Laplacian Flow of Closed $G_2$-Structures Inducing Nilsolitons

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Received: 18 May 2014 / Published online: 7 April 2015 © Mathematica Josephina, Inc. 2015

Abstract We study the existence of left invariant closed $G_2$-structures defining a Ricci soliton metric on simply connected nonabelian nilpotent Lie groups. For each one of these $G_2$-structures, we show long time existence and uniqueness of solution for the Laplacian flow on the noncompact manifold. Moreover, considering the Laplacian flow on the associated Lie algebra as a bracket flow on $\mathbb{R}^7$ in a similar way as in Lauret (Commun Anal Geom 19(5):831–854, 2011) we prove that the underlying metrics $g(t)$ of the solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in the nilpotent Lie group, as $t$ goes to infinity.

Keywords Closed $G_2$-structure · Nilsoliton · Laplacian flow

Mathematics Subject Classification 53C38 · 53C25 · 22E25

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1 Introduction

A $G_2$-structure on a 7-dimensional manifold $M$ can be characterized by the existence of a globally defined 3-form $\varphi$, which is called the $G_2$ form or the fundamental 3-form and it can be described locally as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

with respect to some local basis $\{e^1, \ldots, e^7\}$ of the 1-forms on $M$.

There are many different $G_2$-structures attending to the behavior of the exterior derivative of the $G_2$ form [2,13]. In the following, we will focus our attention on closed $G_2$-structures which are characterized by the closure of the $G_2$ form.

The existence of a $G_2$ form $\varphi$ on a manifold $M$ induces a Riemannian metric $g_\varphi$ on $M$ given by

$$g_\varphi(X, Y)_{vol} = \frac{1}{6} \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi,$$

(1)

for any vector fields $X, Y$ on $M$, where $vol$ is the volume form on $M$.

By [3,6] a closed $G_2$-structure on a compact manifold cannot induce an Einstein metric, unless the induced metric has holonomy contained in $G_2$. It is still an open problem to see if the same property holds on noncompact manifolds. For the homogeneous case, a negative answer has been recently given in [12]. Indeed, we showed that if a solvable Lie algebra has a closed $G_2$-structure then the induced inner product is Einstein if and only if it is flat.

Natural generalizations of Einstein metrics are given by Ricci solitons, which have been introduced by Hamilton in [14]. A natural question is thus to see if a closed $G_2$-structure on a noncompact manifold induces a (non-Einstein) Ricci soliton metric. In this paper we give a positive answer to this question, showing that there exist 7-dimensional simply connected nonabelian nilpotent Lie groups with a closed $G_2$-structure which determines a left invariant Ricci soliton metric.

All known examples of nontrivial homogeneous Ricci solitons are left invariant metrics on simply connected solvable Lie groups, whose Ricci operator satisfies the condition

$$\text{Ric}(g) = \lambda I + D,$$

for some $\lambda \in \mathbb{R}$ and some derivation $D$ of the corresponding Lie algebra. The left invariant metrics satisfying the previous condition are called nilsolitons if the Lie groups are nilpotent [19]. Not all nilpotent Lie groups admit nilsoliton metrics, but if a nilsoliton exists, then it is unique up to automorphism and scaling [19]. The nilsolitons metrics are strictly related to left invariant Einstein metrics on solvable Lie groups. Indeed, by [20], a simply connected nilpotent Lie group $N$ admits a nilsoliton metric if and only if its Lie algebra $n$ is an Einstein nilradical, which means that $n$ has an inner product $\langle \cdot, \cdot \rangle$ such that there is a metric solvable extension of $(n, \langle \cdot, \cdot \rangle)$ which is Einstein. According to [15,21], such an Einstein metric has to be of standard type and it is unique, up to isometry and scaling.

Seven dimensional nilpotent Lie algebras admitting a closed $G_2$-structure have been recently classified in [8], showing that there are twelve isomorphism classes, including
the abelian case which has a trivial nilsoliton because it is flat. A classification of 7-dimensional nilpotent Lie algebras admitting a nilsoliton has been recently given in [11], but the explicit expression of the nilsoliton is not written in all the cases that we need.

Using the classification in [8] and Table 1 in [9], we have that, up to isomorphism, there is a unique nilpotent Lie algebra with a closed $G_2$ form but not admitting nilsolitons. It turns out that all the other ten nilpotent Lie algebras have a nilsoliton, and we can determine explicitly the nilsoliton except for the Lie algebra $n_{10}$ which is 4-step nilpotent (see also [9–11]). In Proposition 3.4 we prove that the Lie algebra $n_i$ ($i = 3, 5, 7, 8, 11$) has a nilsoliton but no closed $G_2$-structure inducing the nilsoliton. Moreover, as we mentioned before, the existence of a nilsoliton on the Lie algebra $n_{10}$ was shown in [9, Example 2], but we cannot explicit its nilsoliton. Therefore, it remains open the question of whether the Lie algebra $n_{10}$ admits a closed $G_2$ form inducing a nilsoliton or not. This is the reason why the result of Theorem 3.6 is restricted to $s$-step nilpotent Lie algebras, with $s = 2, 3$. In fact, in Theorem 3.6, we show that, up to isomorphism, there are exactly four $s$-step nilpotent Lie algebras ($s = 2, 3$) with a closed $G_2$ form defining a nilsoliton.

The Ricci flow became a very important issue in Riemannian geometry and has been deeply studied. The same techniques are also useful in the study of the flow involving other geometrical structures, like for example, the Kähler Ricci flow that was studied by Cao in [5].

For any closed $G_2$-structure on a manifold $M$, in [3] Bryant introduced a natural flow, the so-called Laplacian flow, given by

$$\begin{align*}
\frac{d}{dt}\varphi(t) &= \Delta_t \varphi(t), \\
\varphi(0) &= \varphi_0,
\end{align*}$$

where $\varphi(t)$ is a closed $G_2$ form on $M$, and $\Delta_t$ is the Hodge Laplacian operator of the metric determined by $\varphi(t)$. If the initial 3-form $\varphi_0$ is closed, then a solution $\varphi(t)$ of the flow remains closed, and the de Rham cohomology class $[\varphi(t)]$ is constant in $t$. The short time existence and uniqueness of solution for the Laplacian flow of any closed $G_2$-structure, on a compact manifold $M$, has been proved by Bryant and Xu in the unpublished paper [4]. Also, long time existence and convergence of the Laplacian flow starting near a torsion-free $G_2$-structure was proved in the unpublished paper [29].

In Sect. 4 (Theorems 4.2, 4.7, 4.8 and 4.10) we show long time existence of the solution for the Laplacian flow on the four nilpotent Lie groups admitting an invariant closed $G_2$-structure which determines the nilsoliton (see Theorem 3.6).

To our knowledge, these are the first examples of noncompact manifolds having a closed $G_2$-structure with long time existence of solution.

Since the Laplacian flow is invariant by diffeomorphisms and the initial $G_2$-form $\varphi_0$ is invariant, the solution $\varphi(t)$ of the Laplacian flow has to be also invariant. Therefore, we show that the Laplacian flow is equivalent to a system of ordinary differential equations which admits a unique solution. We prove that the solution for the four manifolds is defined for any $t \in [0, +\infty)$. Moreover, considering the Laplacian flow on the associated Lie algebra as a bracket flow on $\mathbb{R}^7$, in a similar way as Lauret did in [23] for the Ricci flow, we show that the underlying metrics $g(t)$ of the solution
converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in the nilpotent Lie group as $t$ goes to infinity. Indeed, by [23, Proposition 2.1] the convergence of the metrics in the $C^\infty$ uniformly on compact sets in $\mathbb{R}^7$ is equivalent to the convergence of the nilpotent Lie brackets $\mu_t$ in the algebraic subset of nilpotent Lie brackets $\mathcal{N} \subset (\Lambda^2 \mathbb{R}^7)^* \otimes \mathbb{R}^7$ with the usual vector space topology.

2 Preliminaries on Nilsolitons

In this section, we recall some definitions and results about nontrivial homogeneous Ricci soliton metrics and, in particular, on nilsolitons. For more details, see for instance [7,17,19].

A complete Riemannian metric $g$ on a manifold $M$ is said to be a Ricci soliton if its Ricci curvature tensor $\text{Ric}(g)$ satisfies the following condition

$$\text{Ric}(g) = \lambda g + \mathcal{L}_X g,$$

for some real constant $\lambda$ and a complete vector field $X$ on $M$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to $X$. If in addition $X$ is the gradient vector field of a smooth function $f : M \to \mathbb{R}$, then the Ricci soliton is said to be of gradient type. Ricci solitons are called expanding, steady or shrinking depending on whether $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively.

In the next section we will focus our attention on nilsolitons, that is, a particular type of nontrivial homogeneous Ricci soliton metrics.

A Ricci soliton metric $g$ on $M$ is called trivial if $g$ is an Einstein metric or $g$ is the product of a homogeneous Einstein metric with the Euclidean metric; and $g$ is said to be homogeneous if its isometry group acts transitively on $M$, and hence $g$ has bounded curvature [22].

In order to characterize the nontrivial homogeneous Ricci soliton metrics, we note that any homogeneous steady or shrinking Ricci soliton metric $g$ of gradient type is trivial. Indeed, if $g$ is steady, one can check that $g$ is Ricci flat, and so by [1] $g$ must be flat. If $g$ is shrinking, then by the results in [25, Theorem 1.2] and in [27], $(M, g)$ is isometric to a quotient of $P \times \mathbb{R}^k$, where $P$ is some homogeneous Einstein manifold with positive scalar curvature. Now, we should notice that this last result for shrinking homogeneous Ricci soliton metrics is also true for homogeneous Ricci solitons of gradient type [27]. Moreover, if a homogeneous Ricci soliton $g$ on a manifold $M$ is expanding, then by [16] $M$ must be noncompact; and from [26] all Ricci solitons (homogeneous or nonhomogeneous) on a compact manifold are of gradient type. Therefore, as it was noticed by Lauret in [22] we have the following

**Lemma 2.1** ([22]) Let $g$ be a nontrivial homogeneous Ricci soliton on a manifold $M$. Then, $g$ is expanding and it cannot be of gradient type. Moreover, $M$ is noncompact.

All known examples of nontrivial homogeneous Ricci solitons are left invariant metrics on simply connected solvable Lie groups whose Ricci operator is a multiple of the identity modulo derivations, and they are called solsolitons or, in the nilpotent case, nilsolitons.
Let $N$ be a simply connected nilpotent Lie group, and denote by $n$ its Lie algebra. A left invariant metric $g$ on $N$ is called a Ricci nilsoliton metric (or simply nilsoliton metric) if its Ricci endomorphism $\text{Ric}(g)$ differs from a derivation $D$ of $n$ by a scalar multiple of the identity map $I$, i.e., if there exists a real number $\lambda$ such that

$$\text{Ric}(g) = \lambda I + D.$$  \hspace{1cm} (2)

Clearly, any left invariant metric which satisfies (2) is automatically a Ricci soliton.

Nilsoliton metrics have properties that make them preferred left invariant metrics on nilpotent Lie groups in the absence of Einstein metrics. Indeed, nonabelian nilpotent Lie groups do not admit left invariant Einstein metrics ([24]).

From now on, we will always identify a left invariant metric on a Lie group $N$ with an inner product $\langle \cdot, \cdot \rangle_n$ on the Lie algebra $n$ of $N$. A Lie algebra $n$ endowed with an inner product is usually called in the literature a metric Lie algebra and is denoted as the pair $(n, \langle \cdot, \cdot \rangle_n)$.

We will say that a metric nilpotent Lie algebra $(n, \langle \cdot, \cdot \rangle_n)$ is a nilsoliton if there exists a real number $\lambda$ and a derivation $D$ of $n$ such that

$$\text{Ric}(n, \langle \cdot, \cdot \rangle_n) = \lambda I + D.$$  \hspace{1cm} (3)

Not all nilpotent Lie algebras admit nilsoliton inner products, but if a nilsoliton inner product exists, then it is unique up to automorphism and scaling [19]. A computational method for classifying nilpotent Lie algebras having a nilsoliton inner product in a large subclass of the set of all nilpotent Lie algebras, has been recently introduced in [18]. By Lauret’s results it turns out that nilsoliton metrics on simply connected nilpotent Lie groups $N$ are strictly related to Einstein metrics on the so-called solvable rank-one extensions of $N$.

**Definition 2.2** Let $(n, \langle \cdot, \cdot \rangle_n)$ be a metric nilpotent Lie algebra. A metric solvable extension of $(n, \langle \cdot, \cdot \rangle_n)$ is a metric solvable Lie algebra $(s = n \oplus a, \langle \cdot, \cdot \rangle_s)$ such that $n = [s, s]$ and $\langle \cdot, \cdot \rangle_s|_{n \times n} = \langle \cdot, \cdot \rangle_n$. The metric solvable Lie algebra $(s, \langle \cdot, \cdot \rangle_s)$ is standard, or has standard type, if $a$ is an abelian subalgebra of $s$; in this case, the dimension of $a$ is called the rank of the metric solvable extension.

Heber showed in [15] that a simply connected solvable Lie group admits at most one Einstein left invariant metric up to isometry and scaling. Moreover, he proved that the study of Einstein metrics on simply connected solvable Lie groups, of standard type, can be reduced to the rank-one case, that is, $\dim a = 1$.

Recently, Lauret in [20,21] proved the following

**Theorem 2.3** ([20,21]) Any Einstein metric solvable Lie algebra $(s, \langle \cdot, \cdot \rangle_s)$ has to be of standard type. Moreover, a simply connected nilpotent Lie group $N$ admits a nilsoliton metric if and only if its Lie algebra $n$ is an Einstein nilradical, that is, $n$ possesses an inner product $\langle \cdot, \cdot \rangle$ such that $(n, \langle \cdot, \cdot \rangle)$ has a metric solvable extension which is Einstein.
3 Nilsoliton Metrics Determined by Closed $G_2$ Forms

In this section we prove that, up to isomorphism, there are only four (nonabelian) $s$-step nilpotent Lie groups ($s = 2, 3$) with a nilsoliton inner product determined by a left invariant closed $G_2$-structure. We also show that, up to isomorphism, there is a unique 7-dimensional nilpotent Lie group with a left invariant closed $G_2$-structure but not having nilsoliton metrics.

Let $N$ be a 7-dimensional simply connected nilpotent Lie group with Lie algebra $n$. Then, a $G_2$-structure on $N$ is left invariant if and only if the corresponding 3-form is left invariant. Thus, a left invariant $G_2$-structure on $N$ corresponds to an element $\varphi$ of $\Lambda^3(n^*)$ that can be written as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{236} - e^{146} - e^{245},$$

with respect to some coframe $\{e^1, \ldots, e^7\}$ on $n^*$, and we shall say that $\varphi$ defines a $G_2$-structure on $n$. A $G_2$-structure on $n$ is said to be closed if $\varphi$ is closed, i.e.,

$$d\varphi = 0,$$

where $d$ denotes the Chevalley–Eilenberg differential on $n^*$.

From now on, given a 7-dimensional Lie algebra $n$ whose dual is spanned by $\{e^1, \ldots, e^7\}$, we will write $e^{ij} = e^i \wedge e^j$, $e^{ijk} = e^i \wedge e^j \wedge e^k$, and so forth. Moreover, by the notation

$$n = (0, 0, 0, 0, e^{12}, e^{13}, 0),$$

we mean that the dual space $n^*$ of the Lie algebra $n$ has a fixed basis $\{e^1, \ldots, e^7\}$ such that

$$de^5 = e^{12}, \quad de^6 = e^{13}, \quad de^1 = de^2 = de^3 = de^4 = de^7 = 0.$$

The classification of nilpotent Lie algebras admitting a closed $G_2$-structure is given in [8] as follows.

**Theorem 3.1** Up to isomorphism, there are exactly 12 nilpotent Lie algebras that admit a closed $G_2$-structure. They are:

- $n_1 = (0, 0, 0, 0, 0, 0, 0),$
- $n_2 = (0, 0, 0, 0, e^{12}, e^{13}, 0),$
- $n_3 = (0, 0, 0, e^{12}, e^{13}, e^{23}, 0),$
- $n_4 = (0, 0, e^{12}, 0, 0, e^{13} + e^{24}, e^{15}),$
- $n_5 = (0, 0, e^{12}, 0, 0, e^{13} + e^{14}, e^{25}),$
- $n_6 = (0, 0, 0, e^{12}, e^{13}, e^{14}, e^{15}),$
- $n_7 = (0, 0, 0, e^{12}, e^{13}, e^{14} + e^{23}, e^{15}),$
- $n_8 = (0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{34}),$
- $n_9 = (0, 0, e^{12}, e^{13}, e^{23}, e^{15} + e^{24}, e^{16} + e^{34} + e^{25}).$
\[ n_{10} = (0, 0, e^{12}, 0, e^{13} + e^{24}, e^{14}, e^{46} + e^{34} + e^{15} + e^{23}), \]
\[ n_{11} = (0, 0, e^{12}, 0, e^{13}, e^{24} + e^{23}, e^{25} + e^{34} + e^{15} + e^{16} - 3e^{26}), \]
\[ n_{12} = (0, 0, 0, e^{12}, e^{23}, -e^{13}, 2e^{26} - 2e^{34} - 2e^{16} + 2e^{25}). \]

Using Table 1 in [9] we can determine which indecomposable Lie algebras \( n_i \) (4 \( \leq i \leq 12 \)) do not have nilsoliton inner products. Note that the existence of nilsolitons on \( n_2 \) and \( n_3 \) is not studied in [9] since they are decomposable. Concretely, the correspondence between the indecomposable Lie algebras of Theorem 3.1 and Table 1 in [9] is the following:

\[ n_4 \cong 3.8, \quad n_5 \cong 3.11, \quad n_6 \cong 3.20, \quad n_7 \cong 2.39, \quad n_8 \cong 2.5, \quad n_9 \cong 1.1(i_4), \quad \text{and} \quad n_{10} \cong 1.3(i_1). \]

Moreover, \( n_{11} \) and \( n_{12} \) are respectively isomorphic to the real form of 1.2(\( i_{-3} \)) and 3.1(\( i_2 \)). In particular, we have that \( n_9 \) is the only 7-dimensional nilpotent Lie algebra with a closed \( G_2 \) form but not admitting a nilsoliton.

**Remark 3.2** Note that the abelian Lie algebra \( n_1 \) admits as rank-one Einstein solvable extension the Lie algebra \( sl_1 \) with structure equations

\[ (ae^{18}, ae^{28}, ae^{38}, ae^{48}, ae^{58}, ae^{68}, ae^{78}, 0), \]

for some real number \( a \neq 0 \), and the nilsoliton inner product on \( n_1 \) is trivial because it is flat. Since we are interested in nontrivial nilsolitons inner products, in the sequel when we refer to a nilpotent Lie algebra we will mean a nonabelian nilpotent Lie algebra.

In order to classify the Lie algebras \( n_i \) admitting a (nontrivial) nilsoliton but with no closed \( G_2 \) forms inducing the nilsoliton, we need to recall the following obstruction proved in [8] for the existence of a closed \( G_2 \)-structure on a 7-dimensional Lie algebra.

**Lemma 3.3** ([8]). Let \( g \) be a 7-dimensional Lie algebra. If there is a non-zero \( X \in g \) such that \( (\iota_X \phi)^3 = 0 \) (where \( \iota_X \) denotes the contraction by \( X \)) for every closed 3-form \( \phi \) on \( g \), then \( g \) has no closed \( G_2 \)-structures.

By [28, Proposition 4.5] if \( \phi \) is a \( G_2 \)-structure on a 7-dimensional Lie algebra and we choose a vector \( X \in g \) of length one with respect to \( g_\phi \), then on the orthogonal complement of the span of \( X \) one has an \( SU(3) \)-structure given by the 2-form \( \alpha = \iota_X \phi \) and the 3-form \( \beta = \phi - \alpha \wedge \eta \), where \( \eta = \iota_X (g_\phi) \). So in particular \( \alpha \wedge \beta = 0 \).

By using these results we can prove the following proposition

**Proposition 3.4** The Lie algebra \( n_i \) (\( i = 3, 5, 7, 8, 11 \)) has a nilsoliton inner product but no closed \( G_2 \)-structure inducing the nilsoliton inner product.

**Proof** To prove that \( n_3 \) has a nilsoliton, we consider the Lie algebra \( n_3 \) defined by the equations given in Theorem 3.1. Let \( \langle \cdot, \cdot \rangle_{n_3} \) be the inner product on \( n_3 \) such that \( \{e^1, \ldots, e^7\} \) is orthonormal. Then, \( \langle \cdot, \cdot \rangle_{n_3} \) is a nilsoliton because its Ricci tensor

\[ Ric = diag \left( -1, -1, -1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0 \right) \]
satisfies (3), for \( \lambda = -5/2 \) and

\[
D = \text{diag} \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 3, 3, 3, \frac{5}{2} \right).
\]

Since the nilsoliton inner product is unique (up to isometry and scaling) it suffices to prove that there is no closed \( G_2 \) form on \( n_3 \) inducing such an inner product.

Suppose that \( n_3 \) has a closed \( G_2 \) form \( \phi \) such that

\[
g_\phi = \langle \cdot, \cdot \rangle_{n_3} = \sum_{i=1}^{7} (e_i^l)^2.
\]

Thus, \( g_\phi \) has to satisfy

\[
\prod_{i=1}^{7} g_\phi(e_i, e_i) = 1.
\]

A generic closed 3-form \( \gamma \) on \( n_3 \) has the following expression

\[
\gamma = c_{123}e^{123} + c_{124}e^{124} + c_{125}e^{125} + c_{126}e^{126} + c_{127}e^{127} + c_{134}e^{134} + c_{135}e^{135} + c_{136}e^{136} + c_{137}e^{137} + c_{145}e^{145} + c_{146}e^{146} + c_{147}e^{147} + c_{156}e^{156} + c_{157}e^{157} + c_{167}e^{167} + c_{234}e^{234} + c_{235}e^{235} + c_{236}e^{236} + c_{237}e^{237} + c_{245}e^{245} + c_{246}e^{246} + c_{247}e^{247} + c_{256}e^{256} + c_{257}e^{257} + c_{267}e^{267} + c_{345}e^{345} + c_{256}e^{256} + (c_{257} - c_{167})e^{347} + c_{356}e^{356} + c_{357}e^{357} + c_{367}e^{367},
\]

where \( c_{ijk} \) are arbitrary real numbers.

Now, we show conditions on the coefficients \( c_{ijk} \) so that \( \phi = \gamma \) is a closed \( G_2 \) form such that \( g_\phi \) satisfies (4). To this end, we apply the aforementioned result of [28, Proposition 4.5] for \( X = e_i \) (\( 1 \leq i \leq 7 \)) and so \( \eta = e^i \) by (4). For \( X = e_1 \), thus \( \eta = e^1 \), we have

\[
\alpha_1 = \iota_{e_1} \phi = c_{123}e^{23} + c_{124}e^{24} + c_{125}e^{25} + c_{126}e^{26} + c_{127}e^{27} + c_{134}e^{34} + c_{135}e^{35} + c_{136}e^{36} + c_{137}e^{37} + c_{145}e^{45} + c_{146}e^{46} + c_{147}e^{47} + c_{156}e^{56} + c_{157}e^{57} + c_{167}e^{67},
\]

and

\[
\beta_1 = \phi - \iota_{e_1} \phi \wedge e^1 = c_{234}e^{234} + c_{235}e^{235} + c_{236}e^{236} + c_{237}e^{237} + c_{245}e^{245} + c_{246}e^{246} + c_{247}e^{247} + c_{256}e^{256} + c_{257}e^{257} + c_{267}e^{267} + c_{345}e^{345} + c_{256}e^{256} + (c_{257} - c_{167})e^{347} + c_{356}e^{356} + c_{357}e^{357} + c_{367}e^{367},
\]

But, \( \alpha_1 \wedge \beta_1 = 0 \) describes a system of 6 equations. Hence, after apply the result of [28, Proposition 4.5] for \( X = e_2, \ldots, e_7 \), we obtain a system of 42 equations. This system and condition (4) imply that any closed \( G_2 \) form on \( n_3 \) satisfying (5) is expressed as follows

\[
\phi = c_{123}e^{123} + c_{145}e^{145} + c_{167}e^{167} + c_{246}e^{246} + c_{257}e^{257} + (c_{257} - c_{167})e^{347} + c_{356}e^{356}.
\]
Because $\phi$ should be a closed $G_2$ form on $n_3$, at least for certain coefficients $c_{ijk}$, Lemma 3.3 implies that the coefficients appearing on (6) cannot vanish. In particular, $c_{257} - c_{167} \neq 0$. Now, denote by $G_\phi$ the matrix associated with the inner product on $n_3$ induced by the 3-form $\phi$ given by (6). Then, (4) implies that $G_\phi = I_7$, for some $c_{ijk}$ and then

$$S = G_\phi - I_7 = 0,$$

for those coefficients. From now on, we denote by $G_{\phi}$ the matrix associated with the inner product on $n_3$ induced by the 3-form $\phi$ given by (6). Then, (4) implies that $G_{\phi} = I_7$, for some $c_{ijk}$ and then

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for those coefficients. From now on, we denote by $G_{\phi}$ the matrix associated with the inner product on $n_3$ induced by the 3-form $\phi$ given by (6). Then, (4) implies that $G_{\phi} = I_7$, for some $c_{ijk}$ and then

$$S = G_{\phi} - I_7 = 0,$$

for those coefficients. From now on, we denote by $G_{\phi}$ the matrix associated with the inner product on $n_3$ induced by the 3-form $\phi$ given by (6). Then, (4) implies that $G_{\phi} = I_7$, for some $c_{ijk}$ and then

$$S = G_{\phi} - I_7 = 0,$$

for those coefficients. From now on, we denote by $G_{\phi}$ the matrix associated with the inner product on $n_3$ induced by the 3-form $\phi$ given by (6). Then, (4) implies that $G_{\phi} = I_7$, for some $c_{ijk}$ and then

$$S = G_{\phi} - I_7 = 0,$$

for those coefficients. From now on, we denote by $G_{\phi}$ the matrix associated with the inner product on $n_3$ induced by the 3-form $\phi$ given by (6). Then, (4) implies that $G_{\phi} = I_7$, for some $c_{ijk}$ and then

$$S = G_{\phi} - I_7 = 0,$$

for those coefficients. From now on, we denote by $G_{\phi}$ the matrix associated with the inner product on $n_3$ induced by the 3-form $\phi$ given by (6). Then, (4) implies that $G_{\phi} = I_7$, for some $c_{ijk}$ and then

$$S = G_{\phi} - I_7 = 0,$$

for those coefficients. From now on, we denote by $G_{\phi}$ the matrix associated with the inner product on $n_3$ induced by the 3-form $\phi$ given by (6). Then, (4) implies that $G_{\phi} = I_7$, for some $c_{ijk}$ and then

$$S = G_{\phi} - I_7 = 0,$$

for those coefficients. From now on, we denote by $G_{\phi}$ the matrix associated with the inner product on $n_3$ induced by the 3-form $\phi$ given by (6). Then, (4) implies that $G_{\phi} = I_7$, for some $c_{ijk}$ and then

$$S = G_{\phi} - I_7 = 0,$$

for those coefficients. From now on, we denote by $G_{\phi}$ the matrix associated with the inner product on $n_3$ induced by the 3-form $\phi$ given by (6). Then, (4) implies that $G_{\phi} = I_7$, for some $c_{ijk}$ and then

$$S = G_{\phi} - I_7 = 0,$$
\[ \gamma = c_{123}e^{123} + c_{124}e^{124} + c_{125}e^{125} + c_{126}e^{126} + c_{127}e^{127} + c_{134}e^{134} + c_{135}e^{135} + c_{136}e^{136} + c_{137}e^{137} + c_{145}e^{145} + c_{146}e^{146} + c_{147}e^{147} + c_{156}e^{156} + c_{157}e^{157} + c_{167}e^{167} + c_{234}e^{234} + c_{235}e^{235} + c_{236}e^{236} + c_{237}e^{237} + c_{245}e^{245} + \frac{1}{2}c_{237}e^{246} + c_{247}e^{247} - \frac{1}{2}\sqrt{3}c_{137}e^{256} + \sqrt{3}(c_{345} - c_{147})e^{257} + c_{345}e^{345} - c_{167}e^{356} + c_{457}e^{457}, \]

where \( c_{ijk} \) are arbitrary real numbers. Now we show conditions on the coefficients \( c_{ijk} \) so that \( \phi = \gamma \) is a closed \( G_2 \) form such that \( g_\phi = \langle \cdot, \cdot \rangle_n^5 \). Lemma 3.3 (applied for \( X = e_7 \)) implies that \( c_{167}c_{237}c_{457} \neq 0 \). (8)

Now, we denote by \( G_\phi \) the matrix associated with the inner product on \( n_5 \) induced by the generic closed 3-form \( \phi \). Then the condition \( g_\phi = \langle \cdot, \cdot \rangle_n^5 \) implies (7) for some coefficients \( c_{ijk} \). From the equations \( S_{66} = S_{77} = S_{67} = S_{37} = S_{46} = S_{33} = S_{36} = S_{47} = 0 \) we have that

\[
\begin{align*}
  c_{237} &= \frac{2}{c_{167}}, \\
  c_{457} &= \frac{1}{2}c_{167}, \\
  c_{236} &= -2c_{247}, \\
  c_{136} &= -2c_{147}, \\
  c_{345} &= 0, \\
  c_{134} &= \frac{1}{2}c_{167}, \\
  c_{137} &= 2c_{146}, \\
  c_{234} &= 0.
\end{align*}
\]

Therefore, \( S_{44} = -\frac{3}{8}c_{167}c_{237} \) which by (8) cannot vanish and so \( S \neq 0 \), which is a contradiction with (7).

Consider now the Lie algebra \( n_7 \) defined by the structure equations

\[
n_7 = \left( 0, 0, 0, e^{12}, \frac{\sqrt{6}}{2}e^{13}, e^{14} + \frac{\sqrt{6}}{2}e^{23}, \sqrt{2}e^{15} \right).
\]

Let \( \langle \cdot, \cdot \rangle_{n_7} \) be the inner product on \( n_7 \) such that the basis \( \{ e^1, \ldots, e^7 \} \) is orthonormal. Then, \( \langle \cdot, \cdot \rangle_{n_7} = \sum_{i=1}^7 (e^i)^2 \) is a nilsoliton since

\[
Ric = \left( -\frac{11}{4}, -\frac{5}{4}, -\frac{3}{2}, 0, -\frac{1}{4}, \frac{5}{4}, 1 \right) = -4I_7 + D,
\]

where

\[
D = \text{diag} \left( \frac{5}{4}, \frac{11}{4}, \frac{5}{2}, 4, \frac{15}{4}, \frac{21}{4}, 5 \right),
\]

is a derivation of \( n_7 \). As before, since the nilsoliton inner product is unique (up to isometry and scaling) it suffices to prove that there is no closed \( G_2 \) form on \( n_7 \) inducing such an inner product.
Suppose that \( n_7 \) has a closed \( G_2 \) form \( \phi \) such that \( g_\phi = \langle \cdot, \cdot \rangle_{n_7} \). A generic closed 3-form \( \gamma \) on \( n_7 \) has the following expression

\[
\gamma = c_{123}e^{123} + c_{124}e^{124} + c_{125}e^{125} + c_{126}e^{126} + c_{127}e^{127} + c_{134}e^{134} + c_{135}e^{135} + c_{136}e^{136} + c_{137}e^{137} + c_{145}e^{145} + c_{146}e^{146} + c_{147}e^{147} + c_{156}e^{156} + c_{157}e^{157} + c_{167}e^{167} + c_{234}e^{234} + c_{235}e^{235} + \left( \frac{\sqrt{6}}{2} c_{245} - \frac{\sqrt{6}}{2} c_{146} \right) e^{236} + c_{237}e^{237} + c_{245}e^{245} + c_{246}e^{246} + \frac{\sqrt{6}}{2} c_{256}e^{247} + c_{256}e^{256} + \left( c_{167} + \frac{\sqrt{6}}{3} c_{347} \right) e^{257} + \left( \frac{\sqrt{6}}{2} c_{156} + \sqrt{2} c_{237} \right) e^{345} + \frac{\sqrt{6}}{2} c_{256}e^{346} + c_{347}e^{347} + \sqrt{2} c_{347}e^{356} + c_{357}e^{357},
\]

where \( c_{ijk} \) are arbitrary real numbers. Now, we show conditions on the coefficients \( c_{ijk} \) so that \( \phi = \gamma \) be a closed \( G_2 \) form such that \( g_\phi = \langle \cdot, \cdot \rangle_{n_7} \). Lemma 3.3 applied for \( X = e_7 \) implies that

\[
c_{167} \neq 0. \tag{9}
\]

Now we apply the result of [28, Proposition 4.5] for \( X = e_i \) (1 \( \leq i \leq 7 \)) and so \( \eta = e^i \) by (4). For \( X = e_1 \), we have

\[
\alpha_1 = i_{e_1} \phi = c_{123}e^{23} + c_{124}e^{24} + c_{125}e^{25} + c_{126}e^{26} + c_{127}e^{27} + c_{134}e^{34} + c_{135}e^{35} + c_{136}e^{36} + c_{137}e^{37} + c_{145}e^{45} + c_{146}e^{46} + c_{147}e^{47} + c_{156}e^{56} + c_{157}e^{57} + c_{167}e^{67}
\]

and

\[
\beta_1 = \phi - i_{e_1} \phi \wedge e^1 = c_{234}e^{234} + c_{235}e^{235} + \left( \frac{\sqrt{6}}{2} c_{245} - \frac{\sqrt{6}}{2} c_{146} \right) e^{236} + c_{237}e^{237} + c_{245}e^{245} + c_{246}e^{246} + \frac{\sqrt{6}}{2} c_{256}e^{247} + c_{256}e^{256} + \left( c_{167} + \frac{\sqrt{6}}{3} c_{347} \right) e^{257} + \left( \frac{\sqrt{6}}{2} c_{156} + \sqrt{2} c_{237} \right) e^{345} + \frac{\sqrt{6}}{2} c_{256}e^{346} + c_{347}e^{347} + \sqrt{2} c_{347}e^{356} + c_{357}e^{357}.
\]

Therefore, \( \alpha_1 \wedge \beta_1 = 0 \) describes a system of 6 equations. Hence, after apply the result of [28, Proposition 4.5] for \( X = e_2, \ldots, e_7 \), we obtain a system of 42 equations. This system together with the fact that \( c_{167} \neq 0 \) and the condition \( g_\phi = \langle \cdot, \cdot \rangle_{n_7} \) imply that any closed \( G_2 \) form on \( n_7 \) satisfying (5) is expressed as follows

\[ 
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\]
\[ \phi = c_{123}e^{123} + c_{145}e^{145} + c_{167}e^{167} + c_{246}e^{246} + \left( c_{167} + \frac{\sqrt{6}}{3}c_{347} \right)e^{257} \\
+ c_{347}e^{347} + \sqrt{2}c_{347}e^{356}. \]  

Now we denote by \( G_\phi \) the matrix associated with the inner product on \( n_7 \) induced by the 3-form \( \phi \) given by (10). Then, the condition \( g_\phi = \langle \cdot, \cdot \rangle_{n_7} \) implies (7) is satisfied for some coefficients \( c_{ijk} \). From equations \( S_{11} = S_{33} = S_{44} = S_{66} = 0 \) we have

\[ c_{123} = \frac{\sqrt{2}}{2c_{347}^3}, \quad c_{145} = -\frac{\sqrt{2}}{c_{347}}, \quad c_{167} = -c_{347}, \quad \text{and} \quad c_{246} = \frac{\sqrt{2}}{2c_{347}^3}. \]

Therefore \( S_{55} = 1 \) and so \( S \neq 0 \) which is a contradiction with (7).

Let \( n_8 \) be the Lie algebra described by the structure equations

\[ n_8 = (0, 0, e^{12}, -e^{13}, -e^{23}, e^{15} + e^{24}, -e^{16} - e^{34}), \]

and let \( \langle \cdot, \cdot \rangle_{n_8} \) be the inner product on \( n_8 \) such that \( \{e^1, \ldots, e^7\} \) is orthonormal. Then, \( \langle \cdot, \cdot \rangle_{n_8} = \sum_{i=1}^7 (e^i)^2 \) is a nilsoliton because its Ricci tensor

\[ Ric = \text{diag} \left( -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1 \right) \]

satisfies (3), for \( \lambda = -\frac{5}{2} \) and

\[ D = \text{diag} \left( \frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2} \right). \]

The nilsoliton inner product is unique (up to isometry and scaling) therefore it suffices to prove that there is no closed \( G_2 \) form on \( n_8 \) inducing such an inner product. A generic closed 3-form \( \gamma \) on \( n_8 \) has the following expression

\[ \gamma = c_{123}e^{123} + c_{124}e^{124} + c_{125}e^{125} + c_{126}e^{126} + c_{127}e^{127} + c_{134}e^{134} + c_{135}e^{135} \]
\[ + \left( c_{167} + \frac{\sqrt{6}}{3}c_{347} \right)e^{257} \]
\[ + c_{347}e^{347} + \sqrt{2}c_{347}e^{356}. \]

where \( c_{ijk} \) are real numbers. Now, we show conditions on the coefficients \( c_{ijk} \) so that \( \phi = \gamma \) is a closed \( G_2 \) form such that \( g_\phi = \langle \cdot, \cdot \rangle_{n_8} \). We apply the result previously mentioned [28, Proposition 4.5] for \( X = e_i \) (1 ≤ \( i \) ≤ 7) and so \( \eta = e^i \) by the condition \( g_\phi = \langle \cdot, \cdot \rangle_{n_8} \). After solving the system of 42 equations we have that any closed \( G_2 \) form on \( n_8 \) satisfying (5) is expressed as follows
\[
\phi = c_{123}e^{123} + c_{124}e^{124} + c_{125}e^{125} + c_{126}e^{126} + c_{135}e^{135} + c_{136}e^{136} - c_{136}e^{145} + c_{234}e^{234} + c_{235}e^{235} + c_{236}e^{236} + c_{236}e^{245} + c_{256}e^{256}.
\]

(11)

Now denote by \(G_\phi\) the matrix associated with the inner product on \(n_8\) induced by the 3-form \(\phi\) given by (11). Then \(G_\phi = 0\) obtaining a contradiction with (7).

It only remains to study the Lie algebra \(n_{11}\). According to Theorem 3.1, \(n_{11}\) is defined by the equations

\[
n_{11} = \left(0, 0, f^{12}, 0, f^{13}, f^{24} + f^{23}, f^{25} + f^{34} + f^{15} + f^{16} - 3f^{26}\right).
\]

We consider the new basis \(\{e^j\}^7_{j=1}\) of \(n_{11}'\) with

\[
\begin{align*}
e^1 &= f^2, e^2 = -\frac{\sqrt{3}}{3} f^1, e^3 = \frac{\sqrt{39}}{39} f^3 + \frac{\sqrt{39}}{78} f^4, e^4 = -\frac{\sqrt{78}}{78} f^4, \\
e^5 &= \frac{\sqrt{3}}{39} f^6, e^6 = -\frac{1}{3} f^5, e^7 = -\frac{\sqrt{3}}{1014} f^7.
\end{align*}
\]

Thus, the Lie algebra \(n_{11}\) can also be described by the structure equations

\[
n_{11} = \left(0, 0, \frac{\sqrt{13}}{13} e^{12}, 0, \frac{\sqrt{13}}{13} e^{13} - \frac{\sqrt{26}}{26} e^{14}, \frac{\sqrt{26}}{26} e^{24} + \frac{\sqrt{13}}{13} e^{23}, \frac{\sqrt{13}}{26} e^{25} + \frac{\sqrt{26}}{26} e^{34} + \frac{\sqrt{39}}{26} e^{15} + \frac{\sqrt{13}}{26} e^{16} - \frac{\sqrt{39}}{26} e^{26}\right).
\]

Let \(\langle \cdot, \cdot \rangle_{n_{11}}\) be the inner product on \(n_{11}\) such that \(\{e^1, \ldots, e^7\}\) is orthonormal. Then, \(\langle \cdot, \cdot \rangle_{n_{11}} = \sum_{i=1}^7 (e^i)^2\) is a nilsoliton because its Ricci tensor

\[
Ric = \frac{1}{52} diag(-7, -7, -3, -3, 1, 1, 5)
\]

satisfies \(Ric = -\frac{11}{52} Id + D\), where \(D\) is the derivation of the Lie algebra \(n_{11}\) given by

\[
D = \frac{1}{13} diag(1, 1, 2, 2, 3, 3, 4).
\]

It suffices to prove that there is no closed \(G_2\) form on \(n_{11}\) inducing such an inner product. Let’s suppose that \(n_{11}\) has a closed \(G_2\) form \(\phi\) such that \(g_\phi = \langle \cdot, \cdot \rangle_{n_{11}}\). A generic closed 3-form \(\gamma\) on \(n_{11}\) has the following expression

\[
\gamma = c_{123}e^{123} + c_{124}e^{124} + c_{125}e^{125} + c_{126}e^{126} + c_{127}e^{127} + c_{134}e^{134} + c_{135}e^{135} + c_{136}e^{136} + c_{137}e^{137} + c_{145}e^{145} + c_{146}e^{146} + c_{147}e^{147} + c_{156}e^{156} - \frac{\sqrt{3}}{2} c_{347}e^{157} + \frac{c_{347}e^{167}}{\sqrt{2}} + c_{234}e^{234} + c_{235}e^{235} + c_{236}e^{236} + \left(\frac{c_{137}}{\sqrt{3}} - \frac{2c_{156}}{\sqrt{3}}\right) e^{237}
\]

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such that the Ricci tensor of the inner product

\[ g_{i}^{cijk} \]

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Equations $S_{66} = S_{77} = 0$ imply that

\[ c_{246} = -\frac{1}{2} c_{347}, \quad \text{and} \quad c_{347} = 2^{-1/3}. \]

Therefore, $S_{44} = -\frac{1}{2}$ and so $S \neq 0$ which contradicts (7). \hfill \Box

Remark 3.5 Note that the 4-step nilpotent Lie algebra $n_{10}$ is isomorphic in the classification given in [11] to the Lie algebra $L.3(i)\{\lambda = 1\}$ and the existence of the nilsoliton was shown in [9, Example 2]. Since an explicit expression of the nilsoliton is not known, we cannot apply the argument used in the proof of Proposition 3.4. Thus, it remains open the question of whether the Lie algebra $n_{10}$ admits a closed $G_2$ form inducing a nilsoliton or not. Moreover, the explicit expression of the nilsolitons for $n_{11}$ and $n_{12}$ have been already determined in [11] (see there page 20, Remark 3.5), but our basis is different for the nilsoliton on the other Lie algebras.

Theorem 3.6 Up to isomorphism, $n_{2}$, $n_{4}$, $n_{6}$ and $n_{12}$ are the unique $s$-step nilpotent Lie algebras ($s = 2$, 3) with a nilsoliton inner product determined by a closed $G_2$-structure.

Proof We will show that the Lie algebra $n_{i}$ ($i = 2, 4, 6, 12$) has a closed $G_2$ form $\varphi_{i}$ such that the Ricci tensor of the inner product $g_{\varphi_{i}}$ satisfies (3), for some derivation $D$ of $n_{i}$ and some real number $\lambda$.

For $n_{2}$ we consider the closed $G_2$ form $\varphi_{2}$ defined by

\[ \varphi_{2} = e^{147} + e^{267} + e^{357} + e^{123} + e^{156} + e^{245} - e^{346}. \]
The inner product $g_{\phi_2}$ given by (1) is the one making orthonormal the basis \{$e^1, \ldots, e^7$\}, and it is a nilsoliton since $Ric = -2I_7 + D$, where

$$D = \text{diag} \left( 1, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}, 2 \right)$$

is a derivation of $n_2$.

On the Lie algebra $n_4$, we define the $G_2$ form $\phi_4$ by

$$\phi_4 = -e^{124} - e^{456} + e^{347} + e^{135} + e^{167} + e^{257} - e^{236}. \quad (14)$$

Then, $\phi_4$ is closed, the inner product $g_{\phi_4}$ makes the basis \{$e^1, \ldots, e^7$\} orthonormal and $g_{\phi_4}$ is a nilsoliton since $Ric = -\frac{5}{2}I_7 + D$, where $D$ is the derivation of $n_4$ given by

$$D = \text{diag} \left( 1, \frac{3}{2}, 2, 2, 7, 3, 3 \right).$$

For the Lie algebra $n_6$ we consider the closed $G_2$-structure defined by the 3-form

$$\phi_6 = e^{123} + e^{145} + e^{167} + e^{257} - e^{246} + e^{347} + e^{356}. \quad (15)$$

Therefore, the inner product $g_{\phi_6}$ is such that the basis \{$e^1, \ldots, e^7$\} is orthonormal and it is a nilsoliton since $Ric = -\frac{5}{2}I_7 + D$, where $D$ is the derivation of $n_6$ given by

$$D = \text{diag} \left( \frac{1}{2}, 2, 2, \frac{5}{2}, 3, 3 \right).$$

Theorem 3.1 implies that the Lie algebra $n_{12}$ is defined by the equations

$$n_{12} = \left( 0, 0, h^{12}, h^{23}, -h^{13}, 2h^{26} - 2h^{34} - 2h^{16} + 2h^{25} \right).$$

We consider the basis \{$e^i\}_{i=1}^7$ of $n_{12}^*$ given by

$$\left\{ e^1 = \frac{\sqrt{3}}{2} h^2, e^2 = h^1 - \frac{1}{2} h^2, e^3 = h^3, e^4 = -\frac{1}{4} h^4, e^5 = \frac{1}{4} h^5 + \frac{1}{4} h^6, \\
e^6 = -\frac{\sqrt{3}}{12} h^5 + \frac{\sqrt{3}}{12} h^6, e^7 = -\frac{\sqrt{3}}{48} h^7 \right\}. \quad (16)$$

Then, $n_{12}$ is defined as follows

$$n_{12} = \left( 0, 0, 0, \frac{\sqrt{3}}{6} e^{12}, -\frac{1}{4} e^{23} + \frac{\sqrt{3}}{12} e^{13}, -\frac{\sqrt{3}}{12} e^{23} - \frac{1}{4} e^{13}, \\
-\frac{\sqrt{3}}{6} e^{34} + \frac{\sqrt{3}}{12} e^{25} + \frac{1}{4} e^{26} + \frac{\sqrt{3}}{12} e^{16} - \frac{1}{4} e^{15} \right).$$
We define the $G_2$ form $\varphi_{12}$ by
\[
\varphi_{12} = -e^{124} + e^{135} + e^{167} - e^{236} + e^{257} + e^{347} - e^{456}.
\] (17)

Clearly $\varphi_{12}$ is closed. Moreover, $\varphi_{12}$ defines the inner product $g_{\varphi_{12}}$ which makes the basis $\{e^1, \ldots, e^7\}$ orthonormal, and $g_{\varphi_{12}}$ is a nilsoliton since $Ric = -\frac{1}{4}Id + \frac{1}{8}D$, where $D$ is the derivation of $n_{12}$ given by
\[
D = diag(1, 1, 1, 2, 2, 2, 3).
\]

4 Laplacian Flow

Let us consider the nilpotent Lie algebra $n_{i}$ ($i = 2, 4, 6$) defined in Theorem 3.1, and the Lie algebra $n_{12}$ defined by (16). Let $N_{i}$ be the simply connected nilpotent Lie group with Lie algebra $n_{i}$, and let $\varphi_{i}$ be the closed $G_2$ form on $N_{i}$ ($i = 2, 4, 6, 12$) given by (13), (14), (15) and (17), for $i = 2, 4, 6$ and 12, respectively.

The purpose of this section is to prove long time existence and uniqueness of solution for the Laplacian flow of $\varphi_{i}$ on $N_{i}$, and that the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in $N_{i}$, as $t$ goes to infinity.

Let $M$ be a 7-dimensional manifold with an arbitrary $G_2$ form $\varphi$. The Laplacian flow of $\varphi$ is defined to be
\[
\begin{align*}
\frac{d}{dt} \varphi(t) &= \Delta_{t} \varphi(t), \\
\varphi(0) &= \varphi,
\end{align*}
\] (18)

In the case of closed $G_2$-structures on compact manifolds, Bryant and Xu [4] gave a result of short time existence and uniqueness of solution.

**Theorem 4.1** [4] If $M$ is compact, then (18) has a unique solution for a short time $0 \leq t < \epsilon$, with $\epsilon$ depending on $\varphi = \varphi(0)$. 
In the following theorem we determine a global solution of the Laplacian flow of the closed $G_2$ form $\varphi_2$ on $N_2$.

**Theorem 4.2** The family of closed $G_2$ forms $\varphi_2(t)$ on $N_2$ given by

$$\varphi_2(t) = e^{147} + e^{267} + e^{357} + f(t)^3 e^{123} + e^{156} + e^{245} - e^{346}, \quad t \in \left(-\frac{3}{10}, +\infty\right),$$  

is the solution of the Laplacian flow (18) of $\varphi_2$, where $f = f(t)$ is the function

$$f(t) = \left(\frac{10}{3}t + 1\right)^{\frac{1}{5}}.$$  

Moreover, the underlying metrics $g(t)$ of this solution converge smoothly, up to pullback by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in $N_2$, as $t$ goes to infinity.

**Proof** Let $f_i = f_i(t) (i = 1, \ldots, 7)$ be some differentiable real functions depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1$ and $f_i(t) \neq 0$, for any $t \in I$, where $I$ is a real open interval. For each $t \in I$, we consider the basis $\{x^1, \ldots, x^7\}$ of left invariant 1-forms on $N_2$ defined by

$$x^i = x^i(t) = f_i(t) e^i, \quad 1 \leq i \leq 7.$$  

From now on we write $f_{ij} = f_{ij}(t) = f_i(t) f_j(t)$, $f_{ijk} = f_{ijk}(t) = f_i(t) f_j(t) f_k(t)$, and so forth. Then, the structure equations of $N_2$ with respect to this basis are

$$dx^i = 0, \quad i = 1, 2, 3, 4, 7, \quad dx^5 = \frac{f_5}{f_{12}} x^{12}, \quad dx^6 = \frac{f_6}{f_{13}} x^{13}. \quad (20)$$

Now, for any $t \in I$, we consider the $G_2$ form $\varphi_2(t)$ on $N_2$ given by

$$\varphi_2(t) = x^{147} + x^{267} + x^{357} + x^{123} + x^{156} + x^{245} - x^{346}$$

$$= f_{147} e^{147} + f_{267} e^{267} + f_{357} e^{357} + f_{123} e^{123} + f_{156} e^{156} + f_{245} e^{245} - f_{346} e^{346}, \quad (21)$$

Note that $\varphi_2(0) = \varphi_2$ and, for any $t$, the 3-form $\varphi_2(t)$ on $N_2$ determines the metric $g_t$ such that the basis $\{x_i = \frac{1}{f_i} e_i; \ i = 1, \ldots, 7\}$ of $n_2$ is orthonormal. So,

$$g(t)(e_i, e_i) = f_i^2.$$  

Using (20), one can check that $d\varphi_2(t) = 0$ if and only if

$$f_{26}(t) = f_{35}(t), \quad (22)$$

for any $t$. Assuming $f_i(0) = 1$ and (22), to solve the flow (18) of $\varphi_2$, we need to determine the functions $f_i$ and the interval $I$ so that $\frac{d}{dt} \varphi_2(t) = \Delta_t \varphi_2(t)$, for $t \in I$. Using (21) we have
\[
\frac{d}{dt} \varphi_2(t) = \left( f_{147} \right)' e^{147} + \left( f_{267} \right)' e^{267} + \left( f_{357} \right)' e^{357} + \left( f_{123} \right)' e^{123} \\
+ \left( f_{156} \right)' e^{156} + \left( f_{245} \right)' e^{245} - \left( f_{346} \right)' e^{346}.
\]  

(23)

Now, we calculate \( \Delta_t \varphi_2(t) = - \frac{d}{dt} \star_t \varphi_2(t) \). On the one hand, we have

\[
\star_t \varphi_2(t) = x^{2356} - x^{1345} - x^{1246} + x^{4567} + x^{2347} - x^{1367} + x^{1257}.
\]  

(24)

So, \( x^{4567} \) is the unique nonclosed summand in \( \star_t \varphi_2(t) \). Then, taking into account (22), we obtain

\[
d(\star_t d \star_t \varphi_2(t)) = \frac{f_6}{f_{13}} \left( - \frac{f_5}{f_{12}} x^{123} - \frac{f_6}{f_{13}} x^{123} \right) = -2 \left( \frac{f_6}{f_{13}} \right)^2 x^{123}.
\]

Therefore, in terms of the forms \( e^{ijk} \), the expression of \( -d(\star_t d \star_t \varphi_2(t)) \) is

\[
- d(\star_t d \star_t \varphi_2(t)) = 2 f_{123} \left( \frac{f_6}{f_{13}} \right)^2 e^{123} = 2 \left( \frac{f_2 (f_6)^2}{f_{13}} \right) e^{123}.
\]  

(25)

Comparing (23) and (25) we see that, in particular, \( f_{156}(t) = 1 \), for any \( t \in I \). Then, using (22), we have

\[
\frac{f_2 (f_6)^2}{f_{13}} = \frac{1}{(f_1)^2}.
\]

This equality and (25) imply that \( -d(\star_t d \star_t \varphi_2(t)) \) can be expressed as follows

\[
- d(\star_t d \star_t \varphi_2(t)) = 2 \frac{1}{(f_1)^2} e^{123}.
\]  

(26)

Then, from (23) and (26) we have that \( \frac{d}{dt} \varphi_2(t) = \Delta_t \varphi_2(t) \) if and only if the functions \( f_i(t) \) satisfy the following system of differential equations

\[
\left( f_{147} \right)' = \left( f_{267} \right)' = \left( f_{357} \right)' = \left( f_{156} \right)' = \left( f_{245} \right)' = \left( f_{346} \right)' = 0, \\
\left( f_{123} \right)' = 2 \frac{1}{(f_1)^2}.
\]

(27)

Because \( \varphi_2(0) = \varphi_2 \), the equations in the first line of (27) imply

\[
f_{147}(t) = f_{267}(t) = f_{357}(t) = f_{156}(t) = f_{245}(t) = f_{346}(t) = 1,
\]  

(28)

for any \( t \in I \). From the equations (28) we obtain

\[
f_1^2 = f_2^2 = f_3^2.
\]
Let us consider $f = f_1 = f_2 = f_3$. Using again (28) we have

$$f_i(t) = \left( f(t) \right)^{-\frac{1}{2}}, \quad i = 4, 5, 6, 7.$$ 

Now, the last equation of (27) implies that $f^4 f' = \frac{2}{3}$. Integrating this equation, we obtain

$$f^5 = \frac{10}{3} t + B, \quad B = \text{constant}.$$ 

But $\varphi_2(0) = \varphi_2$ implies $f^3(0) = f_{123}(0) = 1$, that is, $B = 1$. Hence,

$$f(t) = \left( \frac{10}{3} t + 1 \right)^{\frac{5}{2}},$$

and so the one-parameter family of 3-forms $\{\varphi_2(t)\}$ given by (19) is the solution of the Laplacian flow of $\varphi_2$ on $N_2$, and it is defined for every $t \in (-\frac{3}{10}, +\infty)$.

To complete the proof, we study the behavior of the underlying metric $g(t)$ of such a solution in the limit for $t \to +\infty$. Indeed, if we think of the Laplacian flow as a one-parameter family of $G_2$ manifolds with a closed $G_2$-structure, it can be checked that, in the limit, the resulting manifold has vanishing curvature. For every $t \in (-\frac{3}{10}, +\infty)$, denote by $g(t)$ the metric on $N_2$ induced by the $G_2$ form $\varphi_2(t)$ given by (19). Then,

$$g(t) = \left( \frac{10}{3} t + 1 \right)^{2/5} (e^1)^2 + \left( \frac{10}{3} t + 1 \right)^{2/5} (e^2)^2 + \left( \frac{10}{3} t + 1 \right)^{2/5} (e^3)^2$$

$$+ \left( \frac{10}{3} t + 1 \right)^{-1/5} (e^4)^2 + \left( \frac{10}{3} t + 1 \right)^{-1/5} (e^5)^2 + \left( \frac{10}{3} t + 1 \right)^{-1/5} (e^6)^2$$

$$+ \left( \frac{10}{3} t + 1 \right)^{-1/5} (e^7)^2.$$ 

Concretely, taking into account the symmetry properties of the Riemannian curvature $R(t)$ we obtain

$$R_{1212} = R_{1313} = -\frac{3}{4(1 + \frac{10}{3} t)},$$

$$R_{1515} = R_{1616} = R_{3636} = R_{2525} = \frac{1}{4(1 + \frac{10}{3} t)},$$

$$R_{2356} = -\frac{1}{4(1 + \frac{10}{3} t)}; \quad R_{ijkl} = 0 \quad \text{otherwise},$$

where $R_{ijkl} = R(t)(e_i, e_j, e_k, e_l)$. Therefore, $\lim_{t \to +\infty} R(t) = 0$. \qed

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Remark 4.3 Note that, for every $t \in \left(-\frac{3}{10}, +\infty\right)$, the metric $g(t)$ is a nilsoliton on the Lie algebra $n_2$ of $N_2$. In fact, with respect to the orthonormal basis $(x_1(t), \ldots, x_7(t))$, we have

$$Ric(g(t)) = -\frac{6}{(3+10t)}Id + \frac{3}{(3+10t)}diag\left(1, \frac{3}{2}, 2, \frac{3}{2}, 2, \frac{5}{2}, 2\right)$$

$$= \frac{3}{(3+10t)}Ric(g(0))$$

with $\frac{3}{(3+10t)}diag\left(1, \frac{3}{2}, 2, \frac{3}{2}, 2, \frac{5}{2}, 2\right)$ a derivation of $n_2$ for every $t$.

Remark 4.4 The limit can be also computed fixing the $G_2$-structure and changing the Lie bracket as in [23]. We evolve the Lie brackets $\mu(t)$ instead of the 3-form defining the $G_2$-structure and we can show that the corresponding bracket flow has a solution for every $t$. Indeed, if we fix on $\mathbb{R}^7$ the 3-form $x^{147}+x^{267}+x^{357}+x^{123}+x^{156}+x^{245}-x^{346}$, the basis $(x_1(t), \ldots, x_7(t))$ defines for every positive $t$ a nilpotent Lie algebra with bracket $\mu(t)$ such that $\mu(0)$ is the Lie bracket of $n_2$. Moreover, the solution converges to the null bracket corresponding to the abelian Lie algebra.

In order to prove long time existence of solution for the Laplacian flow (18) of the closed $G_2$ form $\varphi_4$ on $N_4$, we need to study the (nonlinear) system of ordinary differential equations

$$\begin{cases}
  u' = +\frac{2}{3} \frac{2-u^3}{u^3 v^3}, \\
  v' = -\frac{2}{3} \frac{1-2u^3}{u^4 v^2},
\end{cases} \quad (29)$$

with initial conditions

$$u(0) = v(0) = 1, \quad (30)$$

where $u = u(t)$ and $v = v(t)$ are differentiable real functions such that are both positive. Note that the first equation of (29) implies that $u' > 0$ since $u(0) = 1$, $u = u(t) > 0$ and $v = v(t) > 0$. Moreover, we note also that the functions at the second member of (29) are $C^\infty$ in the domain

$$\Omega = \{(u, v) \in \mathbb{R}^2 \mid 0 < u < 2^{1/3}, \ v > 0\},$$

in the phase plane. Then, for every point $(u_0, v_0) \in \Omega$, there exists a unique maximal solution $(u, v)$, which has $(u_0, v_0)$ as initial condition, and with existence domain a certain open interval $I$ such that either

$$\lim_{t \to \inf I} \left(\frac{u(t)}{2} + v(t)^2\right) = +\infty,$$

or

$$\lim_{t \to \inf I} (u(t), v(t)) \in \partial \Omega,$$
and either
\[
\lim_{t \to \sup I} \left( u(t)^2 + v(t)^2 \right) = +\infty,
\]
or
\[
\lim_{t \to \sup I} (u(t), v(t)) \in \partial \Omega,
\]
where \(\partial \Omega\) denotes the boundary of \(\Omega\).

**Proposition 4.5** The maximal solution \((u(t), v(t))\) of (29), satisfying the initial conditions (30), belongs to the trajectory of equation
\[
v = \frac{1}{\sqrt{u(2 - u^3)}}, \tag{31}
\]

**Proof** From (29) we obtain
\[
\frac{dv}{du} = -\frac{v(1 - 2u^3)}{u(2 - u^3)},
\]
that is,
\[
\frac{dv}{v} = -\frac{1 - 2u^3}{u(2 - u^3)} \, du.
\]
Integrating this equation and using (30), we have
\[
\log v = \log(u(2 - u^3)^{-1/2}).
\]
Therefore,
\[
v = \frac{1}{\sqrt{u(2 - u^3)}}.
\]

As a consequence we have the following corollary.

**Corollary 4.6** The maximal solution of (29)–(30),
\[
I \ni t \mapsto (u(t), v(t)) \in \Omega
\]
parameterizes the whole curve (31). Moreover, the maximal solution is defined in the interval
\[
I = (t_{\min}, +\infty),
\]
where
\[
t_{\min} = -\frac{3}{2} \int_{0}^{1} \frac{x^{3/2}}{(2 - x^3)^{5/2}} \, dx, \tag{32}
\]
and
\[
\begin{align*}
\lim_{t \to t_{\text{min}}} u(t) &= 0, \\
\lim_{t \to t_{\text{min}}} v(t) &= +\infty, \\
\lim_{t \to +\infty} u(t) &= 2^{1/3}, \\
\lim_{t \to +\infty} v(t) &= +\infty.
\end{align*}
\]

**Proof** Let \( I = (t_{\text{min}}, t_{\text{max}}) \) the existence interval of the maximal solution \((u(t), v(t))\) of (29) satisfying the initial conditions (30). Using the previous proposition and the first equation of (29) we see that
\[
v(t) = (2u(t) - u(t)^4)^{-1/2}, \quad u'(t) = -\frac{2u(t)^3 - 4}{3u(t)^3v(t)^3},
\]
which imply
\[
u'(t) = \frac{2}{3} \frac{(2 - u(t)^3)^{5/2}}{u(t)^{3/2}}.
\]
We define the functions \(x(t)\) and \(f(x)\) by
\[
x(t) = u(t), \quad f(x) = \frac{2}{3} \frac{(2 - x^3)^{5/2}}{x^{3/2}}.
\]
In order to find \(t_{\text{max}}\), we can use that \(\frac{dx}{dt} = f(x(t))\) or, equivalently,
\[
\frac{dx}{f(x)} = dt.
\]
So, in particular, we have
\[
\frac{dt}{dx} = \frac{3}{2} x^{3/2} (2 - x^3)^{-5/2}.
\]
Note that the function \(\frac{3}{2} x^{3/2} (2 - x^3)^{-5/2}\) is increasing from 0, for \(x = 0\), to \(+\infty\), for \(x = 2^{1/3}\). Then, integrating \(\frac{dx}{f(x)} = dt\) between \(t_{\text{min}}\) and 0, and using that \(x(t_{\text{min}}) = 0\) and \(x(0) = 1\), we have that \(t_{\text{min}}\) is finite and equal to the real number
\[
t_{\text{min}} = -\frac{3}{2} \int_0^1 x^{3/2} (2 - x^3)^{-5/2} dx.
\]
Similarly, in order to find \(t_{\text{max}}\) we integrate again \(\frac{dx}{f(x)} = dt\) between 0 and \(t_{\text{max}}\). Since \(x(t_{\text{max}}) = 2^{1/3}\) we get
\[
t_{\text{max}} = -\frac{3}{2} \int_1^{2^{1/3}} x^{3/2} (2 - x^3)^{-5/2} dx,
\]
which implies that \(t_{\text{max}}\) is \(+\infty\) because this integral is not defined in \(x = 2^{1/3}\). \(\square\)
Theorem 4.7 There exists a solution \( \varphi_4(t) \) of the Laplacian flow of \( \varphi_4 \) on \( N_4 \) defined in the interval \( I = (t_{\text{min}}, +\infty) \), where \( t_{\text{min}} \) is the negative real number given by the elliptic integral

\[
t_{\text{min}} = -\frac{3}{2} \int_0^1 x^{3/2} (2 - x^3)^{-5/2} dx.
\]

Moreover, the underlying metrics \( g(t) \) of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in \( N_4 \), as \( t \) goes to infinity.

Proof Let us consider some differentiable real functions \( f_i = f_i(t) \) (\( i = 1, \ldots, 7 \)) and \( h_j = h_j(t) \) (\( j = 1, 2, 3 \)) depending on a parameter \( t \in I \subset \mathbb{R} \) such that \( f_i(0) = 1, h_j(0) = 0 \) and \( f_i(t) \neq 0 \), for any \( t \in I \) and for any \( i \) and \( j \). For each \( t \in I \), we consider the basis \( \{x^1, \ldots, x^7\} \) of left invariant 1-forms on \( N_4 \) defined by

\[
\begin{align*}
x^i &= x^i(t) = f_i(t) e^i, \quad 1 \leq i \leq 4, \quad x^5 &= x^5(t) = f_5(t) e^5 + h_1(t) e^1, \\
x^6 &= x^6(t) = f_6(t) e^6 + h_2(t) e^2, \quad x^7 &= x^7(t) = f_7(t) e^7 + h_3(t) e^4.
\end{align*}
\]

The structure equations of \( N_4 \) with respect to this basis are

\[
\begin{align*}
dx^i &= 0, \quad i = 1, 2, 4, 5, \quad dx^3 = \frac{f_3}{f_{12}} x^{12}, \\
dx^6 &= \frac{f_6}{f_{13}} x^{13} + \frac{f_6}{f_{24}} x^{24}, \quad dx^7 = \frac{f_7}{f_{15}} x^{15}.
\end{align*}
\]

For any \( t \in I \), we define the \( G_2 \) form \( \varphi_4(t) \) on \( N_4 \) by

\[
\varphi_4(t) = -x^{124} - x^{456} + x^{347} + x^{135} + x^{167} + x^{257} - x^{236}
\]

\[
= \left(-f_{124} - f_{4h_12} - f_{2h_13} + f_{1h_23}\right) e^{124} - f_{456} e^{456} + f_{347} e^{347} + f_{135} e^{135} + f_{167} e^{167} + f_{257} e^{257} - f_{236} e^{236} + \left( f_{4h_1} - f_{1h_3}\right) e^{146} + \left(-f_{27h_1} + f_{17h_2}\right) e^{127}.
\]

Clearly \( \varphi_4(0) = \varphi_4 \) since \( f_i(0) = 1 \) and \( h_j(0) = 0 \). Moreover, using (33) and (34), one can check that \( d\varphi_4(t) = 0 \) if and only if

\[
f_{16}(t) = f_{34}(t), \quad f_{37}(t) = f_{56}(t),
\]

for any \( t \).
To study the flow (18) of \( \varphi_4 \), we need to determine the functions \( f_i, h_j \) and the interval \( I \) so that \( \frac{d}{dt} \varphi_4(t) = \Delta_t \varphi_4(t) \), for \( t \in I \). On the one hand, using (34) we have

\[
\frac{d}{dt} \varphi_4(t) = \left( -f_{124} - f_4 h_{12} - f_2 h_{13} + f_1 h_{23} \right) e^{124} - \left( f_{456} \right)' e^{456} + \left( f_{347} \right)' e^{347} + \left( f_{135} \right)' e^{135} + \left( f_{167} \right)' e^{167} + \left( f_{257} \right)' e^{257} - \left( f_{236} \right)' e^{236} + \left( f_{46} h_1 - f_{16} h_3 \right)' e^{146} - \left( f_{45} h_2 + f_{25} h_3 \right)' e^{245} + \left( -f_{27} h_1 + f_{17} h_2 \right)' e^{127}.
\]

(35)

On the other hand,

\[
*_* \varphi_4(t) = x^{3567} + x^{1237} + x^{1256} - x^{2467} + x^{2345} + x^{1457} + x^{1346}.
\]

So, \( x^{3567} \) and \( x^{2467} \) are the nonclosed summands in \(*_* \varphi_4(t)\).

Then, for \( \Delta_t \varphi_4(t) = -d \ *_* \varphi_4(t) \) we obtain

\[
\Delta_t \varphi_4(t) = - \left( f_{124} \left( f_2^2 f_1^2 - f_2^2 f_2^2 \right) + f_6^2 \overline{\frac{f_3 h_3}{f_1 h_3}} - f_6^2 h_1 - f_6^2 h_3 \right) e^{124} + f_{135} \left( f_2^2 f_1^2 + f_2^2 f_2^2 \right) e^{135} + f_5 f_6^2 f_1^2 f_3 e^{245} + f_3 f_6^2 f_1^2 f_3 e^{127}.
\]

(36)

Comparing (35) and (36) we see that the functions \( f_i, h_1 \) and \( h_3 \) satisfy

\[
f_{167}(t) = f_{236}(t) = f_{257}(t) = f_{347}(t) = f_{456}(t) = 1, \quad f_{46}(t) h_1(t) - f_{16}(t) h_3(t) = 0,
\]

for any \( t \in I \). But these equations are satisfied if

\[
f_1 = f_{23}^2, \quad f_4 = f_2, \quad f_5 = f_3, \quad f_6 = f_7 = \frac{1}{f_{23}}, \quad h_1 = f_2 f_3 h_3.
\]

(37)

Using (37), we write (35) and (36) in terms of \( f_i, h_1 \) and \( h_3 \). Then, we see that

\[
\frac{d}{dt} \varphi_4(t) = \Delta_t \varphi_4(t) \]

if and only if

\[
f_1 = u \cdot v, \quad f_2 = f_4 = v^{1/2}, \quad f_3 = f_5 = u^{1/2}, \quad f_6 = f_7 = (uv)^{-1/2},
\]

\[
h_1 = \frac{1}{2} u^{5/2} v - \frac{1}{2} u^{1/2}, \quad h_2 = 0, \quad h_3 = \frac{1}{2} u^{3/2} v^{1/2} - \frac{1}{2} (uv)^{-1/2}.
\]

(38)

where \( u = u(t) \) and \( v = v(t) \) are differentiable real functions satisfying the system of ordinary differential equations (29) with initial conditions (30). By Corollary 4.6, we know that the system (29)–(30) has a solution \( u = u(t), v = v(t) \) defined in \( I = (t_{min}, +\infty) \). Then, taking into account (34) and (38), the family of closed \( G_2 \) forms \( \varphi_4(t) \) solving (18) for \( \varphi_4 \) is given by
\[
\varphi_4(t) = \frac{1}{4}e^{124}\left(-u^4v^2 + 2u^2v - 4uv^2 - 1 \right) + \frac{1}{2}e^{127}(u^2v - 1) + u^2ve^{135} + e^{167} - e^{236} + \frac{1}{2}e^{245}(u^2v - 1) + e^{257} + e^{347} - e^{456},
\]

for \(t \in (t_{\min}, +\infty)\). The underlying metric \(g(t)\) of this solution converges to a flat metric. To check that the corresponding manifold in the limit is flat, we note that all non-vanishing coefficients of the Riemannian curvature \(R(t)\) of \(g(t)\) are proportional to the function \(2u(t) - u^4(t)\). According with Corollary 4.6, we have that the function \(u(t)\) satisfies

\[
\lim_{t \to +\infty} u(t) = 2^{1/3},
\]

and so

\[
\lim_{t \to +\infty} R(t) = 0.
\]

\(\Box\)

Concerning the Laplacian flow (18) of the closed \(G_2\) form \(\varphi_6\) on \(N_6\) we have the following.

**Theorem 4.8** The Laplacian flow of \(\varphi_6\) has a solution \(\varphi_6(t)\) on \(N_6\) defined in the interval \(I = (t_{\min}, +\infty)\), where \(t_{\min}\) is the negative real number given by (32). Moreover, the underlying metrics \(g(t)\) of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in \(N_6\), as \(t\) goes to infinity.

**Proof** We take differentiable real functions \(f_i = f_i(t)\) \((i = 1, \ldots, 7)\) and \(h_j = h_j(t)\) \((j = 1, 2)\) depending on a parameter \(t \in I \subset \mathbb{R}\) such that \(f_i(0) = 1, h_j(0) = 0\) and \(f_i(t) \neq 0\), for any \(t \in I\) and for any \(i\) and \(j\). Now, for each \(t \in I\), we consider the basis \(\{x^1, \ldots, x^7\}\) of left invariant 1-forms on \(N_6\) defined by

\[
x^i = x^i(t) = f_i(t)e^i, \quad 1 \leq i \leq 5,
\]

\[
x^6 = x^6(t) = f_6(t)e^6 + h_1(t)e^2,
\]

\[
x^7 = x^7(t) = f_7(t)e^7 + h_2(t)e^3.
\]

For any \(t \in I\), let \(\varphi_6(t)\) the \(G_2\) form on \(N_6\) defined by

\[
\varphi_6(t) = x^{123} + x^{145} + x^{167} + x^{257} - x^{246} + x^{347} + x^{356}.
\]

In order to study the flow (18) of \(\varphi_6\), we proceed as in the proof of Theorem 4.7. We see that the forms \(\varphi_6(t)\) defined by (39) are a solution of (18) if and only if the functions \(f_i, h_1\) and \(h_2\) satisfy
\[ f_1 = u \cdot v, \quad f_2 = f_3 = v^{1/2}, \quad f_4 = f_5 = u^{1/2}, \]
\[ f_6 = f_7 = (uv)^{-1/2}, \quad h_1 = h_2 = -\frac{1}{2}(uv)^{-1/2} + \frac{1}{2}u^3/2v^{1/2}, \]

where \( u = u(t) \) and \( v = v(t) \) are differentiable real functions satisfying the system of ordinary differential equations

\[
\begin{cases}
  u' = \frac{2}{3} \frac{2 - u^3}{u^3 v^3}, \\
  v' = -\frac{2}{3} \frac{1 - 2u^3}{u^4 v^2},
\end{cases}
\] (40)

with initial conditions

\[ u(0) = v(0) = 1. \] (41)

Clearly, the systems (40)–(41) and (29)–(30) are the same. Thus, the maximal solution of (40)–(41) satisfies the properties expressed in Corollary 4.6 for the maximal solution of (29)–(30).

To finish the proof we see that, for \( t \in (t_{\text{min}}, +\infty) \), the expression of \( \varphi_6(t) \) is given by

\[ \varphi_6(t) = \frac{1}{4} \left( 1 + 4uv^2 - 2u^2v + u^4v^2 \right) e^{123} + e^{347} + e^{356} + e^{167} - e^{246} + e^{257} \\
+ u^2v e^{145} + \frac{1}{2} \left( 1 - u^2v \right) \left( e^{136} - e^{127} \right). \]

The underlying metric \( g(t) \) of this solution converges to a flat metric. To check that the limit metric is flat, we note that all non-vanishing coefficients of the Riemannian curvature \( R(t) \) of \( g(t) \) are proportional to the function

\[ u^p(t)(2 - u^3(t))^q, \]

where \( p \) and \( q \) are real numbers satisfying that \( q > 0 \). According with Corollary 4.6), we have that the function \( u(t) \) satisfies

\[ \lim_{t \to +\infty} u(t) = 2^{1/3}, \]

and so

\[ \lim_{t \to +\infty} R(t) = 0. \]

\[ \square \]

**Remark 4.9** Note that surprising in the \( N_4 \) and \( N_6 \) cases we get the same system of equations.

Finally, for the Laplacian flow of the closed \( G_2 \) form \( \varphi_{12} \) on \( N_{12} \) we have the following.
Theorem 4.10  The family of closed $G_2$ forms $\varphi_{12}(t)$ on $N_{12}$ given by

$$\varphi_{12}(t) = -e^{124} + e^{167} + f(t)^6 e^{135} - f(t)^6 e^{236} + e^{257} + e^{347} - e^{456}, \ t \in (-3, +\infty)$$

is the solution of the Laplacian flow of $\varphi_{12}$, where $f = f(t)$ is the function

$$f(t) = \left(\frac{1}{3} t + 1\right)^{1/8}.$$

Moreover, the underlying metrics $g(t)$ of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in $N_{12}$, as $t$ goes to infinity.

Proof  Let $f_i = f_i(t)$ ($i = 1, \ldots, 7$) be some differentiable real functions depending on a parameter $t \in I \subset \mathbb{R}$ such that $f_i(0) = 1$ and $f_i(t) \neq 0$, for any $t \in I$, where $I$ is an open interval. For each $t \in I$, we consider the basis $\{x^1, \ldots, x^7\}$ of left invariant 1-forms on $N_{12}$ defined by

$$x^i = x^i(t) = f_i(t)e^i, \ 1 \leq i \leq 7.$$

Then, from (16) the structure equations of $N_{12}$ with respect to this basis are

$$\begin{align*}
\frac{dx^i}{dt} &= 0, \ i = 1, 2, 3, \\
\frac{dx^4}{dt} &= \frac{\sqrt{3}}{6} \frac{f_4}{f_{12}} x^{12}, \\
\frac{dx^5}{dt} &= -\frac{1}{4} \frac{f_5}{f_{23}} x^{23} + \frac{\sqrt{3}}{12} \frac{f_5}{f_{13}} x^{13}, \\
\frac{dx^6}{dt} &= -\frac{\sqrt{3}}{12} \frac{f_6}{f_{23}} x^{23} - \frac{1}{4} \frac{f_6}{f_{13}} x^{13}, \\
\frac{dx^7}{dt} &= -\frac{\sqrt{3}}{6} \frac{f_7}{f_{34}} x^{34} + \frac{\sqrt{3}}{12} \frac{f_7}{f_{25}} x^{25} + \frac{1}{4} \frac{f_7}{f_{26}} x^{26} + \frac{\sqrt{3}}{12} \frac{f_7}{f_{16}} x^{16} - \frac{1}{4} \frac{f_7}{f_{15}} x^{15}.
\end{align*}$$

(43)

Now, for any $t \in I$, we consider the $G_2$ form $\varphi_{12}(t)$ on $N_{12}$ given by

$$\varphi_{12}(t) = -x^{124} + x^{167} + x^{135} - x^{236} + x^{257} + x^{347} - x^{456} =$$

$$= -f_{124}e^{124} + f_{167}e^{167} + f_{135}e^{135} - f_{236}e^{236} + f_{257}e^{257} + f_{347}e^{347} - f_{456}e^{456}.$$

(44)

Note that $\varphi_{12}(0) = \varphi_{12}$ and, for any $t$, the 3-form $\varphi_{12}(t)$ on $N_{12}$ determines the metric $g_t$ such that the basis $\{x_i = \frac{1}{f_i} e_i; i = 1, \ldots, 7\}$ of $n_{12}$ is orthonormal. So, $g_t(e_i, e_i) = f_i^2$.

We need to determine the functions $f_i$ and the interval $I$ so that $\frac{d}{dt} \varphi_{12}(t) = \Delta_t \varphi_{12}(t)$, for $t \in I$. Using (44) we have

$$\begin{align*}
\frac{d}{dt} \varphi_{12}(t) &= -(f_{124})' e^{124} + (f_{167})' e^{167} + (f_{135})' e^{135} - (f_{236})' e^{236} + \\
&\quad + (f_{257})' e^{257} + (f_{347})' e^{347} - (f_{456})' e^{456}.
\end{align*}$$

(45)
Now, we calculate $\Delta_t \varphi_{12}(t) = -d \ast_t d \ast_t \varphi_{12}(t)$. On the one hand, we have
\[ \ast_t \varphi_{12}(t) = x^{3567} - x^{2467} + x^{2345} + x^{1457} + x^{1346} + x^{1256} + x^{1237}. \]
(46)

So, $x^{2467}$ and $x^{1457}$ are the unique non closed summands in $\ast_t \varphi_{12}(t)$. Then, taking into account the structure equations (43) and that $x^i(t) = f_i(t)e^i$, $1 \leq i \leq 7$ we obtain
\[ \Delta_t \varphi_{12}(t) = -\frac{(f_{15} + f_{26})(f_2^2 f_6^2 + f_3^2 f_7^2)}{16 f_1 f_2 f_3 f_5 f_6} (e^{236} - e^{135}) + \frac{(f_{15} + f_{26})(f_2^2 f_6^2 - f_3^2 f_7^2)}{16 \sqrt{3} f_1 f_2 f_3 f_5 f_6} (e^{136} + e^{235}). \]
(47)

Comparing (45) and (47), in particular, we have that
\[(f_{124})' = (f_{167})' = (f_{257})' = (f_{347})' = (f_{456})' = 0, \]
and since $\varphi_{12}(0) = \varphi_{12}$ this imply that
\[ f_{124}(t) = f_{167}(t) = f_{257}(t) = f_{347}(t) = f_{456}(t) = 1, \]
(48)

for any $t \in I$. From the equation (48) we obtain that
\[ f_1 = f_1; \quad f_2 = f_2; \quad f_3 = (f_1 f_2)^2; \quad f_4 = \frac{1}{f_1 f_2}; \quad f_5 = f_1; \quad f_6 = f_2; \quad f_7 = \frac{1}{f_1 f_2}. \]

Let us consider $f = f_1 = f_2$. With these concrete values (45) and (47) become
\[ \frac{d}{dt} \varphi_{12}(t) = (f^6(t))' (e^{135} - e^{236}), \]
(49)

and
\[ \Delta_t \varphi_{12}(t) = \frac{f(t)^{-2}}{4} (e^{135} - e^{236}), \]
(50)

respectively. From (49) and (50) finding a solution of the Laplacian flow is equivalent to solve $f^7 f' = \frac{1}{\Delta_t}$. Integrating this equation, we obtain
\[ f^8 = \frac{1}{3} t + B, \quad B = constant. \]

But $\varphi(0) = \varphi_{12}$ implies that $f(0) = 1$, that is, $B = 1$. Hence
\[ f(t) = \left( \frac{1}{3} t + 1 \right)^{1/8}, \]
and so the one-parameter family of 3-forms $\{ \varphi_{12}(t) \}$ given by (42) is the solution of the Laplacian flow of $\varphi_{12}$ on $N_{12}$, and it is defined for every $t \in (-3, +\infty)$. 

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Finally, we study the behavior of the underlying metric \( g(t) \) of such a solution in the limit. If we think of the Laplacian flow as a one-parameter family of \( G_2 \) manifolds with a closed \( G_2 \)-structure, it can also be checked that, in the limit, the resulting manifold has vanishing curvature. Denote by \( g(t), t \in (-3, +\infty) \), the metric on \( N_{12} \) induced by the \( G_2 \) form \( \varphi_{12}(t) \) defined by (42). Then, \( g(t) \) has the following expression

\[
\begin{align*}
g(t) &= \left(\frac{1}{3}t + 1\right)^{1/4} (e^1)^2 + \left(\frac{1}{3}t + 1\right)^{1/4} (e^2)^2 + \left(\frac{1}{3}t + 1\right)^{-1} (e^3)^2 \\
&\quad + \left(\frac{1}{3}t + 1\right)^{-1/2} (e^4)^2 + \left(\frac{1}{3}t + 1\right)^{1/4} (e^5)^2 + \left(\frac{1}{3}t + 1\right)^{1/4} (e^6)^2 \\
&\quad + \left(\frac{1}{3}t + 1\right)^{-1/2} (e^7)^2.
\end{align*}
\]

Concretely, every non vanishing coefficient appearing in the expression of the Riemannian curvature \( R(t) \) of \( g(t) \) is proportional to \((t + 3)^{-1}\). Therefore, \( \lim_{t \to +\infty} R(t) = 0 \).

\[\square\]

**Remark 4.11** Note that, for every \( t \in (-3, +\infty) \), the metric \( g(t) \) is a nilsoliton on the Lie algebra \( n_{12} \) of \( N_{12} \). In fact, with respect to the orthonormal basis \((x_1(t), \ldots, x_7(t))\), we have

\[
\text{Ric}(g(t)) = -\frac{3}{4(3+t)} Id + \frac{3}{8(3+t)} \text{diag}(1, 1, 1, 2, 2, 2, 3) = \frac{3}{(3+t)} \text{Ric}(g(0))
\]

with \( \frac{3}{8(3+t)} \text{diag}(1, 1, 1, 2, 2, 2, 3) \) a derivation of \( n_{12} \) for every \( t \).

**Acknowledgments** We would like to thank Vivina Barutello, Ernesto Buzano, Diego Conti, Edison Fernández, Jorge Lauret and Mario Valenzano for useful conversations. Moreover, we are grateful to the anonymous referees for useful comments and improvements. This work has been partially supported by (Spanish) MINECO Project MTM2011-28326-C02-02, Project UPV/EHU ref. UFI11/52 and by (Italian) GNSAGA of INdAM.

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