INDEX MAPS IN THE $K$-THEORY OF GRAPH ALGEBRAS

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Abstract. Let $C^*(E)$ be the graph $C^*$-algebra associated to a graph $E$ and let $J$ be a gauge-invariant ideal in $C^*(E)$. We compute the cyclic six-term exact sequence in $K$-theory associated to the extension

$$0 \to J \to C^*(E) \to C^*(E)/J \to 0$$

in terms of the adjacency matrix associated to $E$. The ordered six-term exact sequence is a complete stable isomorphism invariant for several classes of graph $C^*$-algebras, for instance those containing a unique proper nontrivial ideal. Further, in many other cases, finite collections of such sequences comprise complete invariants.

Our results allow for explicit computation of the invariant, giving an exact sequence in terms of kernels and cokernels of matrices determined by the vertex matrix of $E$.

1. INTRODUCTION

The cyclic six-term exact sequence

$$
\begin{array}{cccccc}
K_0(J) & \xrightarrow{i_*} & K_0(C^*(E)) & \xrightarrow{\pi_*} & K_0(C^*(E)/J) \\
\partial_1 & & \partial_0 & & \\
K_1(C^*(E)/J) & \xleftarrow{\pi_*} & K_1(C^*(E)) & \xleftarrow{i_*} & K_1(J)
\end{array}
$$

is a complete stable isomorphism invariant for a graph $C^*$-algebra $C^*(E)$ of real rank zero containing a proper nontrivial ideal $J$ when any of the following are satisfied

- $J$ is the unique proper nontrivial ideal of $C^*(E)$ ([7, Theorem 4.5]),
- $J$ is a smallest proper nontrivial ideal of $C^*(E)$, and $C^*(E)/J$ is AF ([6, Corollary 6.4]),
- $J$ is a largest proper nontrivial ideal of $C^*(E)$, and $J$ is AF ([7, Theorem 4.7]).

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In other cases (cf. [6]) a complete invariant may be obtained by combining several six-term exact sequences associated to $\mathcal{C}^*(E)$ and its ideals.

It is therefore important to address how to compute sequences of the form in (1.1). In the existing literature it is shown that if $E$ is a row-finite graph with no sinks, then

$$K_0(\mathcal{C}^*(E)) \cong \text{coker}(A^t - I) \quad \text{and} \quad K_1(\mathcal{C}^*(E)) \cong \ker(A^t - I),$$

where $A^t - I : \mathbb{Z}^{E_0} \to \mathbb{Z}^{E_0}$ is the linear map given by the transpose of the vertex matrix $A$ of $E$ minus the identity matrix $I$. This description of the $K_0$-group also includes a description of its order, and a similar computation exists when sinks and infinite emitters are allowed. Since gauge-invariant ideals of graph $C^*$-algebras and the corresponding quotients are naturally isomorphic to graph $C^*$-algebras, this allows one to compute the $K_0$-groups and $K_1$-groups in the above exact sequence. Moreover, since the $C^*$-algebra of a graph satisfying Condition (K) has real rank zero [9, Theorem 3.5], it follows from [3] that the descending connecting map $\partial_0 : K_0(\mathcal{C}^*(E)/J) \to K_1(J)$ is the zero map. All that remains is to describe a method for computing the other connecting group homomorphisms.

The purpose of this paper is to provide explicit formulae for computing the six-term exact sequence, the main challenge being to compute the connecting map $\partial_1 : K_1(\mathcal{C}^*(E)/J) \to K_0(J)$. We shall also show that $\partial_0 : K_0(\mathcal{C}^*(E)/J) \to K_1(J)$ is the zero map regardless of whether the graph $E$ satisfies Condition (K) or not. All our calculations hold for an arbitrary graph algebra $\mathcal{C}^*(E)$ and an arbitrary gauge-invariant ideal $J$ in $\mathcal{C}^*(E)$, even in the case of so-called breaking vertices.

To compute $\partial_1$, we need to choose generators for the $K$-groups involved. There is a canonical (and well-known) way to do this in $K_0$; one can choose an isomorphism of $K_0(\mathcal{C}^*(E))$ with $\text{coker}(A^t - I)$ taking $[p_v]$ to $e_v + \text{Im}(A^t - I)$, where $e_v$ is the vector with a 1 in the $v$th position and zeroes elsewhere. However, for the $K_1$-group the calculation is substantially harder. Descriptions of $K_1$ can be found in [2] and [5], but we need a more explicit description and therefore choose a different approach, choosing explicit generators for $K_1$ based on a slightly intricate indexing of the entries in a matrix over $\mathcal{C}^*(E)$. Although any quotient of a graph $C^*$-algebra by a gauge-invariant ideal is isomorphic to a graph $C^*$-algebra, it will be more convenient for us to use that such a quotient is isomorphic to a relative graph $C^*$-algebra (cf. [11]), and we will therefore find generators of $K_0$ and $K_1$, not just for graph $C^*$-algebras, but for relative graph $C^*$-algebras.

We prove that the generators we choose for $K_1$ are indeed generators by computing the index map of the canonical Toeplitz extension of $\mathcal{C}^*(E)$, using methods developed by Katsura in that framework. Our approach involves computing the index map using the canonical method (cf. [14]) of lifting the generating unitaries to partial isometries and computing defects. This method has similarities with the approach for Cuntz-Krieger algebras outlined by Cuntz himself in [4], and discussed with a few more details
in \([\mathbb{D}]\). After describing how to choose generators for \(K_0\) and \(K_1\) of any relative graph \(C^\ast\)-algebra, we determine the index map \(\partial_1 : K_1(C^\ast(E)/J) \to K_0(J)\) by, in a new extension, again lifting our generating unitaries to partial isometries, and computing defects.

In Section 2 we briefly introduce graph \(C^\ast\)-algebras, relative graph \(C^\ast\)-algebras, and gauge-invariant ideals of graph \(C^\ast\)-algebras. In Section 3 we find generators of \(K_0\) and \(K_1\) of any relative graph \(C^\ast\)-algebra. Section 4 states the main result of the paper, allowing the computation of the index map \(\partial_1 : K_1(C^\ast(E)/J) \to K_0(J)\) and the other maps in the six-term exact sequence (1.1), and this result is proved in Section 5.

2. Preliminaries

A (directed) graph \(E = (E^0, E^1, r, s)\) consists of a countable set \(E^0\) of vertices, a countable set \(E^1\) of edges, and maps \(r, s : E^1 \to E^0\) identifying the range and source of each edge. A vertex \(v \in E^0\) is called a sink if \(|s^{-1}(v)| = 0\), and \(v\) is called an infinite emitter if \(|s^{-1}(v)| = \infty\). A graph \(E\) is said to be row-finite if it has no infinite emitters. If \(v\) is either a sink or an infinite emitter, then we call \(v\) a singular vertex. We write \(E^0_{\text{reg}}\) for the set of singular vertices. Vertices that are not singular vertices are called regular vertices and we write \(E^0_{\text{reg}}\) for the set of regular vertices.

If \(E\) is a graph, a Cuntz-Krieger \(E\)-family is a set of mutually orthogonal projections \(\{p_v : v \in E^0\}\) and a set of partial isometries \(\{s_e : e \in E^1\}\) with mutually orthogonal ranges which satisfy the Cuntz-Krieger relations:

\[
\begin{align*}
&\text{(CK1)} \quad s_e^* s_e = p_{r(e)} \text{ for every } e \in E^1; \\
&\text{(CK2)} \quad p_v = \sum_{s(e) = v} s_e s_e^* \text{ for every } v \in E^0_{\text{reg}}; \\
&\text{(CK3)} \quad s_e s_e^* \leq p_{s(e)} \text{ for every } e \in E^1.
\end{align*}
\]

The graph algebra \(C^\ast(E)\) is defined to be the \(C^\ast\)-algebra generated by a universal Cuntz-Krieger \(E\)-family.

It will in this paper also be relevant to work with relative graph \(C^\ast\)-algebras introduced in \([\mathbb{D}]\). To define a relative graph \(C^\ast\)-algebra we must, in addition to a graph \(E\), specify a subset \(R\) of \(E^0_{\text{reg}}\). A Cuntz-Krieger \((E, R)\)-family is then a set of mutually orthogonal projections \(\{p_v : v \in E^0\}\) and a set of partial isometries \(\{s_e : e \in E^1\}\) with mutually orthogonal ranges which satisfy the relations (CK1) and (CK3) above together with the following relative Cuntz-Krieger relation:

\[
\text{(RCK2)} \quad p_v = \sum_{s(e) = v} s_e s_e^* \text{ for every } v \in R.
\]

The relative graph algebra \(C^\ast(E, R)\) is defined to be the \(C^\ast\)-algebra generated by a universal Cuntz-Krieger \((E, R)\)-family. If \(R = E^0_{\text{reg}}\), then a Cuntz-Krieger \((E, R)\)-family is the same as a Cuntz-Krieger \(E\)-family and \(C^\ast(E, R) = C^\ast(E)\). If \(R = \emptyset\), then \(C^\ast(E, R)\) is the Toeplitz algebra \(T(E)\) defined in \([\mathbb{B}]\) Theorem 4.1. We will call a Cuntz-Krieger \((E, \emptyset)\)-family a Toeplitz-Cuntz-Krieger \(E\)-family.
A path in $E$ is a sequence of edges $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$ with $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i < n$, and we say that $\alpha$ has length $|\alpha| = n$. We let $E^n$ denote the set of all paths of length $n$, and we let $E^* := \bigcup_{n=0}^{\infty} E^n$ denote the set of finite paths in $E$. Note that vertices are considered paths of length zero. The maps $r, s$ extend to $E^*$, and for $v, w \in E^0$ we write $v \geq w$ if there exists a path $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = w$. Also for a path $\alpha := \alpha_1 \ldots \alpha_n$ we define $s_\alpha := s_{\alpha_1} \ldots s_{\alpha_n}$, and for a vertex $v \in E^0$ we let $s_v := p_v$. It is a consequence of the relations (CK1) and (CK3) that $C^*(E, R) = \overline{\text{span}} \{ s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) \}$.

We say that a path $\alpha := \alpha_1 \ldots \alpha_n$ of length 1 or greater is a cycle if $r(\alpha) = s(\alpha)$, and we call the vertex $s(\alpha) = r(\alpha)$ the base point of the cycle. A cycle is said to be simple if $s(\alpha_i) \neq s(\alpha_1)$ for all $1 < i \leq n$. The following is an important condition in the theory of graph $C^*$-algebras.

**Condition (K):** No vertex in $E$ is the base point of exactly one simple cycle; that is, every vertex is either the base point of no cycles or at least two simple cycles.

For any graph $E$ a subset $H \subseteq E^0$ is hereditary if whenever $v, w \in E^0$ with $v \in H$ and $v \geq w$, then $w \in H$. A hereditary subset $H$ is saturated if whenever $v \in E^0_{\text{reg}}$ with $r(s^{-1}(v)) \subseteq H$, then $v \in H$. For any saturated hereditary subset $H$, the breaking vertices corresponding to $H$ are the elements of the set

\[ B_H := \{ v \in E^0 : |s^{-1}(v)| = \infty \text{ and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty \}. \]

An admissible pair $(H, S)$ consists of a saturated hereditary subset $H$ and a subset $S \subseteq B_H$. For a fixed graph $E$ we order the collection of admissible pairs for $E$ by defining $(H, S) \leq (H', S')$ if and only if $H \subseteq H'$ and $S \subseteq H' \cup S'$. For any admissible pair $(H, S)$ we define $J_{(H, S)}$ to be the ideal in $C^*(E)$ generated by

\[ \{ p_v : v \in H \} \cup \{ p_v^H : v_0 \in S \}, \]

where $p_v^H$ is the gap projection defined by

\[ p_v^H := p_v - \sum_{s(e)=v_0 \atop r(e) \notin H} s_es_e^*. \]

Note that the definition of $B_H$ ensures that the sum on the right is finite.

For any graph $E$ there is a canonical gauge action $\gamma : \mathbb{T} \rightarrow \text{Aut} \ C^*(E)$ with the property that for any $z \in \mathbb{T}$ we have $\gamma_z(p_v) = p_v$ for all $v \in E^0$ and $\gamma_z(s_e) = zs_e$ for all $e \in E^1$. We say that an ideal $J \triangleleft C^*(E)$ is gauge invariant if $\gamma_z(J) \subseteq J$ for all $z \in \mathbb{T}$.

There is a bijective correspondence between the lattice of admissible pairs of $E$ and the lattice of gauge-invariant ideals of $C^*(E)$ given by $(H, S) \mapsto J_{(H, S)}$ [2, Theorem 3.6]. When $E$ satisfies Condition (K), all ideals of $C^*(E)$
are gauge invariant [2, Corollary 3.8] and the map \((H, S) \mapsto J_{(H, S)}\) is onto the lattice of ideals of \(C^*(E)\). When \(B_H = \emptyset\), we write \(J_H\) in place of \(J_{(H, \emptyset)}\) and observe that \(J_H\) equals the ideal generated by \(\{p_v : v \in H\}\). Note that if \(E\) is row-finite, then \(B_H\) is empty for every saturated hereditary subset \(H\).

3. \(K\)-theory for relative graph algebras

For a graph \(E\), the adjacency matrix is the \(E^0 \times E^0\) matrix \(A_E\) with

\[
A_E(v, w) := \# \{ e \in E^1 : s(e) = v \text{ and } r(e) = w \}.
\]

Note that the entries of \(A_E\) are elements of \(\{0, 1, 2, \ldots\} \cup \{\infty\}\). Writing the adjacency matrix with respect to the decomposition \(E^0 = E^0_{\text{reg}} \sqcup E^0_{\text{sing}}\), where the regular vertices are listed first, we obtain a (possibly infinite) block matrix

\[
A_E = \begin{bmatrix}
A & \alpha \\
H & \eta
\end{bmatrix}
\]

in which all entries of \(A\) and \(\alpha\) are finite, but the entries in \(H\) and \(\eta\) may be infinite. We will often just substitute “\(*\)” for \(H\) and \(\eta\), as they turn out to be irrelevant for the \(K\)-theory. Indeed, by [2] and [5] we know that the map

\[
\begin{bmatrix}
A^t - I \\
\alpha^t
\end{bmatrix} : \mathbb{Z} E^0_{\text{reg}} \to \mathbb{Z} E^0
\]

contains the needed information, as

\[
K_0(C^*(E)) \simeq \text{coker } \begin{bmatrix}
A^t - I \\
\alpha^t
\end{bmatrix} \quad K_1(C^*(E)) \simeq \ker \begin{bmatrix}
A^t - I \\
\alpha^t
\end{bmatrix}.
\]

This result can be generalized to relative graph \(C^*\)-algebras. In fact, we prove in Proposition 3.8 that if \(E\) is a graph, \(R \subseteq E^0_{\text{reg}}\), and \(A_E = \begin{bmatrix} A & \alpha \\ H & \eta \end{bmatrix}\) is the adjacency matrix of \(E\) written with respect to the decomposition \(E^0 = R \sqcup (E^0 \setminus R)\), where the vertices belonging to \(R\) are listed first, then there exists a group isomorphism \(\chi_0 : \text{coker } \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix} \to K_0(C^*(E, R))\) given for any \(v \in R\) by

\[
\chi_0 \left( e_v + \text{im } \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix} \right) = [p_v]_0,
\]

and we construct a similar group isomorphism \(\chi_1\) between \(\ker \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix}\) and \(K_1(C^*(E, R))\). For this we first introduce some notation:

Given \(x \in \ker \begin{bmatrix} A^t - I \\ \alpha^t \end{bmatrix}\), first note that by definition \(x\) has only finitely many nonzero entries \(x_{v_1}, \ldots, x_{v_k}\). We define

\[
L^+_x := \{(e, i) : e \in E^1, 1 \leq i \leq -x_{s(e)}\} \cup \{(v, i) : v \in E^0, 1 \leq i \leq x_v\}
\]

\[
L^-_x := \{(e, i) : e \in E^1, 1 \leq i \leq x_{s(e)}\} \cup \{(v, i) : v \in E^0, 1 \leq i \leq -x_v\}
\]

and note, using the convention that \(r(v) = v\) for any \(v \in E^0\), that
Lemma 3.1. When $x \in \ker \left[ A^\alpha - I \right]$, then for any vertex $v \in E^0$ the sets

$$L^+_v = \{(x, i) : (x, i) \in L^+_x : r(x) = v\}$$

and

$$L^-_v = \{(x, i) : (x, i) \in L^-_x : r(x) = v\}$$

are finite and have the same number of elements.

Proof. We need to consider three cases separately.

Case I: $v \in R$ and $x_v \geq 0$.

The number of elements in $L^+_v$ is

$$x_v + \sum_{x_w < 0} \#\{e \in E^1 : s(e) = w, r(e) = v\} \cdot (-x_w) = x_v - \sum_{x_w < 0} A^t_{v, w} x_w$$

and the number of elements in $L^-_v$ is

$$\sum_{x_w > 0} \#\{e \in E^1 : s(e) = w, r(e) = v\} \cdot x_w = \sum_{x_w > 0} A^t_{v, w} x_w$$

so the claim follows by inspecting the $v$ coordinate of the equality $A^t \mathbf{x} = \mathbf{x}$.

Case II: $v \in R$ and $x_v < 0$.

As above.

Case III: $v \in E^0 \setminus R$.

The number of elements in $L^+_v$ is

$$\sum_{x_u < 0} \#\{e \in E^1 : s(e) = u, r(e) = v\} \cdot (-x_u) = -\sum_{x_u < 0} A^t_{u, v} x_u$$

and the number of elements in $L^-_v$ is

$$\sum_{x_u > 0} \#\{e \in E^1 : s(e) = u, r(e) = v\} \cdot x_u = \sum_{x_u > 0} A^t_{u, v} x_u$$

so the claim follows by inspecting the $v$ coordinate of the equality $A^t \mathbf{x} = 0$.

Lemma 3.2. $L^+_x$ and $L^-_x$ are finite sets, and have the same number of elements.

Proof. This follows from Lemma 3.1 as indeed $L^+_v \neq \emptyset$ only when $v$ lies in the set

$$\{v : x_v \neq 0\} \cup \{v : x_w \neq 0 \text{ for some } w \in s(r^{-1}(v))\}$$

which is finite since no $w$ is an infinite emitter.

Denote the common number of elements in $L^+_x$ and $L^-_x$ by $h$. Because of Lemma 3.1 we can define bijections

$$[\cdot] : L^+_x \to \{1, \ldots, h\} \quad \langle \cdot \rangle : L^-_x \to \{1, \ldots, h\}$$

with the property that

$$[x, i] = \langle y, j \rangle \implies r(x) = r(y)$$
with the convention \( r(v) = v \).

When \( \mathfrak{A} \) is a \( C^* \)-algebra then we let \( \mathcal{M}_h(\mathfrak{A}) \) denote the \( C^* \)-algebra of \( h \times h \)-matrices over \( \mathfrak{A} \). We are ready for our key definitions:

**Definition 3.3.** Suppose that \( \mathfrak{A} \) is a \( C^* \)-algebra which contains a Toeplitz-Cuntz-Krieger \( E \)-family \( \{p_v : v \in E^0\} \cup \{s_e : e \in E^1\} \). With notation as above, we define the two elements \( V, P \in \mathcal{M}_h(\mathfrak{A}) \) by

\[
V = \sum_{1 \leq i \leq x_w, s(e) = w} s_e E_{[w,i],(e,i)} + \sum_{1 \leq i \leq -x_w, s(e) = w} s_e^* E_{[e,i],(w,i)}
\]

and

\[
P = \sum_{1 \leq i \leq x_w} p_w E_{[w,i],[w,i]} + \sum_{1 \leq i \leq -x_w} p_v E_{[e,i],[e,i]}.
\]

Here \( E_{*,*} \) denote the standard matrix units in \( \mathcal{M}_h(M(\mathfrak{A})) \) where \( M(\mathfrak{A}) \) is the multiplier algebra of \( \mathfrak{A} \).

**Lemma 3.4.** If \( \{s_e,p_v : e \in E^1, v \in E^0\} \) is a Toeplitz-Cuntz-Krieger \( E \)-family, then

\[
P = \sum_{1 \leq i \leq -x_w} p_w E_{[w,i],[w,i]} + \sum_{1 \leq i \leq x_w} p_v E_{[e,i],[e,i]},
\]

\[
V^* = \sum_{1 \leq i \leq x_w} s_e^* E_{[w,i],[e,i]} + \sum_{1 \leq i \leq -x_w} s_e E_{[e,i],[w,i]},
\]

\[
VV^* = \sum_{1 \leq i \leq x_w} s_e s_e^* E_{[w,i],[w,i]} + \sum_{1 \leq i \leq -x_w} p_e E_{[e,i],[e,i]},
\]

\[
V^*V = \sum_{1 \leq i \leq -x_w} s_e s_e^* E_{[w,i],[w,i]} + \sum_{1 \leq i \leq x_w} p_v E_{[e,i],[e,i]}.
\]

**Proof.** It follows from Lemma 3.2 and Equation (3.1) that

\[
\sum_{(x,i) \in L^+_k} p_{r(x)} E_{[x,i],[x,i]} = \sum_{(x,i) \in L^-_k} p_{r(x)} E_{[x,i],[x,i]},
\]

and it is easy to check that

\[
P = \sum_{(x,i) \in L^+_k} p_{r(x)} E_{[x,i],[x,i]}
\]

and that

\[
\sum_{(x,i) \in L^-_k} p_{r(x)} E_{[x,i],[x,i]} = \sum_{1 \leq i \leq -x_w} p_w E_{[w,i],[w,i]} + \sum_{1 \leq i \leq x_w} p_v E_{[e,i],[e,i]}.
\]
and (3.4) holds from (CK1) and the fact that the $s_e$’s have mutually orthogonal ranges. The computation for $V^*V$ is similar.

Lemma 3.5. If $\{s_e, p_v : e \in E^1, v \in E^0\}$ is a Toeplitz-Cuntz-Krieger $E$-family, then $V$ is a partial isometry with $PV = VP = V$.

Proof. Using Equation (3.5), the definition of $V$, and the fact that the $s_e$’s are partial isometries, we see that $V^*V = V$, so that $V$ is a partial isometry. Furthermore, (CK3) implies $PV = V$ and $VP = V$ by Equation (3.2).

We now let $\{s_e, p_v : e \in E^1, v \in E^0\}$ be the universal Cuntz-Krieger $(E, R)$-family generating $C^*(E, R)$ and write $V_x$ and $P_x$ for the corresponding elements $V$ and $P$ in $M_h(C^*(E, R))$ defined in Definition 3.3, using the added subscript to emphasize the dependence of each of $V$ and $P$ on $x \in \ker \left[ \frac{A^*-1}{\alpha^*} \right]$. In addition, we define $U_x := V_x + (1 - P_x)$.

Fact 3.6. We have that $V_x V^*_x = V^*_x V_x = P_x$, and hence that $U_x$ is a unitary.

Proof. It follows from Equation (3.4) and (RCK2) that

$$V_x V^*_x = \sum_{1 \leq i \leq x_w} \left( \sum_{s(e) = w} s_e s^*_e \right) E_{[w, i], [w, i]} + \sum_{1 \leq i \leq x_w} p_v E_{[e, i], [e, i]}$$

$$= \sum_{1 \leq i \leq x_w} p_v E_{[w, i], [w, i]} + \sum_{1 \leq i \leq x_w} p_v E_{[e, i], [e, i]}$$

$$= P_x$$

showing the first claim. Likewise, Equation (3.5) and (RCK2) show that $V^*_x V_x = P_x$. The fact that $U_x$ is a unitary follows.

Remark 3.7. Notice that although $U_x$ does depend on the choice of bijections $[\cdot] : L^+_x \to \{1, \ldots, h\}$ and $(\cdot) : L^-_x \to \{1, \ldots, h\}$, the element $[U_x]_{1}$ of $K_1(C^*(E, R))$ does not.

Proposition 3.8. Let $E$ be a graph, let $V$ be a subset of $E_{\text{reg}}$ and let

$$A_E = \begin{bmatrix} A & 0 \\ H & \eta \end{bmatrix}$$

be the adjacency matrix of $E$ written with respect to the decomposition $E^0 = V \cup (E^0 \setminus V)$ where the vertices belonging to $V$ are listed first.
There exists a group isomorphism \( \chi_0 : \text{coker} \left[ \frac{A^t - I}{\alpha^t} \right] \to K_0(\mathcal{C}^*(E, R)) \)
given for any \( v \in E^0 \) by

\[
\chi_0 \left( e_v + \text{im} \left[ \frac{A^t - I}{\alpha^t} \right] \right) = [p_v]_0.
\]

The preimage of the positive cone of \( K_0(\mathcal{C}^*(E, R)) \) is generated by

\[
\left\{ e_v : v \in E^0 \right\} \cup \left\{ e_v - \sum_{e \in F} e_{r(e)} : v \in E_{\text{sing}}^0, \ F \subseteq s^{-1}(v), \ F \text{ finite} \right\}.
\]

The map \( \chi_1 : \ker \left[ \frac{A^t - I}{\alpha^t} \right] \to K_1(\mathcal{C}^*(E, R)) \) given by

\[
\chi_1(x) = [U_x]_1
\]
is group isomorphism.

**Proof.** As noted in [11], we can realize \( \mathcal{C}^*(E, R) \) as a relative Cuntz-Pimsner algebra over a Hilbert bimodule \( \mathcal{X}_E \). It is not difficult to check that the corresponding Toeplitz algebra \( T_{\mathcal{X}_E} \) is isomorphic to the Toeplitz algebra \( T(E) \). We let \( \pi : T(E) \to \mathcal{C}^*(E, R) \) denote the canonical map, so that

\[
0 \to \ker \pi \xrightarrow{\iota} T(E) \xrightarrow{\pi} \mathcal{C}^*(E, R) \to 0
\]
is exact. The associated \( K \)-theory is then

\[
K_0(\ker \pi) \xrightarrow{\iota_*} K_0(T(E)) \xrightarrow{\pi_*} K_0(\mathcal{C}^*(E, R))
\]

\[
\partial_1
\]

\[
K_1(\mathcal{C}^*(E, R)) \xleftarrow{\pi_*} K_1(T(E)) \xleftarrow{\iota_*} K_1(\ker \pi).
\]

Now we appeal to Katsura’s work. It follows from the results of [10, §8], that \( \ker \pi \) and \( T(E) \) are \( KK \)-equivalent to the commutative \( AF \)-algebras \( c_0(R) \) and \( c_0(E^0) \), respectively, and that there are group isomorphisms \( \kappa : K_0(\ker \pi) \to \mathbb{Z}^R \) and \( \lambda : K_0(T(E)) \to \mathbb{Z}^{E^0} \) such that the diagram

\[
0 \to K_1(\mathcal{C}^*(E, R)) \xrightarrow{\partial_1} K_0(\ker \pi) \xrightarrow{\iota_*} K_0(T(E)) \xrightarrow{\pi_*} K_0(\mathcal{C}^*(E, R)) \to 0
\]

\[
\kappa
\]

\[
\mathbb{Z}^R \xrightarrow{\left[ \frac{A^t - I}{\alpha^t} \right]} \mathbb{Z}^{E^0}
\]

commutes with the top row exact. In [10] concrete \( * \)-homomorphisms are given inducing \( \kappa \) and \( \lambda \), but we do not need them here. All we need is the fact that \( \lambda(p_v) = e_v \) and

\[
\kappa \left( p_w - \sum_{s(e) = w} s_es_e^* \right)_0 = e_w.
\]
for \( v \in E^0 \) and \( w \in R \). It follows that \( \pi_v \circ \lambda \circ \gamma^{-1} \) is a surjective group homomorphism from \( \mathbb{Z}E^0 \) to \( K_0(C^*(E, R)) \) which for any \( v \in E^0 \) maps \( \epsilon_v \) to \( [p_v]_0 \) and whose kernel is im \( [A^t - I]_{\alpha^t} \). The existence of a group isomorphism \( \chi_0 : \text{coker } [A^t - I]_{\alpha^t} \rightarrow K_0(C^*(E, R)) \) which for any \( v \in E^0 \) satisfies Equation (3.8) follows from this. The description of the positive cone in the row-finite case was given in [11, Theorem 7.1]. For the general situation, it is shown in [15, Theorem 2.2] that the process of desingularization can be used to extend the result from the row-finite case to the general case.

To see that \( \chi_1 : \ker [A^t - I]_{\alpha^t} \rightarrow K_1(C^*(E, R)) \) is a group isomorphism, fix \( \epsilon \in \ker [A^t - I]_{\alpha^t} \) and lift \( U_\epsilon = V_\epsilon + (1 - P_\epsilon) \in M_h(C^*(E, R)) \) to \( \tilde{U}_\epsilon = \tilde{V}_\epsilon + (1 - P_\epsilon) \in M_h(\mathcal{T}(E)) \) where \( \tilde{V}_\epsilon \) and \( \tilde{P}_\epsilon \) are the elements \( V \) and \( P \) in \( M_h(\mathcal{T}(E)) \) we get by using the universal Toeplitz-Cuntz-Krieger \( E \)-family which generates \( \mathcal{T}(E) \) in Definition 3.3. By Lemma 3.5, \( \tilde{V}_\epsilon \) is a partial isometry with \( \tilde{P}_\epsilon \tilde{V}_\epsilon = \tilde{V}_\epsilon \tilde{P}_\epsilon = \tilde{V}_\epsilon \). It follows that \( U_\epsilon \) is also a partial isometry. We need to compute the defect of \( \tilde{U}_\epsilon \) as an element of \( K_0(\ker \pi) \).

We have by Lemma 3.4 that

\[
1 - \tilde{U}_\epsilon \tilde{U}_\epsilon^* = \tilde{P}_\epsilon - \tilde{V}_\epsilon \tilde{V}_\epsilon^* = \sum_{1 \leq i \leq x_\epsilon} \left( p_w - \sum_{s(e) = w} s_e s_e^* \right) E_{[w,i],[w,i]}
\]

and a similar equation for \( 1 - \tilde{U}_\epsilon^* \tilde{U}_\epsilon \). Hence, in \( K_0(\ker \pi) \) we have that

\[
\begin{align*}
[1 - \tilde{U}_\epsilon \tilde{U}_\epsilon^*]_0 - [1 - \tilde{U}_\epsilon^* \tilde{U}_\epsilon]_0 &= \sum_{x_\epsilon \neq 0} x_\epsilon \left( p_w - \sum_{s(e) = w} s_e s_e^* \right) \\
\text{which together with Equation } &3.8 \text{ and Equation } 3.9 \text{ implies that }
\end{align*}
\]

\[
\kappa \circ \partial_1 \circ \chi_1(x) = x
\]

for any \( x \in [A^t - I]_{\alpha^t} \). This shows that \( \chi_1 \) is injective. Let us also prove that \( \chi_1 \) is a group isomorphism. Fix \( y \in K_1(C^*(E, R)) \) and note that

\[
\left[ A^t - I \right]_{\alpha^t} \circ \kappa \circ \partial_1 (y) = \lambda \circ \iota_\epsilon \circ \partial_1 (y) = 0
\]

so that \( z := \kappa \circ \partial_1 (y) \) lies in ker \( [A^t - I]_{\alpha^t} \). Since \( \kappa \circ \partial_1 \) is injective, it follows from Equation (3.10) that \( \chi_1(z) = y \). We conclude that \( \kappa \circ \partial_1 \) is actually an inverse to \( \chi_1 \), and hence \( \chi_1 \) is a group isomorphism.

**4. The index map**

Let \( E \) be a graph and let \( J \) be a gauge-invariant ideal in \( C^*(E) \). It follows from [2] that \( J \) is of the form \( J_{(H,S)} \) for an admissible pair \( (H,S) \). Writing
the adjacency matrix of $E$ with respect to the decomposition

$$E^0_{reg} \cap H, \quad E^0_{\text{sing}} \cap H, \quad E^0_{reg} \setminus H, \quad E^0_{\text{sing}} \setminus (H \cup S), \quad S$$

we arrive at the matrix

$$\begin{bmatrix}
A & \alpha & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
X & \xi & B & \beta & \eta \\
* & * & * & * & * \\
* & * & \Gamma & \gamma & Z
\end{bmatrix}.$$ 

We are now ready to state our main result. Here and below, whenever $T : G_1 \to G_2$ is a group homomorphism between abelian groups and $H_1$ and $H_2$ are subgroups of $G_1$ and $G_2$, respectively, such that $T(H_1) \subseteq H_2$, then we also use $T$ to denote the group homomorphism from $G_1/H_1$ to $G_2/H_2$ induced by $T$, and we denote by $I_{a_1 \ldots a_k}$ the canonical inclusion of the indicated components of a direct sum into a larger direct sum, and by $P_{a_1 \ldots a_k}$ the corresponding projection.

**Theorem 4.1.** Let $E$ be a graph and let $(H,S)$ be an admissible pair. The six term exact sequence in $K$-theory

$$K_0(J_{(H,S)}) \xrightarrow{\iota_*} K_0(C^*(E)) \xrightarrow{\pi_*} K_0(C^*(E)/J_{(H,S)})$$

$$K_1(C^*(E)/J_{(H,S)}) \xrightarrow{\pi_*} K_1(C^*(E)) \xrightarrow{\iota_*} K_1(J_{(H,S)})$$

is isomorphic to

$$\begin{array}{c}
\text{coker } \begin{bmatrix}
A^t - I \\
\alpha^t \\
0
\end{bmatrix} \xrightarrow{\iota} \text{coker } \begin{bmatrix}
A^t - I \\
\alpha^t & X^t & \xi^t \\
0 & B^t - I & \beta^t \\
0 & 0 & \eta^t
\end{bmatrix} \\
\text{ker } \begin{bmatrix}
B^t - I \\
\beta^t \\
\gamma^t \\
\eta^t \\
Z^t - I
\end{bmatrix} \xrightarrow{I_1 \circ P_2} \text{ker } \begin{bmatrix}
A^t - I \\
\alpha^t & X^t & \xi^t \\
0 & B^t - I & \beta^t \\
0 & 0 & \eta^t
\end{bmatrix}
\end{array}$$

$$\begin{array}{c}
P_{345} \xrightarrow{P_{345}} \text{coker } \begin{bmatrix}
B^t - I \\
\beta^t & \gamma^t \\
\eta^t & Z^t - I
\end{bmatrix} \\
\text{ker } \begin{bmatrix}
\alpha^t \\
0
\end{bmatrix} \xrightarrow{I_1} \text{ker } \begin{bmatrix}
\alpha^t \\
0
\end{bmatrix}
\end{array}$$

where $\iota$ is given by the block matrix

$$\begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & -\Gamma^t \\
0 & 0 & -\gamma^t \\
0 & 0 & I - Z^t
\end{bmatrix} = I_{125} - \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \Gamma^t \\
0 & 0 & \gamma^t \\
0 & 0 & Z^t
\end{bmatrix}.$$
Each cokernel is ordered as described in Theorem 3.8. We postpone the proof of the theorem to the ensuing section, but remark here that the isomorphism between the two six term exact sequences is given by explicit defined maps which are described in the proof.

For now, let us record a number of examples and specializations:

**Remark 4.2.** If the saturated hereditary subset \( H \) has no breaking vertices (this is always the case if \( E \) is row-finite), or if \( S = \emptyset \), then the six term exact sequence of Theorem 4.1 reduces to

\[
\begin{align*}
\text{coker} \left[ A^t - I \atop \alpha^t \right] & \xrightarrow{I_1} \text{coker} \left[ A^t - I \atop X^t \atop \xi^t \atop 0 \atop 0 \right] \xrightarrow{P_34} \text{coker} \left[ B^t - I \atop \beta^t \right], \\
\text{ker} \left[ B^t - I \atop \beta^t \right] & \xleftarrow{P_2} \text{ker} \left[ A^t - I \atop X^t \atop \xi^t \atop 0 \atop 0 \right] \xrightarrow{I_1} \text{ker} \left[ A^t - I \atop \alpha^t \right].
\end{align*}
\] (4.1)

**Remark 4.3.** Let \( E \) be a row-finite graph with no sinks. Then any gauge-invariant ideal in \( \text{C}^*(E) \) has the form \( J_H \) for some saturated hereditary subset \( H \) and the six term exact sequence of Theorem 4.1 reduces in this case to

\[
\begin{align*}
\text{coker} \left[ A^t - I \right] & \xrightarrow{I_1} \text{coker} \left[ A^t - I \atop X^t \atop 0 \atop B^t - I \right] \xrightarrow{P_2} \text{coker} \left[ B^t - I \right] \\
\text{ker} \left[ B^t - I \right] & \xleftarrow{P_2} \text{ker} \left[ A^t - I \atop X^t \atop 0 \atop B^t - I \right] \xrightarrow{I_1} \text{ker} \left[ A^t - I \right].
\end{align*}
\]

**Corollary 4.4.** Let \( E \) be a graph such that the associated graph \( \text{C}^*-\text{algebra} \) \( \text{C}^*(E) \) contains a unique proper nontrivial ideal. Then this ideal has the form \( J_H \) for some saturated hereditary subset \( H \) with no breaking vertices. Consequently, the cyclic six term exact sequence determined by the short exact sequence \( 0 \to J_H \to \text{C}^*(E) \to \text{C}^*(E)/J_H \to 0 \) is isomorphic to the cyclic exact sequence described in (4.1).

**Proof.** If \( E \) has a unique proper nontrivial ideal, then it follows from [7, Lemma 3.1] that the ideal has the form \( J_H \) for a saturated hereditary subset \( H \) with no breaking vertices. \( \square \)

**Example 4.5.** Consider the class of graphs \( E_{x,y,z} \) given by the adjacency matrix

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
x & 1 & 1 & 0 \\
y & 1 & 1 & 1 \\
z & 0 & 1 & 1
\end{bmatrix}
\]
where $x, y, z \in \mathbb{N}$. These graphs all satisfy Condition (K) and have one non-trivial saturated hereditary subset (the subset consisting of the first vertex). Thus we are in the situation of Corollary 4.4, with $E_{\text{reg}}^0 = \{v_2, v_3, v_4\}$ and $E_{\text{reg}}^0 = H = \{v_1\}$. Hence the adjacency matrix has the block form

$$
\begin{bmatrix}
\alpha & 0 \\
\xi & B
\end{bmatrix}
$$

and the six-term exact sequence is

$$
\begin{array}{ccccccc}
\text{coker} & 0 \times 0 & \xrightarrow{I_1} & \text{coker} & x & y & z \\
\text{ker} & 0 \times 0 & \xleftarrow{P_{234}} & \text{ker} & x & y & z \\
\end{array}
$$

which simplifies to

$$
\begin{array}{ccccccc}
0 & \xrightarrow{x - z} & \mathbb{Z} & \xrightarrow{x - z} & \mathbb{Z} & \xrightarrow{x - z} & \mathbb{Z} \\
\end{array}
$$

when $x \neq z$ and to

$$
\begin{array}{ccccccc}
0 & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\
\end{array}
$$

when $z = x$.

The $K_0$-group of the ideal is canonically ordered, and the order of the $K_0$-group of the quotient is trivial, irrespective of $x, y, z$. We may hence apply [7] to prove that $C^*(E_{x,y,z}) \otimes K \simeq C^*(E_{x',y',z'}) \otimes K$ precisely when $x - z = \pm(x' - z')$.

**Example 4.6.** Consider the class of graphs $F_{y,z}$ given by the adjacency matrix

$$
\begin{bmatrix}
0 & 0 & 0 \\
y & 3 & 1 \\
\infty & z & 3
\end{bmatrix}
$$

where $y, z \in \mathbb{N}$. These graphs all satisfy Condition (K) and have one non-trivial saturated hereditary subset $\{v_1\}$ for which $\{v_3\}$ is breaking. We furthermore have that $E_{\text{reg}}^0 = \{v_2\}$ and $E_{\text{sing}}^0 = \{v_1, v_3\}$. If we consider the ideal $J_{\{v_1\}, \{v_3\}}$, then the adjacency matrix has the block form

$$
\begin{bmatrix}
* & 0 & 0 \\
\xi & B & \eta \\
* & 1 & Z
\end{bmatrix}
$$
which gives

\[
\begin{array}{c}
\text{coker } 0_{2\times 0} \\
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
\begin{bmatrix} y \\ 0 \end{bmatrix} \\
\end{array}
\begin{array}{c}
\text{coker } \begin{bmatrix} y \\ 1 \end{bmatrix} \\
\begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
\end{array}
\begin{array}{c}
\text{coker } 0_{2\times 0} \\
\begin{bmatrix} z \\ 1 \end{bmatrix} \\
\end{array}
\end{array}
\]

simplifying to

\[
0 \longrightarrow \mathbb{Z}^2 \begin{bmatrix} -1 & -2y \\ 0 & 4-z \end{bmatrix} \mathbb{Z}^2 \longrightarrow \mathbb{Z}_{z-4} \longrightarrow 0
\]

when \( z \neq 4 \) and to

\[
0 \longrightarrow \mathbb{Z} \begin{bmatrix} -2y \\ 1 \end{bmatrix} \longrightarrow \mathbb{Z}^2 \begin{bmatrix} -1 & -2y \\ 0 & 0 \end{bmatrix} \longrightarrow \mathbb{Z} \longrightarrow 0
\]

when \( z = 4 \).

In both cases, the \( K_0 \)-group of the ideal is ordered by

\[
\{ (x_1, x_3) : x_3 > 1 \text{ or } [x_3 = 0, x_1 \geq 0] \},
\]

having only the trivial automorphism, so the computations combine with [7 Theorem 4.7] to show that \( C^* (F_{y', z'}) \otimes \mathbb{K} \simeq C^* (F_{y, z}) \otimes \mathbb{K} \) precisely when \( 4 - z = \pm (4 - z') \) and \( y - y' \in (4 - z) \mathbb{Z} \).

5. Proof of main result

The isomorphism of the two six-term exact sequences in Theorem 4.1 is given by the six group isomorphisms \( \chi_0', \chi_0, \chi_0'', \chi_1', \chi_1, \chi_1'' \) defined as follows. If we let \( E(H, S) \) be the subgraph of \( E \) with vertices \( H \cup S \) and edges \( s^{-1}(H) \cup (s^{-1}(S) \cap r^{-1}(H)) \), then the graph \( C^* \)-algebra \( C^* (E(H, S)) \) is isomorphic to a full corner of \( J(H, S) \) via an embedding \( \phi : C^* (E(H, S)) \rightarrow J(H, S) \) with \( \phi(p_v) = p_v \) for \( v \in H \), \( \phi(p_{v_0}) = p^H_{v_0} \) for \( v_0 \in S \) and \( \phi(s_e) = s_e \) for \( e \in E_1 (H, S) \) (cf. [2]). Notice that \( (E(H, S))^{0}_{\text{reg}} = E^{0}_{\text{reg}} \cap H \) and that \( (E(H, S))^{0}_{\text{sing}} = (E^{0}_{\text{sing}} \cap H) \cup S \). It follows (for example by [12 Proposition 1.2]) that \( \phi \) induces an isomorphism \( \phi_* : K_* (C^* (E(H, S))) \rightarrow K_* (J(H, S)) \). Thus if we let \( \chi_*^{E(H, S)} \) denote the group isomorphisms given by Proposition 3.3 for \( C^* (E(H, S)) \), then

\[
\chi_0' := \phi_* \circ \chi_0^{E(H, S)} : \text{coker } \begin{bmatrix} A'_{1} - I \\ 0 \end{bmatrix} \rightarrow K_0(J(H, S))
\]

and

\[
\chi_1' := \phi_* \circ \chi_1^{E(H, S)} : \ker \begin{bmatrix} A'_{1} - I \\ 0 \end{bmatrix} \rightarrow K_1(J(H, S))
\]

are group isomorphisms. Similarly, if we let \( E \setminus H \) be the subgraph of \( E \) with vertices \( E^0 \setminus H \) and edges \( r^{-1}(E^0 \setminus H) \), then there is an isomorphism \( \psi : C^* (E \setminus H, S) \rightarrow C^* (E)/J(H, S) \) which for any \( v \in E^0 \setminus H \) maps \( p_v \) to
Claim 5.1. If \( p_v + J_{(H,S)} \) and for any \( e \in r^{-1}(E^0 \setminus H) \) maps \( s_e \) to \( s_e + J_{(H,S)} \) (cf. Example 3.10). Notice that \( (E \setminus H)^{0}_{\reg} = E^0_{\reg} \setminus H \) and that \( (E \setminus H)^{0}_{\sing} = E^0_{\sing} \setminus H \). Thus if we let \( \chi^E \) denote the group isomorphisms given by Proposition 3.8 for \( C^*(E \setminus H, S) \), then

\[
\chi''_0 := \psi_0 \circ \chi^E_0 : \ker \left[ \begin{array}{ccc}
B^t - I & \Gamma^t & Z^t - I \\
\beta^t & \eta^t & 0
\end{array} \right] \to K_0 \left( C^* (E) / J_{(H,S)} \right)
\]

and

\[
\chi''_1 := \psi_0 \circ \chi^E_1 : \ker \left[ \begin{array}{ccc}
B^t - I & \Gamma^t & Z^t - I \\
\beta^t & \eta^t & 0
\end{array} \right] \to K_1 \left( C^* (E) / J_{(H,S)} \right)
\]

are group isomorphisms. Finally we let \( \chi^E \) denote the group isomorphisms given directly by Proposition 3.8 for \( C^* (E) \).

The theorem then follows from the ensuing six claims.

Claim 5.2. \( \tau_* \circ \chi''_0 = \chi''_0 \circ \bar{I} \).

Proof. If \( v \in H \), then we have that

\[
\chi_0 \left( e_v + \im \left[ \begin{array}{c}
A^t - I \\
\alpha^t
\end{array} \right] \right) = \chi_0 \left( e_v + \im \left[ \begin{array}{cc}
A^t - I & X^t \\
\alpha^t & \xi^t
\end{array} \right] \right) = [p_v]_0 = [[\tau(\phi(p_v))]_0 = \tau_* \circ \chi''_0 \left( e_v + \im \left[ \begin{array}{c}
A^t - I \\
\alpha^t
\end{array} \right] \right),
\]

and if \( v_0 \in S \), the left hand side equals

\[
\chi_0 \left( e_{v_0} - \sum_{\begin{array}{c}s(e) = v_0 \cr r(e) \notin H \end{array}} e_r + \im \left[ \begin{array}{cc}
A^t - I & X^t \\
\alpha^t & \xi^t
\end{array} \right] \right) = [p_{v_0}]_0 - \sum_{\begin{array}{c}s(e) = v_0 \cr r(e) \notin H \end{array}} [s_e s^*_e]_0 = [[\tau(\phi(p_v))]_0 = \tau_* \circ \chi''_0 \left( e_v + \im \left[ \begin{array}{c}
A^t - I \\
\alpha^t
\end{array} \right] \right).
\]

Claim 5.2. \( \pi_* \circ \chi_0 = \chi''_0 \circ P_{345} \).

Proof. As above, we check the claim of each class given by \( e_v \). If \( v \in H \), then both sides vanish. If \( v \notin H \), both sides equal \( [p_v]_0 \).

Claim 5.3. \( \pi_* \circ \chi_0 = \chi''_0 \circ I_1 \circ P_2 \).

\[\square\]
Proof. Fix
\[ x = \begin{bmatrix} y \\ z \end{bmatrix} \in \ker \begin{bmatrix} A^t - I & X^t \\ \alpha^t & \xi^t \end{bmatrix}. \]

Then \( \left[ \begin{smallmatrix} z \\ 0 \end{smallmatrix} \right] \in \ker \begin{bmatrix} B^t - I & \Gamma^t \\ \beta^t & \eta^t \end{bmatrix} \) and we furthermore have that \( L^+_x [0] \subseteq L^+_x \) and \( L^-_x [0] \subseteq L^-_x \). Thus if we let \( h \) be the number of elements in \( L^+_x \) (and in \( L^-_x \)), and we let \( h' \) denote the number of elements in \( L^+_x [0] \) (and in \( L^-_x [0] \)), then we can choose the bijections
\[ \cdot : L^+_x \to \{1, \ldots, h\} \quad \langle \cdot \rangle : L^-_x \to \{1, \ldots, h\} \]
and
\[ \cdot' : L^+_x [0] \to \{1, \ldots, h'\} \quad \langle \cdot' \rangle : L^-_x [0] \to \{1, \ldots, h'\} \]
such that \( \cdot \) is an extension of \( \cdot' \), and \( \langle \cdot \rangle \) is an extension of \( \langle \cdot' \rangle \). We then have that
\[
\pi(V_x) = \pi \left( \sum_{1 \leq i \leq x_w, s(e) = w} s_e E_{[w, i], [e, i]} + \sum_{1 \leq i \leq -x_w, s(e) = w} s_e^* E_{[e, i], [w, i]} \right)
\]
\[
= \sum_{1 \leq i \leq z_w, s(e) = w} \pi(s_e) E_{[w, i], [e, i]} + \sum_{1 \leq i \leq z_w, s(e) = w} \pi(s_e^*) E_{[e, i], [w, i]}
\]
\[
= \psi \left( \sum_{1 \leq i \leq z_w, s(e) = w} s_e E_{[w, i], [e, i]} + \sum_{1 \leq i \leq -z_w, s(e) = w} s_e^* E_{[e, i], [w, i]} \right)
\]
\[
= \psi \left( V_x [0] \right).
\]
since \( s_e \in J_{(H,S)} = \ker \pi \) when \( s(e) \) (and thus \( r(e) \)) lies in \( H \), and \( z_w = x_w \) when \( w \notin H \). A similar computation for \( P_x \) shows that \( \pi(P_x) = \psi \left( P_x [0] \right) \).

Thus \( \pi(U_x) = \psi \left( U_x [0] \right) \) and
\[
\pi_* \circ \chi_1(x) = \left[ \pi(U_x) \right]_1 = \left[ \psi(U_x [0]) \right]_1 = \chi''([\delta]) = \chi'' \circ I_1 \circ P_2(x).
\]

Claim 5.4. \( \iota_* \circ \chi_1 = \chi_1 \circ I_1 \).
Proof. Fix \( x \in \ker \begin{bmatrix} A^t - I \\ \alpha^t I \\ 0 \end{bmatrix} \). This follows like in Claim 5.3 by choosing the bijections
\[
[\cdot] : L^+_X \to \{1, \ldots, h\} \quad [\cdot] : L^+_\bar{X} \to \{1, \ldots, h\}
\]
and
\[
\langle \cdot \rangle : L^-_X \to \{1, \ldots, h\} \quad \langle \cdot \rangle : L^-_\bar{X} \to \{1, \ldots, h\}
\]
to be pairwise equal.

\[\square\]

Claim 5.5. \( \partial_0 = 0 \).

Proof. It follows from Claim 5.3 that \( \iota_* : K_1(J_{(H,S)}) \to K_1(C^*(E)) \) is injective. Thus \( \text{im}(\partial_0) = 0 \) from which it follows that \( \partial_0 = 0 \).

Claim 5.6. \( \partial_1 \circ \chi''_1 = \chi'_0 \circ \begin{bmatrix} X^t 0 \\ \xi^t 0 \\ 0 I \end{bmatrix} \).

Proof. Fix \( x = \begin{bmatrix} y \end{bmatrix} \in \ker \begin{bmatrix} B^t - I \\ \beta^t \eta^t \\ Z^t - I \end{bmatrix} \). We lift \( \psi(V_x) \) and \( \psi(P_x) \) to
\[
\hat{V}_x = \sum_{1 \leq i \leq x_w, s(e) \in w, r(e) \notin H} s_e E_{[w,i],(e,i)} + \sum_{1 \leq i \leq -x_w, s(e) = w, r(e) \notin H} s_e^* E_{[e,i],(e,i)}
\]
and
\[
\hat{P}_x = \sum_{1 \leq i \leq x_w} p_{w} E_{[w,i],[w,i]} + \sum_{1 \leq i \leq -x_w, s(e) = w, r(e) = v, v \notin H} p_v E_{[e,i],[e,i]},
\]
respectively, in \( M_h(C^*(E)) \). Since \( \{ s_e, p_v : e \in r^{-1}(E^0 \setminus H), v \in E^0 \setminus H \} \) is a Toeplitz-Cuntz-Krieger \((E \setminus H)\)-family, it follows from Lemma 3.4 that \( \hat{V}_x \) is a partial isometry and that \( \hat{P}_x \hat{V}_x = \hat{V}_x \hat{P}_x = \hat{V}_x \). It follows that also \( \hat{U}_x := \hat{V}_x + (1 - \hat{P}_x) \) is a partial isometry. Hence, to compute the value of the index map on \([U_x]_1\), we just need to compute the defect of \( \hat{U}_x \) in \( K_0(J_{(H,S)}) \), cf. [13] Proposition 9.2.2. We have, using Lemma 3.4 that
\[
1 - \hat{U}_x^* \hat{U}_x^* = \hat{P}_x - \hat{V}_x \hat{V}_x^*
\]
\[
= \sum_{1 \leq i \leq x_w} \left( p_{w} - \sum_{s(e) = w, r(e) = v, v \notin H} s_e s_e^* \right) E_{[w,i],[w,i]}
\]
\[
= \sum_{1 \leq i \leq y_w} \left( \sum_{s(e) = w, r(e) = v} s_e s_e^* - \sum_{s(e) = w, r(e) = v, v \notin H} s_e s_e^* \right) E_{[w,i],[w,i]}
\]
By comparison, completing the proof. □
INDEX MAPS IN THE $K$-THEORY OF GRAPH ALGEBRAS

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