Abstract

The quantization of a real massless scalar field in a spacetime produced in a collision of two electromagnetic plane waves with constant wave fronts is considered. The background geometry in the interaction region, the Bell-Szekeres solution, is locally isometric to the conformally flat Bertotti-Robinson universe filled with a uniform electric field. It is shown that before the waves interact the Bogoliubov coefficients relating different observers are trivial and no vacuum polarization takes place. In the non-singular interaction region neutral scalar particles are produced with number of created particles and spectrum typical of gravitational wave collision.

PACS numbers: 04.62 + v, 04.20Jb, 04.30. – w
1 Introduction

It is known that quantum particles are not produced in the vicinity of plane electromagnetic waves due to the high degree of symmetry \[1, 2\]. Neither, one expects particle creation when such waves scatter in a flat spacetime, without gravity taken into account, because of the linearity of the process. If the electromagnetic waves are self-gravitating, however, they interact non-linearly producing a region with a non-zero Coulomb component of the electric field. One then expects particles with charge to be created. Thus, the effect of quantum particle creation when two self-gravitating electromagnetic waves scatter can be considered as a purely curvature effect caused by the non-linear interaction of the fields governed by the coupled Einstein-Maxwell equations.

The simplest example of the collision of the plane electromagnetic waves was given by Bell and Szekeres \[3\] and studied later by Matzner and Tipler \[4\] and by Clarke and Hayward \[5\]. The example considered by Bell and Szekeres involves a collision of two step plane waves with constant wave-fronts which, unlike most of the waves after the collision, do not focus to a curvature singularity, but rather, to a Killing-Cauchy horizon. It was shown by Clarke and Hayward further on, that the solution in the interaction region is extendible across the focusing surface similarly to the previously studied cases of non-singular collisions of pure gravitational plane waves \[6, 7\].

Matter field quantization on the colliding wave background was first studied by Yurtsever \[8\] with the background produced by the collision of two plane impulsive gravitational waves, the Khan-Penrose solution \[9\]. Because of the peculiar property of the Khan-Penrose solution with flat regions in the single-wave propagating parts of the spacetime, Yurtsever has managed to construct, in a relatively simple way, an unambiguous “out”-vacuum related to these flat regions behind wave fronts. Generally, however, the spacetime regions behind the wave fronts have nonzero curvature and it is not easy to construct the “out”-modes with the procedure outlined by Yurtsever.

Dorca and Verdaguer \[10, 11\] have noticed recently that the presence of the Killing-Cauchy horizon, instead of the strong curvature singularity, makes the task of solving the quantum field theory on the background of colliding waves technically more plausible. The presence of the Killing-Cauchy horizon and the symmetries associated with it can be used to define the unique preferred vacuum state, as pointed out by Kay and Wald \[12\] in a general context.

In this paper we consider the quantization of massless neutral scalar
field on the background geometry produced by the collision of step electromagnetic plane waves with constant wave fronts - the Bell-Szekeres solution. The interaction region of the Bell-Szekeres solution is isometric to the Bertotti-Robinson universe \[13, 14\] - the static conformally flat solution of the Einstein-Maxwell equations with uniform electric field.

Quantum test electrodynamics on the Bertotti-Robinson background, within the context of the Euclidean quantum field theory, was previously considered by Lapedes\[15\]. The Bertotti-Robinson universe has three non-commuting timelike Killing vectors and one may associate three different observers related to these vector fields experiencing different accelerations \[15\]. The observers “see” different spectra of created particles according as to whether the acceleration experienced by the observers exceeds a certain critical value \(a_{cr}\). If the acceleration exceeds \(a_{cr}\) then the spectra of the created particles has a form typical to a Hawking thermal spectrum, whereas if the acceleration is equal or smaller than the \(a_{cr}\) the spectrum is nonthermal approaching the Schwinger spectrum \[1\] in the limit of high electric field strength.

In this work we are interested in the creation of neutral scalar pairs, rather than charged particles, because the production of the latter ones would not come in as a surprise in a region of spacetime (the interaction region of the Bell-Szekeres solution) which can be thought of as filled with uniform electric field. The effect of neutral particle creation in the Bell-Szekeres solution, however, can be considered as not only due to the nonlinear interaction of the waves as stated at the beginning but also as a result of the dynamical evolution of the spacetime.

While maybe somewhat simplified, the Bell-Szekeres example of the scattering of two electromagnetic waves represents an interesting theoretical laboratory to study the quantum field theory. This is mainly due to the simplicity of the metric in each of the different regions defined in the problem of plane wave collision. Nevertheless, the global compositeness of the spacetime preserves all the features of more complicated problems involving plane wave scattering.

In the following Section 2 we briefly discuss some relevant geometrical properties of the Bell-Szekeres solution defining different coordinate patches to be used in this paper. In Section 3 we solve exactly the Klein-Gordon equation in all four regions associated with the wave collision. We also discuss the nonexistence of vacuum polarization in the case of a single plane electromagnetic wave. We then follow Dorca’s and Verdaguer’s idea \[\text{[10]}\] propagating the “in”-vacuum state from the initially Minkowskian back-
ground into the interaction region via the regions of single plane waves and then define the “out”-vacuum relating it to the observer at rest at the Killing-Cauchy horizon. In Section 4 we evaluate the Bogoliubov coefficients relating the “in” and the “out” modes and calculate the number of particle created as seen by the observer at the horizon.

## 2 Geometric Properties

The Bell-Szekeres solution represents the collision of two electromagnetic plane waves with different amplitudes, and constant polarization \cite{[3, 14]}. The metric tensor for this spacetime is:

\[ ds^2 = 2dudv - \cos^2(au) dx^2 - \cos^2(bv) dy^2 \] (1)

where \( au < \pi/2, \ bx < \pi/2, \ -\infty < x, y < \infty \); here the positive constants \( a \) and \( b \) are related to the strengths of the electromagnetic plane waves.

The metric coefficients have square-integrable weak derivatives, so that the curvature tensor can be split into a regular and distributional parts:

\[ R^\alpha_{\beta\gamma\delta} = \hat{R}^\alpha_{\beta\gamma\delta} + \overline{R}^\alpha_{\beta\gamma\delta} \] (2)

Here the distributional part \( \overline{R}^\alpha_{\beta\gamma\delta} \) is a linear combination of the following distributions: \( \delta(au) \sin(bv) \) and \( \delta(bv) \sin(au) \). The regular part of the Ricci tensor is zero, and the curvature scalar is zero globally:

\[ R = 0 \] (3)

The components of the Weyl tensor as well as the electromagnetic tensor are given in reference \cite{[3]}. The spacetime represents two electromagnetic and gravitational impulsive colliding plane waves along \( u = 0, \ v = 0 \) hypersurfaces. Note, that there are no gravitational waves before the collision and these are induced only after the scattering has taken place.

The spacetime is split into four different regions:

\[ ds^2_{IV} = 2dudv - dx^2 - dy^2, \quad u, v < 0 \]
\[ ds^2_{III} = 2dudv - \cos^2(bv)(dx^2 + dy^2), \quad u < 0, 0 < v < \pi/2b \]
\[ ds^2_{II} = 2dudv - \cos^2(au)(dx^2 + dy^2), \quad v < 0, 0 < u < \pi/2a \]
\[ ds^2_{I} = 2dudv - \cos^2(au - bv)dx^2 - \cos^2(au + bv)dy^2, \quad au + bv < \pi/2. \] (4)
The metric is regular in each region with apparent singularities in: \((u = \frac{\pi}{2}, v < 0), (v = \frac{\pi}{2}, u < 0),\) and \((u > 0, v > 0, au + bv = \frac{\pi}{2})\) which can be removed by a change of coordinates. The first two are fold singularities and the last one is the Killing-Cauchy horizon where the plane waves focalize \[3, 5\].

Region IV is Minkowski spacetime. One can show that regions II and III are conformally flat by performing the following coordinate transformation (in the region II, for example):

\[
\bar{u} = \tan au
\]

then the metric in region II takes the form:

\[
ds^2 = \frac{1}{1 + \bar{u}^2} \left( \frac{2}{a} d\bar{u} dv - dx^2 - dy^2 \right)
\]

Region I is also conformally flat. It can be brought to the explicitly conformally flat form by the following coordinate transformation \[16\]:

\[
t + r = \coth\left[\frac{1}{2} \sech^{-1}(\cos(au + bv)) - \frac{q}{2q}\right]
\]

\[
t - r = -\tanh\left[\frac{1}{2} \sech^{-1}(\cos(au + bv)) + \frac{q}{2q}\right]
\]

\[
\theta = \pi/2 - (bv - au)
\]

\[
\phi = x/q,
\]

where \(q = \frac{1}{\sqrt{2ab}}\). Then, the metric tensor is simply:

\[
ds^2 = \frac{q^2}{r^2}(dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2 \theta d\phi^2).
\]

One can immediately recognize the line element \(8\) as the Bertotti-Robinson spacetime which has a geometry similar to that one of the throat of the Reissner-Nordstrom solution for the special case \(Q = M \[17\].

In the Bertotti-Robinson solution, the coordinate \(\phi\) is cyclic: \(0 \leq \phi < 2\pi\), while in the Bell-Szekeres solution, however, the corresponding \(x\) coordinate is defined over the whole range: \(-\infty < x < \infty\).

The Bertotti-Robinson coordinates are not well adapted to describe the Killing-Cauchy horizon. It is convenient, therefore, to introduce a new set of coordinates in order to describe the interaction region. One then defines a set of Kruskal-Szekeres-like coordinates in the following way \[10\]:

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First we define a new dimensionless time and space coordinate \((\xi, \eta)\):
\[
\begin{align*}
\xi &= au + bv & 0 \leq \xi < \pi/2 \\
\eta &= bv - au & -\pi/2 \leq \eta < \pi/2,
\end{align*}
\]
(introduce a new dimensionless time-like coordinate \(\xi^*\):
\[
\xi^* = \frac{1}{\sqrt{2ab}} \log \left(\frac{1 + \sin \xi}{\cos \xi}\right),
\]
and a new set of null coordinates:
\[
\tilde{U} = \xi^* - y \\
\tilde{V} = \xi^* + y,
\]
Finally, we define the Kruskal-Szekeres-like null coordinates:
\[
U' = -q e^{-\tilde{U}/q} \\
V' = -q e^{-\tilde{V}/q}
\]
so that the metric in the interaction region becomes:
\[
ds^2 = (1 + \sin \xi)^2 dU' dV' - \frac{1}{2ab} d\eta^2 - \cos^2 \eta \, dx^2,
\]
with
\[
U' V' = q^2 \frac{1 - \sin \xi}{1 + \sin \xi},
\]
and
\[
\frac{U'}{V'} = e^{2y/q}.
\]
The curves \(\xi = \text{const.}\) are hyperbolae and \(y = \text{const.}\) are straight lines in the \((U', V')\) plane. When \(\xi \to \pi/2\) we obtain hyperbolae \(U' V' = \epsilon, \ \epsilon > 0\).
And the hypersurface which is the Killing-Cauchy horizon \(\xi = \pi/2\) is then \(\{(U' = 0, V' \leq 0) \cup (V' = 0, U' \leq 0)\}\).

3 Quantization of the scalar field

The quantization of a scalar field and the production of particles on a given curved background is done in a standard manner (see for example [13]). We
will proceed as follows: first construct a complete orthonormal set of modes for a massless real scalar field related to the Minkowskian region IV of the spacetime before the collision. In this region, one is able to build a Fock space related to an inertial observer. These modes will be considered as the “in”-modes and, since the region is flat, all the inertial observers in this region would agree on the definition of particles [18].

Next, we propagate these modes throughout all the spacetime up to horizon by solving the Klein-Gordon equation in each region of the spacetime and matching the modes across different hypersurfaces separating the regions. At each state we will be able to solve exactly the Klein-Gordon equation in every region of spacetime.

In passing, we study the case of a single electromagnetic plane wave. Using two different observers, one related to the “in”-vacuum state propagated from the region IV into the region II and the other which is related to the modes constructed using the conformal symmetry of the single wave region we explicitly show that the Bogoliubov coefficients are trivial and there is no particle creation in the vicinity of the plane wave. The triviality of these Bogoliubov coefficients prevents to construct a different set of modes and to proceed the quantization in the interaction region in the way done by Yurtsever [8]. Neither it is simple to define a different set of modes using the harmonic coordinates due to the nonflatness of the single wave region. One therefore is bounded to use the procedure of Dorca and Verdaguer [10] which is most suitable to our case due to the presence of two null Killing vector fields at the horizon.

Formally, we could have tried to use different symmetries associated with the geometry of the interaction region: the conformal flatness or the existence of three non-commuting timelike Killing vectors [15]. This, however, leads to serious difficulties with the definition of particles due to compositeness of the Bell-Szekeres spacetime which limits the range of the time coordinate related to these symmetries.

3.1 Region IV

The metric in the region IV is the Minkowski spacetime:

\[ ds^2 = \frac{2}{ab}dudv - dx^2 - dy^2 \quad u < 0, \quad v < 0, \quad (16) \]

where we have rescaled the coordinates:

\[ u' = au \quad (17) \]
and then abolished the primes.

The general solution of the Klein-Gordon equation for a massless scalar field can be expanded into the following orthonormal set of modes:

\[ u_{k}^{inIV}(u,v,x,y) = \frac{1}{\sqrt{2k_{-}(2\pi)^{3}}} \exp \left\{ -i \frac{k_{-}}{a} u - i \frac{k_{-}}{b} v \right\} \]  

\[ x_{k}^{inIV}(u,v,x,y) = \frac{1}{\sqrt{2k_{-}(2\pi)^{3}}} \exp \left\{ -i \frac{k_{-}}{a} u + i k_{+} x + i k_{y} y \right\} \]  

where \( k_{-} k_{+} = 1/2 (k_{x}^{2} + k_{y}^{2}) \).

In spite of the fact that the single plane wave spacetimes do not contain a global Cauchy surface \( [14] \) in order to construct the scalar product, it was shown by Gibbons \( [20] \) that one can use the null surfaces \( u = const \) instead. One can then see that the modes given by the equation \( [14] \) are well normalized on the “roof” hypersurface \( \Sigma : \{(u = 0, v < 0) \cup (u < 0, v = 0)\} \):

\[ (u_{k}^{inIV}, u_{k'}^{inIV}) = \delta(k_{x} - k'_{x}) \delta(k_{y} - k'_{y}) \delta(k_{-} - k'_{-}). \]  

These modes impose the following boundary conditions on the “in”-modes in the single wave region II (all expressions in III can be obtained by interchanging \( u \leftrightarrow v, a \leftrightarrow b \) and \( k_{-} \leftrightarrow k_{+} \)) due to the continuity:

\[ u_{k}^{inII} \big|_{u=0} = u_{k}^{inIV} \big|_{u=0} = \frac{1}{\sqrt{2k_{-}(2\pi)^{3}}} \exp \left\{ -i \frac{k_{-}}{b} v + i k_{x} x + i k_{y} y \right\} \]  

### 3.2 Regions II and III.

The metric in the region II takes the following form:

\[ ds^{2} = \frac{2}{ab} du dv - \cos^{2}(u)(dx^{2} + dy^{2}) \quad v < 0, 0 < u < \frac{\pi}{2} \]  

Representing the passage of a single electromagnetic plane wave.

The corresponding Klein-Gordon equation for the region II becomes:

\[ \phi_{uv} - \tan u \phi_{v} - \frac{\phi_{xx} + \phi_{yy}}{2ab \cos^{2} u} = 0 \]  

For the region III the corresponding Klein-Gordon equation is basically the same equation as \( [23] \) changing \( u \) for \( v \). The “in”-mode solutions for the region II satisfying the boundary conditions \( [21] \) are:

\[ u_{k}^{inII}(u,v,x,y) = \frac{1}{\sqrt{2k_{-}(2\pi)^{3}}} f(u) \exp \left\{ -i \frac{k_{-}}{b} v + i k_{x} x + i k_{y} y \right\} \]
where \( f(u) \) is:

\[
f(u) = \frac{1}{\cos(u)} \exp \left[-i \frac{k_+}{a} \tan(u) \right]. \tag{25}
\]

We now choose the null hypersurface \( \Sigma : \{(v = 0, 0 \leq u < \pi/2) \cup (u = 0, 0 \leq v < \pi/2)\} \), which is the characteristic surface of Klein-Gordon equation in these regions, to orthonormalize the “in”-modes:

\[
(u_k^{in}, u_{k'}^{in}) = -i \int dxdy \int_0^{\pi/2} \cos^2(u) (u_k^{inII} \partial_u u_{k'}^{inII}) \left|_{v=0}^{u=0} \right. \tag{26}
\]

\[
= -i \int dxdy \int_0^{\pi/2} \cos^2(v) (u_k^{inIII} \partial_v u_{k'}^{inIII}) \left|_{u=0}^{v=0} \right.,
\]

which gives:

\[
(u_k^{in}, u_{k'}^{in}) = \delta(k_x - k'_x) \delta(k_y - k'_y) \delta(k_- - k'_-). \tag{27}
\]

The “in”-modes then are orthonormal considering these two propagation regions, and induce, in turn, the following boundary conditions for the “in”-modes in the interaction region across the hypersurfaces:

\[
u_k^{inI} |_{v=0} = u_k^{inII} |_{v=0} = \frac{1}{\sqrt{2k_- (2\pi)^3} \cos(u)} \exp \left[-i \frac{k_+}{a} \tan(u) + ik_xx + ik_yy \right]. \tag{28}
\]

And the corresponding for the \( u = 0 \) hypersurface related to the “in”-modes in the region III.

Although the normal “in”-modes in the region II (III) diverge at the points of the fold singularities \( \{u = \pi/2, v = 0\}, \{v = \pi/2, u = 0\} \), this divergence does not influence the scalar product in this region because, as argued by Dorca and Verdaguer \[10\], only a set of null measure of these modes arrive at these points.

### 3.3 Conformally flat modes in the single wave region

We have seen that the regions II and III corresponding to the propagation of single electromagnetic waves, are conformally flat. Therefore, an observer adapted to the symmetries of the region i.e., an observer who sees the region as a conformally flat one would be a physically meaningful observer (see for example \[18\]). Following the coordinate change which transforms the metric into an explicitly conformally flat form for the region II (for example):

\[
\bar{u} = \tan(u), \tag{29}
\]
the metric becomes:

$$ds^2 = \frac{1}{1 + \bar{u}^2} \left( \frac{2}{ab} \bar{d}u \bar{d}v - dx^2 - dy^2 \right) = \Omega^2(u) \left( \frac{2}{ab} \bar{d}u \bar{d}v - dx^2 - dy^2 \right)$$  \hspace{1cm} (30)

The solutions for a massless scalar field of the corresponding Klein-Gordon equation in our case are simply the plane waves with the amplitude multiplied by the inverse of the conformal factor: $\Omega(u)$

$$u_{k}^{conf}(\bar{u}, v, x, y) = \frac{\sqrt{1 + \bar{u}^2}}{\sqrt{(2\pi)^3} 2k_-} \exp \left[ -i \frac{k_-}{b} v - i \frac{k_+}{a} \bar{u} + i k_x x + i k_y y \right], \hspace{1cm} (31)$$

where the coefficients are related by $k_- k_+ = (k_x^2 + k_y^2)/2$. The conformal modes for the region III are obtained by changing $(a, \bar{u})$ by $(b, \bar{v})$.

We now look at the Bogoliubov transformation between these conformal modes (31) and the “in”-modes (24) in the single wave regions:

$$u_{k}^{in} = \sum_{k'} \left( \alpha_{kk'} u_{k'}^{conf} + \beta_{kk'} u_{k'}^{conf*} \right). \hspace{1cm} (32)$$

After some simple algebra, introducing the inverse of the transformation equation (29) in the definition of the conformal modes (31), it can be easily shown that the Bogoliubov coefficients are trivially:

$$\alpha_{kk'} = \delta^3(k - k'), \hspace{1cm} \beta_{kk'} \equiv 0 \hspace{1cm} (33)$$

So that, the two sets of modes are in fact the same set.

This implies that there is no particle creation in the single electromagnetic wave region, which is in full agreement with earlier studies ([2], [20], [21], [22]). Also due to a particular case of the electromagnetic wave the “conformal” observer and the “in” observer that arrives from the Minkowskian region have the same definition of particles.

3.4 Interaction region.-

The metric tensor in the interaction region is given by:

$$ds^2 = \frac{2}{ab} d^2(v - u) - \cos^2(u - v) dx^2 - \cos^2(u + v) dy^2$$ \hspace{1cm} (34)

$$u > 0, \hspace{0.5cm} v > 0, \hspace{0.5cm} u + v \leq \pi/2$$

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In these coordinates the Klein-Gordon equation for the scalar field is non separable. We thus change to the following dimensionless coordinates:

\[
\xi = u + v \\
\eta = v - u,
\]

and the line element becomes:

\[
ds^2 = \frac{1}{2ab} d\xi^2 - \frac{1}{2ab} d\eta^2 - \cos^2 \eta \, dx^2 - \cos^2 \xi \, dy^2
\]  

(35)

The Klein-Gordon equation in this region then reads:

\[
2ab\Phi_{,\xi\xi} - 2ab\Phi_{,\eta\eta} - 2ab \tan \xi \Phi_{,\xi} + 2ab \tan \eta \Phi_{,\eta} - \frac{\Phi_{,xx}}{\cos^2 \eta} - \frac{\Phi_{,yy}}{\cos^2 \xi} = 0.
\]  

(36)

It is convenient to separate the solution in the following form:

\[
\Phi(\xi, \eta, x, y) = e^{ik_y y} \varphi(\eta, x) \psi(\xi),
\]  

(37)

obtaining two decoupled differential equations:

\[
\ddot{\varphi} - \tan \eta \, \dot{\varphi} + \frac{\alpha}{2ab} \varphi + \frac{\varphi_{,xx}}{\cos^2 \eta} = 0
\]  

(38)

\[
\ddot{\psi} - \tan \xi \, \dot{\psi} + \left(\frac{\hat{k}_y^2}{\cos^2 \xi} + \frac{\alpha}{2ab}\right) \psi = 0.
\]  

(39)

Here \(\alpha\) is the separation constant and \(\hat{k}_i = k_i/\sqrt{2ab}\).

The solutions of the first differential equation (38) are given by the product of exponentials and associated Legendre functions:

\[
\varphi(x, \eta) \propto e^{ik_x x} P^k_x (\cos \eta),
\]  

(40)

where \(\mu\) is defined by \(\frac{\alpha}{2ab} = \mu(\mu + 1)\).

We now identify the coordinate \(x\) with the angular coordinate with the range: \(0 < x \leq 2\pi L\) (see also [10] and [23]), and imposing the regularity conditions on the axis one can show that \(L = q\) independently on which possible analytic extension one wishes to perform across the Killing-Cauchy horizon. The solution thus can be written in spherical harmonic functions:

\[
\varphi_{l,m}(\eta) \propto Y^m_l \left(\frac{\pi}{2} - \eta, \frac{x}{q}\right)
\]  

(41)
where \( m \equiv \hat{k}_x \) and \( l \equiv \mu \) are integers related in the usual form \( (l = 0, 1, \ldots, \infty; m = -l, \ldots, l) \).

The second equation can be put into the hypergeometric form by the following transformations:

\[
z = \sin \xi \quad (42)
\]
\[
\psi(z) = (1 - z^2)^i \hat{k}_y \phi(z) \quad (43)
\]
\[
u = \frac{1}{2}(1 - z) \quad (44)
\]

we then have:

\[
u(1 - \nu) \ddot{\phi} + (1 + i \hat{k}_y)(1 - 2\nu)\dot{\phi} + \left( \frac{k_y^2 + l(l + 1)}{2ab} - i \hat{k}_y \right) \phi = 0. \quad (45)
\]

Using the properties of the hypergeometric functions, the general solution of the equation (45) is a linear combination of the following solutions:

\[
\psi_{l, \hat{k}_y, 1}(\xi) = \left( \frac{1 + \sin \xi}{1 - \sin \xi} \right)^{-i|\hat{k}_y|/2} \, _2F_1 \left[ 1 + l, -l; 1 + i \hat{k}_y; \frac{1}{2}(1 - \sin \xi) \right], \quad (46)
\]

and

\[
\psi_{l, \hat{k}_y, 2}(\xi) = \left( \frac{1 + \sin \xi}{1 - \sin \xi} \right)^{i|\hat{k}_y|/2} \, _2F_1 \left[ 1 + l, -l; 1 - i \hat{k}_y; \frac{1}{2}(1 - \sin \xi) \right] \quad (47)
\]

The general solution in the interaction region finally is:

\[
\Phi(\xi, \eta, x, y) = e^{ik_y y} \sum_l V^m_l \left( \frac{\pi}{2} - \eta, \frac{x}{\eta} \right) \left( C^{(1)}_{l, \hat{k}_x} \psi^{(1)}_{l, \hat{k}_x}(\xi) + C^{(2)}_{l, \hat{k}_x} \psi^{(2)}_{l, \hat{k}_x}(\xi) \right) \quad (48)
\]

The coefficients \( C^{(1)} \) and \( C^{(2)} \) depend on the separation constant and are subject to the boundary conditions (28).

### 3.4.1 The “in”-modes near the horizon

It is usually difficult to define the “out”-modes in the interaction region of the general colliding plane wave spacetime. Even in our case, in spite of the presence of certain symmetries, the definition of the “out”- modes is rather complicated. Fortunately however, as pointed out by Dorca and Verdaguer [11], the existence of a Killing-Cauchy horizon helps one to define an unambiguous “out”-vacuum. One thus is interested in the asymptotic behaviour of the “in”-modes at the horizon.
To study the behaviour of the “in”-modes near the horizon we can proceed in two different manners. We can either look at the asymptotic form of the equation (39) near the horizon and then solve it or look directly at the asymptotic behaviour of the solutions given by the equation (48) at the horizon. Both give the same result. The equation (39) can be written as:

$$\psi_{\xi^* \xi^*} + (k_y^2 + \alpha \cos^2 \xi) \psi = 0 \quad (49)$$

where $\xi^*$ is defined by (10).

Near the horizon at $\xi \to \pi/2$ this equation can be simplified to an oscillator equation:

$$\psi_{\xi^* \xi^*} + k_y^2 \psi = 0 \quad (50)$$

and the solutions are:

$$\psi_1(\xi^*) \propto e^{-i|k_y|\xi^*/2}$$

$$\psi_2(\xi^*) \propto e^{i|k_y|\xi^*} \quad (51)$$

Here $\xi^*$ is a time coordinate, so that the first term represents a purely ingoing wave to the horizon while the second term represents a purely outgoing wave.

If the solutions are expressed in the original $\xi$ coordinate, these take the form:

$$\psi_1(\xi \to \pi/2) \propto \left( \frac{1 + \sin \xi}{1 - \sin \xi} \right)^{-i|k_y|/2} \quad (52)$$

and

$$\psi_2(\xi \to \pi/2) \propto \left( \frac{1 + \sin \xi}{1 - \sin \xi} \right)^{i|k_y|/2} \quad (53)$$

One can easily see that $\psi_1(\xi^*)$ and $\psi_2(\xi^*)$ of the equation (53) are the asymptotic forms of $\psi^{(1)}_{\mu,k_s}(\xi)$ and $\psi^{(2)}_{\mu,k_s}(\xi)$ of the equation (17) near the horizon respectively. We can also see that the potential in the equation (49) vanishes at the horizon $\xi \to \pi/2$, which allows one [10] to consider the ingoing modes only ($C^{(2)} = 0$).

In this region the “in” modes are defined as:

$$u^{inI}_k(\xi, \eta, x, y) = \frac{e^{i k_y y}}{\sqrt{(2\pi)^2|k_y|}} \sum_l C_l^{(1)} Y_l^m(\frac{\pi}{2} - \eta, \frac{x}{q}), \quad (54)$$

and near the horizon they behave as:

$$u^{inI}_k(\xi \simeq \pi/2) = \frac{e^{i k_y y}}{\sqrt{(2\pi)^2|k_y|}} \sum_l C_l^{(1)} e^{-i|k_y|\xi^*} Y_l^m(\frac{\pi}{2} - \eta, \frac{x}{q}) \quad (55)$$
In the \( (\tilde{U}, \tilde{V}) \) coordinates this expression takes the form:

\[
\begin{align*}
    u^{in}_k(\xi \simeq \pi/2) &= \frac{1}{\sqrt{(2\pi)^2|k_y|}} \sum_l C_l^{(1)} Y^m_l(\frac{\pi}{2} - \eta, \frac{x}{q}) \left\{ e^{-i|k_y|\tilde{U}} , k_y \geq 0 \right\} \left\{ e^{-i|k_y|\tilde{V}} , k_y \leq 0 \right\} \\
    &\quad \left\{ (-U''/q)^i|k_y| , k_y \geq 0 \right\} \left\{ (-V''/q)^i|k_y| , k_y \leq 0 \right\}.
\end{align*}
\]  

Finally, using the Kruskal-Szekeres-like null coordinates defined by (12), we obtain:

\[
\begin{align*}
    u^{in}_k(\xi \simeq \pi/2) &= \frac{1}{\sqrt{(2\pi)^2|k_y|}} \sum_l C_l^{(1)} Y^m_l(\frac{\pi}{2} - \eta, \frac{x}{q}) \left\{ (-U''/q)^i|k_y| , k_y \geq 0 \right\} \left\{ (-V''/q)^i|k_y| , k_y \leq 0 \right\}.
\end{align*}
\]  

The equation (57) defines the “in”-modes in the Kruskal-Szekeres-like coordinates at the horizon. These coordinates are important because of their relation to the null Killing vector fields at the horizon which will be further used to define a new set of normal modes.

The coefficients \( C_l^{(1)} \) are subject to the following orthonormalization relation at the horizon:

\[
(u^{in}_k, u^{in}_{k'}) = q^2 \delta(k_y - k'_y) \delta_{mm'} \sum_l |C_l^{(1)}|^2 \equiv \delta(k_y - k'_y) \delta(k_x - k'_x) \delta(k_\pm - k'_\pm).
\]  

3.5 The “out” modes

The metric (35) can be transformed, using the Kruskal-Szekeres-like coordinates defined by (12) into the following form:

\[
ds^2 = (1 + \sin \xi)^2 dU' dV' - \frac{1}{2ab} d\eta^2 - \cos^2 \eta \, dx^2
\]

Near the horizon \( \xi \to \pi/2 \) the line element is:

\[
ds^2 = 4 dU' dV' - \frac{1}{2ab} d\eta^2 - \cos^2 \eta \, dx^2
\]

One can see that at the horizon the line element possesses two null Killing vector fields: \( \partial_{U'} \) and \( \partial_{V'} \), so that the “out”-modes will be taken as those with positive frequency with respect to these Killing vectors, and will have the form:

\[
\Phi(U', V', \eta, x) = e^{-i\omega_\pm U' - i\omega_0 V'} \phi(\eta, x)
\]

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The Klein-Gordon equation reduces to:
\[
\phi,\eta - \tan \eta \phi, \eta + \frac{\phi,xx}{2ab \cos^2 \eta} + \frac{\omega_+ - \omega_-}{2ab} \phi = 0
\] (62)

Identifying the \( x \) coordinate with the angular one, as in the subsection 3.3, the general solution of this equation will be a linear combination of the modes:
\[
u_{\text{out}}^k(U', V', \eta, x) = \frac{1}{\sqrt{2\pi^2 \omega}} e^{-i\omega_t U' - i\omega_- V'} Y_l m (\frac{\pi}{2} - \eta, \frac{x}{q})
\] (63)

where:
\[
l(l + 1) = \frac{\omega_+ - \omega_-}{2ab}.
\] (64)

These modes are orthonormal at the horizon:
\[
(u_{\text{out}}^k, u_{\text{out}}^{k'}) = q^2 \delta_{ll} \delta_{mm} \delta(\omega_+ - \omega_-').
\] (65)

The \( q^2 \) factor insures the correct dimensions.

4 Particle creation

We now evaluate the Bogoliubov coefficients between the “in”-modes and the “out”-modes at the horizon. The Bogoliubov transformation between these will be:
\[
u_k^i = \sum_{k'} \left( \alpha_{kk'} \nu_k' \right) + \beta_{kk'} \nu_k'^* \]
(66)
\[
u_k^o = \sum_{k'} \left( \alpha_{kk'} \nu_k' \right) - \beta_{kk'} \nu_k'^* \]
(67)

where \( \alpha_{kk'} \) and \( \beta_{kk'} \) can be found from:
\[
\alpha_{kk'} = (\nu_k^i, \nu_k'^*) \]
(68)
\[
\beta_{kk'} = -(\nu_k^i, \nu_k'^*) \]
(69)

Substituting the expressions of the modes (57) and (63) to evaluate the scalar products at the horizon, we obtain:
\[
\alpha_{kk'} = \frac{q^2 C_l^{(1)} |\hat{k}_y| \delta_{m',m}}{2\pi \sqrt{|\hat{k}_y| |\omega_+|}} \Gamma[i|\hat{k}_y|] \left\{ (i\omega_+) - i|\hat{k}_y| \right\} \left\{ (i\omega_) - i|\hat{k}_y| \right\} \]
(70)
\[
\beta_{kk'} = -\frac{q^2(-1)^l |\hat{k}_y| C^{(1)}_y \delta_{m',m}}{2\pi \sqrt{|\hat{k}_y|\omega_{\pm}}} \Gamma[i|\hat{k}_y|] \left\{ \begin{array}{l}
(-i\hat{\omega}_+) - i|\hat{k}_y| k_y \geq 0 \\
(-i\hat{\omega}_-) - i|\hat{k}_y| k_y \leq 0
\end{array} \right. 
\] (71)

It can be easily shown that the relation between these coefficients is:

\[
|\alpha_{kk'}|^2 = \exp(2\pi|\hat{k}_y|) |\beta_{kk'}|^2, 
\] (72)

and if one looks at the colliding wave problem in a time reversal manner, i.e.

the waves running away from the initial caustic singularity finally producing

a flat background region, then the exponential term of the equation (72)

would give rise to a thermal spectrum of particles (the number of particles

as seen by the “in” observer when the field is in the “out” vacuum).

In the colliding wave problem, one is interested to calculate the number

of “out” particles in the “in” vacuum at the horizon. The number of “out”

particles with frequencies in a range between \(\omega_{\pm}\) and \(\omega_{\pm} + d\omega_{\pm}\), that the

static “out” observer sees at the horizon if the field is in the “in” vacuum

state |0, in> is given by:

\[
N_{k'}^{\text{out}} \equiv a_{k'}^{\text{out},\dagger} a_{k'}^{\text{out}} = \int d^3 k |\beta_{kk'}|^2 = \frac{1}{q^2} \sum_m \sum_l \int dk_y |\beta_{kk'}|^2. 
\] (73)

Introducing the Bogoliubov coefficients \(\beta\) given by the equation (71) one can

get the following expression for the number of created particles:

\[
N_{\omega_{\pm}}^{\text{out}} = \frac{q^2 \delta_{m',m}}{2\pi^2 \omega_{\pm}} \sum_l \int \frac{dk}{e^k - 1} |C^{(1)}_y|^2 
\] (74)

where \(k\) is a dimensionless variable defined by \(k \equiv 2\pi k_y\). The coefficients

\(C^{(1)}_y\) depend on \(m, l\) and \(k\) and can be in principle evaluated explicitly by

comparing the expression given by the equation (48) and the “in” modes

in the regions II (24) and III at the wave fronts as well as imposing the

orthonormalization relation (58).

Comparing the expression (74) with those obtained by Yurtsever [8] and

Dorca and Verdaguer [10] in the cases of pure gravitational wave scattering,

one can see that the results are similar. The equation (74) is consistent with

the long wavelength limit of a thermal distribution of scalar particles with

a temperature given by:

\[
T = \frac{\hbar c}{k_B 2\pi q} = \frac{\hbar c \sqrt{2ab}}{k_B 2\pi}. 
\] (75)
We have limited ourselves in this work to the case of the neutral particles. In the case where the particles would have some charge, one would expect the spectrum of created particles to be characterised not only by a temperature but by a chemical potential as well. At any rate, one would not expect results qualitatively different from those obtained by Lapedes [15] in the Bertotti-Robinson universe.

An interesting question to address is whether the quantum field theory has some implications on the arrow of time in the plane wave collisions, i.e. to ask as to whether there is an entropy increase in one way or another. Related question, as well, would be as to whether there is a preferable analytic extension across the horizon: a possible extension could be a static one or the time symmetric one as pointed in reference [23]. One hopes that the quantum field theory could help to answer this question. These and some other questions are currently being considered by the authors and will be discussed elsewhere.

Acknowledgements

We are grateful to Enric Verdaguer for helpful suggestions and discussions. A.F. would like to thank Chaim Charach for many useful conversations in the past. M.A.P.S. is grateful to Miquel Dorca, Manuel A. Valle and Iñigo Egusquiza for helpful discussions.

This work is supported by the Spanish Ministry of Education Grant (CICYT) No. PB-93 – 0507. M.A.P.S. is also supported by a Spanish Ministry of Education pre-doctoral fellowship No. AP92 30619396.

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