THE WEYL BOUND FOR TRIPLE PRODUCT L-FUNCTIONS

VALENTIN BLOMER, SUBHAJIT JANA, AND PAUL D. NELSON

Abstract. Let \( \pi_1, \pi_2, \pi_3 \) be three cuspidal automorphic representations for the group \( \text{SL}(2, \mathbb{Z}) \), where \( \pi_1 \) and \( \pi_2 \) are fixed and \( \pi_3 \) has large analytic conductor. We prove a subconvex bound for \( L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3) \) of Weyl-type quality. Allowing \( \pi_3 \) to be an Eisenstein series we also obtain a Weyl-type subconvex bound for \( L(1/2 + it, \pi_1 \otimes \pi_2) \).

1. Introduction

1.1. Weyl-type subconvexity. Subconvexity estimates belong to the core topics in the theory of \( L \)-functions and are one of the most challenging testing grounds for the strength of existing technology. If \( C \) denotes the analytic conductor of the relevant \( L \)-function (restricted to the parameters of interest), then the Phragmén–Lindelöf principle gives the bound \( C^{1/4+\varepsilon} \) for the \( L \)-function on the central line \( \Re s = 1/2 \). In the most favourable cases, one can obtain an upper bound \( C^{1/6+\varepsilon} \), which we refer to as a Weyl-type subconvex bound. For instance, a classical result states that the Riemann zeta function satisfies the bound

\[
\zeta(1/2 + it) \ll \varepsilon (1 + |t|)^{1/6+\varepsilon}
\]

on the critical line. Based on work of Weyl [We], it was proved first by Hardy and Littlewood (cf. [Li]), and first written down by Landau [La] in a slightly refined form and generalized to all Dirichlet \( L \)-functions. Results of similar strength exist in the archimedean aspect for automorphic \( L \)-functions of degree 2, starting with the work of Good [Go] and culminating in the hybrid bound of Jutila and Motohashi [JM1]. We also have a Weyl-type bound in degree 4 in some limited cases pertaining to Rankin-Selberg \( L \)-functions, such as [JM2, LLY]

\[
L(1/2, f \otimes g) \ll_{g,\varepsilon} C(f)^{1/3+\varepsilon}
\]

for two cusp forms \( f, g \) for \( \text{SL}_2(\mathbb{Z}) \), where \( C(f) \) denotes the conductor of \( f \) as defined in §2.2.5 below. Although slightly better bounds are available for \( \text{GL}(1) \) [Bo, Mi], the Weyl exponent marks a natural barrier that has

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never been improved, and rarely been reached, beyond GL(1). We note that for some applications (see [GS], [Ma]), the essential input is a Weyl-type subconvex bound (or something approaching it), rather than merely any nontrivial subconvex bound.\footnote{Indeed, to show that the number of zeros on $i[1, \infty)$ of a holomorphic Hecke eigenform $f$ of weight $k$ tends to infinity as $k \to \infty$, the proof in [GS] needs $L(1/2 + it, f) \ll k^{1.335}$ with polynomial dependence in $t$.} We note also that Petrow and Young recently established Weyl-type subconvex bounds for GL(1) in the level aspect, improving spectacularly upon the decades-old Burgess-type bounds (see [PY1, PY2, Ne2]).

A celebrated result of Bernstein and Reznikov established for the first time subconvex bounds for certain $L$-functions of degree 8. For two fixed spherical cuspidal automorphic representations $\pi_1, \pi_2$ (i.e. generated by Maass forms for $\text{SL}_2(\mathbb{Z})$) and another spherical cuspidal automorphic representation $\pi_3$ of large analytic conductor $C(\pi_3)$ they proved [BR1, BR2]

\begin{equation}
\sum_{T \leq C(\pi_3)^{1/2} \leq T + T^{1/3}} L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3) \ll_{\pi_1, \pi_2, \varepsilon} T^{5/3 + \varepsilon},
\end{equation}

which implies (by non-negativity of the central value) in particular $L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3) \ll_{\pi_1, \pi_2, \varepsilon} C(\pi_3)^{5/6 + \varepsilon}$. The proof employs a beautiful combination of representation theory, invariant norms and asymptotic analysis of oscillatory Airy-type integrals. In fact, their result is really an estimate for triple product periods, as $L$-functions enter only through the period formula of Watson–Ichino [Ic].

We observe, however, that (1.1) is not optimal. The Lindelöf hypothesis suggests the Weyl-type bound with an exponent $4/3$ instead of $5/3$. That such a result might be within reach was indicated by Suvitie [Su2]. For a fixed holomorphic cusp form $F$ of weight $k$ and a Maass form $h$ she showed

\begin{equation}
\sum_{T \leq C(h)^{1/2} \leq T + T^{1/3}} e^{\pi r_h} |\langle y^k |F|^2, h \rangle|^2 \ll_{F, \varepsilon} T^{2k - \frac{2}{3} + \varepsilon}
\end{equation}

which via the formula of Watson–Ichino translates into

\begin{equation}
\sum_{T \leq C(h)^{1/2} \leq T + T^{1/3}} L(1/2, F \otimes F \otimes h) \ll_{F, \varepsilon} T^{4/3 + \varepsilon}.
\end{equation}

This $L$-function is not primitive, as it factorizes into a degree 6 and a degree 2 $L$-function, but the same argument would work for $FG$ instead of $|F|^2$ in (1.2). More seriously, however, the proof starts by replacing $F$ with a holomorphic Poincaré series and unfolding the inner product, a route that is not available in general. In fact, an attempt to generalize this to Maass forms remained incomplete [Su1] and seems not to work. In particular, the work of Bernstein–Reznikov remained unimproved.

In this paper we establish the Weyl-type bound for triple product $L$-functions in a uniform fashion for all combinations of local types at infinity, i.e., any of the three factors can be holomorphic or Maass. As mentioned
before, the Weyl-type bound marks the natural limit of all present day approaches to subconvexity. The key novelty in our paper is the method. We combine in a substantial way representation theory, local harmonic analysis, and analytic number theory to establish a robust method for the subconvexity problem for triple product $L$-functions.

**Theorem 1.** Let $\pi_1, \pi_2$ be two fixed cuspidal automorphic representations for the group $\text{SL}_2(\mathbb{Z})$. Let $\pi_3$ run over cuspidal automorphic representations for $\text{SL}_2(\mathbb{Z})$ with conductor satisfying $T \leq C(\pi_3)^{1/2} \leq T + T^{1/3}$. Then

$$
\sum_{T \leq C(\pi_3)^{1/2} \leq T + T^{1/3}} L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3) \ll_{\pi_1, \pi_2, \epsilon} T^{4/3 + \epsilon},
$$

in particular

$$L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3) \ll_{\pi_1, \pi_2, \epsilon} C(\pi_3)^{2/3 + \epsilon}$$

for every $\epsilon > 0$.

An inspection of the proof shows that the dependence on the analytic conductors of $\pi_1, \pi_2$ is polynomial. Under Watson’s formula [Wat], the latter estimate translates to bounds for triple product integrals of Maaß forms $\varphi_j$ of eigenvalue $1/4 + t_j^2$ ($j = 1, 2, 3$):

$$
\left| \int_{\text{SL}_2(\mathbb{Z}) \backslash \mathbb{R}} \varphi_1 \varphi_2 \varphi_3 \right|^2 \ll_{\varphi_1, \varphi_2, \epsilon} |t_3|^{-2/3 + \epsilon} \exp(-\pi |t_3|),
$$

giving a further improvement beyond that in [BR2] on the general exponential decay bounds of [Sa].

We can allow $\pi_3$ to be an Eisenstein series, and our proof yields as a by-product a Weyl-type bound for Rankin-Selberg $L$-functions.

**Theorem 2.** Let $\pi_1, \pi_2$ be two fixed cuspidal automorphic representations for the group $\text{SL}_2(\mathbb{Z})$. Then

$$
\int_{T}^{T + T^{1/3}} |L(1/2 + it, \pi_1 \otimes \pi_2)|^2 dt \ll_{\pi_1, \pi_2, \epsilon} T^{4/3 + \epsilon},
$$

in particular

$$L(1/2 + it, \pi_1 \otimes \pi_2) \ll_{\pi_1, \pi_2, \epsilon} (1 + |t|)^{2/3 + \epsilon}$$

for every $\epsilon > 0$ and $t \in \mathbb{R}$.

For the rest of the paper, all implied constants may depend on $\epsilon$, and we suppress it in subsequent formulæ. The weaker bound $L(1/2 + it, \pi_1 \otimes \pi_2) \ll_{\pi_1, \pi_2} (1 + |t|)^{5/6 + \epsilon}$ is implicit in [BR2, Remarks 7.2.2.2] and was the best known result until now. By a method purely based on analytic number theory, the bound $L(1/2 + it, \pi_1 \otimes \pi_2) \ll_{\pi_1, \pi_2} (1 + |t|)^{15/16 + \epsilon}$ was recently shown in [ASS]. For bounds of triple product $L$-functions in the level aspect see [Ve, Hu].
1.2. Remarks.

1) Our results feature “pure” exponents of Weyl-type quality that are independent of bounds towards the Ramanujan conjecture or the Selberg eigenvalue conjecture. The proof uses at one place that the smallest non-zero Laplace eigenvalue is larger than $3/16$.

2) In principle, the proof produces an asymptotic formula. If $\psi$ is a sufficiently regular test function with “essential support” in $[T, T + H]$, e.g. $\psi(t) = \exp(-(t - T)^2H^{-2})$, and $t_{\pi_3} \asymp \sqrt{C(\pi_3)}$ (cf. (2.20)) denotes the spectral parameter of $\pi_3$, then with the same notation and under the same assumptions as in Theorem 1, one can relate

$$\sum_{\pi_3} \psi(t_{\pi_3}) \frac{L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3)}{L(1, \text{Ad}^2 \pi_3)} + \int_{t \in \mathbb{R}} \psi(t) \left| L(1/2 + it, \pi_1 \otimes \pi_2) \right| \frac{dt}{2\pi}$$

(1.3)

to

$$cL(1, \text{Ad}^2 \pi_1)L(1, \text{Ad}^2 \pi_2)TH + O(T^{3/2+\varepsilon}H^{-1/2})$$

for a suitable constant $c$ (depending on $\psi$).

3) The proof of Theorem 1 has the shape of a reciprocity formula as for instance in [BLM]. A spectral sum in a window $[T, T + H]$ as in (1.3) is ultimately transformed into a spectral sum of similar shape with spectral parameter up to $\ll T/H$ (see (5.30), (5.28), (5.13)). This is analogous to the discussion after (1.11) in [BLM], and a new instance of a reciprocity phenomenon. The optimal choice is $H = T^{1/3}$, in which case both spectral sums have length $T^{4/3}$. This yields the Weyl bound, and we see that the Weyl bound is indeed the natural limit from the point of view of spectral analysis. An abstract version of the underlying reciprocity formula is displayed in (1.7) below which features central $L$-values as well as their “canonical square roots”.

4) Implicit in the proof of Theorem 1 is an alternative description of the central triple product $L$-value $L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3)$ in terms of a certain shifted convolution problem very roughly of the shape

$$L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3) \approx \left| \frac{1}{t_{\pi_3}^{1/2}} \sum_{\nu \ll 1} \sum_{m \ll t_{\pi_3}^2} \lambda_{\pi_3}(m) \lambda_{\pi_2}(m + \nu) \lambda_{\pi_1}(\nu) m^{1/4} \exp \left( \pm 2it_{\pi_3} \sqrt{\frac{\nu}{m}} \right) \right|^2.$$  

(1.4)

The $\approx$ sign has to be interpreted in a broad sense (see §6 for details). In the generic range $m \asymp t_{\pi_3}^2$ the oscillatory factor is flat (see §2.4 for definition of flatness).
1.3. **Comparison with Bernstein–Reznikov and Michel–Venkatesh.**

It is instructive to compare our approach for studying the moment \((1.1)\) with those of Bernstein–Reznikov [BR1, BR2] and Michel–Venkatesh [MV].

To begin, we briefly sketch the approach to the subconvexity problem introduced by Bernstein–Reznikov. We borrow some presentation features from Michel–Venkatesh, see especially [MV, §1.1.1, §1.1.3]. The starting point of this approach is the triple product formula [Ic]: for unit vectors \(v_j \in \pi_j\) we have

\[
L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3) \frac{L(1, \text{Ad} \pi_1)L(1, \text{Ad} \pi_2)L(1, \text{Ad} \pi_3)}{L(1, \text{Ad}^2 \pi_1)L(1, \text{Ad}^2 \pi_2)L(1, \text{Ad}^2 \pi_3)} L_\infty(v_1, v_2, v_3)
= \left| \int_{g \in \text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R})} v_1 v_2 v_3(g) \, dg \right|^2
\]

for a suitable local factor \(L_\infty(v_1, v_2, v_3)\) (a constant multiple of a matrix coefficient integral).

For unit vectors \(v_1 \in \pi_1\) and \(v_2 \in \pi_2\) we consider the inner product identity

\[
\langle v_1 v_2, v_1 v_2 \rangle = \int_{g \in \text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R})} |v_1 v_2|^2(g) \, dg = \langle |v_1|^2, |v_2|^2 \rangle.
\]

By expanding each of these inner products over the spectrum of \(L^2(\text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R}))\) and applying \((1.5)\), we obtain a spectral identity of families of \(L\)-functions, roughly of the shape

\[
\sum_{\pi_3} h(\pi_3) L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3) 
\approx 1 + \sum_{\sigma} \tilde{h}(\sigma) \sqrt{L(1/2, \pi_1 \otimes \pi_1 \otimes \sigma)L(1/2, \pi_2 \otimes \pi_2 \otimes \sigma)}.
\]

Here \(\pi_3\) and \(\sigma\) run over cuspidal automorphic representations of \(\text{SL}_2(\mathbb{Z})\), the square roots of \(L\)-values are “canonical square roots” (in the sense [MV, §1.1.3]) and the meaning of “\(\approx\)” is that

- we have suppressed adjoint \(L\)-factors and other proportionality constants, and
- we have elided the contribution of the continuous spectrum and all degenerate terms except for the “expected main term” \(1\), which arises from the inner product \(\langle |v_1|^2, 1 \rangle \langle 1, |v_2|^2 \rangle = 1\) (up to volume factors).

The weight functions \(h\) and \(\tilde{h}\) depend upon the choice of vectors \(v_1\) and \(v_2\).

The weights \(h(\pi_3)\) and the \(L\)-values \(L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3)\) are known to be nonnegative, so if we can bound the right hand side of \((1.7)\) by \(O(1)\) (the natural limit, in view of the expected main term), then we deduce by dropping all but one term the estimate

\[
L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3) \ll 1/h(\pi_3).
\]
Given some $\pi_3$ with $\text{cond}(\pi_3)^{1/2} \asymp T$, we now face the optimization problem of choosing unit vectors $v_1$ and $v_2$ for which $h(\pi_3)$ is as large as possible, so that the bound (1.8) is as strong as possible. Bernstein–Reznikov [BR2, (2.6.3), Prop 9.1] showed (for $\pi_j$ spherical) that one may choose $v_1$ and $v_2$ so that $h(\pi_3)$ is roughly $T^{-5/3}$. This choice and a suitable bound for the global period eventually yields their estimate (1.1). Michel–Venkatesh [MV, §3.6-3.7] (for $\pi_1$ tempered and spherical) employed a simpler choice of vectors for which $h(\pi_3)$ is of size $T^{-2}$; for this choice, the estimate (1.8) only recovers the convexity bound, but Michel–Venkatesh managed to apply the amplification method to save a further small power of $T$ (in a more general “all aspects” setting).

To approach the Weyl bound $L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3) \ll T^{4/3+\varepsilon}$ using (1.7) would seem to require producing $v_1$ and $v_2$ for which $h(\pi_3)$ is at least $T^{-4/3}$, but the analysis of Bernstein–Reznikov strongly suggests that their lower bound $T^{-5/3}$ is best possible. To obtain a stronger lower bound thus requires a more flexible class of weight functions $h(\pi_3)$. Such a class may be obtained from the generalization of (1.6) to higher-rank tensors $\sum_j v_{1,j} \otimes v_{2,j} \in \pi_1 \otimes \pi_2$, namely

$$\sum_{j,k} \langle v_{1,j} v_{2,j}, v_{1,k} v_{2,k} \rangle = \sum_{j,k} \langle v_{1,j} \overline{v_{1,k}}, v_{2,k} \overline{v_{2,j}} \rangle.$$ 

Such tensors yield more flexible forms of (1.7). One could hope to find a tensor $\sum_j v_{1,j} \otimes v_{2,j}$ for which the corresponding weight $h(\pi_3)$ localizes on $\pi_3$ satisfying $T \leq \text{cond}(\pi_3)^{1/2} \leq T + T^{1/3}$ and for which the right hand side of the corresponding spectral identity as in (1.7) may be effectively bounded. To implement this idea in practice would require a careful study of the spaces of test functions $\{h\}$ and $\{\tilde{h}\}$ as well as the transform relating them. Unfortunately, such a study has not yet been carried out, and does not seem straightforward in the generality of Theorem 1 (which, we should emphasize, imposes no local conditions on the representations $\pi_j$).

The method of this paper consists of two stages. We first use a somewhat crude choice of $v_1$ and $v_2$ (like in the work of Michel–Venkatesh) and an unfolding technique (see §1.5, in particular (1.11)) to express the $L$-values of interest as bilinear forms in the Hecke eigenvalues of the varying form $\pi_3$. We then average over the spectral window $T \leq C(\pi_3)^{1/2} \leq T + T^{1/3}$ of interest by means of analytic number theory, and in particular the Kuznetsov formula.

Our approach has in common with the works of Bernstein–Reznikov and Michel–Venkatesh that we make use of the well-developed theory of integral representations of $L$-functions to produce and analyze our test vectors. In this respect, our basic framework owes much to those works. The essential difference is that we analyze sums over much narrower spectral windows, and the more technical difference is that we implement this analysis using the Kuznetsov formula rather than the spectral theory of triple product periods.
The main advantage of our approach is that we can make full use of available technology related to the Kuznetsov formula (abundance of test functions, explicit integral transforms, Bessel function asymptotics, large sieve estimates ...), whose avatars are not available at the level of the triple product periods. To the best of our knowledge, this paper is the first to employ such a combination of the theory of integral representations and analytic number theory. We hope that this methodology will be useful more broadly.

1.4. Analytic number theory. Having discussed the representation theoretic ideas in the previous subsection, we now give a brief sketch of how analytic number theory can handle expressions like (1.4) that can be extracted from the triple product formula. This is a precursor to the analysis in §5. The trivial bound in the $m$-sum recovers convexity, and if we had square-root cancellation in the $m$-sum we would obtain Lindelöf. We now consider the sum

$$L(1/2, f \otimes g \otimes h).$$

An application of the Kuznetsov and Voronoi formulæ gives a dual shifted convolution problem that can be treated by a $\delta$-symbol method. With a final application of the spectral large sieve, the sum (1.9) can be shown to be $\ll (TH)^{1+\varepsilon}$ for $H = T^{1/3}$, which establishes the Weyl bound. For convenience, we describe a toy version of this argument, restricting each parameter to the generic range. This is somewhat misleading because smaller ranges of $m$ in (1.4) are punished by an additional oscillation which complicates matters considerably, but it nevertheless gives a flavour for what is happening. Restricting (1.4) to $\nu = 1$ (for simplicity) and applying the Kuznetsov formula, we obtain an expression roughly of the shape

$$\frac{H^{3/2}}{T^{5/2}} \sum_{m_1,m_2 \ll T^2} \sum_{c \ll T/H} \lambda_f(m_1)\lambda_f(m_2)S(m_1-1,m_2-1,c) \times e\left(\frac{2\sqrt{m_1 m_2}}{c} - \frac{T^2 c}{\sqrt{m_1 m_2}}\right),$$

cf. (5.14), provided that $H \geq T^{1/3}$. (For smaller $H$, more terms in the exponential would be necessary.) The complicated exponential is reminiscent of the uniform asymptotic expansion of the $J$-Bessel function at imaginary index. The leading term of the Kuznetsov kernel equals the Voronoi kernel, a feature that is now crucially exploited: applying the Voronoi summation formula to $m_2$, the dual variable will be close to $m_1$ and a large portion of the oscillation disappears. We obtain roughly

$$\frac{H^{5/2}}{T^{9/2}} \sum_{c \ll T/H} \sum_{h \ll T^2/H^2} S(h-1,-1,c) \sum_{m \ll T^2} \lambda_f(m)\lambda_f(m+h)e\left(\frac{2Th^{1/2}}{m^{1/2}}\right),$$
The inner sum is now a shifted convolution problem with a moderately oscillatory factor of size $T(h/m)^{1/2} \asymp T/H$. It can be spectrally decomposed by a delta-symbol method (cf. §5.7). Another application of the Voronoi and Kuznetsov formulæ (cf. §5.8 and §5.9) leads to a spectral sum of length $T/H$ having $(T/H)^2$ terms by Weyl’s law. Note that this is the length of the $h$-sum, so the large sieve (which is itself an application of the Kuznetsov formula) can show its full power on the $h$-sum, leading to the desired final bound. As an aside, we see that this analysis employs the Kuznetsov formula three times, in various directions.

We finally remark that a direct approach to (1.9), by an approximate functional equation followed by the Kuznetsov formula, appears to be hopeless: we would obtain sums over $\lambda_f(n)\lambda_g(n)$ with $n \asymp T^4$ against oscillatory functions of (combined arithmetic and analytic) conductor of size $T^2$ (regardless of the choice of $H$), so that a $\text{GL}(2) \times \text{GL}(2)$ Voronoi summation formula would not reduce the length of summation. In other words, after applying the Kuznetsov formula, we run out of moves immediately.

1.5. Unfolding. Following [Su2], we now sketch a beautiful, but completely different approach to the formula (1.4), specific to the case of discrete series representations $\pi_1, \pi_2$. Suppose that $\pi_1, \pi_2$ are generated by holomorphic forms $f, g$ and $\pi_3$ is generated by a Maaß form $h$ of spectral parameter $t_h \geq 0$. By Watson’s formula, we have

$$L(1/2, f \otimes g \otimes h) \approx e^{\pi t_h t_h^2/2} \left| \int_{z \in \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \tilde{f}(z) g(z) h(z) y^k d\mu(z) \right|^2.$$

We write $g$ as a linear combination of Poincaré series, and without much loss of generality we assume that

$$g(z) = P_n(z) = \sum_{(c \, \gamma + d) \in \Gamma \backslash \text{SL}_2(\mathbb{Z})} (cz + d)^{-k} e(n\gamma z)$$

is the $n$-th Poincaré series. We insert the Fourier expansions and unfold. This gives a $y$-integral

$$\int_0^\infty y^k \sqrt{y \cosh \left( \frac{\pi t}{2} \right) K_{it} \left( 2\pi my \right) e^{-2\pi (m+n)y}} \frac{dy}{y^2} \approx \frac{e^{-\pi t/2} 2^{2k-2}}{m^{k-1/2}} \exp \left( \pm 2it \sqrt{\frac{\log m}{m}} \right) \min \left( \frac{m^{k/2-1/4}}{t k^{-1/2}}, 1 \right)$$

for large $t$ and fixed $n$. This integral was first analyzied by Good [Go, Section 4] in terms of hypergeometric $2F_1$ functions; the analysis is long and difficult. At least if the weight $k$ is fixed but relatively large (for small $k$, one needs to work a little harder), the typical range is $m \ll t^2$, and we obtain that $L(1/2, f \otimes g \otimes h)$ is essentially the absolute-square of a linear combination
of
\begin{equation}
\frac{1}{t^{1/2}} \sum_{m \ll t^2} \frac{\lambda_h(m) \lambda_f(m + n)}{m^{1/4}} \exp \left( \pm 2\pi \sqrt{\frac{n}{m}} \right)
\end{equation}
for a fixed number of $n$’s. It is very interesting to note that this resembles closely (1.4). The previous argument is due to [Su2].

The unfolding step is obviously not applicable in the set-up of Bernstein and Reznikov. Following an idea of Zagier [Za], the formal identity
\begin{equation}
\int_{z \in \text{SL}_2(\mathbb{Z})/\mathbb{H}} f(z) d\mu(z) = \frac{\pi}{3} \int_{z \in \text{SL}_2(\mathbb{Z})/\mathbb{H}} f(z) \text{res}_s E(z, s) d\mu(z)
\end{equation}
for an $\text{SL}_2(\mathbb{Z})$-invariant function $f$ can be used to mimic unfolding in more general situations (see [Ho, Appendix A] for a related idea). Applied to the triple product of three classical Maaß forms it yields a $y$-integral involving three $K$-Bessel functions
\[ \int_0^\infty K_{it_1}(m_1 y) K_{it_2}(m_2 y) K_{it_3}(m_3 y) y^{s-1/2} dy \]
for $m_1 + m_2 + m_3 = 0$. This can still be analyzed to some extent, but the resulting highly oscillatory shifted convolution problems become untreatable with the required precision. This is the reason why the attempt on the Maaß case in [Su1] remained incomplete. Nevertheless, the unfolding step (1.11) is also present in our argument (cf. §6). It acts as a hinge between the triple product identity and sums over Fourier coefficients, leading eventually to the description (1.4) for the central $L$-value in terms of shifted convolution sums.

1.6. The analytic test vector problem. The art in using the triple product formula (1.5) to bound $L$-functions consists of choosing appropriate test vectors. The traditional test vector problem asks for explicit $v_1, v_2, v_3$ for which the local factor $L_\infty(v_1, v_2, v_3)$ is nonzero. Spherical vectors are often test vectors in this sense, but are usually not the best choice due to the exponential decay of the local factor. For analytic applications, it is useful to work with analytic test vectors: vectors for which the local factor is not merely nonvanishing, but enjoys (informally speaking) a reasonable quantitative lower bound. Michel–Venkatesh [MV, §3.6.1] gave a robust supply of test vectors under local assumptions relevant for Rankin–Selberg subconvexity. We will revisit and extend their approach to the triple product setting, removing all local assumptions in a uniform way.

For our analysis of test vectors, we adopt the language of analytic newvectors [JN], which is well-suited for keeping track of the essential invariance properties. Analytic newvectors are approximate archimedean analogues of the classical $p$-adic newvectors introduced by Casselman [C] (see also
Let $K_0(p^N)$ denote the standard congruence subgroup consisting of matrices in $\text{PGL}_2(\mathbb{Z}_p)$ whose lower-left entry is divisible by $p^N$. Let $\xi$ be a generic irreducible representation of $\text{PGL}_2(\mathbb{Q}_p)$. Denote by $c(\xi)$ the conductor exponent of $\xi$, so that $p^{c(\xi)}$ is the usual arithmetic conductor of $\xi$. The main result of local newvector theory is that there is a unique (up to scalar) nonzero vector $v \in \xi$ such that $\xi(g)v = v$ for all $g \in K_0(p^{c(\xi)})$. Such vectors $v$ are called newvectors.

An archimedean analogue of the family of congruence subgroups $K_0(p^N) \subseteq \text{PGL}_2(\mathbb{Z}_p)$ is the family of subsets $K_0(X,\tau)$ that is defined by the image in $\text{PGL}_2(\mathbb{R})$ of the set

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{R}) : \begin{array}{c} |a - 1| < \tau, \\
|b| < \tau, \\
|c| < \frac{\tau}{X}, \\
|d - 1| < \tau \end{array} \right\}. $$

Here $X$ is a large positive parameter, thought of as tending off to infinity, while $\tau \in (0,1)$ is taken small but fixed. An archimedean analogue of local newvector theory is given by [JN, Theorem 1]: for each fixed $0 \leq \vartheta < 1/2$ and arbitrary $\delta > 0$, there is a $\tau > 0$ so that for every generic irreducible unitary $\vartheta$-tempered (see §2.3) representation $\pi$ of $\text{PGL}_2(\mathbb{R})$, there is a unit vector $v \in \pi$ such that

$$\|\pi(g)v - v\| < \delta \quad \text{for all } g \in K_0(C(\pi),\tau).$$

We refer to such vectors as analytic newvectors (suppressing, for terminological brevity, the dependence of this notion upon the parameters $\delta$ and $\tau$). Such vectors $v$ may be constructed explicitly as fixed bump functions in the Kirillov model (see [JN, Theorem 7]).

Inspired by the construction of [MV, §3.6.1], we approach the analytic test vector problem for the local triple product periods $L_\infty(v_1,v_2,v_3)$ by choosing $v_1$ and $v_3$ to be analytic newvectors. The choice of $v_2$ is simplest to describe when $\pi_2$ is a principal series representation. In that case, we describe $v_2$ in the induced model by a function on the lower-triangular subgroup supported within $O(1/X)$ of the identity. More generally, we make use of the fact that $\pi_2$ may be embedded in a (not necessarily unitary) principal series representation.

### 1.7. Plan for the paper.

Having chosen test vectors $v_1,v_2,v_3$ as indicated above, we need to solve three main problems:

- We need to compute (a lower bound for) the matrix coefficient integral $L_\infty(v_1,v_2,v_3)$. This will be done in §3. The idea of the proof, as in [MV, §3.7.2], is to write the matrix coefficient integral as the square of a Rankin–Selberg integral and then to estimate the latter by playing the support properties of $v_1$ in its Kirillov model and $v_2$ in its induced model against the invariance properties of $v_3$. One subtlety is that we have not assumed that any of our representations belongs to the principal series. For this reason, the reduction
to Rankin–Selberg integrals is achieved in general only after embedding $\pi_2$ into a principal series representation and using the standard intertwining operator to normalize its inner product.

• We use (a refined version of) the formula (1.11) to compute the right hand side of (1.5). This leads to an integral of three archimedean Whittaker functions that will be computed asymptotically in §4. The proof involves several applications of the local functional equation and stationary phase analysis, but no input concerning special functions beyond Stirling’s formula. This yields the expression (1.4). In other words, choosing test vectors as above has the exact same effect as using Poincaré series in the holomorphic case and unfolding, cf. (1.10). This is a rather remarkable feature.

• We need to bound the shifted convolution problem (1.4). This can be done by analytic number theory, roughly as indicated in §1.4. This is the content of §5. It is here that we implement the “hard analysis” required by our short spectral summation.

Theorems 1 and 2 are then an easy consequence and will be derived in §6.

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2. Preliminaries

2.1. Basic notation. Throughout we work with the group $G := \text{PGL}_2(\mathbb{R})$, and its subgroup $N$ of unipotent upper triangular matrices which are equipped with the usual Haar measures. We use the notation

$$n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad n'(x) := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \quad a(y) := \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix},$$

$$k(\theta) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We view $k(\theta)$ as a function of $\theta \in \mathbb{R}/\pi\mathbb{Z}$. For convenience, we may assume $\theta$ taken in the interval $[-\pi/2, \pi/2]$.

We write $d^x y = dy/|y|$ for the Haar measure on $\mathbb{R}^\times$ and $dx$ for the Lebesgue measure on $\mathbb{R}$. We fix a $G$-invariant measure $dg$ on $N \setminus G$ given in Iwahori coordinates by

$$N \setminus G \ni g = a(y)n'(x), \quad dg = \frac{d^x y}{|y|} dx,$$
and in Iwasawa coordinates by

\[ N \backslash G \ni g = a(y)k(\theta), \quad dg = \frac{d^x y}{|y|}d\theta. \]

We equip \( \text{GL}_2(\mathbb{R}) \) with the Haar measure compatible with the chosen Haar measures on \( \mathbb{R} \times \) and \( G \) via the short exact sequence \( 1 \rightarrow \mathbb{R} \times \rightarrow \text{GL}_2(\mathbb{R}) \rightarrow G \rightarrow 1 \).

Let \( \mathcal{X}(\mathbb{R}^\times) \) denote the character group of \( \mathbb{R}^\times \). Each \( \chi \in \mathcal{X}(\mathbb{R}^\times) \) is uniquely of the form \( \chi = |.|^s \text{ sgn}^a \) for some \( s \in \mathbb{C} \) and \( a \in \{0, 1\} \). We set \( \Re(\chi) := \Re(s), \Im(\chi) := \Im(s), \) and \( C(\chi) := (1 + |\Im(s)|)/(2\pi), \) cf. (2.14). The group \( \mathcal{X}(\mathbb{R}^\times) \) is a complex manifold with respect to the coordinate charts \( \chi \mapsto s \).

For a smooth function \( f : \mathcal{X}(\mathbb{R}^\times) \rightarrow \mathbb{C} \) of sufficient decay, we then have the Mellin inversion formula

\[
\int_{\Re(\chi) = \sigma} f(\chi) d\chi := \frac{1}{2} \sum_{a \in \{0, 1\}} \int_{\Re(s) = \sigma} f(|.|^s \text{ sgn}^a) \frac{ds}{2\pi i}.
\]

For a smooth function \( f : \mathbb{R}^\times \rightarrow \mathbb{C} \) of sufficient decay, we have the Stirling’s formula, and the analytic conductor.

Let \( \rho \) be a finite-dimensional representation of the Weil group \( W_\mathbb{R} \). Let \( \psi(x) := e^{2\pi ix} \) be the standard additive character of \( \mathbb{R} \). The local \( \gamma \)-factor of \( \rho \) is defined as usual by

\[
\gamma(s, \rho) := \gamma(s, \rho, \psi) := \epsilon(s, \rho, \psi)\frac{L(1-s, \bar{\rho})}{L(s, \rho)},
\]

where \( \epsilon \) and \( L \) denote the \( \epsilon \)-factor and \( L \)-factor, respectively, a description of which can be found in [Wat, §3.1] for the cases relevant in this paper. We regard \( \psi \) as fixed once and for all, and for this reason we drop it from the notation.

The analytic conductor \( C(\rho) \) has been defined in various slightly different ways (see for instance [IS, §2], [IK, §5] [MV, §3.1.8]). For many applications, it is unimportant precisely which definition is used: what matters is just that \( C(\rho) \) controls the local \( \gamma \)-factor in the sense that for small enough \( s \), and under favorable conditions, one has at least the rough approximation

\[
\gamma(s, \rho) \approx C(\rho)^{1/2-s}.
\]

For the purposes of this paper, it will be convenient to normalize the definition of \( C(\rho) \) somewhat more precisely, so that a correspondingly more precise form of (2.3) holds. While we could work with ad hoc definitions, it is useful to present this in a slightly more general context. The purpose of the following computation is to give a uniform asymptotic formula for the local gamma factors in the cases relevant for our application. This is achieved in (2.20) and (2.21) below and used in §4.5.
2.2.1. Stirling’s formula. With the principal branch of the logarithm, we have
\[
\Gamma(z) = z^{-1/2} \left( \frac{Z}{e} \right)^z \left( G_N(z) + O_N,ε(|z|^{-N}) \right), \quad |\arg(z)| \leq π - ε, \quad |z| \geq ε
\]
for some smooth function \( G_N \) satisfying
\[
|z|^j \frac{d^j}{dz^j} G_N(z) \ll_{j,N} 1
\]
for all \( N, j \in \mathbb{Z}_{\geq 0} \).

2.2.2. Characters of \( \mathbb{R}^\times \). Set \( Γ_\mathbb{R}(s) := π^{-s/2}Γ(s/2) \). The basic archimedean local \( γ \)-factors over \( \mathbb{R} \) are given (with respect to the standard character \( ψ \) of \( \mathbb{R} \), as above) by
\[
γ(s, sgn^a) = i^a \frac{Γ(1 - s + a)}{Γ(s + a)} \quad (s \in \mathbb{C}, a \in \{0, 1\}),
\]
corresponding to the characters \( |.|^s sgn^a \) of \( \mathbb{R}^\times \). For \( s = \sigma + iτ \), we define \( g_{σ,a}(τ) \) by writing
\[
γ(s, sgn^a) = \left( \frac{|τ|}{2π} \right)^{1/2 - σ - w} \exp(iφ_τ) g_{σ,u,a}(τ,v),
\]
where
\[
φ_τ := -(v + τ) \log \left( \frac{1 + τ/v}{e} \right)
\]
and \( g_{σ,u,a}(τ,v) = (e^{-1}|1 + τ/v|)^{1/2 - σ - u} g_{σ+u,a}(τ + v) \). For \( |v| \geq \max(1, 2|τ|) \) and \( σ, u \ll 1 \), we conclude from (2.4) that
\[
∂_τ^j φ_τ \ll_σ |v|^{-j_1 - j_2}.
\]
2.2.3. Characters of \( \mathbb{C}^\times \). We now record the analogous discussion over \( \mathbb{C} \).

Set \( \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s) \). The basic local \( \gamma \)-factors over \( \mathbb{C} \) are given with respect to the standard additive character \( \psi_{\mathbb{C}}(x) := e^{2\pi i(x+\bar{x})} \) of \( \mathbb{C} \) by

\[
\gamma_{\mathbb{C}}(s, \text{sgn}_C^a) = i^{a+1} \frac{\Gamma_C(1 - s + |a|/2)}{\Gamma_C(s + |a|/2)} \quad (s \in \mathbb{C}, a \in \mathbb{Z}),
\]

(2.8)

corresponding to the character \( |.\text{sgn}_C^a| \) of \( \mathbb{C}^\times \); here \( |z|_\mathbb{C} := z \bar{z} \), \( \text{sgn}_C(z) := z/|z| \). We extend the definition (2.8) to arbitrary \( a \in \mathbb{R} \) by taking \( i^{a+1} := \exp(\frac{i\pi}{2}(a+1)) \). We suppose henceforth that \( a \geq 0 \). For \( s = \sigma + i\tau \), we define \( g_{\mathbb{C},\sigma}(\frac{a}{2} + i\tau) \) by writing

\[
\gamma_{\mathbb{C}}(s, \text{sgn}_C^a) = \left( \frac{|a/2+i\tau|}{2\pi e} \right)^{1-2s} \left( \frac{a/2+i\tau}{|a/2+i\tau|} \right)^{a} g_{\mathbb{C},\sigma}(\frac{a}{2} + i\tau).
\]

Again, for \( \sigma \) restricted to a fixed interval and \( (s,a) \) a fixed distance away from poles of \( \gamma_{\mathbb{C}}(s, \text{sgn}_C^a) \), Stirling’s formula implies

\[
\partial^j_1 \partial^j_2 g_{\mathbb{C},\sigma}(z) \ll_D j_1, j_2 (1 + |\tau| + |a|)^{-j_1-j_2}
\]

(2.9)

for \( j_1, j_2 \geq 0 \). We write

\[
\gamma_{\mathbb{C}}(s+w, \text{sgn}_C^a) = \left( \frac{|a/2+i\nu|}{2\pi} \right)^{1-2s-2w} \exp(i\phi_{\mathbb{C},\nu,a}(\tau)) g_{\mathbb{C},\sigma,u}(\tau, \frac{a}{2} + i\nu),
\]

(2.10)

where

\[
\phi_{\mathbb{C},\nu,a}(\tau) := -2(v+\tau) \log \left( \frac{1}{e} \left| 1 + \frac{i\tau}{a/2+i\nu} \right| \right) - a \arg \left( \frac{a}{2} + i(\sigma + v) \right)
\]

(2.11)

and \( g_{\mathbb{C},\sigma,u}(\tau, \frac{a}{2} + i\nu) = (e^{-1}|1 + i\tau/(\frac{a}{2} + i\nu)|)^{1-2\sigma-2u} g_{\mathbb{C},\sigma+u}(\frac{a}{2} + i(v + \tau)) \).

For \( |a/2+i\nu| \geq \max(1,2|\tau|) \) and \( \sigma, u \ll 1 \), we obtain from (2.9) that

\[
\partial^j_1 \partial^j_2 \partial^l_1 g_{\mathbb{C},\sigma,u}(\tau, \frac{a}{2} + i\nu) \ll (|\nu| + |a|)^{-j_1-j_2-j_3}.
\]

(2.12)

2.2.4. The general definition. Any \( n \)-dimensional representation of the Weil group \( \mathbb{W}_\mathbb{R} \) is isomorphic to a direct sum

\[
\rho = (\oplus_{j=1}^{n_1} |w_j \text{sgn}_{C}^{a_j}|) \oplus (\oplus_{j=1}^{n_2} |z_j \text{sgn}_{C}^{b_j}|)
\]

(2.13)

where \( n = n_1 + 2n_2 \), \( w_j, z_j \in \mathbb{C}, a_j \in \{0,1\} \) and \( b_j \in \mathbb{Z}_{\geq 1} \). Here we identify the indicated characters of \( \mathbb{C}^\times \approx W_\mathbb{C} \) with the corresponding two-dimensional induced representations of \( \mathbb{W}_\mathbb{R} \). The local \( \gamma \)-factor of \( \rho \) is now given by

\[
\gamma(s, \rho) = \prod_{j=1}^{n_1} \gamma(s + w_j, \text{sgn}_{C}^{a_j}) \prod_{j=1}^{n_2} \gamma(s + z_j, \text{sgn}_{C}^{b_j}).
\]

Write \( w_j = u_j + iv_j \) and \( z_j = x_j + iy_j \). We define the analytic conductor

\[
C(\rho) := \prod_{j=1}^{n_1} \max(1, |v_j|) \prod_{j=1}^{n_2} \max(1, |b_j/2 + iy_j|)^2
\]

(2.14)

and, using (2.6) and (2.11), the phase function
\[ \phi_\rho(\tau) := \sum_{j=1}^{n_1} \phi_{v_j}(\tau) + \sum_{j=1}^{n_2} \phi_{C,y_j,b_j}(\tau) \]
and the factors
\[ g_\sigma(\tau, \rho) := \prod_{j=1}^{n_1} g_{\sigma,u_j,a_j}(\tau, v_j) \prod_{j=1}^{n_2} g_{C,\sigma,x_j}(\tau, \frac{b_j}{2} + iy_j), \]
\[ e_\rho := \prod_{j=1}^{n_1} \left( \frac{|v_j|}{2\pi} \right)^{-u_j} \prod_{j=1}^{n_2} \left( \frac{b_j/2 + iy_j}{2\pi} \right)^{-x_j}. \]

By the dual (resp. conjugate) of \( \rho \), we mean the representation obtained by negating (resp. by conjugating) the parameters \( w_j, z_j \) in (2.13). We summarize the previous discussion in the following lemma.

**Lemma 1.** Suppose that \( \rho \) is isomorphic to its conjugate dual. Then
\[ \gamma(s, \rho) = e_\rho C(\rho)^{1/2-s} \exp(i\phi_\rho(\tau)) g_\sigma(\tau, \rho). \]
If moreover \( \rho \) is self-dual (equivalently, self-conjugate), then \( e_\rho = 1 \).

**Proof.** The content of our hypothesis is that we have the equalities of multisets
\[ \{(w_1, a_1), \ldots, (w_{n_1}, a_{n_1})\} = \{(-w_1, a_1), \ldots, (-w_{n_1}, a_{n_1})\}, \]
\[ \{(z_1, b_1), \ldots, (z_{n_2}, b_{n_2})\} = \{(-z_1, b_1), \ldots, (-z_{n_2}, b_{n_2})\}. \]
It follows that
\[ \prod_{j=1}^{n_1} \left( \frac{|v_j|}{2\pi} \right)^{-u_j} = \prod_{j=1}^{n_2} \left( \frac{|b_j/2 + iy_j|}{2\pi} \right)^{-x_j} = 1. \]
We deduce (2.15) by multiplying together the identities (2.5) and (2.10). Assuming moreover that \( \rho \) is self-dual, we obtain the additional equalities of multisets as in (2.16) and (2.17), but without the conjugations, which in turn give that \( e_\rho = 1 \). \( \square \)

The primary hypothesis of Lemma 1 is satisfied if, for instance, \( \rho \) corresponds to a unitary representation \( \pi \) of \( \text{GL}_n(\mathbb{R}) \), while the full hypotheses are satisfied if \( \pi \) is self-dual.
2.2.5. The examples of interest. We consider in this paper cuspidal automorphic representations \( \pi \) for \( \text{SL}_2(\mathbb{Z}) \). Each such \( \pi \) defines a generic irreducible unitary representation of \( \text{PGL}_2(\mathbb{R}) \), hence a two-dimensional representation \( \rho_\pi \) of \( \mathbb{R} \). We set

\[
(2.18) \quad \gamma(s, \pi) := \gamma(s, \rho_\pi),
\]

and similarly define \( C(\pi), \phi_\pi(\tau) \) and \( g_\sigma(\tau, \pi) \). The possibilities for \( \rho_\pi \) are as follows:

1. \( \pi \) is a principal series representation \( \pi = \pi(r, a) \) obtained by normalized induction of the character \( |.|^r \text{sgn}^a \) for some \( r \in \mathbb{R} \cup (-1/2, 1/2) i \) and \( a \in \{0,1\} \), in which case \( \rho_\pi = |.|^r \text{sgn}^a \oplus |.-i|^r \text{sgn}^a \), or

2. \( \pi \) is a discrete series representation \( \pi = \pi(k) \) of lowest weight \( k \in 2\mathbb{Z}_{\geq 1} \), in which case \( \rho_\pi = \text{sgn}^{k-1} \).

We note that any such \( \pi \) is self-dual, hence any such \( \rho \) is both self-dual and self-conjugate; this property is evident in each example. Thus for \( s = \sigma + i\tau \), we have

\[
(2.19) \quad \gamma(s, \pi) = C(\pi)^{1/2-s} \exp(i\phi_\pi(\tau)) g_\sigma(\tau, \pi)
\]

where

- for \( \pi = \pi(r, a) \), we have
  \[
  C(\pi) = \left( \frac{\max(1, |R(r)|)}{2\pi} \right)^2,
  \]
  \[
  \phi_\pi(\tau) = -(r + \tau) \log \left( \ln \left| 1 + \frac{\tau}{r} \right| \right) - (r + \tau) \log \left( \ln \left| 1 - \frac{\tau}{r} \right| \right) + 2\tau,
  \]

- for \( \pi = \pi(k) \),
  \[
  C(\pi) = \left( \frac{\max(1, (k - 1)/2)}{2\pi} \right)^2,
  \]
  \[
  \phi_\pi(\tau) = -2\tau \log \left( \frac{1}{e} \ln \left| 1 + \frac{i\tau}{(k - 1)/2} \right| \right) - (k - 1) \text{arg} \left( \frac{k - 1}{2} + i\tau \right)
  \]
and \( g_\sigma(\tau, \pi) \) varies mildly in the sense given by the estimates (2.7) and (2.12), namely

- for \( |r| \geq \max(1, 2|\tau|) \) and \( \sigma \ll 1 \),
  \[
  \partial_{\tau}^{j_1} \partial_{\tau}^{j_2} g_\sigma(\tau, \pi(r, a)) \ll |r|^{-j_1-j_2},
  \]

- for \( |k - 1| \geq 4|\tau| \) and \( \sigma \ll 1 \),
  \[
  \partial_{\tau}^{j_1} \partial_{\tau}^{j_2} g_\sigma(\tau, \pi(k)) \ll |k|^{-j_1-j_2}.
  \]

We note that, while \( \pi(k) \) is not defined as a representation for non-integral \( k \), each of the factors \( \gamma(s, \pi(k)), C(\pi(k)), \phi_\pi(k) \) and hence also \( g_\sigma(\tau, \pi(k)) \) is defined for any \( k \in \mathbb{R}_{\geq 1} \); see after (2.8). For this reason, it makes sense to differentiate with respect to \( k \) in (2.22).

On one occasion, we will apply Lemma 1 to a Rankin–Selberg convolution \( \pi_1 \otimes \pi_2 \otimes \chi \) of a pair of generic irreducible unitary representations of \( \text{PGL}_2(\mathbb{R}) \),
twisted further by a character $\chi$ of $\text{GL}_1(\mathbb{R})$. Writing $\gamma = |\chi_0| \cdot |\Re(\chi)|$ with $\chi_0$ unitary, we have
\begin{equation}
\gamma(1/2, \pi_1 \otimes \pi_2 \otimes \chi) = \gamma(1/2 + \Re(\chi), \pi_1 \otimes \pi_2 \otimes \chi_0) \ll C(\pi_1 \otimes \pi_2 \otimes \chi)^{-\Re(\chi)},
\end{equation}
where in the second step we invoke Lemma 1 and the accompanying Stirling estimates, using the unitarity of $\pi_1, \pi_2$ and $\chi_0$ to verify its hypotheses.

2.3. General bounds for Whittaker functions. Let $\pi$ be a generic irreducible unitary representation of $G := \text{PGL}_2(\mathbb{R})$. We recall that “generic” means that there is a $G$-equivariant embedding, necessarily unique, $\pi \hookrightarrow \{ W : G \to \mathbb{C} \text{ smooth } | W(n(x)g) = e(x)W(g) \}, v \mapsto Wv,$
where $G$ acts on the space on the right hand side by right translation. The image of $\pi$ under such an embedding is called the Whittaker model of $\pi$ with respect to $\psi$. An invariant inner product on $\pi$ is given by
\begin{equation}
\langle v_1, v_2 \rangle_\pi := \int_{\mathbb{R} \times W} W_{v_1}(a(y))W_{v_2}(a(y)) \, d^x y.
\end{equation}
When we speak below of $\pi$ being realized in its Whittaker model, we mean that we identify $\pi$ with its image under such an embedding, with inner product normalized as in (2.24).

Fix $\vartheta \in [0, 1/2)$. We say that $\pi$ is $\vartheta$-tempered if it lies in the discrete series or if, writing $\pi$ as a Langlands quotient of an isobaric sum $\sigma_1 \otimes | \det |^{s_1} \boxplus \sigma_2 \otimes | \det |^{s_2}$, we have that each $|\Re(s_i)| \leq \vartheta$. Then $\pi$ is 0-tempered in the above sense if and only if it is tempered in the usual sense, i.e., its matrix coefficients lie in $L^{2+\varepsilon}(G)$ for each $\varepsilon > 0$.

In what follows, we work exclusively with smooth vectors in such representations. Thus “let $v \in \pi^n$” is shorthand for “let $v$ be a smooth vector in $\pi$.”

We denote by $S_d$ the Sobolev norm on $\pi$ defined in [MV, (2.6)]. It takes finite values on smooth vectors.

**Lemma 2.** Let $\pi$ be a $\vartheta$-tempered generic irreducible unitary representation of $G$, realized in its Whittaker model. For each $W \in \pi$ and all $y \in \mathbb{R}^\times$ and $z \in \mathbb{R}$ with $|z| \leq 1000$, we have
\begin{equation}
(y \partial_y)^{j_2} \partial_z^{j_1} W(a(y)wn(z)) \ll S_d(W) \min(|y|^{1/2-\vartheta}, |y|^{-N}).
\end{equation}
and
\begin{equation}
(y \partial_y)^{j_2} \partial_z^{j_1} W(a(y)n'(z)) \ll S_d(W) \min(|y|^{1/2-\vartheta}, |y|^{-N})
\end{equation}
for all $j_1, j_2, N \in \mathbb{Z}_{\geq 0}$, where $d \in \mathbb{Z}_{\geq 0}$ and the implied constants depend at most upon $j_1, j_2, N$.

**Proof.** [MV, §2.4.1 and §3.2.3].
2.4. Smooth weight functions. Let $X$ be a large parameter which will be clear from the context. We adopt the convention that $\varepsilon$ denotes a fixed (i.e., independent of $X$) positive quantity, whose precise meaning may change from line to line. As usual, the notation $A \ll B$ means that $|A| \leq C|B|$ for some fixed $C$; we introduce subscripts as in $A \ll_j B$ to signify that $C$ may depend upon $j$. We use the notation $A \ll B$ to denote that $A$ and $B$ are nonzero real numbers for which $A/B$ lies in some fixed compact subset of $(0, \infty)$; we then have $A \ll B \ll A$. We introduce the abbreviation

$$A \ll_j B \iff A \ll \varepsilon X \ll_j B.$$ 

We call an expression negligible if it is $\ll N X^{-N}$ for any $N > 0$. We call a smooth function $V : \mathbb{R}^n \to \mathbb{C}$ flat if

$$(2.27) \quad x_1^{j_1} \cdots x_n^{j_n} V^{(j_1, \ldots, j_n)}(x_1, \ldots, x_n) \ll_j 1$$

for all $j \in \mathbb{Z}_+^n$. Clearly if $V$ is flat, then so is $\exp(iV)$. If in addition $V$ has fixed compact support in $(0, \infty)^n$, we call it nice. We generally let $V$ denote a nice function in one or more variables, not necessarily the same at every occurrence. In practice, $V$ may depend on some additional parameters having certain prescribed sizes; it will always be clear from the context with respect to which variables “flatness” is applied (in which case all implied constants are uniform in these parameters).

For a nice function $V$, we may separate variables in $V(x_1, \ldots, x_n)$ by first inserting a redundant function $V(x_1) \cdots V(x_n)$ that is 1 on the support of $V$ and then applying Mellin inversion

$$V(x_1, \ldots, x_n) = \int_{\mathbb{R}(s_1)=0} \cdots \int_{\mathbb{R}(s_n)=0} \frac{\hat{V}(s_1, \ldots, s_n)(V(x_1) \cdots V(x_n)x_1^{-s_1} \cdots x_n^{-s_n}) ds_1 \cdots ds_n}{(2\pi i)^n}.$$ 

Here we can truncate the vertical integrals at height $|\Im s| \ll 1$ at the cost of a negligible error. We will often separate variables in this way without explicit mention.

2.5. Integration by parts and stationary phase. We quote the following lemmas from [BKY, Section 8] and its extension in [KPY, Section 3].

**Lemma 3.** Let $Y \geq 1$, $X,P,U,S > 0$, and suppose that $w$ is a smooth function with support on $[\alpha, \beta]$, satisfying

$$w^{(j)}(t) \ll_j XU^{-j}.$$

Suppose $h$ is a smooth function on $[\alpha, \beta]$ such that

$$|h'(t)| \gg S$$

for some $S > 0$, and

$$h^{(j)}(t) \ll_j YP^{-j}, \quad \text{for } j = 2, 3, \ldots.$$
Then
\[ \int_{t \in \mathbb{R}} w(t) e^{ih(t)} \, dt \ll A (\beta - \alpha) X [(PS/\sqrt{Y})^{-A} + (SU)^{-A}] . \]

**Lemma 4.** Let \( X \) be a large parameter. Let \( V \) be a flat function in the sense of §2.4 with support in \( \times_{j=1}^{d} [c_{1j}, c_{2j}] \) for some fixed intervals \( [c_{1j}, c_{2j}] \subseteq \mathbb{R} \) not containing 0. Let \( X_1, \ldots, X_d > 0, Y \geq X^{\varepsilon} \). Write \( \mathcal{S} = \times_{j=1}^{d} [c_{1j} X_j, c_{2j} X_j] \subseteq \mathbb{R}^d \). Let \( \phi : \mathbb{R}^d \to \mathbb{R} \) be a smooth function satisfying the derivative upper bounds
\[ \phi^{(j_1, \ldots, j_d)}(t_1; t_2, \ldots, t_d) \ll Y \prod_{i=1}^{d} X_i^{-j_i} \]
for \( j \in \mathbb{N}^d \) and \( (t_1, \ldots, t_d) \in \mathcal{S} \), as well as the following second derivative lower bound in the first variable:
\[ \phi^{(2,0,\ldots,0)}(t_1; t_2, \ldots, t_d) \gg Y X_1^{-2} . \]

Suppose that there exists \( t^* = t^*(t_2, \ldots, t_d) \) such that \( \phi^{(1,0,\ldots,0)}(t^*, t_2, \ldots, t_d) = 0 \). Then for any \( N > 0 \) we have
\[ \int_{\mathbb{R}} V\left(\frac{t_1}{X_1}, \ldots, \frac{t_d}{X_d}\right) e^{i\phi(t_1; \ldots, t_d)} \, dt_1 = \frac{X_1}{Y^{1/2}} e^{i\phi(t^*, t_2, \ldots, t_d)} W\left(\frac{t_2}{X_2}, \ldots, \frac{t_d}{X_d}\right) + O_N(X^{-N}) \]
for a flat function \( W = W_N \) with support in \( \times_{j=2}^{d} [c_{1j}, c_{2j}] \).

### 3. The local triple product factor

Let \( \pi_i \) for \( i = 1, 2, 3 \) be generic irreducible unitary representations of \( G \) such that
- \( \pi_1 \) and \( \pi_2 \) are \( \vartheta \)-tempered, while
- \( \pi_3 \) is tempered.

We regard \( \pi_1 \) and \( \pi_2 \) as fixed. We write \( Q = C(\pi_3) \) for the conductor of \( \pi_3 \) and think of \( Q \) as a large parameter. The aim of this section is to obtain a lower bound for the local triple product integral \( L_\infty(v_1, v_2, v_3) \) in (1.5) for a certain choice of vectors \( v_j \in \pi_j \). The choice will be made at the beginning of §3.3 and the result will be stated in Theorem 3 at the end of this section.

Let \( \psi \) denote the additive character of \( \mathbb{N} \) given by \( n(x) \mapsto e(x) \). We realize \( \pi_1 \) (resp. \( \pi_3 \)) in its Whittaker model with respect to \( \psi \) (resp. \( \overline{\psi} \)), with inner products normalized as in (2.24).

In this section we abbreviate \( \chi_s := \chi \otimes |\cdot|^s \) for \( \chi \in \mathcal{X}(\mathbb{R}^\times) \) and \( s \in \mathbb{C} \).

---

2 The main result in [KPY, Section 3] states this with \( \ll \) instead of \( \ll \), but our conclusion on \( W \) is insensitive to \( Q^\varepsilon \)-powers.
3.1. Embedding via intertwiners. Let \( \chi \in \mathfrak{X}(\mathbb{R}^\times) \). Let \( \mathcal{I}(\chi) \) denote the unitarily normalized induction of \( \chi \) from the standard upper-triangular Borel subgroup in \( G \), consisting of smooth \( f : G \to \mathbb{C} \) satisfying \( f(n(x)a(y)g) = |y|^{1/2} \chi(y)f(g) \). Let \( M(\chi) \) denote the standard intertwining operator from the principal series \( \mathcal{I}(\chi) \) to \( \mathcal{I}(\chi^{-1}) \), defined by the integral

\[
(f \mapsto \int_{x \in \mathbb{R}} f(wn(x))dx)
\]

for \( \Re(\chi) > 0 \) and then meromorphically continued to all of \( \mathfrak{X}(\mathbb{R}^\times) \).

Let \( \chi = |.|^{1/2+k} \) for \( k \in \mathbb{Z}_{\geq 0} \) and consider a \( K \)-type basis \( \{f_l\}_{l \in \mathbb{Z}} \) on \( \mathcal{I}(\chi) \). From the computation of [Bu, Proposition 2.6.3] we see that \( M(\chi)f_l = 0 \) for \( |l| \geq 1 + k \). Thus \( M(\chi) \) has a unique infinite dimensional kernel which is isomorphic to the discrete series \( D_k \) of weight \( k \). We normalize \( M(\chi) \) as

\[
M^*(\chi) := \gamma(0, \chi^2)M(\chi),
\]

where \( \gamma \) is the local Tate gamma factor as in §2.2. Then \( M^*(\chi) \) is non-zero for all \( \Re(\chi) > 0 \) and is meromorphic for all \( \chi \). In other words \( D_k \) can be embedded into the principal series representation \( \mathcal{I}(\chi) \) with \( \chi = |.|^{k+1/2} \) via the normalized intertwining operator \( M^*(\chi) \). A similar embedding can be done for the complementary series representation as well, see [Bu, §2.6].

Let \( \chi \) with \( \Re(\chi) \geq 0 \) not be a pole of \( M^*(\chi) \). From now on we will only consider \( \chi \) for which either \( \Re(\chi) = 0 \) or \( \Im(\chi) = 0 \). Note that if \( \mathcal{I}(\chi) \) is unitary then \( \chi \) satisfies this property. We can define a \( G \)-invariant sesquilinear pairing on \( \mathcal{I}(\chi) \) by

\[
(f_1, f_2)_0 := \begin{cases} 
\int_{x \in \mathbb{R}} f_1(n'(x))\overline{f_2(n'(x))}dx, & \text{if } \Re(\chi) = 0, \\
\int_{x \in \mathbb{R}} f_1(n'(x))M^*(\chi)f_2(n'(x))dx, & \text{if } \Im(\chi) = 0.
\end{cases}
\]

for \( f_1, f_2 \in \mathcal{I}(\chi) \).

There is a principal series representation \( \pi_2^T = \mathcal{I}(\chi) \), with \( \chi \) of nonnegative real part, into which \( \pi_2 \) embeds. Explicitly:

- If \( \pi_2 \) is a tempered principal series \( \mathcal{I}(\chi_0) \), i.e. if \( \chi_0 \) is unitary, then we choose \( \chi = \chi_0 \).
- If \( \pi_2 \) is the weight \( k \) discrete series \( D_k \), then we choose \( \chi = |.|^{1/2+k} \).
- If \( \pi_2 \) is the complementary series attached to \( 0 < \sigma < 1/2 \) then we choose \( \chi = |.|^\sigma \).

In each case, we have a \( G \)-invariant embedding \( \pi_2 \hookrightarrow \pi_2^T = \mathcal{I}(\chi) \) and \( c(\chi) \in \mathbb{C}^\times \) such that

\[
\langle v_1, v_2 \rangle_{\pi_2} = c(\chi)(f_{v_1}, f_{v_2})_0 =: (f_{v_1}, f_{v_2}),
\]

where \( f_{v_i} \) are the images of \( v_i \) under the above embedding and \( \langle ., . \rangle_{\pi_2} \) is as defined in (2.24). We refer to [Bu, §2.6] for details.

3.2. Local Rankin–Selberg zeta integral. Let \( W_1 \in \pi_1 \) and \( W_3 \in \pi_3 \) and let \( f_2 \in \mathcal{I}(\chi) \) for some \( \chi \in \mathfrak{X}(\mathbb{R}^\times) \). We may parametrize \( f_2 \) in terms
of a Schwartz function, as follows. Let $e_2 := (0, 1) \in \mathbb{R}^2$ and $\Phi \in \mathcal{S}(\mathbb{R}^2)$ a Schwartz function. We define
\[
f_2(g) := \int_{t \in \mathbb{R}^\times} \Phi(e_2tg)\chi_{1/2}(\det(tg))d^x t.
\]
The above integral converges absolutely for $\Re(\chi) > -1/2$ and continues meromorphically to all $\chi \in \mathcal{X}(\mathbb{R}^\times)$.

The local $GL(2) \times GL(2)$ Rankin–Selberg zeta integral of $\pi_1$ and $\pi_3$ is defined by
\[
\Psi(W_1, f_2, W_3) := \int_{g \in \mathcal{N}\backslash G} W_1(g)f_2(g)W_3(g)dg
\]
\[
= \int_{g \in \mathcal{N}\backslash GL_2(\mathbb{R})} W_1(g)W_3(g)\Phi(e_2g)\chi_{1/2}(\det(g))dg,
\]
for $\Re(\chi)$ sufficiently large and in general by meromorphic continuation. The $GL(2) \times GL(2)$ local functional equation (see [Cog, Theorem 3.2]) asserts, using the notation (2.18) and (2.2),
\[
(3.3) \quad \gamma(1/2, \pi_1 \otimes \pi_3 \otimes \chi) \int_{g \in \mathcal{N}\backslash GL_2(\mathbb{R})} W_1(g)W_3(g)\Phi(e_2g)\chi_{1/2}(\det(g))dg
\]
\[
= \int_{g \in \mathcal{N}\backslash GL_2(\mathbb{R})} \tilde{W}_1(g)\tilde{W}_3(g)\tilde{\Phi}(e_2g)\chi_{-1/2}(\det(g))dg,
\]
where $\tilde{W}_i \in \tilde{\pi}_i$ is the contragredient of $W_i$ defined by $\tilde{W}_i(g) = W(wg^{-T})$ and $\tilde{\Phi}$ is the Fourier transform of $\Phi$ defined by
\[
\tilde{\Phi}(y) := \int_{x \in \mathbb{R}^2} \Phi(x)e(y^\top x)dx.
\]
For $\chi$ a fixed distance away from a pole or zero of $\gamma(1/2, \pi_1 \otimes \pi_3 \otimes \chi)$, we have
\[
(3.4) \quad \gamma(1/2, \pi_1 \otimes \pi_3 \otimes \chi)^{-1} \asymp \gamma(1/2, \tilde{\pi}_1 \otimes \tilde{\pi}_3 \otimes \chi^{-1})
\]
\[
\ll_{\Re(\chi)} C(\pi_1 \otimes \pi_3 \otimes \chi)^{\Re(\chi)} \ll_{\pi_1, \chi} C(\pi_3)^{2\Re(\chi)},
\]
when $\chi$ is fixed with $\Re(\chi) \geq 0$. The first estimate above follows from the definition of the gamma factor. The second estimate follows from (2.23).

The third estimate follows from repeated application of [IB, Lemma A.2].

We record a variant of the local functional equation:

**Lemma 5.** We have
\[
\Psi(W_1, f_2, W_3)\gamma(1/2, \pi_1 \otimes \pi_3 \otimes \chi) = \Psi(W_1, M^*(\chi)f_2, W_3),
\]
where $M^*(\chi)$ is as in (3.2).

**Proof.** Let $\Re(\chi)$ be sufficiently large. By expanding the definition (3.1) of the intertwining operator, we see that
\[
M(\chi)f_2(g) = \chi_{1/2}(\det(g)) \int_{x \in \mathbb{R}} \int_{t \in \mathbb{R}^\times} \Phi((t,x)g)\chi^2(t)d^x t dx.
\]
We use local Tate functional equation to evaluate the above as
\[
\gamma(0, \chi^2)\chi^{-1}(\det(g))|\det(g)|^{-1/2} \int_{t \in \mathbb{R}^\times} \hat{\Phi}(t, 0)g^{-\top} \chi^{-2}(t)|t|d^\times t
\]
(compare with [GJ, p. 225]). Recalling (3.2), we may thus write
\[
M^*(\chi)f_2(g) = \int_{t \in \mathbb{R}^\times} \hat{\Phi}(e_2twg^{-\top})\chi_1^{-1}(\det(tg^{-\top}))d^\times t.
\]

We use the definition of \( \hat{\text{W}}_i \) and change variables \( g \mapsto wg^{-\top} \) in the right hand side of the local functional equation (3.3) to write
\[
\gamma(1/2, \pi_1 \otimes \pi_3 \otimes \chi) \int_{g \in \mathbb{N} \setminus \text{GL}_2(\mathbb{R})} W_1(g)W_3(g)\Phi(e_2g)\chi_{1/2}(\det(g))dg
\]
\[
= \int_{g \in \mathbb{N} \setminus \text{GL}_2(\mathbb{R})} W_1(g)W_3(g)\hat{\Phi}(e_2wg^{-\top})\chi_{-1/2}(\det(g))dg.
\]

Folding the above integrals over \( \mathbb{R}^\times \), the identity follows for \( \Re(\chi) \) large. We conclude the proof by meromorphic continuation of the zeta integrals and the intertwiner. \( \square \)

**Lemma 6.** Let \( \pi_1 \) and \( \pi_2 \) be \( \vartheta \)-tempered with \( \vartheta < 1/4 \) and \( \pi_3 \) be tempered. Let \( \pi_i \ni v_i \mapsto W_i \) for \( i = 1, 3 \) be realized in their respective Whittaker models equipped with the inner products as defined in (2.24). Also let \( v_2 \mapsto f_2 \) under \( \pi_2 \mapsto \pi_2^0 = \mathcal{I}(\chi) \) as described in §3.1. Then
\[
\int_{g \in G} \prod_{i=1}^3 \langle \pi_i(g)v_i, v_i \rangle_{\pi_i} dg = c(\chi)\Psi(W_1, f_2, W_3)\Psi(W_1, f_2, W_3),
\]
where
\[
\tilde{f}_2 = \begin{cases} 
    f_2, & \text{if } \pi_2^0 \text{ is a tempered principal series}, \\
    M^*(\chi)f_2, & \text{otherwise}.
\end{cases}
\]

From [MV, §2.5.1] we have the bound of the matrix coefficients
\[
\langle \pi_i(g)v_i, v_i \rangle \ll_{\pi_i} \Xi(g)^{1-2\vartheta} \text{ for } i = 1, 2, \quad \langle \pi_3(g)v_3, v_3 \rangle \ll_{\pi_3} \Xi(g).
\]

Here \( \Xi \) is the Harish-Chandra \( \Xi \)-function, which satisfies \( \int_{g \in G} \Xi(g)^{2+\epsilon}dg < \infty \). Thus from the assumption that \( \vartheta < 1/4 \) we see that the local triple product integral is absolutely convergent.

**Proof.** The proof is essentially given in [Ne1, 2.14.3 Lemma], but in an analogous metaplectic setting. We modify the relevant part of the proof. Note that the left hand side of the equation in the lemma is
\[
\int_{g \in G} \langle \pi_1(g)W_1, W_1 \rangle_{\pi_2^0(g)f_2} \langle \pi_2^0(g)f_2, f_2 \rangle \langle \pi_3(g)W_3, W_3 \rangle dg.
\]

We define
\[
\xi_1 := W_1f_2, \quad \xi_2 := W_1\tilde{f}_2,
\]
and note that
\[ \xi_i(n g) = \psi(n) \xi_i(g), \quad n \in N, g \in G. \]

Using Iwahori coordinates \( g = a(y)n'(x) \in N \setminus G \) and the transformation of \( f_2 \) under the Borel subgroup we compute the absolutely convergent integral
\[
\int_{h \in N \setminus G} \xi_1(hg) \overline{\xi_2(h)} \, dh
\]
\[
= \int_{x \in \mathbb{R}} \pi_2^n(g) f_2(n'(x)) \overline{f_2(n'(x))} \int_{y \in \mathbb{R}^x} \pi_1(g) W_1(a(y)n'(x)) W_1(a(y)n'(x)) d^x y \, dx.
\]

The inner integral evaluates to \( \langle \pi_1(g) W_1, W_1 \rangle \) and consequently, we have
\[
\int_{h \in N \setminus G} \xi_1(hg) \overline{\xi_2(h)} \, dh = \langle \pi_1(g) W_1, W_1 \rangle \langle \pi_2^n(g) f_2, f_2 \rangle_0.
\]

Hence, the left hand side of the equation in the lemma equals
\[
c(\chi) \int_{g \in G} \int_{h \in N \setminus G} \xi_1(hg) \overline{\xi_2(h)} \langle \pi_3(g) W_3, W_3 \rangle \, dh \, dg.
\]

The above double integral is only conditionally convergent. We proceed exactly as in the proof of the identity \([Ne1, (2.29)]\) to evaluate the above integral as
\[
c(\chi) \int_{h \in N \setminus G} \xi_1(h) W_3(h) \, dh \int_{h \in N \setminus G} \overline{\xi_2(h) W_3(h)} \, dh.
\]

The proof is now complete. \( \square \)

3.3. Choice of the vectors. We choose \( f_2 \in \pi_2^n \) as before
\[
f_2(g) := \int_{t \in \mathbb{R}^x} \Phi(e_2 t g) \chi_{1/2}(\det(tg)) d^x t,
\]
where \( \Phi \) is a smooth non-negative bump function on \( \mathbb{R}^2 \) sufficiently concentrated around the point \( e_2 = (0,1) \) in terms of \( \pi_1 \) and \( \pi_2 \) only. Such a vector has a non-zero preimage \( v'_2 \in \pi_2 \). We choose
\[
v_2 := a(Q)v'_2
\]
where, as we recall, \( Q = C(\pi_3) \) is the conductor of \( \pi_3 \) as in §2.2. We choose \( v_i \in \pi_i \) for \( i = 1, 3 \) such that \( v_i \) are analytic newvectors, in the sense of §1.6, i.e. \( v_i \) in their Kirillov models (with conjugate additive characters of \( N \)) are given by a fixed bump functions in \( C_\infty^c(\mathbb{R}^x) \) sufficiently concentrated around 1. We denote by \( W_i \) the images of \( v_i \) in their Whittaker models for \( i = 1, 3 \). We note that
\[
\int_{x \in \mathbb{R}} f_2(n'(x)) \, dx \asymp 1, \quad \int_{y \in \mathbb{R}^x} W_1(a(y)) W_3(a(y)) \chi_{-1/2}(y) d^x y \asymp 1.
\]
We normalize \( v_1, v'_2, v_3 \) so that both of the above integrals are 1.
Lemma 7. Let \( \pi_1 \) be \( \vartheta \)-tempered with \( \vartheta < 1/2 \) and \( \pi_3 \) be tempered. Let \( \chi \) with \( \Re(\chi) \geq 0 \) and \( f_2 \in \mathcal{I}(\chi) \) be as chosen above. Then for \( C(\pi_3) = Q \) sufficiently large we have

\[
\Psi(W_1, f_2(a(Q)), W_3) \gg_{\pi_1, \pi_2} Q^{-1/2 + \Re(\chi)}.
\]

Proof. We write the zeta integral with the Iwahori coordinates and change variables to obtain

\[
(3.5) \quad \chi_{1/2}(Q)\Psi(W_1, f_2(a(Q)), W_3)
= \chi_{-1/2}(Q) \int_{y \in \mathbb{R}^\times} \int_{x \in \mathbb{R}} W_1(a(y)n'(\frac{x}{Q})) W_3(a(y)n'(\frac{x}{Q}))
\times f_2(a(yQ)n'(x)) dx \frac{dy}{|y|}.
\]

Note that

\[
f_2(a(yQ)n'(x)) = \chi_{1/2}(yQ)f_2(n'(x)),
\]

and the support condition of \( \Phi \) confirms that \( f_2(n'(x)) \) is supported in a sufficiently small neighbourhood of 0. We rewrite the right hand side of (3.5) as

\[
(3.6) \quad \int_{y \in \mathbb{R}^\times} \int_{x \in \mathbb{R}} \left( W_3(a(y)n'(\frac{x}{Q})) - W_3(a(y)) \right)
\times W_1(a(y)n'(\frac{x}{Q})) f_2(n'(x)) \chi_{-1/2}(y) dx \frac{dy}{y}
+ \int_{y \in \mathbb{R}^\times} \int_{x \in \mathbb{R}} W_1(a(y)n'(\frac{x}{Q})) f_2(n'(x)) \chi_{-1/2}(y) dx \frac{dy}{y}
= \int_{y \in \mathbb{R}^\times} W_1(a(y)) W_3(a(y)) \chi_{-1/2}(y) dx \frac{dy}{y} \int_{x \in \mathbb{R}} f_2(n'(x)) dx.
\]

Note that the third integral equals one by the choice of normalizations of the vectors.

As \( \pi_1 \) is fixed, we may apply (2.26) to conclude that

\[
W_1(a(y)n'(x/Q)) \ll_N \min \left( |y|^{1/2-\theta}, |y|^{-N} \right)
\]

for \( x \ll 1 \). Moreover, given some sufficiently small constant \( c > 0 \), then for \( x \) sufficiently small (in terms of \( c \) only) and \( C(\pi_1) \leq Q \) we have

\[
W_i(a(y)n'(x/Q)) - W_i(a(y)) \ll \frac{C(\pi_1)}{Q} c |y|^{1/2-\theta-\eta},
\]

for any \( \eta > 0 \) and \( \theta = \vartheta, 0 \) if \( i = 1, 3 \), respectively. This estimate is essentially contained in [JN, §2.1] and can be seen as follows: using Mellin inversion for \( W_i(a(y)n'(x/Q)) |y|^{-\sigma} \) for \( \sigma = 1/2 - \theta - \eta \) and the PGL(2) \times GL(1) local

\[
(3.7)
\]
functional equation, we write the above difference (as in \[JN, \S 2.1\])

\[
\int_{\mathbb{R}(\chi')=0} \chi'(y) \frac{\chi'(C(\pi_i))}{\gamma(1/2 - \sigma, \pi_i \otimes \chi'^{-1})} \times \int_{t \in \mathbb{R}} \left( e \left( -\frac{txC(\pi_i)}{Q} \right) - 1 \right) W_i(a(C(\pi_i)t)\chi'(t) d^x t d\chi',
\]

and (3.7) follows as in \([JN, \S 2.1]\).

Now we define

\[
I_2(\chi) := \int_{x \in \mathbb{R}} |f_2(n'(x))| dx \leq \int_{(t,x) \in \mathbb{R}^2} \Phi(t,x, t|^{2\Re(\chi)} dt dx \ll 1.
\]

The first integral in (3.6) is therefore

\[
\ll cI_2(\chi) \int_{y \in \mathbb{R}^x} |y|^{1/2 - \eta} \min(|y|^{1/2 - \eta}, |y|^{-\eta}) |y|^{\Re(\chi)-1/2} d^x y \ll c,
\]
as \(\Re(\chi) \geq 0\) and \(\vartheta < 1/2\). The second integral in (3.6) is similarly

\[
\ll Q^{-1}I_2(\chi) \int_{y \in \mathbb{R}^x} W_3(a(y)) |y|^{1/2 - \eta} |y|^{\Re(\chi)-1/2} d^x y \ll Q^{-1}.
\]

Thus we estimate

\[
\Psi(W_1, f_2(a(Q)), W_3) \gg Q^{-1/2 + \Re(\chi)}(1 + O(c) + O(1/Q))
\]

which concludes the proof upon choosing \(c\) sufficiently small in terms of the implied constants. \(\Box\)

**Theorem 3.** Let \(\pi_i\) for \(i = 1, 2, 3\) be generic irreducible unitary representations of \(G\) such that \(\pi_1\) and \(\pi_2\) are \(\vartheta\)-tempered with \(\vartheta < 1/4\) and \(\pi_3\) is tempered with sufficiently large conductor \(Q\). The (smooth) vectors \(v_i \in \pi_i\) specified at the beginning of \(\S 3.3\) have the following properties:

(i) \(\|v_i\| \asymp 1\) for \(i = 1, 2, 3\).

(ii) We have \(v_1 = v_1^0\) and \(v_2 = a(Q)v_2^0\) where \(v_1^0, v_2^0\) are fixed (independent of \(Q\)).

(iii) For any fixed nontrivial unitary character \(\psi\) of \(N\), the vector \(v_3\) is given in the \(\psi\)-Kirillov model by a fixed bump function.

(iv) We have

\[
\int_{g \in G} \prod_{i=1}^3 \langle \pi_i(g)v_i, v_i \rangle dg \gg_{\pi_1, \pi_2} Q^{-1}.
\]

**Proof.** Assertions (i), (ii), and (iii) are clear from the construction. (The description of \(v_3\) in the Kirillov model is independent of the choice of \(\psi\): different choices give rise to models that are isomorphic to one another via left translation by a suitable diagonal matrix.)

---

3In that paper authors took \(y = 1\).
To verify (iv), we embed \( \pi_2 \hookrightarrow \pi_2^0 = I(\chi) \) for some \( \chi \) with \( \Re(\chi) \geq 0 \) such that either \( \Re(\chi) = 0 \) or \( \Im(\chi) = 0 \). For \( \Re(\chi) = 0 \), the integral in question evaluates to
\[
c(\chi)|\Psi(W_1, f_2(a(Q)), W_3)|^2
\]
by Lemma 6, and Lemma 7 implies the required bound. For \( \Im(\chi) = 0 \) and \( \Re(\chi) > 0 \), we apply Lemma 6 and Lemma 5 to see that the integral in question is
\[
c(\chi)\gamma(1/2, \pi_1 \otimes \pi_3 \otimes \chi)|\Psi(W_1, f_2(a(Q)), W_3)|^2.
\]
An appeal to (3.4) and Lemma 7 then completes the proof. \( \square \)

4. A triple Whittaker integral

4.1. Setting and statement of results. In this section we evaluate asymptotically an integral containing three Whittaker functions which is the crucial ingredient for an understanding of the right hand side of (1.5). We will not need any knowledge on special functions, but we do use extensively Stirling’s formula and stationary phase analysis as described in Sections 2.2 and 2.5.

We retain the basic notation of §2.1. We continue to adopt the following setting (as in the previous section):

- \( \pi_1 \) and \( \pi_2 \) are fixed \( \vartheta \)-tempered generic irreducible unitary representations of \( G \), with \( 0 \leq \vartheta < 1/2 \) fixed.
- \( \pi_3 \) is a varying tempered generic irreducible unitary representation of \( G \), whose analytic conductor (normalized as in §2.2) we denote by \( Q := C(\pi_3) \).
- Recall that we realize each \( \pi_j \) in its Whittaker model as a space of functions \( W \) satisfying \( W(n(x)g) = e(x)W(g) \), where \( e(x) := e^{2\pi ix} \). Also, recall from (2.24) that we normalize this realization so that the inner product on \( \pi_j \) is given in the Kirillov model by integration over the diagonal subgroup: \( \|W\|^2 = \int_{y \in \mathbb{R} \times \mathbb{R}} |W(a(y))|^2 \, dx \).
- We let \( W_j \in \pi_j \) be the image of the vector \( v_j \) as in Theorem 3. Thus \( W_1 = W_1^0 \) and \( W_2 = a(Q)W_2^0 \) with \( W_1^0, W_2^0 \) fixed (independent of \( Q \)).

The basic bounds from Lemma 2 can be used for \( W_1 = W_1^0 \) and \( W_2^0 \). We will derive useful bounds for \( W_3 \) in §4.3 below.

We recall the notation and conventions of smooth weight functions in §2.4. Our basic large parameter here is \( Q \), so \( A \ll B \) means \( A \ll_{\varepsilon} Q^{\varepsilon}B \). As usual, the value of \( \varepsilon \) may change from line to line.

For \( y_1, y_2 \in \mathbb{R}^\times \) with \( y_1 + y_2 \neq 0 \), we define \( y_3 \in \mathbb{R}^\times \) by requiring that
\[
y_1 + y_2 + y_3 = 0.
\]
We set
\[
F(y_1, y_2) := \int_{\theta \in \mathbb{R}/\pi\mathbb{Z}} \prod_{j=1,2,3} W_j(a(y_j)k(\theta)) \, d\theta.
\]
The main result of this section is the following estimate for $F$ and its derivatives.

**Theorem 4.** We have

$$F(y_1, y_2) = \sum e \left( \pm 2\sqrt{Q} \Psi \left( \frac{y_1}{y_2} \right) \right) N_{\pm}(y_1, y_2) + \mathcal{E}(y_1, y_2),$$

where $\Psi$ is a smooth function satisfying the estimates

$$\Psi(y) = |y|^{1/2} + O(|y|^{3/2}), \quad \Psi^{(j)}(y) \ll |y|^{1/2-j} \quad (j \in \mathbb{N})$$

and for fixed $j_1, j_2, N \geq 0$ we have

$$(y_1 \partial_{y_1})^{j_1} (y_2 \partial_{y_2})^{j_2} N_{\pm}(y_1, y_2) \leq \left( \frac{|y_2|}{Q} \right)^{3/4} (1 + |y_1|)^{-N} (1 + |y_2|)^{-N}$$

and $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$, where for some absolute constant $c > 0$ we have

$$\mathcal{E}_1(y_1, y_2) = Q^{-1/2} (1 + |y_1| + |y_2|)^{-N},$$

$$\mathcal{E}_2(y_1, y_2) = Q^c (1 + |y_1| + |Qy_2|)^{-N},$$

$$\mathcal{E}_3(y_1, y_2) = (1 + |y_1| + |y_2/Q|)^{-N} Q^{-N}.$$

Here and in the following, all implied constants may depend on $N$ and $j$, with or without subscript. The proof gives an exact formula for $\Psi$ which however is irrelevant for our application. It depends mildly on whether $\pi_3$ belongs to the principal series or to the discrete series. The key point of Theorem 4 is that it produces the desired and expected oscillatory factor, cf. (1.4).

4.2. Preliminary decomposition. It will be convenient first to switch to Iwahori coordinates. We may find an even function $\phi_0 \in C^\infty_c(\mathbb{R}/\pi \mathbb{Z})$, supported on $(-\pi/3, \pi/3) + \pi \mathbb{Z}$, so that $\phi_0(\theta) + \phi_0(\pi/2 + \theta) = 1$ for all $\theta$. Setting $z := \tan \theta$ and using the matrix identities

$$k(\theta) = n(-z)a(z^2 + 1)n'(z),$$

$$k(\pi/2 - \theta) = n(z)a(z^2 + 1)wn(z)$$

and the relation $d\theta = (1 + z^2)^{-1} dz$, we see that $F(y_1, y_2) = F_1(y_1, y_2) + F_w(y_1, y_2)$, where, with the abbreviation $y'_j := y_j(z^2 + 1)$, we have

$$F_1(y_1, y_2) := \int_{z \in \mathbb{R}} \left( \prod_{j=1}^{2} W_j(a(y'_j)n'(z)) \right) \phi_0(\text{arctan}(z)) \frac{dz}{z^2 + 1},$$

$$F_w(y_1, y_2) := \int_{z \in \mathbb{R}} \left( \prod_{j=1}^{2} W_j(a(y'_j)wn(z)) \right) \phi_0(\text{arctan}(z)) \frac{dz}{z^2 + 1}.$$ 

Note that $z \mapsto \phi_0(\text{arctan}(z))$ defines a smooth compactly-supported function on $\mathbb{R}$.

We now further decompose $F_1$. We write $1$ as a sum $\phi^x + \phi^y$ of smooth functions on $\mathbb{R}^\times$ with

- $\phi^x(z)$ supported on $z \ll Q^{-1}$,
\( \phi^\dagger(z) \) supported on \( Q^{-1} \ll z \),
and with each function \( \phi \) in this decomposition satisfying \( (z\partial_z)^j \phi(z) \ll 1 \).
We accordingly decompose \( F_1 = F_1^\phi + F_1^\sharp \) by weighting the \( z \)-integral. In summary, we have decomposed
\[
F = F_w + F_1^\phi + F_1^\sharp. 
\]
We will verify that each of the three terms on the right hand side of (4.4) satisfies the conclusions of Theorem 4. The first two terms are fairly straightforward to analyze. We treat them in §4.4. Indeed, we will see that the contribution of \( F_1^\phi \) can be absorbed into the error term \( \mathcal{E} \), while \( F_w \) is non-negligible only if \( y_1 \ll 1 \) and \( y_2 \) is roughly of size \( Q \), in which case the oscillatory factor \( e(\pm Q^{1/2} \Psi(y_1/y_2)) \) is flat in the sense of §2.4. The somewhat more intricate analysis of \( F_1^\sharp \) is carried out in §4.5.

4.3. Interlude: bounds for newvectors. For the relevant asymptotic analysis we will need certain uniform bounds of the test vectors. We record them here.

**Lemma 8.** Let \( W \in \pi \), \( y \in \mathbb{R}^\times \), \( z \in \mathbb{R} \) and \( \sigma > -1/2 + \vartheta \). Then \( W(a(y)wn(z)) \) admits the absolutely convergent integral representation
\[
W(a(y)wn(z)) = \int_{\Re(\chi)=\sigma} \chi(y)\gamma(1/2, \pi \otimes \chi) \left( \int_{t \in \mathbb{R}^\times} e(tz)W(a(t))\chi(t)dt^\times \right) d\chi.
\]

**Proof.** By Mellin inversion (2.1) – using the estimate (2.25) to verify its hypotheses – we have
\[
W(a(y)wn(z)) = \int_{\Re(\chi)=\sigma} \chi(y) \int_{t \in \mathbb{R}^\times} W(a(t)wn(z))\chi^{-1}(t)dt^\times t d\chi.
\]
By the PGL(2) \( \times \) GL(1) local functional equation (see [Cog, Theorem 3.1]), the inner integral evaluates to
\[
\gamma(1/2, \pi \otimes \chi) \int_{t \in \mathbb{R}^\times} W(a(t)n(z))\chi(t)dt^\times t.
\]
We conclude by calculating that \( W(a(t)n(z)) = W(n(tz)a(t)) = e(tz)W(a(t)) \). □

**Lemma 9.** Let \( \pi \) be tempered of conductor \( Q \) and \( W \in \pi \) an analytic newvector. For fixed \( N > 0 \) and \( j \in \mathbb{Z}_{\geq 0} \) we have
\[
(x\partial_x)^jW(a(Qx)wn(z)) \ll \min \left( |x|^{-N}, |x|^N + |x|^{1/2}Q^{-N} \right)
\]
uniformly in \( z \ll 1 \).

**Proof.** We write the proof in the case \( j = 0 \). The general case is treated similarly, using that \( (y\partial_y)^j \chi(y) = s^j \chi(y) \) for \( \chi = |.|^s \text{ sgn}^a \).
By (4.5), we have the Mellin expansion
\[
W(a(Qx)wn(z)) = \int_{\chi} \chi(Qx)\gamma(1/2, \pi \otimes \chi)V_\chi(z) d\chi,
\]
where

\[ V_z(\chi) := \int_{t \in \mathbb{R}} e(tz)W(t)\chi(t)d^x t. \]

Since \( z \ll 1 \) and \( W \) is supported in a fixed compact set, we see that \( V_z \) is entire and \( V_z(\chi) \ll C(\chi)^{-N} \) in vertical strips. By Lemma 1 and \[ \text{IIIB, Lemma A.2} \] we have

\[ \gamma(1/2, \pi \otimes \chi) \ll C(\pi \otimes \chi)^{-\Re(\chi)} \ll C(\pi)^{-\Re(\chi)}C(\chi)^{2|\Re(\chi)|}, \]

which holds uniformly in any fixed vertical strip and for any \( \chi \) separated by \( \gg 1 \) from any pole of \( \gamma(1/2, \pi \otimes \chi) \) (in more detail, we apply Lemma 1 by writing \( \chi = \chi_0 \cdot |\cdot|^{\Re(\chi)} \), with \( \chi_0 \) unitary, and using that \( \gamma(1/2, \pi \otimes \chi) = \gamma(1/2 + \Re(\chi), \pi \otimes \chi_0) \) and \( C(\pi \otimes \chi_0) \ll C(\pi \otimes \chi) \). We obtain an adequate estimate in the case \( |x| \geq 1 \) by shifting the contour to \( \Re(\chi) = -N \), passing no poles.

It remains to consider the case \( |x| \leq 1 \). We shift the contour to \( \Re(\chi) = N \). Recall that \( Q \) is assumed sufficiently large in terms of “fixed” quantities; in particular, \( Q \) is much larger than any fixed power of \( N \). It follows that if \( \pi \) belongs to the discrete series, then this contour shift passes no poles.

The required estimate follows (in the stronger form obtained by omitting the term \( Q^{-N}|x|^{-1/2} \)). Suppose now that \( \pi \) belongs to the principal series. By hypothesis, \( \pi \) is tempered. Thus each pole that we cross is of the form \( \chi = \text{sgn}^a \cdot |\cdot|^{\sigma + it} \) with \( \sigma \geq 1/2 - \vartheta \) and \( t \asymp Q^{1/2} \). The required estimate follows from the rapid decay of \( V_z \).

\[ \square \]

4.4. The easy cases: estimates for \( F_w, F_1^2 \).

**Proposition 10.** The function \( F_w \) satisfies the conclusions of Theorem 4.

**Proof.** By the matrix identity

\[ a(y_2')wn(z)a(Q) = a(y_2/Q)wn(z/Q) \in G \]

and the relations \( W_1 = W_0^0, W_2 = a(Q)W_2^0 \), we write \( F_w(y_1, y_2) \) as

\[
\int_{z \in \mathbb{R}} W_1^0 (a(y_1')wn(z))W_2^0 (a(y_2/Q)wn(z))
\times W_3 (a(y_3')wn(z))\phi_0 (\arctan(z)) \frac{dz}{z^2 + 1}.
\]

We use (2.25) to bound the first two factors of the integrand as

\[
(4.7) \quad (y_1\partial_{y_1})^{N_1}(y_2\partial_{y_2})^{N_2}W_1^0 (a(y_1')wn(z))W_2^0 (a(y_2/Q)wn(z))
\ll (1 + |y_1|)^{-N} (1 + |y_2/Q|)^{-N}
\]

uniformly for \( z \ll 1 \). We bound the third factor by (4.6) with \( x = y_3/Q \) getting

\[
(y_3\partial_{y_3}) W_3 (a(y_3')wn(z)) \ll \min \left( \frac{|y_3|}{Q}^{-N}, \frac{|y_3|}{Q}^N + Q^{-N} \right).
\]
Recalling (4.1), it is easy to see that

$$(y_1 \partial_y_1)^{j_1} (y_2 \partial_y_2)^{j_2} F_w(y_1, y_2) \ll (1 + |y_1|)^{-N} \min \left(\frac{|y_2/Q|^{-N}}{Q}, \frac{1 + |y_2|^N}{Q^N}\right).$$

If $\Psi$ satisfies (4.3), then $e(\pm 2\sqrt{Qz\Psi(y_1/y_2)})$ is flat for $y_1 \ll 1$ and $Q^{1-\varepsilon} \ll |y_2| \ll Q^{1+\varepsilon}$, so up to a contribution that can go into $\mathcal{E}_3$ we have

$$(y_1 \partial_y_1)^{j_1} (y_2 \partial_y_2)^{j_2} e(\pm 2\sqrt{Qz\Psi(y_1/y_2)}) F_w(y_1, y_2)$$

$$\ll (1 + |y_1|)^{-N} \min \left(\frac{|y_2/Q|^{-N}}{Q}, \frac{1 + |y_2|^N}{Q^N}\right)$$

which is admissible for Theorem 4. □

**Proposition 11.** The function $F_1^\phi$ satisfies the conclusions of Theorem 4.

**Proof.** By definition,

$$F_1^\phi(y_1, y_2) = \int_{z \in \mathbb{R}} \left( \prod_{j=1, 2, 3} W_j(a(y_j')n'(z)) \right) \phi^\phi(z) \phi_0(\arctan(z)) \frac{dz}{z^2 + 1}.$$ 

For $|z| \ll 1/Q$, we have

$$W_1(a(y_1')n'(z)) \ll (1 + |y_1|)^{-N},$$

$$W_2(a(y_2')n'(z)) = W_2^0(a(Qy_2')n'(Qz)) \ll (1 + |y_2|Q)^{-N},$$

where we used (2.26). The required estimate $F_1^\phi \ll \mathcal{E}_2 \ll \mathcal{E}$ follows now from the weak a priori bound $\|W_3\|_\infty \ll Q^{O(1)}$.

To deduce the latter it suffices by the Iwasawa decomposition to estimate $W_3(a(y)k(\theta))$. We appeal first to Lemma 2, which gives for any $W \in \pi_3$ the estimate $W(a(y)) \ll \mathcal{S}_d(W)$ for some fixed $d$. Taking $W = k(\theta)W_3$, we obtain $W(a(y)) = W_3(a(y)k(\theta)) \ll \mathcal{S}_d(W_3) = \mathcal{S}_d(W_3)$. On the other hand, by [MV, §3.2.5] and the fact that $y \mapsto W_3(a(y))$ is a fixed bump function, we have $\mathcal{S}_d(W_3) \ll Q^d$ for some fixed $d'$. The required a priori bound follows. □

**4.5. The critical case: estimates for $F_1^\phi$.**

**Proposition 12.** The function $F_1^\phi$ satisfies the conclusions of Theorem 4.

**Proof.** Let $Q^{-1} \ll z$. We use the identity

$$a(y)n'(z)a(Q) = n(y/z)a(y/Qz^2)wn(1/Qz) \in G$$

to write

$$W_2(a(y_2')n'(z)) = e(y_2'/z)W_2^0(a(y_2'/Qz^2)wn(1/Qz)).$$

Similarly, we have

$$W_3(a(y_3')n'(z)) = W_3(n(y_3'/z)a(y_3'/z^2)wn(1/z))$$

$$= e(y_3'/z)W_3(a(y_3'/z^2)wn(1/z)).$$
Using the consequence $e(y_2/z)e(y_3/z) = e(-y_1/z)$ of the hypothesis (4.1), we may write $E^z_1(y_1, y_2)$ in the form

$$
\int_{z \in \mathbb{R}} W\left(y_1, \frac{y_2}{Qz^2}, z\right) e\left(-\frac{y_1(z^2 + 1)}{z}\right)
\times W_3\left(a\left(\frac{y_3}{z^2}\right) \Phi\left(\frac{1}{z}\right)\right) \phi(z) \phi_0(\arctan(z)) \frac{dz}{z^2 + 1}
$$

where

$$
W(x_1, x_2, z) := W_1^0\left(a(x_1(1 + z^2))n'(z)\right) W_2^0\left(a(x_2(1 + z^2))\right) \Phi(1/Qz).
$$

From now on we will use extensively the conventions on smooth weight functions stated in §2.4. In particular, $V$ denotes generally a flat function, not necessarily the same at every occurrence.

We apply the substitution $z \mapsto 1/z$ and a smooth dyadic partition of unity localizing $\pm z \times Z$ where $Z$ runs over $\ll 1$ values (e.g. powers of 2) satisfying $1 \ll Z \ll Q$. For notational simplicity we restrict to $z > 0$, the case $z < 0$ being essentially identical. Since $W$ is flat in $z$, it suffices to estimate

$$
I_0(y_1, y_2, y_3) = W\left(y_1, \frac{y_2Z^2}{Q}\right)
\times \int_{z \in \mathbb{R}} e\left(-y_1(z + z^{-1})\right) W_3\left(a(y_3(1 + z^2))\right) V\left(\frac{z}{Z}\right) \frac{dz}{|z|^2}
$$

where $W$ satisfies the estimates

$$
(x_1 \partial_{x_1})^j (x_2 \partial_{x_2})^j W(x_1, x_2) \ll \prod_{j=1}^2 (1 + |x_j|)^{-N}.
$$

At this point we can easily deal with the contribution $Z \gg Q^{1/2 - \varepsilon}$. A trivial estimate returns the bound

$$
\ll Q^{-1/2} (1 + |y_1|)^{-N} (1 + |y_2|)^{-N} = \mathcal{E}_1(y_1, y_2)
$$

which is acceptable.

We can also easily deal with the contribution $Z \ll 1$. In this case, (4.6) yields

$$
(y_3 \partial_{y_3})^j W_3\left(a(y_3(1 + z^2))\right) \ll \min\left(\left(\frac{|y_3|}{Q}\right)^N + Q^{-N}, \left(\frac{|y_3|}{Q}\right)^{-N}\right).
$$

Recalling (4.1), we argue as in the proof of Proposition 10 that under the present assumption $Z \ll 1$, up to an error of size $\mathcal{E}_3$, we have

$$
(y_1 \partial_{y_1})^j (y_2 \partial_{y_2})^j I_0(y_1, y_2, y_3) \ll (1 + |y_1|)^{-N} \min\left(\left(\frac{|y_2|}{Q}\right)^{-N}, \frac{1 + |y_2|}{Q^N}\right).
$$
but then also

\[ (y_1 \partial_{y_1})^j_1(y_2 \partial_{y_2})^j_2 e(\pm 2\sqrt{Q\Psi(y_1/y_2)})I_0(y_1, y_2, y_3) \]

\[ \ll (1 + |y_1|)^{-N} \min \left( \frac{|y_2|}{Q}, \frac{1 + |y_2|^N}{Q^N} \right) \]

as required for the bound for \( \mathcal{N}_{\pm} \). So from now on we assume

\[ Q^\varepsilon \ll Z \ll Q^{1/2 - \varepsilon}. \]

We appeal to the (rapidly-convergent) integral formula (4.5) to rewrite the integral over \( z \) in (4.8) as

\[ I := \int_{\mathbb{R}(\chi) = 0} \gamma(1/2, \pi_3 \otimes \chi) \]

\[ \times \int_{z \in \mathbb{R}} \chi(y_3(z^2 + 1)) e(-y_1(z + z^{-1})) V \left( \frac{z}{Z} \right) I_1(z, \chi) \frac{dz}{|z|^2} d\chi \]

where

\[ I_1(z, \chi) := \int_{t \in \mathbb{R}} e(tz) \chi(t) W_3(a(t)) \frac{dt}{|t|}, \]

We keep in mind that the bound (4.9) allows us to assume \( y_1 \ll 1 \), otherwise we can bound \( I \) trivially by \( \mathcal{E}_3 \).

Let us write \( \chi = \|t\chi\| \text{ sgn } a_{\chi} \) with \( t_{\chi} \in \mathbb{R} \) and \( a_{\chi} \in \{0, 1\} \). We split the \( \chi \)-integral in (4.10) according to the value of \( a_{\chi} \) and regard that value as fixed from now on. We have expressed \( I \) as a triple integral in \( t, z, t_{\chi} \), and we will apply stationary phase in each of the variables, one at a time. Here \( z \asymp Z \), \( t \asymp 1 \), and we will see in a moment that the \( t \)-integral is negligible unless \( t_{\chi} \asymp Z \). Stationary phase saves a factor \( Z^{1/2} \) in each of these integrals, so we expect that \( I \) is of size \( Z^{-3/2} \), and we can explicitly compute the oscillatory behavior.

**Step 1: the \( t \)-integral.** We apply stationary phase analysis to find, for each fixed \( N \in \mathbb{Z}_{\geq 0} \), a nice function \( V \) so that

\[ I_1(z, \chi) = z^{-1/2} V \left( \frac{-t_{\chi}}{z}, \frac{z}{Z} \right) \chi \left( \frac{t_{\chi}}{2\pi ez} \right) + O(Q^{-N}). \]

To see this, we apply Lemma 4 if \(-t_{\chi}/z \asymp 1\) by taking

\[ \phi(t) = \phi(t; z, t_{\chi}) = 2\pi z t + t_{\chi} \log |t|, \quad t^* = -\frac{t_{\chi}}{2\pi z}, \]

\[ (X_1, X_2, X_3) = (1, Z, Z), \quad Y = Z \]

and otherwise the integral is negligible by Lemma 3 with \( U = P = 1, S = Y = Z \). Here and henceforth the contribution of all negligible error terms is covered by \( \mathcal{E}_3 \) in the statement of Theorem 4. We deduce that \( I \) is
given up to acceptable error (that can go into $\mathcal{E}_3$) by

$$
\int_{\Re(\chi)=0} \gamma(1/2, \pi_3 \otimes \chi) \chi \left( \frac{t_\chi y_3}{2\pi e} \right) \chi(z) e(-y_1 z) V \left( \frac{z}{Z}, -\frac{t_\chi}{Z} \right) \chi(1 + z^{-2}) e(-y_1 z^{-1}) \frac{dz}{|z|^{5/2}} d\chi.
$$

Note that for $|t_\chi| \asymp z \gg Q^\varepsilon$ and $y_1 \ll Q^\varepsilon$ the function $\chi(1 + z^{-2}) e(-y_1 z^{-1})$ is flat in both $z$ and $t_\chi$, so we can incorporate this factor into $V$ (and continue to call this new function $V$). Thus we may reduce further to estimating

$$
(4.12) \quad \int_{\Re(\chi)=0} V \left( \frac{-t_\chi}{Z} \right) \gamma(1/2, \pi_3 \otimes \chi) \chi \left( \frac{t_\chi y_3}{2\pi e} \right) I_2(\chi, y_1) d\chi
$$

with

$$
I_2(\chi, y_1) := \int_{z \in \mathbb{R}} V \left( \frac{z}{Z} \right) \chi(z) e(-y_1 z) \frac{dz}{|z|^{5/2}}.
$$

**Step 2: the $z$-integral.** By stationary phase analysis, we may find for each fixed $N$ a nice function $V$ (again potentially different from previous versions of $V$) so that

$$
(4.13) \quad I_2(\chi, y_1) = Z^{-2} V \left( \frac{-t_\chi}{Z y_1} \right) \chi \left( \frac{t_\chi}{2\pi e y_1} \right) + O(Q^{-N}),
$$

whenever $-t_\chi \asymp Z$. We apply Lemma 4, taking

$$
\phi(t) = \phi(t; y_1, t_\chi) = -2\pi y_1 t + t_\chi \log |t|, \quad t^* = \frac{t_\chi}{2\pi y_1},
$$

$$(X_1, X_2, X_3) = (Z, 1, Z), \quad Y = Z
$$

(if $|y_1| \asymp 1$, and otherwise the integral is negligible by Lemma 3 with $U = Y = P = Z = S = 1$). Thus we reduce to studying

$$
(4.14) \quad Z^{-2} V(-y_1) \int_{\Re(\chi)=0} V \left( \frac{-t_\chi}{Z} \right) \gamma(1/2, \pi_3 \otimes \chi) \chi \left( \frac{t_\chi^2 y_3}{(2\pi e)^2 y_1} \right) d\chi,
$$

for certain nice functions $V$.

**Step 3: the $t_\chi$-integral.** We now evaluate $\gamma(1/2, \pi_3 \otimes \chi)$ asymptotically using Stirling’s formula, and appeal in particular to (2.19) and the subsequent explicit formulæ. Since $|t_\chi| \asymp Z \ll Q^{1/2-\varepsilon}$, both are applicable. We start with the principal series case. In view of (2.20), the relevant phase function may be written

$$
\phi(t_\chi) = \phi(t_\chi; r, y_1, y_3)
$$

$$
= t_\chi \log \left| \frac{t_\chi^2 y_3}{(2\pi e)^2 y_1} \right| + (r - t_\chi) \log(r - t_\chi) - (r + t_\chi) \log(r + t_\chi) + 2t_\chi \log(2\pi e)
$$
For convenience, we record some relevant derivatives:

\[
\frac{\partial}{\partial t_\chi} \phi(...) = \log \left| \frac{t_\chi^2 y_3}{y_1 (r^2 - t_\chi^2)} \right|, \quad \frac{\partial}{\partial r} \phi(...) = \log \frac{r - t_\chi}{r + t_\chi},
\]

\[
\frac{\partial}{\partial y_1} \phi(...) = -\frac{t_\chi}{y_1}, \quad \frac{\partial}{\partial y_3} \phi(...) = \frac{t_\chi}{y_3},
\]

\[
\frac{\partial^2}{\partial t_\chi^2} \phi(...) = \frac{2r^2}{t_\chi (r^2 - t_\chi^2)}, \quad \frac{\partial^2}{\partial r \partial t_\chi} \phi(...) = -\frac{2r}{r^2 - t_\chi^2}.
\]

Using (4.1), we deduce that

\[
|t_\chi^*| = r |y_1/y_2|^{1/2}.
\]

The integral (4.14) is negligible by an application of Lemma 3 with \( U = Y = P = Z, S = 1 \) unless \( \text{sgn}(y_1) = -\text{sgn}(y_2) \) and \( |y_2| \asymp Q/Z^2 \). In this case, we apply Lemma 4 with

\[
(X_1, X_2, X_3, X_4) = (Z, Q^{1/2}, 1, Q/Z^2), \quad Y = Z.
\]

We compute

\[
\phi \left( r \left| \frac{y_1}{y_2} \right|^{1/2} ; r, y_1, y_3 \right) = 2r \Psi \left( \frac{y_1}{y_2} \right), \quad \Psi(y) = |y|^{1/2} \arctan(|y|) - \arctanh(|y|^{1/2})
\]

where

\[
\Psi(y) = -|y|^{1/2} + O(|y|^{3/2}), \quad \Psi(y) \ll |y|^{1/2-j}.
\]

The analysis in the discrete series case is similar. We write \( \kappa = (k - 1)/2 \) and apply Lemma 4 and (2.21) with

\[
\phi(t_\chi) = \phi(t_\chi; \kappa, y_1, y_3) = t_\chi \log \left| \frac{t_\chi^2 y_3}{e^{2y_1 \kappa^2}} \right| - 2\kappa \arctan \frac{t_\chi}{\kappa} - t_\chi \log \frac{\kappa^2 + t_\chi^2}{e^{2\kappa^2} y_3},
\]

\[
\frac{\partial}{\partial t_\chi} \phi(...) = \text{log} \left| \frac{t_\chi^2 y_3}{y_1 (\kappa^2 + t_\chi^2)} \right|, \quad \frac{\partial}{\partial \kappa} \phi(...) = -2\arctan \frac{t_\chi}{\kappa}, \quad \frac{\partial}{\partial y_1} \phi(...) = -\frac{t_\chi}{y_1},
\]

\[
\frac{\partial}{\partial y_3} \phi(...) = \frac{t_\chi}{y_3}, \quad \frac{\partial^2}{\partial t_\chi^2} \phi(t_\chi) = \frac{2\kappa^2}{t_\chi (\kappa^2 + t_\chi^2)}.
\]

We have \( \phi^{(1,0,0,0)}(t_\chi^*; \kappa, y_1, y_3) = 0 \) if and only if \( \text{sgn}(y_1) = \text{sgn}(y_3) \) and

\[
|t_\chi^*| = \kappa \sqrt{\frac{y_1}{y_3 - y_1}}.
\]

Again this is in the support of the integrand if and only if \( |y_2| \asymp Q/Z^2 \), otherwise the integral is negligible. We compute

\[
\phi \left( \kappa \sqrt{\frac{y_1}{y_3 - y_1}} ; \kappa, y_1, y_3 \right) = -2\kappa \arctan \sqrt{\frac{y_1}{y_3 - y_1}} = 2\kappa \tilde{\Psi} \left( \frac{y_1}{y_2} \right),
\]

\[
\tilde{\Psi}(y) = -\arctan \left( \left( \frac{1}{|x|} + 2 \right)^{-1/2} \right)
\]

where \( \tilde{\Psi} \) satisfies the same formul\ae\, as in (4.15).
By the definition of the conductor in (2.20) and (2.21) we conclude that (4.14) equals (up to a negligible error)

\[ Z^{-3/2}V(-y_1, \frac{y_2Z^2}{Q}) e\left(-2\sqrt{Q}\Psi\left(\frac{y_1}{y_2}\right)\right) \]

in the principal series case and

\[ Z^{-3/2}V(-y_1, \frac{y_2Z^2}{Q}) e\left(-2\sqrt{Q}\tilde{\Psi}\left(\frac{y_1}{y_2}\right)\right) \]

in the discrete series case.

We made in the beginning the assumption \( z > 0 \). The case \( z < 0 \) leads to an analogous expression, with the minus-sign in the exponential removed. This completes the proof. \( \square \)

5. A shifted convolution problem

5.1. Some preparation.

5.1.1. Bessel functions. We need the following uniform asymptotic formulæ for Bessel functions. For \( t \in \mathbb{R} \), \( |t| > 1 \) and \( x > 0 \), we have [EMOT, 7.13.2(17)]

\[
\frac{J_{2it}(2x)}{\cosh(\pi t)} = \sum_{\pm} e^{\pm 2i\omega(x,t)} \frac{f_N^{\pm}(x,t)}{x^{1/2} + |t|^{1/2}} + O_N((x + |t|)^{-N}),
\]

\[
\omega(x,t) = |t| \cdot \arcsinh \frac{|t|}{x} - \sqrt{t^2 + x^2},
\]

where for any fixed \( N > 0 \) the function \( f_N^{\pm} \) is flat. The error term estimate stated in [EMOT] is \( O(x^{-N}) \), but for \( x \leq t^{1/3} \), say, the estimate \( O(|t|^{-N}) \) follows from the power series expansion [GR, 8.402] of \( J_{2it}(x) \). When we apply this formula in practice, we first extract the negligible error \( O_N((x + |t|)^{-N}) \) in the series expansion given by [EMOT, 7.13.2(17)] and [GR, 8.402], without pausing to estimate any derivatives of that error. We then differentiate the remaining series expansion to verify the flatness condition.

For future reference we note the identities

\[
\frac{\partial}{\partial t} \omega(x,t) = \arcsinh \frac{t}{x}, \quad \frac{\partial^2}{\partial t^2} \omega(x,t) = \frac{1}{x^2 + t^2},
\]

\[
\frac{\partial}{\partial x} \omega(x,t) = -\sqrt{x^2 + t^2}, \quad \omega(x,t) = -x + \frac{t^2}{2x} + O\left(\frac{t^4}{x^3}\right).
\]

For \( |t| \geq 2x > 0 \) we have by [EMOT, 7.13.2(19)] (again coupled with the power series expansion [GR, 8.445 & 8.485] for \( x \leq t^{1/3} \) or [Ba, (20) with \( z \leq 1/2 \)]

\[
K_{2it}(2x) \cosh(\pi t) = |t|^{-1/2} \sum_{\pm} e^{\pm 2i\omega^*(x,t)} g_N^{\pm}(x,t) + O_N((x + |t|)^{-N}),
\]

\[
\omega^*(x,t) = |t| \cdot \arccosh \frac{|t|}{x} - \sqrt{t^2 - x^2},
\]
where for any fixed $N > 0$ the function $g_N^\pm$ is flat.
Similarly, for $x \geq 2k$ we have [Ol, (4.24)]

\[ J_{2k-1}(2x) = \sum_{\pm} e^{\pm 2ik\tilde{\omega}(x,k)} \frac{h_N^\pm(x,k)}{x^{1/2}} + O_N((x+k)^{-N}), \]

\[ \tilde{\omega}(x,k) = -k \arctan \sqrt{\frac{x^2}{k^2} - 1 + \sqrt{x^2 - k^2}}, \]

where $h_N^\pm$ is flat. For fixed index, these formulæ simplify greatly, and we have

\[ J_{2it}(2x) = \frac{1}{x^{1/2} + x^{2|3t|}} \sum_{\pm} e^{\pm 2ix f^\pm(x)}, \]

\[ K_{2it}(2x) = \frac{e^{-2x}}{x^{1/2} + x^{2|3t|}} g(x) \]

for $t \in \mathbb{C}$, $k \in \mathbb{N}$, $x > 0$ with $|3t| \leq 1/4$ (for simplicity) and $t,k$ in a fixed compact set, where $f^\pm, g, h^\pm$ can be chosen to be flat (depending on $t$ or $k$). See [BM, Lemma 15] for details on how to glue together the asymptotic formulæ for $x > 1$ and $x < 1$.

**Remark:** As the referee remarked, [EMOT] contains no proofs. The expansions (5.1), (5.3), (5.4) are all relatively simple to obtain, since we are in the so-called oscillatory range away from possible degenerate points ($t = x$ for the $K$-function and $k = x$ for the $J$-function with real order). All three uniform asymptotic expansions can be obtained from the integral representations [GR, 8.421.1/2, 8.405], [GR, 8.432.4] and [GR, 8.411]

\[ K_{it}(x) = \frac{1}{2 \cosh(\pi t/2)} \int_{-\infty}^{\infty} \cos(x \sinh v) \exp(itv) dv, \]

\[ J_{it}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} (\cosh(\pi t/2) \sin(x \cosh v) - i \sin(\pi t/2) \cos(x \cosh v)) \exp(itv) dv, \]

\[ J_k(x) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \exp(-ki\theta + ix \sin \theta) d\theta \]

by an application of Lemma 4. For the improper integrals, note that the tail can be estimated by partial integration using Lemma 3, cf. e.g. [BB, Section 4.4].

5.1.2. *Jutila’s circle method.* We quote Jutila’s circle method [Ju1].

**Lemma 13.** Let $Q \geq 1$ and $V$ be a smooth, nonnegative, nonzero function with support in $[1,2]$. For $r \in \mathbb{Q}$ write $I_r(\alpha)$ for the characteristic function
of the interval \([r - 1/Q, r + 1/Q]\) and define

\[
\Lambda := \sum_{q} V\left(\frac{q}{Q}\right) \phi(q), \quad I(\alpha) = \frac{Q}{2\Lambda} \sum_{q} V\left(\frac{q}{Q}\right) \sum_{d \mod q \atop (d, q) = 1} I_{d/q}(\alpha).
\]

Then \(I(\alpha)\) is a good approximation to the characteristic function on \([0, 1]\) in the sense that

\[
\int_{0}^{1} (1 - I(\alpha))^{2} d\alpha \ll \varepsilon Q^{-1}
\]

for any \(\varepsilon > 0\).

5.2. Notation and set-up. Let \(T\) be a large positive real number. We recall once again the notation and conventions from §2.4, in particular with respect to weight functions \(V\). Moreover, \(A \ll B\) denotes \(A \ll T^\varepsilon B\). We consider two more parameters \(M\) and \(H\) satisfying

\[
M \ll T^2, \quad H = T^{1/3 + \varepsilon},
\]

and \(\nu \in \mathbb{Z} \setminus \{0\}\) with \(\nu \approx 1\). The choice of \(H\) will eventually turn out to be the optimal choice, and it simplifies the argument if we make it right away at this point. With slightly more extra work, we could run the same argument for \(T^{1/3} \ll H \ll T^{1-\varepsilon}\).

Let \(g\) a fixed holomorphic or Maaß Hecke eigenform for \(SL_2(\mathbb{Z})\) with Hecke eigenvalues \(\lambda_g(n)\) of weight \(k_g\) or spectral parameter \(t_g\). Let \(\sigma_t(m) = \sum_{ab=n} a^t b^{-it}\). We fix a choice of sign \(\pm\). With these notations we consider

\[
\mathcal{L} := \frac{1}{TM^{1/2}} \sum_{2\pi T \leq t_j \leq 2\pi(T + H)} \frac{1}{L(Ad^2 u_j, 1)} \times \left| \sum_m V\left(\frac{m}{M}\right) \lambda_j(m) \lambda_g(m + \nu) \exp \left( \pm 2i \frac{t_j \sqrt{|\nu|}}{\sqrt{m}} \right) \right|^2 + \frac{1}{TM^{1/2}} \int_{2\pi T}^{2\pi(T + H)} \frac{1}{\zeta(1 + 2it)^2} \times \left| \sum_m V\left(\frac{m}{M}\right) \sigma_t(m) \lambda_g(m + \nu) \exp \left( \pm 2i \frac{t \sqrt{|\nu|}}{\sqrt{m}} \right) \right|^2 dt \frac{2\pi}{2\pi}
\]

where \(u_j\) runs through Hecke Maaß cusp forms for \(SL_2(\mathbb{Z})\) with Hecke eigenvalues \(\lambda_j(n)\) and spectral parameter \(t_j \in [2\pi T, 2\pi(T + H)]\). The right hand side of (5.7) is essentially a combination of (1.4) and (1.9); this explains its relevance.
Analogously we also consider the holomorphic analogue

\[
\tilde{L} := \frac{1}{TM^{1/2}} \sum_{4\pi T \leq k \leq 4\pi (T + H)} \sum_{u_j \in B_k} \frac{1}{L(\text{Ad}^2 u_j, 1)} \times \sum_{m} V\left(\frac{m}{M}\right) \lambda_j(m) \lambda_g(m + \nu) \exp \left( \pm 2i \frac{k \sqrt{\nu}}{\sqrt{m}} \right)^2
\]

where \( u_j \) runs over a Hecke eigenbasis \( B_k \) of cusp forms of weight \( k \). For simplicity let us assume \( \nu > 0 \), the other case being essentially identical.

The aim of this section is to prove the following theorem.

**Theorem 5.** Let \( T \) be a large parameter and let \( \mathcal{L}, \tilde{L} \) be defined as in (5.7), (5.8) with \( M, H \) as in (5.6). Then \( \mathcal{L}, \tilde{L} \ll TH \).

The proof of Theorem follows the steps outlined in §1.4. In particular, we will eventually transform the spectral sum (5.7) into a “reciprocal” spectral sum of length \( T/H \) in §5.9 to which we apply the large sieve.

For \( M \leq T^{2/3+\varepsilon} \) we can estimate trivially (using a standard Rankin–Selberg bound)

\[
\mathcal{L} \ll \frac{1}{TM^{1/2}} THM^2 = HM^{3/2} \ll TH
\]

in agreement with Theorem 5. From now on we assume \( M > T^{2/3+\varepsilon} \). Then

\[
\exp \left( \pm 2i \frac{(t - 2\pi T) \sqrt{\nu}}{\sqrt{m}} \right)
\]

is flat for \( t \in [2\pi T, 2\pi (T + H)] \) by our choice of \( H \) in (5.6), so we can replace \( t_j \) and \( t \) with \( 2\pi T \) in the exponential (using the by now familiar device to separate variables in nice functions after having multiplied by a suitable function with compact support in \( 2\pi T + O(H) \)). We restrict to the positive sign in the exponential, the negative sign being essentially identical.

We can majorize the characteristic function on \( [2\pi T, 2\pi (T + H)] \) by \( h(t) = \exp\left(-\frac{2\pi t}{H}\right)^2 \) and then symmetrize with respect to \( t \mapsto -t \) in order to make the expression amenable for the Kuznetsov formula. The exact shape of the function plays no role.

### 5.3. Application of the spectral summation formula.

We open the square in (5.7) and (5.8) and apply the Kuznetsov formula [IK, Theorem 16.3, along with a conversion from Fourier coefficients to Hecke eigenvalues]. Since \( \int h(t) \tanh(\pi t) dt \ll TH \) by our choice of \( h \), the diagonal term is bounded by

\[
\ll \frac{1}{TM^{1/2}} TH \cdot M = M^{1/2}H \ll TH
\]

by (5.6) in agreement with Theorem 5. For the off-diagonal term we must understand the Bessel transform

\[
\int_{t \in \mathbb{R}} \exp \left( -\left( \frac{t - 2\pi T}{H} \right)^2 \right) \frac{J_{2\nu}(x)}{\cosh(\pi t)} dt
\]
As usual, we use holomorphicity to shift the contour a bit; in this way, we can truncate the $c$-sum in (5.12) by some large power of $T$ (cf. [JM1, p. 75]). Having this done, we may smoothly truncate the integral to the interval $[2\pi T - HT^c, 2\pi T + HT^c]$. For $x \leq T^{1-\varepsilon}H$, we apply Lemma 3 and the uniform asymptotic formula (5.1) (along with (5.2)) with

$$ S = \min(1, T/x), \quad Q = Y = T + x, \quad U = H $$

to see that the integral is negligible. Having recorded the condition

(5.10) \[ x \geq T^{1-\varepsilon}H, \]

we don’t exploit any further cancellation in the integral. Using (5.1) along with a Taylor expansion we have, for $t \in [2\pi T - HT^c, 2\pi T + HT^c]$ and $x \geq T^{1-\varepsilon}H$, the approximation

(5.11) \[ J_{2\pi t}(2x) \cosh(\pi t) = x - \frac{1}{2} + \sum_{\pm} e^{\pm 2i(-x + \frac{1}{2}(2\pi T)^2/x)}F_{\pm}^2(x, t) + O(|t|^{-N}) \]

for a flat function $F_{\pm}^2$. We substitute this into (5.9) and integrate trivially over $t$. Thus it suffices to estimate the following off-diagonal term

(5.12) \[
\frac{TH}{TM^{1/2}} \sum_{m_1, m_2} V\left(\frac{m_1}{M}, \frac{m_2}{M}\right) \sum_{c} \lambda_g(m_1 + \nu)\lambda_g(m_2 + \nu) \frac{S(m_1, m_2, c)}{C} \times \frac{C^{1/2}}{M^{1/2}} e^{{\pm 2\sqrt{m_1 m_2} c - \frac{T^2 c}{\sqrt{m_1 m_2}}} + 2T^{1/2} \frac{\sqrt{m_2} - \sqrt{m_1}}{\sqrt{m_1 m_2}}}
\]

by $\ll TH$ (with all sign combinations), as required in Theorem 5, where $C$ runs through $\ll 1$ numbers (e.g. powers of 2) satisfying

(5.13) \[ 1 \leq C \leq \frac{M}{TH} \]

and each weight function $V$ is nice.

We pause for a moment and consider the average $\overline{L}$ over holomorphic forms, in which we replace the Kuznetsov formula with the Petersson formula [IK, Theorem 14.5]. Here the analogue of (5.9) is

$$ \sum_{k \in \mathbb{Z}} i^k V\left(\frac{k - 4\pi T}{H}\right) k J_{k-1}(2\pi x). $$

Using the Fourier representation [GR, 8.411.1] of the Bessel function coupled with Poisson summation, this equals

$$ \sum_{k \in \mathbb{Z}} i^k V\left(\frac{k - 4\pi T}{H}\right) k \int_{-1/2}^{1/2} e((1 - k)\theta + x \sin 2\pi \theta) d\theta 
= \frac{1}{2} \sum_{h \in \mathbb{Z}, h \text{ odd}} \int_{-1/2}^{1/2} \int_{-\infty}^{\infty} V\left(\frac{y - 4\pi T}{H}\right) ye\left(\frac{yh}{4} + (1 - y)\theta + x \sin 2\pi \theta\right) dy d\theta. $$
The $y$-integral is negligible unless $h = \pm 1$ and $\theta = \pm 1/4 + O(T^\varepsilon H^{-1})$, but then the remaining $\theta$-integral, smoothly truncated to the latter range, is negligible unless $x \geq T^{1-\varepsilon} H$. This last condition is the analogue of (5.10). The analogue of (5.11), derived from (5.4), is

$$i^k J_{2k}(x) = x^{-1/2} \sum e^{\pm 2i(x + \frac{1}{4}(2\pi T)^2/x)} \tilde{F}^\pm(x,k) + O(k^{-N})$$

for the present range of variables which leads to the same expression as (5.12) except for a sign in the exponential which won’t play a role later.

We now continue with the analysis of (5.12). In preparation for an application of Voronoi summation, we shift the variables $m_1, m_2$ by $\nu$. By a Taylor expansion, this makes no difference in the exponential, the resulting correction term being flat. It therefore suffices to bound

$$\frac{H}{M} \sum_{m_1,m_2} V\left( \frac{m_1}{M}, \frac{m_2}{M} \right) \sum_c V\left( \frac{c}{C} \right) S(m_1 - \nu, m_2 - \nu, c) \sqrt{C} \lambda_g(m_1) \lambda_g(m_2)$$

$$\times e \left( \pm \left( \frac{2\sqrt{m_1 m_2}}{c} - \frac{T^2 c}{\sqrt{m_1 m_2}} \right) \pm \frac{2T^1/2(\sqrt{m_2} - \sqrt{m_1})}{\sqrt{m_1 m_2}} \right).$$

5.4. Preparatory interlude. We pause to recall the Voronoi formula (see e.g. [BM, Lemma 25 & Lemma 6]): for $c \in \mathbb{N}$ and $(b,c) = 1$, we have

$$\sum_n V(n) \lambda_g(n) \exp\left( \frac{bn}{c} \right) = \frac{1}{c} \sum \sum_n V^\pm\left( \frac{n}{c^2} \right) \lambda_g(n) \exp\left( \pm \frac{bn}{c} \right),$$

$$V^\pm(y) = \int_0^\infty V(x) J^\pm(4\pi\sqrt{xy}) \, dx,$$

where $J^\pm = J^\pm_g$ is given for $g$ a Maass form of eigenvalue $1/4 + t_g^2$ by

$$J^+(x) = \pi \frac{J_{2t_g}(x) - J_{-2t_g}(x)}{\sinh(\pi t_g)}, \quad J^-(x) = 4 \cosh(\pi t_g) K_{2t_g}(x)$$

and for $g$ a holomorphic form of weight $k_g$ by

$$J^+(x) = 2\pi^k e^{i\pi x} J_{k_g-1}(x), \quad J^-(x) = 0.$$

In the following sections we will twice have to compute integrals of the form

$$\int_{x \in \mathbb{R}} V\left( \frac{x}{M} \right) e^{(\alpha x^{1/2} + \beta x^{-1/2})} \, dx$$

for certain $\alpha, \beta \in \mathbb{R}$ satisfying

$$|\alpha| M^{1/2} + |\beta| M^{-1/2} \gg M^\varepsilon.$$

In this case it follows from Lemma 3 with

$$U = M^{1-\varepsilon/2}, \quad P = M, \quad S = |\alpha| M^{-1/2} + |\beta| M^{-3/2}, \quad Y = MS.$$
that (5.15) is negligible unless $\beta/\alpha \asymp M$ (which, by the conventions of §2.4, implies in particular that $\text{sgn}(\alpha) = \text{sgn}(\beta)$) in which case by Lemma 4 (after restricting to dyadic ranges $\alpha \asymp A$ and $\beta \asymp B$ and also possibly restricting the support of $V$ to a neighbourhood of $t^*$) with

$$
\phi(t) = \phi(t; \alpha, \beta) = \alpha t^{1/2} + \beta t^{-1/2}, \quad t^* = \frac{\beta}{\alpha}, \quad \frac{\partial}{\partial t}\phi(t^*; \alpha, \beta) = \frac{\beta}{2 t^{5/2}} \asymp \frac{\beta}{M^{5/2}},
$$

it equals

$$
(5.17) \quad \frac{M^{5/4}}{|\beta|^{1/2}} V\left(\frac{\alpha}{A}\right) V\left(\frac{\beta}{B}\right) e\left(2 \text{sgn}(\alpha) \sqrt{\alpha \beta}\right),
$$
as usual with different functions $V$, and up to a negligible error.

5.5. Application of Voronoi summation. We open the Kloosterman sum in (5.14) and apply the Voronoi formula to the following $m_2$-sum:

$$
\sum_{m_2} V\left(\frac{m_2}{M}\right) \lambda_g(m_2) e\left(\frac{d m_2}{c}\right) e\left(\pm \left(\frac{2 \sqrt{m_1 m_2}}{c} - \frac{T^2 c}{\sqrt{m_1 m_2}} \pm \frac{2 T \nu^{1/2}}{\sqrt{m_2}}\right)\right)
$$

where $(d, c) = 1$. We analyze the integral transforms using (5.4) and (5.5).

In the Maaß case, we see that the $J^-$-term is negligible thanks to the rapid decay of the Bessel $K$-function and the consequence $M/C^2 \gtrsim H^2 \geq T^{2/3+\epsilon}$ of our hypotheses (5.6) and (5.13). In either case, the $J^+$ term contributes (5.18)

$$
\sum_{m_2} \lambda_g(m_2) e\left(-\frac{d m_2}{c}\right) \frac{1}{c} \int_{x \in \mathbb{R}} V\left(\frac{x}{M}\right) \left(\frac{\sqrt{m_2 x}}{c}\right)^{1/2} + \left(\frac{\sqrt{m_2 x}}{c}\right)^{2|\text{sgn}|} e\left(\sigma_1 \frac{2 \sqrt{m_1 x}}{\sqrt{m_1}} + \sigma_2 \frac{2 T \nu^{1/2}}{\sqrt{x}}\right) d x
$$

with $\sigma_1, \sigma_2, \sigma_3 \in \{\pm 1\}$, and where as usual the meaning of $V$ may have changed. The $x$-integral is of the shape (5.15) with

$$
\alpha = \frac{2(\sigma_1 \sqrt{m_1} + \sigma_3 \sqrt{m_2})}{c}, \quad \beta = -\sigma_1 \frac{T^2 c}{\sqrt{m_1}} + \sigma_2 2 T \nu^{1/2}.
$$

For the analysis of this integral it is important to note that

$$
\frac{\sqrt{m_1} x}{c} \asymp \frac{M}{C}
$$
is by at least a factor $H^2 T^{-\epsilon}$ larger than

$$
\frac{T^2 c}{\sqrt{m_1 x}} \asymp \frac{T^2 C}{M}
$$

by (5.6) and (5.13), and the latter is larger than $T \nu^{1/2} x^{-1/2} \ll T M^{-1/2}$ unless $C \leq 1$ and $T^2 \ll M$. Let us define (5.19)

$$
R := T^2 C^2 / M.
$$
If $R \geq T^\varepsilon$, then (5.16) is satisfied, and we conclude from the discussion in the previous subsection that the $x$-integral is negligible unless $\sigma_1 = -\sigma_3$ and

$$m_2 - m_1 \asymp R.$$

If $R \ll 1$, the same argument still shows that $m_2 - m_1 \ll R \ll 1$ (which allows in particular $m_1 = m_2$). In particular, $\sqrt{m_2x}/c \asymp M/C \gg T^{1-\varepsilon}H \gg 1$ is large and therefore we can drop the term containing $|3t_g|$ in (5.18).

Before we proceed, we estimate the total contribution of $R \ll 1$ in (5.18) when substituted into (5.14) trivially by

$$\lesssim \frac{H}{M} \cdot M \cdot M^{1/2} \approx HM^{1/2} \ll HT,$$

which is acceptable for Theorem 5. So from now on we assume that

(5.20) $R \geq T^\varepsilon$.

In view of (5.6) we can then also assume

(5.21) $T^2C/M \geq T^\varepsilon$.

For such $R$ and $m_2 - m_1 \asymp R$ and $\sigma_1 = -\sigma_3$ we see from (5.17) that (5.18) can be recast as

$$\frac{M/T}{c} \sum_{m_2} V\left(\frac{m_2 - m_1}{R}\right) \lambda_g(m_2)e\left(-\frac{dm_2}{c}\right)$$

$$\times e\left(\pm \sqrt{8(\sqrt{m_2} - \sqrt{m_1})\frac{T^2}{\sqrt{m_1}} \pm \frac{2T\nu^{1/2}}{c}}\right),$$

up to a negligible error and for suitable sign combinations. Plugging back into (5.14) and calling $r = m_2 - m_1$, $m = m_1$, we obtain a total contribution

$$\frac{H}{TC^{1/2}} \sum_r \sum_m \sum_c V\left(\frac{r}{R}\right) \frac{m}{M} \cdot \frac{c}{C} S(-r - \nu, -\nu, c) \lambda_g(m)\lambda_g(m + r)$$

$$\times e\left(\pm \sqrt{8(\sqrt{m + r} - \sqrt{m})\frac{T^2}{\sqrt{m}} \pm \frac{2T\nu^{1/2}}{c}} \pm \frac{2T\nu^{1/2}}{\sqrt{m}}\right).$$

For notational simplicity the previous display deals only with the case $r > 0$. The case $r < 0$, coming from the holomorphic average $\tilde{\mathcal{L}}$ can be treated in the same way. We simplify the exponential a bit using suitable Taylor expansions. First, using the expansion $\sqrt{1 + 2x} - 1 = x^{1/2} - x^{3/2}/4 + \cdots$,
we replace $\sqrt{|m + r - \sqrt{m}|}$ with $(r/2)^{1/2}m^{-1/4}$, the error being flat since, by (5.6) and (5.13),

$$\frac{1}{M^{1/4}} \cdot \frac{R^{3/2}}{M^{3/2}} \cdot \frac{T}{M^{1/4}} = \frac{T^4C^3}{M^3} \lesssim \frac{T}{H^3} \lesssim 1.$$ 

Similarly, we can replace

$$\left| \frac{T^2}{\sqrt{m}} \pm \frac{2T^{1/2}}{c} \right|^{1/2}$$

with

$$\left| \frac{T}{m^{1/4}} \pm \frac{m^{1/4}T^{1/2}}{c} \right|$$

up to a flat function. Thus it suffices to bound (with various sign combinations)

$$\frac{H}{T^2C^1/2} \sum_r \sum_m \sum_c V\left( \frac{r}{R}, \frac{m}{M} \cdot \frac{c}{C} \right) S(-r - \nu, -\nu, c)$$

$$\times \lambda_g(m) \lambda_g(m + r) e\left( \pm \frac{2T^{1/2} r^{1/2}}{m^{1/2}} \pm \frac{2T^{1/2} \nu^{1/2}}{c} \right)$$

by $\lesssim TH$. We write the previous display as

(5.22)

$$\frac{H}{T^2C^1/2} \sum_r F(r) \sum_m V\left( \frac{r}{R}, \frac{m}{M} \cdot \frac{c}{C} \right) \lambda_g(m) \lambda_g(m + r) e\left( \pm \frac{2T(r^{1/2} \pm \nu^{1/2})}{m^{1/2}} \right)$$

where

$$F(r) = \sum_c V\left( \frac{c}{C} \right) S(-r - \nu, -\nu, c) e\left( \pm \frac{2(r\nu)^{1/2}}{c} \right).$$

5.6. An average of Kloosterman sums. We pause for a moment and prove that

(5.23)

$$\sum_{r \approx R} |F(r)|^2 \lesssim R.$$ 

It is tempting to use the Kuznetsov formula, but we can argue in an elementary way. We insert a smooth weight and open the square getting

$$\sum_r V\left( \frac{r}{R} \right) |F(r)|^2 = \sum_{c_1, c_2 \approx C} \frac{1}{c_1 c_2} V\left( \frac{c_1}{C}, \frac{c_2}{C} \right)$$

$$\times \sum_{d_1 \pmod{c_1}} \sum_{d_2 \pmod{c_2}}^* e\left( -\nu\left( \frac{d_1 + \bar{d_1}}{c_1} + \frac{d_2 + \bar{d_2}}{c_2} \right) \right)$$

$$\times \sum_r V\left( \frac{r}{R} \right) e\left( \frac{d_1 c_2 - d_2 c_1}{c_1 c_2} r \right) e\left( \pm 2(\nu^{1/2} c_2 - c_1) \right).$$

We split the $r$-sum into residue classes modulo $c_1 c_2$ and apply Poisson summation. The combined conductor of the exponentials is $\lesssim C^2 R^{1/2}/C$, and
VALENTIN BLOMER, SUBHAJIT JANA, AND PAUL D. NELSON

since \( R \gg T^{-\varepsilon}C^2R^{1/2}/C \), it is easy to see that the dual sum picks up at most \( \ll 1 \) terms. Hence we obtain the upper bound

\[
\sum_{r \geq R} |F(r)|^2 \ll R \sum_{h \leq \varepsilon, c_1, c_2 \leq C} \frac{1}{c_1 c_2} \sum_{d_1 \equiv h \pmod{c_1 c_2}} \sum_{d_2 \equiv h \pmod{c_1 c_2}} 1.
\]

If \( h = 0 \), then the inner double sum vanishes unless \( c_1 = c_2 \). If \( h \neq 0 \), then the congruence fixes \( d_j \) modulo \( c_j/(c_j, h) \). In either case we confirm (5.23).

\[\textbf{Remark:}\] It is clear from the proof that the smooth weight function \( V(c/C) \) in the definition of \( F(r) \) plays no role here and could be replaced with arbitrary bounded weights \( \alpha_c \ll 1 \) for \( c \approx C \). The only assumption on \( R \) and \( C \) used in the proof is \( C \ll R^{1/2} \).

5.7. Application of the circle method. We now return to (5.22) and treat the \( m \)-sum as a shifted convolution problem:

\[
\sum_{n,m} V\left( \frac{n}{M}, \frac{m}{M} \right) \lambda_g(n) \lambda_g(m) e\left( \pm \frac{2T(r^{1/2} \pm \nu^{1/2})}{n^{1/2}} \right) \int_0^1 e((m - n - r)\alpha) d\alpha.
\]

We choose a gigantic parameter \( Q = T^{1000} \) and replace the characteristic function on \([0, 1]\) with \( I(\alpha) \), defined in Lemma 13. By Cauchy-Schwarz and trivial bounds, this introduces an error at most \( M^2Q^{\varepsilon^{-1}/2} \) which can be neglected. In this way, the \( \alpha \)-integral becomes

\[
\frac{Q}{2\Lambda} \sum_q V\left( \frac{q}{Q} \right) \sum_{d \equiv 0 \pmod{q}} \int_{-1/Q}^{1/Q} e\left( \frac{d}{q} \alpha (m - n - r) \right) d\alpha.
\]

The portion \( e(\alpha(m - n - r)) \) is obviously flat, so we end up with bounding

\[
\frac{1}{\Lambda} \sum_q V\left( \frac{q}{Q} \right) \sum_{d \equiv 0 \pmod{q}} \sum_{n,m} V\left( \frac{n}{M}, \frac{m}{M} \right) \lambda_g(n) \lambda_g(m) e\left( \pm \frac{2T(r^{1/2} \pm \nu^{1/2})}{n^{1/2}} \right) e\left( \frac{d}{q} (m - n - r) \right)
\]

where \( \Lambda = \sum_q V(q/Q)\phi(q) = Q^{2+o(1)} \). Recall that this represents the \( m \)-sum in (5.22).

5.8. Voronoi again. Having separated the variables \( m, n \) by the circle method, we apply the Voronoi formula (§5.5) to both sums in (5.24). This is simple for the \( m \)-sum

\[
\sum_m V\left( \frac{m}{M} \right) \lambda_g(m) e\left( \frac{dm}{q} \right)
\]
because it contains no archimedean oscillation. If \( g \) is a Maass form, we get two terms, one with a Bessel \( J \)-function and one with a Bessel \( K \)-function, and as before we use (5.5) for the analysis. The dual variable can be truncated at \( \ll Q^2/M \) at the cost of a negligible error, by the oscillatory behaviour of the Bessel \( J \)-function (5.1) with \( t \ll 1 \) and the rapid decay of the Bessel \( K \)-function. Using a smooth partition of unity, we obtain \( \ll 1 \) partial sums of the shape

\[
(5.25) \quad \frac{M}{Q} \sum_m V\left(\frac{m}{M'}\right) \left(\frac{M'M}{Q^2}\right)^{-|3t_\varrho|} \lambda_g(m) e\left( \pm \frac{\bar{d}m}{q} \right), \quad M' \ll Q^2/M.
\]

The same analysis (slightly simpler) applies if \( g \) is holomorphic.

For the \( n \)-sum

\[
\sum_n V\left(\frac{n}{M}\right) \lambda_g(n) e\left( \pm \frac{2T(r^{1/2} \pm \nu^{1/2})}{n^{1/2}} \right) e\left( \frac{dn}{q} \right)
\]

we argue as follows. We note that \( T^{1/2}r^{-1/2} \times TR^{1/2}M^{-1/2} \geq T^c \) by (5.6) and (5.20), so there is sizeable oscillation. In particular, we see from the last formula in (5.5) and Lemma 3 with \( U = P = M, Y = TR^{1/2}M^{-1/2}, S = Y/M \) (so that \( PS/\sqrt{Y} \) and \( SU \) are both \( \gg T^c \) that the Bessel \( K \)-term is negligible. For the Bessel \( J \)-term again by (5.5) we need to understand the transform

\[
\int_{x \in \mathbb{R}} V\left(\frac{x}{M}\right) e\left( \pm \frac{2T(r^{1/2} \pm \nu^{1/2})}{x^{1/2}} \right) e\left( \frac{2\sqrt{nx}}{q} \right)
\times \left( \left( \frac{\sqrt{nx}}{q} \right)^{1/2} + \left( \frac{\sqrt{nx}}{q} \right)^{1/2} \right)^{-1} dx,
\]

which is of the shape (5.15) with

\[
\alpha = \pm \frac{2\sqrt{n}}{q}, \quad \beta = \pm 2T(r^{1/2} \pm \nu^{1/2}).
\]

The condition (5.16) is satisfied in view of (5.6) and (5.20). From the discussion in §5.4 we conclude that the \( x \)-integral is negligible unless

\[
n \asymp \frac{Q^2T^2R}{M^2},
\]

in which case it equals, up to a negligible error,

\[
V\left(\frac{n}{Q^2T^2R/M^2}\right) e\left( \pm \frac{4T^{1/2}(r^{1/2} \pm \nu^{1/2})^{1/2}n^{1/4}}{q^{1/2}} \right)
\times \left( \left( \frac{TR^{1/2}}{M^{1/2}} \right)^{1/2} + \left( \frac{TR^{1/2}}{M^{1/2}} \right)^{2|3t_\varrho|} \right)^{-1} \left( \frac{TR^{1/2}}{M^{5/2}} \right)^{-1/2}.
\]
Here we can afford to drop the term involving $|\Im g|$ and see that the $n$-sum is of the form
\[
\frac{1}{q} \left( \frac{TR^{1/2}}{M^{3/2}} \right)^{-1} \sum_n V \left( \frac{n}{Q^2T^2R/M^2} \right) \lambda_g(n) e \left( \frac{\tilde{d}n}{q} \right) \times e \left( \pm \frac{4T^{1/2}(r^{1/2} + \nu^{1/2})^{1/2}n^{1/4}}{q^{1/2}} \right).
\]
Substituting this and (5.25) back into (5.24), we can replace (5.24) with terms of the form
\[
\frac{M}{Q} \left( \frac{M'M}{Q_2} \right)^{-|\Im g|} \frac{M^{3/2}}{TR^{1/2}Q^2} \frac{1}{M'} \sum_m \sum_n V \left( \frac{m}{M'} \right) V \left( \frac{n}{Q^2T^2R/M^2} \right) \lambda_g(n) \lambda_g(m) \times \sum_q V \left( \frac{q}{Q} \right) \frac{S(\pm m + n, -r, q)}{q} e \left( \pm \frac{4T^{1/2}(r^{1/2} + \nu^{1/2})^{1/2}n^{1/4}}{q^{1/2}} \right).
\]
We note that $n$ is always substantially bigger than $m,$ since $R \geq T^e$, so the arguments of the Kloosterman sum never vanish.

5.9. Kuznetsov again. Eventually (5.26) has to be inserted into (5.22), but before we do this we focus on the $q$-sum which calls for an application of the Kuznetsov formula $[IK,$ Theorem 16.5, 16.6]. Before we carry this out, we simplify the exponential a bit. By a Taylor expansion we can replace $n^{1/4}$ with $(n + m)^{1/4},$ the error being flat since
\[
\frac{T^{1/2}R^{1/2}m}{n^{3/4}Q^{1/2}} \lesssim \frac{T^{1/2}R^{1/2}Q^2M^{3/2}}{M^{3/2}T^{3/2}R^{3/4}Q^{1/2}} = \frac{M^{1/2}}{TR^{1/4}} \lesssim \frac{1}{R^{1/4}} \lesssim 1.
\]
We can also replace $(r^{1/2} + \nu^{1/2})^{1/2}$ with $r^{1/4} + \frac{1}{2} \nu^{1/2}r^{-1/4},$ the total error being flat since
\[
\frac{T^{1/2}n^{1/4}}{Q^{1/2}R^{3/4}} \lesssim \frac{T^{1/2}Q^{1/2}T^{1/2}R^{1/4}}{M^{1/2}Q^{1/2}R^{3/4}} = \frac{1}{C} \lesssim 1.
\]
Therefore the $q$-sum in (5.26) becomes
\[
\sum_q V \left( \frac{q}{Q} \right) e \left( \pm \frac{4T^{1/2}r^{1/4}(n + m)^{1/4}}{q^{1/2}} \right) e \left( \pm \frac{2(\nu T)^{1/2}(n + m)^{1/4}}{r^{1/4}q^{1/2}} \right).
\]
The first exponential fits very well into the shape of the Kuznetsov formula, the second does not. Unfortunately it is not (always) flat, but we can afford to open it by Mellin inversion. We first add a redundant weight function.
\[ V(r/R) \text{ and then write} \]
\[ V \left( \frac{r}{R} \right) e \left( \pm \frac{2(\nu T)^{1/2}(n \pm m)^{1/4}}{r^{1/4}q^{1/2}} \right) \]
\[ = \int_{\Re(s)=0} \left( \int_{x\in\mathbb{R}>0} V \left( \frac{x}{R} \right) e \left( \pm \frac{2(\nu T)^{1/2}r^{1/4}(n \pm m)^{1/4}}{x^{1/2}q^{1/2}} \right) x^s \frac{dx}{x} \right) r^{-s} \frac{ds}{2\pi i}. \]

The outer s-integral can be truncated at
\[ \Im s \approx \frac{T^{1/2}(Q^2T^2R/M^2)^{1/4}}{R^{1/4}Q^{1/2}} = \frac{T}{M^{1/2}}. \]

We sacrifice all cancellation in the \(x, s\)-integrals and pull them outside of all sums, including the \(r, m, n\)-sums in (5.26) and (5.22) which are currently not displayed. (We remark that in the “generic” range \(M \approx T^2\), we sacrifice nothing here.) The remaining \(q\)-sum is of the form
\[ (5.27) \quad \frac{T}{M^{1/2}} \sum_{q} S(\pm m + n, -r,q) \frac{\Phi_x}{q} \left( \sqrt{(n \pm m)r} \right) \]
where \(x \asymp R\) and
\[ \Phi_x(z) = V \left( \frac{z}{Z} \right) e \left( \pm 4(Tz)^{1/2} \left( 1 \pm \frac{\nu^{1/2}}{2z^{1/2}} \right) \right) \]
with
\[ (5.28) \quad Z = \frac{R^{1/2}(Q^2T^2R/M^2)^{1/2}}{Q} = \frac{RT}{M} = \frac{T^3C^2}{M^2} \]
by (5.19). It is important that \(\Phi_x\) does not depend on any of the variables \(n, m, r, q\). We can now apply the Kuznetsov formula to the \(q\)-sum. For a quantitative analysis, we need to understand the three integral transforms
\[ \tilde{\Phi}_+(t) = \int_{0}^{\infty} \Phi_x(z) J_{2it}(z) - J_{-2it}(z) \frac{dz}{\sinh(\pi t)} z, \]
\[ \tilde{\Phi}_-(t) = \int_{0}^{\infty} \Phi_x(z) K_{2it}(z) \cosh(\pi t) \frac{dz}{z}, \]
\[ \tilde{\Phi}_{\text{hol}}(k) = \int_{0}^{\infty} \Phi_x(z) J_{k-1}(z) \frac{dz}{z}. \]
To this end, it is important to note that \(\Phi_x\) has sizeable oscillation since
\[ T^{1/2}z^{1/2} \asymp T^{1/2}Z^{1/2} = T^2C/M \gg T^c \]
by (5.21). We note also, by (5.13) and (5.28), that
\[ (5.29) \quad Z/T = (TC/M)^2 \ll 1/H^2, \]
so that \(Z\) is much smaller than \(T\).
We start with an analysis of $\tilde{\Phi}(t)$, recalling the formula (5.1). First we observe that it is negligible unless $t \asymp T^{1/2}Z^{1/2}$, otherwise we may apply Lemma 3 with

$$U = ZT^{-\varepsilon}, \quad P = Z, \quad Y = \max(|t|, T^{1/2}Z^{1/2}), \quad S = Y/Z.$$ 

If $t \asymp T^{1/2}Z^{1/2}$, we can apply Lemma 4 (or in fact the one-dimensional version [BKY, Proposition 8.2]). We will not compute the stationary point $x_0$ and the shape of the resulting phase (although it can be done algebraically and leads to a quadratic equation with a unique solution if potentially the support of $V$ is slightly restricted), but only bound the size of the integral to be

$$\ll \left(\frac{T^{1/2}}{Z^{3/2}}\right)^{-1/2} \frac{1}{(TZ)^{1/4}} \frac{1}{Z} = \frac{1}{(TZ)^{1/2}}.$$ 

Here the first factor comes from the stationary phase analysis, the second factor from (5.1) (noting that $Z^{1/2} + |t|^{1/2} \asymp |t|^{1/2} \asymp (TZ)^{1/4}$) and the last factor from the measure $dz/z$.

For $\tilde{\Phi}^- (t)$ we can argue in the same way, using (5.3), if $|t| \gg Z$ with a sufficiently large implied constant. For $|t| \ll Z$ with a sufficiently small implied constant, the Bessel $K$-function is negligible (cf. e.g. [BLM, (A.3)]), and for $|t| \asymp Z$ we simply regard the Bessel kernel as part of the weight function and use Lemma 3 with

$$U = \min(ZT^{-\varepsilon}, 1), \quad P = Z, \quad Y = T^{1/2}Z^{1/2}, \quad S = Y/Z$$

(cf. e.g. [BLM, (A.1) & (A.3)] for the relevant bounds for Bessel functions) to show that this contribution is negligible, too.

Similarly we see that $\tilde{\Phi}_\text{hol}(k)$ is negligible in all cases.

As an aside, we note that this analysis is independent of potential exceptional eigenvalues, whose contribution would always be negligible (because of (5.29)).

Summarizing the previous discussion, we can re-write (5.27), up to a negligible error, as

$$\sum_{t_j \asymp (TZ)^{1/2}} \rho_j(n \pm m)\rho_j(-r) \frac{\Psi(t_j)}{\cosh(\pi t_j)} + \text{continuous spectrum}$$

where $\Psi$ is some function of which we only need to know $\Psi \ll 1$, and the sum runs over the Fourier coefficients (in usual normalization) of Hecke Maass cusp forms for $\text{SL}_2(\mathbb{Z})$ with spectral parameter $t_j$. We do not need to be more precise since the spectral sum (including the continuous contribution) will disappear in a moment when we apply the Cauchy–Schwarz inequality and the large sieve.

5.10. Cauchy–Schwarz and the large sieve. We recall that the previous display represents the $q$-sum in (5.26) which itself is the $m$-sum in (5.22).
Applying the Cauchy–Schwarz inequality, we deduce that the total contribution to $\mathcal{L}$ and $\tilde{\mathcal{L}}$ is $\ll \Delta \sqrt{\Sigma_1 \Sigma_2}$, where

\[
\Delta := \frac{H}{TC^{1/2}} \frac{M}{Q} \left( \frac{M'M}{Q^2} \right)^{-\left|S_t\right|} \frac{M^{3/2}}{TR^{1/2} Q^2} \frac{T}{M^{1/2}} \frac{1}{TZ},
\]

\[
\Sigma_1 := \sum_{t_j \gg T^2C/M} \frac{1}{\cosh(\pi t_j)} \left| \sum_r V \left( \frac{r}{R} \right) F(r) \rho_j(r) \right|^2 + (\cdots),
\]

\[
\Sigma_2 := \sum_{t_j \gg T^2C/M} \frac{1}{\cosh(\pi t_j)} \left| \sum_r \rho_j(s) G(s) \right|^2 + (\cdots),
\]

where $(\cdots)$ denotes the continuous spectrum contribution and

\[
G(s) := \sum_{n=0}^{\infty} \frac{V \left( \frac{m}{M} \right) V \left( \frac{M^2n}{Q^2T^2R} \right)}{\lambda_j(n) \lambda_j(m)}.
\]

We prepare for the large sieve by estimating the 2-norm

\[
\sum_s \left| G(s) \right|^2 = \sum_{n_1 \pm m_1 = n_2 \pm m_2} V \left( \frac{m_1}{M'}, \frac{m_2}{M'} \right) \times V \left( \frac{M^2n_1}{Q^2T^2R}, \frac{M^2n_2}{Q^2T^2R} \right) \lambda_j(n_1) \lambda_j(m_1) \lambda_j(n_2) \lambda_j(m_2).
\]

We detect the condition $n_1 \pm m_1 = n_2 \pm m_2$ by a Fourier integral $\int_0^1 e((n_1 \pm m_1 - (n_2 \pm m_2)\alpha)dx$ and use Wilton’s bound to conclude that

\[
\sum_s \left| G(s) \right|^2 \ll M' Q^2 T^2 R / M^2.
\]

Now the scene has been prepared for the endgame with the spectral large sieve of Deshoulers–Iwaniec [DI]. Using this and recalling that $Q$ is very large, we deduce that $\Delta \sqrt{\Sigma_1 \Sigma_2}$ is

\[
\ll \frac{H}{TC^{1/2}} \frac{M'}{Q} \left( \frac{M'M}{Q^2} \right)^{-\left|S_t\right|} \frac{M^{3/2}}{TR^{1/2} Q^2} \frac{T}{M^{1/2}} \frac{1}{TZ} \times \left( \frac{T^4C^2}{M^2} + R \right)^{1/2} \left( \frac{Q^2T^2 R}{M^2} \cdot \frac{Q^2T^2 R}{M^2} \cdot M' \right)^{1/2}.
\]

Since $|S_t| < 1/2$, this expression is increasing in $M'$, so we can replace $M'$ with its largest value $Q^2 / M$ (up to $T^e$), cf. (5.25), so that we can drop the term $(M'M/Q^2)^{-\left|S_t\right|}$. Simplifying and using (5.28), (5.19), (5.13) and (5.6), we obtain the final upper bound

\[
\ll \frac{H M^3}{Q^3 R^{3/2} C^{3/2}} \cdot \frac{T^2 C}{M} R^{1/2} \cdot \frac{Q^3 T^2 R}{M^{5/2}} \cdot \frac{M^3}{H^{1/2}} \approx \frac{TH}{T^{3/2} H^{1/2}} \approx TH.
\]
This finishes the proof of Theorem 5.

6. Proof of the main results

We deduce Theorems 1 and 2 by applying the combination of Theorems 3, 4 and 5 to the triple product formula (1.5). Recall the set-up and the choice of test vectors from the beginning of §4. Ichino’s formula [Ic] says that the identity (1.5) holds with \( L_\infty \) a constant multiple of the matrix coefficient integral as in Theorem 3. Therefore

\[
\frac{L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3)}{L(1, \text{Ad}^2 \pi_1)L(1, \text{Ad}^2 \pi_2)L(1, \text{Ad}^2 \pi_3)} \ll Q \left| \int_{g \in \Gamma \backslash G} v_1 v_2 v_3(g) dg \right|^2
\]

with \( \Gamma = \text{PGL}_2(\mathbb{Z}) \). We now appeal to the following regularized version of (1.11), cf. [Ju2, (2.10)] or [Ne3, Thm 5.6]. Recall from §2.1 our notation for Iwasawa coordinates on \( G \). For a \( \Gamma \)-invariant function \( \Phi \) on \( G \) of rapid decay near the cusps, we have

\[
\int_{g \in \Gamma \backslash G} \Phi(g) dg = 2 \int_{\Re(s) = a} \pi^{-s} \Gamma(s) \zeta(2s)(2s-1) \times \left( \int_{\Gamma_N \backslash G^+} \Phi(n(x)a(y)k(\theta)) y^s dx \frac{dy}{y^2} d\theta \right) \frac{ds}{2\pi i}
\]

where \( \Gamma_N := \Gamma \cap N \) denotes the upper-triangular unipotent subgroup of \( \Gamma \), \( G^+ \) denote the positive-determinant subgroup of \( G \), and the parameter \( a > 1 \) is at our disposal. Note that the \( s \)-integral is rapidly convergent due to the decay of \( \Gamma(s) \). We apply this formula with \( \Phi = v_1 v_2 v_3 \), insert the Fourier expansion and integrate over \( x \). This gives

\[
\int_{\Gamma_N \backslash G} v_1 v_2 v_3(n(x)a(y)k(\theta)) y^s dx \frac{dy}{y^2} d\theta = \sum_{n_1,n_2,n_3 \neq 0 \atop n_1 + n_2 + n_3 = 0} \frac{\lambda_{\pi_1}(n_1)\lambda_{\pi_2}(n_2)\lambda_{\pi_3}(n_3)}{\sqrt{n_1n_2n_3}} \int_0^\infty F(n_1 y, n_2 y) y^s \frac{dy}{y^2}
\]

with \( F \) as in (4.2). Shifting the \( s \)-contour to the far right, we can restrict the \( y \)-integral to \( y \gg Q^{-\varepsilon} \), the remaining error being \( O(Q^{-N}) \). On the other hand, since \( n_1 \in \mathbb{Z} \setminus \{0\} \), the upper bound for \( F \) in Theorem 4 implies that we can also restrict to \( y \ll Q^{\varepsilon} \) and \( n_1 \ll 1 \) at the cost of a negligible error. We insert the asymptotic formula from Theorem 4. The error terms \( \mathcal{E}_2, \mathcal{E}_3 \) contribute negligibly, while \( \mathcal{E}_1 \) contributes \( \ll Q^{-1/2} \). It remains to consider the contribution of \( \mathcal{N} \). We smoothly decompose the sum over \( n_2 \) into dyadic ranges \( n_2 \asymp M \ll Q \). We focus on the contribution from \( M > 0 \); the case \( M < 0 \) may be treated similarly. We estimate the contribution from \( M \ll Q^{1/3} \) trivially (using Cauchy–Schwarz and standard Rankin–Selberg
follows from a standard argument.

For $M \geq Q^{1/3+\varepsilon}$ we insert the full asymptotic formula for $\mathcal{N}$, giving

$$
\frac{L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3)}{L(1, \text{Ad}^2 \pi_1)L(1, \text{Ad}^2 \pi_2)L(1, \text{Ad}^2 \pi_3)} \ll 1 + \sup_{Q^{1/3+\varepsilon} \leq M \leq Q} \left| \sum_{\nu \leq 1} \sum_{m} V\left(\frac{m}{M}\right) \frac{\lambda_{\pi_2}(m)\lambda_{\pi_3}(m+\nu)}{\sqrt{|m(m+\nu)|}} \right|^2 \left(\pm 2\sqrt{Q}\Psi\left(\frac{\nu}{m}\right)\right)^2
$$

for some nice function $V$ (cf. §2.4). We implicitly restrict the sum over $m$ to the support of $V(m/M)$; in particular, $m \neq 0, -\nu$. In the given range of $M$, we can replace $\Psi(y)$ with $|y|^{1/2}$, the error being flat. By the usual procedure (§2.4) of separating variables and changing the weight function, we arrive at the upper bound

$$
1 + \sup_{Q^{1/3+\varepsilon} \leq M \leq Q} \frac{1}{(MQ)^{1/2}} \sum_{\nu \leq 1} \left| \sum_{m} V\left(\frac{m}{M}\right) \lambda_{\pi_2}(m)\lambda_{\pi_3}(m+\nu)e\left(\pm 2\sqrt{Q}\nu/m\right)\right|^2.
$$

For $\pi_1, \pi_2$ fixed, we average this over $\pi_3$ in a spectral window $T \leq \sqrt{Q} \leq T + H$ with $H = T^{1/3+\varepsilon}$. From Theorem 5 we obtain the first bounds in Theorems 1 and 2. The second bound in Theorem 1 follows directly by dropping all but one term (using positivity of central triple product values), while the second bound in Theorem 2 follows from a standard argument based on the functional equation (see e.g. [Go, p. 63]).

References

[ASS] R. Acharya, P. Sharma, S. Singh, $t$-aspect subconvexity for GL($2 \times GL(2)$) L-function, arXiv:2011.01172

[Ba] C. Balogh, Asymptotic expansions of the modified Bessel function of the third kind of imaginary order, SIAM J. Appl. Math. 15 (1967), 1315-1323

[BR1] J. Bernstein, A. Reznikov, Periods, subconvexity of L-functions and representation theory, J. Diff. Geom. 70 (2005), 129-142

[BR2] J. Bernstein, A. Reznikov, Subconvexity bounds for triple L-functions and representation theory, Ann. of Math. 172 (2010), 1679-1718.

[BB] V. Blomer, J. Buttcane, On the subconvexity problem for L-functions on GL(3), Ann. Sci. Éc. Norm. Supér. 53 (2020), 1441-1500

[BKY] V. Blomer, R. Khan, M. Young, Distribution of mass of holomorphic cusp forms, Duke Math. J. 162 (2013), 2609-2644.

[BLM] V. Blomer, X. Li, S. Miller, A spectral reciprocity formula and non-vanishing for L-functions on GL(4) $\times$ GL(2), J. Number Theory Prime 205 (2019), 1-43.

[BM] V. Blomer, D. Milićević, The second moment of twisted modular L-functions, Geom. Funct. Anal. 25 (2015), 453-516.

[Bo] J. Bourgain, Decoupling, exponential sums and the Riemann zeta function, J. Amer. Math. Soc. 30 (2017), 205-224

[Bu] D. Bump, Automorphic forms and representations, Cambridge Studies in Advanced Mathematics 55 (1997), Cambridge University Press
[C] W. Casselmann, *On some results of Atkin and Lehner*, Math. Ann. 201 (1973), 301-314.

[Cog] J. Cogdell, *Analytic theory of L-functions for GLn*, An introduction to the Langlands program (Jerusalem, 2001), 197-228, Birkhäuser Boston, Boston, MA, 2003.

[DI] J.-M. Deshouillers, H. Iwaniec, *Kloosterman sums and Fourier coefficients of cusp forms*, Invent. Math. 70 (1982), 219-288.

[EMOT] A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi, *Higher transcendental functions II*, McGraw-Hill 1953

[GJ] S. Gelbart, H. Jacquet, *A relation between automorphic representations of GL(2) and GL(3)*, Ann. Sci. École Norm. Sup. 11 (1978), 471–542

[Go] A. Good, *The square mean of Dirichlet series associated with cusp forms*, Mathematika 29 (1982/83), 278-295

[GR] I. S. Gradshteyn, I. M. Ryzhik, *Tables of integrals, series, and products*, 7th edition, Academic Press, New York, 2007.

[GS] A. Ghosh, P. Sarnak, *Real zeros of holomorphic Hecke cusp forms*, J. Eur. Math. Soc. 14 (2012), 465–487

[Ho] R. Holowinksy, *A sieve method for shifted convolution sums*, Duke Math. J. 146 (2009), 401-448.

[Hu] Y. Hu, *Triple product formula and the subconvexity bound of triple product L-function in level aspect*, Amer. J. Math. 139 (2017), 215-259

[HB] P. Humphries, *Standard zero-free regions for Rankin-Selberg L-functions via sieve theory*, with an appendix by F. Brumley, Math. Z. 292 (2019), 1105-1122.

[IC] A. Ichino, *Trilinear forms and the central values of triple product L-functions*, Duke Math. J. 145 (2008), 281-307

[IK] H. Iwaniec, E. Kowalski, *Analytic Number Theory*, Colloquium Publication 53 (2004), AMS, Providence, RI

[IS] H. Iwaniec, P. Sarnak, *Perspectives on the analytic theory of L-functions*, Geom. Funct. Anal. 2000, Special Volume, Part II, 705-741.

[JN] S. Jana, P. Nelson, *Analytic newvectors for GLn(R)*, arxiv:1911.01880

[JPSS] H. Jacquet, I. Piatetski-Shapiro, J. Shalika, *Conducteur des représentations du groupe linéaire*, Math. Ann. 256 (1981), no. 2, 199-214.

[Ju1] M. Jutila, *A Variant of the Circle Method*, in: Sieve methods, exponential sums, and their application in number theory (Cardiff 1995), 245-254

[Ju2] M. Jutila, *The additive divisor problem and its analogs for Fourier coefficients of cusp forms*. I., Math. Z. 223 (1996), 435-461.

[JM1] M. Jutila, Y. Motohashi, *Uniform bound for Hecke L-functions*, Acta Math. 195 (2005), 61-115

[JM2] M. Jutila, Y. Motohashi, *Uniform bounds for Rankin-Selberg L-functions*, in: Multiple Dirichlet series, automorphic forms, and analytic number theory, 243-256, Proc. Sympos. Pure Math. 75, Amer. Math. Soc., Providence, RI, 2006.

[KPY] M. Kiral, I. Petrov, M. P. Young, * Oscillatory integrals with uniformity in parameters*, J. Théor. Nombres Bordeaux 31 (2019), 145-159

[La] E. Landau, *Über die ζ-Funktion und die L-Funktionen*, Math. Z. 20 (1924), 105-125

[LLY] Y.-K. Lau, J. Liu, Y. Ye, *A new bound k^{2/3+ε} for Rankin-Selberg L-functions for Hecke congruence subgroups*, Int. Math. Res. Pap. 2006, Art. ID 35090, 78 pp.

[Li] J. E. Littlewood, *Researches in the theory of the Riemann ζ-function*, Proc. Lond. Math. Soc. (2) 20 (1922), Records, page xxiv

[Ma] K. Matomäki, *Real zeros of holomorphic Hecke cusp forms and sieving short intervals*, J. Eur. Math. Soc. 18 (2016), 123-146
[Mi] D. Milićević, *Sub-Weyl subconvexity for Dirichlet L-functions to prime power moduli*, Compos. Math. 152 (2016), 825–875

[MV] P. Michel, A. Venkatesh, *The subconvexity problem for GL₂*, Publ. Math. IHES 111 (2010), 171-271

[Ne1] P. D. Nelson, *Subconvex equidistribution of cusp forms: reduction to Eisenstein observables*, Duke Math. J. 168 (2019), 1665-1722.

[Ne2] P. D. Nelson, *Eisenstein series and the cubic moment for PGL₂*, arxiv:1911.06310

[Ne3] P. D. Nelson, *Evaluating modular forms on Shimura curves*, Math. Comp. 84 (2015), 2471–2503

[Ol] F. W. J. Olver, *The asymptotic expansion of Bessel functions of large order*, Phil. Trans. R. Soc. Lond. A 247 (1954), 328-368.

[PY1] I. Petrow, M. P. Young, *The Weyl bound for Dirichlet L-functions of cube-free conductor*, Ann. of Math. (2) 192 (2020), 437–486

[PY2] I. Petrow, M. P. Young, *The fourth moment of Dirichlet L-functions along a coset and the Weyl bound*, arXiv:1908.10346

[Sa] P. Sarnak, *Integrals of products of eigenfunctions*, Internat. Math. Res. Notices 6 (1994)

[Su1] E. Suivitie, *On inner products involving holomorphic cusp forms and Maass forms*, ˘Siauliai Math. Semin. 3(11) (2008), 221-233

[Su2] E. Suivitie, *On a short spectral sum involving inner products of a holomorphic cusp form and Maass forms*, Acta Arith. 144 (2010), 395–418

[Ve] A. Venkatesh, *Sparse equidistribution problems, period bounds and subconvexity*, Ann. of Math. 172 (2010), 989-1094

[Wat] T. C. Watson, *Rankin triple products and quantum chaos*, arXiv:0810.0425

[We] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. 77 (1916), 313-352.

[Za] D. Zagier, *The Rankin-Selberg method for automorphic functions which are not of rapid decay*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981/82), 415-437.