EXTREMAL LENGTH FUNCTIONS ARE LOG-PLURISUBHARMONIC

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Abstract. In this paper, we show that the extremal length functions on Teichmüller space are log-plurisubharmonic. As a corollary, we obtain an alternative proof of L. Liu and W. Su’s results on the plurisubharmonicity of extremal length functions. We also obtain alternative proofs of S. Krushkal’s results that a function defined by the Teichmüller distance from a reference point is plurisubharmonic, and the Teichmüller space is hyperconvex. To show the log-plurisubharmonicity, we give an explicit formula of the Levi form of the extremal length functions in generic case.

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1. Introduction

1.1. Results. A positive function $U$ on a complex manifold $N$ is said to be log-plurisubharmonic if $\log U$ is plurisubharmonic. Any log-plurisubharmonic function is plurisubharmonic (cf. §5.2).

Let $T_{g,m}$ be the Teichmüller space of Riemann surfaces of analytically finite type $(g, m)$ with $2g - 2 + m > 0$. One of the aims of this paper is to give the following theorem, which improves a result by L. Liu and W. Su in [23].

Theorem 1.1 (Log-plurisubharminicity). Extremal length functions on $T_{g,m}$ are log-plurisubharmonic.

Moreover, we will also observe that for generic measured foliations, the log-extremal length functions are real analytic strictly plurisubharmonic functions on $T_{g,m}$ (cf. Theorem 5.3).

2010 Mathematics Subject Classification. 32G15, 30F45, 31C10.
Key words and phrases. Teichmüller theory, Teichmüller distance, Extremal length, Plurisubharmonic functions.

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When \(3g - 3 + m = 1\), Theorem 1.1 follows from the direct calculation (cf. \(\S7.2\)). Hence, in the discussion in the later section, we always assume that \(3g - 3 + m \geq 2\).

We will prove Theorem 1.1 in \(\S5.2\). In fact, Theorem 1.1 is derived from a more stronger property of the extremal length functions than that given in Theorem 1.1 (cf. Theorem 5.2). However, the property in Theorem 1.1 is also important as we see below (cf. Corollaries 1.1 and 1.2). To show Theorem 1.1, we will give an explicit formula of the Levi forms of extremal length functions in generic case. (cf. Theorem 5.1).

The following immediately follows from Theorem 1.1 (cf. \([17\), Colollary 2.6.9]).

**Corollary 1.1** (Log-plurisubharmonicity of positive polynomials). Let \(F_1, \cdots, F_n\) be measured foliations. For any polynomial \(P\) of \(n\)-variables with positive coefficients, the function

\[
T_{g,m} \ni x \mapsto P(\text{Ext}_x(F_1), \cdots, \text{Ext}_x(F_n)) \in \mathbb{R}
\]

is log-plurisubharmonic.

1.2. **Teichmüller distance is plurisubharmonic.** It is known that the least upper semicontinuous majorant of a family of plurisubharmonic functions is either a constant function \(+\infty\) or plurisubharmonic (cf. Theorem 5 in \([21\), Chapter II, \(\S2\)). Kerckhoff’s formula \([22\) implies that the Teichmüller distance \(d_T(x_0, \cdot)\) from a reference point \(x_0 \in T_{g,m}\) coincides with the half of the least upper semicontinuous majorant of the family of log-extremal length functions. Therefore, we obtain an alternative proof of Krushkal’s result \([20\), Corollary 3] as follows.

**Corollary 1.2** (Teichmüller distance is plurisubharmonic). For \(x_0 \in T_{g,m}\), the distance function

\[
T_{g,m} \ni x \mapsto d_T(x_0, x)
\]

is plurisubharmonic.

1.3. **Convexities for Teichmüller space.** A subset \(K\) in a complex manifold \(N\) is said to be **disk-convex** in \(N\) if for any continuous mapping \(g: \mathbb{D} \to N\), holomorphic in \(\mathbb{D}\), \(f(\partial \mathbb{D}) \subset K\) implies \(f(\mathbb{D}) \subset K\) (cf. \([24\)). For \(\epsilon > 0\) and a measured foliation \(F\), we call the set of the form \(\{x \in T_{g,m} \mid \text{Ext}_x(F) < \epsilon\}\) the \(\epsilon\)-**horoball** for \(F\). From the maximum modulus principle for (pluri)subharmonic functions, we obtain the following.

**Corollary 1.3** (Disk convexity). For \(\epsilon > 0\) and a measured foliation \(F\), the \(\epsilon\)-horoball for \(F\) is disk-convex in \(T_{g,m}\). The metric ball with respect to the Teichmüller distance is also disk-convex in \(T_{g,m}\).

About the convexity of metric balls, A. Lenzhen and K. Rafi \([22\) observed that the metric ball with respect to the Teichmüller distance is quasi-convex. Furthermore, they also showed that the extremal length function is not convex along a Teichmüller geodesic in general. This means that the intersection between a horoball of extremal length and a Teichmüller disk is simply connected, but may not be hyperbolically convex in general.

A complex manifold \(N\) is said to be **hyperconvex** (in the sense of Stehlé) if it admits a continuous plurisubharmonic exhaustion function \(\rho: N \to [-\infty, 0)\) in the sense that \(\{x \in N \mid \rho(x) < c\}\) is relatively compact in \(M\) for every \(c < 0\) (cf. \([32\).
See also [28]. Take two essentially complete measured foliations $F$ and $G$ which are transverse in the sense that

\[(1.1) \quad i(F,H) + i(G,H) > 0\]

for all $H \in \mathcal{MF} - \{0\}$. In Theorem 5.2 given later, we will check that

\[(1.2) \quad \rho: T_{g,m} \ni x \mapsto -\frac{1}{\text{Ext}_x(F) + \text{Ext}_x(G) + 1}\]

is a real analytic, negative, strictly plurisubharmonic exhaustion function with lower bounds. Thus we obtain the following.

**Corollary 1.4 (Hyperconvexity).** Teichmüller space is hyperconvex.

Corollary [14] is already given by Krushkal in [19]. Notice that the hyperconvexity of Teichmüller space also follows from the completeness of the Carathéodory distance (cf. [1] and [7]). As an immediate corollary of the hyperconvexity (or Corollary 1.2), by virtue of Oka’s theorem, we deduce the following, which is first proven by L.Bers and L.Ehrenpreis (see [2]. See also [7], [22], [31], [36]).

**Corollary 1.5 (Holomorphic convexity).** Teichmüller space is a domain of holomorphy.

2. Notation

2.1. **Teichmüller space.** Let $\Sigma_{g,m}$ be a compact orientable surface of genus $g$ with $m$-disks removed with $2g - 2 + m > 0$. The Teichmüller space $T_{g,m}$ of Riemann surfaces of analytically finite type $(g,m)$ is the set of equivalence classes of pairs $(M,f)$ consisting of a Riemann surface $M$ of analytically finite type $(g,m)$ and an orientation preserving homeomorphism $f: \text{Int}(\Sigma_{g,m}) \to M$. Two marked Riemann surfaces $(M_1,f_1)$ and $(M_2,f_2)$ are equivalent if there is a conformal mapping $h: M_1 \to M_2$ which homotopic to $f_2 \circ f_1^{-1}$.

The Teichmüller space admits a canonical distance $d_T$, which we call the Teichmüller distance. The Teichmüller distance is originally defined as the logarithm of the infimum of the maximal dilatations of quasiconformal mappings respecting the markings (cf. [14, §5]). S. Kerckhoff [16] gave a geometric description of the Teichmüller distance via the extremal length, which we will recall in 2.2 below.

2.2. **Measured foliations.** Let $\mathcal{S}$ be the set of homotopy classes of non-trivial and non-peripheral simple closed curves on $\Sigma_{g,m}$. Let $\mathcal{WS}$ be the set of weighted simple closed curves, that is, the set of formal products $ta$ of non-negative number $t$ and $\alpha \in \mathcal{S}$. The closure $\mathcal{MF}$ of the embedding

\[\mathcal{WS} \ni ta \mapsto [\mathcal{S} \ni \beta \mapsto t i(\alpha, \beta)] \in \mathbb{R}_+^\mathcal{S}\]

is called the space of measured foliations on $\Sigma_{g,m}$. We consider $\mathcal{WS}$ as a subset of $\mathcal{MF}$. We identify $1 \cdot \alpha \in \mathcal{WS}$ with $\alpha \in \mathcal{S}$. Any measured foliation is described as a pair consisting of a singular foliation and a transverse measure (cf. [5, Exposé 5]). For instance, $ta \in \mathcal{MF}$ is a foliated annulus with core $\alpha$ which identified with an annulus $[0,t] \times [0,1]/(x,0) \sim (x,1)$ and a transverse measure associated to $|dx|$.

It is known that the geometric intersection number $i(\alpha, \beta)$ for $\alpha, \beta \in \mathcal{S}$ extends continuously to $\mathcal{MF} \times \mathcal{MF}$ (cf. [3]).

If we fix a complete hyperbolic structure on the interior of $\Sigma_{g,m}$, any measured foliation canonically corresponds to a measured lamination. A *measured lamination*
is a pair of a geodesic lamination, which called the support, and a transverse measure, where a geodesic lamination \( a \) is a compact set in the interior of \( \Sigma_{g,m} \) which consists of disjoint complete geodesics (cf. [29, §1.7]). A measured foliation is said to be essentially complete if the support of the corresponding measured lamination is maximal, that is, the complement consists of ideal triangles or ideal punctured monogon if \((g,m) \neq (1,1)\), a punctured bigon otherwise (cf. [33, Definition 9.5.1, Propositions 9.5.2 and 9.5.4]). In the interior of \( \Sigma_{g,m} \), each singularity of the associated foliation of an essentially complete measured foliation is a three prong singularity. At any puncture, the associated foliation has a one prong singularity (cf. [29, Epilogue]).

2.3. Extremal length. Let \( M \) be a Riemann surface and let \( A \) be a doubly connected domain on \( M \). If \( A \) is conformally equivalent to a round annulus \( \{ 1 < |z| < R \} \), we define the modulus of \( A \) by

\[
\text{Mod}(A) = \frac{1}{2\pi} \log R.
\]

The extremal length of a simple closed curve \( \alpha \) on \( M \) is defined by

\[
\text{Ext}_M(\alpha) = \inf \left\{ \frac{1}{\text{Mod}(A)} \mid \text{the core curve of } A \subset M \text{ is homotopic to } \alpha \right\}.
\]

In [16], Kerckhoff showed that if we define the extremal length of \( t\alpha \in WS \) by

\[
\text{Ext}_M(t\alpha) = t^2 \text{Ext}_M(\alpha),
\]

then the extremal length function \( \text{Ext}_M \) on \( WS \) extends continuously to \( \mathcal{MF} \). For \( F \in \mathcal{MF} \), the extremal length function \( \text{Ext}(F) \) on \( T_{g,m} \) is defined by

\[
\text{Ext}_x(F) = \text{Ext}_M(f(F)).
\]

It is known that the function

\[
T_{g,m} \times \mathcal{MF} \ni (x,F) \mapsto \text{Ext}_x(F)
\]

is continuous. Furthermore, the extremal length function \( \text{Ext}_x : \mathcal{MF} \rightarrow \mathbb{R} \) is a proper function and satisfies the quasiconformal distortion property:

\[
\text{Ext}_y(F) \leq e^{2d_T(x,y)} \text{Ext}_x(F)
\]

for any \( F \in \mathcal{MF} \) and \( x, y \in T_{g,m} \).

2.4. Kerckhoff’s formula and Minsky’s inequality. The Teichmüller distance \( d_T \) is expressed with the extremal length, which we call Kerckhoff’s formula:

\[
d_T(x_1,x_2) = \frac{1}{2} \log \sup_{\alpha \in S} \frac{\text{Ext}_{x_2}(\alpha)}{\text{Ext}_{x_1}(\alpha)}
\]

\[
= \frac{1}{2} \sup_{\alpha \in S} \left( \log \text{Ext}_{x_2}(\alpha) - \log \text{Ext}_{x_1}(\alpha) \right)
\]

for \( x_1, x_2 \in T_{g,m} \) (see [16]). Y. Minsky [27] observed the following inequality, which we recently call Minsky’s inequality:

\[
i(F,G)^2 \leq \text{Ext}_x(F) \text{Ext}_x(G)
\]

for \( x \in T_{g,m} \) and \( F, G \in \mathcal{MF} \). If two measured foliation \( F \) and \( G \) are transverse in the sense of ([11]), we have

\[
\text{Ext}_x(F) + \text{Ext}_x(G) \geq \frac{i(F,H)^2 + i(G,H)^2}{\text{Ext}_x(H)}
\]
Henceforth, for all zeroes of \( q \) poles at punctures. A quadratic differential with the space of holomorphic quadratic differentials. The natural pairing between holomorphic tangent space and cotangent space is given by denoted it by \( v \) for all \( \alpha \in S \), where \((M, f) = \pi(q) \in \mathcal{T}_{g,m} \). We call \( F \) the vertical foliation and denote it by \( v(q) \). We call \( h(q) = v(q) \) the horizontal foliation of \( q \). It is known that for any \( F \in \mathcal{MF} \) and \( x = (M, f) \in \mathcal{T}_{g,m} \), there is a unique holomorphic quadratic differential \( q_{F,x} \) such that \( v(q_{F,x}) = F \). We call \( q_{F,x} \) the Hubbard-Masur differential for \( F \) on \( x \) (cf. [2.7]). If \( F \) is essentially complete, the Hubbard-Masur differential \( q_{F,x} \) is generic for all \( x \in \mathcal{T}_{g,m} \), since any Whitehead equivalent measured foliation to \( F \) is isotopic to \( F \).

The Hubbard-Masur differential \( q_{F,x} = q_{F,x}(z)dz^2 \) for \( F \) on \( x = (M, f) \) satisfies

\[
\text{Ext}_x(F) = \|q_{F,x}\| = \int_M |q_{F,x}| = \int_M |q_{F,x}(z)| \cdot \frac{i}{2} dz \wedge d\bar{z}.
\]

2.5. Quadratic differentials. Let \( Q_{g,m} \) be the space of holomorphic quadratic differentials with finite \( L^1 \)-norm. Namely, the space \( Q_{g,m} \) is a holomorphic vector bundle over \( \mathcal{T}_{g,m} \) and the fiber \( Q_x \) at \( x = (M, f) \in \mathcal{T}_{g,m} \) is the space of holomorphic quadratic differentials on \( M \) with finite \( L^1 \)-norm (cf. [2.7]). Any \( q \in Q_x \) extends meromorphically to the completion \( \overline{M} \) by filling punctures and has (at most) simple poles at punctures. A quadratic differential \( q \in Q_{g,m} \) is said to be generic if it satisfies the following two conditions:

1. \( q \) has a simple pole at every puncture.
2. all zeroes of \( q \) are simple.

We can easily check that the set of all generic holomorphic quadratic differentials is an open and dense subset in \( Q_{g,m} \).

2.6. Hubbard-Masur differentials. For \( q \in Q_{g,m} \), there is a unique measured foliation \( F \) such that

\[
i(F, \alpha) = \inf_{\alpha' \sim f(\alpha)} \int_{\alpha'} |\text{Re} q|.
\]

for all \( \alpha \in S \), where \((M, f) = \pi(q) \in \mathcal{T}_{g,m} \). We call \( F \) the vertical foliation and denote it by \( v(q) \). We call \( h(q) = v(q) \) the horizontal foliation of \( q \). It is known that for any \( F \in \mathcal{MF} \) and \( x = (M, f) \in \mathcal{T}_{g,m} \), there is a unique holomorphic quadratic differential \( q_{F,x} \) such that \( v(q_{F,x}) = F \). We call \( q_{F,x} \) the Hubbard-Masur differential for \( F \) on \( x \) (cf. [2.7]). If \( F \) is essentially complete, the Hubbard-Masur differential \( q_{F,x} \) is generic for all \( x \in \mathcal{T}_{g,m} \), since any Whitehead equivalent measured foliation to \( F \) is isotopic to \( F \).

The Hubbard-Masur differential \( q_{F,x} = q_{F,x}(z)dz^2 \) for \( F \) on \( x = (M, f) \) satisfies

\[
\text{Ext}_x(F) = \|q_{F,x}\| = \int_M |q_{F,x}| = \int_M |q_{F,x}(z)| \cdot \frac{i}{2} dz \wedge d\bar{z}.
\]

2.7. Infinitesimal theory. Teichmüller space also has a canonical complex structure, which induced from the deformation by quasiconformal mappings. A Beltrami differential is, by definition, an \( L^\infty \)-measurable \((-1, 1)\)-form. The holomorphic tangent space is described as the quotient of the complex Banach space of the Beltami differentials (cf. [13] Theorem 7.5]). The holomorphic cotangent space is identified with the space of holomorphic quadratic differentials. The natural pairing between holomorphic tangent space and cotangent space is given by

\[
\langle v, q \rangle = \int_M \mu q = \int_M \mu(z)q(z) \cdot \frac{i}{2} dz \wedge d\bar{z}
\]

for \( v = [\mu] \in T_x \mathcal{T}_{g,m} \) and \( q \in T_x^* \mathcal{T}_{g,m} \cong Q_x \) and \( x = (M, f) \in \mathcal{T}_{g,m} \).

**Convention 1.** Henceforth, for \((p_i, q_i)\) forms \( \varphi_i = \varphi_i(z)dz^{p_i}d\bar{z}^{q_i} \) \((i = 1, 2)\) on a Riemann surface \( M \) with \( p_1 + p_2 = q_1 + q_2 = 1 \), we write

\[
\int_M \varphi_1 \varphi_2 = \int_M \varphi_1(z)\varphi_2(z) \cdot \frac{i}{2} dz \wedge d\bar{z}.
\]
3. Coordinates via representations of the odd cohomology

3.1. The tangent spaces to quadratic differentials. Following [13], we describe the tangent space $T_{q_0}Q_{g,m}$ as the first hypercohomology group $H^1(L^*)$ of a complex of sheaves (cf. [11] or [12]). We will need the Kodaira-Spencer identification of the tangent space of Teichmüller space with the first cohomology group of the sheaf of holomorphic vector fields (for instance, see [18]. See also [14] and [15]).

Let $X$ and $q$ be a holomorphic vector field and a holomorphic quadratic differential on an open set of a Riemann surface $M$. Denote by $L_Xq$ the Lie derivative of $q$ along $X$. Let $\Theta_M$ and $\Omega^2_M$ be the sheaves of germs of holomorphic vector fields with zeroes at punctures and meromorphic quadratic differentials on $M$ with (at most) first order poles at punctures, respectively. Let $q_0 \in Q_{g,m}$ ($q_0$ need not to be generic). The tangent space $T_{q_0}Q_{g,m}$ is identified with the first hypercohomology group of the complex of sheaves

$$L^*: 0 \longrightarrow \Theta_M \xrightarrow{L_q} \Omega^2_M \longrightarrow 0$$

(cf. [13 Proposition 4.5]). For the convenience, we recall the definition of the (first) hypercohomology group which we use here. The first cochain group is the direct sum $C^0(M,\Omega^2_M) \oplus C^1(M,\Theta_M)$. Consider an appropriate covering $U = \{U_i\}_{i}$ on $M$ such that $\mathbb{H}^1(L^*) \cong \mathbb{H}^1(\mathcal{U},L^*)$ (see the proof of [13 Proposition 4.5]). A cochain $((\phi_i)_{i}, (X_{ij})_{i,j})$ in $C^0(\mathcal{U},\Omega^2_M) \oplus C^1(\mathcal{U},\Theta_M)$ is said to be cocycle if it satisfies

$$\delta(X_{ij})_{i,j} = X_{ij} + X_{jk} + X_{ki} = 0, \quad \delta(\phi_i)_i = \phi_i - \phi_j = L_{X_{ij}}(q_0).$$

A coboundary is a cochain $((\phi_i)_{i}, (X_{ij})_{i,j})$ of the form

$$X_{ij} = Z_i - Z_j = \delta(Z_i)_i, \quad \phi_i = L_{Z_i}(q_0)$$

for some 0-cochain $\{Z_i\}_i \in C^0(\mathcal{U},\Theta_M)$ (cf. Figure 1).

3.2. Branched covering space associated with generic differentials. Let $x = (M,f) \in T_{g,m}$ and $q_0$ a generic holomorphic quadratic differential on $M$. We consider $M$ as a pair of topological surface $\Sigma_g$ and a complex structure on $\Sigma_g - \{\text{punctures of } M\}$.

Let $U$ be a contractible neighborhood of $q_0$ in $Q_{g,m}^0$. For $q \in U$, let $A_q = \text{Zero}(q) \cup \text{Pole}(q) \subset \Sigma_g$ and $M_q$ the underlying surface of $q$. Then, we have a homomorphism

$$(3.1) \quad \rho_q : \pi_1(\Sigma_g - A_q) \to \mathbb{Z}/2\mathbb{Z} \subset \text{Isom}(\mathbb{C})$$
associated to the double covering space

$$\pi_q: \tilde{M}_q \to M_q$$

of Riemann surface of $\sqrt{\mathfrak{t}}$ over $\Sigma_q - A_q$, where $\mathbb{Z}/2\mathbb{Z}$ in (3.1) is recognized as a subgroup of the isometry group Isom($\Sigma$) of $\mathbb{C}$ with the standard Euclidean metric generated by the $\pi$-rotation. The square root $\sqrt{\mathfrak{t}}$ lifts to a well-defined holomorphic 1-form $\omega_q$ on $\tilde{M}_q$.

Let $W = \Sigma_q - A_q$, and $W'$ be a compact subsurface of $W$ with smooth boundary such that the inclusion $W' \hookrightarrow W_0$ is homotopy equivalent and the complement $\Sigma_q - W'$ is a union of tiny open $|q_0|$-disks of centers in $A_0$ By taking sufficiently small $U$ if necessary, we may assume that $A_q \subset \Sigma_q - W'$ for any $q \in U$. Hence, by passing the inclusions $W' \hookrightarrow W$ and $W' \hookrightarrow \Sigma_q - A_q$, we get a canonical identification between $\pi_1(W)$ and $\pi_1(\Sigma_q - A_q)$ so that $\rho_q = \rho_0$ as a homomorphism from $\pi_1(W)$ to $\mathbb{Z}/2\mathbb{Z}$ for all $q \in U$.

3.3. Backgrounds. Let $q_0 \in \mathcal{Q}_{g,m}$ be a generic differential on $M_{q_0}$. Take a neighborhood $U$ of $q_0$ as the previous section. Then, $\mathcal{M} = \cup_{q \in U} \tilde{M}_q$ is a holomorphic family of closed Riemann surfaces over $U$, and the arrangement of the covering transformation $r_q: \tilde{M}_q \to M_q$ defines the holomorphic automorphism $r$ of $\tilde{M}$ of order 2. Taking a sufficiently small $U$ if necessary, we may assume that there is a covering $\{V_i\}_i$ of $\tilde{M}$ such that

1. each $V_i$ admits a biholomorphic mapping $A_i$ onto the product domain $\mathbb{D} \times U$ such that its restriction to $V_{i,q} = V_i \cap \tilde{M}_q$ is a biholomorphism onto $\mathbb{D} \times \{q\}$ for each $q \in U$;
2. for any finite $i_1, \ldots, i_k$ and $q \in U$, the intersection $V_{i_1,q} \cap \cdots \cap V_{i_k,q}$ is connected and simply connected;
3. each $V_{i,q}$ contains at most one branch point of $\tilde{M}_q \to M_q$ and any branch point of $\tilde{M}_q \to M_q$ is contained in at most one open set in $\{V_{i,q}\}_i$; and
4. the covering $\{V_i\}_i$ is equivariant under $r$ in the sense that $r(V_i)_i = \{V_i\}_i$.

Let $D = \{\lambda = s + it \mid |s|, |t| < \delta_0\}$. Let $\{q_{\lambda}\}_{\lambda \in D}$ be a differential family of holomorphic quadratic differentials with $q_0 = q_0$ and $q_\lambda \in U$ for $\lambda \in D$. For any $\lambda \in D$, we let $\alpha_{\lambda}(\lambda) : V_{i,q_{\lambda}} \to \mathbb{D}$ by $\alpha_{\lambda}(\lambda) = A_i |\tilde{M}_{\lambda,q_{\lambda}}$ and set $V_{\lambda;i} = \alpha_{\lambda}(\lambda)(V_{i,q_\lambda} \cap V_{i,q_{\lambda}})$, and $\alpha_{\lambda;i}(\lambda) = \alpha_{\lambda;i}(\lambda) \circ \alpha_i(\lambda)^{-1} : V_{\lambda;i} \to V_{\lambda;i}$. For simplicity, set $\alpha_i = \alpha_{0;i}$ and $V_{ij} = V_{0;j}$. Consider the image of $\alpha_{\lambda}$ in the $z_j$-plane.

Let $\tilde{q}_{\lambda;i} dz_i^2$ be the representation of $(\pi_q)_*'(q_{\lambda})$ under the coordinate $\alpha_i(\lambda)$. Let

$$\phi_{\lambda;i} = \frac{\partial \tilde{q}_{\lambda;i}}{\partial \lambda}, \quad \tilde{X}_{\lambda;i} = -\frac{\partial \alpha_i(\lambda)}{\partial \lambda} \circ \alpha_i(\lambda)^{-1},$$

on $\mathbb{D}$ for all $i$ and set

$$\tilde{X}_{\lambda;i}(z_j) = \frac{\partial \alpha_{\lambda;j}(\lambda)}{\partial \lambda}(z_i),$$

where $z_j = \alpha_{\lambda;j}(\lambda)(z_i)$ and $z_i \in V_{\lambda;i}$ (tildes mean objects (differentials or vector fields etc.) on the covering space $M_{q_0}$ which obtained as lifts of objects on $M_{q_0}$). Then,

$$\tilde{X}_{\lambda;i}(z_i) \frac{\partial}{\partial z_i}, \quad \tilde{X}_{\lambda;j}(z_j) \frac{\partial}{\partial z_j}$$
are vector fields on appropriate sets of \( \tilde{M}_q \). Set \( \tilde{q}_i = q_{0,i} \) and \( \tilde{\phi}_i = \phi_{0,i} \). Notice that

\[
\alpha_{ij}(\lambda)_*(\tilde{X}_{\lambda;i}) - \tilde{X}_{\lambda;i} = \tilde{X}_{\lambda;ij},
\]

\[
\alpha_{ij}(\lambda)^*(\tilde{\phi}_{\lambda;i}) - \tilde{\phi}_{\lambda;i} = L_\tilde{X}_{\lambda;ij}(\tilde{q}_{\lambda,j})
\]
on \( V_{\lambda;ij} \). We abbreviate these equations to

\[
\tilde{X}_{\lambda;ij} = \tilde{X}_{\lambda;i} - \tilde{X}_{\lambda,j}, \quad \tilde{\phi}_{\lambda;i} - \tilde{\phi}_{\lambda;j} = L_{\tilde{X}_{\lambda;ij}}(\tilde{q}_{\lambda,j}).
\]

for instance. Notice again that \( \omega_q^\lambda = (\pi_q)_*^\lambda(q_\lambda) \) for all \( \lambda \in D \).

Notice that for any cocycle \( \{X_{ij}\}_{ij} \in C^1(M, \Theta_M) \), we can always find a 0-cochain \( \{z\}_i \) of the sheaf of \( C^\infty \)-vector fields which satisfies

\[
X_i - X_j = X_{ij} \quad \text{on} \quad U_i \cap U_j
\]

applying a partition of unity. Then, from (3.3), the \( \tau \)-derivative

\[
\dot{\nu} = -(\tilde{X}_i)^\tau
\]
defines a Beltrami differential on \( M \) which represents the infinitesimal deformation corresponding to the tangent vector associated to the cohomology class of \( \{X_{ij}\}_{i,j} \) in \( H^1(M, \Theta_M) \). Compare with Equation (7.27) in \([4]\) §7.2.4).

### 3.4. Local coordinates via homomorphisms

Let \( H_1(\tilde{M}_q)^- \) is the odd part of the homology of \( \tilde{M}_q \) with coefficient in \( \mathbb{Z} \). Namely, \( H_1(\tilde{M}_q)^- \) is the the eigen space of the eigen value \(-1\) of the linear automorphism \( (r_q)_* \) on the first cohomology group of \( M_q \) defined by the covering transformation \( r_q \). In \([4]\), Douady and Hubbard consider the following mapping

\[
DH: U \ni \lambda \mapsto \chi_\lambda \in \text{Hom}(H_1(\tilde{M}_q), \mathbb{C})
\]
deﬁned by

\[
\chi_\lambda(c) = \int c \omega_q
\]

where we identify \( H_1(\tilde{M}_q)_- \simeq H_1(\tilde{M})^-_1 \) in a canonical way as discussed in \([4]\).

Notice that \( \text{Hom}(H_1(\tilde{M}_q)^-, \mathbb{C}) \) is a \( \mathbb{C} \)-vector space of complex dimension \( 6g - 6 + 2m \). We shall check the following.

**Lemma 3.1** (Differential of \( DH \)). Let \( q_0 \in Q_{g,m} \) be a generic differential. Let \( v \in T_{q_0}Q_{g,m} \) be a tangent vector associated to a 1-cochain \( \{\phi_i\}_i, \{X_{ij}\}_{i,j} \). Let \( \{X_i\}_i \) be a 0-cochain of the sheaf of \( C^\infty \)-vector fields on \( X_{q_0} \). Then, the derivative of the mapping (3.7) along \( v \) satisfies

\[
(DH)_*[v](c) = \int c \Phi
\]

for all \( c \in H_1(\tilde{M}_q)^- \), where

\[
\Phi = \left( \frac{\tilde{\phi}_i}{2\omega_{q_0}} - \omega'_{q_0}\tilde{X}_i - \omega_{q_0}(\tilde{X}_i)_z \right) dz - \omega_{q_0}(\tilde{X}_i)\sigma d\tau
\]
on \( \mathcal{V}_{i,0} \).

One can easily check that \( \Phi \) in Lemma 3.1 is a well-defined closed one-form on \( \tilde{M}_q \) which satisfies \( (r_{q_0})^*(\Phi) = -\Phi \).
Proof. We calculate the $\lambda$-derivative at $\lambda = 0$ with the notation in \[3.3\] Notice that
\[
\alpha_i(\lambda) \circ \alpha_i^{-1}(z_i) = z_i - \lambda \ddot{X}_i(z_i) - \ddot{Y}_i(z_i) + o(\lambda)
\]
\[
\dot{q}_i, (z_i) = \dot{q}_i(z_i) + \lambda \dot{\phi}_i(z_i) + \ddot{\psi}_i(z_i) + o(\lambda)
\]
on $\mathbb{D} = \alpha_i(V_{i,0})$ as $\lambda \to 0$, where
\[
\dot{Y}_i = \frac{\partial \alpha_i(\lambda)}{\partial \lambda} \bigg|_{\lambda=0} \circ \alpha_i^{-1}, \quad \ddot{\psi}_i = \frac{\partial q_{i,1}}{\partial \lambda} \bigg|_{\lambda=0},
\]
and the convergence is valid uniformly on any compact sets of $\mathbb{D}$. In our case, $\lambda$-derivative of $D \ni \lambda \mapsto DH(q_\lambda)(c)$ at $\lambda = 0$ will be the desired formula.

Notice that the partial $\lambda$-derivative of $\omega_\lambda$ at $\lambda = 0$ is
\[
\frac{\partial \omega_\lambda}{\partial \lambda} \bigg|_{\lambda=0} = \frac{\dot{\phi}_i}{2\omega_q}
\]
on $\mathbb{D} = \alpha_i(V_{i,0})$. Notice also in general that the derivative of a function defined by the integration
\[
\int_{\alpha(s)}^{b(s)} f(s, x) \, dx
\]
at $s = 0$ is equal to
\[
\int_{\alpha(0)}^{b(0)} \frac{\partial f}{\partial s}(0, x) \, dx + f(0, b(0)) \frac{db}{ds}(0) - f(0, a(0)) \frac{da}{ds}(0).
\]
Hence, by the standard argument, like as the discussion in Proposition 1 of \[3.7\], one can check the equations \[3.8\] hold.

Though the following is well-known (cf. \[34\], \[35\], \[25\]), we confirm for the following for the completeness. Indeed, the calculation in the following proof will be a guide in the later discussion.

Lemma 3.2 (DH defines a local coordinate). Let $q_0 \in Q_{g, m}$ be as above. When we take the above neighborhood $U$ of $q_0$ to be sufficiently small, the mapping \[3.7\] defines a holomorphic local coordinate around $q_0$.

Proof. Let $v \in T_{q_0}Q_{g, m}$ be a tangent vector which is represented by the 1 cocycle $\{\phi_i\}, \{X_{ij}\}$ in $C^0(V_0, \Omega^2_{M_{q_0}}) \oplus C^1(V_0, \Theta_{M_{q_0}})$, where $V_0 = \{\pi_{q_0}(V_{i,0})\}$, is a covering of $M_{q_0}$ (cf. \[3.3\]).

Suppose that $DH_*[v] = 0$. Let $\Phi$ be a closed one form defined in Lemma 3.1. Since $(r_0)^*((\Phi) = -\Phi$,
\[
\int_c \Phi = \int_{r_0(c)} \Phi = -\int_c \Phi
\]
for all $c \in H_1(\hat{M}_{q_0})^+$, where $H_1(\hat{M}_{q_0})^+$ is the even part of the homology of $\hat{M}_q$ with coefficient in $\mathbb{Z}$. Hence, there is a smooth function $f$ on $\hat{M}_{q_0}$ such that $df = \Phi$ from de Rham’s theorem. Namely,
\[
\frac{\dot{\phi}_i}{2\omega_q} \cdot \dot{X}_i - \omega_{q_0} \cdot (\dot{X}_i) = f_z
\]
\[
-\omega_{q_0} \cdot (\dot{X}_i) = f_{\tau}
\]
hold on $\mathbb{D} = \alpha_i(\mathcal{V}_{i,0})$, where $\tilde{\phi}_i$ and $\tilde{X}_i$ are lifts of $\phi_i$ and $X_i$ to $\tilde{M}_{g_0}$, respectively. Since $\{\{\mathcal{V}_{i,\lambda}, \alpha_i(\lambda)\}\}$ is a system of holomorphic local coordinates on $\tilde{M}_{g_0}$, for each $\lambda \in D$, $\tilde{\nu} = -\tilde{X}_i$ defines a Beltrami differential on $\tilde{M}_{g_0}$ which descends to a Beltrami differential $\tilde{\mu}$ on $M_{g_0}$. Notice that for any holomorphic quadratic differential $\varphi$ on $M_{g_0}$, the quotient $(\pi_{g_0})^*(\varphi)/\omega_{g_0}$ is a holomorphic 1-form on $\tilde{M}_{g_0}$ and

$$
\int_{M_{g_0}} \tilde{\mu} \varphi = \frac{1}{2} \int_{M_{g_0}} \tilde{\nu} \cdot (\pi_{g_0})^*(\varphi) = \frac{1}{2} \int_{M_{g_0}} \frac{f(z)}{\omega_{g_0}(z)} dz
$$

$$
= \frac{1}{2} \int_{M_{g_0}} \frac{f(z)(\pi_{g_0})^*(\varphi)(z)}{\omega_{g_0}(z)} dz = \frac{1}{2} \int_{M_{g_0}} d \left( f(z) \frac{\varphi(z)}{\omega_{g_0}(z)} \right) = 0.
$$

Hence, $\tilde{\mu}$ is an infinitesimally trivial Beltrami differential on $M_{g_0}$ by Teichmüller lemma (cf. Lemma 7.6 of [15]). This means that the cohomology class

$$
\{\{X_{ij}\}_{i,j}\} \in H^1(V_{0,\nu}, \Theta_{M_{g_0}}) \cong H^1(M_{g_0}, \Theta_{M_{g_0}})
$$

is trivial from (3.3) since $\mathcal{V}_0$ is a Leray covering (see also Theorem 3.5 of [18]). Therefore, there is a 0-cochain $\{Z_i\}_{i} \in C^0(V_{0,i}, \Theta_{M_{g_0}})$ such that $Z_i - Z_j = X_{ij}$ for all $i,j$.

Let $\zeta_i = L_{Z_i}(q_0) \in \Gamma(V_{0,i}, \Omega^2_{M_{g_0}})$ and $\tilde{\zeta}_i = L_{\tilde{Z}_i}(\omega_{g_0}^2)$ be the lift of $\zeta_i$. Then,

$$
\tilde{\zeta}_i - \tilde{\zeta}_j = L_{\tilde{X}_{ij}}(\omega_{g_0}^2)
$$

and hence $\{\tilde{\phi}_i - \tilde{\zeta}_i\}_{i}$ defines a holomorphic quadratic differential $\tilde{\varphi}$ on $\tilde{M}_{g_0}$ which descends to a holomorphic quadratic differential $\varphi$ on $M_{g_0}$ associated to $\{\phi_i - \zeta_i\}_{i}$.

Since $\tilde{X}_i - \tilde{Z}_i = \tilde{X}_j - \tilde{Z}_j$ on $\mathcal{V}_{i,0} \cap \mathcal{V}_{j,0}$, $\{\tilde{X}_i - \tilde{Z}_i\}_{i}$ defines a vector field $\tilde{W}$ on $M_{g_0}$ which satisfies

$$
((\omega_{g_0})'(\tilde{X}_i) + \omega_{g_0} \cdot (\tilde{X}_i)_z)dz + \omega_{g_0} \cdot (\tilde{X}_i)_\zeta d\zeta = \frac{L_{\tilde{Z}_i}(\omega_{g_0}^2)}{2\omega_{g_0}}
$$

$$
= ((\omega_{g_0})'(\tilde{X}_i - \tilde{Z}_i) + \omega_{g_0} \cdot (\tilde{X}_i - \tilde{Z}_i)_z)dz + \omega_{g_0} \cdot (\tilde{X}_i - \tilde{Z}_i)_\zeta d\zeta
$$

$$
= d(\omega_{g_0} \cdot \tilde{W})
$$

since each $\tilde{Z}_i$ is a holomorphic section. Consequently, we deduce from the assumption that

$$(3.9) \quad 0 = \int_c \Omega = \int_c \left( \Omega - \frac{\tilde{\zeta}_i - L_{\tilde{Z}_i}(\omega_{g_0}^2)}{2\omega_{g_0}} \right) = \int_c \frac{\tilde{\varphi}}{2\omega_{g_0}} - \int_c d(\omega_{g_0} \cdot \tilde{W}) = \int_c \frac{\tilde{\varphi}}{2\omega_{g_0}}$$

for all $c \in H_1(M_{g_0})$, and hence $\tilde{\varphi} = 0$ and $\varphi = 0$. Therefore $\phi_i = \zeta_i$ and

$$
((\{\phi_i\}_i, \{X_{ij}\}_{i,j}) = ((L_{Z_i}(q_0)), \{Z_i - Z_j\}_{i,j})
$$

Thus, the cohomology class of the cochain $(\{\phi_i\}_i, \{X_{ij}\}_{i,j})$ vanishes.

As a consequence, the derivative of the map DH on the tangent space $T_{g_0}Q_{g,m} \cong \mathbb{H}^1(L^*)$ is injective. Since the dimensions of $Q_{g,m}$ and $\text{Hom}(H_1(M_{g_0})^-, \mathbb{C})$ are same, the derivative of DH is an isomorphism from $T_{g_0}Q_{g,m}$ onto $\text{Hom}(H_1(M_{g_0})^-, \mathbb{C})$. \quad \square
3.5. Homomorphisms on slices. For $F \in \mathcal{MF} - \{0\}$, we define a slice

$$E^v(F) = \{ q \in \mathcal{Q}_{g,m} \mid v(q) = F \}.$$ 

for $F$ in $\mathcal{Q}_{g,m}$. Hubbard and Masur showed that the restriction of the projection

$$E^v(F) \ni q \mapsto \pi(q) \in \mathcal{T}_{g,m}$$

is a homeomorphism ([13 §2]). The following two lemmas are well-known (cf. [13 Lemma 4.3] and [24 Proposition 1]. See also [6]).

**Lemma 3.3.** Let $q_0 \in \mathcal{Q}_{g,m}$ be a generic differential. When we take $U$ sufficiently small,

$$E^v(v(q_0)) \cap U = \{ q \in \mathcal{Q}_{g,m} \mid \text{Re}(DH(q)) = \text{Re}(DH(q_0)) \}$$

holds.

**Lemma 3.4.** Let $x_0 \in \mathcal{T}_{g,m}$ and $F \in \mathcal{MF}$. If $q_{F,x_0}$ is generic, then the map

$$\mathcal{T}_{g,m} \ni x \mapsto q_{F,x} \in \mathcal{Q}_{g,m}$$

is real analytic around $x_0$.

4. Family of quadratic differentials with prescribed vertical foliation

Fix $F \in \mathcal{MF}$. Consider a family $\{q_\lambda\}_{\lambda \in D}$ of holomorphic quadratic differentials such that $v(q_\lambda) = F$. Suppose that $q_0$ is generic and $x_\lambda = \pi(q_\lambda) \in \mathcal{T}_{g,m}$ varies holomorphically. Then, the family $\{q_\lambda\}_{\lambda \in D}$ is a real analytic family of quadratic differentials (cf. Lemma 3.4). Take a neighborhood $U$ of $q_0$ as the previous sections.

4.1. Differential via the complex conjugate. We first notice a simple observation. Let $\pi: E \to M$ be a holomorphic vector bundle. Let $g: D \to M$ be a holomorphic mapping and $G: D \to E$ be a $C^1$ mapping such that $\pi \circ G = g$ on $D$. Then,

$$\pi_* \left( G_* \left( \frac{\partial}{\partial \lambda} \right) \right) = g_* \left( \frac{\partial}{\partial \lambda} \right) = 0$$

since $g$ is holomorphic. Hence,

$$G_* \left( \frac{\partial}{\partial \lambda} \right) \in \text{Ker}(\pi_*) \cong E_{g(\lambda)}$$

for all $\lambda \in D$.

**Lemma 4.1 (First derivatives).** Let $\{q_\lambda\}_{\lambda \in D}$ be the family of holomorphic quadratic differentials defined as above. Let $\eta_\lambda$ be a holomorphic quadratic differential on $X_{q_\lambda}$ such that the infinitesimal Bertrami differential $\tilde{\mu}$ associated to the tangent vector for the $\lambda$-derivative of the mapping

$$D \ni \lambda \mapsto \pi(q_\lambda) \in \mathcal{T}_{g,m}$$

at $\lambda$ satisfies

$$\int_{M_{q_\lambda}} \tilde{\mu} \psi = \int_{M_{q_\lambda}} \frac{\pi_*}{|\pi_*|} \psi$$

for all holomorphic quadratic differential $\psi$ on $M_{q_\lambda}$. Then,

$$\frac{\partial X_{q_\lambda}(c)}{\partial \lambda} = \int_c \frac{(\pi_{q_\lambda})^* (\eta_\lambda)}{\omega_{q_\lambda}}, \quad \frac{\partial X_{q_\lambda}(c)}{\partial \lambda} = - \int_c \frac{(\pi_{q_\lambda})^* (\eta_\lambda)}{\omega_{q_\lambda}}$$

for any $c \in H_1(M_{q_0})$ and $\lambda \in D$. 
Proof. The existence and the uniqueness of $\eta_\lambda$ follows from an argument by David Dumas [6, Theorem 5.3] since each $q_\lambda$ is generic. Indeed, the pairing

$$(\eta, \psi) \mapsto \int_{M_{q_\lambda}} \frac{\overline{\eta_\psi}}{|q_\lambda|}$$

is a positive definite hermitian inner product on the space of holomorphic quadratic differentials on $M_{q_\lambda}$. Notice that

$$\partial \chi_{q_\lambda}(c) \bigg|_{\lambda=0} = C c \omega_{q_\lambda}$$

for any $c \in H_1(M_{q_0})^+$ and $\lambda \in D$. It suffices to show the case where $\lambda = 0$ and $c \in H_1(M_{q_0})^-$. As we discussed around (4.1), from Lemma 3.1 and the proof of Lemma 3.2 there is a holomorphic quadratic differential $\varphi$ on $X_{q_0}$ such that

$$\partial \chi_{q_\lambda}(c) = C c \omega_{q_\lambda}$$

for any $c \in H_1(M_{q_0})^-$ (see (3.9)). Since the vertical foliation $|\text{Re}(\omega_{q_\lambda})|$ of $\omega_{q_\lambda}$ is unchanged on $D$, for all $c \in H_1(M_{q_0})^-$, the real part of $\chi_{q_\lambda}(c)$ is a constant function on $D$ by Lemma 3.3. Therefore,

$$\frac{\partial \chi_{q_\lambda}(c)}{\partial \lambda} = -\frac{\partial \chi_{q_\lambda}(c)}{\partial \lambda}$$

on $D$. Let $(\{\phi_i\}, \{X_{ij}\}_{i,j})$ be the cocycle representing the tangent vector associated to the $\lambda$-derivative of the mapping $D \ni \lambda \mapsto q_\lambda$ at $\lambda = 0$. Let $(X_i)_i$ be a 0-cochain of the sheaf of $C^\infty$-vector fields with $X_i - X_j = X_{ij}$. Then,

$$\Omega = \left( \frac{\tilde{\varphi}_i}{2\omega_{q_0}} - \omega_{q_0}' \tilde{X}_i - \omega_{q_0} (\tilde{X}_i)z \right) dz - (\omega_{q_0}(\tilde{X}_i)\overline{\varphi})d\overline{z}$$

on $U_i$ defines a closed 1-form on $\tilde{M}_{q_0}$ satisfies

$$\frac{\partial \chi_{q_\lambda}(c)}{\partial \lambda} = \int_c \Omega$$

for all $c \in H_1(M_{q_0})$. Therefore, there is a $C^\infty$-function $f$ on $M_{q_0}$ such that

$$\Omega = -\frac{(\pi_{q_0})^*(\varphi)(z)}{\omega_{q_0}(z)} d\overline{z} + df,$$

that is,

(4.2) \[ \frac{\tilde{\varphi}_i}{2\omega_{q_0}} - \omega_{q_0}' \tilde{X}_i - \omega_{q_0} (\tilde{X}_i)z = f_z \]

(4.3) \[ -\omega_{q_0}(\tilde{X}_i)\overline{\varphi} = -\left( \frac{(\pi_{q_0})^*(\varphi)(z)}{\omega_{q_0}(z)} \right) + f_{\overline{z}} \]

on $\mathcal{V}_{1,0}$. From (4.3), the lift $(\pi_{q_0})^*(\tilde{\mu})$ of $\tilde{\mu} = -(X_i)_{\overline{z}}$ satisfies

(4.4) \[ (\pi_{q_0})^*(\tilde{\mu}) = -\frac{(\pi_{q_0})^*(\varphi)}{\omega_{q_0}} d\overline{z} + f_{\overline{z}} \]
on $\tilde{M}_{q_0}$. Let $\psi$ be a holomorphic quadratic differential on $M_{q_0}$. From (4.4)

$$\int_{\tilde{M}_{q_0}} (\pi_{q_0})^*(\mu)(\pi_{q_0})^*(\psi) = -\int_{M_{q_0}} \frac{\pi_{q_0})^*(\varphi)}{\omega_{q_0}} (\pi_{q_0})^*(\psi) + \int_{\tilde{M}_{q_0}} \frac{\partial}{\partial \lambda} (\pi_{q_0})^*(\psi)$$

$$= -\int_{M_{q_0}} \frac{\pi_{q_0})^*(\varphi)}{\omega_{q_0}} (\pi_{q_0})^*(\psi) + \int_{\tilde{M}_{q_0}} d \left( \frac{(\pi_{q_0})^*(\psi)}{\omega_{q_0}} \right)$$

$$= -\int_{M_{q_0}} \frac{\pi_{q_0})^*(\varphi)}{\omega_{q_0}} (\pi_{q_0})^*(\psi)$$

By descending to $M_{q_0}$, we have

$$\int_{M_{q_0}} \varphi = \int_{M_{q_0}} \mu \psi = -\int_{M_{q_0}} \varphi$$

for all $\psi$. Therefore, $\varphi = -\eta_0$ from (4.3) Theorem 5.3] again.

### 4.2. Laplacian of homomorphisms

This section is devoted to calculating the Laplacian of Douady-Hubbard map (4.7). The author must confess that for the proof of the plurisubharmonicity of extremal length functors, we need the equation (4.6) rather than the formula (4.5) in the following lemma. However, we give the following lemma for its own interests.

**Lemma 4.2 (Laplacian).** Under the assumption in Lemma 4.2, we have

$$\frac{\partial^2 (\chi_{q_0}(c))}{\partial \lambda \partial \lambda} \bigg|_{\lambda=0} = -4i \operatorname{Im} \int_{\tilde{M}_{q_0}} \frac{\omega_{q_0})^*(\eta_0)}{\omega_{q_0}}^2 \sigma_c$$

for $c \in H_1(\tilde{M}_{q_0})$, where $\sigma_c = \sigma_c(z)dz$ is the holomorphic part of the reproducing harmonic differential associated to $c$.

**Proof.** When the homology class $c$ is in the even homology group, $\chi_{q_0}(c) = 0$. The integrand of the right-hand side of (4.5) satisfies

$$r^0 \left( \frac{|(\pi_{q_0})^*(\eta_0)|^2}{\omega_{q_0}} \right) \sigma_c = \frac{|(\pi_{q_0})^*(\eta_0)|^2}{\omega_{q_0}} \sigma_c$$

and hence, the integral over $\tilde{M}_{q_0}$ vanishes. This means that (4.5) holds for homology classes in the even homology group.

By applying the same argument as that of Lemma 3.1 one obtain

$$\frac{\partial^2 \chi_{q_0}(c)}{\partial \lambda \partial \lambda} \bigg|_{\lambda=0} = \int_c$$

for any $c \in H_1(\tilde{M}_{q_0})$, where $\Omega_0$ is a closed differential of the form

$$\left( \eta_0 + \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right) (X_i) + \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right) (X_i) \right) dz + \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right) (X_i) \right) dz \right.$$

and

$$\eta_0 = -\frac{\partial}{\partial \lambda} \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right) \bigg|_{\lambda=0}.$$
Since the real part of the function $D \ni \lambda \mapsto \chi_{q_\lambda}(c)$ is constant and the Laplacian is the real operator,
\[
\left. \frac{\partial^2 \chi_{q_\lambda}(c)}{\partial \lambda \partial \bar{\lambda}} \right|_{\lambda=0} = - \left. \frac{\partial^2 \chi_{q_\lambda}(c)}{\partial \lambda \partial \bar{\lambda}} \right|_{\lambda=0} = - \left( \int_c \Omega_0 \right) = - \int_c \Omega_0
\]
for all $c \in H_1(\tilde{M}_{q_0})$. Hence, there is a $C^\infty$-function $f$ on $\tilde{M}_{q_0}$ such that
\[
\Omega_0 = -\bar{\Omega}_0 + df.
\]
Namely, we have
\[
\bar{\eta}_0 + \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right)'(X_i) + \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right)(\bar{X}_i)z = - \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right)(\bar{X}_i)z + f_z
\]
\[
= \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right)(\pi_{q_0})^*(\bar{\mu}) + f_z,
\]
where $\bar{\mu} = -(X_i)z$ is the infinitesimal Beltrami differential representing the tangent vector of the mapping $D \ni \lambda \mapsto \pi(q_\lambda)$ at $\lambda = 0$ (cf. [10]). Applying the above equation, we also obtain
\[
\Omega_0 = \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right)(\pi_{q_0})^*(\bar{\mu}) + f_z
\]
\[
= \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right)(\pi_{q_0})^*(\bar{\mu})dz - \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right)(\pi_{q_0})^*(\bar{\mu})d\bar{z}
\]
\[
= \frac{1}{2} df + i \frac{1}{2} df.
\]
Let $\sigma(c) = \sigma(c)(dz + \bar{\sigma}(z)d\bar{z})$ be the reproducing harmonic differential on $\tilde{M}_{q_0}$ associated to $c \in H_1(\tilde{M}_{q_0})$, where $\sigma(c)(dz)$ is a holomorphic 1-form on $\tilde{M}_{q_0}$ (we call this holomorphic 1-form the \textit{holomorphic part of $\sigma(c)$}). Namely, the differential $\sigma(c)$ is a unique harmonic differential on $\tilde{M}_{q_0}$ which satisfies
\[
\int_{\tilde{M}_{q_0}} \omega = \int_{\tilde{M}_{q_0}} \omega \wedge ^* \sigma(c)
\]
for all closed 1-form $\omega$ on $\tilde{M}_{q_0}$. Then, by applying the orthogonal decomposition theorem for the space of $L^2$-closed forms, we have
\[
\frac{\partial^2 \chi_{q_\lambda}(c)}{\partial \lambda \partial \bar{\lambda}} \bigg|_{\lambda=0} = \int_{\tilde{M}_{q_0}} \Omega_0 = \int_{\tilde{M}_{q_0}} \Omega_0 \wedge ^* \sigma(c)
\]
\[
= \int_{\tilde{M}_{q_0}} \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right)(\pi_{q_0})^*(\bar{\mu})dz - \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right)(\pi_{q_0})^*(\bar{\mu})d\bar{z}
\]
\[
\wedge ^* \left( \sigma_c(z)dz + \overline{\sigma_c(z)d\bar{z}} \right)
\]
\[
= \int_{\tilde{M}_{q_0}} \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right)(\pi_{q_0})^*(\bar{\mu})dz - \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right)(\pi_{q_0})^*(\bar{\mu})d\bar{z}
\]
\[
\wedge (-i\overline{\sigma_c}dz + i\sigma_c dz)
\]
\[
= -4i \text{Im} \int_{\tilde{M}_{q_0}} \left( \frac{(\pi_{q_0})^*(\eta_0)}{\omega_{q_0}} \right)(\pi_{q_0})^*(\bar{\mu})\sigma_c.
\]
Now, we suppose that the homology class \( c \) is in \( H_1(\tilde{M}_{g_0}) \). Since \( (r_0)_*(c) = -c, \ r_0^*\sigma(c) = -\sigma(c) \). This means that \( \sigma(c) \) has zeroes at each branch points of \( \pi_{g_0} : \tilde{M}_{g_0} \to M_{g_0} \), and the holomorphic quadratic differential
\[
\frac{(\pi_{g_0})^*(\eta_0)}{\omega_{g_0}} \sigma_c = \frac{(\pi_{g_0})^*(\eta_0)(z)}{\omega_{g_0}(z)} \sigma_c(z)dz^2
\]
descends to a holomorphic quadratic differential on \( M_{g_0} \). Therefore, we obtain from the definition of \( \eta_0 \) that
\[
\frac{\partial^2 x_{g_0}(c)}{\partial \lambda \partial \chi} \bigg|_{\lambda=0} = -4i \text{ Im} \int_{\pi_{g_0}^{-1}} \frac{(\pi_{g_0})^*(\eta_0)}{\omega_{g_0}} \frac{(\pi_{g_0})^*(\eta_0)}{\omega_{g_0}} \sigma_c
\]
(4.7)
\[
= -4i \text{ Im} \int_{\pi_{g_0}^{-1}} \frac{|(\pi_{g_0})^*(\eta_0)|^2}{\omega_{g_0}} \sigma_c
\]
(cf. Lemma 4.1). This is what we wanted. \( \square \)

5. Levi forms and Plurisubharmonicity of subspecies

5.1. Levi forms of extremal length functions. For a real-valued \( C^2 \)-function \( U \) on a complex manifold \( N \), the Levi form of \( U \) is an hermitian inner product on the holomorphic tangent bundle \( TN \) of \( N \) defined as
\[
\mathcal{L}(U)[v,\overline{v}] = \sum_{i,j=1}^{n} \frac{\partial^2 U}{\partial z_i \partial \overline{z}_j} v_i \overline{v_j}
\]
for \( v = \sum_{i=1}^{n} v_i(\partial/\partial z_i) \in TN \). Let \( p \in N \) and \( v \in T_p N \). Let \( F : D \to N \) be a holomorphic mapping with \( f(0) = p \) and \( f_*(\partial/\partial \lambda) \big|_{\lambda=0} = v \), then
\[
\mathcal{L}(U)[v,\overline{v}] = \frac{\partial^2 (U \circ f)}{\partial \lambda \partial \chi} \bigg|_{\lambda=0}
\]

Theorem 5.1 (Levi forms). Let \( x_0 = (M_0, f_0) \in \mathcal{T}_{g,m} \) and \( v = [\tilde{\mu}] \in T_{x_0} \mathcal{T}_{g,m} \). Let \( F \in \mathcal{M}F \). Suppose that the Hubbard-Masur differential \( q_0 = q_{F,x_0} \) is generic. Then, the extremal length function is real analytic around \( x_0 \), and the Levi form of the extremal length function \( T_{g,m} \ni x \mapsto \text{Ext}_x(F) \) at \( x_0 \) satisfies
\[
\mathcal{L}(\text{Ext}(F))[v,\overline{v}] = 2 \int_{M_0} \frac{|\eta_v|^2}{|q_0|},
\]
where \( \eta_v \) is a unique holomorphic quadratic differential on \( M_0 = M_{g_0} \) satisfying that
\[
(v,\psi) = \int_{M_0} \tilde{\mu} \psi = \int_{M_0} \frac{\eta_v}{|q_0|} \psi
\]
for all \( \psi \in \mathcal{Q}_{x_0} \).

Remark 5.1. In \( \text{10} \) we will discuss a geometric interpretation of the anti-complex linear map
\[
T_{x_0} \mathcal{T}_{g,m} \ni v \mapsto \eta_v \in \mathcal{Q}_{x_0}.
\]

Proof of Theorem 5.1. We continue to use the symbols defined in the previous sections. Let \( \tilde{g} = 4g - 2 + m \ (\geq 2) \) be the genus of \( \tilde{M}_{g_0} \). Let \( \{\alpha_k, \beta_k\}_{k=1}^{g} \) be a
symplectic generator of $H_1(\tilde{M}_{q_0})$. Let $x_\lambda = \pi(q_\lambda)$ for instance. By the bilinear relation, we have

$$\text{Ext}_{x_\lambda}(F) = \frac{1}{2} \|\omega_{q_\lambda}\|^2 = \frac{1}{4} \int_{\tilde{M}_{q_\lambda}} \omega_{q_\lambda} \wedge \overline{\omega_{q_\lambda}}$$

$$= \frac{i}{4} \sum_{k=1}^{\tilde{g}} \left( \chi_{q_\lambda}(\alpha_k)\chi_{q_\lambda}(\beta_k) - \chi_{q_\lambda}(\beta_k)\chi_{q_\lambda}(\alpha_k) \right)$$

for $\lambda \in D$ (cf. [§3 Corollary in §III.2.3]). From Lemmas 3.2 and 3.3, $\chi_{qF,x}(\alpha_k)$ and $\chi_{qF,x}(\beta_k)$ are real analytic around $x_0$ for each $k$, and hence so is the extremal length function. From Lemma 4.2, we have

$$\left. \frac{\partial^2}{\partial \lambda \partial \overline{\lambda}} \left( \chi_{q_\lambda}(c_1)\chi_{q_\lambda}(c_2) \right) \right|_{\lambda=0}$$

$$= \left. \frac{\partial^2 \chi_{q_\lambda}(c_1)}{\partial \lambda \partial \overline{\lambda}} \right|_{\lambda=0} \chi_{q_\lambda}(c_2) + \left. \frac{\partial^2 \chi_{q_\lambda}(c_2)}{\partial \lambda \partial \overline{\lambda}} \right|_{\lambda=0} \chi_{q_\lambda}(c_1)$$

$$+ \left. \frac{\partial \chi_{q_\lambda}(c_1)}{\partial \lambda} \right|_{\lambda=0} \left. \frac{\partial \chi_{q_\lambda}(c_1)}{\partial \overline{\lambda}} \right|_{\lambda=0} \frac{\partial \chi_{q_\lambda}(c_2)}{\partial \lambda}$$

$$+ \left. \frac{\partial \chi_{q_\lambda}(c_1)}{\partial \overline{\lambda}} \right|_{\lambda=0} \left. \frac{\partial \chi_{q_\lambda}(c_1)}{\partial \lambda} \right|_{\lambda=0} \frac{\partial \chi_{q_\lambda}(c_2)}{\partial \overline{\lambda}}$$

for $c_1, c_2 \in H_1(\tilde{M}_{q_\lambda})$, since the real part of $\chi_{q_\lambda}(c)$ is constant for all $c \in H_1(\tilde{M}_{q_0})$ by Lemma 3.3. Notice that the sum of the last two terms of above calculation is a real number. Therefore, we obtain

$$\left. \frac{\partial^2}{\partial \lambda \partial \overline{\lambda}} \left( \chi_{q_\lambda}(\alpha_k)\chi_{q_\lambda}(\beta_k) - \chi_{q_\lambda}(\beta_k)\chi_{q_\lambda}(\alpha_k) \right) \right|_{\lambda=0}$$

$$= \left( \Omega_0 \int_{\beta_k} \omega_{q_0} - \int_{\beta_k} \Omega_0 \int_{\alpha_k} \omega_{q_0} - \int_{\alpha_k} \Omega_0 \int_{\beta_k} \omega_{q_0} - \int_{\beta_k} \Omega_0 \int_{\alpha_k} \omega_{q_0} \right)$$

$$= \left( \Omega_0 \int_{\beta_k} \omega_{q_0} - \int_{\beta_k} \Omega_0 \int_{\alpha_k} \omega_{q_0} + \int_{\alpha_k} \Omega_0 \int_{\beta_k} \omega_{q_0} - \int_{\beta_k} \Omega_0 \int_{\alpha_k} \omega_{q_0} \right)$$

where $\Omega_0$ is a closed form defined as (4.6). Since $\omega_{q_0}$ is a holomorphic differential, $*\omega_{q_0}$ and $\overline{*\omega_{q_0}}$ are harmonic differentials and they are perpendicular to the exact and co-exact differentials in the $L^2$-inner product of differential forms (cf. [§II.3]).
From (4.6), we obtain
\[ L(\text{Ext}(F))[v, \overline{v}] = \frac{\partial^2}{\partial \lambda \partial \overline{\lambda}} \text{Ext}_{x, \lambda}(F) \bigg|_{\lambda=0} \]
\[ = \frac{i}{4} \sum_{k=1}^g \left( \int_{\alpha_k} \Omega_0 \int_{\beta_k} \omega_{q_0} - \int_{\beta_k} \Omega_0 \int_{\alpha_k} \omega_{q_0} \right) \]
\[ + \frac{i}{4} \sum_{k=1}^g \left( \int_{\beta_k} \Omega_0 \int_{\alpha_k} \omega_{q_0} - \int_{\alpha_k} \Omega_0 \int_{\beta_k} \omega_{q_0} \right) \]
\[ = \frac{i}{4} \int_{M_0} \Omega_0 \wedge \omega_{q_0} - \frac{i}{4} \int_{M_0} \Omega_0 \wedge \omega_{q_0} \]
\[ = \text{Re} \int_{M_0} \frac{\pi_{q_0}}{\omega_{q_0}} \cdot (\eta_k^*) (\eta_k^*) (\mu) \omega_{q_0} \]
\[ = 2 \text{Re} \int_{M_0} \eta_\mu = 2 \int_{M_0} \frac{\eta_\mu}{|q_0|} \]

from the definition of the differential \( \eta_\mu \) (cf. [9 Proposition III.2.3]). This implies what we wanted. \( \square \)

From Theorem 5.1 we deduce the following inequality.

**Corollary 5.1 (Gradients and Levi forms).** Let \( x_0 = (M_0, f_0) \in T_{g,m} \) and \( v = [\mu] \in T_{x_0} T_{g,m} \). Let \( F \in \mathcal{M}_F \). Suppose that \( q_0 = q_{F,x_0} \) is generic. Then,
\[ 2 |(\text{Ext}(F))_* [v]|^2 \leq \text{Ext}_{x_0}(F) \cdot L(\text{Ext}(F))[v, \overline{v}] \]

holds.

**Proof.** It follows from Gardiner’s formula that
\[ (\text{Ext}(F))_* [v] = - \int_{M_0} \mu q_0 = - \int_{M_0} \frac{\eta_\mu}{|q_0|} q_0. \]

See [9] §11.8 (See also [7.1]). Notice in comparing the Gardiner’s original formula with (5.1) that our differential \( q_0 = q_{F,x_0} \) has \( F \) as the vertical foliation, while Gardiner considers the quadratic differential with horizontal foliation \( F \). Hence, we have to put the minus sign in (5.1). From Cauchy-Schwarz inequality, one can see from Theorem 5.1 that
\[ \text{Ext}_{x_0}(F) L(\text{Ext}(F))[v, \overline{v}] - 2 |(\text{Ext}(F))_* [v]|^2 \]
\[ = \text{Ext}_{x_0}(F) \cdot 2 \int_{M_0} \frac{\eta_\mu}{|q_0|} - 2 \int_{M_0} \frac{\eta_\mu}{|q_0|} q_0 \]
\[ = 2 \left( \int_{M_0} \frac{\eta_\mu}{|q_0|} \cdot \int_{M_0} \frac{\eta_\mu}{|q_0|} - \int_{M_0} \frac{\eta_\mu}{|q_0|} q_0 \frac{q_0}{|q_0|^{1/2}} \right)^2 \geq 0, \]

which implies what we wanted. \( \square \)

### 5.2. Plurisubharmonicity for subspecies

Let \( N \) be a complex manifold. A function \( U \) on \( N \) is said to be **plurisubharmonic** if \( U \) is upper semi-continuous, and \( U \circ f \) is subharmonic for all holomorphic mapping \( f : \mathbb{D} \rightarrow N \). A \( C^2 \)-function on \( N \) is plurisubharmonic if the Levi form is positive semi-definite. We call a \( C^2 \)-function **strictly plurisubharmonic** if the Levi form is positive-definite. When a function \( U \)
is plurisubharmonic, so is \( g \circ U \) for any increasing convex function \( g \) on \( \mathbb{R} \) such that the limit \( \lim_{t \to -\infty} g(t) \) exists.

In the proof of the following theorem, we remind that if \( U(z) \) is a real-valued positive \( C^2 \)-function on a domain on \( \mathbb{C} \), \( \log U \) and \( -1/U \) satisfy

\[
(\log U)_{\lambda \lambda} = \frac{U \cdot U_{\lambda\lambda} - |U_{\lambda}|^2}{U^2},
\]

\[
\left(-\frac{1}{U}\right)_{\lambda \lambda} = \frac{U \cdot U_{\lambda\lambda} - 2|U_{\lambda}|^2}{U^3}.
\]

Especially, for a positive \( C^2 \)-function \( U \) on a complex manifold, if \( -1/U \) is plurisubharmonic, \( \log U \) and \( U \) are also plurisubharmonic. From (5.2) and (5.3), Theorem 1.1 follows from the following theorem.

**Theorem 5.2** (Plurisuperharmonicity of reciprocals). Let \( F_1, F_2, \cdots, F_n \) be measured foliations. Let \( a_k, c \geq 0 \) \((k = 1, \cdots, n)\) with \( \sum_{k=1}^n a_k > 0 \). If \( n \geq 2 \), \( a_k, c > 0 \) \((k = 1, \cdots, n)\) and some two of them are transverse in the sense of (1.1), the function

\[
\rho(x) = -\frac{1}{c + \sum_{k=1}^n a_k \text{Ext}_x(F_k)}
\]

is a negative, plurisubharmonic exhaustion function of \( T_{g,m} \) with lower bound. When \( n \geq 1 \) or \( c \geq 0 \), \( \rho \) is a negative, plurisubharmonic function on \( T_{g,m} \) with lower bound.

**Proof.** We only show the case \( n = 2 \). The other case can be treated in the same way. Furthermore, we may also assume that each \( F_k \) are essentially complete, and \( F_1 \) and \( F_2 \) are transverse in the sense of (1.1). The general case follows from a standard approximating procedure (cf. [17, Theorem 2.6.1], [21, Theorem 2 in Chapter II] or [30, 3.3 in Chapter II]). Under the assumption, each extremal length function \( T_{g,m} \ni x \mapsto \text{Ext}_x(F_k) \) is real analytic.

Let \( x = (M, f) \in T_{g,m} \) and \( v \in T_{x,0}T_{g,m} \). Let \( h: \mathbb{D} \to T_{g,m} \) be a holomorphic mapping with \( h(0) = x_0 \) and \( h_*((\partial/\partial \lambda) |\lambda=0) = v \). Set \( E^1(\lambda) = \text{Ext}_{h(\lambda)}(F_1) \), \( E^2(\lambda) = \text{Ext}_{h(\lambda)}(F_2) \) and \( E = a_1E^1 + a_2E^2 \) for simplicity.

Then, the Levi form of \( \rho \) is

\[
\mathcal{L}(\rho)[v, \tau] = \frac{c}{(c + E(0))^3} \left( E(0) E_{\lambda\lambda}(\lambda=0) - 2|E_{\lambda}(\lambda=0)|^2 \right)
\]

\[
\frac{1}{(c + E(0))^3}.
\]
Therefore, we conclude from Corollary 5.1 that
\[
E(0) E\lambda |_{\lambda=0} - 2 | E\lambda |_{\lambda=0}|^2
= a_1^2 \left( E^1(0) E\lambda |_{\lambda=0} - 2 | E\lambda |_{\lambda=0}|^2 \right) + a_2^2 \left( E^2(0) E\lambda |_{\lambda=0} - 2 | E\lambda |_{\lambda=0}|^2 \right)
+ a_1 a_2 \left( E^1(0) E\lambda |_{\lambda=0} + E^2(0) E\lambda |_{\lambda=0} - 4 \Re \left( E\lambda |_{\lambda=0} \right) \right)
\geq a_1 a_2 \left( E^1(0) E\lambda |_{\lambda=0} + E^2(0) E\lambda |_{\lambda=0} - 4 | E\lambda |_{\lambda=0}| \right)
\geq a_1 a_2 \left( E^1(0) E\lambda |_{\lambda=0} + E^2(0) E\lambda |_{\lambda=0} - \right)
-2E^1(0)^{1/2} \left( E\lambda |_{\lambda=0} \right)^{1/2} E^1(0)^{1/2} \left( E\lambda |_{\lambda=0} \right)^{1/2}
= a_1 a_2 \left( E^1(0)^{1/2} \left( E\lambda |_{\lambda=0} \right)^{1/2} - E^1(0)^{1/2} \left( E\lambda |_{\lambda=0} \right)^{1/2} \right)^2 \geq 0
\]
and hence
\[
\mathcal{L}(\rho)[v, \eta] \geq \frac{c E\lambda |_{\lambda=0}}{(c + E(0))^3} = \frac{c a_1^2 \mathcal{L}(\text{Ext.}(F_1))[v, \eta] + a_2^2 \mathcal{L}(\text{Ext.}(F_2))[v, \eta]}{(c + a_1 \text{Ext}_{x_0}(F_1) + a_2 \text{Ext}_{x_0}(F_2))^3}
\]
when \(a + b > 0, c > 0\). Indeed, since \(q_{F,x} \) is generic, the anti-complex linear map
\[
T_{x_0} T_{g,m} \ni v \mapsto \eta_v \in \mathcal{Q}_{x_0}
\]
is an isomorphism. Therefore, \(\rho\) is (real analytic) strictly plurisubharmonic on \(T_{g,m}\).
From the definition, \(\rho\) satisfies that \(\rho(x) < 0\) for \(x \in T_{g,m}\). Since \(F_1\) and \(F_2\) are transverse, for any \(\epsilon > 0\), there is a compact set \(K \subset T_{g,m}\) such that \(\rho > -\epsilon\) for all \(x \in T_{g,m} - K\) (cf. 2.4). Since \(\rho(x) > -1/a\) for all \(x \in T_{g,m}\), the function \(\rho(x)\)
 satisfies the desired condition. \(\square\)

We also obtain the following comparizon.

**Theorem 5.3 (Gradients and Levi forms for log-extremal length functions).** For any measured foliation \(F\),
\[
d \log \text{Ext.}(F) \wedge d^c \log \text{Ext.}(F) \leq \frac{1}{2 \text{Ext.}(F)} - dd^c \text{Ext.}(F) \leq dd^c \log \text{Ext.}(F)
\]
holds in the sense of currents, where \(d = \partial + \overline{\partial}\) and \(d^c = i(\partial - \overline{\partial})\). Especially, the log-extremal length function \(\log \text{Ext.}(F)\) is a real analytic strictly plurisubharmonic function on \(T_{g,m}\), when \(F\) is essentially complete.

**Proof.** By approximation, we may assume that \(F\) is essentially complete. Then, the inequalities follow from the combinations of (5.2) and Corollary 5.1. \(\square\)

6. Geometric interpretation of \(\eta_v\)

6.1. **Motivation.** Let \(x_0 = (M_0, f_0) \in T_{g,m}\). Suppose \(q_0 = q_{F,x_0}\) is generic. Let \(v = [\mu] \in T_{x_0} T_{g,m}\). We defined the quadratic differential \(\eta_v \in \mathcal{Q}_{g,m}\) to satisfy
\[
\langle v, \psi \rangle = \int_{M_0} \frac{\eta_v}{|q_0|} \psi
\]
for all \(\psi \in \mathcal{Q}_{g,m}\) in Lemma 1.1 and Theorem 5.1. The definition of \(\eta_v\) implies as if the infinitesimal Beltrami differential \(\mu\) were infinitesimally equivalent to the “infinitesimal deformation” along \(\overline{\eta_v}/|q_0|\) on \(M_{q_0}\) (cf. 14). However, \(\overline{\eta_v}/|q_0|\) is not
Figure 2. A schematic of the map $\mathcal{J}_{x_0}$.

a Beltrami differential in general, because it may have $1/|z|$-singularities at zeroes of $q_0$ and hence it could happen $\|\eta_v/q_0\|_\infty = \infty$. Therefore, the above discussion does not interpret correctly the geometry of the anti-complex line at map (6.2)

$$T_{x_0}T_{g,m} \ni v \mapsto \eta_v \in Q_{x_0}$$

in general. In this section, we shall try to give a geometric description of the correspondence (6.2).

6.2. Geometric interpretation. From [6] Theorem 5.8] and [10], the map

(6.3) $$Q_{g,m} \ni q \mapsto (h(q), v(q)) \in MF \times MF$$

is a real analytic diffeomorphism around $q_0$. Hence, the map

(6.4) $$\mathcal{J} : T_{g,m} \times T_{g,m} \ni (x, y) \mapsto -q_{h(q_F,x),y} \in Q_{g,m}$$

is also a real analytic diffeomorphism on a neighborhood $N_1 \times N_1$ of $(x_0, x_0)$ onto the image, where $N_1$ is a neighborhood of $x_0 = \pi(q_0) \in T_{g,m}$. Notice that $h(\mathcal{J}(x, y)) = h(q_F,x)$, $\mathcal{J}(x, y) \in Q_g$ and $\mathcal{J}(x, x) = q_{F,x}$ for all $x, y \in N_1$. Let $\mathcal{J}_y(x) = \mathcal{J}(x, y)$ for $(x, y) \in N_1 \times N_1$ (Figure 2).

Theorem 6.1 (Geometric interpretation). For $v \in T_{x_0}T_{g,m}$, we have

$$\mathcal{J}_{x_0}(v) = -4\eta_v,$$

where $\eta_v \in Q_{x_0}$ is taken as (6.1).

Proof. Consider the triangulation $\Delta_q$ and the train track $\tau_{q_0}$ on $M_q$ for the horizontal foliation $\text{Im}(\sqrt{q})$ of $q$ (cf. [3] Theorem 5.8). We may assume that $\Delta_q$ contains no horizontal edge (cf. [6] Lemma 5.7). Consider the lifts $\hat{\Delta}_q$ and $\hat{\tau}_q$ of $\Delta_q$ and $\tau_q$ to $\tilde{M}_q$, respectively. We orient each edge of $\hat{\Delta}_q$ so that $\text{Im}(\omega_q)$ is positive along the edge. Each branch $b$ of $\hat{\tau}_q$ intersects a unique edge $e_b$ of $\hat{\Delta}_q$. We orient $b$ so that $b \cdot e_b = +1$, where $\cdot$ means the algebraic intersection number. By taking $U$ sufficiently small, we may also assume that the marking $\tilde{M}_{q_0} \to \tilde{M}_q$ induces isomorphisms from $\hat{\Delta}_{q_0}$ and $\hat{\tau}_{q_0}$ to $\hat{\Delta}_q$ and $\hat{\tau}_q$, respectively.
From [6] Theorem 5.8, the tangential map of the map $h: Q_{x_0} \to MF$ of the horizontal foliations satisfies

$$\text{Im} \int_{M_{q_0}} \frac{\psi_1 \psi_2}{4|q_0|} = \omega_{Th}(h_*(\psi_1), h_*(\psi_2))_{\tau_{q_0}}$$

for $\psi_1, \psi_2 \in T_{q_0} Q_{x_0} \cong Q_{x_0}$, where $\omega_{Th}(\cdot, \cdot)_{\tau_{q_0}}$ is the Thurston’s symplectic form on $W(\tau_{q_0})$, where $W(\tau_{q_0})$ is the space of real valued functions on the branches of $\tau_{q_0}$ satisfying the switch condition. It is known that the space $W(\tau_{q_0})$ is identified with the tangent space of $MF$ at $h(q_0)$ since $\tau_{q_0}$ is maximal (cf. [29] §3.2).

From the definition, $|\text{Im}(\omega_{F,x})|$ is the lift of the horizontal foliation of $q_{F,x}$ to $\tilde{M}_{q_{F,x}}$. By the definition of the orientation of edges of $\tilde{\Delta}_{q_1}$, the image of the map $N_1 \ni x \mapsto h(q_{F,x}) \in MF$ at $x$ is identified with $w_x \in W(\tilde{\tau}_{q_0})$ which is defined by

$$w_x(b) = \int_b |\text{Im}(\omega_{F,x})| = \int_b \text{Im}(\omega_{F,x})$$

From Lemmas [9] and [10] the image of $v \in T_{x_0} T_{g,m}$ under the differential of the map $N_1 \ni x \mapsto \text{Im}(\chi_{q_{F,x}})$ is identified with the homomorphism

$$V_v: H_1(M_{q_0}) \ni c \mapsto -2\text{Im} \int_c \frac{(\pi_{q_0})^*(\eta_c)}{\omega_{q_0}} \in \mathbb{R}.$$ 

Indeed, when $\{q_1\}_l \in D$ is taken as Lemma $[11]$, $[6.6]$ is derived from

$$\frac{\partial \text{Im}(\chi_{q_1}(c))}{\partial t} = -i \left( \frac{\partial \chi_{q_1}(c)}{\partial \lambda} + \frac{\partial \chi_{q_1}(c)}{\partial \lambda^*} \right)$$

$$= -i \int_c \frac{(\pi_{q_1})^*(\eta_\lambda)}{\omega_{q_\lambda}} - \frac{(\pi_{q_1})^*(\eta_\lambda)}{\omega_{q_\lambda}}$$

$$= -2\text{Im} \int_c \frac{(\pi_{q_1})^*(\eta_\lambda)}{\omega_{q_\lambda}}$$

where $\lambda = t + is$, since the real part of $\chi_{q_\lambda}$ is constant on $D$.

Let $p: \tilde{T}_{q_0} \to T_{q_0}$ be the induced covering. For a weight $w \in W(\tau_{q_0})$, we define $C_w \in H_1(M_{q_0}, \mathbb{R})^-$ as the homology class of a cycle

$$\sum_{b \text{branches of } \tilde{T}_{q_0}} w(p(b))b.$$ 

Then,

$$w_x(C_w) = \sum_b w(p(b))w_x(b)$$

$$V_v(C_w) = -2\text{Im} \int_{C_w} \frac{(\pi_{q_0})^*(\eta_c)}{\omega_{q_0}} = -2 \sum_b w(p(b))\text{Im} \int_b \frac{(\pi_{q_0})^*(\eta_c)}{\omega_{q_0}}.$$ 

These equations imply that the differential of the map $N_1 \ni x \mapsto w_x \in W(\tilde{T}_{q_0})$ at $x = x_0$ is equal to the weight (which we abbreviate as)

$$V_v(b) = -2\text{Im} \int_b \frac{(\pi_{q_0})^*(\eta_c)}{\omega_{q_0}}.$$ 

The weight $V_v$ is invariant under the action of covering transformation $r_0$ of $\tilde{M}_{q_0} \to M_{q_0}$ on $\tilde{T}_{q_0}$, because for any branch $b$ of $\tilde{T}_{q_0}$, the orientation of $r_0(b)$ is opposite to the orientation induced from $b$. Thus, $V_v$ descends to a weight $p_*V_v$ on $\tau_{q_0}$.
Let \( \psi_0 = (J_{x_0})_* [v] \). For all \( \psi \in Q_{x_0} \subset T_{q_0} Q_{g,m} \), we define \( w(\psi) \in W(\tilde{r}_{q_0}) \) by

\[
w(\psi)(b) = \text{Im} \int_b \frac{(\pi_{q_0})^*(\psi)}{2\omega_{q_0}}.
\]

Then, we have \( h_* [\psi] = p_*(w(\psi)) \) (cf. [6] Proof of Theorem 5.8]). By the definition of \( J_{x_0} \), \( h \circ J_{x_0}(x) = h(q_{F,x}) \) for all \( x \in N_1 \). Hence, \( h_* [\psi_0] = p_*(V_0) \) as elements of \( W(\tilde{r}_{q_0}) \cong T_{h(q_0)} MF \). From [6.3] and [6] (5.4), the following holds for all \( \psi \in Q_{x_0} \):

\[
\text{Im} \int_{M_{q_0}} \frac{\psi_0 \psi_0^*}{4|q_0|} = \frac{1}{2} \text{Im} \int_{M_{q_0}} \frac{(\pi_{q_0})^*(\psi_0)(\pi_{q_0})^*(\psi)}{4|\omega_{q_0}|^2} = \frac{1}{2} \omega_{T} (V_0, w(\psi)) \tilde{\tau}_{q_0}
\]

\[= -2 \int_{M_{q_0}} \text{Im} \frac{(\pi_{q_0})^*(\eta_\tau)}{2\omega_{q_0}} \wedge \text{Im} \frac{(\pi_{q_0})^*(\psi)}{2\omega_{q_0}}
\]

\[-= -2 \text{Re} \int_{M_{q_0}} \frac{(\pi_{q_0})^*(\eta_\tau)}{2\omega_{q_0}} \wedge \frac{(\pi_{q_0})^*(\psi)}{2\omega_{q_0}}
\]

\[-= -2 \text{Im} \int_{M_{q_0}} \frac{(\pi_{q_0})^*(\eta_\tau)(\pi_{q_0})^*(\psi)}{4|\omega_{q_0}|^2} = -\text{Im} \int_{M_{q_0}} \frac{\eta_\tau \psi}{|q_0|}.
\]

Therefore, we have \( (J_{x_0})_* [v] = \psi_0 = -4\eta_{\psi} \).

\[\square\]

7. Appendix

7.1. Alternative approach to Gardiner’s formula. Applying our method in Theorem 5.1, one can also get an alternative proof of Gardiner’s formula for the derivative of extremal length in some case. Indeed, this is the case where we need in the proof of Theorem 5.1.

Let \( F \in MF \) and \( x_0 = (M, f) \in T_{g,m} \). Suppose that the Hubbard-Masur differential \( q_0 = q_{F,x_0} \) is generic. Let \( v = [\hat{\mu}] \) and \( \{q_\lambda\}_{\lambda \in \Omega} \) as the proof of Theorem 5.1. Suppose \( x_\lambda = \pi(q_\lambda) \) varies holomorphically. By applying the discussion in the proof of Theorem 5.1 from Lemma 1.1, we have

\[
\frac{\partial}{\partial \lambda} \left. \text{Ext} x_\lambda(F) \right|_{\lambda = 0}
\]

\[= i \sum_{k=1}^5 \left( \int_{\alpha_k} (\pi_{q_0})^*(\eta_\tau) \int_{\beta_k} \omega_{q_0} - \int_{\beta_k} (\pi_{q_0})^*(\eta_\tau) \int_{\alpha_k} \omega_{q_0} \right)
\]

\[= i \int_{M_{q_0}} \frac{(\pi_{q_0})^*(\eta_\tau)}{2\omega_{q_0}} \wedge \omega_{q_0} = -\frac{i}{4} \int_{M_{q_0}} \omega_{q_0} \wedge \frac{(\pi_{q_0})^*(\eta_\tau)}{\omega_{q_0}}
\]

\[= -\frac{1}{2} \int_{M_{q_0}} \frac{(\pi_{q_0})^*(\eta_\tau)}{|\omega_{q_0}|^2} \omega_{q_0}^2 = -\int_{M_{q_0}} \frac{\eta_\tau}{|q_0|} dV = -\int_M \hat{\mu} q_{F,x_0}
\]

because \( M = M_{q_0} \) by definition.

7.2. Examples. In this section, we treat the exceptional cases in the proofs of pluriharmonicity in the previous sections.
7.2.1. Levi forms for the case of flat tori. We start to consider the Levi forms of extremal length functions on the Teichmüller space of flat tori (once punctured tori), and check our formula in Theorem 5.1 also valid in this case. This case is excluded from our assumption in the theorem. However, this is a motivated example of our result.

The Teichmüller space of a flat torus is identified with the upper half plane $\mathbb{H}$. A point $\tau \in \mathbb{H}$ corresponds to a marked Riemann surface $M_\tau$ which is defined as the quotient space of $\mathbb{C}$ by a lattice generated by $z + 1$ and $z + \tau$. Notice that the space $\mathcal{PMF} = \mathcal{PMF}(M_i)$ of projective measured foliations on $M_i$ ($i = \sqrt{-1} \in \mathbb{H}$) is identified with the projective space $\mathbb{RP}^1 = \{(x: y) \mid (x, y) \neq (0, 0)\}$. A point $[x: y] \in \mathbb{RP}^1$ corresponds to a measured foliation whose leaves are parallel to the line of direction $x + yi$.

Let $\alpha$ be a closed curve on $M_i$ corresponding to $[1: 0] \in \mathcal{PMF}$. Then, the Jenkins-Strebel differential $q_{\alpha, \tau}$ on $M_\tau$ for $\alpha$ and the extremal length function of $\alpha$ at $M_\tau$ are obtained as

$$q_{\alpha, \tau} = \frac{-1}{\text{Im}(\tau)^2} d\tau^2$$

$$\text{Ext}_\tau(\alpha) = \frac{1}{\text{Im}(\tau)}.$$ 

Fix $\tau_0 \in \mathbb{H}$. For $V \in \mathbb{C}$, let $v$ be the tangent vector at $\tau_0$ corresponding to the infinitesimal quasiconformal deformation from $M_{\tau_0}$ to $M_{\tau_0 + \lambda V}$ as $\lambda \to 0$. The Levi form (Laplacian) of the extremal length function is

$$\mathcal{L}(\text{Ext}(\alpha))[v, \tau] = \left. \frac{\partial^2}{\partial \lambda \partial \lambda} \text{Ext}_{\tau_0 + \lambda V}(\lambda) \right|_{\lambda = 0} = \frac{|V|^2}{2 \text{Im}(\tau_0)^3}.$$ 

The Beltrami differential $\mu(\lambda)$ of the quasiconformal deformation from $M_{\tau_0}$ to $M_{\tau_0 + \lambda V}$ behaves

$$\mu(\lambda) = \lambda \cdot \hat{\mu} + o(|\lambda|) = \lambda \cdot iV \frac{d\bar{z}}{2 \text{Im}(\tau_0)} + o(|\lambda|)$$

as $t \to 0$. By definition, $v = [\hat{\mu}]$. We define a holomorphic quadratic differential $\eta_v$ on $M_{\tau_0}$ by

$$\eta_v = -\frac{iV}{2 \text{Im}(\tau_0)^3} d\tau^2.$$ 

This differential satisfies that

$$\langle v, \psi \rangle = \int_{M_{\tau_0}} \hat{\mu} \psi = \int_{M_{\tau_0}} \eta_v \psi$$

for all holomorphic quadratic differential $\psi$ on $M_{\tau_0}$. Therefore, the right-hand side of the formula in Theorem 5.1 is equal to

$$2 \int_{M_{\tau_0}} \frac{|\eta_v|^2}{|q_{\alpha, \tau_0}|} = \frac{|V|^2}{2 \text{Im}(\tau_0)^3},$$

which coincides with the right-hand side of (7.1).

Let $F$ be a measured foliation corresponding to $[a: b] \in \mathcal{PMF}$ with $b \neq 0$. We normalize $F$ with $i(\alpha, F) = 1$. Then the Hubbard-Masur differential $q_{F, \tau}$ on $M_\tau$
and the extremal length function for $F$ are
\[ q_{F, \tau} = \frac{(a + b\tau)^2}{b^2 \text{Im}(\tau)^2} dz^2 \]
\[ \text{Ext}_\tau(F) = \frac{|a + b\tau|^2}{b^2 \text{Im}(\tau)}. \]

Let $v$ be the tangent vector at $\tau_0$ corresponding to $\dot{\tau} = \frac{IV}{2\text{Im}(\tau_0)^2}$. Then,
\[ (7.3) \quad L(\text{Ext}(\alpha))[v, \tau] = \frac{\partial^2}{\partial \lambda \partial \overline{\lambda}} \text{Ext}_{\tau_0 + \lambda \text{V}(\lambda)} \bigg|_{\lambda = 0} = \frac{|a + b\tau_0|^2}{2b^2 \text{Im}(\tau_0)^3} |\text{V}|^2 \]

In this case, the differential $\eta_v \in Q_{\tau_0}$ is
\[ (7.4) \quad \eta_v = \frac{i|a + b\tau_0|^2\text{V}}{2b^2 \text{Im}(\tau_0)^3} \quad \text{dz}. \]

and hence
\[ 2 \int_{M_{\tau_0}} \frac{|\eta_v|^2}{q_{F, \tau_0}} = \frac{|a + b\tau_0|^2}{2b^2 \text{Im}(\tau_0)^3} |\text{V}|^2, \]
which coincides with the right-hand side of (6.3).

7.2.2. The geometric interpretation of $\eta_v$ for the case of tori. We check Theorem 6.1 in the case of flat tori (once punctured tori). We use the notation defined in §7.2.1 frequently.

First we consider the case $\alpha = [1: 0]$. Fix $\tau_0 \in \mathbb{H}$. Since the projective class of the horizontal foliation of $q_{a, \tau}$ corresponds to a projective class $[-\text{Re}(\tau): 1] \in \mathbb{R}P^1$, the underlying foliation of the horizontal foliation $J_{\tau_0}(\tau)$ is foliated by the lines (leaves) directed to $-(\text{Re}(\tau)) + \tau_0$. Since the horizontal foliations of $J_{\tau_0}(\tau)$ and $q_{a, \tau}$ are same,
\[ \int_{\alpha} \text{Im} \sqrt{J_{\tau_0}(\tau)} = i(h(J_{\tau_0}(\tau)), \alpha) = i(h(q_{a, \tau}), \alpha) = \int_{\alpha} \text{Im} \sqrt{q_{a, \tau}} \]
holds. Therefore, we can check that
\[ J_{\tau_0}(\tau) = \left( \frac{-\text{Re}(\tau) + \tau_0}{\text{Im}(\tau) \text{Im}(\tau_0)} \right)^2 dz^2 \in Q_{\tau_0} \]
for $\tau \in \mathbb{H}$. Let $\nu = [\dot{\mu}]$ be a tangent vector at $\tau_0$, where $\dot{\mu} = \frac{iV}{2\text{Im}(\tau_0)^2} dz$ with $V \in \mathbb{C}$ as (7.2.1). Then, from the above calculation, we obtain
\[ (J_{\tau_0})_*[v] = \left( 2 \frac{iV}{\text{Im}(\tau_0)^3} \right) dz^2 \in Q_{\tau_0}. \]

Comparing with (7.2), we get $(J_{\tau_0})_*[v] = -4\eta_v$ as we appeared in Theorem 6.1.

Let $F \in \mathcal{PMF}$ be a measured foliation corresponding to $[a: b] \in \mathcal{PMF}$ with $b \neq 0$ and $i(\alpha, F) = 1$. Then, the underlying foliation of the horizontal foliation of $q_{F, \tau}$ is foliated by the lines of direction $[a\text{Re}(\tau) + b|\tau|^2: a + b\text{Re}(\tau)]$. By the similar argument as above, one can check that
\[ J_{\tau_0}(\tau) = \left( \frac{-a\text{Re}(\tau) + b|\tau|^2 + (a + b\text{Re}(\tau))\tau_0}{b\text{Im}(\tau) \text{Im}(\tau_0)} \right)^2 dz^2 \]
\[ (J_{\tau_0})_*[v] = \left( 2 \frac{i|a + b\tau_0|^2V}{b^2 \text{Im}(\tau_0)^3} \right) dz^2 \in Q_{\tau_0}. \]
From (7.4), we also have $(J_{τ_{0}})_{*}[v] = -4η_{v}$.

7.2.3. A comment on the case of $3g - 3 + m = 1$. The case of once punctured tori is treated in the same way as that in the case of flat tori. Indeed, the canonical completion at the puncture gives an isomorphism between the Teichmüller space of once punctured tori and that of flat tori. The extremal length functions in the both spaces are identified under the isomorphism. The case of fours punctured sphere is treated in the similar way.

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