Magnetic impurities make superconductivity in 3D Dirac semi-metal triplet.

Baruch Rosenstein$^{1,2}$, B.Ya. Shapiro$^3$, Dingping Li$^{4,5}$ and I. Shapiro$^3$

$^1$ Electrophysics Department, National Chiao Tung University, Hsinchu 30050, Taiwan, R. O. C
$^2$ Physics Department, Ariel University, Ariel 40700, Israel
$^3$ Physics Department, Bar-Ilan University, 52900 Ramat-Gan, Israel
$^4$ School of Physics, Peking University, Beijing 100871, China
$^5$ Collaborative Innovation Center of Quantum Matter, Beijing, China

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Abstract – Conventional electron-phonon coupling induces either odd (triplet) or even (singlet) pairing states in a time reversal and inversion invariant Dirac semi-metal. In certain range of the chemical potential $\mu$ and parameters characterizing the pairing attraction (effective electron-electron coupling constant $\lambda$ and the Debye energy $T_D$) the energy of the singlet although always lower, prevails by a very slim margin over the triplet. This means that interactions that are small but discriminate between the spin singlet and the spin triplet determine the nature of the superconducting order there. It shown that in materials close enough to the Dirac point ($\mu \lesssim T_D$) magnetic impurities stabilize the odd pairing superconducting state.

Introduction. Recently solids with electronic states described by the Bloch wave functions, obeying the "pseudo-relativistic" Dirac equation (with Fermi velocity $v_F$ replacing the velocity of light) attracted widespread attention. One outstanding example is graphene, a two-dimensional (2D) hexagonal lattice made of carbon atoms. The 2D Dirac bi-spinor (spin in this case is actually pseudospin/sublattice) incorporates excitations near its $K$ and $K'$ points in the Brillouin zone, so that the model is in fact of the two band variety. Although a similar two band electronic structure of bismuth was described by a nearly massless Dirac fermion in 3D, this time caused by strong spin-orbit interactions, long ago [1] (with spin replacing pseudospin), only recently several systems were demonstrated to exhibit the 3D Dirac quasiparticles [2,3]. Their discovery followed recent exploration of the topological band theory [8].

A systematic proposal [6] to make a 3D Dirac semi-metal is to close the insulating gap by tuning a topological insulator towards the quantum phase transition to trivial insulators led to their discovery. The time reversal invariant 3D Dirac point in materials like $Na_3Bi$ was theoretically investigated [7] and observed [2]. A well known compound $Cd_3As_2$ is a symmetry-protected 3D Dirac semi-metal with a single pair of Dirac points in the bulk [4]. Most recently conductivity and magnetoabsorption of a zinc-blende crystal, $HgCdTe$ was measured [8] and is in agreement with theoretical expectations in Dirac semimetal [8]. The discovery of the 3D Dirac materials makes it possible to investigate their physics including remarkable electronic properties. This is reach in new phenomena, not seen in 2D Dirac semi - metals like graphene. Examples include the giant diamagnetism that diverges logarithmically when chemical potential approaches the 3D Dirac point, slow dynamics [9], linear in frequency AC conductivity that has an imaginary part [8], quantum magnetoresistance showing linear field dependence in the bulk [9]. Most of the properties of these new materials were measured at relatively high temperatures. However some of topological insulators and suspected 3D Dirac semi-metals exhibit superconductivity at about the liquid $He$ temperature.

The well known topological insulator $Bi_2Se_3$ doped with $Cu$, becomes superconducting at $T_c = 3.8K$ [10]. When subjected to pressure [11], $T_c$ increases to $7K$ at $30GPa$. Quasilinear temperature dependence of the upper critical field $H_{c2}$ that exceeds the orbital and Pauli limits for the singlet pairing points to the triplet super-
Fig. 1: Schematic picture of the impurity spin $s$ interacting with electrons composing a singlet (left) or triplet (right) Cooper pair. As usual the magnetic impurities suppress the singlet, while it is not pair breaking for the triplet.

Clarify theoretically two questions. (i) Does a conventional phononic superconductivity exists in these materials with just a minute density of states compared even with high $T_c$, cuprates that apparently utilize very different pairing mechanism than phonons offer? (ii) Is it possible that phonons in 3D Dirac materials lead to triplet pairing that even becomes dominant under certain circumstances?

In the present letter we construct the theory of the superconducting transition in 3D Dirac semi-metal at arbitrary chemical potential including zero assuming the local (probably, but not necessarily, phonon mediated) pairing. The possible pairing channels are classified in this rather unusual situation using symmetries of the system. In contrast to the 2D case, the odd parity (triplet) pairing is not only possible, but with small concentration of magnetic impurities the odd parity is the preferred channel taking over the more "conventional" even parity one.

**Symmetry classification of pairing channels.** Electrons in 3D Dirac semi-metal are described by field operators $\psi_{fs}(r)$, where $f = L, R$ are the valley index for the left/right chirality bands with spin projections taking the values $s = \uparrow, \downarrow$. These are combined into a four component bi-spinor creation operator, whose index $\gamma = \{f, s\}$ takes four values. The non-interacting massless Hamiltonian with chemical potential $\mu$ reads $[7]$.

$$K = \int_{r} \psi_{\gamma}^{\dagger} (-i\hbar v_{F} \nabla \alpha_{\gamma}^{\dagger} \psi_{\gamma} - \mu \delta_{\gamma, \uparrow} \psi_{\gamma}, \quad (1)$$

$$\alpha = \left( \begin{array}{cc} \sigma & 0 \\ 0 & -\sigma \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \gamma,$$

where $\sigma^i$ and $1$ are the Pauli and the unit matrices respectively. We assume the time reversal, inversion and 3D rotational symmetry that in particular requires an isotropic Fermi velocity. Electrons interact electrostatically via the density - density potential. The effective electron-electron interaction due to both electron - phonon attraction and density - density potential. The effective electron-electron interaction due to both electron - phonon attraction and Coulomb repulsion (pseudopotential) can be taken local

$$V_{eff} = -\frac{q}{2} \int_{r} \psi_{\gamma}^{\dagger} \psi_{\beta}^{\dagger} \psi_{\beta} \psi_{\gamma}. \quad (2)$$

Unlike the free Hamiltonian $K$, Eq.(1), this interaction Hamiltonian does not mix different spin components. Such a coupling implicitly restricts the spin independent local interaction to be symmetric under the band permutation and the additional term is not generated. A more general case with additional independent term was considered in ref. $[19]$. The strength of the phonon pairing depends on the cutoff: the Debye temperature $T_D$.

Since we consider the local interactions as dominant, the superconducting order parameter will be local $\hat{\Delta} = \int_{r} \psi_{\gamma}^{\dagger} (r) M_{\alpha \beta} \psi_{\gamma} (r)$, where the constant matrix $M$ should be antisymmetric. Due to the rotation symmetry they transform covariantly under infinitesimal rotations generated by the spin rotation generators $S^\alpha$, whose density is

$$S (r) = \psi^{\dagger} (r) \Sigma \psi (r); \quad \Sigma = \left( \begin{array}{cc} \sigma & 0 \\ 0 & \sigma \end{array} \right). \quad (3)$$
The representations of the rotation group therefore characterize various possible superconducting phases. Out of 16 possible matrices $M$ six are antisymmetric. One finds one vector of the rotation group triplet $M^T = \{ \gamma z, -\gamma x\gamma y, \gamma x \}$ and three scalar multiplets: $M^2_F = i\sigma_y$; $M^2_S = i\Sigma_y$; $M^2_T = -i\gamma_x\gamma_z$ (see Supplemental Materials (SM) for details [18]). In the odd parity superconductivity state the rotational $O(3)$ symmetry is spontaneously broken, leading to weak ferromagnetism that has already been considered (on level of the Ginzburg - Landau approach) for to heavy fermion superconductor $UPt_3$ [19][20]. Which one of the condensates is realized depends on energy determined by the interplay of the interactions and disorder. Let us first consider clean homogeneous Dirac semi-metal.

**Singlet vs triplet**. The gap function for a channel $M$ can be written [21] as

$$\Delta_{\beta\gamma} = gF_{\beta\gamma}(r,\tau; r,\tau) = \Delta_M M_{\beta\gamma},$$  

(4)

where $F(r,\tau; r',\tau') \equiv \langle T\psi_\beta(r,\tau)\psi_\gamma(r',\tau') \rangle$ is the anomalous Matsubara Green’s function ($\Delta_M$ can be chosen real).

The Fourier transform of $F$ satisfies the matrix Gor’kov equation:

$$F^+(p,\omega) = -D^l(p,\omega) L^+(p,\omega) G(p,\omega),$$

(5)

where $D^{-1}_{\beta\gamma} = (i\omega - \mu)\delta_{\beta\gamma} + v_F p^\alpha\alpha_{\alpha\beta}\gamma^\alpha$, while the Green’s function $G$ obeys the Dyson equation,

$$G^{-1} = D^{-1} + \Delta D^*\Delta^*.$$  

(6)

For the local phonon interaction, Eq.(2), the operator

$$L^+_{ph} = -g \sum_{q,\nu} F^+(q,\nu),$$

(7)

is independent of momenta and frequency and in view of Eq.(4) leads to an integrated form of the Gor’kov equation,

$$\Delta^* = -g \sum_{p,\omega} D^l(p,\omega) \Delta^* G(p,\omega).$$

(8)

The equation is solved with UV cutoff $T_D$ characterizing the electron-phonon interaction along with the phonon-electron coupling $\lambda = gD(\mu) = g\mu^2/8\pi^2v_F^2\hbar^4$ for order parameter $\Delta_M$ for all the channels. In Fig.2 the gap function for singlet $S_1$, $\Delta_S$ in red, and $\Delta_T$ for triplet in blue for chemical potentials $\mu = 2T_D$ (left), $\mu = 4T_D$ (center) and $\mu >> T_D$ (right, the BCS limit given in SM [18]). It turns out that $S_2$ is unstable, while $S_1$ and $S_2$ are degenerate. Examination of the Green’s function reveals, see SM, that the spectrum of excitations for singlet is rotational invariant, while the one of triplet has two nodes. One finds that the singlet has larger gap function for $\mu > 1$, although at small chemical potential and large coupling the triplet order parameter actually is a bit higher than that of the singlet. The value of the gap function itself does not define which channel is stable, so we have calculated energy densities via momentum space Green’s function for all the channels,

$$E_{S,T} = \int_{\Delta = 0}^{\Delta_S,T} d(1/g(\Delta)) = -\frac{1}{2} \sum_{p,\omega} \text{Tr} \left( M^{S,T}D^d(p,\omega) M^{S,T}G(p,\omega) \right),$$

(9)

see SM for details [18]. Limiting cases of BCS when $\mu >> T_D$ can be done analytically, see SM, while experimentally relevant (see below) chemical potentials $\mu = 8T_D$ and $12T_D$ are given for wide range of couplings in Fig.3. Triplet (blue line) has always higher energy than singlet although at $\mu < T_D$ energies of triplet and singlet are close despite the fact that $\Delta_T > \Delta_S$, see SM [18]. Magnetic impurities can strongly affect the relative energy of triplet and especially singlet condensates for which it is pair breaking, see Fig.1.

**Magnetic impurities**. Hamiltonian for impurity spins $s_a$ located at $r_a$ is

$$H_{imp} = -J \sum_a \int_{r} \delta (r - r_a) s_a \cdot S (r),$$

(10)

where itinerary spin density $S$ was defined in Eq.4. Spins are randomly distributed:

$$s^i_a s^j_a \delta (r - r_a) \delta (r' - r_a) \equiv \frac{s(s + 1)}{3} n\delta^{ij} \delta (r - r'),$$

(11)

where $n$ is density of impurities and $sh$ - their spin value.

Due to disorder the singlet, predictably gains energy over triplet and at certain disorder strength a phase transition from the singlet to triplet takes place. At yet large disorder strength the singlet channel becomes unstable and the triplet becomes the only stable channel. As will be shown below the triplet channel is generally not destabilized by this type of impurities, so it is not a pair breaking. After averaging over impurities, see SM, the Gor’kov equations, Eq.4,6 acquires an additional term dependent on
The phase diagram for chemical potential $\lambda$ at two values of chemical potential: $\mu = 8T_D$ (left), $12T_D$ (right).

**Frequency $\omega$ for the singlet:**

$$ L^+_{\text{imp}} (\omega) = -C \sum_q \Sigma^t (q, \omega) \Sigma^i \equiv - \frac{3iC}{g} \Delta_S (\omega) \alpha_y. $$

(12)

The dimensionless disorder strength is

$$ C = s (s + 1) n J^2 T_D / 3 \hbar^3 v_F^3. $$

(13)

and matrices $\Sigma$ were defined in Eq. (3). At a critical dis-

order strength where the singlet channel is suppressed, $\Delta_S = 0$ (so that $G \approx D$), the Gorkov equation integrated over momenta takes a form

$$ \Delta_S (\omega) = f (\omega) \left( -3 C \Delta_S (\omega) + g \Delta_S \right), $$

(14)

where $\Delta_S = \sum_q \Delta (\omega)$ and;

$$ f (\omega) = \frac{1}{4} \text{tr} \sum_q D^t \alpha_y D \alpha_y $$

$$ = \sum_q \left( \frac{\omega^2 + \mu^2 + v_F^2 q^2}{(v_F^2 q^2 + \omega^2 - \mu^2)^2 + 4 \omega^2 \mu^2} \right). $$

(15)

To solve the equations for the critical disorder strength $C^c$, one integrates over $\omega$:

$$ \frac{1}{g} = \sum_\omega \frac{f (\omega)}{1 + 3 C^c f (\omega)}. $$

(16)

The phase diagram for chemical potential $\mu = 4T_D, 8T_D, 12T_D$ in wide range of $\lambda$ and $C$ is presented in Fig. 4. Above the line there is no singlet condensate, while triplet is the ground state. Below the line the singlet pairing exists and possible dominates over the triplet. For the triplet pairing calculation one obtains an equation similar to Eq. (16) with reverse sign in denominator and no solution. This means that magnetic impurities help the pairing rather than destroying it.

**Summary and discussion.** To summarize, we presented a microscopic theory of superconductivity (at zero temperature) in massless Dirac semi-metals. In the framework of the “conventional” phonon mediated local attraction model we classified (under simplifying assumptions of the 3D rotation invariance, inversion and the time reversal) possible pairing channels. There are three even parity (singlet) channels and one odd parity (triplet) channels. In the clean limit the singlet pairing prevails for the arbitrary chemical potential and the electron-electron interaction strength despite the fact that triplet condensate is sometimes higher. This is found by the direct comparison of condensation energies. However a modest concentration of magnetic impurities makes the triplet ground state. Larger impurity concentration suppresses the even parity state all together, while the impurities are not pair breaking for the odd parity weakly ferromagnetic state, Fig. 1.

Here we compare our results with the early work ref. [15] designed to model the symmetries and parameters of Cu doped Bi$_2$Se$_3$. The case that can be directly compared is when the relativistic mass term (denoted by $m$ in ref. [15]) is small compared to chemical potential. In this work more general effective electron - electron interaction was considered with two couplings $V$ and $U$ for local intraband and interband attractions respectively. They are related to our $g$ by $g = 2U = 2V$. Qualitatively indeed for $U/V = 1$ one gets nearly degenerate energies (critical temperatures were compared in ref. [15] instead). This is similar but not identical to our result without impurities. We indeed obtain the near degeneracy of the two gaps, the singlet and the triplet (their $\Delta_1$ and $\Delta_2$ respectively), but only in the limit of large $g$. The gaps are definitely not degenerate when the coupling $g$ is below $20 \pi^2 v_F^2 \hbar^2 / T_D^2$. Even within the BCS regime (SM), $\Delta_T / \Delta_S = \sinh (0.35/\lambda) / \sinh (0.5/\lambda)$. This is consistent with 1 only for quite large coupling and was studied in detail in [22].
To estimate the range of parameters for currently available materials where the odd parity conventional (phonon induced) superconductivity is expected, one should rely on measurements of the electron - phonon coupling. The effective dimensionless electron - electron coupling constant due to phonons \( \lambda \) for materials like \( Bi_2Se_3, Bi_2Te_3 \) reported \[24\] vary widely \[0.1 - 3\]. Taking \[12\] for \( Bi_2Se_3 \) the Debye cutoff energy \( T_D = 150K \) and \( \lambda = 0.2 \) measured at \( \mu = 120 \text{ meV} \), Fermi velocity \( v_F \cdot 10^7 \text{ cm/s} \) one obtains \( 6K \) triplet superconductivity (see Fig.2 and a stronger singlet). To destroy the singlet one, that is to reach the impurity strength \( n \approx 0.005 \) that for the impurity spin \( s = 1 \) and exchange integral of \( J = 0.15 \text{ eV \cdot nm}^3 \) requires the impurity concentration \( n = 2 \cdot 10^{21} \text{ cm}^{-3} \).

The physics of the triplet superconductors of this type is very rich and has already been investigated in connection with heavy fermion superconductors. In particular their magnetic vortices appear as either vector vortices or so called skyrmions \[19\] - coreless topologically nontrivial textures. In particular their magnetic properties like the magnetization are very peculiar and even without magnetic field the system forms a “spontaneous flux state”. The material therefore can be called a “ferromagnetic superconductor”. The superconducting state develops weak ferromagnetism and system of alternating magnetic domains \[20\],

\[
\begin{equation}
\text{REFERENCES}
\end{equation}
\]

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1. Wolff P.A., J. Phys. Chem. Sol., 25 (1964) 1057.
2. Liu Z. K. et al., Science, 343 (2014) 864; Xu S.-Y. et al., Observation of a bulk 3D Dirac multiplet, Lifshitz transition, and nested spin states in Na$_3$Bi, ArXiv, (2013) 1312.7624
3. Orlita M. et al., Nat. Phys., 10 (2014) 233.
4. Xu G. et al., Phys. Rev. Lett., 107 (2011) 186806; Wang Z. et al., Phys. Rev. B, 88 (2013) 125427; Neupane M. et al., Observation of a topological 3D Dirac semimetal phase in high-mobility Cd$_3$As$_2$, ArXiv, (2013) 1309.7892.
5. Hasan M. Z. and Kane C. L., Rev. Mod. Phys., 82 (2010) 3045; Qi X.-L. and Zhang S.-C., Rev. Mod. Phys., 83 (2011) 1057.
6. Young S. M. et al., Phys. Rev. B, 84 (2011) 085106; Young S. M. et al., Phys. Rev. Lett., 108 (2012) 140405.
7. Wang Z. et al., Phys. Rev. B, 85 (2012) 195320; Hosur P. et al., Time-reversal invariant topological superconductivity in doped Weyl semimetals, ArXiv, (2014) 1405.4299.
8. Fuseya Y., Ogata M. and Fukuyama H., Phys. Rev. Lett., 102 (2009) 066601; Hosur P., Parameswaran S. A. and Vishwanath A., Phys. Rev. Lett., 108 (2012) 046602; Lewkowicz M. and Rosenstein B., Phys. Rev. B, 88 (2013) 045108.
9. Kariyado T. and Ogata M., J. Phys. Soc. Jpn., 80 (2011) 083704; 81 (2012) 064701; Delpielc P., Li J. and Carpentier D., Europhys. Lett., 97 (2012) 67004.
10. Hor Y. S. et al., Phys. Rev. Lett., 104 (2010) 057001.
11. Kirshenbaum K. et al., Phys. Rev. Lett., 111 (2013) 087001.
12. Zhu X. et al., Phys. Rev. Lett., 108 (2012) 185501; Kondo T. et al., Phys. Rev. Lett., 110 (2013) 217601; Hatch R. C. et al., Phys. Rev. B, 83 (2011) 241303.
13. Das Sarma S. and Li Q., Phys. Rev. B, 88 (2013) 081404(R); Pan Z.-H. et al., Phys. Rev. Lett., 108 (2012) 187001; Parente V. et al., Phys. Rev. B, 88 (2013) 075432.
14. Ali M. N. et al., Phys. Rev. B, 89 (2014) 020505(R).
15. Fu L. and Berg E., Phys. Rev. Lett., 105 (2010) 097001.
16. Lu C.-K. and Herbut I. F., Phys. Rev. B, 82 (2010) 144505; Roy B., Juricic V. and Herbut I. F., Phys. Rev. B, 87 (2013) 041401.
17. Brydon P. M. R. et al., Odd-parity superconductivity from phonon-mediated pairing, arXiv, (2014) 1402.7061.
18. Supplemental Materials.
19. Knigavko A. and Rosenberg B., Phys. Rev. Lett., 82 (2003) 1261; Knigavko A. and Rosenberg B. and Chen Y.F., Phys. Rev. B, 60 (1999) 5504; Li Q., Toner J. and Belitz D., Phys. Rev. B, 79 (2009) 014517.
20. Bel G., Rosenberg B., Shapiro B. Ya. and Shapiro I., Europhys. Lett., 64 (2003) 503; Knigavko A. and Rosenberg B., Phys. Rev. B, 58 (1999) 9354; Rosenberg B., Shapiro I., Shapiro B. Ya. and Bel G., Phys. Rev. B, 67 (2003) 224507.
21. Abrikosov A. A., Gor’kov L. P. and Dzyaloshinskii I. E., Quantum field theoretical methods in statistical physics (Pergamon Press, New York) 1965.
22. Rosenberg B., Shapiro B. Ya., Li D. and Shapiro I., Triplet superconductivity in 3D Dirac semimetal due to exchange interaction, arXiv, (2014) 1407.0770.
23. Okada Y. et al., Phys. Rev. Lett., 106 (2011) 206805; Checkelsky J. G. et al., Nature Physics., 8 (2012) 729; Zhang J.M. et al., Phys. Rev. B, 88 (2013) 235131; Liu M. et al., Phys. Rev. Lett., 108 (2013) 036805.
24. Howard C. and El-Batanean M., Phys. Rev. B, 89 (2014) 075425;
Supplemental material

1. Symmetry classification of pairing channels. — Electrons in 3D Dirac semi-metal are described by a four component bi-spinor creation operator, \( \psi^\dagger \), whose index \( \gamma \) takes four values. Here we classify the possible local superconducting order parameters, written generally as

\[
\hat{M} = \int_r \psi^\dagger_\alpha(r) M_{\alpha\beta} \psi^\dagger_\beta(r),
\]

(S. 1)

with constant antisymmetric matrix \( M \) according to representations of the 3D rotation group. The representations of the rotation group therefore characterize various possible superconducting phases. Generator of rotations consists of the orbital momentum operator \( L \) and the spin operator

\[
S^i = \int_r \psi^\dagger_\alpha(r) \Sigma^i_{\beta\delta} \psi_\delta(r),
\]

(S. 2)

Due to the rotation symmetry they transform covariantly under the action of \( J = L + S \). The global quantity in Eq. (S. 3) transforms as

\[
\left[ \hat{M}, J^\dagger \right] = \int_{r,r'} \left[ \psi^\dagger_\alpha(r) M_{\alpha\beta} \psi^\dagger_\beta(r'), \psi^\dagger_\gamma(r') \Sigma^i_{\gamma\delta} \psi_\delta(r') \right]
\]

= \(-2 \int_r \psi^\dagger_\gamma(r) \Sigma^i_{\gamma\delta} M_{\delta\kappa} \psi^\dagger_\kappa(r).\)

(S. 3)

Out of 16 possible matrices \( M \) six are antisymmetric. They transform into each other forming the following irreducible representations.

(i) a triplet of matrices \( \{T_x, T_y, T_z\} = \{\beta \alpha, -\gamma z \gamma z, \beta \alpha\} \) transforms as a vector

\[
\left[ \hat{M}^T_k, J^\dagger \right] = i \varepsilon_{kld} \hat{M}^T_m.
\]

(S. 4)

(ii) three scalar multiplets: \( S_1 = i \alpha y; \ S_2 = i \Sigma y; \ S_3 = -i \beta \alpha \gamma \).

Which one of the condensates is realized at zero temperature is determined by the Hamiltonian.

2. Microscopic equations for local pairing. — Gor’kov equations. To treat the pairing the general gaussian approximation can be employed. Using the standard formalism, the Matsubara Green’s functions (\( \tau \) is the Matsubara time),

\[
G_{\alpha\beta}(r, \tau; r', \tau') = -\left< \tau \psi^\dagger_\alpha \left( r, \tau \right) \psi^\dagger_\beta \left( r', \tau' \right) \right> ;
\]

(S. 5)

\[
F^\dagger_{\alpha\beta}(r, \tau; r', \tau') = \left< \tau \psi^\dagger_\beta \left( r, \tau \right) \psi^\dagger_\alpha \left( r', \tau' \right) \right> ,
\]

obey the Gor’kov equations:

\[
-\frac{\partial G_{\gamma\kappa}(r, \tau; r', \tau')}{\partial \tau} = -\int_{r''} \left< \tau \left| \hat{K}_{\gamma\beta} \right| r'' \right> G_{\beta\kappa}(r'', \tau; r', \tau')
\]

(S. 6)

\[
-\frac{\partial F^\dagger_{\beta\kappa}(r, \tau; r', \tau')}{\partial \tau} = -\int_{r''} \left< \tau \left| \hat{K}^\dagger_{\gamma\beta} \right| r'' \right> F^\dagger_{\gamma\kappa}(r'', \tau; r', \tau')
\]

\[
-g F^\dagger_{\gamma\beta}(r, \tau; r, \tau) \frac{\partial F^\dagger_{\beta\kappa}(r, \tau; r, \tau')}{\partial \tau} - \frac{\partial F^\dagger_{\gamma\beta}(r, \tau; r, \tau')}{\partial \tau} \frac{\partial F^\dagger_{\beta\kappa}(r, \tau; r', \tau')}{\partial \tau} = 0.
\]

In the homogeneous case the Gor’kov equations for Fourier components of the Greens functions simplify considerably,

\[
D^\dagger_{\gamma\beta} G_{\beta\kappa}(\omega, p) - D^\dagger_{\gamma\beta} F^\dagger_{\beta\kappa}(\omega, p) = \delta^\kappa; \quad \delta^\gamma
\]

(S. 7)

\[
D^\dagger_{\gamma\beta} F^\dagger_{\beta\kappa}(\omega, p) + \delta^\gamma \Delta^\gamma G_{\beta\kappa}(\omega, p) = 0,
\]

where \( \omega = \pi T (2n + 1) \) is the Matsubara frequency and \( D^\dagger_{\gamma\beta} = (i \omega - \mu) \delta_{\gamma\beta} + v_F p^\beta \alpha_{\beta} \).
The matrix gap function can be chosen as \((\Delta \text{ real})\)

\[
\tilde{\Delta}_{\gamma\beta} = gF_{\gamma\beta}(0) = \Delta M_{\gamma\beta}.
\]  

(S. 8)

These equations are conveniently presented in matrix form (superscript \(t\) denotes transposed and \(I\) - the identity matrix):

\[
D^{-1} G - \Delta F^t = I; \quad D^{-1} F^t + \Delta^* G = 0.
\]  

(S. 9)

Solving these equations, one obtains

\[
G^{-1} = D^{-1} + \Delta D^t \Delta^*; \quad F^t = -D^t \Delta^* G,
\]  

(S. 10)

with the gap function, Eq. (S. 11), found from the consistency condition. Now we find solutions of this equation for each of the possible superconducting phases.

**Singlet representations.** It turns out that the second singlet in Eq. (S.14) gives results identical to that of the first one, while the third singlet does not have a solution in the physically interesting range of parameters. Therefore we assume the order parameter in the matrix form \(\Delta = \Delta_S M_i^S = i\Delta_S \alpha^y\). The relevant matrix element of the matrix gap equation, for real \(\Delta_T > 0\) is:

\[
\frac{1}{g} = \sum_{\omega_p} \frac{\Delta^2_p + p_x^2 - p_y^2 - \mu^2 + \omega^2}{(\Delta^2_p + \omega^2)^2 + (p_x^2 - \mu^2)^2 + 2(p_x^2 + \mu^2)\omega^2 + 2\Delta^2_p (p_x^2 - p_y^2 + \mu^2)};
\]  

(S. 11)

where \(p_x^2 = p_x^2 + p_y^2\). The spectrum of elementary excitations obtained from the four poles of the Greens function, see Fig. SM1, is (in physical units)

\[
E^2_{\pm} = \Delta^2_p + v_F^2 p^2 - \mu^2 \pm 2v_F\sqrt{\Delta^2_p p_x^2 + p_y^2 \mu^2}.
\]  

(S. 12)

There are two nodes at \(p_x = p_y = 0, v_F p_x = \pm \sqrt{\Delta^2_p + \mu^2}\), when the branches \(|E_+|\) and \(|E_-|\) cross, see Fig. SM1a and a section \(p_\perp = 0\) in Fig. SM1b. There is also a saddle points with energy gap, \(2\Delta_T\) on the circle \(p_x^2 + p_y^2 = \mu^2, p_z = 0\) see the section in the \(p_z = 0\) direction in Fig. SM1c. The higher energy band \(E_+\) touches the lower band at \(p = 0\), so that there is a Dirac point for quasiparticles, see Fig. SM1d.

Integration over \(\omega\) gives using polar coordinates for \(p\) and \(x = \cos \theta, \zeta = \sqrt{\Delta^2_p x^2 + \mu^2}\),

\[
\frac{1}{g} = \frac{1}{8\pi^2} \int_{p = \max(\mu, -\mu)}^{\mu + 1} \int_{x = 0}^{1} p_x^2 \zeta \left\{ \frac{\xi + px^2}{\sqrt{\Delta^2_p + p_x^2 + \mu^2 + 2p_x \zeta}} + \frac{\xi - px^2}{\sqrt{\Delta^2_p + p_x^2 + \mu^2 - 2p_x \zeta}} \right\}.
\]  

(S. 13)

The lower bound on the momentum integration is nonzero when chemical potential \(\mu\) exceeds \(T_D\), see Fig. SM2. The integral over \(x\) was performed analytically, while the last integral was done numerically.

**Singlet representations.** It turns out that the second singlet in Eq. (S.14) gives results identical to that of the first one, while the third singlet does not have a solution in the physically interesting range of parameters. Therefore we assume the order parameter in the matrix form \(\Delta = \Delta_S M_i^S = i\Delta_S \alpha^y\). The relevant matrix element of the matrix gap equation, is for real \(\Delta_S\):

\[
\frac{1}{g} = \sum_{\omega_p} \frac{\Delta^2_p + 2\mu^2 + \omega^2}{(\Delta^2_p + \omega^2)^2 + (\mu^2 + \omega^2 + 2\Delta^2_p) (\mu^2 + \omega^2) + 2\mu^2 (\omega^2 - \mu^2)}.
\]  

(S. 14)

Spectrum (in physical units) now is isotropic,

\[
E^2_{\pm} = \Delta^2_S + (v_F |p| \pm \mu)^2.
\]  

(S. 15)

Integration over \(\omega\) gives

\[
\frac{1}{g} = \mu \sum_{\mu - T_D < r_p < \mu + T_D} \frac{p}{r_p + r_- (r_+ - r_-)},
\]  

(S. 16)
Fig. S. 1: Spectrum of triplet excitations. a. section $p_\perp = 0$ in b. There is also a saddle points with energy gap, c. $2\Delta_T$ on the circle $p_\perp^2 + p_\parallel^2 = \mu^2, p_\parallel = 0$ see the section in the $p_\parallel = 0$ direction in . d. The higher energy band $E_+$ touches the lower band at $p = 0$, so that there is a Dirac point for quasiparticles.

where $r_\pm = \sqrt{\Delta_T^2 + (|p| \pm \mu)^2}$, while the $p$ integration results in:

$$\frac{16\pi^2}{g} = \Phi (\mu + 1, \mu) - \Phi (\max [\mu - 1, 0], \mu)$$ (S. 17)

with

$$\Phi (p, \mu) = r_- (p + 3\mu) + r_+ (p - 3\mu) - (\Delta_T^2 - 2\mu^2) \log [(p + r_- - \mu) (p + r_+ + \mu)]$$ (S. 18)

The solution is presented in Fig. 2 of the paper as lines of constant chemical potential. Having found the order parameter, one has to determine what symmetry breaking is realized by comparing energies of the solutions as explained in the text.

3. The BCS and the strong coupling limits. –

**Triplet.** In several limiting cases the integrals can be performed analytically. At zero chemical potential the results are presented in Section IV, while here we list the BCS limit of $\mu >> T_D$ and the strong coupling case of $g\mu^2 >> 1, \Delta_T \propto g$.

(i) In the BCS limit one has

$$\frac{1}{g} = \frac{a_T \mu^2}{4\pi^2} \sinh^{-1} \frac{T_D}{\Delta_T},$$ (S. 19)

with $a_T = 0.69$, leading to exponential gap dependence on $\lambda$ when it is small:

$$\Delta_T = T_D / \sinh (1/2a_T \lambda) \approx 2T_D e^{-1/2a_T \lambda}.$$ (S. 20)

(ii) In the strong coupling one obtains with solution

$$\Delta_T = \frac{g}{12\pi^2} \left\{ \begin{array}{ll} 6\mu^2 + 2 & \text{for } \mu < 1 \\ (\mu + 1)^3 & \text{for } \mu > 1 \end{array} \right.$$ (S. 21)

Usually the local coupling does not prefer the triplet pairing and the singlet channels of coupling are realized. We therefore turn to them.
Fig. S. 2: Chemical potential in Dirac semi-metals and the phonon mediated pairing. (a) Chemical potential relative to Dirac point is smaller than typical energy of phonons, the Debye energy $T_D$. (b) The BCS approximation limit: the chemical potential is much larger than the Debye energy $T_D$.

**Singlet.** For singlet one has

(i) BCS, $\mu >> T_D$

$$\Delta_S = T_D / \sinh (1/2\lambda) \simeq 2T_D e^{-1/2\lambda}.$$  \hspace{1cm} (S. 22)

(ii) Strong coupling

$$\Delta_S = \frac{2\lambda (T_D + \mu)^3}{3\mu^2}.$$  \hspace{1cm} (S. 23)

**Energies.** In limiting cases, one obtains expressions in closed form.

(i) BCS, $\mu > T_D$, using Eq.(S. 19) and Eq.(S. 20) for the triplet and Eq.(S. 22) for the singlet, one has the energy density:

$$F_{T,S} = -\frac{a_{T,S} \mu^2 T_D}{2\pi^2 v_F^3 \hbar^3} \left( \Delta_T^2 + \frac{T_D^2}{T_D^2 - T_D} \right) \simeq -\frac{a_{T,S} \mu^2 T_D^2}{\pi^2 v_F^3 \hbar^3} \exp \left( -\frac{1}{a_{T,S} \lambda} \right),$$

with $a_T = 0.69$, while $a_S = 1$ and assuming $\lambda << 1$. The ratio of the two phases gives

$$\frac{F_T}{F_S} = 0.69 e^{-0.45/\lambda}.$$  \hspace{1cm} (S. 25)

(ii) Strong coupling limit, using Eq.(S. 21) for triplet and Eq.(S. 23) for the singlet,

$$F_T = F_S = -\frac{1}{72\pi^4 v_F^3 \hbar^3} \left\{ \begin{array}{ll} 4 (3\mu^2 + T_D^2)^2 & \text{for } \mu < T_D \\ T_D^{-2} (\mu + T_D)^6 & \text{for } \mu > T_D \end{array} \right.$$  \hspace{1cm} (S. 26)

The difference appears at order $1/g$. To summarize, in most of the parameter range shown triplet is a bit higher than that of the singlet, but the two condensates are nearly degenerate.

4. **Magnetic impurities.** – After averaging over impurities, the Gor’kov equations, Eq.(S. 6) acquires an additional term in components (no Nambu notations)

$$I = D^{-1}G - NG - LF^+$$

$$0 = (D^{+1} - N^+) F^+ - (L^+ - \Delta^+) G$$

where the normal disorder average
lead to the renormalization of the chemical potential and relaxation time that can be safely neglected for our purposes.

The second, anomalous disorder average

\[
N^{\alpha\beta'} (r - r', \tau - \tau') = J^2 \left\langle \sum_{a, b} S^i_a S^i_b \delta (r - r_a) \delta (r' - r_b) \right\rangle_{\text{dis}} \times \tag{S. 28}
\]

\[
\Sigma_{\alpha\beta} \Sigma_{\alpha'\beta'} \left\langle T \psi_\beta (r, \tau) \psi^+_{\alpha'} (r', \tau') \right\rangle = -C \delta (r - r') \Sigma_{\alpha\beta} G_{\beta\alpha'} (0, \tau - \tau') \Sigma_{\alpha'\beta'}
\]
determines the influence of the disorder on the condensate. For singlet one has in Fourier space

\[
\sum_q F^{+\beta\gamma} (q, \omega) = \frac{i}{g} \Delta^S (\omega) \alpha^y_{\beta\gamma},
\]

leading via

\[
i \Sigma^{it} \alpha^y \Sigma^i = i \left( \begin{array}{cc} \sigma_i^t & \sigma_i^t \\ \sigma_i^t & -\sigma_i^t \end{array} \right) \left( \begin{array}{cc} \sigma_i & \sigma_i \\ -\sigma_i & \sigma_i \end{array} \right) = -3i \alpha_y
\]

(S. 31)
to Eq.(L) in the main text from which the bifurcation point is found.

Similarly for triplet

\[
L^{+\beta\gamma} (p, \omega) = -C \Sigma^{it} \sum_q F^{+\beta\gamma} (q, \omega) \Sigma^i = -\frac{C}{g} \Delta^T (\omega) \gamma^\gamma.
\]

since now

\[
\sum_q F^{+\beta\gamma} (q, \omega) = \frac{1}{g} \Delta^T (\omega) \gamma^\gamma
\]

(S. 33)
and

\[
\Sigma^{it} \gamma^\gamma \Sigma^i = \left( \begin{array}{cc} \sigma_i^t & 0 \\ 0 & \sigma_i^t \end{array} \right) \left( \begin{array}{cc} 0 & -\sigma_x \\ \sigma_x & 0 \end{array} \right) \left( \begin{array}{cc} \sigma_i & 0 \\ 0 & \sigma_i \end{array} \right) = \gamma^\gamma
\]

(S. 34)
Note opposite signs of the singlet and triplet. At bifurcation point (destruction of the condensate) the function \( f \) will be now

\[
f_T (\omega) = \frac{1}{4} \text{tr} \sum_q D^T \gamma^\gamma D^T \gamma^\gamma > 0.
\]

(S. 35)
To solve the equations for the critical disorder strength \( C^c \), one integrates over \( \omega \),

\[
\frac{1}{g} = \sum_{\omega} \frac{f_T (\omega)}{1 - C^c f_T (\omega)}
\]

(S. 36)
that has no solution.