FABER–WALSH POLYNOMIALS AND POLYNOMIAL APPROXIMATION PROBLEMS ON TWO DISJOINT INTERVALS

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Abstract. The Faber–Walsh polynomials are a direct generalization of the (classical) Faber polynomials from simply connected sets to sets with several components. The option of working with such sets leads to significantly more flexibility in practical applications. In this paper we analyze certain optimality properties of the Faber–Walsh polynomials (namely their asymptotic optimality) and derive the first, to our knowledge, explicit examples. We focus on Faber–Walsh polynomials for sets consisting of two disjoint real intervals and their application in the convergence theory of iterative methods for solving linear algebraic systems with hermitian indefinite matrices.

Key words. Faber–Walsh polynomials, Chebyshev polynomials, asymptotic convergence factor, conformal maps iterative methods, indefinite matrices, convergence analysis

AMS subject classifications. 30C10, 30C20, 30E10, 65F10

1. Introduction. The construction and analysis of numerous algorithms in applied and computational mathematics involves (best) polynomial approximation problems of the form

\[ \min_{p \in P_k} \| f - p \|_E, \]

where \( E \) is some compact set in the complex plane \( \mathbb{C} \), the function \( f \) is analytic on \( E \), \( P_k \) denotes the set of complex polynomials of degree at most \( k \), and \( \| \cdot \|_E \) denotes the maximum (or Chebyshev) norm on \( E \). The exact polynomial of best approximation which solves the problem (1.1) usually is only available explicitly when the set \( E \) is very simple (e.g. a single interval or disk in \( \mathbb{C} \)). A large body of work in approximation theory and approximation practice therefore is concerned with determining approximations to the exact solution of (1.1) that are almost optimal in some sense and that can be efficiently computed. For a modern presentation of many classical and recent results in this area we highly recommend Trefethen’s monograph from 2013 [22].

When the set \( E \) is simply connected the (classical) Faber polynomials (see, e.g., [2, 21] for their definition) are frequently used in order to obtain approximations to the exact solution of (1.1). For any analytic function \( f \) on \( E \), these polynomials yield a series that converges maximally to \( f \), and they can be efficiently computed when a suitably normalized conformal map from the exterior of \( E \) onto the exterior of the unit disk in \( \mathbb{C} \) is known. For examples using Faber polynomials in the context of numerical linear algebra problems we refer to [1, 3, 12, 14, 15, 20]. Applications of Faber polynomials in a more general context of numerical approximation are described, e.g., in [6, 7].

Many applications in applied and computational mathematics involve approximation problems on sets \( E \) with several components. In such problems one can only use Faber polynomials on a simply connected inclusion set \( \tilde{E} \supset E \). By “connecting” the components of \( E \), however, we usually lose essential information about the given problem. The resulting approximation to the exact solution of as well as the upper bound on the value of (1.1) will then be of little practical interest. Moreover, finding a suitable set \( \tilde{E} \) that allows an efficient computation of the Faber polynomials can be a significant challenge since the function \( f \) must be analytic on \( \tilde{E} \).

From a theoretical and practical point of view it is desirable to work with sets with several components and suitably generalized Faber polynomials. Such a generalization was obtained

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by Walsh in 1958 [23]. The construction of these Faber–Walsh polynomials relies on Walsh’s earlier discovery of a lemniscatic conformal map, which conformally identifies the exterior of a compact set with several components with a certain lemniscatic domain [24]. Walsh’s result on conformal maps is a direct generalization of the Riemann mapping theorem, and the corresponding Faber–Walsh polynomials directly generalize the (classical) Faber polynomials. While the literature on Faber polynomials is quite extensive, the Faber–Walsh polynomials have rarely been studied in the literature. One notable exception is Suetin’s book [21], which contains a proper subsection on Faber–Walsh polynomials as well as a few further references. A major reason for the neglect of the Faber–Walsh polynomials certainly is the fact that until recently no explicit examples of Walsh’s lemniscatic conformal map were known. The first example (to our knowledge) appeared in [17], and we extended it in our previous paper [18]. In this paper we recall the definition of the Faber–Walsh polynomials and give the first (to our knowledge) explicit application of these polynomials to a problem from approximation theory.

The paper is organized as follows. In Section 2 we summarize Walsh’s results from [24, 25] concerning the existence of the Faber–Walsh polynomials. We also state a recursion formula for computing these polynomials that is based on the Laurent series coefficients of the lemniscatic conformal map. In Section 3 we prove that suitably normalized Faber–Walsh polynomials are asymptotically optimal in the sense of Eiermann, Niethammer and Varga [4, 5]. This optimality property is of interest in the convergence analysis of certain iterative methods for solving linear algebraic systems. In Section 4 we consider the Faber–Walsh polynomials and their optimality properties for sets $E$ consisting of two disjoint real intervals. For two intervals of the same length in Section 4.1 we use a special case of a lemniscatic conformal map derived in [18, Section 3]. This yields explicit formulas for the Faber–Walsh polynomials and related quantities such as their asymptotic convergence factor. Furthermore, we prove that the normalized Faber–Walsh polynomials of even degrees are in a certain sense optimal. In Section 4.2 we study the case of two general intervals numerically and relate our results to the convergence theory for the MINRES method applied to discretized Stokes equations developed by Wathen, Fischer and Silvester [27].

2. The Faber–Walsh polynomials. For a given integer $N \geq 1$ let $a_1, \ldots, a_N \in \mathbb{C}$ be pairwise distinct and let the positive real numbers $m_1, \ldots, m_N$ satisfy $\sum_{j=1}^{N} m_j = 1$. Then for any $\mu > 0$ the set

$$(2.1) \quad \mathcal{L} := \{ w \in \hat{\mathbb{C}} : |U(w)| > \mu \}, \quad \text{where} \quad U(w) := \prod_{j=1}^{N} (w - a_j)^{m_j},$$

is called a lemniscatic domain in the extended complex plane $\hat{\mathbb{C}}$. The following theorem of Walsh shows that lemniscatic domains are canonical domains for certain $N$-times connected domains (open and connected sets).

**Theorem 2.1** (see [24, Theorem 3]). Let $E := \bigcup_{j=1}^{N} E_j$, where $E_1, \ldots, E_N \subseteq \mathbb{C}$ are mutually exterior simply connected compact sets (none a single point). Then there exist a lemniscatic domain $\mathcal{L}$ of the form (2.1) with $\mu > 0$ equal to the logarithmic capacity of $E$, and a bijective conformal mapping

$$\psi : \mathcal{L} \to \mathcal{K} := \hat{\mathbb{C}} \setminus E \quad \text{with} \quad \psi(\infty) = \infty, \quad \psi'(\infty) = 1.$$

The function $\psi$ is called a lemniscatic conformal map (for $E$).

For $N = 1$ the set $E$ is simply connected and a lemniscatic domain is the exterior of a disk. Hence in this case Theorem 2.1 is equivalent with the Riemann mapping theorem.
In [25] Walsh used Theorem 2.1 for proving the existence of a direct generalization of the (classical) Faber polynomials for sets $E$ with several components. A major ingredient in his construction is the following result on polynomials associated with a given lemniscatic domain that play the same role as the monomials $w^k$ in case of (the exterior of) the unit disk.

**Lemma 2.2** (see [23, 25, Lemma 2]). Given a lemniscatic domain as in (71), there exists a sequence $(\alpha_j)_{j=1}^\infty$, chosen from the centers $a_1, \ldots, a_N$ of the lemniscate, with the following property: For any closed set $C \subseteq \hat{C}$ not containing any of the points $a_1, \ldots, a_N$ there exist constants $A_1, A_2 > 0$ such that

$$A_1 < \left| \frac{u_k(w)}{U(w)} \right|^k < A_2, \quad \text{for } k = 0, 1, 2, \ldots \text{ and any } w \in C,$$

where $u_k(w) := \prod_{j=1}^{\infty} (w - \alpha_j)$.

The sequence $(\alpha_j)_{j=1}^\infty$ can be chosen constructively from the lemniscate centers $a_1, \ldots, a_N$.

For $N = 1$ we have $\alpha_j = a_1$ for all $j$ and $u_k(w) = (w - a_1)^k$ for all $k$. For $N \geq 2$ the sequence is, however, not unique. For $N = 2$ one possible choice is $\alpha_j = a_1$ if $|jm_1| > |(j-1)m_1|$, and $\alpha_j = a_2$ otherwise, where $[\cdot]$ denotes the integer part. This will be used in Section 4 below.

We can now state Walsh’s main result from [25].

**Theorem 2.2** (see [25, Theorem 3]). Let $E$, $\psi$ and $L$ be as in Theorem 2.1 and define the level curves

$$\Lambda_\sigma := \{ w \in \hat{C} : |U(w)| = \sigma \mu \} \quad \text{and} \quad \Gamma_\sigma := \psi(\Lambda_\sigma)$$

for $\sigma > 1$. Let $\text{ext}$ and $\text{int}$ denote the exterior and interior of a curve (or union of curves), and let $(\alpha_j)_{j=1}^\infty$ and the corresponding polynomials $u_k(w)$ be as in Lemma 2.2. Then the following hold:

1. For any $z \in \Gamma_\sigma$ and $w \in \text{ext}(\Lambda_\sigma)$ we have

$$\psi'(w) \psi(z) \Lambda_\sigma = \sum_{k=0}^{\infty} \frac{b_k(z)}{u_{k+1}(w)} \quad \text{where} \quad b_k(z) = \frac{1}{2\pi i} \int_{\Lambda_{\lambda}} u_k(\tau) \frac{\psi'(\tau)}{\psi(\tau) - z} \, d\tau$$

for any $\lambda > \sigma$. The function $b_k$ is a monic polynomial of degree $k$, which is called the $k$-th Faber–Walsh polynomial for $E$ (and the sequence $(\alpha_j)_{j=1}^\infty$).

2. Let $f$ be analytic on $E$, and let $\rho > 1$ be the largest number such that $f$ is analytic and single-valued in $\text{int}(\Gamma_\rho)$. In $\text{int}(\Gamma_\rho)$ the function $f$ admits the series expansion

$$f = \sum_{k=0}^{\infty} a_k b_k, \quad a_k = \frac{1}{2\pi i} \int_{\Lambda_{\lambda}} \frac{f(\psi(\tau))}{u_{k+1}(\tau)} \, d\tau, \quad 1 < \lambda < \rho,$$

and the series converges maximally to $f$ on $E$, i.e.,

$$\lim_{n \to \infty} \| f - \sum_{k=0}^{n} a_k b_k \|_{E} = \frac{1}{\rho}.$$

For $N = 1$ the Faber–Walsh polynomials reduce to the (classical) Faber polynomials for the (simply connected) set $E$.

Note that the Green’s functions with pole at infinity for the lemniscatic domain $L$ and for the exterior $K$ of $E$ are given by

$$H(w) = \log |U(w)| - \log(\mu) \quad \text{and} \quad G(z) = H(\psi^{-1}(z)),$$
the Faber–Walsh polynomials can also be defined using the coefficients of the Laurent series of the conformal map $\psi$

In our proof of Proposition 3.3 below we will use that for each given $\sigma > 1$ there exists positive constants $C_1, C_2$ independent of $k$ such that

$$0 < C_1 |u_k(w)| \leq |b_k(z)| \leq C_2 |u_k(w)|,$$

Here the upper bound on $|b_k(z)|$ holds for all $k$ and the lower bound holds only for sufficiently large $k$; see [21, p. 253].

In [2,3] the Faber–Walsh polynomials are defined as the (polynomial) coefficients in the expansion of the function $\psi'(w)/\psi(w) - z$. Similar to the (classical) Faber polynomials, the Faber–Walsh polynomials can also be defined using the coefficients of the Laurent series of the conformal map $\psi$ in a neighborhood of infinity. Using this approach one can derive a recursive formula for computing the Faber–Walsh polynomials. In the following result we state the recursion that we have used in our numerical computations that are described in Section 4. A variant of this recursion was first published in the technical report [17].

**Proposition 2.4.** In the notation of Theorem 2.3 suppose that the lemniscatic conformal map $\psi$ has a Laurent series in a neighborhood of infinity of the form

$$\psi(w) = w + c_0 + \sum_{k=1}^{\infty} \frac{c_k}{w^k}.$$ 

Then the Faber–Walsh polynomials $b_k$ are recursively given by

$$b_0(z) = 1$$

$$b_k(z) = (z - c_0 - \alpha_k)b_{k-1}(z) + \beta_{k-1,1}(z), \quad k \geq 1,$$

where the “correction terms” $\beta_{k,\ell}(z)$ are polynomials given by $\beta_{0,1}(z) = 0$ and

$$\beta_{1,\ell}(z) = \alpha_1(\ell - 1)c_{\ell-1} - (\ell + 1)c_{\ell}, \quad \ell \geq 1,$$

$$\beta_{k,\ell}(z) = -c_\ell b_{k-1}(z) - \alpha_k \beta_{k-1,\ell}(z) + \beta_{k-1,\ell+1}(z), \quad k \geq 2, \ell \geq 1.$$ 

3. Asymptotic optimality of the Faber–Walsh polynomials. The convergence analysis of many iterative methods for solving linear algebraic systems $Ax = b$ leads to polynomial approximation problems of the form

$$\min_{p \in \mathcal{P}_k(z_0)} \|p\|_E.$$

Here $\mathcal{P}_k(z_0)$ is the set of polynomials from $\mathcal{P}_k$ with value 1 at some given constraint point $z_0 \in \mathbb{C}$, and the compact set $E$ is associated with the matrix $A$ (e.g., an inclusion set for the spectrum of $A$). For a survey of many results in this area we refer to [13, Sections 5.6–5.7]. A polynomial for which the value (3.1) is attained is called an optimal polynomial for $E$ and $z_0$.

Suppose that $z_0 = 0 \notin E$. Then

$$\min_{p \in \mathcal{P}_k(0)} \|p\|_E = \min_{p \in \mathcal{P}_{k-1}} \|1 - zp\|_E = \min_{p \in \mathcal{P}_{k-1}} \|z(z^{-1} - p)\|_E$$

$$\leq \|z\|_E \min_{p \in \mathcal{P}_{k-1}} \|z^{-1} - p\|_E,$$
which shows that the problems (5.1) with \( z_0 = 0 \) and (1.1) with \( f(z) = z^{-1} \) are closely related. On the other hand, recall that the \( k \)th Chebyshev polynomial \( T_k(z; E) \) for \( E \) is the (uniquely determined) monic polynomial of degree \( k \) which has minimal maximum norm on \( E \). This polynomial gives the simple upper bound

\[
\min_{p \in P(k)} \|p\|_E \leq \frac{\|T_k(z; E)\|_E}{\|T_k(0; E)\|_E}.
\]

The question whether the two sides of this inequality are equal, i.e., whether the normalized Chebyshev polynomial is the optimal polynomial, is highly nontrivial even for the rather simple case of \( E \) consisting of two real intervals; see Section 4 below for details. Since

\[
\|T_k(z; E)\|_E = \min_{p \in P(k-1)} \|z^k - p\|_E,
\]

we again observe a close relation between the problems (5.1) with \( z_0 = 0 \) and (1.1), but here with \( f(z) = z^k \).

The problem (5.1) can be analyzed using the following concepts, which were introduced by Eiermann, Niethammer and Varga [4, 5] in the context of semi-iterative methods.

**Definition 3.1.** For a compact set \( E \subseteq \mathbb{C} \) and \( z_0 \in \mathbb{C} \) we call

\[
R_{z_0}(E) := \limsup_{k \to \infty} \left( \min_{p \in P_k(z_0)} \|p\|_E \right)^{1/k}
\]

the asymptotic convergence factor for polynomials from \( P_k(z_0) \) on \( E \). A sequence of polynomials \( p_k \in P_k(z_0), k = 0, 1, \ldots \) is called asymptotically optimal on \( E \) and with respect to \( z_0 \) if

\[
\lim_{k \to \infty} \|p_k\|_E^{-1} = R_{z_0}(E).
\]

Clearly, for any compact set \( E \) and \( z_0 \in \mathbb{C} \) we have \( R_{z_0}(E) \leq 1 \). If \( z_0 \in E \), then \( R_{z_0}(E) = 1 \). For \( z_0 \in \mathbb{C} \setminus E \) we can prove the following result.

**Proposition 3.2.** Let \( E \subseteq \mathbb{C} \) be a compact set with no isolated points such that its complement \( K = \mathbb{C} \setminus E \) is connected and of finite connectivity, and let \( G \) be the Green’s function with pole at infinity for \( K \) (for the existence of this function see, e.g., [26, p. 65]). Then

\[
R_{z_0}(E) = \frac{1}{\sigma_0} \quad \text{for each } z_0 \in \mathbb{C} \setminus E,
\]

where \( \sigma_0 > 1 \) is defined by \( G(z_0) = \log(\sigma_0) \).

**Proof.** For simply connected \( K \) and \( z_0 = 0 \) the proof is given in [5, Theorem 11]. The authors also state the more general case in [5, p. 524] and refer to the book of Walsh [26]. We now sketch a proof using our notation.

Since \( E \) is compact and \( z_0 \notin E \),

\[
0 < m := \min_{z \in E} |z - z_0| \leq M := \max_{z \in E} |z - z_0| < \infty.
\]

Let \( q_{k-1} \in P_{k-1} \) be arbitrary, then \( p_k(z) := 1 - (z - z_0)q_{k-1}(z) \in P_k(z_0) \) and

\[
|p_k(z)| = |z - z_0| \left| \frac{1}{z - z_0} - q_{k-1}(z) \right| \leq M \left| \frac{1}{z - z_0} - q_{k-1}(z) \right| \leq \frac{M}{m} |p_k(z)|.
\]
for all \( z \in E \), from which we obtain
\[
\min_{p_k \in \mathcal{P}_k(z_0)} \| p_k \|_E \leq M \min_{q_{k-1} \in \mathcal{P}_{k-1}} \| \frac{1}{z-z_0} - q_{k-1}(z) \|_E \leq M \min_{p_k \in \mathcal{P}_k(z_0)} \| p_k \|_E,
\]
and hence
\[
R_{z_0}(E) = \limsup_{k \to \infty} \left( \min_{p_k \in \mathcal{P}_k(z_0)} \| p_k \|_E \right)^{\frac{1}{k}}
\]
(3.2)
\[
= \limsup_{k \to \infty} \left( \min_{q_{k-1} \in \mathcal{P}_{k-1}} \| \frac{1}{z-z_0} - q_{k-1}(z) \|_E \right)^{\frac{1}{k}}.
\]
Note that the function \( \frac{1}{z-z_0} \) is analytic on \( E \) and in \( \text{int}(\Gamma_\sigma) \) (as defined in (2.5)), but it is not analytic in \( \text{int}(\Gamma_\sigma) \) for any \( \sigma > \sigma_0 \). Then, by [16, Theorem 4.1], which follows from the results in [26, Ch. 4], we have
\[
\limsup_{k \to \infty} \left( \min_{q_k \in \mathcal{P}_k} \| \frac{1}{z-z_0} - q_k(z) \|_E \right)^{\frac{1}{k}} = \frac{1}{\sigma_0},
\]
and the assertion follows using (3.2). \( \Box \)

With this characterisation of \( R_{z_0}(E) \) we can now prove the asymptotic optimality of the Faber–Walsh polynomials that are normalized at \( z_0 \).

**Proposition 3.3.** In the notation of Theorem 2.3 let \( z_0 \in \mathbb{C}\setminus E \) and let \( \sigma_0 > 1 \) be such that \( z_0 \in \Gamma_{\sigma_0} \). Then
\[
R_{z_0}(E) = \frac{1}{\sigma_0} \quad \text{and} \quad R_{z_0}(\text{int}(\Gamma_\sigma)) = \frac{\sigma}{\sigma_0} \quad \text{for} \ 1 < \sigma \leq \sigma_0,
\]
and the Faber–Walsh polynomials for \( E \) satisfy
\[
\lim_{k \to \infty} \left( \frac{\| b_k \|_E}{\| b_k(z_0) \|} \right)^{\frac{1}{k}} = \frac{1}{\sigma_0} = R_{z_0}(E),
\]
(3.3)
\[
\lim_{k \to \infty} \left( \frac{\| b_k \|_{\text{int}(\Gamma_\sigma)}}{\| b_k(z_0) \|} \right)^{\frac{1}{k}} = \frac{\sigma}{\sigma_0} \quad \text{for any} \ \sigma > 1.
\]
(3.4)
Hence the normalized Faber–Walsh polynomials \( b_k/z_0 \in \mathcal{P}_k(z_0) \) are asymptotically optimal on \( E \) and on \( \text{int}(\Gamma_\sigma) \) whenever \( 1 < \sigma \leq \sigma_0 \).

**Proof.** The set \( K = \mathbb{C}\setminus E \) has a Green’s function \( G \) with pole at infinity; see the discussion below Theorem 2.3. Since \( z_0 \in K \), there exists a unique \( \sigma_0 > 1 \) with \( z_0 \in \Gamma_{\sigma_0} \), namely \( |U(\psi^{-1}(z_0))| = \sigma_0 \mu \). Thus \( R_{z_0}(E) = \frac{1}{\sigma_0} \) by Proposition 3.2. The Green’s function with pole at infinity for \( \text{ext}(\Gamma_\sigma) \) is \( G_\sigma(z) = G(z) - \log(\sigma) \). Hence, for \( 1 < \sigma \leq \sigma_0 \),
\[
G_\sigma(z_0) = \log\left( \frac{z_0}{\sigma_0} \right) \quad \text{and} \quad R_{z_0}(\text{int}(\Gamma_\sigma)) = \frac{\sigma_0}{\sigma_0} \quad \text{by Proposition 3.2}
\]

Let \( \sigma > 1 \). By (2.7) there exists constants \( C_1, C_2 > 0 \) such that for sufficiently large \( k \) we have
\[
C_1 |u_k(w)| < |b_k(z)| < C_2 |u_k(w)|, \quad w \in \Lambda_\sigma, \quad z = \psi(w) \in \Gamma_\sigma.
\]
Apply Lemma 2.2 to bound \( |u_k(w)| \): There exist \( A_1, A_2 > 0 \) such that (2.7) holds for \( w \in \Lambda_\sigma \). We thus have
\[
C_1 A_1 (\sigma \mu)^k < |b_k(z)| < C_2 A_2 (\sigma \mu)^k, \quad w \in \Lambda_\sigma, \quad z = \psi(w) \in \Gamma_\sigma.
\]
(3.5)
We then have \( b_k(z) \neq 0 \) for \( z \in \Gamma_{\gamma} \) and, in particular, \( b_k(z_0) \neq 0 \) for sufficiently large \( k \). From (3.5) follows \( \lim_{k \to \infty} |b_k(z)|^{\frac{1}{k}} = \sigma \mu \) for any \( z \in \Gamma_{\sigma} \), and, in particular,

\[
\lim_{k \to \infty} |b_k(z_0)|^{\frac{1}{k}} = \sigma_0 \mu.
\]

Further \( \lim_{k \to \infty} \|b_k\|_E^{\frac{1}{k}} = \sigma \mu \), since (3.5) holds uniformly for \( z \in \Gamma_{\sigma} \). This establishes (3.4).

We proceed to show (3.3). Let \( \mu_k := \min \{ \|p\|_E : p \text{ monic of degree } k \} \). From [19, Sect. 1.3.4] it is known that \( (\mu_k^{\frac{1}{k}}) \) converges to the capacity \( \mu \) of \( E \). Since the Faber–Walsh polynomials \( b_k \) are monic of degree \( k \), we have the estimate

\[
\mu_k \leq \|b_k\|_E \leq \|b_k\|_{\inf(\Gamma_{\sigma})} = \|b_k\|_{\Gamma, \sigma} \leq C_2 A_2(\sigma \mu)^k
\]

where \( \sigma > 1 \) is arbitrary; see (3.5). This shows

\[
\mu \leq \lim \inf_{k \to \infty} \|b_k\|_E^{\frac{1}{k}} \leq \lim \sup_{k \to \infty} \|b_k\|_E^{\frac{1}{k}} \leq \sigma \mu
\]

and \( \lim_{k \to \infty} \|b_k\|_E^{\frac{1}{k}} = \mu \), since \( \sigma > 1 \) was arbitrary. Together with (3.6) we obtain (3.3). \( \square \)

4. Asymptotic convergence factors and the Faber–Walsh polynomials on two disjoint intervals. In this section we will consider the problem (3.1), the corresponding asymptotic convergence factor, and the Faber–Walsh polynomials for sets \( E \) consisting of two disjoint real intervals, i.e.,

\[ E = [a, b] \cup [c, d] \quad \text{with } a < b < c < d. \]

The problem (3.1) for such sets \( E \) and \( z_0 \in \mathbb{R} \setminus E \) is of great interest in the analysis of iterative methods for symmetric or hermitian indefinite matrices and it has been intensively studied in this context; see, e.g., [13, Section 5.7.2] for a recent survey of related results.

The theory of asymptotic convergence factors and optimal polynomials that solve the problem (3.1) on two disjoint intervals was treated in detail by Fischer; see in particular [9] and [10, Sections 3.3–3.4]. Based on a representation of the Chebyshev polynomials \( T_k(z; E) \) for \( E \) in terms of the associated Green’s function (originally due to Ahiezer), Fischer obtained necessary and sufficient conditions (on \( E \) and \( T_k(z; E) \)) under which \( T_k(z; E)/T_k(z_0; E) \) is the optimal polynomial for \( E \) and \( z_0 \in \mathbb{R} \setminus E \). Since the Green’s function for \( E \) is written in terms of certain Theta functions, Fischer’s deep theory does not yield simple explicit formulas for the solution of the problem (3.1) or the corresponding asymptotic convergence factor. It also should be pointed out that our approach in this paper is conceptually different from the one described in [10]: The construction of the Faber–Walsh polynomials is based on a lemniscatic conformal map as described in Theorem 2.1 while the (classical) theory in this context conformally identifies \( E \) with an annulus; see [10, pp. 115–119] for the construction of that conformal map.

4.1. Two intervals of same length. We first consider sets \( E \) consisting of two intervals of the same length which are symmetric with respect to the origin, i.e.,

\[
E = [-\delta, -\gamma] \cup [\gamma, \delta] \quad \text{with } 0 < \gamma < \delta.
\]

In this case the solution of the problem (3.1) with \( z_0 \in \mathbb{R} \setminus E \) and for \( k = 2n \) and \( k = 2n + 1 \) is given by

\[
\frac{T_{2n}(z; E)}{T_{2n}(z_0; E)} = \frac{T_n(q(z))}{T_n(q(z_0))}, \quad \text{where } q(z) = \frac{2}{\delta^2 - \gamma^2} z^2 - \frac{\delta^2 + \gamma^2}{\delta^2 - \gamma^2},
\]
and
\[(4.3) \quad T_n(z) = \frac{1}{2}((z + \sqrt{z^2 - 1})^n + (z - \sqrt{z^2 - 1})^n), \quad n \geq 0,\]
is the $n$th (classical) Chebyshev polynomial of the first kind. This result has been stated without proof, for example, in [8, p. 211] and [11, pp. 53–54]. For proving that the result in fact is true one can first show that
\[T_2(z; E) = z^2 - \frac{\delta^2 + \gamma^2}{2}.\]
Since \[\|T_2(z; E)\|_E = \frac{\delta^2 - \gamma^2}{2}\]
we have
\[T_2(z; E)/\|T_2(z; E)\|_E = q(z)\], and the result follows from [10, Cor. 3.3.6] by noting that \(T_2(z; E)\) has the four extremal points \(\pm \gamma, \pm \delta\). We point out that this argument is restricted to real \(z_0\), since the proofs in [10] are based on the alternation property of the Chebyshev polynomials for subsets of the real line.

In particular, for \(z_0 = 0\) the definition of the Chebyshev polynomials of the first kind yields
\[(4.4) \quad \min_{p \in P_k(0)} \|p\|_E = \frac{\|T_{\lfloor k/2 \rfloor}(q(0))\|_E}{\|T_{\lfloor k/2 \rfloor}(q(z))\|_E} \leq 2 \left(\frac{\delta - \gamma}{\delta + \gamma}\right)^{\lfloor \frac{k}{2} \rfloor};\]
see, e.g., [8, p. 211] or [11, pp. 53–54]. Hence the asymptotic convergence factor for polynomials from \(P_k(0)\) on \(E\) satisfies
\[(4.5) \quad R_0(E) \leq \left(\frac{\delta - \gamma}{\delta + \gamma}\right)^{\frac{k}{2}}.\]

In order to compute the asymptotic convergence factor \(R_{z_0}(E)\) and the Faber–Walsh polynomials for \(E\) we need a lemniscatic conformal map as described in Theorem 2.1.

**Proposition 4.1.** Let \(E\) be as in (4.1) and define \(a := \frac{\delta + \gamma}{2}\). Then \(E\) has the logarithmic capacity \(\mu = \frac{\sqrt{\delta^2 - \gamma^2}}{2}\), and the function
\[z = \psi(w) = w \sqrt{1 + \frac{\mu^4}{a^2} \frac{1}{w^2 - a^2}},\]
where we take the principal branch of the square root, conformally maps
\[\mathcal{L} = \{w \in \hat{\mathbb{C}} : |w - a|^{\frac{1}{2}} |w + a|^{\frac{1}{2}} > \mu\}\]
o nto \(K = \hat{\mathbb{C}} \setminus E\), and satisfies the normalisation \(\psi(w) = w + O(\frac{1}{w})\) at infinity. The inverse of \(\psi\) is given by
\[(4.6) \quad w = \Phi(z) = \sqrt{\frac{z^2}{2} + \frac{a^4 - \mu^4}{2a^2} + \frac{1}{2} \sqrt{(z^2 - a^4 + a^2)^2 - 4\mu^4}},\]
with suitably chosen branches of the square root.

Moreover, the Laurent series at infinity of \(\psi\) is
\[(4.7) \quad \psi(w) = w + \sum_{k=1}^{\infty} c_{2k-1} \frac{1}{w^{2k-1}},\]
where the coefficients are given by $c_{2k} = 0$, $k \geq 0$, and

$$c_{2k-1} = \frac{1}{2} \mu^4 a^{2k-4} - \frac{1}{2} \sum_{j=1}^{k-1} c_{2j} c_{2(k-j)-1}, \quad k \geq 1. \quad (4.8)$$

In particular, $c_1 = \frac{1}{2} \mu^4 a^4$. 

**Proof.** The construction of the lemniscatic conformal map $\psi$ and its inverse $\Phi$ can be found in [18, Section 3]; see in particular Corollary 3.2 in that paper. It thus remains to derive the Laurent series of $\psi$.

First note that the function $w \mapsto \sqrt{1 + \frac{\mu^4 a^4}{w^2 - a^2}}$ is analytic and even in $\mathcal{L}$. It thus has a series expansion

$$\sqrt{1 + \frac{\mu^4 a^4}{w^2 - a^2}} = \sum_{k=0}^{\infty} \frac{d_k}{w^{2k}}$$

converging uniformly in a neighbourhood of infinity. Setting $w = \infty$ shows $d_0 = 1$. Note that

$$1 + \frac{\mu^4 a^4}{w^2 - a^2} = 1 + \frac{\mu^4 a^4}{w^2} \sum_{k=0}^{\infty} \frac{a^{2k}}{w^{2k}} = 1 + \sum_{k=0}^{\infty} \mu^4 a^{2k-2} \frac{1}{w^{2k+2}} = 1 + \sum_{k=1}^{\infty} \mu^4 a^{2k-4} \frac{1}{w^{2k}}$$

in a neighbourhood of infinity. Squaring (4.9) yields

$$1 + \sum_{k=1}^{\infty} \mu^4 a^{2k-4} \frac{1}{w^{2k}} = \left( \sum_{k=0}^{\infty} \frac{d_k}{w^{2k}} \right)^2 = \sum_{k=0}^{\infty} \sum_{j=0}^{k} d_j d_{k-j} \frac{1}{w^{2k}}.$$

For $k \geq 1$ we conclude

$$\mu^4 a^{2k-4} = \sum_{j=0}^{k} d_j d_{k-j} = 2d_0d_k + \sum_{j=1}^{k-1} d_j d_{k-j}$$

and thus

$$d_k = \frac{1}{2} \mu^4 a^{2k-4} - \frac{1}{2} \sum_{j=1}^{k-1} d_j d_{k-j}.$$ 

We now have $\psi(w) = w + \sum_{k=1}^{\infty} d_k \frac{1}{w^{2k-1}}$, which shows that $c_{2k} = 0$ for $k \geq 0$ and

$$c_{2k-1} = d_k = \frac{1}{2} \mu^4 a^{2k-4} - \frac{1}{2} \sum_{j=1}^{k-1} d_j d_{k-j} = \frac{1}{2} \mu^4 a^{2k-4} - \frac{1}{2} \sum_{j=1}^{k-1} c_{2j-1} c_{2(k-j)-1}$$

for $k \geq 1$. \( \square \)

Let $E$ be as in (4.11) and let $z_0 \in \mathbb{C} \setminus E$. Since the assumptions of Proposition 3.2 hold for $E$, the asymptotic convergence factor for polynomials from $\mathcal{P}_k(z_0)$ on $E$ is given by $R_{z_0}(E) = \frac{1}{\sigma_0}$, where $\sigma_0 > 1$ is defined by $z_0 \in \Gamma_{\sigma_0}$. The latter condition is equivalent to $\Phi(z_0) = \psi^{-1}(z_0) \in \Lambda_{\sigma_0}$, recall (2.3), i.e.,

$$\sigma_0 \mu = |U(\Phi(z_0))| = |\Phi(z_0)^2 - a^2|^{\frac{1}{2}},$$
and thus

$$R_{z_0}(E) = \frac{\mu}{|\Phi(z_0)^2 - a|^\frac{1}{2}}.$$ 

For the special case $z_0 = 0$ we have $\Phi(0) = 0$ and hence

$$R_0(E) = \frac{\mu}{a} = \left(\frac{\delta - \gamma}{\delta + \gamma}\right)^\frac{1}{2},$$

which shows that the upper bound (4.5) is sharp. In Figure 4.1 we plot the asymptotic convergence factors $R_{z_0}(E_j)$ for the sets

$$E_j := [-1, -2^{-j}] \cup [2^{-j}, 1], \quad j = 1, 2, 3, 4,$$

and real $z_0$ ranging from $-2$ to $2$. Note that when $z_0$ is to the left or the right of the two intervals (i.e. $|z_0| > 1$) the asymptotic convergence factors $R_{z_0}(E_j)$ are almost identical for all $j$, and they decrease quickly with increasing $|z_0|$. On the other hand, when $z_0$ is between the two intervals (i.e. $|z_0| < 2^{-j}$) the asymptotic convergence factors $R_{z_0}(E_j)$ strongly depend on $j$, and for a fixed $z_0$ they increase quickly with increasing $j$. Moreover, $R_{z_0}(E_j)$ for $|z_0| < 2^{-j}$ is minimal when $z_0 = 0$, i.e., when $z_0$ is the midpoint between the two intervals.

Using the coefficients of the Laurent series (4.7)–(4.8) and Proposition 2.4 we can compute the Faber–Walsh polynomials for $E$ and the sequence $(a, -a, a, -a, \ldots)$; cf. our remarks after Lemma 2.2 and recall that $a$ and $-a$ are the two centers of the lemniscate. In Figure 4.2 we plot the polynomials $b_k$ and their normalized counterparts $b_k/b_k(0)$ for the set $E_2 = [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$ and $k = 4, 6$ (left), $k = 5, 7$ (right). We observe that the polynomials $b_k$ of even degrees $k = 4, 6$ have $k + 2$ extremal points on $E_2$. This suggests that their normalized counterparts $b_k/b_k(z_0)$ solve the problem (3.1), which in fact is true.

**Theorem 4.2.** Let $E$ be as in (4.1), let $a = \frac{\delta + \gamma}{2}$ and $z_0 \in \mathbb{R} \setminus E$. Then for each $k \geq 0$ the Faber–Walsh polynomial $b_{2k}$ for $E$ and the sequence $(a, -a, a, -a, \ldots)$ satisfies

$$\frac{b_{2k}(z)}{b_{2k}(z_0)} = \frac{T_{2k}(z; E)}{T_{2k}(z_0; E)}.$$
and hence
\[
\frac{\|b_{2k}\|_E}{|b_{2k}(z_0)|} = \min_{p \in \mathbb{P}_{2k}(z_0)} \|p\|_E, \quad k \geq 0.
\]

**Proof.** We will first show that
\[
b_{2k}(z) = \frac{b_{2k}(0)}{T_k(q(0))} T_k(q(z)), \quad k \geq 0,
\]
with \(T_k\) and \(q\) as in (4.2)–(4.3). For \(k = 0\) this is obvious since \(b_0(z) = 1 = T_0(z)\). Now consider some \(k \geq 1\). Using the definition (2.4) of \(b_{2k}\) and the residue theorem we can compute
\[
b_{2k}(0) = (-1)^k \frac{a^4 k + \mu^4 k}{a^{2k}} = \frac{(-1)^k}{2^{2k}} ((\delta + \gamma)^{2k} + (\delta - \gamma)^{2k}), \quad k \geq 1;
\]
see also [17]. Furthermore, we can show by induction that
\[
T_k(q(0)) = \frac{(\delta + \gamma)^{2k} + (\delta - \gamma)^{2k}}{2(\gamma^2 - \delta^2)^k}, \quad k \geq 0.
\]
Using (4.3) we now find
\[
b_{2k}(0) T_k(q(z)) = \frac{(\delta^2 - \gamma^2)^k}{2^{2k}} ((q(z) + \sqrt{q(z)^2 - 1})^k + (q(z) - \sqrt{q(z)^2 - 1})^k).
\]
Inserting $q(z)$ from (4.2) yields, after some simplifications,

$$
\frac{b_{2k}(0)}{T_k(q(0))} - T_k(q(z)) = \frac{1}{2^k} \left[ \left( z^2 - \frac{\delta^2 + \gamma^2}{2} - \sqrt{(z^2 - \gamma^2)(z^2 - \delta^2)} \right)^k + \left( z^2 - \frac{\delta^2 + \gamma^2}{2} + \sqrt{(z^2 - \gamma^2)(z^2 - \delta^2)} \right)^k \right].
$$

We now show that this is the Faber–Walsh polynomial $b_{2k}$. Here we use the fact, established in [17], that $b_{2k}$ is the polynomial part of the Laurent series at infinity of $u_{2k}(\Phi(z))$. Since we use the sequence $(a, -a, a, -a, \ldots)$, we have $u_{2k}(z) = (z^2 - a^2)^k$. With $\Phi$ as in (4.6) we obtain

$$
\Phi(z)^2 - a^2 = \frac{1}{2} \left[ z^2 - \frac{a^4 + \mu^4}{a^2} + \sqrt{\left( z^2 - \frac{a^4 + \mu^4}{a^2} \right)^2 - 4\mu^4} \right] = \frac{1}{2} \left[ z^2 - \frac{\delta^2 + \gamma^2}{2} + \sqrt{(z^2 - \gamma^2)(z^2 - \delta^2)} \right],
$$
and

$$
\frac{1}{\Phi(z)^2 - a^2} = \left( \frac{4}{\delta^2 - \gamma^2} \right)^2 \frac{1}{2} \left[ z^2 - \frac{\delta^2 + \gamma^2}{2} - \sqrt{(z^2 - \gamma^2)(z^2 - \delta^2)} \right].
$$

Now, since $\Phi(z) = O(z)$ at infinity, the Laurent series at infinity of $u_{2k}(\Phi(z))$ and $u_{2k}(\Phi(z)) + (\delta^2 - \gamma^2)^{2k} T_{2k}(q(z))$ have the same polynomial parts. The latter expression is equal to

$$
u_{2k}(\Phi(z)) + (\delta^2 - \gamma^2)^{2k} T_{2k}(q(z)) = (\Phi(z)^2 - a^2)^k + \left( \frac{\delta^2 - \gamma^2}{2} \right)^k T_{2k}(q(z)),
$$

which shows (4.11). Since $b_{2k}(z)$ and $T_{2k}(z; E)$ are both monic, (4.2) with $z_0 = 0$ implies that

$$b_{2k}(z) = T_{2k}(z; E).
$$

Now the polynomial $T_{2k}(z; E)/T_{2k}(z_0; E)$ is the optimal polynomial for $E$ and any $z_0 \in \mathbb{R} \setminus E$, which completes the proof. 

Let us continue with our numerical study of the Faber–Walsh polynomials $b_{k,j}$, $k = 1, 2, \ldots$, for the sets $E_j$ of the form (4.10) and the sequence $(a, -a, a, -a, \ldots)$. Again we compute these polynomials using the coefficients of the Laurent series (4.7)–(4.8) and Proposition 2.4. Figure 4.3 shows the values $\|b_{k,j}\|_{|p|}$ for the first 30 Faber–Walsh polynomials $(k = 1, \ldots, 30)$ on the four sets $E_j$ ($j = 1, 2, 3, 4$). We observe that the convergence speed of $\|b_{k,j}\|_{|p|}$, $k = 1, 2, \ldots$, to zero significantly slows down when $j$ increases. The “zigzags” in the curves are due to the fact that for even degrees $b_{k,j}/b_{k,j}(0)$ is the optimal polynomial (as shown in Theorem 4.2), while for odd degrees it is not.

The asymptotic optimality of the normalized Faber–Walsh polynomials on the set $E_j$ means that

$$\|b_{k,j}\|_{|p|} \leq c_j R_j(E_j) = c_j \left( \frac{1 - 2^{-j}}{1 + 2^{-j}} \right)^{\frac{1}{2}}.$$
holds for some constant $c_j > 0$ independent of $k$. In Figure 4.4 we plot the values

$$\frac{\|b_{k,j}\|_E}{|b_{k,j}(0)|} \left( \frac{1 - 2^{-j}}{1 + 2^{-j}} \right)^{-\frac{k}{2}}$$

for $k = 1, \ldots, 30$ and $j = 1, 2, 3, 4$.

The numerical results indicate that the constant $c_j$ is moderate (apparently at most 8) for all $j$. Interestingly, the constant $c_j$ appears to decrease with increasing $j$. In addition, we observe that for small and even $k$ the value (4.12) is less than 2. Hence for these $k$ the inequality (4.4) must be strict.

### 4.2. Two arbitrary real intervals.

In this section we consider the general case of two disjoint real intervals, i.e., a set of the form $[a, b] \cup [c, d]$ with $a < b < c < d$. Using the linear
transformation
\[ w \mapsto \frac{a + d - 2w}{a - d} \]
one can map this set onto
\[(4.13) \quad E := [-1, -\beta] \cup [\gamma, 1] \quad \text{with} \quad -1 < -\beta < \gamma < 1.\]
Therefore it usually suffices to consider sets of the form \((4.13)\).

An important special case arises in the analysis of iterative methods for solving discretized Stokes equations. After a suitable transformation, the spectra of the discretized operators are contained in a set \(E_h\) of the form
\[(4.14) \quad E_h := [-1, -\beta h] \cup [\gamma h^2, 1] \quad \text{with} \quad -1 < -\beta h < 0 < \gamma h^2 < 1,\]
where \(h > 0\) is some asymptotically small parameter that depends on the mesh size of the discretization. Wathen, Fischer and Silvester have shown in \([27]\) that
\[(4.15) \quad R_0(E_h) \leq 1 - (\beta \gamma) \frac{h}{2} + O(h^2) \quad \text{for} \quad h \to 0.\]
This result and its slight generalization obtained in \([28]\) appear to be the only known explicit bounds for sets \(E\) consisting of two intervals of different lengths; also cf. \([27\,\text{p. } 124]\).

Significant technical difficulties also arise when trying to generalize Proposition 4.1 from \((4.13)\) to two intervals of the same length to two intervals of different length. In fact, to our knowledge the lemniscatic conformal maps for sets \((4.13)\) are neither known analytically nor numerically. We will now present a method for constructing a lemniscatic conformal map for an inclusion set \(\tilde{E} \supset E\), which under certain assumptions captures main features of \(E\). We will use this method in particular for a numerical study of the bound \((4.15)\). Our method uses the following lemma, which can be proven by elementary means.

**Lemma 4.3.** Let \(\alpha, \beta \in \mathbb{R}\) with \(\alpha < \beta\). Let \(w_0 = \frac{\alpha + \beta}{2}\) and \(r = \frac{\beta - \alpha}{4}\). Then
\[ z = f(w) = w + \frac{r^2}{w - w_0} \]
is a conformal and bijective map from \(\{w \in \hat{C} : |w - w_0| > r\}\) onto \(\hat{C}\) \([\alpha, \beta]\), normalized at infinity by \(f(w) = w + O\left(\frac{1}{w}\right)\). Its inverse is
\[ f^{-1} : \hat{C}(\alpha, \beta) \to \{w \in \hat{C} : |w - w_0| > r\}, \quad z \mapsto \frac{1}{2}(z + w_0 \pm \sqrt{(z - \alpha)(z - \beta)}), \]
where the branch of the square root is taken such that \(|f^{-1}(z) - w_0| > r\).

Now consider a set \(E\) as in \((4.13)\). Our following construction is illustrated in Figure 4.5 where the first plot (Figure 4.5(a)) shows the set \(E\) defined by \(\beta = h, \gamma = h^2\), and \(h = 0.3\). In order to apply Lemma 4.3 to the interval \([-1, -\beta]\) we define the function
\[ z = f_1(\zeta) = \zeta + \frac{r_1^2}{\zeta - \zeta_1}, \quad \text{where} \quad \zeta_1 = -\frac{1 + \beta}{2}, \quad r_1 = \frac{1 - \beta}{4}. \]
Then \(f_1^{-1}\) maps \(\hat{C}\) \(\backslash E\) onto the exterior of \(C_{r_1}(\zeta_1) \cup [\gamma, \bar{\beta}]\), where \(C_{r_1}(\zeta_1) = \{\zeta : |\zeta - \zeta_1| = r_1\}\) and
\[ \bar{\gamma} = f_1^{-1}(\gamma) = \frac{1}{2}(\gamma - \frac{1 + \beta}{2} + \sqrt{(1 + \gamma)(\beta + \gamma)}), \]
\[ \bar{\delta} = f_1^{-1}(\delta) = \frac{1}{2}(1 - \frac{1 + \beta}{2} + \sqrt{2(1 + \beta)}). \]
with the usual square root; see Figure 4.5(b).

We next want to apply Lemma 4.3 to the interval $[\tilde{\gamma}, \tilde{\delta}]$ and hence define the function

$$\zeta = f_2^{-1}(w) = w + \frac{r_2^2}{w - w_2}, \quad \text{where} \quad w_2 = \frac{\tilde{\gamma} + \tilde{\delta}}{2}, \quad r_2 = \frac{\tilde{\delta} - \tilde{\gamma}}{4}.$$ 

Then $f_2^{-1}$ maps the exterior of $C_{r_1}(\zeta_1) \cup [\tilde{\gamma}, \tilde{\delta}]$ onto the exterior of $f_2^{-1}(C_{r_1}(\zeta_1)) \cup C_{r_2}(w_2)$; see the solid curves in Figure 4.5(c).

In the final step we determine a (generalized) lemniscate consisting of two components and containing the set $f_2^{-1}(C_{r_1}(\zeta_1)) \cup C_{r_2}(w_2)$ its interior. More precisely, we need to find $a_1, a_2 \in \mathbb{C}$ and real $m_1, m_2, \mu > 0$ with $m_1 + m_2 = 1$, such that

$$\{w \in \mathbb{C} : |U(w)| = |w - a_1|^{m_1} |w - a_2|^{m_2} = \mu\}$$

has two components and contains $f_2^{-1}(C_{r_1}(\zeta_1)) \cup C_{r_2}(w_2)$ in its interior; see the dashed curves in Figure 4.5(c). Then, $\mathcal{L} = \{w \in \hat{\mathbb{C}} : |U(w)| > \mu\}$ is a lemniscatic domain and $\psi := f_1 \circ f_2 : \mathcal{L} \to \psi(\mathcal{L}) =: \mathcal{K}$ is a lemniscatic conformal map for the compact set $\hat{E} = \hat{\mathbb{C}} \setminus \mathcal{K}$ which contains $E$; see the dashed curves in Figure 4.5(d).

The goal of the construction is, of course, to have an inclusion set $\hat{E}$ that is “as tight as possible” around the given set $E$. In order to obtain a useful upper bound on the asymptotic
convergence factor \( R_0(E) \) we require that \( 0 \not\in \tilde{E} \). Unfortunately this property cannot be guaranteed by our construction when the distance between the two intervals is small. A numerical illustration is shown in Figure 4.6. In the two pictures on the left we plot the sets \( E_h = [-1, -h] \cup [h^2, 1] \) for \( h = 0.09 \) (top) and \( h = 0.02 \) (bottom) with the corresponding inclusion sets (dashed) and the point \( z = 0 \) (cross). In the two pictures on the right we plot the images of the sets \( E_h \) under the constructed map \( \psi^{-1} = f_2^{-1} \circ f_1^{-1} \) and the enclosing lemniscates obtained by our method that give the best possible fit. For \( h = 0.02 \) we have \( 0 \in \tilde{E} \), and our method fails.

If \( 0 \not\in \tilde{E} \) the asymptotic convergence factor \( R_0(\tilde{E}) \) for the inclusion set \( \tilde{E} \supset E \) is given as in Proposition 3.2, i.e.,

\[
R_0(\tilde{E}) = \frac{1}{\sigma_0},
\]

where \( \sigma_0 > 1 \) is determined by

\[
0 \in \Gamma_{\sigma_0} = \psi(\Lambda_{\sigma_0}), \quad \psi = f_1 \circ f_2, \quad \Lambda_{\sigma_0} = \{ w \in \hat{C} : |U(w)| = \sigma_0 \mu \}.
\]

Now \( 0 \in \Gamma_{\sigma_0} \) is equivalent to \( w_0 = \psi^{-1}(0) \in \Lambda_{\sigma_0} \), where

\[
w_0 := f_2^{-1}(-\zeta_0) = \frac{1}{2} \left( \zeta_0 + w_2 - \sqrt{(\zeta_0 - \tilde{\gamma})(\zeta_0 - \delta)} \right), \quad \zeta_0 := f_1^{-1}(0) = -\frac{1 + \beta}{2} + \frac{1}{2} \sqrt{\beta}.
\]

Let us consider sets \( E_h \) of the form (4.14) with \( \beta = \gamma = 1 \). Using the strategy outlined above we can find an inclusion set \( \tilde{E}_h \supset E_h \) (if \( h \) is not too small) and thus derive an
upper bound on the asymptotic convergence factor $R_0(E_h)$. Under all possible lemniscates we numerically compute the one that “fits best” in the sense that it minimizes the right hand side of (4.16) using the MATLAB function fmincon. The resulting asymptotic convergence factors $R_0(\tilde{E}_h)$ for $h \in [0.05, 0.8]$ are shown by the solid line in Figure 4.7. For comparison we also plot the curves

$$1 - h, \quad 1 - h^{\frac{3}{2}}, \quad 1 - h^2.$$  

Our computed curve with the values $R_0(\tilde{E}_h)$ nicely captures the behavior of the curve $1 - h^{\frac{3}{2}}$, which shows that the asymptotic bound (4.15) is quite exact already for larger values of $h$. Figure 4.8 shows the values of $R_{z_0}(E_h)$ for the inclusion sets $\tilde{E}_h$ of $E_h$ with $h = 2^{-1}, 2^{-2}, 2^{-3}$ and $z_0$ ranging from $-2$ to $2$; cf. Figure 4.1.

For computing the Faber–Walsh polynomials for the inclusion set $\tilde{E} \supset E$ via Proposition 4.4 we need the Laurent series in a neighborhood of infinity of the function $\psi = f_1 \circ f_2$.

**Proposition 4.4.** For $j = 1, 2$ let $f_j(w) = w + \frac{r_j}{w-w_j}$, where $r_j > 0$ and $w_j \in \mathbb{C}$. Then

\begin{equation}
(f_1 \circ f_2)(w) = w + \sum_{k=1}^{\infty} (r_2^k w_2^{k-1} + r_1^k d_k) \frac{1}{w^k}
\end{equation}

in a neighborhood of infinity, where the $d_k$ are given recursively by $d_1 = 1$, $d_2 = w_1$ and

\begin{equation}
d_{k+2} = (w_1 + w_2) d_{k+1} - (w_1 w_2 + r_2^2) d_k, \quad k \geq 1.
\end{equation}

**Proof.** We compute $(f_1 \circ f_2)(w) = w + \frac{r_1^2}{w-w_2} + \frac{r_2^2}{w-(w_1+w_2)}$. Here

$$\frac{1}{w-w_2} = \sum_{k=1}^{\infty} \frac{w_2^{k-1}}{w^k} \quad \text{and} \quad \frac{w-w_2}{r_2^2 + (w-w_1)(w-w_2)} = \sum_{k=1}^{\infty} \frac{d_k}{w^k}.$$
for $w$ near infinity and some coefficients $d_k$, which shows (4.17). Further

$$w - w_2 = \sum_{k=1}^{\infty} d_k \frac{w^2 - (w_1 + w_2)w + w_1 w_2 + r_2^2}{w^k}$$

$$= d_1 w + d_2 - d_1(w_1 + w_2) + \sum_{k=1}^{\infty} \frac{d_{k+2} - (w_1 + w_2)d_{k+1} + (w_1 w_2 + r_2^2)d_k}{w^k}.$$  

Equating coefficients shows $d_1 = 1$, $d_2 = d_1(w_1 + w_2) - w_2 = w_1$ and (4.18). □

We can now numerically compute the Faber–Walsh polynomials for an inclusion set $\tilde{E} \supset E$. Here we take $E = [-1, -h] \cup [h^2, 1]$ with $h = \frac{1}{4}$ and construct an inclusion set $\tilde{E}$ and a conformal map $\psi$ as described in the beginning of this subsection. We compute the Faber–Walsh polynomials with the recursion from Proposition 2.4 and the coefficients of the Laurent series of $\psi$ from Proposition 4.4. The sequence $(\alpha_j)_{j=1}^{\infty}$ is computed from the enclosing lemniscate according to Lemma 2.2. Figure 4.9 shows a few of the Faber–Walsh polynomials $b_k$ and their normalized counterparts $b_k/b_k(0)$. We see that, unlike in the case of two symmetric intervals (cf. Figure 4.2), the normalized Faber–Walsh polynomial $b_k/b_k(0)$ of even degree $k$ has only one extremal point on $\tilde{E}$. Clearly, this polynomial is neither equal to the normalized Chebyshev polynomial for $\tilde{E}$, nor does it solve the problem (3.1). Nevertheless, $b_k/b_k(0)$ is an easily computable polynomial which is asymptotically optimal on $\tilde{E}$.

Remark 4.5. The method we have illustrated in Figure 4.5 aims at an inclusion set $\tilde{E}$ for a given set $E$ consisting of two (general) disjoint real intervals. An alternative approach, which is however of less practical relevance, is to start with a given lemniscatic domain

$$\mathcal{L} = \{ w \in \hat{\mathbb{C}} : |U(w)| = |w - a_1|^m_1 |w - a_2|^m_2 > \mu \},$$

where $a_1, a_2$ are real numbers with $a_1 < a_2$, and $m_1, m_2, \mu$ are positive real numbers with $m_1 + m_2 = 1$, and to construct a conformally equivalent set $\tilde{E}$ which contains two disjoint real intervals. Let $C_{r_1}(w_1)$ be a circle in the left component of $\partial \mathcal{L}$ and apply the Joukowski map $\zeta = f_1(w) = w + \frac{r_2^2}{w - w_1}$ to the exterior of $C_{r_1}(w_1)$. Now place a circle $C_{r_2} (\zeta_2)$ into
the right component of \( f_1(\partial \mathcal{L}) \) and apply the Joukowski map \( z = f_2(\zeta) = \zeta + \frac{r^2}{\zeta - \zeta_0} \) to the exterior of \( C_{r_2}(\zeta_2) \). Then \( \mathcal{K} = (f_2 \circ f_1)(\mathcal{L}) \) is the complement of a compact set \( \tilde{E} \) that contains two disjoint intervals which may have different lengths. Note that unlike in our construction described above, we now always obtain a set \( \tilde{E} \) containing two disjoint intervals, but the lengths of these intervals are in general not known in advance.

Figure 4.10 illustrates this construction for the lemniscate \(|U(w)| = |w + 1|^{0.7} |w - 1|^{0.3} \) and \( \mu = 0.95 \) (solid line). The dashed line is the circle \(|w + 0.9| = 0.65\). Figure 4.10(b) shows the image under \( f_1 \). The dashed circle in the second component is \(|z_1 - 1.2| = 0.14\). Finally, Figure 4.10(c) shows the image under \( f_2 \).

5. Concluding remarks. In this paper we proved the asymptotic optimality of the Faber–Walsh polynomials and derived the first, to our knowledge, explicit examples of these polynomials. We focussed on the important case of sets \( E \) consisting of two disjoint real intervals. For two intervals of equal length we proved in Theorem 4.2 that the normalized Faber–Walsh polynomials \( b_k / b_k(\zeta_0) \) of even degrees \( k \) are optimal, i.e., that they solve the problem (3.1). In this case the (monic) Faber–Walsh polynomial \( b_k \) is in fact the \( k \)th Chebyshev polynomial for \( E \). We thus established a result completely analogous to the (classical) Faber polynomials, which in case of the unit interval \( E = [-1, 1] \) are equal to the Chebyshev polynomials of \( E \); see, e.g., [19, pp. 133–134] or [21, p. 37].

For sets \( E \) consisting of two general intervals the theory of Chebyshev, optimal and Faber–Walsh polynomials is significantly more difficult. We are not aware of an analytically known lemniscatic conformal map for \( E \). Instead we constructed a lemniscatic conformal map for a certain inclusion set of \( E \). Our approach, however, works only when the distance...
between the two intervals is not too small. It may be possible to improve the results in Section 4.2 by refining the conformal mapping strategy.

Moreover, for further advances in the theory and application of the Faber–Walsh polynomials it would be helpful to have additional explicit examples and a numerical method for computing lemniscatic conformal maps for a given set \( E \) consisting of several components.

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