Global Existence and Compact Attractors for the Discrete Nonlinear Schrödinger equation

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Abstract

We study the asymptotic behavior of solutions of discrete nonlinear Schrödinger-type (DNLS) equations. For a conservative system, we consider the global in time solvability and the question of existence of standing wave solutions. Similarities and differences with the continuous counterpart (NLS-partial differential equation) are pointed out. For a dissipative system we prove existence of a global attractor and its stability under finite dimensional approximations. Similar questions are treated in a weighted phase space. Finally, we propose possible extensions for various types of DNLS equations.

1 Introduction

The discrete nonlinear Schrödinger equation (DNLS), is a very popular model with a great variety of applications, ranging from physics to biology. To name but a few, the DNLS has been successfully applied to the modelling of localized pulse propagation in optical fibers and wave guides, to the study of energy relaxation in solids, the behaviour of amorphous material, to the modelling of self-trapping of vibrational energy in proteins or studies related to the denaturation of the DNA double strand (for an account of possible applications see e.g. the recent review [23]). The DNLS serves both as a model in its own right (modelling cases where the nature of the problem is inherently discrete), or as an approximation of the continuous nonlinear Schrödinger through a numerical scheme.

The study of the dynamics of the discrete nonlinear Schrödinger equation, has been an active theme of research in the past decade. The major part of the activity, has focused on the study of localized excitations and nonlinear waves (solitons and breathers), in the conservative DNLS. This activity has started with the ingenious construction introduced in [26] of localized in space time periodic or time-quasiperiodic solutions of discrete lattice systems (which include the DNLS as special case), using the so called anti-integrable limit or anti-continuous limit. Such solutions are essentially found for the weak coupling limit between the lattice sites. Other constructions are available using dynamical systems methods, such as the reduction to a symplectic map and the homoclinic trajectory method. The stability of such solutions has been studied using both analytic and numerical techniques and they have been shown in many cases to be robust under perturbations. They can be applied to the study of diverse physical and biological phenomena such as lattice dynamics of solids, selective bond excitations, localization of electromagnetic waves in photonic crystals with nonlinear response etc. For detailed reviews on this theme we refer to [4], [14], [18]. There have also been recent interesting works on more complicated solutions arising in higher dimensions than one, such as discrete vortex solutions (see eg. [24] or [27]).

In this work we mainly consider the asymptotic behavior of solutions for the following complex lattice dynamical system (LDS)

\[
\begin{align*}
&i\dot{u}_n + \frac{1}{\epsilon}(u_{n-1} - 2u_n + u_{n+1}) + i\delta u_n + F(u_n) = g_n, \quad n \in \mathbb{Z}, \quad \epsilon > 0, \\
&u_n(0) = u_{n,0}, \quad n \in \mathbb{Z}
\end{align*}
\]

The real parameter \(\delta > 0\) introduces weak damping, while the case \(\delta = 0\) and \(g_n = 0\), for all \(n \in \mathbb{Z}\), corresponds to a conservative (NLS) lattice system.
Using techniques from the theory of infinite dimensional systems, we are able to show the global in time existence of solutions to both the conservative and the dissipative DNLS equation. This ensures the well posedness of the model. This result is interesting as it provides a rigorous proof of the fact that blow up is not expected to occur in the discrete models as it may happen in the continuous model, a fact that has been observed numerically by a number of authors [1]. Furthermore, for a finite lattice (assuming Dirichlet boundary conditions), and by using a discrete version of the mountain pass Theorem, we prove the existence of nontrivial standing waves, in the conservative case. For the infinite dimensional case, we show non-existence of nontrivial standing waves for certain values of energy, by using a fixed point argument. For the dissipative case we may further prove a more strong result; the existence of a global attractor, that attracts all bounded sets of the infinite dimensional phase space. It can further be shown that the dynamics on this global attractor may be approximated by the dynamics of finite dimensional systems. Results of this kind, for lattice dynamical systems of first and second order are presented in the recent works [8, 38, 39].

Many of the physically interesting solutions of the DNLS equation present strong spatial localization properties. This is true for instance for soliton solutions or breathers. To cover such situations, we study the DNLS equation in weighted spaces, with properly chosen weight functions. For the dissipative case we may prove global existence of solutions in weighted spaces as well as the existence of a global attractor. Finally, we comment on possible generalizations of the above results in higher spatial dimensions and for more general models.

2 Formulation of the problem

Problem (1)-(2) is an infinite system of ordinary differential equations (infinite dimensional dynamical system). We will formulate the problem in an appropriate functional setting. In what follows, a complex Hilbert space $X$ will be considered as a real Hilbert space with the scalar product $(\cdot, \cdot)_X = \text{Re} B_X(\cdot, \cdot)$. Let $T : D(T) \subseteq X \to X$ a $C$-linear, self-adjoint $\leq 0$ operator with dense domain $D(T)$ on the Hilbert space $X$, equipped with the scalar product $(\cdot, \cdot)_X$. The space $X_T$ is the completion of $D(T)$ in the norm $||u||^2_X = ||u||^2_X - (Tu, u)_X$, for $u \in X_T$, and we denote by $X^*_T$ its dual and by $T^*$ the extension of $T$ to the dual of $D(T)$, denoted by $D(T)^*$. A function $F : X \to X$ is Lipschitz continuous on bounded subsets of $X$, if for all $M > 0$ there exists a constant $L(M)$ such that $||F(u) - F(v)||_X \leq L(R)||u - v||_X$, for all $u, v \in B_M$ the closed ball of $X$, of center 0 and of radius $R$.

For the treatment of (1)-(2), we shall use complexifications of the usual real sequence spaces denoted by

$$
\ell^p := \left\{ u = (u_n)_{n \in \mathbb{Z}} \in \mathbb{C} : ||u||_{\ell^p} := \left( \sum_{n \in \mathbb{Z}} |u_n|^p \right)^{\frac{1}{p}} < \infty \right\}.
$$

Let us recall that between $\ell^p$, spaces the following elementary embedding relation [19, pg. 145] holds,

$$
\ell^q \subset \ell^p, \quad ||u||_{\ell^p} \leq ||u||_{\ell^q}, \quad 1 \leq q \leq p \leq \infty.
$$

We will see in the sequel that relation (4) has important effects concerning the global in time solvability of DNLS equation, especially in the conservative case. For $p = 2$ we get the usual Hilbert space of square-summable (complex) sequences endowed with the real scalar product

$$
(u, v)_{\ell^2} = \text{Re} \sum_{n \in \mathbb{Z}} u_n \overline{v_n}, \quad u, v \in \ell^2.
$$

We describe next the discrete differential operators employed. For any $u, v \in \ell^2$ we consider the linear operators $A, B, B^* : \ell^2 \to \ell^2$,

$$
(Bu)_n = u_{n+1} - u_n, \quad (B^*u)_n = u_{n-1} - u_n,
$$

$$
(Au)_n = (u_{n-1} - 2u_n + u_{n+1}).
$$

Moreover it holds that

$$
(Bu, v)_{\ell^2} = (u, B^*v)_{\ell^2}, \quad (Au, v)_{\ell^2} = -(Bu, Bv)_{\ell^2}, \quad u, v \in \ell^2.
$$

The functional set up will also include (see [38]) the following bilinear form and induced norm
\[
(u, v)_{\ell^2} := (Bu, Bv)_{\ell^2} + (u, v)_{\ell^2},
\]
\[
||u||_{\ell^2}^2 := ||Bu||_{\ell^2}^2 + ||u||_{\ell^2}^2.
\]

We denote by $\ell^2$ the Hilbert space with scalar product (10) and norm (11). The usual norm of $\ell^2$ and (11) are equivalent, since
\[
||u||_{\ell^2}^2 \leq ||u||_{\ell_1^2}^2 \leq c||u||_{\ell^2}^2, \quad c > 0.
\]

For local existence, we examine the following examples of nonlinearities $F : \mathbb{C} \rightarrow \mathbb{C}$:
\(\text{(N)} \) $F(0) = 0$ and there exist constants $c > 0$, $\beta \geq 0$ such that $|F(z_1) - F(z_2)| \leq c(|z_1|^\beta + |z_2|^\beta)|z_1 - z_2|$, or alternatively
\(\text{(N2)} \) $F(z) = f(|z|^2)z$ where $f, f' : \mathbb{R} \rightarrow \mathbb{R}$, are continuous.

Both cases include the power-law nonlinearity $F(z) = |z|^{2\sigma}z$ (for (N2) we must have $1 < \sigma < \infty$). Since we intend to apply the general theory of abstract Schrödinger equations [11, 12], the following Lemma will be useful.

**Lemma 2.1** Let $F : \mathbb{C} \rightarrow \mathbb{C}$ satisfy (N1) or (N2). Then the function $F$ defines an operator (still denoted by $F$)
\[
F : \ell^2 \rightarrow \ell^2, \quad (F(u))_{n \in \mathbb{Z}} := F(u_n),
\]
which is Lipschitz continuous on bounded sets of $\ell^2$.

**Proof:** We consider the case (N1). Let $u \in B_R$, a closed ball in $\ell^2$ of center 0 and radius $R$. We have from (13) that
\[
||F(u)||_{\ell^2}^2 \leq c^2 \sum_{n \in \mathbb{Z}} |u_n|^{2\beta+2} = c^2 ||u||_{\ell^2}^{2\beta+2} \leq c^2 ||u||_{\ell^2}^{2\beta+2}
\]

hence $F : \ell^2 \rightarrow \ell^2$, is bounded on bounded sets of $\ell^2$.

For $u, v \in B_R$, we observe that
\[
||F(u) - F(v)||_{\ell^2}^2 \leq c^2 \sum_{n \in \mathbb{Z}} (|u_n|^\beta + |v_n|^\beta)^2 |u_n - v_n|^2
\]
\[
\leq c^2 \sup_{n \in \mathbb{Z}} [(|u_n|^\beta + |v_n|^\beta)^2] \sum_{n \in \mathbb{Z}} |u_n - v_n|^2 \leq c^2 4R^{2\beta} ||u - v||_{\ell^2}^2,
\]
justifying that the map $F : \ell^2 \rightarrow \ell^2$ is Lipschitz continuous on bounded sets of $\ell^2$ with Lipschitz constant $L(R) = c2R^\beta$.

For the case (N2) we have
\[
||F(u)||_{\ell^2}^2 = \sum_{n \in \mathbb{Z}} |f(|u_n|^2)|^2 |u_n|^2.
\]

Since $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, there exists a monotone increasing $C^1$-function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that
\[
|f(\rho)| \leq g(|\rho|), \quad \text{for all } \rho \in \mathbb{R}.
\]
(see e.g [37], p.g 796]). Note also that $|u_n|^2 \leq ||u_n||_{\ell^2}^2$ for all $n \in \mathbb{Z}$. Now (14) and (17) imply the inequality
\[
||F(u)||_{\ell^2}^2 \leq \sum_{n \in \mathbb{Z}} g(|u_n|^2)^2 |u_n|^2 \leq \sum_{n \in \mathbb{Z}} g(||u||_{\ell^2}^2)^2 |u_n|^2
\]
\[
\leq \left( \max_{\rho \in [0,R^2]} |g(\rho)| \right)^2 \sum_{n \in \mathbb{Z}} |u_n|^2
\]
\[
\leq K(R)^2 ||u||_{\ell^2}^2.
\]
with \( K(R) = g(R^2) \).

To check the Lipschitz property, we may argue as for the proof of (18). For some \( \theta \in (0, 1) \) and an appropriate \( C^1 \)-function \( g_1 : \mathbb{R} \rightarrow \mathbb{R} \), we get the inequality

\[
\|F(u) - F(v)\|_{\ell^2} \leq 2 \sum_{n \in \mathbb{Z}} |f(|u_n|^2)|^2|u_n - v_n|^2 + 2 \sum_{n \in \mathbb{Z}} |f(|v_n|^2)|^2|v_n|^2, \quad (19)
\]

\[
\sum_{n \in \mathbb{Z}} |f(|u_n|^2) - f(|v_n|^2)|^2|v_n|^2 = \sum_{n \in \mathbb{Z}} |f'(|u_n|^2 + (1 - \theta)|v_n|^2)|(|u_n| + |v_n|)^2(|u_n - |v_n|)^2|v_n|^2 \leq \left\{ \max_{\rho \in [0,2R^2]} g_1(\rho) \right\}^2 \sup_{n \in \mathbb{Z}} \left[ (|u_n| + |v_n|)^2|v_n|^2 \right] \sum_{n \in \mathbb{Z}} |u_n - v_n|^2. \quad (20)
\]

A combination of (19) and (20), implies that there exists a constant \( L(R) = (2g^2(R^2) + 8R^4 g_1(2R^2))^{1/2} \) such that

\[
\|F(u) - F(v)\|_{\ell^2} \leq L(R)\|u - v\|_{\ell^2}. \quad (21)
\]

The Lemma is proved. \( \diamond \)

We observe by \( \mathbf{10} \), that the operator \( A \) satisfies the relations

\[
(Au, u)_{\ell^2} = -\|Bu\|_{\ell^2}^2 \leq 0, \quad (22)
\]

\[
(Au, v)_{\ell^2} = (u, Av)_{\ell^2}, \quad (23)
\]

therefore defines a self-adjoint operator on \( D(A) = X = \ell^2 \) and \( A \leq 0 \). Note that the graph norm

\[
\|u\|_{D(A)} = \|Au\|_{\ell^2} + \|u\|_{\ell^2},
\]

is also an equivalent norm with the \( \ell^2 \)-norm since

\[
\|u\|_{\ell^2}^2 \leq \sum_{n \in \mathbb{Z}} |u_{n+1} - 2u_n + u_{n-1}|^2 + \sum_{n \in \mathbb{Z}} |u_n|^2 \leq c\|u\|_{\ell^2}^2.
\]

In our case, as it is indicated by \( \mathbf{10} \), \( \mathbf{22} \), and \( \mathbf{23} \), we may choose \( X_A = \ell^2 \) equipped with the norm \( \|u\|_{\ell^2}^2 = \|u\|^2_A - (Au, u)_X \equiv \|u\|_{\ell^2}^2 \), for \( u \in \ell^2 \). Moreover, \( D(A) = X = \ell^2 = D(A)^* \). Obviously \( A^* = A \) and the operator \( iA : \ell^2 \rightarrow \ell^2 \) defined by \( (iA)u = iAu \) for \( u \in \ell^2 \), is \( \mathbb{C} \)-linear and self-adjoint and \( iA \) generates a group \( (T(t))_{t \in \mathbb{R}} \), of isometries on \( \ell^2 \). It is easy to check that the same properties hold for the operator \( \epsilon^{-1}A \) with \( \epsilon > 0 \) (i.e., we may consider the equivalent norm \( \epsilon^{-1}\|Bu\|_{\ell^2}^2 + \|u\|_{\ell^2}^2 \)).

Thus, for fixed \( T > 0 \) and \( (u_{n,0})_{n \in \mathbb{Z}} := u_0 \in \ell^2 \), a function \( u \in C([0, T], \ell^2) \) is a solution of (1)-2 if and only if

\[
u(t) = T(t)u_0 + i\int_0^t T(t - s)F_1(u(s)) ds, \quad (F_1(u))_{n \in \mathbb{Z}} := F_1(u_n) = i\delta u_n + F(u_n) - g_n. \quad (24)\]

Our local existence result can be stated as follows: \( \textbf{Theorem 2.1} \)

Let assumptions \( (N_1) \) or \( (N_2) \) be satisfied, assume that \( g := (g_n)_{n \in \mathbb{Z}} \in \ell^2 \). Then there exists a function \( T^* : \ell^2 \rightarrow (0, \infty) \) with the following properties:

(a) For all \( u_0 \in \ell^2 \), there exists \( u \in C([0, T^*(u_0)), \ell^2) \) such that for all \( 0 < T < T^*(u_0) \), \( u \) is the unique solution of (1)-2 in \( C([0, T], \ell^2) \) (well posedness).

(b) For all \( t \in [0, T^*(u_0)) \),

\[
T^*(u_0) - t \geq \frac{1}{2(L_1(R) + 1)} := T_R, \quad R = \|g\|_{\ell^2} + 2\|u(t)\|_{\ell^2}, \quad L_1(R) = \delta + L(R), \quad (25)
\]

where \( L(R) \) is the Lipschitz constant for the map \( F : \ell^2 \rightarrow \ell^2 \). Moreover the following alternative holds: \( i \) \( T^*(u_0) = \infty \), or \( ii \) \( T^*(u_0) < \infty \) and \( \lim_{t \uparrow T^*(u_0)} \|u(t)\|_{\ell^2} = \infty \) (maximality).

(c) \( T^* : \ell^2 \rightarrow (0, \infty) \) is lower semicontinuous. In addition, if \( \{u_{n_0}\}_{n_0} \) is a sequence in \( \ell^2 \) such that \( u_{n_0} \rightarrow u_0 \) and if \( T < T^*(u_0) \), then \( S(t)u_{n_0} \rightarrow S(t)u_0 \) in \( C([0, T], \ell^2) \), where \( S(t)u_0 = u(t), t \in (0, T^*(u_0)) \), denotes the solution operator (continuous dependence on initial data).
Proof: Since the result is an application of the results in \cite{11} p.g. 56-59, we present only an outline of the proof, for the sake of completeness. It follows from Lemma 2.1 that \( F_1 : \ell^2 \to \ell^2 \) is Lipschitz continuous on bounded sets of \( \ell^2 \) with Lipschitz constant \( L_1(R) = \delta + L(R) \). For all \( u_0 \in \ell^2 \) with \( ||u_0||_{\ell^2} \leq M \) we set \( R = 2M + ||F_1(0)||_{\ell^2} = 2M + ||g||_{\ell^2} \). Following closely the lines of \cite{11} Lemma 4.3.2-Proposition 4.3.3 and using \cite{24} we may show that the map

\[
\Phi_t(u) := T(t)u_0 + i \int_0^t T(t-s)F_1(u(s))ds,
\]

is a contraction on the complete metric space \((X_R, d)\) where

\[
X_R := \{ u \in C([0, T_R], \ell^2) : ||u(t)||_{\ell^2} \leq R, \forall t \in [0, T_R] \}, \quad T_R = \frac{1}{2L_1(R) + 1}, \quad d = \max_{t \in [0, T_R]} ||u(t) - v(t)||_{\ell^2}.
\]

For \( u_0 \in \ell^2 \) we define \( T^*(u_0) = \sup \{ T > 0 : \exists S(t)u_0 = u(t) \in C([0, T], \ell^2) \text{ solution of} \ (24) \} \}. \) Inequality \cite{26} follows by the contradiction argument of \cite{11} Theorem 4.3.4): We assume instead of \( (25) \), that for \( R = ||u(t)||_{\ell^2} \), \( T^*(u_0) - t < T_R \) and we consider the solution \( v \in C([0, T_R], \ell^2) \) of

\[
v(s) = T(s)u(t) + i \int_0^s T(s-\sigma)F_1(v(\sigma))d\sigma, \quad s \in [0, T_R].
\]

Then we can define a solution \( w \in C([0, t + T_R], \ell^2) \) of \( (24) \) at \( T = t + T_R \), defined by

\[
w(s) = \begin{cases} u(s), & s \in [0, t], \\ v(s - t), & s \in [t, t + T_R]. \end{cases}
\]

which is in contradiction with the assumption \( T^*(u_0) < t + T_R \). To prove (c), as in \cite{11} Proposition 4.3.7, we set

\[
R = 2 \sup_{t \in [0, T]} ||u(t)||_{\ell^2}
\]

and define \( \tau_n = \sup \{ t \in [0, T^*(u_0n)] : ||S(s)u_0n||_{\ell^2} \leq 2R, \forall s \in [0, t] \} \). Since \( u_{0n} \to u_0 \) in \( \ell^2 \), it follows that for sufficiently large \( n \), \( ||u_{0n}||_{\ell^2} \leq R \). Therefore \( \tau_n > T_R \) for \( R > 0 \). Moreover, for all \( t \leq T \) and \( t \leq \tau_n \), we get from \cite{24} and Gronwall’s Lemma the inequality

\[
||S(t)u_0n - S(t)u_0||_{\ell^2} \leq e^{L_1(2R)T}||u_0n - u_0||_{\ell^2}.
\]

Letting \( n \to \infty \), we get that \( S(t)u_0n \to S(t)u_0 \) in \( \ell^2 \) and \( \tau_n > T \), which implies that \( T^*(u_0n) > T \). \( \diamond \)

3 Global existence for the conservative case

In this section we discuss the global in time solvability of \cite{11} for the case \( \delta = 0 \) and \( g_n = 0 \), for all \( n \in \mathbb{Z} \) (conservative case). We consider as a model problem, the following DNLS equation with arbitrary power nonlinearity

\[
iu_n + \frac{1}{\epsilon}(u_{n-1} - 2u_n + u_{n+1}) + |u|^2u_n = 0, \quad n \in \mathbb{Z}, \quad 0 \leq \sigma < \infty, \quad \epsilon > 0,
\]

\[
\epsilon u_n(0) = u_{n,0}, \quad n \in \mathbb{Z}
\]

We also consider the (NLS) partial differential equation

\[
i\partial_t u + u_{xx} + |u|^{2\sigma}u = 0, \quad x \in \mathbb{R}, \quad t > 0,
\]

\[
u(x, 0) = u_0(x).
\]

We will show that there is a vast difference concerning the global solvability of the discrete (NLS) equation \cite{26} with \( \epsilon > 0 \) and its continuum limit as \( \epsilon \to 0 \) given by \cite{28}. Concerning \cite{26}, we have the following Theorem.

**Theorem 3.1** Let \( u_0 \in \ell^2 \). For any \( \epsilon > 0 \), and \( 0 \leq \sigma < \infty \), there exists a unique solution of \( (24) \) such that \( u(t) \in C^{1}([0, \infty), \ell^2) \).

**Proof:** The result is a consequence of discrete conservation laws satisfied by the solutions of \( (26) \). We take the scalar product of \( (26) \) with \( iu \). By using \( (22) \) we obtain

\[
\frac{d}{dt} ||u(t)||_{\ell^2} = 0, \quad \text{for every} \quad t \in [0, T^*(u_0)),
\]

i.e. \( ||u(t)||_{\ell^2} \) is uniformly bounded on the maximal interval of existence. It follows then by Theorem \cite{24} (b)-(c) that \( T^*(u_0) = \infty \) and \( \sup \{ ||u(t)||_{\ell^2}, t \in [0, \infty) \} < \infty \). \( \diamond \)
Remark 3.1 Theorem 3.1 can be also established by a discrete version of the conservation of energy, satisfied by solutions of (26). This approach elucidates the role of the nonlinearity exponent, in the global in time solvability of DNLS: It can be easily checked that

\[
E(u(t)) = E(u_0), \quad E(u(t)) := \frac{1}{\epsilon^2} ||u(t)||_{L^2}^2 - \frac{1}{\sigma + 1} ||u(t)||_{L^{2\sigma+2}}^{2\sigma+2}, \quad t \in [0, T^*(u_0)).
\]

Then, by using (31) and (32) we may derive the estimate

\[
||u(t)||_{L^2}^2 \leq ||u_0||_{L^2}^2 + \frac{2\epsilon}{\sigma + 1} ||u_0||_{L^{2\sigma+2}}^{2\sigma+2}. \tag{31}
\]

As a consequence of (31) we obtain once again that \( T^*(u_0) = \infty \) and \( \sup \left\{ ||u(t)||_{L^2}^2, \ t \in [0, \infty) \right\} < \infty \).

Clearly, Theorem 3.1 covers the case of DNLS with a nonlinearity satisfying (N_2). The case of a globally Lipschitz continuous function \( F : [0, \infty) \to \mathbb{R} \) with \( F(0) = 0 \) is also included. Following [11, Section 7.2], [12, p.g 53], such a function can be extended to the complex function \( F : \mathbb{C} \setminus \{0\} \to \mathbb{C} \) by setting \( F(z) = \frac{\epsilon^2}{z^2} F(|z|^2) \). It can be shown as in Lemma 2.1 that \( F : \ell^2 \to \ell^2 \) is locally Lipschitz and \( (F(u), u_0)|_{\ell^2} = 0 \) which implies (26). Setting \( F_\epsilon(z) = \int_0^{|z|^2} F(s) ds \) it can be shown as in Section 4, Lemma 4.2 that the functional \( (V(u))_n := -\sum_{n \in \mathbb{Z}} F_\epsilon(u_n) \) is a \( C^1 \)-functional on \( \ell^2 \) and \( V'(u) = -F(u) \). In this case, we observe conservation of energy defined by \( E(u(t)) := \frac{1}{\epsilon^2} ||u(t)||_{L^2}^2 + V(u) \). \bullet

For a comparison between the discrete (NLS) equation (26) and its continuous counterpart (28), we refer to the main results concerning (28) (see [11, 12, 28]).

For \( u_0 \in H^1(\mathbb{R}) \) and \( 0 \leq \sigma < \infty \) there exists a unique maximal solution of (28), \( u(t) \in C([0,T_{max}),H^1(\mathbb{R})) \cap C^1([0,T_{max}),L^2(\mathbb{R})) \). In addition: If \( \sigma < 2 \) then \( T_{max} = \infty \) and \( u \) is bounded in \( H^1(\mathbb{R}) \). Let \( \sigma \geq 2 \). Assume that \( u_0 \in H^1(\mathbb{R}) \) such that \( \int_\mathbb{R} |x|^2 |u_0|^2 dx < \infty \) (initial data with finite variance) and \( E(u_0) < \infty \). Then \( T_{max} < \infty \).

On the other hand if \( ||u_0||_{H^1} \) is sufficiently small, \( T_{max} = \infty \) and \( u \) is bounded in \( H^1(\mathbb{R}) \).

It follows from Theorem 3.1 that solutions of (26) with \( \epsilon > 0 \), exist globally in \( \ell^2 \)-norm unconditionally with respect to the degree of the nonlinearity the size of the initial data and the sign of the initial energy \( E(u_0) \). This is in agreement with the numerical observations in [11, 12]. According to [11, 12], for a discrete system of the form (26), numerical simulations provide evidence that the solution of the discrete equation still exists after localization, while the solution of the continuum system blows-up in finite time. Note that the discrete version of initial data with finite variance reads as \( \sum_{n \in \mathbb{Z}} n^2 |u_{n,0}|^2 < \infty \) and such data belong to \( \ell^2 \).

4 Existence of standing wave solutions for the conservative DNLS

We conclude the discussion on the conservative DNLS, with a discussion on the existence of standing wave solutions. The standing wave solution for fixed \( \omega^2 \) is given by the ansatz

\[
u_n(t) = e^{i\omega t} \phi_n, \quad n \in \mathbb{Z}, \quad \phi_n \text{ independent of } t. \tag{32}\]

We do not restrict \( \phi_n \) to be real. We are focused on DNLS (26). It can be easily seen, that any standing wave solution of (26), satisfies the following equation

\[
-\frac{1}{\epsilon}(\phi_{n-1} - 2 \phi_n + \phi_{n+1}) + \omega^2 \phi_n = |\phi_n|^{2\sigma} \phi_n, \quad n \in \mathbb{Z}. \tag{33}\]

We will study in this section two related problems. The first problem we will study is given by equation (33), with \( n \) taking finite values. We consider the finite dimensional subspace of \( \ell^2 \)

\[
\mathbb{C}^{2m+1} := \{ \psi_n \in \ell^2 : \psi_{-(m+1)} = \psi_{m+1} = 0 \}.
\]

We consider the following nonlinear system in \( \mathbb{C}^{2m+1} \),

\[
-\frac{1}{\epsilon}(\phi_{n-1} - 2 \phi_n + \phi_{n+1}) + \omega^2 \phi_n = |\phi_n|^{2\sigma} \phi_n, \quad |n| \leq m, \tag{34}\]

\[
\phi_{-(m+1)} = \phi_{m+1} = 0. \tag{35}\]

This is a finite dimensional problem, and is related to the problem of existence of standing wave solutions for the DNLS equation in a finite lattice with Dirichlet boundary conditions. Although this problem is finite
dimensional, it is still of interest as any numerical approximation of the continuous space NLS, or even of the infinite dimensional DNLS will necessarily lead to a finite dimensional problem of the above type.

The set $\mathbb{C}^{2m+1}$ endowed with the discrete inner product and induced norm

$$
(\phi, \psi)_2 := \text{Re} \sum_{n=-m}^{n=m} \phi_n \overline{\psi_n}, \quad ||\psi||_2 := \sum_{n=-m}^{n=m} |\psi_n|^2, \quad \phi, \psi \in \mathbb{C}^{2m+1},
$$

is a (finite dimensional) Hilbert space. We consider now the operators

$$(A_1 \psi)|_{n \leq m} := \psi_{n-1} - 2\psi_n + \psi_{n+1}, \quad (B_1 \psi)|_{n \leq m} = \psi_{n+1} - \psi_n.$$  

It can be easily checked (see also [3, pg. 117]) that

$$(-A_1 \psi, \psi)_2 = \sum_{n=-m}^{n=m} |\psi_{n+1} - \psi_n|^2, \quad (-A_1 \phi, \psi)_2 = (B_1 \phi, B_1 \psi)_2.$$  

Hence, we may also consider the inner product and the corresponding norm in $\mathbb{C}^{2m+1}$

$$(\phi, \psi)_{1,2} := (B_1 \phi, B_1 \psi)_2 + (\phi, \psi)_2, \quad ||\psi||_{1,2} := \sum_{n=-m}^{n=m} (|\psi_{n+1} - \psi_n|^2 + |\psi_n|^2).$$  

The norm in (38) is equivalent with (36), and the constants in the equivalence inequality, can be chosen in order to be independent of $m$.

The second is the problem given by equation (33), which is an infinite dimensional system. This is related with the problem of existence of standing wave solutions of the conservative DNLS equation in the infinite lattice.

In what follows we will use a variational principle to show existence of nontrivial standing wave solutions for (34)-(35) and a fixed point argument to show non-existence of non-trivial standing waves, for certain parameter values for the infinite dimensional problem (33).

### 4.1 Existence of non trivial standing wave solutions for the finite dimensional problem

We will treat this problem using a variational approach. We start with some observations and results which are common to both problems. Unless the opposite is explicitly stated, everything that follows will hold for both the finite dimensional and the infinite-dimensional system.

Solutions of (33) are critical points of the functional

$$E(\phi) = \frac{1}{2\epsilon} \sum_{n \in \mathbb{Z}} |(B\phi)_n|^2 + \frac{\omega^2}{2} \sum_{n \in \mathbb{Z}} |\phi_n|^2 - \frac{1}{2\sigma+2} \sum_{n \in \mathbb{Z}} |\phi_n|^{2\sigma+2}.$$  

To establish differentiability of the functional $E : l^2 \rightarrow \mathbb{R}$, we shall use the following discrete version of the dominated convergence Theorem, provided by [9].

**Lemma 4.1** Let $\{\psi_{i,k}\}$ be a double sequence of summable functions (i.e $\sum_{i \in \mathbb{Z}} |\psi_{i,k}| < \infty$) and $\lim_{k \to \infty} \psi_{i,k} = \psi_i$, for all $i \in \mathbb{Z}$. If there exists a summable sequence $\{g_i\}$ such that $|\psi_{i,k}| \leq g_i$ for all $i$, $k$'s, we have that $\lim_{k \to \infty} \sum_{i \in \mathbb{Z}} \psi_{i,k} = \sum_{i \in \mathbb{Z}} \psi_i$.

We then have the following Lemma.

**Lemma 4.2** Let $(\phi_n)_{n \in \mathbb{Z}} = \phi \in l^{2\sigma+2}$ for some $0 < \sigma < \infty$. Then the functional

$$V(\phi) = \sum_{n \in \mathbb{Z}} |\phi_n|^{2\sigma+2},$$

is a $C^1(l^{2\sigma+2}, \mathbb{R})$ functional and

$$< V'(\phi), \psi > = (2\sigma + 2)\text{Re} \sum_{n \in \mathbb{Z}} |\phi_n|^{2\sigma} \phi_n \overline{\psi_n}, \quad \psi = (\psi_n)_{n \in \mathbb{Z}} \in l^{2\sigma+2}.$$
Proof: We assume that \( \phi, \psi \in \ell^{2\sigma+2} \). Then for any \( n \in \mathbb{Z}, 0 < s < 1 \), we get from the mean value Theorem

\[
\frac{V(\phi + s\psi) - V(\psi)}{s} = \frac{1}{s} \text{Re} \sum_{n \in \mathbb{Z}} \int_{0}^{1} \frac{d}{d\theta} |\phi_n + \theta s \psi_n|^{2\sigma+2} d\theta = (2\sigma + 2) \text{Re} \sum_{n \in \mathbb{Z}} \int_{0}^{1} |\phi_n + s\theta \psi_n|^{2\sigma} (\phi_n + s\theta \psi_n) \overline{\psi_n} d\theta.
\]

(41)

For the rhs of (41), we have the estimate

\[
(2\sigma + 2) \sum_{n \in \mathbb{Z}} |\phi_n + \theta s \psi_n|^{2\sigma+1} |\psi_n| \leq (2\sigma + 2) \sum_{n \in \mathbb{Z}} (|\phi_n| + |\psi_n|)^{2\sigma+1} |\psi_n| = \sum_{n \in \mathbb{Z}} z_n
\]

The sequence \((z_n)_{n \in \mathbb{Z}}\) is summable since

\[
\sum_{n \in \mathbb{Z}} |z_n| \leq c \left( \sum_{n \in \mathbb{Z}} (|\phi_n| + |\psi_n|)^{2\sigma+2} \right)^{\frac{1}{2\sigma+2}} \left( \sum_{n \in \mathbb{Z}} |\psi_n|^{2\sigma+2} \right)^{\frac{1}{2\sigma+2}}.
\]

Letting \( s \to 0 \), the existence of the Gateaux derivative (40) of the functional \( V : \ell^{2\sigma+2} \to \mathbb{R} \), follows from Lemma 4.1 (discrete dominated convergence).

We show next that the functional \( V' : \ell^{2\sigma+2} \to \ell^{2\sigma+2} \) is continuous. For \((\phi_n)_{n \in \mathbb{Z}} = \phi \in \ell^{2\sigma+2} \), we set \( F_1(\phi_n) = |\phi_n|^{2\sigma} \phi_n \). We consider a sequence \( \phi^m \in \ell^{2\sigma+2} \) such that \( \phi^m \to \phi \) in \( \ell^{2\sigma+2} \). We get then the inequality

\[
|(V'(\phi_m) - V'(\phi), \psi)| \leq c ||F_1(\phi_m) - F_1(\phi)|| \|\psi\|_{L^p}, \quad q = \frac{2\sigma + 2}{2\sigma + 1}, \quad p = 2\sigma + 2.
\]

(42)

We denote by \((\phi_m)_n\) the \( n \)-th coordinate of the sequence \( \phi_m \in \ell^2 \). Since \( F_1 \) satisfies condition (N1) with \( \beta = 2\sigma \), by setting \( \Phi_n = (|\phi_m|)^{2\sigma} + |\phi_n|^{2\sigma} \), we get from Hölder's inequality that

\[
\sum_{n \in \mathbb{Z}} |F_1((\phi_m)_n) - F_1(\phi_n)|^q \leq c \sum_{n \in \mathbb{Z}} |(\Phi_n)_m - \phi_n|^q \leq c \left( \sum_{n \in \mathbb{Z}} |(\phi_m)_n - \phi_n|^{2\sigma+2} \right)^{\frac{1}{2\sigma+2}} \left( \sum_{n \in \mathbb{Z}} |\phi_n|^{2\sigma+2} \right)^{\frac{1}{2\sigma+2}} \to 0, \text{ as } m \to \infty. \quad \diamond
\]

By (22) (23) and Lemma 4.1, it follows that the functional \( E \) defined by (30) is \( C^1(\ell^2, \mathbb{R}) \). For convenience, we recall [13] Definition 4.1, pg. 130 (PS-condition) and [31] Theorem 6.1, pg. 140 or [31] Theorem 6.1, pg. 109 (Mountain Pass Theorem of Ambrosio-Rabinowitz [11]).

Definition 4.1 Let \( X \) be a Banach space and \( E : X \to \mathbb{R} \) be \( C^1 \). We say that \( E \) satisfies condition (PS) if, for any sequence \( \{u_n\} \subset X \) such that \( |E(u_n)| \) is bounded and \( E'(u_n) \to 0 \) as \( n \to \infty \), there exists a convergent subsequence. If this condition is only satisfied in the region where \( E \geq \alpha > 0 \) (resp \( E \leq -\alpha < 0 \)) for all \( \alpha > 0 \), we say \( E \) satisfies condition (PS+) (resp. (PS-)).

Theorem 4.1 Let \( E : X \to \mathbb{R} \) be \( C^1 \) and satisfy (a) \( E(0) = 0 \), (b) \( \exists \rho > 0, \alpha > 0 : ||u||_X = \rho \implies E(u) \geq \alpha \), (c) \( \exists u_1 \in X : ||u_1||_X \geq \rho \) and \( E(u_1) < \alpha \). Define

\[
\Gamma = \{ \gamma \in C^0([0, 1], X) : \gamma(0) = 0, \gamma(1) = u_1 \}.
\]

Let \( F_\gamma = \{ \gamma(t) \in X : 0 \leq t \leq 1 \} \) and \( \mathcal{L} = \{ F_\gamma : \gamma \in \Gamma \} \). If \( E \) satisfies condition (PS), then

\[
\beta := \inf_{F_\gamma \in \mathcal{L}} \sup \{ E(v) : v \in F_\gamma \} \geq \alpha
\]

is a critical point of the functional \( E \).

We shall verify definition 4.1 and the assumptions of Theorem 4.1 to show that problem (31)-(35) has a nontrivial solution.

Theorem 4.2 Consider the finite dimensional problem (39). Let \( 0 < \sigma < \infty \) and \( \epsilon > 0 \). Then for any \( \omega \neq 0 \), there exists a non-trivial solution of (39).
Proof (Condition (PS)) We observe that the norm
\[ ||\phi||_{\ell^2}^2 = \left( \frac{1}{\epsilon} ||B\phi||_{\ell^2}^2 + \omega^2 ||\phi||_{\ell^2}^2 \right)^{\frac{1}{2}}, \]
(43)
is equivalent with the norm of \( \ell_1 \) defined by \textbf{11} since
\[ \min \left\{ \frac{1}{\epsilon}, \omega^2 \right\} ||\phi||_{\ell^2}^2 \leq ||\phi||_{\ell^2}^2 \leq \max \left\{ \frac{1}{\epsilon}, \omega^2 \right\} ||\phi||_{\ell^2}^2, \]
(44)and is also equivalent with the usual norm of \( \ell^2 \), as it follows from \textbf{12}. Now we let a sequence \( \phi_m \) of \( \ell^2 \) be such that \( |E(\phi_m)| < M \) for some \( M > 0 \) and \( E'(\phi_m) \to 0 \) as \( m \to \infty \). By using \textbf{30} and Lemma \textbf{12} we observe that for \( m \) sufficiently large
\[ M \geq E(\phi_m) - \frac{1}{2\sigma + 2} \langle E'(\phi_m), \phi_m \rangle = \left( \frac{1}{2} - \frac{1}{2\sigma + 2} \right) ||\phi_m||_{\ell^2}^2. \]
(45)Therefore the sequence \( \phi_m \) is bounded. Thus, we may extract a subsequence, still denoted by \( \phi_m \), such that
\[ \phi_m \rightharpoonup \phi \text{ in } \ell^2, \text{ as } m \to \infty. \]
(46)

Focusing now on \textbf{34} - \textbf{35}, we observe that since in the finite dimensional space \( \mathbb{C}^{2m+1} \) the weak convergence coincides with the strong, \( \phi_m \) converges strongly to \( \phi \) in \( \mathbb{C}^{2m+1} \). Therefore the functional \( E \) associated with the boundary value problem \textbf{34} - \textbf{35}, satisfies condition (PS).

(Mountain Pass assumptions) We now check the conditions for the validity of the Mountain Pass Theorem for the functional \( E \). We use the same notations for the equivalent norms in the finite dimensional space \( \mathbb{C}^{2m+1} \), with those of the norms of the infinite dimensional spaces, having in mind that we deal with finite dimensional sums.

For every \( \phi \in \mathbb{C}^{2m+1} \) it holds that
\[ ||\phi||_{\ell^{2\sigma+2}} \leq \kappa_1 ||\phi||_{\ell^2}, \quad \kappa_1 = \frac{1}{\min\{1/\epsilon, \omega\}}. \]
The following inequality holds
\[ E(\phi) = \frac{1}{2} ||\phi||_{\ell^2}^2 - \frac{1}{2\sigma + 2} ||\phi||_{\ell^{2\sigma+2}}^{2\sigma+2} \geq \frac{1}{2} ||\phi||_{\ell^2}^2 - \frac{\kappa_1^{2\sigma+2}}{2\sigma + 2} ||\phi||_{\ell^2}^{2\sigma+2}. \]
Choosing \( \phi \in \mathbb{C}^{2m+1} \) such that \( ||\phi||_{\ell^2} = r \), we observe that if \( 0 < r < \left( \frac{\sigma+1}{\kappa_1} \right)^{\frac{1}{\sigma}} \), then
\[ E(\phi) \geq \alpha > 0, \quad \alpha = r^2 \left( \frac{1}{2} - \frac{\kappa_1^{2\sigma+2}}{2\sigma + 2} r^{2\sigma} \right). \]
This establishes the first condition for the validity of the Mountain Pass Theorem, for the functional \( E \). To check the behaviour of \( E \) at infinity, let us consider \( \phi = te \) where \( t \in \mathbb{R} \) and \( e \in \mathbb{C}^{2m+1}, ||e||_{\ell^2} = 1 \). We have that
\[ E(te) = \frac{t^2}{2} - \frac{1}{2\sigma + 2} ||e||_{\ell^{2\sigma+2}}^{2\sigma+2} \to -\infty, \]
as \( t \to +\infty \). This ensures the validity of the second condition of the Mountain Pass Theorem. For fixed \( \phi \neq 0 \) and choosing \( t \) sufficiently large, we may set \( u_1 = t\phi \) to obtain the existence of a non-trivial solution for \textbf{34} - \textbf{35}.

Remark 4.1 In theorem \textbf{4.2} we have to state explicitly that we deal with the finite dimensional problem, since otherwise we may not prove the validity of the PS condition. In the infinite dimensional case, due to the lack of the Schur property for the space \( \ell^2 \) (in contrast with the space \( \ell^1 \) which posses this property-weak convergence coincides with strong convergence), we may not conclude the strong convergence of the sequence, from its weak convergence. The generalization of the mountain pass argument in the case of the infinite dimensional conservative system is a subject of future research, and will be considered elsewhere. \( \blacksquare \)
4.2 Non-existence of non trivial standing waves

We now provide some non-existence results for nontrivial standing waves. The results that follow are valid for both problems (34)-(35) and (33).

We recall [37] Theorem 18.E, pg. 68 (Theorem of Lax and Milgram), which will be used to establish existence of solutions for an auxiliary linear, non-homogeneous problem, related to (33).

**Theorem 4.3** Let $X$ be a Hilbert space and $A : X \to X$ be a linear continuous operator. Suppose that there exists $c^* > 0$ such that

$$\text{Re}(Au, u) \geq c^* ||u||^2_X, \quad \text{for all } u \in X.$$  \hspace{1cm} (47)

Then for given $f \in X$, the operator equation $Au = f$, $u \in X$, has a unique solution

**Theorem 4.4** There exist no nontrivial standing waves of energy less than $E_c(\omega, \sigma) := \left(\frac{\omega^4}{4}\right)^{1/4\sigma}$.  \hspace{1cm}

**Proof:** Let $\omega \in \mathbb{R}$, $\omega \neq 0$, be fixed. For any $\epsilon > 0$, the operator $A_\omega : \ell^2 \to \ell^2$, defined by

$$(A_\omega \phi)_{n \in \mathbb{Z}} = -\frac{1}{\epsilon}(\phi_{n-1} - 2\phi_n + \phi_{n+1}) + \omega^2 \phi_n,$$  \hspace{1cm} (48)

is linear and continuous and satisfies assumption [47] of Theorem 4.3. Using (44) and (11) we get that

$$(A_\omega \phi, \phi)_{\ell^2} = \frac{1}{\epsilon} ||B\phi||_{\ell^2}^2 + \omega^2 ||\phi||^2 \geq \omega^2 ||\phi||^2_{\ell^2} \quad \text{for all } \phi \in \ell^2.$$  \hspace{1cm} (49)

For given $z \in \ell^2$, we consider the linear operator equation

$$(A_\omega \phi)_{n \in \mathbb{Z}} = |z_n|^{2\sigma} z_n, \quad \phi \in \ell^2.$$  \hspace{1cm} (50)

The function $F_1 : \mathbb{C} \to \mathbb{C}$, $F_1(z) = |z_n|^{2\sigma} z_n$ satisfies Lemma 2.1 with $\beta = 2\sigma$. Therefore assumptions of Theorem 4.3 are satisfied, and (50) has a unique solution $\phi \in \ell^2$. For some $R > 0$, we consider the closed ball of $\ell^2$, $B_R := \{z \in \ell^2 : ||z||_{\ell^2} \leq R\}$, and we define the map $\mathcal{P} : \ell^2 \to \ell^2$, by $\mathcal{P}(z) := \phi$ where $\phi$ is the unique solution of the operator equation (50). Clearly the map $\mathcal{P}$ is well defined.

Let $z, \xi \in B_R$ such that $\phi = \mathcal{P}(z), \psi = \mathcal{P}(\xi)$. The difference $\chi := \phi - \psi$ satisfies the equation

$$(A_\omega \chi)_{n \in \mathbb{Z}} = F_1(z) - F_1(\xi).$$  \hspace{1cm} (51)

Taking now the scalar product of (51) with $\chi$ and using (11), we derive

$$\frac{1}{\epsilon} ||B\chi||_{\ell^2}^2 + \omega^2 ||\chi||_{\ell^2}^2 \leq ||F_1(z) - F_1(\xi)||_{\ell^2}||\chi||_{\ell^2}$$

$$\leq 2R^{2\sigma}||z - \xi||_{\ell^2}||\chi||_{\ell^2}$$

$$\leq \frac{\omega^2}{2} ||\chi||_{\ell^2}^2 + \frac{2}{\omega^2} R^{4\sigma}||z - \xi||_{\ell^2}^2.$$  \hspace{1cm} (52)

From (52), we obtain the inequality

$$||\chi||_{\ell^2}^2 = ||\mathcal{P}(z) - \mathcal{P}(\xi)||_{\ell^2}^2 \leq \frac{4}{\omega^4} R^{4\sigma}||z - \xi||_{\ell^2}^2.$$  \hspace{1cm} (53)

Since $\mathcal{P}(0) = 0$, from inequality (53), we derive that for $R < (\omega^4/4)^{1/4\sigma} := E_c(\omega, \sigma)$, the map $\mathcal{P} : B_R \to B_R$ and is a contraction. Therefore $\mathcal{P}$, satisfies the assumptions of Banach Fixed Point Theorem and has a unique fixed point, the trivial one. Hence, for $R < E_c(\omega, \sigma)$ the only standing wave is the trivial. \quad \diamond

**Remark 4.2** The above theorem is interesting when viewed in the following way: The $\ell^2$ norm corresponds to the energy of the excitations of the lattice. If the energy of the excitation is less than $E_c(\omega, \sigma)$ the lattice may not support a standing wave of frequency $\omega$. This relation may be seen as some kind of dispersion relation of frequency vs energy for the standing wave solutions. It contains information on the type of nonlinearity, through its dependence on the nonlinearity exponent $\sigma$. Let us note that a similar computation to (52) shows that $\mathcal{P}$ maps also the ball $B_R$ with $R \leq (\omega^4/4)^{1/4\sigma} := R_1$, to itself and obviously $E_c < R_1$, but within this range, the map is not essentially a contraction. \quad \bullet

**Remark 4.3** Arguments of the same type as above, may be used for the proof of existence of unique steady states for the case of nonlinearities that do not vanish when $\phi = 0$ (see the end of Remark 3.1). \quad \bullet
5 Existence of global attractor in $\ell^2$ for the dissipative case

In this section we study the asymptotic behavior of solutions of (1)-(2) in the dissipative case, $\delta > 0$ and $g_n \neq 0$, i.e. taking into account the effect of a weak dissipation and of an external excitation. This time, as a model problem we shall employ the dissipative case of (26)-(27)

\[ iu_n + \frac{1}{\epsilon}(u_{n-1} - 2u_n + u_{n+1}) + i\delta u_n + |u_n|^{2\sigma} u_n = g_n, \quad n \in \mathbb{Z}, \quad 0 \leq \sigma < \infty, \quad \epsilon > 0, \]

\[ u_n(0) = u_{n,0}, \quad n \in \mathbb{Z}. \]

We also consider the weakly damped driven NLS partial differential equation

\[ i\partial_t u + u_{xx} + i\delta u + |u|^{2\sigma} u = g, \quad x \in \mathbb{R}, \quad t > 0, \]

\[ u(x, 0) = u_0(x). \]

It is well known that the dissipation and forcing terms have important effects in the long-time behavior of (56). As an example, we refer to the case $\sigma = 2$, for which has been observed numerically the existence of chaotic attractors [10, 29], in contrast with the conservative case which is completely integrable by the inverse scattering theory, for sufficiently smooth initial data. We refer to [15, 16, 21, 22, 34] for results on the existence, finite dimensionality, and regularity of global attractors for (56).

5.1 Existence of absorbing set

This subsection is devoted to the existence of an absorbing set in $\ell^2$. We have the following Lemma.

**Lemma 5.1** Let $(u_0, n)_{n \in \mathbb{Z}} = (u_0, (g_n)_{n \in \mathbb{Z}} = g \in \ell^2$. For any $\epsilon > 0$ and $0 \leq \sigma < \infty$, the dynamical system defined by (54)-(55)

\[ S(t) : u_0 \in \ell^2 \rightarrow u(t) \in \ell^2, \]

possesses a bounded absorbing set $B_0$ in $\ell^2$: For every bounded set $B$ of $\ell^2$, there exists $t_0(B, B_0)$ such that for all $t \geq t_0(B, B_0)$, it holds $S(t)B \subset B_0$.

**Proof:** We take the scalar product of (54) with $iu(t)$ and we get

\[ \frac{1}{2} \frac{d}{dt} ||u||^2_{\ell^2} + \delta ||u||^2_{\ell^2} = \text{Im} \sum_{n \in \mathbb{Z}} \pi_n g_n \leq \frac{\delta}{2} ||g||^2_{\ell^2} + \frac{1}{2\delta} ||g||^2_{\ell^2}. \]

(57)

and therefore

\[ \frac{d}{dt} ||u||^2_{\ell^2} + \delta ||u||^2_{\ell^2} \leq \frac{1}{\delta} ||g||^2_{\ell^2}. \]

(58)

From (59) we derive that $u \in L^\infty([0, \infty), \ell^2)$: Gronwall’s Lemma implies that

\[ ||u(t)||^2_{\ell^2} \leq ||u_0||^2_{\ell^2} \exp(-\delta t) + \frac{1}{\delta^2} ||g||^2_{\ell^2} \{1 - \exp(-\delta t)\}. \]

(59)

Letting $t \to \infty$ we infer that

\[ \lim_{t \to \infty} ||u(t)||^2_{\ell^2} \leq \frac{1}{\delta^2} ||g||^2_{\ell^2}. \]

Setting $\rho^2 = ||g||^2_{\ell^2}/\delta^2$, it follows that for any number $\rho_1 > \rho$ the ball $B_0$ of $\ell^2$ centered at 0 of radius $\rho_1$ is an absorbing set for the semigroup $S(t)$: If $B$ is a bounded set of $\ell^2$ included in a ball of $\ell^2$ centered at 0 of radius $R$, then for $t \geq t_0(B, B_0)$ where

\[ t_0 = \frac{1}{\delta} \log \frac{R^2}{\rho_1^2 - \rho^2}, \]

(60)

it holds $||u(t)||^2_{\ell^2} \leq \rho_1^2$, i.e. $S(t)B \subset B_0$. Note that in the absence of external excitation, the dynamical system exhibits trivial dynamics, in the sense that $\lim_{t \to \infty} ||u(t)||^2_{\ell^2} = 0$, as (60) clearly shows. \(\diamond\)
Remark 5.1 The existence of the absorbing set, can be shown again by working in a different way, in the equivalent norm of \( \ell_1^2 \): By taking the scalar product of (54) with \(-\dot{u} - \delta u\) we may derive the energy equation

\[
\frac{1}{2} \frac{d}{dt} J(u(t)) = \delta J(u(t)) = \Lambda(u(t)), \quad J(u) := ||u||_{\ell_1^2}^2 - \epsilon \left\{ \frac{1}{\sigma + 1} ||u||_{\ell_1^{2\sigma+2}}^{2\sigma+2} - 2(g, u) \right\}, \quad (62)
\]

\[
\Lambda(u) := \epsilon \delta \left\{ \frac{\sigma}{\sigma + 1} ||u||_{\ell_1^{2\sigma+2}}^{2\sigma+2} + (g, u) \right\} + \text{Im} \sum_{n \in \mathbb{Z}} \varpi_n g_n. \quad (63)
\]

From (41) and (61) we obtain the following estimate for \( \Lambda(u) \)

\[
\Lambda(u) \leq \epsilon \delta \left\{ \frac{\sigma}{\sigma + 1} \rho_1^{2\sigma+2} + ||g||_{\ell_1^2} \rho_1 \right\} + ||g||_{\ell_1^2} \rho_1 = \lambda_1. \quad (64)
\]

Gronwall’s inequality applied in (62) implies that

\[
J(u(t)) \leq J(u_0) \exp(-2\delta t) + \frac{\lambda_1}{\delta} \left\{ 1 - \exp(-2\delta t) \right\} \quad (65)
\]

Letting \( t \to \infty \) we get \( \limsup_{t \to \infty} J(u(t)) \leq \lambda_1/\delta \). Hence, for \( t \geq t_0 \)

\[
||u(t)||_{\ell_1^2}^2 \leq \epsilon \rho_1^{2\sigma+2} + 3\epsilon ||g||_{\ell_1^2} \rho_1 + \frac{1}{\delta} ||g||_{\ell_1^2} \rho_1 := \rho_2^2 \quad (66)
\]

Relation (62), gives an estimate of the radius of the absorbing ball, involving the degree of the nonlinearity and lattice spacing. Note for example, that for sufficiently small

\[
0 < \epsilon < \frac{\rho_1 - ||g||_{\ell_1^2} \delta^{-1}}{\rho_1^{2\sigma+1} + 3||g||_{\ell_1^2}},
\]

we get that \( \rho_2 < \rho_1 \). 

Throughout the rest of the paper, for simplicity reasons we set \( \epsilon = 1 \).

5.2 Asymptotic compactness of the semigroup and existence of global attractor

We now prove the asymptotic compactness of the semigroup generated by the DNLS equation (54)-(55). To establish this property, we follow the approach introduced in [8] and applied also in [38] for a second order lattice dynamical system. This approach is based on the derivation of estimates on the tail ends of solutions of (54)-(55).

Lemma 5.2 Let \((u_0,n)_{n \in \mathbb{Z}} = u_0 \in \mathcal{B}\) where \( \mathcal{B} \) is a bounded set of \( \ell^2 \) and \((g_n)_{n \in \mathbb{Z}} = g \in \ell^2 \). For any \( 0 \leq \sigma < \infty, \) and \( \eta > 0 \), there exist \( T(\eta) \) and \( K(\eta) \) such that the solution \((u_n)_{n \in \mathbb{Z}} = u\) of (54)-(55) satisfies for all \( t \geq T(\eta)\), the estimate

\[
\sum_{|n| > 2M} |u_n(t)|^2 \leq \frac{2\eta}{\delta}, \quad \text{for any } M > K(\eta). \quad (67)
\]

Proof: Choose a smooth function \( \theta \in C^1(\mathbb{R}^+, \mathbb{R}) \) satisfying the following properties

\[
\begin{cases}
\theta(s) = 0, & 0 \leq s \leq 1 \\
0 \leq \theta(s) \leq 1, & 1 \leq s \leq 2 \\
\theta(s) = 1, & s \geq 2.
\end{cases}
\]

and

\[
|\theta'(s)| \leq C_0, \quad s \in \mathbb{R}^+, \quad (68)
\]

for some \( C_0 \in \mathbb{R} \). We shall use the shorthand notation \( \theta_n = \theta \left( \frac{|n|}{M} \right) \) and \( \text{Re}(u_n) = u_{1,n}, \text{Im}(u_n) = u_{2,n} \). We now multiply the DNLS equation with the function \( \theta_n \hat{u}_n, n \in \mathbb{Z} \), and we sum over all sites and keep the imaginary
We may further estimate the effect of the function $\theta$ on $u_n$ is to cut-off all sites within a ball of radius comparable to $M$ and to take into account only the remote sites. This results to

$$\frac{1}{2} \frac{d}{dt} \sum_{n \in \mathbb{Z}} \theta_n | u_n |^2 + \delta \sum_{n \in \mathbb{Z}} \theta_n | u_n |^2 - \sum_{n \in \mathbb{Z}} \{(Bu_2)_n(B\theta u_1)_n - (Bu_1)_n(B\theta u_2)_n \} = \text{Im} \sum_{n \in \mathbb{Z}} g_n \theta_n \bar{u}_n. \quad (69)$$

The nonlinear term contributes a purely real term so it drops out of the above equality. We must now estimate the remaining terms. Using the Cauchy-Schwarz inequality we find for the term containing the forcing that

$$\text{Im} \sum_{n \in \mathbb{Z}} g_n \theta_n \bar{u}_n \leq \frac{\delta}{2} \sum_{n \in \mathbb{Z}} \theta_n | u_n |^2 + \frac{1}{2\delta} \sum_{|n| \geq M} | g_n |^2.$$

We now estimate the last term on the l.h.s. of equation (69). After some algebra we obtain

$$I_1 = \sum_{n \in \mathbb{Z}} \{(Bu_2)_n(B\theta u_1)_n - (Bu_1)_n(B\theta u_2)_n \} = \sum_{n \in \mathbb{Z}} \{(\theta_{n+1} - \theta_n)(u_{1,n}u_{2,n+1} - u_{2,n}u_{1,n+1})\}.$$

We may further estimate $I_1$ as follows,

$$|I_1| \leq \sup_{n \in \mathbb{Z}} | \theta_{n+1} - \theta_n | \sum_{n \in \mathbb{Z}} |(u_{1,n}u_{2,n+1} - u_{2,n}u_{1,n+1})| \leq \frac{C_1}{M} || u ||^2_2 \leq \frac{C_1}{M} \rho_1^2,$$

where $C_1$ depends only on $C_0$. For the last inequality we applied the mean value theorem and the restriction of the solution to the absorbing ball in $\ell^2$ ($\rho_1$ is the radius of the absorbing ball of Lemma 5.1).

Therefore we obtain the following differential inequality

$$\frac{1}{2} \frac{d}{dt} \sum_{n \in \mathbb{Z}} \theta_n | u_n |^2 + \frac{\delta}{2} \sum_{n \in \mathbb{Z}} \theta_n | u_n |^2 \leq \frac{C_1}{M} \rho_1^2 + \frac{1}{2\delta} \sum_{|n| \geq M} | g_n |^2.$$

Using the Gronwall inequality we obtain the following estimate

$$\sum_{n \in \mathbb{Z}} \theta_n | u_n |^2 \leq e^{-\delta(t-t_0)} \sum_{n \in \mathbb{Z}} \theta_n | u_n(t_0) |^2 + \frac{1}{\delta} \left( \frac{2C_1}{M} \rho_1^2 + \frac{1}{\delta} \sum_{|n| \geq M} | g_n |^2 \right),$$

for $t > t_0$ where $t_0$ is the time of entry of initial data bounded in $\ell^2$, to the absorbing ball of radius $\rho_1$ in $\ell^2$. Since $g \in \ell^2$, then for all $\eta > 0$, there exists $K(\eta)$ such that

$$\frac{2C_1}{M} \rho_1^2 + \frac{1}{\delta} \sum_{|n| \geq M} | g_n |^2 \leq \eta, \quad \forall M > K(\eta).$$

Therefore, for all $\eta$ and for $t > t_0$ and $M > K(\eta)$ we obtain that

$$\sum_{n \in \mathbb{Z}} \theta_n | u_n |^2 \leq e^{-\delta(t-t_0)} \rho_1^2 + \frac{1}{\delta} \eta.$$

Choosing $t$ large enough we may then obtain

$$\sum_{|n| \geq 2M} | u_n |^2 \leq \sum_{n \in \mathbb{Z}} \theta_n | u_n |^2 \leq \frac{2\eta}{\delta}.$$

This estimate holds as long as $t \geq T(\eta)$ where

$$T(\eta) = t_0 + \frac{1}{\delta} \ln \left( \frac{\delta \rho_1^2}{\eta} \right)$$

and $M > K(\eta)$. This concludes the proof of the Lemma. $\Diamond$

**Remark** The above Lemma has an additional implication, since it allows us to obtain an estimate for the time scales required for the localization of the energy to the part of the lattice with $|n| \leq M$. $\bullet$
Proposition 5.1 Let $0 \leq \sigma < \infty$. The semigroup $S(t)$ is asymptotically compact in $\ell^2$, that is, if the sequence $\phi_n$ is bounded in $\ell^2$ and $t_n \to \infty$, then $S(t_n)\phi_n$ is precompact in $\ell^2$.

Proof: Let us consider a bounded sequence $\phi_n \in \ell^2$, such that $\| \phi_n \|_{\ell^2} \leq r$, $r > 0$, $n = 1, 2, \ldots$. By Lemma 5.1 there exists $T(r) > 0$ such that $S(t)\phi_n \subset B_0$, $\forall t \geq T(r)$. As $t_n \to \infty$ there exists an integer $N_1(r)$ such that $t_n \geq T(r)$ for $n \geq N_1(r)$ and

$$S(t_n)\phi_n \subset B_0, \ \forall n \geq N_1(r). \quad (70)$$

Therefore $S(t_n)\phi_n$ is weakly relatively compact. Thus, there exists $\phi_0 \in B_0$ and a subsequence of $S(t_n)\phi_n$ (not relabelled) such that $S(t_n)\phi_n \to \phi_0$ weakly in $B_0$. To ensure precompactness we need to ensure that the above convergence is strong, that is for all $\eta > 0$, there exists $N(\eta) \in \mathbb{N}$ such that

$$\| S(t_n)\phi_n - \phi_0 \|_{\ell^2} \leq \eta, \ \forall n \geq N(\eta).$$

From $(70)$ and the tail estimates provided by Lemma 5.2 we may conclude the existence of some $K_1(\eta)$ and $T_1(\eta)$ such that

$$\sum_{|i| \geq K_1(\eta)} | (S(t)S(T_r)\phi_n)_i |^2 \leq \frac{\eta^2}{8}, \ \ t \geq T_1(\eta),$$

where by $(S(t)\phi_n)_i$ we denote the $i$-th coordinate of the infinite sequence $S(t)\phi_n \in \ell^2$. Since $t_n \to \infty$ we may find $N_2(r, \eta) \in \mathbb{N}$ such that $t_n \geq T_r + T(\eta)$ if $n \geq N_2(r, \eta)$. Hence,

$$\sum_{|i| \geq K_1(\eta)} | (S(t_n)\phi_n)_i |^2 = \sum_{|i| \geq K_1(\eta)} | (S(t_n - T_r)S(T_r)\phi_n)_i |^2 \leq \frac{\eta^2}{8}$$

where we used the semigroup property for $S(t)$.

Since $\phi_0 \in \ell^2$ we have the following tail estimate

$$\sum_{|i| \geq K_1(\eta)} | (\phi_0)_i |^2 \leq \frac{\eta^2}{8}.$$

Let us now choose $K(\eta) = max(K_1(\eta), K_2(\eta))$. Since $S(t_n)\phi_n \to \phi_0$ in $\ell^2$, we may see that $(S(t_n)\phi_n)_i \to (\phi_0)_i$ strongly in $\mathbb{C}^{2K(\eta)+1}$. To prove the strong convergence we will break $\| \cdot \|_{\ell^2}$ into two distinct parts, the finite dimensional part (corresponding to $\mathbb{C}^{2K(\eta)+1}$) and the infinite dimensional part corresponding to the tails. Thus,

$$\| S(t_n)\phi_n - \phi_0 \|_{\ell^2}^2 = \sum_{|i| \leq K(\eta)} | (S(t_n)\phi_n - \phi_0)_i |^2 + \sum_{|i| > K(\eta)} | (S(t_n)\phi_n - \phi_0)_i |^2 \leq \frac{\eta^2}{2} + 2 \sum_{|i| > K(\eta)} (| (S(t_n)\phi_n)_i |^2 + | (\phi_0)_i |^2) \leq \eta^2$$

where the first estimate comes from the strong convergence in the finite dimensional space $\mathbb{C}^{2K(\eta)+1}$. \hfill $\diamond$

Using Proposition 5.1 and Theorem 1.1.1 of [33], we may arrive at the main result of this section which can be stated as follows:

Theorem 5.1 Let $0 \leq \sigma < \infty$. The semigroup $S(t)$ associated to $(54)$-$(55)$ possesses a global attractor $A = \omega(B_0) \subset B_0 \subset \ell^2$ which is compact, connected and maximal among the functional invariant sets in $\ell^2$.

6 Finite dimensional approximation of the global attractor

This section is devoted to a result of stability of the global attractor of the dynamical system generated by $(54)$-$(55)$ under its approximation by a global attractor of an appropriate finite dimensional dynamical system.

Let us note first that problem $(54)$-$(55)$ could be viewed as a boundary value problem satisfying the boundary condition that the solutions are bounded at infinity. Since $u(t) \in C^1(\mathbb{R}^+, \ell^2)$ clearly

$$\lim_{n \to \infty} u_n(t) = 0, \ \ t \geq 0.$$
We seek for an approximation of (54)-(55) by a finite dimensional system of ordinary differential equations.

We consider the boundary value problem in $\mathbb{C}^{2m+1}$,

\begin{align*}
i\psi_n + (\psi_{n-1} - 2\psi_n + \psi_{n+1}) + \delta \psi_n + |\psi_n|^{2\sigma} \psi_n &= g_n, \quad |n| \leq m, \quad 0 \leq \sigma < \infty, \\
\psi_{-(m+1)}(\cdot) &= \psi_{(m+1)}(\cdot) = 0, \\
\psi_n(0) &= \psi_{n,0}, \quad |n| \leq m. \tag{73}
\end{align*}

In a similar manner to the infinite dimensional system we may show that for the well posedness and asymptotic behavior of solutions of (71) the following result holds.

**Proposition 6.1** Let $\psi_0 := (\psi_{n,0})_{|n| \leq m} \in \mathbb{C}^{2m+1}$. For $0 \leq \sigma < \infty$, there exists a unique solution of (71)-(73) such that $\psi \in C^1([0, \infty), \mathbb{C}^{2m+1})$. The dynamical system defined by (71)-(73)

\begin{equation}
S_m(t) : \psi_0 \in \mathbb{C}^{2m+1} \rightarrow \psi(t) \in \mathbb{C}^{2m+1}, \tag{74}
\end{equation}

possesses a bounded absorbing set $O_0$ in $\mathbb{C}^{2m+1}$ and a global attractor $A_m \subset O_0 \subset \mathbb{C}^{2m+1}$. For every bounded set $O$ of $\ell^2$, there exists $t_1(O, O_0)$ such that for all $t \geq t_1(O, O_0)$, it holds that $S_m(t)O \subset O_0$ and for every $t \geq 0$ $S_m(t)A_m = A_m$.

Note that the estimates used for the proof of Proposition 6.1 (similar to those used in the proof Lemma 5.1) are independent of $m$.

Following [38, 39] (see also [6, 20] for a similar idea applied to pdes considered in all of $\mathbb{R}^N$), we observe that the $\mathbb{C}^{2m+1}$-solution of (71)-(73) can be extended naturally in the infinite dimensional space $\ell^2$, as

\begin{equation}
(\phi_m(t))_{m \in \mathbb{Z}} := \left\{ \begin{array}{ll}
\psi(t) := (\psi_n(t))_{|n| \leq m}, & |n| \leq m, \\
0, & |n| > m. \end{array} \right. \tag{75}
\end{equation}

The extension (75) will be used in order to verify that the global attractor $A$ of the semigroup $S(t)$ associated with (54)-(55), can be approximated by the global attractor $A_m$ of $S_m(t)$ associated to (71)-(73) as $m \to \infty$. We recall that the semidistance of two nonempty compact subsets of a metric space $X$ endowed with the metric $d_X(\cdot, \cdot)$ is defined as

\begin{equation}
d(B_1, B_2) = \sup_{x \in B_1} \inf_{y \in B_2} d_X(x, y). \tag{76}
\end{equation}

**Theorem 6.1** For any $0 \leq \sigma < \infty$, the global attractor $A_m$ converges to $A$ in the sense of the semidistance related to $\ell^2$: $\lim_{m \to \infty} d(A_m, A) = 0$.

**Proof:** We denote by $U$ an open-neighborhood of the absorbing ball $B_0$ of $S(t)$. Obviously $A$ attracts $U$. For arbitrary $m \in \mathbb{Z}$, we consider the semigroup $S_m(t)$ defined by Proposition 6.1 and its global attractor $A_m$. Exactly as in Lemma 5.1 it can be shown that $B_0 \cap \mathbb{C}^{2m+1}$ is also an absorbing set for $S_m(t)$. Therefore $A_m \subset B_0 \cap \mathbb{C}^{2m+1} \subset U \cap \mathbb{C}^{2m+1}$, and $A_m$ attracts $U \cap \mathbb{C}^{2m+1}$. In the light of Proposition 6.1 and [33] Theorem II.2, pg. 28, it remains to verify that for every compact interval $I$ of $\mathbb{R}^+$,

\begin{equation}
\delta_m(I) := \sup_{\psi_0 \in U \cap \mathbb{C}^{2m+1}} \sup_{t \in I} d(S_m(t)\psi_0, S_m(t)\psi_0) \to 0, \quad \text{as} \quad m \to \infty. \tag{76}
\end{equation}

We consider the corresponding solution $\psi(t) = S_m(t)\psi_0$, $\psi(0) = \psi_0$, in $\mathbb{C}^{2m+1}$ through (71)-(73). Then by Proposition 6.1 it follows that $\psi(t) \in A_m$ for any $t \in \mathbb{R}^+$. Therefore, if $\rho > 0$ is the $m$-independent radius of the absorbing ball $O_m$ in $\mathbb{C}^{2m+1}$, then for every $t \in \mathbb{R}^+\cup\mathbb{R}^+$, $||\psi(t)||_2^2 \leq \rho^2$. Using (76), we clearly observe that $\phi_m(t)$ also satisfies the estimates

\begin{equation}
||\phi_m(t)||_2^2 \leq \rho^2, \quad ||\dot{\phi}_m(t)||_2^2 \leq C(\rho, ||g||), \tag{77}
\end{equation}

the latter derived by [24]. According to [33] Theorem 10.1 pg. 331-332 or [39] Lemma 4, pg. 60, for the justification of (76) it suffices to show that $\phi_m(t)$ converges to a solution $\phi(t)$ of (54)-(55) in an arbitrary
compact interval of \( \mathbb{R}^+ \), and \( \phi_0 = \phi(0) \) in a bounded set of \( \ell^2 \). Let I be an arbitrary compact interval of \( \mathbb{R}^+ \). From estimates \((\ref{77})\), we may extract a subsequence \( \phi_n \) of \( \phi_m \), such that
\[
\phi_n(t) \to \phi(t), \quad \text{in } \ell^2, \quad \text{as } \mu \to \infty, \quad \text{for every } t \in D,
\]
where \( D \) denotes a countable dense subset of \( I \).

For any \( t \in I \) we consider the sequence
\[
\gamma_m(t) := (\phi_m(t), z)_{\ell^2}, \quad z \in \ell^2,
\]
which is differentiable by \((\ref{77})\) as a function of \( t \) and \( \gamma'_m(t) = (\phi_m(t), z)_{\ell^2} \). By the mean-value Theorem, there exists \( \xi \in I \) such that, for fixed \( t, s \in I \)
\[
|\phi_m(t) - \phi_m(s), z| = |\gamma_m(t) - \gamma_m(s)| = |(\phi_m(t), z)_{\ell^2} - t - s| \leq \sup_{\xi \in I} |(\phi_m(\xi), z)_{\ell^2}| t - s | \leq C |t - s|.
\]
It follows then, that there exists an \( m \)-independent constant \( C_1 \) such that
\[
||\phi_m(t) - \phi_m(s)||_{\ell^2} \leq C_1 |t - s|,
\]
i.e the sequence \( \gamma_m \) is equicontinuous. Hence by Ascoli’s Theorem, it follows that the convergence \((\ref{80})\), holds uniformly on \( I \) as \( \mu \to \infty \). Summarizing, we obtain that for the subsequence \( \phi_n \) hold the convergence relations
\[
\phi_n \to \phi \quad \text{in } C(I, \ell^2),
\]
\[
\phi_n \rightharpoonup \phi \quad \text{in } L^\infty(I, \ell^2),
\]
\[
\phi_n \rightharpoonup \phi \quad \text{in } L^\infty(I, \ell^2).
\]

For every \( z \in \ell^2 \) and \( \omega(t) \in C_0^\infty(I) \) we consider the formula (see \cite{12}, p. 59)
\[
\int_I (i \phi_n(t), z)_{\ell^2} \omega(t) dt - \int_I (B \phi_n(t), B z)_{\ell^2} \omega(t) dt + \delta \int_I (i \phi_n(t), z)_{\ell^2} \omega(t) dt + \int_I (f(\phi_n(t)), z)_{\ell^2} \omega(t) dt = \int_I (g(z), z)_{\ell^2} \omega(t) dt,
\]
where \( f(\phi_n) := |\phi_n|^{2\sigma} \phi_n \). Needless to say that any solution of \((\ref{81})\), is in our case, a solution of \((\ref{41}) - (\ref{55})\) and vice versa. Using \((\ref{80})\) we may pass to the limit in \((\ref{81}) \) is Lipschitz continuous on bounded sets of \( \ell^2 \), there exists from \((\ref{80}) \), a constant \( c(\rho) \) such that \( ||f(\phi_n) - f(\phi)||_{\ell^2} \leq c(\rho)||\phi_n - \phi||_{\ell^2} \). Then from \((\ref{80})\) we infer
\[
\left| \int_I (f(\phi_n(t)) - f(\phi(t)), z)_{\ell^2} \omega(t) dt \right| \leq \int_I ||f(\phi_n(t)) - f(\phi(t))||_{\ell^2} ||z||_{\ell^2} \omega(t) dt \leq c \int_I ||\phi_n(t) - \phi(t)||_{\ell^2} ||z||_{\ell^2} \omega(t) dt \leq c \sup_{\xi \in I} ||\phi_n(t) - \phi(t)||_{\ell^2} ||z||_{\ell^2} \int_I \omega(t) dt \to 0, \quad \text{as } \mu \to \infty.
\]

Now we conclude by the same arguments as in \cite{39}, Lemma 4, pg. 61): Since \( I \) is arbitrary, \((\ref{81})\) is satisfied for all \( t \in \mathbb{R}^+ \), i.e. \( \phi(t) \) solves \((\ref{41}) - (\ref{55})\). Moreover by \((\ref{80})\) we get that \( \phi(t) \) is bounded in \( \ell^2 \) for all \( t \in \mathbb{R}^+ \). Therefore \( \phi(t) \in \mathcal{A} \), which implies that \( \phi_n(0) \to \phi(0) \) and \( \phi_n \) is at least, in a bounded set of \( \ell^2 \). Since the convergence holds for any other subsequence having the above formulated properties, by a contradiction argument using uniqueness, we may deduce that the convergence holds for the original sequence \( \phi_m \). Condition \((\ref{10})\) is proved. \( \phi \).

7 The case of weighted spaces: Existence of spatially exponentially localized solutions

We now turn to the properties of the solutions of the DNLS \((\ref{41}) - (\ref{55})\), in weighted spaces. We consider weight function \( w_n \) which is an increasing function of \( |n| \), satisfying for all \( n \in \mathbb{Z} \), the following condition:
\[
(W) \quad \begin{cases}
1 \leq w_n \\
|w_{n+1} - w_n| \leq d_1 w_n \\
d_2 w_n \leq w_{n+1}
\end{cases}
\]
the space $\ell^2_w$, 

$$\ell^2_w = \{ u_n \in \mathbb{C} : \| u \|_{\ell^2_w}^2 := \sum_{n \in \mathbb{Z}} w_n \cdot | u_n |^2 < \infty \}.$$ 

It can easily be seen that the space $\ell^2_w$ is a Hilbert space with the norm $\| \cdot \|_{\ell^2_w}$. Such spaces are the discrete analogue of weighted $L^2$ spaces. A choice for such a function may be the exponential function $w_n = \exp(\lambda \cdot | n |)$ for $\lambda > 0$. Existence of solutions in such spaces will provide us with the existence of (exponentially) localized solutions for the DNLS equation (54)-(55). The use of such spaces is important in the study of existence of soliton solutions or breathers. An instance where such spaces have been used is in [31] where the existence of exponentially localized solutions has been studied in conservative lattices using a continuation argument, related to the anti-integrable limit.

Since the operator $A$ is not symmetric in the space $\ell^2_w$, we cannot apply Theorem 2.1 for the local existence in such spaces. We thus have to resort to general existence Theorems in Banach spaces. To this end we need to show that the operators involved are Lipschitz. This time as a model case we consider (1)-(2) with a nonlinearity satisfying (N_2).

We have the following,

Lemma 7.1 Let condition (N_2) be fulfilled. The operator $T : \ell^2_w \to \ell^2_w$, defined by $T(z) = f(|z|^2)z$ satisfies the following properties:

(i) $T$ is bounded on bounded sets of $\ell^2_w$ and

(ii) $T$ is locally Lipschitz continuous.

Proof: (i) Let $u \in B_R$, a closed ball of $\ell^2_w$ of radius $R$. It follows from condition (W) that $|z_n|^2 \leq \|z\|^2_{\ell^2_w}$ for all $n \in \mathbb{Z}$. Since $f$ is continuous, we may argue as for the proof of Lemma 2.1 in order to get the inequality

$$\| T(z) \|_{\ell^2_w}^2 \leq \sum_{n \in \mathbb{Z}} w_n \cdot f(|z_n|^2) \cdot z_n^2 \leq \sum_{n \in \mathbb{Z}} w_n g(|z_n|^2) \cdot z_n^2 \leq \sum_{n \in \mathbb{Z}} w_n g(||z||^2_{\ell^2_w}) \cdot z_n^2 \leq c(R) \cdot \|z\|_{\ell^2_w}^2,$$

for some positive constant $c(R)$. Thus we conclude that the operator $T$ is bounded on bounded sets of $\ell^2_w$.

(ii) Since (N_2) holds, for some $\theta \in (0,1)$ we have

$$\| T(z) - T(z') \|_{\ell^2_w} \leq 2 \sum_{n \in \mathbb{Z}} w_n \cdot f(|z_n|^2) \cdot |z_n - z'_n|^2$$

$$+ 2 \sum_{n \in \mathbb{Z}} w_n \cdot f'(\theta \cdot |z_n|^2 + (1 - \theta) \cdot |z'_n|^2) \cdot |z_n - z'_n|^2 \cdot |z_n + z'_n|^2 \cdot |z_n - z'_n|^2.$$

A similar inequality to (14) should be obtained: We may see that

$$\| T(z) - T(z') \|_{\ell^2_w}^2 \leq 2 \sum_{n \in \mathbb{Z}} w_n \cdot f(|z_n|^2) \cdot |z_n - z'_n|^2$$

$$+ 2 \left\{ \max_{\rho \in [0,2R]} g_1(\rho) \right\} c(R) \sum_{n \in \mathbb{Z}} w_n \cdot |z_n - z'_n|^2$$

$$\leq c_1(R) \cdot \|z - z'\|_{\ell^2_w}^2.$$

This concludes the proof of the Lemma.

Lemma 7.2 The operator $A : \ell^2_w \to \ell^2_w$, defined by $(Au)_n = u_{n+1} - 2u_n + u_{n-1}$ is globally Lipschitz on $\ell^2_w$.

Proof: Let $u, v \in B_R$. Then $(Au)_n \in \mathbb{Z} - (Av)_n = (u_{n+1} - v_{n+1}) - 2(u_n - v_n) + (u_{n-1} - v_{n-1})$ and it follows that $\|Au - Av\|_{\ell^2_w} \leq 4\|u - v\|_{\ell^2_w}.$

Using the above two Lemmas we may restate the local existence result, in the case of weighted spaces.

Theorem 7.1 Let assumption (N_2) be satisfied, assume that $u_0 \in \ell^2_w$. There exists $T^*(u_0) > 0$, such that for all $0 < T < T^*(u_0)$, there exists a unique solution of the problem (17), (18), $u(t) \in C^1([0,T], \ell^2_w)$. 

**Proof:** This time, we write (1)-(2), as an ordinary differential equation in $\ell^2_w$.

$$\begin{align*}
\dot{u}(t) &+ \Phi(u(t)) = 0, \\
u(0) &= u_0,
\end{align*}$$

where $\Phi(u) = T(u) + A(u) + i\delta u - g$ and $u(t)$ lies in $\ell^2_w$. Lemmas 7.1, 7.2 suffice for the application of standard existence and uniqueness Theorems for ordinary differential equations in Banach spaces [37, pg. 78-82].

In order to prove the existence of global solutions we need to obtain some *a priori* estimates for the solution. In the dissipative case this is achieved by proving the existence of a globally attracting ball of finite radius in $\ell^2_w$.

**Lemma 7.3** Assume condition (W) on the weight function and that the damping coefficient satisfies

$$\frac{\delta}{2} - 2d_1d_2^{-1/2} \geq 0. \quad (82)$$

Let $(w_0,n)_{n \in \mathbb{Z}} = u_0, (g_n)_{n \in \mathbb{Z}} = g \in \ell^2_w$ and condition $(N_2)$ be satisfied. A dynamical system can be defined by (1)-(2),

$$S(t) : u_0 \in \ell^2_w \rightarrow u(t) \in \ell^2_w, \quad (83)$$

possessing a bounded absorbing set $B_0$ in $\ell^2_w$. For every bounded set $B$ of $\ell^2_w$, there exists $t_0(B, B_0)$ such that for all $t \geq t_0(B, B_0)$, it holds $S(t)B \subset B_0$.

**Proof:** We multiply (1) with $w_n, n \in \mathbb{Z}$ add over all lattice sites and keep the imaginary part. Working similarly as in Lemma 7.2 we obtain

$$\frac{1}{2} \frac{d}{dt} \sum_{n \in \mathbb{Z}} w_n |u_n|^2 - \sum_{n \in \mathbb{Z}} (w_{n+1} - w_n)(u_{1,n}u_{2,n+1} - u_{2,n}u_{1,n+1}) + \delta \sum_{n \in \mathbb{Z}} w_n |u_n|^2 \leq \frac{\delta}{2} \sum_{n \in \mathbb{Z}} w_n |u_n|^2 + \frac{1}{2\delta} \sum_{n \in \mathbb{Z}} w_n |g_n|^2$$

Using the assumptions on the weight function (W), we find that

$$\sum_{n \in \mathbb{Z}} |(w_{n+1} - w_n)(u_{1,n}u_{2,n+1} - u_{2,n}u_{1,n+1})| \leq d_1 \sum_{n \in \mathbb{Z}} w_n |(u_{1,n}u_{2,n+1} - u_{2,n}u_{1,n+1})| \leq d_1 \left( \sum_{n \in \mathbb{Z}} w_n |u_{1,n}u_{2,n+1}| + \sum_{n \in \mathbb{Z}} w_n |u_{2,n}u_{1,n+1}| \right) \quad (84)$$

We now estimate the sums in the above inequality as follows:

$$\sum_{n \in \mathbb{Z}} w_n |u_{1,n}u_{2,n+1}| = \sum_{n \in \mathbb{Z}} u_n^{1/2}u_{1,n}^{1/2} u_{1,n}u_{2,n+1} \leq \left( \sum_{n \in \mathbb{Z}} w_n |u_{1,n}|^2 \right)^{1/2} \left( \sum_{n \in \mathbb{Z}} w_n |u_{2,n+1}|^2 \right)^{1/2} \leq d_2^{-1/2} \|u\|_{\ell^2_w}^2,$$

and similarly,

$$\sum_{n \in \mathbb{Z}} w_n |u_{2,n}u_{1,n+1}| \leq d_2^{-1/2} \|u\|_{\ell^2_w}^2.$$

Therefore,

$$\sum_{n \in \mathbb{Z}} |(w_{n+1} - w_n)(u_{1,n}u_{2,n+1} - u_{2,n}u_{1,n+1})| \leq 2d_1d_2^{-1/2} \|u\|_{\ell^2_w}^2.$$
Thus, we obtain the differential inequality

\[ \frac{1}{2} \frac{d}{dt} || u ||_{L^2}^2 + \left( \frac{\delta}{2} - 2d_1d_2^{-1/2} \right) || u ||_{L^2}^2 \leq \frac{1}{2\delta} || g ||_{L^2}^2 \]

Using the Gronwall inequality, if \( \frac{\delta}{2} - 2d_1d_2^{-1/2} \geq 0 \) we obtain the existence of the attracting ball in \( \ell^2_w \). This concludes the proof of the Proposition. \( \diamond \)

**Remark 7.1** In the case of exponential weight \( w_n = \exp(\lambda n) \), it appears that \( d_1 = \exp(\lambda) - 1 \) and \( d_2 = \exp(\lambda) \).

The condition for existence of attracting ball becomes \( 8\sinh(\lambda/2) \leq \delta \).

We may further prove the existence of a global attractor attracting all bounded sets of \( \ell^2_w \). The next Lemma provides us with tail estimates in the weighted space.

**Lemma 7.4** Let \( (u_0,n)_{n \in \mathbb{Z}} = u_0 \in B \) where \( B \) is a bounded set of \( \ell^2_w \) and \( (g_n)_{n \in \mathbb{Z}} = g \in \ell^2_w \). Under assumptions (N2) on the nonlinearity, (W) on the weight function and assumption (\text{S2}) on the dissipation, for any \( \eta > 0 \), there exist \( T(\eta) \) and \( K(\eta) \) such that the solution \( (u_n)_{n \in \mathbb{Z}} = u \) of (\text{1.2}) satisfies for all \( t \geq T(\eta) \), the estimate

\[ \sum_{|n| \geq 2M} w_n \cdot | u_n |^2 \leq \frac{2\eta}{\delta}, \quad \text{for any} \quad M > K(\eta). \]  

**Proof:** The proof follows closely the proof of Lemma 5.2 for the corresponding tail estimates in the space \( \ell^2 \), only that we now multiply the equation by \( w_n \theta_n \pi_n \), instead of multiplying simply by \( \theta_n \pi_n \). We obtain the following inequality

\[ \frac{1}{2} \frac{d}{dt} \sum_{n \in \mathbb{Z}} w_n \theta_n \cdot | u_n |^2 - I_2 + \frac{\delta}{2} \sum_{n \in \mathbb{Z}} w_n \theta_n \cdot | u_n |^2 - \frac{1}{2\delta} \sum_{n \in \mathbb{Z}} w_n \theta_n \cdot | g_n |^2, \]  

where

\[ I_2 = \sum_{n \in \mathbb{Z}} \left\{ (\theta_{n+1}w_{n+1} - \theta_n w_n) (u_{1,n}u_{2,n+1} - u_{2,n}u_{1,n+1}) \right\}. \]

We rewrite

\[ (\theta_{n+1}w_{n+1} - \theta_n w_n) = (\theta_{n+1} - \theta_n) w_{n+1} + \theta_n (w_{n+1} - w_n). \]

Using arguments similar to those used in the proof of Lemma 5.2 and Lemma 7.3 we see that

\[ \sum_{n \in \mathbb{Z}} | \theta_{n+1} - \theta_n | | w_{n+1} | | u_{1,n}u_{2,n+1} - u_{2,n}u_{1,n+1} | \leq C_2 \rho_2 \frac{M}{\delta}, \]  

where \( C_2 \) depends only on \( d_1, d_2 \) and \( \rho_2 \) denotes the absorbing ball in \( \ell^2_w \). We also have the inequality

\[ \sum_{n \in \mathbb{Z}} \theta_n | w_{n+1} - w_n | | u_{1,n}u_{2,n+1} - u_{2,n}u_{1,n+1} | \leq d_1 \sum_{n \in \mathbb{Z}} \theta_n w_n | u_{1,n}u_{2,n+1} | + d_1 \sum_{n \in \mathbb{Z}} \theta_n w_n | u_{2,n}u_{1,n+1} |. \]  

For the first term of the rhs of (88), holds the estimate

\[ d_1 \sum_{n \in \mathbb{Z}} \theta_n w_n | u_{1,n}u_{2,n+1} | \leq d_1 \left( \sum_{n \in \mathbb{Z}} \theta_n w_n | u_{1,n} |^2 \right)^{1/2} \left( \sum_{n \in \mathbb{Z}} \theta_n w_n | u_{2,n+1} |^2 \right)^{1/2} \]

\[ \leq d_1 d_2^{-1/2} \left( \sum_{n \in \mathbb{Z}} \theta_n w_n | u_{1,n} |^2 \right)^{1/2} \left( \sum_{n \in \mathbb{Z}} \theta_n w_n | u_{2,n+1} |^2 \right)^{1/2}. \]  

An application of Young’s inequality to the rhs of (89), implies that

\[ d_1 \sum_{n \in \mathbb{Z}} \theta_n w_n | u_{1,n}u_{2,n+1} | \leq \frac{1}{2} \frac{d_1 d_2^{-1/2}}{2d_1 d_2^{-1/2}} \sum_{n \in \mathbb{Z}} \theta_n w_n | u_{1,n} |^2 + \frac{1}{2} \frac{d_1 d_2^{-1/2}}{2d_1 d_2^{-1/2}} \sum_{n \in \mathbb{Z}} \theta_n w_n | u_{2,n+1} |^2 \]

\[ \leq \frac{1}{2} \frac{d_1 d_2^{-1/2}}{2d_1 d_2^{-1/2}} \sum_{n \in \mathbb{Z}} \theta_n w_n | u_{1,n} |^2 + \frac{1}{2} \frac{d_1 d_2^{-1/2}}{2d_1 d_2^{-1/2}} \sum_{n \in \mathbb{Z}} \theta_n w_n | u_{2,n+1} |^2 \]

\[ + \frac{1}{2} \frac{d_1 d_2^{-1/2}}{2d_1 d_2^{-1/2}} \sum_{n \in \mathbb{Z}} (\theta_n - \theta_{n+1}) w_n | u_{2,n+1} |^2 \]

\[ \leq d_1 d_2^{-1/2} \sum_{n \in \mathbb{Z}} \theta_n w_n | u_{1,n} |^2 + \frac{C_3 \rho_2^2 M}{\delta}. \]
and similarly for the second term of the rhs of (88)

\[ d_1 \sum_{n \in \mathbb{Z}} \theta_n |u_{2,n}u_{1,n+1}| \leq d_1 d_2^{-1/2} \sum_{n \in \mathbb{Z}} \theta_n |u_{2,n}|^2 + \frac{C_4 \rho_2^2}{M}. \]  

(91)

Using (86) - (91) we derive that

\[ \frac{1}{2} \frac{d}{dt} \sum_{n \in \mathbb{Z}} w_n \theta_n |u_n|^2 + \left( \frac{\delta}{2} - 2d_1 d_2^{-1/2} \right) \sum_{n \in \mathbb{Z}} w_n \theta_n |u_n|^2 \leq \frac{C_4 \rho_2^2}{M} + \frac{1}{2d} \sum_{n \in \mathbb{Z}} w_n \theta_n |g_n|^2. \]

with \( C \) depending only on \( d_1, d_2 \). The rest follows by direct generalization of the arguments of Lemma (2).

In complete analogy as before we may prove the asymptotic compactness of the semigroup in the weighted spaces,

**Proposition 7.1** The semigroup \( S(t) \) is asymptotically compact in \( \ell^2_w \), that is if the sequence \( \phi_n \) is bounded in \( \ell^2_w \) and \( t_n \to \infty \) then \( S(t_n) \phi_n \) is precompact in \( \ell^2_w \).

This result leads to the existence of a global attractor in the weighted spaces.

**Theorem 7.2** Let conditions \((N_2), (W)\) and (30) be fulfilled. The semigroup \( S(t) \) associated to (1)-(2) possesses a global attractor \( A = \omega(B_0) \subset B_0 \subset \ell^2_w \) which is compact, connected and maximal among the functional invariant sets in \( \ell^2_w \).

## 8 Remarks on possible extensions

We conclude, by mentioning some other examples of DNLS-type equations for which, extensions of the results on the global solvability and the existence of global attractors, could be investigated.

A first example is provided by the DNLS with potential \( f(u) = |u|^{2n-1}u + V(u) \), i.e \( (1)-(2) \) with a nonlinearity of the form \( F(u)_{n \in \mathbb{Z}} = F(u_n) + V(u) \), where \( V : \mathbb{R} \to \mathbb{R} \) is a real valued potential \( |V(s)| \leq C \) for every \( s \in \mathbb{R} \). The potential function expresses the inhomogeneity properties of the medium.

Another interesting DNLS model, may be given by the spatial discretization of a modified NLS equation with viscosity

\[ iu_t + \frac{1}{2} u_{xx} - |u|^2 u = \delta u_{xx}, \quad \delta > 0 \]  

(92)

which describes the light propagation in an array of optical fibers in a weakly lossy medium. Taking into account some external excitation and considering a general nonlinearity, the discretized counterpart of (92) reads as (see [30])

\[ i u_n + (1 - i \delta)(u_{n+1} - 2u_n + u_{n-1}) + f(|u_n|^2)u_n = g_n. \]  

(93)

Typical DNLS-models include equations of the form \( (17)-(25) \)

\[ iU_n + (\alpha_n U_{n+1} + \beta_n U_{n+1}) + i \delta_n U_n + \gamma_n f(|U_n|^2)U_n = g_n. \]

For the coupling and nonlinear strength and dissipation parameter \( \alpha_n, \beta_n, \gamma_n, \delta_n \in \mathbb{R} \) we assume that there exist some constants \( c_1, c_2, c_3 \) such that

\[ |\alpha_n|, |\beta_n|, |\gamma_n| \leq c_1, \quad c_2 \leq \delta_n \leq c_3, \quad n \in \mathbb{Z}. \]

The validity of the results could be examined for DNLS-type equations considered in \( \mathbb{Z}^N, N > 1 \),

\[ i u_n + (A u)_n + i \delta_n u_n + f(|u_n|^2)u_n = g_n, \]  

(94)

\[ u_n(0) = u_{n,0}, \quad n = (n_1, n_2, ... , n_N) \in \mathbb{Z}^N. \]  

(95)

where dissipation satisfies (94) and the linear operator \( A \) has a decomposition introduced in (39): It is assumed that

\[ A = A_1 + A_2 + ... + A_N \]  

(96)
\[ A_j = B_j^* B_j = B_j^* B_j, \quad \| B_j \|_{\mathcal{L}(\ell^2)} \leq M, \quad j = 1, 2, ..., N, \]  
(97)

for some bounded linear operators \( B_j : \ell^2 \to \ell^2 \), and its adjoint \( B_j^* \), defined as

\[
(B_j u)_n = \sum_{l=-m_0}^{l=m_0} C_{j,l} u_{n+j}, \quad \text{for all } u = (u_n)_{n \in \mathbb{Z}^N}, \quad j = 1, 2, ..., N  
(98)
\]

\[
(B_j^* u)_n = \sum_{l=-m_0}^{l=m_0} C_{j,-l} u_{n+j}, \quad n_{j+l} = (n_1, n_2, ..., n_j-1, n_j + l, n_{j+1}, ..., n_N) \in \mathbb{Z}^N.  
(99)
\]

Under assumptions (96)-(98) similar relations to (8)-(9) are satisfied,

\[
(Bu,v)_{\ell^2} = (u,B^* v)_{\ell^2}, \quad (Au,v)_{\ell^2} = -(Bu,Bv)_{\ell^2}, \quad u,v \in \ell^2.  
\]

Especially for the conservative case, \( \delta_n = 0, n \in \mathbb{Z}^N \), and the power-law nonlinearity, (96)-(98) are sufficient for the extension of Theorem 3.1 to (94)-(95). Although this is a simple observation, it is important, since it is a rigorous verification that spatial dimension does not play a particular role on the global existence of the solutions as it happens in the continuous model [7], [11, 12]. For a discussion on the existence of global attractors for (94)-(95), one could be based on the framework provided by [39].

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References

[1] A. Ambrosetti and P. H. Rabinowitz, Dual Variational Methods in Critical Point Theory and Applications, J. Funct. Anal. 14, (1973), 349-381.
[2] V. S. Afraimovich, S.N. Chow, J. K. Hale, Synchronization in lattices of coupled oscillators, Phys. D 103 (1997), no. 1-4, 442-451.
[3] G. Akrivis, Finite Difference Discretization of the cubic Schrödinger equation, IMA J. Numer. Anal., 13, (1993), 115-124.
[4] S. Aubry, Breathers in nonlinear lattices: Existence, linear stability and quantization, Physica D 103 (1997), 201-250.
[5] C. Baesens and R. S. MacKay, Exponential localization of linear response in networks with exponentially decaying coupling, Nonlinearity 10, (1997) 931-940
[6] A. V. Babin and M. I. Vishik, Attractors for Partial Differential Evolution Equations in an Unbounded Domain, Proc. Roy. Soc. Edinb., 116A, (1990), 221-243.
[7] O. Bang, J. Rasmussen, P. Christiansen, Subcritical localization in the discrete nonlinear Schrödinger equation with arbitrary nonlinearity, Nonlinearity 7, (1994), 205-218.
[8] P. W. Bates, K. Lu, B. Wang, Attractors for lattice dynamical systems, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 11 (2001), no. 1, 143–153.
[9] P. W. Bates, A. Chmaj, A Discrete Convolution Model for Phase Transitions, Arch. Rational Mech. Anal. 150 (1999) 281-305.
[10] K. J. Blow, N. J. Doran, Global and Local Chaos in the Pumped Nonlinear Schrödinger Equation, Physical Review Letters, Vol. 52, No. 7, (1984), 526-539.
[11] T. Cazenave, A. Haraux, Introduction to Semilinear Evolution Equations, Oxford Lecture Series in Mathematics and its Applications 13, 1998.
[12] T. Cazenave, *An introduction to Nonlinear Schrödinger equations*, Textos des Métodos Matemáticos 26, I.M.U.F.R.J., Rio de Janeiro, 1996.

[13] S. N. Chow, J. K. Hale *Methods of Bifurcation Theory*, Grundlehren der mathematischen Wissenschaften-A series of Comprehensive Studies in Mathematics 251, Springer-Verlag, New-York, 1982.

[14] S. Flach and C. R. Willis *Discrete Breathers* Phys. Reports 295 (1998) 181-264

[15] J M Ghidaglia, *Finite Dimensional Behavior for Weakly Damped Driven Schrödinger Equation*, Ann. Inst. Henri Poincaré, 5, (1988), 365-405.

[16] O. Goubet, *Regularity of the attractor for a weakly damped nonlinear Schrödinger equation*, Appl. Anal. 60 (1996), no. 1-2, 99–119.

[17] B. C. Gupta, K. Kundu, Localised states in a 1-D nonlinear chain, Phys. Lett. A 235, (1997), 176-182.

[18] D. Hennig and G. P. Tsironis, *Wave transmission in nonlinear lattices*, Phys. Rep. 307 (1999) 333-432

[19] F. Hirsch, G. Lacombe, *Elements of Functional Analysis*, Graduate Texts in Mathematics 192, Springer-Verlag, New-York, 1999.

[20] N. Karachalios and N. Stavrakakis, *Existence of Global Attractors for Semilinear Dissipative Wave Equations on* $\mathbb{R}^N$, J. Diff. Eqns, 157 (1999), 183-205.

[21] N. Karachalios and N. Stavrakakis, *Global Attractor for the Weakly Damped Driven Schrödinger Equation in* $H^2(\mathbb{R})$, NonDEA Nonlinear Differential Equations Appl., Birkhäuser, 9 (2002), 347-360.

[22] P. Laurençot, *Long Time Behavior for Weakly Damped Driven Nonlinear Schrödinger Equation in* $\mathbb{R}^N$, $N \leq 3$, NonDEA Nonlinear Differential Equations Appl., Birkhäuser, 2 (1995), 357-369.

[23] P. G. Kevrekidis, K. O. Rasmussen and A. R. Bishop, *The discrete nonlinear Schrödinger equation: A survey of recent results*, Int. Journal of Modern Physics B, 15 (2001), 2833-2900

[24] P. G. Kevrekidis, B. A. Malomed, A. R. Bishop and D. J. Frantzeskakis, *Localized vortices with a semi-integer charge in nonlinear dynamical lattices*, Phys. Rev. E 65 (2002) 016605

[25] S. W. Kim, S. Kim, *The structure of eigenmodes and phonon scattering by discrete breathers in the discrete nonlinear Schrödinger chain*, Phys. D 141, (2000), 91-103.

[26] R. S. Mackay and S. Aubry, *Proof of existence of breathers for time reversible or hamiltonian networks of weakly coupled oscillators* Nonlinearity 7 (1994) 1623

[27] B. A. Malomed, P. G. Kevrekidis, D. J. Frantzeskakis, H. E. Nistazakis and A. N. Yannacopoulos, *1 and 2-D solitons in second harmonic generating lattices*, Phys. Rev. E 65 (2002) 056606

[28] Y. Martel, *Blow-up for the nonlinear equation in nonisotropic spaces*, Nonlinear Analysis, TMA 28, no. 12, (1997), 1903-1908.

[29] K. Nozaki, N. Bekki, *Low Dimensional Chaos in a Driven Damped Nonlinear Schrödinger Equation*, Physica D 21, (1986), 381-393.

[30] M. Salerno, B. M. Malomed, V. Konotop, *Shock wave dynamics in a discrete nonlinear Schrödinger equation with internal losses*, Phys. Review E 62, No.6, (2000), 8651-8656.

[31] M. Struwe, *Variational Methods-Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems* (2nd edition), A Series of Modern Surveys in Mathematics, Vol. 34, Springer-Verlag, (1996).

[32] C. Sulem, P. L. Sulem, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*, Applied Mathematical Sciences 139, Springer-Verlag, New-York, 1999.

[33] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd edition*, Springer-Verlag, New York, 1997.

[34] X. Wang, *An Energy Equation for the Weakly Damped Driven Nonlinear Schrödinger equation and its application to their Attractors*, Physica D 88 (1995), 167-175.
[35] A. N. Yannacopoulos, D. J. Frantzeskakis, C. Polymilis and K. Hizanidis, *Conditions for soliton trapping in random potentials using Lyapunov exponents of stochastic ODEs*, Phys. Lett. A 271 (2000) pp. 334-340

[36] A. N. Yannacopoulos, D. J. Frantzeskakis, C. Polymilis and K. Hizanidis, *Motion of Schrödinger solitary waves in the presence of random external potentials*, Physica Scr. 65 (2002) pp. 363-368

[37] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Vols I, II, (Fixed Point Theorems, Monotone Operators)*, Springer-Verlag, Berlin, 1990.

[38] S. Zhou, *Attractors for Second order Dynamical Systems*, J. Diff. Eqns 179, (2002), 605-624.

[39] S. Zhou, *Attractors for first order dissipative lattice dynamical systems*, Physica D 178 (2003), 51-61.