Vibrations of thin piezoelectric shallow shells: Two-dimensional approximation

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Abstract. In this paper we consider the eigenvalue problem for piezoelectric shallow shells and we show that, as the thickness of the shell goes to zero, the eigensolutions of the three-dimensional piezoelectric shells converge to the eigensolutions of a two-dimensional eigenvalue problem.

Keywords. Vibrations; piezoelectricity; shallow shells.

1. Introduction

Lower dimensional models of shells are preferred in numerical computations to three-dimensional models when the thickness of the shells is ‘very small’. A lot of work has been done on the lower dimensional approximation of boundary value and eigenvalue problem for elastic plates and shells (cf. [2,3,4,5,6,8,9]). Recently some work has been done on the lower dimensional approximation of boundary value problem for piezoelectric shells (cf. [1]).

In this paper, we would like to study the limiting behaviour of the eigenvalue problems for thin piezoelectric shallow shells. We begin with a brief description of the problem and describe the results obtained.

Let \( \hat{\Omega}^\varepsilon = \Phi^\varepsilon (\Omega^\varepsilon) \), where \( \Phi^\varepsilon : \Omega^\varepsilon \rightarrow \mathbb{R}^3 \) is given by

\[
\Phi^\varepsilon(x^\varepsilon) = (x_1, x_2, \varepsilon \theta(x_1, x_2)) + x_3 a_3^\varepsilon(x_1, x_2)
\]

for all \( x^\varepsilon = (x_1, x_2, x_3^\varepsilon) \in \Omega^\varepsilon \), where \( \theta \) is an injective mapping of class \( C^3 \) and \( a_3^\varepsilon \) is a unit normal vector to the middle surface \( \Phi^\varepsilon(\partial \Omega^\varepsilon) \) of the shell. Let \( \gamma_0, \gamma_e \subset \partial \omega \) with \( \text{meas}(\gamma_0) > 0 \) and \( \text{meas}(\gamma_e) > 0 \). Let \( \hat{\Gamma}_0^\varepsilon = \Phi^\varepsilon(\gamma_0 \times (\varepsilon, \varepsilon)) \) and let \( \hat{\Gamma}_e^\varepsilon = \Phi^\varepsilon(\gamma_e \times (\varepsilon, \varepsilon)) \). The shell is clamped along the portion \( \hat{\Gamma}_0^\varepsilon \) of the lateral surface.

Then the variational form of the eigenvalue problem consists of finding the displacement vector \( u^\varepsilon \), the electric potential \( \phi^\varepsilon \) and \( \xi^\varepsilon \in \mathbb{R} \) satisfying eq. (2.21). We then show that the component of the eigenvector involving the electric potential \( \phi^\varepsilon \) can be uniquely determined in terms of the displacement vector \( u^\varepsilon \) and the problem thus reduces to finding \((u^\varepsilon, \xi^\varepsilon)\) satisfying equations (2.43) and (2.44).

After making appropriate scalings on the data and the unknowns, we transfer the problem to a domain \( \Omega = \omega \times (-1, 1) \) which is independent of \( \varepsilon \). Then we show that the scaled eigensolutions converge to the solutions of a two-dimensional eigenvalue problem (6.50).
2. The three-dimensional problem

Throughout this paper, Latin indices vary over the set \(\{1, 2, 3\}\) and Greek indices over the set \(\{1, 2\}\) for the components of vectors and tensors. The summation over repeated indices will be used.

Let \(\omega \subset \mathbb{R}^2\) be a bounded domain with a Lipschitz continuous boundary \(\gamma\) and let \(\omega\) lie locally on one side of \(\gamma\). Let \(\gamma_0, \gamma_e \subset \partial \omega\) with \(\text{meas}(\gamma_0) > 0\) and \(\text{meas}(\gamma_e) > 0\). Let \(\gamma_1 = \partial \omega \setminus \gamma_0\) and \(\gamma_2 = \partial \omega \setminus \gamma_e\). For each \(\varepsilon > 0\), we define the sets

\[
\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon), \quad \Gamma^\varepsilon = \gamma_0 \times (-\varepsilon, \varepsilon), \quad \Gamma^e_0 = \gamma_0 \times (-\varepsilon, \varepsilon), \quad \Gamma^e_\varepsilon = \gamma_e \times (-\varepsilon, \varepsilon),
\]

Let \(x^\varepsilon = (x_1, x_2, x_3^\varepsilon)\) be a generic point on \(\Omega^\varepsilon\) and let \(\partial_a = \partial_a^\varepsilon = \frac{\partial}{\partial x_a^\varepsilon}\) and \(\partial_a = \frac{\partial}{\partial x_a}\).

We assume that for each \(\varepsilon\), we are given a function \(\Theta^\varepsilon : \omega \to \mathbb{R}\) of class \(C^3\). We then define the map \(\Phi^\varepsilon : \omega \to \mathbb{R}^3\) by

\[
\phi^\varepsilon(x_1, x_2) = (x_1, x_2, \Theta^\varepsilon(x_1, x_2)) \quad \text{for all } (x_1, x_2) \in \omega. \quad (2.1)
\]

At each point of the surface \(S^\varepsilon = \Phi^\varepsilon(\omega)\), we define the normal vector

\[
a^\varepsilon = \left( |\partial_1 \Theta^\varepsilon|^2 + |\partial_2 \Theta^\varepsilon|^2 + 1 \right)^{-1/2} (\partial_1 \Theta^\varepsilon, -\partial_2 \Theta^\varepsilon, 1).
\]

For each \(\varepsilon > 0\), we define the mapping \(\Phi^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^3\) by

\[
\Phi^\varepsilon(x^\varepsilon) = \phi^\varepsilon(x_1, x_2) + x_3^\varepsilon a^\varepsilon(x_1, x_2) \quad \text{for all } x^\varepsilon \in \Omega^\varepsilon. \quad (2.2)
\]

It can be shown that there exists an \(\varepsilon_0 > 0\) such that the mappings \(\Phi^\varepsilon : \Omega^\varepsilon \to \Phi^\varepsilon(\Omega^\varepsilon)\) are \(C^1\) diffeomorphisms for all \(0 < \varepsilon \leq \varepsilon_0\). The set \(\hat{\Omega}^\varepsilon = \Phi^\varepsilon(\Omega^\varepsilon)\) is the reference configuration of the shell. For \(0 < \varepsilon \leq \varepsilon_0\), we define the sets

\[
\hat{\Gamma}^\varepsilon = \Phi^\varepsilon(\Gamma^\varepsilon) = \Phi^\varepsilon(\Gamma^e_0), \quad \hat{\Gamma}^\varepsilon = \Phi^\varepsilon(\Gamma^e_\varepsilon), \quad \hat{\Gamma}^\varepsilon = \Phi^\varepsilon(\Gamma^e_s), \quad \hat{\Gamma}^\varepsilon = \Phi^\varepsilon(\Gamma^e_{\varepsilon D})
\]

and we define vectors \(g_i^\varepsilon\) and \(g^{i, \varepsilon}\) by the relations

\[
g_i^\varepsilon = \partial_i \Phi^\varepsilon \quad \text{and} \quad g^{j, \varepsilon} \cdot g_i^\varepsilon = \delta_i^j
\]

which form the covariant and contravariant basis respectively of the tangent plane of \(\Phi^\varepsilon(\hat{\Omega}^\varepsilon)\) at \(\Phi^\varepsilon(x^\varepsilon)\). The covariant and contravariant metric tensors are given respectively by

\[
g_{ij}^\varepsilon = g_i^\varepsilon \cdot g_j^\varepsilon \quad \text{and} \quad g^{ij, \varepsilon} = g^{j, \varepsilon} \cdot g^{i, \varepsilon}.
\]

The Christoffel symbols are defined by

\[
\Gamma_{ij}^{\varepsilon} = g^{\varepsilon, \varepsilon} \cdot \partial_j g_i^\varepsilon.
\]

Note however that when the set \(\Omega^\varepsilon\) is of the special form \(\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)\) and the mapping \(\Phi^\varepsilon\) is of the form \((2.2)\), the following relations hold:

\[
\Gamma_{a3}^{\varepsilon} = \Gamma_{33}^{\varepsilon} = 0.
\]
The volume element is given by \( \sqrt{g^e} \mathrm{d}x^e \) where
\[
g^e = \det(g^e_{ij}).
\]

It can be shown that there exist constants \( g_1 \) and \( g_2 \) such that
\[
0 < g_1 \leq g^e \leq g_2
\]
for \( 0 \leq \varepsilon \leq \varepsilon_0 \).

Let \( \hat{A}^{ijkl,e}, \hat{P}^{ijkl,e} \) and \( \hat{\sigma}^{ijkl} \) be the elastic, piezoelectric and dielectric tensors respectively. We assume that the material of the shell is homogeneous and isotropic. Then the elasticity tensor is given by
\[
\hat{A}^{ijkl,e} = \lambda g^{ij} g^{kl} + \mu (g^{ik} g^{jl} + g^{il} g^{jk}),
\]
where \( \lambda \) and \( \mu \) are the Lamé constants of the material.

These tensors satisfy the following coercive relations. There exists a constant \( C > 0 \) such that for all symmetric tensors \( (M_{ij}) \) and for any vector \( (i_j) \in \mathbb{R}^3 \),
\[
\hat{A}^{ijkl,e} M_{kl} M_{ij} \geq C \sum_{i,j=1}^3 (M_{ij})^2, \\
\hat{\sigma}^{ijkl} I_{ij} \geq C \sum_{j=1}^3 i_j^2.
\]

Moreover we have the symmetries
\[
\hat{A}^{ijkl,e} = \hat{A}^{klji,e} = \hat{A}^{ijkl,e}, \quad \hat{\sigma}^{ijkl} = \hat{\sigma}^{klji} = \hat{\sigma}^{ijkl}, \quad \hat{P}^{ijkl} = \hat{P}^{klji} = \hat{P}^{ijkl}.
\]

Then the eigenvalue problem consists of finding \( (\hat{u}^e, \hat{\phi}^e, \xi^e) \) such that
\[
\begin{aligned}
-\text{div} \hat{\sigma}^e(\hat{u}^e, \hat{\phi}^e) &= \xi^e \hat{u}^e \quad \text{in} \ \hat{\Omega}^e \\
\hat{\sigma}^e(\hat{u}^e, \hat{\phi}^e) v &= 0 \quad \text{on} \ \hat{\Gamma}_N^e \\
\hat{u}^e &= 0 \quad \text{on} \ \hat{\Gamma}_0^e,
\end{aligned}
\] (2.7)
\[
\begin{aligned}
\text{div} \hat{D}^e(\hat{u}^e, \hat{\phi}^e) &= 0 \quad \text{in} \ \hat{\Omega}^e \\
\hat{D}^e(\hat{u}^e, \hat{\phi}^e) v &= 0 \quad \text{on} \ \hat{\Gamma}_r^e \\
\hat{\phi}^e &= 0 \quad \text{on} \ \hat{\Gamma}_{ed}^e,
\end{aligned}
\] (2.8)

where
\[
\hat{\sigma}^e_{ij} = \hat{A}^{ijkl,e} \epsilon^e_{ij} - \hat{P}^{ijkl,e} \hat{E}_k,
\]
\[
\hat{D}_k = \hat{P}^{ijkl,e} \epsilon^e_{ij} + \hat{\sigma}^{ijkl,e} E_l,
\]
\[
\hat{E}_k = \frac{1}{\varepsilon_0} (\hat{\sigma}^e_{ij} \hat{u}^e_i + \hat{\sigma}^e_{ij} \hat{u}^e_j), \quad \hat{\sigma}^e_{ij} = \partial_j \hat{\phi}^e \quad \text{and} \quad \hat{E}_k(\hat{\phi}^e) = - \gamma(\hat{\phi}^e).
\]

We define the spaces
\[
\hat{V}^e = \{ \hat{v} \in (H^1(\hat{\Omega}^e))^3 : \hat{v}|_{\hat{\Gamma}_0^e} = 0 \},
\]
\[
\hat{\Psi}^e = \{ \hat{\psi} \in H^1(\hat{\Omega}^e) : \hat{\psi}|_{\hat{\Gamma}_{ed}^e} = 0 \}.
\] (2.11) (2.12)
Then the variational form of systems (2.7) and (2.8) is to find \((\hat{u}^E, \hat{\psi}^E, \xi^E) \in \hat{V}^E \times \hat{\Psi}^E \times \mathbb{R}\) such that
\[
\tilde{a}^E((\hat{u}^E, \hat{\psi}^E), (v^E, \psi^E)) = \xi^E \tilde{F}(v^E, \psi^E) \quad \text{for all } (v^E, \psi^E) \in \hat{V}^E \times \hat{\Psi}^E, \tag{2.13}
\]
where
\[
\tilde{a}^E((\hat{u}^E, \hat{\psi}^E), (v^E, \psi^E)) = \int_{\Omega^E} \hat{A}^{ijkl,E} e^E_{ijkl}(\hat{u}^E) e^E_{ijkl}(\psi^E) \, dx^E + \int_{\Omega^E} \hat{B}^{ij,E} \hat{\psi}^E \hat{\psi}^E \, dx^E + \int_{\Omega^E} \hat{P}^{mij,E} (\hat{\psi}^E e^E_{ijkl}(\hat{u}^E)) \, dx^E,
\]
\[
\hat{F}(v^E, \psi^E) = \int_{\Omega^E} \hat{u}^E \cdot \hat{v}^E \, dx^E. \tag{2.14}
\]

Since the mappings \(\Phi^E : \Omega^E \rightarrow \hat{\Omega}^E\) are assumed to be \(C^1\) diffeomorphisms, the correspondences that associate with every element \(\hat{v}^E \in \hat{V}^E\), the vector
\[
v^E = \hat{v}^E \cdot \Phi^E : \Omega^E \rightarrow \mathbb{R}^3
\]
and with every element \(\psi^E \in \hat{\Psi}^E\), the function
\[
\psi^E = \hat{\psi}^E \cdot \Phi^E : \Omega^E \rightarrow \mathbb{R}
\]
induce bijections between the spaces \(\hat{V}^E\) and \(V^E\), and the spaces \(\hat{\Psi}^E\) and \(\Psi^E\) respectively, where
\[
V^E = \{v^E \in (H^1(\Omega^E))^3 | v^E = 0 \text{ on } \Gamma^E_0\}, \tag{2.16}
\]
\[
\Psi^E = \{\psi^E \in H^1(\Omega^E) | \psi^E = 0 \text{ on } \Gamma^E_\partial\}. \tag{2.17}
\]

Then we have
\[
\hat{\psi}^E(\hat{v}^E) = (\partial^E_{v^E}(v^E^i))_j, \tag{2.18}
\]
\[
\hat{\psi}^E(\hat{v}^E) = (\hat{e}^E_{ijkl}(v^E)^i(g^E)^j)_j, \tag{2.19}
\]
where
\[
e^E_{ijkl}(v^E)^i = \frac{1}{2} (\partial^E_{v^E} v^E_j + \partial^E_{v^E} v^E_i) - \Gamma^E_{ij} v^E_p. \tag{2.20}
\]

Then the variational form (2.13) posed on the domain \(\Omega^E\) is to find \((u^E, \varphi^E, \xi^E) \in V^E \times \Psi^E \times \mathbb{R}\) such that
\[
a^E((u^E, \varphi^E), (v^E, \psi^E)) = \xi^E F(v^E, \psi^E) \quad \text{for all } (v^E, \psi^E) \in V^E \times \Psi^E, \tag{2.21}
\]
where
\[
a^E((u^E, \varphi^E), (v^E, \psi^E)) = \int_{\Omega^E} A^{ijkl,E} e^E_{ijkl}(u^E) e^E_{ijkl}(\psi^E) \sqrt{g^E} \, dx^E + \int_{\Omega^E} B^{ij,E} \varphi^E \varphi^E \sqrt{g^E} \, dx^E + \int_{\Omega^E} P^{mij,E} (\varphi^E e^E_{ijkl}(u^E)) \sqrt{g^E} \, dx^E,
\]
\[
- \partial^E_m \psi^E e^E_{ijkl}(u^E) \sqrt{g^E} \, dx^E. \tag{2.22}
\]
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\[ l^E(\psi^E, \phi^E) = \int_{\Omega^E} u^E \cdot v^E \sqrt{g^E} \, dx, \quad (2.23) \]

\[ A^{ijkl,E} = A^{ijkl}(g^{E}, e_{ij}) \cdot (g^{E})_k \cdot (g^{E})_l, \quad (2.24) \]

\[ \theta_{pq,E} = \theta_{pq}(g^{E}, (g^{E})_j \cdot (g^{E})_i), \quad (2.25) \]

\[ p^{pq,E} = p^{pq}(g^{E}, (g^{E})_i \cdot (g^{E})_j). \quad (2.26) \]

Using the relations (2.3), (2.5) and (2.6), it can be shown that there exists a constant \( C > 0 \) such that for all symmetric tensor \( (M_{ij}) \) and for any vector \( (t_i) \in \mathbb{R}^3 \),

\[ A_{ijkl} M_{kl} M_{ij} \geq C \sum_{i,j=1}^{3} (M_{ij})^2, \quad (2.27) \]

\[ \theta_{ij,E} t_{ij} \geq C \sum_{i=1}^{3} t_{ii}^2. \quad (2.28) \]

Clearly the bilinear form associated with the left-hand side of (2.24) is elliptic. Hence by Lax–Milgram theorem, given \( f^E \in V^E \) and \( h^E \in \Psi^E \), there exists a unique \( (u^E, \phi^E) \in V^E \times \Psi^E \) such that

\[ a^E((u^E, \phi^E), (v^E, \psi^E)) = \langle (f^E, h^E), (v^E, \psi^E) \rangle \quad \forall (v^E, \psi^E) \in V^E \times \Psi^E. \quad (2.29) \]

In particular, for each \( f^E \in (L^2(\Omega^E))^3 \), there exists a unique solution \( (u^E, \phi^E) \in V^E \times \Psi^E \) such that

\[ a^E((u^E, \phi^E), (v^E, \psi^E)) = \int_{\Omega^E} f^E v^E \sqrt{g^E} \, dx \quad \forall (v^E, \psi^E) \in V^E \times \Psi^E. \quad (2.30) \]

This is equivalent to the following equations.

\[
\int_{\Omega^E} A^{ijkl,E} e_{ij}(u^E) e_{lj}(v^E) \sqrt{g^E} \, dx + \int_{\Omega^E} p^{mn,E} e_{mn}(\phi^E) e_{lj}(v^E) \sqrt{g^E} \, dx \\
= \int_{\Omega^E} f^E v^E \sqrt{g^E} \, dx \quad \forall v^E \in V^E
\]

and

\[
\int_{\Omega^E} \theta_{ij,E} \psi^E \partial^E \psi^E \sqrt{g^E} \, dx = \int_{\Omega^E} p^{mn,E} e_{ij}(u^E) \sqrt{g^E} \, dx \quad \forall \psi^E \in \Psi^E.
\]

From relation (2.28), it follows that the bilinear form associated with the left-hand side of (2.32) is \( \Psi^E \)-elliptic.

Also for each \( h^E \in V^E \), the mapping

\[ \psi^E \rightarrow \int_{\Omega^E} p^{mn,E} e_{ij}(h^E) \sqrt{g^E} \, dx \]

defines a linear functional on \( \Psi^E \). Hence for each \( h^E \in V^E \), there exists a unique \( T^E(h^E) \in \Psi^E \) such that

\[
\int_{\Omega^E} \theta_{ij,E} \partial^E T^E(h^E) \partial^E \psi^E \sqrt{g^E} \, dx = \int_{\Omega^E} p^{mn,E} e_{ij}(h^E) \sqrt{g^E} \, dx \quad \forall \psi^E \in \Psi^E
\]
Lemma 2.1. For each \( h^e \in (L^2(\Omega^e))^3 \), there exists a unique \( G^e(h^e) \in V^e \) such that

\[
\int_{\Omega^e} A^{ijkl}_e e^e_{kli}(u^e) e^e_{jil}(v^e) \sqrt{g^e} \, dx^e + \int_{\Omega^e} B^{ijkl}_e \partial^e_m(T^e(u^e)) e^e_{kli}(v^e) \sqrt{g^e} \, dx^e = \int_{\Omega^e} h^e v^e \sqrt{g^e} \, dx^e \quad \forall v^e \in V^e \quad (2.36)
\]

and that \( G^e : (L^2(\Omega^e))^3 \to V^e \) is continuous.

Proof. Let \( B^e(u^e, v^e) \) denotes the bilinear form associated with the left-hand side of eq. (2.34). Using (2.35), we have

\[
B^e(u^e, v^e) = \int_{\Omega^e} A^{ijkl}_e e^e_{kli}(u^e) e^e_{jil}(v^e) \sqrt{g^e} \, dx^e
+ \int_{\Omega^e} B^{ijkl}_e \partial^e_m(T^e(u^e)) e^e_{kli}(v^e) \sqrt{g^e} \, dx^e
= \int_{\Omega^e} A^{ijkl}_e e^e_{kli}(u^e) e^e_{jil}(v^e) \sqrt{g^e} \, dx^e
+ \int_{\Omega^e} \delta^{ijkl}_e \partial^e_j(T^e(u^e)) \partial^e_j(T^e(v^e)) \sqrt{g^e} \, dx^e
= B^e(v^e, u^e). \quad (2.37)
\]
Also, using (2.35) and the relations (2.27) and (2.28), we have

\[
B^\varepsilon(u^\varepsilon, u^\varepsilon) = \int_{\Omega^e} A^{ijkl, e}_{k||l}(e^\varepsilon_{i||j} e^\varepsilon_{i||j}(u^\varepsilon)) \sqrt{g^\varepsilon} \, dx^\varepsilon \\
+ \int_{\Omega^e} p^{mij, e}_{m||n}(T^e(u^\varepsilon)) e^\varepsilon_{i||j}(u^\varepsilon) \sqrt{g^\varepsilon} \, dx^\varepsilon \\
= \int_{\Omega^e} A^{ijkl, e}_{k||l}(e^\varepsilon_{i||j}(u^\varepsilon)) \sqrt{g^\varepsilon} \, dx^\varepsilon \\
+ \int_{\Omega^e} \varepsilon^{ij, e}_{i||j} \partial_i^e(T^e(u^\varepsilon)) \partial^i_j(T^e(u^\varepsilon)) \sqrt{g^\varepsilon} \, dx^\varepsilon \\
\geq C\|u^\varepsilon\|^2_{V^e}. \tag{2.38}
\]

Hence \(B^\varepsilon(\cdot, \cdot)\) is symmetric and \(V^e\)-elliptic. Hence by Lax–Milgram theorem, there exists a unique \(G^e(h^\varepsilon)\) satisfying (2.36). Letting \(v^e = G^e(h^\varepsilon)\) in (2.36), we get

\[
\int_{\Omega^e} A^{ijkl, e}_{k||l}(G^e(h^\varepsilon)) e^\varepsilon_{i||j}(G^e(h^\varepsilon)) \sqrt{g^\varepsilon} \, dx^\varepsilon \\
+ \int_{\Omega^e} p^{mij, e}_{m||n}(G^e(h^\varepsilon)) e^\varepsilon_{i||j}(G^e(h^\varepsilon)) \sqrt{g^\varepsilon} \, dx^\varepsilon \\
= \int_{\Omega^e} h^\varepsilon G^e(h^\varepsilon) \sqrt{g^\varepsilon} \, dx^\varepsilon. \tag{2.39}
\]

Using (2.35), it becomes

\[
\int_{\Omega^e} A^{ijkl, e}_{k||l}(G^e(h^\varepsilon)) e^\varepsilon_{i||j}(G^e(h^\varepsilon)) \sqrt{g^\varepsilon} \, dx^\varepsilon \\
+ \int_{\Omega^e} \varepsilon^{ij, e}_{i||j} \partial_i^e(G^e(h^\varepsilon)) \partial^i_j(G^e(h^\varepsilon)) \sqrt{g^\varepsilon} \, dx^\varepsilon \\
= \int_{\Omega^e} h^\varepsilon G^e(h^\varepsilon) \sqrt{g^\varepsilon} \, dx^\varepsilon. \tag{2.40}
\]

Using the relations (2.27) and (2.28), we have

\[
\|G^e(h^\varepsilon)\|^2_{V^e} \leq C^e\|G^e(h^\varepsilon)\|_{V^e}\|h^\varepsilon\|_{(L^2(\Omega^e))^3}. \tag{2.41}
\]

Hence

\[
\|G^e(h^\varepsilon)\|_{V^e} \leq C^e\|h^\varepsilon\|_{(L^2(\Omega^e))^3} \tag{2.42}
\]

which implies that \(G^e\) is continuous.

It follows from (2.27) and the above lemma that \(u^\varepsilon = G^e(f^\varepsilon)\). Since the inclusion \((H^1(\Omega^e))^3 \rightarrow (L^2(\Omega^e))^3\) is compact, it follows that \(G^e : (L^2(\Omega^e))^3 \rightarrow (L^2(\Omega^e))^3\) is compact. Also since the bilinear form \(B^\varepsilon(\cdot, \cdot)\) is symmetric, it follows that \(G^e\) is self-adjoint. Hence from the spectral theory of compact, self-adjoint operators, it follows that there
exists a sequence of eigenpairs \((u^{m,e}, \xi^{m,e})\)\(^{\infty}_{m=1}\) such that
\[
\begin{align*}
\int_{\Omega^e} A_{ijlk}^e & \epsilon_{ijl}^e (u^{m,e})^e_{ij} (v^e) \sqrt{g^e} \, dx^e + \int_{\Omega^e} p_{mij}^e \partial_m^e (T^e (u^{m,e})) e_{ij}^e (v^e) \sqrt{g^e} \, dx^e \\
& = \xi^{m,e} \int_{\Omega^e} u^{m,e} \sqrt{g^e} \, dx^e \quad \forall \psi^e \in V^e, \\
& = \xi^{m,e} \int_{\Omega^e} u^{m,e} \sqrt{g^e} \, dx^e \quad \forall \psi^e \in V^e,
\end{align*}
\] (2.43)

The sequence \((u^{m,e})\) forms a complete orthonormal basis for \((L^2(\Omega))^3\).

Define the Rayleigh quotient \(R(\epsilon)(v^e)\) for \(v^e \in V^e\) by
\[
R^e (v^e) = \frac{\int_{\Omega^e} A_{ijlk}^e \epsilon_{ijl}^e (v^e) e_{ij}^e (v^e) \sqrt{g^e} \, dx^e + \int_{\Omega^e} p_{mij}^e \partial_m^e (T^e (v^e)) e_{ij}^e (v^e) \sqrt{g^e} \, dx^e}{\int_{\Omega^e} v^e_i \sqrt{g^e} \, dx^e}.
\] (2.47)

Then
\[
\xi^{m,e} = \min_{W^e \in W^e_m} \max_{v^e \in W^e \setminus \{0\}} R^e (v^e),
\] (2.48)

where \(W^e_m\) denotes the collection of all \(m\)-dimensional subspaces of \(V^e\).

3. The scaled problem

We now perform a change of variable so that the domain no longer depends on \(\epsilon\). With \(x = (x_1, x_2, x_3) \in \Omega\), we associate \(x^e = (x_1, x_2, \epsilon x_3) \in \Omega^e\). Let
\[
\begin{align*}
\Gamma_0 &= \gamma_0 \times (-1, 1), \quad \Gamma_1 = \gamma_1 \times (-1, 1), \quad \Gamma^\pm = \omega \times \{\pm 1\}, \\
\Gamma_\epsilon &= \gamma_\epsilon \times (-1, 1), \quad \Gamma_s = \gamma_s \times (-1, 1), \\
\Gamma_N &= \Gamma_1 \cup \Gamma^+ \cup \Gamma^-, \quad \Gamma_{\epsilon D} = \Gamma^+ \cup \Gamma^- \cup \Gamma_\epsilon.
\end{align*}
\]

With the functions \(\Gamma^p(\epsilon), g^e, A_{ijkl}^e, p_{ijkl}^e, \sigma_{ijkl}^e : \Omega^e \to \mathbb{R}\), we associate the functions \(\Gamma^p(\epsilon), g^e, A_{ijkl}^e, p_{ijkl}^e, \sigma_{ijkl}^e : \Omega \to \mathbb{R}\) defined by
\[
\begin{align*}
\Gamma^p(\epsilon)(x) &:= \Gamma^p(\epsilon)(x^e), \quad g(\epsilon)(x) = g^e(x^e), \quad A_{ijkl}^e(\epsilon)(x) = A_{ijkl}^e(x^e), \\
p_{ijkl}^e(\epsilon)(x) &:= p_{ijkl}^e(x^e), \quad \sigma_{ijkl}^e(\epsilon)(x) = \sigma_{ijkl}^e(x^e).
\end{align*}
\] (3.1)
Assumption. We assume that the shell is a shallow shell, i.e. there exists a function $\theta \in C^3(\omega)$ such that
\[
\phi^\varepsilon(x_1, x_2) = (x_1, x_2, \varepsilon \theta(x_1, x_2)) \quad \text{for all} \quad (x_1, x_2) \in \omega, \tag{3.3}
\]
i.e., the curvature of the shell is of the order of the thickness of the shell.

We make the following scalings on the eigensolutions.
\[
\begin{align*}
\psi^m_{\alpha}(x^\varepsilon) &= \varepsilon^2 \psi^m_{\alpha}(x), \\
\psi^m_3(x^\varepsilon) &= \varepsilon^2 \psi^m_3(x),
\end{align*}
\]
\[
T^\varepsilon(u^m_{\alpha}(x^\varepsilon)) = \varepsilon^3 T(\varepsilon)(u^m_{\alpha}(x)),
\]
\[
\delta^m_{\alpha} = \varepsilon^2 \delta^m_{\alpha}(\varepsilon).
\]

With the tensors $e^{ij}_{ij}(\varepsilon)$, we associate the tensors $e^{ij}_{ij}(\varepsilon)$ through the relation
\[
e^{ij}_{ij}(\varepsilon) = \varepsilon^2 e^{ij}_{ij}(\varepsilon, \psi(x)). \tag{3.8}
\]

We define the spaces
\[
\begin{align*}
V(\Omega) &= \{ \psi \in H^1(\Omega), \psi|_{\Gamma_0} = 0 \}, \\
\Psi(\Omega) &= \{ \psi \in H^1(\Omega), \psi|_{\Gamma_{x0}} = 0 \}. \tag{3.9-3.10}
\end{align*}
\]

We denote $\phi^m(\varepsilon) = T(\varepsilon)(u^m(\varepsilon))$. Then the variational equations (eqs 2.43–2.46) become
\[
\int_\Omega A^{ijkl}(\varepsilon) e_{k||j}(\varepsilon, u^m(\varepsilon)) e_{i||j}(\varepsilon, \psi) \sqrt{\varepsilon} \, dx \\
+ \int_\Omega P^{3kl} \partial_3 \phi^m(\varepsilon) e_{k||j}(\varepsilon, \psi) \sqrt{\varepsilon} \, dx \\
+ \varepsilon \int_\Omega P^{aikl}(\varepsilon) \partial_3 \phi^m(\varepsilon) e_{k||j}(\varepsilon, \psi) \sqrt{\varepsilon} \, dx \\
= \delta^m_{\alpha} \int_\Omega \left[ \varepsilon^2 u^m_{\alpha}(\varepsilon) v_{\alpha} + u^m_3(\varepsilon) v_3 \right] \sqrt{\varepsilon} \, dx \quad \text{for all} \quad \psi \in V(\Omega). \tag{3.11}
\]
\[
\int_\Omega \delta^{33}(\varepsilon) \partial_3 \phi^m(\varepsilon) \partial_3 \psi \sqrt{\varepsilon} \, dx \\
+ \varepsilon \int_\Omega \left[ \delta^{3a}(\varepsilon) (\partial_3 \phi^m(\varepsilon) \partial_3 \psi + \partial_3 \phi^m(\varepsilon) \partial_\alpha \psi) \right] \sqrt{\varepsilon} \, dx \\
+ \varepsilon^2 \int_\Omega \delta^{a3}(\varepsilon) \partial_3 \phi^m(\varepsilon) \partial_\beta \psi \sqrt{\varepsilon} \, dx \\
= \int_\Omega P^{3kl}(\varepsilon) \partial_3 \psi e_{k||j}(\varepsilon, u^m(\varepsilon)) \sqrt{\varepsilon} \, dx \\
+ \varepsilon \int_\Omega \left[ P^{aikl}(\varepsilon) \partial_3 \psi e_{k||j}(\varepsilon, u^m(\varepsilon)) \right] \sqrt{\varepsilon} \, dx \quad \text{for all} \quad \psi \in \Psi(\Omega), \tag{3.12}
\]
\[
\int_\Omega \left[ \varepsilon^2 u^m_{\alpha}(\varepsilon) u^m_{\alpha}(\varepsilon) + u^m_3(\varepsilon) u^m_3(\varepsilon) \right] \sqrt{\varepsilon} \, dx = \delta_{mn}. \tag{3.13}
\]
4. Technical preliminaries

The following two lemmas are crucial; they play an important role in the proof of the convergence of the scaled unknowns as $\varepsilon \to 0$. In the sequel, we denote by $C_1, C_2, \ldots, C_n$ various constants whose values do not depend on $\varepsilon$ but may depend on $\theta$.

**Lemma 4.1.** The functions $e_{\alpha|\beta}(\varepsilon; v)$ defined in (3.8) are of the form

$$
e_{\alpha|\beta}(\varepsilon; v) = \varepsilon_{\alpha\beta}(v) + \varepsilon^2 \tilde{e}_{\alpha\beta}(\varepsilon; v), \quad (4.1)$$

$$e_{\alpha|3}(\varepsilon; v) = \frac{1}{\varepsilon} \{ \tilde{e}_{\alpha3}(v) + \varepsilon^2 \tilde{e}_{\alpha|3}(\varepsilon; v) \}, \quad (4.2)$$

$$e_{3|3}(\varepsilon; v) = \frac{1}{\varepsilon^2} \tilde{e}_{33}(v), \quad (4.3)$$

where

$$\tilde{e}_{\alpha\beta}(v) = \frac{1}{2} (\partial_{\alpha} v_{\beta} + \partial_{\beta} v_{\alpha} - v_3 \partial_{\alpha\beta} \theta), \quad (4.4)$$

$$\tilde{e}_{\alpha3}(v) = \frac{1}{2} (\partial_{\alpha} v_3 + \partial_3 v_{\alpha}), \quad (4.5)$$

$$\tilde{e}_{33}(v) = \partial_3 v_3 \quad (4.6)$$

and there exists constant $C_1$ such that

$$\sup_{0 < \varepsilon \leq \varepsilon_0} \max_{\alpha, j} ||e_{\alpha|j}(\varepsilon; v)||_{0, \Omega} \leq C_1 ||v||_{1, \Omega} \quad \text{for all } v \in V. \quad (4.7)$$

Also there exist constants $C_2, C_3$ and $C_4$ such that

$$\sup_{0 < \varepsilon \leq \varepsilon_0} \max_{x \in \Omega} |g(x) - 1| \leq C_2 \varepsilon^2, \quad (4.8)$$

$$\sup_{0 < \varepsilon \leq \varepsilon_0} \max_{x \in \Omega} |A^{ijkl}(\varepsilon) - A^{ijkl}| \leq C_3 \varepsilon^2, \quad (4.9)$$

where

$$A^{ijkl} = \lambda \delta^{ij} \delta^{kl} + \mu (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) \quad (4.10)$$

and

$$A^{ijkl} M_{kl} M_{ij} \geq C_4 M_{ij} M_{ij} \quad (4.11)$$

for $0 < \varepsilon \leq \varepsilon_0$ and for all symmetric tensors $(M_{ij})$.

**Proof.** The proof is based on Lemma 4.1 of [2].

From relation (2.6) and definition (3.2), it follows that there exists a constant $C_5$ such that for any vector $(t_i) \in \mathbb{R}^3$,

$$\varepsilon^{ij}(\varepsilon) t_i t_j \geq C_5 \sum_{j=1}^{3} t_j^2. \quad (4.12)$$
We assume that there exists functions $P_{kij}$ and $E_{ij}$ such that

\[
\sup_{0<\varepsilon \leq \varepsilon_0} \max_{x \in \Omega} |P_{kij}(\varepsilon) - P_{kij}| \leq C_6 \varepsilon, \tag{4.13}
\]

\[
\sup_{0<\varepsilon \leq \varepsilon_0} \max_{x \in \Omega} |E_{ij}(\varepsilon) - E_{ij}| \leq C_7 \varepsilon. \tag{4.14}
\]

**Lemma 4.2.** Let $\theta \in C^3(\omega)$ be a given function and let the functions $\tilde{e}_{ij}$ be defined as in (4.4)–(4.6). Then there exists a constant $C_8$ such that the following generalised Korn’s inequality holds:

\[
\|v\|_{1,\Omega} \leq C_8 \left\{ \sum_{i,j} \|\tilde{e}_{ij}(v)\|_{0,\Omega}^2 \right\}^{1/2} \tag{4.15}
\]

for all $v \in V(\Omega)$ where $V(\Omega)$ is the space defined in (3.9).

**Proof.** The proof is based on Lemma 4.2 of [2].

**5. A priori estimates**

In this section, we show that for each positive integer $m$, the scaled eigenvalues $\{\xi^m(\varepsilon)\}$ are bounded uniformly with respect to $\varepsilon$.

Let $\phi \in H_0^2(\omega)$. Then

\[
v_{\phi} := (-x_3 \partial_1 \phi, -x_3 \partial_2 \phi, \phi) \in V(\Omega) \tag{5.1}
\]

and

\[
\tilde{e}_{\alpha\beta}(v_{\phi}) = -x_3 \partial_{\alpha\beta} \phi - \phi \partial_{\alpha\beta} \theta, \quad \tilde{e}_{33}(v_{\phi}) = 0. \tag{5.2}
\]

Hence

\[
e_{\alpha\beta}(\varepsilon, v_{\phi}) = -x_3 \partial_{\alpha\beta} \phi - \phi \partial_{\alpha\beta} \theta + O(\varepsilon^2), \tag{5.3}
\]

\[
e_{\alpha3}(\varepsilon, v_{\phi}) = O(\varepsilon), \tag{5.4}
\]

\[
e_{33}(\varepsilon, v_{\phi}) = 0. \tag{5.5}
\]

We need the following lemma to prove the boundedness of the scaled eigenvalues.

**Lemma 5.1.** There exists a constant $C_9 > 0$ such that

\[
|\partial_3(T(\varepsilon)(v_{\phi}))|_{0,\Omega} \leq C_9 |\phi|_{2,\omega}, \tag{5.6}
\]

\[
|\varepsilon \partial_3(T(\varepsilon)(v_{\phi}))|_{0,\Omega} \leq C_9 |\phi|_{2,\omega}. \tag{5.7}
\]

**Proof.** With the scalings (3.3)–(3.7), the variational equation (eq. (2.33)) posed on the domain $\Omega$ reads as follows:
For each $h \in (H^1(\Omega))^3$, there exists a unique solution $T(\varepsilon)(h) \in (H^1(\Omega))^3$ such that

\[
\int_{\Omega} \varepsilon^{33}(\varepsilon) \partial_3 T(\varepsilon)(h) \partial_3 \psi \sqrt{g(\varepsilon)} d\mathbf{x} \\
+ \varepsilon \int_{\Omega} [\varepsilon^{33}(\varepsilon)(\partial_\alpha T(\varepsilon)(h) \partial_3 \psi + \partial_3 T(\varepsilon)(h) \partial_\alpha \psi)] \sqrt{g(\varepsilon)} d\mathbf{x} \\
+ \varepsilon^2 \int_{\Omega} \varepsilon^{\alpha \beta}(\varepsilon) \partial_\alpha T(\varepsilon)(h) \partial_\beta \psi \sqrt{g(\varepsilon)} d\mathbf{x} \\
= \int_{\Omega} P^{3k}(\varepsilon) \partial_3 \psi e_{k||}(\varepsilon, h) \sqrt{g(\varepsilon)} d\mathbf{x} \\
+ \varepsilon \int_{\Omega} P^{\alpha k}(\varepsilon) \partial_\alpha \psi e_{k||}(\varepsilon, h) \sqrt{g(\varepsilon)} d\mathbf{x} \quad \forall \psi \in \Psi.
\]

(5.8)

Taking $h = v_\varphi$ and $\psi = T(\varepsilon)(v_\varphi)$ in the above equation, we have

\[
\int_{\Omega} \varepsilon^{33}(\varepsilon) \partial_3 T(\varepsilon)(v_\varphi) \partial_3 T(\varepsilon)(v_\varphi) \sqrt{g(\varepsilon)} d\mathbf{x} \\
+ \varepsilon \int_{\Omega} [\varepsilon^{33}(\varepsilon)(\partial_\alpha T(\varepsilon)(v_\varphi) \partial_3 T(\varepsilon)(v_\varphi)) \\
+ \partial_3 T(\varepsilon)(v_\varphi) \partial_\alpha T(\varepsilon)(v_\varphi)] \sqrt{g(\varepsilon)} d\mathbf{x} \\
+ \varepsilon^2 \int_{\Omega} \varepsilon^{\alpha \beta}(\varepsilon) \partial_\alpha T(\varepsilon)(v_\varphi) \partial_\beta T(\varepsilon)(v_\varphi) \sqrt{g(\varepsilon)} d\mathbf{x} \\
= \int_{\Omega} P^{3k}(\varepsilon) \partial_3 T(\varepsilon)(v_\varphi) e_{k||}(\varepsilon, v_\varphi) \sqrt{g(\varepsilon)} d\mathbf{x} \\
+ \varepsilon \int_{\Omega} P^{\alpha k}(\varepsilon) \partial_\alpha T(\varepsilon)(v_\varphi) e_{k||}(\varepsilon, v_\varphi) \sqrt{g(\varepsilon)} d\mathbf{x}.
\]

(5.9)

Using the relations (4.12) and (5.2)–(5.5), it follows that there exists a constant $C_0 > 0$ such that

\[
|\partial_3 T(\varepsilon)(v_\varphi)|_{0, \Omega}^2 + |\varepsilon \partial_\alpha T(\varepsilon)(v_\varphi)|_{0, \Omega}^2 \\
\leq C_0 \{ |\partial_3 T(\varepsilon)(v_\varphi)|_{0, \Omega} \varphi_{|2, \omega} + |\varepsilon \partial_\alpha T(\varepsilon)(v_\varphi)|_{0, \Omega} \varphi_{|2, \omega} \}
\]

(5.10)

and hence the result follows.

**Theorem 5.2.** For each positive integer $m$, there exists a constant $C(m) > 0$ such that

\[
\xi^m(\varepsilon) \leq C(m).
\]

(5.11)

**Proof.** Since problem (3.11) was derived from (2.43) after a change of scale, we still have the variational characterization of the scaled eigenvalues $\xi^m(\varepsilon)$. Let $V_m$ denote the collection of all $m$-dimensional subspaces of $V(\Omega)$. Then

\[
\xi^m(\varepsilon) = \min_{W \in V_m} \max_{v \in W} \frac{N(\varepsilon)(v, v)}{D(\varepsilon)(v, v)}.
\]

(5.12)
where

\[
N(\varepsilon)(v, v) = \int_{\Omega} A^{ijkl} e_{ijkl}(\varepsilon, v) e_{ijkl}(\varepsilon, v) \sqrt{g(\varepsilon)} \, dx \\
+ \int_{\Omega} P^{ijkl} \partial_3 T(\varepsilon)(v) e_{ijkl}(\varepsilon, v) \sqrt{g(\varepsilon)} \, dx \\
+ \varepsilon \int_{\Omega} P^{ijkl} \partial_3 T(\varepsilon)(v) e_{ijkl}(\varepsilon, v) \sqrt{g(\varepsilon)} \, dx,
\]

(5.13)

\[
D(\varepsilon)(v, v) = \int_{\Omega} \left[ \varepsilon^2 v_\alpha v_\alpha + v_3 v_3 \right] \sqrt{g(\varepsilon)} \, dx.
\]

(5.14)

Let \( W_m \) be the collection of all \( m \)-dimensional subspaces of \( H^2_0(\omega) \). Let \( W \in W_m \). Define

\[
W = \{ v_\varphi \mid \varphi \in W \}.
\]

(5.15)

It follows that \( W \in V_m \). Hence, it follows from (5.12) that

\[
\xi_m(\varepsilon) \leq \min_{W \in W_m} \max_{\varphi \in W} \frac{N(\varepsilon)(v, v_\varphi)}{D(\varepsilon)(v, v_\varphi)}.
\]

(5.16)

Now,

\[
D(\varepsilon)(v, v_\varphi) = \int_{\Omega} \left[ \varepsilon^2 x_3^2 |\partial_3 \varphi|^2 + |\varphi|^2 \right] \sqrt{g(\varepsilon)} \, dx \\
\geq \int_\omega \varphi^2 \, d\omega.
\]

(5.17)

Using the relations (5.3)–(5.5) and Lemma 5.1, it follows that

\[
\int_{\Omega} A^{ijkl} e_{ijkl}(\varepsilon, v_\varphi) e_{ijkl}(\varepsilon, v_\varphi) \sqrt{g(\varepsilon)} \, dx \leq C \int_\omega |\varphi|^2 \, d\omega,
\]

(5.18)

\[
\int_{\Omega} P^{ijkl} \partial_3 T(\varepsilon)(v_\varphi) e_{ijkl}(\varepsilon, v_\varphi) \sqrt{g(\varepsilon)} \, dx \leq C \int_\omega |\varphi|^2 \, d\omega,
\]

(5.19)

\[
\varepsilon \int_{\Omega} P^{ijkl} \partial_3 T(\varepsilon)(v_\varphi) e_{ijkl}(\varepsilon, v_\varphi) \sqrt{g(\varepsilon)} \, dx \leq C \int_\omega |\varphi|^2 \, d\omega.
\]

(5.20)

Hence

\[
\xi_m(\varepsilon) \leq C \min_{W \in W_m} \max_{\varphi \in W} \frac{\int_\omega |\varphi|^2 \, d\omega}{\int_\omega \varphi^2 \, d\omega} \\
\leq C \lambda^m,
\]

(5.21)

where \( \lambda^m \) is the \( m \)th eigenvalue of the two-dimensional elliptic eigenvalue problem

\[
\triangle^2 u = \lambda u \quad \text{in} \ \omega \\
u = \partial_\nu u = 0 \quad \text{on} \ \partial \omega.
\]

(5.22)

This completes the proof of the theorem on setting \( C(m) = C \lambda^m \).
6. The limit problem

**Theorem 6.1.** (a) For each positive integer \(m\), there exists \(u^m \in H^1(\Omega)\), \(\varphi^m \in L^2(\Omega)\) and \(\xi^m \in \mathbb{R}\) such that

\[
\begin{align*}
  u^m(\varepsilon) &\to u^m \text{ in } H^1(\Omega), & \varphi^m(\varepsilon) &\to \varphi^m \text{ in } L^2(\Omega), \\
  (\varepsilon \partial_1 \varphi^m(\varepsilon), \varepsilon \partial_2 \varphi^m(\varepsilon), \partial_3 \varphi^m(\varepsilon)) &\to (0, 0, \partial_3 \varphi^m) \text{ in } L^2(\Omega), \\
  \xi^m(\varepsilon) &\to \xi^m. 
\end{align*}
\]

(b) Define the spaces

\[
\begin{align*}
  V_H(\omega) &= \{(\eta_\alpha) \in (H^1(\omega))^2; \eta_\alpha = 0 \text{ on } \gamma_0\}, \\
  V_3(\omega) &= \{\eta_3 \in H^2(\omega); \eta_3 = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\}, \\
  V_{KL} &= \{v \in H^1(\Omega); v = \eta_\alpha - x_3 \partial_\alpha \eta_3, (\eta_\alpha) \in V_H(\omega) \times V_3(\omega)\}, \\
  \Psi_{\ell} &= \{\psi \in L^2(\Omega), \partial_\psi \psi \in L^2(\Omega)\}, \\
  \Psi_{\ell_0} &= \{\psi \in L^2(\Omega), \partial_\psi \psi \in L^2(\Omega), \psi|_{\Gamma^\pm} = 0\}. 
\end{align*}
\]

Then there exists \((\zeta_\alpha^m, \xi_3^m) \in V_H \times V_3(\omega)\) such that

\[
\begin{align*}
  u_\alpha^m &= \zeta_\alpha^m - x_3 \partial_\alpha \zeta_3^m \quad \text{and} \quad u_3^m = \xi_3^m, \\
  \varphi^m &= (1 - x_3^2) \frac{p_{3\alpha\beta}}{p_{33}} \partial_\alpha \xi_3^m 
\end{align*}
\]

and \((\zeta_\alpha^m, \xi_3^m) \in V_H \times V_3 \times \mathbb{R}\) satisfies

\[
\begin{align*}
  &- \int_\omega m_{\alpha\beta}(\zeta_\alpha^m) \partial_\alpha \eta_3 d\omega + \int_\omega n_{\alpha\beta}(\zeta_\alpha^m) \partial_\alpha \eta_3 d\omega + \frac{2}{3} \int_\omega p_{3\alpha\beta} \partial_\alpha \xi_3^m \partial_\beta \eta_3 d\omega \\
  &= \xi_3^m \int_\omega \zeta_3^m \eta_3 d\omega \quad \forall \eta_3 \in V_3(\omega), \\
  \int_\omega n_{\alpha\beta} \partial_\beta \eta_\alpha d\omega &= 0 \quad \forall \eta_\alpha \in V_H(\omega), \\
\end{align*}
\]

where

\[
\begin{align*}
  m_{\alpha\beta}(\zeta) &= -\left\{ \frac{4\lambda}{3(\lambda + 4\mu)} \xi_3 \delta_{\alpha\beta} + \frac{4\mu}{3} \partial_\alpha \zeta_3 \right\}, \\
  n_{\alpha\beta}(\zeta) &= \frac{4\lambda}{\lambda + 2\mu} \xi_3 \sigma(\xi) \delta_{\alpha\beta} + 4\mu \epsilon_{\alpha\beta}(\zeta), \\
  p_{33} &= \frac{1}{\mu} p_{3\alpha\beta} p_{3\beta} + \frac{1}{\lambda + 2\mu} p_{333} p_{333} + \sigma_{33}, \\
  p_{3\alpha\beta} &= p_{3\alpha\beta} - \frac{\lambda}{\lambda + 2\mu} p_{333} \delta_{\alpha\beta}. 
\end{align*}
\]
Letting \( \varphi = \rho \varphi^m(\varepsilon) \) and the tensor \( \tilde{K}^m(\varepsilon) = (\tilde{K}^m_{ij}(\varepsilon)) \) by

\[
\varphi^m(\varepsilon) = (\varepsilon \partial_1 \varphi^m(\varepsilon), \varepsilon \partial_2 \varphi^m(\varepsilon), \partial_3 \varphi^m(\varepsilon)),
\]

\[
\tilde{K}^m_{ij}(\varepsilon) = \tilde{K}^m_{ij}(\varepsilon), \quad \tilde{K}^m_{jj}(\varepsilon) = \frac{1}{\varepsilon} \tilde{K}^m_{jj}(\varepsilon), \quad \tilde{K}^m_{33}(\varepsilon) = \frac{1}{\varepsilon^2} \tilde{K}^m_{33}(\varepsilon).
\]

Then there exists a constant \( C_{10} > 0 \) such that

\[
\|u^m(\varepsilon)\|_{1,\Omega} \leq C_{10}, \quad |\tilde{K}^m_{ij}(\varepsilon)|_{0,\Omega} \leq C_{10}, \quad |\varphi^m(\varepsilon)|_{0,\Omega} \leq C_{10}
\]

for all \( 0 < \varepsilon \leq \varepsilon_0 \).

Letting \( \psi = \varphi^m(\varepsilon) \) in (3.11), we have

\[
\int_{\Omega} A^{ijkl}(\varepsilon) e_{kij}(\varepsilon)(u^m(\varepsilon)) e_{lij}(\varepsilon)(u^m(\varepsilon)) \sqrt{g(\varepsilon)} \, dx + \int_{\Omega} P^{ijkl}(\varepsilon) \partial_j \varphi^m(\varepsilon) e_{kij}(\varepsilon)(u^m(\varepsilon)) \sqrt{g(\varepsilon)} \, dx
\]

\[
+ \varepsilon \int_{\Omega} P^{ijkl}(\varepsilon) \partial_i \varphi^m(\varepsilon) e_{kij}(\varepsilon)(u^m(\varepsilon)) \sqrt{g(\varepsilon)} \, dx
\]

\[
= \tilde{\xi}^m(\varepsilon) \int_{\Omega} \left[ e^2 u^m_{ij}(\varepsilon) u^m_{ij}(\varepsilon) + u^m_3(\varepsilon) u^m_3(\varepsilon) \right] \sqrt{g(\varepsilon)} \, dx.
\]

Letting \( \psi = \varphi^m(\varepsilon) \) in (3.12) and using it in the above equation, we get

\[
\int_{\Omega} A^{ijkl}(\varepsilon) e_{kij}(\varepsilon,u^m(\varepsilon)) e_{lij}(\varepsilon,u^m(\varepsilon)) \sqrt{g(\varepsilon)} \, dx + \int_{\Omega} \delta^{ij}(\varepsilon) \tilde{\varphi}^m_j(\varepsilon) \tilde{\varphi}^m_i(\varepsilon) \sqrt{g(\varepsilon)} \, dx
\]

\[
= \tilde{\xi}^m(\varepsilon) \int_{\Omega} \left[ e^2 u^m_{ij}(\varepsilon) u^m_{ij}(\varepsilon) + u^m_3(\varepsilon) u^m_3(\varepsilon) \right] \sqrt{g(\varepsilon)} \, dx.
\]

Using the coerciveness properties (4.11) and (4.12), the inequality \( (a - b)^2 \geq \frac{a^2}{2} - b^2 \) and the generalized Korn’s inequality (4.15), we have for \( \varepsilon \leq \min\{\varepsilon_0,1\}, \)

\[
\int_{\Omega} A^{ijkl}(\varepsilon) e_{kij}(\varepsilon,u^m(\varepsilon)) e_{lij}(\varepsilon,u^m(\varepsilon)) \sqrt{g(\varepsilon)} \, dx
\]

\[
+ \int_{\Omega} \delta^{ij}(\varepsilon) \varphi^m_j(\varepsilon) \varphi^m_i(\varepsilon) \sqrt{g(\varepsilon)} \, dx
\]

\[
\geq C_{11} \sum_{i,j} \|e_{ij}(\varepsilon,u^m(\varepsilon))\|_{0,\Omega}^2 + C_{11} \sum_i \|\varphi^m_i(\varepsilon)\|_{0,\Omega}^2.
\]
\[ = C_{11} \sum_{\alpha, \beta} \left\| \varepsilon_{\alpha\beta}(u^m(\varepsilon)) + \varepsilon^2 \varepsilon_{\alpha\beta}(\varepsilon, u^m(\varepsilon)) \right\|_{0, \Omega} \]
\[ + 2C_{11} \sum_{\alpha} \left\| \frac{1}{\varepsilon} \varepsilon_{\alpha3}(u^m(\varepsilon)) + \varepsilon \varepsilon_{\alpha3}(\varepsilon, u^m(\varepsilon)) \right\|_{0, \Omega}^2 \]
\[ + C_{11} \left\| \frac{1}{\varepsilon^2} \varepsilon_{33}(u^m(\varepsilon)) \right\|_{0, \Omega}^2 + C_{11} \sum_i \| \phi_i^m(\varepsilon) \|_{0, \Omega}^2 \]
\[ \geq C_{11} \left\{ \frac{1}{2} \sum_{i,j} | \bar{K}_{ij}^m(\varepsilon) |_{0, \Omega}^2 - C_1^2 (2\varepsilon^2 + \varepsilon^4) \| \Phi_i^m(\varepsilon) \|_{2, \Omega}^2 \right\} \]
\[ + C_{11} \sum_i \| \phi_i^m(\varepsilon) \|_{0, \Omega}^2 \]
\[ \geq C_{11} \left\{ \frac{1}{2} \sum_{i,j} | \bar{K}_{ij}^m(\varepsilon) |_{0, \Omega}^2 - 3\varepsilon^2 C_1^2 \| \Phi_i^m(\varepsilon) \|_{2, \Omega}^2 \right\} \]
\[ + C_{11} \sum_i \| \phi_i^m(\varepsilon) \|_{0, \Omega}^2 \]
\[ \geq C_{11} \left\{ \frac{1}{2} (C_k)^{-1} - 3\varepsilon^2 C_1^2 \right\} \| \Phi_i^m(\varepsilon) \|_{2, \Omega}^2 + C_{11} \sum_i \| \phi_i^m(\varepsilon) \|_{0, \Omega}^2. \] (6.22)

Combining eqs (6.21) and (6.22) with relations (3.13) and (5.11), we get the relation (6.19).

**Step (ii).** From Step (i) it follows that there exists a subsequence \((\Phi_i^m(\varepsilon))\) and \((\bar{\Phi}_i^m) \in L^2(\Omega)\) such that
\[ (\varepsilon \partial_i \Phi^m(\varepsilon), \varepsilon \partial_2 \Phi^m(\varepsilon), \partial_3 \Phi^m(\varepsilon)) \rightharpoonup (\Phi_1^m, \Phi_2^m, \Phi_3^m) \quad \text{in} \quad (L^2(\Omega))^3. \] (6.23)

Since \(\Gamma_{eD}\) contains \(\Gamma^+\), we have
\[ \Phi^m(\varepsilon)(x_1, x_2, x_3) = \int_{-1}^{x_3} \partial_3 \Phi^m(\varepsilon)(x_1, x_2, s) ds \] (6.24)

and it follows that \(\| \Phi^m(\varepsilon) \|_{0, \Omega} \leq \sqrt{3} \| \partial_1 \Phi^m(\varepsilon) \|_{0, \Omega}\). This implies that \(\Phi^m(\varepsilon)\) is bounded in \(L^2(\Omega)\). Therefore there exists a \(\Phi^m\) in \(L^2(\Omega)\) and a subsequence, still indexed by \(\varepsilon\), such that \(\Phi^m(\varepsilon)\) converges weakly to \(\Phi^m\). Hence it follows from (6.23) that
\[ (\varepsilon \partial_i \Phi^m(\varepsilon), \varepsilon \partial_2 \Phi^m(\varepsilon), \partial_3 \Phi^m(\varepsilon)) \rightharpoonup (0, 0, \partial_3 \Phi^m). \] (6.25)

**Step (iii).** From Step (i) it follows that there exists a subsequence, indexed by \(\varepsilon\) for notational convenience, and functions \(u^m \in V(\Omega)\) and \(\tilde{K}_{ij}^m \in (L^2(\Omega))^9\) such that
\[ u^m(\varepsilon) \rightharpoonup u^m \quad \text{in} \quad H^1(\Omega), \quad \tilde{K}_{ij}^m(\varepsilon) \rightharpoonup \tilde{K}_{ij}^m \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \varepsilon \to 0. \] (6.26)

Then there exist functions \((\zeta^m) \in H^1(\omega)\) and \(\zeta^m \in H^2(\omega)\) satisfying \(\zeta_i^m = \partial_\nu \zeta^m = 0\) on \(\gamma_0\) such that
\[ u^m_{\alpha} = \zeta^m_{\alpha} - x_3 \partial_3 \zeta^m_{\alpha} \quad \text{and} \quad u^m_3 = \zeta^m_3 \] (6.27)
and
\[
\tilde{K}^{m}_{\alpha\beta} = \varepsilon_{\alpha\beta}(u^{m}), \quad \tilde{K}^{m}_{\alpha3} = -\frac{1}{\mu} P^{3\alpha\beta} \partial_{3} \phi^{m},
\]
\[
\tilde{K}^{m}_{33} = -\frac{1}{\lambda + 2\mu} (P^{333} \partial_{3} \phi^{m} + \lambda \tilde{K}^{m}_{\beta\beta}).
\] (6.28)

From definition (6.18) and the boundedness of \((\tilde{K}^{m}_{ij}(\varepsilon))\), we deduce that
\[
\|e_{\alpha3}(u^{m}(\varepsilon))\|_{0,\Omega} \leq \varepsilon C_{13} \quad \text{and} \quad \|e_{33}(u^{m}(\varepsilon))\|_{0,\Omega} \leq \varepsilon^{2} C_{13},
\]
where \(e_{ij}(v) = \frac{1}{2}(\partial_{i} v_{j} + \partial_{j} v_{i})\). Since norm is a weakly lower semicontinuous function
\[
\|e_{ij}(u^{m})\|_{0,\Omega} \leq \liminf_{\varepsilon \to 0} \|e_{ij}(u^{m}(\varepsilon))\|_{0,\Omega} = 0,
\] (6.29)
we obtain \(e_{ij}(u^{m}) = 0\). Then it is a standard argument that the components \(u^{m}_{i}\) of the limit \(u^{m}\) are of the form (6.27).

Since \(u^{m}(\varepsilon) \to u^{m}\) in \(H^{1}(\Omega)\), definition (4.4) of the functions \(\varepsilon_{\alpha\beta}(v)\) shows that the function \(\tilde{K}^{m}_{\alpha\beta}(v) = \varepsilon_{\alpha\beta}(u^{m}(\varepsilon))\) converges weakly in \(L^{2}(\Omega)\) to the function \(\varepsilon_{\alpha\beta}(u^{m})\).

We next note the following result. Let \(w \in L^{2}(\Omega)\) be given; then
\[
\int_{\Omega} w \partial_{3} \varphi \, dx = 0 \quad \text{for all } \varphi \in H^{1}(\Omega) \text{ with } \varphi = 0 \text{ on } \Gamma_{0}, \text{ then } w = 0.
\] (6.30)

Multiplying (3.11) by \(\varepsilon^{2}\), taking \(v_{\alpha} = 0\) and letting \(\varepsilon \to 0\), we get
\[
\int_{\Omega} (\lambda \tilde{K}^{m}_{\alpha\sigma} + (\lambda + 2\mu) \tilde{K}_{33} + P^{333} \partial_{3} \phi^{m}) \partial_{3} \varphi \, dx = 0
\] (6.31)
which implies \((\lambda \tilde{K}^{m}_{\alpha\sigma} + (\lambda + 2\mu) \tilde{K}_{33} + P^{333} \partial_{3} \phi^{m}) = 0\) and hence the third relation in (6.28) follows.

Again, multiplying (3.11) by \(\varepsilon\), taking \(v_{3} = 0\) and letting \(\varepsilon \to 0\), we get
\[
\int_{\Omega} (\mu \tilde{K}^{m}_{\alpha3} + P^{3\alpha\beta} \partial_{3} \phi^{m}) \partial_{3} \varphi \, dx = 0
\] (6.32)
which implies \((\mu \tilde{K}^{m}_{\alpha3} + P^{3\alpha\beta} \partial_{3} \phi^{m}) = 0\) and hence the second relation in (6.28) follows.

Step (iv). The function \(\phi^{m}\) is of the form (6.10).

Letting \(\varepsilon \to 0\) in eq. (6.12), we get
\[
\int_{\Omega} (P^{3\alpha\beta} \tilde{K}^{m}_{\alpha\beta} - \delta^{33} \partial_{3} \phi^{m}) \partial_{3} \psi \, dx = 0 \quad \forall \psi \in \Psi(\Omega).
\] (6.33)

Since \(D(\Omega)\) is dense in \(\Psi(\Omega)\) (and hence in \(\Psi(\Omega)\)) for the norm \(||\cdot||_{\psi}\), eq. (6.33) is equivalent to
\[
\partial_{3}(P^{3\alpha\beta} \tilde{K}^{m}_{\alpha\beta} - \delta^{33} \partial_{3} \phi^{m}) = 0 \quad \text{in } D'(\Omega)
\] (6.34)
which implies \((P^{3\alpha\beta} \tilde{K}^{m}_{\alpha\beta} - \delta^{33} \partial_{3} \phi^{m}) = d^{1}\), with \(d^{1} \in D(\omega)\). Then
\[
\partial_{3} \phi^{m} = \frac{P^{3\alpha\beta}}{p^{33}} [\tilde{e}_{\alpha\beta}(\zeta^{m}) - x_{3} \partial_{3} \zeta^{m}] = \frac{1}{p^{33}} d^{1}
\] (6.35)
Since \( \varphi^m \) satisfies the boundary conditions \( \varphi^m_{\Gamma^+} = \varphi^m_{\Gamma^-} = 0 \), we have

\[
d^0 = \frac{p^3 a^2}{2p^3} \partial_3 \varphi^m, \quad d^1 = \frac{p^3 a^2}{2p^3} \dot{\varphi}_{a\beta}(\varphi^m).
\]  

(6.37)

Thus the conclusion follows.

**Step (v).** The function \( \varphi^m \) satisfies (6.11) and (6.12).

Taking \( v \in V_{KL} \) and letting \( \varepsilon \to 0 \) in (6.11) we get

\[
\int_{\Omega} A^{abkl} \mathcal{K}^m_{ab} (v) dx + \int_{\Omega} p^3 a^2 \partial_3 \varphi^m \mathcal{K}^m_{a\beta} (v) dx = \xi^m \int_{\Omega} u^m_3 \cdot v_3 dx.
\]

(6.38)

Replacing \( u^m \) and \( \mathcal{K}^m \) by the expressions obtained in (6.24) and (6.28), and taking \( v \) of the form

\[
v_\alpha = \eta_\alpha - x_3 \partial_3 \eta_3 \quad \text{and} \quad v_3 = \eta_3
\]

with \( (\eta_\alpha) \in \mathcal{V}_H(\Omega) \times \mathcal{V}_2(\Omega) \), it is verified that (6.38) coincides with eqs (6.11) and (6.12).

**Step (vi).** The convergences \( u^m(\varepsilon) \to u^m \) in \( H^1(\Omega) \) and \( \varphi^m(\varepsilon) \to \varphi^m \) in \( L^2(\Omega) \) are strong.

To show that the family \( (u^m(\varepsilon)) \) converges strongly to \( u^m \) in \( H^1(\Omega) \), by Lemma 4.2, it is enough to show that

\[
\tilde{e}_{ij}(u^m(\varepsilon)) \to \tilde{e}_{ij}(u^m) \quad \text{in} \quad L^2(\Omega).
\]

(6.39)

Since \( \tilde{e}_{ij}(u^m(\varepsilon)) = 0 \) and

\[
\sum_{i,j} \| \tilde{e}_{ij}(u^m(\varepsilon)) - \tilde{e}_{ij}(u^m) \|_{0,\Omega}^2
\]

\[
= \sum_{a,b} \| \mathcal{K}^m_{a\beta}(\varepsilon) - \mathcal{K}^m_{a\beta} \|_{0,\Omega}^2 + 2 \varepsilon^2 \sum_{a} \| \mathcal{K}^m_{a3}(\varepsilon) \|_{0,\Omega}^2 + \varepsilon^4 \| \mathcal{K}^m_{33}(\varepsilon) \|_{0,\Omega}^2,
\]

(6.40)

convergence (6.39) is equivalent to showing that

\[
\mathcal{K}^m(\varepsilon) \to \mathcal{K}^m \quad \text{in} \quad L^2(\Omega).
\]

(6.41)

We define a norm on \( (L^2(\Omega))^9 \times (L^2(\Omega))^3 \) by letting for any matrix \( M \in (L^2(\Omega))^9 \) and any vector \( \chi \in (L^2(\Omega))^3 \),

\[
\| (M, \chi) \| = \left\{ \int_{\Omega} A^{ijkl} M : M \sqrt{g(\varepsilon)} dx + \int_{\Omega} \sigma^{ij} \chi_j \sqrt{g(\varepsilon)} dx \right\}^{1/2}.
\]

(6.42)

Let \( X^m(\varepsilon) \) be the norm of \( (\mathcal{K}^m(\varepsilon), \varepsilon \partial_1 \varphi^m(\varepsilon), \varepsilon \partial_2 \varphi^m(\varepsilon), \partial_3 \varphi^m(\varepsilon)) \) in \( (L^2(\Omega))^{12} \). Using the weak convergence equation (eqs (6.23) and (6.25) and the relation (6.28), it can be shown that

\[
\lim_{\varepsilon \to 0} X^m(\varepsilon) = X^m = \left( \int_{\Omega} A^{ijkl} \mathcal{K}^m : \mathcal{K}^m dx + \int_{\Omega} \sigma^{33} (\partial_3 \varphi^m)^2 dx \right)^{1/2}
\]

(6.43)
which is the norm of \((\tilde{K}^m, 0, 0, \partial_1 \phi^m)\). Since we have already proved that \((\tilde{K}^m(\epsilon), \epsilon \partial_1 \phi^m(\epsilon), \epsilon \partial_2 \phi^m(\epsilon), \partial_3 \phi^m(\epsilon))\) converges weakly to \((\tilde{K}, 0, 0, \partial_3 \phi^m)\) in \((L^2(\Omega))^3\), we have the following strong convergences:

\[
\tilde{K}^m(\epsilon) \to \tilde{K}^m \text{ strongly in } (L^2(\Omega))^9,
\]

\[
(\epsilon \partial_1 \phi^m(\epsilon), \epsilon \partial_2 \phi^m(\epsilon), \partial_3 \phi^m(\epsilon)) \to (0, 0, \partial_3 \phi^m) \text{ strongly in } (L^2(\Omega))^3. 
\]

Hence \(u^m(\epsilon)\) converges strongly to \(u^m\) in \(H^1(\Omega)\) and since \(\phi^m(\epsilon) - \phi^m\) is in \(\Psi_{00}\), the equivalence of norms \(\|\psi\|_{\psi_i}\) and \(\psi \to |\partial_3 \psi|_{\Omega}\) in \(\Psi_{00}\) proves that \(\phi^m(\epsilon)\) converges strongly to \(\phi^m\) in \(L^2(\Omega)\).

Equation (6.12) can be written as

\[
\int_\Omega \left[ \frac{2\lambda \mu}{\lambda + 2\mu} \epsilon_{\rho\rho}(\zeta) \delta_{\alpha \beta} + 2\mu \epsilon_{\alpha \beta}(\zeta) \right] \partial_\beta \eta_\alpha d\omega = \int_\Omega \left[ \frac{2\lambda \mu}{\lambda + 2\mu} \left( \partial_\alpha \theta \partial_\beta \zeta_3 \right) \delta_{\alpha \beta} + \mu \left( \partial_\alpha \theta \partial_\beta \zeta_3 + \partial_\beta \theta \partial_\alpha \zeta_3 \right) \right] \partial_\beta \eta_\alpha d\omega. 
\]

(6.46)

Clearly, the bilinear form

\[
\tilde{b} (\zeta_3, \eta_3) = \int_\Omega \left[ \frac{2\lambda \mu}{\lambda + 2\mu} \epsilon_{\rho\rho}(\zeta) \delta_{\alpha \beta} + 2\mu \epsilon_{\alpha \beta}(\zeta) \right] \partial_\beta \eta_\alpha d\omega
\]

(6.47)

is \(V_H(\omega)\) elliptic. Also for a given \(\zeta_3 \in V_3(\omega)\), the functional

\[
\langle \zeta_3, \eta_3 \rangle = \int_\Omega \left[ \frac{2\lambda \mu}{\lambda + 2\mu} \left( \partial_\alpha \theta \partial_\beta \zeta_3 \right) \delta_{\alpha \beta} + \mu \left( \partial_\alpha \theta \partial_\beta \zeta_3 + \partial_\beta \theta \partial_\alpha \zeta_3 \right) \right] \partial_\beta \eta_\alpha d\omega
\]

(6.48)

is continuous on \(V_H(\omega)\). Thus, given \(\zeta_3 \in V_3(\omega)\), there exists a unique vector \(\langle \zeta_3, \eta_3 \rangle \in V_H(\omega)\) such that

\[
\tilde{b} (\zeta_3, \eta_3) = \langle \zeta_3, \eta_3 \rangle. 
\]

(6.49)

We denote by \(T \zeta_3 \in V_H(\omega) \times V_3(\omega)\) the vector \(\langle \zeta_3, \eta_3 \rangle\). In particular, \(T \zeta_3^m = \langle \zeta_3^m, \zeta_3^m \rangle\).

Substituting this in (6.11), we get

\[
b (\zeta_3^m, \eta_3) = \int_\Omega \zeta_3^m \eta_3 d\omega \quad \text{for all } \eta_3 \in V_3(\omega),
\]

(6.50)

where

\[
b (\zeta_3, \eta_3) = - \int_\Omega m_{a\beta \partial_\alpha \eta_3} d\omega + \int_\Omega n'_{a\beta} (T \zeta_3) \partial_\alpha \theta_3 d\omega
\]

\[
+ \frac{2}{3} \int_\Omega p'_{a\beta \eta_3} p'_{3\tau} \partial_\alpha \zeta_3 \partial_\beta \eta_3 d\omega.
\]

(6.51)
Lemma 6.2. The bilinear form $b(\cdots)$ defined by (6.51) is $V_H(\omega)$-elliptic and symmetric.

Proof. It follows from Lemma 6.2 in [8] that the bilinear form $\tilde{b}(\cdots)$ defined by

$$\tilde{b}(\zeta_3, \eta_3) = -\int_\omega m_{\alpha\beta}(\zeta_3) \partial_{\alpha\beta} \eta_3 d\omega + \int_\omega n_{\alpha\beta}^0(T\zeta_3) \partial_{\alpha\beta} \theta \eta_3 d\omega$$  \hspace{1cm} (6.52)

is $V_H(\omega)$-elliptic and symmetric. Hence it is clear that $b(\cdots)$ is also $V_H(\omega)$-elliptic and symmetric.

Lemma 6.3. Let $(\zeta_3^m, \xi_3^m), m \geq 1,$ be the eigensolutions of problem (6.51) found as limits of the subsequence $(u^m(\epsilon), \xi_3^m(\epsilon)), m \geq 1$ of eigensolutions of the problem (3.11). Then the sequence $(\zeta_3^m)_{m=1}^\infty$ comprises all the eigenvalues, counting multiplicities, of problem (6.51) and the associated sequence $(\xi_3^m)_{m=1}^\infty$ of eigenfunctions forms a complete orthonormal set in the space $V_3(\omega)$.

Proof. The proof is similar to the proof of Lemma 5.4 in [3].

References

[1] Bernadou M and Haenel C, Modelization and numerical approximation of piezoelectric thin shells, Parts I, II, III: Rapport de Recherche, DER-CS (Ecole Supérieure d’Ingénierie Léonard de Vinci, France) (2002) No RR-6, 7, 8
[2] Busse S, Ciarlet P G and Miara B, Justification d’un modèle linéaire bi-dimensional de coques ‘faiblement courbées’ en coordonnées curvilignes, M2NA, 31 (3) (1997) 409–434
[3] Ciarlet P G and Kesavan S, Two-dimensional approximation of three-dimensional eigenvalue problem in plate theory, Comp. Methods Appl. Mech. Engg. 26 (1981) 145–172
[4] Ciarlet P G and Lods V, Asymptotic analysis of linearly elastic shells. I. Justification of membrane shell equation, Arch. Rational Mech. Anal. 136 (1996) 119–161
[5] Ciarlet P G, Lods V and Miara B, Asymptotic analysis of linearly elastic shells. II. Justification of flexural shell equations, Arch. Rational Mech. Anal. 136 (1996) 162–190
[6] Ciarlet P G and Miara B, Justification of the two-dimensional equations of a linearly elastic shell, Comm. Pure Appl. Math. 45 (1992) 327–360
[7] Kesavan S, Homogenization of elliptic eigenvalue problem, Part I, Appl. Math. Optim. 5 (1979) 153–167
[8] Kesavan S and Sabu N, Two-dimensional approximation of eigenvalue problem in shallow shells, Math. Mech. Solids 4 (1999) 441–460
[9] Kesavan S and Sabu N, Two-dimensional approximation of eigenvalue problem for flexural shells, Chinese Ann. Math. 21(B) (2000) 1–16