Majorana CP Violation in Approximately $\mu$-$\tau$ Symmetric Models with $\det(M_\nu)=0$

Teppei Baba$^*$ and Masaki Yasue$^{**}$

Department of Physics, Tokai University,
1117 Kitakaname, Hiratsuka, Kanagawa 259-1292, Japan

We discuss effects of Majorana CP violation in a model-independent way for a given phase structure of flavor neutrino masses. To be more predictive, we confine ourselves to models with $\det(M_\nu)=0$, where $M_\nu$ is a flavor neutrino mass matrix, and to be consistent with observed results of the neutrino oscillation, the models are subject to an approximate $\mu$-$\tau$ symmetry. There are two categories of approximately $\mu$-$\tau$ symmetric models classified as (C1) yielding $\sin^2 \theta_{23} \approx 1$ and $\sin^2 \theta_{13} \ll 1$ and (C2) yielding $\sin^2 \theta_{23} \approx 1$ and $\Delta m^2_{\odot}/|\Delta m^2_{\text{atm}}| \ll 1$, where $\theta_{23(13)}$ stands for the mixing of massive neutrinos $\nu_2$ and $\nu_3$ ($\nu_1$ and $\nu_3$) and $\Delta m^2_{\odot}$ ($\Delta m^2_{\text{atm}}$) stands for the mass squared difference for atmospheric (solar) neutrinos. The Majorana phase can be large for the normal mass hierarchy and for the inverted mass hierarchy with $m_1 \approx -m_2$ only realized in (C1) while they are generically small for the inverted mass hierarchy with $m_1 \approx m_2$ in both (C1) and (C2). These results do not depend on a specific choice of phases in $M_\nu$ but hold true in any models with $\det(M_\nu)=0$ because of the rephasing invariance.

§1. Introduction

Neutrinos are oscillating and mixed with each other among three flavor neutrinos. Such oscillations have been confirmed to occur for the atmospheric neutrinos,$^1$ the solar neutrinos,$^{2,3}$ the reactor neutrinos$^4$ and the accelerator neutrinos.$^5$ Three massive neutrinos have masses $m_{1,2,3}$ measured as mass squared differences defined by $\Delta m^2_{\odot} = m_2^2 - m_1^2$ and $\Delta m^2_{\text{atm}} = m_3^2 - m_1^2$. Three flavor neutrinos $\nu_{e,\mu,\tau}$ are mixed into three massive neutrinos $\nu_{1,2,3}$ during their flight and the mixing can be described by the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix$^6$ parameterized by three mixing angles $\theta_{12,23,13}$, one Dirac CP violating phase $\delta_{\text{CP}}$ and three Majorana phase $\phi_{1,2,3}$,$^7$ where Majorana CP violating phases are given by two combinations of $\phi_{1,2,3}$.

It is CP property of neutrinos that has currently received much attention since the similar CP property of quarks has been observed and successfully described by the Kobayashi-Maskawa mixing matrix.$^8$ If neutrinos exhibit CP violation, there is a new seed to produce the baryon number in the Universe by the Fukugida-Yanagida mechanism of the leptogenesis,$^9$ which favors the seesaw mechanism$^{10}$ of creating tiny neutrino masses. However, there is no direct linkage between CP violation of three flavor neutrinos and that of the leptogenesis since the CP violating phases are associated with heavy neutrinos but not with three flavor neutrinos. If the number of the heavy neutrinos is two, there is one-to-one correspondence between the CP violating phases of flavor neutrinos and that of the seesaw mechanism. The model

$^*$ E-mail: 7atr014@keyaki.cc.u-tokai.ac.jp
$^{**}$ E-mail: yasue@keyaki.cc.u-tokai.ac.jp
with two heavy neutrinos is called minimal seesaw model.\textsuperscript{11}) Even if the minimal seesaw model is adopted, predictions of the leptonic CP violation depend on the choice of various parameters including Dirac neutrino mass terms. It seems of great significance to make predictions independent of the specific parameter choice. The general feature of the minimal seesaw model is that it satisfies $\det(M_\nu) = 0$, where $M_\nu$ represents a flavor neutrino mass matrix. Therefore, we choose this condition $\det(M_\nu) = 0$ as our standing point to investigate effects of leptonic CP violations as general as possible.

To discuss the leptonic CP violation starting from a given phase structure of $M_\nu$, we have to clarify how phases of flavor neutrino masses affect the leptonic CP phases. To do so, we have to mathematically consider 6 phase parameters in the PMNS mixing matrix $U_{PMNS}$ to cover a general phase structure of $M_\nu$. It should be noted that conventional studies utilizing the standard version of the PMNS matrix $U_{PMNS}^{PDG}$ given by the Particle Data Group (PDG)\textsuperscript{12}) and $m_{1,2,3}$ do not provide a clue to see direct effects from phases of the flavor neutrino masses.\textsuperscript{13}) In our study, $U_{PMNS}$ is parameterized by $U_{PMNS} = UK$ with

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\gamma} & 0 \\ 0 & 0 & e^{-i\gamma} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}$$

$$K = \text{diag.}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}),$$

where $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$ ($i,j=1,2,3$). There is the similar phase to $\delta$ and $\rho$, say $\chi$, which can be associated with the $\nu_2$-$\nu_3$ mixing. However, we have confirmed that the phase of the $\nu_2$-$\nu_3$ mixing should be $\gamma$ in place of $\chi$ to consistently describe the neutrino mixings.\textsuperscript{14}) Namely, Eq.(1.1) with $\chi$ included is converted into $UK$ with $\delta = \delta + \chi/2$, $\rho = \rho + \chi/2$, $\gamma = \gamma + \chi/2$, $\varphi_2 = \varphi_2 + \chi/2$ and $\varphi_3 = \varphi_3 + \chi/2$ as obvious replacements. Physically, among $\delta$, $\rho$, and $\gamma$ two phases are redundant. By defining $\phi_1 = \varphi_1 - \rho$ (as well as $\phi_{2,3} = \varphi_{2,3}$) and

$$\delta_{CP} = \delta + \rho,$$

we reach $U_{PMNS}^{PDG} = U^{PDG}K^{PDG}$ consisting of

$$U^{PDG} = \begin{pmatrix} c_{12}c_{13} & -c_{23}s_{12} - s_{23}c_{12}s_{13}e^{i\delta_{CP}} & s_{12}c_{13} \\ -c_{23}s_{12} - s_{23}c_{12}s_{13}e^{i\delta_{CP}} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i\delta_{CP}} & s_{13}e^{-i\delta_{CP}} \\ s_{23}s_{12} - c_{23}c_{12}s_{13}e^{i\delta_{CP}} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i\delta_{CP}} & c_{23}s_{13} \end{pmatrix},$$

$$K^{PDG} = \text{diag.}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}).$$

Furthermore, one Majorana phase is also redundant and the CP violating Majorana phases are given by two combinations of the Majorana phases such as $\phi_i - \phi_3$ ($i = 1, 2$). For the reader’s convenience, we show three typical forms of $U_{PMNS}$ in Appendix A.
The neutrino mixing angles and mass squared differences have been measured by recent neutrino oscillation experiments. The current data of the mixing angles and mass squared differences are shown as:

\[
\sin^2 \theta_{12} = 0.30^{+0.02}_{-0.02}, \quad \sin^2 \theta_{23} = 0.50^{+0.07}_{-0.06}, \quad \sin^2 \theta_{13} < 0.040,
\]
\[
\Delta m^2_\odot = (7.65^{+0.23}_{-0.26}) \times 10^{-5} \text{ eV}^2, \quad |\Delta m^2_{\text{atm}}| = (2.40^{+0.12}_{-0.11}) \times 10^{-3} \text{ eV}^2.
\]

The gross property of the experimental data indicating the almost maximal atmospheric neutrino mixing and the small 1-3 neutrino mixing can be understood as a result of the \(\mu-\tau\) symmetry\(^{16}\) imposed on neutrino interactions, which gives \(\sin^2 \theta_{23} = 0.5\) and \(\sin^2 \theta_{13} = 0\). However, there is no Dirac CP violation. If Dirac CP violation is observed in future neutrino experiments,\(^{17}\) we have to include tiny violation of the \(\mu-\tau\) symmetry.\(^{*}\) If the \(\mu-\tau\) symmetry breaking is included, there are two categories of textures respectively referred to as (C1) and (C2).\(^{18}\) In the category (C1), we have \(\sin 2\theta_{23} \approx \sigma (\sigma = \pm 1)\) and \(\sin^2 \theta_{13} \ll 1\) while in the category (C2), we have \(\sin 2\theta_{23} \approx -\sigma\) and \(\Delta m^2_\odot /|\Delta m^2_{\text{atm}}| \ll 1\). In the category (C2), the \(\mu-\tau\) symmetric limit is signaled by \(\sin \theta_{12} \to 0\) instead of \(\sin \theta_{13} \to 0\). The phenomenologically consistent value of \(\sin \theta_{12}\) is realized by the form of \(\tan 2\theta_{12} \propto \varepsilon / \eta\), where \(\varepsilon\) represents the \(\mu-\tau\) symmetry breaking parameter and another small parameter denoted by \(\eta \sim O(\varepsilon)\) is required.

In this article, we discuss CP property of approximately \(\mu-\tau\) symmetric models satisfying \(\det(M_\nu) = 0\), whose theoretical foundation is supplied by the minimal seesaw model with two heavy right handed neutrinos. We estimate sizes of CP violating phases by using the general phase structure of neutrino mass matrix and by focusing on the rephasing invariance, whose existence in our formalism is discussed in Appendix A. Some of results of the category (C1) are shared by our previous work.\(^{19}\) All CP phases are expressed in terms of flavor neutrino masses so that one can understand that how phases of flavor neutrino masses induce CP violating phases in \(U_{PMNS}\).

In the next section, we define the \(\mu-\tau\) symmetry and explain two categories (C1) and (C2). In Sec.3, we present various formulas to extract general property of the observed neutrino mixings and discuss how the condition of \(\det(M_\nu) = 0\) gives a massless neutrino to exclude the case of \(m_2 = 0\). Detailed discussions to see the appearance of one massless neutrino are given by Appendix B. In Sec.4, we include effects of the \(\mu-\tau\) symmetry breaking to see neutrinos in the categories (C1) and (C2), which are used to construct neutrino mass textures. In Sec.5, we argue how mass hierarchies are realized and show seven viable textures, where we estimate CP violating phases in each texture. The last section is devoted to summary and discussions.

\(^{*}\) The \(\mu-\tau\) symmetry breaking should be present because the charged leptons placed into \(SU(2)_L\)-doublets together with the flavor neutrinos violate the \(\mu-\tau\) symmetry.
§2. $\mu$-$\tau$ symmetry

The $\mu$-$\tau$ symmetry is the symmetry due to the invariance of the lagrangian, especially for the flavor neutrino mass term $M_\nu$, associated with the interchange of $\nu_\mu \leftrightarrow \nu_\tau$. We define the interchange as follows:

$$\nu_\mu \leftrightarrow -\sigma \nu_\tau \ (\sigma = \pm 1),$$

(2.1)

where $\sigma$ will take care of the sign of $\sin \theta_{23}$ as parameterized by Eq.(1.1). The phase $\gamma$ turns out to be of the $\mu$-$\tau$ symmetry breaking type. Our mass term $M_\nu$ is labeled by

$$M_\nu = \begin{pmatrix} M_{ee} & M_{e\mu} & M_{e\tau} \\ M_{e\mu} & M_{\mu\mu} & M_{\mu\tau} \\ M_{e\tau} & M_{\mu\tau} & M_{\tau\tau} \end{pmatrix},$$

(2.2)

which is divided into $M_\nu^{(+)}$ and $M_\nu^{(-)}$ ($M_\nu = M_\nu^{(+)} + M_\nu^{(-)}$):

$$M_\nu^{(+)} = \begin{pmatrix} M_{ee} & M_{e\mu}^{(+)} & -\sigma M_{e\mu}^{(+)} \\ M_{e\mu}^{(+)} & M_{\mu\mu}^{(+)} & M_{\mu\tau}^{(+)} \\ -\sigma M_{e\mu}^{(+)} & M_{\mu\tau}^{(+)} & M_{\tau\tau}^{(+)} \end{pmatrix}, \quad M_\nu^{(-)} = \begin{pmatrix} 0 & M_{e\mu}^{(-)} & \sigma M_{e\mu}^{(-)} \\ M_{e\mu}^{(-)} & M_{\mu\mu}^{(-)} & 0 \\ \sigma M_{e\mu}^{(-)} & 0 & -M_{\mu\mu}^{(-)} \end{pmatrix}.$$ \hspace{1cm} (2.3)

Under the interchange Eq.(2.1), $M_\nu^{(+)}$ is kept intact. From this result, the superscripts ($+$) and ($-$) of $M_\nu$ are, respectively, so chosen to stand for the $\mu$-$\tau$ symmetry preserving and breaking terms.

We find that $\sin \theta_{23} = \sigma/\sqrt{2}$ is determined by $(0, \sigma/\sqrt{2}, 1/\sqrt{2})^T$ as one of the eigenvectors associated with $M_\nu^{(+)}$ if it is assigned to $\nu_3$. One may also assign it to $\nu_2$ giving $\sin \theta_{12} = 0$, even to $\nu_1$ giving $\cos \theta_{12} = 0$. Namely $U_{PMNS}$, respectively, takes the form of

$$\begin{pmatrix} * & * & 0 \\ * & * & \sigma/\sqrt{2} \\ 0 & * & * \end{pmatrix} \ (\sin \theta_{13} = 0), \quad \begin{pmatrix} * & 0 & * \\ * & 1/\sqrt{2} & * \\ * & \sigma/\sqrt{2} & * \end{pmatrix} \ (\sin \theta_{12} = 0),$$

$$\begin{pmatrix} 0 & * & * \\ -1/\sqrt{2} & * & * \\ -\sigma/\sqrt{2} & * & * \end{pmatrix} \ (\cos \theta_{12} = 0).$$ \hspace{1cm} (2.4)

There are in principle three possibilities for $\mu$-$\tau$ symmetric textures. However, for $\cos \theta_{12} = 0$, after the $\mu$-$\tau$ symmetry is broken, it can be shown that $\tan 2\theta_{12} = O(1)$ but $\sin^2 \theta_{12} > \cos^2 \theta_{12}$, which contradicts the data Eq.(1.4). As a plausible choice, we obtain\(^{18}\)

$$\sin \theta_{13} = 0, \quad \sin \theta_{23} = \frac{\sigma}{\sqrt{2}}, \quad \delta = \gamma = 0,$$ \hspace{1cm} (2.5)

as a category (C1) or

$$\sin \theta_{12} = 0, \quad \sin \theta_{23} = -\frac{\sigma}{\sqrt{2}}, \quad \delta + \rho = 0, \quad \gamma = 0,$$ \hspace{1cm} (2.6)
as a category (C2). The phase \( \rho \) is determined as \( \rho = \arg(\sum_{i=e}^{\tau} M_{ei}^1 M_{i\mu}) \) to be shown in Eq.(3.8), where \( M_{ij} \) stands for an \( i-j \) element of \( M_\nu \) (\( i,j=e,\mu,\tau \)) defined in Eq.(2.2). Both categories give no Dirac CP violation signaled by \( \sin \theta_{13} = 0 \) or by \( \delta_{CP} (= \delta + \rho) = 0 \). In other words, the Jarlskog invariant\(^{20}\) vanishes. In models without leptonic CP violation, both cases can produce experimentally allowed results,\(^{18}\) where the \( \mu-\tau \) symmetry breaking is a must for the category (C2).

§3. Various Relations of Masses and Mixings

To extract general property inherent to the observed neutrino mixings, we first derive various formulas expressed in terms of flavor neutrino masses to evaluate neutrino masses, mixing angles and phases.

3.1. Useful Formulas

To get the Dirac CP phase \( \delta \) as well as \( \rho \) and \( \gamma \), it is convenient to use \( M \left( \equiv M_\nu^\dagger M_\nu \right) \) parameterized by

\[
M = \begin{pmatrix}
A & B & C \\
B^* & D & E \\
C^* & E^* & F
\end{pmatrix},
\]

(3.1)

where Majorana phases are removed. From

\[
U_{PMNS}^\dagger M U_{PMNS} = \text{diag.}(m_1^2, m_2^2, m_3^2),
\]

(3.2)

we obtain that

\[
\begin{align*}
\tan 2\theta_{12} e^{i\rho} &= \frac{2X}{A_2 - A_1}, \quad \tan 2\theta_{13} e^{-i\delta} = \frac{2Y}{A_3 - A} \\
\text{Re} \left( e^{-2i\gamma} E \right) \cos 2\theta_{23} - \frac{D - E}{2} \sin 2\theta_{23} + i\text{Im} \left( e^{-2i\gamma} E \right) &= -s_{13} e^{-i(\rho + \delta)} (e^{-i\rho} X)^*,
\end{align*}
\]

(3.3)

\[
\begin{align*}
m_1^2 &= \frac{A_1 + A_2}{2} - \frac{e^{-i\rho} X}{\sin 2\theta_{12}}, \quad m_2^2 = \frac{A_1 + A_2}{2} + \frac{e^{-i\rho} X}{\sin 2\theta_{12}}, \quad m_3^2 = \frac{c_{13}^2 A_3 - s_{13}^2 A}{c_{13}^2 - s_{13}^2},
\end{align*}
\]

(3.4)

\[
\begin{align*}
A_1 &= \frac{c_{13}^2 A - s_{13}^2 A_3}{c_{13}^2 - s_{13}^2}, \quad A_2 = \frac{c_{23}^2 D + s_{23}^2 F - 2s_{23} c_{23} \text{Re} \left( e^{-2i\gamma} E \right)}{c_{13}^2 - s_{13}^2}, \\
A_3 &= s_{23} D + c_{23}^2 F + 2s_{23} c_{23} \text{Re} \left( e^{-2i\gamma} E \right), \\
X &= \frac{c_{23} e^{i\gamma} B - s_{23} e^{-i\gamma} C}{c_{13}}, \quad Y = s_{23} e^{i\gamma} B + c_{23} e^{-i\gamma} C,
\end{align*}
\]

(3.5)

Because \( A \) and \( A_{1,2,3} \) are real numbers, the phases \( \rho \) and \( \delta \) can be determined by

\[
\rho = \arg(X), \quad \delta = -\arg(Y).
\]

(3.6)

The remaining phase \( \gamma \) is determined from Eq.(3.4). It is further obtained that the size of \( |X| \) should be suppressed to realize the hierarchy of \( \Delta m^2_{\odot}/|\Delta m^2_{\text{atm}}| \ll 1 \).
where
\[
\Delta m^2_{\odot} = 2 \frac{e^{-i\rho} X}{\sin 2\theta_{12}} (> 0). \tag{3.9}
\]

To calculate the Majorana phases, we instead diagonalize \( M_\nu \):
\[
U^T_{PMNS} M_\nu U_{PMNS} = \text{diag.}(m_1, m_2, m_3), \tag{3.10}
\]
and find that
\[
\tan 2\theta_{12} = \frac{2x}{\lambda_2 - \lambda_1}, \quad \tan 2\theta_{13} e^{i\rho} = \frac{2y}{\lambda_3 e^{i\delta} - ae^{-i\delta}},
\]
\[
e \cos 2\theta_{23} = \frac{e^{-2i\gamma f} - e^{2i\gamma d}}{2} \sin 2\theta_{23} = -s_{13} e^{-i(\delta + \rho)} x,
\]
\[
m_1 e^{-2i\phi_1} = \frac{\lambda_1 + \lambda_2 - x}{2 \sin 2\theta_{12}}, \quad m_2 e^{-2i\phi_2} = \frac{\lambda_1 + \lambda_2}{2} + \frac{x}{\sin 2\theta_{12}},
\]
\[
m_3 e^{-2i\phi_3} = \frac{c_{13}^2 \lambda_3 - s_{13}^2 e^{-2i\delta} a}{c_{13}^2 - s_{13}^2}, \tag{3.12}
\]
\[
\lambda_1 = e^{2\rho} \frac{c_{13}^2 a - s_{13}^2 e^{2i\delta} \lambda_3}{c_{13}^2 - s_{13}^2}, \quad \lambda_2 = c_{23} e^{2i\gamma d} + s_{23} e^{-2i\gamma f} - 2s_{23} c_{23} e,
\]
\[
\lambda_3 = s_{23}^2 e^{2i\gamma d} + c_{23} e^{-2i\gamma f} + 2s_{23} c_{23} e, \tag{3.13}
\]
\[
x = e^{i\rho} \frac{c_{23} e^{i\gamma b} - s_{23} e^{-i\gamma c}}{c_{13}}, \quad y = e^{i\rho} \left( s_{23} e^{i\gamma b} + c_{23} e^{-i\gamma} \right). \tag{3.14}
\]

If \( x \) is suppressed, \( \lambda_1 \approx \lambda_2 \) to give a sizable \( \theta_{12} \). The Majorana phases \( \phi_{1,2} \) become the similar order for the inverted mass hierarchy with \( m_1 \approx m_2 \). In this case, for \( \det(M_\nu) = 0 \) giving \( m_3 = 0 \), Majorana CP violation is generically small.

3.2. \( \det(M_\nu) = 0 \)

Let us next see how the condition of \( \det(M_\nu) = 0 \) provides a massless neutrino when our formulas of \( m_{1,2,3} \) are used. Since the violation of the \( \mu - \tau \) symmetry is tiny, it does not significantly affect the sizes of the neutrino masses evaluated in the \( \mu - \tau \) symmetric limit although it affects the size of \( \theta_{12} \) for the category (C2). The obtained results are to be used to construct textures either with \( m_1 = 0 \) or \( m_3 = 0 \) but not with \( m_2 = 0 \).

If we parameterize \( M_\nu \) as follows:
\[
M_\nu = \begin{pmatrix}
a & b & c \\
b & d & e \\
c & e & f
\end{pmatrix}, \tag{3.15}
\]
\( \det(M_\nu) = 0 \) is translated into:
\[
e = \frac{bc + s\sigma \sqrt{(bc)^2 - a(b^2f + c^2d - adf)}}{a}, \tag{3.16}
\]
where \( s = \pm 1 \). The factor \( \sigma \) may not be needed in Eq.\,(3.16); however, it intends to take care of \( \sigma \) from \( c = -\sigma b \) in the \( \mu - \tau \) symmetric limit and it is merely our matter.
of convention. In the \(\mu-\tau\) symmetric limit, Eq.(3-16) turns out to be

\[
\sigma e = -d, \tag{3-17}
\]

for \(\sqrt{(b^2 - ad)^2} = s (b^2 - ad)\), and

\[
\sigma e = d - \frac{2b^2}{a}, \tag{3-18}
\]

for \(\sqrt{(b^2 - ad)^2} = -s (b^2 - ad)\). One can also confirm that the condition of \(\det(M_\mu) = 0\) evaluated up to the first order of the \(\mu-\tau\) symmetry breaking coincides with that of \(\det(M_\nu) = 0\) in the \(\mu-\tau\) symmetric limit.

The appearance of a massless neutrino is described in Appendix B using Eq.(3-12). The results are summarized as follows:

- either \(m_1 = 0\) or \(m_2 = 0\) from Eq.(3-18)

\[
m_1 = 0 \text{ if } \sqrt{z^2} = z, \tag{3-19}
\]

\[
m_2 = 0 \text{ if } \sqrt{z^2} = -z, \tag{3-20}
\]

where \(z = d - \sigma e + e^{2i\rho}a\),

- \(m_3 = 0\) from Eq.(3-17),

for the category (C1), where \(\sin\theta_{23} = \sigma/\sqrt{2}\) and \(\sin\theta_{13} = 0\) in the \(\mu-\tau\) symmetric limit, and

- either \(m_1 = 0\) or \(m_3 = 0\) from Eq.(3-18)

\[
m_1 = 0 \text{ if } \sqrt{z^2} = kz, \tag{3-21}
\]

\[
m_3 = 0 \text{ if } \sqrt{z^2} = -kz, \tag{3-22}
\]

where \(k(= \pm 1)\) takes care of the sign of \(\cos 2\theta_{13}\) and \(z = (d - \sigma e)e^{i\delta} + ae^{-i\delta}\),

- \(m_2 = 0\) from Eq.(3-17)

for the category (C2), where \(\sin\theta_{23} = -\sigma/\sqrt{2}\) and \(\sin\theta_{12} = 0\) in the \(\mu-\tau\) symmetric limit. In any cases, the massless \(\nu_2\) should not be realized by textures.

§4. Effect of \(\mu-\tau\) Symmetry Breaking

To specify phase structure of \(M_\nu\), let us first count phases present in \(M_\nu\). There are six complex numbers in \(M_\nu\). Since three phases are removed by the rephasing, among the remaining three phases, one phase can be determined by \(\det(M_\nu) = 0\). We are left with two phases, which are taken to be the phases associated with \(M_{e\mu}\) and \(M_{e\tau}\). For the present discussions, these phases are denoted by \(\alpha\) associated with \(M_{e\mu}^{(+)}\) and by \(\beta\) associated with \(M_{e\mu}^{(-)}\) in place of \(M_{e\mu}\) and \(M_{e\tau}\) for the sake of convenience. Any other choices give the same results of CP violation because of the rephasing invariance as shown in Appendix A. Thus, our results do not depend on our specific choice of phases in \(M_\nu\) and will cover leptonic CP properties in models with \(\det(M_\nu) = 0\), where the charged lepton masses are necessarily taken to diagonal.\(^{21}\)
4.1. Parameterization of $\mu$-$\tau$ Symmetry Breaking

To describe the $\mu$-$\tau$ symmetry breaking flavor neutrino masses, we parameterize

$$M_\nu^{(+)} = \begin{pmatrix} a_0 & e^{i\alpha}b_0 & -\sigma e^{i\alpha}b_0 \\ e^{i\alpha}b_0 & d_0 & e_0 \\ -\sigma e^{i\alpha}b_0 & e_0 & d_0 \end{pmatrix}, \quad M_\nu^{(-)} = \varepsilon \begin{pmatrix} 0 & e^{i\beta}b'_0 & \sigma e^{i\beta}b'_0 \\ e^{i\beta}b'_0 & d'_0 & 0 \\ \sigma e^{i\beta}b'_0 & 0 & -d'_0 \end{pmatrix},$$

(4.1)

for Eq.(2-3) and

$$M^{(+)} = \begin{pmatrix} A & B_+ & -\sigma B_+ \\ B_+^* & D_+ & E_+ \\ -\sigma B_+^* & E_+ & D_+ \end{pmatrix}, \quad M^{(-)} = \begin{pmatrix} 0 & B_- & \sigma B_- \\ B_-^* & D_- & iE_- \\ \sigma B_-^* & -iE_- & -D_- \end{pmatrix},$$

(4.2)

where $M \equiv M^{(+)} + M^{(-)}$. We show results valid up to the terms of $O(\varepsilon)^*$. To do so, we parameterize $\theta_{23}$ as follows:

$$\cos \theta_{23} = \frac{1 + \Delta}{\sqrt{2(1 + \Delta^2)}}, \quad \sin \theta_{23} = \kappa \sigma \frac{1 - \Delta}{\sqrt{2(1 + \Delta^2)}},$$

(4.3)

where $|\Delta| (\ll 1)$ is responsible for the effect of $O(\varepsilon)$. The parameter $\kappa (= \pm 1)$ takes care of the difference of the category: $\kappa = 1$ for the category (C1) and $\kappa = -1$ for the category (C2).

The results are given by

$$A \approx |a_0|^2 + 2|b_0|^2, \quad B_+ \approx b_0 \left(e^{i\alpha}a_0 + e^{-i\alpha}(d_0 - \sigma e_0)\right),$$

$$B_- \approx \varepsilon \left[(a_0e^{i\beta} + e^{-i\beta}(d_0 + \sigma e_0))b'_0 + e^{-i\alpha}b_0d'_0\right],$$

$$D_+ \approx |b_0|^2 + |d_0|^2 + |e_0|^2, \quad D_- = 2\varepsilon \left[b_0b'_0 \cos(\alpha - \beta) + d_0d'_0\right],$$

$$E_+ \approx 2d_0Re \left(e_0\right) - \sigma |b_0|^2, \quad E_- = 2\varepsilon \left[|d'_0| \text{Im} \left(e_0\right) - \sigma b_0b'_0 \sin(\alpha - \beta)\right],$$

$$A_1 \approx A, \quad A_2 \approx \left\{ \begin{array}{ll} 2b_0^2 + |d_0 - \sigma e_0|^2 & (\kappa = 1) \\
|d_0 + \sigma e_0|^2 & (\kappa = -1) \end{array} \right.,$$

$$A_3 \approx \left\{ \begin{array}{ll} |d_0 + \sigma e_0|^2 & (\kappa = 1) \\
2b_0^2 + |d_0 - \sigma e_0|^2 & (\kappa = -1) \end{array} \right.,$$

$$X \approx \left\{ \begin{array}{ll} \sqrt{2}B_+ & (\kappa = 1) \\
\sqrt{2}(B_- + (\Delta + i\gamma)B_+) & (\kappa = -1) \end{array} \right.,$$

$$Y \approx \left\{ \begin{array}{ll} \sqrt{2}\sigma(B_- - (\Delta - i\gamma)B_+) & (\kappa = 1) \\
-\sqrt{2}\sigma B_+ & (\kappa = -1) \end{array} \right.,$$

$$\Delta \approx \frac{\kappa \sigma D_- + s_{13}\Re \left(e^{-i\delta}X^*\right)}{2E_+}, \quad \gamma \approx \frac{E_- + s_{13}\Im \left(e^{-i\delta}X^*\right)}{2E_+},$$

(4.4)

from which the mixing angles in Eq.(3-11) and the Dirac phases in Eq.(3-8) can be estimated, and

$$\lambda_1 \approx e^{2i\rho a_0}, \quad \lambda_2 \approx d_0 - \kappa \sigma e_0, \quad \lambda_3 \approx d_0 + \kappa \sigma e_0,$$

*) It should be noted that the smallness of $\Delta m^2_{atm}/|\Delta m^2_{sun}|$ is naturally $O(\varepsilon)$ in the category (C2) but the smallness of $\sin \theta_{13}$ is also implicitly assumed to be consistent with the experimental observation. It is realized by the smallness of $b_0$ contained in $y$ (See Eq.(4-5) for $\kappa = -1$).
\[ x \approx \begin{cases} 2e^{i(\rho + \alpha)}b_0 \ (\kappa = 1) \\ 2e^{i\rho} ((\Delta + i\gamma) e^{i\alpha}b_0 + \varepsilon e^{i\beta}b'_0) \ (\kappa = -1) \end{cases}, \]
\[ y \approx \begin{cases} 2\sigma e^{i\rho} ((-\Delta + i\gamma) e^{i\alpha}b_0 + \varepsilon e^{i\beta}b'_0) \ (\kappa = 1) \\ -2\sigma e^{i(\rho + \alpha)}b_0 \ (\kappa = -1) \end{cases}, \]
\[ m_1 e^{-2i\varphi_1} \approx \frac{e^{2i\rho} a_0 + d_0 - \kappa \sigma e_0}{2} - \frac{x}{\sin 2\theta_{12}}, \]
\[ m_2 e^{-2i\varphi_2} \approx \frac{e^{2i\rho} a_0 + d_0 - \kappa \sigma e_0}{2} + \frac{x}{\sin 2\theta_{12}}, \]
\[ m_3 e^{-2i\varphi_3} \approx d_0 + \kappa \sigma e_0, \tag{4.5} \]

from which Majorana phases can be estimated.

4.2. Masses, Mixing Angles and Phases

Before we give explicit textures, we calculate masses, mixing angles and phases in terms of the mass parameters of Eq.(4.1) from relations found in the previous subsection for each category. Explicit forms of textures can readily be obtained once we give mass parameters in Eq.(4.1), which are taken to realize the normal mass hierarchy or the inverted mass hierarchy.

4.2.1. Category (C1)

Mixing angles and Dirac phases can be estimated in terms of Eq.(4.4) as

\[ \tan 2\theta_{12} e^{i\rho} \approx \frac{2X}{(|d_0 - \sigma e_0| + |a_0|) (|d_0 - \sigma e_0| - |a_0|)}, \]
\[ \tan 2\theta_{13} e^{-i\delta} \approx \frac{2Y}{(|d_0 + \sigma e_0| + \sqrt{|a_0|^2 + 2|b_0|^2}) (|d_0 + \sigma e_0| - \sqrt{|a_0|^2 + 2|b_0|^2})}, \]
\[ \cos 2\theta_{23} \approx 2\Delta, \tag{4.6} \]

where

\[ X \approx \sqrt{2}b_0 \left( e^{i\alpha}a_0 + e^{-i\alpha} (d_0 - \sigma e_0) \right), \]
\[ Y \approx \sqrt{2} \sigma \left[ \varepsilon \left( (a_0 e^{i\beta} + e^{-i\beta} (d_0 + \sigma e_0)) b'_0 + e^{-i\alpha} b_0 d'_0 \right) - (\Delta - i\gamma) \left( b_0 (e^{i\alpha} a_0 + e^{-i\alpha} (d_0 - \sigma e_0)) \right) \right], \]
\[ \Delta \approx -\frac{2\sigma \varepsilon [ b_0 b'_0 \cos (\alpha - \beta) + d_0 d'_0] - s_{13} \text{Re} \left( e^{-i\delta} X^* \right)}{2 \left( 2d_0 \text{Re} (e_0) - \sigma |b_0|^2 \right)}, \]
\[ \gamma \approx \frac{2 \varepsilon [ d'_0 \text{Im} (e_0) - \sigma b_0 b'_0 \sin (\alpha - \beta)] + s_{13} \text{Im} \left( e^{-i\delta} X^* \right)}{2 \left( 2d_0 \text{Re} (e_0) - \sigma |b_0|^2 \right)}, \tag{4.7} \]

leading to

\[ \tan \varphi \approx \frac{a_0 - (d_0 - \sigma e_0)}{a_0 + (d_0 - \sigma e_0)} \tan \alpha, \tag{4.8} \]
from $X$. Since \( \tan 2\theta_{13} \) involves $s_{13}$ via $\Delta$ and $\gamma$ in its right-hand side, one can further solve $\tan 2\theta_{13}$ to give

$$
\begin{align*}
\tan 2\theta_{13} & \approx \sqrt{2\sigma} \frac{B_- - B_+ \frac{D_{\alpha \beta} - i E_{\alpha \beta}}{2E_+}}{A_3 - A} \\
\end{align*}
$$

(4.9)

where $B_\pm$ and $E_\pm$ are given by Eq.(4.4), which gives $|s_{13}| = \mathcal{O}(\varepsilon)$. Neutrino masses and Majorana phases are estimated to be:

$$
\begin{align*}
m_1 e^{-2i\phi_1} & = 0, \\
m_2 e^{-2i\phi_2} & \approx e^{2i\rho} a_0 + d_0 - \sigma e_0, \\
m_3 e^{-2i\phi_3} & \approx d_0 + \sigma e_0, \\
\end{align*}
$$

(4.10)

for the normal mass hierarchy, and

$$
\begin{align*}
m_1 e^{-2i\phi_1} & \approx \frac{e^{2i\rho} a_0 + d_0 - \sigma e_0}{2} - \frac{x}{\sin 2\theta_{12}}, \\
m_2 e^{-2i\phi_2} & \approx \frac{e^{2i\rho} a_0 + d_0 - \sigma e_0}{2} + \frac{x}{\sin 2\theta_{12}}, \\
m_3 e^{-2i\phi_3} & (= 0) \approx d_0 + \sigma e_0, \\
\end{align*}
$$

(4.11)

for the inverted mass hierarchy, where

$$
x \approx 2e^{i(\rho + \alpha)} b_0. \\
$$

(4.12)

4.2.2. Category (C2)

In this category, mixing angles and Dirac phases can be estimated in terms of Eq.(4.4) as

$$
\begin{align*}
\tan 2\theta_{12} e^{i\rho} & \approx \frac{2X}{(|d_0 + \sigma e_0| - \sqrt{|a_0|^2 + 2 |b_0|^2}) (|d_0 + \sigma e_0| + \sqrt{|a_0|^2 + 2 |b_0|^2})}, \\
\tan 2\theta_{13} e^{-i\delta} & \approx \frac{2Y}{(|d_0 - \sigma e_0| - |a_0|) (|d_0 - \sigma e_0| + |a_0|)}, \\
\cos 2\theta_{23} & \approx 2\Delta, \\
\end{align*}
$$

(4.13)

where

$$
\begin{align*}
X & \approx \sqrt{2} \left[ e \left( \alpha_0 e^{i\beta} + e^{-i\beta} (d_0 + \sigma e_0) \right) b_0 + e^{-i\alpha} b_0 \left( e^{i\alpha} a_0 + e^{-i\alpha} (d_0 - \sigma e_0) \right) \right], \\
Y & \approx -\sqrt{2} \sigma b_0 \left( e^{i\alpha} a_0 + e^{-i\alpha} (d_0 - \sigma e_0) \right), \\
\Delta & \approx \frac{2\sigma \varepsilon [b_0 d_0 \cos (\alpha - \beta) + d_0 d_0] + s_{13} \Re (e^{-i\delta} X^*)}{2 \left( 2d_0 \Re (\varepsilon_0) - \sigma |b_0|^2 \right)}, \\
\end{align*}
$$

(4.14)

with $\gamma$ replaced by $-\gamma$ in Eq.(4.7), leading to

$$
\tan \delta \approx \frac{\alpha_0 - (d_0 - \sigma e_0)}{\alpha_0 + (d_0 - \sigma e_0)} \tan \alpha, \\
$$

(4.15)
from $Y$. Neutrinos exhibit the inverted mass hierarchy (as to be discussed in Sec.5.3), where masses and Majorana phases are given by

$$m_1 e^{-2i\varphi_1} \approx \frac{e^{2i\rho_0} + d_0 + \sigma e_0}{2} - \frac{x}{\sin 2\theta_{12}},$$

$$m_2 e^{-2i\varphi_2} \approx \frac{e^{2i\rho_0} + d_0 + \sigma e_0}{2} + \frac{x}{\sin 2\theta_{12}},$$

$$m_3 e^{-2i\varphi_3} (= 0) \approx d_0 - \sigma e_0,$$

(4.16)

where $d_x \approx 2e^{i\rho}(\Delta + i\gamma)e^{i\alpha_0} + \varepsilon e^{i\beta_0'}$.

One has to finetune parameters to yield a tiny quantity $\eta$ to be used in our textures. This finetuning provides the smallness of

1. $\Delta m^2_{\odot}$ in Eq.(3.9) requiring $|X| \approx 0$ in (4.7) that leads to either $b_0 \approx 0$ or $|e^{i\alpha_0} + e^{-i\alpha_0} (d_0 - \sigma e_0)| \approx 0$ for the category (C1) and
2. $\sin \theta_{13}$ requiring $b_0 \approx 0$ in Eq.(4.14) for the category (C2).

All textures are so parameterized to satisfy these conditions.

§5. Mass Hierarchies and CP Violation

In this section, we select specific mass matrices to see how CP violations depend on phases of flavor neutrino masses. Some of these textures are those extrapolated from textures without no phases, which have been shown to consistently describe the current neutrino oscillations. Other textures are those having nontrivial forms that cannot be extrapolated from the textures without phases. Such nontrivial textures arise in the inverted mass hierarchy.

5.1. Textures and Effect of Phases in the Inverted Mass Hierarchy

In the inverted mass hierarchy, for flavor neutrino masses expressed in terms of Eq.(4.1), we require that $\Delta m^2_{\odot}/|\Delta m^2_{\text{atm}}| \ll 1$ be satisfied and obtain from Eq.(4.5) that

$$(a_0 + 2d_0) \Re (e^{-i\rho}x) \cos \rho + (a_0 - 2d_0) \Im (e^{-i\rho}x) \sin \rho \approx 0,$$

(5.1)

where we have used $d_0 + \kappa \sigma e_0 \approx 0$ ($\kappa = 1$ for the category (C1) and $\kappa = -1$ for the category (C2)) from $m_3 = 0$. There are two solutions:

1. $a_0 + 2d_0 \approx 0$ and $\Im (e^{-i\rho}x) \sin \rho \approx 0$,
2. $a_0 - 2d_0 \approx 0$ and $\Re (e^{-i\rho}x) \cos \rho \approx 0$.

The simplest case is $x \approx 0$, which is the case with $m_1 \approx m_2$. If $x$ is not suppressed, textures with $m_1 \approx -m_2$ are realized and the specific interplay between $\rho$ and $x$ may give the necessary suppression.

For the category (C1), if $x \approx 0$, since $x \propto b_0$ as in Eq.(4.5), $b_0$ should be suppressed while

$$|\cos \rho (a_0 + 2d_0) + i \sin \rho (a_0 - 2d_0)| \neq 0,$$

(5.2)
should be maintained to control the scale of neutrino masses as can be seen from Eq.(4.11). Because tan\(2\theta_{12}\) in Eq.(4.6) contains \(b_0\) in the numerator, the denominator should have a factor to cancel the smallness of \(b_0\). Using Eq.(4.8), we obtain that

\[
\tan 2\theta_{12} \approx \frac{2\sqrt{2}b_0}{2d_0 - a_0} \left| \cos \alpha \right|, \tag{5.3}
\]

where \(d_0\) can be always chosen to be positive. From the constraint of \(b_0 \approx 0\), Eq.(5.3) gives \(\sin^2 2\theta_{12} = O(1)\) if \(2d_0 - a_0 \approx 0\), thus requiring \(a_0 \approx 0\), or if \(\cos \rho \approx 0\) for \(2d_0 - a_0 \neq 0\). This result is consistent with Eq.(5.2). From this observation, we find the following conditions:

1. \(\cos \rho\) is not suppressed and has moderate values including \(\cos \rho = 1\) for \(a_0 \approx 2d_0(> 0)\). The texture is given by Eq.(5.12).
2. \(\cos \rho\) should be suppressed and \(a_0 - 2d_0\) is not suppressed. The texture is given by Eq.(5.20).

For \(x \neq 0\), which is the case with \(m_1 \approx -m_2\), since both \(a_0\) and \(d_0\) are not suppressed, either \(\text{Im}(e^{-i\rho}x)\sin \rho \approx 0\) or \(\text{Re}(e^{-i\rho}x)\cos \rho \approx 0\) should be suppressed. We have to require that

1. \(a_0 + 2d_0 \approx 0\) if \(\text{Im}(e^{-i\rho}x)\sin \rho \approx 0\). The texture is given by Eq.(5.28).
2. \(a_0 - 2d_0 \approx 0\) if \(\text{Re}(e^{-i\rho}x)\cos \rho \approx 0\). The texture is given by Eq.(5.32).

For the category (C2), the case of \(x \approx 0\) can only be satisfied because \(|x| \propto \varepsilon\) as a result of the approximate \(\mu-\tau\) symmetry and Eq.(5.2) should be satisfied. Furthermore, to retain \(\sin^2\theta_{13} \ll 1\), we have \(|b_0| \ll |a_0|\) in \(\tan 2\theta_{13}\) estimated in Eq.(4.13), which in turn further gives

\[
\tan 2\theta_{12} \approx \frac{X}{(2|d_0| - |a_0|)(2|d_0| + |a_0|)}. \tag{5.4}
\]

Since \(|X| = O(\varepsilon)|\), we find that \(|2|d_0| - |a_0|| = O(\varepsilon)|\). Therefore, we obtain conditions:

1. \(a_0 - 2d_0 \approx 0\) as well as \(\cos \rho \neq 0\) from Eq.(5.2). The texture is given by Eq.(5.36),
2. \(a_0 + 2d_0 \approx 0\) as well as \(\sin \rho \neq 0\) from Eq.(5.2). The texture is given by Eq.(5.42).

In both categories, the approximate \(\mu-\tau\) symmetry assures that

1. for the categories (C1) and (C2), \(\cos 2\theta_{23}\) is proportional to \(\varepsilon\) and the almost maximal atmospheric neutrino mixing naturally arises;
2. for the category (C1), \(\tan 2\theta_{13}\) is proportional to \(\varepsilon\) and the smallness of \(\sin^2\theta_{13}\) naturally arises;
3. for the category (C2), \(x\) in \(m_{1,2}\) is proportional to \(\varepsilon\) and the smallness of \(\Delta m_{2,3}\) naturally arises because \(\Delta m_{2,3}\) is proportional to \(x\). It is equivalent to refer to \(X\) instead of \(x\), which obviously gives the suppressed \(\Delta m_{2,3}\) because of Eq.(3.5).

In the next subsections, we estimate sizes of CP phases as functions of \(\alpha\) and \(\beta\) valid up to \(O(\varepsilon)|\). On the other hand, numerical analysis is based on our exact formulas shown in Sec.3 without the perturbation of \(\varepsilon\). In each texture to be discussed, we use \(m_0\) to denote the mass scale of neutrinos, \(p, q\) satisfying \(|p| = O(1)|\).
and $|q| = O(1)$ to respectively denote mass parameters for $M_{ee}$ and $M_{e\mu,e\tau}$ and $\eta$ to denote a tiny parameter, which provides $\Delta m_{\odot}^2/\Delta m_{atm}^2 \ll 1$ for the category (C1) and $\sin^2 \theta_{13} \ll 1$ for the category (C2). Roughly speaking, in the category (C1), $\Delta m_{\odot}^2/\Delta m_{atm}^2 = O(\eta^2)$ for the normal mass hierarchy and $= O(\eta)$ for the inverted mass hierarchy are satisfied. The CP parameters that can be compared with those analyzed by experiments are $\delta_{CP}(=\delta + \rho)$ and $\phi_{1,2,3}$ used in $U_{PMNS}$. We define the CP-violating Majorana phase to be $\phi_{CP} = \phi_3 - \phi_2$ for the normal mass hierarchy and $\phi_{CP} = \phi_2 - \phi_1$ for the inverted mass hierarchy. Estimated Dirac and Majorana phases are illustrated in Fugues as functions of $\sin^2 \theta_{13}$. Masses, mixing angles, and phases in each texture are estimated from Sec. 4.2.

5.2. Category (C1)

5.2.1. Normal Mass Hierarchy

Our mass matrix $M_\nu$ can be parameterized by

$$M_{\nu}^{(C1)N} = m_0 \begin{pmatrix} p \eta & e^{i\alpha} \eta & -\sigma e^{i\alpha} \eta \\ e^{i\alpha} \eta & 1 & e_0/m_0 \\ -\sigma e^{i\alpha} \eta & e_0/m_0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & e^{i\beta} b'_0 & \sigma e^{i\beta} b'_0 \\ e^{i\beta} b'_0 & d'_0 & 0 \\ \sigma e^{i\beta} b'_0 & 0 & -d'_0 \end{pmatrix},$$

(5.5)

where $\sigma e_0/m_0 = 1 - 2e^{2i\alpha} \eta/p + O(\varepsilon^2)$ from $s = 1$ to give $m_1 = 0$ from Eq. (3.18). We obtain that

$$\tan 2\theta_{12} e^{i\rho} \approx \frac{2\sqrt{2} e^{i\alpha}}{2 - p},$$

$$\tan 2\theta_{13} e^{-i\delta} \approx \frac{\sqrt{2}\sigma \varepsilon \left[b'_0 \left[2e^{-i\beta} + \eta \left(pe^{i\beta} - 2\frac{1}{p} e^{i(2\alpha - \beta)}\right]\right] + \eta d'_0 e^{-i\alpha}\right]}{2m_0 \left[1 - 2\frac{p}{2} \cos 2\alpha\right]},$$

(5.6)

where $\Delta$ and $\gamma$ are $O(\varepsilon)$, from which

$$\rho \approx \alpha, \quad \tan \delta \approx \frac{b'_0 \left[(2 - p\eta) \sin \beta + \frac{2p}{p} \sin (2\alpha - \beta)\right] + \eta d'_0 \sin \alpha}{b'_0 \left[(2 + p\eta) \cos \beta - \frac{2p}{p} \cos (2\alpha - \beta)\right] + \eta d'_0 \cos \alpha},$$

(5.7)

are derived. Since $\eta$ in $\delta$ is phenomenologically suppressed, we find that

$$\rho \approx \alpha, \quad \delta \approx \beta.$$  

(5.8)

It should be noted that Dirac CP violation is controlled by $\delta_{CP} \approx \alpha + \beta$ while Majorana CP violation is associated with neutrino masses, which are given by

$$m_2 e^{-2i\phi_2} \approx \left(p + \frac{2}{p}\right) e^{2i\rho} \eta m_0, \quad m_3 e^{-2i\phi_3} \approx 2 \left(1 - \frac{\eta}{p} e^{2i\alpha}\right) m_0,$$

(5.9)

leading to

$$\phi_2 \approx -\rho, \quad \phi_3 \approx 0.$$  

(5.10)
Therefore, Majorana CP violation is controlled by $\phi_{CP} = \phi_3 - \phi_2$:

$$\phi_{CP}(\approx \rho) \approx \alpha, \quad (5.11)$$

which is shown in FIG.1 as function of $\sin^2 \theta_{13}$, where no constraint on the size of $\phi_{CP}$ is found. In other words, the maximal CP violation signaled by $\phi_{CP} \approx \pi/2$ is allowed.

5.2.2. Inverted mass hierarchy I A ($m_1 \approx m_2$)

Our mass matrix $M_\nu$ can be parameterized by

$$M_{\nu}^{(C1)IA} = m_0 \begin{pmatrix} 2 - p\eta & e^{i\alpha} \eta & -\sigma e^{i\alpha} \eta \\ e^{i\alpha} \eta & 1 & e_0/m_0 \\ -\sigma e^{i\alpha} \eta & e_0/m_0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & e^{i\beta} b'_0 & \sigma e^{i\beta} b''_0 \\ e^{i\beta} b'_0 & d'_0 & 0 \\ \sigma e^{i\beta} b''_0 & 0 & -d''_0 \end{pmatrix}, \quad (5.12)$$

where $\sigma e_0/m_0 = -1 + O(\varepsilon^2)$ from $s = -1$ to give $m_3 = 0$ from Eq.(3.17).

We obtain that

$$\tan 2\theta_{12} e^{i\rho} \approx 2\sqrt{2} \left( \frac{\cos \alpha}{p} - \eta \sin \alpha \right),$$

$$\tan 2\theta_{13} e^{-i\delta} \approx -\sqrt{2\sigma} \left[ \varepsilon (2 - p\eta) b'_0 e^{i\beta} + \eta (4 m_0 (\Delta - i\gamma) \cos \alpha) \right] / 2m_0 (1 - p\eta). \quad (5.13)$$

Owing to the phenomenological requirement of $\sin^2 2\theta_{12} = O(1)$, $\cos \alpha = O(1)$ should be realized. It is found that

$$\tan \rho \approx -\frac{p\eta}{4 - p\eta} \tan \alpha, \quad \tan \delta \approx -\frac{\varepsilon (2 - p\eta) b'_0 \sin \beta - \varepsilon d'_0 \sin \alpha - 4 m_0 \gamma \cos \alpha}{\varepsilon (2 - p\eta) b'_0 \cos \beta + \eta (4 d'_0 - 4 m_0 \Delta) \cos \alpha}. \quad (5.14)$$

Since $\cos \alpha = O(1)$ and the terms proportional to $\eta$ in $\tan \delta$ can be neglected, we observe that

$$\rho \approx 0, \quad \delta \approx -\beta, \quad (5.15)$$

leading to

$$\delta_{CP} \approx -\beta. \quad (5.16)$$

Majorana CP violation is associated with neutrino masses, which are calculated to be:

$$m_1 e^{-2i\phi_1} \approx e^{i\rho} \left( \frac{4 \cos \rho - \eta e^{i\rho} p}{2} - \sqrt{2}\eta e^{i\alpha} \sin 2\theta_{12} \right) m_0,$$

$$m_2 e^{-2i\phi_2} \approx e^{i\rho} \left( \frac{4 \cos \rho - \eta e^{i\rho} p}{2} + \sqrt{2}\eta e^{i\alpha} \sin 2\theta_{12} \right) m_0, \quad (5.17)$$

from which

$$\phi_1 \approx \phi_2 \approx 0, \quad (5.18)$$

because of $\rho \approx 0$. Therefore, Majorana CP violation is controlled by $\phi_{CP} = \phi_2 - \phi_1$:

$$\phi_{CP} \approx 0, \quad (5.19)$$

which gives FIG.2.
5.2.3. Inverted mass hierarchy I B \((m_1 \approx m_2)\)

Our mass matrix \(M_\nu\) can be parameterized by

\[
M^{(C1)IB}_\nu = m_0 \begin{pmatrix}
-2 + \eta 
& e^{i\alpha} \eta 
& -\sigma e^{i\alpha} \eta \\
-\sigma e^{i\alpha} \eta 
& 1 
& e_0/m_0 \\
\end{pmatrix}
+ \varepsilon \begin{pmatrix}
0 & e^{i\beta} b_0' & \sigma e^{i\beta} b_0' \\
e^{i\beta} b_0' 
& d_0' 
& 0 \\
\sigma e^{i\beta} b_0' 
& 0 
& -d_0'
\end{pmatrix},
\]  

(5.20)

where \(\sigma e_0/m_0 = -1 + O(\varepsilon^2)\) from \(s = 1\) to give \(m_3 = 0\) from Eq.(3.17). The sign of \(a_0\) differs from the one for Eq.(5.12). This sign difference converts \(a_0 + 2d_0\) into \(-a_0 + 2d_0\) and in turn exchanges the role of \(\cos \rho\) and \(\sin \rho\) in Eq.(5.2);

We obtain that

\[
\tan 2\theta_{12} e^{i\rho} \approx 2\sqrt{2} \left(-\frac{i \sin \alpha}{p} + \frac{\eta \cos \alpha}{4 - p\eta}\right),
\]

\[
\tan 2\theta_{13} e^{-i\delta} \approx -\sqrt{2} \varepsilon \frac{e^{-i\eta} (2 - p\eta) b_0' e^{i\beta} + \eta d_0' e^{-i\alpha}}{2m_0 (1 - p\eta)} + 4m_0 i (\Delta - i\gamma) \eta \sin \alpha.
\]

(5.21)

Owing to the phenomenological requirement of \(\sin^2 2\theta_{12} = O(1), \ |\sin \alpha| = O(1)\) should be realized. It is found that

\[
\tan \rho \approx -\frac{4 - p\eta}{p\eta} \tan \alpha,
\]

\[
\tan \delta \approx -\frac{\varepsilon (2 - p\eta) b_0' \sin \beta + \eta (\varepsilon d_0' - 4m_0 \Delta) \sin \alpha}{\varepsilon (2 - p\eta) b_0' \cos \beta - \eta (\varepsilon d_0' \cos \alpha + 4m_0 \gamma \sin \alpha)}.
\]

(5.22)

Since \(|\sin \alpha| = O(1)\) and \(\eta\) is phenomenologically suppressed, we observe that

\[
\rho \approx \pm \pi/2, \quad \delta \approx -\beta,
\]

(5.23)

leading to

\[
\delta_{CP} \approx -\beta \pm \frac{\pi}{2}.
\]

(5.24)

Majorana CP violation is associated with neutrino masses, which are calculated to be:

\[
m_1 e^{-2i\phi_1} \approx -e^{i\rho} \left(2i \sin \rho - \frac{p\eta e^{i\rho}}{2} + \frac{\sqrt{2} e^{i\alpha}}{\sin 2\theta_{12}} \eta\right) m_0,
\]

\[
m_2 e^{-2i\phi_2} \approx -e^{i\rho} \left(2i \sin \rho - \frac{p\eta e^{i\rho}}{2} - \frac{\sqrt{2} e^{i\alpha}}{\sin 2\theta_{12}} \eta\right) m_0,
\]

(5.25)

from which

\[
\phi_1 \approx \phi_2 \approx 0,
\]

(5.26)

because of Eq.(5.23). Therefore, Majorana CP violation is controlled by \(\phi_{CP} = \phi_2 - \phi_1:\)

\[
\phi_{CP} \approx 0.
\]

(5.27)

which gives FIG.3.
5.2.4. Inverted mass hierarchy II A \( (m_1 \approx -m_2) \)

Our mass matrix \( M_{\nu} \) can be parameterized by

\[
M_{\nu}^{(C1)IIA} = m_0 \left( \begin{array}{ccc}
-2 + \eta & e^{i\alpha} q & -\sigma e^{i\alpha} q \\
e^{i\alpha} q & 1 & e_0/m_0 \\
-\sigma e^{i\alpha} q & e_0/m_0 & 1
\end{array} \right) + \varepsilon \left( \begin{array}{ccc}
0 & e^{i\beta} b_0 & \sigma e^{i\beta} b_0 \\
e^{i\beta} b_0' & d_0' & 0 \\
\sigma e^{i\beta} b_0' & 0 & -d_0'
\end{array} \right),
\]

(5.28)

where \( \sigma e_0/m_0 = -1 + \mathcal{O}(\varepsilon^2) \) from \( s = 1 \) to give \( m_3 = 0 \) from Eq.(3.17).

We obtain that

\[
\tan 2\theta_{12} e^{i\rho} \approx 2\sqrt{2}q \left( \frac{\cos \alpha}{4 - \eta} - i \frac{\sin \alpha}{\eta} \right),
\]

\[
\tan 2\theta_{13} e^{-i\delta} \approx \sqrt{2} \sigma \left[ \varepsilon ((2 - \eta) b_0' e^{i\beta} - q d_0' e^{-i\alpha}) - m_0 q i (\Delta - i\gamma) ((4 - \eta) \sin \alpha + i\eta \cos \alpha) \right] \left( \frac{1}{m_0} \frac{1}{2 (1 - \eta) + q^2} \right).
\]

(5.29)

Owing to the phenomenological requirement of \( \sin^2 2\theta_{12} = \mathcal{O}(1), \) \( |\sin \alpha| \leq \mathcal{O}(\eta) \) should be satisfied. It is found that

\[
\tan \rho \approx \frac{\eta - 4}{\eta} \tan \alpha,
\]

\[
\tan \delta \approx \frac{\varepsilon ((2 - \eta) b_0' \sin \beta + q d_0' \sin \alpha) - m_0 q (4\Delta \sin \alpha + \gamma \eta \cos \alpha)}{\varepsilon ((2 - \eta) b_0' \cos \beta - q d_0' \cos \alpha) + m_0 q (\eta \Delta \cos \alpha - 4\gamma \sin \alpha)}.
\]

(5.30)

Majorana CP violation is associated with neutrino masses, which are calculated to be:

\[
m_1 e^{-2i\phi_1} \approx -e^{i\rho} \left( \frac{\sqrt{2} e^{i\alpha} q}{\sin 2\theta_{12}} \frac{1}{2} \right) m_0,
\]

\[
m_2 e^{-2i\phi_2} \approx e^{i\rho} \left( \frac{\sqrt{2} e^{i\alpha} q}{\sin 2\theta_{12}} \frac{1}{2} \right) m_0.
\]

(5.31)

To satisfy \( \Delta m_{\odot}^2 / |\Delta m_{\text{atm}}^2| \ll 1 \), we require that \( |\sin \rho \sin \alpha| \leq \mathcal{O}(\eta) \). Since \( |\sin \alpha| \leq \mathcal{O}(\eta), \) \( |\sin \rho| \) can be large enough to affect the size of CP violation as far as \( m_1 \approx -m_2 \) is kept. The difference of \( \phi_1 \) and \( \phi_2 \) is enhanced for smaller \( q \) and larger \( \sin \rho \). Therefore, this texture provides larger effect of Majorana CP violation controlled by \( \phi_{CP} = \phi_2 - \phi_1 \) as shown in FIG.4.

5.2.5. Inverted mass hierarchy II B \( (m_1 \approx -m_2) \)

Our mass matrix \( M_{\nu} \) can be parameterized by

\[
M_{\nu}^{(C1)HIB} = m_0 \left( \begin{array}{ccc}
2 - \eta & e^{i\alpha} q & -\sigma e^{i\alpha} q \\
e^{i\alpha} q & 1 & e_0/m_0 \\
-\sigma e^{i\alpha} q & e_0/m_0 & 1
\end{array} \right) + \varepsilon \left( \begin{array}{ccc}
0 & e^{i\beta} b_0 & \sigma e^{i\beta} b_0 \\
e^{i\beta} b_0' & d_0' & 0 \\
\sigma e^{i\beta} b_0' & 0 & -d_0'
\end{array} \right),
\]

(5.32)
where \( \sigma e_0/m_0 = -1 + O(\varepsilon^2) \) from \( s = -1 \) to give \( m_3 = 0 \) from Eq.(3.17). Similarly to the relation between Eq.(5.12) and Eq.(5.20), the sign of \( a_0 \) differs from the one for Eq.(5.28).

The predicted results are very similar to those for the previous texture. We obtain that

\[
\tan 2\theta_{12} e^{i\rho} \approx 2\sqrt{2}q \left( \frac{\cos \alpha}{\eta} - i\frac{\sin \alpha}{4 - \eta} \right),
\]

\[
\tan 2\theta_{13} e^{-i\delta} \approx -\sqrt{2}\sigma \left[ \varepsilon \left( (2 - \eta) e^{i\beta} b_0' + e^{-i\alpha} q d_0' \right) - m_0 q \left( (4 - \eta) \gamma \cos \alpha - \eta \sin \alpha \right) \right] m_0 \left( (2 - \eta)^2 + 2q^2 \right).
\]

(5.33)

Owing to the phenomenological constraint of \( \sin^2 2\theta_{12} = O(1) \), \( |\cos \alpha| \leq O(\eta) \) should be satisfied and is numerically signaled by \( \alpha \approx \pm \pi/2 \) found in FIG.5. It is found that

\[
\tan \rho \approx -\frac{\eta}{4 - \eta} \tan \alpha.
\]

\[
\tan \delta \approx -\frac{\varepsilon \left( (2 - \eta) b_0' \sin \beta - q d_0' \sin \alpha \right) + m_0 q \left( (4 - \eta) \gamma \cos \alpha + \eta \Delta \sin \alpha \right)}{\varepsilon \left( (2 - \eta) b_0' \cos \beta + q d_0' \cos \alpha \right) - m_0 q \left( (4 - \eta) \Delta \cos \alpha - \gamma \sin \alpha \right)}.
\]

(5.34)

Majorana CP violation is associated with neutrino masses, which are calculated to be:

\[
m_1 e^{-2i\phi_1} \approx -e^{i\rho} \left( \frac{\sqrt{2}q e^{i\alpha}}{\sin 2\theta_{12}} - \frac{4 \cos \rho - \eta e^{i\rho}}{2} \right) m_0.
\]

\[
m_2 e^{-2i\phi_2} \approx e^{i\rho} \left( \frac{\sqrt{2}q e^{i\alpha}}{\sin 2\theta_{12}} + \frac{4 \cos \rho - \eta e^{i\rho}}{2} \right) m_0.
\]

(5.35)

To satisfy \( \Delta m^2_{\odot}/|\Delta m^2_{\text{atm}}| \ll 1 \), we require that \( |\cos \rho \cos \alpha| \leq O(\eta) \). Since \( |\cos \alpha| \leq O(\eta) \), \( |\cos \rho| \) can also be large. As far as \( |\cos \rho| = O(1) \) is maintained, the difference of \( \phi_1 \) and \( \phi_2 \) is enhanced for smaller \( q \) and larger \( \cos \rho \). The size of \( \phi_{CP} \) is shown in FIG.5, which indicates larger effect of Majorana CP violation controlled by \( \phi_{CP} = \phi_2 - \phi_1 \).

5.3. Category (C2)

There are only textures giving the inverted mass hierarchy with \( m_1 \approx m_2 \) because of the generic smallness of \( x \) in Eq.(4.16). To ensure another smallness of \( \sin \theta_{13} \) requires \( b_0 \approx 0 \) as indicated by Eqs.(3.11) and (4.5). The inverted mass hierarchy with \( m_1 \approx -m_2 \), corresponding to Eq.(5.28) cannot be accepted because of \( |b_0/m_1| = |q| = O(1) \). A possible texture giving the normal mass hierarchy, namely, corresponding to Eq.(5.5) cannot be accepted. This is because the condition on \( m_1=0 \) for the category (C2) is given by Eq.(3.18) leading to \( \sigma e_0 = d_0 - 2b_0^2/a_0 + O(\varepsilon^2) \),
which, however, gives $\tan 2\theta_{12} \approx 0$ in Eq.(4.13) because its denominator is not suppressed owing to $a_0(=p\eta) \approx 0$ and $b_0 \approx 0$.

5.3.1. Inverted mass hierarchy I A ($m_1 \approx m_2$)

Our mass matrix $M_\nu$ can be parameterized by

\[
M_\nu^{(C2)IA} = m_0 \begin{pmatrix}
2 - p\eta & e^{i\alpha}\eta & -se^{i\alpha}\eta \\
e^{i\alpha}\eta & 1 & e_0/m_0 \\
-se^{i\alpha}\eta & e_0/m_0 & 1
\end{pmatrix} + \varepsilon \begin{pmatrix}
0 & e^{i\beta}b'_0 & se^{i\beta}b'_0 \\
e^{i\beta}b'_0 & d'_0 & 0 \\
se^{i\beta}b'_0 & 0 & -d'_0
\end{pmatrix},
\]

where $se_0/m_0 = 1 - 2e^{2i\alpha}\eta^2/(2 - p\eta) + O(\epsilon^2)$ from $s = 1$ to give $m_3 = 0$ from Eq.(3.18).

We obtain that

\[
\begin{align*}
\tan 2\theta_{12}e^{i\rho} & \approx \frac{[4b'_0 e\beta + \eta (d'_0 e^{-i\alpha} + b'_0 p e^{i\beta})] e^{i\phi_1}}{\sqrt{2m_0p}}, \\
\tan 2\theta_{13}e^{-i\delta} & \approx \sqrt{2}\varepsilon e^{i\alpha}. 
\end{align*}
\]

Owing to the phenomenological constraint of $\sin^2 2\theta_{12} = O(1)$, $\eta \approx \varepsilon$ and $b'_0 \neq 0$ as well as $|\cos \beta| = O(1)$ should be satisfied. The phases becomes

\[
\tan \rho \approx \frac{(\varepsilon d'_0 - 2m_0\Delta) \sin \alpha - 2m_0\gamma \cos \alpha + \varepsilon b'_0p \sin \beta}{4b'_0\sin \beta + (\varepsilon d'_0 + 2m_0\Delta) \cos \alpha - 2m_0\gamma \sin \alpha - \varepsilon b'_0p \cos \beta}, \quad \delta \approx -\alpha.
\]

Because of $\eta \approx \varepsilon$ and $|\cos \beta| = O(1)$, Eq.(5.38) gives $\rho \approx 0$. Since masses are calculated to be:

\[
\begin{align*}
m_1 e^{-2i\phi_1} & \approx e^{i\rho} \left\{ \frac{4\cos \rho - p\eta e^{i\rho}}{2} - \sqrt{2}\varepsilon b'_0e^{i\beta} \right\} m_0, \\
m_2 e^{-2i\phi_3} & \approx e^{i\rho} \left\{ \frac{4\cos \rho - p\eta e^{i\rho}}{2} + \sqrt{2}\varepsilon b'_0e^{i\beta} \right\} m_0.
\end{align*}
\]

Majorana phases become

\[
\phi_1 \approx \phi_2 \approx -\frac{\rho}{2}
\]

Therefore, Majorana CP violation is characterized by

\[
\phi_{CP} \approx 0,
\]

which is shown in FIG.6.

5.3.2. Inverted mass hierarchy I B ($m_1 \approx m_2$)

Our mass matrix $M_\nu$ can be parameterized by

\[
M_\nu^{(C2)IB} = m_0 \begin{pmatrix}
-2 + p\eta & e^{i\alpha}\eta & -se^{i\alpha}\eta \\
e^{i\alpha}\eta & 1 & e_0/m_0 \\
-se^{i\alpha}\eta & e_0/m_0 & 1
\end{pmatrix} + \varepsilon \begin{pmatrix}
0 & e^{i\beta}b'_0 & se^{i\beta}b'_0 \\
e^{i\beta}b'_0 & d'_0 & 0 \\
se^{i\beta}b'_0 & 0 & -d'_0
\end{pmatrix}.
\]
where $\sigma e_\theta/m_0 = 1 + 2e^{2i\eta^2}/(2 - p\eta) + \mathcal{O}(\varepsilon^2)$ from $s = -1$ to give $m_3 = 0$ from Eq.(3.18). Similarly to the relation between Eq.(5.12) and Eq.(5.20), the sign of $a_0$ differs from the one for Eq.(5.36).

We obtain that
\[
\tan 2\theta_{12} e^{i\rho} \approx -\frac{4b'_0 i\sin \beta - \eta \left(d'_0 e^{-i\alpha} - b'_0 p e^{i\beta}\right)}{\sqrt{2m_0}} + 2m_0 (\Delta + i\gamma) e^{i\alpha},
\]
\[
\tan 2\theta_{13} e^{-i\delta} \approx -\sqrt{2}\sigma e^{i\alpha}.
\] (5.43)

Owing to the phenomenological constraint of $\sin^2 2\theta_{12} = \mathcal{O}(1)$, $\eta \approx \varepsilon$ and $b'_0 \neq 0$ as well as $|\sin \beta| = \mathcal{O}(1)$ should be satisfied. The phases becomes
\[
\tan \rho \approx -\frac{4\varepsilon b'_0 \sin \beta + 2m_0 \gamma \cos \alpha + (\varepsilon d'_0 + 2m_0 \Delta) \sin \alpha - \varepsilon b'_0 p \sin \beta}{(\varepsilon d'_0 - 2m_0 \Delta) \cos \alpha + 2m_0 \gamma \sin \alpha + \varepsilon b'_0 p \cos \beta}, \quad \delta \approx -\alpha.
\] (5.44)

Similarly to Eq.(5.38), Eq.(5.44) gives $\rho \approx \pm \pi/2$. Since masses are calculated to be:
\[
m_1 e^{-2i\phi_1} \approx e^{i\rho} \left(\frac{-4i \sin \rho + p \eta e^{i\rho} - \sqrt{2} \varepsilon b'_0 e^{i\beta}}{2m_0 \sin 2\theta_{12}}\right),
\]
\[
m_2 e^{-2i\phi_1} \approx e^{i\rho} \left(\frac{-4i \sin \rho + p \eta e^{i\rho} + \sqrt{2} \varepsilon b'_0 e^{i\beta}}{2m_0 \sin 2\theta_{12}}\right).
\] (5.45)

Majorana phases become
\[
\phi_1 \approx \phi_2 \approx -\frac{1}{2} \left(\rho - \frac{\pi}{2}\right).
\] (5.46)

Therefore, Majorana CP violation is characterized by
\[
\phi_{CP} \approx 0,
\] (5.47)
as shown in FIG.7.

## §6. Summary and Discussions

To discuss leptonic CP violation as direct effects from phases of flavor neutrino masses in a model-independent way, we have focused the general parameterization of $U_{PMNS}$ that can take care of redundant phases originally arising from in the arbitrariness in phases of flavor neutrino masses. As a result, we have found that the Dirac CP phase $\delta_{CP}$ is determined from
\[
\delta_{CP} = \rho + \delta \text{ with } \rho = \arg(X) \text{ and } \delta = -\arg(Y),
\] (6.1)

where $X$ and $Y$ are described by the flavor neutrino masses:
\[
X = (c_{23} + \sigma s_{23}) (B_+ \cos \gamma + iB_- \sin \gamma)
+ (c_{23} - \sigma s_{23}) (B_- \cos \gamma + iB_+ \sin \gamma),
\]
\[
Y = \sigma \left[ (c_{23} + \sigma s_{23}) (B_- \cos \gamma + iB_+ \sin \gamma)
- (c_{23} - \sigma s_{23}) (B_+ \cos \gamma + iB_- \sin \gamma) \right].
\] (6.2)
as can be derived from Eq.(3.7). Under the approximate $\mu$-$\tau$ symmetry, we have obtained $X \approx \sqrt{2}B_+$ for the category (C1) and $Y \approx -\sqrt{2}\sigma B_+$ for the category (C2) and other quantities shows complicated relations among terms of order $\varepsilon$. More precisely, the model is characterized by the two phases $\alpha$ and $\beta$ introduced as phases of $M_{e\mu}^{(\pm)}$ and simple relations between the CP phases $\rho$ and $\delta$ and our specific phases $\alpha$ and $\beta$ turn out to arise, in the category (C1),

1. for the normal mass hierarchy, $\rho \approx \alpha$ and $\delta \approx \beta$,
2. for the inverted mass hierarchy with $m_1 \approx m_2$, $\rho \approx 0$ or $\pm \pi/2$ and $\delta \approx -\beta$,

and, in the category (C2),

1. for the inverted mass hierarchy with $m_1 \approx m_2$, $\delta \approx -\alpha$.

Other cases do not show such simple relations.

Majorana CP violation can only be enhanced in the normal mass hierarchy and the inverted mass hierarchy with $m_1 \approx -m_2$. There are two kinds of the inverted mass hierarchy depending on the relative sign of $m_1$ and $m_2$, namely, with $m_1 \approx m_2$ and $m_1 \approx -m_2$. Majorana CP violation is suppressed for the case with $m_1 \approx m_2$ and is much enhanced for the case with $m_1 \approx -m_2$. Numerically, the enhanced size of the CP violating Majorana phase is given by $-\pi/4 \leq \phi_{CP} \leq \pi/4$ (mod $\pi$). Maximal Dirac CP violation can arise for

1. $\sin^2 \theta_{13} \lesssim 0.001$ in the inverted mass hierarchies I A and I B realized for the category (C2),
2. $\sin^2 \theta_{13} \lesssim 0.01$ in the inverted mass hierarchies II A and II B realized for the category (C1),
3. $\sin^2 \theta_{13} \lesssim 0.04$ in the normal mass hierarchy and the inverted mass hierarchy I

A both realized for the category (C1).

On the other hand, maximal Majorana CP violation is only possible to arise for $\sin^2 \theta_{13} \lesssim 0.03$ in the normal mass hierarchy realized for the category (C1). It has been noted that these predictions do not depend on the choice of our specific phases because of the rephasing invariance in our formalism.

Finally, if det$(M_{\nu}) = 0$ is the result of the minimal seesaw model, these predictions are valid at the seesaw scale and are modified at the weak scale by the renormalization,23 whose effects will be evaluated in the forthcoming paper.24)

Acknowledgements

The authors would like to thank T. Kitabayashi for valuable comments and discussions.
Appendix A
Rephasing invariance in Dirac and Majorana phases

Let us demonstrate the rephasing invariance of \( \delta_{\text{CP}} \) and \( \phi_{\text{CP}} \), which are not trivial when these are expressed in flavor neutrino masses, by considering the induced changes in Eqs. (3.8) for \( \rho \) and \( \delta \) and (3.12) for Majorana phases. One particularly chooses some of phases of flavor neutrino masses to be real by removing theirs phases by the rephasing. To see the rephasing invariance, we first show how \( \delta \) and \( \rho \) vary with the rephasing. Rephasing the charged leptons (\( \ell \)) is caused by \( \ell' = U(\theta)\ell \), where \( U(\theta) = \text{diag}(e^{i\theta_e}, e^{i\theta_\mu}, e^{i\theta_\tau}) \), which in turn calls for the redefinition of the flavor neutrinos: \( \nu'_f = U(\theta)\nu_f \). As a result, the mass term \( \nu'^T M' \nu_f \) is equivalent to \( \nu'^T M'_\nu \nu'_f \) with \( M'_\nu \) defined by

\[
M'_\nu = \begin{pmatrix}
  e^{-2i\theta_e} M_{ee} & e^{-i(\theta_e + \theta_\mu)} M_{e\mu} & e^{-i(\theta_e + \theta_\tau)} M_{e\tau} \\
  e^{-i(\theta_e + \theta_\mu)} M_{e\mu} & e^{-2i\theta_\mu} M_{\mu\mu} & e^{-i(\theta_\mu + \theta_\tau)} M_{\mu\tau} \\
  e^{-i(\theta_e + \theta_\tau)} M_{e\tau} & e^{-i(\theta_\mu + \theta_\tau)} M_{\mu\tau} & e^{-2i\theta_\tau} M_{\tau\tau}
\end{pmatrix}.
\] (A.1)

This mass matrix \( M'_\nu \) yields

\[
X' = e^{i\left(\theta_e - \frac{\theta_\mu + \theta_\tau}{2}\right)} X, \quad Y' = e^{i\left(\theta_e - \frac{\theta_\mu + \theta_\tau}{2}\right)} Y,
\] (A.2)

where

\[
\gamma' = \gamma - \frac{\theta_\tau - \theta_\mu}{2},
\] (A.3)

derived from Eq. (3.4) is used in \( X' \) and \( Y' \), from which

\[
\delta' = \delta - \left(\theta_e - \frac{\theta_\mu + \theta_\tau}{2}\right), \quad \rho' = \rho + \theta_e - \frac{\theta_\mu + \theta_\tau}{2},
\] (A.4)

are obtained. Next, it is, thus, confirmed that \( \delta_{\text{CP}} \) defined by

\[
\delta_{\text{CP}} = \delta + \rho,
\] (A.5)

is a rephasing-invariant quantity. Similarly, we find that Majorana phases become

\[
\varphi'_i = \varphi_i + \frac{\theta_\mu + \theta_\tau}{2},
\] (A.6)

for \( i = 1, 2, 3 \), by using

\[
\lambda'_1 = e^{-i(\theta_\mu + \theta_\tau)} \lambda_1, \quad \lambda'_2 = e^{-i(\theta_\mu + \theta_\tau)} \lambda_2, \quad \lambda'_3 = e^{-i(\theta_\mu + \theta_\tau)} \lambda_3, \quad x' = e^{-i(\theta_\mu + \theta_\tau)} x.
\] (A.7)

The physical Majorana phases defined by \( \varphi'_i - \varphi'_j \) (\( i \neq j \)) for \( i, j = 1, 2, 3 \) turn out to be rephasing-invariant.

It is instructive to note that there are three typical forms of the PMNS unitary matrix depending on how the flavor neutrinos are redefined:
1. $U_{PMNS}$ with $\delta$, $\rho$ and $\gamma$

\[
\begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13}e^{i\rho} & s_{13}e^{-i\delta} \\
    -c_{23}s_{12}e^{-i\rho} + s_{23}c_{12}s_{13}e^{i\delta} & c_{23}c_{12} - s_{23}s_{12}s_{13} & e^{i\gamma} \\
    s_{23}s_{12}e^{-i\rho} - c_{23}c_{12}s_{13}e^{i\delta} & s_{23}c_{12} + c_{23}s_{12}s_{13}e^{i(\delta + \rho)} & e^{-i\gamma} \\
    \end{pmatrix}
\]

\[
\cdot \begin{pmatrix}
    e^{i\varphi_1} & 0 & 0 \\
    0 & e^{i\varphi_2} & 0 \\
    0 & 0 & e^{i\varphi_3}
\end{pmatrix},
\]

for

\[
M_\nu = \begin{pmatrix}
    M_{ee} & M_{e\mu} & M_{e\tau} \\
    M_{e\mu} & M_{\mu\mu} & M_{\mu\tau} \\
    M_{e\tau} & M_{\mu\tau} & M_{\tau\tau}
\end{pmatrix},
\]

(A.9)

2. $U_{PMNS}$ with $\delta$ and $\rho$

\[
\begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13}e^{i\rho} & s_{13}e^{-i\delta} \\
    -c_{23}s_{12}e^{-i\rho} - s_{23}c_{12}s_{13}e^{i\delta} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i(\delta + \rho)} & s_{23}c_{13} \\
    s_{23}s_{12}e^{-i\rho} - c_{23}c_{12}s_{13}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i(\delta + \rho)} & c_{23}c_{13}
\end{pmatrix}
\]

\[
\cdot \begin{pmatrix}
    e^{i\varphi_1} & 0 & 0 \\
    0 & e^{i\varphi_2} & 0 \\
    0 & 0 & e^{i\varphi_3}
\end{pmatrix},
\]

for

\[
M_{\nu}^{\text{Intermediate}} = \begin{pmatrix}
    M_{ee} & e^{i\gamma}M_{e\mu} & e^{-i\gamma}M_{e\tau} \\
    e^{i\gamma}M_{e\mu} & e^{2i\gamma}M_{\mu\mu} & e^{i\gamma}M_{\mu\tau} \\
    e^{-i\gamma}M_{e\tau} & e^{2i\gamma}M_{\mu\tau} & e^{-2i\gamma}M_{\tau\tau}
\end{pmatrix},
\]

(A.11)

3. $U_{PMNS}$ with $\delta_{CP} = \delta + \rho$, $\phi_1 = \varphi_1 - \rho$ and $\phi_{2,3} = \varphi_{2,3}$

\[
\begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13}e^{i\delta_{CP}} & s_{13}e^{-i\delta_{CP}} \\
    -c_{23}s_{12} - s_{23}c_{12}s_{13}e^{i\delta_{CP}} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i(\delta + \rho)} & s_{23}c_{13} \\
    s_{23}s_{12} - c_{23}c_{12}s_{13}e^{i\delta_{CP}} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i(\delta + \rho)} & c_{23}c_{13}
\end{pmatrix}
\]

\[
\cdot \begin{pmatrix}
    e^{i\phi_1} & 0 & 0 \\
    0 & e^{i\phi_2} & 0 \\
    0 & 0 & e^{i\phi_3}
\end{pmatrix},
\]

for

\[
M_{\nu}^{PDG} = \begin{pmatrix}
    e^{2i\rho}M_{ee} & e^{i(\rho + \gamma)}M_{e\mu} & e^{i(\rho - \gamma)}M_{e\tau} \\
    e^{i(\rho + \gamma)}M_{e\mu} & e^{2i\gamma}M_{\mu\mu} & e^{i(\rho - \gamma)}M_{\mu\tau} \\
    e^{i(\rho - \gamma)}M_{e\tau} & e^{2i\gamma}M_{\mu\tau} & e^{-2i\gamma}M_{\tau\tau}
\end{pmatrix}.
\]

(A.13)
In this appendix, we discuss how one massless neutrino arises from our mass formula when \( \det(M_\nu) = 0 \) is applied to it. Because corrections offered by \( M_\nu^{(-)} \) turn out to be \( O(\varepsilon^2) \), we may evaluate \( \det(M_\nu) = 0 \) in the \( \mu - \tau \) symmetric limit to get valid results up to \( (O(\varepsilon)) \). The relations determined by \( \det(M_\nu) = 0 \) are either Eq.(3.17) or Eq.(3.18).

For the category (C1) with \( \sin \theta_{13} = 0 \), neutrino masses are given by

\[
\begin{align*}
    m_1 e^{-2i\varphi_1} &= \frac{e^{2i\rho}a + d - \sigma e}{2} - \frac{\sqrt{2} e^{i\rho} b}{\sin 2\theta_{12}}, \\
    m_2 e^{-2i\varphi_2} &= \frac{e^{2i\rho}a + d - \sigma e}{2} + \frac{\sqrt{2} e^{i\rho} b}{\sin 2\theta_{12}}, \\
    m_3 e^{-2i\varphi_3} &= d + \sigma e.
\end{align*}
\]

If Eq.(3.17) is satisfied, \( m_3 = 0 \) is derived. On the other hand, if Eq.(3.18) is satisfied, we further evaluate \( m_{1,2} \). By evaluating \( \sin 2\theta_{12} \) from \( \tan^2 2\theta_{12} \) in Eq.(3.11), where Eq.(3.18) is used to replace \( b^2 \) in \( \tan^2 2\theta_{12} \), we reach the relation

\[
\frac{\sqrt{2} e^{i\rho} b}{\sin 2\theta_{12}} = \frac{\sqrt{z^2}}{2},
\]

where

\[
z = e^{2i\rho}a + d - \sigma e.
\]

We then find that

\[
\begin{align*}
    m_1 e^{-2i\varphi_1} &= \begin{cases} 0 & (\sqrt{z^2} = z), \\
    e^{2i\rho}a + d - \sigma e & (\sqrt{z^2} = -z), \\
    0 & (\sqrt{z^2} = -z),
\end{cases} \\
    m_2 e^{-2i\varphi_2} &= \begin{cases} 0 & (\sqrt{z^2} = z), \\
    e^{2i\rho}a + d - \sigma e & (\sqrt{z^2} = -z), \\
    0 & (\sqrt{z^2} = -z),
\end{cases} \\
    m_3 e^{-2i\varphi_3} &= d + \sigma e.
\end{align*}
\]

Similarly for the category (C2) with \( \sin \theta_{12} = 0 \), neutrino masses are given by

\[
\begin{align*}
    m_1 e^{-2i\varphi_1} &= \frac{e^{2i\rho}}{2} \left( a + e^{2i\delta} (d - \sigma e) + \frac{a - e^{2i\delta} (d - \sigma e)}{\cos 2\theta_{13}} \right), \\
    m_2 e^{-2i\varphi_2} &= d + \sigma e, \\
    m_3 e^{-2i\varphi_3} &= \frac{e^{2i\rho}}{2} \left( a + e^{2i\delta} (d - \sigma e) - \frac{a - e^{2i\delta} (d - \sigma e)}{\cos 2\theta_{13}} \right).
\end{align*}
\]
derive Eq.(B.2) in the category (C1), we reach the relation

$$\frac{1}{\cos 2\theta_{13}} = k \sqrt{z^2}$$

(B.6)

where $k = \pm 1$ and

$$z = (d - \sigma e) e^{i\delta} + ae^{-i\delta}.$$  

(B.7)

We then find that

$$m_1 e^{-2i\varphi_1} = \begin{cases} 0 & \left(\sqrt{z^2} = kz\right), \\ e^{-2i\delta} (a + e^{2i\delta} (d - \sigma e)) & \left(\sqrt{z^2} = k\right), \\ e^{2i\rho} (a + e^{2i\delta} (d - \sigma e)) & \left(\sqrt{z^2} = -k\right), \\ 0 & \end{cases}$$

$$m_3 e^{-2i\varphi_3} = d + \sigma e.$$  

(B.8)

References

1) Y. Fukuda et al., [Super-Kamiokande Collaboration], Phys. Rev. Lett. 81 (1998) 1562; Phys. Rev. Lett. 82 (1999) 2430; T. Kajita for the collaboration, Nucl. Phys. Proc. Suppl. 77 (1999) 123. See also, T. Kajita and Y. Totsuka, Rev. Mod. Phys. 73 (2001) 85.

2) J.N. Bahcall, W.A. Fowler, I. Iben and R.L. Sears, Astrophys J. 137 (1963) 344; J. Bahcall, Phys. Rev. Lett. 12 (1964) 300; R. Davis, Jr., Phys. Rev. Lett. 12 (1964) 303; R. Davis, Jr., D.S. Harmer and K.C. Hoffman, Phys. Rev. Lett. 20 (1968) 1205; J.N. Bahcall, N.A. Bahcall and G. Shaviv, Phys. Rev. Lett. 20 (1968) 1209; J.N. Bahcall and R. Davis, Jr., Science 191 (1976) 264.

3) Y. Fukuda et al., [Super-Kamiokande Collaboration], Phys. Rev. Lett. 81 (1998) 1158; [Erratum-ibid 81 (1998) 4297]; B.T. Clevel et al., Astrophysics J. 496 (1998) 505; W. Hampel et al., [GALLEX Collaboration], Phys. Lett. B 447 (1999) 127; Q.A. Ahmed. et al., [SNO Collaboration], Phys. Rev. Lett. 87 (2001) 071301; Phys. Rev. Lett. 89 (2002) 011301.

4) M. Apollonio, et al., [CHOOZ Collaboration], Euro. Phys. J. C 27 (2003) 331; K. Eguchi, et al., [KamLAND collaboration], Phys. Rev. Lett. 90 (2003) 021802; K. Inoue, [KamLAND collaboration], New. J. Phys. 6 (2004) 147.

5) S. H. Ahn, et al., [K2K Collaboration], Phys. Lett. B 511 (2001) 178; Phys. Rev. Lett. 90 (2003) 041801.

6) B. Pontecorvo, Sov. Phys. JETP 7 (1958) 172 [Zh. Eksp. Teor. Fiz. 34 (1958) 247]; Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. 28 (1962) 870.

7) S.M. Bilenky, J. Hosek and S.T. Petcov, Phys. Lett. B 94 (1980) 495; J. Schechter and J.W.F. Valle, Phys. Rev. D 22 (1980) 2227; M. Doi, T. Kotani, H. Nishiura, K. Okuda and E. Takasugi, Phys. Lett. B 102 (1981) 323.

8) M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49 (1973) 652.

9) M. Fukugida and T. Yanagida, Phys. Lett. B 174 (1986) 45.

10) P. Minkowski, Phys. Lett. B 67 (1977) 421; T. Yanagida, in Proceedings of the Workshop on the Unified Theory and Baryon Number in the Universe, ed. O. Sawada and A. Sugamoto (KEK report 79-18, 1979)p. 95; Prog. Theor. Phys. 64 (1980) 1870; M. Gell-Mann, P. Ramond, and R. Slansky, in Supergravity edited by P. van Nieuwenhuizen and D.Z. Freedmann (North-Holland, Amsterdam 1979),p.315; R.N. Mohapatra and G. Senjanović, Phys. Rev. Lett. 44 (1980) 912. See also, P. Minkowski, in Proceedings of the XI International Workshop on Neutrino Telescopes in Venice, Venice, 2005, edited by M. Baldo Ceolin (Papergraf S.p.A.,Italy,2005),p.7.

11) L. Lavoura and W. Grimus, JHEP 09 (007) 2000; T. Endoh, S.Kaneko, S.K. Kang, T. Morozumi, and T. Tanimoto, Phys. Rev. Lett. 89 (2002) 231601; P.H. Frampton, S.L.
Glashow, and T. Yanagida, Phys. Lett. B 548 (2002) 119; M. Raidal and A. Strumia, Phys. Lett. B 553 (2003) 72; V. Barger, D.A. Dicus, H-J. He, and T. Li Phys. Lett. B 583 (2004) 173; R.G. Felipe, F.R. Joaquim, and B.M. Nobre, Phys. Rev. D 70 (2004) 085009. For a review, see, for example, W.L. Guo, Z.Z.Xing, and S.Zhou, Int. J. Mod. Phys. E 16 (2007) 1.

12) S. Eidelman et al. (Particle Data Group), Phys. Lett. B 592 (2004) 149. See also, L.-L. Chau and W.-Y. Keung, Phys. Rev. Lett. 53 (1984) 1802.

13) For the recent study, see for example, S. Goswami, S. Petcov, S. Ray and W. Rodejohann, Phys. Rev. D 80 (2009) 053013.

14) T. Baba and M. Yasuè, Phys. Rev. D 75 (2007) 055001;

15) T. Schwetz, M. Tortola and J.W.F. Valle, New. J. Phys. 10 (2008) 113011.

16) T. Fukuyama and H. Nishiura, in Proceedings of International Workshop on Masses and Mixings of Quarks and Leptons, Shizuoka, 1997 edited by Y. Koide (World Scientific, Singapore, 1997), p.252; “Mass Matrix of Majorana Neutrinos”, [arXiv:hep-ph/9702253]; R.N. Mohapatra and S. Nussinov, Phys. Rev. D 60 (1999) 013002; Z.Z. Xing, Phys. Rev. D 61 (2000) 057301; Phys. Rev. D 64 (2001) 093013; Phys. Rev. D 64 (2001) 093013; E. Ma and M. Raidal, Phys. Rev. Lett. 87 (2001) 011802; [Erratum-ibid 87 (2001) 159901]; C.S. Lam, Phys. Lett. B 507 (2001) 214; W. Grimus and L. Lavoura, JHEP 0107 (2001) 045; T. Kitabayashi and M. Yasuè, Phys. Lett. B 524 (2002) 308; P.F. Harrison and W.G. Scott, Phys. Lett. B 547 (2002) 219; T. Ohlsson and G.Seidl Nucl. Phys. B 643 (2002) 247.

17) See for example, G. Altarelli and D. Meloni, NPB 809 (2009) 158.

18) K. Fuki and M. Yasuè, Phys. Rev. D 73 (2006) 055014; Nucl. Phys. B 783 (2007) 31.

19) T. Baba and M. Yasuè, Phys. Rev. D 77 (2008) 075008. See also Shao-Feng Ge, Hong-Jian He and Fu-Rong Yin, “Common Origin of Soft $\mu$-$\tau$ and CP Breaking in Neutrino Seesaw and the Origin of Matter”, [arXiv:1001.0940 [hep-ph]].

20) C. Jarlskog, Phys. Rev. Lett. 55 (1985) 1039.

21) For the case of non-diagonal charged lepton mass matrix, see A.S. Joshipura, B.P. Kodrani, K.M. Patel, Phys. Rev. D 79 (2009) 115017.

22) Riazuddin, JHEP 0310 (2003) 009.

23) See for example, S. Antusch, J. Kersten, M. Lindner and M. Ratz, NPB 674 (2003) 401; S. Antusch, J. Kersten, M. Lindner, M. Ratz and M.A. Schmidt, JHEP 0503 (2005) 024; R.N. Mohapatra, M.K. Parida and G. Rajasekaran, Phys. Rev. D 71 (2005) 057301; J.W. Mei and Z.Z. Xing, Phys. Rev. D 69 (2004) 073003; J.W. Mei, Phys. Rev. D 71 (2005) 073012; S. Luo, J.W. Mei and Z.Z. Xing, Phys. Rev. D 72 (2005) 053014; S. Luo and Z.Z. Xing, Phys. Lett. B 632 (2006) 341.

24) T. Baba and M. Yasuè, paper in preparation.
Figures

Fig. 1. Predictions of the Dirac phase and the Majorana phase as functions of $\sin^2 \theta_{13}$ for the normal mass hierarchy given by $M_{\nu}^{(C1)N}$.

Fig. 2. The same as in FIG.1 but for the inverted mass hierarchy with $m_1 \approx m_2$ given by $M_{\nu}^{(C1)I/A}$. 
Majorana CP Violation in Approximately $\mu$-$\tau$ Symmetric Models with ...
Fig. 6. The same as in FIG.1 but for the inverted mass hierarchy with $m_1 \approx m_2$ given by $M_\nu^{(C^2)1A}$. 

Fig. 7. The same as in FIG.1 but for the inverted mass hierarchy with $m_1 \approx m_2$ given by $M_\nu^{(C^2)1B}$. 