Variational principles for asymptotic variance of general Markov processes

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Abstract
A variational formula for the asymptotic variance of general Markov processes is obtained. As application, we get a upper bound of the mean exit time of reversible Markov processes, and some comparison theorems between the reversible and non-reversible diffusion processes.

Keywords: Markov process, asymptotic variance, variational formula, the mean exit time, comparison theorem, semi-Dirichlet form

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1 Introduction and main results
Asymptotic variance is a popular criterion to evaluate the performance of Markov processes, and widely used in Markov chain Monte Carlo(see e.g. [1, 5, 19, 20, 24]).

There are numerous studies of the asymptotic variance in the literature. For reversible Markov processes, the asymptotic variance can be presented by a spectral calculation, which brings a lot of applications (see [7, 14, 23] etc.). The comparisons on efficiency of reversible Markov processes, in terms of the asymptotic variance, has been extensively researched(see e.g. [11, 17, 20, 27]). Recently, there are also some comparison results between reversible and non-reversible Markov processes, see e.g. [2, 6, 12, 26] for discrete-time Markov chains, and [8, 13, 22] for diffusions. However, the study of the asymptotic variance of non-reversible Markov processes is still a challenge since the lack of spectral theory of non-symmetric operators. Very recently, [11] gives some variational formulas for the asymptotic variance of general discrete-time Markov chains by solving Poisson equation, and obtains some estimates and comparison results of the asymptotic variance.

In this paper we extend the results in [11] to the general Markov process by constructing the weak solution of Poisson equation with the help of the semi-Dirichlet form.

Let \( X = \{X_t\}_{t \geq 0} \) be a positive recurrent (or ergodic) Markov process on a Polish space \((S, \mathcal{S})\), with strongly continuous contraction transition semigroup \( \{P_t\}_{t \geq 0} \) and stationary distribution \( \pi \). Denote \( L^2(\pi) \) by the space of square integrable functions with scalar product \( (u, v) := \int_S u(x)v(x)\pi(dx) \) and norm \( ||u|| = (u, u)^{1/2} \). Let \( L^2_0(\pi) \) be the subspace of functions in \( L^2(\pi) \) with mean-zero, i.e.

\[
L^2_0(\pi) = \{ u \in L^2(\pi) : \pi(u) := \int_S u(x)\pi(dx) = 0 \}.
\]

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Suppose that the semigroup

\[ \text{Theorem 1.1.} \]

is the completion of \( F \) (in addition the associated semi-Dirichlet form \cite[Chapter 1, Theorem 2.15]{18}) for more details.

(1.3)

exponentially ergodic. Then the limit in

\[ \text{Remarkably, if process } X \text{ is reversible:} \]

\[ \pi(dx)P_t(x, dy) = \pi(dy)P_t(y, dx), \quad \text{for all } t \geq 0, \ \pi\text{-a.s.} \ x, y \in S, \]

then the sector condition is always true with \( K = 1 \) by Cauchy-Schwartz inequality.

Under the sector condition, we can obtain a unique semi-Dirichlet form \( (\mathcal{E}, \mathcal{F}) \), where \( \mathcal{F} \) is the completion of \( \mathcal{D}(L) \) with respect to \( \overline{\mathcal{E}}_1^{1/2} \) (\( \overline{\mathcal{E}}_1 \) is the symmetric part of \( \mathcal{E}_1 \)), see \cite[Chapter 1, Theorem 2.15]{18}) for more details.

We say that the semigroup \( \{P_t\}_{t \geq 0} \) is \( L^2 \)-exponentially ergodic, if there exist constants \( C, \lambda_1 > 0 \) such that for \( u \in L_0^2(\pi) \),

\[ \|P_tu\| \leq C\|u\|e^{-\lambda_1 t}. \]

It is well known that when process \( X \) is reversible, \( C \) can be chosen as 1 and (the optimal) \( \lambda_1 \) is nothing but the spectral gap:

\[ \lambda_1 = \inf \{ \mathcal{E}(u, u) : u \in \mathcal{F}, \pi(u) = 0 \text{ and } \pi(u^2) = 1 \}. \quad (1.2) \]

Now for \( f \in L_0^2(\pi) \), we consider the following asymptotic variance for \( X \) and \( f \):

\[ \sigma^2(X, f) = \limsup_{t \to \infty} \mathbb{E}_x \left[ \left( \frac{1}{\sqrt{t}} \int_0^t f(X_s)ds \right)^2 \right]. \quad (1.3) \]

Under the \( L^2 \)-exponential ergodicity and the sector condition, our first main result presents a variational formula for the asymptotic variance as follows.

\textbf{Theorem 1.1.} Suppose that the semigroup \( \{P_t\}_{t \geq 0} \) associated with process \( X \) is \( L^2 \)-exponentially ergodic. Then the limit in (1.3) exists and is finite for \( f \in L_0^2(\pi) \). If in addition the associated semi-Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) satisfies the sector condition, then for \( f \in L_0^2(\pi) \),

\[ 2/\sigma^2(X, f) = \inf_{u \in \mathcal{M}_{f,1}} \sup_{v \in \mathcal{M}_{f,0}} \mathcal{E}(u + v, u - v), \quad (1.4) \]

where \( \mathcal{M}_{f,\delta} = \{ u \in \mathcal{F} : (u, f) = \delta \}, \ \delta = 0, 1. \)

Particularly, if process \( X \) is reversible, then (1.4) is reduced to

\[ 2/\sigma^2(X, f) = \inf_{u \in \mathcal{M}_{f,1}} \mathcal{E}(u, u). \quad (1.5) \]
Remark 1.2. (1) For fixed $f \in L^2_0(\pi)$, from the proof below we will see that functions $Gf := \int_0^\infty P_t f \, dt$ and $G^*f := \int_0^\infty P^*_t f \, dt$ are both in $\mathcal{F}$, here $P^*_t$ is the dual operator of $P_t$ in $L^2(\pi)$. This is a main reason that we need the semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ in (1.4). However, if the generator $L$ is bounded in $L^2(\pi)$, then $D(L) = L^2(\pi)$, so that $Gf, G^*f \in D(L)$. In this case,

$$2/\sigma^2(X, f) = \inf_{u \in L^2_0(\pi), v \in L^2(\pi), \pi(u) = 1, \pi(v) = 0} \sup ((-L)(u + v), u - v).$$

The proof of this result can be obtained immediately by replacing $P - I$ by $L$ in the proof of [11, Theorem 1.1].

(2) The assumption of the $L^2$-exponential ergodicity of $\{P_t\}_{t \geq 0}$ is not too strong for non-reversible Markov processes, since [9] gives a geometrically ergodic Markov chain such that the asymptotic variance is infinite for some $f \in L^2_0(\pi)$.

(3) Variational formula for the asymptotic variance has been studied in [14, Chapter 4]. It is based on a variational formula for positive definite operators in analysis and resolvent equations. Here we obtain a new variational formula.

As a direct application of Theorem 1.1 bound of the mean exit time of the process is obtained. For that, let $\Omega \subset S$ be an open set, denote by $\tau_\Omega = \inf\{t \geq 0 : X_t \notin \Omega\}$ the first exit time from $\Omega$ of process $X$.

Corollary 1.3. Suppose that process $X$ is reversible with $L^2$-exponentially ergodic semi-group $\{P_t\}_{t \geq 0}$ and stationary distribution $\pi$. Let $\Omega \subset S$ be an open set with $\pi(\Omega) \in (0, 1)$, then

$$E_\pi \tau_\Omega \leq \frac{\pi(\Omega)}{2\lambda_1 \pi(\Omega^c)},$$

where $\lambda_1$ is the spectral gap defined in (1.2).

Note that in [10, Remark 3.6(1)], we gave another upper bound for the mean exit time. Explicitly, $E_\pi \tau_\Omega \leq 1/(\lambda_1 \pi(\Omega^c))$ for open set $\Omega \subset S$ satisfying $\pi(\Omega^c) > 0$. It is obvious that the upper bound in Corollary 1.3 is more precise than that.

For the reversible case, similar to [11, Theorem 1.3], we could derive variational formula (1.5) without the assumption of the $L^2$-exponential ergodicity. Since the proof is quite similar, we omit it in this paper.

Theorem 1.4. Suppose that $X$ is a reversible ergodic Markov process with stationary distribution $\pi$. Then for fixed $f \in L^2_0(\pi)$,

$$2/\sigma^2(X, f) = \inf_{u \in \mathcal{A}} \mathcal{E}(u, u).$$

Note that in Theorem 1.4, maybe $\sigma^2(X, f) = \infty$ for some $f \in L^2_0(\pi)$.

The remaining part of this paper is organized as follows. In Section 2 we apply our main result in two situations. The first application is extending the comparison result for the asymptotic variance of one dimensional diffusions in [25, Theorem 1] to multi-dimensional reversible diffusions. We note that [25, Theorem 1] is proved by discrete approximation which is different from our idea, and the less assumptions are requested.
in our proof. Another application is a comparison result between reversible and non-reversible diffusions on Riemannian Manifolds, which shows the asymptotic variance of a non-reversible diffusion is smaller. The similar result can be found in [8, 13] (for example, [13] proves a similar result on compact manifolds by using a spectral theorem), we provide a complete different proof by the new variational formula. Finally, the proofs of Theorem 1.1 and Corollary 1.3 are given in Section 3.

2 Applications

2.1 Reversible diffusions

First, we recall the comparison theorem proved in [25, Theorem 1]. Fix a $C^1$ probability density function $\mu : [I_1, I_2] \rightarrow (0, \infty)$, where $-\infty \leq I_1 < I_2 \leq \infty$. Given a $C^1$ positive function $\eta$ on $[I_1, I_2]$ and consider a one-dimensional Langevin diffusion:

$$dX^n_t = \eta(X^n_t) dB_t + \left(\frac{1}{2} \eta^2(X^n_t) \log \mu(\eta(X^n_t)) + \eta(X^n_t) \eta'(X^n_t)\right) dt.$$ 

Under some additional conditions (see [25, Page 133]), [25] proves that for any $f \in L^2(\mu)$, and two $C^1$ positive functions $\eta, \eta_1$ on $[I_1, I_2]$ such that $\eta_1(x) \leq \eta(x)$ for all $x \in [I_1, I_2]$,

$$\sigma^2(X^{\eta_1}, f) \geq \sigma^2(X^{\eta}, f).$$

Note that in [25], the above conclusion is proved by discrete approximation. In fact, we can obtain the above result by a direct calculation as follows. For convenience, we only consider the case on half-line.

Fix a $C^1$ probability density function $\pi : [0, \infty) \rightarrow (0, \infty)$. Given a $C^1$ positive function $a$ on $[0, \infty)$ and consider a one-dimensional diffusion $X^a$ with reflecting boundary 0 and generator:

$$L_a = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}, \quad (2.1)$$

where $b(x) = a(x)(\pi'(x)/\pi(x)) + a'(x)$. Let $\pi(dx) = \pi(x)dx$. It is easy to see that $L_a$ is symmetric on $L^2(\pi)$. Choose a point $x_0 > 0$ and set

$$c(x) = \int_{x_0}^{x} \frac{b(y)}{a(y)} dy \quad \text{and} \quad \varphi(x) = \int_{0}^{x} e^{-c(y)} dy.$$ 

So we have

$$\pi(x) = e^{c(x)} \pi(x_0) a(x_0)/a(x). \quad (2.2)$$

Assume that $X^a$ is non-explosive, that is,

$$\int_{0}^{\infty} \varphi'(y) \pi([0, y]) dy = \infty,$$

then $X$ is ergodic with stationary distribution $\pi(dx)$ (see e.g. [31, Table 5.1]).

For fixed function $f \in L^2(\pi)$, consider Poisson equation $-L_a u = f$. By some direct calculations and (2.2), the equation has strong solution

$$u(x) = \int_{0}^{x} e^{-c(y)} \left( \int_{y}^{\infty} f(z) e^{c(z)} a(z) dz \right) dy = \frac{1}{\pi(x_0) a(x_0)} \int_{0}^{\infty} f(z) \varphi(x \wedge z) \pi(dz).$$
Since $\sigma^2(X, f) = 2(u, f)$ by Lemma 3.1 and (3.3) below, from $\pi(f) = 0$ and the integration by parts we have that
\[
{\frac{1}{2}}\sigma^2(X^a, f) = \frac{1}{a(x_0)\pi(x_0)} \int_0^\infty \int_0^x f(x)f(y)\varphi(x \land y)\pi(dy)\pi(dx) \nonumber \\
= \frac{2}{a(x_0)\pi(x_0)} \int_0^\infty \varphi(x)f(x) \int_x^\infty f(y)\pi(dy)\pi(dx) \nonumber \\
= -\frac{2}{a(x_0)\pi(x_0)} \int_0^\infty \varphi(x)f(x) \int_0^x f(y)\pi(dy)\pi(dx) \\n= -\frac{1}{a(x_0)\pi(x_0)} \int_0^\infty \varphi(x) \left[ \left( \int_0^x f(y)\pi(dy) \right)^2 \right] ' dx \\
= \int_0^\infty \left( \int_0^x f(y)\pi(dy) \right)^2 \frac{1}{a(x)\pi(x)} dx.
\]

Using the above representation, we obtain the following comparison theorem directly.

**Theorem 2.1.** Let $a, a_1$ be two $C^1$ positive function on $[0, \infty)$. Then Langevin diffusions $X^a$ and $X^{a_1}$, with generators of form (2.1), possess the same stationary distribution $\pi$. Moreover, if $a \geq a_1$, then for any $f \in L^2_0(\pi)$,
\[
\sigma^2(X^a, f) \leq \sigma^2(X^{a_1}, f).
\]

In particular, for fixed $f \in L^2_0(\pi)$, $\sigma^2(X^{ka}, f)$ is non-increasing for $k \in (0, \infty)$.

For multi-dimensional reversible diffusion processes, explicit representation (2.3) for the asymptotic variance is difficult to obtain. However, we could use Theorem 1.4 to get the similar comparison result as follows.

Let $V \in C^2(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} e^{V(x)} dx < \infty$. Consider the reversible diffusion process $X^A$ generated by elliptic operator
\[
L_A = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial}{\partial x_i},
\]
where $A(x) = (a_{ij}(x))_{1 \leq i, j \leq d}, x \in \mathbb{R}^d$ are positive definite matrices with $a_{ij} \in C^2(\mathbb{R}^d)$ and
\[
b_i(x) = \sum_j a_{ij}(x) \frac{\partial}{\partial x_j} V(x) + \sum_j \frac{\partial}{\partial x_j} a_{ij}(x).
\]
Assume that $X^A$ is non-explosive. By [21, Theorem 4.2.1], we see that process $X^A$ is ergodic with stationary distribution
\[
\pi(dx) := \frac{e^{V(x)}}{\int_{\mathbb{R}^d} e^{V(y)} dy} dx.
\]
Denote by $(\mathcal{E}_A(\cdot, \cdot), \mathcal{F}_A)$ the Dirichlet form associated with the process $X^A$. Explicitly, we see that
\[
\mathcal{E}_A(u, v) = \int_{\mathbb{R}^d} \nabla u \cdot A \nabla v \pi dx, \quad \text{for } u, v \in \mathcal{F}_A := \{ u \in L^2(\pi) : \mathcal{E}_A(u, u) < \infty \}.
\]
Theorem 2.2. Let $V \in C^2(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} e^{V(x)} dx < \infty$, $A(x) = (a_{ij}(x))_{1 \leq i, j \leq d}$ and $A_1(x) = (a^1_{ij}(x))_{1 \leq i, j \leq d}$, $x \in \mathbb{R}^d$ be positive definite matrices satisfying $a_{ij}, a^1_{ij} \in C^2(\mathbb{R}^d)$ for $1 \leq i, j \leq d$. Suppose that $A_1 \leq A$ in the sense that $A(x) - A_1(x)$ is non-negative definite for all $x \in \mathbb{R}^d$. Then for any $f \in L^2_0(\pi)$,

$$\sigma^2(X^{A_1}, f) \geq \sigma^2(X^{A}, f).$$

(2.5)

In particular, for fixed $f \in L^2_0(\pi)$, $\sigma^2(X^{kA}, f)$ is non-increasing for $k \in (0, \infty)$.

Proof. Since $A_1 \leq A$, by (2.4) it is easy to check that $F_A \supseteq F_{A_1}$ and $E_A(u, u) \leq E_{A_1}(u, u)$ for all $u \in F_{A_1}$.

Fix $f \in L^2_0(\pi)$. The inequality (2.5) is trivial when $\sigma^2(X^{A_1}, f) = \infty$. Now assume that $\sigma^2(X^{A_1}, f) < \infty$. It follows from Theorem 1.4 that

$$2/\sigma^2(X^{A_1}, f) = \inf_{u \in F_{A_1}, \pi(fu) = 1} \mathcal{E}_{A_1}(u, u) \leq \inf_{u \in F_A, \pi(fu) = 1} \mathcal{E}_A(u, u) = 2/\sigma^2(X^{A}, f).$$

Hence, the proof is completed. □

2.2 Non-reversible diffusions on Riemannian Manifolds

In this section, we turn to non-reversible case. Let $M$ be a connected, complete Riemannian manifold with empty boundary or convex boundary, and $\langle \cdot, \cdot \rangle$ be the inner product under the Riemannian metric. Denote $d\pi$ and $\Delta$ by the Riemannian volume and Laplace operator on $M$, respectively.

Let $\pi(d\pi) := e^{-U(x)} d\pi$ be a probability measure on $M$ with potential function $U \in C^2(M)$. We consider the following diffusion operator:

$$\mathcal{L}_\varphi = \Delta \varphi - \langle \nabla U - Z, \nabla \varphi \rangle,$$

(2.6)

where $Z$ is a $C^1$ vector field on $M$. Denote by $\mathcal{L}^*$ the dual operator of $\mathcal{L}$ on $L^2(\pi)$:

$$\mathcal{L}^* \varphi = \Delta \varphi - \langle \nabla U + Z, \nabla \varphi \rangle - (\text{div } Z - \langle \nabla U, Z \rangle) \varphi,$$

where $\text{div}$ is the divergence operator. It is well known that $\pi$ is the invariant measure of $\mathcal{L}$ if and only if $(\mathcal{L}^*1, \varphi) = 0$ for $\varphi \in C^\infty_0(M)$, i.e.,

$$\int_M (\text{div } Z - \langle \nabla U, Z \rangle) \varphi d\pi = \int_M \text{div}(Ze^{-U}) \varphi d\pi = 0.$$

From now on we assume that

$$\text{div}(Ze^{-U}) \equiv 0.$$  

(2.7)

Then by [3] Corollary 3.6], the diffusion $X$ with generator $\mathcal{L}$ is ergodic with stationary distribution $\pi$.

Denote the symmetric part of $\mathcal{L}$ with respect to $\pi$ by $\overline{\mathcal{L}} := \Delta - \langle \nabla U, \nabla \rangle$, and let $\overline{X}$ be the diffusion generated by $\overline{\mathcal{L}}$. 
Define \((\mathcal{E}, \mathcal{F})\) as the semi-Dirichlet form generated by \(\mathcal{L}\), and denote its symmetric part and antisymmetric part by \(\mathcal{E}^s\), \(\mathcal{E}^a\) respectively. So from the integration by parts and (2.7), we have

\[
\mathcal{E}(\varphi, \phi) = \int_M \langle \nabla \varphi, \nabla \phi \rangle d\pi \quad \text{and} \quad \mathcal{E}^a(\varphi, \phi) = \int_M \phi \langle Z, \nabla \varphi \rangle d\pi \quad \varphi, \phi \in C_0^\infty(M).
\]

Indeed, it is easy to check that \((\mathcal{E}, \mathcal{F})\) is the Dirichlet form generated by \(\mathcal{L}\).

We suppose that the following Assumption A holds:

(A1) \(|\Delta U| \leq \varepsilon_*|\nabla U|^2 + C_U\) for some \(\varepsilon_* < 1\) and \(C_U \geq 0\);

(A2) there is a constant \(K\) such that \(|Z| \leq K(|\nabla U| + 1)\);

(A3) the symmetric Dirichlet form \((\mathcal{E}, \mathcal{F})\) satisfies the Poincaré inequality, i.e., there exists a constant \(\lambda_1 > 0\) such that

\[
\|\varphi\|^2 \leq \lambda_1^{-1} \mathcal{E}(\varphi, \varphi) \quad \text{for all} \ \varphi \in \mathcal{F},
\]

where \(\|\cdot\|\) is \(L^2(\pi)\)-norm.

We note that (A3) is equivalent to the \(L^2\)-exponential ergodicity of semigroup of diffusion \(X\).

**Lemma 2.3.** If Assumption A and (2.7) hold, then \((\mathcal{E}, \mathcal{F})\) satisfies the sector condition (1.1). Therefore, Theorem 1.1 holds for the diffusion \(X\).

**Proof.** Since \((\mathcal{E}, \mathcal{F})\) is symmetric, it satisfies the sector condition, we only need to check the sector condition for the antisymmetric part \(\mathcal{E}^a\).

Fix \(\phi, \varphi \in C_0^\infty(M)\). By Cauchy-Schwarz inequality and (A2) we have

\[
\int_M \langle \phi Z, \nabla \varphi \rangle d\pi \leq K \int_M (|\nabla U| + 1)|\phi| |\nabla \varphi| d\pi
\]

\[
\leq K \int_M |\phi| |\nabla \varphi| d\pi + K \mathcal{E}(\varphi, \varphi)^{1/2}||\nabla U||\phi|. \tag{2.8}
\]

For the last term above, the integration by parts on manifold, Cauchy-Schwarz inequality and (A1) yield that

\[
\|\nabla U\|_{\phi}^2 = - \int_M \langle \phi^2 \nabla U, \nabla e^{-U} \rangle dx = \int_M \text{div} (\phi^2 \nabla U) e^{-U} dx
\]

\[
= \int_M \langle 2\phi \nabla \phi, \nabla U \rangle d\pi + \int_M \Delta U \phi^2 d\pi \tag{2.9}
\]

\[
= \int_M \langle 2\phi \nabla \phi, \nabla U \rangle d\pi + \int_M (\epsilon_*|\nabla U|^2 + C_U) \phi^2 d\pi.
\]

Now fix \(\varepsilon > 0\) such that \(\varepsilon_* + \varepsilon < 1\). Combining inequality \(|xy| \leq (x^2/\varepsilon + \varepsilon y^2)/2\) with (2.9) and (A3) we have

\[
\|\nabla U\|_{\phi}^2 \leq 2 \int |\nabla \phi| |\phi \nabla U| d\pi + \int (\epsilon_*|\nabla U|^2 + C_U) \phi^2 d\pi
\]

\[
\leq \frac{1}{\varepsilon} \mathcal{E}(\phi, \phi) + (\epsilon_* + \varepsilon) \|\phi \nabla U\|^2 + C_U \|\phi\|^2
\]

\[
\leq \frac{1}{\varepsilon} \mathcal{E}(\phi, \phi) + (\epsilon_* + \varepsilon) \|\phi \nabla U\|^2 + C_U \lambda_1^{-1} \mathcal{E}(\phi, \phi),
\]

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which implies that
\[ \| \nabla U \| \phi \|^2 \leq \frac{\lambda_1 + C_U \varepsilon}{(1 - \epsilon_\ast \varepsilon) \varepsilon \lambda_1} \mathcal{E}(\phi, \phi). \]
Combining this with (2.8) and (A3), we obtain that \( \hat{\mathcal{E}} \) satisfies the sector condition on \( \mathcal{F} \).

From Lemma 2.3 and Theorem 1.1, we obtain the following comparison result.

**Theorem 2.4.** Suppose that Assumption A holds. Then for any \( f \in L_0^2(\pi) \),
\[ \sigma^2(X, f) \leq \sigma^2(\mathcal{X}, f). \]

**Proof.** Since the conditions in Theorem 1.1 are satisfied by Lemma 2.3, we obtain by taking \( v = 0 \) that
\[
2/\sigma^2(X, f) = \inf_{u \in \mathcal{M}_f, 1} \sup_{v \in \mathcal{M}_f, 0} \mathcal{E}(u + v, u - v) \\
\geq \inf_{u \in \mathcal{M}_f, 1} \mathcal{E}(u, u) = \inf_{u \in \mathcal{M}_f, 1} \mathcal{E}(u, u) = 2/\sigma^2(\mathcal{X}, f).
\]

**Remark 2.5.** Similar comparison result in Theorem 2.4 can be found in [8, 13]. For example, [13] proves the comparison theorem by using a spectral theorem (see [13, Section 3.4.3]). Here we provide a completely different proof by the new variational formula.

**Example 2.6.** ([16, Example 5.2]) Let \( M = \mathbb{R}^2 \), potential function \( U(x) = (1/2\pi)e^{-|x|^2/2} \) and vector field
\[ Z = -cx_2 \frac{\partial}{\partial x_1} + cx_1 \frac{\partial}{\partial x_2}, \]
where \( c \) is a positive constant. Consider the 2-dimensional Ornstein-Uhlenbeck diffusion with rotation:
\[ \mathcal{L} := \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) - (x_1 + cx_2) \frac{\partial}{\partial x_1} - (x_2 - cx_1) \frac{\partial}{\partial x_2}. \]
Its invariant probability measure is \( \pi(dx) = (1/2\pi)e^{-|x|^2/2}dx \). The symmetric part of \( \mathcal{L} \) with respect to \( \pi \) is
\[ \overline{\mathcal{L}} := \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) - x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}. \]
Since the symmetric Ornstein-Uhlenbeck diffusion generated by \( \overline{\mathcal{L}} \) is exponentially ergodic, (A3) is satisfied. A direct calculation shows that \( \text{div}(Ze^{-U}) = 0 \) and (A1), (A2) are satisfied. Hence, Theorems 1.1 and 2.4 are valid.
3 Proofs of Theorem 1.1 and Corollary 1.3

Recall that $X = \{X_t\}_{t \geq 0}$ is a positive recurrent (or ergodic) Markov process on a Polish space $(S, \mathcal{S})$, with strongly continuous contraction transition semigroup $\{P_t\}_{t \geq 0}$ and stationary distribution $\pi$. $(L, \mathcal{D}(L))$, $(\mathcal{E}, \mathcal{F})$ are its associated infinitesimal generator in $L^2(\pi)$ and semi-Dirichlet form, respectively. For fixed $f \in L^2_0(\pi)$, we want to study the asymptotic variance of $X$ and $f$ defined in (1.3). Indeed, from [13, Section 2.5], we see that the asymptotic variance can be represented by

$$\sigma^2(X, f) = 2 \lim_{t \to \infty} \int_0^t (1 - \frac{s}{t})(P_s f, f) ds. \quad (3.1)$$

To prove Theorem 1.1 first we do some preparations. For any $\alpha > 0$, set $G_\alpha f = \int_0^\infty e^{-\alpha s} P_s f ds$ for $f \in L^2(\pi)$. From [13, Chapter 1, Proposition 1.10] we see that $(G_\alpha)_{\alpha > 0}$ is the strong continuous contraction resolvent associated to $L$ and $G_\alpha f \in \mathcal{D}(L)$ for all $f \in L^2(\pi)$. If the semigroup $\{P_t\}_{t \geq 0}$ is $L^2$-exponentially ergodic, then it is known that $Gf := \int_0^\infty P_s f ds \in L^2(\pi)$ for $f \in L^2_0(\pi)$.

**Lemma 3.1.** Suppose that the semigroup $\{P_t\}_{t \geq 0}$ is $L^2$-exponentially ergodic and its corresponding semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies the sector condition (1.1). Then for all $f \in L^2_0(\pi)$, we have $Gf \in \mathcal{D}(L)$ and

$$\mathcal{E}(Gf, u) = (f, u), \quad u \in \mathcal{F}.$$  

**Proof.** We first prove that $Gf \in \mathcal{D}(L)$ for all $f \in L^2_0(\pi)$. Note that the generator $L$ is closed and densely defined, that is, $\mathcal{D}(L)$ is complete with respect to the graph norm $\|Lu\| + \|u\|, u \in \mathcal{D}(L)$ (see e.g. [13, Chapter 1, Proposition 1.10]). Thus for fixed $f \in L^2_0(\pi)$, we only need to prove that $\|G_{1/n} f - Gf\| \to 0$ as $n \to \infty$ and $\{G_{1/n} f\}_{n \geq 1}$ is a Cauchy sequence under $\|L \cdot \|$ by $G_{1/n} f \in \mathcal{D}(L), n \geq 1$. Indeed, it follows from $L^2$-exponential ergodicity and Hölder inequality that

$$\|G_{1/n} f - Gf\| = \left\| \int_0^\infty (1 - e^{-s/n}) P_s f ds \right\| \leq \int_0^\infty (1 - e^{-s/n}) \|P_s f\| ds$$

$$\leq C \|f\| \int_0^\infty (1 - e^{-s/n}) e^{-\lambda_1 s} ds$$

$$= C \|f\| \frac{1/n}{\lambda_1(\lambda_1 + 1/n)} \to 0, \quad as \ n \to \infty. \quad (3.2)$$

On the other hand, since $Lf = (\alpha - G^{-1}_\alpha) f$ for all $\alpha > 0$ and $f \in L^2_0(\pi)$, we have

$$\|L(G_{1/n} f - G_{1/m} f)\| = \|(1/n - G_{1/n}^{-1}) G_{1/n} f - (1/m - G_{1/m}^{-1}) G_{1/m} f\|$$

$$= \|\frac{1}{n} G_{1/n} f - \frac{1}{m} G_{1/m} f\|$$

$$\leq \frac{1}{n} \|G_{1/n} f - G_{1/m} f\| + \frac{1}{m} \|G_{1/m} f\|$$

$$\to 0, \quad as \ n, m \to \infty.$$  

Therefore $Gf \in \mathcal{D}(L)$. 

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9
To prove (1.4), we set
Thus (3.1), exists and
That is, 
Using this equality and the fact \( Gf \in \mathcal{D}(L) \) shows that for any \( \alpha > 0 \) and \( f \in L^2_0(\pi) \),
That is, \(-LGf = f\) for all \( f \in L^2_0(\pi) \).

From above analysis and [18, Chapter 1, Corollary 2.10] we could obtain that for any \( f \in L^2_0(\pi), u \in \mathcal{F} \),
\[ E(f, u) = (-L)Gf, u = (f, u). \]

We now proceed to prove Theorem 1.1.

Proof of Theorem 1.1 For fixed \( f \in L^2_0(\pi) \), we first claim that the limit in (3.1), i.e. (3.2), exists and \( \sigma^2(X, f) = 2(Gf, f) < \infty \). Indeed, for \( t > 0 \),
\[ 2 \int_0^t (1 - \frac{s}{t})(P_s f, f)ds = 2 \int_0^t (P_s f, f)ds - \frac{2}{t} \int_0^t s(P_s f, f)ds. \]

Since \( \{P_t\}_{t \geq 0} \) is \( L^2 \)-exponentially ergodic, we arrive at
\[ \frac{1}{t} \left| \int_0^t s(P_s f, f)ds \right| \leq \frac{1}{t} \int_0^t s\|P_s f\|\|f\|ds \leq \frac{C\|f\|^2}{t} \int_0^t se^{-\lambda_1 s}ds \]
\[ \leq \frac{1}{t} - (1 + \lambda_1 t)e^{-\lambda_1 t}\frac{C\|f\|^2}{t} \rightarrow 0, \text{ as } t \rightarrow \infty, \]
and
\[ \left| \int_t^\infty (P_s f, f)ds \right| \leq C\|f\|^2 \int_t^\infty e^{-\lambda_1 s}ds \rightarrow 0, \text{ as } t \rightarrow \infty. \]

Therefore, by combining above analysis, we obtain that the limit in (3.1) exists and
\[ \sigma^2(X, f) = 2 \int_0^\infty (P_s f, f)ds < \infty. \]

By the Fubini-Tonelli’s theorem and \( L^2 \)-exponential ergodicity again we get
\[ \int_0^\infty (P_s f, f)ds = \int_0^\infty \int_S fP_s f d\pi ds = \int_S \int_0^\infty fP_s f ds d\pi = (Gf, f). \]

Thus
\[ \sigma^2(X, f) = 2(Gf, f) < \infty. \quad (3.3) \]

To prove (1.4), we set \( w = Gf/(Gf, f) \), \( w^* = G^* f/(Gf, f) \) and \( u_0 = (w + w^*)/2, v_0 = (w - w^*)/2 \). Then \( u_0 \in \mathcal{M}_{f,1} \) and \( v_0 \in \mathcal{M}_{f,0} \) by noting
\[ (Gf, f) = \int_0^\infty (P_s f, f)ds = \int_0^\infty (f, P_s^* f)ds = (G^* f, f). \]
Now let \( v_1 = v - v_0 \) for any \( v \in \mathcal{M}_{f,0} \). By the definition of \( w, w^*, v_0 \) and Lemma 3.1, we have \( \pi(v_1 f) = 0 \) and

\[
\mathcal{E}(v_1, w^*) = \mathcal{E}(w, v_1) = \frac{1}{(Gf,f)} \mathcal{E}(Gf, v_1) = \frac{1}{(Gf,f)} (f, v_1) = 0.
\]

Therefore, using this fact with \( \mathcal{E}(w, w^*) = 1/(Gf, f) \) and \( \mathcal{E}(u, u) \geq 0 \) for all \( u \in \mathcal{F} \) gives that

\[
\mathcal{E}(u_0 + v, u_0 - v) = \mathcal{E}(w - v_1, w^* + v_1) = \mathcal{E}(w, w^*) - \mathcal{E}(v_1, v_1) \leq 1/(Gf, f),
\]

which implies that

\[
1/(Gf, f) \geq \inf_{u \in \mathcal{M}_{f,1}} \sup_{v \in \mathcal{M}_{f,0}} \mathcal{E}(u + v, u - v). \tag{3.4}
\]

For the converse inequality, let \( u_1 = u - u_0 \) for any \( u \in \mathcal{M}_{f,1} \). Since \( u_0 \in \mathcal{M}_{f,1} \), we also have \( \pi(u_1 f) = 0 \). Similar argument shows that

\[
\mathcal{E}(u + v_0, u - v_0) = \mathcal{E}(w + u_1, w^* + u_1) = \mathcal{E}(w, w^*) + \mathcal{E}(u_1, u_1) \geq 1/(Gf, f).
\]

Therefore,

\[
1/(Gf, f) \leq \inf_{u \in \mathcal{M}_{f,1}} \sup_{v \in \mathcal{M}_{f,0}} \mathcal{E}(u + v, u - v). \tag{3.5}
\]

So we obtain (1.4) by combining (3.4), (3.5) and the fact \( \sigma^2(X, f) = 2(Gf, f) \).

When process \( X \) is reversible, \( \mathcal{E}(\cdot, \cdot) \) is symmetric, i.e.,

\[
\mathcal{E}(u, v) = \mathcal{E}(v, u), \quad \text{for } u, v \in \mathcal{F}.
\]

Thus

\[
\mathcal{E}(u + v, u - v) = \mathcal{E}(u, u) - \mathcal{E}(v, v) \leq \mathcal{E}(u, u).
\]

That is, the supremum in (1.4) is attained by \( v = 0 \) for any fixed \( u \in \mathcal{M}_{f,1} \). Hence, we obtain (1.5).

By using Theorem 1.1, we prove Corollary 1.3 as follows.

**Proof of Corollary 1.3.** Fix an open set \( \Omega \subset S \) with \( \pi(\Omega) \in (0, 1) \). It follows from [10, Theorem 3.3] that

\[
1/E_{\pi} \tau_\Omega = \inf_{u \in \mathcal{N}_{\Omega,1}} \mathcal{E}(u, u), \tag{3.6}
\]

where \( \mathcal{N}_{\Omega,1} := \{ u \in \mathcal{F} : u|_{\Omega^c} = 0 \text{ and } \pi(u) = 1 \} \). Take

\[
f = \frac{1_\Omega - \pi(\Omega)}{1 - \pi(\Omega)}.
\]

It is easy to check that \( \pi(f) = 0 \) and \( \|f\|^2 = \pi(\Omega)/\pi(\Omega^c) \). Notice that for any \( u \in \mathcal{N}_{\Omega,1} \), by simple calculation we have \( \pi(u f) = 1 \), thus \( u \in \mathcal{M}_{f,1} \). So we see that \( \mathcal{N}_{\Omega,1} \subset \mathcal{M}_{f,1} \). Combining this fact with (1.5) and (3.6), we obtain that

\[
2/\sigma^2(X, f) = \inf_{u \in \mathcal{M}_{f,1}} \mathcal{E}(u, u) \leq \inf_{u \in \mathcal{N}_{\Omega,1}} \mathcal{E}(u, u) = 1/E_{\pi} \tau_\Omega.
\]
That is, $\mathbb{E}_\pi \tau_\Omega \leq \sigma^2(X, f)/2$. Moreover, from the reversibility and $L^2$-exponential ergodicity we have

$$\sigma^2(X, f)/2 = \int_0^\infty (P_s f, f) ds \leq \int_0^\infty \|P_s f\| \|f\| ds \leq \|f\|^2 / \lambda_1.$$ 

Hence,

$$\mathbb{E}_\pi \tau_\Omega \leq \frac{\|f\|^2}{2\lambda_1} = \frac{\pi(\Omega)}{2\lambda_1 \pi(\Omega^c)}.$$ 

\[ \blacklozenge \]

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