Continuity with respect to parameters of the solutions of time–delayed BSDEs with Stieltjes integral

Luca Di Persio\(^a\), Lucian Maticiuc\(^b\), Adrian Zălinescu\(^c,d\)

\(^a\) Department of Computer Science, University of Verona, Strada le Grazie, no. 15, Verona, 37134, Italy
\(^b\) Faculty of Mathematics, “Alexandru Ioan Cuza” University, Carol I Blvd., no. 11, Iaşi, 700506, Romania
\(^c\) Faculty of Computer Science, “Alexandru Ioan Cuza” University, Carol I Blvd., no. 11, Iaşi, 700506, Romania
\(^d\) “Octav Mayer ” Mathematics Institute of the Romanian Academy, Carol I Blvd., no. 8, Iaşi, 700506, Romania

Abstract
We prove the existence and uniqueness of the solution of a BSDE with time-delayed generator, which employs the Stieltjes integral with respect to an increasing continuous stochastic process. We obtain also a result of continuity of the solution with regard to the increasing process, assuming only uniform convergence, but not in variation.

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1 Introduction
Backward stochastic differential equations (BSDEs for short) were introduced in the linear case by Bismut [1], as adjoint equations involved in the control of SDEs. The nonlinear case was considered by Pardoux and Peng first in [13] and then in [14, 19], where they established a connection between BSDEs and semilinear parabolic partial differential equations (PDEs), by the so-called nonlinear Feynman–Kac formula. It was this kind of applications which triggered an impressive amount of research on the subject. Concerning parabolic PDEs with Neumann boundary conditions, Pardoux and Zhang discovered that their solutions can be linked to BSDEs involving the integral with respect to continuous increasing processes (Stieltjes integral).

This paper represents a first step in establishing a probabilistic representation formula of the solutions of delayed path-dependent parabolic PDEs with Neumann boundary conditions. It consists in studying the well posedness of the associated BSDEs, \textit{i.e.} existence and
uniqueness of solutions, as well as stability with respect to terminal data and coefficients. As already shown in [2] for the case of such PDEs considered on the whole space, the generator of the associated BSDE has to take into account the delayed-path of its solution. As a result, our present work is concerned with the following BSDE:

\[
\begin{align*}
\begin{cases}
  dY(t) &= -F(t, Y(t), Z(t), Y_t, Z_t)dt - G(t, Y(t), Y_t)dA(t) \\
  + Z(t)dW(t), & t \in [0, T]; \\
  Y(T) &= \xi,
\end{cases}
\end{align*}
\]

(1)

where the generators \(F\) and \(G\) depend also on the past of the solution \((Y, Z)\). Here, if \(x : [-\delta, T] \to \mathbb{R}^n\) is a function and \(t \in [0, T]\), \(x_t : [-\delta, 0] \to \mathbb{R}^n\) denotes the delayed-path of \(x\), defined as

\[x_t(\theta) := x(t + \theta), \quad \theta \in [-\delta, 0],\]

where \(\delta > 0\) is a fixed delay. The coefficient \(A\) is a continuous real valued increasing process.

We recall that time-delayed BSDEs were first introduced in [5] and [6] where the authors obtained the existence and uniqueness of the solution of the time–delayed BSDE

\[Y(t) = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z(s)dW(s), \quad 0 \leq t \leq T,\]

(2)

where

\[Y_s := (Y(r))_{r \in [0, s]} \quad \text{and} \quad Z_s := (Z(r))_{r \in [0, s]}.
\]

In particular, the aforementioned existence and uniqueness result holds true if the time horizon \(T\) or the Lipschitz constant for the generator \(f\) are sufficiently small.

This paper is organized as follows. In the remaining of this section, we introduce the notations and set the framework of our problem. In section 2 we derive a result of existence and uniqueness for BSDE (1), based on Banach’s fixed point theorem. Section 3 is devoted to the problem of stability of solutions with respect to terminal data \(Y\) and coefficients \(F, G\) and \(A\). The main difficulty encountered here is to prove the convergence of the solutions of the approximating BSDEs when the increasing process \(A\) is approximated uniformly, but not in variation. In order to tackle this problem, we use a stochastic variant of Helly-Bray theorem, proved in the Appendix section, as it may be an interesting result for use in other applications.

1.1 Problem setting and notations

On the Euclidean space \(\mathbb{R}^n\) we consider the Euclidean norm and scalar product, denoted by \(|\cdot|\) and \(\langle \cdot, \cdot \rangle\), respectively. If \(n, k \in \mathbb{N}^*, \mathbb{R}^{n \times k}\) denotes the space of real \(n \times k\)-matrices, equipped with the Frobenius norm (the Euclidean norm when this space is identified with \(\mathbb{R}^{nk}\), denoted as well by \(|\cdot|\).

For \(s < t\), \(C([s, t]; \mathbb{R}^n)\) represents the set of continuous functions \(x : [s, t] \to \mathbb{R}^d\), endowed with the sup-norm: \(\|x\|_{C([s, t]; \mathbb{R}^n)} := \sup_{r \in [s, t]} |x(r)|\); \(BV([s, t]; \mathbb{R}^n)\) denotes the set of right-continuous functions with bounded variation \(\eta : [s, t] \to \mathbb{R}^n\), i.e. with finite total variation. Recall that the total variation of \(\eta\) on \([s, t]\) is defined as

\[V_s^t(\eta) := \sup \sum_{i=1}^n |\eta(t_i) - \eta(t_{i-1})|,\]
where the sup is taken on all the partitions \( s = t_0 < t_1 < \cdots < t_n = t \). The standard norm on \( BV([s, t]; \mathbb{R}^n) \) is given by

\[
\| \eta \|_{BV([s, t]; \mathbb{R}^n)} := |\eta(s)| + V'_s(\eta).
\]

We will simply denote \( C[s, t], BV[s, t] \) instead of \( C([s, t]; \mathbb{R}), BV([s, t]; \mathbb{R}) \), respectively.

If \( x: [s, t] \to \mathbb{R}^n \) is a Borel-measurable function and \( \eta \in BV([s, t]; \mathbb{R}^n) \), by \( \int_s^t \langle x(r) \rangle d\eta(r) \) we denote the sum

\[
\sum_{i=1}^n \int_s^t (x_i(r)) d\eta_i(r),
\]

where \( x_1, \ldots, x_n \) and \( \eta_1, \ldots, \eta_n \) are the components of \( x \), respectively \( \eta \), in the case where the Lebesgue-Stieltjes integrals are well-defined and the sum makes sense.

We fix now the framework of our problem, to be utilized throughout the article.

Let \( T > 0 \) be a finite horizon of time, \( d, m \in \mathbb{N}^* \) and \( \delta \in (0, T] \) a fixed time-delay. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space, \( W \) a \( d \)-dimensional Brownian motion and \( \mathbb{F} = \{ \mathcal{F}_t \}_{t \in [0, T]} \) the filtration generated by \( W \), augmented by the null-probability subsets of \( \Omega \). The stochastic process \( A : \Omega \times [0, T] \to \mathbb{R}^d \) is an increasing \( \mathbb{F} \)-adapted process with \( A_0 = 0 \), \( \mathbb{P} \)-a.s.

**Definition 1** Let \( p \geq 2 \) and \( \beta \geq 0 \).

(i) \( S_{p, m} \) denotes the space of continuous \( \mathbb{F} \)-progressively measurable processes \( Y : \Omega \times [0, T] \to \mathbb{R}^m \) such that

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |Y(s)|^p \right] < +\infty.
\]

(ii) \( S_{p, m}^\beta \) denotes the space of continuous \( \mathbb{F} \)-progressively measurable processes \( Y : \Omega \times [0, T] \to \mathbb{R}^m \) such that

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} e^{\beta A(s)} |Y(s)|^p \right] + \mathbb{E} \left[ \int_0^T e^{\beta A(s)} |Y(s)|^2 dA(s) \right]^{p/2} < +\infty.
\]

(iii) \( \mathcal{H}_{p, m \times d}^\beta \) denotes the space of \( \mathbb{F} \)-progressively measurable processes \( Z : \Omega \times [0, T] \to \mathbb{R}^{m \times d} \) such that

\[
\mathbb{E} \left[ \int_0^T e^{\beta A(s)} |Z(s)|^2 ds \right]^{p/2} < +\infty.
\]

Instead of \( \mathcal{H}_{0, m \times d}^p \) we will write \( \mathcal{H}_{p, m \times d} \). The space \( S_{p, m}^\beta \times \mathcal{H}_{p, m \times d}^\beta \) (in fact, its quotient with respect to \( \mathbb{P} \times \mathbb{P} dt \)-a.e. equality) is naturally equipped with the following norm

\[
\| (Y, Z) \|_{p, \beta} = \mathbb{E} \left[ \sup_{0 \leq s \leq T} e^{\beta A(s)} |Y(s)|^p \right] + \mathbb{E} \left[ \int_0^T e^{\beta A(s)} |Y(s)|^2 dA(s) \right]^{p/2}
\]

\[
+ \mathbb{E} \left[ \int_0^T e^{\beta A(s)} |Z(s)|^2 ds \right]^{p/2}.
\]
2 Existence and uniqueness

We consider the following BSDE

\[ Y(t) = \xi + \int_t^T F(s, Y(s), Z(s), Y_s, Z_s) ds + \int_t^T G(s, Y(s), Y_s) dA(s) \]
\[ - \int_t^T Z(s) dW(s), \quad t \in [0,T], \] (3)

with \( \xi \in L^2(\Omega, F_T, \mathbb{P}; \mathbb{R}^m) \) and the generators \( F : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times L^2([-\delta,0]; \mathbb{R}^m) \times L^2([-\delta,0]; \mathbb{R}^{m\times d}) \to \mathbb{R}^m \), \( G : \Omega \times [0,T] \times \mathbb{R}^m \times L^2([-\delta,0]; \mathbb{R}^m) \to \mathbb{R}^m \) such that the functions \( F(\cdot,\cdot,\cdot,\cdot,\cdot) \) and \( G(\cdot,\cdot,\cdot,\cdot) \) are \( \mathbb{F} \)-progressively measurable, for any \((y, z, \hat{y}, \hat{z}) \in \mathbb{R}^m \times \mathbb{R}^{m\times d} \times L^2([-\delta,0]; \mathbb{R}^m) \times L^2([-\delta,0]; \mathbb{R}^{m\times d})\), respectively for any \((y, \hat{y}) \in \mathbb{R}^m \times \mathbb{R}^m \).

Recall that, for a function \( x : [-\delta,T] \to \mathbb{R}^n \) and some \( t \in [0,T] \), \( x_t : [-\delta,0] \to \mathbb{R}^n \) denotes the delayed-path of \( x \), defined as

\[ x_t(\theta) := x(t + \theta), \quad \theta \in [-\delta,0]. \]

In order to define \( Y_s \) and \( Z_s \) even for \( s < \delta \), we prolong by convention, \( Y \) by \( Y(0) \) and \( Z \) by \( 0 \) on the negative real axis.

In what follows we present the assumptions required in this section. We suppose that there exist constants \( \beta, L, \tilde{L} > 0 \), bounded progressively measurable stochastic processes \( K, \tilde{K} : \Omega \times [0,T] \to \mathbb{R}_+ \) and \( \rho, \tilde{\rho} \) probability measures on \([[-\delta,0], B([-\delta,0])]\) such that:

\[ (A_0) \quad \mathbb{E}\left[ e^{\beta A(T)} \left( 1 + |\xi|^2 \right) \right] < +\infty; \]
\[ (A_1) \quad \mathbb{E}\left[ \int_0^T e^{\beta A(t)} |F(t,0,0,0,0)|^2 dt + \int_0^T e^{\beta A(t)} |G(t,0,0)|^2 dA(t) \right] < +\infty. \]

\( (A_2) \) for any \( t \in [0,T] \), \((y, z, \hat{y}, \hat{z}) \in \mathbb{R}^m \times \mathbb{R}^{m\times d}, \hat{y}, \hat{y}' \in L^2([-\delta,0]; \mathbb{R}^m) \) and \( \hat{z}, \hat{z}' \in L^2([-\delta,0]; \mathbb{R}^{m\times d}) \), we have

\[
\begin{align*}
(i) & \quad |F(t, y, z, \hat{y}, \hat{z}) - F(t, y', z', \hat{y}, \hat{z})| \leq L(|y - y'| + |z - z'|), \mathbb{P}\text{-a.s.}; \\
(ii) & \quad |F(t, y, z, \hat{y}, \hat{z}) - F(t, y, z, \hat{y}', \hat{z}')|^2 \\
& \quad \leq K(t) \int_{-\delta}^0 \left( |\hat{y}(\theta) - \hat{y}'(\theta)|^2 + |\hat{z}(\theta) - \hat{z}'(\theta)|^2 \right) \rho(\theta) d\theta, \mathbb{P}\text{-a.s.};
\end{align*}
\]

\( (A_3) \) for any \( t \in [0,T] \), \( y, y' \in \mathbb{R}^m \) and \( \hat{y}, \hat{y}' \in L^2([-\delta,0]; \mathbb{R}^m) \), we have

\[
\begin{align*}
(i) & \quad |G(t, y, \hat{y}) - G(t, y', \hat{y})| \leq \tilde{L}|y - y'|, \mathbb{P}\text{-a.s.}; \\
(ii) & \quad |G(t, y, \hat{y}) - G(t, y, \hat{y}')|^2 \leq \tilde{K}(t) \int_{-\delta}^0 |\hat{y}(\theta) - \hat{y}'(\theta)|^2 \tilde{\rho}(\theta) d\theta, \mathbb{P}\text{-a.s.};
\end{align*}
\]

Remark 2 There is a difference between these conditions and those from [5], in that we allow \( T \) to be arbitrary, but different from the delay \( \delta \in [0,T] \). This is done in order to separate the Lipschitz constant \( L \) with respect to \((y, z)\) from the Lipschitz constant \( K \) with respect to \((\hat{y}, \hat{z})\); the restriction on the coefficients can avoid in this way the constant \( L \).
Remark 3 In order to show the existence and uniqueness of a solution to the backward system (3), we will use a standard Banach’s fixed point argument. For that we are obliged to impose that $K$ or $\delta$ should be small enough.

More precisely, by denoting $K_1 := \sup_{s \in [0, T]} K(s)$, $\tilde{K}_1 := \sup_{s \in [0, T]} \tilde{K}(s)$ and

$$\omega_\delta := \sup_{t \in [0, T-\delta]} (A(t + \delta) - A(t)),$$

we will assume that there exists a positive constant $c < c_{\beta, L} := \min \left\{ \frac{\beta^2 - \delta L^2}{4L^2}, \frac{1}{\delta^2} \right\}$ such that

(H$_1$) $K_1 \cdot \max \{1, T\} \cdot \frac{\left( (sL^2 + \frac{1}{L})^{1+\beta}\omega_\delta \right)}{4L^2} \leq c, \quad \mathbb{P}\text{-a.s.}$

(H$_2$) $4\tilde{K}_1 \cdot A(T) \cdot \frac{\left( (sL^2 + \frac{1}{L})^{1+\omega_\delta} \right)}{\beta} \leq c, \quad \mathbb{P}\text{-a.s.}$

Our first result states existence and uniqueness of equation (3).

Theorem 4 Let us assume that (A$_0$)-(A$_3$) hold true and $\beta > 2\sqrt{2L}$. If conditions (H$_1$) and (H$_2$) are satisfied then there exists a unique solution $(Y, Z) \in S_{\beta}^{2,m} \times \mathcal{H}_{\beta}^{2,m \times d}$ for (3).

Proof. The existence and uniqueness will be obtained by the Banach fixed point theorem.

Let us consider the map $\Gamma : S_{\beta}^{2,m} \times \mathcal{H}_{\beta}^{2,m \times d} \to S_{\beta}^{2,m} \times \mathcal{H}_{\beta}^{2,m \times d}$, defined in the following way: for $(U, V) \in S_{\beta}^{2,m} \times \mathcal{H}_{\beta}^{2,m \times d}$, $\Gamma (U, V) = (Y, Z)$, where the couple of adapted processes $(Y, Z)$ is the solution to the equation

$$Y(t) = \xi + \int_t^T F(s, Y(s), Z(s), U_s, V_s)ds + \int_t^T G(s, U(s), U_s)dA(s)$$

$$- \int_t^T Z(s)dW(s), \quad t \in [0, T]. \quad (4)$$

The existence of a unique solution $(Y, Z) \in S^{2,m} \times \mathcal{H}^{2,m \times d}$ is guaranteed by [13]. Indeed, if we denote

$$B(t) := \int_0^t G(s, U(s), U_s)dA(s), \quad t \in [0, T];$$

$$\hat{F}(t, y, z) := F(t, y - B(t), z, U_t, V_t), \quad t \in [0, T], \quad (y, z) \in \mathbb{R}^m \times \mathbb{R}^{m \times d},$$

then $(Y, Z)$ is a solution to equation (3) if and only if $(Y + B, Z)$ solves the equation

$$\hat{Y}(t) = \xi + B(T) + \int_t^T \hat{F}(s, \hat{Y}(s), Z(s), U_s, V_s)ds - \int_t^T Z(s)dW(s), \quad t \in [0, T].$$

Since $\hat{F}$ is Lipschitz with respect to $(y, z)$, it remains to prove that $\mathbb{E} \int_0^T |\hat{F}(t, 0, 0)|^2dt < +\infty$ and $\xi + B(T) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^m)$. We have (remember that $K_1 := \sup_{s \in [0, T]} K(s)$ and $\tilde{K}_1 := \sup_{s \in [0, T]} \tilde{K}(s)$)
\[ \sup_{s \in [0,T]} K(s) : \]
\[ \mathbb{E} \int_0^T |\hat{F}(t,0,0)|^2 dt = \mathbb{E} \int_0^T |F(t,-B(t),0,U_t,V_t)|^2 dt \leq 3\mathbb{E} \int_0^T |F(t,0,0,0)|^2 dt + 3L^2 \mathbb{E} \int_0^T |B(t)|^2 dt + 3\mathbb{E} \int_0^T K(t) \int_{-\delta}^0 \left( |U(t+\theta)|^2 + |V(t+\theta)|^2 \right) \rho(d\theta)dt \]
\[ \leq 3\mathbb{E} \int_0^T |F(t,0,0,0)|^2 dt + 3L^2 \mathbb{E} \int_0^T |B(t)|^2 dt + 3T\mathbb{E} \left[ K_1 \sup_{t \in [0,T]} |U(t)|^2 \right] + 3K_1 \mathbb{E} \int_0^T |V(t)|^2 dt. \]

Since (A_1) holds and K_1 is bounded, we only have to show that \( \mathbb{E} \int_0^T |B(t)|^2 dt < +\infty \) and \( \mathbb{E} |B(T)|^2 < +\infty \). We have

\[ \mathbb{E} \int_0^T \left[ \int_0^t G(s,U(s),U_s)dA(s) \right]^2 dt \]
\[ \leq \mathbb{E} \int_0^T \left[ \int_0^t \int_0^s e^{\beta A(s)} |G(s,U(s),U_s)|^2 dA(s) \cdot \int_0^t e^{-\beta A(s)} dA(s) \right] dt \]
\[ \leq \frac{T}{\beta} \mathbb{E} \int_0^T e^{\beta A(t)} |G(t,U(t),U_t)|^2 dA(t) \leq \frac{2T}{\beta} \mathbb{E} \int_0^T e^{\beta A(t)} |G(t,0,0)|^2 dA(t) \]
\[ + \frac{2T}{\beta} \mathbb{E} \int_0^T e^{\beta A(t)} L^2 |U(t)|^2 dA(t) + \frac{2T}{\beta} \mathbb{E} \int_0^T e^{\beta A(t)} \tilde{K}(t) \int_{-\delta}^0 |U(t+\theta)|^2 \bar{\rho}(d\theta) dA(t) \]
\[ \leq \frac{2T}{\beta} \mathbb{E} \int_0^T e^{\beta A(t)} |G(t,0,0)|^2 dA(t) + \frac{2T L^2}{\beta} \mathbb{E} \int_0^T e^{\beta A(t)} |U(t)|^2 dA(t) \]
\[ + \frac{2T}{\beta} \mathbb{E} \tilde{K}_1 A(T)e^{\beta \omega s} \sup_{t \in [0,T]} e^{\beta A(t)} |U(t)|^2 < +\infty, \]

by (A_1) and (H_2), which proves the claim (along the way we have also proven that \( \mathbb{E} |B(T)|^2 < +\infty \)).

The proof that \((Y,Z) \in S^{2,m}_\beta \times H^{2,m \times d}_\beta\) is very similar to that of Proposition 1.1 from [17], so it is left to the reader.

Let us prove that \( \Gamma \) is a contraction with respect to the equivalent norm

\[ \|(Y,Z)\|_{2,\alpha,\beta,a,b}^2 := \mathbb{E} \left( \sup_{t \in [0,T]} e^{\alpha t + \beta A(t)} |Y(t)|^2 \right) + a\mathbb{E} \int_0^T e^{\alpha s + \beta A(s)} |Y(s)|^2 dA(s) \]
\[ + b\mathbb{E} \int_0^T e^{\alpha s + \beta A(s)} |Z(s)|^2 ds. \]

where \( \alpha := 8L^2 + \frac{1}{2} \) and the constants \( a, b > 0 \) are yet to be chosen.
Let us consider \((U^1, V^1), (U^2, V^2) \in S^2_{\beta} \times H^2_{\beta}\) and \((Y^1, Z^1) := \Gamma (U^1, V^1), (Y^2, Z^2) := \Gamma (U^2, V^2)\). For the sake of brevity, we will denote in what follows
\[
\Delta F (s) := F(s, Y^1 (s), Z^1 (s), U^1_s, V^1_s) - F(s, Y^2 (s), Z^2 (s), U^2_s, V^2_s),
\]
\[
\Delta G (s) := G(s, U^1 (s), U^1_s) - G(s, U^2 (s), U^2_s),
\]
\[
\Delta U (s) := U^1 (s) - U^2 (s), \quad \Delta V (s) := V^1 (s) - V^2 (s),
\]
\[
\Delta Y (s) := Y^1 (s) - Y^2 (s), \quad \Delta Z (s) := Z^1 (s) - Z^2 (s).
\]
Exploiting Itô’s formula we have, for any \(t \in [0, T]\)
\[
e^{\alpha t + \beta A(t)} \left| \Delta Y (t) \right|^2 + \int_t^T e^{\alpha s + \beta A(s)} |\Delta Y (s)|^2 (\alpha ds + \beta dA (s)) + \int_t^T e^{\alpha s + \beta A(s)} |\Delta Z (s)|^2 ds
\]
\[
e^{\alpha T + \beta A(T)} \left| \Delta Y (T) \right|^2 - 2 \int_t^T e^{\alpha s + \beta A(s)} \langle \Delta Y (s), \Delta Z (s) \rangle dW (s)
\]
\[
+ 2 \int_t^T e^{\alpha s + \beta A(s)} \langle \Delta Y (s), \Delta F (s) \rangle ds + 2 \int_t^T e^{\alpha s + \beta A(s)} \langle \Delta Y (s), \Delta G (s) \rangle dA (s).
\]
From assumptions \((A_2)-(A_3)\) we obtain,
\[
2 \left\| \int_t^T e^{\alpha s + \beta A(s)} \langle \Delta Y (s), \Delta F (s) \rangle ds \right\| \leq 2 \int_t^T e^{\alpha s + \beta A(s)} \left| \langle \Delta Y (s), \Delta F (s) \rangle \right| ds
\]
\[
\leq 8L^2 \int_t^T e^{\alpha s + \beta A(s)} |\Delta Y (s)|^2 ds + \frac{1}{8L^2} \int_t^T e^{\alpha s + \beta A(s)} |\Delta F (s)|^2 ds
\]
\[
\leq 8L^2 \int_t^T e^{\alpha s + \beta A(s)} |\Delta Y (s)|^2 ds + \frac{1}{2} \int_t^T e^{\alpha s + \beta A(s)} (|\Delta Y (s)|^2 + |\Delta Z (s)|^2) ds
\]
\[
+ \frac{K_1 T}{4L^2} e^{|\alpha s + \beta A| \cdot \sup_{s \in [0,T]} (e^{\alpha s + \beta A(s)} |\Delta U (s)|^2)}
\]
\[
+ \frac{K_1}{4L^2} e^{|\alpha s + \beta A| \cdot \int_0^T e^{\alpha s + \beta A(s)} |\Delta V (s)|^2 ds}
\]
and
\[
2 \left\| \int_t^T e^{\alpha s + \beta A(s)} \langle \Delta Y (s), \Delta G (s) \rangle dA (s) \right\| \leq 2 \int_t^T e^{\alpha s + \beta A(s)} \left| \langle \Delta Y (s), \Delta G (s) \rangle \right| dA (s)
\]
\[
\leq \frac{\beta}{2} \int_t^T e^{\alpha s + \beta A(s)} |\Delta Y (s)|^2 dA (s) + \frac{2}{\beta} \int_t^T e^{\alpha s + \beta A(s)} |\Delta G (s)|^2 dA (s)
\]
\[
\leq \frac{\beta}{2} \int_t^T e^{\alpha s + \beta A(s)} |\Delta Y (s)|^2 dA (s) + \frac{4L^2}{\beta} \int_t^T e^{\alpha s + \beta A(s)} |\Delta U (s)|^2 dA (s)
\]
\[
+ \frac{4K_1 A(T)}{\beta} e^{|\alpha s + \beta A| \cdot \sup_{s \in [0,T]} (e^{\alpha s + \beta A(s)} |\Delta U (s)|^2)}.
\]
By \((H_1)\) and \((H_2)\), we have
\[
\left( \frac{K_1 T}{4L^2} + \frac{4K_1 A(T)}{\beta} \right) e^{|\alpha s + \beta A|} \leq 2c, \quad \mathbb{P}\text{-a.s.};
\]
\[
\frac{K_1}{4L^2} e^{|\alpha s + \beta A|} \leq c, \quad \mathbb{P}\text{-a.s},
\]
(recall that $\alpha := 8L^2 + \frac{1}{2}$). Therefore,

\[
e^{\alpha t + \beta A(t)} |\Delta Y(t)|^2 + \frac{\beta}{2} \int_t^T e^{\alpha s + \beta A(s)} |\Delta Y(s)|^2 dA(s)
\]

\[
+ \frac{1}{2} \int_t^T e^{\alpha s + \beta A(s)} |\Delta Z(s)|^2 ds
\]

\[
\leq -2 \int_t^T e^{\alpha s + \beta A(s)} (\Delta Y(s), \Delta Z(s) dW(s)) + \frac{4\tilde{L}^2}{\beta} \int_t^T e^{\alpha s + \beta A(s)} |\Delta U(s)|^2 dA(s)
\]

\[
+ 2c\sup_{s \in [0,T]} (e^{\alpha s + \beta A(s)} |\Delta U(s)|^2) + c \int_0^T e^{\alpha s + \beta A(s)} |\Delta V(s)|^2 ds.
\]

Since $e^{\alpha s + \beta A(s)} \Delta Y \in \mathcal{S}^{2,m}$ and $\Delta Z \in \mathcal{H}^{2,m \times d}$, one can show that

\[
E \left[ \int_0^T e^{\alpha s + \beta A(s)} (\Delta Y(s), \Delta Z(s) dW(s)) \right] = 0,
\]

hence

\[
\frac{\beta}{2} E \int_0^T e^{\alpha s + \beta A(s)} |\Delta Y(s)|^2 dA(s) + \frac{1}{2} E \int_0^T e^{\alpha s + \beta A(s)} |\Delta Z(s)|^2 ds
\]

\[
\leq 4\tilde{L}^2 \beta E \int_0^T e^{\alpha s + \beta A(s)} |\Delta U(s)|^2 dA(s) + 2c E \left[ \sup_{s \in [0,T]} (e^{\alpha s + \beta A(s)} |\Delta U(s)|^2) \right]
\]

\[
+ c E \left[ \int_0^T e^{\alpha s + \beta A(s)} |\Delta V(s)|^2 ds \right].
\]

On the other hand, by Burkholder–Davis–Gundy’s inequality, we have

\[
2E \left[ \sup_{t \in [0,T]} \left| \int_t^T e^{\alpha s + \beta A(s)} (\Delta Y(s), \Delta Z(s)) dW(s) \right| \right]
\]

\[
\leq \frac{1}{2} E \left( \sup_{t \in [0,T]} e^{\alpha t + \beta A(t)} |\Delta Y(t)|^2 \right) + 72 \ E \int_0^T e^{\alpha s + \beta A(s)} |\Delta Z(s)|^2 ds.
\]

Hence, by (5),

\[
\frac{1}{2} E \left( \sup_{t \in [0,T]} e^{\alpha t + \beta A(t)} |\Delta Y(t)|^2 \right)
\]

\[
\leq 72 \ E \int_0^T e^{\alpha s + \beta A(s)} |\Delta Z(s)|^2 ds + \frac{4\tilde{L}^2}{\beta} E \int_0^T e^{\alpha s + \beta A(s)} |\Delta U(s)|^2 dA(s)
\]

\[
+ 2c E \left[ \sup_{s \in [0,T]} (e^{\alpha s + \beta A(s)} |\Delta U(s)|^2) \right] + c E \left[ \int_0^T e^{\alpha s + \beta A(s)} |\Delta V(s)|^2 ds \right].
\]
Thus, with \( a := \frac{\lambda \beta}{2} \), \( b := \frac{\lambda}{2} - 144 \) and some \( \lambda > 288 \), by taking into account (6), we obtain

\[
\mathbb{E}(\sup_{t \in [0,T]} e^{\alpha t + \beta A(t)} |\Delta Y(t)|^2) + a \int_0^T e^{\alpha s + \beta A(s)} |\Delta Y(s)|^2 dA(s) \\
+ b \mathbb{E} \int_0^T e^{\alpha s + \beta A(s)} |\Delta Z(s)|^2 ds \\
\leq 2c(2 + \lambda) \mathbb{E} \left[ \sup_{s \in [0,T]} \left( e^{\alpha s + \beta A(s)} |\Delta U(s)|^2 \right) \right] + \frac{4\hat{L}^2}{\beta} (2 + \lambda) \mathbb{E} \int_0^T e^{\alpha s + \beta A(s)} |\Delta U(s)|^2 dA(s) \\
+ c(2 + \lambda) \mathbb{E} \int_0^T e^{\alpha r + \beta A(r)} |\Delta V(r)|^2 dr,
\]

so

\[
\|(\Delta Y, \Delta Z)\|^2_{2,\alpha,\beta,a,b} \leq \mu_\lambda \|(\Delta U, \Delta V)\|^2_{2,\alpha,\beta,a,b},
\]

where

\[
\mu_\lambda := \max \left\{ c(2 + \lambda), \frac{8\hat{L}^2(2 + \lambda)}{\lambda \beta^2}, \frac{2c(2 + \lambda)}{\lambda - 288} \right\}.
\]

Since \( c < c_{\beta, \hat{L}} \), we can take \( \lambda \) slightly bigger than \( \frac{1}{2c_{\beta, \hat{L}}} - 2 \), such that \( 2c(2 + \lambda) \) < 1 and so \( \mu_\lambda \) < 1 (by the definition of \( c_{\beta, \hat{L}} \)).

With this choice of parameters, it follows that the application \( \Gamma \) is a contraction on the Banach space \( S^2_{\beta} \times \mathcal{H}^{2, m \times d}_{\beta} \). Therefore, by Banach fixed point theorem, there exists a unique fixed point \( (Y, Z) = \Gamma(Y, Z) \) in the space \( S^2_{\beta} \times \mathcal{H}^{2, m \times d}_{\beta} \), which completes our proof.

\[\blacksquare\]

### 3 Dependence on parameters

Let us consider, for all \( n \in \mathbb{N}^* \), the following BSDEs which approximate (3):

\[
Y^n(t) = \xi^n + \int_t^T F^n(s, Y^n(s), Z^n(s), Y^n_s, Z^n_s) ds + \int_t^T G^n(s, Y^n(s), Y^n_s) dA^n(s) \\
- \int_t^T Z^n(s) dW(s), \quad t \in [0, T], \quad (7)
\]

In order to unify the notations, we will sometimes denote \( \xi^0 \) instead of \( \xi \), if \( \xi \) is \( \xi, A, F, G, Y \) or \( Z \). We suppose that the coefficients \( \xi^n, A^n, F^n, G^n, n \geq 0 \), satisfy conditions (A\(_2\))-(A\(_3\)), (H\(_1\)), (H\(_2\)) with processes \( K^n, \tilde{K}^n \), but the same constants \( \beta, c, L, \hat{L} \). Moreover, we have to impose that \( \beta > 2 \sqrt{2} \hat{L} \).

We suppose that there exists \( p > 1 \) such that

\[
(A_{0}') \sup_{n \in \mathbb{N}} \mathbb{E} \left[ e^{pA^n(T)} |\xi^n|^{2p} \right] < +\infty;
\]

\[
(A_{0}'') \sup_{n \in \mathbb{N}} \mathbb{E} \left[ e^{rA^n(T)} \right] < +\infty, \text{ for any } r > 0;
\]

\[
(A_{1}') \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \left( \int_0^T e^{\beta A^n(t)} |F^n(t, t, 0, 0, 0)|^2 dt \right)^p \right] < +\infty;
\]

\[
(A_{2}') \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \left| G^n(t, t, 0, 0, 0) \right|^2 \right] < +\infty;
\]

\[
(A_{3}') \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \left( \int_0^T e^{\beta A^n(t)} |F^n(t, t, 0, 0, 0)|^2 dt \right)^{p'} \right] < +\infty;
\]

\[
(A_{4}') \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \left( \int_0^T e^{\beta A^n(t)} |F^n(t, t, 0, 0, 0)|^2 dt \right)^{p''} \right] < +\infty;
\]

\[
(A_{5}') \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \left( \int_0^T e^{\beta A^n(t)} |F^n(t, t, 0, 0, 0)|^2 dt \right)^{p'''} \right] < +\infty.
\]
(A''_1) \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \left( \int_0^T e^{\beta A^n(t)} |G^n(t, 0, 0)|^2 \, dt \right)^p \right] < +\infty.

Under these assumptions, there exists a unique solution \((Y^n, Z^n) \in S_{\beta,2}^{2,m} \times H_{\beta,2}^{p,m \times d}\) to equation (7). Moreover, one can now prove by standard computations that \((Y^n, Z^n) \in S_{\beta,2}^{2,m} \times H_{\beta,2}^{p,m \times d}, \forall n \in \mathbb{N}\) and

\[
\sup_{n \in \mathbb{N}} \|(Y^n, Z^n)\|_{p,\beta} < +\infty. \tag{8}
\]

Our aim is to show that if the coefficients \((\xi^n, A^n, F^n, G^n)\) of equation (7) converge to \((\xi, A, F, G)\), then \((Y^n, Z^n)\) converge to \((Y, Z)\) in \(S_{\beta,2}^{2,m} \times H_{\beta,2}^{p,m \times d}\). Let now specify in what sense the convergence of the coefficients takes place. We define

\[
\Delta_n F := \sup_{t \in [0,T]} \sup_{y \in \mathbb{R}^m} |A^n(t) - A(t)| \to 0 \text{ as } n \to \infty;
\]

\[
\Delta_n G := \sup_{t \in [0,T]} \sup_{y \in \mathbb{R}^m} |G^n(t, y, \hat{\xi}, \hat{\xi}) - F(t, y, \hat{\xi}, \hat{\xi})| \to 0 \text{ as } n \to \infty.
\]

The uniform convergence from assumption \((C_3)\) can be relaxed to the uniform convergence on bounded sets; however, we will work with this hypothesis for the sake of keeping computations as simple as possible.

We will impose one last hypothesis on \(G\):

\[
(A''_1) \sup_{t \in [0,T]} \mathbb{E} \left[ e^{\delta G^n(t, 0, 0)} \right] < +\infty \text{ and } G(\cdot, y, \hat{y}) \text{ is continuous for every } (y, \hat{y}) \in \mathbb{R}^m \times L^2([-\delta, 0]; \mathbb{R}^m)\.
\]

**Theorem 5** Assume that the above assumptions are fulfilled. Then

\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} |Y^n(t) - Y(t)|^2 + \int_0^T |Z^n(t) - Z(t)|^2 \, dt \right] = 0.
\]

**Proof.** Let us denote for short

\[
\Delta_n Y(t) := Y^n(t) - Y(t), \quad \Delta_n Z(t) := Z^n(t) - Z(t); \quad \Delta_n \xi := \xi^n(t) - \xi(t)
\]

\[
\omega_n^\delta := \sup_{t \in [0,T-\delta]} (A^n(t + \delta) - A^n(t)).
\]

By \((H_1)\) and \((H_2)\), we have, exactly as in the proof of Theorem 4,

\[
\left( \frac{K_1 T}{4L^2} + \frac{4K_1 A(T)}{\beta} \right) e^{\alpha \delta + \beta \omega_n^\delta} \leq 2c, \quad \mathbb{P}\text{-a.s.};
\]

\[
\frac{K_1}{4L^2} e^{\alpha \delta + \beta \omega_n^\delta} \leq c, \quad \mathbb{P}\text{-a.s.},
\]

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for all $n \in \mathbb{N}$, where $\alpha = 8L^2 + 1/2$. Let us apply Itô’s formula to $e^{\alpha t + \beta A(t)} |Y^n(t) - Y(t)|^2$:

$$
e^{\alpha t + \beta A^n(t)} |\Delta_n Y(t)|^2 + \int_t^T e^{\alpha s + \beta A^n(s)} |\Delta_n Y(s)|^2 (\alpha ds + \beta dA^n(s)) + \int_t^T e^{\alpha s + \beta A^n(s)} |\Delta_n Z(s)|^2 ds$$

$$= e^{\alpha T + \beta A^n(T)} |\Delta_n Y(T)|^2 - 2 \int_t^T e^{\alpha s + \beta A^n(s)} (\Delta_n Y(s), \Delta_n Z(s) dW(s))$$

$$+ 2 \int_t^T e^{\alpha s + \beta A^n(s)} (\Delta_n Y(s), F^n(s, Y^n(s), Z^n(s), Y^n_s, Z^n_s) - F(s, Y(s), Z(s), Y_s, Z_s)) ds$$

$$+ 2 \int_t^T e^{\alpha s + \beta A^n(s)} (\Delta_n Y(s), G^n(s, Y^n(s), Y^n_s) dA^n(s) - G(s, Y(s), Y_s) dA(s)).$$

From assumptions $(A_2)-(A_3)$ and $(A'_1)$, we have, with $K^n_1 := \sup_{t\in[0,T]} K^n$ and $\bar{K}^n_1 := \sup_{t\in[0,T]} \bar{K}^n$,

$$2 \int_t^T e^{\alpha s + \beta A^n(s)} (\Delta_n Y(s), F^n(s, Y^n(s), Z^n(s), Y^n_s, Z^n_s) - F(s, Y(s), Z(s), Y_s, Z_s)) ds$$

$$\leq 8L^2 \int_t^T e^{\alpha s + \beta A^n(s)} |\Delta_n Y(s)|^2 ds + \frac{1}{2} |\Delta_n F|^2 \int_t^T e^{\alpha s + \beta A^n(s)} ds$$

$$+ \frac{K^n_1 e^{\alpha \delta + \beta \omega^2}}{4L^2} \sup_{s\in[0,T]} (e^{\alpha s + \beta A^n(s)} |\Delta_n Y(s)|^2)$$

$$+ \frac{K^n_1 e^{\alpha \delta + \beta \omega^2}}{4L^2} \int_0^T e^{\alpha r + \beta A^n(r)} |\Delta_n Z(r)|^2 dr$$

and, for all $a > 0$,

$$2 \int_t^T e^{\alpha s + \beta A^n(s)} (\Delta_n Y(s), G^n(s, Y^n(s), Y^n_s) dA^n(s) - G(s, Y(s), Y_s) dA(s))$$

$$= 2 \int_t^T e^{\alpha s + \beta A^n(s)} (\Delta_n Y(s), G^n(s, Y^n(s), Y^n_s) - G^n(s, Y(s), Y_s)) dA^n(s)$$

$$+ 2 \int_t^T e^{\alpha s + \beta A^n(s)} (\Delta_n Y(s), G^n(s, Y^n(s), Y^n_s) - G(s, Y(s), Y_s)) dA^n(s)$$

$$+ 2 \int_t^T e^{\alpha s + \beta A^n(s)} (\Delta_n Y(s), G(s, Y(s), Y_s)) (dA^n(s) - dA(s))$$

$$\leq 2 \int_t^T e^{\alpha s + \beta A^n(s)} (\Delta_n Y(s), G(s, Y(s), Y_s)) (dA^n(s) - dA(s))$$

$$+ a \int_t^T e^{\alpha s + \beta A^n(s)} |\Delta_n Y(s)|^2 dA^n(s) + \frac{4L^2}{a} |\Delta_n G|^2 \int_t^T e^{\alpha s + \beta A^n(s)} dA^n(s)$$

$$+ \frac{\beta}{2} \int_t^T e^{\alpha s + \beta A^n(s)} |\Delta_n Y(s)|^2 dA^n(s) + \frac{4L^2}{\beta} \int_t^T e^{\alpha s + \beta A^n(s)} |\Delta_n Y(s)|^2 dA^n(s)$$

$$+ \frac{4K^n_1 A^n(T) e^{\alpha \delta + \beta \omega^2}}{\beta} \sup_{s\in[0,T]} (e^{\alpha s + \beta A^n(s)} |\Delta_n Y(s)|^2).$$
Since $\alpha = 8L^2 + \frac{1}{2}$ and $\beta > 2\sqrt{2}L$, one can choose $a := \frac{\beta}{2} - \frac{4L^2}{\beta}$ and so we obtain

\[
e^{\alpha t + \beta A^n(t)}|\Delta_n Y(t)|^2 + \frac{1}{2} \int_t^T e^{\alpha s + \beta A^n(s)}|\Delta_n Z(s)|^2 ds
\leq e^{\alpha T + \beta A^n(T)}|\Delta_n \xi|^2 - 2 \int_t^T e^{\alpha s + \beta A^n(s)} \langle \Delta_n Y(s), \Delta_n Z(s) \rangle dW(s)
\]

\[
+ 2 \int_t^T e^{\alpha s + \beta A^n(s)} \langle \Delta_n Y(s), G(s, Y(s), Y_s) \rangle (dA^n(s) - dA(s))
\]

\[
+ \frac{1}{2} |\Delta_n F|^2 \int_t^T e^{\alpha s + \beta A^n(s)} ds + \frac{4L^2}{a} |\Delta_n G|^2 \int_t^T e^{\alpha s + \beta A^n(s)} dA^n(s)
\]

\[
+ \frac{K_1 e^{\alpha \delta + \beta \omega_8^2}}{4L^2} \sup_{s \in [0,T]} (e^{\alpha s + \beta A^n(s)} |\Delta_n Y(s)|^2)
\]

\[
+ \frac{4K_1 A^n(T) e^{\alpha \delta + \beta \omega_8^2}}{\beta} \sup_{s \in [0,T]} (e^{\alpha s + \beta A^n(s)} |\Delta_n Y(s)|^2).
\]

Therefore, by conditions (H_1) and (H_2),

\[
\frac{1}{2} \int_t^T e^{\alpha s + \beta A^n(s)} |\Delta_n Z(s)|^2 ds
\leq 2 \int_t^T e^{\alpha s + \beta A^n(s)} \langle \Delta_n Y(s), G(s, Y(s), Y_s) \rangle (dA^n(s) - dA(s))
\]

\[
+ \frac{1}{2} |\Delta_n F|^2 \int_t^T e^{\alpha s + \beta A^n(s)} ds + \frac{4L^2}{a} |\Delta_n G|^2 \int_t^T e^{\alpha s + \beta A^n(s)} dA^n(s)
\]

\[
+ 2c \sup_{s \in [0,T]} (e^{\alpha s + \beta A^n(s)} |\Delta_n Y(s)|^2) + c \int_t^T e^{\alpha s + \beta A^n(s)} |\Delta_n Z(s)|^2 ds.
\]

Exploiting Burkholder–Davis–Gundy’s inequality, we have that

\[
2E \left[ \sup_{t \in [0,T]} \left| \int_t^T e^{\alpha s + \beta A^n(s)} \langle \Delta_n Y(s), \Delta_n Z(s) \rangle dW(s) \right| \right]
\leq \frac{1}{4} E(e^{\alpha s + \beta A^n(s)} |\Delta_n Y(s)|^2) + 144 E \int_0^T e^{\alpha s + \beta A^n(s)} |\Delta_n Z(s)|^2 ds.
\]

As in the proof of Theorem 4, we obtain

\[
E(\sup_{s \in [0,T]} e^{\alpha s + \beta A^n(s)} |\Delta_n Y(s)|^2) + E \int_0^T e^{\alpha s + \beta A^n(s)} |\Delta_n Z(s)|^2 ds
\leq C E \left[ |\Delta_n \xi|^2 + |\Delta_n F|^2 + |\Delta_n G|^2 \right] \cdot E e^{\beta A^n(T)}
\]

\[
+ CE \sup_{t \in [0,T]} \left| \int_t^T e^{\alpha s + \beta A^n(s)} \langle \Delta_n Y(s), G(s, Y(s), Y_s) \rangle (dA^n(s) - dA(s)) \right|.
\]
where $C$ is a positive constant and $q := \frac{p}{p - r}$.

By conditions $(C_1)$ and $(A''_0)$,

$$\lim_{n \to \infty} \mathbb{E} \left[ |\Delta_n \xi|^{2p} + |\Delta_n F|^{2p} + |\Delta_n G|^{2p} \right] \cdot \mathbb{E} e^{\beta q A^n(T)} = 0.$$  

It remains to prove that

$$\lim_{n \to \infty} \mathbb{E} \sup_{t \in [0, T]} \left| \int_t^T X^n(s) dH^n(s) \right| = 0,$$

where, for $s \in [0, T]$,

$$X^n(s) := e^{\alpha s + \beta A^n(s)} (\Delta_n Y(s), G(s, Y(s), Y_n));$$

$$H^n(s) := A^n(s) - A(s).$$

One can show that $X^n$ is a continuous process and

$$\mathbb{E} \sup_{t \in [0, T]} |X^n(t)|^p$$

is uniformly bounded (with respect to $n$), by $(A''_1)$ and $(8)$. Obviously, by $(A''_0)$,

$$\sup_{n \in \mathbb{N}^*} \mathbb{E} \sup_{t \in [0, T]} |H^n(t)|^2 < +\infty.$$  

Hence, the sequence $(X^n, H^n)_{n \in \mathbb{N}^*}$ is tight in $C [0, T]^2$. By Prokhorov’s theorem, we can extract a sequence, say $(X^n_k, H^n_k)_{k \in \mathbb{N}^*}$, convergent in distribution to some stochastic process $(X, H)$ with continuous paths. Since, by $(C_2)$, $\lim_{n \to \infty} \mathbb{E} \sup_{t \in [0, T]} |H^n(t)| = 0$, $H$ must be $\mathbb{P}$-a.s. equal to 0. The condition $(A''_0)$ also implies that $\sup_{n \in \mathbb{N}} \mathbb{E} \|H^n\|_{BV[0, T]} < +\infty$, for every $r > 1$, so $\|H^n\|_{BV[0, T]}$ is bounded in probability (i.e., it satisfies condition $(9)$). We can now apply Proposition 6, proved as an auxiliary result in the Appendix section, in order to derive the convergence in distribution to 0 of the process

$$\left( \int_0^t X^n(s) dH^n(s) \right)_{t \in [0, T]}.$$  

Since, for some $\nu > 0$, the functional $\phi_\nu : C [0, T] \to \mathbb{R}$, defined by

$$\phi_\nu(x) := \sup_{t \in [0, T]} |x(T) - x(t)| \wedge \nu,$$

is bounded and continuous, it follows that

$$\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_t^T X^n(s) dH^n(s) \right| \wedge \nu \right] = 0,$$

for every $\nu > 0$. Since, by Markov’s inequality, for some $r \in (1, p)$

$$\mathbb{E} \sup_{t \in [0, T]} \left| \int_t^T X^n(s) dH^n(s) \right| \leq \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_t^T X^n(s) dH^n(s) \right| \wedge \nu \right]$$

$$+ \frac{1}{\nu^{p - 1}} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_t^T X^n(s) dH^n(s) \right|^r \right],$$

for every $\nu > 0$. Thus, we conclude that

$$\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_t^T X^n(s) dH^n(s) \right| \wedge \nu \right] = 0.$$
and
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_t^T X^n(s) dH^n(s) \right|^p \right] \leq \mathbb{E} \left[ \left( \sup_{t \in [0,T]} |X^n(t)|^p \right)^{\frac{p}{r}} \right] \leq \left( \mathbb{E} \left[ \sup_{t \in [0,T]} |X^n(t)|^p \right] \right)^{\frac{p}{r}} \left( \mathbb{E} \left[ \|H^n\|_{BV([0,T])}^r \right] \right)^{1 - \frac{r}{p}},
\]
it follows that
\[
\lim_{n \to \infty} \mathbb{E} \sup_{t \in [0,T]} \left| \int_t^T X^n(s) dH^n(s) \right| = 0,
\]
which concludes our proof. ■

4 Appendix

In this section we state the result used in the proof of Theorem 5. It is a variant of the Helly-Bray theorem for the stochastic case and is also stronger than Proposition 3.4 from [21].

Proposition 6 Let \((X_n, H_n) : (\Omega_n, \mathcal{F}_n, \mathbb{P}_n) \to C([0,T]; \mathbb{R}^d)^2, \ n \geq 1\), be a sequence of random variables, converging in distribution to a random variable \((X, H) : (\Omega, \mathcal{F}, \mathbb{P}) \to C([0,T]; \mathbb{R}^d)^2\). If for all \(n \geq 1\), \(H_n\) is \(\mathbb{P}_n\)-a.s. with bounded variation and
\[
\lim_{\nu \to +\infty} \sup_{n \geq 1} \mathbb{P}_n \left( \|H_n\|_{BV([0,T]; \mathbb{R}^d)} > \nu \right) = 0,
\]
then \(H\) is \(\mathbb{P}\)-a.s. with bounded variation and the sequence of \(C[0,T]\)-valued random variables \((\int_0^t (X_n(s), dH_n(s)))_{n \geq 1}\) converges in distribution to \(\int_0^t (X(s), dH(s))\).

As expected, the proof of this result uses a deterministic Helly-Bray type theorem aiming uniform convergence. For the reader’s convenience, we will state and prove this result:

Lemma 7 Let \((x_n)_{n \geq 1} \subseteq C([0,T]; \mathbb{R}^d)\) and \((\eta_n)_{n \geq 1} \subseteq BV([0,T]; \mathbb{R}^d)\) be two sequences of functions such that:

i) \(x_n\) converges uniformly to a function \(x \in C([0,T]; \mathbb{R}^d)\);

ii) \(\eta_n\) converges uniformly to a function \(\eta\);

iii) \(\sup_{n \geq 1} \|\eta_n\|_{BV([0,T]; \mathbb{R}^d)} < +\infty\).

Then \(\eta \in BV([0,T]; \mathbb{R}^d)\), \(\|\eta\|_{BV([0,T]; \mathbb{R}^d)} \leq \liminf_{n \to \infty} \|\eta_n\|_{BV([0,T]; \mathbb{R}^d)}\) and the sequence of continuous functions \((\int_0^t (x_n(s), d\eta_n(s)))_{n \geq 1}\) converges uniformly to \(\int_0^t (x(s), d\eta(s))\).

*In the same time, it corrects an error in the statement of that result: “Let \(X_n, K_n : (\Omega_n, \mathcal{F}_n, P_n) \to \mathbb{W}, \ n \geq 1\), be two sequences of random variables, converging in distribution to \(X\), respectively \(K\)”*, should be replaced with “Let \((X_n, K_n) : (\Omega_n, \mathcal{F}_n, P_n) \to \mathbb{W}^2, \ n \geq 1\), be a sequence of random variables, converging in distribution to \((X, K)\)”*. We emphasize that this doesn’t affect in any way the validity of the other results in that paper, since the arguments involved use in fact this stronger assumption.
Proof. The first two assertions are well-known, so we skip their proof.

Let us prove the last one. We say that a tuple \( \pi = (t_0, \ldots, t_k) \) is a partition of \([0, T]\) if \( 0 = t_0 < t_1 < \cdots < t_{kN} = T \).

We consider \( \pi^N = (t_0^N, \ldots, t_{kN}^N), \) \( N \in \mathbb{N}^* \) partitions of the interval \([0, T]\) such that

\[
\lim_{N \to \infty} \sup_{0 \leq i < t_{kN}^N} |t_{i+1}^N - t_i^N| = 0.
\]

Let \( x^N : [0, T] \to \mathbb{R}^d \) be a step-function approximating \( x \), defined by

\[
x^N := 1_{(0)}(0) + \sum_{i=1}^{kN} 1_{(t_{i-1}^N, t_i^N]}(x(t_i^N)).
\]

Let \( M := \sup_{n \geq 1} \|\eta_n\|_{BV([0,T];\mathbb{R}^d)} \). Then

\[
\begin{align*}
&\left| \int_0^t \langle x_n(s), d\eta_n(s) \rangle - \int_0^t \langle x(s), d\eta(s) \rangle \right| \\
&\quad = \left| \int_0^t \langle x_n(s) - x(s), d\eta_n(s) \rangle \right| \\
&\quad \leq \left| \int_0^t \langle x(s) - x^N(s), d(\eta_n - \eta)(s) \rangle \right| + \left| \int_0^t \langle x^N(s), d(\eta_n - \eta)(s) \rangle \right| \\
&\quad \leq \|x_n - x\|_{C([0,T];\mathbb{R}^d)} \cdot \mathcal{V}_0^T(\eta_n) + \|x^N - x\|_{C([0,T];\mathbb{R}^d)} \cdot \left( \mathcal{V}_0^T(\eta_n) + \mathcal{V}_0^T(\eta) \right) \\
&\quad + \sum_{i=1}^{kN} |x(t_i^N)| \cdot |(\eta_n - \eta)(t_i^N \land t) - (\eta_n - \eta)(t_{i-1}^N \land t)|.
\end{align*}
\]

Therefore,

\[
\sup_{t \in [0,T]} \left| \int_0^t \langle x_n(s), d\eta_n(s) \rangle - \int_0^t \langle x(s), d\eta(s) \rangle \right| \leq M \|x_n - x\|_{C([0,T];\mathbb{R}^d)} \\
+ 2M \|x^N - x\|_{C([0,T];\mathbb{R}^d)} + 2 \left( \sum_{i=1}^{kN} |x(t_i^N)| \right) \|\eta_n - \eta\|_{C([0,T];\mathbb{R}^d)}.
\]

It follows that

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \left| \int_0^t \langle x_n(s), d\eta_n(s) \rangle - \int_0^t \langle x(s), d\eta(s) \rangle \right| \leq 2M \|x^N - x\|_{C([0,T];\mathbb{R}^d)}.
\]

Since \( \lim_{N \to \infty} \|x^N - x\|_{C([0,T];\mathbb{R}^d)} = 0 \), we finally get

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \left| \int_0^t \langle x_n(s), d\eta_n(s) \rangle - \int_0^t \langle x(s), d\eta(s) \rangle \right| = 0.
\]

Let us now proceed with the proof of the main result of this section, which follows the same steps as that of Proposition 3.4 from [21].

Proof of the proposition 6. Let \( W := C([0,T];\mathbb{R}^d), \) \( V := C([0,T];\mathbb{R}^d) \cap BV([0,T];\mathbb{R}^d) \) and, for \( \nu > 0, \)

\[
V_\nu := \left\{ \eta \in V \mid \|\eta\|_{BV([0,T];\mathbb{R}^d)} \leq \nu \right\}.
\]
By the first part of Lemma 7, $V_\nu$ is a closed subset of the Banach space $W$.

Let us consider the function $\Lambda : W \times W \to W$ defined by

$$\Lambda(x, \eta)(t) := \begin{cases} \int_0^t (x(s), d\eta(s)), & (x, \eta) \in W \times V; \\ 0, & (x, \eta) \in W \times (W \setminus V). \end{cases}$$

By the last conclusion of Lemma 7, the restriction $\Lambda|_{W \times V_\nu}$ is continuous.

Let now $R_n := P^n \circ (X^n, H^n)^{-1}$ and $R_0 := P \circ (X, H)^{-1}$, the distribution probabilities of $(X^n, H^n)$, respectively $(X, H)$. By the assumptions of the theorem, $(R_n)_{n \geq 1}$ converges weakly to $R_0$, i.e.

$$\lim_{n \to \infty} \int_{W \times W} \Phi(x, \eta) R_n(dx, d\eta) = \int_{W \times W} \Phi(x, \eta) R_0(dx, d\eta), \quad \text{for every bounded continuous functional } \Phi : W \times W \to \mathbb{R}.$$  

(10)

First of all, by Portmanteau lemma,

$$\limsup_{n \to \infty} R_n (W \times V_\nu) \leq R_0 (W \times V_\nu), \quad \forall \nu > 0.$$  

Since, by condition (9),

$$\lim_{\nu \to +\infty, n \geq 1} R_n (W \times V_\nu) = 1,$$  

(11)

we get $\lim_{\nu \to +\infty} R_0 (W \times V_\nu) = 1$, hence $R_0 (W \times V) = 1$, meaning that $H$ is $P$-a.s. bounded variation.

Let now $\phi : C[0, T] \to \mathbb{R}$ be an arbitrary bounded continuous functional. It remains to prove that $\lim_{n \to \infty} E\phi (\Lambda(X^n, H^n)) = E\phi (\Lambda(X, H))$, which can be written as

$$\lim_{n \to \infty} \int_{W \times W} (\phi \circ \Lambda)dR_n = \int_{W \times W} (\phi \circ \Lambda)dR_0.$$  

Since $\phi \circ \Lambda|_{W \times V_\nu}$ is bounded and continuous, it can be extended to a continuous functional $\Phi_\nu : W \times W \to \mathbb{R}$, bounded by $M := \sup_{z \in C[0, T]} |\phi(z)|$; hence, by (10),

$$\lim_{n \to \infty} \int_{W \times W} \Phi_\nu(x, \eta) R_n(dx, d\eta) = \int_{W \times W} \Phi_\nu(x, \eta) R_0(dx, d\eta).$$  

Let us estimate the term

$$T_{n, \nu} := \left| \int_{W \times W} (\Phi_\nu - \phi \circ \Lambda)(x, \eta) R_n(dx, d\eta) \right|,$$

for $n \in \mathbb{N}$ (including then the case $n = 0$). We have

$$T_{n, \nu} \leq \int_{W \times W} |(\Phi_\nu - \phi \circ \Lambda)(x, \eta)| R_n(dx, d\eta) = \int_{W \times (W \setminus V_\nu)} |(\Phi_\nu - \phi \circ \Lambda)(x, \eta)| R_n(dx, d\eta) \leq 2MR_n (W \times (W \setminus V_\nu)) = 2M (1 - R_n (W \times V_\nu)).$$

Hence, by (11) and its consequence

$$\lim_{\nu \to +\infty, n \geq 0} T_{n, \nu} = 0.$$  

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Finally, for all \( n \geq 1 \) and \( \nu > 0 \),

\[
\left| \int_{W \times W} (\phi \circ \Lambda) dR_n - \int_{W \times W} (\phi \circ \Lambda) dR_0 \right| \\
\leq \left| \int_{W \times W} \Phi_\nu (x, \eta) R_n(dx, d\eta) - \int_{W \times W} \Phi_\nu (x, \eta) R_0(dx, d\eta) \right| + T_{n,\nu} + T_{0,\nu}
\]

and therefore

\[
\limsup_{n \to \infty} \left| \int_{W \times W} (\phi \circ \Lambda) dR_n - \int_{W \times W} (\phi \circ \Lambda) dR_0 \right| \leq 2 \sup_{n \geq 0} T_{n,\nu}, \forall \nu > 0
\]

which, by passing to the limit as \( \nu \to +\infty \), yields the desired conclusion. ■

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