ELLIPTIC GENERA, TORUS ORBIFOLDS AND MULTI-FANS

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Abstract. Multi-fan is an analogous notion of fan in toric theory. Fan is a combinatorial object associated to a toric variety. Multi-fan is associated to an orbifold with an action of half the dimension of the orbifold. In this paper the equivariant elliptic genus and the equivariant orbifold elliptic genus of multi-fans are defined and their character formulas are exhibited. A vanishing theorem concerning elliptic genus of multi-fans of global type and its applications to toric varieties are given.

Keywords: fan, multi-fan, toric variety, torus manifold, elliptic genus, orbifold elliptic genus, $T_y$-genus, rigidity theorem, vanishing theorem

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1. Introduction

A torus orbifold is an oriented closed orbifold of even dimension which admits an action of a torus of half the dimension of the orbifold with some orientation data concerning codimension two fixed point set components of circle subgroups and with some restrictions on isotropy groups of points of the orbifold. Typical examples are complete toric varieties with simplicial fan. To a toric variety there corresponds a fan, and that correspondence is one-to-one. In particular algebro-geometric properties of a toric variety can be described in terms of combinatorial properties of the corresponding fan in principle, see e.g. [14]. In a similar way, to a torus orbifold there corresponds a multi-fan, a generalization of the notion of fan. The notion of multi-fan was introduced in [13] and a combinatorial theory of multi-fans was developed in [6]. In particular it was shown there that every complete simplicial multi-fan of dimension greater than two can be realized as the one associated with a torus orbifold. It should be noticed that a torus orbifold also determines a set of vectors which generates the one dimensional cones of the associated multi-fan. It turns out that many topological invariants of a torus orbifold can be described in terms of the multi-fan and the set of generating vectors associated with it.

The purpose of the present note is to discuss elliptic genera for torus orbifolds and multi-fans. Two sorts of elliptic genus are defined for stably almost complex orbifolds. One is the direct generalization of elliptic genus for stably almost complex manifolds to stably almost complex orbifolds which we shall denote by $\varphi$. The other, denoted by $\hat{\varphi}$, is the so-called orbifold elliptic genus which has its origin in string theory. Correspondingly we can define two sorts of elliptic genus for a pair of complete simplicial multi-fan and generating vectors. Note that, though there may not be a stably almost complex complex structure on a torus orbifold in general, one can still define elliptic genera $\varphi$ and $\hat{\varphi}$ for pairs of complete simplicial multi-fan and generating vectors, and therefore define them for general torus orbifolds vice versa.

Borisov and Libgober gave a beautiful formula for elliptic genus $\varphi$ of complete non-singular toric varieties in [1]. Theorem 3.3 and Theorem 3.4 describe similar formulae...
expressing the equivariant elliptic genera $\varphi$ and $\hat{\varphi}$ of a complete simplicial multi-fan as a virtual character of the associated torus. The starting point of [11] was the sheaf cohomology of toric varieties. Our starting point is the fixed point formula of the Atiyah-Singer type due to Vergne applied to the action of the torus.

A remarkable feature of elliptic genera is their rigidity property. If the circle group acts on a closed almost complex (or more generally stably almost complex) manifold whose first Chern class is divisible by a positive integer $N$ greater than 1, then its equivariant elliptic genus of level $N$ is rigid, that is, it is a constant character of the circle group. It was conjectured by Witten [17] and proved by Taubes [15], Bott-Taubes [3] and Hirzebruch [4]. Liu [12] found a simple proof using the modularity of elliptic genera. Applying this to a non-singular complete toric variety we see that its elliptic genera $\varphi$ with $(-y)^N = 1$, $-y \neq 1$ of level $N$ is rigid if its first Chern class is divisible by $N$. Moreover, using a vanishing theorem due to Hirzebruch [7], we can show that the genus actually vanishes. In this note we shall extend this result to global torus orbifolds and the corresponding type of multi-fans and generating vectors. We call this particular type the global type. Here a torus orbifold $M$ is called a global torus orbifold if $M$ is the quotient of a torus manifold $\tilde{M}$ by a finite subgroup of the torus acting on $\tilde{M}$. The rigidity of elliptic genus and orbifold elliptic genus for general orbifolds do not hold in general. We shall return to this point and related topics in another paper.

The fact that $\varphi$ and $\hat{\varphi}$ are virtual characters of a torus is proved by using the multiplicity formula for Duistermaat-Heckman function for multi-polytopes given in [6]. The Chern class is also defined for a pair of a multi-fan and its generating vectors. The rigidity and vanishing property of elliptic genus of level $N$ can be formulated for a pair of multi-fan and its generating vectors of the global type. One of the main results is Corollary 5.3 which states that, if the first Chern class of a pair of a multi-fan and its generating vectors of the global type is divisible by $N$, then its elliptic genus of level $N$ vanishes. The proof of rigidity and vanishing property follows the idea of the proof given in [7] translated in combinatorial terms. When $N = 2$, the torus manifold is a spin manifold. The corresponding multi-fan might be called a non-singular spin multi-fan. As a corollary we see that its signature vanishes in this case.

The equivariant $T_y$-genus can be considered as a special value of equivariant elliptic genus. It was shown in [6] that the equivariant $T_y$-genus of a complete multi-fan was rigid, and a formula for the $T_y$-genus was given. We shall give another proof of this formula using the character formula mentioned before. Moreover, if the first Chern class is divisible by $N$, then $T_y$-genus vanishes for $(-y)^N = 1$. One can derive some applications from this fact. For example if $\Delta$ is a complete non-singular multi-fan of dimension $n$ with first Chern class $c_1(\Delta)$ divisible by $N$ and with non-vanishing Todd genus, then $N$ must be equal to or less than $n + 1$ (Proposition 6.2). In the extremal case $N = n + 1$, if $\Delta$ is assumed to be a complete non-singular ordinary fan, then $\Delta$ must be isomorphic to the fan of projective space $\mathbb{P}^n$. Hence a complete non-singular toric variety $M$ of dimension $n$ with $c_1(M)$ divisible by $n + 1$ must be isomorphic to $\mathbb{P}^n$ as toric variety (Corollary 6.4). We show furthermore that, in case $c_1(M)$ is divisible by $n$, $M$ is isomorphic to a certain projective space bundle over $\mathbb{P}^1$ (Corollary 6.8). The authors are grateful to T. Oda and T. Fujita for informing them that these results can be obtained by standard arguments in algebraic geometry at least for projective toric varieties. They are also grateful to O. Fujino who communicated to them his proof of these results including the case of singular varieties [5].
The paper is organized as follows. In Section 2 we recall some basic facts about multi-fans from \([6]\). In Section 3 we define the elliptic genus and orbifold elliptic genus of a pair of a multi-fan and its generating vectors and derive the character formulae (Theorem 3.3 and 3.4). A formula for the \(T_r\)-genus is also given. In Section 4 we define equivariant first Chern class of a pair of a multi-fan and its generating vectors and discuss properties concerning its divisibility. In Section 5 the proof of the rigidity of elliptic genus of level \(N\) for the global type is given. The main results are Theorem 5.2 and Corollary 5.3. Section 6 is devoted to applications. In the last section we shall recall the fixed point formula, the formulae for elliptic genus and orbifold elliptic genus for almost complex orbifolds, and give the explicit formulae for elliptic genus and orbifold elliptic genus of torus orbifolds from which the corresponding formulae for multi-fans are deduced.

2. Multi-fans

We refer to \([6]\) for notions and notations concerning multi-fans and torus orbifolds. We shall summarize some of them in the sequel. Let \(L\) be a lattice of rank \(n\) and \(\Delta = (\Sigma, C, w^\pm)\) an \(n\)-dimensional simplicial multi-fan in \(L\) (the notation \(N\) was used in \([6]\) instead of \(L\)). Here \(\Sigma\) is an augmented simplicial set, that is, \(\Sigma\) is a simplicial set with empty set \(* = \emptyset\) added as \((-1)\)-dimensional simplex. \(\Sigma^{(k)}\) denotes the \(k\)-1 skeleton of \(\Sigma\) so that \(* \in \Sigma^{(0)}\). We assume that \(\Sigma = \sum_{k=0}^{n} \Sigma^{(k)}\), and \(\Sigma^{(n)} \neq \emptyset\). \(C\) is a map from \(\Sigma^{(k)}\) into the set of \(k\)-dimensional strongly convex rational polyhedral cones in the vector space \(L_\mathbb{R} = L \otimes \mathbb{R}\) for each \(k\) such that, if \(J\) is a face of \(I\), then \((C(J)\) is a face of \(C(I)\).

\(w^\pm\) are maps \(\Sigma^{(n)} \to \mathbb{Z}_{\geq 0}\) which, when \(\Sigma\) is complete, satisfy certain compatibility conditions, as we shall explain below. We set \(w(I) = w^+(I) - w^-(I)\). A vector \(v \in L_\mathbb{R}\) will be called generic if \(v\) does not lie on any linear subspace spanned by a cone in \(C(\Sigma)\) of dimension less than \(n\). For a generic vector \(v\) we set \(d_v = \sum_{w \in C(I)} w(I)\), where the sum is understood to be zero if there is no such \(I\). We call a multi-fan \(\Delta = (\Sigma, C, w^\pm)\) of dimension \(n\) pre-complete if the integer \(d_v\) is independent of the choice of generic vectors \(v\). We call this integer the degree of \(\Delta\) and denote it by \(\text{deg}(\Delta)\).

For each \(K \in \Sigma\) we set
\[
\Sigma_K = \{ J \in \Sigma \mid K \subset J \}.
\]
It inherits the partial ordering from \(\Sigma\) and becomes an augmented simplicial set where \(K\) is the unique minimum element in \(\Sigma_K\). Let \((L_K)_\mathbb{R}\) be the linear subspace of \(L_\mathbb{R}\) generated by \(C(K)\). Let \(L^K_\mathbb{R}\) be the quotient space of \(L_\mathbb{R}\) by \((L_K)_\mathbb{R}\) and \(L^K\) the image of \(L\) in \(L^K_\mathbb{R}\). \(L^K_\mathbb{R}\) is identified with \(L^K \otimes \mathbb{R}\). For \(J \in \Sigma_K\) we define \(C_K(J)\) to be the cone \(C(J)\) projected on \(L^K_\mathbb{R}\). We define two functions
\[
w^K_{\pm} : \Sigma^{(n-|K|)} K \subset \Sigma^{(n)} \to \mathbb{Z}_{\geq 0}
\]
to be the restrictions of \(w^\pm\) to \(\Sigma^{(n-|K|)} K\). The triple \(\Delta_K := (\Sigma_K, C_K, w^K_{\pm})\) is a multi-fan in \(L^K\) and is called the projected multi-fan with respect to \(K \in \Sigma\). If \(K = \emptyset\) then \(\Delta_K = \Delta\).

A pre-complete multi-fan \(\Delta = (\Sigma, C, w^\pm)\) is said to be complete if the projected multi-fan \(\Delta_K\) is pre-complete for any \(K \in \Sigma\). A multi-fan is complete if and only if the projected multi-fan \(\Delta_J\) is pre-complete for any \(J \in \Sigma^{(n-1)}\).

Let \(M\) be an oriented closed manifold of dimension \(2n\) with an effective action of an \(n\)-dimensional torus \(T\). We assume further that the fixed point set \(MT\) is not empty. There is a finite number of subcircles of \(T\) such that the fixed point set of each subcircle has codimension 2 components. Let \(\{M_i\}_{i=1}^r\) be those components which have non-empty intersection with \(MT\). We call \(M\) a torus manifold if a preferred orientation of
A multi-fan $\Delta(M) = (\Sigma(M), C(M), w^{\pm}(M))$ in the lattice $H_2(BT)$ is associated with $M$, where $BT$ is the classifying space of $T$ (homology is taken with coefficients in the integers, unless otherwise specified). The (augmented) simplicial set $\Sigma(M)$ is defined by

$$\Sigma(M) = \{I \subset \{1, \ldots, r\} | M_I = (\cap_{i \in I} M_i)^T \neq \emptyset\}.$$ 

We make convention that $M_I = M$ for $I = * = \emptyset$. The cones $C(M)(I)$ are defined as follows. Let $v_i$ denote the normal bundle of $M_i$ in $M$. It is an oriented 2-plane bundle with the orientation induced by those of $M_i$ and $M$, and, as such, it is regarded as a complex line bundle. If $S_i$ is the subcircle which fixes $M_i$ pointwise, then $S_i$ acts effectively on each fiber of $v_i$ as complex automorphism. Hence there is a unique isomorphism $\rho : S^1 \to S_i$ such that $\rho_i(t)$ acts as complex multiplication by $t$. Therefore $\rho_i$ defines a primitive element $v_i \in \text{Hom}(S^1, T)$. We identify $\text{Hom}(S^1, T)$ with $L = H_2(BT)$. If $I$ is in $\Sigma(n)$, then $\{v_i\}_{i \in I}$ is a basis of $L$. If $I$ is in $\Sigma$, the cone $C(M)(I)$ is defined to be the cone generated by $\{v_i\}_{i \in I}$ in $L \otimes \mathbb{R}$. The fixed point set $M^T$ coincides with the union $\cup_{i \in \Sigma(M)(n)} M_I$. For each $p \in M^T$ let $\epsilon_p = \pm 1$ be the ratio of two orientations at $p$, one induced from that of $M$ and the other determined as the intersection of oriented submanifolds $\{M_i\}_{i \in I}$. The number $w(M)^+(I)$ (respectively $w(M)^-(I)$) is defined as the number of $p \in M_I$ with $\epsilon_p = +1$ (respectively $\epsilon_p = -1$). $\Delta(M)$ is a complete multi-fan. If $K \in \Sigma(M)$ then the projected multi-fan $\Delta(M)_K$ is closely related to the multi-fan associated with $M_K = \cap_{i \in K} M_i$, where $M_K$ is regarded as a union of torus manifolds, see [3] for details.

Let $\Delta = (\Sigma, C, w^{\pm})$ be a multi-fan in $L$. If $T$ denotes the torus $L_{\mathbb{R}}/L$, then $L$ can be canonically identified with $H_2(BT)$. Then there is a unique primitive vector $v_i \in L = H_2(BT)$ which generates the cone $C(i)$ for each $i \in \Sigma^{(1)}$. $\Delta$ is called non-singular if $\{v_i \mid i \in I\}$ is a basis of the lattice $L = H_2(BT)$ for each $I \in \Sigma^{(n)}$. Thus the multi-fan associated with a torus manifold is a complete non-singular multi-fan.

It is sometimes more convenient to consider a set of vectors $\mathcal{V} = \{v_i \in L\}_{i \in \Sigma^{(1)}}$ such that each $v_i$ generates the cone $C(i)$ in $L_{\mathbb{R}}$ but is not necessarily primitive. This is the case for multi-fans associated with torus orbifolds. A torus orbifold is a closed oriented orbifold of even dimension with an effective action of a torus of half the dimension of the orbifold with some additional condition. We refer to [3] for details. A set of codimension 2 suborbifolds $M_i$ called characteristic suborbifolds is similarly defined as in the case of torus manifolds. To each subcircle $S_i$ which fixes $M_i$ pointwise there is some finite cover $S_i$ and an effective action of $S_i$ on the orbifold cover of each fiber of the normal bundle. This defines a vector $v_i$ in $\text{Hom}(S^1, T) = H_2(BT) = L$ as before. In this way a multi-fan $\Delta(M)$ and a set of vectors $\mathcal{V}(M) = \{v_i\}_{i \in \Sigma^{(1)}}$ are associated to a torus orbifold $M$.

Hereafter multi-fans are assumed to be complete and simplicial, and a set of vectors $\mathcal{V} = \{v_i \in L\}_{i \in \Sigma^{(1)}}$ as above is associated to each multi-fan $\Delta = (\Sigma, C, w^{\pm})$. In case $\Delta$ is non-singular it is further assumed that all the $v_i$ are primitive. If $I$ is in $\Sigma^{(n)}$, then $\{v_i\}_{i \in I}$ becomes a basis of vector space $L_{\mathbb{R}}$. In case $\Delta$ is non-singular it is a basis of the lattice $L$. In general, for $I \in \Sigma^{(n)}$, we define $L_{I, \mathcal{V}}$ to be the sublattice of $L$ generated by $\{v_i\}_{i \in I}$.

Let $L_{I, \mathcal{V}}^*$ be the dual lattice of $L_{I, \mathcal{V}}$ and and $\{u_i^I\}$ the basis of $L_{I, \mathcal{V}}^*$ dual to $\{v_i\}_{i \in I}$. We identify $L_{I, \mathcal{V}}^*$ with the lattice in $L_{\mathbb{R}}^*$ given by

$$\{u \in L_{\mathbb{R}}^* \mid \langle u, v \rangle \in \mathbb{Z}, \text{ for any } v \in L_{I, \mathcal{V}}\}.$$
where \( \langle u, v \rangle \) is the dual pairing. For \( h \in L/L_{1, \gamma} \) and \( u \in L^*_{1, \gamma} \) we define

\[
\chi_I(u, h) = e^{2\pi \sqrt{-1} \langle u, v(h) \rangle},
\]

where \( v(h) \in L \) is a representative of \( h \). If one fixes \( u, h \mapsto \chi_I(u, h) \) gives a character of the group \( L/L_{1, \gamma} \).

The dual lattice \( L^* = H^2(BT) \subset H^2(BT; \mathbb{R}) \) is canonically identified with \( \text{Hom}(T, S^1) \). The latter is embedded in the character ring \( R(T) \). In fact \( R(T) \) can be considered as the group ring \( \mathbb{Z}[L^*] \) of the group \( L^* = \text{Hom}(T, S^1) \). It is convenient to write the element in \( R(T) \) corresponding to \( u \in H^2(BT) \) by \( t^u \). The homomorphism \( v^*: R(T) \rightarrow R(S^1) = \mathbb{Z}[t, t^{-1}] \) induced by an element \( v \in H_2(BT) = \text{Hom}(S^1, T) \) can be written in the form

\[
v^*(t^u) = t^{(u,v)},
\]

where \( t^m \in R(S^1) \) is such that \( t^m(g) = g^m \) for \( g \in S^1 \).

More generally, set \( L_{\gamma} = \bigcap_{I \in \Sigma^{(n)}} L_{I, \gamma} \), and let \( L^*_{\gamma} \) be the dual lattice of \( L_{\gamma} \). \( L^*_{\gamma} \) contains all \( L^*_{I, \gamma} \) and is generated by all the \( u_I^* \)'s. The group ring \( \mathbb{Z}[L^*_{\gamma}] \) contains \( \mathbb{Z}[L^*] = R(T) \) and has a basis \( \{ t^u | u \in L^*_\gamma \} \) with multiplication determined by the addition in \( L^*_\gamma \):

\[
t^u t^{u'} = t^{u+u'}.
\]

If \( v \) is a vector in \( L_{\gamma} \), then \( v \) determines a homomorphism \( v^*: \mathbb{Z}[L^*_{\gamma}] \rightarrow R(S^1) = \mathbb{Z}[t, t^{-1}] \) sending \( t^u \) to \( t^{(u,v)} \). If we vary \( v \) then \( v^*(t^u) \) determines \( t^u \).

Similarly if \( v_1 \) and \( v_2 \) are vectors in \( L \), then they define a homomorphism from a 2-dimensional torus \( T^2 \) into \( T \) and induce a homomorphism \( (v_1, v_2)^*: \mathbb{Z}[L^*] \rightarrow R(T^2) = \mathbb{Z}[t_1, t_2] \) defined by

\[
(v_1, v_2)^*(t^u) = t_1^{(u,v_1)} t_2^{(u,v_2)}.
\]

Moreover if \( v_1 \) and \( v_2 \) belong to \( L_{\gamma} \), then \( (v_1, v_2)^* \) extends to a homomorphism \( \mathbb{Z}[L^*_{\gamma}] \rightarrow R(T^2) \).

We define the equivariant cohomology \( H^*_T(\Delta) \) of a complete multi-fan \( \Delta \) as the face ring of the simplicial complex \( \Sigma \). Namely let \( \{ x_i \} \) be indeterminates indexed by \( \Sigma^{(1)} \), and let \( R \) be the polynomial ring over the integers generated by \( \{ x_i \} \). We denote by \( \mathcal{I} \) the ideal in \( R \) generated by monomials \( \prod_{i \in J} x_i \) such that \( J \notin \Sigma \). \( H^*_T(\Delta) \) is by definition the quotient \( R/\mathcal{I} \). We regard \( H^2(BT) \) as a submodule of \( H^2_T(\Delta) \) by the formula

\[
(1) \quad u = \sum_{i \in \Sigma^{(1)}} \langle u, v_i \rangle x_i.
\]

This determines an \( H^*(BT) \)-module structure of \( H^*_T(\Delta) \). It should be noticed that this module structure depends on the choice of vectors \( \mathcal{V} \) as above. Let \( S \) be the subset of \( H^*(BT) \) multiplicatively generated by non-zero elements in \( H^2(BT) \). If \( M \) is a torus manifold, then \( H^*_T(\Delta(M)) \) can be embedded in \( H^*_T(M) \) divided by \( S \)-torsions and coincides with it provided some additional conditions are satisfied.

For each \( I \in \Sigma^{(n)} \) we define the restriction homomorphism \( t^*_I: H^2_T(\Delta) \rightarrow L^*_\gamma \) by

\[
t^*_I(x_i) = \begin{cases} u_i^I & \text{for} \ i \in I \\ 0 & \text{for} \ i \notin I. \end{cases}
\]

It follows from \((1)\) that \( t^*_I|H^2(BT) \) is the identity map for any \( I \), and \( \sum_{I \in \Sigma^{(n)}} t^*_I \) is injective. Note that, if \( \Delta \) is non-singular, then \( t^*_I \) maps \( H^2_T(\Delta) \) into \( H^2(BT) \).
Lemma 2.1. For any \( x = \sum_{i \in \Sigma^{(1)}} c_i x_i \in H^2_T(\Delta) \), \( c_i \in \mathbb{Z} \), the element
\[
\sum_{i \in \Sigma^{(u)}} \frac{w(I)}{L/L_{I,y}} \sum_{h \in L/L_{I,y}} \frac{\chi_I(t_i^*(x), h)t_i^*(x)}{\prod_{i \in I}(1 - \chi_I(u_i^*, h)^{-1}t_i^{-1}u_i^i)}
\]
in \( \mathbb{C}[L^*_T] \) actually belongs to \( R(T) \).

This was proved in Corollary 7.4 of [6] with a further assumption that \( t_i^*(x) \in H^2(BT) \). The general case can be proved in a similar way. The formula was also given in Corollary 12.10 of [6] when \( \Delta \) is the multi-fan associated with a torus orbifold.

We also use an extended version of Corollary 7.4 in [6]. Let \( K \in \Sigma^{(k)} \) and let \( \Delta_K = (\Sigma_K, C_K, w_K^\pm) \) be the projected multi-fan. If \( I \in \Sigma^{(l)} \) contains \( K \), then \( I \) is considered as lying in \( \Sigma^{(l-k)} \). In order to avoid some notational confusions we introduce the link \( \Sigma'_K \) of \( K \) in \( \Sigma \). It is a simplicial set consisting simplices \( J \) such that \( K \cup J \in \Sigma \) and \( K \cap J = \emptyset \). There is an isomorphism from \( \Sigma'_K \) to \( \Sigma_K \) sending \( J \in \Sigma'_K \) to \( \Delta \cup J \in \Sigma_K \). Let \( K * \Sigma'_K \) be the join of \( K \) (regarded as a simplicial set) and \( \Sigma'_K \). Its simplices are of the form \( J_1 \cup J_2 \) with \( J_1 \subset K \) and \( J_2 \in \Sigma'_K \). The torus \( T^K \) corresponding to \( \Delta_K \) is a quotient of \( T \). We consider the polynomial ring \( R_K \) generated by \( \{ x_i \mid i \in K \cup \Sigma^{(1)}_K \} \) and the ideal \( \mathcal{I}_K \) generated by monomials \( \prod_{i \in J} x_i \) such that \( J \nsubseteq K * \Sigma'_K \). We define the equivariant cohomology \( H^*_T(\Delta_K) \) of \( \Delta_K \) with respect to the torus \( T \) as the quotient ring \( R_K/\mathcal{I}_K \).

Note that \( H^*_T(\Delta_K) \) is different from \( H^*_T(\Delta_K) \).

\( H^2(BT) \) is regarded as a submodule of \( H^2_T(\Delta_K) \) by a formula similar to (1). This defines an \( H^2(BT) \)-module structure on \( H^2_T(\Delta_K) \). The projection \( H^2_T(\Delta) \rightarrow H^2_T(\Delta_K) \) is defined by sending \( x_i \) to \( x_i \) for \( i \in K \cup \Sigma^{(1)}_K \) and putting \( x_i = 0 \) for \( i \notin K \cup \Sigma^{(1)}_K \).

The restriction homomorphism \( t_i^*: H^2_T(\Delta_K) \rightarrow L^*_T \) is also defined for \( I \in \Sigma^{(n-k)}_K \) by \( t_i^*(x_i) = u_i^i \).

If \( M \) is a torus orbifold, then \( H^*_T(\Sigma(M)_K) \) is related to the equivariant cohomology \( H^*_T(M_K) \) with respect to the group \( T \) (not with respect to \( T_K \)), cf. Remark 4.2 in Section 4.

Given \( x = \sum_{i \in K \cup \Sigma^{(1)}_K} c_i x_i \in H^2_T(\Delta_K) \otimes \mathbb{R}, c_i \in \mathbb{R}, \) let \( A^* \) be the affine subspace in the dual space \( L^*_T \) defined by \( \langle u, v \rangle = c_i \) for \( i \in K \). Then we introduce a collection \( \mathcal{F}_K = \{ F_i \mid i \in \Sigma^{(1)}_K \} \) of affine hyperplanes in \( A^* \) by setting
\[
F_i = \{ u \mid u \in A^*, \langle u, v \rangle = c_i \}.
\]

The pair \( (\Delta_K, \mathcal{F}_K) \) will be called a multi-polytope associated with \( x \); see [6] for the case \( K = \emptyset \). For \( I \in \Sigma^{(n-k)}_K \) i.e. \( I \in \Sigma^{(n)}_K \) with \( I \supseteq K \), we put \( u_I = \cap_{i \in I \setminus K} F_i \in A^* \). Note that \( u_I \) is equal to \( t_i^*(x) \). The dual vector space \( (L^*_T)^* \) of \( L^*_T \) is canonically identified with the subspace \( \{ u \mid \langle u, v \rangle = 0, \ i \in K \} \) of \( L^*_T = H^2(\mathbb{C}(BT); \mathbb{R}) \). It is parallel to \( A^* \), and \( u_I \) lies in \( (L^*_T)^* \) for \( I \in \Sigma^{(n-k)}_K \) and \( i \in I \setminus K \). A vector \( v \in L^*_T \) is called generic if \( \langle u_i^i, v \rangle \neq 0 \) for any \( I \in \Sigma^{(n-k)}_K \) and \( i \in I \setminus K \). The image in \( L^*_K \) of a generic vector in \( L^*_T \) is generic. We take a generic vector \( v \in L^*_T \), and define, for \( I \in \Sigma^{(n-k)}_K \) and \( i \in I \setminus K \),
\[
(-1)^i := (-1)^{\# \{ j \in I \cap K \mid \langle u_j^j, v \rangle > 0 \}} \quad \text{and} \quad (u_i^i)^+ := \begin{cases} u_i^i & \text{if } \langle u_i^i, v \rangle > 0 \\ -u_i^i & \text{if } \langle u_i^i, v \rangle < 0 \end{cases}
\]

We denote by \( C^*_K(I)^+ \) the cone in \( A^* \) spanned by \( (u_i^i)^+ \), \( i \in I \setminus K \), with apex at \( u_I \), and by \( \phi_I \) its characteristic function. With these understood, we define a function \( DH_{\mathcal{S}_K} \).
on $A^* \setminus \cup_i F_i$ by
\[
DH_{\mathcal{P}_K} := \sum_{I \in \Sigma_k^{(n-k)}} (-1)^I w(I) \phi_I.
\]
As in [6] we call this function the Duistermaat-Heckman function associated with $\mathcal{P}_K$.

**Lemma 2.2.** The support of the function $DH_{\mathcal{P}_K}$ is bounded, and the function is independent of the choice of generic vector $v$.

The proof is similar to that of Lemma 5.4 in [3]. We shall denote by $\mathcal{P}_K+$ the multipolytope associated with $x_+ = \sum_{i \in K} c_i x_i + \sum_{i \in \Sigma_K^{(1)}} (c_i + \epsilon) x_i$ where $0 < \epsilon < 1$. The following theorem is a generalization of Corollary 7.4 in [6].

**Theorem 2.3.** Let $\Delta$ be a complete simplicial multi-fan. Let $x = \sum_{i \in K \cup \Sigma_K^{(1)}} c_i x_i \in H^2_T(\Delta_K)$ be as above with all $c_i$ integers, and let $\mathcal{P}_K+$ be defined as above. Then
\[
\sum_{u \in A^* \cap L^*} DH_{\mathcal{P}_K+}(u) t^u = \sum_{I \in \Sigma_k^{(n-k)}} \frac{w(I)}{|L/L_{I,v}|} \sum_{h \in L/L_{I,y}} \frac{\chi_I(t^I_1(x), h) t^{I_1(x)}_{1}}{\prod_{l \in I \setminus K} (1 - \chi_I(u^I_l, h)^{-1} t^{-u^I_l})}.
\]
In particular the right hand side belongs to $R(T)$.

The proof is similar to that of Corollary 7.4 in [5]. Applying $(v_1, v_2)^*$ to the both sides of the above equality we get

**Corollary 2.4.** Let $v_1$ and $v_2$ be generic vectors in $L$ such that $\langle u^I_l, v_1 \rangle$ and $\langle u^I_l, v_2 \rangle$ are integers for all $I \in \Sigma_k^{(n-k)}$ and let $x = \sum_{i \in K \cup \Sigma_K^{(1)}} c_i x_i \in H^2_T(\Delta_K)$ with all $c_i$ integers. Then
\[
\sum_{I \in \Sigma_k^{(n-k)}} \frac{w(I)}{|L/L_{I,v}|} \sum_{h \in L/L_{I,y}} \frac{\chi_I(t^I_1(x), h) t^{I_1(x), v_1}_{1} t^{I_2(x), v_2}_{2}}{\prod_{l \in I \setminus K} (1 - \chi_I(u^I_l, h)^{-1} t^{-u^I_l})} \langle -u^I_l, v_1 \rangle \langle -u^I_l, v_2 \rangle
\]
belongs to $R(T^2) = \mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}]$.

For $I \in \Sigma_k^{(n-k)}$ let $G_I$ be the subgroup of the permutation group of $I$ consisting of those elements which are identity on $K$. Let $L_I$ be the set of all linear forms $\sum_{i \in I} m_i u^I_i$ with integer coefficients $m_i$. The group $G_I$ acts on $L_I$. Let $\mathcal{O}_I$ denote the set of orbits of that action. If $I'$ is also in $\Sigma_k^{(n-k)}$, take a bijection $f : I \to I'$ which is the identity on $I \cap I'$. It induces a bijection $f_* : \mathcal{L}_I \to \mathcal{L}_{I'}$ defined by $f_*(u^I_i) = u^{I'}_{f(i)}$. This in turn induces a bijection $f_* : \mathcal{O}_I \to \mathcal{O}_{I'}$. It is easy to see that the latter $f_*$ does not depend on the choice of particular $f$. Thus we shall write it $\tau_{I,I'}$.

The following lemma will be useful later.

**Lemma 2.5.** Let $v_1$ and $v_2$ be generic vectors in $L$ as in Corollary 2.4 and let $\mathcal{A} = \{ \alpha_I \in \mathcal{O}_I \}_{I \in \Sigma_k^{(n-k)}}$ be a collection which satisfies the relations
\[
\tau_{I,I'}(\alpha_I) = \alpha_{I'} \text{ for any } I, I' \in \Sigma_k^{(n-k)}.
\]
Then the expression
\[
\sum_{I \in \Sigma_k^{(n-k)}} \frac{w(I)}{|L/L_{I,v}|} \sum_{l \in \alpha_I} \sum_{h \in L/L_{I,y}} \frac{\chi_I(l, h) t^{l, v_1}_{1} t^{l, v_2}_{2} \prod_{l \in I \setminus K} (1 - \chi_I(u^I_l, h)^{-1} t^{-u^I_l})}{\prod_{l \in I \setminus K} (1 - \chi_I(u^I_l, h)^{-1} t^{-u^I_l})}
\]
belongs to $\mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}]$. 


Proof. We define the support $supp(l)$ of a linear form $l = \sum_{i \in I} m_i u_i^I \in \mathcal{L}_I$ to be the set $\{i \in I | m_i \neq 0\}$. The cardinal number of $supp(l)$ is called the length of $l$ and is denoted by $|l|$. The length is invariant under the action of $G_I$, so that the length $|\alpha_I|$ of $\alpha_I \in \mathcal{O}_I$ is defined as that of a linear form contained in $\alpha_I$. The bijections $\tau_{I,I'}$ preserve length. The proof will proceed by induction on the common length of the $\alpha_I$ in the hypothesis of Lemma.

If $|\alpha_I| = 0$, then $\alpha_I$ consists of 0. Applying Corollary 2.4 to $x = 0$ we see that the statement of Corollary is true in this case.

Suppose that the statement is true for all $\alpha_I$ of length less than $r > 0$. If $\mathcal{L}$ denotes the set of all linear forms $\sum_{i \in K \cup \Sigma_K^{(1)}} m_i x_i$, there is a natural injection $j^I_\ell : \mathcal{L}_I \to \mathcal{L}$ sending $u_i^I$ to $x_i$. Then it is clear that $\tau^I_{I}(l) = l$ for any $l \in \mathcal{L}_I$. On the other hand, $\tau^I_{I'}(f_s(I))$ has length less than that of $l$ for any bijection $f : I \to I'$ fixing elements of $I \cap I'$ unless $supp(l) \subset I \cap I'$. In the latter case we have

$$j^I_{s'}(f_s(l)) = j^I_{s}(l).$$

Let $\mathcal{L}(\mathcal{A})$ be the totality of linear forms of the form $j^I_{s}(l)$ with $l \in \alpha_I$ and $\alpha_I \in \mathcal{A}$. Then, for each $l \in \mathcal{L}(\mathcal{A})$ and $I$, the linear form $\tau^I_{I}(l)$ either belongs to $\alpha_I$ or has length less than $r$. Moreover each $l_1 \in \alpha_I$ appears exactly once in this way. It is also easy to see that the totality of transforms of $\tau^I_{I}(l)$ with length less than $r$ by permutations of $I$ fills some orbits in $\mathcal{O}_I$ with multiplicities. If this set (with multiplicity) is denoted by $\mathcal{L}_{I}(\mathcal{A})$, then one has

$$\sum_{I \in \Sigma_K^{(n-k)}} \sum_{l \in \alpha_I} \frac{w(I)}{|L/L_{I,I',\tau}|} \sum_{h \in L/L_{I,I',\tau}} \frac{\chi_I(l,h) t_1^{(l,v_1)} t_2^{(l,v_2)}}{\prod_{i \in I \setminus K} (1 - \chi_I(u_i^I, h) - t_1^{(-u_i^I,v_1)} t_2^{(-u_i^I,v_2)})}$$

$$= \sum_{I \in \mathcal{L}(\mathcal{A})} \sum_{I \in \Sigma_K^{(n-k)}} \frac{w(I)}{|L/L_{I,I',\tau}|} \sum_{h \in L/L_{I,I',\tau}} \frac{\chi_I(\tau^I_{I}(l), h) t_1^{(\tau^I_{I}(l),v_1)} t_2^{(\tau^I_{I}(l),v_2)}}{\prod_{i \in I \setminus K} (1 - \chi_I(u_i^I, h) - t_1^{(-u_i^I,v_1)} t_2^{(-u_i^I,v_2)})}$$

$$- \sum_{I \in \Sigma_K^{(n-k)}} \sum_{I \in \mathcal{L}(\mathcal{A})} \frac{w(I)}{|L/L_{I,I',\tau}|} \sum_{h \in L/L_{I,I',\tau}} \frac{\chi_I(l,h) t_1^{(l,v_1)} t_2^{(l,v_2)}}{\prod_{i \in I \setminus K} (1 - \chi_I(u_i^I, h) - t_1^{(-u_i^I,v_1)} t_2^{(-u_i^I,v_2)})}$$

Since the first and second terms in the right hand side belong to $\mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}]$ by Corollary 2.4 and induction assumption, the left hand side does so. \qed

3. Elliptic genera of multi-fans

Let $\Delta$ be a complete simplicial multi-fan in a lattice $L$ and $\mathcal{V} = \{v_i\} \in \Sigma^{(1)}$ a set of prescribed vectors as in Section 2. We shall define the (equivariant) elliptic genus $\varphi(\Delta, \mathcal{V})$ and the (equivariant) orbifold elliptic genus $\hat{\varphi}(\Delta, \mathcal{V})$ of the pair $(\Delta, \mathcal{V})$. The definitions of $\varphi(\Delta, \mathcal{V})$ and $\hat{\varphi}(\Delta, \mathcal{V})$ are such that, if $M$ is an almost complex (or more generally stably almost complex) torus orbifold, then $\varphi(\Delta(M), \mathcal{V})$ and $\hat{\varphi}(\Delta(M), \mathcal{V})$ coincide with those of $M$ expressed by the fixed point formula. These facts will be explained in Section 7.

We first consider the function $\Phi(z, \tau)$ of $z \in \mathbb{C}$ and $\tau$ in the upper half plane $\mathcal{H}$ given by the following formula.

$$\Phi(z, \tau) = (t^\frac{1}{2} - t^{-\frac{1}{2}}) \prod_{k=1}^{\infty} \frac{(1 - t^{q^k})(1 - t^{-1}q^k)}{(1 - q^k)^2},$$
where \( t = e^{2\pi \sqrt{-1}z} \) and \( q = e^{2\pi \sqrt{-1}\tau} \). Note that \(|q| < 1\). Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \), and put \( A(z, \tau) = \left( \frac{az+b}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right) \). \( \Phi \) is a Jacobi form and satisfies the following transformation formulae, cf. [3].

\[
\Phi(A(z, \tau)) = (c\tau + d)^{-1} e^{\frac{\tau z^2}{c\tau+d}} \Phi(z, \tau),
\]

\[
\Phi(z + m\tau + n, \tau) = (-1)^{m+n} e^{-\pi \sqrt{-1} \left(2mz + m^2\tau \right)} \Phi(z, \tau)
\]

where \( m, n \in \mathbb{Z} \).

For \( \sigma \in \mathbb{C} \) we set

\[
\phi(z, \tau, \sigma) = \frac{\Phi(z + \sigma, \tau)}{\Phi(z, \tau)} = \zeta^{-\frac{1}{2}} \frac{1 - \zeta t}{1 - t} \prod_{k=1}^{\infty} \frac{(1 - \zeta t q^n)(1 - \zeta^{-1} t^{-1} q^n)}{(1 - t q^n)(1 - t^{-1} q^n)},
\]

where \( \zeta = e^{2\pi \sqrt{-1} \sigma} \). From (2) and (3) the following transformation formulae for \( \phi \) follow:

\[
\phi(A(z, \tau), \sigma) = e^{\pi \sqrt{-1} \left(2z\sigma + (c\tau+d)\sigma^2 \right)} \phi(z, \tau, (c\tau+d)\sigma),
\]

\[
\phi(z + m\tau + n, \tau, \sigma) = e^{-2\pi \sqrt{-1} m\sigma} \phi(z, \tau, \sigma) = \zeta^{-m} \phi(z, \tau, \sigma).
\]

In the sequel we fix the set \( \mathcal{V} \) and put \( H_I = L/L_I, \mathcal{V} \). Let \( v \in L, \mathcal{V} \) be a generic vector. The (equivariant) elliptic genus \( \varphi^v(\Delta, \mathcal{V}) \) along \( v \) and the (equivariant) orbifold elliptic genus \( \tilde{\varphi}^v(\Delta, \mathcal{V}) \) along \( v \) of the pair \( (\Delta, \mathcal{V}) \) are defined by

\[
\varphi^v(\Delta, \mathcal{V}) = \sum_{I \in \Sigma^{(a)}} \frac{w(I)}{|H_I|} \sum_{h \in H_I} \prod_{i \in I} \phi(\langle u_i^I, -zv - v(h) \rangle, \tau, \sigma),
\]

and

\[
\tilde{\varphi}^v(\Delta, \mathcal{V}) = \sum_{I \in \Sigma^{(a)}} \frac{w(I)}{|H_I|} \sum_{(h_1, h_2) \in H_I \times H_I} \prod_{i \in I} \zeta^{\langle u_i^I, v(h_1) \rangle} \phi(\langle u_i^I, -zv + \tau v(h_1) - v(h_2) \rangle, \tau, \sigma).
\]

where \( v(h), v(h_1), v(h_2) \in L \) are representatives of \( h, h_1, h_2 \in H_I \) respectively. The above expressions give well-defined functions independent of the choice of representatives \( v(h), v(h_1), v(h_2) \) as is easily seen from [5]. They are meromorphic functions in the variables \( z, \tau, \sigma \) and sometimes written as \( \varphi^v(\Delta, \mathcal{V}, z, \tau, \sigma) \) and \( \tilde{\varphi}^v(\Delta, \mathcal{V}, z, \tau, \sigma) \) to emphasize the variables.

For each \( K \in \Sigma^{(k)} \) with \( k > 0 \) let \( L_K \) be the kernel of the projection map \( L \rightarrow L^K \) and let \( L_K, \mathcal{V} \) be the sublattice of \( L_K \) generated by \( v_i \in K \). We set \( H_K = L_K / L_K, \mathcal{V} \). If \( J \subset K \) then we have \( L_J \cap L_K, \mathcal{V} = L_J, \mathcal{V} \), and hence \( H_J \) is canonically embedded in \( H_K \). We set

\[
\hat{H}_K = H_K \setminus \bigcup_{J \subset K} H_J.
\]

The subset \( \hat{H}_K \) is characterized by

\[
\hat{H}_K = \{ h \in H_K \mid \langle u^K_i, v(h) \rangle \notin \mathbb{Z} \quad \text{for any } i \in K \},
\]

where \( \{u^K_i\} \) is the basis of \( L^*_{K, \mathcal{V}} \) dual to the basis \( \{v_i\}_{i \in K} \) of \( L_{K, \mathcal{V}} \) and \( v(h) \in L_K \) is a representative of \( h \in H_K \). For the minimum element \( * = \emptyset \in \Sigma^{(0)} \) we set \( \hat{H}_* = H_* = 0 \).
If $K$ is contained in $I \in \Sigma^{(n)}$, then the canonical map $L_{I,Y}^* \to L_{K,Y}^*$ sends $u_i^I$ to $u_i^K$ for $i \in K$ and to 0 for $i \in I \setminus K$. Therefore, if $h$ is in $H_K$, then $\langle u_i^I, v(h) \rangle = 0$ for $i \in I \setminus K$, and $\langle u_i^I, v(h) \rangle = \langle u_i^K, v(h) \rangle$ for $i \in K$. Here $v(h) \in L_K$ is regarded as lying in $L$. This observation leads to the following expression of $\hat{\varphi}^v(\Delta, \mathcal{Y})$ which is sometimes useful.

\begin{equation}
\hat{\varphi}^v(\Delta, \mathcal{Y}) = \sum_{k=0}^n \sum_{K \in \Sigma^{(k)}, h_1 \in H_K} \zeta^{\langle u^K, v(h_1) \rangle} \sum_{I \in \Sigma^{(n-k)}} \frac{w(I)}{|H_I|} \sum_{h_2 \in H_I, i \in I \setminus K} \prod_{i \in I} \phi(-\langle u_i^I, zv + v(h_2) \rangle, \tau, \sigma) \prod_{i \in K} \phi(-\langle u_i^I, zv - \tau v(h_1) + v(h_2) \rangle, \tau, \sigma),
\end{equation}

where $u^K = \sum_{i \in K} u_i^K$.

**Note.** In the sum above with respect to $K \in \Sigma^{(k)}$ and $h_1 \in \mathcal{H}_K$, the term corresponding to $K = * \in \Sigma^{(0)}$ and $h_1 = 0 \in \mathcal{H}_* = 0$ is equal to $\varphi^v(\Delta, \mathcal{Y})$.

It is sometimes useful as well to take a representative $v(h)$ of $h \in H_I$ such that

\begin{equation}
0 \leq \langle u_i^I, v(h) \rangle < 1 \quad \text{for all } i \in I.
\end{equation}

Such a representative is unique. We denote the value $\langle u_i^I, v(h) \rangle$ by $f_{I,h,i}$ for such a representative $v(h)$. If $h$ lies in $H_K$ for $K \in \Sigma^{(k)}$ contained in $I$, then $f_{I,h,i} = 0$ for $i \notin K$, and $f_{I,h,i}$ depends only on $K$ for $i \in K$ which we shall denote by $f_{K,h,i}$. The sum $\sum_{i \in K} f_{K,h,i}$ will be denoted by $\mathcal{H}_K$. Note that (10) can be rewritten as

\[ \mathcal{H}_K = \{ h \in H_K \mid f_{K,h,i} \neq 0 \quad \text{for any } i \in K \}. \]

**Proposition 3.1.** Let $\varphi^v(\Delta, \mathcal{Y}) = \sum_{s=0}^{\infty} \varphi_s(z) q^s$ be the expansion into power series, then $\zeta^\frac{1}{2} \varphi_s(z)$ belongs to $R(S^1) \otimes \mathbb{Z}[\zeta, \zeta^{-1}]$, where $R(S^1)$ is identified with $\mathbb{Z}[t, t^{-1}]$. Let $r$ be the least common multiple of $\{|H_I|\}_{I \in \Sigma^{(n)}}$. Then $\hat{\varphi}^v(\Delta, \mathcal{Y})$ can be expanded in the form $\hat{\varphi}^v(\Delta, \mathcal{Y}) = \sum_{s=0}^{\infty} \hat{\varphi}_s(z) q^s$, where $\zeta^\frac{1}{2} \hat{\varphi}_s(z)$ belongs to $R(S^1) \otimes \mathbb{Z}[\zeta^\frac{1}{2}, \zeta^{-\frac{1}{2}}]$.

**Proof.** We introduce another variable $\tau_1$ with $\Im(\tau_1) > 0$ and put $q_1 = e^{2\pi \sqrt{-1} \tau_1}$. Then, fixing $K \in \Sigma^{(k)}$ and $h_1 \in \mathcal{H}_K$, we consider the function

\[ \hat{\varphi}_{K,h_1}(z, \tau_1, \tau, \sigma) = \sum_{I \in \Sigma^{(n-k)}} \frac{w(I)}{|H_I|} \sum_{h_2 \in H_I, i \in I \setminus K} \prod_{i \in I} \phi(-\langle u_i^I, zv + v(h_2) \rangle, \tau, \sigma) \prod_{i \in K} \phi(-\langle u_i^I, zv - \tau_1 v(h_1) + v(h_2) \rangle, \tau, \sigma). \]

Note that

\[ \zeta^\frac{1}{2} \phi(-\langle u_i^I, zv - \tau v(h_1) + v(h_2) \rangle, \tau, \sigma) = \frac{1 - \zeta_{i,h_2}^{-1} \ell_{i,h_2}^{-1} q^k}{1 - \zeta_{i,h_2}^{-1} q^k} \prod_{k=1}^{\infty} \frac{1 - \zeta_{i,h_2}^{-1} q^k (1 - \zeta_{i,h_2}^{-1} q^k)}{1 - \zeta_{i,h_2}^{-1} q^{k+1}}, \]

where

\[ \zeta_{i,h_2}^{-1} = e^{2\pi \sqrt{-1} \langle u_i^I, zv - \tau_1 v(h_1) + v(h_2) \rangle} = \chi_I(u_i^I, h_2) q_1^{-\langle u_i^I, v(h_1) \rangle}. \]
Since \( \langle u_i^t, v(h_1) \rangle = 0 \) for \( i \in I \setminus K \), we have \( \xi_{i,h_2}^t = \chi_I(u_i^t, h_2) t(u_i^t, v) \) for \( i \in I \setminus K \). We expand \( \hat{\varphi}_{K,h_1}(z, \tau_1, \tau, \sigma) \) with respect to \( \zeta \) and \( q = e^{2\pi \sqrt{-1}r} \) in the following form
\[
\hat{\varphi}_{K,h_1}(z, \tau_1, \tau, \sigma) = \sum_{s_1 \in \mathbb{Z}, s_2 \in \mathbb{Z}_{\geq 0}} \hat{\varphi}_{s_1, s_2}(z, q_1) \zeta^{s_1} q^{s_2}.
\]

Then we see that there is a family \( \{ \alpha_{I,j} \}^r_{j=1} \) for each \( I \) such that \( \tau_{I,i'}(\alpha_{I,j}) = \alpha_{I',j} \) for \( j = 1, \ldots, r \) and
\[
\zeta^{\frac{r}{2}} \hat{\varphi}_{s_1, s_2}(z, q_1) = \sum_{j=1}^{r} \sum_{l \in \bar{K}} \frac{w(l) \chi_I(l, h_2) t(l, v) q_{1}^{-(l, v(h_1))}}{\prod_{i \in I}(1 - \xi_{i,h_2}^t)}.
\]

We further expand each factor \( \frac{1}{1 - \xi_{i,h_2}^t} \) in formal power series in the above expression. Then
\[
\prod_{i \in K}(1 - \xi_{i,h_2}^t)^{-1}
\]

is expanded in an (infinite) sum of expressions of the form
\[
\sum_{l \in \beta I} \chi_I(l, h_2) t(l, v) q_{1}^{-(l, v(h_1))}, \text{ where } \beta I \text{ is an element of } \mathcal{O}_I \text{ and it satisfies } \tau_{I,i'}(\beta_I) = \beta_{I'}.
\]

Hence \( \zeta^{\frac{r}{2}} \hat{\varphi}_{s_1, s_2}(z, q_1) \) is written in a sum of expressions of the following form
\[
\sum_{l \in \bar{K}} \frac{w(l)}{|H_I|} \sum_{l_1 \in \alpha_{I,j}} \sum_{l_2 \in h_2} \prod_{i \in I}(1 - \chi_I(u_i^t, h_2)^{-1} t^{-(l_1 + l_2, v(h_1))}).
\]

Since \( \langle u_i^t, v(h_1) \rangle = 0 \) for \( i \in I \setminus K \), these expressions [11] belong to \( \mathbb{Z}[t, t^{-1}, q_1, q_1^{-1}] \) by Lemma 2.6. Hence \( \zeta^{\frac{r}{2}} \hat{\varphi}_{s_1, s_2}(z, q_1) \) also belongs to \( \mathbb{Z}[t, t^{-1}, q_1, q_1^{-1}] \). Therefore, if we specialize \( \tau_1 \) to \( \tau \) (hence \( q_1 \) to \( q \)) in \( \hat{\varphi}_{K,h_1}(z, \tau, \tau, \sigma) = \sum_{s_1 \in \mathbb{Z}, s_2 \in \mathbb{Z}_{\geq 0}} \hat{\varphi}_{s_1, s_2}(z, q_1) \zeta^{s_1} q^{s_2}, \) we get an expansion
\[
\hat{\varphi}_{K,h_1}(z, \tau, \tau, \sigma) = \sum_{s \in \mathbb{Z}} \hat{\varphi}_{K,h_1,s}(z) q^s
\]

with \( \zeta \frac{r}{2} \hat{\varphi}_{K,h_1,s}(z) \in \mathbb{Z}[t, t^{-1}, \zeta, \zeta^{-1}] \). Then
\[
\varphi^*(\Delta, \mathcal{V}) = \sum_{K \in \Sigma, h_1 \in \mathcal{H}_K} \zeta^{(u^K, v(h_1))} \hat{\varphi}_{K,h_1}(z, \tau, \tau, \sigma) = \sum_{s \in \mathbb{Z}} \zeta^{(u^K, v(h_1))} \hat{\varphi}_{K,h_1,s}(z) q^s
\]

with \( \zeta \frac{r}{2} \zeta^{(u^K, v(h_1))} \hat{\varphi}_{K,h_1,s}(z) \in \mathbb{Z}[t, t^{-1}, \zeta, \zeta^{-1}] \). But \( \varphi^*(\Delta, \mathcal{V}) \) does not have negative powers of \( q \) in its expansion as can be seen by taking a representative \( v(h_1) \) of \( h_1 \in \mathcal{H}_I \) which satisfies [10] for each \( I \) in the expression [7] or [9]. It follows that \( \varphi^*(\Delta, \mathcal{V}) \zeta \frac{r}{2} \) belongs to \( (\mathbb{Z}[t, t^{-1}, \zeta, \zeta^{-1}])([q]) \). This finishes the proof of Proposition 3.1 for \( \hat{\varphi}^*(\Delta, \mathcal{V}) \zeta \frac{r}{2} \).

A similar easier argument shows that \( \varphi^*(\Delta, \mathcal{V}) \zeta \frac{r}{2} \) belongs to \( (R(S^1) \otimes \mathbb{Z}[\zeta, \zeta^{-1}])([q]) \). \( \square \)

The equivariant elliptic genus \( \varphi(\Delta, \mathcal{V}) = \varphi(\Delta, \mathcal{V}; \tau, \sigma) \in (R(T) \otimes \mathbb{Z}[\zeta, \zeta^{-1}])([q]) \) and the equivariant orbifold elliptic genus \( \hat{\varphi}^*(\Delta, \mathcal{V}) = \hat{\varphi}(\Delta, \mathcal{V}; \tau, \sigma) \in (R(T) \otimes \mathbb{Z}[\zeta, \zeta^{-1}])([q]) \) are defined by
\[
v^*(\varphi(\Delta, \mathcal{V})) = \varphi^*(\Delta, \mathcal{V}) \quad \text{and} \quad v^*(\hat{\varphi}(\Delta, \mathcal{V})) = \hat{\varphi}^*(\Delta, \mathcal{V})
\]
where one varies generic vectors \( v \) in \( L_{\mathcal{V}} \).
Remark 3.2. Another but equivalent definition of $\varphi(\Delta, \mathbf{y})$ is the following. Put $L_\mathbb{C} = L \otimes \mathbb{C}$ and let $T_\mathbb{C} = L_\mathbb{C}/L$ be the corresponding complex torus. The pairing $L^* \times L$ extends to a pairing $L^* \times L_\mathbb{C}$.

Then

$$\varphi(\Delta, \mathbf{y}) = \sum_{I \in \Sigma^{(n)}} \frac{w(I)}{|H_I|} \sum_{h \in H_I} \prod_{i \in I} \phi(\langle u_i^I, -w - v(h) \rangle, \tau, \sigma),$$

where $w \in L_\mathbb{C}$, $\varphi(\Delta, \mathbf{y})$ is a meromorphic function in $(w, \tau, \sigma) \in L_\mathbb{C} \times \mathcal{H} \times \mathbb{C}$. Similarly $\hat{\varphi}(\Delta, \mathbf{y})$ may be defined by

$$\hat{\varphi}(\Delta, \mathbf{y}) = \sum_{I \in \Sigma^{(n)}} \frac{w(I)}{|H_I|} \sum_{(h_1, h_2) \in H_I \times H_I} \prod_{i \in I} \zeta^{u_i^I, v(h_1)} \phi(\langle u_i^I, -w + \tau v(h_1) - v(h_2) \rangle, \tau, \sigma).$$

The formulae in the following theorems express explicitly $\varphi(\Delta, \mathbf{y})$ and $\hat{\varphi}(\Delta, \mathbf{y})$ as virtual characters of the torus $T$. When $\Delta$ is the fan of a complete non-singular toric variety, the formula in Theorem 3.3 is due to Borisov and Libgober [1].

**Theorem 3.3.** Let $\Delta = (\Sigma, C, w^\pm)$ be an $n$-dimensional complete simplicial multi-fan in a lattice $L$ and let $\mathbf{y}$ be as above. Then we have the equality

$$\varphi(\Delta, \mathbf{y}) = \sum_{u \in L^*} t^{-u} \left( \sum_{k=0}^{n} \sum_{J \in \Sigma^{(k)}} (-1)^k \deg(\Delta_J) \prod_{j \in J} \frac{1}{1 - \zeta(u, v_j)} \right) \Phi(\sigma, \tau)^n,$$

where $\deg(\Delta_J)$ is the degree of the projected multi-fan $\Delta_J$ of $J$ as defined in Section 2.

**Note.** The above equality is an equality of meromorphic functions as can be seen from the following proof. The right hand side is convergent to a meromorphic function in the domain

$$\Im(\tau) > |\Im(\langle u_i^I, w \rangle)| > 0$$

for all $i \in I$ and $I \in \Sigma^{(n)}$.

Notice that $t^u$ is regarded as a function on $T_\mathbb{C}$ defined by $t^u(\exp(w)) = e^{2\pi \sqrt{-1} \langle u, w \rangle}$, where the projection $L_\mathbb{C} \to T_\mathbb{C}$ is denoted by $\exp$ as usual.

**Example.** The equivariant elliptic genus of the complex projective space $\mathbb{P}^n$ is given by the following formula. The action of $T = T^n$ is the standard one given by

$$(g_1, \ldots, g_n)[z_0, z_1, \ldots, z_n] = [z_0, g_1 z_1, \ldots, g_n z_n].$$

The lattice $L$ is identified with $\mathbb{Z}^n$ and $\mathbf{y} = \{v_i\}$ is given by

$$v_i = e_i \text{ (standard unit vector)} \text{ for } i = 1, \ldots, n \text{ and } v_{n+1} = -(e_1 + \cdots + e_n).$$

Thus

$$\varphi(\mathbb{P}^n) = \left( \sum_{u=(m_1, \ldots, m_n) \in \mathbb{Z}^n} t^{-u} \frac{1 - \zeta^{n+1}}{(1 - \zeta q^{m_1}) \cdots (1 - \zeta q^{m_n})(1 - \zeta^{-m_1 \cdots - m_n})} \right) (-\Phi(\sigma, \tau))^n,$$

where $L^* = H^2(BT)$ is identified with $\mathbb{Z}^n$. 
**Theorem 3.4.** Let \((\Delta, \mathcal{V})\) be as in Theorem 3.3. Then we have the equality
\[
\hat{\phi}(\Delta, \mathcal{V}) = \sum_{u \in L^*} t^{-u} \left( \sum_{k=0}^{n} \sum_{j \in J(k), h \in H_j} (-1)^k \text{deg}(\Delta_j) \zeta_{J,h}^{f_j} q^{(u,v_{J,h})} \prod_{i \in J} \frac{1}{1 - \zeta q^{(u,v_i)}} \right) \Phi(\sigma, \tau)^n,
\]
where \(v_{J,h} = \sum_{i \in J} f_{J,h,i} v_i\).

**Note.** It can be shown that \(v_{J,h}\) belongs to \(L_J\) and hence \(\langle u, v_{J,h}\rangle \in \mathbb{Z}\). This fact shows that \(\hat{\phi}(\Delta, \mathcal{V})\) belongs to \((R(T) \otimes \mathbb{Z}[\zeta^\frac{1}{b}, \zeta^{-\frac{1}{b}}])\mathbb{Z}[q]\). This fact was already proved in Proposition 3.1. We also see that
\[
\zeta_{J,h}^{f_j} q^{(u,v_{J,h})} \prod_{i \in J} \frac{1}{1 - \zeta q^{(u,v_i)}} = \prod_{i \in J} (\zeta q^{(u,v_i)})^{f_{J,h,i}}.
\]

**Example.** Let \(b > 1\) be an integer. Let \(H = \{g \in S^1|g^b = 1\}\) act on \(\mathbb{P}^2\) by \(g[z_0, z_1, z_2] = [z_0, g_1 z_1, g_2 z_2]\) and put \(M = \mathbb{P}^2/H\). The action of \(T = T^2\) on \(M\) is induced from the action on \(\mathbb{P}^2\) given by
\[
(g_1, g_2)[z_0, z_1, z_2] = [z_0, g_1 z_1, g_2 z_2].
\]
The lattice \(L\) is identified with \(\mathbb{Z}^2\) and \(\mathcal{V} = \{v_1\}\) is given by
\[
v_1 = e_1, v_2 = be_2, v_3 = -(e_1 + be_2).
\]
Then a calculation using Theorem 3.4 yields
\[
\hat{\phi}(M) = \sum_{(m_1, m_2) \in \mathbb{Z}^2} a_{m_1, m_2} t^{m_1} t^{m_2}
\]
with
\[
a_{m_1, m_2} = \frac{(1 - \zeta^{3/b})(1 - \zeta^2 q^{-bm_2}) \Phi(\sigma, \tau)^2}{(1 - \zeta^{m_1})(1 - \zeta q^{-m_1 - bm_2})(1 - \zeta^{1/b} q^{m_2})(1 - \zeta^{2/b} q^{-m_2})}.
\]

For the proof of Theorems 3.3 and 3.4 we need lemmas below.

**Lemma 3.5.** Suppose that \(|q| < |t| < 1\). Then we have the equality
\[
(12) \quad \phi(z, \tau, \sigma) = -\Phi(\sigma, \tau) \sum_{m \in \mathbb{Z}} t_1^m \frac{1}{1 - \zeta q^m}.
\]

**Lemma 3.6.** Put \(\alpha = e^{2\pi \sqrt{-1}w}\). Suppose that \(|\alpha| = 1\) and \(|q| < |t| < 1\). If \(l \neq 0\) is an integer, then we have the equality
\[
\phi(lz + w, \tau, \sigma) = \begin{cases} 
-\Phi(\sigma, \tau) \sum_{m \in \mathbb{Z}} a^m t_1^l m_1 \frac{1}{1 - \zeta q^m} & \text{if } l > 0, \\
-\Phi(\sigma, \tau) \sum_{m \in \mathbb{Z}} a^m t_1^l m_1 \left(\frac{1}{1 - \zeta q^m} - 1\right) & \text{if } l < 0.
\end{cases}
\]

**Proof.** First we prove (12). The argument follows that of [1]. We repeat their argument for the sake of completeness. Recall that \(\zeta = e^{2\pi \sqrt{-1}\sigma}\). We regard the both sides of (12) as meromorphic functions of \(\sigma\) and consider the quotient
\[
f(\sigma) = \frac{\phi(z, \tau, \sigma)}{-\Phi(\sigma, \tau) \sum_{m \in \mathbb{Z}} t_1^m \frac{1}{1 - \zeta q^m} \Phi(\sigma, \tau)^n}.
\]
defined for $|q| < |t| < 1$. It is doubly periodic in $\sigma$ with respect to the lattice $\mathbb{Z}\tau + \mathbb{Z}$ and has no poles. Its value at $\sigma = 0$ is 1. This implies the equality (12).

We now prove Lemma 3.6. So suppose $|\alpha| = 1$ and $|q| < |t| < 1$. If $l > 0$ then $|q| < |\alpha t^l| < 1$ and the equality in Lemma 3.6 is (12) itself since $e^{2\pi\sqrt{-1}(lz+w)} = \alpha t^l$. Next we consider the case $l < 0$. Since $|\alpha t^l| > 1$, Lemma 3.6 can not be applied. So we proceed in the following way. First observe that

$$
\phi(-z, \tau, \sigma) = \phi(z, \tau, -\sigma) \quad \text{and} \quad \Phi(\sigma, \tau) = -\Phi(-\sigma, \tau).
$$

Hence, if we put $l' = -l$, we see easily that

$$
\phi(lz + w, \tau, \sigma)/\Phi(\sigma, \tau) = -\phi(l'z - w, \tau, -\sigma)/\Phi(-\sigma, \tau).
$$

We then apply Lemma 3.5 to the right hand side and get

$$
\phi(lz + w, \tau, \sigma)/\Phi(\sigma, \tau) = \sum_{m \in \mathbb{Z}} \alpha^{-mt^lm} \frac{1}{1 - \zeta^{-1}q^m} = -\sum_{m \in \mathbb{Z}} \alpha^{mt^lm} \left( \frac{1}{1 - \zeta q^m} - 1 \right).
$$

This proves Lemma 3.6.

We now proceed to the proof of Theorem 3.3. Take a generic vector $v \in L_\alpha$. Then $\langle u^I_i, v \rangle$ is an integer for any $I \in \Sigma^{(n)}$ and $i \in I$. For $I \in \Sigma^{(n)}$ we put $I(v) = \{ i \in I | \langle u^I_i, v \rangle < 0 \}$. Suppose that $|q| < |\langle u^I_i, v \rangle| < 1$ for all $I \in \Sigma^{(n)}$ and $i \in I$. Using Lemma 3.6 we have

$$
\frac{1}{(-\Phi(\sigma, \tau))^n} \prod_{i \in I} \phi(-\langle u^I_i, zv + v(h) \rangle, \tau, \sigma)
$$

$$
= \prod_{i \in I(v)} \left( \sum_{m_i \in \mathbb{Z}} \chi_I(u^I_i, h)^{-m_i} t^{-m_i} \langle u^I_i, v \rangle \frac{1}{1 - \zeta q^{m_i}} \prod_{i \in I(v)} \left( \sum_{m_i \in \mathbb{Z}} \chi_I(u^I_i, h)^{-m_i} t^{-m_i} \langle u^I_i, v \rangle \left( \frac{1}{1 - \zeta q^{m_i}} - 1 \right) \right) \right)
$$

$$
= \sum_{m_i \in \mathbb{Z}, i \in I} \left( \prod_{i \in I} \chi_I(u^I_i, h)^{-m_i} \prod_{i \in I(v)} t^{-m_i} \langle u^I_i, v \rangle \prod_{i \in I(v)} \frac{1}{1 - \zeta q^{m_i}} \prod_{i \in I(v)} \left( \frac{1}{1 - \zeta q^{m_i}} - 1 \right) \right)
$$

If we put $u = \sum_{i \in I} m_i u^I_i \in L^*_I$, then $m_i = \langle u, v_i \rangle$. Therefore $\prod_{i \in I} t^{-m_i} \langle u^I_i, v \rangle = t^{-\langle u, v \rangle}$. Since $\chi_I(u, ) = e^{2\pi\sqrt{-1}\langle u, \rangle}$ we see that $\prod_{i \in I} \chi_I(u^I_i, h)^{-m_i} = \chi_I(u, h)^{-1}$. Since the 1-dimensional representation of $H_I = L/L_\alpha$ is trivial if and only if $u \in L^*$, it follows that

$$
\sum_{h \in H_I} \chi_I(u, h)^{-1} = \begin{cases} |H_I| & \text{if } u \in L^*, \\ 0 & \text{if } u \not\in L^*. \end{cases}
$$

Furthermore we see easily that

$$
\prod_{i \in I(v)} \frac{1}{1 - \zeta q^{m_i}} \prod_{i \in I(v)} \left( \frac{1}{1 - \zeta q^{m_i}} - 1 \right) = \sum_{k=0}^n \sum_{J \in \Sigma^{(n)}: I(v) \subset J \subset I} (-1)^{n-k} \prod_{j \in J} \frac{1}{1 - \zeta q^{m_j}}.
$$
Lemma 3.7. Let \( I \in \Sigma^{(n)} \). Combining these we have

\[
\sum_{h \in H_1} \frac{1}{\Phi(\sigma, \tau)^n} \prod_{i \in I} \phi(-\langle u^I_i, zv + v(h) \rangle, \tau, \sigma) = |H_1| \sum_{u \in L^*} t^{-\langle u,v \rangle} \left( \sum_{k=0}^{n} \sum_{J \subseteq I} (-1)^k \prod_{j \in J} \frac{1}{1 - \zeta q^{(u,j)}} \right).
\]

Finally, applying this to (6) we have

\[
\frac{1}{\Phi(\sigma, \tau)^n} \varphi^* (\varphi(\Delta, \mathcal{V})) = \frac{1}{\Phi(\sigma, \tau)^n} \varphi^* (\Delta, \mathcal{V}) = \sum_{u \in L^*} t^{-\langle u,v \rangle} \left( \sum_{i \in \Sigma^{(n)}} \prod_{j \in J} w(I) \left( \sum_{k=0}^{n} \sum_{J \subseteq I} (-1)^k \prod_{j \in J} \frac{1}{1 - \zeta q^{(u,j)}} \right) \right).
\]

\[
= \sum_{u \in L^*} t^{-\langle u,v \rangle} \left( \sum_{k=0}^{n} \sum_{J \subseteq I} (-1)^k \deg(\Delta_J) \prod_{j \in J} \frac{1}{1 - \zeta q^{(u,j)}} \right).
\]

since \( \sum_{J \in \Sigma^{(n)}: J \subseteq J \cap I} w(I) = \deg(\Delta_J) \) by definition. Since \( \varphi(\Delta, \mathcal{V}) \) belongs to \( (R(T) \otimes \mathbb{Z}[\zeta, \zeta^{-1}])[q] \) and the last equality holds for any generic vector \( v \), Theorem 3.3 follows.

We next prove Theorem 3.3. We use the following

**Lemma 3.7.** Let \( f \) be a real number with \( 0 < f < 1 \) and \( l \neq 0 \) an integer. If

\[
|q^f|, |q^{1-f}| < |t^{|l|}|, |t| \leq 1
\]

then

\[
\phi(lz + f \tau + w, \tau, \sigma) = -\Phi(\sigma, \tau) \sum_{m \in \mathbb{Z}} \alpha^m t^{|m|} \frac{q^{fm}}{1 - \zeta q^m},
\]

where \( \alpha = e^{2\pi \sqrt{-1}w} \), \( |\alpha| = 1 \) as before.

**Proof.** In view of Lemma 3.5 it is enough to show that

\[
|q| < |q^f t^{|l|}| < 1
\]

regardless of the sign of \( l \).

Suppose that \( l > 0 \). Then \( |q^{1-f}| < |t^{|l|}| \) implies \( |q| < |q^f t^{|l|}| \). Since \( |q^f| < 1 \) and \( |t^{|l|}| \leq 1 \), we have \( |q^f t^{|l|}| < 1 \).

Suppose that \( l < 0 \). Then \( |q^f| < |t^{|-l|}| \) implies \( |q^f t^{|l|}| < 1 \). Also \( |q^{1-f}| < |t^{-|l|}| \) implies \( |q| < |q^f t^{-|l|}| \). But \( |q^f t^{-|l|}| \leq |q^f t^{|l|}| \) since \( l < 0 \) and \( |t| \leq 1 \). Hence \( |q| < |q^f t^{|l|}| \). \( \square \)

We now proceed to the proof of Theorem 3.3. Take a generic vector \( v \in L_f \). Fix \( i \in K \) and take the representative \( v(h_1) \) of \( h_1 \in H_K \) such that

\[
\langle u^I_i, v(h_1) \rangle = \langle u^K_i, v(h_1) \rangle = f_{K, h_1,i}.
\]
If $t \in \mathbb{C}$ satisfies

$$|q^{f_{K, l}}|, |q^{1-f_{K, l}}| < |t|^{(u^l_{J, v})}|, \quad |t| < 1,$$

then by Lemma 3.7, we have

$$\frac{1}{-\Phi(\sigma, \tau)} \phi(-\langle u^l_{i}, zv - \tau v(h_1) + v(h_2) \rangle, \tau, \sigma)
= \sum_{m_i \in \mathbb{Z}} \chi(u^l_{i}, v(h_2))^{-m_i q^{(f_{K, l})} t^{-m_i(u^l_{i}, v)} - 1} \frac{1}{1 - q^{m_i}}.$$

Next fix $i \in I \setminus K$. Then, by Lemma 3.6 we have

$$\frac{1}{-\Phi(\sigma, \tau)} \phi(-\langle u^l_{i}, zv + v(h_2) \rangle, \tau, \sigma)
= \begin{cases}
\sum_{m_i \in \mathbb{Z}} \chi(u^l_{i}, v) - m_i t^{-m_i(u^l_{i}, v)} \frac{1}{1 - q^{m_i}} & \text{for } i \in I(v) \setminus K,
\sum_{m_i \in \mathbb{Z}} \chi(u^l_{i}, v) - m_i t^{-m_i(u^l_{i}, v)} \left( \frac{1}{1 - q^{m_i}} - 1 \right) & \text{for } i \in I \setminus (I(v) \cup K).
\end{cases}$$

Now suppose that $|t| < 1$ and $q$ satisfy

$$|q^{f_{K, l}}|, |q^{1-f_{K, l}}| < |t|^{(u^l_{J, v})}|$$

for all $i \in K$, $h_1 \in \hat{H}_K$, $K \in \Sigma^{(k)}$, and

$$|q| < |t|^{(u^l_{J, v})}|$$

for all $i \in I \setminus K, I \in \Sigma^{(n-k)}_K$ and $K \in \Sigma^{(k)}$. Then we obtain

$$\frac{1}{(-\Phi(\sigma, \tau))^n} \prod_{i \in I \setminus K} \phi(-\langle u^l_{i}, zv + v(h_2) \rangle, \tau, \sigma) \prod_{i \in K} \phi(-\langle u^l_{i}, zv - \tau v(h_1) + v(h_2) \rangle, \tau, \sigma)
= \prod_{i \in I(v) \setminus K} \sum_{m_i \in \mathbb{Z}} \chi(u^l_{i}, h_2) - m_i t^{-m_i(u^l_{i}, v)} \frac{1}{1 - q^{m_i}}
\prod_{i \in I \setminus (I(v) \cup K)} \sum_{m_i \in \mathbb{Z}} \chi(u^l_{i}, h_2) - m_i t^{-m_i(u^l_{i}, v)} \left( \frac{1}{1 - q^{m_i}} - 1 \right).
\prod_{i \in K} \sum_{m_i \in \mathbb{Z}} \chi(u^l_{i}, h_2) - m_i q^{(f_{K, l})} t^{-m_i(u^l_{i}, v)} \frac{1}{1 - q^{m_i}}.$$  

Using the above equality and (10), and arguing as in the proof of Theorem 3.3 we have

$$\hat{\phi}(\Delta, \mathcal{V})/\Phi(\sigma, \tau)^n
= \sum_{u \in L^+} t^{-u} \left( \sum_{k=0}^n \sum_{K \in \Sigma^{(k)}, h \in \hat{H}_K} \zeta^{f_{K, h}} q^{(u, v_k, h)} \sum_{K \subseteq J} (-1)^k \deg(\Delta_J) \prod_{j \in J} \frac{1}{1 - q^{(u, v_j)}} \right).$$

Fix $J \in \Sigma$. It is easy to see that the union of $\{ \hat{H}_K \mid K \in \Sigma, K \subseteq J \}$ is disjoint. Since any $h \in H_J$ is contained in $\hat{H}_{K_h}$ by (8) where $K_h = \{ j \in J \mid f_{J, h, j} \neq 0 \}$, we have $H_J = \cup \hat{H}_K$. Moreover we have

$$f_{J, h} = f_{K_h, h} \quad \text{and} \quad v_{J, h} = v_{K_h, h}.$$
Taking these facts in account the right hand side of the equality (13) is transformed into

$$
\sum_{u \in L^*} t^{-u} \left( \sum_{k=0}^{n} \sum_{J \in \Sigma(k), h \in H_J} (-1)^k \deg(\Delta_J) \zeta_{J,h} q^{(u,v,h)} \prod_{i \in J} \frac{1}{1 - \zeta q^{(u,v)}} \right).
$$

This proves Theorem 3.4.

The elliptic genus \( \varphi(\Delta, \mathcal{V}) \) reduces to the so-called \( T_y \)-genus for \( q = 0 \) if it is multiplied by \( \zeta^{n/2} \) and if \( \zeta \) is substituted by \(-y\). Namely

$$
T_y(\Delta, \mathcal{V}) = \sum_{I \in \Sigma(n)} \frac{w(I)}{|H_I|} \sum_{h \in H_I} \prod_{i \in I} \frac{1 + y \chi_i(u_i, h) - t^{-\langle u_i, v \rangle}}{1 - \chi_i(u_i, h) - t^{-\langle u_i, v \rangle}}.
$$

In [6] it was shown that the equivariant \( T_y \)-genus of a torus manifold was rigid. The same proof is valid for general complete simplicial multi-fans. For the sake of completeness we review the argument briefly. Let \( v \) be a generic vector. We set

$$
\mu(I) = \# \{ i \in I \mid \langle u_i, v \rangle > 0, \}
$$

for \( I \in \Sigma(n) \) and define

$$
h_k(\Delta) := \sum_{I \in \Sigma(n), \mu(I) = k} w(I),
$$

for each integer \( k \) with \( 0 \leq k \leq n \).

We consider \( v^* (T_y(\Delta, \mathcal{V})) \). It is written in the form \( \sum_{I \in \Sigma(n)} R_I(t) \), where

$$
R_I(t) = \frac{w(I)}{|H_I|} \sum_{h \in H_I} \prod_{i \in I} \frac{1 + y \chi_i(u_i, h) - t^{-\langle u_i, v \rangle}}{1 - \chi_i(u_i, h) - t^{-\langle u_i, v \rangle}}.
$$

Regarded as a rational function of \( t \), \( R_I(t) \) takes value

$$
w(I)(-y)^{\mu(I)},
$$

at \( t = 0 \) and similarly

$$
w(I)(-y)^{n - \mu(I)}
$$

at \( t = \infty \). Hence \( v^* (T_y(\Delta, \mathcal{V}))(t) \) takes finite values at \( t = 0 \) and \( t = \infty \). On the other hand it is a Laurent polynomial in \( t \). Hence it must be a constant. Since this is true for any generic vector \( v \), \( T_y(\Delta, \mathcal{V}) \) is a constant. Moreover that constant is equal either to

$$
\sum_{I \in \Sigma(n)} w(I)(-y)^{\mu(I)} = \sum_{k=0}^{n} h_k(\Delta)(-y)^k
$$

or to

$$
\sum_{I \in \Sigma(n)} w(I)(-y)^{n - \mu(I)} = \sum_{k=0}^{n} h_{n-k}(\Delta)(-y)^k.
$$

It follows that \( h_k(\Delta) \) does not depend on \( v \) and that \( h_k(\Delta) = h_{n-k}(\Delta) \). Moreover we see that \( T_y(\Delta, \mathcal{V}) \) is independent of \( \mathcal{V} \), since \( h_k(\Delta) \) depends only on \( \Delta \). So we call it the \( T_y \)-genus of \( \Delta \) and simply write \( T_y(\Delta) \). Thus we have proved

**Proposition 3.8.**

$$
(14) \quad T_y(\Delta) = \sum_{k=0}^{n} h_k(\Delta)(-y)^k.
$$

Here the equality \( h_{n-k}(\Delta) = h_k(\Delta) \) holds.
The class $\sum$ equivariant first Chern class is denoted by $c_1$. Hence $T_0[\Delta]$ equals $\deg(\Delta)$, cf. \cite{6}.

The following Proposition was proved in \cite{6}. We shall give a different proof using Theorem 3.3.

Proposition 3.9.

\begin{equation}
T_0[\Delta] = \sum_{k=0}^{n} e_k(\Delta)(-1 - y)^{n-k}
\end{equation}

where $e_k(\Delta) = \sum_{J \in \Sigma^{(k)}} \deg(\Delta_J)$.

Proof. We look at the coefficient of $t^u$ for $u = 0$ and with $q = 0$ in Theorem 3.3 which is equal to $T_0[\Delta] z^{-\frac{1}{2}}$. Noting that $\Phi(\sigma, \tau)$ is approaching to $(-1 - y) z^{-\frac{1}{2}}$ when $\Re \tau$ approaches to $\infty$, we obtain (15). \hfill $\square$

4. Equivariant first Chern class

Let $\Delta$ be a complete simplicial multi-fan in a lattice $L$ and $\mathcal{V} = \{v_i\}_{i \in \Sigma^{(1)}}$ a set of prescribed edge vectors as before. An $H^*(BT)$-module structure of $H_T^*(\Delta)$ is defined by $\Phi$. The class

$$
\sum_{i \in \Sigma^{(1)}} x_i \in H_T^2(\Delta)
$$

will be called the equivariant first Chern class of the pair $(\Delta, \mathcal{V})$, and will be denoted by $c_T^1(\Delta, \mathcal{V})$. When $\Delta$ is non-singular, $\mathcal{V}$ consists of primitive vectors which is determined by $\Delta$ by our convention. In this case we simply write $c_T^1(\Delta)$ and call it the equivariant first Chern class of the non-singular multi-fan $\Delta$.

The image of $c_T^1(\Delta, \mathcal{V})$ in $H_T^2(\Delta)/H^2(BT)$ is called the first Chern class of $(\Delta, \mathcal{V})$ and is denoted by $c_1(\Delta, \mathcal{V})$. Let $N > 1$ be an integer. The first Chern class $c_1(\Delta, \mathcal{V})$ is divisible by $N$ if and only if $c_T^1(\Delta, \mathcal{V})$ is of the form

$$
c_1^T(\Delta, \mathcal{V}) = N x + u, \ x \in H_T^2(\Delta), \ u \in H^2(BT).
$$

We set $u^I = \iota_1^*(c_1^T(\Delta, \mathcal{V})) = \sum_{i \in I} u_i^I \in L^*_T, \mathcal{V}$. Note that $u^I$ does not belong to $L^* = H^2(BT)$ in general.

Lemma 4.1. The following three conditions are equivalent:

(i) the first Chern class $c_1(\Delta, \mathcal{V})$ is divisible by $N$,

(ii) $u^I \mod N$ regarded as an element of $L^*_T/ NL^*_T$ is independent of $I \in \Sigma^{(n)}$ and belongs to the image of $L^* = H^2(BT)$,

(iii) there is an element $u \in H^2(BT)$ such that $\langle u, v_i \rangle = 1 \mod N$ for all $i \in \Sigma^{(1)}$.

Proof. Suppose $c_1^T(\Delta, \mathcal{V})$ is of the form

$$
c_1^T(\Delta, \mathcal{V}) = N x + u, \ x \in H_T^2(\Delta), \ u \in H^2(BT).
$$

Then $u^I = \iota_1^*(c_1^T(\Delta, \mathcal{V})) \mod N$ is equal to $\iota_1^*(u) = u \mod NL^*_T$, and hence belongs to the image of $H^2(BT)$. Thus (i) implies (ii).

Suppose that $u \in H^2(BT)$ and $u \mod N$ is equal to $u^I \mod N$ for any $I \in \Sigma^{(n)}$. Then $\langle u, v_i \rangle = \langle u^I, v_i \rangle \mod N$, and hence $\langle u, v_i \rangle = \sum_{j \in I} \langle u_j^I, v_i \rangle = 1 \mod N$ for $i \in I$. Thus (ii) implies (iii).
Suppose \( \langle u, v_i \rangle = 1 \mod N \) for any \( i \in \Sigma^{(1)} \). Then, by \((\Pi)\),

\[
c^T_1(\Delta, \mathcal{V}) - u = \sum_{i \in \Sigma^{(1)}} (1 - \langle u, v_i \rangle)x_i = 0 \mod NH^2_T(\Delta).
\]

Hence \( c^T_1(\Delta, \mathcal{V}) \) is of the form \( c^T_1(\Delta, \mathcal{V}) = Nx + u \) with \( u \in H^2(BT) \). Thus (iii) implies (i). \qed

**Remark 4.2.** Let \( M \) be a torus manifold and \( \Delta(M) \) its associated multi-fan. Put \( \hat{H}^2_T(M) = H^2_T(M)/S \)-torsion. In \([13]\) it was shown that there is a canonical embedding of \( H^2_T(\Delta(M)) \) in \( \hat{H}^2_T(M) \), and, in case \( M \) is a stably almost complex torus manifold, \( c^T_1(M) \in \hat{H}^2_T(M) \) descends to \( c^T_1(\Delta(M)) \in \hat{H}^2_T(\Delta(M)) \). It follows that, if \( M \) is a stably almost complex torus manifold and \( c_1(M) \) is divisible by \( N \), then \( c_1(\Delta(M)) \) is also divisible by \( N \). Even if \( M \) is a stably almost complex orbifold it can be shown that \( c^T_1(M) \in \hat{H}^2_T(M; \mathbb{R}) \) descends to \( c^T_1(\Delta(M), \mathcal{V}(M)) \in \hat{H}^2_T(\Delta(M)) \otimes \mathbb{R} \). But the divisibility of the first Chern class has no meaning with real coefficients. We have to work with orbifold cohomology theory with integer coefficients. We understand it in this paper by that of \( c_1(\Delta(M), \mathcal{V}(M)) \).

The following property (P) will be called the global type condition:

(P): \( L_{I, \mathcal{V}} = L_\mathcal{V} \) for all \( I \in \Sigma^{(n)} \).

Typical examples of pairs \((\Delta, \mathcal{V})\) satisfying the condition (P) are provided by global torus orbifolds. In fact, let \( \hat{M} \) be a torus manifold of dimension \( 2n \) and let \( H \) be a finite subgroup of the torus \( \hat{T} \) acting on \( \hat{M} \). The quotient \( \hat{M} = \hat{M}/H \) is a torus orbifold equipped with an orbifold structure for which \((\hat{M}, H, \pi)\) is an orbifold chart where \( \pi : M \to \hat{M} \) is the projection. If \( \hat{M}_i \) is a characteristic submanifold of \( \hat{M} \), then its image \( M_i \) by \( \pi \) is a characteristic suborbifold of \( M \). Conversely every characteristic suborbifold of \( M \) is of the above form. It follows that one can identify the simplicial set \( \Sigma(M) \) with \( \Sigma(\hat{M}) \).

The lattice \( \hat{L} \) for the multi-fan \( \Delta(\hat{M}) \) is identified with \( \pi_1(\hat{T}) \) and similarly the multi-fan \( \Delta(M) \) is defined in the lattice \( L = \pi_1(T) \). Let \( \{ \tilde{v}_i \}_{i \in \Sigma^{(1)}(\hat{M})} \) be the primitive generators corresponding to the oriented characteristic submanifolds \( \hat{M}_i \). Put \( v_i = \pi_*(\tilde{v}_i) \in L \) and \( \mathcal{V} = \{ v_i \} \). Then \( L_{\mathcal{V}} \) coincides with the image \( \pi_*(\hat{L}) \). For any \( I \in \Sigma^{(n)}(M) \) the lattice \( \hat{L} \) is generated by \( \{ \tilde{v}_i \mid i \in I \} \). Hence \( L_{\mathcal{V}} \) is generated by \( \{ v_i \mid i \in I \} \). This shows that the pair \((\Delta(M), \mathcal{V})\) satisfies the condition (P). Note that, when \( \Delta \) is a non-singular multi-fan, we have \( L_{I, \mathcal{V}} = L_{\mathcal{V}} = L \) for all \( I \in \Sigma^{(n)} \) and the condition (P) is automatically satisfied.

**Remark 4.3.** If \((\Delta, \mathcal{V})\) satisfies the condition (P), then there is a complete non-singular multi-fan \( \tilde{\Delta} = (\tilde{\Sigma}, \tilde{C}, \tilde{w}^\pm) \) in a lattice \( \tilde{L} \) and an injective homomorphism \( \pi : \tilde{L} \to L \) such that \( \pi(\tilde{L}) = L_{\mathcal{V}} \). Namely let \( \tilde{L} \) be a copy of \( L_{\mathcal{V}} \) and let \( \tilde{v}_i \) be the copy of \( v_i \) in \( \tilde{L} \). The identification of \( \tilde{L} \) with \( L_{\mathcal{V}} \) composed with the inclusion of \( L_{\mathcal{V}} \) in \( L \) defines the map \( \pi \). We put \( \tilde{\Sigma} = \Sigma, \ \tilde{w}^\pm = w^\pm \) and define \( \tilde{C}(I) \) to be the cone generated by \( \{ \tilde{v}_i \}_{i \in I} \). These define the multi-fan \( \tilde{\Delta} \). We may call it the (ramified) covering of \( \Delta \) with respect to \( \mathcal{V} \). Since \( \{ \tilde{v}_i \mid i \in I \} \) is a basis of \( \tilde{L} \) for any \( I \in \Sigma^{(n)} \), \( \tilde{\Delta} \) is non-singular. We put \( \tilde{T} = \tilde{L}_R/\tilde{L} \). It is the torus associated with the multi-fan \( \tilde{\Delta} \). Then the kernel of the induced map \( \pi : \tilde{T} \to T = L_R/L \) is identified with \( H = L/L_{\mathcal{V}} \). Conversely given a complete non-singular multi-fan \( \tilde{\Delta} = (\tilde{\Sigma}, \tilde{C}, \tilde{w}^\pm) \) in a lattice \( \tilde{L} \) and a sublattice \( L \) of \( \tilde{L}_R \) such that \( \tilde{L} \subset L \) we can construct a new complete simplicial multi-fan \( \Delta \) in the lattice \( L \) and a set of edge vectors \( \mathcal{V} \) such that the covering of \( \Delta \) with respect to \( \mathcal{V} \) coincides
with $\Delta$. Geometric picture of this construction is making quotient torus orbifold $\tilde{M}/H$ out of a torus manifold $\tilde{M}$ by a subgroup $H$ of the torus $\tilde{T}$ acting on $\tilde{M}$.

**Remark 4.4.** The following fact can be proved easily. Under the situation of Remark 4.3 suppose that $c_1(\Delta, \mathcal{V})$ is divisible by $N$, then $c_1(\Delta)$ is also divisible by $N$. Conversely if $c_1(\Delta)$ is divisible by $N$ and the order of $H = L/L_{\mathcal{V}}$ is relatively prime to $N$, then $c_1(\Delta)$ is also divisible by $N$.

Let $v$ be a generic vector in $L_{\mathcal{V}} \subset L = H_2(B\mathcal{T})$. If we fix $I$ and write

$$v = \sum_{i \in I} m_i v_i,$$

then $m_i = \langle u_i^t, v \rangle \in \mathbb{Z}$. Let $m \geq 1$ be an integer. Fixing $v$ and $m$, we put

$$I_{(m)} := \{ i \in I \mid m \text{ does not divide } m_i \}$$

for $I \in \Sigma^{(n)}$. It will be called mod $m$ face of $I$. Note that $I_{(m)}$ depends on $v$.

**Lemma 4.5.** Suppose that the condition (P) is satisfied. Let $v, m$ and $I$ be as above, and let $K = I_{(m)}$ be the mod $m$ face of $I$. If $I' \in \Sigma^{(n)}$ contains $K$, then $K$ is also the mod $m$ face of $I'$. Moreover, $\langle u_i^t, v \rangle = \langle u_i^t, v \rangle \mod m$ for $i \in K$.

**Proof.** We put $m_i = \langle u_i^t, v \rangle$ and $m_i' = \langle u_i^t, v \rangle$. Then we have

$$v = \sum_{i \in I} m_i v_i = \sum_{i' \in I'} m_i' v_i.'$$

By assumption $m_i = 0 \mod m$ for $i \notin K$. Hence

$$\sum_{i \in K} m_i v_i = \sum_{i' \in I'} m_i' v_i' \mod m \text{ in } L_{\mathcal{V}}$$

or

$$\sum_{i \in K} (m_i' - m_i) v_i + \sum_{i' \in I', i' \notin K} m_i' v_i' = 0 \mod m \text{ in } L_{\mathcal{V}}.$$

Since $\{v_{i'} \mid i' \in I'\}$ is a basis of the free module $L_{I', \mathcal{V}} = L_{\mathcal{V}}$, we see that $m_i' = 0 \mod m$ for $i' \in I'$, $i' \notin K$ and $m_i = m_i \mod m$ for $i \in K$. \hfill $\square$

We shall say that $I$ and $I'$ are $(v, m)$-equivalent and write $I \sim I'$ if $I_{(m)} = I'_{(m)}$. This defines an equivalence relation $\sim$ in $\Sigma^{(n)}$. Lemma 4.5 implies that, if $X$ is an equivalence class, the members of $X$ have some $K$ as the common mod $m$ face. We shall call this $K$ the core of the equivalence class $X$.

**Lemma 4.6.** Suppose that the condition (P) is satisfied. Let $X$ be an equivalence class of $(v, m)$-equivalence relation. For $x \in H_2^0(\Delta)$ the value $\langle x^*, \mathcal{V} \rangle \mod m$ does not depend on the choice of $I$ in $X$.

**Proof.** Write $x = \sum_{i \in \Sigma^{(n)}} a_i x_i$. Let $K$ be the core of $X$. Then

$$\langle x^*, \mathcal{V} \rangle = \sum_{i \in K} a_i \langle u_i^t, v \rangle + \sum_{i \notin K} a_i \langle u_i^t, v \rangle.$$

Since $\langle u_i^t, v \rangle = 0 \mod m$ for $i \notin K$ by the definition of the core and $\langle u_i^t, v \rangle \mod m$ for $i \in K$ does not depend on $I$ in $X$ by Lemma 4.5 $\langle x^*, \mathcal{V} \rangle \mod m$ does not depend on the choice of $I$ in $X$. \hfill $\square$
Corollary 4.7. Suppose that the condition (P) is satisfied. Assume that $c_1(\Delta, \mathcal{V})$ is divisible by $N$, and write $c_1^T(\Delta, \mathcal{V}) = Nx + u, u \in H^2(BT)$. Let $v$ and $m$ be as above. If we write $\langle u^I_i, v \rangle$ in the form
\[ \langle u^I_i, v \rangle = mh_i + r_i \text{ with } 0 \leq r_i < m, \]
then the sum $\sum_{i \in I} h_i \mod N$ depends only on the $(v, m)$-equivalence class $X$ of $I$.

Proof. Let $K$ be the core of $X$. Then $\langle u^I_i, v \rangle \equiv 0 \mod m$ for $i \in I \setminus K$ and $\langle u^I_i, v \rangle \mod m$ for $i \in K$ is independent of $I$ in $X$ by Lemma 4.5, so that $r_i = 0$ for $i \in I \setminus K$ and $r_i$ for $i \in K$ is independent of $I$. Hence the sum $\sum_{i \in I} r_i$ is a constant depending only on $X$ which we shall denote by $r$. We put $h_I = \sum_{i \in I} h_i$. Then $\langle u^I, v \rangle = mh_I + r$. By Lemma 4.6 $\langle u^I, v \rangle$ is of the form
\[ \langle u^I, v \rangle = mh_I + r', \]
for $I \in X$, where $r'$ is independent of $I$. Therefore, if we write $\langle u, v \rangle = r''$, then
\[ \langle u^I, v \rangle = N mh_I + Nr' + r''. \]
If we compare this with
\[ \langle u^I, v \rangle = mh_I + r, \]
we see that $Nr' + r''$ is of the form $Nr' + r'' = mh' + r$ and $h_I = Nh'_I + h'$. This shows that $h_I \mod N$ depends only on $X$. \hfill \square

Under the situation of Corollary 4.7, the mod $N$ value of $\sum_{i \in I} h_i$ will be called $(v, m)$-type of $X$ and will be denoted by $h(v, m, X)$. Similarly the mod $N$ value of $\langle u^I, v \rangle$ (which is independent of $I \in \Sigma^{(n)}$) by Lemma 4.4 will be called $v$-type and will be denoted by $h(v)$.

Lemma 4.8. Suppose that the condition (P) is satisfied. Assume $c_1(\Delta, \mathcal{V})$ is divisible by $N$. Any non-zero $b \in \mathbb{Z}/N$ can occur as $v$-type when $v$ varies over generic vectors in $L_\mathcal{V}$.

Proof. This follows readily from the fact that $\{v_i| i \in I\}$ is a basis of $L_\mathcal{V}$ for each $I \in \Sigma^{(n)}$. \hfill \square

5. Rigidity theorem

Let $\Delta = (\Sigma, C, w^\pm)$ be a complete simplicial multi-fan and $\mathcal{V}$ a set of prescribed edge vectors. Let $N$ be an integer greater than 1. When $\zeta_N = 1, \zeta \neq 1, \varphi(\Delta, \mathcal{V})$ and $\hat{\varphi}(\Delta, \mathcal{V})$ are called elliptic genus of level $N$ and orbifold elliptic genus of level $N$ respectively. Let $v \in L_\mathcal{V}$ be a generic vector. Since $\zeta_N = 1, \varphi^v(\Delta, \mathcal{V})$ and $\hat{\varphi}^v(\Delta, \mathcal{V})$ are elliptic functions in $z$ with respect to the lattice $\mathbb{Z}N\tau \oplus \mathbb{Z}$, because $\varphi(z, \tau, \sigma)$ is such a function by [3].

Hereafter the condition (P) is assumed throughout this section.

Lemma 5.1. Suppose that the condition (P) is satisfied and $c_1(\Delta, \mathcal{V})$ is divisible by $N$. Let $v \in L_\mathcal{V}$ be a generic vector and $h(v)$ the $v$-type. Then the elliptic genus $\varphi^v(\Delta, \mathcal{V})$ along $v$ and orbifold elliptic genus $\hat{\varphi}^v(\Delta, \mathcal{V})$ along $v$ of level $N$ transform by
\[ \varphi^v(\Delta, \mathcal{V}; z + \tau, \tau, \sigma) = \zeta^{h(v)}\varphi^v(\Delta, \mathcal{V}; z, \tau, \sigma), \]
\[ \hat{\varphi}^v(\Delta, \mathcal{V}; z + \tau, \tau, \sigma) = \zeta^{h(v)}\hat{\varphi}^v(\Delta, \mathcal{V}; z, \tau, \sigma). \]
Proof. 

\[
\prod_{i \in I} \phi(\langle u_i^I, -(z + \tau)v + \tau v(h_1) - v(h_2) \rangle, \tau, \sigma) \\
= \zeta^{\sum_{i \in I} (u_i^I, v)} \prod_{i \in I} \phi(\langle u_i^I, -zv + \tau v(h_1) - v(h_2) \rangle, \tau, \sigma)
\]

by (5). But \(\sum_{i \in I} (u_i^I, v) = h(v) \mod N\) which is independent of \(I\). Hence we obtain

\[
\varphi^v(\Delta, \mathcal{V}; z + \tau, \tau, \sigma) = \zeta^{h(v)} \varphi^v(\Delta, \mathcal{V}; z, \tau, \sigma), \quad \hat{\varphi}^v(\Delta, \mathcal{V}; z + \tau, \tau, \sigma) = \zeta^{h(v)} \hat{\varphi}^v(\Delta, \mathcal{V}; z, \tau, \sigma),
\]

since \(\zeta^N = 1\). \hfill \Box 

The following theorem and corollary are versions of rigidity theorem and vanishing theorem for multi-fans.

**Theorem 5.2.** Let \(\Delta\) be a complete simplicial multi-fan and \(\mathcal{V}\) a set of prescribed edge vectors satisfying the condition (P). Let \(v \in L\) be a generic vector. Assume that \(c_1(\Delta, \mathcal{V})\) is divisible by an integer \(N > 1\). Then the equivariant elliptic genus \(\varphi^v(\Delta, \mathcal{V}; z, \tau, \sigma)\) of level \(N\) along \(v\) is rigid, i.e. it is constant as a function of \(z\).

**Corollary 5.3.** Under the same situation as in Theorem 5.2, the equivariant elliptic genus \(\varphi(\Delta, \mathcal{V})\) of level \(N\) constantly vanishes. In particular, if \(\Delta\) is a complete non-singular multi-fan whose first Chern class \(c_1(\Delta)\) is divisible by \(N\), then its equivariant elliptic genus \(\varphi(\Delta)\) of level \(N\) constantly vanishes.

Proof. We postpone the proof of Theorem 5.2. As to Corollary 5.3, we take a generic vector \(v\) such that \(\zeta^{h(v)} \neq 1\), which is possible by Lemma 1.8. Since \(\varphi^v(\Delta, \mathcal{V}; z, \tau, \sigma)\) is constant by Theorem 5.2 for any \(v\), \(\varphi(\Delta, \mathcal{V})\) is also constant, which we denote by \(\varphi\).

Since \(\varphi = \zeta^{h(v)} \varphi\) by Lemma 5.1, \(\varphi\) must be equal to 0. \hfill \Box 

The degree 0 term in the \(q\)-expansion of \(\varphi(\Delta, \mathcal{V})\) reduces to the \(T_y\)-genus \(T_y[\Delta]\). It is independent of \(\mathcal{V}\) as was remarked in Section 3. We obtain

**Corollary 5.4.** Suppose that \(\Delta\) is a complete simplicial multi-fan and there is a set of generating vectors \(\mathcal{V}\) which satisfies the condition (P). If \(c_1(\Delta, \mathcal{V})\) is divisible by \(N\), then the \(T_y\)-genus \(T_y[\Delta]\) of \(\Delta\) vanishes for \((-y)^N = 1, -y \neq 1\).

If \(M\) is a stably almost complex closed manifold, then the signature \(\text{Sign}(M)\) equals \(T_1(M)\). After this the \(T_y\)-genus for \(y = 1\) of a complete non-singular multi-fan \(\Delta\) is called the signature of \(\Delta\) and is denoted by \(\text{Sign}(\Delta)\). Hence

**Corollary 5.5.** The signature \(\text{Sign}(\Delta)\) of a complete non-singular multi-fan \(\Delta\) with \(c_1(\Delta) = 0 \mod 2\) vanishes.

Remark 5.6. If \(M\) is a torus manifold, its elliptic genus coincides with that of its associated multi-fan \(\Delta(M)\). If \(c_1(M)\) is divisible by \(N\), \(c_1(\Delta(M))\) is also divisible by \(N\) as was remarked in Section 4 and hence its equivariant elliptic genus of level \(N\) vanishes. In case \(N = 2\) we have the following conclusion. The equivariant Stiefel-Whitney class \(w_2^T(M) \in H^2_B(M; \mathbb{Z}/2)\) is defined and descends to \(c_1^T(\Delta(M)) \mod 2\). If \(M\) is a spin torus manifold, then \(w_2^T(M)\) lies in \(H^2(BT; \mathbb{Z}/2)\). Therefore \(c_1^T(\Delta(M))\) is divisible by 2. It follows from Corollary 5.5 that the signature \(\text{Sign}(M)\) of \(M\) vanishes. This can be also deduced from Corollary in 1.5 of [9]. A complete non-singular multi-fan with \(c_1(\Delta) = 0 \mod 2\) might be called a spin multi-fan.
The rest of this section is devoted to the proof of Theorem 5.2. It is convenient to consider \( \varphi^v(\Delta, \mathcal{V}; z, \tau, \sigma) \) as a function of \( t = e^{2\pi \sqrt{-1}z} \) rather than \( z \). It will be denoted by \( \varphi^v(t) \).

We assume throughout that \( c_1(\Delta, \mathcal{V}) \) is divisible by \( N \), and \( \sigma = \frac{h}{N} \), \( 0 < k < N \), so that \( \zeta^N = 1 \). It suffices to show that \( \varphi^v(t) \) has no poles since it is an elliptic function. It is clear that possible poles \( t = \lambda \) satisfy \( \lambda^m q^{-s} = 1 \) for some integers \( m \geq 1 \) and \( s \). Hence it suffices to show that \( \varphi^v(tq^m) \) has no poles \( t = \lambda \) with \( \lambda^m = 1 \). Let

\[
\Sigma^{(n)} = \bigcup_v \Sigma^{(n)}_{\nu}
\]

be the decomposition into \((v, m)\)-equivalence classes. We fix a class \( \Sigma^{(n)}_{\nu} \) and write

\[
\langle u_i^I, v \rangle = m_i = mh_i + r_i, \quad 0 \leq r_i < m,
\]

for \( I \in \Sigma^{(n)}_{\nu} \). Let \( K_{\nu} \) denote the core of \( \Sigma^{(n)}_{\nu} \). Remember that

\[ r_i = 0 \iff i \notin K_{\nu}, \]

and \( r_i \) is independent of \( I \in \Sigma^{(n)}_{\nu} \) for \( i \in K_{\nu} \). Note that \( \sum_{i \in I} h_i = h(v, m, \Sigma^{(n)}_{\nu}) \mod N \).

We set

\[ \bar{v} = \sum_{i \in K} r_i v_i. \]

Then \( \langle u_i^I, v - \bar{v} \rangle = mh_i \). A straightforward calculation using (5) yields

\[
\varphi^v(tq^m) = \sum_{\nu} \zeta^{sh(v, m, \Sigma^{(n)}_{\nu})} \varphi^v(t),
\]

where

\[
\varphi^v(t) = \sum_{I \in \Sigma^{(n)}_{\nu}} \frac{w(I)}{|H_I|} \prod_{h \in H_I} \phi(-\langle u_i^I, zv + v(h) \rangle, \tau, \sigma) \prod_{i \in K_{\nu}} \phi(-\langle u_i^I, zv + \frac{s}{m} \tau \bar{v} + v(h) \rangle, \tau, \sigma).
\]

**Lemma 5.7.** Fix \( \nu \). Then \( \varphi^v(t) \) can be expanded in the following form:

\[
\varphi^v(t) = \sum_{n=0}^{\infty} \varphi_n(t) q^{\frac{m}{n}}
\]

with \( \zeta^{\frac{m}{n}} \varphi_n(t) \in R(S^1) \otimes \mathbb{Z}[\zeta, \zeta^{-1}]. \)

**Proof.** The proof is similar to that of Proposition 3.1. Put \( \tau_1 = \frac{m}{n} \tau \). We expand \( \varphi^v(t) \) in the form

\[
\varphi^v(t) = \sum_{s_1 \in \mathbb{Z}, s_2 \in \mathbb{Z}_{\geq 0}} \varphi_{s_1, s_2}(z, q_1) \zeta^{s_1} q^{s_2}.
\]

Then \( \zeta^{\frac{m}{n}} \varphi_{s_1, s_2}(z, q_1) \zeta^{s_1} q^{s_2} \) is a sum of expressions of the following form.

\[
\sum_{I \subset K_{\nu}} \frac{w(I)}{|H_I|} \sum_{i \in \alpha} \sum_{h \in H_I} \prod_{i \in \alpha \setminus K_{\nu}} \left( 1 - \chi_I(u_i^I, h) \right) q^{-(i, v)}.
\]
where $\alpha_I \in \mathcal{G}_I$ satisfying $\tau_{I,J}(\alpha_I) = \alpha_J$. It follows that $\zeta^{-\lambda} \varphi_{s_1,s_2}(z, q_1) \zeta^{s_1} q^{s_2}$ belongs to $\mathbb{Z}[t, t^{-1}, q_1, q_1^{-1}]$ by Lemma 2.3. Noting that $q_1 = q^\frac{m}{n}$, we see that $\varphi_\nu(t)$ can be expanded in the form

$$\varphi_\nu(t) = \sum_{n=0}^{\infty} \varphi_n(t) q^{\frac{m}{n}}$$

with $\zeta^{-\lambda} \varphi_n(t) \in R(S^1) \otimes \mathbb{Z}[\zeta, \zeta^{-1}]$. \qed

We shall show that each $\varphi_\nu(t)$ has no poles at $\lambda$ with $\lambda^m = 1$. Suppose that $\lambda$ is a possible pole with $\lambda^m = 1$. Let $\varphi_{n,t}(t)$ be the contribution from $I$ in $\varphi_n(t)$. There is an open set $U$ containing $\lambda$ such that $\varphi_{n,t}(t)$ is holomorphic in $U \setminus \{\lambda\}$ for each $I$. $\sum_{n=0}^{\infty} \varphi_{n,t}(t) q^{\frac{m}{n}}$ converges uniformly in any compact set in $U - \{\lambda\}$. Note that $\sum_{I} \varphi_{n,I}(t) = \varphi_n(t)$ is holomorphic in $U$ because it is a finite Laurent series by Lemma 5.7. We now quote a lemma from \[7\].

**Lemma 5.8** (Hirzebruch). Let $\{b_{n,j}\}$ be a family of meromorphic functions on $U$ with $n$ running over non-negative integers and $j$ running over some finite set $J$. Suppose that they satisfy the following properties:

(i) $b_{n,j}$ is holomorphic in $U \setminus \{\lambda\}$,

(ii) $b_n = \sum_j b_{n,j}$ is holomorphic in $U$,

(iii) $\sum_{n=0}^{\infty} b_{n,j}$ converges uniformly in any compact set in $U \setminus \{\lambda\}$.

Then $\sum_{n=0}^{\infty} b_{n,j}$ converges uniformly in any compact set in $U$ and is a holomorphic extension of $\sum_{j \in J} \sum_{n=0}^{\infty} b_{n,j} | U \setminus \{\lambda\}$.

We apply this Lemma to $\{\varphi_{n,I}(t) q^{\frac{m}{n}}\}$. It follows that $\varphi_\nu(t)$ and hence $\varphi^\nu(t q^{\frac{m}{n}})$ has no pole at $t = \lambda$. Hence $\varphi^\nu(t)$ has no poles and it is a constant. This proves Theorem 5.2.

**Remark 5.9.** Suppose that the pair $(\Delta, \mathcal{V})$ satisfies the condition (P). In Remark 4.8 we introduced a non-singular multi-fan $\tilde{\Delta}$ in a lattice $\tilde{L}$ and an injective homomorphism $\pi : \tilde{L} \to L$ with image $\pi(\tilde{L}) = L_\mathcal{V}$. We identify $L^*$ with the sublattice $\pi^*(L^*)$ of $\tilde{L}$. If we write $\varphi(\tilde{\Delta})$ as

$$\varphi(\tilde{\Delta}) = \sum_{u \in L^*} a_u t^u$$

the coefficients $a_u$ are given by Theorem 3.3. Comparing this with the expression of $\varphi(\Delta, \mathcal{V})$ given by Theorem 3.3 and noting that $\langle u, \tilde{v}_i \rangle = \langle u, v_i \rangle$ we see that

$$\varphi(\Delta, \mathcal{V}) = \sum_{u \in L^*} a_u t^u.$$ 

Thus, if the coefficients of $t^u$ in $\varphi(\tilde{\Delta})$ vanish for all $u \in \tilde{L}^*$, then those of $\varphi(\Delta, \mathcal{V})$ also vanish. This is the case when $c_1(\Delta, \mathcal{V})$ is divisible by $N$ and $\zeta^N = 1$, since $c_1(\tilde{\Delta})$ is also divisible by $N$, as was remarked in Remark 4.3.

**Remark 5.10.** Examples show that the rigidity phenomenon does not necessarily occur for pairs $(\Delta, \mathcal{V})$ without condition (P).

6. Applications

Let $\Delta$ be a complete simplicial multi-fan of dimension $n$. 
Lemma 6.1. If the Todd genus $T_0[\Delta]$ equals 1 and $w(I) = 1$ for all $I \in \Sigma^{(n)}$, then
\begin{equation}
(16) 
    h_k(\Delta) = \#\{I \in \Sigma^{(n)} \mid \mu(I) = k\},
\end{equation}
in (12) and
\begin{equation}
(17) 
    e_k(\Delta) = \#\Sigma^{(k)}
\end{equation}
in (15).

Proof. (16) is immediate. As was remarked in Note after Proposition 3.8, $T_0(\Delta)$ equals $\deg(\Delta)$. Therefore, if $T_0(\Delta) = 1$ then $\deg(\Delta) = 1$, and consequently $\deg(\Delta_J) = 1$ for all $J \in \Sigma$, as is easily seen. Then (17) follows. \qed

Note. Note that these conditions are always satisfied for complete simplicial ordinary fans. In particular, if $M$ is a complete non-singular toric variety, then the fan $\Delta(M)$ associated with $M$ satisfies these conditions. It is known that $T_{-t^2}[\Delta(M)]$ is equal to the Poincaré polynomial $P(t)$ of $M$ if $M$ is a non-singular projective toric variety, see e.g. [4].

Proposition 6.2. Let $\Delta$ be a complete non-singular multi-fan with Todd genus $T_0[\Delta] \neq 0$. If $c_1(\Delta)$ is divisible by a positive integer $N$, then $N$ is equal to or less than $n + 1$. In the cases $N = n + 1$ and $N = n$ the $T_y$- genus must be of the following forms
\begin{equation}
(18) 
    T_y[\Delta] = T_0[\Delta] \sum_{k=0}^{n} (-y)^k \quad (N = n + 1),
\end{equation}
and
\begin{equation}
(19) 
    T_y[\Delta] = T_0[\Delta](1 - y) \sum_{k=0}^{n-1} (-y)^k \quad (N = n).
\end{equation}

Proof. Suppose that $c_1(\Delta)$ is divisible by $N$. Then, by Corollary 5.4, $T_y[\Delta]$ considered as a polynomial in $-y$ has roots at all $N$-th roots of unity other than 1. Hence it must be divisible by $\sum_{k=0}^{N-1}(-y)^k$. On the other hand it is a polynomial of degree $n$ with constant term $T_0[\Delta] \neq 0$ by Proposition 3.8. Therefore we must have $N - 1 \leq n$.

Suppose that $N = n + 1$. Then the same reasoning as above proves (18). If $N = n$, then $T_y[\Delta]$ is divisible by $\sum_{k=0}^{n-1}(-y)^k$. Since the constant term and the coefficient (as a polynomial of $-y$) of the highest term do not vanish by assumption and Proposition 3.8, $T_y[\Delta]$ must be of the form (19). \qed

Lemma 6.3. Let $\Delta$ be a complete non-singular multi-fan with $T_0[\Delta] = 1$ and $w(I) = 1$ for all $I \in \Sigma^{(n)}$. If $T_y[\Delta]$ is of the form (18), then
\begin{equation}
#\Sigma^{(1)} = n + 1 \quad \text{and} \quad #\Sigma^{(n)} = n + 1.
\end{equation}
If $T_y[\Delta]$ is of the form (19), then
\begin{equation}
#\Sigma^{(1)} = n + 2 \quad \text{and} \quad #\Sigma^{(n)} = 2n.
\end{equation}
Moreover, in case $n \geq 3$, $#\Sigma^{(2)} = \frac{1}{2}n(n + 3)$.

Proof. The equality (18) with $T_0[\Delta] = 1$ implies that $h_k(\Delta) = 1$ for all $k$ with $0 \leq k \leq n$ by Proposition 3.8. This implies that $#\Sigma^{(n)} = n + 1$ by (16). Then, using (15) and (17) we see that $#\Sigma^{(1)} = e_1(\Delta) = n + 1$. 


Similarly the equality (19) with $T_0[\Delta] = 1$ implies that $h_k(\Delta) = 1$ for $k = 0, n$ and $h_k(\Delta) = 2$ for $1 \leq k \leq n - 1$. This implies that $\# \Sigma^{(n)} = 2n$ by (16), and yields, together with (15) and (17), the equalities $\# \Sigma^{(1)} = e_1(\Delta) = n + 2$ and $\# \Sigma^{(2)} = e_2(\Delta) = \frac{1}{2}n(n + 3)$ in case $n \geq 3$.

**Corollary 6.4.** Let $M$ be a complete non-singular toric variety of dimension $n$. If $c_1(M)$ is divisible by $n + 1$, then $M$ is isomorphic to the projective space $\mathbb{P}^n$ as a toric variety.

**Proof.** By Proposition 6.2 and Lemma 6.3 the fan associated with $M$ has $n + 1$ 1-dimensional cones and $n + 1$-dimensional cones. Such a fan is unique (up to automorphisms of the lattice $L$) and coincides with the fan associated with $\mathbb{P}^n$. Since a toric variety is determined by its fan, $M$ must be $\mathbb{P}^n$. □

In order to handle the case $N = n$ we investigate the fan associated with a projective space bundle over a projective space. Let $\xi$ denote the hyperplane bundle (dual of the tautological line bundle) over $\mathbb{P}^r$. Let $1 \leq r < n$ and set $\eta = (\bigoplus_{i=r+1}^{n} \xi^{k_i}) \oplus 1$ where $k_i$ are integers and 1 denotes the trivial line bundle. The associated projective space bundle of $\eta$ will be denoted by $M$. It is a complex manifold. A point of $M$ is expressed in homogeneous coordinate

$$[z_0, z_1, \ldots, z_r, w_{r+1}, \ldots, w_n, w_{n+1}]$$

where $z_i, w_j \in \mathbb{C}$, $(z_0, z_1, \ldots, z_r) \neq (0, 0, \ldots, 0)$, $(w_{r+1}, \ldots, w_n, w_{n+1}) \neq (0, \ldots, 0, 0)$, and if, $\alpha \in \mathbb{C}^*$, then

$$[\alpha z_0, \alpha z_1, \ldots, \alpha z_r, \alpha^{k_{r+1}} w_{r+1}, \ldots, \alpha^{k_n} w_n, w_{n+1}]$$

and

$$[z_0, z_1, \ldots, z_r, \alpha w_{r+1}, \ldots, \alpha w_n, \alpha w_{n+1}]$$

are identified with (20).

Let the $(n + 1)$-dimensional torus $T^{n+1} = S^1 \times \cdots \times S^1$ act on $M$ by

$$(t_0, t_1, \ldots, t_n)[z_0, z_1, \ldots, z_r, w_{r+1}, \ldots, w_n, w_{n+1}] = [t_0 z_0, t_1 z_1, \ldots, t_r z_r, t_{r+1} w_{r+1}, \ldots, t_n w_n, w_{n+1}]$$

The action is a holomorphic action. The subgroup $D' = \{ (t, \ldots, t, t_{r+1}^{k_{r+1}}, \ldots, t^{k_n}) \mid t \in S^1 \}$ of $T^{n+1}$ acts trivially on $M$. Hence the quotient $T = T^{n+1}/D'$ acts on $M$. Put

$$M_i = \begin{cases} \{ [z_0, z_1, \ldots, z_r, w_{r+1}, \ldots, w_n, w_{n+1}] \mid z_i = 0 \} & \text{for } 0 \leq i \leq r \\ \{ [z_0, z_1, \ldots, z_r, w_{r+1}, \ldots, w_n, w_{n+1}] \mid w_i = 0 \} & \text{for } r + 1 \leq i \leq n + 1 \end{cases}$$

We also put

$$S_i = \{ (1, \ldots, 1, t_i, 1, \ldots, 1) \in T^{n+1} \} \text{ for } 0 \leq i \leq n$$

and

$$S_{n+1} = \{ (1, 1, \ldots, 1, t_{r+1}, \ldots, t_n) \in T^{n+1} \mid t_{r+1} = \cdots = t_n \}.$$
induced by the projection $\pi : T^{n+1} \to T$. From the definition it follows that there is a relation
\begin{equation}
(21) \quad v_0 + v_1 + \cdots + v_r + k_{r+1}v_{r+1} + \cdots + k_nv_n = 0.
\end{equation}
We also put
\begin{equation}
(22) \quad v_{n+1} = -(v_{r+1} + \cdots + v_n) \in \text{Hom}(S^1, T).
\end{equation}
Then $S_i$ is the image of $v_i : S^1 \to T$. Moreover $S^1$ acts via $v_i$ on each fiber of the normal bundle of $M_i$ in $M$ by standard complex multiplication. Thus \{ $v_i \mid i = 0, \ldots, n, n + 1$ \} coincides with the set of primitive edge vectors of 1-dimensional cones of the multi-fan $\Delta(M) = (\Sigma(M), C(M), w(M)^\pm )$ associated with the torus manifold $M$, and we have $\Sigma(M)^{(1)} = \{ 0, 1, \ldots, n, n + 1 \}$, cf. [D].

To determine the whole augmented simplicial set $\Sigma(M)$, we need to look at the fixed point set $M^\tau$. For $i \in \{ 0, 1, \ldots, r \}$, put $I_i = \{ 0, 1, \ldots, r \} \setminus \{ i \}$, and for $j \in \{ r + 1, \ldots, n + 1 \}$, put $J_j = \{ r + 1, \ldots, n + 1 \} \setminus \{ j \}$. It is not difficult to see that $M^\tau$ consists of points
\[ M_{I_i} \cap M_{J_j}, \quad i \in \{ 0, 1, \ldots, r \}, \quad j \in \{ r + 1, \ldots, n + 1 \}, \]
where $M_{I_i} = \cap_{k \in I_i} M_k$ and $M_{J_j} = \cap_{l \in J_j} M_l$. This implies that
\[ \Sigma(M)^{(n)} = \{ I_i \cup J_j \mid i \in \{ 0, 1, \ldots, r \}, \quad j \in \{ r + 1, \ldots, n + 1 \} \}. \]
In particular
\[ \# \Sigma(M)^{(n)} = (r + 1)(n - r + 1). \]
It follows that
\[ \# \Sigma(M)^{(n)} \geq 2n \quad \text{and} \quad \# \Sigma(M)^{(n)} = 2n \quad \text{if and only if} \quad r = 1 \text{ or } n - 1. \]

Let $\tau$ be the tautological line bundle over the projective space bundle $M$. Its dual $\tau^*$ restricts to the hyperplane bundle on each fiber of $\pi : M \to \mathbb{P}^r$. Let $\omega \in H^2(M)$ be the first Chern class of $\tau^*$. Then, by the Leray-Hirsch theorem, $H^*(M)$ is a free $H^*(\mathbb{P}^r)$-module on generators $1, \omega, \omega^2, \ldots, \omega^{n-r}$. In particular, $H^2(M)$ is a free module on $\omega, \omega'$, where $\omega'$ is the image of the canonical generator of $H^2(\mathbb{P}^r)$ by $\pi^*$. We have

Lemma 6.5.
\[ c_1(M) = (n - r + 1)\omega + \left( \sum_{i=r+1}^{n} k_i + r + 1 \right)\omega'. \]

Proof. The tautological line bundle $\tau$ is a subbundle of $\pi^*\eta$, and the tangent bundle along the fibers $T_f M$ of $\pi : M \to \mathbb{P}^r$ is isomorphic to $\text{Hom}(\tau, \pi^*\eta/\tau) = \tau^* \otimes (\pi^*\eta/\tau)$. Hence
\[ c_1(T_f M) = (n - r)c_1(\tau^*) + c_1(\pi^*\eta/\tau) = (n - r)\omega + c_1(\pi^*\eta) - c_1(\tau) = (n - r)\omega + \left( \sum_{i} k_i \right)\omega' + \omega = (n - r + 1)\omega + \left( \sum_{i} k_i \right)\omega'. \]
Since the tangent bundle $TM$ is isomorphic to $\pi^*T\mathbb{P}^r \oplus T_f M$, and $c_1(\pi^*T\mathbb{P}^r) = (r + 1)\omega'$, we have
\[ c_1(M) = (n - r + 1)\omega + \left( \sum_{i} k_i + r + 1 \right)\omega'. \]
\[ \square \]
As an immediate consequence of Lemma 6.5, we obtain

**Corollary 6.6.** Let $M = \mathbb{P}(\eta)$ be as above. Then $c_1(M)$ is divisible by $n$ if and only if $r = 1$ and $\sum_{i=r+1}^{n} k_i + 2$ is divisible by $n$.

We now consider complete non-singular multi-fans having first Chern class divisible by $n$.

**Lemma 6.7.** Let $\Delta = (\Sigma, C, w^\pm)$ be a complete non-singular multi-fan of dimension $n$ such that

\[ T_0[\Delta] = 1, \ w(I) = 1 \text{ for all } I \in \Sigma^{(n)}, \]
\[ \#\Sigma^{(1)} = n + 2, \ \#\Sigma^{(n)} = 2n, \text{ and } \#\Sigma^{(2)} = \frac{1}{2}n(n + 3) \text{ in case } n \geq 3. \]

Then it is equivalent to the fan associated to a $\mathbb{P}^{n-1}$ bundle over $\mathbb{P}^1$ or a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$.

**Proof.** Let $\{v_i\}_{i=0}^{n+1}$ be the primitive edge vectors of the 1-dimensional cones. In view of (21) and (22) it suffices to show that, under a suitable numbering, they satisfy the relations

\[ v_1 + v_2 + \cdots + v_n = 0, \ v_0 + v_{n+1} + \sum_{i=2}^{n} k_i v_i = 0, \]

or

\[ v_1 + v_2 + \cdots + v_n + k v_{n+1} = 0, \ v_0 + v_{n+1} = 0. \]

We first deal with the case $n \geq 3$. From the completeness we see that each 1-dimensional cone is a face of at least $n$ 2-dimensional cones, and it is a face of at most $n + 1$ 2-dimensional cones because the number of 1-dimensional cones are $n + 2$. Since the number of 2-dimensional cones is $\frac{1}{2}n(n + 3) = n + \frac{1}{2}n(n + 1)$, we conclude that there are two edge vectors, say $v_0$ and $v_{n+1}$ such that $v_0$ ($v_{n+1}$ respectively) spans 2-dimensional cones with each of remaining vectors $v_1, v_2, \ldots, v_n$, and each $v_i$, $1 \leq i \leq n$, spans 2-dimensional cones with $v_j$, $j \neq i$. Thus the projected multi-fan $\Delta_{\{0\}}$ has exactly $n$ 1-dimensional cones. It is complete and non-singular as a projected multi-fan of a complete non-singular multi-fan $\Delta$. It follows that $\Delta_{\{0\}}$ is equivalent to the fan of $\mathbb{P}^{n-1}$, and the projected edge vectors $\bar{v}_i$, $1 \leq i \leq n$, satisfy the relation

\[ \bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_n = 0. \]

This implies the relation

\[ v_1 + v_2 + \cdots + v_n = k v_0. \]

Similarly we have

\[ v_1 + v_2 + \cdots + v_n = k' v_{n+1}. \]

If $k = 0$ then $k' = 0$ since $v_{n+1} \neq 0$, and $v_1, v_2, \ldots, v_n$ lie on a hyperplane $v_1 + v_2 + \cdots + v_n = 0$. Since the multi-fan $\Delta$ is complete and non-singular, the primitive vectors $v_0$ and $v_{n+1}$ lie on the different sides of that hyperplane and must satisfy a relation as described in (23).

If $k \neq 0$ then $k' \neq 0$ and $v_0$ and $v_{n+1}$ are linearly dependent primitive vectors. Therefore we must have $v_0 + v_{n+1} = 0$. Thus (23) holds. This proves Lemma 6.7 in the case $n \geq 3$.

The case $n = 2$ is similar and easier. We see that there are four primitive edge vectors $v_0, v_1, v_2, v_3$ in 2-dimensional vector space $V = L \otimes \mathbb{R}$ such that

\[ v_1 + v_2 = k v_0 = k' v_3. \]
By the same reasoning as above we derive

\[ v_1 + v_2 = 0, \quad v_0 + v_3 + kv_2 = 0 \quad \text{or} \quad v_1 + v_2 + kv_3 = 0, \quad v_0 + v_3 = 0. \]

\[ \square \]

**Corollary 6.8.** Let \( M \) be a complete non-singular toric variety of dimension \( n \). If \( c_1(M) \) is divisible by \( n \), then \( M \) is isomorphic to an \((n - 1)\)-dimensional projective space bundle \( \mathbb{P}(\eta) \) with \( \eta = (\bigoplus_{i=2}^{n} \xi^{k_i}) \oplus 1 \) over \( \mathbb{P}^1 \) such that \((\sum_{i=2}^{n} k_i) + 2\) is divisible by \( n \). One dimensional cones of the fan associated with \( M \) is generated by primitive vectors \( v_0, v_1, v_2, \ldots, v_n, v_{n+1} \) satisfying

\[ v_0 + v_{n+1} + \sum_{i=2}^{n} k_i v_i = 0, \quad v_1 + v_2 + \cdots + v_n = 0. \]

**Proof.** By Proposition 6.2 and Lemma 6.3 the fan \( \Delta(M) \) associated with \( M \) has \( n + 2 \) 1-dimensional cones and \( 2n \) \( n \)-dimensional cones. Moreover the number of 2-dimensional cones is \( \frac{1}{2}n(n + 3) \) in case \( n \geq 3 \). By Lemma 6.7 \( \Delta(M) \) is equivalent to that of a \( \mathbb{P}^{n-1} \)-bundle over \( \mathbb{P}^1 \) or \( \mathbb{P}^1 \) -bundle over \( \mathbb{P}^{n-1} \). Among such manifolds those with \( c_1 \) divisible by \( n \) are of the form given in Corollary 6.6. \[ \square \]

**Note.** Suppose that \((\sum_{i=2}^{n} k_i) + 2 = -kn\). Then the equalities

\[ v_0 + v_{n+1} + \sum_{i=2}^{n} k_i v_i = 0, \quad v_1 + v_2 + \cdots + v_n = 0 \]

are equivalent to

\[ v_0 + v_{n+1} + kv_1 + \sum_{i=2}^{n} (k_i + k) v_i = 0, \quad v_1 + v_2 + \cdots + v_n = 0, \]

and we have \( k + \sum_{i=2}^{n} (k_i + k) = -2 \). If we put \( \bar{k}_i = k \) for \( i = 1 \) and \( \bar{k}_i = k_i + k \), then the projective space bundle \( \mathbb{P}(\bigoplus_{i=2}^{n} \xi^{\bar{k}_i}) \oplus 1 \) is biholomorphic to \( \mathbb{P}(\bigoplus_{i=1}^{n} \xi^{k_i}) \). Thus a complete non-singular toric variety \( M \) of dimension \( n \) with \( c_1(M) \) divisible by \( n \) can be written in the form

\[ M = \mathbb{P}(\bigoplus_{i=1}^{n} \xi^{\bar{k}_i}) \text{ with } \sum_{i} \bar{k}_i = -2. \]

The expression in [5] is given in this form.

7. Appendix; orbifold elliptic genus

In this section we recall the definitions of elliptic genus and of orbifold elliptic genus of an almost complex closed orbifold. For the latter we follow [2]. These genera can be defined also for stably almost complex orbifolds. For the sake of simplicity we shall confine ourselves to almost complex orbifolds. We also show that the genera \( \varphi(\Delta(M), \mathcal{V}) \) and \( \hat{\varphi}(\Delta(M), \mathcal{V}) \) of the pair of multi-fan \( \Delta(M) \) and set of vectors \( \mathcal{V} \) associated with an almost complex torus orbifold \( M \) as was defined in Section 3 coincide with those of \( M \) in the sense of this section.

We first explain some facts about orbifolds needed to define orbifold elliptic genus. We refer to [6] for basic definitions and notations used here. Specifically \((V_x, U_x, H_x, p_x)\) will denote a reduced orbifold chart centered at \( x \in M \), where \( H_x \) is the isotropy group at \( x \) and \( p_x^{-1}(x) \) is a single point in \( V_x \). We always assume that the manifold \( V_x \) is connected.
and small enough so that the fixed point set of $h$, denoted by $V^h_x$, is also connected for each $h \in H_x$. $p_x$ induces a homeomorphism $V_x/H_x \to U_x$. When $p_x$ has an obvious meaning in the context we shall omit $p_x$ and simply write $(V_x, U_x, H_x)$ for orbifold chart.

Let $M$ be a connected closed orbifold. Kawasaki [10] defined an orbifold $\hat{M}$ and an orbifold map $\pi : \hat{M} \to M$ which plays an important role in the index theorem of the Atiyah-Singer type for orbifolds. Some authors call each connected component of $\hat{M}$ by the name sector of $M$. We shall call $\hat{M}$ the total sector of $M$. As a set $\hat{M}$ is defined by

$$\hat{M} = \bigcup_{x \in M} \text{Conj}(H_x)$$

where $H_x$ is the isotropy group at $x \in M$ and $\text{Conj}(H_x)$ denotes the set of conjugacy classes of $H_x$. The projection $\pi$ is the obvious one. The orbifold structure of $\hat{M}$ is given as follows. Let $(V_x, U_x, H_x, p_x)$ be a reduced orbifold chart of $M$ centered at $x$. We put

$$\hat{V}_x = \{(v, h) \in V_x \times H_x \mid hv = v\}.$$ 

$H_x$ acts on $\hat{V}_x$ by

$$h_1(v, h) = (h_1 v, h_1 h^{-1}_h), \quad h_1 \in H_x.$$ 

Set $\hat{U}_x = \hat{V}_x/H_x$. If $(v, h) \in \hat{V}_x$ and $y = p_x(v) \in U_x$, then the isotropy group $H_y$ is contained in $H_x$ and $h$ is contained in $H_y$ since we assumed that $V_x$ is small enough. Therefore we get a map $\hat{V}_x \to \hat{M}$ sending $(v, h)$ to $[h] \in \text{Conj}(H_y)$, the conjugacy class of $h$ in $\text{Conj}(H_y)$. This map factors through $\hat{U}_x = \hat{V}_x/H_x \to \hat{M}$ which is injective. This defines an orbifold chart of $\hat{M}$. Note that $\hat{U}_x$ is not always connected. In fact $\hat{U}_x = \bigcup_{[h] \in \text{Conj}(H_x)} V_x^h/C(h)$ is the decomposition into connected components, where $C(h)$ is the centralizer of $h$ in $H_x$. In particular $(\hat{V}_x, V_x^h/C(h), C(h))$ is a reduced orbifold chart centered at $[h] \in \text{Conj}(H_x) \subset \hat{M}$. The projection $\pi : \hat{M} \to M$ is locally injective and is an orbifold map as follows from the above definition. Each connected component $X$ of $\hat{M}$ is an orbifold. If $[h]$ is in the principal stratum of $X$, then the multiplicity $d(X)$ of $X$ is equal to the order $|C(h)|$ of $C(h)$.

The index theorem of Kawasaki [11] expresses the index of the Dirac operator twisted by a vector bundle as an integral of certain characteristic form over $\hat{M}$ where we assume $M$ to be an almost complex orbifold. If a compact connected group $G$ acts on $M$ preserving the almost complex structure then the index is defined as a virtual character of the group $G$, and its value at $g \in G$ is expressed by Vergne’s fixed point formula [16]. Note that, if $G$ is connected and $(V_x, U_x, H_x)$ is a reduced orbifold chart centered at $x \in M^G$ such that $U_x$ is $G$-invariant, then it can be shown that some finite covering group $\hat{G}$ of $G$ acts on $V_x$ in such a way that the action commutes with the action of $H_x$ and covers the action of $G$ on $U_x$. From this fact it follows that, if $G$ is a compact connected group, then the fixed point set $M^G$ is a suborbifold of $M$.

For the sake of simplicity we shall explain the fixed point formula only for torus actions. Let $M$ be an almost complex closed orbifold and $\xi$ an orbifold complex vector bundle over $M$, both acted on by a torus $T$ topologically generated by $g \in T$. The fixed point set $M^T$ is an almost complex orbifold. Let $\hat{M}^T$ be the total sector of $M^T$ with projection $\pi : \hat{M}^T \to M^T$. We put $\hat{\xi} = \pi^*(\xi)$. Let $(V_x, U_x, H_x)$ be a reduced orbifold chart of the orbifold $M$ centered at $x \in \hat{M}^T$ having properties as explained above. We may suppose that the topological generator $g$ acts on $V_x$. Then it can be shown that its action
commutes with that of $H_x$, and $(V^g_x, U^g_x, H_x)$ is a reduced orbifold chart of the orbifold $M^T$ centered at $x$. We put

$$\hat{V}_x^g = \{(v, h) \in V_x \times H_x \mid gv = v, hv = v\}.$$ 

$H_x$ acts on $\hat{V}_x^g$ and $\hat{U}_x^g = \hat{V}_x^g / H_x$ maps homeomorphically onto an open subset of $\hat{M}^T$ as before. More precisely it can be shown that $\hat{U}_x^g = \bigcup_{[h] \in \text{Conj}(H_x)} (V^g_x)^h / C(h)$ is the decomposition into connected components and $((V^g_x)^h, (V^g_x)^h / C(h), C(h))$ gives a reduced orbifold chart of $\hat{M}^T$ centered at $[h]$.

Let $F$ be a component of $M^T$ and $\hat{F}$ its total sector. We denote by $N_F$ the normal bundle of $F$ in $M$ and put $N_{\hat{F}} = \pi^*(N_F)$. Let $Y$ be a connected component of $\hat{F}$ and $N_Y$ the normal bundle of the immersion $\pi|Y : Y \to F$. Suppose that $[h] \in \text{Conj}(H_x)$ lies in $Y$. The action of $g$ and $h$ on $V_x$ decomposes the tangent bundle $TV_x$ restricted to $(V^g_x)^h$ into a sum of eigen-bundles:

$$TV_x|(V^g_x)^h = T((V^g_x)^h) \oplus \bigoplus_{\lambda} W_{\lambda} \oplus \bigoplus_{\mu, \lambda} W_{\mu, \lambda},$$

where $h$ acts on the fibers of $W_{\lambda}$ and $W_{\mu, \lambda}$ by multiplication by $\lambda$, and $g$ acts trivially on $W_{\lambda}$ and acts on the fibers of $W_{\mu, \lambda}$ by the multiplication by $\mu$. Since the action of $C(h)$ commutes with those of $g$ and $h$, we get orbifold vector bundles $E_{\lambda} = W_{\lambda} / C(h)$ and $E_{\mu, \lambda} = W_{\mu, \lambda} / C(h)$ over $(V^g_x)^h / C(h)$. Thus, locally, we get the decomposition into a sum of eigen-bundles:

$$N_Y = \bigoplus E_{\lambda},$$

$$N_{\hat{F}}|Y = \bigoplus E_{\mu, \lambda}.$$ 

This decomposition is valid throughout the sector $Y$ since the eigenvalues $\lambda$ and $\mu$ are functions of conjugacy classes and it is constant on the connected space $Y$. We also have the decomposition $\hat{\xi}|Y = \bigoplus \xi_{\mu, \lambda}$ in a similar way. For an orbifold complex vector bundle $E$ let $\Omega(E)$ denote the curvature form of a connection on $E$ associated with invariant hermitian metrics on the tangent bundle of the base space and $E$. We put $\Gamma(E) = \frac{1}{2\pi} \Omega(E)$ and

$$\mathcal{T}(Y) = \det \left( \frac{\Gamma(TY)}{1 - e^{-\Gamma(TY)}} \right),$$

$$\text{ch}_g(\hat{\xi}|Y) = \sum_{\mu, \lambda} \mu \lambda \text{tr} e^{\Gamma(\xi_{\mu, \lambda})},$$

$$\mathcal{D}(N_Y) = \prod_{\lambda} \text{det}(1 - \lambda^{-1} e^{-\Gamma(E_{\lambda})}),$$

$$\mathcal{D}(N_{\hat{F}}|Y) = \prod_{\mu, \lambda} \text{det}(1 - (\mu \lambda)^{-1} e^{-\Gamma(E_{\mu, \lambda})}).$$

Let $\partial$ denote the Dirac operator of $M$. Then Vergne’s fixed point formula reads as follows:

$$\text{(25)} \quad \text{ind}(\partial \otimes \xi)_g = \sum_{F} \sum_{Y} \frac{1}{d(Y)} \int_{Y} \frac{\mathcal{T}(Y) \text{ch}_g(\hat{\xi}|Y)}{\mathcal{D}(N_Y) \mathcal{D}(N_{\hat{F}}|Y)},$$
where the left hand side is the value of \( \text{ind}(\partial \otimes \xi) \) regarded as a virtual character of \( T \) at \( g \), and the sums at the right hand side extend over all components \( F \) of \( M^T \) and all components \( Y \) of \( \hat{F} \).

If \( M \) is an almost complex orbifold, the elliptic genus \( \varphi(M) \) of \( M \) is defined as the index of the Dirac operator \( \partial \) twisted by the vector bundle

\[
\Lambda_{-\zeta} T^* M \otimes \bigotimes_{k \geq 1} \left( \Lambda_{-\zeta q^k} T^* M \otimes \Lambda_{-\zeta^{q^k-1}} T M \otimes S_{q^k} T^* M \otimes S_{q^k} T M \right),
\]

where \( T M \) and \( T^* M \) are the tangent bundle and cotangent bundle of \( M \) respectively, and \( \Lambda_y \) and \( S_y \) denote total exterior power and symmetric power respectively.

We next proceed to define the orbifold elliptic genus \( \hat{\varphi}(M) \). Let \( X \) be a component of \( \hat{M} \). It is also an almost complex orbifold. Let \([h] \in \text{Conj}(H_x) \subset X \) be a point in \( X \). Let \( N_X \) denote the orbifold normal bundle of the immersion \( \pi|_X : X \to M \). As before we have the decomposition of \( N_X \) into a sum of eigen-bundles with respect to the action of \( h \):

\[
N_X = \bigoplus_{\lambda} E_{\lambda},
\]

where \( h \) acts on the fibers of \( E_{\lambda} \) by multiplication by \( \lambda = e^{2\pi \sqrt{-1} f_{X,\lambda}} \neq 1 \). We make the convention that \( 0 < f_{X,\lambda} < 1 \). We set \( f_X = \sum_{\lambda} (\text{rank} E_{\lambda}) f_{X,\lambda} \), and

\[
\mathcal{E}_X = \bigotimes_{\lambda \neq 1} \bigotimes_{k = 1}^{\infty} \left( \Lambda_{-\zeta q^k f_X,\lambda q^k} E_{\lambda}^* \otimes \Lambda_{-\zeta^{q^k f_X,\lambda q^k}} E_{\lambda} \otimes S_{q^k f_X,\lambda q^k} E_{\lambda}^* \otimes S_{q^k f_X,\lambda q^k} E_{\lambda} \right).
\]

We now define

\[
\hat{\varphi}(M) = \zeta^{-\frac{\dim}{2}} \sum_{X \subset \hat{M}} \zeta^{f_X} \text{ind} \left( \partial \otimes \Lambda_{-\zeta} T^* X \otimes \bigotimes_{k = 1}^{\infty} \left( \Lambda_{-\zeta q^k f_X} T^* X \otimes \Lambda_{-\zeta^{q^k f_X}} T X \otimes S_{q^k f_X} T^* X \otimes S_{q^k f_X} T X \right) \otimes \mathcal{E}_X \right).
\]

Note that, if we write \( T X = E_1 \) and \( f_{X,1} = 0 \), then we can also write as follows:

\[
\hat{\varphi}(M) = \zeta^{-\frac{\dim}{2}} \sum_{X \subset \hat{M}} \zeta^{f_X} \text{ind} \left( \partial \otimes \Lambda_{-\zeta} T^* X \otimes \bigotimes_{\lambda} \bigotimes_{k = 1}^{\infty} \left( \Lambda_{-\zeta q^k f_X,\lambda q^k} E_{\lambda}^* \otimes \Lambda_{-\zeta^{q^k f_X,\lambda q^k}} E_{\lambda} \otimes S_{q^k f_X,\lambda q^k} E_{\lambda}^* \otimes S_{q^k f_X,\lambda q^k} E_{\lambda} \right) \right).
\]

Suppose that a torus \( T \) acts on the orbifold \( M \). Then \( (26) \) and \( (27) \) can be considered as defining virtual characters of \( T \) which are called equivariant elliptic genus and equivariant orbifold elliptic genus respectively. We shall explicitly write down formulae for the equivariant elliptic genus \( \varphi(M) \) and the equivariant orbifold elliptic genus \( \hat{\varphi}(M) \), assuming that the fixed points are all isolated and the isotropy group at each fixed point is abelian for the sake of simplicity. Write \( M^T = \{ P_j \} \). Let \((V_j, U_j, H_j, p_j)\) be a reduced orbifold chart centered at \( P_j \), and let \( \tilde{x}_j \in V_j \) be the unique point with \( p_j(\tilde{x}_j) = P_j \). Note that the portion of \( M^T \) over \( P_j \) is identified with the set \( \{ P_{j,h} \mid h \in H_j \} \) in this case, where \( P_{j,h} = (\tilde{x}_j, h) \in \tilde{V}_j \).
Let \( g \) be a topological generator of \( T \). The actions of \( g \) and \( H_j \) decompose the tangent space \( T_{\tilde{x}} V_j \) into a sum of one-dimensional eigen-spaces:

\[
T_{\tilde{x}} V_j = \bigoplus_i W_{i,j}
\]

where \( g \) acts on \( W_{i,j} \) by multiplication by \( e^{2\pi \sqrt{-1} m_{i,j} z} \) and \( h \in H_j \) acts by multiplication by \( \chi_{i,j}(h) = e^{2\pi \sqrt{-1} m_{i,j}^H h} \) where \( \chi_{i,j} \) is a character of \( H_j \). We can also view (29) as representing the decomposition of

\[
N_{P_{j,h}} \oplus N_{\pi} P_{j,h}.
\]

Then, under the above situation, we obtain the following formula from (28) and (25).

\[
\varphi(M)_g = \sum_j \frac{1}{|H_j|} \sum_{h \in H_j} \prod_i \phi(-m_{i,j} z - m_{i,j}^H h, \tau, \sigma).
\]

Next let \( X \) be a component of \( \tilde{M} \). We denote by \( J_X \) the set of \( j \) such that \( P_j \in \pi(X) \). Then, for any \( j \in J_X \), there is a unique \( h \in H_j \) such that the point \( P_j \) over \( P_j \) lies in \( X \). This \( h \) will be denoted by \( h_{X,j} \). We see that \( W_{i,j} \) or rather \( W_{i,j}/C(h) \) is tangent to \( \pi(X) \) if and only if \( \chi_{i,j}(h_{X,j}) = 1 \). Moreover, if we write \( \chi_{i,j}(h) = e^{2\pi \sqrt{-1} f_i,j(h)} \) with \( 0 \leq f_i,j(h) < 1 \), then \( f_X = \sum_i f_i,j(h_{X,j}) \). Note that this equality holds for any \( j \in J_X \).

The total sector of the single point \( P_{j,h} \) consists of \( \{ P_{j,h'} \mid h' \in H_j \} \). Then we obtain the following formula from (28) and (25).

\[
\varphi(M)_g = \sum_{X \in \tilde{M}} \zeta^X \sum_{j \in J_X} \sum_{h' \in H_j} \frac{1}{|H_j|} \prod_i \phi(-m_{i,j} z + f_i,j(h_{X,j}) \tau - m_{i,j}^H h', \tau, \sigma).
\]

For \( P_j \) and \( h \in H_j \), we set \( f_{j,h} = \sum_i f_i,j(h) \). The above formula (31) can be transformed in

\[
\varphi(M)_g = \sum_j \sum_{(h_1, h_2) \in H_j \times H_j} \zeta_{f_{j,h_1}} \frac{1}{|H_j|} \prod_i \phi(-m_{i,j} z + f_{i,j}(h_1) \tau - m_{i,j}^H h_2, \tau, \sigma).
\]

In fact, if we set \( \mathcal{A} = \{ (X, j) \mid X \text{ is a component of } \tilde{M}, \ j \in J_X \} \) and \( \mathcal{B} = \bigcup_j H_j \), then there is a bijection \( \rho : \mathcal{A} \rightarrow \mathcal{B} \) which sends \( (X, j) \in \mathcal{A} \) to \( h_{X,j} \in \mathcal{B} \), its inverse being \( H_j \ni h \mapsto (X, j) \in \mathcal{A} \) where \( X \) is the component containing \( P_{j,h} \). Moreover \( f_X = f_{j,h} \) with this correspondence.

Now let \( M \) be an almost complex torus orbifold of dimension \( 2n \) acted on by a torus \( T \). Let \( \Delta(M) = (\Sigma(M), C(M), w(M)^\pm) \) and \( \mathcal{V}(M) = \{ v_i \}_{i \in \Sigma(M)} \) be the multi-fan and the set of generating edge vectors associated with \( M \). \( \Delta(M) \) is a multi-fan in the lattice \( L = \text{Hom}(S^1, T) \). We shall abbreviate the letter \( M \) and simply write \( \Delta \) for \( \Delta(M) \) and so on. The fixed point set \( M^T \) is the union of \( M_I \) with \( I \in \Sigma(n) \).

**Lemma 7.1.** If \( P_j \in M^T \) lies in \( M_I \), then the isotropy group \( H_j \) of \( P_j \) is isomorphic to \( H_I = L/L_I \).

**Proof.** Let \( (V_j, U_j, H_j, p_j) \) be a reduced orbifold chart such that \( U_j \) is invariant under the action of the torus \( T \). The covering group \( \tilde{T}_I = \prod_{i \in I} \tilde{S}_i \) of \( T \) acts effectively on \( V_j \) where \( \tilde{S}_i \) was introduced in Section 2. The isotropy group \( H_j \) is isomorphic to the kernel \( H \) of the natural homomorphism \( \tilde{T}_I \rightarrow T \) as was shown in [6].
On the other hand $L$ and $L_{I',\gamma'}$ are identified with $\pi_1(T)$ and $\pi_1(\tilde{T}_I)$. Therefore the kernel $H$ is isomorphic to $H_I = L/L_{I',\gamma'}$. Hence $H_j$ is isomorphic to $H_I$. $\square$

In the sequel we shall identify $H_j$ with $H_I$ where $P_j \in M_I$. The decomposition (29) of the tangent space $T_{\bar{z}_j}V_j$ into sum of 1-dimensional $\tilde{T}_I$ modules can be written in this case as

\[(33)\quad T_{\bar{z}_j}V_j = \sum_{i \in I} t^{\psi_i^j}.\]

Hence $f_{i,j}(h_1) = f_{I,h_1,i}$ for $h_1 \in H_j = H_I$ where $f_{I,h_1,i}$ is as in Section 3. It follows that $\hat{f}_{i,j}(h_1) = f_{I,h_2}$. Let $v(h_1)$ be the representative of $h_1 \in H_I$ such that $\langle u_i^I, v(h_1) \rangle = f_{I,h_1,i}$. Note also that $w^+(I) = \#\{P_j \in M_I\}$ and $w^-(I) = 0$ in this case since $M$ is an almost complex torus orbifold.

From these observations and from (31) and (32) we obtain

\[(34)\quad \varphi^\psi(M) = \sum_{I \in \Sigma(n)} u(I) \sum_{h \in H_I} \prod_{i \in I} \phi(\langle u_i^I, -zv(v(h)) \rangle, \tau, \sigma),\]

and

\[(35)\quad \hat{\varphi}^\psi(M) = \sum_{I \in \Sigma(n)} u(I) \sum_{(h_1, h_2) \in H_I \times H_I} \zeta^{f_{I,h_1}, h_2} \prod_{i \in I} \phi(\langle u_i^I, -zv(v(h_1)) - v(h_2) \rangle, \tau, \sigma).\]

In order to get the formula for $\hat{\varphi}^\psi(M)$ correspondig to (33), we look for the pair $(X, j) \in \mathcal{A}$ which corresponds via $\rho$ to $P_{j,h}$ with $P_j \in M_I$ and $h \in H_I$. As was remarked just before (31), $t^{\psi_i^j}$ in (33) is tangent to $X$ if and only if $f_{I,h_1,i} = f_{i,j}(h) = 0$. This means that $\pi(X)$ is contained in $M_K$ and is one of its component, where $K = \{i \in I \mid f_{I,h_1,i} \neq 0\}$. Moreover $h$ is contained in $\hat{H}_K$, where $\hat{H}_K$ is defined in Section 3 and is characterized by (3). But (3) is equivalent to

\[\hat{H}_K = \{h \in H_I \mid f_{I,h_1,i} = 0 \text{ for } i \in I \setminus K \text{ and } f_{I,h_1,i} \neq 0 \text{ for } i \in K\}.\]

In particular $f_{K,h} = f_{I,h}$. Thus (35) can be rewritten in the following form.

\[(36)\quad \hat{\varphi}^\psi(M) = \sum_{k=0}^n \sum_{K \in \Sigma(n-k), h_1 \in \hat{H}_K} \zeta^{f_{K,h_1}, h_2} \sum_{I \in \Sigma(n-k)} u(I) \sum_{h_2 \in H_I} \prod_{i \in I \setminus K} \phi(\langle u_i^I, -zv - v(h_2) \rangle, \tau, \sigma) \prod_{i \in K} \phi(\langle u_i^I, -zv + \tau v(v(h_1)) - v(h_2) \rangle, \tau, \sigma)\].

Remark 7.2. So far $M$ is assumed to be an almost complex torus orbifold. Even if $M$ is a stably almost complex torus orbifold the formulae (34), (35) and (36) are valid. We have only to count the points $P_j \in M_I$ with sign and the resulting multiplicity is $w(I)$. The formulae (3) for $\varphi^\psi(\Delta, \gamma')$ and (1) and (2) for $\hat{\varphi}^\psi(\Delta, \gamma')$ given in Section 3 are modelled on (31), (34) and (36) respectively.
Elliptic genera, torus orbifolds and multi-fans

References

[1] L. A. Borisov and A. Libgober, Elliptic genera of toric varieties and applications to mirror symmetry, Invent. math., 140 (2000), 453–485.
[2] ———, Elliptic genera of singular varieties, Duke Math. J., 116 (2003), 319–351.
[3] R. Bott and C. Taubes, On the rigidity theorem of Witten, J. Amer. Math. Soc., 2 (1989), 137–186.
[4] W. Fulton, Introduction to Toric Varieties, Annals Math. Studies, No.131, Princeton UP, 1993.
[5] O. Fujino, Toric varieties whose canonical divisors are divisible by their dimensions, Private Note, 2004.
[6] A. Hattori and M. Masuda, Theory of multi-fans, Osaka J. Math., 40 (2003), 1–68.
[7] F. Hirzebruch, Elliptic genera of level N for complex manifolds, Differential Geometrical Methods in Theoretical Physics, Kluwer, 1988, pp. 37–63; also reproduced with corrections and improvements in [8].
[8] F. Hirzebruch, T. Berger and R. Jung, Manifolds and Modular Forms, Aspects of Mathematics, vol. E20, Vieweg, 1992.
[9] F. Hirzebruch and P. Slodowy, Elliptic genera, involutions, and homogeneous spin manifolds, Geometriae Dedicata, 35 (1990), 309–343.
[10] T. Kawasaki, The signature theorem for V-manifolds, Topology, 17 (1978), 75–83.
[11] ———, The Riemann-Roch theorem for complex V-manifolds, Osaka J. Math., 16 (1979), 151–159.
[12] K. Liu, On elliptic genera and theta-functions, Topology, 35 (1996), 617–640.
[13] M. Masuda, Unitary toric manifolds, multi-fans and equivariant index, Tohoku Math. J., 51 (1999), 237–265.
[14] T. Oda, Convex Bodies and Algebraic Geometry, Springer Verlag, (1988).
[15] C. Taubes, $S^1$ actions and elliptic genera, Comm. Math. Phys., 122 (1989), 455-526.
[16] M. Vergne, Equivariant index formula for orbifolds, Duke Math. J., 82 (1996), 637-652.
[17] E. Witten, The index of the Dirac operator in loop space, Elliptic Curves and Modular Forms in Algebraic Geometry, Lecture Notes in Math., Springer, 1988, pp.161–181.

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