Characterisation of an entanglement-free evolution.

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Abstract: Two or more quantum systems are said to be in an entangled or non-factorisable state if their joint (supposedly pure) wave-function is not expressible as a product of individual wave functions but is instead a superposition of product states. It is only when the systems are in a factorisable state that they can be considered to be separated (in the sense of Bell). We show that whenever two quantum systems interact with each other, it is impossible that all factorisable states remain factorisable during the interaction unless the full Hamiltonian does not couple these systems so to say unless they do not really interact. We also present certain conditions under which particular factorisable states remain factorisable although they represent a bipartite system whose components mutually interact and pay a particular attention to the case where the two particles interact mutually through an action at a distance in the three dimensional space.

Introduction

The term entanglement was first introduced by Schroedinger who described this as the characteristic trait of quantum mechanics, “the one that enforces its entire departure from classical lines of thought” [1]. Bell’s inequalities [2] show that when two systems are prepared in an entangled state, the knowledge of the whole cannot be reduced to the knowledge of the parts, and that to some extent the systems lose their individuality. It is only when their joint wave-function is factorisable that they

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are separable. It is therefore interesting to investigate which are the situations such that two systems, initially prepared in a (pure) product state remain in such a state although they mutually interact.

We shall show (sections 1 and 2) that when the Hilbert spaces associated to the interacting systems $A$ and $B$ are finite dimensional, if we impose that all the product states remain product states during the interaction, the full Hamiltonian can be factorised as follows: $H_{AB}(t) = H_A(t) \otimes I_B + I_A \otimes H_B(t)$, where $H_i(t)$ acts on the “i” system only while $I_j$ is the identity operator on the “j” system ($i, j = A, B$). In other words, in quantum mechanics there is no interaction without entanglement.

We shall also present a sufficient condition under which particular factorisable (non-necessary pure) states remain factorisable during the interaction.

We shall discuss, in the section 3, the situation where the interacting systems are two three-dimensional material points that interact through an action at a distance. We shall show that the factorisability of the full wave-function is preserved (A) in the test-particle limit (when one of the particles is quite more massive than the other one and is localised in a small region of space during the interaction), (B) in the classical limit which is considered here to play relatively to quantum mechanics a role comparable to the one played by geometrical optics relatively to classical wave optics and (C) when the Hartree approximation is valid.

1 Two interacting spin one-halve particles

Firstly, let us consider the most simple situation: the systems $A$ and $B$ are spin one-halve particles. We shall now show the following theorem:

**Theorem 0:**

Let us consider a system that consists of two spin one-halve particles $A$ and $B$. Let us assume that the wave-function of the full system is a pure state of $C^2 \otimes C^2$ which evolves according to Schroedinger’s equation:

$$i\hbar \frac{\partial}{\partial t} \Psi_{AB}(t) = H_{AB}(t)\Psi_{AB}(t) \quad (1)$$

It can be shown that whenever two distant systems are in an entangled (pure) state, there exist well-chosen observables such that the associated correlations do not admit a local realist explanation, which is revealed by the violation of well-chosen Bell’s inequalities [3,4].
Let us assume that any arbitrary initially factorisable state $\Psi_{AB}(t = 0) = \psi_A(t = 0) \otimes \psi_B(t = 0)$ remains factorisable during its temporal evolution: $\Psi_{AB}(t) = \psi_A(t) \otimes \psi_B(t)$ $\forall t \geq 0$.

Then, for each time $t \geq 0$, there exists a “factorisable” Hamiltonian $H_{AB}^{fact} = H_A \otimes I_B + I_A \otimes H_B$ (where $H_{AB}$ is a self-adjoint operator that acts on $\mathbb{C}^2$) which brings the same change at time $t$ as the change due to $H_{AB}(t)$.

Proof of the Theorem 0:

Let us firstly consider that initially the state of the system belongs to a factorisable basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ that consists of the four following states: $\Psi_{AB}(t = 0) = |+_A \rangle \otimes |+_B \rangle$, $\Psi_{AB}^2(t = 0) = |+_A \rangle \otimes |-_B \rangle$, $\Psi_{AB}^3(t = 0) = |-_A \rangle \otimes |+_B \rangle$, $\Psi_{AB}^4(t = 0) = |-_A \rangle \otimes |-_B \rangle$, where $|+ (')_A(B)\rangle$ and $|-(')_A(B)\rangle$ represent up and down spin states along conventional axes of reference $Z_A(Z_B)$ assigned to the systems $A(B)$.

Necessarily, $\Psi_{AB}^1(t) = |+_A \rangle \otimes |+_B \rangle$ for some couple of directions $\vec{Z}_A, \vec{Z}_B$. The unitarity of the evolution law imposes that either $\Psi_{AB}^2(t) = |\psi_A^2(t)\rangle \otimes |-_B \rangle$ or $\Psi_{AB}^2(t) = |-_A \rangle \otimes |\psi_B^2(t)\rangle$ where $|\psi_A^2(t)\rangle$ is undetermined at this level of the proof. Let us consider now the first alternative and assume that the system is initially prepared in the product state $\frac{1}{\sqrt{2}} \cdot (\Psi_{AB}^1(t) + \Psi_{AB}^2(t)) = |+_A \rangle \otimes \frac{1}{\sqrt{2}} \cdot (|+_B \rangle + |-_B \rangle).$

In virtue of the linearity of the evolution law, this state becomes at time $t$ the state $\frac{1}{\sqrt{2}} \cdot (|+_A \rangle \otimes |+_B \rangle + |\psi_A^2(t)\rangle \otimes |-_B \rangle)$ which is not a product state unless $|\psi_A^2(t)\rangle = |+_A \rangle$ up to a global phase-factor that we can consistently take to be equal to unity (up to a redefinition of the phase of $|-_B \rangle$). Then, $\Psi_{AB}^2(t) = |+_A \rangle \otimes |-_B \rangle$. By a similar reasoning, the second alternative leads to the conclusion that $\Psi_{AB}^2(t) = |-_A \rangle \otimes |+_B \rangle$.

By repeating this proof with $\Psi_{AB}^3$ instead of $\Psi_{AB}^2$, we get that either $\Psi_{AB}^3(t) = |+_A \rangle \otimes |+_B \rangle$ or $\Psi_{AB}^3(t) = |-_A \rangle \otimes |+_B \rangle$. In virtue of the unitarity of the evolution law, $\Psi_{AB}^2(t)$ must be orthogonal to $\Psi_{AB}^3(t)$ so that, in conclusion, two alternatives remain possible: either (i) $\Psi_{AB}^2(t) = |+_A \rangle \otimes |-_B \rangle$ and $\Psi_{AB}^3(t) = |-_A \rangle \otimes |+_B \rangle$ or (ii) $\Psi_{AB}^2(t) = |-_A \rangle \otimes |+_B \rangle$ and $\Psi_{AB}^3(t) = |+_A \rangle \otimes |-_B \rangle$. In any case, unitarity imposes that $\Psi_{AB}^4(t) = |+_A \rangle \otimes |+_B \rangle$ up to a global phase. If moreover we require that when the system is initially prepared in the product state $\frac{1}{2} \cdot (\Psi_{AB}^1(0) + \Psi_{AB}^2(0) + \Psi_{AB}^3(0) + \Psi_{AB}^4(0))$ it remains in a product state at time $t$, this global phase factor must be equal to unity. It is easy to check that in both cases all states that are initially product states are still product states at time $t$.

Let us consider firstly that the first alternative is valid. It is easy to find a “factorisable” Hamiltonian $H_{AB}^{fact}(t) = H_A(t) \otimes I_B + I_A \otimes H_B(t)$ that sends $\Psi_{AB}(0)$
on $\Psi_{AB}^{i}(t)$ ($i = 1, 2, 3, 4$) in a time $t$. Obviously, it is sufficient to choose $H_{A(B)}(t)$ in such a way that $|+_{A(B)}\rangle$ is sent onto $|\tilde{+}(t)_{A(B)}\rangle$ in a time $t$. For instance we could take $H_{A(B)}(t)$ to be a multiple of a time independent linear combination of the Pauli matrices that generates on the Bloch sphere a rotation that brings $|+_{A(B)}\rangle$ onto $|\tilde{+}(t)_{A(B)}\rangle$, and modulate the intensity of this Hamiltonian in order to perform the rotation in a time $t$. It is worth noting that such an Hamiltonian sends states that were initially product states on product states for all intermediate times $t'$ ($0 \leq t' \leq t$). Note that we could even let depend $H_{A}$ and $H_{B}$ on time and tailor them in an ad-hoc way in order to generate arbitrary continuous state evolutions for all intermediate times $t'$ ($0 \leq t' \leq t$) in so far the projections of $|+_{A(t')}\rangle$ and $|\tilde{+}_{B(t')}\rangle$ are sufficiently regular curves that can be arbitrarily well approximated by a series of arcs of circles on the Bloch sphere. However, in the present approach, time was discretised, and it is not so simple to quantify properly what we mean by “regular”, a limitation that we shall overcome in the next section, in the proof of our main theorem, where the hypothesis of regularity in time is expressed quite naturally by a requirement on the Taylor development of the temporal state evolution.

Let us now consider the second alternative and assume that for all intermediate times $t'$ ($0 \leq t' \leq t$) the evolution sends states that were initially product states on product states. Then, at time $t' = \frac{t}{2}$ either the first alternative is valid or the second alternative is valid.

If the first alternative is valid at time $t'$ let us consider the time $t'' = \frac{3t}{4}$. At time $t''$ either the first alternative is valid or the second alternative is valid. If the first alternative is valid let us consider the time $t'' = \frac{7t}{8}$. Otherwise let us consider the time $t'' = \frac{5t}{8}$.

If the second alternative is valid at time $t'$ let us consider the time $t'' = \frac{3t}{4}$. At time $t''$ either the first alternative is valid or the second alternative is valid. If the first alternative is valid let us consider the time $t'' = \frac{3t}{8}$. Otherwise let us consider the time $t'' = \frac{t}{8}$ and so on.

By doing so it is easy to show that for any positive integer $N$ there must exist an intermediate time $t_{0}$ such that the states $\Psi_{AB}^{1}(t = t_{0}) = |+_{A}\rangle_{0} \otimes |+_{B}\rangle_{0}$, $\Psi_{AB}^{2}(t = t_{0}) = |+_{A}\rangle_{0} \otimes |-_{B}\rangle_{0}$, $\Psi_{AB}^{3}(t = t_{0}) = |-_{A}\rangle_{0} \otimes +_{B}\rangle_{0}$, $\Psi_{AB}^{4}(t = t_{0}) = |-_{A}\rangle_{0} \otimes -_{B}\rangle_{0}$, (where $|+_{A(t)}\rangle_{0}$ and $|-_{A(t)}\rangle_{0}$ represent up and down spin states along conventional axes of reference $Z_{A}^{0}(Z_{B}^{0})$ assigned to the systems $A(B)$) are sent at time $t_{0} + \epsilon$ onto the states $\Psi_{AB}^{1}(t = t_{0} + \epsilon) = |+_{A}\epsilon\rangle \otimes +_{B}\epsilon\rangle$, $\Psi_{AB}^{2}(t = t_{0} + \epsilon) = |-_{A}\epsilon\rangle \otimes +_{B}\epsilon\rangle$, $\Psi_{AB}^{3}(t = t_{0} + \epsilon) = |+_{A}\epsilon\rangle \otimes -_{B}\epsilon\rangle$, $\Psi_{AB}^{4}(t = t_{0} + \epsilon) = |-_{A}\epsilon\rangle \otimes -_{B}\epsilon\rangle$, with $\epsilon = \frac{1}{2^{N}}$ (where $|+_{A(t)}\rangle_{\epsilon}$ and $|-_{A(t)}\rangle_{\epsilon}$ represent up and
down spin states along conventional axes of reference \( Z_A^\epsilon (Z_B^\epsilon) \). For \( N \) sufficiently large, the continuity of the evolution law (eqn.1) imposes that \(|+_A\rangle, = |+_A\rangle_0 + \tau(\epsilon)\) and \(|+_B\rangle, = |+_B\rangle_0 + \tau(\epsilon)\) where by definition \( \tau(\epsilon) \) decreases at least as fast as the \( m \)th power of \( \epsilon \) when \( \epsilon \) goes to zero. But then the in-product between \( \Psi^i_{AB}(t = t_0) \) and \( \Psi^i_{AB}(t = t_\epsilon) \) is equal to zero (up to \( \tau(\epsilon) \)). In virtue of Pythagoras’s theorem their distance must be equal to \( \sqrt{2} \) (up to \( \tau(\epsilon) \)) which contradicts the continuity of the evolution law. Therefore, the second alternative is not valid and for any time \( t \), we can find a “factorisable” Hamiltonian \( H_{AB}(t) = H_A(t) \otimes I_B + I_A \otimes H_B(t) \) that sends \( \Psi^i_{AB}(t) \) \( i = 1, 2, 3, 4 \) and sends product states on product states for all intermediate times.

Note that continuity in time plays a crucial role in our proof which is similar to the proof of the impossibility of sending a righthand glove onto a lefthand glove by a continuous isometry of the three dimensional physical (Euclidean) space, an intuitively obvious fact that presents deep analogies with the situation encountered here. The transformation described in the second alternative can effectively be obtained by composing the transformation described in the first alternative with a discrete transformation during which the states of the systems \( A \) and \( B \) are interchanged. Similarly, the composition of a continuous isometry (composition of Euclidean rotations and translations) and of a reflection in a mirror sends a righthand glove on a “virtual”, non-physical lefthand glove. It is highly probable that we could refine the theorem 0 and generalise it to arbitrary finite dimensional systems (where the full system is represented in the Hilbert space \( C^{d_A} \otimes C^{d_B} \), with \( d_A \) and \( d_B \) standing for the dimensions of the systems \( A \) and \( B \)) by following the same way of reasoning. Nevertheless, we shall now give a general proof based on a different approach that is less abstract but appeals more to physical intuition.

2 Two interacting finite-dimensional systems

We shall now present our main result. Let us consider two interacting quantum systems \( A \) and \( B \). We assume that the Hilbert spaces associated to these systems are finite dimensional (of dimensions \( d_A \) and \( d_B \) respectively), that the wave-function of the full system is a pure state of \( C^{d_A} \otimes C^{d_B} \) and obeys the Schroedinger equation: 

\[ i\hbar \partial_t \Psi_{AB}(t) = H_{AB}(t) \Psi_{AB}(t) \]

where \( H_{AB}(t) \) is a self-adjoint operator that acts on \( C^{d_A} \otimes C^{d_B} \), that we assume to be sufficiently regular in time in order to ensure that the temporal Taylor development of the wave-function is valid up to the second order in time.
Main Theorem:

All the product states remain product states during the interaction if and only if the full Hamiltonian can be factorised as follows:

$$H_{AB}(t) = H_A(t) \otimes I_B + I_A \otimes H_B(t) \quad (2)$$

where $H_i$ acts on the $i$th system only while $I_j$ is the identity operator on the $j$th system ($i, j = A, B$).

In order to prove this theorem, we shall firstly prove the following lemma:

Lemma:

A pure product state remains product state during the interaction if and only if, during its evolution, the Hamiltonian never couples this product state to a product state that is bi-orthogonal to it.

Proof of the Lemma:

A) Proof of the necessary condition. Let us consider that at time $t$ the system is prepared in a product state $\Psi_{AB}(t) = \psi_A(t) \otimes \psi_B(t)$, and let us choose a basis of product states $|\psi_i^A\rangle \otimes |\phi_j^B\rangle$ ($i : 1...d_A; j : 1...d_B$, and $\langle \psi_i^A|\psi_j^A\rangle = \delta_{ij} = \langle \phi_i^B|\phi_j^B\rangle$) such that $\psi_A(t) = |\psi_1^A\rangle$ and $\phi_B(t) = |\phi_1^B\rangle$. Then, after a short time $\delta t$,

$$\Psi_{AB}(t + \delta t) = (I + \frac{i\delta t}{\hbar} \cdot H_{AB}(t)) \cdot \Psi_{AB}(t) + \tau(\delta t^2)$$

where $\tau(\epsilon'^m)$ was defined in the previous section. In a matricial form, the previous equation becomes:

$$\Psi_{AB}(t + \delta t) = |\psi_1^A\rangle \otimes |\phi_1^B\rangle + \frac{i\delta t}{\hbar} \sum_{i:1...d_A; j:1...d_B} H_{i1j1} |\psi_i^A\rangle \otimes |\phi_j^B\rangle + \tau(\delta t^2) \quad (3)$$

where

$$H_{ikjl} = \langle \psi_i^A| \otimes \langle \phi_j^B| H_{AB}(t) |\psi_k^A\rangle \otimes |\phi_l^B\rangle + \tau(\delta t^2) \quad (4)$$

Equivalently,

$$\Psi_{AB}(t + \delta t) = |\psi_1^A\rangle \otimes |\phi_1^B\rangle + \frac{i\delta t}{\hbar} (\sum_{i:1...d_A} H_{i111} |\psi_i^A\rangle \otimes |\phi_1^B\rangle)

+ \sum_{j:2...d_B} H_{11j1} |\psi_1^A\rangle \otimes |\phi_j^B\rangle

+ \sum_{i:2...d_A; j:2...d_B} H_{i1j1} |\psi_i^A\rangle \otimes |\phi_j^B\rangle) + \tau(\delta t^2)$$
All the components of $\Psi_{AB}(t + \delta t)$ that are bi-orthogonal to $\Psi_{AB}(t)$ are contained in the last term of the previous equation: $\Sigma_{i2...d_A;j2...d_B} H_{1i1j1} |\psi^i_A\rangle \otimes |\phi^j_B\rangle$, up to $\tau(\delta t^2)$. We can rewrite this equation as follows:

$$\Psi_{AB}(t + \delta t) = \left( |\psi^1_A\rangle + \frac{i\delta t}{\hbar} \sum_{i1...d_A} H_{i111} |\psi^i_A\rangle \right) \otimes \left( |\phi^1_B\rangle + \frac{i\delta t}{\hbar} \sum_{j2...d_B} H_{1i1j1} |\phi^j_B\rangle \right)$$

$$+ \frac{i\delta t}{\hbar} \sum_{i2...d_A;j2...d_B} H_{1i1j1} |\psi^i_A\rangle \otimes |\phi^j_B\rangle + \tau(\delta t^2)$$

(5)

Let us assume that the Hamiltonian couples $\Psi_{AB}(t = 0)$ to states that are bi-orthogonal to it, which means that $\Sigma_{i2...d_A;j2...d_B} |H_{i1j1}|^2 \neq 0$. We shall now show that then the development of the first order in $\delta t$ of the bi-orthogonal or Schmidt decomposition [5] of $\Psi_{AB}(t + \delta t)$ contains more than one product state, which means that $\Psi_{AB}(t + \delta t)$ is entangled for $\delta t$ small enough. In order to do so, let us consider the components of $\Psi_{AB}(t + \delta t)$ that are bi-orthogonal to $\Psi_{AB}(t)$. In virtue of Schmidt’s theorem of the bi-orthogonal decomposition [5], we can find $d_A - 1$ normalized states $|\tilde{\psi}^1_A\rangle$ of $C^{d_A}$ mutually orthogonal and orthogonal to $|\psi^1_A\rangle$ and $d_B - 1$ normalized states $|\tilde{\phi}^1_B\rangle$ of $C^{d_B}$ mutually orthogonal and orthogonal to $|\phi^1_B\rangle$ such that

$$\frac{i\delta t}{\hbar} \sum_{i2...d_A;j2...d_B} H_{i1j1} |\psi^i_A\rangle \otimes |\phi^j_B\rangle = \sum_{i\leq \text{min}(d_A,d_B)} \alpha_i |\tilde{\psi}^i_A\rangle \otimes |\tilde{\phi}^i_B\rangle$$

Let us now define the state $|\tilde{\psi}^1_A\rangle$ of $C^{d_A}$ as follows: $|\tilde{\psi}^1_A\rangle = \frac{1}{N_1}( |\psi^1_A\rangle + \frac{i\delta t}{\hbar} \sum_{i1...d_A} H_{i111} |\psi^i_A\rangle )$, where $N_1$ is a normalisation factor, and let us replace the orthonormal basis $\{|\psi^1_A\rangle, |\psi^2_A\rangle, ..., |\psi^d_A\rangle\}$ of $C^{d_A}$ by the orthonormal basis $\{|\tilde{\psi}^1_A\rangle, |\tilde{\psi}^2_A\rangle, ..., |\tilde{\psi}^d_A\rangle\}$ of $C^{d_A}$ that we obtain by the Gram-Schmidt orthonormalisation procedure:

$$|\tilde{\psi}^2_A\rangle = \frac{1}{N_2} \cdot ( |\psi^2_A\rangle - \langle \psi^2_A | \tilde{\psi}^1_A \rangle \cdot |\tilde{\psi}^1_A\rangle )$$

where $N_2$ is a normalisation factor.

$$|\tilde{\psi}^3_A\rangle = \frac{1}{N_3} \cdot ( |\psi^3_A\rangle - \langle \psi^3_A | \tilde{\psi}^1_A \rangle \cdot |\tilde{\psi}^1_A\rangle - \langle \psi^3_A | \tilde{\psi}^2_A \rangle \cdot |\tilde{\psi}^2_A\rangle )$$

where $N_3$ is a normalisation factor, and so on. It is easy to check that $|\tilde{\psi}^i_A\rangle = |\psi^i_A\rangle + \tau(\delta t)$. Note that this is no longer true when the dimension $d_A$ is not finite. We can repeat the same operation in order to replace the orthonormal basis $\{|\psi^1_B\rangle, |\psi^2_B\rangle, ..., |\psi^d_B\rangle\}$ of $C^{d_B}$ by the orthonormal basis $\{|\tilde{\psi}^1_B\rangle, |\tilde{\psi}^2_B\rangle, ..., |\tilde{\psi}^d_B\rangle\}$ of $C^{d_B}$. Then, after substitution in the eqn.5, we obtain that:

$$\Psi_{AB}(t + \delta t) = \sum_{i\leq \text{min}(d_A,d_B)} \alpha_i |\tilde{\psi}^i_A\rangle \otimes |\tilde{\phi}^i_B\rangle + \tau(\delta t^2)$$
\[
|\alpha_1|^2 = 1 + \tau(\delta t^2), \sum_{i:2\cdot\min(d_A,d_B)}|\alpha_i|^2 = \frac{\delta t^2}{\hbar^2} \sum_{i:2\cdot d_A;j:2\cdot d_B}|H_{ij}|^2 + \tau(\delta t^3) \tag{6}
\]

The previous equation expresses that the development up to the first order in \(\delta t\) of the bi-orthogonal Schmidt decomposition of \(\Psi_{AB}(t + \delta t)\) contains more than one product state. It is well known that then \(\Psi_{AB}(t + \delta t)\) is an entangled state. Nevertheless, for those who are not familiar with this property, we shall prove directly the result by making use of the reduced density matrix. By definition, the reduced density matrix \(\rho_A\) of the system \(A\) is equal to \(Tr_B\rho\) where \(\rho\) is the projector on \(\Psi_{AB}\). Obviously, when the state of the system is a product state \((\Psi_{AB} = \psi_A \otimes \psi_B)\), \(\rho_A\) is the projector on \(\psi_A\), and we have that \(\rho_A = \rho_A^2\), and \(Tr\rho_A = Tr\rho_A^2 = 1\). Actually, \(Tr\rho_A - Tr\rho_A^2\) provides a good measure of the degree of the entanglement of the full system. If the Schmidt bi-orthogonal decomposition of the state \(\Psi_{AB}\) is equal to \(\sum_{i:1\cdot\min(d_A,d_B)}|\psi_A^i\rangle \otimes |\phi_B^i\rangle\), then it is easy to check that \(\rho_A = \sum_{i:1\cdot\min(d_A,d_B)}|\phi_A^i\rangle \langle \phi_A^i|\), \(Tr\rho_A = \sum_{i:1\cdot\min(d_A,d_B)}|\alpha_i|^2\) by normalisation and \(Tr\rho_A^2 = \sum_{i:1\cdot\min(d_A,d_B)}|\alpha_i|^2 \leq (Tr\rho_A)^2 = 1^2 = 1\). The last inequality is saturated for product states only. Note that \(Tr\rho_A^2 = Tr\rho_B^2\) which shows that this parameter expresses properties of the system considered as a whole, as it must be when we are dealing with entanglement. Obviously \(Tr\rho_A^2(t + \delta t) \leq |\alpha_1|^4 + (\sum_{i:2\cdot\min(d_A,d_B)}|\alpha_i|^2)^2\). But

\[
|\alpha_1|^4 = (1 - \sum_{i:2\cdot\min(d_A,d_B)}|\alpha_i|^2)^2 = (1 - \frac{\delta t^2}{\hbar^2} \sum_{i:2\cdot d_A;j:2\cdot d_B}|H_{ij}|^2 + \tau(\delta t^3))^2 \text{ and } (\sum_{i:2\cdot\min(d_A,d_B)}|\alpha_i|^2)^2 = (\frac{\delta t^2}{\hbar^2} \sum_{i:2\cdot d_A;j:2\cdot d_B}|H_{ij}|^2 + \tau(\delta t^3))^2\text{ in virtue of the eqn.}\tag{6}\]

so that \(Tr\rho_A^2(t + \delta t) \leq 1 - 2 \cdot \frac{\delta t^2}{\hbar^2} \sum_{i:2\cdot d_A;j:2\cdot d_B}|H_{ij}|^2 + \tau(\delta t^3) < 1\) for \(\delta t\) small enough\(^3\), which proves the necessary condition of the lemma.

B) Proof of the sufficient condition. Let us consider that at time \(t\) the system is prepared in a product state \(\Psi_{AB}(t) = \psi_A(t) \otimes \psi_B(t)\), and let us choose a basis of product states \(|\psi_A^i\rangle \otimes |\phi_B^i\rangle\) similar to the basis introduced in the proof of the necessary condition. When the Hamiltonian does not couple \(\Psi_{AB}(t)\) to states that are bi-orthogonal to it, \(\sum_{i:2\cdot d_A;j:2\cdot d_B}|H_{ij}|^2 = 0\) (where \(H_{ik,jl}\) is defined in the eqn.\(^4\)) and, in virtue of the eqn.\(^5\),

\[
i\hbar \frac{d}{dt} \Psi_{AB}(t) = H_{AB}(t)\Psi_{AB}(t) = (\sum_{i:d_A}H_{i111}^i|\psi_A^i\rangle \otimes |\phi_B^i\rangle + |\psi_A^i\rangle \otimes (\sum_{j:d_B}H_{11j1}^j|\phi_B^j\rangle))
\]

\(^3\)It can be shown by direct computation that when the state of the system is a product state \((\Psi_{AB}(t) = \psi_A(t) \otimes \psi_B(t))\), then the following identity \(\frac{d}{dt}Tr\rho_A^2(t) = 0\) is necessarily satisfied, independently of the form of the Hamiltonian \(H_{AB}\). This explains why no term of the first order in time appears in the previous development.
We can rewrite this equation as follows:

\[ \text{i} \hbar \partial_t \Psi_{AB}(t) = (H_{A}^{ff}(t) \cdot \psi_A(t)) \otimes \psi_B(t) + \psi_A(t) \otimes (H_{B}^{ff}(t) \cdot \psi_B(t)) \] (8)

where the effective Hamiltonians \( H_{eff} \) are defined as follows:

\[ H_{A}^{eff}(t) \cdot \rho_A(t) = Tr_B(H_{AB}(t)\rho_{AB}(t)) \] (9)

and

\[ H_{B}^{eff}(t) \cdot \rho_B(t) = Tr_A(H_{AB}(t)\rho_{AB}(t)) - (Tr_A(H_{AB}(t)\rho_{AB}(t))) \cdot \rho_B(t) \] (10)

In these expressions \( Tr_i \) represents the partial trace over the degrees of freedom assigned to the system \( i \) while \( \rho_{AB}(t) \) is the projector onto \( \Psi_{AB}(t) \), \( \rho_A(t) = Tr_B\rho_{AB}(t) \), and \( \rho_B(t) = Tr_A\rho_{AB}(t) \). For instance, we have that

\[ Tr_B(H_{AB}(t)\rho_{AB}(t)) = \sum_{i:1...d_A}\langle \phi^i_B | H_{AB} | \psi^i_A \rangle | \phi^i_B \rangle \langle \psi^i_A | + \sum_{i:1...d_A,j:1...d_B}H_{i1j1} | \psi^i_A \rangle \otimes | \phi^j_B \rangle \delta_{ij} \langle \psi^i_A | \]

\[ = \sum_{i:1...d_A} H_{i111} | \psi^i_A \rangle \langle \psi^i_A | \]

so that \( H_{A}^{eff}(t) \cdot \psi_A(t) = \sum_{i:1...d_A} H_{i111} | \psi^i_A \rangle \langle \psi^i_A | \).

Let us consider the product state \( \psi_{A}^{red}(t') \otimes \psi_{B}^{red}(t') \), where \( \psi_{A(B)}^{red}(t') \) is a solution of the reduced Schroedinger equation \( \text{i} \hbar \partial_t \psi_{A(B)}^{red}(t') = H_{A(B)}^{eff}(t') \cdot \psi_{A(B)}^{red}(t') \) for the initial condition \( \psi_{A(B)}^{red}(t) = \psi_{A(B)}(t) \). Obviously, \( \text{i} \hbar \partial_t \psi_{A}^{red}(t') \otimes \psi_{B}^{red}(t') = H_{AB} \psi_{A}^{red}(t') \otimes \psi_{B}^{red}(t') \) and \( \Psi_{AB}(t') = \psi_{A}^{red}(t') \otimes \psi_{B}^{red}(t') \) so that \( \Psi_{AB}(t') = \psi_{A}^{red}(t') \otimes \psi_{B}^{red}(t') \), \( \forall t' \geq t \) which ends the proof of the lemma.

We shall now prove the main theorem.

**Proof of the Main Theorem:**

A) Proof of the necessary condition. Let us choose a basis of product states \( | \psi^i_A \rangle \otimes | \phi^j_B \rangle \ (i : 1...d_A; j : 1...d_B \) and \( \langle \psi^i_A | \psi^j_A \rangle = \delta_{ij} = \langle \phi^i_B | \phi^j_B \rangle \). If we impose that all the product states remain product states during the interaction, then, in virtue
of the lemma, the full Hamiltonian never couples a product state to a product state that is bi-orthogonal to it. Then, at any time \( t \), \( \Sigma_{i=2 \ldots d_A,j=2 \ldots d_B} |H_{i,j}|^2 = 0 \) (where \( H_{i,j} \) is defined in the eqn 4) so that we have that:

\[
H_{AB}(t) \cdot |\psi^i_A \rangle \otimes |\phi^i_B \rangle = |\Delta_{ij}^A \psi^i_A \rangle \otimes |\phi^i_B \rangle + |\psi^i_A \rangle \otimes |\Delta_{ij}^B \phi^i_B \rangle
\]

where

\[
|\Delta_{ij}^A \psi^i_A \rangle = \Sigma_{k:1 \ldots d_A} H_{kij} |\psi^k_A \rangle
\]

and

\[
|\Delta_{ij}^B \phi^i_B \rangle = \Sigma_{k:1 \ldots d_B,k \neq j} H_{ikj} |\phi^k_B \rangle
\]

Let us consider that at time \( t \) the system is prepared along one of the first four states \( \Psi^i_{AB} (i : 1, \ldots 4) \) of this basis: \( \Psi^1_{AB}(t) = |\psi^1_A \rangle \otimes |\phi^1_B \rangle \), \( \Psi^2_{AB}(t) = |\psi^1_A \rangle \otimes |\phi^2_B \rangle \), \( \Psi^3_{AB}(t) = |\psi^2_A \rangle \otimes |\phi^2_B \rangle \), \( \Psi^4_{AB}(t) = |\psi^1_A \rangle \otimes |\phi^3_B \rangle \). Then,

\[
H_{AB}(t) \cdot \Psi^1_{AB}(t) = |\Delta_{ij}^{11} \psi^1_A \rangle \otimes |\phi^1_B \rangle + |\psi^1_A \rangle \otimes |\Delta_{ij}^{11} \phi^1_B \rangle
\]

\[
H_{AB}(t) \cdot \Psi^2_{AB}(t) = |\Delta_{ij}^{12} \psi^1_A \rangle \otimes |\phi^2_B \rangle + |\psi^1_A \rangle \otimes |\Delta_{ij}^{12} \phi^2_B \rangle
\]

\[
H_{AB}(t) \cdot \Psi^3_{AB}(t) = |\Delta_{ij}^{21} \psi^2_A \rangle \otimes |\phi^1_B \rangle + |\psi^2_A \rangle \otimes |\Delta_{ij}^{21} \phi^2_B \rangle
\]

\[
H_{AB}(t) \cdot \Psi^4_{AB}(t) = |\Delta_{ij}^{22} \psi^2_A \rangle \otimes |\phi^2_B \rangle + |\psi^2_A \rangle \otimes |\Delta_{ij}^{22} \phi^2_B \rangle
\]

By linearity,

\[
H_{AB}(t) \cdot \frac{1}{\sqrt{2}} (\Psi^1_{AB}(t) + \Psi^3_{AB}(t)) = H_{AB}(t) \cdot \frac{1}{\sqrt{2}} (|\psi^1_A \rangle + |\psi^2_A \rangle) \otimes |\phi^1_B \rangle
\]

\[
= \frac{1}{\sqrt{2}} ((|\Delta_{ij}^{11} \psi^1_A \rangle + |\Delta_{ij}^{21} \psi^2_A \rangle) \otimes |\phi^1_B \rangle + |\psi^1_A \rangle \otimes |\Delta_{ij}^{11} \phi^1_B \rangle + |\psi^2_A \rangle \otimes |\Delta_{ij}^{21} \phi^1_B \rangle)
\]

\[
= \frac{1}{\sqrt{2}} ((|\psi^1_A \rangle - |\psi^2_A \rangle) \otimes (|\Delta_{ij}^{11} \phi^1_B \rangle - |\Delta_{ij}^{21} \phi^1_B \rangle))
\]

\[
\frac{1}{\sqrt{2}} (|\psi^1_A \rangle - |\psi^2_A \rangle)
\]

is orthogonal to \( \frac{1}{\sqrt{2}} (|\psi^1_A \rangle + |\psi^2_A \rangle) \), so that \( H_{AB}(t) \cdot \frac{1}{\sqrt{2}} (\Psi^1_{AB}(t) + \Psi^3_{AB}(t)) \) couples \( \frac{1}{\sqrt{2}} (\Psi^1_{AB}(t) + \Psi^3_{AB}(t)) \) to a bi-orthogonal state unless \( (|\Delta_{ij}^{11} \phi^1_B \rangle - |\Delta_{ij}^{21} \phi^1_B \rangle) \) is parallel to \( |\phi^1_B \rangle \). Now, \( \frac{1}{\sqrt{2}} (\Psi^1_{AB}(t) + \Psi^3_{AB}(t)) \) is a product state so that, in virtue of the lemma, the following constraint must be satisfied:

\[
(|\Delta_{ij}^{11} \phi^1_B \rangle - |\Delta_{ij}^{21} \phi^1_B \rangle) = \lambda |\phi^1_B \rangle
\]
The same reasoning is valid with the states $\frac{1}{\sqrt{2}}(\Psi_{AB}^2(t) + \Psi_{AB}^4(t))$, $\frac{1}{\sqrt{2}}(\Psi_{AB}^1(t) + \Psi_{AB}^3(t))$ and $\frac{1}{\sqrt{2}}(\Psi_{AB}^2(t) + \Psi_{AB}^4(t))$ and leads to the following constraints:

$$(|\Delta_B^{12}\phi_B^2| - |\Delta_B^{22}\phi_B^2|) = \lambda'\phi_B^2$$

$$(|\Delta_A^{11}\psi_A^1| - |\Delta_A^{12}\psi_A^1|) = \lambda''\psi_A^1$$

$$(|\Delta_A^{21}\psi_A^2| - |\Delta_A^{22}\psi_A^2|) = \lambda''\psi_A^2$$

By definition (eqn. 12), $|\Delta_B^{ij}\phi_B^j|$ is orthogonal to $|\phi_B^j|$ so that necessarily $\lambda = \lambda' = 0$. Let us now consider the product state $(\Psi_{AB}^1(t) + \Psi_{AB}^2(t) + \Psi_{AB}^3(t) + \Psi_{AB}^4(t))$. By linearity:

$$H_{AB}(t) \cdot \frac{1}{2}(\Psi_{AB}^1(t) + \Psi_{AB}^2(t) + \Psi_{AB}^3(t) + \Psi_{AB}^4(t)) = H_{AB}(t) \cdot \frac{1}{2}(|\psi_A^1| + |\psi_A^2|) \otimes (|\phi_B^1| + |\phi_B^2|)$$

$$= \frac{1}{\sqrt{2}}((|\Delta_A^{11}\psi_A^1| + |\Delta_A^{21}\psi_A^2|) \otimes |\phi_B^1| + (|\Delta_B^{12}\psi_A^1| + |\Delta_B^{22}\psi_A^2|) \otimes |\phi_B^2|)$$

$$+ |\psi_A^1| \otimes (|\Delta_B^{11}\phi_B^1| + |\Delta_B^{21}\phi_B^2|) + |\psi_A^2| \otimes (|\Delta_B^{12}\phi_B^1| + |\Delta_B^{22}\phi_B^2|))$$

In virtue of the constraints, we get that:

$$H_{AB}(t) \cdot \frac{1}{2}(\Psi_{AB}^1(t) + \Psi_{AB}^2(t) + \Psi_{AB}^3(t) + \Psi_{AB}^4(t)) =$$

$$= \frac{1}{\sqrt{2}}(\lambda''|\psi_A^1| + \lambda''|\psi_A^2|) \otimes |\phi_B^1| + (|\Delta_A^{11}\psi_A^1| + |\Delta_A^{22}\psi_A^2|) \otimes (|\phi_B^1| + |\phi_B^2|)$$

$$+ (|\psi_A^1| + |\psi_A^2|) \otimes (|\Delta_B^{11}\phi_B^1| + |\Delta_B^{22}\phi_B^2|))$$

Such a state does not contain any state bi-orthogonal to $\frac{1}{2}(\Psi_{AB}^1(t) + \Psi_{AB}^2(t) + \Psi_{AB}^3(t) + \Psi_{AB}^4(t))$ only if $\lambda''|\psi_A^1| + \lambda''|\psi_A^2| = \lambda''(|\psi_A^1| + |\psi_A^2|)$, which imposes that $\lambda'' = \lambda'' = \lambda''$. We can repeat this proof with the indices $ii'$ for the system $A$ and $1j$ for the system $B$ instead of 12 as it was the case in the previous proof, and we obtain that $|\Delta_B^{ij}\phi_B^j| = |\Delta_B^{ij}\phi_B^j| = |\Delta_B^{ij}\phi_B^j|$, and $|\Delta_B^{ij}\psi_A^j| = |\Delta_B^{ij}\psi_A^j| - \lambda(j)|\psi_A^j| = |\Delta_B^{ij}\psi_A^j| - \lambda(j)|\psi_A^j|$ (where $|\Delta_B^{ij}\phi_B^j|$ does not depend on $j$ while $\lambda(j)$ and $|\Delta_B^{ij}\phi_B^j|$ do not depend on $i$). Therefore:

$$H_{AB}(t) \cdot |\psi_A^1| \otimes |\phi_B^j| = |\Delta_A^{ij}\psi_A^j| \otimes |\phi_B^j| + |\psi_A^j| \otimes |\Delta_B^{ij}\phi_B^j| - \lambda(j)|\psi_A^j| \otimes |\phi_B^j|$$
which fulfills the eqn.2 provided we proceed to the following identifications: $H_A(t) \cdot |\psi_A^i\rangle = |\Delta_A^i \psi_A^i\rangle$ and $H_B(t) \cdot |\phi_B^j\rangle = |\Delta_B^j \phi_B^j\rangle - \lambda(j)|\phi_B^j\rangle$. This ends the proof of the necessary condition of the main theorem.

B) Proof of the sufficient condition. Let us assume that the full Hamiltonian can be factorised according to the eqn.2. Let us consider the product state $\rho_{AB}(t) = \rho_A(t) \otimes \rho_B(t)$, where

$$\rho_{AB}(t) = \rho_A(t) \otimes \rho_B(t)$$

and

$$\rho_A(t) = \rho_A(t) \otimes (H_A^{eff}(t) \cdot \rho_A(t))$$

(13)

where

$$H_A^{eff}(t) \cdot \rho_A(t) = Tr_B(H_{AB}(t)\rho_{AB}(t))$$

and

$$H_B^{eff}(t) \cdot \rho_B(t) = Tr_A(H_{AB}(t)\rho_{AB}(t)) - (Tr_A(H_{AB}(t)\rho_{AB}(t))) \cdot \rho_B(t),$$

then, necessarily, the state remains factorisable during the interaction: $\rho_{AB}(t) = \rho_A(t) \otimes \rho_B(t)$ $\forall t \geq 0$.

**Proof of the Theorem 2:**

When we describe the state of the system by a density matrix, its evolution obeys the von Neumann equation:

$$i\hbar \partial_t \rho_{AB}(t) = [H_{AB}(t), \rho_{AB}(t)]$$

(14)

where $[X, Y]$ represents the commutator of two operators $X$ and $Y$. $H_{AB}(t)$ and $\rho_{AB}(t)$ are self-adjoint operators so that

$$[H_{AB}(t), \rho_{AB}(t)] = H_{AB}(t) \cdot \rho_{AB}(t) - \rho_{AB}(t) \cdot (H_{AB}(t)).$$
\( \rho_{AB}(t) \) where \( O^+ \) represents the self-adjoint operator of \( O \). Moreover, in virtue of the eqn.13, we have that:

\[
\begin{align*}
\text{i}\hbar \partial_t \rho_{AB}(t) &= (H_A^{\text{eff.}}(t) \cdot \rho_A(t)) \otimes \rho_B(t) + \rho_A(t) \otimes (H_B^{\text{eff.}}(t) \cdot \rho_B(t)) \\
&\quad - ((H_A^{\text{eff.}}(t) \cdot \rho_A(t)) \otimes \rho_B(t) + \rho_A(t) \otimes (H_B^{\text{eff.}}(t) \cdot \rho_B(t)))^+
\end{align*}
\]

Let us consider the product state \( \rho_{AB}^\text{red}(t) \otimes \rho_B^\text{red}(t) \), where \( \rho_{A(B)}^\text{red}(t) \) is a solution of the reduced von Neumann equation \( \text{i}\hbar \partial_t \rho_{A(B)}^\text{red}(t) = [H_{A(B)}^{\text{eff.}}(t), \rho_{A(B)}^\text{red}(t)] \) for the initial condition \( \rho_{A(B)}^\text{red}(t = 0) = \rho_{A(B)}(t = 0) \). Obviously, \( \text{i}\hbar \partial_t \rho_{A}^\text{red}(t) \otimes \rho_B^\text{red}(t) = H_{AB}(t)\rho_{AB}^\text{red}(t) \otimes \rho_B^\text{red}(t) \) and \( \rho_{AB}(t = 0) = \rho_{A}^\text{red}(t = 0) \otimes \rho_B^\text{red}(t = 0) \) so that \( \rho_{AB}(t) = \rho_{A}^\text{red}(t) \otimes \rho_B^\text{red}(t) \), \( \forall t \geq 0 \) which proves the theorem.

Note that the eqn.13 is linear in the coupling Hamiltonian \( H_{AB} \) and is automatically satisfied when the eqn.2 is satisfied. Nevertheless it is non-linear in \( \rho_{AB} \). Moreover, the effective potential that acts onto say the \( A \) particle is likely to depend on the state of the \( B \) particle, a situation that does not occur if we impose that all product states remain product states.

Beside, it is worth noting that the sufficient condition expressed by the eqn.13 is also necessary in the case of pure states. Effectively, if, initially, the bipartite system is prepared in a factorisable pure state: \( \Psi_{AB}(t = 0) = \psi_A(t = 0) \otimes \psi_B(t = 0) \), and that the state remains factorisable during the interaction, then, in virtue of the necessary condition of the lemma, the Hamiltonian may not couple the state \( \Psi_{AB}(t) = \psi_A(t) \otimes \psi_B(t) \) at any time \( t \geq 0 \) to a bi-orthogonal state so that, following the proof of the lemma, the eqn.8 must be valid at any time. Therefore, in virtue of the eqn.13, the eqn.13 must be valid too.

Now, the sufficient condition expressed by the eqn.13 is in general not necessary in the case of non-pure states as shows the following counterexample. If initially, the bipartite system is prepared in a factorisable state: \( \rho_{AB}(t = 0) = \rho_A(t = 0) \otimes \rho_B(t = 0) \), and that \( \forall t \geq 0, H_{AB}(t) = \rho_{AB}(t = 0) \), then it is easy to check that \( \rho_{AB}(t = 0) = \rho_{A}(t) \forall t \geq 0 \), \( H_A^{\text{eff.}}(t) \cdot \rho_A(t) = \text{Tr}_B(H_{AB}(t)\rho_{AB}(t)) = \rho_A(t) \cdot \text{Tr}_B\rho_B^2(t = 0) \), \( H_B^{\text{eff.}}(t) \cdot \rho_B(t) = \text{Tr}_A(H_{AB}(t)\rho_{AB}(t)) - (\text{Tr}_A(H_{AB}(t)\rho_{AB}(t))) \cdot \rho_B(t) = \text{Tr}_A\rho_A^2(t = 0) \cdot \rho_B(t = 0) - \text{Tr}_A\rho_A^2(t = 0) \cdot \text{Tr}_B\rho_B^2(t = 0) \cdot \rho_B(t = 0) \) and it is easy to check that in general the eqns. 8 or 13 are not valid when the initial state is not pure so to say when it is not a product of pure states.

\footnote{Note that this proof as well as the proof of the sufficient condition of the main theorem are also valid when the systems \( A \) and \( B \) are infinite dimensional, for instance when they are localised particles that interact through a central potential.}
3 The infinite dimensional case

The proofs of the necessary conditions of the lemma (and thus of the necessary condition of the main theorem) are not valid when the systems $A$ and $B$ are infinite dimensional. Nevertheless, we conjecture that these conditions are still true in that case, so to say that there is no interaction without entanglement. Let us for instance consider that $A$ and $B$ are two distinguishable particles, and that their interaction potential is an action a distance that is time-independent and invariant under spatial translations (a Coulombian interaction for instance). They fulfill thus (in the non-relativistic regime) the following Schrödinger equation:

$$i\hbar \partial_t \Psi(r_A, r_B, t) = \left(-\frac{\hbar^2}{2m_A} \Delta_A + \frac{\hbar^2}{2m_B} \Delta_B \right)\Psi(r_A, r_B, t)$$

$$+ V_{AB}(r_A - r_B) \Psi(r_A, r_B, t)$$

where $\Delta_A(B)$ is the Laplacian operator in the $A(B)$ coordinates. As the potential does depend on the relative position $r_{\text{rel}} = r_A - r_B$ only, it is convenient to pass to the center of mass coordinates:

$$i\hbar \partial_t \Psi(r_{\text{CM}}, r_{\text{rel}}, t) = \left(-\frac{\hbar^2}{2(m_A + m_B)} \Delta_{\text{CM}} + \frac{\hbar^2}{2\mu} \Delta_{\text{rel}} \right)\Psi(r_{\text{CM}}, r_{\text{rel}}, t)$$

$$+ V_{AB}(r_{\text{rel}}) \Psi(r_{\text{CM}}, r_{\text{rel}}, t)$$

where $r_{\text{CM}} = \frac{m_A r_A + m_B r_B}{m_A + m_B}$ and $\mu = \frac{m_A m_B}{m_A + m_B}$. As it is well-known, the previous equation is separable which means that if, initially, the wave-function is factorisable in these coordinates, it will remain so during the evolution. Now, we are interested in situations for which the wave-function is initially factorisable according to the partition of the Hilbert space that is induced by the systems $A$ and $B$. In general, such a wave-function is not factorisable in the coordinates of the center of mass. Formally, if $\Psi(r_A, r_B, t = 0) = \psi_A(r_A, t = 0) \cdot \psi_B(r_B, t = 0)$, $\Psi(r_{\text{CM}}, r_{\text{rel}}, t = 0) = \int d\omega A(\omega) \psi_{\text{CM}}(r_{\text{CM}}, t = 0) \cdot \psi_{\text{rel}}(r_{\text{rel}}, t = 0)$ where $A(\omega)$ is a generally non-peaked amplitude distribution. Then, at time $t$, $\Psi(r_{\text{CM}}, r_{\text{rel}}, t) = \int d\omega A(\omega) \psi_{\text{CM}}(r_{\text{CM}}, t) \cdot \psi_{\text{rel}}(r_{\text{rel}}, t)$, where $\psi_{\text{CM}}(r_{\text{CM}}, t)$ obeyed during the time interval $[0, t]$ a free Schrödinger evolution for the initial condition $\psi_{\text{CM}}(r_{\text{CM}}, t = 0)$ while $\psi_{\text{rel}}(r_{\text{rel}}, t)$ was submitted to the interaction potential $V_{AB}(r_{\text{rel}})$. In general, $\Psi(r_A, r_B, t)$ is no longer factorisable into a product of the form $\psi_A(r_A, t) \cdot \psi_B(r_B, t)$. Actually, this is not astonishing because, in virtue of Noether’s theorem the full momentum is conserved during the evolution. Therefore the recoil of one of the two particles could be used in order to determine (up to the initial undeterminacy of
the centre of mass) what is the recoil of the second particle. The existence of such correlations is expressed by the entanglement of the full wave-function. On the basis of such general considerations we expect that entanglement is very likely to occur due to the interaction between the two particles.

Nevertheless, it is interesting to investigate in which situations it is a good approximation to consider that the systems $A$ and $B$ remain in a factorisable state during time. We shall distinguish three typical situations.

### 3.1 Scattering of a light particle by a heavy and well localized target (the test-particle limit)

Let us assume that $m_A << m_B$, and that the $B$ particle is initially at rest and well localized. The particle $A$ is assumed to be initially prepared in such a way that it will pass in the vicinity of the heavy particle $B$, that its trajectory will undergo a deviation due to the influence of the interaction $V_{AB}$, and that it will finally escape to infinity without exerting any significant back action onto the particle $B$. This situation is often encountered during scattering experiment. If we let coincide the origin of the system of coordinates associated to the particle $B$ with its location, and that we neglect its recoil as well as its dispersion (this approximation is only valid during a limited period of time), the following approximations are valid: $r_{CM} \approx r_B \approx 0$, $r_{rel} \approx r_A - 0 = r_A$, $\psi_A(r_A, t) \approx \psi_{rel}(r_{rel}, t)$ and $\psi_B(r_B, t) \approx \psi_{CM}(r_{CM}, t)$. Moreover, $\Psi(r_A, r_B, t = 0) = \psi_A(r_A, t = 0) \cdot \psi_B(r_B, t = 0) \approx \psi_{rel}(r_{rel}, t = 0) \cdot \psi_{CM}(r_{CM}, t = 0) \approx \Psi(r_{rel}, t = 0)$. At time $t$, $\Psi(r_{CM}, r_{rel}, t) \approx \psi_{rel}(r_{rel}, t = 0) \cdot \psi_{CM}(r_{CM}, t) \approx \psi_A(r_A, t) \cdot \psi_B(r_B, t) \approx \Psi(r_A, r_B, t)$. The separability of the full system into its components $A$ and $B$ is thus ensured, in good approximation, during the scattering process.

### 3.2 Mutual scattering of two well localized wave packets (the classical limit-interacting material points)

Another interesting limiting case is the situation during which we can neglect the quantum extension of the interacting particles. This will occur when the interaction potential $V_{AB}$ is smooth enough and that the particles $A$ and $B$ are described by wave packets the extension of which is small in comparison to the typical length of variation of the potential. It is well known that in this regime, when the de Broglie
wave lengths of the wave packets are large enough, it is consistent to approximate quantum wave mechanics by its geometrical limit, which is classical mechanics. For instance the quantum differential cross sections converge in the limit of short wave-lengths to the corresponding classical cross section. Ehrenfest’s theorem also predicts that when we can neglect the quantum fluctuations, which is the case here, the average motions are nearly classical and provide a good approximation to the behaviour of the full wave-packet in so far we consider it as a material point. In this regime, we can in good approximation replace the interaction potential by the first order term of its Taylor development around the centers of the wave-packets associated to the particles $A$ and $B$:

$$V_{AB}(\mathbf{r}_A - \mathbf{r}_B) \approx V_{AB}(<\mathbf{r}_A>_t - <\mathbf{r}_B>_t) + \nabla_A V_{AB}(<\mathbf{r}_A>_t - <\mathbf{r}_B>_t) \cdot (\mathbf{r}_A - <\mathbf{r}_A>_t)$$

$$+ \nabla_B V_{AB}(<\mathbf{r}_A>_t - <\mathbf{r}_B>_t) \cdot (\mathbf{r}_B - <\mathbf{r}_B>_t).$$

Then the evolution equation is in good approximation separable into the coordinates $\mathbf{r}_A, \mathbf{r}_B$ and we have that, when $\Psi(\mathbf{r}_A, \mathbf{r}_B, t=0) = \psi_A(\mathbf{r}_A, t=0) \cdot \psi_B(\mathbf{r}_B, t=0)$, then, at time $t$, $\Psi(\mathbf{r}_A, \mathbf{r}_B, t) \approx \psi_A(\mathbf{r}_A, t) \cdot \psi_B(\mathbf{r}_B, t)$ where

$$i\hbar \partial_t \psi_A(\mathbf{r}_A, t) \approx -\frac{\hbar^2}{2m_A} \Delta_A \psi_A(\mathbf{r}_A, t)$$

$$+ (V_{AB}(<\mathbf{r}_A>_t - <\mathbf{r}_B>_t) + \nabla_A V_{AB}(<\mathbf{r}_A>_t - <\mathbf{r}_B>_t) \cdot (\mathbf{r}_A - <\mathbf{r}_A>_t)) \psi_A(\mathbf{r}_A, t)$$

(18)

$$i\hbar \partial_t \psi_B(\mathbf{r}_B, t) \approx -\frac{\hbar^2}{2m_B} \Delta_B \psi_B(\mathbf{r}_B, t)$$

$$+ (\nabla_B V_{AB}(<\mathbf{r}_A>_t - <\mathbf{r}_B>_t) \cdot (\mathbf{r}_B - <\mathbf{r}_B>_t)) \psi_B(\mathbf{r}_B, t)$$

(19)

Note that the Bohmian velocities associated to the particles $A$ and $B$ are factorisable only when the full state is factorisable. Otherwise, the velocity of a particle depends non-locally on the location of both particles.

### 3.3 Bound states: the Hartree approximation

When the energy of the full system is negative, we expect that it will remain in a well localised bound state. When one particle is quite more massive than the rest of the system as is the case with the sun in the solar system or with the nucleus inside the atom, it is a very good approximation to neglect its recoil and its extension for all times. Indeed, if we think to the nucleus for instance, its recoil is zero in
average over an orbit, and its Compton wave length is very small. Therefore it is consistent in a first approach to reduce the study of the energy levels of atoms to the study of the energy levels of the electrons that are assumed to undergo an external central Coulombian potential due to the presence of the nucleus and to factorize the full wave function into a product of an electronic wave function and of a nuclear one. For sure this approximation is valid to the extent that we can neglect other degrees of freedom as the nuclear spin and so on which is not always the case. If moreover we assume that the electronic wave function is itself factorisable, which is certainly a crude approximation because of the presence of exchange terms due the undistinguishability of the electrons and because the Coulombian interaction between the electrons is likely to generate entanglement, we are performing the so called Hartree approximation [6]. Let us consider the Helium atom for instance, and let us neglect the fermionic exchange contributions, the spins of the electrons and of the nucleus and so on. The time independent (electronic) Schroedinger equation is then the following:

\[
E_{AB} \cdot \Psi(r_A, r_B) = (-\frac{\hbar^2}{2m_A} \Delta_A + V_A - \frac{\hbar^2}{2m_B} \Delta_B + V_B) \Psi(r_A, r_B) + V_{AB}(r_A - r_B) \Psi(r_A, r_B)
\]

(20)

where \(V_A\) and \(V_B\) represent the external fields (for instance the Coulombian nuclear field), while \(V_{AB}\) represents the Coulombian repulsion between the electrons \(A\) and \(B\). Let us assume that this equation admits a factorisable solution \(\Psi(r_A, r_B) = \psi_A(r_A) \cdot \psi_B(r_B)\); then:

\[
E_{AB} \cdot \psi_A(r_A) \cdot \psi_B(r_B)
\]

\[
= ((-\frac{\hbar^2}{2m_A} \Delta_A + V_A)\psi_A(r_A)) \cdot \psi_B(r_B) + \psi_A(r_A) \cdot (-\frac{\hbar^2}{2m_B} \Delta_B + V_B)\psi_B(r_B)
\]

\[
+ V_{AB}(r_A - r_B) \psi_A(r_A) \cdot \psi_B(r_B)
\]

(21)

Let us now take the in-product of this equation with \(\psi_A(r_A)\) and multiply the resulting equation by \(\psi_A(r_A)\) respectively. We obtain:

\[
E_{AB} \cdot \psi_A(r_A) \cdot \psi_B(r_B)
\]

\[\text{Note that when the Hartree approximation is valid, particles behave as if they were discernable, and constituted of a dilute, continuous medium distributed in space according to the quantum distribution in } \psi_{AB}^2(r_A, B, t), \text{ which is close to the interpretation of the wave-function originally adopted by Schroedinger.}\]
\[
\psi_A(\mathbf{r}_A) \cdot (\mathbf{E}_A \cdot \psi_B(\mathbf{r}_B)) + \psi_B(\mathbf{r}_B) \\
\psi_A(\mathbf{r}_A) \cdot (\mathbf{E}_B \cdot \psi_A(\mathbf{r}_A)) + \psi_B(\mathbf{r}_B)
\]

Similarly, we get that:

\[
E_{AB} \cdot \psi_A(\mathbf{r}_A) \cdot \psi_B(\mathbf{r}_B)
\]

\[
= < (-\frac{\hbar^2}{2m_A} \Delta_A + V_A) \cdot \psi_B(\mathbf{r}_B) >_A + < (-\frac{\hbar^2}{2m_B} \Delta_B + V_B) \cdot \psi_A(\mathbf{r}_A) >_B + < V_{AB}(\mathbf{r}_A - \mathbf{r}_B) \cdot \psi_A(\mathbf{r}_A) \cdot \psi_B(\mathbf{r}_B) >_{AB}
\]

Let us now sum the two previous equations and subtract the eqn.21. We obtain the following consistency condition:

\[
(E_{AB} - < (-\frac{\hbar^2}{2m_A} \Delta_A + V_A) >_A - < (-\frac{\hbar^2}{2m_B} \Delta_B + V_B) >_B) \cdot \psi_A(\mathbf{r}_A) \cdot \psi_B(\mathbf{r}_B)
\]

\[
= < V_{AB}(\mathbf{r}_A - \mathbf{r}_B) >_A + < V_{AB}(\mathbf{r}_A - \mathbf{r}_B) >_B - < V_{AB}(\mathbf{r}_A - \mathbf{r}_B) >_{AB}
\]

Equivalently, when the wave-function does not vanish, the following condition must be satisfied:

\[
V_{AB}(\mathbf{r}_A - \mathbf{r}_B) = < V_{AB}(\mathbf{r}_A - \mathbf{r}_B) >_A + < V_{AB}(\mathbf{r}_A - \mathbf{r}_B) >_B - < V_{AB}(\mathbf{r}_A - \mathbf{r}_B) >_{AB}
\]

This is nothing else than the condition 13 in a static form. Note that here it appears to be a necessary condition, which does not infirm our conjecture that the necessary condition of the lemma is true in the infinite dimensional case. Reciprocally, it is easy to check that if the eqn.13 is satisfied, that the full state is factorisable and that the reduced states of the particles A and B are eigenstates of their respective effective Hamiltonians, the full state is eigenstate of the full Hamiltonian, in agreement with the sufficient condition of the lemma (which is a special case of the theorem 2). We see thus that the Hartree approximation is valid when the interaction factorises into the sum of two effective potentials that act separately on both particles, and express the average influence due to the presence of the other particle (which is not true in general and certainly not inside the atom). Similarly, in the test-particle limit, the effective potential undergone by the massive particle is close to zero, and when the heavy particle is well localised, its average, effective, potential is close to the real potential undergone by the light “test-particle”. In the classical limit (material points), the quantum internal structure of the interacting particles can be neglected and the potential is equivalent to the sum of the effective potentials.

In the three cases, the necessary condition of the lemma is not infirmed, and its sufficient condition is confirmed. In all the cases, the systems are separated only in first approximation.
4 Conclusions and comments

A conclusion of this work could be: in quantum mechanics to interact means nearly always to entangle. We showed that real interactions do necessarily generate entanglement (the inverse result, that it is impossible to generate entanglement without turning on an interaction, is rather trivial).

Considered so, the degree of entanglement of the universe ought to increase with time, which would indicate some analogy between entanglement and entropy. Note however that the temporal reversibility of the Schroedinger equation implies that the degree of entanglement could also decrease in time so that we face a paradox analog to the famous Loschmidt paradox which emphasises the apparent contradiction between the temporal asymmetry of the second principle of thermodynamics and the temporal symmetry of fundamental interactions. Obviously, such considerations are out of the scope of this paper and we invite the interested reader to consult the reference [7] and references therein.

Beside, it would be worth investigating the generalisation of our results to infinite dimensional systems. We conjecture that the necessary conditions of our lemma and of our main theorem are still true when we deal with infinite dimensional systems, as is the case for the corresponding sufficient conditions and for the theorem 2.

Let us briefly reconsider the three situations during which the interaction between two mutually interacting particles (in the three dimensional, physical space) is entanglement-free, at least in first approximation (see section 3). These are the test-particle limit (no feedback), the geometrical limit of quantum wave mechanics (narrow wave-packets) and the Hartree approximation (particles seen as a dilute gas). Each of these situations has a counterpart in “classical” physics: idealised test-particles play an important role in classical mechanics and in general relativity, the geometrical limit of quantum wave mechanics is Hamiltonian mechanics, while the image according to which charged particles are characterised by a spatial distribution (spherical or other) motivated important works in classical electro-magnetism at the beginning of our century. Entanglement really marks a departure from such lines of thought, which confirms the deep intuition of Schroedinger, already mentioned in the introduction, who described entanglement as the characteristic trait of quantum mechanics, “the one that enforces its entire departure from classical lines of thought” [1].

The present work was motivated by the results presented in the references [8,9]. In these papers it is argued and shown that retrievable, usable quantum information
can be transferred in a scheme which, in striking contrast to the quantum teleportation schemes, requires no external channel and does not involve the transfer of a quantum state from one subsystem to the other. Entanglement-free interaction between two mutually scattering particles (in the three dimensional, physical space) plays a crucial role in this scheme. The previous remarks suggest that localisation of at least one of the particles is a necessary ingredient of such protocols for quantum information transfer. For instance, in the test-particle limit the massive particle is localised while in the classical limit, both particles are localised. It is easy to show that if at least one of the two interacting particles is not well localised (bilocated for instance), and that the particles interact through a position-dependent potential (action at a distance), they are highly likely to end up in an entangled state.

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References

[1] E. Schroedinger, *Discussion of probability relations between separated systems*, Proc. Cambridge Philos. Soc. 31, 555 (1935).

[2] J. S. Bell, *On the EPR paradox*, Physics, 1, 165 (1964).

[3] N. Gisin, *Bell’s inequality holds for all non-product states*, Phys. Lett. A 154, n° 5,6, 201 (1991).

[4] D. Home and F. Selleri, *Bell’s theorem and the EPR paradox*, La Rivista del Nuovo Cimento della Societa Italiana di fisica, 14, n° 9 (1991) p 24.

[5] A. Peres, *Quantum Theory: Concepts and Methods*, Kluwer Dordrecht (1993) p123.

[6] L. Landau and E. M. Lifshitz, *Non-Relativistic Quantum Mechanics*, Pergamon Press Oxford (1962) p234.
[7] J. Gemmer, A. Otte and G. Mahler, *Quantum approach to a derivation of the second law of thermodynamics*, Phys. Rev. Lett. **86**, 1927 (2001).

[8] J. Corbett and D. Home, *Quantum effects involving interplay between unitary dynamics and kinematic entanglement*, Phys. Rev. A, **62**, 062103 (2000).

[9] J. Corbett and D. Home, *Ipso-Information-transfer*, quant-ph/0103146.