Lectures on Lie Groups over Local Fields

Helge Glöckner†

Abstract
The goal of these notes is to provide an introduction to $p$-adic Lie groups and Lie groups over fields of Laurent series, with an emphasis on the dynamics of automorphisms and the specialization of Willis’ theory to this setting. In particular, we shall discuss the scale, tidy subgroups and contraction groups for automorphisms of Lie groups over local fields. Special attention is paid to the case of Lie groups over local fields of positive characteristic.

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Introduction

Lie groups over local fields (notably $p$-adic Lie groups) are among those totally disconnected, locally compact groups which are both well accessible and arise frequently. For example, $p$-adic Lie groups play an important role in the theory of pro-$p$-groups (i.e., projective limits of finite $p$-groups), where they are called “analytic pro-$p$-groups.” In ground-breaking work in the 1960s, Michel Lazard obtained deep insights into the structure of analytic pro-$p$-groups and characterized them within the class of pro-$p$-groups [32] (see [10] and [11] for later developments).

It is possible to study Lie groups over local fields from various points of view and on various levels, taking more and more structure into account. At the most basic level, they can be considered as mere topological groups. Next, we can consider them as analytic manifolds, enabling us to use ideas from Lie theory and properties of analytic functions. At the highest level, one

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can focus on those Lie groups which are linear algebraic groups over local fields (in particular, reductive or semi-simple groups), and study them using techniques from the theory of algebraic groups and algebraic geometry.

While much of the literature focusses either on pro-$p$-groups or algebraic groups, our aims are complementary: We emphasize aspects at the borderline between Lie theory and the structure theory of totally disconnected, locally compact groups, as developed in [46], [47] and [3]. In particular, we shall discuss the scale, tidy subgroups and contraction groups for automorphisms of Lie groups over local fields.

For each of these topics, an algebraic group structure does not play a role, but merely the Lie group structure.

**Scope and structure of the lectures.** After an introduction to some essentials of Lie theory and calculus over local fields, we discuss the scale, contraction groups, tidy subgroups (and related topics) in three stages:

- In a first step, we consider linear automorphisms of finite-dimensional vector spaces over local fields.

A sound understanding of this special case is essential.

- Next, we turn to automorphisms of $p$-adic Lie groups.

- Finally, we discuss Lie groups over local fields of positive characteristic and their automorphisms.

For various reasons, the case of positive characteristic is more complicated than the $p$-adic case, and it is one of our goals to explain which additional ideas are needed to tackle this case.

As usual in an expository article, we shall present many facts without proof. However, we have taken care to give at least a sketch of proof for all central results, and to explain the main ideas. In particular, we have taken care to explain carefully the ideas needed to deal with Lie groups over local fields of positive characteristic, and found it appropriate to give slightly more details when they are concerned (the more so because not all of the results are available yet in the published literature).

Mention should be made of what these lectures do not strive to achieve. First of all, we do not intend to give an introduction to linear algebraic groups.
over local fields, nor to Bruhat-Tits buildings or to the representation theory of \( p \)-adic groups (and harmonic analysis thereon). The reader will find nothing about these topics here.

Second, we shall hardly speak about the theory of analytic pro-\( p \)-groups, although this theory is certainly located at a very similar position in the spectrum ranging from topological groups to algebraic groups (one of Lazard’s results will be recalled in Section 3, but no later developments). One reason is that we want to focus on aspects related to the structure theory of totally disconnected, locally compact groups—which is designed primarily for the study of non-compact groups.

It is interesting to note that also in the area of pro-\( p \)-groups, Lie groups over fields (and suitable pro-\( p \)-rings) of positive characteristic are attracting more and more attention. The reader is referred to the seminal work [33] and subsequent research (like [1], [2], [7], [28] and [29]; cf. also [11]).

Despite the importance of \( p \)-adic Lie groups in the area of pro-\( p \)-groups, it is not fully clear yet whether \( p \)-adic Lie groups (or Lie groups over local fields) play a fundamental role in the theory of general locally compact groups (comparable to that of real Lie groups, as in [25]). Two recent studies indicate that this may be the case, at least in the presence of group actions. They describe situations where Lie groups over local fields are among the building blocks for general structures:

**Contraction groups** Let \( G \) be a totally disconnected, locally compact group and \( \alpha: G \to G \) be an automorphism which is contractive, i.e., \( \alpha^n(x) \to 1 \) as \( n \to \infty \), for each \( x \in G \) (where \( 1 \in G \) is the neutral element). Then the torsion elements form a closed subgroup \( \text{tor}(G) \) and

\[
G = \text{tor}(G) \times G_{p_1} \times \cdots \times G_{p_n}
\]

for certain \( \alpha \)-stable \( p \)-adic Lie groups \( G_p \) (see [22]).

**\( \mathbb{Z}^n \)-actions** If \( \mathbb{Z}^n \) acts with a dense orbit on a locally compact group \( G \) via automorphisms, then \( G \) has a compact normal subgroup \( K \) invariant under the action such that

\[
G/K \cong \mathbb{L}_1 \times \cdots \times \mathbb{L}_m ,
\]

where \( \mathbb{L}_1, \ldots, \mathbb{L}_m \) are local fields of characteristic 0 and the action is diagonal, via scalar multiplication by field elements [9].
In both studies, Lazard’s theory of analytic pro-$p$-groups accounts for the occurrence of $p$-adic Lie groups.

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1 Generalities

This section compiles elementary definitions and facts concerning local fields, analytic functions and Lie groups.

Basic information on local fields can be found in many books, e.g. [44] and [39]. Our main sources for Lie groups over local fields are [40] and [6].

Local fields

By a local field, we mean a totally disconnected, locally compact, non-discrete topological field $K$. Each local field admits an ultrametric absolute value $|.|$ defining its topology, i.e.,

(a) $|t| \geq 0$ for each $t \in K$, with equality if and only if $t = 0$;

(b) $|st| = |s| \cdot |t|$ for all $s, t \in K$;
(c) The *ultrametric inequality* holds, i.e., $|s + t| \leq \max\{|s|, |t|\}$ for all $s, t \in K$.

An example of such an absolute value is what we call the *natural absolute value*, given by $|0| := 0$ and

$$
|x| := \Delta_K(m_x) \quad \text{for} \quad x \in K \setminus \{0\} \quad (1)
$$

(cf. [44, Chapter II, §2]), where $m_x : K \rightarrow K, y \mapsto xy$ is scalar multiplication by $x$ and $\Delta_K(m_x)$ its module (the definition of which is recalled in [13]).

It is known that every local field $K$ either is a field of formal Laurent series over some finite field (if char($K$) > 0), or a finite extension of the field of $p$-adic numbers for some prime $p$ (if char($K$) = 0). Let us fix our notation concerning these basic examples.

**Example 1.1** Given a prime number $p$, the field $\mathbb{Q}_p$ of $p$-adic numbers is the completion of $\mathbb{Q}$ with respect to the $p$-adic absolute value,

$$
\left| p^{k} \frac{n}{m} \right|_p := p^{-k} \quad \text{for} \quad k \in \mathbb{Z} \text{ and } n, m \in \mathbb{Z} \setminus p\mathbb{Z}.
$$

We use the same notation, $|\cdot|_p$, for the extension of the $p$-adic absolute value to $\mathbb{Q}_p$. Then the topology coming from $|\cdot|_p$ makes $\mathbb{Q}_p$ a local field, and $|\cdot|_p$ is the natural absolute value on $\mathbb{Q}_p$. Every non-zero element $x$ in $\mathbb{Q}_p$ can be written uniquely in the form

$$
x = \sum_{k=n}^{\infty} a_k p^k
$$

with $n \in \mathbb{Z}$, $a_k \in \{0, 1, \ldots, p-1\}$ and $a_n \neq 0$. Then $|x|_p = p^{-n}$. The elements of the form $\sum_{k=0}^{\infty} a_k p^k$ form the subring $\mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \}$ of $\mathbb{Q}_p$, which is open and also compact, because it is homeomorphic to $\{0, 1, \ldots, p-1\}^{\mathbb{N}_0}$ via $\sum_{k=0}^{\infty} a_k p^k \mapsto (a_k)_{k \in \mathbb{N}_0}$.

**Example 1.2** Given a finite field $\mathbb{F}$ (with $q$ elements), we let $\mathbb{F}(\!(X) \!)$ be the field of formal Laurent series $\sum_{k=n}^{\infty} a_k X^k$ with $a_k \in \mathbb{F}$. Here addition

2Note that if $K$ is an extension of $\mathbb{Q}_p$ of degree $d$, then $|p| = p^{-d}$ depends on the extension.
is pointwise, and multiplication is given by the Cauchy product. We endow $F((X))$ with the topology arising from the ultrametric absolute value

$$
\left| \sum_{k=n}^{\infty} a_k X^k \right| := q^{-n} \quad \text{if} \ a_n \neq 0. \quad (2)
$$

Then the set $F[[X]]$ of formal power series $\sum_{k=0}^{\infty} a_k X^k$ is a compact and open subring of $F((X))$, and thus $F((X))$ is a local field. Its natural absolute value is given by (2).

Beyond local fields, we shall occasionally use ultrametric fields $(\mathbb{K}, |.|)$. Thus $\mathbb{K}$ is a field and $|.|$ an ultrametric absolute value on $\mathbb{K}$ which defines a non-discrete topology on $\mathbb{K}$. For example, we occasionally use an algebraic closure $\overline{\mathbb{K}}$ of a local field $\mathbb{K}$ and exploit that an ultrametric absolute value $|.|$ on $\mathbb{K}$ extends uniquely to an ultrametric absolute value on $\overline{\mathbb{K}}$ (see, e.g., [39, Theorem 16.1]). The same notation, $|.|$, will be used for the extended absolute value. An ultrametric field $(\mathbb{K}, |.|)$ is called complete if $\mathbb{K}$ is a complete metric space with respect to the metric given by $d(x, y) := |x - y|$.

**Basic consequences of the ultrametric inequality**

Let $(\mathbb{K}, |.|)$ be an ultrametric field and $(E, \|\|)$ be a normed $\mathbb{K}$-vector space whose norm is ultrametric in the sense that $\|x + y\| \leq \max\{|x|, |y|\}$ for all $x, y \in E$.

Since $\|x\| = \|x + y - y\| \leq \max\{|x + y|, |y|\}$, it follows that $\|x + y\| \geq |x|$ if $|y| < |x|$ and hence

$$\|x + y\| = \|x\| \quad \text{for all} \ x, y \in E \ \text{such that} \ |y| < |x| \quad (3)$$

(“the strongest wins,” [38, p. 77]). Hence, small perturbations have much smaller impact in the ultrametric case than they would have in the real case (a philosophy which will be made concrete below).

The ultrametric inequality has many other useful consequences. For example, consider the balls

$$B_r^E(x) := \{ y \in E : \|y - x\| < r \}$$

and

$$\overline{B}_r^E(x) := \{ y \in E : \|y - x\| \leq r \}$$
for $x \in E$, $r \in [0, \infty[$. Then $B_r^E(0)$ and $\overline{B_r^E}(0)$ are subgroups of $(E, +)$ with non-empty interior (and hence are both open and closed). Specializing to $E = \mathbb{K}$, we see that

$$\mathcal{O} := \{t \in \mathbb{K} : |t| \leq 1\}$$

(4)

is an open subring of $\mathbb{K}$, its so-called valuation ring. If $\mathbb{K}$ is a local field, then $\mathcal{O}$ is a compact subring of $\mathbb{K}$ (which is maximal and hence independent of the choice of absolute value).

Calculation of indices

Indices of compact subgroups inside others are omnipresent in the structure theory of totally disconnected groups. We therefore perform some basic calculations of indices now. Haar measure $\lambda_G$ on a locally compact group $G$ will be used as a tool and also the notion of the module of an automorphism, which measures the distortion of Haar measure. We recall:

1.3 Let $G$ be a locally compact group and $\mathcal{B}(G)$ be the $\sigma$-algebra of Borel subsets of $G$. We let $\lambda_G : \mathcal{B}(G) \to [0, \infty]$ be a Haar measure on $G$, i.e., a non-zero Radon measure which is left-invariant in the sense that $\lambda_G(gA) = \lambda_G(A)$ for all $g \in G$ and $A \in \mathcal{B}(G)$. It is well-known that a Haar measure always exists, and that it is unique up to multiplication with a positive real number (cf. [24]). If $\alpha : G \to G$ is a (bicontinuous) automorphism, then also

$$\mathcal{B}(G) \to [0, \infty], \quad A \mapsto \lambda_G(\alpha(A))$$

is a left-invariant non-zero Radon measure on $G$ and hence a multiple of Haar measure: There exists $\Delta(\alpha) > 0$ such that $\lambda_G(\alpha(A)) = \Delta(\alpha)\lambda_G(A)$ for all $A \in \mathcal{B}(G)$. If $U \subseteq G$ is a relatively compact, open, non-empty subset, then

$$\Delta(\alpha) = \frac{\lambda_G(\alpha(U))}{\lambda_G(U)}$$

(5)

(cf. [24] (15.26)], where however the conventions differ). We also write $\Delta_G(\alpha)$ instead of $\Delta(\alpha)$, if we wish to emphasize the underlying group $G$.

Remark 1.4 Let $U$ be a compact open subgroup of $G$. If $U \subseteq \alpha(U)$, with index $[\alpha(U) : U] =: n$, we can pick representatives $g_1, \ldots, g_n \in \alpha(U)$ for the
left cosets of $U$ in $\alpha(U)$. Exploiting the left-invariance of Haar measure, (5) turns into

$$\Delta(\alpha) = \frac{\lambda_G(\alpha(U))}{\lambda_G(U)} = \sum_{j=1}^{n} \frac{\lambda_G(g_jU)}{\lambda_G(U)} = [\alpha(U) : U]. \quad (6)$$

If $\alpha(U) \subseteq U$, applying (6) to $\alpha^{-1}$ instead of $\alpha$ and $\alpha(U)$ instead of $U$, we obtain

$$\Delta(\alpha^{-1}) = [U : \alpha(U)]. \quad (7)$$

In the following, the group $\text{GL}_n(\mathbb{K})$ of invertible $n \times n$-matrices will frequently be identified with the group $\text{GL}(\mathbb{K}^n)$ of linear automorphisms of $\mathbb{K}^n$.

**Lemma 1.5** Let $\mathbb{K}$ be a local field, $|.|$ its natural absolute value, and $A \in \text{GL}_n(\mathbb{K})$. Then

$$\Delta_{\mathbb{K}^n}(A) = |\det A| = \prod_{i=1}^{n} |\lambda_i|, \quad (8)$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$ in an algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$.

**Proof.** Let $\lambda_{\mathbb{K}^n}$ and $\lambda_{\mathbb{K}}$ be Haar measures on $(\mathbb{K}^n, +)$ and $(\mathbb{K}, +)$, respectively, such that $\lambda_{\mathbb{K}^n} = \lambda_{\mathbb{K}} \otimes \cdots \otimes \lambda_{\mathbb{K}}$. Let $\mathcal{O} \subseteq \mathbb{K}$ be the valuation ring and $U := \mathcal{O}^n$. Since both $A \mapsto \Delta_{\mathbb{K}^n}(A)$ and $A \mapsto |\det A|$ are homomorphisms from $\text{GL}_n(\mathbb{K})$ to the multiplicative group $]0, \infty[\,$, it suffices to check the first equality in (8) for diagonal matrices and elementary matrices of the form $1 + e_{ij}$ with matrix units $e_{ij}$, where $i \neq j$ (because these generate $\text{GL}_n(\mathbb{K})$ as a group). For such an elementary matrix $A$, we have $A(U) = U$ and thus

$$\Delta_{\mathbb{K}^n}(A) = \frac{\lambda_{\mathbb{K}^n}(A(U))}{\lambda_{\mathbb{K}^n}(U)} = 1 = |\det A|. \quad (9)$$

If $A$ is diagonal with diagonal entries $t_1, \ldots, t_n$, we also have

$$\Delta_{\mathbb{K}^n}(A) = \frac{\lambda_{\mathbb{K}^n}(A(U))}{\lambda_{\mathbb{K}^n}(U)} = \prod_{i=1}^{n} \frac{\lambda_{\mathbb{K}}(t_i \mathcal{O})}{\lambda_{\mathbb{K}}(\mathcal{O})} = \prod_{i=1}^{n} \Delta_{\mathbb{K}}(m_{t_i}) = \prod_{i=1}^{n} |t_i| = |\det A|, \quad (10)$$

as required. The final equality in (8) is clear if all eigenvalues lie in $\mathbb{K}$. For the general case, pick a finite extension $\mathbb{L}$ of $\mathbb{K}$ containing the eigenvalues, and let
\[ d := [L : \mathbb{K}] \text{ be the degree of the field extension. Then } \Delta_{L^n}(A) = (\Delta_{\mathbb{K}^n}(A))^d. \]

Since the extended absolute value is given by
\[ |x| = \sqrt[\lvert d \rvert]{\Delta_L(m_x)} \]
(see [27] Chapter 9, Theorem 9.8 or [39] Exercise 15.E), the final equality follows from the special case already treated (applied now to \( \mathbb{L} \)). \[ \square \]

1.6 For later use, consider a ball \( B_r(0) \subseteq \mathbb{Q}_p^n \) with respect to the maximum norm, where \( r \in ]0, \infty[ \). Let \( m \in \mathbb{N} \) and \( A \in \text{GL}_n(\mathbb{Q}_p) \) be the diagonal matrix with diagonal entries \( p^m, \ldots, p^m \). Then (7) and (8) imply that
\[ [B_r(0) : B_{p^{-m}r}(0)] = [B_r(0) : A.B_r(0)] = |\det(A^{-1})| = p^{mn}. \] (9)

Analytic functions, manifolds and Lie groups

Given a local field \((\mathbb{K}, |.|)\) and \( n \in \mathbb{N} \), we equip \( \mathbb{K}^n \) with the maximum norm,
\[ \|(x_1, \ldots, x_n)\| := \max\{|x_1|, \ldots, |x_n|\} \] (the choice of norm does not really matter because all norms are equivalent; see [39] Theorem 13.3). If \( \alpha \in \mathbb{N}_0^n \) is a multi-index, we write \( |\alpha| := \alpha_1 + \cdots + \alpha_n \). We mention that confusion with the absolute value \(|.|\) is unlikely; the intended meaning of \(|.|\) will always be clear from the context. If \( \alpha \in \mathbb{N}_0^n \) and \( y = (y_1, \ldots, y_n) \in \mathbb{K}^n \), we abbreviate \( y^\alpha := y_1^{\alpha_1} \cdots y_n^{\alpha_n} \), as usual. See [40] for the following concepts.

Definition 1.7 Given an open subset \( U \subseteq \mathbb{K}^n \), a map \( f : U \to \mathbb{K}^m \) is called \textit{analytic} if it is given locally by a convergent power series around each point \( x \in U \), i.e.,
\[ f(x + y) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha y^\alpha \quad \text{for all } y \in \overline{B}_r^{\mathbb{K}^n}(0), \]
with \( a_\alpha \in \mathbb{K}^m \) and some \( r > 0 \) such that \( \overline{B}_r(\mathbb{K}^n)(x) \subseteq U \) and
\[ \sum_{\alpha \in \mathbb{N}_0^n} \|a_\alpha\| r^{\lvert \alpha \rvert} < \infty. \]

\[ ^3 \text{In other parts of the literature related to rigid analytic geometry, such functions are called \textit{locally analytic} to distinguish them from functions which are globally given by a power series.} \]
It can be shown that compositions of analytic functions are analytic [40, Theorem, p. 70]. We can therefore define an \( n \)-dimensional analytic manifold \( M \) over a local field \( \mathbb{K} \) in the usual way, namely as a Hausdorff topological space \( M \), equipped with a set \( \mathcal{A} \) of homeomorphisms \( \phi \) from open subsets of \( M \) onto open subsets of \( \mathbb{K}^n \) such that the transition map \( \psi \circ \phi^{-1} \) is analytic, for all \( \phi, \psi \in \mathcal{A} \).

Analytic mappings between analytic manifolds are defined as usual (by checking analyticity in local charts).

A Lie group over a local field \( \mathbb{K} \) is a group \( G \), equipped with a (finite-dimensional) analytic manifold structure which turns the group multiplication
\[
m: G \times G \to G, \quad m(x, y) := xy
\]
and the group inversion
\[
j: G \to G, \quad j(x) := x^{-1}
\]
into analytic mappings.

Besides the additive groups of finite-dimensional \( \mathbb{K} \)-vector spaces, the most obvious examples of \( \mathbb{K} \)-analytic Lie groups are general linear groups.

**Example 1.8** \( \text{GL}_n(\mathbb{K}) = \text{det}^{-1}(\mathbb{K}^\times) \) is an open subset of the space \( \text{M}_n(\mathbb{K}) \cong \mathbb{K}^{n^2} \) of \( n \times n \)-matrices and hence is an \( n^2 \)-dimensional \( \mathbb{K} \)-analytic manifold. The group operations are rational maps and hence analytic.

More generally, one can show (cf. [34, Chapter I, Proposition 2.5.2]):

**Example 1.9** Every (group of \( \mathbb{K} \)-rational points of a) linear algebraic group defined over \( \mathbb{K} \) is a \( \mathbb{K} \)-analytic Lie group, viz. every subgroup \( G \leq \text{GL}_n(\mathbb{K}) \) which is the set of joint zeros of a set of polynomial functions \( \text{M}_n(\mathbb{K}) \to \mathbb{K} \).

For example, \( \text{SL}_n(\mathbb{K}) = \{ A \in \text{GL}_n(\mathbb{K}) : \text{det}(A) = 1 \} \) is a \( \mathbb{K} \)-analytic Lie group.

**Remark 1.10** There are much more general Lie groups than linear Lie groups. For instance, in Remark 9.8 we shall encounter an analytic Lie group \( G \) over \( \mathbb{K} = \mathbb{F}_p((X)) \) which does not admit a faithful, continuous linear representation \( G \to \text{GL}_n(\mathbb{K}) \) for any \( n \).
The Lie algebra functor. If \( G \) is a \( \mathbb{K} \)-analytic Lie group, then its tangent space \( L(G) := T_1(G) \) at the identity element can be made a Lie algebra via the identification of \( x \in L(G) \) with the corresponding left invariant vector field on \( G \) (noting that the left invariant vector fields form a Lie subalgebra of the Lie algebra \( \mathcal{V}^\omega(G) \) of all analytic vector fields on \( G \)).

If \( \alpha : G \to H \) is an analytic group homomorphism between \( \mathbb{K} \)-analytic Lie groups, then the tangent map \( L(\alpha) := T_1(\alpha) : L(G) \to L(H) \) is a linear map and actually a Lie algebra homomorphism (cf. [6, Chapter 3, §3.7 and §3.8], Lemma 5.1 on p. 129 in [40, Part II, Chapter V.1]). An analytic automorphism of a Lie group \( G \) is an invertible group homomorphism \( \alpha : G \to G \) such that both \( \alpha \) and \( \alpha^{-1} \) are analytic.

Ultrametric inverse function theorem

Since small perturbations do not change the size of a given non-zero vector in the ultrametric case (as “the strongest wins”), the ultrametric inverse function theorem has a much nicer form than its classical real counterpart. Around a point with invertible differential, an analytic map behaves on balls simply like an affine-linear map (namely its linearization).

In the following three propositions, we let \((\mathbb{K},|.|)\) be a complete ultrametric field and equip \( \mathbb{K}^n \) with any ultrametric norm (e.g., the maximum norm). Given \( x \in \mathbb{K}^n \) and \( r > 0 \), we abbreviate \( B_r(x) := B^\mathbb{K}^n_r(x) \). The total differential of \( f \) at \( x \) is denoted by \( f'(x) \). Now the ultrametric inverse function theorem (for analytic functions) reads as follows.

**Proposition 1.11** Let \( f : U \to \mathbb{K}^n \) be an analytic map on an open set \( U \subseteq \mathbb{K}^n \) and \( x \in U \) such that \( f'(x) \in \text{GL}_n(\mathbb{K}) \). Then there exists \( r > 0 \) such that \( B_r(x) \subseteq U \),

\[
f(B_s(y)) = f(y) + f'(x)B_s(0) \quad \text{for all } y \in B_r(x) \text{ and } s \in [0,r],
\]

and \( f|_{B_s(y)} \) is an analytic diffeomorphism onto its open image. If \( f'(x) \) is an isometry, then so is \( f|_{B_r(x)} \) for small \( r \). \( \square \)

It is useful that \( r \) can be chosen uniformly in the presence of parameters. As a special case of [14, Theorem 8.1 (b)], (which only requires that \( f \) be \( C^k \) in
diffeomorphism for each $q$ the current work. We also mention that the results from [14] were strengthened further in [19].

However, for each $0 \neq r > 0$ let $\delta$ be an isometry, then
\[ f_q(B_r(y)) = f_q(y) + f'_p(x)B_r(0) \]  
for all $q \in Q$, $y \in B_r(x)$ and $s \in ]0, r[$. If $f'_q(x)$ is an isometry, then also $f_q|_{B_r(x)}$ is an isometry for all $q \in Q$, if $Q$ and $r$ are chosen sufficiently small. \hfill \square

We mention that the group of linear isometries is large (an open subgroup).

**Proposition 1.13** The group $\text{Iso}(\mathbb{K}^n)$ of linear isometries is open in $\text{GL}_n(\mathbb{K})$. If $\mathbb{K}^n$ is equipped with the maximum norm, then
\[ \text{Iso}(\mathbb{K}^n) = \text{GL}_n(\mathbb{K}) = \{ A \in \text{GL}_n(\mathbb{K}) : A, A^{-1} \in M_n(\mathbb{O}) \} , \]
where $\mathbb{O}$ (as in (4)) is the valuation ring of $\mathbb{K}$.

**Proof.** The subgroup $\text{Iso}(\mathbb{K}^n)$ will be open in $\text{GL}_n(\mathbb{K})$ if it is an identity neighbourhood. The latter is guaranteed if we can prove that $1 + A \in \text{Iso}(\mathbb{K}^n)$ for all $A \in M_n(\mathbb{K})$ of operator norm
\[ \| A \|_{\text{op}} := \sup \{ \frac{\| Ax \|}{\| x \|} : 0 \neq x \in \mathbb{K}^n \} < 1 . \]
However, for each $0 \neq x \in \mathbb{K}^n$ we have $\| Ax \| \leq \| A \|_{\text{op}} \| x \| < \| x \|$ and hence $\| (1 + A)x \| = \| x + Ax \| = \| x \|$ by (3), as the strongest wins.

Now assume that $\| \cdot \|$ is the maximum norm on $\mathbb{K}^n$, and $A \in \text{GL}_n(\mathbb{K})$. If $A$ is an isometry, then $\| Ae_i \| = 1$ for the standard basis vector $e_i$ with $j$-th component $\delta_{ij}$, and hence $A \in M_n(\mathbb{O})$. Likewise, $A^{-1} \in M_n(\mathbb{O})$ and hence $A \in \text{GL}_n(\mathbb{O})$. If $A \in \text{GL}_n(\mathbb{O})$, then $\| Ax \| \leq \| x \|$ and $\| x \| = \| A^{-1} Ax \| \leq \| Ax \|$, whence $\| Ax \| = \| x \|$ and $A$ is an isometry. \hfill \square

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5Combining the cited theorem and Proposition[11.11] one can also show that, for $Q$ and $r$ small enough, $f_q|_{B_r(x)} : B_r(x) \to f_q(x) + f'_p(x)B_r(0) = f_p(x) + f'_p(0)B_r(0)$ is an analytic diffeomorphism for each $q \in Q$. However, this additional information shall not be used in the current work. We also mention that the results from [14] were strengthened further in [19].
Construction of small open subgroups

Let $G$ be a Lie group over a local field $\mathbb{K}$, and $|.|$ be an absolute value on $\mathbb{K}$ defining its topology. Fix an ultrametric norm $\|\|$ on $L(G)$ and abbreviate $B_r(x) := B_r^{L(G)}(x)$ for $x \in L(G)$ and $r > 0$. Using a chart $\phi: G \supseteq U \to V \subseteq L(G)$ around 1 such that $\phi(1) = 0$, the group multiplication gives rise to a multiplication $\mu: W \times W \to V$, $\mu(x, y) = x \ast y$ for an open 0-neighbourhood $W \subseteq V$, via

$$x \ast y := \phi(\phi^{-1}(x)\phi^{-1}(y)).$$

It is easy to see that the first order Taylor expansions of multiplication and inversion in local coordinates read

$$x \ast y = x + y + \cdots$$

and

$$x^{-1} = -x + \cdots$$

(compare [40, p. 113]). Applying the ultrametric inverse function theorem with parameters to the maps $(x, y) \mapsto x \ast y$ and $(x, y) \mapsto y \ast x$, we find $r > 0$ with $B_r(0) \subseteq W$ such that

$$x \ast B_s(0) = x + B_s(0) = B_s(0) \ast x$$

for all $x \in B_r(0)$ and $s \in [0, r]$ (exploiting that all relevant differentials or partial differentials are isometries in view of (12) and (13)). In particular, (14) implies that $B_s(0) \ast B_s(0) = B_s(0)$ for each $s \in [0, r]$, and thus also $y^{-1} \in B_s(0)$ for each $y \in B_s(0)$.

Summing up:

**Lemma 1.14** $(B_s(0), \ast)$ is a group for each $s \in [0, r]$ and hence $\phi^{-1}(B_s(0))$ is a compact open subgroup of $G$, for each $s \in [0, r]$. Moreover, $B_s(0)$ is a normal subgroup of $(B_r(0), \ast)$. □

Thus small balls in $L(G)$ correspond to compact open subgroups in $G$. These special subgroups are very useful for many purposes. In particular, we shall see later that for suitable choices of the norm $\|\|$, the groups $\phi^{-1}(B_s(0))$ will be tidy for a given automorphism $\alpha$, as long as $\alpha$ is well-behaved (exceptional cases where this goes wrong will be pinpointed as well).

**Remark 1.15** Note that (14) entails that the indices of $B_s(0)$ in $(B_r(0), +)$ and $(B_r(0), \ast)$ coincide (as the cosets coincide). This observation will be useful later.
2 Basic facts concerning $p$-adic Lie groups

For each local field $\mathbb{K}$ of characteristic 0, the exponential series $\sum_{k=0}^{\infty} \frac{1}{k!} A^k$ converges on some 0-neighbourhood $U \subseteq M_n(\mathbb{K})$ and defines an analytic mapping $\exp : U \to \text{GL}_n(\mathbb{K})$. More generally, it can be shown that every $\mathbb{K}$-analytic Lie group $G$ admits an exponential function in the following sense:

**Definition 2.1** An analytic map $\exp_G : U \to G$ on an open 0-neighbourhood $U \subseteq L(G)$ with $\mathcal{O}U \subseteq U$ is called an exponential function if $\exp_G(0) = 1$, $T_0(\exp_G) = \text{id}$ and

$$\exp_G((s + t)x) = \exp_G(sx)\exp_G(tx) \quad \text{for all } x \in U \text{ and } s, t \in \mathcal{O},$$

where $\mathcal{O} := \{ t \in \mathbb{K} : |t| \leq 1 \}$ is the valuation ring.

Since $T_0(\exp_G) = \text{id}$, after shrinking $U$ one can assume that $\exp_G(U)$ is open and $\exp_G$ is a diffeomorphism onto its image. By Lemma 1.14, after shrinking $U$ further if necessary, we may assume that $\exp_G(U)$ is a subgroup of $G$. Hence also $U$ can be considered as a Lie group. The Taylor expansion of multiplication with respect to the logarithmic chart $\exp_G^{-1}$ is given by the Baker-Campbell-Hausdorff (BCH-) series

$$x \ast y = x + y + \frac{1}{2}[x, y] + \cdots \quad (15)$$

(all terms of which are nested Lie brackets with rational coefficients), and hence $x \ast y$ is given by this series for small $U$ (cf. Lemma 3 and Theorem 2 in [8, Chapter 3, §4, no. 2]). In this case, we call $\exp_G(U)$ a BCH-subgroup of $G$.

For later use, we note that

$$x^n = x \ast \cdots \ast x = nx \quad (16)$$

for all $x \in U$ and $n \in \mathbb{N}_0$ (since $[x, x] = 0$ in (15)). As a consequence, the closed subgroup of $(U, \ast)$ generated by $x \in U$ is of the form

$$\langle x \rangle = \mathbb{Z}_p x. \quad (17)$$

We also note that

$$\exp_U := \text{id}_U \quad (18)$$
is an exponential map for $U$.

Next, let us consider homomorphisms between Lie groups. Recall that if $\alpha: G \to H$ is an analytic homomorphism between real Lie groups, then the diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\alpha} & H \\
\exp_G & & \uparrow \exp_H \\
L(G) & \xrightarrow{L(\alpha)} & L(H)
\end{array}
$$

commutes (a fact referred to as the “naturality of $\exp$”). If $\alpha: G \to H$ is an analytic homomorphism between Lie groups over a local field $\mathbb{K}$ of characteristic 0, we can still choose $\exp_G: U_G \to G$ and $\exp_H: U_H \to H$ with $L(\alpha).U_G \subseteq U_H$ and

$$
\exp_H \circ L(\alpha)|_{U_G} = \alpha \circ \exp_G
$$

(see Proposition 8 in [6, Chapter 3, §4, no. 4]).

The following fact (see [6, Chapter 3, Theorem 1 in §8, no. 1]) is essential.

**Proposition 2.2** Every continuous homomorphism between $p$-adic Lie groups is analytic.

As a consequence, there is at most one $p$-adic Lie group structure on a given topological group. Following the general custom, we call a topological group a $p$-adic Lie group if it admits a $p$-adic Lie group structure.

We record standard facts; for the proofs, the reader is referred to Proposition 7 in §1, no. 4; Proposition 11 in §1, no. 6; and Theorem 2 in §8, no. 2 in [6, Chapter 3].

**Proposition 2.3** Closed subgroups, finite direct products and Hausdorff quotient groups of $p$-adic Lie groups are $p$-adic Lie groups.

It is also known that a topological group which is an extension of a $p$-adic Lie group by a $p$-adic Lie group is again a $p$-adic Lie group. In fact, the case of compact $p$-adic Lie groups is known in the theory of analytic pro-$p$-groups (one can combine [43, Proposition 8.1.1 (b)] with Proposition 1.11 (ii) and Corollary 8.33 from [10]). The general case then is a well-known consequence (see, e.g., [22, Lemma 9.1 (a)]).

The following fact is essential for us (cf. Step 1 in the proof of Theorem 3.5 in [43]).
Proposition 2.4 Every $p$-adic Lie group $G$ has an open subgroup $U$ which satisfies the ascending chain condition on closed subgroups.

Proof. We show that every BCH-subgroup $U$ has the desired property. It suffices to discuss $U \subseteq L(G)$ with the BCH-multiplication. It is known that the Lie algebra $L(H)$ of a closed subgroup $H \leq U$ can be identified with the set of all $x \in L(U) = L(G)$ such that

$$\exp_U(Wx) \subseteq H$$

for some 0-neighbourhood $W \subseteq \mathbb{Q}_p$ (see Corollary 1 (ii) in [6, Chapter 3, §4, no. 4]). Since $\exp_U = \text{id}_U$ here (see (18)) and $\mathbb{Z}_p x = \langle x \rangle \subseteq H$ for each $x \in H$ (by (17)), we deduce that

$$L(H) = \text{span}_{\mathbb{Q}_p}(H)$$

is the linear span of $H$ in the current situation. Now consider an ascending series $H_1 \leq H_2 \leq \cdots$ of closed subgroups. We may assume that each $H_n$ has the same dimension and thus $\mathfrak{h} := L(H_1) = L(H_2) = \cdots$ for each $n$. Then $H := \mathfrak{h} \cap U$ is a compact group in which $H_1$ is open, whence $[H : H_1]$ is finite and the series has to stabilize. $\blacksquare$

We record an important consequence:

Proposition 2.5 If $G$ is a $p$-adic Lie group and $H_1 \subseteq H_2 \subseteq \cdots$ an ascending sequence of closed subgroups of $G$, then also $H := \bigcup_{n \in \mathbb{N}} H_n$ is closed in $G$.

Proof. Let $V \subseteq G$ be an open subgroup satisfying an ascending chain condition on closed subgroups. Then

$$V \cap H_1 \subseteq V \cap H_2 \subseteq \cdots$$

is an ascending sequence of closed subgroups of $V$ and thus stabilizes, say at $V \cap H_m$. Then

$$V \cap H = V \cap H_m$$

is closed in $G$. Being locally closed, the subgroup $H$ is closed (compare [24, Theorem (5.9)]). $\blacksquare$

Two important applications are now described. The second one (Corollary [23] is part of [3] Theorem 3.5 (ii)]. The first application might also be deduced from the second using [3] Theorem 3.32].
2.6 We recall from [46] and [47]: If $G$ is a totally disconnected, locally compact group and $\alpha$ an automorphism of $G$, then a compact open subgroup $V \subseteq G$ is called tidy for $\alpha$ if it has the following properties:

TA $V = V_+ V_-$, where $V_+ := \bigcap_{n \in \mathbb{N}_0} \alpha^n(V)$ and $V_- := \bigcap_{n \in \mathbb{N}_0} \alpha^{-n}(V)$; and

TB The ascending union $V_{++} := \bigcup_{n \in \mathbb{N}_0} \alpha^n(V_+)$ is closed in $G$.

If $V$ satisfies TA, it is also called tidy above. If $V$ satisfies TB, it is also called tidy below.

**Corollary 2.7** Let $G$ be a $p$-adic Lie group, $\alpha : G \to G$ be an automorphism and $V \subseteq G$ be a compact open subgroup. Then $V$ is tidy below.

**Proof.** The subgroup $\bigcup_{n \in \mathbb{N}_0} \alpha^n(V_+)$ is an ascending union of closed subgroups of $G$ and hence closed, by Proposition 2.5. □

The second application of Proposition 2.5 concerns contraction groups.

**Definition 2.8** Given a topological group $G$ and automorphism $\alpha : G \to G$, we define the contraction group $^6$ of $\alpha$ via

$$U_\alpha := \{ x \in G : \alpha^n(x) \to 1 \text{ as } n \to \infty \} .$$

(20)

**Corollary 2.9** Let $G$ be a $p$-adic Lie group. Then the contraction group $U_\alpha$ is closed in $G$, for each automorphism $\alpha : G \to G$.

**Proof.** Let $V_1 \supseteq V_2 \supseteq \cdots$ be a sequence of compact open subgroups of $G$ which form a basis of identity neighbourhoods (cf. Lemma 1.14). Then an element $x \in G$ belongs to $U_\alpha$ if and only if

$$\forall n \in \mathbb{N} \, (\exists m \in \mathbb{N}) \, (\forall k \geq m) \, \alpha^k(x) \in V_n .$$

Since $\alpha^k(x) \in V_n$ if and only if $x \in \alpha^{-k}(V_n)$, we deduce that

$$U_\alpha = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{k \geq m} \alpha^{-k}(V_n) .$$

Note that $W_n := \bigcup_{m \in \mathbb{N}} \bigcap_{k \geq m} \alpha^{-k}(V_n)$ is an ascending union of closed subgroups of $G$ and hence closed, by Proposition 2.5. Consequently, $U_\alpha = \bigcap_{n \in \mathbb{N}} W_n$ is closed. □

---

$^6$Some authors may prefer to call $\alpha$ the contraction subgroup of $\alpha$. Another recent notation for $U_\alpha$ is $\operatorname{con}(\alpha)$. 

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Remark 2.10 We mention that contraction groups of the form $U_\alpha$ arise in many contexts: In representation theory in connection with the Mautner phenomenon (see [34] Chapter II, Lemma 3.2) and (for the $p$-adic case) [43]; in probability theory on groups (see [23], [41], [42] and (for the $p$-adic case) [8]); and in the structure theory of totally disconnected, locally compact groups [3].

3 Lazard’s characterization of $p$-adic groups

Lazard [32] obtained various characterizations of $p$-adic Lie groups within the class of locally compact groups, and many more have been found since in the theory of analytic pro-$p$-groups (see [10], pp. 97–98). These characterizations (and the underlying theory of analytic pro-$p$-groups) are of great value for the structure theory of totally disconnected groups. For example, they explain the occurrence of $p$-adic Lie groups in the areas of $\mathbb{Z}^n$-actions and contractive automorphisms mentioned in the introduction. We recall one of Lazard’s characterizations in a form recorded in [40] p. 157:

Theorem 3.1 A topological group $G$ is a $p$-adic Lie group if and only if it has an open subgroup $U$ with the following properties:

(a) $U$ is a pro-$p$-group;

(b) $U$ is topologically finitely generated, i.e., $U = \langle F \rangle$ for a finite subset $F \subseteq U$;

(c) $[U,U] := \langle xyx^{-1}y^{-1}: x, y \in U \rangle \subseteq \{x^{p^n}: x \in U\}$.

Although the proof of the sufficiency of conditions (a)–(c) is non-trivial, their necessity is clear. In fact, (a) is immediate from [9] and Remark 1.15, while (c) can easily be proved using the second order Taylor expansion of the commutator map and the inverse function theorem (applied to the map $x \mapsto x^{p^n}$). To obtain (b), one picks an exponential map $\exp_G: V \to G$ as well as a basis $x_1, \ldots, x_d \in V$ of $L(G)$, and notes that the analytic map

$$\phi: (\mathbb{Z}_p)^d \to G, \quad (t_1, \ldots, t_d) \mapsto \exp_G(t_1x_1) \cdot \ldots \cdot \exp_G(t_dx_d)$$

has an invertible differential at the origin (the map $(t_1, \ldots, t_d) \mapsto \sum_{j=1}^d t_jx_j$), and hence restricts to a diffeomorphism from $p^n\mathbb{Z}_p^d$ (for some $n \in \mathbb{N}_0$) onto an
open identity neighbourhood $U$, which can be chosen as subgroup of $G$ (by Lemma 1.14). Replacing the elements $x_j$ with $p^n x_j$, we can always achieve that $\phi$ is a diffeomorphism from $\mathbb{Z}_p^d$ onto a compact open subgroup $U$ of $G$ (occasionally, one speaks of “coordinates of the second kind” in this situation). Then
\[
\langle \exp_G(x_1), \ldots, \exp_G(x_d) \rangle = U,
\]
establishing (b).

4 Iteration of linear automorphisms

A good understanding of linear automorphisms of vector spaces over local fields is essential for an understanding of automorphisms of general Lie groups.

Decomposition of $E$ and adapted norms

For our first consideration, we let $(\mathbb{K}, |.|)$ be a complete ultrametric field, $E$ be a finite-dimensional $\mathbb{K}$-vector space, and $\alpha : E \to E$ be a linear automorphism. We let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$, and use the same symbol, $|.|$, for the unique extension of the given absolute value to $\overline{\mathbb{K}}$ (see [39, Theorem 16.1]). We let $R(\alpha)$ be the set of all absolute values $|\lambda|$, where $\lambda \in \overline{\mathbb{K}}$ is an eigenvalue of the automorphism $\alpha_{\overline{\mathbb{K}}} := \alpha \otimes \text{id}_{\overline{\mathbb{K}}}$ of $E_{\overline{\mathbb{K}}} := E \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ obtained by extension of scalars. For each $\lambda \in \overline{\mathbb{K}}$, we let
\[
(E_{\overline{\mathbb{K}}})_{(\lambda)} := \{ x \in E_{\overline{\mathbb{K}}}: (\alpha_{\overline{\mathbb{K}}} - \lambda)^d x = 0 \}
\]
be the generalized eigenspace of $\alpha_{\overline{\mathbb{K}}}$ in $E_{\overline{\mathbb{K}}}$ corresponding to $\lambda$ (where $d$ is the dimension of the $\mathbb{K}$-vector space $E$). Given $\rho \in R(\alpha)$, we define
\[
(E_{\overline{\mathbb{K}}})_{\rho} := \bigoplus_{|\lambda| = \rho} (E_{\overline{\mathbb{K}}})_{(\lambda)} \subseteq E_{\overline{\mathbb{K}}},
\]
where the sum is taken over all $\lambda \in \overline{\mathbb{K}}$ such that $|\lambda| = \rho$. As usual, we identify $E$ with $E \otimes 1 \subseteq E_{\overline{\mathbb{K}}}$.

The following fact (cf. (1.0) on p. 81 in [34, Chapter II]) is essential: \footnote{In [34] p. 81, $\mathbb{K}$ is a local field, but the proof works also for complete ultrametric fields.}
Lemma 4.1 For each $\rho \in R(\alpha)$, the vector subspace $(E_{\overline{K}})_\rho$ of $E_{\overline{K}}$ is defined over $\mathbb{K}$, i.e., $(E_{\overline{K}})_\rho = (E_{\rho})_{\overline{K}}$, where $E_{\rho} := (E_{\overline{K}})_\rho \cap E$. Thus

$$E = \bigoplus_{\rho \in R(\alpha)} E_{\rho},$$

and each $E_{\rho}$ is an $\alpha$-invariant vector subspace of $E$. \hfill \Box

It is useful for us that certain well-behaved norms exist on $E$ (cf. [12, Lemma 3.3] and its proof for the $p$-adic case).

Definition 4.2 A norm $\|\cdot\|$ on $E$ is adapted to $\alpha$ if the following holds:

A1 $\|\cdot\|$ is ultrametric;

A2 $\|\sum_{\rho \in R(\alpha)} x_{\rho}\| = \max\{\|x_{\rho}\| : \rho \in R(\alpha)\}$ for each $(x_{\rho})_{\rho \in R(\alpha)} \in \prod_{\rho \in R(\alpha)} E_{\rho}$;

A3 $\|\alpha(x)\| = \rho \|x\|$ for each $\rho \in R(\alpha)$ and $x \in E_{\rho}$.

Proposition 4.3 Let $E$ be a finite-dimensional vector space over a complete ultrametric field $(\mathbb{K}, |\cdot|)$ and $\alpha : E \to E$ be a linear automorphism. Then $E$ admits a norm $\|\cdot\|$ adapted to $\alpha$.

This follows from the next lemma.

Lemma 4.4 For each $\rho \in R(\alpha)$, there exists an ultrametric norm $\|\cdot\|_{\rho}$ on $E_{\rho}$ such that $\|\alpha(x)\|_{\rho} = \rho \|x\|_{\rho}$ for each $x \in E_{\rho}$.

Proof. Assume $\rho \geq 1$ first. We choose a $\overline{\mathbb{K}}$-basis $w_1, \ldots, w_m$ of $(E_{\overline{K}})_\rho =: V$ such that the matrix $A_\rho$ of $\alpha_{\overline{K}}|V : V \to V$ with respect to this basis has Jordan canonical form. We let $\|\cdot\|$ be the maximum norm on $V$ with respect to this basis.

If $\rho = 1$, then $A_\rho \in \text{GL}_m(\overline{\mathcal{O}})$, where $\overline{\mathcal{O}} = \{t \in \overline{\mathbb{K}} : |t| \leq 1\}$ is the valuation ring of $\overline{\mathbb{K}}$, and hence $\alpha_{\overline{K}}|V$ is an isometry with respect to $\|\cdot\|$ (by Proposition [1.13]). Thus $\|\alpha_{\overline{K}}(x)\| = \|x\| = \rho \|x\|$ for all $x \in V$.

If $\rho > 1$, we note that for each $k \in \{1, \ldots, m\}$, there exists an eigenvalue $\mu_k$ such that $w_k \in (E_{\overline{K}})_{\mu_k}$. Then $\alpha_{\overline{K}}(w_k) = \mu_k w_k + (\alpha_{\overline{K}}(w_k) - \mu_k w_k)$ for each $k$, with $\|\mu_k w_k\| = \rho > 1 \geq \|\alpha_{\overline{K}}(w_k) - \mu_k w_k\|$. As a consequence, $\|\alpha_{\overline{K}}(x)\| = \rho \|x\|$ for each $x \in V$, using (3).
In either of the preceding cases, we define \( \| \cdot \|_{\rho} \) as the restriction of \( \| \cdot \| \) to \( E^1_{\rho} \). To complete the proof, assume \( \rho < 1 \) now. Then \( E^1_{\rho} = E^{1-1}_{\rho^{-1}}(\alpha^{-1}) \), where \( \rho^{-1} > 1 \). Thus, by what has already been shown, there exists an ultrametric norm \( \| \cdot \|_{\rho} \) on \( E^1_{\rho} \) such that \( \| \alpha^{-1}(x) \|_{\rho} = \rho^{-1} \| x \|_{\rho} \) for each \( x \in E^1_{\rho} \). Then \( \| \alpha(x) \|_{\rho} = \rho \| x \|_{\rho} \) for each \( x \in E^1_{\rho} \), as required. \( \square \)

**Proof of Proposition 4.3.** For each \( \rho \in R(\alpha) \), we choose a norm \( \| \cdot \|_{\rho} \) on \( E^1_{\rho} \) as described in Lemma 4.4. Then
\[
\left\| \sum_{\rho \in R(\alpha)} x_{\rho} \right\| := \max \left\{ \| x_{\rho} \|_{\rho} : \rho \in R(\alpha) \right\} \quad \text{for } (x_{\rho})_{\rho \in R(\alpha)} \in \prod_{\rho \in R(\alpha)} E^1_{\rho}
\]
defines a norm \( \| \cdot \| : E \to [0, \infty[ \) which, by construction, is adapted to \( \alpha \). \( \square \)

**Contraction group and Levi factor for a linear automorphism**

For \( \alpha \) a linear automorphism of a finite-dimensional vector space \( E \) over a local field \( \mathbb{K} \), we now determine the contraction group (as introduced in Definition 2.8) and the associated Levi factor, defined as follows:

**Definition 4.5** Let \( G \) be a topological group and \( \alpha : G \to G \) be an automorphism. Following [3], we define the **Levi factor** \( M_\alpha \) as the set of all \( x \in G \) such that the two-sided orbit \( \alpha^{\pm}(x) \) is relatively compact in \( G \).

It is clear that both \( U_\alpha \) and \( M_\alpha \) are subgroups. If \( G \) is locally compact and totally disconnected, then \( M_\alpha \) is closed, as can be shown using tidy subgroups as a tool (see [3, p. 224]).

**Proposition 4.6** Let \( \alpha \) be a linear automorphism of a finite-dimensional \( \mathbb{K} \)-vector space \( E \). Then
\[
U_\alpha = \bigoplus_{\rho \in R(\alpha), \rho < 1} E^1_{\rho}, \quad M_\alpha = E^1_1 \quad \text{and} \quad U^{-1}_{\alpha} = \bigoplus_{\rho \in R(\alpha), \rho > 1} E^1_{\rho}.
\]
Furthermore, \( E = U_\alpha \oplus M_\alpha \oplus U^{-1}_{\alpha} \) as a \( \mathbb{K} \)-vector space.

**Proof.** Using an adapted norm on \( E \), the characterizations of \( U_\alpha, M_\alpha \) and \( U^{-1}_{\alpha} \) are clear. That \( E \) is the indicated direct sum follows from (22). \( \square \)
Tidy subgroups and the scale for a linear automorphism

In the preceding situation, define

\[ E_+ := \bigoplus_{\rho \geq 1} E_\rho \quad \text{and} \quad E_- := \bigoplus_{\rho \leq 1} E_\rho. \]  \hfill (23)

**Proposition 4.7** For each \( r > 0 \) and norm \( \| \cdot \| \) on \( E \) adapted to \( \alpha \), the ball \( B_r := \{ x \in E : \| x \| < r \} \) is tidy for \( \alpha \), with

\[ (B_r)_\pm = B_r \cap E_\pm. \]  \hfill (24)

The scale of \( \alpha \) is given by

\[ s_E(\alpha) = \Delta_{E_+}(\alpha|_{E_+}) = |\det(\alpha|_{E_+})| = \prod_{j \in \{1, \ldots, d\} : |\lambda_j| \geq 1} |\lambda_j|, \]  \hfill (25)

where \( \lambda_1, \ldots, \lambda_d \) are the eigenvalues of \( \alpha \) (occurring with multiplicities).

**Proof.** Let \( x \in E \). It is clear from the definition of an adapted norm that \( \alpha^{-n}(x) \in B_r \) for all \( n \in \mathbb{N}_0 \) (i.e., \( x \in (B_r)_+ \)) if and only if \( x \in \bigoplus_{\rho \geq 1} E_\rho = E_+ \). Thus \( (B_r)_+ \) is given by \((24)\), and \( (B_r)_- \) can be discussed analogously. It follows from property A2 in the definition of an adapted norm that

\[ B_r = \bigoplus_{\rho \in R(\alpha)} (B_r \cap E_\rho), \]  \hfill (26)

and thus \( B_r = (B_r)_+ + (B_r)_- \). Hence \( B_r \) is tidy above. Since

\[ (B_r)_+^+ = \bigcup_{n \in \mathbb{N}_0} \alpha^n((B_r)_+) = (B_r \cap E_1) \oplus \bigoplus_{\rho > 1} E_\rho \]

is closed, \( B_r \) is also tidy below and thus \( B_r \) is tidy.

Let \( \lambda \) be a Haar measure on \( E_+ \). Since \( K := (B_r)_+ \) is open in \( E_+ \), compact, and \( K \subseteq \alpha(K) \), we obtain

\[ \Delta_{E_+}(\alpha|_{E_+}) = \frac{\lambda(\alpha(K))}{\lambda(K)} = \frac{[\alpha(K) : K] \lambda(K)}{\lambda(K)} = [\alpha(K) : K] = s_E(\alpha), \]

using the translation invariance of Haar measure for the second equality. Here

\[ \Delta_{E_+}(\alpha|_{E_+}) = |\det(\alpha|_{E_+})| = \prod_{j \in \{1, \ldots, d\} : |\lambda_j| \geq 1} |\lambda_j|, \]

by (8) in Lemma L5. \hfill \Box
5 Scale and tidy subgroups for automorphisms of \(p\)-adic Lie groups

In this section, we determine tidy subgroups and calculate the scale for automorphisms of a \(p\)-adic Lie group.

5.1 Applying \((19)\) to an automorphism \(\alpha\) of a \(p\)-adic Lie group \(G\), we find an exponential function \(\exp_G: V \to G\) which is a diffeomorphism onto its image, and an open 0-neighbourhood \(W \subseteq V\) such that \(L(\alpha).W \subseteq V\) and

\[
\exp_G \circ L(\alpha)|_W = \alpha \circ \exp_G|_W.
\]  

(27)

Thus \(\alpha\) is locally linear in a suitable (logarithmic) chart, which simplifies an understanding of the dynamics of \(\alpha\). Many aspects can be reduced to the dynamics of the linear automorphism \(\beta := L(\alpha)\) of \(L(G)\), as discussed in Section \([2]\). After shrinking \(W\), we may assume that also \(L(\alpha^{-1}).W \subseteq V\) and

\[
\exp_G \circ L(\alpha^{-1})|_W = \alpha^{-1} \circ \exp_G|_W.
\]  

(28)

To construct subgroups tidy for \(\alpha\), we pick a norm \(\| \cdot \|\) on \(E := L(G)\) adapted to \(\beta := L(\alpha)\), as in Proposition \([4,3]\). There is \(r > 0\) such that \(B_r := B_r^E(0) \subseteq W\) and

\[
V_t := \exp_G(B_t)
\]

is a compact open subgroup of \(G\), for each \(t \in ]0, r]\) (see Lemma \([14]\)). Abbreviate \(Q_t := B_t \cap \bigoplus_{\rho < 1} E_{\rho}\). Since \(B_t = (B_t)_+ \oplus Q_t\) by \((26)\) (applied to the linear automorphism \(\beta\)), the ultrametric inverse function theorem (Proposition \([11]\) and \([12]\) imply that \(V_t = \exp_G((B_t)_+) \exp_G(Q_t)\) and thus

\[
V_t = \exp_G((B_t)_+) \exp_G((B_t)_-)
\]  

(29)

for all \(t \in ]0, r]\), after shrinking \(r\) if necessary. Then we have (as first recorded in \([15]\) Theorem 3.4 (c)):

**Theorem 5.2** The subgroup \(V_t\) of \(G\) is tidy for \(\alpha\), for each \(t \in ]0, r]\). Moreover,

\[
s_G(\alpha) = s_{L(G)}(L(\alpha)),
\]

where \(s_{L(G)}(L(\alpha))\) can be calculated as in Proposition \([17]\).  

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Proof. Since \( \beta((B_t)_-) \subseteq (B_t)_- \) and \( \beta^{-1}((B_t)_+) \subseteq (B_t)_+ \), it follows with (27), (28) and induction that
\[
\alpha^{T_n} (\exp_G((B_t)_\pm)) = \exp_G(\beta^{T_n}((B_t)_\pm)) \subseteq \exp_G((B_t)_\pm)
\]
for each \( n \in \mathbb{N}_0 \) and hence
\[
\exp_G((B_t)_\pm) \subseteq (V_t)_\pm.
\]
Therefore \( V_t = (V_t)_+(V_t)_- \) (using (29)) and thus \( V_t \) is tidy above. By Corollary 2.7 \( V_t \) is also tidy below and thus \( V_t \) is tidy for \( \alpha \).

Closer inspection shows that \( L((V_t)_\pm) = L(G)_\pm \) (see [12, Theorem 3.5] or proof of Theorem 9.3 below), where \( L(G)_\pm \) is defined as in (23), using \( \beta \). Since the module of a Lie group automorphism can be calculated on the Lie algebra level (using Proposition 55 (ii) in [6, Chapter 3, §3, no.16]), it follows that \( s_G(\alpha) = \Delta_{(V_t)_+}(\alpha|_{(V_t)_+}) = \Delta_{L(G)_+}(L(\alpha)|_{L(G)_+}) = s_{L(G)}(L(\alpha)) \). \( \square \)

6 \( p \)-adic contraction groups

Exploiting the local linearity of \( \alpha \) (see (27)) and the closedness of \( p \)-adic contraction groups (see Corollary 2.9), it is possible to reduce the discussion of \( p \)-adic contraction groups to contraction groups of linear automorphisms, as in Proposition 4.6. We record the results, due to Wang [43, Theorem 3.5].

\textbf{Theorem 6.1} Let \( G \) be a \( p \)-adic Lie group and \( \alpha: G \to G \) be an automorphism. Abbreviate \( \beta := L(\alpha) \). Then \( U_\alpha, M_\alpha \) and \( U_{\alpha^{-1}} \) are closed subgroups of \( G \) with Lie algebras
\[
L(U_\alpha) = U_\beta, \quad L(M_\alpha) = M_\beta \quad \text{and} \quad L(U_{\alpha^{-1}}) = U_{\beta^{-1}},
\]
respectively. Furthermore, \( U_\alpha M_\alpha U_{\alpha^{-1}} \) is an open identity neighbourhood in \( G \), and the product map
\[
U_\alpha \times M_\alpha \times U_{\alpha^{-1}} \to U_\alpha M_\alpha U_{\alpha^{-1}}, \quad (x,y,z) \mapsto xyz
\]
is an analytic diffeomorphism. \( \square \)

In Part (ii) of his Theorem 3.5, Wang also obtained essential information concerning the groups \( U_\alpha \):
**Theorem 6.2** Let \( G \) be a \( p \)-adic Lie group admitting a contractive automorphism \( \alpha \). Then \( G \) is a unipotent linear algebraic group defined over \( \mathbb{Q}_p \), and hence nilpotent. □

We remark that Theorem 6.1 becomes false in general for Lie groups over local fields of positive characteristic (as illustrated in 7.1). However, we shall see later that its conclusions remain valid if closedness of \( U_\alpha \) is made an extra hypothesis. Assuming closedness of \( U_\alpha \), a certain analogue of Theorem 6.1 can even be obtained in a purely topological setting:

**Proposition 6.3** Let \( G \) be a totally disconnected, locally compact group and \( \alpha: G \to G \) be an automorphism. If \( U_\alpha \) is closed, then \( U_\alpha M_\alpha U_\alpha^{-1} \) is an open identity neighbourhood in \( G \) and the product map
\[
\pi: U_\alpha \times M_\alpha \times U_\alpha^{-1} \to U_\alpha M_\alpha U_\alpha^{-1}, \quad (x, y, z) \mapsto xyz
\]
(30)
is a homeomorphism.

**Proof.** If \( U_\alpha \) is closed, then every identity neighbourhood in \( G \) contains a compact open subgroup tidy for \( \alpha \) (see [3] if \( G \) is metrizable; the results from [30] can be used to remove the metrizability condition). Thus \( \alpha \) is “tidy” in the terminology of [13]. Therefore, the proposition is covered by part (f) of the theorem in [13] (the proof of which heavily uses results from [3]). □

We conclude the section with an example for the calculations in Sections 4–6.

**Example 6.4** We consider the \( p \)-adic Lie group \( G := \text{GL}_2(\mathbb{Q}_p) \) and its inner automorphism \( \alpha: A \mapsto gAg^{-1} \) given by
\[
g := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.
\]

For
\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,
\]
we have
\[
g^nA g^{-n} = \begin{pmatrix} a & p^{-n}b \\ p^n c & d \end{pmatrix}
\]
for all \( n \in \mathbb{Z} \), entailing that \( M_\alpha \subseteq G \) is the subgroup of all invertible diagonal matrices,

\[
U_\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} : c \in \mathbb{Q}_p \right\} \quad \text{and} \quad U_\alpha^{-1} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Q}_p \right\}.
\]

Since \( G \) is an open subset of the space \( M_2(\mathbb{Q}_p) \) of \( p \)-adic \((2 \times 2)\)-matrices, we can identify the tangent space \( L(G) \) at the identity element with \( M_2(\mathbb{Q}_p) \) (as usual). Then \( L(\alpha) \) corresponds to the linear automorphism

\[
\beta : M_2(\mathbb{Q}_p) \to M_2(\mathbb{Q}_p), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & p^{-1}b \\ pc & d \end{pmatrix}.
\]

Note that the matrix units \( E_{j,k} \) with only one non-zero-entry, 1, in the \( j \)th row and \( k \)th column, form a basis of \( \beta \)-eigenvectors for \( M_2(\mathbb{Q}_p) \) with eigenvalues \( p \) and \( p^{-1} \) (of multiplicity one) and 1 as an eigenvalue of multiplicity 2. Thus

\[
s_G(\alpha) = s_{L(G)}(\beta) = |1| \cdot |1| \cdot |p^{-1}| = p,
\]

by Theorem 5.2 and Proposition 4.7. It is clear from the preceding decomposition in eigenspaces that the maximum-norm on \( M_2(\mathbb{Q}_p) \), \( \|A\|_\infty := \max\{|a|, |b|, |c|, |d|\} \) for \( A \) as above, is adapted to \( \beta \). Therefore

\[
B_r := \{A \in M_2(\mathbb{Q}_p) : \|A\|_\infty < r\}
\]

is tidy for \( \beta \) for each \( r > 0 \) (see Proposition 4.7). As the matrix exponential function

\[
\exp : U \to \text{GL}_2(\mathbb{Q}_p)
\]

(defined on some open 0-neighbourhood \( U \subseteq M_2(\mathbb{Q}_p) \)) is an exponential function for \( G \) (see \[9\]), we deduce with Theorem 5.2 that

\[
V_r := \exp(B_r)
\]

is a tidy subgroup for \( \alpha \) for all sufficiently small \( r > 0 \). We mention that \( V_r = 1 + B_r \) coincides with the ball of radius \( r \) around the identity matrix for small \( r \), by Proposition 1.11, using that the derivative \( \exp'(0) = \text{id}_{M_2(\mathbb{Q}_p)} \) is an isometry of \( (M_2(\mathbb{Q}_p), \|\cdot\|_\infty) \).
7 Pathologies in positive characteristic

Most of our discussion of automorphisms of $p$-adic Lie groups becomes false for Lie groups over local fields of positive characteristic (without extra assumptions). Suitable extra assumptions will be described later. For the moment, let us have a look at some of the possible pathologies.

7.1 Non-closed contraction groups

We describe an analytic automorphism of an analytic Lie group over a local field of positive characteristic, with a non-closed contraction group.

Given a finite field $\mathbb{F}$ with prime order $|\mathbb{F}| = p$, consider the compact group $G := \mathbb{F}^\mathbb{Z}$ and the right shift $\alpha : G \rightarrow G$, $\alpha(f)(n) := f(n - 1)$.

Since the sets \( \{ f \in \mathbb{F}^\mathbb{Z} : f(-n) = f(-n+1) = \cdots = f(n-1) = f(n) = 0 \} \) (for $n \in \mathbb{N}$) form a basis of 0-neighbourhoods in $G$, it is easy to see that the contraction group $U_\alpha$ consists exactly of the functions with support bounded below, i.e.,

\[ U_\alpha = \mathbb{F}^{(-\mathbb{N})} \times \mathbb{F}^{\mathbb{N}_0}. \]

This is a dense, non-closed subgroup of $G$. Also note that $G$ does not have arbitrarily small subgroups tidy for $\alpha$: in fact, $G$ is the only tidy subgroup.

We now observe that $G$ can be considered as a 2-dimensional Lie group over $\mathbb{K} := \mathbb{F}((X))$: The map

\[ G \rightarrow \mathbb{F}[[X]] \times \mathbb{F}[[X]], \quad f \mapsto \left( \sum_{n=1}^{\infty} f(-n)X^{n-1}, \sum_{n=0}^{\infty} f(n)X^n \right) \]

is a global chart. The automorphism of $\mathbb{F}[[X]]^2$ corresponding to $\alpha$ coincides on the open 0-neighbourhood $X\mathbb{F}[[X]] \times \mathbb{F}[[X]]$ with the linear map

\[ \beta : \mathbb{K}^2 \rightarrow \mathbb{K}^2, \quad \beta(v, w) = (X^{-1}v, Xw). \]

Hence $\alpha$ is an analytic automorphism.

---

8 Compiled from [20].
7.2 An automorphism whose scale cannot be calculated on the Lie algebra level

We retain $G, \alpha$ and $\beta$ from the preceding example.

Since $G$ is compact, we have $s_G(\alpha) = 1$. However, $L(\alpha) = \beta$ and therefore

$$s_{L(G)}(L(\alpha)) = s_{K[[X]]^2}(\beta) = |X^{-1}| = p.$$ 

Thus

$$s_G(\alpha) \neq s_{L(G)}(L(\alpha)),$$

in stark contrast to the $p$-adic case, where equality did always hold!

7.3 An automorphism which is not locally linear in any chart

For $\mathbb{F}$, $\mathbb{K}$ and $p$ as before, consider the 1-dimensional $\mathbb{K}$-analytic Lie group $G := \mathbb{F}[[X]]$. Then

$$\alpha : G \to G, \quad z \mapsto z + Xz^p$$ 

is an analytic automorphism of $G$ with $\alpha'(0) = \text{id}$. If $\alpha$ could be locally linearized, we could find a diffeomorphism $\phi : U \to V$ between open 0-neighbourhoods $U, V \subseteq G$ such that $\phi(0) = 0$ and

$$\alpha \circ \phi|_W = \phi \circ \beta|_W$$ (31)

for some linear map $\beta : \mathbb{K} \to \mathbb{K}$ and 0-neighbourhood $W \subseteq G$. Then $\phi'(0) = \alpha'(0) \circ \phi'(0) = \phi'(0) \circ \beta$, and hence $\beta = \text{id}$. Substituting this into (31), we deduce that $\alpha|_{\phi(W)} = \text{id}_{\phi(W)}$, which is a contradiction.

8 Tools from non-linear analysis: invariant manifolds

As an automorphism $\alpha$ of a Lie group $G$ over a local field of positive characteristic need not be locally linear (see 7.3), one has to resort to techniques from non-linear analysis to understand the iteration of such an automorphism. The essential tools are stable manifolds and centre manifolds around a fixed point, which are well-known tools in the classical case of dynamical systems on real manifolds. The analogy is clear: We are dealing with the
(time-) discrete dynamical system \((G, \alpha)\), and are interested in its behaviour around the fixed point \(1\).

We shall use the following terminology (cf. \([6]\) and \([40]\)): A subset \(N\) of an \(m\)-dimensional analytic manifold \(M\) if for each \(x \in N\), there is an analytic diffeomorphism \(\phi = (\phi_1, \ldots, \phi_m): U \to V\) from an open \(x\)-neighbourhood \(U \subseteq M\) onto an open subset \(V \subseteq \mathbb{K}^m\) such that \(\phi(U \cap N) = V \cap (\mathbb{K}^n \times \{0\}) \subseteq \mathbb{K}^n \times \mathbb{K}^{m-n}\). Then the maps \((\phi_1, \ldots, \phi_n)|_{U \cap N}\) form an atlas of charts for an analytic manifold structure on \(N\). An immersed submanifold of \(M\) is a subset \(N \subseteq M\), together with an analytic manifold structure on \(N\) for which the inclusion map \(\iota: N \to M\) is an analytic immersion (i.e., an analytic map whose tangent maps \(T_x(\iota): T_xN \to T_xM\) are injective for all \(x \in N\)). If a subgroup \(H\) of an analytic Lie group \(G\) is a submanifold, then the inherited analytic manifold structure turns it into an analytic Lie group. A subgroup \(H\) of \(G\), together with an analytic Lie group structure on \(H\), is called an immersed Lie subgroup of \(G\), if the latter turns the inclusion map \(H \to G\) into an immersion.

To discuss stable manifolds in the ultrametric case, we consider the following setting: \((\mathbb{K}, |.|)\) is a complete ultrametric field and \(M\) a finite-dimensional analytic manifold over \(\mathbb{K}\), of dimension \(m\). Also, \(\alpha: M \to M\) is a given analytic diffeomorphism and \(z \in M\) a fixed point of \(\alpha\). We let \(a \in \]0, 1[\ set of all \(x \in M\) with the following property\footnote{The letter “s” in \(W^s_\alpha(\alpha, z)\) stands for “stable.”}

Now define \(W^s_\alpha(\alpha, z)\) as the set of all \(x \in M\) with the following property\footnote{The letter “s” in \(W^s_\alpha(\alpha, z)\) stands for “stable.”}

For some (and hence each) chart \(\phi: U \to V \subseteq \mathbb{K}^m\) of \(M\) around \(z\) such that \(\phi(z) = 0\), and some (and hence each) norm \(\|.|\|\) on \(\mathbb{K}^m\), there exists \(n_0 \in \mathbb{N}\) such that \(\alpha^n(x) \in U\) for all integers \(n \geq n_0\) and

\[
\lim_{n \to \infty} \frac{\|\phi(\alpha^n(x))\|}{\alpha^n} = 0 .
\]

It is clear from the definition that \(W^s_\alpha(\alpha, z)\) is an \(\alpha\)-stable subset of \(M\). The following fact is obtained if we combine \([17\] Theorem 1.3] and \([18\] Theorem A1] (along with its proof):

**Theorem 8.1** For each \(a \in \]0, 1[\ set \(W^s_\alpha(\alpha, z)\) is an immersed submanifold of \(M\). Its tangent space at \(z\) is \(\bigoplus_{\rho < a} T_z(M)_{\rho}\), using notation as in Lemma 4.1 with \(\beta := T_z(\alpha): T_z(M) \to T_z(M)\) in place of \(\alpha\) and \(E := T_z(M)\).
As in the real case, we call \( W^s_\alpha(z) \) the a-stable manifold around \( z \).

**Some ideas of the proof.** The main point is to construct a local a-stable manifold, i.e., an \( \alpha \)-invariant submanifold \( N \) of \( M \) tangent to \( \bigoplus_{\rho \leq a} T_{\rho}(M) \) such that \( \alpha|_N : N \to N \) is an analytic map. In the real case, M. C. Irwin showed how the construction of a (local) a-stable manifold can be reduced to the implicit function theorem, applied to a Banach space of sequences (cf. also [15]). Since implicit function theorems are available also for analytic mappings between ultrametric Banach spaces (see [5]), it is possible to adapt Irwin’s method to the ultrametric case (see [17]).

Other classical types of invariant manifolds are also available in the ultrametric case. In particular:

**8.2** For \( M \) and \( \alpha \) as before, there always exists a so-called centre manifold, i.e., an immersed analytic submanifold \( C \) of \( M \) which is tangent to \( M_\beta \subseteq T_z(M) \) at \( z \), \( \alpha \)-stable (i.e., \( \alpha(C) = C \)), and such that the restriction \( f|_C : C \to C \) is analytic [17, Theorem 1.10]. By Proposition 1.11 after shrinking \( C \) we may assume that \( C \) is diffeomorphic to a ball and hence compact.

The neutral element 1 of a Lie group \( G \) is a fixed point of each automorphism \( \alpha \) of \( G \), but it need not be a hyperbolic fixed point, i.e., it may very well happen that \( 1 \in R(T_1(\alpha)) \). Nonetheless, it is always possible to make \( U_\alpha = W^s_1(\alpha, 1) \) a manifold (see [18, Theorem D]):

**Proposition 8.3** Let \( G \) be an analytic Lie group over a complete valued field, \( \alpha : G \to G \) be an analytic automorphism and \( \beta := L(\alpha) \) the associated Lie algebra automorphism of \( L(G) \). Then \( U_\alpha \) can be made an immersed Lie subgroup modelled on \( U_\beta = W^s_1(\beta, 0) \subseteq L(G) \), such that \( \alpha|_{U_\alpha} : U_\alpha \to U_\alpha \) is an analytic automorphism and contractive.

**Proof.** The idea is to show that \( U_\alpha = W^s_\alpha(\alpha, 1) \) for \( a \in ]0, 1[ \setminus R(\beta) \) close to 1. Details can be found in [18].

Although it always is an immersed Lie subgroup, \( U_\alpha \) need not be a Lie subgroup (as the example in Section 7 shows).

---

10 Then \( W^s_\alpha(\alpha, z) = \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(N) \), which is easily made an analytic manifold in such a way that \( N \) becomes an open subset and \( \alpha \) restricts to an analytic diffeomorphism of \( W^s_\alpha(\alpha, z) \).

11 The idea is to construct not the points \( x \) of the manifold, but their orbits \( (\alpha^n(x))_{n \in \mathbb{N}_0} \).
9 The scale, tidy subgroups and contraction groups in positive characteristic

We begin with a discussion of the contraction group for a well-behaved automorphism $\alpha$ of a Lie group $G$ over a local field, and the associated local decomposition of $G$ adapted to $\alpha$. Using the tools from non-linear analysis just described, one obtains the following analogue of Theorem 6.1 (see [20]):

**Theorem 9.1** Let $G$ be an analytic Lie group over a local field and $\alpha : G \to G$ be an analytic automorphism. Abbreviate $\beta := L(\alpha)$. If $U_\alpha$ is closed, then $U_\alpha, M_\alpha$ and $U_\alpha^{-1}$ are closed Lie subgroups of $G$ with Lie algebras

$$L(U_\alpha) = U_\beta, \quad L(M_\alpha) = M_\beta \quad \text{and} \quad L(U_\alpha^{-1}) = U_{\beta^{-1}},$$

respectively. Moreover, $U_\alpha M_\alpha U_\alpha^{-1}$ is an open identity neighbourhood in $G$, and the product map

$$\pi : U_\alpha \times M_\alpha \times U_\alpha^{-1} \to U_\alpha M_\alpha U_\alpha^{-1}, \quad (x, y, z) \mapsto xyz \quad (33)$$

is an analytic diffeomorphism.

**Some ideas of the proof.** By Proposition 8.3, $U_\alpha$ and $U_\alpha^{-1}$ are immersed Lie subgroups of $G$ with Lie algebras $U_\beta$ and $U_{\beta^{-1}}$, respectively. By 8.2 there exists a compact centre manifold $C$ for $\alpha$ around 1, modelled on $M_\beta$. Since $\alpha^n(C) = C$ for each $n \in \mathbb{Z}$, we have $C \subseteq M_\alpha$. Now $L(G) = U_\beta \oplus M_\beta \oplus M_{\beta^{-1}}$ (see Proposition 4.6). By the inverse function theorem, the product map

$$U_\alpha \times C \times U_\alpha^{-1} \to G$$

is a local diffeomorphism at $(1, 1, 1)$. Using Proposition 6.3 we deduce that $C$ is open in $M_\alpha$, and now a standard argument (Proposition 18 in [6] Chapter 3, §1, no. 9) can be used to make $M_\alpha$ a Lie group. Then $\pi$ from (30) is a local diffeomorphism at $(1, 1, 1)$, and one deduces as in the proof of Part (f) in the theorem in [13] that $\pi$ is an analytic diffeomorphism onto its image. 

**Remark 9.2** Even if $U_\alpha$ is not closed, its stable manifold structure makes it an immersed Lie subgroup of $G$, and $\alpha |_{U_\alpha}$ is a contractive automorphism of this Lie group (see Proposition 8.3). Therefore Section 10 below provides structural information on $U_\alpha$ (no matter whether it is closed or not).

31
Next, we discuss tidy subgroups and the scale for well-behaved automorphisms of Lie groups over local fields.

Let $G$ be an analytic Lie group over a local field, and $\alpha$ be an analytic automorphism of $G$. Let $\| \cdot \|$ be a norm on $L(G)$ adapted to $\beta := L(\alpha)$ and $\phi: U \to V \subseteq L(G)$ a chart of $G$ around 1 such that $\phi(1) = 0$ and $T_1\phi = \text{id}_{L(G)}$. We know from Lemma 1.14 that $V_r := \phi^{-1}(B_r(0))$ is a subgroup of $G$ if $r$ is small (where $B_r(0) \subseteq L(G)$). Then the following holds:

**Theorem 9.3** If $U_\alpha$ is closed, then the subgroup $V_r$ of $G$ (as just defined) is tidy for $\alpha$, for each sufficiently small $r > 0$. Moreover,

$$s_G(\alpha) = s_{L(G)}(L(\alpha)),$$

where $s_{L(G)}(L(\alpha))$ can be calculated as in Proposition 4.7. If $U_\alpha$ is not closed, then $s_G(\alpha) \neq s_{L(G)}(L(\alpha))$; more precisely, $s_G(\alpha)$ is a proper divisor of $s_{L(G)}(L(\alpha))$.

**Sketch of proof.** We explain why $V_r$ is tidy. Combining Theorem 9.1 with the Ultrametric Inverse Function Theorem, we obtain a diffeomorphic decomposition

$$V_r = (V_r \cap U_\alpha)(V_r \cap M_\alpha)(V_r \cap U_{\alpha^{-1}})$$

for small $r$. Since $\beta$ takes $B_r(0) \cap U_\beta$ inside $B_r\|\beta\|_{\text{op}}(0) \cap U_\beta$ for each $r$ (where $\|\beta\|_{\text{op}} < 1$), the Ultrametric Inverse Function Theorem implies that

$$\bigcap_{n \in \mathbb{N}_0} \alpha^n(V_r \cap U_\alpha) = \{1\}$$

for all small $r > 0$. Similarly, $\beta(B_r(0) \cap M_\beta) = B_r(0) \cap M_\beta$ and $\beta(B_r(0) \cap U_{\beta^{-1}}) \supseteq B_r(0) \cap U_{\beta^{-1}}$ imply that

$$\bigcap_{n \in \mathbb{N}_0} \alpha^n(V_r \cap M_\alpha) = V_r \cap M_\alpha \quad \text{and} \quad \bigcap_{n \in \mathbb{N}_0} \alpha^n(V_r \cap U_{\alpha^{-1}}) = V_r \cap U_{\alpha^{-1}}$$

for small $r$. Combining this information with (35) and (33), we see that

$$(V_r)_+ := \bigcap_{n \in \mathbb{N}_0} \alpha^n(V_r) = (V_r \cap M_\alpha)(V_r \cap U_{\alpha^{-1}}).$$
As a consequence, \((V_r)_+\) has the Lie algebra \(M_\beta \oplus U_{\beta^{-1}}\). Likewise, \((V_r)_- = (V_r \cap U_\alpha)(V_r \cap M_\alpha)\) and \(L((V_r)_-) = U_\beta \oplus M_\beta\). Hence \(V_r = (V_r)_+(V_r)_-\), by (35). Thus \(V_r\) is tidy above (as in 2.6). Since \((V_r)_++ = (V_r \cap M_\alpha)U_{\alpha^{-1}}\) as a consequence of (33) and the \(\alpha\)-stability of \(V_r \cap M_\alpha\), we see that \((V_r)_++\) is closed (whence \(V_r\) is also tidy below and hence tidy). For the remaining assertions, see [20].

\[\square\]

**Remark 9.4** At a first glance, the discussion of the scale and tidy subgroups for automorphisms of Lie groups are easier in the \(p\)-adic case than in positive characteristic because any automorphism is locally linear in the \(p\)-adic case. But a more profound difference is the automatic validity of the tidiness property \(TB\) for automorphisms of \(p\)-adic Lie groups, which becomes false in positive characteristic.\[12\]

As observed in [20], results from [3] imply the following closedness criterion for \(U_\alpha\), which is frequently easy to check.

**Proposition 9.5** Let \(G\) be a totally disconnected, locally compact group and \(\alpha\) be an automorphism of \(G\). If there exists an injective continuous homomorphism \(\phi: G \to H\) to a totally disconnected, locally compact group \(H\) and an automorphism \(\beta\) of \(H\) such that \(U_\beta\) is closed and \(\beta \circ \phi = \phi \circ \alpha\), then also \(U_\alpha\) is closed.

**Proof.** It is known that \(U_\beta \cap M_\beta = \{1\}\) if and only if \(U_\beta\) is closed (see [3, Theorem 3.32]) if \(H\) is metrizable; the metrizability condition can be removed using techniques from [30]), which holds by hypothesis. Since \(\phi(U_\alpha) \subseteq U_\beta\) and \(\phi(M_\alpha) \subseteq M_\beta\), the injectivity of \(\phi\) entails that \(U_\alpha \cap M_\alpha = \{1\}\). Hence \(U_\alpha\) is closed, using [3, Theorem 3.32] again. \[\square\]

Since \(U_\beta \cap M_\beta = \{1\}\) holds for each inner automorphism of \(GL_\alpha(K)\) (compare Proposition [4,6]), the preceding proposition immediately entails:

**Proposition 9.6** If a totally disconnected, locally compact group \(G\) admits an injective, continuous homomorphism into \(GL_n(K)\) for some \(n \in \mathbb{N}\) and some local field \(K\), then \(U_\alpha\) is closed in \(G\) for each inner automorphism \(\alpha\) of \(G\). \[\square\]

**Remark 9.7** In particular, \(U_\alpha\) is closed for each inner automorphism of a closed subgroup \(G\) of the general linear group \(GL_n(K)\) over a local field \(K\), as

\[12\]Note that also the pathological automorphism \(\alpha\) in Section 7 was locally linear.
already observed in [3, Remark 3.33 (3)]. Also an analogue of (34) can already be found in [3, Proposition 3.23], in the special case of Zariski connected reductive linear algebraic groups defined over $K$. While our approach is analytic, the special case is discussed in [3] using methods from the theory of linear algebraic groups.

**Remark 9.8** Let $K = \mathbb{F}((X))$, $G$ and its automorphism $\alpha$ be as in [7,1] and $H := G \rtimes \langle \alpha \rangle$. Then $H$ is a 2-dimensional $K$-analytic Lie group, and conjugation with $\alpha$ restricts to $\alpha$ on $G$. In view of the considerations in [7,1] we have found an inner automorphism of $H$ with a non-closed contraction group. By Proposition 9.6 $H$ is not a linear Lie group, and more generally it does not admit a faithful continuous linear representation over a local field.

For recent studies of linearity questions on the level of pro-$p$-groups, the reader is referred to [1], [2], [7], [28], [29], [33] and the references therein.

### 10 The structure of contraction groups in the case of positive characteristic

Wang’s structural result concerning $p$-adic contraction groups (as recalled in Theorem 6.2) can partially be adapted to Lie groups over local fields of positive characteristic (see Theorems A and B in [16]):

**Theorem 10.1** Let $G$ be an analytic Lie group over a local field $K$ of positive characteristic. If $G$ admits an analytic automorphism $\alpha : G \to G$ which is contractive, then $G$ is a torsion group of finite exponent. Moreover, $G$ is nilpotent and admits a central series

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

where each $G_j$ is a Lie subgroup of $G$.

**Some ideas of the proof.** Let $p := \text{char}(K)$. It is known from [22] that $G = \text{tor}(G) \times H_{p_1} \times \cdots \times H_{p_m}$ with certain $\alpha$-stable $p_j$-adic Lie groups $H_{p_j}$, for suitable primes $p_1, \ldots, p_m$. Since $G$ is locally pro-$p$, we must have $G = \text{tor}(G) \times H_p$. If $H_p$ was non-trivial, then the size of the sets of $p^k$-th powers of balls in the $p$-adic Lie group $H_p$ and such in $G$ would differ too much as $k$ tends to $\infty$, and one can reach a contradiction (if one makes this idea more
precise, as in the proof of [16, Theorem A]). Hence $G = \text{tor}(G)$ is a torsion group.

To see that $G$ is nilpotent, we exploit that the $a$-stable submanifolds $W^s_a(\alpha, 1)$ are Lie subgroups for all $a \in ]0, 1[ \setminus R(\alpha)$ (see [18, Proposition 6.2]). Since

$$[W^s_a(\alpha, 1), W^s_b(\alpha, 1)] \subseteq W^s_{ab}(\alpha, 1)$$

holds for the commutator subgroups whenever $a, b, ab \in ]0, 1[ \setminus R(\alpha)$ (as follows from the second order Taylor expansion of the commutator map), one can easily pick numbers $a_1 < \cdots < a_n$ in $]0, 1[ \setminus R(L(\alpha))$ for some $n$, such that the Lie subgroups $G_j := W^s_{a_j}(\alpha, 1)$ (for $j \in \{1, \ldots, n\}$) form the desired central series.

\[ \square \]

11 Further results that can be adapted to positive characteristic

We mention that a variety of further results can be generalized from $p$-adic Lie groups to Lie groups over arbitrary local fields, if one assumes that the relevant automorphisms have closed contraction groups. The ultrametric inverse function theorems (Propositions 1.11 and 1.12) usually suffice as a replacement for the naturality of exp (although we cannot linearize, at least balls are only deformed as by a linear map). And Theorem 9.1 gives control over the eigenvalues of $L(\alpha)$.

We now state two results which can be generalized to positive characteristic following this pattern.

11.1 To discuss these results, introduce more terminology. First, we recall from [35] that a totally disconnected, locally compact group $G$ is called unis\-calar if $s_G(x) = 1$ for each $x \in G$. This holds if and only if each group element $x \in G$ normalizes some compact, open subgroup $V_x$ (which may depend on $x$). It is natural to ask whether this condition implies that $V_x$ can be chosen independently of $x$, i.e., whether $G$ has a compact, open, normal subgroup. A suitable $p$-adic Lie group (see [21, §6]) shows that a positive answer can only be expected if $G$ is compactly generated. But even in the compactly generated case, the answer is negative in general (see [4] and preparatory work in [31]), and one has to restrict attention to particular classes of groups to obtain a positive answer (like compactly generated
$p$-adic Lie groups). If $G$ has the (even stronger) property that every identity neighbourhood contains an open, compact, normal subgroup of $G$, then $G$ is called pro-discrete. Finally, a Lie group $G$ over a local field is of type $R$ if the eigenvalues of $L(\alpha)$ have absolute value 1, for each inner automorphism $\alpha$ (see [37]).

Using the above strategy, one can generalize results from [37] and [21] (see [20]):

**Proposition 11.2** Let $G$ be an analytic Lie group over a local field $\mathbb{K}$, and $\alpha$ be an analytic automorphism of $G$. Then the following properties are equivalent:

(a) $U_\alpha$ is closed, and $s_G(\alpha) = s_G(\alpha^{-1}) = 1$;

(b) All eigenvalues of $L(\alpha)$ in $\mathbb{K}$ have absolute value 1;

(c) Every identity neighbourhood of $G$ contains a compact, open subgroup $U$ which is $\alpha$-stable, i.e., $\alpha(U) = U$.

In particular, $G$ is of type $R$ if and only if $G$ is uniscalar and $U_\alpha$ is closed for each inner automorphism $\alpha$ of $G$ (in which case $U_\alpha = \{1\}$).

Using a fixed point theorem for group actions on buildings by A. Parreau, it was shown in [36] (and [21]) that every compactly generated, uniscalar $p$-adic Lie group is pro-discrete. More generally, one can prove (see [20]):

**Proposition 11.3** Every compactly generated analytic Lie group of type $R$ over a local field is pro-discrete.

As mentioned in [11.1] results of the preceding form cannot be expected for general totally disconnected, locally compact groups. The author is grateful to the referee for the following additional example (cf. also [4] and [31]). Consider the Grigorchuk group $H$. Recall that $H$ is a certain infinite (discrete) group which is a 2-group (in the sense that every element has order a power of 2) and finitely generated (by four elements). It is known that $H$ admits a transitive left action

$$H \times \mathbb{Z} \to \mathbb{Z}, \quad (h, n) \mapsto h.n$$

on the set of integers such that each $h \in H$ commensurates $\mathbb{N}$ (in the sense that $h.\mathbb{N}$ and $\mathbb{N}$ have finite symmetric difference); such an action exists
since $H$ admits Schreier graphs with several ends. The preceding action induces a left action\footnote{for $(a_n)_{n \in \mathbb{Z}} \in \mathbb{F}^\mathbb{Z}_2$ with support bounded below.} by automorphisms of the topological group $(\mathbb{F}_2((X)), +)$, endowed with its usual topology (such that $\mathbb{F}_2[X]$ is a compact open subgroup). We use this action to form the semi-direct product $G := \mathbb{F}_2((X)) \rtimes H$.

**Proposition 11.4** $G := \mathbb{F}_2((X)) \rtimes H$ is a compactly generated, totally disconnected locally compact torsion group. For every $g \in G$ and identity neighbourhood $W \subseteq G$, there exists a compact open subgroup $V$ of $G$ such that $V \subseteq W$ and $V$ is normalized by $g$; moreover, the contraction group $U_{\alpha_g}$ of the inner automorphism $\alpha_g : G \to G, x \mapsto gxg^{-1}$ is trivial. However, $G$ does not have a compact, open, normal subgroup; in particular, $G$ is not pro-discrete.

**Proof.** After shrinking $W$, we may assume that $W$ is a compact open subgroup of $G$. Since $\mathbb{F}_2((X))$ and $G/\mathbb{F}_2((X)) \cong H$ are torsion groups, also $G$ is a torsion group. If $m \in \mathbb{N}_0$ is the order of $g \in G$, then $V := \bigcap_{n=0}^m g^n(W)$ is a compact open subgroup of $G$ which is normalized by $g$ and contained in $W$. If $e \neq x \in G$, after shrinking $W$ we may assume that $x \notin W$ and hence $x \notin V$. As the complement of $V$ in $G$ is stable under conjugation with $g$, we deduce that $g^kxg^{-k} \notin V$ for all $k \in \mathbb{N}$ and hence $g \notin U_{\alpha_g}$, whence $U_{\alpha_g} = \{e\}$ is trivial. Suppose we could find a compact, open, normal subgroup $V \subseteq G$. Then $X^n\mathbb{F}_2[X] \subseteq V$ for some $n \in \mathbb{Z}$ and thus $X^n \in V$. Since $H$ acts transitively on $\mathbb{Z}$, we deduce that $X^m \in V$ for each $m \in \mathbb{Z}$ and thus $\mathbb{F}_2((X)) \subseteq V$, as $V$ is a closed subgroup of $G$. But then $\mathbb{F}_2((X))$ would be a closed subgroup of $V$ and so $\mathbb{F}_2((X))$ would be compact, a contradiction. 



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Helge Glöckner, Universität Paderborn, Institut für Mathematik, Warburger Str. 100, 33098 Paderborn, Germany;
e-mail: glockner@math.upb.de