About the finding of independent vertices of a graph

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Abstract

We examine the Maximum Independent Set Problem in an undirected graph.

The main result is that this problem can be considered as the solving the same problem in a subclass of the weighted normal twin-orthogonal graphs.

The problem is formulated which is dual to the problem above. It is shown that, for trivial twin-orthogonal graphs, any of its maximal independent set is also maximum one.

1 Statement of the problem

Consider the class \( L \) of undirected graphs without loops and multiple edges with weighted vertices.

Assume that there is a graph \( G = (X, \Gamma, M) \in L \), where \( X = \{x_1, \ldots, x_n\} \) be the set of the graph vertices, \( \Gamma \) is the mapping \( X \) into \( X \), and \( M = \{\mu(x_1), \ldots, \mu(x_n)\} \) is the set of the non-negative integers – weights of the graph vertices. If \( X_1 = \{x_{i_1}, \ldots, x_{i_m}\} \subset X \) then \( \Gamma X_1 = \Gamma x_{i_1} \cup \cdots \cup \Gamma x_{i_m} \).

A graph \( G = (X, \Gamma, M) \in L \) is called isometric if \( \mu(x_i) = \mu(x_j) \) \( (i \neq j) \) for all \( x_i, x_j \in X \).

For any \( A \subset X \) we shall designate

\[
\mu(A) = \sum_{\forall x_i \in A} \mu(x_i).
\]

As a problem \( Z \), given on a graph \( G = (X, \Gamma, M) \in L \), we shall call the problem of finding of vertex set \( U \subset X \) such that satisfies conditions

\[
U \cap \Gamma U = \emptyset, \quad (1)
\]

\[
U \cup \Gamma U = X \quad (2)
\]

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and supplies the maximum of a function
\[ \mu(U). \] (3)

Any vertex set \( U \subset X \), satisfying the condition (1), is called independent. An independent set \( U \subset X \), satisfying (2), is called the maximal independent set (MIS) of the graph \( G \).

A MIS \( \hat{U} \subset X \), supplying the maximum of the function (3), is called the maximum independent set (MMIS) of the graph \( G \) (the optimum solution of the problem \( Z \)).

The problem \( Z \) has the different applications \([1, 4, 5, 3]\). It has the special significance in Computation Complexity Theory, as it is NP-complete \([2]\). From the point of view of applications, the significance of any NP-complete problem is that it can be considered as a mathematical model of all discrete problems.

The existing methods for solving of the problem \( Z \) (as a rule, in an isometric graph) consist in basic in finding all MIS of the graph \( G \in L \) and selection of them the maximum independent set \([1, 3]\).

The inefficiency of such approach to solving the problem \( Z \) is proved by that the maximum number of the MISs \( \sigma_G(n) \), a graph \( G \in L \) can has, is equal to \( \sigma_G(n) = \gamma(s)3^{r-1} \) \([1, 3]\), where \( n = 3r + s \), \( \gamma(0) = 3 \), \( \gamma(1) = 4 \), \( \gamma(2) = 6 \). Hence, complexity of any algorithm, based on searching of all MIS of a graph \( G \), can not have an evaluation better than \( O(3^{n/3}) \).

With the problem \( Z \), given on a graph \( G \), it is usually connected a problem of finding the maximum complete subgraph (the maximal clique) in the additional graph \( \overline{G} = (X, \Gamma, M) \in L \) as the subset of vertices \( \hat{U} \in X \), inducing the maximum clique of a graph \( \overline{G} \), is MMIS of a graph \( G \) \([1]\).

Notice that the Maximum Clique Problem is a maximize problem, and from the point of view of the approach, accepted in Operations Research, is not dual to the problem \( Z \).

Unfortunately, the difficulties, connected with finding MMIS of a graph \( G \in L \), can not are overcome by development of a polynomial algorithm, enabling to find approximate solution of the problem \( Z \) with a guaranteed deviation from the optimum solution \([3]\). Therefore, for development of the solution methods of the problem \( Z \), it is necessary either to try to create an algorithm, discovering its exact solution (in this case it will be proven that \( P=NP \)), or to find the exact solution of the problem for separate subclasses of graphs of \( L \) (the majority authors go to the last way).

The main result of the given work is that the problem \( Z \), given on an arbitrary graph \( G \in L \), can be considered as the solving the same problem in subclass of the normal conjugate-orthogonal graphs. A problem is formu-
lated that is dual to the problem $Z$. It is shown that, for trivial conjugate-

orthogonal graphs, any of its MIS is also a MMIS.

## 2 A normal graph

Divide a set of all vertices of a graph $G = (X, \Gamma, \epsilon) \in L$ into classes $j$ ($j = 1, \ldots, s$) such that if $i_1, i_2 \in K_j$, then $\Gamma_i = \Gamma_{i_2}$. The set of all such classes of the graph is designated by $H_G$.

**Theorem 2.1** If $i_1 \in U$ and $i_1 \in j$ then $j \in U$, where $U$ be a MIS of a graph $G = (X, \Gamma, \epsilon)$.

Assume the conditions of Theorem 2.1 are satisfied and we allow that there exists a vertex $i_2 \in j$ such that $i_2 \notin U$ ($i_2 \neq i_1$).

As $\Gamma x_{i_2} = \Gamma x_{i_1}$ then $i_2 \notin \Gamma U$ owing to (1). But, it takes into considering (2), we have $i_2 \in U$. The contradiction have obtained. Q.E.D.

**Theorem 2.2** For any vertex $x \in (i = 1, n)$ of a graph $G = (X, \Gamma, \epsilon)$

$$\Gamma x_i = \bigcup_r K'_{jr} \quad (K'_{jr} \in H_G)$$

It is clear that the vertex set $\Gamma x_i$ can be divided into the classes $K'_{j_1}, \ldots, K'_{jt}$ as it is mentioned above. Assume that these classes of vertices are distinct from similar vertex classes of the graph $G$, that is, $K'_{j_1} \subset K'_{j_r}$ and $K'_{j_1} \neq K'_{j_r}$ ($= 1, t$).

It follows from here that there are the vertices $k_1 \in K'_{j_1} \subset K_j$ and $x_{k_2} \in j \setminus K'_{j_1}$ such that $k_1 \in \Gamma x_i$ and $x_{k_2} \notin \Gamma x_i$.

As vertices $k_1, k_2 \in j$, then $\Gamma x_{k_1} = \Gamma x_{k_2}$ by the definition. If $k_1 \in \Gamma x_i$ then $i \in K_{k_1}$, it signifies, $i \in K_{k_2}$. Then we have $x_{k_2} \notin \Gamma x_i$. The contradiction have obtained. Q.E.D.

Thus, it is established that $\Gamma x_i$, for any vertex $i \in (i = 1, n)$ of a graph $G = (X, \Gamma, \epsilon)$, be an union of some classes $j \in H_G$.

**Corollary 2.1** For any class $j \in H_G$ of a graph $G = (X, \Gamma, \epsilon)$

$$\Gamma K_j = \bigcup_r K'_{jr} \quad (K'_{jr} \in H_G)$$

A graph $G_1 = (X_1, \Gamma_1, \epsilon_1) \in L$ is called normal if for any two vertices $y_{j_1}, j_2 \in X_1$ ($j_1 \neq j_2$) the relation $\Gamma y_{j_1} \neq \Gamma y_{j_2}$ takes place.

Obviously, that for any graph $G = (X, \Gamma, M) \in L$ can be found a mapping $\phi$: $G_1 = \phi(G)$, where $G_1 = (X_1, \Gamma_1, M_1) \in L$ be the normal graph. Thus,
\[ G_1 = \phi(G) \text{ if } y_j = \phi(K_j) \ (K_j \in H_G) \text{ and } \Gamma_1 y_j = \phi(\Gamma K_j), \ \mu(y_j) = \mu(K_j) \text{ for all } y_j \in X_1 \ (j = 1, s). \]

Fig. 1 (a) shows the graph \( G \in L \) with the unit weights of its vertices, and Fig. 1 (B) shows the normal graph \( G_1 \) that corresponds it (weights of its vertices are put in brackets).

Designate by \( L_H \) the set of all normal graphs with the weighted vertices that correspond graphs of the class \( L \).

Further, speaking about a graph \( G = (X, \Gamma_\cdot) \), we shall mean that \( G \in L_H \). Besides, we assume that \( \text{Card}X = n. \)

### 3 A twin-orthogonal graph

Let \( G = (X, \Gamma_\cdot) \in L_H. \)

The adjacent vertices \( x, x_2 \in X \) of the graph \( G \) is called **orthogonal** if for all vertices \( x_i \in \Gamma x_1 \setminus \{x_2\} \) and \( x_{i_2} \in \Gamma_2 \setminus \{x_1\} \), when they exist, the relation are fulfilled:

\[ \Gamma x_1 \subseteq \Gamma x_{i_2}, \quad \Gamma x_2 \subseteq \Gamma x_{i_1}. \]  

**Theorem 3.1** If at least one of adjacent vertices \( x, x_2 \in X \) of a graph \( G \) is dangling then the vertices \( x_1 \) and \( x_2 \) are orthogonal.

Really, suppose, for example, a vertex \( x_1 \in X \) of a graph \( G \), adjacent with a vertex \( x_2 \in X \), is dangling. Hence, \( \Gamma_1 = \{2\}. \)

Then we shall have \( \Gamma_1 \setminus \{x_2\} \neq \emptyset \) and \( \Gamma_1 \subseteq \Gamma x_{i_2} \) for all \( x_{i_2} \in \Gamma_2 \setminus \{1\} \) when \( \Gamma x_2 \setminus \{x_1\} \neq \emptyset \) (as \( x_2 \in \Gamma x_{i_2} \)).

\[ \text{Q.E.D.} \]
**Theorem 3.2** Let \( U \subset \) be an arbitrary MIS of a graph \( G = (X, \Gamma, M) \). If \( x_1, x_2 \in \) is the orthogonal vertices of \( G \) either \( 1 \in U \) or \( 2 \in U \).

Assume that the conditions of Theorem 3.2 are satisfied, and suppose that \( x_1, x_2 \in \Gamma U \). Then there exists at least vertex \( 3 \in \Gamma x_1 \) such that \( x_3 \in U \), and at least vertex \( 4 \in \Gamma x_2 \) such that \( x_4 \in U \).

By the condition (4), for the orthogonal vertices \( x_1, x_2 \in X \), we shall have \( x_3 \in \Gamma_4 \) and \( x_4 \in \Gamma x_3 \), that is, the vertices \( x_3, x_4 \in U \) are adjacent. We have received the contradiction. Q.E.D.

A graph \( \tilde{G} = (\tilde{X}, \tilde{\Gamma}, \tilde{M}) \in L_H \) is called **twin-orthogonal** if graph vertices can divide into pairs of the orthogonal vertices. It is clear that \( rd() = n = 2k \), where \( k \) is a non-negative integer.

![Fig. 2: A twin-orthogonal graph](image)

Fig. 2 is represented of a twin-orthogonal graph with the unit weights of the vertices.

We shall be to say that the twin-orthogonal graph \( \tilde{G} = (\tilde{X}, \tilde{\Gamma}, \tilde{\gamma}) \) corresponds a graph \( G = (X, \Gamma, \gamma) \), if:

a) \( X \subset \tilde{X} \);

b) \( \mu(x_i) = 0 \) for any vertex \( x_i \in \tilde{X} \setminus X \);

c) any MIS \( U \subset X \) of \( G \) can be obtained from some MIS \( \tilde{U} \subset \tilde{X} \) of \( \tilde{G} \) by removal of all vertices \( x_i \in \tilde{X} \) such that \( x_i \notin X \).

It is easy to see that one of twin-orthogonal graphs \( \tilde{G} = (\tilde{X}, \tilde{\Gamma}, \tilde{\gamma}) \), corresponding a graph \( G = (X, \Gamma, M) \), can be constructed as follows.
Let \( X_1 \subset G \) be the set of all vertices of the graph \( G \), not being orthogonal for one vertex of this graph. We join a set of vertices \( 2 = \{x_{n+1}, \ldots, x_{n+p}\} \) (\( p = rd(X_1) \)) to the graph \( G \), and each of vertices \( k \in 2 \) we connect by an edge with one and only one of vertices \( j \in X_1 \). We assume that \( \mu(x_k) = 0 \) for all \( k \in X_2 \).

It is clear that, as a result, a twin-orthogonal graph \( \tilde{G} = (\tilde{X}, \tilde{\Gamma}, \tilde{\mu}) \) will be obtained that is induced on a vertex set \( \tilde{X} = \cup X_1 \cup 2 \), where \( \tilde{\Gamma}_i = \Gamma_i \cup \{j\} \) for any vertex \( i \in X_1 \) and \( \tilde{\Gamma}_x = \{x\} \) for all \( x \in X_2 \).

It is easy to be convinced that the constructed twin-orthogonal graph \( \tilde{G} \) corresponds the initial graph \( G \).

More simple way for a construction of the twin-orthogonal graph \( \tilde{G} = (\tilde{X}, \tilde{\Gamma}, \tilde{\mu}) \), corresponding a graph \( G = (X, \Gamma, M) \), is based on Theorem 3.1.

We shall join a vertex set \( 1 = \{x_{n+1}, \ldots, x_{n+p}\} \) (\( p = rd(X_1) \)) to a graph \( G \) such that each vertex \( j \in 1 \) we shall connect by an edge with one and only one of vertices \( i \in X \). We assume \( \mu(j) = 0 \) for all \( j \in X_1 \). As a result, obviously, it be also obtained a twin-orthogonal graph \( \tilde{G} = (\tilde{X}, \tilde{\Gamma}, \tilde{\mu}) \) induced on a set of vertices \( \tilde{X} = \cup \cup X_1 \), where \( \tilde{\Gamma}_i = \Gamma_i \cup \{j\} \) for any vertex \( i \in X \) and \( \tilde{\Gamma}_x = \{x\} \) for all \( x \in X_2 \).

**Theorem 3.3** If \( \tilde{U} \subset \tilde{X} \) be an optimum solution of the problem \( \tilde{U} \) on the twin-orthogonal graph \( \tilde{\Gamma}_i = \Gamma_i \cup \{j\} \) corresponding a graph \( \tilde{G} = (X, \Gamma, M) \) then an optimum solution \( U \in X \) of the problem \( \tilde{U} \), given on the graph \( \tilde{G} \), can be obtained by removal from \( \tilde{U} \) of all vertices \( i \in X \) such that \( i \in \tilde{X} \), and, besides, \( \mu(U) = \mu(\tilde{U}) \).

It follows from the definition of a twin-orthogonal graph \( \tilde{G} \), corresponding a graph \( G \).

\[ \text{Q.E.D.} \]

## 4 Some properties of a twin-orthogonal graph

Let \( L_0 \) be the set of the normal twin-orthogonal graphs.

**Theorem 4.1** If \( U_1, U_2 \in \tilde{X} \) be the different MISs of a twin-orthogonal graph \( G = (X, \Gamma, M) \in L_0 \) then \( rd(U_1) = rd(U_2) = k \), where \( rd() = n = 2k \), \( k \) be a non-negative integer.

It follows from Theorem 3.2. Q.E.D.

A twin-orthogonal graph \( G = (X, \Gamma, M) \) is called trivial if for any orthogonal vertices \( i, j \in X \) the relation is fulfilled: \( \mu(i) = \mu(x_j) \).

**Theorem 4.2** If \( G = (X, \Gamma, M) \) be a trivial twin-orthogonal graph then any MIS is also MMIS.
It follows from Theorems 3.2 and 4.1. Q.E.D.

**Theorem 4.3** If \( x_1, x_2 \in \) be the orthogonal vertices of a graph \( G = (X, \Gamma, ) \) then any pair of vertices from a set \( \Gamma x_1 \setminus \{x_2\} \neq \emptyset \) \( \Gamma x_2 \setminus \{x_1\} \neq \emptyset \) is not orthogonal.

Assume that the conditions of Theorem 4.3 are satisfied, and we suppose that the vertices \( i_1, i_2 \in \Gamma x_1 \setminus \{2\} \) are orthogonal. Then we have \( 1 \in \Gamma x_{i_1} \), and \( 1 \in \Gamma i_2 \), that is, the relation (4) are not fulfilled for vertices \( x_{i_1}, x_{i_2} \in \). We have obtained the contradiction. Q.E.D.

**Corollary 4.1** If the vertices \( 1, x_2 \in \) of a graph \( G = (X, \Gamma, ) \) are orthogonal then they do not form a three-vertex clique with any vertex \( i \in (i \neq 1, i \neq 2). \)

**Corollary 4.2** If the vertices \( 1, 2 \in \) of a graph \( G = (X, \Gamma, ) \) are orthogonal then \( (\Gamma x_1 \setminus \{x_2\}) \cap (\Gamma x_2 \setminus \{x_1\}) = \emptyset. \)

## 5 A dual problem

Further, for convenience, any two orthogonal vertices of a graph \( G = (X, \Gamma, ) \in L_0 \) we shall designate by \( _i \) and \( _*i \).

A graph \( G^* = (X, \Gamma^*, M) \in L_0 \), obtained from a graph \( G = (X, \Gamma, ) \in L_0 \) by renaming of pairs of orthogonal vertices, is called conjugate for the graph \( G \).

Thus, any orthogonal vertices \( _i, _j \in \) are adjacent in graphs \( G \) and \( G^* \). The vertices \( _i, _j \in \) if they are not orthogonal in the graph \( G \), are adjacent in the graph \( G^* \) if and only if corresponding vertices \( x_i ^*, x_j^* \in X \) are adjacent in the graph \( G \).

Obviously, that \( (G^*)^* = G \).

A problem of finding of a vertex set \( U \subset \) of a graph \( G^* = (X, \Gamma^*, ) \), satisfying conditions (1), (2) and supplying the minimum of the function (3), we shall call dual to the problem \( Z \).

A MIS \( U \subset X \), supplying the minimum of the function (3), is called the minimum independent set of vertices (MNMIS) of a graph \( G^* \).

The following statements are proved easily.

**Theorem 5.1** If \( U \subset \) be a MIS of a graph \( G = (X, \Gamma, M) \) then \( \Gamma U = \setminus U \) be a MIS of the conjugate graph \( G^* = (X, \Gamma^*, ) \).
Theorem 5.2 If \( U_1, U_2 \in \text{MISs of a graph } G = (X, \Gamma) \) then \( \mu(U_1) \geq \mu(U_2) \) if and only if \( \mu(\Gamma U_1) \leq \mu(\Gamma U_2) \).

The following theorem is a corollary of Theorems 5.1 and 5.2.

Theorem 5.3 MIS \( \hat{U} \subset \) is MNMIS of a graph \( G = (X, \Gamma) \) if and only if \( \Gamma \hat{U} = \setminus \hat{U} \) is MNMIS of a conjugate graph \( G^* = (X, \Gamma^*) \).

Theorem 5.4 Let \( \hat{U}, \check{U} \subset \) be MMIS and MNMIS of a graph \( G = (X, \Gamma, M) \) respectively. Then a relation takes place

\[
0 \leq \mu(\hat{U}) - \mu(\check{U}) \leq \sum_{\forall x_i, x_i^* \in X} |\mu(x_i) - \mu(x_i^*)|.
\]

It is also easy to be convinced in a validity of this statement.

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