Nonlinear Capillary Waves on the Boundary of a Fluid Disk

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(Dated: May 6, 2022)

This is a first draft of a publication on nonlinear traveling waves on the boundary of a droplet. In the present work we numerically compute exact solutions of an integro-differential equation for traveling waves that exist on the boundary of a 2D blob of an ideal fluid subject to surface tension. We find that the solutions with many lobes resemble the Crapper waves on the surface of flat water, whereas the solutions with few lobes tend to become elongated as they become more nonlinear. We conjecture that waves with few lobes can become arbitrarily long.

I. INTRODUCTION

The purpose of the present section is to introduce the reader to complex variables approach to dynamics of free surface. The ideas of application of conformal variables can be traced back to the seminal work of G.G. Stokes [1] for infinite depth fluid, and later in the twentieth century to the extension of the conformal variables approach to the time-dependent problem by Ovsyannikov [2]. Extensive work on dynamics of a fluid in closed domain (blob) as well as on the exterior of an air bubble submerged in a 2D ideal fluid was done in the works of D. Crowdy [3]. Oscillations of 3D drops have been a subject of study since the works of Rayleigh [4], and have been studied both experimentally and analytically, see e.g. [5]. The shapes of traveling waves at the boundary of the disk are intimately connected to the periodic capillary waves at the free surface of water, see the relevant results in the work of Crapper [6] and later by Longuet-Higgins [7].

In this section we consider the motion of an ideal fluid in a bounded domain in 2D, whose flow is potential with fluid velocity potential \( \phi(r,t) \), where \( r = (x,y)^T \). The system is Hamiltonian, which is given by the formula:

\[
H = \frac{1}{2} \iint_D (\nabla \phi)^2 \, dx \, dy + \sigma \int_{\partial D} dl,
\]

where \( \nabla \) is 2D gradient, and \( \sigma \) is the surface tension coefficient. Here \( D \) is a bounded 2D domain occupied by the fluid whose boundary \( \partial D \) is a closed time-dependent curve in 2D often referred to as the free surface.

The global minimum of the potential energy subject to a fixed fluid mass is attained when \( D \) has a shape of the disk which is a well-known geometric result. We define \( \mu \) to be the total mass (area) of the fluid as follows:

\[
\mu = \iint_D dx \, dy.
\]

The motion of the free boundary \( \partial D \) relative to an inertial reference frame is considered. The natural choice of the inertial reference frame corresponds to the origin \( z = x + iy = 0 \) where we identify the \( \mathbb{R}^2 \) with the complex plane \( \mathbb{C} \).

The center of mass \( (z = 0) \) is stationary in this reference frame, and the total momentum of fluid \( P \) is zero:

\[
P = P_x + iP_y = \iint_D (\phi_x + i\phi_y) \, dx \, dy.
\]

It is convenient to introduce the angular momentum \( J \) of the fluid as

\[
J = \iint_D [r \times \nabla \phi] \, dx \, dy,
\]

which is a constant of motion, as well.

II. MECHANICS OF DROPLET AND THE CONFORMAL MAP

We introduce a conformal mapping of a semi–infinite periodic dimensionless strip \( w = u + iv \in \{-\pi \leq u < \pi, v \leq 0\} \) to the fluid domain \( z = x+iy \in D \) using a time-dependent complex function \( z(w,t) \), see also [3]. A complete description of the conformal map requires an additional mapping of a point at \( w \rightarrow -i\infty \), i.e. \( z(w \rightarrow -i\infty) = z_0 \). A convenient choice is \( z_0 = 0 \), however it is not unique and arbitrary point (or trajectory) can be selected instead. We begin by expressing kinematic constants through the velocity potential at the free surface in the conformal plane.
A. The Hamiltonian, momentum and angular momentum

The Hamiltonian $H$ in conformal variables is given by the formula:

$$H = \frac{1}{2} \int_D (\nabla \varphi)^2 dxdy + \sigma \int_{\partial D} \hat{\psi} \psi du + \sigma \int_{-\pi}^{\pi} |z_u| du$$

(5)

where $\hat{H}$ is the Hilbert transform, and $\hat{k} = -\hat{H} \partial u$. Here $\psi(u, t) = \varphi(x(u, t), y(u, t), t)$ is the velocity potential at the free surface.

The total volume of an incompressible fluid is proportional to the mass of the fluid $\mu$, a trivial constant of motion.

The fluid volume and momentum $P_x + i P_y$ in conformal variables are given by the formulas:

$$\mu = \int_D dxdy = \frac{1}{4i} \int |z \bar{z} - \bar{z} z_u| du,$$

$$P_x + i P_y = \int_D \nabla \varphi dxdy = i \int \psi z_u du.$$

(6)

(7)

The angular momentum of the fluid $J$ is a constant of motion. In the $(x, y)$-plane it is given by the formula:

$$J = \int \left[ r \times \nabla \phi \right] dxdy = \int \left( x \phi_y - y \phi_x \right) dxdy = -\frac{1}{2} \int r^2 \partial_{\theta} \psi du,$$

(8)

where $r = \sqrt{x^2 + y^2}$ and $n$ is the unit normal to the free surface. Expressed in conformal domain, the angular momentum is given by the formula:

$$J = -\frac{1}{2} \int |z|^2 \psi_u du, \quad \frac{dJ}{dt} = 0.$$

(9)

The moment of inertia $I$ is introduced as follows:

$$I = \int_D r^2 dxdy = \frac{1}{8} \int |z|^2 |z|^2 du = \frac{1}{8i} \int |z|^2 (z \bar{z}_u - \bar{z} z_u) du.$$

(10)

We note that the moment of inertia satisfies an ordinary differential equation:

$$\frac{dI}{dt} = \int |z|^2 \psi du.$$

(11)

In section [IV] we seek exact solutions that preserve the geometry of the droplet, and hence keep the moment of inertia fixed.

B. The Center of Mass

The center of mass of the fluid is located at the origin, and in conformal variables it may also be determined from integrating vector $r = (x, y)^T$ over the fluid domain, namely

$$R_{cm} = \int_D r dxdy = \frac{1}{2} \int_D \nabla (r^2) dxdy = \frac{1}{2} \int r^2 dl,$$

(12)

whereas in the conformal plane, this expression becomes:

$$R_{cm} = \frac{i}{2} \int |z|^2 z_u du.$$

(13)

The conformal mapping $z(w)$ can be represented by a Fourier series:

$$z(w, t) = z_0(t) + \sum_{k=1}^{\infty} z_k(t) e^{-ikw},$$

(14)

where the first term $z_0$ is determined from the relation:

$$2i z_0 = \frac{1}{\mu} \int |z - z_0|^2 z_u du,$$

(15)

derived from [13].
III. THE COMPLEX EQUATIONS OF MOTION

The equations of motion for the fluid are obtained from the Bernoulli equation and the kinematic condition at the free surface:

\[
\frac{\partial F}{\partial t} + (\nabla \varphi \cdot \nabla F) \bigg|_{F=0} = 0,
\]

\[
\frac{\partial \varphi}{\partial t} \bigg|_{F=0} + \frac{|\nabla \varphi|^2}{2} \bigg|_{F=0} + \sigma \kappa = 0,
\]

where \( F(x, y, t) \) is the implicit form of the free surface, and \( \kappa \) is the local curvature. Alternatively, the equations of motion can be obtained from the least-action principle. The Lagrangian \( \mathcal{L} \) is formed from the Hamiltonian \( \mathcal{H} \), while noting that in the physical plane the surface potential and elevation are canonical variables as first discovered in the work [9]:

\[
\mathcal{L} = \frac{1}{2i} \int \psi (z_t \bar{z}_u - \bar{z}_t z_u) - \mathcal{H} + i \int f \left( \hat{P}^+ z_u - \hat{P} \bar{z}_u \right),
\]

whereas in terms of the real and imaginary parts of \( z \) it can also be written as

\[
\mathcal{L} = \int \psi (y_t x_u - y_u x_t) \, du - \int f (y_u + \hat{k} x) \, du - \mathcal{H},
\]

where \( f(u, t) \) is the Lagrange multiplier that ensures analyticity of \( z(w, t) \) in \( w \in \mathbb{C}^- \). The action \( S \) is formed from the Lagrangian, and its variational derivative must vanish:

\[
S = \int \mathcal{L} \, dt, \quad \text{and} \quad \frac{\delta S}{\delta \psi} = 0, \quad \frac{\delta S}{\delta f} = 0, \quad \frac{\delta S}{\delta x} = 0, \quad \frac{\delta S}{\delta y} = 0.
\]

A. Kinematic Equation

We start with the implicit form of the kinematic equation for the conformal map:

\[
z_t \bar{z}_u - \bar{z}_t z_u = \bar{\Phi}_u - \Phi_u,
\]

or, in terms of the real and imaginary parts of \( z \),

\[
y_t x_u - y_u x_t = \hat{k} \psi.
\]

One may verify that the trivial solution, a unit disk, is represented by a one-parameter family of conformal maps:

\[
z(w) = e^{-iw} \frac{1 + \bar{A}e^{iw}}{1 + A e^{-iw}},
\]

where \( A \) is a free complex parameter with \(|A| < 1\). It is convenient to set the image of \( w \to -i\infty \) to be the center of mass of fluid at every instant of time:

\[z(w \to -i\infty) = 0 \quad \text{and} \quad z_t(w \to -i\infty) = 0.\]

For convenience we will denote:

\[
z_t(w \to -i\infty) = \frac{dz_0}{dt} = i \bar{v}_0(t).
\]

The complex transport velocity \( U \) is given by the formula:

\[
U = \hat{P} \left[ \frac{i (\Phi_u - \bar{v}_0 z_u + c.c.)}{|z_u|^2} \right],
\]

where \( 2\hat{P} = 1 + i \hat{H} \) is the projection operator. The kinematic condition can now be resolved for the time–derivative and yields:

\[
\frac{z_t}{z_u} - \frac{i \bar{v}_0}{z_u} = i U, \quad \text{or} \quad z_t = i \bar{v}_0 + iU z_u.
\]
B. Dynamic Condition

In order to derive the dynamic boundary condition from least action principle, we variate action w.r.t. \( \bar{z} \), and obtain the following implicit relations for the potential at the surface:

\[
\psi_t z_u - \psi_u z_t + i \sigma \partial_u \left( \frac{z_u}{|z_u|} \right) = \left( 1 - i \hat{H} \right) f_u.
\]  

(28)

Separating the real and the imaginary parts, we get

\[
\begin{align*}
\psi_t x_u - \psi_u x_t - \sigma \partial_u \left( \frac{y_u}{|z_u|} \right) &= f_u, \\
\psi_t y_u - \psi_u y_t + \sigma \partial_u \left( \frac{x_u}{|z_u|} \right) &= \hat{k} f.
\end{align*}
\]

(29)

(30)

These equations can be solved for the Lagrange multiplier \( \Lambda_u \) to find:

\[
\Lambda_u = f_u + i \hat{H} f_u = - \Phi_u^2 z_u.
\]

(31)

The implicit form of the dynamic condition becomes

\[
\psi_t \bar{z}_u - \psi_u \bar{z}_t + \frac{\Phi_u^2}{2 z_u} = i \sigma \partial_u \left( \frac{\bar{z}_u}{|z_u|} \right),
\]

(32)

and the Bernoulli equation is obtained:

\[
\left( \Phi_t - \Phi_u \frac{z_t}{z_u} \right) + \left( \Phi_t - \Phi_u \frac{\bar{z}_t}{\bar{z}_u} \right) + \left| \Phi_u \right|^2 \frac{\bar{z}_u z_{uu} - z_u \bar{z}_{uu}}{|z_u|^3} = 0.
\]

(33)

We apply the projection operator, \( \hat{P} \) to obtain explicit form of the equation for complex potential. We define auxiliary analytic function \( B \) as follows:

\[
B = \hat{P} \left[ |\Phi_u|^2 \frac{2}{|z_u|^2} + 2 i \sigma \frac{\bar{z}_u z_{uu} - z_u \bar{z}_{uu}}{|z_u|^3} \right],
\]

(42)

and

\[
\Phi_t - \Phi_u \frac{z_t}{z_u} + B = 0.
\]

(35)

We substitute equation (27) into (33) and write the full system of hydrodynamic equations:

\[
\begin{align*}
z_t - i \nu_0 &= i U z_u, \\
\Phi_t - \nu_0 \frac{i \Phi_u}{z_u} &= i U \Phi_u - B.
\end{align*}
\]

(36)

(37)

Using standard \( R \) and \( V \) variables discovered by A. I. Dyachenko (10),

\[
R = \frac{1}{z_u}, \quad V = \frac{i \Phi_u}{z_u},
\]

(38)

the hydrodynamic equations (36), (37) are now recast as

\[
\begin{align*}
R_t &= i \left[ U R_u - U_u R \right], \\
V_t &= i \left[ (U + \nu_0 R) V_u - B_u R \right],
\end{align*}
\]

(39)

(40)

where the auxiliary analytic functions \( U, B \) are given by (here \( Q = \sqrt{Re^{-iu}} \))

\[
\begin{align*}
U &= \hat{P} \left[ (V - \nu_0) R + (\bar{V} - \bar{\nu}_0) \bar{R} \right], \\
B &= \hat{P} \left[ |V|^2 + 2 \sigma |Q|^2 + 2 i \sigma (Q \bar{Q}_u - \bar{Q} Q_u) \right].
\end{align*}
\]

(41)

(42)

The explicit equations for \( R \) and \( V \) are convenient for numerical simulation of a 2D droplet of ideal fluid.
FIG. 1: (Left) The shape of a perturbed droplet with \( k = 3, k_0 = 1, \) and \( a_1 = 3.54 \times 10^{-2} \) (green), and \( a_1 = 0.1 \) (red) in the formula (50). (Right) The \(|z(u, t = 0)|\) as a function of conformal variable \( u \) for ideal disk (black dotted line), and four values of amplitude \( a_1 \). These conformal maps are the initial data for a sequence of numerical simulations to demonstrate linear standing and traveling waves.

IV. TRAVELING WAVES AROUND A DISK

A traveling wave on the free surface of a disk is obtained by seeking conformal map and potential in the form:

\[
z(u, t) = e^{-i\Omega t} z(u - \Omega t) \quad \text{and} \quad \Phi(u, t) = i\Omega \hat{P} |z|^2 - \beta t,
\]

where \( \beta \) is the Bernoulli constant. We note that the equations of motion are invariant under the change of variables \( u \to u - \Omega t \), and thus the solution may be sought in the form \( z = z(u) \). Substitution in the equations (29) and (30) leads to an equation for traveling waves:

\[
2\beta y_x - \frac{\Omega^2}{2} \left[ x\hat{k}|z|^2 - \hat{H} \left( y\hat{k}|z|^2 \right) \right] - \sigma \partial_x \left[ \frac{x_u}{|z_u|} - \hat{H} \left( \frac{y_u}{|z_u|} \right) \right] = 0,
\]

or, in the complex form,

\[
2i\beta z_u + \Omega^2 \hat{P} \left[ z\hat{k}|z|^2 \right] + 2\sigma \partial_u \hat{P} \left[ \frac{z_u}{|z_u|} \right] = 0.
\]

For a traveling wave solution, kinematic constants are related via the formula

\[
\mu \beta = \Omega J + \frac{\sigma}{2} \langle |z_u| \rangle,
\]

where angular brackets denote integral over one period, and the last term is the perimeter of the droplet.

V. LINEAR WAVES

We will now investigate small amplitude traveling waves by writing conformal map \( z \) and potential \( \Phi \) in the form:

\[
z(u, t) = z_0 + \frac{i}{k_0} e^{-iu} (1 + \delta z(u, t)), \quad \Phi(u, t) = -\beta t + \delta \Phi(u, t),
\]

where \( \delta z(u, t), \delta \Phi(u, t) \) are small. We write the kinematic equation (21) while keeping only the terms linear in \( \delta z, \delta \Phi \):

\[
\frac{i}{k_0} \left( \partial_t \delta z + \partial_u \delta \bar{z} \right) = \delta \Phi_u - \delta \Phi_u.
\]
By applying the projection operator to the linearized equation, we find that
\[ \partial_t \delta z = ik_0^2 \partial_u \delta \Phi, \]  
and by substituting \([47] - [48]\) in the dynamic equation \([35]\) and keeping only the terms of leading order we obtain:
\[ -\beta + \sigma k_0 + \delta \Phi_t - \sigma k_0 \left( \partial_u^2 + 1 \right) \delta z = 0. \]
We find that \( \beta = \sigma k_0 \), and write down the linearization system for perturbation \( \delta \Phi, \delta z \):
\[ \frac{\partial}{\partial t} \left( \begin{array}{c} \delta \Phi \\ \delta z \end{array} \right) = \left( \begin{array}{cc} 0 & \sigma k_0 (1 + \partial_u^2) \\ ik_0^2 \partial_u & 0 \end{array} \right) \left( \begin{array}{c} \delta \Phi \\ \delta z \end{array} \right). \]  
When \( k > 1 \) the eigenfunctions of the linearization matrix can be written in the form:
\[ \delta \Phi, \delta z \sim e^{-i(ku - \omega t)}, \]  
and the dispersion relation of linear waves is given by:
\[ \omega^2 = \sigma k_0^3 k \left( k^2 - 1 \right). \]
Since the admissible perturbations are integers, \( \omega^2(k) \) is non-negative and the general form of a solution with linear perturbation is given by:
\[ \Phi(u, t) = -\sigma k_0 t + \frac{i \omega}{k_0^2 k} \left[ a_1 e^{-i(ku - \omega t)} - a_2 e^{-i(ku + \omega t)} \right], \]
\[ z(u, t) = \frac{i}{k_0} e^{-iu} \left[ 1 + a_1 e^{-i(ku - \omega t)} + a_2 e^{-i(ku + \omega t)} \right], \]  
where the perturbed conformal map satisfies both \( z(w \to -i\infty, t) = 0 \) and \([15]\). The constants \( a_1, a_2 \) are free. By setting one of the constants to zero we find a linear traveling wave, and by setting \( a_1 = a_2 \) we find a linear standing wave solution. The standing wave does not result in rotation of the surface of the droplet, however, in a traveling wave the surface shape rotates with the angular velocity \( \Omega \) in the \( z = x + iy \) plane. The angular velocity \( \Omega \) is determined from \([43]\) and is given by:
\[ \Omega^2 = \frac{\omega^2}{k^2} = \sigma k_0^3 \left( k - \frac{1}{k} \right). \]
For the conformal wavenumber \( k = 1 \), the frequency \( \omega \) vanishes, and there is no wave motion. The eigenfunction of the linearization matrix is sought in the form:
\[ \delta z = A(t) e^{-iu} \quad \text{and} \quad \delta \Phi = B(t) e^{-iu}. \]
The linearization matrix becomes singular with a \( 2 \times 2 \) Jordan block and zero eigenvalue:
\[ \frac{d}{dt} \left( \begin{array}{c} A \\ B \end{array} \right) = \left( \begin{array}{cc} 0 & k_0^2 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} A \\ B \end{array} \right), \]  
and the general solution is given by:
\[ A(t) = \left( c_1 k_0^2 t + c_2 \right) \quad \text{and} \quad B(t) = c_1, \]  
where \( c_1 \) and \( c_2 \) are free constants. Using equations \([47], [3], \) and \([15]\), we find that the conformal map becomes
\[ z(u, t) = \frac{i}{k_0} \left[ -\bar{A} + e^{-iu} + A e^{-2iu} \right] + O(g^2). \]
Note that a conformal remapping of the stationary disk \([23]\) with a time-dependent parameter \( A \) coincides up to quadratic terms in \( A \) with the generalized eigenfunction for the zero eigenvalue:
\[ z(u, t) = \frac{ie^{-iu}}{k_0} \frac{1 - \bar{A} e^{iu}}{1 - Ae^{-iu}} = \frac{i}{k_0} \left[ -\bar{A} + e^{-iu} + A e^{-2iu} \right] + O(A^2). \]
FIG. 2: The magnitude of the conformal map $|z(u, t)|$ for a traveling wave (left) of small amplitude $a_1 = 0.025\sqrt{2}$ (left top), and amplitude $a_1 = 0.05$ (left bottom); and the magnitude of the conformal map $|z(u, t)|$ for a standing wave (right) of small amplitude $a_1 = a_2 = 0.0125\sqrt{2}$ (right top) and $a_1 = a_2 = 0.025$ (right bottom).

FIG. 3: The kinetic energy as a function of time in five simulations with $k = 3$, and various values of amplitude: $0.025\sqrt{2}$ (green), $a = 0.05$ (gold), $a = 0.05\sqrt{2}$ (blue). (Left) The kinetic energy of a true linear traveling waves is constant, however since the amplitude is small but finite, we see small amplitude beats coming from nonlinear coupling of Fourier modes. The larger the amplitude is the stronger are the deviations from constant given by the equation (63). (Right) The kinetic energy of a true standing wave is at double the frequency of the oscillations of the surface, yet because the amplitude is small but finite some nonlinear corrections are present. The vertical lines at $t = T$, $t = 2T$ and $t = 3T$ mark the ends of the first, second and third period of linear wave. As evident from the figure, the frequency decreases due to a nonlinear frequency shift, especially evident for the largest amplitude wave (blue).

This is no coincidence, because variation of the function $A(t)$ does not lead to a change in the shape of the surface of the disk in the physical plane, but only moves the image of the point at $w \rightarrow -i\infty$ to $-i\bar{A}/k_0$.

The kinetic energy of a linear wave can be computed exactly:

$$ K = \frac{\sigma}{2k_0} \left( k^2 - 1 \right) |a_1|^2, \quad (63) $$

and the potential energy may be computed approximately up to the terms of the order $|a_1|^2$:

$$ P = \frac{2\pi \sigma}{k_0} + \frac{\sigma}{2k_0} (k+1)^2 |a_1|^2. \quad (64) $$
VI. NUMERICAL SIMULATION

We solve the equations (59)–(60) numerically using a pseudospectral method to approximate the functions $R$ and $V$. The projection operator and derivatives with respect to $u$–variable are applied as Fourier multipliers. The Runge–Kutta method of fourth order is used for time integration.

We illustrate linear waves by solving the time–dependent equations with the initial data given by the equations (65)–(66). In the simulations of traveling waves (see the left panels of Fig. 2 and Fig. 3) the amplitude $a_2 = 0$, and

$$a_1 = 0.025\sqrt{2}, \ 0.05, \text{ and } a_1 = 0.05\sqrt{2}. \tag{65}$$

In the simulations of standing waves (see the right panel of Fig. 2 and Fig. 3) the amplitudes $a_1$ and $a_2$ are equal:

$$a_1 = a_2 = 0.0125\sqrt{2}, \ 0.025, \text{ and } a_1 = a_2 = 0.025\sqrt{2}. \tag{66}$$

The accuracy of simulations is controlled by measuring the total energy and mass in the course of the simulation and ensuring that the Fourier spectrum of $R$ and $V$ is resolved to machine precision.

VII. SERIES SOLUTION

Recall that $z = e^{-iu}$ describes the conformal map of the interior of the droplet onto the strip in lower complex half-plane. At $w = u \ (v = 0)$, we obtain a map of the droplet surface onto the interval $[-\pi, \pi]$ on the real line. Adding a perturbation to $z = e^{-iu}$, we can seek linear waves:

$$z = e^{-iu}(1 + ae^{-iku}), \tag{67}$$

where $k \in \mathbb{Z}_+$ is a positive integer. Plugging it into the dynamic condition (28), and keeping $O(a)$ terms, we get

$$\beta = \sigma + O(a^2), \quad \Omega^2 = \frac{(k^2 - 1)\sigma}{k} + O(a^2). \tag{68}$$

Fixing a small value of $a$, and utilizing parameters in (68), we obtain linear waves rotating with frequency $\Omega$.

We will now investigate small amplitude nonlinear traveling waves for dynamic condition (28). Choosing $k = 2$, we look for the conformal map $z$ in the form

$$z = e^{-iu}(1 + ae^{-2iu} + be^{-4iu}) + O(a^3). \tag{69}$$

Utilizing it in equation (15) for traveling waves and collecting corresponding terms, the first three frequencies give

$$e^{-iu} \left[2(\beta - \sigma) + \frac{1}{128}(64(9\sigma + 4\Omega^2)a^2 + O(a^4)) \right] \quad \text{(determines $\beta$)}$$

$$+ e^{-3iu}a \left[(6\beta - 9\sigma + 2\Omega^2) + \left(\frac{9}{8}(-9a^2 + 20b)\sigma + 4ca^2\Omega^2\right) + O(a^4)\right] \quad \text{(determines $\Omega^2$)}$$

$$+ e^{-5iu} \left[128(5(8b\beta + 9a^2\sigma - 20b\sigma)) + 8(a^2 + 2b)\Omega^2 + O(a^4)\right], \quad \text{(determines $b$)}$$

from where we obtain values of parameters:

$$\beta = \sigma \left(1 - \frac{15}{4}a^2\right) + O(a^4), \quad \Omega^2 = \sigma \left(\frac{3}{2} - \frac{25}{4}a^2\right) + O(a^4), \quad b = \frac{19}{12}a^2 + O(a^4).$$

Series (69) leads to consistent equations on solution’s parameters and therefore gives us a good initial guess on nonlinear wave solution of equation (15). Plot of the solution (69) with various frequencies added is given in Figure. 4.

More generally, for an arbitrary $k > 1$, we look for conformal map in the form

$$z = e^{-iu}(1 + ae^{-iku} + be^{-2iku}) + O(a^3). \tag{70}$$

Repeating the same steps and keeping orders up to $O(a^2)$, we obtain

$$\beta = \left(1 - \frac{a^2}{4}(k + 1)(3k - 1)\right)\sigma + O(a^4), \tag{71a}$$

$$\Omega^2 = \frac{k^2 - 1}{4k(1 + 2k^2)}(4 + 8k^2 + a^2(1 + k)(6 - k(22 + k + k^2)))\sigma + O(a^4), \tag{71b}$$

$$b = \frac{(1 + k)((7 + 2k)k - 3)}{4(1 + 2k^2)}a^2 + O(a^4). \tag{71c}$$
FIG. 4: Series solution of the equation (45) with \( k = 2, \alpha = 0.1 \) and 1-3 frequencies used in the formula (69).

Conformal map (70) with parameters (71) can serve as an initial guess for numerical iterative procedure such as Newton’s method.

VIII. NONLINEAR WAVES

In order to find nonlinear waves numerically, we apply the Newton’s method to solve the nonlinear equation (45) iteratively. Suppose \( z_{\text{exact}}(u) \) is the exact solution, and \( z(u) \) is a given approximate solution:

\[
z_{\text{exact}} = z + \delta z,
\]

and expanding the equation (45) to linear terms in \( \delta z \), we find that the correction \( \delta z \) satisfies the linearization equation:

\[
2i\beta \delta z + \Omega^2 \hat{P} \left[ \delta \hat{k} \hat{|z|^2} + \hat{z}\hat{k}(\overline{\delta z}) + \hat{z}(\overline{\delta z}) \right] + \sigma \partial u \hat{P} \left[ \frac{\hat{z}_{u}}{\hat{|z_u|^2}} (\overline{z_u \delta z_u} - z_u \delta z_u) \right] + N(z) = 0,
\]

where \( N(z) \) is given by (45). Since \( z = x + iy, \delta z = \delta x + i\delta y, \) and Hilbert operator relates real and imaginary parts: \( x = -\hat{H}y, \delta x = -\hat{H}\delta y, \) we only solve for \( \delta y \), and recover \( \delta z \) via the Hilbert transform. The equation for \( \delta y \) is given by the formula:

\[
-2\beta y_u + \Omega^2 \left( -\frac{1}{2} (\hat{H}y_u) \hat{|z|^2} - (\hat{H}y) \hat{k}(\hat{|y_u|})(\hat{H}\delta y) + y \delta y \right) - \hat{H} \left[ \frac{1}{2} \delta y \hat{k} \hat{|z|^2} + y \hat{k}(\hat{|y_u|})(\hat{H}\delta y) + y \delta y \right] + \sigma \partial u \left( -\frac{y_u}{|z_u|^2} (\hat{|y_u|} - y_u \hat{|\delta y_u|}) \right) - 2\beta y_u + \frac{\Omega^2}{2} (\hat{H}y_u \hat{|z|^2} - \hat{H} \left[ \frac{y_u}{|z_u|^2} \right]\hat{|z|^2}) + \sigma \partial u \left( -\frac{y_u}{|z_u|^2} (\hat{|y_u|} - y_u \hat{|\delta y_u|}) \right) = 0,
\]

or, writing it in compact form,

\[
L_1(y, \delta y) \delta y + L_0(y) = 0.
\]

Applying Hilbert operator to both sides, we obtain a linear equation for perturbation \( \delta y \):

\[
\hat{H}L_1(y, \delta y) = -\hat{H}L_0(y)
\]

whose operator is self-adjoint with respect to the dot product

\[
(f, g) = \int_{-\pi}^{\pi} f(x)g(x)dx,
\]
for real-valued \( f(x) \) and \( g(x) \), and we use the Conjugate-Residual method \[13\] to solve the linear operator equation (72) to find \( \delta y \). The matrix vector product is done pseudospectrally avoiding the formation of the matrix of operator in order \( N \log N \) flops, where \( N \) is the number of Fourier modes. In practice, the value of \( N \) does not exceed 16384 Fourier modes. In order to converge to the solution of the equation (45), we need no more than 10 Newton iterations and converge to the solution with the desired tolerance \( \| L_0(y) \|_2 \lesssim 10^{-9} \).

The weakly nonlinear waves are used as the initial guess for Newton’s method, and we obtain the fully nonlinear solutions to (45) by using numerical continuation in either the parameter \( \beta \), or \( \Omega^2 \). We illustrate nonlinear waves for \( k = 2, 3, 4, 25 \) by showing their profiles in \( xy \)-plane (see Figure 5, left). In addition, we show parameter curves \( \beta \) vs \( \Omega^2 \) (see Figure 5, right).

**FIG. 5:** Left: The shape of a perturbed droplet with \( k = 2, 3, 4 \) and \( k = 25 \). Right: \( \Omega^2 \) as a function of \( \beta \).

**IX. CONCLUSION**

We considered a problem of deformation of a fluid disk with a free boundary subject to the force of surface tension. We derived that a conformal map associated with such a flow satisfies a pseudodifferential equation that is similar to Babenko equation for the Stokes wave. We have shown that the motion of small amplitude deformations is subject to the linear dispersion relation given by (54). We demonstrate the results of numerical simulation with initial data close to linear waves, and observe excellent agreement for small amplitude waves, and report significant deviations as amplitude grows.

The nonlinear equation (44), or its complex form (45), are solved by the Conjugate Residual \[13\] method that is also applicable to the Stokes wave problem. The present work is a precursor to further investigation of nonlinear waves, and of particular interest is the question of the existence of the limiting wave, and its singularities. One may speculate that the limiting wave will not form an angle on the surface, since it would make the potential energy grow; yet the numerical simulations suggest the breaking of a droplet (for small number of lobes), and a tendency to develop a self-touching solution like the Crapper wave (for large number of lobes). The study of limiting scenarios is the subject of ongoing work.

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