On Cui-Kano’s Characterization Problem on Graph Factors

Hongliang Lu\(^1\) and David G. L. Wang\(^2\)

\(^1\)SCHOOL OF MATHEMATICS AND STATISTICS
XIAN JIAOTONG UNIVERSITY
XIAN 710049, P. R. CHINA
E-mail: luhongliang@mail.xjtu.edu.cn

\(^2\)BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH
PEKING UNIVERSITY
BEIJING 100871, P. R. CHINA
E-mail: wgl@math.pku.edu.cn

Received September 27, 2011; Revised August 25, 2012

Published online 2 January 2013 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jgt.21712

Abstract: An \(H_n\)-factor of a graph \(G\) is defined to be a spanning subgraph \(F\) of \(G\) such that each vertex has a degree belonging to the set \{1, 3, 5, \ldots, 2n - 1, 2n\} in \(F\), where \(n \geq 2\). In this article, we investigate \(H_n\)-factors of graphs by using Lovász’s structural descriptions to the degree prescribed subgraph problem. We find some sufficient conditions for the existence of an \(H_n\)-factor of a graph. In particular, we make progress on the characterization problem for a special family of graphs proposed by Cui and Kano in 1988.

\(\copyright\) 2013 Wiley Periodicals, Inc. J. Graph Theory 74: 335–343, 2013

Keywords: Lovász’s structural description; \(H_n\)-factor; \(H_n\)-decomposition

2010 AMS Classification: 05C75

1. INTRODUCTION

Let \(G = (V(G), E(G))\) be a simple graph. Let \(f, g : V(G) \rightarrow \mathbb{Z}\) be two functions with \(g(v) \leq f(v)\) for all vertices \(v\). A \((g, f)\)-factor of \(G\) is a spanning subgraph \(F\) such...
that

\[ g(v) \leq d_F(v) \leq f(v) \quad \text{for every } v \in V(G), \]

where \( d_F(v) \) denotes the degree of \( v \) in \( F \). In particular, if there exist integers \( a \) and \( b \) such that \( g(v) = a \) and \( f(v) = b \) for all vertices \( v \), then the \((g, f)\)-factor is called an \([a, b]\)-factor. There is a large amount of literature on graph factors, see Plummer [12], Liu and Yu [14], and Akiyama and Kano [1] for surveys. For connected factors, we refer the reader to Kouider and Vestergaard [7].

Let \( H \) be a function associating a subset of \( \mathbb{Z} \) to each vertex of \( G \). It is natural to generalize \((g, f)\)-factors to \( H \)-factors, which are spanning subgraphs \( F \) such that

\[ d_F(v) \in H(v) \quad \text{for every } v \in V(G). \tag{1} \]

Let \( F \) be a spanning subgraph of \( G \). Following Lovász [10], one may measure the “deviation” of \( F \) from the condition (1) by

\[ \delta_H(F) = \sum_{v \in V(G)} \min\{|d_F(v) - h| : h \in H(v)\}. \tag{2} \]

Moreover, the “solvability” of (1) can be characterized by

\[ \delta(H) = \min \{ \delta_H(F) : F \text{ is a spanning subgraph of } G \}. \]

The subgraph \( F \) is said to be \( H \)-optimal if \( \delta_H(F) = \delta(H) \). It is clear that \( F \) is an \( H \)-factor if and only if \( \delta_H(F) = 0 \), and any \( H \)-factor (if exists) is \( H \)-optimal.

In [9], Lovász proposed the problem of determining the value of \( \delta(H) \), called the degree prescribed subgraph problem. Let

\[ H(v) = \{ h_1, h_2, \ldots, h_m \} \quad \text{for all } v \in V(G), \]

where \( h_1 < h_2 < \cdots < h_m \). It is said to be allowed (following [10]) if each of the gaps of \( H(v) \) has at most one integer, i.e.,

\[ h_{i+1} - h_i \leq 2 \quad \text{for all } 1 \leq i \leq m - 1. \]

In [10], Lovász built up a whole theory to the degree prescribed subgraph problem in case that \( H \) is allowed, and showed that the problem is NP-complete if \( H \) is not restricted to be allowed. As given in the preface of the famous book [11] of Lováasz, “a property \( T \) of graphs is in NP if we are able to efficiently prove (exhibit) \( T \) if it holds. (Technically, ‘efficiently’ means that the length of the proof is bounded by a polynomial in the size of the graph.)” We refer the reader to also Garey and Johnson [5] for the notion of NP.

Cornuéjols [3] provided the first polynomial algorithm for the problem with \( H \) allowed.

Let \( S \) be a subset of \( V(G) \). Denote by \( G - S \) the subgraph of \( G \) obtained by removing all vertices in \( S \). Denote by \( o(G) \) the number of odd components of \( G \). Let \( n \geq 2 \) be an integer. Let \( H_o \) be the allowed function associating the first \( n \) positive odd integers to each vertex, i.e.,

\[ H_o(v) = \{ 1, 3, 5, \ldots, 2n - 1 \}. \]

A special case of the degree prescribed subgraph problem is the so-called \( f \)-parity subgraph problem, i.e., the problem with

\[ H(v) = \{ \ldots, f(v) - 4, f(v) - 2, f(v) \} \]
for some function \( f : V(G) \to \mathbb{Z} \). The first investigation of the \( f \)-parity subgraph problem is due to Amahashi [2], who gave a Tutte-type characterization for graphs having a global odd factor.

**Theorem 1.1** (Amahashi). A graph \( G \) has an \( H_o \)-factor if and only if
\[
o(G - S) \leq (2n - 1) |S| \quad \text{for all subsets } S \subset V(G).
\]

For general odd value functions \( f \), Cui and Kano [4] established a Tutte-type theorem. Noticing the form of the condition (3), they asked the question of characterizing graphs \( G \) in terms of graph factors such that
\[
o(G - S) \leq 2n |S| \quad \text{for all subsets } S \subset V(G).
\]

For more studies on the \( f \)-parity subgraph problem, see Topp and Vestergaard [13], and Kano, Katona, and Szabó [6].

Motivated by Cui-Kano’s problem, we consider the degree prescribed subgraph problem for the special set function
\[
H_n(v) = H_o(v) \cup \{2n\} = \{1, 3, 5, \ldots, 2n - 1, 2n\}.
\]

We shall study the structure of graphs without \( H_n \)-factor by using Lovász’s factorization theory [10]. In Corollary 2.6, we obtain that any graph satisfying the condition (4) contains an \( H_n \)-factor.

Besides many applications of Lovász’s structural description for special families of graphs and special allowed set functions, there is much attention paid to find sufficient conditions for the existence of an \( H \)-factor of a graph for special allowed set functions \( H \), see [12]. In this article, we also give some sufficient conditions for the existence of an \( H_n \)-factor.

## 2. THE MAIN RESULT

In this section, we study \( H_n \)-factors of graphs based on Lovász’s structural description to the degree prescribed subgraph problem.

Let \( H \) be an allowed set function. Denote by \( I_H(v) \) the set of vertex degrees in all \( H \)-optimal subgraphs, i.e.,
\[
I_H(v) = \{d_F(v) : \text{All } H \text{-optimal subgraphs } F\}.
\]

Comparing the set \( I_H(v) \) with \( H(v) \), one may partition the vertex set \( V(G) \) into the following four classes:
\[
C_H = \{v \in V(G) : I_H(v) \subseteq H(v)\},
A_H = \{v \in V(G) \setminus C_H : \min I_H(v) \geq \max H(v)\},
B_H = \{v \in V(G) \setminus C_H : \max I_H(v) \leq \min H(v)\},
D_H = V(G) \setminus A_H \setminus B_H \setminus C_H.
\]

It is clear that the 4-tuple \((A_H, B_H, C_H, D_H)\) is a pairwise disjoint partition of \( V(G) \). We call it the \( H \)-decomposition of \( G \). In fact, the four subsets can be distinguished according
to the contributions of their members to the deviation (2). A graph \(G\) is said to be \(H\)-critical if it is connected and \(D_H = V(G)\).

In [10, Theorem (2.1)], Lovász gave the following property for the subset \(D_H\).

**Lemma 2.1** (Lovász). If \(D_H \neq \emptyset\), then the intersection

\[
\{\min I_H(v), \max I_H(v)\} \cap H(v)
\]

contains no consecutive integers for any vertex \(v \in D_H\).

In [10, Corollary (2.4)], Lovász gave the next structural result.

**Lemma 2.2** (Lovász). There is no edge between \(C_H\) and \(D_H\).

For any subset \(S \subseteq V(G)\), denote by \(G[S]\) the subgraph induced by \(S\). Denote the number of components of \(G[S]\) by \(c(S)\), and the number of odd components of \(G[S]\) by \(o(S)\). In [10, Theorem (4.3)], Lovász established the formula

\[
\delta(H) = c(D_H) + \sum_{v \in B_H} \min H(v) - \sum_{v \in A_H} \max H(v) - \sum_{v \in B_H} d_{G-A_H}(v).
\]

By definition, \(G\) contains no \(H\)-factor if and only if \(\delta(H) > 0\). This yields the following lemma immediately.

**Lemma 2.3** (Lovász). A graph \(G\) contains no \(H\)-factor if and only if

\[
c(D_H) + \sum_{v \in B_H} \min H(v) > \sum_{v \in A_H} \max H(v) + \sum_{v \in B_H} d_{G-A_H}(v). \tag{6}
\]

Let \(X \subseteq V(G)\). For any vertex \(v\), define

\[
H_X(v) = \{h - e(v, X) : h \in H(v)\},
\]

where \(e(v, X)\) is the number of edges from \(v\) to \(X\). In [10, Theorem (4.2)], Lovász showed that each component of \(G[D_H]\) is \(H'\)-critical where \(H' = H_{B_H}\). He [10, Lemma (4.1)] also obtained that \(\delta(H) = 1\) if \(G\) is \(H\)-critical. This leads to the lemma below.

**Lemma 2.4** (Lovász). If \(D_H \neq \emptyset\), then for any component \(T\) of the subgraph \(G[D_H]\), and any \(H\)-optimal subgraph \(F\) of \(T\), we have \(\delta(F) = 1\).

This article concerns \(H_n\)-decompositions where \(H_n\) is defined by (5). For convenience, we often use another set function \(H^*_n\) defined by

\[
H^*_n(v) = H_n(v) \cup \{-1\} = \{-1, 1, 3, 5, \ldots, 2n - 1, 2n\}. \tag{8}
\]

As will be seen, the \(B_{H^*_n}\) part in the \(H^*_n\)-decomposition of certain graph is empty, and the structure becomes clear in this way. Here is the main result of this article.

**Theorem 2.5.** Let \(G\) be a connected graph of even order. If \(G\) contains no \(H_n\)-factor, then there exists a nonempty subset \(S \subseteq V(G)\) such that the subgraph \(G - S\) contains at least \(2n|S| + 1\) odd components, each of which contains no \(H_n\)-factor.

**Proof.** Assume that \(G\) has no \(H_n\)-factor. By the definition (8), the graph \(G\) contains no \(H_n^*\)-factor. Let \((A, B, C, D)\) be the \(H_n^*\)-decomposition of \(G\). We will show that the subset \(A\) can be taken as the required \(S\).

Since \(\min H_n^* = -1\), we have \(B = \emptyset\), and the inequality (6) reduces to

\[
c(D) > 2n|A|. \tag{9}
\]

*Journal of Graph Theory* DOI 10.1002/jgt
This implies $D \neq \emptyset$. Let $T$ be a component of the subgraph $G[D]$, and $F$ an $H_n^*$-optimal subgraph of $T$. Since $B = \emptyset$, we see that $H_B = H_n^*$. So, $\delta_{H_n^*}(F) = 1$ by Lemma 2.4. Therefore, there exists a vertex, say $v_0 \in V(T)$, such that

$$
\min \{|d_F(v) - h| : h \in H_n^*(v)\} = \begin{cases} 
1, & \text{if } v = v_0; \\
0, & \text{if } v \in V(T) \setminus \{v_0\}.
\end{cases} \tag{10}
$$

On the other hand, assume that $\max I_{H_n^*}(v) \geq 2n$ for some $v \in D$. By Lemma 2.1, we have

$$
\min I_{H_n^*}(v) \geq 2n.
$$

It follows immediately that $v \in A$, a contradiction. Thus, $\max I_{H_n^*}(v) \leq 2n - 1$, namely

$$
d_F(v) \leq 2n - 1 \text{ for any } v \in T.
$$

Consequently, the formula (10) implies that the degree $d_F(v_0)$ is even, while $d_F(v)$ is odd for any $v \in V(T) \setminus \{v_0\}$. Since the sum $\sum_{v \in V(T)} d_F(v)$ is even, we deduce that $T$ is an odd component of $G[D]$.

Assume that $A = \emptyset$. Since $B = \emptyset$, by Lemma 2.2, we see that $T$ is an odd component of $G$. But $G$ is connected graph of even order, a contradiction. So, $A \neq \emptyset$.

By Lemma 2.2 and the inequality (9), we have

$$
2n|A| < c(D) = o(D) \leq o(C) + o(D) = o(C \cup D) = o(G - A). \tag{11}
$$

Namely, the subgraph $G - A$ has at least $2n|A| + 1$ odd components. In view of (10), any component $T$ of $G[D]$ has no $H_n^*$-factor. Hence, $T$ has no $H_n$-factor. This completes the proof.

For graphs satisfying Cui-Kano’s condition (4), we obtain the following corollary immediately since (4) implies that $G$ is of even order by setting $S = \emptyset$.

**Corollary 2.6.** Every connected graph $G$ satisfying the condition (4) contains an $H_n$-factor.

For graphs of odd order, we have the following result.

**Theorem 2.7.** Let $G$ be a connected graph of odd order. Suppose that

$$
o(G - S) \leq 2n|S| \text{ for all } \emptyset \neq S \subset V(G). \tag{12}
$$

Then, either $G$ contains an $H_n$-factor, or $G$ is $H_n^*$-critical.

**Proof.** Suppose that $G$ contains no $H_n$-factor. Let $(A, B, C, D)$ be the $H_n^*$-decomposition of $G$. From the proof of Theorem 2.5, we see that $B = \emptyset$, and obtain the inequality (11). Together with the condition (12), we see that $A = \emptyset$. Since $G$ is connected, we find that $C = \emptyset$ by Lemma 2.2. Hence, $G$ is $H_n^*$-critical. This completes the proof.

We remark that the condition (4) is not necessary for the existence of an $H_n$-factor in a graph. Consider the graph

$$
G = v_0 + (2n + 1)K_{2n+1},
$$

obtained by linking a vertex $v_0$ to all vertices in $2n + 1$ copies of the complete graph $K_{2n+1}$. Denote by $C_j$ ($1 \leq j \leq 2n + 1$) the $j$-th copy of $K_{2n+1}$. Let $v_j \in V(C_j)$. Let $F$ be
the factor consisting of the following $2n + 2$ components:
\[ C_1 - v_1, C_2 - v_2, \ldots, C_{2n} - v_{2n}, C_{2n+1}, G \{ v_0, v_1, \ldots, v_{2n} \}. \]

It is easy to verify that $F$ is an $H_n$-factor. However, taking the subset $S$ to be the single vertex $v_0$, we see that the condition (4) does not hold for $G$.

To end this section, we point out that the coefficient $2n$ in the condition (4) is a sharp bound in the sense that for any $\epsilon > 0$, there exists a graph $G$ with a subset $S \subset V(G)$ satisfying
\[ o(G - S) < (2n + \epsilon) |S|, \]
and that $G$ contains no $H_n$-factor. We need Las Vergnas’s theorem [8].

**Theorem 2.8** (Las Vergnas). A graph $G$ contains a $[1, n]$-factor if and only if for all subsets $S \subset V(G)$, the number of isolated vertices in the subgraph $G - S$ is at most $n|S|$.

**Theorem 2.9.** For any $\epsilon > 0$, there exists a graph $G$ with a subset $S \subset V(G)$ satisfying the $\epsilon$-condition (13) but with no $H_n$-factor.

**Proof.** Let $m$ be an integer such that $m > 1/\epsilon$. Let $V_m$ be a set of $m$ isolated vertices, and $V_{2nm+1}$ be a set of $2nm + 1$ isolated vertices. Denote by $K_{m, 2nm+1}$, the complete bipartite graph obtained by connecting each vertex in $V_m$ with each vertex in $V_{2nm+1}$. Setting $S = V_m$ in Theorem 2.8, we deduce that $K_{m, 2nm+1}$ contains no $[1, 2n]$-factor, and thus no $H_n$-factor. Moreover,
\[ o(K_{m, 2nm+1} - V_m) = 2nm + 1 < (2n + \epsilon) |V_m|. \]

### 3. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF AN $H_n$-FACTOR

In this section, we present some sufficient conditions for the existence of an $H_n$-factor. For any vertex $v$ of $G$, denote by $N_G(v)$ the set of neighbors of $v$. Denote the number of vertices of $G$ by $|G|$.

**Theorem 3.1.** Let $G$ be a connected graph of even order. If for any nonadjacent vertices $u$ and $v$,
\[ |N_G(u) \cup N_G(v)| > \max \left\{ \frac{|G| - 2}{2n}, 1, \frac{2|G| - 4}{4n + 1}, \frac{|G| - 1}{2n + 1}, 4n - 3 \right\}, \]
then $G$ contains an $H_n$-factor.

**Proof.** Suppose the contrary that $G$ contains no $H_n$-factor. By Theorem 2.5, there exists a nonempty subset $S \subset V(G)$ such that the subgraph $G - S$ has at least $2ns + 1$ odd components, say, $C_1, C_2, \ldots, C_{2ns+1}$, where no $C_i$ has $H_n$-factor, and $s = |S|$. Let $c_i = |V(C_i)|$. Suppose that
\[ 1 \leq c_1 \leq c_2 \leq \cdots \leq c_{2ns+1}. \]

It is clear that
\[ 2ns + 1 \leq c_1 + c_2 + \cdots + c_{2ns+1} \leq |G| - s. \]
Therefore,
\[ c_1 \leq \frac{|G| - s}{2ns + 1}, \]
\[ c_2 \leq \frac{|G| - s - c_1}{2ns}. \]

It also follows
\[ c_1 + c_2 \leq \frac{2(|G| - s)}{2ns + 1}, \quad (17) \]
\[ c_2 \leq \frac{|G| - s - 1}{2ns}. \quad (18) \]

Moreover, the inequality (16) implies that
\[ s \leq s^*, \quad \text{where} \]
\[ s^* = \frac{|G| - 1}{2n + 1}. \]

Let \( u \in V(C_1) \) and \( v \in V(C_2) \). Then,
\[ |N_G(u) \cup N_G(v)| \leq s + (c_1 - 1) + (c_2 - 1). \quad (19) \]

By (17), we find that
\[ |N_G(u) \cup N_G(v)| \leq h(s), \quad \text{where} \]
\[ h(s) = \frac{2(|G| - s)}{2ns + 1} + s - 2. \]

Note that the second derivative \( h''(s) > 0 \). If \( s \geq 2 \), then we have
\[ |N_G(u) \cup N_G(v)| \leq \max\{|h(2), h(s^*)| = \max\left\{ \frac{2|G| - 4}{4n + 1}, \frac{|G| - 1}{2n + 1} \right\}, \]
contradicting the condition (14). Otherwise \( s = 1 \). In this case, if \( c_2 \leq 2n - 1 \), then
\[ c_1 \leq 2n - 1 \text{ by (15)}. \]
By (19), we have
\[ |N_G(u) \cup N_G(v)| \leq c_1 + c_2 - 1 \leq 4n - 3, \]
contradicting (14). So \( c_2 \geq 2n \). It is easy to verify that any complete graph \( K_m \) with \( m \geq 2n \) has an \( H_n \)-factor. Since \( C_2 \) contains no \( H_n \)-factor, we deduce that \( C_2 \) is not complete. So, there exist vertices \( u' \) and \( v' \) that are not adjacent in \( C_2 \). By (18), we have
\[ |N_G(u') \cup N_G(v')| \leq s + c_2 - 2 \leq \frac{|G| - 2}{2n} - 1, \]
contradicting (14). This completes the proof.

Observe that when \( |G| \geq 8n^2 + 2n + 2 \), one has
\[ \max\left\{ \frac{|G| - 2}{2n} - 1, \frac{2|G| - 4}{4n + 1}, \frac{|G| - 1}{2n + 1}, 4n - 3 \right\} = \frac{|G| - 2}{2n} - 1. \]
This gives the following corollary immediately.

**Corollary 3.2.** Let \( G \) be a connected graph of even order. If \( |G| \geq 8n^2 + 2n + 2 \), and for any nonadjacent vertices \( u \) and \( v \),
\[ |N_G(u) \cup N_G(v)| > \frac{|G| - 2}{2n} - 1, \]
then \( G \) contains an \( H_n \)-factor.

*Journal of Graph Theory* DOI 10.1002/jgt
Let $H$ be an allowed set function. In [10, Lemma (3.5)], Lovász gave the following result describing the $H$-decomposition of a graph when a vertex in $A_H$ is removed.

**Lemma 3.3** (Lovász). Let $(A, B, C, D)$ be the $H$-decomposition of $G$. Let $v$ be a vertex in $A$, and $(A', B', C', D')$ the $H$-decomposition of the subgraph $G - v$. Then,

$$A' = A - v, \quad B' = B, \quad C' = C, \quad D' = D.$$ 

We give an analogue of this for $H_n$-factors.

**Theorem 3.4.** Let $G$ be a connected graph of even order. Then, $G$ contains an $H_n$-factor if the subgraph $G - v$ contains an $H_n$-factor for all vertices $v$.

**Proof.** Suppose that $G$ has no $H_n$-factor. Let $(A, B, C, D)$ be the $H_n$-decomposition of $G$. From the proof of Theorem 2.5, we see that $B = \emptyset$ and

$$2n|A| < c(D). \tag{20}$$

Moreover, every component of $G[D]$ is odd.

Assume that $A \neq \emptyset$. Let $v \in A$, and $(A', B', C', D')$ be the $H_n^*$-decomposition of $G - v$. By Lemma 2.3, we have

$$c(D') \leq 2n|A'|. \tag{21}$$

By (20), (21), and Lemma 3.3, we deduce that

$$2n|A| \leq c(D) - 1 = c(D') - 1 \leq 2n|A'| - 1 = 2n(|A| - 1) - 1,$$

a contradiction. So, $A = \emptyset$. By Lemma 2.2, any component of $G[D]$ is an odd component of $G$. But $G$ is a connected graph of even order, a contradiction. This completes the proof. 

\[\square\]

**ACKNOWLEDGMENTS**

Lu was supported by the National Natural Science Foundation of China (Grant No. 11101329) and the Fundamental Research Funds for the Central Universities. Wang was supported by the National Natural Science Foundation of China (Grant No. 11101010). He expresses his gratitude to University of Haifa for the hospitality and support during his stay there when the final version of this article was completed. Both authors are grateful to Mikio Kano for his comments on this article and for help with references. Thanks are also given to the anonymous referees for helpful suggestions.

**REFERENCES**

[1] J. Akiyama and M. Kano, Factors and Factorizations of Graphs: Proof Techniques in Factor Theory, Lecture Notes in Math., vol. 2031, Springer-Verlag, Berlin Heidelberg, 2011.

[2] A. Amahashi, On factors with all degree odd, Graphs Combin 1 (1985), 111–114.
[3] G. Cornuéjols, General factors of graphs, J. Combin Theory Ser B 45 (1988), 185–198.

[4] Y. Cui and M. Kano, Some results on odd factors of graphs, J Graph Theory 12 (1988), 327–333.

[5] M. R. Garey and D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, A Series of Books in the Mathematical Sciences, W. H. Freeman and Co., San Francisco, Calif., 1979.

[6] M. Kano, G. Y. Katona, and J. Szabó, Elementary graphs with respect to f-parity factors, Graphs Combin 25 (2009), 717–726.

[7] M. Kouider and P. D. Vestergaard, Connected factors in graphs—a survey, Graphs Combin 21 (2005), 1–26.

[8] M. Las Vergnas, An extension of Tutte’s 1-factor theorem, Discrete Math 23 (1978), 241–255.

[9] L. Lovász, The factorization of graphs, Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969), Gordon and Breach, New York (1970), 243–246.

[10] L. Lovász, The factorization of graphs. II, Acta Math Hungar 23 (1972), 223–246.

[11] L. Lovász, Combinatorial Problems and Exercises, 2nd. ed., AMS Chelsea Publishing, AMS, Province, RI, 2007.

[12] M. D. Plummer, Graph factors and factorization: 1985–2003: a survey, Discrete Math 307 (2007), 791–821.

[13] J. Topp and P. D. Vestergaard, Odd factors of a graph, Graphs Combin 9 (1993), 371–381.

[14] Q. Yu and G. Liu, Graph Factors and Matching Extensions, Higher Education Press, Springer-Verlag, Berlin Heidelberg, 2009.