Convergence of Kähler-Ricci flow on lower dimensional algebraic manifolds of general type

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Abstract
In this paper, we prove that the $L^4$-norm of Ricci curvature is uniformly bounded along a Kähler-Ricci flow on any minimal algebraic manifold. As an application, we show that on any minimal algebraic manifold $M$ of general type and with dimension $n \leq 3$, any solution of the normalized Kähler-Ricci flow converges to the unique singular Kähler-Einstein metric on the canonical model of $M$ in the Cheeger-Gromov topology.

1 Introduction
The purpose of this note is to prove the following

Theorem 1.1. Let $M$ be a smooth minimal model of general type with dimension $n \leq 3$ and $\omega(t)$ be a solution to the normalized Kähler-Ricci flow

$$\frac{\partial}{\partial t} \omega = -\text{Ric} - \omega. \tag{1.1}$$

Then $(M, \omega(t))$ converges in the Cheeger-Gromov sense to the unique singular Kähler-Einstein metric on the canonical model of $M$.

Here, by a smooth minimal model, we mean an algebraic manifold $M$ with nef canonical bundle $K_M$. The theorem should remain true in higher dimensional case; cf. Conjecture 4.1 in [27]. Assuming the uniform bound of Ricci curvature, the conjecture is confirmed by Guo [18]. On the other hand, it has been known since Tsuji [32] and Tian-Zhang [31] the convergence of the Kähler-Ricci flow in the current sense and the smooth convergence on the ample locus of the canonical class.

Applying the $L^2$ bound of Riemannian curvature (cf. Section 3) and Kähler-Einstein condition, we can say more about the limit singular space $M_\infty$. When $n = 2$, by a

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classical argument of removing isolated singularities following Anderson [1], Bando-Kasue-Nakajima [4] and Tian [29], we can show that the limit is a smooth Kähler-Einstein orbifold with finite orbifold points. When \( n = 3 \), by the argument of Cheeger-Colding-Tian [11] or Cheeger [7], we have that the 2-dimensional Hausdorff measure of the singular set is finite. Moreover, it follows from the parabolic version of the partial \( C^0 \)-estimate (see [30]) that the limit \( M_\infty \) is a normal variety and there is a natural holomorphic map from \( M \) onto \( M_\infty \). This actually implies that \( M_\infty \) is the canonical model of \( M \).

The proof of our theorem relies on a uniform \( L^4 \) bound of Ricci curvature under the Kähler-Ricci flow on a smooth minimal model together with the diameter boundedness of the limit singular Kähler-Einstein space (in the case of general type) proved by Song [26]. From these we derive a uniform local noncollapsing estimate of Kähler-Ricci flow on a minimal model of general type and the Gromov-Hausdorff convergence follows.

In Section 2, we present a short discussion on manifolds with integral bounded Ricci curvature, with emphasis on a uniform local noncollapsing estimate which is essential in extending the regularity theory of Cheeger-Condong and Cheeger-Colding-Tian. Then, in Section 3, we give a proof of our theorem by establishing a uniform \( L^4 \) Ricci curvature estimate under the Kähler-Ricci flow.

After we completed the first draft of this paper, we learned that Guo-Song-Weinkove obtained a different proof for the 2-dimensional case of Theorem 1.1 (see [19]).

2 Manifolds with \( L^p \) bounded Ricci curvature

We recall the relative volume comparison formula relative to the \( L^p \) Ricci curvature, \( p \) bigger than half dimension, which is due to Petersen-Wei [21]. It implies a uniform local noncollapsing property for the Kähler-Ricci flow on lower dimensional minimal models of general type. We will use this to prove the convergence of Kähler-Ricci flow on such manifolds.

Let \((M, g)\) be a complete Riemannian manifold of (real) dimension \( m \). For any \( \kappa \in \mathbb{R} \) denote by \( B^\kappa_r \) a metric ball of radius \( r \) in the space form of dimension \( m \) with sectional curvature \( \kappa \) and by \( \text{vol}(B^\kappa_r) \) its volume. Then we have

\[
\frac{d}{dr} \left( \frac{\text{vol}(B_r(x))}{\text{vol}(B^\kappa_r)} \right)^{\frac{1}{p}} \leq \frac{C r^{2p}}{\text{vol}(B^\kappa_r)} \left( \int_{B_r(x)} |(\text{Ric}-(m-1)\Lambda g)_-|^p \, dv \right)^{\frac{1}{p}},
\]

where we define

\[
(\text{Ric}-(m-1)\Lambda g)_- = \max_{|v|=1} \left(0, -\text{Ric}(v,v) + (m-1)\Lambda\right)
\]

pointwisely. In particular, for any \( r_2 > r_1 > 0 \),

\[
\left( \frac{\text{vol}(B_{r_2}(x))}{\text{vol}(B^\kappa_{r_2})} \right)^{\frac{1}{p}} - \left( \frac{\text{vol}(B_{r_1}(x))}{\text{vol}(B^\kappa_{r_1})} \right)^{\frac{1}{p}} \leq C \left( \int_{B_{r_2}(x)} |(\text{Ric}-(m-1)\Lambda g)_-|^p \, dv \right)^{\frac{1}{p}}.
\]
Then, by letting \( r_1 \to 0 \) it gives, for any \( r > 0 \),

\[
\frac{\text{vol}(B_r(x))}{\text{vol}(B_1^\Lambda)} \leq 1 + C \int_{B_r(x)} |(\text{Ric} - (m - 1)\Lambda g)|^p dv.
\] (2.3)

The following corollary gives a kind of uniform local noncollapsing property on manifolds with integral Ricci curvature bound; see [22] and [16] for similar volume doubling estimates.

**Corollary 2.2.** For any \( \Lambda < 0 \) and \( p > \frac{m}{2} \), there exists \( \varepsilon = \varepsilon(m, p, \Lambda) > 0 \) such that the following holds. Fix a base point \( x_0 \in M \). For any \( x \in M \) with \( \text{dist}(x_0, x) = d \), if

\[
\frac{1}{\text{vol}(B_1(x_0))} \int_{B_{2d+1}(x_0)} |(\text{Ric} - (m - 1)\Lambda g)|^p dv \leq \frac{\varepsilon}{\text{vol}(B_{d+1}^\Lambda)},
\] (2.4)

then,

\[
\frac{\text{vol}(B_r(x))}{r^m} \geq \frac{\text{vol}(B_1(x_0))}{2 \text{vol}(B_{d+1}^\Lambda)}, \quad \forall r \leq 1.
\] (2.5)

**Proof.** By (2.2), for any \( r \leq 1 \),

\[
\left( \frac{\text{vol}(B_r(x))}{\text{vol}(B_r^\Lambda)} \right)^{\frac{1}{2p}} \geq \left( \frac{\text{vol}(B_{d+1}(x))}{\text{vol}(B_{d+1}^\Lambda)} \right)^{\frac{1}{2p}} - C \left( \int_{B_{d+1}(x)} |(\text{Ric} - (m - 1)\Lambda g)|^p dv \right)^{\frac{1}{2p}}
\]

\[
\geq \left( \frac{\text{vol}(B_1(x_0))}{\text{vol}(B_{d+1}^\Lambda)} \right)^{\frac{1}{2p}} - C \left( \int_{B_{2d+1}(x_0)} |(\text{Ric} - (m - 1)\Lambda g)|^p dv \right)^{\frac{1}{2p}}.
\]

where \( C = C(m, p, \Lambda) \). If (2.4) holds for some \( \varepsilon = \varepsilon(m, p, \Lambda) \) sufficiently small, then

\[
\left( \frac{\text{vol}(B_r(x))}{\text{vol}(B_r^\Lambda)} \right)^{\frac{1}{2p}} \geq \frac{1}{2^{2p}} \left( \frac{\text{vol}(B_1(x_0))}{\text{vol}(B_{d+1}^\Lambda)} \right)^{\frac{1}{2p}},
\]

which is exactly the estimate (2.5).

The lemma suggests a condition for Gromov-Hausdorff convergence. Suppose we have a sequence of complete Riemannian manifolds \((M_i, g_i)\) of dimension \( m \) such that

\[
\int_M |(\text{Ric}_{g_i} - (m - 1)\Lambda g_i)|^p dv_{g_i} \to 0
\] (2.6)

for some \( p > \frac{m}{2} \), \( \Lambda > 0 \), and

\[
\text{vol}_{g_i}(B_1(x_i)) \geq v
\] (2.7)

uniformly for some \( v > 0 \) and \( x_i \in M_i \), then along a subsequence, the manifolds \((M_i, g_i)\) are uniformly locally noncollapsing on \( B_r(x_i) \) for any fixed \( r > 0 \). Thus, we can apply Gromov precompactness theorem to get a noncollapsing limit of \((M_i, g_i, x_i)\) in the Gromov-Hausdorff topology. Furthermore, as showed in [30], one can extend the regularity theory
of Colding [12], Cheeger-Colding [8, 9], Cheeger-Colding-Tian [11] and Colding-Naber [13] in this setting. If in addition we replace (2.6) by
\[ \int_M |\text{Ric}_{g_i} - (m-1)\Lambda g_i|^p dv_{g_i} \to 0, \] (2.8)
then Anderson’s harmonic radius estimate [2] can also be applied. In summation, we can follow the arguments in [22] and [30] to prove

**Theorem 2.3.** Let \((M_i, g_i)\) be a sequence of Riemannian manifolds satisfying (2.7) and (2.8) for some \(p > \frac{m}{2}\) and \(\Lambda, v > 0\). Then, passing to a subsequence if necessary, \((M_i, g_i, x_i)\) converges in the Cheeger-Gromov sense to a limit length space \((M_\infty, d_\infty, x_\infty)\) such that

1. for any \(r > 0\) and \(y_i \in M_i\) with \(y_i \to y_\infty \in M_\infty\) we have
   \[ \text{vol}(B_r(y_i) \to \mathcal{H}^m(B_r(y_\infty))), \] (2.9)
   where \(\mathcal{H}^m\) denotes the \(m\)-dimensional Hausdorff measure;
2. \(M_\infty = R \cup S\) such that \(S\) is a closed set of codimension \(\geq 2\) and \(R\) is convex in \(M_\infty\);
   \(R\) consists of points whose tangent cones are \(\mathbb{R}^m\);
3. the convergence on \(R\) takes place in the \(C^\alpha\) topology for any \(0 < \alpha < 2 - \frac{m}{p}\);
4. the tangent of any \(y \in M_\infty\) is a metric cone which splits off lines isometrically; the tangent cone has the same properties presented in (2) and (3);
5. the singular set of \(S\) has codimension \(\geq 4\) if each \(M_i\) is Kählerian.

3 Kähler-Ricci flow on minimal models

3.1 \(L^4\) bound of Ricci curvature under Kähler-Ricci flow on minimal models

Let \(M\) be a smooth minimal model. If the Kodaira dimension equals 0, the manifold is Calabi-Yau and any Kähler-Ricci flow on \(M\) converges smoothly to a Ricci flat metric [6]. We assume from now on that the Kodaira dimension of \(M\) is positive. Then we consider the normalized Kähler-Ricci flow
\[ \frac{\partial}{\partial t}\omega(t) = -\text{Ric}_\omega(t) - \omega(t), \omega(0) = \omega_0. \] (3.1)
It is well-known that the solution exists for all time \(t \in [0, \infty)\) [31].

We shall prove the following theorem.

**Theorem 3.1.** Suppose \(M\) has positive Kodaira dimension and \(K_M\) is semi-ample. Then there is a constant \(C\) depending on \(\omega_0\) such that
\[ \int_t^{t+1} \int_M |\text{Ric}_{\omega(s)}|^{\frac{4}{n}} \omega(s)^n ds \leq C, \forall t \in [0, \infty). \] (3.2)
Moreover, for any $0 < p < 4$ we have
\[
\int_t^{t+1} \int_M |\text{Ric}_{\omega(s)} + \omega(s)|^{p} \omega(s) ds \to 0, \text{ as } t \to \infty. \quad (3.3)
\]

We start with some general set-up following [28]. Since $K_M$ is semi-ample, a basis of $H^0(M; K_M^\ell)$ for some large $\ell$ gives rise to a holomorphic map
\[
\pi : M \to \mathbb{C}P^N, N = \dim H^0(M; K_M^\ell) - 1. \quad (3.4)
\]
Let $\omega_{FS}$ be the Fubini-Study metric on $\mathbb{C}P^N$ and put
\[
\chi = \frac{1}{\ell} \pi^* \omega_{FS} \in \mathcal{K}_M. \quad (3.5)
\]
Choose a smooth volume form $\Omega$ such that $\text{Ric}(\Omega) = -\chi$. Put
\[
\hat{\omega}_t = e^{-t} \omega_0 + (1 - e^{-t}) \chi.
\]
It represents a Kähler metric in the class $[\omega(t)]$ and write
\[
\omega(t) = \hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi(t)
\]
for a family of smooth functions $\varphi(t)$. Then the Kähler-Ricci flow (3.1) is equivalent to
\[
\frac{\partial \varphi}{\partial t} = \log \frac{e^{(n-k)t}(\hat{\omega}_t + \sqrt{-1} \partial \bar{\partial} \varphi)^n}{\Omega} - \varphi. \quad (3.7)
\]

**Lemma 3.2** ([28]). There exists $C_i = C_i(\omega_0, \chi), i = 1, 2$, such that
\[
\|\varphi(t)\|_{C^0} + \|\varphi'(t)\|_{C^0} \leq C_1, \forall t \geq 0 \quad (3.8)
\]
and
\[
\chi \leq C_2 \omega(t), \forall t \geq 0. \quad (3.9)
\]
Let $u = \varphi + \varphi'$ for any time $t$.

**Lemma 3.3** ([28]). There exists $C_3 = C_3(\omega_0, \chi)$ such that
\[
\|\nabla u(t)\|_{C^0} + \|\Delta u(t)\|_{C^0} \leq C_3, \forall t \geq 0. \quad (3.10)
\]

When the manifold is of general type, these estimates are proved in [35]. We also observe that, from (3.7),
\[
\text{Ric}_{\omega(t)} + \sqrt{-1} \partial \bar{\partial} u(t) = -\chi. \quad (3.11)
\]
So, by the uniform bound of $\chi$ in terms of $\omega(t)$, to prove the $L^4$ bound of Ricci curvature, it suffices to show
\[
\int_t^{t+1} \int_M |\partial \bar{\partial} u(s)|^4 \omega(s) ds \leq C_4, \forall t \geq 0
\]
for some $C_4$ independent of $t$. We follow the line in [30] to prove this estimate.

As in [30] we let $\nabla$ and $\bar{\nabla}$ denote the $(1,0)$ and $(0,1)$ part of the Levi-Civita connection of $\omega(t)$. Then, by the calculations in [30], we have the following lemmas
Lemma 3.4. There exists $C_5 = C_5(\omega_0, \chi)$ such that
\[
\int_M \left( |\nabla \nabla u|^2 + |\nabla u|^2 + |\text{Ric}|^2 + |\text{Rm}|^2 \right) \omega^n \leq C_5, \forall t \geq 0. \tag{3.13}
\]

Lemma 3.5 ([30]). There exists a universal constant $C_6 = C_6(n)$ such that
\[
\int_M |\nabla \nabla u|^4 \leq C_6 \int_M |\nabla u|^2 \left( |\nabla \nabla u|^2 + |\nabla u|^2 \right); \tag{3.14}
\]
\[
\int_M \left( |\nabla \nabla u|^2 + |\nabla u|^2 \right) \leq C_6 \int_M \left( |\nabla u|^2 + |\nabla u|^2 \cdot |\text{Rm}|^2 \right). \tag{3.15}
\]

The estimates in the last lemma are general facts which remain hold for any smooth function.

The following theorem gives the required estimate (3.2).

Theorem 3.6. There exists $C_7 = C_7(\omega_0, \chi)$ such that
\[
\int_t^{t+1} \int_M \left( |\nabla \nabla u|^4 + |\nabla \nabla u|^2 + |\nabla \nabla u|^2 \right) \leq C_7, \forall t \geq 0. \tag{3.16}
\]

Proof. By the previous two lemmas we are sufficient to prove a uniform $L^2$ bound of $\nabla \Delta u$.

Recall the evolution of $\Delta u$, cf. (3.22) in [28],
\[
\frac{\partial}{\partial t} \Delta u = \Delta \Delta u + \Delta u + \langle \text{Ric}, \partial \bar{\partial} u \rangle_\omega + \Delta \left( \text{tr}_\omega \chi \right).
\]

Thus,
\[
\frac{\partial}{\partial t} (\Delta u)^2 = \Delta (\Delta u)^2 - 2 |\nabla \Delta u|^2 + 2 (\Delta u)^2 + 2 \Delta u \langle \text{Ric}, \partial \bar{\partial} u \rangle_\omega + 2 \Delta u \Delta \left( \text{tr}_\omega \chi \right). \tag{3.17}
\]

Integrating over the manifolds gives
\[
\int_M |\nabla \Delta u|^2 \leq \int_M \left( (\Delta u)^2 + |\Delta u| |\text{Ric}| |\nabla \nabla u| - \nabla_i \Delta u \nabla_i \left( \text{tr}_\omega \chi \right) - \frac{1}{2} \frac{\partial}{\partial t} (\Delta u)^2 \right)
\]
\[
\leq \int_M \left( \frac{1}{2} |\nabla \Delta u|^2 + (\Delta u)^2 + (\Delta u)^2 |\text{Ric}|^2 + |\nabla \nabla u|^2 + 2 |\nabla \left( \text{tr}_\omega \chi \right)|^2 \right)
\]
\[
- \frac{1}{2} \int_M (\Delta u)^2 (s + n) - \frac{1}{2} \frac{d}{dt} \int_M (\Delta u)^2.
\]

Applying the uniform bound of $\Delta u$ and above lemma, we get
\[
\int_M |\nabla \Delta u|^2 \leq C \int_M (1 + |\nabla (\text{tr}_\omega \chi)|^2) - \frac{d}{dt} \int_M (\Delta u)^2.
\]

Integrating over the time interval we have
\[
\int_t^{t+1} \int_M |\nabla \Delta u|^2 \leq C \int_t^{t+1} \int_M (1 + |\nabla (\text{tr}_\omega \chi)|^2) + C, \forall t \geq 0. \tag{3.18}
\]
The term $|\nabla (\text{tr}_\omega \chi)|^2$ can be estimated through the Schwarz lemma. Recall the following formula
\[
\Delta (\text{tr}_\omega \chi) \geq -|\text{Ric}| \text{tr}_\omega \chi - C(\text{tr}_\omega \chi)^2 + \frac{|\nabla (\text{tr}_\omega \chi)|^2}{\text{tr}_\omega \chi} \tag{3.19}
\]
where $C$ is a universal constant given by the upper bound of the bisectional curvature of $\omega_{FS}$ on $\mathbb{C}P^n$. Because $0 \leq \text{tr}_\omega \chi \leq C$ under the flow, we have
\[
|\nabla (\text{tr}_\omega \chi)|^2 \leq C(\Delta (\text{tr}_\omega \chi) + |\text{Ric}| + C).
\]
Thus,
\[
\int_M |\nabla (\text{tr}_\omega \chi)|^2 \leq C \int_M (1 + |\text{Ric}|) \leq C
\]
uniformly. Substituting into (3.18) we get the desired estimate. \(\square\)

To prove (3.3) we use the $L^2$ estimate to traceless Ricci curvature following the calculation by Y. Zhang [34].

**Lemma 3.7.** Under the Kähler-Ricci flow,
\[
\int_t^{t+1} \int_M |\text{Ric}_{\omega(s)} + \omega(s)|^2 \omega(s)^n ds \to 0, \text{ as } t \to \infty. \tag{3.20}
\]

**Proof.** Recall the evolution of scalar curvature $R = \text{tr}_\omega \text{Ric}$, cf. [34],
\[
\frac{\partial}{\partial t} R = \Delta R + |\text{Ric}|^2 + R = \Delta R + |\text{Ric} + \omega|^2 - (R + n).
\]
The maximum principle shows that $\bar{R} = \inf R$ satisfies $\frac{d}{dt} \bar{R} \geq -(\bar{R} + n)$, which implies immediately
\[
\bar{R}(t) + n \geq e^{-t} \min (\bar{R}(0) + n, 0) \geq -Ce^{-t} \tag{3.21}
\]
for some $C = C(\omega_0) > 0$. Then,
\[
\int |\text{Ric} + \omega|^2 \omega^n = \int \left( \frac{\partial}{\partial t} R + R + n \right) \omega^n
\]
\[
= \frac{d}{dt} \int R \omega^n + \int (R + n)(R + 1) \omega^n
\]
\[
= \frac{d}{dt} \int R \omega^n + \int (R + n + Ce^{-t})(R + 1) \omega^n - Ce^{-t} \int (R + 1) \omega^n
\]
\[
\leq \frac{d}{dt} \int R \omega^n + C \int (R + n) \omega^n + Ce^{-t}
\]
where we used the uniform bound of scalar curvature and volume of $\omega(t)$. The integration of $R + n$ can be computed as
\[
\int (R + n) \omega^n = n \int (\text{Ric} + \omega) \wedge \omega^{n-1} = n \int (-\chi + \hat{\omega}) \wedge \hat{\omega}^{n-1} = ne^{-t} \int (\omega_0 - \chi) \wedge \hat{\omega}^{n-1}.
\]
Thus, \( \int (R + n)\omega^n \leq Ce^{-t} \). Then we have
\[
\int_0^\infty \int_M |\text{Ric} + \omega|^2 \omega(t)^n dt \leq \lim_{t \to \infty} \int R(t) \omega(t)^n - \int R(0) \omega_0^n + C \leq C.
\]
The lemma is proved by this estimate.

The estimate (3.3) when \( 2 \leq p < 4 \) then is a direct consequence of the Hölder inequality
\[
\int_t^{t+1} \int_M |\text{Ric} + \omega|^p \leq \left( \int_t^{t+1} \int_M |\text{Ric} + \omega|^4 \right)^{\frac{p-2}{4}} \left( \int_t^{t+1} \int_M |\text{Ric} + \omega|^2 \right)^{\frac{4-p}{4}}.
\]
When \( 0 < p < 2 \) the estimate (3.3) is obvious.

**Remark 3.8.** M. Simon presented in [25] an \( L^4 \) Ricci curvature estimate under Ricci flow on a four-manifold. Combined with the Kähler condition, his arguments can be adapted to give an alternative proof of our estimate. Another related integral bound of curvature can be found in [3].

### 3.2 Cheeger-Gromov convergence

When the manifold \( M \) is of general type, the Kähler-Ricci flow (3.1) should converge in the Gromov-Hausdorff topology to a singular Kähler-Einstein metric on the canonical model; cf. Conjecture 4.1 in [27]. In this subsection we confirm this convergence without any curvature assumption in the case of the dimension less than or equal to 3.

Recall the holomorphic map \( \pi : M \to \mathbb{C}P^N \) by a basis of \( H^0(M; K_\ell M) \) for some large \( \ell \). Its image \( M_{can} = \pi(M) \) is called the canonical model of \( M \). Let \( E \subset M \) be the exceptional locus of \( \pi \). Then we have

**Theorem 3.9 (31, 26).** Let \( M \) be a smooth minimal model of general type and \( \omega(t) \) be any solution to the Kähler-Ricci flow (3.1). Then,

1. \( \omega(t) \) converges in the current sense to a Kähler-Einstein metric \( \omega_\infty \), the convergence takes place smoothly outside the exceptional locus \( E \);
2. the metric completion of \( (M \setminus E, \omega_\infty) \) is homeomorphic to \( M_{can} \), so it is compact.

**Remark 3.10.** It is known that the exceptional locus \( E \) coincides with the non-ample or non-Kähler locus of the canonical class; cf. [31] and [14].

Suppose \( \dim \mathbb{C}M \leq 3 \). Let \( t_i \to \infty \) be a sequence of times such that
\[
\int_M |\text{Ric}_{\omega(t_i)} + \omega(t_i)|\tilde{\omega}(t_i)^n \to 0, \quad i \to \infty.
\]
Choose a regular point \( x_0 \in M \setminus E \). The volume of the unit ball \( \text{vol}(B_1(\omega(t_i)) (x_0)) \) has a uniform lower bound. By the \( L^p \) extension of Cheeger-Colding-Tian theory, Theorem 2.3, we may assume that \( (M, \omega(t_i), x_0) \) converges in the Cheeger-Gromov sense to a limit metric space \( (M_\infty, d_\infty, x_\infty) \). Since the metric \( \omega(t) \) converges smoothly on \( M \setminus E \), we may view \( (M \setminus E, \omega_\infty) \) as a subset of \( (M_\infty, d_\infty) \) through a locally isometric embedding.

Let \( M_\infty = \mathcal{R} \cup \mathcal{S} \) the regular/singular decomposition of \( M_\infty \).
Lemma 3.11. Suppose $x \in \mathcal{R}$, $0 < \alpha < 2 - \frac{4n}{\ell}$. There exists $r = r(x) > 0$ such that any $x_i \in M$ converging to $x$ has a holomorphic coordinate $(z^1, \cdots, z^n)$ on $B_{r,\omega(t_i)}(x_i)$ which satisfies
\[
\frac{1}{2} \leq (g_{k\bar{\ell}}) \leq 2, \|g_{k\bar{\ell}}\|_{C^\alpha} \leq 2
\]
where $g_{k\bar{\ell}} = \omega(t_i)(\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{\zeta}^\ell})$.

Proof. The existence of holomorphic coordinates is well-known. It can be constructed by a slight modification of the local harmonic coordinates. We include a short proof for the reader's convenience. First of all, by the $C^\alpha$ convergence on $\mathcal{R}$, there is a sequence of harmonic coordinate $v_i = (v_i^1, \cdots, v_i^{2n})$ on $B_{r,\omega(t_i)}(x_i)$ for some $r > 0$ independent of $i$ such that
\[
\frac{3}{4} \leq (h_{pq}) \leq \frac{4}{3} \text{ and } r^{-\alpha}\|h_{pq}\|_{C^\alpha} \leq \frac{4}{3},
\]
where $h_{pq} = g_i(\frac{\partial}{\partial v_i^p}, \frac{\partial}{\partial v_i^q})$, $g_i$ is the Kähler metric of $\omega(t_i)$, and \[22\]
\[
\int_{B_{r,\omega(t_i)}(x_i)} |(\frac{\partial}{\partial \bar{\zeta}^p}, \frac{\partial}{\partial v^q}) - \delta_{pq}| \omega(t_i)^n \to 0, \text{ as } i \to \infty.
\]
Moreover, since each $\omega(t_i)$ is Kähler, we may assume that
\[
\int_{B_{r,\omega(t_i)}(x_i)} |(\frac{\partial}{\partial \bar{\zeta}^p}, \frac{\partial}{\partial v^q}) - \delta_{pq}| \omega(t_i)^n \to 0, \text{ as } i \to \infty,
\]
for any $1 \leq p, q \leq n$, where $J$ is the complex structure of $M$; see \[11\] Section 9 for a discussion. Then we choose a pseudoconvex domain $B_{\frac{9}{10}r,\omega(t_i)}(x_i) \subset \Omega_i \subset B_{r,\omega(t_i)}(x_i)$ and solve the $\bar{\partial}$ equation in $\Omega_i$:
\[
\bar{\partial}f_i^p = \bar{\partial}(v_i^p + \sqrt{-1}v^{n+p}), \quad p = 1, \cdots, n.
\]
The domain can be chosen as the Euclidean ball in the local coordinate. The equation has solution satisfying the $L^2$ estimate \[24\] Theorem 4.3.4
\[
\int_{B_{\frac{9}{10}r,\omega(t_i)}(x_i)} |f_i^p|^2 \omega(t_i)^n \leq Cr^2 \int_{B_{r,\omega(t_i)}(x_i)} |\bar{\partial}(v_i^p + \sqrt{-1}v^{n+p})|^2 \omega(t_i)^n
\]
for a universal constant $C$. This implies $\int_{B_{\frac{9}{10}r,\omega(t_i)}(x_i)} |f_i^p|^2 \omega(t_i)^n \to 0$ for all $1 \leq p \leq n$. Then applying the elliptic regularity to $\triangle\omega(t_i)f_i^p = 0$ we get
\[
\sup_{B_{\frac{9}{10}r,\omega(t_i)}(x_i)} (|\partial f_i^p| + |\bar{\partial}f_i^p|) \to 0, \forall 1 \leq p \leq n.
\]
In particular, the function $w_i^p = \nu^p + \sqrt{-1}v^{n+p} - f_i^p$, $1 \leq p \leq n$, gives rise to the desired holomorphic coordinate on $B_{\frac{9}{10}r,\omega(t_i)}(x_i)$ whenever $i$ is large enough.

Lemma 3.12. $M_\infty \setminus (M \setminus E)$ is a closed subset of $M_\infty$ with Hausdorff codimension $\geq 2$. 

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Proof. Notice that $M \setminus E \subset \mathcal{R}$, $M_{\infty} \setminus (M \setminus E) = \mathcal{S} \cup (\mathcal{R} \setminus (M \setminus E))$ where $\text{Codim} \mathcal{S} \geq 4$ by Theorem 2.3 (5). Therefore, it suffices to show that $\text{Codim} (\mathcal{R} \setminus (M \setminus E)) \geq 2$.

For any $x \in \mathcal{R} \setminus (M \setminus E)$ there exists a sequence of points $x_i \in E$ which converges to $x$. By above lemma, there exists local holomorphic coordinate in $B_{r,\omega(t_i)}(x_i)$ for some $r = r(x) > 0$ independent of $i$ with required $C^\alpha$ estimate. The intersection $E \cap B_{r,\omega(t_i)}(x_i)$ is a local subvariety with finite volume, so passing to a subsequence, $E \cap B_{r,\omega(t_i)}(x_i)$ converges to a limit analytical set $E_\infty \subset B_{r,\omega_\infty}(x)$. Thus, $\mathcal{R} \setminus (M \setminus E)$ is an analytical set, $\text{Codim} (\mathcal{R} \setminus (M \setminus E)) \geq 2$ as desired. 

Lemma 3.13. $(M_{\infty}, d_{\infty})$ is isometric to the metric completion of $(M \setminus E, \omega_\infty)$.

Proof. The lemma follows from the fact that $\text{Codim} (\mathcal{R} \setminus (M \setminus E)) \geq 2$ and the local isoperimetric constant estimate; see [10] or [23] for details. For the estimate of local isoperimetric constant, one can apply the same argument as Croke [15] by using the volume comparison of geodesic balls by Petersen-Wei (cf. [30] Corollary 2.4) or (2.3)).

By the compactness of the limit space by Song [26], see Theorem 3.9 (2) above, the diameters of the sequence $(M, \omega(t_i))$ are uniformly bounded.

Lemma 3.14. The Kähler-Ricci flow $\omega(t)$ is uniformly noncollapsing in the sense that: there exists $\kappa = \kappa(n, \omega_0) > 0$ such that

$$\text{vol}_{\omega(t)} (B_{r,\omega(t)}(x)) \geq \kappa r^{2n}, \forall x \in M, r \leq 1. \quad (3.23)$$

Proof. The lemma follows from Perelman’s noncollapsing estimate to Ricci flow [20] together with the scalar curvature estimate by Z. Zhang [35]. Suppose that

$$r_i^{-2n} \text{vol}_{\omega(t_i)} (B_{r,\omega(t_i)}(x_i)) \to 0 \quad (3.24)$$

for a sequence of times $t_i \to \infty$ and $x_i \in M, r_i \leq 1$. Choose $t_i' \in [t_i - 2, t_i - 1]$ such that (3.22) hold at $t_i'$. Then by above lemma we have $(M, \omega(t_i'))$ converges in the Gromov-Hausdorff topology to the unique limit $(M_{\infty}, d_{\infty})$. In particular we have

$$R(\omega(t_i')) \leq C, \text{diam}(M, \omega(t_i')) \leq C, C_S (M, \omega(t_i')) \leq C$$

for some $C$ independent of $i$, where $R$ is the scalar curvature, $C_S$ is the Sobolev constant.

Let $\tilde{\omega}_i(\bar{t}) = (1 + \bar{t})\omega(t_i' + \log(1 + \bar{t}))$ be the solution to the Ricci flow

$$\frac{\partial}{\partial \bar{t}} \tilde{\omega}_i = -\text{Ric}(\tilde{\omega}_i), \tilde{\omega}_i(0) = \omega(t_i') \quad (3.25)$$

Under this rescalings, the metric $\omega(t_i) = (1 + \bar{t}_i)^{-1}\tilde{\omega}_i(\bar{t}_i)$ for some $\bar{t}_i \in [e - 1, e^2 - 1]$ and $B_{r,\omega(t_i)}(x_i) = B_{\tilde{r}_i,\tilde{\omega}(\tilde{t}_i)}(x_i)$ for some $\tilde{r}_i \leq e$. Recall the $\mu$ functional of Perelman [20]

$$\mu(g, \tau) = \inf_M \left[ \tau (R + |\nabla f|^2) + f - 2n \right] (4\pi\tau)^{-n} e^{-f} dv_g$$

for any Riemannian metric $g$ and $\tau > 0$, where the infimum is taken over all $f \in C^\infty(M; \mathbb{R})$ with restriction $\int_M (4\pi\tau)^{-n} e^{-f} dv_g = 1$. The condition (3.24) implies that $\mu(\tilde{\omega}_i(\bar{t}_i), \bar{t}_i^2) \to
−∞ as i → ∞; see [20, Section 4.1]. Then Perelman’s monotonicity formula shows
µ(˜ω_i(0), ˜t_i + ˜r_i^2) → −∞ as i → ∞, where ˜t_i + ˜r_i^2 ∈ [e^{−1}, 2e^{2} − 1]. But this can never
happen because of the lower estimate of µ (cf. the estimate in [36]):

\[ \mu(\tilde{\omega}_i(0), \tau) \geq \tau \inf_{\tilde{\omega}_i(0)} R(\tilde{\omega}_i(0)) - \frac{n}{2} \ln \tau - n \ln CS(\tilde{\omega}_i(0)) - C(n), \forall \tau \geq \frac{n}{8}. \] (3.26)

So (3.24) can not hold, the lemma is proved.

The global Cheeger-Gromov convergence is a direct corollary of the following proposition.

**Proposition 3.15.** For any sequence \( t_j \to \infty \), \((M, \omega_{t_j})\) converges along a subsequence in
the Cheeger-Gromov sense to the limit \((M_{\infty}, d_{\infty})\).

**Proof.** By the regularity of manifolds with \( L^p \) bounded Ricci curvature, Theorem 2.3, and
the uniform \( L^p \) estimate of Ricci curvature under the Kähler-Ricci flow, (3.2) and (3.3),
we can find another sequence of times \( t'_j \) such that

\[ t_j - \varepsilon_j \leq t'_j \leq t_j \]

where \( \varepsilon_j \to 0 \) as \( j \to \infty \), and

\[ (M, \omega_{t'_j}) \xrightarrow{d_G} (M_{\infty}, d_{\infty}) \]

along a subsequence. On the other hand, by Gromov precompactness theorem [17] together
with the local noncollapsing estimate (3.23), after passing to a subsequence, the manifolds
\((M, \omega(t_j))\) also converge in the Gromov-Hausdorff topology to a compact limit

\[ (M, \omega(t_j)) \xrightarrow{d_G} (M'_\infty, d'_{\infty}). \]

It is remained to show that \((M'_\infty, d'_{\infty})\) is isometric to \((M_{\infty}, d_{\infty})\). Actually we have

**Claim 3.16.** There is a sequence of positive numbers \( \delta_j \to 0 \) as \( j \to \infty \) such that the
identity map defines a \( \delta_j \)-Gromov-Hausdorff approximation from \((M, \omega(t'_j))\) to \((M, \omega(t_j))\).

**Proof of the Claim.** Recall that by the smooth convergence of \( \omega(t) \) on \( M \setminus E \) and uniform
volume noncollapsing (3.23) we have for any \( \delta > 0 \) one compact subset \( K \subset M \setminus E \) and
\( \epsilon > 0 \), \( j_0 \gg 1 \) such that

\[ \epsilon \leq \inf_{t} \text{dist}_{\omega(t)}(K, E) \leq \sup_{t} \sup_{x \in E} \text{dist}_{\omega(t)}(x, K) \leq \delta, \] (3.27)

and,

\[ \text{vol}_{\omega(t)}(M \setminus K) \leq \delta, \forall t \geq t_{j_0}. \] (3.28)

The later can be seen by simply the derivative estimate to volume

\[ \frac{d}{dt} \int_{M \setminus K} \omega(t)^n = - \int_{M \setminus K} (\inf R + n)\omega(t)^n \leq C(\omega_0)e^{-t} \int_{M \setminus K} \omega(t)^n \]
connecting which gives the upper estimate to the distance distortion 
\[ \varepsilon \]
intersect the \( M \) a natural isomorphism from 
\[ M \]
normal projective variety. On the other hand, using the Kähler-Ricci flow, we can produce \( j \) whenever \( 1 \) such that 
\[ H \]
the above result by J. Song. This can be done by choosing a family of orthonormal basis \( \gamma \) whenever \( j \) such that \( H \) is large enough such that 
\[ K \]
Moreover, by the convexity of the regular set \( M \setminus E \) in \( M_{\infty} \) we may assume in addition that any minimal geodesic in \( (M, \omega(t)) \) with endpoints in \( K \) does not intersect the \( \varepsilon \) neighborhood of \( E \). Also observe that there exists \( C = C(\varepsilon) \) independent of \( j \) such that 
\[ \| \text{Ric}(\omega(t)) \|(x) \leq C, \forall \text{dist}_{\omega(t)}(x, E) \geq \frac{\varepsilon}{2}. \quad (3.29) \]
Then, by the derivative estimate to distance function, cf. Lemma 8.3 in \([20]\),
\[ \frac{d}{dt} \text{dist}_{\omega(t)}(x, y) \geq -2(2n - 1)(C\varepsilon + \varepsilon^{-1}), \forall x, y \in K. \]
Thus, we obtain the distance lower variation estimate
\[ \text{dist}_{\omega(t_j)}(x, y) \geq \text{dist}_{\omega(t_j')}(x, y) - 2(2n - 1)(C\varepsilon + \varepsilon^{-1})\varepsilon_j \geq \text{dist}_{\omega(t_j')}(x, y) - \sqrt{\varepsilon_j} \]
whenever \( j \) is large enough. On the other hand, let \( \gamma \) be any minimal geodesic in \( (M, \omega(t_j')) \) connecting \( x, y \in K \). By assumption \( \text{dist}_{\omega(t)}(\gamma, E) \geq \varepsilon \) for any \( t \), so
\[ \frac{d}{dt} \int_{\gamma} |\dot{\gamma}| ds \leq \int_{\gamma} |\text{Ric} + \omega||\dot{\gamma}| ds \leq (C + n) \int_{\gamma} |\dot{\gamma}| ds \]
which gives the upper estimate to the distance distortion
\[ \text{dist}_{\omega(t_j)}(x, y) \leq e^{(C+n)\varepsilon_j} \text{dist}_{\omega(t_j')}(x, y) \leq \text{dist}_{\omega(t_j)}(x, y) + \sqrt{\varepsilon_j} \]
whenever \( j \) is large enough. Finally, \((3.27)\) shows that \( K \) is \( \delta \) dense in any \((M, \omega(t))\). Combining with the upper and lower distance variation estimate we get that the identity map defines an \( \delta \)-Gromov-Hausdorff approximation between \((M, \omega(t_j))\) and \((M, \omega(t_j'))\) whenever \( j \) is large enough such that \( \sqrt{\varepsilon_j} \leq \delta \). The claim is proved.

Proposition \(3.15\) follows directly from the Claim.

We end the paper with a brief discussion about the algebraic structure of \( M_{\infty} \). By Song’s work \([20]\), the limit space \( M_{\infty} \) coincides with the canonical model \( M_{can} \), so it is a normal projective variety. On the other hand, using the Kähler-Ricci flow, we can produce a natural isomorphism from \( M_{\infty} \) to \( M_{can} \) through holomorphic sections of \( K_M^\ell \) for some \( \ell \gg 1 \) such that \( K_M^\ell \) is base point free, and consequently, give an alternative proof of the above result by J. Song. This can be done by choosing a family of orthonormal basis of \( H^0(M, K_M^\ell) \) with respect to the Hermitian metric \( h(t) = \omega(t)^{-n\ell} \), say \( \{s_i, t\}_{i=0}^{N_\ell} \) where \( N_\ell = \dim H^0(M; K_M^\ell) - 1 \), which satisfies the ODE system
\[ \frac{\partial}{\partial t} s_{i, t} = \sum_j b_{ij}(t)s_{j, t}. \quad (3.30) \]
In order to preserve the orthonormal property we choose
\[ b_{ij} = \overline{b_{ji}} = -\frac{\ell - 1}{2} \int_M (R + n)\langle s_i, s_j \rangle h(t)^n. \quad (3.31) \]
Lemma 3.17. There exists $C = C(\omega_0, \ell)$ such that
\[ |b_{ij}| \leq Ce^{-t}. \]

Proof. First of all we notice that the section $s_{i,t}$ admits a uniform $L^\infty$ bound. This can be seen by the uniform equivalence of the Hermitian metric $h(t) = e^{-\ell u(t)}\Omega^{-t}$, the volume form $\omega(t)^n = e^{u(t)}\Omega$ and the $L^\infty$ estimate of holomorphic sections at time $t = 0$. Here we used the uniform $C^0$ estimate of $u (3.8)$. Then we can estimate the integration as in the proof of Lemma 3.7, by \( R + n \geq -Ce^{-t} \) and \( \int (R + n)\omega^n \leq Ce^{-t}, \)
\[ |b_{ij}| \leq C \int |R + n|\omega^n \leq C \int (R + n + Ce^{-t})\omega^n + Ce^{-t} \leq Ce^{-t} \]
where $C = C(\omega_0, \ell)$. \qed

Suppose $s_{i,t} = \sum_j a_{ij}(t)s_{j,0}$ and denote by $A(t) = (a_{ij}(t))$ a Hermitian matrix, then $A(t) = e^{\int_0^t B(s)ds}$ where $B(t) = (b_{ij}(t))$. Thus the sections $\{s_{i,t}\}_{i=0}^{N_\ell}$ converge to a set of holomorphic sections $\{s_{i,\infty}\}_{i=0}^{N_\ell}$ which forms another basis of $H^0(M; K_M^\ell)$. It induces a morphism
\[ \Phi : M \to M_{can} \subset \mathbb{C}P^{N_\ell}. \]
On the other hand, we also have a uniform gradient estimate to each $s = s_{i,0}$, cf. [26],
\[ |\nabla^h(t)s|_{h(t)\otimes\omega(t)} = |\nabla^{\Omega^{-t}}s + \partial u \otimes s|_{h(t)\otimes\omega(t)} \leq C|\nabla^{\Omega^{-t}}s|_{\Omega^{-t}\otimes\chi} + C|\partial u|_{\omega(t)}|s|_{\Omega^{-t}} \leq C \]
where we used the uniform $C^1$ estimate of $u$ and $\chi \leq C\omega(t)$, this leads to a convergence of $\{s_{i,t}\}_{i=0}^{N_\ell}$ under the Cheeger-Gromov convergence, so $\{s_{i,\infty}\}_{i=0}^{N_\ell}$ can also be seen as a set of holomorphic sections of $K_M^{\ell\infty}$. It is obvious that $\{s_{i,\infty}\}_{i=0}^{N_\ell}$ is base point free on $M_{\infty}$, so it defines a map
\[ \Phi_{\infty} : M_{\infty} \to \mathbb{C}P^{N_\ell}. \]
Finally, through construction of local peak sections, up to rising a power of $\ell$, the map $\Phi_{\infty}$ separates points of $M_{\infty}$, so it defines a homeomorphism.

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