Two linear transformations each tridiagonal with respect to an eigenbasis of the other; an overview

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Abstract

Let \( \mathbb{K} \) denote a field and let \( V \) denote a vector space over \( \mathbb{K} \) with finite positive dimension. We consider an ordered pair of linear transformations \( A : V \to V \) and \( A^* : V \to V \) that satisfy conditions (i), (ii) below.

(i) There exists a basis for \( V \) with respect to which the matrix representing \( A \) is irreducible tridiagonal and the matrix representing \( A^* \) is diagonal.

(ii) There exists a basis for \( V \) with respect to which the matrix representing \( A \) is diagonal and the matrix representing \( A^* \) is irreducible tridiagonal.

We call such a pair a Leonard pair on \( V \). We give an overview of the theory of Leonard pairs.

1 Leonard pairs

We begin by recalling the notion of a Leonard pair. We will use the following terms. Let \( X \) denote a square matrix. Then \( X \) is called tridiagonal whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume \( X \) is tridiagonal. Then \( X \) is called irreducible whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

We now define a Leonard pair. For the rest of this paper \( \mathbb{K} \) will denote a field.

Definition 1.1 [37] Let \( V \) denote a vector space over \( \mathbb{K} \) with finite positive dimension. By a Leonard pair on \( V \) we mean an ordered pair of linear transformations \( A : V \to V \) and \( A^* : V \to V \) that satisfy conditions (i), (ii) below.

(i) There exists a basis for \( V \) with respect to which the matrix representing \( A \) is irreducible tridiagonal and the matrix representing \( A^* \) is diagonal.

(ii) There exists a basis for \( V \) with respect to which the matrix representing \( A \) is diagonal and the matrix representing \( A^* \) is irreducible tridiagonal.
Note 1.2 According to a common notational convention, $A^*$ denotes the conjugate transpose of $A$. We are not using this convention. In a Leonard pair $A, A^*$ the linear transformations $A, A^*$ are arbitrary subject to (i), (ii) above.

Note 1.3 Our use of the name “Leonard pair” is motivated by a connection to a theorem of D. Leonard [3, p. 260], [33] involving the $q$-Racah and related polynomials of the Askey Scheme.

2 An example of a Leonard pair

Here is an example of a Leonard pair. Set $V = \mathbb{K}^4$ (column vectors), set

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \quad A^* = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

and view $A$ and $A^*$ as linear transformations on $V$. We assume the characteristic of $\mathbb{K}$ is not 2 or 3 to ensure $A$ is irreducible. Then the pair $A, A^*$ is a Leonard pair on $V$. Indeed condition (i) in Definition 1.1 is satisfied by the basis for $V$ consisting of the columns of the 4 by 4 identity matrix. To verify condition (ii), we display an invertible matrix $P$ such that $P^{-1}AP$ is diagonal and $P^{-1}A^*P$ is irreducible tridiagonal. Set

$$P = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}.$$ 

By matrix multiplication $P^2 = 8I$, where $I$ denotes the identity, so $P^{-1}$ exists. Also by matrix multiplication,

$$AP = PA^*.$$ 

(1)

Apparently $P^{-1}AP$ is equal to $A^*$ and is therefore diagonal. By (1) and since $P^{-1}$ is a scalar multiple of $P$, we find $P^{-1}A^*P$ is equal to $A$ and is therefore irreducible tridiagonal. Now condition (ii) of Definition 1.1 is satisfied by the basis for $V$ consisting of the columns of $P$.

3 Leonard systems

When working with a Leonard pair, it is often convenient to consider a closely related and somewhat more abstract concept called a Leonard system. In order to define this we recall a few terms. Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $\text{End}(V)$ denote the $\mathbb{K}$-algebra consisting of the linear transformations from $V$ to $V$. For $A \in \text{End}(V)$, by the eigenvalues of $A$ we mean the roots of the characteristic polynomial of
A. These eigenvalues are contained in the algebraic closure of $\mathbb{K}$. We say $A$ is multiplicity-free whenever the eigenvalues of $A$ are mutually distinct and contained in $\mathbb{K}$. Assume for the moment that $A$ is multiplicity-free. Let $\theta_0, \theta_1, \ldots, \theta_d$ denote an ordering of the eigenvalues of $A$. For $0 \leq i \leq d$ let $v_i$ denote a nonzero vector in $V$ that is an eigenvector for $A$ with eigenvalue $\theta_i$. Observe the sequence $v_0, v_1, \ldots, v_d$ is a basis for $V$. For $0 \leq i \leq d$ define $E_i \in \text{End}(V)$ so that $E_i v_j = \delta_{ij} v_j$ for $0 \leq j \leq d$. We call $E_i$ the primitive idempotent of $A$ associated with $\theta_i$.

**Definition 3.1** [37] Let $d$ denote a nonnegative integer and let $V$ denote a vector space over $\mathbb{K}$ with dimension $d + 1$. By a Leonard system on $V$ we mean a sequence $\Phi = (A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ that satisfies conditions (i)–(v) below.

(i) Each of $A, A^*$ is a multiplicity-free element of $\text{End}(V)$.

(ii) $E_0, E_1, \ldots, E_d$ is an ordering of the primitive idempotents of $A$.

(iii) $E_0^*, E_1^*, \ldots, E_d^*$ is an ordering of the primitive idempotents of $A^*$.

(iv) $E_i^* A E_j^* = \begin{cases} 0, & \text{if } |i - j| > 1; \\ \neq 0, & \text{if } |i - j| = 1 \end{cases} (0 \leq i, j \leq d)$.

(v) $E_i A^* E_j = \begin{cases} 0, & \text{if } |i - j| > 1; \\ \neq 0, & \text{if } |i - j| = 1 \end{cases} (0 \leq i, j \leq d)$.

We call $d$ the diameter of $\Phi$.

Leonard pairs and Leonard systems are related as follows.

**Theorem 3.2** [43] For $A, A^*$ in $\text{End}(V)$, the pair $A, A^*$ is a Leonard pair on $V$ if and only if the following (i), (ii) hold.

(i) Each of $A, A^*$ is multiplicity-free.

(ii) There exists an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of $A$ and there exists an ordering $E_0^*, E_1^*, \ldots, E_d^*$ of the primitive idempotents of $A^*$ such that $(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system on $V$.

4 Leonard pairs from 24 points of view

Let $A, A^*$ denote a Leonard pair on $V$. We describe 24 bases for $V$ on which $A, A^*$ act in an attractive fashion.

To describe the bases we will use the following terms. Let $X$ denote a square matrix. Then $X$ is called lower bidiagonal whenever each nonzero entry lies on either the diagonal or the subdiagonal. We say $X$ is upper bidiagonal whenever the transpose of $X$ is lower bidiagonal. Let $v_0, v_1, \ldots, v_d$ denote a basis for $V$. By the inversion of this basis we mean the basis $v_d, v_{d-1}, \ldots, v_0$.

The 24 bases are described by the diagram below. In that diagram each vertex represents one of the 24 bases. The shading on the vertex indicates the nature of the $A, A^*$ action.
(i) Black: $A$ is diagonal and $A^*$ is irreducible tridiagonal.
(ii) Green: $A$ is lower bidiagonal and $A^*$ is upper bidiagonal.
(iii) Red: $A$ is upper bidiagonal and $A^*$ is lower bidiagonal.
(iv) Yellow: $A$ is irreducible tridiagonal and $A^*$ is diagonal.

For each pair of bases in the diagram that are connected by an arc, consider the transition matrix from one of these bases to the other. The shading on the arc indicates the nature of this transition matrix.

(i) Solid arc: Transition matrix is diagonal.
(ii) Dashed arc: Transition matrix is lower triangular.
(iii) Dotted arc: The bases are the inversion of one another.

See [39] for more information concerning the 24 bases.

5 The classifying space

We will shortly give a classification of the Leonard systems. In order to describe the result we recall the notion of a parameter array.

**Definition 5.1** Let $d$ denote a nonnegative integer. By a parameter array over $\mathbb{K}$ of diameter $d$ we mean a sequence of scalars $(\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ taken from $\mathbb{K}$ which satisfy the following conditions (PA1)–(PA5).

(\text{PA1}) \quad \theta_i \neq \theta_j, \quad \theta^*_i \neq \theta^*_j \quad \text{if} \quad i \neq j, \quad (0 \leq i, j \leq d).

(\text{PA2}) \quad \varphi_i \neq 0, \quad \phi_i \neq 0 \quad (1 \leq i \leq d).

(\text{PA3}) \quad \varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) \quad (1 \leq i \leq d).
\[ \phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d). \]

(\text{PA5}) The expressions

\[ \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \]

are equal and independent of \( i \) for \( 2 \leq i \leq d - 1 \).

6 The classification of Leonard systems

In this section we give a bijection from the set of parameter arrays to the set of isomorphism classes of Leonard systems. Let \((\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)\) denote a parameter array over \( K \). Define

\[
\begin{align*}
A &= \begin{pmatrix}
\theta_0 & 0 & & & \\
1 & \theta_1 & 0 & & \\
& 1 & \theta_2 & \ddots & \\
& & \ddots & \ddots & 0 \\
0 & & & 1 & \theta_d
\end{pmatrix}, & A^* &= \begin{pmatrix}
\theta_0^* & \varphi_1 & 0 & & \\
\theta_1^* & \varphi_2 & 0 & & \\
& \ddots & \ddots & \ddots & \\
0 & & \ddots & \ddots & \varphi_d \\
& & & \ddots & \theta_d^*
\end{pmatrix}.
\end{align*}
\]

Observe \( A \) (resp. \( A^* \)) is multiplicity free with eigenvalues \( \theta_0, \theta_1, \ldots, \theta_d \) (resp. \( \theta_0^*, \theta_1^*, \ldots, \theta_d^* \)). For \( 0 \leq i \leq d \) let \( E_i \) (resp. \( E_i^* \)) denote the primitive idempotent of \( A \) (resp. \( A^* \)) associated with \( \theta_i \) (resp. \( \theta_i^* \)). Then the sequence \((A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)\) is a Leonard system on \( K^{d+1} \). This construction induces a bijection from the set of parameter arrays to the set of isomorphism classes of Leonard systems \([37]\).

7 Leonard pairs \( A, A^* \) with \( A \) lower bidiagonal and \( A^* \) upper bidiagonal

Let \( A, A^* \) denote matrices in \( \text{Mat}_{d+1}(K) \). Let us assume \( A \) is lower bidiagonal and \( A^* \) is upper bidiagonal. We give a necessary and sufficient condition for \( A, A^* \) to be a Leonard pair.

Theorem 7.1 \([40]\) Let \( d \) denote a nonnegative integer and let \( A, A^* \) denote matrices in \( \text{Mat}_{d+1}(K) \). Assume \( A \) lower bidiagonal and \( A^* \) is upper bidiagonal. Then the following (i), (ii) are equivalent.

(i) The pair \( A, A^* \) is a Leonard pair.

(ii) There exists a parameter array \((\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)\) over \( K \) such that

\[
\begin{align*}
A_{ii} &= \theta_i, & A_{ii}^* &= \theta_i^* \quad (0 \leq i \leq d), \\
A_{i,i-1}A_{i-1,i}^* &= \varphi_i \quad (1 \leq i \leq d).
\end{align*}
\]
8 Leonard pairs $A, A^*$ with $A$ tridiagonal and $A^*$ diagonal

Let $A, A^*$ denote matrices in $\text{Mat}_{d+1}(K)$. Let us assume $A$ is tridiagonal and $A^*$ is diagonal. We give a necessary and sufficient condition for $A, A^*$ to be a Leonard pair.

Theorem 8.1 Let $d$ denote a nonnegative integer and let $A, A^*$ denote matrices in $\text{Mat}_{d+1}(K)$. Assume $A$ is tridiagonal and $A^*$ is diagonal. Then the following (i), (ii) are equivalent.

(i) The pair $A, A^*$ is a Leonard pair.

(ii) There exists a parameter array $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ over $K$ such that

\[
A_{ii} = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*} \quad (0 \leq i \leq d),
\]

\[
A_{i,i-1}A_{i-1,i} = \varphi_i \varphi_{i-1} \prod_{h=0}^{i-2}(\theta_{i-1}^* - \theta_h^*) \prod_{h=i+1}^{d}(\theta_i^* - \theta_h^*) \prod_{h=i}^{d-1}(\theta_{i-1}^* - \theta_h^*) \quad (1 \leq i \leq d),
\]

\[
A_{i,i}^* = \theta_i^* \quad (0 \leq i \leq d).
\]

9 A characterization of the parameter arrays I

In this section we characterize the parameter arrays in terms of bidiagonal matrices. We will refer to the following set-up.

Definition 9.1 Let $d$ denote a nonnegative integer and let $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ denote a sequence of scalars taken from $K$. We assume this sequence satisfies PA1 and PA2.

Theorem 9.2 With reference to Definition 9.1, the following (i), (ii) are equivalent.

(i) The sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ satisfies PA3–PA5.

(ii) There exists an invertible matrix $G \in \text{Mat}_{d+1}(K)$ such that both

\[
G^{-1} \begin{pmatrix} \theta_0 & 1 & 0 & \cdots & 0 \\ 1 & \theta_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \theta_d & \cdots & 0 \\ \theta_0^* & \varphi_1 & \theta_1^* & \cdots & \theta_d^* \end{pmatrix} G = \begin{pmatrix} \theta_d & 1 & \theta_{d-1} & \cdots & 0 \\ 1 & \theta_{d-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \theta_0 & \cdots & 0 \\ \theta_0^* & \phi_1 & \theta_1^* & \cdots & \phi_d \end{pmatrix},
\]

\[
G^{-1} \begin{pmatrix} \theta_0^* & \varphi_1 & \theta_1^* & \cdots & \theta_d^* \\ \theta_0^* & \varphi_1 & \theta_1^* & \cdots & \theta_d^* \\ \theta_1^* & \varphi_2 & \theta_2^* & \cdots & \theta_{d-1}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \varphi_d & \theta_d & \cdots & \theta_0^* \end{pmatrix} G = \begin{pmatrix} \theta_0^* & \phi_1 & \theta_1^* & \cdots & \phi_d \\ \theta_0^* & \phi_1 & \theta_1^* & \cdots & \phi_d \\ \theta_1^* & \phi_2 & \theta_2^* & \cdots & \phi_{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \phi_d & \theta_d & \cdots & \theta_0^* \end{pmatrix}.
\]
10 A characterization of the parameter arrays II

In this section we characterize the parameter arrays in terms of polynomials. We will use the following notation. Let \( \lambda \) denote an indeterminate, and let \( K[\lambda] \) denote the \( K \)-algebra consisting of all polynomials in \( \lambda \) which have coefficients in \( K \). From now on all polynomials which we discuss are assumed to lie in \( K[\lambda] \).

**Theorem 10.1** [44] With reference to Definition 9.1, the following (i), (ii) are equivalent.

(i) The sequence \((\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)\) satisfies PA3–PA5.

(ii) For \(0 \leq i \leq d\) the polynomial

\[
\sum_{n=0}^{i} \frac{(\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{n-1})(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{n-1}^*)}{\varphi_1\varphi_2 \cdots \varphi_n}
\]  

is a scalar multiple of the polynomial

\[
\sum_{n=0}^{i} \frac{(\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-n+1})(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{n-1}^*)}{\phi_1\phi_2 \cdots \phi_n}.
\]

11 Some orthogonal polynomials of the Askey scheme

There is a natural correspondence between Leonard systems and a class of orthogonal polynomials consisting of the \( q \)-Racah polynomials and some related polynomials of the Askey scheme. This correspondence is described as follows [44].

Let \( \Phi \) denote a Leonard system. By the *polynomials which correspond to \( \Phi \)* we mean the polynomials from (2), where the parameter array from that line is the one associated with \( \Phi \). The polynomials which correspond to a Leonard system are listed below:

\( q \)-Racah, \( q \)-Hahn, dual \( q \)-Hahn, \( q \)-Krawtchouk, dual \( q \)-Krawtchouk, quantum \( q \)-Krawtchouk, affine \( q \)-Krawtchouk, Racah, Hahn, dual-Hahn, Krawtchouk, Bannai/Ito, and orphan polynomials.

The Bannai/Ito polynomials can be obtained from the \( q \)-Racah polynomials by letting \( q \) tend to \(-1\). The orphan polynomials have maximal degree 3 and exist for char(\( K \)) = 2 only. See [27] for information on the Askey scheme.

12 The Askey-Wilson relations

We turn our attention to the representation theoretic aspects of Leonard pairs.

**Theorem 12.1** [46] Let \( V \) denote a vector space over \( K \) with finite positive dimension. Let \( A, A^* \) denote a Leonard pair on \( V \). Then there exists a sequence of scalars \( \beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^* \) taken from \( K \) such that both

\[
A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \varrho A^* = \gamma^* A^2 + \omega A + \eta I,
\]

\[
A^* 2 A - \beta A^* A A^* + A A^* 2 - \gamma^* (A^* A + A A^*) - \varrho^* A = \gamma A^2 + \omega A^* + \eta^* I.
\]
The sequence is uniquely determined by the pair $A, A^*$ provided the dimension of $V$ is at least 4.

The following theorem is a kind of converse to Theorem 12.1.

**Theorem 12.2** [46] Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $A : V \to V$ and $A^* : V \to V$ denote linear transformations. Suppose that:

- There exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ taken from $\mathbb{K}$ which satisfies (3), (3).
- $q$ is not a root of unity, where $q + q^{-1} = \beta$.
- Each of $A$ and $A^*$ is multiplicity-free.
- There does not exist a subspace $W \subseteq V$ such that $W \neq 0$, $W \neq V$, $AW \subseteq W$, $A^*W \subseteq W$.

Then $A, A^*$ is a Leonard pair on $V$.

### 13 Leonard pairs and the Lie algebra $sl_2$

In this section we assume the field $\mathbb{K}$ is algebraically closed with characteristic zero.

We recall the Lie algebra $sl_2 = sl_2(\mathbb{K})$. This algebra has a basis $e, f, h$ satisfying

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h,$$

where $[,]$ denotes the Lie bracket.

We recall the irreducible finite dimensional modules for $sl_2$.

**Lemma 13.1** [26, p. 102] There exists a family $V_d$ of irreducible finite dimensional $sl_2$-modules with the following properties. The module $V_d$ has a basis $v_0, v_1, \ldots, v_d$ satisfying $hv_i = (d - 2i)v_i$ for $0 \leq i \leq d$, $fv_i = (i + 1)v_{i+1}$ for $0 \leq i \leq d - 1$, $fv_d = 0$, $ev_i = (d - i + 1)v_{i-1}$ for $1 \leq i \leq d$, $ev_0 = 0$. Every irreducible finite dimensional $sl_2$-module is isomorphic to exactly one of the modules in line (3).

**Theorem 13.2** [24, Ex. 1.5] Let $A$ and $A^*$ denote semi-simple elements in $sl_2$ and assume $sl_2$ is generated by these elements. Let $V$ denote an irreducible finite dimensional module for $sl_2$. Then the pair $A, A^*$ acts on $V$ as a Leonard pair.

We remark the Leonard pairs in Theorem 13.2 correspond to the Krawtchouk polynomials [27].
14 Leonard pairs and $U_q(sl_2)$

In this section we assume $\mathbb{K}$ is algebraically closed. We fix a nonzero scalar $q \in \mathbb{K}$ which is not a root of unity. We recall the quantum algebra $U_q(sl_2)$.

**Definition 14.1** [26, p.122] Let $U_q(sl_2)$ denote the associative $\mathbb{K}$-algebra with 1 generated by symbols $e, f, k, k^{-1}$ subject to the relations

$$kk^{-1} = k^{-1}k = 1,$$

$$ke = q^2ek, \quad kf = q^{-2}fk,$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

We recall the irreducible finite dimensional modules for $U_q(sl_2)$. We use the following notation.

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{Z}.$$

**Lemma 14.2** [26, p. 128] With reference to Definition 14.1, there exists a family $V_{\varepsilon, d}$ for $\varepsilon \in \{1, -1\}$, $d = 0, 1, 2 \ldots$ (4) of irreducible finite dimensional $U_q(sl_2)$-modules with the following properties. The module $V_{\varepsilon, d}$ has a basis $u_0, u_1, \ldots, u_d$ satisfying $ku_i = \varepsilon q^{d-2i}u_i$ for $0 \leq i \leq d$, $fu_i = [i + 1]_q u_{i+1}$ for $0 \leq i \leq d-1$, $fu_d = 0$, $eu_i = \varepsilon [d-i+1]_q u_{i-1}$ for $1 \leq i \leq d$, $eu_0 = 0$. Every irreducible finite dimensional $U_q(sl_2)$-module is isomorphic to exactly one of the modules $V_{\varepsilon, d}$. (Referring to line (4), if $\mathbb{K}$ has characteristic 2 we interpret the set $\{1, -1\}$ as having a single element.)

**Theorem 14.3** [30], [31], [42] Referring to Definition 14.1 and Lemma 14.2, let $\alpha, \beta$ denote nonzero scalars in $\mathbb{K}$ and define $A, A^*$ as follows.

$$A = \alpha f + \frac{k}{q - q^{-1}}, \quad A^* = \beta e + \frac{k^{-1}}{q - q^{-1}}.$$  

Let $d$ denote a nonnegative integer and choose $\varepsilon \in \{1, -1\}$. Then the pair $A, A^*$ acts on $V_{\varepsilon, d}$ as a Leonard pair provided $\varepsilon \alpha \beta$ is not among $q^{d-1}, q^{d-3}, \ldots, q^{1-d}$.

We remark the Leonard pairs in Theorem 14.3 correspond to the quantum $q$-Krawtchouk polynomials [27], [29].
15 Leonard pairs in combinatorics

Leonard pairs arise in many branches of combinatorics. For instance they arise in the theory of partially ordered sets (posets). We illustrate this with a poset called the subspace lattice $L_n(q)$.

In this section we assume our field $\mathbb{K}$ is the field $\mathbb{C}$ of complex numbers.

To define the subspace lattice we introduce a second field. Let $GF(q)$ denote a finite field of order $q$. Let $n$ denote a positive integer and let $W$ denote an $n$-dimensional vector space over $GF(q)$. Let $P$ denote the set consisting of all subspaces of $W$. The set $P$, together with the containment relation, is a poset called $L_n(q)$.

Using $L_n(q)$ we obtain a family of Leonard pairs as follows. Let $\mathbb{C}P$ denote the vector space over $\mathbb{C}$ consisting of all formal $\mathbb{C}$-linear combinations of elements of $P$. We observe $P$ is a basis for $\mathbb{C}P$ so the dimension of $\mathbb{C}P$ is equal to the cardinality of $P$.

We define three linear transformations on $\mathbb{C}P$. We call these $K$, $R$ (for “raising”), $L$ (for “lowering”).

We begin with $K$. For all $x \in P$,

$$Kx = q^{n/2 - \dim x} x.$$ 

Apparently each element of $P$ is an eigenvector for $K$.

To define $R$ and $L$ we use the following notation. For $x, y \in P$ we say $y$ covers $x$ whenever (i) $x \subseteq y$ and (ii) $\dim y = 1 + \dim x$.

The maps $R$ and $L$ are defined as follows. For all $x \in P$,

$$Rx = \sum_{y \text{ covers } x} y.$$ 

Similarly

$$Lx = q^{(1-n)/2} \sum_{x \text{ covers } y} y.$$ 

(The scalar $q^{(1-n)/2}$ is included for aesthetic reasons.)

We consider the properties of $K, R, L$. From the construction we find $K^{-1}$ exists. By combinatorial counting we verify

$$KL = qLK,$$

$$KR = q^{-1}RK,$$

$$LR - RL = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}}.$$ 

We recognize these equations. They are the defining relations for $U_{q^{1/2}}(sl_2)$. Apparently $K, R, L$ turn $\mathbb{C}P$ into a module for $U_{q^{1/2}}(sl_2)$. 
We now see how to get Leonard pairs from \( L_n(q) \). Let \( \alpha, \beta \) denote nonzero complex scalars and define \( A, A^* \) as follows.

\[
A = \alpha R + \frac{K}{q^{1/2} - q^{-1/2}}, \quad A^* = \beta L + \frac{K^{-1}}{q^{1/2} - q^{-1/2}}.
\]

To avoid degenerate situations we assume \( \alpha \beta \) is not among \( q^{(n-1)/2}, q^{(n-3)/2}, \ldots, q^{(1-n)/2} \).

The \( U_{q^{1/2}}(sl_2) \)-module \( CP \) is completely reducible \([26, p. 144]\). In other words \( CP \) is a direct sum of irreducible \( U_{q^{1/2}}(sl_2) \)-modules. On each irreducible module in this sum the pair \( A, A^* \) acts as a Leonard pair. This follows from Theorem \([14,3]\).

We just saw how the subspace lattice gives Leonard pairs. It is implicit in \([35]\) that the following posets give Leonard pairs in a similar fashion: the subset lattice, the Hamming semi-lattice, the attenuated spaces, and the classical polar spaces. Definitions of these posets can be found in \([35]\).

16 Further reading

We mention some additional topics which are related to Leonard pairs.

Earlier in this paper we obtained Leonard pairs from the irreducible finite dimensional modules for the Lie algebra \( sl_2 \) and the quantum algebra \( U_q(sl_2) \). We cite some other algebras whose modules are related to Leonard pairs. These are the Askey-Wilson algebra \([12, 13, 14, 15, 16, 47, 48, 49]\), the Onsager algebra \([1, 8, 9, 10]\), and the Tridiagonal algebra \([24, 37, 38]\).

We discussed how certain classical posets give Leonard pairs. Another combinatorial object which gives Leonard pairs is a \( P \)- and \( Q \)-polynomial association scheme \([3, 4, 36]\). Leonard pairs have been used to describe certain irreducible modules for the subconstituent algebra of these schemes \([5, 6, 7, 24, 36]\).

The topic of Leonard pairs is closely related to the work of Grünbaum and Haine on the "bispectral problem" \([19, 20]\). See \([17, 18, 21, 22, 23]\) for related work.

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