FINITENESS PROPERTIES FOR SELF-SIMILAR CONTINUA

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ABSTRACT. We consider self-similar continua possessing finite intersection property and prove intersection graph criterion for self-similar dendrites, finite order Theorem for such continua satisfying open set condition in \( \mathbb{R}^n \), and parameter matching Theorem which states that all Jordan arcs starting from a intersection point in such continuum on a plane should have the same slope parameter at that point.

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INTRODUCTION

Let \( S = \{S_1, ..., S_m\} \) be a system of injective contraction maps in a complete metric space \( X \). A non-empty compact set \( K \) satisfying the equation \( K = \bigcup_{i=1}^{m} S_i(K) \) is called the attractor of the system \( S \) and the sets \( K_i = S_i(K) \), where \( i = 1, ..., m \), are called the pieces of the set \( K \). We will focus mostly on a case in which the space \( X = \mathbb{R}^n \), the maps \( S_i \) are the similarities of \( \mathbb{R}^n \) and the attractor \( K \) is connected. In this case we say \( K \) is a self-similar continuum.

We say the self-similar continuum \( K \) (as well as the system \( S \)) has finite intersection property if for any non-equal \( i, j \), the intersection of pieces \( K_i \cap K_j \) is finite. If \( s \) is an upper bound for all \( \#(K_i \cap K_j) \), we say that \( S \) is a FI(s)-system of contractions.

For a long time it seemed highly likely that finite intersection property could imply open set condition (see Definition [1.5]), at least in case of one-point intersections. C. Bandt and H. Rao proved in [11] that FI(s)-systems of similarities in \( \mathbb{R}^2 \) with connected attractor satisfy OSC. From the other side, it was proved in [11] that in \( \mathbb{R}^3 \) this does...
not hold and in [12] it was also shown that there are one-point intersection systems (with totally disconnected attractor) in $\mathbb{R}$ which violate OSC. Therefore the interplay between open set condition and finite intersection property is a bit more intricate than it could be supposed.

In this paper we prove three theorems which are useful tool for analyzing self-similar continua which possess finite intersection property.

The first is the intersection graph criterion for self-similar dendrites (Theorem 2.6). It states that if $S$ is a FI(s)-system of contractions in a complete metric space $X$, then its attractor $K$ is a dendrite if and only if the intersection graph $\Gamma(S)$ of the system $S$ is a tree.

It has a simple topological proof and is applicable to different kinds of systems $S = \{S_1, ..., S_m\}$ of injective contractions in complete metric spaces.

The second one is finite order Theorem 3.1 for FI(s)-self-similar continua in $\mathbb{R}^n$ which satisfy open set condition. It says there is a finite uniform upper bound for the cardinality of the set of addresses of boundary points, for the number $N_C$ of components of $V \setminus \{x\}$ (resp. $V \setminus K_j$), where $V$ is a connected neighborhood of a given set, and for the topological order of any point $x \in K$ and of any piece $K_j$.

The third is parameter matching Theorem 4.5 for the intersection points, which extends our result for polygonal dendrites [5, Theorem 3.12] to FI(s)-self-similar continua. It states that if a point $p \in K_i \cap K_j$ has a periodic address, then all Jordan arcs $\gamma \subset K$ with the end point $p$ have the same slope parameter at the point $p$.

In Section 2 we define a FI(s)-system $S$ of contractions and its intersection graph $\Gamma(S)$. This graph is a bipartite graph whose "white" vertices are the pieces $K_i$ and "black" vertices are the intersection points $p \in K_i \cap K_j$. In case when $\Gamma(S)$ is a tree we prove Theorem 2.6.

In Section 3 we consider FI(s)-systems $S$ of contracting similarities which satisfy open set condition and prove Theorem 3.1. As an outcome of this Theorem we find that each $x \in K$ and each $K_j \subset K$ has a fundamental system of stable neighborhoods $V_j$ (Theorem 3.2). Finally we prove a ramification order estimate for self-similar dendrites which need not satisfy finite intersection property (Theorem 2.4).

In Section 4 we consider the fixed points $z_j$ of the similarities $S_j \in G_S$ and prove the existence of invariant arcs in each of the components $Q_k$ of the set $K \setminus \{z_j\}$. This yields the parameter matching theorem for the points $x \in K$ which have multiple preperiodic addresses.
1. Preliminaries

Definition 1.1. Let \( S = \{S_1, S_2, \ldots, S_m\} \) be a system of injective contractions on a complete metric space \((X, d)\). A nonempty compact \( K \subset X \) is called the attractor of the system \( S \), if \( K = \bigcup_{i=1}^{m} S_i(K) \).

The system \( S \) defines its Hutchinson operator \( T \) by the equation \( T(A) = \bigcup_{i=1}^{m} S_i(A) \). By Hutchinson’s Theorem [6], the attractor \( K \) always exists and is unique for \( S \). We also call the set \( K \) self-similar with respect to \( S \).

Given the system \( S \), the set \( I = \{1, 2, \ldots, m\} \) is called the set of indices, the set \( I^* = \bigcup_{n=1}^{\infty} I^n \) is the set of multiindices \( j = j_1j_2...j_n \).

If \( j \in I^* \), we write \( S_j = S_{j_1}S_{j_2}...S_{j_n} \) and denote \( S_j(K) \) by \( K_j \).

The set of all infinite sequences \( I^\infty = \{\alpha = \alpha_1\alpha_2...\alpha_i \in I\} \) is called the index space of the system \( S \); the map \( \pi : I^\infty \rightarrow K \) which sends each \( \alpha \) to the point \( \pi(\alpha) = \bigcap_{n=1}^{\infty} K_{\alpha_1...\alpha_n} \) is called the index map.

If \( \pi(\alpha) = x \), then \( \alpha \) is called an address of the point \( x \).

By \( ij \) we denote the concatenation of respective multiindices. We say \( i \sqsubset j \), if \( j = il \) for some \( l \in I^* \); if \( i \not\sqsubset j \) and \( j \not\sqsubset i \), we say that \( i \) and \( j \) are incomparable.

Similarly, by \( i\alpha \) we denote the concatenation of \( i \) and \( \alpha \) and we write \( i \sqsubset \alpha \), if \( \alpha = i\beta \) for some \( \beta \in I^* \). If \( \alpha = i\alpha \), we write \( \alpha = \bar{i} \) and say that \( \alpha \) is periodic. We say that \( \alpha \) is preperiodic if \( \alpha = j\bar{i} \) for some \( i, j \in I^* \).

We denote \( iI^\infty = \{i\alpha, \alpha \in I^\infty\} \), and \( JI^\infty = \{i\alpha : i \in J, \alpha \in I^\infty\} \). Along with the system \( S \) we consider its \( n \)-th refinement \( S^{(n)} = \{S_j, j \in I^n\} \). The Hutchinson operator of the system \( S^{(n)} \) is equal to \( T^n \).

Definition 1.2. Let \( J' \subset I^* \) be a set of incomparable multiindices, and let \( A \subset I^\infty \) be a set of addresses. We write \( J < J' \) (resp. \( J < A \)), if \( J \) is a set of incomparable multiindices and if there is a surjection \( \psi : J' \rightarrow J \) or \( \psi : A \rightarrow J \) such that \( j' \sqsupset \psi(\alpha) \) (resp. \( \alpha \sqsupset \psi(\alpha) \)).

Suppose \( B \subset K \) and \( J < \pi^{-1}(B) \). We denote by \( V_J(B) \) the set \( \bigcup_{j \in J} K_j \).

Lemma 1.3. If \( B \subset K \) is a compact set and \( J < \pi^{-1}(B) \), then \( V_J \) is a neighborhood of \( B \) in \( K \).
Proof. Notice that the family \(\{jI^\infty, j \in J\}\) is an open covering of \(\pi^{-1}(B)\). Since the multiindices \(j \in J\) are incomparable, this covering is a disjoint one and therefore the set \(J\) is finite. There is a finite \(J' \subset I^*\) such that \(J'I^\infty = I^\infty \setminus JI^\infty\). The set \(\pi(J'I^\infty)\) is compact and \(\pi(J'I^\infty) \cap B = \emptyset\). There is such \(\varepsilon > 0\) that for any \(x \in B\) and \(y \in \pi(J'I^\infty)\), \(d(x, y) > \varepsilon\). Therefore \(V_j(B)\) contains a \(\varepsilon\)-neighborhood of \(B\) in \(K\), therefore \(V_j(B)\) is a neighborhood of \(B\).  

Definition 1.4. Let \(S\) be a contracting injective map on a metric space \(X\) and \(x = \text{fix}(S)\). A Jordan arc \(\gamma \subset X\) with an endpoint at \(x\) is called an invariant arc for \(S\), if there is \(k \in \mathbb{N}\) such that \(S^k(\gamma) \subset \gamma\).

Definition 1.5. The system \(S\) satisfies the open set condition (OSC) if there exists a non-empty open set \(O \subset X\) such that the sets \(S_i(O), \{1 \leq i \leq m\}\) are pairwise disjoint and are contained in \(O\).

For any \(i, j \in I^*\), \(i \sqsubset j\) iff \(S_i(O) \supseteq S_j(O)\) and \(i\) and \(j\) are incomparable, iff \(S_i(K) \cap S_j(K) = \emptyset\). The union \(\mathcal{C}\) of all intersections \(S_i(K) \cap S_j(K), i, j \in I^*, i \neq j\) is called the critical set of the system \(S\). The set of all predecessors of the points in \(\mathcal{C}\), \(\partial K = \{x \in K: \text{for some } j \in I^*, S_j \in \mathcal{C}\}\), is called the self-similar boundary of the set \(K\).

We denote by \(G_S = \{S_j, j \in I^*\}\) the semigroup, generated by \(S\). In case when \(S\) is a system of similarities in \(\mathbb{R}^n\) then \(\mathcal{F} = G_S^{-1} \circ G_S\), or a set of all compositions \(S_j^{-1}S_i\), \(i, j \in I^*\), is the associated family of similarities\([3]\). The system \(S\) has the weak separation property (WSP) iff \(\text{Id} \notin \mathcal{F} \setminus \text{Id}[S]\), \([13]\).

Definition of \(M_a\). According to Zerner’s Theorem \([13]\), if the system \(S\) of contraction similarities satisfies the Weak Separation Condition, then for any \(a > 0\) there is a positive number

\[
1. M_a = \sup_{U \subset \mathbb{R}^n} \# \{S_j : a|U| r_{\min} < |K_j| \leq a|U| \ & K_j \cap U \neq \emptyset\}.
\]

If the system \(S\) possesses the finite intersection property it has no exact overlaps, and \(S_i = S_j\) iff \(i = j\), then \((1)\) becomes

\[
2. M_a = \sup_{U \subset \mathbb{R}^n} \# \{j : a|U| r_{\min} < |K_j| \leq a|U| \ & K_j \cap U \neq \emptyset\}.
\]
2. THE INTERSECTION GRAPH OF A SYSTEM WITH FINITE INTERSECTION PROPERTY

In this section we start with a definition of a FI(s)-set system \( \mathcal{K} \) and define its intersection graph \( \Gamma(\mathcal{K}) \). Then we define the \((\mathcal{L}, f)\)-refinement operation for such systems and show that the class of tree-like systems is closed under this operation. After all, we apply the established properties to systems \( S \) of contractions to prove Theorem 2.6.

**Definition 2.1.** Let \( \mathcal{K} = \{K_i, i \in I = \{1, \ldots, m\}\} \) be a finite system of continua in a Hausdorff topological space \( X \). We say \( \mathcal{K} \) possesses finite intersection property, if for any \( i \neq j \in I \), the intersection \( P_{ij} = K_i \cap K_j \) is finite. We call \( \mathcal{K} \) a FI(s)-set system, if \( \max \# P_{ij} = s \).

In the settings of the Definition 2.1 we denote \( K = \bigcup_{i \in I} K_i, P = \bigcup_{i \neq j} P_{ij} \) and \( P_i = \bigcup_{j \in I \setminus \{i\}} P_{ij} \). Considering \( K \) as the subspace of \( X \) provided by the induced topology, we see that the set \( P_i \) is the boundary \( \partial K_i \) of the set \( K_i \) in \( K \), and that its interior is \( \overset{\cdot}{K}_i = K_i \setminus P_i \). Observe that for any \( i \in I \), \( \# \partial K_i \leq (m - 1)s \).

**Definition 2.2.** Let \( S = \{S_1, \ldots, S_m\} \) be a system of injective contraction maps on a complete metric space \( X \) and \( K \) be its attractor. Let \( \mathcal{K}(S) = \{K_1, \ldots, K_m\} \). \( S \) is called a FI(s)-system of contractions if the system \( \mathcal{K}(S) \) is a FI(s)-set system.

Along with the FI(s)-system \( S \) we consider its \( n \)-th refinement \( S^n = \{S_i, i \in I^n\} \) and the system \( \mathcal{K}_n(S) = \{K_i : i \in I^n\} \). For each \( K_i \) its boundary is \( \partial K_i = \bigcup_{j \in I^n \setminus \{i\}} K_j \cup K_i \). We define \( P^n = \bigcup_{i \in I^n} \partial K_i \), and the set of all boundary points of the pieces of \( K \) is \( P^* = \bigcup_{i \in I^*} \partial K_i \).

If \( i, j \in I^* \) and \( i_1 \neq j_1 \) then \( \#(K_i \cap K_j) \leq \#(K_{i_1} \cap K_{j_1}) \leq s \). This implies that for any incomparable \( i, j \in I^* \) we obtain \( \#(K_i \cap K_j) \leq s \). Therefore for each \( n \) the system \( S^n \) is also a FI(s)-system of contractions.

2. For a FI(s)-set system \( \mathcal{K} \) (resp. FI(s)-system \( S \)) we define its intersection graph \( \Gamma(\mathcal{K}) \) (resp. \( \Gamma(S) \)) as a bipartite graph \( (\mathcal{K}, P; E) \) with parts \( \mathcal{K} \) and \( P \), for which an edge \( \{K_i, p\} \in E \) iff \( p \in K_i \).
We call $K_i \in \mathcal{K}$ white vertices and $p \in P$ black vertices of the graph $\Gamma$. The set $N(K_i)$ of the neighbors of any white vertex $K_i$ is $P_i$, whereas for any black vertex $p$, $N(p) = \{K_i : p \in K_i\}$. Each $p \in P$ is the intersection point of at least two of the sets $K_i$, therefore $\deg(p) \geq 2$.

Given two FI(s)-set systems $\mathcal{K} = \{K_i, i \in I\}$, $\mathcal{L} = \{L_i, i \in J\}$, we define the following operation which produces a $(\mathcal{L}, f)$-refinement $\mathcal{K}'$ of $\mathcal{K}$ whose intersection graph $\Gamma(\mathcal{K}')$ is an expansion of the graph $\Gamma(\mathcal{K})$: We take some $l \in I$ and some continuous injection $f : L \to K_l$ of the set $L = \bigcup_{\mathcal{L}} L_j$ which satisfies $P_l \subseteq f(L)$. The map $f$ transforms the system $\mathcal{L}$ to a FI(s)-set system $\mathcal{L}_f = \{f(L_i), i \in J\}$. Then the system $\mathcal{K}' = (\mathcal{K}\setminus\{K_l\}) \cup \mathcal{L}_f$ is a FI(s)-set system, whose intersection graph $\Gamma(\mathcal{K}')$ is an expansion of $\Gamma(\mathcal{K})$.

3. If the intersection graph $\Gamma(\mathcal{K})$ of a FI(s)-set system $\mathcal{S}$ is a tree, then $s = 1$. Moreover, if the intersection graphs of FI(1)-set systems $\mathcal{K}, \mathcal{L}$ are trees, the same is true for any $(\mathcal{L}, f)$-refinement $\mathcal{K}'$ of $\mathcal{K}$:

**Proposition 2.3.** Let $\mathcal{K} = \{K_i, i \in I\}$, $\mathcal{L} = \{L_i, i \in J\}$, and $l \in I$. Let the intersection graphs $\Gamma(\mathcal{K}), \Gamma(\mathcal{L})$ be trees. Then the intersection graph of any $(\mathcal{L}, f)$-refinement $\mathcal{K}'$ of the system $\mathcal{K}$ is a tree.

**Proof:** Let $l \in I$ and let $f : L \to K_l$ be a continuous injection which maps the set such that $P_l \subseteq f(L)$.

Consider the graph $\Gamma(\mathcal{K}\setminus\{K_l\})$ obtained by deletion of the star of $K_l$ in $\Gamma(\mathcal{K})$. It is a disjoint union of finite number of connected components $Q_p$ each being a non-degenerate tree containing one of the vertices $p \in P_l$. Since $f : L \to K_l$ is injective, there are exactly $n_l = \#P_l$ points $p' = f^{-1}(p), p \in P_l$, contained in $\bigcup_{i \in J} L_i$. 

![Figure 1. A FI(3)-set system and its intersection graph.](image-url)
These points are of two kinds. First are the points \( p' \in f^{-1}(P_k) \cap P_L \) which correspond to black vertices of \( \Gamma(L) \). The second type are the points \( p' \in f^{-1}(P_k) \setminus P_L \). Each of these points is contained in some unique \( L_i \in L \). We construct an extension \( \tilde{\Gamma} \) of the graph \( \Gamma_L \), adding the points of second kind to the set of black vertices \( P_L \) and respective edges \( (L_i, p') \) to the edge set \( E_L \). Such extension does not produce cycles and the graph \( \tilde{\Gamma} \) is again a tree with two parts.

Now we identify each of the points \( f^{-1}(p_i) \), \( p_i \in P_l \), with the point \( p_i \in Q_i \). Thus we paste the tree \( Q_i \) to the graph \( \tilde{\Gamma} \).

The resulting graph \( \Gamma(K'_{(S)}) \) is a tree with two parts, \( K' = K \setminus \{K_l\} \cup L \) and \( P_{K'} = P_{K_l} \cup (P_L \setminus f^{-1}(P_l)) \). The degree of each vertex \( p \in P_{K'} \) is \( \geq 2 \). ■

4. The refinement operation can be applied to a FI(s)-system \( S \) of contractions and to its \( n \)-th intersection graphs \( \Gamma_n(S) = \Gamma(K_n(S)) \).

**Lemma 2.4.** Let \( S = \{S_i, i \in I\} \) be a system of injective contraction maps, such that the intersection graph \( \Gamma_1(S) \) is a tree. Then for any \( n \in \mathbb{N} \), the intersection graph \( \Gamma_n(S) \) is a tree.

**Proof:** Suppose that the intersection graph \( \Gamma_{n-1}(S) \) is a tree. Applying \((K, S_i)\)-refinement operation to the piece \( K_i \) successively for all \( i \in I^{n-1} \), we finally obtain that the intersection graph \( \Gamma_n \) is also a tree. ■

If the intersection graph of a FI(1)-set system \( K \) is a tree, then a simple loop in \( K \) cannot pass through any of the boundary points between the sets \( K_i \). So, we get the following result.

**Lemma 2.5.** Let \( K \) be a FI(1)-system of sets and \( \Gamma(A) \) be a tree. Let \( \gamma \) be a simple closed curve in \( K \). Then there is unique \( K_i \in \mathcal{K} \) such that \( \gamma \subset K_i \).

**Proof:** Let \( p \) be some point in \( P \) and let \( Q_i, Q_j \) be the components of \( K \setminus \{p\} \). Suppose \( \gamma \) is a closed curve containing some \( a \in Q_i \) and \( b \in Q_j \). Since each path connecting \( a \) and \( b \) passes through \( p \), the point \( p \) is a multiple point of \( \gamma \). Therefore if \( \gamma \cap K_i \neq \emptyset \), then \( \gamma \in K_i \). ■

**Theorem 2.6.** Let \( S \) be a system of injective contraction maps in a complete metric space \( X \) which possesses finite intersection property. The attractor \( K \) of the system \( S \) is a dendrite if and only if the intersection graph of the system \( S \) is a tree.

**Proof:** Let \( \gamma \in K \) be a simple closed curve. Since for any \( n \in \mathbb{N} \) the graph \( \Gamma_n \) is a tree, there is unique \( j \in I^n \) such that \( \gamma \in K_j \). Therefore \( |\gamma| = 0 \). ■
3. Finiteness theorems for ramification points and pieces

Throughout this and further sections, unless otherwise is stated, we restrict ourself to the case when \( S = \{S_1, \ldots, S_m\} \) is a FI(s)-system of contracting similarities in \( \mathbb{R}^d \). Therefore its attractor \( K \) is a continuum and for any non-equal \( i, j \in I \), \( \#(K_i \cap K_j) \leq s \).

Consider the following notation:
\( \#\pi^{-1}(x) \) (resp. \( \#\pi^{-1}(\partial K_j) \)) denotes the cardinality of the set of addresses of a point \( x \in K \) (resp. of boundary points of a piece \( K_j \)).
\( N_C(x) \) (resp. \( N_C(K_j) \)) denotes the upper bound for the cardinality of the set of connected components of \( V(A) \setminus A \). Here \( V(A) \) denotes a compact connected neighborhood of \( A \), and \( A \) stands for \( \{x\} \) or \( K_j \).

By Ord\((A, X)\) we denote the topological order of the set \( A \) in the space \( X \). \([7], [9] \S 9.3\)

We prove that if the FI(s)-system \( S \) satisfies OSC, then each of the above numbers has a uniform upper bound independent of the choice of \( K_j, j \in I^* \) or \( x \in K \):

**Theorem 3.1.** If a FI(s)-system \( S = \{S_1, \ldots, S_m\} \) of similarities satisfies open set condition, then each of the numbers:
\( \#\pi^{-1}(x); \#\pi^{-1}(\partial K_j); N_C(\{x\}) \); \( N_C(K_j) \); Ord\((x,K)\); Ord\((K_j,K)\)
has a uniform finite upper bound independent of \( x \in K \) and \( j \in I^* \).

**Proof:**
(i) Cases \( \#\pi^{-1}(x) \) and \( \#\pi^{-1}(\partial K_j) \).
If \( x \notin P^* \), then \( \#\pi^{-1}(x) = 1 \). Let \( x \in P^* \). Take some \( \rho > 0 \) and consider the set
\( C_\rho = \{j \in I^*: \rho r_{\min} < |K_j| \leq \rho \text{ or } K_j \cap B(x, \rho/2) \neq \emptyset\} \).
By \([13]\), Theorem 1, \( \#C_\rho \leq \mathcal{M}_1 \). For any \( \rho > 0 \) and any \( \alpha \in \pi^{-1}(x) \) there is \( j \supseteq \alpha \) such that \( j \in C_\rho \), therefore
\( \#\pi^{-1}(x) \leq \sup\{\#C_\rho, \rho > 0\} \leq \mathcal{M}_1 \)
For each \( x \in \partial K_i \) choose \( j \in I^* \) such that \( x \in K_j \cap K_i \) and \( |K_i| r_{\min} \leq |K_j| \leq |K_i| \). The number of all such \( j \) is at most \( \mathcal{M}_1 \), whereas the number \( \#(K_i \cap K_j) \leq s \). This implies \( \#\partial K_i \leq \mathcal{M}_1 \cdot s \). Therefore \( \#\pi^{-1}(\partial K_i) \leq \mathcal{M}_1^4 s \).

(ii) Cases \( N_C(\{x\}) \) and \( N_C(K_j) \).
Let \( Q = \{Q_1, \ldots, Q_n\} \) be some finite set of connected components of \( W \setminus \{x\} \). Take such \( \rho \), that for any \( Q_k \in Q \), \( Q_k \setminus B(x, \rho) \neq \emptyset \). Each component \( Q_k \in Q \) contains such \( y_k \), that \( d(x, y_k) = 3\rho/4 \). Let \( j_k \)
be such that $y_k \in K_{j_k}$, and $\rho r_{\min}/2 < |K_{j_k}| \leq \rho/2$. Since $x \notin K_{j_k}$, $K_{j_k} \subset Q_k$. Therefore all $j_k$ are incomparable and the number of such $j$ is no greater than $\mathcal{M}_{1/3}$. Thus, $\#Q \leq \mathcal{M}_{1/3}$, which gives the bound for $N_C(\{x\})$.

Notice that $N_C(K_i)$ is no greater the sum of all $N_C(x)$, where $x \in \partial K_i$, we obtain $N_C(K_i) \leq \mathcal{M}_{1/3} \mathcal{M}_{1/3}$.

(iii) Cases Ord$(x, K)$ and Ord$(K_j, K)$.

By Lemma 1.3, for each $x \in K$ and $J \prec \pi^{-1}(x)$, the set $V_J(x) = \bigcup_{j \in J} K_j$ is a connected neighborhood of the point $x$.

The same is true for each $i \in I^*$. If $J \prec \pi^{-1}(\partial K_i)$, the set $V_J(K_i) = \bigcup_{j \in J} K_j \cup K_i$ is a connected neighborhood of $K_i$.

Moreover, the family $V(x) := \{V_J(x) : J \prec \pi^{-1}(x)\}$ (resp. $V(K_i) := \{V_J(K_i) : J \prec \pi^{-1}(\partial K_i)\}$) is a neighborhood basis in $K$ for the point $x$ (resp. the set $K_i$), which consists entirely of connected sets.

Taking into account the inclusion $\partial V_J \subset \bigcup_{j \in J} \partial K_j$, we obtain the inequalities $\text{Ord}(x, K) \leq \mathcal{M}_{1/3}^2$ and $\text{Ord}(K_i, K) \leq \mathcal{M}_{1/3}^3$.

**Proposition 3.2.** Let $S = \{S_1, ..., S_m\}$ be a FL(s)-system satisfying OSC. Then

1) for any $x \in K$ there is a family $J \prec \pi^{-1}(x)$ such that for any non-equal $j, k \in J, K_j \cap K_k = \{x\}$ and the number of connected components of $V_J \backslash \{x\}$ is equal to $N_C(x)$;

2) for any $i \in I^*$ there is a family $J \prec \pi^{-1}(\partial K_i)$ such that for any $j \in J, K_j \cap \partial K_i$ is a unique point, the sets $K_j \backslash K_i$ are disjoint and the number of connected components of $V_J(x) \backslash K_i$ is equal to $N_C(K_i)$.

**Proof:** Let $\pi^{-1}(x) = \{\alpha_1, ..., \alpha_n\}$. There is $J \prec \pi^{-1}(x)$ which consists of incomparable $j \in J$. Therefore the set $P_J = \bigcup_{j \in J} \partial K_j$ is finite.

Take $\rho > 0$ such that $B(x, \rho) \cap P_J = \{x\}$. If $J' \supset J$ satisfies the relation $J \prec J' \prec \pi^{-1}(x)$ and for any $j \in J'$, diam$(K_j) < \rho$, then for any non-equal $j, k \in J', K_j \cap K_k = \{x\}$. This proves (i).

Take $\alpha \in \pi^{-1}(x)$. Denote the initial substring of length $k$ in $\alpha$ by $j_k$.

The sequence $\{n_k := \#Q_{j_k}\}$ is non-decreasing. By Theorem 3.1, it has an upper bound. Therefore there is $k_0$ such that if $k \geq k_0$, then $n_k = n_{k_0}$. For any $l > k \geq k_0$, $Q \in Q_{j_k}$ implies $Q \cap K_{j_k} \in Q_{j_k}$. Choosing respective $j \sqsubset \alpha$ for each $\alpha \in \pi^{-1}(x)$ we get the desired $J \prec \pi^{-1}(x)$. 


Let $A = \pi^{-1}(\partial K_i)$. Since $A$ is finite, we can take some set of incomparable multiindices $J \prec A$. Let $\rho = 1/2 \min \{d(x, y), x, y \in P_J\}$. Take such $J' \prec A$, that $J \prec J'$ and for any $j \in J'$, $|K_j| < \rho$. Then for any $j \in J'$, the intersection $K_j \cap P_J$ is a singleton and therefore is contained in $\partial K_i$. ■

**Definition 3.3.** The set $\bigcup_{j \in J} Q_j$ defined by Proposition 3.2(ii) is called a stable set of components for the point $x$. The set $V_J(x)$ is called a stable neighborhood of the point $x$.

If $S$ satisfies the assumptions of Theorem 3.1 and its attractor $K$ is a dendrite, then for any $x \in K$ and any stable neighborhood $V_J(x)$ of the point $x$, $\# \partial V_J = N_C(x)$, therefore by Theorem 3.1(3), $\text{Ord}(x, K) \leq M_{1/3}$.

Surprisingly, we can prove a similar estimate without the requirement of finite intersection property:

**Theorem 3.4.** Suppose a system $S = \{S_1, ..., S_m\}$ of contracting similarities in $\mathbb{R}^n$ satisfies WSP, and its attractor $K$ is a dendrite. Then for any $x \in K$, $\text{Ord}(x, K) \leq M_{1/2}$.

**Proof:** Let $Q_1, ..., Q_n$ be some finite set of connected components of $K \setminus \{x\}$. Let $\rho = \min_{1 \leq k \leq n} \text{diam}(Q_k)$. For each $1 \leq k \leq n$ take some $z_k \in \partial B(x, \rho) \cap Q_k$. Take such $j_k \in I^*$ that $z_k \in K_{j_k}$ and $\text{diam}(K_{j_k}) < \rho$. Since $K$ is a dendrite and $x \notin K_{j_k}$, the sets $K_{j_k}$ are disjoint. Therefore by [13, Theorem 1], $n \leq M_{1/2}$ for any $x \in K$. ■

4. Boundary points and parameter matching for FI(s) continua in the plane

If a FI(s)-system of similarities $S$ satisfies OSC, then the fixed points of maps $S_i, i \in I^*$ possess some special properties.

**Proposition 4.1.** If a FI(s)-system $S = \{S_1, ..., S_m\}$ of similarities in $\mathbb{R}^d$ satisfies OSC, and $x = \text{fix} S_i$ for some $i \in I^*$, then:

(i) $\pi^{-1}(x) = \{i\}$;
(ii) The set $Q$ of connected components of the set $K \setminus \{x\}$ is a stable set of components for the point $x$;
(iii) for any component $Q \in Q$ there is a point $y \in S_i^{-1}(\partial K_i) \cap Q$ and an invariant Jordan arc $\gamma \subset Q$ for $S_i$ with endpoints $x$ and $y$.  

Proof: (i) Suppose there is \( \alpha \in \pi^{-1}(x) \) such that \( \alpha \neq \tilde{z} \). Then for any \( k \in \mathbb{N} \), \( i^k \alpha \in \pi^{-1}(x) \). Since all these addresses are different, this contradicts Theorem 3.1.

(ii) Let \( Q = \{ Q_k, k = 1, \ldots, s \} \) be a stable set of components for the point \( x \). The equality \( S_i(Q_k) = S_i(K) \cap Q_{\sigma(k)} \) defines a permutation \( \sigma \) of the set \( \{ 1, \ldots, s \} \). Take such \( l \) that \( \sigma^l = \text{Id} \). Then for any \( k = 1, \ldots, s \), \( S_i(Q_k) = S_i^l(K) \cap Q_k \).

Since \( K \setminus \{ x \} = \bigcup_{k=1}^s (S_i^{-l}(Q_k \cap K)) \), the family \( \{ S_i^{-l}(Q_k \cap K) \} \) is also a stable set of components for \( x \).

(iii) To make the notation more simple we use \( j = i^l \) in our further argument.

Let \( Q \) be a component of \( K \setminus \{ x \} \) and let \( D_Q = S_i^{-1}(\partial S_i(K) \cap Q) \). Then it follows from (ii) that for any \( k \in \mathbb{N} \), \( S_i^k(D) = \partial S_i^k(K) \cap Q \).

We define a map \( \varphi : D \to D \) the following way.

Since \( Q \) is arcwise connected, for any \( z \in D \) there is a Jordan arc \( \delta \in Q \) with endpoints \( z \) and \( x \). Let \( \delta(z) \) be the closure of a maximal subarc of \( \delta \cap (Q \setminus S_j(Q)) \) containing \( z \). We denote the second endpoint of \( \delta(z) \) by \( z' \) and define a map \( \varphi : D \to D' \) by the equation \( \varphi(z) = S_j^{-1}(z') \).

There are \( n \leq \# D \) and \( y \in D \) such that \( \varphi^n(y) = y \). For any non-negative integer \( k < n \) and \( y_k = \varphi^k(y) \), \( \delta(y_k) \) is a Jordan arc in \( Q \setminus S_i(Q) \) connecting \( y_k \) and \( S_j(y_{k+1}) \).

Therefore \( \gamma' = \bigcup_{k=0}^{n} S_j^k(\delta(y_k)) \) is a Jordan arc connecting \( y \) and \( S_j^n(y) \) which lies in \( Q \setminus S_j^n(Q) \) and \( \gamma = \bigcup_{k=0}^{\infty} S_j^{kn}(\gamma') \cup \{ x \} \) is the desired invariant arc in \( Q \) with endpoints \( x \) and \( y \).

Remark. The proof of Proposition 4.1 is purely combinatorial and is based on finite intersection property and on the assumption that both \( \pi^{-1}(x) \) and the set of components of \( K \setminus \{ x \} \) are finite. This allows us to extend this Proposition to FI(s)-systems of contractions in metric spaces.

Corollary 4.2. If Let \( S = \{ S_1, \ldots, S_m \} \) be a FI(s)-system of contractions in a complete metric space \( X \). If for any \( x \in K \) the set \( \pi^{-1}(x) \) and the set of components of \( K \setminus \{ x \} \) are finite, then the statements (i), (ii), (iii) of Proposition 4.1 are fulfilled.

If \( S \) is a FI(s)-system of similarities on a plane, then it follows from [II], that it satisfies open set condition. In dimension 2, the invariant
arcs are more tame and we show that they obey certain restrictions. To analyze their behavior, we use complex notation and consider FI(s)-systems $S$ as acting in $\mathbb{C}$.

Let $z_0 \in \mathbb{C}$ and let $\gamma \subset \mathbb{C} \setminus \{z_0\}$ be a Jordan arc with endpoints $z_1, z_2$. We denote by $\Delta(\arg(z - a))|_{\gamma}$ the increment of $\arg(z - a)$ along the arc $\gamma$ as $z$ travels from $z_1$ to $z_2$.

**Definition 4.3.** We say that a Jordan arc $\gamma = \gamma(z_0, b)$ in $\mathbb{C}$ has a slope parameter $\lambda$ at the endpoint $z_0$ if there is $M > 0$ such that for any subarc $\gamma' = \gamma(z_1, z_2) \subset \gamma$ the increment $\Delta \arg(z - z_0)|_{\gamma'}$ of the argument of $z - z_0$ along the arc $\gamma'$ satisfies the inequality

$$|\Delta \arg(z - z_0)|_{\gamma'} - \lambda(\log |z_2 - z_0| - \log |z_1 - z_0|)| \leq M.$$ 

**Proposition 4.4.** If $\gamma(z_0, y)$ is an invariant arc for $S_i$ and $S_i^\alpha(\gamma) \subset \gamma$, then the slope parameter of $\gamma$ at the point $z_0$ is equal to

$$\lambda = \frac{\Delta(\arg(z - z_0))|_{\gamma \setminus S_i(\gamma)}}{k \log \text{Lip} S_i}.$$ 

If the point $z_0$ is contained in boundary $\partial A$ of some unbounded component $A$ of $\mathbb{C} \setminus K$, then for any arc $\gamma' \subset K$ whose endpoint is $z_0$, this arc has the slope parameter $\lambda$ at $z_0$.

**Proof:** Replacing, if necessary, the multiindex $i$ by its n-th multiple, we suppose from now on that $S_j(\gamma) \subset \gamma$ whenever we deal with fixed points and their invariant arcs.

Denote $R = -\log \text{Lip} S_i$ and $\alpha = \Delta(\arg(z - z_0))|_{\gamma \setminus S_i(\gamma)}$ and let $T$ be a translation of the complex plane defined by $T(w) = w - R - i\alpha$.

We slightly modify the proof of [1, Lemma 3.1]. Consider the map $\varphi$ defined by the formula $z = z_0 + e^w$ of the complex plane $w = \rho + i\theta$ as a universal covering map of $\mathbb{C} \setminus \{z_0\}$. For any lifting $\tilde{\gamma}$ of the arc $\gamma$ with respect to $\varphi$, $T(\tilde{\gamma}) \subset \tilde{\gamma}$. Fix one of the liftings and let $\tilde{y} = \rho_0 + i\theta_0 \in \varphi^{-1}(y)$ be its endpoint and let $\tilde{\delta} = \tilde{\gamma} \setminus T(\tilde{\gamma})$ be the respective lifting of $\delta$. There is $M > 0$ such that the arc $\tilde{\delta}$ lies in a strip $|\theta - \theta_0 - \lambda(\rho - \rho_0)| < M$. By its $T$-invariance, the same is true for $\tilde{\gamma}$, thus proving the first statement of the Proposition.

Now let $\gamma'$ be a Jordan arc in $A$, whose endpoints are $z_0$ and some $y' \in A$. Without loss of generality, we suppose that for any $z \in \gamma'$, $|z - z_0| < |y - z_0|$. Since $\gamma' \cap \gamma = \emptyset$, one of the liftings $\tilde{\gamma}'$ of the arc $\gamma'$ lies in some half-strip bounded by the lines $\tilde{\gamma}'$ and $\tilde{\gamma}' + 2i\pi$ and a segment $[\tilde{y}, \tilde{y} + 2i\pi]$ of the line $Re(w) = \rho_0$. Therefore, for any $w \in \tilde{\gamma}'$, $|\theta - \theta_0 - \pi - \lambda(\rho - \rho_0)| < M + \pi$. Applying similar argument to a subarc
\[ \gamma'^' \subset Q \text{ with the endpoint } z_0 \text{ and the arc } \gamma', \text{ we obtain the inequality for any } w \in \gamma'^', |\theta - \theta_0 - 2\pi - \lambda(\rho - \rho_0)| < M + 2\pi. \]

Thus we arrive to the following theorem for FI(s)-systems of similarities on the plane.

**Theorem 4.5.** Let \( S = \{S_1, \ldots, S_m\} \) be a FI(s)-system of similarities in \( \mathbb{R}^2 \). If one of the addresses of a point \( p \in P \) is preperiodic, then Jordan arcs \( \gamma \subset K \) with the end point \( p \) have a slope parameter \( \lambda(p) \), which is the same for all these arcs.

**Proof:** Let \( j \bar{i} \) be the preperiodic address of the point \( p \). Then the point \( p' = S_{\bar{i}}^{-1}(p) \) is a fixed point of \( S_i \), so \( \pi^{-1}(p') \ni \bar{i} \).

There is \( l \neq j_1 \) such that \( p \in K_{j_l} \cap K_j \). Notice that there is a Jordan arc \( \gamma'' \subset (K_1 \setminus p) \subset \mathbb{C} \setminus K_l \) whose endpoint is \( p \). Therefore, \( p' \) lies on the boundary of some component \( A \) of \( \mathbb{C} \setminus K_j \) which contains \( S_{\bar{i}}^{-1}(\gamma'') \). By Proposition 4.4, all the Jordan arcs \( \gamma \subset K \) with endpoint \( p' \) have a slope parameter \( \lambda \), which implies that all Jordan arcs \( \gamma \in K \) which have the endpoint at \( p \) also have the slope parameter \( \lambda \). \( \blacksquare \)

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