ABSOLUTELY CONTINUOUS SPECTRUM OF A
SCHRÖDINGER OPERATOR ON A TREE

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Abstract. We give sufficient conditions for the presence of the absolutely continuous spectrum of a Schrödinger operator on a regular rooted tree without loops (also called regular Bethe lattice or Cayley tree).

INTRODUCTION AND RESULTS

The spectral properties of Schrödinger operators on graphs have numerous applications in physics and they have been intensively studied since late 90’s.

We will be mainly interested in the properties of the absolutely continuous component of the spectral measure of a discrete Schrödinger operator $H_V$ on a tree, see (0.2) for an example. Probably, the first specific results in this direction were obtained by Klein [11] who proved the presence of the absolutely continuous component for $H_V$’s with random iid potential on a regular Bethe lattice. Recently, Aizenman-Sims-Warzel [1] obtained the result with the help of a new general method. They also handled quasi-periodic operators [3] and a Laplacian on a random quantum tree [2]; see Aizenman-Sims-Warzel [4] for a nice overview of the topic. We also mention interesting papers by Froese-Hasler-Spitzer [8, 9] and Breuer [6].

Almost simultaneously to the above-mentioned works, Killip-Simon [10], Nazarov-Peherstrofer-Volberg-Yuditskii [15] obtained important results in the spectral theory of one-dimensional (1D) Schrödinger operators and, more generally, Jacobi matrices. These and subsequent papers [12, 13, 14, 18, 19] gave a fairly complete picture of the spectral behavior of these 1D objects.

It was hence very tempting to apply the well-developed methods of the one-dimensional analysis to the spectral problems for Schrödinger operators on trees. The first step in this direction was made by Denisov [7], who succeeded to carry over methods of Simon [16] to $H_V$ described in (0.2).

However, the general picture remained quite unclear. In particular, we did not understand to what extent the construction for 1D worked for Schrödinger operators on trees. This gap is fixed by the present paper. Amongst other results, we prove the Main Lemma (see Section 1) which expresses the Jost solutions for $H_V$ in terms of corresponding perturbation determinants. This observation implies immediately that there are strong parallels between spectral behavior of 1D Schrödinger operators and similar objects on trees, and we recover a big part of the 1D theory for these $H_V$’s. In particular, the sum rules of higher order for 1D Jacobi matrices

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become the “sum inequalities” for $H_V$. These higher order sum inequalities are proved in Theorems 0.2, 0.3.

Let $T = T_0$ be a regular binary tree of without loops (also called Cayley tree or Bethe lattice). Its root vertex is denoted by 0. The set of all vertices of the tree is denoted by $V(T)$. The distance $|x_1 - x_2|$ between two vertices $x_1, x_2 \in V(T)$ is the number of edges of the (unique) path leading from $x_1$ to $x_2$. The sphere of radius $n$ and centered at 0 is

$$S(0,n) = \{x \in V(T) : |x| = |x - 0| = n \}.$$ 

Every vertex in the tree has one ascendant and two descendants. The descendants of different vertices are different since the tree does not have closed loops. So, the sphere $S(0,n)$ contains $2^n$ vertices $x = (n,k)$, $k = 1, \ldots, 2^n$. That is,

- $V(T) = \{0, (n,k) : n \in \mathbb{N}, k = 1, \ldots, 2^n \}$,
- the descendants of 0 are vertices $(1,1), (1,2)$,
- for $n \geq 1$, the descendants of $(n,j), j = 1, \ldots, 2^n$, are $(n+1, 2j-1)$ and $(n+1, 2j)$, see Figure 1.

Let

$$(0.1) \quad \ell^p(T) = \{\{u(x)\}_{x \in V(T)} : \sum_{x \in V(T)} |u(x)|^p < \infty \}$$

with $1 \leq p \leq \infty$. The standard “basis” vectors are $\{e_x\}_{x \in V(T)}$, where $e_x(x) = 1$ and $e_x(y) = 0$ for $y \neq x, y \in V(T)$. The free Laplacian $H_0 = H_{0,T}$ is defined as

$$(H_0 f)(x) = \sum_{x' : |x' - x| = 1} f(x'),$$

where $f \in \ell^2(T)$. We also set

$$m_{H_0}(z) = \langle (H_0 - z)^{-1} e_0, e_0 \rangle = \int \frac{d\mu_0}{x - z}$$

to be the Weyl-Titchmarsh function of the operator. The Borel measure $\mu_0$ is called the spectral measure of $H_0$ (with respect to $e_0$). The spectrum $\sigma(H_0)$ coincides with supp $\mu_0$, and $\sigma(H_0) = \sigma_{ac}(H_0) = [-2\sqrt{2}, 2\sqrt{2}]$, see for example [7, Sect. 2].
A Schrödinger operator on $l^2(T)$ is a diagonal perturbation of $H_0$,
\begin{equation}
(H_V f)(x) = \sum_{x' : |x' - x| = 1} f(x') + V(x).
\end{equation}
We always assume that $V = \{V(x)\}_{x \in \mathcal{V}(T)}$ lies in $c_0(T)$, where
\[ c_0(T) = \{ \{u(x)\}_{x \in \mathcal{V}(T)} : \lim_{|x| \to +\infty} u(x) = 0 \}. \]
Then the operator $H_V$ is self-adjoint and, once again, we define its spectral measure $\mu = \mu_{H_V}$ as
\[ m_{H_V}(z) = ((H_V - z)^{-1} e_0, e_0) = \int_{\mathbb{R}} \frac{d\mu}{x - z}. \]
Since $H_V - H_0$ is compact, the Weyl-von Neumann theorem says that the essential spectrum $\sigma_{\text{ess}}(H_V)$ of the operator $H_V$ equals $[-2\sqrt{2}, 2\sqrt{2}]$, and the point spectrum $\sigma_p(H_V) \subseteq \mathbb{R} \setminus \sigma_{\text{ess}}(H_V)$ accumulates to the points $\pm 2\sqrt{2}$ only. It is convenient to enumerate $\sigma_p(H_V) = \{x^\pm_{V,T;1}\}$ as follows
\begin{equation}
x^-_{V,T;1} \leq \cdots \leq x^-_{V,T;2} \leq \cdots \leq -2\sqrt{2},
\end{equation}
and
\begin{equation}
2\sqrt{2} < \cdots \leq x^+_{V,T;2} \leq \cdots \leq x^+_{V,T;1}.
\end{equation}
The numbering takes into account the (geometric) multiplicities of the eigenvalues.

For a given potential $V = \{V(x)\}_{x \in \mathcal{V}(T)}$, define its “truncation” as
\[ V(n) = \{V(n; x)\}_{x \in \mathcal{V}(T)} = \begin{cases} V(x), & |x| \leq n, \\ 0, & |x| > n. \end{cases} \]
Let $T_x, x \in \mathcal{V}(T)$, be a subtree of $T$ growing from the vertex $x$. By $H_{V,T_x}$ we mean the Schrödinger operator with potential $V|_{T_x}$, the restriction of the original potential $V$ to $T_x$. The notation $\sigma_p(H_{V,T_x}) = \{x^\pm_{V,T_x;1}\}$ are self-obvious and stay for the point spectrum and eigenvalues of the operator $H_{V,T_x}$.

We give sufficient conditions for the support $\sigma_{\text{ac}}(H_V)$ of the absolutely continuous part of the measure $\mu_{H_V}$ to fill in the interval $[-2\sqrt{2}, 2\sqrt{2}]$. For instance, the following theorem is proved in Denisov [7].

**Theorem 0.1.** Let $H_V$ be a Schrödinger operator (0.2), $V \in c_0(T)$, and
\[ \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x : |x| = n} V(x)^2 < \infty. \]
Then
\[ \int_{-2\sqrt{2}}^{2\sqrt{2}} \log \mu'(x) \cdot \sqrt{8 - x^2} \, dx > -\infty, \]
\begin{equation}
\limsup_n \left( EV_{V(n),T}^{3/2} - \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{x : |x| = k} EV_{V(n),T}^{3/2} \right) < \infty,
\end{equation}
Remark 0.4. As in [7], the above theorem can be modified to assert that relations (0.9) (with $V$ (0.11)) then we have (0.8)

Theorem 0.2. Let $H_V$ be a Schrödinger operator (0.2), $V \in c_0(T)$, and

$$
\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} V(x)^4 < \infty, \quad \sum_{n=2}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} (\delta V)(x)^2 < \infty,
$$

Then

$$
\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} V(x)^6 < \infty, \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} (\delta V)(x)^2 < \infty.
$$

Then we have

$$
\int_{-2\sqrt{2}}^{2\sqrt{2}} \log \mu'(x) \cdot (8 - x^2)^{3/2} \, dx > -\infty.
$$

Theorem 0.3. Let $H_V$ be as in (0.2), $V \in c_0(T)$, and

$$
\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} V(x)^6 < \infty, \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} (\delta V)(x)^2 < \infty.
$$

Then we have

$$
\int_{-2\sqrt{2}}^{2\sqrt{2}} \log \mu'(x) \cdot (8 - x^2)^{5/2} \, dx > -\infty.
$$

These theorems lead to conjectures stated in Section 2.

Remark 0.4. As in [7], the above theorem can be modified to assert that that relations (0.9) (with $V \in l^\infty(T)$) yield (0.10), and, consequently, $[-2\sqrt{2}, 2\sqrt{2}] \subset \sigma_{ac}(H_V)$. The same applies to Conjecture 2.1.p with odd $p$'s.

Using results of Borichev-Golinskii-Kupin [5], we can express relations (0.5), (0.7) and (0.11) in terms of $\sigma_p(H_V)$ and the point spectra $\sigma_p(H_{V,T_s})$ of the corresponding operators. Of course, the assumptions on the potential $V$ become considerably more stringent.

Proposition 0.5. For $p \geq 1$, let $V \in l^p(T)$, $p' < p + 1/2$. Then the limit below exists and

$$
\lim_{n \to \infty} \left\{ EV_{V(n),T}^{p+1/2} \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{x:|x|=k} EV_{V(n),T_s}^{p+1/2} \right\} = EV_{V,T}^{p+1/2} - \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{x:|x|=k} EV_{V,T_s}^{p+1/2}.
$$
We also have
\[ EV_{V,T}^{p+1/2} \leq C(p,p', ||V||_\infty)||V||_{p'}, \]
where \( ||.||_\infty, ||.||_p \) are the norms of \( L^\infty(T), L^p(T) \), respectively.

1. Sketch of the proof of Theorem 0.2

As usual, we uniformize the domain \( \mathcal{C} \setminus [-2\sqrt{2}, 2\sqrt{2}] \) with the help of the maps \( z(\zeta) = \sqrt{2}(\zeta + 1/\zeta), \exists (z) = \frac{1}{2\sqrt{2}}(z - \sqrt{z^2 - 8}), \) where \( \zeta \in \mathbb{D} = \{ \zeta : |\zeta| < 1 \}. \)

Let for the moment \( \text{rank} V < \infty \). We put for an arbitrary subtree \( T_x \subset T, x \in \mathcal{V}(T), \)
\[ L_{T_x}(\zeta) = L_{T_x}(z(\zeta)) = \det(H_{V,T_x} - z)(H_{0,T_x} - z)^{-1}, \]
where \( H_{0,T_x} \) is the free Laplacian on \( T_x \). Furthermore, let \( \tilde{X} \) be a finite subset of vertices of the tree \( T \) with the property \( T_x \cap T_y = \emptyset \) for \( x, y \in \tilde{X} \) and \( x \neq y \). Obviously, we can speak about \( H_{T,\tilde{X}} = \oplus_{x \in \tilde{X}} H_{T_x} \) and
\[ L_{T,\tilde{X}}(\zeta) = L_{T,\tilde{X}}(z(\zeta)) = \det(H_{V,T,\tilde{X}} - z)(H_{0,T,\tilde{X}} - z)^{-1} = \prod_{x \in \tilde{X}} L_{T_x}(\zeta). \]

Consider the path \( \gamma_y \) leading from 0 to \( y \in \mathcal{V}(T) \) and denote by \( \tilde{X}(y) \) the set of vertices lying on the distance one from vertices of the path, that is,
\[ \tilde{X}(y) = \{ x \in \mathcal{V}(T) : \exists w \in \mathcal{V}(T) \cap \gamma_y, |x - w| = 1 \}. \]

It is easy that \( \tilde{X}(y) \) has the above-mentioned disjointness property and, moreover, \( (\mathcal{V}(T) \cap \gamma_y) \cup \mathcal{V}(T_{\tilde{X}(y)}) = \mathcal{V}(T) \).

The next lemma is the key to the proofs of Theorems 0.1-0.3. It is new and it expresses the Jost solution of the operator \( H_{V} \) in terms of \( L_{T,\tilde{X}(y)} \), compare to [10] Theorem 2.16]. It goes without saying that the lemma holds also for “sparse” trees considered in [6].

Main Lemma. Let \( \text{rank} V < \infty \) and \( H_{V} \) be the Schrödinger operator (0.2). Let \( f(\zeta) = \{ f_y(\zeta) \}_{y \in T} \in L^2(T) \) and \( f = (H_{V} - z(\zeta))^{-1}e_0 \). Then, for \( n = |y|, \)
\[ f_y(\zeta) = \left( \frac{\zeta}{\sqrt{2}} \right)^n L_{T,\tilde{X}(y)}(\zeta)/L_T(\zeta). \]

The proofs of the theorem use the techniques developed in [10, 15, 12, 13] and the lemma.

Sketch of the proof of Theorem 0.2. Let the potential \( V \in c_0(T) \) satisfy the assumptions of the theorem. We do the computations for the operator \( H_{V(N)} \), and then pass to the limit with respect to \( N \rightarrow \infty \).

Make the change of variables \( z(\zeta) = \sqrt{2}(\zeta + 1/\zeta) \) and transfer the spectral measure \( \mu_N = \mu_{H_{V(N)}} \) to the unit disk \( \mathbb{D} \) and its boundary. The absolutely continuous part of the image of the measure is then supported on the unit circle and its density is still denoted \( \mu'_N \). We write \( \{ \zeta_{V(N),T_x;s} \} \) for the images of \( \{ x_{V(N),T_x;s} \} \). Then relations (0.7), (0.8) read as
\[ \int_{0}^{2\pi} \log \mu'_N(e^{i\theta}) \sin^4 \theta d\theta > -\infty, \limsup_N \left( F_{V(N),T} - \sum_{k=1}^{\infty} \frac{1}{2\pi} \sum_{x:|x|=k} F_{V(N),T_x} \right) < \infty, \]
where

\[ F_{V(N),T} = \sum_s (1 - |\zeta_{V(N)},T_z,s|)^5. \]

For a given vertex \( x \in V(T) \), we consider \( H_{V(N),T_x} \) and the perturbation determinant \( L_{T_x}(\zeta) = \det(H_{V(N),T_x} - z)(H_{0,T_x} - z)^{-1} \). The eigenvalues \( \{\zeta_{V(N),T_z,s}\} \) coincide with the zeros of the determinant up to multiplicities. We have in a neighborhood of \( \zeta = 0 \)

\[ \log L_{T_x}(\zeta) = -\sum_{k=1}^\infty \frac{1}{k} \operatorname{tr} \left( T_k(\frac{1}{\sqrt{2}}H_{V(N),T_x} - T_k(\frac{1}{\sqrt{2}}H_{0,T_x}) \right) \zeta^k, \]

where \( T_k(2 \cos \theta) = 2 \cos k \theta, k = 0, 1, 2, \ldots \) are properly normalized Chebyshev polynomials of the first kind. The following identities hold: for \( n \geq 1 \),

\[ \frac{1}{2\pi} \int_0^{2\pi} |L_{T_x}(e^{i\theta})|^2 |d\theta = \sum_s \log 1/|\zeta_{V(N),T_z,s}|, \]

for \( n \geq 1 \),

\[ \frac{1}{2\pi} \int_0^{2\pi} |L_{T_x}(e^{i\theta})|^2 |d\theta = \frac{1}{n} \sum_s (1/|\zeta_{V(N),T_z,s}|)^n - |\zeta_{V(N),T_z,s}|^n \]

\[ - \frac{1}{n} \operatorname{tr} \left( T_n(\frac{1}{\sqrt{2}}H_{V(N),T_x}) - T_n(\frac{1}{\sqrt{2}}H_{0,T_x}) \right). \]

Combining these equalities, we get

(1.1)

\[ \frac{1}{2\pi} \int_0^{2\pi} |L_{T_x}(e^{i\theta})|^2 (16 \sin^4 \theta) |d\theta = \sum_s G(|\zeta_{V(N),T_z,s}|) \]

\[ - \frac{1}{8} \operatorname{tr} \left\{ (H^4_{V(N),T_x} - 24H^2_{V(N),T_x} - (H^4_{0,T_x} - 24H^2_{0,T_x}) \right\}, \]

where

\[ G(|\zeta|) = \frac{1}{2} \left\{ \left( \frac{1}{|\zeta|} - |\zeta|^4 \right) - 8 \left( \frac{1}{|\zeta|^2} - |\zeta|^2 \right) + 24 \log |\zeta| \right\}. \]

This relation readily implies that

\[ \sum_s G(|\zeta_{V(N),T_z,s}|) \geq \sum_s (1 - |\zeta_{V(N),T_z,s}|)^5. \]

Turning back to the operator \( H_{V(N)} \) and its spectral characteristics, we observe that

\[ M_{H_{V(N)}}(\zeta) = -m_{H_{V(N)}}(\zeta(\zeta)) = f_0(\zeta), \]

and the computation for \( \operatorname{Im} M_{H_{V(N)}}(\zeta), \zeta = e^{i\theta} \in T, \) gives

\[ \operatorname{Im} M_{H_{V(N)}}(\zeta) = \sqrt{2} \sin \theta \frac{\sum_{y:|y|=N} |L_{T_{\tilde{\xi}(y)}}(\zeta)|^2}{2^N |L_T(\zeta)|^2}. \]

The inequality between the arithmetic and the geometric mean \( \left( \frac{1}{n} (a_1 + \cdots + a_n) \right)^{1/n} \geq (a_1a_2 \cdots a_n)^{1/n} \) with \( a_j \geq 0 \) and some simple combinatorics yield

\[ \log \frac{\operatorname{Im} M_{H_{V(N)}}(\zeta)}{\sqrt{2} \sin \theta} \geq \frac{1}{2N} \sum_{y:|y|=N} \log |L_{T_{\tilde{\xi}(y)}}(\zeta)|^2 - \log |L_T(\zeta)|^2 \]
= \sum_{j=1}^{N} \frac{1}{2j} \sum_{x:|x|=j} \log |L_{T_{x}}(\zeta)|^2 - \log |L_{T}(\zeta)|^2.

We now apply equality (1.1) to the logarithms in the RHS and transfer the sums corresponding to the point spectra to the LHS of the inequality. So we come to

(1.2)

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \log \frac{\text{Im} M_{H_{V}(N)}(e^{i\theta})}{\sqrt{2\sin \theta}} (16 \sin^4 \theta) d\theta + \left\{ \sum_{s} G(|\zeta_{V(N)},T_{x};s|) - \sum_{j=1}^{N} \frac{1}{2j} \sum_{x:|x|=j} \sum_{s} G(|\zeta_{V(N)},T_{x};s|) \right\} \geq \frac{1}{8} \text{tr} \left[ K(H_{V(N)},T) - K(H_{0,T}) - \sum_{j=1}^{N} \frac{1}{2j} \sum_{x:|x|=j} (K(H_{V(N)},T_{x}) - K(H_{0,T_{x}})) \right],
\]

where \( K(H) = H^4 - 24H^2 \). An elementary calculation shows that the expression in the RHS of the relation satisfies the inequality

\[
\ldots \geq -C \left\{ \sum_{j=1}^{N} \frac{1}{2j} \sum_{x:|x|=j} V^4(x) + \sum_{j=2}^{N} \frac{1}{2j} \sum_{x:|x|=j} (\delta V)(x)^2 \right\}.
\]

Now, take \( \limsup \) of the both sides of inequality (1.2). Its RHS is finite by the assumptions of the theorem. Use the semi-continuity of the entropy (see [10]) to get

\[
\limsup_{N} \int_{0}^{2\pi} \log \frac{\mu'_{N}(e^{i\theta})}{\sqrt{2\sin \theta}} \sin^4 \theta d\theta \leq \int_{0}^{2\pi} \log \frac{\mu'(e^{i\theta})}{\sqrt{2\sin \theta}} \sin^4 \theta d\theta.
\]

Hypothesis (0.7) of the theorem says that

\[
\limsup_{N} \left\{ \sum_{s} G(|\zeta_{V(N)},T_{x};s|) - \sum_{j=1}^{N} \frac{1}{2j} \sum_{x:|x|=j} \sum_{s} G(|\zeta_{V(N)},T_{x};s|) \right\} < \infty.
\]

The proof is complete. \( \square \)

2. Some open questions and conjectures

The following conjecture seems very natural.

**Conjecture 2.1.p.** Let \( H_{V} \) be a Schrödinger operator (0.2) on a tree and \( V \in c_{0}(T) \). Let, for an odd \( p \geq 1 \),

(2.1)

\[
\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} V(x)^{2p} < \infty, \quad \sum_{n=2}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} (\delta V)(x)^2 < \infty,
\]

Then

\[
\int_{-2\sqrt{2}}^{2\sqrt{2}} \log \mu'(x) (8 - x^2)^{p-1/2} dx > -\infty,
\]
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\[
\begin{aligned}
(2.2) \quad \limsup_{n} \left\{ EV_{V(n),T}^{(p+1)/2} - \sum_{k=1}^{\infty} \frac{1}{2k} \sum_{x:|x|=k} EV_{V(n),T}^{(p+1)/2} \right\} < \infty.
\end{aligned}
\]

For an even \( p \geq 1 \), condition (2.2) becomes a hypothesis. The proof of this conjecture should follow the arguments of the theorem obtained in Section I. The difficulties are mainly of computational character.

The next observation is that the above-mentioned difference derivatives have depend more on the “tree structure” of the operator \( H_{V} \). Namely, the conjecture is that \( \delta V (\delta^{k} V) \) have to be replaced by \( \tilde{\delta} V (\tilde{\delta}^{k} V \), respectively), where

\[
(\tilde{\delta} V)(n,j) = V(n,j) - \frac{1}{2} (V(n+1,2j-1) + V(n+1,2j)).
\]

The following conjecture contains the previous ones as a very particular case. This is a carry over of Simon’s conjecture [17, Sect. 2.8], supplemented by Nazarov-Peherstorfer-Volberg-Yuditskii [15, Lemma 6.8]. To formulate it, we define an isometry \( W = \text{diag} \{ A_{n} \}_{n} : l^{2} \rightarrow l^{2}(T) \), where

\[
A_{n}^{t} = \left( \frac{1}{2} \right)^{n/2}, \ldots, \left( \frac{1}{2} \right)^{n/2} \right) \] 2\(^{n}\) entries

or \( J_{V/\sqrt{2}} \) is unitarily equivalent to a proper restriction of \( H_{V} \).

We rewrite the conjecture for \( J_{V/\sqrt{2}} \) given in [15] in the following way. Let \( A(x) = \sum_{k=0}^{2m} a_k x^k \) be a polynomial of degree \( 2m \) and \( A(x) \geq 0 \) on \( \mathbb{R} \). Let also

\[
(2.3) \quad dH_{V} = \left[ V(0): \underbrace{0,0}_{2 \text{ entries}}: \frac{1}{2} (V(1,1), V(1,2)), \frac{1}{2} (V(1,1), V(1,2)) \right]
\]

\[
0, \ldots, 0: \underbrace{\frac{1}{4} (V(2,1), V(2,2), V(2,3), V(2,4)), \ldots : \ldots \right)^{t} 4 \times 4 = 16 \text{ entries}.
\]

Consider an operator-valued polynomial

\[
B(H_{0}) = \sum_{k=0}^{2m} \frac{a_k}{2^{k/2}} H_{0} W \cdot W^{*} H_{0} W \cdot \ldots \cdot W^{*} H_{0}.
\]

**Conjecture 2.2.** Let

\[
\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} |V(x)|^{2m+2} < \infty, \quad |(B(H_{0}) dH_{V} , dH_{V})| < \infty.
\]
Then
\[ \int_{-2\sqrt{2}}^{2\sqrt{2}} \log \mu'(x) A(x/\sqrt{2}) \sqrt{8 - x^2} \, dx > -\infty, \]

\[ \limsup_n \left\{ F_{V(n),T}^A - \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{x:|x|=k} F_{V(n),T_s}^A \right\} < \infty, \]

where
\[ F_{V(n),T}^A = \sum_s F_{x_{V(n),T_s}^A}^A + \sum_s F_{x_{V(n),T_s}^{-A}}, \]

and, for \( \pm x > \pm 2\sqrt{2}, \)
\[ F_{x_{V(n),T_s}^A}^A = \pm \int_{\pm 2\sqrt{2}}^{x_{V(n),T_s}^A} A(s/\sqrt{2}) \sqrt{8 - s^2} \, ds. \]

The notation \( \{ x_{V(n),T_s}^\pm \}_{s \in [0,1]} \) is introduced in (0.3), (0.4).

Several remarks are in order. First, we can formulate a similar conjecture for Jacobi operators on trees. The increments of the coefficients associated to the edges of the tree then fill in zero entries in (2.3). Second, consider the usual shift
\[ S e_k = e_{k+1} \]
on \( l^2, \) \( \{ e_k \}_{k \in \mathbb{Z}} \) being the standard basis of the space. We now fix the standard basis \( \{ e_x \}_{x \in V(T)} \) in \( l^2(T), \) see (0.1). The binary shift is defined as
\[ S_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ B_1 & 0 & 0 & \cdots \\ 0 & B_2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \]

where \( B_1 = [1,1]^t, \) and \( B_{k+1} \) is the \( 2k+1 \times 2k \) matrix \( \begin{bmatrix} B_k & 0 \\ 0 & B_k \end{bmatrix}. \) It is plain that
\[ H_0 = S_1 + S_1^* \]
and
\[ S_1 W = \sqrt{2} W S, \quad W^* S_1 = \sqrt{2} S^* W, \]
\[ W^* S_1 = \sqrt{2} W S^*, \quad S_1^* W = \sqrt{2} W S. \]

For instance, we see for Theorem 0.1
\[ A(x) = 1, \quad B(H_0) = I, \quad (B(H_0) \, dH_V, \, dH_V) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} V(x)^2 < \infty, \]
and for Theorem 0.2
\[ A(x) = x^2 - 4, \quad B(H_0) = \frac{1}{2}(H_0 W W^* H_0 - 8) = -\frac{1}{2}(S_1 - S_1^*)^* W W^* (S_1 - S_1^*), \]
\[ (B(H_0) \, dH_V, \, dH_V) = -\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \sum_{x:|x|=n} \delta V(x)^2 > -\infty. \]

Roughly speaking, Conjecture 2.2 says that the role of \( J_0 \) is played by \( H_0 \) and the usual shift \( S \) is replaced by \( S_1 \) as compared to conjectures [17, Sect. 2.8] and [15, Lemma 6.8]. The presence of the binary shift leads to the “binary” derivatives appearing in the formulations of the theorems.

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