BOREL FRACTIONAL COLORINGS OF SCHREIER GRAPHS

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Abstract. Let $\Gamma$ be a countable group and let $G$ be the Schreier graph of the free part of the Bernoulli shift $\Gamma \subset 2^\Gamma$ (with respect to some finite subset $F \subseteq \Gamma$). We show that the Borel fractional chromatic number of $G$ is equal to $1$ over the measurable independence number of $G$. As a consequence, we asymptotically determine the Borel fractional chromatic number of $G$ when $\Gamma$ is the free group, answering a question of Meehan.

1. Definitions and results

All graphs in this paper are undirected and simple. Recall that for a graph $G$, a subset $I \subseteq V(G)$ is $G$-independent if no two vertices in $I$ are adjacent in $G$. The chromatic number of $G$, denoted by $\chi(G)$, is the least $k \in \mathbb{N}$ such that there exist $G$-independent sets $I_1, \ldots, I_k$ whose union is $V(G)$. (If no such $k$ exists, we set $\chi(G) := \infty$.) The sequence $I_1, \ldots, I_k$ is called an $k$-coloring of $G$, where we think of the vertices in $I_i$ as being assigned the color $i$.

Fractional coloring is a well-studied relaxation of graph coloring. For an introduction to this topic, see the book [SU97] by Scheinerman and Ullman. Given $k \in \mathbb{N}$, the $k$-fold chromatic number of $G$, denoted by $\chi^k(G)$, is the least $\ell \in \mathbb{N}$ such that there are $G$-independent sets $I_1, \ldots, I_\ell$ which cover every vertex of $G$ at least $k$ times (such a sequence $I_1, \ldots, I_\ell$ is called a $k$-fold $\ell$-coloring). Note that the sets $I_1, \ldots, I_\ell$ need not be distinct. In particular, if $I_1, \ldots, I_{\chi(G)}$ is a $\chi(G)$-coloring of $G$, then, by repeating each set $k$ times, we obtain a $k$-fold $k\chi(G)$-coloring, which shows that

$$\chi^k(G) \leq k\chi(G) \quad \text{for all } k.$$  

This inequality can be strict; for example, the $5$-cycle $C_5$ satisfies $\chi(C_5) = 3$ but $\chi^2(C_5) = 5$. It is therefore natural to define the fractional chromatic number $\chi^*(G)$ of $G$ by the formula

$$\chi^*(G) := \inf_{k \geq 1} \frac{\chi^k(G)}{k}. $$

In this note we investigate fractional colorings from the standpoint of Borel combinatorics. For a general overview of Borel combinatorics, see the surveys [KM20] by Kechris and Marks and [Pik20] by Pikhurko. The study of fractional colorings in this setting was initiated by Meehan [Mee18]; see also [KM20, §8.6]. We say that a graph $G$ is Borel if $V(G)$ is a standard Borel space and $E(G)$ is a Borel subset of $V(G) \times V(G)$. The Borel chromatic number $\chi_B(G)$ of $G$ is the least $\ell \in \mathbb{N}$ such that there exist Borel $G$-independent sets $I_1, \ldots, I_\ell$ whose union is $V(G)$. The Borel $k$-fold chromatic number $\chi^k_B(G)$ is defined analogously, and the Borel fractional chromatic number $\chi^*_B(G)$ is

$$\chi^*_B(G) := \inf_{k \geq 1} \frac{\chi^k_B(G)}{k}. $$

A particularly important class of Borel graphs are Schreier graphs of group actions. Let $\Gamma$ be a countable group with identity element $1$ and let $F \subseteq \Gamma$ be a finite subset. The Cayley graph $G(\Gamma, F)$ of $\Gamma$ is the graph with vertex set $\Gamma$ in which two distinct group elements $\gamma, \delta$ are adjacent if

\begin{itemize}
\item $\gamma = \delta$, or
\item $\gamma = \delta F$, or
\item $\delta = \gamma F$.
\end{itemize}
and only if $\gamma = \sigma \delta$ for some $\sigma \in F \cup F^{-1}$. This definition can be extended as follows. Let $\Gamma \circ X$ be a Borel action of $\Gamma$ on a standard Borel space $X$. The action $\Gamma \circ X$ is free if

$$\gamma \cdot x \neq x \quad \text{for all } x \in X \text{ and } 1 \neq \gamma \in \Gamma.$$ 

The Schreier graph $G(X, F)$ of an action $\Gamma \circ X$ is the graph with vertex set $X$ in which two distinct points $x, y \in X$ are adjacent if and only if $y = \sigma \cdot x$ for some $\sigma \in F \cup F^{-1}$. Note that the Cayley graph $G(\Gamma, F)$ is a special case of this construction corresponding to the left multiplication action $\Gamma \circ \Gamma$. More generally, when the action $\Gamma \circ X$ is free, $G(X, F)$ is obtained by putting a copy of the Cayley graph $G(\Gamma, F)$ onto each orbit.

A crucial example of a Borel action is the (Bernoulli) shift $\Gamma \circ 2^\Gamma$, given by the formula

$$(\gamma \cdot x)(\delta) := x(\delta \gamma) \quad \text{for all } x: \Gamma \to 2 \text{ and } \gamma, \delta \in \Gamma.$$

We use $\beta$ to denote the “coin flip” probability measure on $2^\Gamma$, obtained as the product of countably many copies of the uniform probability measure on $2 = \{0, 1\}$. Note that $\beta$ is invariant under the shift action. The free part of $2^\Gamma$, denoted by $\text{Free}(2^\Gamma)$, is the set of all points $x \in 2^\Gamma$ with trivial stabilizer. In other words, $\text{Free}(2^\Gamma)$ is the largest subspace of $2^\Gamma$ on which the shift action is free. It is easy to see that the shift action $\Gamma \circ 2^\Gamma$ is free $\beta$-almost everywhere, i.e., $\beta(\text{Free}(2^\Gamma)) = 1$.

Let $G$ be a Borel graph and let $\mu$ be a probability (Borel) measure on $V(G)$. The $\mu$-independence number of $G$ is the quantity $\alpha_\mu(G) := \sup I, \mu(I)$, where the supremum is taken over all $\mu$-measurable $G$-independent subsets $I \subseteq V(G)$. Note that if $I_1, \ldots, I_\ell$ is a Borel $\ell$-fold coloring of $G$, then

$$\ell \alpha_\mu(G) \geq \mu(I_1) + \cdots + \mu(I_\ell) \geq \kappa,$$

which implies $\chi^*_B(G) \geq 1/\alpha_\mu(G)$. Our main result is a matching upper bound for Schreier graphs:

**Theorem 1.1.** Let $\Gamma$ be a countable group and let $F \subseteq \Gamma$ be a finite set. If $\Gamma \circ X$ is a free Borel action on a standard Borel space, then

$$\chi^*_B(G(X, F)) = \frac{1}{\alpha_\beta(G(\text{Free}(2^\Gamma), F))}. \quad (1.1)$$

In particular,

$$\chi^*_B(G(\text{Free}(2^\Gamma), F)) = \frac{1}{\alpha_\beta(G(\text{Free}(2^\Gamma), F))}. \quad (1.2)$$

While (1.2) is a special case of (1.1), it is possible to deduce (1.1) from (1.2) using a theorem of Seward and Tucker-Drob [ST16], which asserts that every free Borel action of $\Gamma$ admits a Borel $\Gamma$-equivariant map to $\text{Free}(2^\Gamma)$. Nevertheless, we will give a simple direct proof of (1.1) in §2.

An interesting feature of Theorem 1.1 is that it establishes a precise relationship between a Borel parameter $\chi^*_B$ and a measurable parameter $\alpha_\beta$. We find this somewhat surprising, since ignoring sets of measure 0 usually significantly reduces the difficulty of problems in Borel combinatorics. For instance, given a Borel graph $G$ and a probability measure $\mu$ on $V(G)$, one can consider the $\mu$-measurable chromatic number $\chi_\mu(G)$, i.e., the least $\ell \in \mathbb{N}$ such that there exist $\mu$-measurable $G$-independent sets $I_1, \ldots, I_\ell$ whose union is $V(G)$. By definition, $\chi_\mu(G) \leq \chi_B(G)$, and it is often the case that this inequality is strict—see [KM20, §6] for a number of examples. By contrast, as an immediate consequence of Theorem 1.1 we obtain the opposite inequality $\chi^*_B(G) \leq \chi_\beta(G)$, where $G$ is the Schreier graph of the free part of the shift action:

**Corollary 1.2.** Let $\Gamma$ be a countable group and let $F \subseteq \Gamma$ be a finite set. Set $G := G(\text{Free}(2^\Gamma), F)$. Then $\chi^*_B(G) \leq \chi_\beta(G)$.

**Proof.** The result follows from Theorem 1.1 and the obvious inequality $\alpha_\beta(G) \geq 1/\chi_\beta(G)$. 

As a concrete application of Theorem 1.1, consider the free group case. For $n \geq 1$, let $F_n$ be the free group of rank $n$ generated freely by elements $\sigma_1, \ldots, \sigma_n$ and let $G_n$ denote the Schreier graph of the free part of the shift action $F_n \circ 2^{F_n}$ with respect to the set $\{\sigma_1, \ldots, \sigma_n\}$. Then every
connected component of $G_n$ is an (infinite) 2n-regular tree. In particular, the chromatic number of $G_n$ is 2. On the other hand, Marks [Mar16] proved that $\chi_B(G_n) = 2n + 1$. Meehan inquired where between these two extremes the Borel fractional chromatic number of $G_n$ lies:

**Question 1.3** ([Mee18, Question 4.6.3]; see also [KM20, Problem 8.17]). What is the Borel fractional chromatic number of $G_n$? Is it always equal to 2?

Using Theorem 1.1 together with some known results we asymptotically determine $\chi_B^*(G_n)$ (and, in particular, give a negative answer to the second part of Question 1.3):

**Corollary 1.4.** For all $n \geq 1$, we have

$$\chi_B^*(G_n) = (2 + o(1)) \frac{n}{\log n},$$

where $o(1)$ denotes a function of $n$ that approaches 0 as $n \to \infty$.

In other words, the Borel fractional chromatic number of $G_n$ is less than its ordinary Borel chromatic number roughly by a factor of $\log n$. We present the derivation of Corollary 1.4 in §3.

2. **Proof of Theorem 1.1**

We shall use the following theorem of Kechris, Solecki, and Todorcevic:

**Theorem 2.1** (Kechris–Solecki–Todorcevic [KST99, Proposition 4.6]). If $G$ is a Borel graph of finite maximum degree $d$, then $\chi_B(G) \leq d + 1$.

Fix a countable group $\Gamma$ and a finite subset $F \subseteq \Gamma$. Without loss of generality, we may assume that $1 \notin F$. Say that a set $I \subseteq 2^\Gamma$ is independent if $I \cap (\sigma \cdot I) = \emptyset$ for all $\sigma \in F$ (when $I \subseteq \text{Free}(2^\Gamma)$, this exactly means that $I$ is $\text{G(Free}(2^\Gamma), F)$-independent). For brevity, let

$$\alpha_{\beta} := \alpha_{\beta}(\text{G(Free}(2^\Gamma), F)).$$

**Lemma 2.2.** For every $\alpha < \alpha_{\beta}$, there is a clopen independent set $I \subseteq 2^\Gamma$ such that $\beta(I) \geq \alpha$.

**Proof.** Let $J \subseteq \text{Free}(2^\Gamma)$ be a $\beta$-measurable independent set with $\beta(J) > \alpha$. Since $\beta$ is regular [Kec95, Theorem 17.10] and $2^\Gamma$ is zero-dimensional, there is a clopen set $C \subseteq 2^\Gamma$ with

$$\mu(J \triangle C) \leq \frac{\beta(J) - \alpha}{|F| + 1}.$$ 

Set $I := C \setminus \bigcup_{\sigma \in F}(\sigma \cdot C)$. By construction, $I$ is clopen and independent. Moreover, if $x \in J \setminus I$, then either $x \in J \setminus C$ or $x \in (\sigma \cdot C) \setminus (\sigma \cdot J)$ for some $\sigma \in F$. Therefore,

$$\beta(I) \geq \beta(J) - (|F| + 1)\beta(J \triangle C) \geq \alpha.$$  

Let $\Gamma \acts X$ be a free Borel action on a standard Borel space. Fix an arbitrary clopen independent set $I \subseteq 2^\Gamma$. We will prove that $\chi_B^*(G(X, F)) \leq 1/\beta(I)$, which yields Theorem 1.1 by Lemma 2.2. Since $I$ is clopen, there exist finite sets $D \subseteq \Gamma$ and $\Phi \subseteq 2^D$ such that

$$I = \{x \in 2^\Gamma : x|D \in \Phi\},$$

where $x|D$ denotes the restriction of $x$ to $D$. Note that

$$\beta(I) = \frac{|\Phi|}{2^{|D|}}.$$ 

Let $N := |DD^{-1}|$ and consider the graph $H := G(X, DD^{-1})$. Every vertex in $H$ has precisely $N - 1$ neighbors (we are subtracting 1 to account for the fact that a vertex is not adjacent to itself). By Theorem 2.1, this implies that $\chi_B(H) \leq N$. In other words, we may fix a Borel function $f : X \to N$ such that $f(u) \neq f(v)$ whenever $u, v \in X$ are distinct points satisfying $v \in DD^{-1} \cdot u$. This implies
that for each $x \in X$, the restriction of $f$ to the set $D \cdot x$ is injective. Now, to each mapping $\varphi: N \to 2$, we associate a Borel $\Gamma$-equivariant function $\pi_\varphi: X \to 2^\Gamma$ as follows:

$$\pi_\varphi(x)(\gamma) := (\varphi \circ f)(\gamma \cdot x)$$

for all $x \in X$ and $\gamma \in \Gamma$.

Let $I_\varphi := \pi_\varphi^{-1}(I)$. Since $\pi_\varphi$ is $\Gamma$-equivariant, $I_\varphi$ is $G(X, F)$-independent. Consider any $x \in X$ and let

$$S_x := \{f(\gamma \cdot x) : \gamma \in D\}.$$

By the choice of $f$, $S_x$ is a subset of $N$ of size $|D|$. Whether or not $x$ is in $I_\varphi$ is determined by the restriction of $\varphi$ to $S_x$; furthermore, there are exactly $|\Phi|$ such restrictions that put $x$ in $I_\varphi$. Thus, the number of mappings $\varphi: N \to 2$ for which $x \in I_\varphi$ is

$$|\Phi|2^{N-|D|} = \beta(I)2^N.$$

Since this holds for all $x \in X$, we conclude that the sets $I_\varphi$ cover every point in $X$ exactly $\beta(I)2^N$ times. Therefore, $\chi_B^*(G(X, F)) \leq 1/\beta(I)$, as desired.

### 3. Proof of Corollary 1.4

Thanks to Theorem 1.1, in order to establish Corollary 1.4 it is enough to verify that

$$\alpha_\beta(G_n) = \left(\frac{1}{2} + o(1)\right) \frac{\log n}{n}.$$

There are a number of known constructions that witness the lower bound

$$\alpha_\beta(G_n) \geq \left(\frac{1}{2} + o(1)\right) \frac{\log n}{n};$$

see, e.g., [LW07] by Lauer and Wormald and [GG10] by Gamarnik and Goldberg. Moreover, by [Ber19, Corollary 1.2], even the inequality $\chi_\beta(G_n) \leq (2 + o(1))n/\log n$ holds. For the upper bound

$$\alpha_\beta(G_n) \leq \left(\frac{1}{2} + o(1)\right) \frac{\log n}{n},$$

we shall use a theorem of Rahman and Virág [RV17], which says that the largest density of a factor of i.i.d. independent set in the $d$-regular tree is at most $(1 + o(1))\log d/d$. In the remainder of this section we describe their result and explain how it implies the desired upper bound on $\alpha_\beta(G_n)$.

Fix an integer $n \geq 1$. For our purposes, it will be somewhat more convenient to work on the space $[0, 1]^{F_n}$ instead of $2^{F_n}$, where $[0, 1]$ is the unit interval equipped with the usual Lebesgue probability measure. The product measure on $[0, 1]^{F_n}$ is denoted by $\lambda$. Let $H_n$ be the Schreier graph of the shift action $F_n \subset [0, 1]^{F_n}$ corresponding to the standard generating set of $F_n$. We remark that, by a theorem of Abért and Weiss [AW13] (see also [KM20, Theorem 6.46]), $\alpha_\beta(G_n) = \alpha_\lambda(H_n)$, so it does not really matter whether we are working with $G_n$ or $H_n$.

Set $d := 2n$ and let $T_d$ denote the Cayley graph of the free group $F_n$ with respect to the standard generating set. In other words, $T_d$ is an (infinite) $d$-regular tree. We view $T_d$ as a rooted tree, whose root is the vertex $1$, i.e., the identity element of $F_n$. Let $\mathfrak{A}$ be the automorphism group of $T_d$, i.e., the set of all bijections $a: F_n \to F_n$ that preserve the edges of $T_d$, and let $\mathfrak{A}_* \subseteq \mathfrak{A}$ be the subgroup comprising the root-preserving automorphisms, i.e., those $a \in \mathfrak{A}$ that map $1$ to $1$. The space $[0, 1]^{F_n}$ is equipped with a natural right action $[0, 1]^{F_n} \curvearrowright \mathfrak{A}$. Namely, for $a \in \mathfrak{A}$ and $x \in [0, 1]^{F_n}$, the result of acting by $a$ on $x$ is the function $x \cdot a: F_n \to [0, 1]$ given by

$$(x \cdot a)(\delta) := x(a(\delta))$$

for all $\delta \in F_n$.

For each $\gamma \in F_n$, there is a corresponding automorphism $a_\gamma \in \mathfrak{A}$ sending every group element $\delta \in F_n$ to $\delta \gamma$. The mapping $F_n \to \mathfrak{A}: \gamma \mapsto a_\gamma$ is an antihomomorphism of groups, that is, we have

$$a_{\gamma \sigma} = a_\sigma \circ a_\gamma$$

for all $\gamma, \sigma \in F_n$, where
where \( \circ \) denotes composition. In particular, \( \{ a_\gamma : \gamma \in F_n \} \) is a subgroup of \( \mathfrak{A} \) isomorphic to \( F_n \). The right action \( [0,1]^{F_n} \triangleright \mathfrak{A} \) and the left action \( \mathfrak{A} \triangleright [0,1]^{F_n} \) are related by the formula

\[
x \cdot a_\gamma = \gamma \cdot x \quad \text{for all } x \in [0,1]^{F_n}.
\]

A set \( X \subseteq [0,1]^{F_n} \) is called \( \mathfrak{A}_* \)-invariant if \( x \cdot a \in X \) for all \( x \in X \) and \( a \in \mathfrak{A}_* \). The Rahman–Virág theorem can now be stated as follows:

**Theorem 3.1** (Rahman–Virág [RV17, Theorem 2.1]). If \( I \subseteq [0,1]^{F_n} \) is an \( \mathfrak{A}_* \)-invariant \( \lambda \)-measurable \( H_n \)-independent set, then

\[
\lambda(I) \leq (1 + o(1)) \frac{\log d}{d} = \left( \frac{1}{2} + o(1) \right) \frac{\log n}{n}.
\]

Theorem 3.1 is almost the result we want, except that we need an upper bound on the measure of every (not necessarily \( \mathfrak{A}_* \)-invariant) \( \lambda \)-measurable \( H_n \)-independent set \( I \). To remove the \( \mathfrak{A}_* \)-invariance assumption, we use the following consequence of Theorem 3.1:

**Corollary 3.2.** There exists a Borel graph \( Q \) with a probability measure \( \mu \) on \( V(Q) \) such that:

- every connected component of \( Q \) is a \( d \)-regular tree; and
- \( \alpha_\mu(Q) \leq (1/2 + o(1)) \log n/n \).

**Proof.** We use a construction that was studied by Conley, Kechris, and Tucker-Drob in [CKT13]. Let \( \Omega \) be the set of all points \( x \in [0,1]^{F_n} \) such that \( x \cdot a \neq x \) for every non-identity automorphism \( a \in \mathfrak{A} \). Let us make a couple observations about \( \Omega \). Notice that, by definition, the set \( \Omega \) is invariant under the action \( [0,1]^{F_n} \triangleright \mathfrak{A} \); in particular, it is invariant under the shift action \( F_n \triangleright [0,1]^{F_n} \).

Furthermore, the induced action of \( F_n \) on \( \Omega \) is free (indeed, even the action \( \Omega \triangleright \mathfrak{A} \) is free). Since every injective mapping \( F_n \rightarrow [0,1] \) belongs to \( \Omega \), we conclude that \( \lambda(\Omega) = 1 \). Now consider the quotient space \( V := \Omega/\mathfrak{A}_* \). As the group \( \mathfrak{A}_* \) is compact, the space \( V \) is standard Borel [KT13, paragraph preceding Lemma 7.8]. Let \( \mu \) be the push-forward of \( \lambda \) under the quotient map \( \Omega \rightarrow V \), and let \( Q \) be the graph with vertex set \( V \) in which two vertices \( x, y \in V \) are adjacent if and only if there are representatives \( x \in \mathfrak{X} \) and \( y \in \mathfrak{Y} \) that are adjacent in \( H_n \). Conley, Kechris, and Tucker-Drob [KT13, Lemma 7.9] (see also [Tho21, Proposition 1.9]) showed that every connected component of \( Q \) is a \( d \)-regular tree. Furthermore, by construction, a set \( I \subseteq V \) is \( Q \)-independent if and only if its preimage under the quotient map is \( H_n \)-independent. Since the quotient map establishes a one-to-one correspondence between subsets of \( V \) and \( \mathfrak{A}_* \)-invariant subsets of \( \Omega \), Theorem 3.1 is equivalent to the assertion that \( \alpha_\mu(Q) \leq (1/2 + o(1)) \log n/n \), as desired.

In view of Corollary 3.2, the following lemma completes the proof of (3.1):

**Lemma 3.3.** Let \( Q \) be a Borel graph in which every connected component is a \( d \)-regular tree and let \( \mu \) be a probability measure on \( V(Q) \). Then \( \alpha_\mu(Q) \geq \alpha_\beta(G_n) \).

In the case when \( Q \) is the Schreier graph of a free measure-preserving action of \( F_n \), the conclusion of Lemma 3.3 follows from the Abért–Weiss theorem [AW13]. To handle the general case, we rely on a strengthening of a recent result of Tóth [Tót21] due to Grebík [Greb20], which, roughly, asserts that every \( d \)-regular Borel graph is “approximately” induced by an action of \( F_n \).

To state this result precisely, we introduce the following terminology. A **Borel partial action** \( p \) of \( F_n \) on a standard Borel space \( X \), in symbols \( p : F_n \triangleright * X \), is a sequence of Borel partial injections \( p_1, \ldots, p_n : X \rightarrow X \). Given a Borel graph \( Q \), we say that a Borel partial action \( p : F_n \triangleright * V(Q) \) is a **partial Schreier decoration** of \( Q \) if \( p_i(x) \) is adjacent to \( x \) for all \( 1 \leq i \leq n \) and \( x \in \text{dom}(p_i) \). If \( p \) is a partial Schreier decoration of a graph \( Q \), then we let \( C(Q,p) \) be the set of all vertices \( x \in V(Q) \) such that \( x \) belongs to both the domain and the image of every \( p_i \) and the neighborhood of \( x \) in \( Q \) is equal to the set \( \{ p_1(x), \ldots, p_n(x), p_1^{-1}(x), \ldots, p_n^{-1}(x) \} \). A **Schreier decoration** of \( Q \) is a
partial Schreier decoration $p$ such that $C(Q, p) = V(Q)$. It is easy to see that $Q$ admits a Schreier decoration if and only if it is the Schreier graph of a Borel action of $\mathbb{F}_n$.

Now we can state Grebik’s result:

**Theorem 3.4** (Grebik [Gre20, Theorem 0.2(III)]). Let $Q$ be a $d$-regular Borel graph and let $\mu$ be a probability measure on $V(Q)$. Then for every $\varepsilon > 0$, $Q$ admits a partial Schreier decoration $p$ such that $\mu(C(Q, p)) \geq 1 - \varepsilon$.

With Theorem 3.4 in hand, we are ready to establish Lemma 3.3.

**Proof of Lemma 3.3.** Recall that we denote the generators of $\mathbb{F}_n$ by $\sigma_1, \ldots, \sigma_n$. Let $Q$ be a Borel graph in which every connected component is a $d$-regular tree and let $\mu$ be a probability measure on $V(Q)$. Thanks to Lemma 2.2, it suffices to show that $\alpha_\mu(Q) \geq \beta(I)$ for every clopen independent set $I \subseteq 2^{\mathbb{F}_n}$, where, as in §2, we say that $I$ is independent if $I \cap (\sigma_i \cdot I) = \emptyset$ for each $1 \leq i \leq n$.

Fix a clopen independent set $I \subseteq 2^{\mathbb{F}_n}$. Since $I$ is clopen, we can write

$$I = \{ x \in 2^{\mathbb{F}_n} : x|_D \in \Phi \},$$

where $D \subseteq \mathbb{F}_n$ and $\Phi \subseteq 2^D$ are finite sets. Furthermore, we may assume without loss of generality that $D = \{ \gamma \in \mathbb{F}_n : |\gamma| \leq k \}$ for some $k \in \mathbb{N}$, where $|\gamma|$ denotes the word norm of $\gamma$. For a vertex $x \in V(Q)$, we let $N^k(x)$ be the set of all vertices that are joined to $x$ by a path of length at most $k$. Since every connected component of $Q$ is a $d$-regular tree, we have $|N^k(x)| = |D|$ for all $x \in V(Q)$. This allows us to define a probability measure $\mu_k$ on $V(Q)$ via

$$\mu_k(A) := \int \frac{|A \cap N^k(x)|}{|D|} \, d\mu(x) \quad \text{for all Borel } A \subseteq V(Q).$$

We have now prepared the ground for an application of Theorem 3.4. Fix $\varepsilon > 0$ and let $p$ be a partial Schreier decoration of $Q$ such that

$$\mu_k(C(Q, p)) \geq 1 - \frac{\varepsilon}{|D|},$$

which exists by Theorem 3.4. Let $C_k$ be the set of all $x \in V(Q)$ such that $N^k(x) \subseteq C(Q, p)$. Then

$$1 - \frac{\varepsilon}{|D|} \leq \mu_k(C(Q, p)) = \int \frac{|C(Q, p) \cap N^k(x)|}{|D|} \, d\mu(x) \leq \mu(C_k) + \left( 1 - \frac{1}{|D|} \right) \left( 1 - \mu(C_k) \right) = \frac{1}{|D|} \mu(C_k) + 1 - \frac{1}{|D|},$$

which implies that $\mu(C_k) \geq 1 - \varepsilon$. The importance of the set $C_k$ lies in the fact that for each $x \in C_k$ and $\gamma \in D$, there is a natural way to define the notation $\gamma \cdot x$. Namely, we write $\gamma$ as a reduced word:

$$\gamma = \sigma_{i_1}^{s_1} \cdots \sigma_{i_\ell}^{s_\ell},$$

where $0 \leq \ell \leq k$, each index $i_j$ is between 1 and $n$, and each $s_j$ is 1 or $-1$. Since $N^k(x) \subseteq C(Q, p)$, there is a unique sequence $x_0, x_1, \ldots, x_\ell$ of vertices with

$$x_0 = x \quad \text{and} \quad x_j = p_{i_j}^{s_j}(x_{j-1}) \quad \text{for all } 1 \leq j \leq \ell.$$

We then set $\gamma \cdot x := x_\ell$. Note that we have $N^k(x) = \{ \gamma \cdot x : \gamma \in D \}$.

The remainder of the argument utilizes a construction similar to the one in the proof of Theorem 1.1 given in §2. Consider the graph $R$ with the same vertex set as $Q$ in which two distinct vertices are adjacent if and only if they are joined by a path of length at most $2k$ in $Q$. Since every connected component of $Q$ is a $d$-regular tree, each vertex in $R$ has the same finite number of neighbors, so, by Theorem 2.1, the Borel chromatic number $\chi_B(R)$ is finite. Let $N := \chi_B(R)$ and fix a Borel function $f : V(Q) \to N$ such that $f(u) \neq f(v)$ whenever $u$ and $v$ are adjacent in $R$. Then for each $x \in V(Q)$,
the restriction of \( f \) to the set \( N^k(x) \) is injective. Now, to each mapping \( \varphi: N \to 2 \), we associate function \( \pi_\varphi: C_k \to 2^D \) as follows:

\[
\pi_\varphi(x)(\gamma) := (\varphi \circ f)(\gamma \cdot x) \quad \text{for all } x \in C_k \text{ and } \gamma \in D.
\]

Let \( I_\varphi := \{ x \in C_k : \pi_\varphi(x) \in \Phi \} \). The independence of \( I \) implies that the set \( I_\varphi \) is \( Q \)-independent. We will show that for some choice of \( \varphi: N \to 2 \), \( \mu(I_\varphi) \geq (1 - \varepsilon)\beta(I) \). Since \( \varepsilon \) is arbitrary, this yields the desired bound \( \alpha(Q) \geq \beta(I) \) and completes the proof of Lemma 3.3.

Consider any \( x \in C_k \) and let

\[
S_x := \{ f(\gamma \cdot x) : \gamma \in D \}.
\]

Since \( f \) is injective on \( N^k(x) \), \( S_x \) is a subset of \( N \) of size \(|D|\). Whether or not \( x \) is in \( I_\varphi \) is determined by the restriction of \( \varphi \) to \( S_x \); furthermore, there are exactly \(|\Phi|\) such restrictions that put \( x \) in \( I_\varphi \).

Thus, the number of mappings \( \varphi: N \to 2 \) for which \( x \in I_\varphi \) is

\[
|\Phi|2^{N - |D|} = \beta(I)2^N.
\]

Since this holds for all \( x \in C_k \), we conclude that

\[
\sum_{\varphi: N \to 2} \mu(I_\varphi) \geq \mu(C_k)\beta(I)2^N \geq (1 - \varepsilon)\beta(I)2^N,
\]

where the second inequality uses that \( \mu(C_k) \geq 1 - \varepsilon \). In other words, the average value of \( \mu(I_\varphi) \) over all \( \varphi: N \to 2 \) is at least \((1 - \varepsilon)\beta(I)\). Thus, the maximum is at least \((1 - \varepsilon)\beta(I)\) as well, and the proof is complete. \( \square \)

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