Sling effect in collisions of water droplets in turbulent clouds

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We describe and evaluate the contribution of sling effect into the collision rate of the same-size water droplets in turbulent clouds. We show that already for Stokes numbers exceeding 0.2 the sling effect gives a contribution comparable to Saffman-Turner contribution, which may explain why the latter consistently underestimates collision rate (even with the account of preferential concentration).

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Let us first briefly discuss the latter effect called preferential concentration. It has been identified long ago (see Maxey 1987; Squires and Eaton 1991; Sundaram and Collins 1997; Reade and Collins 2000; Kostinski Shaw 2001; Jaczewski and Malinowski 2005; McFarquhar 2004; Franklin et al 2005; Grits et al 2006 and the references therein). Still, the proper quantification of this effect and of its role in the collision rate enhancement in warm clouds remains to be done. It has been inferred from the data (Sundaram and Collins, 1997) that $P(l)$ has a power-law dependence and argued theoretically (Balkovsky et al, 2001) that the dependence must be of the form $P(l) \sim (l/\lambda)^{-\alpha}$ where $\alpha$ is of the order of the viscous scale of turbulence and the dimensionless quantity $\alpha$ depends on the dimensionless numbers that characterize air turbulence, particle inertia and gravity — Reynolds, Stokes and Froude numbers respectively:

$$Re = (L/\eta)^{1/3}, \quad St = \eta \nu/\eta^2 = \lambda \tau, \quad F = \lambda \eta / g \tau.$$  

Here $L$ is the size of the largest turbulent eddies (typically, the size of the cloud), $\nu$ is the air viscosity, $g$ is the acceleration of gravity and $\tau = (2/9)(\rho_0/\rho)(a^2/\nu)$ is called the reaction (Stokes) time with $\rho, \rho_0$ being the air and water density respectively.

The behavior of $\alpha$ is well-understood for small $St$ where $\alpha \sim b(Re, F) St^2$ as predicted by Balkovsky et al (2001), Falkovich et al (2002), and confirmed by Falkovich and Pumir (2004) and Chiu and Koch (2005) for moderate $Re$. Here we establish $\alpha$ for arbitrary $St$ and substantially higher $Re$ than before. Note in passing that a generalization for different-size droplets value, $P(a, a')$, has been suggested in Falkovich et al (2002) and Bec et al (2005).

The main subject of the paper is the proper evaluation of the contribution of the relative velocity into the collision rate, particularly, the (generally nonlocal) relation between the air flow and the droplet velocities. Saffman and Turner (1956) assumed that the relative velocity of the droplets is determined locally by the air velocity which is spatially smooth at such scales (since $\eta \ll \eta$) so that $\Delta v \simeq 2\lambda a$ where $\lambda$ is the rms air-velocity gradient according to (2). However, the droplet velocity is determined not only by a local air velocity but also by the previous history because of inertia. As was first noticed by Falkovich et al (2002), this leads to an extra contribution to the collision rate not captured by the Saffman-Turner formula. Figure 1 illustrates this (so-called sling) effect: The right droplet passed through an intense vortex and had been thrown away as if by a sling. As a result, the relative velocity of the droplets at the point of collision may be determined not by the air-flow gradient at this point but rather by a distant vortex. Let us give here some numbers adopted from the calculations to be described below. Consider turbulence with $\lambda = 80 s^{-1}$. For droplets with $a = 10 \mu m$, $St = 0.08$ and sling effects are negligible; for $a = 15 \mu m$, $St = 0.2$ and the frequency of sling effects is $\simeq \lambda / 1000$ while their contribution into the collision rate is about 20%; for $a = 20 \mu m$, $St = 0.35$ and the frequency of sling effects is $\simeq \lambda / 50$ while their contribution into the collision rate is about 35% (the numbers correspond to Figs. 3, 5, 6 below). An independent confirmation that Saffman-Turner formula (which disregards sling events) consistently underestimates the collision rate even at relatively small Stokes numbers has been obtained recently by direct numerical simulations (Franklin et al, 2005).
must be described by the Saffman-Turner formula with a proper preferential concentration correction) and a sling-effect contribution. That impedes the progress towards a proper parametrization of the collision rate. Here we propose another, complementary way of modelling based on the continuous description of the flow of droplets. The equation for the droplet velocity $\mathbf{v}$,

$$\frac{dv}{dt} = (\mathbf{u} - \mathbf{v})/\tau + \mathbf{g},$$

(3)
can be considered as defining everywhere in space the field $\mathbf{v}(\mathbf{r})$ from the known air-flow field $\mathbf{u}(\mathbf{r})$. In the continuous description we are able to see clearly the sling contribution. Indeed, sling events appear in the equations of motion for particles as crossing of trajectories or finite-time singularities of the continuous equations. For the matrix of droplet-flow gradients, $\sigma_{ij} = \partial v_i/\partial r_j$, one gets in the co-moving (Lagrangian) reference frame:

$$\dot{\sigma}_{ij} + \sigma_{ik} \sigma_{kj} + \sigma_{ij} / \tau = s_{ij} / \tau,$$

(4)

where $s_{ij} = \partial u_i / \partial r_j$. Smoothness of the air flow means finiteness of $s_{ij}(t)$. On the contrary, nonlinearity (which corresponds to inertia) in the equation for $\dot{\sigma}$ leads to the possibility of explosions when some component of $\sigma_{ij}$ turns into $-\infty$ in a finite time by the law $\sigma \propto (t - t_0)^{-1}$ (Falkovich et al. 2002).

Because of spatial and temporal non-locality, analytical description of the sling-effect contribution into the collision rate is difficult. Using simple models, it has been predicted that probabilities of such events have a very sharp dependence on the Stokes number: $\exp[-c(Re, Fr) / St]$ (Falkovich et al. 2002; Wilkinson and Mehlig 2003, 2005), where neither the value of the factor $c(Re, Fr)$ nor its dependence on the Reynolds and Froude numbers is known. Particularly important is understanding the dependence on Reynolds number which varies by many orders of magnitude in atmospheric flows.

In this paper, we perform direct numerical simulations of the air-flow turbulence at such high $Re$ that were never reached before in collision rate calculations. We pay the price by not performing kinetic droplet simulations (i.e. following separate droplets) but by using continuous field description based on (3,4) and the equation for particle concentration in the Lagrangian reference frame,

$$dn/dt = -n \sigma_{ii}(t).$$

(5)

To this end, we generate a statistically stationary turbulent flow in a cube with periodic boundary conditions. The equations are solved with a pseudo spectral code, see Pumir (1994) for details. All the appropriate length scales of the flow are adequately resolved. We work in the range $21 \leq R_\lambda \leq 105$. In this turbulent flow, we follow the motion of inertial particles by solving the equation for the position, $d\mathbf{x}/dt = \mathbf{v}$, along with (3). The equation (4) for the tensor of droplet velocity derivatives is integrated along the way. The integration of Eq. (4) requires the interpolation of the fluid velocity, $\mathbf{u}$, or its derivative, $s$, from the numerical mesh to the particle position. This is done by using spline interpolation techniques, see Girimaji and Pope (1990). The equations are solved by using algorithms that are second order accurate in time, or higher.

For each run, we integrate the equations of motion for several Stokes numbers (ranging from $St = 0.05$ to $St \approx 5$). Gravity is also taken into account, by taking a finite value of the Froude number. In this work, as in Falkovich and Pumir (2004), the only property of the droplets which varies is the radius, $a$. Accordingly, the Froude and the Stokes number vary in such a way that the product of the Stokes number by the Froude number is constant:

$$\epsilon_0 \equiv St \times Fr$$

(6)

Numerically, it was found that $\epsilon_0 \gtrsim 5$ corresponds to a vanishingly weak gravity. The effect of gravity becomes appreciable for values of $\epsilon_0$ smaller than 1.

The occurrence of a sling effect results in a singularity of $\sigma$ in a finite time, $\sigma \propto (t - t_0)^{-1}$, which in turn, leads to a divergence of the particle density $n \propto (t - t_0)^{-1}$. Physically, neither the droplet velocity gradient nor the droplet density can grow unrestricted since droplets have a finite size, $a$, and cannot come arbitrarily close to one another. Therefore, the droplet velocity gradients do not grow by more than by a factor $\sim (\eta/a)$. Once the gradient has reached this predetermined threshold value, the equation is regularized, by flipping the sign of $\sigma$. This simply corresponds to a fast droplet passing a slow one so that their velocity difference changes sign (as well as the velocity gradient). Similarly, the density increases until the time of the flip and then decreases. This algorithm leads to a numerically well-posed problem.

To infer the properties of the coarse grained distribution at a given scale $r$, we use the method developed by Balkovsky et al. (2001), Falkovich et al. (2002) and implemented numerically by Falkovich and Pumir (2004) in the restricted case where the Stokes number was small, and where no sling contribution was expected. That method
requires one to follow (in addition to $\mathbf{x}, \mathbf{v}, \sigma$ and $n$) the deformation of a volume, seeded with bubbles, and carried by the $v$-flow. Specifically, one needs to determine the contraction rate of the volume along particle trajectories. This is effectively done by monitoring the growth of the inverse of the deformation tensor, $W$, which describes how a line element is transported by the flow: $\delta \mathbf{l}(t) = W(t) \cdot \delta \mathbf{l}(0)$, where $W^{-1}$ satisfies:

\[ \frac{dW^{-1}}{dt} = -(W^{-1} \cdot \sigma + \sigma^T \cdot W^{-1}) \quad (7) \]

To estimate the contribution of a trajectory to the coarse grained particle density at scale $\tau$, the integration is carried out until $W^{-1}$ reaches $(\eta/\tau)$. At this point, the value of $n$ is recorded. The Eulerian value of the $k^{th}$ moment is obtained by averaging the values of $n^{k-1}$ over all the trajectories computed.

In such an approach, the collision rate along a trajectory consists of two contributions. The first one is the ‘continuous contribution’, determined by a local velocity gradient (like in the Saffman-Turner approach). Given the value of $n$ and of $\sigma$ at a given time, the instantaneous flux of incoming particles towards a given particle is as follows:

\[ \Phi_{\text{cont}}(t) = -(2a)^3 n(t) \int_{\hat{\mathbf{e}} \cdot \sigma < 0} (\hat{\mathbf{e}} \cdot \sigma \cdot \hat{\mathbf{e}}) \, d\Omega \quad (8) \]

The contribution to the collision rate, $K_{\text{cont}}$, along a trajectory is simply obtained by integrating $\Phi_{\text{cont}}$ over time. Because the growth of $n$ is accounted for in the resulting contribution for the collision term, the influence of the sling effect on concentration (because of caustics left after sling events) is taken into account in this formula. However, as has been pointed out above, the sling effect also results in an additional contribution to the velocity difference, which is not proportional to the local velocity gradient. As suggested by Fig. 1, a sling event involves a number of particles incoming in a small region with significantly different velocities, a situation leading to an outburst of collisions, which we estimate as follows. The source term in Eq. 8 necessary to start a blow-up process, $s/\tau$, should be large enough to drive $\sigma$ to start the blow up: $|s| > 1/\tau$. Based on the Kolmogorov picture of fully developed turbulence, the extent of the region where gradients reach the value $s$ of the order or larger than $1/\tau$ is of the order of $l \sim (v\tau)^{1/2} = \eta St^{1/2}$. The time over which the collision takes place is estimated to be of the order of the droplet relaxation time, $\tau$. Last, the range of particle velocities involved during the collision can be estimated as $|\delta \mathbf{v}| \sim 1/\tau$. Based on these estimates, the number of collisions that occur in the wake of a sling event that have happened at time $t_s$ can be estimated as follows:

\[ N_{\text{sling}}(t) = 4\pi (2a)^2 \times n(t_s + \tau/2) \times |\delta \mathbf{v}| \times \tau \quad (9) \]

The relaxation time of droplet velocities is $\tau$ and this is a typical duration of the time when droplets have substantially different velocities and sling-effect contribution appears. That is why in estimating $N_{\text{sling}}$ the density of particles is taken at a time $\tau/2$ after the sling event time $t_s$, which provides a reasonable estimate for the particle density during the entire process. As it is clear from the derivation, the value of $N_{\text{sling}}$ obtained in such a way is up to a numerical factor of order unity, whose precise value could be estimated by kinetic numerical simulations, which is beyond the scope of this work. In the rest of this paper, we estimate separately the contributions from the regular term and the sling contribution, given by Eq. 8 and Eq. 9 respectively.

The method sketched above to estimate the coarse grained properties at a scale $\tau$ is then used to evaluate the collision rate as a function of scale. More precisely, consider droplets of radius $a$. To compute the collision rate, the regular and sling contributions to the collision rate are computed along trajectories, until the compression, given by $|W^{-1}|$, see Eq. 4, reach the value $\eta/a$. The various contributions coming from different trajectories are then accumulated, and the mean value of the collision rate extracted.

The way the flow leads to the compression of an ensemble of particles, described by Eq. 4, plays a crucial importance in the physical processes controlling particle collision rates. The numerical results indicate that $ln(|W^{-1}|)$ grows linearly with time. The growth rate gives a direct access to the smallest Lyapunov exponent $\lambda_3$ that corresponds to the strongest contraction in the flow. At a given Reynolds number, the value of $-\lambda_3 \times \tau_K$ has a non trivial dependence on the Stokes number. It starts at a value $-\lambda_3 \times \tau_K \approx 0.16$ at $St = 0$ (Falkovich et al., 2002), then increases up to $St \sim 0.5$, leading to a twofold increase of $\lambda_3$ compared to its value when $St = 0$, before decreasing at higher values of $St$, see Fig. 2. Qualitatively similar dependence has been found for a model short-correlated flow (Wilkinson and Mehlig, 2003). The values of $\lambda_3$ are not affected much by the effect of gravity, at the moderate values of the Froude numbers considered here, as shown in the figure. No significant dependence of the product $\tau_K \times \lambda_3$ as a function of the Reynolds number was found over the range covered here.

Sling events, in our approach, are manifested by divergences of the particle velocity derivative tensor, $\sigma$. The blow-up frequency, $f_{bu}$, defined as the total number of sling events, divided by the integration time, plays here a crucial role. Fig. 3 shows $f_{bu}$ multiplied by the Kolmogorov time $\tau_K$. The data shown correspond to two values of $\epsilon_0$ ($\epsilon_0 = 5$ and $\epsilon_0 = 0.4$). The blow-up frequency, made dimensionless with the Kolmogorov time, $\tau_K$, shows a fairly weak dependence on the Reynolds number. No blow-up is observed at very low value of $St$ (for $St \lesssim 0.15$). The value of $f_{bu}$ then raises to a maximum value for $St \sim 1.5$, then decreases slowly. Upon increasing gravity (decreasing $\epsilon_0$), the blow-up frequency generally goes down, with a similar Stokes number dependence. The recent works on simple versions of the problem (1-dimensional and short-correlated flows suggest the dependence of the blow-up frequency as a func-
FIG. 2: The contraction rate along trajectories of particles at $R_\lambda = 105$ and at a very low gravity ($\epsilon_0 = 5$, upper curve) and at a moderate gravity ($\epsilon_0 = 0.4$, lower curve).

The dependence of $St$ of the form: $f_{bu} \approx \exp(-A/\text{St})$ (Wilkinson and Mehlig 2005, Derevyanko et al 2006). Here we find empirically that the curve could be fit pretty well by the dependence of the form:

$$f_{bu} \times \tau_K = St^{-2} \times \exp(-A/\text{St}) \times (B + C \text{St}^\alpha) \quad (10)$$

The coefficient $A$ is found to decrease slightly as the Reynolds number increases ($A = 2.1$ for $R_\lambda = 45$, $A = 1.85$ for $R_\lambda = 83$ and $A = 1.70$ for $R_\lambda = 105$), consistent with the fact that as turbulence becomes more intense, higher gradients appear in the flow, which are able to induce blow-up of $\sigma$ at increasingly low values of $St$. Yet, at higher values of the Stokes number, the blow-up frequency seems to decrease as the value of the Reynolds number increases, a somewhat surprising effect.

The dependence of the coarse-grained particle density, $\langle n^2 \rangle_r$, as a function of $\eta/r$, is very similar to the one obtained by Falkovich and Pumir, 2004. Namely, $\langle n^2 \rangle_r$ has essentially a power-law dependence as a function of $\eta/r$. The exponent $\alpha$ of the exponent is plotted here as a function of $St$ at $R_\lambda = 83$, and for the values of $\epsilon_0 = 5$ (very low gravity) and $\epsilon_0 = 0.4$ (moderate gravity). The value of the exponent increases sharply as a function of $St$ up to $St \approx 1$, where it starts to saturate and decrease slightly. The qualitative aspect of the dependence of $\alpha$ as function of $St$ does not depend on the precise value of the Reynolds number in the range of $R_\lambda$ studied. We find that at values of $St \geq 0.1$, the value of the exponent is somewhat higher than the one found by Falkovich et al, 2002, see Fig. 4. This difference can be attributed to the fact that the exponent there was computed by studying contraction along the fluid trajectory, which in the limit $St \to 0$ differs very little from the particles trajectories. Quantitative differences remain even for values of $St$ as small as $St \approx 0.1$. At moderate values of $St$, the exponent is larger when gravity is small, as expected (Falkovich et al 2002). At larger values of the Stokes number ($St \gtrsim 2$), the two exponents obtained with a very low gravity ($\epsilon_0 = 5$) and with a moderate gravity ($\epsilon_0 = 0.4$) become very close to one another.

Finally, Fig. 5 shows the continuous contribution to the collision rates, normalized by $32\pi a^3 \tau_K$, as a function of the Stokes numbers. The continuous contribution starts at a low value at $St \to 0$, close to the value predicted by...
the Saffman and Turner formula (indicated by the horizontal dashed line), and increases to a maximum value at \( St \approx 1.0 \), before decreasing very slowly. Increasing gravity (decreasing \( \epsilon_0 \)) tends to decrease the collision rate. Over the range of parameters studied here, it was found that the continuous part of the collision rate increases when the Reynolds number increases.

The sling contribution to the collision rate, shown in Fig. 6 at the value of the Reynolds number \( R_\lambda = 105 \), starts from essentially zero at very small values of the Stokes number (the probability of having a sling effect is practically zero at \( St \ll 1 \)). Again, similarly to what has been observed for the continuous contribution to the collision term, it increases to a maximum at \( St \approx 0.8 \).

The phenomenological description of the sling collision rate used in this work is not expected to hold at values of the Stokes numbers larger than \( \sim 1 \). In fact, our approach is based on the implicit assumption that particles can be described by an essentially smooth hydrodynamic representation. This assumption becomes questionable as soon as \( St \gg 1 \). For this reason, only the part of the curve corresponding to values of \( St \leq 1 \) has been shown. Fig. 5 is meant to show the main trend, at moderate Stokes numbers (the formula used to define this term, Eq. 9 is defined up to a constant). These data do not allow to test the simple approximation \( St^{-1/2} \exp(-A/St) \) for the sling contribution suggested by Wilkinson et al. (2006) for the case without gravity.

Our method to estimate collision rates, although based on procedures which can be formally completely justified, is very indirect. It is therefore appropriate to compare our estimates for the collision rates with the results obtained by Franklin et al., 2005, by using direct numerical simulations, and by estimating in a straightforward manner the collision rate among particles. We ran a simulation at a value of \( R_\lambda = 45 \) (run 3a) comparable to the run 3 of Franklin et al., at \( R_\lambda = 48 \). The values of their ‘Geometric collision rates’, estimated for two particles’ sizes : \( a = 10 \mu m \) (\( \Gamma_{10} \)) and \( a = 20 \mu m \) (\( \Gamma_{20} \)), see their Table 5, are to be compared directly to the value of \( K_{cont} \) and of \( K_{sling} \), computed in the present work. The values obtained with our method, after proper rescaling, are shown in Table II. At the lowest value of \( St \), the value of \( \Gamma \) and \( K_{cont} \) are very close. Part of the difference can be explained by the fact that our Reynolds is slightly less than the one obtained by Franklin et al. No sling contribution is expected in this case. At the highest value of the Stokes number, one finds that the continuous part of the collision rate, \( K_{cont} \), underestimates the value found in Franklin et al. On the other hand, a significant sling effect is expected at this value of the Stokes number, so the difference can be interpreted as resulting from the sling contribution.

In conclusion, we have studied the collision rates in-
duced by turbulent air motion. The method used in this work is essentially lagrangian. We follow particles advected in the flow, compute directly the flux of incoming particles (continuous contribution) and estimate the number of collisions occurring in the aftermath of a 'sling' effect.

The ratio of the collision rate to the Saffman-Turner formula is found to increase significantly from 1 to $\sim 10$ when the Stokes number increases from $St = 0$ to $St \approx 1$. The increase becomes more pronounced as the Reynolds number becomes larger.

In the range of Reynolds number studied here, sling contributions are negligible at very small Stokes numbers: their probability goes as $\exp(-A/St)$ as a function of $St$, with a coefficient $A$ of order 1. In practice, they become significant for Stokes numbers $St \gtrsim 0.20$.

The actual collision rates computed in this work are consistent, at $R_\Lambda \approx 45$, with the recent results obtained by Franklin et al. (2005). In particular, our results allow us to disentangle the contributions due to the sling events, which we find to be quite significant for particles of size $a = 20\mu m$.

This work should help clarify the origin of the enhancement of the collision rates of inertial particles due to turbulence, and also ultimately, to devise a parametrization of this collision rate.

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[28] There is a common misconception that the gravitational collision rate is zero for equal-size droplets. This is not so since hydrodynamic interaction changes settling velocities (for example, a pair of close droplets falls faster). For a dilute set of droplets, such effects can be neglected though.