A TORELLI THEOREM FOR MODULI SPACES OF PRINCIPAL BUNDLES ON CURVES DEFINED OVER $\mathbb{R}$

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Abstract. Let $X$ be a geometrically irreducible smooth projective curve, of genus at least three, defined over the field of real numbers. Let $G$ be a connected reductive affine algebraic group, defined over $\mathbb{R}$, such that $G$ is nonabelian and has one simple factor. We prove that the isomorphism class of the moduli space of principal $G$–bundles on $X$ determine uniquely the isomorphism class of $X$.

1. Introduction

Let $X$ be a geometrically irreducible smooth projective curve defined over the field of real numbers, of genus $g$, with $g \geq 3$. Let $L \in \text{Pic}^d(X)$ be a point defined over $\mathbb{R}$. We note that $L$ need not correspond to a line bundle over $X$. For example, the unique $\mathbb{R}$–point of $\text{Pic}^1$ of the anisotropic conic does not correspond to a line bundle over the anisotropic conic. Let $\mathcal{N}_X(r, L)$ denote the moduli space of semistable vector bundles on $X$ of rank $r$ and determinant $L$, where $r \geq 2$.

We prove that the isomorphism class of the variety $\mathcal{N}_X(r, L)$ uniquely determines the isomorphism class of the real curve $X$ (Theorem 2.1).

When the base field is complex numbers, this was proved in [MN] for rank two, and in [Tj], [KP, p. 229, Theorem E] for general $r$ and $d$.

Let $X_{\mathbb{C}}$ be the complexification of $X$. Let $G_{\mathbb{C}}$ be a connected reductive affine algebraic group defined over $\mathbb{C}$, and let $G$ be a real form of $G_{\mathbb{C}}$. We assume that $G_{\mathbb{C}}$ is nonabelian and it has exactly one simple factor. The anti-holomorphic involution of $G_{\mathbb{C}}$ corresponding to $G$ will be denoted by $\sigma_G$. Let $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$ denote the moduli space of topologically trivial semistable principal $G_{\mathbb{C}}$–bundles on $X_{\mathbb{C}}$. The variety $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$ is the complexification of the component of the moduli space of principal $G$–bundles on $X$ that contains the trivial $G$–bundle. The involution $\sigma_G$ and the anti-holomorphic involution of $X_{\mathbb{C}}$ together produce the anti-holomorphic involution $\sigma_M$ of $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$.

We prove that the isomorphism class of the real variety $(\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}}), \sigma_M)$ uniquely determines the isomorphism class of $X$ (Theorem 3.4).

The proof of Theorem 3.4 crucially uses a result of [BHo] which says that the isomorphism class of $\mathcal{M}_{X_{\mathbb{C}}}(G_{\mathbb{C}})$ uniquely determines the isomorphism class of $X_{\mathbb{C}}$.

2010 Mathematics Subject Classification. 14D20, 14P99, 14C34.
Key words and phrases. Curve over $\mathbb{R}$, principal bundle, moduli space, semistability, Torelli theorem.
2. Moduli spaces of vector bundles

Let $X$ be a geometrically irreducible smooth projective curve defined over $\mathbb{R}$. Let $g$ denote the genus of $X$. We will assume that $g \geq 3$. For any $d \in \mathbb{Z}$ and any integer $r \geq 2$, let $\mathcal{M}_X(r, d)$ be the moduli space of semistable vector bundles on $X$ of rank $r$ and degree $d$; see [BH], [BG], [BHH], [Sc1], [Sc2], [Sc3] for moduli spaces of bundles over $X$. Let

$$\det : \mathcal{M}_X(r, d) \to \text{Pic}^d(X)$$

be the morphism defined by $E \mapsto \bigwedge^r E$. Take any $\mathbb{R}$–point $\mathcal{L} \in \text{Pic}^d(X)$. Define

$$N_X(r, \mathcal{L}) := \det^{-1}(\mathcal{L}) \subset \mathcal{M}_X(r, d).$$

This $N_X(r, \mathcal{L})$ is a geometrically irreducible normal projective variety defined over $\mathbb{R}$, of dimension $(r^2 - 1)(g - 1)$.

Let $X_{\mathbb{C}} := X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$ be the complex projective curve obtained from $X$ by extending the base field to $\mathbb{C}$. Let $\mathcal{L}_{\mathbb{C}} \in \text{Pic}^d(X_{\mathbb{C}})$ be the pull-back of $\mathcal{L}$ to $X_{\mathbb{C}}$ by the natural morphism $\xi : X_{\mathbb{C}} \to X$. The nontrivial element of the Galois group $\text{Gal}(\xi) = \text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ produces an antiholomorphic involution

$$\sigma : X_{\mathbb{C}} \to X_{\mathbb{C}}.$$

The conjugate vector bundle of a holomorphic vector bundle $E$ on $X_{\mathbb{C}}$ will be denoted by $\overline{E}$. We recall that the underlying real vector bundle for $\overline{E}$ is identified with that of $E$, while the multiplication on $\overline{E}$ by any $c \in \mathbb{C}$ coincides with the multiplication by $\overline{c}$ on $E$. The $C^\infty$ vector bundle $\sigma^*\overline{E}$ has a natural holomorphic structure which is uniquely determined by the condition that the natural $\mathbb{R}$–linear identification of it with $E$ is anti-holomorphic. Note that we have a commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\sim} & \sigma^*\overline{E} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sigma} & X
\end{array}
$$

It is easy to see that $E$ is semistable (respectively, stable) if and only if $\sigma^*\overline{E}$ is semistable (respectively, stable). Similarly, $E$ is polystable if and only if $\sigma^*\overline{E}$ is polystable.

The above hypothesis that $\mathcal{L} \in \text{Pic}^d(X)$ means that the line bundle $\mathcal{L}_{\mathbb{C}}$ is holomorphically isomorphic to the line bundle $\sigma^*\overline{\mathcal{L}}_{\mathbb{C}}$.

Let $N_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}})$ be the moduli space of semistable vector bundles on $X_{\mathbb{C}}$ of rank $r$ and determinant $\mathcal{L}_{\mathbb{C}}$. The complex variety $N_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}})$ coincides with the complexification $N_X(r, \mathcal{L}) \times_{\mathbb{R}} \mathbb{C}$ of $N_X(r, \mathcal{L})$; the resulting antiholomorphic involution

$$\sigma_N : N_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}}) \to N_{X_{\mathbb{C}}}(r, \mathcal{L}_{\mathbb{C}})$$

sends a vector bundle $E$ on $X_{\mathbb{C}}$ to the vector bundle $\sigma^*\overline{E}$.

**Theorem 2.1.** The isomorphism class of the $\mathbb{R}$–variety $N_X(r, \mathcal{L})$ uniquely determines the isomorphism class of the real curve $X$. 
Proof. First note that the isomorphism class of the complex variety $N_{X_C}(r, L_C)$ uniquely
determines the complex curve $X_C$ \[\{7\}, \{KP\} \text{ p. 229, Theorem E}\]. We have to prove that
the antiholomorphic involution $\sigma_N$ determines $\sigma$.

Let $\tau$ be an antiholomorphic involution of $X_C$ such that the involution
$E \mapsto \tau^*E$ preserves $N_{X_C}(r, L_C)$. The resulting antiholomorphic involution of $N_{X_C}(r, L_C)$ will be
denoted by $\tau_N$. The two real varieties $(N_{X_C}(r, L_C), \tau_N)$ and $(N_{X_C}(r, L_C), \sigma_N)$ are isomorphic
if and only if there exists a complex algebraic automorphism $f$ of $N_{X_C}(r, L_C)$ such that

$$
\tau_N = f^{-1} \sigma_N f. \tag{2.2}
$$

Assume that the two real varieties $(N_{X_C}(r, L_C), \tau_N)$ and $(N_{X_C}(r, L_C), \sigma_N)$ are isomorphic.
Fix an automorphism $f$ of $N_{X_C}(r, L_C)$ satisfying \[\{2.2\}\].

The dual a vector bundle $E$ will be denoted by $E^\vee$; the dual of a line bundle $\nu$ will also
be denoted by $\nu^{-1}$.

Take any algebraic automorphism $h$ of $N_{X_C}(r, L_C)$. It follows from \[\{KP\} \text{ p. 228, Theorem B}\] and \[\{KP\} \text{ p. 228, remark 0.1}\] that $h$ is either of the form $E \mapsto H^*E \otimes \nu$ or $E \mapsto H^*E^\vee \otimes \nu_1$, where $H$ is an automorphism of $X_C$ uniquely determined by $h$ while
$\nu$ a line bundle on $X_C$ with $\nu^{\sigma r} = \mathcal{O}_X$ and $\nu_1$ a line bundle on $X_C$ with $\nu_1^{\sigma r} = L_C^{\sigma 2}$; it
should be clarified both $\nu$ and $\nu_1$ are independent of $E$. Therefore, we get a map

$$
\Psi : \text{Aut}(N_{X_C}(r, L_C)) \rightarrow \text{Aut}(X_C), \ h \mapsto H^{-1}. \tag{2.3}
$$

It is straight-forward to check that $\Psi$ is a homomorphism of groups.

We will denote $\Psi(f) \in \text{Aut}(X_C)$ by $\varphi$, where $\Psi$ is defined in \[\{2.3\}\] and $f$ is the
automorphism in \[\{2.2\}\]. First assume that

$$
f(V) = A \otimes \varphi^*V,
$$

where $A$ is a line bundle on $X_C$. Therefore, we have

$$
f^{-1}(V) = ((\varphi^{-1})^* A^{-1}) \otimes (\varphi^{-1})^* V.
$$

Hence the automorphism $\tau_N^{-1} \circ f^{-1} \circ \sigma_N \circ f$ of $N_{X_C}(r, L_C)$ is the morphism defined by

$$
V \mapsto \tau^*((\varphi^{-1})^* A^{-1}) \otimes (\varphi^{-1})^*((\sigma^* A) \otimes (\sigma^* \varphi^* V))
= B \otimes \tau^*(\varphi^{-1})^* \sigma^* \varphi^* V = B \otimes (\varphi \circ \sigma \circ \varphi^{-1} \circ \tau)^* V,
$$

where $B$ is a line bundle which does not depend on $V$. This implies that

$$
\eta := \Psi(\tau_N^{-1} \circ f^{-1} \circ \sigma_N \circ f) = \varphi \circ \sigma \circ \varphi^{-1} \circ \tau. \tag{2.4}
$$

Now from \[\{2.2\}\] we conclude that $\eta = \text{Id}_{X_C}$. So from \[\{2.4\}\] we have

$$
\tau = \varphi \circ \sigma \circ \varphi^{-1}.
$$

Therefore, $\varphi$ produces an isomorphism between the two curves $(X_C, \sigma)$ and $(X_C, \tau)$.

Next assume that

$$
f(V) = A \otimes \varphi^* V^\vee,
$$

where $A$ is a line bundle on $X_C$. Then

$$
f^{-1}(V) = ((\varphi^{-1})^* A) \otimes (\varphi^{-1})^* V^\vee.
$$
Therefore, the automorphism \( \tau_N^{-1} \circ f^{-1} \circ \sigma_N \circ f \) of \( \mathcal{N}_{X_C}(r, L) \) is the morphism defined by

\[
V \mapsto \tau^*((\varphi^{-1})^*A) \otimes ((\varphi^{-1})^*(\sigma^*A) \otimes (\sigma^*\varphi^*V)^\vee)) = B \otimes \tau^*(\varphi^{-1})^*A \otimes (\varphi^{-1})^*(\sigma^*\varphi^*V),
\]

where \( B \) is a line bundle which does not depend on \( V \). This implies that

\[
\Psi(\tau_N^{-1} \circ f^{-1} \circ \sigma_N \circ f) = \varphi \circ \sigma \circ \varphi^{-1} \circ \tau.
\]

Hence, as before, \( \tau = \varphi \circ \sigma \circ \varphi^{-1} \). This completes the proof of the theorem. \( \square \)

3. Moduli spaces of principal bundles

Let \( G_C \) be a connected nonabelian reductive group over \( C \) with only one simple factor and let

\[
\sigma_G : G_C \rightarrow G_C
\]

be an antiholomorphic automorphism of order two. We denote by \( G \) the real form of \( G_C \) corresponding to \( \sigma_G \).

Let \( \mathcal{M}_{X_C}(G_C) \) denote the moduli space of topologically trivial semistable principal \( G_C \)-bundles on \( X_C \). It is an irreducible normal projective variety defined over \( C \). For any holomorphic principal \( G_C \)-bundle \( E \) on \( X_C \), let

\[
\overline{E} = E(\sigma_G) = E \times^\sigma G_C \rightarrow X_C
\]

be the \( C^\infty \) principal \( G_C \)-bundle obtained by twisting the action of \( G_C \) using the homomorphism \( \sigma_G \). So the total space of \( \overline{E} \) is identified with that of \( E \), but the action of any \( y \in G_C \) on \( \overline{E} \) is the action of \( \sigma_G(y) \) on \( E \) in terms of the identification of \( E \) with \( \overline{E} \). The pullback \( \sigma^*\overline{E} \) has a holomorphic structure uniquely determined by the condition that the above identification between the total spaces of \( E \) and \( \sigma^*\overline{E} \) is anti-holomorphic; since the total spaces of \( E \) and \( \sigma^*\overline{E} \) are naturally identified, the above identification between the total spaces of \( E \) and \( \overline{E} \) produces an identification of the total spaces of \( E \) and \( \sigma^*\overline{E} \). The complex projective variety \( \mathcal{M}_{X_C}(G_C) \) carries a real structure associated to the antiholomorphic involution

\[
\sigma_M : \mathcal{M}_{X_C}(G_C) \rightarrow \mathcal{M}_{X_C}(G_C), \ E \mapsto \sigma^*\overline{E}.
\]

Let \( \mathcal{M}_X(G) \) denote the variety over \( \mathbb{R} \) defined by the above pair \( (\mathcal{M}_{G_C}(X_C), \sigma_M) \).

A Zariski closed connected subgroup \( P \subset G_C \) is called a parabolic subgroup if \( G_C/P \) is a complete variety. A Levi subgroup of \( P \) is a maximal connected reductive subgroup of \( P \) containing a maximal torus. Any two Levi subgroups of \( P \) are conjugate by some element of \( P \). A proper parabolic subgroup \( P \subset G_C \) is called maximal if there is no proper parabolic subgroup of \( G_C \) containing \( P \).

**Lemma 3.1.** There exists a maximal parabolic subgroup \( P \subset G_C \) and a Levi subgroup \( L \subset P \), such that the two subgroups \( \sigma_G(L) \) and \( L \) are conjugate by some element of \( G_C \).
Proof. For any parabolic subgroup $P \subset G_C$, the image $\sigma_G(P)$ is also a parabolic subgroup of $G_C$. Since $\sigma_G(y^{-1}Py) = \sigma_G(y)^{-1}\sigma_G(P)\sigma_G(y)$, we get a self-map of the conjugacy classes of parabolic subgroups of $G_C$ that sends the conjugacy class of any $P$ to the conjugacy class of $\sigma_G(P)$. Therefore, the involution $\sigma_G$ also acts on the Dynkin diagram $D$ of $G_C$ as an involution $\tau$. Examining the Dynkin diagrams we observe that an involution of the Dynkin diagram of $G_C$ must have a fixed point unless $G_C$ is of type $A_n$ for $n$ even.

If $G_C$ is not of type $A_n$, let $P$ be a maximal parabolic subgroup corresponding to a vertex of $D$ fixed by the above constructed involution $\tau$. Then $P$ and $\sigma_G(P)$ are conjugate in $G_C$. Let $y \in G_C$ be such that $\sigma_G(P) = y^{-1}Py$. Then for any Levi subgroup $L$ of $P$,

$$y^{-1}Ly \subset y^{-1}Py = \sigma_G(P)$$

is a Levi subgroup of $\sigma_G(P)$.

If $G_C$ is of type $A_n$, then $\sigma_G(L)$ and $L$ are conjugate for every Levi subgroup of every maximal parabolic subgroup of $G_C$. It is enough to check this for $G_C = \text{SL}(n+1, \mathbb{C})$, in which case this is obvious. □

Remark 3.2. We can be more precise as follows. The two subgroups $\sigma_G(L)$ and $L$ are conjugate for every Levi subgroup of every maximal parabolic subgroup of $G_C$. It is enough to check this for $G_C = \text{SL}(n+1, \mathbb{C})$, in which case this is obvious.

Lemma 3.3. Let $L$ be any Levi subgroup of a parabolic subgroup $P$ of $G_C$, and let

$$L' = [L, L]$$

be its derived subgroup. Then the homomorphism

$$\pi_1(L') \rightarrow \pi_1(G_C)$$

induced by the inclusion $L' \hookrightarrow G_C$ is injective.

Proof. Consider the fibration $L' \rightarrow G_C \rightarrow G_C/L'$. Let

$$\pi_2(G_C) \rightarrow 0 \rightarrow \pi_2(G_C/L') \rightarrow \pi_1(L') \rightarrow \pi_1(G_C)$$

be the long exact sequence of homotopy groups associated to it. From (3.2) we conclude that the homomorphism in (3.1) is injective if

$$\pi_2(G_C/L') = 0.$$  

(3.3)

Since $\pi_2(G_C) = 0$ and $\pi_1(L')$ is a finite group (recall that $L'$ is semisimple), from (3.2) it follows that $\pi_2(G_C/L')$ is a finite group.

Now consider the fibration $P/L' \rightarrow G_C/L' \rightarrow G_C/P$. Let

$$\pi_2(P/L') \rightarrow \pi_2(G_C/L') \rightarrow \pi_2(G_C/P) \rightarrow \pi_1(G_C)$$

be the long exact sequence of homotopy groups associated to it. Since $G_C/P$ is simply connected, the second homotopy group $\pi_2(G_C/P)$ is isomorphic to $H_2(G_C/P, \mathbb{Z})$, which is a free abelian group. Therefore, there is no nonzero homomorphism from the finite group $\pi_2(G_C/L')$ to $\pi_2(G_C/P)$. Hence, the homomorphism

$$\pi_2(P/L') \rightarrow \pi_2(G_C/L')$$

(3.5)
in (3.4) is surjective.

Finally, consider the long exact sequence of homotopy groups

$$\pi_2(L/L') \to \pi_2(P/L') \to \pi_2(P/L)$$

(3.6)

associated to the fibration

$$L/L' \to P/L' \to P/L.$$ 

Since $P/L$ is diffeomorphic to the unipotent radical of $P$, which is contractible, we have $\pi_2(P/L) = 0$. Also, $\pi_2(L/L') = 0$ because $L/L'$ is a Lie group. Hence from (3.6) it follows that $\pi_2(P/L') = 0$. This implies that (3.3) holds because the homomorphism in (3.5) is surjective. □

**Theorem 3.4.** The real variety $M_X(G)$ uniquely determines the real curve $X$.

**Proof.** We already know that the complex variety $M_{X_C}(G_C) = M_X(G) \times_R \mathbb{C}$ determines the complex curve $X_C$ [BHo].

Let $\text{Sing}(M_{X_C}(G_C))$ denote the singular locus of the variety $M_{X_C}(G_C)$. Recall from [BHo] that the strictly semi–stable locus $\Delta_G \subset M_{X_C}(G_C)$ is the Zariski closure of the set of closed points $[E] \in \text{Sing}(M_{X_C}(G_C))$ with the property that every Euclidean neighborhood $U$ of $[E]$ contains an open neighborhood $U' \ni [E]$ such that $U' \setminus (U' \cap \text{Sing}(M_{X_C}(G_C)))$ is connected and simply connected. Moreover, this closed subset $\Delta_G$ is the union of irreducible components corresponding to the conjugacy classes of Levi subgroups of maximal parabolic subgroups of $G_C$. More precisely, the decomposition of $\Delta_G$ into irreducible components is the union

$$\Delta_G = \bigcup_L M_L$$

where $L$ ranges over conjugacy classes of Levi subgroups of maximal parabolic subgroups of $G_C$, and $M_L$ is the image of the morphism $M_{X_C}(L) \to M_{X_C}(G_C)$ given by the inclusion of $L$ in $G_C$. This can be deduced from [BHo, Proposition 3.1] and Lemma 3.3. Indeed, every closed point in $\Delta_G$ is defined by a principal $G_C$–bundle $E$ admitting a reduction of structure group $E_L$ to a Levi subgroup $L$ of a maximal parabolic subgroup of $G_C$. Moreover, this $L$–bundle $E_L$ is semistable and its topological type $\delta \in \pi_1(L)$ is torsion, which means that $\delta$ belongs to $\pi_1([L, L])$; this is because $\pi_1(L/[L, L])$ is free abelian. Now, since $E$ is topologically trivial, $\delta$ must be trivial (follows from Lemma 3.3), i.e., $[E] \in M_{X_C}(G_C)$ belongs to $M_L$.

Moreover, $M_L$ is never empty, since there always exist semi-stable principal $L$–bundles which are topologically trivial, for example the trivial holomorphic principal $L$–bundle. The fact that the subvarieties $M_{L_1}$ and $M_{L_2}$ of $M_{X_C}(G_C)$ are distinct when $L_1$ and $L_2$ are not conjugate by some element of $G_C$ is contained in the last part of the proof of [BHo, Proposition 3.1].
The antiholomorphic involution $\sigma_M$ maps the strictly semi–stable locus $\Delta_G$ into itself, permuting its irreducible components. It follows from Lemma 3.1 that there exists at least one component $M_L$ which is fixed by $\sigma_M$. The restriction of $\sigma_M$ to this component $M_L$ has at least one fixed point, namely the closed point corresponding to the trivial bundle.

We now proceed as in the proof of [BHo, Theorem 4.1] to recover the involution $\sigma$ defining the real curve $X$.

First, one can assume $G_C$ to be semi–simple. To see this, let $Z^0_{G_C}$ be the connected component of the center of $G_C$ containing the identity element. Let us denote by $G'$ the quotient $G_C/Z^0_{G_C}$ of $G_C$. Note that $\sigma_G$ preserves $Z^0_{G_C}$, so it produces a real structure on the quotient $G'$. The canonical line bundle of $\calm_{X_C}(G_C)$ (respectively, $\calm_{X_C}(G')$) will be denoted by $\omega_{\calm_{X_C}(G_C)}$ (respectively, $\omega_{\calm_{X_C}(G')}$. We note that $\omega_{\calm_{X_C}(G')}$ pulls back to $\omega_{\calm_{X_C}(G_C)}$ under the morphism $\calm_{X_C}(G_C) \to \calm_{X_C}(G')$ given by the quotient map $G_C \to G'$. There exists an integer $m$ such that the pluri–anti–canonical system $|-m\omega_{\calm_{X_C}(G_C)}|$ factors into the natural map $\calm_{X_C}(G_C) \to \calm_{X_C}(G')$ followed by the embedding

$$\calm_{X_C}(G') \hookrightarrow |-m\omega_{\calm_{X_C}(G_C)}|^* = |-m\omega_{\calm_{X_C}(G')}|^*.$$  

Since the dualizing sheaves are defined over the reals, we have real structures on $|-m\omega_{\calm_{X_C}(G_C)}|^*$ and $|-m\omega_{\calm_{X_C}(G')}|^*$. All the maps above are defined over $\mathbb{R}$. Therefore, it is enough to prove the theorem for $G'$.

So let us assume that $G_C$ is semi–simple. We have seen above that $\Delta_G \subset \calm_{X_C}(G_C)$ contains at least one irreducible component fixed by $\sigma_M$, which is equal to the variety $M_L$ associated to a Levi subgroup $L$ of a maximal parabolic subgroup of $G_C$. Let

$$\alpha: \tilde{M}_L \to M_L$$

be the normalization of $M_L$, and let $\sigma_L$ be the antiholomorphic involution of $M_L$. Since normalization commutes with the base change of field of definition, the variety $\tilde{M}_L$ is also defined over $\mathbb{R}$, and the morphism $\alpha$ is also defined over $\mathbb{R}$. Hence the antiholomorphic involution $\sigma_L$ of $M_L$ lifts to $\tilde{M}_L$. Moreover, $\tilde{M}_L$ is isomorphic to the quotient $\calm_{X_C}(L)/\Gamma_L$, where $\Gamma_L$ is the image of $N_{G_C}(L)$ in $\text{Out}(L)$, which is either trivial or $\mathbb{Z}/2\mathbb{Z}$ (see [BHo]), and this quotient map is compatible with the real structures on $\calm_{X_C}(L)$ and $\tilde{M}_L$.

Let $Z^0_L$ be the connected component of the center of $L$ containing the identity element. Let us denote by $L'$ the quotient $L/Z^0_L$. Then the above group $\Gamma_L$ also acts on $\calm_{X_C}(L')$, and the morphism (defined over the real numbers)

$$\theta: \tilde{M}_L \simeq \calm_{X_C}(L)/\Gamma_L \to \calm_{X_C}(L')/\Gamma_L \quad (3.7)$$

can be recovered from the second tensor power of the canonical line bundle on the smooth locus of $\tilde{M}_L$. Indeed, this second tensor power extends to a line bundle on the whole variety, and a sufficiently negative power of it gives the morphism $\theta$ (see [BHo]).

Let $\beta: \calm_{X_C}(L') \to \calm_{X_C}(L')/\Gamma_L$ be the quotient map. Consider $\theta$ in (3.7). For any point $y \in \calm_{X_C}(L')/\Gamma_L$, the fiber $\theta^{-1}(y)$ is $J_{X_C}$ (respectively, $J_{X_C}/(\mathbb{Z}/2\mathbb{Z})$ if $\#\beta^{-1}(y) = 2$ (respectively, $\#\beta^{-1}(y) = 1$).
Now take any smooth point
\[ y \in \mathcal{M}_{X_C}(L')/\Gamma_L \]
fixed by the antiholomorphic involution. As noted above, the fiber \( \theta^{-1}(y) \) is isomorphic to either \( J_{X_C} \) or the singular Kummer variety \( J_{X_C}/(\mathbb{Z}/2\mathbb{Z}) \). The real structure on \( \theta^{-1}(y) \) induced by that of \( \tilde{M}_L \) comes from the real structure on the Jacobian associated to the curve. So in both cases we recover the Jacobian variety together with its natural real structure: when \( \theta^{-1}(y) \) is isomorphic to \( J_{X_C}/(\mathbb{Z}/2\mathbb{Z}) \), then \( J_X \) is obtained from the two-sheeted cover of the smooth locus of the Kummer variety defined by the unique maximal torsion-free subgroup in its fundamental group. The antiholomorphic involution can be lifted to this cover, and this lift extends to \( J_{X_C} \) because its construction is over \( \mathbb{R} \).

Finally, the class of the canonical principal polarization on \( J_{X_C} \) is determined as in [BHo]. Now the theorem follows from the real analog of Torelli theorem [GH, Theorem 9.4].

\[ \square \]

Acknowledgements

We thank the referee for helpful comments. The first author acknowledges support of a J. C. Bose Fellowship.

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