The Lattice Structure of the Subgroups of order 21 in the subgroup lattices of 3 X 3 Matrices over Z₂

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Abstract: Let 𝒢 be the set of all 3 X 3 non-singular matrices \( \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \), where a,b,c,d,e,f,g,h,i are integers modulo p. Then 𝒢 is a group under matrix multiplication modulo p, of order \( (p^n - 1)(p^n - p)(p^n - p^2) \ldots (p^n - p^{n-1}) \). Let G be the subgroup of 𝒢 defined by \( G = \{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in 𝒢 : \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1 \} \). Then G is of order \( (p^n - 1)(p^n - p)(p^n - p^2) \ldots (p^n - p^{n-1}) \). Let L(G) be the lattice formed by all subgroups G. In this paper, we give the structure of the subgroups of order 21 of L(G) in the case when P=2.

Keywords: Matrix group, Subgroups, Poset, Lattice, Atom.

1. Introduction
Let L(G) be the Lattice of Subgroups of G, where G is a group of 3x3 matrices over Z_p having determinant value 1 under matrix multiplication modulo p, where p is a prime number.

Let \( G = \{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} : a, b, c, d, e, f, g, h, i \in \mathbb{Z}_p, \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0 \} \)

Then G is a group under matrix multiplication modulo p.

Let \( G = \{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \in G : \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1 \} \)

Then G is a subgroup of G.

We have, \( o(G) = (p^n - 1)(p^n - p)(p^n - p^2) \ldots (p^n - p^{n-1}) \)

\( \text{and} \ o(G) = \frac{(p^n - 1)(p^n - p)(p^n - p^2) \ldots (p^n - p^{n-1})}{p-1} \).

In this paper, we give the structure of the subgroups of order 21 of L(G) in the case when P=2.

2. Preliminaries
In this section we give the definition needed for the development of the paper.

Definition 2.1 (Poset): A partial order on a non-empty set P is a binary relation \( \leq \) on P that is reflexive, anti-symmetric and transitive. The pair (P, \( \leq \)) is called a partially ordered set or poset. A poset (P, \( \leq \)) is totally ordered if every x, y \( \in P \) are comparable, that is either \( x \leq y \) or \( y \leq x \). A non-empty subset S of P is a chain in P if S is totally ordered by \( \leq \).
Definition 2.2: Let \((P, \leq)\) be a poset and let \(S \subseteq P\). An upper bound of \(S\) is an element \(x \in P\) for which \(s \leq x\) for all \(s \in S\). The least upper bound of \(S\) is called the **supremum or join** of \(S\). A lower bound for \(S\) is an element \(x \in P\) for which \(x \leq s\) for all \(s \in S\). The greatest lower bound of \(S\) is called the **infimum or meet** of \(S\).

Definition 2.3 (Lattice): Poset \((P, \leq)\) is called a lattice if every pair \(x, y\) elements of \(P\) has a supremum and an infimum, which are denoted by \(x \sqcup y\) and \(x \sqcap y\) respectively.

Definition 2.4 (Atom): An element ‘a’ is an atom, if \(a > 0\) and a dual atom, if \(a < 1\).

3. Arrangement of elements of \(G\) according to their orders

Let \(G\) be the subgroup of \(G\) defined by

\[
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{bmatrix}
\in G : \begin{bmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{bmatrix} = 1
\]

Then \(|G| = \frac{(p^n-1)(p^n-p)(p^n-p^2)\ldots(p^n-p^{n-1})}{p-1}\)

\[= \frac{(2^3-1)(2^3-2)(2^3-2^2)}{2-1} = \frac{(8-1)(8-2)(8-4)}{1} = (7)(6)(4) = 168\]

3.1 Element of order 1(one element)

\[
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

3.2 Elements of order 2 (21 elements)

\[
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
  0 & 0 & 1 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
  0 & 1 & 0 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
  1 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
  1 & 1 & 0 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
  1 & 1 & 0 \\
  1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
  1 & 1 & 1 \\
  0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
  1 & 1 & 1 \\
  0 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
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  1 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
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  0 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
  1 & 1 & 1 \\
  0 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
  1 & 1 & 1 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & 0 & 0 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & 0 & 0 \\
  1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & 0 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & 0 \\
  1 & 1 & 0
\end{bmatrix}
\]

3.3 Elements of order 3 (56 elements)

\[
\begin{bmatrix}
  1 & 0 & 1 \\
  0 & 1 & 0 \\
  1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  0 & 0 & 1 \\
  1 & 0 & 0 \\
  0 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  0 & 0 & 1 \\
  0 & 0 & 1 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 0 & 0 \\
  1 & 0 & 0 \\
  0 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 0 & 0 \\
  1 & 0 & 0 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 0 & 0 \\
  1 & 0 & 1 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 0 & 0 \\
  1 & 0 & 1 \\
  1 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
  1 & 0 & 0 \\
  1 & 0 & 1 \\
  1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 0 & 0 \\
  1 & 0 & 1 \\
  1 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
  1 & 0 & 0 \\
  1 & 0 & 1 \\
  1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  1 & 0 & 0 \\
  1 & 0 & 1 \\
  1 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
  1 & 0 & 0 \\
  1 & 0 & 1 \\
  1 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
  0 & 0 & 1 \\
  1 & 0 & 0 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  0 & 0 & 1 \\
  0 & 0 & 1 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  0 & 0 & 1 \\
  0 & 1 & 0 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  0 & 0 & 1 \\
  0 & 1 & 0 \\
  0 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  0 & 0 & 1 \\
  1 & 0 & 0 \\
  1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
  0 & 0 & 1 \\
  1 & 0 & 0 \\
  1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  0 & 0 & 1 \\
  1 & 0 & 0 \\
  1 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
  0 & 0 & 1 \\
  1 & 0 & 0 \\
  1 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
  0 & 0 & 1 \\
  1 & 0 & 0 \\
  1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  0 & 0 & 1 \\
  1 & 0 & 0 \\
  1 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
  0 & 0 & 1 \\
  1 & 0 & 0 \\
  1 & 1 & 1
\end{bmatrix}
\]
### 3.4 Elements of order 4 (42 elements)

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{pmatrix}
\]

### 3.5 Elements of order 7 (48 elements)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
4. We find the subgroups of $G$ of different orders namely 3, 7 and 21

4.1 Subgroups of order 3

Since 3 is a prime number, any subgroup of $G$ of order 3 is cyclic and hence it is generated by an element of order 3.

Thus, all the subgroups of $G$ of order 3 are obtained as follows:

$$K_1 = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$K_2 = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$K_3 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$K_4 = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$K_5 = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, e \right\},$$

$$K_6 = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, e \right\},$$

$$K_7 = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, e \right\},$$

$$K_8 = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$K_9 = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$K_{10} = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, e \right\},$$

$$K_{11} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, e \right\},$$

$$K_{12} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$K_{13} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$K_{14} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$K_{15} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$K_{16} = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$K_{17} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, e \right\},$$

$$K_{18} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$K_{19} = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \right\},$$

$$K_{20} = \left\{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, e \right\},$$

$$K_{21} = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e \right\}.$$
\( K_{23} = \{ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, e \} \)
\( K_{24} = \{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, e \} \)
\( K_{25} = \{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, e \} \)
\( K_{26} = \{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e \} \)
\( K_{27} = \{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, e \} \)
\( K_{28} = \{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, e \} \)

Since \( |G| = 2^3 \times 3 \times 7 \), \( |G| = 2^3 \times 7 \).

\[ \text{Therefore, G has a } 3\text{ – sylow subgroup of order 3.} \]

The number of 3-sylow subgroups is of the form \( 1+3k \) \ and \( 1+3k | |G| \).

\[ \text{Therefore, } 1 + 3k | 2^3 \times 7. \]

The possible values for \( k \) are 0, 1, 2 and 9.

\[ \text{Therefore, the maximum number of 3-sylow subgroups of G of order 3 is 28 when } k=9. \]

So, these are the only subgroups of order 3.

### 4.2 Subgroups of order 7

\( |G| = 2^3 \times 3 \times 7 | |G| = 2^3 \times 7 \).

\[ \text{Therefore, G has a } 7\text{ – sylow subgroup of order 7.} \]

The number of 7-sylow subgroups is of the form \( 1+7k \) \ and \( 1+7k | |G| \).

\[ \text{Therefore, } 1 + 7k | 2^3 \times 3. \]

The possible values for \( K \) are 0 and 1.

\[ \text{Therefore, the maximum number of 7-sylow subgroups of G of order 7 is 8 when } k=1. \]

So, these are the only subgroups of order 7.

Thus, all the subgroups of G of order 7 are obtained as follows:

\( N_1 = \{ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e \} \)
\( N_2 = \{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, e \} \)
\( N_3 = \{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e \} \)
\( N_4 = \{ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, e \} \)
\[ N_5 = \left\{ \begin{array}{cccccccc}
1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
e
\end{array} \right\}, \]

\[ N_6 = \left\{ \begin{array}{cccccccc}
0 & 0 & 1 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
e
\end{array} \right\}, \]

\[ N_7 = \left\{ \begin{array}{cccccccc}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
e
\end{array} \right\}, \]

\[ N_8 = \left\{ \begin{array}{cccccccc}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
e
\end{array} \right\}, \]

### 4.3 Subgroups of order 21

Let \( S \) be an arbitrary subgroup of order 21. Since \( |S| = 7 \times 3 \), the number of 7-sylow subgroups of order 7 in \( S \) is \( 1 + 7k \) and \( 1 + 7k \mid 3 \). The possible value of \( k \) is 0 only.

Hence the number of subgroups of \( S \) of order 7 is 1.

Similarly, the number of 3-sylow subgroups of order 3 in \( S \) is \( 1 + 3k \) and \( 1 + 3k \mid 7 \). The possible values of \( k \) are 0, 2. Hence the number of subgroups of order 3 in \( S \) is either 1 or 7.

There are two possibilities:

(i). The number of subgroups of order 7 is 1 and of order 3 is 1.

(ii). The number of subgroups of order 7 is 1 and of order 3 is 7.

**Case: (I)**

Let the one subgroup of order 7 in \( S \) be \( N \) and the one subgroup of order 3 in \( S \) be \( K \).

Then \( N \) and \( K \) are normal in \( S \).

Hence \( T = NK \) must be abelian, but which is not true by checking all possibilities of \( N \) and \( K \). Thus, we get the conclusion that this case cannot occur.

**Case: (II)**

Taking a subgroup of order 7 at a time, combining this with seven subgroups of order 3, we are able to determine the following eight subgroups of order 21 by trial.

Since each subgroup of order 21 contains one subgroup of order 7 and we have only eight subgroups of order 7, there are exactly eight subgroups of order 21.

Thus, all the subgroups of \( G \) of order 21 are obtained as follows:

\[ S_1 = \left\{ \begin{array}{cccccccc}
0 & 0 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 0 \\
e
\end{array} \right\}, \]

\[ S_2 = \left\{ \begin{array}{cccccccc}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
e
\end{array} \right\}, \]

\[ S_3 = \left\{ \begin{array}{cccccccc}
0 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
e
\end{array} \right\}, \]

\[ S_4 = \left\{ \begin{array}{cccccccc}
0 & 0 & 1 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
e
\end{array} \right\}, \]

\[ S_5 = \left\{ \begin{array}{cccccccc}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
e
\end{array} \right\}, \]

\[ S_6 = \left\{ \begin{array}{cccccccc}
0 & 0 & 1 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0 \\
e
\end{array} \right\}, \]

\[ S_7 = \left\{ \begin{array}{cccccccc}
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
e
\end{array} \right\}, \]

\[ S_8 = \left\{ \begin{array}{cccccccc}
0 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
e
\end{array} \right\} \]
\[
S_2 = \left[
\begin{array}{cccccccc}
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
\end{array}
\right]
\]

\[
S_3 = \left[
\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\end{array}
\right]
\]

\[
S_4 = \left[
\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
\end{array}
\right]
\]

\[
S_5 = \left[
\begin{array}{cccccccc}
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\right]
\]

\[
S_6 = \left[
\begin{array}{cccccccc}
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\right]
\]
Here, \( N_1, K_4, K_8, K_{11}, K_{18}, K_{21}, K_{26}, K_{28} \subseteq S_1; \)
\( N_2, K_{28}, K_{13}, K_{12}, K_6, K_{14}, K_{22}, K_{24} \subseteq S_2; \)
\( N_3, K_{27}, K_{25}, K_{12}, K_{18}, K_3, K_{16}, K_7 \subseteq S_3; \)
\( N_4, K_{27}, K_{17}, K_6, K_{10}, K_5, K_{24} \subseteq S_4; \)
\( N_5, K_{26}, K_{23}, K_5, K_{19}, K_{22}, K_3, K_{15} \subseteq S_5; \)
\( N_6, K_{25}, K_{21}, K_{15}, K_{10}, K_2, K_6, K_9 \subseteq S_6; \)
\( N_7, K_{23}, K_{20}, K_4, K_{13}, K_{17}, K_{16}, K_2 \subseteq S_7; \)
\( N_8, K_1, K_7, K_9, K_{11}, K_{14}, K_{19}, K_{20} \subseteq S_8; \)

5. Lattice Structure of some lower interval of subgroups of order 21 in \( L(G) \) over \( \mathbb{Z}_2 \)

We name all the subgroups of order 21, by the symbols \( S_k, 1 \leq k \leq 8 \). We observe that \( S_k \)'s are of only one type. For example we take \( S_1 \) and it contains seven subgroups of order 3 and one subgroup of order 7.

- Table 5.1: Subgroups of \( S_1 \)

| Order | Subgroups |
|-------|-----------|
| 7     | \( N_1 \) |
| 3     | \( K_4, K_8, K_{11}, K_{18}, K_{21}, K_{26}, K_{28} \) |
| 1     | \( \{e\} \) |
6. Conclusion

In this paper, we produced the lattice structure of the subgroups of order 21 in the subgroup lattices of 3x3 matrices over \( \mathbb{Z}_2 \).

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