | |z| \leq 1\}.$$ 

Clearly $\partial D$ equals the circle (4.8); $D$ does not intersect $\bar{P}^\circ$; and does intersect $\bar{Q}$ in exactly two points $(\pm i\sqrt{\varepsilon}, 0)$, in both of them the intersection is transversal. The latter implies that the monodromy of $\bar{M}_0$ around $\partial D$ is equal to the product of monodromies of the two loops, each surrounding once exactly one of the the two points of intersection. But the monodromy of each such loop is equal to $-1$, hence their product is +1. Hence the monodromy of $\bar{M}_0$ around (4.8) is +1. Hence the monodromy of $N$ around $0 \in \bar{P}$ is +1. Lemma 4.7 is proved.

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