Fixed-Treewidth-Efficient Algorithms for Edge-Deletion to Interval Graph Classes∗

Toshiki Saitoh† Ryo Yoshinaka‡ Hans L. Bodlaender§

Abstract

For a graph class $C$, the $C$-Edge-Deletion problem asks for a given graph $G$ to delete the minimum number of edges from $G$ in order to obtain a graph in $C$. We study the $C$-Edge-Deletion problem for $C$ the class of interval graphs and other related graph classes. It follows from Courcelle’s Theorem that these problems are fixed parameter tractable when parameterized by treewidth. In this paper, we present concrete FPT algorithms for these problems. By giving explicit algorithms and analyzing these in detail, we obtain algorithms that are significantly faster than the algorithms obtained by using Courcelle’s theorem.

1 Introduction

Intersection graphs are represented by geometric objects aligned in certain ways so that each object corresponds to a vertex and two objects intersect if and only if the corresponding vertices are adjacent. Intersection graphs are well-studied in the area of graph algorithms since there are many important applications and we can solve many NP-hard problems in general graphs in polynomial time on such graph classes. Interval graphs are intersection graphs which are represented by intervals on a line. CLIQUE, INDEPENDENT SET, and COLORING on interval graphs can be solved in linear time and interval graphs have many applications in bioinformatics, scheduling, and so on. See [14, 3, 26] for more details of interval graphs and other intersection graphs.

Graph modification problems on a graph class $C$ are to find a graph in $C$ by modifying a given graph in certain ways. $C$-Vertex-Deletion, $C$-Edge-Deletion, and $C$-Completion are to find a graph in $C$ by deleting vertices, deleting edges, and adding edges, respectively, with the minimum cost. These problems can be seen as generalizations of many NP-hard problems. CLIQUE is equivalent to COMPLETE-VERTEX-DELETION: we find a complete graph
by deleting the smallest number of vertices. Modification problems on intersection graph classes also have many applications. For example, INTERVAL-VERTEX/EDGE-DELETION problems have applications to DNA (physical) mapping [13, 12, 27]. Lewis and Yannakakis showed that $C$-VERTEX-DELETION is NP-complete for any nontrivial hereditary graph class [17]. A graph class $C$ is hereditary if for any graph in $C$, every induced subgraph of the graph is also in $C$. Since the class of intersection graphs are hereditary, $C$-VERTEX-DELETION is NP-complete for any nontrivial intersection graph class $C$. The problems $C$-EDGE-DELETION are also NP-hard when $C$ is the class of perfect, chordal, split, circular arc, chain [23], interval, proper interval [13], trivially perfect (a.k.a. nested interval) [25], threshold [20], permutation, weakly chordal, or circle graphs [4]. See the lists in [19, 4].

Parameterized complexity is well-studied in the area of computer science. A problem with a parameter $k$ is fixed parameter tractable, FPT for short, if there is an algorithm running in $f(k)n^c$ time where $n$ is the size of input, $f$ is a computable function and $c$ is a constant. Such an algorithm is called an FPT algorithm. The treewidth $tw(G)$ of a graph $G$ represents treelikeness and is one of the most important parameters in parameterized complexity concerning graph algorithms. For many NP-hard problems in general, there are tons of FPT algorithms with parameter $tw(G)$ by dynamic programming on tree decompositions. Finding the treewidth of an input graph is NP-hard and it is known that CHORDAL-COMPLETION with minimizing the size of the smallest maximum clique is equivalent to the problem. There is an FPT algorithm for computing the treewidth of a graph by Bodlaender [2] which runs in $O(f(tw(G))(n + m))$ time where $n$ and $m$ are the numbers of vertices and edges of a given graph: i.e., the running time is linear in the size of input. Courcelle showed that every problem that can be expressed in monadic second order logic (MSO$_2$) has a linear time algorithm on graphs of bounded treewidth [9]. Some intersection graph classes, for example interval graphs, proper interval graphs, chordal graphs, and permutation graphs, can be represented by MSO$_2$ [8] and thus there are FPT algorithms for EDGE-DELETION problems on such graph classes. However, the algorithms obtained by Courcelle’s theorem have a very large hidden constant factor even when the treewidth is very small, since the running time is the exponential tower of the coding size of the MSO$_2$ expression.

Our results: We propose concrete FPT algorithms for EDGE-DELETION to interval graphs and other related graph classes, when parameterized by the treewidth of the input graph. Our algorithms virtually compute a set of edges $S$ with the minimum size such that $G - S$ is in a graph class $C$ by using dynamic programming on a tree-decomposition. We maintain possible alignments of geometric objects corresponding to vertices in the bag of each node of the tree-decomposition. Alignments of the objects of forgotten vertices are remembered only relatively to the objects of the current bag. If two forgotten objects have the same relative position to the objects of the current bag, we remember only the fact that there is at least one forgotten object at that position. In this way, we achieve the fixed-parameter-tractability, while guaranteeing that
no object pairs of non-adjacent vertices of the input graph will intersect in our
dynamic programming algorithm. Our algorithms run in $O(f(tw(G)) \cdot (n + m))$
time where $n$ and $m$ are the numbers of vertices and edges of the input graph.
Our explicit algorithms are significantly faster than those obtained by using
Courcelle’s theorem. We also analyze the time complexity of our algorithms pa-
rameterized by pathwidth which is analogous to treewidth. The relation among
the graph classes for which this paper provides $\mathcal{C}$-EDGE-DELETION algorithms
is shown in Figure 1.

```
Circular Arc
    /\
   /  \
Interval       Permutation
  /       \       /       \
Proper Interval Trivially Perfect
                        /\
                        /  \
                        Threshold
```

Figure 1: The graph classes of which this paper presents algorithms for the
edge-deletion problems.

**Related works:** Another kind of common parameters considered in parame-
terized complexity of graph modification problems is the number of vertices or
edges to be removed or to be added. Here we review preceding studies on those
problems for intersection graphs with those parameters.

Concerning parameterized complexity of $\mathcal{C}$-VERTEX-DELETION, Hof et al.
proposed an FPT algorithm for PROPER-INTERVAL-VERTEX-DELETION [28],
and Marx proposed an FPT algorithm for CHORDAL-VERTEX-DELETION [21].
Heggernes et al. showed PERFECT-VERTEX-DELETION and WEAKLY-CHORDAL-
VERTEX-DELETION are $\mathcal{W}[2]$-hard [15]. Cai showed that $\mathcal{C}$-VERTEX/EDGE-
DELETION are FPT when $\mathcal{C}$ is characterized by a finite set of forbidden induced
subgraphs [5].

For modification problems on interval graphs, Villanger et al. presented an
FPT algorithm for INTERVAL-COMPLETION [29], and Cao and Marx presented
an FPT algorithm for INTERVAL-VERTEX-DELETION [2]. Cao improved these
algorithms and developed an FPT algorithm for EDGE-DELETION [6].

It is known that THRESHOLD-EDGE-DELETION, CHAIN-EDGE-DELETION
and TRIVIALLY-PERFECT-EDGE-DELETION are FPT, since threshold graphs,
chain graphs and trivially perfect graphs are characterized by a finite set of
forbidden induced subgraphs [5]. Nastos and Gao presented faster algorithms for
the problems [22], and Liu et al. improved their algorithms to $O(2.57^k(n + m))$
and $O(2.42^k(n + m))$ using modular decomposition trees [18], where $k$ is the
number of deleted edges. There are algorithms to find a polynomial kernel for
CHAIN-EDGE-DELETION and TRIVIALLY PERFECT-EDGE-DELETION [1][11].

**Organization of this article:** Section 2 prepares the notation and defini-
tions used in this paper. We propose an FPT algorithm for INTERVAL-EDGE-
DELETION in Section 3. We then extend the algorithm related to the interval
graphs in Section 4. We conclude this paper and provide some open questions in Section 6.

2 Preliminaries

For a set $X$, its cardinality is denoted by $|X|$. A partition of $X$ is a tuple $(X_1, \ldots, X_k)$ of subsets of $X$ such that $X = X_1 \cup \cdots \cup X_k$ and $X_i \cap X_j = \emptyset$ if $1 \leq i < j \leq k$, where we allow some of the subsets to be empty. For entities $x, y, z \in X$, we let $x[y/z] = y$ if $x = z$, and $x[y/z] = x$ otherwise. For a subset $Y \subseteq X$, define $Y[y/z] = \{ x[y/z] \mid x \in Y \}$. For a linear order $\pi$ over a finite set $X$, the maximum and the minimum elements of $X\text{ w.r.t. } \pi$ are denoted by $\max_\pi X$ and $\min_\pi X$, respectively. We denote the successor of $x \in X\text{ w.r.t. } \pi$ by $\text{succ}_\pi(x) = \min_\pi \{ y \in X \mid x <_\pi y \}$. Note that $\text{succ}_\pi(\max_\pi X)$ is undefined. Similarly $\text{pred}_\pi(x) = \max_\pi \{ y \in X \mid y <_\pi x \}$ denotes the predecessor of $x$.

A simple graph $G = (V, E)$ is a pair of vertex and edge sets, where each element of $E$ is a subset of $V$ consisting of exactly two elements.

A tree-decomposition of $G = (V, E)$ is a tree $T$ such that\footnote{We use the terms “vertices” for an input graph and “nodes” for a tree-decomposition.}

- to each node of $T$ a subset of $V$ is assigned,
- if the assigned sets of two nodes of $T$ contain a vertex $u \in V$, then so does every node on the path between the two nodes,
- for each $\{u, v\} \in E$, there is a node of $T$ whose assigned set includes both $u$ and $v$.

The width of a tree-decomposition is the maximum cardinality of the assigned sets minus one and the treewidth of a graph is the smallest width of its tree-decompositions. A tree-decomposition is said to be nice if it is rooted, the root is assigned the empty set, and its nodes are grouped into the following four:

- leaf nodes, which have no children and are assigned the empty set,
- introduce nodes, each of which has just one child, where the set assigned to the node adds one vertex to the child’s set,
- forget nodes, each of which has just one child, where the set assigned to the node removes one vertex from the child’s set,
- join nodes, each of which has just two children, where the same vertex set is assigned to the node and its children.

It is known that every tree-decomposition has a nice tree-decomposition of the same width whose size is $O(k|V|)$, where $k$ is the treewidth of the tree-decomposition \cite{Leung1981, Tinhofer1986}. Hereafter, under a fixed graph $G$ and a fixed nice tree-decomposition $T$, we let $X_s$ denote the subset of $V$ assigned to a node $s$ of a
tree-decomposition and $X_{\leq s}$ denote the union of all the subsets assigned to the node $s$ and its descendant nodes. We call vertices in $X_s$ and in $X_{\leq s} - X_s$ active and forgotten, respectively. Moreover, we define $E_s = \{ \{u, v\} \in E \mid u, v \in X_s\}$ and $E_{\leq s} = \{ \{u, v\} \in E \mid u, v \in X_{\leq s}\}$. Given a tree decomposition of treewidth $k$, we can compute a nice tree-decomposition with treewidth $k$ and $O(kn)$ nodes in $O(k^2(n + m))$ time [11].

A tree-decomposition is called a path-decomposition if the tree is a path. The pathwidth of a graph is the smallest width of its path-decompositions. Every path-decomposition has a nice path-decomposition of the same pathwidth, which consists of leaf, introduce, forget, but not join nodes.

The problem we tackle in this paper is given as follows.

**Definition 1.** For a graph class $C$, the $C$-Edge-Deletion is a problem to find the minimum natural number $c$ such that there is a subgraph $G' = (V, E')$ of $G$ with $G' \in C$ and $|E| - |E'| = c$ for an input simple graph $G = (V, E)$.

In the succeeding sections, for different classes $C$ of intersection graphs, we present algorithms for $C$-Edge-Deletion that run in linear time in the input graph size when the treewidth is bounded. We assume that the algorithm takes a nice tree-decomposition $T$ of $G$ in addition as input [2, 16]. Our algorithms are dynamic programming algorithms that recursively compute solutions (and some auxiliary information) for the subproblems on $(X_{\leq s}, E_{\leq s})$ for each node $s$ in the given tree-decomposition from leaves to the root.

### 3 Finding a Largest Interval Subgraph

An interval representation $\pi$ over a set $X$ is a linear order over the set $LR_X = L_X \cup R_X \cup \{\bot, \top\}$ with $L_X = \{l_x \mid x \in X\}$ and $R_X = \{r_x \mid x \in X\}$ such that \( \bot <_{\pi} l_x <_{\pi} r_x <_{\pi} \top \) for all $x \in X$. An interval is a pair \((p, q) \in LR_X \times LR_X\). We say that two intervals \((p_1, q_1)\) and \((p_2, q_2)\) intersect, denoted by \((p_1, q_1) \parallel_{\pi} (p_2, q_2)\) if $p_1 <_{\pi} q_2$ and $p_2 <_{\pi} q_1$. Otherwise, we write \((p_1, q_1) \perp_{\pi} (p_2, q_2)\).

The interval graph $G_\pi$ of an interval representation $\pi$ on $V$ is $(V, E_\pi)$ where

$$E_\pi = \{\{u, v\} \subseteq V \mid (l_u, r_u) \parallel_{\pi} (l_v, r_v) \text{ and } u \neq v\}.$$ 

Figure 2 (a) and (b) show an example of an interval representation and the defined interval graph, respectively.

This section presents an FPT algorithm for the interval edge deletion problem w.r.t. the treewidth. Let $G = (V, E)$ be an input graph and $s$ a node of a nice tree-decomposition $T$ of $G$. On each node $s$ of $T$, for each interval representation $\rho$ over $X_{\leq s}$ that gives an interval subgraph of $(X_{\leq s}, E_{\leq s})$, we would like to remember some pieces of information about $\rho$, which we call the “abstraction” of $\rho$. The abstraction is the triple $(\pi, I, c)$, where the linear order $\pi$ over $LR_X$ is the restriction of $\rho$ to $X_s$, forgotten vertices are represented in $I$ by anchoring the ends of intervals formed by forgotten vertices to active vertices, and $c$ counts the number of edges of $E - E_\rho$ such that at least one of their
ends is forgotten. For each forgotten vertex \( w \in X_{\leq s} - X_s \), let the intersection closure of \( w \) (w.r.t. \( \rho \) and \( s \)) be the smallest set \([w]_\rho^s \subseteq LR_{X_{\leq s} - X_s}\) such that

- \( l_w, r_w \in [w]_\rho^s \)
- if \( l_u, r_u \in [w]_\rho^s \), \( v \in X_{\leq s} - X_s \) and \((l_u, r_u) \not\subseteq (l_v, r_v)\), then \( l_v, r_v \in [w]_\rho^s \).

For an interval representation \( \rho \) over \( X_{\leq s} \) such that \( G_\rho \) is a subgraph of \((X_{\leq s}, E_{\leq s})\), we define the abstraction \( \mathcal{A}(\rho, s) \) of \( \rho \) for \( s \) to be the triple \((\pi, I, c)\) such that

- \( \pi \) is the restriction of \( \rho \) to \( LR_{X_s} \),
- \( I = \{ (p, q) \in LR_{X_s} \times LR_{X_s} \mid \) there is \( w \in X_{\leq s} - X_s \) such that \( p <_\rho \min_{\rho} [w]_\rho^s <_\rho \operatorname{suc}_\pi(p) \) and \( q <_\rho \max_{\rho} [w]_\rho^s <_\rho \operatorname{suc}_\pi(q) \}\),
- \( c = \left| E_{\leq s} - E_{\rho} - E_s \right| = \left| \{ (u, v) \in E_{\leq s} \mid u \notin X_s \text{ and } (l_u, r_u) \not\subseteq (l_v, r_v) \} \right| \).

We call elements of \( I \) forbidden intervals because introducing a new vertex interval intersecting a forgotten interval means making the new vertex and a forgotten vertex adjacent, which is forbidden. Figure 2(c) shows an example of \( I \). Since forbidden intervals are formed by intersection closures, no distinct forbidden intervals may intersect. If \( s \) is the root of a nice tree-decomposition, the abstraction will be \( \mathcal{A}(\rho, s) = (o, \{(\bot, \top)\}, c_\rho) \) where \( o \) is the trivial order on \{\( \bot, \top \)\} such that \( \bot <_o \top \) and \( c_\rho = |E - E_\rho| \). That is, the smallest \( c_\rho \) for an interval subgraph \( G_\rho \) is the solution to our problem.

However, we do not have to compute \( \mathcal{A}(\rho, s) \) of all the possible interval representations \( \rho \) over \( X_{\leq s} \) on each node \( s \). We say that \((\pi, I, c)\) dominates \((\pi', I', c')\) if and only if \( \pi = \pi' \), \( I \sqsubseteq_\pi I' \) and \( c \leq c' \), where we write \( I \sqsubseteq_\pi I' \) if every forbidden interval of \( I \) is inside some forbidden interval of \( I' \): i.e., for

\[\frac{\text{Figure 2: (a) Visualization of interval representation } \rho. \text{ (b) Interval graph } G_\rho. \text{ (c) For } X_s = \{x, y, z\} \text{ and } X_{\leq s} - X_s = \{u, v, w\}, \text{ we have } [u]_\rho^s = \{l_u, r_u\}, [v]_\rho^s = [w]_\rho^s = \{l_v, r_v, l_w, r_w\}, \text{ and } \mathcal{A}(\rho, s) = (\pi, I, 0) \text{ where } I = \{(l_x, l_z), (r_x, r_z)\}.}{\text{ends is forgotten. For each forgotten vertex } w \in X_{\leq s} - X_s, \text{ let the intersection closure of } w \text{ (w.r.t. } \rho \text{ and } s) \text{ be the smallest set } [w]_\rho^s \subseteq LR_{X_{\leq s} - X_s} \text{ such that}}\]

\[\begin{align*}
\bullet & \ l_w, r_w \in [w]_\rho^s \text{ and } \\
\bullet & \text{ if } l_u, r_u \in [w]_\rho^s, v \in X_{\leq s} - X_s \text{ and } (l_u, r_u) \not\subseteq (l_v, r_v), \text{ then } l_v, r_v \in [w]_\rho^s.
\end{align*}\]

\[\text{For an interval representation } \rho \text{ over } X_{\leq s} \text{ such that } G_\rho \text{ is a subgraph of } (X_{\leq s}, E_{\leq s}), \text{ we define the abstraction } \mathcal{A}(\rho, s) \text{ of } \rho \text{ for } s \text{ to be the triple } (\pi, I, c) \text{ such that}\]

\[\begin{align*}
\bullet & \ \pi \text{ is the restriction of } \rho \text{ to } LR_{X_s}, \\
\bullet & \ I = \{ (p, q) \in LR_{X_s} \times LR_{X_s} \mid \text{ there is } w \in X_{\leq s} - X_s \text{ such that } p <_\rho \min_{\rho} [w]_\rho^s <_\rho \operatorname{suc}_\pi(p) \text{ and } q <_\rho \max_{\rho} [w]_\rho^s <_\rho \operatorname{suc}_\pi(q) \}, \\
\bullet & \ c = \left| E_{\leq s} - E_{\rho} - E_s \right| = \left| \{ (u, v) \in E_{\leq s} \mid u \notin X_s \text{ and } (l_u, r_u) \not\subseteq (l_v, r_v) \} \right|.
\end{align*}\]

\[\text{We call elements of } I \text{ forbidden intervals because introducing a new vertex interval intersecting a forgotten interval means making the new vertex and a forgotten vertex adjacent, which is forbidden. Figure 2(c) shows an example of } I. \text{ Since forbidden intervals are formed by intersection closures, no distinct forbidden intervals may intersect. If } s \text{ is the root of a nice tree-decomposition, the abstraction will be } \mathcal{A}(\rho, s) = (o, \{(\bot, \top)\}, c_\rho) \text{ where } o \text{ is the trivial order on } \{\bot, \top\} \text{ such that } \bot <_o \top \text{ and } c_\rho = |E - E_\rho|. \text{ That is, the smallest } c_\rho \text{ for an interval subgraph } G_\rho \text{ is the solution to our problem.}\]

\[\text{However, we do not have to compute } \mathcal{A}(\rho, s) \text{ of all the possible interval representations } \rho \text{ over } X_{\leq s} \text{ on each node } s. \text{ We say that } (\pi, I, c) \text{ dominates } (\pi', I', c') \text{ if and only if } \pi = \pi' \text{, } I \sqsubseteq_\pi I' \text{ and } c \leq c', \text{ where we write } I \sqsubseteq_\pi I' \text{ if every forbidden interval of } I \text{ is inside some forbidden interval of } I': \text{ i.e., for}\]

\[\text{Actually presence of intervals } (\bot, \bot) \text{ and } (\text{pred}_x(\top), \text{pred}_x(\top)) \text{ in } I \text{ does not matter.}\]

\[\text{6}\]
every \((p, q) \in I\), there is \((p', q') \in I'\) such that \(p' \leq \pi p\) and \(q \leq \pi q'\). In this case, every possible way of introducing new intervals to \(\rho'\) is also possible for \(\rho\) by cheaper or equivalent cost. Therefore, it is enough to remember \(\mathcal{A}(\rho, s)\) discarding \(\mathcal{A}(\rho', s)\). For a set of abstractions, the process of removing ones which are dominated by others is called reduction. We call a set of abstractions reduced if it has no pair of distinct elements such that one dominates the other.

**Lemma 1.** Suppose \(\mathcal{A}(\rho, s) = (\pi, I, c)\) dominates \(\mathcal{A}(\rho', s) = (\pi', I', c')\). For any extension \(\sigma'\) of \(\rho'\) such that \(E_{\sigma'} \subseteq E_{\leq s}\), there is an extension \(\sigma\) of \(\rho\) such that \(E_{\sigma'} \subseteq E_{\sigma} \subseteq E_{\leq s}\).

Our algorithm calculates a reduced set \(\mathcal{I}_s\) of abstractions of interval representations of interval subgraphs of \((X_{\leq s}, E_{\leq s})\) for each node \(s\) of \(T\) which satisfies the following invariant.

**Condition 1.**

- Every element \((\pi, I, c) \in \mathcal{I}_s\) is the abstraction of some interval representation of an interval subgraph of \((X_{\leq s}, E_{\leq s})\) for \(X_s\),
- Any interval representation \(\rho\) of any interval subgraph of \((X_{\leq s}, E_{\leq s})\) has an element of \(\mathcal{I}_s\) that dominates its abstraction \(\mathcal{A}(\rho, X_s)\).

Since \(\mathcal{I}_s\) is reduced, if \(X_s = \emptyset\), we have \(\mathcal{I}_s = \{(\perp, \perp)\}\) for some \(I \subseteq \{(\perp, \perp)\}\) and \(c \in \mathbb{N}\), where \(\perp\) is the trivial order such that \(\perp <_\perp \top\). Particularly for the root node \(s\), the number \(c\) is the least number such that one can obtain an interval subgraph by removing \(c\) edges from \(G\). That is, \(c\) is the solution to our problem. If \(s\) is a leaf, \(\mathcal{I}_s = \{(\emptyset, 0, 0)\}\) by definition. It remains to show how to calculate \(\mathcal{I}_s\) from the child(ren) of \(s\), while preserving the invariant (Condition 1).

**Introduce Node:** Suppose that \(s\) is an introduce node. It has just one child \(t\) such that \(X_s = X_t \cup \{x\}\). For an extension \(\pi'\) of \(\pi\) for \((\pi, I, c) \in \mathcal{I}_t\), let us anchor the new points \(l_x\) and \(r_x\) to points \(p_0\) and \(q_0\) in \(LR_{X_t}^*\):

\[
\begin{align*}
p_0 &= \text{pred}_{\pi'}(l_x), \\
q_0 &= \begin{cases} 
\text{pred}_{\pi'}(r_x) & \text{if } \text{pred}_{\pi'}(r_x) \neq l_x, \\
\text{pred}_{\pi'}(l_x) & \text{otherwise}.
\end{cases}
\end{align*}
\]

We say \(\pi'\) respects \(E\) and \(I\) if

- \(\{x, u\} \notin E\) for \(u \in X_t\) implies \((l_x, r_x) \parallel_{\pi'} (l_u, r_u)\),
- \((p, q) \in I\) implies \((p_0, q_0) \parallel_{\pi} (p, q)\),

respectively. If \(\pi'\) does not respect \(E\) (resp. \(I\)), then it means that we are creating an edge between \(x\) and a vertex \(u\) in \(X_s\) (resp. in \(X_{\leq s} - X_s\)) which are not adjacent in the input graph \(G\). Figure 3 shows examples of extensions that
Example 1. Figure 3 shows several possibilities of extending $\pi$ ($m <_{\pi} n <_{\pi} p <_{\pi} q$) to $\pi'$ by introducing $l_x$ and $r_x$. Defining $\pi'$ by letting $m <_{\pi'} n <_{\pi'} l_x <_{\pi'} p <_{\pi'} r_x <_{\pi'} r_z <_{\pi'} q$ is illustrated by the interval of $x_3$, where $I_L = \{(m, n)\}$, $I_R = \{(p, p), (p, q)\}$, and $I_M = \emptyset$, in which case the anchors $(p, p), (p, q) \in I_R$ of forbidden intervals will be updated to $(r_x, r_z), (r_x, q)$. Defining $\pi'$ by letting $m <_{\pi'} n <_{\pi'} r_z <_{\pi'} r_x <_{\pi'} q$ gives $I_L = \{(m, n)\}$, $I_R = \{(p, q)\}$, and $I_M = \{(p, p)\}$. We consider two ways to put the interval $(l_x, r_x)$ relative to the forbidden interval anchored to $(p, p) \in I_M$. One is to put $(l_x, r_x)$ left to those forbidden intervals, like the interval of $x_1$ shows, in which case we update $(p, p) \in I_M$ to $(r_x, r_x)$, so we obtain $I_1 = I_L \cup I_M' \cup I_R' = \{(m, n), (r_x, r_x), (r_x, q)\}$. The other is to put $(l_x, r_x)$ right to them, as $x_2$ shows, in which case we keep $(p, p) \in I_M$, so we obtain $I_2 = I_L \cup I_M \cup I_R' = \{(m, n), (p, p), (r_x, q)\}$. We note that there can be several non-intersecting forbidden intervals between $p$
we make \( I' = \{(m, n), (p, p), (r_x, r_x), (r_x, q)\} \) but we ignore this possibility, due to \( I_1, I_2 \subseteq I' \).

We then obtain \( \mathcal{F}_s \) by reducing \( \mathcal{F}'_s \).

**Forget Node:** Suppose that \( s \) is a forget node. It has just one child \( t \) such that \( X_t = X_s \cup \{x\} \). For each \((\pi, I, c)\) in \( \mathcal{F}_t \), in accordance with the definition of abstractions, we add to \( \mathcal{F}'_s \) the triple \((\pi', I', c)\) where

- \( \pi' \) is the restriction of \( \pi \),
- \( c' = c + |\{ \{x, u\} \in E \mid (l_u, r_u) \not\sqsubset (l_x, r_x) \text{ and } u \in X_s \}|\).

We make \( I' \) from \( I \) by

- making a new forbidden interval involving \( (l_x, r_x) \) and
- re-anchoring forbidden intervals if they have an anchor \( l_x \) or \( r_x \).

Let us anchor the points \( l_x \) and \( r_x \) to points \( p_0 \) and \( q_0 \) in \( X_s \):

\[
p_0 = \max_{\pi'} \{ p \in X_s \mid p <_{\pi} l_x \} = \text{pred}_{\pi}(l_x),
\]

\[
q_0 = \max_{\pi'} \{ p \in X_s \mid p <_{\pi} r_x \} = \begin{cases} \text{pred}_{\pi}(r_x) & \text{if } \text{pred}_{\pi}(r_x) \neq l_x, \\ \text{pred}_{\pi}(l_x) & \text{otherwise}. \end{cases}
\]

The set \( I_X \subseteq I \) of forbidden intervals that intersect with \((l_x, r_x)\) will be given below. They will be merged into one in \( I' \).

\[
I_X = \{(p, q) \in I \mid (p_0, r_x) \not\sqsubset (p, q) \}
\]

\[
p_* = \min_{\pi} (\{p_0\} \cup \{ p \mid (p, q) \in I_X \})
\]

\[
q_* = \max_{\pi} (\{r_x\} \cup \{ q \mid (p, q) \in I_X \})
\]

Figure 4 may help understanding why we take \((p, q) \in I \) “intersecting” with \((p_0, r_x)\) rather than with \((l_x, r_x)\). This comes from the gap of the meanings of position pairs \((l_x, r_x)\) and \((p, q) \in I \). While the interval \((l_x, r_x)\) of \( x \) starts exactly at \( l_x \) and ends at \( r_x \), the forbidden interval anchored to \((p, q) \in I \) starts after \( p \) and ends after \( q \). Although forbidden intervals anchored to \((p, l_x)\) for some \( p \leq_{\pi} l_x \) must intersect with the interval \((l_x, r_x)\) of \( x \), we have \((p, l_x) \not\sqsubset (l_x, r_x)\).
For each compatible pair \((p, q)\) in \(I\), we will make two vertices adjacent which are not adjacent in the input graph \(G\). Abstractions are any interval representations \(\pi\) of forbidden intervals. Suppose that \(x\) is not compatible, any interval representation \(I\) of \(\pi\) to \(\mathcal{F}_s\) holds. Then we obtain \(\mathcal{F}_s\) by reducing \(\mathcal{F}_s\).

**Join Node:** Suppose that \(s\) has two children \(t_1\) and \(t_2\), where \(X_s = X_{t_1} = X_{t_2}\). We say that \(A_1 = (\pi_1, I_1, c_1) \in \mathcal{F}_{t_1}\) and \(A_2 = (\pi_2, I_2, c_2) \in \mathcal{F}_{t_2}\) are compatible if \(\pi_1 = \pi_2\) and there are no \((p_1, q_1) \in I_1\) and \((p_2, q_2) \in I_2\) such that \((p_1, q_1) \cup p_2\) is not compatible, which is witnessed by the pair of \((n, n) \in I_1\) and \((m, p) \in I_2\). If one of them was absent, they were compatible.

According to the definition. On the other hand, forbidden intervals anchored to \((r_x, q)\) for some \(q \geq r_x\) do not intersect with \((l_x, r_x)\) and \((r_x, q) / /_\pi (l_x, r_x)\) holds. The new forbidden interval will be

\[
I' = (I - I_X \cup \{(p, q_s)\})[q_0/r_x],
\]

Then we obtain \(\mathcal{F}_s\) by reducing \(\mathcal{F}_s\).

**Theorem 1.** The edge deletion problem for interval graphs can be solved in \(O(|V(2k)! \cdot 2^{0.76k} \text{poly}(k))\) time for the treewidth \(k\) of \(G\) and some polynomial function \(\text{poly}\). If \(k\) is the pathwidth, it can be solved in \(O(|V(2k)! \cdot 2^{3.38k} \text{poly}(k))\) time.

**Proof.** Let \(k\) be the maximum size of the assigned set \(X_s\) to a node of a nice tree-decomposition. We first estimate the number \(N\) of possible forbidden interval sets \(I\) for a fixed \(\pi\) in \((\pi, I, c) \in \mathcal{F}_s\). Recall that \(I\) must satisfy that

- \((p, q) \in I\) implies \(p \leq q\),
- \((p, q_1), (p, q_2) \in I\) and \(p \neq q_1, q_2\) implies \(q_1 = q_2\),
- \((p_1, q), (p_2, q) \in I\) and \(q \neq p_1, p_2\) implies \(p_1 = p_2\).

That is, there are at most three intervals in \(I\) that involves each \(p \in LR_{X_s} - \{\top\}\):

1. ending a forbidden interval started earlier: \((q, p) \in I\) for some \(q < p\).
Figure 6: Possible ways to extend an interval set over \( n \) points to one over \( n+1 \) points. (a) If the last interval is right-bounded, there are four ways to add \( p \) to the set. (b) If the last interval is right-unbounded, there are five ways to add \( p \) to the set.

(2) starting and ending a minimal forbidden interval: \((p, p) \in I\),

(3) starting a new forbidden interval: \((p, q) \in I\) for some \( q > \pi p \).

Those possibilities are neither mutually exclusive nor independent. To count the number of possible forbidden intervals inductively, hereafter we will also count interval sets that may contain a right-unbounded interval. Let \( A_b(n) \) and \( A_u(n) \) be the numbers of possible ways of drawing forbidden intervals with \( n \) points in addition to \( \bot \) where the last intervals are right-bounded and right-unbounded, respectively. Solving the recurrence equations

\[
\begin{align*}
A_b(n + 1) &= 2A_b(n) + 2A_u(n), & A_b(0) &= 2, \\
A_u(n + 1) &= 2A_b(n) + 3A_u(n), & A_u(0) &= 1,
\end{align*}
\]

we obtain \( A_b(n) \in O(2^{2.19n}) \). Therefore, since we have \( 2k \) points, there are at most \( N \in O(2^{4.38k}) \) possible sets of forbidden intervals. There can be at most \( (2k)!/2^k \) varieties of \( \pi \). Recall that \( \mathcal{I} \) is reduced, where if \( \mathcal{I} \) has two elements of the form \((\pi, I, c)\) and \((\pi, I, c')\), then \( c = c' \). We conclude that each \( \mathcal{I} \) may contain at most \((2k)!/2^k N \in O(2^{3.38k}(2k)!))\) elements.

Suppose \( s \) is a forget node, whose child is \( t \). Then, from each element \((\pi, I, c) \in \mathcal{I} \), we calculate just one element \((\pi', I', c') \in \mathcal{I}' \). Therefore, the calculation can be done in \( O(N(2k)!/2^k \text{Poly}(k)) \) time.

Suppose \( s \) is an introduce node, whose child is \( t \). Then, each element \((\pi', I', c') \in \mathcal{I} \) is derived from a unique element \((\pi, I, c) \in \mathcal{I} \). Therefore, the calculation can be done in \( O(N(2k)!/2^k \text{Poly}(k)) \) time.

Suppose \( s \) is a join node with children \( t_1 \) and \( t_2 \). Recall that \((\pi_1, I_1, c_1) \in \mathcal{I}_{t_1} \) and \((\pi_2, I_2, c_2) \in \mathcal{I}_{t_2} \) are compatible only when \( \pi_1 = \pi_2 \). Checking the compatibility of \((\pi, I_1, c_1)\) and \((\pi, I_2, c_2)\) and computing their “join” \((\pi, I_1 \cup I_2, c_1 + c_2)\) takes \( \text{Poly}(k) \) time. Since we have at most \( N^2 \) pairs to examine for each \( \pi \), it takes \( O((2k)!/2^k N^2 \text{Poly}(k)) \) time.

Since the nice tree-decomposition has \( O(|V|) \) nodes, we obtain the conclusion.
Figure 7: (a) Permutation representation $\rho$. (b) Permutation graph $G_\rho$. (c) Illustration of the abstraction $\mathcal{A}(\rho, s) = (\pi, I, c)$ where $X_s = \{u_1, u_2, u_3\}$ and $X_{\leq s} - X_s = \{w_1, w_2, w_3, w_4\}$. The forbidden areas are represented by $I = \{(w_2, u_1, u_1, u_1), (u_1, u_1, u_1, u_1)\}$, where the intersection closure $[w_1]_\rho^s = \{w_1, w_2, w_3\}$ is anchored to $(u_2, u_1, u_1, u_1)$ and $[w_4]_\rho^s = \{w_4\}$ is anchored to $(u_1, u_1, u_1, u_1)$ in $\mathcal{A}(\rho, s)$.

4 Finding a Largest Permutation Subgraph

A permutation representation over a set $X$ is a pair $\pi = (\pi_1, \pi_2)$ of linear orders on $X^+ = X \cup \{\top, \bot\}$ such that $\bot <_{\pi_i} x <_{\pi_i} \top$ for all $x \in X$ and $i \in \{1, 2\}$. For two points $u$ and $v$ in $X^+$, we write $u \parallel _\pi v$ if $\pi_1$ and $\pi_2$ agree on the order of $u$ and $v$: that is, either $u \leq_{\pi_1} v$ and $u \leq_{\pi_2} v$ or $v <_{\pi_1} u$ and $v <_{\pi_2} u$. Otherwise, we write $u \smallfrown _\pi v$ and say that $u$ and $v$ intersect in $\pi$. The permutation graph $G_\pi$ of a permutation representation $\pi$ on $V$ is $(V, \mathcal{E}_\pi)$ where

$$\mathcal{E}_\pi = \{ \{u, v\} \subseteq V \mid u \text{ and } v \text{ intersect in } \pi \}.$$ 

Figure 7 (a) shows an example of a permutation representation $\rho$ and (b) shows the induced permutation graph $G_\rho$. For each vertex $u$ of $V$, we put a point on each of the two parallel lines respecting the linear orders $\pi_1$ and $\pi_2$ and draw a line, called the $u$-line, between the points. Then we make an edge between $u$ and $v$ if and only if the $u$-line and the $v$-line intersect. This section gives an FPT algorithm for the edge deletion problem for permutation graphs.

4.1 Algorithm Invariant

We anchor the areas bordered with the lines of forgotten vertices $X_{\leq s} - X_s$ to the points of the current bag $X_s$. For each $w \in X_{\leq s} - X_s$, let the intersection closure of $w$ (w.r.t. $\rho$ and $s$) be the smallest set $[w]_\rho^s$ such that

- $w \in [w]_\rho^s$ and
- if $w_1 \in [w]_\rho^s$, $w_2 \in X_{\leq s} - X_s$, and $w_1 \smallfrown _\rho w_2$, then $w_2 \in [w]_\rho^s$.

Each closure forms a forbidden area in the sense that we may introduce no new lines that intersect the area.

For each permutation representation $\rho = (\rho_1, \rho_2)$ over $X_{\leq s}$, we define the abstraction $\mathcal{A}(\rho, s) = ((\pi_1, \pi_2), I, c)$ of $\rho$ for $s$ as follows:
• π₁ and π₂ are the restrictions of ρ₁ and ρ₂ to X_s, respectively.

• I = \{ (p₁, q₁, p₂, q₂) ∈ (X^+_s)^4 \mid \text{there is } w ∈ X_{≤s} − X_s \text{ s.t. for } i ∈ \{1, 2\}, p_i < ρ_i, \min_{ρ_i} [w]_p < ρ_i, \text{suc}_π_i(p_i) \text{ and } q_i < ρ_i, \max_{ρ_i} [w]_p < ρ_i, \text{suc}_π_i(q_i) \}\,

• c = |E_{≤s} − E_ρ − E_s| = \{ (u, v) ∈ E_{≤s} \mid (u, v) ∉ X_s \text{ and } u \not\sim ρ v \}.

Intuitively, I anchors occurrences of forgotten vertices in ρ to occurrences of active vertices in π = (π₁, π₂). Figure 7 (c) illustrates an example of forbidden areas. We note that, two distinct elements (p₁, q₁, p₂, q₂) ∈ I and (p'₁, q'₁, p'₂, q'₂) ∈ I cannot “intersect”, due to the definition of intersection closures. In other words, there are no (p₁, q₁, p₂, q₂), (p'₁, q'₁, p'₂, q'₂) ∈ I such that p_i < π_i, q_i and p'_j < π_j, q_j for some i, j ∈ \{1, 2\}. Elements of I will be linearly ordered by extending π so that (p₁, q₁, p₂, q₂) < π (p'₁, q'₁, p'₂, q'₂) if and only if p₁ < q'₁ or p₂ < q'₂. The integer c is the number of the edges deleted from (X_{≤s}, E_{≤s}) except those among active vertices.

We say that \((π, I, c)\) dominates \((π', I', c')\) if and only if π = π', I ⊆ π I' and c < c', where we write I ⊆ π I' if every forbidden area of I is inside some forbidden area of I': i.e., for every (p₁, q₁, p₂, q₂) ∈ I, there is (p'₁, q'₁, p'₂, q'₂) ∈ I' such that p'_i ≤ π_i, p_i ≤ π_j, q'_j < π_j, q_j such that p'_i ≤ π_i, p_i ≤ π_j, q'_j < π_j, q_j.

**Lemma 2.** Suppose \(A(ρ, s) = (π, I, c)\) dominates \(A(ρ', s) = (π', I', c')\). For any extension \(σ'\) of \(ρ'\) such that \(E_{σ'} ⊆ E_{≤s}\), there is an extension \(σ\) of \(ρ\) such that \(E_{σ'} ⊆ E_σ ⊆ E_{≤s}\).

We call a set of abstractions reduced if it has no pair of distinct elements of which one dominates the other. The reduced form of a set of abstractions is obtained by removing abstractions that are dominated by others. Our algorithm for PERMUTATION-EDGE-DELETION calculates, for each node s of the tree-decomposition, a reduced set \(I_s\) of abstractions of permutation representations of permutation subgraphs of \((X_{≤s}, E_{≤s})\) for \(X_s\) satisfying the following invariant properties.

**Condition 2.**

• Every element of \(I_s\) is the abstraction of some permutation representation of a permutation subgraph of \((X_{≤s}, E_{≤s})\) for \(X_s\),

• Any permutation representation \(ρ\) of any permutation subgraph of \((X_{≤s}, E_{≤s})\) has an element of \(I_s\) that dominates its abstraction \(A(ρ, s)\).

Clearly \(I_s\) satisfies the above condition if and only if so is its reduced form. We make each \(I_s\) reduced. Particularly if s is the root node, we have \(I_s = \{(o, o), \{⊥, ⊥, ⊥, ⊥\}, c\}\), where o is the trivial order over \{⊥, ⊥\} such that ⊥ < o ⊥ and c is the least number such that one can obtain a permutation subgraph by removing c edges from \(G\). That is, the number c will be the solution to our problem. If s is a leaf, by definition \(I_s = \{(o, o), \emptyset, \emptyset\}\). Our algorithm computes those values \(I_s\) recursively from leaves to the root. In what follows, we show how to calculate \(I_s\) from \(I_t\) for child(ren) t of s, while preserving the invariant (Condition 2).
Figure 8: Introduce Node. (a) Extending $\pi$ by introducing $x_1$ or $x_2$ respects $I$ and introducing $x_3$ or $x_4$ does not. We see $\text{succ}_{\pi_1'}(p_1) <_{\pi_1'} x_3 <_{\pi_2'} q_2$ and $\text{succ}_{\pi_1'}(p_2) <_{\pi_2'} x_4 <_{\pi_2'} q_2$. (b) Relation between newly introduced vertex $x$ and forbidden areas. Since forbidden areas should not intersect with $x$, they are grouped into three: ones ($I_L$) that must be left to $x$, ones ($I_R$) that must be right to $x$, and the other ($I_M \subseteq \{(y_1, y_1, y_1, y_1)\}$).

4.2 Algorithm

Leaf Node: If $s$ is a leaf, we let $\mathcal{F}_s = \{((a, o), \emptyset, 0)\}$.

Introduce Node: Suppose $s$ is an introduce node. It has just one child $t$ such that $X_s = X_t \cup \{x\}$. Figure 8 would help to understand the behavior of our algorithm in this case. For each $(\pi, I, c) \in \mathcal{F}_t$, we say that an extension $\pi'$ of $\pi$ to $X_s$ respects $E$ and $I$ precisely in the cases where

- if $\{u, v\} /\not\in E$ for $u \in X_t$, then $u /\not\parallel_{\pi'} x$,
- there are no $\langle p_1, q_1, p_2, q_2 \rangle \in I$ and $i, j \in \{1, 2\}$ such that $\text{succ}_{\pi_1'}(p_i) <_{\pi_1'} x$ and $x <_{\pi_j'} q_j$,

respectively. If $\pi'$ does not respect $E$, we would wrongly make two non-adjacent vertices $u \in X_s$ and $x$ of $G$ intersect in $\pi'$, which gives a non-subgraph of the input graph. Otherwise, they will be non-adjacent. Since $\{x, w\} /\not\in E$ for any forgotten vertex $w \in X_s - X_t$, we must avoid $x$ and $w$ intersect in an extension of $\rho$. This requires $\pi'$ to respect $I$. If $\pi'$ respects $I$, there are $\rho$ such that $\mathcal{A}(\rho, t) = (I, \pi, c)$ and an extension $\rho'$ of both $\rho$ and $\pi'$ such that $E_{\rho'} \subseteq E_{\rho}$. We note that here we use $\text{succ}_{\pi_1'}(p_i)$ rather than $p_i$ to judge whether $\pi'$ respects $I$. This is because the quadruple $\langle p_1, q_1, p_2, q_2 \rangle \in I$ consists of anchors rather than the exact points of the forbidden area.

For each extension $\pi'$ respecting $E$ and $I$, we add at most two triples to $\mathcal{F}'_s$ as described below. Since $X_s \supseteq X_t$, $c$ need not be updated by definition.

Some forbidden areas that had an anchor $y_i = \text{pred}_{s'}(x)$ may be re-anchored to $x$. Forbidden areas are partitioned into three, where the ones in $I_L$ and $I_R$ are certainly left to and right to $x$, respectively, while $(y_1, y_1, y_2, y_2)$ can go to
We then define $I$ which is illustrated in Figure 9. We obtain

$$I_L = \{ (p_1, q_1, p_2, q_2) \in I \mid p_1 <_{\pi_1} y_1 \text{ or } p_2 <_{\pi_2} y_2 \},$$

$$I_R = \{ (p_1, q_1, p_2, q_2) \in I \mid y_1 <_{\pi_1} q_1 \text{ or } y_2 <_{\pi_2} q_2 \},$$

$$I_M = \{ (p_1, q_1, p_2, q_2) \in I \mid p_1 = y_1 = q_1 \text{ and } p_2 = y_2 = q_2 \}.$$

Note that $I_M$ is either empty or a singleton and that $I_L$ and $I_R$ are mutually exclusive, since $\pi_i$ respects $I$. If a forbidden area has a point anchored to $y_i$ and is right to $x$, it will be re-anchored:

$$I'_R = \{ (p_1[x/y_1], q_1[x/y_1], p_2[x/y_2], q_2[x/y_2]) \mid (p_1, q_1, p_2, q_2) \in I_R \},$$

$$I'_M = \{ (p_1[x/y_1], q_1[x/y_1], p_2[x/y_2], q_2[x/y_2]) \mid (p_1, q_1, p_2, q_2) \in I_M \}.$$

Corresponding to the possible choices, we put both $(\pi', I_L \cup I_M \cup I'_R, c)$ and $(\pi', I_L \cup I_M \cup I'_R, c)$ into $\mathcal{F}'$. We will not add $(\pi', I_L \cup I_M \cup I'_M \cup I'_R, c)$ since it is dominated by the other two.

We then obtain $\mathcal{F}_s$ by reducing $\mathcal{F}'$.

**Forget Node:** Suppose $s$ is a forget node. It has just one child $t$ such that $X_t = X_s \cup \{ t \}$. For each $(\pi, I, c)$ in $\mathcal{F}_i$, we add the following triple $(\pi', I', c')$ to $\mathcal{F}'_s$ where

- $\pi'$ is the restriction of $\pi$ for $X_s$,
- $c' = c + |\{ u \in X_s \mid \{ x, u \} \in E_{\pi_s} \text{ and } u \parallel x \}|$.

Let

$$I_X = \{ (p_1, q_1, p_2, q_2) \in I \mid p_i <_{\pi_i} x \text{ and } x \leq_{\pi_j} q_j \text{ with } \{ i, j \} = \{ 1, 2 \} \}$$

be the set of forbidden areas of $I$ intersecting with $x$. We note that the asymmetry between $p_i <_{\pi_i} x$ and $x \leq_{\pi_j} q_j$ is due to the fact that the actual forbidden area starts after $p_1$ and $p_2$ and ends after $q_1$ and $q_2$, while the points of $x$ are exact.

The new forbidden area formed by $x$ and $I_X$ will be $\alpha = (y_1, z_1, y_2, z_2)$ where for $x_1 = \text{pred}_{\pi_1}(x)$ and $x_2 = \text{pred}_{\pi_2}(x)$,

$$y_1 = \min_{\pi_1}(\{ x_1 \} \cup \{ p_1[x_1/x] \mid (p_1, q_1, p_2, q_2) \in I_X \}),$$

$$z_1 = \max_{\pi_1}(\{ x_1 \} \cup \{ q_1[x_1/x] \mid (p_1, q_1, p_2, q_2) \in I_X \}),$$

$$y_2 = \min_{\pi_2}(\{ x_2 \} \cup \{ p_2[x_2/x] \mid (p_1, q_1, p_2, q_2) \in I_X \}),$$

$$z_2 = \max_{\pi_2}(\{ x_2 \} \cup \{ q_2[x_2/x] \mid (p_1, q_1, p_2, q_2) \in I_X \}).$$

We then define $I'$ by

$$I' = \{ (p_1[x_1/x], q_1[x_1/x], p_2[x_2/x], q_2[x_2/x]) \mid (p_1, q_1, p_2, q_3) \in I - I_X \} \cup \{ \alpha \}$$

This is illustrated in Figure 9. We obtain $\mathcal{F}_s$ by reducing $\mathcal{F}'_s$.
I, q
interval (I
the case where
Suppose
Join Node: Suppose s has two children t and t', where X_s = X_t = X_{t'}. We say that a pair of (π, I, c) ∈ \mathcal{F}_t and (π', I', c') ∈ \mathcal{F}_{t'} is compatible precisely in the case where

- π = π', say π = π' = (π_1, π_2), and
- no members of I and I' intersect: that is, there are no pairs of (p_i, q_1, p_2, q_2) ∈ I and (p'_i, q'_1, p'_2, q'_2) ∈ I' such that p_i <_π q'_i and p'_j <_π q_j for some i, j ∈ \{1, 2\}.

If they are compatible, we add the triple ((π_1, π_2), I ∪ I', c + c') to \mathcal{F}'_s. We then obtain \mathcal{F}_s by reducing \mathcal{F}'_s.

**Theorem 2.** The edge deletion problem for permutation graphs can be solved in O(|V|(|k|!)^2N^2poly(k)) time where N = 2^{7.34k} for the treewidth k of G. If k is the pathwidth, it can be solved in O(|V|(|k|!)^2N^{poly(k)}) time.

**Proof.** Let k be the maximum size of the assigned set X_s to a node of the tree-decomposition. There are at most (k!)^2 variants of π of abstractions (π, I, c).

In order to count the number N of possible forbidden interval sets I for a fixed π in (π, I, c) ∈ \mathcal{F}_s, we give a transformation of I below. Let I_i for i \in \{1, 2\} be the interval projections of I:

\[ I_i = \{(p, q) \mid (p_1, q_1, p_2, q_2) \in I\}. \]

The proof of Theorem 1 has shown that there can be at most 2^{2.19k} possibilities for each I_i. However, since there is no one-one corresponding between I_1 and I_2, the pair (I_1, I_2) is not informative enough to recover I. For example, there can be distinct (p, q) and (p', q') in I such that (p, q, n, n), (p', q', n, n) ∈ I. We call an interval (p, p) ∈ I_1 a hub if there are two (or more) distinct intervals (p_2, q_2), (p'_2, q'_2) ∈ I_2 such that (p, p, p_2, q_2), (p, p, p'_2, q'_2) ∈ I. We call an interval (p, q) ∈ I_1 a terminal if it is not a hub and there is no n such that (p, q, n, n), (\text{succ}_{\pi_1}(p, q), n, n) ∈ I where succ_{\pi_1}(p, q) ∈ I_1 is the successor of (p, q) in I_1. We use the same names for intervals in I_2. Let I'_1 enrich I_1 so that each interval (p, q) ∈ I_1 has a 3-valued variable that indicate whether it is a hub, a terminal, or neither. Figure 10 illustrates how I will be transformed into I'_1 and I'_2.

Figure 9: Forget Node. The forbidden areas in I_X = \{(m, m, p, p), (n, n, q, x)\} will be merged into the single area α = (x_1, n, p, x_2) by forgetting x.
We will show that \( N \) is at most the number of possible enriched sets \( I_1' \) and \( I_2' \). Algorithm 1 recovers elements of \( I \) from left to right using \( I_1' \) and \( I_2' \), where elements of \( I_1' \) and \( I_2' \) are sorted and stored in stacks. From the first intervals \( (p_i, q_i) = \min_{\pi} I_i \) of the stacks, we obtain the first forbidden area \( (p_1, q_1, p_2, q_2) \in I \). If \( (p_1, q_1) \) is a hub, we keep it as the active hub. Note that it is impossible that both \( (p_1, q_1) \) and \( (p_2, q_2) \) are hubs simultaneously. Suppose we have reconstructed a forbidden area \( (p_1, q_1, p_2, q_2) \in I \) with no active hub. In this case, the next forbidden area will be \( (\text{suc}_{\pi_1}(p_1, q_1), \text{suc}_{\pi_2}(p_2, q_2)) \). Suppose \( (p_1, q_1) \) is the active hub. If \( (p_2, q_2) \) is neither a hub nor a terminal, then the next forbidden area will be \( (p_1, q_1, \text{suc}_{\pi_2}(p_2, q_2)) \) and we keep the active hub \( (p_1, q_1) \). If \( (p_2, q_2) \) is a terminal, then the next forbidden area will be \( (\text{suc}_{\pi_1}(p_1, q_1), \text{suc}_{\pi_2}(p_2, q_2)) \) and the active hub will be null. If \( (p_2, q_2) \) is a hub, then the next forbidden area will be \( (\text{suc}_{\pi_1}(p_1, q_1), p_2, q_2) \) and the new active hub is \( (p_2, q_2) \in I_2 \).

A point \( p \) may appear in \( I_i \) in the following ways:

1. ending a forbidden interval started earlier: \( (q, p) \in I_i \) for some \( q <_{\pi} p \),
2. starting and ending a minimal forbidden interval: \( (p, p) \in I_i \), which may be a hub or a terminal,
3. starting a new forbidden interval: \( (p, q) \in I_i \) for some \( q >_{\pi} p \), which may be a terminal but not a hub.

We do not care whether or not it is a terminal in the first case, leaving counting different possibilities to the interval starting point \( q \). Solving the recurrence equations

\[
A_h(n + 1) = 4A_h(n) + 4A_u(n), \quad A_h(0) = 4, \\
A_u(n + 1) = 8A_h(n) + 9A_u(n), \quad A_u(0) = 3,
\]

we obtain \( A_h(n) \in O(2^{3.67n}) \). All in all, we have at most \( N \in O(2^{7.34k}) \) possibilities for \( I \).

Arguments similar to the proof of Theorem 1 derive the theorem. \( \square \)
Algorithm 1: Reconstruction of $I$

1. $I \leftarrow \emptyset$ and $ActiveHub \leftarrow \text{Null}$;
2. repeat
3. if $ActiveHub = \text{Null}$ then
4.  $(p_1, q_1, r_1) \leftarrow I_1'\cdot \text{pop}$ and $(p_2, q_2, r_2) \leftarrow I_2'\cdot \text{pop}$;
5.  $I \leftarrow I \cup \{(p_1, q_1, p_2, q_2)\}$;
6.  if $r_1 = h$ then $ActiveHub \leftarrow (1, p_1, q_1)$;
7. else if $r_2 = h$ then $ActiveHub \leftarrow (2, p_2, q_2)$;
8. else if $ActiveHub = (1, p_1, q_1)$ for some $p_1, q_1$ then
9.  $(p_2, q_2, r_2) \leftarrow I_2'\cdot \text{pop}$;
10. $I \leftarrow I \cup \{(p_1, q_1, p_2, q_2)\}$;
11. if $r_2 = h$ then $ActiveHub \leftarrow (2, p_2, q_2)$;
12. else if $r_2 = t$ then $ActiveHub \leftarrow \text{Null}$;
13. else if $ActiveHub = (2, p_2, q_2)$ for some $p_2, q_2$ then
14.  $(p_1, q_1, r_1) \leftarrow I_1'\cdot \text{pop}$;
15.  $I \leftarrow I \cup \{(p_1, q_1, p_2, q_2)\}$;
16. if $r_1 = h$ then $ActiveHub \leftarrow (1, p_1, q_1)$;
17. else if $r_1 = t$ then $ActiveHub \leftarrow \text{Null}$;
18. until the stacks $I_1'$ and $I_2'$ are empty;

5 Other Classes Related to Interval Graphs

The algorithm presented in the previous section can be applied to the C-EDGE-DELETION for some subclasses and a superclass of interval graphs with a slight modifications.

5.1 Proper interval graphs

An interval representation $\pi$ is said to be proper if there are no $u, v \in V$ such that $l_u <_{\pi} l_v <_{\pi} r_v <_{\pi} r_u$. An interval graph is proper if it admits a proper interval representation. In accordance with the definition of the graph class, we simply require $\pi'$ in Introduce Node of the algorithm in Section 3 to be a proper interval representation.

Corollary 1. The edge deletion problem for proper interval graphs can be solved in $O(|V|N^2\text{poly}(k))$ time where $N = 2^3.91k (\frac{2k!}{(k+1)!})$ for the treewidth $k$ of $G$. If $k$ is the pathwidth, it can be solved in $O(|V|N\text{poly}(k))$ time.

Proof. Let $k$ be the maximum size of the assigned set $X_s$ to a node of a nice tree-decomposition. The number of possible proper interval representations over $k$ vertices is $k!C_k$ for the $k$-th Catalan number $C_k = \frac{(2k)!}{(k+1)!}$. Let $A(n)$ be the numbers of possible ways of drawing forbidden intervals with $n$ points in addition to $\perp$ where the last intervals may be right-unbounded. We observed in the proof of Theorem 1 (Figure 6) that $A(n+1) \leq 5A(n)$. However, in the case
of proper interval representations, we never have \((l_u, l_u) \in I\). If the last \((n+1)\)st point is \(l_u\), we have at most three times more possible forbidden interval sets than the number of possible forbidden interval sets over \(n\) points. Therefore, the number of possible abstractions in each \(\mathcal{I}_s\) is at most \(O\left(\frac{(2k)!}{(k+1)!} \cdot 3^k \cdot 5^k\right) \subseteq O(2^{1.91k} \left(\frac{(2k)!}{(k+1)!}\right))\). \(\Box\)

### 5.2 Trivially perfect graphs

An interval representation \(\pi\) is said to be nested if there are no \(u, v \in V\), such that \(l_u <_\pi l_v <_\pi r_u <_\pi r_v\). A trivially perfect graph (a.k.a. nested interval graph) is an interval graph that admits a nested interval representation. The algorithm presented in Section 3 can easily be modified so that it solves the edge deletion problem for trivially perfect graphs. In accordance with the definition of the graph class, we simply require \(\pi'\) in Introduce Node of the algorithm in Section 3 to be a nested interval representation.

**Corollary 2.** The edge deletion problem for trivially perfect graphs can be solved in \(O(|V|N^{2\text{poly}(k)})\) time where \(N = 2^{\frac{4k(2k)!}{k}}\) for the treewidth \(k\) of \(G\). If \(k\) is the pathwidth, it can be solved in \(O(|V|N\text{poly}(k))\) time.

**Proof.** In this case, forbidden intervals and active intervals are nested. Let \(D_n\) be the number of possible pairs \((\pi, I)\) in abstractions with \(n\) active vertices modulo renaming of vertices. Then, the maximum cardinality of \(\mathcal{I}_s\) is at most \(n! \cdot D_n\). Hereafter we assume that neither \((\bot, \bot)\) nor \((\text{pred}_s(\top), \text{pred}_s(\top))\) appears in \(I\). Let us say that \((\pi, I)\) is splittable if \(LR_{X_s} - \{\bot, \top\}\) can be partitioned into two non-empty subsets \(Y_1\) and \(Y_2\) so that

- \(\max_\pi Y_1 < \min_\pi Y_2\),
- for any \(u \in X_s\), \(l_u\) and \(r_u\) belong to the same component \(Y_1\) or \(Y_2\),
- if \((p, q) \in I\) and \(p \neq q\), \(\text{succ}_s(p)\) and \(q\) belong to the same component \(Y_1\) or \(Y_2\).

Otherwise, \((\pi, I)\) is non-splittable. Let \(D'_n\) and \(D''_n\) count the numbers of non-splittable and splittable pairs \((\pi, I)\), respectively, where \(n = |X_s|\). We have \(D_0 = D'_0 = 1\), \(D'_1 = 0\), \(D_1 = D'_1 = 3\), and for \(n \geq 2\),

\[
\begin{align*}
D_n &= D'_n + D''_n, \\
D'_n &= C_n + 4D_{n-1}, \\
D''_n &= \sum_{i=1}^{n-1} 2D'_i D_{n-i}.
\end{align*}
\]

The first term \(C_n\) of (2) is the \(n\)th Catalan number, which counts the possible permutations \(\pi\) when \((\bot, \text{pred}_s(\top)) \in I\), i.e., when \(I = \{\bot, \text{pred}_s(\top)\}\). The second term \(4D_{n-1}\) counts cases where \((\bot, \text{pred}_s(\top)) \notin I\) and \((\text{pred}_s(\bot), \text{pred}_s(\top)) = (l_u, r_u)\) for some vertex \(u\). The coefficient 4 counts the number of subsets of

\[19\]
{\{l_u, l_v\}, \{\text{pred}_u(r_u), \text{pred}_v(r_v)\}\}. Note that when \(n = 1\), \(l_u = \text{pred}_u(r_u)\) and thus we have \(D'_1 = C_1 + 2D_0\). Each term \(2D'_iD_{n-i}\) of \(B\) counts the number of pairs where the first splitting point is at the \(i\)th active vertex interval. In other words, \(2i\) is the minimum cardinality of the first splitting component \(Y_i\). The coefficient \(2\) counts the cases where \((r_u, r_v)\) presents and absents in \(I\) where \(r_v = \max_\pi Y_i\).

We show \(D_n \leq 2^{2n}C_n\) for \(\alpha = 2.4\) by induction on \(n\). One can confirm it is true for \(n \leq 22\) by calculation. For \(n \geq 23\),

\[
D_n = C_n + 4D_{n-1} + 2\sum_{i=1}^{n-1} (C_i + 4D_{i-1})D_{n-i} - 4D_0D_{n-1}
\]

\[
\leq C_n + 2\sum_{i=1}^{n-1} (C_i + 2^{2(i-1)+2}C_{i-1})2^{\alpha(n-i)}C_{n-i}
\]

\[
= C_n + 2\sum_{i=1}^{n-1} 2^{\alpha(n-i)}C_iC_{n-i} + 2^{\alpha(n-1)+3}\sum_{i=1}^{n-1} C_{i-1}C_{n-i}
\]

\[
= C_n + \sum_{i=1}^{n-1} (2^{\alpha(n-i)} + 2^{\alpha i})C_iC_{n-i} + 2^{\alpha(n-1)+3}(C_0C_{n-1} + C_{n-1})
\]

\[
\leq C_n + (2^{\alpha(n-1)} + 2^\alpha)C_n + 2^{\alpha(n-1)+4}C_{n-1}
\]

\[
= (1 + 2^{\alpha(n-1)} + 2^\alpha + \frac{n+1}{4n+2}2^{\alpha(n-1)+4})C_n.
\]

Noting that \(n \geq 23\) and \(\alpha = 2.4\), we obtain

\[
D_n \leq 2^{2.4n}C_n.
\]

Hence, the number of possible abstractions is at most \((k+1)!D_{k+1} \leq (k+1)! \cdot 2^{2.4(k+1)} \cdot \frac{(2k+1)!}{(k+2)!} \in O(2^{2.4k}(2k)!/k!)\).

### 5.3 Circular-arc graphs

Circular-arc graphs are a generalization of interval graphs which have an arc model, which can be seen as a “circular” interval representation. For a linear order \(\pi\) over a set \(S\) and four elements \(p_1, q_1, p_2, q_2 \in S\), we write \((p_1, q_1) \blacktriangleright_\pi (p_2, q_2)\) if either

- \(p_1 <_\pi m <_\pi q_1\) for some \(m \in \{p_2, q_2\}\), or

- \(q_1 <_\pi p_1 <_\pi m\) or \(m <_\pi q_1 <_\pi p_1\) for some \(m \in \{p_2, q_2\}\).

Otherwise, we write \((p_1, q_1) \blacktriangleleft_\pi (p_2, q_2)\). A circular-arc graph is a graph \(G_\pi = (V, E)\) such that

\[
E = \{ \{u, v\} \subseteq V \mid (l_u, r_u) \blacktriangleright_\pi (l_v, r_v) \}
\]

for some linear order \(\pi\) over \(L_V \cup R_V\). Note that this set contains neither \(\top\) nor \(\bot\). The algorithm presented in Section 3 can easily be modified so that it solves
the edge deletion problem for circular-arc graphs by replacing the definitions of \(//\) and \(\backslash\) as above, and defining \(\text{suc}_\pi(\max X) = \min X \) and \(\text{pred}_\pi(\min X) = \max X\). Since we allow \(r_u <_\pi l_u\), the number of admissible arc model is bigger than that of (ordinary) interval representations. This affects the computational complexity.

**Corollary 3.** The edge deletion problem for circular-arc graphs can be solved in \(O(|V|^2N^{\text{poly}(k)})\) time where \(N = (2k)! \cdot 2^{4.38k}\) for the treewidth \(k\) of \(G\). If \(k\) is the pathwidth, it can be solved in \(O(|V|^2N^{\text{poly}(k)})\) time.

**Proof.** There can be \((2k + 2)!\) varieties of \(\pi\). One can argue that each \(\pi\) has at most \(O(2^{4.38k})\) sets of forbidden intervals similarly to the proof of Theorem \(\blacksquare\)

### 5.4 Threshold graphs

Threshold graphs are special cases of trivially perfect graphs, which can be defined in several different ways. Here we use a pair of a vertex subset \(W \subseteq V\) and a linear order \(\pi\) over \(R_V\) as a *threshold interval representation*. We say that vertices \(u\) and \(v\) intersect on \((W,\pi)\) if and only if \(u \in W\) and \(r_u <_\pi r_v\) or the other way around. A *threshold graph* is a graph \(G_{W,\pi} = (V, E_{W,\pi})\) where \((W,\pi)\) is a threshold interval representation on \(V\) and \(E_{W,\pi} = \{\{u, v\} \subseteq V \mid u\) and \(v\) intersect on \((W,\pi)\}\). By extending \(\pi\) to \(\pi'\) over \(L_R V\) so that \(l_w <_\pi r_v\) for all \(w \in W\) and \(v \in V\) and \(\text{suc}_{\pi'}(l_u) = r_u\) for all \(u \in V - W\), then the induced interval graph coincides with the threshold graph. To attain drastic improvement on the complexity, we design an algorithm for the edge deletion problem for threshold graphs from scratch, rather than modifying the one for interval graphs.

For a threshold representation \((Y, \rho)\) of a subgraph \(G_{Y,\rho} = (X_{\leq s}, E_{Y,\rho})\), we define its abstraction \(\mathcal{A}'((Y, \rho), s) = (Y', \pi, b, p, c)\) as follows: (1) \(\pi\) is the restriction of \(\rho\) to \(R_{X_s}\), (2) \(Y' = Y \cap X_s\), (3) if \(Y' = Y\), then \(b = 0\) and \(p = \max\{p \in R_{X_s} \mid p <_\rho r_y \text{ for all } y \in X_{\leq s} - X_s\}\), (4) if \(Y' \neq Y\), then \(b = 1\) and \(p = \max\{p \in R_{X_s} \mid p <_\rho r_y \text{ for some } y \in Y - Y'\}\), and (5) \(c = |E_{Y_s} - E_{Y,\rho}| = |\{\{u, v\} \in E \mid \{u, v\} \notin X_s \text{ and } u\) and \(v\) do not intersect on \((Y, \rho)\}|\).

We say that \((Y', \pi', b', p', c')\) dominates \((Y, \pi, b, p, c)\) if \(Y' = Y, \pi' = \pi, c' \leq c\), and either (a) \(b' = b = 0\) and \(p' \geq p\), (b) \(b' = b = 1\) and \(p' \leq p\), or (c) \(b' = 0\) and \(b = 1\). Using the above invariant, we can provide an algorithm for **Threshold-Edge-Deletion**.

Our algorithm assigns a set \(\mathcal{I}_s\) for each node \(s\) of \(T\) so that the threshold counterpart of Condition \(\blacksquare\) holds. Accordingly \(\mathcal{I}_s = \{(\emptyset, 0, 0, \perp, 0)\}\) for leaf nodes \(s\).

**Introduce Node:** Suppose \(s\) has just one child \(t\) such that \(X_s = X_t \cup \{x\}\). For each \((Y, \pi, b, p, c) \in \mathcal{I}_t\), we add to \(\mathcal{I}_s\) all triples \((Y', \pi', b, p', c)\) fulfilling the following conditions:

- \(\pi'\) is an extension of \(\pi\) to \(R_{X_s}\),

21
• $Y \subseteq Y' \subseteq Y \cup \{x\}$.

• if $\{x, u\} \notin E$ for $u \in X_t$, then $x$ and $u$ do not intersect in $(Y', \pi')$.

• if $b = 0$, then $x \notin Y'$ or $r_x < \pi' \text{ suc}_\pi(p)$,
  and moreover $p' = \begin{cases} r_x & \text{if } p < \pi' \wedge x \in Y' \\
  p & \text{otherwise}, \end{cases}$

• if $b = 1$, then $x \notin Y'$ and $p' = p < \pi' r_x$.

We then obtain $\mathcal{F}_s$ by reducing $\mathcal{F}_s'$.

**Forget Node:** Suppose $s$ has just one child $t$ such that $X_t = X_s \cup \{x\}$. For each $(Y, \pi, b, p, c) \in \mathcal{F}_t$, we add the tuple $(Y', \pi', b', p', c')$ to $\mathcal{F}_s'$ where

• $\pi'$ is the restriction of $\pi$ for $R_{X_s}$,

• $Y' = Y - \{x\}$,

• $b' = 1$ if $x \in Y$, and $b' = b$ otherwise,

• if $b' = 0$, then $p' = \min\{p, \text{ pred}_s(r_x)\}$,

• if $b' = 1$, then $p' = \begin{cases} \text{pred}_\pi(r_x) & \text{if } b = 0 \text{ or } p < \pi' r_x \wedge x \in Y \text{ or } r_x = p, \\
  p & \text{otherwise}. \end{cases}$

• $c' = c + |\{ \{u, x\} \in E \mid u \in X_s \text{ and } x \text{ do not intersect on } (Y, \pi) \}|$

We then obtain $\mathcal{F}_s$ by reducing $\mathcal{F}_s'$.

**Join node:** Suppose $s$ has two children $t_1$ and $t_2$, where $X_s = X_{t_1} = X_{t_2}$. We add $(Y, \pi, b, p, c)$ to $\mathcal{F}_s'$ if there are $(Y, \pi, b_1, p_1, c_1) \in \mathcal{F}_{t_1}$ and $(Y, \pi, b_2, p_2, c_2) \in \mathcal{F}_{t_2}$ such that $c = c_1 + c_2$, and either

• $b = b_1 = b_2$ and $p = \min_{\pi_1}\{p_1, p_2\}$,

• $b = b_1 = 1, b_2 = 0$ and $p = p_1 \leq \pi p_2$, or

• $b = b_2 = 1, b_1 = 0$ and $p = p_2 \leq \pi p_1$.

We then obtain $\mathcal{F}_s$ by reducing $\mathcal{F}_s'$.

**Theorem 3.** The edge deletion problem for threshold graphs can be solved in $O(|V|^2 N^2 \text{poly}(k))$ time where $N = k! \cdot 2^k$ for the treewidth $k$ of $G$. If $k$ is the pathwidth, it can be solved in $O(|V|^2 N \text{poly}(k))$ time.
6 Conclusion

We have proposed FPT algorithms for Edge-Deletion to some intersection graphs parameterized by treewidth in this paper. Our algorithms maintain partial intersection models on a node of a tree decomposition with some restrictions and extend the models consistently for the restrictions in the next step. We expect that the ideas in our algorithms can be applied to other intersection graphs whose intersection models can be represented as linear-orders, for example circle graphs, chain graphs and so on, and to Vertex-Deletion of intersection graphs.

We have the following questions as future work:

• Do there exist single exponential time algorithms for the considered problems, that is, $O^*(2^{tw(G)})$ time, or can we show matching lower bounds assuming the Exponential Time Hypothesis?

• Are there FPT algorithms parameterized by treewidth for $C$-Completion which is to find the minimum number of adding edges to obtain a graph in an intersection graph class $C$? We can naturally apply the idea of our algorithms to $C$-Completion problems. While $C$-Edge-Deletion algorithms do not allow introduced objects to intersect with forgotten objects, $C$-Completion algorithms do allow it with the cost of addition of new edges. Thus $C$-Completion algorithms based on this naive approach will be XP algorithms since we have to remember the number of forgotten objects in the representation to count the number of intersections between the introduced objects and forgotten objects.

• Are there FPT algorithms for Edge-Deletion to intersection graphs defined using objects on a plane, like unit disk graphs? The intersection graph classes discussed in this paper are all defined using objects aligned on a line. Going up to a geometric space of higher dimension is a challenging topic.

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