A fast alternating projection method for complex frequency estimation.

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Abstract

The problem of approximating a sampled function using sums of a fixed number of complex exponentials is considered. We use alternating projections between fixed rank matrices and Hankel matrices to obtain such an approximation. Convergence, convergence rates and error estimates for this technique are proven, and fast algorithms are developed. We compare the numerical results obtain with the MUSIC and ESPRIT methods.

1 Introduction

The present paper is devoted to the problem of approximating a sampled function with a sum of a given number of complex exponentials. Our approach is based on the fact that if such a sum is used as a generating function for a Hankel matrix, then that Hankel matrix will (generically) be of rank \( k \). Using this fact, we develop a method for the detection of complex frequencies from a signal by alternating projections: we project the corresponding Hankel matrix onto the class of symmetric rank \( k \)-matrices, project the projection on the class of Hankel matrices, and so on. By a complex frequency, we refer to the coefficient \( \zeta \in \mathbb{C} \) in an exponential of the form \( e^{\zeta t} \).

There are several alternative techniques for the estimation of (complex) frequencies from a signal. Two of the most commonly used ones are multiple signal classification (MUSIC) \[24\] and estimation using rotational invariance (ESPRIT) \[23\]. The MUSIC method is a generalization of the Pisarenko method \[21\]. Recently, complex frequency estimation has been used in the construction of close to optimal quadratures, for instance for bandlimited functions \[6\]. This work is related to the work of Adamjan, Arov and Krein \[1\], and the algorithms described in \[6\] have been investigated in more detail in \[3\].

The technique of alternating projections is generally described as follows: Given two manifolds \( M_1, M_2 \subset K \) (where \( K \) is some Hilbert space) and some point \( x_0 \in K \), find a point \( x \in M_1 \cap M_2 \) that is close to \( x_0 \), by projecting alternately onto \( M_1 \) and \( M_2 \), respectively. It was proven by von Neumann \[18\] that if \( M_1 \) and \( M_2 \) are affine linear subspaces, then the sequence of alternating projections

\[ \pi_1(x_0), \pi_2(\pi_1(x_0)), \pi_1(\pi_2(\pi_1(x_0))), \ldots \]

converges to an optimal solution \( x \in M_1 \cap M_2 \), i.e., one that minimizes \( \|x - x_0\| \).
The extension to the case where \( M_1 \) and \( M_2 \) are convex sets has been extensively treated for a number of applications, cf. [4, 5] and the references therein. Another generalization was given in [13], where the convergence of the alternating projection scheme was proven for the case where, loosely speaking, the tangent spaces of \( M_1 \) and \( M_2 \) together span \( K \). Note that only convergence to some point in \( M_1 \cap M_2 \) can be proven, but that this point is not necessarily the point in \( M_1 \cap M_2 \) that is closest to \( x_0 \).

Moreover, neither of the cases above apply to the case which we are interested in, as the space of rank \( k \)-matrices is not convex, and the spanning condition is typically far from satisfied. In [2], convergence of alternating projections between two manifolds is proved under much milder conditions than the ones given in [13]. In this paper we prove that these conditions are generically satisfied in our case; complex symmetric rank \( k \)-matrices and Hankel matrices. Moreover, through the framework of [2] we can provide estimates for how far away from the initial (sampled) function the approximating \( k \)-term complex exponential sum will be.

The idea of using alternating projections for frequency estimation has appeared in a number of different settings. The method of alternating projection is commonly referred to as Cadzow’s method in the signal processing community. In [8], Zangwill’s global convergence theorem is used to prove convergence for algorithms with alternately projects onto (possibly more than two) manifolds. However, Zangwill’s theorem only provides the existence of a convergent subsequence, and the results in [8] do not give any information on whether or not the point of convergence is close to the original one, cf. [9]. In the paper [8], several applications are mentioned; one of them is the projection between finite rank matrices and Toeplitz matrices. Toeplitz matrices appear in the estimation of exponentials by using infinite measurement (or expected value) of autocorrelation matrices. For an (infinitely dense) sampling of a function consisting of \( k \) complex exponentials, it is possible to form a Toeplitz matrix from which the \( k \) frequencies can be recovered. It is worth mentioning that for a finite sampling of a function with \( k \) complex frequencies, the resulting autocorrelation function will not have a Toeplitz structure, and hence the frequencies can not be exactly recovered with this method, even in the absence of noise. A survey of problems of approximations using a combination of structured matrices and low-rank matrices is given in [17]. Alternating projections is mentioned as one of the numerical methods for finding approximate solutions.

The use of alternating projections (Cadzow’s method) between Hankel and low-rank matrices has appeared several times in the signal processing literature [14, 15, 22]. The approaches differ in the way the complex frequencies are estimated, once the alternative projection method has converged.

In this paper we develop fast methods for the projection steps. We make use of the fact that multiplication by a Hankel matrix, as well as the projection of low rank matrices onto Hankel matrices, can be computed in a fast manner by the use of FFT. For the projection onto low-rank representations, we will use a customized complex symmetric version of the Lanzcos algorithm.

Finally, we consider the approximation by exponentials for a particular class of weighted spaces – including (approximate) Gaussian weights. Let \( w \) be a nonnegative function on \( \mathbb{R} \) with support \([-1, 1]\) and let

\[
\omega = w \ast w.
\]  

Let \( L^2(\omega) \) denote the set of functions for which \( \|f\|_{L^2(\omega)}^2 = \int_{-2}^{2} |f(t)|^2 \omega(t) \, dt < \infty \). Given \( f \in L^2(\omega) \) and \( k \in \mathbb{N} \), we are interested in computationally efficient methods for finding the best (or close to best) approximation of \( f \) by functions of the form \( \sum_{j=1}^{k} c_j e^{\zeta_j t} \). In this paper we develop a theory for finite sequences rather than functions on a continuum. Using techniques similar to those developed in [3], it seems to be possible to develop a similar technique for the approximation of functions on a continuum by a finite number of complex exponentials.

## 2 Preliminaries

In section 2.1 we give the necessary tools for projection onto matrices of a certain rank and set up the spaces we will work with. In section 2.2 we describe how to go from a Hankel matrix to its symbol and back, in these spaces.
2.1 Takagi factorization and the Eckart-Young theorem

We use the notation $\mathbb{M}_{M,N}$ to denote the Hilbert space of $M \times N$ matrices with complex entries, equipped with the Frobenius norm, given by

$$\|A\|^2 = \sum_{j=1}^{M} \sum_{l=1}^{N} |A(j,l)|^2. \quad (2)$$

Complex symmetric matrices satisfy the symmetry condition $A = A^T$, which is different from the usual (Hermitian) self-adjointness condition $A = A^\ast$. Similarly to real symmetric matrices, which are always diagonalizable, complex symmetric matrices can be decomposed as

$$A = \sum_{m=1}^{N} s_m u_m^* u_m, \quad s_m \in \mathbb{R}^+, \quad u_m \in \mathbb{C}^N,$$

where the vectors $\{u_m\}_m$ are mutually orthogonal. (As usual, elements of $\mathbb{C}^N$ are identified by column matrices, and $u^*$ is the adjoint, i.e. the transpose of the complex conjugate of $u$. This decomposition of $A$ is called a Takagi factorization. Note that in contrast to the Hermitian case, the numbers $s_m$ are nonnegative. Moreover, the vectors $u_m$ satisfy the relation

$$A u_m = s_m u_m. \quad (3)$$

In [12], the vectors $u_m$ are referred to as con-eigenvectors and the positive numbers $s_m$ are referred to as con-eigenvalues. However, the con-eigenvectors are simply singular vectors (obtained from the Singular Value Decomposition), and the con-eigenvalues are the singular values. This is seen by noting that

$$s_m^2 u_m = A^\ast A u_m.$$

The converse is not true, since it is easily seen that e.g. $i u_m$ fails to be a con-eigenvector but is still a singular vector. However, in the case where the $s_m$’s are distinct and $(u_m)_{m=1}^N$ is any basis of singular vectors, then one can choose $\theta_m \in [0, 2\pi)$, such that $(e^{i\theta_m} u_m)_{m=1}^N$ are con-eigenvectors. For the purposes of this paper, we are only interested in the corresponding polynomials, and hence the $\theta_m$’s have no importance, but it will be computationally more convenient to extract the con-eigenvectors, and we have thus chosen to use this terminology.

We recall the Eckart-Young theorem (see e.g. [12, p 205], [10]), (usually stated using the singular vectors):

**Theorem 1** Let $A \in \mathbb{M}_{N,N}$ be a complex symmetric matrix with distinct con-eigenvalues. Given a positive integer $k \leq N$, the best rank $k$ approximation of $A$ (in $\mathbb{M}_{N,N}$) is given by

$$\sum_{m=1}^{k} s_m u_m^* u_m, \quad (4)$$

where $s_m$ and $u_m$ are the (decreasingly ordered) con-eigenvalues and con-eigenvectors of $A$, respectively.

The above theorem can clearly be used to project a given matrix onto the closest rank $k$ matrix (with respect to the Frobenius norm). We will also make use of approximations in weighted spaces. Given a positive weight $w \in \mathbb{R}^N$, we denote by $\mathbb{M}_{N,N}^w$ the Hilbert space of matrices with the weighted Frobenius norm, given by

$$\|A\|_w^2 = \sum_{j,k=1}^{N} w(j)|A(j,k)|^2. \quad (5)$$

**Theorem 2** Let $A \in \mathbb{M}_{N,N}^w$, and let $s_m$ and $q_m$ denote con-eigenvalues and con-eigenvectors of $B = \text{diag}(\sqrt{w}) A \text{diag}(\sqrt{w})$. Then the best rank $k$-approximation of $A$ (in $\mathbb{M}_{N,N}^w$) is given by

$$\sum_{m=1}^{k} s_m u_m^* u_m, \quad (6)$$

where $u_m(l) = q_m(l)/\sqrt{w(l)}$, $1 \leq l \leq N$. 


Proof: By definition

\[
\left\| A - \sum_{m=1}^{k} s_m u_m u_m^* \right\|_w = \left\| \text{diag}(\sqrt{w}) \left( A - \sum_{m=1}^{k} s_m u_m u_m^* \right) \text{diag}(\sqrt{w}) \right\|_w
\]

\[
= \left\| B - \sum_{m=1}^{k} s_m (\text{diag}(\sqrt{w}) u_m) (\text{diag}(\sqrt{w}) u_m)^* \right\|
\]

which according to Theorem 1 is minimized by choosing \( u_m = (\text{diag}(\sqrt{w}))^{-1} q_m \) and by choosing \( s_m \) as the con-
eigenvalues of \( B \).

There are different ways to compute Takagi factorizations. We indicate one method, the first step of which is the
following proposition.

**Proposition 1** Let \( A \) and \( B \) be real symmetric \((N \times N)\)-matrices and let

\[
W = \begin{pmatrix} A & -B \\ -B & -A \end{pmatrix}.
\]

Let \( d_1 \geq d_2 \geq \ldots \geq d_{2N} \) be the eigenvalues of \( W \). Then \( d_j + d_{2N+1-j} = 0 \) for \( j = 1, 2, \ldots, 2N \), and an orthonormal
basis of eigenvectors can be chosen as

\[
\begin{pmatrix} X_j \\ Y_j \end{pmatrix}, \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}, \ldots, \begin{pmatrix} X_{2n} \\ Y_{2n} \end{pmatrix},
\]

where \( X_j, Y_j \in \mathbb{R}^N \), \( X_{2n+1-j} = -Y_j \) and \( Y_{2n+1-j} = X_j \) for \( j = 1, 2, \ldots, N \).

The proof is given as an exercise in [12].

### 2.2 Hankel matrices

A **Hankel** matrix \( A \) has constant entries on the anti-diagonals, i.e. it satisfies the relation

\[
A(j, l) = A(j', l'), \quad \text{if } j + l = j' + l'.
\]

Every Hankel matrix \( A \in \mathbb{M}_{N,N} \) can thus be generated from some vector \( f = (f_j)_{j=1}^{2N} \) by

\[
A(j, l) = Hf(j, l) = f(j + l), \quad 1 \leq j, l \leq N.
\]

(5)

An orthonormal basis for the Hankel matrices in \( \mathbb{M}_{N,N} \) is given by

\[
e_m(j, l) = \begin{cases} \frac{1}{\sqrt{\omega(m)}}, & \text{if } j + l = m; \\ 0, & \text{otherwise}. \end{cases}
\]

(6)

for \( 2 \leq m \leq 2N \), where the normalization factor originates from the number of elements along anti-diagonal \( m \).

When considering Hankel matrices in weighted spaces we need to use proper normalization; the basis elements should
be normalized with respect to the induced (matrix) weights along the anti-diagonal. We associate the weights

\[
\omega(m) = \sum_{j+l=m, 1 \leq j \leq N} w(j)w(l), \quad 2 \leq m \leq 2N;
\]

(7)

to \( w \), and note that this can be written as a discrete convolution \( \omega = \tilde{w} * \tilde{w} \), where \( \tilde{w} \) denotes the zero padded version
of \( w \). A basis for Hankel matrices in the weighted space \( \mathbb{M}_{N,N}^w \) is then given by

\[
e_m^w(j, l) = \begin{cases} \frac{1}{\sqrt{\omega(m)}}, & \text{if } j + l = m; \\ 0, & \text{otherwise}. \end{cases}
\]
for \(2 \leq m \leq 2N\). Note that in the case \(w = 1\) we get the “triangle weight” which appeared in (6). We let \(\ell^2_{2N-1}\) be the space of complex sequences \(f = (f_j)_{j=2}^{2N}\), equipped with the norm defined by

\[
\|f\|_2^2 = \sum_j |f_j|^2 \omega(j).
\]

The mapping \(H\) (given by (5)) will in the sequel be considered as a mapping from \(\ell^2_{2N-1}\) to \((\mathbb{M}_{N,N}, \| \cdot \|_w)\). It is a unitary map (isometric isomorphism), whose adjoint is the weighted averaging operator

\[
H^* A(m) = \frac{1}{\omega(m)} \sum_{j+l=m} w(j) A(j,l) w(l),
\]

and \(H^* H = I\). The following proposition is now immediate.

**Proposition 2** Let \(w \in \mathbb{R}^N_+\) be given and let \(\omega\) be the associated weight defined by (4). Let \(f = (f_j)_{j=2}^{2N}\) and let \(S\) be any set of Hankel matrices. Then the problem

\[
\arg\min_{\tilde{H} \in S} \|H f - \tilde{H}\|_w
\]

is equivalent to the problem

\[
\arg\min_{g \in H^*(S)} \|f - g\|_{\ell^2_{2N-1}}.
\]

The solutions are related by \(Hg = \tilde{H}\).

### 3 Properties of fixed-rank and Hankel matrices

The key observation behind the algorithms of this paper is that a rank \(k\) Hankel operator generically has a symbol which is a sum of \(k\) exponentials. However, this is not always true, and neither is the projection onto rank \(k\) matrices, given by Theorem 1 well defined at all points. In this section we show that the exceptional set is very small. We introduce the concept of a thin set, and show that the exceptional points are confined to thin sets.

We denote by \(\mathcal{H}_N\) the set of Hankel matrices in \(\mathbb{M}_{N,N}\), and \(\mathcal{R}_{N,k}\) will denote the set of matrices in \(\mathbb{M}_{N,N}\) of rank at most \(k\).

#### 3.1 Manifold structure

In this entire section, we will work with subsets of \(\mathbb{M}_{N,N}\), consisting of matrices whose entries are ordered from 1 to \(N\). \(\mathcal{H}\) is a linear subspace of \(\mathbb{M}_{N,N}\) and, hence, a differentiable manifold of (real) dimension \(2(2N-1)\). By identifying \(\mathbb{C}\) with \(\mathbb{R}^2\) in the obvious way, a simple modification of \(H\) (defined in (5)) provides a natural chart. The structure of \(\mathcal{R}_{N,k}\) is more complicated; we will show that it is a manifold of (real) dimension \(2(2Nk-k^2)\) outside a small exceptional set. Suppose \(A \in \mathcal{R}_{N,k}\), and use the singular value decomposition of \(A\) to find \(\sigma_A \in (\mathbb{R}^+)^k\) and \(U_A, V_A\) such that \(U_A^* U_A = V_A^* V_A = I_k\) (where \(I_k\) is the \(k \times k\) identity matrix and \(U_A, V_A\) are \(N \times k\)-matrices) and

\[
A = V_A \begin{pmatrix} \sigma_{A,1} & 0 & \cdots & 0 \\ 0 & \sigma_{A,2} & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_{A,k} \end{pmatrix} U_A^*.
\]

A typical matrix in \(\mathcal{R}_{N,k}\) satisfies

\[
\sigma_{A,1} > \sigma_{A,2} > \ldots > \sigma_{A,k} > 0,
\]

and, if this is not the case, an arbitrary small numerical perturbation will yield distinct singular values. The subset of \(\mathcal{R}_{N,k}\), consisting of \(N \times N\)-matrices satisfying (10), will be denoted \(\mathcal{R}_{N,k}^d\), where \(d\) stands for “distinct”. If \(\mathcal{M}\) is a manifold and \(E\) is a set, contained in the union of finitely many manifolds of dimension lower than the dimension of \(\mathcal{M}\), we will say that \(E\) is thin relatively to \(\mathcal{M}\).
**Proposition 3** \( \mathcal{R}^d_{N,k} \subset \mathbb{M}_{N,N} \) is a manifold of (real) dimension \( 2(2Nk - k^2) \). Moreover, \( \mathcal{R}_{N,k} = \mathcal{R}^{d^2}_{N,k} \) and \( \mathcal{R}_{N,k} \setminus \mathcal{R}^d_{N,k} \) is thin relatively to \( \mathcal{R}_{N,k} \).

**Proof:** We start by remarking that the set \( \mathcal{U}(N,k) \) of complex \( N \times k \)-matrices \( U \) satisfying \( U^*U = I_k \) is a real manifold of dimension \( 2Nk - k^2 \). Namely, the columns \( U(:,1), U(:,2), \ldots, U(:,k) \) of such a matrix can be identified with points on \( S^{2N-1} \), and thus \( \mathcal{U}(N,k) \) can be identified with the subset of elements \( U \in (S^{2N-1})^k \), satisfying the functionally independent equations

\[
\text{Re} U^*(-j)U(:,l) = \text{Im} U^*(-j)U(:,l) = 0, \quad 1 \leq l \leq j \leq k.
\]

The number of these equations is

\[2 + 4 + \ldots + 2(k-1) = k^2 - k,
\]

and thus \( \mathcal{U}(N,k) \) is a manifold of dimension \( k(2N-1) - (k^2 - k) = 2Nk - k^2 \).

Now let \( A \in \mathcal{R}^d_{N,k} \). Then there are matrices \( U_A \) and \( V_A \) in \( \mathcal{U}(N,k) \) and a vector \( \sigma_A \in \mathbb{R}^k_+ \) with \( \sigma_{A,1} > \sigma_{A,2} > \ldots > \sigma_{A,k} \), such that

\[A = V_A \text{diag}(\sigma_A) U^*_A = \sum_{j=1}^k \sigma_{A,j} V_A(:,j) U_A^*(\cdot, j).
\]

In this representation, the numbers \( \sigma_{A,j} \) are uniquely determined by \( A \), and so are the products \( V_A(:,j) U_A^*(\cdot, j) \), but the vectors \( U_A(:,j) \) and \( V_A(:,j) \) are not; each vector \( U_A(:,j) \) can be multiplied by a complex unit factor \( e^{i\theta_j} \in S^1 \) and \( V_A(:,j) \) by the same factor, whence the product \( V_A(:,j) U_A^*(\cdot, j) \) remains unaffected. We can thus define a mapping

\[F : \mathcal{R}^d_k \times (S^1)^k \to \mathcal{U}(N,k) \times \{ \sigma \in \mathbb{R}^k_+ ; \sigma_1 > \sigma_2 > \ldots > \sigma_k \geq 0 \} \times \mathcal{U}(N,k)
\]

by

\[F(A, (e^{i\theta_j})_{j=1}^k) = \left( V_A \text{diag}(e^{i\theta_j})_{j=1}^k, \text{diag}(\sigma_A), U_A \text{diag}(e^{i\theta_j})_{j=1}^k \right).
\]

It is easily verified that this mapping is a diffeomorphism, and hence

\[\dim \mathcal{R}^d_k + k = (2Nk - k^2) + k + (2Nk - k^2), \quad \text{i.e.} \quad \dim \mathcal{R}^d_k = 2(2Nk - k^2)
\]

We omit a proof of the remaining statements, which can be obtained by standard matrix theory and differential geometry.

Given a matrix \( A \in \mathcal{M}_{N,N} \), the closest point in \( \mathcal{R}_{N,k} \) is given by the Eckart-Young theorem, and it is unique whenever the singular values are distinct. By the above theorem, it is very improbable that this would not be the case for an arbitrary matrix \( A \). Indeed, when working with “real numerical” data this never happens, so we will for simplicity treat the projection onto \( \mathcal{R}_{N,k} \) as a well defined map which we denote by \( \pi_{\mathcal{R}_{N,k}} \). A more stringent approach would be to work with “point to set”-maps, as in [38] and [31].

**Infinite Hankel matrices of finite rank**

To understand the structure of Hankel matrices, it seems indispensable to consider infinite Hankel matrices, by which we mean complex-valued functions on \( \mathbb{N} \times \mathbb{N} \), where \( \mathbb{N} = \{ 0, 1, 2, \ldots \} \) (in this section we include \( 0 \) in the index set for convenience). For a complex valued function \( f \) on \( \mathbb{N} \), we denote by \( Hf \) the infinite Hankel matrix with \( Hf(j,l) = f(j+l) \). This means that \( H \) is an operator from \( \mathbb{C}^\mathbb{N} \) to \( \mathbb{C}^{\mathbb{N}^2} \).

The rank of an infinite matrix is the dimension of its column space (the linear space generated by its columns).

**Assume that** \( A = Hf \) **is an infinite Hankel matrix, such that some column is a (complex) linear combination of the preceding ones (i.e. rank \( A < \infty \)).** Låt \( A(:,r) \) be the first one of these. It thus holds

\[A(j,r) + \sum_{l=0}^{r-1} \lambda_l A(j,l) = 0
\]

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Theorem 3

Let $A$ be an infinite Hankel matrix of rank $r < \infty$. Then

$$\begin{vmatrix}
A(0, 0) & A(0, 1) & \cdots & A(0, r - 1) \\
A(1, 0) & A(1, 1) & \cdots & A(1, r - 1) \\
\vdots & \vdots & \ddots & \vdots \\
A(r - 1, 0) & A(r - 1, 1) & \cdots & A(r - 1, r - 1)
\end{vmatrix} \neq 0. $$

Proof: Assume that the determinant vanishes. We have seen that every column, starting with $A(\cdot, r)$, is a linear combination (with the same coefficients) of the $r$ preceding columns, and we conclude that $r$ is the rank of the matrix.

We now study the generating function for $f$:

$$F(x) = \sum_{k=0}^{\infty} f(k)x^k$$

Using (11), we get

$$F(x) = \frac{f(0) + \sum_{k=1}^{r-1} (f(k) + \sum_{l=r-k}^{r-1} \lambda_l f(k - r + l))x^k}{1 + \sum_{k=1}^{\infty} \lambda_{r-k}x^k}. $$

In this quotient, the degree of the numerator is at most $r - 1$. If $\lambda_0 \neq 0$, the degree of the denominator is $r$, and there is an expansion

$$F(x) = \sum_{\nu=1}^{p} \sum_{\mu=0}^{m_{\nu} - 1} \frac{a_{\nu, \mu}}{(1 - \zeta_{\nu}x)^{\mu+1}}, $$

where $m_1 + m_2 + \ldots + m_p = r$, and $a_{\nu, \mu}$ are constants with $a_{\nu, m_{\nu}} \neq 0$. Hence

$$F(x) = \sum_{\nu=1}^{p} \sum_{\mu=0}^{m_{\nu} - 1} \frac{a_{\nu, \mu}}{\mu!} \frac{d^\mu}{dx^\mu} \frac{1}{1 - \zeta_{\nu}x} \sum_{k=0}^{\infty} \zeta_{\nu}^{k-\mu} x^k
= \sum_{\nu=1}^{p} \sum_{\mu=0}^{m_{\nu} - 1} A_{\nu, \mu} \sum_{k=\mu}^{\infty} \frac{k}{\mu} (\zeta_{\nu}x)^{k-\mu} = \sum_{\nu=1}^{p} \sum_{\mu=0}^{m_{\nu} - 1} a_{\nu, \mu} \sum_{\mu=0}^{\infty} \left( \frac{k + \mu}{\mu} \right) (\zeta_{\nu}x)^k,
We find that
$$f(k) = \sum_{\nu=1}^{p} Q_{\nu}(k) c_{k}^{k},$$
where the $Q_{\nu}$ are polynomials of degree $m_{\nu} - 1$, $\nu = 1, 2, \ldots, p$.

If $\lambda_0 = 0$, the numerator in (12) is of degree $r - 1$, because otherwise the columns $A(\cdot, r - 1)$ would be a linear combination of the preceding ones, contrary to our choice of $r$. In this case we let $d$ be the first number with $\lambda_d \neq 0$, and a polynomial division yields
$$F(x) = Q(x) + \sum_{\nu=1}^{p} \sum_{\mu=0}^{m_{\nu}-1} \frac{a_{\nu,\mu}}{(1 - \zeta_{\nu}x)^{\mu+1}}, \quad \deg Q(x) = d - 1.$$
where $m_1 + m_2 + \ldots + m_p = r - d$, and the $a_{\nu,\mu}$ are constants with $a_{\nu,m_\nu} \neq 0$.

If we define $\delta(j)$ as 0 when $j \neq 0$ and 1 when $j = 0$, we can write

$$\text{Theorem 4} \quad \text{Let } A = Hf \text{ be an infinite Hankel matrix of finite rank } r. \text{ Then}$$
$$f(k) = \sum_{\mu=0}^{d-1} c_{\mu}\delta(k - \mu) + \sum_{\nu=1}^{p} Q_{\nu}(k) \zeta_{\nu}^{k},$$
where $c_{d-1} \neq 0$ (in case $d \geq 1$), $Q_{\nu}$ are polynomials with $\deg Q_{\nu} = m_{\nu} - 1$, and $d + m_1 + m_2 + \ldots + m_p = r$.

We can also write
$$f(j + l) = \sum_{\mu=0}^{d-1} c_{\mu}\delta(j + l - \mu) + \sum_{\nu=1}^{p} \sum_{\mu=0}^{m_{\nu}-1} q_{\nu,\mu}(j) \zeta_{\nu}^{j} l^{\mu} \zeta_{\nu}^{l}, \quad \deg q_{\nu,\mu} = m_{\nu} - 1 - \mu.$$

We will now investigate how the nodes $\zeta_{\nu}$ can be determined. We put
$$P(x) = x^{d} \prod_{\nu=1}^{p} (x - \zeta_{\nu})^{m_{\nu}} = \sum_{l=0}^{r} \lambda_{l} x^{l}.$$Then, for $\nu = 1, 2, \ldots, p$,
$$\left(\frac{d}{dx}\right)^{\mu} P(x) |_{x=\zeta_{\nu}} = 0, \quad \mu = 0, 1, \ldots, m_{\nu} - 1,$$i.e.
$$\sum_{l=0}^{r} \lambda_{l} l^{\mu} \zeta_{\nu}^{l} = 0, \quad \mu = 0, 1, \ldots, m_{\nu} - 1.$$It also holds $\lambda_{l} = 0$ if $l < d$. Hence, for $l = 0, 1, 2, \ldots, r$,
$$\sum_{l=0}^{r} f(j + l) \lambda_{l} = \sum_{\mu=0}^{d-1} c_{\mu} \sum_{l=0}^{r} \delta(j + l - \mu) \lambda_{l} + \sum_{\nu=1}^{p} \sum_{\mu=0}^{m_{\nu}-1} q_{\nu,\mu}(j) \zeta_{\nu}^{l} \sum_{l=0}^{r} \lambda_{l} l^{\mu} \zeta_{\nu}^{l} = 0.$$Now, define, for any nonnegative integer $k$, the upper left corner submatrix of order $k + 1$ by
$$A_{k} = \begin{pmatrix}
A(0, 0) & A(0, 1) & \ldots & A(0, k) \\
A(1, 0) & A(1, 1) & \ldots & A(1, k) \\
\vdots & \vdots & \ddots & \vdots \\
A(k, 0) & A(k, 1) & \ldots & A(k, k)
\end{pmatrix}.$$
We know that \( \det A_{r-1} \neq 0 \) and \( \det A_r = 0 \). Hence the kernel for \( A_r \) is one-dimensional, and we have characterized it: It is generated by the vector \((\lambda_0, \lambda_1, \ldots, \lambda_r)\), where

\[
\sum_{l=0}^r \lambda_l x^l = x^d \prod_{\nu=1}^p (x - \zeta_\nu)^{m_\nu}.
\]

We now observe that the numbers \( \lambda_0, \lambda_1, \ldots, \lambda_r \) are exactly the numbers appearing in (11) (with \( \lambda_r = 1 \)), and using that recursion equation, it is easily seen that for \( k \geq r \), the \((k+1-r)\)-dimensional kernel of \( A_k \) is generated by the vectors \((0, \ldots, 0, \lambda_0, \lambda_1, \ldots, \lambda_{r-1}, \lambda_r, 0, \ldots, 0)\). The coordinates of these vectors are the coefficients in the polynomials \( x^l P(x), \ l = 0, 1, \ldots, k - r \). We now summarize the observations made:

**Proposition 4** Let \( A = Hf \) be an infinite Hankel matrix of rank \( r < \infty \). Then

\[
f(k) = \sum_{\mu=0}^{d-1} c_\mu \delta(k - \mu) + \sum_{\nu=1}^p Q_\nu(k) \zeta_\nu^k,
\]

where \( d \geq 0, \ c_{d-1} \neq 0 \) (in case \( d \geq 1 \)), \( Q_\nu \) are polynomials with \( \deg Q_\nu = m_\nu - 1 \) and \( d + m_1 + m_2 + \ldots + m_p = r \).

If \( k \geq r \), the vector \((\mu_0, \mu_1, \ldots, \mu_k)\) belongs to the kernel of \( A_k \) if and only if there is a polynomial \( Q \) of degree at most \( k - r \), such that

\[
\sum_{l=0}^k \mu_l x^l = Q(x) x^d \prod_{\nu=1}^p (x - \zeta_\nu)^{m_\nu}.
\]

We call the polynomial

\[
P(x) = x^d \prod_{\nu=1}^p (x - \zeta_\nu)^{m_\nu}
\]

the central polynomial for \( A \).

**Finite Hankel matrices**

For an infinite Hankel matrix of finite rank \( r \), we have seen that the upper left corner matrix of order \( r \) is non-singular. For finite Hankel matrices, this will not always be the case. Let \( A \) be a Hankel matrix of size \( N \times N \), i.e. a complex-valued function on \( \{(j, l) \in \mathbb{N}^2; \ 0 \leq j \leq N - 1, \ 0 \leq l \leq N - 1\} \), such that \( A(j, l) = A(j', l') \) whenever \( j + l = j' + l' \). Then there are infinitely many functions \( f \) on \( \mathbb{N} \), such that \( A_{j, l} = f(j + l) \). Such a function \( f \) is determined by \( A = Hf \) only on the set \( \{0, 1, \ldots, 2N - 2\} \). We will now discuss “canonical” extensions of \( A \) to \( \mathbb{N}^2 \).

**Theorem 5** Let \( A = Hf \) be a \( N \times N \) Hankel matrix of rank \( r < N \) and assume that its upper left corner submatrix \( A_r \) of order \( r \) is non-singular. Then there are uniquely determined constants \( \lambda_0, \lambda_1, \ldots, \lambda_{r-1} \), such that

\[
f(k) + \sum_{l=0}^r \lambda_l f(k - r + l) = 0, \ \ \ k = r, r + 1, \ldots, 2N - 2.
\]

**Proof:** We have

\[
f(k) = A(0, k), \ \ 0 \leq k \leq r - 1
\]

and, since the column \( A(:, r) \) is a linear combination of the linearly independent columns \( A(:, l), \ l = 0, 1, \ldots, r - 1 \), there are uniquely determined constants \( \lambda_0, \lambda_1, \ldots, \lambda_{r-1} \), such that

\[
f(j + r) = A(j, r) = -\sum_{l=0}^{r-1} \lambda_l A(j, l) = -\sum_{l=0}^{r-1} \lambda_l f(j + l), \ \ \ 0 \leq j \leq N - 1.
\]
The relation (14) is thus valid for \( k \leq N - 1 + r \). Consequently, for \( k = r + 1, r + 2, \ldots, N - 1 \),

\[
A(j, k) = f(j + k) = - \sum_{l=0}^{r-1} \lambda_l f(j + k - r + l) = - \sum_{l=0}^{r-1} \lambda_l A(j, k - r + l), \quad j = 0, 1, \ldots, r - 1,
\]

Following the same relation holds for \( j = r, \ldots, N - 1 \), and thus the recursion formula (14) holds for \( k = r, r + 1, \ldots, 2N - 2 \).

A function \( f \), satisfying (14), has of course a unique extension to a function on \( \mathbb{N} \), satisfying the same relation. We conclude that if the condition on the upper left corner submatrix is fulfilled, then \( A \) has a canonical rank-preserving extension to an infinite Hankel matrix. If not, any extension to an infinite Hankel matrix is necessarily of a strictly higher rank. The first case is of course generic, and the latter case is exceptional. We will limit our attention to the generic case.

**Definition 1** A matrix \( A \in \mathcal{H}_{N,r} := \mathcal{R}_{N,r} \cap \mathcal{H}_N \) belongs to the class \( \mathcal{H}_{N,r}^0 \) if

1. The upper left corner submatrix of order \( r \) is non-singular,
2. In the central polynomial \( P(x) = x^d \prod_{\nu=1}^{p} (x - \zeta_{\nu})^{m_{\nu}} \), we have \( d = 0 \) and \( m_{\nu} = 1 \) for all \( \nu \) (and, consequently, \( p = r \)).

**Theorem 6**

\( \mathcal{H}_N \) is a real \( 2(2N - 1) \)-dimensional linear subspace of \( \mathcal{M}_{N,N} \).

\( \mathcal{H}_{N,r}^0 \) is a real differentiable manifold of dimension \( 4r \) which is dense in \( \mathcal{H}_{N,r} \). Its complement \( \mathcal{H}_{N,k} \setminus \mathcal{H}_{N,r}^0 \) is
thin relatively to \( \mathcal{H}_{N,r}^0 \).

The map \( \pi_{\mathcal{R}_{N,r}} \) is well defined at all points of \( \mathcal{H}_{N,r}^0 \).

**Proof:** The first statement is obvious. For the second, it is easily seen that the complex numbers \( f(0), f(1), \ldots, f(r - 1), \lambda_0, \lambda_1, \ldots, \lambda_{r-1} \) in (14) serve as complex coordinates on \( \mathcal{H}_{N,r}^0 \), and that the exceptional points (corresponding to matrices not in \( \mathcal{H}_{N,r}^0 \)) are given by restrictions, confining them to a thin set. The third statement is immediate by the Eckart-Young theorem.

### 3.2 Extracting frequencies from low rank Hankel matrices

We note that the second statement in Theorem 6 can be seen as a finite-dimensional version of Kronecker’s theorem. We will exploit it in order to approximate functions by sums of \( k \) exponentials:

\[
f(l) = \sum_{p=1}^{k} c_pe^{\zeta_p l}, \quad c_p, \zeta_p \in \mathbb{C}.
\] (15)

We choose some positive weight \( w \) that gives rise to a weight \( \omega \) through (7). The problem of approximating \( f \) by a sum of \( k \) exponentials in \( \ell^\omega \) is then according to Proposition 2 equivalent to finding the matrix \( Hf_{opt} \in \mathcal{R}_{N,k} \cap \mathcal{H}_N \) that minimizes \( \| Hg - Hf \|_w \).

Let us turn our focus to how to find \( c_p \) and \( \zeta_p \) in (15) given \( Hf \in \mathcal{H}_{N,k}^0 \). If \( u = (u_0, u_1, \ldots, u_k) \) is a vector in \( \mathbb{C}^{k+1} \), we define the polynomial \( P_u \), generated by \( u \), by

\[
P_u(x) = \sum_{j=0}^{k} u_j x^j.
\]

From Proposition 3 it follows that the nodes \( e^{\zeta_p} \) in (15) are precisely the zeroes of the central polynomial \( P(x) \) for \( Hf \), and this polynomial is the last common divisor of all the polynomials generated by vectors in the nullspace of \( Hf \).
Alternatively, it is the polynomial generated by a single vector, generating the nullspace of \((Hf)_{k+1}\). This approach is relatively fast (time \(O(k^3)\)), but it does not have good numerical stability. The reason for this is that we use only local data, i.e. only \(k+1\) elements from each co-eigenvector \(u_m\).

A better method is to observe that if \(f\) has the form \((15)\), then, due to \((3)\), the co-eigenvectors \(u_1, u_2, \ldots, u_k\) of \(Hf\) span the same subspace of \(\mathbb{C}^N\) as the vectors

\[
Z_l = (1, e^{\zeta_1}, e^{2\zeta_1}, \ldots, e^{(N-1)\zeta_1}), \quad l = 1, 2, \ldots, k.
\]

Let \(U = (u_1 \ldots u_k) \in \mathbb{M}_{N,k}\). We then have \(U = ZG\), where \(Z = (Z(\zeta_1), Z(\zeta_2), \ldots, Z(\zeta_k)) \in \mathbb{M}_{k,k}\) is the Vandermonde matrix generated by \(e^{\zeta_l}\) \((Z(p) = e^{\zeta p})\) and \(G\) is an invertible matrix in \(\mathbb{M}_{k,k}\). For any matrix \(A\), we denote by \(A(i)\) the matrix that appears when the \(j\)-th row of \(A\) is removed. Clearly, we have that \(U_1 = ZG\) and \(U(N) = Z(N)G\). We also note that

\[
Z(1) = Z(N)\text{diag}(e^{\zeta_1}, e^{2\zeta_1}, \ldots, e^{k\zeta_k}).
\]

Recall that \(U(N)\) has a natural left inverse given by \(U^\dagger(N) = (U(N)U(N))^{-1}U^*\). From the relations above, it follows that

\[
U^\dagger(N)U(1) = (U(N)U(N))^{-1}U^*Z(N)\text{diag}(e^{\zeta_1}, \ldots, e^{k\zeta_k})G.
\]

Now

\[
(U(N)U(N))^{-1}U^*Z(N)G = (U(N)U(N))^{-1}U^*U(N) = I_k,
\]

and thus

\[
U^\dagger(N)U(1) = G^{-1}\text{diag}(e^{\zeta_1}, \ldots, e^{k\zeta_k})G.
\]

Hence we can compute the nodes \(\zeta_m\) by computing the eigenvalues of \(U^\dagger(N)U(1)\). This method is numerically stable and can be computed in \(O(Nk^2 + k^3)\) time.

Once the nodes \(\zeta_m\) are found, the problem of finding \(c_m\) becomes linear, and again it will be sufficient to consider \(k\) consecutive elements solve the corresponding linear system.

\section{Alternating projections}

Given \(f \in \ell_{2N-1}^\omega\), the problem of finding the best approximation in \(\ell_{2N-1}^\omega\) of the form \(f_{opt}(l) = \sum_{j=1}^{k} c_je^{\zeta_jl}\) is hard.

Instead, our aim is to find an \(f_{opt}\) that is close to optimal. We will do this by employing alternating projections. By Proposition\ref{proposition} we know that this problem is equivalent to

\[
\arg \min_{Hg \in \mathcal{R}_{N,k} \cap \mathcal{H}_N} \|Hf - Hg\|_w. \tag{16}
\]

By starting with \(Hf_0 = Hf\) and alternatively projecting onto the subsets \(\mathcal{R}_{N,k}\) and \(\mathcal{H}_N\), the idea is that the so arising sequence \(Hf_m\) will converge to an intersection point \(Hf_\infty \in \mathcal{R}_{N,k} \cap \mathcal{H}_N\), and moreover that \(Hf_\infty\) is in fact close to the optimal one, \(Hf_{opt}\). This idea was investigated in a general framework in \(\ref{framework}\). The main result of \(\ref{framework}\) roughly says that the above scheme indeed works if we start not too far away from \(\mathcal{H}_N\) and avoid the thin set of bad points related to \(\mathcal{R}_{N,k}\) and \(\mathcal{H}_N\), (which in practice does not seem to be an issue). As an example we studied the case of projections between rank \(k\) matrices and Hankel matrices in non-weighted spaces. In this paper we make a more thorough study of this particular application and extend it to weighted spaces. Moreover, we discuss how to use the weighted spaces for approximating functions by sums of Gaussians, we discuss how to construct fast implementations of this idea, and finally we will also prove that the framework of \(\ref{framework}\) indeed applies.

We now state the main result of \(\ref{framework}\) in the current framework. Let \(P_{\mathcal{R}_{N,k}}, P_{\mathcal{H}}\) and \(P_{\mathcal{H}_k}\) denote the maps taking a given matrix \(B\) onto the closest point in the respective manifolds. Already here we hit some technical issues. We clearly have a formula for \(P_{\mathcal{H}_k}\) since \(\mathcal{H}_N\) is linear, so \(P_{\mathcal{H}_k}\) is an orthogonal projection and an explicit formula is given by \(\ref{formula}\). Concerning \(P_{\mathcal{R}_{N,k}}\) we do have a formula for computing it, but the drawback is that if \(B\) has singular values of higher multiplicity, then the map is not well defined. This is a common feature in algorithmic frameworks, and can be dealt with by introducing point-to-set maps, following \(\ref{approach}\). However, this seems over-ambitious in the current
framework, since matrices with singular values of multiplicity $> 1$ constitute a thin set (Theorem 6), and arbitrary small (numerical) perturbation yields distinct singular values. Moreover, in [2] we prove that $P_{H_N}$ is well defined near “regular non-tangential” points of $H_N$, and we will prove in Appendix 10 that the complement of such points is thin as well. With this in mind, we will from now on treat $P_{\mathcal{R}_{N,k}}$ and $P_{H_N}$ as well defined maps. Note that there is no simple way of computing $P_{H_N}$. We will prove in Appendix 10 that the theory developed in [2] applies in the present setting. Combined with this, the Theorem 6.1 of [2] reads;

**Theorem 7** For all $A \in H_N$ outside a thin subset, the following is true. Given any $\epsilon > 0$, there exists an $s > 0$ such that, for all $Hf$ with $\|Hf - A\| \leq s$, the sequence of alternating projections given by $B_0 = Hf$ and

$$B_{j+1} = \begin{cases} 
P_{\mathcal{R}_{N,k}}(B_j) & j \text{ is even} \\
P_H(B_j) & j \text{ is odd}
\end{cases} \quad (17)$$

(i) converges to a point $Hf_{\infty} \in H_{N,k}^n$

(ii) $\|Hf_{\infty} - Hf_{opt}\| \leq \epsilon \|Hf - Hf_{opt}\|$

A few remarks: (i) combined with Theorem 6 says that we will achieve an approximation of $f$ of the form $f_{\infty}(l) = \sum_{j=1}^{k} c_j f_j^l$. Moreover, note that if we had 0 on the right hand side of (ii), then $f_{\infty} = f_{opt}$. (ii) says that the error $\|f_{\infty} - f_{opt}\|_2$ can be made arbitrarily small relative to the distance $\|f - f_{opt}\|_2$. Finally, the full theorem in [2] has a third post, but to define this we need to discuss angles between manifolds, which we like to avoid. Basically, the third post says that there exists a number $0 < c < 1$, whose lower bound is related to the angle between $\mathcal{R}_{N,k}$ and $H_N$ at $A$, such that

$$(iii) \|Hf_{\infty} - B_j\| < c \|Hf - Hf_{opt}\|.$$ 

For practical purposes, this is an important observation, since it says that the algorithm has so called $c$-linear convergence.

Let us now briefly discuss what happens if we are not close enough to $H_N$ for the above theorem to apply. First of all, we have never encountered a situation where the algorithm does not converge. Secondly, it is easy to see that both $P_{H_N}$ and $P_{\mathcal{R}_{N,k}}$ are contractions, so $(B_j)_{j=0}^{\infty}$ is a bounded sequence. It thus has a convergent subsequence by basic properties of compact sets. Moreover, it is easy to see that the distance $\|B_{l+1} - B_l\|$ is strictly decreasing with $l$, and hence the limit point of the convergent subsequence is in $H_N$. (However, there is of course no indication that the corresponding $f_{\infty}$ is at all close to $f_{opt}$, so this observation has limit value.) In literature treating similar topics as in this article, one is usually content with concluding that the algorithm in question has the property that it generates a sequence with a convergent subsequence having a limit point in the desired set, and attributes this to Zangwill’s theorem, [31]. Clearly, Theorem 7 provides much more information in our setting; every point in $H_N$, outside some thin subset, has a neighborhood such that, if any $B_l$ enters that neighborhood, the sequence $(B_j)_{j=0}^{\infty}$ will converge. Since the sequence necessarily has more than one accumulation point if it does not converge, the only possibility for divergence is that $(B_j)_{j=0}^{\infty}$ wanders back and forth along the valleys of the thin pathological set, between the hills constituting the open set formed by all nice neighborhoods mentioned above. This seems highly unlikely, but we leave it as an open question to rule out this possibility. Clearly, it would be interesting to have some concrete values of the parameters $\epsilon$ and $s$ in Theorem 7. We will return to this issue in what follows.

Below is an algorithm that specifically describes how to apply the alternating projection scheme in our case.

**Algorithm 1**

1. Let $f_0 = f$, $l = 0$

2. (Application of $P_{\mathcal{R}_{N,k}}$) Compute the first $l$ con-eigenvalues $s_m$ and the con-eigenvectors $u_m$ of $Hf_l$ using Theorem 2. The projection $P_{\mathcal{R}_{N,k}} Hf_l$ is then given

$$\sum_{m=0}^{k} s_m u_m^* u_m^* \quad (18)$$
3. (Application of $P_{H,z}$) Compute

$$f_{l+1} = H^* \left( \sum_{m=1}^{k} s_m u_m u_m^* \right)$$

4. Increase $l$ and repeat from (2).

5. The root–MUSIC and ESPRIT methods

We briefly recapitulate the two most widely used methods for “high accuracy” frequency estimation. Our description will follow the implementation given in \cite{27}.

In a previous section we noted that we can find the nodes for a function $f$ of the form \eqref{15}, by considering the null space of a Hankel matrix that is generated from $f$. Recall that it was sufficient to consider a submatrix of size $(k+1) \times (k+1)$ to accomplish this. The nodes can in principle be found by finding the roots of the central polynomial, which is the polynomial generated by a the vector generating $\ker H_{(k+1)}$. However, just as discussed previously, this would lead to numerical instabilities, even when $f$ is a pure a sum of $k$ exponentials. From Theorem 5 it is easily seen that we can find the nodes by considering a singular value decomposition of a rectangular Hankel matrix, also generated from $f$. Let $H^* f \in \mathbb{C}^{N-M,M}$, with $M > k$, be such a Hankel matrix. and suppose that \eqref{15} holds. Then the nodes can in principle be found by finding the roots of any polynomial generated by a $u \in \ker H^* f$. Such a $u$ is in the kernel of

$$(H^* f)^*(H^* f)(j,k) = Rf(j,k)$$

where $0 \leq j,k < M$. The matrix $Rf$ is sometimes referred to as the sample covariance matrix. It may seem to be beneficial to work with $Rf$ instead of with the full Hankel matrices, since it is in principle possible to choose $M$ much smaller than $N$ It appears tractable that we make eigenvalue decomposition on a smaller matrix, and that the root finding step is also done with smaller matrices. The standard implementations of root-MUSIC and ESPRIT in \cite{27} work on for instance on $Rf$ rather than $Hf$. However, just as discussed previously, this too small $M$ can lead to numerical instabilities, even when $f$ is purely a sum of $k$ exponentials. Moreover, the matrix $Rf$ needs to be computed. It is not hard to see that this can be achieved in $O(N \log(N+M^2))$ time by splitting $H^*$ into two parts and employing FFT. For large $M$ this is not particularly advantageous. Another drawback is the loss of precision when forming $(H^* f)^*(H^* f)$.

The discussion so far has been conducted under the assumption that \eqref{15} is valid. In the typical situation this is not quite true; the standard assumption is that $f$ contains additive noise as well. Alternatively, we could be interested in the compression problem of representing a function using only frequencies and coefficients, in which the additive part has more structure than white noise.

Let

$$H_r f = V \Sigma U^*.$$ 

We will as before denote the columns of $U$ by $u_1, \ldots, u_M$. In the noiseless case, we did see that we had a great deal of flexibility, as any $u_m, k < m \leq M$ could be selected to find the nodes. The root-MUSIC method exploits this property, and tries to use all of the vectors $u_m, k < m \leq M$ to reduce the influence of noise. In the root–MUSIC method, roots are found by solving

$$P_{\text{MUSIC}}(z) = \sum_{m=k+1}^{M} P_{u_m}(z) \overline{P_{u_m}(z)} = 0,$$

where $^*$ reverses the order of the elements in a vector. Loosely speaking, this choice is motivated by the facts that the roots will appear in pairs when $f$ is a linear combination of purely oscillatory exponentials. There will be $2M - 2$ roots to $P_{\text{MUSIC}}(z) = 0$. The pairs associate with the true nodes, will have $\text{Re}(\zeta) \approx 0$, with one slightly larger than zero and one slightly smaller. For a more detailed justification on the choice of $P_{\text{MUSIC}}$, cf. \cite{27}.

In the general case, where there is no constraint on the nodes $\zeta_p$, $k$ roots need to be selected out of the $2M - 2$ that are given from $P_{\text{MUSIC}}(z) = 0$. In the simulations performed in the later sections, we have used the MUSIC code.
provided in [27], and added a selection step where we approximate \( f \) using all \( 2M - 2 \) nodes using a least squares approach, and then selecting the \( k \) nodes with largest coefficients. It appears unnecessary to compute nodes that have to be neglected.

The ESPRIT method avoids the step of computing unnecessary nodes. Instead, a similar approach as in Section 3 is used. For the noiseless case, it is readily verified that the eigenvalues of

\[
(U^*_M U_M)^{-1} (U^*_1 U_1)
\]

will coincide with the eigenvalues of \( e^{\zeta_m}, m = 1, \ldots, k \). In the ESPRIT method the eigenvalues of the expression above are used to compute nodes also in the case where noise is present, cf. [23]

We end this by section by a few remarks about the connection to autocorrelation and Toeplitz matrices. For a function of the form (15) where the exponentials are purely harmonic (zero real part of \( \zeta_p \)), it holds that

\[
\lim_{N \to \infty} \frac{1}{N} R f
\]

is the self-adjoint Toeplitz matrix generated by the autocorrelation of \( f \) (where \( f \) is the \( 4N + 1 \)-point sampling of a fixed function on a fixed interval). According to a Theorem by Carathéodory [28], if the self-adjoint Toeplitz matrix generated by a function has rank \( k \), then that function can be expressed as a sum of \( k \) purely oscillatory exponentials. This motivates alternating projection schemes between the manifolds of Toeplitz matrices and low rank matrices, for the approximation of the autocorrelation of a function. However, the effect of a finite sample length cannot be neglected, and the Toeplitz matrix generated from the autocorrelation of a pure sum of \( k \) oscillatory exponentials, will fail to have rank \( k \). For that sake, the approach we have chosen seems preferable.

## 6 Fast algorithms

There are two operations for which we will need fast numerical methods in the alternating projections approach for frequency detection; (low rank) Takagi decomposition, and the application of the averaging operator (18). It turns out that both operations can be implemented in a fast manner, but the first one will require some more effort than the second.

**Proposition 5** The application of a Hankel matrix to a vector can be done in \( \mathcal{O}(N \log(N)) \) time by means of FFT.

**Proof:** This is a standard result [11], and makes use of the fact that circular matrices are diagonalized by the discrete Fourier transform, and that it easy to construct a circular matrix from a Hankel matrix by permutation and periodic extension. The \( \mathcal{O}(N \log(N)) \) time complexity can then be achieved by employing FFT.

**Proposition 6** The weighted averaging operator \( H^* \) in (18) can be applied to a rank 1 matrix in \( \mathcal{O}(N \log(N)) \) time.

**Proof:** By definition

\[
H^*(uu^T)(l) = \frac{1}{\omega(l)} \sum_{j+k=l} w(j)u(j)u(k)w(k),
\]

\[
= \frac{1}{\omega(l)} \sum_{j+k=l} v(j)v(k), \quad 2 \leq l \leq 2N,
\]

where \( v = uu \). It is easy to see that the sum above can we written as a discrete convolution \( (v(j)v(l-j)) \) using zero padding to avoid boundary effects. The discrete convolutions can then be computed in \( \mathcal{O}(N \log(N)) \) time using FFT.
6.1 Lanczos method for complex symmetric matrices

We will use a modified Lanczos method for finding the $k$ first con-eigenvalues/con-eigenvectors. The Lanczos method is a way to perform a unitary transformation of a Hermitian matrix to tridiagonal form, i.e., given $A = A^*$, compute $T = Q^*AQ$, where $Q$ is unitary and $T = T^*$ is tridiagonal. We need a similar decomposition for complex symmetric matrices. The usage of a modified Lanczos method has been addressed in [16, 29, 26]. As we only need to compute the first $k$ con-eigenvalues/con-eigenvectors, we develop a method customized to that purpose.

The basic step in the Lanczos method is simple. However, it is notorious for the loss of precision, sometimes in a counterintuitive way. This issue must be addressed carefully. The columns in the unitary matrix $Q$ are orthonormal. If, at some step before the last one, $\beta_j = 0$, then the subspace $X$ has the property $q \in X \Rightarrow \overline{AQ} \in X$. We will make use of ingredients from both these methods in our particular setup.

For a given symmetric matrix $A \in \mathbb{M}_{N,N}$, we look for a unitary matrix $Q$, complex numbers $\alpha_1, \alpha_2, \ldots$ and nonnegative real numbers $\beta_1, \beta_2, \ldots$, such that $T = \overline{Q}AQ^*$, where

$$T = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 & 0 \\ 0 & \beta_2 & \alpha_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{N-1} & \beta_{N-1} \\ 0 & 0 & 0 & \cdots & \beta_{N-1} & \alpha_N \end{pmatrix}$$

with $\beta_j > 0$. The matrices $Q$ and $T$ can be constructed as follows: We want to achieve $AQ = \overline{QT}$, which means that

$$Aq_1 = \alpha_1q_1 + \beta_1q_2$$

and, for $2 \leq j \leq m - 1$,

$$Aq_j = \beta_{j-1}q_{j-1} + \alpha_jq_j + \beta_jq_{j+1}$$

and finally

$$Aq_N = \beta_{N-1}q_{N-1} + \alpha_Nq_N,$$

where $q_1, q_2, \ldots, q_N$ are the columns of $Q$. We choose a unit vector $q_1$ and define

$$\alpha_1 = (Aq_1, q_1^T) = q_1^TAq_1, \quad \beta_1 = \|Aq_1 - \alpha_1q_1^T\|, \quad q_2 = \frac{1}{\beta_1}(Aq_1 - \alpha_1q_1)$$

if $\beta_1 \neq 0$, and then, recursively

$$\alpha_j = (Aq_j, q_j^T), \quad \beta_j = \|Aq_1 - \beta_{j-1}q_{j-1} - \alpha_jq_j\|, \quad q_{j+1} = \frac{1}{\beta_j}(Aq_j - \beta_{j-1}q_{j-1} - \alpha_jq_j)$$

as long as $\beta_j \neq 0$. One readily verifies, by induction, that the vectors $q_j$ are orthonormal. If, at some step before the last one, $\beta_m = 0$, then the subspace $X = \text{span}(q_1, q_2, \ldots, q_m)$ of $\mathbb{C}^N$ has the property

$$q \in X \Rightarrow \overline{AQ} \in X.$$
decomposition, and the vanishing of $\beta_m$ at some step generically implies that we do not need to proceed further. Now let $Q_m \in \mathbb{R}^{N \times m}$ consist of the first $m$ columns of $Q$ and let $T_m$ be the upper left corner $m \times m$-submatrix of $T$. Write $Q_m = (q_1, \ldots , q_m)$ and let $T_m$ denote the $m \times m$ upper left corner submatrix of $T$. By standard arguments one sees that the con-eigenvectors of $A_m$ converge to those of $A$, and moreover that for con-eigenvalues with a low subindex, this convergence is obtained (within certain precision) with high probability (depending on $x_0$) for $m << N$.

The immediate application of the modified Lanczos-method outlined above is that we can compute con-eigenvector and con-eigenvalues for $T$ instead of for $A$ (cf. discussion about con-similarity [12, p.244, p251]), which due to the tridiagonal structure of $T$ it is beneficial. Moreover, since in our setting we are only interested in the first $k$ con-eigenvalues and con-eigenvectors, we briefly provide the details.

Since we are working with con-eigenvalues and con-eigenvectors instead eigenvalues and eigenvectors, we briefly state Lemma 1.

**Lemma 1** Let $A = A^T$ be given and let $T_m$ be as above. Denote the con-eigenvectors of $T_m$ by $v_j$ and the corresponding con-eigenvalues by $\beta_j$, $1 \leq j \leq m$. Then, for each $j$, there is a con-eigenvalue $\lambda_j$ of $A$ such that

$$|\beta_j - \lambda_j| \leq \beta_m |u_j(m)|.$$  

**Proof:** Set $e_m = (0, \ldots , 1) \in \mathbb{R}^m$ and note that $AQ_m - \bar{Q}_m T_m = \beta_m q_{m+1}^T e_m$ by (22). We apply this to $u_j$ to get

$$\|AQ_m u_j - \bar{Q}_m T_m u_j\| = \|AQ_m u_j - \beta_j Q_u_j\| = \|\beta_m q_{m+1}^T u_j\| = \beta_m |u_j(m)|,$$

and introduce $w_j = Q_m u_j$. Since $\|w_j\| = 1$, it follows that $\|w_j\| = 1$. Denote the con-eigenvectors of $A$ by $v_l$, and represent $w_j = \sum_l s_l v_l$, $\sum_l |s_l|^2 = 1$. We then have

$$\|AQ_m u_j - \bar{Q}_m T_m u_j\|^2 = \|Aw_j - \mu_{\beta_j} \| \|w_j\|^2 = \|Aw_j - \mu_{\beta_j} \|_2^2 = \| Aw_j - \mu_{\beta_j} w_j \|_2^2 \geq \min_l |\lambda_l - \mu_{\beta_j}|^2 \sum_l |s_l|^2 = \min_l |\lambda_l - \mu_{\beta_j}|^2.$$

This a well known result for the case of Hermitian symmetry, see for instance [20, p. 69]. A similar result is given in [26, Proposition 2.2].

Lemma 1 provides a way to control the convergence of con-eigenvectors. When the quantities in (23) are small, then $w_j$ will be a good approximation of the con-eigenvector to $A$ that is associated with $\lambda_j$. In many cases, convergence for the first con-eigenvalues are reached for comparatively small $m$. In particular, for the case where the (con)spectrum of $A$ has a large gap after, say $k$ terms, it is typically only necessary to use $m$ slightly larger than $k$. This will be the case for all but the first step in our alternating projection algorithm.

As mentioned before, in a straightforward Lanczos implementation the orthogonality of $Q$ will quickly be lost due to finite precision arithmetics. Moreover, and somewhat counterintuitively, the loss of orthogonality will grow as the con-eigenvectors converge, cf. [19]. The simple remedy to this problem is to reorthogonalize $q_{m+1}$ to all previous $q_j$ at each iteration. However, this increases the algorithmic complexity of the method. Instead, we want to have a criterion on when reorthogonalization is needed. The loss of orthogonality is also indicative of con-eigenvalue convergence.

Two suggestions on reorthogonalization criteria are given in [25,19]. We will follow the approach given in [25]. Since we are working with con-eigenvalues and con-eigenvectors instead eigenvalues and eigenvectors, we briefly provide the details.

Due to finite precision arithmetics, we model (22) as

$$\beta_m \bar{Q}_m q_{m+1} = AQ_m - \bar{Q}_m Q_m - \bar{Q}_{m-1} \beta_{m-1} + \epsilon_m,$$

where $\epsilon_m$ describes the error introduced by the finite precision. We now let $q_j$ denote the vectors computed from the relation (22). Due to the errors $\epsilon_m$, these vectors will not be orthogonal. Let $\omega_j = q_j^* q_k$. Then $\omega_j$ will satisfy the recursion relation

$$\omega_{m+1,m} = 1, \quad \omega_{m+1,m} = q_{m+1}^* q_m = \psi_{m+1},$$

$$\omega_{m+1,j} = \frac{1}{\beta_m} \left( \alpha_{m,j} \omega_{m,j} + \beta_{j-1} \omega_{m,j-1} + \beta_{j} \omega_{m,j+1} - \alpha_m \omega_{m,j} - \beta_{m-1} \omega_{m-1,j} \right) + \theta_{m,j}.$$

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where \( \vartheta_{m,j} = \beta_m^{-1}(q_j^T \epsilon_m - q_m^T \epsilon_j) \). The last equality follows from multiplying \( (24) \) by \( q_j^T \), and subtracting the same quantity with the indices \( j \) and \( m \) interchanged. Since \( A = A^T \) the quantity \( q_j^T A q_m \), then cancels.

Using the recursion formula above, we can monitor the level of lost orthogonality without explicitly having to compute inner products of the columns of \( Q \). In analogy with the empirical results in \( [25, 19] \), we simulate the error quantities as

\[
\vartheta_{m,j} \in N \left( 0, 0.3 \varepsilon (\beta_m + \beta_j) \right),
\]

\[
\psi_{m+1} \in N \left( 0, 0.6 \varepsilon (2N + 1) \frac{\beta_1}{\beta_m} \right),
\]

where \( N(0, \sigma) \) denotes the complex normal distribution with standard deviation \( \sigma \), zero mean and independent real and imaginary parts. Above, \( \varepsilon \) denotes the machine precision.

The maximum loss of precision that can be tolerated without loss of precision in the coefficients \( \alpha_{m+1} \) and \( \beta_m \) is \( \sqrt{\varepsilon} \). Once some \( \omega_{m+1,j} \) exceeds that level, it is necessary to reorthogonalize. As seen from \( (25) \), each \( \omega_{m+1,j} \) is strongly influenced by its neighbors. Hence, it will not be efficient to only reorthogonalize against the vectors \( q_j \) where \( \omega_{m+1,j} \), since for isolated \( j \)'s the orthogonalization would immediately get lost in the next iteration. Instead, it is beneficial to reorthogonalize against a batch of \( q_j \)'s. Hence (and in accordance with \( [25] \)) we reorthogonalize against the set of \( q_j \) which have \( |\omega_{m+1,j}| > \varepsilon^{3/4} \) once \( |\omega_{m+1,j}| > \sqrt{\varepsilon} \) for some \( j \).

After a reorthogonalization has taken place, we need to reset the quantities \( \omega_{m+1,j} \). Again following \( [25] \), we choose \( \omega_{m+1,j} \in N(0, 1.5 \varepsilon) \).

The final ingredient is a rule for when to utilize Lemma\( 1 \) for convergence monitoring of con-eigenvalues. Clearly \( m \geq k \) in order to find \( k \) con-eigenvalues that have converged. Since the convergence of con-eigenvalues and the loss of orthogonality are coupled, we compute a Takagi factorization of \( T_m \), once loss of orthogonality is indicated by \( |\omega_{m+1,j}| > \sqrt{\varepsilon} \) for some \( j \), given that \( m \leq k \). Moreover, we can monitor the behavior of \( \beta_m \) to check for convergence. If \( \beta_m \) becomes very small for some \( m \), then it means that \( Q_m \) defines an almost invariant (con)subspace under \( A \), which implies convergence of the (non-zero) con-eigenvalues. We let \( \varepsilon_L \) denote the desired resolution of con-eigenvalues, and impose the convergence criterion

\[
\frac{\beta_m}{\beta_1} < \varepsilon_L.
\]

As always with numerical implementations, it can be difficult to determine how small \( \beta_m \) has to be in order to consider it to have almost vanished, i.e., if \( \varepsilon_L \) is chosen very small. A typical feature of this case is that the last value \( \beta \) jumps dramatically in size. This behavior also serves as a good criterion for when to check for convergence by means of Lemma\( 1 \).

In the procedure above, we need to compute the Takagi factorization of \( T_m \). The cost of that step when using Proposition\( 1 \) is \( O(m^3) \). However, due to the tridiagonal structure there are methods to compute this in \( O(m^2) \) time, cf. \( [16, 31, 29] \). These methods are based on straightforward modifications of methods for eigenvalue decomposition of tridiagonal Hermitian matrices.

The most expensive step in the Lanczos procedure described above is the matrix vector multiplication \( A q_m \) in \( (22) \). However, this step can be computed in \( O(N \log N) \) time by Proposition\( 5 \).

**Proposition 7** The time complexity for computing the first \( k \) con-eigenvectors and con-eigenvalues of a Hankel matrix to accuracy \( \varepsilon \) using the modified Lanczos method described above is \( O(mN \log N + m^2) \), where \( m \) denotes the total number Lanczos steps, and where \( m \geq k \), but where \( m \) is typically of the same order as \( k \).
7 Numerical simulations

7.1 Performance analysis

In this section we compare the performance of our approach against the ESPRIT and root-MUSIC methods. We simulate functions of the form

\[ f_0(t) = \sum_{p=1}^{k} c_pe^{\zeta_p t}, \]

and where \( n \) is a noise component. The coefficients \( c_p \) are chosen as complex normal distributed variables, and the nodes as \( \zeta_p = 1/(4N + 1)(50Z_{\bar{r}}^p + iZ_{\bar{i}}^p) \), where \( Z_{\bar{r}}^p \) and \( Z_{\bar{i}}^p \) are normally distributed.

The noise component is constructed by letting \( \tilde{n}(k) = n_r(k) + in_i(k) \), where \( n_r \) and \( n_i \) are normally distributed noise, and where

\[ n = \sqrt{\frac{\|f_0\|^2}{\|\tilde{n}\|^2} 10^{-\text{SNR}/10} \tilde{n}}, \]

for some signal to noise parameter \( \text{SNR} \). By this construction, the signal to noise ratio will be exactly equal to the parameter \( \text{SNR} \) when measured in dB. Throughout the tests, we have chosen to work with a signal length of 511, i.e. \( N = 256 \). This is chosen to make the FFT routines run fast. All simulations have been run in a MATLAB environment, without any compiled optimizations. For the ESPRIT and root-MUSIC, we have used the routines provided in [27], with minor modifications to make them work for the case \( \text{Re}(\zeta_p) \neq 0 \). The accuracy parameter \( \epsilon \) used in the alternating projection method has been chosen to be a factor 100 lower than the noise magnitude.

In figure 1, we show some simulation results for the different methods. We conduct a small number of simulations for \( \text{SNR} = 10 \text{ dB} \) and \( k = 10 \), and consider the performance in terms of the errors generated by the different methods. We display the errors are displayed in two ways; in relation to the pure signal \( f \) and in relation to the noise one \( f_0 \).

Figure 1: The solid curves show the error \( \|g - f\|/\|f\| \) with \( g \) obtained with MUSIC (green), ESPRIT (red) and our propose method (blue), respectively, for \( \text{SNR}=10 \). The dashed ones show the counterparts for \( \|g - f_0\|/\|f_0\| \).

We see that our proposed method systematically has a smaller error in both ways of measurement. We also note that for all methods we have a substantially smaller error when compared to the pure signal \( f_0 \) instead of the noisy one. Hence, all three methods successfully filter out a large part of the noise. It is also notable how close the error in relation to \( f \) is to the signal to noise ratio for our proposed method. This is also implied by Theorem[7]. Basically, in the notation of Section[4] we have \( g = g_0 \), and it is reasonable to assume that \( f_0 \approx g_{opt} \). This is because the noise...
has a high probability of being orthogonal to \( f_0 \), and that \( P_{M^n \cap H} \) locally acts as an orthogonal projection, (which is further elaborated on in \([2]\)). Thus, Figure 1 can be interpreted as the upper blue line shows \( \| f - g_{\text{opt}} \| / \| f \| \), whereas the lower blue line gives an indication of the size of \( \| g_{\infty} - g_{\text{opt}} \| / \| f \| \). In terms of Theorem 7 with \( A = H f_0 \) of norm 1 and \( s = 0.1 \), this means that we can pick \( \epsilon \) around 0.1 as well. Although the above images are constructed using standard \( \ell^2 \)-norm, not the weighted one required for Theorem 7 to kick in, it is interesting to observe that this is in line with the observations in \([2]\). There, using more carefully conducted examples to test Theorem 7, it seems that one can take \( s \approx \epsilon \) when working with \( k \approx 10 \).

It is interesting to see how these result depend on the different parameters, i.e., the number of nodes \( k \) and the noise level \( SNR \). In Figure 2 we have conducted more thorough investigations. For each \( k = 1, 2, \ldots, 30 \) we have done 100 simulations and computed average results. The averaging has been made in dB, in order to limit the effect of outlier results. As for Figure 1 we display errors in two ways, using solid lines for errors in comparison to the noise signal \( f \) and dashed lines for the comparison to the original one, \( f_0 \). We also display some of the impact that the choice of size \( (M) \) of \( R \) in (19) has. The thin lines in red and green show errors for \( M = 4k \) and the thick lines show the counterpart for \( M = N = 256 \).

There are a few interesting conclusions that can be drawn from the results depicted in Figure 2. First, we note that for the cases were \( k \) is small, all three methods perform comparably well, given that the size \( (M) \) of the sample covariance matrix used in MUSIC and ESPRIT is sufficiently large. However, as \( k \) increases, the alternating projection method starts to outperform the other two. We can again note that the errors (compared to \( f \)) produced by the alternating projection method almost coincide with the signal to noise ratio. Moreover, in terms of Theorem 7, Figure 2 seems to indicate that \( \epsilon \approx s \) is a good rule of thumb, although the ratio gets slightly worse as the complexity of the manifold \( M^n \cap H \) increases with increasing \( k \).

From the results we have seen so far we can conclude that the alternating projection method should be the method of choice unless \( k \) is very small, given that the prime concern is to minimize the estimation errors. The other criterion for method selection is speed. The computational times for the different methods is displayed in Figure 4. As mentioned before, the MUSIC and ESPRIT algorithms that are used are slightly modified versions of the ones given in \([27]\). In Figure 3 the fast alternating projection method is the fastest. The MUSIC and ESPRIT algorithms are substantially slower for high \( M \). For the MUSIC algorithm, the most time consuming step is the root solving step. We note, however, that by using our fast method for finding the first \( k \) con-eigenvector / con-eigenvalues, we can construct a method that would have much resemblance with the ESPRIT method as described in \([23]\). It seems advantageous...
Figure 3: Average execution time for the different methods in milliseconds. The line notation as in Figure 2 is used.

to work directly with Hankel matrices rather than the covariance sample matrix of (19). Using such an approach, we would be able to construct a computational method that would provide similar results as the ESPRIT algorithm described in [27], but substantially faster than the one based on the sample covariance matrix. From the results in Figure 2 we can conclude that the results would not be as good as the ones obtained by the fast alternating projection method proposed here. However, it would be faster, as it would only involve one decomposition step. In other words, it would be equivalent to using the alternating projection scheme with only one iteration.

A natural question would then be how much faster such “fast” ESPRIT algorithm would be. A first guess would be that the speed ratio would be proportional to the number of alternating projections performed before the target accuracy $\epsilon$ is reached. It turns out that the fast alternating projection method is faster than that. The reason for this is that fewer Lanczos iterations are required in each alternating projection iteration. In Figure 4 we display the ratio between the total time and the time for the first iteration for the alternating projection method. We see that the ratio typically lies around 2. This means that the fast alternating projection method would only be about twice as expensive as a fast implementation of ESPRIT, while providing smaller errors. Again, we note that the proposed fast alternating

Figure 4: The ratio between the time for total number of iterations compared to the first one for the alternating projection method for 1000 simulations.
projection method is substantially faster than the standard implementation of ESPRIT and MUSIC.

7.2 Approximations with Gaussians

As a final example, we show some results concerning the approximation of functions with Gaussians with fixed half-width, using the fast alternating projection method with Gaussian weights. There are two possible interesting cases. The first one concerns the case where the functions are of the form

$$\sum_p c_p e^{-\alpha(x-x_p)^2+i\xi x},$$

with fixed (and known) constant $\alpha$. The second case concerns the approximation of functions using a Gaussian window, for example as done in time–frequency analysis. Using a non-linear approach may be beneficial compared to short-time Fourier transform representations with overlapping windows. However, we will in this section only show some results concerning (26).

In Figure 5 we show the result from one simulation using a function of the form (26), using 10 Gaussians. In order to approximate this function using exponentials, we choose the weights $w$ such that $\sqrt{\omega_l}$ approximates $e^{-\alpha l/2N}$, $l = -2N, \ldots, 2N$. To this end, we choose

$$w_k = \sqrt{\frac{2}{4N+1}} \sqrt{\frac{8\alpha}{\pi}} e^{-4\alpha k/4N}, \quad k = -N, \ldots, N.$$  

For sufficiently narrow Gaussians (large $\alpha$), we will then have that $\sqrt{\omega_l} \approx e^{-\alpha l/2N}$. Just as before we let $f_0$ be of the form (26) and use additive noise to obtain $f$. One simulation is shown in Figure 5 for $SNR = 10$. Before we start the alternating projection scheme, we divide $f$ pointwise with $1/\sqrt{\omega_l}$. This will boost the amplitude at the endpoints of $f$.

Figure 5: In the top left panel the noisy signal $f$ is shown, the top right shows the original $f_0$, the bottom left shows the reconstruction, and the bottom right shows errors. The errors are shown unscaled in gray, and scaled with respect to $\sqrt{\omega}$ in black.
substantially, but since we approximate using $\omega_l$ as a weight, we will obtain a uniform approximation. The noise will, however, not be uniform with this approach, but larger at the end-points.

The result from one simulation is shown in Figure 5. The noise signal is depicted in the top left panel, while the original is displayed in the top right panel. In the bottom left we see the obtained reconstruction. We can see that most of the features from the original signal is captured. In the bottom right panel we show pointwise errors; in black the error weighted with $\omega_l$ and in grey the unweighted pointwise error.

8 Conclusions

We have developed a method for the fast estimation of complex frequencies using an alternating projection scheme between Hankel matrices and rank $k$ Takagi representations. The method has a time complexity of $O(nN \log N + n^3)$. FFT routines are used both to get fast matrix–vector multiplications, and to project rank $k$ representations to Hankel matrices. In order to compute the first $k$ Takagi vectors, we employ a modified Lanczos scheme for self-adjoint matrices. The number of necessary alternating projection steps depends on an accuracy parameter, but in typical situations the total time is only twice as large as the time needed for the first iteration. The reason for this is that fewer Lanczos steps are needed when the matrix we obtain is closer to being both Hankel, and rank $k$.

In our simulations we see that the proposed method performs better both with regards to speed and approximation accuracy, compared to standard implementations like root-MUSIC and ESPRIT. We also verify that the errors that we obtain behave in the manner theoretically predicted in [2]. The method works also for some weighted representations. A particular case of weights that can be used are Gaussian weights, for which case some numerical examples are provided.

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10 Appendix; the set of tangential points in $\mathcal{H}_k$ is thin

Following the terminology of [2], a point $A \in \mathcal{H}_k$ is called regular if the dimension of $\mathcal{R}_k \cap \mathcal{H}_k$ and $\mathcal{H}$ are constant in a neighborhood of $A$. Thus theorem 6 says that the set of non-regular points is thin. Moreover, recall that a point $A \in \mathcal{H}_k$ is called non-tangential if

$$T_{\mathcal{R}_k}(A) \cap T_{\mathcal{H}}(A) = T_{\mathcal{R}_k \cap \mathcal{H}}.$$  \hspace{1cm} (27)

In order to prove Theorem 7 we need to show that the set of tangential points in $\mathcal{H}_k$ is thin, and then apply Theorem 6.1 of [2].

Theorem 8 The set of tangential points is thin in $\mathcal{H}_k$.

Proof: By Theorem 6 we immediately get that all points in $\mathcal{H}_k^O$ are regular and that $\mathcal{H} \setminus \mathcal{H}_k^O$ is thin. To verify that $A \in \mathcal{H}_k^O$ is non-tangential, it thus suffices to establish (27), e.g. that $T_{\mathcal{H}}(A) \cap T_{\mathcal{R}_k^O}(A) = T_{\mathcal{H}_k^O}(A)$, since $\mathcal{H}_k^O \subset \mathcal{R}_k^O$. Clearly

$$T_{\mathcal{H}}(A) \cap T_{\mathcal{R}_k^O}(A) \ni T_{\mathcal{H}_k^O}(A).$$  \hspace{1cm} (28)

By Theorem 6 and the fact that $A$ is regular we have $\dim(T_{\mathcal{H}_k^O}(A)) = 4k$ and

$$\dim(T_{\mathcal{H}}(A) \cap T_{\mathcal{R}_k^O}(A)) = \dim(T_{\mathcal{H}}(A)) + \dim(T_{\mathcal{R}_k^O}(A)) - \dim(T_{\mathcal{H}}(A) + T_{\mathcal{R}_k^O}(A)) =$$

$$= 2(2n - 1) + 2(2kn - k^2) - \dim(T_{\mathcal{H}}(A) + T_{\mathcal{R}_k^O}(A)).$$

To establish the reverse inclusion to (28), it thus suffices to show that $\dim(T_{\mathcal{H}}(A) \cap T_{\mathcal{R}_k^O}(A)) \leq 4k$, or equivalently

$$\dim(T_{\mathcal{H}}(A) + T_{\mathcal{R}_k^O}(A)) \geq 2(2n - 1) + 2(2kn - k^2) - 4k.$$
Moreover, since both subspaces are closed under multiplication by \( \mathbb{C} \), it suffices to verify
\[
\dim_{\mathbb{C}}(T_H(A) + T_{R_k}(A)) \geq 2n - 1 + 2kn - k^2 - 2k,
\]
(29)
where \( \dim_{\mathbb{C}} \) denotes the dimension over \( \mathbb{C} \). To this end, note that the map \( \mathfrak{M} : (M_{n,k})^2 \to M_{n,n} \) given by
\[
\mathfrak{M}(U, V) = VU^* = \sum_{j=1}^{k} u_j u_j^*,
\]
where \( u_j, v_j \) denote the columns of \( U \) and \( V \) respectively, is an immersion onto \( R_k \). By this we mean that for each \( A \in R_k \) there exists \( U_A, V_A \) such that \( A = \mathfrak{M}(U_A, V_A) \) and, if \( A \in R_k \), then
\[
T_{R_k}(A) = \text{Ran } \partial \mathfrak{M},
\]
where \( \partial \mathfrak{M} \) denotes the derivative of \( \mathfrak{M} \). In this section we define \( \mathfrak{U} : \mathbb{C}^n \to M_{n,k} \) and \( \mathfrak{M} : \mathbb{C}^n \times \mathbb{C}^n \to M_{n,k} \) via
\[
\mathfrak{U}(\alpha) = \begin{pmatrix} 1 & \cdots & 1 \\
\alpha_1 & \cdots & \alpha_k \\
\alpha_1^2 & \cdots & \alpha_k^2 \\
& \ddots & \ddots \\
\alpha_1^n & \cdots & \alpha_k^n \end{pmatrix}, \quad \mathfrak{M}(c, \alpha) = \begin{pmatrix} c_1 & \cdots & c_k \\
c_1 \alpha_1 & \cdots & c_k \alpha_k \\
c_1 \alpha_1^2 & \cdots & c_k \alpha_k^2 \\
& \ddots & \ddots \\
c_1 \alpha_1^n & \cdots & c_k \alpha_k^n \end{pmatrix}.
\]
It is easily seen that, given any \( \alpha \in \mathbb{C} \), the matrix
\[
H(\alpha) = \begin{pmatrix}
1 & \alpha & \alpha^2 & \cdots & \alpha^{N-1} \\
\alpha & \alpha^2 & \cdots & \alpha^{N-1} & \alpha^N \\
\alpha^2 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
\alpha^{N-1} & \cdots & \cdots & \cdots & \alpha^{2N-3} \\
\alpha^N & \cdots & \cdots & \alpha^{2N-3} & \alpha^{2N-2}
\end{pmatrix}
\]
defines a rank 1 Hankel matrix. Thus
\[
\mathfrak{H}(c, \alpha) = \sum_{j=1}^{k} c_j H(\alpha_j)
\]
is a rank \( k \) Hankel matrix. It is clear that
\[
\mathfrak{H}(c, \alpha) = \mathfrak{M}(\mathfrak{U}(\alpha), \mathfrak{M}(c, \alpha)).
\]
Thus, whenever \( A = \mathfrak{H}(c, \alpha) \in \mathcal{H}_k^n \), we have
\[
T_{R_k}(A) = \text{Ran } \partial \mathfrak{M}(\mathfrak{U}(\alpha), \mathfrak{M}(c, \alpha)).
\]
(32)
Now, it is not hard to see that \( \partial \mathfrak{M}(\mathfrak{U}(\alpha), \mathfrak{M}(c, \alpha)) \) is a polynomial in the variables \( c \) and \( \alpha \). To visualize, say that \( n = 3 \) and \( k = 2 \). Then the right hand side is given as the span of the 12 matrices
\[
\begin{pmatrix}
1 & \alpha_j & \alpha_j^2 \\
0 & \alpha_j & \alpha_j^2 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \alpha_j & \alpha_j^2
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & \alpha_j & \alpha_j^2
\end{pmatrix}, \quad (j = 1, 2),
\]
and
\[
\begin{pmatrix}
c_j & 0 & 0 \\
c_j \alpha_j & 0 & 0 \\
c_j \alpha_j^2 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & c_j & 0 \\
0 & c_j & 0 \\
0 & c_j & \alpha_j 
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & c_j \\
0 & 0 & c_j \\
0 & 0 & \alpha_j 
\end{pmatrix}, \quad (j = 1, 2).
\]
Moreover, picking a basis for $\mathbb{M}_{n,n}$ (for example, the standard one which we order lexicographically), the right hand side of (32) can be identified with the range of a matrix with polynomial entries which we denote by $\partial\mathbb{M}(\mu(\alpha), \mathcal{V}(c, \alpha))$.

To continue the example, we get

$$\widetilde{\partial\mathbb{M}}(\cdots) = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & c_1 & 0 & 0 & c_2 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & c_1 \alpha_1 & 0 & 0 & c_2 \alpha_2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & c_1 \alpha_1^2 & 0 & 0 & c_2 \alpha_2^2 & 0 & 0 \\
\alpha_1 & 0 & 0 & \alpha_2 & 0 & 0 & c_1 & 0 & 0 & c_2 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 & 0 & c_1 \alpha_1 & 0 & 0 & c_2 \alpha_2 & 0 & 0 \\
0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 & c_1 \alpha_1^2 & 0 & 0 & c_2 \alpha_2^2 & 0 & 0 \\
\alpha_1^2 & 0 & 0 & \alpha_2^2 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & c_2 & 0 \\
0 & \alpha_1^2 & 0 & 0 & \alpha_2^2 & 0 & 0 & 0 & c_1 \alpha_1 & 0 & 0 & c_2 \alpha_2 & 0 \\
0 & 0 & \alpha_1^2 & 0 & 0 & \alpha_2^2 & 0 & 0 & c_1 \alpha_1^2 & 0 & 0 & c_2 \alpha_2^2 & 0
\end{pmatrix}$$  \hspace{1cm} (33)

With

$$E_j(m, l) = \begin{cases} 1, & \text{if } m + l = j; \\ 0, & \text{otherwise}. \end{cases}$$

$\tilde{\mathcal{H}}$ is spanned by $\tilde{E}_1, \ldots, \tilde{E}_{2n-1}$, where the notation is self-explanatory. In our example we get

$$\mathcal{H} = \text{Ran} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}$$  \hspace{1cm} (34)

Let us denote the matrix obtained by adjoining $\partial\mathbb{M}(\cdots)$ and $\tilde{\mathcal{H}}$ by $[\partial\mathbb{M} \tilde{\mathcal{H}}]$. To verify (29), it thus suffices to show that

$$\text{Rank } [\partial\mathbb{M} \tilde{\mathcal{H}}] \geq 2n - 1 + 2kn - k^2 - 2k,$$  \hspace{1cm} (35)

holds, evaluated at $(\mu(\alpha), \mathcal{V}(c, \alpha))$ for some $c, \alpha$ such that $\delta(c, \alpha) \in \mathcal{H}_k^{\alpha}$. Note that

(i) If we can establish (35) for one point $A = \delta(c, \alpha)$, then it easily follows that (35) holds at all but a thin set of points $A$. To see this, set $q = 2n - 1 + 2kn - k^2 - 2k$ and first note that we can pick a $q \times q$ submatrix of $[\partial\mathbb{M} \tilde{\mathcal{H}}]$ whose determinant is a non-zero polynomial. Thus, by standard algebraic geometry, the set of points $(c, \alpha)$ where the determinant is zero is thin in $\mathbb{C}^{2k}$. Finally, it is also clear that the image of a thin set under a chart, in this case $\delta$, is again thin.

(ii) Let $B = \mathbb{M}(U_B, V_B) \in \mathcal{H}_k$ be a point such that

$$\text{Rank } \partial\mathbb{M}(U_B, V_B) = 2kn - k^2,$$  \hspace{1cm} (36)

but where $(U_B, V_B)$ is not necessarily in the closure of the range of $(\mu, \mathcal{V})$. We claim that in order to establish (i), it suffices to establish (35) at the point $(U_B, V_B)$. To see this, first note that by (36), $\mathcal{R}_k$ is locally a manifold (of dimension $2kn - k^2$) around $B$, and we can take an affine subspace of $N \subset \mathbb{M}_{n,k}^2$ containing $(U_B, V_B)$ such that $\mathbb{M}|_N$ becomes a local chart for $\mathcal{R}_k$. If (35) holds for $(U_B, V_B)$, then arguing as above with determinants, it holds in a neighborhood of $(U_B, V_B)$. By Theorem 6, $\mathcal{H}_k^{\alpha}$ is dense in $\mathcal{H}_k$, so in particular we can pick a $C \in \mathcal{H}_k^{\alpha}$ and corresponding $U_C$, $V_C \in N$ and $c_C, \alpha_C \in \mathbb{C}^{n}$ such that $C = \mathbb{M}(U_C, V_C) = \delta(c_C, \alpha_C)$ and (35) is satisfied for $[\partial\mathbb{M}(U_C, V_C) \tilde{\mathcal{H}}]$. By (32) and (36) we have

$$\text{Ran } \partial\mathbb{M}(U_C, V_C) = T_{\mathcal{R}_k^{\alpha}}(C) = \text{Ran } \partial\mathbb{M}(\mu(\alpha_C), \mathcal{V}(c_C, \alpha_C)),$$

which shows that (35) is satisfied at $(\mu(\alpha_C), \mathcal{V}(c_C, \alpha_C))$, as desired.
So, it remains to verify (35) and (36) for some point $B = \mathcal{M}(U_B, V_B) \in \mathcal{H}_k$. In terms of our example, we pick

$$U_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad V_B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

so that $B$ becomes the rank 2 Hankel operator

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$

Then $\partial \mathcal{M}(U_B, V_B)$ is spanned by the 6 "V-derivatives"

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and the 6 "U-derivatives;

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is clearly an 8-dimensional space not including

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which happens to be $E_5$ in the basis for $\mathcal{H}$, and thus

$$\text{Rank } [\partial \mathcal{M}(U_B, V_B) \mathcal{H}] = 9 = 2 \times 3 - 1 + 2 \times 2 \times 3 - 2^2 - 2 \times 2,$$

establishing (35) in this particular case. The reason for working with this simple example, is that it is easy to generalize the idea to arbitrary $k, n$, but hard to write down and we want to omit the details. Roughly, in the general case the "V-derivatives" will span the first $k$ columns of $M_{n,n}$, whereas the $U$-derivatives will span the first $k$ rows. Thus $\text{Rank } [\partial \mathcal{M}(U_B, V_B)] = 2kn - k^2$, as required in (36). Moreover, it is easy to see that $\{E_1, \ldots, E_{2k}\}$ is a subset of $\text{Ran } \partial \mathcal{M}(U_B, V_B)$, whereas $\{E_{2k+1}, \ldots, E_{2n-1}\}$ form a basis for a disjoint subspace, (except for the point zero), see Fig 10. In general we thus get

$$\text{Rank } [\partial \mathcal{M}(U_B, V_B) T_H] = 2kn - k^2 + 2n - 1 - 2k,$$

as desired.

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Figure 6: Rank illustration. The filled dots represents the Hankel basis elements which are included in $\text{Ran} \partial \mathcal{W}(U_B, V_B)$.

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