Research Article

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The Hahn-Exton $q$-Bessel function as the characteristic function of a Jacobi matrix

Abstract: A family $T^{(\nu)}$, $\nu \in \mathbb{R}$, of semiinfinite positive Jacobi matrices is introduced with matrix entries taken from the Hahn-Exton $q$-difference equation. The corresponding matrix operators defined on the linear hull of the canonical basis in $l^2(\mathbb{Z}_+)$ are essentially self-adjoint for $|\nu| \geq 1$ and have deficiency indices $(1, 1)$ for $|\nu| < 1$. A convenient description of all self-adjoint extensions is obtained and the spectral problem is analyzed in detail. The spectrum is discrete and the characteristic equation on eigenvalues is derived explicitly in all cases. Particularly, the Hahn-Exton $q$-Bessel function $J_\nu(z; q)$ serves as the characteristic function of the Friedrichs extension. As a direct application one can reproduce, in an alternative way, some basic results about the $q$-Bessel function due to Koelink and Swarttouw.

Keywords: Jacobi matrix, Hahn-Exton $q$-Bessel function, self-adjoint extension, spectral problem

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1 Introduction

There exist three commonly used $q$-analogues of the Bessel function $J_\nu(z)$. Two of them were introduced by Jackson in the beginning of the 20th century and are mutually closely related, see [6] for a basic overview and original references. Here we shall be concerned with the third analogue usually named after Hahn and Exton. Its most important features like properties of the zeros and the associated Lommel polynomials including orthogonality relations were studied not so long ago in [9–11]. The Hahn-Exton $q$-Bessel function is defined as follows

$$J_\nu(z; q) \equiv f_\nu^{(3)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} z^{\nu} \phi_1(0; q^{\nu+1}; q, qz^2).$$

(1)

Here $\phi_1(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z)$ stands for the basic hypergeometric series (see, for instance, [6]). It is of importance that $J_\nu(z; q)$ obeys the Hahn-Exton $q$-Bessel difference equation

$$J_\nu(qz; q) + \frac{q^{-\nu/2}(qz^2 - 1 - q^\nu)}{q^{1/2}z^2}J_\nu(q^{1/2}z; q) + J_\nu(z; q) = 0.$$  

(2)

Using the coefficients from (2) one can introduce a two-parameter family of real symmetric Jacobi matrices

$$T \equiv T^{(\nu)} = \begin{pmatrix} \beta_0 & a_0 & a_1 & \ldots \\ a_0 & \beta_1 & a_1 & \ldots \\ a_1 & \beta_2 & a_2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(3)

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depending on $v \in \mathbb{R}$ and also on $q$, $0 < q < 1$. But $q$ is treated below as having been fixed and is not indicated explicitly in most cases. Matrix entries are supposed to be indexed by $m, n = 0, 1, 2, \ldots$. More formally, we put $T_{n,n} = \beta_n$, $T_{n,n+1} = T_{n+1,n} = \alpha_n$ and $T_{m,n} = 0$ otherwise, where
\[ \alpha_n \equiv \alpha_n(v) = -q^{-n+(v-1)/2}, \quad \beta_n \equiv \beta_n(v) = (1 + q^v)q^{-n}, \quad n \in \mathbb{Z}_+, \] (4) and with $\mathbb{Z}_+$ standing for nonnegative integers. In order to keep notations simple we will also suppress the superscript $(v)$ provided this cannot lead to misunderstanding.

Our main goal in this paper is to provide a detailed analysis of those operators $T$ in $\ell^2 \equiv \ell^2(\mathbb{Z}_+)$ whose matrix in the canonical basis equals $T$. This example has that interesting feature that it exhibits a transition between the indeterminate and determinate cases depending on $v$. In more detail, denote by $C^\infty$ the linear space of all complex sequences indexed by $\mathbb{Z}_+$ and by $D$ the subspace of those sequences having at most finitely many nonvanishing entries. One may also say that $D$ is the linear hull of the canonical basis in $\ell^2$. It turns out that the matrix operator induced by $T$ on the domain $D$ is essentially self-adjoint in $\ell^2$ if and only if $|v| \geq 1$. For $|v| < 1$ there exists a one-parameter family of self-adjoint extensions.

Another interesting point is a close relationship between the spectral data for these operators $T$ and the Hahn-Exton $q$-Bessel function. It turns out that, for an appropriate (Friedrichs) self-adjoint extension, $J_v(q^{-1/2}\sqrt{\pi}; q)$ serves as the characteristic function of $T$ in the sense that its zero set on $\mathbb{R}$, exactly coincides with the spectrum of $T$. There also exists an explicit formula for corresponding eigenvectors. Moreover, $T^{-1}$ can be shown to be compact. This makes it possible to reproduce, in a quite straightforward but alternative way, some results originally derived in [9, 11].

Finally we remark that recently we have constructed, in [14, 15], a number of examples of Jacobi operators with discrete spectra and characteristic functions explicitly expressed in terms of special functions, a good deal of them comprising various combinations of $q$-Bessel functions. That construction confines, however, only to a class of Jacobi matrices characterized by a convergence condition imposed on the matrix entries. For this condition is readily seen to be violated in the case of $T$, as defined in (3) and (4), in the present paper we have to undertake another approach whose essential part is a careful asymptotic analysis of formal eigenvectors of $T$.

\section{Self-adjoint operators induced by $T$}

\subsection{A *-algebra of semiinfinite matrices}

Denote by $M_{\text{fin}}$ the set of all semiinfinite matrices indexed by $\mathbb{Z}_+ \times \mathbb{Z}_+$ such that each row and column of a matrix has only finitely many nonzero entries. For instance, $M_{\text{fin}}$ comprises all band matrices and so all finite-order difference operators. Notice that $M_{\text{fin}}$ is naturally endowed with the structure of a *-algebra, matrices from $M_{\text{fin}}$ act linearly on $C^\infty$ and $D$ is $M_{\text{fin}}$-invariant.

Choose $A \in M_{\text{fin}}$ and let $A^H$ stand for its Hermitian adjoint. Let us introduce, in a fully standard manner, operators $\hat{A}$, $A_{\text{min}}$ and $A_{\text{max}}$ on $\ell^2$, all of them being restrictions of $A$ to appropriate domains. Namely, $\hat{A}$ is the restriction $A|_D$, $A_{\text{min}}$ is the closure of $\hat{A}$ and
\[ \text{Dom} A_{\text{max}} = \{ f \in \ell^2 ; Af \in \ell^2 \}. \]
Clearly, $\hat{A} \subseteq A_{\text{max}}$. Straightforward arguments based just on systematic application of definitions show that
\[ (\hat{A})^* = (A_{\text{min}})^* = A_{\text{max}}^H, \quad (A_{\text{max}})^* = A_{\text{min}}^H. \]
Hence $A_{\text{max}}$ is closed and $A_{\text{min}} \subseteq A_{\text{max}}$.

\textbf{Lemma 1.} Suppose $p, w \in \mathbb{C}$ and let $A \in M_{\text{fin}}$ be defined by
\[ A_{n,n} = p^n, \quad A_{n+1,n} = -wp^{n+1} \quad \text{for all } n \in \mathbb{Z}_+, \quad A_{m,n} = 0 \text{ otherwise}. \] (5)
Then $A_{\text{min}} \neq A_{\text{max}}$ if and only if $1/|p| < |w| < 1$, and in that case

$$\text{Dom} \ A_{\text{min}} = \{ f \in \text{Dom} \ A_{\text{max}} ; \lim_{n \to \infty} w^n f_n = 0 \}.$$

Proof. Choose arbitrary $f \in \text{Dom} \ A_{\text{max}}$. Then $f \in \text{Dom} \ A_{\text{min}}$ iff

$$\forall g \in \text{Dom} \ A_{\text{max}}^H, \ 0 = \langle A^H g, f \rangle - \langle g, Af \rangle = - \lim_{n \to \infty} A_{n,n} g_n f_n. \quad (6)$$

Since both $f$ and $g$ in (6) are supposed to belong to $l^2$ this condition is obviously fulfilled if $|p| \leq 1$. Furthermore, the situation becomes fully transparent for $w = 0$. In that case the sequences $\{p^n g_n\}$ and $\{p^n f_n\}$ are square summable and (6) is always fulfilled. In the remainder of the proof we assume that $|p| > 1$ and $w \neq 0$.

Consider first the case when $|w| \geq 1$. Relation $Af = h$ can readily be inverted, even in $C^\infty$, and one finds that

$$p^n f_n = \sum_{k=0}^{n} (pw)^k h_{n-k} = (pw)^n \sum_{k=0}^{n} (pw)^{-k} h_k, \ \forall n.$$ Denote temporarily by $\tilde{h}$ the sequence with $\tilde{h}_n = (\overline{pw})^{-n}$. It is square summable since, by our assumptions, $|pw| > 1$. For $f \in \text{Dom} \ A_{\text{max}}$ one has $h \in l^2$ and

$$f_n = w^n (\tilde{h}, h) - \zeta_n$$ where $\zeta_n = \sum_{k=n+1}^{\infty} (pw)^{-k} h_k$.

Assumption $f \in l^2$ clearly implies $\langle \tilde{h}, h \rangle = 0$ and then, by the Schwarz inequality,

$$|A_{n,n} f_n| \leq \|\tilde{h}\| \sqrt{|pw| - 1}, \ \forall n.$$ Whence $A_{n,n} g_n f_n \to 0$ as $n \to \infty$ for all $g \in \text{Dom} \ A_{\text{max}}^H$ and so $f \in \text{Dom} \ A_{\text{min}}$.

Suppose now that $|w| < 1$. If $A^H g = h$ in $C^\infty$ and $h$ is bounded then, as an easy computation shows,

$$(\overline{p})^n g_n = \gamma (\overline{w})^{-n} + \sum_{k=n+1}^{\infty} (\overline{w})^k h_{n-k} \quad (7)$$

for all $n$ and some constant $\gamma$. Observe that, by the Schwarz inequality,

$$\left| \sum_{k=n+1}^{\infty} (\overline{w})^k h_{n-k} \right| \leq \frac{1}{\sqrt{1 - |w|^2}} \left( \sum_{k=n}^{\infty} |h_k|^2 \right)^{1/2}, \quad (8)$$

and this expression tends to zero as $n$ tends to infinity provided $h \in l^2$.

In the case when $|pw| \leq 1$ the property $g \in l^2$ and $A^H g = h \in l^2$ implies that the constant $\gamma$ in (7) is zero, and from (8) one infers that $A_{n,n} g_n \to 0$ as $n \to \infty$. Thus one finds condition (6) to be always fulfilled meaning that $f \in \text{Dom} \ A_{\text{min}}$.

If $|pw| > 1$ then the sequence $g$ defined in (7) is square summable whatever $\gamma \in \mathbb{C}$ and $h \in l^2$ are. Condition (6) is automatically fulfilled, however, for $\gamma = 0$. Hence (6) can be reduced to the single nontrivial case when we choose $\tilde{g} \in \text{Dom} \ A_{\text{max}}^H$ with $\tilde{g}_n = (\overline{pw})^{-n}$. Then $A^H \tilde{g} = 0$ and condition $\langle \tilde{g}, Af \rangle = 0$ means that $w^n f_n \to 0$ as $n \to \infty$. It remains to show that there exists $f \in \text{Dom} \ A_{\text{max}}$ not having this property. However the sequence $\tilde{f}$, with $\tilde{f}_n = w^n$, does the job since $A \tilde{f} = (1, 0, 0, \ldots) \in l^2$.

### 2.2 Associated orthogonal polynomials, self-adjoint extensions

The tridiagonal matrix $T$ defined in (3), (4) belongs to $M_{\text{fin}}$. With $T$ there is associated a sequence of monic orthogonal polynomials [7], called $\{P_n(x) \equiv P_n^{(0)}(x)\}$ and defined by the recurrence

$$P_n(x) = (x - \beta_{n-1}) P_{n-1}(x) - \alpha_{n-2} P_{n-2}(x), \ n \geq 1, \quad (9)$$
with \( P_-(x) = 0, P_0(x) = 1 \). Put

\[
P_n(x) = \hat{P}_n(x) = (-1)^\nu q^{\nu(n-\nu)/2} P_n(x).
\]

(10)

Then \( \hat{P}_0(x), \hat{P}_1(x), \hat{P}_2(x), \ldots \) is a formal eigenvector of \( T \) (\( \equiv \) an eigenvector of \( T \) in \( \mathbb{C}^\infty \)), i.e.

\[
(\beta_0 - x) \hat{P}_0(x) + a_0 \hat{P}_1(x) = 0,
\]

\[
(\beta_n - x) \hat{P}_n(x) + a_n \hat{P}_{n-1}(x) = 0 \quad \text{for } n \geq 1.
\]

(11)

Observe that \( \tau^{(\nu)} = q^\nu \tau^{(\nu)} \). Since we are primarily interested in spectral properties of \( \tau^{(\nu)} \) in the Hilbert space \( \ell^2 \) we may restrict ourselves, without loss of generality, to nonnegative values of the parameter \( \nu \). The value \( \nu = 0 \) turns out to be somewhat special and will be discussed separately later on, in Subsection 3.2. Thus, if not stated otherwise, we assume from now on that \( \nu > 0 \).

Given \( T \in \mathcal{M}_{\mathbb{R}_+} \) we again introduce the operators \( \hat{T}, T_{\min}, T_{\max} \) as explained in Subsection 2.1. Notice that

\[
\beta_n = q^{(\nu-1)/2} |a_{n-1}| + q^{-(\nu-1)/2} |a_n|.
\]

It follows at once that the operators \( \hat{T} \) and consequently \( T_{\min} \) are positive. In fact, for any real sequence \( \{f_n\} \in \mathcal{D} \) one has

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} T_{m,n} f_m f_n = |a_{-1}| q^{(\nu-1)/2} f_0^2 + \sum_{n=1}^{\infty} |a_{n-1}| (q^{(\nu-1)/4} f_n - q^{-(\nu-1)/4} f_{n-1})^2 \geq 0.
\]

This is equivalent to the factorization \( T = A^H A \) where the matrix \( A \equiv A^{(\nu)} \in \mathcal{M}_{\mathbb{R}_+} \) is defined by the prescription:

\[
(\mathcal{A} f)_0 = |a_{-1}|^{1/2} q^{-(\nu-1)/4} f_0, \quad (\mathcal{A} f)_n = |a_{n-1}|^{1/2} (q^{(\nu-1)/4} f_n - q^{-(\nu-1)/4} f_{n-1}) \quad \text{for } n \geq 1.
\]

That is, \( \forall n \geq 0 \),

\[
\mathcal{A}_{n,n} = |a_{n-1}|^{1/2} q^{-(\nu-1)/4} = q^{-(\nu-1)/2}, \quad \mathcal{A}_{n+1,n} = -|a_n|^{1/2} q^{-(\nu-1)/4} = -q^{-n/2},
\]

(12)

and \( \mathcal{A}_{m,n} = 0 \) otherwise.

Thus \( T \) induces a positive form on the domain \( \mathcal{D} \) with values \( \langle f, T f \rangle = |\mathcal{A} f|^2, \forall f \in \mathcal{D} \). Let us call \( t \) its closure. Then \( \text{Dom } t = \text{Dom } A_{\min} \) and \( t(x) = |A_{\min} x|^2, \forall x \in \text{Dom } t \). The positive operator \( T^F \) associated with \( t \) according to the representation theorem is the Friedrichs extension of the closed positive operator \( T_{\min} \). One has

\[
T^F = A_{\min}^* A_{\min}.
\]

It is known that \( T^F \) has the smallest form-domain among all self-adjoint extensions of \( T_{\min} \), and also that this is the only self-adjoint extension of \( T_{\min} \), with its domain contained in \( \text{Dom } t \), see [8, Chapter VI].

One can apply Lemma 1, with \( p = q^{1/2} \) and \( w = q^{(1-\nu)/2} \), to obtain an explicit description of the form-domain of \( T^F \). Using still \( A \) defined in (12) one has \( \text{Dom } t = \{ f \in \ell^2; A f \in \ell^2 \} \) for \( \nu \geq 1 \), and

\[
\text{Dom } t = \{ f \in \ell^2; A f \in \ell^2 \text{ and } \lim_{n \to \infty} q^{(\nu-1)n/2} f_n = 0 \}
\]

(13)

for \( 0 < \nu < 1 \).

In [5] one finds a clear explicit description of the domain of the Friedrichs extension of a positive Jacobi matrix which can be applied to our case. To this end, consider the homogeneous three-term recurrence equation

\[
a_n Q_{n+1} + \beta_n Q_n + a_{n-1} Q_{n-1} = 0
\]

(14)

on \( \mathbb{Z} \). It simplifies to a recurrence equation with constant coefficients,

\[
q^{(\nu-1)/2} Q_{n+1} - (1 + q^\nu) Q_n + q^{(\nu+1)/2} Q_{n-1} = 0.
\]

(15)

One can distinguish two independent solutions, \( \{ Q_n^{(1)} \} \) and \( \{ Q_n^{(2)} \} \), where

\[
Q_n^{(1)} = \frac{q^{(1-\nu)n/2} - q^{(\nu+1)n/2}}{1 - q^\nu}, \quad Q_n^{(2)} = Q_n^{(1+\nu)n/2}, \quad n \in \mathbb{Z}.
\]

(16)
Notice that \( \{Q_n^{(1)}\} \) satisfies the initial conditions \( Q_{n-1}^{(1)} = 0, Q_n^{(1)} = 1 \), and so \( Q_n^{(1)} = \tilde{P}_n(0), \forall n \geq 0 \). On the other hand, \( \{Q_n^{(2)}\} \) is always square summable over \( \mathbb{Z} \), and this is the so-called minimal solution at \(+\infty\) since

\[
\lim_{n \to +\infty} \frac{Q_n^{(2)}}{Q_n} = 0
\]

for every solution \( \{Q_n\} \) of (14) which is linearly independent of \( \{Q_n^{(2)}\} \). The Wronskian of \( Q^{(1)} \) and \( Q^{(2)} \) equals

\[
W_n(Q^{(1)}, Q^{(2)}) = 1, \forall n \in \mathbb{Z},
\]

where \( W_n(f, g) := a_n(f_n g_{n+1} - g_n f_{n+1}) \). Theorem 4 in [5] tells us that

\[
\text{Dom } T^F = \{ f \in \ell^2 ; T f \in \ell^2 \text{ and } W_\infty(f, Q^{(2)}) = 0 \}
\]

(17)

where we put

\[
W_\infty(f, g) = \lim_{n \to \infty} W_n(f, g)
\]

for \( f, g \in C^\infty \) provided the limit exists. It is useful to note, however, that discrete Green’s formula implies existence of the limit whenever \( f, g \in \text{Dom } T_{\max} \), and then

\[
\langle T_{\max} f, g \rangle - \langle f, T_{\max} g \rangle = -W_\infty(T, g).
\]

We wish to determine all self-adjoint extensions of the closed positive operator \( T_{\min} \). This is a standard general fact that the deficiency indices of \( T_{\min} \) for any real symmetric Jacobi matrix \( J \) of the form (3), with all \( a_n’s \) nonzero, are either \((0, 0)\) or \((1, 1)\). The latter case happens if and only if for some \( x \in \mathbb{C} \) all solutions of the second-order difference equation

\[
\alpha_n Q_{n+1} + (\beta_n - x) Q_n + \alpha_n Q_{n-1} = 0
\]

(18)

are square summable on \( \mathbb{Z}_+ \), and in that case this is true for any value of the spectral parameter \( x \) (see, for instance, a detailed discussion in Section 2.6 of [16]).

Let us remark that a convenient description of the one-parameter family of all self-adjoint extensions is also available if the deficiency indices are \((1, 1)\). Fix \( x \in \mathbb{R} \) and any couple \( Q^{(1)}, Q^{(2)} \) of independent solutions of (18). Then all self-adjoint extensions of \( T_{\min} \) are operators \( \tilde{T}(k) \) defined on the domains

\[
\text{Dom } \tilde{T}(k) = \{ f \in \ell^2 ; T f \in \ell^2 \text{ and } W_\infty(f, Q^{(1)}) = k W_\infty(f, Q^{(2)}) \}
\]

(19)

with \( k \in \mathbb{R} \cup \{-\infty\} \). Moreover, all of them are mutually different. Of course, \( \tilde{T}(k) = T f \), \( \forall f \in \text{Dom } \tilde{T}(k) \).

In our case we know, for \( x = 0 \), a couple of solutions of (18) explicitly, cf. (16). From their form it becomes obvious that \( T_{\min} = T_{\max} \) is self-adjoint if and only if \( \nu \geq 1 \). With this choice of \( Q^{(1)}, Q^{(2)} \) and sticking to notation (19), it is seen from (17) that the Friedrichs extension \( T^F \) coincides with \( \tilde{T}(\infty) \).

Lemma 2. Suppose \( 0 < \nu < 1 \). Then every sequence \( f \in \text{Dom } T_{\max} \) has the asymptotic expansion

\[
f_n = C_1 q^{(1-\nu)n/2} + C_2 q^{(1+\nu)n/2} + o(q^n) \text{ as } n \to \infty,
\]

(20)

where \( C_1, C_2 \in \mathbb{C} \) are some constants.

Proof. Let \( f \in \text{Dom } T_{\max} \). That means \( f \in \ell^2 \) and \( \mathcal{A} H \mathcal{A}^* f = h \in \ell^2 \) where \( \mathcal{A} \) is defined in (5), with \( p = q^{-1/2}, \ w = q^{(1-\nu)/2} \) (then \( H = q^\nu \mathcal{A}^* \mathcal{A} \)). Denote \( g = \mathcal{A} f \). Hence \( \mathcal{A}^* g = h \) and, as already observed in the course of the proof of Lemma 1, there exists a constant \( \gamma \) such that

\[
g_n = \gamma q^{\nu n/2} + q_n^{\nu n/2} \sum_{k=0}^\infty q^{(1-\nu)k/2} h_{n+k}, \forall n.
\]

Furthermore, the relation \( \mathcal{A} f = g \) can be inverted,

\[
f_n = q^{(1-\nu)n/2} \sum_{k=0}^n q^{\nu k/2} g_k, \forall n.
\]
Whence
\[ f_n = \frac{\gamma}{1-q^2} \left( q^{(1-v)n/2} - q^{v+(1+v)n/2} \right) + q^{(1-v)n/2} \sum_{k=0}^{n} q^{(1+v)k/2} \sum_{j=0}^{\infty} q^{(1-v)j/2} h_{k+j} \]
\[ = C_1 q^{(1-v)n/2} + C_2 q^{(1+v)n/2} + q^n \zeta_n \]
where
\[ C_1 = \frac{\gamma}{1-q^2} + \sum_{k=0}^{\infty} q^{(1+v)k/2} \sum_{j=0}^{\infty} q^{(1-v)j/2} h_{k+j}, \quad C_2 = -\frac{\gamma q^v}{1-q^2}, \]
and
\[ \zeta_n = -\sum_{k=1}^{\infty} q^{(1+v)k/2} \sum_{j=0}^{\infty} q^{(1-v)j/2} h_{n+k+j}. \]

Bearing in mind that \( h \in \ell^2 \) one concludes, with the aid of the Schwarz inequality, that \( \zeta_n \to 0 \) as \( n \to \infty \). \( \square \)

With the knowledge of the asymptotic expansion established in Lemma 2 one can formulate a somewhat simpler and more explicit description of self-adjoint extensions of \( T_{\min} \).

**Proposition 3.** The operator \( T_{\min} \equiv T_{\min}^{(v)} \), with \( v > 0 \), is self-adjoint if and only if \( v \geq 1 \). If \( 0 < v < 1 \) then all mutually different self-adjoint extensions of \( T_{\min} \) are parametrized by \( \kappa \in P^1(\mathbb{R}) \equiv \mathbb{R} \cup \{ \infty \} \) as follows. For \( f \in \text{Dom} \ T_{\max} \) let \( C_1(f), C_2(f) \) be the constants from the asymptotic expansion (20), i.e.
\[ C_1(f) = \lim_{n \to \infty} f_n q^{-(1-v)n/2}, \quad C_2(f) = \lim_{n \to \infty} (f_n - C_1(f)q^{(1-v)n/2})q^{(1+v)n/2}. \]

For \( \kappa \in P^1(\mathbb{R}) \), a self-adjoint extension \( T(\kappa) \) of \( T_{\min} \) is a restriction of \( T_{\max} \) to the domain
\[ \text{Dom} \ T(\kappa) = \{ f \in \ell^2; \exists f \in \ell^2 \text{ and } C_2(f) = \kappa C_1(f) \}. \] (21)

In particular, \( T(\infty) \) equals the Friedrichs extension \( T^F \).

**Proof.** Let \( 0 < v < 1 \), \( \{ \zeta_n \} \) be a sequence converging to zero (bounded would be sufficient) and \( g^{(1)}, g^{(2)}, h \in C^\infty \) be the sequences defined by
\[ h_n^{(1)} = q^{(1-v)n/2}, \quad g_n^{(2)} = q^{(1+v)n/2}, \quad h_n = q^n \zeta_n, \forall n. \]

Hence, referring to (16),
\[ Q^{(1)} = \frac{1}{1-q^2} (g^{(1)} - q^v g^{(2)}), \quad Q^{(2)} = g^{(2)}. \]

One finds at once that \( W_{\infty}(g^{(1)}, h) = W_{\infty}(g^{(2)}, h) = 0 \) and
\[ W_n(g^{(1)}, g^{(2)}) = 1 - q^v, \forall n. \]

After a simple computation one deduces from (19) that \( f \in \text{Dom} \ T_{\max} \) belongs to \( \text{Dom} \ \tilde{T}(\kappa) \) for some \( \tilde{\kappa} \in P^1(\mathbb{R}) \), i.e. \( W_{\infty}(f, Q^{(1)}) = \tilde{\kappa} W_{\infty}(f, Q^{(2)}) \), if and only if \( C_2(f) = \kappa C_1(f) \), with \( \kappa = -q^v - (1 - q^v)\tilde{\kappa} \). In other words, \( \tilde{T}(\kappa) = T(\kappa) \). Since the mapping
\[ P^1(\mathbb{R}) \to P^1(\mathbb{R}); \tilde{\kappa} \mapsto \kappa = -q^v - (1 - q^v)\tilde{\kappa} \]
is one-to-one, \( P^1(\mathbb{R}) \ni \kappa \mapsto T(\kappa) \) is another parametrization of self-adjoint extensions of \( T_{\min} \). Particularly, \( \tilde{\kappa} = \infty \) maps to \( \kappa = \infty \) and so \( T(\infty) = T^F \). \( \square \)

**Remark 4.** One can also describe \( \text{Dom} \ T_{\min} \). For \( v \geq 1 \) we simply have \( T_{\min} = T_{\max} = T^F \). In the case when \( 0 < v < 1 \) it has been observed in [5] that a sequence \( f \in \text{Dom} \ T_{\max} \) belongs to \( \text{Dom} \ T_{\min} \) if and only if \( W_{\infty}(f, g) = 0 \) for all \( g \in \text{Dom} \ T_{\max} \). But this is equivalent to the requirement \( C_1(f) = C_2(f) = 0 \). Thus one has
\[ \text{Dom} \ T_{\min} = \{ f \in \ell^2; \exists f \in \ell^2 \text{ and } \lim_{n \to \infty} f_n q^{-(1+v)n/2} = 0 \}. \] (22)
2.3 The Green function and spectral properties

For \( \nu \geq 1 \) we shall write shortly \( T \equiv T^{(\nu)} \) instead of \( T_{\min} = T_{\max} = T^F \). Referring to solutions (16) we claim that the Green function (matrix) of \( T \), if \( \nu \geq 1 \), or \( T^F \), if \( 0 < \nu < 1 \), reads

\[
G_{j,k} = \begin{cases} 
Q_j^{(1)}Q_k^{(2)} & \text{for } j \leq k, \\
Q_k^{(1)}Q_j^{(2)} & \text{for } j > k.
\end{cases} \tag{23}
\]

**Proposition 5.** The matrix \( (G_{j,k}) \) defined in (23) represents a Hilbert-Schmidt operator \( G \equiv G^{(\nu)} \) on \( \ell^2 \) with the Hilbert-Schmidt norm

\[
\|G\|_{HS}^2 = \frac{1 + q^{2+\nu}}{1-q^2(1-q^{1+\nu})^2(1-q^{2+\nu})}. \tag{24}
\]

The operator \( G \) is positive and one has, \( \forall f \in \ell^2 \),

\[
(f, Gf) = \sum_{k=0}^{\infty} q^k \sum_{j=0}^{\infty} q^{(1+\nu)j/2} f_{k+j}^2.
\]

Moreover, the inverse \( G^{-1} \) exists and equals \( T \), if \( \nu \geq 1 \), or \( T^F \), if \( 0 < \nu < 1 \).

**Proof.** As is well known, if \( T_{\min} \) is not self-adjoint then the resolvent of any of its self-adjoint extensions is a Hilbert-Schmidt operator [16, Lemma 2.19]. But in our case the resolvent is claimed to be Hilbert-Schmidt for \( \nu \geq 1 \) as well. One can directly compute the Hilbert-Schmidt norm of \( G \) for any \( \nu > 0 \),

\[
\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} G_{j,k}^2 = \sum_{j=0}^{\infty} (Q_j^{(1)})^2(Q_j^{(2)})^2 + 2 \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} (Q_k^{(1)})^2(Q_k^{(2)})^2
\]

\[
= \frac{1 + q^{1+\nu}}{1-q^{2+\nu}} \sum_{j=0}^{\infty} \left( \frac{1-q^j}{1-q} \right)^2 q^{2j}.
\]

Thus one obtains (24). Hence the Green matrix unambiguously defines a self-adjoint compact operator \( G \) on \( \ell^2 \).

Concerning the formula for the quadratic form one has to verify that, for all \( m, n \in \mathbb{Z}_+ \), \( m \leq n \),

\[
Q_n^{(1)}Q_n^{(2)} = \sum_{k=0}^{\infty} q^k \left( \sum_{j=0}^{\infty} q^{(1+\nu)j/2} \delta_{m,k+j} \right) \left( \sum_{j=0}^{\infty} q^{(1+\nu)j/2} \delta_{n,k+j} \right).
\]

But this can be carried out in a straightforward manner.

A simple computation shows that for any \( f \in \mathbb{C}^\infty \) and \( n, N \in \mathbb{Z}_+, n < N \),

\[
Q_n^{(2)} \sum_{k=0}^{n} Q_k^{(1)}(\mathcal{T}f)_k + Q_n^{(1)} \sum_{k=n+1}^{N} Q_k^{(2)}(\mathcal{T}f)_k = f_n - Q_n^{(1)}a_N(Q_n^{(2)}f_N - Q_n^{(2)}f_{N+1}).
\]

Considering the limit \( N \to \infty \) one finds that, for a given \( f \in \text{Dom } T_{\max} \), the equality \( G\mathcal{T}f = f \) holds iff \( W_\infty(f, Q^{(2)}) = 0 \). According to (17), this condition determines the domain of the Friedrichs extension \( T^F \). Hence \( G T^F \subset I \) (the identity operator).

Furthermore, one readily verifies that, for all \( f \in \ell^2 \), \( \mathcal{T}Gf = f \). We still have to check that \( \text{Ran } G \subset \text{Dom } T^F \). But using the equality \( W_\infty(Q^{(1)}, Q^{(2)}) = 1 \) one computes, for \( f \in \ell^2 \) and \( n \in \mathbb{Z}_+ \),

\[
W_n(Gf, Q^{(2)}) = \sum_{k=n+1}^{\infty} Q_k^{(2)}f_k \to 0 \text{ as } n \to \infty,
\]

since \( Q^{(2)} \in \ell^2 \). Hence \( T^F G = I \). We conclude that \( G^{-1} = T^F \) (remember that we have agreed to write \( T^F = T \) for \( \nu \geq 1 \)). \( \square \)
Proposition 6. The spectrum of any of the operators \( T \), if \( v \geq 1 \), or \( T(\kappa) \), with arbitrary \( \kappa \in P^1(\mathbb{R}) \), if \( 0 < v < 1 \), is pure point and bounded from below, with all eigenvalues being simple and without finite accumulation points. Moreover, the operator \( T \), for \( v \geq 1 \), or \( T^\xi \), for \( 0 < v < 1 \), is positive definite, and one has the following lower bound on the spectrum, i.e. on the smallest eigenvalue \( \xi_1 \equiv \xi_{1|v}^1 \),

\[
\xi_1^2 \geq \frac{(1 - q^2)(1 - q^{1 + v})(1 - q^{2 + v})}{1 + q^{2 + v}}.
\]

Proof. This is a simple general fact that all formal eigenvectors of the Jacobi matrix \( J \) are unique up to a multiplier [3]. By Proposition 5, \((T^\xi)^{-1}\) is compact and therefore the spectrum of \( T^\xi \) is pure point and with eigenvalues accumulating only at infinity.

For \( 0 < v < 1 \), the deficiency indices of \( \text{min} \) are \((1, 1)\). Whence, by the general spectral theory, if \( T^\xi \) has an empty essential spectrum then the same is true for all other self-adjoint extensions \( T(\kappa) \), \( \kappa \in \mathbb{R} \). Moreover, there is at most one eigenvalue of \( T(\kappa) \) below \( \xi_1 := \text{min spec}(T^\xi) \), see [18, § 8.3]. Referring once more to Proposition 5 one has

\[
\text{min spec}(T^\xi) = (\text{max spec}(G))^{-1} \geq \|G\|_{HS}^{-1}.
\]
In view of (24), one obtains the desired estimate on \( \xi_1 \).

\[ \square \]

2.4 More details on the indeterminate case

In this subsection we confine ourselves to the case \( 0 < v < 1 \) and focus on some general spectral properties of the self-adjoint extensions \( T(\kappa) \), \( \kappa \in P^1(\mathbb{R}) \), in addition to those already mentioned in Proposition 6. The spectra of any two different self-adjoint extensions of \( \text{min} \) are known to be disjoint (see, for instance, proof of Theorem 4.2.4 in [3]). Moreover, the eigenvalues of such a couple of self-adjoint extensions interlace (see [12] and references therein, or this can also be deduced from general properties of self-adjoint extensions with deficiency indices \((1, 1)\) [18, § 8.3]). It is useful to note, too, that every \( \chi \in \mathbb{R} \) is an eigenvalue of a unique self-adjoint extension \( T(\kappa) \), \( \kappa \in P^1(\mathbb{R}) \) [13, Theorem 4.11].

For positive symmetric operators there exists another powerful theory of self-adjoint extensions due to Birman, Krein and Vishik based on the analysis of associated quadratic forms. A clear exposition of the theory can be found in [2]. Its application to our case, with deficiency indices \((1, 1)\), is as follows. A crucial role is played by the null space of \( \text{max} = \text{min} \) which we denote by

\[
\mathcal{N} := \text{Ker} \ T_{\text{max}} = CQ^{(1)}
\]
(recall that \( Q^{(1)} = P_n(0), \forall n \in \mathbb{Z}_\tau \)). Let \( t_\infty = t \) be the quadratic form associated with the Friedrichs extension \( T^\xi \). Remember that the domain of \( t \) has been specified in (13). All other self-adjoint extensions of \( \text{min} \), except for \( T^\xi \), are in one-to-one correspondence with real numbers \( \tau \). The corresponding associated quadratic forms \( t_\tau \), \( \tau \in \mathbb{R} \), have all the same domain,

\[
\text{Dom} \ t_\tau = \text{Dom} \ t_\infty + \mathcal{N}
\]
(a direct sum), and for \( f \in \text{Dom} \ t_\infty \), \( \lambda \in \mathbb{C} \), one has

\[
t_\tau(f + \lambda Q^{(1)}) = t_\infty(f) + \tau |\lambda|^2.
\]

Our next task is to relate the self-adjoint extensions \( T(\kappa) \) described in Proposition 3 to the quadratic forms \( t_\tau \).

Proposition 7. The quadratic form associated with a self-adjoint extension \( T(\kappa) \), \( \kappa \in \mathbb{R} \), is \( t_\tau \) defined in (25), (26), with \( \tau = (\kappa + q^*)/(1 - q^*) \).
Proof. Let $x \in \mathbb{R}$ and $\sigma$ be the real parameter such that $t_0$ is the quadratic form associated with $T(x)$. Recall (16). One has $\mathcal{T}Q^{(1)} = 0$ and $(\mathcal{T}Q^{(2)})_n = \delta_{n,0}, \forall n \in \mathbb{Z}$. According to (17),

$$Q^{(2)} \in \text{Dom } T(\infty) \subset \text{Dom } t_\infty.$$  

One computes $t_\infty(Q^{(2)}) = (Q^{(2)}, \mathcal{T}Q^{(2)}) = 1$. Let $\tau = (x + q^+)/ (1 - q^+)$ and

$$h = \tau Q^{(2)} + Q^{(1)} \in \text{Dom } t_\infty + \mathbb{C}Q^{(1)} = \text{Dom } t_\infty.$$  

Then $(1 - q^+)h_n = q^{(1-v)n/2} + \kappa q^{(1+v)n/2}, \forall n \in \mathbb{Z}$. Hence, in virtue of (21), $h \in \text{Dom } T(x)$, and, referring to (26),

$$\tau^2 + \sigma = t_0(h) = \langle h, T(x)h \rangle = \langle h, Th \rangle = \tau(\tau + 1).$$  

Whence $\sigma = \tau$. \hfill $\square$

Now we are ready to describe the announced additional spectral properties of $T(x)$. The terminology and basic results concerning quadratic (sesquilinear) forms used below are taken from Kato [8].

**Lemma 8.** Let $S$ and $B$ be linear subspaces in a Hilbert space $\mathcal{H}$ such that $S \cap B = \{0\}$, and let $s$ and $b$ be positive quadratic forms on $S$ and $B$, respectively. Denote by $\bar{s}$ and $\bar{b}$ the extensions of these forms to $S + B$ defined by

$$\forall \varphi \in S, \forall \eta \in B, \bar{s}(\varphi + \eta) = s(\varphi) \text{ and } \bar{b}(\varphi + \eta) = b(\eta),$$

and assume that, for every $\rho \in \mathbb{R}$, the form $\bar{s} + \rho \bar{b}$ is semibounded and closed. Then, for any $\tau \in \mathcal{C}$, the form $\bar{s} + \tau \bar{b}$ is sectorial and closed. In particular, if $S + B$ is dense in $\mathcal{H}$ then $\bar{s} + \tau \bar{b}, \tau \in \mathcal{C}$, is a holomorphic family of forms of type (a) in the sense of Kato.

**Proof.** Fix $\tau \in \mathcal{C}, \theta \in (\pi/4, \pi/2)$, and choose $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$ so that

$$\bar{s} + \Re(\tau)\bar{b} \geq \gamma_1, \quad (\tan(\theta) - 1)\bar{s} + \tan(\theta)\Re(\tau)\bar{b} \geq \gamma_2, \quad \bar{s} - |\Im(\tau)|\bar{b} \geq \gamma_3.$$  

Let $\gamma = \min\{\gamma_1, \cot(\theta)(\gamma_2 + \gamma_3)\}$. Then $\Re(\bar{s} + \tau \bar{b}) \geq \gamma$ and, for any couple $\varphi \in S, \eta \in B$,

$$|\Im(\bar{s} + \tau \bar{b})(\varphi + \eta)| = |\Im(\tau)\bar{b}(\eta)| \leq s(\varphi) - \gamma_3 \|\varphi + \eta\|^2 \leq \tan(\theta)(s(\varphi) + \Re(\tau)b(\eta)) - (\gamma_2 + \gamma_3)|\varphi + \eta\|^2 \leq \tan(\theta)(s(\varphi) + \Re(\tau)b(\eta) - \gamma|\varphi + \eta\|^2).$$

This estimates show that $\bar{s} + \tau \bar{b}$ is sectorial. Finally, a sectorial form is known to be closed if and only if its real part is closed. \hfill $\square$

**Proposition 9.** Let $\{\xi_n(x); n \in \mathbb{N}\}$ be the eigenvalues of $T(x), x \in P^1(\mathbb{R})$, ordered increasingly. Then for every $n \in \mathbb{N}, \xi_n(x)$ is a real-analytic strictly increasing function on $\mathbb{R}$, and one has, $\forall k \in \mathbb{R},$

$$\xi_1(x) < \xi_1(\infty) < \xi_2(x) < \xi_2(\infty) < \xi_3(x) < \xi_3(\infty) < \ldots.$$  

(27)

Moreover,

$$\lim_{k \to -\infty} \xi_1(x) = -\infty, \quad \lim_{k \to -\infty} \xi_n(x) = \xi_{n-1}(\infty) \text{ for } n \geq 2, \quad \lim_{k \to -\infty} \xi_n(x) = \xi_n(\infty) \text{ for } n \geq 1.$$  

(28)

**Proof.** The Friedrichs extension of a positive operator is maximal in the form sense among all self-adjoint extensions of that operator [2]. Particularly,

$$\xi_1(x) = \min(\text{spec } T(x)) \leq \xi_1(\infty) = \min(\text{spec } T(\infty)).$$

But as already remarked above, the eigenvalues of $T(x)$ and $T(\infty)$ interlace and so we have (27).

Referring to (25), (26), the property $\kappa_1, \kappa_2 \in \mathbb{R}, \kappa_1 < \kappa_2$ clearly implies $t_{\tau(\kappa_1)} < t_{\tau(\kappa_2)}$ where $\tau(x) = (x + q^+)/ (1 - q^+)$. In virtue of Proposition 7 and the min-max principle, $\xi_n(\kappa_1) \leq \xi_n(\kappa_2), \forall n \in \mathbb{N}$. But the spectra of $T(\kappa_1)$ and $T(\kappa_2)$ are disjoint and so the functions $\xi_n(x)$ are strictly increasing on $\mathbb{R}$. 


One can admit complex values for the parameter \( \tau \) in (25), (26). Then, according to Lemma 8, the family of forms \( \langle \cdot, \cdot \rangle \), \( \tau \in \mathbb{C} \), is of type (a) in the sense of Kato. Referring once more to Proposition 7 one infers from [8, Theorem VII-4.2] that the family of self-adjoint operators \( T(\kappa), \kappa \in \mathbb{R} \), extends to a holomorphic family of operators on \( \mathbb{C} \). This implies that for any bounded interval \( K \subset \mathbb{R} \) there exists an open neighborhood \( D \) of \( K \) in \( \mathbb{C} \) and \( \rho \in \mathbb{R} \) sufficiently large so that the resolvents \( (T(\kappa) + \rho)^{-1}, \kappa \in K \), extend to a holomorphic family of bounded operators on \( D \). In addition we know that, for every fixed \( n \in \mathbb{N} \) and \( \kappa \in \mathbb{R} \), the \( n \)th eigenvalue of \( T(\kappa) \) is simple and isolated. By the analytic perturbation theory [8, § VII.3], \( \xi_n(\kappa) \) is an analytic function on \( \mathbb{R} \).

Finally we note that every \( \chi \in \mathbb{R} \) is an eigenvalue of \( T(\kappa) \) for some (in fact, unambiguous) \( \kappa \in \mathbb{R} \) and so the range \( \xi_n(\mathbb{R}) \) must exhaust the entire interval either \((-\infty, \xi_1(\infty))\), if \( n = 1 \), or \((\xi_{n-1}(\infty), \xi_n(\infty))\), if \( n > 1 \). This clearly means that (28) must hold.  

\[ \Box \]

Remark 10. As noted in [2, Theorem 2.15], \( \xi_1(\kappa) \) is a concave function.

### 3 The characteristic function

#### 3.1 A construction of the characteristic function for \( \nu > 0 \)

Recall (9), (10). Observe that the sequence \( \{\hat{P}_n(x)\} \) obeys the relation

\[
\hat{P}_n(x) = Q_n^{(1)} - xq^{(1-v)/2} \sum_{k=0}^{n} Q_{n-k}^{(1)} q^k \hat{P}_k(x) \quad \text{for } n \geq -1.
\]

This relation already implies that \( \hat{P}_{-1}(x) = 0, \hat{P}_0(x) = 1 \). Notice also that the last term in the sum, with \( k = n \), is zero and so (29) is in fact a recurrence for \( \{\hat{P}_n(x)\} \). Equation (29) is pretty standard. Nevertheless, one may readily verify it by checking that this recurrence implies the original defining recurrence, i.e. the formal eigenvalue equation (11) which can be rewritten as follows

\[
q^{(v-1)/2} \hat{P}_{n+1}(x) - (1 + q^n)\hat{P}_n(x) + q^{(v+1)/2} \hat{P}_{n-1}(x) = -xq^n \hat{P}_n(x), \quad \forall n \geq 0.
\]

Actually, from (29) one derives that

\[
q^{(v-1)/2} \hat{P}_{n+1}(x) - \hat{P}_n(x) = q^{(v-1)/2}Q_{n+1}^{(1)} - Q_n^{(1)} - x \sum_{k=0}^{n} (Q_{n-k}^{(1)} - q^{(1-v)/2}Q_{n-k-1}^{(1)}) q^k \hat{P}_k(x)
\]

\[
= q^{v+(1+v)m/2} - xq^{(1+v)m/2} \sum_{k=0}^{n} q^{(1-v)k/2} \hat{P}_k(x)
\]

and so

\[
q^{(v-1)/2} \hat{P}_{n+1}(x) - \hat{P}_n(x) - q^{v+1/2}(q^{(v-1)/2} \hat{P}_n(x) - \hat{P}_{n-1}(x)) = -xq^n \hat{P}_n(x),
\]

as claimed.

**Proposition 11.** The sequence of polynomials \( \{q^{(v-1)n/2} \hat{P}_n(x); \ n \in \mathbb{Z}_+\} \) converges locally uniformly on \( C \) to an entire function \( \Phi(x) \equiv \Phi(x; q) \). Moreover, \( \Phi(x) \) fulfills

\[
\Phi(x) = \frac{1}{1 - q^x} \left( 1 - x \sum_{k=0}^{\infty} q^{(1+v)k/2} \hat{P}_k(x) \right),
\]

and one has, \( \forall n \in \mathbb{Z}, \ n \geq -1, \)

\[
\hat{P}_n(x) = (1 - q^n)\Phi(x)Q_n^{(1)} + xQ_n^{(1)} \sum_{k=0}^{n} Q_{n-k}^{(1)} \hat{P}_k(x) + xQ_n^{(1)} \sum_{k=n+1}^{\infty} Q_{n-k}^{(2)} \hat{P}_k(x).
\]

(32)
Proof. Denote (temporarily) \( H_n(x) = q^{(n-1)n/2} \tilde{P}_n(x), n \in \mathbb{Z}_+. \) Then (29) means that
\[
(1 - q^n)H_n(x) = 1 - q^{n(n+1)} - x \sum_{k=0}^{n-1} (1 - q^{(n-k)})q^k H_k(x), \quad n \in \mathbb{Z}_+. \tag{33}
\]

Proceeding by mathematical induction in \( n \) one can show that, \( \forall n \in \mathbb{Z}_+ \),
\[
|H_n(x)| \leq \frac{(-a;q)_n}{1 - q^a} \quad \text{where} \quad a = \frac{|x|}{1 - q^a} \quad \text{and} \quad (-a;q)_n = \prod_{k=0}^{n-1} (1 + q^k a) \tag{34}
\]
is the \( q \)-Pochhammer symbol. This is obvious for \( n = 0 \). For the the induction step it suffices to notice that (33) implies
\[
|H_n(x)| \leq \frac{1}{1 - q^a} + a \sum_{k=0}^{n-1} q^k |H_k(x)|. \]
Moreover,
\[
1 + a \sum_{k=0}^{n-1} q^k (-a;q)_k = (-a;q)_n.
\]

From the estimate (34) one infers that \( \{H_n(x)\} \) is locally uniformly bounded on \( \mathbb{C} \). Consequently, from (33) it is seen that the RHS converges as \( n \to \infty \) and so \( H_n(x) \to \Phi(x) \) pointwise. This leads to identity (31). Furthermore, one can rewrite (29) as follows
\[
\tilde{P}_n(x) = Q_n^{(1)} - x \sum_{k=0}^{n} \frac{q^{(1-v)n/2}q^{(1+v)k/2} - q^{(1+v)n/2}q^{(1-v)k/2}}{1 - q^a} \tilde{P}_k(x)
\]
\[
= Q_n^{(1)} + xQ_n^{(2)} \sum_{k=0}^{n} Q_k^{(1)} \tilde{P}_k(x) - xQ_n^{(1)} \sum_{k=0}^{n} Q_k^{(2)} \tilde{P}_k(x)
\]
\[
= \left( 1 - x \sum_{k=0}^{\infty} Q_k^{(2)} \tilde{P}_k(x) \right) Q_n^{(1)} + xQ_n^{(2)} \sum_{k=0}^{n} Q_k^{(1)} \tilde{P}_k(x) + xQ_n^{(1)} \sum_{k=n+1}^{\infty} Q_k^{(2)} \tilde{P}_k(x)
\]
Taking into account (31) one arrives at (32).

Finally, from the locally uniform boundedness and Montel’s theorem it follows that the convergence of \( \{H_n(x)\} \) is even locally uniform and so \( \Phi(x) \) is an entire function. \( \square \)

It turns out that \( \Phi(x) \) may be called the characteristic function of the Jacobi operator \( T \), if \( \nu \geq 1 \), or the Friedrichs extension \( T^\ast \), if \( \nu < 1 \).

**Lemma 12.** Assume that \( \nu \geq 1 \). Suppose further that \( f \in \mathbb{C}^\infty, \{q^{-\sigma_0} fn\} \) is bounded for some \( \sigma_0 > -(\nu + 1)/2 \), and \( f = xGf \) for some \( x \in \mathbb{R} \) where
\[
(Gf)_n : = Q_n^{(2)} \sum_{k=0}^{n} Q_k^{(1)} f_k + Q_n^{(1)} \sum_{k=n+1}^{\infty} Q_k^{(2)} f_k, \quad n \in \mathbb{Z}_+. \tag{35}
\]

Then the sequence \( \{q^{-\sigma_0} fn\} \) is bounded for every \( \sigma < (\nu + 1)/2 \). In particular, \( f \in \ell^2 \).

**Proof.** Put
\[
S = \{ \sigma > -(\nu + 1)/2; \{q^{-\sigma} fn\} \in \ell^\infty \}, \quad \sigma_* = \sup S.
\]

Notice that, by the assumptions, \( S \neq \emptyset \) and the definition of \( Gf \) makes good sense. We have to show that \( \sigma_* \geq (1 + \nu)/2 \). Let us assume the contrary.

We claim that if \( \sigma \in S \) and \( \sigma < (\nu - 1)/2 \) then \( \sigma + 1 \in S \). In particular, \( \sigma_* \geq (\nu - 1)/2 \). In fact, write \( f_n = q^\sigma h_n, h \in \ell^\infty \). From (35) one derives the estimate
\[
|(Gf)_n| \leq \frac{||h||_{\ell^\infty}}{1 - q^a} \left( q^{(\nu+1)n/2} \sum_{k=0}^{n-1} q^{(\sigma+(1-\nu)/2)k} + q^{(\sigma+1)n/2} \right). \]
From here one deduces that there exists a constant $C > 0$ such that
\[ |f_n| = |x(\hat{G}f_n)| \leq C q^{(\sigma_1^*)n}, \]
as claimed.

Choose $\sigma$ such that $\sigma_* < \sigma < (\nu + 1)/2$. Then
\[ \frac{-\nu + 1}{2} \leq \frac{\nu - 1}{2} - 1 < \sigma - 1 < \frac{\nu - 1}{2}, \]
and so $\sigma - 1 \in S$. But in that case $\sigma \in S$ as well, a contradiction.

**Proposition 13.** If $\nu \geq 1$, the spectrum of $T$ coincides with the zero set of $\Phi(x)$. If $0 < \nu < 1$ then $\text{spec } T(x)$, $x \in P^1(\mathbb{R})$, consists of the roots of the characteristic equation
\[ \kappa \Phi(x) + \Psi(x) = 0 \]
where
\[ \Psi(x) \equiv \Psi^{(\nu)}(x; q) = \frac{1}{1-q^\nu} \left( q^{\nu} - x \sum_{k=0}^{\infty} q^{(1-\nu)k/2} \hat{P}_k(x) \right). \] (36)

In particular, the spectrum of $T^\nu = T(\infty)$ equals the zero set of $\Phi(x)$.

**Proof.** From Proposition 6 we already know that the spectrum of $T$ (or $T(x)$) is pure point and with no finite accumulation points. Assume first that $\nu \geq 1$. According to Proposition 3, we are dealing with the determinate case and so $x$ is an eigenvalue of $T$ if and only if the formal eigenvector $\hat{P}(x) = \{ \hat{P}_n(x) \}$ is square summable. If $\hat{P}(x) \in l^2$ then $q^{(\nu-1)n/2} \hat{P}_n(x) \to 0$ as $n \to \infty$ and so $\Phi(x) = 0$ (see Proposition 11). Conversely, if $\Phi(x) = 0$ then (32) tells us that $\hat{P} = x \hat{G} \hat{P}$, cf. (35). By Lemma 12, $\hat{P}(x) \in l^2$.

Assume now that $0 < \nu < 1$. This the indeterminate case meaning that $\hat{P}(x)$ is square summable for all $x \in \mathbb{C}$. Hence $x$ is an eigenvalue of $T(x)$ iff $\hat{P}(x) \in \text{Dom } T(x)$. Recall that $T(x)$ is defined in Proposition 3. From (32) one derives the asymptotic expansion
\[ \hat{P}_n(x) = \Phi(x) (q^{(1-\nu)n/2} - q^{\nu+(1+\nu)n/2}) + x q^{(1-\nu)n/2} \sum_{k=0}^{\infty} Q_k^{(1)} \hat{P}_k(x) + o(q^n) \text{ as } n \to \infty. \]

From here it is seen that $\hat{P}(x)$ fulfills the boundary condition in (21) if and only if $x$ solves the equation
\[ (x + q^\nu) \Phi(x) - x \kappa \hat{P}^{(1)}(x) = 0. \]

Referring to (16) one finds that $x(\kappa \hat{P}^{(1)}(x), \hat{P}(x)) = q^\nu \Phi(x) - \Psi(x)$. \hfill \qed

**Proposition 14.** For $\nu > 0$ one has
\[ \Phi(x) = \frac{1}{1-q^\nu} \phi_1(0; q^{\nu+1}; q, x) = \frac{(q; q)_\infty}{(q^\nu; q)_\infty} q^{\nu/2} x^{\nu/2} J_\nu(q^{-1/2} \sqrt{x}; q), \]
and for $0 < \nu < 1$,
\[ \Psi(x) = \frac{q^\nu}{1-q^\nu} \phi_1(0; q^{1-\nu}; q, q^{-\nu} x) = -\frac{(q; q)_\infty}{(q^{-\nu}; q)_\infty} q^{\nu(v+1)/2} x^{\nu/2} J_\nu(q^{(\nu+1)/2} \sqrt{x}; q). \] (37)

If $\Phi(x) = 0$ and so $x$ is an eigenvalue of $T$, provided $\nu > 0$, or $T^\nu$, provided $0 < \nu < 1$, then $x > 0$ and the components of a corresponding eigenvector can be chosen as
\[ u_k(x) = q^{k/2} J_\nu(q^{1/2} \sqrt{x}; q) = C q^{(\nu+1)/2} \phi_1(0; q^{\nu+1}; q, q^{k+1} x), \quad k \in \mathbb{Z}_+, \] (38)
where $C = x^{\nu/2} (q^{\nu+1}; q)_\infty/(q; q)_\infty$. 

If $0 < \nu < 1$, $\kappa \in \mathbb{R}$ and $\kappa \Phi(x) + \Psi(x) = 0$ and so $x$ is an eigenvalue of $T(\kappa)$ then the components of a corresponding eigenvector can be chosen as

$$u_k(x, \kappa) = q^{k/2} \left( \kappa f(q^{k/2} \sqrt{x}; q) - \frac{(q')^\infty}{(q; q')^\infty} q^{-(\nu+2)/2} x^{\nu} f(q^{(k-\nu)/2} \sqrt{x}; q) \right)$$

$$= C \left( \kappa q^{(1+\nu)/2} \chi_1(0; q^{\nu+1}; q, q^{\nu+1} x) + q^{1-\nu} f(q^{(1-\nu)/2} \sqrt{x}; q, q, q^{1-\nu}), \right),$$

with $k \in \mathbb{Z}$, ($C$ is the same as above).

**Lemma 15.** For every $m \in \mathbb{Z}_+$ and $\sigma > 0$,

$$\sum_{k=0}^{\infty} q^{m+\nu+1/2} k \frac{d^m \tilde{P}_k(0)}{dx^m} = (1 - \nu) m! \frac{q^{m+\nu+1/2}}{(q; q)_{m+1} (q^{\nu+1}; q)_{m+1}}.$$  \hspace{1cm} (39)

**Proof.** For a given $m \in \mathbb{N}$, one derives from (30) the three-term inhomogeneous recurrence relation

$$q^{(\nu+1)/2} \frac{d^m \tilde{P}_{m+1}(0)}{dx^m} = (1 + q^{\nu}) \frac{d^m \tilde{P}_m(0)}{dx^m} + q^{(\nu+1)/2} \frac{d^m \tilde{P}_{m-1}(0)}{dx^m} = -mq^\nu \frac{d^m \tilde{P}_m(0)}{dx^m}, \quad m \geq 0,$$  \hspace{1cm} (40)

with the initial conditions

$$\frac{d^m \tilde{P}_0(0)}{dx^m} = 0, \quad \frac{d^m \tilde{P}_0(0)}{dx^m} = \delta_{m,0} \text{ for all } m \geq 0.$$  \hspace{1cm} (41)

Recall that, by Proposition 11, the sequence $\{q^{(\nu+1)/2} \tilde{P}_n(x)\}$ converges on $\mathbb{C}$ locally uniformly and hence it is locally uniformly bounded. Combining this observation with Cauchy’s integral formula one justifies that, for any $m \in \mathbb{Z}_+$ fixed, the sequence $\{q^{(\nu+1)/2} \frac{d^m \tilde{P}_m(0)}{dx^m}\}$ is bounded as well. Therefore the LHS of (39) is well defined. Let us call it $S_{m,\sigma}$. Applying summation in $n$ to (40) and bearing in mind (41) one derives the recurrence

$$S_{m,\sigma} = -\frac{m \nu q^{\nu}}{(1 - q^{\nu})(1 - q^{\nu+1})} S_{m-1,\sigma+1} \text{ for } m \geq 1, \sigma > 0.$$  \hspace{1cm} (42)

Particularly, for $m = 0$ we know that $\tilde{P}_n(0) = Q_n^{(1)}$, $n \in \mathbb{Z}_+$. Whence

$$\forall \sigma > 0, \quad S_{0,\sigma} = \frac{1}{(1 - q^\nu)(1 - q^{\nu+1})}$$

(cf. (16)). A routine application of mathematical induction in $m$ proves (39). \hspace{1cm} \Box

**Proof of Proposition 14.** Letting $\sigma = 1$ in (39) and making use of the locally uniform convergence (cf. Proposition 11) one has

$$\frac{1}{m!} \frac{d^m}{dx^m} \sum_{k=0}^{\infty} q^{(\nu+1)/2} \tilde{P}_k(x) \bigg|_{x=0} = \frac{(-1)^m m^{m+1/2}}{(q; q)_{m+1} (q^{\nu+1}; q)_{m+1}} \text{ for } m \in \mathbb{Z}_+.$$  \hspace{1cm} (31)

Now, since $\Phi(x)$ is analytic it suffices to refer to formula (31) to obtain

$$\Phi(x) = \frac{1}{1 - q^\nu} \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)/2} x^n (q; q)_n (q^{\nu+1}; q)_n}{(q; q)_{m+1} (q^{\nu+1}; q)_{m+1}}, \forall m \in \mathbb{Z}_+.$$  \hspace{1cm} (37)

Letting $\sigma = 1 - \nu$ in (39), a fully analogous computation can be carried out to evaluate the RHS of (36) thus getting formula (37) for $\Psi(x)$.

From (2) it is seen that the sequences $\{u_k(x); k \in \mathbb{Z}\}$ and $\{v_k(x); k \in \mathbb{Z}\}$, where

$$u_k(x) = q^{k/2} f(q^{k/2} \sqrt{x}; q) \quad \text{and} \quad v_k(x) = q^{k/2} f(q^{k/2} \sqrt{x}; q),$$

obey both the difference equation

$$\alpha_k u_{k+1} + \beta_k u_k + \alpha_{k-1} u_{k-1} = xu_k$$

with $\alpha_k, \beta_k$ being defined in (4)). In the case of the former sequence, $\nu$ can be arbitrary positive, and in the case of the latter one we assume that $0 < \nu < 1$. Hence the sequence $(u_0(x), u_1(x), u_2(x), \ldots)$ is a formal eigenvector of the Jacobi matrix $T$ if and only if $u_{-1}(x) = 0$. A similar observation holds true if we replace $u_k(x)$ by $u_k(x)$. In view of Proposition 14, it suffices to notice that $u_{-1}(x)$ is proportional to $\Phi(x)$ and $u_{-1}(x, \kappa)$ to $\kappa \Phi(x) + \Psi(x)$. \hspace{1cm} \Box
3.2 The case $\nu = 0$

The case $\nu = 0$ is very much the same thing as the case when $0 < \nu < 1$. First of all, this is again an indeterminate case, i.e. $T_{\text{min}}$ is not self-adjoint. On the other hand, there are some differences causing the necessity to modify several formulas, some of them rather substantially. Perhaps the main reason for this is the fact that the characteristic polynomial of the difference equation with constant coefficients, (15), has one double root if $\nu = 0$ while it has two different roots if $0 < \nu$. Here we summarize the basic modifications but without going into details since the arguing remains quite analogous.

For $\nu = 0$ one has $\text{Dom} t = \{ f \in \ell^2; \ A f \in \ell^2 \}$, and two distinguished solutions of (14) are

$$ Q^{(1)}_n = (n + 1)q^{n/2}, \ Q^{(2)}_n = q^{n/2}, \ n \in \mathbb{Z}, $$

where again $Q^{(1)}_n = \hat{P}_n(0)$ for $n \geq 0$ and $\{Q^{(2)}_n\}$ is a minimal solution, $W_n(Q^{(1)}_n, Q^{(2)}_n) = 1$. The asymptotic expansion of a sequence $f \in \text{Dom} T_{\max}$ reads

$$ f_n = (C_1(n + 1) + C_2) q^{n/2} + o(q^n) \text{ as } n \to \infty, $$

with $C_1, C_2 \in \mathbb{C}$. The one-parameter family of self-adjoint extensions of $T_{\text{min}}$ is again denoted $T(\kappa), \kappa \in \mathbb{P}^1(\mathbb{R})$. Definition (21) of Dom $T(\kappa)$ formally remains the same but the constants $C_1(f), C_2(f)$ in the definition are now determined by the limits

$$ C_1(f) = \lim_{n \to +\infty} f_n(n + 1)^{-1} q^{-n/2}, \ C_2(f) = \lim_{n \to +\infty} (f_n - C_1(n + 1) q^{n/2}) q^{-n/2}. $$

One still has $T(\kappa) = T^\kappa$. Similarly, $f \in \text{Dom} T_{\max}$ belongs to Dom $T_{\text{min}}$ if and only if $C_1(f) = C_2(f) = 0$ meaning that (22) is true for $\nu = 0$, too. Furthermore, everything what is claimed in Propositions 5 and 6 about the values $0 < \nu < 1$ is true for $\nu = 0$ as well.

Proposition 7 should be modified so that the quadratic form associated with a self-adjoint extension $T(\kappa), \kappa \in \mathbb{R}$, i.e. for $\nu = 0$ one lets $\tau \equiv \tau(\kappa) = \kappa + 1$. On the other hand, Proposition 9 holds verbatim true also for $\nu = 0$.

Relation (29) is valid for $\nu = 0$ as well but more substantial modifications are needed in Proposition 11. One has

$$ \frac{q^{-n/2}}{n + 1} \hat{P}_n(x) \to \Phi(x) = 1 - x \sum_{k=0}^{\infty} k^{1/2} \hat{P}_k(x) \text{ as } n \to \infty, $$

and the convergence is locally uniform on $\mathbb{C}$ for one can estimate

$$ \left| \frac{q^{-n/2}}{n + 1} \hat{P}_n(x) \right| \leq \prod_{k=0}^{n} (1 + (k + 1)q^k|x|), \ n \in \mathbb{Z}_+. $$

Equation (32) should be replaced by

$$ \hat{P}_n(x) = \Phi(x)Q^{(1)}_n + xQ^{(2)}_n \sum_{k=0}^{n} Q^{(1)}_k \hat{P}_k(x) + xQ^{(2)}_n \sum_{k=n+1}^{\infty} Q^{(2)}_k \hat{P}_k(x). $$

From here one infers the asymptotic expansion

$$ \hat{P}_n(x) = \Phi(x)(n + 1)q^{n/2} + xq^{n/2} \sum_{k=0}^{\infty} Q^{(1)}_k \hat{P}_k(x) + o(q^n) \text{ as } n \to \infty. $$

One concludes that what is claimed in Proposition 13 about the values $0 < \nu < 1$ is true for $\nu = 0$ as well but instead of (36) one should write

$$ \Psi(x) = -x \sum_{k=0}^{\infty} (k + 1)q^{k/2} \hat{P}_k(x). $$

Finally let us consider modifications needed in Proposition 14. For $\nu = 0$ one has

$$ \Phi(x) = \int \phi_1(0; q; g, x) = J_0(q^{-1/2} \overline{\xi}; q), $$

where
and
\[
\Psi(x) = \frac{\partial}{\partial p} \phi_2(0, q; p, q, p, q, x) = 2q \frac{\partial}{\partial p} \phi_1(0; p, q, q, x) + \frac{\partial}{\partial x} \phi_1(0; q, q, x).
\]
Let
\[
u_k(x) = (k+1)q^{(k+1)/2} \phi_1(0; q, q, q^{k+1}x) + 2q^{(k+3)/2} \frac{\partial}{\partial p} \phi_1(0; p, q, q^{k+1}x) + q^{(k+1)/2} x \frac{\partial}{\partial x} \phi_1(0; q, q, q^{k+1}x),
\]
k ∈ ℤ. Then both sequences \{u_k(x)\} and \{v_k(x)\} solve (42) on ℤ and \(u_{-1}(x) = \Phi(x), v_{-1}(x) = \Psi(x)\). Consequently, if \(\Phi(x) = 0\) then components of an eigenvector of \(T(\infty) = T^F\) corresponding to the eigenvalue \(x\) can be chosen to be \(u_k(x), k ∈ ℤ\). Similarly, if \(\kappa \Phi(x) + \Psi(x) = 0\) for some \(\kappa \in ℜ\) then components of an eigenvector of \(T(x)\) corresponding to the eigenvalue \(x\) can be chosen to be \(\kappa u_k(x) + v_k(x), k ∈ ℤ^+\).

### 4 Some applications to the \(q\)-Bessel functions

In this section, if \(0 < \nu < 1\), we will consider the Friedrichs extension only. To simplify the formulations below we will unify the notation and use the same symbol \(T^F\) for the corresponding self-adjoint Jacobi operator for all positive values of \(\nu\), this is to say even in the case when \(\nu ≥ 1\). Making use of the close relationship between the spectral data for \(T^F\) and the \(q\)-Bessel functions, as asserted in Propositions 13 and 14, we are able to reproduce, in an alternative way, some results from [9, 11].

**Proposition 16** (Koëlink, Swarttouw). Assume that \(\nu > 0\). The zeros of \(z \mapsto J_\nu(z; q)\) are all real (arranged symmetrically with respect to the origin), simple and form an infinite countable set with no finite accumulation points. Let \(0 < w_1 < w_2 < w_3 < \ldots\) be the positive zeros of \(J_\nu(z; q)\). Then the sequences
\[
u(n) = (J_\nu(q^{1/2}w_1; q), q^{1/2}J_\nu(qw_2; q), qJ_\nu(q^{3/2}w_3; q), \ldots), \quad n ∈ ℤ,
\]
form an orthogonal basis in \(ℓ^2\). In particular, the orthogonality relation
\[
∑_{k=0}^{∞} q^k J_\nu(q^{(k+1)/2}w_m; q) J_\nu(q^{(k+1)/2}w_n; q) = \frac{q^{-1+\nu^2}}{2w_n} J_\nu(q^{1/2}w_n; q) \frac{∂J_\nu(w_n; q)}{∂z} \delta_{m,n}
\]
holds for all \(m, n ∈ ℤ\).

**Remark.** It is not difficult to show that the proposition remains valid also for \(-1 < \nu ≤ 0\). To this end, one can extend the values \(\nu > 0\) to \(\nu = 0\) following the lines sketched in Subsection 3.2, and employ Propositions 13 and 14 while letting \(\kappa = 0\) in order to treat the values \(-1 < \nu < 0\). But we omit the details. An original proof of this proposition can be found in [11, Section 3].

**Proof.** All claims, except the simplicity of zeros and the normalization of eigenvectors, follow from the known spectral properties of \(T^F\). Namely, \(T^F\) is positive definite, \((T^F)^{-1}\) is compact, \(\text{spec } T^F = \{qw_n^2; n ∈ ℤ\}\) and corresponding eigenvectors are given by formula (38); cf. Propositions 5, 6, 13 and 14.

The remaining properties can be derived, in an entirely standard way, with the aid of discrete Green’s formula. Suppose a sequence of differentiable functions \(u_n(x), n ∈ ℤ\), obeys the difference equation (42). Then Green’s formula implies that, for all \(m, n ∈ ℤ, m ≤ n,\)
\[
∑_{k=m}^{n} u_k(x)^2 = α_{m-1}(u_{n-1}(x)u_m(x) - u_{m-1}(x)u_{m}(x)) - α_n(u_{n}(x)u_{n+1}(x) - u_{n-1}(x)u_{n}(x))
\]
More generally, in Theorem 2.2 and Remark 2.3 in [4] it is shown that for any \( m \), one immediately infers the asymptotic behavior

\[
u_k(x) = C(x)(1 + \mathcal{O}(q^{m+1}) q^{q^{m+1}/2}), \quad \nu_k(x) = C'(x)(1 + \mathcal{O}(q^{m+1}) q^{q^{m+1}/2}, \quad \text{as } k \to \infty, \tag{45}\]

where \( C(x) = x^{m/2}(q^{-1}; q)_{\infty}/(q; q)_{\infty} \). It follows that one can send \( n \to \infty \) in Green's formula. For \( x = q w_n^2 \) we have \( u_{-1}(x) = 0 \) and the formula reduces to the equality

\[
\sum_{k=0}^{\infty} q^k j_k(q^{k+1/2} w_n; q)^2 = -q^{v/2} j_v(q^{1/2} w_n; q) \frac{\partial J_v(q^{1/2} \sqrt{x}; q)}{\partial x} \bigg|_{x = q w_n^2}.
\]

Whence (44). From the asymptotic behavior (45) it is also obvious that \( u_k(x) \neq 0 \) for sufficiently large \( k \). Necessarily, \( \partial J_v(w_n; q)/\partial z \neq 0 \).

In addition, one obtains at once an orthogonality relation for the sequence of orthogonal polynomials \( \{\hat{P}_n(x)\} \). As is well known from the general theory [3] and Proposition 3, the orthogonality relation is unique when \( v \geq 1 \) and indeterminate if \( 0 < v < 1 \). It was originally derived in [9, Theorem 3.6].

**Proposition 17** (Koelink). Assume that \( v > 0 \) and let \( \{\hat{P}_n(x)\} \) be the sequence of orthogonal polynomials defined in (9), (10), and \( 0 < w_1 < w_2 < w_3 < \ldots \) be the positive zeros of \( z \mapsto J_v(z; q) \). Then the orthogonality relation

\[
-2q^{1-v/2} \sum_{k=1}^{\infty} \frac{w_k j_v(q^{1/2} w_k; q) j_v(q^{1/2} w_k; q)}{\partial J_v(q^{1/2} w_k; q)/\partial z} \hat{P}_m(q w_k^2) \hat{P}_n(q w_k^2) = \delta_{m,n} \tag{46}
\]

holds for all \( m, n \in \mathbb{Z}_+ \).

**Proof.** Let \( u(k), k \in \mathbb{N} \), be the orthogonal basis in \( \ell^2 \) introduced in (43), i.e. we put

\[
u(k)_n = q^{n/2} j_v(q^{n+1/2} w_k; q), \quad k \in \mathbb{N}, \ n \in \mathbb{Z}_+.
\]

Notice that the norm \( \|u(k)\| \) is known from (44). The vectors \( u(k) \) and \( \hat{P}(x) = (\hat{P}_0(x), \hat{P}_1(x), \hat{P}_2(x), \ldots) \), with \( x = q w_n^2 \), are both eigenvectors of \( T^* \) corresponding to the same eigenvalue. Hence these vectors are linearly dependent and one has

\[
u^{n/2} j_v(q^{n+1/2} w_k; q) = j_v(q^{1/2} w_k; q) \hat{P}_n(q w_k^2), \quad k \in \mathbb{N}, \ n \in \mathbb{Z}_+.
\]

One concludes that Parseval's identity

\[
\sum_{k=1}^{\infty} \frac{u(k)_m u(k)_n}{\|u(k)\|^2} = \delta_{m,n}, \quad m, n \in \mathbb{Z}_+,
\]

yields (46). \( \square \)

**Remark 18.** To complete the picture let us mention two more results which are known about the Hahn-Exton \( q \)-Bessel functions and the associated polynomials. First, denote again by \( w_n^{(j)} \equiv w_n, \ n \in \mathbb{N} \), the increasingly ordered positive zeros of \( J_v(z; q) \). In [1] it is proved that if \( q \) is sufficiently small, more precisely, if \( q^{v+1} < (1-q)^2 \) then

\[
u^{m-2} > w_m > q^{m/2} \left( 1 - \frac{q^{m+1}}{1-q} \right), \quad \forall m \in \mathbb{N}.
\]

More generally, in Theorem 2.2 and Remark 2.3 in [4] it is shown that for any \( q, 0 < q < 1 \), one has

\[
u = q^{m/2} (1 + \mathcal{O}(q^m)) \quad \text{as } m \to \infty.
\]

Second, in [9, 11] one can find an explicit expression for the sequence of orthogonal polynomials \( \{\hat{P}_n(x)\} \), namely

\[
u^{n/2} \sum_{j=0}^{n} \frac{q^{n+1/2}(q^{-n}; q)^j}{(q; q)_j} {}_2\phi_1(q^{j-n}; q^{j+1}; q^{n}; q, q^{j+1}) x^j, \quad n \in \mathbb{Z}_+.
\]

Let us remark that a relative formula in terms of the Al-Salam–Chihara polynomials has been derived in [17, Theorem 2].
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