Lewenstein-Sanpera Decomposition for $2 \otimes 2$ Systems

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Abstract

As it is well known, every bipartite $2 \otimes 2$ density matrix can be obtained from Bell decomposable states via local quantum operations and classical communications (LQCC). Using this fact, the Lewenstein-Sanpera decomposition of an arbitrary bipartite $2 \otimes 2$ density matrix has been obtained through LQCC action upon Lewenstein-Sanpera decomposition of Bell decomposable states of $2 \otimes 2$ quantum systems, where the product states introduced by Wootters in [W. K. Wootters, Phys. Rev. Lett. 80 2245 (1998)] form the best separable approximation ensemble for Bell decomposable states. It is shown that in these systems the average concurrence of the Lewenstein-Sanpera decomposition is equal to the concurrence of these states.

Keywords: Quantum entanglement, Lewenstein-Sanpera decomposition, Concurrence, LQCC, Bell decomposable states

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1 Introduction

Perhaps, quantum entanglement is the most non classical features of quantum mechanics \[1, 2\] which has recently been attracted much attention although it was discovered many decades ago by Einstein and Schrödinger \[1, 2\]. It plays a central role in quantum information theory and provides potential resource for quantum communication and information processing \[3, 4, 5\]. Entanglement is usually arise from quantum correlations between separated subsystems which can not be created by local actions on each subsystems. By definition, a bipartite mixed state $\rho$ is said to be separable if it can be expressed as

$$\rho = \sum_i w_i \rho_i^{(1)} \otimes \rho_i^{(2)}, \quad w_i \geq 0, \quad \sum_i w_i = 1,$$

where $\rho_i^{(1)}$ and $\rho_i^{(2)}$ denote density matrices of subsystems 1 and 2 respectively. Otherwise the state is entangled.

The central tasks of quantum information theory is to characterize and quantify entangled states. A first attempt in characterization of entangled states has been made by Peres and Horodecki family \[6, 7\]. Peres showed that a necessary condition for separability of a two partite system is that its partial transposition be positive. Horodeckis have shown that this condition is sufficient for separability of composite systems only for dimensions $2 \otimes 2$ and $2 \otimes 3$.

There is also an increasing attention in quantifying entanglement, particularly for mixed states of a bipartite system, and a number of measures have been proposed \[5, 8, 9, 10\]. Among them the entanglement of formation has more importance, since it intends to quantify the resources needed to create a given entangled state.

An interesting description of entanglement is Lewenstein-Sanpera decomposition \[11\]. Lewenstein and Sanpera in \[11\] showed that any two partite density matrix can be represented optimally
as a sum of a separable state and an entangled state. They have also shown that for 2-qubit systems
the decomposition reduces to a mixture of a mixed separable state and an entangled pure state,
thus all non-separability content of the state is concentrated in the pure entangled state. This leads
to an unambiguous measure of entanglement for any 2-qubit state as entanglement of pure state
multiplied by the weight of pure part in the decomposition.

In the Ref. [11], the numerical method for finding the BSA has been reported. Also in $2 \otimes 2$
systems some analytical results for special states were found in [12]. In [13] we have been able to
obtain an analytical expression for L-S decomposition of Bell decomposable (BD) states. We have
also obtained the optimal decomposition for a particular class of states obtained from BD states
via some restricted LQCC actions.

In this paper using the fact that, every bipartite $2 \otimes 2$ density matrix can be obtained from
Bell decomposable states via local quantum operations and classical communications (LQCC)[15,
16, 17, 18], we obtain the optimal Lewenstein-Sanpera decomposition of an arbitrary bipart 2 \otimes 2
density matrix through general LQCC action upon the optimal Lewenstein-Sanpera decomposition
of BD states of $2 \otimes 2$ quantum systems, where the product states introduced by Wootters in [W. K.
Wootters, Phys. Rev. Lett. 80 2245 (1998)] form the best separable approximation ensemble for
BD states. We also show that in these systems the average concurrence of the Lewenstein-Sanpera
decomposition is equal to the concurrence of these states.

The paper is organized as follows. In section 2 we give a brief review of Bell decomposable states
together with their separability properties. The concurrence of these states is evaluated in section
3, via the method presented by Wootters in [10]. In section 4 we obtain L-S decomposition of
these states. By using product states defined by Wootters in [10] we prove that the decomposition
is optimal. In section 4 we obtain the optimal decomposition for an arbitrary $2 \otimes 2$ states by using
a general LQCC action which is the main result of this paper. The paper is ended with a brief conclusion in section 5.

2 Bell decomposable states

In this section we review Bell decomposable (BD) states and some of their properties. A BD state is defined by

\[
\rho = \sum_{i=1}^{4} p_i \ket{\psi_i} \bra{\psi_i}, \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^{4} p_i = 1.
\]  

(2-1)

where \( \ket{\psi_i} \) are Bell states given by

\[
\ket{\psi_1} = \ket{\uparrow\uparrow} + \ket{\downarrow\downarrow},
\]  

(2-2)

\[
\ket{\psi_2} = \ket{\uparrow\uparrow} - \ket{\downarrow\downarrow},
\]  

(2-3)

\[
\ket{\psi_3} = \ket{\uparrow\downarrow} + \ket{\downarrow\uparrow},
\]  

(2-4)

\[
\ket{\psi_4} = \ket{\uparrow\downarrow} - \ket{\downarrow\uparrow}.
\]  

(2-5)

These states form a four simplex (tetrahedral) with its vertices defined by \( p_1 = 1, p_2 = 1, p_3 = 1 \) and \( p_4 = 1 \) [14].

A necessary condition for separability of composite quantum systems is presented by Peres [6]. He showed that if a state is separable then the matrix obtained from partial transposition must be positive. Horodecki family [7] have shown that Peres criterion provides sufficient condition only for separability of mixed quantum states of dimensions \( 2 \otimes 2 \) and \( 2 \otimes 3 \). This implies that the state given in Eq. (2-1) is separable if and only if the following inequalities are satisfying

\[
p_i \leq \frac{1}{2}, \quad \text{for } i = 1, 2, 3, 4.
\]  

(2-6)
In the next sections we consider entangled states for which $p_1 \geq \frac{1}{2}$.

### 3 Concurrence

In this section we first give a brief review of concurrence of mixed states. From the various proposed measures of quantification of entanglement, the entanglement of formation has a special position which in fact intends to quantify the resources needed to create a given entangled state [5]. Wootters in [10] has shown that for a 2-qubit system entanglement of formation of a mixed state $\rho$ can be defined as

$$E(\rho) = H \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - C^2} \right) ,$$

where $H(x) = -x \ln x - (1 - x) \ln (1 - x)$ is binary entropy and $C(\rho)$, called concurrence, is defined by

$$C(\rho) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} ,$$

where $\lambda_i$ are the non-negative eigenvalues, with $\lambda_1$ being the largest one, of the Hermitian matrix $R = \sqrt{\sqrt{\rho} \rho \sqrt{\rho}}$ and

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y) ,$$

where $\rho^*$ is the complex conjugate of $\rho$ when it is written in a standard basis such as $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle\}, \{|\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ and $\sigma_y$ represent Pauli matrix in local basis $\{|\uparrow\rangle, |\downarrow\rangle\}$.

In order to obtain the concurrence of BD states we follow the method presented by Wootters in [10]. Starting from spectral decomposition for BD states, given in (2-1), we define subnormalized orthogonal eigenvectors $|v_i\rangle$ as

$$|v_i\rangle = \sqrt{p_i} |\psi_i\rangle , \quad \langle v_i | v_j \rangle = p_i \delta_{ij} .$$


Now, we can define states $|x_i\rangle$ by

$$|x_i\rangle = \sum_{j}^{4} U_{ij}^* |v_i\rangle,$$

for $i = 1, 2, 3, 4,$ (3-11)

such that

$$\langle x_i | \tilde{x}_j \rangle = (U\tau U^T)_{ij} = \lambda_i \delta_{ij},$$

(3-12)

where $\tau_{ij} = \langle v_i | v_j \rangle$ is a symmetric but not necessarily Hermitian matrix. To construct $|x_i\rangle$ we use the fact that for any symmetric matrix $\tau$ one can always find a unitary matrix $U$ in such a way that $\lambda_i$ are real and non-negative, that is, they are the square roots of eigenvalues of $\tau\tau^*$ which are same as eigenvalues of $R$. Moreover one can always find $U$ such that $\lambda_i$ appear in decreasing order.

By using the above protocol we get for the state of $\rho$ given in Eq. (2-1)

$$\tau = \begin{pmatrix}
-p_1 & 0 & 0 & 0 \\
0 & p_2 & 0 & 0 \\
0 & 0 & p_3 & 0 \\
0 & 0 & 0 & -p_4 \\
\end{pmatrix},$$

(3-13)

Now it is easy to evaluate $\lambda_i$ which yields

$$\lambda_1 = p_1, \quad \lambda_2 = p_2, \quad \lambda_3 = p_3, \quad \lambda_4 = p_4.$$  

(3-14)

Then one can evaluate the concurrence of BD states as

$$C = p_1 - p_2 - p_3 - p_4 = 2p_1 - 1.$$  

(3-15)
Finally we introduce the unitary matrix $U$ which is going to be used later

$$U = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$  
(3-16)

4 Lewenstein-Sanpera decomposition

According to Lewenstein-Sanpera decomposition [11], any 2-qubit density matrix $\rho$ can be written as

$$\rho = \lambda \rho_{sep} + (1 - \lambda) |\psi\rangle \langle \psi|, \quad \lambda \in [0, 1],$$  
(4-17)

where $\rho_{sep}$ is a separable density matrix and $|\psi\rangle$ is a pure entangled state. The Lewenstein-Sanpera decomposition of a given density matrix $\rho$ is not unique and, in general, there is a continuum set of L-S decomposition to choose from. The optimal decomposition is, however, unique for which $\lambda$ is maximal and

$$\rho = \lambda^{(opt)} \rho_{sep}^{(opt)} + (1 - \lambda^{(opt)}) |\psi^{(opt)}\rangle \langle \psi^{(opt)}|, \quad \lambda^{(opt)} \in [0, 1].$$  
(4-18)

Lewenstein and Sanpera in [11] have shown that any other decomposition of the form $\rho = \tilde{\lambda} \tilde{\rho}_{sep} + (1 - \tilde{\lambda}) |\tilde{\psi}\rangle \langle \tilde{\psi}|$ with $\tilde{\rho} \neq \rho^{(opt)}$ necessarily implies that $\tilde{\lambda} < \lambda^{(opt)}$ [11]. One should notice that Eq. (4-18) is the required optimal L-S decomposition, that is, $\lambda$ is maximal and $\rho_s$ is the best separable approximation (BSA).

Here in this section we obtain L-S decomposition for BD states. Let us consider entangled state $\rho$ which belongs to entangled region defined by $p_1 \geq \frac{1}{2}$. We start by writing $\rho$ as a convex sum of
pure state $|\psi_1\rangle$ and separable state $\rho_s$ as
\[
\rho = \lambda \rho_s + (1 - \lambda) |\psi_1\rangle \langle \psi_1|.
\] (4-19)

Expanding separable state $\rho_s$ as $\rho_s = \sum_{i=1}^{4} p'_i |\psi_i\rangle \langle \psi_i|$ and using Eq. (2-1) for $\rho$ we arrive at the following results
\[
p'_1 = \frac{1}{2}, \quad p'_i = \frac{p_i}{2(1-p_1)} \quad \text{for} \quad i = 2, 3, 4,
\] (4-20)
and
\[
\lambda = 2(1-p_1).
\] (4-21)

In the rest of this section we will prove that the decomposition (4-19) is the optimal one. To do so we have to find a decomposition for $\rho_s$ in terms of product states $|e_\alpha, f_\alpha\rangle$, i.e.
\[
\rho_s = \sum_\alpha \Lambda_\alpha |e_\alpha, f_\alpha\rangle \langle e_\alpha, f_\alpha|
\] (4-22)
such that the following conditions are satisfied [11]

i) All $\Lambda_\alpha$ are maximal with respect to $\rho_\alpha = \Lambda_\alpha |e_\alpha, f_\alpha\rangle \langle e_\alpha, f_\alpha| + (1 - \lambda) |\psi_1\rangle \langle \psi_1|$ and projector $P_\alpha = |e_\alpha, f_\alpha\rangle \langle e_\alpha, f_\alpha|$.

ii) All pairs $(\Lambda_\alpha, \Lambda_\beta)$ are maximal with respect to $\rho_{\alpha\beta} = \Lambda_\alpha |e_\alpha, f_\alpha\rangle \langle e_\alpha, f_\alpha| + \Lambda_\beta |e_\beta, f_\beta\rangle \langle e_\beta, f_\beta| + (1 - \lambda) |\psi_1\rangle \langle \psi_1|$ and the pairs of projector $(P_\alpha, P_\beta)$.

Then according to [11] $\rho_s$ is BSA and the decomposition given in Eq. (4-19) is optimal.

Lewenstein and Sanpera in [11] have shown that $\Lambda$ is maximal with respect to $\rho$ and $P = |\psi\rangle \langle \psi|$ iff a) if $|\psi\rangle \not\in \mathcal{R}(\rho)$ then $\Lambda = 0$, and b) if $|\psi\rangle \in \mathcal{R}(\rho)$ then $\Lambda = \langle \psi| \rho^{-1} |\psi\rangle^{-1} > 0$. They have also shown that a pair $(\Lambda_1, \Lambda_2)$ is maximal with respect to $\rho$ and a pair of projectors $(P_1, P_2)$ iff: a) if $|\psi_1\rangle, |\psi_2\rangle$ do not belong to $\mathcal{R}(\rho)$ then $\Lambda_1 = \Lambda_2 = 0$; b) if $|\psi_1\rangle$ does not belong, while $|\psi_2\rangle \in \mathcal{R}(\rho)$ then $\Lambda_1 = 0, \Lambda_2 = \langle \psi_2| \rho^{-1} |\psi_2\rangle^{-1}$; c) if $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{R}(\rho)$ and $\langle \psi_1| \rho^{-1} |\psi_2\rangle = 0$ then $\Lambda_i = \langle \psi_i| \rho^{-1} |\psi_i\rangle^{-1}$, $i = 1, 2$; d) finally, if $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{R}(\rho)$ and $\langle \psi_1| \rho^{-1} |\psi_2\rangle \neq 0$ then
\[ \Lambda_1 = \frac{\langle \psi_2 | \rho^{-1} | \psi_2 \rangle - | \langle \psi_1 | \rho^{-1} | \psi_2 \rangle |}{D}, \]

\[ \Lambda_2 = \frac{\langle \psi_1 | \rho^{-1} | \psi_1 \rangle - | \langle \psi_1 | \rho^{-1} | \psi_2 \rangle |}{D}, \] (4-23)

where \( D = \langle \psi_1 | \rho^{-1} | \psi_1 \rangle \langle \psi_2 | \rho^{-1} | \psi_2 \rangle - | \langle \psi_1 | \rho^{-1} | \psi_2 \rangle |^2. \)

Now let us return to show that the decomposition given in Eq. (4-19) is optimal. Wootters in [10] has shown that any \( 2 \otimes 2 \) separable density matrix can be expanded in terms of following product states

\[ |z_1\rangle = \frac{1}{2} \left( e^{i \theta_1} |x_1\rangle + e^{i \theta_2} |x_2\rangle + e^{i \theta_3} |x_3\rangle + e^{i \theta_4} |x_4\rangle \right), \]

\[ |z_2\rangle = \frac{1}{2} \left( e^{i \theta_1} |x_1\rangle + e^{i \theta_2} |x_2\rangle - e^{i \theta_3} |x_3\rangle - e^{i \theta_4} |x_4\rangle \right), \]

\[ |z_3\rangle = \frac{1}{2} \left( e^{i \theta_1} |x_1\rangle - e^{i \theta_2} |x_2\rangle + e^{i \theta_3} |x_3\rangle - e^{i \theta_4} |x_4\rangle \right), \]

\[ |z_4\rangle = \frac{1}{2} \left( e^{i \theta_1} |x_1\rangle - e^{i \theta_2} |x_2\rangle - e^{i \theta_3} |x_3\rangle + e^{i \theta_4} |x_4\rangle \right), \]

(4-24) (4-25) (4-26) (4-27)

provided that \( \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \leq 0. \) Now, the zero concurrence is guaranteed by choosing phases \( \theta_i, i = 1, 2, 3, 4 \) to satisfy the relation \( \sum_{j=1}^4 e^{2i \theta_j} \lambda_j = 0. \)

Now using the fact that for marginal states \( \rho_s \) (located at the boundary of separable region) the eigenvalues \( \lambda_i \) satisfy constraint \( \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 = 0, \) we can choose the phase factors \( \theta_i \) as \( \theta_2 = \theta_3 = \theta_4 = \theta_1 + \frac{\pi}{2}. \) Choosing \( \theta_1 = 0 \) we arrive at the following product ensemble for \( \rho_s \)

\[ |z_1\rangle = \frac{1}{2} \left( -i \sqrt{p_1' | \psi_1 \rangle - i \sqrt{p_2' | \psi_2 \rangle} - i \sqrt{p_3' | \psi_3 \rangle} - \sqrt{p_4' | \psi_4 \rangle} \right), \]

\[ |z_2\rangle = \frac{1}{2} \left( -i \sqrt{p_1' | \psi_1 \rangle - i \sqrt{p_2' | \psi_2 \rangle} + i \sqrt{p_3' | \psi_3 \rangle} + \sqrt{p_4' | \psi_4 \rangle} \right), \]

\[ |z_3\rangle = \frac{1}{2} \left( -i \sqrt{p_1' | \psi_1 \rangle + i \sqrt{p_2' | \psi_2 \rangle} - i \sqrt{p_3' | \psi_3 \rangle} + \sqrt{p_4' | \psi_4 \rangle} \right), \]

\[ |z_4\rangle = \frac{1}{2} \left( -i \sqrt{p_1' | \psi_1 \rangle + i \sqrt{p_2' | \psi_2 \rangle} + i \sqrt{p_3' | \psi_3 \rangle} - \sqrt{p_4' | \psi_4 \rangle} \right), \]

(4-28)

where \( p_i' \) are defined in Eq. (4-20).

Let us consider the set of four product vectors \( \{|z_\alpha\rangle\} \) and one entangled state \( |\psi_1\rangle. \) In Ref. [10] it is shown that the ensemble \( \{|z_\alpha\rangle\} \) are linearly independent. Evaluating Wronskian determinant
of vectors $|\psi\rangle$ and $|z_\alpha\rangle$ we get $W_\alpha = \frac{1}{8}$. This implies that vector $|\psi\rangle$ is linearly independent with respect to all vectors $|z_\alpha\rangle$. Also evaluating the Wronskian of three vectors $|\psi\rangle$, $|z_\alpha\rangle$ and $|z_\beta\rangle$ we get

$$W_{12} = W_{34} = \frac{1}{8} p'_{2}(1 - 2p'_{2}) , \quad W_{13} = W_{24} = \frac{1}{8} p'_{3}(1 - 2p'_{3}), \quad W_{14} = W_{23} = \frac{1}{8} p'_{4}(1 - 2p'_{4}). \quad (4-29)$$

Equations (4-29) shows that in the cases that $\rho$ has full rank three vectors $|z_\alpha\rangle$, $|z_\beta\rangle$ and $|\psi\rangle$ are linearly independent. Now we consider the matrices $\rho_\alpha = \Lambda_\alpha |z_\alpha\rangle \langle z_\alpha| + (1 - \lambda) |\psi\rangle \langle \psi|$. Due to independence of $|z_\alpha\rangle$ and $|\psi\rangle$ we can deduce that the range of $\rho_\alpha$ is two dimensional. Thus after restriction to its range and defining their dual basis $|\hat{z}_\alpha\rangle$ and $|\hat{\psi}_1\rangle$, we can expand restricted inverse $\rho^{-1}_\alpha$ as $\rho^{-1}_\alpha = \Lambda^{-1}_\alpha |\hat{z}_\alpha\rangle \langle \hat{z}_\alpha| + (1 - \lambda)^{-1} |\hat{\psi}_1\rangle \langle \hat{\psi}_1|$. Using Eq. (4-49) it is easy to see that $\langle z_\alpha | \rho^{-1}_\alpha | z_\alpha \rangle = \Lambda^{-1}_\alpha$. This shows that $\Lambda_\alpha$ are maximal with respect to $\rho_\alpha$ and the projector $P_{\alpha}$.

Similarly by considering the matrices $\rho_{\alpha\beta} = \Lambda_\alpha |z_\alpha\rangle \langle z_\alpha| + \Lambda_\beta |z_\beta\rangle \langle z_\beta| + (1 - \lambda) |\psi\rangle \langle \psi|$ and taking into account the independence of three vectors $|z_\alpha\rangle$, $|z_\beta\rangle$ and $|\psi\rangle$ we see that rang of $\rho_{\alpha\beta}$ is three dimensional, where after restriction to its range and defining dual basis $|\hat{z}_\alpha\rangle$, $|\hat{z}_\beta\rangle$ and $|\hat{\psi}_1\rangle$ we can write restricted inverse $\rho^{-1}_{\alpha\beta}$ as $\rho^{-1}_{\alpha\beta} = \Lambda^{-1}_\alpha |\hat{z}_\alpha\rangle \langle \hat{z}_\alpha| + \Lambda^{-1}_\beta |\hat{z}_\beta\rangle \langle \hat{z}_\beta| + (1 - \lambda)^{-1} |\hat{\psi}_1\rangle \langle \hat{\psi}_1|$. Then it is straightforward to get $\langle \hat{e}_\alpha | \rho^{-1}_{\alpha\beta} | \hat{z}_\alpha \rangle = \Lambda^{-1}_\alpha$, $\langle \hat{z}_\beta | \rho^{-1}_{\alpha\beta} | \hat{z}_\beta \rangle = \Lambda^{-1}_\beta$ and $\langle \hat{z}_\alpha | \rho^{-1}_{\alpha\beta} | \hat{z}_\beta \rangle = 0$.

This implies that the pairs $(\Lambda_\alpha, \Lambda_\beta)$ are maximal with respect to $\rho_{\alpha\beta}$ and the pairs of projectors $(P_\alpha, P_\beta)$, hence we can conclude that the decomposition given in Eq. (4-19) is optimal.

We now consider cases that $\rho$ has not full rank. Let $p_\alpha = 0$ for $\alpha \neq 1$. In this case Eq. (4-29) shows that the pairs $\{ |z_1\rangle , |z_\alpha\rangle \}$ and also $\{ |z_\beta\rangle , |z_\gamma\rangle \}$ for $\beta, \gamma \neq 1, \alpha$ are no longer independent with respect to $|\psi\rangle$. In the former case we express $|\psi\rangle$ in terms of $|z_1\rangle$, $|z_\alpha\rangle$ then matrix $\rho_{1\alpha}$ can be written in terms of two basis $|z_1\rangle$, $|z_\alpha\rangle$ and after some calculations we get $\langle z_1 | \rho_{1\alpha}^{-1} | z_1 \rangle = \frac{\Lambda_1 + 2(1 - \lambda)}{\Gamma_{1\alpha}}$, $\langle z_\alpha | \rho_{1\alpha}^{-1} | z_\alpha \rangle = \frac{\Lambda_1 + 2(1 - \lambda)}{\Gamma_{1\alpha}}$ and $\langle z_1 | \rho_{1\alpha}^{-1} | z_\alpha \rangle = -\frac{2(1 - \lambda)}{\Gamma_{1\alpha}}$ where $\Gamma_{1\alpha} = \Lambda_1 \Lambda_\alpha + 2(1 - \lambda)(\Lambda_1 + \Lambda_\alpha)$. By
using the above results together with Eqs. (4-23) we obtain the maximality of pair \((\Lambda_1, \Lambda_\alpha)\) with respect to \(\rho_{1\alpha}\) and the pair of projectors \((P_1, P_\alpha)\).

Similarly for latter case we express \(|\psi_1\rangle\) in terms of \(|z_\beta\rangle, |z_\gamma\rangle\) then matrix \(\rho_{\beta\gamma}\) can be written in terms of two basis \(|z_\beta\rangle, |z_\gamma\rangle\) and we get \(\langle z_\beta | \rho_{\beta\gamma}^{-1} | z_\beta \rangle = \frac{\Lambda_\gamma + 2(1 - \lambda)}{\Gamma_{\beta\gamma}}\) and \(\langle z_\gamma | \rho_{\beta\gamma}^{-1} | z_\gamma \rangle = \frac{\Lambda_\beta + 2(1 - \lambda)}{\Gamma_{\beta\gamma}}\) and \(\langle z_1 | \rho_{1\alpha}^{-1} | z_1 \rangle = \frac{\Lambda_\alpha + 2(1 - \lambda)}{\Gamma_{1\alpha}}\) and \(\langle z_\alpha | \rho_{1\alpha}^{-1} | z_\alpha \rangle = \frac{\Lambda_1 + 2(1 - \lambda)}{\Gamma_{1\alpha}}\) where \(\Gamma_{\beta\gamma} = \Lambda_\beta \Lambda_\gamma + 2(1 - \lambda)(\Lambda_\beta + \Lambda_\gamma)\). Again using the above results together with Eqs. (4-23) we obtain the maximality of pairs \((\Lambda_\beta, \Lambda_\gamma)\) with respect to \(\rho_{\beta\gamma}\) and the pairs of projectors \((P_\beta, P_\gamma)\).

Finally let us consider cases that rank \(\rho\) is 2. Let \(p_\alpha = p_\beta = 0\) for \(\alpha, \beta \neq 1\). In this cases we have \(|z_\alpha\rangle = |z_\beta\rangle\) and \(|z_1\rangle = |z_\gamma\rangle\) for \(\gamma \neq 1, \alpha, \beta\). It is now sufficient to take \(|z_1\rangle\) and \(|z_\alpha\rangle\) as product ensemble. But Eq. (4-29) shows that these vectors are not independent any more, so that we can express \(|\psi_1\rangle\) in terms of \(|z_1\rangle\) and \(|z_\alpha\rangle\), therefore, matrix \(\rho_{1\alpha}\) can be written in terms of two vectors \(|z_1\rangle\) and \(|z_\alpha\rangle\) and we get after some calculations \(\langle z_1 | \rho_{1\alpha}^{-1} | z_1 \rangle = \frac{\Lambda_\alpha + 2(1 - \lambda)}{\Gamma_{1\alpha}}\) and \(\langle z_\alpha | \rho_{1\alpha}^{-1} | z_\alpha \rangle = \frac{\Lambda_1 + 2(1 - \lambda)}{\Gamma_{1\alpha}}\) and \(\langle z_1 | \rho_{1\alpha}^{-1} | z_1 \rangle = \frac{2(1 - \lambda)}{\Gamma_{1\alpha}}\) where \(\Gamma_{1\alpha} = \Lambda_1 \Lambda_\alpha + 2(1 - \lambda)(\Lambda_1 + \Lambda_\alpha)\). Using the above results together with Eqs. (4-23) we deduce the maximality of pairs \((\Lambda_1, \Lambda_\alpha)\) with respect to \(\rho_{1\alpha}\) and the pairs of projectors \((P_1, P_\alpha)\).

5 Behavior of L-S decomposition under LQCC

In this section we study the behavior of L-S decomposition under local quantum operations and classical communications (LQCC). A general LQCC is defined by \([13, 16]\)

\[
\rho' = \frac{(A \otimes B) \rho (A \otimes B)^\dagger}{Tr((A \otimes B) \rho (A \otimes B)^\dagger)}, \quad (5-30)
\]

where operators \(A\) and \(B\) can be written as

\[
A \otimes B = U_A f^{\mu,a,m} \otimes U_B f^{\nu,b,n}, \quad (5-31)
\]
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where $U_A$ and $U_B$ are unitary operators acting on subsystems $A$ and $B$, respectively and the filtration $f$ is defined by

$$f^{\mu,a,m} = \mu(I_2 + a m \sigma),$$

$$f^{\nu,b,n} = \nu(I_2 + b n \sigma).$$

As it is shown in Refs. [15, 16], the concurrence of the state $\rho$ transforms under LQCC of the form given in Eq. (5-30) as

$$C(\rho') = \frac{\mu^2 \nu^2 (1-a^2)(1-b^2)}{Tr((A \otimes B)\rho(A \otimes B)\dagger)} C(\rho).$$

Performing LQCC on L-S decomposition of BD states we get

$$\rho' = \frac{(A \otimes B)\rho(A \otimes B)\dagger}{Tr((A \otimes B)\rho(A \otimes B)\dagger)} = \lambda' \rho'_{s} + (1-\lambda')|\psi'\rangle \langle \psi'|,$$

with $\rho'_{s}$ and $|\psi'\rangle$ defined as

$$\rho'_{s} = \frac{(A \otimes B)\rho_{s}(A \otimes B)\dagger}{Tr((A \otimes B)\rho_{s}(A \otimes B)\dagger)},$$

$$|\psi'\rangle = \frac{(A \otimes B)|\psi_{1}\rangle}{\sqrt{\langle \psi_{1}|(A A^\dagger \otimes B B^\dagger)|\psi_{1}\rangle}},$$

respectively, and $\lambda'$ is

$$\lambda' = \frac{Tr((A \otimes B)\rho_{s}(A \otimes B)\dagger)}{Tr((A \otimes B)\rho(A \otimes B)\dagger)} \lambda.$$

Using Eq. (5-37), we get for the weight of entangled part in the decomposition (5-34)

$$(1-\lambda') = \frac{\langle \psi_{1}|(A A^\dagger \otimes B B^\dagger)|\psi_{1}\rangle}{Tr((A \otimes B)\rho(A \otimes B)\dagger)} (1-\lambda).$$

Now we can easily evaluate the average concurrence of $\rho'$ in the L-S decomposition given in (5-34)

$$(1-\lambda')C(|\psi'\rangle) = \frac{\mu^2 \nu^2 (1-a^2)(1-b^2)}{Tr((A \otimes B)\rho(A \otimes B)\dagger)} (1-\lambda)C(|\psi_{1}\rangle),$$

where, by comparing the above equation with Eq. (5-33) we see that $(1-\lambda)C(|\psi\rangle)$ (the average concurrence in the L-S decomposition) transforms like the concurrence under LQCC.
Now we would like to show that the decomposition given in Eq. (5-34) is optimal. To do so, we perform LQCC action on matrices $\rho_{\alpha} = \Lambda_{\alpha} \left| z_{\alpha} \right\rangle \langle z_{\alpha} \right| + (1 - \lambda) \left| \psi_{1} \right\rangle \langle \psi_{1} \right|$ and get

$$\rho'_{\alpha} = \frac{(A \otimes B)\rho_{\alpha}(A \otimes B)^\dagger}{\text{Tr}((A \otimes B)\rho_{\alpha}(A \otimes B)^\dagger)} = \Lambda_{\alpha}' \left| z'_{\alpha} \right\rangle \langle z'_{\alpha} \right| + (1 - \lambda') \left| \psi'_{1} \right\rangle \langle \psi'_{1} \right|$$  \hspace{1cm} (5-40)

where

$$\left| z'_{\alpha} \right\rangle = \frac{(A \otimes B)\left| z_{\alpha} \right\rangle}{\sqrt{\langle z_{\alpha}|(AA^\dagger \otimes BB^\dagger)|z_{\alpha}\rangle}},$$  \hspace{1cm} (5-41)

and

$$\Lambda_{\alpha}' = \frac{\langle z_{\alpha}|(AA^\dagger \otimes BB^\dagger)|z_{\alpha}\rangle}{\text{Tr}((A \otimes B)\rho_{\alpha}(A \otimes B)^\dagger)} \Lambda_{\alpha}.$$  \hspace{1cm} (5-42)

Using the fact that LQCC transformations are invertible \cite{16, 17, 18}, we can evaluate $\rho_{\alpha}'^{-1}$ as

$$\rho_{\alpha}'^{-1} = \text{Tr}((A \otimes B)\rho_{\alpha}(A \otimes B)^\dagger)(A^\dagger \otimes B^\dagger)^{-1}\rho_{\alpha}^{-1}(A \otimes B)^{-1}.$$  \hspace{1cm} (5-43)

Using the above equation and Eq. (5-41) we get

$$\langle z'_{\alpha}|\rho_{\alpha}'^{-1}|z'_{\alpha}\rangle = \frac{\text{Tr}((A \otimes B)\rho_{\alpha}(A \otimes B)^\dagger)}{\langle z_{\alpha}|(AA^\dagger \otimes BB^\dagger)|z_{\alpha}\rangle} \langle z_{\alpha}|\rho_{\alpha}^{-1}|z_{\alpha}\rangle = \Lambda_{\alpha}'^{-1}.$$  \hspace{1cm} (5-44)

Equation (5-44) shows that $\Lambda_{\alpha}'$s are maximal with respect to $\rho_{\alpha}'$ and the projector $P_{\alpha}'$.

Matrices $\rho_{\alpha\beta} = \Lambda_{\alpha} \left| z_{\alpha} \right\rangle \langle z_{\alpha} \right| + \Lambda_{\beta} \left| z_{\beta} \right\rangle \langle z_{\beta} \right| + (1 - \lambda) \left| \psi_{1} \right\rangle \langle \psi_{1} \right|$ transform under LQCC as

$$\rho'_{\alpha\beta} = \frac{(A \otimes B)\rho_{\alpha\beta}(A \otimes B)^\dagger}{\text{Tr}((A \otimes B)\rho_{\alpha\beta}(A \otimes B)^\dagger)} = \Lambda_{\alpha\beta}' \left| z'_{\alpha,\beta} \right\rangle \langle z'_{\alpha,\beta} \right| + \Lambda_{\beta}' \left| z'_{\beta} \right\rangle \langle z'_{\beta} \right| + (1 - \lambda') \left| \psi'_{1} \right\rangle \langle \psi'_{1} \right|$$  \hspace{1cm} (5-45)

where

$$\left| z'_{\alpha,\beta} \right\rangle = \frac{(A \otimes B)\left| z_{\alpha,\beta} \right\rangle}{\sqrt{\langle z_{\alpha,\beta}|(AA^\dagger \otimes BB^\dagger)|z_{\alpha,\beta}\rangle}},$$  \hspace{1cm} (5-46)

and

$$\Lambda_{\alpha\beta}' = \frac{\langle z_{\alpha,\beta}|(AA^\dagger \otimes BB^\dagger)|z_{\alpha,\beta}\rangle}{\text{Tr}((A \otimes B)\rho_{\alpha\beta}(A \otimes B)^\dagger)} \Lambda_{\alpha,\beta}.$$  \hspace{1cm} (5-47)
We now consider cases that $\rho$ is full rank. In these cases we have already showed that all vectors $|z_\alpha\rangle$, $|z_\beta\rangle$ and $|\psi_1\rangle$ are linearly independent. Using the above results together with invertibility of LQCC actions we arrive at the following results

\[
\langle z'_\alpha | \rho_{\alpha\beta}^{-1} | z'_\alpha \rangle = \frac{\text{Tr}((A \otimes B) \rho_{ab}(A \otimes B)^\dagger)}{\langle z_\alpha | (AA^\dagger \otimes BB^\dagger) | z_\alpha \rangle} \langle z_\alpha | \rho_{\alpha\beta}^{-1} | z_\alpha \rangle = \Lambda'_\alpha, \\
\langle z'_\beta | \rho_{\alpha\beta}^{-1} | z'_\beta \rangle = \frac{\text{Tr}((A \otimes B) \rho_{ab}(A \otimes B)^\dagger)}{\langle z_\beta | (AA^\dagger \otimes BB^\dagger) | z_\beta \rangle} \langle z_\beta | \rho_{\alpha\beta}^{-1} | z_\beta \rangle = \Lambda'_\beta, \\
\langle z'_\alpha | \rho_{\alpha\beta}^{-1} | z'_\beta \rangle = \frac{\text{Tr}((A \otimes B) \rho_{ab}(A \otimes B)^\dagger)}{\sqrt{\langle z_\alpha | (AA^\dagger \otimes BB^\dagger) | z_\alpha \rangle \langle z_\beta | (AA^\dagger \otimes BB^\dagger) | z_\beta \rangle}} \langle z_\alpha | \rho_{\alpha\beta}^{-1} | z_\beta \rangle = 0.
\]

Equations (5-48) show that the pair $(\Lambda'_\alpha, \Lambda'_\beta)$ are maximal with respect to $\rho'_{\alpha,\beta}$ and the pair of projectors $(P'_\alpha, P'_\beta)$. For other cases that $\rho$ is not full rank we saw that there is some dependency between three vectors $|z_\alpha\rangle$, $|z_\beta\rangle$ and $|\psi_1\rangle$ such that $\langle z_\alpha | \rho_{\alpha\beta}^{-1} | z_\beta \rangle \neq 0$. This implies that in general $\langle z'_\alpha | \rho_{\alpha\beta}^{-1} | z'_\beta \rangle \neq 0$. In this cases in [13] we have shown that under restricted LQCC actions for which $A = B$, the optimality of the decomposition given in (5-34) will be achieved.

6 Conclusion

We have derived Lewenstein-Sanpera decomposition for BD states and have showed that for these states the average concurrence of the decomposition is equal to their concurrence. It is also shown that product states introduced by Wootters in [10] form BSA ensemble for these states. By performing LQCC action on these states we have been able to obtain optimal decomposition for all $2 \otimes 2$ systems. It is also shown that for these states the average concurrence of the decomposition is equal to their concurrence.

Appendix
Let us consider the set of linearly independent vectors $\{|\phi_i\rangle\}$, then one can define their dual vectors $\{|\hat{\phi}_i\rangle\}$ such that the following relation

$$\langle \hat{\phi}_i | \phi_j \rangle = \delta_{ij} \quad (6-49)$$

hold. It is straightforward to show that the $\{|\phi_i\rangle\}$ and their dual $\{|\hat{\phi}_i\rangle\}$ possess the following completeness relation

$$\sum_i |\hat{\phi}_i\rangle\langle \phi_i| = I, \quad \sum_i |\phi_i\rangle\langle \hat{\phi}_i| = I. \quad (6-50)$$

Consider an invertible operator $M$ which is expanded in terms of states $|\phi_i\rangle$ as

$$M = \sum_i a_{ij} |\phi_i\rangle\langle \phi_j| \quad (6-51)$$

Then the inverse of $M$ denoted by $M^{-1}$ can be expanded in terms of dual bases as

$$M^{-1} = \sum_i b_{ij} |\hat{\phi}_i\rangle\langle \hat{\phi}_j| \quad (6-52)$$

where $b_{ij} = (A^{-1})_{ij}$ and $A_{ij} = a_{ij}$.

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