Bianchi type I model with two interacting scalar fields

Vladimir Folomeev
Institute of Physics of NAS KR, 265 a, Chui str., Bishkek, 720071, Kyrgyz Republic

The flat anisotropic model of the early Universe is considered. Two interacting scalar fields with special form of potential energy are a source of matter fields. Analytic solutions for inflationary and scalaron stages are found. It is shown that the solutions tend asymptotically to isotropic Friedmann model.

I. INTRODUCTION

Investigation of models of the early Universe has a long history (see e.g. Refs. [1, 2]). One of the basic directions is consideration of classical scalar fields. For such type of models the necessary inflationary evolution can be ensured by a corresponding choice of energy-momentum tensor which is equivalent to the hydrodynamic equation of state $p = -\epsilon$. In this case the exponential expansion of the Universe is realized, and its size increase from the Planck value up to macroscopic one. The subsequent redistribution of the field energy between kinetic and potential parts leads to fast oscillations of the field (the so-called scalaron stage [3]) and to realization of the effective equation of state $p \ll \epsilon$.

Such fast oscillations with the Planck energy density provide mass creation of elementary particles with subsequent filling of the Universe by hot plasma. At the corresponding choice of initial parameters, such a simple model solves the principal problems cosmology (causality, homogeneity, flatness, e.t.c).

In this Letter we offer the model of the early Universe using the special form of the potential energy of scalar field:

$$V(\phi, \chi) = \frac{\lambda_1}{4}(\phi^2 - m_1^2)^2 + \frac{\lambda_2}{4}(\chi^2 - m_2^2)^2 + \phi^2 \chi^2 - V_0.$$  (1)

Here $\phi, \chi$ are two real scalar fields with masses $m_1, m_2$ and self-coupling constants $\lambda_1, \lambda_2$ accordingly, $V_0$ is a normalization constant. This potential was obtained in Ref. [4] at approximate modeling of condensate of a gauge field in the SU(3) Yang-Mills theory. The potential (1) has two global minima at $\phi = 0, \chi = \pm m_2$, local maximum at $\phi = 0, \chi = 0$ and two local minima at $\chi = 0, \phi = \pm m_1$ at values of the parameters $\lambda_1, \lambda_2$ used in this paper. The conditions for existence of the local minima are: $\lambda_1 > 0$, $m_1^2 > \lambda_2 m_2^2/2$, and for the global minima: $\lambda_2 > 0$, $m_2^2 > \lambda_1 m_1^2/2$.

The inflationary models with several scalar fields are known (for example, the hybrid inflation model [5], see also the review [6]). The potential (1) was used at description of different models: in Ref. [7] for a model of boson star, in Ref. [8] at consideration of the cosmic string, in Ref. [9] at modeling of the phantom energy in the early Universe, in Ref. [10] for a model of thick brane.

In this Letter we consider inflationary and scalaron stages within the framework of the flat anisotropic model of the early Universe with the potential (1).

II. GENERAL EQUATIONS

We start from the following Lagrangian:

$$L = -\frac{R}{16\pi G} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - V(\phi, \chi).$$  (2)

The anisotropic metric is choosing in the form:

$$ds^2 = dt^2 - a_1^2(t)dx^2 - a_2^2(t)dy^2 - a_3^2(t)dz^2,$$  (3)

where $a_1, a_2, a_3$ are the scale factors on directions $x$, $y$, and $z$ accordingly. For carrying out of calculations, it is convenient to represent the scale factors in the following form [11] (see also [12]):

$$a_1(t) = r(t)q_1, \quad a_2(t) = r(t)q_2, \quad a_3(t) = r(t)q_3.$$  (4)
where $q_1, q_2, q_3$ are dimensionless variables satisfying the requirements:

$$
\prod_{\alpha=1}^{3} q_{\alpha} = 1, \quad \sum_{\alpha=1}^{3} (q_{\alpha}/q_{\alpha}) = 0, \quad \dot{q}_{\alpha} = \frac{dq_{\alpha}}{dt}.
$$

whence it follows that $\prod_{\alpha=1}^{3} a_{\alpha} = r^3$. For the line element (4), with account of (4), the components of the Ricci tensor will be:

$$
-R_{00}^0 = 3 \frac{\ddot{r}}{r} + 3 \sum_{\alpha=1}^{3} \left( \frac{\dot{q}_{\alpha}}{q_{\alpha}} \right)^2,
$$

$$
-R_{\alpha\alpha}^\alpha = \frac{\ddot{r}}{r} + 2 \left( \frac{\dot{r}}{r} \right)^2 + 3 \frac{\dot{r}}{r} q_{\alpha} + \left( \frac{\dot{q}_{\alpha}}{q_{\alpha}} \right)^2,
$$

$$
-R = 6 \left[ \frac{\ddot{r}}{r} + \left( \frac{\dot{r}}{r} \right)^2 \right] + 3 \sum_{\alpha=1}^{3} \left( \frac{\dot{q}_{\alpha}}{q_{\alpha}} \right)^2.
$$

Let us use the last expressions for obtaining of (1-1) and (2-2) components of the Einstein tensor:

$$
G_{11} = 2 \frac{\ddot{r}}{r} + \left( \frac{\dot{r}}{r} \right)^2 - 3 \frac{\dot{r}}{r} q_{1} - \left( \frac{\dot{q}_{1}}{q_{1}} \right)^2 + \frac{1}{2} \sum_{\alpha=1}^{3} \left( \frac{\dot{q}_{\alpha}}{q_{\alpha}} \right)^2,
$$

$$
G_{22} = 2 \frac{\ddot{r}}{r} + \left( \frac{\dot{r}}{r} \right)^2 - 3 \frac{\dot{r}}{r} q_{2} - \left( \frac{\dot{q}_{2}}{q_{2}} \right)^2 + \frac{1}{2} \sum_{\alpha=1}^{3} \left( \frac{\dot{q}_{\alpha}}{q_{\alpha}} \right)^2.
$$

Subtracting the component $G_{22}^2$ from $G_{11}^1$, one obtains:

$$
3 \frac{\dot{r}}{r} \left( \frac{\dot{q}_{2}}{q_{2}} - \frac{\dot{q}_{1}}{q_{1}} \right) + \left( \frac{\dot{q}_{2}}{q_{2}} - \frac{\dot{q}_{1}}{q_{1}} \right)^2 = 0.
$$

Introducing in the last equation a notification $Q_{\alpha\beta} = (\dot{q}_{2}/q_{2} - \dot{q}_{1}/q_{1})$, we have:

$$
3 \frac{\dot{r}}{r} \frac{Q_{\alpha\beta}}{Q_{\alpha\beta}} = 0,
$$

that gives after integration:

$$
Q_{\alpha\beta} = C_{\alpha\beta}/r^3,
$$

where $C_{\alpha\beta}$ are integration constants. Hence it appears:

$$
\frac{\dot{q}_{\alpha}}{q_{\alpha}} = \frac{C_{\alpha}}{r^3},
$$

where $C_{\alpha}$ are integration constants and, according to the requirements (4), $\sum_{\alpha=1}^{3} C_{\alpha} = 0$. Integrating the last equation, one can obtain:

$$
q_{\alpha} = A_{\alpha} \exp \left\{ C_{\alpha} \int \frac{dt}{r^3} \right\},
$$

where $A_{\alpha}$ are integration constants and $\prod_{\alpha=1}^{3} A_{\alpha} = 1$. Now, using the relation (7), we have from (6):

$$
- R_{00}^0 = 3 \frac{\ddot{r}}{r} + \frac{1}{r^6} \sum_{\alpha=1}^{3} C_{\alpha}^2, \quad -R = 6 \left[ \frac{\ddot{r}}{r} + \left( \frac{\dot{r}}{r} \right)^2 \right] + \frac{1}{r^6} \sum_{\alpha=1}^{3} C_{\alpha}^2.
$$
where the coefficient \( \sum_{\alpha=1}^{3} C_{\alpha}^{2} \) determines an anisotropy of the model. Then one can obtain from Eqs. (9) and (2) the (0-0) component of the Einstein’s equations (here and further we use the geometric units \( 8\pi G = 1, c = 1 \)):

\[
3 \left( \frac{\dot{r}}{r} \right)^{2} - \frac{1}{2r^{2}} \sum_{\alpha=1}^{3} C_{\alpha}^{2} = \frac{1}{2} \left[ \dot{\phi}^{2} + \dot{\chi}^{2} \right] + V(\phi, \chi). \tag{10}
\]

The corresponding equations for the scalar fields can be obtained from the Lagrangian (2). They will be:

\[
\ddot{\phi} + 3 \frac{r}{r} \dot{\phi} = -\phi \left[ 2\chi^{2} + \lambda_{1} \left( \phi^{2} - m_{1}^{2} \right) \right], \tag{11}
\]

\[
\ddot{\chi} + 3 \frac{r}{r} \dot{\chi} = -\chi \left[ 2\phi^{2} + \lambda_{2} \left( \chi^{2} - m_{2}^{2} \right) \right]. \tag{12}
\]

One can see that the system of equations (10)-(12) has solutions beginning in the local maximum of the potential (1), i.e. at \( \phi = 0, \chi = 0 \), and tending asymptotically to the global minimum at \( \phi = 0, \chi = \pm m_{2} \) (Fig. 1). (Here we choose the normalization constant \( V_{0} = (\lambda_{1}/4)m_{1}^{4} \) in order to provide \( V(\phi, \chi) = 0 \) in the global minima.) These solutions are similar to solutions in systems with spontaneous symmetry breaking [1]. In such systems, scalar fields roll down from maximum of a potential (inflationary stage) toward minimum with subsequent fast oscillations (scalaron stage).

![Fig. 1: The graphs for the scalar fields \( \chi \) (top) and \( \phi \) with \( m_{1} = 1, m_{2} = 1, \lambda_{1} = 0.1, \lambda_{2} = 1 \). Both fields tend asymptotically to the global minimum \( \phi = 0, \chi = m_{2} \).](image)

In the next section we will consider behavior of the model on these two stages.

### III. INFLATONARY AND SCALARON STAGES

#### A. Inflationary stage

As is well-known [1], the conditions \( \dot{\phi}^{2}, \dot{\chi}^{2} \ll V(\phi, \chi), \ddot{\phi}, \ddot{\chi} \ll \partial V(\phi, \chi)/\partial(\phi, \chi) \) should be fulfilled on the stage of inflation. It allows to rewrite Eqs. (10)-(12) as follows:

\[
3 \left( \frac{\dot{r}}{r} \right)^{2} - \frac{1}{2r^{2}} \sum_{\alpha=1}^{3} C_{\alpha}^{2} = V(\phi, \chi),
\]

\[
3 \frac{\dot{r}}{r} \dot{\phi} = -\phi \left[ 2\chi^{2} + \lambda_{1} \left( \phi^{2} - m_{1}^{2} \right) \right], \tag{13}
\]

\[
3 \frac{\dot{r}}{r} \dot{\chi} = -\chi \left[ 2\phi^{2} + \lambda_{2} \left( \chi^{2} - m_{2}^{2} \right) \right].
\]
We will search for inflationary solutions near the maximum of the potential (1) in the following form:

\[ \phi = \delta \phi, \quad \chi = \delta \chi, \quad \delta \phi, \delta \chi \ll 1. \]  

(14)

The first equation from (13) will then be:

\[ \left( \frac{\dot{r}}{r} \right)^2 - \frac{\sigma}{r^6} = H^2, \quad \sigma = \frac{1}{6} \sum_{\alpha=1}^{3} C_{\alpha}^2, \quad H^2 = \frac{\lambda_2 m_2^2}{12}, \]  

(15)

where \( \sigma \) is an anisotropy parameter and \( H \) is an effective Hubble constant. The last equation has a solution:

\[ r^3 = \sqrt{\frac{\sigma}{H}} \sinh 3H(t - t_0), \]  

(16)

where \( t_0 \) is an integration constant.

Let us show that for isotropic case, i.e. at \( \sigma \to 0 \), Eq. (16) describes usual exponential expansion. Indeed, if we take the arbitrary constant \( t_0 \) in the form \( t_0 = \ln \left( \sqrt{\sigma H^2 / 2} / 3H \right) \), then:

\[ r^3 = H^{-3} e^{3Ht}, \]  

(17)

i.e. one has usual inflationary solution for a flat isotropic model.

Let us further find evolution of the scalar fields. Using (14), we can rewrite the second and the third equations from (13) as follows:

\[ 3 \frac{\dot{r}}{r} \dot{\delta \phi} - \lambda_1 m_1^2 \delta \phi = 0, \quad 3 \frac{\dot{r}}{r} \dot{\delta \chi} - \lambda_2 m_2^2 \delta \chi = 0, \]

or, taken into account (16), we have:

\[ \delta \dot{\phi} - \frac{1}{3H \coth 3H(t - t_0)} \lambda_1 m_1^2 \delta \phi = 0, \]  

(18)

\[ \delta \dot{\chi} - \frac{1}{3H \coth 3H(t - t_0)} \lambda_2 m_2^2 \delta \chi = 0, \]  

(19)

with solutions:

\[ \delta \phi = B_1 \left[ \cosh 3H(t - t_0) \right]^{\lambda_1 m_1^2 / 9H^2}, \]  

(20)

\[ \delta \chi = B_2 \left[ \cosh 3H(t - t_0) \right]^{\lambda_2 m_2^2 / 9H^2}, \]  

(21)

where a sign of the integration constants \( B_1, B_2 \) depends on whether the solution should appear in the global minimum at \( \phi = 0, \chi = -m_2 \) or at \( \phi = 0, \chi = +m_2 \).

Now we can find a form of the metric coefficients \( a_\alpha \). Eqs. (8) and (16) yield:

\[ q_\alpha = A_\alpha \left[ \frac{e^{3H(t - t_0)} - 1}{e^{3H(t - t_0)} + 1} \right]^{C_{\alpha} / 3\sqrt{\sigma}}, \]  

(22)

and, using (11), (16) and (22), we finally have:

\[ a_\alpha = A_\alpha \left[ \frac{e^{3H(t - t_0)} - 1}{e^{3H(t - t_0)} + 1} \right]^{C_{\alpha} / 3\sqrt{\sigma}} \left[ \frac{\sqrt{\sigma}}{H} \sinh 3H(t - t_0) \right]^{1/3}. \]  

(23)

### B. Scalaron stage

On this stage, there are fast oscillations of the fields near the minimum of the potential (1) with loss of the energy for creation of particles (hot plasma). We will search for a solution of Eqs. (10)-(12) in the following form:

\[ \phi = \delta \phi, \quad \chi = m_2 + \delta \chi, \]

(24)
keeping in the potential terms up to second order on \( \delta \phi, \delta \chi \). Then Eqs. (10)-(12) and (11) yield:

\[
\begin{align*}
\delta \ddot{\phi} + 3 \frac{\dot{r}}{r} \delta \dot{\phi} + 2 \left[ m_2^2 - \frac{\lambda_1}{2} m_1^2 \right] \delta \phi &= 0, \\
\delta \ddot{\chi} + 3 \frac{\dot{r}}{r} \delta \dot{\chi} + 2 \lambda_2 m_2^2 \delta \chi &= 0, \\
3 \left( \frac{\dot{r}}{r} \right)^2 - \frac{1}{2 r^2} \sum_{\alpha=1}^{3} C_{\alpha}^2 &= \frac{1}{2} \left[ \delta \phi^2 + \delta \chi^2 \right] + \left[ m_2^2 - \frac{\lambda_1}{2} m_1^2 \right] \delta \phi^2 + \lambda_2 m_2^2 \delta \chi^2.
\end{align*}
\] (25) (26) (27)

(Note that the expression \( m_2^2 - (\lambda_1/2) m_1^2 \) > 0 due to the condition for existence of the global minima, see the Introduction.)

Let us search for solutions of Eqs. (25)-(26) by the following method (12):

\[
\delta \phi = A(t) \sin \sqrt{2(m_2^2 - (\lambda_1/2) m_1^2)}(t - t_0), \quad \delta \chi = B(t) \sin \sqrt{2\lambda_2 m_2^2}(t - t_0),
\] (28)

where \( A(t), B(t) \) are some arbitrary functions obeying the following conditions on the scalaron stage:

\[
\dot{A}/A \ll \sqrt{2(m_2^2 - (\lambda_1/2) m_1^2)}, \quad \dot{B}/B \ll \sqrt{2\lambda_2 m_2^2}.
\]

Then, inserting (28) into Eqs. (25)-(26) and neglecting terms with second derivatives, one can obtain:

\[
A = D_1 r^{-3/2}, \quad B = D_2 r^{-3/2},
\] (29)

where \( D_1, D_2 \) are integration constants. Now, inserting (28) into Eq. (27) and keeping only zero terms on derivative, we have:

\[
3 \left( \frac{\dot{r}}{r} \right)^2 - \frac{1}{2 r^2} \sum_{\alpha=1}^{3} C_{\alpha}^2 = \left[ m_2^2 - \frac{\lambda_1}{2} m_1^2 \right] A^2 + \lambda_2 m_2^2 B^2,
\] (30)

or, taken into account (29),

\[
r^2 - \sigma r^{-4} - (\gamma_1 + \gamma_2) r^{-1} = 0,
\] (31)

where

\[
\gamma_1 = \frac{1}{3} D_1 \left[ m_2^2 - \frac{\lambda_1}{2} m_1^2 \right], \quad \gamma_2 = \frac{1}{3} D_2 \lambda_2 m_2^2.
\]

Eq. (31) has a solution:

\[
r^3 = \left( \frac{9}{4} (\gamma_1 + \gamma_2)^2 (t - t_0)^2 - \sigma \right) / (\gamma_1 + \gamma_2),
\] (32)

where \( t_0 \) is an integration constant. Then \( q_\alpha \) from (8) will take on form:

\[
q_\alpha = A_\alpha \left[ \frac{3(\gamma_1 + \gamma_2)(t - t_0)/2 - 1}{3(\gamma_1 + \gamma_2)(t - t_0)/2 + 1} \right]^{C_{\alpha}/3\sqrt{\sigma}},
\] (33)

and, using (11), (32) and (33), one can obtain:

\[
a_\alpha = A_\alpha \left[ \frac{3(\gamma_1 + \gamma_2)(t - t_0)/2 - 1}{3(\gamma_1 + \gamma_2)(t - t_0)/2 + 1} \right]^{C_{\alpha}/3\sqrt{\sigma}} \left[ \left( \frac{9}{4} (\gamma_1 + \gamma_2)^2 (t - t_0)^2 - \sigma \right) / (\gamma_1 + \gamma_2) \right]^{1/3}.
\] (34)

Finally, from Eqs. (28), (29) and (32), we have:

\[
\begin{align*}
\delta \phi &= D_1 \left[ \left( \frac{9}{4} (\gamma_1 + \gamma_2)^2 (t - t_0)^2 - \sigma \right) / (\gamma_1 + \gamma_2) \right]^{-1/2} \sin \sqrt{2(m_2^2 - (\lambda_1/2) m_1^2)}(t - t_0), \\
\delta \chi &= D_2 \left[ \left( \frac{9}{4} (\gamma_1 + \gamma_2)^2 (t - t_0)^2 - \sigma \right) / (\gamma_1 + \gamma_2) \right]^{-1/2} \sin \sqrt{2\lambda_2 m_2^2}(t - t_0).
\end{align*}
\] (35) (36)
IV. CONCLUSIONS

Summarizing, the anisotropic model of the early Universe on the inflationary and scalaron stages was considered. It was shown that there are solutions beginning in the local maximum of the potential \( V \) and tending asymptotically to one of the global minima. The solution near the local maximum describes inflationary stage, and the solutions near the global minima - scalaron stage. It is also obvious from (23) and (34) that the solutions tend asymptotically to isotropic case.

Note here that a type of the obtained solutions differs from the solutions from Refs. \[7\]-\[10\]. In the case under consideration, we have exponential (near the local maximum) or oscillating solutions (near the global minima) with arbitrary values of the parameters \( m_1, m_2, \lambda_1, \lambda_2 \). (They are restricted only by the conditions on existence of the local and global minima.) But in the referred above papers, the problems boiled down to finding of eigenvalues of the parameters \( m_1, m_2 \): at other values of these parameters, the solutions are singular.

---

[1] A. Linde, *Particle Physics and Inflationary Cosmology*, Harwood, Chur (1990); hep-th/0503203.
[2] A. R. Liddle and D. H. Lyth, *Cosmological inflation and large-scale structure*, Cambridge University Press (2000).
[3] A.A. Starobinsky, *Phys. Lett.* B91, 99 (1980).
[4] V. Dzhunushaliev, “Color defects in a gauge condensate”, hep-ph/0605070.
[5] A. D. Linde, *Phys. Rev.* D49, 748 (1994).
[6] D. Lyth and A. Riotto, *Phys. Rept.* 314, 1 (1999).
[7] V. Dzhunushaliev, “Boson stars from a gauge condensate”, gr-qc/0604110.
[8] V. Dzhunushaliev, V. Folomeev, K. Myrzakulov, R. Myrzakulov, *Mod. Phys. Lett.* A22, 407 (2007); gr-qc/0610111.
[9] V. Dzhunushaliev, V. Folomeev, K. Myrzakulov, R. Myrzakulov, “Bouncing off and inflation of the Universe with phantom fields”, gr-qc/0608025.
[10] V. Dzhunushaliev, “Thick brane solution in the presence of two interacting scalar fields”, gr-qc/0603020.
[11] V.Ts. Gurovich and A. A. Starobinsky, *Sov. Phys. JETP*, 50, 844 (1979).
[12] V.N. Folomeev, V.Ts. Gurovich, *Gen. Rel. Grav.* 32, 1255 (2000).