SOME MULTIDIMENSIONAL INTEGRALS IN NUMBER THEORY AND CONNECTIONS WITH THE PAINLEVÉ V EQUATION

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Abstract. We study piecewise polynomial functions $\gamma_k(c)$ that appear in the asymptotics of averages of the divisor sum in short intervals. Specifically, we express these polynomials as the inverse Fourier transform of a Hankel determinant that satisfies a Painlevé V equation. We prove that $\gamma_k(c)$ is very smooth at its transition points, and also determine the asymptotics of $\gamma_k(c)$ in a large neighbourhood of $k = c/2$. Finally, we consider the coefficients that appear in the asymptotics of elliptic aliquot cycles.

1. Introduction

Asymptotics of the mean square of sums of the $k$-th divisor function over short intervals. Let $d_k(n)$ be the $k$-th divisor numbers, i.e. the Dirichlet coefficients of the $k$-th power of the Riemann zeta function:

$$\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}, \quad \Re s > 1. \quad (1.1)$$

The Dirichlet coefficient $d_k(n)$ is equal to the number of ways of writing $n$ as a product of $k$ factors. Define

$$S_k(X) = \sum_{n \leq X} d_k(n). \quad (1.2)$$

Let $XP_{k-1}(\log X)$ be the residue, at $s = 1$ of $\zeta(s)^k X^s/s$, with $P_{k-1}(\log X)$ being a polynomial in $\log X$ of degree $k - 1$. Then

$$S_k(X) = XP_{k-1}(\log X) + \Delta_k(X), \quad (1.3)$$

with $\Delta_k(X)$ denoting the remainder term.

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The $k$ divisor problem states that the true order of magnitude for $\Delta_k$ is:

$$\Delta_k(X) = O \left( X^{(k-1)/2k+\epsilon} \right). \quad (1.4)$$

When $k = 2$, the traditional Dirichlet divisor problem is

$$D_2(X) = X \log X + (2\gamma - 1)X + \Delta_2(X), \quad (1.5)$$

with a conjectured remainder

$$\Delta_2(X) = O \left( X^{1/4+\epsilon} \right). \quad (1.6)$$

The estimate for the remainder term $\Delta_k(X)$ is based on expected cancellation in Voronoi-type formulas for $\Delta_k(X)$ and also on estimates, due to Cramér [3] ($k = 2$) and Tong [10] ($k > 2$), for the mean square of $\Delta_k$.

Let

$$\Delta_k(x; H) = \Delta_k(x + H) - \Delta_k(x) \quad (1.7)$$

be the remainder term for sums of $d_k$ over the interval $[x, x + H]$.

Define

$$a_k = \prod_p \left\{ (1 - \frac{1}{p})^k \sum_{j=0}^{\infty} \left( \frac{\Gamma(k+j)}{\Gamma(k)j!} \right)^2 \frac{1}{p^j} \right\}. \quad (1.8)$$

Keating, Rodgers, Roditty-Gershon, and Rudnick conjectured [7]:

**Conjecture 1.1.** If $0 < \alpha < 1 - \frac{1}{k}$ is fixed, then for $H = X^\alpha$,

$$\frac{1}{X} \int_X^{2X} \left( \Delta_k(x, H) \right)^2 dx \sim a_k P_k(\alpha) H(\log X)^{k^2-1}, \quad X \to \infty \quad (1.9)$$

where $P_k(\alpha)$ is given by

$$P_k(\alpha) = (1 - \alpha)^{k^2-1} \gamma_k \left( \frac{1}{1 - \alpha} \right). \quad (1.10)$$

Here

$$\gamma_k(c) = \frac{1}{k! G(1+k)^2} \int_{[0,1]^k} \delta(t_1 + \ldots + t_k - c) \prod_{i<j} (t_i - t_j)^2 dt_1 \ldots dt_k,$$

$$G is the Barnes G-function, so that for positive integers $k$, $G(1+k) = 1! \cdot 2! \cdot 3! \cdots (k-1)!$.

For $1 - \frac{1}{k-1} < \alpha < 1 - \frac{1}{k}$, the conjecture is consistent with a theorem of Lester [8].
Let $U$ be an $N \times N$ matrix. The secular coefficients $S_{c_j}(U)$ are the coefficients of the characteristic polynomial of $U$:

$$\det(I + xU) = \sum_{j=0}^{N} S_{c_j}(U)x^j$$

(1.12)

Thus $S_{c_0}(U) = 1$, $S_{c_1}(U) = \text{tr } U$, $S_{c_N}(U) = \det U$. The secular coefficients are the elementary symmetric functions in the eigenvalues of $U$.

Define the matrix integrals, with respect to Haar measure, over the group $U(N)$ of $N \times N$ unitary matrices:

$$I_k(m; N) := \int_{U(N)} \left| \sum_{j_1 + \ldots + j_k = m, 0 \leq j_1, \ldots, j_k \leq N} S_{c_{j_1}}(U) \ldots S_{c_{j_k}}(U) \right|^2 dU.$$  \hspace{1cm} (1.13)

**Theorem 1.1 (KR$^3$).** Let $c := m/N$. Then for $c \in [0, k]$,

$$I_k(m; N) = \gamma_k(c) N^{k^2 - 1} + O_k(N^{k^2 - 2}),$$

(1.14)

with

$$\gamma_k(c) = \frac{1}{k! \cdot G(1 + k)^2 \int_{[0,1]^k} \delta(t_1 + \ldots + t_k - c) \prod \limits_{i<j} (t_i - t_j)^2 \, dt_1 \ldots dt_k},$$

(1.15)

KR$^3$ also proved the matrix integral satisfies a functional equation $I_k(m; N) = I_k(kN - m; N)$, from which it follows that

$$\gamma_k(c) = \gamma_k(k - c),$$

(1.16)

and also that

**Theorem 1.2 (KR$^3$).**

$$\gamma_k(c) = \sum_{0 \leq \ell < c} \binom{k}{\ell}^2 (c - \ell)^{(k-\ell)^2 + \ell^2 - 1} g_{k,\ell}(c - \ell)$$

(1.17)

where $g_{k,\ell}(c - \ell)$ are (complicated) polynomials in $c - \ell$.

For a fixed $k$, $\gamma_k(c)$ is a piecewise polynomial function of $c$. Specifically, it is a fixed polynomial for $r \leq c < r + 1$ ($r$ integer), and each time the value of $c$ passes through an integer it becomes a different polynomial.

For example,

$$\gamma_2(c) = \frac{1}{2!} \int_{0 \leq t_1 \leq 1} \left( t_1 - (c - t_1) \right)^2 \, dt_1 = \begin{cases} \frac{c^3}{3!}, & 0 \leq c \leq 1 \\ \frac{(2-c)^3}{3!}, & 1 \leq c \leq 2 \end{cases}$$

(1.18)
and
\[
\gamma_3(c) = \begin{cases} 
\frac{1}{8!} c^8, & 0 < c < 1 \\
\frac{1}{8!} (3 - c)^8, & 2 < c < 3 
\end{cases}
\]
(1.19)
while for \(1 < c < 2\) we get
\[
\gamma_3(c) = \frac{1}{8!} \left( -2c^8 + 24c^7 - 252c^6 + 1512c^5 - 4830c^4 \\
+ 8568c^3 - 8484c^2 + 4392c - 927 \right).
\]
(1.20)

2. Relationship to a Hankel determinant

Our starting point is to derive an expression for \(\gamma_k(c)\) as the Fourier transform of a Hankel determinant. In (1.11), we substitute for the Dirac delta function:
\[
\delta(x) = \int_{-\infty}^{\infty} \exp(2\pi ixy)dy.
\]
(2.1)
One can be rigorous by writing \(\delta(x)\) as the limit of a highly peaked Gaussian, i.e. as the inverse Fourier transform of a highly spread out Gaussian, but for convenience we proceed as above.

Thus
\[
\gamma_k(c) = \frac{1}{k!} \frac{1}{G(1+k)^2} \int_{-\infty}^{\infty} \exp(2\pi iuc) \int_{[0,1]^k} \exp \left( -2\pi iu \sum t_j \right) \\
\times \prod_{i<j} (t_i - t_j)^2 dt_1 \ldots dt_k du.
\]
(2.2)
We also note a more symmetric form of the above by substituting \(t_j = x_j + 1/2\), so that
\[
\gamma_k(c) = \frac{1}{k!} \frac{1}{G(1+k)^2} \int_{-\infty}^{\infty} \exp(2\pi iuc - k/2) \int_{[-1/2,1/2]^k} \exp \left( -2\pi iu \sum x_j \right) \\
\times \prod_{i<j} (x_i - x_j)^2 dx_1 \ldots dx_k du.
\]
(2.3)
We will prove the following two formulas for \(\gamma_k(c)\).

**Theorem 2.1.**
\[
\gamma_k(c) = \frac{1}{G(1+k)^2(2\pi i)^{k(k-1)}} \int_{-\infty}^{\infty} \exp(2\pi iuc) \det_{k \times k} \left( f^{(i+j-2)}(u) \right) du
\]
where \(f(u) = \int_0^1 \exp(-2\pi iut)dt = (1 - \exp(-2\pi iu))/(2\pi iu)\). The determinant is a Hankel determinant.
A similar, but more symmetric, identity is:

\[
\gamma_k(c) = \frac{1}{G(1+k)^2(2\pi i)^{k(k-1)}} \int_{-\infty}^{\infty} \exp(2\pi iuc(c-k/2)) \det_{k \times k} \left( h^{i+j-2}(u) \right) du
\]

(2.5)

where \( h(u) = \int_{-1/2}^{1/2} \exp(-2\pi iux)dx = \sin(\pi u)/(\pi u) \).

Our proof will use the Andreief identity:

**Lemma 2.2** (Andreief). Let \( A_k(t), B_k(t), r(t) \) be integrable functions on the interval \([a, b]\). Then

\[
\frac{1}{N!} \int_{[a,b]^N} \prod_{j=1}^{N} r(t_j) \det_{N \times N} (A_k(t_j)) \det_{N \times N} (B_k(t_j)) dt_1 \ldots dt_N \tag{2.6}
\]

\[
= \det_{N \times N} \left( \int_a^b r(t) A_j(t) B_k(t) dt \right). \tag{2.7}
\]

**Proof of Theorem 2.1.** To prove the first identity in 2.1, apply Andreief’s identity to equation (2.2), with \( A \) and \( B \) two Vandermonde determinants, and \( r(t) = \exp(-2\pi iut) \), to get:

\[
\gamma_k(c) = \frac{1}{G(1+k)^2} \int_{-\infty}^{\infty} \exp(2\pi iuc) \det_{k \times k} \left( \int_0^1 \exp(-2\pi iut) t^{i+j-2} dt \right) du \tag{2.8}
\]

The entries of the matrix can be expressed as derivatives, with respect to \( u \), of \( \int_0^1 \exp(-2\pi iut) dt \), and we can then correct for the extra powers of \(-2\pi iu\) by dividing the \( l \)-th row by \((-2\pi iu)^{l-1}\) and the \( j \)-th column by \((-2\pi iu)^{j-1}\), thus by \((-2\pi iu)^{k(k-1)}\) in total (and then dropping the \(-1\) since \( k(k-1) \) is even).

Using the second form (2.3), we similarly have (2.5) where \( h(u) = \int_{-1/2}^{1/2} \exp(-2\pi iux)dx = \sin(\pi u)/(\pi u) \). \qed

Some of the basic properties of \( \gamma_k(c) \) can be read from (2.4). For example, the inverse Fourier transform of \( f^{(j)} \) is equal to \((-2\pi i)^j c^j\) on the interval \((0, 1)\) and 0 outside this interval. Expanding the determinant as a permutation sum, each summand thus has inverse Fourier transform a convolution of such terms, and is thus supported on \( c \in (0, k) \).

It also shows that \( \gamma_k(c) \) is a polynomial in \( c \) on each interval \([j, j+1]\), \( 0 \leq j \leq k - 1 \) of degree at most \( k^2 - 1 \), because the \( i, j \) entry has inverse Fourier Transform a polynomial in \( c \) on \((0, 1)\) of degree \( i+j-2 \). Multiply out the determinant as a permutation sum. Each summand, when integrated with respect to \( c \), is the inverse Fourier transform of
a product of $k$ functions, and hence consists of $k - 1$ convolutions of the individual inverse Fourier transforms. Each convolution increases the degree of the polynomial by 1. Hence, each permutation $\sigma$ has its resulting degree bounded by $(k - 1) + \sum_{i=1}^{k}(i + \sigma_i - 2) = k^2 - 1$.

We can thus use (2.4) to compute the polynomials $\gamma_k(c)$ by evaluating it at $\geq k^2$ rational values of $c$, say, in each unit interval and interpolating. In this manner, we determined the polynomials $\gamma_k(c)$ listed in Table 1 and 2.

In the symmetric form (2.5), one also sees that $\gamma_k(c) = \gamma_k(c - k)$, by substituting $-u$ for $u$, and using the fact that the determinant in that formula is an even function of $u$.

Setting

$$g(t) = \int_{0}^{1} \exp(-tx)dx,$$

so that

$$g^{(n)}(t) = \int_{0}^{1} (-x)^n \exp(-tx)dx,$$

and letting

$$D_k(t) = \det(g^{(i+j-2)}(t)),$$

we have that (2.4) can be written as

$$\gamma_k(c) = \frac{1}{G(k+1)^2} \int_{-\infty}^{\infty} \exp(2\pi icu) D_k(2\pi iu) du. \tag{2.12}$$

$D_k(t)$ also satisfies a Painlevé V equation. This is proven in more generality in a paper of Basor, Chen and Ehrhardt [1] (4.38 of that paper, with $a = 0$, $b = t$, $\alpha = 0$). Specifically, the following holds.

**Theorem 2.3.** Let

$$H_k(t) = \frac{D'(k)(t)}{D_k(t)} + k^2.$$ \hspace{1cm} (2.13)

Then

$$(tH'_k(t))^2 = (H_k(t) + (2k - t)H'_k(t))^2 - 4(H'_k(t))^2(k^2 - H_k(t) + tH'_k(t)). \tag{2.14}$$

Another interesting feature, is that, while $\gamma_k(c)$ is given by a different polynomial on each $[j, j + 1]$, $0 \leq j \leq k - 1$, $\gamma_k(c)$ can be differentiated $j^2 + (k - j)^2 - 2$ times at $c = j$, i.e. is very smooth.

**Theorem 2.4.** Let $j$ be an integer and $0 < j < k$. Define

$$\nu(c, k) = c^2 + (k - c)^2.$$ \hspace{1cm} (2.15)

Then $\gamma_k(c)$ is $(\nu(j, k) - 2)$-times differentiable at $c = j$. 
Note that $\nu(c, k)$ reaches its minimum at $c = \lfloor \frac{k+1}{2} \rfloor$, in which case
\[
\nu \left( \left\lfloor \frac{k+1}{2} \right\rfloor, k \right) = \left\lfloor \frac{k^2 + 1}{2} \right\rfloor. \tag{2.16}
\]
Thus, we have

**Corollary 2.5.** The function $\gamma_k(c)$ is $(\lfloor \frac{k^2+1}{2} \rfloor - 2)$-times differentiable for all $0 < c < k$.

The following lemma is essentially proved in Section 4 of [6].

**Lemma 2.6.** Let
\[
I_k(u) = \frac{1}{k!} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i u \sum_j t_j} \prod_{j<\ell} (t_j - t_\ell)^2 dt_1 \cdots dt_k. \tag{2.17}
\]
Then
\[
I_k(u) = \sum_{c=0}^{k} e^{i\pi u (k-2c)} \left( \frac{a(c, k)}{u^{\nu(c,k)}} + O \left( \frac{1}{u^{\nu(c,k)+1}} \right) \right) \tag{2.18}
\]
where
\[
\nu(c, k) = c^2 + (k - c)^2 \tag{2.19}
\]
and
\[
a(c, k) = (-1)^c (2\pi i)^{-\nu(c,k)} G(c+1)^2 G(k-c+1)^2. \tag{2.20}
\]

Note that $I_k$ above is essentially the inner multidimensional integral in the expression (2.3) for $\gamma_k$.

**Lemma 2.7.** We have
\[
\gamma_2(c) = \frac{1}{(2\pi i)^2} \int_{-\infty}^{\infty} e^{2\pi i u (c-1)} \left( \frac{1}{u^2} + \frac{\sin(\pi u)^2}{\pi^2 u^4} \right) du \tag{2.21}
\]
\[
= \begin{cases} 
\frac{c^3}{3!}, & \text{if } 0 \leq c \leq 1, \\
\frac{(2-c)^3}{3!}, & \text{if } 1 \leq c \leq 2.
\end{cases} \tag{2.23}
\]
In particular, $\gamma_2(c)$ is not differentiable at $c = 1$.

**Proof of Theorem 2.4.** Substituting (2.17) into equation (2.3),
\[
\gamma_k(c) = \frac{1}{G(1+k)^2} \int_{-\infty}^{\infty} e^{2\pi i u (c-\frac{k}{2})} I_k(u) du. \tag{2.24}
\]
Moreover, from its multi-integral definition we see that $I_k(u)$ is continuous for all real $u$. In particular, $I_k(u)$ is bounded near the origin.
Therefore, to prove that \( \gamma_k(c) \) is \((\nu(j,k) - 2)\)-times differentiable at \( c = j \), it suffices to show that

\[
J_k(c) := \int_{|u| > 1} e^{2\pi i u (c - j)} I_k(u) du \tag{2.25}
\]

is \((\nu(j,k) - 2)\)-times differentiable at \( c = j \).

By Lemma [2.6],

\[
J_k(c) = \sum_{\ell = 0}^{k} \int_{|u| > 1} e^{2\pi i u (c - j)} \left( \frac{a(\ell, k)}{u^{\nu(\ell,k)}} + O \left( \frac{1}{u^{\nu(\ell,k) + 1}} \right) \right) du.
\]

We show that for each \( \ell \),

\[
J_{\ell,k}(c) := \int_{|u| > 1} e^{2\pi i u (c - \ell)} \left( \frac{a(\ell, k)}{u^{\nu(\ell,k)}} + O \left( \frac{1}{u^{\nu(\ell,k) + 1}} \right) \right) du, \tag{2.26}
\]

is \((\nu(j,k) - 2)\)-times differentiable at \( c = j \).

**Case 1: \( \ell = j \).** In this case, we observe that, for \( n = 1, 2, \ldots, \nu(j,k) - 2 \), the integrals

\[
\int_{|u| > 1} \frac{\partial^n}{\partial c^n} \left[ e^{2\pi i u (c-j)} \cdot \left( \frac{a(j, k)}{u^{\nu(j,k)}} + O \left( \frac{1}{u^{\nu(j,k) + 1}} \right) \right) \right] du
\]

\[
= \int_{|u| > 1} e^{2\pi i u (c-j)} \cdot (2\pi i u)^n \left( \frac{a(j, k)}{u^{\nu(j,k)}} + O \left( \frac{1}{u^{\nu(j,k) + 1}} \right) \right) du
\]

\[
\ll \int_{|u| > 1} u^n \left( \frac{a(j, k)}{u^{\nu(j,k)}} + O \left( \frac{1}{u^{\nu(j,k) + 1}} \right) \right) du
\]

are uniformly convergent in \( c \). Therefore, \( J_{j,k} \) is \((\nu(j,k) - 2)\)-times differentiable at \( c = j \) and, in addition,

\[
\frac{d^n}{dc^n} J_{j,k}(c) = \int_{|u| > 1} e^{2\pi i u (c-j)} \cdot (2\pi i u)^n \left( \frac{a(j, k)}{u^{\nu(j,k)}} + O \left( \frac{1}{u^{\nu(j,k) + 1}} \right) \right) du \tag{2.27}
\]

for \( n = 1, 2, \ldots, \nu(j,k) - 2 \).

**Case 2: \( \ell \neq j \).** In this case, we show that \( J_{\ell,k}(c) \) is in fact \( C^\infty \) at \( c = j \). To prove this, it suffices to show that

\[
\int_{|u| > \delta} e^{2\pi i u \delta} \frac{du}{u} \tag{2.28}
\]

is \( C^\infty \) at \( \delta \neq 0 \).
Using integration by parts repeatedly we see that
\[
\int_{|c|>1} e^{2\pi i u c} \frac{du}{u} = \frac{m!}{(2\pi i \delta)^m} \int_{|c|>1} e^{2\pi i u c} \frac{du}{u^{m+1}} + O_m(\delta^{-1} + \delta^{-m}) \quad (2.29)
\]
for any \( m \in \mathbb{N} \) and real \( \delta \neq 0 \), where the Big-O term is a \( C^\infty \) function for \( \delta \neq 0 \). Also, by uniform convergence (see a similar argument in Case 1)
\[
\frac{m!}{(2\pi i \delta)^m} \int_{|c|>1} e^{2\pi i u c} \frac{du}{u^{m+1}} \quad (2.30)
\]
is \((m-1)\)-times differentiable at \( \delta \neq 0 \). It follows that
\[
\int_{|c|>1} e^{2\pi i u c} \frac{du}{u} \quad (2.31)
\]
is \((m-1)\)-times differentiable at \( \delta \neq 0 \). Since \( m \) is arbitrary, we have
\[
\int_{|c|>1} e^{2\pi i u c} \frac{du}{u} \quad (2.32)
\]
is \( C^\infty \) at \( \delta \neq 0 \).

Combining Case 1 and Case 2 we obtain that
\[
J_k(c) := \int_{|u|>1} e^{2\pi i u (c-k^2)} I_k(u) du \quad (2.33)
\]
is \((\nu(j,k) - 2)\)-times differentiable at \( c = j \), and therefore, so is \( \gamma_k(c) \).

Lastly, we show that
\[
\left( \frac{d}{dc} \right)^{\nu(j,k)-2} \gamma_k(c) \quad (2.34)
\]
is not differentiable at \( c = j \). It suffices to show that
\[
\left( \frac{d}{dc} \right)^{\nu(j,k)-2} J_{j,k} \quad (2.35)
\]
is not differentiable at \( c = j \). By equation (2.27) we have
\[
\left( \frac{d}{dc} \right)^{\nu(j,k)-2} J_{j,k} = \int_{|u|>1} e^{2\pi i u (c-j)} \cdot (2\pi i u)^{\nu(j,k)-2}
\]
\[
\left( \frac{a(j,k)}{u^{\nu(j,k)}} + O \left( \frac{1}{u^{\nu(j,k)+1}} \right) \right) du.
\]
Again, by the uniform convergence argument we see that
\[
\int_{|u|>1} e^{2\pi i u (c-j)} \cdot (2\pi i u)^{\nu(j,k)-2} \cdot O \left( \frac{1}{u^{\nu(j,k)+1}} \right) du
\]
is differentiable at \( c = j \). Therefore, it remains to show that
\[
\int_{|u|>1} e^{2\pi i u(c-j)} \cdot (2\pi i u)^{\nu(j,k)-2} \cdot \frac{a(j,k)}{u^{\nu(j,k)}} \, du
\]
is not differentiable at \( c = j \), or equivalently,
\[
\int_{|u|>1} e^{2\pi i u(c-1)} \cdot \frac{du}{u^2}
\]
is not differentiable at \( c = 1 \).

It follows from Lemma 2.7 that
\[
\int_{|u|>1} e^{2\pi i u(c-1)} \left( -\frac{1}{u^2} + \frac{\sin(\pi u)^2}{\pi^2 u^4} \right) \, du
\]
is not differentiable at \( c = 1 \). Since
\[
\int_{|u|>1} e^{2\pi i u(c-1)} \cdot \frac{\sin(\pi u)^2}{\pi^2 u^4} \, du
\]
is differentiable at \( c = 1 \), we see that
\[
\int_{|u|>1} e^{2\pi i u(c-1)} \cdot \frac{du}{u^2}
\]
is not differentiable at \( c = 1 \). This ends our proof of Theorem 2.4.

\[\square\]

The highly smooth nature of \( \gamma_k(c) \) was first observed empirically by Conrey in the related problem of determining the asymptotics of the second moment of Dirichlet polynomials whose coefficients are \( k \)-th divisor numbers. Specifically, he defines
\[
M_k(c) = \lim_{T \to \infty} \frac{(k^2)!}{a_k T (\log T)^{k^2}} \int_0^T \left| \sum_{n=1}^N \frac{d_k(n)}{n^{1/2+it}} \right|^2 \, dt
\]
for integer values of \( k \) and \( N = T^c \) with \( c > 0 \), and determined \( M_k(c) \) for \( k \leq 4 \) (conjecturally for \( k = 3, 4 \)). By comparing Conrey’s tables (personal communication) for \( M_k(c) \) with our tables for \( \gamma_k(c) \), it appears to be the case that the derivative of \( M_k(c) \) is equal to \( (k^2)! \gamma_k(c) \). Bettin [2] has proven the analogous smoothness for the polynomials \( M_k(c) \).

3. Expansion for \( \log D_k(t) \) and the limiting behaviour of \( \gamma_k(c) \)

Notice that
\[
g^{(n)}(0) = \int_0^1 (-x)^n \, dx = (-1)^n / (n+1).
\]
Thus, pulling out powers of \(-1\) from the determinant, of which there are an even number, we have
\[
D_k(0) = \det_{i \times i}(1/(i + j - 1)),
\]
which is a special case of the Cauchy determinant and thus
\[
D_k(0) = G(k + 1)^4/G(2k + 1). \tag{3.2}
\]

Now, \(D_k(t)\) satisfies the Toda equation [9]:
\[
\frac{D_{k-1}(t)D_{k+1}(t)}{D_k(t)^2} = \frac{D_k''(t)}{D_k(t)} - \frac{(D_k'(t))^2}{D_k(t)^2} = (\log(D_k(t)))'' \tag{3.3}
\]
This follows from a recursion of Dodgson (aka Lewis Carroll) for computing determinants [5]. Define \(c_m(k)\) by:
\[
D_k(t) = D_k(0) \exp \left( \sum_{m=1}^{\infty} \frac{c_m(k)}{m} t^m \right). \tag{3.4}
\]
Take the log derivative of the lhs and rhs of the above identity, substitute the series for \(\log(D_k(t))\), and clear the denominator of the rhs. Comparing coefficients gives the recursion, for \(M > 2\):
\[
c_M(k) = \frac{1}{(M - 1)(M - 2)} \sum_{m=0}^{M-3} (m + 1)c_{m+2}(k) \times (c_{M-m-2}(k - 1) + c_{M-m-2}(k + 1) - 2c_{M-m-2}(k)) \tag{3.5}
\]
This recursion determines the coefficients \(c_M(k)\) in terms of \(c_1(k), \ldots, c_{M-2}(k)\).

To get \(c_1(k)\):
\[
c_1(k) = \frac{D_k'(0)}{D_k(0)}. \tag{3.6}
\]
One can differentiate \(D_k(t)\) by using the product rule to get a sum of determinants where we differentiate the \(i\)-th row. However, because the entries of \(D_k(t)\) are derivatives, differentiating the \(i\)-th row produces a row that matches the one below it, and the determinant vanishes. Thus, only the last of these terms, where we differentiate the last row, survives. However, that determinant is also a Cauchy determinant with \(i, j\) entry \((-1)^{i+j-1}/(i + j - 1)\) as before, except for the last row where the entry is \((-1)^{i+j}/(i + j)\).

Using the formula for the Cauchy determinant, a lot of cancellation occurs and we get
\[
c_1(k) = -k/2. \tag{3.7}
\]
To determine \(c_2(k)\), substitute \(t = 0\) into identity (3.3). On the lhs:
\[
\frac{D_{k-1}(0)D_{k+1}(0)}{D_k(0)^2} = \frac{G(k)^4G(k + 2)^4G(2k + 1)^2/(G(2k - 1)G(2k + 3)G(k + 1)^8)}{k^2/(4(4k^2 - 1))}. \tag{3.8}
\]
On the rhs, the constant term of \((\log(D_k(t)))''\) is \(c_2(k)\), so
\[
c_2(k) = k^2/(4(4k^2 - 1)). \tag{3.9}
\]
The recursion, along with the initial two terms determine all the \(c_m(k)\)'s. For example, \(c_3(k) = 0\), and
\[
c_4(k) = \frac{k^2}{16 (4k^2 - 1)^2 (4k^2 - 9)}. \tag{3.10}
\]
We can apply the above to determine the asymptotic expansion of \(\gamma_k(c)\) in a large neighbourhood of \(k/2\). To do so, isolate the \(m = 1, 2\) terms from the series (3.4), substitute into (2.12) with \(t = 2\pi i u\), and compose the series for \(\exp\) with that of the terms \(m \geq 3\) of (3.4), to get that the integrand of (2.12) equals:
\[
\exp \left( -\frac{(k\pi u)^2}{2(4k^2 - 1)} + 2\pi i(c - k/2)u \right) \left( 1 + \frac{k^2(\pi u)^4}{4 (4k^2 - 1)^2 (4k^2 - 9)} + \ldots \right). \tag{3.11}
\]
One can obtain more terms, if desired, from the recursion for \(c_M(k)\). We thus have the following asymptotic expansion:

**Theorem 3.1.** Let \(b_k = 8(1 - 1/(4k^2))\) and \(c = k/2 + o(k)\). Then
\[
\gamma_k(c) \sim \frac{G(k + 1)^2}{G(2k + 1)} \sqrt{\frac{b_k}{\pi}} \exp(-b_k(c - k/2)^2)
\]
\[
\times \left( 1 + \frac{1}{4k^2 - 9} \left( \frac{64(c - k/2)^4 - 24(c - k/2)^2 + 3/4}{k^2}
\right.ight.
\]
\[
- 2 \left( \frac{(c - k/2)^2(16(c - k/2)^2 - 3)}{k^4} + 4 \left( \frac{(c - k/2)^4}{k^6} \right) \right) + \ldots \right). \tag{3.12}
\]
i.e. Gaussian near the centre.

4. **Elliptic aliquot cycles**

The basic method used to pass from (1.11) to equation (2.2) can be used in the context of elliptic aliquot cycles.

Let \(p = (p_1, \ldots, p_d)\) be a \(d\)-tuple of distinct primes. Let \(\alpha(p)\) be the probability of choosing random and independently \(d\) elliptic curves \(E_1, \ldots, E_d\) over \(\mathbb{F}_{p_1}, \ldots, \mathbb{F}_{p_d}\), respectively, with the property that \(|E(\mathbb{F}_{p_j})| = p_{j+1}\), for \(j \in \{1, \ldots, d\}\). Here, \(p_{d+1} = p_1\). We are choosing the curves \(E_j\) uniformly from the set of isomorphism classes of elliptic curves over \(\mathbb{F}_p\).
David, Koukoulopoulos, and Smith [4] gave an asymptotic for the average of $\alpha(p)$ over the set
\[ P_d(x) = \{(p_1, \ldots, p_d) : p_1 \leq x \}. \] (4.1)

(Hasse’s bound implies that $\alpha(p) = 0$ unless $|p_{j+1} - p_j - 1| < 2\sqrt{p_j}$ for $1 \leq j \leq d$).

**Theorem 4.1 (DKS).** For any fixed $A > 0$,
\[ \sum_{p \in P_d(x)} \alpha(p) = C_{\text{aliquot}}^{(d)} \int d^d u \frac{\sqrt{x}}{2\sqrt{u}(\log u)^d} + O_A \left( \frac{\sqrt{x}}{(\log x)^A} \right) \sim C_{\text{aliquot}}^{(d)} \frac{\sqrt{x}}{(\log x)^d}, \]

where
\[ C_{\text{aliquot}}^{(d)} := \frac{I_{\text{aliquot}}^{(d)}}{\ell_d} \int \prod_{\ell} \frac{\# \{ \sigma \in \text{GL}_2(\mathbb{Z}/\ell \mathbb{Z})^d : \text{det}(\sigma_j) + 1 - \text{tr}(\sigma_j) \equiv \text{det}(\sigma_{j+1})^{(\ell)} \} \}}{|\text{GL}_2(\mathbb{Z}/\ell \mathbb{Z})|^d} \]

with
\[ I_{\text{aliquot}}^{(d)} := \frac{2^d}{\pi^d} \int \cdots \int \sqrt{1 - (1 - t_1^2)^{1/2} \prod_{j=1}^{d-1} \sqrt{1 - t_j^2}} dt_1 \cdots dt_{d-1}. \] (4.2)

$I(1) = 1$, $I(2) = 4/3$. One might wonder if $I(d)$ persists in being rational. We will show, for $d = 3$, that this seems unlikely.

Replacing the Dirac delta function by the integral in (2.1), we have
\[ I(d) = \int \cdots \int \sqrt{1 - (1 - t_1^2)^{1/2} \prod_{j=1}^{d-1} \sqrt{1 - t_j^2}} dt_1 \cdots dt_{d-1}. \] (4.5)

But
\[ \int_{-1}^{1} (1 - t^2)^{1/2} \exp(2\pi iy) dt = J_1(2\pi y)/(2y), \] (4.4)

($J$-Bessel function on the rhs). Separating the integral, we get
\[ I(d) = \int_{-\infty}^{\infty} \left( \frac{J_1(2\pi y)}{(2y)} \right)^d dy, \] (4.5)
i.e. a one dimensional integral.

This formula can be used to efficiently evaluate $I(d)$ for, say, $d = 3, 4, \ldots$, for example with Poisson summation.
Let \( f \in L^1(\mathbb{R}) \) and let
\[
\hat{f}(y) = \int_{-\infty}^{\infty} f(t)e^{-2\pi iyt} \, dt.
\] (4.6)
denote its Fourier transform. The Poisson summation formula asserts, for, say, \( f \) continuous, that
\[
\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)
\] (4.7)
provided the rhs converges absolutely and that \( \sum f(n + v) \) converges uniformly in \( v \) on compact sets.

Let \( \Delta > 0 \). By a change of variable
\[
\Delta \sum_{n=-\infty}^{\infty} f(n\Delta) = \sum_{n=-\infty}^{\infty} \hat{f}(n/\Delta) = \hat{f}(0) + \sum_{n \neq 0} \hat{f}(n/\Delta),
\] (4.8)
so that
\[
\int_{-\infty}^{\infty} f(t) \, dt - \Delta \sum_{n=-\infty}^{\infty} f(n\Delta) = - \sum_{n \neq 0} \hat{f}(n/\Delta)
\] (4.9)
tells us how closely the Riemann sum \( \Delta \sum_{n=-\infty}^{\infty} f(n\Delta) \) approximates the integral \( \int_{-\infty}^{\infty} f(t) \, dt \).

Apply, with
\[
f(y) = \left( \frac{J_1(2\pi y)}{(2y)} \right)^d.
\] (4.10)

Note that
\[
\int_{-\infty}^{\infty} J_1(2\pi y) \exp(-2\pi iux) \, dx = \begin{cases} 
(1 - u^2)^{1/2}, & |u| \leq 1, \\
0, & \text{otherwise}.
\end{cases}
\] (4.11)
Therefore, the Fourier transform of \( \left( \frac{J_1(2\pi y)}{(2y)} \right)^d \), being the \( d \)-fold convolution of \( (1 - u^2)^{1/2} \) with itself, is supported in \( |u| \leq d \).

Hence, in the Poisson sum method, any choice of \( \Delta \geq 1/d \) gives no remainder in the Poisson formula (i.e. 0 contribution from them \( |n| \geq 1 \) terms). Thus, taking \( \Delta = 1/d \) gives:
\[
I(d) = \int_{-\infty}^{\infty} \left( \frac{J_1(2\pi y)}{(2y)} \right)^d \, dy = \frac{1}{d} \sum_{n=-\infty}^{\infty} \left( \frac{J_1(2\pi n/d)}{(2n/d)} \right)^d.
\] (4.12)
Furthermore, \( J_1(z) \sim \sqrt{\frac{2}{\pi z}} \cos(z - 3\pi/4) \), hence the sum on the right has terms that are \( \ll (2\pi)^{-d} (n/d)^{-3d/2} \). Thus with \( d = 3 \), the first million terms of the sum gives more than twenty digits accuracy.
One can accelerate the convergence of the sum further using the asymptotics of the $J$-Bessel function, and algorithms for the evaluation of the polylogarithm $\text{Li}_s(z) = \sum_{1}^{\infty} z^n/n^s$. Or one can cheat and just use a blackbox like Maple to evaluate (4.5), with $d = 3$:

$$I(3) = 1.7053570421915038354985956872898996791331386909$$

$$7890590667136169819331192007797559594679011\ldots$$  

(4.13)

Let $A_n/B_n$ be the $n-th$ convergent of the continued fraction of the real number $\alpha$. If $p, q \in \mathbb{Z}$ satisfies:

$$|\alpha - p/q| < |\alpha - A_n/B_n|$$  

(4.14) then $q > B_n$. Therefore, computing the continued fraction for $I(3)$, the 85-th convergent is:

$$14703927951211792459205597491632973549428444428$$

$$8622199098152613288048825699460716423721576467$$  

(4.15)

(and $|I(3) - A_{85}/B_{85}| \neq 0$. With given precision, there is a limit to how many convergents we can meaningfully use).

Thus, if $I(3)$ is rational, then it has denominator at least $10^{45}$. It would not be too difficult to increase the denominator to hundreds or thousands of digits (millions of digits with some effort), assuming $I(3)$ is irrational.

Maple’s identify command did not turn up any obvious expressions for $I(3)$ in terms of algebraic numbers and known constants.

One can also determine the behaviour of $I(d)$ for large $d$. Writing

$$\left(\frac{J_1(2\pi y)}{(2y)}\right)^d = \left(\frac{\pi}{2}\right)^d \exp(d \log(J_1(2\pi y)/(\pi y))),$$  

(4.16)

expanding $J_1$ in its Maclaurin series, and pulling out the $y^2$ term, the above becomes

$$\left(\frac{\pi}{2}\right)^d \exp\left(-\frac{d\pi^2 y^2}{2}\right) \times \exp\left(-\frac{d\pi^4 y^4}{24} - \frac{d\pi^6 y^6}{144} - \frac{d\pi^8 y^8}{720} - \frac{13d\pi^{10} y^{10}}{43200} + \ldots\right).$$  

(4.17)
Taking the Maclaurin series of the latter exponential (truncated with remainder term), we thus get the asymptotic expansion

\[
I(d) = \int_{-\infty}^{\infty} \left( \frac{J_1(2\pi y)}{2y} \right)^d \ dy
\sim \left( \frac{\pi}{2} \right)^{d-1/2} \frac{1}{d^{1/2}} \left( 1 - \frac{1}{8d} - \frac{5}{384d^2} + \frac{7}{3072d^3} + \frac{3829}{491520d^4} + \ldots \right).
\]

(4.18)
| $k$ | $j$ | \((k^2 - 1)!\gamma_k(c)\) |
|-----|-----|------------------|
| 2   | 0   | $c^3$            |
| 3   | 0   | $c^5$            |
| 4   | 0   | $c^{13}$         |
| 5   | 0   | $c^{24}$         |
| 1   |     | $-2c^8 + 24c^6 - 252c^4 + 1512c^3 - 4830c^2 + 8568c - 8484c^2 + 4392c - 927$ |
| 2   |     | $(c - 3)^6$      |

**Table 1.** The polynomials \((k^2 - 1)!\gamma_k(c)\) for \(k \leq 5\) and 
\(j \leq c \leq j + 1\).
Table 2. \((k^2 - 1)!\gamma_k(c)\) for \(k = 6\) and \(j \leq c \leq j + 1, j = 0, 1, 2\). The polynomials for \(j = 3, 4, 5\) can be determined from the above using \(\gamma_k(c) = \gamma_k(k - c)\).

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