Dynamic Random Subjective Expected Utility

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Abstract

Dynamic Random Subjective Expected Utility (DR-SEU) allows to model choice data observed from an agent or a population of agents whose tastes and beliefs about objective payoff-relevant states can evolve stochastically. Our observable, the augmented Stochastic Choice Function (aSCF), allows for a direct test of whether the agents’ beliefs reflect the true data-generating process conditional on their private information as well as identification of the possibly incorrect beliefs. We give an axiomatic characterization of when an agent satisfies the model, both in static and dynamic settings. We also prove natural comparative static results on the degree of belief incorrectness as well as on the speed of learning about taste.

1 Introduction

The study of stochastic choice has found renewed popularity in economics. Along with a considerable amount of research on static stochastic choice models, several recent works have pioneered foundational work into dynamic stochastic choice models.1 In a dynamic setting the agent solves a dynamic decision problem and learns as time passes about either the environment she is facing or her own evolution of preferences or both. In many applications an analyst only observes choices of an agent as well as some (possibly public) signals about payoff-relevant objective states. He doesn’t have information about the stochastic process of the preferences of the agent (the private information of the agent).

In this paper we consider such a general environment: there are payoff-relevant objective states every period, an agent has every period standard subjective expected utility (SEU) preferences, comprised of beliefs about the objective, payoff-relevant state as well as a Bernoulli utility over a set of prizes. The subjective state of an agent in each period consists of her realized SEU. We assume that these follow an exogenously given stochastic process which is well-known to the agent (albeit unknown to the analyst). We assume the agent can’t influence the given stochastic process and allow for both stochastic tastes and stochastic beliefs. In many real life examples this two-fold randomness is present, e.g. investment and saving behavior may depend both on exogenous, objective randomness such as market conditions as well as on the stochastic evolution of the risk aversion of the agent.

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1 [Fudenberg, Strzalecki ’15], [Frick, Iijima, Strzalecki ’17], [Steiner, Stewart, Matejka ’17] to name a few.
We assume that after each history of choices and realizations of the objective states the analyst observes limiting frequencies of the choice of the agent in decision problems/menus of the current period as well as the realization of the objective states in the current period. The data also reflect variation of the decision problems/menus. Thus, the observable is in every period, after each history of choices and objective states, a *probability distribution* over choices from a menu in the current period and over realizations of the objective state.\(^2\) Many situations in real life deliver such data, from employment situations in the labor market, consumption and investment decisions, to educational choices of students, loan practices, etc.\(^3\)

Our focus is axiomatic throughout. Under the assumption that the distribution of the private information of the agent doesn’t depend on the decision problem she faces and that the analyst has access to a rich observable featuring variation in the decision problems, we give conditions on the observable which allow the analyst to uncover the distribution of the private information of the agent regardless of its arbitrariness. Under these conditions the analyst can also study whether the agent’s beliefs when making choices reflect the correct data-generating process conditional on her private information and whenever that is not the case he can identify the biases conditional on the agent’s private information. While the study of misspecified learning is not new, this is the first work, to the best of knowledge, where there are no a priori assumptions on the origin of the misspecification. The misspecified beliefs may be because of misspecified priors, because of imprecise observation of private signals by the agent or because conditional on her private information the agent has some arbitrary behavioral biases in beliefs.\(^4\)

The model we consider is still falsifiable as we require the agent to be Bayesian with respect to the stochastic process describing the evolution of her private information, even though she may be non-Bayesian with respect to the true data generating process of the objective states. Moreover, we don’t allow any misspecified learner to receive hard evidence about misspecification, such as the occurrence of an unforeseen contingency. Thus, in this paper the agent is able to explain any observed string of objective states within her model, even though as time passes her beliefs might diverge more and more from the true data-generating process.\(^5\)

The richer observable allows comparative static results about the degree of biasedness of beliefs. We show how an analyst can use the data to construct a precise estimator of the extent of the belief biasedness of the agent and how he can compare different agents using this estimator. Moreover, since our model allows for both stochastic taste and beliefs, we show what an analyst can say about the relative speed with which two different agents learn their taste, given their respective datasets.

This paper is most related to [Lu '16] – who studies the same static model but with unobservable objective states, and [Frick, Iijima, Strzalecki '17] – who study a fully non-

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\(^2\)Our identification results are valid under more general conditions – see Remark 1 in section 2.

\(^3\)This type of data also allows an alternative heterogeneous population interpretation: there is a population of agents facing similar choice situations. The analyst observes in many instances the choice of an agent as well as the realization of some payoff-relevant objective state. We focus on the single-agent case in the exposition, but intuitions and results can be readily translated.

\(^4\)E.g. this model allows for the case of confirmatory bias studied in [Rabin, Schrag ’99] where an agent may misread signals in a way favorable to her current hypothesis. The agent in their model is not Bayesian with respect to the correct prior but is so within her model.

\(^5\)The time horizon is assumed to be finite. Thus the agent cannot resort to statistical tests of arbitrary accuracy to determine that her beliefs might indeed be misspecified.
parametric dynamic model as here but without payoff-relevant objective states. Relatedly, [Dillenberger et al '14] study the ex-ante menu preference of the agent modeled by [Lu '16]. Among other things we extend their work to allow for stochastic taste. Conceptually the paper is also related to [Lu '17] who shows how a combination of ex-ante preference over acts and post-signal random choice can overcome the classical issue of identification in the Expected State-dependent Utility model. Our model illustrates the strong identification properties of random choice data for the case of state-independent utilities in a rich dynamic environment allowing for stochastic taste. Finally, the observable in this paper can be interpreted as a likelihood function of a dynamic choice model in the spirit of [Rust '87] and the literature that it inspired. Whereas that literature has focused on identification and inference of controlled stochastic processes, this paper offers an axiomatic treatment of such likelihood functions for choice behavior in a general set up with both observable and unobservable states.

In the following we explain in detail the organization of the paper mentioning its contribution at each step.

Section 2 focuses on the static model. For each decision problem $A$ an analyst observes the frequency of an agent’s choice and the realization of a payoff-relevant objective state $s$ (we say agent picks act $f$ from menu $A$ and objective state $s$ is realized with a certain probability $\rho(f, A, s)$). We call this observable an augmented stochastic choice function (aSCF). We show how the analyst can identify from this observable the space of the subjective states of the agent. We call this the revealed subjective support of the data. We impose axioms similar to the ones in [Lu '16] to ensure that the revealed subjective support consists of SEUs that are identified by a belief $q$ about the realization of $s$ as well as a Bernoulli utility $u$. Furthermore, we show how the analyst can use the concept of the revealed subjective support to test whether the agent is using the correct data-generating process of objective states, conditional on her private information. This corresponds to the classical statistical concept of well-calibrated beliefs originating in [Dawid '82] but now in a general setting which allows for stochastic taste. Intuitively, an agent has correct interim beliefs only if the observed frequency of the realization of $s$ conditional on observing $f$ chosen from $A$ is a mixture of beliefs in the subjective support of the data which can rationalize the choice of $f$ from $A$. Whenever this condition fails the analyst can identify the incorrect beliefs as well as the true data-generating process, conditional on the private information. We also give a relaxation of the correct interim beliefs condition which restricts the extent of belief incorrectness: the agent never receives hard evidence that her beliefs may be incorrect because the realization of $s$ is always in the support of her belief $q$.

Section 3 introduces the dynamic model. The observable is now a history-dependent aSCF: for every history $h^{t-1}$ occurring with positive probability, the analyst observes frequencies of the choice in a subsequent decision problem $A_t$ together with the realization of the objective state in the respective period (we say agent picks $f_t$ from menu $A_t$ after history $h^{t-1}$ and objective state $s_t$ is realized with probability $\rho_t(f_t, A_t, s_t|h^{t-1})$). Histories

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6Our proofs modify and extend the proofs of [Lu '16] and [Frick, Iijima, Strzalecki '17] in multiple directions as well as extending several other models in the literature. E.g. we extend [Ahn, Sarver '13] to include objective states and stochastic beliefs. Details are in the online appendix.

7See [Rust '94] and [Aricidiacono, Ellickson '11] for surveys on the dynamic discrete choice literature.

8The last section of [Lu '16] also studies the property of well-calibrated beliefs but in a setting of non-stochastic taste.
have empirical content, as they help the analyst identify the serial correlation in the private information of the agent, i.e. in her tastes and beliefs.

We assume these history-dependent aSCFs satisfy the assumptions of the static model. In contrast to the static case there is now limited observability: not every menu is observable after every history. This is a similar observability problem as in [Frick, Iijima, Strzalecki '17] and technically its solution in this paper adapts theirs to our more general setting with payoff-relevant states. It relies in identifying two classes of histories which reveal the same private information. Whenever the observable satisfies the history-dependent version of the static model and the two history equivalence properties the analyst can identify the stochastic evolution of the private information of the agent as well as the true data-generating process of the objective states. This is the DR-SEU model, the namesake of the paper.

After establishing the main characterization result we focus on two special cases of DR-SEU whose static versions are indistinguishable: Evolving SEU, where the evolution of agent’s Bernoulli utility is given through a Bellman equation and its specialization, Gradual Learning, where the agent is learning about a fixed but unknown taste. Additionally, and because we need it for the dynamic characterization results, we describe when a menu preference comes from an agent who is subjectively learning both about objective states and about her Bernoulli utility/taste through a new axiom called Weak Dominance. Intuitively, such an agent would always prefer to exchange any menu of acts $A$ for a menu $\bar{A}$ which allows her to pick any of the prizes occurring in $A$ with positive probability irrespective of the realization of the uncertainty she’s facing ex-ante.

Section 4 leverages the characterization theorems to prove comparative statics results. In a setting of non-stochastic taste we address the question of how an analyst can compare agents with respect to their biasedness of beliefs. Namely, given a commonly observable characteristic, e.g. gender, race or letter grades, if the analyst fixes a direction of biased beliefs for every characteristic, he can tell from stochastic choice data when an agent is more biased than another agent. Intuitively, the choice data give evidence that the more biased agent values menus uniformly more differently to a fictitious unbiased agent than the less biased agent. Finally, in the special case of the Gradual Learning representation, we show how an analyst may distinguish when an agent’s uncertainty for taste fully resolves and how the analyst may compare different agents with respect to the speed of learning their taste. Intuitively, agent 2 learns her taste more slowly than agent 1 whenever the data suggests that agent 2 satisfies Weak Dominance whenever agent 1 does.

Section 5 concludes and comments on avenues for future work. The appendix contains the proofs of the main theorem for the static setting as well as of the main theorem for the dynamic setting accompanied by a set of auxiliary results necessary to understand the main proofs. Other characterization theorems as well as technical extensions of results from several papers in the literature which are needed for the proofs are relegated to the online appendix. The latter also contains a section considering the case when the analyst does not observe the realization of objective states.

Before continuing with the theoretical set up and the results we note two examples which illustrate the questions and issues this paper addresses.

\footnote{The two equivalence properties are called Contraction history independence and Linear history independence.}
1.1 Examples

1.1.1 A model of discrimination

Consider an employer at a job fair looking at applications for a job vacancy. The job consists of performing a task, after the job fair is concluded, whose outcome has two potential values coming from $S_1 = \{g, b\}$ ($g$ stands for *good* and $b$ for *bad*). We assume that whether $g$ or $b$ is realized depends on both the ability of the employee as well as other randomness outside of the control of the employee.

During the job fair, in the first period of the model ($t = 0$) some characteristic $s_0 \in S_0 = \{s'_0, s''_0\}$ of the applicant is revealed to the employer, say ethnicity, gender, education level, etc. We assume the distribution of $s_0$ over $S_0$ is known to the employer. This may be justified e.g. if the data about the prevalence of the characteristic $s_0$ in the population of the applicants at the job fair is public. In the second period ($t = 1$) the employer has beliefs about the outcome of the task, conditional on the revealed characteristic $s_0$. These are coded by $(\hat{q}_1, \hat{q}_2) = (\hat{q}_1(g|s'_0), \hat{q}_1(g|s''_0)) \in (0, 1)^2$. These can potentially be different from the true data generating process which here for simplicity is given by $q_1(g|s'_0) = q_1(g|s''_0) = \frac{1}{2}$. Assume here for simplicity that the analyst knows this data-generating process. In our example we say that the employer has incorrect beliefs if the following holds.

**Incorrect Beliefs:** $1 > \hat{q}_1(g|s'_0) > \frac{1}{2} > \hat{q}_1(g|s''_0) > 0$.\(^{11}\)

We assume in the following that the objective state $s_1$ (task outcome) is also observable to the analyst after the choice of the employer.

Given the observed characteristic $s_0$ the employer can choose in $t = 1$ whether to hire the candidate (formally, act $h_{s_0} : S_1 \to \mathbb{R}$ ) or not hire (act $nh_{s_0} : S_1 \to \mathbb{R}$). In the case of not hiring, the utility of the employer is always zero $u_{s_0}(nh_{s_0}(s_1)) = 0$ for all $s_0 \in S_0, s_1 \in S_1$.

In the case of hiring the employer has (possibly) stochastic utility $u_{s_0} : \mathbb{R} \to \mathbb{R}$ which satisfies

$$u_{s_0}(h_{s_0}(g)) = g_{s_0}, \quad u_{s_0}(h_{s_0}(b)) = b_{s_0} \text{ with } g_{s_0} > 0 > b_{s_0} \text{ almost surely.}$$

Stochastic utility conditional on the realization of $s_0$ is meant to capture the possibility that the utility of a successful task for the employer may depend on the specific task to be solved, here assumed unobservable to the analyst, besides on the characteristic $s_0$ of the employee. It may also happen due to other characteristics of the candidate besides $s_0$ which are unobservable to the analyst but relevant to the employer.\(^{12}\) Finally, we assume that whenever the employer is indifferent between hiring and not hiring a candidate he uses an unbiased coin to break ties.

Besides biases in beliefs we allow for the possibility that the employer cares about the realization of $s_0$ as well. We require for the random variables $g_i, b_i, i = 1, 2$ to be jointly continuously distributed and to fulfill the following condition.

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10 Many situations have the same structure: lending activity of a bank, university applications, etc.

11 Other assumptions are possible. These here are for definiteness.

12 The employer may have lexicographic preferences; she cares about $s_0$ first and foremost but given $s_0$ also takes into account other unobservable features of the candidate.
(C) \( g_{s_0} \geq g_{s_0''} > 0 > b_{s_0} \geq b_{s_0''} \) almost surely.

A successful task benefits the employer more – and a failed one hurts him less – if it is the deed of an agent of characteristic \( s_0 \) rather than \( s_0'' \). That is, the employer incurs uniformly lower payoffs from \( s_0'' \) for each outcome.

We say that the employer cares about \( s_0 \) if the following holds.

**Preference for \( s_0' \):** \( g_{s_0'} > g_{s_0''} > 0 > b_{s_0'} > b_{s_0''} \) almost surely.

Here we ask for the ‘extreme’ inequalities in condition \((C)\) to hold strictly almost surely.\(^{13}\)

Assume now that an analyst has frequency data on both hiring decisions at the job fair and on the outcome of the task, even though she may not observe the precise type of the task in every instance. Thus for all \( s_0 = s_0', s_0'' \) and \( s_1 = g, b \) the analyst observes the limiting frequency that candidate \( s_0 \) is hired, and that state \( s_1 \) is realized, denoted by \( \rho_{s_0}(h_{s_0}, \{h_{s_0}, nh_{s_0}\}, s_1) \). This paper gives conditions on stochastic choice data which allows the following.

- As a first step the analyst can confirm that the true data-generating process is unbiased, i.e. that \( q_1(g|s_0') = q_1(g|s_0'') = \frac{1}{2} \) holds. This corresponds to the constraint
  \[
  \rho_{s_0'}(f_{s_0'}, \{f_{s_0'}, h_{s_0'}\}, g) = \rho_{s_0''}(f_{s_0''}, \{f_{s_0''}, h_{s_0''}\}, g) = \frac{1}{2}.
  \]

- The analyst can also discern from stochastic choice data whether there is bias in beliefs, whether the employer cares about the realization of \( s_0 \) or whether both are occurring simultaneously.

Namely, whenever the employer is unbiased in beliefs and doesn’t care about the realization of \( s_0 \) per se he chooses to hire either candidate with the same positive probability. This corresponds to the constraint

\[
\sum_{s_1} \rho_{s_0'}(f_{s_0'}, \{f_{s_0'}, h_{s_0'}\}, s_1) = \sum_{s_1} \rho_{s_0''}(f_{s_0''}, \{f_{s_0''}, h_{s_0''}\}, s_1).
\]

Whenever there is either bias in beliefs or the employer has preference for \( s_0' \) he hires candidate \( s_0' \) with strictly higher probability than candidate \( s_0'' \).

\[
\sum_{s_1} \rho_{s_0'}(f_{s_0'}, \{f_{s_0'}, h_{s_0'}\}, s_1) > \sum_{s_1} \rho_{s_0''}(f_{s_0''}, \{f_{s_0''}, h_{s_0''}\}, s_1) \tag{1}
\]

Finally, whenever the employer has incorrect beliefs and has preference for \( s_0' \), all else equal he hires candidate \( s_0' \) with a (weakly) higher probability than in the case of either bias in beliefs only or preference for \( s_0' \) only. This corresponds to a larger gap in (1).

This example shows that stochastic choice data coming from standard subjective expected utility (SEU) maximizers can be used to identify biases, whenever the analyst gets information for the realization of the objective state (here whether the task is successful or not). As we show, stochastic choice data allow comparisons of different employers in terms of their biases in much more complicated examples than the current one.

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\(^{13}\) Just as for beliefs other assumptions are here possible.
1.1.2 Educational choices

Consider an undergraduate student who adheres to subjective expected utility (SEU) and has beliefs about the final outcome in the job market once she graduates. This outcome comes from a finite objective state space, say,

\[ S = \{ \text{job in finance, job in tech industry, job in government, graduate school, start-up} \}. \]

At the beginning of the undergraduate education the student is also learning about her taste regarding possible careers and so has stochastic tastes \( \tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_\tau \) about the final outcome. At the end of some student-specific year \( \tau \geq 1 \), learning about taste ceases: the student has a fixed Bernoulli utility \( v \) about the final outcome \( S \) even though her beliefs \( q_t \) about the final outcome in \( S \) remain stochastic throughout the whole higher education experience.

Formally, let school years be encoded by \( t \in \{ 0, 1, \ldots, T \} \). Let \( s_t \) be a period-\( t \) signal about final outcome coming from a finite space of objective signals \( S_t \). These can be grades or feedback from faculty, experiences in internships, etc. Let acts (decisions of a student) correspond to jobs/projects/classes she engages with in each year and menus \( A_t \) be finite collections of such acts the student can choose from in each education year. Denote the set of menus available in period \( t \) by \( A_t \). Given a realized signal \( s_t \), each act \( f_t \) in period \( t \) delivers a lottery over pairs consisting of an instantaneous prize from a finite set of prizes \( Z \) and a continuation decision problem \( A_{t+1} \) from \( A_{t+1} \). The realization of the continuation problem \( A_{t+1} \) corresponds to jobs/internships/classes possibly available to the student, after she has taken a current class corresponding to the act \( f_t \). Say that an act \( f_t \) is constant, if the lottery over pairs of current prize and continuation decision doesn’t depend on the realization of the signal \( s_t \), i.e. it is the same for all \( s_t \) in \( S_t \). E.g. a constant act is a summer job a student may take only due to financial reasons and which doesn’t enhance her intellectual skills in the job market for any possible career.

The analyst observes past choices of, say act \( f_t \) chosen from menu \( A_t \) as well as the realization of signal \( s_t \in S_t \), and for the current period \( t \in \{ 1, \ldots, T \} \) she observes after the history \( h_{t-1} = (f_0, A_0, s_0; \ldots; f_{t-1}, A_{t-1}, s_{t-1}) \) the frequencies of triples \((f_t, A_t, s_t)\). These history-dependent frequencies, denoted by \( \rho_t(f_t, A_t, s_t|h_{t-1}) \), are to be interpreted as after history \( h_{t-1} \) student chose \( f_t \) when facing \( A_t \) and the objective signal \( s_t \) was realized.

If the history-dependent preference of the student over menus/decision problems from \( A_t, t = 0, \ldots, T \) were observable, it is intuitive to expect it satisfies the following properties.

A. **Preference for Flexibility:** Every year the student prefers menus which are larger rather than subsets thereof. That is, \( B_t \in A_t \) is less valuable than \( A_t \in A_t \) if \( B_t \subset A_t \). This is because a strict subset offers less option value for a SEU agent than a full menu.

B. **Weak Dominance for \( t \leq \tau \):** At \( \tau = 0 \), say, she prefers to replace a single act \( f_1 \) whose utility depends on the realization of the signal \( s_1 \) with a menu of constant acts \( \bar{A} = \{ f_1(s_1) : s_1 \in S_1 \} \) offering the same outcomes (lotteries over \( Z \times A_2 \)) as every \( s_1 \)-dependent outcome of \( f_1 \). This is because menu \( \bar{A} \) offers insurance against her stochastic taste in \( t = 1 \). Intuitively, summer jobs where the student doesn’t learn

\[ ^{14} \text{There is no continuation problem in period } t = T. \]
new specialized skills for the job market may be more valuable to a student who is still unsure of her taste about different careers than committing to an internship whose outcome is highly dependent on what she learns about her career taste at the end of the current period.

C. **Strong Dominance for** $t > \tau$: From the end of period $\tau$ on, whenever the act $f_{t+1}$ delivers weakly better utility for each realization of the signal in period $t + 1$ than $g_{t+1}$, from the perspective of the end of year $t$, the menu $\{f_{t+1}, g_{t+1}\}$ is as good as $\{f_{t+1}\}$. This unambiguous comparison of continuation problems in the end of year $t$ becomes possible because at the end of period $\tau$ the career tastes of the student have stabilized and are deterministic.\(^{15}\) Given a fixed taste about distinct careers she is able to at least determine when an act is uniformly more valuable than another, no matter the realization of the objective signal in the current period $t$.

We show how the properties A-C can be derived from ex-post stochastic choice from menus without knowing anything about the preference over menus of the student. Moreover, our methods allow the analyst to also determine the speed with which an agent, such as the student in this example, learns her final taste $v$ (e.g. to determine the $\tau$ of the student). For example, if the act $f_1$ is taking an internship which requires substantial investment in learning new skills in a very specific field like finance, i.e. an act whose outcome is highly dependent on $s_1$ as well as the realization of the future taste $\tilde{v}_1$, we should expect an agent who knows by the end of period $t = 0$ that her taste is so that she likes to get a job in finance, to prefer committing to $f_1$ at the end of $t = 0$. This should be especially the case if the alternative is to face a menu which offers acts whose outcomes don’t depend much on $s_1$ or the realization of $\tilde{v}_1$ such as helping out with grading an undergrad class, taking up a summer job in the library, etc, even though they might be as financially profitable as picking the internship in finance $f_1$.

Finally, given richness of the data, our characterization results show how an analyst is able to compare different agents according to their speed of learning about taste in similar situations.\(^{16}\)

## 2 Static Random Subjective Expected Utility with observable objective states

In this section we introduce and characterize the static model. This is the crucial building block of the dynamic model of section 3.

**Set up in the static model.** Let $Z$ be a prize space assumed to have a separable, metric topological structure. Let $S$ be a finite set of objective states\(^{17}\) and $\mathbb{F}$ the set of Anscombe-Aumann acts (AA acts) with a typical element given by $f : S \rightarrow \Delta(Z)$ where

\(^{15}\)The names are justified: after formally introducing the technical set up and the axioms in the main body of the paper we show in online appendix section 5 that under Preference for Flexibility, Strong Dominance implies Weak Dominance but not the other way around.

\(^{16}\)Intuitively in our example, student 1 learns her taste faster than student 2, if stochastic choice data give evidence that student 1 satisfies Strong Dominance whenever student 2 does.

\(^{17}\)The wording objective means that the state $s$ is verifiable by both agent and analyst after it occurs.
\[ \Delta(Z) \text{ denotes the space of simple lotteries over prizes in } Z. \] Finally, denote by \( \mathcal{A} \) the collection of finite, nonempty subsets of \( \mathbb{F} \). A typical element in \( \mathcal{A} \) is called a menu and denoted by a capital letter, e.g. \( A \in \mathcal{A} \). \( \mathcal{A} \) is equipped with the Hausdorff topology.

Given a belief of the agent over \( S \), i.e. an element \( q \) from \( \Delta(S) \) and an Expected Utility function \( u : \Delta(Z) \to \mathbb{R} \) to evaluate simple lotteries, we say the agent satisfies Subjective Expected Utility (SEU) with beliefs \( q \) and taste \( u \) if the utility of an act \( f \) is given by
\[
q \cdot (u \circ f) := \sum_{s \in S} q(s)u(f(s)).
\]

Define
\[
N(A, f) = \{(q, u) \in \Delta(S) \times \mathbb{R}^Z : q \cdot (u \circ f) \geq q \cdot (u \circ g), g \in A\}.
\]
This is the set of SEUs which can rationalize the choice of \( f \) from menu \( A \).

Denote \( N^+(A, f) \) the respective subset of \( N(A, f) \) where \( f \) is not tied to other acts from \( A \).

Moreover, define
\[
M(A; u, q) = \{f \in A : q \cdot (u \circ f) \geq q \cdot (u \circ g), g \in A\}.
\]
This is the set of maximizers when the agent's belief about objective state of the world is \( q \) and her Bernoulli utility is \( u \).

The timeline of the one-period model is the following.

\begin{center}
\begin{tikzpicture}
\draw[->] (-5,0) -- (5,0);
\draw (0,0) -- (0,-1) node[anchor=east] {\small SEU \((q,u)\) realized};
\draw (2,-1) node[anchor=west] {\small Agent picks \( f \) out of \( A \)};
\draw (5,-1) node[anchor=west] {\small \( s \) realized, payoff \( f(s) \)};
\end{tikzpicture}
\end{center}

Figure 1: Timeline for the static setting.

Let \( \mu \) be a probability measure over \( \Delta(S) \times \mathbb{R}^X \), equipped with the sigma-algebra \( \mathcal{F} \) generated by sets of the form \( N^+(A, f), N(A, f) \) or alternatively with the Borel sigma-algebra of \( \Delta(S) \times \mathbb{R}^X \).

We say that \( \mu \) is regular if \( \mu(N^+(A, f)) = \mu(N(A, f)) \) for any \( A \in \mathcal{A}, f \in \mathcal{F} \). In this paper regular measures \( \mu \) have the following form: whenever there are ties, i.e. \( M(A; u, q) \) is not a singleton for some \( A \) and SEU pair \((q, u)\) the agent randomly picks an auxiliary SEU pair \((p, v)\) such that \( M(M(A; u, q); p, v) \) is a singleton.

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18A lottery is called simple if only finitely many prizes can happen with positive probability. \( \Delta(Z) \) is equipped with the topology of weak convergence of probability measures. The set of acts \( \mathbb{F} \) is equipped with the product-topology over \( \Delta(Z)^S \).

19In the following we often identify the EU-functional \( u : \Delta(Z) \to \mathbb{R} \) with its Bernoulli utility from \( \mathbb{R}^Z \).

20This is constructed as a product sigma-Algebra of the respective Borel sigma-Algebra of weak convergence on \( \Delta(S) \) and the Borel one from \( \mathbb{R}^X \) (the latter again a product sigma-Algebra).

21The interested reader can peruse the proofs in Section 1 in the online appendix for the mathematical details. This tie-breaking rule is special and it will be reflected in the properties of the data in the form of a specific axiom: Extremeness-type of Axioms (see next subsections) imply tie-breaking through SEUs.
Observable in the static model. We assume the analyst observes an augmented stochastic choice function defined as in part 1) of the following definition.

**Definition 1.** 1) An augmented stochastic choice function (aSCF) is a map \( \rho : \mathcal{F} \times \mathcal{A} \times S \rightarrow [0, 1] \) with the properties

\[
(a) \quad \sum_s \sum_{f \in \mathcal{A}} \rho(f, A, s) = 1, \quad \forall A \in \mathcal{A}.
\]

\[
(b) \quad \rho(s) := \sum_{f \in \mathcal{A}} \rho(f, A, s) = \sum_{f \in \mathcal{B}} \rho(f, B, s) > 0, \quad \forall A, B \in \mathcal{A}, s \in S.
\]

2) A stochastic choice function (SCF) \( \zeta \) is a map \( \zeta : \mathcal{F} \times \mathcal{A} \rightarrow [0, 1] \) with the property

\[
\sum_{f \in \mathcal{A}} \zeta(f, A) = 1, \quad \forall A \in \mathcal{A}
\]

The second requirement in the definition of aSCF makes sure that we can define the observed frequency of objective state \( s \) independently of the decision problem the agent is facing. This says that objective uncertainty is fully exogenous and independent from the problem the agent is facing in addition to being outside the influence of the analyst. Formally, it allows the definition of \( \rho(s) = \sum_{f \in \mathcal{A}} \rho(f, A, s) \) for any \( A \) and \( s \in S \), i.e. the probability of observing \( s \) in the data.

For a given aSCF \( \rho \) we denote in the following by \( \bar{\rho} \) the SCF derived from summing each \( \rho(f, A, s) \) across states. Formally,

\[
\bar{\rho}(f, A) := \sum_{s \in S} \rho(f, A, s), \quad f \in \mathcal{A}, A \in \mathcal{A}.
\]

Discussion of the Observable. Assuming that the data of the analyst comes in the form of aSCFs characterizes an analyst with superior information compared to the set up of [Lu ’16]. In many realistic situations this is a viable assumption: loan performance data, how students perform in school or how an employee performs in some task is often observable to an outside analyst.\(^{22}\)

[Ellis ‘18] and [Caplin, Dean ’15] also consider state-dependent choice data but have a different focus: that of information acquisition in a static setting. They don’t study the question of misspecified learning, either because the analyst doesn’t get to see the realization of the objective state or because they assume from the start that the agent is using the correct prior. [Caplin, Martin ’14] considers state-dependent stochastic choice data in a passive learning model similar to ours but assume that the taste of the agent is deterministic and known to the analyst.

\(^{22}\)Section 2 of the online appendix considers extensively the case when the observable corresponds to SCF, that is the realization of \( s \) is not observable by the analyst. For the static setting the whole theory, up to explicit modeling of the tie-breaking is contained in [Lu ’16], whereas the dynamic version of his model can be derived easily using the approach of [Frick, Iijima, Strzalecki ’17]. See the online appendix for more details.
Remark 1. The observable in Definition 1 has more general applicability, e.g. it can be used even if there is partial observability of $s$ as long as there is full identification in the aggregate.

In more detail, assume the analyst observes a signal $y \in Y$ about the true realization of the objective state $s \in S$ instead of its realization. If $\hat{\mu}(y|s)$ gives the (menu-independent) conditional probability of observing signal $y$ when the realized state is $s$ the assumption of aSCFs as observable is valid for the analysis if the following two conditions hold:
- $\hat{\mu}$ is known by the analyst,
- The matrix $(\hat{\mu}(y|s))_{y \in Y, s \in S}$ is quadratic and has full rank.

2.1 Representation in the Static Setting

We now introduce the Random Subjective Expected Utility representation for an aSCF $\rho$ we are after. An agent has private information about both beliefs over the realization of the objective state $s$ as well as her taste $u \in \mathbb{R}^Z$. The analyst observes only aggregate frequencies of choice data and realizations of the objective state from the same agent in many choice instances or similar aggregate data choices from a population of agents.

Definition 2. A Random SEU representation (R-SEU) of the aSCF $\rho$ is a tuple $(\Omega, \mathcal{F}^*, \mu, (q, u, s), (\hat{q}, \hat{u}))$ such that

A. $(\Omega, \mathcal{F}^*, \mu)$ is a finitely additive probability space,

B. $(q, u, s) : \Omega \rightarrow \Delta(S) \times \mathbb{R}^Z \times S$ is an injective map, has non-constant SEU $(q(\omega), u(\omega))$

C. Either

C1. The representation has correct interim beliefs (cib): $\mu(s \in \cdot | q, u) = q(\cdot)$

or otherwise

C2. The representation has no unforeseen contingencies (nuc):

$\text{supp}(\mu(s \in \cdot | q, u)) \subset \text{supp}(q(\cdot))$.

D. the $(q, u)$-measurable tiebreaking process $(\hat{q}, \hat{u}) : \Omega \rightarrow \mathbb{R}^Z$ is regular and for all $f \in A$,

$\rho(f, A, s) = \mu(C(f, A, s))$.

Here, $C$ is defined as

$C(f, A, s) = \{\omega \in \Omega : f \in M(M(A, q(\omega), u(\omega)), \hat{q}(\omega), \hat{u}(\omega)) \land s(\omega) = s\}$.

In the following $\omega$ are called states of the world. $C(f, A, s)$ denotes then the collection of states of the world where the agent chooses $f$ from $A$ and the objective state $s$ is realized.

Before continuing, we note down the true data-generating process (DGP) derived from the representation.

Definition 3. For an aSCF $\rho$ that satisfies a R-SEU representation define the DGP, a $\Delta(S)$-valued random variable $\bar{q} : \Omega \rightarrow \Delta(S)$ as

$\bar{q}(\omega)(\cdot) = \mu(s \in \cdot | q, u)(\omega)$.
Given this definition the property of correct interim beliefs (cib) can be written as

\[ \bar{q} = q \]

whereas that of unforeseen contingencies (nuc) is written as

\[ \text{supp}(\bar{q}) \subset \text{supp}(q). \]

### 2.2 The revealed subjective support of a SCF

We look at an SEU agent and introduce a concept which is helpful in the characterization results of this paper.\(^{23}\) For this part we assume a unique normalization of the Bernoulli utilities appearing in the definition of an SEU. Fix two distinct prizes \(z_0, z_1 \in \mathbb{Z}\) and define \(U_0 := \{u \in \mathbb{R}^Z : u(z_0) = 0, u(z_1) = 1\}\). That is, we encode in the following a SEU preference through a belief \(q \in \Delta(S)\) and a taste \(u \in U_0\).

**Definition 4.** For a SCF \(\zeta\) let \(RSSupp(\zeta)\), the revealed subjective support of \(\zeta\), be defined through

\[
RSSupp(\zeta) = \{(q, u) \in \Delta(S) \times U_0 : \forall A \in A, f \in A, \text{if } (q, u) \in N(A, f) \text{ then there exists } (f_n, A_n) \to (f, A) \text{ with } \zeta(f_n, A_n) > 0 \}. 
\]

Here convergence \((f_n, A_n) \to (f, A)\) is in the product topology of \(\mathbb{F} \times A\).

This says that a SEU \((q, u)\) is in the revealed subjective support of \(\zeta\) if every choice that can be rationalized by the SEU appears in the data encoded by \(\zeta\), up to tie-breaking.\(^{24}\)

**Aside.** Another compact and suggestive way to write down the revealed subjective support of a SCF \(\zeta\) is as follows.

For a SEU preference \((q, u)\) over \(\mathbb{F}\) denote the set of choices it can rationalize as \(R^{(q,u)}\), that is

\[
R^{(q,u)} = \{(f, A) \in \mathbb{F} \times A : (q, u) \in N(f, A)\}.
\]

This is the set of choice data that are consistent with maximization of the SEU \(R^{(q,u)}\). The set of choices explained by the data represented by some SCF \(\zeta\) is given by

\[
N(\zeta) = \{(f, A) : f \in A, \exists (f_n, A_n) \to (f, A) \text{ with } \zeta(f_n, A_n) > 0 \text{ for all } n\}.
\]

Then \(RSSupp(\zeta)\) can be characterized as follows.

\[
RSSupp(\zeta) = \{(q, u) \in \Delta(S) \times U_0 : R^{(q,u)} \subset N(\zeta)\}.
\]

\(^{23}\)In fact, the concept of Revealed Subjective Support of a SCF is more general and can be applied to any SCF whose support is on continuous preferences over acts in \(\mathbb{F}\).

\(^{24}\)In more detail: \((q, u)\) occurs in the data if for every choice pair \((f, A)\) either (1) \(\zeta(f, A) > 0\) and \((q, u) \in N(A, f)\) or if (2) \(\zeta(f, A) = 0\) and \((q, u) \in N(A, f)\) then \(\zeta(f, A) = 0\) only happens due to tie-breaking.
2.3 Axiomatization of aSCFs

The following axiomatization of aSCFs is based on previous results about the axiomatization of SCFs in [Lu ’16] and [Ahn, Sarver ’13].

Axioms 0-1 till 0-5 below are adaptations to our setting of aSCFs of the standard axioms from Theorem S.1 of [Lu ’16]. They imply that an aSCF comes from an underlying RUM whose revealed subjective support contains only SEUs. Axiom 0-6 is adapted from [Ahn, Sarver ’13] and ensures that there can only occur finitely many such SEUs.

**Standard Axioms in statewise form.** For all $s \in S$ it holds

**Axiom 0-1: Statewise Monotonicity.** $\rho(f, A, s) \geq \rho(f, B, s)$ for $A \subset B$.

**Axiom 0-2: Statewise Linearity.** $\rho(\lambda f + (1 - \lambda)g, \lambda A + (1 - \lambda)\{g\}, s) = \rho(f, A, s)$ for any $A \in \mathcal{A}, g \in \mathcal{F}$ and $\lambda \in (0, 1)$.

**Axiom 0-3: Statewise Extremeness.** $\rho(\text{ext}(A), A, s) = 1$ for all $A \in \mathcal{A}$.

**Axiom 0-4: Statewise Continuity.** $\mathcal{A} \ni A \mapsto \rho(\cdot, A|s)$ is continuous.  

**Axiom 0-5: State Independence.** To explain this axiom we first introduce some terminology: a menu $A$ is called constant if it contains only constant acts. Given a menu $A$ and a state $r \in S$ let $A(r) = \{f(r) : f \in A\}$ be the constant menu containing all lotteries from acts in $A$ which happen at state $r$.

Then State Independence says: Suppose $f(s_1) = f(s_2), A_1(s_1) = A_2(s_2)$ and $A_i(s) = \{f(s)\}, s \neq s_i, i = 1, 2$. Then $\rho(f, A_i, s) = \rho(f, A_1 \cup A_2, s)$.

Intuitively, if an act $f$ yields the same payoff in states $s_1$ and $s_2$, payoffs of menu $A_1$ in $s_1$ are the same as those of menu $A_2$ in $s_2$ and acts in $A_i$ only differ in $s_i$ then the probability of choosing $f$ in $A_1$ is the same as choosing $f$ in $A_1 \cup A_2$, unless the realization of the Bernoulli utility of the agent depends on whether $s_1$ or $s_2$ is realized.

**Axiom 0-6: Statewise Finiteness.** There is $K > 0$ such that for all $A \in \mathcal{A}$, there is $B \subset A$ with $|B| \leq K$ independent of $s$ such that for every $f \in A \setminus B$ there are sequences $f^n \to^m f$ and $B^n \to^m B$ with $\rho(f_n, \{f_n\} \cup B^n, s) = 0$.

To state the axiom of correct beliefs we define for a SEU pair $(q, u)$ where $q$ is the belief of the agent and $u$ her Bernoulli utility as $\pi_q(p, u) = p$. That is, the projection to the belief used from the agent. Furthermore, in the following $\rho(s|f, A)$ is the conditional probability of observing the realization of the objective state $s$ in the data conditional on the agent choosing $f$ from menu $A$.

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25Note that $\mathcal{F}$ has a mixture structure in the usual way. In particular, one can form $\text{conv}(A)$, the convex hull of $A$ for any menu $A$. Then $\text{ext}(A)$ is identified with the set of extremum points of $\text{conv}(A)$.

26The image of the mapping is the space of simple lotteries on $\mathcal{F}$, equipped with the topology of weak convergence of probability measures.
Axiom 0-7: Correct Interim Beliefs (CIB). For all $f \in \mathbb{F}$ and $A \in \mathcal{A}$ with $\bar{\rho}(f, A) > 0$ we have
\[ \rho(\cdot | f, A) \in \pi_q \left( \text{conv} \left( N(f, A) \cap \text{RSSupp}(\bar{\rho}) \right) \right). \]

The axiom says that the DGP of the objective state $s$ conditional on observed choice $(f, A)$ is a mixture of beliefs which correspond to some SEU that fulfill two natural conditions simultaneously: 1) the SEU is contained in the revealed subjective support of the data and 2) the SEU rationalizes the choice $f$ from $A$.

Incorrect beliefs can arise due to different reasons: the agent may observe objective signals with noise, she may have a misspecified prior or otherwise have subjectively biased beliefs even though they average out to the correct prior. We exclude in this paper the case when incorrect beliefs originate from non-Bayesian updating with respect to any prior.

In contrast to section 6 of [Lu '16] here the analyst gets information about the realization of the objective state and can glean out the true DGP from data. This allows her to make a direct comparison between the true DGP and the beliefs of the agent. Section 7 of [Lu '16] constructs a test of CIB based on test acts. His methods require non-stochastic taste whereas our axiom is robust to stochasticity of tastes.

Now we present a relaxation of the Correct Interim Beliefs Axiom which allows for incorrect beliefs but so that the incorrectness remains undetected by the agent ex-post. This is inconsequential in a static setting but has repercussions in the dynamic setting of Section 3 where we study an agent who passively learns about objective states as well as her taste in every period.

Axiom 0-7*: No Unforeseen Contingencies (NUC) For all $f \in \mathbb{F}$ and $A \in \mathcal{A}$ with $\bar{\rho}(f, A) > 0$ it holds
\[ \text{supp} \left( \rho(\cdot | f, A) \right) \subset \bigcup \{ \text{supp}(q) : q \in \pi_q \left( N(A, f) \cap \text{RSSupp}(\bar{\rho}) \right) \}. \]

Our first main result gives the axiomatization of aSCFs in a static setting.

Theorem 0. The aSCF $\rho$ on $\mathcal{A}$ admits a R-SEU representation with CIB satisfied if and only if it satisfies Axioms 0-1 till 0-7. It admits a R-SEU representation with NUC satisfied if and only if it satisfies Axioms 0-1 till 0-6 together with Axiom 0-7'.

In the following whenever for an aSCF $\rho$ the Axioms 0-1- till 0-6 together with 0-7' are satisfied, we say Axiom 0 is satisfied for $\rho$.

2.3.1 Informational Representation for aSCFs

We consider here the special case of Theorem 0 where all possible Bernoulli utilities in the representation are equal up to positive affine transformations of each other. This implies that stochasticity in choice only comes from randomness in beliefs.

To facilitate analysis, we require the existence of a best constant act. This requirement is easily expressed in terms of stochastic choice.

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27 The right hand side of (2) is equal to $\text{conv} \left( \pi_q \left( N(f, A) \cap \text{RSSupp}(\bar{\rho}) \right) \right)$, as can be easily established.

28 Moreover, in the dynamic model in Section 3 we assume that the agent is sophisticated and thus our model doesn’t allow any prospective overconfidence/underconfidence as in [Lu '16].
Axiom: Existence of a constant best act. There exists a constant act $\bar{f} \in \mathbb{F}$ such that for every act $f \in \mathbb{F}$ it holds

$$f \neq \bar{f} \implies \rho(f, \{f, \bar{f}\}) = 0.$$ 

The existence of a best constant act is assured for example if $Z$ consists of monetary prizes and the preferences of the agent over money are strictly increasing. Whenever this Axiom is satisfied, it becomes easier to eschew tie-breaking considerations when writing down other Axioms on data.

The axiom on data which ensures that the agent has a deterministic taste is the following.\footnote{This is an adaptation of the C-Determinism Axiom from the [Lu '16] who doesn’t consider tie-breaking explicitly as we do.}

**Axiom: C-Determinism*.** For any menu $A$ consisting of constant acts it holds true

$$\lim_{a \to 1} \rho\left(af + (1-a)\bar{f}; A \setminus \{f\} \cup \{af + (1-a)\bar{f}\}\right) \in \{0, 1\}.$$ 

This says that except for possible stochastic tie-breaking, constant acts are chosen deterministically. On the other hand, if taste is stochastic then choice from constant menus should be stochastic, even after taking into account possible stochastic tie-breaking. Given this intuition the following characterization result is not surprising.

**Proposition 1 (Informational Representation for aSCFs).** Assume that an aSCF $\rho$ has a R-SEU representation with regular measure $\mu$. Assume that there exists a constant best act.

Then the following are equivalent.

A. For all $(q, u), (p, v) \in \text{RSSupp}(\bar{\rho})$ $u$ is a positive affine transformation of $v$.

B. $\rho$ satisfies C-Determinism*.

### 3 Dynamic Random Subjective Expected Utility

This section is devoted to the dynamic model. We introduce the general representation and two interesting specializations of it. After that, we give axioms for all three representations.

**Set up in the dynamic model.** Let $Z$ be a finite prize space, $\infty > T \geq 1$ and for each $t = 0, \ldots, T$ let $S_t$ be finite spaces of objective states. The objective states evolve according to a DGP which cannot be influenced by the agent (passive learner situation).

Define recursively the spaces of consequences for every period as follows. Let $X_T = Z$ and the set of acts $\mathbb{F}_T$ with a typical element $f_T : S_T \to \Delta(Z)$. Let $\mathcal{A}_T$ be the collection of finite sets from $\mathbb{F}_T$. Then continue inductively by defining $X_t = Z \times \mathcal{A}_{t+1}$, where $\mathcal{A}_{t+1}$ is the collection of finite menus from $\mathbb{F}_{t+1}$. $\mathbb{F}_t$ is then the set of acts $f_t : S_t \to \Delta(X_t)$.\footnote{Furthermore we denote in the following by $\mathcal{A}_c^t$ the collection of period−$t$ menus consisting of constant acts.}
Thus, an act $f_t$ at time $t < T$ gives for each possible objective state $s_t$ a lottery over current consumption and a continuation decision problem/menu. We denote $f_t^A$ the marginal act on menus $A_{t+1}$ and $f_t^Z$ the marginal act on $Z$ induced by $f_t$.

We assume in each period $(q_t,u_t)$ is private information of the agent whereas the realization of $s_t$ is observed by both the agent and the analyst. Thus stochasticity in choice comes from the information asymmetry between the agent and the analyst in the single-agent interpretation, whereas in the population interpretation the analyst is observing dynamic data from a population of SEU agents whose preference characteristics are unknown.

Visually the timeline is depicted in Figure 2.

![Figure 2: Timeline for the dynamic setting.](image)

**The observable in the dynamic setting.** The analyst observes histories with a typical element $h^t$ as well as history-dependent aSCFs $\rho_t(\cdot|\cdot^t)$. The collection of the former is denoted by $\mathcal{H}_t$ whereas of the latter simply by $\rho$ and called a dynamic augmented stochastic choice function (dynamic aSCF). These are described recursively as follows. For $t = 0$ the analyst observes an aSCF $\rho_0$ as in Definition 1. The set $\mathcal{H}_0$ collects all histories $h^0 = (f_0,A_0,s_0) \in F \times A_0 \times S_0$ such that $\rho_0(h^0) > 0$. For $h^0 \in \mathcal{H}_0$ denote $A_1(h^0) := \text{supp}(f_0^A)$ the set of period−1 menus that follow $h^0$ with positive probability. The construction is continued recursively: for any history $h^t \in \mathcal{H}_t$ there is an aSCF $\rho_{t+1}(\cdot|h^t)$ which can be used to define the set of possible continuation menus $A_{t+1}(h^t)$. The set of period−$(t+1)$ histories is then $\mathcal{H}_{t+1} := \{(h^{t+1},f_{t+1},A_{t+1},s_{t+1}): A_{t+1} \in A_{t+1}(h^{t+1}),\rho_{t+1}(f_{t+1},A_{t+1},s_{t+1}|h^{t+1}) > 0\}$.

In simple words: histories are finite sequences of triplets $(f_i,A_i,s_i)$ with the interpretation that the data shows that with positive probability $f_i$ is chosen from menu $A_i$ and $s_i$ is the realized objective state in period $i$. Moreover, a history can only happen if the elements $(f_i,A_i,s_i)$ of its sequence happen successively with positive probability starting from the ‘oldest’ one $(f_0,A_0,s_0)$ to the most recent.

The data reflects limited observability in the sense that $\rho_t$ is defined only conditional on histories which happen with positive probability in the data. We show below how this can be overcome.

### 3.1 Representations

We first define properties shared by all representations. The focus is on having properties which are tractable but still allow for a general enough representation.
3.1.1 Simplicity, regularity and preference-based tie-breaking.

Say that the triple \((F_t, q_t, u_t, s_t)_{0 \leq t \leq T}\) is simple w.r.t.\(^{31}\) the probability space \((\Omega, \mathcal{F}^*, \mu)\) if

A. each \(F_t\) is generated by a finite partition such that \(\mu(F_t(\omega)) > 0\) for all \(\omega \in \Omega\). Here \(F_t(\omega)\) is the partition cell of \(F_t\) which contains \(\omega\).

B. the map \((q_t, u_t, s_t) : \Omega \rightarrow \Delta(S_t) \times \mathbb{R}^{X_t} \times S_t\) has non-constant SEU \((q_t(\omega), u_t(\omega))\) for all \(\omega\) and is adapted to the filtration \(F_t, t \leq T\). Moreover, whenever \(\omega' \not\in F_t(\omega)\) it holds \((q_t(\omega), u_t(\omega), s_t(\omega)) \neq (q_t(\omega'), u_t(\omega'), s_t(\omega'))\).

The tiebreakers \((\hat{q}_t, \hat{u}_t)_{0 \leq t \leq T}\) are regular and preference-based, i.e.

A. \(\mu(\omega \in \Omega : |M(A_t, \hat{q}_t, \hat{u}_t)| = 1) = 1\) for all \(A_t \in \mathcal{A}_t\).

B. conditional on \(F_T(\omega)\) the sequence \((\hat{q}_t, \hat{u}_t), \ldots, (\hat{q}_T, \hat{u}_T)\) is independent and

C. \(\mu((\hat{q}_t, \hat{u}_t) \in \cdot |F_T(\omega)) = \mu((\hat{q}_l, \hat{u}_l) \in \cdot |q_l(\omega), u_l(\omega), l \leq t)\) for all \(t\).

Simplicity and regularity are necessary for a parsimonious representation, whereas the preference-based condition incorporated in C. ensures that the tie-breaking of the agent depends only on her realized SEU in the period at hand (and through it also on past history) but not on the realization of the objective state in the current period.

We define for a triple \((f_k, A_k, s_k)\) the set

\[C(f_k, A_k, s_k) = \{\omega \in \Omega : f_k \in \mathcal{M}(M(A_k, q_k(\omega), u_k(\omega)), \hat{q}_k(\omega), \hat{u}_k(\omega)), s_k(\omega) = s_k\}.\]

These are the states of the world which rationalize the observable \((f_k, A_k, s_k)\) in period \(k\). Similarly one defines for a history \(h^t = (A_0, f_0, s_0; \ldots; A_t, f_t, s_t)\) the set of states of the world which rationalize the occurrence of the history.

\[C(h^t) = \cap_{t \leq l} C(A_l, f_l, s_l).\]

3.1.2 The general representation.

We are now ready to write down the most general representation of a dynamic aSCF. It doesn’t impose any functional restrictions on the Bernoulli utilities of the agents and only a minimal restriction on the evolution of beliefs.

**Definition 5.** A Dynamic Random SEU representation (DR-SEU) of the dynamic aSCF \(\rho\) is a tuple \((\Omega, \mathcal{F}^*, \mu, (F_t, (q_t, u_t), s_t, (\hat{q}_t, \hat{u}_t))_{0 \leq t \leq T})\) such that

A. \((\Omega, \mathcal{F}^*, \mu)\) is a finitely additive probability space,

B. the filtration \((F_t) \subset \mathcal{F}^*\) and the \(\mathcal{F}_t\)-adapted process \((q_t, u_t, s_t) : \Omega \rightarrow \Delta(S_t) \times \mathbb{R}^{X_t} \times S_t\) is simple,

\(^{31}\)w.r.t. stands for with respect to.
C. the $\mathcal{F}^*$-measurable tiebreaking process $(\hat{q}_t, \hat{u}_t) : \Omega \to \mathbb{R}^{X_t}$ is regular and preference-based and for all $f_t \in A_t, h^{t-1} \in \mathcal{H}_{t-1}(A_t)$,

$$\rho_t(f_t, A_t, s_t|h^{t-1}) = \mu(C(f_t, A_t, s_t)|C(h^{t-1})).$$

D. Either

D.1. The representation has correct interim beliefs (CIB):

$$\mu(s_t \in \cdot | q_t) = q_t(\cdot)$$

for all $t \in \{0, \ldots, T\}$,

or otherwise

D.2. The representation has no unforeseen contingencies (NUC):

$$\text{supp}\left(\mu(s_t \in \cdot | q_t, u_t)\right) \subset \text{supp}(q_t(\cdot)).$$

Some explanations are in order. History $h^{t-1}$ happens with the probability $\mu(C(h^{t-1}))$: the state of the world has to be so that for each $l \leq t$ the realized subjective state/SEU $(q_l, u_l)$ picks $f_l$ from $A_l$, $f_l$ survives any possible tie-breaking and finally, in period $l$ the objective state $s_l$ is realized.

Conditional on $C(h^{t-1})$ occurring, $f_t$ is chosen from $A_t$ only if the realized subjective state in period $t$ given by the pair $(q_t, u_t)$ is so that a SEU-maximizing choice from $A_t$ is $f_t$ and $f_t$ survives any possible tie-breaking.

Note that the stochastic process of the objective and subjective states is unconstrained, except for the condition D: the agent uses the correct data-generating process conditional on her private information (correct interim beliefs) or otherwise she respects the requirement of (no unforeseen contingencies), i.e. the agent never gets hard evidence that her belief process is misspecified. The only other requirement embodied in the definition is that the agent uses Bayes rule to update her beliefs.

3.1.3 Two special cases: Evolving SEU vs. Gradual Learning.

As noted before, the general representation doesn’t include any behavioral restrictions on the evolution of the beliefs and tastes of the agent besides the SEU assumptions and that the agent remains Bayesian after every history with respect to her beliefs about the future evolution of tastes and objective states. In particular, her beliefs about the future SEU realizations may be incorrect. In this subsection we exclude this possibility.

**Evolving SEU.** This specialization of DR-SEU captures a dynamically sophisticated agent who correctly takes into account the evolution of her future SEU preferences. There is an $\mathcal{F}_t$-adapted process of random EU-functionals $v_t, t = 0, \ldots, T$, the felicity functions, over instantaneous consumption lotteries $l \in \Delta(Z)$ and a discount factor $\delta > 0$ such that $u_T = v_T$ and $u_t$ for $t \leq T$ is given by the following Bellman equation.

$$u_t(f_t(s_t)) = v_t(f_t^Z(s_t)) + \delta \mathbb{E}_{A_{t+1} \sim f_t^Z(s_t), q_{t+1} \cdot u_{t+1}} \left[ \max_{f_{t+1} \in A_{t+1}} (q_{t+1} \cdot u_{t+1})(f_{t+1}) \bigg| \mathcal{F}_t \right].$$

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32 This model of sophisticated behavior still doesn’t encompass all possible sophisticated behaviors allowed by the general DR-SEU representation – see Example 3 concerning [Epstein '06] and [Epstein et al '08] in subsection 3.2.
Here the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t]$ takes into account the randomness coming from the lottery $f^k_t(s_t)$ of the continuation problem as well as from the uncertainty about the SEU of the agent in period $t+1$. The agent makes the correct inference about the future SEU $q_{t+1} \cdot u_{t+1}$, given her current information in $\mathcal{F}_t$.

**Definition 6.** An Evolving SEU representation of the dynamic aSCF $\rho$ is a tuple $(\Omega, \mathcal{F}^*, \mu, (\mathcal{F}_t, q_t, u_t, s_t)_{0 \leq t \leq T})$ such that

A. $(\Omega, \mathcal{F}^*, \mu, (\mathcal{F}_t, q_t, u_t, s_t)_{0 \leq t \leq T})$ is a DR-SEU representation.

B. (3) holds true for the stochastic process of Bernoulli utilities $u_t, t = 0, \ldots, T$.

If we assume there is only one period ($T = 0$) then Evolving SEU collapses to the static model of section 2. The same holds trivially true for the following special case of Evolving SEU.

**Gradual Learning.** This is a specialization of the Evolving SEU representation which captures an agent who is learning about her taste. This results in a martingale condition on the evolution of the felicities $v_t, t = 0, \ldots, T$.

**Definition 7.** A Gradual Learning (GL-SEU) representation of the dynamic augmented stochastic choice rule $\rho$ is a tuple $(\Omega, \mathcal{F}^*, \mu, (\mathcal{F}_t, q_t, u_t, s_t)_{0 \leq t \leq T})$ such that

A. $(\Omega, \mathcal{F}^*, \mu, (\mathcal{F}_t, q_t, u_t, s_t)_{0 \leq t \leq T})$ is a Evolving-SEU representation.

B. There exists an EU-function $v$ for lotteries in $\Delta(Z)$ such that for all $t = 0, \ldots, T$ it holds

$$v_t = \mathbb{E}[v|\mathcal{F}_t].$$

As we show in the following subsection dynamic stochastic choice data are enough to distinguish the two special cases Evolving SEU and Gradual Learning even though the two models coincide in the static setting.33

### 3.2 Axiomatic Characterizations

The first axiomatization concerns the most general representation.

#### 3.2.1 Axioms for DR-SEU

Axioms for the general representation in Definition 5 can be classified in two groups. The first group identifies two types of *observationally equivalent* histories. The second group comprises requiring Axiom 0 from the static setting after each history together with a technical axiom of *history continuity*.

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33[Frick, Iijima, Strzalecki ‘17] showed the same insight in a setting of lotteries and without objective payoff-relevant states.
Overcoming limited observability. Similar to [Frick, Iijima, Strzalecki ’17] we characterize histories which are equivalent with respect to the information they reveal through two axioms: Contraction History Independence and Linear History Independence. This allows to overcome the limited observability problem.

Given a history \( h^{t-1} = (A_0, f_0, s_0; \ldots, A_{t-1}, f_{t-1}, s_{t-1}) \) let \( (h^{t-1}_{\lambda B_k}, (A'_{k}, f'_{k}, s'_{k})) \) be the history of the form \( (A_0, f_0, s_0; \ldots, A'_{k}, f'_{k}, s'_{k}; \ldots; A_{t-1}, f_{t-1}, s_{t-1}) \). That is, the history is changed only in period \( k \).

**Definition 8.** We say that \( g^{t-1} \in \mathcal{H}^{t-1} \) is contraction equivalent to \( h^{t-1} \) if for some \( k \) we have \( g^{t-1} = (h^{t-1}_{\lambda B_k}, (B_k, f_k, s_k)) \) where \( A_k \subset B_k \) and \( \rho_k(f_k, A_k, s_k|h^{k-1}) = \rho_k(f_k, B_k, s_k|h^{k-1}). \)

That is, when expanding the set of opportunities at a period \( k \) but otherwise holding the history \( h^{t-1} \) intact, the same stochastic choice results in the period of the expansion.

**Axiom 1: Contraction History Independence (CHI)** For all \( t \leq T \), if \( g^{t-1} \in \mathcal{H}_{t-1}(A_t) \) is contraction equivalent to \( h^{t-1} \in \mathcal{H}_{t-1}(A_t) \) then for all \( s_t \in S_t \)

\[
\rho_t(\cdot, A_t, s_t|h^{t-1}) = \rho_t(\cdot, A_t, s_t|g^{t-1}).
\]

Intuitively, if the distribution of the preferences is stable, two contraction equivalent histories should give the same stochastic choice in the future as well, all else equal. This is because in the Definition 8 above, elements from \( B_k \setminus A_k \) were not attractive to any SEU in the underlying distribution of preferences which has induced either of the histories \( h^{t-1} \) and \( g^{t-1} \), and given the stability of the underlying distribution of preferences the content of private information revealed from the two histories \( h^{t-1} \) and \( g^{t-1} \) is the same. This implies that the continuation stochastic choice should be the same.

The other class of equivalent histories is the following.

**Definition 9.** A finite set of histories \( G^{t-1} \subset \mathcal{H}^{t-1} \) is linearly equivalent to \( h^{t-1} = (A_0, f_0, s_0; \ldots, A_{t-1}, f_{t-1}, s_{t-1}) \) if

\[
G^{t-1} = \{(h^{t-1}_{\lambda A_k} + (1 - \lambda)B_k, \lambda f_k + (1 - \lambda)g_k, s_k) : g_k \in B_k\}.
\]

That is, a history is changed only at a single period by having the revealed choice \( f_k \) from \( A_k \) mixed with all possible choices \( g_k \) from a menu \( B_k \).

One can calculate from the history-dependent aSCF, the probability choices conditional on a set of histories \( G^{t-1} \) by the formula

\[
\rho(f_t, A_t, s_t|G^{t-1}) = \sum_{g^{t-1} \in G^{t-1}} \rho_t(f_t, A_t, s_t|g^{t-1}) \cdot \frac{\rho(g^{t-1})}{\sum_{h^{t-1} \in G^{t-1}} \rho(h^{t-1})}.
\]

**Axiom 2: Linear History Independence (LHI)** For all \( t \leq T \) if \( G^{t-1} \subset \mathcal{H}_{t-1}(A_t) \) is linearly equivalent to \( h^{t-1} \in \mathcal{H}_{t-1}(A_t) \), then \( \rho_t(f_t, A_t, s_t|h^{t-1}) = \rho_t(f_t, A_t, s_t|G^{t-1}). \)

Intuitively, if we have a set of histories \( G^{t-1} \) linearly equivalent to history \( h^{t-1} \) with the mixing happening in period \( k \), because of SEU-properties, \( f_k \) is optimal from \( A_k \) if and only if a mixture of the type \( \lambda f_k + (1 - \lambda)g_k \) with some \( g_k \) is optimal from the mixed menu \( \lambda f_k + (1 - \lambda)B_k \). Therefore, the mixing doesn’t reveal anything new regarding
the private information of the agent and so continuation stochastic choice should be the same.

Now let Axioms 1 and 2 hold for the observable and assume the menu $A_t$ is not possible with positive probability after history $h^{t-1}$. Define
\[ \rho^{h^{-1}}(f_t, A_t, s_t) := \rho_t(f_t, A_t, s_t|\lambda h^{t-1} + (1 - \lambda)d^{t-1}), \]
for some history $d^{t-1} = (g_k, \{g_k\}, s_k)_{0 \leq k \leq t-1}$ which leads to menu $A_t$ with probability one. LHI ensures that the construction is well-defined and coincides with $\rho_t(f_t, A_t, s_t|h^{t-1})$ whenever $A_t \in \mathcal{A}_t(h^{t-1})$. Note here that histories of the type $d^{t-1}$ don’t reveal anything about the private information of the agent. They should be interpreted as tools for the analyst to obtain variation in the data, much needed for identification of the underlying parameters.

**History-Dependent R-SEU and History Continuity.** We model agents who in every period are SEU but have private information about their preferences. Therefore, the data need to satisfy Axiom 0 from the static setting. This is the content of the next Axiom.

**Axiom 3: R-SEU in every period** For all $t \leq T$ and $h^{t-1}$, each of the history-dependent aSCFs $\rho_t(\cdot|h^{t-1})$ satisfies Axiom 0 from the static setting, i.e. it has a R-SEU representation.

The last axiom needed to characterize DR-SEU is a technical form of Continuity. The following definition gives our concept of continuity for histories and is adapted from [Frick, Iijima, Strzalecki ’17].

**Definition 10.** 1) For a sequence of acts $f_n$ say that $f_n$ converges in mixture to the act $f$, written as $f_n \to^m f$, if there exists $h \in \mathbb{F}$ and $\alpha_n \to 0$ with $f_n = \alpha_n h + (1 - \alpha_n)f$.

2) For a sequence of menus $(B^n)_n \subset \mathcal{A}$ say that $B_n$ converges in mixture to the act $f$, written $B^n \to^m f$, if there exists $B \in \mathcal{A}$ and $\alpha_n$ with $B^n = \alpha_n B + (1 - \alpha_n)\{f\}$.

3) For a sequence of menus $(A^n)_n \subset \mathcal{A}$ say that $A_n$ converges in mixture to the menu $A$, written $A^n \to^m A$, if for each $f \in \mathcal{A}$ there is a sequence $(B^n_f)_n \subset \mathcal{A}$ such that $B^n_f \to^m \{f\}$ and $A^n = \bigcup_{f \in \mathcal{A}} B^n_f$.

We next define menus and histories without ties, a concept we also come across later.

**Definition 11.** For any $0 \leq t \leq T$ and $h^{t-1} \in \mathcal{H}_{t-1}$ the set of period $t$-menus without ties conditional on $h^{t-1}$ is denoted by $\mathcal{A}^*_t(h^{t-1})$ and consists of all $A_t \in \mathcal{A}_t$ such that for any $f_t \in A_t$ and any sequences $f^n_t \to^m f_t$, $s_t \in S_t$ and $B^n_t \to^m A_t \setminus \{f_t\}$ we have
\[ \lim_{n} \rho_t(f^n_t, B^n_t \cup \{f^n\}, s_t|h^{t-1}) = \rho_t(f_t, A_t, s_t). \]

For $t = 0$ we write $\mathcal{A}^*_0 := \mathcal{A}^*_0(h^{t-1})$. The set of period $t$ histories without ties is $\mathcal{H}^*_t := \{h^t = (A_0, f_0, s_0; \ldots; A_t, f_t, s_t) \in \mathcal{H}_t : A_k \in \mathcal{A}^*_k(h^{k-1}), \text{ for all } k \leq t\}$.

Intuitively, a menu $A_t$ without ties is so that no matter the SEU of the agent, she never needs to perform tie-breaking. Therefore the menu can be perturbed in any direction and the probabilities of observing the perturbed act $f^n_t$ chosen from the perturbed menu $B^n_t$ converge to the probability of observing $f_t$ chosen from $A_t$. A history without ties is so that every menu occurring in it is without ties.

The technical Continuity axiom reads then as follows.
Axiom 4: History Continuity  For all $t \leq T, A_t, f_t$ and $h^{t-1} \in \mathcal{H}_{t-1}$, 

$$\rho_t(f_t, A_t, s_t|h^{t-1}) \in \text{co}\{\lim_n \rho_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}|h^{t-1,n}) : h^{t,n} \rightarrow \text{m} h^t, h^{t-1,n} \in \mathcal{H}^*_t \}.$$ 

Whenever a history $h^{t-1}$ is perturbed slightly, the change is in choices and decision problems as the objective states $s_k, k \leq t - 1$ come from a finite set. If the perturbation comes from menus without ties so that the agent doesn’t need to perform tie-breaking along the path of the history, the probabilities of observing $f_t$ chosen from $A_t$ as well as $s_t$ realized should change continuously with the history.

Theorem 1. For a dynamic aSCF $\rho$ the Axioms 1-4 are equivalent to the existence of a DR-SEU representation.

If we add Existence of a Best Act and C-Determinism* from subsection 2.3.1 to Axiom 0, we get a characterization of the special case of DR-SEU representation where the agent knows her Bernoulli utility $u_t$ for certain in every period. That is, she is learning only about the objective states.

Proposition 2 (Informational Representation for aSCFs). Assume that a dynamic aSCF $\rho$ has a DR-SEU representation with regular measure $\mu$. Assume that there exists a constant best prize.

Then the following are equivalent after every history $h^t$ observed with positive probability.

A. For all $(q_t, u_t), (p_t, v_t) \in \text{RSSupp}(\bar{\rho}_t(\cdot|h^t))$ $u$ is a positive affine transformation of $v_t$.

B. $\rho_t(\cdot|h^t)$ satisfies C-Determinism*.

3.2.2 Evolving SEU

History-dependent revealed preference. Stochastic choice coupled with the SEU assumption imposes enough structure on data to allow the identification of a history-dependent preference relation $\succeq_{h^t}$ on acts. Intuitively, if the ‘tail’ of the history $h^t$ is $(f_t, A_t, s_t)$, the SEU draw $(q_t, u_t)$ in period $t$ has to rationalize the choice of $f_t$ from $A_t$. For every pair of acts $g_t, r_t$ we can then define $g_t \succeq_{h^t} r_t$ if $g_t$ is weakly better than $r_t$ for every possible draw of SEU from $N(f_t, A_t)$ that happens with positive probability under the respective DR-SEU representation. Note that this implies that $\succeq_{h^t}$ is potentially incomplete. The following definition adds tie-breaking considerations to the intuition we just explained.

Definition 12. For each $t \leq T - 1$ and $h^t = (h^{t-1}, A_t, f_t, s_t) \in \mathcal{H}_t$ we define the relation $\succeq_{h^t}$ on $\mathbb{F}_t$ as follows: For any $g_t, g'_t \in \mathbb{F}_t$ we have $g_t \succeq_{h^t} r_t$ if there exist sequences in $\mathbb{F}_t$ with $g_t^n \rightarrow^m g_t$ and $r_t^n \rightarrow^m r_t$ such that

$$\rho_t\left(\frac{1}{2} f_t + \frac{1}{2} r_t^n, \frac{1}{2} A_t + \frac{1}{2} \{g_t^n, r_t^n\}, s_t \parallel h^{t-1}\right) = 0, \text{ for all } n.$$ 

Finally, let $\sim_{h^t}, \succ_{h^t}$ be the indiscernibility and strict part of $\succeq_{h^t}$.

Because of Axiom 0 in DR-SEU, specifically the no unforeseen contingencies (NUC) assumption, the preference $\succeq_{h^t}$ doesn’t depend on the realization of the period $t$ objective state $s_t$ as long as that state has positive probability under $h^{t-1}$.

We now put the additional axioms characterizing Evolving SEU on $\succeq_{h^t}$. 

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Axiom 4: Separability. For all $t \leq T - 1, g^t, r^t \in \mathbb{F}_t$ we have $g_t \sim_{h^t} r_t$ whenever $g^t(s_t) =^d r^t(s_t)$ and $g^t_z(s_t) =^d r^t_z(s_t)$ for all $s_t \in S_t$.

This says that whenever the marginal distributions over the current prize lottery and continuation menu of two acts after a history $h^t$ are the same then the two acts are indifferent under the revealed preference after the history. It ensures that Bernoulli utility $u_t$ has the form

$$u_t(z_t, A_{t+1}) = v_t(z_t) + \delta V_t(A_{t+1}). \tag{5}$$

Axiom 4 allows the definition of a history-dependent menu preference over continuation menus.

Definition 13. Fix a $z_t \in Z$. Take a $h^t \in H_t$ and define an ex-post menu preference $\succeq_{h^t}$ over $A_{t+1}$ by

$$A_{t+1} \succeq_{h^t} B_{t+1}, \text{ if } \delta(z_t, A_{t+1}) \succeq_{h^t} \delta(z_t, B_{t+1}).$$

We now add other menu preference axioms to shape the menu preference $V$ from (5) into the form needed for (3). The next three Axioms are standard.

Axiom 5: Monotonicity. Whenever $A_{t+1} \subseteq B_{t+1}$ it holds $B_{t+1} \succeq_{h^t} A_{t+1}$.

Axiom 6: Indifference to Timing. For any $A_{t+1}, B_{t+1}$ and $\alpha \in (0, 1)$ we have

$$\alpha A_{t+1} + (1 - \alpha) B_{t+1} \sim_{h^t} \alpha A_{t+1} + (1 - \alpha) B_{t+1}.$$

Axiom 7: Menu Non-Degeneracy. There exists $A_{t+1}, B_{t+1}$ such that $\delta(z_t, B_{t+1}) \succeq_{h^t} \delta(z_t, A_{t+1})$ for all $z_t$.

Before stating the next axiom, we introduce an operation on menus which produces for every menu a constant menu containing all the lotteries in its acts. Formally, in a setting with AA-acts from $\mathbb{F}$ for a menu $A \subseteq \mathbb{F}$ define the menu of constant acts from $\tilde{A}$ as follows.

$$\tilde{A} = \{ g \in \mathbb{F} : g \text{ constant act with } g(s) = f(s') \text{ for some } f \in A, s, s' \in S \}.$$

The following axiom ensures that the menu preference $\succeq_{h^t}$ of Definition 13 can be represented by Expected Utility preferences with stochastic but state-independent Bernoulli utilities.

Axiom 8: Weak Dominance. For any $A_{t+1} \in \mathbb{A}_{t+1}$ it holds $\tilde{A}_{t+1} \succeq_{h^t} A_{t+1}$.

Intuitively, from the perspective of the end of period $t$ and compared to the menu $A_{t+1}$, the menu $\tilde{A}_{t+1}$ offers insurance w.r.t. the stochasticity of both beliefs and tastes as ex-post in $t + 1$ the agent can choose her best lottery from any act in $A_{t+1}$ whereas in $A_{t+1}$ which lottery the agent ultimately faces depends on the realization of the objective state $s_{t+1}$.

Menu Finiteness (technical). Next we define what it means for a menu preference to be finite. This is a technical property we need for tractability.

Definition 14. For $\succeq$ a menu preference over some set of prizes $X$ say that it satisfies Finiteness if there exists $K \in \mathbb{N}$ such that for menu $A$ there exists $B \subseteq A$ with $|B| \leq K$ and so that $B \sim A$. 

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Axiom 9: Finiteness of Menu preference  For all $h^t \in \mathcal{H}_t$, the menu preference on $\mathcal{A}_{t+1}$ derived from $\succeq_{h^t}$ satisfies Finiteness as in Definition 14.

Finally, we add the sophistication axiom which ensures that the agent correctly predicts her future beliefs and tastes. Intuitively, if enlarging the menu $\mathcal{A}_{t+1}$ to $\mathcal{B}_{t+1}$ is valuable for the agent just after the realization of history $h^t$ and her beliefs about the future evolution of her preferences are correct, this is because there are possible draws of SEUs in period $t+1$ for which elements in $\mathcal{B}_{t+1} \setminus \mathcal{A}_{t+1}$ are optimal. This should be then reflected in the $h^t$-dependent stochastic choice from $\mathcal{B}_{t+1}$.

Axiom 10: Sophistication  For all $t \leq T-1$, $h^t \in \mathcal{H}_t$ and $\mathcal{A}_{t+1} \subset \mathcal{B}_{t+1} \in \mathcal{A}^*_{t+1}(h^t)$, the following are equivalent

A. $\rho_{t+1}(f_{t+1}, B_{t+1}, s_{t+1}|h^t) > 0$ for some $f_{t+1} \in B_{t+1} \setminus A_{t+1}$ and some $s_{t+1} \in S_{t+1}$.

B. $B_{t+1} \succ_h A_{t+1}$.

Theorem 2. For a dynamic aSCF $\rho$ satisfying a DR-SEU representation the Axioms 4-10 are equivalent to the existence of an Evolving SEU representation.

Next, we note down a special cases of the Evolving SEU representation which can be used to model data from a population of agents with deterministic but heterogeneous tastes who are learning about payoff-relevant objective states. Thus, uncertainty about taste resolves in the first period, i.e. after an agent from the population is ‘drawn’, but there is persistent uncertainty about payoff-relevant objective states.

Example 3: Stochastic taste only in period zero.  If we replace Axiom 8 with the following Strong Dominance axiom\textsuperscript{34} then we get a version of Evolving SEU, where tastes are stochastic only in $t = 0$ and the profile of future tastes is completely determined after every period-0 history.

Axiom 8’: Strong Dominance  For all $0 \leq t \leq T-1$ and $h^t \in \mathcal{H}_t$ we have:

If $f_{t+1} \in A_{t+1}$ and $\{f_{t+1}(s_{t+1})\} \succeq_{h^t} \{g_{t+1}(s_{t+1})\}$ for all $s_{t+1} \in S_{t+1}$ then $A_{t+1} \sim_{h^t} A_{t+1} \cup \{g_{t+1}\}$.

Intuitively, if the Bernoulli utility is deterministic and if an act is better than another uniformly across all states, adding the dominated act to a menu which contains the dominating act doesn’t make the menu more valuable.

Proposition 3. For a dynamic aSCF $\rho$ satisfying a DR-SEU representation the Axioms 4-7,8’,9 and 10 are equivalent to the existence of an Evolving SEU representation where stochasticity of tastes is resolved at the end of period 0.

Finally, we note a special case of DR-SEU involving a sophisticated agent but which doesn’t have an Evolving SEU representation.

\textsuperscript{34}This is what [Dillenberger et al '14] call Dominance in their main theorem.
Example 4. [Epstein ’06] and [Epstein et al ’08] consider a sophisticated agent who experiences temptation in beliefs and therefore updates her beliefs about objective states in a subjective way not necessarily conforming to Bayesian updating with respect to the true data-generating process. The ex-post choice versions of these models are special cases of DR-SEU and satisfy C-Determinism*, but they violate Axiom 5 (Monotonicity), which is necessary for an Evolving SEU representation.\textsuperscript{35}

3.2.3 Gradual Learning

Gradual Learning imposes additional restrictions on the evolution of Bernoulli utilities of an Evolving SEU representation: the agent is learning about a fixed taste.

To explain the three additional Axioms which lead to the Gradual Learning representation we introduce some notation.

For some \( t \leq T - 1 \) and given a sequence \( l_t, \ldots, l_T \in \Delta(Z) \) of consumption lotteries, let the stream of lotteries \( (l_t, \ldots, l_T) \in \Delta(X_t) \subset F_t \) be the period-\( t \) lottery that at every period \( \tau \geq t \) yields consumption according to \( l_\tau \). Formally, for any consumption lottery \( l \in \Delta(Z) \) and menu of constant acts \( A_{t+1} \in A_{t+1}^c \) define \( (l, A_{t+1}) \in \Delta(X_{t+1}) \) to be the lottery which has stochastic consumption now and fixed continuation with probability one.\textsuperscript{36}

Then \( (l_t, \ldots, l_T) = (l_t, A_{t+1}) \in \Delta(X_t) \) is defined recursively from period \( T \) backwards by \( A_T = \{l_T\} \in A_T \) and \( A_s = \{(l_s, A_{s+1})\} \in A_s \) for all \( s = t + 1, \ldots, T - 1 \). We write \( (l_t, \ldots, l_\tau, m, \ldots, m) \) if \( l_{t+1} = \cdots = l_T \) for some \( m \in \Delta(Z) \) and \( \tau \geq t \).

Axiom 11: Stationary Preference over Lotteries [FIS]. For all \( t \leq T - 1, l, m, n \in \Delta(Z) \) and \( h^t \) we have

\[
(l, n, \ldots, n) \succ_{h^t} (m, n, \ldots, n) \text{ if and only if } (n, l, \ldots, n) \succ_{h^t} (n, m, n, \ldots, n).
\]

Intuitively, if and only if the felicity \( v_t \) today is just the average of the future felicity \( v_{t+1} \) tomorrow, it holds true from today’s perspective that postponing the choice between two lotteries by a period results in the same ranking as for the case that the choice is made immediately.

For the second axiom, just as in [Frick, Iijima, Strzalecki ’17] for lotteries \( l, m \in \Delta(Z) \), we say they are \( h^t \)-non-indifferent if \( (l, n, \ldots, n) \not\sim_{h^t} (m, n, \ldots, n) \) for some \( n \in \Delta(Z) \).

Moreover, to avoid tautologies we require a non-degeneracy condition.

Condition 1: Consumption Non-degeneracy For all \( t \leq T - 1 \) and \( h^t \), there exists \( h^t \)-non-indifferent \( l, m \in \Delta(Z) \).

Axiom 12: Constant Intertemporal Trade-off [FIS]. For all \( t, \tau \leq T - 1 \), if \( l, m \) are \( h^t \)-non-indifferent and \( \hat{l}, \hat{m} \) are \( g^\tau \)-non-indifferent, then for all \( \alpha \in [0, 1] \) and \( n \in \Delta(Z) \):

\[
(l, m, n, \ldots, n) \sim_{h^t} (\alpha l + (1 - \alpha)m, \alpha l + (1 - \alpha)m, n, \ldots, n)
\]

\[
\iff (\hat{l}, \hat{m}, n, \ldots, n) \sim_{g^\tau} (\alpha \hat{l} + (1 - \alpha)\hat{m}, \alpha \hat{l} + (1 - \alpha)\hat{m}, n, \ldots, n).
\]

\textsuperscript{35}The model in [Epstein et al ’08] features infinite horizon so the statement above holds for its finite horizon version.

\textsuperscript{36}This is similar to the definition in section 4.3 of [Frick, Iijima, Strzalecki ’17].
This ensures that the discounting factor \( \delta \) from the Evolving SEU representation is unique.

Finally, we note down the classical axiom which gives \( \delta < 1 \).

**Axiom 13: Impatience [FIS].** For all \( t \leq T - 1, h_t \) and \( l, m, n \in \Delta(Z) \), if \((l, n, \ldots, n) \succ_{h_t} (m, n, \ldots, n) \), then \((l, m, n, \ldots, n) \succ_{h_t} (m, l, n, \ldots, n) \).

The characterization result for Gradual Learning is then as follows.

**Theorem 3.** Assume the aSCF \( \rho \) satisfies an Evolving SEU model and assume Condition 1 is satisfied. Then Axioms 11-13 are equivalent to the existence of a Gradual Learning representation for \( \rho \).

### 3.3 Uniqueness

The following Proposition proved in Section 4 of the online appendix shows that all three representations are unique up to positive affine transformations of the Bernoulli utilities the agent uses to evaluate lotteries over the respective consequence spaces \( X_t \) as well as up to relabeling of the states of the world \( \omega \) and of the objective states \( s_t \).

The characterization of uniqueness is a prerequisite for the comparative static exercises of Section 4. The results mirror closely the identification in [Frick, Iijima, Strzalecki '17] adapted to our more general setting with agents who hold (possibly incorrect) beliefs about payoff-relevant states.

**Proposition 4.** 1) Suppose that a dynamic aSCF \( \rho \) admits two DR-SEU representations \((\Omega, \mathcal{F}_t, \mu, (\mathcal{F}_t, (q_t, u_t), s_t, (\hat{q}_t, \hat{u}_t))_{0 \leq t \leq T})\) and \((\Omega', \mathcal{F}'_t, \mu', (\mathcal{F}'_t, (q'_t, u'_t), s'_t, (\hat{q}'_t, \hat{u}'_t))_{0 \leq t \leq T})\).

Then there exists a bijection \( \phi_t : \mathcal{F}_t \rightarrow \mathcal{F}'_t \) and \( \mathcal{F}_t \)-measurable functions \( \alpha_t : \Omega \rightarrow \mathbb{R}^+ \) and \( \beta_t : \Omega \rightarrow \mathbb{R} \) such that for all \( \omega \in \Omega \):

1. \( \mu(\mathcal{F}_0(\omega)) = \mu'(\phi_0(\mathcal{F}_0(\omega))) \) and \( \mu(\mathcal{F}_t(\omega)|\mathcal{F}_{t-1}(\omega)) = \mu'(\phi_t(\mathcal{F}_t(\omega))|\phi_{t-1}(\mathcal{F}_0(\omega))) \) if \( t \geq 1 \);
2. \( q'_t \equiv q_t \) for all \( t \geq 1 \), \( u_t(\omega) = \alpha_t(\omega)u'_t(\omega') + \beta_t(\omega) \) whenever \( \omega' \in \phi_t(\mathcal{F}_t(\omega)) \);
3. \( \mu((\hat{q}_t, \hat{u}_t) \in B_t(\omega)|\mathcal{F}_t(\omega)) = \mu'(((\hat{q}'_t, \hat{u}'_t) \in \phi_t(B_t(\omega))|\mathcal{F}'_t(\phi_t(\mathcal{F}_t(\omega)))) \) for any \( B_t(\omega) = \{(p_t, v_t) \in \Delta(S_t) \times \mathbb{R}^{X_t} : f_t \in M(M(A_t, (q_t(\omega), q_t(\omega)), p_t, v_t)) \} \) for some \( f_t \in A_t, A_t \in A_t \).

2) If \( \rho \) admits two Evolving-SEU representations then in addition to (i)-(iii) above we have

1. \( \alpha_t(\omega) = \alpha_0(\omega) \left( \frac{1}{\delta} \right)^t \), for all \( \omega \in \Omega \) and \( t \geq 0 \);
2. \( v_t(\omega) = \alpha_t(\omega)v'_t(\omega') + \gamma_t(\omega) \) whenever \( \omega' \in \phi_t(\mathcal{F}_t(\omega)) \), where \( \gamma_t(\omega) = \beta_t(\omega) \) and \( \gamma_t(\omega) = \beta_t(\omega) - \delta E[\beta_{t+1}|\mathcal{F}_t(\omega)] \) if \( t \leq T - 1 \).

3) If \( \rho \) has two Gradual Learning Representations and satisfies Condition 1, then in addition to (i)-(v) the following holds

1. \( \delta = \delta' \)
\[(vii) \beta_t(\omega) = \frac{1-\delta^{T-t+1}}{1-\delta} \mathbb{E}[\beta_T | \mathcal{F}_t(\omega)].\]

1) shows that agent’s choices uniquely identify the evolution of her private information in both relevant dimensions: tastes and beliefs. The lack of identification for the Bernoulli utility functions \(u_t\) is unavoidable. Intuitively, when one rescales the Bernoulli utilities by a factor which depends only on information up to time \(t\), the sets of maximal elements \(M(A_t; q_t, u_t)\) don’t change.

2) shows that the Evolving SEU model allows for stronger identification of the Bernoulli utilities. The scaling factor of Bernoulli utilities needs to be measurable with respect to the information available at \(t = 0\). This is because in the Evolving SEU model the utility of the continuation problem enters cardinally into the overall utility of choosing an act from a menu. One can then use the same information, namely that available in period \(t = 0\), to build a measuring rod with which utilities can be compared across periods. Obviously, the scaling factor \(\alpha_t\) still depends on the state of the world \(\omega\). In a population interpretation of the observable aSCF this means that different agents may use different information available at \(t = 0\) to compare utils intertemporally.

3) shows that the Gradual Learning model improves on the identification properties of the Evolving SEU model because the discount factor is identified uniquely. This is a consequence of the Constant Intertemporal Trade-Off Axiom. Under that Axiom any possible scaling of the Bernoulli utilities in addition to depending on time \(t = 0\) information only, has to be constant over time.

## 4 Comparative Statics Results

This section offers simple comparative statics results under varying assumptions about the representations of the observable aSCF. The characterizations are simple because aSCFs represent very rich data sources.

### 4.1 A measure of belief biasedness

If the analyst doesn’t observe anything about the realization of objective states, it is impossible to discuss correctness of beliefs of the agents. Most of the canonical models of behavior based only on menu choice as an observable, as in [Dillenberger et al '14] and [Krishna, Sadowski '14] and many others, as well as models of stochastic choice without observable objective states as in [Lu '16] cannot address questions of belief biasedness. In this part we illustrate what is possible if the observable of the analyst consists of aSCFs.

For simplicity we assume there are best and worst prizes which coincide for all agents considered: that is, constant acts \(f, \bar{f}\) such that for every aSCF \(\rho\) considered it holds:

\[
\text{for every } f \neq f \text{ we have } \bar{\rho}(f, \{f, f\}) = 1 \text{ and for every } f \neq \bar{f} \text{ we have } \bar{\rho}(f, \{\bar{f}, f\}) = 0.
\]

Moreover, for simplicity we assume the agents have the same non-stochastic taste \(u\) and focus on comparative statics related to beliefs.\(^{37}\)

\(^{37}\)Formally speaking all aSCF/SCF-s in this subsection satisfy C-determinism* – choice is stochastic because beliefs of an agent are stochastic, besides possible randomness coming from tie-breaking. In this setting all the machinery of [Lu '16], esp. the related test acts can be used (see online appendix). The conditions on the SCFs which imply that the taste of distinct agents are the same are available upon request.
We assume there is an underlying state of the world $\omega$ coming from a finite set $\Omega$. For example in Example 2 $\omega$ may encode gender or ethnicity. An analyst observes two agents $i = 1, 2$ who are interested in the realization of an objective payoff-relevant state $s \in S$. A state of the world $\omega$ goes hand in hand with a set of beliefs about the possible realizations of $s$ for each agent and a true data-generating-process (DGP). The analyst observes the aSCFs of the agents which are assumed to have the following form.

$$\rho_i(f, A, s) = \sum_{\omega \in \Omega} \mu(\omega, s) \tau_{q_i(\omega), u}(f, A), \quad i = 1, 2.$$  \hfill (6)

Here $\mu \in \Delta(\omega \times S)$ and the tie-breakers $\tau_{q_i(\omega), u}$ depend only on the realized SEU $(q_i(\omega), u)$ of agent $i$.

We assume $\mu$ is either known by the analyst (e.g. an experiment in a lab) or the analyst gleans it from the data $\rho_i$ using Theorem 0.

Now assume the analyst fixes a direction $q(\omega) \in \Delta(S)$ for possible biases for every $\omega \in \Omega$ and is interested in finding out how biased, if at all, the beliefs of the agents are in the direction $\{q(\omega)\}_{\omega \in \Omega}$. The analyst might think that a possible bias for $\omega$ corresponds to some ‘extreme’ $q(\omega) \neq \mu(\cdot|\omega)$.

A natural way in terms of the aSCF to say that an agent is biased in the direction $\{q(\omega)\}_{\omega \in \Omega}$ and that, say, agent 1 has uniformly less biased beliefs than agent 2 is to require the following in terms of the representation.

**Definition 15.** 1) Agent $i$’s beliefs are biased toward the direction $q := \{q(\omega)\}_{\omega \in \Omega}$ if and only if there exists a vector of weights $\{a(\omega)\}_{\omega \in \Omega} \in [0, 1]^\Omega$ such that the following holds

$$q_i(\omega) = a_i(\omega) q(\omega) + (1 - a_i(\omega)) \mu(\cdot|\omega) \textrm{ for some } a(\omega) \in [0, 1].$$

2) Agent 1’s beliefs are uniformly less biased toward $q$ than agent 2’s beliefs if and only if it holds for every $\omega \in \Omega$ that $0 \leq a_1(\omega) \leq a_2(\omega) \leq 1$.

Figure 3 helps describe the definition.

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38For example, if $\Omega$ encodes gender and the true DGP is that $\mu(\cdot|\omega)$ is independent of $\omega$, a possible extreme bias might be to assume that for $\omega = \text{male}$, $q(\omega)$ is ‘tilted’ towards more favorable realizations of the objective state $s$ whereas for $\omega = \text{female}$, $q(\omega)$ is ‘tilted’ towards more unfavorable realizations of the objective state $s$. As Example 1 illustrates, this might be the case with employment data depending on the vocation and job properties.
The associated menu preference approach from [Lu ‘16] provides a way to identify the weights of the bias in some direction $q$.

**Definition 16 ([Lu ‘16]).** Given $\bar{\rho}$, let the associated menu preference $\succeq_{\bar{\rho}}$ be given by the utility function on menus $V_{\bar{\rho}} : A \rightarrow [0, 1]$ with

$$V_{\bar{\rho}}(A) = \int_0^1 \bar{\rho}(A, A \cup \{\alpha f + (1 - \alpha) \bar{f}\}) da.$$ 

For a fixed weight in $\alpha \in [0, 1]$ the value $\bar{\rho}(A, A \cup \{\alpha f + (1 - \alpha) \bar{f}\})$ gives the probability that an element of $A$ beats the act $\alpha f + (1 - \alpha) \bar{f}$, that is, the probability that the agent prefers items out of the menu $A$ instead of the test act with weight $\alpha$ on the worst prize. Intuitively speaking, a menu is more valuable in the associated menu preference of a SCF if in the aggregate its elements are more preferred than test acts $\alpha f + (1 - \alpha) \bar{f}$. [Lu ‘16] shows that, up to tie-breaking considerations, every stochastic choice function as $\bar{\rho}$ can be characterized through its associated menu preference $V_{\bar{\rho}}$. Thus, except for tie-breaking, $\bar{\rho}$ contains no more information about the agent than $V_{\bar{\rho}}$ does.

Given the direction of bias $q$ define for every weight of biases $a : \Omega \rightarrow [0, 1]$ the associated menu preference where the agent gives weight $a(\omega)$ to the belief $q(\omega)$ whenever the state of the world $\Omega$ is realized.

$$V_a(A) = \int_{\Omega} \max_{f \in A} [a(\omega)q(\omega) + (1 - a(\omega))\mu(\cdot | \omega)] \cdot (u \circ f)\mu(d\omega).$$

This gives a map $\psi_q : [0, 1]^\Omega \rightarrow \{\text{menu preferences}\}$.\(^{39}\) Intuitively, one can interpret any element $a \in [0, 1]^\Omega$ as a vector of degrees of biasedness towards $q$.

Note that the construction of the map $\psi_q$ comes directly from the data: the aSCF-s $\rho_i, i = 1, 2$ give $\mu(\cdot | \omega)$ (or the analyst knows this already) and the analyst picks the bias vector $q$. One can show that once a bias direction $q$ is fixed, every weight vector $a$ defines a unique menu preference $V_a$.

This allows the following characterization of the degree of belief-biasedness in direction $q$ in terms of observables/data. Here, recall that the induced menu preference from the stochastic choice function $\bar{\rho}$ is also completely constructed from stochastic choice data.

**Proposition 5.** Assume that the two aSCF $\rho_i, i = 1, 2$ are as in (6) and consider a vector of biases $q \in \Delta(S)^\Omega$. It holds:

A. Agent $i$’s beliefs are uniformly biased toward the direction $q$ with degree $a \in [0, 1]^\Omega$ if and only if $\psi_q^{-1}(V_{\rho_i}) = a$, i.e. if and only if $a$ is the image under $\psi_q$ of the menu preference induced from stochastic choice.

B. Agent 1’s beliefs are uniformly less biased toward the direction $q$ than agent 2’s beliefs if and only if $\psi_q^{-1}(V_{\rho_1}) \leq \psi_q^{-1}(V_{\rho_2})$.

Note that by varying $q$, an analyst can use the induced menu preference of $\bar{\rho}_i$ (from Definition 16) to identify the actual bias direction of an agent whenever her aSCF doesn’t satisfy the Axiom of Correct Interim Beliefs from Definition 2.

\(^{39}\)The image of this map can naturally be identified with value functions of menu preferences.
**Example 1 continued.** In the context of Example 1 from the Introduction, subsection 1.1.1 this Proposition states that stochastic choice data are enough for the analyst to identify the incorrect beliefs $\hat{q}_i$, $i = 1, 2$. Namely, assume directions for the biases $q(s'_0) = (1, 0)$ and $q(s''_0) = (0, 1)$. These correspond to the ‘extreme’ beliefs that a candidate with $s_0 = s'_0$ will always deliver outcome $s_1 = g$ and a candidate $s_0 = s''_0$ will always deliver outcome $s_1 = b$. The Proposition delivers then $a(s'_0) = 2\hat{q}_1 - 1$ and $a(s''_0) = 1 - 2\hat{q}_2$ so that whenever $a : S_0 \to [0, 1]$ is identified from data the analyst can recover the incorrect beliefs $\hat{q}_i$, $i = 1, 2$.

An alternative to the vector of weights $a \in [0, 1]^\Omega$ on biases is to require instead a uniform weight $a \in [0, 1]$ on biases which is independent of the realization of the characteristic $\omega$. The conditions on the induced menu preferences identifying the bias $a$ are then simpler than in Proposition 15. Nevertheless, in applications, the bias weights will usually differ according to the realization of the characteristic $\omega$. For example, one might expect in some cases the agent to use the correct conditional DGP $\mu(\cdot|\omega)$ and in other cases of realized $\omega$-s to use a very biased belief much closer to an ‘extreme’ $q(\omega) \neq \mu(\cdot|\omega)$. Therefore, here we have focused on the concept of Definition 15 which allows for this additional flexibility.

### 4.2 The speed of learning about taste

In this subsection we consider agents in a dynamic setting ($T \geq 1$) whose stochastic choice data satisfy the Gradual Learning model and discuss measures across agents of the speed of learning about taste. We assume for all agents considered in this subsection that at time $t = 0$ their taste is not deterministic. Formally, we require the following conditions on any aSCF of this section.

**Assumptions** For all aSCFs in this subsection it holds true:

A. $\rho$ satisfies a Gradual Learning (GL) representation with $T \geq 1$ and sequence of felicities $v_t$, $t \in \{0, \ldots, T\}$.

B. $\bar{\rho}_0$ doesn’t satisfy C-Determinism*.

B. ascertains that there is non-trivial learning about taste for an agent. On the other hand, due to Sophistication (assumed as part of A.), if an agent learns her future taste at the end of a period $t$, her taste remains deterministic in all future periods.

Recall that the preferences $\succeq_{h^t}$ on continuation menus $A_{t+1}$ for some history $h^t \in \mathcal{H}_t$ from Definition 13 are derived solely from stochastic choice data. If for an agent her uncertainty about future taste is resolved after a history $h^t$ the derived menu preference on $A_{t+1}$ derived from $\succeq_{h^t}$ will satisfy Strong Dominance. On the other hand, Strong Dominance will be violated for $\succeq_{h^t}$ whenever an agent’s uncertainty about future taste doesn’t get resolved after history $h^t$. The same holds if instead of looking at whether Strong Dominance is satisfied we look at whether C-Determinism* is satisfied.

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40Defining the menu preference of an unbiased agent (a counterfactual) and of a fully biased agent, the condition of biasedness is that the induced menu preference of the agent is a convex combination of the menu preferences of the unbiased and fully biased agent and that $a$ corresponds to the weight on the biased agent.
This suggests a simple way to define the speed of learning about taste of an agent who satisfies a Gradual Learning model as well as an equally simple way to rank such agents according to their speed of learning about taste.

**Definition 17.** 1) Say that an agent learns her future taste after history $h^t$ if her derived menu preference on $A_{t+1}$ from $\succeq_{h^t}$ satisfies Strong Dominance or equivalently, if $\rho_{t+1}(|h^t)$ satisfies C-Determinism.\(^{41}\)

2) Say that an agent becomes certain of her future taste at time $t$ if she learns her future taste after every history $h^t \in \mathcal{H}_t$.

3) Say that agent 1 learns her taste faster than agent 2 if the following implication holds true for every $t \leq T - 1$:

agent 2 becomes certain of her taste at $t \implies$ agent 1 becomes certain of her taste at $t$.

The characterization of these concepts in terms of the GL representation (Definition 7) is as follows.

**Proposition 6.** 1) Suppose an agent has a GL representation with probability space $(\Omega, \mathcal{F}^*, \mu)$. Then an agent learns her future taste after history $h^t$ if and only if conditional on $C(h^t)$ her felicity is deterministic, i.e.

\[ v_{t+1} \text{ is a constant function on } C(h^t). \]

2) Suppose an agent has a GL representation with underlying probability space $(\Omega, \mathcal{F}^*, \mu)$. An agent becomes certain of her future taste at time $t$ if and only if her felicity at time $t$ is independent of the state of the world $\omega$, i.e.

\[ v_{t+1} \text{ is a constant function on all of } \Omega. \]

3) Suppose two agents $i = 1, 2$ have GL representations with underlying probability space $(\Omega, \mathcal{F}^*, \mu)$ but otherwise may have different filtrations $\{\mathcal{F}^i_t\}_{t \leq T}$ and different evolution of SEUs $\{(q^i_t, w^i_t)\}_{t \leq T}$ for $i = 1, 2$. Then agent 1 learns her taste faster than agent 2 if and only if the following implication holds true for every $t \leq T - 1$:

\[ v^2_{t+1} \text{ is a constant function on all of } \Omega \implies v^1_{t+1} \text{ is a constant function on all of } \Omega. \]

**Example 4.** Assume that we have two investors $i = 1, 2$ facing the same market conditions whose CARA Bernoulli utility over monetary outcomes has the form $x \mapsto 1 - e^{-\gamma_i x}$ where $\gamma_i$ is random according to a discrete distribution taking positive values from a finite set $\Gamma \subset [1, +\infty)$. In every period each investor decides whether to invest in a risky project $f$, whose outcome is strongly dependent on market conditions (objective state $s_t \in \mathbb{R}_+$ drawn anew each period) through $f(s_t) \sim \sqrt{s_t} \times \text{Uniform}\{-1, 1\} + s_t$ or to pick investments $h(\alpha)$ whose $s_t$-independent outcome satisfies $h(\alpha) \sim \sqrt{\alpha} \times \text{Uniform}\{-1, 1\} + \alpha$. Then according to the above Proposition an analyst has two ways of telling who of the two investors has learned her parameter $\gamma_i$ the earliest. If she only has data on choices from menus containing only acts of the type $h(\alpha)$ she finds the first time when the choice of each investor on such menus becomes deterministic. If she only has data of choice among menus, an indicator that investor 1 learned her preference parameter earlier is that she starts preferring menus where $f$ is present to menus where $f$ isn’t present earlier in time than investor 2 does.

\(^{41}\)Equivalence holds under the assumption that the data satisfy the GL representation.
5 Conclusion

We have introduced a dynamic stochastic choice model general enough to encompass situations where a subjective expected utility agent has both stochastic taste as well as stochastic beliefs about the realization of objective payoff-relevant states. Under the assumption that the analyst has access to data which reveal the agent’s history-dependent choices as well as the sequence of realizations of objective states we have characterized axiomatically the case when the analyst can uncover the otherwise arbitrary evolution of the private information of the agent.

The assumed richness of the data allows the analyst to test whether the agent is using correctly specified beliefs about objective states conditional on her private information and if not, to determine the bias of the agent as well as to compare different agents according to their biasedness of beliefs. We have also characterized special cases of the general representation, Evolving SEU and Gradual Learning, which would have been otherwise indistinguishable in the static setting. Finally, in the case of Gradual Learning, we have shown how an analyst is able to detect from data that the agent has stopped learning about her taste and that therefore the randomness in choice only comes from randomness in beliefs.

Information acquisition is outside the scope of this model and constitutes the natural next step in research. E.g. we shouldn’t expect the student in Example 2 not to try and actively learn early about her final job market outcome. So it natural to expect Indifference to Timing to be violated; if an agent tries to actively learn about future tastes by spending resources after history $h^t$ we should expect her to satisfy instead the weaker condition:

$$\text{if } A_{t+1} \sim_{h_{t+1}} B_{t+1} \text{ then } \alpha A_{t+1} + (1 - \alpha) B_{t+1} \preceq_{h^t} A_{t+1}.$$ 

That is, since contingent planning costs utility, the agent is averse to it whenever she is ex-ante indifferent between two decision problems. Introducing information acquisition in this framework would also allow a better study of misspecified learning.

Other directions to pursue are as follows. We haven’t considered consumption dependence as [Frick, Iijima, Strzalecki ‘17] do in their DREU model of stochastic taste only. Developing ‘systems’ of DR-SEUs coming from agents in strategic situations is also left for future research, as is characterizing meaningful relaxations of the Sophistication assumption in the Evolving SEU model.

Finally, on another perspective, this paper is about identification and not inference. In applications data sets are naturally finite. We leave for future research characterizations of stochastic dynamic behavior when data sets are finite.

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Appendices

The Appendix is organized as follows. Appendix A is devoted to the proof of Theorem 0. Appendix B describes the Ahn-Sarver representations in the dynamic setting. These are more convenient for proofs and their equivalence to the Filtration-based representations from the main text of the paper is proved in the online appendix. Appendix C proves the existence of so-called separating histories. These are an essential tool in the proof of the main characterization theorems. Most of Appendix D is devoted to the proof of Theorem 1, the rest of it to the proofs of Section 4. The proof of the rest of the characterization theorems is in the online appendix. Besides the rest of the auxiliary results, the latter also contains most of the technical work needed to extend the menu choice literature to the setting of SEUs, add explicit tie-breaking to [Lu ’16] and beliefs about objective payoff-relevant states to [Ahn, Sarver ’13].

A  Random Subjective Expected Utility with observable objective states (AS-version)

A.1 Separation property for acts - static setting

We prove a separation property for menus of acts, similar to Lemma 1 in [Ahn, Sarver ’13] (separation property for lotteries).

We start with a trivial remark which will be used extensively in the following.

Remark 2. 1) A SEU preference encoded by $(q,u)$ is constant (i.e. consists of only indifferences) if and only if $u$ is constant.

2) Two SEU representations $(q,u)$ and $(q',u')$ represent the same SEU preference if and only if $q = q'$ and $u \approx u'$.

The separation property for acts is as follows.

Lemma 1 (Separation property in the AA setting). Let $Z'$ be any set of prizes (possibly infinite) and let $(q_k, u_k) : k = 1, \ldots, K \subset \Delta(S) \times \mathbb{R}^{Z'}$ be a set of pairwise distinct SEU representations s.t. $u_k$ is non-constant for all $k = 1, \ldots, K$. Then there is a collection of acts $\{f_k : k = 1, \ldots, K\} \subset \mathbb{F}$ s.t. $q_k \cdot u_k(f_k) > q_k \cdot u_l(f_l)$ for any distinct $l, k \in \{1, \ldots, K\}$.

Proof. We divide the proof in three steps.

Step 1. Assume first that $u_k \not\approx u_l$ for all $l \neq k$ and that $Z'$ is finite. Then we are in the setting of Lemma 13 from [Frick, Iijima, Strzalecki ’17] and can use a menu of constant acts to realize the separation property required.

Step 2. Assume now that $u_k \approx u_l$ for all $l \neq k$ and that $Z'$ is finite. W.l.o.g. we can assume that $u_k = u_l = u$ and that $\text{im}(u) = [0, 1]$. Note that in this case it also holds $q_k \neq q_l$ for all $l \neq k$. It is enough in this case to solve the following problem:

\[(P) \quad \text{For all } k \text{ find } p_k \in \Delta(S) \text{ s.t. } q_k \cdot p_k > q_k \cdot p_l, \quad l \neq k.\]

Now we are again in the setting of Lemma 13 in [Frick, Iijima, Strzalecki ’17], if we take as Bernoulli utilities the $q_k$-s. Formally, it follows $q_k \not\approx q_l$ whenever $S$ has more than one element as one can check using uniqueness result in the classical vNM Theorem.
Thus, Lemma 13 in [Frick, Iijima, Strzalecki '17] gives probability distributions $p_k, k = 1, \ldots, K$ satisfying (P). Now, we can easily construct the acts needed by the formula $u(f_k(s)) = p_k(s), s \in S, k = 1, \ldots, K$. Note that this trick works because $\Delta(S) \subseteq [0, 1]^S$.

**Step 3.** Assume now that we are in the general case $(q_k, u_k) \neq (q_l, u_l), l \neq k$. There exists a finite $Z \subset Z'$ s.t. all $u_k$ are non-constant in $\Delta(Z)$. We are going to choose acts $f : S \rightarrow \Delta(Z)$. Assume w.l.o.g. that for all $k$ we have $\text{im}(u_k) \subseteq [0, 1]$. Divide the Bernoulli utilities $u_k$ in classes $r = 1, \ldots, R \leq K$ s.t. if $l, k$ are so that $u_k \approx u_l$ they belong to the same class. Within the same class, normalize the Bernoulli utilities to be equal. Thus, we can rewrite the SEU preferences given as

$$\{(q_{rl}, u_r) : r = 1, \ldots, R, l = 1, \ldots, K_r\}.$$ 

Now pick constant acts $h_r, r = 1, \ldots, R$ as in **Step 1** with $u_r(h_r) > u_{r'}(h_{r'}), r \neq r'$. Pick also within each group $r \in \{1, \ldots, R\}$ acts $f_{rl} , l = 1, \ldots, K_r$ with image in $\Delta(Z)$ s.t. $q_{rl} \cdot u_r(f_{rl}) > q_{rl} \cdot u_r(f_{rl'}), l \neq l'$. We claim that the separating acts we are after can be taken of the form

$$\lambda f_{rl} + (1 - \lambda)h_r, \quad r = 1, \ldots, R; l = 1, \ldots, K_r$$

whenever $\lambda > 0$ small enough.

We need to show that there exists $\lambda \in (0, 1)$ with

$$(P1) \quad q_{rl} \cdot u_r(\lambda f_{rl} + (1 - \lambda)h_r) > q_{rl} \cdot u_r(\lambda f_{rl'} + (1 - \lambda)h_r'), \quad \text{whenever} \quad (r, l) \neq (r', l').$$

Consider first the case $r = r'$. Then $l \neq l'$ and (P1) is true for all $\lambda$ by linearity of the Bernoulli functions and the choice of $f_{rl}$.

Consider then the case $r \neq r'$. Given that $u_r(h_r) > u_{r'}(h_{r'})$ and the linearity of the Bernoulli functions, for a fixed pair of tuples $(r, l) \neq (r', l')$ (P1) becomes true whenever $\lambda$ is small enough for that pair. This gives a positive upper bound on $\lambda$. Since the number of pairs $(r, l)$ is finite, overall there exists a $1 > \lambda > 0$ for which (P1) is satisfied for all distinct pairs $(r, l) \neq (r', l')$.

$\square$

### A.2 Proof for the Axiomatization of aSCFs (AS-version)

Pick an element $y^* \in X$ and set $U = \{u \in \mathbb{R}^X : u(y^*) = 0\}$.

We first define the AS-version (Ahn-Sarver version) of the representation.

**Definition 18.** 1) Let $\rho$ be an aSCF for acts in $\mathbb{F}$ over $\Delta(X)$ where $X$ is a separable metric space and $S$, the set of objective states is finite.

We say that $\rho$ admits an AS-version R-SEU representation if there is a triple

$$(\text{SubS}, \mu, \{(q, u), \tau_{q,u} : (q, u) \in \text{SubS}\})$$

such that

A. SubS is a finite subjective state space of distinct and non-constant SEUs and $\mu$ is a probability measure on $\text{SubS} \times S$.

B. For each $(q, u) \in \text{SubS}$ the tie-breaking rule $\tau_{q,u}$ is a regular sigma-additive probability measure on $\Delta(S) \times U$ endowed with the respective product Borel sigma-Algebra.
C. For all \( f \in \mathcal{F}, A \in \mathcal{A} \) and \( s \in S \) we have

\[
\rho(f, A, s) = \sum_{(q, u) \in \text{SubS}} \mu(q, u)\tau_{q, u}(f, A),
\]

where \( \tau_{q, u}(f, A) := \tau_{q, u}(\{(p, w) \in \Delta(S) \times U : f \in M(M(A; u, q); w, p)\}). \)

2) We say that the AS-version R-SEU representation has no unforeseen contingencies if \( \text{supp}(\mu|_{q(u)}) \subseteq \text{supp}(q) \) for all \( (q, u) \in \text{SubS} \).

3) We say that the AS-version R-SEU representation has correct interim beliefs if \( \mu(\cdot|q, u) = q(\cdot) \) for all \( (q, u) \in \text{SubS} \).

The next Theorem gives the axiomatization of aSCFs which have an AS-version R-SEU representation.

**Theorem 4.** The aSCF \( \rho \) on \( \mathcal{A} \) admits an AS-version R-SEU representation if and only if it satisfies

A. Statewise Monotonicity

B. Statewise Linearity

C. Statewise Extremeness

D. Statewise Continuity

E. Statewise State Independence

F. Statewise Finiteness

Moreover, it additionally has a No Unforeseen Contingencies representation if and only if it additionally satisfies No Unforeseen Contingencies. Finally, it has a Correct Interim Beliefs representation if and only if it additionally satisfies Correct Interim Beliefs.

**Proof of Theorem 4. Necessity.** Checking this is routine. In particular, one checks easily that \( \text{RSSupp}(\bar{\rho}) = \text{supp}(\mu) \).

**Sufficiency.** We prove this in several steps.

**Step 1.** We construct the SCFs \( \bar{\rho} \) from \( \rho \) as well as \( \rho(\cdot, \cdot|s) \) for all \( s \in S \). Due to the axioms on \( \rho \) all of \( \bar{\rho} \) as well as \( \rho(\cdot, \cdot|s), s \in S \) satisfy all axioms from Theorem 1 in the online appendix. In particular, we have the following representations: for all \( f \in \mathcal{F}, A \in \mathcal{A} \)

\[
\bar{\rho}(f, A) = \sum_{(q, u) \in \text{SubS}} \psi(q, u)\tau_{q, u}(f, A),
\]

and

\[
\rho(f, A|s) = \sum_{(q, u) \in \text{SubS}(s)} \psi^s(q, u)\tau_{q, u}^s(f, A).
\]

with appropriate probability measures \( \psi \) and \( \psi^s \) on finite sets of SEUs.

**Step 2.** Due to simple probability accounting it holds
\[ \bar{\rho}(f, A) = \sum_{s \in S} \rho(f, A|s) \rho(s). \] (10)

If it were true that \( \text{supp}(\psi^s) \not\subseteq \text{supp}(\psi) \) for some \( s \in S \) then by use of separating menus as constructed in Lemma 1 one could come to a contradiction to (10). The same kind of contradiction argument and use of Lemma 1 leads to exclusion of the case \( \text{supp}(\psi) \setminus \cup_{s \in S} \text{supp}(\psi^s) \neq \emptyset \). In all we have established

\[ \text{supp}(\psi) = \cup_{s \in S} \text{supp}(\psi^s). \]

In particular, we can extend w.l.o.g. \( \psi^s \) for all \( s \) to all of \( \text{supp}(\psi) \) by setting it to zero outside of \( \text{supp}(\psi^s) \).

**Step 3.** By a similar mixing argument as in Proposition 2 in the online appendix (see step 3 there) one can easily show that whenever \( (q, u) \in \text{supp}(\psi) \cap \text{supp}(\psi^s) \) we have \( \tau_{q,u}^s = \tau_{q,u} \). In particular, we can write the representations for \( \rho(\cdot, \cdot|s) \) as

\[ \rho(f, A|s) = \sum_{(q,u) \in \text{Sub}_S} \psi^s(q,u) \tau_{q,u}(f, A). \] (11)

By plugging (11) in (10), rearranging and using the uniqueness result for the AS-representation of \( \rho \) from Proposition 2 in the online appendix we get

\[ \psi(q,u) = \sum_{s \in S} \psi^s(q,u) \rho(s), \quad (q,u) \in \text{supp}(\psi). \] (12)

By setting \( \mu(q,u,s) = \psi^s(q,u) \rho(s) \) we define a probability measure over \( \text{Sub}_S \times S \) whose marginal over \( \text{Sub}_S \) is full support and which satisfies (7).

**Step 4.** Take a separating menu \( \bar{A} = \{f(q,u) : (q,u) \in \text{supp}(\psi)\} \) for \( \text{supp}(\psi) \). We show that the following property (P) gives us the representation for correct interim beliefs.

\[ (P) \quad \rho(\cdot|f(q,u), \bar{A}) = q(\cdot), \quad (q,u) \in \text{supp}(\psi). \]

**Claim.** (P) implies the representation with correct interim beliefs.

**Proof of Claim.** For the menu \( \bar{A} \) and each \( (q,u) \in \text{supp}(\psi) \) we have

\[ \psi^s(q,u) = \rho(f(q,u), \bar{A}|s) = \frac{\rho(f(q,u), \bar{A})}{\rho(s)} \frac{\rho(s|f(q,u), \bar{A})}{\rho(s)} \frac{\rho(f(q,u), \bar{A})}{\rho(s)} = \frac{q(s)\psi(q,u)}{\rho(s)}. \] (13)

Here, only in the last equality we have used (P) and the definition and representation of \( \bar{\rho} \) from Theorem 1 in the online appendix. We write this as the identity

\[ (!) \quad \rho(s)\psi^s(q,u) = q(s)\psi(q,u). \]

Summing (!) w.r.t. \( (q,u) \) we get the identity (!!!) \( \rho(s) = \sum_{(q,u) \in \text{Sub}_S} \psi(q,u)q(s) \) for all \( s \in S \) and thus a unique solution for \( \psi^s \) in (13). It is then trivial to see that the representation holds because of (11) and (!!). \( \square \)
**Step 5.** In this step we show that (P) is implied by Correct Interim Beliefs.
Denote in general for each \( q \in \Delta(S) \) such that \( (q, u) \in \text{supp}(\mu) \) for some \( u \), \( \rho(\cdot|f(q,u), \bar{A}) = \hat{q}(q,u)(\cdot) \).

Suppose by contradiction that there exists some \( (q,u) \in \text{supp}(\mu) \) with \( \rho(\cdot|f(q,u), \bar{A}) \neq q(\cdot) \). If it holds for some \( \hat{u} \) that \( (\hat{q}(q,u), \hat{u}) \in \text{supp}(\mu) = \text{RSSupp}(\hat{\rho}) \) then we know that \( (\hat{q}(q,u), \hat{u}) \notin \mathcal{N}(\bar{A}, f(q,u)) \cap \text{RSSupp}(\hat{\rho}) = \{(q,u)\} \) as \( \bar{A} \) is separating for \( \text{RSSupp}(\hat{\rho}) \) and \( \hat{q}(q,u) \neq q \), which implies \( (q,u) \not\approx (\hat{q}(q,u), \hat{u}) \). But clearly \( |\mathcal{N}(\bar{A}, f(q,u)) \cap \text{RSSupp}(\hat{\rho})| = |\{(q,u)\}| = 1 \).

Overall it follows that Correlated Interim Belief axiom is violated at the choice data \( (f(q,u), \bar{A}) \).

**Step 6.** We show that the following property \((P!)\) gives us the representation for no unforeseen contingencies.

\[
(P!) \quad \text{supp}(\rho(\cdot|f(q,u), \bar{A})) \subset \text{supp}(q(\cdot)), \quad (q, u) \in \text{supp}(\psi).
\]

**Claim.** \((P!)\) implies the representation with unforeseen contingencies.

We look at (13), but leave out the final equality. The Claim follows immediately.

**Step 7.** In this step we show that \((P!)\) is implied by No Unforeseen Contingencies.

Suppose by contradiction that there exists some \( (q,u) \in \text{supp}(\psi) = \text{RSSupp}(\hat{\rho}) \) with \( \text{supp}(\rho(\cdot|f(q,u), \bar{A})) \not\subset \text{supp}(q(\cdot)) \). Pick again a separating menu for \( \text{RSSupp}(\hat{\rho}) \) and note that \( |\mathcal{N}(\bar{A}, f(q,u)) \cap \text{RSSupp}(\hat{\rho})| = |\{(q,u)\}| = 1 \). Overall it follows that the No Unforeseen Contingencies axiom is violated for the choice data \( (f(q,u), \bar{A}) \).

\(\Box\)

We note down uniqueness.\(^{/43}\)

**Proposition 7.** The AS-version REU-representation for an aSCF \( \rho \) is essentially unique in the sense that for each two representations the only degree of freedom is positive affine transformations of the Bernoulli utilities of elements in the support of the measures over SEUs.

**Proof.** For the case of CIB this follows directly from Proposition 2 in the online appendix applied to the SCF corresponding to the aSCF.

For the case of NUC, if there are two different representations for \( \rho \) with respective measures \( \mu, \mu' \) it follows from Proposition 2 in the online appendix that the marginals are equal: \( \sum_s \mu(q,u,s) = \sum_s \mu'(q,u,s) \) for all \( (q,u,s) \). In particular, up to equivalence classes of positive affine transformations of the Bernoulli utility functions the support of these two marginals in \( \Delta(S) \times \mathbb{R}^X \) is equal for the two measures. Assume then w.l.o.g. the same normalization for both supports. Taking now a separating menu \( \bar{A} \) for the SEUs in the support of the two measures \( \mu, \mu' \), we have from the representation property that

\[
\rho(f(q,u), \bar{A}, s) = \mu(q,u,s) = \mu'(q,u,s) \quad \text{for all } s.
\]

This concludes the proof. \(\Box\)

**Proof for Proposition 1.** **Sufficiency.** Define the SCF on \( \Delta(X) \) by the formula\(^{/44}\)

\(^{/43}\)The online appendix shows equivalence between AS-based representations and Filtration-based representations.

\(^{/44}\)Here a slight abuse of notation as we haven’t written down the isomorphism between constant menus of acts and menus of lotteries, but the context gives clarity.
\[ \tau(f, A) = \rho(f, A), \quad A \text{ is menu of constant acts.} \]

Note that Theorem 1 in the online appendix gives with some slight abuse of notation

\[ \tau(f, A) = \sum_{(q,u) \in \text{SubS for some } q} \mu(q, u) \tau_u(q, u) \{ (p, w) \in \Delta(S) \times U : f \in M(A; q, u; p, w) \}. \]

Since the beliefs play no role in the decision of the agent (all acts are constant), one can rewrite this as

\[ \tau(f, A) = \sum_{u \in \pi_u(\text{SubS})} \mu(u) \tau'_u\{ w \in U : f \in M(A; u; w) \}, \]

where \( \mu(u) = \sum_{q: (q, u) \in \text{SubS}} \mu(q, u) > 0 \) and \( \tau'_u = \sum_{q: (q, u) \in \text{SubS}} \frac{\mu(q, u)}{\mu(u)} \tau_q(u) \). Note that \( \tau'_u \) is a regular tie-breaker for lotteries.\(^{45}\)

Obviously this gives an S-based REU representation as in Theorem 4 of [Frick, Iijima, Strzalecki '17]. C-Determinism* implies then directly that \( \tau \) has only one state in the sense of the S-based representation from [Frick, Iijima, Strzalecki '17].\(^{46}\) In particular, \( u \approx v \) for all \( u, v \in U \) such that \( (q, u), (p, v) \in \text{supp}(\mu) \) for some \( q, p \in \Delta(S) \).

**Necessity.** Consider a menu of constant acts \( A \). Then for all \( (q, u), (p, u) \in \text{supp}(\mu) \) we have \( M(A; u, q) = M(A; v, u) =: M(A, q) \), so that by a small abuse of notation which uses the fact that the menu \( A \) is constant we can write

\[ \rho(f, A) = \sum_{(q, u) \in \text{SubS}} \mu(q) \tau_q\{ w \in U : f \in M(A; u; w) \}. \]

The existence of a best constant act \( f \) means \( u(\bar{f}) > u(f) \) whenever \( f \neq \bar{f} \) and \( f \) also constant.

Note now that for each \( g \in A, g \neq f \) we have for either \( u(af + (1 - a)\bar{f}) > u(g) \) or \( u(af + (1 - a)\bar{f}) < u(g) \) for all \( a < 1 \) near enough to \( a \). It follows that

\[ \rho(af + (1 - a)\bar{f}; A \setminus \{f\} \cup \{af + (1 - a)\bar{f}\}) \in \{0, 1\}, \quad \text{for all } a < 1 \text{ near enough to } 1. \]

Thus C-Determinism* is satisfied. \( \Box \)

We skip writing down a statement and proof of a Proposition connecting AS-version representations with the representations in Definition 2 (filtration form) since it will be subsumed in the more general arguments in Section 4 of the online appendix.

### B AS-Based Representations for the dynamic setting

The proofs in this appendix are done in the AS-version of the representations. Here we explain what these are. The online appendix then establishes the equivalence between the two types of representations.

\[^{45}\text{Here, the } w \text{ breaking ties from } M(A, u) \text{ is drawn as follows: first draw a } (q, u) \text{ where } (q, u) \text{ has probability } \frac{\mu(q, u)}{\mu(u)} \text{ and then, draw (conditionally independently across the } (q, u)-\text{s) } w \text{ according to the marginal of } \tau_q(u) \text{ on } U. \text{ This works because the tie-breakers are preference-based.} \]

\[^{46}\text{Otherwise one arrives easily at a contradiction through separating lotteries to either } \mu(u) > 0 \text{ for all } u \text{ or to the C-Determinism* Axiom.} \]
B.1 Dynamic Random Subjective Expected Utility (DR-SEU)

Definition 19. We say that a history-dependent family of aSCF $\rho = (\rho_0, \ldots, \rho_T)$ has a DR-SEU representation if there exists

- a finite objective state space $S$ and a collection of partitions $S_t, t = 1, \ldots, T$ of $S$ such that $S_t$ is a refinement of $S_{t-1}$.

- a finite collection of states of the world $\Theta_t, t = 0, \ldots, T$ (an element is of the type $(q_t, u_t, s_t) \in \Delta(S_t) \times \mathbb{R}^{X_t} \times S_t$). The sequence $\Theta_t, t \leq T$ has a partitional structure and there are no repetitions: each element $(q_t, u_t, s_t)$ is indexed by the predecessors $(q_0, u_0, s_0; \ldots; q_{t-1}, u_{t-1}, s_{t-1})$.\(^{47}\) Moreover we have the restriction that $s_k \in \text{supp}(q_k)$.

- a collection of probability kernels

$$\psi_k : \Theta_{k-1} \to \Delta(\Theta_k)$$

for $k = 0, \ldots, T$\(^{48}\) with a typical element in the image written as $\psi_k^\theta_{k-1}(s_{k-1}, u_{k-1})$. In particular, the probability that $(q_k, u_k, s_k)$ is realized after $\theta_{k-1}$ occurs is $\psi_k^\theta_{k-1}(q_k, u_k, s_k)$.

- a sequence of tie-breakers: for all $t = 0, \ldots, T$ a regular probability measure $\tau(q_t, u_t)$ over $\Delta(S_t) \times \mathbb{R}^{X_t}$, for all $(q_t, u_t) = \pi qu(\theta)$ for some $\theta_t \in \Theta_t$.

such that the following two conditions hold.

**DR-SEU 1**

(a) every $(q_t, u_t) \in \pi qu \left( \text{supp}(\psi_t^\theta_{t-1}) \right)$ represents a non-constant SEU preference.

(b) $\text{supp}(\psi_t^\theta_{t-1}) \cap \text{supp}(\psi_t^\theta'_{t-1}) = \emptyset$ whenever $\theta_{t-1} \neq \theta'_{t-1}$, both in $\Theta_{t-1}$.\(^{49}\)

(c) $\cup_{\theta_{t-1}} \text{supp}(\psi_t^\theta_{t-1}) = \Theta_t$.

(d) either (correct interim beliefs) The kernels $\psi$ satisfy $\psi_k^\theta_{k-1}(s_k|q_k, u_k) = q_k(s_k)$

or otherwise (no unforeseen contingencies) $\text{supp}(\psi_k^\theta_{k-1}(\cdot|q_k, u_k)) \subset \text{supp}(q_k)$.

**DR-SEU 2**

The SCF $\rho_t$ after a history $h^{t-1} = (A_0, f_0, s_0; \ldots; A_{t-1}, f_{t-1}, s_{t-1})$ is given by

$$
\rho_t(s_t, f_t, A_t|h^{t-1}) = \frac{\sum_{s_0, \ldots, s_{t-1}} \pi_s(\theta_0, \ldots, \theta_{t-1}) |(s_0, \ldots, s_{t-1})| \prod_{k=0}^{t-1} \psi_k^\theta_{k-1}(\theta_k) \tau_{q_u}(\theta_k) f_k(A_k) \psi_{k-1}^\theta(\theta_{t-1}) \tau_{q_u}(\theta_{t-1}) f_{t-1}(A_{t-1})}{\sum_{s_0, \ldots, s_{t-1}} \pi_s(\theta_0, \ldots, \theta_{t-1}) |(s_0, \ldots, s_{t-1})| \prod_{k=0}^{t-1} \psi_k^\theta_{k-1}(\theta_k) \tau_{q_u}(\theta_k) f_k(A_k)}.
$$

\(^{47}\)This means that there can be repetitions in terms of the SEUs $(q_t, u_t)$ but whenever this happens a different $s_t$ is realized.

\(^{48}\)With the obvious conventions for $k = 0$.

\(^{49}\)This implies, that whenever $\pi_s(\theta_{t-1}) = \pi_s(\theta'_{t-1})$ and two elements $\theta_t \in \text{supp}(\psi_t^\theta_{t-1}), \theta'_t \in \text{supp}(\psi_t^\theta'_{t-1})$ with $\pi_{q_u}(\theta_t) = \pi_{q_u}(\theta'_t)$ we must have $u_t \neq u'_t$. 

\[41\]
B.2 Evolving Subjective Utility (Evolving SEU)

The Evolving Subjective Expected Utility representation is a special case of DR-SEU.

In the pre-choice situation in period $t$ when the agent knows $(q_t, u_t) = \pi_{qu}(\theta_t)$ and satisfies the Evolving SEU representation she evaluates acts according to the following SEU functional

$$\mathbb{E}_{q_t}[u_t(f_t)] = \mathbb{E}_{q_t \sim q_t}[u_t(f_t(s_t))] = \mathbb{E}_{q_t \sim q_t}[v_t(f_Z^s(s_t))] + \delta V_t^{\pi_{qu}(\theta_t)}(f^A).$$

(14)

Here $V_t^{\pi_{qu}(\theta_t)}(f^A)$ is defined in two steps. First we define

$$V_t^{\theta_t}(A_{t+1}) = \int_{f_{t+1} \in A_{t+1}} \max_{f_{t+1} \in A_{t+1}} \mathbb{E}_{q_{t+1}}[u_{t+1}(f_{t+1})] d\psi_{t+1}^{\theta_t}(q_{t+1}, u_{t+1}).$$

(15)

This gives the value of a menu when the agent knows the menu, but not the SEU with which it will evaluate the acts. This is the situation just after $(z_t, A_{t+1})$ is known to the agent at the end of period $t$.

A moment before, i.e. when the agent doesn’t know $s_t$ yet the value of $f_t^A$ is given by

$$V_t^{\pi_{qu}(\theta_t)}(f_t^A) := \sum_{s_t} \sum_{A_{t+1} \in \text{supp} f_t^A(s_t)} q_t(s_t) f_t^A(s_t)(A_{t+1}) V_t^{\theta_t}(A_{t+1}) =: \sum_{s_t} q_t(s_t) V_t^{\theta_t}(f_t^A(s_t)).$$

(16)

Note that the uncertainty that is integrated out in (16) is the objective one concerning $s_t$ and that we have used equation (15) to define the extension of $V_t^{\theta_t}$ to lotteries over menus.50

We can rewrite this in integral form as follows.

$$V_t^{\pi_{qu}(\theta_t)}(f_t^A) = \int_{f_{t+1} \in A_{t+1}} \max_{f_{t+1} \in A_{t+1}} \mathbb{E}_{q_{t+1}}[u_{t+1}(f_{t+1})] d\psi_{t+1}^{\pi_{qu}(\theta_t)}(q_{t+1}, u_{t+1}),$$

where $\psi_{t+1}^{\pi_{qu}(\theta_t)}(q_{t+1}, u_{t+1}) := \sum_{s_t} q_t(s_t) \psi_{t+1}^{\theta_t}(q_{t+1}, u_{t+1}) = \sum_{s_t, s_{t+1}} q_t(s_t) \psi_{t+1}^{\theta_t}(q_{t+1}, u_{t+1}, s_{t+1}).$

B.3 Gradual SEU-Learning.

Gradual SEU-learning is the case of Evolving SEU with the additional requirement that her sequence of expected utility functionals from consumption $v_t$, $t = 0, \ldots, T$ form a Martingale. In the following we use the projection $\pi_v$, which for a $u_t$ as in (14) gives the corresponding $v_t$.

Normalize $v_t(\bar{p}) = 0$ for all $t$ where $\bar{p}$ is the uniform lottery over $Z$. This is possible because $Z$ is assumed to be finite for the dynamic setting. After a $\theta_t = (q_t, u_t, s_t)$ it has to hold for the sequence $\pi_v(\theta_t)$ from the Evolving SEU representation

$$\pi_v(\theta_t) = \frac{1}{\delta} \sum_{(q_{t+1}, u_{t+1}) \in \pi_{qu}(\theta_{t+1})} \psi_{t+1}^{\theta_t}(q_{t+1}, u_{t+1}) \cdot \pi_v(u_{t+1}) = \frac{1}{\delta} \mathbb{E}[\pi_v(\theta_{t+1}) | \theta_t].$$

(17)

50I.e. agent is Expected Utility w.r.t. lotteries over menus.
C  Separating histories

We first define histories consistent with a state $\theta_t$. Then we define separating histories for a fixed state $\theta_t$. The main result of this section establishes the existence of separating histories (Lemma 7).

Let us assume that we have an aSCF $\rho$ which satisfies DR-SEU 1. We define the predecessor of a state $\theta$ as $\text{pred}(\theta) = (\theta_0, \ldots, \theta_{t-1})$.

**Definition 20.** For a state $\theta_t = (q_t, u_t, s_t)$ denote by $\text{pred}(\theta_t) = (\theta_0, \ldots, \theta_{t-1})$ the unique predecessor of $\theta_t$ from $\prod_{i=0}^{t-1} \Theta_i$.

The concept is well-defined because of DR-SEU 1 (a)-(b).

**Definition 21.** Given a history $h^t = (A_0, f_0, s_0; A_1, f_1, s_1; \ldots; A_t, f_t, s_t)$ say that $\theta_t$ is consistent with $h^t$ if for the unique predecessor of $\theta_t$, given by $(\theta_0, \ldots, \theta_{t-1})$ we have

$$\prod_{k=0}^{t} \tau_{\psi_q}(\theta_k)(f_k, A_k) \cdot \psi_k^{\theta_{k-1}}(\theta_k) > 0.$$  

Here we use the convention $\psi_0^{\theta_{-1}} := \psi_0$.

Note that multiple states $\theta_t$ can be consistent with the same history $h^t$.

Define

$$QU_{\theta_k}(A_{k+1}, f_{k+1}, s_{k+1}) = \{(q_{k+1}, u_{k+1}) : (q_{k+1}, u_{k+1}, s_{k+1}) \in \text{supp}(\psi_k^{\theta_k}) \text{ and } f_{k+1} \in M(A_{k+1}; q_{k+1}, u_{k+1})\}.$$  

This is the set of SEU-s $(q_{k+1}, u_{k+1})$ occurring right after $\theta_k$ which can rationalize the data $(A_{k+1}, f_{k+1}, s_{k+1})$.

For time $t = 0$ define

$$QU_0(A_0, f_0, s_0) = \{(q_0, u_0) : (q_0, u_0, s_0) \in \text{supp}(\psi_0^{\theta_0}) \text{ and } f_0 \in M(A_0; q_0, u_0)\}.$$  

We prove first the following Lemma.

**Lemma 2** (Pendant to Lemma 1 in [Frick, Iijima, Strzalecki '17]). Fix any $\theta_t$ and its predecessor $(\theta_0, \ldots, \theta_{t-1})$. Suppose $h^t = (B_0, g_0, s_0; \ldots; B_t, g_t, s_t)$ satisfies $QU_{\theta_{t-1}}(B_k, g_k, s_k) = \{\pi_{\psi_q}(\theta_k)\}$. Then for all $k = 0, \ldots, t$, only $\theta_k$ in $\Theta_k$ can be consistent with $h^t$.

**Proof.** Fix any $l = 0, \ldots, t$ and consider $\theta'_l \in \Theta_l \setminus \{\theta_l\}$ with $\text{pred}(\theta'_l) = (\theta'_0, \ldots, \theta'_{l-1})$.

Let $k \leq l$ be smallest such that $\theta'_k \neq \theta_k$. Then $\pi_{\psi_q}(\theta'_k) \in \pi_{\psi_q}(\text{supp}(\psi_k^{\theta_k-1}))$. So $QU_{\theta_{l-1}}(B_k, g_k, s_k) = \{(q_k, u_k)\}$ (which is assumed) implies either (A) $(q_k, u_k) \neq (q'_k, u'_k)$ or (B) $(q_k, u_k) = (q'_k, u'_k), s_k = s'_k$ (otherwise contradiction to $\theta_k \neq \theta'_k$).

In the case of (B) the definition of the QU-sets implies then that $s'_k \notin \text{supp}(g_k)$, i.e. $q'_k(s'_k) = 0$. In the case of (A) the definition of the QU-sets implies $g_k \notin M(B_k; q'_k, u'_k)$, i.e. $\tau_{q'_k,u'_k}(g_k, B_k) = 0$. Overall we have that $\theta'_l$ is not consistent with $h^t$.

Next we show that $\theta_l$ is consistent with $h^t$. Note that from the definition of histories w.r.t. to some aSCF it follows that $\rho(g_t, B_t|h^t) > 0$. DR-SEU 2 then implies

$$\sum_{\pi_{\psi_q}(\theta_0, \ldots, \theta_t) \in \times_{i \leq l} SEU_i} \left[ \prod_{k=0}^{l-1} \psi_k^{\theta_{k-1}}(\theta_k) \tau_{\psi_q}(\theta_k)(f_k, A_k) \right] \cdot \psi_l^{\theta_{l-1}}(\theta_l) \tau_{\psi_q}(\theta_l)(f_l, A_l) q_l(s_l) > 0.$$  

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Lemma 3.
If it happens that pred(θt) ≠ (θ0, . . . , θt−1) then \( \prod_{k=0}^{t-1} \psi_k^{θ_k-1}(θ_k) \cdot \psi_t^{θ_t-1}(θ_t) = 0 \) just by the definition of DR-SEU 1. If otherwise pred(θt) = (θ0, . . . , θt−1) but θt ≠ θt′ then we showed above that \( \prod_{k=0}^{t-1} g_k(π_k(θ_k))τ_{π_k(θ_k)}(f_k, A_k) \cdot τ_{π_k(θ_t)}(f_t, A_t)g_t(s_t) = 0. \)

\[\square\]

Definition 22. A separating history for θt with pred(θt) = (θ0, . . . , θt−1) is a history \( h^t = (B_0, g_0, s_0; . . . ; B_t, g_t, s_t) \in \mathcal{H}_t^s \) such that \( QU_{θ_{k-1}}(B_k, g_k, s_k) = \{π_k(θ_k)\} \) for all \( k \leq t \). For the case \( k = 0 \) we abuse notation and write \( QU_0(B_0, g_0, s_0) = QU_0(B_0, g_0, s_0) \).

Remark 3. 1) Let \( A_t \in \mathcal{A}_t \) arbitrary. After introducing LHI below, one sees easily, that when mixing a separating history for \( θ_t \) with a deterministic history such that it has the same projection on objective states as \( h_t^t \) one can assume that \( h_t^t \) is so that \( A_t \in \mathcal{A}_t^s(h_t^t) \). In particular separating histories are not unique.

2) By definition, \( θ_t \) is the only state in \( Θ_t-1 \) consistent with \( h_t^t \) if \( h_t^t \) is a separating history for \( θ_{t-1} \).

Write

\[ D_{t-1} = \{d_t \in \mathcal{H}_{t-1} : d_t = (\{f_0\}, f_0, s_0; . . . \{f_{t-1}\}, f_{t-1}, s_{t-1}), f_i \in \mathbb{F}_t\}, \]

for the set of histories such that the menu is degenerate in each period and look at its subset

\[ \mathcal{D}C_{t-1} = \{d_t \in \mathcal{H}_{t-1} : d_t = (\{h_0\}, h_0, s_0; . . . \{h_{t-1}\}, h_{t-1}, s_{t-1}), h_i, i \leq t-1 \text{ are constant acts}\}. \]

The latter consists of deterministic histories where the agent faces only constant acts and thus objective states don’t matter.

Note that given a menu \( A_t \not\in \mathcal{A}_t(h_t^t) \) we can always choose a \( h_t^t \in \mathcal{D}C_{t-1} \) with \( A_t \in \text{supp}(h_t^t) \). Then we can define the extended aSCF as follows.

Definition 23. For a history \( h_t^t \in \mathcal{H}_{t-1}, A_t \in \mathcal{A}_t \) and \( s_t \in S_t \) define

\[ \rho_t^{h_t^t}(\cdot, A_t, s_t) = \rho_t(\cdot, A_t, s_t | λh_t^t + (1 - λ)d_t^t), \]

for some \( λ \in (0, 1] \), where \( d_t^t \in \mathcal{D}C_{t-1} \) is so that \( λh_t^t + (1 - λ)d_t^t \in \mathcal{H}_{t-1}(A_t). \)

We prove the extension is well-defined.

Lemma 3. Suppose that \( ρ \) satisfies LHI. Fix \( t ≥ 1, A_t \in \mathcal{A}_t, h_t^t = (A_0, f_0, s_0; . . . , A_{t-1}, f_{t-1}, s_{t-1}) \in \mathcal{H}_{t-1} \) and \( (λ_0, . . . , λ_{t-1}), (\hat{λ}_0, . . . , \hat{λ}_{t-1}) \in (0, 1]^t \).

Suppose \( d_t^t = (h_0, \{h_0\}, s_0; . . . ; h_{t-1}, \{h_{t-1}\}, s_{t-1}) \), \( \hat{d}_t^t = (\hat{h}_0, \{\hat{h}_0\}, s_0; . . . ; \hat{h}_{t-1}, \{\hat{h}_{t-1}\}, s_{t-1}) \) \in \mathcal{D}C_{t-1}(A_t). \) Then we have

\[ ρ_t(\cdot, A_t, s_t | λh_t^t + (1 - λ)d_t^t) = ρ_t(\cdot, A_t, s_t | \hat{λ}h_t^t + (1 - \hat{λ})\hat{d}_t^t). \]

In particular, \( ρ_t^{h_t^t} \) is well-defined.

\[ ^{51}\text{For this to hold it suffices that } A_t \in \text{supp}(h_t^t). \]

\[ ^{52}\text{Here the mixture operation for histories is valid for every pair of histories which share the same sub-history of objective states – the mixture operation only acts on the sub-history of acts and menus. Recall that mixture of menus is defined through the Minkowski sum.} \]
**Proof.** Let \( k = \max\{n = 0, \ldots, t - 1 : h_n \neq \hat{h}_n\} \).

Suppose that \( k = -1 \). This means that \( d^{t-1} = \hat{d}^{t-1} \). If \( \lambda_i > \hat{\lambda}_i \) for \( i = 0, \ldots, t - 1 \) then the \( i \)-th entry of \( \lambda h^{t-1} + (1 - \lambda)d^{t-1} \) can be rewritten as an appropriate mixture of the \( i \)-th entry of \( \hat{\lambda} h^{t-1} + (1 - \hat{\lambda})\hat{d}^{t-1} \) and \((A_i, f_i, s_i)\). If on the other hand \( \lambda_i \leq \hat{\lambda}_i \) for \( i = 0, \ldots, t - 1 \) then the \( i \)-th entry of \( \lambda h^{t-1} + (1 - \lambda)d^{t-1} \) can be rewritten as an appropriate mixture of the \( i \)-th entry of \( \hat{\lambda} h^{t-1} + (1 - \hat{\lambda})\hat{d}^{t-1} \) and \((A_i, f_i, s_i)\). Starting from \( i = 0 \) and using LHI and working our way up the index \( i = 0, \ldots, t - 1 \) we see that the aSCF is unaffected by replacing each entry of \( \lambda h^{t-1} + (1 - \lambda)d^{t-1} \) with its corresponding entry from \( \lambda h^{t-1} + (1 - \lambda)d^{t-1} \). This shows the result for the case \( k = -1 \).

Assume now for the induction step that the statement is true for all \( k \leq m - 1 \) for some \( 0 \leq m \leq t - 1 \). We show that the claim still holds for \( k = m \).

Define the following objects.

\[
B_m = \frac{1}{2} A_m + \frac{1}{2} \{ h_m \}, \quad \hat{B}_m = \frac{1}{2} A_m + \frac{1}{2} \{ \hat{h}_m \}, \quad r_m = \frac{1}{2} f_m + \frac{1}{2} h_m, \quad \hat{r}_m = \frac{1}{2} f_m + \frac{1}{2} \hat{h}_m
\]

\[
g_n = \frac{1}{2} h_n + \frac{1}{2} l_n, \quad \hat{g}_n = \frac{1}{2} \hat{h}_n + \frac{1}{2} l_n,
\]

for soon to be specified \( l_n, n = 1, \ldots, t - 1 \). Namely, define \( l_n \) recursively so that they satisfy

\[
\lambda_n A_n + (1 - \lambda_n) \{ g_n \}, \hat{\lambda}_n A_n + (1 - \hat{\lambda}_n) \{ \hat{g}_n \}, \frac{1}{2} A_n + \frac{1}{2} \{ h_n \}, \frac{1}{2} A_n + \frac{1}{2} \{ \hat{h}_n \}, \{ g_n \}, \{ \hat{g}_n \} \in \text{supp}(l^A_{n-1}).
\]

Finally augment the constant act \( l_{m-1} \) so that

\[
\frac{2}{3} B_m + \frac{1}{3} \{ \hat{g}_m \}, \frac{2}{3} \hat{B}_m + \frac{1}{3} \{ g_m \}, \frac{1}{2} \{ g_m \} + \frac{1}{2} \{ \hat{g}_m \} \in \text{supp}(l^A_{m-1}).
\]

Denote \( c^t := (g_n, \{ g_n \}, s_n)_{n=0}^{t-1} \) and \( \hat{c}^t := (\hat{g}_n, \{ \hat{g}_n \}, s_n)_{n=0}^{t-1} \) both in \( D\mathcal{C}_{t-1} \). Note that we have \( \lambda h^{t-1} + (1 - \lambda)c^t - 1, \lambda h^{t-1} + (1 - \hat{\lambda})\hat{c}^t - 1 \in \mathcal{H}_{t-1}(A_t) \) by construction. Also, the last entry at which \( c^t - 1 \) and \( \hat{c}^t - 1 \) differ is \( m \). Thus by repeated application of LHI we can replace \( \lambda h^{t-1} + (1 - \lambda)d^{t-1} \) by \( \lambda h^{t-1} + (1 - \lambda)c^t - 1 \) and \( \hat{\lambda} h^{t-1} + (1 - \hat{\lambda})\hat{d}^{t-1} \) by \( \hat{\lambda} h^{t-1} + (1 - \hat{\lambda})\hat{c}^t - 1 \). \( c^t - 1, \hat{c}^t - 1 \) and also satisfy the following relations.

\[
(a) : \quad \frac{1}{2} h^{t-1} + \frac{1}{2} d^{t-1}, \frac{1}{2} h^{t-1} + \frac{1}{2} \hat{d}^{t-1} \in \mathcal{H}_{t-1}(A_t),
\]

\[
(b) : \quad \frac{2}{3} B_m + \frac{1}{3} \{ \hat{h}_m \}, \{ \frac{1}{2} h_m + \frac{1}{2} \hat{h}_m \} \in \text{supp}(h^A_{m-1}),
\]

\[
(c) : \quad \frac{2}{3} \hat{B}_m + \frac{1}{3} \{ h_m \}, \{ \frac{1}{2} h_m + \frac{1}{2} \hat{h}_m \} \in \text{supp}(\hat{h}^A_{m-1}).
\]

These imply immediately

\[
(d) : \quad \left( \frac{2}{3} B_m + \frac{1}{3} \{ \hat{g}_m \}, \frac{2}{3} \hat{r}_m + \frac{1}{3} \{ g_m \} \right) = \left( \frac{2}{3} \hat{B}_m + \frac{1}{3} \{ g_m \}, \frac{2}{3} r_m + \frac{1}{3} \{ \hat{g}_m \} \right)
\]

\[
= \left( \frac{1}{3} A_m + \frac{2}{3} \{ \frac{1}{2} h_m + \frac{1}{2} \hat{h}_m \}, \frac{1}{3} f_m + \frac{2}{3} \{ \frac{1}{2} h_m + \frac{1}{2} \hat{h}_m \} \right).
\]

\[\text{53} \text{The argument is the same as in the proof of Lemma 15 in [Frick, Iijima, Strzalecki '17]. It is based on the fact that when mixing a history } h^{t-1} \text{ with a degenerate history from } D_{t-1}, \text{ then the sets of maximizers } N(A_i, f_i) \text{ doesn’t change.}\]
Now (a)-(c) imply that the histories
\[
\left(\frac{1}{2} h^{t-1} + \frac{1}{2} c^{t-1}\right)_{-m}, \left(\frac{2}{3} B_m + \frac{1}{3} \{g_m\}, \frac{2}{3} r_m + \frac{1}{3} \{\hat{g}_m\}, s_m\right)
\]
and
\[
\left(\frac{1}{2} h^{t-1} + \frac{1}{2} c^{t-1}\right)_{-m}, \left(\frac{2}{3} \hat{B}_m + \frac{1}{3} \{g_m\}, \frac{2}{3} \hat{r}_m + \frac{1}{3} \{g_m\}, s_m\right)
\]
are in \( \mathcal{H}_{t-1}(A_t) \). Moreover, (d) implies that the first history is an entry-wise mixture of 
\( h^{t-1} = (e^{t-1}_m, \{\frac{1}{2} h_m + \frac{1}{2} \hat{h}_m\}) \), whereas the second is an entry-wise mixture of \( c^{t-1} \) with \( \hat{c}^{t-1} = (d^{t-1}_m, \{\frac{1}{2} h_m + \frac{1}{2} \hat{h}_m\}) \).

The base case of the induction \((k = -1)\) gives
\[
\rho_t(\cdot; A_t, s_t | \lambda h^{t-1} + (1 - \lambda)c^{t-1}) = \rho_t(\cdot; A_t, s_t | \frac{1}{2} h^{t-1} + \frac{1}{2} c^{t-1})
\]
and
\[
\rho_t(\cdot; A_t, s_t | \lambda h^{t-1} + (1 - \lambda)\hat{c}^{t-1}) = \rho_t(\cdot; A_t, s_t | \frac{1}{2} h^{t-1} + \frac{1}{2} \hat{c}^{t-1}).
\]
But note that the entry where \( e^{t-1}, \hat{c}^{t-1} \) first differ is strictly less than \( m \). Hence applying the inductive hypothesis we have
\[
\rho_t(\cdot; A_t, s_t | \left(\frac{1}{2} h^{t-1} + \frac{1}{2} c^{t-1}\right)_{-m}, \left(\frac{2}{3} B_m + \frac{1}{3} \{g_m\}, \frac{2}{3} r_m + \frac{1}{3} \{\hat{g}_m\}, s_m\right))
\]
\[
\rho_t(\cdot; A_t, s_t | \left(\frac{1}{2} h^{t-1} + \frac{1}{2} \hat{c}^{t-1}\right)_{-m}, \left(\frac{2}{3} \hat{B}_m + \frac{1}{3} \{g_m\}, \frac{2}{3} \hat{r}_m + \frac{1}{3} \{g_m\}, s_m\right)).
\]
Combining this together with the implication from the base case we get the result. \( \Box \)

In the next Lemma we show that the extended aSCF satisfies the formula in DR-SEU 2.

**Lemma 4.** Suppose that we have an aSCF \( \rho \) which has a DR-SEU representation as in Definition 19 till some period \( T \in \mathbb{N} \). Then the extended version of \( \rho \) as in Definition 23 will satisfy DR-SEU 2, i.e. for all \( t \leq T, \forall f'_t, A'_t \) and \( h^{t-1} = (A_0, f_0, s_0; \ldots; A_{t-1}, f_{t-1}, s_{t-1}) \) and \( f_t, A_t \) we have
\[
\rho_t(s_t, f_t, A_t | h^{t-1}) = \sum_{\pi_s(\theta_0, \ldots, \theta_{t-1}) = (s_0, \ldots, s_{t-1})} \frac{\psi_{t-1}^{\theta_{t-1}}(\theta_t) \tau_{\pi qu}(\theta_t) \pi_{f_t}(f_t, A_t)}{\sum_{\pi_s(\theta_0, \ldots, \theta_{t-1}) = (s_0, \ldots, s_{t-1})} \Pi_{k=0}^{t-1} \psi_{\theta_k}^{\theta_{k-1}}(\theta_k) \tau_{\pi qu}(\theta_k) \pi_{f_k}(f_k, A_k)}.
\]

**Proof.** If \( h^{t-1} \in \mathcal{H}_{t-1}(A_t) \) then the claim follows directly from DR-SEU2. Assume thus that \( h^{t-1} \notin \mathcal{H}_{t-1}(A_t) \) and take \( d^{t-1} = (h_0, h_0, s_0; \ldots; h_{t-1}, h_{t-1}, s_{t-1}) \in \mathcal{D}_{t-1} \) with \( d^{t-1} \in \mathcal{H}_{t-1}(A_t) \) and compatible with the sub-history of objective states so that according to Definition 23 we can define for some \( \lambda \in (0, 1) \)
\[
\rho_t(f_t, A_t, s_t | h^{t-1}) := \rho_t(f_t, A_t, s_t | \lambda h^{t-1} + (1 - \lambda)d^{t-1}).
\]
Note that
(1) the formula depends on the menus and acts chosen only through the tiebreakers \( \tau \).

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(2) $d^{t-1} \in D^{t-1}$ implies that for all $s \leq t$

$$f_s \in M(M(A_s; q_s, u_s), p_s, w_s) \iff \lambda f_s + (1 - \lambda) h_s \in M(M(\lambda A_s + (1 - \lambda)\{h_s\}; q_s, u_s), p_s, w_s).$$

1) and 2) imply immediately that for all $s \leq t$

$$\tau_{q_s, u_s}(f_s, A_s) = \tau_{q_s, u_s}(\lambda f_s + (1 - \lambda) h_s, \lambda A_s + (1 - \lambda)\{h_s\}).$$

From here the result follows from applying DR-SEU 2 to the history $\lambda h^{t-1} + (1 - \lambda)d^{t-1}$. \hfill \Box

We define $\Theta(h^{t-1}) \subset \Theta_{t-1}$ as the set of states $\theta_{t-1}$ consistent with $h^{t-1}$ in the sense of Definition 21.

**Lemma 5.** [Pendant to Lemma 14 in Frick, Iijima, Strzalecki ’17] Fix $t \in \{0, \ldots, T\}$ and suppose that we have a DR-SEU representation up to time $t$. Take any $h^{t-1} = (A_0, f_0, s_0; \ldots; (A_{t-1}, f_{t-1}, s_{t-1}) \in \mathcal{H}_{t-1}$ and $A_t \in \mathcal{A}_t$. Then the following are equivalent.

A. $A_t \in \mathcal{A}_t^*(h^{t-1})$.

B. For each $\theta_{t-1} \in \Theta(h^{t-1})$ and $(q_t, u_t) \in \pi_{q_u}(\supp(\psi_t^{\theta_{t-1}}))$ we have $|M(A_t; q_t, u_t)| = 1$.

**Proof.** From A. to B.: We prove the contrapositive. Suppose that there is $\theta_{t-1} \in \Theta(h^{t-1})$ and $(q_t, u_t) \in \pi_{q_u}(\supp(\psi_t^{\theta_{t-1}}))$ with $|M(A_t; q_t, u_t)| > 1$. Pick any $f_t \in M(A_t; q_t, u_t)$ with $\tau_{q_t, u_t}(f_t, A_t) > 0$. Since $u_t$ is non-constant by DR-SEU 1, we can find lotteries $\Delta(X_t)$ with $u_t(\omega) < u_t(\bar{\omega})$. Fix a sequence $\alpha = (0, 1)$ with $\alpha_0 \rightarrow 0$ and let $f_t^n = \alpha_n \delta_\omega + (1 - \alpha_n) f_t$ as well as $\psi_t^n = \alpha_n \delta_\omega + (1 - \alpha_n) g_t$ and $\bar{f}_t^n = \alpha_n \delta_\omega + (1 - \alpha_n) g_t$ for all $g_t \in A_t \setminus \{f_t\}$. Let $\bar{B}_t^n = \{g_t^n : g_t \in A_t \setminus \{f_t^n\}\}$ and $\bar{B}_t^n = \{g_t^n : g_t \in A_t \setminus \{f_t^n\}\}$. Finally let $B_t^n = \bar{B}_t^n \cup \bar{B}_t^n$. Then we have $B_t^n \rightarrow^m A_t \setminus \{f_t^n\}$ and $f_t^n \rightarrow^m f_t$. Furthermore, since $|M(A_t; q_t, u_t)| > 1$ we can pick $g_t \in A_t \setminus \{f_t^n\}$ such that $q_t \cdot u_t(\bar{f}_t^n) > q_t \cdot u_t(f_t^n)$. This implies $\tau_{q_t, u_t}(f_t^n, B_t^n \cup \{f_t^n\}) = 0$.

Furthermore, note that for $(q_t^n, u_t^n) \in \pi_{q_u}(\Theta_t) \setminus \{(q_t, u_t)\}$ we always have

$$N(M(A_t; q_t^n, u_t^n; f_t^n)) = N(M(B_t^n \cup \{f_t^n\}; q_t^n, u_t^n; f_t^n)) \geq N(M(B_t^n \cup \{f_t^n\}; q_t^n, u_t^n; f_t^n),$$

which implies $\tau_{q_t^n, u_t^n}(f_t^n, A_t) \geq \tau_{q_t^n, u_t^n}(f_t^n, B_t^n \cup \{f_t^n\})$ for all $n$. Letting $\mathrm{pred}(\theta_{t-1}) = (\theta_0, \ldots, \theta_{t-2})$ Lemma 4 implies that for all $n$ and all $s_t \in S_t$\(^54\)

$$\rho_t(f_t, A_t, s_t|h^{t-1}) - \rho_t(f_t^n, B_t^n \cup \{f_t^n\}, s_t|h^{t-1}) =$$

$$\Sigma_{\pi(s'_{0}, \ldots, s'_{t}) = (s'_0, \ldots, s'_t)} \left[ \prod_{k=0}^{t-1} \psi_k^{\theta_k-1}(\theta_{k}) \tau_{\pi_{q_u}(\theta_{k})}(f_k, A_k) \right] \psi_t^{\theta_t-1}(\theta_{t}) \left( \tau_{\pi_{q_u}(\theta_{t})}(f_t, A_t) - \tau_{\pi_{q_u}(\theta_{t})}(f_t^n, B_t^n \cup \{f_t^n\}) \right)$$

$$\geq \frac{\prod_{k=0}^{t-1} \psi_k^{\theta_k-1}(\theta_{k}) \tau_{\pi_{q_u}(\theta_{k})}(f_t, A_t)}{\sum_{\pi(s'_{0}, \ldots, s'_{t-1}) = (s'_0, \ldots, s'_{t-1})} \prod_{k=0}^{t-1} \psi_k^{\theta_k-1}(\theta'_{k}) \tau_{\pi_{q_u}(\theta'_{k})}(f_t, A_t)} > 0.$$}

The last line doesn’t depend on $n$ so we get

$$\limsup_{n \rightarrow \infty} \rho_t(f_t^n, B_t^n \cup \{f_t^n\}, s_t|h^{t-1}) < \rho_t(f_t, A_t, s_t|h^{t-1}).$$

\(^54\)Note that we need Lemma 4 here because the history $h^{t-1}$ is not assured to lead to $B_t^n \cup \{f_t^n\}$ with positive probability.
By Definition 11 we have $A_t \notin \mathcal{A}_t^\ast(h^{t-1})$.

From B. to A.: Suppose $A_t$ satisfies B. Consider any $f_t \in A_t$, $f_t^m \rightarrow f_t$, $B_t^{n} \rightarrow A_t \setminus \{f_t\}$. Consider a $\theta_{t-1} \in \Theta(h_{t-1})$ and $(q_t, u_t) \in \pi_{q_u}(\text{supp}(\psi^{\theta_{t-1}}_{t-1}))$.

By 2. we either have $M(A_t; q_t, u_t) = \{f_t\}$ or $f_t \notin M(A_t; q_t, u_t)$. In the former case $q_t \cdot u_t(f_t) > q_t \cdot u_t(g_t)$ for all $A_t \supseteq g_t \neq f_t$. By linearity we have $q_t \cdot u_t(f_t^n) > q_t \cdot u_t(g_t^n)$ for all $g_t^n \in B_t^n$ for all $n$ large enough.

This implies $\tau_{q_t, u_t}(f_t, A_t) = \lim_n \tau_{q_t, u_t}(f_t^n, B_t^n \cup \{f_t^n\}) = 1$. In the case that $f_t \notin M(A_t; q_t, u_t)$ we have similarly $q_t \cdot u_t(f_t) < q_t \cdot u_t(g_t)$ for some $A_t \supseteq g_t \neq f_t$. But then linearity implies $\tau_{q_t, u_t}(f_t, A_t) = \lim_n \tau_{q_t, u_t}(f_t^n, B_t^n \cup \{f_t^n\}) = 0$.

Overall, for all $\theta_{t-1} \in \Theta(h^{t-1})$ and $(q_t, u_t) \in \pi_{q_u}(\text{supp}(\psi^{\theta_{t-1}}_{t-1}))$ it holds $\tau_{q_t, u_t}(f_t, A_t) = \lim_n \tau_{q_t, u_t}(f_t^n, B_t^n \cup \{f_t^n\})$. By looking at the formula in Lemma 4 we see that this implies for all $\theta_{t-1} \in \Theta(h^{t-1})$ and all $n$ large enough

$$\rho_t(f_t^n, B_t^n \cup \{f_t^n\}, s_t|h^{t-1}) = \rho_t(f_t, A_t, s_t|h^{t-1}).$$

This finishes the proof. □

Before continuing, we register the piece of notation for an arbitrary $f_t \in \mathbb{F}_t$: $\text{supp}^{Z}(f_t) := \cup_{q \in \text{supp}(f_t)} \text{supp}(q)$.

Lemma 6. [Pendant to Lemma 17 in [Frick, Iijima, Strzalecki ’17].] Suppose we have a DR-SEU representation till time $T$. Fix any $\theta_{t-1} \in \Theta_{t-1}$, separating history $h^{t-1}$ for $\theta_{t-1}$ and $A_t \in A_t$. Then there exists a sequence $A_t^m \rightarrow A_t$ with $A_t^m \in A_t^\ast(h^{t-1})$. Moreover, given a $(q_t', u_t') \in \pi_{q_u}(\text{supp}(\psi^{\theta_{t-1}}_{t-1}))$ and $f_t \in M(A_t; q_t, u_t)$ we can ensure in this construction that there is $f_t^n(q_t', u_t') \in A_t^n$ with $f_t^n(q_t', u_t') \rightarrow m f_t$ such that $QU_{\theta_{t-1}}(A_t^n, f_t^n(q_t', u_t'), s_t) = \{q_t', u_t'\}$ for all $s_t \in \text{supp}(q_t')$.\n
Proof. Let $QU(\theta_{t-1}) := \pi_{q_u}(\text{supp}(\psi^{\theta_{t-1}}_{t-1}))$. By Definition 19 there exists a finite set $Y_t \subseteq X_t$ such that (i) for any $(q_t, u_t) \in QU(\theta_{t-1})$, $u_t$ is non-constant over $Y_t$; (ii) for any distinct $(q_t, u_t) \neq (q_t', u_t')$, both in $\text{supp}(\psi^{\theta_{t-1}}_{t-1})$, $(q_t, u_t) \neq (q_t', u_t')$ on $\mathbb{F}(Y_t)$ and (iii) $\cup_{f_t \in A_t, \text{supp}^{Z}(f_t) \subseteq Y_t}$.

By (i) and (ii) and Lemma 1 we can find a separating menu $C_t = \{f_t(q_t, u_t) : (q_t, u_t) \in QU(\theta_{t-1})\}$, i.e. such that for all $(q_t, u_t) \in QU(\theta_{t-1})$ we have $M(C_t; q_t, u_t) = \{f_t(q_t, u_t)\}$.

Pick $z(q_t, u_t) \in \arg\max_{y \in Y_t} u_t(y)$ for all $(q_t, u_t) \in QU(\theta_{t-1})$, write by a small abuse of notation again $z(q_t, u_t)$ for the constant act paying out $z(q_t, u_t)$ with probability one at each state of the world and define the constant act $b_t = \frac{1}{|Y_t|} \sum_{y \in Y_t} \delta_y \in \Delta(Y_t)$. Again, we denote by $b_t$ with a small abuse of notation the constant act which pays out the lottery $b_t$ in each state of the world.

By (i) we have $q_t \cdot u_t(z(q_t, u_t)) > q_t \cdot u_t(b_t)$ for all $(q_t, u_t) \in QU(\theta_{t-1})$. If we then define $\hat{f}_t(q_t, u_t) = \alpha f_t(q_t, u_t) + (1 - \alpha) z(q_t, u_t)$ we still have $q_t \cdot u_t(\hat{f}_t(q_t, u_t)) > q_t \cdot u_t(b_t)$ if we choose $\alpha \in (0, 1)$ small enough. This is because of the ‘finiteness’ of all the data going into the problem. Note also, that if we define $\hat{C}_t = \{\hat{f}_t(q_t, u_t) : (q_t, u_t) \in QU(\theta_{t-1})\}$ we still have $M(C_t; q_t, u_t) = \{\hat{f}_t(q_t, u_t)\}$.

Now pick for each $(q_t, u_t) \in QU(\theta_{t-1})$ a $f_t(q_t, u_t) \in M(A_t; q_t, u_t)$. To also prove the ‘moreover’ part, pick $f_t(q_t, u_t)$ as required in the ‘moreover’ part. Fix a sequence $\epsilon_n \in (0, 1)$ going to zero. For each $n$ and $(q_t, u_t) \in QU(\theta_{t-1}) := \text{supp}(\psi^{\theta_{t-1}}_{t-1})$ let $f_t^n(q_t, u_t) =$

\footnote{Note that because of Remark 3 this is w.l.o.g.}

\footnote{Recall this denotes the set of acts whose images are contained in $\Delta(Y_t)$.}
\[(1 - \epsilon_n)\hat{f}_t(q_t, u_t) + \epsilon_n f_t(q_t, u_t)\]. Moreover, for each \(g_t \in A_t\) define \(g^n_t = (1 - \epsilon_n)g_t + \epsilon_n b_t\). Finally, take
\[A^n_t = \{f^n_t(q_t, u_t) : (q_t, u_t) \in QU(\theta_{t-1})\} \cup \{g^n_t : g_t \in A_t\}\].

Note that \(A^n_t \rightarrow A_t\). Finally, note that by construction we have \(M(A^n_t; q_t, u_t) = \{f^n_t(q_t, u_t)\}\).

Since by Remark 3, part 2) \(\theta_{t-1}\) is the only state consistent with \(h^{t-1}\) Lemma 5 and the construction here imply \(A^n_t \in A^*_t(h_{t-1})\), as required. The last required property, i.e. \(QU_{\theta_{t-1}}(A^n_t, f^n_t(q_t, u_t), s_t) = \{(q_t, u_t)\}\) for any \(s_t \in supp(q_t)\) is true by construction.

The next result proves the existence of separating histories.

**Lemma 7.** [Pendant to Lemma 2 in [Frick, Iijima, Strzalecki '17].] For any \(\theta_t \in \Theta_t\) with \(pred(\theta_t) = (\theta'_0, \ldots, \theta'_{t-1})\) there always exists a separating history.

**Proof.** By Lemma 1 and DR-SEU 1 we can construct for \(\Theta_0\) a menu \(B_0 = \{f^{\pi_{qu}(\theta_0)}_0 : \theta_0 \in \Theta_0\}\) \(\in A_0\) such that \(QU_0(B_0, f^{\pi_{qu}(\theta_0)}_0; \pi_s(\theta_0)) = \{\pi_{qu}(\theta_0)\}\) for all \(\theta_0 \in \Theta_0\). Proceeding inductively, again using Lemma 1 and DR-SEU 1, we can find a menu \(B_k(\theta_{k-1}) = \{f^{\pi_{qu}(\theta_k)}_k : \pi_{qu}(\theta_k) \in \pi_{qu}(supp(\psi^\theta)_{k-1}))\}\) for all \(\theta_{k-1} \in \Theta_{k-1}\) such that (!) \(QU_{\theta_{k-1}}(B_k(\theta_{k-1}), f^{\pi_{qu}(\theta_k)}_k; \pi_s(\theta_k)) = \{\pi_{qu}(\theta_k)\}\) for all \(\pi_{qu}(\theta_k) \in \pi_{qu}(supp(\psi^\theta)_{k-1}))\).

Moreover, we can assume that \(B_{k+1}(\theta_k) \in supp(A^{\pi_{qu}(\theta_k)}_k)\) for all \(k = 0, \ldots, t - 1\) and \(\theta_k \in \Theta_k\) by mixing each \(f^{\pi_{qu}(\theta_k)}_k\) with the constant act delivering \((z, B_{k+1}(\theta_k))\) for a \(z \in Z\) fixed throughout. If the mixing puts small enough probability on the constant act in question, then (!) is preserved.

This implies in particular that \(h^t := (B_0, f^0_0, s'_0; \ldots; B_t(\theta'_{t-1}), f^{\pi_{qu}(\theta')}_t; \pi_s(\theta_t)) \in \mathcal{H}_t\). Moreover, since \(QU_{\theta_{k-1}}(B_k(\theta_{k-1}), f^{\pi_{qu}(\theta')}_k; \pi_s(\theta'_k)) = \{\pi_{qu}(\theta'_k)\}\), it follows by Lemma 2 that only the state \(\theta_k\) is consistent with \(h_k\) for \(k = 0, \ldots, t\). Additionally, by construction for all \((q_k, u_k) \in \pi_{qu}(supp(\psi_{k-1}))\) we have \(M(B_k(\theta'_{k-1}); q_k, u_k) = \{f^{\pi_{qu}(\theta_k)}_k\}\). Hence, by Lemma 5 we have \(B_k(\theta'_{k-1}) \in A^*_k(h_{k-1})\). Since this holds for all \(k\) we have overall \(h^t \in \mathcal{H}^*_t\). In summary it follows that \(h^t\) is a separating history for \(\theta_t\).

\[\square\]

**D Proof of the main result in the dynamic setting**

Here we prove the representation theorem in its AS-version for DR-SEU. The proofs for the special cases Evolving SEU and Gradual Learning are in the online appendix.

**D.1 Proof for DR-SEU**

**D.1.1 Sufficiency**

We proceed by induction on \(t \leq T\). First consider \(t = 0\). Because of the axioms and \(X_0\) being a separable metric space we have the existence of an AS-version R-SEU representation for \(\rho\) on \(\mathcal{H}^0\). Depending on the version of the representation we are looking at, i.e. whether CIB or NUC is satisfied, we also have the respective property for the representation at time \(t = 0\). Set \(SEU_0 = \{\pi_{qu}(\theta_0) : \theta_0 \in \Theta_0\}\).

Suppose next that we have the representation for all \(t' \leq t\). We now construct the representation for \(t + 1\).
To this end, pick a subjective state \( \theta_t \in \Theta_t \) and pick an arbitrary separating history \( h^t(\theta_t) \) for \( \theta_t \). This exists by Lemma 7. Define

\[
\rho^\theta_{t+1}(\cdot, A_{t+1}, s_{t+1}) = \rho(\cdot, A_{t+1}, s_{t+1} | h^t(\theta_t)).
\]

Here we use for the right-hand side the extended aSCF, which is well-defined as per Lemma 23. As per axioms we get a representation

\[
\rho^\theta_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}) = \sum_{(q_{t+1}, u_{t+1}) \in SEU^\theta_{t+1}} \psi^\theta_{t+1}(q_{t+1}, u_{t+1}, s_{t+1}) \tau(q_{t+1}, u_{t+1})(f_{t+1}, A_{t+1}). \tag{18}
\]

Again, depending on the respective property required by the axioms on beliefs, CIB or NUC, the kernel \( \psi^\theta_{t+1} \) satisfies the respectively required property in DR-SEU 1.

We set \( SEU_{t+1} = \cup_{\theta_t} SEU^\theta_{t+1} \) and define \( \Theta_{t+1} \) accordingly by the collection of all \( (q_{t+1}, u_{t+1}, s_{t+1}) \) such that \( (q_{t+1}, u_{t+1}) \in SEU_{t+1} \) and \( s_{t+1} \in \text{supp}(q_{t+1}) \).\(^{57}\) We extend the measures \( \psi^\theta_{t+1} \) to all of \( SEU_{t+1} \) by setting them to zero outside of \( SEU^\theta_{t+1} \).

We see that DR-SEU 1 is satisfied by Definition.

With this definition we can rewrite (18) as

\[
\rho^\theta_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}) = \sum_{\theta_t \in \Theta_{t+1}} \psi^\theta_{t+1}(\cdot, A_{t+1}, s_{t+1}) \tau_{\pi_{qu}(\cdot)}(f_{t+1}, A_{t+1}).
\]

Before showing DR-SEU 2, we show that the definition of \( \rho^\theta_{t+1} \) doesn’t depend on the particular separating history for \( \theta_t \) picked in its definition.

**Lemma 8.** Fix any \( \theta_t \in \Theta_t \) with \( \text{pred}(\theta_t) = (\theta_0, \ldots, \theta_{t-1}) \). Suppose \( h^t = (f_0, A_0, s_0; \ldots; f_t, A_t, s_t) \in H_t \) satisfies \( QU_{\theta_{k-1}}(A_k, f_k, s_k) = \{\pi_{qu}(\theta_t)\} \) for all \( k = 0, 1, \ldots, t \). Then for any \( A_{t+1} \in A_{t+1} \) and \( s_{t+1} \in S_{t+1} \) it holds \( \rho_{t+1}(\cdot, A_{t+1}, s_{t+1} | h^t) = \rho^\theta_{t+1}(\cdot, A_{t+1}, s_{t+1}) \).

**Proof.** Step 1. Let \( \tilde{h}^t = (\tilde{f}_0, \tilde{A}_0, \tilde{s}_0; \ldots; \tilde{f}_t, \tilde{A}_t, \tilde{s}_t) \in H_t \) denote the separating history for \( \theta_t \) used to define \( \rho^\theta_{t+1} \). We first prove the Lemma under the assumption that \( h^t \in H_t^* \), i.e. that \( h^t \) is itself a separating history for \( \theta_t \). Note that since \( h^t, \tilde{h}^t \in H_t^* \) and \( QU_{\theta_{k-1}}(A_k, f_k, s_k) = QU_{\theta_{k-1}}(\tilde{A}_k, \tilde{f}_k, \tilde{s}_k) = \{g_k, u_k\} \) Lemma 5 implies that \( M(A_k, g_k, u_k) = \{f_k\} \) and \( M(\tilde{A}_k, g_k, u_k) = \{\tilde{f}_k\} \).

Pick lotteries \( (r_0, \ldots, r_t) \in \Delta(X_0) \times \cdots \times \Delta(X_t) \) such that \( A_{t+1} \in \text{supp}(r^A_t) \) and so that for all \( k = 0, \ldots, t-1 \) it holds

\[
\{B_{k+1}, \tilde{B}_{k+1}, B_{k+1} \cup \tilde{B}_{k+1}\} \subset \text{supp}(r^A_k),
\]

where \( B_l = \frac{1}{3} A_l + \frac{1}{3} \{\tilde{f}_l\} + \frac{1}{3} \{r_l\} \) and \( \tilde{B}_l = \frac{1}{3} \tilde{A}_l + \frac{1}{3} \{f_l\} + \frac{1}{3} \{r_l\} \) for \( l = 0, \ldots, t \). Here we have identified lotteries with their respective constant acts. Define also the mixture act \( g_l = \frac{1}{3} f_l + \frac{1}{3} \tilde{f}_l + \frac{1}{3} r_l \).

Linearity of SEU functionals implies

\[
QU_{\theta_{k-1}}(B_k, g_k, s_k) = QU_{\theta_{k-1}}(\tilde{B}_k, g_k, \tilde{s}_k) = QU_{\theta_{k-1}}(\tilde{B}_k \cup B_k, g_k, \tilde{s}_k) = \{g_k, u_k\}.\tag{58}
\]

\(^{57}\)The symbol \( \cup \) means we join them into a union of disjoint sets, i.e. if a SEU \( (q, u) \) appears in the support of two distinct \( \theta_t, \theta_t' \) then we count it twice.

\(^{58}\)Note that in the last equality it is irrelevant whether we write \( \tilde{s}_k \) or \( s_k \) because of the argument in the first paragraph of the first step of the proof.
We also have
\[ M(B_k, q_k, u_k) = M(\tilde{B}_k, q_k, u_k) = M(\tilde{B}_k \cup B_k, q_k, u_k) = \{g_k\}. \]

This implies that for all \(k = 0, \ldots, t\) and \((q'_k, u'_k) \in \pi_{qu}(\text{supp}(\psi_{k-1}^{\theta_k}))\) we have
\[ \tau_{q'_k, u'_k}(g_k, B_k) = \tau_{q'_k, u'_k}(g_k, \tilde{B}_k) = \tau_{q'_k, u'_k}(g_k, \tilde{B}_k \cup B_k) = \begin{cases} 1, & \text{if } \pi_{qu}(\theta_k) = \pi_{qu}(\theta'_k) \\ 0, & \text{if } \pi_{qu}(\theta_k) \neq \pi_{qu}(\theta'_k). \end{cases} \]

By DR-SEU 2 of the inductive hypothesis it follows for all \(k = 0, \ldots, t - 1\) that
\[
\psi_t^{\theta_t-1}(q_t, u_t, s_t) = \rho_t(g_t, \tilde{B}_t, s_t|\tilde{B}_0, g_0, s_0; \ldots; \tilde{B}_{t-1}, g_{t-1}, s_{t-1}) \\
= \rho_t(g_t, \tilde{B}_t, s_t|B_0, g_0, s_0, \ldots; B_t, g_t, s_t) \\
= \rho_t(g_t, \tilde{B}_t \cup B_t, s_t|B_0, g_0, s_0, \ldots; \tilde{B}_0 \cup B_0, g_0, s_0; \ldots; \tilde{B}_{t-1} \cup B_{t-1}, g_{t-1}, s_{t-1}) \\
= \rho_t(g_t, \tilde{B}_t \cup B_t, s_t|B_0, g_0, s_0, \ldots; \tilde{B}_k \cup B_k, g_k, s_k, \ldots; \tilde{B}_{t-1} \cup B_{t-1}, g_{t-1}, s_{t-1}).
\]

Note that in these relations we could have replaced everywhere \(s_k\) with \(\tilde{s}_k\), since both are in the support of \(q_k\) by the definition of the operator \(QU_{\theta_{k-1}}\).

Since all the histories considered above are compatible with \(A_{t+1}\) we apply CHI recursively to get
\[
\rho_{t+1}(\cdot, A_{t+1}, s_{t+1}|B_0, g_0, s_0; \ldots; B_t, g_t, s_t) = \rho_{t+1}(\cdot, A_{t+1}, s_{t+1}|\tilde{B}_0 \cup B_0, g_0, s_0; \ldots; \tilde{B}_t \cup B_t, g_t, s_t) \\
= \rho_{t+1}(\cdot, A_{t+1}, s_{t+1}|\tilde{B}_0, g_0, s_0; \ldots; \tilde{B}_t, g_t, s_t).
\] (19)

Here \(s_{t+1} \in S_{t+1}\) is arbitrary. Use LHI and Lemma 3 (well-definiteness of the extended aSCF) to get
\[
\rho_{t+1}(\cdot, A_{t+1}, s_{t+1}|h^t) = \rho_{t+1}(\cdot, A_{t+1}, s_{t+1}|B_0, g_0, s_0; \ldots; B_t, g_t, s_t), \\
\rho_{t+1}(\cdot, A_{t+1}, s_{t+1}|\tilde{h}^t) = \rho_{t+1}(\cdot, A_{t+1}, \tilde{s}_{t+1}|\tilde{B}_0, g_0, \tilde{s}_0; \ldots; \tilde{B}_t, g_t, \tilde{s}_t). \\
\] (20)

Finally, we put (19) and (20) together to get
\[
\rho_{t+1}(\cdot, A_{t+1}, s_{t+1}|h^t) = \rho_{t+1}(\cdot, A_{t+1}, s_{t+1}|\tilde{h}^t).
\]

This establishes the proof for the case that \(h^t \in \mathcal{H}^t\).

Step 2. Now suppose that \(h^t \notin \mathcal{H}^t\). Take any sequence of (valid) histories \(h^{t,n} \in \mathcal{H}^{t,n}\) with \(h^{t,n} \rightarrow^n h^t\) with \(h^{t,n} = (A_0^n, f_0^n, s_0^n; \ldots; A_t^n, f_t^n, s_t^n)\) for each \(n\). Existence is ensured by the Axiom of History Continuity.

Claim. For all large \(n\) we have \(QU_{\theta_{k-1}}(A_k^n, f_k^n, s_k^n) = \pi_{qu}(\theta_k)\) for all \(k = 0, \ldots, t\).

Proof of Claim. Take some subsequence \((h^{t,n})_{n \geq 1}\) of \((h^{t,n})_{n \geq 1}\). We have \(\rho_k(f_k^n, A_k^n, s_k^n|h^{k-1,n}) > 0\) for all \(k = 0, \ldots, t\) by the definition of histories. Assume that by DR-SEU 2 for \(k \leq t\) we can find \(\theta_{t,n}^l \in \Theta_t\) with \(\text{pred}(\theta_{t,n}^l) = (\theta_{0,n}^l, \ldots, \theta_{t-1,n}^l)\) and \((\theta_{0,n}^l, \ldots, \theta_{t,n}^l) \neq (\theta_{0,n}^l, \ldots, \theta_{t,n}^l)\) such that \(\pi_{qu}(\theta_{k,n}^l) \in QU_{\theta_{k-1,n}^l}(f_k^n, A_k^n, s_k^n)\) for all \(k = 0, \ldots, t\). Since \(S_0 \times \ldots \times S_t\) is finite, by choosing an appropriate subsequence we can assume \(\theta_{0,n}^l, \ldots, \theta_{t,n}^l) = (\theta_{0,l}^l, \ldots, \theta_{t,l}^l)\) for all \(l\). Pick the smallest \(k\) such that \(\theta_{k,n}^l \neq \theta_k\) and pick any \(g_k \in A_k\). Since \(A_k^n \rightarrow^m A_k\) we can find \(g_k^n \in A_k^n\) with \(g_k^n \rightarrow^m g_k\). Since we have for
all \( l \) that \( \pi_{qu}(\theta'_l) \in QU_{\theta_{l-1}}(f^m_k, A^m_k, s^m_k) \), so \( \pi_{qu}(\theta'_l)(f^m_k) \geq \pi_{qu}(\theta'_l)(g^m_k) \) and thus also \( \pi_{qu}(\theta'_l)(f_k) \geq \pi_{qu}(\theta'_l)(g_k) \) by linearity of the SEU represented by \( \pi_{qu}(\theta'_l) \).

Moreover, by choice of \( k \) we have \( \pi_{qu}(\theta'_l) \in \pi_{qu}(\text{supp}(\psi_{k-1}^{\theta'_l-1})) = \pi_{qu}(\text{supp}(\psi_{k-1}^{\theta'_l-1})) \). But the fact that \( QU_{\theta_{k-1}}(f_k, A_k, s_k) = \{ \pi_{qu}(\theta_k) \} \) implies that \( \pi_{qu}(\theta'_l) = \pi_{qu}(\theta_k) \) for all \( k \). We have thus shown that each subsequence \( (h_{l,m}^u)_{l \geq 1} \) of \( (h_{l,m}^u)_{n \geq 1} \) has a subsequence with the property required by the claim. A simple argument by contradiction now establishes the claim.

**End of Proof of Claim.**

The Claim establishes that for all large enough \( n \), \( h_{l,m}^u \) satisfies the assumption of the Lemma. Since \( h_{l,m}^u \in \mathcal{H}^*, \) Step 1 then shows that \( \rho_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}|h_{l,m}^u) = \rho_{t+1}^u(f_{t+1}, A_{t+1}, s_{t+1}) \) for all large enough \( n \) and all \( f_{t+1}, s_{t+1} \). History Continuity now allows to close the argument and prove that

\[
\rho_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}|h_{t}) = \rho_{t+1}^u(f_{t+1}, A_{t+1}, s_{t+1}).
\]

\( \square \)

As a next step we establish that \( \rho_{t+1}(\cdot|h_{t}) \) is a weighted average of the \( \rho_{t+1}^u \) for \( \theta_t \) consistent with \( h_{t} \).

**Lemma 9.** [Pendant of Lemma 4 in [Frick, Iijima, Strzalecki '17]] For any \( f_{t+1} \in \mathcal{A}_{t+1} \) and \( h_{t} = (A_0, f_0, s_0; \ldots; A_t, f_t, s_t) \in \mathcal{H}_t(A_{t+1}) \) we have

\[
\rho_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}|h_{t}) = \frac{\sum_{\pi_{l}(\theta_0, \theta_1, \ldots, \theta_t)=(s_0, \ldots, s_t)} \prod_{k=0}^{t} \psi_{k}^{\theta_{k}-1}(\theta_k) \pi_{qu}(\theta_k)(f_k, A_k) \rho_{t+1}^u(f_{t+1}, A_{t+1}, s_{t+1})}{\sum_{\pi_{l}(\theta_0, \theta_1, \ldots, \theta_t)=(s_0, \ldots, s_t)} \prod_{k=0}^{t} \psi_{k}^{\theta_{k}-1}(\theta_k) \pi_{qu}(\theta_k)(f_k, A_k)}.
\]

**Proof.** Let \( (\theta^1_t, \ldots, \theta^m_t) \) be the set of elements from \( \Theta_t \) that are consistent with history \( h^t \), as defined in Definition 21. For each \( j = 1, \ldots, m \) let \( h^t(j) = (B^0_j, f^0_j, s_0; \ldots; B^t_j, f^t_j, s_t) \) be a separating history for \( \theta^j_t \). Note that such a history exists because under \( \theta^j_t \) and its predecessors the ‘right’ sub-history of objective states \( (s_0, \ldots, s_t) \) has positive probability.

We can assume w.l.o.g. that for each \( k = 1, \ldots, t \) in all objective states \( s_{t-1} \) there is a positive probability (albeit possibly small) for \( (z, \frac{1}{2} A_k + \frac{1}{2} B^j_k) \) for some \( z \). This can be achieved by mixing with constant acts. Thus, w.l.o.g. we can ensure that \( h^t(j) := \frac{1}{2} h^t + \frac{1}{2} h^t(j) \in \mathcal{H}_t(A_{t+1}) \).

Note first that it holds for all \( j = 1, \ldots, m \)

\[
\rho(h^t(j)) = \prod_{k=0}^{t} \psi_{k}^{\theta^j_k}(\theta_k) \pi_{qu}(\theta_k)(f_k, A_k).
\]

(21)

This follows from the following calculation.

\[
\rho(h^t(j)) = \prod_{k=0}^{t} \rho_k(\frac{1}{2} f_k + \frac{1}{2} f^j_k; \frac{1}{2} A_k + \frac{1}{2} B^j_k, s_k|h^k(j))
\]

\[
= \sum_{(\theta^0_k, \ldots, \theta^t_k)} \prod_{k=0}^{t} \psi_{k}^{\theta^0_k-1}(\theta^0_k) \pi_{qu}(\theta^0_k)(\frac{1}{2} f_k + \frac{1}{2} f^j_k; \frac{1}{2} A_k + \frac{1}{2} B^j_k)
\]

\[
= \prod_{k=0}^{t} \psi_{k}^{\theta^0_k-1}(\theta^0_k) \pi_{qu}(\theta^0_k)(f_k, A_k).
\]
Here the second equality follows from DR-SEU2 and the inductive hypothesis for Sufficiency. The final two equalities follow from the fact that \( h^t(j) \) is a separating history for \( \theta^t_j \) (see Lemma 5). Since \( \theta^t_j \) is consistent with \( h^t \) it follows
\[
\psi^t_{\theta^t_j^{-1}}(\theta^t_j) \cdot \tau_{\pi_{\text{qu}}(\theta^t_j)}(f_k, A_k) > 0 \quad \text{for all } k = 0, \ldots, t \text{ and therefore also:}
\]

for every \( \pi_{\text{qu}}(\theta^t_j) \in \pi_{\text{qu}}(\text{supp}(\psi^t_{\theta^t_j^{-1}})) \), \( \tau_{\pi_{\text{qu}}(\theta^t_j)}(\frac{1}{2} f_k + \frac{1}{2} f^t_j, \frac{1}{2} A_k + \frac{1}{2} B^t_j) > 0 \) if and only if \( \pi_{\text{qu}}(\theta^t_k) = \pi_{\text{qu}}(\theta^t_j) \). This yields the third equality above.

Define now \( H^t = \{ h^t(j) : j = 1, \ldots, m \} \subseteq \mathcal{H}_t(A_{t+1}) \). By repeated application of LHI we have that
\[
\rho_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}|h^t) = \rho_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}|H^t). \tag{22}
\]
Moreover, we have that
\[
\rho_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}|H^t) = \frac{\sum_{j=1}^{m} \rho(h^t(j)) \rho_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}|h^t(j))}{\sum_{j=1}^{m} \rho(h^t(j))}
\]
\[
= \frac{\sum_{j=1}^{m} \prod_{k=0}^{t} \psi^t_{\theta^t_j^{-1}}(\theta^t_j) \tau_{\pi_{\text{qu}}(\theta^t_j)}(f_k, A_k) \cdot \rho_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}|h^t(j))}{\sum_{j=1}^{m} \prod_{k=0}^{t} \psi^t_{\theta^t_j^{-1}}(\theta^t_j) \tau_{\pi_{\text{qu}}(\theta^t_j)}(f_k, A_k)}
\]
\[
= \frac{\sum_{j=1}^{m} \prod_{k=0}^{t} \psi^t_{\theta^t_j^{-1}}(\theta^t_j) \tau_{\pi_{\text{qu}}(\theta^t_j)}(f_k, A_k) \rho_{t+1}(f_{t+1}, A_{t+1}, s_{t+1})}{\sum_{j=1}^{m} \prod_{k=0}^{t} \psi^t_{\theta^t_j^{-1}}(\theta^t_j) \tau_{\pi_{\text{qu}}(\theta^t_j)}(f_k, A_k)} \tag{23}
\]

Here the first equality holds by definition of choice conditional on a set of histories. The second follows from (21). Note that \( h^t(j) \), being a separating history for \( \theta^t_j \) and consistent with \( h^t \), implies \( QU_{\theta^t_j} (\frac{1}{2} f_k + \frac{1}{2} f^t_j, \frac{1}{2} A_k + \frac{1}{2} B^t_j, s_k) = \{ \pi_{\text{qu}}(\theta^t_j) \} \) for each \( k \). Hence, Lemma 8 implies that \( \rho_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}|h^t(j)) = \rho_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}) \). This yields the third equality.

Finally, note that if \( (\theta_0, \ldots, \theta_t) \in \Theta_0 \times \cdots \times \Theta_t \) has \( (\theta_0, \ldots, \theta_t) \neq (\theta^t_0, \ldots, \theta^t_t) \) for all \( j \), then either \( \theta_{j} \not\in \{ \theta^t_j : j = 1, \ldots, m \} \) or \( \theta_{j} = \theta^t_j \) for some \( j \) but \( \text{pred}(\theta^t_j) \neq (\theta_0, \ldots, \theta_{t-1}) \). In either case we have \( \prod_{k=0}^{t} \psi^t_{\theta^t_j^{-1}}(\theta^t_j) \tau_{\pi_{\text{qu}}(\theta^t_j)}(f_k, A_k) = 0 \) by the inductive step up to \( t \). This justifies the last equality in (23).

Combining (22) and (23), we obtain the desired conclusion. \( \square \)

We show that our construction satisfies DR-SEU2 at step \( t+1 \) as well. We recall the representation in (18) and combine it with Lemma 9 to get for any \( h^t = (A_0, f_0, s_0; \ldots; A_t, f_t, s_t) \in \mathcal{H}_t(A_{t+1}) \)
\[
\rho_{t+1}(f_{t+1}, A_{t+1}, s_{t+1}|h^t) =
\]
\[
= \frac{\sum_{\pi_{\theta}(\theta_0, \ldots, \theta_t) = (s_0, \ldots, s_t)} \prod_{k=0}^{t} \psi^t_{\theta^t_k^{-1}}(\theta^t_k) \tau_{\pi_{\text{qu}}(\theta^t_k)}(f_k, A_k) \cdot \left( \sum_{\theta_{t+1}} \psi^t_{\theta^t_{t+1}}(\theta_{t+1}) \tau_{\pi_{\text{qu}}(\theta_{t+1})}(f_{t+1}, A_{t+1}) \right)}{\sum_{\pi_{\theta}(\theta_0, \ldots, \theta_{t+1}) = (s_0, \ldots, s_{t+1})} \prod_{k=0}^{t+1} \psi^t_{\theta^t_k^{-1}}(\theta^t_k) \tau_{\pi_{\text{qu}}(\theta^t_k)}(f_k, A_k)}
\]
D.1.2 Necessity

Suppose that $\rho$ admits a DR-SEU representation as in Definition 19. From the representation in DR-SEU 2 and from Lemma 3 we have that for a fixed $h^t \in H_t$ the static aSCF rule $\rho_t(\cdot|\hat{h}^t)$ satisfies the static axioms.

**Claim 1.** $\rho$ satisfies CHI.

**Proof.** Take any $h^{t-1} = (h_{t-k}^{t-1}, (A_k, f_k, s_k))$ and $\hat{h}^{t-1} = (h_{t-k}^{t-1}, (B_k, f_k, s_k))$ with $A_k \subseteq B_k$ and $\rho_k(f_k, A_k, s_k|h^{k-1}) = \rho_k(f_k, B_k, s_k|h^{k-1})$. From DR-SEU 2 this implies

$$
\sum_{(\theta_0, \ldots, \theta_k)} \left( \prod_{l=0}^k \psi_l^{\theta_l-1}(\theta_l) \tau_{\pi_{qu}(\theta_l)}(f_l, A_l) \right) = \sum_{(\theta_0, \ldots, \theta_k)} \left( \prod_{l=0}^k \psi_l^{\theta_l-1}(\theta_l) \tau_{\pi_{qu}(\theta_l)}(f_l, B_l) \right). \tag{24}
$$

It follows from $\tau_{\pi_{qu}(\theta_l)}(f_k, A_k) \leq \tau_{\pi_{qu}(\theta_l)}(f_k, B_k)$ that equality in (24) can hold if and only if $\tau_{\pi_{qu}(\theta_l)}(f_k, A_k) = \tau_{\pi_{qu}(\theta_l)}(f_k, B_k)$ whenever $\theta_k$ is consistent with $h_k$. This implies immediately due to DR-SEU 2 that

$$
\rho_t(\cdot|h^{t-1}) = \rho_t(\cdot|\hat{h}^{t-1}).
$$

\[\square\]

**Claim 2.** $\rho$ satisfies LHI.

**Proof.** Take any $A_t, s_t$ and $h^{t-1} = (a_0, f_0, s_0; \ldots; a_{t-1}, f_{t-1}, s_{t-1}) \in H_t - H_{t-1}(A_t)$ and $h^{t-1} \subseteq H_t - H_{t-1}(A_t)$ of the form $H^{t-1} = \{h_{t-k}^{t-1}, (\lambda A_k + (1 - \lambda)B_k, \lambda f_k + (1 - \lambda)g_k, s_k) : g_k \in B_k\}$ for some $k < t, \lambda \in (0, 1)$ and $B_k = \{g_k^j : j = 1, \ldots, m\} \subset A_k$. Let $\hat{A}_k = \lambda A_k + (1 - \lambda)B_k$ and for each $j = 1, \ldots, m$ let $\tilde{f}_k^j = \lambda f_k + (1 - \lambda)g_k^j$ and $\bar{h}^{t-1}(j) = (h_{t-k}^{t-1}, (\hat{A}_k, \tilde{f}_k^j, s_k))$.

By DR-SEU 2, for all $f_t$ we have

$$
\rho_t(f_t, A_t, s_t|h^{t-1}) = \frac{\sum_{(\theta_0, \ldots, \theta_{t-1})=(s_0, \ldots, s_{t-1})} \prod_{l=0}^{t-1} \psi_l^{\theta_l-1}(\theta_l) \tau_{\pi_{qu}(\theta_l)}(f_l, A_l)}{\sum_{(\theta_0, \ldots, \theta_{t-1})=(s_0, \ldots, s_{t-1})} \prod_{l=0}^{t-1} \psi_l^{\theta_l-1}(\theta_l) \tau_{\pi_{qu}(\theta_l)}(f_l, A_l)}
$$

and by definition also

$$
\rho_t(f_t, A_t, s_t|H^{t-1}) = \frac{\sum_{j=1}^m \rho(\bar{h}^{t-1}(j)) \rho_t(f_t, A_t, s_t|\bar{h}^{t-1}(j))}{\sum_{j=1}^m \rho(\bar{h}^{t-1}(j))}
$$

For each $j = 1, \ldots, m$ DR-SEU 2 yields

$$
\rho(\bar{h}^{t-1}(j)) = \sum_{(\theta_0, \ldots, \theta_{t-1})=(s_0, \ldots, s_{t-1})} \prod_{l=0}^{t-1} \psi_l^{\theta_l-1}(\theta_l) \tau_{\pi_{qu}(\theta_l)}(f_l, A_l) \psi_l^{\theta_l-1}(\theta_l) \tau_{\pi_{qu}(\theta_l)}(f_l, A_l) \tau_{\pi_{qu}(\theta_l)}(f_l, \tilde{A}_k)
$$

as well as

$$
\rho(\tilde{f}_k^j, A_k, s_k|h^{k-1}) = \sum_{(\theta_0, \ldots, \theta_{t-1})=(s_0, \ldots, s_{t-1})} \prod_{l=0}^{t-1} \psi_l^{\theta_l-1}(\theta_l) \tau_{\pi_{qu}(\theta_l)}(f_l, A_l) \psi_l^{\theta_l-1}(\theta_l) \tau_{\pi_{qu}(\theta_l)}(f_l, A_l) \tau_{\pi_{qu}(\theta_l)}(f_l, \tilde{A}_k).
$$

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We put the last three formulas together and rearrange to obtain
\[
\rho_t(f_t, A_t, s_t|H^{t-1}) = \frac{\sum_{s_0, \ldots, s_t} \prod_{k=0}^{t-1} \psi^{	heta_k-1} f_k \tau_{\pi_{qu}(\theta)}(f_k, A_k) \psi^{\theta_k-1} \tau_{\pi_{qu}(\theta)}(f_k, A_k)}{\sum_{s_0, \ldots, s_t} \prod_{k=0}^{t-1} \psi^{\theta_k-1} \tau_{\pi_{qu}(\theta)}(f_k, A_k)}.
\]

But note that for all \(\theta_k \in \Theta_k\) it holds
\[
\sum_{j=1}^{m} \tau_{\pi_{qu}(\theta)}(\tilde{j}, \tilde{A}_k) = \sum_{j=1}^{m} \tau_{\pi_{qu}(\theta)}((q', u') \in \Delta(S_k) \times \mathbb{R}^{X_k} : f_k \in M(M(\tilde{A}_k; \pi_{qu}(\theta_k))); (q', u'))
\]
\[
= \sum_{\theta_k \in \mathcal{B}} \tau_{\pi_{qu}(\theta)}((q', u') \in \Delta(S_k) \times \mathbb{R}^{X_k} : f_k \in M(M(A_k; \pi_{qu}(\theta_k))); (q', u'))
\]
\[
= \tau_{\pi_{qu}(\theta)}(f_k, A_k).
\]

By plugging this into the formula for \(\rho_t(f_t, A_t, s_t|H^{t-1})\) we see that
\[
\rho_t(f_t, A_t, s_t|H^{t-1}) = \rho_t(f_t, A_t, s_t|H^{t-1}).
\]

\[\square\]

**Claim 3.** \(\rho\) satisfies History Continuity.

**Proof.** Fix any \((f_t, A_t, s_t)\) and \(h^{t-1} = (f_0, A_0, s_0; \ldots; f_{t-1}, A_{t-1}, s_{t-1}) \in H^{t-1}\). Let \(\Theta_{t-1}(h^{t-1}) \subseteq \Theta_{t-1}\) denote the set of period-\((t-1)\) states that are consistent with \(h^{t-1}\). Define \(\rho^{\theta_t-1}(f_t, A_t, s_t) = \sum_{\theta_t} \psi^{\theta_t-1} (\theta_t) \tau_{\pi_{qu}(\theta_t)}(f_t, A_t)\). By Lemma 4 we have
\[
\rho_t(f_t, A_t, s_t|h^{t-1}) = \frac{\sum_{s_0, \ldots, s_t} \prod_{k=0}^{t-1} \psi^{\theta_k-1} f_k \tau_{\pi_{qu}(\theta)}(f_k, A_k) \sum_{s_0, \ldots, s_{t-1}} \prod_{k=0}^{t-1} \psi^{\theta_k-1} \tau_{\pi_{qu}(\theta)} (f_k, A_k)}{\sum_{s_0, \ldots, s_t} \prod_{k=0}^{t-1} \psi^{\theta_k-1} \tau_{\pi_{qu}(\theta)}(f_k, A_k)}.
\]

We see that \(\rho_t(f_t, A_t, s_t|h^{t-1}) \in \text{co}\{\rho^{\theta_t-1}(f_t, A_t, s_t) : \theta_t \in \Theta_{t-1}(h^{t-1})\}\). Fix any \(\theta_t \in \Theta_{t-1}(h^{t-1})\). To prove the claim it suffices to show that
\[
\rho^{\theta_t-1}(f_t, A_t, s_t) \in \{\lim_n \rho_t(f_t, A_t, s_t|h^{t-1}) : h^{t-1} \rightarrow m h^{t-1}, h^{t-1} \in \mathcal{H}^t\}.
\]

To this end, let \(\text{pred}(\theta_{t-1}) = (\theta_0, \ldots, \theta_{t-2})\) and let \(\bar{h}^{t-1} = (B_0, g_0, s_0; \ldots; B_{t-1}, g_{t-1}, s_{t-1}) \in \mathcal{H}^t\) be a separating history for \(\theta_{t-1}\). By Lemma 6 for each \(k = 0, \ldots, t-1\) we can find sequences \(A^n_k \in A^n_k(\bar{h}^{k-1})\) and \(f^n_k \in A^n_k\) with \(f^n_k \rightarrow m f_k\) and \(QU_{\theta_{t-1}}(A^n_k, f^n_k, s_k) = \{\pi_{qu}(\theta_k)\}\) for all \(n\) and all \(k = 0, \ldots, t-1\). Working backwards from \(k = t-2\) we can inductively replace \(A^n_k\) and \(f^n_k\) with a mixture putting small weight on a constant act yielding \((z, A^a_{k+1})\) for some \(z\) so as to ensure that \(A^a_{k+1} \in \text{supp}^A(f^n_k(s_k))\), irrespective of \(s_k \in S_k\). This can be done maintaining the previous properties of \(A^n_k\) and \(f^n_k\).

By construction it follows \(h^{t-1} = (A^n_0, f^n_0, s_0; \ldots; A^n_{t-1}, f^n_{t-1}, s_{t-1}) \in \mathcal{H}^t(A_t)\) and this is also a separating history for \(\theta_{t-1}\).
By Lemma 4 the latter fact implies for each $n$ that
\[
\rho_t(f_t, A_t, s_t|h_{n}^{t-1}) = \sum_{\theta_t} \left( \prod_{k=0}^{t-1} \psi_{k}^{\theta_{k-1}}(\theta_{k})\tau_{\pi_u(\theta_{k})}(f_{k}^{n}, A_{k}^{n}) \right) \cdot \prod_{k=0}^{t-1} \psi_{k}^{\theta_{k-1}}(\theta_{k})\tau_{\pi_u(\theta_{k})}(f_{k}^{n}, A_{k}^{n}) \psi_{\theta_t}^{t-1}(\theta_t)\tau_{\pi_u(\theta_t)}(f_t, A_t) \\
= \psi_{\theta_t}^{t-1}(\theta_t)\tau_{\theta_t}(f_t, A_t) \\
= \rho_t^{\theta_t-1}(f_t, A_t, s_t).
\]

The desired claim follows since $h_{n}^{t-1} \rightarrow^{m} h_{t-1}^{t-1}$.

\[\square\]

D.2 Proofs for the Comparative Statics part

D.2.1 Proof of Proposition 5

This is a trivial application of Lemma 24 in the online appendix.

D.2.2 Proof of Proposition 6

This is a direct implication of the Proof of the Representation Theorems for Evolving SEU and Gradual Learning (Theorems 2 and 3 in the main body of the paper) as well as Theorem 1 in [Dillenberger et al ’14].