Locally solvable subnormal and quasinormal subgroups
in division rings

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Abstract. In this paper, we show that every locally solvable subnormal subgroup or
locally solvable quasinormal subgroup of the multiplicative group of a division ring is
central.

1. Introduction

A subgroup \(N\) of a group \(G\) is said to be subnormal in \(G\) if there is a finite
chain of subgroups

\[
N = N_r \leq N_{r-1} \leq \cdots \leq N_0 = G,
\]

for which \(N_i\) is normal in \(N_{i-1}\) for all \(1 \leq i \leq r\). Also, if \(Q\) is a subgroup of \(G\)
such that the relation \(QH = HQ\) holds for any subgroup \(H\) of \(G\), then we say
that \(Q\) is quasinormal (or permutable) in \(G\). It is pointed out in [11, Chapter 7]
that there are close relations between these types of subgroups. It was shown
by S. E. Stonehewer that if \(G\) is a finitely generated group, then every quasi-
normal subgroup of \(G\) is subnormal ([12, Theorem B]). However, the con-
verse does not hold up. As an example, let \(G\) be the dihedral group of
order 8 generated by subgroups \(A\) and \(B\) which are of order 2. It follows that
\(AB \neq BA\) since \(|AB| = 4\) and \(G \neq AB\), implying that \(A\) and \(B\) are not quasi-
normal subgroups of \(G\). On the other hand, the nilpotency of \(G\) implies that
both \(A\) and \(B\) are subnormal. (Recall that every subgroup of a nilpotent group
is subnormal.) In this paper, we study subnormal subgroups and quasinormal
subgroups of the multiplicative group of a division ring. Relating to this, note
that in [1] there is an example of a division ring which contains quasinormal
subgroups that are not subnormal.

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normal subgroup.
In the literature, there are very rich results concerning the algebraic structure of multiplicative subgroups in a division ring (e.g., see [6]). As a direction of the study, in 1950’s and 1960’s, many authors paid attention to an interesting question of how far the multiplicative group $D^*$ of a division ring $D$ is from being abelian. In this direction, a well-known result of L. K. Hua says that if $D^*$ is solvable, then $D$ is a field. Motivated by this result, several other authors examined various aspects of subnormal subgroups of $D^*$, instead of the whole group $D^*$. For example, it was shown that every subnormal subgroup of $D^*$ must be central in $D$ if it is locally nilpotent, solvable, or $n$-Engel (see [13], [8], [10], respectively). Now, we consider the same problem in which the subnormal subgroup is assumed to be locally solvable. By definition, a group is called locally solvable if its every finitely generated subgroup is solvable. In [4], B. X. Hai and D. V. P. Ha proved that if $D^*$ is locally solvable, then $D$ is a field. Moreover, it was proved by A. E. Zalesskii in [17] that every locally solvable normal subgroup of $D^*$ is contained in the center $F$ of $D$. It is natural to ask whether every locally solvable subnormal subgroup, say $G$, of $D^*$ is also contained in $F$. A positive answer to this question was given for some particular cases where $D$ is supposed to be algebraic over $F$ ([5]), or where the derived subgroup $G^{(i)}$ of $G$ is assumed to be algebraic over $F$ for some $i \geq 1$ ([9]). The first purpose of the present paper is to give the affirmative answer to the question in the general setting; that is, we shall show that every locally solvable subnormal subgroup of $D^*$ is contained in $F$ (Theorem 1). The second purpose is to prove that every locally solvable quasinormal subgroup of $D^*$ is also central; and this goal will be achieved in Theorem 3.

Throughout this paper, the word “ring” always refers to a ring with an identity element $1 \neq 0$. For a ring $R$, the symbol $R^*$ denotes the group of units in $R$. If $D$ is a division ring with center $F$ and $S$ is a subset of $D$, then $F[S]$ (resp. $F(S)$) denotes the subring (resp. the division subring) of $D$ generated by $F \cup S$. For a group $G$, the Hirsch-Plotkin radical of $G$ is defined to be the subgroup generated by all locally nilpotent normal subgroups of $G$. If $H$ and $K$ are two subgroups of $G$, then the symbol $[H, K]$ stands for the subgroup of $G$ generated by the set of all commutators $[a, b] = a^{-1}b^{-1}ab$, where $a \in H$ and $b \in K$. We say that $G$ is radical over a subgroup $Q$ if for each $g$ in $G$, there is a positive integer $n$ depending on $g$ such that $g^n$ belongs to $Q$.

2. Locally solvable subnormal subgroups

We begin with a group-theoretic lemma which, despite its apparent simplicity, will be frequently applied in the sequel.
**Lemma 1.** Every group contains a unique maximal periodic normal subgroup. Moreover, such a subgroup is characteristic in the whole group.

**Proof.** Our proof shall be obtained by mainly using Zorn’s Lemma. First, we define a family of subgroups of a group \( G \) by taking

\[ \mathcal{A} = \{ H \mid H \text{ is a periodic normal subgroup of } G \} \]

This family is obviously non-empty since the identity subgroup belongs to \( \mathcal{A} \). Now, we consider an arbitrary chain \( \{ H_i \} \) of subgroups in \( \mathcal{A} \). Our task, of course, is to show that \( \bigcup H_i \) is again a member of \( \mathcal{A} \); that is, to prove that \( \bigcup H_i \) forms a periodic normal subgroup of \( G \). For this purpose, pick any two elements \( a, b \in \bigcup H_i \). Then, there exist indices \( i \) and \( j \) for which \( a \in H_i \) and \( b \in H_j \). Since the collection \( \{ H_i \} \) forms a chain, either \( H_i \subseteq H_j \) or \( H_j \subseteq H_i \). It is clear that we may assume that \( H_i \subseteq H_j \) and so \( a.b^{-1} \in H_j \subseteq \bigcup H_i \). This implies that \( \bigcup H_i \) is a subgroup of \( G \). The normality as well as the periodicity of \( \bigcup H_i \) may be obtained by the same way. All of this shows that \( \bigcup H_i \) is a member of \( \mathcal{A} \), completing our task. Therefore, on the basis of Zorn’s Lemma, the family \( \mathcal{A} \) contains a maximal element \( M \).

Next, we shall prove that \( M \) is maximal with respect to being periodic and normal. Let \( N \) be a periodic normal subgroup of \( G \) for which \( M \subseteq N \). Since \( M \) is a maximal element of \( \mathcal{A} \) and \( N \in \mathcal{A} \), we must have \( M = N \), which implies the maximality of \( M \).

To see the uniqueness of \( M \), take any periodic normal subgroup \( A \) of \( G \). The normality of \( M \) and \( A \) in \( G \) permits us to form the product subgroup \( A.M \), which is obviously a periodic normal subgroup of \( G \). But then, the maximality of \( M \) reveals that \( A.M = M \), or \( A \subseteq M \). This argument shows that every periodic normal subgroup of \( G \) is contained in \( M \), proving the uniqueness of \( M \).

It remains only to show that \( M \) is characteristic in \( G \). For this purpose, we pick \( \varphi \in \text{Aut}(G) \), then \( \varphi(M) \) is certainly a periodic normal subgroup of \( G \). The uniqueness of \( M \) implies that \( \varphi(M) = M \). Our proof is finally finished.

For any group \( G \), let us denote by \( \tau(G) \) the unique maximal periodic normal subgroup of \( G \) and by \( B(G) \) the subgroup of \( G \) such that \( B(G)/\tau(G) \) is the Hirsch-Plotkin radical of \( G/\tau(G) \). It is easy to see that \( B(G) \) is a normal subgroup of \( G \).

**Proposition 1.** Let \( D \) be a division ring with center \( F \). If \( G \) is a subnormal subgroup of \( D^* \), then \( B(G) \) is contained in \( F \).

**Proof.** Being a normal subgroup of \( G \), the subgroup \( \tau(G) \) is a periodic subnormal subgroup of \( D^* \). With reference to [7, Theorem 8], we conclude that \( \tau(G) \) is contained in \( F \).

**Proposition 1.**
Our next step is to assert that $B(G)$ is indeed a locally nilpotent group. For this purpose, we take an arbitrary finitely generated subgroup $H$ of $B(G)$, and our aim is to show that this is a nilpotent group. It is a simple matter to see that $H\tau(G)/\tau(G)$ is a finitely generated subgroup of $B(G)/\tau(G)$. Accordingly, the local nilpotence of $B(G)/\tau(G)$ implies that $H\tau(G)/\tau(G)$ is nilpotent. We set

$$H_1 = [H, H], \quad H_2 = [H_1, H], \quad H_3 = [H_2, H], \ldots$$

Now, as $H\tau(G)/\tau(G)$ is nilpotent, we can find an integer $n$ for which $H_n \subseteq \tau(G) \subseteq F$. This fact says that any element of $H_n$ commutes element-wise with $H$ and, in consequence, we have $H_{n+1} = [H_n, H] = 1$, from which it follows that $H$ is nilpotent. In other words, we obtain that $B(G)$ is locally nilpotent, as asserted.

As we have pointed out before, $B(G)$ is a normal subgroup of $G$. This assures us to conclude that $B(G)$ is a locally nilpotent subnormal subgroup of $D^*$. By virtue of Huzurbazar’s result ([8]), we finally obtain that $B(G) \subseteq F$. Our proof is finished.

The following lemma, which provides the key to later success, gives us a way to calculate the normalizer of a locally solvable subgroup in a division ring.

**Lemma 2 ([15, Point 20]).** Let $R = F[G]$ be an algebra over the field $F$ that is a domain. If $G$ is a locally solvable, then $R$ is an Ore domain. Moreover, if we assume that $D$ is the skew field of fractions of $R$ and that $B(G) = F^* \cap G$, then $N_D \cdot (G) = GF^*$.

**Lemma 3.** Let $D$ be a division ring with center $F$. If $G$ is a locally solvable non-central subnormal subgroup of $D^*$, then $F(G) = D$ and $N_D \cdot (G)$ is locally solvable.

**Proof.** With reference to the previous lemma, the local solvability of $G$ assures us to conclude that $R = F[G]$ is an Ore domain. Accordingly, its skew field of fractions is exactly $F(G)$, the division subring of $D$ generated by $G$ over $F$. Since $F(G)$ contains $G$ which is assumed to be non-central, in the light of Stuth’s Theorem ([13, Theorem 1]), we obtain that $F(G) = D$.

Next, we argue that $B(G) = F^* \cap G$. First, it follows directly from Proposition 1 that $B(G) \subseteq F^* \cap G$, which implies that $B(G)/\tau(G) \subseteq (F^* \cap G)/\tau(G)$. In regard to the reverse inclusion, we note that, being the Hirsch-Plotkin radical of $G/\tau(G)$, the factor group $B(G)/\tau(G)$ is the largest locally nilpotent normal subgroup of $G/\tau(G)$. On the other hand, it is clear that $(F^* \cap G)/\tau(G)$
is an abelian normal subgroup of $G/\tau(G)$, which yields that $(F^* \cap G)/\tau(G) \subseteq B(G)/\tau(G)$. In other words, we must have $(F^* \cap G)/\tau(G) = B(G)/\tau(G)$, from which it follows that $B(G) = F^* \cap G$. Our argument is now finished. Finally, the last assertion follows immediately from the proceeding lemma.

Before presenting the main theorem, we need a result of Zalesskii:

**Lemma 4 ([17]).** Let $D$ be a division ring with center $F$. If $G$ is a locally solvable normal subgroup of $D^*$, then $G$ is contained in $F$.

Here now is the main result of this section.

**Theorem 1.** Let $D$ be a division ring with center $F$. If $G$ is a locally solvable subnormal subgroup of $D^*$, then $G$ is contained in $F$.

**Proof.** There is nothing to prove if $D$ is commutative. Therefore, we may suppose that $D$ is non-commutative. Assume that $G$ is not contained in $F$. Since $G$ is a subnormal subgroup of $D^*$, there exists a finite chain of subgroups

$$G = G_r \leq G_{r-1} \leq \cdots \leq G_0 = D^*,$$

in which $G_i$ is normal in $G_{i-1}$ for $0 \leq i \leq r$. By virtue of Lemma 3, we conclude that $N_{D^*}(G)$, the normalizer of $G$ in $D^*$, is a locally solvable group. The normality of $G_r$ in $G_{r-1}$ implies that $G_{r-1}$ is contained in $N_{D^*}(G)$ and, in consequence, the subgroup $G_{r-1}$ is locally solvable and non-central.

Repeat this procedure, now starting with $G_{r-1}$, we obtain that $G_{r-2}$ is locally solvable, too. This process must eventually terminate after finite steps, and at the final stage, we have the fact that $D^*$ is locally solvable. It follows immediately from Lemma 4 that $D$ is commutative, which is a contradiction. Our proof is finally completed.

3. **Locally solvable quasinormal subgroups**

We prepare the way by first establishing a few results concerning groups which are radical over subgroups.

**Lemma 5 ([3, Theorem 2]).** Let $G$ be a group and $Q$ a quasinormal subgroup of $G$. If $C$ is an infinite cyclic subgroup of $G$ such that $Q \cap C = 1$, then $Q$ is a normal subgroup of $Q^G$ and $Q/Q_G$ is abelian.

**Lemma 6.** Let $G$ be a group. If $Q$ is a quasinormal subgroup of $G$, then either $G$ is radical over $Q$ or $Q$ is subnormal in $G$ of defect at most 2.

**Proof.** To start, we assume that $G$ is not radical over $Q$. As such, we can find an element $g \in G$ such that $g^n$ does not belong to $Q$ for every integer
number \( n \). Let \( C \) be the cyclic subgroup of \( G \) generated by the element \( g \). Then, the fact that \( g^n \not\in Q \) for any choice of \( n \) ensures that \( Q \cap C = 1 \). By virtue of the above lemma, we obtain that \( Q \) is normal in \( Q^G \), which is a normal subgroup of \( G \). Phrased in another way, \( Q \) is a subnormal subgroup of \( G \) with the correspondent series \( Q \unlhd Q^G \unlhd G \). This completes our proof.

The next lemma, which is an interesting result of C. Faith, provides the key to establish the main result of this section.

**Lemma 7** ([2, Theorem B]). *Every division ring which is radical over a proper subring is a field.*

By an analogy with C. Faith’s result, a ring which is radical over a subgroup may be characterized in the following manner.

**Proposition 2.** Let \( R \) be a ring and \( G \) a subgroup of \( R^* \). If \( R \setminus \{0\} \) is radical over \( G \), then \( R \) is a division ring.

**Proof.** To prove that \( R \) is a division ring, it suffices to show that each nonzero element of \( R \) is right invertible. For this purpose, we take \( x \) to be an arbitrary nonzero element of \( R \). The radicality over \( G \) of \( x \) permits us to find an integer \( n \geq 1 \) for which \( x^n \in G \). As \( G \) is a group, we can find an element \( g \in G \) such that \( x^n g = 1 \). Or, equivalently, we have \( x(x^{n-1}g) = 1 \). This relation shows that \( x \) is right invertible with the right inverse \( x^{-1} = x^{n-1}g \). Therefore, the ring \( R \) is indeed a division ring and our proposition is proved.

**Lemma 8.** Let \( D \) be a division ring, and \( G \) a non-abelian subgroup of \( D^* \). Assume that \( D^* \) is radical over \( G \). Then, every subring of \( D \) containing \( G \) is coincided with \( D \).

**Proof.** For a proof by contradiction, we assume that \( E \) is a proper subring of \( D \) containing \( G \). It is a fairly simple matter to see that \( E = E[G] \). The assumption on \( D^* \) assures us to deduce that \( D \) is radical over \( E \) and so \( D \) is a field by Lemma 7. But this contrasts to the fact that \( G \) is assume to be non-abelian.

The following theorem illustrates how the multiplicative group of a division ring is affected by certain subgroups over which it is radical.

**Theorem 2.** Let \( D \) be a division ring, and \( G \) a locally solvable subgroup of \( D^* \). If \( D^* \) is radical over \( G \), then \( D \) is a field.

**Proof.** Suppose, to the contrary, that \( D \) is non-commutative. If \( G \) is abelian, then \( F(G) \) is a proper subfield over which \( D \) is radical. It follows from previous lemma that \( D \) is a field, which violates our supposition. We
may therefore assume that $G$ is non-abelian. In the light of Lemma 8, we obtain that $F[G] = D$ and so $G$ possesses an abelian normal subgroup $A$ for which $G/A$ is locally finite ([14, Point 3]). This last fact ensures that $G$ is radical over $A$ and, in consequence, so is $D^*$. As a result, the division ring $D$ is radical over the subfield $F(A)$, from which it follows that $D$ is a field. Again, we arrive at a desired contradiction, proving our theorem.

This may be a good place to give the main result of this section.

**THEOREM 3.** Let $D$ be a division ring with center $F$. If $Q$ is a locally solvable quasinormal subgroup of $D^*$, then $Q$ is contained in $F$.

**Proof.** With reference to Lemma 6, we have either $Q$ is subnormal in $D^*$ or $D^*$ is radical over $Q$. In the first event, our result follows immediately from Theorem 1. It remains to examine the case where $D^*$ is radical over $Q$. In this case, previous theorem says that $D$ is commutative, and our result certainly holds. Our proof is now completed.

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