CP T-odd Resonances in Neutrino Oscillations

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Abstract

We consider the consequences for future neutrino factory experiments of small CP T-odd interactions in neutrino oscillations. The $\nu_\mu \rightarrow \nu_\mu$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu$ survival probabilities at a baseline $L = 732$ km can test for CP T-odd contributions at orders of magnitude better sensitivity than present limits. Interference between the CP T-violating interaction and CP T-even mass terms in the Lagrangian can lead to a resonant enhancement of the oscillation amplitude. For oscillations in matter, a simultaneous enhancement of both neutrino and antineutrino oscillation amplitudes is possible.
I. INTRODUCTION

The discrete symmetries $C$, $P$, and $T$ have fundamental importance in elementary particle theory. Violations of $C$, $P$, $CP$ and $T$ by the weak interactions have all been observed \[1\]. $CPT$ invariance is a basic property of local quantum field theory \[2\] and no evidence of deviations from $CPT$ invariance have been found so far. The most stringent limit on $CPT$ violation is obtained from the difference of the $K^0$ and $\bar{K}^0$ masses \[3\],

\[ m_K - m_{\bar{K}} < 0.44 \times 10^{-18} \text{ GeV}. \] (1)

In string theory, the $CPT$ invariance may not be manifest due to the extended nature of strings \[4\]. Mechanisms by which string theories could spontaneously break $CPT$ have been formulated \[4\]. The search for $CPT$ violation is thus of considerable theoretical interest as a means of searching for purely string effects. Neutrino oscillations have been considered as phenomena that could probe $CPT$ non-conservation \[5\]. With growing interest in the construction of neutrino factories to make high-precision measurements of neutrino mass-squared differences and of the $CP$ violating phase in the neutrino sector \[6\], it is appropriate to undertake a more extensive study of the ability to measure $CPT$-violating effects in neutrino oscillations. We find that a comparison of $\nu_\mu \rightarrow \nu_\mu$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu$ oscillation probabilities at neutrino factories would give precision tests of $CPT$. A significant result of our study, as reported below, is that $CPT$ violating resonance effects can occur that can magnify small $CPT$ violation into a measurable oscillation amplitude.

II. BASIC FORMALISM

Consequences of $CP$, $T$ and $CPT$ violation for neutrino oscillations have been written down before \[6\]. We summarize them briefly for the $\nu_\alpha \rightarrow \nu_\beta$ flavor oscillation probabilities $P_{\alpha\beta}$ at a distance $L$ from the source. If

\[ P_{\alpha\beta}(L) \neq P_{\bar{\alpha}\bar{\beta}}(L), \quad \beta \neq \alpha, \] (2)

then $CP$ is not conserved. If

\[ P_{\alpha\beta}(L) \neq P_{\beta\alpha}(L), \quad \beta \neq \alpha, \] (3)

then $T$-invariance is violated. If

\[ P_{\alpha\beta}(L) \neq P_{\bar{\beta}\bar{\alpha}}(L), \quad \beta \neq \alpha, \] (4)

or

\[ P_{\alpha\alpha}(L) \neq P_{\bar{\alpha}\bar{\alpha}}(L), \] (5)

then $CPT$ is violated. When neutrinos propagate in matter, matter effects give rise to apparent $CP$ and $CPT$ violation even if the mass matrix is $CP$ conserving.

The $CPT$ violating terms can be Lorentz-invariance violating (LV) or Lorentz invariant. The Lorentz-invariance violating, $CPT$ violating case has been discussed by Coleman and Glashow \[6\] and by Colladay and Kostelecky \[9\]. We will consider this first.

The effective LV $CPT$ violating interaction for neutrinos is of the form
$\bar{\nu}_L^\alpha b_{\mu}^{\alpha\beta} \gamma_{\mu} \nu_L^\beta$, \hfill (6)

where $\alpha$ and $\beta$ are flavor indices. We assume rotational invariance in the “preferred” frame, in which the cosmic microwave background radiation is isotropic (following Coleman and Glashow [3]). The energies of ultra-relativistic neutrinos with definite momentum $p$ are eigenvalues of the matrix

$$m^2/2p + b_0,$$ \hfill (7)

where $b_0$ is a hermitian matrix, hereafter labeled $b$.

In the two-flavor case the neutrino phases may be chosen such that $b$ is real, in which case the interaction in Eq. (3) is CPT-odd. The survival probabilities for flavors $\alpha$ and $\bar{\alpha}$ produced at $t = 0$ are given by [3]

$$P_{\alpha\alpha}(L) = 1 - \sin^2 2\theta \sin^2(\Delta L/4),$$ \hfill (8)

and

$$P_{\bar{\alpha}\bar{\alpha}}(L) = 1 - \sin^2 2\bar{\theta} \sin^2(\bar{\Delta} L/4),$$ \hfill (9)

where

$$\Delta \sin 2\theta = \left| (\delta m^2/E) \sin 2\theta_m + 2\delta be^{i\eta} \sin 2\theta_b \right|,$$ \hfill (10)

$$\Delta \cos 2\theta = (\delta m^2/E) \cos 2\theta_m + 2\delta b \cos 2\theta_b.$$ \hfill (11)

$\bar{\Delta}$ and $\bar{\Theta}$ are defined by similar equations with $\delta b \to -\delta b$. Here $\theta_m$ and $\theta_b$ define the rotation angles that diagonalize $m^2$ and $b$, respectively, $\delta m^2 = m_2^2 - m_1^2$ and $\delta b = b_2 - b_1$, where $m_i^2$ and $b_i$ are the respective eigenvalues. We use the convention that $\cos 2\theta_m$ and $\cos 2\theta_b$ are positive and that $\delta m^2$ and $\delta b$ can have either sign. The phase $\eta$ in Eq. (10) is the difference of the phases in the unitary matrices that diagonalize $\delta m^2$ and $\delta b$; only one of these two phases can be absorbed by a redefinition of the neutrino states.

Observable CPT-violation in the two-flavor case is a consequence of the interference of the $\delta m^2$ terms (which are CPT-even) and the LV terms in Eq. (3) (which are CPT-odd); if $\delta m^2 = 0$ or $\delta b = 0$, then there is no observable CPT-violating effect in neutrino oscillations. If $\delta m^2/E \gg 2\delta b$ then $\Theta \simeq \theta_m$ and $\Delta \simeq \delta m^2/E$, whereas if $\delta m^2/E \ll 2\delta b$ then $\Theta \simeq \theta_b$ and $\Delta \simeq 2\delta b$. Hence the effective mixing angle and oscillation wavelength can vary dramatically with $E$ for appropriate values of $\delta b$.

We note that a CPT-odd resonance for neutrinos ($\sin^2 2\theta = 1$) occurs whenever $\cos 2\Theta = 0$ or

$$(\delta m^2/E) \cos 2\theta_m + 2\delta b \cos 2\theta_b = 0;$$ \hfill (12)

similar to the resonance due to matter effects [11,12]. The condition for antineutrinos is the same except $\delta b$ is replaced by $-\delta b$. The resonance occurs for neutrinos if $\delta m^2$ and $\delta b$ have

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1 An experimental limit on $b_{ee}^{\nu}$ of $10^{-29}$ GeV has been obtained [4] in studies of torques on a spin polarized torsion pendulum. This translates into a bound on the $ee$ element of the matrix $b_0$ of $5 \times 10^{-25}$ GeV; if $SU(2)_L$ symmetry holds, a similar bound is implied on the $\nu_e \nu_e$ element of $b_0$, but there are no similar bounds on other $\nu\nu$ elements of $b_0$. 

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the opposite sign, and for antineutrinos if they have the same sign. A resonance can occur even when \( \theta_m \) and \( \theta_b \) are both small, and for all values of \( \eta \); if \( \theta_m = \theta_b \), a resonance can occur only if \( \eta \neq 0 \).

If one of \( \nu_\alpha \) or \( \nu_\beta \) is \( \nu_e \), then the neutrino propagation is modified in the presence of matter. Then Eq. (11) becomes

\[
\Delta \cos 2\Theta = \left( \frac{\delta m^2}{E} \right) \cos 2\theta_m + 2\delta b \cos 2\theta_b - 2\sqrt{2}G_FN_e ,
\]

for neutrinos, where \( N_e \) is the number density of electrons in matter. For antineutrinos, \( \delta b \to -\delta b \) and \( N_e \to -N_e \) in Eq. (13).

### III. EXAMPLES OF CPT-VIOLATION AND CPT-ODD RESONANCES

Hereafter, for simplicity, we assume that \( m^2 \) and \( b \) are diagonalized by the same angle \( \theta \), i.e., \( \theta_m = \theta_b \equiv \theta \).

#### A. \( \eta = 0 \)

For \( \eta = 0 \) we have

\[
\Theta = \theta , \quad \Delta = \left( \frac{\delta m^2}{E} \right) + 2\delta b .
\]

In this \( \theta_m = \theta_b \), \( \eta = 0 \) case a resonance is not possible. The oscillation probabilities become

\[
P_{\alpha\alpha}(L) = 1 - \sin^2 2\theta \sin^2 \left( \frac{\delta m^2 L}{4E} + \frac{\delta b}{2} \right) ,
\]

\[
P_{\tilde{\alpha}\tilde{\alpha}}(L) = 1 - \sin^2 2\theta \sin^2 \left( \frac{\delta m^2 L}{4E} - \frac{\delta b}{2} \right) .
\]

For fixed \( E \), the \( \delta b \) terms act as a phase shift in the oscillation argument; for fixed \( L \), the \( \delta b \) terms act as a modification of the oscillation wavelength.

An approximate direct limit on \( \delta b \) when \( \alpha = \mu \) can be obtained by noting that in atmospheric neutrino data the flux of downward going \( \nu_\mu \) is not depleted whereas that of upward going \( \nu_\mu \) is. Hence, the oscillation arguments in Eqs. (16) and (17) cannot have fully developed for downward neutrinos. Taking \( |\delta bL/2| < \pi/2 \) with \( L \sim 20 \) km for downward events leads to the upper bound \( |\delta b| < 3 \times 10^{-20} \) GeV; upward going events could in principle test \( |\delta b| \) as low as \( 5 \times 10^{-23} \) GeV. Since the CPT-odd oscillation argument depends on \( L \) and the ordinary oscillation argument on \( L/E \), improved direct limits could be obtained by a dedicated study of the energy and zenith angle dependence of the atmospheric neutrino data.

The difference between \( P_{\alpha\alpha} \) and \( P_{\tilde{\alpha}\tilde{\alpha}} \)

\[
P_{\alpha\alpha}(L) - P_{\tilde{\alpha}\tilde{\alpha}}(L) = -2\sin^2 2\theta \sin \left( \frac{\delta m^2 L}{2E} \right) \sin(\delta bL) ,
\]
can be used to test for CPT-violation. In a neutrino factory, the ratio of $\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu$ to $\nu_\mu \rightarrow \nu_\mu$ events will differ from the standard model (or any local quantum field theory model) value if CPT is violated. Figure 1 shows the event ratios $N(\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu)/N(\nu_\mu \rightarrow \nu_\mu)$ versus $\delta b$ for a neutrino factory with $10^{19}$ stored muons and a 10 kt detector at several values of stored muon energy, assuming $\delta m^2 = 3.5 \times 10^{-3} \text{ eV}^2$ and $\sin^2 2\theta = 1.0$, as indicated by the atmospheric neutrino data [13]. The error bars in Fig. 1 are representative statistical uncertainties. The node near $\delta b = 8 \times 10^{-22} \text{ GeV}$ is a consequence of the fact that $P_{\alpha\alpha} = P_{\bar{\alpha}\bar{\alpha}}$, independent of $E$, whenever $\delta bL = n\pi$, where $n$ is any integer; the node in Fig. 1 is for $n = 1$. A 3$\sigma$ CPT violation effect is possible in such an experiment for $\delta b$ as low as $3 \times 10^{-23} \text{ GeV}$ for stored muon energies of 20 GeV. Although matter effects also induce an apparent CPT-violating effect, the dominant oscillation here is $\nu_\mu \rightarrow \nu_\tau$, which has no matter corrections in the two-neutrino limit; in any event, the matter effect is in general small for distances much shorter than the Earth’s radius.

We have also checked the observability of CPT violation at other distances, assuming the same neutrino factory parameters used above. For $L = 250 \text{ km}$, the $\delta bL$ oscillation argument in Eq. (18) has not fully developed and the ratio of $\bar{\nu}$ to $\nu$ events is still relatively close to the standard model value. For $L = 2900 \text{ km}$, a $\delta b$ as low as $10^{-23} \text{ GeV}$ may be observable at the 3$\sigma$ level. However, longer distances may also have matter effects that simulate CPT violation.

B. $\eta = \pi/2$

For $\eta = \pi/2$ we have

$$P_{\alpha\alpha} = 1 - \sin^2 2\Theta \sin^2 \{\Delta L/4\},$$

$$P_{\bar{\alpha}\bar{\alpha}} = 1 - \sin^2 2\bar{\Theta} \sin^2 \{\bar{\Delta} L/4\},$$

where

$$\tan 2\Theta = \frac{\sqrt{\delta m^2/E^2 + (2\delta b)^2}}{(\delta m^2/E) + 2\delta b} \tan 2\theta,$$

$$\Delta^2 = \left[(\delta m^2/E) + 2\bar{\delta} b\right]^2 - 4(\delta m^2/E)\delta b \sin^2 2\theta,$$

and $\bar{\Theta}$ and $\bar{\Delta}$ are defined similarly with $\delta b \rightarrow -\delta b$. In this case the resonance condition for neutrinos is $\delta m^2/E + 2\delta b = 0$. Figure 2 shows the effective oscillation amplitude $\sin^2 2\Theta$ and oscillation argument $\Delta$ versus $\delta b$ with $\delta m^2 = 3.5 \times 10^{-3} \text{ eV}^2$ and $\sin^2 2\theta = 0.1$ (which may be appropriate for $\nu_e \rightarrow \nu_e$ oscillations) for several values of neutrino energy. Although the above example assumed $\eta = \pi/2$, such a resonance can occur in this $\theta_b = \theta_m$ example for any value of $\eta$ in the open interval $(0, 2\pi)$.

C. CPT-odd term with matter

In the presence of matter, the effective $\nu_e$ oscillation amplitude and argument are defined by Eqs. (14) and (13). Again assuming $\theta_b = \theta_m \equiv \theta$ and $\eta = 0$, we have
\[ \tan 2\Theta = \frac{[(\delta m^2 / E) + 2\delta b] \sin 2\theta}{[(\delta m^2 / E) + 2\delta b] \cos 2\theta - 2\sqrt{2}G_FN_e}, \]  
\[ \Delta^2 = \left\{ \left[ (\delta m^2 / E) + 2\delta b \right] \cos 2\theta - 2\sqrt{2}G_FN_e \right\}^2 + \left\{ [(\delta m^2 / E) + 2\delta b] \sin^2 2\theta \right\}, \]  
for neutrinos, with \( \delta b \rightarrow -\delta b \) and \( N_e \rightarrow -N_e \) for antineutrinos. Thus a resonance \( (\sin^2 2\Theta = 1) \) occurs for neutrinos when \( [(\delta m^2 / E) + 2\delta b] \cos 2\theta = 2\sqrt{2}G_FN_e \), and for antineutrinos when \( [(\delta m^2 / E) - 2\delta b] \cos 2\theta = -2\sqrt{2}G_FN_e \). A resonance can occur simultaneously for neutrinos and antineutrinos only in the limit when \( \delta m^2 / E \ll 2\delta b \) and the CPT-odd effects dominate. However, it is possible to have an effective oscillation amplitude that is significantly enhanced for both neutrinos and antineutrinos even when \( \delta m^2 / E \) is not small compared to \( 2\delta b \). For \( N_e = 1.67N_A/cm^3 \) (the electron density appropriate for the upper mantle of the Earth) and vacuum amplitude \( \sin^2 2\Theta = 0.1 \), the effective oscillation amplitudes \( \sin^2 2\Theta \) for \( \nu_e \rightarrow \nu_e \) and \( \sin^2 2\Theta \) for \( \bar{\nu}_e \rightarrow \bar{\nu}_e \) can both be greater than 0.5 when \( \delta b \) and \( \delta m^2 \) satisfy both \( 0.0002 \text{eV}^2/\text{GeV} < 2\delta b + (\delta m^2 / E) < 0.0004 \text{eV}^2/\text{GeV} \) and \( 0.0002 \text{eV}^2/\text{GeV} < 2\delta b - (\delta m^2 / E) < 0.0004 \text{eV}^2/\text{GeV} \). These conditions are satisfied when \( \delta b \approx 1 - 2 \times 10^{-22} \text{GeV} \) and with \( |\delta m^2 / E| \) as large as \( 10^{-4} \text{eV}^2/\text{GeV} \). Assuming \( \delta m^2 \approx 3.5 \times 10^{-3} \text{eV}^2 \), such enhancements in \( \nu_e \rightarrow \nu_e \) and \( \bar{\nu}_e \rightarrow \bar{\nu}_e \) are possible for \( E > 35 \text{GeV} \), provided that \( \delta b > 0 \). Although here we have considered the case \( \eta = 0 \), similar enhancements are possible for any value of \( \eta \) since they rely on the denominator of Eq. \[23\] being small, which is independent of \( \eta \).

IV. LORENTZ-IN Variant CASE

We can simulate a possible Lorentz-invariant CPT-odd effective interaction by allowing the mass matrix for \( \nu \)'s to be different from the one for \( \bar{\nu} \)'s. If we assume, for simplicity, that the CPT-violating effects are more important in \( \delta m^2 \) than in mixing, then there is only one CPT-odd parameter, namely \( \delta m^2 - \delta m^2 = \epsilon \), and the oscillation probabilities are
\[ P_{\alpha\alpha} = 1 - \sin^2 2\theta \sin^2 [\delta m^2 L / (4E)], \]  
\[ P_{\alpha\alpha} = 1 - \sin^2 2\theta \sin^2 [(\delta m^2 - \epsilon)L / (4E)]. \]  
From the lack of large disappearance of downward going atmospheric muons, we obtain an approximate upper bound of \( |\epsilon| < 0.1 \text{eV}^2 \) when \( \alpha = \mu \). A fit to the total number of muon and antimuon events in the SuperK atmospheric neutrino data sample would greatly improve this bound.

V. SUMMARY

We have shown that small CPT-odd interactions of neutrinos can have measureable consequences in neutrino oscillations. Resonant enhancements of the oscillation amplitude for either neutrinos or antineutrinos (but not both) are possible if the unitary matrices which diagonalize the neutrino mass term and the CPT-odd term are not the same. A resonance can occur for any relative phase between the CPT-even mass term and the CPT-odd interaction, but if the rotation angles in the two sectors are the same, a resonance
is possible only if the relative phase is not zero. In matter, significant enhancements are possible for both neutrinos and antineutrinos. Measurement of $\nu_\mu \rightarrow \nu_\mu$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_\mu$ oscillation probabilities in neutrino factories can place stringent limits on the $CPT$-odd interaction.

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FIG. 1. The ratio of $\bar{\nu}_\mu \to \bar{\nu}_\mu$ to $\nu_\mu \to \nu_\mu$ event rates in a 10 kt detector for a neutrino factory with $10^{19}$ stored muon with energies $E_\mu = 10, 20, 30, 50$ GeV for baseline $L = 732$ km versus the $CPT$-odd parameter $\delta b$ with $\theta_m = \theta_b \equiv \theta$ and phase $\eta = 0$. The neutrino mass and mixing parameters are $\delta m^2 = 3.5 \times 10^{-3}$ eV$^2$ and $\sin^2 2\theta = 1.0$. The dotted line indicates the result for $\delta b = 0$, which is given by the ratio of the $\bar{\nu}$ and $\nu$ charge-current cross sections. The error bars are representative statistical uncertainties.
FIG. 2. Resonance effects in $\nu \rightarrow \nu$ and $\bar{\nu} \rightarrow \bar{\nu}$ oscillations shown versus $CPT$-odd parameter $\delta b$ for various values of neutrino energy $E$ with $\delta m^2 = 3.5 \times 10^{-3} \text{eV}^2$, $\sin^2 2\theta = 0.1$ and phase $\eta = \pi/2$: (a) oscillation amplitude $\sin^2 2\Theta$ in Eq. (21) and (b) oscillation argument $\Delta$ in Eq. (22).