The algebraic theory of matrix of matrices polynomials

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Abstract In this paper, we introduce some important basic notions and operations of a matrix of matrices (for shorting, MMs) polynomials. We use these notions to motivate the definitions of the division of MMs polynomial and to illustrate their properties. Also, we study the generalized Bezout theorem and the Cayley-Hamilton theorem over MMs. The block companion MMs is established. Furthermore, this paper is concerned with the construction of the right and left solvents of an MMs polynomial from latent roots and vectors.

Keywords matrix of matrices · division polynomial · solvent · block companion

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1 Introduction

In the last few years, there has been a growing interest in the theory of matrices such as [1, 2, 10, 12, 25, 26, 30, 33], some sets of spaces of matrices have been discussed, including the structure of commutative matrices, group matrices, and other commutative algebras of simultaneously diagonalizable matrices. The block matrices and the Drazin (group) inverse of block matrices in many areas have been considered (c.f. [5, 6, 4, 3, 7, 11]). Especially, in the articles [4, 5, 8, 9], the contribution and the representations of the group inverse for the block matrices are studied. The definition of the set of matrices over commutative matrices and determinant are discussed in [19]. Recently, Kishka et al. (see [22–24]) studied the set of matrices over a ring of matrices and the resulting set is called a matrix of matrices. This study differs from previous studies in many properties and calculations. The proposed MMs differ from the block matrix in many properties. Also, in this study, we consider the determinant of MMs is a matrix.

Polynomials and polynomial matrices arise naturally as modeling tools in many areas of applied mathematics, sciences and engineering, especially
in systems theory and mathematical description of the dynamics of multivariable systems [17,18,21,28,29,31–33]. Many authors study the algebraic properties of matrix polynomials (see [13–15]).

Our first contribution in this work to benefit from the concept of MMs is to define MMs polynomials. Our second contribution concerns to define and study the division of MMs polynomial and to illustrate their properties. The third contribution in this paper is to show that right and left solvents can be constructed from latent roots.

The paper is organized as follows. In the next section, we give some definitions that we are going to use later. In section 3, we introduce a definition of MMs polynomial and give some important basic notions. Section 4 is devoted to showing the division of MMs polynomials and generalizations of Bezout Theorem and the relation between left and right solvents. In section 5, we present block companion MMs. In section 6, we give the construction of solvent. Finally, we draw some concluding remarks in section 7.

2 Preliminaries

We will use the following standard terminology and notations throughout this article:

Let $K$ be a field and $M_l(K)$ be the set of all $l \times l$ matrices defined on $K$, $I$ and $O$ stand for the identity matrix and the zero matrix in $M_l(K)$, respectively. We denote by $M_{m \times n}(M_l(K))$ the set of all $m \times n$ MMs over $M_l(K)$. The elements of $M_{m \times n}(M_l(K))$ are denoted by $A, B, C, \ldots, \text{etc}$. Two important special MMs are the identity MMs $I$ and the zero MMs $O$.

**Definition 2.1** [22] Let $A \in M_{m \times n}(M_l(K))$, then it can be written as a rectangular table of elements $A_{ij}; i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, as follows:

$$A = \begin{pmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \ldots & A_{mn} \end{pmatrix},$$

where $A_{ij} \in M_l(K)$.

**Definition 2.2** [20] The determinant of $n-$square $A = (A_{ij})$, where $A_{ij} \in M_l'(K)$, denoted by $\det(A)$ or $|A|$, is

$$\det(A) = \sum_\sigma (\text{sgn } \sigma) A_{1j_1}A_{2j_2}...A_{nj_n},$$

or

$$\det(A) = \sum_\sigma (\text{sgn } \sigma) A_{1\sigma(1)}A_{2\sigma(2)}...A_{n\sigma(n)}.$$

Observe that each term where $A_{1j_1}A_{2j_2}...A_{nj_n}$ contains exactly one entry from each row and each column of $A$, $\sigma = j_1j_2...j_n \in S_n$, where $S_n$ is the set of all permutations on $\{1, 2, \ldots, n\}$ (see [27,16]).
Definition 2.3 [14] Given $n \times n$ matrices $S_1, S_2, \ldots, S_m$ the block Vandermonde matrix is

\[
V(S_1, S_2, \ldots, S_m) \equiv \begin{pmatrix}
I & I & \cdots & I \\
S_1 & S_2 & \cdots & S_m \\
\vdots & \vdots & \ddots & \vdots \\
S_1^{m-1} & S_2^{m-1} & \cdots & S_m^{m-1}
\end{pmatrix}.
\]

3 The notion of an MMs polynomial

In this section, we define the MMs polynomial and introduce some important basic notions and operations of the MMs polynomial.

Definition 3.1 Let $A_i \in M_n (M_l (\mathbb{K}))$ and $X \in M_n (M_l (\mathbb{K}[x]))$, $i = 0, 1, 2, \ldots, m$. We say,

\[
F(X) = A_0X^m + A_1X^{m-1} + \cdots + A_m,
\]

is an MMs polynomial of degree $m$. We call $F(X)$ a monic MMs polynomial if $A_0 = I_n$.

Definition 3.2 The MMs polynomial $F(X)$ is called

(i) **Regular** if $\det A_0 \neq 0$,

(ii) **Comonic** if the leading coefficient $A_m = I_n$,

(iii) **Unimodular** if $\det F(X)$ is a nonzero constant matrix independent of $x$ (element of $X$).

If the coefficients $A_i$, $i = 0, 1, 2, \ldots, m$ are scalar MMs, $A_i = A_i I$, then (3.1) reduces to

\[
F(X) = A_0X^m + A_1X^{m-1} + \cdots + A_m I,
\]

where $A_i \in M_l (\mathbb{K})$, $i = 0, 1, 2, \ldots, m$. The same case if we replace $A_i = a_i \in \mathbb{K}$, i.e.,

\[
F(X) = a_0X^m + a_1X^{m-1} + \cdots + a_m I,
\]

If $X$ is a scalar MMs, $X = E I$, then (3.1) reduces to

\[
F(E) = A_0E^m + A_1E^{m-1} + \cdots + A_m I,
\]

where $E \in M_l (\mathbb{K}[x])$. Also, if $X$ is a scalar MMs, $X = \lambda I$, then (3.1) reduces to

\[
F(\lambda) = A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_m I,
\]

where $\lambda \in \mathbb{K}$, this is called a lambda-MMs.
Example 3.1 Let $F(X) = A_0X^m + A_1X^{m-1} + \ldots + A_m$, then we find
(i) If $\det A_0 \neq O$ and $A_0^{-1}$ exists, $A_0^{-1}F(X)$ is a monic MMs polynomial.
(ii) If $\det A_m \neq O$ and $A_m^{-1}$ exists, $A_m^{-1}F(X)$ is a comonic MMs polynomial.
(iii) Let $F(X) = I_2X^2 + \left( \begin{array}{cc} I & O \\ 2I & O \end{array} \right)X$, then $F(X)$ is a regular MMs polynomial.
(iv) Let $F(X) = I_2 + X$, where $X = \left( \begin{array}{cc} I & O \\ A & O \end{array} \right)$ and $A \in M_l(\mathbb{K}[x])$, then $F(X)$ is unimodular MMs polynomial.

Example 3.2 The MMs polynomial
$$F(E) = \left( \begin{array}{cc} I & -2I \\ -I & 2I \end{array} \right)E^2 + \left( \begin{array}{cc} I & I \\ 2I & O \end{array} \right)E + \left( \begin{array}{cc} O & O \\ 2I & O \end{array} \right),$$
of degree 2 and $E \in M_l(\mathbb{K}[x])$.

Definition 3.3 Let $F(X) = \sum_{k=0}^q A_kX^k$ and $G(X) = \sum_{k=0}^t B_kX^k$, then the sum of two MMs polynomial is given by
$$F(X) + G(X) = \begin{cases} \sum_{k=0}^t (A_k + B_k)X^k + \sum_{k=q+1}^t A_kX^k & q > t \\ \sum_{k=0}^q (A_k + B_k)X^k & q = t \\ \sum_{k=0}^q (A_k + B_k)X^k + \sum_{k=t+1}^t A_kX^k & t > q \end{cases},$$
where $\deg(F) = q$ and $\deg(G) = t$.

Remark 3.1 If $q = t$ and $F_q + G_q \neq O$, then the sum in (3.6) is an MMs polynomial of degree $q$, and if $F_q + G_q = O$, then this sum is an MMs polynomial of a degree less than $q$. Thus, we have
$$\deg(F(X) + G(X)) \leq \max[\deg(F(X)), \deg(G(X))].$$

The same way, we can define the difference of two MMs polynomial.

Definition 3.4 The product of an MMs polynomial $F(X)$ by matrix $E \in M_l(\mathbb{K})$.
$$EF(X) = F(EX).$$
From this definition for $E \neq O$, we have $\deg(EF(X)) = \deg(F(EX))$.
Remark 3.2 The above Definition 3.4 is true if we replace $E = \alpha$, where $\alpha \in \mathbb{K}$.

Definition 3.5 Multiplication of two MMs polynomials $F(X)$ and $G(X)$ in $\mathcal{M}_n \left( M_l \left( \mathbb{K} \left[ x \right] \right) \right)$, is given by

$$H(X) = F(X) \cdot G(X) = \sum_{k=0}^{q+t} C_k X^k,$$

where

$$C_k = \sum_{l=0}^{k} A_l B_{k-l}; \quad k = 0, 1, ..., q + t.$$

Remark 3.3 From (3.8) it follows that $C_{q+t} = A_q B_t$ and it is a nonzero one if at least one of the MMs $A_q$ and $B_t$ is non-singular. In other words, one of the MMs $F(X)$ and $G(X)$ is a regular one. Thus, we have the relationship

$$\{\begin{align*}
\deg (F(X) G(X)) &= \deg (F(X)) + \deg (G(X)) \text{ if at least one of these MMs is a regular} \\
\deg (F(X) G(X)) &\leq \deg (F(X)) + \deg (G(X)) \text{ otherwise.}
\end{align*}\}$$

Example 3.3 The product of the MMs polynomials

$$F(\lambda) = \left( \begin{array}{cc} I & -2I \\ I & 2I \end{array} \right) \lambda^2 + \left( \begin{array}{cc} I & I \\ I & 4I \end{array} \right) \lambda + \left( \begin{array}{cc} O & O \\ O & 2I \end{array} \right),$$

$$G(\lambda) = \left( \begin{array}{cc} I & I \\ I & 2I \end{array} \right) \lambda + \left( \begin{array}{cc} I & O \\ O & I \end{array} \right),$$

is the following MMs polynomial

$$F(\lambda) G(\lambda) = \left( \begin{array}{cc} -I & -3I \\ 3I & 5I \end{array} \right) \lambda^3 + \left( \begin{array}{cc} 3I & I \\ 3I & 4I \end{array} \right) \lambda^2 + \left( \begin{array}{cc} I & I \\ 4I & 4I \end{array} \right) \lambda + \left( \begin{array}{cc} O & O \\ O & 2I \end{array} \right),$$

whose degree equals the sum $\deg F(\lambda) + \deg G(\lambda)$.

Definition 3.6 An $n \times n$ MMs polynomial $F(X)$ is invertible if there exists an $n \times n$ MMs polynomial $G(X)$ such that

$$F(X) G(X) = G(X) F(X) = I.$$

As a matrix polynomial, an $n \times n$ MMs polynomial $F(X)$ is invertible if and only if it is unimodular.

Definition 3.7 An MMs $S$ is a right solvent of $F(X)$ if

$$F(S) = O.$$

We say MMs $W$ is a weak solvent of $F(X)$ if $F(W)$ is singular.
A special case of the weak solvent problem is the important \( \lambda \)-MMs problem (3.5). The \( \lambda \)-MMs problem is that of finding a scalar \( \lambda \) such that \( F(\lambda) \) is singular. Such a scalar is called a latent root of \( \lambda \)-MMs polynomial \( F(\lambda) \).

A vector of \( \lambda \)-MMs \( U \) and \( D \) are right and left latent vectors of \( \lambda \)-MMs, respectively, associated with \( \lambda \) satisfies \( F(\lambda)U = O_{m \times 1} \) and \( D^t F(\lambda) = O_{1 \times m} \).

Now, we define the block Vandermonde matrix.

**Definition 3.8** Let \( A_i \in M_n (M_l (K)) \); \( i = 1, 2, \ldots, m \), then the block Vandermonde \( \lambda \)-MMs is

\[
V (A_1, A_2, \ldots, A_m) \equiv \begin{pmatrix}
I & I & \ldots & I \\
A_1 & A_2 & \ldots & A_m \\
\vdots & \vdots & \ddots & \vdots \\
A_{m-1}^m & A_{m-1}^{m-1} & \ldots & A_{m-1}^1 
\end{pmatrix}.
\]

3.1 Construction of \( \lambda \)-MMs coefficients of an \( \lambda \)-MMs

For each right solvent \( S_i; \ i = 1, 2, \ldots, n \) of \( F(\lambda) \), we have:

\[
S_i^m + A_1 S_i^{m-1} + \ldots + A_{m-1} S_i + A_m = O_m,
\]

then

\[
A_1 S_i^{m-1} + \ldots + A_{m-1} S_i + A_m = -S_i^m,
\]

or for \( i = 1, 2, \ldots, m \).

\[
\begin{pmatrix}
A_m & A_{m-1} & \ldots & A_1 \\
S_i^m & \vdots & \vdots & \vdots \\
S_i^{m-1} & \vdots & \ddots & \vdots \\
S_i^1 & S_i^2 & \ldots & S_i^n
\end{pmatrix} = -S_i^m.
\]

Then

\[
\begin{pmatrix}
A_m & A_{m-1} & \ldots & A_1 \\
S_i^m & \vdots & \vdots & \vdots \\
S_i^{m-1} & \vdots & \ddots & \vdots \\
S_i^1 & S_i^2 & \ldots & S_i^n
\end{pmatrix} = -S_i^m
\]

where \( V_S \) is a right block Vandermonde \( \lambda \)-MMs

\[
V_S = \begin{pmatrix}
I_m & I_m & \ldots & I_m \\
S_1 & S_2 & \ldots & S_n \\
\vdots & \vdots & \ddots & \vdots \\
S_{m-1}^m & S_{m-1}^{m-1} & \ldots & S_{m-1}^1
\end{pmatrix}.
\]

For each left solvent \( R_i; \ i = 1, 2, \ldots, n \) of \( F(\lambda) \) we have:

\[
R_i^m + R_i^{m-1} A_1 + \ldots + R_i A_{m-1} + A_m = O_m,
\]
then the same development:
\[
\begin{pmatrix}
A_m \\
A_{m-1} \\
\vdots \\
A_1
\end{pmatrix}
= -V_R^{-1}
\begin{pmatrix}
R_1^m \\
R_2^m \\
\vdots \\
R_n^m
\end{pmatrix},
\]
where $V_R$ is a right block Vandermonde MMs:
\[
V_R = 
\begin{pmatrix}
I_m & R_1 & \cdots & R_1^{m-1} \\
I_m & R_2 & \cdots & R_2^{m-1} \\
\vdots & \vdots & \ddots & \vdots \\
I_m & R_n & \cdots & R_n^{m-1}
\end{pmatrix}.
\]

4 Division of MMs polynomials

In this section, we define and study the division of MMs polynomials and generalizations of Bezout theorem. Moreover, the relation between left and right solvents is established.

Let $F(X) = X^2 + A_1X + A_2$. It might seem natural to define the division of $F(X)$ by the linear MMs polynomial $X + C_1$ by

$$X^2 + A_1X + A_2 = (X + C_1)(X + C_2) + C_3.$$  

This cannot be done since the right side is not an MMs polynomial. We introduce generalized division for the class of MMs polynomials such that the class is closed under this operation. It reduces to the scalar division if $n = 1, l = 1$.

**Theorem 4.1** Let $F(X) = X^m + A_1X^{m-1} + \cdots + A_m$ and $G(X) = X^p + B_1X^{p-1} + \cdots + B_p$, with $p \leq m$. Then there exists a unique monic MMs polynomial $H(X)$ of degree $m - p$ and a unique MMs polynomial $L(X)$ of degree $p - 1$ such that

$$F(X) = H(X)X^p + B_1H(X)X^{p-1} + \cdots + B_pH(X) + L(X).$$ \hspace{1cm} (4.1)

**Proof.** Let $H(X) = X^{m-p} + V_1X^{m-p-1} + \cdots + V_{m-p}$ and $L(X) = N_0X^{p-1} + N_1X^{p-2} + \cdots + N_{p-1}$. Equating coefficients of equation (4.1), $V_1, V_2, \ldots, V_{m-p}$ and $N_0, N_1, \ldots, N_{p-1}$ can be successively and uniquely determined from the $m$ equations.

\[ \square \]

Equation (4.1) is the MMs polynomial division of $F(X)$ on the left by $G(X)$ with quotient $H(X)$ and remainder $L(X)$.  

77
Definition 4.2 Associated with the MMs polynomial,
\[ F(X) = X^m + A_1X^{m-1} + \ldots + A_m, \]
is the commuted MMs polynomial
\[ \hat{F}(X) = X^m + X^{m-1}A_1 + \ldots + A_m, \quad (4.2) \]
If \( \hat{F}(\mathcal{R}) = 0 \), then \( \mathcal{R} \) is a left solvent of \( F(X) \).

An important association between the remainder, \( L(X) \), and the dividend, \( F(X) \), in (4.1) will now be given. It generalizes the fact that for scalar polynomials the dividend and remainder are equal when evaluated at the roots of the divisor.

Corollary 4.3 If \( \mathcal{R} \) is a left solvent of \( G(X) \), then \( \hat{L}(\mathcal{R}) = \hat{F}(\mathcal{R}) \).

Proof. Let \( Q(X) = F(X) - L(X) \). Then it is easily shown that
\[ \hat{Q}(X) = X^{m-p}\hat{G}(X) + X^{m-p-1}\hat{G}(X)V_1 + \ldots + \hat{G}(X)V_{m-p}, \quad (4.3) \]
The result then follows immediately since \( \hat{Q}(X) = 0 \) for all left solvents of \( G(X) \).

The case where \( p = 1 \) in Theorem 4.1 is of special importance in this paper. Here we have \( G(X) = X - \mathcal{R} \) where \( \mathcal{R} \) is both a left and right solvent of \( G(X) \). Then Theorem 4.1, shows that
\[ F(X) = H(X)(X - \mathcal{R}) + L, \quad (4.4) \]
where \( L \) is a constant matrix. Now Corollary 4.3 shows that \( L = \hat{F}(\mathcal{R}) \), and thus
\[ F(X) = H(X)(X - \mathcal{R}) + \hat{F}(\mathcal{R}), \quad (4.5) \]
There is a corresponding theory for \( \hat{F}(X) \). In this case, (4.1) replaced by
\[ \hat{F}(X) = X^p\hat{H}(X) + X^{p-1}\hat{H}(X)B_1 + \ldots + \hat{H}(X)B_p + \hat{N}(X), \quad (4.6) \]
and Corollary 4.3 becomes the following.

Corollary 4.4 If \( \mathcal{S} \) is a right solvent of \( G(X) \), then \( \hat{N}(\mathcal{S}) = F(\mathcal{S}) \).

We again consider the case of \( p = 1 \). Let \( G(X) = X - \mathcal{S} \), then equation (4.5) becomes
\[ \hat{F}(X) = (X - \mathcal{S})\hat{H}(X) + \hat{F}(\mathcal{S}). \quad (4.7) \]
Restricting \( X \) to a scalar MMs \( E \) and noting \( F(ET) = \hat{F}(ET) \), we get Bezout’s theorem over MMs from equation (4.5) and (4.7)
\[ F(ET) = (IE - \mathcal{R})H(E) + \hat{F}(\mathcal{R}) = H(E)(IE - \mathcal{S}) + F(\mathcal{S}). \quad (4.8) \]
If \( R \) and \( S \) are left and right solvents respectively, of \( F(\lambda) \), then
\[
F(\lambda) = H(\lambda) \lambda - RH(\lambda), \tag{4.9}
\]
and
\[
\hat{F}(\lambda) = XH(\lambda) - \hat{H}(\lambda)S, \tag{4.10}
\]
and
\[
F(EX) = (IE - R)H(\lambda) = \hat{H}(\lambda)(IE - S).
\]
Thus, Corollaries 4.3 and 4.4 are generalizations of Bezout’s theorem.

**Remark 4.1** The above results are true if we replace \( E \) by \( \lambda \).

**Example 4.1** For the MMs
\[
F(x) = Ix^2 + \begin{pmatrix} I & I \\ 2I & 2I \end{pmatrix} x + \begin{pmatrix} I & -2I \\ -I & 2I \end{pmatrix},
\]
\[
G(x) = Ix + \begin{pmatrix} I & O \\ O & 2I \end{pmatrix},
\]
determine the MMs polynomials \( H(x) \) and \( L(x) \) satisfying the equality (4.1)
\[
H(x) = Ix + \begin{pmatrix} O & I \\ 2I & O \end{pmatrix}
\]
\[
L(x) = \begin{pmatrix} I & -3I \\ -5I & 2I \end{pmatrix}.
\]

A matrix polynomial exactly divides another matrix polynomial if all the roots of the divisor are roots of the dividend. A generalization of the matrix polynomial is given next.

**Corollary 4.5** If \( G(\lambda) \) has \( p \) left solvents \( R_1, \ldots, R_p \) which are also left solvents of \( F(\lambda) \) and if \( \mathcal{V}(R_1, \ldots, R_p) \) is non-singular then the remainder \( L(\lambda) = O \).

**Proof.** Corollary 3.3 shows that \( \hat{L}(R_i) = O, i = 1, \ldots, p \). Since \( \mathcal{V}(R_1, \ldots, R_p) \) is non-singular, and since
\[
\begin{pmatrix}
I & \mathcal{R}_1 & \ldots & \mathcal{R}_p \\
I & \mathcal{R}_2 & \ldots & \mathcal{R}_p \\
\vdots & \vdots & \ddots & \vdots \\
I & \mathcal{R}_p & \ldots & \mathcal{R}_p
\end{pmatrix}
\begin{pmatrix}
L_{p-1} \\
L_{p-2} \\
\vdots \\
L_0
\end{pmatrix}
= \begin{pmatrix}
\hat{L}(R_1) \\
\hat{L}(R_2) \\
\vdots \\
\hat{L}(R_p)
\end{pmatrix},
\]
it follows that \( L(\lambda) = O \). Thus
\[
F(\lambda) = H(\lambda) \lambda^p + B_1 H(\lambda) \lambda^{p-1} + \cdots + B_p H(\lambda).
\]
\(\square\)
From equation (4.9) it follows the eigenvalue of any solvent (left or right) of $F(X)$ are latent roots of $F(\lambda)$ (when replace $X$ by $\lambda$). These equations allow us to think of right or left solvents of $F(X)$ as right or left factors of $F(\lambda)$.

The next corollary gives the relation between left and right solvents.

**Corollary 4.6** If $R$ and $S$ are left and right solvents of $F(X)$, respectively, $S$ and $R$ have no common eigenvalue then $H(S) = O$, where $H(X)$ is the quotient of division of $F(X)$ on the left by $X - R$ (see equation (4.9)).

**Proof.** Since equation (4.9) explains

$$H(S)S - RH(S) = O, \quad (4.11)$$

for $S$ and $R$ have no common eigenvalues $H(S) = O$ uniquely.

Now, the relation between left and right solvents appear from equation (4.11) i.e.,

$$R = H(S)S H^{-1}(S). \quad (4.12)$$

### 5 Block companion MMs

In this section, we define block companion MMs and study the eigenvalues of a companion MMs which are the roots of its associated polynomial. Useful in the study of scalar polynomials is the companion matrix also the study of matrix polynomials is the block companion matrix. The eigenvalues of a companion matrix (block companion matrix) which are the roots of its associated polynomial (matrix polynomial). A generalization of this is given below.

**Definition 5.1** Let $F(X) \in M_n(M_l(\mathbb{K}[x]))$ an MMs polynomial

$$F(X) = \lambda^m + A_1\lambda^{m-1} + ... + A_m,$$

the block companion MMs with it is

$$C = \begin{pmatrix}
O & I & O & ... & O \\
\vdots & I & \ddots & \vdots & \\
O & O & \ldots & I \\
-A_m & -A_{m-1} & \ldots & -A_1
\end{pmatrix}. \quad (5.1)$$

The results are a direct generalization of the scalar (matrix) case which is easily verified. The relation between eigenvalue of the block companion MMs and roots of the associated lambda-MMs explain by the following theorem.

**Theorem 5.2** $\det(C - \lambda I) \equiv (-1)^m \det(\lambda^m + A_1\lambda^{m-1} + ... + A_m).$
Proof. From the definition of $C$.

The next corollary, it is easy to prove.

**Corollary 5.3** $\mathcal{F}(\lambda)$ has exactly $mn$ finite latent roots.

### 6 Construction of solvent

As a matrix polynomial, the fundamental theorem of algebra does not hold for MMs polynomials. A sufficient condition for the existence of a solvent is given by the following theorems:

**Theorem 6.1** If $\mathcal{F}(\lambda)$ has $n$ linearly independent right latent vectors $U_1, U_2, \ldots, U_n$ corresponding to latent roots $\lambda_i$, $i = 1, 2, \ldots, n$. Then $S = U \Lambda U^{-1}$ is a right solvent, where $\Lambda = \text{diag}(\lambda_i I)$ and $U = (U_1 \ U_2 \ \ldots \ U_n)$

Proof. Since

$$\mathcal{F}(S) = U \Lambda^m U^{-1} + A_1 U \Lambda^{m-1} U^{-1} + \ldots + A_{m-1} U \Lambda U^{-1} + A_m,$$

It can easily be verified that

$$U \Lambda^k U^{-1} = \sum_{i=1}^{n} \lambda^k U_i P_i,$$

where

$$U^{-1} = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix},$$

which gives

$$\mathcal{F}(S) = \sum_{i=1}^{n} \lambda^m U_i P_i + A_1 \sum_{i=1}^{n} \lambda^{m-1} U_i P_i + \ldots + A_{m-1} \sum_{i=1}^{n} \lambda U_i P_i + A_m,$$

Since $\lambda_i, U_i$ is a latent pair it follow that:

$S = O$. 

\qed
Example 6.1 Consider

\[ F(\lambda) = \begin{pmatrix} I & O \\ O & I \end{pmatrix} \lambda^3 + \begin{pmatrix} O & I \\ O & 5I \end{pmatrix} \lambda^2 + \begin{pmatrix} -I & 5I \\ O & 6I \end{pmatrix} \lambda + \begin{pmatrix} O & 4I \\ O & O \end{pmatrix}, \]

where \( I, O \in M_{10}(\mathbb{K}) \), the latent roots are 0, 1, –1, –2, –3 with multiplicity 10 except 0 has 20 multiplicity. The right latent vectors corresponding to these latent roots are respectively:

\[ \left\{ \begin{pmatrix} I \\ O \end{pmatrix}, \begin{pmatrix} I \\ O \end{pmatrix}, \begin{pmatrix} I \\ -3I \end{pmatrix}, \begin{pmatrix} I \\ -12I \end{pmatrix} \right\}, \]

the right solvent being an 2-MMs. Now, we form right solvents by pairing latent roots with corresponding linearly independent right latent vectors of matrices such as \{0, –2\}, \{0, –3\}, \{1, –2\}, ..., etc. The right solvent, say involving the \{0, –3\}

\[ S = \begin{pmatrix} I & I \\ O & -12I \end{pmatrix} \begin{pmatrix} O & O \\ O & -3I \end{pmatrix} \begin{pmatrix} I & I \\ O & -12I \end{pmatrix}^{-1}, \]

so

\[ S = \begin{pmatrix} O & \frac{1}{4}I \\ O & -\frac{3}{4}I \end{pmatrix}. \]

In a similar manner, we shall establish a left solvent. Let \( R \in M_n(M_l(\mathbb{K}[x])) \) can be constructed from a set \{\lambda_i : i = 1, 2, ..., m\} of m latent roots and a corresponding set of m linearly independent left latent vector of matrices \{Y_i : i = 1, 2, ..., m\}.

Theorem 6.2 The n-MMs \( R = DAD^{-1} \) is a left solvent of \( F(\lambda) \), where

\[ D = \begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix}. \]

7 Conclusion

In this work, we have introduced the definition of MMs polynomials. Also, the division of MMs polynomial and properties are studied. Furthermore, the Bezout theorem and the Cayley-Hamilton theorem over MMs are generalized. Moreover the block companion MMs was presented. Finally, solvents of a MMs polynomial can be constructed from latent roots and latent vectors.

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