Primitive Normal completions of the affine plane I

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Abstract

In this article we study normal compactifications of \( \mathbb{C}^2 \) from the point of view of (discrete) valuations associated to the curves at infinity, or equivalently, pencils of ‘jets of curve-germs’ centered at points at infinity. We give an explicit (and easy to calculate) characterization of discrete valuations which correspond to normal compactifications of \( \mathbb{C}^2 \) which are primitive (i.e. the curve at infinity is irreducible). We also calculate several invariants of the primitive compactifications of \( \mathbb{C}^2 \) in terms of the corresponding curve-jets. As a consequence of these calculations we derive a new proof of Jung’s theorem on polynomial automorphisms of \( \mathbb{C}^2 \).

1 Introduction

Disclaimer: This is an unpolished draft of the article. A clearer exposition is in order and this submission will be updated in a few days.

In this first of a series of two articles we study normal compactifications of \( \mathbb{C}^2 \), i.e. complete normal analytic surfaces containing \( \mathbb{C}^2 \), from the point of view of (discrete) valuations associated to the curves at infinity. We associate to every such valuation a pencil of ‘jets of curve-germs’ centered at a point at infinity. This is a reformulation of the classical treatment of valuations of \( \mathbb{C}(x,y) \) in terms of Puiseux series (e.g. as in [FJ04, Chapter 4]) which seems to have been overlooked in the literature. In this series of articles we hope to convey parts of the picture that emerges from looking at normal compactifications of \( \mathbb{C}^2 \) from this perspective. For example, from this picture one immediately sees that normal compactifications of \( \mathbb{C}^2 \) which are also primitive (i.e. the complement of \( \mathbb{C}^2 \) consists of one irreducible curve) contains at most two singular points, and from this observation it is easy to find the answer to a question in [Fur97] about existence of a rational Gorenstein ‘homology plane’ which is not a compactification of \( \mathbb{C}^2 \); we should note that the fact that our example answers this question affirmatively can also be seen from [MZ88, Theorem 2] - see the discussion following Proposition 1.6.

Note. From now on, for the sake of avoiding repetitions, we will omit the word ‘normal’ when talking about compactifications of \( \mathbb{C}^2 \); all the compactifications will be normal unless otherwise stated.

Primitive projective compactifications of \( \mathbb{C}^2 \) have been the subject of study of a number of works (see e.g. [Bre80], [BDPS1], [Fur97], [Koj01], [KT09], [Oht01]). Our main result is that a primitive compactification of \( \mathbb{C}^2 \) is projective iff it is algebraic iff its associated pencil of curve-gets has a representative which is a plane curve with one place at infinity. It follows from this...
result that there are primitive compactifications of \( \mathbb{C}^2 \) which are not algebraic. While both these results (and explicit criteria for determining when they occur) will be shown in part II, in this part we give an explicit (and easy to calculate) characterization of which pencil of curve-jets (or equivalently, discrete valuations) correspond to primitive compactifications of \( \mathbb{C}^2 \). We also calculate several invariants of the primitive compactifications of \( \mathbb{C}^2 \) in terms of the corresponding curve-jets. As a consequence of these calculations we derive a new proof of Jung’s theorem on polynomial automorphisms of \( \mathbb{C}^2 \).

The predominant way (in the literature) to study primitive compactifications of \( \mathbb{C}^2 \) has been to consider certain ‘good’ (i.e. ‘normal’ in the sense of \([?]\)) nonsingular compactifications of \( \mathbb{C}^2 \) which arise from desingularization of the original surfaces and use the classification of ‘minimal normal completions’ of \( \mathbb{C}^2 \) from \([?]\). We, on the other hand, study these surfaces via the valuations associated to the curves at infinity. Let us fix polynomial coordinates \((x,y)\) on \( \mathbb{C}^2 \). Then for every primitive compactification \( X \) of \( \mathbb{C}^2 \), the order of vanishing along the curve at infinity induces a discrete valuation \( \nu \) on \( \mathbb{C}(x,y) \). It is straightforward to see that

\[
\nu(f) < 0 \text{ for all } f \in \mathbb{C}[x,y] \setminus \mathbb{C}, \text{ and }
\nu \text{ is divisorial, i.e. the transcendence degree of the valuation ring of } \nu \text{ over } \mathbb{C} \text{ is 1.} (\ast)
\]

Moreover, it is not hard to see (e.g. by the universal property of normal varieties) that \( \nu \) uniquely determines \( X \), i.e. if \( Z \) is another primitive normal compactification of \( \mathbb{C}^2 \) such that the order of vanishing along \( Z \setminus \mathbb{C}^2 \) equals \( \nu \), then \( Z \cong X \). This gives rise to a natural question:

**Question 1.1.** Does every discrete valuation on \( \mathbb{C}(x,y) \) that satisfies \((\ast)\) come from a primitive compactification of \( \mathbb{C}^2 \)?

More generally, given a divisorial discrete valuation \( \nu \) on \( \mathbb{C}[x,y] \) which is centered at infinity (i.e. \( \nu(f) < 0 \) for some \( f \in \mathbb{C}[x,y] \)), let us say that \( \nu \) is \( k \)-realizable at infinity if there exists a compactification \( X \) of \( \mathbb{C}^2 \) with \( k \) irreducible curves at infinity such that \( \nu \) is the discrete valuation corresponding to one of the curves at infinity. Then, in a similar vein as in Question 1.1, one can ask:

**Question 1.2.**

(a) What is the least \( k \) such that \( \nu \) is \( k \)-realizable at infinity?

(b) What is the least \( k \) such that \( \nu \) is \( k \)-realizable at infinity by a projective surface?

Let us now show that in general the answer to Question 1.2 is 2. Indeed, since \( \nu \) is centered at infinity, after a linear change of coordinates if necessary we may assume that \( \nu(y) < 0 \) and \( \nu(x) < 0 \). Let \( X^{(0)} := \mathbb{P}^1 \times \mathbb{P}^1 \) and embed \( X := \mathbb{C}^2 \hookrightarrow X^{(0)} \) via \( (x,y) \mapsto ([1:x],[1:y]) \). Let \( O := ([0:1],[0:1]) \in X^{(0)} \setminus X \). Then \( O \) has a Zariski open neighborhood \( U \) such that \( (U,O) \cong (\mathbb{C}^2,0) \) with coordinates \((1/x,1/y)\). Since \( \nu \) is centered at \( O \), i.e. the maximal ideal of \( O \) is contained in the maximal ideal of the valuation ring of \( \nu \), it follows from the classical theory of valuations (see, e.g. \([\text{Zar39}]\)) that after a finite sequence of blow-ups centered at \( O \) or points in the exceptional divisors (accumulated from preceding blow-ups) we end up with a variety \( X^{(1)} \) and an exceptional divisor \( E \) on \( X^{(1)} \) such that \( \nu \) is precisely the order of vanishing along \( E \). Let \( C \) be the (possibly reducible) curve consisting of all the exceptional divisors on \( X^{(1)} \) other than \( E \). Then the intersection matrix of the components of \( C \) is negative definite, so by results of Grauert and Artin, \( X^{(1)} \) can be blown down along \( C \) to a normal analytic
surface $\tilde{X}$. Consequently, $\nu$ is 2-realizable at infinity by $\tilde{X}$, as claimed. To answer Question 2b one has to determine when $\tilde{X}$ is projective, for which there is no general criterion. In Section 3 we give an alternate construction of $\tilde{X}(1)$ as a projective surface and consequently show that the answer to Question 2b is also 2. We note that the construction of $\tilde{X}(1)$ in Section 3 is much more elementary (the ingredient being basic algebraic geometry) and does not require the contractibility theorem. The answer to Question 1.1 is more subtle and we answer it (negatively) in part II, where we also give an effective algorithm to determine if $\nu$ satisfies $(\ast)$.

We now introduce pencils of jets of curve-germs corresponding to divisorial valuations centered at a point. Let $O$ be the origin in $\mathbb{C}^2$ and $(u,v)$ be analytic coordinates near $O$.

**Definition 1.3.** Given a positive rational number $\omega$ and two irreducible analytic germs $C_1, C_2$ of curves at $O$, we say that $C_1 \equiv_{u,\omega} C_2$ iff there are Puiseux series expansions $v = \phi_i(u)$ for $C_i$, $1 \leq i \leq 2$, such that $\text{ord}_u(\phi_1 - \phi_2) > \omega$. A $(u,\omega)$-jet of curve-germs at $O$ is an equivalence class $[\phi]$ of analytic germs $C$ at $O$ modulo the equivalence relation $\equiv_{u,\omega}$. Frequently we will simply say $\omega$-jet when the variable for the Puiseux series expansions is clear from the context. By a pencil of $\omega$-jets at $O$ we mean a family of jets of curve-germs of the form $\{[\phi(u) + \xi u^r] : \xi \in \mathbb{C}\}$ for a finite Puiseux series $\phi$ in $u$ such that $\deg_u(\phi) < \omega$ (note that here we identified curves with their Puiseux series expansions in $u$).

A simple reformulation of the theory of discrete valuations and Puiseux series (as developed in [FJ04]) yields

**Theorem 1.4** (cf. [FJ04, Proposition 4.1, Theorem 4.17]). There is a one-to-one correspondence among the following three families:

1. Divisorial discrete valuations $\nu$ on $\mathbb{C}(u,v)$ centered at the origin and normalized in the sense that $\gcd(\nu(f) : f \in \mathbb{C}(u,v)) = 1$,

2. Finite Puiseux series $\phi$ in $u$ and a rational number $r > \deg_u(\phi)$, and

3. Pencils $\mathcal{L}$ of jets of analytic curve-germs at the origin.

The correspondence $[\phi] \leftrightarrow \nu_{\phi,r}$ is given by $(\phi, r) \mapsto \nu_{\phi,r}$, where for every $f \in \mathbb{C}(u,v)$,

$$\nu_{\phi,r}(f) := \text{ord}_u(f(u, \phi(u) + \xi u^r))$$

(1)

where $\xi$ is an indeterminate. The correspondence $[\phi] \leftrightarrow \mathcal{L}$ is given by $\mathcal{L} \mapsto \nu_{\mathcal{L}}$, where for every $f \in \mathbb{C}[u,v]$,

$$\nu_{\mathcal{L}}(f) := \text{intersection multiplicity at the origin of } V(f) \text{ and a generic curve in } \mathcal{L}.$$

(2)

An immediate implication of theorem 1.4 is the following characterization of the exceptional divisor corresponding to divisorial discrete valuations centered at the origin (note that we say that an exceptional divisor $E$ in an infinitesimal neighborhood of the origin corresponds to a divisorial discrete valuation $\nu$ centered at the origin iff $\nu$ is precisely the order of vanishing along $E$).

**Corollary 1.5.** Let $\nu$ be a divisorial discrete valuation centered at the origin and $\mathcal{L}_\nu$ be the corresponding pencil of the jets of curve-germs. Then
1. If \( E \) is the exceptional divisor corresponding to \( \nu \) on a blow up \( \tilde{U} \) of a neighborhood \( U \) of the origin, then the strict transforms of (representatives of) generic curve-germs in \( \mathcal{L}_\nu \) intersect \( E \) at generic points.

2. Conversely, if \( E \) is an exceptional divisor on a blow up \( \tilde{U} \) of a neighborhood \( U \) of the origin such that the strict transforms of (representatives of) generic curve-germs in \( \mathcal{L}_\nu \) intersect \( E \) at generic points, then \( E \) is precisely the exceptional divisor on \( \tilde{U} \) corresponding to \( \nu \).

Proof. At first assume that \( E \) is as in assertion \([1]\) and (with a view towards a proof by contradiction) assume that there is a point \( P \in E \) such that infinitely many curve-germs in \( \mathcal{L} \) intersects \( E \) at \( P \). Then one can find a polynomial \( f \in \mathbb{C}[u, v] \) such that \( f \) does not vanish along \( E \), but the strict transform of \( V(f) \) intersects \( E \) at \( P \). Since \( E \) is the exceptional divisor corresponding to \( \nu \), it follows that \( \nu(f) = \text{ord}_E(f) = 0 \). On the other hand, since \( V(f) \) intersects infinitely many elements of \( \mathcal{L}_\nu \), it follows from theorem \([1,4]\) that \( \nu(f) > 0 \). This contradiction proves assertion \([1]\).

Now assume \( E \) is as in assertion \([2]\). Let \( E_\nu \) be the exceptional divisor corresponding to \( E \) (on an appropriate infinitesimal neighborhood \( U' \) of the origin). Then assertion \([1]\) implies that the birational correspondence \( \phi : \tilde{U} \to U' \) generic points of \( E \) to generic points of \( E_\nu \) and vice versa. Consequently \( \phi \) restricts to an isomorphism on a neighborhood of an open subset of \( E \) and assertion \([2]\) follows.

An immediate application of Corollary \([1,5]\) to the case of valuations centered at infinity yields the following proposition. This proof essentially follows an idea of Dmitry Kerner:

**Proposition 1.6.** A primitive compactification of \( \mathbb{C}^2 \) contains at most two singular points.

Proof. Let \( \bar{X}^{(0)} := \mathbb{P}^2 \) be the usual compactification of \( \mathbb{C}^2 \) via the embedding \( (x, y) \mapsto ([x : y : 1]) \) and let \( \bar{X} \) be an arbitrary primitive compactification of \( \mathbb{C}^2 \). Let \( \nu \) be the discrete valuation on \( \mathbb{C}(x, y) \) corresponding to the curve at infinity on \( \bar{X} \). W.l.o.g. we may assume that \( \nu \neq - \text{deg} \) (where \( \text{deg} \) is the usual degree in \( (x, y) \) coordinates), since then \( \bar{X} \cong \bar{X}^{(0)} \) and the proposition is trivially true. Then the center of \( \nu \) on \( \bar{X}^{(0)} \) is a unique point \( O \) at infinity. After a linear change of coordinates if necessary, we may assume that \( O \) has coordinates \([1 : 0 : 0]\). Then \( u := 1/x \) and \( v := y/x \) gives a system of coordinates near \( O \). Let \( \mathcal{L}_\nu \) be the pencil of jets of curve-germs at \( O \) corresponding to \( \nu \). Then \( \mathcal{L}_\nu = \{ [\phi(u) + \xi u^\omega] : \xi \in \mathbb{C} \} \) for a finite Puiseux series \( \phi \) in \( u \) and a rational number \( \omega > \text{deg}_u(\phi) \). After a finite sequence of blow-ups centered at \( O \) and points infinitely near to \( O \), we arrive at a surface \( \bar{X} \) and a coordinate chart \( U \) (with coordinates \((\bar{u}, \bar{v})\)) on \( \bar{X} \) such that \( E := \{ \bar{u} = 0 \} \) is the ‘last’ exceptional divisor and the strict transform of the curve-branches \( v = \phi(u) + \xi u^\omega \) is of the form \( \bar{v} = \xi \). Corollary \([1,5]\) then implies that \( E \) is precisely the exceptional divisor corresponding to \( \nu \), and therefore, \( \bar{X} \) is the blow-down of \( \bar{X} \) along the union \( C \) of all other exceptional divisors and the strict transform of the line at infinity on \( \bar{X}^{(0)} \). Now, keeping track of the blow-up process from \( \bar{X}^{(0)} \) to \( \bar{X} \) shows that at every step the ‘newest’ exceptional divisor intersects at most two of the ‘old’ exceptional divisors. It follows that \( C \) has at most two connected components and blows down to at most two points on \( \bar{X} \). Since the complement of the image of \( C \) in \( \bar{X} \) is isomorphic to \( \bar{X} \setminus C \), we are done!

The following question was asked in \([1,5]\):
Question 1.7 ([Fur97, Question 1.1]). Does there exist a projective rational Gorenstein surface \( Y \) such that \( H_i(Y, \mathbb{Z}) = H_i(\mathbb{P}^2, \mathbb{Z}) \) for all \( i \), but \( Y \) is not a compactification of \( \mathbb{C}^2 \)?

Proposition 1.6 suggests a simple approach to find answers for Question 1.7: find \( Y \) such that \( Y \) satisfies the hypotheses of Question 1.7 but has more than two singular points. For example, let \( Y \) be the toric surface determined by the triangle \( P \) which is the convex hull of \((1,1), (-1,1)\) and \((0,-1)\). Then \( P \) is reflexive, which implies that \( Y \) is a Gorenstein surface (see e.g. [CLSon, Section 8.3]). But \( Y \) has three singular points, namely the three points fixed by the action of the torus, so that \( Y \) can not be a compactification of \( \mathbb{C}^2 \). On the other hand, the integral homologies of \( Y \) are equal to those of \( \mathbb{P}^2 \) (which can be seen, e.g. by [?], Lemma 3.1), so that \( Y \) answers Question 1.7 affirmatively. Note that one can also see that \( Y \) is not a compactification of \( \mathbb{C}^2 \) from [MZ88, Theorem 2] which states that if the smooth locus \( S^0 \) of a Gorenstein log del Pezzo surface \( S \) does not contain any \((-1)\) curves, then \( S \) is a compactification of \( \mathbb{C}^2 \) iff \( S^0 \) is simply connected.

Now we describe the organization of the rest of the article. In Section 2 we give the explicit criterion for determining if a pencil of jets of curve-germs correspond to a primitive compactification of \( \mathbb{C}^2 \) and compute some of the invariants of the primitive compactifications in terms of these jets. In Section 3 we give an explicit construction of a projective compactification of \( \mathbb{C}^2 \) that 2-realizes a given discrete valuation at infinity. We also find an explicit description of curves that approach the points at infinity on such compactifications.

I heartily thank Professor Pierre Milman. This work started as I was trying to understand some of his questions in a simple case. Very special thanks go to Dmitry Kerner - I had to find the formulation (of divisorial discrete valuations) in terms of jets of curve-germs in order to answer his questions, and this made the proofs (especially of Proposition 1.6) considerably simpler.

2 Criterion for primitive compactifications and some of their properties

Definition 2.1. Let \( k \) be a field and \( A \) be a \( k \)-algebra. A semidegree \( \delta \) on \( A \) is a function \( \delta : A \setminus \{0\} \rightarrow \mathbb{Z} \) such that \(-\delta\) is an integral multiple of a discrete valuation \( \nu \). We say that \( \delta \) is divisorial if \( \nu \) is a divisorial valuation.

Proposition 2.2. Let \( \delta \) be a divisorial semidegree on \( \mathbb{C}[x,y] \) such that \( \delta(x) > 0 \). Then there is a finite Puiseux series \( \psi_\delta \) in \( x \) and a rational number \( r_\delta < \operatorname{ord}_x(\psi) \) such that for all \( f \in \mathbb{C}[x,y], \)

\[
\delta(f) = \delta(x) \deg_x(f(x, \psi_\delta(x) + \xi x^{r_\delta})).
\] (3)

Proof. Pick a positive integer \( q \) such that \( \delta(x^q) = q\delta(x) > \delta(y) \). Let \( u := 1/x, v := y/x^q \) and \( \nu \) be the restriction of \(-\delta\) to \( \mathbb{C}[u,v] \). Since \( \mathbb{C}(u,v) = \mathbb{C}(x,y) \), it follows that \( \operatorname{tr.deg}(\nu) = \operatorname{tr.deg}(-\delta) = 1 \). Therefore \( \nu \) is a divisorial valuation, and Theorem 1.4 implies that there is a finite Puiseux series \( \psi \) in \( u \) and a rational number \( r > \deg_u(\psi) \) such that

\[
\frac{\nu(\phi(u,v))}{\nu(u)} = \operatorname{ord}_u(\phi(u, \psi(u) + \xi u^r))
\]
Therefore the proof of the corollary is complete with \( \psi \).

Moreover, \( \text{ord}_y(\Phi) \) for all \( \phi \) weighted degree in \((\Phi(x,y))\text{coordinates. In this case } r = \delta(y)/\delta(x). \]

**Remark - example 2.5.** Let \( \delta, \psi_5 \) and \( r_5 \) be as in corollary 2.2. Then \( \delta \) determines \( r_5 \) uniquely, but \( \psi_5 \) only up to conjugacy. For example, let \( \delta \) be the semidegree defined by identity 3 with \( \psi_5 := x^{3/2} \) and \( r_5 < 3/2 \). Then \(-x^{3/2} \) is a conjugate of \( \psi_5 \) and it follows that setting \( \psi_5 := x^{-3/2} \) (and keeping \( r_5 \) unchanged) also defines the same semidegree as \( \delta \).

**Definition 2.6.** A formal (possibly infinite) \( \mathbb{C} \)-linear combination \( \phi(t) := \sum_{i=1}^{\infty} a_i t^i \) is called a degree-wise Puiseux series in a variable \( t \) if \( q_1 > q_2 > \cdots \) is a decreasing sequence of rational numbers and \( \phi(1/t) \) is a usual Puiseux series in \( t \). We write \( \mathcal{C}((t)) \) for the field of degree-wise Puiseux series in \( t \). The polydromy order of \( \phi \) is defined to be that of \( \phi(1/t) \). Similarly, the conjugates of \( \phi \) are precisely those degree-wise Puiseux series \( \eta \) such that \( \eta(1/t) \) are conjugates of \( \phi(1/t) \).

**Remark 2.7.** If \( \phi \) is a Puiseux series with finitely many terms, then \( \phi \) is also a degree-wise Puiseux series. It follows that in the statement of Corollary 2.2 we may replace ‘Puiseux series’ by ‘degree-wise Puiseux series’. In this article we use Corollary 2.2 mainly in this modified form.

**Definition 2.8.** Let \( \delta \) be a divisorsial semidegree on \( \mathbb{C}(x,y) \) such that \( \delta(x) > 0 \) and let \( \psi_5 \) and \( r \) be as in 3. Let \( \Psi_\delta \in \mathbb{C}[x,x^{-1},y] \) be the unique irreducible element such that \( \Psi_\delta \) is monic in \( y \) and \((y - \psi_5(x)) \) divides \( \Psi_\delta \), i.e. \( \Psi_\delta \) is just the product of all conjugates of \( y - \psi_5(x) \).

**Theorem 2.9.** Let \( \delta \) be a divisorsial semidegree on \( \mathbb{C}(x,y) \) such that \( \delta(x) > 0 \). Then \( \delta \) corresponds to a primitive normal compactification of \( \mathbb{C}^2 \) iff \( \delta(\Psi_\delta(x,y)) > 0 \).

**Proof.** We prove this theorem assuming the result of Theorem 3.1. W.l.o.g. we may assume that \( \delta \) is normalized, i.e. \( \gcd(\delta(f) : f \in \mathbb{C}[x,y]) = 1 \). Let

\[
p := \deg_y(\Psi_\delta) = \text{the polydromy order of } \psi_5.
\]
Let \( \delta := \max\{r \cdot \deg, s \delta\} \) for suitable integers \( r, s \) to ensure that \( \delta \) has two associated semidegrees. Let \( C \) and \( D \) be the curves at infinity on \( X^\delta \) corresponding to \( \delta \) and \( \deg \) respectively. It follows from Theorem 3.1 that there is a regular map \( \pi : X^\delta \to Y := \mathbb{P}^2 \) which is isomorphic on the complement of \( C \). Let \( D' \) be the line at infinity on \( Y \). Then \( \pi^*(D') = D + pC \) and therefore

\[
(D + pC, C) = 0, \quad \text{and} \quad (D + pC, D) = 1
\]

Now let \( k \) be the minimal integer such that \( x \cdot \Psi_\delta \in \frac{1}{x^{q-1}} \mathbb{C}[x, y] \), and \( \Psi^* := x^k \Psi_\delta \). Then \( \Psi^* \) has two points at infinity, namely \( P := [0 : 1 : 0] \) and \( Q := [1 : 0 : 0] \). Let \( Z \) be the curve in \( \mathbb{C}^2 \) defined by \( \Psi^* \). Then

\[
(\bar{Z}^X, D) = \text{intersection multiplicity of } \bar{Z}^Y \text{ and } D' \text{ at } P = k
\]

It follows that

\[
((e + kp)C + (p + k)D, D) = k
\]

where \( e := \delta(\Psi_\delta(x, y)) \). Solving this system of equations yield: \( (D, D) = -\frac{e}{p+q} \), which implies the theorem by Artin’s criterion. \( \square \)

**Example 2.10.** If \( \delta \) is a weighted degree on \( \mathbb{C}[x, y] \) such that \( p := \delta(x) > 0 \) and \( q := \delta(y) \in \mathbb{Z} \), then \( \psi_\delta(x) = 0 \) and \( r = q/p \). It follows that \( \Psi_\delta(x, y) = y \) and theorem 2.9 reduces to the fact that \( \delta \) gives a compactification of \( \mathbb{C}^2 \) if \( q > 0 \) (in which case the corresponding completion \( \bar{X} \) is just the weighted projective surface \( \mathbb{P}^2(1, p, q) \)).

**Example 2.11.** Let \( p, q \) be relatively prime positive integers and \( \psi(x) := x^{p/q} \). Pick \( r < p/q \in \mathbb{Q} \) and define \( \delta \) as in (3) with \( \psi_\delta := \psi \). Let \( \zeta \) be a \( q \)-th root of identity. Then the conjugates of \( y - \psi(x) \) are precisely \( y - \zeta^i x^{p/q}, 0 \leq i < q \), and therefore

\[
\Psi_\delta = \prod_{i=0}^{q-1} (y - \zeta^i x^{p/q}) = y^q - x^p,
\]

and

\[
\delta(\Psi_\delta) = \deg_x \left( \prod_{i=0}^{q-1} (x^{p/q} + \zeta^i x^{p/q}) \right) = \deg_x \left( 1 - \zeta \cdot \cdots \cdot (1 - \zeta^{q-1}) \xi (x^{p/q} + r) \right) = \frac{p(q-1)}{q} + r
\]

where l.o.t. is a shorthand for ‘lower order terms’ and refers to monomials with lower \( x \)-degree. It follows that \( \delta \) is the semidegree corresponding to a minimal normal compactification of \( \mathbb{C}^2 \) iff \( \frac{p(q-1)}{q} + r > 0 \).

**Lemma 2.12.** One of the two singular points of a primitive compactification \( \bar{X} \) of \( X := \mathbb{C}^2 \) is at most a cyclic quotient singularity.

**Proof.** Consider the coordinates near the point at infinity corresponding to the jet on \( L_\delta \) corresponding to \( \xi = 0 \). The order of the quotient singularity is precisely the factor of the division of the polydromy order of \( \psi_\delta \) by the denominator of \( r \). \( \square \)

**Lemma 2.13.** If the non-rigid point at infinity is not singular, then \( \delta(\Psi_\delta) = 1 \).

**A proof of Jung’s theorem on automorphisms on \( \mathbb{C}[x, y] \):** Let \( \sigma : Z := \mathbb{C}^2 \ni (x, y) \mapsto (u, v) \in W := \mathbb{C}^2 \) be a polynomial automorphism. Let \( \delta \) be the semidegree on \( \mathbb{C}[u, v] \) induced by the usual degree in \( \mathbb{C}[x, y] \). Since \( \sigma \) is a polynomial automorphism, it follows that \( Z^\delta \cong \mathbb{F}^2 \). In particular, \( Z^\delta \) is non-singular. Now apply the preceding two lemmas.
3 Projective 2-realization of a divisorial discrete valuation at infinity

Given a divisorial discrete valuation $\nu$ on $\mathbb{C}(x,y)$ centered at infinity, in this section we construct (with two irreducible components of the curve at infinity) that realizes $\nu$ at infinity. The motivation for the construction comes from toric geometry. Assume $\delta$ is a weighted degree in $(x,y)$-coordinates. How can we realize it?

**Theorem 3.1.** Let $\delta_1$ and $\delta_2$ be divisorial semidegrees on $A := \mathbb{C}[x,y]$. If $\delta_1$ is a positive weighted degree, then $\delta := \max\{\delta_1, \delta_2\}$ is a finitely generated subdegree.

**Proof.** At first we prove the theorem in a special case and then we reduce the general case to that special case. So assume that

H1. $\delta_1(x) = q$ and $\delta_1(y) = qr$, where $q, r$ are positive integers,

H2. the degree-wise Puiseux series $\psi(x)$ associated to $\delta_2$ is either identically zero or has the form

$$\psi(x) = \sum_{k=1}^{l} a_k x^{s_k},$$  \hspace{1cm} (4)

where $s_k$ is an integer for each $k$, $1 \leq k \leq l$ and $s_1 < r$, and

H3. the generic term associated with $\delta_2$ is $\xi x^s$, where $s$ is an integer with $s < s_l$.

Note that $\psi$ is identically zero if and only if $\delta_2$ is a weighted degree in $(x,y)$-coordinates. Since the maxima of weighted degrees in same coordinates are finitely generated (these are degree-like functions corresponding to toric varieties determined by convex rational polytopes, see [Mon10, Example 3.5]), it follows that $\delta$ is finitely generated if $\psi$ is identically zero. So assume $\psi$ is as in (4). If $s_1$ is positive, then $m$ be the largest integer such that $s_m$ is non-negative and let $\tilde{y} := y - \sum_{k=1}^{m} a_k x^{s_k}$. Since $s_1 < r$, it follows that $\delta_1$ is a weighted degree in $(x, \tilde{y})$-coordinates as well, and H1 is also satisfied with $y$ substituted by $\tilde{y}$. Therefore, replacing $y$ by $\tilde{y}$, we may assume that the $\psi(x)$ has the form

$$\psi(x) = \sum_{k=1}^{l} a_k x^{-s_k},$$  \hspace{1cm} (4')

where $\{s_k : 1 \leq k \leq l\}$ is an increasing sequence of positive integers, and the generic term associated to $\delta_2$ is $\xi x^{-s}$, where $s$ is an integer with $s > s_l$. W.l.o.g. we may also assume that $p := \delta_2(x) > q$ (for otherwise $\delta = \delta_1$ and therefore $\delta$ is finitely generated).

Let $X^{\delta_1}$ be the completion of $X := \mathbb{C}^2$ determined by $\delta_1$. Then $X^{\delta_1}$ is a weighted projective space. We construct $X^{\delta}$ as the normalization of a ‘weighted blow-up’ of $X^{\delta_1}$. Two of the components of the blow-up map are polynomials of the form $f_1 := x^{\alpha_1}y^{\beta_1}$ and $f_2 := x^{\alpha_2}(y - \psi(x))^{\beta_2}$ such that $\delta_1(f_j) = \delta_2(f_j)$ for $j = 1$ and $j = 2$. Note that

$$\delta_1(x^{\alpha_1}y^{\beta_1}) = \delta_2(x^{\alpha_1}y^{\beta_1}) \iff q\alpha_1 + qr\beta_1 = p\alpha_1 - ps_1\beta_1 \iff (p - q)\alpha_1 = (qr + ps_1)\beta_1,$$  \hspace{1cm} (5)
and for each \((\alpha_1, \beta_1)\) satisfying \([3]\),
\[
\delta_1(x^{\alpha_1}y^{\beta_1}) = q\beta_1\left(\frac{qr + ps_1}{p - q} + r\right) = pq\beta_1 \frac{r + s_1}{p - q}
\]
Similarly,
\[
\delta_1(x^{\alpha_2}(y - \psi(x))^{\beta_2}) = \delta_2(x^{\alpha_2}(y - \psi(x))^{\beta_2}) \iff \alpha_2 = p\alpha_2 - ps\beta_2
\]
\[
\iff (p - q)\alpha_2 = (qr + ps)\beta_2 \tag{[7]}
\]
and for each \((\alpha_2, \beta_2)\) satisfying \([5]\),
\[
\delta_2(x^{\alpha_2}(y - \psi(x))^{\beta_2}) = q\beta_2\left(\frac{qr + ps}{p - q} + r\right) = pq\beta_2 \frac{r + s}{p - q}
\]
\[
\tag{[8]}
\]
Let \(\beta_1 := r(p - q)(r + s)\) and \(\beta_2 := r(p - q)(r + s_1)\). Then
\[
\alpha_1 = r(r + s)(qr + ps_1), \quad \alpha_2 = r(r + s_1)(qr + ps), \quad \text{and}
\]
\[
\delta_i(f_j) = pq(r + s)(r + s_1) \text{ for all } i, j, \ 1 \leq i, j \leq 2.
\]
Let \(\Psi : \mathbb{C}^2 \to \mathbb{P}^4\) be the map defined by
\[
(x, y) \mapsto (1 : y^{(r+s)(r+s_1)} : x^{qr(r+s)(r+s_1)} : f_1 : f_2).
\]
Note that \(X^{\delta_1} \cong \mathbb{P}^2(1, q, qr) \cong \mathbb{P}^2(1, 1, r)\) (the latter isomorphism is a consequence of the fact that \(\mathbb{P}^2(1, 1, r)\) is the image of \(\mathbb{P}^2(1, q, qr)\) under the \(q\)-uple Veronese embedding, see e.g. [7, Section 1.3.1]). Let the weighted homogeneous coordinates of \(\mathbb{P}^2(1, 1, r)\) be \((W : X : Y)\), so that \(x = X/W\) and \(y = Y/W^r\). Identifying \(X^{\delta_1}\) with \(\mathbb{P}^2(1, 1, r)\) we see that \(\Psi\) extends to a rational map (which we also denote by \(\Psi\)) from \(X^{\delta_1}\) to \(\mathbb{P}^4\) such that
\[
\Psi(W : X : Y) = \left(W^{cpr} : Y^{cpr} : W^{cr(p-q)} : X^{cr} : X^{\alpha_1}Y_{\beta_1}
\right.
\]
\[
\left. : X^{\alpha_2 - \beta_2}((X^{s_1}Y - \sum_{k=1}^l a_k X^{s_1 - s_k}W^{r+s_k})^{\beta_2})\right),
\]
where \(c := (r + s)(r + s_1)\). It follows from the above expression for \(\Psi\) that there is only one point of indeterminacy of \(\Psi\), namely the point \(O\) with weighted homogeneous coordinates \([0 : 1 : 0]\). Consequently, the closure \(\bar{\Gamma}\) of the graph \(\Gamma\) of \(\Psi|_{X^{\delta_1} \setminus \{O\}}\) in \(X^{\delta_1} \times \mathbb{P}^4\) is a disjoint union of \(\Gamma\) and a closed subvariety \(E\) such that \(E\) projects to \(P\) under the natural projection \(\Gamma \to X^{\delta_1}\). We next show that \(E\) is an irreducible curve and find a family of curves in \(X^{\delta_1}\) approaching \(O\) such that their images in \(\Gamma\) approach generic points of \(E\).

**Claim 3.1.1.**

1. \(E\) is an irreducible curve.
2. For all \(b \in \mathbb{C}\), let \(\psi_b(x) := \psi(x) + bx^{-s}\) and \(\gamma_b(t)\) be the curve on \(\mathbb{C}^2\) with associated degree-wise Puiseux series \(y = \psi_b(x)\) (i.e. \(\gamma_b(t) = (t, \psi_b(t))\) for all \(t \in \mathbb{C} \setminus \{0\}\)). Then as \(t \to \infty\), \(\Psi(\gamma_b(t))\) approaches a point on \(E\).
3. Let \(X^{\delta_1}_\infty := X^{\delta_1} \setminus X\) and \(P\) be the point of intersection of \(E\) and the closure of \(\Psi(X^{\delta_1}_\infty)\) in \(\bar{\Gamma}\). Then for all \(z \in E \setminus \{P\}\), \(z = \lim_{t \to \infty} \Psi(\gamma_b(t))\) for some \(b \in \mathbb{C}\).
4. Conversely, for all \( z \in E \setminus \{ P \} \), if \( \gamma(t) \) is any curve on \( \mathbb{C}^2 \) such that \( \lim_{t \to \infty} \Psi(\gamma(t)) = z \), then there exists \( b \in \mathbb{C} \) such that the degree-wise Puiseux expansion (in terms of \( x \)) for \( \gamma \) near \( z \) is

\[
y = \psi_0(x) + \text{l.o.t.}
\]

where l.o.t. is a shortcut for `lower order terms' (in \( x \)).

5. Moreover, for each \( b \in \mathbb{C} \), if the degree-wise Puiseux expansion for \( \gamma \) is as in (9), then

\[
\lim_{t \to \infty} \Psi(\gamma(t)) = \lim_{t \to \infty} \Psi(\gamma_b(t)).
\]

Proof. There is an open affine neighborhood \( U \) of \( O \) such that \( U \cong \mathbb{C}^2 \) with coordinates \( w_0 := W/X = 1/x \) and \( y_0 := Y/X^r = y/x^r \). With respect to \((w_0, y_0)\) coordinates, \( \Psi \) has the following form:

\[
\Psi(w_0, y_0) = \left( w_0^{\beta p} : y_0^{\beta p} : u_0^{cpr(p-q)} : y_0^{\beta_1} : (y_0 - \sum_{k=1}^l a_k w_0^{-r+s_k})^{\beta_2} \right),
\]

(10)

For every \( z \in E \), there is an algebraic curve \( \gamma(t) \in U \) such that \( \gamma(0) = O \) and \( \lim_{t \to 0} \Psi(\gamma(t)) = z \). Since \( O \) has the coordinates \((0, 0)\), we may assume that near \( O \), either

L1. the image of \( \gamma \) lies on the \( y_0 \)-axis, in which case

\[
z = \lim_{t \to 0} \Psi(0, t) = \lim_{t \to 0} \left( 0 : t^{c p} : 0 : t^{\beta_1} : t^{\beta_2} \right) = (0 : 0 : 0 : 0 : 1) =: P,
\]

since \( cp = p(r + s)(r + s_1) > (p - q)(r + s)r = \beta_1 > (p - q)(r + s_1)r = \beta_2 \), or

L2. \( \gamma \) has a parametrization of the form \((t, \phi(t))\), where \( \phi(t) \) is a Puiseux series in \( t \).

Let \( z \in E \setminus \{ P \} \). Let \( \phi(t) := \sum_{k=1}^\infty b_k t^{r_k} \) be as in assertion [12]. Then

\[
\Psi(\gamma(t)) = \left( t^{c p} : k_t^{cp} : \text{h.o.t.} : t^{c r(p-q)} : b_1^{\beta_1} t^{\beta_1 r_1} : \text{h.o.t.} : \left( \sum_{k=1}^\infty b_k t^{r_k} - \sum_{k=1}^l a_k t^{r+s_k} \right)^{\beta_2} \right),
\]

where h.o.t. is a shortcut for `higher order terms' (in \( t \)). Note that \( cr(p-q) < cpr \). Moreover, as proved in assertion [11] above, \( cp > \beta_1 \) and therefore, \( cpr_1 > \beta_1 r_1 \) (since \( r_1 > 0 \)). Consequently,

\[
z = \lim_{t \to 0} \Psi(\gamma(t)) = \lim_{t \to 0} \left( 0 : t^{cr(p-q)} : b_1^{\beta_1} t^{\beta_1 r_1} : \left( \sum_{k=1}^\infty b_k t^{r_k} - \sum_{k=1}^l a_k t^{r+s_k} \right)^{\beta_2} \right).
\]

Let \( \tilde{\phi}(t) := \sum_{k=1}^\infty b_k t^{r_k} - \sum_{k=1}^l a_k t^{r+s_k} \). We now show that \( r_1 = r + s_1 \) and \( b_1 = a_1 \). Indeed, if \( r_1 < r + s_1 \), then \( \text{ord}(\tilde{\phi}(t)) = r_1 \). Moreover, \( cr(p-q) = r(r+s)(r+s_1)(p-q) > r(r+s)r_1(p-q) = \beta_1 r_1 > \beta_2 r_1 = \beta_2 \text{ord}(\tilde{\phi}(t)) \) (the last inequality is a consequence of the fact that \( \beta_1 > \beta_2 \), which we have shown in assertion [11] above). It follows that in this case \( z = \lim_{t \to 0} \Psi(\gamma(t)) = P \), which is a contradiction to our choice of \( z \). Similarly, if \( r_1 > r + s_1 \), or if \( r_1 = r + s_1 \) and \( b_1 \neq a_1 \), then \( \text{ord}(\tilde{\phi}(t)) = r + s_1 \) and therefore \( \beta_1 r_1 = r(r+s)(p-q)r_1 \geq r(r+s)(p-q)(r+s_1) = cr(p-q) > r(p-q)(r+s_1)(r+s_1) = \beta_2 \text{ord}(\tilde{\phi}(t)) \) (here the last inequality is the consequence of the fact that \( s > s_1 \geq s_1 \)). Consequently, in
this case as well, \( z = \lim_{t \to 0} \Psi(\gamma(t)) = P \) which is a contradiction as noted above.

Since \( r_1 = r + s_1 \), it follows that \( cr(p - q) = \beta_1 r_1 = r(p - q)(r + s)(r + s_1) = \beta_2(r + s) \).
Therefore \( \text{ord } \phi(t) \geq r + s \), for otherwise we will again have that \( z = P \). It follows that

\[
\phi(t) = \sum_{k=1}^{l} a_k t^{r+s_k} + bt^{r+s} + \text{h.o.t.} \tag{11}
\]

for some \( b \in \mathbb{C} \). Consequently,

\[
z = \lim_{t \to 0} \left( 0 : 0 : t^{cr(p-q)} : a_1^{\beta_1} t^{cr(p-q)} : \beta_2 t^{cr(p-q)} + \text{h.o.t.} \right) = \left( 0 : 0 : 1 : a_1^{\beta_1} : \beta_2 \right) \tag{12}
\]

It follows that

\[
E = \{(0 : 0 : 1 : a_1^{\beta_1} : \tilde{b}) : \tilde{b} \in \mathbb{C} \} \cup \{P\}. \tag{13}
\]

Then \( E \cong \mathbb{P}^1 \) and therefore \( E \) is irreducible, which proves assertion \( 1 \) of the claim.

It follows from (11) that the Puiseux expansion for \( y_0 \) (in terms of \( w_0 \)) associated with \( \gamma \) is

\[
y_0 = \sum_{k=1}^{l} a_k w_0^{s+k} + bw_0^{s} + \text{h.o.t.}
\]

In terms of \( y \) and \( x \) the above expression becomes

\[
\frac{y}{x^r} = \sum_{k=1}^{l} a_k x^{-(r+s_k)} + bx^{-(r+s)} + \text{l.o.t.}
\]

\[
\Rightarrow y = \sum_{k=1}^{l} a_k x^{-s_k} + bx^{-s} + \text{l.o.t.}
\]

\[
\Rightarrow y = \psi_b(x) + \text{l.o.t.} \tag{14}
\]

Identities (12), (13) and (14) prove assertions [1], [4] and [5] of the claim. It remains to prove that \( P \) is the intersection of \( E \) and the closure of \( \Psi(X_\infty^\delta) \) in \( \bar{\Gamma} \). But we have shown in [11] that \( P \) is the point of intersection of \( E \) and the closure in \( \bar{\Gamma} \) of the image of \( y_0 \)-axis under \( \Psi \). Since \( y_0 \)-axis is precisely \( X_\infty^\delta \), the proof of the claim is complete.

We now show that \( E \) remains irreducible in the normalization of \( \bar{\Gamma} \). More precisely, let \( Z \) be the normalization of \( \bar{\Gamma} \), with the normalization map \( \pi : Z \to \bar{\Gamma} \). Then

**Claim 3.1.2.** \( \bar{E} := \pi^{-1}(E) \) is an irreducible curve. The discrete valuation on \( A \) associated to \( \bar{E} \) is \( -\delta_2/p \).

**Proof.** Let \( E' \) be an irreducible component of \( \bar{E} \) and \( \nu' \) be the discrete valuation on \( A \) corresponding to \( E' \). Let

\[ h := x^s(y - \psi(x)) \in \mathbb{C}[x, y]. \]

At first we show that for every \( a \in \mathbb{C}, \nu'(h - a) = 0 \). Indeed, if \( \nu'(h - a) > 0 \) for some \( a \in \mathbb{C} \), then pick a Zariski open subset \( U' \) of \( Z \) such that \( U' \cap E' \neq \emptyset \) and \( h - a \) is regular on \( U' \).
Then $h$ is also a regular function on $U$ and $h(z) = a$ for all $z \in E' \cap U'$. Now, assertion 2 of claim 3.1.1 implies that there is $z \in E' \cap U'$ such that $\pi(z) = \lim_{t \to \infty} \gamma_b(t)$ for some $b \in \mathbb{C}$, $b \neq a$, where $\gamma_b(t) := (t, \psi(t) + bt^{-s})$ for all $t \in \mathbb{C} \setminus \{0\}$. Pick a curve $\tilde{\gamma}$ on $\pi^{-1}(\mathbb{C}^2)$ such that $\lim_{t \to \infty} \tilde{\gamma}(t) = z$. Assertion 5 of claim 3.1.1 implies that in $(x, y)$-coordinates $\pi(\tilde{\gamma}(t))$ has the following form:

$$\pi(\tilde{\gamma}(t)) = (t, \psi(t) + bt^{-s} + \text{l.o.t.}).$$

It follows that $h(z) = \lim_{t \to \infty} h(\tilde{\gamma}(t)) = \lim_{t \to \infty} h(\pi(\tilde{\gamma}(t))) = b$, which is a contradiction, since $b \neq a$. Therefore $\nu'(h-a) \leq 0$ for all $a \in \mathbb{C}$.

Similarly, if $\nu'(h-a)$ is negative for some $a \in \mathbb{C}$, then $\nu'(h)$ is also negative and we may pick a Zariski open subset $U'$ of $Z$ such that $U' \cap E' \neq \emptyset$ and $h^{-1}$ is regular on $U'$ and $h^{-1}(z) = 0$ for all $z \in E' \cap U'$. Choose $z \in E' \cap U'$ such that $\pi(z) = \lim_{t \to \infty} \gamma_b(t)$ for some $b \in \mathbb{C}$, $b \neq 0$. Then exactly as in the previous paragraph, we may show that $h^{-1}(z) = b^{-1} \neq 0$, which is a contradiction. It follows that for all $a \in \mathbb{C}$, $\nu'(h-a) = 0$, as required.

It follows that for all $a \in \mathbb{C}$,

$$\nu'(y - \psi(x) - ax^{-s}) = -sv'(x). \quad (15)$$

In particular, setting $a = 0$, we see that

$$\nu'(y - \psi(x)) = -sv'(x). \quad (16)$$

Recall that the projection from $\bar{\Gamma}$ to $\bar{X}$ maps $E$ to $O \in \bar{X}$ and $w_0 := 1/x$ is regular at $O$. It follows that $1/x$ is regular at every point of $\bar{E}$ and therefore $\nu'(x) \leq 0$. We claim that $\nu'(x) < 0$. Indeed, if $\nu'(x) = 0$, then $(15)$ implies that $\nu'(f) \geq 0$, which implies that $\nu'(f) \geq 0$ for all $f \in \mathbb{C}[x, y]$. But this is impossible since $\bar{E}$ is a curve at infinity with respect to $\mathbb{C}^2$ (see, e.g., [Goo69, Proposition 1]). Therefore $\nu'(x) < 0$.

Let $\delta' := -\nu'$. Then $\delta'$ satisfies the hypothesis of corollary 2.2 and therefore there is a finite Puiseux series $\tilde{\psi}$ in $x$ and a rational number $\tilde{\delta} < \text{ord}_x(\tilde{\psi})$ such that for all $f \in \mathbb{C}[x, y]$,

$$\delta'(f) = \delta'(x) \deg_x(f(x, \tilde{\psi}(x) + \xi x^{\tilde{\delta}})). \quad (17)$$

Identities $(10)$ and $(15)$ then imply that

$$\deg_x(\tilde{\psi}(x) + \xi x^{\tilde{\delta}} - \psi(x)) = -s, \quad \text{and}$$

$$\deg_x(\tilde{\psi}(x) + \xi x^{\tilde{\delta}} - \psi(x) - ax^{-s}) = -s \quad \text{for all } a \in \mathbb{C}. \quad (19)$$

Since $\psi(x) = \sum_{k=1}^{l} a_k x^{-s_k}$ with $s_1 < s_2 < \cdots < s_l < s$, identities $(15)$ and $(19)$ imply that $\tilde{\psi} = \psi$ and $\tilde{\delta} = s$. It follows that $\delta' = \delta_2$ and $\delta_2$ are proportional (remark ??). Also note that for each $f \in \mathbb{C}[x, y]$, $\delta'(f)$ is a multiple of $\delta'(x)$ (this follows from $(17)$ since $\psi(x) \in \mathbb{C}[x, x^{-1}, y]$ and $s$ is an integer). It follows that $\delta'(x) = 1$ and consequently

$$\delta' = \frac{\delta'(x)}{\delta_2(x)} \delta_2 = \frac{1}{p} \delta_2.$$

Therefore we have shown that every irreducible component of $\bar{E}$ corresponds to the same discrete valuation on $\mathbb{C}[x, y]$. Since the discrete valuation associated to a curve uniquely distinguishes the curve from other curves on a surface, it follows that $\bar{E}$ is irreducible and the discrete valuation associated to $\bar{E}$ is precisely $-\delta_2/p$. \qed
Claim 3.1.2 implies that $Z_{\infty} := Z \setminus \mathbb{C}^2$ has precisely two components - one corresponding to each $\delta_i, 1 \leq i \leq 2$. Consequently, if $\tilde{\delta}$ is a subdegree on $\mathbb{C}[x, y]$ such that the corresponding completion $X^{\tilde{\delta}}$ of $\mathbb{C}$ is isomorphic to $Z$, then $\tilde{\delta} = \max\{r_1\delta_1, r_2\delta_2\}$ for some $r_1, r_2 \in \mathbb{Z}_{>0}$. The special case of the theorem will be proved if we show that there is a projective embedding of $Z$ for which $r_1 = r_2$.

Recall that $Z$ is the normalization of $\bar{\Gamma}$, where $\bar{\Gamma}$ is the closure of

$$\Gamma := \{(z, \Psi(z)) : z \in \mathbb{C}^2\} \subset X^{\delta_1} \times \mathbb{P}^4.$$ 

Therefore, choosing different projective embeddings of $X^{\delta_1}$, we get different projective embeddings of $X^{\delta_1} \times \mathbb{P}^4$, and these in turn induces different projective embeddings of $\bar{\Gamma}$. In particular, let $\phi_d$ be the $dr$-uple embedding of $X^{\delta_1}$, i.e.

$$\phi_d(W : X : Y) := (\cdots : W^\omega X^\alpha Y^\beta : \cdots)$$

where the monomials which appear as homogeneous coordinates in the right hand side of the above expression are precisely $S := \{W^\omega X^\alpha Y^\beta : \omega + \alpha + r\beta = dr\}$. Then composing $\phi_d \times \bar{\Gamma}$ is isomorphic to the closure in $\mathbb{P}^{\delta_1-1} \times \mathbb{P}^4$ of

$$\tilde{\phi}_d(x, y) = \sigma_d(\phi_d(x, y), \Psi(x, y))$$

$$= \sigma_d((1 : y : \cdots : y^d : \cdots : x^{dr}), (1 : g_1 : g_2 : g_3 : g_4))$$

$$= (1 : g_1 : g_2 : g_3 : g_4 : \cdots : x^{dr} g_1 : x^{dr} g_2 : x^{dr} g_3 : x^{dr} g_4)$$

Let $U_\alpha := \mathbb{P}^{\delta_1-1} \setminus V(z_\alpha)$ and $V_j := \mathbb{P}^6 \setminus V(u_j)$. Recall that in the proof of claim 3.1.1 we computed the coordinates of the point $P$ of intersection of the components of $\bar{\Gamma} \setminus \mathbb{C}^2$. In particular, $P \in U_{(0,dr,0)} \times V_4$. Let $\tilde{\delta}^{(d)}$ be a normalization of the degree-like function on $\mathbb{C}[x, y]$ corresponding to the embedding $\tilde{\phi}_d : \mathbb{C}^2 \hookrightarrow \bar{\Gamma}$. Then there exists a positive integer $k_d$ such that the $k_d$-uple divisor at infinity on $X^{\tilde{\delta}^{(d)}}$ is generated by $1/(h_\alpha g_2)$ on $\pi^{-1}(\bar{\Gamma} \cap (U_\alpha \times V_j))$. This implies that

$$D^{\tilde{\delta}^{(d)}}_{k_d, \infty} = -\nu_1(g_4 x^{dr})[X_{\infty,1}] + -\nu_2(g_4 x^{dr})[X_{\infty,2}]$$

Since $g_4 = f_2$, it follows from [5] that $\nu_1(g_4) = -pr(r + s)(r + s + 1)$ and $\nu_2(g_4) = -qr(r + s)(r + s + 1)$. Recall that $\nu_1(x) = \nu_2(y) = -1$. It follows that

$$D^{\tilde{\delta}^{(d)}}_{k_d, \infty} = r(d + p(r + s)(r + s + 1))[X_{\infty,1}] + r(d + q(r + s)(r + s + 1))[X_{\infty,2}]$$

(20)

Now pick two positive integers $p'$ and $q'$ such that $p' > q'$. Note that replacing $p$ and $q$ respectively by $p'$ and $q'$ gives rise to a variety isomorphic to $Z$ [Mon10, Proposition 6.5]. Replacing $p$ and $q$ by suitable $p'$ and $q'$ and choosing a suitable $d$ in (20) we may ensure that

$$D^{\tilde{\delta}^{(d)}}_{k_d, \infty} = mp[X_{\infty,1}] + mq[X_{\infty,2}]$$

for some positive integer $m$. But then $\delta^{(d)} = \max\{k\delta_1, k\delta_2\} = k\delta$ for some $k \geq 1$. Hence $k\delta$ and therefore $\delta$ is finitely generated. This completes the proof of the theorem in the special case. \[\square\]
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