A NOTE ON GENERALIZED MYERS-TYPE THEOREMS FOR 
H-ALMOST RICCI TENSORS AND GENERALIZED 
QUASI-EINSTEIN TENSORS

SANGHUN LEE

Abstract. In this paper, we prove some compactness theorems of Myers, 
Ambrose, and Galloway for complete Riemannian manifold in the concept 
of $h$–almost Ricci tensors and generalized quasi–Einstein tensors. Also, we 
extend the previous theorems when $h$ has at most linear growth in the distance 
function.

1. Introduction

The concept of $h$-almost Ricci solitons is introduced by Gomes, Wang, and Xia [10]. This soliton is a natural extension of an almost Ricci soliton [3] [15]. An almost 
Ricci soliton is an $n$-dimensional Riemannian manifold $(M, g)$ with a vector field $V$ 
on $M$ and a soliton function $\lambda : M \to \mathbb{R}$ satisfying

\[ \text{Ric}_{V} = \lambda g. \]

Here,$$Ric_{V} := \text{Ric} + \frac{1}{2} \mathcal{L}_{V} g,$$where $\mathcal{L}$ denotes the Lie derivative.$\text{Ric}_{V}$ is called Bakry–Emery Ricci tensor and it is related to diffusion processes (see [2]). The Bakry–Emery Ricci tensor is 
studied in [2] and occurs naturally in many different subjects (cf. [8] [12] [13] [20]). Recently, Gomes, Wang, and Xia come up with the following definition, which is a 
generalization of the almost Ricci soliton.

Definition 1.1 ([10]). An $h$-almost Ricci soliton is an $n$-dimensional Riemannian 
manifold $(M, g)$ with a vector field $V$ on $M$ and a soliton function $\lambda : M \to \mathbb{R}$ and a function $h : M \to \mathbb{R}$ which are smooth and satisfy the equation:

\[ h \text{Ric}_{V} = \lambda g. \]

Here,$$h \text{Ric}_{V} := h \text{Ric} + \frac{h}{2} \mathcal{L}_{V} g.$$

2010 Mathematics Subject Classification. 53C20; 53C21.

Key words and phrases. $h$–almost Ricci tensor, Generalized quasi–Einstein tensor, Myers theorem, Diameter estimate, Riccati inequality.
We say that $Ric^h_V$ is an $h$–almost Ricci tensor. When $V = \nabla u$ for some smooth function $u : M \to \mathbb{R}$, we call this a gradient $h$–almost Ricci soliton with a potential function $u$. In this case, the equation (1.2) can be written as

$$Ric + h \text{Hess } u = \lambda g,$$

where Hess $u$ denotes the Hessian of $u$.

It should be mentioned that an $h$-almost Ricci soliton is expanding, steady, or shrinking if $\lambda$ is negative, zero, or positive, respectively, on $M$. When $\mathcal{L}_V g = cg$ for some constant $c$, an $h$-almost Ricci soliton is said to be trivial. Otherwise it is nontrivial. We remark that the traditional Ricci soliton is a 1-Ricci soliton with constant $\lambda$. Moreover, 1-almost Ricci soliton is just the almost Ricci soliton. It is said that $h$ has defined to be signal if either $h > 0$ on $M$ or $h < 0$ on $M$.

In [14] Maschler studied equation (1.4) and he referred to equation (1.4) as Ricci-Hessian equation. The Ricci-Hessian equation is related to a new class of Riemannian metrics, introduced by Catino [6], which are natural generalizations of Einstein metrics. In more detail, he called a generalized quasi-Einstein manifold, if there are smooth functions $f$, $\lambda$, $\mu$ on $M$ satisfying

$$Ric^\mu_f = \lambda g.$$  

Here,

$$Ric^\mu_f := Ric + Hess f - \mu df \otimes df$$

and we say that $Ric^\mu_f$ is a generalized quasi–Einstein tensor. When $\mu = \frac{1}{m}$, where $m$ is a positive integer, the above generalized quasi–Einstein manifold is called a generalized $m$-quasi–Einstein manifold (cf. [4]) and simply $m$-quasi–Einstein manifold when $\lambda$ is constant. It has been proved in [5, 11] that $m$-quasi–Einstein manifolds are directly related to the warped product Einstein manifolds.

The purpose of this paper is to investigate which geometric and topological results for manifolds with a lower bound on the Ricci tensor extend to smooth metric measure spaces with $h$–almost Ricci tensor or generalized quasi–Einstein tensor bounded below. Myers studied this problem in [15]. More precisely, Myers stated that if an $n$-dimensional complete Riemannian manifold $(M, g)$ satisfies $Ric \geq (n - 1)H$ with $H > 0$, then $M$ is compact and $\text{diam}(M) \leq \frac{\pi}{\sqrt{H}}$. Moreover, the fundamental group $\pi_1(M)$ is finite. This theorem has been widely generalized in various directions by many authors [1, 7, 8, 9, 12, 17, 18, 19, 20]. In [1], Ambrose first generalized. Ambrose replaced lower bound on the Ricci curvature with an integral condition on the Ricci curvature.

**Theorem 1.1** ([1]). Let $M$ be an $n$-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0, \infty) \to M$ emanating from $p$ satisfies

$$\int_0^\infty Ric(\gamma'(s), \gamma'(s)) ds = \infty.$$

Then $M$ is compact.

On the other hand, motivated by relativistic cosmology, Galloway [9] proved compactness theorem by perturbing the positive lower bound on the Ricci curvature by the derivative in the radial direction of some bounded function.
The following theorem generalizes original Myers theorem via $h$-almost Ricci tensor and generalized quasi-Einstein tensor. In this paper, we study Myers-type theorems for complete Riemannian manifolds in the context of the $h$-almost Ricci tensor. We prove the following compactness theorem:

**Theorem 1.2** (1.2). Let $M$ be an $n$-dimensional complete Riemannian manifold and $\gamma$ be a minimal geodesic joining two points of $M$. Assume that

$$\text{Ric}(\gamma', \gamma') \geq (n-1)H + \frac{d\phi}{dt}$$

holds along $\gamma$, where $H$ is a positive constant and $\phi$ is any smooth function satisfying $|\phi| \leq L$. Then $M$ is compact and its diameter is bounded from above by

$$\text{diam}(M) \leq \frac{\pi}{(n-1)H} (L + \sqrt{L^2 + (n-1)^2H}).$$

In this paper, we study Myers-type theorems for complete Riemannian manifolds. Suppose that there exist some constants $H > 0$ such that the $h$-almost Ricci tensor satisfies

(1.7) $$\text{Ric}_h(\gamma', \gamma') \geq (n-1)H,$$

$|V| \leq k_1,$ $|h| \leq k_2,$ and $|h'| \leq k_3$ for some constants $k_1 k_3 < (n-1)H$, $k_2 > 0$. Then $M$ is compact and

$$\text{diam}(M) \leq \frac{1}{((n-1)H - k_1 k_3)} \left(2k_1 k_2 + \sqrt{4k_1^2 k_2^2 + ((n-1)H - k_1 k_3)((n-1)^2)}\right).$$

As to the theorem via $h$-almost Ricci tensor, we prove the following compactness theorem:

**Theorem 1.4.** Let $M$ be an $n$-dimensional complete Riemannian manifold. Suppose that there exist some constants $H > 0$ and $L \geq 0$ such that for every pair of points in $M$ and minimal geodesic $\gamma$ joining those points, the $h$-almost Ricci tensor satisfies

(1.8) $$\text{Ric}_h(\gamma', \gamma') \geq (n-1)H + \frac{d\phi}{dt},$$

where $\phi$ is some smooth function of the arc length satisfying $|\phi| \leq L$ along $\gamma$. If the vector field $V$ and a smooth function $h$ satisfy $|V| \leq k_1$, $|h| \leq k_2$, and $|h'| \leq k_3$ for some constants $k_1 k_3 < (n-1)H$, $k_2 > 0$, then $M$ is compact and

$$\text{diam}(M) \leq \frac{2(L + k_1 k_2)}{((n-1)H - k_1 k_3)} + \frac{\sqrt{4(L + k_1 k_2)^2 + ((n-1)H - k_1 k_3)((n-1)^2)}}{((n-1)H - k_1 k_3)}.$$

**Remark 1.5.** By taking $L = 0$, theorem is reduced to the Myers theorem via $h$-almost Ricci tensor (theorem 1.3) with the diameter estimate

$$\text{diam}(M) \leq \frac{1}{((n-1)H - k_1 k_3)} \left(2k_1 k_2 + \sqrt{4k_1^2 k_2^2 + ((n-1)H - k_1 k_3)((n-1)^2)}\right).$$

Moreover, we prove the Myers theorem of a complete Riemannian manifold with a lower bound on the $h$-almost Ricci tensor under the condition that the smooth function $h$ in the $h$-almost Ricci tensor has at most linear growth in the distance function.
**Theorem 1.6.** Let $M$ be an $n$-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $H > 0$ such that the $h$–almost Ricci tensor satisfies
\[
\text{Ric}^h V (\gamma', \gamma') \geq (n - 1)H,
\]
where $|V| \leq k_1$, $|h| \leq k_2(d(x, p) + 1)$, and $|h'| \leq k_2$ for some constant $k_1k_2 < \frac{(n-1)H}{2}$, where $d(x, p)$ is the distance function from some fixed $p$ to $x$. Then $M$ is compact.

Furthermore, we prove the Myers-type theorem for a complete Riemannian manifold when $h$ has at most linear growth.

**Theorem 1.7.** Let $M$ be an $n$-dimensional complete Riemannian manifold. Suppose that there exist some constants $H > 0$ and $L \geq 0$ such that for every pair of points in $M$ and minimal geodesic $\gamma$ joining those points, the $h$–almost Ricci tensor satisfies
\[
\text{Ric}^h V (\gamma', \gamma') \geq (n - 1)H + \frac{d\phi}{dt},
\]
where $\phi$ is some smooth function of the arc length satisfying $\phi \geq -L$ along $\gamma$. If the vector field $V$ and a smooth function $h$ satisfy $|V| \leq k_1$, $|h| \leq k_2(d(x, p) + 1)$, and $|h'| \leq k_2$ for some constant $k_1k_2 < \frac{(n-1)H}{2}$, where $d(x, p)$ is the distance function from some fixed $p$ to $x$, then $M$ is compact.

Now we consider generalized quasi–Einstein tensor. First, we study original Myers theorem via generalized quasi–Einstein tensor.

**Theorem 1.8.** Let $M$ be an $n$-dimensional complete Riemannian manifold. Suppose that there exists some positive constant $H > 0$ such that a generalized quasi–Einstein tensor satisfies
\[
\text{Ric}^\mu (\gamma', \gamma') \geq (n - 1)H,
\]
where $\mu \geq \frac{1}{k_4}$ for some positive constant $k_4$. Then $M$ is compact and
\[
\text{diam}(M) \leq \frac{\pi}{(n - 1)H} \sqrt{(n + k_4 - 1)(n - 1)H}.
\]

Secondly, we prove the following compactness theorem for generalized quasi–Einstein tensor which generalizes theorem 1.2.

**Theorem 1.9.** Let $M$ be an $n$-dimensional complete Riemannian manifold. Suppose that there exist some constants $H > 0$ and $L \geq 0$ such that for every pair of points in $M$ and minimal geodesic $\gamma$ joining those points, a generalized quasi–Einstein tensor satisfies
\[
\text{Ric}^\mu (\gamma', \gamma') \geq (n - 1)H + \frac{d\phi}{dt},
\]
where $\phi$ is some smooth function of the arc length satisfying $|\phi| \leq L$ along $\gamma$. If a smooth function $\mu$ satisfies $\mu \geq \frac{1}{k_4}$ for some positive constant $k_4$, then $M$ is compact and
\[
\text{diam}(M) \leq \frac{1}{(n - 1)H} \left(2L + \sqrt{4L^2 + (n - 1)H(n + k_4 - 1)\pi^2}\right).
\]

Finally, we generalize Theorem 1.1 to generalized quasi–Einstein tensors.
Theorem 1.10. Let $M$ be an $n$-dimensional complete Riemannian manifold. Suppose that there exists some point $p \in M$ for which every geodesic $\gamma : [0, \infty) \to M$ emanating from $p$ satisfies

$$\int_0^\infty \text{Ric}^\mu_f(\gamma'(s), \gamma'(s))ds = \infty,$$

where $\mu \geq \frac{1}{k_4}$ for some positive constant $k_4$. Then $M$ is compact.

This paper is organized as follows: In Section 2 and Section 3, we prove Myers-type theorems for $h$–almost Ricci tensor. In Section 4, we study Myers-type theorems for generalized quasi–Einstein tensor.

2. When $h$ is bounded to a constant

In this section, we prove Theorem 1.3 and Theorem 1.4. First, we prove Theorem 1.3. Let $p, q \in M$ and $\gamma$ be a minimizing unit speed geodesic segment from $p$ to $q$ of length $\ell$. Consider a parallel orthonormal frame $\{E_1 = E_1, E_2, \ldots, E_n = \gamma'\}$ along $\gamma$ and a smooth function $b \in C^\infty([0, \ell])$ such that $b(0) = b(\ell) = 0$. By the index form, we have

$$\sum_{i=1}^{n-1} I(bE_i, bE_i) = \int_0^\ell (n-1)(b')^2 - b^2 \text{Ric}(\gamma', \gamma')dt,$$

where $I$ denotes the index form of $\gamma$. Using the assumption (1.7) and definition of $h$–almost Ricci tensor, we get

$$\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq (n-1) \int_0^\ell (b')^2 dt - (n-1)H \int_0^\ell b^2 dt + \int_0^\ell \frac{b^2 h}{2} \mathcal{L}_V g(\gamma', \gamma') dt.$$

Note that

$$\int_0^\ell \frac{b^2 h}{2} \mathcal{L}_V g(\gamma', \gamma') dt = \int_0^\ell b^2 h \frac{d}{dt} \langle V, \gamma' \rangle dt$$

$$= \int_0^\ell \frac{d}{dt} (b^2 h \langle V, \gamma' \rangle) dt - 2 \int_0^\ell b' h \langle V, \gamma' \rangle dt - \int_0^\ell b^2 h' \langle V, \gamma' \rangle dt$$

$$\leq 2 \int_0^\ell |bb'||h| \langle V, \gamma' \rangle | dt + \int_0^\ell |b^2|h'| \langle V, \gamma' \rangle | dt$$

$$\leq 2k_1 k_2 \int_0^\ell |bb'| dt + k_1 k_3 \int_0^\ell |b^2| dt.$$

So we have

$$\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq (n-1) \int_0^\ell (b')^2 dt - (n-1)H \int_0^\ell b^2 dt + 2k_1 k_2 \int_0^\ell |bb'| dt$$

$$+ k_1 k_3 \int_0^\ell |b^2| dt.$$
By setting the function \( b \) to be \( b(t) = \sin\left(\frac{\pi t}{\ell}\right) \), we have \( b'(t) = \frac{\pi}{\ell} \cos\left(\frac{\pi t}{\ell}\right) \). Thus,

\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq \frac{(n-1)\pi^2}{\ell^2} \int_0^\ell \cos^2\left(\frac{\pi t}{\ell}\right) dt - (n-1)H \int_0^\ell \sin^2\left(\frac{\pi t}{\ell}\right) dt \\
+ \frac{k_1k_2\pi}{\ell} \int_0^\ell \sin\left(\frac{2\pi t}{\ell}\right) dt + k_1k_3 \int_0^\ell \sin^2\left(\frac{\pi t}{\ell}\right) dt \\
\leq \frac{(n-1)\pi^2}{2\ell} - \frac{(n-1)H\ell}{2} + 2k_1k_2 + \frac{k_1k_3\ell}{2} \\
= -\frac{1}{2\ell} ((n-1)H - k_1k_3)\ell^2 - 4k_1k_2\ell - (n-1)^2. 
\]

Since \( \gamma \) is a minimizing geodesic, we must take

\[
((n-1)H - k_1k_3)\ell^2 - 4k_1k_2\ell - (n-1)^2 \leq 0.
\]

If \( k_1k_3 < (n-1)H \), then \( (n-1)H - k_1k_3 > 0 \), so we have

\[
\ell \leq \frac{1}{(n-1)H - k_1k_3} \left( 2k_1k_2 + \sqrt{4k_1^2k_3^2 + ((n-1)H - k_1k_3)((n-1)^2)} \right).
\]

Hence, \( M \) is compact and

\[
diam(M) \leq \frac{1}{(n-1)H - k_1k_3} \left( 2k_1k_2 + \sqrt{4k_1^2k_3^2 + ((n-1)H - k_1k_3)((n-1)^2)} \right).
\]

This completes the proof. \( \square \)

Second, we prove Theorem 1.4. Its proof is similar to the previous proof; thus, the setting is identical. The index form implies

\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) = \int_0^\ell (n-1)(b')^2 - b^2 Ric(\gamma', \gamma')dt.
\]

From (1.8), we have

\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq (n-1) \int_0^\ell (b')^2 dt - (n-1)H \int_0^\ell b^2 dt - \int_0^\ell b^2 \frac{d\phi}{dt} dt \\
+ \int_0^\ell \frac{1}{2} b^2 h\mathcal{L}_V g(\gamma', \gamma') dt.
\]

Note that

\[
\int_0^\ell \frac{1}{2} b^2 h\mathcal{L}_V g(\gamma', \gamma') dt \leq 2k_1k_2 \int_0^\ell |bb'| dt + k_1k_3 \int_0^\ell |b^2| dt
\]

and

\[
- \int_0^\ell b^2 \frac{d\phi}{dt} dt = - \left( \int_0^\ell \frac{d}{dt}(b^2\phi) dt - 2 \int_0^\ell bb'\phi dt \right) \\
\leq 2 \int_0^\ell |bb'| |\phi| dt \leq 2L \int_0^\ell |bb'| dt.
\]
Thus,
\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq (n - 1) \int_0^\ell (b')^2 dt - (n - 1)H \int_0^\ell b^2 dt + 2L \int_0^\ell |bb'| dt + 2k_1k_2 \int_0^\ell |bb'| dt + k_1k_3 \int_0^\ell |b^2| dt.
\]

If the function $b$ is taken to be $b(t) = \sin(\frac{n\pi}{\ell})$, then we obtain
\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq \frac{(n - 1)\pi^2}{\ell^2} \int_0^\ell \cos^2(\frac{n\pi t}{\ell}) dt - (n - 1)H \int_0^\ell \sin^2(\frac{n\pi t}{\ell}) dt + \frac{\pi(L + k_1k_2)}{\ell} \int_0^\ell \sin(\frac{2\pi t}{\ell}) dt + k_1k_3 \int_0^\ell \sin^2(\frac{\pi t}{\ell}) dt
\]
\[
= \frac{(n - 1)\pi^2}{2\ell} - \frac{(n - 1)H\ell}{2} + 2(L + k_1k_2) + \frac{k_1k_3\ell}{2}
\]
\[
= -\frac{1}{2\ell} ((n - 1)H - k_1k_3)\ell^2 - 4\ell(L + k_1k_2) - (n - 1)\pi^2.
\]

Since $\gamma$ is a minimizing geodesic, we must take
\[
((n - 1)H - k_1k_3)\ell^2 - 4\ell(L + k_1k_2) - (n - 1)\pi^2 \leq 0.
\]

If $k_1k_3 < (n - 1)H$, then $(n - 1)H - k_1k_3 > 0$, so we get
\[
\ell \leq \frac{2(L + k_1k_2)}{(n - 1)H - k_1k_3} + \frac{\sqrt{4(L + k_1k_2)^2 + ((n - 1)H - k_1k_3)((n - 1)\pi^2)}}{(n - 1)H - k_1k_3}.
\]

This proves Theorem 1.7

\[\Box\]

3. When $h$ has at most linear growth

In this section, we prove Myers-type theorems (Theorem 1.6 and 1.7) under the condition that $h$ has at most linear growth. Our proofs use the Riccati inequality.

First, we prove Theorem 1.6. Suppose that $M$ is non-compact. Fix a point $p \in M$, there exists a unit speed ray $\gamma(t)$ starting from $p$ satisfying $\gamma(0) = p$. For every $t > 0$, the mean curvature function $m(t)$ defined by $m(x) = \Delta r(x)$, where $r(x) = d(p, x)$ satisfies the Riccati inequality:
\[
Ric(\gamma', \gamma') \leq -m'(t) - \frac{1}{n-1}m^2(t).
\]
By adding \( \frac{h(t)}{2} \mathcal{L}_V g(\gamma', \gamma') \) to both sides of this inequality, we have
\[
\text{Ric}(\gamma'(t), \gamma'(t)) + \frac{h(t)}{2} \mathcal{L}_V g(\gamma'(t), \gamma'(t)) \leq -m'(t) - \frac{1}{n-1} m^2(t) + \frac{h(t)}{2} \mathcal{L}_V g(\gamma'(t), \gamma'(t))
\]

From (1.19) and definition of \( \text{Ric}_V^b \), we obtain
\[
(n - 1)H \leq -m'(t) - \frac{1}{n-1} m^2(t) + \frac{h(t)}{2} \mathcal{L}_V g(\gamma'(t), \gamma'(t)).
\]

Integrating both sides of (3.1), we have
\[
\int_1^t (n - 1)H \, ds \leq -m(t) + m(1) - \frac{1}{n-1} \int_1^t m^2(s) \, ds + \frac{h(t)}{2} \int_1^t \mathcal{L}_V g(\gamma'(s), \gamma'(s)) \, ds.
\]

Note that
\[
\int_1^t \frac{h(s)}{2} \mathcal{L}_V g(\gamma'(s), \gamma'(s)) \, ds = \int_1^t \frac{h(s) \, dV}{ds} \langle V, \gamma'(s) \rangle \, ds
\]
\[
= \int_1^t \frac{d}{ds} (h(s) \langle V, \gamma'(s) \rangle) \, ds - \int_1^t h'(s) \langle V, \gamma'(s) \rangle \, ds
\]
\[
= h(t) \langle V, \gamma'(t) \rangle - h(1) \langle V, \gamma'(1) \rangle - \int_1^t h'(s) \langle V, \gamma'(s) \rangle \, ds.
\]

So we get
\[
(n - 1)Ht - (n - 1)H \leq -m(t) + m(1) - \frac{1}{n-1} \int_1^t m^2(s) \, ds + h(t) \langle V, \gamma'(t) \rangle
\]
\[
- h(1) \langle V, \gamma'(1) \rangle - \int_1^t h'(s) \langle V, \gamma'(s) \rangle \, ds
\]
\[
\leq -m(t) + m(1) - \frac{1}{n-1} \int_1^t m^2(s) \, ds + |h(t)| \langle V, \gamma'(t) \rangle
\]
\[
- h(1) \langle V, \gamma'(1) \rangle + \int_1^t |h'(s)| \langle V, \gamma'(s) \rangle \, ds
\]
\[
\leq -m(t) + m(1) - \frac{1}{n-1} \int_1^t m^2(s) \, ds + 2k_1k_2t - h(1) \langle V, \gamma'(1) \rangle.
\]

It follows that
\[
-m(t) - \frac{1}{n-1} \int_1^t m^2(s) \, ds \geq ((n - 1)H - 2k_1k_2)t + C_1,
\]
where \( C_1 := +h(1) \langle V, \gamma'(1) \rangle - (n - 1)H - m(1) \). Since \( k_1k_2 < \frac{(n-1)H}{2} \), there exists \( t_1 > 1 \) such that for all \( t \geq t_1 \), we have
\[
-m(t) - \frac{1}{n-1} \int_1^t m^2(s) \, ds \geq 2.
\]

Now we consider the increasing sequence \( \{t_\ell\} \) defined by
\[
t_{\ell+1} = t_\ell + (n - 1)2^{1-\ell}, \quad \text{for } \ell \geq 1.
\]
Note that \( \{t_\ell\} \) converges to \( T := t_1 + 2(n - 1) \) as \( \ell \to \infty \).

We claim that \( -m(t) \geq 2^\ell \) for all \( t \geq t_\ell \). To prove this claim, we use induction. If \( \ell = 1 \), the claim is trivially true from the inequality in (3.2). Now, for all \( t \geq t_{\ell+1} \), we have

\[
-m(t) \geq 2 + \frac{1}{n-1} \int_3^t m^2(s) \, ds \\
\geq \frac{1}{n-1} \int_{t_\ell}^{t_{\ell+1}} m^2(s) \, ds \\
\geq \frac{1}{n-1} 2^{2\ell}(t_{\ell+1} - t_\ell) = 2^{\ell+1}.
\]

Hence, the claim is true for all \( t \geq t_{\ell+1} \). Therefore,

\[
\lim_{\ell \to \infty} -m(t_\ell) = -m(T) \geq \lim_{\ell \to \infty} 2^{\ell+1}.
\]

This contradicts the smoothness of \( m(t) \), which completes the proof of Theorem 1.6.

Second, we prove Theorem 1.7. Setting is the same as the above proof. We have

\[
\text{Ric}(\gamma'(t), \gamma'(t)) + \frac{h(t)}{2} \mathcal{L}_V g(\gamma'(t), \gamma'(t)) \leq -m'(t) - \frac{1}{n-1} m^2(t) + h(t) \mathcal{L}_V g(\gamma'(t), \gamma'(t))
\]

From (1.10), we obtain

\[
(n-1)H + \frac{d\phi}{dt} \leq -m'(t) - \frac{1}{n-1} m^2(t) + \frac{h(t)}{2} \mathcal{L}_V g(\gamma'(t), \gamma'(t)).
\]

Integrating both sides of (3.3) from 1 to \( t \), we get

\[
(n-1)Ht - (n-1)H + \phi(t) - \phi(1) \leq -m(t) + m(1) - \frac{1}{n-1} \int_1^t m^2(s) \, ds \\
+ \int_1^t \frac{h(s)}{2} \mathcal{L}_V g(\gamma'(s), \gamma'(s)) \, ds.
\]

Note that

\[
\int_1^t \frac{h(s)}{2} \mathcal{L}_V g(\gamma'(s), \gamma'(s)) \, ds = \int_1^t h(s) \frac{d}{ds} \langle V, \gamma'(s) \rangle \, ds \\
= h(t)\langle V, \gamma'(t) \rangle - h(1)\langle V, \gamma'(1) \rangle - \int_1^t h'(s)\langle V, \gamma'(s) \rangle \, ds \\
\leq +|h(t)||\langle V, \gamma'(t) \rangle| - h(1)\langle V, \gamma'(1) \rangle + \int_1^t |h'(s)||\langle V, \gamma'(s) \rangle| \, ds \\
\leq 2k_1k_2t - h(1)\langle V, \gamma'(1) \rangle.
\]

If \( \phi \geq -L \), then we have

\[
(n-1)Ht - (n-1)H - L - \phi(1) \leq -m(t) + m(1) - \frac{1}{n-1} \int_1^t m^2(s) \, ds \\
+ 2k_1k_2t - h(1)\langle V, \gamma'(1) \rangle.
\]
Thus,
\[-m(t) - \frac{1}{n-1} \int_1^t m^2(s) \, ds \geq (n - 1)Ht - (n - 1)H - L - \phi(1) - m(1) - 2k_1k_2t + h(1)(V, \gamma')(1).\]

Let \(C_2 := -(n - 1)H - L - \phi(1) + h(1)(V, \gamma')(1)\). Then we obtain
\[-m(t) - \frac{1}{n-1} \int_1^t m^2(s) \, ds \geq (n - 1)H - 2k_1k_2t + C_2.\]

Since \(k_1k_2 < \frac{(n-1)H}{2}\), the above inequality implies that there exists \(t_1 > 1\) such that for all \(t \geq t_1\), we get
\[-m(t) - \frac{1}{n-1} \int_1^t m^2(s) \, ds \geq 2.\]

Now we can complete this proof by using the same argument as in the proof of Theorem 1.6.

\[\Box\]

4. Generalized Quasi–Einstein Tensor

In this section, we prove Theorems 1.8, 1.9, and 1.10. First, we prove Theorem 1.8. The setting is identical with proof of Theorem 1.3. By the index form, we have
\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) = \int_0^t (n - 1)(b')^2 - b^2Ric(\gamma', \gamma') \, dt.
\]

By the assumption (1.11) and definition of generalized quasi–Einstein tensor (1.6), we obtain
\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq (n - 1)\int_0^t (b')^2 \, dt - (n - 1)H\int_0^t b^2 \, dt + b^2 Hess f(\gamma', \gamma') \, dt - \int_0^t b^2 \mu \, d\langle \gamma', \gamma' \rangle \, dt.
\]

Note that
\[
\int_0^t b^2 Hess f(\gamma', \gamma') \, dt = \int_0^t b^2 \frac{d}{dt} \langle \nabla f, \gamma' \rangle(t) \, dt
= \int_0^t \frac{d}{dt} (b^2 \langle \nabla f, \gamma' \rangle(t)) \, dt - 2 \int_0^t b\langle \nabla f, \gamma' \rangle(t) \, dt
\leq 2 \int_0^t |b'| |b \langle \nabla f, \gamma' \rangle(t)\, dt
\leq 2 \left( \int_0^t \frac{1}{\mu} (b')^2 \, dt \right)^{1/2} \left( \int_0^t \mu (b \langle \nabla f, \gamma' \rangle(t))^2 \, dt \right)^{1/2}
\leq \int_0^t \frac{1}{\mu} (b')^2 \, dt + \int_0^t \mu (b \langle \nabla f, \gamma' \rangle(t))^2 \, dt.
\]
So we have
\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq (n-1) \int_0^\ell (b')^2 dt - (n-1)H \int_0^\ell b^2 dt + \int_0^\ell \frac{1}{\mu} (b')^2 dt
\]
\[
\leq (n-1) \int_0^\ell (b')^2 dt - (n-1)H \int_0^\ell b^2 dt + k_4 \int_0^\ell (b')^2 dt.
\]
If \( b(t) = \sin(\frac{t}{2}) \), then we obtain
\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq \frac{\pi^2(n-1)}{2\ell} - \frac{(n-1)H\ell}{2} + \frac{\pi^2 k_4}{2\ell}
\]
\[
\leq -\frac{1}{2\ell} ((n-1)H\ell^2 - \pi^2(n + k_4 - 1)) .
\]
We must take
\[ (n-1)H\ell^2 - \pi^2(n + k_4 - 1) \leq 0. \]
Hence, we get
\[
\ell \leq \frac{\pi\sqrt{(n-1)H(n+k_4-1)}}{(n-1)H}.
\]
So we complete the proof of Theorem 1.8.

Now we prove Theorem 1.9. Setting is the same as the above proof. From the index form, generalized quasi–Einstein tensor (1.6), and the assumption (1.12), we have
\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq (n-1) \int_0^\ell (b')^2 dt - (n-1)H \int_0^\ell b^2 dt - \int_0^\ell b^2 \frac{d\phi}{dt} dt
\]
\[+ \int_0^\ell b^2 \text{Hess} f(\gamma', \gamma') dt - \int_0^\ell b^2 \mu dt \otimes df(\gamma', \gamma') dt. \]
Note that
\[
\int_0^\ell b^2 \text{Hess} f(\gamma', \gamma') dt \leq \int_0^\ell \frac{1}{\mu} (b')^2 dt + \int_0^\ell \mu (b(\nabla f, \gamma')(t))^2 dt,
\]
and
\[- \int_0^\ell b^2 \frac{d\phi}{dt} dt \leq 2L \int_0^\ell |bb'| dt. \]
It follows that
\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq (n-1) \int_0^\ell (b')^2 dt - (n-1)H \int_0^\ell b^2 dt + 2L \int_0^\ell |bb'| dt + \int_0^\ell \frac{1}{\mu} (b')^2 dt
\]
\[\leq (n-1) \int_0^\ell (b')^2 dt - (n-1)H \int_0^\ell b^2 dt + 2L \int_0^\ell |bb'| dt + k_4 \int_0^\ell (b')^2 dt.
\]
If \( b(t) = \sin(\frac{t}{2}) \), then we have
\[
\sum_{i=1}^{n-1} I(bE_i, bE_i) \leq -\frac{1}{2\ell} ((n-1)H\ell^2 - 4L\ell - \pi^2(n + k_4 - 1)) .
\]
So we must take
\[ (n-1)H\ell^2 - 4L\ell - \pi^2(n + k_4 - 1) \leq 0. \]
Therefore, we obtain
\[
\ell \leq \frac{1}{(n-1)H} \left( 2L + \sqrt{4L^2 + \pi^2(n + k_4 - 1)(n - 1)H} \right).
\]
This proves Theorem 1.9. \hfill \Box

Finally, we prove Theorem 1.10. Before the proof, we fix several notation. Let \((M, g, e^{-f}dv_g)\) be a smooth metric measure space on an \(n\)-dimensional complete Riemannian manifold \(M\). For the measure \(e^{-f}dv_g\), the \(f\)-mean curvature is \(m_f = m - \partial_r f\), where \(m\) is the mean curvature of the geodesic sphere with inward pointing normal vector. Then the \(f\)-Laplacian is defined by \(\Delta_f := \Delta - \langle \nabla f, \nabla \rangle\). Note that \(m_f = \Delta_f(r)\) and \(m = \Delta(r)\), where \(r\) is the distance function.

Fix a point \(p \in M\) and take a unit speed ray \(\gamma = \gamma(s)\) emanating from \(p\) satisfying \(\gamma(0) = p\). Let \(r(x) = d(x, p)\) be the distance between \(x\) and \(p\). By Bochner formula and Schwarz inequality, the distance function \(r\) satisfies the Riccati inequality
\[
\frac{\partial}{\partial r}(\Delta_f(r)) \leq -\frac{1}{n-1}(\Delta_f(r))^2 - \text{Ric}(\nabla r, \nabla r).
\]
We know that \(\Delta_f(r) = \Delta r - \langle \nabla f, \nabla r \rangle\). So we have
\[
\frac{\partial}{\partial r}(\Delta_f(r)) \leq -\frac{1}{n-1}(\Delta_f(r))^2 + \frac{\langle \nabla f, \nabla r \rangle^2}{(n-1)\alpha} - \text{Ric}(\nabla r, \nabla r) - \text{Hess}_r(\nabla r, \nabla r).
\]
Recall the elementary inequality
\[
(a + b)^2 \geq \frac{1}{\alpha + 1} a^2 - \frac{1}{\alpha} b^2
\]
for \(\alpha > 0\). By (4.1), we obtain
\[
\frac{\partial}{\partial r}(\Delta_f(r)) \leq -\frac{\Delta_f(r)^2}{(n-1)(\alpha + 1)} + \frac{\langle \nabla f, \nabla r \rangle^2}{(n-1)\alpha} - \text{Ric}(\nabla r, \nabla r) - \text{Hess}_r(\nabla r, \nabla r).
\]
Let \((n-1)\alpha = k_4\). Then we have
\[
\frac{\partial}{\partial r}(\Delta_f(r)) \leq -\frac{(\Delta_f(r))^2}{k_4 + n - 1} + \frac{\langle \nabla f, \nabla r \rangle^2}{k_4} - \text{Ric}(\nabla r, \nabla r) - \text{Hess}_r(\nabla r, \nabla r)
\]
\[
\leq -\frac{(\Delta_f(r))^2}{k_4 + n - 1} + \mu \langle \nabla f, \nabla r \rangle^2 - \text{Ric}(\nabla r, \nabla r) - \text{Hess}_r(\nabla r, \nabla r)
\]
\[
= -\frac{(\Delta_f(r))^2}{k_4 + n - 1} - \text{Ric}_f^\mu (\nabla r, \nabla r).
\]
It follows that
\[
\text{Ric}_f^\mu (\gamma', \gamma') \leq -m_f'(s) - \frac{m_f^2(s)}{k_4 + n - 1},
\]
where \(m_f(s) = (\Delta_f r)(\gamma(s))\).

Integrating both sides of (4.2), we have
\[
\lim_{t \to \infty} \int_1^t \text{Ric}_f^\mu (\gamma'(s), \gamma'(s))ds \leq \lim_{t \to \infty} \int_1^t \left( -m_f'(s) - \frac{m_f^2(s)}{k_4 + n - 1} \right) ds.
\]
If
\[
\lim_{t \to \infty} \int_1^t \text{Ric}_f^\mu (\gamma'(s), \gamma'(s))ds = \infty,
\]
then we obtain
\[
\lim_{t \to \infty} \left( -m_f(t) - \frac{1}{k_4 + n - 1} \int_1^t m_f^2(s) ds \right) = \infty.
\]
So there exists \( t_1 > 1 \) such that for all \( t \geq t_1 \), we get
\[
-m_f(t) - \frac{1}{k_4 + n - 1} \int_1^t m_f^2(s) ds > 2.
\]
Using the same argument as in the proof of Theorem 1.6, we may complete the proof.

\[
\square
\]

References

[1] W. Ambrose, A Theorem of Myers, Duke Math. J. 24 (1957), 345–348.
[2] D. Bakry and M. Emery, Diffusions hypercontractives, In: Seminarie de probabilites XIX, 1983/1984. Lecture Notes in Math., Springer, Berlin 1123 (1985), 177-206.
[3] A. Barros and E. Ribeiro, Jr., Some characterizations for compact almost Ricci solitons, Proc. Amer. Math. Soc. 140 (2012), 1033–1040.
[4] A. Barros and E. Ribeiro, Jr., Characterizations and integral formulæ for generalized m-quasi-Einstein metrics, Bull. Braz. Math. Soc. (N.S.) 45 (2014), 325–341.
[5] J. Case, Y. Shu, G. Wei, Rigity of quasi-Einstein metrics, Differ. Geom. Appl. 29 (2011), 93-100.
[6] G. Catino, Generalized quasi-Einstein manifolds with harmonic weyl tensor, Math. Z. 271 (2012), 751–756.
[7] J. Cheeger, M. Gromov, M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, J. Differ. Geom. 17 (1982), 15–53.
[8] M. Fernandez-Lopez and E. Garcia-Rio, A remark on compact Ricci solitons, Math. Ann. 340 (2008), 893-896.
[9] G. J. Galloway, A Generalization of Myers Theorem and an application to relativistic cosmology, J. Differ. Geom. 14 (1979), 105–116.
[10] J. N. Gomes, Q. Wang, and C. Xia On the h-almost Ricci soliton, J. Geom. Phys. 114 (2017), 216–222.
[11] D. -S. Kim, Y. H. Kim Compact Einstein warped product spaces with nonpositive scalar curvature, Proc. Amer. Math. Soc. 131 (2003), 2573-2576.
[12] M. Limoncu, The Bakry–Emery Ricci tensor and its applications to some compactness theorems, Math. Z. 271 (2012) 715–722.
[13] J. Lott, Some geometric properties of the Bakry–Emery Ricci tensor, Comment. Math. Helv. 78 (2003), 865-883.
[14] G. Maschler, Special Kahler-Ricci potentials and Ricci solitons, Ann. Glob. Anal. Geom. 34 (2008), 367-380.
[15] S. B. Myers, Riemannian Manifold with Positive mean curvature, Duke Math. J. 8 (1941), 401–404.
[16] S. Pigola, M. Rigoli, M. Rimoldi, A. Setti, Ricci almost solitons, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 10 (2011), 757–799.
[17] C. Sprouse, Integral Curvature Bounds and Bounded Diameter, Commun. Anal. Geom. 8 (2000), 531–543.
[18] H. Tadano Some Ambrose- and Galloway-type theorems via Bakry-Emery and modified Ricci curvatures, Pac. J. Math. 294 (2018), 213-232.
[19] L. F. Wang Diameter estimate for compact quasi–Einstein metrics, Math. Z. 273 (2013), 801-809.
[20] G. Wei and W. Wylie Comparison geometry for the Bakry–Emery Ricci tensor, J. Differ. Geom. 83 (2009), 377-405.
Department of Mathematics, Chung-Ang University, 84 Heukseok-ro, Dongjak-gu, Seoul, Republic of Korea

Email address: kazauye@cau.ac.kr