How Objective is Black Hole Entropy?

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Abstract

The objectivity of black hole entropy is discussed in the particular case of a Schwarzschild black hole. Using Jaynes’ maximum entropy formalism and Euclidean path integral evaluation of partition function, it is argued that in the semiclassical limit when the fluctuation of metric is neglected, the black hole entropy of a Schwarzschild black hole is equal to the maximal information entropy of an observer whose sole knowledge of the black hole is its mass. Black hole entropy becomes a measure of number of its internal mass eigenstates in accordance with the Boltzmann principle only in the limit of negligible relative mass fluctuation. From the information theoretic perspective, the example of a Schwarzschild black hole seems to suggest that black hole entropy is no different from ordinary thermodynamic entropy. It is a property of the experimental data of a black hole, rather than being an intrinsic physical property of a black hole itself independent of any observer. However, it is still weakly objective in the sense that different observers given the same set of data of a black hole will measure the same maximal information entropy.
Given a black hole with certain mass, charge and momentum. From the consideration of particle creation of a black hole\(^1\), it may be assigned an entropy like quantity given by the Bekenstein Hawking formula as

\[ S_{BH} = \frac{1}{4}A \]

where \( A \) is the area of the event horizon. The units in which \( G = \hbar = c = k = 1 \) will be used throughout where all symbols have their usual meanings. \( S_{BH} \) is usually associated with the loss of information of the microstates of a black hole for an observer outside the event horizon who, according to the no hair theorem, knows only the mass, charge and angular momentum of the hole\(^2\). However, the association of \( S_{BH} \) with loss of information remains at a heuristic level so far, nothing more than as a guide of thinking.

On the other hand, black hole entropy may be identified up to a constant with the area of the event horizon. Since the area of the event horizon is geometric in character, it does not seem to depend on any coarse graining process or the presence of an observer. This led to the speculation \(^4\) that black hole entropy is more fundamental than its counterparts for other physical systems. It is an objective, intrinsic physical attribute of a black hole, in equal footing with the mass, charge and angular momentum.

The aim of this note is to examine these two problems in the particular case of a Schwarzschild black hole, in the hope that this will shed some light on the issues in general. We shall show that in the semiclassical limit when the fluctuation of the metric is negligible,

(i) Black hole entropy given by the Bekenstein Hawking formula is indeed equal to the maximal information entropy (in a sense to be explained below)
of an observer whose sole knowledge of the black hole is its mass. In doing so, we also see that black hole entropy is a measure of the number of internal mass eigenstates of a Schwarzschild black hole, in accordance with the Boltzmann principle only in the limit of negligible relative fluctuation of mass.

(ii) The claim of objectivity in the sense described above of black hole entropy does not seem to stand. However, a weaker form of objectivity is still preserved.

**Information Entropy of Schwarzschild Black Hole**

Consider a Schwarzschild black hole with mass $M$. From a thermodynamic perspective, $M$ is the average mass of the black hole. The real mass fluctuates around $M$. To simplify discussion, assume the black hole has discrete mass spectrum and denote its mass eigenstates by the mass eigenvalues $M_i > 0, i = 1 \cdots N$ for some positive integer $N$. The consideration of continuous mass spectrum is similar but mathematically more complicated. It does not seem to add any further physical insight in the present context.

Let $p_i$ be the probability that the black hole is in the $M_i$ state subject to the constraint that $\sum p_i M_i = M$ and $\sum p_i = 1$. Throughout summation sign is understood to be a sum from $i = 1$ to $N$. Given only $M$, the assignment of $p_i$ is quite arbitrary. However, if we invoke the Jaynes’ principle$^{5,6}$, then the most unbiased assignment would be $p_i$ which maximises the quantity

$$\sum p_i \ln p_i + \lambda M + \mu \sum p_i$$
where $\lambda, \mu$ are Lagrange multipliers. This gives

$$p_i = \frac{1}{Z} e^{-\lambda M_i}$$

for $i = 1 \cdots N$. $Z$ is the partition function given as

$$Z = \sum e^{-\lambda M_i}$$

In addition, we have

$$M = -\frac{\partial \ln Z}{\partial \lambda}$$

$$S_I = \lambda M + \ln Z$$

and

$$\lambda = M = \frac{\partial S_I}{\partial M}$$

where $S_I = -\sum p_i \ln p_i$ is the maximal information entropy. The word maximal is used in the sense that for any other arbitrary assignment of $p'_i$ to $M_i$ subject to the same constraint $\sum p'_i M_i = M$ and $\sum p'_i = 1$, the information entropy $-\sum p'_i \ln p'_i$ is less than or equal to $S_I$. With $M_i > 0$ for $i = 1, 2 \cdots N$, it may be shown from (5) that $\lambda > 0$.

So far, the above argument is true for any system characterised by a parameter whose parameter has mean value $M$. To proceed, we need to evaluate further the partition function in (2). Note that (2) may be expressed as a path integral 7

$$\int e^{iI} Dg$$

where the functional integral is over all Euclidean, asymptotically flat metrics whose imaginary time variables have period $\lambda$. $I$ is the action for a metric $g$
given as
\[ I = -\frac{1}{16\pi} \int_M R + \frac{1}{8\pi} \int_{\partial M} [K] \]

Here \( R \) is the Ricci scalar, \( M \) is a compact region in an asymptotically flat Riemannian manifold bounded by the boundary \( \partial M \) which is topologically \( S^1 \times S^2 \). \([K]\) is the difference between the trace of the second fundamental form of \( M \) in \( g \) and the Euclidean flat space metric.

Observe that among all Euclidean, asymptotically flat metrics satisfying the prescribed periodic boundary condition, the classical action is given by the Euclidean Schwarzchild metric with mass \( \frac{\lambda}{8\pi} \). Since the path integral in (3) is dominated by contribution from the classical action, using the stationary phase approximation and neglecting fluctuation of the metric around the classical background, one may readily show that \(^7\)
\[ \ln Z = -\frac{\lambda^2}{16\pi} \] \hspace{1cm} (7)

(3) and (7) together imply
\[ \lambda = 8\pi M \] \hspace{1cm} (8)

From (4) and (8), we may then infer
\[ S_I = S_{BH} \] \hspace{1cm} (9)

The path integral evaluation of partition function is essentially due to Gibbons and Hawking\(^7\). However, unlike that in their work, the classical black hole metric concerned is not a priori included in the path integral. Instead, the relation between the path integral and the black hole concerned is given by (3). In doing so, we may see more clearly the connection between information entropy in statistical mechanics and black hole entropy.
**Boltzmann principle**

Black hole entropy is a measure of number of microstates of a black hole was first conjectured by Bekenstein\(^2\). It was later confirmed in an affirmative way by Zurek\(^3\). Here if we write (1)

\[
p_i = \frac{1}{Z} e^{-\lambda M(1 + \frac{\Delta M}{M})}
\]

where \(\Delta M_i = M_i - M\). It is clear that in the limit the relative fluctuation of mass \(\frac{\Delta M_i}{M} \to 0\) for \(i = 1 \cdots N\), we have from (10), (4) and (9) that

\[
S_{BH} = S = \ln N
\]

in accordance with the Boltzmann principle. In the present context, apart from seeing (11), it also shows that Boltzmann formula is a limit expression for the black hole entropy when the thermal fluctuation of mass is negligible compared with the mass itself, as it should be the case for the thermodynamic description of a physical system to be valid.

However, one also has to admit that the above argument is not entirely satisfactory since it does not involve the thermodynamic limit \(N \to \infty\). One would like a more rigorous limit theorem which asserts that, subject to some restriction on the variance on \(\Delta M_i\), \(\frac{\Delta M_i}{M} \to 0\) as \(N \to \infty\) for \(i = 1 \cdots N\). However, due to the negative heat capacity for a Schwarzchild black hole, the canonical ensemble constructed by Jaynes' variational principle is not stable under fluctuation of \(M\). This manifests in the absurdity that the variance of \(\Delta M_i\) is negative. At the same time, the mean of \(\Delta M_i\) is zero. This renders the formulation of such a theorem along the line of the law of large numbers difficult.
How Objective is Black Hole Entropy?

Ever since the inception of black hole entropy into physics, due to its geometric character, there has been speculation that black hole entropy is more fundamental than its counterparts for other physical systems. It is an intrinsic, objective physical property of a black hole, in equal footing with for example the mass. We shall examine this belief in what follows in view of the equality between \( S_I \) and \( S_{BH} \) established in (9).

From the perspective of Jaynes’ maximum entropy formalism, equilibrium thermodynamic entropy is merely a special case of information entropy, being the maximal information entropy subject to a given set of time independent experimental data. Therefore equilibrium thermodynamic entropy is defined relative to a set of experimental data, rather than being a physical characteristic of a system, like energy, pressure etc. This view was first put forward by Jaynes \(^8\) and subsequently given careful linguistic formulation by Denbigh \(^9\). Different set of experimental data will give rise to different maximal information entropy. Even so, the notion of equilibrium thermodynamic entropy still preserves weak objectivity in the sense that different observers given the same set of experimental data will agree on the maximal information entropy assigned to the data. From this information theoretic perspective, in view of (9), black hole entropy of a Schwarzschild black hole is no different from the equilibrium thermodynamic entropy of other thermodynamical systems. It is defined relative to the given mass data of the black hole. Being able to identify \( S_I \) with a geometric quantity in spacetime should be looked on as merely an aesthetically appealing coincidence. This explains why an
observer travelling beyond the event horizon into a black hole will not measure any black hole entropy\textsuperscript{1}. Depending on the information in hand about the microstates of a black hole, an observer inside the event horizon will assign a maximal information entropy in accordance with the data in hand. This maximal information entropy will in general be different from that of an observer who only knows the mass of the black hole. In view of the different assignment of maximal information entropy to the same black hole for observers inside and outside the event horizon, it seems that, contrary to many would like to believe, black hole entropy of a Schwarzchild black hole provides another example which supports Jaynes’ thesis that thermodynamic entropy is not a physical property of a thermodynamic system, it is defined relative to a set of experimental data.

**Concluding Remarks**

We have shown that, in the particular case of a Schwarzchild black hole, in the semiclassical limit, the black hole entropy may be identified with the maximal information entropy of an observer who knows only the mass of the black hole. Various implications of this result are also discussed. In particular, it throws considerable doubt that entropy will acquire an objective status, similar to other physical attributes of a physical system in the context of black hole physics. It remains to investigate whether in the generic case of a Kerr Newman black hole, the maximal information entropy of an observer who knows only the mass, charge and angular momentum of the black hole may be identified with the Bekenstein Hawking black hole entropy. It seems that
more sophisticated argument than that presented here is needed to achieve
this general result. The difficulty lies not so much in the evaluation of the
partition function but in the fact that, instead of (3), we have a system of
PDE whose solution is sought for.

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