A simple proof of the exactness of expanding maps of the interval with an indifferent fixed point

MARCO LENCI

November 2015

Final version to be published in Chaos Solitons & Fractals

Abstract
Expanding maps with indifferent fixed points, a.k.a. intermittent maps, are popular models in nonlinear dynamics and infinite ergodic theory. We present a simple proof of the exactness of a wide class of expanding maps of [0, 1], with countably many surjective branches and a strongly neutral fixed point in 0.

Mathematics Subject Classification (2010): 37E05, 37D25, 37A40, 37A25.

1 Introduction
Expanding maps with indifferent fixed points are very popular models in nonlinear dynamics. Not only are they among the simplest chaotic dynamical systems whose physical measure may be infinite, they have also been used to model anomalous transport in deterministic settings (see, e.g., [GT], [GNZ], [BG §1.2.3.3], [ZK], [K] and references therein).

Constructing one such scheme is simple. Starting, say, with a sufficiently regular map \( T : S^1 \to S^1 \) with an indifferent fixed point in \( \bar{x} \), we lift \( T \) to a map \( \tau : [0, 1) \to \mathbb{R} \) (that is, after choosing an identification \( S^1 \cong [0, 1) \), \( \tau \) is such that \( T(x) = \tau(x) \mod 1 \)). Then we define \( \mathcal{T} : \mathbb{R} \to \mathbb{R} \) as the translation invariant version of \( \tau \) (namely, for \( x \in [k, k + 1[, \mathcal{T}(x) := \tau(x - k) + k \)). Maps like \( \mathcal{T} \) are called quasi-lifts in [L2].

---

* Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy. E-mail: marco.lenci@unibo.it.
† Istituto Nazionale di Fisica Nucleare, Sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy.
One is interested in the diffusive properties of the trajectories of $\mathcal{T}$, for example as functions of an initial condition $x \in [0,1)$, chosen w.r.t. the physical measure normalized to $[0,1)$.

Assume that $\tau$ was chosen so that $\tau(\bar{x}) = \bar{x}$. Then $\mathcal{T}$ has countably many indifferent fixed points in $\bar{x}_k := \bar{x} + k$, $k \in \mathbb{Z}$. When $\mathcal{T}^n(x)$ gets very close to one of them, the trajectory will remain around it for a long while. After that, it will reach a region where the modulus of the derivative of $\mathcal{T}$ is substantially different from 1, causing it to undergo an erratic, or chaotic, motion. This will end when $\mathcal{T}^{n'}(x)$, with $n' > n$, gets again very close to a fixed point, and so on. So the typical trajectory will alternate between almost constant stretches and random-looking stretches. In jargon, it will have an intermittent behavior (whence the name intermittent maps for expanding maps with indifferent fixed points). The statistical properties of $(\mathcal{T}^n(x))_n$, seen as a random process, may be very different from the case of a similar map with expanding fixed points only: for example, the scaling rate that is observed in the (generalized) CLT and/or the mean square displacement might be $n^\gamma$, with $\gamma \neq 1/2$. In this case one speaks of anomalous diffusion \cite{Berg1983}.

This phenomenon can also be studied in terms of the original map $\mathcal{T}$, which we now view as a map $[0, 1) \rightarrow [0, 1)$ (via the aforementioned identification $\mathbb{S}^1 \cong [0, 1)$). Choose an initial condition $x \in [0,1)$ and let $\lfloor \cdot \rfloor$ denote the integer part of a real number. The definitions of $\tau$ and $\mathcal{T}$ easily imply that $\mathcal{T}(x) = \tau(x) = \lfloor \tau(x) \rfloor + T(x)$. Setting $f(x) := \lfloor \tau(x) \rfloor$ and iterating the procedure, we obtain

$$
\mathcal{T}^n(x) = \sum_{j=0}^{n-1} f(T^j(x)) + T^n(x),
$$

(1.1)

Since $T^n(x)$ is bounded, the diffusive properties of $\mathcal{T}^n(x)$ are completely revealed by the Birkhoff sum of the observable $f : [0, 1) \rightarrow \mathbb{Z}$.

The latter may be called discrete displacement, as it specifies in what copy of the unit interval the dynamics is going to take place at the next iteration. One can also study Birkhoff sums of more general observables $f$, taking values in $\mathbb{Z}$ or $\mathbb{R}$, regular or not around $\bar{x}$, vanishing or not there. Each choice gives rises to different statistical properties of the random variables $\sum_j f \circ T^j$, which can always be viewed as the trajectories of a given extended dynamical system (called a group extension or skew product \cite{Alpern, §8.1}).

This preamble was meant to illustrate the importance, from the point of view of applications, of the stochastic properties of interval maps with indifferent fixed points. Exactness is one of the strongest of these properties: for a non-singular dynamical system, it means that the system eventually loses all initial information-encoded in the form of an absolutely continuous probability measure for the choice of the initial conditions. (The reader unfamiliar with this or other notions of the theory of dynamical systems is referred to the brief recapitulation of Section A.1 of the Appendix.) If the system preserves a finite absolutely continuous measure, exactness implies mixing of all orders \cite{Ruelle, Quas}. If the relevant invariant measure

---

M. Lenci
is instead infinite, exactness is perhaps the only notion of strong mixing whose
definition works well in infinite ergodic theory too (see the discussion in [L1,L2]).

In this paper we deal with expanding Markov maps of the interval with a finite
number of indifferent fixed points. These maps are always non-singular w.r.t. the
Lebesgue measure and, in great generality, possess a unique absolutely continuous
invariant measure [T1]. Under some conditions on the nature of the fixed points,
such measure is infinite. Famous examples are the Pomeau-Manneville maps [PM]
and the Farey map (see, e.g., [I] or [KS]).

In 1983, Thaler proved that if a map as described above has surjective branches
only, then it is exact under very mild technical conditions [L2]. This theorem is
partly based on previous work by the same author [T1]. In his celebrated 1997
book, Aaronson extended the result to a large class of Markov maps in a general
setting [A, §4]. Understandably, such general proof is rather cumbersome. On the
other hand, Thaler’s original papers are not straightforward either, as they involve
non-standard types of induced maps, a martingale convergence theorem and so on.
(Recent proofs of the exactness of specific maps, such as the Farey map [KS] and
α-Farey maps [KMS], are not easily generalizable, or especially simple either.)

The purpose of this note is to present a hands-on and relatively short proof of
the exactness of the simplest kinds of Markov maps of [0,1) preserving an infinite
measure: those defined by an indifferent fixed point at 0 and a countable number
of uniformly expanding surjective branches. This is not a serious restriction within
Thaler’s family, as will become clear below.

What makes our key argument rather immediate is the use of a recent criterion
for exactness by Miernowski and Nogueira [MN] (a generalization of which we present
in the Appendix). Understandably, the argument needs distortion estimates. The
ones we give here are transparent—at least in this author’s view—for they are based
on a simple estimate by Young [Y, §6]. To make this paper self-contained, Young’s
proof is reported in the Appendix too.

Acknowledgments. I would like to thank Claudio Bonanno, Sara Munday and
Lai-Sang Young for useful discussions, and Roberto Artuso for pointing out some
relevant references. This work is part of my activities within the Gruppo Nazionale
di Fisica Matematica (INdAM, Italy). It was also partially supported by PRIN
Grant 2012AZS52J_001 (MIUR, Italy).

2 Setup and result

Many of the least common mathematical terms used in this section are defined in
Section A.1 of the Appendix.

We assume that there is a partition $\mathcal{P} := \{I_j\}_{j \in \mathcal{J}}$ of $I := [0,1]$. The partition
can be finite, in which case $\mathcal{J} := \mathbb{Z}_N := \{0,1,\ldots,N-1\}$ or countable, in which
case $\mathcal{J} := \mathbb{N}$ (in our notation $0 \in \mathbb{N}$). The elements of the partition are defined to be
$I_j = [a_j, a_{j+1}]$, with $0 = a_0 < a_1 < \ldots < a_k < \ldots$.
Let $T : I \to I$ be a Markov map w.r.t. $\mathcal{P}$, with the following properties:

(A1) $T|_{(a_j, a_{j+1})}$ possesses an extension $\tau_j : I_j \to I$ which is bijective and $C^2$ up to the boundary.

(A2) There exists $\Lambda > 1$ such that $|\tau_j'(x)| \geq \Lambda$, for all $x \in I_j$ with $j \geq 1$.

(A3) There exists $K > 0$ such that $\frac{|\tau_j''(x)|}{|\tau_j'(x)|^2} \leq K$, for all $x$ and $j$.

(A4) $\tau_0$ is convex with $\tau_0(0) = 0$, $\tau_0'(0) = 1$, $\tau_0'(x) > 1$, for $x > 0$, and $\tau_0''(x) \sim x^\beta$, for $x \to 0^+$, for some $\beta \geq 0$.

It is proved in [T1] that, under the above conditions, $T$ possesses an infinite invariant measure $\mu$, absolutely continuous w.r.t. the Lebesgue measure $m$ and such that $h := d\mu/dm$ is bounded on every $[\varepsilon, 1]$. The arguments there, as well as in [A §1.5], prove that $\mu$ is unique up to factors. In any event, the point of view of this note is that the map $T$ is given together with the measure $\mu$ it preserves, as is the case in many applications. This way, none of our proofs depend on [T1].

**Terminology and conventions.**

1. Unless it is important and clearly specified, neither our notation nor our language will mention null-measure sets. For example, we liberally say that $\mathcal{P}$ is a partition of $I$ even though $I_j$ and $I_{j+1}$ intersect in a point; or we write $TI_j = I$ even though this might be true only mod $\mu$.

2. Throughout the paper, the $\sigma$-algebra of reference for $I$ will be its Borel $\sigma$-algebra $\mathcal{B}$. In fact, every time a $\sigma$-algebra is implied in the arguments, we shall always intend the Borel $\sigma$-algebra of the space at hand.

This is our main result:

**Theorem 2.1** $T : I \to I$ is conservative and exact, w.r.t. $\mu$, or, equivalently, $m$.

**Proof.** It is easy to check that $\forall x \in (0, a_1)$, $\exists n \in \mathbb{N}$ such that $x < T(x) < \ldots < T^n(x) \notin I_0$. So $J := \bigcup_{j \in \mathcal{J} \setminus \{0\}} I_j$ is a global cross-section, in the sense that almost every orbit of the system intersects it. Moreover $\mu(J) < \infty$. Therefore, via the Poincaré Recurrence Theorem applied to the map induced by $T$ on $J$, the dynamical system is conservative.

As for the exactness, we are going to use the Miernowski-Nogueira criterion [MN]:

**Proposition 2.2** The non-singular and ergodic dynamical system $(X, \mathcal{A}, \nu, T)$ is exact if, and only if, $\forall A \in \mathcal{A}$ with $\nu(A) > 0$, $\exists n = n(A)$ such that $\nu(T^{n+1}A \cap T^n A) > 0$. 

A generalization of this criterion to the case of non-ergodic maps is given in Section A.2 of the Appendix.

We need to define a more refined Markov partition for $T$. Let $(b_k)_{k \in \mathbb{N}} \subset I_0$ be uniquely defined by $b_0 := a_1$ and $T(b_{k+1}) = b_k$, with $b_{k+1} < b_k$. Now, for $k \in \mathbb{Z}^+$, set $I_{-k} := [b_k, b_{k-1}]$. Then, $\mathcal{P}_- := \{I_j\}_{j \in \mathbb{Z}^-}$ is a partition of $I_0$. So $\mathcal{P}_o := \mathcal{P}_- \cup \mathcal{P} \setminus \{I_0\}$ is a partition of $I$. Its index set will be denoted $\mathcal{J}_o := \mathbb{Z}^- \cup \mathcal{J} \setminus \{0\}$. $\mathcal{P}_o$ is a Markov partition because $T(I_{-1}) = J$ and, for $k \geq 2$, $T(I_{-k}) = I_{-k+1}$.

Let $\mathcal{P}_o^n := \bigvee_{k=0}^{n-1} T^{-k} \mathcal{P}_o$ denote the refinement of $\mathcal{P}_o$ relative to $T$ up to time $n$. For $j^n := (j_0, \ldots, j_{n-1}) \in (\mathcal{J}_o)^n$, its elements are denoted

$$I_{j^n} := I_{j_0} \cap T^{-1} I_{j_1} \cap \cdots \cap T^{-n+1} I_{j_{n-1}} \quad (2.1)$$

(see that there are many $j^n$ for which $I_{j^n} = \emptyset$). Since $T$ is uniformly expanding away from 0, and since a.a. orbits visit $J$ infinitely often, it is easily seen that, for any sequence $(j_n)_{n \in \mathbb{N}} \subset \mathcal{J}_o$,

$$\lim_{n \to \infty} m(I_{(j_0, \ldots, j_{n-1})}) = 0. \quad (2.2)$$

We now enter the core of the proof. Let $A$ be any positive-measure set. Among the infinitely many density points of $A$, relative to $m$, let us choose $x_0$ so that its orbit intersects $J$ infinitely many times (this is possible because $J$ is a global cross-section). Let $(j_n)$ describe the itinerary of $x_0$ w.r.t. $\mathcal{P}_o$, namely, $T^n(x_0) \in I_{j_n}$, $\forall n \in \mathbb{N}$; equivalently, $x_0 \in I_{(j_0, \ldots, j_{n-1})} = I_{j^n}$, $\forall n \in \mathbb{N}$. By (2.2), using the notation of conditional measure,

$$\lim_{n \to \infty} m(A \mid I_{j^n}) = 1. \quad (2.3)$$

Moreover, we can assume that there exist $\bar{j} \in \mathbb{Z}^+$ and a subsequence $(j_{n_k})$ such $j_{n_k} = \bar{j}$. In fact, keeping in mind that $J = \bigcup_{j \in \mathcal{J} \setminus \{0\}} I_j$ is a global cross-section, if the orbit of $x_0$ intersected each $I_j$, with $j \geq 1$, only a finite number of times, then necessarily $T^n(x_0) \to 1$, as $n \to \infty$. But $T$ is conservative and 1 is not an atom of $\mu$, so there can only be a null-measure set of such points, and we can pick a different $x_0$.

We need a distortion lemma, which will be proved in Section 3.

**Lemma 2.3** There exists $D > 1$ such that, for any $n \in \mathbb{N}$; any $j^{n+1} = (j_0, \ldots, j_n) \in (\mathcal{J}_o)^{n+1}$ with $m(I_{j^{n+1}}) > 0$ and such that at least one of its components $j_k > 0$; and any $B \subset I_{j^{n+1}}$, one has:

(i) $T^n B \subset I_{j^n}$;

(ii) $m(T^n B \mid I_{j_n}) \leq D m(B \mid I_{j^{n+1}})$;

(iii) $m(T^{n+1} B) \leq D m(B \mid I_{j^{n+1}})$. 


From now till the end of the proof, to comply with one of the hypotheses of the lemma, we always take \( n \geq n_1 \): that way, for \( k = n_1, j_k = \bar{j} > 0 \).

Applying Lemma 2.3 (ii) to \( B := I_{j_{n+1}} \setminus A \), whose conditional Lebesgue measure in \( I_{j_{n+1}} \) vanishes by (2.3), and observing that \( I_{j_n} \cap (T^nA) \supseteq I_{j_n} \setminus T^nB \), we see that

\[
\lim_{n \to \infty} m(T^nA \mid I_{j_n}) = 1. \tag{2.4}
\]

Now we notice that \( \exists \delta \in (0, 1) \) such that, if \( C \subseteq I_{\bar{j}} \) with \( m(C \mid I_{\bar{j}}) > \delta \), then \( m(C \cap TC) > 0 \). (This is not hard to prove, using Lemma 2.3 (iii) with \( n = 0, j_0 = \bar{j}, \) and \( B = I_{\bar{j}} \setminus C \). The optimal estimate for \( \delta \) is found to be \( D/(D + m(I_{\bar{j}})) \).

Therefore, choosing a sufficiently large \( k \) such that, by (2.4), \( m(T^{nk}A \mid I_{j_{nk}}) > \delta \), and since \( j_{nk} = \bar{j} \), we obtain \( m(T^{nk}A \cap T^{nk+1}A) > 0 \), which is the exactness condition of Proposition 2.2.

In order to apply that proposition, we still need to verify that \( T \) is ergodic. But this follows immediately from the above arguments. In fact, if \( A \) is an invariant set with \( m(A) > 0 \), (2.4) gives

\[
m(A \mid I_{\bar{j}}) = \lim_{k \to \infty} m(T^{nk}A \mid I_{j_{nk}}) = 1. \tag{2.5}
\]

Using that \( TA = A, TI_{\bar{j}} = I \) and \( T \) is non-singular, (2.5) implies that \( m(A) = m(TA \mid TI_{\bar{j}}) = 1 \). Q.E.D.

### 3 Distortion

This section is entirely devoted to the proof of Lemma 2.3. We will use standard techniques and variations thereof.

Firstly, (i) follows from (2.1) since \( B \subseteq I_{j_{n+1}} \). Secondly, (ii) comes from the following distortion inequality: \( \forall x, y \in I_{j_{n+1}} \),

\[
D^{-1} \leq \frac{|(T^n)'(x)|}{|(T^n)'(y)|} \leq D. \tag{3.1}
\]

In fact,

\[
m(T^nB \mid I_{j_n}) = \frac{m(T^nB)}{m(T^nI_{j_{n+1}})} = \frac{\int_B |(T^n)'(x)| \, dm}{\int_{I_{j_{n+1}}} |(T^n)'(x)| \, dm} \leq \frac{\max_B |(T^n)'| m(B)}{\min_{I_{j_{n+1}}} |(T^n)'| m(I_{j_{n+1}})} \leq D m(B \mid I_{j_{n+1}}). \tag{3.2}
\]

Assertion (iii) is derived in the same way from the inequality

\[
D^{-1} \leq \frac{|(T^{n+1})'(x)|}{|(T^{n+1})'(y)|} \leq D, \tag{3.3}
\]
Exactness of maps with an indifferent fixed point

using that $T^{n+1}B \subseteq T^{n+1}I_{j_{n+1}} = I$.

Thus, we need to prove (3.1) and (3.3). We will only write the proof of the latter, since the former is completely analogous (and in fact implied by our proof, as will be clear). Denoting for short $x_k := T^k(x)$ and $y_k := T^k(y)$, an easy sufficient condition for (3.3) is

$$\left| \sum_{k=0}^{n} \log \left| \frac{T'(x_k)}{T'(y_k)} \right| \right| \leq C, \quad (3.4)$$

where $C$ is a positive constant (whence $D := e^C$).

Observe that, by definition of $I_{j_{n+1}}$, the orbit segments $(x_k)_{k=0}^{n}$ and $(y_k)_{k=0}^{n}$ have the same itinerary w.r.t $P_\circ$. We are going to parse them by grouping excursions in $I_0$, where we define an excursion in $I_0$ to be an orbit segment $\{x_i, x_{i+1}, \ldots, x_j\}$ such that $x_i \in J$ and $x_k \in I_0$, for all $i < k \leq j$. The excursion is said to be complete, respectively partial, if $x_{j+1} \in J$, respectively $x_{j+1} \in I_0$.

Set $k_0 := 0$, and, recursively for $i \geq 1$, $k_i := \min\{k > k_{i-1} \mid x_k \in J\}$ (the definition would be equivalent with $y_k$ in place of $x_k$). This process stops when there are no more $k_i \leq n$ to define. We denote by $\ell$ the last index $i$ for which $k_i$ has been defined, and also set $k_{\ell+1} := n + 1$.

So each time frame $\{k_i, k_i + 1, \ldots, k_{i+1} - 1\}$ corresponds to one of following four types of orbit segments:

**Type 1**: The first segment of the parsing, which might not be an actual excursion in $I_0$, if $x_0, y_0 \notin J$.

**Type 2**: Bona fide complete excursions in $I_0$, that is, complete excursions of cardinality bigger than 1.

**Type 3**: Degenerate excursions, that is, single points in $J$ followed by points in $J$.

**Type 4**: The last segment of the parsing, which might only be a partial excursion, if $x_{n+1}, y_{n+1} \notin J$.

**Remark 3.1** The hypothesis that at least one of the $j_k$ is positive means that, for at least one $k$, $x_k, y_k \in J$. This implies that the parsing is not trivial, i.e., it cannot comprise just one segment. Otherwise, as will be clear below, certain estimates might be arbitrarily bad.

We are going to show that there exist constants $\eta \in (0, 1)$ and $\kappa > 0$ such that, in each time frame of type 1-3, we have:

$$|x_{k_i} - y_{k_i}| \leq \eta |x_{k_{i+1}} - y_{k_{i+1}}|; \quad (3.5)$$

$$\sum_{k=k_i}^{k_{i+1}-1} \log \left| \frac{T'(x_k)}{T'(y_k)} \right| \leq \kappa |x_{k_{i+1}} - y_{k_{i+1}}|. \quad (3.6)$$
For the type 4 segment, we have
\[
\left| \sum_{k=k_{\ell}}^{n} \log \left| \frac{T'(x_k)}{T'(y_k)} \right| \right| \leq \kappa. \tag{3.7}
\]

The estimates (3.5)-(3.7) yield (3.4) because
\[
\left| \sum_{k=0}^{n} \log \left| \frac{T'(x_k)}{T'(y_k)} \right| \right| \leq \sum_{i=0}^{\ell-1} \sum_{k=k_i}^{k_{i+1}^{-1}} \log \left| \frac{T'(x_k)}{T'(y_k)} \right| + \sum_{k=k_{\ell}}^{n} \log \left| \frac{T'(x_k)}{T'(y_k)} \right| \leq \left( \sum_{i=0}^{\ell-1} \kappa \eta^i \right) \left| x_{k_{\ell+1}} - y_{k_{\ell+1}} \right| + \kappa \leq \frac{\kappa}{1 - \eta} \leq C,
\tag{3.8}
\]
where we have used that \( \left| x_{k_{\ell+1}} - y_{k_{\ell+1}} \right| = \left| x_{n+1} - y_{n+1} \right| \leq 1. \)

Let us prove (3.5)-(3.6) for each of the first three types of orbit segments, starting with the easiest.

**Type 3.** Since \( x_k, y_k \in J \), (A2) yields
\[
\left| x_{k+1} - y_{k+1} \right| \geq \Lambda \left| x_k - y_k \right|.
\tag{3.9}
\]
Furthermore, let \( j \in J \) be such that \( x_k, y_k \in I_j \). For some \( \xi \) between \( x_{k+1} \) and \( y_{k+1} \) one has
\[
\left| \log \frac{T'(x_k)}{T'(y_k)} \right| = \left| \log \frac{T'(\tau_j^{-1}(x_{k+1}))}{T'(\tau_j^{-1}(y_{k+1}))} \right| \leq K \left| x_{k+1} - y_{k+1} \right|. \tag{3.10}
\]
by (A3). Since in this case \( x_{k+1} = x_{k+1} \) and \( y_{k+1} = y_{k+1} \), (3.5)-(3.6) are shown.

**Type 2.** In this case too \( x_k, y_k \in J \), therefore (3.5) comes from (3.9) and the trivial inequality
\[
\left| x_{k+1} - y_{k+1} \right| \geq \left| x_{k+1} - y_{k+1} \right|. \tag{3.11}
\]
To show (3.6) we need the following lemma, which is practically the same as [Y, §6.2, Lem. 5]. For the sake of completeness, we give a proof in Section A.3 of the Appendix.

**Lemma 3.2** There exists \( C' > 0 \) such that, for all \( j \geq 1, 0 \leq p \leq j, \) and \( x, y \in I_{-j}, \)
\[
\left| \log \frac{(T^p)'(x)}{(T^p)'(y)} \right| \leq C' \left| T^p(x) - T^p(y) \right| \leq C',
\]
where, for \( p \leq j - 1, L_{p-j} := \left| I_{p-j} \right| = b_{j-p-1} - b_{j-p} and, for p = j, \) \( L_0 := \left| J \right| = 1 - a_1 \) (observe that \( T^p(x), T^p(y) \) belong to \( I_{p-j} \) or \( J \), respectively).
The l.h.s. of (3.6) can be estimated by

$$\left| \sum_{k=k_0}^{k_1-1} \log \frac{|T'(x_k)|}{|T'(y_k)|} \right| \leq \left| \log \frac{|T'(x_{k_0})|}{|T'(y_{k_0})|} \right| + \left| \log \frac{(T^{k_1})'(x_{k_1})}{(T^{k_1})'(y_{k_1})} \right|,$$

(3.12)

with \( p := k_{i+1} - k_i - 1 \). (Notice that \( (T^p)'(x_{k_1}) > 0 \), because \( \{x_{k_1+1}, \ldots, x_{k_{i+1}}\} \subset I_0 \), where the map is increasing by (A4).) By (3.10)-(3.11), the first term of the r.h.s. of (3.12) is bounded above by \( K|x_{k_{i+1}} - y_{k_1+1}| \). For the second term we apply Lemma 3.2 in fact, \( (T^p)(x_{k_1}) = x_{k_1+1} \), and the same for \( y_{k_1+1} \). Also, since \( \{x_{k_1}, \ldots, x_{k_{i+1}}\} \) is a complete excursion, \( x_{k_{i+1}}, y_{k_{i+1}} \in J \) by construction: this means we apply the lemma in the case \( p = j \). The second term in the assertion of Lemma 3.2 now reads \((C'/|J|)|x_{k_{i+1}} - y_{k_1+1}| \). So (3.6) is proved for all \( \kappa \geq K + C'/|J| \).

Type 1. In this case, (3.5) is given by

$$|x_{k_1} - y_{k_1}| \geq \left( \sup_{I_1} |T'| \right) |x_{k_1-1} - y_{k_1-1}| \geq |x_0 - y_0|.$$  

(3.13)

As in the previous case, (3.6) follows from Lemma 3.2 with \( p = j := k_1 \):

$$\left| \sum_{k=0}^{k_1-1} \log \frac{|T'(x_k)|}{|T'(y_k)|} \right| = \left| \log \frac{(T^{k_1})'(x_0)}{(T^{k_1})'(y_0)} \right| \leq \frac{C'}{|J|} |x_{k_1} - y_{k_1}|.$$  

(3.14)

Type 4. It remains to verify (3.7) for the last segment of the parsing. In analogy with (3.12),

$$\left| \sum_{k=k_\ell}^{n} \log \frac{|T'(x_k)|}{|T'(y_k)|} \right| \leq \left| \log \frac{|T'(x_{k_\ell})|}{|T'(y_{k_\ell})|} \right| + \left| \log \frac{(T^p)'(x_{k_{\ell+1}})}{(T^p)'(y_{k_{\ell+1}})} \right|,$$

(3.15)

with \( p := k_{\ell+1} - k_\ell - 1 = n - k_\ell \). By (3.10), the first term of the above r.h.s. is bounded above by \( K|x_{k_{\ell+1}} - y_{k_\ell+1}| \leq K \). The second term is bounded by \( C' \) via the second inequality of Lemma 3.2.

This concludes the proof of (3.5)-(3.7) in all cases, yielding (3.8), thus (3.4), thus Lemma 2.3.

A Appendix

A.1 Basic notions

We recall some basic notions of the mathematical theory of dynamical systems that have been used in the paper. Most of this material is presented, e.g., in [A].

A dynamical system \((X, \mathcal{A}, \nu, T)\) is defined by a measure space \((X, \mathcal{A}, \nu)\) and a map \(T : X \to X\). We assume that \(TX = X\). \(\mathcal{A}\) is a \(\sigma\)-algebra defined on \(X\) and
\( \nu \) is a \( \sigma \)-finite measure for \((X, \mathcal{A})\) (this means that \( \exists (A_n)_{n \in \mathbb{N}} \subset \mathcal{A} \), with \( \nu(A_n) < \infty \) such that \( \bigcup_n A_n = X \)). The measure of the space, \( \nu(X) \), can be either finite or infinite; in the former case, it is conventional to normalize \( \nu \) so that \( \nu(X) = 1 \). The map \( T : X \rightarrow X \) is bi-measurable in the sense that, \( \forall A \in \mathcal{A} \), both \( T^{-1}A \) and \( TA \) belong to \( \mathcal{A} \).

The dynamical system, or the map, is called **non-singular** if \( A \in \mathcal{A}, \nu(A) = 0 \) implies \( \nu(T^{-1}A) = 0 \). It is called **measure-preserving** if \( \nu(T^{-1}A) = \nu(A), \forall A \in \mathcal{A} \) (equivalently, \( \nu \) is said to be an **invariant measure** for \( T \)). Clearly, the latter property implies the former.

For a non-singular dynamical system, a **wandering set** is a measurable set \( W \) such that all the sets \( \{ T^{-n}W \}_{n \in \mathbb{N}} \) are disjoint. Points in \( W \) have a non-recurrent behavior, insofar as, by definition, \( x \in W \) implies \( T^n(x) \notin W \), for all \( n \geq 0 \). It is always possible to partition \( X \) into two parts \( D \) and \( C \), defined up to null-measure sets, such that every wandering set is contained in \( D \) (mod \( \nu \)). \( D \) is called the **dissipative part** of \( X \) and can be always be represented as a countable disjoint union of wandering sets. \( C \) is called the **conservative part** of \( X \) and it is where the recurrent behavior takes place. By definition, in fact, every \( A \subseteq C \) is recurrent in the sense of Poincaré, i.e., almost every \( x \in A \) is such that \( T^n(x) \in A \), at a countable number of times \( n \). A dynamical system is called **conservative** if \( X = C \) and **dissipative** if \( X = D \). A finite-measure-preserving system is always conservative (Poincaré Recurrence Theorem).

The dynamical system is called **ergodic** if all the invariant sets are trivial, namely, \( T^{-1}A = A \) mod \( \nu \) implies that either \( \nu(A) = 0 \) or \( \nu(X \setminus A) = 0 \). This is equivalent to saying that the invariant \( \sigma \)-algebra

\[
\mathcal{I} := \{ A \in \mathcal{A} \mid T^{-1}A = A \text{ mod } \nu \} \tag{A.1}
\]

is trivial (i.e., it contains only zero-measure sets and their complements). Observe that, in the infinite-measure case, this is a stronger notion than the classical definition whereby the time (i.e., Birkhoff) average of any observable is constant almost everywhere. For example, in the case where \( \nu \) is an infinite invariant measure, the fact that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = 0, \tag{A.2}
\]

for any integrable function \( f : X \rightarrow \mathbb{R} \) and almost every \( x \in X \) (depending on \( f \)), does not imply that \( T \) is ergodic.

The **tail \( \sigma \)-algebra** of a non-singular dynamical system is defined to be

\[
\mathcal{F} := \bigcap_{n=0}^{\infty} T^{-n} \mathcal{A}. \tag{A.3}
\]

If \( T : X \rightarrow X \) is a bijection, clearly \( \mathcal{F} = \mathcal{A} \), so this quantity is only relevant for non-invertible maps. In rough terms, we might say that the structure of \( \mathcal{F} \) represents
the order that persists when the dynamics evolves and chaos is produced. Equiva-
larly, the information that we obtain by observing the system via $T$-measurable
functions is the information that comes from the infinite past of the dynamics and
is not increased by further observations in time. Regardless, it is easy to see that
$\mathcal{I} \subseteq \mathcal{T}$. A non-singular dynamical system is called exact if $\mathcal{T}$ is trivial. Hence,
exactness implies ergodicity.

If $T$ is non-singular and $X$ admits a finite or countable partition $\mathcal{P}$ such that,
for any element $E \in \mathcal{P}, T E$ is a union of elements of $\mathcal{P}$ and the restriction $T|_E$ is
invertible mod $\nu$, then $T$ is called a Markov map relative to the Markov parti-
tion $\mathcal{P}$. (Depending on the context and the author, more technical conditions are
required in the definition of Markov map. The overly general definition that we give
here is sufficient for our illustrative purposes.)

The most common examples of Markov maps are those defined on an interval,
with a Markov partition made up of sub-intervals in whose interior $T$ is smooth. Such
are the systems discussed in this paper. A map of this kind is called expanding if
$|T'(x)| > 1$, for all $x$ where $T'(x)$ is defined. It is called uniformly expanding if
$\exists \lambda > 1$ such that $|T'(x)| \geq \lambda$ for all $x$ as above. A fixed point $\bar{x}$ (i.e., $T(\bar{x}) = \bar{x}$)
is called expanding, respectively contracting, if $|T'(\bar{x})| > 1$, respectively $< 1$. It
is called indifferent, or neutral, if $|T'(\bar{x})| = 1$. It is sometimes said that a fixed
point is strongly neutral if $T''(x)$ is regular around $\bar{x}$.

### A.2 A criterion for the exact components

In this section we generalize Proposition 2.2 to the case of non-ergodic maps, ob-
taining a characterization of the exact components of non-singular maps.

**Definition A.1** Let $(X, \mathcal{A}, \nu, T)$ be a non-singular dynamical system on a $\sigma$-finite
measure space (cf. Section [A.1]). We say that $A \in \mathcal{A}$, with $\nu(A) > 0$, is asympto-
tically intersecting w.r.t. the given dynamical system if $\exists n = n(A)$ such that
$\nu(T^{n+1}A \cap T^n A) > 0$. By the non-singularity of $T$, this is equivalent to $\nu(T^{k+1}A \cap
T^k A) > 0$, for all $k \geq n$.

**Proposition A.2** In the framework of Definition [A.1], let $\mathcal{I}$ and $\mathcal{T}$ denote, re-
spectively, the invariant and the tail $\sigma$-algebras (cf. [A.1], [A.3]). The following holds:

(i) if every positive-measure $A \in \mathcal{T}$ is asymptotically intersecting, then $\mathcal{I} = \mathcal{T}$;

(ii) if $\mathcal{I} = \mathcal{T}$, then every positive-measure $A \in \mathcal{A}$ is asymptotically intersecting.

**Remark A.3** Observe that (i) is a stronger statement than the converse of (ii): in parti-
cular, combining (i) and (ii), we see that if every set in the tail $\sigma$-algebra is
asymptotically intersecting, so is every measurable set. Also, using the fact that any
power of an exact map is exact, and vice-versa, it is easy to show that if $\mathcal{I} = \mathcal{T}$ then,
\( \forall A \in \mathcal{A}, \) with \( \nu(A) > 0, \) and \( \forall \ell \in \mathbb{Z}^+, \exists n = n(A, \ell) \) such that \( \nu(T^{k+j}A \cap T^kA) > 0, \) whenever \( k \geq n \) and \( 1 \leq j \leq \ell. \) In any event, Proposition \ref{prop: completeness} is now a corollary of Proposition \ref{prop: completeness}.

**Proof of Proposition \ref{prop: completeness}**

We remark that the techniques used here come entirely from [MN, Lem. 2.1].

The proof of (i) is already contained in [L2]. We report it here for the sake of completeness. Take \( B \in \mathcal{I}. \) We set out to prove that \( B \in \mathcal{I}. \) If \( \nu(B) = 0, \) then \( \nu(T^{-1}B) = 0 \) and \( B \in \mathcal{I}. \) So we assume that \( \nu(B) > 0. \) It is a known simple fact that, for all \( B \in \mathcal{I} \) and \( k \in \mathbb{N}, \)

\[
B = T^{-k}T^kB. \tag{A.4}
\]

We want to show that \( B = TB \mod \nu. \) This and (A.4) will imply that \( T^{-1}B = T^{-1}TB = B \mod \nu, \) whence \( B \in \mathcal{I}, \) as desired.

Set \( A := B \setminus TB \in \mathcal{I}. \) By (A.4), for all \( n \geq 0, \)

\[
A = T^{-n}T^nB \setminus T(T^{-n-1}T^{n+1}B) = T^{-n}(T^nB \setminus T^{n+1}B), \tag{A.5}
\]

whence

\[
T^nA = T^nB \setminus T^{n+1}B. \tag{A.6}
\]

Applying (A.6) with \( n+1 \) in lieu of \( n \) implies that \( T^{n+1}A \subseteq T^{n+1}B, \) which, compared again to (A.6), gives \( T^{n+1}A \cap T^nA = \emptyset. \) Since this holds for all \( n \in \mathbb{N}, \) the hypotheses imply that \( \nu(A) = 0. \) Thus, \( B \subseteq TB \mod \nu. \)

Analogously, setting \( A' := TB \setminus B, \) we get that, for all \( n \geq 0, \) \( T^nA' = T^{n+1}B \setminus T^nB, \) whence \( T^nA' \subseteq T^{n+1}B \) and \( T^{n+1}A' = T^{n+2}B \setminus T^{n+1}B. \) Therefore, \( T^{n+1}A' \cap T^nA' = \emptyset. \) For the same reasons as before, \( TB \subseteq B \mod \nu, \) which completes the proof of assertion (i).

As for (ii), assume by contradiction that there exists \( A \in \mathcal{A} \) with \( \nu(A) > 0 \) such that \( \nu(T^{n+1}A \cap T^nA) = 0, \) for all \( n \in \mathbb{N}. \) We want to show that this is incompatible with \( \mathcal{I} = \mathcal{I}. \)

Since \( T \) is non-singular, the above assumption implies that

\[
\nu(T^{-n}T^{n+1}A \cap A) \leq \nu(T^{-n}T^{n+1}A \cap T^{-n}T^nA) = 0. \tag{A.7}
\]

Therefore, setting

\[
B := \bigcup_{n \in \mathbb{N}} T^{-n}T^{n+1}A, \tag{A.8}
\]

we have

\[
\nu(B \cap A) = 0. \tag{A.9}
\]

The sequence of sets in the r.h.s. of (A.8) is increasing, so \( B = \bigcup_{n \geq k} T^{-n}T^{n+1}A \in T^{-k} \mathcal{A}, \) for all \( k \in \mathbb{N}, \) whence \( B \in \mathcal{I}. \) On the other hand, if \( \mathcal{I} = \mathcal{I}, \)

\[
B = T^{-1}B = \bigcup_{n \in \mathbb{N}} T^{-n-1}T^{n+1}A \supseteq A, \tag{A.10}
\]

which contradicts (A.9) because \( \nu(A) > 0. \) Q.E.D.
A.3 Young’s distortion estimate

Here we prove Lemma 3.2 copying almost verbatim the proof of Lemma 5, §6.2 in [Y].

Terminology. In what follows, we write \( f_n \sim g_n \) to mean that \( \exists \kappa_2 > \kappa_1 > 0 \) such that \( \kappa_1 < |f_n/g_n| < \kappa_2 \), for all \( n \) (possibly with some restrictions, if so specified); the same goes for other integer indices, such as \( k \), or \( i \). Also, we write \( f(x) \sim g(x) \) to mean that \( \kappa_1 < |f(x)/g(x)| < \kappa_2 \) holds for all \( x \in (0, a_1) \) (in all the cases below, this will be equivalent to the asymptotics \( x \to 0^+ \)).

Set \( \alpha := 1/(\beta + 1) \) and, for \( n \geq 1 \), \( \Delta n^{-\alpha} := n^{-\alpha} - (n + 1)^{-\alpha} \). Observe that

\[
\Delta n^{-\alpha} \sim n^{-(\alpha+1)} = \frac{n^{-\alpha}}{(\alpha+1)}. \tag{A.11}
\]

For \( k \in \mathbb{N} \), let \( n_k \) be the unique index such that

\[
b_k \in [(n_k + 1)^{-\alpha}, n_k^{-\alpha}). \tag{A.12}
\]

By (A4), \( T(x) - x \sim x^{\beta+2} \). This, the definition of \( b_k \), and (A11)-(A12) imply that

\[
\Delta b_k := b_{k-1} - b_k = T(b_k) - b_k \\
\sim b_k^{\beta+2} \sim (n_k^{-\alpha})^{\beta+2} \\
\sim (\Delta n_k^{-\alpha})^{(\beta+2)/(\beta+1)} \\
= \Delta n_k^{-\alpha}. \tag{A.13}
\]

Since, for every fixed positive integer \( l \), \( \Delta(n + l)^{-\alpha} \sim \Delta n^{-\alpha} \) and \( \Delta b_{k+l} \sim \Delta b_k \) (respectively, as functions of \( n \) and \( k \)), the above shows that each \( I_{-k} = [b_k, b_{k-1}] \) intersects at most a bounded number of intervals \([(n + 1)^{-\alpha}, n^{-\alpha})\), and vice-versa.

Recall that \( x, y \in I_{-j} \). For \( 0 \leq i \leq p-1 \), there exists \( \xi_i \) between \( T^i(x) \) and \( T^i(y) \) (hence \( \xi_i \in I_{i-j} \)) such that

\[
\log T'(T^i(x)) - \log T'(T^i(y)) = \frac{T''(\xi_i)}{T'(\xi_i)} (T^i(x) - T^i(y)). \tag{A.14}
\]

But, for all \( i \in \{0, \ldots, p-1\} \), \( T''(\xi_i) \sim \xi_i^\beta \sim b_{j-i}^{\beta} \); \( T'(\xi_i) \sim 1 \); and \( |T^i(x) - T^i(y)| \leq L_{i-j} \sim \Delta b_{j-i} \sim b_{j-i}^{\beta+2} \). All this implies that, for any \( 0 \leq q \leq p \),

\[
\left| \log \left( \frac{(T^q)'(x)}{(T^q)'(y)} \right) \right| \leq \sum_{i=0}^{q-1} \frac{T''(\xi_i)}{T'(\xi_i)} |T^i(x) - T^i(y)| \\
\leq C_1 \sum_{i=0}^{q-1} b_{j-i}^{2\beta+2} \leq C_2 \sum_{i=0}^{q-1} n_{j-i}^{-\alpha(2\beta+2)} \tag{A.15} \\
\leq C_2 \sum_{n=1}^{\infty} n^{-2} := C_3,
\]

Exactness of maps with an indifferent fixed point
where we have used the considerations of the first part of the proof and $C_1, C_2$ are suitable positive constants.

The distortion inequality (A.15) holds for a generic pair $x, y \in I_{j-1}$, not necessarily the one given in the statement of Lemma 3.2. By standard arguments—as in (3.1)–(3.4)—it gives

$$e^{-C_3} \frac{|x - y|}{L_{q-j}} \leq \frac{|T^q(x) - T^q(y)|}{L_{q-j}} \leq e^{C_3} \frac{|x - y|}{L_{q-j}}.$$  \hspace{1cm} \text{(A.16)}

Comparing the above expression for a generic $q = i \in \{0, \ldots, p-1\}$ with the same for $q = p$, we see that, for all $0 \leq i \leq p-1$,

$$\frac{|T^i(x) - T^i(y)|}{L_{i-j}} \sim \frac{|T^p(x) - T^p(y)|}{L_{p-j}}.$$  \hspace{1cm} \text{(A.17)}

Using (A.17) in the first line of (A.15), with $q = p$, together with some of the above estimates, yields

$$\left| \log \left( \frac{(T^p)'(x)}{(T^p)'(y)} \right) \right| \leq C' \sum_{i=0}^{p-1} b_{i-j}^p L_{i-j} \frac{|T^p(x) - T^p(y)|}{L_{p-j}} \leq C' \frac{|T^p(x) - T^p(y)|}{L_{p-j}},$$  \hspace{1cm} \text{(A.18)}

where $C'$ is another positive constant, and the last inequality follows from $b_{k}^p < 1$ and $\sum_{k \in \mathbb{N}} L_{-k} = 1$. Q.E.D.

References

[A] J. Aaronson, An introduction to infinite ergodic theory, Mathematical Surveys and Monographs, 50. American Mathematical Society, Providence, RI, 1997.

[BG] J.-P. Bouchaud and A. Georges, Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications, Phys. Rep. 195 (1990), nos. 4-5, 127–293.

[GNZ] T. Geisel, J. Nierwetberg and A. Zacherl, Accelerated diffusion in Josephson junctions and related chaotic systems, Phys. Rev. Lett. 54 (1985), no. 7, 616–619.

[GT] T. Geisel and S. Thomae, Anomalous diffusion in intermittent chaotic systems, Phys. Rev. Lett. 52 (1984), no. 22, 1936–1939.

[I] S. Isola, From infinite ergodic theory to number theory (and possibly back), Chaos Solitons Fractals 44 (2011), no. 7, 467–479.

[KMS] M. Kesseböhmer, S. Munday and B. Stratmann, Strong renewal theorems and Lyapunov spectra for $\alpha$-Farey and $\alpha$-Lüroth systems, Ergodic Theory Dynam. Systems 32 (2012), no. 3, 989–1017.

[KS] M. Kesseböhmer and B. Stratmann, On the asymptotic behaviour of the Lebesgue measure of sum-level sets for continued fractions, Discrete Contin. Dyn. Syst. 32 (2012), no. 7, 2437–2451.
Exactness of maps with an indifferent fixed point

[K] R. Klages, From deterministic chaos to anomalous diffusion, in: Reviews of Nonlinear Dynamics and Complexity, Vol. 3, edited by H. G. Schuster, pp. 169–227, Wiley, 2010.

[L1] M. Lenci, Exactness, K-property and infinite mixing, Publ. Mat. Urug. 14 (2013), 159–170.

[L2] M. Lenci, Uniformly expanding Markov maps of the real line: exactness and infinite mixing, arXiv:1404.2212, preprint (2014).

[MN] T. Miernowski and A. Nogueira, Exactness of the Euclidean algorithm and of the Rauzy induction on the space of interval exchange transformations, Ergodic Theory Dynam. Systems 33 (2013), no. 1, 221–246.

[PM] Y. Pomeau and P. Manneville, Intermittent transition to turbulence in dissipative dynamical systems, Comm. Math. Phys. 74 (1980), no. 2, 189–197.

[Q] A. Quas, Ergodicity and mixing properties, in: R. E. Meyers (ed.), Encyclopedia of Complexity and Systems Science, pp. 2918–2933, Springer, 2009.

[R] V. A. Rohlin, Exact endomorphisms of a Lebesgue space, Izv. Akad. Nauk SSSR Ser. Mat. 25 (1961), 499–530; English translation in Amer. Math. Soc. Transl. (2) 39 (1964), 1–36.

[T1] M. Thaler, Estimates of the invariant densities of endomorphisms with indifferent fixed points, Israel J. Math. 37 (1980), 303–314.

[T2] M. Thaler, Transformations on [0,1] with infinite invariant measures, Israel J. Math. 46 (1983), no. 1-2, 67–96.

[Y] L.S. Young, Recurrence times and rates of mixing, Israel J. Math. 110 (1999), 153–188.

[ZK] G. Zumofen and J. Klafter, Scale-invariant motion in intermittent chaotic systems, Phys. Rev. E 47 (1993), no. 2, 851–863.