THE SPECTRAL DECOMPOSITION OF SHIFTED
CONVOLUTION SUMS

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Abstract. Let $\pi_1, \pi_2$ be cuspidal automorphic representations of $\text{PGL}_2(\mathbb{R})$ of conductor 1 and Hecke eigenvalues $\lambda_{\pi_1,2}(n)$, and let $h > 0$ be an integer. For any smooth compactly supported weight functions $W_{1,2} : \mathbb{R}^\times \to \mathbb{C}$ and any $Y > 0$ a spectral decomposition of the shifted convolution sum

$$\sum_{m \pm n = h} \frac{\lambda_{\pi_1}(|m|)\lambda_{\pi_2}(|n|)}{\sqrt{|mn|}} W_1 \left(\frac{m}{Y}\right) W_2 \left(\frac{n}{Y}\right)$$

is obtained. As an application, a spectral decomposition of the Dirichlet series

$$\sum_{m, n \geq 1 \atop m - n = h} \frac{\lambda_{\pi_1}(m)\lambda_{\pi_2}(n)}{(m + n)^s} \left(\frac{\sqrt{mn}}{mn}\right)^{100}$$

is proved for $\Re s > 1/2$ with polynomial growth on vertical lines in the $s$ aspect and uniformity in the $h$ aspect.

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1. INTRODUCTION

Let $G = \text{PGL}_2(\mathbb{R})$ and $\Gamma = \text{PGL}_2(\mathbb{Z})$. There is a spectral decomposition

$$L^2(\Gamma \backslash G) = \int_{\tau} V_\tau \, d\tau,$$

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where \((\tau, V_\tau)\) are irreducible automorphic representations of \(\Gamma \backslash G\) (including the trivial representation) and \(d\tau\) is the spectral measure defined as follows: The trivial representation \((\tau_0, C)\) has spectral measure 1. Each nontrivial representation \((\tau, V_\tau)\) is generated by a modular form on \(H\) with respect to the full modular group \(\text{SL}_2(\mathbb{Z})\), hence it corresponds to some Laplacian eigenvalue
\[
\lambda_\tau = 1/4 - \nu_\tau^2 \in \mathbb{R} \quad \text{with} \quad \Re \nu_\tau \geq 0, \quad \Im \nu_\tau \geq 0.
\]
We shall use the notation
\[
\check{\lambda}_\tau := 1 + |\lambda_\tau|.
\]
A representation \((\tau, V_\tau)\) generated by a holomorphic or Maass cusp form has spectral measure 1. We can assume that the underlying cusp form is a Hecke eigenform, and we denote by \(\lambda_\tau(n)\) its \(n\)-th Hecke eigenvalue. A representation \((\tau, V_\tau)\) generated by an Eisenstein series with \(\Re \nu_\tau = 0\) and \(\Im \nu_\tau > 0\) has spectral measure \(d\nu_\tau/(2\pi i)\), and we denote by \(\lambda_\tau(n)\) the divisor sum \(\sum_{ab=n} (a/b)^{\nu_\tau}\).

For a function \(W \in C^d(\mathbb{R}^\times)\) we denote
\[
\|W\|_{A^d} := \sum_{j=0}^d \left( \int_{\mathbb{R}^\times} \left( |u| + |u|^{-1} \right)^d \left| \frac{d^j W}{d u^j} \right|^2 d^\times u \right)^{1/2},
\]
provided the integral is finite. Here \(d^\times u := du/|u|\) is the Haar measure on \(\mathbb{R}^\times\).

In this paper we obtain a spectral decomposition for shifted convolution sums of Hecke eigenvalues of two arbitrary cusp forms, as well as a spectral decomposition of the corresponding Dirichlet series with polynomial growth estimates on vertical lines and uniform dependence with respect to the shift parameter.

**Theorem 1.** Let \(\pi_1\) and \(\pi_2\) be arbitrary cuspidal automorphic representations of \(\Gamma \backslash G\). Let \(a, b, c \geq 0\) be arbitrary integers and let \(W_{1,2} : \mathbb{R}^\times \to \mathbb{C}\) be arbitrary functions such that \(\|W_{1,2}\|_{A^d}\) exist for \(d = 18 + 2a + 2b + 4c\). Then there exist functions \(W_\tau : (0, \infty) \to \mathbb{C}\) depending only on \(\pi_{1,2}, W_{1,2}\), and \(\tau\) such that the following two properties hold:

- If \(h > 0\) is an arbitrary integer and \(Y > 0\) is arbitrary then one has the decomposition over the full spectrum (excluding the trivial representation)
\[
\sum_{m+n=h} \frac{\lambda_{\pi_1}(|m|) \lambda_{\pi_2}(|n|)}{\sqrt{|mn|}} W_1 \left( \frac{m}{Y} \right) W_2 \left( \frac{n}{Y} \right) = \int_{\tau \neq \tau_0} \frac{\lambda_\tau(h)}{\sqrt{h}} W_\tau \left( \frac{h}{Y} \right) d\tau.
\]

- If \(0 < \varepsilon < 1/4\) is arbitrary then for \(y > 0\) one has the uniform bound
\[
\int_{\tau \neq \tau_0} \check{\lambda}_\tau^c \left( \frac{d}{dy} \right)^a W_\tau(y) \frac{d\tau}{\sqrt{h}} \ll_{\varepsilon, a, b, c} C_{a, b, c} \min(y^{1/2 - \varepsilon}, y^{1/2 - b - \varepsilon}),
\]
where
\[
C_{a, b, c} := (\check{\lambda}_{\pi_1} + \check{\lambda}_{\pi_2})^{11 + a + b + 2c} \sum_{d_1 + d_2 = 18 + 2a + 2b + 4c} \|W_1\|_{A^{d_1}} \|W_2\|_{A^{d_2}}.
\]

**Remark 1.** The condition on \(W_{1,2}\) is satisfied as long as these functions decay rapidly near 0 and \(\pm \infty\). One can view \(W\) as a function on \((0, \infty)\) times (a subset of) the unitary dual of \(\Gamma \backslash G\). Implicit in (3) is the fact that this function is integrable in the second variable with respect to the spectral measure. By explicating \(W\) stronger regularity properties would follow. Finally, the right-hand side of (3) captures also the summation condition \(m - n = h\) if the support of \(W_2\) is on the negative axis.
By a result of Kim–Sarnak [KS, Appendix 2] we have
\[ |\lambda_\tau(h)| \leq d(h)h^\theta \]
for \( \theta = 7/64 \) and any cuspidal automorphic representation \( \tau \) of \( \Gamma \backslash G \), where \( d(n) \) is the number of divisors of \( n \). Thus for two smooth compactly supported functions \( W_{1,2} \), the bound (4) gives immediately
\begin{equation}
\sum_{m \pm n = h} \lambda_{\pi_1}(|m|)\lambda_{\pi_2}(|n|) W_1 \left( \frac{m}{Y} \right) W_2 \left( \frac{n}{Y} \right) \ll_{\epsilon,\pi_1,\pi_2,W_1,W_2} h^\theta Y^{-1/2}(hY)^\epsilon
\end{equation}
for any \( \epsilon > 0 \), uniformly in \( h > 0 \), cf. e.g. [BL]. The novelty of Theorem 1 is to obtain an exact spectral decomposition of the left side of (3) rather than an upper bound. Thus Theorem 1 develops its full strength when the left side of (3) is averaged over \( h \), as is necessary, for example, to prove subconvex estimates for certain families \( L \)-functions. If \( \pi \) is a cuspidal automorphic representation of \( G \) of arbitrary conductor and \( \chi \) is a primitive Dirichlet character of conductor \( q \), then without much effort we can deduce the Burgess-type bound
\begin{equation}
L(1/2, \pi \otimes \chi) \ll_{\epsilon,\tau} q^{\frac{1}{4} + \epsilon + \epsilon}
\end{equation}
by combining a slight generalization of Theorem 1 with amplification and a large sieve inequality (cf. Remark 2 below). In fact, a similar result can be derived more generally over totally real number fields, where classical methods are much harder to implement. We postpone this discussion to [BH] and give here another application of Theorem 1 that was our initial motivation.

**Theorem 2.** Let \( \pi_1 \) and \( \pi_2 \) be arbitrary cuspidal automorphic representations of \( \Gamma \backslash G \), and let \( c, k \geq 0 \) be arbitrary integers satisfying \( k > 60 + 12c \). There are holomorphic functions \( F_{k,\tau} : \{ s : \frac{1}{2} < \Re s < \frac{3}{2} \} \rightarrow \mathbb{C} \) depending only on \( \pi_1, \pi_2, c, k, \) and \( \tau \) such that the following two properties hold:

1. If \( h > 0 \) is an arbitrary integer then one has the decomposition over the full spectrum (excluding the trivial representation)
\begin{equation}
\sum_{\substack{m,n \geq 1 \atop m-n=h}} \frac{\lambda_{\pi_1}(m)\lambda_{\pi_2}(n)(mn)^{\frac{1}{s+k-1}}}{(m+n)^{s+k-1}} = h^{\frac{1}{2}-s} \int_{\tau \neq \tau_0} \lambda_{\tau}(h) F_{k,\tau}(s) \, d\tau, \quad \Re s > 1.
\end{equation}

2. One has the uniform bound
\begin{equation}
\int_{\tau \neq \tau_0} \Lambda_{\tau}^c |F_{k,\tau}(s)| \, d\tau \ll_{\epsilon,k} (\lambda_{\pi_1} + \lambda_{\pi_2})^{12 + 4c} |s|^{22 + 4c} |\theta|_{k,\tau}^4, \quad \frac{1}{2} + \epsilon < \Re s < \frac{3}{2}.
\end{equation}

**Remark 2.** Theorems 1 and 2 extend to arbitrary level and to the more general additive constraints \( \ell_1 m \pm \ell_2 n = h \) in a straightforward fashion with good control in the \( \ell_{1,2} \) parameters. Here small technical complications arise from the possible presence of complementary series representations \( \tau \) and the presence of additional cusps. By [KS, Appendix 2] complementary series representations satisfy \( 0 < \nu_\tau \leq \theta \) for \( \theta = 7/64 \). Accordingly, the right hand side of (4) becomes (cf. (26))
\[ \ll_{\epsilon,a,b,c} C_{a,b,c}' (\ell_1 \ell_2)^{1/2 + \epsilon} \min(y^{1/2 - a - \epsilon}, y^{1/2 - a - b - \epsilon}) \]
for any \( 0 < \epsilon < 1/4 - \theta \), where \( C_{a,b,c}' \) is a similar constant as \( C_{a,b,c} \) in Theorem 1 but with a larger exponent for \( \lambda_{\pi_1} + \lambda_{\pi_2} \) and a sum over larger \( d_1 + d_2 \). For the
Eisenstein spectrum the exponent of $\ell_1 \ell_2$ can be lowered to $1/4 + \varepsilon$. Similarly, the right hand side of (8) becomes, for suitable constants $A, B, C > 0$,
\[
\ll_{\varepsilon, k} (\ell_1 \ell_2)^A \left( \hat{\lambda}_{\pi_1} + \hat{\lambda}_{\pi_2} \right)^{B+4c} |s|^{C+4c}, \quad \frac{1}{2} + \theta + \varepsilon < \Re s < \frac{3}{2}.
\]
These extensions enable considerable simplifications in arguments leading to sub-convexity results such as (6) which appeared originally in [BHM] or the more difficult case of Rankin–Selberg $L$-functions treated in [HM]. In this paper we have decided to present the theorems in a special case in order to keep the notational burden minimal and to emphasize the conceptual simplicity of the approach.

Selberg [Se] proved in 1965 that for holomorphic cusp forms (that is, when $\pi_1$ and $\pi_2$ are in the discrete series), the left hand side of (7) is meromorphic in $s$ and holomorphic for $\Re(s) > 1/2$ (note that for $\Gamma$ poles on the segment $1/2 < s \leq 1$ do not occur).\footnote{He adds regretfully [Se, p.14]: “We cannot make much use of this function at present [...]”} Good [Go1, Go2] was the first to use the spectral decomposition of shifted convolution sums coming from the Fourier coefficients of a holomorphic cusp form for estimating the second moment of the corresponding modular $L$-function on the critical line. His new key ingredient was to show the polynomial growth on vertical lines.

The spectral decomposition in the non-holomorphic case has resisted all attempts so far. Good’s method, based on the fact that holomorphic cusp forms are linear combination of Poincaré series, is not applicable here. Jutila [Ju1] and Sarnak [Sa1, Sa2] independently considered an approximation of the Dirichlet series (7), along with a spectral decomposition, that could be continued to the half plane $\Re s > 1/2$ with a bound $O(h^{1-\sigma+\varepsilon}|s|^A)$ on vertical lines $\Re s = \sigma$. However, this approximation introduces an error of $O(h^{1-\sigma+\varepsilon}|s|^B)$ which one would like to remove.\footnote{For a striking recent application of the approximate spectral decomposition see [LLY]. For a careful analysis of the error term see [Ju3].} The second author found in his thesis [Ha1, Chapter 5] that the error signifies missing harmonics and anticipated that analysis on $\Gamma \backslash G$ will be key to obtaining a complete system. Independently, Motohashi [Mo2, Mo3] brought the representation theory of $\Gamma \backslash G$ and the Kirillov model into the discussion. In this paper, we pursue his approach further. One of the new ideas we employ is the use of Sobolev norms for smooth vectors inspired by the recent work of Venkatesh [Ve] which allows for a soft treatment and avoids the difficulties imposed by Poincaré series and the estimation of triple product periods. Sobolev norms also play important roles in related works of Bernstein and Reznikov [BR1, BR2, BR3, BR4] and Cogdell and Piatetski-Shapiro [CoPS]. Shortly after this paper was finished, Motohashi [Mo4] gave an alternative proof of Theorem 2 which makes regularity properties of the weight functions $F_{k, \tau}(s)$ more transparent.

Finally we remark that there are other important techniques for understanding shifted convolution sums. The archetype of shifted convolution sums are the additive divisor sums which have been studied extensively.\footnote{In fact one can trace back history to various elegant identities published by Jacobi in 1829.} These special sums arise from Eisenstein series rather than cusp forms and their spectral decomposition is very explicitly known by the work of Motohashi [Mo1] (cf. [JM, Lemma 4]). In the general case variants of the circle method with the Kloosterman refinement have been particularly successful [DFI1, DFI2, Ju2, Ha2, HM, Bl, BHM]. Recently
Venkatesh [Ve] developed a geometric method, based on equidistribution and mixing, which can be applied for shifted convolution sums of higher rank. In our context, the left hand side of (3) is an automorphic period over a closed horocycle, so that [Ve, Theorem 3.2] implies a weaker but nontrivial version of (5).

2. Bounds for the discrete spectrum

2.1. Kirillov model and Sobolev norms. We shall use the notation

\[ n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a(u) := \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \quad k(\theta) := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \]

and we shall think of these matrices as elements of \( G = \text{PGL}_2(\mathbb{R}) \). In addition, we shall write

\[ e(x) := e^{2\pi i x}, \quad x \in \mathbb{R}. \]

Let \( (\pi, V_\pi) \) be a representation generated by a cusp form on \( \text{SL}_2(\mathbb{Z})\setminus \mathcal{H} \). Then \( (\pi, V_\pi) \) is contained in \( L^2(\Gamma\setminus G) \), hence it is equipped with a canonical inner product given by

\[ \langle \phi_1, \phi_2 \rangle := \int_{\Gamma\setminus G} \phi_1(g) \overline{\phi_2(g)} \, dg, \]

where \( dg := dx(du/u^2)(d\theta/\pi) \) for \( g = n(x) a(u) k(\theta) \) is the Haar measure on \( G \). The Kirillov model \( K(\pi) \) realizes \( \pi \) in \( V_{K(\pi)} := L^2(\mathbb{R}^\times, d^\times u) \) which is equipped with its own canonical inner product given by

\[ \langle W_1, W_2 \rangle := \int_{\mathbb{R}^\times} W_1(u) \overline{W_2(u)} \, d^\times u. \]

Let \( \| \cdot \| \) denote the norms determined by these inner products.

For a smooth vector \( \phi \in V_\pi^\infty \) we define the corresponding smooth vector \( W_\phi \in V_{K(\pi)}^\infty \) as

\[ W_\phi(u) := \int_0^1 \phi(n(x) a(u)) e(-x) \, dx, \quad u \in \mathbb{R}^\times. \]

It gives rise to the Fourier decomposition

\[ \phi(n(x) a(u)) = \sum_{n \in \mathbb{Z}} \frac{\lambda_\pi(n)}{\sqrt{|n|}} W_\phi(nu) e(nx), \quad x \in \mathbb{R}, \quad u \in \mathbb{R}^\times, \]

where \( \lambda_\pi(n) \) denotes the \( n \)-th Hecke eigenvalue of the cusp form on \( \text{SL}_2(\mathbb{Z})\setminus \mathcal{H} \) that generates \( (\pi, V_\pi) \). This follows from Shalika’s multiplicity one theorem combined with standard facts about Hecke operators, see [CoPS, Sections 4.1 and 6.2] and [DFI3, (6.14)–(6.15)]. We have the uniform bound [DFI3, Proposition 19.6]

\[ \sum_{1 \leq n \leq x} |\lambda_\pi(n)|^2 \ll_{\varepsilon} x(\nu_\pi)^\varepsilon. \]

By Kirillov’s theorem [CoPS, Sections 4.2–4.4], the canonical inner products of \( V_\pi \) and \( V_{K(\pi)} \) are related by a proportionality constant depending only on \( \pi \),

\[ \langle \phi_1, \phi_2 \rangle = C_\pi(W_{\phi_1}, W_{\phi_2}), \quad \phi_1, \phi_2 \in V_\pi^\infty. \]

The relations (12) and (14) can also be verified by classical means, see [DFI3, Section 4] and [BrMo, Sections 2 and 4].
We can evaluate the proportionality constant \( C_\pi \) as follows. Let

\[
E(g, s) := \frac{1}{2} \sum_{\gamma \in \Gamma} U_s(\gamma g)
\]

with \( U_s(n(x)a(u)k(\theta)) := |u|^s \) and \( \Gamma := \{ n(x) : x \in \mathbb{Z} \} \) denote the standard weight 0 Eisenstein series on \( G \) and let \( \phi \in V^\infty_\pi \) be any vector of pure weight (cf. Section 2.2). Then for \( \Re s > 1 \) we have, by the Rankin–Selberg unfolding technique, (12), and Parseval,

\[
\langle |\phi(g)|^2, E(g, s) \rangle = \frac{1}{2} \int_{\mathbb{R}^\times} \int_{0}^{1} |\phi(n(x)a(u))|^2 |u|^{s-2} dx du
\]

\[
= \frac{1}{2} \sum_{n \in \mathbb{Z}} \frac{|\lambda_\pi(n)|^2}{|n|} \int_{\mathbb{R}^\times} |W_\phi(nu)|^2 |u|^{s-1} d^\times u
\]

\[
= \sum_{n=1}^{\infty} \frac{|\lambda_\pi(n)|^2}{n^s} \int_{\mathbb{R}^\times} |W_\phi(u)|^2 |u|^{s-1} d^\times u.
\]

Taking residues of both sides at \( s = 1 \) yields

\[
\| \phi \|^2_{s=1} E(g, s) = \| W_\phi \|^2_{s=1} \frac{L(s, \pi \otimes \bar{\pi})}{\zeta(2s)} \|L(1, \operatorname{Ad}^2 \pi). \]

Using (14) we conclude that

\[
C_\pi = \frac{\text{vol}(\Gamma \backslash G)}{\zeta(2)} L(1, \operatorname{Ad}^2 \pi).
\]

In particular, by (13) and [HL, Theorem 0.2] we know that

\[
\lambda_\pi \ll \varepsilon C_\pi \ll \varepsilon \lambda_\pi
\]

for any \( \varepsilon > 0 \).

We shall introduce Sobolev norms for vectors in \( V^\infty_\pi \) and \( V^\infty_{\mathcal{K}(\pi)} \) in terms of the derived action of the Lie algebra \( \mathfrak{g} \) of \( G \). We consider the usual basis of \( \mathfrak{g} \) consisting of

\[
H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

and note the commutation relations \([H, R] = 2R, [H, L] = -2L, [R, L] = H \). Then for smooth vectors \( \phi \in V^\infty_\pi \) and \( W \in V^\infty_{\mathcal{K}(\pi)} \) we define

\[
\| \phi \|_{S^d} := \sum_{\text{ord}(D) \leq d} \| D\phi \| \quad \text{and} \quad \| W \|_{S^d} := \sum_{\text{ord}(D) \leq d} \| DW \|
\]

where \( D \) ranges over all monomials in \( H, R, L \) of order at most \( d \) in the universal enveloping algebra \( U(\mathfrak{g}) \). We can see how \( U(\mathfrak{g}) \) acts on \( \mathcal{K}(\pi) \): \( H \) acts by \( 2u \frac{d^2}{d\tau^2} \) and \( R \) acts by \( 2\pi i u \), cf. [Bu, p.155]. The Casimir element \( H^2 + 2RL + 2LR = H^2 - 2H + 4RL \) acts by \( -4\lambda_\pi \), hence \( RL \) acts by \(-\lambda_\pi + u^2 \frac{d^2}{d\tau^2} \), therefore \( L \) acts by \((2\pi i)^{-1}(-\lambda_\pi u^{-1} + u \frac{d^2}{d\tau^2}) \). This way we obtain the following estimate [Ve, Lemma 8.4]:

\[
\| W \|_{S^d} \ll_d \lambda^d_\pi \| W \|_{A^d}.
\]

Here the norm \( \| \cdot \|_{A^d} \) was defined in (2).

\[\text{\footnote{Here the adjoint square is the same as the symmetric square.}}\]
2.2. Normalized Whittaker functions. In order to understand the behavior of $W_\phi(u)$ near 0 we decompose $\phi \in V_\pi$ into pure weight pieces
\[
\phi = \sum_{p \in \mathbb{Z}} \phi_p,
\]
where $\phi_p \in V_\pi^\infty$ satisfies
\[
\phi_p(gk(\theta)) = e^{2ip\theta} \phi_p(g), \quad g \in G, \quad \theta \in \mathbb{R}.
\]
Convergence of (18) is understood in $L^2$-norm. Correspondingly, $W_\phi$ decomposes in $V_{K(\pi)}$ as
\[
W_\phi = \sum_{p \in \mathbb{Z}} W_{\phi_p}.
\]
Note that Parseval gives
\[
\|\phi\|^2 = \sum_{p \in \mathbb{Z}} \|\phi_p\|^2.
\]
It is known (cf. \cite[2.14–2.27]{BrMo}) that each $W_{\phi_p}(u)$ is a constant multiple of some normalized Whittaker function
\[
\tilde{W}_{p,\pi}(u) := \frac{\varepsilon_{p,\pi}(\text{sgn}(u)) W_{\text{sgn}(u)p,\nu_\pi}(|u|)}{|\Gamma\left(\frac{\lambda + \nu_\pi + \text{sgn}(u)p}{2}\right)\Gamma\left(\frac{\lambda + \nu_\pi + \text{sgn}(u)p}{2} + 1\right)|^{1/2}}, \quad u \in \mathbb{R}^*,
\]
where $\varepsilon_{p,\pi} : \{\pm 1\} \to \{z : |z| = 1\}$ is a suitable phase factor, $W_{\alpha,\beta}$ is the standard Whittaker function (see \cite[Chapter XVI]{WW}), and the right hand side is understood as zero if one of $\frac{\lambda}{2} \pm \nu_\pi + \text{sgn}(u)p$ is a nonpositive integer. By \cite[Section 4]{BrMo}, the functions $\tilde{W}_{p,\pi} : \mathbb{R}^* \to \mathbb{C}$ form an orthonormal basis of $L^2(\mathbb{R}^*, d^\times u)$, therefore
\[
|W_{\phi_p}(u)| = \|W_{\phi_p}\| |\tilde{W}_{p,\pi}(u)|, \quad u \in \mathbb{R}^*.
\]
We can choose the parameter $\nu_\pi$ so that $\Re \nu_\pi \geq 0$. Then for $\tilde{W}_{p,\pi}(u) \neq 0$ we have
\[
\frac{\Gamma\left(\frac{\lambda + \nu_\pi + \text{sgn}(u)p}{2}\right)\Gamma\left(\frac{\lambda + \nu_\pi + \text{sgn}(u)p}{2} + 1\right)}{\Gamma\left(\frac{\lambda + \nu_\pi + \text{sgn}(u)p}{2}ight)} \ll (|p| + |\nu_\pi| + 1)^{2\Re \nu_\pi},
\]
so that the uniform bound \cite[(4.5)]{BrMo} (whose proof applies in all cases) yields
\[
\tilde{W}_{p,\pi}(u) \ll |u|^{1/2} \left( \frac{|u|}{|p| + |\nu_\pi| + 1} \right)^{-1-2\Re \nu_\pi} \exp\left(-\frac{|u|}{|p| + |\nu_\pi| + 1}\right), \quad u \in \mathbb{R}^*.
\]
If $(\pi, V_\pi)$ belongs to the principal series (i.e. $\Re \nu_\pi = 0$) then for $0 < \varepsilon < 1$ we also have the uniform bound (cf. \cite[(4.3)]{BrMo})
\[
\tilde{W}_{p,\pi}(u) \ll_{\varepsilon} (|p| + |\nu_\pi| + 1)|u|^{1/2-\varepsilon}, \quad u \in \mathbb{R}^*.
\]
Indeed, for $|u| \geq 1$ the bound (24) is stronger, while for $|u| < 1$ it is an immediate consequence of \cite[(4.2)]{BrMo} and \cite[Appendix]{HM}. We shall show below that for $0 < \varepsilon < 1/4$ this bound holds true even when $(\pi, V_\pi)$ belongs to the discrete series. We note that representations belonging to the complementary series (i.e. $0 < \nu_\pi < 1/2$) do not occur in (1), but for completeness we record an analogue of (25) for this case, valid for $0 < \varepsilon < 1$ (cf. (24), \cite[(4.2)]{BrMo}, \cite[Appendix]{HM}):
\[
\tilde{W}_{p,\pi}(u) \ll_{\varepsilon} (|p| + |\nu_\pi| + 1)^{1+\nu_\pi} |u|^{1/2-\nu_\pi-\varepsilon}, \quad u \in \mathbb{R}^*.
\]
If \((\pi, V_\pi)\) belongs to the discrete series then \(\nu_\pi = \ell - \frac{1}{2}\), where \(\ell \geq 1\) is an integer. For \(|p| < \ell\) we have \(\tilde{W}_{p,\pi} = 0\). For \(|p| \geq \ell\) it follows from (24) via \(\exp(-t) \ll t^{-1}\) (or from [BrMo, (2.16)]) by two integration by parts that
\[
\tilde{W}_{p,\pi}(u) \ll |3p|^{\ell+3/2} |u|^{-\ell-1}, \quad u \in \mathbb{R}^\times.
\]

The Mellin transform of \(W_{p,\pi}(u)\) satisfies the Jacquet–Langlands local functional equation (see [BrMo, (4.11)]) which is reflected in the convolution identity (see [BrMo, (4.9)])
\[
\tilde{W}_{p,\pi}(u) = (-1)^p \int_0^\infty j_{\ell-\frac{1}{2}}(y) \tilde{W}_{p,\pi}(y/u) \, dy, \quad u \in \mathbb{R}^\times,
\]
where
\[
j_{\ell-\frac{1}{2}}(y) := (-1)^\ell 2\pi \sqrt{y} J_{2\ell-1}(4\pi \sqrt{y}), \quad y > 0.
\]

Using the bound [BrMo, (2.46)] for the kernel \(j_{\ell-\frac{1}{2}}(y)\), we can conclude
\[
\tilde{W}_{p,\pi}(u) \ll \int_0^\infty \min(y^{1/4}, y^\ell) \, |\tilde{W}_{p,\pi}(y/u)| \, dy, \quad u \in \mathbb{R}^\times.
\]

We split the integral at \(y = |3pu|\) and estimate the two pieces separately. On the one hand, by Cauchy–Schwarz and \(\|\tilde{W}_{p,\pi}\| = 1\),
\[
\int_0^{|3pu|} \ldots \ll \left\{ \int_0^{|3pu|} \min(y^{1/2}, y^{2\ell}) \, dy \right\}^{1/2}
\ll \min(|3pu|^{1/4}, |3pu|^{\ell}).
\]

On the other hand, by (27),
\[
\int_{|3pu|}^\infty \ldots \ll |3p|^{\ell+3/2} \int_{|3pu|}^\infty \min(y^{1/4}, y^\ell) \, (y/|u|)^{-\ell-1} \, dy
\ll |p|^{1/2} \min(|3pu|^{1/4}, |3pu|^{\ell}).
\]

All in all we see that for any \(0 < \varepsilon < 1/4\) we have
\[
\tilde{W}_{p,\pi}(u) \ll |p|^{1/2} \min(|3pu|^{1/4}, |3pu|^{\ell}) \ll |p|^{1/2} |3pu|^{1/2-\varepsilon}, \quad u \in \mathbb{R}^\times.
\]

This implies (25) as claimed.

2.3. Bounds for smooth vectors. We can derive a bound for \(\|\phi\|_\infty\) in terms of a suitable Sobolev norm of \(\phi\) (cf. (16)). Let \(y_0 := |p| + |\nu_\pi| + 1\). By (12), (19), (23), (24), and (25) we have
\[
\|\phi_p\|_\infty \leq \sup_{y > 1/2} \sum_{n \neq 0} \frac{|\lambda_\pi(n)|}{\sqrt{|n|}} |W_{\phi_p}(ny)|
\ll \varepsilon \|W_{\phi_p}\| \sup_{y > 1/2} \left\{ \sum_{1 \leq n \leq y_0/y} \frac{|\lambda_\pi(n)|}{\sqrt{n}} \right\}
\ll \varepsilon \|W_{\phi_p}\| \sup_{y > 1/2} \left\{ \sum_{n \leq y_0/y} \frac{y_0^2}{\sqrt{ny}} \right\}
\ll \varepsilon \|W_{\phi_p}\|. \quad (13), (14), and (15)
\]

Together with (13), (14), and (15) we find
\[
\|\phi_p\|_\infty \ll \varepsilon \|W_{\phi_p}\| \sup_{y > 1/2} \frac{y_0^2}{\sqrt{y}} (y_0(1 + |\nu_\pi|))^{\varepsilon} \ll ([p] + |\nu_\pi| + 1)^{2+\varepsilon} \|\phi_p\|.
\]
Finally, by replacing \( \phi \) by \((\pm 1 + H^2 + 2RL + 2LR)\epsilon R^n H^a \phi \) we obtain the more general inequality (cf. comments after (16))

\[
\left( \frac{d}{du} \right)^a W_\phi(u) \ll_{\varepsilon, a, b, c} \lambda_\pi^{1/2-c+\varepsilon} \| \phi \|_{S^2} \min(|u|^{1/2-\varepsilon}, |u|^{1/2-b-\varepsilon}), \quad u \in \mathbb{R}^\times,
\]

for any \( 0 < \varepsilon < 1/4 \) and any integers \( a, b, c \geq 0 \).

3. Bounds for the continuous spectrum

Let \((\pi, V_\pi)\) be a representation generated by an Eisenstein series on \( \text{SL}_2(\mathbb{Z}) \setminus \mathcal{H} \). As \((\pi, V_\pi)\) is not contained in \( L^2(\Gamma \setminus G) \), we cannot use the definition (9) as in the cuspidal case. Nevertheless, we can still define the Kirillov model \( \mathcal{K}(\pi) \) and use the definitions (10), (11). It turns out that for the purpose of spectral decomposition the right analogue of (9) reads\(^6\)

\[
\langle \phi_1, \phi_2 \rangle := \pi^{-1} |\zeta(1 + 2\nu_\pi)|^2 \langle W_{\phi_1}, W_{\phi_2} \rangle,
\]

then we have (14) and the lower bound part of (15) with

\[
C_\pi := \pi^{-1} |\zeta(1 + 2\nu_\pi)|^2.
\]

Note that \( \nu_\pi = 0 \) does not occur for a nonzero Eisenstein series. Let \( \| \cdot \| \) denote the norms on \( V_\pi \) and \( V_{\mathcal{K}(\pi)} \) determined by these inner products, then (16) defines the corresponding Sobolev norms \( \| \cdot \|_{S^d} \) on \( V_\pi^\infty \) and \( V_{\mathcal{K}(\pi)}^\infty \).

\(^5\)Less explicit versions of this bound were derived by Bernstein–Reznikov and Venkatesh in more general contexts, without recourse to Whittaker functions, see [BR2, Proposition 4.1] and [Ve, Lemma 9.3].

\(^6\)We apologize to the reader that \( \pi \) denotes a constant and a representation at the same time.
For a smooth vector \( \phi \in V^\infty_x \) we have the following analogue of (12):

\[ \phi(n(x)a(u)) = W_{\phi,0}(y) + \sum_{n \in \mathbb{Z}_{\neq 0}} \frac{\lambda_n(|n|)}{\sqrt{|n|}} W_n(nu) e(nx), \quad x \in \mathbb{R}, \ u \in \mathbb{R}^x, \]

where

\[ W_{\phi,0}(u) := \int_0^1 \phi(n(x)a(u)) \, dx, \quad u \in \mathbb{R}^x, \]

and

\[ \lambda_n(n) := \sum_{a/b=n} (a/b)^{\nu*}. \]

The equations (18), (19), (20), (21) hold true as in the cuspidal case.

It is known (cf. [BrMo, (3.31), (2.16)]) that each \( W_{\phi_n}(u) \) is a constant multiple of some normalized Whittaker function of the form (22), therefore we can conclude, exactly as in the cuspidal case,

\[ \left( a \frac{d}{du} \right)^{\lambda} W_{\phi}(u) \ll_{\varepsilon,a,b,c} \lambda^{1/2-c+\varepsilon} \| \phi \|_{S^{2+a+b+2c}} \min(|u|^{1/2-\varepsilon}, |u|^{1/2-b-\varepsilon}), \quad u \in \mathbb{R}^x, \]

for any \( 0 < \varepsilon < 1/4 \) and any integers \( a, b, c \geq 0 \).

4. Proof of Theorem 1

Let \((\pi_1, V_{\pi_1})\) and \((\pi_2, V_{\pi_2})\) be arbitrary cuspidal automorphic representations of \( \Gamma \setminus G \), and let \( W_{1,2} : \mathbb{R}^x \to \mathbb{C} \) be arbitrary smooth functions of compact support. There are unique smooth vectors \( \phi_i \in V_{\pi_i}^\infty \) such that \( W_{\phi_i} = W_i \). If \( h > 0 \) is an arbitrary integer and \( Y > 0 \) is arbitrary then we have, by (12),

\[ \sum_{m+n=h} \frac{\lambda_{\pi_1}(|m|)\lambda_{\pi_2}(|n|)}{\sqrt{mn}} W_1 \left( \frac{m}{Y} \right) W_2 \left( \frac{n}{Y} \right) = \int_0^1 (\phi_1\phi_2)(n(x)a(Y^{-1})) e(-hx) \, dx. \]

Let us decompose spectrally the smooth vector \( \phi_1\phi_2 \in L^2(\Gamma \setminus G) \) according to (1),

\[ \phi_1\phi_2 = \int_\tau \psi_\tau \, d\tau. \]

This decomposition is unique and converges in the topology defined by the Sobolev norms \( \| \cdot \|_{S^k} \), see Propositions 1.3 and 1.4 in [CoPS]. In particular, \( \psi_\tau \in V_{\pi}^\infty \), and the decomposition is compatible with the actions of \( G \) and \( \mathfrak{g} \). The explicit knowledge of the projections \( L^2(\Gamma \setminus G) \to V_{\tau} \) yields (cf. [BrMo, (2.14), (3.31), Lemma 2]), in combination with Plancherel, (9), and (30),

\[ ||D(\phi_1\phi_2)||^2 = \int_\tau ||D\psi_\tau||^2 \, d\tau, \quad D \in U(\mathfrak{g}). \]

By Corollary to Lemma 1.1 in [CoPS], the functional \( \phi \mapsto W_{\phi}(h) \) is continuous in the topology defined by the Sobolev norms \( || \cdot ||_{S^k} \) (this also follows from (29), (32), and Plancherel), hence the above imply, in combination with (12) and (31),

\[ \sum_{m+n=h} \frac{\lambda_{\pi_1}(|m|)\lambda_{\pi_2}(|n|)}{\sqrt{mn}} W_1 \left( \frac{m}{Y} \right) W_2 \left( \frac{n}{Y} \right) = \int_{\tau \neq \tau_0} \frac{\lambda_{\tau}(h)}{\sqrt{n}} W_{\tau} \left( \frac{h}{Y} \right) \, d\tau \]
where \( W_\tau := W_{\psi_\tau} \). This is just (3). On the right hand side we have, by (29) and (32),
\[
\tilde{\lambda}_\tau \left( y \frac{d}{dy} \right)^a W_\tau(y) \ll_{\varepsilon,a,b,c} \lambda_\tau^{-3/2+\varepsilon} \| \psi_\tau \|_{S^{a+b+c}} \min(y^{1/2-\varepsilon}, y^{1/2-b-\varepsilon}), \quad y > 0.
\]
A combination of Cauchy–Schwarz, Weyl’s law, and (33) then shows
\[
\int_{\tau \neq \tau_0} \tilde{\lambda}_\tau \left| \left( y \frac{d}{dy} \right)^a W_\tau(y) \right| d\tau \ll_{\varepsilon,a,b,c} \lambda_\tau^{-3/2+\varepsilon} \| \psi_\tau \|_{S^{a+b+c}} \min(y^{1/2-\varepsilon}, y^{1/2-b-\varepsilon}), \quad y > 0.
\]
Using (28) and the Leibniz rule for derivations we see that
\[
\| \phi_1 \phi_2 \|_{S^{a+b+c}} \lesssim_{\varepsilon,d} (\tilde{\lambda}_{\pi_1} + \tilde{\lambda}_{\pi_2})^{1+\varepsilon} \sum_{d_1+d_2=d+3} \| \phi_1 \|_{S^{d_1}} \| \phi_2 \|_{S^{d_2}}
\]
for any integer \( d \geq 0 \), which implies, by (14), (15), and (17),
\[
\| \phi_1 \phi_2 \|_{S^d} \lesssim_{\varepsilon,d} (\tilde{\lambda}_{\pi_1} + \tilde{\lambda}_{\pi_2})^{d+4+\varepsilon} \sum_{d_1+d_2=2d+6} \| W_1 \|_{A^{d_1}} \| W_2 \|_{A^{d_2}}.
\]
We combine this inequality with (34) to arrive at (4).

In retrospect we can see that this proof works for all test functions \( W_{1,2} : \mathbb{R}^\times \to \mathbb{C} \) whose norms \( \| W_{1,2} \|_{A^d} \) exist for \( d = 18 + 2a + 2b + 4c \).

## 5. Proof of Theorem 2

Let \((\pi_1, V_{\pi_1})\) and \((\pi_2, V_{\pi_2})\) be arbitrary cuspidal automorphic representations of \( \Gamma \setminus G \), and let \( c, k \geq 0 \) be arbitrary integers. Let \( q > 0 \) be an integer to be determined later in terms of \( c \) and \( k \). Consider the function \( G : [0, \infty) \to \mathbb{C} \) defined by
\[
G(t) := \begin{cases} \{t(1-t)\}^q, & 0 \leq t \leq 1, \\ 0, & t > 1, \end{cases}
\]
and its Laplace transform (composed with \( z \mapsto -z \)) defined by
\[
\tilde{G}(z) := \int_0^\infty G(t) e^{-zt} dt, \quad z \in \mathbb{C}.
\]
Note that \( \tilde{G}(z) \) is entire and by successive integration by parts it satisfies the uniform bound
\[
\tilde{G}(z) \ll_q |z|^{-q-1}, \quad \Re z = 1.
\]

If \( m, n \geq 1 \) are arbitrary integers and \( Y > 0 \) is arbitrary then by (35) and the theory of the Laplace transform we have the identity
\[
\left( \frac{mn}{Y^2} \right)^{k/2} G \left( \frac{m+n}{Y} \right) = \frac{1}{2\pi i} \int_{(1)} \tilde{G}(z) W_k \left( \frac{m}{Y}, z \right) W_k \left( \frac{n}{Y}, z \right) dz,
\]
where \( W_k : [0, \infty) \times \mathbb{C} \to \mathbb{C} \) is the function defined by
\[
W_k(t, z) := t^{k/2} e^{-zt}, \quad t \geq 0, \quad z \in \mathbb{C}.
\]
By Theorem 1 we can see that if \( k \) is sufficiently large in terms of \( c \) then there exist functions \( W_{k,\tau} : (0, \infty) \times \{ z : \Re z > 0 \} \to \mathbb{C} \) depending only on \( \pi_{1,2}, k, \) and
\( \tau \) such that the following two properties hold for all \( z \) with \( \Re z > 0 \). If \( h > 0 \) is an arbitrary integer and \( Y > 0 \) is arbitrary then we have the decomposition over the full spectrum (excluding the trivial representation)

\[
\sum_{m,n \geq 1 \atop m-n=h} \frac{\lambda_{\pi_1}(m)\lambda_{\pi_2}(n)}{\sqrt{mn}} W_k \left( \frac{m}{\sqrt{h}}, z \right) W_k \left( \frac{n}{\sqrt{h}}, z \right) = \int_{\tau \neq \tau_0} \frac{\lambda_{\tau}(h)}{\sqrt{h}} W_{k,\tau} \left( \frac{h}{\sqrt{h}}, z \right) \, d\tau,
\]

and we have the uniform bounds

\[
\int_{\tau \neq \tau_0} \lambda_{\tau}^c |W_{k,\tau}(y,z)| \, d\tau \ll \epsilon \, c \quad \text{for all} \quad y > 0, \quad \Re z > 0,
\]

where \( 0 < \epsilon < 1/4 \) is arbitrary and

\[
C_{0,1,c}(z) := (\lambda_{\pi_1} + \lambda_{\pi_2})^{1/2+\epsilon} \sum_{d_1+d_2=20+4c} \|W_k(\cdot, z)\|_{A^d_1} \|W_k(\cdot, z)\|_{A^d_2}.
\]

Here we use the convention that \( W_k(u,z) = 0 \) for \( u < 0 \). By (2) and (37) the right hand side exists as long as \( k > 60 + 12c \). Under this condition (38) is justified and we conclude that

\[
\int_{\tau \neq \tau_0} \lambda_{\tau}^c |W_{k,\tau}(y,z)| \, d\tau \ll \epsilon, k
\]

\[
(\lambda_{\pi_1} + \lambda_{\pi_2})^{1/2+\epsilon} |z|^{20+4c} \min(y^{1/2-\epsilon}, y^{-1/2-\epsilon}), \quad y > 0, \quad \Re z = 1.
\]

The functions \( W_{k,\tau}(y,z) \) furnished by the proof of Theorem 1 also have good behavior for individual \( \tau \). To see this, denote by \( \phi_{i,z} \in V_{\tau} \), the two vectors corresponding to the left hand side of (38) in the proof of Theorem 1. By (29) and (32) we have

\[
W_{k,\tau}(y,z) \ll_{\epsilon, \tau} \|\psi_{\tau,z}\|_{S^3} \min(y^{1/2-\epsilon}, y^{-1/2-\epsilon}), \quad y > 0, \quad \Re (z) > 0,
\]

where \( \psi_{\tau,z} \in V_{\tau} \) is the projection of \( \phi_{1,z} \phi_{2,z} \) on \( V_{\tau} \). Applying (14) and (30), then (17),

\[
\|\psi_{\tau,z}\|_{S^3} \ll_{\tau} \|W_{k,\tau}(\cdot, z)\|_{S^3} \ll_{\tau} \|W_{k,\tau}(\cdot, z)\|_{A^d}, \quad \Re (z) > 0,
\]

whence by (37) we can conclude

\[
W_{k,\tau}(y,z) \ll_{\epsilon, \tau, \tau} |z|^{6} \min(y^{1/2-\epsilon}, y^{-1/2-\epsilon}), \quad y > 0, \quad \Re z = 1.
\]

In an almost identical fashion,

\[
y \frac{d}{dy} W_{k,\tau}(y,z) \ll_{\epsilon, \tau, \tau} |z|^{6} y^{1/2-\epsilon}, \quad y > 0, \quad \Re z = 1.
\]

Finally, by (29), (32), (14), (30), and (17),

\[
W_{k,\tau}(y,z) - W_{k,\tau}(y,z') \ll_{\tau,y} \|\psi_{\tau,z} - \psi_{\tau,z'}\|_{S^2}
\]

\[
\ll_{\tau,y} \|W_{k,\tau}(\cdot, z) - W_{k,\tau}(\cdot, z')\|_{S^2}
\]

\[
\ll_{\tau,y} \|W_{k,\tau}(\cdot, z) - W_{k,\tau}(\cdot, z')\|_{A^d}, \quad \Re z, \Re z' > 0,
\]

whence by (37) we can conclude that

\[
\lim_{z' \to z} W_{k,\tau}(y,z') = W_{k,\tau}(y,z), \quad y > 0, \quad \Re z > 0.
\]

By (35), (40), (41), and (42), the integral

\[
H_{k,\tau}(y) := \frac{1}{2\pi i} \int_{(1)} \tilde{G}(z) W_{k,\tau}(y,z) \, dz, \quad y > 0,
\]
defines a differentiable function \( H_{k,\tau} : (0, \infty) \to \mathbb{C} \) for all \( \tau \) and satisfies the uniform bound
\[
H_{k,\tau}(y) \ll_{\varepsilon,k,\tau} \min(y^{1/2-\varepsilon}, y^{-1/2-\varepsilon}), \quad y > 0,
\]
as long as \( q > 6 \). Then (35), (36), (37), (38), (39), and two applications of Fubini's theorem show that
\[
\sum_{m,n \geq 1 \atop m-n=h} \frac{\lambda_{\pi_1}(m)\lambda_{\pi_2}(n)}{\sqrt{mn}} \left( \frac{mn}{Y^2} \right)^{k/2} G \left( \frac{m+n}{Y} \right) = \int_{\tau \neq \tau_0} \frac{\lambda_{\tau}(h)}{\sqrt{h}} H_{k,\tau} \left( \frac{h}{Y} \right) d\tau,
\]
as long as \( q > 20 + 4c \). We specify \( q := 21 + 4c \) and evaluate the Mellin transform in \( Y \) at \( 1 - s \) of both sides:
\[
\begin{align*}
\int_0^\infty Y^{1-s-k} \sum_{m,n \geq 1 \atop m-n=h} \lambda_{\pi_1}(m)\lambda_{\pi_2}(n) \left( \frac{mn}{Y^2} \right)^{k/2} G \left( \frac{m+n}{Y} \right) dY \\
= \int_0^\infty Y^{1-s} \int_{\tau \neq \tau_0} \frac{\lambda_{\tau}(h)}{\sqrt{h}} \frac{h}{Y} H_{k,\tau} \left( \frac{h}{Y} \right) d\tau \frac{dY}{Y}.
\end{align*}
\]
The left hand side is absolutely convergent for \( \Re s > 1 \), and by Fubini's theorem it equals
\[
\hat{G}(s-1) \sum_{m,n \geq 1 \atop m-n=h} \frac{\lambda_{\pi_1}(m)\lambda_{\pi_2}(n)(mn)^{k/2}}{(m+n)^s} = \Re(s) > 1.
\]
The right hand side is absolutely convergent for \( \frac{1}{2} < \Re s < \frac{3}{2} \) by (43), (35), (39), and by Fubini's theorem it equals
\[
h^{1-s} \int_{\tau \neq \tau_0} \lambda_{\tau}(h) \hat{H}_{k,\tau}(s-1) d\tau, \quad \frac{1}{2} < \Re s < \frac{3}{2}.
\]
The Mellin transforms \( \hat{H}_{k,\tau}(s-1) \) are holomorphic functions in the strip \( \frac{1}{2} < \Re s < \frac{3}{2} \) by (44), and by (43), (35), (39) they satisfy the uniform bound
\[
(46) \quad \int_{\tau \neq \tau_0} \frac{\lambda_{\tau}(h)}{\sqrt{h}} \frac{h}{Y} H_{k,\tau}(s-1) H_{k,\tau}(s-1) d\tau \ll_{\varepsilon,k} \left( \lambda_{\pi_1} + \lambda_{\pi_2} \right)^{1+4c}, \quad \frac{1}{2} + \varepsilon < \Re s < \frac{3}{2}.
\]
Finally we observe that
\[
\hat{G}(s-1) = \int_0^1 \{t(1-t)\}^q t^{s-2} dt = \frac{\Gamma(s+q-1)\Gamma(q+1)}{\Gamma(s+2q)}, \quad \Re s > 1-q,
\]
therefore
\[
\hat{G}(s-1) \gg_q |s|^{-q-1}, \quad \frac{1}{2} < \Re s < \frac{3}{2}.
\]
We put
\[
F_{k,\tau}(s) := \frac{\hat{H}_{k,\tau}(s-1)}{\hat{G}(s-1)}, \quad \frac{1}{2} < \Re s < \frac{3}{2},
\]
then the statements of Theorem 2 are immediate. In particular, (7) follows from the comparison of the two sides in (45), while (8) is a consequence of (46).
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