ON THE TRANSCENDENCE OF SPECIAL VALUES OF GOSS
L-FUNCTIONS ATTACHED TO DRINFELD MODULES

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Abstract. Let \( \mathbb{F}_q \) be the finite field with \( q \) elements and consider the rational function field \( K := \mathbb{F}_q(\theta) \). For a Drinfeld module \( \phi \) defined over \( K \), we study the transcendence of special values of the Goss \( L \)-function attached to the abelian \( t \)-motive \( M_\phi \) of \( \phi \). Moreover, when \( \phi \) is a Drinfeld module of rank \( r \geq 2 \) defined over \( K \) which has everywhere good reduction, we prove that the value of the Goss \( L \)-function attached to the \((r - 1)\)-st exterior power of \( M_\phi \) at any positive integer is transcendental over \( K \).

1. Introduction

1.1. Motivation and Background. After Grothendieck discovered the notion of motives in the 1960s, the \( L \)-functions attached to them have gotten enormous attention over the following years. These are the objects which can be seen as generalizations of several well-known functions such as the Riemann zeta function \( \zeta(s) \) defined for \( s \) with \( \Re(s) > 1 \). In particular, setting \( \mathbb{Q}(-1) \) to be the Lefschetz motive, the motivic \( L \)-function \( L(\mathbb{Q}(-1), \cdot) \) of \( \mathbb{Q}(-1) \) has the property that

\[
L(\mathbb{Q}(-1), s) = \zeta(s - 1) = \sum_{m=1}^{\infty} \frac{1}{m^{s-1}}, \quad s = 3, 4, \ldots.
\]

One interesting phenomenon concerning these \( L \)-functions is to understand the transcendence of their special values. When \( s \) is a positive integer, it is known that \( \zeta(2s) \) is transcendental over \( \mathbb{Q} \) and the same result is expected to hold for \( \zeta(s) \) where \( s \in \mathbb{Z}_{\geq 2} \). Regarding more general motives, Deligne [Del79] conjectured that for a particular integer \( s \) and a pure motive \( \mathcal{M} \), \( L(\mathcal{M}, s) \) is equal to the product of a rational number and the determinant of a matrix consisting of periods arising from the comparison isomorphism between Betti and de Rham cohomologies of \( \mathcal{M} \). Later on, generalizing the ideas of Deligne, Beilinson [Bel85] conjectured that for a pure motive \( \mathcal{M} \) with weight less than -2, \( L(\mathcal{M}, 0) \) can be written, modulo \( \mathbb{Q}^\times \), as the determinant of the Beilinson regulator (see also [BK07] for a refinement of Beilinson’s conjecture). We refer the reader to [Nek94] and [RSS88] for more details and the recent progress on the aforementioned conjectures.

In this paper, we focus on the motivic \( L \)-functions which are defined over a field of prime characteristic by following a close analogy with the classical setting and study the transcendence of their special values.

In what follows, we introduce a few notation necessary to define our main objects. Let \( p \) be a prime and \( \mathbb{F}_q \) be the finite field with \( q = p^m \) elements where \( m \in \mathbb{Z}_{\geq 1} \). Considering a variable \( \theta \) over \( \mathbb{F}_q \), we let \( A := \mathbb{F}_q[\theta] \), the set of polynomials in \( \theta \) with coefficients in \( \mathbb{F}_q \) and
denote it by $K$. Let $|·|$ be the norm corresponding to the infinite place normalized so that $|θ| = q$. Consider the formal Laurent series ring $\mathbb{F}_q((1/θ))$ which is the completion of $K$ with respect to $|·|$ and denote it by $K_∞$. We let $\mathbb{C}_∞$ be the completion of a fixed algebraic closure of $K_∞$ and set $\overline{K}$ to be the algebraic closure of $K$ in $\mathbb{C}_∞$.

Let $R$ be a subring of $\mathbb{C}_∞$ containing $A$. We define the non-commutative polynomial ring $R[τ]$ subject to the condition

$$τc = c^qτ, \ c ∈ R.$$  

Let $t$ be a variable over $\mathbb{C}_∞$ and set $A := \mathbb{F}_q[t]$. A Drinfeld $A$-module $φ$ of rank $r ∈ \mathbb{Z}_{≥1}$ defined over $R$ is an $\mathbb{F}_q$-linear ring homomorphism $φ : A → R[τ]$ which is uniquely given by

$$φt := θ + c1τ + ··· + crτ^r, \ c_r ≠ 0.$$  

We call two Drinfeld $A$-modules $φ$ and $φ'$ isomorphic over $R$ if there exists an element $u ∈ R^×$ so that $φtu = uφ'$. We further say that a Drinfeld $A$-module $φ$ of rank $r$ defined over $K$ has everywhere good reduction if $φ$ is isomorphic, over $K$, to a Drinfeld $A$-module $ψ$ given by

$$ψt = θ + a1τ + ··· + arτ^r,$$

so that $a_1, \ldots, a_{r−1} ∈ A$ and $a_r ∈ \mathbb{F}_q^×$ (see [Bjo98, Sec. 1]). As an example, the Carlitz module $C$ given by

$$Ct = θ + τ$$

is a Drinfeld $A$-module of rank 1 defined over $A$ and has everywhere good reduction. In addition to its tremendous importance for the present work, one can also see [Car35], [Car38], and [Hay74] for its role to study class field theory for global function fields.

### 1.2. Special values of Goss $L$-functions.

In [And86], Anderson introduced a function field analogue of motives in the classical setting which form a category closed under taking tensor products and direct sums (see [Tha04, Sec. 7.1]). We emphasize that Anderson called such objects “$t$-motives” but we will instead follow Goss’s terminology [Gos96, Def. 5.4.12] and call them “abelian $t$-motives” in the rest of the paper. For any given abelian $t$-motive, Anderson further showed that one can associate, unique up to isomorphism, an abelian $t$-module which can be seen as a higher dimensional generalization of Drinfeld $A$-modules. In particular, to make the reader more familiar with the objects in use, we emphasize that any Drinfeld $A$-module is a one dimensional abelian $t$-module (see §2.1 for more details).

For the abelian $t$-motive $M_φ$ corresponding to $φ$ and $s ∈ \mathbb{Z}_{≥1}$, Goss [Gos92], closely following ideas of Gekeler [Gek91] Rem. 5.10], defined the motivic $L$-function $L(M_φ, s)$ as well as the $L$-function $L(∧^k_{K[θ]} M_φ, s)$ associated to the $k$-th exterior power of $M_φ$ (see §3.1 for more details and the explicit construction of these $L$-functions). Let us analyze an example of such $L$-functions in what follows. When $φ = C$, the Carlitz module, one can have

$$L(M_C, n) = \sum_{a ∈ A_+} \frac{1}{a^{n−1}} ∈ K∞, \ n ∈ \mathbb{Z}_{≥2}.$$  

These values are also known as Carlitz zeta values at $n − 1$ whose construction dates back to Carlitz. One can also note, by comparing [1] with (3), the immediate analogy of $L(M_C, n)$ with the Riemann zeta values at positive integers as well as the analogy between $M_C$ and $\mathbb{Q}(−1)$. Due to the rich transcendence theory established in the function field setting (see for example [CPT12, Pap08, Yu91]), many remarkable results on the special values of $L(M_C, n)$ have been obtained. Just to name a few, when $n − 1$ is divisible by $q − 1$, Wade [Wad11] was
able to prove that $L(M_C, n)$ is transcendental over $\overline{K}$. In 1990, Anderson and Thakur were able to write $L(M_C, n)$ as an $A$-linear combination of polylogarithms $[AT90]$ Thm. 3.8.3. Later on, building on the work of Anderson and Thakur and proving a version of Hermite-Lindemann theorem for tensor powers of the Carlitz module, Yu [Yu91 Thm. 3.1] showed that $L(M_C, n)$ is transcendental over $\overline{K}$ for any $n \in \mathbb{Z}_{\geq 2}$.

When $C$ is replaced by any Drinfeld $A$-module $\psi$ of arbitrary rank defined over $K$, explicit formulas for $L(M_\psi, n)$ are not known yet except for some special cases studied in [Gez20]. However, using the methods developed in the present paper, we are able to conclude the transcendence of special values stated as follows in our main result.

**Theorem 1.1.** Let $n$ be a positive integer and $\phi$ be a Drinfeld $A$-module of rank $r \geq 2$ defined over $K$. Moreover, for any $b \in K \setminus \{0\}$, let $C^{(b)}$ be the Drinfeld $A$-module of rank one given by $C^{(b)} := \theta + b\tau$.

(i) The special values $L(M_\phi, n)$ and $L(M_{C^{(b)}}, m)$ for $m \in \mathbb{Z}_{\geq 2}$ are transcendental over $\overline{K}$.

(ii) Assume that $\phi$ has everywhere good reduction. Then, $L(\Lambda_{K[\theta]}^{-1} M_\phi, n)$ is transcendental over $\overline{K}$.

**Remark 1.2.**

(i) Our strategy for the proof of Theorem 1.1 relies on showing the transcendence of particular families of Taelman $L$-values (see §3.4 for their definition). Hence we provide an affirmative answer to [ANDTR20b, Problem 4.1] for some certain cases (Corollary 4.3).

(ii) In [CY07], using the transcendence theory of Papanikolas, Chang and Yu determined all the $\overline{K}$-algebraic relations among Carlitz zeta values $L(M_C, n)$, which are explicitly the Frobenius $p$-th power relations and Euler-Carlitz relations. Let $\phi$ be a Drinfeld $A$-module of rank $r \geq 2$ defined over $K$. It would be natural to ask a similar question for the values $L(M_\phi, n)$. However, due to the intricate nature of the multiplicative function appearing in the infinite sum expansion of $L(M_\phi, n)$ (see [CEGP18, Sec. 3] for more details), it is complicated to obtain non-trivial algebraic relations among the values $L(M_\phi, n)$ which would shed light to generalize the results known for the Carlitz zeta values. We hope to come back to this question in the near future.

1.3. **Outline of the paper.** In what follows, we describe the outline of the paper and briefly explain the method of our proof for the main results:

- In §2, we discuss $t$-modules and provide explicit examples to them. In §2.1, we introduce $C^{(b)}_n$ and $\phi \otimes C^{\otimes n}$ explicitly where the latter will be also denoted by $G_n$ throughout the paper to ease the notation. After introducing Taelman $t$-motives, a generalization of abelian $t$-motives, we finish this section by defining $(\Lambda^{r-1} \phi) \otimes C^{\otimes n}$ for any $n \in \mathbb{Z}_{\geq 0}$.

- In §3, we analyze Goss $L$-functions attached to abelian $t$-motives and some explicitly determined Taelman $t$-motives as well as their relation with Taelman $L$-values. Studying Taelman $t$-motives rather than abelian $t$-motives enables us to define the notion of duality and relate them to special values of Goss $L$-functions (see §3.1). In §3.2, we define the abelian $t$-modules $\mathcal{E}_n^{(r)}$ and $\widetilde{G}_n$ corresponding to certain abelian $t$-motives we introduce. Analyzing local factors of Goss $L$-functions as well as using Gardeyn’s work in [Gar02 Sec. 8.4], we obtain (33), (34), and (35) which enable us
to interpret Theorem 3.6 in terms of Taelman $L$-values in Theorem 3.5 and Theorem 3.6.

Finally in §4, we briefly introduce the work of Anglès, Ngo Dac and Tavares Ribeiro [ANDTR20a] on proving Taelman’s conjecture [Tae09b, Conj. 1] which essentially allows us to realize our Taelman $L$-values as a determinant of a matrix consisting of certain coordinates of logarithms of $t$-modules (Theorem 1.1). Then, we prove Theorem 3.5 (Theorem 3.6 resp.) by using the previous results of authors established in [GN24, Thm. 1.1 and 1.2], Yu’s theorem [Yu91, Thm. 2.3] and Fang’s result [Fan15, Thm. 1.10] generalizing Taelman’s class number formula [Tae12, Thm. 1].

Acknowledgments. To reduce technicality and improve the presentation of the results in [GN21], it has been divided into two parts so that the first part is [GN24] with additional results and the second part is the present paper.

The authors would like to thank Chieh-Yu Chang and Yen-Tsung Chen for fruitful discussions and valuable suggestions. We are also grateful to Shih-Chieh Liaw for his presentation results and the second part is the present paper.

2. $t$-modules and Taelman $t$-motives

In this section, we first review the notion of $t$-modules introduced by Anderson [And86], which are higher dimensional generalizations of Drinfeld. We also review some objects corresponding to them such as effective $t$-motives and Taelman $t$-motives. Then, we briefly explain Taelman $t$-motives and their fundamental properties.

2.1. $t$-modules. Let us assume that $L$ is a field extension of $F_q$. We say that $L$ is an $A$-field if there exists a ring homomorphism $\text{ch} : A \rightarrow L$. Throughout the paper, we consider any extension $L$ of $K$ in $\mathbb{C}_\infty$ as an $A$-field where $\text{ch}$ sends $t \mapsto \theta$. In this case, we say that $L$ has generic characteristic.

Let $m, l \in \mathbb{Z}_{\geq 1}$. For any matrix $B = (B_{ik}) \in \text{Mat}_{m \times l}(L)$ and $j \in \mathbb{Z}$, set $B^{(j)} := (B_{ik}^{(j)}) \in \text{Mat}_{m \times l}(L)$. We also let $\text{Mat}_{m \times l}(L)[[\tau]]$ be the set of polynomials of $\tau$ with coefficients in $\text{Mat}_{m \times l}(L)$. Moreover, when $m = n$, we define the ring $\text{Mat}_{m}(L)[[\tau]]$ of power series of $\tau$ with coefficients in $\text{Mat}_{m}(L) := \text{Mat}_{m \times m}(L)$ with respect to the condition

$$\tau B = B^{(1)} \tau, \quad B \in \text{Mat}_{m}(L).$$

We also let $\text{Mat}_{m}(L)[[\tau]] \subset \text{Mat}_{m}(L)[[\tau]]$ to be the subring of polynomials of $\tau$. Furthermore, for any $B = B_0 + B_1 \tau + \cdots + B_k \tau^k \in \text{Mat}_{m \times l}(L)[[\tau]]$, we set $dB := B_0 \in \text{Mat}_{m \times l}(L)$.

**Definition 2.1.**

(i) Let $L$ be an $A$-field. A $t$-module of dimension $s \in \mathbb{Z}_{\geq 1}$ is a tuple $G = (G_{s/a/L, \varphi})$ of the $s$-dimensional additive algebraic group $\mathbb{G}_a^s$ over $L$ and an $F_q$-linear ring homomorphism $\varphi : A \rightarrow \text{Mat}_{s}(L)[[\tau]]$ given by

$$\varphi(t) := A_0 + A_1 \tau + \cdots + A_m \tau^m$$

for some $m \in \mathbb{Z}_{\geq 0}$ so that $d\varphi(t) := d\varphi(t) = A_0 = \text{ch}(t) \text{Id}_s + N$ where $\text{Id}_s$ is the $s \times s$ identity matrix and $N$ is a nilpotent matrix.
(ii) For each $0 \leq i \leq m$, if all the entries of $A_i$ lie in an $A$-algebra $R$, then we say $G$ is defined over $R$.

(iii) Let $G_1 = (G_{a/L}^s, \varphi_1)$ and $G_2 = (G_{a/L}^s, \varphi_2)$ be two $t$-modules. A morphism $\Pi : G_1 \to G_2$ defined over $L$ is an element $\Pi \in \text{Mat}_{s \times s_2}(L[\tau])$ satisfying

$$\Pi \varphi_1(t) = \varphi_2(t) \Pi.$$

We set $\text{Hom}(G_1, G_2)$ to be the set of morphisms $G_1 \to G_2$ defined over $L$. Moreover, when $s_1 = s_2$, we call $G_1$ and $G_2$ isomorphic over $L$ if there exists an element $u \in \text{GL}_{s_1}(L)$ so that $u \varphi_1(t) = \varphi_2(t) u$. Let $\text{End}(G) := \text{Hom}_K(G, G)$ be the ring of endomorphisms of $G = (G_{a/L}^s, \varphi)$ over $K$. Observe that $\text{End}(G)$ has an $A$-module structure given by

$$a \cdot \Pi := \varphi(a) \Pi, \quad a \in A, \quad \Pi \in \text{End}(G).$$

**Example 2.2.** Recall the Drinfeld $A$-module $C(b)$ given in Theorem 1.1 for any non-zero $b \in K$. For any $n \in \mathbb{Z}_{\geq 1}$, we define $C_n(b) := (G_{a/K}^n, \rho)$ where the $\mathbb{F}_q$-linear homomorphism $\rho : A \to \text{Mat}_{n}(K)[[\tau]]$ is defined by

$$\rho(t) = \begin{bmatrix} \theta & 1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \theta & 1 & \cdots & \cdots & \cdots \end{bmatrix} + \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \tau.$$

Note that when $n = 1$, we have $C_1(b) = C(b)$ and moreover, when $b = 1$, we call $C_{\otimes n} := C_n^{(1)}$, the $n$-th tensor power of the Carlitz module (see [AT90] for more details on $C_{\otimes n}$).

For any $A$-algebra $R \subset L$ and a $t$-module $G = (G_{a/L}^s, \varphi)$ defined over $R$, we set $\text{Lie}(G)(R) := \text{Mat}_{s \times 1}(R)$ and equip it with the $A$-module structure given by

$$t \cdot x := d_\varphi(t)x = A_0x, \quad x \in \text{Lie}(G)(R).$$

Furthermore, we also define $G(R) := \text{Mat}_{s \times 1}(R)$ whose $A$-module structure is given by

$$t \cdot x := \varphi(t)x = A_0x + A_1x^{(1)} + \cdots + A_m x^{(m)}, \quad x \in G(R).$$

Let $L$ be an $A$-field which has generic characteristic. For any $t$-module $G$ defined over $L$, there exists a unique infinite series $\text{Exp}_G := \sum_{i \geq 0} \alpha_i \tau^i \in \text{Mat}_s(L)[[\tau]]$ satisfying $\alpha_0 = 1$ and

$$\text{Exp}_G d_\varphi(t) = \varphi(t) \text{Exp}_G.$$  \hspace{1cm} (5)

It leads to the exponential function of $G$, which is an everywhere convergent $\mathbb{F}_q$-linear homomorphism $\text{Exp}_G : \text{Lie}(G)(C_{\infty}) \to G(C_{\infty})$, given by $\text{Exp}_G(x) = \sum_{i \geq 0} \alpha_i x^{(i)}$ for any $x \in \text{Lie}(G)(C_{\infty})$.

We now set $\phi$ to be the Drinfeld $A$-module of rank $r \geq 2$ defined over $K$ defined by

$$\phi(t) = \theta + a_1 \tau + \cdots + a_r \tau^r \in K[\tau].$$  \hspace{1cm} (6)

We warn the reader that $\phi$ will also be denoted by $G_0 := (G_{a/K}, \phi)$ whenever $r \geq 2$ throughout the paper to be compatible with the notation for family of (abelian) $t$-modules studied in [GN24] §4.
Assume that \( r \geq 2 \). For any positive integer \( n \), we define the \( t \)-module \( G_n := (\mathcal{C}_{a_i/K}^n, \phi_n) \) constructed from \( \phi \) and \( C_{n^2} \) where \( \phi_n : A \to \text{Mat}_{rn+1}(K)[r] \) is given by
\[
\phi_n(t) = \theta \cdot \text{Id}_{rn+1} + N + E_t
\]
so that \( N \in \text{Mat}_{rn+1}(F_q) \) and \( E \in \text{Mat}_{rn+1}(K) \) are defined as
\[
N := \begin{pmatrix}
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\end{pmatrix}
\]
and
\[
E := \begin{pmatrix}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & \cdots & \cdots & 0 \\
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \cdots & \cdots & \cdots & \cdots & \vdots \\
a_1 & \cdots & a_r & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
\]

Note that the last \( r \)-rows of \( N \) contain only zeros.

**Remark 2.3.** We refer the reader to [Ham93] for further details on the tensor products of Drinfeld \( A \)-modules of arbitrary rank.

2.2. **Effective \( t \)-motives and abelian \( t \)-modules.** In this subsection, we introduce effective \( t \)-motives defined by Taelman [Tae09a]. We refer the reader to [Tae09a] and [Tae09b] for further details on the subject.

Let \( L \) be an extension of \( F_q \) which is also an \( A \)-field. We define \( L[t] \) to be the set of polynomials in the variable \( t \) with coefficients in \( L \) and \( L(t) \) to be its fraction field. For any \( f = \sum_{i \geq 0} a_i t^i \in L[t] \) and \( j \in \mathbb{Z} \), we set \( f^{(j)} := \sum_{i \geq 0} a_i^j t^i \in L[t] \). Furthermore, we set \( L[t, \tau] := L[t][\tau] \) to be the non-commutative ring defined subject to the condition
\[
\tau f = f^{(1)} \tau, \quad f \in L[t].
\]

Unless otherwise stated, the tensor products in this subsection are over \( L[t] \).

**Definition 2.4.**

(i) An effective \( t \)-motive \( M \) over \( L \) is a left \( L[t, \tau] \)-module which is free and finitely generated over \( L[t] \) with the property that the determinant of the matrix representing the \( \tau \)-action on \( M \) is equal to \( c(t - \text{ch}(t))^n \) for some \( c \in L^\times \) and \( n \in \mathbb{Z}_{\geq 0} \).

(ii) The set \( \mathfrak{M}_L \) of effective \( t \)-motives over \( L \) forms a category where the morphisms between any two effective \( t \)-motives are left \( L[t, \tau] \)-module homomorphisms. The set of morphisms between \( M_1 \in \mathfrak{M}_L \) and \( M_2 \in \mathfrak{M}_L \) is denoted by \( \text{Hom}_{\mathfrak{M}_L}(M_1, M_2) \).

(iii) The tensor product of effective \( t \)-motives \( M_1 \) and \( M_2 \) is an effective \( t \)-motive denoted by \( M_1 \otimes M_2 \) on which \( \tau \) acts diagonally.
Now considering the abelian t-motive $\Phi \in \text{Mat}_s(L)[\tau]$ such that
\[
t \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_s \end{bmatrix} = \Phi \theta \begin{bmatrix} v_1 \\ \vdots \\ v_s \end{bmatrix}.
\]

Now considering the $\mathbb{F}_q$-linear homomorphism $\psi : A \to \text{Mat}_s(L)[\tau]$ given by $\psi(t) = \Phi \theta$, one sees that $(\mathbb{G}^s_{a/L}, \psi)$ forms a t-module. A t-module constructed in this way is called an abelian t-module.

We remark that there is an anti-equivalence between the category of abelian t-motives over $L$ and the category of abelian t-modules defined over $L$ (see [Tae09b, Thm. 1]). Although we will not go into the details, in what follows, we introduce several examples of this correspondence.

**Example 2.5.**

(i) Let $\phi$ be a Drinfeld $A$-module of rank $r$ defined as in (6). Let $M_\phi := \bigoplus_{i=1}^r K[t]m_i$ be the free $K[t]$-module with some chosen $K[t]$-basis $\{m_1, \ldots, m_r\}$ and equip it with the left $K[\tau]$-module structure given by
\[
cr \tau \cdot \sum_{i=1}^r f_i m_i = c(f_{r_1}^{(1)}(t - \theta)a_{r-1}^{-1}m_1 + (f_1^{(1)} - f_{r_1}^{(1)}a_{r-1}^{-1}m_2 + \cdots + (f_{r-1}^{(1)} - f_{r_1}^{(1)}a_{r-1}^{-1}m_r)
\]
where $f_1, \ldots, f_r \in K[t]$ and $c \in K$. Then, $M_\phi$ is free of rank one over $K[\tau]$ with $K[\tau]$-basis $\{m_1\}$ and hence is the abelian t-motive over $K$ corresponding to $C^{(a_1)}$ if $r = 1$ and $G_0$ otherwise.

(ii) For any $n \in \mathbb{Z}_{\geq 1}$ and $b \in K \setminus \{0\}$, let $C^{(b)}_n := K[t]m$ for some $K[t]$-basis $\{m\}$ so that $\tau \cdot fm = f^{(1)}(b(t - \theta)^nm$ where $f \in K[t]$. It forms a left $K[t, \tau]$-module and an easy computation yields that it is the abelian t-motive over $K$ corresponding to $C^{(b)}_n$. When $n = 1$, we simply set $C^{(b)} := C^{(b)}_1$ and for $b = 1$ and $n \in \mathbb{Z}_{\geq 1}$, we let $C^{(n)} := C^{(1)}$ and $C := C^{(1)}$.

(iii) For any effective t-motive $M$ over $K$ which has rank $r$ over $K[t]$ and for any $1 \leq i \leq r$, we denote by $\wedge^i_{K[t]}M$ the $i$-th exterior power of $M$. The action of $\tau$ on $\wedge^i_{K[t]}M$ is induced from the action of $\tau$ on the $i$-th tensor power of $M$ and hence it can be realized as a left $K[t, \tau]$-module. Moreover, by [Boc05] Prop. 1, we know that $\wedge^i_{K[t]}M$ is an effective t-motive. Furthermore, we set $\det(M)$ to be the $r$-th exterior power of $M$. By construction, $\det(M)$ is a $K[t]$-module of rank one. If we consider $M = M_\phi$ as above, then we see that $\det(M_\phi)$ is the effective t-motive over $K$ given by $\det(M_\phi) = K[t](m_1 \wedge \cdots \wedge m_r)$ so that
\[
\tau(f(m_1 \wedge \cdots \wedge m_r)) = f^{(1)}(-1)^{r-1}a_{r-1}^{-1}(t - \theta)(m_1 \wedge \cdots \wedge m_r), \quad f \in K[t].
\]
This also implies that $\det(M_\phi)$ is a free left $K[\tau]$-module with the basis $\{m_1 \wedge \cdots \wedge m_r\}$ and hence is an abelian t-motive over $K$.
For \( n \in \mathbb{Z}_{\geq 1} \) and a Drinfeld \( A \)-module \( \phi \) of rank \( r \geq 2 \) given as in (6), we now consider the left \( K[t, \tau] \)-module \( M_n := M_\phi \otimes \mathcal{C}^s \) on which \( \tau \) acts diagonally. Example [2,5] implies that \( M_n \) has a \( K[t] \)-basis given by \( \{ m_1 \otimes m, \ldots, m_r \otimes m \} \). For \( 1 \leq i \leq r \) and \( 0 \leq j \leq n - 1 \), we set \( v_{i,j} := m_i \otimes (t - \theta)^j m \) and furthermore, we define \( v_{1,n} := m_1 \otimes (t - \theta)^n m \). Using the left \( K[\tau] \)-module structure on \( M_\phi \) and \( \mathcal{C}^s \), one can see that the set \( \{ v_{i,j} \mid 1 \leq i \leq r, \ 0 \leq j \leq n - 1 \} \cup \{ v_{1,n} \} \) forms a left \( K[\tau] \)-module basis for \( M_n \) and moreover, one can obtain

\[
(t - \theta)v_{i,j} = v_{i,j+1} \quad \text{for} \quad 0 \leq j \leq n - 2, \quad \text{and} \quad 1 \leq i \leq r, \\
(t - \theta)v_{1,n-1} = v_{1,n}, \\
(t - \theta)v_{k,n-1} = \tau v_{k-1,0} \quad \text{for} \quad 2 \leq k \leq r, \\
(t - \theta)v_{1,n} = a_1 \tau v_{1,0} + \cdots + a_r \tau v_{r,0}.
\]

Hence, we have that

\[
t^j[v_{1,0}, \ldots, v_{r,0}, \ldots, v_{1,n-1}, \ldots, v_{r,n-1}, v_{1,n}]^{tr} = \phi_n(t) [v_{1,0}, \ldots, v_{r,0}, \ldots, v_{1,n-1}, \ldots, v_{r,n-1}, v_{1,n}]^{tr}
\]

where \( \phi_n : A \to \text{Mat}_{r \times r}(K)[\tau] \) is the \( \mathbb{F}_p \)-linear homomorphism defined as in \( \S 2.1 \). Thus, the abelian \( t \)-module corresponding to \( M_n \) is given by \( G_n = (G^r_{a/K}, \phi_n) \).

We finish this subsection with the following definition due to Anderson [And86, Sec. 1.9] which will be used in \( \S 3 \).

**Definition 2.6.**  
(i) Assume that \( L \) is an algebraically closed field and let \( L((1/t)) \) be the ring of formal Laurent series. Let \( M \) be a left \( L[t, \tau] \)-module and consider the \( L[t, \tau] \)-module \( M((1/t)) := M \otimes L((1/t)) \) so that

\[
\tau \cdot \left( m \otimes \sum_{i=0}^{\infty} b_i t^i \right) := \tau m \otimes \sum_{i=0}^{\infty} b_i^t t^i.
\]

(ii) Let \( L[[1/t]] \) be the ring of power series of \( t \) with coefficients in \( L \). An \( L[[1/t]] \)-submodule \( J \) of \( M((1/t)) \) is called a lattice if \( J \otimes_{L[[1/t]]} L((1/t)) \cong M((1/t)) \).

(iii) We say that \( M \) is pure if it is free and finitely generated over \( L[t] \) and there exists a lattice \( J \subseteq M((1/t)) \) and \( s_1, s_2 \in \mathbb{Z}_{\geq 2} \) such that \( \tau^{s_1} J = t^{s_2} J \).

(iv) An abelian \( t \)-module \( G \) is pure if its corresponding abelian \( t \)-motive over \( L \) is pure.

2.3. Taelman \( t \)-motives. For \( M_1, M_2 \in \mathfrak{M}_L \), we define \( \text{Hom}_{L[t]}(M_1, M_2) \) to be the set of \( L[t] \)-module homomorphisms between \( M_1 \) and \( M_2 \). Let \( \overline{L} \) be a fixed algebraic closure of \( L \). One can define a left \( \overline{L}[\tau] \)-module action on the \( \overline{L}(t) \)-module \( \text{Hom}_{L[t]}(M_1 \otimes \overline{L}(t), M_2 \otimes \overline{L}(t)) \) by setting

\[
\tau f = \tau_2 \circ f \circ \tau_1^{-1}
\]

for all \( f \in \text{Hom}_{L[t]}(M_1 \otimes \overline{L}(t), M_2 \otimes \overline{L}(t)) \) where \( \tau_1^{-1} \) (\( \tau_2 \) respectively) is the semi-linear map \( \tau^{-1} \) (\( \tau \) respectively) on \( M_1 \otimes \overline{L}(t) \) (\( M_2 \otimes \overline{L}(t) \) respectively). Furthermore, one can obtain that (see [Tae99a, Sec. 2.2.3] for details) for an arbitrarily large \( n \), \( \text{Hom}_L(M_1, M_2 \otimes \mathcal{C}^s) \subseteq \text{Hom}_{L[t]}(M_1 \otimes \overline{L}(t), M_2 \otimes \mathcal{C}^s \otimes \overline{L}(t)) \) has a left \( L[\tau] \)-module structure and hence it can be regarded as an effective \( t \)-motive over \( L \).

**Definition 2.7.**  
(i) A Taelman \( t \)-motive \( \mathfrak{M} \) over \( L \) is a tuple \( \mathfrak{M} = (M, n) \) consisting of an effective \( t \)-motive \( M \) over \( L \) and \( n \in \mathbb{Z} \).
(ii) Let $\mathcal{M}_L$ be the set of all Taelman $t$-motives over $L$. For any $M_1 = (M_1, n_1)$ and $M_2 = (M_2, n_2)$, the set of morphisms between $M_1$ and $M_2$ are given by

$$\text{Hom}_{\mathcal{M}_L}(M_1, M_2) := \text{Hom}_{\mathcal{M}_L}(M_1 \otimes C^{\otimes (n_1+1)}, M_2 \otimes C^{\otimes (n_2+1)})$$

where $n \geq \max\{-n_1, -n_2\}$. We note that the definition is independent of the choice of $n$ (see [Tag09a, Sec. 2.3]).

(iii) Continuing with the notation in (ii), we define the operation $\otimes_{\mathcal{M}_L}$ on $\mathcal{M}_L$ by

$$M_1 \otimes_{\mathcal{M}_L} M_2 := (M_1 \otimes M_2, n_1 + n_2).$$

(iv) The internal hom $\text{Hom}_L(M_1, M_2)$ is given by

$$\text{Hom}_{L[t]}(M_1, M_2) := \text{Hom}_{L[t]}(M_1, M_2 \otimes C^{\otimes (n_2-1+i)}), -i)$$

for some integer $i$. Note that the definition is independent of the choice of $i$ when it is sufficiently large.

For any $M_1, \ldots, M_4 \in \mathcal{M}_L$, one can obtain (see [Tag09a, Sec. 2.3.5]) that

$$\text{Hom}_{L[t]}(M_1, M_2) \otimes_{\mathcal{M}_L} \text{Hom}_{L[t]}(M_3, M_4) \cong \text{Hom}_{L[t]}(M_1 \otimes M_3, M_2 \otimes M_4).$$

**Remark 2.8.** Let $M \in \mathcal{M}_L$ be an effective $t$-motive. Using the fully faithful functor between the categories $\mathcal{M}_L$ and $\mathcal{M}_L$ sending $M \mapsto (M, 0)$, we denote the element $(M, 0)$ by $M$ throughout the paper and it should not lead to any confusion.

For any given effective $t$-motive $M$, one can define the dual $M^\vee$ of $M$ to be $\text{Hom}_{L[t]}(M, 1)$. We remark that, if we assume that the $\tau$-action on $M$ with respect to a $K[t]$-basis $\mathbf{m}$ is represented by $\Phi$ whose determinant is equal to $c(t-\theta)^\ell$ for some $\ell \in \mathbb{Z}_{\geq 0}$ and $c \in \mathbb{C}_\infty^*$, then there exists a $K[t]$-basis $\mathbf{n}$ of $\text{Hom}_{K[t]}(M, C^{\otimes \ell})$ such that the $\tau$-action on $M$ with respect to $\mathbf{n}$ $\text{Hom}_{K[t]}(M, C^{\otimes \ell})$ is represented by $(\Phi^{-1})^\tau$. Observe that in general $M^\vee$ is not an element in $\mathcal{M}_L$. However, it can be always represented by an element $(M', i)$ in $\mathcal{M}_L$ for some effective $t$-motive $M'$ over $L$ and for some $i \in \mathbb{Z}$. To see an example of this case, we use the abelian $t$-motive $M_\phi$ given as in Example 1.3(i) and prove the next proposition.

**Proposition 2.9** (cf. [Lag96] Eqn. (1.2.1)). Let $\phi$ be the Drinfeld $A$-module of rank $r \geq 2$ given as in (10). We have the following isomorphism of effective $t$-motives over $K$:

$$\text{Hom}_{K[t]}(M_\phi, C) \cong (\wedge_{K[t]}^{r-1} M_\phi) \otimes \text{Hom}_{K[t]}(\det(M_\phi), C).$$

Moreover, $\wedge_{K[t]}^{r-1} M_\phi$ is an abelian $t$-motive whose corresponding abelian $t$-module $E_0 := (\mathbb{G}_{a/K}, \varphi_0)$ is defined by $\varphi_0(t) := \theta \text{Id}_{r-1} + E_1' \tau + E_2' \tau^2$ so that the matrices $E_1' \in \text{Mat}_{r-1}(K)$ and $E_2' \in \text{Mat}_{r-1}(K)$ are given as

$$E_1' := (-1)^{r-1} \begin{bmatrix} -a_{r-1} & a_r & \cdots \\ \vdots & \ddots & \vdots \\ -a_2 & 0 & \cdots & a_r \\ -a_1 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad E_2' := \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_r & \cdots & 0 \end{bmatrix}.$$  

**Proof.** Since $M_\phi$ is a free $K[t]$-module of rank $r$, $\wedge_{K[t]}^{r-1} M_\phi$ is also free over $K[t]$ of rank $r$. Let us pick the $K[t]$-basis $\{g_1, \ldots, g_r\}$ for $\wedge_{K[t]}^{r-1} M_\phi$ given by $g_1 := m_1 \wedge \cdots \wedge \tau^{r-2} m_1$, $g_r := \tau m_1 \wedge \cdots \wedge \tau^{r-1} m_1$ and

$$g_i := m_1 \wedge \cdots \wedge \tau^{r-(i+1)} m_1 \wedge \tau^{r-(i-1)} m_1 \wedge \cdots \wedge \tau^{r-1} m_1$$

for $2 \leq i \leq r - 1$. Since the left $\tau$-action on $\wedge_{K[t]}^{r-1} M_\phi$ is diagonal, one can obtain

$$\tau g_1 = g_r$$

and

$$\tau g_i = (-1)^{r-2}a_r^{-1}(t - \theta)g_{i-1} + (-1)^{r-1}a_{r-(i-1)}g_r, \quad 2 \leq i \leq r.$$ 

Recall that the left $K[t, \tau]$-module $\det(M_\phi)$ has rank one as a $K[t]$-module with the basis $m' \in \det(M_\phi)$ where $\tau$ acts on $m'$ by $\tau m' = (-1)^{r-1} a_r^{-1}(t - \theta)m'$. Therefore, the left $K[t, \tau]$-module $\text{Hom}_{K[t]}(\det(M_\phi), C)$ has a $K[t]$-basis $\{\tilde{m}\}$ such that

$$\tau \tilde{m} = (-1)^{r-1}a_r \tilde{m}$$

and hence is isomorphic to the effective $t$-motive $1_{(-1)^{r-1}a_r}$ as left $K[t, \tau]$-modules. Note that $\{g_1 \otimes \tilde{m}, \ldots, g_r \otimes \tilde{m}\}$ forms a $K[t]$-basis for $\wedge_{K[t]}^{r-1} M_\phi \otimes \text{Hom}_{K[t]}(\det(M_\phi), C)$ and furthermore, considering the diagonal $\tau$-action on $\wedge_{K[t]}^{r-1} M_\phi \otimes \text{Hom}_{K[t]}(\det(M_\phi), C)$, we have

$$\tau(g_1 \otimes \tilde{m}) = (-1)^{r-1}a_r (g_r \otimes \tilde{m})$$

and for $2 \leq i \leq r$,

$$\tau(g_i \otimes \tilde{m}) = (-1)^{r-2}(-1)^{r-1}(t - \theta)g_{i-1} \otimes \tilde{m} - (-1)^{r-1}(-1)^{r-2}a_{r-(i-1)}g_r \otimes \tilde{m}$$

$$= -(t - \theta)g_{i-1} \otimes \tilde{m} + (-1)^{r-1}a_{r-(i-1)}g_r \otimes \tilde{m}.$$ 

On the other hand, consider the $K[t]$-basis $\{f_1, \ldots, f_r\}$ of the left $K[t, \tau]$-module $\text{Hom}_{K[t]}(M_\phi, C)$ where each $f_i$ is a $K[t]$-linear map defined by $f_i(m_k) = m$ if $i = k$ and $f_i(m_k) = 0$ otherwise, where $m$ is the $K[t]$-basis for $C$. Then, we have that

$$\tau f_i = a_i f_i + (t - \theta)f_{i+1}$$

for $1 \leq i \leq r - 1$, and

$$\tau f_r = a_r f_1.$$ 

Then, (13) and (16) now imply that $\text{Hom}_{K[t]}(M_\phi, C)$ is an effective $t$-motive. Let

$$\iota : (\wedge_{K[t]}^{r-1} M_\phi) \otimes \text{Hom}_{K[t]}(\det(M_\phi), C) \to \text{Hom}_{K[t]}(M_\phi, C)$$

be the $K[t]$-module homomorphism defined as $\iota(g_i \otimes \tilde{m}) = (-1)^{r-i} f_{r-(i-1)}$. It is clear that $\iota$ is a $K[t]$-module isomorphism. Moreover, by using the left $\tau$-action (13)–(16) on the $K[t]$-bases $\{g_1 \otimes \tilde{m}, \ldots, g_r \otimes \tilde{m}\}$ of $\wedge_{K[t]}^{r-1} M_\phi \otimes \text{Hom}_{K[t]}(\det(M_\phi), C)$ and $\{f_1, \ldots, f_r\}$ of $\text{Hom}_{K[t]}(M_\phi, C)$, we have for any $\alpha \in K[t]$

$$\iota(\tau(\alpha g_1 \otimes \tilde{m})) = \alpha(1)\iota((-1)^{r-1} a_r g_r \otimes \tilde{m}) = \alpha(1)(-1)^{r-1} a_r f_1 = \tau(\iota(\alpha g_1 \otimes \tilde{m}))$$

and for $2 \leq i \leq r$

$$\iota(\tau(\alpha g_i \otimes \tilde{m})) = \iota(\alpha(1)(-t - \theta)g_{i-1} \otimes \tilde{m} + (-1)^{r-i}a_{r-(i-1)}g_r \otimes \tilde{m}))$$

$$= \alpha(1)(-1)^{r-i}((t - \theta)f_{r-(i-2)} + a_{r-(i-1)}f_1)$$

$$= \tau(\iota(\alpha g_i \otimes \tilde{m})$$

which imply that $\iota$ is a left $K[t, \tau]$-module isomorphism.
Using the $K[t]$-bases $\{f_1, \ldots, f_r\}$ and $\{m'\}$ of the left $K[t, \tau]$-modules $\text{Hom}_{K[t]}(M_\phi, C)$ and $1_{(-1)^{r-1}a\tau^{-1}} \cong (\text{Hom}_{K[t]}(\text{det}(M_\phi), C))^{\ast}$, one can obtain a $K[t]$-basis $\{f'_1, \ldots, f'_r\}$ of $\wedge_{K[t]}^{r-1}M_\phi$ so that
\begin{equation}
\tau \begin{bmatrix} f'_1 \\ f'_2 \\ \vdots \\ f'_{r-1} \\ f'_r \end{bmatrix} = (-1)^{r-1} \begin{bmatrix} a_{r-1}^{-1}a_1 & a_r^{-1}(t - \theta) & 0 & \ldots & 0 \\ a_{r-1}^{-1}a_2 & 0 & a_{r-1}^{-1}(t - \theta) & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \ldots & \ldots & 0 \end{bmatrix} \begin{bmatrix} f'_1 \\ f'_2 \\ \vdots \\ f'_{r-1} \\ f'_r \end{bmatrix}.
\end{equation}

Using (17), we have that $\{f'_r, \ldots, f'_{2}\}$ is a left $K[\tau]$-basis for $\wedge_{K[t]}^{r-1}M_\phi$ and we obtain
\[ t \cdot [f'_r, \ldots, f'_2] = \varphi_0(t)[f'_r, \ldots, f'_2]^{\tau}. \]

Thus, the latter assertion follows.

In what follows, we define another family of (abelian) $t$-modules studied in [GN24 §3]. Let $n$ be a positive integer. Using the $K[\tau]$-basis $\{f'_r, \ldots, f'_1, (t - \theta)f'_r, \ldots, (t - \theta)^{n-1}f'_1, (t - \theta)^{n-1}f'_2, \ldots, (t - \theta)^{n-1}f'_2\}$ of $(\wedge_{K[t]}^{r-1}M_\phi) \otimes C^\otimes n$, one can see that its corresponding abelian $t$-module $\mathcal{E}_n := (\wedge_{t}^{r-1}C) \otimes C^\otimes n := (C_{a/K}^{rn+r-1}, \varphi_n)$ is given by
\[ \varphi_n(t) := \theta \text{Id}_{rn+r-1} + N' + E' \tau \]
so that the matrices $N' \in \text{Mat}_{rn+r-1}(\mathbb{F}_q)$ and $E' \in \text{Mat}_{rn+r-1}(K)$ are defined as
\begin{equation}
N' := \begin{bmatrix}
0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
\end{bmatrix}^{rn-1}
\end{equation}
and
\begin{equation}
E' := (-1)^{r-1} \begin{bmatrix}
0 & \ldots & \ldots & \ldots & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & \ldots & 0 \\
1 & 0 & \ldots & \ldots & \ldots & 0 \\
-a_{r-1} & a_r & \ddots & \ddots & \ddots & \vdots \\
-a_1 & 0 & \ldots & a_r & 0 & 0 \end{bmatrix}^{rn-1}
\end{equation}

Here, the last $r$-rows of $N'$ contain only zeros which will be fundamental in proving our main results (see §5.2 and §6). By a similar calculation to the one in §2.3, one can see that $\mathcal{E}_n$ is the abelian $t$-module corresponding to the abelian $t$-motive $\wedge_{K[t]}^{r-1}M_\phi \otimes C^\otimes n$. We leave the details to the reader.
Example 2.10. (i) Let $n \in \mathbb{Z}_{\geq 1}$ and $b \in K \setminus \{0\}$. One can see that $\text{Hom}_{K[t]}(C_n^{(b)}, C^{(n)}) \subseteq \text{Hom}_{K(t)}(C_n^{(b)} \otimes \overline{K}(t), C^{(n)} \otimes \overline{K}(t))$ is isomorphic to $1_b$ as left $K[t, \tau]$-modules. Using (9), we see that $(C_n^{(b)})^\vee = \text{Hom}_{K[t]}(C_n^{(b)}, 1) = (\text{Hom}_{K[t]}(C_n^{(b)}, C^{(n)}), -n) = (1_{b^{-1}}, -n) = C^{(b^{-1})} \otimes (C^{(n+1)})^\vee$.

(ii) Let $M$ be an effective $t$-motive over $K$. The natural left $K[t, \tau]$-module isomorphism between $M \otimes C \otimes C^{\otimes n}$ and $M \otimes C^{\otimes (n+1)}$ implies that

$$\left( M \otimes C, i \right) \cong (M, i + 1), \ i \in \mathbb{Z}.$$ In particular, $C^{\otimes n}$ can also be represented as $(1, n)$ in $\mathcal{M}_K$.

(iii) By (9) and Proposition 2.9 when $\phi$ is a Drinfeld $A$-module of rank $r \geq 2$, we have

$$M_\phi^\vee = (\text{Hom}_{K[t]}(M_\phi, C), -1) \cong ((\Lambda_{K[t]}^{r-1} M_\phi) \otimes \text{Hom}_{K[t]}(\det(M_\phi), C), -1)$$

$$= (\Lambda_{K[t]}^{r-1} M_\phi) \otimes \det(M_\phi)^\vee.$$  

3. An interpretation of Theorem 1.1 in terms of Taelman $L$-values

In this section, we introduce the $L$-function of a given abelian $t$-motive and relate its special values to Taelman $L$-values for an explicit family of abelian $t$-modules isomorphic to $\Delta_n$ or $G_n$ described in §2 (see [Gos94, Sec. 2.3] and [Tae09b, Sec. 2.8] for $L$-functions attached to more general effective $t$-motives). Our purpose is to state Theorem 3.5 and Theorem 3.6 which are equivalent to the first and the second part of Theorem 1.1 respectively.

3.1. $L$-function of abelian $t$-motives. Consider an irreducible polynomial $\beta \in A_+$ and set $L := A/\beta A$. For any abelian $t$-module $G$ of dimension $s$ defined over $K$, by clearing out the denominators of the entries of the coefficients if necessary, one can obtain an abelian $t$-module $\mathfrak{G} = (G_s/K, \varphi)$ defined over $A$ which is isomorphic to $G$ over $K$. Let us set $\overline{\mathfrak{G}} := (G_s/L, \overline{\varphi})$ to be the $t$-module where $\overline{\varphi}(a)$ is the reduction of $\varphi(a)$ modulo $\beta$ for each $a \in A$. Let $M \in \mathcal{M}_K$ be the abelian $t$-motive corresponding to $G$ which has rank $r$ over $K[t]$. We say that $M$ has good reduction at $\beta$ if there exists an abelian $t$-module $\mathfrak{G}$ defined over $A$ isomorphic to $G$ over $K$ so that the abelian $t$-motive $\overline{\mathfrak{G}}$ over $L$ corresponding to the abelian $t$-module $\mathfrak{G}$ has rank $r$ over $L[t]$.

Let $K^{\text{sep}}$ be the separable closure of $K$ in $C_\infty$ and define $M_{K^{\text{sep}}} := M \otimes_K K^{\text{sep}}$. Let $v$ be a monic irreducible element in $A$. Consider $A_v := \lim_{\leftarrow i} A/v^i A$ and set $K_v := A_v \otimes A \mathbb{F}_q(t)$. We define

$$\mathcal{H}_v(M) := \lim_{\leftarrow i} (M_{K^{\text{sep}}} / v^i M_{K^{\text{sep}}})^\tau$$

where, for each $i$, $(M_{K^{\text{sep}}} / v^i M_{K^{\text{sep}}})^\tau$ denotes the elements of $M_{K^{\text{sep}}} / v^i M_{K^{\text{sep}}}$ fixed by the action of $\tau$. Note that $\mathcal{H}_v(M)$ is a free $A_v$-module of rank $r$ with a continuous action of $\text{Gal}(K^{\text{sep}}/K)$ [Gos96, Sec.5.6].

Let $\beta'$ be a prime of $K^{\text{sep}}$ lying above $\beta$ and $I_{\beta'} \subset \text{Gal}(K^{\text{sep}}/K)$ be the inertia group at $\beta'$. We set $\mathcal{H}_v(M)^I_{\beta'}$ to be the $A_v$-module consisting of elements of $\mathcal{H}_v(M)$ invariant under the action of $I_{\beta'}$. Let $\overline{T}$ be a fixed algebraic closure of $L$ and let $\text{Frob}_{\beta}^{-1} \in \text{Gal}(\overline{T}/L)$ be the geometric Frobenius at $\beta$. We define the local factor of $M$ at $\beta$ to be the following polynomial:

$$P^M_{\beta}(X) := \det_{A_v}(1 - X \text{Frob}_{\beta}^{-1}) \mid \mathcal{H}_v(M)^I_{\beta'}) \in A_v[X].$$
Note that the definition of $P^M_{\beta}(X)$ is indeed independent of the choice of $\beta'$ [Gos96, Sec. 8.6]. The following result is due to Gardeyn.

**Theorem 3.1.** [Gar02, Thm. 1.1, Thm. 7.3] For any irreducible element $v \in A$ so that $v(\theta)$ is not equal to $\beta$, $I_\beta$ acts trivially on $\mathcal{H}_v(M)$ if and only if $M$ has good reduction at $\beta$. Moreover, for such $v$ and $\beta$, the polynomial $P^M_{\beta}(X)$ has coefficients in $A$ and is independent of the choice of $v$.

There exists a finite set $S$ of primes in $A$ so that $M$ does not have good reduction. Consider the following infinite product

$$L_S(M, n) := \prod_{\beta \in S} P^M_{\beta}(\beta^{-n})^{-1}.$$ 

Now for any natural number $n$, we construct the $L$-function $L(M, n)$ of $M$ by

$$L(M, n) := L_S(M, n) \prod_{\beta \in S} P^M_{\beta}(\beta^{-n})^{-1}.$$ 

Let $\beta \not\in S$. Assume that $P^M_{\beta}(X) = (1 - \alpha_1 X) \cdots (1 - \alpha_r X)$ for some $\alpha_1, \ldots, \alpha_r \in K_v$ where $K_v$ is a fixed algebraic closure of $K_v$. We further define $P^M_{\beta'}(X) := (1 - \alpha_1^{-1} X) \cdots (1 - \alpha_r^{-1} X) \in K_v[X]$ and consider

$$L_S(M', n) := \prod_{\beta \in S} P^M_{\beta'}(\beta^{-n})^{-1}.$$ 

**Remark 3.2.** Let the notation be as above. For later use, we also consider the polynomial $Q^M_{\beta}(X)$ defined by the relation $P^M_{\beta}(X) = X^r Q^M_{\beta}(X^{-1})$. Indeed, we have

$$Q^M_{\beta}(X) = (X - \alpha_1) \cdots (X - \alpha_r).$$

We note that that

$$\frac{Q^M_{\beta}(1)}{Q^M_{\beta}(0)} = \frac{(1 - \alpha_1) \cdots (1 - \alpha_r)}{(-1)^r \alpha_1 \cdots \alpha_r} = (1 - \alpha_1^{-1}) \cdots (1 - \alpha_r^{-1}) = P^M_{\beta'}(1).$$

**Remark 3.3.** It is important to note that due to the construction, we need to distinguish the variable $t$ and $\theta$ which causes our $L$-functions to converge in $F_q((1/t))$. However, for the rest of the paper, we instead consider these values in $K_\infty$ by simply replacing $t$ with $\theta$.

As an example, one can see that the value of the $L$-function $L(C, s) = L_\emptyset(C, s)$ at $s \in \mathbb{Z}_{\geq 2}$ corresponding to the abelian $t$-motive $C$ can be given by

$$L(C, s) = \sum_{a \in A_+} \frac{1}{a^{s-1}} \in K_\infty.$$

Consider a Taelman $t$-motive $M = (M, i)$ so that $M$ is an abelian $t$-motive over $K$ and has good reduction at primes of $A$ outside of $S$. Observe that

$$L_S(M \otimes_K C, n + 1) = L_S(M, n)$$

for any $M \in \mathcal{M}$ and sufficiently large $n$ which simply follows from the tensor compatibility of the functor $\mathcal{H}_v$ (see [HJ20, Sec. 2.3.5]) and the fact that the polynomial $P_{\beta}(X)$ for $C$ is given by $1 - X_{\beta|_{\theta=t}}$.

We continue to assume that $v$ is a monic irreducible polynomial so that $v(\theta) \neq \beta$. We finish this subsection by defining the $v$-adic Tate module which will be fundamental to analyze...
local factors of $L$-functions. Let $\mathfrak{G} = (\mathbb{G}_{a/K}, \varphi)$ be an abelian $t$-module defined over $A$ whose corresponding abelian $t$-motive has rank $r$ over $K[t]$. Assume that $\overline{\mathfrak{G}}$ is an abelian $t$-module over $L$ whose corresponding abelian $t$-motive has rank $r$ over $L[t]$.

Let $L^{\text{sep}}$ be a fixed separable closure of $L$. For any $n \geq 1$, we define the set of $v^n$-torsion points of $\overline{\mathfrak{G}}$ by

$$\overline{\mathfrak{G}}[v^n] := \{ f \in \overline{\mathfrak{G}}(L^{\text{sep}}) \mid v^n \cdot f = 0 \}.$$ 

By [Gos96, Cor. 5.6.4], we know that $\overline{\mathfrak{G}}[v^n]$ is a finite $A$-module isomorphic to $(A/v^n A)^{\oplus r}$. We define the $v$-adic Tate module of $\mathfrak{G}$ by

$$T_v(\mathfrak{G}) := \varprojlim_n \overline{\mathfrak{G}}[v^n].$$

By [Gos96, Thm. 5.6.8], it is a free $A_v$-module of rank $r$ which also has a continuous action of $\text{Gal}(L^{\text{sep}}/L)$.

### 3.2. Explicitly constructed abelian $t$-motives

Recall the Drinfeld $A$-module $G_0 = (\mathbb{G}_{a/K}, \phi)$ defined in (10) and assume that it has rank $r \geq 2$. Recall also its abelian $t$-motive $M_0 \in \mathcal{M}_K$ from Example 2.3(i).

Let $n$ be a non-negative integer. In what follows, we investigate families of abelian $t$-motives and their corresponding abelian $t$-modules. Moreover, we analyze their associated $L$-functions by assuming that they are well-defined elements in $K_\infty$ which will be clear to the reader in §3.4 after their comparison with certain Taelman $L$-values.

#### 3.2.1. Construction of $M'_n$ and $M'_{n,A}$

Consider the effective $t$-motive $M'_n$ given by

$$M'_n := (\wedge^{r-1}_{K[t]} M_0) \otimes \mathbb{C}^{(n+1)} \otimes \mathbb{A} \otimes \det(M_0)^{\vee} \in \mathcal{M}_K$$

which is indeed isomorphic to an effective $t$-motive over $K$. Recall the abelian $t$-module $\wedge^{r-1} \phi = (\mathbb{G}_{a/K}^{r-1}, \varphi_0)$ from the statement of Proposition 2.9 as well as the matrices $E'_1$ and $E'_2$. Set $\gamma := ((-1)^{-1} a^{-1}_{r-1} 1/(a-1)) \in K$. Using a similar approach as in §2.4, one can see that $M'_n$ is an abelian $t$-motive over $K$ and when $n = 0$, the abelian $t$-module $E'_0 := (\mathbb{G}_{a/K}^{r-1}, \varphi_0)$ attached to $M'_0$ is given by $\gamma^{-1} \varphi_0 \gamma =: \phi'_0 : A \to \text{Mat}_{r-1}(K)[\tau]$ which can be explicitly stated as

$$\phi'_0(t) := \theta \text{Id}_{r-1} + (-1)^{r-1} a^{-1}_r E'_1 \tau + a^{-1}_r E'_2 \tau^2 \in \text{Mat}_{r-1}(K)[\tau].$$

Recall the abelian $t$-module $E_n = (\mathbb{G}_{a/K}^{r+n-1}, \varphi_n)$ and the matrices $N'$ and $E'$ from §2.4. When $n \geq 1$, the abelian $t$-module $E'_n := (\mathbb{G}_{a/K}^{r+n-1}, \varphi'_n)$ attached to $M'_n$ is given by $\gamma^{-1} \varphi_n \gamma =: \phi'_n : A \to \text{Mat}_{rn+r-1}(K)[\tau]$ where

$$\phi'_n(t) := \theta \text{Id}_{rn+r-1} + N' + (-1)^{r-1} a^{-1}_r E' \tau \in \text{Mat}_{rn+r-1}(K)[\tau].$$

Moreover, for each $n \geq 0$, there exists an element $a \in A \setminus \{0\}$ so that the abelian $t$-module $E'_{n,A} := (\mathbb{G}_{a/K}^{r+n-1}, \varphi'_n)$ given by $\varphi'_n(t) := a^{-1} \phi'_n(t) a$ is defined over $A$. Let $M'_{n,A} \in \mathcal{M}_K$ be its corresponding abelian $t$-motive. It is clear that $M'_{n,A} \cong M'_n$ as $K[t, \tau]$-modules.

Since taking the dual of an effective $t$-motive is reflexive (see [Tae09a, Sec. 2.3.5]), by (20), we have an isomorphism of Taelman $t$-motives

$$\left(\wedge^{r-1}_{K[t]} M_0\right)^{\vee} \cong M_0 \otimes \mathcal{M}_K \otimes \det(M_0)^{\vee}.$$(23)
On the other hand, using \([10]\) and \([23]\), one can see that
\[
(M'_n)^\vee = \text{Hom}(M'_n, 1)
\]
\[
\cong \text{Hom}(\wedge^{r-1}_{K[t]} M_\phi, 1) \otimes_{\mathcal{A}_K} (\mathbb{C}^\otimes(n+1))^\vee \otimes_{\mathcal{A}_K} \det(M_\phi)
\]
\[
\cong M_\phi \otimes_{\mathcal{A}_K} \det(M_\phi)^\vee \otimes_{\mathcal{A}_K} (\mathbb{C}^\otimes(n+1))^\vee \otimes_{\mathcal{A}_K} \det(M_\phi)
\]
\[
\cong M_\phi \otimes_{\mathcal{A}_K} (\mathbb{C}^\otimes(n+1))^\vee.
\]

Assume that \(S\) is the subset of primes of \(A\) where \(M_\phi\) does not have good reduction. Applying \([22]\) repeatedly together with the fact that \(C^\vee \otimes_{\mathcal{A}_K} C = 1\), we obtain
\[
(24) \quad L_S((M_{\phi_{n,A}})^\vee, 0) = L_S((M'_n)^\vee, 0) = L_S(M_\phi \otimes_{\mathcal{A}_K} (\mathbb{C}^\otimes(n+1))^\vee, 0) = L_S(M_\phi, n + 1).
\]

### 3.2.2. Construction of \(C_n^{(b')}\).

For any \(b \in K \setminus \{0\}\) and \(n \in \mathbb{Z}_{\geq 1}\), there exists \(b' \in K \setminus \{0\}\) so that \(C_n^{(b')}\) is defined over \(A\) and isomorphic to \(C_n^{(b-1)}\) over \(K\). Moreover, \(C_n^{(b-1)} \cong C_n^{(b')}\) as \(K[t, \tau]-\text{modules}\). In this case, as it is worked out in Example 2.10, we obtain
\[
(C_n^{(b')})^\vee = C_n^{((b')-1)} \otimes (\mathbb{C}^\otimes(n+1))^\vee
\]

Consider the finite set \(S'\) consisting of primes of \(A\) where \(C^{(b)}\) does not have good reduction. Finally, letting \(n \in \mathbb{Z}_{\geq 1}\) and using Example 2.12(i), we also obtain
\[
(25) \quad L_{S'}((C_n^{(b')})^\vee, 0) = L_{S'}((C_n^{(b-1)})^\vee, 0) = L_{S'}(C^{(b)} \otimes_{\mathcal{A}_K} (\mathbb{C}^\otimes(n+1))^\vee, 0) = L_{S'}(C^{(b)}, n + 1).
\]

### 3.2.3. Construction of \(\widetilde{M}_n\).

In this case, we let \(\phi\) be a Drinfeld \(A\)-module which is isomorphic, over \(K\), to \(\psi\) given as in \([2]\) and let \(M_\psi \in \mathcal{M}_K\) be its corresponding abelian \(t\)-motive. Consider the Taelman \(t\)-motive \(\widetilde{M}_n\) given by
\[
\widetilde{M}_n := M_\psi \otimes_{\mathcal{A}_K} C^\otimes(n+1) \otimes_{\mathcal{A}_K} \det(M_\psi)^\vee \in \mathcal{M}_K.
\]

Indeed, one can check that \(\widetilde{M}_n\) is an abelian \(t\)-motive over \(K\) whose corresponding abelian \(t\)-module \(\widetilde{G}_n := (\mathbb{G}_a^{r+1}, \tilde{\psi}_n)\) is given as
\[
(26) \quad \tilde{\psi}_n(t) := \gamma^{-1} \psi_n(t) \gamma = \theta \text{Id}_{r+1} + N + (-1)^{r-1} a_r^{-1} E \tau \in \text{Mat}_{r+1}(A)[\tau]
\]

where the matrices \(N\) and \(E\) are as in \([7]\) and \([8]\) respectively. Using \([10]\) and \([20]\), one can immediately see that
\[
\widetilde{M}_n^\vee \cong (\wedge^{r-1}_{K[t]} M_\psi) \otimes_{\mathcal{A}_K} (\mathbb{C}^\otimes(n+1))^\vee.
\]

On the other hand, since \(a_r \in \mathbb{F}_q^\times\), it can be seen by \([17]\) that \(\wedge^{r-1}_{K[t]} M_\psi\) has good reduction at any prime of \(A\). Hence applying \([22]\) repeatedly, we obtain
\[
(27) \quad L(M_n^\vee, 0) = L(\wedge^{r-1}_{K[t]} M_\psi \otimes_{\mathcal{A}_K} (\mathbb{C}^\otimes(n+1))^\vee, 0) = L(\wedge^{r-1}_{K[t]} M_\psi, n + 1) = L(\wedge^{r-1}_{K[t]} M_\phi, n + 1)
\]

where the last equality follows from the fact that \(M_\phi \cong M_\psi\) by assumption.

We finish this subsection with the following proposition which is a consequence of \([\text{Tha04}]\) Rem. 7.3.5).

**Proposition 3.4.** Assume that \(C_n^{(b')}, M_\phi_{n,A}\) and \(M_\phi\) have good reduction at a prime \(\beta \in A_+\). Then, the abelian \(t\)-modules \(C_n^{(b')}, \overline{\mathcal{E}}_{n,A}\) and \(\overline{G}_n\) defined over \(L = A/\beta A\) are pure.
3.3. Local factors at places with good reduction. For any finite \( A \)-module \( N \) so that \( N \cong \bigoplus_{i=1}^{m} A/f_i A \) for some elements \( f_1, \ldots, f_m \in A \), we define \( |N|_A \) to be the monic generator of the principal ideal \((f_1 \cdots f_m) \subseteq A\).

Recall that \( \beta \) is an irreducible polynomial in \( A_+ \) and let \( v \in A \) so that \( v(\theta) \neq \beta \). As we proceed in the previous subsection, we divide our analysis into three cases which correspond to the calculation for the local factors of the \( L \)-functions corresponding to the abelian \( t \)-motives defined in \( \S 3.2 \).

3.3.1. Local factors of the \( L \)-function of \( M_{\mathfrak{g}_{\ell,n,A}}^{M_{\mathfrak{g}_{\ell,n,A}}} \). We first analyze the local factors of the \( L \)-function of \( M_{\mathfrak{g}_{\ell,n,A}} \) defined as in \( \S 3.2.1 \). Assume that \( M_{\mathfrak{g}_{\ell,n,A}} \) has good reduction at \( \beta \) and recall that \( M_{\mathfrak{g}_{\ell,n,A}} \) is the abelian \( t \)-motive over \( L \) corresponding to \( \mathfrak{g}_{\ell,n,A} \). Let \( Q_{\beta}^{M_{\mathfrak{g}_{\ell,n,A}}}(X) \in A_v[X] \) be the characteristic polynomial of the \( \mathfrak{g}^{\deg_\theta(\beta)} \)-th power map \( \tau^{\deg_\theta(\beta)} \) on \( T_v(\mathfrak{g}_{\ell,n,A}) \). We have

\[
P_{\beta}^{M_{\mathfrak{g}_{\ell,n,A}}}(X) = \det(1 - X \tau^{\deg_\theta(\beta)} | M_{\mathfrak{g}_{\ell,n,A}}) = \det(1 - X \tau^{\deg_\theta(\beta)} | T_v(\mathfrak{g}_{\ell,n,A})) = X^r Q_{\beta}^{M_{\mathfrak{g}_{\ell,n,A}}}(X^{-1})
\]

where the first equality follows from [Tae09b, Prop. 7] (see also [Gar02, Sec. 7]) and the second equality from [TW96, Sec. 6] (see also [Gos96, Prop. 5.6.9]).

Note that, in this case, by (11), we have

\[
P_{\beta}^{M_{\mathfrak{g}_{\ell,n,A}}}(X) = d(1 - \alpha_1^{-1}\beta^n X) \cdots (1 - \alpha_r^{-1}\beta^{n+1} X)
\]

for some \( d \in \mathbb{F}_q^\times \). Thus, by [Gek91, Thm. 5.1] (see also [CEGP18, Thm. 3.1, Cor. 3.2]), we obtain

\[
Q_{\beta}^{M_{\mathfrak{g}_{\ell,n,A}}}(X) = b_0 + b_1 X + \cdots + b_{r-1} X^{r-1} + X^r \in A[X]
\]

where \( \deg_i(b_i) < \deg_\theta(\beta^{r+n+r-1}) \) for \( 1 \leq i \leq r - 1 \) and \( b_0 = c\beta^{r+n+r-1} \) for some \( c \in \mathbb{F}_q^\times \).

For any \( a \in A \), let \( (a)_v \) be the \( v \)-primary part of the principal ideal \((a) \subseteq A \) generated by \( a \). By Proposition 3.4 and [Gos96, Thm. 5.6.10] (see also [BH09, Thm. 7.8]), we know that the eigenvalues of \( \text{Frob}_\beta \) has norm larger than one. This implies that the map \( 1 - \text{Frob}_\beta \) is injective on \( T_v(\mathfrak{g}_{\ell,n,A}) \). Therefore by [Deb16, Prop. 2.15] and the fact that the set of \( v^m \)-torsion points of \( \mathfrak{g}_{\ell,n,A} \) is finite for each \( m \in \mathbb{Z}_{\geq 1} \), we have

\[
(P_{\beta}^{M_{\mathfrak{g}_{\ell,n,A}}}(1))_v = (Q_{\beta}^{M_{\mathfrak{g}_{\ell,n,A}}}(1))_v = (\mathfrak{g}_{\ell,n,A}(A/\beta A)|_A)_v.
\]

On the other hand, by (28), we have \( \deg_i(Q_{\beta}^{M_{\mathfrak{g}_{\ell,n,A}}}(1)) = \deg_i([\mathfrak{g}_{\ell,n,A}(A/\beta A)|_A] = \deg_i(\beta)(rn + r - 1) \). Hence we obtain

\[
c^{-1}Q_{\beta}^{M_{\mathfrak{g}_{\ell,n,A}}}(1) = [\mathfrak{g}_{\ell,n,A}(A/\beta A)|_A]
\]

Since \( \beta^{rn+r-1} \) annihilates the \( A \)-module \( \text{Lie}(\mathfrak{g}_{\ell,n,A}) \), one can also see that

\[
(Q_{\beta}^{M_{\mathfrak{g}_{\ell,n,A}}}(0)) = (\beta^{rn+r-1}) = ([\text{Lie}(\mathfrak{g}_{\ell,n,A})(A/\beta A)|_A]).
\]
It implies that \( c^{-1}Q^\mu_{\beta^n}(0) = |\text{Lie}(G_n)(A/\beta A)|_A \). Hence, we finally obtain
\[
(29) \quad \frac{|\text{Lie}(G_n)(A/\beta A)|_A}{|\text{Lie}(G_n)(A/\beta A)|_A} = \frac{Q^\mu_{\beta^n}(1)}{Q^\mu_{\beta^n}(0)} = P^{M^\vee}_{\beta^n}(1) = P^{M^\vee}_{\beta^n}(1)
\]
where the second equality follows from (21) and the last equality from the fact that \( M_{\beta^n} \cong M^n \).

3.3.2. Local factors of the \( L \)-function of \((C_n^{(b^{-1})})^\vee\). If we assume that \( \beta \) is not in the set \( S' \) defined in §3.2, for any given \( b \in K \setminus \{0\} \), using the above strategy, one can obtain that
\[
(30) \quad \frac{|C_n^{(b)}(A/\beta A)|_A}{|\text{Lie}(C_n^{(b)})(A/\beta A)|_A} = P^{(C_n^{(b^{-1})})^\vee}(1)
\]
which we leave the details to the reader.

3.3.3. Local factors of the \( L \)-function of \( \tilde{M}^\vee_n \). Now we analyze, \( Q^\tilde{M}_n(X) \in A_v[X] \), the characteristic polynomial of the \( q^{\text{deg}(\beta)} \)th power map on \( T_v(\tilde{G}_n) \) using the idea as above. Note that, in this case, we have
\[
P^{\tilde{M}_n}(X) = d(1 - \alpha_1\beta^n X) \cdots (1 - \alpha_r\beta^n X)
\]
for some \( d \in F_q \). Thus, again by [Gek91, Thm. 5.11], we obtain
\[
Q^{\tilde{M}_n}(X) = \tilde{b}_0 + \tilde{b}_1X + \cdots + \tilde{b}_{r-1}X^{r-1} + X^r \in A[X]
\]
where \( \text{deg}(\tilde{b}_i) < \text{deg}(\beta^{r+1}) \) for \( 1 \leq i \leq r - 1 \) and \( \tilde{b}_0 = \tilde{c}\beta^{(r+1)} \) for some \( \tilde{c} \in F_q \). Moreover, using the same idea in §3.3.1, we also obtain
\[
\tilde{c}^{-1}Q^\tilde{M}_n(1) = |\text{Lie}(\tilde{G}_n)(A/\beta A)|_A
\]
and
\[
\tilde{c}^{-1}Q^\tilde{M}_n(0) = |\text{Lie}(\tilde{G}_n)(A/\beta A)|_A.
\]
Hence, by again (21), we see that
\[
(31) \quad \frac{|\text{Lie}(\tilde{G}_n)(A/\beta A)|_A}{|\text{Lie}(\tilde{G}_n)(A/\beta A)|_A} = \frac{Q^{\tilde{M}_n}(1)}{Q^{\tilde{M}_n}(0)} = P^{\tilde{M}_n}(1).
\]

3.4. Taelman \( L \)-values. We start by introducing Taelman \( L \)-values which were firstly defined by Taelman [Tae12] for Drinfeld \( A \)-modules and later by Fang [Fan15] for abelian \( t \)-modules.

Let \( G = (\mathbb{G}_a/K, \varphi) \) be an abelian \( t \)-module of dimension \( s \) defined over \( A \). For any finite \( A \)-module \( N \), we now set \( |N|_A := \langle |N| \rangle_{t=\beta} \). Let \( \beta \in A^+ \) be a prime element. The Taelman \( L \)-value \( L(G/A) \) is defined by the infinite product
\[
L(G/A) := \prod_{\beta \in A^+, \beta \text{ prime}} \frac{|\text{Lie}(\tilde{G})(A/\beta A)|_A}{|G(A/\beta A)|_A} \in 1 + \frac{1}{\theta} F_q \left[ \left[ \frac{1}{\theta} \right] \right].
\]
Consider the Goss \( L \)-function \( L(M_{\phi}, n) \) defined as above and the finite set \( S \) of places of \( A \) where \( M_{\phi} \) does not have good reduction. By [Gar02, Cor. 8.4], we know that
\[
P_{\beta \in S} P_{\beta}^{M_{\beta}} (\beta|_{\theta=t})^{-1} \text{ is an element in } \mathbb{F}_q(t). \text{ This implies, after evaluating the product at } t = \theta, \text{ that}
\]
\[
L(M_{\phi}, n + 1) = \alpha L_{\mathcal{S}}(M_{\phi}, n + 1)
\]
for some \( \alpha \in K^\times \) whenever both sides converge in \( K_\infty \). Hence combining (24), (29) and (32), we obtain
\[
L(M_{\phi}, n + 1) = \alpha L_{\mathcal{S}}(M_{\phi}, n + 1) = \alpha L_{\mathcal{S}}((M_{\phi, A})^\vee, 0) = \alpha' L(\mathcal{E}'_{n, A}/A)
\]
for some \( \alpha' \in K^\times \).

On the other hand, [Gar02, Cor. 8.4] implies (see also [Gos02, Ex. 7]) that for \( \mathfrak{b} \in K \setminus \{0\} \), the local factors of the Goss \( L \)-function of \( C_n^{(b)} \) at primes in \( S' \) are all equal to 1. Hence, for any \( n \in \mathbb{Z}_{\geq 1} \), using (25) and (30), we obtain
\[
L(C_n^{(b)}, n + 1) = \alpha'' L(C_n^{(b')}/A), \quad \alpha'' \in K^\times.
\]
We now obtain the following equivalent statement of the first part of Theorem 1.1.

**Theorem 3.5.** For any non-negative integer \( n \), let \( \mathcal{E}'_{n, A} = (G_{n, A}^{r n + r - 1}, \mathcal{F}_n') \) be the abelian \( t \)-module defined as in §3.2.1 and \( L(\mathcal{E}'_{n, A}/A) \) be its Taelman \( L \)-value. Then, \( L(\mathcal{E}'_{n, A}/A) \) is transcendental over \( \overline{K} \). Furthermore, for any \( n \in \mathbb{Z}_{\geq 1} \) and \( \mathfrak{b} \in K \setminus \{0\} \), \( L(C_n^{(b')}/A) \) is also transcendental over \( \overline{K} \).

Observe that (27) and (31) imply
\[
L((\Lambda_{K[t]}^{-1}M_{\phi}, n + 1) = L(\tilde{M}'_{n}, 0) = L(\tilde{G}_n/A).
\]
Hence we may also write an equivalent statement for the second part of Theorem 1.1 as follows.

**Theorem 3.6.** Let \( \tilde{G}_n = (G_{n, A}^{r n + 1}, \tilde{\mathcal{F}}_{n}) \) be the abelian \( t \)-module defined as in §2.6 and \( L(\tilde{G}_n/A) \) be its Taelman \( L \)-value. Then, for any non-negative integer \( n \), \( L(\tilde{G}_n/A) \) is transcendental over \( \overline{K} \).

### 4. The Proof of Theorem 3.5 and Theorem 3.6

In this section, our aim is to prove Theorem 3.5 and Theorem 3.6 which are equivalent to Theorem 1.1(i) and Theorem 1.1(ii) respectively. Since the methods we use to obtain the results are similar, we only explain the details of the proof of Theorem 3.5 and explicitly state how to obtain Theorem 3.6 from the ideas of the proof of Theorem 3.5.

Let \( G = (G_{\mathbb{A}/K}, \mathcal{F}) \) be an abelian \( t \)-module so that \( \mathcal{F}(t) \in \text{Mat}_d(A)[\tau] \). We define the unit module \( U(G/A) \) of \( G \) by
\[
U(G/A) := \{ x \in \text{Lie}(G)(K_\infty) \mid \text{Exp}_G(x) \in G(A) \}.
\]
Using (5), one can obtain that \( U(G/A) \) is an \( A \)-submodule of \( \text{Lie}(G)(K_\infty) \). Moreover, by [Pan15, Thm. 1.10], \( U(G/A) \) is indeed a free \( A \)-module of rank \( d \).

We consider the map \( d_{\phi_n} : \mathbb{F}_q((1/t)) \to K_\infty \) defined by
\[
d_{\phi_n} \left( \sum_{i \geq i_0} c_i t^{-i} \right) = \sum_{i \geq i_0} c_i d_{\phi_n}(t)^{-i}, \quad c_i \in \mathbb{F}_q
\]
and we further equip $\text{Lie}(G)(K_\infty)$ with the $\mathbb{F}_q((1/t))$-vector space structure given by
\[ f \cdot x = d_{\varphi_n}(f)x, \quad f \in \mathbb{F}_q((1/t)), \quad x \in \text{Lie}(G)(K_\infty). \]
We set $W_G(K_\infty) := \frac{\text{Lie}(G)(K_\infty)}{(d_{\varphi_n}(t) - \theta \text{Id}) \text{Lie}(G)(K_\infty)}$. It has an $A$-module structure induced from the $A$-action on $\text{Lie}(G)(K_\infty)$. We also define $W_G(A)$ to be the $A$-submodule of $W_G(K_\infty)$ consisting of equivalence classes with coefficients in $A$.

Extending the definition of a monic polynomial in $A$, we say that an element $\sum_{i \geq i_0} c_i \theta^{-i} \in K_\infty$ is monic, if the leading coefficient $c_{i_0} \in \mathbb{F}_q^\times$ is equal to 1. Let $\text{proj} : \text{Lie}(G)(K_\infty) \to W_G(K_\infty)$ be the natural projection.

Let $n$ be a positive integer. Using [Mau24, Thm. A], [ANDTR20a, Thm. 4.4] and [ANDTR20a, Cor. 4.5] for our abelian $t$-modules $\mathcal{E}_{n,A}'$ and $G_n$, we state our next result.

**Theorem 4.1.** Let $G$ be either $\mathcal{E}_{n,A}'$ or $G_n$. Then, the following statements hold.

(i) There exists an $\mathbb{F}_q((1/t))$-vector subspace $\mathcal{Z}'$ of $\text{Lie}(G)(K_\infty)$ which is free of rank $r$ and isomorphic to $W_G(K_\infty)$ via the natural projection $\text{proj}$.

(ii) The intersection $Z := U(G/A) \cap \mathcal{Z}'$ is a free $A$-module of rank $r$. In particular, for any $A$-basis $\{g_1, \ldots, g_r\}$ of $Z$, we have $\text{Exp}_G(g_i) \in G(A)$ for any $1 \leq i \leq r$.

(iii) There exists a non-zero element $a \in K$ such that
\[ a \wedge_A \text{proj}(Z) = L(G/A) \wedge_A W_G(A) \]
where $\wedge_A \text{proj}(Z)$ ($\wedge_A W_G(A)$ respectively) is the monic generator of the $r$-th exterior power of the free $A$-module $\text{proj}(Z)$ ($W_G(A)$ respectively).

**Remark 4.2.** We refer the interested reader to the proof of [ANDTR20a, Prop. 4.3] for the explicit construction of the $\mathbb{F}_q((1/t))$-vector space $\mathcal{Z}'$.

Note that, for $n \geq 1$, $a \in A$ and $x_{r_1}, \ldots, x_{r_1+r-1} \in K_\infty$, the $A$-module structure on $W_{\mathcal{E}_{n,A}'}(K_\infty)$ can be explicitly given by
\[ a(t) \cdot (0, \ldots, 0, x_{r_1}, \ldots, x_{r_1+r-1})^\text{tr} + (d_{\varphi_n}(t) - \theta \text{Id}) \text{proj}(\mathcal{E}_{n,A}') (K_\infty) = (0, \ldots, 0, ax_{r_1}, \ldots, ax_{r_1+r-1})^\text{tr} + (d_{\varphi_n}(t) - \theta \text{Id}) \text{proj}(\mathcal{E}_{n,A}') (K_\infty). \]

If we write $Z := \oplus_{i=1}^r A \bar{g}_i$ for some $A$-basis $\{g_1, \ldots, g_r\}$ where $g_i = [g_i,1, \ldots, g_i,1, r_i+1] \in \text{Lie}(\mathcal{E}_{n,A}') (K_\infty)$ for $1 \leq i \leq r$, then $\text{proj}(Z) = \oplus_{i=1}^r A \bar{g}_i$ where
\[ \bar{g}_i := [0, \ldots, 0, g_i, r_i, \ldots, g_i, r_i, r_i+1] \quad \text{and} \quad (d_{\varphi_n}(t) - \theta \text{Id}) \text{proj}(\mathcal{E}_{n,A}') (K_\infty). \]

Thus, we see that $\wedge_A \text{proj}(Z) = c \text{det}(\mathcal{R})$ where $\mathcal{R}$ is the matrix
\[ \mathcal{R} := \begin{bmatrix} g_{1,r_1} & \cdots & g_{r,r_1} \\ \vdots & \ddots & \vdots \\ g_{1,r_1+r-1} & \cdots & g_{r,r_1+r-1} \end{bmatrix} \in \text{Mat}_r(K_\infty) \]
and $c \in \mathbb{F}_q^\times$ so that $c \text{det}(\mathcal{R})$ is monic in $K_\infty$. In a similar way, one can easily calculate that
\[ \wedge_A W_{\mathcal{E}_{n,A}'}(A) = 1. \]

Recall that $\gamma = ((-1)^{-1} a_r^{-1})^{1/(q-1)} \in K$ and for any $n \geq 0$, observe the following identity in $\text{Mat}_{r+1}(K)[[\tau]]$:
\[ (36) \quad \text{Exp}_{\mathcal{E}_{n,A}'} = (a \gamma)^{-1} \text{Exp}_{\mathcal{E}_n} a \gamma. \]
Recall the endomorphism ring $\text{End}(\mathfrak{e}_n)$ (End($G_n$) resp.) of $\mathfrak{e}_n$ ($G_n$ resp.). Since, by [GN24] Prop. 3.7, Prop. 4.6], $\text{End}(\mathfrak{e}_n)$ (End($G_n$) resp.) is an integral domain, one can consider the fraction field $K_n$ ($K_n^{\text{tens}}$ resp.) of $\text{End}(\mathfrak{e}_n)$ (End($G_n$) resp.). Before stating the proof of Theorem 3.5, we say that the elements $z_1, \ldots, z_k \in \text{Lie}(\mathfrak{e}_n)(C_\infty)$ are linearly independent over $K_n$ if whenever $dP_1z_1 + \cdots + dP_kz_k = 0$ for some $P_1, \ldots, P_k \in K_n$, we have $P_1 = \cdots = P_k = 0$. The analogous definition can be made for linearly independency over $K_n^{\text{tens}}$ similarly.

**Proof of Theorem 3.5.** When $n = 0$, we know by the class number formula of Fang [Fan15 Thm. 1.10] for abelian $t$-modules that $L(\mathfrak{e}_{0,A}^t/A)$ may be written as a product of a non-zero polynomial in $A$ and the determinant $\mathfrak{D}$ of a matrix consisting of the entries of some $y_1', \ldots, y_{r-1}' \in \text{Lie}(\mathfrak{e}_{0,A}^t)(K_\infty)$ so that $\text{Exp}_{\mathfrak{e}_{0,A}^t}(y_i') \in \mathfrak{e}_{0,A}^t(A)$ for each $1 \leq i \leq r - 1$. Up to rearrangement if necessary, after setting $y_i = [y_{i,1}, \ldots, y_{i,r-1}]^\text{tr}$, let $\{y_1, \ldots, y_k\}$ be the maximal linearly independent subset of $\{y_1', \ldots, y_{r-1}'\}$ over $K_0$, where $k \leq r - 1$. Then one can write

$$\mathfrak{D} = \tilde{f}(y_{1,1}, \ldots, y_{1,r-1}, \ldots, y_{k,1}, \ldots, y_{k,r-1}),$$

for some non-constant polynomial $\tilde{f} \in K[X_1, \ldots, X_{r-1}]$. Since $L(\mathfrak{e}_{0,A}^t/A)$ is non-zero, using (36) and [GN24 Thm. 1.1(ii)], we conclude the proof of this case by using (36) and [GN24 Thm. 1.1(i)]. Now let $n \in \mathbb{Z}_{\geq 1}$ and $\mathfrak{e}_n$ be the abelian $t$-module defined as in §3.2. By Theorem 4.1(iii), we have that

$$L(\mathfrak{e}_{n,A}^t/A) = a \det(\mathfrak{R})$$

for some $a \in K \setminus \{0\}$. Since $L(\mathfrak{e}_{n,A}^t/A)$ is non-zero, using the same reasoning above to show the transcendence of the determinant in the $n = 0$ case, by [GN24 Thm. 1.1(ii)], we obtain that $L(\mathfrak{e}_{n,A}^t/A) \not\subseteq K$. We finally prove the last assertion. For any $\mathfrak{b} \in K \setminus \{0\}$, by [ANDTR20a Cor. 4.5], one can apply [ANDTR20a Thm. 4.4] to $C^t_\mathfrak{e}(\mathfrak{b})$ and hence repeat the above argument to obtain that there exists an element $\mathfrak{g} \in \text{Lie}(C^t_\mathfrak{e}(\mathfrak{b}))(K_\infty)$ so that $\text{Exp}_{C^t_\mathfrak{e}(\mathfrak{b})}(\mathfrak{g}) \in C^t_\mathfrak{e}(\mathfrak{b})(A)$ and $L(C^t_\mathfrak{e}(\mathfrak{b})/A) = a' \mathfrak{g}_r$, where $\mathfrak{g}_r$ is the last coordinate of $\mathfrak{g}$ and $a' \in K$. Since $\mathfrak{g}$ is non-zero and $\text{Exp}_{C^t_\mathfrak{e}(\mathfrak{b})} = (b')^{-1/q-1} \text{Exp}_{C^t_\mathfrak{e}(\mathfrak{b})}^{1/q-1}$ for some fixed $(q - 1)$-st root $(b')^{1/q-1}$ of $b'$, [Yu91 Thm. 2.3] implies that $L(C^t_\mathfrak{e}(\mathfrak{b})/A)$ is transcendental over $K$. \hfill \Box

**Proof of Theorem 3.6.** When $n = 0$, note that the corresponding abelian $t$-module to $\tilde{M}_0$ is a Drinfeld $A$-module defined over $A$. Thus, [CEGP18 Cor. 4.6] implies the desired result. We now prove the case $n \in \mathbb{Z}_{\geq 1}$. Since the matrix $N$ given in (7) has only zeros in its last $r$ rows, repeating the same argument above by using Theorem 4.1 one can see that there exist elements $\tilde{g}_1, \ldots, \tilde{g}_r \in \text{Lie}(\tilde{G}_n)(K_\infty)$ satisfying $\text{Exp}_{\tilde{G}_n}(\tilde{g}_\ell) \in G_n(A)$ for each $1 \leq \ell \leq r$ and a non-zero $\tilde{a} \in K$ so that

$$L(\tilde{G}_n/A) = \tilde{a} \det(\mathfrak{R})$$

where $\tilde{g}_\ell = [\tilde{g}_{\ell,1}, \ldots, \tilde{g}_{\ell,n+1}]^\text{tr}$ and $\mathfrak{R} := (\tilde{g}_{i,j}, r_{(n-1)+1}^{(n+1)}, \ldots, r_{(n+1)+1}^{(n+1)}) \in \text{Mat}_r(K_\infty)$. Up to rearrangement if necessary, let $\{\tilde{g}_1, \ldots, \tilde{g}_k\}$ be the maximal linearly independent subset of $\{g_1, \ldots, g_r\}$ over $K_n^{\text{tens}}$, where $k \leq r - 1$. Then one can write

$$\det(\mathfrak{R}) = \tilde{f}(g_{1,1}, \ldots, g_{1,n+1}, \ldots, g_{k,1}, \ldots, g_{k,n+1}),$$

for some non-constant polynomial $\tilde{f} \in K[X_1, \ldots, X_{rk}]$. Let $\psi'$ be a Drinfeld $A$-module of rank $r$ defined over $A$ so that $\psi' = \gamma^{-1} \psi \gamma$. Observe that $\tilde{G}_n$ is the abelian $t$-module constructed from $\psi'$ and $C^\otimes$ as in §2.1. Thus, by (37) and [GN24 Thm. 1.3], $\det(\mathfrak{R}) \in K_\infty$ is either
zero or transcendental over $\overline{K}$. Since $\tilde{a}$ is non-zero and $L(\tilde{G}_n/A)$ is a 1-unit in $K_\infty$, we conclude that $L(\tilde{G}_n/A) \notin \overline{K}$. □

The following corollary, giving a positive answer to [ANDTR20b, Problem 4.1] in the case of the tensor product of Drinfeld $A$-modules of rank $r$ defined over $A$ and their $(r - 1)$-st exterior powers with Carlitz tensor powers, immediately follows from Theorem 3.5 and Theorem 3.6.

Corollary 4.3. Let $\phi$ be a Drinfeld $A$-module of rank $r$ defined over $A$ and $n$ be a nonnegative integer.

(i) The Taelman $L$-value $L(\phi \otimes C^{\otimes n}/A)$ of $\phi \otimes C^{\otimes n}$ is transcendental over $K$.

(ii) Assume that $r \geq 2$. The Taelman $L$-value $L(\wedge^{r-1}\phi \otimes C^{\otimes n}/A)$ of $\wedge^{r-1}\phi \otimes C^{\otimes n}$ is transcendental over $K$.

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