Sharp minimax tests for large Toeplitz covariance matrices with repeated observations

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June 5, 2015

Abstract

We observe a sample of $n$ independent $p$-dimensional Gaussian vectors with Toeplitz covariance matrix $\Sigma = [\sigma_{|i-j|}]_{1 \leq i,j \leq p}$ and $\sigma_0 = 1$. We consider the problem of testing the hypothesis that $\Sigma$ is the identity matrix asymptotically when $n \to \infty$ and $p \to \infty$. We suppose that the covariances $\sigma_k$ decrease either polynomially ($\sum_{k \geq 1} k^{2\alpha} \sigma_k^2 \leq L$ for $\alpha > 1/4$ and $L > 0$) or exponentially ($\sum_{k \geq 1} e^{2Ak} \sigma_k^2 \leq L$ for $A, L > 0$).

We consider a test procedure based on a weighted U-statistic of order 2, with optimal weights chosen as solution of an extremal problem. We give the asymptotic normality of the test statistic under the null hypothesis for fixed $n$ and $p \to +\infty$ and the asymptotic behavior of the type I error probability of our test procedure. We also show that the maximal type II error probability, either tend to 0, or is bounded from above. In the latter case, the upper bound is given using the asymptotic normality of our test statistic under alternatives close to the separation boundary. Our assumptions imply mild conditions: $n = o(p^{2\alpha-1/2})$ (in the polynomial case), $n = o(e^p)$ (in the exponential case).

We prove both rate optimality and sharp optimality of our results, for $\alpha > 1$ in the polynomial case and for any $A > 0$ in the exponential case.

A simulation study illustrates the good behavior of our procedure, in particular for small $n$, large $p$.

\textbf{Key Words:} Toeplitz matrix, covariance matrix, high-dimensional data, U-statistic, minimax hypothesis testing, optimal separation rates, sharp asymptotic rates.

\textbf{MSC 2000:} 62G10, 62H15, 62G20, 62H10
1 Introduction

In the last decade, both functional data analysis (FDA) and high-dimensional (HD) problems have known an unprecedented expansion both from a theoretical point of view (as they offer many mathematical challenges) and for the applications (where data have complex structure and grow larger every day). Therefore, both areas share a large number of trends, see [12] and the review by [11], like regression models with functional or large-dimensional covariates, supervised or unsupervised classification, testing procedures, covariance operators.

Functional data analysis proceeds very often by discretizing curve datasets in time domain or by projecting on suitable orthonormal systems and produces large dimensional vectors with size possibly larger than the sample size. Hence methods and techniques from HD problems can be successfully implemented (see e.g. [1]). However, in some cases, HD vectors can be transformed into stochastic processes, see [8], and then techniques from FDA bring new insights into HD problems. Our work is of the former type.

We observe independent, identically distributed Gaussian vectors $X_1, ..., X_n$, $n \geq 2$, which are $p$-dimensional, centered and with a positive definite Toeplitz covariance matrix $\Sigma$. We denote by $X_k = (X_{k,1}, ..., X_{k,p})^\top$ the coordinates of the vector $X_k$ in $\mathbb{R}^p$ for all $k$.

Our model is that of a stationary Gaussian time series, repeatedly and independently observed $n$ times, for $n \geq 2$. We assume that $n$ and $p$ are large. In functional data analysis, it is quite often that curves are observed in an independent way: electrocardiograms of different patients, power supply for different households and so on, see other data sets in [12]. After modelisation of the discretized curves, the statistician will study the normality and the whiteness of the residuals in order to validate the model. Our problem is to test from independent samples of high-dimensional residual vectors that the standardized Gaussian coordinates are uncorrelated.

Let us denote by $\sigma_{|j|} = \text{Cov}(X_{k,h}, X_{k,h+j})$, for all integer numbers $h$ and $j$, for all $k \in \mathbb{N}^*$, where $\mathbb{N}^*$ is the set of positive integers. We assume that $\sigma_0 = 1$, therefore $\sigma_j$ are correlation coefficients. We recall that $\{\sigma_j\}_{j \in \mathbb{N}}$ is a sequence of non-negative type, or, equivalently, the associated Toeplitz matrix $\Sigma$ is non-negative definite. We assume that the sequence $\{\sigma_j\}_{j \in \mathbb{N}}$ belongs to $\ell_1(\mathbb{N}) \cap \ell_2(\mathbb{N})$, where $\ell_1(\mathbb{N})$ (resp. $\ell_2(\mathbb{N})$) is the set all absolutely (resp. square) summable sequences. It is therefore possible to construct a positive, periodic function

$$f(x) = \frac{1}{2\pi} \left( 1 + 2 \sum_{j=1}^{\infty} \sigma_j \cos(jx) \right), \quad \text{for } x \in (-\pi, \pi),$$

belonging to $L_2(-\pi, \pi)$ the set of all square-integrable functions $f$ over $(-\pi, \pi)$. This function is known as the spectral density of the stationary series $\{X_{k,i}, i \in \mathbb{Z}\}$.
We solve the following test problem,

$$H_0 : \Sigma = I$$  \hspace{1cm} (1)\

versus the alternative

$$H_1 : \Sigma \in T(\alpha, L) \text{ such that } \sum_{j \geq 1} \sigma_j^2 \geq \psi^2,$$

for $\psi = (\psi_{n,p})_{n,p}$ a positive sequence converging to 0. From now on, $C_{>0}$ denotes the set of squared symmetric and positive definite matrices. The set $T(\alpha, L)$ is an ellipsoid of Sobolev type

$$T(\alpha, L) = \{ \Sigma \in C_{>0}, \Sigma \text{ is Toeplitz} ; \sum_{j \geq 1} \sigma_j^2 j^{2\alpha} \leq L \text{ and } \sigma_0 = 1 \} , \alpha > 1/4, L > 0.$$ We shall also test against

$$H_1 : \Sigma \in E(A, L) \text{ such that } \sum_{j \geq 1} \sigma_j^2 \geq \psi^2,$$

for $\psi > 0$,

$$E(A, L) = \{ \Sigma \in C_{>0}, \Sigma \text{ is Toeplitz} ; \sum_{j \geq 1} \sigma_j^2 e^{2 Aj} \leq L \text{ and } \sigma_0 = 1 \} , A, L > 0.$$\hspace{1cm} (3)

This class contains the covariance matrices whose elements decrease exponentially, when moving away from the diagonal. We denote by $G(\psi)$ either $G(T(\alpha, L), \psi)$ the set of matrices under the alternative \hspace{1cm} (2) or $G(E(A, L), \psi)$ under the alternative \hspace{1cm} (3).

We stress the fact that a matrix $\Sigma$ in $G(\psi)$ is such that $1/(2p) \| \Sigma - I \|_F^2 \geq \sum_{j \geq 1} \sigma_j^2 \geq \psi^2,$ i.e. $\Sigma$ is outside a neighborhood of $I$ with radius $\psi$ in Frobenius norm.

Our test can be applied in the context of model fitting for testing the whiteness of the standard Gaussian residuals. In this context, it is natural to assume that the covariance matrix under the alternative hypothesis has small entries like in our classes of covariance matrices. Such tests have been proposed by \hspace{1cm} [15], where it is noted that weighted test statistics can be more powerful.

Note that, most of the literature on testing the null hypothesis \hspace{1cm} (1), either focus on finding the asymptotic behavior of the test statistic under the null hypothesis, or control in addition the type II error probability for one fixed unknown matrix under the alternative, whereas our main interest is to quantify the worst type II error probabilities, i.e. uniformly over a large set of possible covariance matrices.

Various test statistics in high dimensional settings have been considered for testing \hspace{1cm} (1), as it was known for some time that likelihood ratio tests do not converge when dimension
grows. Therefore, a corrected Likelihood Ratio Test is proposed in [2] when $p/n \to c \in (0,1)$, and its asymptotic behavior is given under the null hypothesis, based on the random matrix theory. In [25] the result is extended to $c = 1$. An exact test based on one column of the covariance matrix is constructed by [20]. A series of papers propose test statistics based on the Frobenius norm of $\Sigma - I$, see [26], [32], [33] and [9]. Different test statistics are introduced and their asymptotic distribution is studied. In particular in [9] the test statistic is a U-statistic with constant weights. An unbiased estimator of $tr(\Sigma - B_k(\Sigma))^2$ is constructed in [29], where $B_k(\Sigma) = (\sigma_{ij} \cdot I \{ |i-j| \leq k \})$, in order to develop a test statistic for the problem of testing the bandedness of a given matrix. Another extension of our test problem is to test the sphericity hypothesis $\Sigma = \sigma^2 I$, where $\sigma^2 > 0$ is unknown. [16] introduced a test statistic based on functionals of order 4 of the covariance matrix. Motivated by these results, the test $H_0 : \Sigma = I$ is revisited by [14]. The maximum value of non-diagonal elements of the empirical covariance matrix was also investigated as a test statistic. Its asymptotic extreme-value distribution was given under the identity covariance matrix by [5] and for other covariance matrices by [34]. We propose here a new test statistic to test (1) which is a weighted U-statistic of order 2 and study its probability errors uniformly over the set of matrices given by the alternative hypothesis.

The test problem with alternative (2) and with one sample ($n=1$) was solved in the sharp asymptotic framework, as $p \to \infty$, by [13]. Indeed, [13] studies sharp minimax testing of the spectral density $f$ of the Gaussian process. Note that under the null hypothesis we have a constant spectral density $f_0(x) = 1/(2\pi)$ for all $x$ and the alternative can be described in $L_2$ norm as we have the following isometry $\| f - f_0 \|^2_2 = (2\pi)^{-1} \| \Sigma - I \|^2_F$. Moreover, the ellipsoid of covariance matrices $T(\alpha, L)$ are in bijection with Sobolev ellipsoids of spectral densities $f$. Let us also recall that the adaptive rates for minimax testing are obtained for the spectral density problem by [18] by a non constructive method using the asymptotic equivalence with a Gaussian white noise model. Finding explicit test procedures which adapt automatically to parameters $\alpha$ and/or $L$ of our class of matrices will be the object of future work. Our efforts go here into finding sharp minimax rates for testing.

Our results generalize the results in [13] to the case of repeatedly observed stationary Gaussian process. We stress the fact that repeated sampling of the stationary process $(X_{1,1}, ..., X_{1,p})$ to $(X_{n,1}, ..., X_{n,p})$ can be viewed as one sample of size $n \times p$ under the null hypothesis. However, this sample will not fit the assumptions of our alternative. Indeed, under the alternative, its covariance matrix is not Toeplitz, but block diagonal. Moreover, we can summarize the $n$ independent vectors into one $p$-dimensional vector $X = n^{-1/2} \sum_{k=1}^n X_k$ having Gaussian distribution $N_p(0, \Sigma)$. The results by [13] will produce a test procedure with rate that we expect optimal as a function of $p$, but more biased and suboptimal as a function
of \(n\). The test statistic that we suggest removes cross-terms and has smaller bias. Therefore, results in \([13]\) do not apply in a straightforward way to our setup.

A conjecture in the sense of asymptotic equivalence of the model of repeatedly observed Gaussian vectors and a Gaussian white noise model was given by \([7]\). Our rates go in the sense of the conjecture.

The test of \(H_0 : \Sigma = I\) against \((2)\), with \(\Sigma\) not necessary Toeplitz, is given in \([3]\). Their rates show a loss of a factor \(p\) when compared to the rates for Toeplitz matrices obtained here. This can be interpreted heuristically by the size of the set of unknown parameters which is \(p(p-1)/2\) for \([3]\) whereas here it is \(p\). We can see that the family of Toeplitz matrices is a subfamily of general covariance matrices in \([3]\). Therefore, the lower bounds are different, they are attained through a particular family of Toeplitz large covariance matrices. The upper bounds take into account as well the fact that we have repeated information on the same diagonal elements. The test statistic is different from the one used in \([3]\).

The test problem with alternative hypothesis \((3)\) has not been studied in this model. The class \(E(A,L)\) contains matrices with exponentially decaying elements when further from the main diagonal. The spectral density function associated to this process belongs to the class of functions which are in \(L^2\) and admit an analytic continuation on the strip of complex numbers \(z\) with \(|\text{Im}(z)| \leq A\). Such classes of analytic functions are very popular in the literature of minimax estimation, see \([19]\).

In times series analysis such covariance matrices describe among others the linear ARMA processes. The problem of adaptive estimation of the spectral density of an ARMA process has been studied by \([17]\) (for known \(\alpha\)) and adaptively to \(\alpha\) via wavelet based methods by \([28]\) and by model selection by \([10]\). In the case of an ARFIMA process, obtained by fractional differentiation of order \(d \in (-1/2,1/2)\) of a casual invertible ARMA process, \([31]\) gave adaptive estimators of the spectral density based on the log-periodogram regression model when the covariance matrix belongs to \(E(A,L)\).

Before describing our results let us define more precisely the quantities we are interested in evaluating.

### 1.1 Formalism of the minimax theory of testing

Let \(\chi\) be a test, that is a measurable function of the observations \(X_1, \ldots, X_n\) taking values in \(\{0,1\}\) and recall that \(G(\psi)\) corresponds to the set of covariance matrice under the alternative hypothesis. Let

\[
\eta(\chi) = E_I(\chi) \quad \text{be its type I error probability, and}
\]

\[
\beta(\chi, G(\psi)) = \sup_{\Sigma \in G(\psi)} E_\Sigma(1 - \chi) \quad \text{be its maximal type II error probability.}
\]
\[
\begin{array}{|c|c|c|c|}
\hline
\Sigma & \mathcal{T}(\alpha, L) & \mathcal{E}(A, L) & \text{not Toeplitz and } \mathcal{T}(\alpha, L) \\
\hline
\tilde{\psi} & (C(\alpha, L) \cdot n^2 p^2)^{-\frac{\alpha}{2\alpha+1}} \left( \frac{2 \ln(n^2 p^2)}{An^2 p^2} \right)^{1/4} & (C(\alpha, L) \cdot n^2 p^2)^{-\frac{\alpha}{4\alpha+1}} & \\
\hline
b(\psi)^2 & C(\alpha, L) \cdot \psi^{\frac{4\alpha+1}{\alpha}} \left( \frac{A\psi^4}{2 \ln \left( \frac{1}{\psi} \right)} \right) & C(\alpha, L) \cdot \psi^{\frac{4\alpha+1}{\alpha}} & \\
\hline
\end{array}
\]

Table 1: Separation rates \(\tilde{\psi}\) and \(b(\psi)\) in the sharp asymptotic bounds where \(C(\alpha, L) = (2\alpha + 1)(4\alpha + 1)^{-\left(1 + \frac{1}{2\alpha}\right)} L^{-\frac{1}{2\alpha}}\).

We consider two criteria to measure the performance of the test procedure. The first one corresponds to the classical Neyman-Pearson criterion. For \(w \in (0, 1)\), we define,

\[
\beta_w(G(\psi)) = \inf_{\chi : \eta(\chi) \leq w} \beta(\chi, G(\psi)).
\]

The test \(\chi_w\) is asymptotically minimax according to the Neyman-Pearson criterion if

\[
\eta(\chi_w) \leq w + o(1) \quad \text{and} \quad \beta(\chi_w, G(\psi)) = \beta_w(G(\psi)) + o(1).
\]

The second criterion is the total error probability, which is defined as follows:

\[
\gamma(\chi, G(\psi)) = \eta(\chi) + \beta(\chi, G(\psi)).
\]

Define also the minimax total error probability \(\gamma\) as \(\gamma(G(\psi)) = \inf_{\chi} \gamma(\chi, G(\psi))\), where the infimum is taken over all possible tests.

Note that the two criteria are related since \(\gamma(G(\psi)) = \inf_{w \in (0, 1)} (w + \beta_w(G(\psi)))\) (see Ingster and Suslina [23]).

A test \(\chi\) is asymptotically minimax if: \(\gamma(G(\psi)) = \gamma(\chi, G(\psi)) + o(1)\). We say that \(\tilde{\psi}\) is a (asymptotic) separation rate, if the following lower bounds hold

\[
\gamma(G(\psi)) \to 1 \quad \text{as} \quad \frac{\psi}{\tilde{\psi}} \to 0
\]

together with the following upper bounds: there exists a test \(\chi\) such that,

\[
\gamma(\chi, G(\psi)) \to 0 \quad \text{as} \quad \frac{\psi}{\tilde{\psi}} \to +\infty.
\]

The sharp optimality corresponds to the study of the asymptotic behavior of the maximal type II error probability \(\beta_w(G(\psi))\) and the total error probability \(\gamma(G(\psi))\). In our study we obtain asymptotic behavior of Gaussian type, i.e. we show that, under some assumptions,

\[
\beta_w(G(\psi)) = \Phi(z_{1-w} - npb(\psi)) + o(1) \quad \text{and} \quad \gamma(G(\psi)) = 2\Phi(-npb(\psi)) + o(1),
\]

(4)
where $\Phi$ is the cumulative distribution function of a standard Gaussian random variable, 
$z_{1-w}$ is the $1-w$ quantile of the standard Gaussian distribution for any $w \in (0,1)$, and $b(\psi)$ 
has an explicit form for each ellipsoid of Toeplitz covariance matrices.

Separation rates and sharp asymptotic results for different testing problem were studied 
under this formalism by [22]. We refer for precise definitions of sharp asymptotic and non 
asymptotic rates to [27]. Note that throughout this paper, asymptotics and symbols $o$, $O$, $\sim$ 
and $\asymp$ are considered as $p$ tends to infinity, unless we specify that $n$ tends to infinity. Recall 
that, given sequences of real numbers $u$ and real positive numbers $v$, we say that they are 
asymptotically equivalent, $u \sim v$, if $\lim u/v = 1$. Moreover, we say that the sequences are 
asymptotically of the same order, $u \asymp v$, if there exist two constants $0 < c \leq C < \infty$ such 
that $c \leq \lim \inf u/v$ and $\lim \sup u/v \leq C$.

1.2 Overview of the results

In this paper, we describe the separation rates $\bar{\psi}$ and sharp asymptotics for the error proba-
bilities for testing the identity matrix against $G(\mathcal{T}(\alpha, L), \psi)$ and $G(\mathcal{E}(A, L), \psi)$ respectively.

We propose here a test procedure whose type II error probability tends to 0 uniformly 
over the set of $G(\psi)$, that is even for a covariance matrix that gets closer to the identity 
matrix at distance $\bar{\psi} \to 0$ as $n$ and $p$ increase. The radius $\bar{\psi}$ in Table 1 is the smallest vicinity 
around the identity matrix which still allows testing error probabilities to tend to 0. Our test 
statistic is a weighted quadratic form and we show how to choose these weights in an optimal 
way over each class of alternative hypotheses.

Under mild assumptions we obtain the sharp optimality in (4), where $b(\psi)$ is described 
in Table 1 and compared to the case of non Toeplitz matrices in [3].

This paper is structured as follows. In Section 2, we study the test problem with alterna-
tive hypothesis defined by the class $G(\mathcal{T}(\alpha, L), \psi)$, $\alpha > 1/4$, $L, \psi > 0$. We define explicitly 
the test statistic and give its first and second moments under the null and the alternative 
hypotheses. We derive its Gaussian asymptotic behavior under the null hypothesis and under 
the alternative submitted to the constraints that $\psi$ is close to the separation rate $\bar{\psi}$ and that 
$\Sigma$ is closed to the solution of an extremal problem $\Sigma^*$. We deduce the asymptotic separation 
rates. Their optimality is shown only for $\alpha > 1$. Our lower bounds are original in the 
literature of minimax lower bounds, as in this case we cannot reduce the proof to the vector 
case, or diagonal matrices. We give the sharp rates for $\psi \asymp \bar{\psi}$. Our assumptions imply 
that necessarily $n = o(p^{2\alpha-1/2})$ as $p \to \infty$. That does not prevent $n$ to be larger than $p$ for 
sufficiently large $\alpha$.

In Section 3, we derive analogous results over the class $G(\mathcal{E}(A, L), \psi)$, with $A, L, \psi > 0$. 
We show how to choose the parameters in this case and study the test procedure similarly. We
give asymptotic separation rates. The sharp bounds are attained as $\psi \asymp \tilde{\psi}$. Our assumptions involve that $n = o(\exp(p))$ which allows $n$ to grow exponentially fast with $p$. That can be explained by the fact that the elements of $\Sigma$ decay much faster over exponential ellipsoids than over the polynomial ones. In Section 4, we implement our procedure and show the power of testing over two families of covariance matrices.

The proofs of our results are postponed to the Section 5 and to the Supplementary material.

2 Testing procedure and results for polynomially decreasing covariances

We introduce a weighted U-statistic of order 2, which is an estimator of the functional $\sum_{j \geq 1} \sigma_j^2$ that defines the separation between a Toeplitz covariance matrix under the alternative hypothesis from the identity matrix under the null. Indeed, in nonparametric estimation of quadratic functionals such as $\sum_{j \geq 1} \sigma_j^2$ weighted estimators are often considered (see e.g. [4]). These weights have finite support of length $T$, where $T$ is optimal in some sense. Intuitively, as the coefficients $\{\sigma_j\}_{j}$ belong to an ellipsoid, they become smaller when $j$ increases and thus the bias due to the truncation and the weights becomes as small as the variance for estimating the weighted finite sum.

2.1 Test Statistic

Let us denote by $T_p(\{\sigma_j\}_{j \geq 1})$ the symmetric $p \times p$ Toeplitz matrix $\Sigma = [\sigma_{lk}]_{1 \leq l,k \leq p}$ such that the diagonal elements of $\Sigma$ are equal to 1, and $\sigma_{lk} = \sigma_{kl} = \sigma_{|l-k|}$, for all $l \neq k$. Now we define the weighted test statistic in this setup

$$\hat{A}_n := \tilde{A}_n = \frac{1}{n(n-1)(p-T)^2} \sum_{1 \leq k \neq l \leq n} \sum_{j=1}^{T} w_j^* \sum_{T+1 \leq i_1, i_2 \leq p} X_{k,i_1} X_{k,i_1-j} X_{l,i_2} X_{l,i_2-j} \quad (5)$$

where the weights $\{w_j^*\}_{j \geq 1}$ and the parameters $T, \lambda, b^2(\psi)$ are obtained by solving the following extremal problem:

$$b(\psi) := \sum_{j \geq 1} w_j^* \sigma_j^2 = \sup \left\{ (w_j)_{j} : w_j \geq 0; \sum_{j \geq 1} w_j^2 = \frac{1}{2}, \sum_{j \geq 1} \sigma_j^2 \geq \psi \right\}.$$  \quad (6)

This extremal problem appears heuristically as we want that the expected value of our test statistic for the worst parameter $\Sigma$ under the alternative hypothesis (closest to the null) to be as large as possible for the weights we use. This problem will provide the optimal weights $\{w_j^*\}_{j \geq 1}$ in order to control the worst type II error probability, but also the critical matrix.
\( \Sigma^* = T_p(\{\sigma_j^*\}) \) that will be used in the lower bounds. Indeed, \( \Sigma^* \) is positive definite for small enough \( \psi \) (see [3]).

The solution of the extremal problem [6] can be found in [23]:

\[
\begin{align*}
   w_j^* &= \frac{\lambda}{2b(\psi)} \left( 1 - \left( \frac{j}{T} \right)^{2\alpha} \right), \quad \sigma_j^{*2} = \lambda \left( 1 - \left( \frac{j}{T} \right)^{2\alpha} \right), \quad T = \left( (L(4\alpha + 1))^\frac{1}{2\alpha} \cdot \psi^{-\frac{1}{2\alpha}} \right) \\
   \lambda &= \frac{2\alpha + 1}{2\alpha (L(4\alpha + 1))^{\frac{1}{2\alpha}}} \cdot \psi^{\frac{2\alpha + 1}{2\alpha}}, \quad b^2(\psi) = \frac{1}{2} \sum_j \sigma_j^{*4} = \frac{2\alpha + 1}{L^{\frac{1}{2\alpha}} (4\alpha + 1)^{\frac{1}{2\alpha}}} \cdot \psi^{\frac{4\alpha + 1}{2\alpha}}
\end{align*}
\] (7)

Remark that \( T \) is a finite number but grows to infinity as \( \psi \to 0 \). Moreover, the test statistic will have optimality properties under the additional condition that \( T/p \to 0 \) which is equivalent to \( p\psi^{1/\alpha} \to \infty \). It is obvious that in practice it might happen that \( T \geq p \) and then we have no solution but to use \( T = p - 1 \), with the inconvenient that the procedure does not behave as well as the theory predicts.

**Proposition 1**  Under the null hypothesis, the test statistic \( \hat{A}_n \) is centered, \( E_T(\hat{A}_n) = 0 \), with variance :

\[ Var_T(\hat{A}_n) = \frac{1}{n(n - 1)(p - T)^2}. \]

Moreover, under the alternative hypothesis with \( \alpha > 1/4 \), if we assume that \( \psi \to 0 \) we have:

\[ E_\Sigma(\hat{A}_n) = \sum_{j=1}^T w_j^* \sigma_j^2 \geq b(\psi) \quad \text{and} \quad Var_\Sigma(\hat{A}_n) = \frac{R_1}{n(n - 1)(p - T)^4} + \frac{R_2}{n(p - T)^2}, \]

uniformly over \( \Sigma \) in \( G(T(\alpha, L), \psi) \), where

\[
\begin{align*}
   R_1 &\leq (p - T)^2 \cdot \{ 1 + o(1) + E_\Sigma(\hat{A}_n) \cdot (O(\sqrt{T}) + O(T^{3/2 - 2\alpha})) + E_\Sigma^2(\hat{A}_n) \cdot O(T^2) \} \\
   R_2 &\leq (p - T) \cdot \{ E_\Sigma(\hat{A}_n) \cdot o(1) + E_\Sigma^2(\hat{A}_n) \cdot (O(T^{1/4}) + O(T^{3/4 - \alpha})) + E_\Sigma^2(\hat{A}_n) \cdot O(T) \}
\end{align*}
\] (8)

In the next Proposition we prove asymptotic normality of the test statistic under the null and under the alternative hypothesis with additional assumptions. More precisely, we need that \( \psi \) is of the same order as the separation rate and that the matrix \( \Sigma \) is close to the optimal \( \Sigma^* \). This is not a drawback, since the asymptotic constant for probability errors are attained under the same assumptions or tend to 0 otherwise.

**Proposition 2**  Suppose that \( n, p \to +\infty, \alpha > 1/4, \psi \to 0, p\psi^{1/\alpha} \to +\infty \) and moreover assume that \( n(p - T)b(\psi) \asymp 1 \), the test statistic \( \hat{A}_n \) defined by [5] with parameters given in [7], verifies :

\[ n(p - T) \left( \hat{A}_n - E_\Sigma(\hat{A}_n) \right) \to \mathcal{N}(0, 1) \]

for all \( \Sigma \in G(T(\alpha, L), \psi) \), such that \( E_\Sigma(\hat{A}_n) = O(b(\psi)) \).

Moreover, \( n(p - T)\hat{A}_n \) has asymptotical \( \mathcal{N}(0, 1) \) distribution under \( H_0 \), as \( p \to \infty \) for any fixed \( n \geq 2 \).
2.2 Separation rate and sharp asymptotic optimality

Based on the test statistic \( \hat{A}_n \), we define the test procedure

\[
\chi^* = \chi^*(t) = \mathbb{1}(\hat{A}_n > t),
\]

for conveniently chosen \( t > 0 \), where \( \hat{A}_n \) is the estimator defined in (5) with parameters in (7).

The next theorem gives the separation rate under the assumption that \( T = o(p) \), or equivalently, that \( p\psi^{1/\alpha} \to \infty \). The upper bounds are attained for arbitrary \( \alpha > 1/4 \), but the lower bounds require \( \alpha > 1 \).

**Theorem 1** Suppose that asymptotically

\[
\psi \to 0 \quad \text{and} \quad p\psi^{1/\alpha} \to +\infty
\]

**Lower bound.** If \( \alpha > 1 \) and \( n^2 p^2 b^2(\psi) = C(\alpha, L)n^2 p^2 \psi^{4\alpha+1/\alpha} \to 0 \) then

\[
\gamma = \inf_{\chi} \gamma(\chi, G(T(\alpha, L), \psi)) \to 1,
\]

where the infimum is taken over all test statistics \( \chi \).

**Upper bound.** The test procedure \( \chi^* \) defined in (10) with \( t > 0 \) has the following properties:

Type I error probability : if \( np \cdot t \to +\infty \) then \( \eta(\chi^*) \to 0 \).

Type II error probability : if

\[
\alpha > 1/4 \quad \text{and} \quad n^2 p^2 b^2(\psi) = C(\alpha, L)n^2 p^2 \psi^{4\alpha+1/\alpha} \to +\infty
\]

then, uniformly over \( t \) such that \( t \leq c \cdot C^{1/2}(\alpha, L) \cdot \psi^{4\alpha+1/\alpha} \), for some constant \( 0 < c < 1 \), we have

\[
\beta(\chi^*, G(T(\alpha, L), \psi)) \to 0.
\]

Under the assumptions given in (11) and (12), with \( t \) verifying the assumptions of Theorem 1, we get :

\[
\gamma(\chi^*, G(T(\alpha, L), \psi)) \to 0
\]

As a consequence of the previous theorem, we get that \( \chi^* \) is an asymptotically minimax test procedure if \( \psi/\tilde{\psi} \to +\infty \). From the lower bounds we deduce that, if \( \psi/\tilde{\psi} \to 0 \), there is no test procedure to distinguish between the null and the alternative hypotheses, with errors tending to 0. The minimax separation rate \( \tilde{\psi} \) is therefore :

\[
\tilde{\psi} = \left( \frac{2\alpha + 1}{L^{1/\alpha} (4\alpha + 1)^{1+1/\alpha} \cdot n^2 p^2} \right)^{-\frac{\alpha}{4\alpha+1}}
\]
It is obtained from the relation \( n^2 p^2 b^2(\psi) = 1 \). Naturally the constant does not play any role here. Remark that the condition \( T/p \to 0 \times p \tilde{\psi}^{1/\alpha} \to +\infty \) implies that \( n = o(p^{2\alpha-\frac{1}{2}}) \).

The maximal type II error probability either tends to 0, see Theorem \( \ref{thm:lower_bound} \) or is less than \( \Phi(np(t - b(\psi))) + o(1) \) when \( npt < npb(\psi) \times 1 \). The latter case is the object of the next theorem giving sharp bounds for the asymptotic errors. The upper bounds are attained for arbitrary \( n \geq 2 \) and for \( \alpha > 1/4 \), while our proof of the sharp lower bounds requires additionally that \( n \to \infty \) and \( \alpha > 1 \).

**Theorem 2** Suppose that \( \psi \to 0 \) such that \( p/T \asymp p \tilde{\psi}^{1/\alpha} \to +\infty \) and, moreover, that

\[
\inf_{\chi: \eta(\chi) \leq w} \beta(\chi, G(T(\alpha, L), \psi)) \geq \Phi(z_{1-w} - npb(\psi)) + o(1),
\]

where the infimum is taken over all test statistics \( \chi \) with type I error probability less than or equal to \( w \). Moreover,

\[
\gamma = \inf_{\chi} \gamma(\chi, G(T(\alpha, L), \psi)) \geq 2\Phi(-npb(\psi) / 2) + o(1).
\]

**Lower bound.** If \( \alpha > 1 \), then

\[
\inf_{\chi: \eta(\chi) \leq w} \beta(\chi, G(T(\alpha, L), \psi)) \geq \Phi(z_{1-w} - npb(\psi)) + o(1),
\]

**Upper bound.** The test procedure \( \chi^* \) defined in (14) with \( t > 0 \) has the following properties.

Type I error probability : \( \eta(\chi^*) = 1 - \Phi(np \cdot t) + o(1) \).

Type II error probability : under the assumption (14), and for all \( \alpha > 1/4 \), we have that, uniformly over \( t \):

\[
\beta(\chi^*, G(T(\alpha, L), \psi)) \leq \Phi(np \cdot (t - b(\psi))) + o(1).
\]

In particular, for \( t = t^w \), such that \( np \cdot t^w = z_{1-w} \), we have \( \eta(\chi^*(t^w)) \leq w + o(1) \) and also,

\[
\beta(\chi^*(t^w), G(T(\alpha, L), \psi)) = \Phi(z_{1-w} - np \cdot b(\psi)) + o(1).
\]

Another important consequence of the previous theorem, is that the test procedure \( \chi^* \), with \( t^* = b(\psi)/2 \) is such that

\[
\gamma(\chi^*(t^*), G(T(\alpha, L), \psi)) = 2\Phi\left(-np \frac{b(\psi)}{2}\right) + o(1).
\]

Then we can deduce that the minimax separation rate \( \tilde{\psi} \) defined in (13) is sharp.
3 Exponentially decreasing covariances

In this section we want to test (1) against (3), where the alternative set is $G(\mathcal{E}(A, L), \psi)$, for some $A, L, \psi > 0$. It is well known in the nonparametric minimax theory that $\mathcal{E}(A, L)$ is in bijection with ellipsoids of analytic spectral densities admitting analytic continuation on the strip $\{z \in \mathbb{C} : \text{Im}(z) \leq A\}$ of the complex plane. On this class nearly parametric rates are attained for testing in the Gaussian noise model, see Ingster \[24\].

Let us define $\widehat{\mathcal{A}}^n_L$ in (15)

$$\widehat{\mathcal{A}}^n_L = \frac{1}{n(n-1)(p-T)^2} \sum_{1 \leq k \neq l \leq n} \sum_{j=1}^T w_j^* \sum_{1 \leq i_1, i_2 \leq p} X_{k,i_1} X_{k,i_1-j} X_{l,i_2} X_{l,i_2-j}, \quad (15)$$

where the weights $\{w_j^*\}_{j \geq 1}$, are obtained by solving the optimization problem (6), with the class $\mathcal{T}(\alpha, L)$ replaced by $\mathcal{E}(A, L)$. The solution given in [24] is as follows:

$$w_j^* = \frac{\lambda}{2b(\psi)} \left(1 - \left(\frac{e^j}{e^T}\right)^{2A}\right)_+, \quad \sigma_j^* = \sqrt{X(1 - \left(\frac{e^j}{e^T}\right)^{2A})_{+}}, \quad T = \left\lfloor \frac{1}{A} \ln \left(\frac{1}{\psi}\right) \right\rfloor,$$

$$\lambda = \frac{A\psi^2}{\ln \left(\frac{1}{\psi}\right)}, \quad b^2(\psi) = \frac{A\psi^4}{2\ln \left(\frac{1}{\psi}\right)}.$$

Note that all parameters above are free of the radius $L > 0$. Moreover, we have:

$$\sup_j w_j^* \leq \frac{\lambda}{2b(\psi)} < \frac{1}{2(\ln(1/\psi))^{1/2}} \to 0.$$

Under the null hypothesis, we still have $E_1(\widehat{\mathcal{A}}_n^L) = 0$, $\text{Var}_1(\widehat{\mathcal{A}}_n^L) = 1/(n(n-1)(p-T)^2)$ and

$$n(p-T)\widehat{\mathcal{A}}_n^L \xrightarrow{L} \mathcal{N}(0, 1) \quad \text{for fixed } n \geq 2 \text{ and } p \to +\infty.$$

In the following proposition, we see how the upper bounds of the variance have changed under $\Sigma$ in $G(\mathcal{E}(A, L), \psi)$.

**Proposition 3** Under the alternative, for all $\Sigma \in G(\mathcal{E}(A, L), \psi)$, we have:

$$E_{\Sigma}(\widehat{\mathcal{A}}_n^L) = \sum_{j=1}^T w_j^* \sigma_j^2 \geq b(\psi) \quad \text{and} \quad \text{Var}_{\Sigma}(\widehat{\mathcal{A}}_n^L) = \frac{R_1}{n(n-1)(p-T)^4} + \frac{R_2}{n(p-T)^2},$$

where, for all $A > 0$, and as $\psi \to 0$:

$$R_1 \leq (p-T)^2 \cdot \{1 + o(1) + E_{\Sigma}(\widehat{A}_n^L) \cdot O(\sqrt{T}) + E_{\Sigma}^2(\widehat{A}_n^L) \cdot O(T^2)\} \quad (17)$$

$$R_2 \leq (p-T) \cdot \{E_{\Sigma}(\widehat{A}_n^L) \cdot o(1) + E_{\Sigma}^3(\widehat{A}_n^L) \cdot O(T^{1/4}) + E_{\Sigma}^2(\widehat{A}_n^L) \cdot O(T)\} \quad (18)$$

Moreover, if $n(p-T)b(\psi) \asymp 1$, we show that $n(p-T)(\widehat{\mathcal{A}}_n^L - E_{\Sigma}(\widehat{A}_n^L)) \to \mathcal{N}(0, 1)$, for all $\Sigma \in \mathcal{E}(A, L)$, such that $E_{\Sigma}(\widehat{A}_n^L) = O(b(\psi))$. 

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Now we define the test procedure as follows,

\[ \Delta^* = \Delta^*(t) = 1(\hat{A}_n > t). \]

We describe next the separation rate. We stress the fact that Lemma 2 shows that the optimal sequence \( \{\sigma_j^*\} \) in (16) provides a Toeplitz positive definite covariance matrix. The sharp results are obtained under the additional assumption that \( \psi \asymp \tilde{\psi} \) and the lower bounds require that \( n \) tends also to infinity.

**Theorem 3** Suppose that asymptotically \( \psi \to 0 \) and \( p/T \asymp p/\ln(1/\psi) \to \infty \).

1. Separation rate. **Lower bound:** if \( n^2p^2b^2(\psi) = n^2p^2 \cdot A\psi^4/(2\ln(1/\psi)) \to 0 \) then

\[ \gamma = \inf_{\Delta} \gamma(\Delta, G(\psi)) \to 1, \]

where the infimum is taken over all test statistics \( \Delta \).

**Upper bound:** the test procedure \( \Delta^* \) defined previously with \( t > 0 \) has the following properties:

Type I error probability: if \( np \cdot t \to +\infty \) then \( \eta(\Delta^*) \to 0 \).

Type II error probability: if \( n^2p^2b^2(\psi) = n^2p^2 \cdot A\psi^4/(2\ln(1/\psi)) \to +\infty \) then, uniformly over \( t \) such that \( t \leq c \cdot A^{1/2} \psi^2/(2\ln(1/\psi))^{1/2} \), for some constant \( c < 1 \),

\[ \beta(\Delta^*, G(\psi)) \to 0. \]

2. Sharp asymptotic bounds. **Lower bound:** suppose that \( n \to +\infty \) and that

\[ n^2p^2b^2(\psi) \asymp 1, \]

then we get

\[ \inf_{\Delta, |\gamma(\Delta)| \leq w} \beta(\Delta, G(\psi)) \geq \Phi(z_{1-w} - npb(\psi)) + o(1), \]

where the infimum is taken over all test statistics \( \Delta \) with type I error probability less than or equal to \( w \) for \( w \in (0,1) \). Moreover,

\[ \gamma = \inf_{\Delta} \gamma(\Delta, G(\psi)) \geq 2\Phi(-np \frac{b(\psi)}{2}) + o(1). \]

**Upper bound:** we have

Type I error probability: \( \eta(\Delta^*) = 1 - \Phi(np\cdot t) + o(1) \).

Type II error probability: under the condition (19), we get that, uniformly over \( t \),

\[ \beta(\Delta^*, G(\psi)) \leq \Phi(np \cdot (t - b(\psi))) + o(1). \]

In particular, the test procedure \( \Delta^*(b(\psi)/2) \), is such that \( \gamma(\Delta^*(b(\psi)/2), G(\psi)) = 2\Phi(-np \frac{b(\psi)}{2})+ o(1) \). We get the sharp minimax separation rate: \( \tilde{\psi} = \left(\frac{2\ln(n^2p^2)}{An^2p^2}\right)^{1/4} \). Remark that, in this case the condition \( T/p \to 0 \) implies that \( n = o(e^p) \), which is considerably less restrictive than the condition \( n = o(p^{2\alpha-1}) \) of the previous case and allows for exponentially large \( n \), e.g. \( n = e^{p/2} \).
4 Numerical implementation and extensions

In this section we implement the test procedure $\chi$ in (10) with empirically chosen threshold $t > 0$ and study its numerical performance over two families of covariance matrices. We estimate the type I and type II errors by Monte Carlo sampling with 1000 repetitions. First, we choose $\Sigma = \Sigma(M) = [\sigma_{jj}]_{j}^{j}, \sigma_{jj} = j^{-2}/M$ under the alternative hypothesis, for various values of $M \in \{2, 2.5, 3, 4, 6, 8, 16, 30, 60, 80\}$. We implement the test statistic $\hat{A}_{n}^{T}$ defined in (5) and (7), for parameters $\alpha = 1, L = 1$ and $\psi = \psi(M) = \left(\sum_{j=1}^{p-1} j^{-4}\right)^{\frac{1}{2}} / M$. Our choice of the values for $M$ provides positive definite matrices. We denote by $A(M)$ the random variable $n(p - T)\hat{A}_{n}^{T}$ when $\Sigma = \Sigma(M)$, and by $A(0)$ when $\Sigma = I$. Note that large values of $M$ give $\Sigma(M)$ with small off-diagonal entries, which is very close to the identity matrix.

Figure 1: Distributions of $A(M) = n(p - T)\hat{A}_{n}^{T}$ for $I = \Sigma(0)$ and $\Sigma = \Sigma(M)$, when $p = 60$ and $n = 40$.

Figure 1 shows that $n(p - T)\hat{A}_{n}^{T}$ is distributed as a standard normal random variable, when $\Sigma = I$ and $\Sigma(M)$ close enough to the identity. And as a non-centered normal distribution when $\Sigma(M)$ is far from the identity matrix.

To evaluate the performance of our test procedure we compute it’s power. For each value of $n$ and $p$, we estimate the 95th percentile $t$ of the distribution of $n(p - T)\hat{A}_{n}^{T}$ under the null hypothesis $\Sigma = I$. We use $t$ previously defined to estimate the type II error probability, and then plot the associated power. In Figure 2 we plot the power function of our test procedure $\chi$-test as function of $\psi(M)$, for a fixed value of $n$ and different values of $p$. 
Figure 2: Power curves of the $\chi$-test as function of $\psi(M)$ for $n = 10$ and $p \in \{10, 30, 50, 70\}$

The vertical lines in figure 2 represent the different $\tilde{\psi}(n, p)$ associated to different values of $p$ and $n = 10$. We remark that, on the one hand the power grows with $\psi(M)$ for all $p \in \{10, 30, 50, 70\}$. On the other hand the power is an increasing function of $p$ for a fixed covariance matrix $\Sigma(M)$.

We also compare our test procedure with the one defined in [6]. Recall that the test statistic defined by [6] is given by:

$$\hat{T}_{CM}^n = \frac{2}{n(n-1)} \sum_{1 \leq k < l \leq n} \left( (X_k^\top X_l)^2 - X_k^\top X_k - X_l^\top X_l + p \right).$$

Figure 3: Power curves of the $\chi$-test and the CM-test as functions of $\psi(M)$, when the alternative consists of matrices whose elements decrease polynomially when moving away from the main diagonal
Note that for matrices \( \Sigma \in \mathcal{T}(1,1) \), we have \( (1/p)\|\Sigma - I\|^2_F \sim \sum_{j=1}^{p-1} \sigma_j^2 \), thus we implement \( \hat{T}_{n}^{CM}/p \) as CM-test statistic. To have fair comparison, we estimate the 95th percentile under the null hypothesis for both tests. Figures 3 shows that when \( n \) is bigger than or equal to \( p \) the powers of the \( \chi \)-test and the CM-test take close values. While when \( n \) is smaller then \( p \), the gap between the power values of the two tests is large, and the \( \chi \)-test is more powerful than the CM-test.

Second, we consider tridiagonal matrices under the alternative. We define \( \Sigma = \Sigma(\rho) = [\sigma_j]_j ; \sigma_j = \rho \cdot 1\{j = 1\}, \) for \( \rho \in (0,1) \). In this case the parameter \( \psi \) is \( \psi(\rho) = \rho \), for a grid of 10 points \( \rho \) belonging to the interval \( (0,0.35) \) and as previously we take \( \alpha = 1 \) and \( L = 1 \).

![Figure 4: Power curves of the \( \chi \)-test and the CM-test as functions of \( \psi(\rho) \), when the alternative consists of tridiagonal matrices](image)

Figure 4 shows that, the \( \chi \)-test performs better than the U-test, in the three cases : \( p \) smaller than \( n \), \( p \) equal \( n \) and \( p \) larger than \( n \). Moreover, we see that the power curves of the \( \chi \)-test and the CM-test are closer, when the ratio \( p/n \) is smaller. We expect even better results in this particular example if we use a larger value of \( \alpha \), or the procedure defined by (15) and (16). The question arises of a test statistic free of parameters \( \alpha \), respectively \( A \), which is beyond the scope of this paper.

5 Proofs

**Proof of Theorems 1 and 2.** Recall the assumptions \( n, p \to +\infty, \psi \to 0 \) and \( T/p \asymp 1/(p\psi^{1/\alpha}) \to 0 \).

**Lower bounds:** In order to show the lower bound, we first reduce the set of parameters...
to a convenient parametric family. Let $\Sigma^* = T_p(\{\sigma_k^*\}_{k \geq 1})$ be the Toeplitz matrix such that,

$$
\sigma_k^* = \sqrt{\lambda} \left(1 - \left(\frac{k}{T}\right)^{2\alpha}\right)^\frac{1}{2} \quad \text{for } 1 \leq k \leq p - 1,
$$

with $\lambda$ and $T$ are given by (7).

Let us define $G^*$ a subset of $G(T(\alpha, L), \psi)$ as follows

$$
G^* = \{\Sigma_U^* : \Sigma^*_U = T_p(\{u_k\sigma_k\}_{k \geq 1}), U \in \mathcal{U}\},
$$

where

$$
\mathcal{U} = \{U = T_p(\{u_k\}_{k \geq 1}) - I_p \text{ and } u_k = \pm 1 \cdot I(k \leq T - 1), \text{ for } 1 \leq k \leq T - 1\}.
$$

The cardinality of $\mathcal{U}$ is $2^{T-1}$.

From Proposition 3 in [3], we can see that if $\alpha > 1/2$, for all $U \in \mathcal{U}$, the matrix $\Sigma_U^*$ is positive definite, for $\psi > 0$ small enough. In contrast with [3], we change the signs randomly on each diagonal of the upper triangle of $\Sigma^*$ and not of all its elements. That allows us to stay into the model of Toeplitz covariance matrices and will actually change the rates of these lower bounds.

Assume that $X_1, \ldots, X_n \sim N(0, I)$ under the null hypothesis and denote by $P_I$ the likelihood of these random variables. Moreover assume that $X_1, \ldots, X_n \sim N(0, \Sigma_U^*)$ under the alternative, and we denote $P_U$ the associated likelihood. In addition let

$$
P_\pi = \frac{1}{2^{T-1}} \sum_{U \in G^*} P_U
$$

be the average likelihood over $G^*$.

The problem can be reduced to the test $H_0 : X_1, \ldots, X_n \sim P_I$ against the averaged distribution $H_1 : X_1, \ldots, X_n \sim P_\pi$, in the sense that

$$
\inf_{\chi : \eta(\chi) \leq w} \beta(\chi, G(T(\alpha, L), \psi)) = \inf_{\chi : \eta(\chi) \leq w} \sup_{\Sigma \in G(T(\alpha, L), \psi)} \mathbb{E}_\Sigma(1 - \chi) \geq \inf_{\chi : \eta(\chi) \leq w} \sup_{\Sigma \in G^*} \mathbb{E}_\Sigma(1 - \chi)
$$

$$
\geq \inf_{\chi : \eta(\chi) \leq w} \frac{1}{2^{T-1}} \mathbb{E}_\Sigma(1 - \chi) = \inf_{\chi : \eta(\chi) \leq w} \mathbb{E}_\pi(1 - \chi) := \inf_{\chi : \eta(\chi) \leq w} \beta(\chi, \{P_\pi\})
$$

and that

$$
\inf \gamma(\chi, G(T(\alpha, L), \psi)) \geq \inf \gamma(\chi, \{P_\pi\}) + o(1)
$$

where, with an abuse of notation, $\beta(\chi, \{P_\pi\}) = \mathbb{E}_\pi(1 - \chi)$ and $\gamma(\chi, \{P_\pi\}) = \mathbb{E}_I(\chi) + \mathbb{E}_\pi(1 - \chi)$.

It is therefore sufficient to show that, when $u_n \asymp 1$,

$$
\inf_{\chi : \eta(\chi) \leq w} \beta(\chi, \{P_\pi\}) \geq \Phi(z_1 - w - npb(\psi)) + o(1)
$$

(21)
and that
\[
\inf_{\chi} \gamma(\chi, \{P_\pi\}) \geq 2\Phi(-np \frac{b(\psi)}{2}) + o(1),
\]  
while, for \(u_n = o(1)\), we need that
\[
\gamma(\chi, \{P_\pi\}) \to 1.
\]  

**Lemma 1** Assume that \(\psi \to 0\) such that \(p\psi^{1/\alpha} \to \infty\) and let \(f_\pi\) be the probability density associated to the likelihood \(P_\pi\) previously defined. Then
\[
L_{n,p} := \log \frac{f_\pi}{f_I}(X_1, \ldots, X_n) = u_n Z_n - \frac{u_n^2}{2} + o_P(1), \quad \text{in } P_I \text{ probability},
\]  
where \(Z_n\) is asymptotically distributed as a standard Gaussian distribution and \(u_n = npb(\psi)\) is such that either \(u_n \to 0\) or \(u_n \asymp 1\). Moreover, \(L_{n,p}\) is uniformly integrable.

In order to obtain (21) and (22), we apply results in Section 4.3.1 of [23] giving the sufficient condition is (24).

It is known that \(\gamma(\chi, \{P_\pi\}) = 1 - \frac{1}{2} \|P_I - P_\pi\|_1\) and we bound the \(L_1\) norm by the Kullback-Leibler divergence
\[
\frac{1}{2} \|P_I - P_\pi\|_1^2 \leq K(P_I, P_\pi).
\]  
Therefore to show (23), we apply Lemma 1 to see that the log likelihood \(\log f_\pi / f_I(X_1, \ldots, X_n)\) is an uniformly integrable sequence. This implies that \(K(P_I, P_\pi) = \mathbb{E}_I(\log f_\pi / f_I(X_1, \ldots, X_n)) \to 0\).

**Upper bounds** : By the Proposition 1 we have that under the null hypothesis \(n(p - T)\hat{\chi} \to \mathcal{N}(0, 1)\). Then we can deduce that the Type I error probability of \(\chi^*\) has the following form:
\[
\eta(\chi^*) = \mathbb{P}(\hat{\chi} > t) = 1 - \Phi(npt) + o(1).
\]  
For the Type II error probability of \(\chi^*\), we shall distinguish two cases, when \(n^2p^2b^2(\psi)\) tends to infinity or is bounded by some finite constant. First, assume that \(\psi / \bar{\psi} \to +\infty\) or, equivalently, that \(n^2p^2b^2(\psi) \to +\infty\). Then by the Markov inequality,
\[
\mathbb{P}_\Sigma(\hat{\chi} \leq t) \leq \mathbb{P}_\Sigma(|\hat{\chi} - \mathbb{E}_\Sigma(\hat{\chi})| \geq \mathbb{E}_\Sigma(\hat{\chi}) - t) \leq \frac{\text{Var}_\Sigma(\hat{\chi})}{(\mathbb{E}_\Sigma(\hat{\chi}) - t)^2}
\]  
for all \(\Sigma \in G(T(\alpha, L), \psi)\) and \(t \leq c \cdot b(\psi)\) such that \(0 < c < 1\). Recall that under the alternative, we have \(\mathbb{E}_\Sigma(\hat{\chi}) \geq b(\psi)\) which gives:
\[
\mathbb{E}_\Sigma(\hat{\chi}) - t \geq (1 - c)\mathbb{E}_\Sigma(\hat{\chi}) \geq (1 - c)b(\psi).
\]  
Therefore from the first part of the inequality (25) and the variance expression of \(\hat{\chi}\) under \(H_1\), given in Proposition 1, we have:
\[
\mathbb{P}_\Sigma(\hat{\chi} \leq t) \leq \frac{R_1}{n(n - 1)(p - T)^4(1 - c)^2\mathbb{E}_\Sigma^2(\hat{\chi})} + \frac{R_2}{n(p - T)^2(1 - c)^2\mathbb{E}_\Sigma^2(\hat{\chi})} := U_1 + U_2.
\]
Let us bound from above $U_1$, using (8) and the second part of the inequality (25):

$$U_1 \leq \frac{1 + o(1)}{n(n-1)(p-T)^2(1-c)^2b^2(\psi)} + \frac{O(\sqrt{T}) + O(T^{\frac{3}{2} - 2\alpha})}{n(n-1)(p-T)^2b(\psi)} + \frac{O(T^2)}{n(n-1)(p-T)^2}.$$ 

We have $T^{\frac{3}{2} - 2\alpha}b(\psi) \asymp T^2b^2(\psi) \asymp \psi^{1 - \frac{1}{\alpha}} = o(1)$, for all $\alpha > 1/4$, which proves that:

$$U_1 \leq \frac{1 + o(1)}{n(n-1)(p-T)(1-c)^2b^2(\psi)} = o(1).$$

Indeed, $n^2(p-T)^2b^2(\psi) \rightarrow +\infty$, since $n^2p^2b^2(\psi) \rightarrow +\infty$ and $T/p \rightarrow 0$.

We can check using (9) that the term $U_2$ tends to zero as well:

$$U_2 \leq \frac{o(1)}{n(p-T)b(\psi)} + \frac{O(T^{1/4}) + O(T^{3/4 - \alpha})}{n(p-T)b^{1/2}(\psi)} + \frac{O(T)}{n(p-T)}$$

$$= o(1) \text{ for all } \alpha > 1/4, \text{ as soon as } n^2p^2b^2(\psi) \rightarrow +\infty.$$

Finally, when $\psi$ is of the same order of the separation rate, i.e. $n^2p^2b^2(\psi) \asymp 1$, we may have either $E_{\Sigma}(\hat{A}_n)/b(\psi)$ tends to infinity, or $E_{\Sigma}(\hat{A}_n) = O(b(\psi))$. In the first case it is easy to see that $U_1 + U_2 \rightarrow 0$. In the latter the Proposition 2 gives the asymptotic normality of $n(p-T)(\hat{A}_n - E_{\Sigma}(\hat{A}_n))$. Thereby,

$$\sup_{\Sigma \in G(T(\alpha,L),\psi)} \mathbb{P}_{\Sigma}(\hat{A}_n \leq t) \leq \sup_{\Sigma \in G(T(\alpha,L),\psi)} \Phi(n \cdot (t - E_{\Sigma}(\hat{A}_n))) + o(1)$$

$$\leq \Phi(n \cdot (t - \inf_{\Sigma \in G(T(\alpha,L),\psi)} E_{\Sigma}(\hat{A}_n))) + o(1)$$

$$= \Phi(n \cdot (t - b(\psi))) + o(1).$$

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6 Supplementary material

6.1 Additional proofs for the results in Section 2

Proof of Lemma 1. We need to study the log-likelihood ratio:

\[ L_{n,p} := \log \frac{f_{\pi}}{f_{I}}(X_1, ..., X_n) = \log E_U \exp \left( -\frac{1}{2} \sum_{k=1}^{n} X_k^\top ((\Sigma^*_U)^{-1} - I)X_k - \frac{n}{2} \log \det(\Sigma^*_U) \right), \]

where \( U \) is seen as a randomly chosen matrix with uniform distribution over the set \( U \).

Moreover, let us denote \( \Delta_U = \Sigma^*_U - I \) which is a symmetric matrix with null diagonal. Recall that for all \( U \in U \), \( tr(\Delta_U) = 0 \) and that \( \|\Delta_U\| = O(p^{1-1/(2\alpha)}) \). Remember also that \( \sigma^*_k = 0 \) for all \( |k| \geq T \).

The matrix Taylor expansion gives

\[ (\Sigma^*_U)^{-1} - I = -\Delta_U + \Delta^2_U + O(1) \cdot \Delta^3_U, \]

\[ \log \det(\Sigma^*_U) = -\frac{1}{2} tr(\Delta^2_U) + O(1) \cdot tr(\Delta^3_U). \]

On the one hand, \( tr(\Delta^2_U) = \sum_{1 \leq i \neq j \leq p} (\sigma^*_{|i-j|})^2 \), does not depend on \( U \). Moreover,

\[ tr(\Delta^3_U) \leq \|\Delta_U\| \cdot \|\Delta_U\|_F^2 = O(p\psi^{3-\frac{1}{2\alpha}}) = O(np\psi^{2+\frac{1}{2\alpha}} \cdot \frac{\psi^{1-\frac{1}{\alpha}}}{n}) = o(1) \text{ for } \alpha > 1. \tag{26} \]

Thus we get

\[ \frac{n}{2} \log \det(\Sigma^*_U) = \frac{n}{2} \sum_{1 \leq i \neq j \leq p} (\sigma^*_{|i-j|})^2 + o(1). \tag{27} \]

On the other hand, we see that

\[ X_k^\top \Delta_U X_k = \sum_{1 \leq i, j \leq p} X_k,i u_{i-j}^{\sigma^*_{|i-j|}} X_k,j = 2 \sum_{1 \leq r < T} u_{r}^{\sigma^*_r} \sum_{i=1+r}^{p} X_{k,i} X_{k,i-r} \tag{28} \]

and that

\[ X_k^\top \Delta^2_U X_k = \sum_{1 \leq i, j \leq p} X_k,i X_k,j \sum_{h=1}^{p} \sum_{h \neq \{i,j\}} u_{|i-h|}^{\sigma^*_{|i-h|}} u_{|j-h|}^{\sigma^*_{|j-h|}} \]

\[ = \sum_{i=1}^{p} X_{k,i}^2 \sum_{h=1}^{p} (\sigma^*_{|i-h|})^2 + \sum_{1 \leq i \neq j \leq p} X_{k,i} X_{k,j} \sum_{h=1}^{p} \sum_{h \neq \{i,j\}} u_{|i-h|}^{\sigma^*_{|i-h|}} u_{|j-h|}^{\sigma^*_{|j-h|}} \]

\[ := S_1 + S_2. \]

In the term \( S_2 \), we change the variables \( i \) and \( j \) into \( l = i - h \) and \( m = j - h \) and due to the constraints we have \( |l|, |m| \in \{1, \ldots, T - 1\} \) and \( l \neq m \), while \( h \) varies in the set...
\{1 \lor (1 - l) \lor (1 - m), p \land (p - l) \land (p - m)\} for each fixed pair \((l, m)\). Therefore,

\[
S_2 = \sum_{l \neq m}^{p \land (p - l) \land (p - m)} \sum_{1 \leq |l|, |m| < T}^{p \land (p - l) \land (p - m)} u_{l|m}[\sigma_1^* \sigma_2^*] X_{k,l+h}X_{k,m+h}.
\]

We split the previous sums over \(l \neq m\) such that \(\text{sign}(l \cdot m) > 0\) and get

\[
S_{2,1} := \sum_{1 \leq l \neq m < T}^{p \land (p - l) \land (p - m)} u_{l|m} \sigma_1^* \sigma_2^* \left( \sum_{h=1}^{p - l} X_{k,h+l}X_{k,h+m} + \sum_{h=(1+l) \lor (1+m)}^{p} X_{k,h-l}X_{k,h-m} \right)
\]

respectively, over \(l, m\) of opposite signs: \(\text{sign}(l \cdot m) < 0\) and get

\[
S_{2,2} = 2 \sum_{T-1}^{p - l} \sum_{h=1+l}^{p - l} u_{l|m} \sigma_1^* \sigma_2^* X_{k,h+l}X_{k,h-m} + 2 \sum_{1 \leq l \neq m < T}^{p \land (p - l) \land (p - m)} u_{l|m} \sigma_1^* \sigma_2^* X_{k,h+l}X_{k,h-m}.
\]

In conclusion, we can group terms differently and write

\[
X_k^T \Delta_U^2 X_k = \sum_{1 \leq l \neq m < T}^{(p-l) \land (p-m)} u_{l|m} \sigma_1^* \sigma_2^* \left( \sum_{h=1}^{p - l} X_{k,h+l}X_{k,h+m} + \sum_{h=(1+l) \lor (1+m)}^{p} X_{k,h-l}X_{k,h-m} \right)
\]

\[
+ 2 \sum_{h=1+l}^{p - l} X_{k,h+l}X_{k,h-m} \right) + \sum_{i=1}^{T-1} \sum_{h=1}^{p} X_{k,i}^2 \sum_{h=1}^{p} (\sigma_i^* \sigma_i^*)^2 + 2 \sum_{i=1}^{T-1} \sum_{h=1+l}^{p} \sigma_i^* \sigma_i^* X_{k,h+l}X_{k,h-l},
\]

(29)

where

\[
V_p(l, m, k) := \sum_{h=1}^{(p-l) \land (p-m)} X_{k,h+l}X_{k,h+m} + \sum_{h=(1+l) \lor (1+m)}^{p} X_{k,h-l}X_{k,h-m} + 2 \sum_{h=1+m}^{p - l} X_{k,h+l}X_{k,h-m}.
\]

Now, let us see that:

\[
\mathbb{E}_I(X_k^T \Delta_U^3 X_k) = \mathbb{E}_I(tr(X_k^T \Delta_U^3 X_k)) = \mathbb{E}_I(tr(X_k X_k^T \Delta_U^3)) = tr(\Delta_U^3 \mathbb{E}_I(X_k X_k^T)) = tr(\Delta_U^3)
\]

and recall (26) to get

\[
\mathbb{E}_I\left(\sum_{k=1}^{n} X_k^T \Delta_U^3 X_k\right) = O(np\psi^3 - \frac{1}{np}) = o(1).
\]
Moreover, we have $$\mathbb{E}_l(X_k^T \Delta^3_U X_k)^2 = tr^2(\Delta^3_U) + 2tr(\Delta^6_U)$$ by Proposition A.1 in [9], which implies that

$$\text{Var}_l(\sum_{k=1}^n X_k^T \Delta^3_U X_k) = 2ntr(\Delta^6_U) \leq 2n\|\Delta_U\|^4\|\Delta_U\|_F^2 = O(n^6\psi^{-4/(2\alpha)}) = o(1).$$

Then, using Chebyshev’s inequality we obtain,

$$\sum_{k=1}^n X_k^T \Delta^3_U X_k = o_P(1).$$

(30)

Thus we replace (27) to (30) in $$L_{n,p}$$ and get

$$L_{n,p} = \log \mathbb{E}_U \exp \left( \sum_{1 \leq r < T} u_r \sigma_r^* \sum_{i=1+r}^p \sum_{k=1}^n X_{i,k} \sum_{i \neq i-r} - \frac{1}{2} \sum_{1 \leq i \neq m < T} u_i u_m \sigma_i^* \sigma_m^* \sum_{k=1}^n V_p(l, m, k) \right)
- \frac{1}{2} \sum_{i=1}^p \sum_{k=1}^n X_{i,k}^2 \sum_{h=1}^p \sigma_{i-l}^2 \sum_{1 \leq l \neq m < T} \sum_{h=1+i}^p \sum_{k=1}^n X_{k,h} \sum_{h \neq h-l} + \frac{n}{4} \sum_{1 \leq i \neq j \leq n} (\sigma_{i-j}^*)^2 + o_P(1).$$

Denote by $$W_{l,m} := \sum_{k=1}^n X_{k,l}X_{k,m}$$. Now, we evaluate the expected value with respect to the i.i.d. Rademacher variables $$u_r, u_i u_m$$ for all $$1 \leq r < T$$ and $$1 \leq l \neq m < T$$ to get

$$L_{n,p} = \log \left( \prod_{1 \leq r \leq T-1} \cosh(\sigma_r^* \sum_{i=r+1}^p W_{i,i-r}) \right) + \log \left( \prod_{1 \leq i \neq m < T} \cosh \left( \frac{1}{2} \sigma_i^* \sigma_m^* \sum_{k=1}^n V_p(l, m, k) \right) \right)
- \frac{1}{2} \sum_{i=1}^p \sum_{j \neq i} (\sigma_{i-j}^*)^2 \sum_{1 \leq l \neq m < T} \sum_{h=1+i}^p \sum_{k=1}^n X_{k,h} \sum_{h \neq h-l} + \frac{n}{4} \sum_{1 \leq i \neq j \leq p} (\sigma_{i-j}^*)^2 + o_P(1).$$

We get that

$$L_{n,p} = \sum_{1 \leq r \leq T-1} \log \cosh(\sigma_r^* \sum_{i=r+1}^p W_{i,i-r}) + \sum_{1 \leq l \neq m < T} \log \cosh \left( \frac{1}{2} \sigma_i^* \sigma_m^* \sum_{k=1}^n V_p(l, m, k) \right)
- \frac{1}{2} \sum_{i=1}^p \sum_{j \neq i} (\sigma_{i-j}^*)^2 \sum_{1 \leq l \neq m < T} \sum_{h=1+i}^p \sum_{k=1}^n X_{k,h} \sum_{h \neq h-l} + \frac{n}{4} \sum_{1 \leq i \neq j \leq p} (\sigma_{i-j}^*)^2 + o_P(1).$$

Note that

$$\sum_{k=1}^n V_p(l, m, k) = \sum_{h=1}^p W_{h,l,h+m} + \sum_{h=(1+l)\lor(1+m)}^p W_{h-l,h-m} + 2 \sum_{h=1+m}^p W_{h,l,h-m}.$$
On the other hand, let us treat each term of (31) separately. We first decompose \( \sum_{i=r+1}^{p} W_{i,i-r} \), as soon as \( \psi \rightarrow 0 \). Then,

\[
\log \cosh(\sigma_{r}^{*} \sum_{i=r+1}^{p} W_{i,i-r}) = \frac{1}{2} \sigma_{r}^{*2}(\sum_{i=r+1}^{p} W_{i,i-r})^{2} - \frac{(1 + o_{P}(1))}{12} \cdot \sigma_{r}^{*4}(\sum_{i=r+1}^{p} W_{i,i-r})^{4}.
\]

On the other hand,

\[
\frac{\sigma_{l}^{*}\sigma_{m}^{*}}{2} \left( \sum_{h=1}^{(p-l)\land(p-m)} W_{h+l, h+m} + \sum_{h=(1+l)\vee(1+m)}^{p} W_{h-l, h-m} + 2 \sum_{h=1+m}^{p-l} W_{h+l, h-m} \right)
\]

\[
\leq \frac{\lambda}{2} \cdot \left( \sum_{h=1}^{(p-l)\land(p-m)} W_{h+l, h+m} \right) + \frac{\lambda}{2} \cdot \left( \sum_{h=(1+l)\vee(1+m)}^{p} W_{h-l, h-m} \right) + \lambda \cdot \left( \sum_{l=1+m}^{p-l} W_{h+l, h-m} \right)
\]

\[
\leq O_{P}(\lambda \sqrt{np}) = O_{P}(\psi^{1/2} \sqrt{npb(\psi)}) = o_{P}(1).
\]

Thus we have to study now

\[
L_{n,p} = \log \frac{f_{\pi}}{f_{T}}(X_{1}, ..., X_{n})
\]

\[
= \sum_{1 \leq r < T} \left\{ \frac{1}{2} \cdot \sigma_{r}^{*2}(\sum_{i=r+1}^{p} W_{i,i-r})^{2} - \frac{(1 + o_{P}(1))}{12} \cdot \sigma_{r}^{*4}(\sum_{i=r+1}^{p} W_{i,i-r})^{4} \right\}
\]

\[
+ \frac{1}{4} \sum_{1 \leq l \neq m < T} \sigma_{l}^{*2} \sigma_{m}^{*2} \left( \sum_{k=1}^{n} V_{p}(l, m, k) \right)^{2} (1 + o_{P}(1))
\]

\[
- \frac{1}{2} \sum_{i=1}^{p} W_{i,i} \sum_{j:j \neq i} (\sigma_{i-j}^{*})^{2} - \sum_{1 \leq l \leq T-1} \sigma_{l}^{*2} \sum_{h=1+l}^{p-l} W_{h+l, h-l} + \frac{n}{4} \sum_{1 \leq i \neq j \leq p} (\sigma_{i-j}^{*})^{2} + o_{P}(1).
\]

Let us treat each term of (31) separately. We first decompose \((\sum_{i=r+1}^{p} W_{i,i-r})^{2}\) as follows,

\[
A := (\sum_{i=r+1}^{p} W_{i,i-r})^{2}
\]

\[
= \sum_{1+r \leq i_{1}, i_{2} \leq p} \left( \sum_{k=1}^{n} \sum_{l=1}^{n} X_{k,i_{1}} X_{k,i_{1}-r} X_{l,i_{2}} X_{l,i_{2}-r} + \sum_{k=1}^{n} X_{k,i_{1}} X_{k,i_{1}-r} X_{k,i_{2}} X_{k,i_{2}-r} \right)
\]

\[
= \sum_{1+r \leq i_{1}, i_{2} \leq p} \sum_{k=1}^{n} X_{k,i_{1}} X_{k,i_{1}-r} X_{l,i_{2}} X_{l,i_{2}-r}
\]

\[
+ \sum_{1+r \leq i_{1} \neq i_{2} \leq p} \sum_{k=1}^{n} X_{k,i_{1}} X_{k,i_{1}-r} X_{k,i_{2}} X_{k,i_{2}-r} + \sum_{1+r \leq i \leq p} \sum_{k=1}^{n} X_{k,i}^{2} X_{k,i}^{2} - r
\]

\[
:= A_{1} + A_{2} + A_{3}
\]

The term \( A_{3} \) will be taken into account as it is later on.
Therefore, we have, and similarly, the dominant term giving the asymptotic distribution is:

\[
\frac{1}{2} \sum_{1 \leq r < T} \sigma_r^2 \cdot A_1 = \frac{1}{2} \sum_{1 \leq r < T} \sigma_r^2 \sum_{1 + r \leq i_1, i_2 \leq p} \sum_{k=1}^{n} \sum_{l=1}^{n} X_{k, i_1} X_{k, i_1 - r} X_{l, i_2} X_{l, i_2 - r}
\]

\[
= \frac{1}{2} \sum_{1 \leq r < T} \sigma_r^2 \sum_{1 + r \leq i_1, i_2 \leq p} \sum_{k=1}^{n} \sum_{l=1}^{n} X_{k, i_1} X_{k, i_1 - r} X_{l, i_2} X_{l, i_2 - r}
\]

\[
+ \sum_{1 \leq r < T} \sigma_r^2 \sum_{1 + r \leq i_1, i_2 \leq T} \sum_{1 + r \leq i_1, i_2 \leq T} \sum_{k=1}^{n} \sum_{l=1}^{n} X_{k, i_1} X_{k, i_1 - r} X_{l, i_2} X_{l, i_2 - r}
\]

\[
+ \frac{1}{2} \sum_{1 \leq r < T} \sum_{1 + r \leq i_1, i_2 \leq T} \sum_{1 \leq r < T} \sum_{1 \leq r < T} \sum_{k=1}^{n} \sum_{l=1}^{n} X_{k, i_1} X_{k, i_1 - r} X_{l, i_2} X_{l, i_2 - r}
\]

\[
:= A_{1,1} + A_{1,2} + A_{1,3}, \quad \text{say.}
\]

Recall that \( \sigma_r^2 = 2u_r^* b(\psi) \) and then \( A_{1,1} = n(p - T)\hat{A}_n \cdot n(p - T)b(\psi) \). By Proposition \( n(p - T)\hat{A}_n \overset{d}{\rightarrow} \mathcal{N}(0, 1) \) and thus \( A_{1,1} \) can be written \( u_n Z_n \) with \( Z_n \overset{d}{\rightarrow} \mathcal{N}(0, 1) \).

Next, under \( P_I \) all variables in the multiple sums of \( A_{1,2} \) are uncorrelated (as well as for \( A_{1,3} \)). Thus,

\[
\text{Var}_I(A_{1,2}) = 2 \sum_{1 \leq r < T} \sigma_r^4 \sum_{1 \leq i_1 \leq T} \sum_{1 \leq i_2 \leq T} \sum_{k=1}^{n} \sum_{l=1}^{n} \text{Var}_I(X_{k, i_1} X_{k, i_1 - r} X_{l, i_2} X_{l, i_2 - r})
\]

\[
= 2 \sum_{1 \leq r < T} \sigma_r^4 (p - T)(T - r) n(n - 1) \leq n^2 p T \sum_{1 \leq r < T} \sigma_r^4 = 2n^2 p T b^2(\psi)
\]

\[
= 2 \cdot \frac{T}{p} \cdot u_n^2 = o(u_n^2), \quad \text{as } T/p \to 0.
\]

And, similarly,

\[
\text{Var}_I(A_{1,3}) = \frac{1}{4} \sum_{1 \leq r < T} \sigma_r^4 \sum_{1 \leq i_1, i_2 \leq T} \sum_{k=1}^{n} \sum_{l=1}^{n} \text{Var}_I(X_{k, i_1} X_{k, i_1 - r} X_{l, i_2} X_{l, i_2 - r})
\]

\[
\leq \frac{1}{2} \cdot T^2 n(n - 1) b^2(\psi) = O\left(\left(\frac{T}{p}\right)^2 \cdot u_n^2\right) = o(u_n^2).
\]

Therefore, \( A_{1,1} + A_{1,2} + A_{1,3} = u_n Z_n + o_p(u_n) \), where \( Z_n \overset{d}{\rightarrow} \mathcal{N}(0, 1) \). For the same reason, we have,

\[
\text{Var}_I\left(\frac{1}{2} \sum_{1 \leq r < T} \sigma_r^2 \cdot A_2\right) = \frac{1}{4} \sum_{1 \leq r < T} \sigma_r^4 \sum_{1 + r \leq i_1 \neq i_2 \leq p} \sum_{k=1}^{n} \sum_{l=1}^{n} \text{Var}_I(X_{k, i_1} X_{k, i_1 - r} X_{l, i_2} X_{l, i_2 - r})
\]

\[
\leq \frac{1}{4} \cdot n p^2 b^2(\psi) = O\left(\frac{1}{n} \cdot u_n^2\right) = o(1),
\]

as soon as \( n \to \infty \) or \( u_n \to 0 \). We want to show that

\[
B = \frac{1}{12} \sum_{1 \leq r < T} \sigma_r^4 \sum_{i=r+1}^{p} W_{i, i-r}^4 = \frac{u_n^2}{2} + o_p(1).
\]
Indeed,

\[
\mathbb{E}_I(B) = \frac{1}{12} \sum_{1 \leq r < T} \sigma_r^4 \cdot \mathbb{E}_I(\sum_{i=r+1}^p W_{i,i-r})^4 = \frac{1}{12} \sum_{i=r+1}^p \sigma_r^4 \cdot \mathbb{E}_I(\sum_{k=1}^n X_{k,i} X_{k,i-r})^4
\]

\[
= \frac{1}{12} \sum_{1 \leq r < T} \sigma_r^4 \sum_{k=1}^n \left( \sum_{i=r+1}^p \mathbb{E}_I(X_{k,i}^4 X_{k,i-r}^4) + 3 \sum_{1+r \leq i_1 \leq i_2 \leq p} \mathbb{E}_I(X_{k,i_1}^2 X_{k,i_1-r}^2) \mathbb{E}_I(X_{k,i_2}^2 X_{k,i_2-r}^2) \right)
\]

\[
+ \frac{3}{12} \sum_{1 \leq r < T} \sigma_r^4 \sum_{1 \leq k_1 \neq k_2 \leq n} \sum_{1+r \leq i_1 \neq i_2 \leq p} \mathbb{E}_I(X_{k_1,i_1}^2 X_{k_1,i_1-r}^2) \mathbb{E}_I(X_{k_2,i_2}^2 X_{k_2,i_2-r}^2)
\]

\[
= \frac{3}{4} \sum_{1 \leq r < T} \sigma_r^4 \cdot n(p-r) + \frac{1}{4} \sum_{1 \leq r < T} \sigma_r^4 \cdot n(p-r)^2 + \frac{1}{4} \sum_{1 \leq r < T} \sigma_r^4 \cdot n^2(p-r)^2
\]  \hfill (32)

Recall that \(2\beta^2(\psi) = \sum_j \sigma_j^4\), thus

\[
\mathbb{E}_I(B) = \frac{3}{2} \cdot npb^2(\psi)(1+o(1)) + \frac{1}{2} \cdot npb^2(\psi)(1+o(1)) + \frac{1}{2} \cdot n^2 p^2 b^2(\psi)(1+o(1)) = \frac{u_n^2}{2} (1+o(1)).
\]

Moreover,

\[
\text{Var}_I(B) = \frac{1}{12^2} \sum_{1 \leq r < T} \sigma_r^8 \cdot \text{Var}_I(\sum_{i=r+1}^p W_{i,i-r})^4
\]

\[
+ \frac{1}{12^2} \sum_{1 \leq r \neq r' < T} \sigma_r^4 \sigma_r'^4 \text{Cov}_I(\sum_{i=r+1}^p W_{i,i-r})^4, (\sum_{i'=r'+1}^p W_{i',i'-r'})^4).\]

As in the calculation of the expected value of \(B\), we can see that the term of higher order is obtained when we gather the indices into distinct pairs. Thus following the same reasoning we get

\[
\text{Var}_I(\sum_{i=r+1}^p \sum_{k=1}^n X_{k,i} X_{k,i-r})^4 = O(n^4 p^4).
\]

Through a very technical calculation, and using similar arguments as previously, we can prove that, for \(r \neq r'\),

\[
\text{Cov}_I((\sum_{i=r+1}^p \sum_{k=1}^n X_{k,i} X_{k,i-r})^4, (\sum_{i'=r'+1}^p \sum_{k'=1}^n X_{k',i'} X_{k',i'-r'})^4) = O(n^3 p^4).
\]

Thus,

\[
\text{Var}_I(B) = O(\lambda^4 Tn^4 p^4) + O(b^4(\psi)n^3 p^4) = O(\psi^2 n^4 p^4 b^4(\psi)) + O\left(\frac{1}{n} \cdot n^4 p^4 b^4(\psi)\right) = o(1).
\]

By Chebyshev’s inequality we deduce that

\[
\frac{1}{12} \sum_{r=1}^{T-1} \sum_{i=r+1}^p W_{i,i-r})^4 = \mathbb{E}_I\left( \frac{1}{12} \sum_{r=1}^{T-1} \sum_{i=r+1}^p W_{i,i-r})^4 \right) + o_P(1)
\]

\[
= \frac{3(1+o(1))}{12} \cdot 2n^2(p-r)^2 b^2(\psi) + o_P(1) = \frac{u_n^2}{2} (1+o_P(1)).
\]
Also using that \( \mathbb{E}_I(\sum_{i=r+1}^{p} W_{i,i-r})^4 = O(n^2p^2) \), we get

\[
C := \sum_{1 \leq l \neq m < T} \frac{\sigma_l^2 \sigma_m^2}{4} \left( \sum_{h=1}^{(p-1)/2} W_{h,l,h-m} + \sum_{h=1+1}^{p} W_{h-l,h-m} + 2 \sum_{h=1+m}^{p-l} W_{h+l,h-m} \right)^2
\]

\[
= O_p(\chi^2 T^2 np) = o_P(\psi(2^{1/2n}) \cdot u_n) = o_P(1) \quad \text{for } \alpha > 1/4 \text{ and since } \psi \to 0.
\]

Moreover,

\[
F := - \sum_{1 \leq l \leq T-1} \sum_{h=1+l}^{p-l} W_{h+l,h-l} = O_P(\sqrt{npb(\psi)}) = o_P(u_n) = o_P(1).
\]

Finally, we group the remaining terms of (31) as follows,

\[
G := \frac{1}{2} \sum_{r=1}^{T-1} \sigma_r^2 A_3 - \frac{1}{2} \sum_{i=1}^{p} W_{i,i} \sum_{j \neq i} (\sigma_{i-j})^2 + \frac{n}{4} \sum_{1 \leq i \neq j \leq p} (\sigma_{i,j})^2
\]

\[
= \frac{1}{4} \sum_{1 \leq i \neq j \leq p} (\sigma_{i,j})^2 \sum_{k=1}^{n} X_{k,i}^2 X_{k,j}^2 - \frac{1}{2} \sum_{1 \leq i \neq j \leq p} (\sigma_{i,j})^2 \sum_{k=1}^{n} X_{k-i}^2 + \frac{n}{4} \sum_{1 \leq i \neq j \leq p} (\sigma_{i,j})^2
\]

\[
= \frac{1}{4} \sum_{1 \leq i \neq j \leq p} (\sigma_{i,j})^2 \sum_{k=1}^{n} (X_{k,i}^2 - 1)(X_{k,j}^2 - 1) = O_P(\sqrt{np} \cdot b(\psi)) = o_P(u_n) = o_P(1).
\]

Let us note that throughout the previous proof we also showed that the likelihood ratio \( L_n,p \) has a variance which tends to 0, for all \( n \geq 2 \), when \( u_n \to 0 \).

**Proof of Proposition 1**. Under the null hypothesis, \( \hat{\alpha}_n \) is centered, and

\[
\text{Var}_I(\hat{\alpha}_n) = \frac{2}{n(n-1)(p-T)^4} \text{Var}_I \left( \sum_{j=1}^{T} w_j^* \sum_{1 \leq i_1, i_2 \leq p} X_{1,i_1} X_{1,i_1-j} X_{2,i_2} X_{2,i_2-j} \right)
\]

\[
= \frac{2}{n(n-1)(p-T)^4} \sum_{j=1}^{T} w_j^{*2} \sum_{1 \leq i_1, i_2 \leq p} \mathbb{E}_I(X_{1,i_1}^2 X_{1,i_1-j}^2 X_{2,i_2}^2 X_{2,i_2-j}^2)
\]

\[
= \frac{2}{n(n-1)(p-T)^2} \sum_{j=1}^{T} w_j^{*2}
\]

Recall that \( \sum_{j=1}^{T} w_j^{*2} = 1/2 \) to get the desired result. Under the alternative, for all \( \Sigma \in G(\alpha, L, \psi) \), we decompose \( \hat{\alpha}_n - \mathbb{E}_\Sigma(\hat{\alpha}_n) \) into a sum of two uncorrelated terms.

\[
\hat{\alpha}_n - \mathbb{E}_\Sigma(\hat{\alpha}_n) = \frac{1}{n(n-1)(p-T)^2} \sum_{1 \leq k \neq l \leq n} \sum_{j=1}^{T} w_j^* \sum_{1 \leq i_1, i_2 \leq p} (X_{k,i_1} X_{k,i_1-j} - \sigma_j)(X_{l,i_2} X_{l,i_2-j} - \sigma_j)
\]

\[
+ \frac{2}{n(p-T)} \sum_{k=1}^{n} \sum_{j=1}^{T} w_j^* \sum_{i=T+1}^{p} (X_{k,i_1} X_{k,i_1-j} - \sigma_j)\sigma_j.
\]
Then the variance of $\hat{A}_n$ will be given as a sum of two terms,

$$\text{Var}_\Sigma(\hat{A}_n) = \frac{R_1}{n(n-1)(p-T)^2} + \frac{R_2}{n(p-T)^2},$$

where

$$R_1 = 2E_\Sigma \left( \sum_{j=1}^{T} \sum_{T+1 \leq i_1, i_2 \leq p} (X_{1,i_1}X_{1,i_1-j} - \sigma_j)(X_{2,i_2}X_{2,i_2-j} - \sigma_j) \right)^2,$$

$$R_2 = 4E_\Sigma \left( \sum_{j=1}^{T} \sum_{i_1 = T+1}^{p} (X_{1,i_1}X_{1,i_1-j} - \sigma_j)\sigma_j \right)^2.$$

Let us deal first with $R_1$:

$$R_1 = 2 \sum_{1 \leq j, j' < T} w_j^* w_{j'}^* \sum_{T+1 \leq i_1, i_2 \leq p} E_\Sigma [(X_{1,i_1}X_{1,i_1-j} - \sigma_j)(X_{1,i_3}X_{1,i_3-j'} - \sigma_{j'})] \cdot E_\Sigma [(X_{2,i_2}X_{2,i_2-j} - \sigma_j)(X_{2,i_4}X_{2,i_4-j'} - \sigma_{j'})]$$

$$= 2 \sum_{1 \leq j, j' < T} w_j^* w_{j'}^* \left( \sum_{T+1 \leq i_1, i_2 \leq p} (\sigma_{|i_1-i_3|}\sigma_{|i_1-i_3-j+j'|} + \sigma_{|i_1-i_3-j|}\sigma_{|i_1-i_3+j'|}) \right)^2$$

$$= 2 \sum_{1 \leq j, j' < T} w_j^* w_{j'}^* \left( \sum_{r = -p+T+1}^{p-(T+1)} (p-T-|r|)(\sigma_{|r|}\sigma_{|r-j+j'|} + \sigma_{|r-j|}\sigma_{|r+j'|}) \right)^2.$$

Our aim here is to find an upper bound of $R_1$. In $R_1$ we distinguish two cases: the first one when for $j = j'$ and the second one when $j \neq j'$. Let us begin with the case when $j = j'$:

$$R_{1,1} := 2 \sum_{j=1}^{T} w_j^2 \left( \sum_{r = -p+T+1}^{p-(T+1)} (p-T-|r|)(\sigma_{0}^2 + \sigma_{|r-j+j'|}^2) \right)^2$$

$$= 2 \sum_{j=1}^{T} w_j^2 \left( (p-T)(\sigma_{0}^2 + \sigma_{j}^2) + 2 \sum_{r = 1}^{p-(T+1)} (p-T-r)(\sigma_{r}^2 + \sigma_{|r-j|}\sigma_{|r+j|}) \right)^2$$

$$= 2 \sum_{j=1}^{T} w_j^2 \left[ (p-T)^2(\sigma_{0}^2 + \sigma_{j}^2) + 4 \left( \sum_{r = 1}^{p-(T+1)} (p-T-r)(\sigma_{r}^2 + \sigma_{|r-j|}\sigma_{|r+j|}) \right)^2 \right.$$

$$\left. + 4(p-T)(\sigma_{0}^2 + \sigma_{j}^2) \sum_{r = 1}^{p-(T+1)} (p-T-r)(\sigma_{r}^2 + \sigma_{|r-j|}\sigma_{|r+j|}) \right].$$

Let us bound from above each term on the right-hand side of the previous equality:

$$R_{1,1} := 2 \sum_{j=1}^{T} w_j^2 (p-T)^2(\sigma_{0}^2 + \sigma_{j}^2) = 2(p-T)^2 \left( \sum_{j=1}^{T} w_j^2 \sigma_{0}^2 + 2 \sum_{j=1}^{T} w_j^2 \sigma_{j}^2 \right)$$

$$\leq 2(p-T)^2 \left( \frac{1}{2} + 3L \cdot (\sup_j w_j^*)^2 \right) = (p-T)^2(1 + o(1)). \quad (34)$$

30
Now we give an upper bound for the second term of (38). Using Cauchy-Schwarz inequality we get,

\[
R_{1,1,2} := 8 \sum_{j=1}^{T} w_j^2 \left[ \sum_{r=1}^{p-(T+1)} (p - T - r) \left( \sigma_r^2 + \sigma_{|r-j|} \right) \right]^2 \\
\leq 8(p - T)^2 \sum_{j=1}^{T} w_j^2 \left[ \sum_{r=1}^{p-(T+1)} \sigma_r^2 + \left( \sum_{r=1}^{p-(T+1)} \sigma_{|r-j|} \right)^{1/2} \left( \sum_{r=1}^{p-(T+1)} \sigma_{|r-j|} \right)^{1/2} \right]^2 \\
\leq 16(p - T)^2 \sum_{j=1}^{T} w_j^2 \left[ \left( \sum_{r=1}^{p-(T+1)} \sigma_r^2 \right)^2 + \left( \sum_{r=1}^{p-(T+1)} \sigma_{|r-j|} \right) \left( \sum_{r=1}^{p-(T+1)} \sigma_{|r-j|} \right) \right].
\]

Again we will treat each term of the previous inequality apart. Let us see first, that if \( r \leq j \Rightarrow w_j^* \leq w_r^* \). In addition to the previous remark we use the class property to get:

\[
R_{1,1,2,1} := \sum_{j=1}^{T} w_j^2 \left( \sum_{r=1}^{p-(T+1)} \sigma_r^2 \right)^2 \leq \sum_{j=1}^{T} w_j^2 \left( \sum_{r=1}^{j} \sigma_r^2 + \sum_{r=j+1}^{p-(T+1)} \frac{p-2\alpha}{j^{2\alpha}} \sigma_r^2 \right)^2 \\
\leq 2 \sum_{j=1}^{T} \left( \sum_{r=1}^{j} w_r^* \sigma_r^2 \right)^2 + 2(\sup_j w_j^*)^2 \sum_{j=1}^{T} \frac{1}{j^{4\alpha}} \left( \sum_{r=j+1}^{p-(T+1)} r^{2\alpha} \sigma_r^2 \right)^2 \\
\leq 2 \cdot T \cdot \mathbb{E}_{\Sigma}(\hat{A}_n) + (\sup_j w_j^*)^2 \cdot k_0(\alpha, L). \tag{35}
\]

Indeed, for \( \alpha > 1/4 \), we have, \( \sum_{j=1}^{T} j^{-4\alpha} \leq (4\alpha - 1)^{-1} \) and we can take \( k_0(\alpha, L) = 2L^2(4\alpha - 1)^{-1} \). Using similar arguments we prove that,

\[
R_{1,1,2,2} := \sum_{j=1}^{T} w_j^2 \left( \sum_{|r-j| < 1} \sigma_{|r-j|}^2 + \sum_{|r-j| \geq 1} \sigma_{|r-j|}^2 \right) \left( \sum_{r=1}^{p-(T+1)} \sigma_r^2 \right) \\
\leq \sum_{j=1}^{T} w_j^2 \left( \sum_{|r-j| < 1} w_r^* \sigma_{|r-j|}^2 + \sum_{|r-j| \geq 1} \sigma_{|r-j|} \right) + \sum_{j=1}^{T} w_j^2 \left( \sum_{|r-j| \geq 1} \left| \frac{r-j}{j^{2\alpha}} \right|^{2\alpha} \sigma_{|r-j|} \right) \left( \sum_{r=1}^{p-(T+1)} \frac{(r+j)^{2\alpha}}{j^{2\alpha}} \sigma_r^2 \right) \\
\leq (\sup_j w_j^*) \cdot T \cdot \mathbb{E}_{\Sigma}(\hat{A}_n) \cdot L + (\sup_j w_j^*)^2 \cdot k_0(\alpha, L). \tag{36}
\]
The third term in $R_{1,1}$ is treated by similar arguments:

\[
R_{1,1,3} = (p - T) \sum_{j=1}^{T} w_j^2 (\sigma_0^2 + \sigma_j^2) \sum_{r=1}^{p-(T+1)} (p - T - r)(\sigma_r^2 + \sigma_{|r-j|}\sigma_{|r+j|})
\]

\[
\leq (p - T)^2 \cdot \sup_{j} (\sigma_0^2 + \sigma_j^2) \cdot \left\{ \sum_{j=1}^{T} w_j^2 \sum_{r=1}^{j} w_r^2 \sigma_r^2 + (\sup_{j} w_j^2) \sum_{r=j}^{T-1} \frac{p-(T+1)}{j^{2\alpha}} \sum_{r=j+1}^{p-(T+1)} r^{2\alpha} \sigma_r^2 \right\}
\]

\[
+ (\sup_{j} w_j^2) \sum_{j=1}^{T} \left( \sum_{r=1}^{p-(T+1)} (r + j)^{2\alpha} \sigma_r^2 \right)^{1/2} \left\{ \sum_{r=1}^{p-(T+1)} (r + j)^{2\alpha} \sigma_r^2 \right\}^{1/2}
\]

\[
\leq 2(p - T)^2 \left\{ O(\sqrt{T}) \cdot \mathbb{E}_\Sigma(\hat{A}_n) + (\sup_{j} w_j^2) \cdot \left( O(\max\{1,T^{-2\alpha+1}\}) + O(\max\{1,T^{-\alpha+1}\}) \right) \right\}
\]

\[
\leq 2(p - T)^2 \cdot \left\{ O(\sqrt{T}) \cdot \mathbb{E}_\Sigma(\hat{A}_n) + o(1) \right\}. \quad (37)
\]

Put together bounds in (34) to (37), we can deduce that,

\[
R_{1,1} \leq (p - T)^2(1 + o(1)) + (p - T)^2 \cdot \mathbb{E}_\Sigma(\hat{A}_n) \cdot O(\sqrt{T}) + (p - T)^2 \cdot \mathbb{E}_\Sigma^2(\hat{A}_n) \cdot O(T). \quad (38)
\]

Now, we will treat the case when, $j \neq j'$.

\[
R_{1,2} := \sum_{1 \leq j \neq j' \leq T} w_j^* w_{j'}^* \left( \sum_{r=1}^{p-(T+1)} (p - |r|)(\sigma_{|r|}\sigma_{|r-j+j'|} + \sigma_{|r-j|}\sigma_{|r+j'|}) \right)^2
\]

\[
\leq 4(p - T)^2 \sum_{1 \leq j \neq j' \leq T} w_j^* w_{j'}^* \left[ \left( \sum_{r=1}^{p-(T+1)} \sigma_{|r|}\sigma_{|r-j+j'|} \right)^2 + \left( \sum_{r=-p+T+1}^{p-(T+1)} \sigma_{|r-j|}\sigma_{|r+j'|} \right)^2 \right].
\]

These last two terms are treated similarly, so let us deal with the first one. By using the same arguments as previously, we have

\[
R_{1,2,2} := \sum_{1 \leq j \neq j' \leq T} w_j^* w_{j'}^* \left( \sigma_{|j'-j|}^2 + \sum_{r=-p+T+1}^{p-(T+1)} \sigma_{|r|}\sigma_{|r-j+j'|} \right)^2
\]

\[
\leq 2 \sum_{1 \leq j \neq j' \leq T} w_j^* w_{j'}^* \sigma_{|j'-j|}^2 + 4 \sum_{1 \leq j \neq j' \leq T} w_j^* w_{j'}^* \left( \sum_{r=1}^{p-(T+1)} \sigma_r^2 \right) \left( \sum_{r=-p+T+1}^{p-(T+1)} \sigma_{|r-j+j'|}^2 \right). \quad (39)
\]

We decompose the sum over $j \neq j'$ over sets where $\{|j' - j| \leq j\}$ and $\{|j' - j| > j\}$ and use
1 \leq |j' - j|^{2\alpha}/j^{2\alpha} over the later, then similarly for sums over $r$:

\[ R_{1,2,2} \leq 2 \sum_{1 \leq j, j' \leq T \atop j' \neq j} w_j^* w_{j'}^* \sigma_j^2 \left| \frac{j'}{j} \right|^{2\alpha} \sigma_j' \sigma_{j'}' + 2 \sum_{1 \leq j, j' \leq T \atop j' > j} w_j^* w_{j'}^* \left| \frac{j'}{j} \right|^{2\alpha} \sigma_j' \sigma_{j'}' + 4 \sum_{r=1}^{\infty} \left( \sum_{j} w_j^* \sigma_r^2 \right) + w_j^* \left( \sum_{r=p+T+1}^{p-(T+1)} \frac{2\alpha}{j^{2\alpha}} \sigma_r \right) \] 

\begin{align*}
&\leq 4 \cdot (\sup_j w_j^*) \cdot T \cdot \mathbb{E}_\Sigma(\hat{\mathcal{A}}_n) + 4L \cdot (\sup_j w_j^*)^2 \cdot O(\max\{1, T^{-2\alpha+1}\}) + O(T^2) \cdot \mathbb{E}_\Sigma(\hat{\mathcal{A}}_n) \\
&+ 16L \cdot (\sup_j w_j^*) \cdot T \cdot O(\max\{1, T^{-2\alpha+1}\}) \cdot \mathbb{E}_\Sigma(\hat{\mathcal{A}}_n) + 16L^2 \cdot (\sup_j w_j^*)^2 \cdot O(\max\{1, T^{-4\alpha+2}\}).
\end{align*}

As consequence, for all $\alpha > 1/4$,

\[ R_{1,2} \leq (p - T)^2 \{ \mathbb{E}_\Sigma(\hat{\mathcal{A}}_n) \cdot O(\sqrt{T}) + \mathbb{E}_\Sigma^2(\hat{\mathcal{A}}_n) \cdot O(T^2) + \mathbb{E}_\Sigma(\hat{\mathcal{A}}_n) \cdot O(T^{3/2-2\alpha}) + o(1) \}. \] (40)

Finally put together (38) and (40) to get (8). In order to find an upper bound for the variance of $\hat{\mathcal{A}}_n$ we still have to bound from above $R_2$.

\[ R_2 = 4 \sum_{1 \leq j, j' \leq T} w_j^* w_{j'}^* \sigma_j \sigma_{j'} \sum_{T+1 \leq i_1, i_2 \leq p} \mathbb{E}_\Sigma[(X_{i_1i_1}X_{i_1i_2}X_{i_2i_2} - \sigma_j)(X_{i_2i_1}X_{i_1i_2} - \sigma_{j'})]
\]

\[ = 4 \sum_{1 \leq j, j' \leq T} w_j^* w_{j'}^* \sigma_j \sigma_{j'} \sum_{T+1 \leq i_1, i_2 \leq p} \sum_{p-(T+1)}^{p-(T+1)} \mathbb{E}_\Sigma[(\sigma_{i_1i_2}\sigma_{i_1i_2} - \sigma_j)(\sigma_{i_1i_2}\sigma_{i_1i_2} - \sigma_{j'})]
\]

\[ = 4 \sum_{1 \leq j, j' \leq T} w_j^* w_{j'}^* \sigma_j \sigma_{j'} \sum_{r=-p+T+1}^{p-(T+1)} (p - T - |r|)(\sigma_{i_1i_2}\sigma_{i_1i_2} - \sigma_j)(\sigma_{i_1i_2}\sigma_{i_1i_2} - \sigma_{j'})
\]

Let us begin by the first case when $j = j'$. It is easily seen that,

\[ R_{2,1} := 4 \sum_{j=1}^{p-(T+1)} w_j^* \sigma_j \sum_{r=-p+T+1}^{p-(T+1)} (p - T - |r|)(\sigma_{i_1i_2}\sigma_{i_1i_2} - \sigma_j)(\sigma_{i_1i_2}\sigma_{i_1i_2} - \sigma_{j'})
\]

\[ \leq 8L \cdot p \cdot (\sup_j w_j^*) \cdot \mathbb{E}_\Sigma(\hat{\mathcal{A}}_n) \] (41)

While, when $j \neq j'$, we can prove that,

\[ R_{2,2} := 4 \sum_{1 \leq j, j' \leq T} w_j^* w_{j'}^* \sigma_j \sigma_{j'} \sum_{r=-p+T+1}^{p-(T+1)} (p - T - |r|)(\sigma_{i_1i_2}\sigma_{i_1i_2} - \sigma_j)(\sigma_{i_1i_2}\sigma_{i_1i_2} - \sigma_{j'})
\]

\[ \leq 4 \left( \sum_{1 \leq j, j' \leq T} w_j^* w_{j'}^* \frac{\sigma^2}{j^2} \right)^{1/2} \left( \sum_{1 \leq j, j' \leq T} w_j^* w_{j'}^* \left( \sum_{r=-p+T+1}^{p-(T+1)} (p - T - |r|)(\sigma_{i_1i_2}\sigma_{i_1i_2} - \sigma_j)(\sigma_{i_1i_2}\sigma_{i_1i_2} - \sigma_{j'}) \right)^{1/2} \right)
\]

\[ \leq 4 \mathbb{E}_\Sigma(\hat{\mathcal{A}}_n) \cdot (R_{1,2})^{1/2}.
\]

We use the bound obtained in (40) to deduce that:

\[ R_{2,2} \leq (p - T) \left( \mathbb{E}_\Sigma^2(\hat{\mathcal{A}}_n) \cdot O(T) + \mathbb{E}_\Sigma^2(\hat{\mathcal{A}}_n) \cdot O(T^{1/4}) + \mathbb{E}_\Sigma(\hat{\mathcal{A}}_n) \cdot o(1) \right). \] (42)
Therefore we should check that $H_k$ kernel differences, see e.g. [30].

The result is an application of the more general theorem of asymptotic normality for martingale

In order to prove this previous convergence, we are led to apply theorem 1 of [21]. This

To show that,

According to (9), and under the assumption that $np \cdot \mathbb{E}(\hat{A}_n) \to 1$, we can see that,

Which involves by Slutsky theorem that for proving the asymptotic normality it is sufficient to show that,

In order to prove this previous convergence, we are led to apply theorem 1 of [21]. This

$A_n, \ldots$ is a centered, 1-degenerate, U-Statistic of second order, with kernel $H_n(X_1, X_2)$ defined by,

Therefore we should check that $\mathbb{E}(H_n^2(X_1, X_2)) < +\infty$ and

where $G_n(\mathbb{x, y}) := \mathbb{E}(H_n(X_1, \mathbb{x, y})H_n(X_1, \mathbb{y}))$, for $x, y \in \mathbb{R}^p$. The proof of (44) is given separately hereafter.

The asymptotic normality under $\Sigma = I$ (the null hypothesis) is only simpler as $\sigma_j = 0$ for all $j \geq 1$, for $n, p \to \infty$. However, under the null hypothesis we prove separately (hereafter) that

(45)
Proof of (44). To show (44), we first calculate $G_n(x, y)$ and $E_\Sigma(H^2_n(X_1, X_2))$. That is,

$$
G_n(x, y) = \frac{1}{n^2(p - T)^2} \sum_{1 \leq j_1, j_2 < T} w^*_1 w^*_{j_1} \sum_{r=-p+T+1}^{p-(T+1)} (p - T - |r|)(\sigma_{|r|} \sigma_{r-j_1+j_2} + \sigma_{r-j_1} \sigma_{r+j_2})
$$

$$
\sum_{1 \leq i_1, i_2 \leq p} (x_{i_1} x_{i_1-j_1} - \sigma_{j_1})(y_{i_2} y_{i_2-j_2} - \sigma_{j_2})
$$

(46)

Note that, under the assumption $np \cdot E(\tilde{A}_n) \asymp 1$, $\alpha > 1/4$, $p^{1/\alpha} \to +\infty$ and using (48), we have,

$$
E_\Sigma(H^2_n(X_1, X_2)) = \frac{1 + o(1)}{2n^2}
$$

Now, let us verify that, uniformly over $\Sigma$,

$$
E_\Sigma(G^2_n(X_1, X_2)) / E_\Sigma(H^2_n(X_1, X_2)) = o(1).
$$

(47)

We write

$$
\frac{E_\Sigma(G^2_n(X_1, X_2))}{E_\Sigma(H^2_n(X_1, X_2))} = 4n^4 \cdot E_\Sigma(G^2_n(X_1, X_2))
$$

$$
= \frac{4}{(p - T)^4} \sum_{1 \leq j_1, j_2, j_3, j_4 < T} w^*_1 w^*_{j_1} w^*_{j_2} w^*_{j_3} \sum_{-p+T+1 \leq r_1, r_2 \leq p-(T+1)} (p - T - |r_1|)(p - T - |r_2|)
$$

$$
\cdot (\sigma_{|r_1|} \sigma_{r_1-j_1+j_2} + \sigma_{|r_1-j_1|} \sigma_{r_1+j_2}) (\sigma_{|r_2|} \sigma_{r_2-j_3+j_4} + \sigma_{r_2-j_3} \sigma_{r_2+j_4})
$$

$$
\cdot \sum_{T+1 \leq i_1, i_3 \leq p} E_\Sigma[(X_{1, i_1} X_{1, i_1-j_1} - \sigma_{j_1})(X_{1, i_3} X_{1, i_3-j_3} - \sigma_{j_3})]
$$

$$
\cdot \sum_{T+1 \leq i_2, i_4 \leq p} E_\Sigma[(X_{2, i_2} X_{2, i_2-j_2} - \sigma_{j_2})(X_{2, i_4} X_{2, i_4-j_4} - \sigma_{j_4})]
$$

(48)

We calculate each expected value, and bound from above by the absolute value, we obtain:

$$
4n^4 \cdot E_\Sigma(G^2_n(X_1, X_2)) \leq 4 \sum_{1 \leq j_1, j_2, j_3, j_4 < T} w^*_1 w^*_{j_1} w^*_{j_2} w^*_{j_3} \sum_{-p+T+1 \leq r_1, r_2, r_3, r_4 \leq p-(T+1)} (|\sigma_{|r_1|} \sigma_{r_1-j_1+j_2}| + |\sigma_{|r_1-j_1|} \sigma_{r_1+j_2}|)(|\sigma_{|r_2|} \sigma_{r_2-j_3+j_4}| + |\sigma_{r_2-j_3} \sigma_{r_2+j_4}|)
$$

$$
\cdot (|\sigma_{|r_3|} \sigma_{r_3-j_3+j_4}| + |\sigma_{r_3-j_3} \sigma_{r_3+j_4}|)(|\sigma_{|r_4|} \sigma_{r_4-j_2+j_4}| + |\sigma_{r_4-j_2} \sigma_{r_4+j_4}|)
$$

(49)

In (49), there are sixteen terms, that are all treated the same way, then we deal with,

$$
G := 4 \sum_{1 \leq j_1, j_2, j_3, j_4 < T} w^*_1 w^*_{j_1} w^*_{j_2} w^*_{j_3} \sum_{-p+T+1 \leq r_1, r_2, r_3, r_4 \leq p-(T+1)} (|\sigma_{|r_1|} \sigma_{r_1-j_1+j_2}|)(|\sigma_{|r_2|} \sigma_{r_2-j_3+j_4}|)(|\sigma_{|r_3|} \sigma_{r_3-j_3+j_4}|)(|\sigma_{|r_4|} \sigma_{r_4-j_2+j_4}|)
$$

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To bound from above this previous quantity, we distinguish four cases, based on the indices 
j_1, j_2, j_3 and j_4. Let us begin by the first case, when \( j_1 = j_2 = j_3 = j_4 \):

\[
G_1 := 4 \sum_{j_1=1}^{T} w_{j_1}^4 \sum_{-p+T+1 \leq r_1, r_2, r_3, r_4 \leq p-T} \sigma_{r_1}^2 \sigma_{r_2}^2 \sigma_{r_3}^2 \sigma_{r_4}^2 \leq 4 \left( \sup_j w_j^4 \right) \cdot T \cdot (2L)^4 = O\left( \frac{1}{T} \right) = o(1)
\]

We consider the second case, where there are two different values of indices, either two groups of 
two, or one group of three and one separate index. For the first one, let us assume that 
\( j_1 = j_4, j_2 = j_3 \text{ and } j_1 \neq j_2 \),

\[
G_2 := 4 \sum_{1 \leq j_1 \neq j_2 < T} w_{j_1}^2 w_{j_2}^2 \sum_{-p+T+1 \leq r_1, r_2, r_3, r_4 \leq p-T} \sigma_{r_1}^2 \sigma_{r_2}^2 \sigma_{r_3}^2 \sigma_{r_4}^2 \leq 4 \left( \sup_j w_j^4 \right) \cdot T \cdot (2L)^4 = O\left( \frac{1}{T} \right) = o(1)
\]

We apply the Cauchy-Schwarz inequality with respect to \( r_1 \) and \( r_2 \) separately to get:

\[
G_2 \leq 4 \cdot (2 + 2L)^2 \sum_{1 \leq j_1 \neq j_2 < T} w_{j_1}^2 w_{j_2}^2 \cdot \left\{ 4 \sigma_{|j_1 - j_2|}^2 \right\} \leq 16 \cdot (2 + 2L)^2 \left( \sum_{1 \leq j_1 \neq j_2 < T} w_{j_1}^2 w_{j_2}^2 \sigma_{|j_1 - j_2|}^2 + 2L \sum_{1 \leq j_1 \neq j_2 < T} w_{j_1}^2 w_{j_2}^2 \sum_{r_1 \neq 0} \sigma_{r_1}^2 \right)
\]

\[
\leq 16 \cdot (2 + 2L)^2 \left( \sup_{j_2} \sum_{j_1=1}^{T} w_{j_1}^2 \sum_{j_2=1}^{T} \sigma_{|j_1 - j_2|}^2 \right) + 2L \cdot \sum_{j_1=1}^{T} \sum_{r_1 \neq 0} |r_1|^{2\alpha} \sum_{j_1=1}^{T} \sum_{r_1 \neq 0} |r_1|^{2\alpha} \sigma_{|j_1 - j_2|}^2
\]

\[
\leq O\left( \frac{1}{T} \right) + O(\sqrt{T}) \cdot E(\hat{A}_n) + O\left( \frac{1}{T} \right) \cdot \max\{1, T^{-2\alpha+1}\} = o(1) \text{ since } E(\hat{A}_n) \asymp 1/np \text{ and } T/p \to 0.
\]

Similar argument to prove that for \( j_1 = j_3 = j_4 \text{ and } j_1 \neq j_2 \), we have,

\[
4 \sum_{1 \leq j_1 \neq j_2 < T} w_{j_1}^3 w_{j_2}^* \sum_{-p+T+1 \leq r_1, r_2, r_3, r_4 \leq p-T} \sigma_{r_1}^2 \sigma_{r_2}^2 \sigma_{r_3}^2 \sigma_{r_4}^2 \sigma_{|r_1 - r_2 + r_3 - r_4|}^2 = o(1)
\]

which finishes the second case. Now let us assume that we have three different values, \( j_1 = j_4 \)
and $j_1 \neq j_2 \neq j_3$), we obtain,

$$G_3 := 4 \sum_{1 \leq j_1 \neq j_2 \neq j_3 < T} w_{j_1}^* w_{j_2}^* w_{j_3}^* \sum_{r_1 = -p+T+1, r_2 \neq -p+T+1, r_3 \neq -p+T+1} \left| \sigma_{|r_1|} \sigma_{|r_1-j_2+j_3|} \sigma_{|r_2+j_3-j_1|} \right|$$

$$= 4 \sum_{1 \leq j_1 \neq j_2 \neq j_3 < T} w_{j_1}^* w_{j_2}^* w_{j_3}^* \left( 2 |\sigma_{j_2-j_1}| + \sum_{r_1 = -p+T+1, r_1 \neq -p+T+1, r_1 \neq j_2-j_1} \left| \sigma_{|r_1|} \sigma_{|r_1-j_2+j_3|} \right| \right)$$

and hence

$$G_{3,1} := \sum_{1 \leq j_1 \neq j_2 \neq j_3 < T} w_{j_1}^* w_{j_2}^* w_{j_3}^* |\sigma_{j_2-j_1}| |\sigma_{j_1-j_3}| |\sigma_{j_3-j_2}| \leq \left( \sup_{j} w_j^* \right)^2 \sum_{j_1=1}^{T} w_{j_1}^2 \sum_{j_2=1}^{T} w_{j_2}^2 \sum_{j_3=1}^{T} |\sigma_{j_1-j_3}|$$

$$\leq \left( \sup_{j} w_j^* \right)^2 \cdot \frac{1}{2} \cdot 4L^2 = o(1).$$

Note that $\sup \sigma_r \leq 1$ and by Cauchy-Schwarz we have

$$\sum_{r_4 = -p+T+1, r_4 \neq -p+T+1, r_4 \neq j_2-j_1} \left| \sigma_{|r_4|} \sigma_{|r_4-j_2+j_1|} \right| \leq \sum_{r_4 = -p+T+1, r_4 \neq -p+T+1, r_4 \neq j_2-j_1} \sigma_{r_4}^2.$$

Thus we get,

$$G_{3,2} := \sum_{1 \leq j_1 \neq j_2 \neq j_3 < T} w_{j_1}^2 w_{j_2}^2 w_{j_3}^2 |\sigma_{j_2-j_1}| |\sigma_{j_1-j_3}| |\sigma_{j_3-j_2}| \sum_{r_4 = -p+T+1, r_4 \neq -p+T+1, r_4 \neq j_2-j_1} \left| \sigma_{|r_4|} \sigma_{|r_4-j_2+j_1|} \right|$$

$$\leq \left( \sup_{j} w_j^* \right) \cdot \sum_{j_1} w_{j_1}^2 \sum_{j_2} w_{j_2}^2 \sum_{j_3} |\sigma_{j_1-j_3}| \left( \sum_{r_4 \leq j_2} \sigma_{r_4}^2 + \sum_{r_4 > j_2} \sigma_{r_4}^2 \right)$$

$$\leq \left( \sup_{j} w_j^* \right) \cdot \frac{1}{2} \cdot 2L \cdot \left( T \cdot \mathbb{E}(\tilde{A}_n) + \left( \sup_{j} w_j^* \right) \cdot \max \{1, T^{-2\alpha+1} \} \cdot 2L \right) = o(1).$$

Moreover,

$$G_{3,3} := \sum_{1 \leq j_1 \neq j_2 \neq j_3 < T} w_{j_1}^2 w_{j_2}^2 w_{j_3}^2 |\sigma_{j_2-j_1}| |\sigma_{j_1-j_3}| |\sigma_{j_3-j_2}| \sum_{r_2 = -p+T+1, r_2 \neq -p+T+1, r_2 \neq j_1-j_3} \sum_{r_3 = -p+T+1, r_3 \neq -p+T+1, r_3 \neq j_2-j_3} \left| \sigma_{|r_2|} \sigma_{|r_2-j_1+j_3|} \right|$$

$$\leq \sum_{j_1} w_{j_1}^2 \sum_{j_2} w_{j_2}^2 \left( \sum_{r_2 \leq j_2} \sum_{r_2 > j_2} \sigma_{r_2}^2 \sum_{r_2 \neq j_2} \frac{|r_4|^{2\alpha}}{j_2^3} \sigma_{r_4}^2 \right) \sum_{j_3} w_{j_3}^2 \left( \sum_{r_4 \leq j_3} \sum_{r_4 > j_3} \sigma_{r_4}^2 \sum_{r_4 \neq j_3} \frac{|r_4|^{2\alpha}}{j_3^3} \sigma_{r_4}^2 \right)$$

$$\leq \frac{1}{2} \left( T \cdot \mathbb{E}(\tilde{A}_n) + \left( \sup_{j} w_j^* \right) \cdot \max \{1, T^{-2\alpha+1} \} \cdot 2L \right)^2 = o(1)$$

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and
\[
G_{3,4} := \sum_{1 \leq j_1 \neq j_2, j_3 < T} w_{j_1}^* w_{j_2}^* w_{j_3}^* \sigma_{j_2-j_1} \sum_{r_2=-p+T+1}^{p-(T+1)} |\sigma_{r_2}| \max\{1, T\} \sum_{r_3=0}^{p-(T+1)} |\sigma_{r_3}| \sigma_{r_3-j_2-j_1} \mid \\
\sum_{r_4=-p+T+1}^{p-(T+1)} |\sigma_{r_4}| \sigma_{r_4-j_2-j_1} \mid \leq \sum_{j_1} w_{j_1}^2 \sum_{j_2} w_{j_2}^2 \sum_{j_3} w_{j_3}^2 \sum_{r_2=0}^{\sigma_{r_2} \leq 2} |\sigma_{r_2}|^2 \sum_{j_1} w_{j_3}^2 \sum_{j_2} \sum_{1 \leq j_2, j_3 < T} w_{j_2}^2 w_{j_3}^2 \sigma_{j_2-j_1} \mid \\
= o(1).
\]

Similarly we show that
\[
G_{3,5} := \sum_{1 \leq j_1 \neq j_2, j_3 < T} w_{j_1}^* w_{j_2}^* w_{j_3}^* \sum_{r_1=-p+T+1}^{p-(T+1)} |\sigma_{r_1}| \sigma_{r_1-j_1-j_2} \mid \\
\sum_{r_2=-p+T+1}^{p-(T+1)} |\sigma_{r_2}| \sigma_{r_2-j_1-j_2} \mid = o(1).
\]

Finally, when all indices are pairwise distinct. We use the same arguments as previously, and we get,
\[
G_4 := 4 \left( \sum_{1 \leq j_1 \neq j_2, j_3 < T} w_{j_1}^* w_{j_2}^* w_{j_3}^* w_{j_4}^* \sum_{r_1=-p+T+1}^{p-(T+1)} |\sigma_{r_1}| \sigma_{r_1-j_1-j_2} | \mid \\
\sum_{r_2=-p+T+1}^{p-(T+1)} |\sigma_{r_2}| \sigma_{r_2-j_1-j_2} | \mid + \sum_{r_3=-p+T+1}^{p-(T+1)} |\sigma_{r_3}| \sigma_{r_3-j_1-j_2} | \mid \\
+ \sum_{r_4=-p+T+1}^{p-(T+1)} |\sigma_{r_4}| \sigma_{r_4-j_1-j_2} | \mid \right) \leq \left( \sum_{j_1, j_2} w_{j_1}^* w_{j_2}^* \sigma_{j_1-j_2}^2 \sum_{j_3, j_4} w_{j_3}^* w_{j_4}^* \sigma_{j_3-j_4}^2 \right) \frac{1}{2} \left( \sum_{j_1, j_2} w_{j_1}^* w_{j_2}^* \sigma_{j_1-j_2}^2 \sum_{j_3, j_4} w_{j_3}^* w_{j_4}^* \sigma_{j_3-j_4}^2 \right) \frac{1}{2} \leq \left( \sum_{j_1} w_{j_1}^* \sum_{j_2 \geq 1} w_{j_2}^* \sigma_{j_2-j_1}^2 \right) + \left( \sup_{j_0} w_{j_0}^* \right) \sum_{j_1} \sum_{j_2 \geq 1} w_{j_2}^* \sigma_{j_2-j_1}^2 \leq \left( O(\sqrt{T}) \cdot \mathcal{E}_\Sigma(\hat{A}_n) + O\left( \frac{1}{\sqrt{T}} \right) \cdot \max\{1, T^{-2a+1}\} \cdot L \right)^2 = o(1)
\]

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and

\[ G_{4,2} := \sum_{1 \leq j_1 \neq j_2 \neq j_3 \neq j_4 < T} w_{j_1}^* w_{j_2}^* w_{j_3}^* w_{j_4}^* |\sigma_{j_1-j_2}| |\sigma_{j_3-j_4}| |\sigma_{j_1-j_3}| \sum_{r_4 \neq 0, r_4 \neq j_2-j_4} |\sigma_{r_4}| |\sigma_{r_4-j_2+j_4}| \]

\[ \leq \sum_{j_2, j_4} \sum_{j_1} w_{j_2}^* w_{j_4}^* \left( \sum_{j_3} \sum_{j_1} w_{j_1}^* w_{j_3}^* \sigma_{j_1-j_3}^2 \right)^{\frac{1}{2}} \left( \sum_{j_1} w_{j_1}^* \sigma_{j_1-j_1}^2 \right)^{\frac{1}{2}} \left( \sum_{j_3} w_{j_3}^* \sigma_{j_3-j_3}^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{r_4 \neq 0, r_4 \neq j_2-j_4} |\sigma_{r_4}| + \sum_{r_4 \neq 0, r_4 \neq j_2-j_4} |\sigma_{r_4}| \right) \]

\[ \leq \left( \sum_{j_1} \sum_{j_3} w_{j_1}^* w_{j_3}^* \sigma_{j_1-j_3}^2 \right)^{\frac{1}{2}} \left( \sup_j w_j^* \right) \cdot L \]

\[ \cdot \left( \sum_{j_2, j_4} \sum_{r_4 \neq 0} w_{j_2}^* \sigma_{r_4}^2 + \sum_{j_2, j_4} \sum_{r_4 \neq 0} w_{j_2}^* w_{j_4}^* \sum_{r_4 \neq 0} \frac{|r_4|^{2|\sigma_2|}}{2^2 \sigma_2^2} \right) \]

\[ \leq \left( O(\sqrt{T}) \cdot \mathbb{E}_\Sigma(\tilde{A}_n) + O \left( \frac{1}{\sqrt{T}} \right) \cdot \max \{1, T^{-2\alpha+1}\} \right)^{\frac{1}{2}} \cdot O \left( \frac{1}{\sqrt{T}} \right) \cdot \left( T \cdot \mathbb{E}_\Sigma(\tilde{A}_n) + T \cdot \max \{1, T^{-2\alpha+1}\} \cdot L \right) \]

\[ \leq \left( O(\sqrt{T}) \cdot \mathbb{E}_\Sigma(\tilde{A}_n) + O \left( \frac{1}{\sqrt{T}} \right) \cdot \max \{1, T^{-2\alpha+1}\} \cdot 2L \right)^{\frac{1}{2}} \]

\[ \cdot \left( T \cdot \mathbb{E}_\Sigma(\tilde{A}_n) + O \left( \frac{1}{\sqrt{T}} \right) \cdot \max \{1, T^{-2\alpha+1}\} \cdot L \right) \]

\[ = o(1) \quad \text{since } \mathbb{E}(\tilde{A}_n) \asymp 1/np \text{ and for all } \alpha > 1/4. \]

We use similar argument as previously to show that the remaining terms in \( G_4 \) tend to zero.

To complete the proof, we need to verify that,

\[ \mathbb{E}_\Sigma(H_n^4(X_1, X_2))/\mathbb{E}_\Sigma^2(H_n^2(X_1, X_2)) = o(n). \]  

(51)

We write

\[ \frac{\mathbb{E}_\Sigma(H_n^4(X_1, X_2))}{\mathbb{E}_\Sigma^2(H_n^2(X_1, X_2))} = \frac{1}{(p-T)^4} \sum_{j_1, j_2, j_3, j_4} w_{j_1}^* w_{j_2}^* w_{j_3}^* w_{j_4}^* \sum_{T+1 \leq i_3, i_4} \sum_{T+1 \leq i_5, i_6} \mathbb{E}_\Sigma[(X_{i_1, i_3} X_{i_1, i_3} - \sigma_{i_1, i_3})(X_{i_1, i_3} X_{i_1, i_3} - \sigma_{i_1, i_3}) (X_{i_1, i_3} X_{i_1, i_3} - \sigma_{i_1, i_3}) (X_{i_1, i_3} X_{i_1, i_3} - \sigma_{i_1, i_3})] \]

\[ \cdot \mathbb{E}_\Sigma[(X_{i_1, i_3} X_{i_1, i_3} - \sigma_{i_1, i_3})(X_{i_1, i_3} X_{i_1, i_3} - \sigma_{i_1, i_3}) (X_{i_1, i_3} X_{i_1, i_3} - \sigma_{i_1, i_3}) (X_{i_1, i_3} X_{i_1, i_3} - \sigma_{i_1, i_3})] \]

To bound from above the previous sum, we replace the expected value by it's value, which is a sum of many terms, that are all treated similarly. So let us give an upper bound for the
followed by:

\[ H := \frac{1}{(p-T)^4} \sum_{j_1,j_2,j_3,j_4} w_{j_1}^* w_{j_2}^* w_{j_3}^* w_{j_4}^* \sum_{i_1,i_2,i_3,i_4 \leq p} \sigma_{|i_1-i_2|} \sigma_{|i_1-i_3-j_1+j_2|} \]

\[ \leq \sum_{j_1,j_2,j_3,j_4} w_{j_1}^* w_{j_2}^* w_{j_3}^* w_{j_4}^* \sum_{i_1,i_2,i_3,i_4 \leq p} \sigma_{|i_1-i_2|} \sigma_{|i_1-i_3-j_1+j_2|} \]

We see that \( H \) can be treated in the same way as \( G \). However, we show that \( H = O(1) = o(n) \).

Let us deal with one of the terms of \( H \), consider the term for which we have \( j_1 = j_2, j_3 = j_4 \), and \( j_1 \neq j_3 \) thus we get

\[
\sum_{1 \leq j_1 \neq j_3 < T} \sum_{-p+1 \leq r_1, r_2, r_3, r_4 \leq p-1} w_{j_1}^* w_{j_3}^* \sigma_{|r_1|}^2 \sigma_{|r_2|}^2 \sigma_{|r_3|}^2 \sigma_{|r_4|}^2 = \sum_{1 \leq j_1 \neq j_3 < T} w_{j_1}^* w_{j_3}^* \left( \sigma_0^2 + \sum_{r_1 \neq 0} \sigma_{|r_1|}^2 \right)^2 
\]

It is easily seen that \( \sum_{1 \leq j_1 \neq j_3 < T} w_{j_1}^2 w_{j_3}^2 = O(1) \). And so on, we show that all terms in \( H \) are \( O(1) \) and thus we get the desired result. Together with \( \text{(47)} \), this proves \( \text{(41)} \). In consequence, we apply theorem 1 of \( \text{[21]} \), to get \( \text{(43)} \).

\[ \text{Proof of (45).} \]

We define \( \hat{B}_{n,p} \) as follows,

\[
\hat{B}_{n,p} = \frac{2}{\sqrt{n(n-1)(p-T)(p-T-1)}} \sum_{i=T+1}^{p} \sum_{h=i+1}^{p} \sum_{1 \leq k \neq j}^{T-1} w_j^* X_{k,i} X_{k,-j} X_{l,h} X_{l,-j} 
\]

We set

\[
D_{n,p,i} = \frac{2}{\sqrt{n(n-1)(p-T)(p-T-1)}} \sum_{h=i+1}^{p} \sum_{1 \leq k \neq j}^{T-1} w_j^* X_{k,i} X_{k,-j} X_{l,h} X_{l,-j} 
\]

Note that the sequence of martingale differences \( \{D_{n,p,i}\}_{T+1 \leq i \leq p} \) is a sequence of martingale differences with respect to the sequence of \( \{F_i, i \geq T + 1\} \) such that \( F_i = \sigma \{X, r, r \leq i\} \), we denote by \( \mathbb{E}_i(\cdot) = \mathbb{E}(\cdot | F_i) \), where \( \mathbb{E} \) is the expected value under the null hypothesis. Indeed, for all \( T+1 \leq i \leq p \), we have, \( \mathbb{E}_{i-1}(D_{n,p,i}) = 0 \). We use sufficient conditions to show the asymptotic normality of a sum of martingale differences \( \hat{B}_{n,p} \) for all \( n \geq 2 \), as \( (p-T) \to \infty \), see e.g. \( \text{[30]} \). Thus it suffices to show that,

\[
\mathbb{E} \left( \sum_{i=T+1}^{p} \mathbb{E}_{i-1}(D_{n,p,i}^2) - 1 \right)^2 \to 0 \quad \text{and} \quad \sum_{i=T+1}^{p} \mathbb{E}(D_{n,p,i}^4) \to 0. \quad (52)
\]
We first show the first part of (52).

\[
E_{i-1}(D_{n,p,i}^2) = (c(n, p, T))^2 \sum_{h=i+1}^{p} \sum_{1 \leq k \neq l \leq n} \sum_{1 \leq j, j_1 \leq T-1} w_j w_{j_1}^* X_{k,i-j} X_{k,i-j_1} E_{i-1}(X_{l,h-j} X_{l,h-j_1})
\]

\[
= (c(n, p, T))^2 \left( \sum_{1 \leq j, j_1 \leq T-1} \sum_{1 \leq k \neq l \leq n} w_j w_{j_1}^* X_{k,i-j} X_{k,i-j_1} \sum_{h=i+1}^{p} X_{l,h-j} X_{l,h-j_1} \right)
\]

\[
+ (n - 1) \sum_{k=1}^{n} \sum_{j=1}^{T} w_j^2 X_{k,i-j}^2 (p - i - j + 1)
\]

giving

\[
\mathbb{E}( \sum_{i=T+1}^{p} E_{i-1}(D_{n,p,i}^2) )
\]

\[
= c^2(n, p, T) \cdot \left( n(n - 1) \sum_{i=T+1}^{p} \sum_{j=1}^{T-1} w_j^2 (j - 1) + n(n - 1) \sum_{i=T+1}^{p} \sum_{j=1}^{T-1} w_j^2 (p - i - j + 1) \right)
\]

\[
= \frac{4}{(p-T)(p-T-1)} \sum_{j=1}^{T-1} w_j^2 \sum_{i=T+1}^{p} (p - i) = 1.
\]

Thus, to show that \( \mathbb{E}\left( \sum_{i=T+1}^{p} E_{i-1}(D_{n,p,i}^2) \right) \to 0 \), it is sufficient to show that \( \mathbb{E}\left( \sum_{i=T+1}^{p} E_{i-1}(D_{n,p,i}^2) \right)^2 = 1 + o(1) \). Indeed,

\[
\mathbb{E}\left( \sum_{i=T+1}^{p} E_{i-1}(D_{n,p,i}^2) \right)^2 = (c(n, p, T))^4 \cdot \left( E_1 + E_2 + E_3 + E_4 \right). \tag{53}
\]

where \( E_1, E_2, E_3 \) and \( E_4 \) are given by the following.

\[
E_1 = \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq k \neq l \leq n} \sum_{1 \leq j, j_1 \leq T} \sum_{1 \leq j', j'_1 \leq T-1} \sum_{h=i+1}^{T} \sum_{h'=i'+1}^{T-1} w_j^* w_{j_1}^* w_{j'}^* w_{j'_1}^* \mathbb{E}(X_{k,i-j} X_{k,i-j_1} X_{l,h-j} X_{l,h-j_1} X_{k',i'-j'} X_{l',h'-j'_1} X_{k',i'-j'_1} X_{l',h'-j_1})
\]

Now we decompose \( E_1 \) into five sums that depends on the indices \( k, k', l \) and \( l' \). We begin by
the first case when \( k = k' \) and \( l = l' \),

\[
E_{1,1} := \sum_{1 \leq k \neq l \leq n} \left( \sum_{i=T+1}^{p} \left( \sum_{j=1}^{T-1} w_j^4 \cdot (3(j-1) + (j-1)(j-2)) + \sum_{1 \leq j \neq j' \leq T} w_{j'}^2 w_{j}^2 (j-1)(j'-1) \right) \right.
\]
\[
+ 2 \sum_{1 \leq j \neq j' \leq T-1} (j-1) \wedge (j-1)
\]
\[
+ \sum_{T+1 \leq i \neq i' \leq p} \left( \sum_{j=1}^{T-1} w_j^4 (3(j-1) + (j-1)(j-2)) + \sum_{1 \leq j \neq j' \leq T} w_{j'}^2 w_{j}^2 (j-1)(j'-1) \right)
\]
\[
= n(n-1) \left( 2 \sum_{i=T+1}^{p} \sum_{j=1}^{T-1} w_j^4 (j-1)(j+1) + 2 \sum_{T+1 \leq i, i' \leq p} \sum_{j=1}^{T-1} w_j^4 (j-1) \right)
\]

When \( k = l' \) and \( l = k' \), we have using similar arguments as previously that,

\[
E_{1,2} = n(n-1) \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j, j' \leq T} w_j^2 w_{j'}^2 (j-1)(j'-1)
\]

We move to the term, when \( k = k' \) and \( l \neq l' \),

\[
E_{1,3} := \sum_{1 \leq k, l, l' \leq n} \left\{ \sum_{j=1}^{T-1} w_j^4 \left( \sum_{i=T+1}^{p} 3(j-1)^2 + \sum_{T+1 \leq i \neq i' \leq p} (j-1)^2 \right) \right. \]
\[
+ \sum_{1 \leq j \neq j' \leq T} w_j^2 w_{j'}^2 \sum_{T+1 \leq i, i' \leq p} (j-1)(j'-1) \right\}
\]
\[
= n(n-1)(n-2) \left( 2 \sum_{i=T+1}^{p} \sum_{j=1}^{T-1} w_j^4 (j-1)^2 + \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j, j' \leq T-1} w_j^2 w_{j'}^2 (j-1)(j'-1) \right)
\]

Now we treat the case when \( k \neq k' \) and \( l = l' \),

\[
E_{1,4} := \sum_{1 \leq k, k', l \leq n} \left\{ \sum_{j=1}^{T-1} w_j^4 \sum_{T+1 \leq i, i' \leq p} (3(j-1) + (j-1)(j-2)) \right. \]
\[
+ \sum_{1 \leq j \neq j' \leq T} w_j^2 w_{j'}^2 \cdot (j-1)(j'-1) \right\}
\]
\[
= n(n-1)(n-2) \sum_{T+1 \leq i, i' \leq p} \left\{ \sum_{j=1}^{T-1} w_j^4 (j-1)(j+1) + \sum_{1 \leq j \neq j' \leq T} w_j^2 w_{j'}^2 \cdot (j-1)(j'-1) \right\}
\]
\[
= n(n-1)(n-2) \sum_{T+1 \leq i, i' \leq p} \left( \sum_{1 \leq j \leq T-1} w_j^2 w_{j'}^2 (j-1)(j'-1) + 2 \sum_{j=1}^{T-1} w_j^4 (j-1) \right)
\]
Finally, we treat the term for \( k \neq k' \) and \( l \neq l' \),

\[
E_{1,5} := \sum_{1 \leq k \neq l} \sum_{1 \leq k' \neq l'} \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j, j' \leq T} w_j w_{j'}^2 (j - 1)(j' - 1)
\]

\[
= n(n - 1)^2 (n - 2) \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j, j' \leq T} w_j w_{j'}^2 (j - 1)(j' - 1)
\]

We group the previous result to get,

\[
E_1 = \left( 2n(n - 1) + 2n(n - 1)(n - 2) + n(n - 1)^2 (n - 2) \right) \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j, j' \leq T} w_j w_{j'}^2 (j - 1)(j' - 1)
\]

\[
+ R_1(n, p, T)
\]

where,

\[
R_1(n, p, T)
\]

\[
= 2\left(n(n - 1) + n(n - 1)(n - 2)\right) \sum_{T+1 \leq i, i' \leq p} \sum_{j=1}^{T-1} w_j^4 (j - 1) + 2n(n - 1) \sum_{i=T+1}^p \sum_{j=1}^{T-1} w_j^4 (j - 1)(j + 1)
\]

\[
+ 2n(n - 1)(n - 2) \sum_{i=T+1}^p \sum_{j=1}^{T-1} w_j^4 (j - 1)^2 + 2n(n - 1) \sum_{i=T+1}^p \sum_{j=1}^{T-1} w_j^2 w_{j'}^2 ((j - 1) \wedge (j' - 1))
\]

\[
= o((c(n, p, T)^{-4})
\]

Now, let us bound from above the term \( E_2 \) in \((53)\):

\[
E_2 := (n - 1) \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq k \neq l} \sum_{1 \leq j, j' \leq T} w_j w_{j'}^2 E(X_{k,i-j}X_{k,i-j_1}X_{k',i' - j'}) (p - i' - j' + 1)
\]

\[
\sum_{h=i+1}^{(i+j-1) \wedge (i+j_1-1)} E(X_{t,h-j}X_{t,h-j_1})
\]

We treat the two cases \( k = k' \) and \( k \neq k' \) each one apart. We begin by the case when \( k \neq k' \),

\[
E_{2,1} := n(n - 1)^2 \left\{ \sum_{j=1}^{T-1} w_j^4 \left( \sum_{i=T+1}^p (p - i - j + 1)(j - 1) + \sum_{T+1 \leq i \neq i' \leq p} (p - i' - j + 1)(j - 1) \right) \right.
\]

\[
+ \sum_{1 \leq j \neq j' \leq T} w_j^2 w_{j'}^2 \left( \sum_{i=T+1}^p (p - i - j' + 1)(j - 1) + \sum_{T+1 \leq i \neq i' \leq p} (p - i' - j' + 1)(j - 1) \right)
\]

\[
= n(n - 1)^2 \left\{ 2 \sum_{i=T+1}^p \sum_{j=1}^{T-1} w_j^4 (p - i - j + 1)(j - 1)
\right.
\]

\[
+ \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j, j' \leq T} w_j^2 w_{j'}^2 (p - i' - j' + 1)(j - 1)
\]

When \( k \neq k' \),

\[
E_{2,2} := n(n - 1)^3 \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j, j' \leq T} w_j^2 w_{j'}^2 (p - i' - j' + 1)(j - 1).
\]
As consequence
\[
E_2 = \left( n(n-1)^2 + n(n-1)^3 \right) \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j, j' \leq T-1} w_j^2 w_{i'}^2 (p - i' - j + 1)(j - 1) + o((c(n, p, T)^{-4}).
\]

Similarly we get,
\[
E_3 = (n-1)^2 \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq k, k' \leq n} \sum_{1 \leq j, j' \leq T} w_j^2 w_{i'}^2 \mathbb{E}(X_{k,i-j} X_{k',i'-j'} X_{k,i-j}) (p - i - j + 1)
\]

\[
	imes \sum_{h=1}^{T-1} \mathbb{E}(X_{h',i-h} X_{h',j-h})
\]
\[
= n^2(n-1)^2 \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j, j' \leq T-1} w_j^2 w_{i'}^2 (p - i - j + 1)(j' - 1)
\]
\[
+ 2n(n-1)^2 \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j, j' \leq T} w_j^4 (p - i - j + 1)(j - 1)
\]

The term \(E_4\) of (58) is treated as follows,
\[
E_4 := (n-1)^2 \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq k, k' \leq n} \sum_{1 \leq j, j' \leq T} w_j^2 w_{i'}^2 \mathbb{E}(X_{k,i-j} X_{k',i'-j'}) (p - i - j + 1)(p - i' - j' + 1)
\]
\[
= n(n-1)^2 \left\{ \sum_{i=T+1}^{p} \left( \sum_{1 \leq j \leq T} w_j^4 \mathbb{E}(X_{k,i-j} X_{k',i'-j'}) \right) + \sum_{1 \leq j \neq j' \leq T} w_j^2 w_{i'}^2 (p - i - j + 1)(p - i' - j' + 1) \right\}
\]
\[
+ n(n-1)^3 \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j, j' \leq T-1} w_j^2 w_{i'}^2 (p - i - j + 1)(p - i' - j' + 1)
\]
\[
= n^2(n-1)^2 \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j, j' \leq T-1} w_j^2 w_{i'}^2 (p - i - j + 1)(p - i' - j' + 1)
\]
\[
+ 2n(n-1)^2 \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j \leq T} w_j^4 (p - i - j + 1)^2
\]

Finally we group all the previous terms and obtain,
\[
\mathbb{E} \left( \sum_{i=T+1}^{p} E_{i-1} (D_{n,p,i}^2) \right)^2 = c^4(n, p, T) \left\{ n^2(n-1)^2 \sum_{T+1 \leq i, i' \leq p} \sum_{1 \leq j, j' \leq T-1} w_j^2 w_{i'}^2 (j - 1)(j' - 1)
\right.
\]
\[
+ (p - i' - j' + 1)(j - 1) + (p - i - j + 1)(j' - 1)
\]
\[
+ (p - i - j + 1)(p - i' - j' + 1) + o((c(n, p, T)^{-4}) \right) \}
\]
\[
= \frac{16}{(p - T)^2(p - T - 1)^2} \sum_{1 \leq j, j' \leq T-1} w_j^2 w_{i'}^2 \sum_{T+1 \leq i, i' \leq p} (p - i)(p - i') + o(1).
\]
\[
= \frac{16}{(p - T)^2(p - T - 1)^2} \cdot \frac{1}{4} \left( \frac{(p - T - 1)(p - T)}{2} \right)^2 + o(1) = 1 + o(1)
\]
To achieve the proof, we show that the second condition given in (52) is also verified. Indeed,

\[
\sum_{i=T+1}^{p} \mathbb{E}(D_{n,p,i}^4) = (c(n, p, T))^4 \sum_{i=T+1}^{p} \sum_{i+1 \leq k_1, h_2, h_3, h_4 \leq p} \sum_{1 \leq k_1 \neq l_1 \leq 1 \leq k_2 \neq l_2 \leq 1 \leq k_3 \neq l_2 \leq 1 \leq k_4 \neq l_4 \leq n} w_{j_1}^* w_{j_2}^* w_{j_3}^* w_{j_4}^* \mathbb{E}(X_{k_1,i} X_{k_2,i} X_{k_3,i} X_{k_4,i} X_{l_1,j_1} X_{l_2,j_2} X_{l_3,j_3} X_{l_4,j_4})
\]

\[
\sum_{1 \leq j_1, j_2, j_3, j_4 \leq T-1} w_{j_1}^2 w_{j_2}^2 w_{j_3}^2 w_{j_4}^2 \mathbb{E}(X_{l_1,j_1} X_{l_2,j_2} X_{l_3,j_3} X_{l_4,j_4})
\]

\[
= O(1) \cdot (c(n, p, T))^4 \cdot \sum_{i=T+2}^{p} \sum_{T+1 \leq k_1, h_2, h_3, h_4 \leq p} \sum_{1 \leq k_1 \neq l_1 \leq 1 \leq k_2 \neq l_2 \leq 1 \leq k_3 \neq l_2 \leq 1 \leq k_4 \neq l_4 \leq n} w_{j_1}^2 w_{j_2}^2 w_{j_3}^2 w_{j_4}^2
\]

\[
= \frac{O(1)}{(p-T)^2(p-T-1)^2} \cdot (p-T)^3 = o(1).
\]

6.2 Proofs of results in Section 3

**Proof of Proposition 3.** To show the upper bound for the variance of \( \hat{A}_n^c \), we follow the line of proof of Proposition 1. We use that \( \sum_{j \geq 1} 1/\left(n^2 A_j\right) = 1/(e^{nA} - 1) \) for all \( A > 0 \) and \( n \) finite integer. As an example, let us bound from above one term of the variance of \( \hat{A}_n^c \):

\[
R_{1,2,2} := \sum_{1 \leq j \neq j' \leq T} w_j^* w_{j'}^* \left| \sigma_{|j'-j|} \right|^2 + \sum_{r=-p+T+1}^{-p} \sum_{r 
eq 0} |\sigma_{|r|} \sigma_{|r+j'|}|^2
\]

\[
\leq 2 \sum_{1 \leq j \neq j' \leq T} w_j^* w_{j'}^* \sigma_{|j'-j|}^2 + 2 \sum_{1 \leq j \neq j' \leq T} w_j^* w_{j'}^* \sigma_{|j'-j|}^2
\]

\[
\leq 2 \sum_{1 \leq j \neq j' \leq T} w_j^* w_{j'}^* \sigma_{|j'-j|}^2 + 2 \sum_{r=-p+T+1}^{-p} \sum_{r 
eq 0} \sigma_{|r|} \sigma_{|r+j'|}^2
\]

\[
\leq 4 \cdot \left( \sup_j w_j^* \right) \cdot T \cdot \mathbb{E}_\Sigma(\hat{A}_n^c) + 4L \cdot (\sup_j w_j^*)^2 \cdot (1/(e^{2A} - 1) + O(T^2) \cdot \mathbb{E}_\Sigma(\hat{A}_n^c)
\]

\[
+ 16L^2 \cdot (\sup_j w_j^*) \cdot T \cdot (1/(e^{2A} - 1)) \cdot \mathbb{E}_\Sigma(\hat{A}_n^c) + 16L^2 \cdot (\sup_j w_j^*)^2 \cdot (1/(e^{2A} - 1))^2.
\]

The proof of the asymptotic normality of \( n(p-T)(\hat{A}_n^c - \mathbb{E}_\Sigma(\hat{A}_n^c)) \), when \( n(p-T)b(\psi) \sim 1 \) and for \( \Sigma \in G(E, A, \psi) \) such that \( \mathbb{E}_\Sigma(\hat{A}_n^c) = O(b(\psi)) \), is also due to Theorem 1 of [21]. That is, we have to check (44) as in Proposition 2. As an example, let us bound from above...
the term $G_2$ in (50) with the parameters given in (16):

$$G_2 := 4 \sum_{1 \leq j_1 \neq j_2 < T} w_{j_1}^2 w_{j_2}^2 \sum_{-p+T+1 \leq r_1, r_2, r_3, r_4 \leq p-(T+1)} |\sigma_{r_1}| |\sigma_{r_1-j_1+j_2}| |\sigma_{r_2}| |\sigma_{r_2-j_2+j_1}|$$

$$\leq 4 \sum_{1 \leq j_1 \neq j_2 < T} w_{j_1}^2 w_{j_2}^2 \left( \sum_{r_1} \sigma_{r_1}^2 \right) \left( \sum_{r_1} \sigma_{r_1-j_1+j_2}^2 \right) \left( \sum_{r_2} \sigma_{r_2-j_2+j_1}^2 \right)$$

$$\leq 16L^2 \sum_{1 \leq j_1 \neq j_2 < T} w_{j_1}^2 w_{j_2}^2 \left( \sum_{r_1} \sigma_{r_1}^2 \right) + \sum_{|r_1| > j_1} \sigma_{r_1}^2$$

$$\leq 16L^2 \left\{ \sum_{1 \leq j_1 \neq j_2 < T} w_{j_1}^2 \left( \sum_{|r_1| \leq j_1} w_{r_1}^2 \sigma_{r_1}^2 \right) \right\} + \sum_{1 \leq j_1 \neq j_2 < T} w_{j_1}^2 w_{j_2}^2 \left( \sum_{|r_1| > j_1} \sigma_{r_1}^2 \right)$$

$$\leq 16L^2 \left\{ \sum_{j_1} \left( \sum_{j_2} w_{j_2}^2 \right) \cdot \mathbb{E}_\Sigma(\hat{A}_n^c) + 4L^2 \left( \sum_{j_2} w_{j_2}^* \right) \cdot \left( \sum_{j_1} w_{j_1}^* \right) \cdot \frac{1}{e^{2A_j}} \right\}$$

$$= 16L^2 \left\{ \frac{T}{2} \cdot \mathbb{E}_\Sigma(\hat{A}_n^c) + 4L^2 \cdot \frac{1}{2} \cdot \left( \sup_j w_{j_1}^2 \right) \cdot \frac{1}{e^{2A_j}} \right\}$$

$$\leq \mathbb{E}_\Sigma(\hat{A}_n^c) \cdot O(T) + o(1) = O \left( \frac{T}{n^2(p-T)^2} \right) + o(1) = o(1).$$

(54)

**Proof of Theorem 3.** To show the upper bound, we use first the asymptotic normality of the $n(p-T)\hat{A}_n^c$ under $H_0$ to prove that the type I error probability of $\Delta^* : \eta(\Delta^*) = 1 - \Phi(npb(\psi)) + o(1)$.

To bound from above the type II error probability, we shall distinguish 2 cases. First, when $n^2 p^2 b^2(\psi) \to +\infty$, we use the Markov inequality, (17) and (18), to show that $\beta(\Delta^*, G(\psi)) \to 0$. Then, when $n^2 p^2 b^2(\psi) \approx 1$, we have two possibilities: either $\mathbb{E}_\Sigma(\hat{A}_n^c)/b(\psi) \to \infty$, or $\mathbb{E}_\Sigma(\hat{A}_n^c) = O(b(\psi))$. We show respectively that either type II error probability tends to zero, or we use the asymptotic normality of $n(p-T)(\hat{A}_n^c - \mathbb{E}_\Sigma(\hat{A}_n^c))$ to get that $\beta(\Delta^*, G(\psi)) \leq \Phi(np(t-b(\psi)) + o(1)$.

To show the lower bound, we follow the same sketch of proof of lower bounds of Theorems 1 and 2. The key point for ellipsoids $E(A, L)$ is to check the positivity of the matrix

$$\Sigma^* = T_p(\{\sigma_j^*\}_{j \geq 1}) \quad \text{where} \quad \sigma_j^* = \sqrt{\lambda} \left( 1 - \left( \frac{c_j}{e^T} \right)^2 A_j \right)^{1/2} \quad \text{for all } j \geq 1.$$

Then we create a parametric family of matrices by changing the sign randomly on each diagonal of $\Sigma^*$, with parameters given in (16).

**Lemma 2** For $A > 0$, the symmetric Toeplitz matrix $\Sigma^*_U = T_p(\{u_j \sigma_j^*\}_{j \geq 1})$, where $U = \{u_j\}_{j \geq 0}$ with $u_0 = 1$, $u_j = \pm 1$ for all $j \geq 1$, and $\sigma_j^*$ defined as previously, is positive definite, for $\psi > 0$ small enough. Moreover, denote by $\lambda^{*}_{1,U}, \ldots, \lambda^{*}_{p,U}$ the eigenvalues of $\Sigma^*_U$, then $|\lambda^{*}_{i,U} - 1| \leq O(\psi \cdot \sqrt{\ln(1/\psi)})$, for all $i$ from 1 to $p$. 46
**Proof of Lemma 2.** Using Gershgorin’s Theorem we get that each eigenvalue of $\Sigma_U^* = T_p(\{u_j\sigma_j^*\}_{j \geq 1})$ verifies, $|\lambda_{i,U}^* - u_0\sigma_0^*| \leq 2 \sum_{j \geq 1} |u_j\sigma_j^*| = 2 \sum_{j \geq 1} \sigma_j^*$. We have,

$$\sum_{j \geq 1} \sigma_j^* = \sqrt{\lambda} \sum_{j \geq 1} \left(1 - \left(\frac{e_j}{e^T A}\right)^2\right)^{1/2} \leq \sqrt{\lambda} \sum_{j=1}^T \left(1 - \left(\frac{e_j}{e^T A}\right)^2\right)^{1/2} = O(1) \sqrt{\lambda} \cdot T \times \psi \cdot \sqrt{\ln(1/\psi)}.$$

We deduce that the smallest eigenvalue is bounded from below by

$$\min_{i=1,\ldots,p} \lambda_{i,U}^* \geq \sigma_0^* - 2 \sum_{j \geq 1} \sigma_j^* \geq 1 - O(1) \psi \cdot \sqrt{\ln(1/\psi)}.$$

which is strictly positive for $\psi > 0$ small enough. ■

To complete the proof, we follow the steps of the proof of the lower bound in Section 2.2.