Throughput Optimized Multi-Source Cooperative Networks With Compute-and-Forward

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Abstract—In this work, we investigate a multi-source multicast network with the aid of multiple relays, where it is assumed that no direct link is available at each S-D pair. The aim is to maximum the common multicast throughput of all source nodes. A transmission protocol employing the relaying strategy, namely, compute-and-forward (CPF), is proposed. Both the delay-stringent transmission and the delay-tolerant transmission applications are investigated. The associated optimization problems to maximum the short-term and long-term common multicast throughput are formulated and solved. Performance comparisons show that the CPF strategy outperforms conventional decode-and-forward (DF) strategy.

Index Terms—Compute-and-forward, resource allocation, delay-stringent, delay-tolerant, fading

I. INTRODUCTION

Network coding, an efficient way to mitigate network interference and improve throughput, was firstly proposed by Yeung et al in [1][2] for wireline networks. Employing it, relay nodes capability is expanding from only simply forwarding messages to forwarding some functions of different messages to multiple destinations nodes. The intended nodes can then detract the message required as long as they have prior knowledge of the rest messages. In this way, network throughput is improved.

However, due to broadcast nature in wireless communications, the performance potential of conventional digital network coding (DNC) is strictly constrained by the interference from transmissions of other irrelevant transmitters in wireless networks. For instance, for a simple three-node, two-way relay network (TWRN), the relay node needs to jointly decode two individual messages in the multi-access phase with DNC [3], whereas the performance is degraded due to the fact that one user's message is regarded as interference to the message of the other user at the relay node in the multi-access uplink. As it is, this interference constrains the achievable rate pair especially in high signal-to-noise ratio (SNR) regime. Other strategies, like amplify-and-forward (AF) and compress-and-forward (CF), has its intrinsic advantages without decoding individual messages, but the noise term will be amplified along with messages delivery throughout the network, resulting in degradation of network performance.

To this end, a smart way of network coding, which was referred to as physical layer network coding (PNC) [4], i.e., compute-and-forward (CPF) [5] for general multi-source multi-relay networks, attracted increasing attention. Typically for a TWRN, in the uplink phase, the two source nodes simultaneously transmit their messages to the relay node, and the relay node merely decodes a linear combination of these two messages with integer coefficients, other than employing joint decoding. In the downlink phase, the relay node can transmit to the two source nodes the linear combined message. In this case, the two sources can subtract their transmitted messages and then decode the intended message. In this way, the relay node mitigates the interference coming from joint decoding in the uplink phase by employing digital network coding, and also avoids the noise amplification when using analog network coding (ANC) [6]. In [7], PNC was shown to perform close to a capacity upper bound and its performance gain over digital network coding and analog-network coding was demonstrated.

More importantly, PNC is shown to achieve high performance for more general multi-source multi-relay networks in [5][8]. In the celebrated work [5], PNC was referred to as compute-and-forward (CPF) strategy. With this strategy, each relay will decode a function message formed by a linear combination of the messages from all source nodes with integer coefficients. Each destination node then obtains different function messages from various relay nodes and decodes all the source messages as long as the integer coefficients constructed matrix is in full rank. In these works, the outage probability performance of CPF is demonstrated to outperform other relaying strategies, such as decode-and-forward (DF) and amplify-and-forward (AF). The authors also proved that by employing a type of linear codes, lattice codes, the derived capacity region of compute-and-forward is achievable. However, Nazer et al mainly investigated the computation rates over S-R links in [5] whereas the transmission over R-D links was not discussed. In [9], How to obtain the locally optimal integer coefficient vector at each relay node to maximum its computation rate was addressed, however the full rank of the matrix constructed by all the integer coefficient vectors of all relays was not guaranteed hence the destination nodes may still not be able to decode all source messages. In [10] it jointly optimized the integer coefficient vector at each relay to finally obtain the optimal common computation rate with the guaranteed full rank matrix constructed by these integer coefficient vectors, at the cost that the achievable computing...
rates at some relay nodes may be reduced to satisfy the full rank requirement.

To the best of our knowledge, all the previous works in the literature only considered the outage performance of CPF strategy with fixed resource utilization strategy and no resource allocation was studied so far. Therefore, we are motivated to investigate the achievable throughput with CPF with flexible resource allocation strategies, assuming that full channel state information (CSI) is available at all transmitters.

In this work, a general multi-source multi-relay multicast network is considered. The performance with CPF in terms of the achievable common multicast throughput is investigated, with both the S-R links as well as the R-D links taken into account. Two cases of interest will be studied. One is a delay-stringent scenario, where each multicast transmission from all sources to all destinations should be finished within one slot. The other is a delay-tolerant transmission scenario, where the multicast transmission from all sources to all destinations can be finished within arbitrary finite number of slots, i.e., no delay constraints are imposed. are listed as follows.

The remainder of this work is organized as follows. In Section II, we present the system model of a multi-source multi-relay multicast network with the aid of multiple relay nodes, briefly describe the CPF strategy as well as review some theorems of CPF in the literature. In Section III and Section IV, delay-limited scenario and delay-tolerant scenario are investigated, respectively. The associated problems to maximize the common multicast throughput of the entire network are formulated and solved, by jointly allocating time and energy resources for each transmit phase. Simulation results are presented in Section V. Finally, we conclude this work in Section VI.

II. SYSTEM MODEL

In this work, we mainly focus on a multi-source, multi-relay, multicast network, as shown in Fig. 1, which consists of $M$ sources ($S_1, \ldots, S_M$), $M$ relays ($R_1, \ldots, R_M$) and $L$ destinations ($D_1, \ldots, D_L$). Each source node or relay node is equipped with one antenna and works in half-duplex mode, i.e., cannot transmit and receive data simultaneously. A block fading channel model is assumed for each link. The stochastic and instantaneous channel gain of each link are assumed to be known at both the transmitter and the receivers, which can be realized via feedback.

We assume that $S_i$ is with average power constraint $P_s$, and $R_i$ is with average power constraint $P_r$. It is also assumed that there are no direct links between any S-D pair. Hence, transmission must be aided by relay nodes and indeed consists of $M+1$ phases, as described below.

- In Phase 1, all source nodes transmit their data simultaneously to all relay nodes. In this phase, each relay node tries to decode a linear equation of the combination of individual transmitted messages from all sources with integer coefficients.
- In Phase $i$ ($i = 2, \ldots, M + 1$), the $(i - 1)$th relay will deliver its decoded function message to all destination nodes. These destination nodes can then recover all original messages at the end of Phase $M + 1$, as long as sufficient amount of equations can be received reliably, i.e., rank $M$ (the number of source nodes) of the matrix constructed by all these integer coefficients is achieved at each destination node.

![Fig. 1. System model for a multi-source multicast network with the aid of multiple relay nodes. The direct link between each S-D pair is assumed to be unavailable.](W3.2.pdf)

In this work, we are interested in addressing the symmetric traffic application, i.e., each source node is assumed to have equal amount of messages to be delivered to all destination nodes. The aim then is to optimize the common multicast rate for both the delay-stringent case and the delay-tolerant case. For clarity, before formulating the corresponding problems, we would like to review the key ideas of CPF and the attained theorems in the literature.

A. Review on Compute-and-Forward Strategy

In this section, we briefly describe how compute-and-forward works and present some theorems in [5][9].

Employing compute-and-forward strategy, at the end of Phase I, each relay will observe a noisy linear combination of the transmitted messages, which is,

$$y_m = \sum_{i=1}^{M} h_{ml} x_i + z_m \quad m = 1, \ldots, M$$

(1)

where $x_i$ is the transmitted signal vector from $S_i$ and $h_{ml}$ is channel fading coefficient of the link from $S_i$ to $R_m$. $z_m$ is the i.i.d. additive white Gaussian noise vector, $z_m \sim N(0, I_n)$. We also denote $h_m = [h_{m1} \ldots h_{mM}]^T$ as the channel coefficient vector consisting of the links from all sources to $R_m$. Similarly, we denote $g_m = [g_{m1} \ldots g_{mL}]^T$ as the channel coefficient vector consisting of the links from the $m$th relay to all destinations and $g_r = \min_{i} |g_{mi}|^2$ as the minimum channel gain of the links from the $m$th relay to all destinations.

Employing compute-and-forward, each relay node will select a scalar $\beta_m \in \mathbb{R}$ to obtain a scaled version of the received mixed messages, i.e., $\beta_m y_m$, as well as an integer coefficient vector $a_m \in \mathbb{Z}^L$ to form a probably decodable integer-combined version, i.e., $\sum_{i=1}^{M} a_{mi} x_i$ from the scaled version $\beta_m y_m$, which is given by,

$$\beta_m y_m = \sum_{i=1}^{M} a_{mi} x_i + \sum_{i=1}^{M} (\beta_m h_{ml} - a_{ml}) x_i + \beta_m z_m$$

(2)
where the second term and the third term can be regarded as effective noise dominating throughput performance. The former rate penalty follows from the non-integer coefficient and the latter rate penalty is from the scaled noise. With CPF, we only manage to decode the integer-combined version $\sum_{i=1}^{M} a_{m,i}x_i$, other than jointly decoding all individual messages, hence has the potential to greatly improve network performance. It is also noted that due to linearity of lattice codes, this integer-combined version is also a lattice code. In this way, we are interested in the computation rate of the network-coded codeword $\sum_{i=1}^{M} a_{m,i}x_i$. Based on the observation in (2), some important results from [5][9] on the achievable computation rates with CPF are given as follows.

**Theorem 1:** The maximum achievable computation rate is given by

$$R_{CPF}(\mathbf{h}_m, a_m) = \max_{\beta_m \in \mathbb{R}} \frac{1}{2} \log^+ \left( |a_m|^2 - \frac{P(h_m^T a_m)^2}{1 + P||h_m||^2} \right)^{-1}$$

(3)

**Theorem 2:** For a given channel coefficient vector $\mathbf{h}_m \in \mathcal{R}^L$, $R_{m}(\mathbf{h}_m, a_m)$ is maximized by choosing the integer coding coefficient vector $a_m \in \mathbb{Z}^L$ as,

$$a_m = \min_{a_m \in \mathbb{Z}^L, a_m \neq 0} (a_m^T G_m a_m)$$

(4)

where $G_m = \mathbf{I} - \frac{P}{||h_m||^2} \mathbf{H}_m$, and $\mathbf{H}_m = [H(m)]_{ij}, H(m) = h_{mi} h_{mj}, 1 \leq i, j \leq L$.

It is observed that $(a_m^T G_m a_m)$ is a function that determines how good an integer coefficient vector candidate is. Hence, we define a criterion function $\text{Cret}(a_m, h_m) = (a_m^T G_m a_m)$.

From Theorem 2, we obtain the achievable common rate of all relays, which is given by

$$R_{CPF} = \min_i R_{CPF}^i \quad i = 1, \ldots, M$$

(5)

Correspondingly, the required transmit power at the $m$th relay with the common transmit rate $R_{CPF}$, i.e., $P_{m,CPF}$, is given by,

$$P_{m,CPF} = \frac{1 - 2 R_{CPF} b_m}{2 R_{CPF} c_m - a_m}$$

(6)

where $a_m = |h_m|^2$, $b_m = |a_m|^2$, $c_m = |h_m|^2 |a_m|^2 - |h_m^T a_m|^2$, $d_m = |h_m^T a_m|^2$. In the sequel, we set them as given above throughout this work. The required power for the compute-and-forward phase is given by,

$$P_{CPF} = \max_m P_{m,CPF}$$

At this power level, all relays can enjoy the common computing rate $R_{CPF}$.

**B. Integer Coefficient Vector**

Here we briefly present the three adopted methods to obtain the integer coefficient vectors for each channel coefficient vector at the relay nodes. For a given $h_m$ at the $m$th relay,

- method a): Obtain the integer coefficient vector $a_m$ individually by solving $a_{m,j} = \text{round}(h_{m,j})$ $(j = 1, \ldots, M)$ at the $m$th relay, where the function $\text{round}(\cdot)$ returns the closest integer to $\{\cdot\}$.

- method b): Obtain the locally-optimal integer coefficient vector $a_m$, individually by solving (4) in Theorem 2 with the respective $h_m$ at the $m$th relay.

- method c): Obtain all the integer coefficient vectors at all relays by employing the jointly optimization method in [10] to guarantee the full rank property of the matrix constructed by the integer coefficient vectors of all relays.

**III. DELAY-STRINGENT TRANSMISSION**

In this section, we shall address the achievable common multicast throughput employing compute-and-forward relaying with stringent delay constraints, i.e., data must be delivered within one slot from the source nodes to the destination nodes. In this case, the aim is to maximize the achievable common multicast throughput by optimally allocating time resource to each phase within each slot. This can be applied to some real time communication applications with the minimum rate requirement within each slot.

First, we denote that $f_{CPF}$ is the time fraction assigned to the first phase for S-R transmission employing compute-and-forward and $f_1$ is the time fraction allotted to the $(i + 1)$th phase or the $i$th relay for transmission. Let $P_{sm}$ be the minimum average power constraint of all source nodes, i.e., $P_{sm} = \min_i P_{s,i}$. In addition, $P_{CPF}$ and $P_i$ are denoted as the transmit powers consumed of each source node at the first phase (compute-and-forward phase) and the $i$th relay, respectively.

The optimization problem within one slot for given channel coefficients of all links, termed as $\textbf{P1}$, is then formulated as follows,

$$\max_{f_1, f_{CPF}} \min_i (f_{CPF} R_{CPF}(P_{CPF}), f_i R_i(P_i))$$

(7)

where $i = 1, \ldots, M$. The associated power constraint and physical constraint are given as follows.

$$P_{CPF} \leq P_{sm}$$

(8)

$$P_i \leq P_{r,i}$$

(9)

$$f_{CPF} + \sum_{i=1}^{M} f_i \leq 1$$

(10)

This is a simple problem and the solution to $\textbf{P1}$ is given by,

$$f_{CPF}^* = \frac{L^*}{R_{CPF}(P_{sm})}$$

(11)

$$f_i^* = \frac{L^*}{R_i(P_{r,i})}$$

(12)

$$L^* = \frac{1}{R_{CPF}(P_{sm})} + \sum_{i=1}^{M} \frac{1}{R_i(P_{r,i})}$$

(13)

where the asterisk denotes optimality and $L^*$ is the optimal achievable common multicast throughput with given channel coefficients in delay-stringent applications.

Note that the average power constraint at each node is imposed as the constraint for each node for the discussed slot, since we seek to optimize the throughput within each slot.
IV. Delay-Tolerant Transmission

In this section, we shall address the performance achievable by employing compute-and-forward with full channel state information available at the transmitters. We also assume that the derived integer coefficient vectors of all relays are available at all the destination nodes. As it is, all these messages can be obtained via a feedback channel in a time-sharing manner or frequency-sharing manner for each node.

For clarity, we denote \( P_{\text{CPF}} \) as the average power consumed at each source node for the CPF phase and \( \bar{P}_i \) the average power consumed of the \( i \)th relay, respectively. All these values are averaged over the associated channel coefficient distributions. By way of power and time resources allocation for each phase, the problem to maximize the average common multicast throughput, referred to as \( \textbf{P2} \), is formulated as follows.

\[
\max_{R_{\text{CPF}}(\mathbf{h}_i), R_i(\mathbf{g}_i)} \min(f_{\text{CPF}} \bar{R}_{\text{CPF}}(\mathbf{h}_i), f_i \bar{R}_i(\mathbf{g}_i)) \quad (14)
\]

where \( l, i = 1, \ldots M \). The associated constraints are given as follows.

\[
\begin{align*}
\bar{P}_{\text{CPF}} & \leq P_{\text{sm}} \quad (15) \\
\bar{P}_i & \leq P_t, \quad i = 1, \ldots, M \quad (16) \\
f_{\text{CPF}} + \sum_{i=1}^{M} f_i & \leq 1 \quad (17)
\end{align*}
\]

The optimization problem above aims to optimally allocate time resources to different phases in order to mitigate the performance degradation caused by bottleneck links. In this sense, \( \textbf{P2} \) can be transformed into an equivalent optimization problem below

\[
\max_{R_{\text{CPF}}(\mathbf{h}_i), R_i(\mathbf{g}_i)} f_{\text{CPF}} \bar{R}_{\text{CPF}} \quad (18)
\]

subject to the following constraints

\[
\begin{align*}
\bar{P}_{\text{CPF}} & \leq P_{\text{sm}} \quad (19) \\
\bar{P}_i & \leq P_t, \quad i = 1, \ldots, M \quad (20) \\
f_{\text{CPF}} \bar{R}_{\text{CPF}} & = f_i \bar{R}_i \quad \forall i \quad (21)
\end{align*}
\]

\[
f_{\text{CPF}} + \sum_{i=1}^{M} f_i \leq 1 \quad (22)
\]

where (21) reveals the fact that the product of average rate of each phase and the time resource allotted to that phase should be made equal for each phase for optimality. Note that it cannot be readily observed that \( \textbf{P2} \) is a convex optimization problem due to the relationship between \( P_{\text{CPF}} \) and \( R_{\text{CPF}} \) in (6). It can be verified that \( P_{\text{CPF}} \) is actually a convex function of \( R_{\text{CPF}} \) [11] and therefore \( \textbf{P2} \) can be solved by gradient methods.

V. SIMULATION

We now present some simulation results to compare the achievable rates by employing CPF and DF. Full channel state information is assumed to be available at the associated transmitters. The average power constraint at all nodes are assumed to be the same in the simulation setting for simplicity. The channels are assumed to be real valued channels and their gains are modeled by zero mean and unit gain Gaussian variables. In addition, the noises at all nodes are zero mean and unit variance Gaussian variables. For ease of computation, we shall mainly focus on a multicast network with two source nodes, two relay nodes and two destination nodes (\( L = M = 2 \), if not specified).

For comparison, here we briefly give a DF protocol for a multi-source multi-relay multicast network with \( 2M \) phases, which is,

- In the first \( i \)th (\( i = 1, \ldots, M \)) phase, the \( i \)th source node transmits its data to the \( i \)th relay nodes at rate \( R_i \).
- In the \( (M + i) \)th phase (\( i = 1, \ldots, M \)), the \( i \)th relay node broadcasts the data from the \( i \)th source node to all destination nodes at rate \( R_{M+i} \).

For delay-stringent applications, the problem to maximize the common multicast throughput within one slot with DF, is formulated as \( \textbf{P3} \) below.

\[
\max_{f_i} \min_i (f_i R_i) \quad (23)
\]

s.t.

\[
\begin{align*}
P_i & \leq P_{s_i}, \quad i = 1, \ldots, M \quad (24) \\
P_i & \leq P_{r_i}, \quad i = M + 1, \ldots, 2M \quad (25) \\
\sum_{i=1}^{2M} f_i & \leq 1 \quad (26)
\end{align*}
\]

The solution to \( \textbf{P3} \) is similar to that of \( \textbf{P1} \) and is hence omitted. In addition, the average throughput for delay-stringent applications can be readily obtained by averaging over the channel distributions.

Similarly, the problem to optimize the averaged common multicast throughput for delay-tolerant applications is formulated as \( \textbf{P4} \) below.

\[
\max_{f_i, R_i(\mathbf{h}_i), R_i(\mathbf{g}_i)} f_i \bar{R}_i \quad (27)
\]

subject to the following constraints

\[
\begin{align*}
\bar{P}_i & \leq P_{s_i}, \quad i = 1, \ldots, M \quad (28) \\
\bar{P}_{M+i} & \leq P_{r_i}, \quad i = 1, \ldots, M \quad (29) \\
f_i \bar{R}_i & = f_i \bar{R}_i \quad \forall i \quad (30)
\end{align*}
\]

\[
\sum_{i=1}^{2M} f_i \leq 1 \quad (31)
\]

Note that \( \textbf{P4} \) is a standard convex optimization problem and can be readily solved by KKT conditions. The solution to it is however omitted for brevity.

For clarity, a table describing the associated transmit strategies linked to different applications is presented below.

Before presenting the performance of the proposed strategies, we would like to show the probability that the network integer vector constructed matrix is not in full rank by employing the proposed methods. It is worth mentioning that, if
this matrix is not in full rank, each destination node can not recover all messages from the source nodes. Hence, full rank requirement of the network integer vectors plays a crucial role in applying CPF strategy.

In Fig. 2, the optimal averaged common multicast throughput by using CPF strategy as well as DF strategy with delay-stringent constraints are compared over 10000 randomly generated channel realizations. It is shown that CPF-DS with the integer network channel coefficient vectors found by the global optimal method, i.e., method c), outperforms that with the local optimal method (method b)) and the naive method (method a)). It is also observed that CPF-DS employing method c) outperforms DF in terms of achievable throughput. However, it is seen that CPF-DS with method a) or b) performs worse than DF strategy, due to their non-negligible rank failure probabilities.

In Fig. 3, the optimal common multicast throughput by using different strategies for delay-stringent applications are shown. It is shown that CPF-DT with method c) outperforms DF strategy in terms of common throughput. For instance, in the regime of 30 dB transmit power, employing CPF-DT with method c), over 10% throughput improvement is achievable, compared with DF-DT. It is also interesting to note that CPF-DT with method b) is slightly better than DF, whereas CPF-DT with method a) is worse than DF, which is due to the high probability of rank failure and the far-from-optimal integer coefficient vectors sorted by adopting method a).

**VI. CONCLUSION**

In this work, we considered a multi-source multicast network with the aid of multiple relay nodes. We aim to maximize common multicast throughput of all S-D pairs. To this end, a transmission protocol employing compute-and-forward strategy was proposed. Delay-stringent transmission as well as delay-tolerant transmission applications were both investigated. The associated optimization problems were formulated and solved, through the allocation of time and energy resources. Simulation results indicated that, our proposed protocol outperform the conventional DF strategy.

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