Binary CEO Problem under Log-Loss with BSC Test-Channel Model

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Abstract—In this paper, we propose an efficient coding scheme for the two-link binary Chief Executive Officer (CEO) problem under logarithmic loss criterion. The exact rate-distortion bound for a two-link binary CEO problem under the logarithmic loss has been obtained by Courtade and Weissman. We propose an encoding scheme based on compound LDGM-LDPC codes to achieve the theoretical bounds. In the proposed encoding, a binary quantizer using LDGM codes and a syndrome-coding employing LDPC codes are applied. An iterative joint decoding is also designed as a fusion center. The proposed CEO decoder is based on the sum-product algorithm and a soft estimator.

I. INTRODUCTION

The Chief Executive Officer (CEO) problem is defined by Berger et al. for distributed source coding of multi-observations of a source corrupted by independent noises [1]. The CEO problem empirically emerges in wireless sensor networks, where a particular phenomenon is measured by some separate and independent sensors in a noisy environment. By using the compressed observations, a fusion center makes an estimation of the source at the receiver with an acceptable distortion between the original and the estimated source. In this paper, a soft reconstruction is considered.

In the last two decades, an abundant flurry of studies has been paid to address the theoretical bounds of the quadratic Gaussian CEO problem [2], [3]. A tight upper bound on the sum-rate distortion function of the quadratic Gaussian CEO problem and the optimal rate allocation scheme are provided in [4]. Alternatively, studies like [5] and [6] present various coding schemes to achieve any point of the achievable rate-distortion region.

The case of a binary source with observations corrupted by the binary noises, called the binary CEO problem, has been paid less attention during these years. In general, its exact rate-distortion bound is an open problem in the information theory. The binary CEO problem appears in cooperative digital communication networks where some correlated remote sources are being sent to a central receiver via parallel channels with independent noises. The most common criterion for measuring distortion in the binary case is the Hamming distortion measure. A lower bound for the achievable rate-distortion region of a two-link binary CEO problem is established in [7] under the Hamming distortion. The Berger-Tung inner and outer bounds [8], are exploited for this case which are not tight under the Hamming distortion. The prior studies on the binary CEO in [7] and [9] consider that the correlated observations are transmitted through AWGN channels, and hence their encoders apply a channel coding to protect the transmitted data. Our goal is to achieve the maximum compression of the correlated noisy observations for sending through noiseless channels with minimum distortion.

Due to increasing demand for developing deep learning in upcoming complex networks, the logarithmic loss, or simply log-loss, has emerged as a useful criterion to measure distortion in many applications like machine learning, classification, and estimation theory. In this paper, we focus on the binary CEO problem under the log-loss criterion. This loss has been interpreted as the conditional entropy and the estimated symbols of the fusion center are soft. Moreover, it has been also shown that the log-loss is a universal criterion for measuring the performance of lossy source coding in [10] and [11].

Our main contributions in this paper can be considered in the context of coding and information theory. We assume a binary symmetric channel (BSC) as a test-channel for the lossy encoders in the binary CEO problem. Then, we obtain the optimal values of crossover probabilities of the test-channels for each BSC. Finally, an efficient coding scheme is proposed by utilizing the compound LDGM-LDPC codes and iterative message-passing algorithms. We show that the rate-distortion performance of the proposed coding scheme is close to the theoretical bounds. A full version of this paper including proofs and details are provided in [12].

The organization of this paper is as follows. In Section II, the system model and information theoretic aspect of the problem are provided. The designed encoding and decoding schemes are presented in Section III. Numerical results and discussions are given in Section IV. Finally, Section V draws the conclusion.

II. PRELIMINARIES

Throughout this paper, the logarithm is to base 2. Random variables, their realizations, and their alphabet set are depicted by the uppercase, lowercase, and calligraphic letters, respectively.

A. System model

Consider a communication system consisting of an independent and identically distributed (i.i.d.) binary symmetric source (BSS) and its two noisy observations being transmitted via two parallel links as depicted in Fig. 1.
where $\hat{x}_j(x_j)$ generally depends on $(c_1,c_2)$. The total value of log-loss between $x^n$ and $\hat{x}^n$ is obtained by uniform averaging over all symbols,

$$d(x^n, \hat{x}^n) = \frac{1}{n} \sum_{j=1}^{n} \log \left( \frac{1}{\hat{x}_j(x_j)} \right).$$  \hspace{1cm} (2)

**B. An Information Theory Perspective**

The rate-distortion theory objective implies to minimize the rate (distortion) subject to a fixed distortion (rate) value. By using the provided theoretical bounds in [13], the achievable rate-distortion region under the log-loss criterion for encoded sequences $U_1^n$ and $U_2^n$ is as follows:

$$R_i \geq I(Y_i;U_i|U_{i-1},Q), \quad i = 1, 2, \hspace{1cm} (3)$$

$$R_1 + R_2 \geq I(Y_1,Y_2;U_1,U_2|Q), \hspace{1cm}$$

$$D \geq H(X|U_1,U_2,Q).$$

The cardinality bound implies that $|U_i| \leq |Y_i| = 2$ for $i = 1, 2$ and $|Q| \leq 4$. The dominant face of the achievable rate region is the sum-rate bound, hence the mentioned optimization problem of the rate-distortion theory is as follows:

$$\min_{p(u_1|y_1,q)p(u_2|y_2,q)p(q)} I(U_1,U_2;Y_1,Y_2|Q), \hspace{1cm} \text{s.t.} \ H(X|U_1,U_2,Q) = D_0,$$  \hspace{1cm} (4)

where $H(X|Y_1,Y_2) \leq D_0 \leq 1$. This optimization problem can be written in the following unconstrained form:

$$\min_{p(u_1|y_1,q)p(u_2|y_2,q)p(q)} H(X|U_1,U_2,Q) + \mu I(U_1,U_2;Y_1,Y_2|Q),$$  \hspace{1cm} (5)

where $\mu$ is the Lagrangian multiplier. Note that

$$H(X|U_1,U_2,Q) + \mu I(U_1,U_2;Y_1,Y_2|Q) = \sum_{q\in Q} p(q)[H(X|U_1,U_2,Q = q) + \mu I(U_1,U_2;Y_1,Y_2|Q = q)] \geq \min_{q\in Q} H(X|U_1,U_2,Q = q) + \mu I(U_1,U_2;Y_1,Y_2|Q = q).$$

Therefore, for the purpose of characterizing the sum-rate-distortion function, there is no loss of generality in assuming that $Q$ is a constant, which eliminates $Q$ in (5).

We shall assume that $p(u_i|y_i)$ is a BSC with crossover probability $d_i$, $i = 1, 2$, which is satisfied by the extensive numerical solution to (5). Consequently, after some calculus manipulations, the rate and the distortion bounds are expressed by:

$$R_i \geq h_b(p \ast d) - h_b(d_i), \quad i = 1, 2,$$  \hspace{1cm} (7)

$$R \triangleq R_1 + R_2 \geq 1 + h_b(d \ast p) - h_b(d_1) - h_b(d_2),$$

$$D \geq h_b(p_1 \ast d_1) + h_b(p_2 \ast d_2) - h_b(p \ast d),$$

where $d \triangleq d_1 \ast d_2 = d_1(1 - d_2) + d_2(1 - d_1),$ $p \triangleq p_1 \ast p_2,$ and $h_b(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$ is the binary entropy function.

**Theorem 1.** Bounds of distortion $D$ and sum-rate $R$ in (7) are neither convex nor concave in terms of variables $(d_1,d_2)$.

**Theorem 2.** The maximum value of $\mu$ occurs in $(R,D) = (0,1)$ when $(d_1^*,d_2^*) = (0.5,0.5)$ and it equals:

$$\mu_{\text{max}} = \max\{(1 - 2p_1)^2, (1 - 2p_2)^2\}. \hspace{1cm} (8)$$

Obviously, its minimum value equals 0 and occurs in $(R,D) = (1 + h_b(p), h_b(p_1) + h_b(p_2) - h_b(p))$ when $(d_1^*,d_2^*) = \ldots$
Fig. 2: Location of the optimum points \((d_1^*, d_2^*)\).

\((0, 0)\). If \((d_1^*, d_2^*) \to (0, 0)\), then the location of optimum points follow from an explicit relation.

**Theorem 3.** For the optimization problem with BSC test-channel models, if \(d_1 \to 0\) and \(d_2 \to 0\), then:

\[
d_2 \approx e^{-\frac{K(k_2-k)}{K(k_1-k)}} d_1^{k_1-k}. \tag{9}
\]

**Remark 1.** The slope of the tangent lines to the curve of location of optimum points in \((d_1^*, d_2^*) = (0, 0)\) are, respectively, 1, \(\infty\), or 0 when \(p_1 = p_2\), \(p_1 < p_2\), or \(p_1 > p_2\).

## III. The Proposed Coding Scheme

In this section, a practical coding scheme is proposed to achieve the rate-distortion bound of the binary CEO problem under BSC assumption for the encoders in each link. Actually, the encoders 1 and 2 are respectively modeled by BSCs with the crossover probabilities \(d_1\) and \(d_2\). In our proposed coding scheme, the Bias-Propagation (BiP) algorithm [14] is utilized for the binary quantization, and the Sum-Product (SP) algorithm and its modified version are employed for the syndrome-decoding.

### A. The Encoding Scheme

A conventional rate-distortion quantizer can asymptotically achieve the compression rate \(1 - h_b(d_i)\) when distortion is assumed to be \(d_i\); hence, it is impossible to get close to (7) only by using a rate-distortion quantizer. Therefore, another lossless compressor should be employed after the quantizer for achieving the rate of (7). We use an LDGM quantizer concatenated with a Syndrome-Generator (SG), inspired by the “quantize-and-bin” idea in the information theory.

For the two-link binary CEO problem, a binary Wyner-Ziv coding is considered in each link. Our proposed encoding includes two steps; first is mapping the observations \(Y_1\) and \(Y_2\) to the nearest codewords \(U_1\) and \(U_2\) in terms of the Hamming distance from two LDGM codebooks 1 and 2, respectively. This step is implemented by utilizing the BiP algorithm. In this step, the rate and distortion in \(i\)-th link are, respectively, denoted by \(R_{i,1}^* = \frac{m_i}{N_i}\) and \(d_{i,1}\), \(i = 1, 2\). In the second step, syndromes of the codewords \(U_i\) are generated by using the LDPC codes of rates \(R_{i,2} = \frac{m_i - k_i}{n_i}\), \(i = 1, 2\), and their non-zero entries are sent to the decoder as the lossy encoded sequences \(S_1\) and \(S_2\). These two steps are executed by using the compound LDGM-LDPC codes [14]. A compound LDGM-LDPC code in the \(i\)-th link includes nested LDGM and LDPC codes with the following parity-check matrices:

\[
H_{LDPC}^{(i)} = \left[ \begin{array}{c} H_{LDGM}^{(i)} \\ \Delta H^{(i)} \end{array} \right], \tag{10}
\]

where \(H_{LDPC}^{(i)}\) and \(H_{LDGM}^{(i)}\) are, respectively, parity-check matrices of the LDPC and LDGM codes. Let assume their sizes are \((n - m_i + k_i) \times n\) and \((n - m_i) \times n\), respectively. Hence, the total rate of the \(i\)-th link is equal to \(R_i = R_{i,1} - R_{i,2} = \frac{m_i}{N_i}\).

### B. The Joint CEO Decoder

In the decoder, we propose a Joint Sum-Product (JSP) algorithm which is a modified version of the SP algorithm. In this algorithm, the received syndromes \(s_{1}^{k_1}\) and \(s_{2}^{k_2}\) are located in the check nodes of the LDPC codes. The JSP includes \(r\) rounds and each round includes \(l\) iterations. At the starting point of the JSP, initial LLRs in the variable nodes are set based on a random side information in each SP. At the end of each round, which includes update equations in the check and the variable nodes, the bit values are calculated according to the maximum likelihood (ML) decision rule of the SP algorithm. In the next round, these updated bit values in the variable nodes are used as a new side information for calculating successive initial LLR values. Finally, after \(r\) rounds, \(u_1^*\) and \(u_2^*\) are decoded based on the ML decision rule of the SP algorithm in the variable nodes.

Consider the BER of the syndrome-decoding part is \(d_{i,2}\), for \(i = 1, 2\). Hence, the total distortion in each link equals \(d_i = d_{i,1} + d_{i,2}\). An EXIT chart analysis is presented in [15] for a similar JSP decoder which shows the capacity approaching property with two parallel and collaborative SP decoders. The soft estimation \(\hat{x}_j = \Pr\{x_j|u_{1,j}, u_{2,j}\}\) accomplishes the decoding process. The block diagram of the proposed coding scheme is shown in Fig. 3.

### C. A Practical Analysis for the Proposed Coding Scheme

Some coding parameters are affected by the information theoretical limits, each of which should be considered in the code designing. In the following, any \(\epsilon\) denotes a sufficiently small positive value. Assume that achieving the following intermediate point in the dominant face of the achievable rate region is desired,

\[
(R_1^*, R_2^*) = (h_b(p \ast d) - h_b(d_1) + \delta, 1 - h_b(d_2) - \delta), \tag{11}
\]

where \(0 < \delta < 1 - h_b(p \ast d)\). In the proposed coding scheme for achieving (11), the relation between the rate-distortion
The above approximation expresses that achieving the empirical distortion decreases in a range between 0 and 0.02 to 0.03 for the noise parameters \( (p_1, p_2) = (0.15, 0.15) \). In the BiP algorithm, the parameters \( t = 0.8, \gamma_i \approx 2R_{i,1} \) are selected for \( i = 1, 2 \). Maximum number of iterations in each round of this algorithm is set to be 25. In a single SP algorithm, maximum number of iterations is set to be 100. Also, in the JSP algorithm, \( r = 15 \) and \( l = 40 \). All of the reported values for the empirical distortions are averaged over 50 runs. Parameters of the employed codes and their results are presented in Table I. The first and the second parts of this table are dedicated to the intermediate and the corner points of the sum-rate bound, respectively. The gap value is equal to difference between the empirical distortion \( D_{em} \) and the theoretical distortion \( D_{th} \), and it evaluates performance of the designed codes. For this case, the gap values is about 0.03 to 0.06 for the block length 10^4, as indicated in Table I. It is obvious that by increasing the target distortions, the gap values increase. As the block length is set to 10^5, the gap value decreases in a range between 0.02 to 0.03. Performance of the sum-rate versus distortion is depicted in Fig. 4 for the proposed coding scheme. The simulation results confirm that performance of the sum-rate in terms of distortion is very close to the theoretical bounds for the empirically achieved points. The theoretical bounds are 1.00 to 1.02 for the noise parameters \( (p_1, p_2) = (0.15, 0.15) \).
are asymptotically achievable by employing the proposed coding scheme as well.

V. Conclusion

In this paper, we investigated the two-link binary CEO problem under the log-loss, where the exact achievable rate-distortion region of the two-link binary CEO problem is known. By assuming the BSC test-channel model for the encoders, we found optimal values of the crossover probabilities. Next, we proposed a practical coding scheme based on the graph-based codes and message-passing algorithms. In the encoding side, a binary quantization and a syndrome-generation are utilized in each link for construction of the lossy compressed sequences. This was realized by the compound LDGM-LDPC codes. In the decoding, the SP algorithms based on the optimized LDPC codes for the BSCs were employed at the first step. Then, a soft decoder calculated the final reconstruction value in the form of a probability distribution. Our experimental simulation results confirmed that the proposed coding scheme asymptotically achieves the theoretical bound of the two-link binary CEO problem under the log-loss. For a finite block length, there remains a slight gap between the rate-distortion of the proposed scheme and the associated theoretical bound.

Table 1: Numerical results of the proposed encoding and decoding methods.

| (p1, p2) | Region | n | m1, m2 | k1, k2 | d_{1,1}, d_{1,2} | d_{2,1}, d_{2,2} | p | R_{th} | D_{th} | D_{exp} | Gap |
|----------|--------|---|-------|-------|----------------|----------------|---|--------|--------|--------|-----|
| (0.15, 0.15) | 1 | 10^4 | 5400, 5400 | 5100, 5100 | 0.0114, 0.0115 | 0.0114, 0.0115 | 10^5 | 0.0114 | 0.0115 | 0.0114 | 0.0114 |
| (0.15, 0.15) | 2 | 10^4 | 5400, 5400 | 5100, 5100 | 0.0128, 0.0171 | 0.0128, 0.0171 | 10^3 | 0.0128 | 0.0171 | 0.0128 | 0.0171 |
| (0.15, 0.15) | 3 | 10^4 | 5400, 5400 | 5100, 5100 | 0.1003, 0.1043 | 0.1003, 0.1043 | 10^2 | 0.1003 | 0.1043 | 0.1003 | 0.1043 |
| (0.15, 0.15) | 4 | 10^4 | 5400, 5400 | 5100, 5100 | 0.3018, 0.3034 | 0.3018, 0.3034 | 10^1 | 0.3018 | 0.3034 | 0.3018 | 0.3034 |

Fig. 4: Performance of the sum-rate versus distortion for the proposed coding scheme.

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