ON THE SECOND-ORDER TANGENT BUNDLE WITH DEFORMED 2-ND LIFT METRIC

ABDULLAH MAĞDEN, KUBRA KARACA, AND AYDIN GEZER

Abstract. Let \((M, g)\) be a pseudo-Riemannian manifold and \(T^2M\) be its the second-order tangent bundle equipped with the deformed 2–nd lift metric \(\tilde{g}\) which obtained from the 2–nd lift metric by deforming the horizontal part with a symmetric \((0, 2)\)–tensor field \(c\). In the present paper, we first compute the Levi-Civita connection and its Riemannian curvature tensor field of \((T^2M, g)\). We give necessary and sufficient conditions for \((T^2M, g)\) to be semi-symmetric. Secondly, we show that \((T^2M, g)\) is a plural-holomorphic \(B\)–manifold with the natural integrable nilpotent structure. Finally, we get the conditions under which \((T^2M, g)\) with the 2–nd lift of an almost complex structure is an anti-Kähler manifold.

AMS Mathematics Subject Classification (2010): 53C07, 53C15, 53C35.

Keywords: Anti-Kähler manifold, Deformed 2–nd lift metric, Killing vector field, plural-holomorphic \(B\)–manifold, Semi-symmetry.

1. Introduction

Given an \(n\)–dimensional manifold \(M\), a second-order tangent bundle \(T^2M\) over \(M\) can be constructed from the equivalent classes of curves on \(M\) which agree up to their acceleration (for details, see [5] and [15]). Moreover, in [5], it is proved that a second-order tangent bundle \(T^2M\) becomes a vector bundle over \(M\) if and only if \(M\) has a linear connection. The prolongations of tensor fields and connections given on \(M\) to its second-order tangent bundle \(T^2M\) was studied in [15]. Let \((M, g)\) be an \(n\)–dimensional pseudo-Riemannian manifold and \(T^2M\) be its second-order tangent bundle. The 2–nd lift metric on \(T^2M\) was defined and studied by Yano and Ishihara in [15].

We point out here and once that all geometric objects considered in this paper are supposed to be of class \(C^\infty\). In this section, we recall some fundamental facts on the second-order tangent bundle that are needed later.

The second-order tangent bundle \(T^2M\) of a differentiable manifold \(M\) is the \(3n\)–dimensional manifold as the set of all 2–jets of \(M\) determined by mappings of the real line \(\mathbb{R}\) into \(M\). The canonical projection \(\pi_2 : T^2M \rightarrow M\) defines the natural bundle structure of \(T^2M\) over \(M\). If we introduce the canonical projection \(\pi_{12} : T^2M \rightarrow TM\), then \(T^2M\) has a bundle structure over the tangent bundle \(TM\) with projection \(\pi_{12}\). In the paper, we use Einstein’s convention on repeated indices.

Let \((U, x^i)\) be a system of coordinates in \(M\) and \(F\) be a curve in \(U\) which locally expressed as \(x^i = F^i(t)\). If we take a 2–jet \(j^2F\) belonging to \(\pi_2^{-1}(U)\) and define

\[
    x^i = F^i(0), \quad y^i = \frac{dF^i}{dt}(0), \quad z^i = \frac{1}{2} \frac{d^2F^i}{dt^2}(0),
\]

then we have...
then the 2–jet \( j^2 F \) is expressed uniquely by the set \((x^i, y^j, z^j)\). Thus, \((x^i, y^j, z^j)\) is the system of coordinates induced in \(\pi_2^{-1}(U)\) from \((U, x^i)\). The coordinates \((x^i, y^j, z^j)\) in \(\pi_2^{-1}(U)\) are called the induced coordinates and sometimes denote them by \(\{\xi^A\}\), that is, by putting

\[
\xi^i = x^i, \quad \xi^j = y^j, \quad \xi^1 = z^1.
\]

The indices \(A, B, C, \ldots \) run over the range \(\{1, 2, \ldots, n; n+1, 2n+2, 3n+2, \ldots, 2n+1, 2n+2, 3n+2, \ldots, 3n\}\).

For a function \(f\) locally expressed by \(f = f(x)\) on \(M\), there corresponds on \(T^2 M\) the 0–th, the 1–st and the 2–nd lifts of the function \(f\) respectively defined by

\[
0 f = f(x); \quad 1 f = y^j \partial_j f(x); \quad 2 f = z^1 \partial_1 f(x) + \frac{1}{2} y^j y^i \partial_j \partial_i f(x)
\]

with respect to the induced coordinates \(\{\xi^A\}\), where \(\partial_i = \frac{\partial}{\partial x^i}\).

Given a vector field \(X = X^i \partial_i\) on \(M\), we define the the 0–th, the 1–st and the 2–nd lifts of \(X\) to \(T^2 M\) as follows: [15]

\[
0 X = X^i \partial_i,
\]

\[
1 X = X^i \partial_i + y^s \partial_s X^j \partial_j
\]

and

\[
2 X = X^i \partial_i + y^s \partial_s X^j \partial_j + \{z^s \partial_s X^j + \frac{1}{2} y^t y^s \partial_t \partial_s X^j\} \partial_j
\]

with respect to the induced coordinates \(\{\xi^A\}\), where \(\partial_i = \frac{\partial}{\partial x^i}\), \(\partial_x = \frac{\partial}{\partial y^x}\), \(\partial_s = \frac{\partial}{\partial z^s}\).

By (1.2)–(1.4) and (1.1) we have directly

\[
0 X^0 f = 0^0 X f = 0^0 X^1 f = 0^0 (X f),
\]

\[
1 X^0 f = 1^0 X f = 0^0 X^1 f = 1^0 (X f),
\]

\[
2 X^0 f = 1^0 X^1 f = 2^0 (X f),
\]

for any vector field \(X\) and function \(f\) on \(M\).

For the Lie bracket on \(T^2 M\) in terms of the lifts of vector fields \(X, Y\) on \(M\), we have the following formulas: [15]

\[
[0 X, 0 Y] = 0,
\]

\[
[0 X, 1^0 Y] = 0, \quad [1^1 X, 0 Y] = 0, \quad [1^1 X, 1^1 Y] = 1^1 [X, Y]
\]

\[
[1^0 X, 1^0 Y] = 0, \quad [1^1 X, 1^0 Y] = 1^0 [X, Y].
\]

**Remark 1.** The 2–nd lift defined by (1.4) determines an isomorphism of the Lie algebra of vector fields on \(M\) into the Lie algebra of vector fields on \(T^2 M\).

2. THE DEFORMED 2–ND LIFT METRIC ON THE SECOND-ORDER TANGENT BUNDLE

The 0–th, the 1–st and the 2–nd lifts of a pseudo-Riemannian metric on a manifold \(M\) to the second-order tangent bundle \(T^2 M\) is respectively given by

\[
0 g = \begin{pmatrix}
g_{ij} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
ON THE SECOND-ORDER TANGENT BUNDLE WITH DEFORMED 2-ND LIFT METRIC

(2.1) \[ L^g = \begin{pmatrix} y^*\partial_{s}g_{ij} & g_{ij} & 0 \\ g_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \geq \]

(2.2) \[ II^g = \begin{pmatrix} z^*\partial_{s}g_{ij} + \frac{1}{2}y^*y^*\partial_{s}g_{ij} & y^*\partial_{s}g_{ij} & g_{ij} \\ y^*\partial_{s}g_{ij} & g_{ij} & 0 \\ g_{ij} & 0 & 0 \end{pmatrix} \]

with respect to the induced coordinates \( \{\xi^A\} \), where \( g_{ij} \) denote local components of \( g \) on \( M \). By using the 0-th lift of a symmetric (0, 2)-tensor field \( c \) on \( (M, g) \) to \( T^2M \), the deformed 2-nd lift metric on \( T^2M \) is defined by \( \overline{g} = II^g + 0_c \), that is,

\[ \overline{g} = II^g + 0_c = \begin{pmatrix} z^*\partial_{s}g_{ij} + \frac{1}{2}y^*y^*\partial_{s}g_{ij} + c_{ij} & y^*\partial_{s}g_{ij} & g_{ij} \\ y^*\partial_{s}g_{ij} & g_{ij} & 0 \\ g_{ij} & 0 & 0 \end{pmatrix} \]

with respect to the induced coordinates \( \{\xi^A\} \), where \( c_{ij} \) are local components of \( c \) on \( M \). Also note that the deformed 2-nd lift metric is a pseudo-Riemannian metric.

Using (1.2), (1.4) and (2.2), we get:

**Proposition 1.** Let \( (M, g) \) be a pseudo-Riemannian manifold and \( T^2M \) be its second-order tangent bundle equipped with the deformed 2-nd lift metric \( \overline{g} \). For any vector field \( X, Y \) on \( M \),

i) \( \overline{g}(0X,0Y) = 0 \), ii) \( \overline{g}(0X,IY) = 0 \), iii) \( \overline{g}(0X,IIY) = 0 \) \( (g(X, Y)) \)

iv) \( \overline{g}(X,X) = 0 \) \( (g(X, X)) \), v) \( \overline{g}(X,Y) = I (g(X, Y)) \),

vi) \( \overline{g}(X,Z) = II (g(X, Y)) + 0 \) \( (c(X, Z)) \).

Let us denote by \( L_{X,Y} \) the operator of Lie derivation with respect to any vector field \( X \) on \( T^2M \). Consider \( L_{X,Y} \) for arbitrary vector fields \( \nabla, Z \) on \( T^2M \), we have

\[ (L_{0,Y} \nabla)(IIY, IIZ) = L_{0,Y}(\overline{g}(IIY, IIZ)) \]

\[ = L_{0,Y}(I_{II}Y, IIZ) - \overline{g}(0IIY, I_{II}Z) \]

\[ = L_{0,Y}(I_{II}Y, IIZ) - \overline{g}(0IIY, I_{II}Z) \]

\[ = 0(\nabla(g(Y, Z))) - 0(g(L_{X,Y} Z)) - 0(g(Y, L_{X,Y} Z)) \]

\[ (L_{1,Y} \nabla)(IIY, IIZ) = L_{1,Y}(\overline{g}(IIY, IIZ)) \]

\[ = L_{1,Y}(I_{II}Y, IIZ) - \overline{g}(I_{II}Y, I_{II}Z) \]

\[ = L_{1,Y}(I_{II}Y, IIZ) - \overline{g}(I_{II}Y, I_{II}Z) \]

\[ = I_{II}(L_{X,Y} Z) = I_{II}(L_{X,Y} Z) \]

\[ (L_{II,Y} \nabla)(IIY, IIZ) = L_{II,Y}(\overline{g}(IIY, IIZ)) \]

\[ = L_{II,Y}(I_{II}Y, IIZ) - \overline{g}(IIY, I_{II}Z) \]

\[ = L_{II,Y}(I_{II}Y, IIZ) - \overline{g}(IIY, I_{II}Z) \]

\[ = II(L_{X,Y} Z) - II(g(L_{X,Y} Z)) - II(g(Y, L_{X,Y} Z)) \]

\[ = II(L_{X,Y} Z) + 0(L_{X,Y} Z) \]
As is known, any vector field \( X \) on a (pseudo-)Riemannian manifold \((M, g)\) is a Killing vector field if and only if \( L_X g = 0 \). Hence, from the relations above we obtain the following result.

**Proposition 2.** Let \((M, g)\) be a pseudo-Riemannian manifold and \( T^2M \) be its second-order tangent bundle equipped with the deformed 2–nd lift metric \( \mathcal{F} \).

i) The 0–th and 1–st lifts of a vector field \( X \) on \( M \) are both Killing vector fields on \((T^2M, \mathcal{F})\) if and only if \( X \) is a Killing vector field on \((M, g)\).

ii) The 2–nd lift of a vector field \( X \) on \( M \) is a Killing vector field on \((T^2M, \mathcal{F})\) if and only if \( X \) is a Killing vector field on \((M, g)\) and \( L_X c = 0 \).

3. The Levi-Civita Connection and its curvature tensor field

Let \( M \) be a pseudo-Riemannian manifold with a pseudo-Riemannian metric and \( \nabla \) be the Levi-Civita connection determined by \( g \). Now consider a global linear connection \( \nabla \) on \( T^2M \) denoted by

\[
\nabla_X^Y = {}^H \nabla_X^Y + {}^0 H(X, Y)
\]

for all vector fields \( X, Y \) on \( T^2M \), where \( H \) is a \((1, 2)\)–tensor field on \( M \). The 2–nd lift of the Levi-Civita connection \( \nabla \) to \( T^2M \) satisfies

\[
{}^H \nabla_{\sigma X}^0 Y = 0, {}^H \nabla_{\sigma X}^I Y = 0, {}^H \nabla_{\sigma X}^{II} Y = 0(\nabla_X Y),
\]

\[
{}^H \nabla_{i X}^0 Y = 0, {}^H \nabla_{i X}^I Y = 0(\nabla_X Y), {}^H \nabla_{i X}^{II} Y = l(\nabla_X Y),
\]

\[
{}^H \nabla_{i I X}^0 Y = 0(\nabla_X Y), {}^H \nabla_{i I X}^I Y = l(\nabla_X Y),
\]

\[
{}^H \nabla_{i I X}^{II} Y = {}^H \nabla_{i I X}^0 Y + {}^0 H(\nabla_X Y).
\]

Here, we also use the following relations: [15]

\[
{}^0 H(0 X, 0 Y) = {}^0 H(0 X, I Y) = {}^0 H(0 X, II Y) = 0,
\]

\[
{}^0 H(I X, 0 Y) = {}^0 H(I X, I Y) = {}^0 H(I X, II Y) = 0,
\]

\[
{}^0 H(II X, 0 Y) = {}^0 H(II X, I Y) = 0, {}^0 H(II X, II Y) = {}^0 H(\nabla_X Y).
\]

We shall calculate the torsion tensor of the linear connection \( \nabla \). The torsion tensor \( \mathcal{T} \) of \( \nabla \) is, by definition, given by

\[
\mathcal{T}(I I X, I I Y) = \nabla_{I I X}^{II} Y - \nabla_{I I Y}^{II} X - [I I X, I I Y]
\]

\[
= {}^I I (\nabla_X Y) + {}^0 H(\nabla_X Y) - {}^I I (\nabla_Y X)
\]

\[
= {}^0 H(\nabla_X Y) - {}^I I (\nabla_Y X)
\]

\[
= {}^0 H(\nabla_X Y) - {}^I I (\nabla_Y X).
\]
Next, taking covariant derivation of the deformed 2–nd lift metric \( g \) with respect to the linear connection \( \nabla \), we get

\[
(3.4) \quad \left( \nabla_{i\ell} g \right) ^{II} (Y, Z) = \nabla^{II} (g(Y, Z)) - \nabla_{i\ell} (\nabla^{II} Y, Z) - \nabla^{II} (\nabla_{i\ell} Y, Z) - \nabla^{II} (\nabla_{i\ell} Y, Z) - \nabla^{II} (\nabla_{i\ell} Y, Z)
\]

Hence, we get

\[
(3.5) \quad g(H(X, Y), Z) = \frac{1}{2} \left[ (\nabla_{X} c)(Y, Z) + (\nabla_{Y} c)(X, Z) + (\nabla_{Z} c)(X, Y) \right].
\]

Hence, we get

**Proposition 3.** Let \((M, g)\) be a pseudo-Riemannian manifold and \( T^2M \) be its second-order tangent bundle equipped with the deformed 2–nd lift metric \( g \). Under the condition \( (3.3) \), the linear connection \( \nabla \) is the Levi-Civita connection of \( g \).

Let \( X \) be a vector field on \( M \) with a linear connection \( \nabla \). It is well-known that \( X \) is an affine Killing vector field if and only if \( L_{X} \nabla = 0 \). Taking the Lie derivation of the Levi-Civita connection \( \nabla \) with respect to the vector fields \( ^0X, ^{1}X \) and \( ^{II}X \), we have

\[
(L_{0}X \nabla) ^{II} (Y, Z) = L_{0}X \left( \nabla^{II} Y, Z \right) - \nabla^{II}_{0}(X, Y) + \nabla^{II}_{0}(X, Y) - \nabla^{II}_{0}(X, Y) + \nabla^{II}_{0}(X, Y)
\]

\[
(L_{1}X \nabla) ^{II} (Y, Z) = L_{1}X \left( \nabla^{II} Y, Z \right) - \nabla^{II}_{1}(X, Y) + \nabla^{II}_{1}(X, Y) - \nabla^{II}_{1}(X, Y) + \nabla^{II}_{1}(X, Y)
\]
\[(L_{\iota X} \nabla)^{II} (Y, Z) = L_{\iota X} (\nabla_{\iota Y}^{II} Z) - \nabla_{\iota Y} (L_{\iota X}^{II} Z) - \nabla_{[\iota X, \iota Y]}^{II} Z \]
\[= L_{\iota X} (\nabla_{\iota Y}^{II} Z + 0(H(Y, Z))) - \nabla_{\iota Y}^{II} (L_{\iota X} Z) - \nabla_{[\iota X, \iota Y]}^{II} Z \]
\[= L_{\iota X}^{II} (\nabla_{\iota Y} Z) + L_{\iota X} 0(H(Y, Z)) - \nabla_{\iota Y}^{II} (L_{\iota X} Z) \]
\[= 0(H(Y, L_{\iota X} Z)) - \nabla_{[\iota X, \iota Y]}^{II} Z - 0(H([X, Y], Z)) \]
\[= L_{\iota X}^{II} (\nabla_{\iota Y} Z) - \nabla_{\iota Y}^{II} (L_{\iota X} Z) - \nabla_{[\iota X, \iota Y]}^{II} Z + L_{\iota X} 0(H(Y, Z)) \]
\[= 0(H(Y, L_{\iota X} Z)) - \nabla_{[\iota X, \iota Y]}^{II} Z - 0(H([X, Y], Z)) \]
\[= L_{\iota X}^{II} (\nabla_{\iota Y} Z) + 0(L_{\iota X} (H(Y, Z))) - H(Y, L_{\iota X} Z) - H(L_{\iota X} Y, Z) \]
\[= L_{\iota X}^{II} (\nabla_{\iota Y} Z) + 0((L_{\iota X} H)(Y, Z)). \]

Thus, the above relations give the following.

**Proposition 4.** Let \((M, g)\) be a pseudo-Riemannian manifold and \(T^2 M\) be its second-order tangent bundle equipped with the deformed \(2\text{-nd lift metric} \overline{g}\).

i) The \(0\text{-th and } 1\text{-st lifts of a vector field } X \text{ on } M \text{ are both affine Killing vector fields on } (T^2 M, \overline{g}) \text{ if and only if } X \text{ is an affine Killing vector field on } (M, g).

ii) The \(2\text{-nd lift of a vector field } X \text{ on } M \text{ is an affine Killing vector field on } (T^2 M, \overline{g}) \text{ if and only if } X \text{ is an affine Killing vector field on } (M, g) \text{ and } L_X H = 0.\)

For the Riemannian curvature tensor field \(\overline{R}\) of the Levi-Civita connection \(\nabla\), we obtain

\[
\overline{R}^{II}(X, \iota Y) = \nabla_{\iota Y} [L_{\iota X}^{II}(\nabla_{\iota Y} Z)] - \nabla_{\iota Y} [L_{\iota X}^{II} Z] - \nabla_{[\iota X, \iota Y]}^{II} Z
\]
\[
\overline{R}^{II}(X, \iota Y) = L_{\iota X}^{II}(\nabla_{\iota Y} Z) + L_{\iota X} 0(H(Y, Z)) - \nabla_{\iota Y}^{II} (L_{\iota X} Z) \]
\[
\overline{R}^{II}(X, \iota Y) = 0(H(Y, L_{\iota X} Z)) - \nabla_{[\iota X, \iota Y]}^{II} Z - 0(H([X, Y], Z)) \]
\[
\overline{R}^{II}(X, \iota Y) = L_{\iota X}^{II}(\nabla_{\iota Y} Z) - \nabla_{\iota Y}^{II} (L_{\iota X} Z) - \nabla_{[\iota X, \iota Y]}^{II} Z + L_{\iota X} 0(H(Y, Z)) \]
\[
\overline{R}^{II}(X, \iota Y) = 0(H(Y, L_{\iota X} Z)) - \nabla_{[\iota X, \iota Y]}^{II} Z - 0(H([X, Y], Z)) \]
\[
\overline{R}^{II}(X, \iota Y) = L_{\iota X}^{II}(\nabla_{\iota Y} Z) + 0(L_{\iota X} (H(Y, Z))) - H(Y, L_{\iota X} Z) - H(L_{\iota X} Y, Z) \]
\[
\overline{R}^{II}(X, \iota Y) = L_{\iota X}^{II}(\nabla_{\iota Y} Z) + 0((L_{\iota X} H)(Y, Z)). \]

from which, using \(\nabla_X Y - \nabla_Y X = [X, Y]\) we get

\[
\overline{R}^{II}(X, \iota Y) = L_{\iota X}^{II}(R(Y, Z)) + 0((\nabla_{\iota X} H)(Y, Z) - (\nabla_{\iota Y} H)(X, Z)). \]

Thus we state following result.

**Proposition 5.** Let \((M, g)\) be a pseudo-Riemannian manifold and \(T^2 M\) be its second-order tangent bundle equipped with the deformed \(2\text{-nd lift metric} \overline{g}.\) \((T^2 M, \overline{g})\) is flat if and if the base manifold \(M\) is flat and the condition \((\nabla_{\iota X} H)(Y, Z) = (\nabla_{\iota Y} H)(X, Z)\) is fulfilled.
The $\text{(0,4)}$—Riemannian curvature tensor of the Levi-Civita connection $\nabla$ is as follows:

\[
\begin{align*}
\tilde{R}(I^I X, I^I Y, I^I Z, I^I W) &= g(\tilde{R}(I^I X, I^I Y) I^I Z, I^I W) \\
 &= g(I^I (R(X,Y)Z), I^I W) + g(0(\nabla_X H)(Y,Z) - (\nabla_Y H)(X,Z), I^I W) \\
 &= I^I (g(R(X,Y)Z,W)) + 0(c(R(X,Y)Z,W)) \\
&\quad + 0(g((\nabla_X H)(Y,Z) - (\nabla_Y H)(X,Z), W)) \\
&\quad + 0(\nabla_X H)(Y,Z,W) - (\nabla_Y H)(X,Z,W). \\
\end{align*}
\]

Given a manifold $M$ (dim$(M) \geq 3$) endowed with a linear connection $\nabla$ whose curvature tensor is signed as $\tilde{R}$, for any tensor field of type $(0,k), k \geq 1$, the tensor field $R(X,Y)S$ is expressed in the form:

\[
(R(X,Y)S)(X_1, X_2, ..., X_k) = -S(R(X,Y)X_1, X_2, ..., X_k) \\
- ... - S(X_1, X_2, ..., X_{k-1}, R(X,Y)X_k)
\]

for any vector fields $X_1, X_2, ..., X_k, Y, X, Y$ on $M$, where $R(X,Y)$ acts as a derivation on $S$. If $R(X,Y)S = 0$, then the manifold $M$ is said to be $S$ semi-symmetric with respect to the linear connection $\nabla$. A (pseudo-) Riemannian manifold $(M, g)$ such that its curvature tensor $\tilde{R}$ satisfies the condition

\[
R(X,Y)\tilde{R} = 0
\]

is called a semi-symmetric manifold. Also, note that locally symmetric manifold (\nabla R = 0) are semi-symmetric, but in general the converse is not true. The semi-symmetric manifold was first studied by Cartan. Nevertheless, Sinjukov first used the name “semi-symmetric” for manifolds satisfying the above curvature condition \cite{Sinjukov}. Later, Szabo gave the full local and global classification of semi-symmetric manifolds \cite{Szabo1, Szabo2}. Now we are interested in the semi-symmetry property of $T^2M$ with the deformed 2—nd lift metric $\overline{g}$. For the sake of simplicity we shall choose $c = g$ in the deformed 2—nd lift metric $\overline{g}$. In this case, the relation \cite{Simo} reduces to

\[
\tilde{R}(I^I X, I^I Y, I^I Z, I^I W) = I^I (R(X,Y,Z,W)) + 0(R(X,Y,Z,W)).
\]

**Theorem 1.** Let $(M, g)$ be a pseudo-Riemannian manifold and $T^2M$ be its second-order tangent bundle equipped with the deformed 2—nd lift metric $\overline{g} = I^I g + 0g$. $(T^2M, \overline{g})$ is semi-symmetric if and only if $(M, g)$ is semi-symmetric.

**Proof.** We consider the condition $\overline{R}(\overline{X}, \overline{Y})\tilde{R} = 0$ for all vector field $\overline{X}, \overline{Y}$ on $(T^2M, \overline{g})$. We calculate

\[
\begin{align*}
\big(\overline{R}(I^I X, I^I Y)\big)(I^I X_1, I^I X_2, I^I X_3, I^I X_4) &= -\tilde{R}(\overline{R}(I^I X, I^I Y) I^I X_1, I^I X_2, I^I X_3, I^I X_4) - \tilde{R}(I^I X_1, \overline{R}(I^I X, I^I Y) I^I X_2, I^I X_3, I^I X_4) \\
&\quad - \tilde{R}(I^I X_1, I^I X_2, \overline{R}(I^I X, I^I Y) I^I X_3, I^I X_4) - \tilde{R}(I^I X_1, I^I X_2, I^I X_3, \overline{R}(I^I X, I^I Y) I^I X_4) \\
&= -\tilde{R}(I^I (\overline{R}(X,Y) X_1), I^I X_2, I^I X_3, I^I X_4) - \tilde{R}(I^I X_1, I^I (\overline{R}(X,Y) X_2), I^I X_3, I^I X_4) \\
&\quad - \tilde{R}(I^I X_1, I^I X_2, I^I (\overline{R}(X,Y) X_3), I^I X_4) - \tilde{R}(I^I X_1, I^I X_2, I^I X_3, I^I (\overline{R}(X,Y) X_4))
\end{align*}
\]
\[ -\mathcal{L}_X \mathcal{L}_Y \text{g} + \mathcal{L}_Y \mathcal{L}_X \text{g} - 2 \mathcal{L}_{[X,Y]} \text{g} = 0 \]

which completes the proof. \qed

4. Plural-holomorphic $B$–manifolds

A nilpotent structure on $M$ is a $(1,1)$–tensor field $\gamma$ such that $\gamma^3 = 0$ ($\gamma \neq 0$). A pure metric (for pure tensors, see [6]) with respect to the nilpotent structure is a pseudo-Riemannian metric $g$ such that

\[ g(\gamma X, Y) = g(X, Y) \]

for any vector fields $X, Y$ on $M$. Metrics of this type have also been studied under the name $B$-metrics [11, 12, 13, 14], since the metric tensor $g$ with respect to the structure $\gamma$ is a $B$-tensor according to the terminology accepted in [2]. If $(M, \gamma)$ is a manifold with a $B$-metric and a nilpotent structure, we say that $(M, \gamma, g)$ is an almost $B$-manifold. If $\gamma$ is integrable, we say that $(M, \gamma, g)$ is a $B$-manifold. A plural-holomorphic $B$–manifold [4] can be defined as a triple $(M, \gamma, g)$ which consists of a smooth manifold $M$ endowed with a nilpotent structure $\gamma$ and a $B$-metric $g$ such that $\Phi_\gamma g = 0$, where $\Phi_\gamma$ is the Tachibana operator [10, 16]:

\[ (\Phi_\gamma g)(X, Y, Z) = (\gamma X)(g(Y, Z)) - X(g(\gamma Y, Z)) + g((L_Y \gamma)X, Z) + g(Y, (L_Z \gamma)X). \]

Recall that there exists a $(1,1)$–tensor field $\tilde{\gamma}$ on $T^2M$ which has components of the form

\[ \tilde{\gamma} = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \]

with respect to the induced coordinates $\{\xi^A\}$, where $I$ being unit matrix. The tensor field satisfies $\tilde{\gamma}^3 = 0$, that is, $0 T^2M$ has a natural integrable nilpotent structure. The natural integrable nilpotent structure has the properties

\[ \tilde{\gamma}^0 X = 0, \tilde{\gamma}^I X = 0, \tilde{\gamma}^{II} X = \xi^I X \]

which characterize $\tilde{\gamma}$. We compute, for any vector fields $X, Y$ on $M$

\[ \overline{g}(\tilde{\gamma}^{II} X, \xi^I Y) = \overline{g}(\xi^I X, \tilde{\gamma}^{II} Y) = \xi^I (g(X, Y)) \]

\[ \overline{g}(\tilde{\gamma}^{II} X, \tilde{\gamma}^{II} Y) = \overline{g}(\tilde{\gamma}^{II} X, \xi^I Y) = \xi^I (g(X, Y)) \]

that is, the the deformed 2–nd lift metric $\overline{g}$ is a $B$–metric with respect to $\tilde{\gamma}$. Hence $(T^2M, \tilde{\gamma}, \overline{g})$ is a $B$–manifold. Applying the Tachibana operator $\Phi_\gamma$ to $\overline{g}$, we get

\[ (\Phi_\gamma \overline{g})(\xi^I X, \tilde{\gamma}^{II} Y, \tilde{\gamma}^{II} Z) = (\tilde{\gamma}^{II} X)(\overline{g}(\tilde{\gamma}^{II} Y, \tilde{\gamma}^{II} Z)) = \xi^I (X(g(Y, Z))) \]

Hence we state the following theorem.
**Theorem 2.** Let \((M, g)\) be a pseudo-Riemannian manifold and \(T^2M\) be its second-order tangent bundle equipped with the deformed 2-nd lift metric \(\hat{\mathcal{g}}\) and the natural integrable nilpotent structure \(\hat{\gamma}\). The triple \((T^2M, \hat{\gamma}, \hat{\mathcal{g}})\) is a plural-holomorphic \(B\)-manifold.

**5. Anti-Kähler structures on \(T^2M\)**

An almost complex anti-Hermitian manifold \((M, J, g)\) is a real \(2k\)-dimensional differentiable manifold \(M\) with an almost complex structure \(J\) and a pseudo-Riemannian metric \(g\) such that:

\[ g(JX, Y) = g(X, JY) \]

for all vector fields \(X, Y\) on \(M\). An anti-Kähler manifold can be defined as a triple \((M, J, g)\) which consists of a smooth manifold \(M\) endowed with an almost complex structure \(J\) and an anti-Hermitian metric \(g\) such that \(\nabla J = 0\), where \(\nabla\) is the Levi-Civita connection of \(g\). It is well known that the condition \(\nabla J = 0\) is equivalent to \(C\)-holomorphicity (analyticity) of the anti-Hermitian metric \(g\), i.e. \(\Phi_Jg = 0\) [3]. Since in dimension 2 an anti-Kähler manifold is flat, we assume in the sequel that \(\text{dim } M \geq 4\).

The 2-nd lift of a \((1, 1)\)-tensor field \(J\) to \(T^2M\) has the followings

\[
II J(II X) = II (JX), \quad II J(I X) = I (JX), \quad II J(0 X) = 0 (JX)
\]

for any \(X\) on \(M\). Moreover, it is well known that if \(J\) is an almost complex structure on \((M, g)\), then \(II J\) is an almost complex structure on \(T^2M\) [15]. Now we prove the following theorem.

**Theorem 3.** Let \((M, J, g)\) be an anti-Kähler manifold. Then \(T^2M\) is an anti-Kähler manifold equipped with the deformed 2-nd lift metric \(\mathcal{g}\) and the almost complex structure \(II J\) if and only if the symmetric \((0, 2)\)-tensor field \(c\) on \(M\) is a holomorphic tensor field with respect to the almost complex structure \(J\).

**Proof.** Let \((M, J, g)\) be a anti-Kähler manifold. Then we have

\[
\mathcal{g}(II J(II X), II Y) - \mathcal{g}(II X, II J(II Y)) = \mathcal{g}(II (JX), II Y) - \mathcal{g}(II X, II (JY)) = c(JX, Y) - c(X, JY).
\]

From the last equations, the deformed 2-nd lift metric \(\mathcal{g}\) is anti-Hermitian with respect to \(II J\) if and only if the symmetric \((0, 2)\)-tensor field \(c\) is pure with respect to \(J\).
Now, we are interested in the holomorphy property of the deformed 2–nd lift metric $\mathcal{J}$ with respect to $J$. We calculate
\[
(\Phi_{\mathcal{J}} \mathcal{J})(II X, II Y, II Z) \\
= (II JX)(\mathcal{J}(II Y, II Z)) - II X(\mathcal{J}(II JY, II Z)) \\
+ \mathcal{J}(L_{II Y} II J)II X, II Z) + \mathcal{J}(II Y, (L_{II Z} II J)II X) \\
= \{\mathcal{J}(\mathcal{J})(g(X, Y)) + (c(X, Y))\} - II X\{\mathcal{J}(g(JY, Z)) + (c(JY, Z))\} \\
+ II \{g((L_{II Y} J)X, Z)) + (c((L_{II Y} J)X, Z))\} + II \{g(Y, (L_{II Z} J)X)) + (c(Y, (L_{II Z} J)X))\} \\
= II \{(g(X, Y)) - X(g(JY, Z)) + g((L_{II Y} J)X, Z) + g(Y, (L_{II Z} J)X)\} \\
\]
Hence, from the relation above, since $(\Phi_{\mathcal{J}} g) = 0$, it follows that $\Phi_{\mathcal{J}} \mathcal{J} = 0$ if and only if $\Phi_{\mathcal{J}} c = 0$, that is, $c$ is holomorphic. This completes the proof. \hfill \Box

References

[1] M. de Leon, E. Vazquez, On the geometry of the tangent bundle of order 2. An. Univ. \textit{Bucuresti} Mat. \textbf{34} (1985), 40–48.
[2] A. P. Norden, On a certain class of four-dimensional A-spaces. Izv. Vuzov. Mat. \textbf{4} (1960), 145–157.
[3] M. Iscan, A. A. Salimov, On Kähler-Norden manifolds. Proc. Indian Acad. Sci. Math. Sci. \textbf{119} (2009), no. 1, 71–80.
[4] M. Iscan, A. Magden, On B-manifolds defined by algebra of plural numbers. Arab. J. Sci. \textit{Eng.} \textbf{35} (2010), Number 1D, 57-63.
[5] C.T.J. Dodson and M.S. Radivoiovici, Tangent and frame bundles of order two. Analele \textit{Stiintifice ale Universitatii "Al. I. Cuza"} \textbf{28} (1982), 63-71.
[6] A. Salimov, On operators associated with tensor fields, J. Geom. \textbf{99} (1–2) (2010), 107–145.
[7] N. S. Sinjukov, Geodesic mappings of Riemannian spaces (Russian). Publishing House “Nauka”, Moscow, 1979.
[8] Z. I. Szabo, Structure theorems on Riemannian spaces satisfying $R(X, Y).R = 0$. I. The local version, J. Differential Geom. \textbf{17} (1982), 531–582.
[9] Z. I. Szabo, Structure theorems on Riemannian spaces satisfying $R(X, Y).R = 0$. II. Global version, Geom. Dedicata \textbf{19} (1985), 65-108.
[10] S. Tachibana, Analytic tensor and its generalization. Tohoku Math. J. \textbf{12} (1960) no.2 208–221.
[11] V. V. Vishnevskii, Structures of projective spaces generated by affinor, Izv. Vuzov. Mat. \textbf{6} (1969), 35–46.
[12] V. V. Vishnevskii, Affinor structures of affine connection spaces, Izv. Vuzov. Mat. \textbf{1} (1970), 12–23.
[13] V. V. Vishnevskii, Integrable affinor structures and their plural interpretations, J. Math. Sci. \textbf{108} (2) (2002), 151–187.
[14] V. V. Vishnevskii, A. P. Shirokov, V. V. Shurygin, Spaces over algebras. Kazan Gos. University, Kazan, Russian, 1985.
[15] K. Yano, S. Ishihara, Tangent and cotangent bundles: differential geometry. Pure and Applied Mathematics, No. 16. Marcel Dekker, Inc., New York, 1973.
[16] K. Yano, M. Ako, On certain operators associated with tensor field, Kodai Math. Sem. Rep. \textbf{20} (1968) 414-436.
Ataturk University, Faculty of Science, Department of Mathematics, 25240, Erzurum-Turkey.
E-mail address: amagden@atauni.edu.tr

Ataturk University, Faculty of Science, Department of Mathematics, 25240, Erzurum-Turkey.
E-mail address: kubrakaraca91@gmail.com

Ataturk University, Faculty of Science, Department of Mathematics, 25240, Erzurum-Turkey.
E-mail address: agezer@atauni.edu.tr