Estimation of Optimal Dynamic Treatment Assignment Rules under Policy Constraints

Shosei Sakaguchi†

May 30, 2022

Abstract

This paper studies statistical decisions for dynamic treatment assignment problems. Many policies involve dynamics in their treatment assignments where treatments are sequentially assigned to individuals across multiple stages and the effect of treatment at each stage is usually heterogeneous with respect to the prior treatments, past outcomes, and observed covariates. We consider estimating an optimal dynamic treatment rule that guides the optimal treatment assignment for each individual at each stage based on the individual’s history. This paper proposes an empirical welfare maximization approach in a dynamic framework. The approach estimates the optimal dynamic treatment rule from panel data taken from an experimental or quasi-experimental study. The paper proposes two estimation methods: one solves the treatment assignment problem at each stage through backward induction, and the other solves the whole dynamic treatment assignment problem simultaneously across all stages. We derive finite-sample upper bounds on the worst-case average welfare-regrets for the proposed methods and show $n^{-1/2}$-minimax convergence rates. We also modify the simultaneous estimation method to incorporate intertemporal budget/capacity constraints.

Keywords: Dynamic treatment effect, dynamic treatment regime, individualized treatment rule, empirical welfare maximization.

JEL codes: C22, C44, C54.

* I would like to thank Toru Kitagawa, Aleksey Tetenov, Ryo Okui, Jeff Rowley, and participants in seminars at UCL and University of Tokyo and Cemmap/WISE Workshop on Advances in Econometrics in Xiamen, the 2019 Asian Meeting of the Econometric Society in Xiamen, and the 2020 World Congress of the Econometric Society for their comments and suggestions. I acknowledge financial support from ERC Grant (number 715940).

† Faculty of Economics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan. Email: sakaguchi@e.u-tokyo.ac.jp.
1 Introduction

Many policies involve dynamics in their treatment assignments. Some policies assign a series of treatments to each individual across multiple stages, e.g., job training programs that consist of multiple stages (e.g., Lechner (2009); Rodríguez et al. (2018)). Some policies are characterized by when to start or stop consecutive treatment assignment, e.g., unemployment insurance programs with reduced benefit level after unemployment duration exceeds a certain length (e.g., Meyer (1995); Kolsrud et al. (2018)). Examples of dynamic treatment assignment also include sequential medical interventions, educational interventions, and online advertisements, among others.

For a dynamic treatment policy, the effect of treatment at each stage is usually heterogeneous with respect to the past treatments, past outcomes, and individual characteristics. Thus, when implementing a dynamic treatment policy, a policy-maker wants to know how to assign a series of treatments across multiple stages to maximize social welfare. The decision of treatment assignment to an individual at each stage should depend upon his/her accumulated information at the corresponding stage. In the context of the sequential job training program, the policy-maker’s interest is in which training regimen to assign to each individual at each stage depending on his/her history of prior training participation, associated labor outcomes, and other observed characteristics. In the context of the unemployment insurance policy, an important question is when and whom to reduce the insurance level given a recipient’s characteristics and past effort towards job search.

This paper proposes a statistical decision approach to solve dynamic treatment choice problems using panel data from experimental or quasi-experimental study. We assume dynamic unconfoundedness (Robins (1997)) holds, meaning that the treatment assignment at each stage is independent of current and future potential outcomes given the history of treatment assignments, associated outcomes, and observed covariates. Under this assumption, we construct an approach to estimate the optimal Dynamic Treatment Regime (DTR)\(^1\) building on the idea of Empirical Welfare Maximization (EWM) (Kitagawa and Tetenov (2018)). The EWM approach is based on similarity between treatment choice by

\(^{1}\)Borrowing the terminology of statistics literature, we call the dynamic treatment assignment rule DTR.
maximizing empirical welfare and classification by minimizing empirical misclassification risk. We call the proposed approach Dynamic Empirical Welfare Maximization (DEWM) approach. The DEWM approach estimates the optimal DTR by maximizing empirical welfare, which is expressed in terms of a sample mean of propensity score weighted outcomes and a given DTR, over a pre-specified class of feasible DTRs. True (estimated) propensity scores are used in the experimental (observational) data setting.

The DEWM approach can accommodate exogenous policy constraints (e.g., interpretability or fairness) by restricting the class of feasible DTRs. Moreover, the DEWM approach can be applied to different types of dynamic treatment choice problems, such as start/stop time decision problems, where the interest is in when consecutive treatment assignment should be started or stopped for each individual, and one-shot decision problem, where the interest is in when each individual should receive a one-shot treatment. Various types of dynamic treatment assignment problems can be specified by constraining the relationship of treatment assignments across stages.

Adapting the EWM approach to the dynamic treatment framework is a nontrivial task. In the dynamic framework, the effect of treatment at each stage can vary with respect to the past treatments and outcomes. Hence, the treatment at each stage should be decided by taking account of not only the direct effect of the treatment on current and future outcomes but also its indirect effect on future outcomes through changing the effects of future treatments. We solve this problem by providing two approaches. One approach is to estimate the optimal DTR through backward induction, which solves the treatment choice problem from the final stage to the first stage supposing at each stage that the optimal treatments are chosen in subsequent stages. The other approach is to estimate the optimal DTR simultaneously over all stages, solving the whole empirical welfare maximization problem at once with respect to the entire DTR.

The two approaches are complementary each other. The backward estimation method is computationally efficient; however, consistency is ensured only when the pre-specified class of DTRs contains the first-best policy that assigns the best treatment to any individual with any history at all stages (except for the first stage). In contrast, the simultaneous estimation method can consistently estimate the optimal DTR on the pre-specified class of DTRs irrespective of the feasibility of the first-best policy, at the cost of computational
efficiency.

Practically, some dynamic policies have budget/capacity constraints on their treatment assignments, and these constraints are often imposed intertemporally. Then a favorable DTR should effectively allocate the limited budget/capacity to each stage so that welfare is maximized subject to the limited budget/capacity. For this reason, the simultaneous estimation method is modified in a way that the estimated DTR maximizes the empirical welfare criteria under empirical budget/capacity constraints.

The statistical performance of the DEWM approach is evaluated in terms of average welfare-regret that is the average welfare loss relative to the maximum welfare achievable in the pre-specified class of DTRs. We derive finite-sample and distribution-free upper bounds on the average welfare-regrets of the backward and simultaneous estimation methods, which depend on the sample size $n$ and a measure of complexity of the class of DTRs. Our main theorem shows that the average welfare-regrets of the two methods converge to zero at rate $n^{-1/2}$ in the experimental data setting. This rate is shown to be the optimal convergence rate. In the observational data setting, the rate of convergence depends on that of the estimated propensity scores. When the budget/capacity constraints are imposed, we also analyze the excess implementation cost of the estimated DTR over the actual budget/capacity. We derive finite sample and distribution-free upper bounds on both the welfare-regret and the excess cost of the estimated DTR which hold with high probability.

**Related Literature**

This paper is related to the econometric literature on the statistical decision of treatment choice, although most works in that literature focus on the static treatment assignment problem. A partial list of works in that literature is Manski (2004), Dehejia (2005), Hirano and Porter (2009), Stoye (2009, 2012), Bhattacharya and Dupas (2012), Chamberlain (2012), Tetenov (2012), Kitagawa and Tetenov (2018), Athey and Wager (2021), Kitagawa et al. (2021), and Mbakop and Tabord-Meehan (2021). The policy learning methods of Kitagawa and Tetenov (2018), Athey and Wager (2021), and Mbakop and Tabord-Meehan (2021) build on the similarity of the empirical welfare maximizing treatment choice and the empirical risk minimizing classification. Mbakop and Tabord-Meehan
Athey and Wager (2021) apply doubly-robust estimators to the static policy learning, and show that an $n^{-1/2}$-upper bound on regret can be achieved even in the observational data setting.

In the dynamic treatment framework, Han (2020a) relaxes the sequential randomization assumption, allowing for noncompliance, and studies point identification of the average dynamic treatment effects and optimal non-additive DTR, in which treatment assignment at each stage does not depend on the history. Han (2020b) proposes a way to characterize the sharp partial ordering of the counterfactual welfares of DTRs and the sharp set of the optimal DTRs in an instrumental variable setting by using a set of linear programs. Heckman and Navarro (2007) and Heckman et al. (2016) use exclusion restrictions to identify the dynamic treatment effect by extending the literature on the marginal treatment effect, but do not study identification of the optimal DTRs.

Estimation of the optimal DTR has been widely studied in the biostatistics and statistics literatures under the labels of dynamic treatment regime, adaptive strategies, and adaptive interventions. Chakraborty and Moodie (2013), Chakraborty and Murphy (2014), Laber et al. (2014), and Tsiatis et al. (2019) review the developments in this field. There are some dominant approaches: G-estimation (Robins (1989); Robins et al. (1992)) and Q-learning (Murphy (2005); Moodie et al. (2012)). The G-estimation method specifies the Structural Nested Mean Model (SNMM) that is the stage-specific difference of the conditional mean of outcomes between two treatments. The G-estimation method estimates the parameters of the SNMM, and then predicts the optimal DTR from the estimated model (see, e.g., Murphy (2003), Vansteelandt and Goetghebeur (2003), and Robins (2004) for further details about G-estimation). The Q-learning method models the so-called Q-function, a function of the stage-specific expected reward with respect to the current treatment and history given that the optimal treatment is assigned in subsequent stages. The Q-learning method estimates the parameters of the Q-function via backward induction, and then estimates the optimal DTR as the sequence of stage-specific treatment assignment rules that maximize the estimated Q-functions. A potential drawback of these approaches is the risk of misspecification of the models relevant to the counter-
factual outcomes. In contrast, the DEWM approach does not need to specify any model relevant to the counterfactual outcomes; instead, it specifies the propensity scores, which are known in the experimental data setting.

Several works have proposed estimation methods of the optimal DTR based on the similarity between the statistical treatment choice and machine learning classification problems. Zhao et al. (2015) develop estimation methods of the optimal DTR using the Support Vector Machine with propensity score weighted outcomes. They show the convergence of the welfare-regret towards zero. Their approach is computationally efficient because of the use of a convex surrogate loss function. However, their approach is not consistent if the pre-specified class of DTRs does not contain the first-best treatment for every stage. Zhang et al. (2018) propose estimating the list form of the optimal DTR, converting an unconstrained treatment rule estimated by Q-learning to the list form of the decision rule at each stage. Nie et al. (2020) propose an estimation method for the start/stop time decision problem based on the doubly-robust estimator, and show that an $n^{-1/2}$ welfare-regret bound can be achieved even in the observational data setting. While they focus on the start/stop time decision problem, this paper covers a wider class of dynamic treatment choice problems.

Finally, we note that the dynamic framework studied in this paper is different from the framework of the bandit problem studied by, for example, Kock and Thyrsgaard (2018). In the bandit problem framework, different individuals arrive at each stage and each of them receives treatment only once. In contrast, in the dynamic framework studied in this paper, the same individuals arrive at each stage and each of them receives multiple treatments across stages.

**Structure of the Paper**

The remainder of the paper is structured as follows. Section 2 describes the dynamic treatment framework and defines the dynamic treatment choice problem. Section 3 presents the two types of DEWM methods (backward and simultaneous estimation methods) and shows their statistical properties. Section 4 extends the simultaneous estimation method to accommodate the intertemporal budget/capacity constraints. Section 5 modifies the backward and simultaneous methods for the observational data setting. Section 6 presents
a simulation study to evaluate the finite sample performance of the proposed methods. In Section 7, the proposed methods are applied to the Project STAR (Steps to Achieving Resilience) data, where we estimate an optimal DTR to allocate each student to a class with or without a full-time teacher aide in kindergarten and first grade. We conclude this paper with some remarks in Section 8. All proofs are given in Appendix A.

2 Setup

We first introduce the dynamic treatment framework, following Robins’s dynamic counterfactual outcomes framework (Robins (1986, 1997)), in Section 2.1. Subsequently, we define the dynamic treatment choice problem in Section 2.2. Throughout the paper, we denote by \( E_P \) the expectation with respect to a distribution function \( P \) and by \( E_n \) the sample average.

### 2.1 Dynamic Treatment Framework

We suppose that there are \( T (T \geq 2) \) stages of binary treatment assignment. Let \( D_t \in \{0, 1\} \), for \( t = 1, \ldots, T \), denote the binary treatment at stage \( t \). Throughout this paper, for any variable \( A_t \), we denote by \( \mathbf{A_t} \equiv (A_1, \ldots, A_t) \) a history of the variable up to stage \( t \), and denote by \( \mathbf{A_{t,s}} \equiv (A_s, \ldots, A_t) \), for \( s \leq t \), a partial history of the variable from stage \( s \) up to stage \( t \). For example, the history of treatment up to stage \( t \) is denoted by \( \mathbf{D_t} = (D_1, \ldots, D_t) \). Depending on the history of treatment up to stage \( t \), we observe an outcome \( Y_t \) at each stage \( t \). Let \( Y_t (\mathbf{d_t}) \) be a potential outcome at stage \( t \) that is realized when the history of treatment up to stage \( t \) coincides with \( \mathbf{d_t} \in \{0, 1\}^t \). The observed outcome at each stage \( t \) is

\[
Y_t \equiv \sum_{\mathbf{d_t} \in \{0, 1\}^t} (1 \{ \mathbf{D_t} = \mathbf{d_t} \} \cdot Y_t (\mathbf{d_t})) ,
\]

where \( 1 \{ \cdot \} \) denotes the indicator function. Let \( X_t \) be a \( k \)-dimensional vector of covariates that are observed before a treatment is assigned at stage \( t \). The distribution of \( X_t \) may depend on the past treatments, outcomes, and covariates. The covariates at the first stage, \( X_1 \), represent pre-treatment information, which contain individuals’ demographic
characteristics observed prior to the policy implementation. Let $H_t \equiv (D_{t-1}, Y_{t-1}, X_t)$ denote the history of all the observed variables up to stage $t$. $H_t$ is available information for the policy-maker when she chooses a treatment assignment at stage $t$. Note that $H_s \subseteq H_t$ for any $s \leq t$ and that $H_1 = (X_1)$. We denote the support of $H_t$ and $Z \equiv (D_t, X_t, Y_t)_{t=1}^T$, respectively, by $H_t$ and $Z$, and denote by $P$ the distribution of all the defined variables $\left( D_t, \{ Y_t(d_t) \}_{d_t \in \{0,1\}^t}^T, X_t \right)_{t=1}^T$.

From an experimental or observational study, we observe $Z_i \equiv (D_{it}, X_{it}, Y_{it})_{t=1}^T$ for individuals $i = 1, \ldots, n$, where $Y_{it} = \sum_{d_t \in \{0,1\}^t} (1 \{ D_{it} = d_t \} \cdot Y_{it}(d_t))$ with $Y_{it}(d_t)$ being a potential outcome for individual $i$ at stage $t$ that is realized when $D_{it} = d_t$. We suppose that the vectors of random variables $\left( D_{it}, \{ Y_{it}(d_t) \}_{d_t \in \{0,1\}^t}^T, X_{it} \right)_{t=1}^T$, $i = 1, \ldots, n$, are independent and identically distributed (i.i.d) under the distribution $P$. Let $P^n$ denote the join distribution of $\left\{ \left( D_{it}, \{ Y_{it}(d_t) \}_{d_t \in \{0,1\}^t}^T, X_{it} \right)_{t=1}^T : i = 1, \ldots, n \right\}$ generated by $P$.

Let $e_t(d_t, h_t) \equiv \Pr (D_t = d_t \mid H_t = h_t)$ be a propensity score of treatment at stage $t$ given the history up to that point. We suppose that the propensity scores are known under the experimental study, but they are unknown and need to be estimated under the observational study. We consider the experimental and observational study settings in Sections 3-4 and Section 5, respectively.

Throughout the paper, we suppose that the following assumptions hold.

Assumption 2.1 (Sequential Independence Assumption). For any $t = 1, \ldots, T$ and $d_T \in \{0,1\}^T$, $D_t \perp (Y_t(d_t), \ldots, Y_T(d_T)) \mid H_t$ holds.

Assumption 2.2 (Bounded Outcomes). There exists $M_t < \infty$ such that the support of $Y_t$ is contained in $[-M_t/2, M_t/2]$ for $t = 1, \ldots, T$.

Assumption 2.1 is what is called a dynamic unconfoundedness assumption or sequential/dynamic conditional independence assumption elsewhere, which is commonly used in the literature on dynamic treatment effect analysis (Robins (1997); Murphy (2003)). This assumption means that the treatment assignment at each stage is independent of the contemporaneous and future potential outcomes conditional on the history up to that point. This is usually satisfied in sequential randomization experiments. In observational studies, this assumption is often controversial, but can hold if a sufficient set of confounders is
available. Assumption 2.2 is a common assumption in the literature on treatment effect analysis.

2.2 Dynamic Treatment Choice Problem

The goal of this paper is to provide methods to estimate the optimal DTR from experimental or observational panel data. We denote a treatment rule at each stage $t$ by $g_t : \mathcal{H}_t \mapsto \{0, 1\}$, a mapping from the history up to stage $t$ to the binary treatment. We define the DTR by $g \equiv (g_1, \ldots, g_T)$, a sequence of stage-specific treatment rules. The DTR guides the policy-maker to choose a treatment for each individual at each stage depending on the individual’s history up to that point.

We suppose that the welfare that the policy-maker wants to maximize is the population mean of a weighted sum of outcomes: $E_P \left[ \sum_{t=1}^{T} \gamma_t Y_t \right]$, where the weight $\gamma_t$, for $t = 1, \ldots, T$, lies in $[0, 1]$ and is chosen by the policy-maker. If the policy-maker targets a time-discounted welfare, the weight at each stage is $\gamma_t = \gamma^{T-t}$, where $\gamma$ is a time-discount factor and lies in $(0, 1)$. If the policy-maker targets the outcome at the last stage only, the weights are $\gamma_T = 1$ and $\gamma_t = 0$ for all $t \neq T$.

Under a DTR $g$, the realized welfare takes the following form:

$$W(g) = E_P \left[ \sum_{d \in \{0, 1\}^T} \left( \prod_{t=1}^{T} 1 \{g_t(H_t) = d_t\} \cdot \sum_{t=1}^{T} \gamma_t Y_t(d_t) \right) \right]$$

$$= \sum_{t=1}^{T} E_P \left[ \sum_{d \in \{0, 1\}^t} \left( \prod_{s=1}^{t} 1 \{g_s(H_s) = d_s\} \cdot \gamma_t Y_t(d_t) \right) \right].$$

Given the propensity scores $\{e_t(d_t, h_t)\}_{t=1}^{T}$ and Assumption 2.1, the welfare function can be expressed by the observables only:

$$W(g) = \sum_{t=1}^{T} E_P \left[ \frac{\prod_{s=1}^{t} 1 \{g_s(H_s) = D_s\} \gamma_t Y_t}{\prod_{s=1}^{t} e_s(D_s, H_s)} \right].$$

(1)

We suppose that the policy-maker chooses a DTR from a pre-specified class of feasible DTRs, which we denote by $\mathcal{G} \equiv \mathcal{G}_1 \times \cdots \times \mathcal{G}_T$, where $\mathcal{G}_t$ is a class of feasible treatment rules at stage $t$ (i.e., a class of measurable functions $g_t : \mathcal{H}_t \mapsto \{0, 1\}$). Then the ultimate goal of the analysis is to choose an optimal DTR that maximizes the welfare $W(\cdot)$ over
In this paper, we restrict the complexity of the class of feasible DTRs in terms of VC-dimension. The following definition gives the definition of VC-dimension of a class of indicator functions and relevant concepts.

**Definition 2.1** (VC-dimension of a Class of Indicator Functions). Let $\mathcal{Z}$ be an arbitrary space and $\mathcal{F}$ be a class of indicator functions from $\mathcal{Z}$ to $\{0,1\}$. For a finite sample $S = (z_1,\ldots,z_m)$ of $m \geq 1$ points in $\mathcal{Z}$, we define the set of dichotomies by $\Pi_{\mathcal{F}}(S) \equiv \{(f(z_1),\ldots,f(z_m)) : f \in \mathcal{F}\}$, which is all possible assignments of $S$ by functions in $\mathcal{F}$. We say that $S$ is shattered by $\mathcal{F}$ when $|\Pi_{\mathcal{F}}(S)| = 2^m$; that is, $\mathcal{F}$ realizes all possible dichotomies of $S$. Then the VC-dimension of $\mathcal{F}$, denoted by $VC(\mathcal{F})$, is defined to be the size of the largest sample $S$ shattered by $\mathcal{F}$, i.e.,

$$VC(\mathcal{F}) \equiv \max\left\{ m : \max_{S=(z_1,\ldots,z_m) \subseteq \mathcal{Z}} |\Pi_{\mathcal{F}}(S)| = 2^m \right\}.$$ 

We say that $\mathcal{F}$ is a VC-class of indicator functions if $VC(\mathcal{F}) < \infty$.

The following assumption restricts the complexity of the class of feasible DTRs $\mathcal{G}$ in terms of the VC-dimension of $\mathcal{G}_t$ for each $t = 1,\ldots,T$.

**Assumption 2.3** (VC-class). For $t = 1,\ldots,T$, $\mathcal{G}_t$ is a VC-class of functions and has VC-dimension $v_t < \infty$.

This assumption restricts the complexity of the class of whole DTRs $\mathcal{G}$ by restricting the class of feasible treatment rules $\mathcal{G}_t$ at each stage. By restricting the complexity, we can choose a simple to explain/interpret DTR and keep estimated DTRs from overfitting to the data. We can also incorporate arbitrary exogenous policy constraints from ethical or political reasons into DTRs by specifying the form of $\mathcal{G}_t$ for each $t$. Some examples of practically relevant classes of DTRs are linear eligibility score rules and decision tree rules.
**Example 2.2 (Linear Eligibility Score).** The class of DTRs based on the Linear Eligibility Score is $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_T$ with $\mathcal{G}_t$, $t = 1, \ldots, T$, having the following form:

$$
\mathcal{G}_t = \left\{ 1 \left\{ \beta_{1t}^t x_t + \beta_{2t}^t d_{t-1} + \beta_{3t}^t y_{t-1} \geq c_t \right\} : (\beta_{1t}^t, \beta_{2t}^t, \beta_{3t}^t, c_t) \in \mathbb{R}^{k+2t-1} \right\}.
$$

Under this class of DTRs, treatment assignment at each stage is decided based on whether or not the linear eligibility score ($\beta_{1t}^t x_t + \beta_{2t}^t d_{t-1} + \beta_{3t}^t y_{t-1}$) exceeds a certain threshold value $c_t$. The main objective of data analysis using this class is to construct an eligibility score (i.e., to decide $(\beta_{1t}^t, \beta_{2t}^t, \beta_{3t}^t, c_t)_{t=1,\ldots,T}$) such that the obtained DTR maximizes the welfare $W(\cdot)$ over $\mathcal{G}$. In this example, each $\mathcal{G}_t$ has VC-dimension at most $k + 2t - 1$.

**Example 2.3 (Decision Tree).** The decision tree representation of $g_t$ is a tree representing partition of $\mathcal{H}_t$, which predicts treatment 0 or 1 by traveling from a root node to a leaf node of a tree. For any integer $L \geq 1$, a depth-$L$ decision tree has $L$ layers, in which the first $L-1$ layers consist of branch nodes, and the $L$-th layer consists of leaf nodes. Let $p_t$ denote the dimension of $H_t$. Each branch node in the $\ell$-th layer is specified by a splitting variable $H_{tj} \in H_t$ for some $j = 1, \ldots, p_t$, a threshold value $c \in \mathbb{R}$, and left and right child nodes in the $(\ell + 1)$-th layer. If $H_{tj} \geq c$, the left child node is followed; otherwise, the right child node is followed. Every path terminates at a leaf node, each of which assigns $g(h_t) = 0$ or 1. From Lemma 4 of Zhou et al. (2018), the depth-$L$ decision tree class of $\mathcal{G}_t$ on $\mathcal{H}_t$ has VC-dimension bounded on the order of $VC(\mathcal{G}_t) = \tilde{O}(2^L \log(p_t))$.\footnote{The notation $f(n) = \tilde{O}(g(n))$ means that there is a function $h(\cdot)$ that scales poly-logarithmically in its argument for which $f(n) \leq h(g(n))g(n)$. See Zhou et al. (2018) and Athey and Wager (2021) for detail.}

Aside from the constraint on the functional form, by restricting the intertemporal relationship of treatment rules across stages, we can specify a type of dynamic treatment choice problem.

**Example 2.4 (One-shot Treatment).** If the policy-maker wants to decide when to assign a one-shot treatment to each individual, the analyst should impose the restriction $\sum_{s=1}^{t-1} d_s + g_t(\cdot) \leq 1$ on $\mathcal{G}_t$ for each $t$. 
Example 2.5 (Start/Stop Time Decision). If the problem is to decide when to start or stop consecutive treatment assignment for each individual, the restriction $d_s \leq g_t(\cdot)$ or $d_s \geq g_t(\cdot)$, for all $s \leq t$, should be imposed on $G_t$, respectively.

Note that the VC-dimension of an additionally restricted class is not larger than that of the original class.

Given a possibly constrained class of DTRs $G$, we suppose that the following overlap condition holds on the propensity scores $\{e_t(d_t, h_t)\}_{t=1}^T$.

**Assumption 2.4** (Overlap Condition). For $t = 1, \ldots, T$, there exists $\kappa_t \in (0, 1)$ for which $\kappa_t \leq e_t(d_t, h_t)$ holds for any pair $(d_t, h_t) \in \{0, 1\} \times H_t$ such that there exists $g_t \in G_t$ that satisfies $g_t(h_t) = d_t$.

When $G$ is structurally constrained, Assumption 2.4 requires less strict overlap condition than a common overlap condition which requires $e_t(d_t, h_t) \in (0, 1)$ for all $(d_t, h_t) \in \{0, 1\} \times H_t$ and $t = 1, \ldots, T$.

We denote the highest welfare that is attainable on the class of feasible DTRs $G$ by

$$W^*_G \equiv \max_{g \in G} W(g).$$

We consider estimating the optimal DTR that maximizes the welfare $W(\cdot)$ over $G$ from the sample $\{Z_i\}_{i=1}^n$. In the following section, we provide two methods to estimate the optimal DTR and evaluate their statistical properties in terms of the maximum average welfare-regret.

### 3 Dynamic Empirical Welfare Maximization

This section proposes two DEWM methods. One is based on backward induction (dynamic programming) to solve the dynamic treatment choice problem, sequentially, from the final stage to the initial stage. The other is based on simultaneous optimization of
W (·) over the whole class of DTRs \( \mathcal{G} \). The backward estimation method is computationally efficient, but it can consistently estimate the optimal DTR only when \( \mathcal{G}_t \) contains the first-best treatment rule for all \( t \geq 2 \). In contrast, the simultaneous estimation method can consistently estimate the optimal DTR irrespective of whether \( \mathcal{G}_t \) contains the first-best treatment rule at each stage \( t \), though it is computationally less efficient. We explain the backward and simultaneous estimation methods in Sections 3.1 and 3.2, respectively. We subsequently evaluate the statistical properties of these methods in terms of the maximum welfare-regret in Section 3.3.

### 3.1 Backward Dynamic Empirical Welfare Maximization

We first explain the backward estimation method. To guarantee consistent estimation with a given distribution \( P \), the following assumption requires that the first-best treatment rule is attainable at all but the first stage.

**Assumption 3.1 (First-Best Treatment Rule).** *For any \( t = 2, \ldots, T \), there exists \( g^*_{t,FB} \in \mathcal{G}_t \) such that*

\[
E_P \left[ \sum_{s=t}^{T} \gamma_s Y_s \left( \Delta_{s-1}, g^*_{t,FB}(H_t), \ldots, g^*_{T,FB}(H_T) \right) \mid H_t = h_t \right] \\
\geq E_P \left[ \sum_{s=t}^{T} \gamma_s Y_s \left( \Delta_{s-1}, 1 - g^*_{t,FB}(H_t), g^*_{t+1,FB}(H_{t+1}), \ldots, g^*_{T,FB}(H_T) \right) \mid H_t = h_t \right]
\]

*for all \( h_t \in \mathcal{H}_t \).*

We call \( g^*_{t,FB} \) that satisfies Assumption 3.1 the first-best treatment rule at stage \( t \). This assumption is satisfied when \( \mathcal{G}_t, t = 2, \ldots, T \), are rich enough or are correctly specified in the sense that they contain the first-best treatment rule. Note that the treatment class for the first stage \( \mathcal{G}_1 \) is not required to contain the first-best treatment rule.

To present the idea of the backward estimation method, we now suppose that the generative distribution function \( P \) is known and that the pair \((P, \mathcal{G})\) satisfies Assumptions 2.1 and 3.1. Then we can solve the dynamic treatment choice problem with dynamic
programming. Firstly, for the final stage $T$, let

$$g_T^* (h_T) \in \arg \max_{g_T \in \mathcal{G}_T} Q_T (h_T, g_T)$$

for any $h_T \in \mathcal{H}_T$, where

$$Q_T (h_T, g_T) \equiv E_P \left[ \gamma_T Y_T \mid H_T = h_T, D_T = g_T (h_T) \right]$$

is the conditional mean of the weighted final outcome $\gamma_T Y_T$ given that the history is $h_T$ and that the treatment rule $g_T$ is followed. Under Assumptions 2.1 and 3.1, $g_T^* (h_T)$ is an optimal treatment assignment for any $h_T \in \mathcal{H}_T$, meaning that the function $g_T^* : \mathcal{H}_T \mapsto \{0, 1\}$ coincides with the first-best treatment rule (i.e., $g_T^* = g_{T, FB}^*$). Thus, under Assumptions 2.1 and 3.1, the first-best treatment rule $g_T^*$ can be obtained by solving

$$g_T^* \in \arg \max_{g_T \in \mathcal{G}_T} E_P \left[ Q_T (H_T, g_T) \right], \quad (3)$$

where the objective function is the expectation of $Q_T (H_T, g_T)$ over $\mathcal{H}_T$.

Recursively, from $t = T - 1$ to 1, we obtain

$$g_t^* \in \arg \max_{g_t \in \mathcal{G}_t} E_P \left[ Q_t (H_t, g_t) \right], \quad (4)$$

where

$$Q_t (h_t, g_t) \equiv E_P \left[ \gamma_t Y_t + \max_{g_{t+1} \in \mathcal{G}_{t+1}} Q_{t+1} (H_{t+1}, g_{t+1}) \mid H_t = h_t, D_t = g_t (h_t) \right]$$

$$= E_P \left[ \gamma_t Y_t + Q_{t+1} (H_{t+1}, g_{t+1}^*) \mid H_t = h_t, D_t = g_t (h_t) \right].$$

The function $Q_t (h_t, g_t)$ represents the expected welfare that realizes when the history is $h_t$, $g_t$ is followed at stage $t$, and the optimal treatment rules are followed at the subsequent stages. Under Assumptions 2.1 and 3.1, $g_t^*$ corresponds to the first-best treatment rule $g_{t, FB}^*$ for $t = 2, \ldots, T - 1$, and $g_1^*$ is the optimal treatment rule that maximizes $W (g_1, g_{2, FB}^*, \ldots, g_{T, FB}^*)$ over $g_1 \in \mathcal{G}_1$.

This procedure obtains the optimal treatment rule at each stage by solving the welfare maximization problem given that the optimal treatment rules at the subsequent stages
are known and followed. Since $g_t^*$ corresponds to the first-best treatment rule $g^*_{t,FB}$ for all $t = 2, \ldots, T$ and $g_t^*$ is also the optimal treatment rule given $(g_2^*, \ldots, g_T^*)$, the sequentially identified DTR $g^* \equiv (g_1^*, \ldots, g_T^*)$ corresponds to the solution of the whole welfare maximization problem (2).

If the first-best treatment rule is not achievable in $G_t$ for some stage $t \geq 2$, then $g_s^*$ for $s \leq t$ do not necessary correspond to the optimal treatment rule. We illustrate this problem in the following remark with a simple example and in simulation study in Section 6.

**Remark 3.1.** Suppose that $T = 3$ and that the data generating process $P$ satisfies the following:

$$E_P[Y_2(0, 0, 0)] = 0.2, \ E_P[Y_2(1, 0, 0)] = 0.3, \ E_P[Y_2(0, 1, 0)] = 0.4, \ E_P[Y_2(1, 1, 0)] = 0.5;$$

$$E_P[Y_2(0, 0, 1)] = 0, \ E_P[Y_2(1, 0, 1)] = 0, \ E_P[Y_2(0, 1, 1)] = 0, \ E_P[Y_2(1, 1, 1)] = 1;$$

$D_1$, $D_2$, and $D_3$ are independently distributed as $Ber(1/2)$.

We suppose $H_1 = \emptyset$, $H_2 = (D_1)$, and $H_3 = (D_1, D_2)$. We use a class of DTRs $G = G_1 \times G_2 \times G_3$ with classes of constants $G_t = \{0, 1\}$ for $t = 1, 2, 3$. The target welfare to maximize is

$$W(g) = E_P[Y_3(D_1, D_2, D_3) \mid D_1 = g_1(H_1), D_2 = g_2(H_2), D_3 = g_3(H_3)].$$

In this setting, the first best DTR $g^*_{FB} = (g_{1,FB}^*, g_{2,FB}^*(d_1), g_{3,FB}^*(d_1, d_2))$ satisfies $g_{1,FB}^* = 1$, $g_{2,FB}^*(1) = 1$, and $g_{3,FB}^*(1, 1) = 1$; however, such a DTR is not contained by $G$. The optimal DTR over $G$ is $g^*_opt = (g_{opt,1}^*, g_{opt,2}^*, g_{opt,3}^*) = (1, 1, 1)$ and leads to $W(g^*_opt) = 1$.

We next consider to solve the population welfare maximization problem by the backward induction approach. Firstly, solving (3) with $T = 3$ leads to $g_3^* = 0$. Second, given $g_3^* = 0$, solving (4) with $t = 2$ leads to $g_2^* = 1$. Third, given $g_3^* = 0$ and $g_2^* = 1$, solving (4) with $t = 1$ leads to $g_1^* = 1$. Hence the backward induction solution is $g^* = (1, 1, 0)$ which differs from the optimal solution $g^*_opt$ and leads to a lower welfare 0.5.

The above example implies that when the first best rule is not feasible in $G_t$ ($t \geq 2$), the backward induction solution $g_t^*$ is subject to the data generating process from an
experiment. This data generating process is different from data generating processes that arise when the treatment assignments except for stage $t$ follow the optimal treatment rules. This discrepancy causes the suboptimality of the backward induction approach. When the first best rule is feasible in $G_t$ for each $t \geq 2$, the backward induction solution $g_t^*$ can be the optimal rule irrespective of the data generating process from an experiment, because the first best rule leads to the best treatment assignment for any history information.

Given the propensity scores $\{e_t(D_t, H_t)\}_{t=1}^T$ and under Assumption 2.1, $E_P[Q_t(H_t, g_t)]$ can be written equivalently as

$$E_P[Q_t(H_t, g_t)] = E_P[q_t(Z, g_t; g_{t+1}, \ldots, g_T)],$$

where

$$q_t(Z, g_t; g_{t+1}, \ldots, g_T) \equiv \sum_{s=t}^T \left\{ \prod_{s=t}^1 \{g_{t+1}(H_t) = D_t\} \gamma_s Y_s \prod_{s=t}^T e_t(D_t, H_t) \right\}. $$

Hence, the objective function $E_P[Q_t(H_t, g_t)]$ is expressible in terms of the observables.

Using the propensity score weighting, the first estimation method we propose is based on the empirical analogue of the above backward induction procedure. We call this method the Backward DEWM method. The Backward DEWM method first estimates $g_T^*$ by

$$\hat{g}_T^B \in \arg \max_{g_T \in G_T} \frac{1}{n} \sum_{i=1}^n q_T(Z_i, g_T).$$

Then, recursively, from $t = T-1$ to $1$, $g_t^*$ is estimated by

$$\hat{g}_t^B \in \arg \max_{g_t \in G_t} \frac{1}{n} \sum_{i=1}^n q_t(Z_i, g_t; \hat{g}_{t+1}^B, \ldots, \hat{g}_T^B).$$

We denote by $\hat{g}^B \equiv (\hat{g}_1^B, \ldots, \hat{g}_T^B)$ the DTR obtained from this procedure.

Remark 3.2. The Q-learning method is also based on the idea of backward induction (Murphy (2005); Moodie et al. (2012)). Q-learning uses regression models to estimate the $Q$-functions that correspond to $Q_t(h_t, g_t)$ for $t = 1, \ldots, T$, and then predicts the optimal
DTR from the estimated Q-functions. Q-learning specifies the Q-functions but not the class of feasible DTRs. Linear models are typically used to approximate the Q-functions.

3.2 Simultaneous Dynamic Empirical Welfare Maximization

The second approach is a sample analogue of the whole welfare maximization problem (2). We call the proposed method Simultaneous DEWM method as it simultaneously estimates the optimal treatment rules across all stages. Let \( g \equiv (g_1, \ldots, g_T) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_T \) be the vector of treatment rules up to stage \( t \). The Simultaneous DEWM method estimates the optimal DTR though the maximization of the sample analogue of (1):

\[
(\hat{g}_1^S, \ldots, \hat{g}_T^S) \in \arg \max_{g \in \mathcal{G}} \sum_{t=1}^{T} \left[ \frac{1}{n} \sum_{i=1}^{n} w_t^S(Z_i, g_t) \right],
\]

(6)

where

\[
\begin{align*}
  w_t^S(Z_i, g_t) &\equiv \frac{\prod_{s=1}^{t} 1 \{ g_s(H_{is}) = D_{is} \} \gamma_i Y_{it}}{\prod_{s=1}^{t} e_s(D_{is}, H_{is})}.
\end{align*}
\]

In the maximization problem (6), \( n^{-1} \sum_{i=1}^{n} w_t^S(Z_i, g_t) \) corresponds to the sample analogue of the \( t \)-th term in (1). We denote by \( \hat{g}^S \equiv (\hat{g}_1^S, \ldots, \hat{g}_T^S) \) the DTR obtained from this procedure. Theorem 3.4 below shows that this method can consistently estimate the optimal DTR on \( \mathcal{G} \) even when the class of treatment rule \( \mathcal{G}_t \) does not contain the first-best treatment rule for some \( t \) (i.e., Assumption 3.1 does not hold).

Remark 3.3. When \( \mathcal{G}_t (t = 1, \ldots, T) \) are classes of the linear eligibility scores, the optimization problems (5) for the Backward DEWM and (6) for the Simultaneous DEWM can be formulated as Mixed Integer Linear Programming (MILP) problems. See Appendix B for details.

3.3 Statistical Properties

As in much of the literature that follows Manski (2004), we evaluate the statistical properties of the two DEWM methods in terms of the maximum average welfare-regret that is the maximum average welfare loss relative to the maximum feasible welfare \( W_\mathcal{G}^* \). Following
Kitagawa and Tetenov (2018), we focus on the non-asymptotic upper bounds of the worst-case average welfare-regret, \( \sup_{P \in \mathcal{P}(M, \kappa, \mathcal{G})} E_P [W^*_G - W(\hat{g})] \), for the Backward DEWM method \((\hat{g} = \hat{g}^B)\) and the Simultaneous DEWM method \((\hat{g} = \hat{g}^S)\), where \( \mathcal{P}(M, \kappa, \mathcal{G}) \) is a class of distributions of \((D_t, \{Y_t(d_t)\}_{d_t \in \{0,1\}^t}, X_t)_{t=1}^T\) that satisfy Assumptions 2.1, 2.2, and 2.4 with \( M \equiv (M_1, \ldots, M_T)' \), \( \kappa \equiv (\kappa_1, \ldots, \kappa_T)' \), and a fixed \( \mathcal{G} \).

The following theorem provides a finite-sample upper bound on the worst-case average welfare-regret and shows its dependence on the sample size \( n \), the VC-dimension of \( \mathcal{G}_t \) for each \( t \), and the number of stages \( T \).

**Theorem 3.4.** Suppose that Assumptions 2.1, 2.2, and 2.4 hold for any distribution \( P \in \mathcal{P}(M, \kappa, \mathcal{G}) \) and Assumption 2.3 holds for \( \mathcal{G} \).

(i) For the Simultaneous DEWM method, there holds

\[
\sup_{P \in \mathcal{P}(M, \kappa, \mathcal{G})} E_P [W^*_G - W(\hat{g}^S)] \leq C_n \left( \sum_{t=1}^T \gamma_t M_t \prod_{s=1}^t \kappa_s \sqrt{\sum_{s=1}^t v_s n} \right),
\]

where \( C \) is some universal constant.

(ii) Suppose, in addition, that Assumption 3.1 holds for a pair of \( \mathcal{G} \) and any \( P \in \mathcal{P}(M, \kappa, \mathcal{G}) \). Then, for the Backward DEWM method, there holds

\[
\sup_{P \in \mathcal{P}(M, \kappa, \mathcal{G})} E_P [W^*_G - W(\hat{g}^B)] \leq C_n \left( \sum_{t=1}^T \gamma_t M_t \prod_{s=1}^t \kappa_s \sqrt{\sum_{s=1}^t v_s n} \right) + C_n \left( \sum_{t=2}^T \frac{2^{t-2}}{\prod_{s=1}^{t-1} \kappa_s} \left( \sum_{s=t}^T \frac{\gamma_s M_s \prod_{\ell=t}^{s-1} \kappa_\ell \sqrt{\sum_{\ell=t}^{s-1} v_\ell n}}{n} \right) \right),
\]

where \( C \) is the same universal constant.

**Proof.** See Appendix A.1. \( \square \)

This theorem shows that the convergence rates of the worst-case average welfare-regrets of the two methods are not slower than \( n^{-1/2} \). The upper bounds increase with the VC-dimension of \( \mathcal{G}_t \), implying that as the candidate treatment rules become more complex in terms of VC-dimension, the estimated DTR tends to overfit the data (the distribution of welfare-regret becomes more dispersed). The upper bound corresponding
to the Backward DEWM method is greater than the upper bound corresponding to the Simultaneous DEWM method, though neither bound is necessarily sharp. Technically, the difference between these upper bounds arises because the optimization problem (5) in each step of the Backward DEWM method depends on the previous estimators \( \hat{g}_{t+1}^B, \ldots, \hat{g}_T^B \), which leads to additional uncertainty in the estimation.

The next theorem shows a lower bound on the maximum average welfare-regret for any data-driven DTR. To present the theorem formally, let \( v_{s:t} \), for \( s \leq t \), denote the VC-dimension of the following class of indicator functions on \( Z \):

\[
\{ f(z) = 1 \{ g_s(h_s) = d_s, \ldots, g_t(h_t) = d_t \} : (g_s, \ldots, g_t) \in \mathcal{G}_s \times \cdots \times \mathcal{G}_t \}. 
\]

Note that \( v_{s:t} \leq \sum_{t=s}^T v_\ell \) holds (see Lemma A.3 in Appendix A).

**Theorem 3.5.** Suppose that Assumptions 2.1, 2.2, and 2.4 hold for any distribution \( P \in \mathcal{P}(M, \kappa, \mathcal{G}) \) and Assumption 2.3 holds for \( \mathcal{G} \). Then, for any DTR \( \hat{g} \in \mathcal{G} \) as a function of \( (Z_1, \ldots, Z_n) \), there holds

\[
\sup_{P \in \mathcal{P}(M, \kappa, \mathcal{G})} E_P^n [W_{\hat{g}}^* - W(\hat{g})] \geq \frac{1}{2} \exp \left( -4 \right) \max_{t \in \{1, \ldots, T\}} \left\{ \gamma_t M_t \sqrt{\frac{v_{1:t}}{n}} \right\}
\]

for all \( n \geq 16v_{1:T} \). This result holds irrespective of whether or not Assumption 3.1 additionally holds for a pair of \( \mathcal{G} \) and any \( P \in \mathcal{P}(M, \kappa, \mathcal{G}) \).

**Proof.** See Appendix A.1.

This theorem, along with Theorem 3.4, shows that both \( \hat{g}^S \) and \( \hat{g}^B \) are minimax rate optimal over the class of data generating processes \( \mathcal{P}(M, \kappa, \mathcal{G}) \). Optimality is in the sense that the convergence rates of the upper bounds of the worst-case average welfare-regrets in Theorem 3.4 agree with the convergence rate of the universal lower bound with respect to the sample size \( n \). The convergence rate is also optimal with respect to the VC-dimension if the largest VC-dimension \( v_{1:T} \) in Theorem 3.5 corresponds to \( \sum_{s=1}^T v_s \) in Theorem 3.4. In Theorem 3.5, the maximum of \( \gamma_t M_t \sqrt{\frac{v_{1:t}}{n}} \) over \( t = 1, \ldots, T \), rather than its summation over \( t = 1, \ldots, T \), appears in the lower bound. This is due to the simplicity of the derivation of the lower bound in the proof.
Remark 3.6. The finite sample optimization problems (5) and (6) are not invariant to adding a constant. This can be problematic in applied work because the researcher can change the estimated DTR by manipulating the outcome variables (as pointed out by Kitagawa and Tetenov (2018)). Let \( Y_{t}^{dm} = Y_t - E_n[Y_t] \), for \( t = 1, \ldots, T \), be the outcomes less their sample means. The demeaned outcomes \( Y_{t}^{dm} \) are invariant to adding a constant to the original outcomes \( Y_t \). Following Kitagawa and Tetenov (2018), we suggest using the demeaned outcomes \( Y_{t}^{dm} \) instead of the original ones \( Y_t \) in the optimization problems (5) and (6). The same suggestion is also applied to all estimation methods given later.

4 Budget/Capacity Constraints

We consider budget/capacity constraints that restrict the proportion of the population to receive treatment at each stage. In the dynamic treatment policy, the budget/capacity constraints may be imposed intertemporally, meaning that the constraints are imposed on treatment rules across multiple stages. The policy-maker faces an intertemporal budget/capacity constraint if she has a budget that can be expended across multiple stages or limited amount of treatment that can be allocated across multiple stages. These constraints are formalized as follows.

We suppose that the policy-maker faces the following \( B \) constraints:

\[
\sum_{t=1}^{T} K_{tb} E_P [g_t (H_t) \mid D_1 = g_1 (H_1), \ldots, D_{t-1} = g_{t-1} (H_{t-1})] \leq C_b \quad \text{for } b = 1, \ldots, B, \quad (7)
\]

where \( K_{tb} \in [0, 1] \) and \( C_b \geq 0 \). As a scale normalization, we assume \( \sum_{t=1}^{T} K_{tb} = 1 \) for all \( b \). The weights \( K_{1b}, \ldots, K_{Tb} \) represent the relative costs of treatments across stages, and \( C_b \) represents the total budget or capacity. If at least two of \( K_{1b}, \ldots, K_{Tb} \) take non-zero values, the \( b \)-th constraint is an intertemporal budget/capacity constraint (otherwise, the \( b \)-th constraint is an temporal budget/capacity constraint).

We suppose that the policy-maker wants to maximize the welfare under the budget/capacity constraints (7) and over the class of feasible DTRs \( \mathcal{G} \). Then the population welfare maximization problem is formulated as

\[
W^*_\mathcal{G} = \max_{g \in \mathcal{G}} W (g) \quad (8)
\]
s.t. $\sum_{t=1}^{T} K_{tb} E_P[g_t(H_t) \mid D_1 = g_1(H_1), \ldots, D_{t-1} = g_{t-1}(H_{t-1})] \leq C_b$ for $b = 1, \ldots, B$.

The goal of the analysis is then to choose a DTR from $G$ that maximizes the welfare $W(\cdot)$ subject to the budget/capacity constraints (7).

To this end, we incorporate the sample analogues of the budget/capacity constraints (7) into the Simultaneous DEWM. The Simultaneous DEWM method with the budget/capacity constraints solves the following problem:

$$
\left(\hat{g}_1^S, \ldots, \hat{g}_T^S\right) \in \arg\max_{g \in G} \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{T} w_t^S \left(Z_i, g_t\right) \tag{9}
$$

s.t. $\sum_{t=1}^{T} K_{tb} \hat{E} [g_t(H_t) \mid D_1 = g_1(H_1), \ldots, D_{t-1} = g_{t-1}(H_{t-1})] \leq C_b + \alpha_n$ \hspace{1cm} (10)

for $b = 1, \ldots, B$,

where

$$
\hat{E} [g_t(H_t) \mid D_1 = g_1(H_1), \ldots, D_{t-1} = g_{t-1}(H_{t-1})] = \frac{\sum_{i=1}^{n} \left(\prod_{s=1}^{t-1} 1 \{D_{is} = g_s(H_{is})\}\right) g_t(H_{it})}{\sum_{i=1}^{n} \left(\prod_{s=1}^{t-1} 1 \{D_{is} = g_s(H_{is})\}\right)}.
$$

The inequality constraints (10) are empirical budget/capacity constraints, in which $\alpha_n$ is a tuning parameter that takes a non-negative value, depends on the sample size $n$, and converges to zero as $n$ becomes large. As $\alpha_n$ becomes small, the empirical budget/capacity constraints become tighter. A sufficiently large value of $\alpha_n$ ensures that the optimal DTR (a solution of (8)) is attainable under the sample budget/capacity constraints with high probability. When $G_t$ is the class of linear eligibility scores for all $t = 1, \ldots, T$, the optimization problem (9) can be also formulated as a MILP problem (see Appendix B).

The following theorem shows the finite-sample properties of the worst-case welfare-regret of the modified Simultaneous DEWM method as well as the excess implementation costs of the estimated DTR over the actual budget/capacity.

**Theorem 4.1.** Suppose that Assumptions 2.1, 2.2, and 2.4 hold for any distribution $P \in \mathcal{P}(M, \kappa, G)$ and Assumption 2.3 holds for $G$. Let $W^*_G$ be defined in (8) and $\hat{g}^S$ be a solution of (9) subject to (10). Let $\delta$ be any value in $(0, 1)$ and $C$ be the same constant as
in Theorem 3.4.

(i) If \( \alpha_n \geq \sqrt{\log(6B/\delta)/(2n)} \), then the following holds for any distribution \( P \in \mathcal{P}(M, \kappa, G) \) with probability at least \( 1 - \delta \):

\[
|W_G^* - W(\hat{g}^S)| \leq \frac{1}{\sqrt{n}} \sum_{t=1}^{T} \left[ \left( \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s} \right) \left( 4C \sqrt{\sum_{s=1}^{t} v_s + \sqrt{8 \log(6/\delta)}} \right) \right]
\]

and, for any \( b \in \{1, \ldots, B\} \),

\[
\sum_{t=1}^{T} K_t E_P \left[ \hat{g}^S_t(H_t) \mid D_1 = \hat{g}^S_1(H_1), \ldots, D_{t-1} = \hat{g}^S_{t-1}(H_{t-1}) \right] - C_b
\leq \frac{1}{\sqrt{n}} \sum_{t=1}^{T} \left[ \left( \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s} \right) \left( 4C \sqrt{\sum_{s=1}^{t} v_s + \sqrt{2 \log(2B/\delta)}} \right) \right] + \alpha_n.
\] (11)

(ii) Let \( k_n = \sqrt{\log(6B/\delta)/(2n)} \), and \( \tilde{W}_{G,k_n}^* \) be the optimal value of the optimization problem (8) with \( C_b \) replaced by \( C_b - k_n \). If \( \alpha_n = 0 \), the following holds for any distribution \( P \in \mathcal{P}(M, \kappa, G) \) with probability at least \( 1 - \delta \):

\[
|W_G^* - W(\hat{g}^S)| \leq (W^*_G - \tilde{W}^*_{G,k_n}) + \frac{1}{\sqrt{n}} \sum_{t=1}^{T} \left[ \left( \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s} \right) \left( 4C \sqrt{\sum_{s=1}^{t} v_s + \sqrt{8 \log(6/\delta)}} \right) \right]
\]

and, for any \( b \in \{1, \ldots, B\} \),

\[
\sum_{t=1}^{T} K_t E_P \left[ \hat{g}^S_t(H_t) \mid D_1 = \hat{g}^S_1(H_1), \ldots, D_{t-1} = \hat{g}^S_{t-1}(H_{t-1}) \right] - C_b
\leq \frac{1}{\sqrt{n}} \sum_{t=1}^{T} \left[ \left( \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s} \right) \left( 4C \sqrt{\sum_{s=1}^{t} v_s + \sqrt{2 \log(2B/\delta)}} \right) \right].
\] (13)

Proof. (ii) follows immediately from (i) by noting that \( |W_G^* - W(\hat{g}^S)| \leq (W^*_G - \tilde{W}^*_{G,k_n}) + |\tilde{W}^*_{G,k_n} - W(\hat{g}^S)| \) and that \( |\tilde{W}^*_{G,k_n} - W(\hat{g}^S)| \leq |W^*_G - W(\hat{g}^S)| \). For the proof of (i), see Appendix A.2. \( \square \)

The left-hand sides of (11) and (13) represent the excess implementation costs of the
estimated DTR over the $b$-th budget/capacity. The theorem implies that when the sample size is large, the worst-case welfare-regret and the worst-case budget excess is likely to be small. They diminish at the rate of $n^{-1/2}$. Note that, in contrast to Theorem 3.4, this theorem evaluates actual values of the welfare-regret and the budget excess rather than the expected values of them. We do this because the actual value of the budget excess should be practically of greater concern than its expected value. The result also guides how to choose the sample size $n$ in a way that the budget excess is constrained to a certain level with a particular probability. When we set $\alpha_n = 0$, the upper bound on the budget excess in (13) no longer depends on $\alpha_n$, while the upper bound on the welfare-regret in (12) depends on $W^*_G - \tilde{W}^*_{G,k_n}$, the difference in the maximum feasible welfare under the budgets $C_b$ and $C_b - k_n$ ($b = 1, \ldots, B$), for which a remark is given below.

Remark 4.2. Whether $W^*_G - \tilde{W}^*_{G,k_n}$ in (12) converges to zero or not depends on the properties of the class of DTR $G$ and distribution $P$. Suppose for simplicity that $K_{1b} = \cdots = K_{Tb} = 1$ for all $b = 1, \ldots, B$. If the difference $W^*_G - \tilde{W}^*_{G,k_n}$ is not greater than the product of the budget difference $k_n$ and the maximum feasible value of unit-welfare loss $\sum_{t=1}^{T} (\gamma_t M_t / \prod_{s=1}^{t} \kappa_s)$, i.e.,

$$W^*_G - \tilde{W}^*_{G,k_n} \leq \sum_{t=1}^{T} \left( \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s} \right) \cdot k_n,$$

then $W^*_G - \tilde{W}^*_{G,k_n}$ has an $n^{-1/2}$-upper bound because $k_n$ converges to zero at rate $n^{-1/2}$. However, not all pairs $(P, G)$ let $W^*_G - \tilde{W}^*_{G,k_n}$ converge to zero. For example, when $G$ consists of a finite number of functions, the maximum welfare subject to budget constraints may not be continuous with respect to the budget $C_b$ under some $P$, and $W^*_G - \tilde{W}^*_{G,k_n}$ may not converge to zero.

5 Estimated Propensity Score

In this section, we consider the case where the propensity scores are not known and must be estimated from data. This case occurs when the analyst has access to data from an observational study rather than an experimental study. We modify the Backward and Simultaneous DEWM methods to use the estimated propensity scores based on the e-
hybrid EWM rule proposed by Kitagawa and Tetenov (2018). For the static case, Athey and Wager (2021) argue that estimating the welfare objective using the doubly robust estimator can improve the convergence rate of the welfare-regret relative to the e-hybrid EWM rule. Nie et al. (2020) extend this approach to the problem of deciding when to start/stop a consecutive treatment assignment.\footnote{Nie et al. (2020) also discuss the difficulty in extending this approach to the more general dynamic treatment setting considered in this paper. The difficulty comes from the fact that the Q-function, \( Q_t(h_t, g_t) \), depends on the treatment rules not only of the corresponding stage but also of future stages.}

Let \( \hat{e}_t(d_t, h_t) \) be an estimated version of the propensity score \( e_t(d_t, h_t) \). For the estimators of the propensity scores, we suppose that the following high-level assumption holds.

**Assumption 5.1.** (i) Define

\[
\tau_t(d_t, H_t) \equiv \left\{ \prod_{s=1}^{t} 1 \{ D_s = d_s \} \gamma_t Y_t \prod_{s=1}^{t} e_s(d_s, H_s) \right\},
\]

and

\[
\hat{\tau}_t(d_t, H_t) \equiv \left\{ \prod_{s=1}^{t} 1 \{ D_s = d_s \} \gamma_t Y_t \prod_{s=1}^{t} \hat{e}_s(d_s, H_s) \right\},
\]

where \( \hat{e}_t(d_t, H_t) \) is an estimated propensity score taking a value in \((0,1)\). For a class of data generating processes \( P_e \), there exists a sequence \( \phi_n \to \infty \) such that

\[
\sup_{P \in P_e} \sup_{t \in \{1, \ldots, T\}} \sum_{d_t \in \{0,1\}^T} E_{P^n} \left[ \frac{1}{n} \sum_{i=1}^{n} \left| \hat{\tau}_t(d_t, H_{iT}) - \tau_t(d_t, H_{iT}) \right| \right] = O\left( \phi_n^{-1} \right).
\]

(ii) Define

\[
\eta_t(d_{t:T}, H_T) \equiv \sum_{s=t}^{T} \left\{ \prod_{\ell=t}^{s} 1 \{ D_\ell = d_\ell \} \gamma_s Y_s \prod_{\ell=t}^{s} e_\ell(d_\ell, H_\ell) \right\},
\]

\[
\hat{\eta}_t(d_{t:T}, H_T) \equiv \sum_{s=t}^{T} \left\{ \prod_{\ell=t}^{s} 1 \{ D_\ell = d_\ell \} \gamma_s Y_s \prod_{\ell=t}^{s} \hat{e}_\ell(d_\ell, H_\ell) \right\}.
\]

For a class of data generating processes \( \tilde{P}_e \), there exists a sequence \( \psi_n \to \infty \) such that

\[
\sup_{P \in \tilde{P}_e} \sup_{t \in \{1, \ldots, T\}} \sum_{d_t \in \{0,1\}^{T-t+1}} E_{P^n} \left[ \frac{1}{n} \sum_{i=1}^{n} \left| \hat{\eta}_t(d_{t:T}, H_{iT}) - \eta_t(d_{t:T}, H_{iT}) \right| \right] = O\left( \psi_n^{-1} \right).
\]
Note that

\[
E_P [\gamma_t (d_t, H_t)] = E_P [\gamma_t Y_t (d_t)],
\]

\[
E_P [\eta_t (d_{t:T}, H_T)] = E_P \left[ \sum_{s=t}^T \gamma_s Y_s (d_s) \right],
\]

hold under Assumption 2.1 and that \( n^{-1} \sum_{i=1}^n \hat{\tau}_i (d_i) \) and \( n^{-1} \sum_{i=1}^n \hat{\eta}_i (d_{i:T}) \) are estimators of these, respectively. We do not explore lower-level conditions that satisfy the conditions in Assumption 5.1. When the propensity scores are parametrically specified, they are estimated at rate \( n^{-1/2} \).

When the estimated propensity scores are used, the Backward DEWM method solves the following problem, recursively, from \( t = T \) to 1:

\[
\hat{g}_{t,e}^B \in \arg \max_{g_t \in G_t} \frac{1}{n} \sum_{i=1}^n \hat{q}_t (H_i t, g_t; \hat{g}_{t+1,e}^B, \ldots, \hat{g}_{T,e}^B),
\]

where

\[
\hat{q}_t (h_t, g_t; g_{t+1}, \ldots, g_T) \equiv \sum_{s=t}^T \left\{ \frac{\prod_{\ell=t}^s 1 \{ g_\ell (H_\ell) = D_\ell \} \cdot \gamma_s Y_s \prod_{s=t}^t \hat{e}_s (D_s, H_s) }{\prod_{\ell=t}^s \hat{e}_\ell (D_\ell, H_\ell) } \right\}.
\]

The difference between \( \hat{q}_t (\cdot) \) and \( q_t (\cdot) \) given in Section 3.1 is that the estimated propensity scores are used instead of the true ones. We denote by \( \hat{g}_e^B \equiv (\hat{g}_{1,e}^B, \ldots, \hat{g}_{T,e}^B) \) the DTR obtained by this procedure.

Similarly, the Simultaneous DEWM method solves the following problem:

\[
(\hat{g}_{1,e}^S, \ldots, \hat{g}_{T,e}^S) \in \arg \max_{g \in G} T \sum_{t=1}^T \left[ \frac{1}{n} \sum_{i=1}^n \hat{w}_t^S (Z_{i t}, g_t) \right],
\]

where

\[
\hat{w}_t^S (Z_{i t}, g_t) \equiv \frac{\prod_{s=1}^t 1 \{ g_s (H_i s) = D_i s \} \cdot \gamma_t Y_{i t} }{\prod_{s=1}^t \hat{e}_s (D_i s, H_i s) } \] uses the estimated propensity scores instead of the true ones. We denote the estimated DTR by \( \hat{g}_e^S \equiv (\hat{g}_{1,e}^S, \ldots, \hat{g}_{T,e}^S) \).

The following theorem shows the uniform convergence rate bounds on the worst-case
average welfare-regret for the two estimation methods.

**Theorem 5.1.** Suppose that Assumptions 2.1, 2.2, and 2.4 hold for any distribution \( P \in \mathcal{P}(M, \kappa, \mathcal{G}) \) and that Assumption 2.3 holds for \( \mathcal{G} \).

(i) Suppose further that Assumption 5.1 (i) holds for any distribution \( P \in \mathcal{P}_e \). For the Simultaneous DEWM method, there holds

\[
\sup_{P \in \mathcal{P}_e \cap \mathcal{P}(M, \kappa, \mathcal{G})} E_{P^n} \left[ W^* - W \left( \hat{g}_e^S \right) \right] \leq C \sum_{t=1}^T \left\{ \frac{\gamma_t M_t}{\prod_{s=1}^t \kappa_s} \sqrt{\frac{\sum_{s=1}^t v_s}{n}} \right\} + O \left( \phi_n^{-1} \right),
\]

where \( C \) is the same universal constant as that introduced in Theorem 3.4.

(ii) Suppose further that Assumption 5.1 (ii) holds for any distribution \( P \in \tilde{\mathcal{P}}_e \) and that Assumption 3.1 holds for a pair \((P, \mathcal{G})\) for any \( P \in \mathcal{P}(M, \kappa, \mathcal{G}) \). Then, for the Backward DEWM method, there holds

\[
\sup_{P \in \tilde{\mathcal{P}}_e \cap \mathcal{P}(M, \kappa, \mathcal{G})} E_{P^n} \left[ W^* - W \left( \hat{g}_e^B \right) \right] \leq C \sum_{t=1}^T \left\{ \frac{\gamma_t M_t}{\prod_{s=1}^t \kappa_s} \sqrt{\frac{\sum_{s=1}^t v_s}{n}} \right\} + C \sum_{t=2}^T \frac{2^{t-2}}{\prod_{s=1}^{t-1} \kappa_s} \left( \sum_{s=1}^T \left\{ \frac{\gamma_s M_s}{\prod_{t=1}^s \kappa_t} \sqrt{\frac{\sum_{t=1}^s v_t}{n}} \right\} \right) + O \left( \psi_n^{-1} \right).
\]

**Proof.** See Appendix A.1. \( \square \)

The theorem implies that the convergence rate of the worst-case average welfare-regret for each method depends on that of the estimators of the propensity scores. If parametric estimators are used to estimate the propensity scores, both methods achieve an \( n^{-1/2} \)-convergence rate of the worst-case average welfare-regret.

### 6 Simulation Study

We conduct a simulation study to examine the finite sample performance of the proposed estimation methods. We compare the performance of the Backward DEWM, Simultaneous DEWM, and Q-learning.
We consider data generating processes (DGPs) that consist of two stages of treatment assignment \((D_1, D_2)\), associated potential outcomes \((Y_1(d_1), Y_2(d_1, d_2))_{d_1,d_2 \in \{0,1\}^2}\), and a covariate \(X_1\) observed at the first stage. The potential outcomes are generated as
\[
Y_1(d_1) = \phi_{01} + \phi_{11}X_1 + (\psi_{01} + \psi_{11}X_1)d_1 + U_1,
\]
\[
Y_2(d_1, d_2) = \phi_{02} + \phi_{12}Y_1(d_1) + \left(\psi_{02} + \psi_{12}d_1 + \sum_{j=1}^{3} \psi_{j+1,2}(Y_1(d_1))^j\right)d_2 + U_2
\]
for \((d_1, d_2) \in \{0,1\}^2\). We consider three DGPs labeled DGPs 1-3. In all the DGPs, \(X_1, U_1,\) and \(U_2\) are independently drawn from \(N(0,1)\); \(D_1\) and \(D_2\) are independently drawn from \(Ber(1/2)\); and \((\phi_{01}, \phi_{11}, \psi_{01}, \psi_{11}) = (0.5, -1.0, 1.0, 1.5)\) and \((\phi_{02}, \phi_{12}, \psi_{02}, \psi_{12}) = (0.5, 0.5, 0.5, 0.5)\). Regarding the other parameters, we set \((\psi_{22}, \psi_{32}, \psi_{42}) = (0, 0, 0)\) in DGP1, \((\psi_{22}, \psi_{32}, \psi_{42}) = (1, 0, 0)\) in DGP2, and \((\psi_{22}, \psi_{32}, \psi_{42}) = (0.3, 0.3, -0.4)\) in DGP3. We suppose that the target welfare to maximize is
\[
W(g_1, g_2) = E_P[Y_2(D_1, D_2) \mid D_1 = g_1(H_1), D_2 = g_2(H_2)].
\]
Note that the treatment effect of \(D_2\) does not depend on the past outcome in DGP1, but it does in DGPs 2 and 3.

For the Backward and Simultaneous DEWM methods, we use a class of DTRs \(\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2\) that consists of the following linear index classes of treatment rules:
\[
\mathcal{G}_1 = \{ 1 \{ (1, X_1)' \beta_1 \geq 0 \} : \beta_1 = (\beta_{01}, \beta_{11})' \in \mathbb{R}^2 \},
\]
\[
\mathcal{G}_2 = \{ 1 \{ (1, D_1, Y_1)' \beta_2 \geq 0 \} : \beta_2 = (\beta_{02}, \beta_{12}, \beta_{22})' \in \mathbb{R}^3 \}.
\]
\(\mathcal{G}_1\) contains the first-best treatment rule only under DGP1; \(\mathcal{G}_2\) does under DGPs 1 and 2, but not under DGP3. Thus, the Backward DEWM method can consistently estimate the optimal DTR under DGPs 1 and 2, but cannot under DGP3.

For Q-learning, we assume that the conditional outcomes are specified as
\[
E[ Y_1 \mid H_1, D_1; \alpha_1, \gamma_1] = \alpha_{01} + \alpha_{11}X_1 + (\gamma_{01} + \gamma_{11}X_1)D_1,
\]
\[
E[ Y_2 \mid H_2, D_2; \alpha_2, \gamma_2] = \alpha_{02} + \alpha_{12}Y_1 + (\gamma_{02} + \gamma_{12}D_1 + \gamma_{22}Y_1)D_2,
\]
where $\mathbf{\alpha}'_t = (\alpha_{0t}, \alpha_{1t})'$ for each $t = 1, 2$, $\mathbf{\gamma}'_1 = (\gamma_{01}, \gamma_{11})'$, and $\mathbf{\gamma}'_2 = (\gamma_{02}, \gamma_{12}, \gamma_{22})'$. This specification is correct under DGPs 1 and 2, but is not under DGP3.

Table 1 reports the results of 500 simulations with sample sizes $n = 200, 400$, and 600, where we calculate the mean and median welfare achieved by each estimated DTR with 3,000 observations randomly drawn from the same DGP used in the estimation. The results show that Q-learning performs better than the Backward and Simultaneous DEWM methods in DGPs 1 and 2 in terms of the population mean welfare. However, both the Backward and Simultaneous DEWM methods show better performance than Q-learning in DGP3, where the outcome model used by Q-learning is misspecified. In DGPs 1 and 2, the Backward DEWM method achieve slightly higher mean welfare than the Simultaneous DEWM method, whereas in DGP3 the Simultaneous DEWM method achieves clearly higher welfare than the Backward DEWM method.

| Table 1: Monte Carlo Simulation Results |
|----------------------------------------|
| DGP | n=200 | n=400 | n=600 |
|-----|-------|-------|-------|
|     | Mean  | Median| SD    | Mean  | Median| SD    | Mean  | Median| SD    |
| Q-learning 1 | 2.272 | 2.274 | 0.042 | 2.284| 2.281 | 0.070 | 2.274| 2.271 | 0.069 |
| B-DEWM 1 | 2.049 | 2.147 | 0.269 | 2.154| 2.205 | 0.211 | 2.148| 2.214 | 0.231 |
| S-DEWM 1 | 2.006 | 2.121 | 0.288 | 2.148| 2.211 | 0.245 | 2.178| 2.220 | 0.184 |
| Q-learning 2 | 3.969 | 3.972 | 0.070 | 3.975| 3.971 | 0.122 | 3.978| 3.977 | 0.118 |
| B-DEWM 2 | 3.560 | 3.717 | 0.498 | 3.711| 3.780 | 0.383 | 3.694| 3.788 | 0.426 |
| S-DEWM 2 | 3.497 | 3.666 | 0.526 | 3.710| 3.795 | 0.374 | 3.737| 3.823 | 0.388 |
| Q-learning 3 | 1.637 | 1.625 | 0.107 | 1.629| 1.612 | 0.150 | 1.626| 1.612 | 0.145 |
| B-DEWM 3 | 1.765 | 1.799 | 0.130 | 1.785| 1.799 | 0.126 | 1.794| 1.808 | 0.092 |
| S-DEWM 3 | 1.856 | 1.886 | 0.154 | 1.888| 1.899 | 0.167 | 1.906| 1.924 | 0.128 |

Note: Mean and Median represent the mean and median of the population mean welfares achieved by the estimated DTRs across the simulations; SD is the standard deviation of the population mean welfares across the simulations. The population mean welfare is calculated using 3,000 observations randomly drawn from the corresponding DGP. B-DEWM and S-DEWM mean the Backward and Simultaneous DEWM methods, respectively.

7 Empirical Application

We apply the proposed methods to the data from Project STAR (e.g., Krueger (1999); Gerber et al. (2001); Schanzenbach (2006); Chetty et al. (2011)). In this experimental project, among 1,346 kindergarten students who did not belong to small classes, 672 students were randomly allocated to regular-size classes with a full-time teacher aide and the others were allocated to regular-size classes without the teacher aide. Upon their
progression to grade 1, the enrolled students were randomly shuffled to regular class-size classes with or without a teacher aide and remained in these allocated classes until the end of grade 3.

We study how to optimally allocate each student to the two types of classes (regular-size classes with or without a teacher aide) in grades K and 1 depending on their socio-economic information and intermediate academic achievement.\footnote{We focus on allocation to a regular-size class with a teacher aide, rather than allocation to a small-size class, because the allocation of students to regular-size classes with or without a teacher aide in the experiment matches the sequential randomization design, but the allocation to small-size classes in the experiment does not.} We suppose that the welfare that the policy-maker wants to maximize is a population average of a sum of scores of reading and mathematics tests that students take at the end of grade 1.\footnote{We focus on the test score at the end of grade 1 rather than at the end of grade 3 because we found attending a class with a teacher aide at kindergarten has little effect on the test score at the end of the grade 3 even when treatment effect heterogeneity is taken into account.} We set the first and second stages ($t = 1$ and $2$) to grades K and 1, respectively. The treatment variable $D_t$, for $t = 1, 2$, takes the value one if the student belongs to a class with a teacher aide at stage $t$ and zero otherwise. The intermediate outcome $Y_1$ and final outcome $Y_2$ are the sum of reading and mathematics test scores at the ends of grades K and 1, respectively. In the following estimation, we use demeaned outcomes $Y_{1dm}$ and $Y_{2dm}$, instead of the original ones, as suggested in Remark 3.6. The standard deviations (SDs) of $Y_{1dm}$ and $Y_{2dm}$ in the sample are 67.69 and 83.27, respectively. The socioeconomic information we use are qualification for free or reduced price school lunch and type of school location (rural or non-rural location). A binary variable $X_{1t}$ indicates by 1 that the student is eligible for free or reduced price school lunches at stage $t$ and by 0 otherwise; $X_{2t}$ indicates by 1 that a school that the student attends at stage $t$ is located in a rural area and by 0 otherwise. We also have access to student information such as sex and race, but use of this information in the treatment assignment decision is discriminatory and prohibited.

As a set of class-allocation policies, we use a set of linear eligibility scores $G = G_1 \times G_2$ where $G_1$ and $G_2$ have the following forms:

\[
G_1 = \left\{1 \{\beta_1 x_{11} + \beta_2 x_{21} \geq c_1\} : \beta_1 \geq 0, (\beta_2, c_1)' \in \mathbb{R}^2\right\},
\]
\[
G_2 = \left\{1 \{\gamma_1 x_{12} + \gamma_2 x_{22} + \gamma_3 (1 - d_1) y_{1dm} + \gamma_4 d_1 y_{1dm} \geq c_2\} : \gamma_1 \geq 0, (\gamma_2, \gamma_3, \gamma_4, c_2)' \in \mathbb{R}^4\right\}.
\]
In the forms of $\mathcal{G}_1$ and $\mathcal{G}_1$, the signs of the coefficients of $x_{11}$ and $x_{12}$ are restricted to be non-negative so that students who are eligible for free or reduced price school lunches are not less likely to be allocated to a class with a teacher aide (given that the other information is fixed). The interaction terms $(1 - d_1) y_{1}^{dm}$ and $d_1 y_{1}^{dm}$ in $\mathcal{G}_2$ enables the eligibility score to evaluate the intermediate outcome differently depending on the class allocation at kindergarten. For a DTR $g \in \mathcal{G}$, we define the welfare gain of $g$ as $W(g) - W(0, 0)$, a welfare increase by allocating students to each class subject to the DTR $g$ rather than allocating every student to the regular classes without any additional teacher aide at all stages. Applying the Backward and Simultaneous DEWM methods, we estimate the optimal DTR over $\mathcal{G}$ and their welfare gains. The same sample (1,346 observations) is used to estimate both the optimal DTR and welfare gain.

The DTR estimated by the Backward DEWM method is $\hat{g}^B = (\hat{g}_1^B, \hat{g}_2^B)$ where

$$
\begin{align*}
\hat{g}_1^B(H_1) &= 1, \\
\hat{g}_2^B(H_2) &= 1 \{0.702X_{12} - 0.056X_{22} - 0.073 (1 - D_1) Y_{1}^{dm} + 0.152D_1 Y_{1}^{dm} \geq -0.016\}.
\end{align*}
$$

The DTR estimated by the Simultaneous DEWM method is $\hat{g}^S = (\hat{g}_1^S, \hat{g}_2^S)$ where

$$
\begin{align*}
\hat{g}_1^S(H_1) &= 1 \{X_{21} = 1\}, \\
\hat{g}_2^S(H_2) &= 1 \{0.288X_{12} + 0.686X_{22} + 0.096D_1 Y_{1}^{dm} \geq -0.007\}.
\end{align*}
$$

$\hat{g}_1^B$ assigns every student to a class with a teacher aide in grade K, while $\hat{g}_1^S$ assigns only students in rural areas to classes with a teacher aide in grade K. Under both treatment rules $\hat{g}_2^B$ and $\hat{g}_2^S$, a student who attends a class with a teacher aide and attains a high test score in grade K is more likely to be assigned to a class with a teacher aide in grade 1.

Table 2 reports the estimated welfare gains and shares of population to be treated at each stage by $\hat{g}^B$ and $\hat{g}^S$ and the other three DTRs $(g_1, g_2) = (1, 0), (0, 1), (1, 1)$. For example, the DTR $(g_1, g_2) = (1, 0)$ assigns every student to a class with a teacher aide in kindergarten but assigns no students to a class with a teacher aide in grade 1. The results show that both the Backward and Simultaneous DEWMs lead to higher welfare gains than the primitive DTRs $(g_1, g_2) = (1, 0), (0, 1), (1, 1)$. The Simultaneous DEWM method leads to a higher welfare gain than the Backward DEWM method. This would be
because the constrained $G_2$ does not contain the first-best treatment rule (i.e., Assumption 3.1 does not hold).

Table 2: Estimated welfare gains

| Dynamic treatment regime | Share of population to be treated | 1st stage | 2nd stage | Estimated welfare gain |
|--------------------------|----------------------------------|-----------|-----------|------------------------|
| $(g_1, g_2) = (1, 0)$    |                                  | 1         | 0         | 9.50                   |
| $(g_1, g_2) = (0, 1)$    |                                  | 0         | 1         | 17.82                  |
| $(g_1, g_2) = (1, 1)$    |                                  | 1         | 1         | 13.58                  |
| $(\hat{g}_B^1, \hat{g}_B^2)$ |                              | 0.50      | 0.74      | 20.88                  |
| $(\hat{g}_S^1, \hat{g}_S^2)$ |                              | 0.63      | 0.74      | 27.24                  |

Notes: The SD of $Y_{dm}^2$ in the sample is 83.27. The same 1,346 observations are used to calculate the DTRs $\hat{g}^B$ and $\hat{g}^S$, shares of population to be treated, and welfare gains of $\hat{g}^B$ and $\hat{g}^S$.

Next, we impose a restriction $g_2(h_2) \geq d_1$ for all $h_2 \in \mathcal{H}_2$; thereby, we consider the decision problem of when each student should start attending a class with a teacher aide. Under this constraint, the DTR estimated by the Backward DEWM method is

$$\hat{g}^B = (\hat{g}_1^B, \hat{g}_2^B)$$

with

$$\hat{g}_1^B(H_1) = 0,$$
$$\hat{g}_2^B(H_2) = 1 \left\{0.702X_{12} - 0.056X_{22} - 0.073(1 - D_1)Y_{1dm}^1 + 0.152D_1Y_{1dm}^1 \geq -0.016\right\};$$

the DTR estimated by the Simultaneous DEWM method is $\hat{g}^S = (\hat{g}_1^S, \hat{g}_2^S)$ with

$$\hat{g}_1^S(H_1) = 1 \left\{X_{21} = 1\right\},$$
$$\hat{g}_2^S(H_2) = 1.$$

Table 3 reports the estimated welfare gains and shares of population to be treated by $\hat{g}^B$ and $\hat{g}^S$ and the other two DTRs $(g_1, g_2) = (0, 1), (1, 1)$, which satisfy the constraint $g_2(h_2) \geq d_1$ for all $h_2 \in \mathcal{H}_2$. In contrast to the results without the additional constraint in Table 2, the Backward DEWM method leads to less welfare gains than the constant DTRs $(g_1, g_2) = (0, 1), (1, 1)$, while the Simultaneous DEWM method leads to the highest welfare gain.

Finally, we consider estimating an optimal DTR under the following fictitious budget
Table 3: Estimated welfare gains for the start-time decision problem

| Dynamic treatment regime | Share of population to be treated | 1st stage | 2nd stage | Estimated welfare gain |
|--------------------------|----------------------------------|----------|----------|------------------------|
| \((g_1, g_2) = (0, 1)\) | 0                                | 1        | 17.82    |
| \((g_1, g_2) = (1, 1)\) | 1                                | 1        | 13.58    |
| \((\hat{g}^B_1, \hat{g}^B_2)\) | 0          | 0.50     | 6.78     |
| \((\hat{g}^S_1, \hat{g}^S_2)\) | 0.63       | 1.00     | 23.59    |

Note: The SD of \(Y_{2dm}\) in the sample is 83.27. The same 1,346 observations are used to calculate the DTRs \(\hat{g}^B\) and \(\hat{g}^S\), shares of population to be treated, and welfare gains of \(\hat{g}^B\) and \(\hat{g}^S\).

The modified Simultaneous DEWM method estimates the DTR to be \(\hat{g}^S = (\hat{g}^S_1, \hat{g}^S_2)\) with

\[
\hat{g}^S_1(H_1) = 1 \{X_{21} = 1\},
\hat{g}^S_2(H_2) = 1 \{0.004X_{12} - 0.104X_{22} + 0.052 (1 - d_1) y_{1dm} + 0.106 d_1 y_{1dm} \geq 0.817\}.
\]

The estimates of the shares of the population that are treated in grades K and 1 are 0.63 and 0.37, respectively. The estimated welfare gain is 25.15, which is less than the welfare gain without the budget constraint, 27.52. The estimated excess cost of the estimated policy is 0, meaning that the estimated policy exhausts the budget in the sample.

8 Conclusion

This paper proposed empirical methods to estimate the optimal DTR over a pre-specified class of feasible DTRs based on the empirical welfare maximization approach. We proposed two estimation methods, the Backward DEWM method and the Simultaneous
DEWM method, which estimate the optimal DTR through backward induction and simultaneous maximization, respectively. The former is computationally efficient, but it can consistently estimate the optimal DTR only when the class of feasible DTRs contains the first-best treatment rule at all stages except for the first stage. In contrast, the Simultaneous DEWM can consistently estimate the optimal DTR without such a condition, though it is computationally less efficient. These methods can accommodate exogenous constraints on the class of feasible DTRs and, furthermore, can specify different types of dynamic treatment choice problems. The upper and lower bounds on the worst-case average welfare-regrets for the two methods are derived. The main result of the paper is that each method can achieve the optimal convergence rate of $n^{-1/2}$ toward zero in the experimental data setting. In the observational data setting, the convergence rates depend on that of the estimated propensity scores. We also modified the Simultaneous DEWM to accommodate intertemporal budget/capacity constraints, and derived the $n^{-1/2}$-upper bounds on the worst-case welfare-regret and the excess implementation cost of the estimated DTR over the actual budget that hold with high probability. The proposed estimation methods with the linear eligibility policy class can be formulated as MILPs (see Appendix B).
Appendix

A Proofs

In this appendix, Section A.1 provides the proofs of Theorems 3.4, 3.5, and 5.1 with some auxiliary lemmas. Section A.2 provides the proof of Theorem 4.1 with some auxiliary lemmas and a proposition.

A.1 Proofs of Theorems 3.4, 3.5, and 5.1.

This section provides the proofs of Theorems 3.4, 3.5, and 5.1. Many concepts and techniques used in the proofs are taken from the literature on statistical learning theory (e.g., Devroye et al. (1996); Lugosi (2002); Mohri et al. (2012)).

We first introduce the VC-dimension for a class of subsets. Let $Z$ be any space, and let $z_\ell = (z_1, \ldots, z_\ell)$ be a finite set of $\ell \geq 1$ points in $Z$. Given a class of subsets $\mathcal{G} \subseteq 2^Z$ and a subset $\tilde{z}$ of $z_\ell$, we say that $\mathcal{G}$ picks out $\tilde{z}$ when $\tilde{z} \cap G = \tilde{z}$ holds for some $G \in \mathcal{G}$. We say that $\mathcal{G}$ shatters $z_\ell$ when $\left| \{z_\ell \cap G : G \in \mathcal{G} \} \right| = 2^\ell$ holds, that is all subsets of $z_\ell$ are picked out by $\mathcal{G}$. The VC-dimension of the class of subsets $\mathcal{G}$, denoted by $VC(\mathcal{G})$, is defined as the cardinality of the largest subset $z_\ell$ contained in $Z$ and shattered by $\mathcal{G}$, i.e.,

$$VC(\mathcal{G}) \equiv \max \{ \ell : \max_{z_\ell \subseteq Z} \left| \{z_\ell \cap G : G \in \mathcal{G} \} \right| = 2^\ell \}.$$  

We say that a class of subsets $\mathcal{G}$ is a VC-class of subsets if $VC(\mathcal{G}) < \infty$.

We next introduce a concept of the subgraph of a real-valued function $f : Z \to \mathbb{R}$ that is the set

$$SG(f) \equiv \{(z, t) \in Z \times \mathbb{R} : t \leq f(z)\}.$$  

Let $SG(\mathcal{F}) \equiv \{SG(f) : f \in \mathcal{F}\}$ be a collection of subgraphs over a class of functions $\mathcal{F}$. We here consider the VC-dimension of $SG(\mathcal{F})$ as a complexity measure of $\mathcal{F}$. Note that in the case of $\mathcal{F}$ being a class of indicator functions, the VC-dimension of $SG(\mathcal{F})$ corresponds to the VC-dimension of $\mathcal{F}$ in the sense of Definition 2.1. We say that a class of functions $\mathcal{F}$ is a VC-subgraph class of functions if $VC(SG(\mathcal{F})) < \infty$.  

34
We first consider establishing a link between the concepts of VC-dimension of a class of feasible DTRs and VC-dimension of a collection of subgraphs of functions on \( \mathcal{Z} \). The following lemmas are auxiliary lemmas for this purpose.

**Lemma A.1.** *(Sauer’s lemma; see, for example, Theorem 3.6.2 of Giné and Nickl (2016))*

Let \( \mathcal{Z} \) be any space, and \( (z_1, \ldots, z_\ell) \) be a finite set of \( \ell \geq 1 \) points in \( \mathcal{Z} \). Let \( \mathcal{G} \) be a VC-class of subsets in \( \mathcal{Z} \) with \( \text{VC}(\mathcal{G}) = v < \infty \). Let \( \Delta_\ell(\mathcal{G}, (z_1, \ldots, z_\ell)) \) denote the number of subsets of \( (z_1, \ldots, z_\ell) \) that are picked out by \( \mathcal{G} \), i.e.,

\[
\Delta_\ell(\mathcal{G}, (z_1, \ldots, z_\ell)) \equiv |\{(z_1, \ldots, z_\ell) \cap G : G \in \mathcal{G}\}|.
\]

Then the following holds:

\[
\max_{(z_1, \ldots, z_\ell) \subseteq \mathcal{Z}} \Delta_\ell(\mathcal{G}, (z_1, \ldots, z_\ell)) \leq \sum_{j=0}^{v} \binom{\ell}{j} \leq \left(\frac{\ell e}{v}\right)^v.
\]

**Lemma A.2.** Let \( \mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2 \) be any product space, and \( \mathcal{G} \) be a class of indicator functions from \( \mathcal{Z}_2 \) to \( \{0, 1\} \). Suppose that \( \mathcal{G} \) has VC-dimension \( v \geq 0 \) in the sense of Definition 2.1. Fix a function \( f \) on \( \mathcal{Z} \), and define a class of functions on \( \mathcal{Z} \):

\[
\mathcal{F}_\mathcal{G} = \{f \cdot g : g \in \mathcal{G}\}.
\]

Then \( \mathcal{F}_\mathcal{G} \) is a VC-subgraph class of functions with \( \text{VC}(\text{SG}(\mathcal{F}_\mathcal{G})) \leq v \).

**Proof.** We prove the statement by contradiction. Suppose that there exist some \( (v+1) \)-points \( \{(z_1, t_1), \ldots, (z_{v+1}, t_{v+1})\} \equiv \{(z_{1,1}, z_{2,1}, t_1), \ldots, (z_{1,v+1}, z_{2,v+1}, t_{v+1})\} \subset \mathcal{Z}_1 \times \mathcal{Z}_2 \times \mathbb{R} \) that are shattered by \( \text{SG}(\mathcal{F}_\mathcal{G}) \).

When \( t \leq f(z) \wedge 0 \) or \( t > f(z) \vee 0 \) for some \( (z, t) \in \{(z_1, t_1), \ldots, (z_{v+1}, t_{v+1})\} \), \( \text{SG}(\mathcal{F}_\mathcal{G}) \) cannot pick out \( \{(z_1, t_1), \ldots, (z_{v+1}, t_{v+1})\} \setminus \{(z, t)\} \) or \( \{(z, t)\} \). Thus, we need to consider only the case that \( (f(z) \wedge 0) < t < (f(z) \vee 0) \) for all \( (z, t) \in \{(z_1, t_1), \ldots, (z_{v+1}, t_{v+1})\} \). In the remaining case, we indicate \( \delta_j = 1 \) if \( t_j \leq f(z_j) \) and \( \delta_j = 0 \) otherwise. Since the VC-dimension of \( \mathcal{G} \) is at most \( v \) in the sense of Definition 2.1, there exists a subset \( S \equiv (\tilde{z}_{2,1}, \ldots, \tilde{z}_{2,m}) \) (for some \( m > 0 \)) of \( \{z_{2,1}, \ldots, z_{2,v+1}\} \) such that \( (g(\tilde{z}_{2,1}), \ldots, g(\tilde{z}_{2,m})) \neq (f(\tilde{z}_{2,1}), \ldots, f(\tilde{z}_{2,m})) \) for some (...)
(1, . . . , 1) and (g(z_{2,1}), . . . , g(z_{2,v+1}) \backslash (g(\tilde{z}_{2,1}), . . . , g(\tilde{z}_{2,m})) \neq (0, . . . , 0) for any g ∈ G. Then SG (F_G) cannot pick out the following subset:

\[ \{ (z_j, t_j) : (z_{2,j} ∈ S and δ_j = 1) or (z_{2,j} \not∈ S and δ_j = 0) \}, \]

because this set of points could be contained in SG (f · g) only when sign(t_j) = sign (g (z_{2,j}) − 1/2) for all j = 1, . . . , v+1. This contradicts the assumption that \{ (z_1, t_1), . . . , (z_{v+1}, t_{v+1}) \} ⊂ Z × R is shattered by SG (F_G). □

The following lemma establishes the link between the VC-dimension of a class of feasible DTRs and the VC-dimension of a class of subgraphs of functions on Z.

**Lemma A.3.** Suppose that Assumption 2.3 holds. Let r : Z → R be any function. For any integers s, t with 1 ≤ s ≤ t ≤ T, a class of functions from Z to R

\[ F_{s,t} ≡ \{ f(z) = 1 \{ g_s(h_s) = d_s, . . . , g_t(h_t) = d_t \} · r(z) : (g_s, . . . , g_t) ∈ G_s × · · · × G_t \} \]

is a VC-subgraph class of functions with VC(SG(F_{s,t})) ≤ \sum_{j=s}^{t} v_j.

**Proof.** We prove for the case that s = 1 and t = T. The result follows for the remaining cases by a similar argument. Let m be an arbitrary integer and (z_1, . . . , z_m) be m arbitrary points on Z. For each t, fixing g_s ∈ G_s for all s ≠ t, define a class of functions

\[ \tilde{F}_t ≡ \{ f(z) = 1 \{ g_1(h_1) = d_1, . . . , g_T(h_T) = d_T \} · r(z) : g_t ∈ G_t \}, \]

and, fixing g_s ∈ G_s for all s > t, define

\[ \tilde{F}_{1,t} ≡ \{ f(z) = 1 \{ g_1(h_1) = d_1, . . . , g_T(h_T) = d_T \} · r(z) : (g_1, . . . , g_t) ∈ G_1 × · · · × G_t \}. \]

We first consider \( \tilde{F}_1 \), or equivalently \( \tilde{F}_{1,1} \). Applying Lemma A.2 to \( \tilde{F}_1 \) shows that \( \tilde{F}_1 \) is a VC-subgraph of functions with VC(SG(\( \tilde{F}_1 \))) ≤ v_1. Therefore, from Lemma A.1, SG(\( \tilde{F}_1 \)) can pick out at most \( O(m^v) \) subsets from (z_1, . . . , z_m).

Next we study \( \tilde{F}_2 \) and then \( \tilde{F}_{1,2} \). Let (z_1, . . . , z_{m'}) be an arbitrary subset picked out by SG(\( \tilde{f}_1 \)) where \( \tilde{f}_1 ∈ \tilde{F}_{1,1} \) has a fixed \( g_1 ∈ G_1 \). Lemmas A.1 and A.2 show that SG(\( \tilde{F}_2 \))
can pick out at most $O(m^2)$ subsets from $(z_1, \ldots, z_m)$. Because $\text{SG}(\tilde{F}_2)$ can pick out at most $O(m^2)$ subsets from each subset of $(z_1, \ldots, z_m)$ and $\text{SG}(\tilde{F}_{1:1})$ can pick out at most $O(m^2)$ subsets from $(z_1, \ldots, z_m)$, by varying $(g_1, g_2)$ over $G_1 \times G_2$, $\text{SG}(\tilde{F}_{1:2})$ picks out at most $O(m^2)$ subsets from $(z_1, \ldots, z_m)$.

For $s \geq 2$, suppose that $\tilde{F}_{1:s-1}$ can pick out at most $O(m^{\sum_{t=1}^{s-1} v_t})$ subsets from $(z_1, \ldots, z_m)$. Let $(z_1, \ldots, z_m')$ be an arbitrary subset picked out by $\text{SG}(\tilde{f}_{s-1})$ where $\tilde{f}_{s-1} \in \tilde{F}_{1:s-1}$ has fixed $g_1, \ldots, g_{s-1}$. From $(z_1, \ldots, z_m')$, $\text{SG}(\tilde{F}_s)$ can pick out at most $O(m^{v_s})$ subsets. Combining this result with the fact that $\tilde{F}_{1:s-1}$ can pick out at most $O(m^{\sum_{t=1}^{s-1} v_t})$ subsets from $(z_1, \ldots, z_m)$ leads to the conclusion that $\tilde{F}_{1:s}$ picks out at most $O(m^{\sum_{t=1}^{s} v_t})$ subsets from $(z_1, \ldots, z_m)$.

Recursively, we can prove that $\tilde{F}_{1:T}$ picks out at most $O(m^{\sum_{t=1}^{T} v_t})$ subsets from $(z_1, \ldots, z_m)$. Hence, $\text{SG}(\tilde{F}_{1:T})$ is a VC-subgraph class of functions with VC-dimension less than or equal to $\sum_{t=1}^{T} v_t$.

The next lemma, which corresponds to Lemma A.4 of Kitagawa and Tetenov (2018), gives a uniform upper bound for the mean of a supremum of centered empirical processes indexed by a VC-subgraph class of functions. This is a fundamental result in the literature on empirical process theory and its proof can be found, for example, in van der Vaart and Wellner (1996) and Kitagawa and Tetenov (2018). The following lemma will be used in the proofs of Lemmas A.5 and A.7 below.

**Lemma A.4.** Let $F$ be a class of uniformly bounded functions on $Z$, that is, there exists $\bar{F} < \infty$ such that $\|f\|_{\infty} \leq \bar{F}$ for all $f \in F$. Assume that $F$ is a VC-subgraph of functions with VC-dimension $v < \infty$. Then there is a universal constant $C$ such that

$$E_{P^n} \left[ \sup_{f \in F} \left| E_n (f) - E_P (f) \right| \right] \leq C \bar{F} \sqrt{\frac{v}{n}}$$

holds for all $n \geq 1$.  

37
Before proceeding to the proofs of the main theorems, we define

\[ \tilde{Q}_t(g_t, \ldots, g_T) \equiv E_P \left[ q_t(Z, g_t, \ldots, g_T) \right] = E_P \left[ \sum_{s=t}^{T} \left\{ \prod_{\ell=t}^{s} 1 \{ g_\ell(H_\ell) = D_\ell \} \gamma_s Y_s \right\} \prod_{\ell=t}^{s} e_\ell(D_\ell, H_\ell) \right] \]

and

\[ \tilde{Q}_{nt}(g_t, \ldots, g_T) \equiv E_n \left[ q_t(Z, g_t, \ldots, g_T) \right] = E_n \left[ \sum_{s=t}^{T} \left\{ \prod_{\ell=t}^{s} 1 \{ g_\ell(H_\ell) = D_\ell \} \gamma_s Y_s \right\} \prod_{\ell=t}^{s} e_\ell(D_\ell, H_\ell) \right]. \]

We further define

\[ \Delta \tilde{Q}_t \equiv \tilde{Q}_t(g^*_t, \ldots, g^*_T) - \tilde{Q}_t(g^*_t, \ldots, g^*_s, \hat{g}^B_{s+1}, \ldots, \hat{g}^B_T), \]

\[ \Delta \tilde{Q}^+_t \equiv \tilde{Q}_t(g^*_t, \hat{g}^B_{t+1}, \ldots, \hat{g}^B_T) - \tilde{Q}_t(g^*_t, \ldots, \hat{g}^B_T). \]

The following lemma will be used in the proof of Theorem 3.4 (ii) for the Backward DEWM method.

**Lemma A.5.** Suppose that Assumptions 2.1, 2.4, and 3.1 hold for a pair \((P, \mathcal{G})\). Then the following hold:

(i) for any \(t = 1, \ldots, T\),

\[ E_P \left[ |\Delta \tilde{Q}^+_t| \right] \leq C \left( \sum_{s=t}^{T} \frac{\gamma_s M_s}{\prod_{\ell=t}^{s} \kappa_\ell} \right) \sqrt{\frac{\sum_{s=t}^{T} v_s}{n}}, \]

where \(C\) is the same constant term introduced in Lemma A.4;

(ii) for any \(t = 1, \ldots, T - 1\) and \(s = t + 1, \ldots, T\),

\[ \tilde{Q}_t(g^*_t, \ldots, g^*_T) - \tilde{Q}_t(g^*_t, \ldots, g^*_s, \hat{g}^B_{s+1}, \ldots, \hat{g}^B_T) \leq \frac{1}{\prod_{\ell=t}^{s} \kappa_\ell} \Delta \tilde{Q}_{s+1}; \]
\[ \Delta \tilde{Q}_1 \leq \Delta \tilde{Q}_1^j + \sum_{s=1}^{T-1} \frac{2^{s-1}}{\prod_{t=1}^s \kappa_t} \Delta \tilde{Q}_{s+1}^j. \]

**Proof.** (i) It follows that

\[ \Delta \tilde{Q}_1^j = \tilde{Q}_t (g_t^*, g_{t+1}^B, \ldots, g_T^B) - \hat{Q}_t (\hat{g}_t^B, \ldots, \hat{g}_T^B) \]

\[ + \tilde{Q}_{nt} (g_t^*, \hat{g}_{t+1}^B, \ldots, \hat{g}_T^B) - \hat{Q}_t (\hat{g}_t^B, \ldots, \hat{g}_T^B) \]

\[ \leq \tilde{Q}_t (g_t^*, \hat{g}_{t+1}^B, \ldots, \hat{g}_T^B) - \hat{Q}_{nt} (g_t^*, \hat{g}_{t+1}^B, \ldots, \hat{g}_T^B) + \tilde{Q}_{nt} (\hat{g}_t^B, \ldots, \hat{g}_T^B) - \hat{Q}_t (\hat{g}_t^B, \ldots, \hat{g}_T^B) \]

\[ \leq 2 \sup_{(g_t, \ldots, g_T) \in \mathcal{G}_t \times \ldots \times \mathcal{G}_T} \left| \tilde{Q}_{nt} (g_t, \ldots, g_T) - \hat{Q}_t (g_t, \ldots, g_T) \right|, \]

where the first inequality follows from the fact that \( \hat{g}_t^B \) maximizes \( \tilde{Q}_{nt} (\cdot, \hat{g}_{t+1}^B, \ldots, \hat{g}_T^B) \) over \( \mathcal{G}_t \). Because

\[ \left\| \hat{Q}_t (g_t, \ldots, g_T) \right\|_\infty \leq \sum_{s=t}^T \frac{\gamma_s M_s/2}{\prod_{t=1}^s \kappa_t} \]

holds under Assumptions 2.2 and 2.4, by applying Lemmas A.3 and A.4 to the following class of functions:

\[ \left\{ \sum_{s=t}^T \left\{ \frac{1 \{g_t (H_t) = D_t\}}{e_t (D_t, H_t)} \right\} : (g_t, \ldots, g_T) \in \mathcal{G}_t \times \ldots \times \mathcal{G}_T \right\}, \]

we have

\[ \mathbb{E}_P \left[ \sup_{(g_t, \ldots, g_T) \in \mathcal{G}_t \times \ldots \times \mathcal{G}_T} \left| \tilde{Q}_{nt} (g_t, \ldots, g_T) - \hat{Q}_t (g_t, \ldots, g_T) \right| \right] \leq C \left( \sum_{s=t}^T \frac{\gamma_s M_s/2}{\prod_{t=1}^s \kappa_t} \right) \sqrt{\sum_{s=t}^T v_s / n}. \]

Combining this with equation (15) leads to the result.

(ii) For any integers \( s \) and \( t \) such that \( 1 \leq t < s \leq T \), it follows that

\[ \hat{Q}_t (g_t^*, \ldots, g_T^*) - \hat{Q}_t (g_t^*, \ldots, g_{s-1}^*, \hat{g}_s^B, \ldots, \hat{g}_T^B) \]

39
Since
\[ E_P \left[ \tilde{Q}_{s+1} \left( g_{s+1}^*, \ldots, g_T^* \right) - \tilde{Q}_{s+1} \left( \hat{g}_{s+1}^B, \ldots, \hat{g}_T^B \right) \mid H_{s+1} = h_{s+1} \right] \geq 0 \]
holds for all \( h_{s+1} \in \mathcal{H}_{s+1} \) under Assumptions 2.1 and 3.1, and
\[
\left\| \prod_{\ell=t}^{s} \frac{1\{D_\ell = g_\ell^* (H_\ell)\}}{e_\ell (D_\ell, H_\ell)} \right\|_\infty \leq \frac{1}{\prod_{\ell=t}^{s} \kappa_\ell}
\]
holds from Assumption 2.4, the following holds:
\[
E_P \left[ \prod_{\ell=t}^{s} \frac{1\{D_\ell = g_\ell^* (H_\ell)\}}{e_\ell (D_\ell, H_\ell)} \right] E_P \left[ \tilde{Q}_{s+1} \left( g_{s+1}^*, \ldots, g_T^* \right) - \tilde{Q}_{s+1} \left( \hat{g}_{s+1}^B, \ldots, \hat{g}_T^B \right) \mid H_{s+1} \right] \leq \frac{1}{\prod_{\ell=t}^{s} \kappa_\ell} \Delta \tilde{Q}_{s+1}.
\]
Combining this with (16) leads to the result.

(iii) Note that it holds that
\[ \Delta \tilde{Q}_T = \tilde{Q}_T (g_T^*) - \tilde{Q}_T (\hat{g}_T^B) = \Delta \tilde{Q}_T^+. \]

Then, for \( t = T - 1 \), we have
\[
\Delta \tilde{Q}_{T-1} = \tilde{Q}_{T-1} (g_{T-1}^*, g_T^*) - \tilde{Q}_{T-1} (\hat{g}_{T-1}^B, \hat{g}_T^B)
= \tilde{Q}_{T-1} (g_{T-1}^*, g_T^*) - \tilde{Q}_{T-1} (g_{T-1}^*, \hat{g}_T^B) + \tilde{Q}_{T-1} (g_{T-1}^*, \hat{g}_T^B) - \tilde{Q}_{T-1} (\hat{g}_{T-1}^B, \hat{g}_T^B)
\leq \frac{1}{\kappa_{T-1}} \Delta \tilde{Q}_T^+ + \Delta \tilde{Q}_{T-1}^+,
\]
where the inequality follows from Lemma A.5 (ii).
Generally, for any $k = 1, \ldots, T - 1$, it follows that

$$
\Delta \tilde{Q}_{T-k} = \tilde{Q}_{T-k}(g^*_{T-k}, \ldots, g^*_T) - \tilde{Q}_{T-k}(\tilde{g}^B_{T-k}, \ldots, \tilde{g}^B_T)
$$

$$
= \sum_{s=T-k}^{T} \left[ \tilde{Q}_{T-k}(g^*_{T-k}, \ldots, g^*_s, \tilde{g}^B_{s+1}, \ldots, \tilde{g}^B_T) - \tilde{Q}_{T-k}(g^*_{T-k}, \ldots, g^*_s, \hat{g}^B_{s+1}, \ldots, \hat{g}^B_T) \right]
$$

$$
\leq \sum_{s=T-k}^{T} \left[ \tilde{Q}_{T-k}(g^*_{T-k}, \ldots, g^*_s) - \tilde{Q}_{T-k}(g^*_{T-k}, \ldots, \hat{g}^B_s, \ldots, \hat{g}^B_T) \right] + \Delta \tilde{Q}^\dagger_{T-k}
$$

where the second line follows by taking a telescope sum; the third line follows from the fact that $(g^*_{s+1}, \ldots, g^*_T)$ maximizes $\tilde{Q}_{T-k}(g^*_{T-k}, \ldots, g^*_s)$ over $G_{s+1} \times \cdots \times G_T$ under Assumption 3.1; the last line follows from Lemma A.5 (ii).

Then, recursively, the following hold:

$$
\Delta \tilde{Q}_{T-1} \leq \frac{1}{\kappa_{T-1}} \Delta \tilde{Q}_{T} + \Delta \tilde{Q}^\dagger_{T-1} = \frac{1}{\kappa_{T-1}} \Delta \tilde{Q}^\dagger_{T} + \Delta \tilde{Q}^\dagger_{T-1},
$$

$$
\Delta \tilde{Q}_{T-2} \leq \frac{1}{\kappa_{T-2}} \Delta \tilde{Q}_{T-1} + \frac{1}{\kappa_{T-2} \kappa_{T-1}} \Delta \tilde{Q}_{T} + \Delta \tilde{Q}^\dagger_{T-2}
$$

$$
\leq \frac{2}{\kappa_{T-2} \kappa_{T-1}} \Delta \tilde{Q}^\dagger_{T} + \frac{1}{\kappa_{T-2}} \Delta \tilde{Q}^\dagger_{T-1} + \Delta \tilde{Q}^\dagger_{T-2},
$$

$$
\vdots
$$

$$
\Delta \tilde{Q}_{T-k} \leq \sum_{s=1}^{k} \frac{2^{k-s}}{\prod_{t=T-k}^{T-s} \kappa_t} \Delta \tilde{Q}^\dagger_{T-s} + \Delta \tilde{Q}^\dagger_{T-k}.
$$

Therefore, when $k = T - 1$, we have

$$
\Delta \tilde{Q}_1 \leq \Delta \tilde{Q}^\dagger_1 + \sum_{s=1}^{T-1} \frac{2^{T-1-s}}{\prod_{t=1}^{T-s} \kappa_t} \Delta \tilde{Q}^\dagger_{T-s+1}
$$

$$
= \Delta \tilde{Q}^\dagger_1 + \sum_{s=1}^{T-1} \frac{2^{s-1}}{\prod_{t=1}^{s} \kappa_t} \Delta \tilde{Q}^\dagger_{s+1}.
$$
We are now prepared to give the proof of Theorem 3.4. We first give the proof for the Simultaneous DEWM method.

Proof of Theorem 3.4 (i): Let \( P \in \mathcal{P}(M, \kappa, G) \) be fixed. Define

\[
W_t(g_t) \equiv E_P \left[ \sum_{d_t \in \{0, 1\}^{t}} \left( \prod_{s=1}^{t} 1 \{ g_s(H_s) = d_s \} \cdot \gamma_t Y_t(d_t) \right) \right].
\]

Note that \( W_t(g_t) = E_P[w_{st}(Z, g_t)] \) and \( W(g) = \sum_{t=1}^{T} W_t(g_t) \) hold, where \( w_{st}(Z, g_t) \) is defined in Section 3.2. Let \( W_{nt}(g_t) \) and \( W_n(g) \) be sample analogies of \( W_t(g_t) \) and \( \sum_{t=1}^{T} W_t(g_t) \), respectively:

\[
W_{nt}(g_t) \equiv \frac{1}{n} \sum_{i=1}^{n} w_{st}(Z_i, g_t)
\]

\[
W_n(g) \equiv \sum_{t=1}^{T} W_{nt}(g_t).
\]

It follows, for any \( g \in G \), that

\[
E_P^n \left[ |W(g) - W(\hat{g}^S)| \right] = E_P^n \left[ |W(g) - W_n(g) + W_n(g) - W(\hat{g}^S)| \right]
\]

\[
\leq E_P^n \left[ |W(g) - W_n(g) + W_n(\hat{g}^S) - W(\hat{g}^S)| \right]
\]

\[
\leq 2E_P^n \left[ \sup_{g \in G} |W_n(g) - W(g)| \right]
\]

\[
= 2E_P^n \left[ \sup_{g \in G} \sum_{t=1}^{T} \left( W_{nt}(g_t) - W_t(g_t) \right) \right]
\]

\[
\leq 2 \sum_{t=1}^{T} E_P^n \left[ \sup_{g_t \in G_t^{\times t}} \left| W_{nt}(g_t) - W_t(g_t) \right| \right], \tag{17}
\]

where the second line follows from the fact that \( \hat{g}^S \) maximizes \( W_n(\cdot) \) over \( G \), and the fourth line follows from the definition of \( W_n(\cdot) \) and equation (1).

Applying Lemma A.4, combined with Lemma A.3, to each term in (17) leads to the following: for each \( t = 1, \ldots, T \),

\[
E_P^n \left[ \sup_{g_t \in G_t^{\times t}} \left| W_{nt}(g_t) - W_t(g_t) \right| \right] \leq C \frac{\gamma_t M_t/2}{\prod_{s=1}^{t} \kappa_s} \sqrt{\frac{\sum_{s=1}^{t} v_s}{n}}.
\]
where $C$ is the same universal constant that appears in Lemma A.4. Combining this result with (17), we obtain

$$E_{P^n} \left[ \left| W^*_g - W (\hat{g}^B) \right| \right] \leq C \sum_{t=1}^{T} \left\{ \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s} \sqrt{\frac{\sum_{s=1}^{t} v_s}{n}} \right\}. $$

Since this upper bound does not depend on $P \in \mathcal{P}(M, \kappa, G)$, the upper bound is uniform over $\mathcal{P}(M, \kappa, G)$.

We next provide the proof for the Backward DEWM method.

Proof of Theorem 3.4 (ii): Let $P \in \mathcal{P}(M, \kappa, G)$ be fixed. Let $g^*$ be defined in Section 3.1. It follows under Assumptions 2.1 and 3.1 that

$$W^*_g - W (\hat{g}^B) = \tilde{Q}_1 (g^*) - \hat{Q}_1 (\hat{g}^B) \leq \Delta \tilde{Q}_1.$$

Then, from Lemma A.5 (iii),

$$W^*_g - W (\hat{g}^B) \leq \Delta \tilde{Q}_1^t + \sum_{s=1}^{T-1} \frac{2^{s-1}}{\prod_{t=1}^{s} \kappa_t} \Delta \tilde{Q}_s^t. $$

Thus, we have

$$E_{P^n} \left[ \left| W^*_g - W (\hat{g}^B) \right| \right] \leq E_{P^n} \left[ \left| \Delta \tilde{Q}_1^t \right| \right] + \sum_{s=1}^{T-1} \frac{2^{s-1}}{\prod_{t=1}^{s} \kappa_t} E_{P^n} \left[ \left| \Delta \tilde{Q}_s^t \right| \right]. \quad (18)$$

Since

$$E_{P^n} \left[ \left| \Delta \tilde{Q}_1^t \right| \right] \leq E_{P^n} \left[ \sup_{g_t \in G_t} \left| \tilde{Q}_t (g_t, \hat{g}^B_{t+1}, \ldots, \hat{g}^B_T) - \tilde{Q}_t (\hat{g}^B_t, \ldots, \hat{g}^B_T) \right| \right]$$

for each $t$, applying Lemma A.5 (i) to each term in (18) leads to

$$E_{P^n} \left[ \left| W^*_g - W (\hat{g}^B) \right| \right] \leq C \sum_{t=1}^{T} \left\{ \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s} \sqrt{\frac{\sum_{s=1}^{t} v_s}{n}} \right\} + \sum_{t=2}^{T} \frac{2^{t-2}}{\prod_{s=1}^{t-1} \kappa_s} \left( C \sum_{s=t}^{T} \left\{ \frac{\gamma_s M_s}{\prod_{\ell=1}^{s} \kappa_{\ell}} \sqrt{\frac{\sum_{\ell=1}^{s} v_{\ell}}{n}} \right\} \right),$$

43
where $C$ is the same universal constant that appears in Lemma A.4. Since this upper bound does not depend on $P \in \mathcal{P}(M, \kappa, \mathcal{G})$, the upper bound is uniform over $\mathcal{P}(M, \kappa, \mathcal{G})$.

The following is an auxiliary lemma for the proof of Theorem 3.5, where we use the same strategy as the proofs of Theorem 2 of Massart et al. (2006) and Theorem 2.2 of Kitagawa and Tetenov (2018), but extend it to the dynamic treatment setting.

**Lemma A.6.** Suppose that Assumptions 2.1, 2.2, and 2.4 hold for any distribution $P \in \mathcal{P}(M, \kappa, \mathcal{G})$ and Assumption 2.3 holds for $\mathcal{G}$. Fix $t \in \{1, \ldots, T\}$, and let $\gamma_t = 1$ and $\gamma_s = 0$ for all $s \neq t$. Then, for any DTR $\hat{g} \in \mathcal{G}$ as a function of $(Z_1, \ldots, Z_n)$,

$$
\sup_{P \in \mathcal{P}(M, \kappa, \mathcal{G})} \mathbb{E}_P \left[ W^*_p - W(\hat{g}) \right] \geq 2^{-1} \exp \left( -2 M t \frac{\sqrt{v_{1,t}}}{n} \right)
$$

holds for all $n \geq 16v_{1,t}$. This result holds irrespective of whether Assumption 3.1 additionally holds for a pair of $\mathcal{G}$ and any $P \in \mathcal{P}(M, \kappa, \mathcal{G})$ or not.

**Proof.** The proof follows by constructing a specific subclass of $\mathcal{P}(M, \kappa, \mathcal{G})$, for which the worst-case average welfare-regret can be bounded from below. We here prove the statement for the lemma in the case that $t = T$ (i.e., $\gamma_T = 1$ and $\gamma_s = 0$ for $s \neq T$). The proof follows for the remaining cases by a similar argument. For simplicity, we normalize the support of the potential outcomes to $Y_t(d) \in [-1/2, 1/2]$ for all $d \in \{0,1\}^t$ and $t = 1, \ldots, T$. Let $1_T$ denote a $T$-dimensional vector of ones.

We construct a specific subclass $\mathcal{P}^* \subset \mathcal{P}(1_T, \kappa)$ as follows. Let $\tilde{Z} \equiv ((D_t, X_t, Y_t)_{t=1}^{T-1}, D_T, X_T)$, which is a vector of all the observed variables excluding $Y_T$, and denote its space by $\tilde{Z}$. Let $\tilde{z}_1, \ldots, \tilde{z}_{v_{1,T}}$ be $v_{1,T}$ points in $\tilde{Z}$ such that a set $\{(\tilde{z}_1, 1/2), \ldots, (\tilde{z}_{v_{1,T}}, 1/2)\}$ is shattered by a collection of indicator functions

$$
\{ f(z) = 1 \{ g_1(h_1) = d_1, \ldots, g_T(h_T) = d_T \} : (g_1, \ldots, g_T) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_T \}
$$

in the sense of Definition 2.1. For $j = 1, \ldots, v_{1,T}$, denote $\tilde{z}_j = ((d_{ij}, x_{ij}, y_{ij})_{i=1}^{T-1}, d_T, x_T) \in \tilde{Z}$. For any $P \in \mathcal{P}^*$, we suppose for the marginal distributions of $\tilde{Z}$ on $\tilde{Z}$ that $P\left( \tilde{Z} = \tilde{z}_j \right) = 1/v_{1,T}$ for each $j = 1, \ldots, v_{1,T}$. Let $b = (b_1, \ldots, b_{v_{1,T}}) \in \{0,1\}^{v_{1,T}}$ be a bit vector that indexes a member of $\mathcal{P}^*$. Hence $\mathcal{P}^*$ consists of $2^{v_{1,T}}$ distinct DGPs. For each $j = 1, \ldots, v_{1,T}$, depending on $b$, we construct the following conditional distribution of $Y_T(d_T)$ given
\( \tilde{Z} = \tilde{z}_j \): if \( b_j = 1 \),

\[
Y_T (d_{Tj}) = \begin{cases} 
1/2 & \text{w.p. } 1/2 + \delta \\
-1/2 & \text{w.p. } 1/2 - \delta 
\end{cases}
\]

otherwise

\[
Y_T (d_{Tj}) = \begin{cases} 
1/2 & \text{w.p. } 1/2 - \delta \\
-1/2 & \text{w.p. } 1/2 + \delta 
\end{cases}
\]

where \( d_{Tj} \) is the history of the realized treatments from stage 1 to \( T \) when \( \tilde{Z} = \tilde{z}_j \), and \( \delta \in [0, 1/2] \) is chosen properly in a later step of the proof. When \( b_j = 1 \), \( E_P \left[ Y_T (d_{Tj}) \mid \tilde{Z} = \tilde{z}_j \right] = \delta \); otherwise, \( E_P \left[ Y_T (d_{Tj}) \mid \tilde{Z} = \tilde{z}_j \right] = -\delta \). For conditional distributions of the other potential outcomes \( Y_T (d_T) \) given \( \tilde{Z} = \tilde{z}_j \), we set \( Y_T (d_T) = 0 \) with probability \( 1/2 \) if \( d_T \neq d_{Tj} \).

When \( b \) is known, an optimal DTR, denoted by \( g_b^* = (g_{1,b}^*, \ldots, g_{T,b}^*) \), is such that

\[
(g_{1,b}^*(h_{1j}), \ldots, g_{T,b}^*(h_{Tj})) = \begin{cases} 
d_{Tj} & \text{if } b_j = 1 \\
(1 - d_{1j}, \ldots, 1 - d_{Tj}) & \text{otherwise}
\end{cases}
\]

for \( j = 1, \ldots, v_{1:T} \), where \( h_{tj} \) is the history information in \( \tilde{z}_j \) up to stage \( t \). Such a DTR is feasible in \( \mathcal{G} \). Then, the optimized social welfare given \( b \) is

\[
W (g_b^*) = \frac{1}{v_{1:T}} \sum_{j=1}^{v_{1:T}} b_j.
\]

Let \( \hat{g} = (\hat{g}_1, \ldots, \hat{g}_T) : \mathcal{H}_1 \times \cdots \times \mathcal{H}_T \mapsto \{0, 1\}^T \) be an arbitrary DTR depending on the sample \( (Z_1, \ldots, Z_n) \), and let \( \hat{b} \in \{0, 1\}^{v_{1:T}} \) be a binary vector such that its \( j \)-th element is given by

\[
\hat{b}_j = 1 \{ \hat{g}_1 (h_{1j}) = d_{1j}, \ldots, \hat{g}_T (h_{Tj}) = d_{Tj} \}.
\]

We define by \( \pi (b) \) a prior of \( b \) such that \( b_1, \ldots, b_{v_{1:T}} \) are i.i.d and \( b_1 \sim \text{Ber}(1/2) \).

Then the maximum average welfare-regret on \( \mathcal{P} (1_T, \kappa) \) satisfies the following:

\[
\sup_{P \in \mathcal{P} (1_T, \kappa)} E_{P^n} \left[ W_G^* - W (\hat{g}) \right] \geq \sup_{P_b \in \mathcal{P}_b^*} E_{P_b^n} \left[ W (g_b^*) - W (\hat{g}) \right] \geq \int_b E_{P_b^n} \left[ W (g_b^*) - W (\hat{g}) \right] d \pi (b)
\]
\[
\geq \delta \int \int_{Z_1,\ldots,Z_n} P_Z\left(\{b(\tilde{Z}) \neq \hat{b} (\tilde{Z})\}\right) \, dP^n (Z_1,\ldots,Z_n) \, d\pi (b),
\]

where \( P_Z \) is a probability measure of \( \tilde{Z} \), and \( b(\tilde{Z}) \) and \( \hat{b} (\tilde{Z}) \) are elements of \( b \) and \( \hat{b} \) such that \( b (\tilde{z}_j) = b_j \) and \( \hat{b} (\tilde{z}_j) = \hat{b}_j \), respectively. Note that the above minimization problem can be seen as the minimization of the Bayes risk when the loss function corresponds to the classification error for predicting the binary random variable \( b(\tilde{Z}) \). Hence, the risk is minimized by the Bayes classifier such that for each \( j = 1,\ldots,J \),

\[
\hat{b}^* (\tilde{z}_j) = \begin{cases} 
1 & \text{if } \pi (b_j = 1 \mid Z_1,\ldots,Z_n) \geq 1/2 \\
0 & \text{otherwise}
\end{cases},
\]

where \( \pi (b_j = 1 \mid Z_1,\ldots,Z_n) \) is the posterior distribution for \( b_j = 1 \). This Bayes classifier is achieved by a DTR \( \hat{g}^* = (\hat{g}^*_1,\ldots,\hat{g}^*_T) \) that satisfies for \( j = 1,\ldots,J \),

\[
(\hat{g}^*_1 (h_{1j}),\ldots,\hat{g}^*_T (h_{Tj})) = \begin{cases} 
d_{Tj} & \text{if } \pi (b_j = 1 \mid Z_1,\ldots,Z_n) \geq 1/2 \\
(1 - d_{1j},\ldots,1 - d_{Tj}) & \text{otherwise}
\end{cases}.
\]

Note that \( \hat{g}^* \) is feasible in \( G \).

Then, using \( \hat{g}^* \), the minimized risk is given by

\[
\delta \int_{Z_1,\ldots,Z_n} E_Z \left[ \min \left\{ \pi \left( b \left( \tilde{Z} \right) = 1 \mid Z_1,\ldots,Z_n \right) , 1 - \pi \left( b \left( \tilde{Z} \right) = 1 \mid Z_1,\ldots,Z_n \right) \right\} \right] \, d\hat{P}^n
= \frac{1}{v_{1:T}} \delta \int_{Z_1,\ldots,Z_n} \sum_{j=1}^{v_{1:T}} \left[ \min \left\{ \pi \left( b_j = 1 \mid Z_1,\ldots,Z_n \right) , 1 - \pi \left( b_j = 1 \mid Z_1,\ldots,Z_n \right) \right\} \right] \, d\hat{P}^n,
\]

where \( \hat{P} \) is the marginal likelihood of \( \left\{ Y_{it} (d_t) \right\}_{d_t \in \{0,1\}^T}, \left\{ D_{it}, X_{it} \right\} \) with prior \( \pi (b) \).

For each \( j = 1,\ldots,v_{1:T} \), let

\[
k^+_{j} = \# \left\{ i : \tilde{Z}_i = \tilde{z}_j, Y_i = 1/2 \right\}, \\
k^-_{j} = \# \left\{ i : \tilde{Z}_i = \tilde{z}_j, Y_i = -1/2 \right\}.
\]
Then the posteriors for $b_j = 1$ can be written as

$$
\pi (b_j = 1 \mid Z_1, \ldots, Z_n) = \begin{cases} \\
\frac{1}{2} \left( \frac{1}{2} + \delta \right)^{k_j^+} \left( \frac{1}{2} - \delta \right)^{k_j^-} & \text{if } k_j^+ + k_j^- = 0 \\
(\frac{1}{2} + \delta)^{k_j^+} (\frac{1}{2} - \delta)^{k_j^-} + (\frac{1}{2} + \delta)^{k_j^-} (\frac{1}{2} - \delta)^{k_j^+} & \text{otherwise.}
\end{cases}
$$

Hence, the following holds:

$$
\min \{ \pi (b_j = 1 \mid Z_1, \ldots, Z_n), 1 - \pi (b_j = 1 \mid Z_1, \ldots, Z_n) \} = \\
\min \left\{ \left( \frac{1}{2} + \delta \right)^{k_j^+} \left( \frac{1}{2} - \delta \right)^{k_j^-}, \left( \frac{1}{2} + \delta \right)^{k_j^-} \left( \frac{1}{2} - \delta \right)^{k_j^+} \right\} = \\
\min \left\{ 1, \left( \frac{1}{2} + \delta \right)^{k_j^+ - k_j^-} \right\} = \\
1 + \frac{1}{1 + a |k_j^+ - k_j^-|}, \text{ where } a = \frac{1 + 2\delta}{1 - 2\delta} > 1.
$$

(20)

Since

$$
k_j^+ - k_j^- = \sum_{i: \tilde{Z}_i = \tilde{z}_j} 2Y_{Ti},
$$

plugging (20) into (19) yields

$$
(19) = \frac{1}{v_{1:T}} \delta \sum_{j=1}^{v_{1:T}} E_{\tilde{p}_n} \left[ \frac{1}{1 + a \sum_{i: \tilde{Z}_i = \tilde{z}_j} 2Y_{Ti}} \right] \geq \frac{\delta}{2v_{1:T}} \sum_{j=1}^{v_{1:T}} E_{\tilde{p}_n} \left[ \frac{1}{a \sum_{i: \tilde{Z}_i = \tilde{z}_j} 2Y_{Ti}} \right] \geq \frac{\delta}{2v_{1:T}} \sum_{j=1}^{v_{1:T}} a^{-E_{\tilde{p}_n} \left[ \sum_{i: \tilde{Z}_i = \tilde{z}_j} 2Y_{Ti} \right]},
$$

where $E_{\tilde{p}_n} \left[ \cdot \right]$ is the expectation with respect to the marginal likelihood of

$$
\left\{ \{Y_{it} (d_{it})\}_{d_{it} \in \{0,1\}^T}, D_{it}, X_{it} \right\}_{i=1,\ldots,n; t=1,\ldots,T}.
$$

The first inequality follows by $a > 1$, and the second inequality follows by Jensen’s inequality. Given our prior distribution for $b$, for each $d_{it} \in \{0,1\}^T$, the marginal distribution of $Y_{iT} (d_{iT})$ is $P (Y_{iT} (d_{iT}) = 1/2) = P (Y_{iT} (d_{iT}) = -1/2) = 1/2$ if there exist $d_{iT,j}$ among
\(d_{r1}, \ldots, d_{rv_1}\) such that \(d_{rj} = d_r\); otherwise, \(P(Y_{iT}(d_r) = 0) = 1\). Thus, we have

\[
E_{\tilde{P}_n} \left| \sum_{i: \tilde{Z}_i = \tilde{z}_j} 2Y_{iT} \right| = E_{\tilde{P}_n} \left| \sum_{i: \tilde{Z}_i = \tilde{z}_j} 2Y_{iT}(d_{rj}) \right|
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{v_{1:T}} \right)^k \left( 1 - \frac{1}{v_{1:T}} \right)^{n-k} E \left| B \left( k, \frac{1}{2} \right) - \frac{k}{2} \right|,
\]

where \(B(k,1/2)\) is the binomial random variable with parameters \(k\) and 1/2. By the Cauchy-Schwarz inequality, it follows that

\[
E \left| B \left( k, \frac{1}{2} \right) - \frac{1}{2} \right| \leq \sqrt{E \left( B \left( k, \frac{1}{2} \right) - \frac{k}{2} \right)^2} = \sqrt{k/4}.
\]

Thus, we obtain

\[
E_{\tilde{P}_n} \left| \sum_{i: \tilde{Z}_i = \tilde{z}_j} 2Y_{iT} \right| \leq \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{v_{1:T}} \right)^k \left( 1 - \frac{1}{v_{1:T}} \right)^{n-k} \sqrt{k/4}
\]

\[
= E \sqrt{\frac{B(n, \frac{1}{v_{1:T}})}{4}}
\]

\[
\leq \sqrt{\frac{n}{4v_{1:T}}},
\]

where the last inequality follows by Jensen’s inequality. Hence, the Bayes risk is bounded from below by

\[
\frac{\delta}{2} a - \sqrt{\frac{v_{1:T}}{n}} \geq \frac{\delta}{2} \exp \left\{ - (a - 1) \sqrt{\frac{n}{4v_{1:T}}} \right\}
\]

\[
= \frac{\delta}{2} \exp \left\{ - \frac{4\delta}{1 - 2\delta} \sqrt{\frac{n}{4v_{1:T}}} \right\},
\]

where the inequality follows from the fact that \(1 + x \leq e^x\) for any \(x\). This lower bound on the Bayes risk has the slowest convergence rate when \(\delta\) is set to be proportional to \(n^{-1/2}\).

Specifically, letting \(\delta = \sqrt{v_{1:T}/n}\), we have

\[
\frac{\delta}{2} \exp \left\{ - \frac{4\delta}{1 - 2\delta} \sqrt{\frac{n}{4v_{1:T}}} \right\} = \frac{1}{2} \sqrt{\frac{v_{1:T}}{n}} \exp \left\{ - \frac{2}{1 - 2\delta} \right\} \geq \frac{1}{2} \sqrt{\frac{v_{1:T}}{n}} \exp \left( -4 \right) \text{ if } 1 - 2\delta \geq \frac{1}{2}.
\]

The condition \(1 - 2\delta \geq 1/2\) is equivalent to \(n \geq 16v_{1:T}\). Multiplying the lower bound by
MT gives
\[ \sup_{P \in \mathcal{P}(M, \kappa, G)} E_{P^n} \left[ W_G^* - W(\hat{g}) \right] \geq \frac{1}{2} \exp \left( -4 \right) MT \sqrt{\frac{v_{1:T}}{n}} \]
for all \( n \geq 16v_{1:T} \).

The proof is valid irrespective of whether Assumption 3.1 holds for a pair \((P, G)\) with any \( P \in \mathcal{P}(M, \kappa, G) \) or not. \qed

Proof of Theorem 3.5. The result immediately follows by setting \( t = \arg \max_{s \in \{1, \ldots, T\}} \gamma_s M_s \sqrt{\frac{v_{1:s}}{n}} \) in the statement of Lemma A.6. \qed

In the rest of this section, We derive uniform upper bounds on the worst-case average welfare-regrets of the two DEWM methods in the case where estimated propensity scores are used instead of true ones.

Proof of Theorem 5.1 (i): Let \( P \in \mathcal{P} \cap \mathcal{P}(M, \kappa, G) \) be fixed. Define \( \hat{W}_{nt}(g_j) = \frac{1}{n} \sum_{i=1}^{n} \hat{w}_{it}^S(Z_i, g_j) \) and \( \hat{W}(g) = \sum_{t=1}^{T} \hat{W}_{nt}(g) \), which are estimators of \( W_t(g) \) and \( W(g) \), respectively. It follows for any \( g \in G \) that
\[
W(g) - W\left(\hat{g}_c^S\right) \leq W_n(g) - \hat{W}_n \left(\hat{g}_c^S\right) + \hat{W}_n \left(\hat{g}_c^S\right) - W_n \left(\hat{g}_c^S\right) + W \left(\hat{g}_c^S\right) - W_n \left(\hat{g}_c^S\right) + W_n \left(\hat{g}_c^S\right) - W_n(g)
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{T} \sum_{d_t \in \{0,1\}^t} \left[ \left( \gamma_t Y_{it} \cdot 1 \{ D_{it} = d_t \} \prod_{s=1}^{t} \epsilon_s (d_s, H_{is}) - \gamma_t Y_{it} \cdot 1 \{ D_{it} = d_t \} \prod_{s=1}^{t} \hat{\epsilon}_s (d_s, H_{is}) \right) \right]
\]
\[
\times \left( \prod_{s=1}^{t} \{ g_s(H_{is}) = d_s \} - \prod_{s=1}^{t} \{ \hat{g}_c^S_{c,s}(H_{is}) = d_s \} \right)
\]
\[
+ W(g) - W_n(g) + W_n \left(\hat{g}_c^S\right) - W \left(\hat{g}_c^S\right) \leq \sum_{t=1}^{T} \sum_{d_t \in \{0,1\}^t} \left( \frac{1}{n} \sum_{i=1}^{n} | \tau_t (d_t, H_{it}) - \hat{\tau}_t (d_t, H_{it}) | \right) + 2 \sup_{g \in \mathcal{G}} | W_n(g) - W(g) |
\]
\[
\leq \sum_{t=1}^{T} \sum_{d_t \in \{0,1\}^t} \left( \frac{1}{n} \sum_{i=1}^{n} | \tau_t (d_t, H_{it}) - \hat{\tau}_t (d_t, H_{it}) | \right) + 2 \sum_{t=1}^{T} \sup_{g \in \mathcal{G}_t \times \cdots \times \mathcal{G}_t} \left| W_{nt}(g) - W_t(g) \right|.
\]
The first inequality follows from the fact that \( \hat{g}_e^S \) maximizes \( \hat{W}_n (\cdot) \) over \( \mathcal{G} \). The second inequality follows from the fact that

\[
\prod_{s=1}^t \mathbb{1}\{g_s (H_{is}) = d_s\} - \prod_{s=1}^t \mathbb{1}\{\hat{g}_{e,s}^S (H_{is}) = d_s\} \leq 1.
\]

Thus, the average welfare-regret can be bounded from above by

\[
E_{P^n} \left[ W^*_G - W (\hat{g}_e^S) \right] \leq \sum_{t=1}^T \sum_{d \in \{0,1\}^t} E_{P^n} \left[ \frac{1}{n} \sum_{i=1}^n |\tau_t (d, H_{it}) - \hat{\tau}_t (d, H_{it})| \right]
\]

\[+ 2 \sum_{t=1}^T \sup_{\hat{g}_{\cdot}, \cdot \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_t} E_{P^n} \left[ |W_{nt}(\hat{g}_{\cdot}, \cdot) - W_t(\hat{g}_{\cdot}, \cdot)| \right].
\]

Therefore, by the same argument as in the proof of Theorem 3.4 (i) and from Assumption 5.1 (i), the average welfare-regret is bounded from above as

\[
E_{P^n} \left[ W^*_G - W (\hat{g}_e^S) \right] \leq C \sum_{t=1}^T \left\{ \frac{\gamma_t M_t}{\prod_{s=1}^t \kappa_s} \sqrt{\sum_{s=1}^t v_s} \right\} + O \left( \rho^{-1} \right).
\]

Since this upper bound does not depend on \( P \in \mathcal{P}_e \cap \mathcal{P}(M, \kappa, \mathcal{G}) \), the upper bound is uniform over \( \mathcal{P}_e \cap \mathcal{P}(M, \kappa, \mathcal{G}) \).

Before proceeding to the proof of Theorem 5.1 (ii), we define

\[
\Delta \hat{Q}_{t,e} \equiv \hat{Q}_t (g^*_{t,1}, \ldots, g^*_{t,T}) - \hat{Q}_t (\hat{g}^B_{t,e}, \ldots, \hat{g}^B_{T,e}) ,
\]

\[
\Delta \hat{Q}^\dagger_{t,e} \equiv \hat{Q}_t (g^*_{t,1}, \hat{g}^B_{t+1,e}, \ldots, \hat{g}^B_{T,e}) - \hat{Q}_t (\hat{g}^B_{t,e}, \ldots, \hat{g}^B_{T,e}) ,
\]

\[
\hat{Q}_{nt} (g_t, \ldots, g_T) \equiv E_n [\hat{q}_t (Z, g_t, \ldots, g_T)]
\]

\[
= \sum_{s=t}^T E_n \left[ \frac{\prod_{t'=t}^s 1\{g_{t'} (H_{t'}) = D_{t'}\} \gamma_s Y_s}{\prod_{t'=t}^s \hat{e}_{t'} (D_{t'}, H_{t'})} \right],
\]

where \( \hat{Q} \) is defined in (14). The following lemmas will be used in the proof of Theorem 5.1 (ii).

**Lemma A.7.** Suppose that Assumptions 2.1, 2.2, and 2.4 hold for any \( P \in \mathcal{P}(M, \kappa, \mathcal{G}) \), that Assumption 2.3 holds for \( \mathcal{G} \), that Assumption 3.1 holds for a pair \( (P, \mathcal{G}) \) with any \( P \in \mathcal{P}(M, \kappa, \mathcal{G}) \), and that Assumption 5.1 (ii) holds for any \( P \in \mathcal{P}_e \). Then, for any \( P \in \mathcal{P}(M, \kappa, \mathcal{G}) \cap \mathcal{P}_e \), the following hold:
(i) for $t = 1, \ldots, T$,
\[
E_{\pi^n} \left[ \left| \Delta \tilde{Q}_{t,e}^1 \right| \right] \leq C \left( \sum_{s=1}^{T} \frac{\gamma_s M_s}{\prod_{t=1}^{T} \kappa_{t}} \right) \sqrt{\sum_{s=1}^{T} v_s} \frac{1}{n} + O \left( \psi_n^{-1} \right),
\]
where $C$ is the same constant term as introduced in Lemma A.4;

(ii) for $t = 1, \ldots, T - 1$ and $s = t + 1, \ldots, T$,
\[
\tilde{Q}_t \left( g_t^*, \ldots, g_T^* \right) - \tilde{Q}_t \left( g_t^*, \ldots, \bar{g}_{s+1}^B, \ldots, \bar{g}_{T,e}^B \right) \leq \frac{1}{\prod_{t=1}^{T} \kappa_{t}} \Delta \tilde{Q}_{s+1,e};
\]

(iii)
\[
\Delta \tilde{Q}_{1,e} \leq \Delta \tilde{Q}_{1,e}^1 + \sum_{s=1}^{T-1} \frac{2^{s-1}}{\prod_{t=1}^{T} \kappa_{t}} \Delta \tilde{Q}_{s+1,e}^1.
\]

Proof. (i) It follows for any $\tilde{g}_t \in \mathcal{G}_t$ that
\[
\tilde{Q}_t \left( \tilde{g}_t, \tilde{g}_{t+1,e}^B, \ldots, \tilde{g}_{T,e}^B \right) - \tilde{Q}_t \left( \tilde{g}_{t,e}^B, \ldots, \tilde{g}_{T,e}^B \right)
\leq \tilde{Q}_{nt} \left( \tilde{g}_t, \tilde{g}_{t+1,e}^B, \ldots, \tilde{g}_{T,e}^B \right) - \tilde{Q}_{nt} \left( \tilde{g}_{t,e}^B, \tilde{g}_{t+1,e}^B, \ldots, \tilde{g}_{T,e}^B \right) - \tilde{Q}_{nt} \left( \tilde{g}_{t,e}^B, \tilde{g}_{t+1,e}^B, \ldots, \tilde{g}_{T,e}^B \right) + \tilde{Q}_{nt} \left( \tilde{g}_{t,e}^B, \tilde{\tilde{g}}_{t+1,e}^B, \ldots, \tilde{g}_{T,e}^B \right)
\leq \frac{1}{n} \sum_{s=1}^{T} \sum_{D_{t,s} \in \mathcal{D}_{t,\epsilon}^{T-e-1}} \left( \frac{\sum_{t=1}^{T} \gamma_s M_s}{\prod_{t=1}^{T} \kappa_{t}} \right) \left( \frac{1}{\prod_{t=1}^{T} \kappa_{t}} \right) \Delta \tilde{Q}_{s+1,e}^1
\leq \sum_{D_{t,s} \in \mathcal{D}_{t,\epsilon}^{T-e-1}} \left( \frac{1}{n} \sum_{i=1}^{n} \left| \eta_i \left( D_{t,T}, H_{1T} \right) - \eta_i \left( D_{t,T}, H_{0T} \right) \right| \right)
+ 2 \sup_{g_t \times \ldots \times g_T} \left| \tilde{Q}_{nt} \left( g_t, \ldots, g_T \right) - \tilde{Q}_t \left( g_t, \ldots, g_T \right) \right|.
\]

The first inequality follows from the fact that $\tilde{g}_{t,e}^B$ maximizes $\tilde{Q}_{nt} \left( \cdot, \tilde{g}_{t+1,e}^B, \ldots, \tilde{g}_{T,e}^B \right)$ over $\mathcal{G}_t$. 

51
Then we have
\[ E_{p_n} \left[ \left\| \Delta \tilde{Q}_{T,e}^i \right\| \right] \leq 2 E_{p_n} \left[ \sup_{(g_t,\ldots,g_T) \in G_t \times \cdots \times G_T} \left| \hat{Q}_n (g_t,\ldots,g_T) - \hat{Q}_l (g_t,\ldots,g_T) \right| \right] + \frac{1}{n} \sum_{t=1}^{n} E_{p_n} \left[ \frac{1}{n} \sum_{i=1}^{n} |\hat{\eta}_t (d_{e,T}, H_{\delta T}) - \eta_l (d_{e,T}, H_{\delta T})| \right] \]

Therefore, applying Lemma A.4 to the first term in the right hand side (as in the proof of Lemma A.5 (i)) and Assumption 5.1 (ii) to the second term in the right hand side leads to the result.

(ii) The proof of Lemma A.7 follows from the same argument with the proof of Lemma A.5 (ii).

(iii) We follow the same strategy as in Lemma A.5 (iii). First, note that
\[ \Delta \tilde{Q}_{T,e} = \tilde{Q}_T (g_T^*) - \tilde{Q}_T (\hat{g}_T^*) = \Delta \tilde{Q}_{T,e}^i. \]

Then, for \( t = T - 1 \), we have
\[
\Delta \tilde{Q}_{T-1,e} = \tilde{Q}_{T-1} (g_{T-1}^*, g_T^*) - \tilde{Q}_{T-1} (\hat{g}_{T-1,e}^*, \hat{g}_T^*) \\
= \tilde{Q}_{T-1} (g_{T-1}^*, g_T^*) - \tilde{Q}_{T-1} (\hat{g}_{T-1,e}^*, \hat{g}_T^*) + \tilde{Q}_{T-1} (g_{T-1}^*, \hat{g}_T^*) - \tilde{Q}_{T-1} (\hat{g}_{T-1,e}^*, \hat{g}_T^*) \\
\leq \frac{1}{k_{T-1}} \Delta \tilde{Q}_{T,e}^i + \Delta \tilde{Q}_{T-1,e}^i,
\]

where the inequality follows from Lemma A.7 (ii).

Generally, for any \( k = 1, \ldots, T - 1 \), it follows that
\[
\Delta \tilde{Q}_{T-k,e} = \tilde{Q}_{T-k} (g_{T-k}^*, \ldots, g_T^*) - \tilde{Q}_{T-k} (\hat{g}_{T-k,e}^*, \ldots, \hat{g}_T^*) \\
= \sum_{s=T-k}^{T} \left[ \tilde{Q}_{T-k} (g_{T-k}^*, \ldots, g_s^*, \hat{g}_{s+1,e}^*, \ldots, \hat{g}_T^*) - \tilde{Q}_{T-k} (g_{T-k}^*, \ldots, g_s^*, \hat{g}_{s,e}^*, \ldots, \hat{g}_T^*) \right] \\
\leq \sum_{s=T-k}^{T} \left[ \tilde{Q}_{T-k} (g_{T-k}^*, \ldots, g_s^*) - \tilde{Q}_{T-k} (g_{T-k}^*, \ldots, g_s^*, \hat{g}_{s,e}^*, \ldots, \hat{g}_T^*) \right] \\
= \sum_{s=T-k+1}^{T} \left[ \tilde{Q}_{T-k} (g_{T-k}^*, \ldots, g_s^*) - \tilde{Q}_{T-k} (g_{T-k}^*, \ldots, g_s^*, \hat{g}_{s,e}^*, \ldots, \hat{g}_T^*) \right] + \Delta \tilde{Q}_{T-k,e}^i \\
\leq \sum_{s=T-k+1}^{T} \frac{1}{\prod_{\ell=T-k}^{\delta \ell} \kappa_{\ell}} \Delta \tilde{Q}_{s,e} + \Delta \tilde{Q}_{T-k,e}^i
\]

where the second line follows by taking a telescope sum; the third line follows from the
fact that \((g_{s+1}^*, \ldots, g_T^*)\) maximizes \(\tilde{Q}_{T-k}(g_{T-k}^*, g_{s+1}^*, \ldots, \cdot)\) over \(G_{s+1} \times \cdots \times G_T\) under Assumption 3.1; the last line follows from Lemma A.7(ii).

Then, recursively, the following hold:

\[
\Delta \tilde{Q}_{T-k, e} \leq \frac{1}{\kappa_{T-k}} \Delta \tilde{Q}_{T, e} + \frac{1}{\kappa_{T-k}} \Delta \tilde{Q}_{T-k, e} + \Delta \tilde{Q}_{T-k}^{\dagger, e},
\]

\[
\Delta \tilde{Q}_{T-2, e} \leq \frac{2}{\kappa_{T-2} \kappa_{T-1}} \Delta \tilde{Q}_{T-1, e} + \frac{1}{\kappa_{T-1}} \Delta \tilde{Q}_{T-2, e} + \Delta \tilde{Q}_{T-2}^{\dagger, e},
\]

\cdots

\[
\Delta \tilde{Q}_{k, e} \leq \sum_{s=1}^{k} \frac{2^{k-s}}{\prod_{t=T-k}^{T-s} \kappa_t} \Delta \tilde{Q}_{T-s+1, e} + \Delta \tilde{Q}_{k, e}^{\dagger, e}.
\]

Therefore, when \(k = T - 1\), we have

\[
\Delta \tilde{Q}_{1, e} \leq \Delta \tilde{Q}_{1, e}^{\dagger, e} + \sum_{s=1}^{T-1} \frac{2^{T-s}}{\prod_{t=1}^{T-s} \kappa_t} \Delta \tilde{Q}_{T-s+1, e}^{\dagger, e} + \Delta \tilde{Q}_{T-1, e}^{\dagger, e}.
\]

Proof of Theorem 5.1 (ii): Let \(P \in \tilde{P}_e \cap P(M, \kappa, G)\) be fixed. By the same argument as in the proof of Theorem 3.4 (ii), it follows for \(g^* \in \arg \max_g W(g)\) that

\[
W_g^* - W(\hat{g}_e^B) = \tilde{Q}_1(g^*) - \tilde{Q}_1(\hat{g}_e^B) \leq \Delta \tilde{Q}_{1, e} \leq \Delta \tilde{Q}_{1, e}^{\dagger, e} + \Delta \tilde{Q}_{T-1, e}^{\dagger, e}.
\]

where the second inequality follows from Lemma A.7 (iii). Thus, since \(W_g^* - W(\hat{g}_e^B) \geq 0\), we have

\[
E_{P^n} \left[ W_g^* - W(\hat{g}_e^B) \right] \leq E_{P^n} \left[ \Delta \tilde{Q}_{1, e}^{\dagger, e} \right] + \sum_{s=1}^{T-1} \frac{2^{s-1}}{\prod_{t=1}^{s-1} \kappa_t} E_{P^n} \left[ \Delta \tilde{Q}_{s+1, e}^{\dagger, e} \right] .
\]

53
Applying Lemma A.7 (i) to each term in the right hand side gives

\[
E_P \left[ |W_g^p - W(\hat{g}_E^B)| \right] \leq C \sum_{t=1}^{T} \left\{ \frac{\gamma_t M_t \sqrt{\sum_{s=1}^{t} v_s}}{n} \right\}
\]

\[
+ \sum_{t=2}^{T} \frac{2^{t-2}}{\prod_{s=1}^{t-1} \kappa_s} \left( C \sum_{s=1}^{T} \left\{ \frac{\gamma_s M_s \sqrt{\sum_{t=s}^{T} v_t}}{n} \right\} \right) + O(\psi_n^{-1}).
\]

Since this upper bound does not depend on \( P \in \mathcal{P}_e \cap \mathcal{P}(M, \kappa, \mathcal{G}) \), the upper bound is uniform over \( \mathcal{P}_e \cap \mathcal{P}(M, \kappa, \mathcal{G}) \).

\[\square\]

### A.2 Proof of Theorem 4.1

This appendix provides the proof of Theorem 4.1. We first introduce lemmas and a proposition that will be used in the proof of Theorem 4.1.

The following lemma provides a concentration inequality that is frequently used in the literature on statistical learning theory, the proof of which can be found, for example, in Mohri et al. (2012).

**Lemma A.8.** \((\text{McDiarmid’s Inequality}): \) Let \( S = (Z_1, \ldots, Z_n) \in \mathcal{Z}^n \) be a set of \( n \) independent random variables, and \( g \) be a mapping from \( \mathcal{Z}^n \) to \( \mathbb{R} \) such that there exist \( c_1, \ldots, c_n > 0 \) that satisfy the following conditions:

\[
|g(z_1, \ldots, z_i, \ldots, z_n) - g(z_1, \ldots, z'_i, \ldots, z_n)| < c_i
\]

for any \( n+1 \) points \( z_1, \ldots, z_n, z'_i \) in \( \mathcal{Z} \) and all \( i \in \{1, \ldots, n\} \). Let \( g(S) \) denote \( g(Z_1, \ldots, Z_n) \). Then the following inequalities hold for all \( \epsilon > 0 \):

\[
\Pr \left[ g(S) - E[g(S)] \geq \epsilon \right] \leq \exp \left( \frac{-2\epsilon^2}{\sum_{i=1}^{n} c_i^2} \right),
\]

\[
\Pr \left[ g(S) - E[g(S)] \leq -\epsilon \right] \leq \exp \left( \frac{-2\epsilon^2}{\sum_{i=1}^{n} c_i^2} \right).
\]

The following proposition gives a finite-sample upper bound on the exact maximum welfare-regret of the Simultaneous DEWM method that holds with a high probability.
Proposition A.1. Suppose that any distribution $P \in \mathcal{P}(M, \kappa, G)$ satisfies Assumptions 2.1, 2.2, and 2.4 and that $G$ satisfies Assumption 2.3. Then, for any $\delta \in (0, 1)$, the following holds for any distribution $P \in \mathcal{P}(M, \kappa, G)$ with probability at least $1 - \delta$:

$$|W^*_G - W(\hat{g}^S)| \leq \frac{1}{\sqrt{n}} \sum_{t=1}^{T} \gamma_t M_t \prod_{s=1}^{t} \kappa_s \left\{ 2C \sqrt{\frac{1}{\delta}} \sum_{s=1}^{t} v_s + \sqrt{2 \log (1/\delta)} \right\}.$$  

Proof. The proof follows a similar argument as that of Corollary 3.4 of Mohri et al. (2012). Let $P \in \mathcal{P}(M, \kappa, G)$ be fixed. From the proof of Theorem 3.4, 

$$|W^*_G - W(\hat{g}^S)| \leq 2 \sup_{g \in G} |W_n(g) - W(g)|. \quad (21)$$

We will evaluate $\sup_{g \in G} |W_n(g) - W(g)|$. Let $S = (Z_1, \ldots, Z_n)$ be the sample and define 

$$A(S) \equiv \sup_{g \in G} \{ W(g) - W_S(g) \},$$

where, for any sample $S$ with size $n$, $W_S(g)$ is defined as $W_n(g)$ that uses the sample $S$. Introduce $S' = (Z_1, \ldots, Z_{n-1}, Z'_n)$, a sample that is different from $S$ with respect to the final component.

Then, it follows that 

$$A(S) - A(S') = \sup_{g \in G} \inf_{g' \in G} \{ W(g) - W_S(g) - W(g') + W_{S'}(g') \}$$

$$\leq \sup_{g \in G} \{ W(g) - W_S(g) - W(g) + W_{S'}(g) \}$$

$$= \frac{1}{n} \sup_{g \in G} \left\{ \sum_{t=1}^{T} w_t^S(Z_n, \hat{g}_t) - \sum_{t=1}^{T} w_t^S(Z'_n, \hat{g}_t) \right\}$$

$$\leq \frac{1}{n} \sum_{t=1}^{T} \sup_{g \in G} \left\{ w_t^S(Z_n, \hat{g}_t) - w_t^S(Z'_n, \hat{g}_t) \right\}$$

$$\leq \frac{1}{n} \sum_{t=1}^{T} \left( \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s} \right),$$

where the last inequality follows from the fact that under Assumptions 2.2 and 2.4, $w_t^S(Z, \hat{g})$ is bounded from above by $(\gamma_t M_t/2) / (\prod_{s=1}^{t} \kappa_s)$. Since we have 

$$|A(S) - A(S')| \leq \frac{1}{n} \sum_{t=1}^{T} \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s},$$

55
applying Lemma A.8 leads to

\[ P \left( |A(S) - E_{P^n}[A(S)]| \geq \epsilon \right) \leq \exp \left( \frac{-2n\epsilon^2}{\left( \sum_{t=1}^{T} \frac{\gamma_t M_t}{\Pi_{s=1}^{t} \kappa_s} \right)^2} \right) \]

for any \( \epsilon > 0 \). This is equivalent to the following inequality: for any \( \delta \in (0, 1) \),

\[ P \left( |A(S) - E_{P^n}[A(S)]| \leq \left( \sum_{t=1}^{T} \frac{\gamma_t M_t}{\Pi_{s=1}^{t} \kappa_s} \right) \sqrt{\frac{\log (1/\delta)}{2n}} \right) \geq 1 - \delta. \tag{22} \]

Subsequently, we will evaluate \( E_{P^n}[A(S)] \). Since

\[ E_{P^n}[A(S)] = E_{P^n} \left[ \sum_{t=1}^{T} \left( W_{nt}(g_t) - W_t(g_t) \right) \right] \leq \sum_{t=1}^{T} E_{P^n} \left[ \sup_{g_t \in \mathcal{G}_t} \left| W_{nt}(g_t) - W_t(g_t) \right| \right], \]

applying Lemma A.4 combined with Lemma A.3 leads to

\[ E_{P^n}[A(S)] \leq C \sum_{t=1}^{T} \left\{ \frac{\gamma_t M_t}{\Pi_{s=1}^{t} \kappa_s} \sqrt{\frac{\sum_{s=1}^{t} v_s}{n}} \right\}, \tag{23} \]

where \( C \) is the same constant that appears in Lemma A.4.

Consequently, combining (21), (22), and (23), for any \( \delta \in (0, 1) \), it follows with probability at least 1 – \( \delta \) that

\[ |W^*_G - W(\hat{g}^S)| \leq 2C \sum_{t=1}^{T} \left[ \left( \frac{\gamma_t M_t}{\Pi_{s=1}^{t} \kappa_s} \right) \sqrt{\frac{\sum_{s=1}^{t} v_s}{n}} + 2 \left( \sum_{t=1}^{T} \frac{\gamma_t M_t}{\Pi_{s=1}^{t} \kappa_s} \right) \sqrt{\frac{\log (1/\delta)}{2n}} \right], \]

Since this upper bound does not depend on \( P \in \mathcal{P}(M, \kappa, \mathcal{G}) \), this probability inequality is uniform over \( \mathcal{P}(M, \kappa, \mathcal{G}) \).

With some abuse of notation, let \( g^* = (g^*_1, \ldots, g^*_T) \) be a solution of the constrained maximization problem (8). The following lemma shows that a class of feasible DTRs that satisfy the empirical budget/capacity constraints (10) contains \( g^* \) with high probability.
Lemma A.9. Suppose that a distribution $P$ satisfies Assumption 2.1 and that $\sum_{t=1}^{T} K_{tb} = 1$ holds for all $b = 1, \ldots, B$. Define

$$G_{\alpha_n}^S = \left\{ g \in \mathcal{G} : \sum_{t=1}^{T} K_{tb} \hat{E} \left[ g_t(H_t) \mid D_1 = g_1(H_1), \ldots, D_{t-1} = g_{t-1}(H_{t-1}) \right] \leq C_b + \alpha_n \text{ for } b = 1, \ldots, B \right\},$$

which is a subset of DTRs that satisfy the sample budget constraints (10). Then, for any $\delta \in (0, 1)$, if $\alpha_n \geq \sqrt{\log(B/\delta)/(2n)}$, $g^* \in G_{\alpha_n}^S$ holds with probability at least $1 - \delta$ under the distribution $P$.

Proof. It follows that

$$P\left( g^* \notin G_{\alpha_n}^S \right) = P\left( \max_{b=1,\ldots,B} \left\{ \sum_{t=1}^{T} K_{tb} \hat{E} \left[ g_t(H_t) \mid D_1 = g_1(H_1), \ldots, D_{t-1} = g_{t-1}(H_{t-1}) \right] - C_b \right\} > \alpha_n \right)$$

$$\leq \sum_{b=1}^{B} P \left[ \sum_{t=1}^{T} K_{tb} \hat{E} \left[ g_t(H_t) \mid D_1 = g_1(H_1), \ldots, D_{t-1} = g_{t-1}(H_{t-1}) \right] - C_b > \alpha_n \right]$$

$$\leq \sum_{b=1}^{B} \left( \sum_{t=1}^{T} K_{tb} \hat{E} \left[ g_t(H_t) \mid D_1 = g_1(H_1), \ldots, D_{t-1} = g_{t-1}(H_{t-1}) \right] - \sum_{t=1}^{T} K_{tb} E_P \left[ g_t(H_t) \mid D_1 = g_1(H_1), \ldots, D_{t-1} = g_{t-1}(H_{t-1}) \right] \right) > \alpha_n \right).$$

The second inequality follows from the fact that $g^*$ satisfies the population budget/capacity constraints (7).

By Hoeffding’s inequality, it follows that

$$P \left( \sum_{t=1}^{T} K_{tb} \hat{E} \left[ g_t(H_t) \mid D_1 = g_1(H_1), \ldots, D_{t-1} = g_{t-1}(H_{t-1}) \right] \right)$$

$$- \sum_{t=1}^{T} K_{tb} E_P \left[ g_t(H_t) \mid D_1 = g_1(H_1), \ldots, D_{t-1} = g_{t-1}(H_{t-1}) \right] > \alpha_n \right) \leq \exp \left\{ -\frac{2n\alpha_n^2}{\left( \sum_{t=1}^{T} K_{tb} \right)^2} \right\} = \exp \left( -2n\alpha_n^2 \right)$$

for each $b = 1, \ldots, B$, where the equality follows from the scale normalization: $\sum_{t=1}^{T} K_{tb} =$

57
1. Thus, we have

\[ P\left(g^* \notin \mathcal{G}_n^S\right) \leq B \exp\left(-2n\alpha_n^2\right). \]

Therefore, if \( \alpha_n \geq \sqrt{\log (B/\delta) / (2n)} \), \( g^* \in \mathcal{G}_n^S \) holds with probability at least \( 1 - \delta \).

**Proof of Theorem 4.1 (i):** Let \( P \in \mathcal{P}(M, \kappa, \mathcal{G}) \) be fixed. We use the notation \( A \leq \delta B \) to denote that \( A \leq B \) holds with probability at least \( 1 - \delta \). From the proof of Proposition A.1, we have, for any \( g \in \mathcal{G} \), that

\[
|W(g) - W_n(g)| \leq \frac{1}{\sqrt{n}} \sum_{t=1}^{T} \left( \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s} \right) \left( 2C \sqrt{\sum_{s=1}^{t} v_s + \sqrt{2 \log (1/\delta)}} \right). \tag{24}
\]

Applying the same argument as in the proof of Proposition A.1, it follows for each \( b = 1, \ldots, B \) that

\[
\left| E_n \left[ \sum_{t=1}^{T} K_{tb} \hat{g}_t^S (H_t) \right] - E_P \left[ \sum_{t=1}^{T} K_{tb} \hat{g}_t^S (H_t) \right] \right| \leq \frac{1}{\sqrt{n}} \sum_{t=1}^{T} \left[ K_{tb} \left( 2C \sqrt{\sum_{s=1}^{t} v_s + \sqrt{2 \log (1/\delta)}} \right) \right]. \tag{25}
\]

By Lemma A.9, when \( \alpha_n \geq \sqrt{\log (6B/\delta) / (2n)} \), we have \( W_n(g^*) \leq \delta/6 W_n(\hat{g}^S) \) because \( g^* \) is contained in \( \mathcal{G}_n^S \) with probability at least \( 1 - \delta/6 \) and \( \hat{g}^S \) maximizes \( W_n(\cdot) \) over \( \mathcal{G}_n^S \).

By combining the fact that \( W_n(g^*) \leq \delta/6 W_n(\hat{g}^S) \) with (24), it follows that

\[
W(g^*) \leq \delta/6 W_n(g^*) + \frac{1}{\sqrt{n}} \sum_{t=1}^{T} \left( \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s} \right) \left( 2C \sqrt{\sum_{s=1}^{t} v_s + \sqrt{2 \log (6/\delta)}} \right) \leq \delta/6 W_n(\hat{g}^S) + \frac{1}{\sqrt{n}} \sum_{t=1}^{T} \left( \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s} \right) \left( 2C \sqrt{\sum_{s=1}^{t} v_s + \sqrt{2 \log (6/\delta)}} \right) \leq \delta/6 W(\hat{g}^S) + \frac{2}{\sqrt{n}} \sum_{t=1}^{T} \left( \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s} \right) \left( 2C \sqrt{\sum_{s=1}^{t} v_s + \sqrt{2 \log (6/\delta)}} \right). \]

The first inequality follows from the inequality in (24); the second inequality follows from the fact that \( \hat{g}^S \) maximizes \( W_n(\cdot) \) over \( \mathcal{G}_n^S \) and \( g^* \in \mathcal{G}_n^S \) holds with probability at least
\[ W(g^*) \leq \delta/2 \cdot W(\hat{g}^S) + \frac{1}{\sqrt{n}} \sum_{t=1}^{T} \left( \frac{\gamma_t M_t}{\prod_{s=1}^{t} \kappa_s} \right) \left\{ 4C \sqrt{\sum_{s=1}^{t} v_s + \sqrt{8 \log (6/\delta)}} \right\}. \tag{26} \]

Furthermore, combining the fact that \[ W_n(g^*) \leq \delta/6 \cdot W_n(\hat{g}^S) \] with equation (25), it follows for each \( b = 1, \ldots, B \),

\[
\mathbb{E}_{P} \left[ \sum_{t=1}^{T} K_b \hat{g}^S_t (H_t) \right] \leq \delta/(2B) \cdot \mathbb{E}_n \left[ \sum_{t=1}^{T} K_b \hat{g}^S_t (H_t) \right] + \frac{1}{\sqrt{n}} \sum_{t=1}^{T} K_{tb} \left\{ 2C \sqrt{\sum_{s=1}^{t} v_s + \sqrt{2 \log (2B/\delta)}} \right\} \leq C_b + \alpha_n + \frac{1}{\sqrt{n}} \sum_{t=1}^{T} K_{tb} \left\{ 2C \sqrt{\sum_{s=1}^{t} v_s + \sqrt{2 \log (2B/\delta)}} \right\},
\]

where the first inequality follows from the inequality in (25), and the second inequality follows from the fact that \( \hat{g}^S \in \mathcal{G}_{\alpha_n} \). Thus, the following holds with probability at least \( 1 - \delta \): for any \( b \in \{1, \ldots, B\} \),

\[
\mathbb{E}_{P} \left[ \sum_{t=1}^{T} K_b \hat{g}^S_t (H_t) - C_b \right] \leq \delta/(2B) \cdot \alpha_n + \frac{1}{\sqrt{n}} \sum_{t=1}^{T} K_{tb} \left\{ 2C \sqrt{\sum_{s=1}^{t} v_s + \sqrt{2 \log (2B/\delta)}} \right\}. \tag{27} \]

Since the upper bounds in (26) and (27) do not depend on \( P \in \mathcal{P}(M, \kappa, \mathcal{G}) \), the probability inequalities (26) and (27) hold uniformly over \( \mathcal{P}(M, \kappa, \mathcal{G}) \). Therefore, the theorem follows from combining the probability inequalities (26) and (27).

\[ \Box \]

## B Computation

In this appendix, we explain computation of the Backward and Simultaneous DEWMs with the linear eligibility score rule introduced in Example 2.2. The non-convexity of the objective functions make these computations challenging. However, the optimization problems of both methods can be formulated as Mixed Integer Linear Programming (MILP) problems, for which some efficient softwares (e.g., CPLEX; Gurobi) are available. In the following subsections, we illustrate the MILP formalization in the case of \( T = 2 \).
We suppose that the class of feasible treatment rules at each stage \( t = 1, 2 \) takes the form of \( \mathcal{G}_t = \{ \{ (1, H'_t) \beta_t \geq 0 \} : \beta_t \in \mathbb{R}^{(k+2)t-1} \} \).

### B.1 Backward DEWM

Using slightly different notation from Section 3.1, the first step of the Backward DEWM method is

\[
\max_{g_2 \in \mathcal{G}_2} \sum_{i=1}^{n} m^B_{i2} g_2,
\]

where

\[
m^B_{i2} = \left( \frac{D_{i2}}{e_2 (1, H_{i2})} - \frac{1 - D_{i2}}{e_2 (0, H_{i2})} \right) \gamma_2 Y_{i2}.
\]

Let \( \hat{g}^B_{i2} \) be a maximizer of the above problem. Then, the second step of the Backward DEWM method is

\[
\max_{g_1 \in \mathcal{G}_1} \sum_{i=1}^{n} m^B_{i1} g_1,
\]

where

\[
m^B_{i1} = \left( \frac{D_{i1}}{e_1 (1, H_{i1})} - \frac{1 - D_{i1}}{e_2 (0, H_{i1})} \right) \times \left( \frac{D_{i2} \hat{g}^B_{i2} (H_{i2})}{e_2 (1, H_{i2})} - \frac{1 - D_{i2}}{e_2 (0, H_{i2})} \left( 1 - \hat{g}^B_{i2} (H_{i2}) \right) \right) (\gamma_1 Y_{i1} + \gamma_2 Y_{i2}).
\]

When the class of DTRs is constrained to the class of linear eligibility rules, each step of the Backward DEWM method described in Section 3.1 can be formulated as MILP problem. The optimization problem in the first step is equivalent to the following MILP problem:

**First step**

\[
\max_{\beta_2 \in \mathbb{R}^{2k+3}} \sum_{i=1}^{n} m^B_{i2} z_{i2},
\]

s.t. \( \frac{(1, H'_{i2}) \beta_2}{C_{i2}} < z_{i2} \leq 1 + \frac{(1, H'_{i2}) \beta_2}{C_{i2}} \) for \( i = 1, \ldots, n \),

where \( C_{i2} \) are constants that should satisfy \( C_{i2} > \sup_{\beta_2 \in \mathbb{R}^{2k+3}} \| (1, H'_{i2}) \beta_2 \| \).
Subsequently, the optimization problem in the second step is equivalent to the following MILP problem:

\[
\begin{align*}
\max_{\beta_1 \in \mathbb{R}^{k+1}} & \quad \sum_{i=1}^{n} m^B_{i1} z_{i1} \\
\text{s.t.} & \quad \frac{(1, H'_1)^{\beta_1}}{C_{i1}} < z_{i1} \leq 1 + \frac{(1, H'_1)^{\beta_1}}{C_{i1}} \text{ for } i = 1, \ldots, n,
\end{align*}
\]

where \( C_{i1} \) are constants that should satisfy \( C_{i1} > \sup_{\beta_1 \in \mathbb{R}^{k+1}} |(1, H'_1)^{\beta_1}|. \)

When we specify the dynamic treatment choice problem as the start (stop) time decision problem discussed in Section 2.2, the linear constraints \( z_{i2} \geq D_{i1} \) and \( D_{i2} \geq z_{i1} \) (\( z_{i2} \leq D_{i1} \) and \( D_{i2} \leq z_{i1} \)) should be added into the MILP problems for the first and second steps, respectively. When we specify the problem as the one-shot treatment decision problem discussed in Section 2.2, the linear constraints \( z_{i2} + D_{i1} \leq 1 \) and \( D_{i2} + z_{i1} \leq 1 \) should be added into the MILP problems for the first and second steps, respectively.

### B.2 Simultaneous DEWM

In the case of \( T = 2 \), the optimization problem of the Simultaneous DEWM method is equivalent to

\[
\max_{(g_1, g_2) \in G} \sum_{i=1}^{n} [m^S_{i1} g_1 + m^S_{i2} g_2 + m^S_{i3} g_1 g_2],
\]

where \( m^S_{is} \) for \( s = 1, 2, 3 \) are defined as

\[
\begin{align*}
m^S_{i1} &= \frac{D_{i1}}{e_1 (1, H_{i1})} \left( \gamma_1 Y_{i1} + \frac{(1 - D_{i2}) \gamma_2 Y_{i2}}{e_2 (0, H_{i2})} \right) - \frac{1 - D_{i1}}{e_1 (0, H_{i1})} \left( \gamma_1 Y_{i1} + \frac{(1 - D_{i2}) \gamma_2 Y_{i2}}{e_2 (0, H_{i2})} \right), \\
m^S_{i2} &= \left( \frac{(1 - D_{i1}) D_{i2}}{e_1 (0, H_{i1}) e_2 (1, H_{i2})} - \frac{(1 - D_{i1}) (1 - D_{i2})}{e_1 (0, H_{i1}) e_2 (0, H_{i2})} \right) \gamma_2 Y_{i2}, \\
m^S_{i3} &= \sum_{(d_1, d_2) \in \{0, 1\}^2} \frac{1 \{ D_{i1} = d_1, D_{i2} = d_2 \} \gamma_2 Y_{i2}}{e_1 (d_1, H_{i1}) e_2 (d_2, H_{i2})}.
\end{align*}
\]
When the class of DTRs is constrained to the class of linear eligibility rules, the above optimization problem is equivalent to the following MILP problem:

\[
\max_{(\beta'_1, \beta'_2) \in \mathbb{R}^{3k+4}} \sum_{i=1}^{n} \left[ m_{i1}^S z_{i1} + m_{i2}^S z_{i2} + m_{i3}^S z_{i3} \right]
\]

\[
(z_{i1}, \ldots, z_{it})_{t=1}^3 \in \{0,1\}^3 n
\]

s.t. \( \frac{(1, H'_{it}) \beta_t}{C_{it}} < z_{it} \leq 1 + \frac{(1, H'_{it}) \beta_t}{C_{it}} \) for \( i = 1, \ldots, n \) and \( t = 1, 2 \),

\[
z_{i3} = z_{i1} z_{i2} \text{ for } i = 1, \ldots, n,
\]

where \( C_{it} \) are constants that should satisfy \( C_{it} > \sup_{\beta_t \in \mathbb{R}^{(k+2)t-1}} |(1, H'_{it}) \beta_t| \).

When we specify the dynamic treatment choice problem as the start (stop) time decision problem discussed in section 2.2, the linear constraints \( z_{i2} \geq z_{i1} \) (\( z_{i2} \leq z_{i1} \)) should be added into the MILP problem. When we specify the problem as the one-shot treatment decision problem discussed in section 2.2, the linear constraint \( z_{i1} + z_{i2} \leq 1 \) should be added into the MILP problem.

### B.3 Budget/Capacity Constraint

The budget/capacity constraints studied in Section 4 can be incorporated into the MILP problem for the Simultaneous DEWM. The optimization problem (9) with the class of linear eligibility score rules is formulated as the following MILP problem:

\[
\max_{(\beta'_1, \beta'_2) \in \mathbb{R}^{3k+4}} \sum_{i=1}^{n} \left[ m_{i1}^S z_{i1} + m_{i2}^S z_{i2} + m_{i3}^S z_{i3} \right]
\]

\[
(z_{i1}, \ldots, z_{it})_{t=1}^3 \in \{0,1\}^3 n
\]

s.t. \( \frac{(1, H'_{it}) \beta_t}{C_{it}} < z_{it} \leq 1 + \frac{(1, H'_{it}) \beta_t}{C_{it}} \) for \( i = 1, \ldots, n \) and \( t = 1, 2 \),

\[
z_{i3} = z_{i1} z_{i2} \text{ for } i = 1, \ldots, n,
\]

\[
\frac{1}{n} \sum_{t=1}^{2} \sum_{i=1}^{n} K_{it} z_{it} \leq C_b + \alpha_n \text{ for } b = 1, \ldots, B \text{ and } t = 1, 2,
\]

where \( C_{it} \) are constants that should satisfy \( C_{it} > \sup_{\beta_t \in \mathbb{R}^{(k+2)t-1}} |(1, H'_{it}) \beta_t| \). The linear constraints in the last line correspond to the budget/capacity constraints.
References

ATHEY, S. AND S. WAGER (2021): “Policy learning with observational data,” Econometrica, 89, 133–161.

BHATTACHARYA, D. AND P. DUPAS (2012): “Inferring welfare maximizing treatment assignment under budget constraints,” Journal of Econometrics, 167, 168–196.

CHAKRABORTY, B. AND E. E. M. MOODIE (2013): Statistical Methods for Dynamic Treatment Regimes, New York: Springer.

CHAKRABORTY, B. AND S. A. MURPHY (2014): “Dynamic treatment regimes,” Annual Review of Statistics and Its Application, 1, 447–464.

CHAMBERLAIN, G. (2012): “Bayesian aspects of treatment choice,” in The Oxford Handbook of Bayesian Econometrics, ed. by J. Geweke, G. Koop, and H. van Dijk, Oxford: Oxford University Press.

CHETTY, R., J. N. FRIEDMAN, N. HILGER, E. SAEZ, D. W. SCHANZENBACH, AND D. YAGAN (2011): “How does your kindergarten classroom affect your earnings? Evidence from Project STAR,” Quarterly journal of economics, 126, 1593–1660.

DEHEJIA, R. H. (2005): “Program evaluation as a decision problem,” Journal of Econometrics, 125, 141–173.

DEVROYE, L., L., GYÖRFI, AND G. LUGOSI (1996): A Probabilistic Theory of Pattern Recognition, vol. 31 of Stochastic Modelling and Applied Probability, New York: Springer.

GERBER, S. B., J. D. FINN, C. M. ACHILLES, AND J. BOYD-ZAHARIAS (2001): “Teacher aides and students’ academic achievement,” Educational Evaluation and Policy Analysis, 23, 123–143.

GINÉ, E. AND R. NICKL (2016): Mathematical Foundations of Infinite-Dimensional Statistical Models, New York: Cambridge University Press.

HAN, S. (2020a): “Identification in nonparametric models for dynamic treatment effects,” Journal of Econometrics, forthcoming.

——— (2020b): “Optimal dynamic treatment regimes and partial welfare ordering,” arXiv preprint arXiv:1912.10014.
Heckman, J. J., J. E. Humphries, and G. Veramendi (2016): “Dynamic treatment effects,” *Journal of Econometrics*, 191, 276–292.

Heckman, J. J. and S. Navarro (2007): “Dynamic discrete choice and dynamic treatment effects,” *Journal of Econometrics*, 136, 341–396.

Hirano, K. and J. Porter (2009): “Asymptotics for statistical treatment rules,” *Econometrica*, 77, 1683–1701.

Kitagawa, T., S. Sakaguchi, and A. Tetenenov (2021): “Constrained classification and policy learning,” *arXiv preprint arXiv:2106.12886*.

Kitagawa, T. and A. Tetenenov (2018): “Who should be treated? Empirical welfare maximization methods for treatment choice,” *Econometrica*, 86, 591–616.

Kock, A. B. and M. Thyrgsgaard (2018): “Optimal sequential treatment allocation,” *arXiv preprint arXiv:1705.09952*.

Kolsrud, J., C. Landais, P. Nilsson, and J. Spinnewyn (2018): “The optimal timing of unemployment benefits: Theory and evidence from Sweden,” *American Economic Review*, 108, 985–1033.

Krueger, A. B. (1999): “Experimental estimates of education production functions,” *Quarterly Journal of Economics*, 114, 497–532.

Laber, E. B., D. J. Lizotte, M. Qian, W. E. Pelham, and S. A. Murphy (2014): “Dynamic treatment regimes: Technical challenges and applications,” *Electronic journal of statistics*, 8, 1225–1272.

Lechner, M. (2009): “Sequential causal models for the evaluation of labor market programs,” *Journal of Business & Economic Statistics*, 27, 71–83.

Lugosi, G. (2002): “Pattern classification and learning theory,” in *Principles of Nonparametric Learning*, ed. by L. Gyrfi, Vienna: Springer, 1–56.

Manski, C. F. (2004): “Statistical treatment rules for heterogeneous populations,” *Econometrica*, 72, 1221–1246.

Massart, P., É. Nédélec, et al. (2006): “Risk bounds for statistical learning,” *Annals of Statistics*, 34, 2326–2366.

Mbakop, E. and M. Tabord-Meehan (2021): “Model selection for treatment choice: Penalized welfare maximization,” *Econometrica*, 89, 825–848.
Meyer, B. D. (1995): “Lessons from the U.S. unemployment insurance experiments,” *Journal of Economic Literature*, 33, 91–131.

Mohri, M., A. Rostamizadeh, and A. Talwalkar (2012): *Foundations of Machine Learning*, Cambridge, MA: MIT Press.

Moodie, E., B. Chakraborty, and M. S. Kramer (2012): “Q-learning for estimating optimal dynamic treatment rules from observational data,” *Canadian Journal of Statistics*, 40, 629–645.

Murphy, S. A. (2003): “Optimal dynamic treatment regimes,” *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 65, 331–355.

——— (2005): “A generalization error for Q-learning.” *Journal of Machine Learning Research*, 6, 1073–1097.

Nie, X., E. Brunskill, and S. Wager (2020): “Learning when-to-treat policies,” *Journal of the American Statistical Association*, 116, 392–409.

Robins, J. M. (1986): “A new approach to causal inference in mortality studies with a sustained exposure period—application to control of the healthy worker survivor effect,” *Mathematical Modelling*, 7, 1393–1512.

——— (1989): “The analysis of randomized and nonrandomized AIDS treatment trials using a new approach to causal inference in longitudinal studies,” in *Health Service Research Methodology: A Focus on AIDS*, ed. by L. Sechrest, H. Freeman, and A. Mulley, Washington D.C.: U.S. Public Health Service, National Center for Health Services Research, 113–159.

——— (1997): “Causal inference from complex longitudinal data in latent variable modeling and applications to causality,” in *Lecture Notes in Statistics*, ed. by M. Berkane, New York: Springer, 69–117.

——— (2004): “Optimal structural nested models for optimal sequential decisions,” in *Proceedings of the Second Seattle Symposium in Biostatistics. Lecture Notes in Statistics*, ed. by D. Y. Lin and P. J. Heagerty, New York: Springer, 189–326.

Robins, J. M., D. Blevins, G. Ritter, and M. Wulfsohn (1992): “G-estimation of the effect of prophylaxis therapy for Pneumocystis carinii pneumonia on the survival of AIDS patients,” *Epidemiology*, 3, 319–336.
RODRÍGUEZ, J., F. SALTIEL, AND S. S. URZÚA (2018): “Dynamic treatment Effects of job training,” National Bureau of Economic Research Working Paper Series, No. 25408.

SCHANZENBACH, D. W. (2006): “What have researchers learned from Project STAR?” Brookings Papers on Education Policy, 205–228.

STOYE, J. (2009): “Minimax regret treatment choice with finite samples,” Journal of Econometrics, 151, 70–81.

——— (2012): “Minimax regret treatment choice with covariates or with limited validity of experiments,” Journal of Econometrics, 166, 138–156.

TETENOV, A. (2012): “Statistical treatment choice based on asymmetric minimax regret criteria,” Journal of Econometrics, 166, 157–165.

TSIATIS, A. A., M. DAVIDIAN, S. T. HOLLOWAY, AND E. B. LABER (2019): Dynamic Treatment Regimes: Statistical Methods for Precision Medicine, CRC press.

VAN DER VAART, A. AND J. A. WELLNER (1996): Weak Convergence and Empirical Processes, New York: Springer.

VANSTEELANDT, S. AND E. GOETGHEBEUR (2003): “Causal inference with generalized structural mean models,” Journal of the Royal Statistical Society: Series B (Statistical Methodology), 65, 817–835.

ZHANG, Y., E. B. LABER, M. DAVIDIAN, AND A. A. TSIATIS (2018): “Interpretable dynamic treatment regimes,” Journal of the American Statistical Association, 113, 1541–1549.

ZHAO, Y. Q., D. ZENG, E. B. LABER, AND M. R. KOSOROK (2015): “New statistical learning methods for estimating optimal dynamic treatment regimes,” Journal of the American Statistical Association, 110, 583–598.

ZHOU, Z., S. Athey, AND S. WAGER (2018): “Offline multi-action policy learning: Generalization and optimization,” arXiv preprint arXiv:1810.04778.