The Failure Probability at Sink Node of Random Linear Network Coding

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Abstract—In practice, since many communication networks are huge in scale or complicated in structure even dynamic, the predesigned network codes based on the network topology is impossible even if the topological structure is known. Therefore, random linear network coding was proposed as an acceptable coding technique. In this paper, we further study the performance of random linear network coding by analyzing the failure probabilities at sink node for different knowledge of network topology and get some tight and asymptotically tight upper bounds of the failure probabilities. In particular, the worst cases are indicated for these bounds. Furthermore, if the more information about the network topology is utilized, the better upper bounds are obtained. These bounds improve on the known ones. Finally, we also discuss the lower bound of this failure probability and show that it is also asymptotically tight.

I. INTRODUCTION

Network coding was first introduced by Yeung and Zhang in [1] and then profoundly developed in Ahlswede et al. in [2]. In the latter paper [2], the authors showed that if coding is applied at the nodes instead of routing alone, the source node can multicast the information to all sink nodes at the theoretically maximum rate. Li et al. [3] indicated that linear network coding with finite alphabet size is sufficient for multicast. In [4], Koetter and Médard presented an algebraic characterization of network coding. Although network coding allows the higher information rate than classical routing, Jaggi et al. [5] still proposed a deterministic polynomial-time algorithm to construct a linear network code. For a detail and comprehensive discussion of network coding, refer to [6], [7], [8], [9], and [10].

Random linear network coding was originally proposed and analyzed in the papers Ho et al. [11] and [12], where the main results are upper bounds for failure probabilities of the code. Balli, Yan, and Zhang [13] improved on these bounds and the tightness of the new bounds was studied by analyzing the asymptotic behavior of the failure probability as the field size goes to infinity. However, the upper bounds of failure probabilities proposed either by Ho et al. [12] or by Balli et al. [13] are not tight. In this paper, we further study the random linear network coding and improve on the bounds of the failure probabilities for different cases. In particular, if the more knowledge about the topology of the network is known, we can get the better bounds. Further, we indicate that these bounds are either tight or asymptotically tight.

II. LINEAR NETWORK CODING AND PRELIMINARIES

A communication network is defined as a finite acyclic directed graph $G = (V, E)$, where the vertex set $V$ stands for the set of the nodes and the edge set $E$ represents the set of communication channels of the network. The nodes set $V$ consists of three disjoint subsets $S$, $T$, and $J$, where $S$ is the set of source nodes, $T$ is the set of sink nodes, the other nodes in $J = V - S - T$ are called internal nodes and thus the subset $J$ is called the set of internal nodes. A direct edge $e = (i, j) \in E$ represents a channel leading from node $i$ to node $j$. Node $i$ is called the tail of the channel $e$, node $j$ is called the head of the channel $e$, and they are written as $i = tail(e)$, $j = head(e)$, respectively. Correspondingly, the channel $e$ is called an outgoing channel of $i$ and an incoming channel of $j$. For each node $i$, define

$Out(i) = \{e \in E : e$ is an outgoing channel of $i\}$,

$In(i) = \{e \in E : e$ is an incoming channel of $i\}$.

For each channel $e \in E$, there exists a positive number $R_e$ called the capacity of the channel $e$. We allow the multiple channels between two nodes and then assume reasonably that all capacity of the channel is unit 1. That is, one field symbol can be transmitted over a channel in one unit time. The source nodes generate messages and transmit them to all sink nodes over the network by network coding.

In this paper, we sequentially consider single source multicast networks, i.e. $|S| = 1$, and the unique source node is denoted by $s$. The source node $s$ has no incoming channels and any sink node has no outgoing channels, but we use the concept of the imaginary incoming channels of the source node $s$ and assume that these imaginary channels provide the source messages to $s$. Let the information rate be $w$ symbols per unit time which means that the source node $s$ has $w$ imaginary incoming channels $d_1, d_2, \ldots, d_w$ and let $In(s) = \{d_1, d_2, \ldots, d_w\}$. The source messages are $w$ symbols $X = (X_1, X_2, \ldots, X_w)$ arranged in a row vector where each $X_i$ is an element of the finite base field $\mathcal{F}$. Assume that they are transmitted to $s$ through the $w$ imaginary channels. Using network coding, these messages are multicast to each sink node and decoded at each sink node.

We use $U_e$ to denote the message transmitted over channel
$e = (i, j)$ and $U_e$ is calculated by the following formula

$$U_e = \sum_{d \in I_n(i)} k_{d,e} U_d,$$

where at the source node $s$, assume that the message transmitted over $i$th imaginary channel $d_i$ is the $i$th source message, i.e. $U_{d_i} = X_i$. And, by the definition of the global kernels of the channel $e$, we have $U_e = X \cdot f_e$.

The linear network coding discussed above was designed based on the global topology of the network. However, in most communication networks, we cannot utilize the global topology because the network is huge in scale, or complicated in structure, even dynamic, or some another reasons. In other words, it is impossible to use the predesigned codes based on the global topology. Thus random linear network coding was proposed as an acceptable coding technique. The main idea of random network coding is that when a node (may be the source node $s$) receives the messages from its all incoming channels, for each outgoing channel, it randomly and uniformly picks the encoding coefficients from the base field $\mathcal{F}$, uses them to encode the messages, and transmits the encoded messages over the outgoing channel. In other words, the local coding coefficients $k_{d,e}$ are independently and uniformly distributed random variables in the base field $\mathcal{F}$. Since random linear network coding does not consider the network global topology or does not coordinate codings at different nodes, it may not achieve the best possible performance of network coding, that is, some sink nodes may not decode correctly. Therefore, the performance analysis of random linear network coding is important in theory and application.

Before further discussion, we introduce some notation and definitions as follows.

Let $A$ be a set of vectors from a linear space. $(A)$ represents a linear subspace spanned by the vectors in $A$. In addition, we give the definition of the failure probability at sink node which was introduced exactly in [13].

**Definition 1:** Let $G$ be a single source multicast network, and the information rate be $w$ symbols per unit time. $P_{e_t} \triangleq Pr(\text{Rank}(F_t) < w)$ is called the failure probability of the random linear network coding at sink node $t$, that is the probability that the source messages cannot be decoded correctly at sink node $t \in T$.

**III. Failure Probabilities of Random Linear Network Coding at Sink Node**

We have known that the performance analysis of random linear network coding is very important in theory and application. In particular, the random linear network coding is an acceptable coding technique for non-coherent networks. However, many coherent networks are huge and complicated, and thus the random linear network coding are often used for the coherent networks. In this section, we study the failure probability $P_{e_t}$ from coherent to non-coherent networks. At first, we give the following lemma.

**Lemma 1:** Let $\mathcal{L}$ be a $n$-dimensional linear space over finite field $\mathcal{F}$, $\mathcal{L}_0$, $\mathcal{L}_1$ be two subspaces of $\mathcal{L}$ of dimensions $k_0$, $k_1$, respectively, and $\langle \mathcal{L}_0 \cup \mathcal{L}_1 \rangle = \mathcal{L}$. Let $l_1$, $l_2$, \ldots, $l_m$ $(m = n - k_0)$ be $m$ independently uniformly distributed random vectors taking values in $\mathcal{L}_1$. Then

$$Pr(\text{dim}(\langle \mathcal{L}_0 \cup \{l_1, l_2, \ldots, l_m\} \rangle) = n) = \prod_{i=1}^{n-k_0} \left(1 - \frac{1}{|\mathcal{F}|}\right).$$

Therefore,

$$\frac{1}{|\mathcal{F}|} \leq Pr(\text{dim}(\langle \mathcal{L}_0 \cup \{l_1, l_2, \ldots, l_m\} \rangle) < n) < \frac{1}{|\mathcal{F}| - 1}.$$

**Remark 1:** We can observe that under the condition of Lemma 1, $Pr(\text{dim}(\langle \mathcal{L}_0 \cup \{l_1, l_2, \ldots, l_m\} \rangle) = n)$ is not related to the dimension of $\mathcal{L}_1$.

Let $G$ be a single source multicast network, where the single source node is denoted by $s$, the set of the sink nodes is denoted by $T$, and the minimum cut capacity between $s$ and $t \in T$ is $C_t$. The information rate is $w \leq \min_{t \in T} C_t$ symbols per unit time.

For each sink node $t \in T$, since $w \leq C_t$ and Menger’s Theorem, there exist $w$ channel-disjoint paths from $s$ to $t$. Let the arbitrarily chosen $w$ channel-disjoint paths from $s$ to $t$ be $P_t = \{P_{t,1}, P_{t,2}, \ldots, P_{t,w} \}$ and let $P_t = \{e_{i,1}, e_{i,2}, \ldots, e_{i,m_t} \}$ satisfying $\text{tail}(e_{i,1}) = s$, $\text{head}(e_{i,m_t}) = t$, and $\text{head}(e_{i,j-1}) = \text{tail}(e_{i,j})$ for others. The set of all channels in $P_t$ is denoted by $E_{P_t}$. Furthermore, assume that the number of the nodes in $P_t = r + 2$, where one is the source node $s$, one is the sink node $t$, and another $r$ are internal nodes, which are denoted by $i_1, i_2, \ldots, i_r$. There is a topological order ancestrally, and without loss of generality, let the order be

$$s \triangleq i_0(i_1 < i_2 < \cdots < i_r < i_{r+1} = t).$$

During our discussion, we use the concept of cuts of the paths from $s$ to $t$ proposed in [13], which is different from the concept of cuts of the network in graph theory. The first cut is $\text{CUT}_{t,0} = \text{In}(s)$, i.e. the set of the $w$ imaginary channels. Through the node $i_0 = s$, the next cut $\text{CUT}_{t,1}$ is the set of the first channels of all $w$ paths, i.e. $\text{CUT}_{t,1} = \{e_{i,1} : 1 \leq i \leq w \}$. Through the node $i_1$, the next cut $\text{CUT}_{t,2}$ is formed from $\text{CUT}_{t,1}$ by replacing those channels in $\text{In}(i_1) \cap \text{CUT}_{t,1}$ by their respective next channels in the paths. These new channels are in $\text{Out}(i_1) \cap E_{P_t}$. Other channels remain the same as in $\text{CUT}_{t,1}$. Subsequently, once $\text{CUT}_{t,k}$ is defined, $\text{CUT}_{t,k+1}$ is formed from $\text{CUT}_{t,k}$ by the same method as above. By induction, all cuts $\text{CUT}_{t,k}$ for $k = 0, 1, \ldots, r + 1$ can be defined. Furthermore, for each $\text{CUT}_{t,k}$, we divide $\text{CUT}_{t,k}$ into two disjoint parts $\text{CUT}_{t,k}^{\text{out}}$ and $\text{CUT}_{t,k}^{\text{in}}$, where

$$\text{CUT}_{t,k}^{\text{out}} = \{ e : e \in \text{CUT}_{t,k} \setminus \text{In}(i_k) \},$$

$$\text{CUT}_{t,k}^{\text{in}} = \{ e : e \in \text{CUT}_{t,k} \cap \text{In}(i_k) \}.$$

**Theorem 2:** For this network $G$ mentioned as above, the failure probability of random linear network coding at sink node $t \in T$ satisfies

$$P_{e_t} \leq 1 - \prod_{k=0}^{r} \prod_{i=1}^{w-|\text{CUT}_{t,k}^{\text{out}}|} \left(1 - \frac{1}{|\mathcal{F}|}\right).$$
Proof: For sink node \( t \in T \), the decoding matrix \( F_t = (f_e : e \in In(t)) \) is a \( w \times |In(t)| \) matrix over the field \( \mathcal{F} \). Define a \( w \times w \) matrix \( F_t^{w} = (f_{e_1,m_1}, f_{e_2,m_2}, \ldots, f_{e_w,m_w}) \). It is not hard to see that \( F_t^{w} \) is a submatrix of \( F_t \). It follows that the event “\( \text{Rank}(F_t) < w \)” \( \subseteq \) the event “\( \text{Rank}(F_t^{w}) < w \)”.

This means that

\[
\Pr(\text{Rank}(F_t) < w) \leq \Pr(\text{Rank}(F_t^{w}) < w).
\]

Further define \( w \times w \) matrices \( F_t^{(k)} = (f_e : e \in CUT_t,k) \) for \( k = 0, 1, \ldots, r + 1 \). If \( \text{Rank}(F_t^{(k)}) < w \), we call that we have a failure at \( CUT_t,k \). We use \( \Gamma_{t,k} \) to denote the event “\( \text{Rank}(F_t^{(k)}) = w \)”.

Obviously, \( F_t^{(r+1)} = F_t^{w} \) because \( CUT_{t,r+1} = \{e_1,m_1, e_2,m_2, \ldots, e_w,m_w\} \). This implies

\[
\Pr(\text{Rank}(F_t^{(k)}) < w) = \Pr(\text{Rank}(F_t^{(r+1)}) < w) = \Pr(\Gamma_{t,r+1}) = 1 - \Pr(\Gamma_{t,r+1}).
\]

In addition, since encoding at any node is independent of what happened before this node as long as no failure has occurred up to this node, we have

\[
\Pr(\Gamma_{t,r+1}) \geq \Pr(\Gamma_{t,r+1}|\Gamma_{t,r} \cap \ldots \cap \Gamma_{t,1}|\Gamma_{t,0}) = \Pr(\Gamma_{t,r+1}|\Gamma_{t,r}|\Gamma_{t,r-1}) \cdot \ldots \cdot \Pr(\Gamma_{t,1}|\Gamma_{t,0}) \Pr(\Gamma_{t,0})
\]

Combining (1) and (2), it follows that

\[
\Pr(\Gamma_{t,r+1}) \geq \prod_{i=1}^{\text{dim}(CUT_{t,k})} \frac{w - \text{dim}(CUT_{t,k})}{|\mathcal{F}|} \geq \prod_{i=1}^{w - \text{dim}(CUT_{t,k})} \left(1 - \frac{1}{|\mathcal{F}|} \right) \geq \prod_{i=1}^{w - \text{dim}(CUT_{t,k})} \left(1 - \frac{1}{|\mathcal{F}|} \right) \geq \prod_{i=1}^{w - \text{dim}(CUT_{t,k})} \left(1 - \frac{1}{|\mathcal{F}|} \right).
\]

That is, we get the upper bound of the failure probability at the sink node \( t \),

\[
P_e \leq 1 - \frac{\prod_{i=1}^{w - \text{dim}(CUT_{t,k})} \left(1 - \frac{1}{|\mathcal{F}|} \right)}{\prod_{i=1}^{w - \text{dim}(CUT_{t,k})} \left(1 - \frac{1}{|\mathcal{F}|} \right)} = 1 - \frac{(\text{dim}(CUT_{t,k}) + 1)(|\mathcal{F}| - 1)^6}{|\mathcal{F}|^7}.
\]

The proof is completed.

Remark 2: This upper bound of the failure probability at the sink node \( t \) in Theorem 3 is tight.

Example 1: For the well-known butterfly network, by Theorem 3 we know

\[
P_e \leq 1 - \frac{2}{\prod_{i=1}^{|\mathcal{F}|} \left(1 - \frac{1}{|\mathcal{F}|} \right)}(1 - \frac{1}{|\mathcal{F}|}) = 1 - \frac{(\text{dim}(CUT_{t,k}) + 1)(|\mathcal{F}| - 1)^6}{|\mathcal{F}|^7}.
\]

On the other hand, Guang and Fu [13] have shown that for the butterfly network \( P_e = 1 - \frac{\text{dim}(CUT_{t,k}) + 1}{|\mathcal{F}|} \). This means that this upper bound is tight for the butterfly network.

However, this upper bound is too complicated in practice. Thus, we have to give a simpler in form but looser upper bound.

**Theorem 3:** For this network \( G \), the failure probability of the random linear network coding at sink node \( t \in T \) satisfies

\[
P_e \leq 1 - \prod_{i=1}^{w} \left(1 - \frac{1}{|\mathcal{F}|} \right)^{r+1}.
\]

In particular, if we choose the \( w \) channel-disjoint paths with the minimum number of the internal nodes among the collection of all \( w \) channel-disjoint paths from \( s \) to \( t \) over network \( G \), and denote this minimum number by \( R_e \), then we get a smaller upper bound with the same simple form.

**Corollary 4:** For this network \( G \), the failure probability of the random linear network coding at sink node \( t \in T \) satisfies

\[
P_e \leq 1 - \prod_{i=1}^{w} \left(1 - \frac{1}{|\mathcal{F}|} \right)^{R_e+1}.
\]

**Remark 3:** Both upper bounds of the failure probability at the sink node in Theorem 3 and Corollary 4 are tight, and we can show the tightness by the same way. Therefore, we only construct a network to show the tightness of the upper bound in Theorem 3. In other words, we will give a network as the worst case.

**Example 2:** For the given information rate \( w \), the network

\[
\text{Fig. 1. Plait Network with } r \text{ internal nodes}
\]

\( G_1 \) shown by Fig 1 can be constructed as follows. Let the source node be \( s \), the sink node be \( t \), the number of the internal nodes be \( r \), and denote these internal nodes by \( i_1, i_2, \ldots, i_r \). Let the topological order of all nodes be \( s \prec i_1 \prec i_2 \prec \cdots \prec i_r \prec t \).

Draw \( w \) parallel channels from \( s \) to \( i_1 \), \( w \) parallel channels from \( i_1 \) to \( i_2 \), in succession, \( w \) parallel channels from \( i_r \) to \( t \). The total \((r+1)w\) channels are all channels of the network \( G_1 \). For this type of networks, we call them plait networks.

For this constructed network \( G_1 \), we will show that the failure probability \( P_e \) at sink node \( t \) is

\[
P_e = 1 - \frac{\prod_{i=1}^{w} \left(1 - \frac{1}{|\mathcal{F}|} \right)^{r+1}}{\prod_{i=1}^{w} \left(1 - \frac{1}{|\mathcal{F}|} \right)^{R_e+1}}.
\]

It is not difficult to see that the event “\( \text{Rank}(F_t) < w \)” is equivalent to the event “\( \text{Rank}(F_t^{(r+1)}) < w \)” because of \( F_t = F_t^{(r+1)} \). This implies

\[
P_e = \Pr(\text{Rank}(F_t^{(r+1)}) < w) = 1 - \Pr(\Gamma_{t,r+1}).
\]
Furthermore, for $G_1$,
\[
Pr(\Gamma_{t,r+1}) = Pr(\Gamma_{t,r+1}\Gamma_{t,r}\cdots\Gamma_{t,1}\Gamma_{t,0})
\]
\[=Pr(\Gamma_{t,r+1}|\Gamma_{t,r})Pr(\Gamma_{t,r}|\Gamma_{t,r-1})\cdots Pr(\Gamma_{t,1}|\Gamma_{t,0}).
\]
And, for any $k = 0, 1, \ldots, r$,
\[
Pr(\Gamma_{t,k+1}|\Gamma_{t,k})
\]
\[=Pr(\{f_{e_{k,1}} \notin \emptyset, f_{e_{k,2}} \notin \{f_{e_{k,1}}\}, f_{e_{k,3}} \notin \{f_{e_{k,1}}, f_{e_{k,2}}\}, \ldots, f_{e_{k,w}} \notin \{f_{e_{k,1}}, \ldots, f_{e_{k,w-1}}\}\})
\]
\[= \prod_{i=1}^{w} \left(1 - \frac{1}{|F|}\right),
\]
where $In(i_{k+1}) = Out(i_{k}) = \{e_{k,1}, e_{k,2}, \ldots, e_{k,w}\}$ and $\emptyset$ is a zero vector.
Combining the above, we get
\[
Pr(\Gamma_{t,r+1}) = \left[\prod_{i=1}^{w} \left(1 - \frac{1}{|F|}\right)\right]^{r+1},
\]
that is,
\[
P_{e_t} = 1 - Pr(\Gamma_{t,r+1}) = 1 - \left[\prod_{i=1}^{w} \left(1 - \frac{1}{|F|}\right)\right]^{r+1}.
\]
This means that the upper bound of the failure probability at the sink node is tight, and the type of plaint networks is the worst case.

As mentioned above, sometimes, it is hard to use the pre-designed linear network coding based on the network topology even through the topology of the network is known. But usually we still can get some information about the network topology more or less. For instance, we can know the number of the internal nodes $|J|$ at least. In these cases, we also can analyze the performance of the random linear network coding.

**Theorem 5:** Let $G$ be a single source multicast network. Using the random linear network coding, the failure probability at the sink node $t \in T$ satisfies
\[
P_{e_t} \leq 1 - \left[\prod_{i=1}^{w} \left(1 - \frac{1}{|F|}\right)\right]^{[J]+1}.
\]
**Remark 4:** This upper bound is still tight and we can also give an example to indicate the tightness.

**Example 3:** For a given information rate $w$, construct a plait network $G_2$, where the unique source node is $s$, the sink node is $t$, and all internal nodes are $i_1, i_2, \ldots, i_{|J|}$. Let the topological order of all nodes be
\[
s \triangleq i_0 < i_1 < i_2 < \cdots < i_{|J|} < i_{|J|+1} \triangleq t.
\]
There are $w$ parallel channels from $i_j$ to $i_{j+1}$, $0 \leq j \leq |J|$. Similar to the example above, we obtain that the failure probability $P_{e_t}$ for plait network $G_2$ is
\[
P_{e_t} = 1 - \left[\prod_{i=1}^{w} \left(1 - \frac{1}{|F|}\right)\right]^{[J]+1}.
\]

**IV. THE LOWER BOUNDS OF THE FAILURE PROBABILITIES**

In the last section, we give some upper bounds of the failure probability at sink node in order to analyze performance of random linear network coding. In this section, we give the lower bound of this failure probability.

**Theorem 6:** For a single source multicast network $G$, using random linear network coding, the failure probability at the sink node satisfies $P_{e_t} \geq 1/|F|^{|J|+1}$, where $\delta_t = C_t - w$.

**Remark 5:** The lower bound in this theorem is also asymptotically tight.

**V. CONCLUSION**

The performance of random linear network coding is important for theory and application. In the present paper, we further analyze the upper bounds of failure probability at sink node. In particular, the more information about the network topology is utilized, the better upper bounds are obtained. We further discuss the lower bound of this failure probability and indicate that it is also asymptotically tight.

In addition, other probabilities, such as failure probability for network and average failure probability, can also be defined to characterize the performance of random linear network coding. We have also analyzed these probabilities. But due to limited pages, we omit them.

**REFERENCES**

[1] R. W. Yeung and Z. Zhang, "Distributed Source Coding for Satellite Communications," IEEE Transactions on Information Theory, vol. 45, no. 4, pp. 1111-1120, May 1999.

[2] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, “Network Information Flow,” IEEE Transactions on Information Theory, vol. 46, no. 4, pp. 1204-1216, Jul. 2000.

[3] S.-Y. R. Li, R. W. Yeung, and N. Cai, “Linear Network Coding,” IEEE Transactions on Information Theory, vol. 49, no. 2, pp. 371-381, Jul. 2003.

[4] R. Koetter and M. Médard, “An Algebraic Approach to Network Coding,” IEEE/ACM Transactions on Networking, vol. 11, no. 5, pp. 782-795, Oct. 2003.

[5] S. Jaggi, P. Sanders, P. A. Chou, M. Effros, S. Egner, K. Jain, and L. M. G. M. Tulliozien, “Polynomial Time Algorithms for Multicast Network Code Constructions,” IEEE Transactions on Information Theory, vol. 51, no. 6, pp. 1973-1982, Jun. 2005.

[6] R. W. Yeung, S.-Y. R. Li, N. Cai, and Z. Zhang, “Network Coding Theory,” Foundations and Trends in Communications and Information Theory, vol. 2, nos.4 and 5, pp. 241-381, 2005.

[7] R. W. Yeung, “Information Theory and Network Coding,” Springer.

[8] C. Fragouli and E. Soljanin, “Network Coding Fundamentals,” Foundations and Trends in Networking, vol. 2, no.1, pp. 1-133, 2007.

[9] C. Fragouli and E. Soljanin, “Network Coding Applications,” Foundations and Trends in Networking, vol. 2, no.2, pp. 135-269, 2007.

[10] T. Ho and D. S. Lun, “Network Coding: An Introduction,” Cambridge University Press, 2008.

[11] T. Ho, M. Médard, J. Shi, M. Effros, and D. R. Karger, “On randomized network coding,” in Proc. 41st Annu. Allerton Conf. Communication, Control, and Computing, Monticello, IL, Oct. 2003.

[12] T. Ho, R. Koetter, M. Médard, M. Effros, J. Shi, and D. Karger, “A Random Linear Network Coding Approach to Multicast,” IEEE Transactions on Information Theory, vol. 52, no. 10, pp. 4413-4430, Oct. 2006.

[13] H. Balli, X. Yan, and Z. Zhang, “On Randomized Linear Network Codes and Their Error Correction Capabilities,” IEEE Transactions on Information Theory, vol. 55, no. 7, pp. 3148-3160, Jul. 2009.

[14] X. Guan, and F. W. Fu, “On Random Linear Network Coding for Butterfly Network,” submitted to Chinese Journal of Electronics, 2010.