The Vortex Lattice in Ginzburg-Landau Superconductors

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Abstract

Abrikosov’s solution of the linearized Ginzburg-Landau theory describing a periodic lattice of vortex lines in type-II superconductors at large inductions, is generalized to non-periodic vortex arrangements, e.g., to lattices with a vacancy surrounded by relaxing vortices and to periodically distorted lattices that are needed in the nonlocal theory of elasticity of the vortex lattice. Generalizations to lower magnetic inductions and to three-dimensional arrangements of curved vortex lines are also given. Finally, it is shown how the periodic vortex lattice can be computed for bulk superconductors and for thick and thin films in a perpendicular field for all inductions $B$ and Ginzburg-Landau parameters $\kappa$.

1 Introduction

From Ginzburg-Landau (GL) theory [1] Landau’s thesis student Alexei Abrikosov predicted that superconductors with a GL parameter $\kappa > 1/\sqrt{2}$ may contain a lattice of vortices of supercurrent, or flux lines (fluxons) with quantized magnetic flux $\Phi_0 = h/2e = 2 \cdot 10^{-15}$ Tm$^2$. Abrikosov had linearized the GL equations with respect to a small order parameter $|\psi|^2$ and discovered a solution $\psi(x, y)$ possessing a regular lattice of zero lines. This lattice solution appears when the applied magnetic field $B_a$ (along $z$) is decreased below the upper critical field $B_{c2} = \Phi_0/(2\pi \xi^2)$, where $\xi = \lambda/\kappa$ is the GL coherence length. At $B_a = B_{c2}$ one has the average induction $\bar{B} = B_{c2}$. With decreasing $B_a$, the induction decreases and reaches $\bar{B} = 0$ at the lower critical field $B_a = B_{c1} = \Phi_0/(\ln \kappa + \alpha)/(4\pi \lambda^2)$ with $\alpha(\kappa) \approx 0.5$ for $\kappa \gg 1$ (see below). At the same time the vortex lattice spacing $a \approx (\Phi_0/\bar{B})^{1/2}$ increases and diverges when $\bar{B} \to 0$. Abrikosov tells that he had obtained this vortex solution in 1953 but Landau didn’t like it, stating that there are no line-like singularities in electrodynamics. Only when Feynman [2] had published his paper on vortices in superfluid Helium, did Landau agree and Abrikosov could publish his solution in 1957 [3]. For his prediction of the vortex lattice Abrikosov 50 years later in 2003 received the Nobel Prize in Physics together with Vitalii Ginzburg and Anthony Leggett.

After first evidence of the triangular vortex lattice in superconducting Niobium by small-angle neutron scattering in Saclay [4], Träuble and Essmann in Stuttgart succeeded [5, 6] to observe the vortex lattice directly by decorating the surface of a superconductor with iron microcrystallites (“magnetic smoke”). At that time I joined this research group headed by A. Seeger and wrote my thesis on the theory of defects in the vortex lattice [7]. Parts 1 and 2 deal with low inductions $\bar{B} \ll B_{c2}$ when London theory may be used and the vortices interact with each other pairwise, similar to 2D atomic lattices. Parts 3 and 4 consider high inductions $\bar{B} \approx B_2$, where the shape of the GL solutions $\psi(x, y)$ and $B(x, y)$ may be obtained from linearized GL theory, while the nonlinear GL terms determine the amplitudes of this $\psi$ and $B$. My thesis extended Abrikosov’s theory of periodic vortex lattices to non-periodic vortex arrangements, see below. Such distorted-lattice solutions are required to calculate the elastic energy of the vortex lattice and the energy of lattice defects like vacancies and dislocations. They are also helpful to visualize where the solutions of the linearized GL theory apply and how they have to be modified at lower inductions.
2 Abrikosov’s ideal vortex lattice near $B_{c2}$

In the usual reduced units (length $\lambda$, induction $\sqrt{2}B_c$, energy density $B_c^2/\mu_0$, where $B_c = B_{c2}/\sqrt{2}\kappa$ is the thermodynamic critical field) the spatially averaged free energy density $F$ of the GL theory referred to the Meissner state ($\psi = 1, B = 0$) within the superconductor reads

$$F = \exp\left(\frac{(1 - |\psi|^2)^2}{2} + \left(\frac{\nabla}{i\kappa} - A\right)^2 + B^2\right).$$

Here $\psi(r) = f \exp(i\varphi)$ is the complex GL function, $B(r) = \nabla \times A$ the magnetic induction, $A(r)$ the vector potential, and $\langle \ldots \rangle = (1/V) \int V d^3r \ldots$ means spatial averaging over the superconductor with volume $V$. Introducing the super velocity $Q(r) = A - \nabla \varphi/\kappa$ and the magnitude $f(r) = |\psi|$ one may write $F$ as a functional of the real and gauge-invariant functions $f$ or $f^2 = \omega$ and $Q$,

$$F = \exp\left(\frac{(1 - f^2)^2}{2} + \left(\frac{\nabla f}{\kappa^2}\right)^2 + f^2 Q^2 + (\nabla \times Q)^2\right).$$

In the presence of vortices $Q(r)$ has to be chosen such that $\nabla \times Q$ has the appropriate singularities along the vortex cores, where $f$ vanishes. By minimizing this $F$ with respect to $\psi$, $A$ or $f$, $Q$, one obtains the GL equations together with the appropriate boundary conditions. For the superconducting film considered in Sec. 5, one has to add the energy of the magnetic stray field outside the film, which makes the perpendicular component $B_z$ of $B$ continuous at the film surface, see below.

The two GL equations are obtained by minimization of $F$ (1) with respect to $\psi$ and $A$, $\delta F/\delta \psi = 0$ and $\delta F/\delta A = 0$, yielding

$$(\nabla / i - \kappa A)^2 \psi = \kappa (1 - |\psi|^2) \psi,$$

$$\nabla \times [\nabla \times A] = |\psi|^2 Q,$$

With $B_a$ and $\bar{B}$ chosen along the $z$ axis and in the gauge $A_x = -\bar{B}y + \tilde{A}_x(x, y)$, $A_y = \tilde{A}_y(x, y)$ ($\tilde{A}_x$, $\tilde{A}_y$ are terms of higher order) the linearized first GL equation, obtained by omitting the term $|\psi|^2$ in (3), has the general solution

$$\psi(x, y) = \exp(-\kappa \bar{B}y^2/2) g(x, y),$$

$$\frac{\partial g}{\partial x} + i \frac{\partial g}{\partial y} = 0. $$

This means $g(x, y) = g(z)$, $z = x + iy$, can be any analytical function. For a periodic solution satisfying $|\psi|^2(r + R_{mn}) = |\psi|^2(r)$, $r = (x, y)$, with real and reciprocal lattice vectors

$$R_{mn} = (mx_1 + nx_2; ny_2),$$

$$K_{mn} = (2\pi/x_1y_2)(my_2; -mx_2 + nx_1),$$

$(m, n = 0, \pm 1, \pm 2, \ldots$; triangular lattice: $x_1 = a$, $x_2 = x_1/2$, $y_2 = x_1\sqrt{3}/2$; square lattice: $x_1 = y_2 = a$, $x_2 = 0$) and with a zero at $r = 0$, one obtains for $g(z)$ the function $\vartheta_1$ defined as

$$\vartheta_1(z, \tau) = 2 \sum_{n=0}^{\infty} (-)^n \exp\left[i\pi\tau(n + \frac{1}{2})^2\right] \sin(2n + 1)z.$$

Thus, the periodic Abrikosov solution with zeros at the $r = R_{mn}$ may be written as

$$\psi_A(x, y) = \exp\left(-\frac{\pi y^2}{x_1y_2}\right) \vartheta_1\left(\frac{\pi}{x_1}(x + iy), \frac{x_2 + iy_2}{x_1}\right).$$
This solution has the mean induction \( B = \Phi_0/(x_1y_2) \), normalized order parameter \( \langle |\psi_A|^2 \rangle = 1 \), and the Fourier series \( |\psi_A|^2 = \omega_A(x,y) \),

\[
\omega_A(r) = \sum_{K_{mn}} (-1)^m \exp \left( - \frac{K_{mn}x_1y_2}{8\pi} \right) e^{iKr}.
\] (11)

From the zero \( \omega_A(0,0) = 0 \) follows that the sum over all Fourier coefficients in (11) is zero for all lattice symmetries. (Abrikosov \[3\] chose a different position for \( \omega_A = 0 \) and thus obtained the function \( \vartheta_3 \)).

### 3 Distorted vortex lattice near \( B_{c2} \)

The GL solution for a distorted vortex lattice near \( B_{c2} \) is obtained as follows. Assume that each of the straight and parallel vortex lines is displaced from its ideal lattice positions \( R_{mn} = R_\nu = (X_\nu, Y_\nu) \) by displacements \( s_\nu = (s_{\nu x}, s_{\nu y}) \), \( R_\nu = (x_\nu, y_\nu) = R_\nu + s_\nu \), such that the displacement field itself is periodic with a super lattice \( N \) times larger than the vortex lattice, but with same symmetry, \( s(r + NR_{mn}) = s(r) \). Where needed we use a continuous displacement field \( s(r) \) defined such that it has the same Fourier transform as the discrete \( s_\nu = s(R_\nu) \). A distorted triangular vortex lattice with spacing \( x_1 = a \) then has the solution, Eq. (5),

\[
\psi(x,y) = c_1 \exp \left( - \frac{2\pi y^2}{\sqrt{3}a^2} \right) \vartheta_1(\pi/a, \tau) \prod \vartheta_1((\pi/Na)(z-z_\nu-s_\nu), \tau) \] (12)

with \( z = x + iy \), \( z_\nu = X_\nu + iY_\nu \), \( \tau = (1+i\sqrt{3})/2 \), \( s_\nu = s_{\nu x} + is_{\nu y} \), the product is over one super cell, and \( c_1 \approx 1 \) is a normalization constant. When all \( s_\nu = 0 \), the product in (12) is unity, \( \prod = 1 \), thus the first two factors in (12) are the ideal lattice solution with \( c_1 = 1 \), cf. Eq. (10). Each factor of the product shifts one zero from \( r = R_\nu \) to \( r = R_\nu + s_\nu \). The absolute value of \( |\psi|^2 = \omega \) of this GL function may also be expressed in terms of the Fourier series \( \omega_A(r) \) (11),

\[
\omega(r) = c_1^2 \omega_A(r) \prod \frac{\omega_A((r-R_\nu-s_\nu)/N)}{\omega_A((r-R_\nu)/N)}.
\] (13)

In the limit of infinite super cell, \( N \to \infty \), one may use \( \vartheta_1(z/N, \tau) \propto z/N \) for \( |z|/N \ll 1 \), thus one may replace the function \( \vartheta_1 \) by its argument since all the constant factors cancel or combine to a normalization factor that follows from numerics. One then obtains simply

\[
\omega(r) = c_1^2 \omega_A(r) \prod \frac{|r-R_\nu-s_\nu|^2}{|r-R_\nu|^2}.
\] (14)

### 4 Vortex lattice vacancy near \( B_{c2} \)

Removing the central vortex at \( R_\nu = 0 \) adds a factor \( 1/r^2 \) to the linearized solution \( \omega(r) \). Obviously, if the other vortices are not allowed to relax, this solution at large distances vanishes as \( 1/r^2 \); it cannot be normalized and its energy is infinite. However, if the relaxation of the other vortices is chosen appropriately it will minimize the defect energy and make it finite. This can be seen from the solution

\[
\omega(r) = c_1^2 \frac{\omega_A(r)}{r^2} \left| \frac{h(z)}{h(0)} \right|^2, \quad h(z) = \prod_{\nu \neq 0} \left( 1 - \frac{s_\nu}{z-z_\nu} \right).
\] (15)
Figure 1: Order parameter $\omega(x, 0)$ for the ideal vortex lattice (dashed line) and for vortex lattice with vacancy, Eq. (15), with simple relaxation field (19) (dotted line) and with better relaxation field that minimizes the defect energy (solid line, see text).

The constant factor $|1/h(0)|^2$ was added to force convergence of the infinite product. The solution for a super lattice of vacancies positioned at the $NR_\nu$, is given by expression (13) divided by $\omega(r/N)$ that removes the zeros at positions $NR_\nu$...

The energy of both the ideally periodic and the distorted vortex lattices is calculated via the Abrikosov parameter

$$\beta = \frac{\langle |\psi|^4 \rangle}{\langle |\psi|^2 \rangle^2} = \frac{\langle \omega^2 \rangle}{\langle \omega \rangle} \geq 1. \quad (16)$$

This $\beta$ enters the free energy of the linearized GL theory (referred to the normal state), that has to be minimized when $\bar{B}$ is held constant,

$$F = \frac{\bar{B}^2}{2\mu_0} - \frac{(B_{c2} - \bar{B})^2}{2\mu_0[1 + (2\kappa^2 - 1)\beta]} \quad (17)$$

and the free enthalpy that has to be minimized when $B_a$ is held constant,

$$G = F - \frac{BB_a}{\mu_0} = -\frac{(B_{c2} - B_a)^2}{2\mu_0(2\kappa^2 - 1)\beta}. \quad (18)$$

The elastic energy of the distorted vortex lattice is the product of the derivatives $\partial F/\partial \beta$ or $\partial G/\partial \beta$ times the change of $\beta\{\psi\}$ times the volume, with the limit of infinite volume taken. This means that all elastic energies and energies of structural defects near $B_{c2}$ vanish as $(B_{c2} - \bar{B})^2 \propto (B_{c2} - B_a)^2$. This is true also for the shear modulus $c_{66}$ of the vortex lattice, which can be obtained using Abrikosov’s periodic lattice solution \[7\, 9\].

In the case of the vortex vacancy, the resulting defect energy is finite only if the vortices relax (shift towards the removed vortex) such that at large distances (and after numerical minimization practically at all distances) the vortex displacements are

$$s_\nu = -\frac{R_\nu}{2\pi nR_\nu^2} \quad (19)$$
with \( n = \frac{\bar{B}}{\Phi_0} = 1/(x_1y_2) \) the vortex density. If the radial displacements were chosen smaller (larger) than in (19), the order parameter (15) would vanish (diverge) at large distances \( r \). But with the correct displacements that minimize \( \beta \) and thus the defect energy, the amplitude of the oscillating order parameter stays almost constant, even near the vacancy. This can be seen in Fig. 1, where the profiles of \( \omega(x, y) \) along \( y = 0 \) are plotted for ideal triangular vortex lattice and for the lattice with a central vacancy with the simple relaxation (19) and with an improved relaxation field \( s_\nu = -R_\nu \left[ \sqrt{3}a^2/(4\pi R_0^4) + 0.068a^4/R_0^4 \right] \). Figure 2 shows the contour lines of the fully relaxed order parameter \( \omega(x, y) \), which has a maximum at the origin (vacancy position) and minima (zeros) at the vortex positions.

The displacement (19) means that the area of the “relaxing ring”, \( 2\pi R_\nu s_\nu = 1/n = x_1y_2 = \Phi_0/\bar{B} \), exactly equals the area of one lattice cell. In other words, the removal of one vortex at \( R_\nu = 0 \) is compensated by the relaxation of the surrounding vortices such that the average vortex density in any contour (containing or not containing the vacancy) stays constant and equals the density \( n \) that was there before the vacancy was introduced. Note that the field (19) satisfies \( \nabla \cdot s(r) = 0 \), and thus describes a pure shear deformation. More precisely, one has \( \nabla \cdot s(r) = (1/n)\delta_2(r) \) (\( \delta_2 \) is the 2D delta function), i.e., the displacement field (19) “remembers” that one vortex cell area was removed. Further interesting properties of structural defects in vortex lattices and other two- or three-dimensional soft lattices are discussed in [10].

5 Distorted vortex lattice away from \( B_{c_2} \)

As shown with the vacancy example, the distorted-lattice solution (14) of the linearized GL equations yields finite energies of lattice defects only if most of the vortex displacements are allowed to relax appropriately. But without this relaxation, the elastic energy is infinite. For example, if only the one vortex at the origin is displaced, \( s_\nu = s_0\delta_{\nu0}\hat{x} \) (\( \delta_{\nu0} = 1 \) if \( \nu = 0 \), else \( \delta_{\nu0} = 0 \)) one has from (14)

\[
\omega(r) = \omega_A(r)|r - s|^2/r^2 = \omega_A \cdot (1 - 2xs_0/r^2) + O(s^2),
\]
i.e., the periodic order parameter is modulated by a slowly decreasing function. The Abrikosov $\beta$ of this defect times the volume, $\lim_{V \to \infty} (\beta - \beta_0) V$, diverges and so the defect energy diverges.

This unphysical divergence of defect energies of the vortex lattice is removed when the influence of the nonlinear GL terms on the solutions $\omega(x, y)$ and $B(x, y)$ is accounted for. This calculation was performed in a series of 4 papers [11]: Parts 1 and 2 deal with the linear elastic energy of the vortex lattice at low and high inductions $\tilde{B}$. Parts 3 and 4 derive the GL solutions for the distorted vortex lattice when the vortex lines are straight and parallel or arbitrarily curved. The essential result is that the long-ranging modulation factors like (14) become exponentially damped over a new length $\xi' = 1/k_\psi = \xi/\sqrt{2(1-b)}$ with $b = \tilde{B}/B_{c2}$. As $b \to 1$, this screening length becomes infinite and the linearized solution (14) is recovered. At $b < 1$, the distorted-lattice solution (14) should be replaced by

$$\omega(r) = \omega_A(r)[1 + \sum_\nu s_\nu \nabla K_0(|r - R_\nu|/k_\psi)]^2 + O(s^2),$$

where $K_0(x)$ is a modified Bessel function with the limits $K_0(0) = -\ln x (x \ll 1)$, $K_0(x) \approx (\pi/2x)^{1/2} e^{-x} (x \gg 1)$. This generalized expression up to terms linear in the vortex shifts $s_\nu$ reproduces the linearized solution (14), (20) when $k_\psi \to 0$, but it does not possess the correct zeros at $r_\nu = R_\nu + s_\nu$. This may be corrected by replacing in (21) the periodic order parameter $\omega_A(r)$ by the “phase modulated” $\omega_A[r - s(r)]$ and cutting the infinity of $K_0$ off. The resulting solution is still exact up to linear terms in $s_\nu$, since $\nabla \omega_A(r)$ vanishes at the $R_\nu$ and thus the expansion of $\omega_A[r - s(r)]$ contains no linear term.

The screening length $\xi' = 1/k_\psi$ may be derived by considering only one Fourier component of the displacement field,

$$s_\nu = \text{Re}\{s_0 \exp(ikR_\nu)\}$$

with $k = (k_x, k_y, 0)$ and $\text{Re} = \text{real part}$. One may then write the linearized solution as

$$\omega(r) = \omega_A(r)[1 + \frac{1}{2} \eta(r)]^2 + O(s^2),$$

$$\eta(r) = 2 \sum_\nu s_\nu \frac{r - R_\nu}{(r - R_\nu)^2} = \frac{2b}{\xi^2} \text{Re}\left\{\sum_\nu s_\nu \sum_\K \frac{i(k + \K)}{(k + \K)^2} \exp[i(k + \K)r]\right\}.$$

In $\eta(r)$ (24) the terms with reciprocal lattice vectors $\K \neq 0$ shift the zeros of $\omega(r)$ (“phase modulation”), while the term $\K = 0$ yields an “amplitude modulation” of $\omega(r)$. This term diverges as $1/k^2$, i.e., it yields a diverging amplitude modulation when the wavelength of the displacement field is large.

From physical reasons it is clear that this term $\propto 1/k^2$ has to be cut off, e.g., replaced by $1/(k^2 + k_\psi^2)$. Accounting for all the GL terms nonlinear in $\omega \propto 1 - b$ ($b = \tilde{B}/B_{c2}$, terms like $\omega^2$, $B^2$, $Q^2$) indeed yields such a cut off, with $k_\psi^2 = 2(1-b)/\xi^2$. The resulting solution for periodic $s(r)$ may be written as

$$\omega(r) = \omega_A[r - s(r)]\left[1 + \frac{2b}{\xi^2} \frac{\nabla s(r)}{k^2 + k_\psi^2}\right] + O(s^2).$$

In a similar way, the solution for the induction $B(x, y)$ of the linearized GL theory,

$$B(r) = \tilde{B} + B_{c2}\frac{\langle \omega \rangle - \omega(r)}{2k^2}$$

is modified by the nonlinear terms to give for periodic $s(r)$

$$B(r) = B_0[r - s(r)] - \frac{\tilde{B}}{1 + k^2/k_\psi^2} \frac{\nabla s(r)}{2k^2} + O(s^2).$$
with \( k_h^2 = 1/\lambda^2 = \langle \omega \rangle /\lambda^2 \approx (1 - b)/\lambda^2 \) and \( B_0(x,y) \) the ideal periodic solution for \( s \equiv 0 \). In deriving (27) all terms containing \( k_\psi \) have cancelled. From the solutions (25) and (27) for periodic \( s(\mathbf{r}) \), the generalization to arbitrary displacement fields is obtained by Fourier transform.

6 Curved vortices

The above method can be extended to 3D displacement fields \( s_\nu(z) = [s_{\nu x}(z), s_{\nu y}(z), 0] \) describing distorted lattices of curved vortices,

\[
\begin{align*}
\mathbf{s}_\nu(z) &= \int_{\text{BZ}} \frac{d^3 k}{8\pi^3 n} \tilde{s}(\mathbf{k}) \exp(i\mathbf{kR}_\nu) , \\
\tilde{s}(\mathbf{k}) &= \sum_\nu \int dz \mathbf{s}_\nu(z) \exp(-i\mathbf{kR}_\nu) ,
\end{align*}
\]

where now \( \mathbf{r} = (x, y, z) \), \( \mathbf{R}_\nu = (x_\nu, y_\nu, z) \), \( n = B/\Phi_0 \), and the \( \mathbf{k} \) integration extends over the first Brillouin zone of the ideal vortex lattice [since \( \tilde{s}(\mathbf{k} + \mathbf{K}) = \tilde{s}(\mathbf{k}) \) and over \(-\infty < k_z < \infty \). The coordinate \( z \) plays here the role of a line parameter. The order parameter which solves the GL equations near \( B_{c2} \) and has zeros at the vortex positions \( \mathbf{r}_\nu(z) = \mathbf{R}_\nu + \mathbf{s}_\nu(z) \) is

\[
\omega(\mathbf{r}) = \omega_A(\mathbf{r}) \left[ 1 + \sum_\nu \int dz' \mathbf{s}_\nu(z') \frac{\nabla_2 \exp\left(-|\mathbf{r} - \mathbf{R}_\nu'|/\rho k_h\right)}{2|\mathbf{r} - \mathbf{R}_\nu'|} \right] + O(s^2) .
\]

The 3D solution for the induction \( \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) \) for periodic \( \mathbf{s}(\mathbf{r}) \) after averaging over a vortex cell may be written as

\[
\begin{align*}
\mathbf{B}(\mathbf{r}) &= \hat{z}\hat{B} + \hat{B} \frac{\hat{z} \nabla \mathbf{s}(\mathbf{r})}{1 + k^2/k_h^2} + O(s^2) , \\
\mathbf{A}(\mathbf{r}) &= \frac{1}{2} \hat{B} \hat{z} \times \mathbf{r} + \hat{B} \mathbf{s}(\mathbf{r}) \times \hat{z} + O(s^2) .
\end{align*}
\]

These expressions coincide with the first-order expansion terms (in \( s \)) of a linear superposition of spherical “source fields” centered at each vortex element:

\[
\begin{align*}
\mathbf{B}(\mathbf{r}) &= \Phi_0 k_h^2 \sum_\nu \int d\mathbf{r}_\nu \frac{\exp(-\rho/k_h)}{4\pi \rho} ,
\end{align*}
\]

where \( \rho \approx [(\mathbf{r} - \mathbf{r}_\nu)^2 + a^2/4]^{1/2} \) has an inner cut-off \( \approx a/2 \), half the vortex spacing in our derivation from the vortex lattice. The expression (32) is also the solution of London theory for arbitrarily arranged curved or straight vortices if one puts \( k_h = 1/\lambda \) (i.e. \( b \to 0 \)) and the vortex core radius \( r_c \approx \xi \) for the inner cutoff. The line element of the path integral in (32) may be parameterized with \( z \) as line parameter and integration variable,

\[
d\mathbf{r}_\nu = \frac{d\mathbf{r}_\nu(z)}{dz} dz = \left( \hat{z} + \frac{d\mathbf{s}_\nu(z)}{dz} \right) dz .
\]

7 Nonlocal elasticity of the vortex lattice

The distorted-lattice solution up to terms linear in the displacements \( \mathbf{s} \) can be used to calculate the linear elastic energy of the vortex lattice, \( F_{\text{elast}} = F\{\mathbf{s}_\nu\} - F\{\mathbf{s}_\nu \equiv 0\} \), referred to the perfect lattice (the equilibrium state). The most general expression quadratic in the 2D displacements \( \mathbf{s}_\nu(z) \), or in their Fourier transforms \( \tilde{s}(\mathbf{k}) = (\tilde{s}_x, \tilde{s}_y, 0) \), (28), is

\[
F_{\text{elast}} = \frac{1}{2} \int_{\text{BZ}} \frac{d^3 k}{8\pi^3 n} \tilde{s}_\alpha(\mathbf{k}) \Phi_{\alpha\beta}(\mathbf{k}) \tilde{s}_\beta(-\mathbf{k}) ,
\]
where the sum over the indices $\alpha$, $\beta = (x, y)$ is taken. The $2 \times 2$ matrix $\Phi_{\alpha\beta}$ is the elastic matrix. This expression applies for both an elastic continuum and for a lattice. For a lattice $\Phi_{\alpha\beta}$ is periodic, $\Phi_{\alpha\beta}(k + K) = \Phi_{\alpha\beta}(k)$, and thus the integral should be restricted to the first Brillouin zone (BZ). The BZ for the triangular lattice is a hexagon, and for the square lattice a square. Where required, the BZ may be approximated by a circle with radius $k_B = (2b)^{1/2}/\xi$, $b = \tilde{B}/B_{c2}$, and area $\pi k_B^2 = 4\pi^2 n$, $n = \tilde{B}/\phi_0$.

For a uniaxial elastic continuum the elastic matrix $\Phi_{\alpha\beta}(k_x, k_y, k_z)$ has the form

$$n\Phi_{\alpha\beta}(k) = (c_{11} - c_{66})k_\alpha k_\beta + \delta_{\alpha\beta}[(k_x^2 + k_y^2)c_{66} + k_z^2 c_{44}] . \tag{35}$$

In it the coefficients are the elastic moduli: $c_{11} - c_{66}$ the isotropic compression modulus, $c_{11}$ the uniaxial compression modulus, $c_{66}$ the shear modulus, and $c_{44}$ the tilt modulus. The elastic moduli of the vortex lattice are obtained by deriving the elastic energy, e.g., from GL theory and comparing it at $k_x^2 + k_y^2 \ll k_B^2$ with the continuum limit (35). This yields

$$c_{11}(k) = \frac{B^2}{\mu_0} \frac{\partial B_a}{\partial B} \left(1 + k_x^2/k_B^2 \right) \left(1 + k_y^2/k_B^2 \right) + c_{66} \tag{36}$$

$$c_{66} = \frac{B B_{c2}}{8k^2\mu_0} \left(1 - b\right)^2 \frac{(2\kappa^2 - 1)2k^2}{[2k^2 - 1 + 1/\beta_A]^2} (1 - 0.3b) \tag{37}$$

$$c_{44}(k) = \frac{B^2}{\mu_0} \frac{1}{1 + k^2/k_B^2} + \frac{\tilde{B}(B_a - \tilde{B})}{\mu_0} . \tag{38}$$

These expressions are exact at large reduced induction $b = \tilde{B}/B_{c2} \rightarrow 1$ and for all $\kappa$, but they are written such that they reduce to the correct values also in the limit of small induction $\tilde{B} \ll B_{c2}$. In $c_{66}$, $\beta_A = 1.160$ is the Abrikosov parameter of the triangular lattice (the square lattice is unstable and thus has negative $c_{66}$); the third factor reduces to 1 for $2\kappa^2 \gg 1$ and to $(2\kappa^2 - 1)\beta_A^2 \rightarrow 0$ for $\kappa \rightarrow 1/\sqrt{2}$, which means the shear stiffness of the vortex lattice is zero in superconductors with $\kappa = 0.71$; the factor $1 - 0.3b$ interpolates between the the correct limits at $b \rightarrow 1$ and $b \rightarrow 0$. In particular, for $b \ll 1$ and $2\kappa^2 \gg 1$, (37) reproduces the London result $c_{66} = \tilde{B}B_{c2}/(8\kappa^2 \mu_0)$.

An interesting result is the dependence of $c_{11}$ (36) and $c_{44}$ (38) on $k = |k|$, which means the elasticity of the vortex lattice is non-local. In the limit of uniform stress, $k \rightarrow 0$, these expressions reproduce the known values of the compression and tilt moduli obtained by thermodynamics, $c_{11} - c_{66} = (B^2/\mu_0)\partial B_a/\partial \tilde{B}$, $c_{44} = \tilde{B}B_a/\mu_0$. However, when the wavelength of the periodic compression or tilt decreases, i.e., the wave vector $k$ increases, these moduli decrease. This means, the vortex lattice is softer for short-wavelengths compression and tilt than it is for long wavelengths. The two characteristic lengths or wave vectors were already introduced above, $k_h = 1/\lambda' \approx \sqrt{1-b}/\lambda$ and $k_\psi = 1/\xi' = \sqrt{2(1-b)}/\xi$.

This dispersion or elastic non-locality means, e.g., that a point force exerted by a small pinning center on the vortex lattice, deforms the vortex on which it acts not like plugging a string but more, causing a sharp cusp since a local deformation costs little energy. If the interaction of the vortices with the pinning center is via the order parameter $|\psi|^2$ or via the gradient term in the GL functional, then this interaction itself is nonlocal, smeared over the length $\xi' = 1/k_\psi$. In the expressions for the elastic force and the elastic energy there is thus a factor $1 + k^2/k_\psi^2$ in the numerator that compensates the same factor in the denominator originating from $c_{11}(k)$, (36). Therefore, the factor $1/(1 + k^2/k_\psi^2)$ in $c_{11}$ has no physical meaning in pinning problems since near $B_{c2}$ where $\xi'$ can be larger than the vortex spacing $a$, it is not possible to exert a pinning force on one single zero of the order parameter but only on an area with radius $\xi'$ containing several such zeros. The nonlocality factor $1/(1 + k^2/k_\psi^2)$ in $c_{44}$, however, is important in pinning theories since it strongly enhances the elastic deformations caused by small pins acting on the
vortex cores. In (not very realistic) models where the pinning force acts only the magnetic field of the vortex but not on the vortex cores, this enhancement of the elastic displacement may vanish, cancelled by the non-locality of this model force.

The correct, non-local elasticity thus effectively softens the vortex lattice and leads to large, pinning-caused distortions and disorder of the vortex lattice. Furthermore, the thermal fluctuations of the vortex lattice are strongly enhanced by this non-local elasticity. In both cases the lattice softening is caused mainly by the dispersion of $c_{44}(k)$, while the dispersion and reduction of $c_{11}(k)$ is not so important since the shear modulus $c_{66}$ is typically much smaller than $c_{11}(k)$ and the shear modes of the elastic deformation thus dominate over the compressional modes.

Figure 3: The magnetization curves of the triangular vortex lattice (solid lines, numerical result), coinciding within line thickness with those of the square lattice. Shown are $h = B_a/B_{c2}$ versus $b = B/B_{c2}$ (upper left triangle) and $-m = h - b$ versus $h$ (lower right triangle). The dots show the fit, Eq. (59), good for $\kappa \leq 20$.

8 Vortex arrangements at low inductions

At low inductions $b < 0.2$ and not too small $\kappa > 2$, the GL theory for arbitrary 3D arrangements of vortices reduces to the London theory, which may be expressed by the energy functional

$$F\{\mathbf{B}\} = \frac{\mu_0}{2} \int d^3r [B^2 + \lambda^2 (\nabla \times \mathbf{B})^2].$$

(39)

Here $\lambda$ is the London depth equal to the GL magnetic penetration depth. Minimizing $F\{\mathbf{B}\}$ with respect to the induction $\mathbf{B}(\mathbf{r})$ using $\nabla \mathbf{B} = 0$, and adding appropriate singularities along the positions $\mathbf{r}_\nu(z)$ of the vortex cores, one obtains the modified London equation [12],

$$(-\lambda^2 \nabla^2 + 1)\mathbf{B}(\mathbf{r}) = \Phi_0 \sum_\nu \int d\mathbf{r}_\nu \delta_3(\mathbf{r} - \mathbf{r}_\nu)$$

(40)
Figure 4: The magnetic field $B(r)$ and order parameter $|\psi(r)|^2$ of an isolated vortex line calculated from Ginzburg-Landau theory for GL parameters $\kappa = 2, 5,$ and 20. For such large $\kappa$ the field in the vortex center is twice the applied equilibrium field, $B(0) \approx 2B_{c1} = 2B_a.$

with $\delta_3$ the 3D delta function. From this one obtains the energy of an arbitrary arrangement of straight or curved vortices,

$$F\{r_\nu(z)\} = \frac{\Phi_0^2}{8\pi \lambda^2 \mu_0} \sum_\mu \sum_\nu \int dr_\mu \int dr_\nu \frac{\exp(-r_{\mu \nu}/\lambda)}{r_{\mu \nu}} .$$

(41)

In this double sum the terms $\mu \neq \nu$ describe the pairwise interaction of the vortex line elements $dr_\mu, dr_\nu$ over the distance $r_{\mu \nu} = |r_\mu - r_\nu|$. The term $\mu = \nu$ is the self-energy of the $\mu$th vortex line, which depends on the shape of this vortex. In it an inner cut off is needed, obtained, e.g., by putting $r_{\mu \mu}^2(z, z') = |r_\mu(z) - r_\mu(z')|^2 + r_c^2$ with $r_c \approx \xi$ the vortex core radius, to avoid divergence when in the integral the parameters equal, $z = z'$.

From the GL nonlocal elastic energy (34)-(38) one may construct an effective interaction potential between vortex line elements such that the full nonlocal linear elastic energy is reproduced at small displacements [13]. At the same time, this interaction at low $b \ll 1$ reproduces the London interaction for arbitrary vortex arrangements, and an approximate GL interaction valid at all $b$ and $\kappa$,

$$F\{r_\nu(z)\} = \frac{\Phi_0^2}{8\pi \lambda^2 \mu_0} \sum_\mu \sum_\nu \left[ \int dr_\mu \int dr_\nu \frac{\exp(-r_{\mu \nu}/\lambda')}{r_{\mu \nu}} - \int |dr_\mu| \int |dr_\nu| \frac{\exp(-r_{\mu \nu}/\xi')}{r_{\mu \nu}} \right]$$

(42)

with $r_{\mu \nu} = |r_\mu - r_\nu|$. For $b \ll 1$ the first term in (42) reproduces the magnetic repulsion of London vortices with $\lambda' = \lambda/\sqrt{1-b} \approx \lambda$; this magnetic interaction is vectorial due to the product $dr_\mu \cdot dr_\nu$ containing the cosine of the angle between two line elements. The second term of shorter range $\xi' = \xi/\sqrt{2(1-b)}$ may be interpreted as an attraction caused by the overlap of the vortex cores, the regions where the order parameter is reduced: two overlapping cores require less (positive) condensation energy than two separated cores, thus the cores attract.
This attraction has scalar character, hence the product $|d r_\mu| |d r_\nu|$. The attractive second term in (42) removes the logarithmic divergence of the magnetic repulsion at zero distance, since both terms have the same singularity but of opposite sign, such that they cancel. The total potential (42) is thus a smooth function that at $r_{\mu\nu} = 0$ starts with a finite value and then decreases monotonically to zero with increasing distance $r_{\mu\nu} \to \infty$.

For straight parallel vortex lines the general 3D energy expression (42) simplifies to the sum of the vortex self energies, $F_{\text{self}} = \Phi_0 B_{c1} / \mu_0$ per unit length and per vortex, and the interaction energy $F_{\text{int}}$ of all vortices per unit length,

$$F_{\text{int}} \{ r_\nu \} = \frac{\Phi_0^2}{2\pi \lambda^2 \mu_0} \sum_\mu \sum_{\nu > \mu} \left[ K_0 \left( \frac{|r_\mu - r_\nu|}{\lambda} \right) - K_0 \left( \frac{|r_\mu - r_\nu|}{\xi'} \right) \right]. \tag{43}$$

Here $K_0(x)$ is a modified Bessel function, see Eq. (21). The effective 2D interaction potential in (43) is a smooth, monotonically decreasing function with a finite value at $r_{\mu\nu} = 0$ since the two logarithmic singularities of the $K_0$ functions cancel each other. As in the 3D expression (42), the first term in (43) is the magnetic repulsion of the straight vortices, and the second term is an attraction due to gain in condensation energy during the overlap of vortex cores.

### 9 Vortex lattice solution for all $\kappa$ and $\bar{B}$

Abrikosov’s solution method for the periodic vortex lattice starts from the linearized GL theory and is thus valid only at large inductions $\bar{B}$ near the upper critical field $B_{c2}$. First numerical solutions for all $\bar{B}$ and $\kappa$ were obtained by the “circular cell method” \cite{14} that approximates the
hexagonal Wigner-Seitz cell of the triangular vortex lattice by a circle and solves a cylindrically symmetric problem, see also Ref. [15]. The periodic solution in the entire ranges of reduced induction \(0 < b = B/B_{c2} < 1\) and GL parameter \(1/\sqrt{2} < \kappa < \infty\) may be obtained for bulk superconductors by the following numerical method [15, 16, 17]. We start from the free energy functional \(F\), Eq. (2), and minimize it with respect to the real and periodic functions \(\omega(x,y) = f^2 = |\psi|^2\) (order parameter) and \(Q(x,y) = A - \nabla \varphi / \kappa\) (negative super velocity) or \(\hat{z}B(x,y) = \nabla \times Q\) (induction). We consider periodic lattices with one flux quantum per vortex. In the sense of a Ritz variational method we use Fourier series for the periodic trial functions with a finite number of Fourier coefficients \(a_K\) and \(b_K\),

\[
\omega(r) = \sum_K a_K (1 - \cos K r),
\]

\[
B(r) = \bar{B} + \sum_K b_K \cos K r,
\]

\[
Q(r) = Q_A(r) + \sum_K b_K \hat{z} \times \frac{K}{K^2} \sin K r,
\]

where \(r = (x,y)\) and \(K = (K_x, K_y)\) are the reciprocal lattice vectors (8) of the vortex lattice with positions (7). In all sums here and below the term \(K = 0\) is excluded. In (46) \(Q_A(x,y)\) is the super velocity of the Abrikosov \(B_{c2}\) solution, which satisfies

\[
\nabla \times Q_A = [\bar{B} - \Phi_0 \sum_R \delta_2(r - R)] \hat{z},
\]

where \(\delta_2(r) = \delta(x)\delta(y)\) is the 2D delta function. This relation shows that \(Q_A\) is the velocity field of a lattice of ideal vortex lines but with zero average rotation. Close to each vortex center one has \(Q_A(r) \approx r' \times \hat{z} / (2 \kappa r'^2)\) and \(\omega(r) \propto r'^2\) with \(r' = r - R\). In principle \(Q_A(r)\) may be expressed as a slowly converging Fourier series by integrating (47) using div\(Q = \text{div}Q_A = 0\) as in Ref. [16]. But it is more convenient to take \(Q_A\) from the exact relation

\[
Q_A(r) = \frac{\nabla \omega_A \times \hat{z}}{2 \kappa \omega_A},
\]

where \(\omega_A(x,y)\) is the Abrikosov \(B_{c2}\) solution given by the rapidly converging series (11). With (48) the numerical method becomes highly accurate. Note that the ansatz (46) assumes that

Figure 6: Contour lines of \(\omega(x,y) = |\psi|^2\) and \(B(x,y)\) for \(\kappa = 5, b = 0.5\).
Figure 7: The magnetic field variance \( \sigma = \langle [B(x, y) - B]^2 \rangle \) of the triangular FLL for \( \kappa = 0.85, 1, \ldots, 200 \) plotted in units of \( B_c^2 \) as \( \sqrt{\sigma} \cdot (\kappa^2 - 0.069)/B_c^2 \) (solid lines) such that the curves for all \( \kappa \) collapse near \( b = 1 \). The dashed lines show the same functions divided by \((1 - b)^{-1}\) such that they tend to a finite constant 0.172 at \( b = 1 \). All curves are plotted versus \( \sqrt{b} = \sqrt{\bar{B}/B_c^2} \) to stretch them at small \( b \) values and show that they go to zero linearly. The upper frame 0.383 is the usual London approximation. The limit for very small \( b \) is shown as two dash-dotted straight lines for \( \kappa = 5 \) and \( \kappa = 10 \). The upper frame 0.383 shows the usual London approximation.

div\(Q = 0\). This assumption can be shown to be exact at high and low inductions, but I did not find a proof that it is true in the general case, though it is satisfied numerically with high precision for the periodic vortex lattice at all \( B \) and \( \kappa \).

The solutions \( \omega(r) \) and \( B(r) \) may be computed by using a finite number of Fourier coefficients \( a_K \) and \( b_K \) and minimizing the free energy \( F(B, \kappa, a_K, b_K) \) with respect to these coefficients as done in [16]. However, a much faster and more accurate solution method [15, 17] is to iterate the two GL equations \( \delta F/\delta \omega = 0 \) and \( \delta F/\delta Q = 0 \) written in appropriate form. The iteration is stable and converges rapidly if one isolates a term \((-\nabla^2 + \text{const})(\omega, Q)\) on the l.h.s. and puts the remaining terms to the r.h.s. as an “inhomogeneity” of such London-like equations, e.g.,

\[
(-\nabla^2 + 2\kappa^2)\omega = 2\kappa^2(2\omega - \omega^2 - \omega Q^2 - g), \tag{49}
\]

\[
(-\nabla^2 + \bar{\omega}) Q_b = -\omega Q_A - (\omega - \bar{\omega})Q_b, \tag{50}
\]

with the abbreviations \( g(r) = (\nabla \omega)^2/(4\kappa^2 \omega) \), \( Q_b = Q - Q_A \), \( \nabla \times Q_b = B(r) - \bar{B} \), and \( \bar{\omega} = \langle \omega \rangle = \sum_K a_K \). Equations (49), (50) introduce some “penetration depths” \((2\kappa^2)^{-1/2} = \xi/\sqrt{2}\) and \(\bar{\omega}^{-1/2} = \lambda/\bar{\omega}^{1/2}\) (in real units), which stabilize the convergence of the iteration. Acting on the Fourier series \( \omega \) (44) and \( Q_b \) (46) the Laplacian operator \( \nabla^2 \) yields a factor \(-K^2\), which facilitates the inversion of (49) and (50). Using the orthonormality

\[
2 \langle \cos K r \cos K' r \rangle = \delta_{KK'}, \tag{51}
\]

valid for \( K \neq 0 \), one obtains from (44), (45) \( a_K = -2\langle \omega(r) \cos Kr \rangle \) and \( b_K = 2\langle B(r) \cos Kr \rangle \). The convergence of the iteration is considerably improved by adding a third equation which
Figure 8: The shear modulus $c_{66}$ of the triangular vortex lattice in bulk superconductors as function of the reduced induction $b = \bar{B}/B_{c2}$ for GL parameters $\kappa = 0.4, 0.5, 0.6, 0.707, 0.75, 1, 1.4, 2, 3, 5, 7, 10, 100$, in units $B_{c2}^2/(1000\mu_0)$. For $\kappa < 2^{-1/2} = 0.707$ one formally has negative shear modulus $c_{66} < 0$, though vortices and a vortex lattice are energetically not favorable in bulk type-I superconductors.
Figure 9: Magnetic field lines for a superconductor film calculated from Ginzburg-Landau theory for the triangular vortex lattice. Shown is the example $b = B/B_{c2} = 0.04$, $\kappa = 1.4$, triangular lattice with vortex spacing (unit length) $x_1 = 3^{-1/4}(2\Phi_0/B)^{1/2} = 5x_1(B_{c2}) \approx 10\lambda$, film thickness $d = 0.8x_1 \approx 8\lambda$. The left half shows the field lines that would apply if the field inside the film would not change near the surfaces $z = \pm d/2$ marked by dashed lines. The right half shows the correct solution. The density of the depicted field lines is proportional to $|\mathbf{B}(\mathbf{r})|$.

The equilibrium field $B_a = \mu_0 \partial F/\partial \bar{\mathbf{B}}$ (the applied field) are easily computed from Doria’s virial theorem \[18\], which in our reduced units reads

$$B_a = \frac{\langle f^2 - f^4 + 2B(x,y)^2 \rangle}{2 \langle B \rangle}. \quad (55)$$

In this way we find the lower critical field \[15\], $B_{c1}(\kappa) = \lim_{\bar{B} \to 0} B_a(\bar{B}, \kappa)$,

$$B_{c1}(\kappa) = \frac{\Phi_0}{4\pi\lambda^2} [\ln \kappa + \alpha(\kappa)], \quad h_{c1} = \frac{B_{c1}}{B_{c2}} = \frac{\ln \kappa + \alpha(\kappa)}{2\kappa^2},$$

$$\alpha(\kappa) = \alpha_\infty + \exp[-c_0 - c_1 \ln \kappa - c_2(\ln \kappa)^2] \pm \epsilon \quad (56)$$

with $\alpha_\infty = 0.49693$, $c_0 = 0.41477$, $c_1 = 0.775$, $c_2 = 0.1303$, and $\epsilon \leq 0.00076$. This expression yields at $\kappa = 1/\sqrt{2}$ the correct value $h_{c1} = 1$ and for $\kappa \gg 1$ it has the limit $\alpha = 0.49693$. A simpler expression for $\alpha(\kappa)$, yielding an $h_{c1}$ with error still less than 1% and with the correct limits at $\kappa = 1/\sqrt{2}$ and $\kappa \gg 1$, is

$$\alpha(\kappa) = 0.5 + (1 + \ln 2)/(2\kappa - \sqrt{2} + 2). \quad (57)$$

The resulting magnetization curves $M = \bar{B} - B_a$ are shown in Fig. 3. They are well fitted by

$$h(b, \kappa) = \frac{B_a}{B_{c2}} \approx h_{c1} + \frac{c_1 b^3}{1 + c_2 b + c_3 b^2},$$

$$c_1 = \frac{(1 - h_{c1})^3/(h_{c1} - p)^2},$$

$$c_2 = \frac{(1 - 3h_{c1} + 2p)/(h_{c1} - p)},$$

$$c_3 = 1 + (1 - h_{c1})(1 - 2h_{c1} + p)/(h_{c1} - p)^2 \quad (58)$$
with \( h_{c1} \) from Eq. (56) and \( p = -d\mu dB_{b=1} = 1/[(2\kappa^2 - 1)\beta_A + 1] \) \((m = b - h = M/B_{c2})\), \( \beta_A = 1.15960 \) (1.18034) for the triangular (square) vortex lattice. This \( h(b) \) satisfies the exact relations: \( h(0) = h_{c1}, h'(0) = h''(0) = h''(1) = 0, h(1) = 1, h'(1) = 1 - p(\kappa) \). The \( M(B_a) \) from the fit (58) applies for not too large \( \kappa \leq 10 \ldots 20 \). For larger \( \kappa \), better fits are given in Ref. [15], where it is also shown that the often used “logarithmic law at \( B_{c1} \ll B_a \ll B_{c2} \)” for the magnetization \( M(B_a) = \bar{B} - B_a \) has very limited range of validity.

Figure 4 shows the profiles \( B(r) \) and \( \omega(r) = |\psi(r)|^2 \) for an isolated vortex line \( \bar{B} \to 0 \) from GL theory for \( \kappa = 2, 5, \) and 20. Profiles for the vortex lattice are shown in Fig. 5 for \( \kappa = 5 \) and for two inductions at which the vortex spacing is \( a = 2\lambda \) and \( a = 4\lambda \). Contour lines for \( \omega(x,y) \) and \( B(x,y) \) are plotted in Fig. 6 for \( \kappa = 5 \) at \( b = 0.5 \). At \( b > 0.7 \) the contours of \( \omega(x,y) \) (see right hand part in Fig. 2) and \( B(x,y) \) practically coincide, and at \( b < 0.3 \) the contours are nearly circular, around well separated vortex cores and field peaks.

The variance of the magnetic field
\[
\sigma = \langle |B(x,y) - \bar{B}|^2 \rangle = \sum_{\textbf{k} \neq 0} B_{\textbf{k}}^2
\]

is shown in Fig. 7. In the low-field range \( 0.13/\kappa^2 \ll b \ll 1 \) one has for the triangular lattice the London limit \( \sigma = 0.00371\Phi_0^2/\lambda^4 \) (upper frame in Fig. 7), at very small \( b \ll 0.13/\kappa^2 \) one has \( \sigma = (b\kappa^2/8\pi^2)\Phi_0^2/\lambda^4 \) (dash-dotted straight lines in in Fig. 7), and near \( b = 1 \) one has the Abrikosov limit \( \sigma = 7.52 \times 10^{-4}(\Phi_0^2/\lambda^4)[\kappa^2(1-b)/(\kappa^2 - 0.069)]^2 \). This field variance is needed, e.g., for the interpretation of Muon Spin Rotation (\( \mu \)SR) experiments [19, 20, 21, 22].

In Fig. 8 the shear modulus of the bulk triangular vortex lattice is plotted versus the reduced induction for various GL parameters \( \kappa \). Note that for \( \kappa = 1/\sqrt{2} \) where \( B_{c1} = B_c = B_{c2} = \Phi_0/(4\pi\lambda^2) \), one has \( c_{66} = 0 \). One can show that in the particular case \( \kappa = 1/\sqrt{2} \) all possible vortex configurations have the same free energy \( F = BB_{c1}/\mu_0 \), e.g., triangular and square lattice, lattices with two flux quanta per vortex, or all vortices merged into one giant vortex.
Figure 11: The shear modulus $c_{66}$ of the triangular vortex lattice in films with thicknesses $d/\xi = 0.1, 0.32, 0.56, 1, 1.8, 3.2, 5.6, 10, 32$, plotted versus $b$ for $\kappa = 0.5$. This $c_{66}$ is positive, i.e., the triangular vortex lattice is stable, for sufficiently thin films or for small inductions. For $d \gg \xi$ the bulk $c_{66}$ at the same $\kappa = 0.5$ is reached (dash-dotted line), and for $d \ll \xi$ the bulk $c_{66}$ in the limit $\kappa \gg 1$ is reached (dashed line).

10 Vortex lattice for thin and thick films

The 2D Fourier method for the bulk vortex lattice in Sec. 9 can be generalized to the 3D problem of vortex lattices in infinite films of arbitrary thickness $d$ put into a uniform magnetic field $B_a$, since then the functions $\omega(x, y, z)$ and $B(x, y, z)$ are still periodic in the $(x, y)$ plane. I consider here the case when $B_a = B_0\hat{z}$ is perpendicular to the film plane [23], though in principle the Fourier method applies also to tilted applied field. The total free energy $F_{\text{tot}}$ per unit volume of the infinite film is the free energy, Eq. (2), plus the stray-field energy $F_{\text{stray}}$, i.e., the energy of the magnetic field variations outside the film,

$$F_{\text{tot}} = F + \frac{F_{\text{stray}}}{d}, \quad F_{\text{stray}} = 2\int_{d/2}^{\infty}(B(r)^2 - \bar{B}^2)_{x,y}dz. \quad (60)$$

One has $\bar{B} = B_a$ since all field lines have to cross the infinite film. The factor of 2 in (60) comes from the two half spaces above and below the film, which contribute equally to $F_{\text{stray}}$. The stray field $B(x, y, z > d/2)$ with constant planar average $\langle B(x, y, z) \rangle_{x,y} = \bar{B}\hat{z}$ is determined by the Laplace equation $\nabla^2 B = 0$ (since $\nabla \cdot B = 0$ and $\nabla \times B = 0$ in vacuum) and by its perpendicular component at the film surface $z = d/2$, since $B_z$ has to be continuous across this surface. The trial functions for $\omega(r), B(r) = B\hat{z} + b(r)$, and $Q(r) = Q_A(x, y) + q(r)$ inside the film $(|z| \leq d/2)$ are now 3D Fourier series [23],

$$\omega(r) = \sum_{\mathbf{K}} a_{\mathbf{K}} (1 - \cos \mathbf{K}_\perp \mathbf{r}_\perp) \cos K_z z,$$

$$b_z(r) = \sum_{\mathbf{K}} b_{\mathbf{K}} \cos \mathbf{K}_\perp \mathbf{r}_\perp \cos K_z z,$$
$\frac{-M}{B_{c2}^2} = \frac{b}{B_{c2}} = h = \frac{B_a}{B_{c2}}$ for $\kappa = 0.5, 0.707, 1, 1.5$.

$\kappa = 0.707$

$\kappa = 1$

$\kappa = 1.5$

Here $r = (x, y, z)$, $r_\perp = (x, y)$, $K = (K_x, K_y, K_z)$ with $K_\perp = (K_x, K_y)$ from Eq. (8), and $K_z = (2\pi/d)l, l = 0, 1, 2, \ldots$ In all sums here and below the term $K_\perp = 0$ is excluded. Minimizing $F_{\text{tot}}$ with respect to the coefficients $a_K$ and $b_K$ and using the appropriate orthonormality relations one arrives at iteration equations for the $a_K$ and $b_K$ similar to Eqs. (52)-(54). The solution is then obtained by first finding the 2D bulk solution as in Sec. 10 by considering only the terms with $K_z = 0$. The magnetic field lines then still have an unphysical sharp bend at the surface, see left part of Fig. 9. Next we allow for the terms with $K_z \neq 0$. This yields a “mushrooming” of the field lines of each vortex when it approaches the film surface such that these lines smoothly cross the surface with no bend, see right part of Fig. 9.

Profiles $\omega(x, 0, z)$ and $B_z(x, 0, z)$ are shown in Fig. 10 for $z = 0$ (middle plane of the film) and $z = d/2$ (film surface). One can see that the spatial variation of $B$ at the surface is reduced from its bulk value by nearly 1/2. Outside the film, the transverse field components $B_x, B_y$ rapidly decrease as $\exp(-|K_10|z') \approx \exp(-2\pi z'/a)$ where $z' = |z| - d/2$ is the distance from the surface. Interestingly, the profile of the order parameter $\omega$ in films is almost independent of $z$.

The shear modulus $c_{66}$ of the triangular vortex lattice in films can be positive even when $\kappa < 1/\sqrt{2}$, provided the film thickness $d$ is smaller than the coherence length $\xi$, see Fig. 11. This means that a stable vortex lattice may exist in thin type-I superconductor films. Our numerical result confirms the $c_{66}$ of films that was calculated analytically near $B_{c2}$ by Albert Schmid [24].

In such infinitely extended films one has $\bar{B} = B_a$ since all field lines have to pass the film. Therefore, the magnetization $M$ of the film cannot be calculated as a difference of fields, but one...
has to take the derivative of the total free energy, \( M = \bar{B} - \partial F_{\text{tot}} / \partial \bar{B} \). A more elegant method calculates \( M \) by Doria’s virial theorem [18], which for bulk superconductors yields Eq. (55). Indeed, it was shown recently [25] that this virial idea can be generalized to films of arbitrary thickness and \( M \) may be calculated directly from the GL solution for the film, with no need to take an energy derivative. The resulting magnetization of the film is plotted versus the applied field in Fig. 12 for various film thicknesses \( d \) and various GL parameters \( \kappa \). The curves for various \( d \) cut each other at \( b \approx 0.1/\kappa \). For \( \kappa = 0.707 \) the thick film limit (\( d \gg \xi \) but still film width \( w \gg d \)) yields a straight line, \(-M/B_{c2} = 1 - b \). This result is valid for large demagnetization factor \( N \), \( 1 - N \ll 1 \); it differs from the bulk result for \( M \) shown in Fig. 3, which is valid for the demagnetization-free limit of \( N \ll 1 \).

### 11 Remarks on the magnetization

One may ask why the magnetization \( M = m/V \) is not calculated via the general definition of the magnetic moment \( m = VM = \frac{1}{2} \int_V d^3r \times j \) [26, 27] from the current density \( j(r) \), which is easily calculated as a periodic function by our Fourier method, both for bulk and film superconductors. However, into this definition enters not only the periodic part, i.e., the vortex currents circulating inside each vortex cell; this contribution even would give the wrong sign of \( m \). The main contribution to the magnetic moment of a superconductor of any shape comes from the screening currents that flow near the surface of the specimen. The magnetization in superconductors is thus not a volume property as it is in magnets. For example, for a long cylinder in parallel field \( B_a \) the magnetization \( M = \bar{B} - B_a < 0 \) is composed of the positive contribution \( \bar{B} \) of the vortex currents and the negative (and larger) contribution \( B_a \) of the surface currents that screen the cylinder from the applied field \( B_a \) before vortices are allowed to penetrate. Near \( B_{c2} \), both terms nearly compensate and \( |M| \) is a small difference of two big terms. In the film geometry, and for most other shapes of the superconductor, the screening current is not easily known but has to be computed; such computations for thin and thick strips, disks, and plates of macroscopic size \( \gg \xi \) are presented in [26]. Landau knew this problem, since he worked on demagnetization factors and on the intermediate state in type-I superconductors [27], and he used the thermodynamic definition of the magnetization as an energy derivative. In my view, the simple formula (55) derived by Doria, Gubernatis and Rainer [18] by scaling the coordinates in GL theory and finding a novel virial relationship between kinetic and potential energies from which the equilibrium field \( B_a = \bar{B} - M \) follows, was a fundamental discovery that occurred long after the publication of GL theory [1].

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