AN ANALOGUE OF THE KAC-WEISFEILER
CONJECTURE

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Abstract. In this paper we discuss an analogue of the Kac-Weisfeiler conjecture for a certain class of almost commutative algebras. In particular, we prove the Kac-Weisfeiler type statement for rational Cherednik algebras.

1. INTRODUCTION

Throughout $k = \overline{k}$ we be an algebraically closed field of characteristic $p > 2$. Let $A$ be an affine $k$-algebra which if finite over its center $\mathbb{Z}(A)$. In this setting one is interested in studying simple modules, in particular their dimensions. By Schur’s lemma all simple $A$-modules are finite dimensional and they are parametrized by the corresponding characters of $\mathbb{Z}(A)$: we have a surjective map with finite fibers \{Irr\} $\rightarrow$ Spec $\mathbb{Z}(A)$, from the set of isomorphism classes of simple $A$-modules to the set of characters of $\chi \in \mathbb{Z}(A)$. For each character $\chi \in \text{Spec} \mathbb{Z}(A)$, we will denote by $A_\chi$ the algebra $A \otimes \mathbb{Z}(A) k$, where $k$ is viewed as a $\mathbb{Z}(A)$ module via $\chi$. We would like to study the largest power of $p$ that divides dimensions of all simple $A_\chi$-modules. Let us denote this number by $i(\chi)$. We would like to relate function $i : \text{Spec} \mathbb{Z}(A) \rightarrow \mathbb{Z}_+$ to geometry of Spec $\mathbb{Z}(A)$. More specifically, let $\cup S_i = \text{Spec} \mathbb{Z}(A)$ be the smooth stratification of Spec $\mathbb{Z}(A)$, and for $\chi \in \text{Spec} \mathbb{Z}(A)$ denote by $s(\chi)$ the dimension of the smooth stratum containing $\chi$.

Motivated by the Kac-Weisfeiler conjecture [KW], which is now a theorem of Premet [P], it is tempting to state the following.

Conjecture 1. Supposed that $A$ is a nonnegatively filtered $k$-algebra, such that $\text{gr} A$ is a finitely generated commutative domain over $k$. Assume that $(\text{gr} A)^p \subset \text{gr} \mathbb{Z}(A)$ and that Spec $\text{gr} A$ is a union of finitely many symplectic leaves and is a Cohen-Macaulay variety, then for any central character $\chi \in \text{Spec} \mathbb{Z}(A)$, we have $i(\chi) \geq \frac{1}{2} s(\chi)$.

Let us explain how does the above relate to the Kac-Weisfeiler conjecture.

Let $g$ be a Lie algebra of a semisimple simply-connected algebraic group $G$ over $k$. Then given that $p$ is large enough, we have Veldkamp’s theorem ([V]) describing the center of the enveloping algebra of $g$: $Z(\mathfrak{u}g) = Z_p(g) \otimes Z_{HC}$, where $Z_p$ is the $p$-center, generated by elements $g^p - g^{p^2}, g \in g$, and $Z_{HC} = \Lambda g^G$ is the Harish-Chandra part of the center. A character of $Z(\mathfrak{u}g)$ can be thought of as a pair $(\chi, \lambda), \chi \in g^*, \lambda : Z_{HC} \rightarrow k$. Put $A = \Lambda g/\text{ker}(\lambda)$. 
Then $A$ inherits the filtration from $\mathfrak{U}g$, and $\text{gr} A = k[\mathcal{N}]$, where $\mathcal{N}$ is the nilpotent cone of $g^*$. Thus, $\text{gr} A$ is a Cohen-Macaulay domain, and $\text{Spec gr} A$ is a union of finitely many symplectic leaves. Thus, assumptions of the conjecture are met. We have that $\chi \in \text{Spec} Z(A)$. Now it follows that $i(\chi) = \dim G\chi$. Thus Conjecture [1] in this case says that any simple $A$-module affording character $\chi$ has dimension divisible by $p^{\frac{1}{2}\dim G\chi}$, which is the statement of the Kac-Weisfeiler conjecture.

As a supporting evidence for the above conjecture, we will show it for $\chi$ in the smooth locus of $\text{Spec} Z(A)$ (Proposition 3.2). Also, we will show that for all but finite $\chi \in \text{Spec} Z(A)$ any irreducible representation affording $\chi$ has dimension divisible by $p$ (Corollary 3.2).

Following the approach of Premet and Skryabin [PS], we prove the Kac-Weisfeiler type statement (which is weaker than Conjecture [1]) for a large class of filtered algebras which includes rational Cherednik algebras (Corollary 4.1).

2. Codimensions of Poisson ideals

We start by recalling the definition of algebraic symplectic leaves.

**Definition 2.1.** Let $A$ be a Poisson algebra over $k$. Then a closed symplectic leaf of $\text{Spec} A$ is a closed subvariety defined by a prime Poisson ideal $I$ such that the Poisson variety $\text{Spec} A/I$ is a symplectic variety. A closed symplectic leaf of a Poisson variety $X$ is a closed subvariety $Z \subset X$ such that for any affine open subset $\text{Spec} A \subset X$, $Z \cap \text{Spec} A$ is a closed symplectic leaf of $\text{Spec} A$. Finally, an algebraic symplectic leaf of a Poisson variety $X$ is a closed symplectic leaf of an open subvariety of $X$.

We will recall the definition of Poisson orders by Brown-Gordon [BG0].

**Definition 2.2.** A Poisson order is a pair of an affine $k$-algebra $A$ and its central subalgebra $Z_0$, such that $A$ is finitely generated module over $Z_0$, and $Z_0$ is a Poisson algebra, $A$ is a Poisson $Z_0$-module such that for $a \in Z_0$, the Poisson bracket $\{a, \cdot\}$ is a derivation of $A$. In this case $A$ is called Poisson $Z_0$-order.

**Definition 2.3.** Let $A$ be a Poisson $Z_0$-order. A Poisson $A$-module is a left $A$-module $M$ equipped with a $k$-bilinear map $\{\cdot, \cdot\} : Z_0 \otimes M \to M$ such that

\[
\{\{a, a'\}, m\} = \{a, \{a', m\\}\} - \{a', \{a, m\}\},
\]

\[
\{a, bm\} = \{a, b\} m + b\{a, m\}
\]

for all $a, a' \in Z_0, b \in A, m \in M$.

Let $A$ be an associative $k$-algebra, and let $Z$ be its central subalgebra. Assume moreover that $A$ is a finitely generated $Z$-module. Let $L \subset \text{Der}_Z(A)$ be a $Z$-submodule which contains all inner derivations and is closed under the commutator bracket and taking to the $p$-th power. Thus $L$ is a restricted
Lie subalgebra of $\text{Der}_Z(A)$. Let $D_L(A)$ denote the following algebra. $D'_L(A)$ is generated by $L$ as an algebra over $A$ subject to the following relations:

\[ i_1 a - a i_1 = l(a), i_1 i_2 - i_2 i_1 = i_1 [i_1, i_2]; \]
\[ (i_1)^p = i_1^p, a \in A, l_1, l_2 \in L, \text{ad}_a - a = 0 \]

It is immediate that if $L$ is the set of all inner derivations, then $D'_L(A) = A$.

Given a Poisson $\mathbb{Z}_0$-order we will define algebra $D_{Z_0}'(A)$ as follows. Let $L \subset \text{Der}_k(A)$ be the restricted Lie subalgebra of $\text{Der}_k(A)$ generated by all $\{a, -\}, a \in \mathbb{Z}_0$ and all inner derivations. Then $D_{Z_0}'(A)$ denotes $D'_L(A)$.

We will be primarily interested in two-sided Poisson ideals of $A$, i.e. two sided ideals $I \subset A$, such that $\{z, a\} \in I$ for all $z \in \mathbb{Z}, a \in I$. Clearly given such an ideal $I$, both $I, A/I$ are $D_{Z_0}'(A)$-modules. The following is very standard

**Lemma 2.1.** Let $A$ be a Poisson $\mathbb{Z}_0$-order. Then $A$ embeds in $D_{Z_0}'(A)$ and $D_{Z_0}'(A)$ is finite over its central subalgebra $(\mathbb{Z}_0)^p$.

**Proof.** Let $J$ be the kernel of the map $A \rightarrow D_{Z_0}'(A)$. Since $A$ is naturally a $D_{Z_0}'(A)$-module, we have that $JA = 0$, so $J = 0$. It is immediate that $(\mathbb{Z}_0)^p$ is central in $D_{Z_0}'(A)$, and since $\text{Der}_k(A)$ is a finite $(\mathbb{Z}_0)^p$-module, we get that $D_{Z_0}'(A)$ is finite over $(\mathbb{Z}_0)^p$.

Let $X$ be a variety over $k$. Recall the definition of the quasi-coherent sheaf of algebras of the crystalline differential operators $D(X)$. Locally it is generated by $\mathcal{O}_X$ and vector fields $TX$ subject to the condition

\[ [\xi, f] = \xi(f), \xi_1 \cdot \xi_2 - \xi_2 \cdot \xi_1 = [\xi_1, \xi_2], \]

where $f, \xi_1$ and $\xi_2$ are local sections of $\mathcal{O}_X, TX$ respectively.

Given an affine Poisson $k$-algebra $S$. We will denote $D'_S(S)$ simply by $D'(S)$. Also, given a Poisson ideal $I \subset S$ we will denote by $D'(S, I)$ the quotient of $D'(S)$ by the two sided ideal generated by the image of $I$ in $D'(S)$. Finally, for any closed point $y \in \text{Spec} S/I$, $D'(S, I)_y$ will denote the quotient of $D'(S, I)$ by the two-sided ideal generated by the image of $m^p_y$ (where $m_y$ denotes the maximal ideal corresponding to $y$).

Clearly, $D'(S, I)_y$ is a finite dimensional algebra over $k$.

We will use the following result of Bezrukavnikov-Mirkovic-Rumynin [BMR]

**Theorem 2.1** ([BMR] Theorem 2.2.3). Let $X$ be a smooth variety over $k$, then the ring of crystalline differential operators $D(X)$ is an Azumaya sheaf of algebras over $T^*X^{(1)}$ (the Frobenius twist of the cotangent bundle of $X$).

**Corollary 2.1.** Let $B$ is a finitely generated Poisson domain over $k$, such that $\text{Spec} B$ is a symplectic variety, then $D'(B)_y$ is isomorphic to the matrix algebra of dimension $p^{2 \dim B}$ over $k$. 

Proof. Since Spec $B$ is a symplectic variety, it follows that $D'(B)_y$ is isomorphic to the stalk of $D(X)$ at $(y,0) \in T^*X^{(1)}$. Therefore, by Theorem 2.1 we are done.

Recall that for any closed point $y$ of a Poisson variety $X$, we denote by $d(y)$ the dimension of the symplectic leaf of $X$ which contains $y$. The following will be crucial.

**Theorem 2.2.** Let $A$ be a Poisson $S$-order, and let $y \in \text{Spec} S$ be a closed point. Then any finite dimensional Poisson $A$-module which as an $S$-module is supported on $y$ has dimension divisible by $p^{d(y)}$.

**Proof.** Let us put $d(y) = d$. For a Poisson algebra $S$, we will denote by $\text{Der}'(S) \subset \text{Der}(S)$ the $k$-span of all derivations of the form $a\{b,-\}, a, b \in S$. Let $Y \subset \text{Spec} S$ be the symplectic leaf containing $y$. Let $I$ be a Poisson ideal corresponding to $Y$. Let $f \in S$ be an element which does not vanish on $y$ and vanishes on $\overline{Y} - Y$. Let us put $S' = S_f$, then $D'(S', I)_y = D'(S, I)_y$ and Spec$S'$ contains the closed symplectic leaf through $y$. Any Poisson $S$-module $M$ supported on $y$ is $D(S', I)_y$-module. Consider a descending filtration $M \supset IM \supset I^2M \supset \ldots \supset 0$. Since $I \subset m_y$, we have $I^lM = 0$ for some $l$. Each quotient $I^lM/I^{l+1}M$ is a module over $D(S', I)_y$. Therefore, it suffices to prove that any finite dimensional $D(S', I)_y$-module has dimension divisible by $p^{d(y)}$.

Denote $S'/I$ by $B$. Thus, Spec $B$ is a symplectic variety. We have a natural projection $j : D'(S', I)_y \to D'(B)_y$. Since the localization of $B$ at $m_y$ is a regular local ring with the residue field $k$, we have that $B/m_y^p = k[x_1, \ldots, x_d]/m^p$, where $m_y = (x_1, \ldots, x_d)$. Remark that $B/(m_y)^p$ is the image of $S'$ in $D'(S, I)_y$. We will denote the images of $x_1, \ldots, x_d$ in $D'(S, I)$ again by $x_1, \ldots, x_d$. Let $y_1, \ldots, y_d \in \text{Der}'(B/m_y^p)$ be such that

$$[y_i, x_j] = \delta_{ij}; [y_i, y_j] = 0.$$ 

Thus, $D'(B)_y$ is generated by $\sum_{a_i, \ldots, a_d} y_1^{a_1} \cdots y_d^{a_d}$ as a free left (or right) $B/m_y^p$-module. Let $\xi_1, \ldots, \xi_d$ be any lifts of $y_1, \ldots, y_d$ in the image of $\text{Der}'(S')$ in $D'(S, I)_y$. Notice that $[\xi_i, x_j] = \delta_{ij}$. Let us denote by $J$ a $k$-subalgebra of $D'(S, I)_y$ generated by elements of $\text{Der}'(S)$ whose images are in $I$. Then $J$ is an ideal of the Lie algebra $D'(S, I)_y$ and $[B/m_y^p, J] = 0$. Denote by $N \subset D'(S, I)_y^p$ the $k$-span of elements of the form $\xi_1^{a_1} \cdots \xi_d^{a_d}, a_i < p$, so $\dim N = p^d$. For any $\xi \in \text{Der}'(S)$, its image in $D'(B)_y$ can be written as $\sum_{i=1}^d b_iy_i$ for some $b_i \in B/m_y^p$. Thus $\xi \equiv -\sum b_i\xi_i \in J$, so $\text{Der}'(S) \subset N(J(B/m_y^p))$. Therefore, $D'(S, I)_y^p y = N(J(B/m_y^p))$.

Let $V$ be a simple $D'(S, I)_y^p$-module. Let $v \neq 0, v \in V$ be such that $m_y v = 0$. Then $m_yJV = 0$, so $NJv = V$. Now we claim that if $v_1, \ldots, v_k \in JV$ are linearly independent and $\sum_{i=1}^k n_i v_i = 0, n_i \in N$, then $n_i = 0$ for all $i$. Indeed, we have that $\sum_i [x_j, n_i] v_i = \sum_i \frac{\partial n_i}{\partial x_j} v_i = 0$. Proceeding by
induction on the total degrees of $n_i$ in $\xi_1, \ldots, \xi_n$, we are done. Therefore, $\dim V = \dim N \dim (Jv)$ is a multiple of $p^d$.

\[ \square \]

**Proposition 2.1.** Let $M$ be a finite dimensional Poisson module over a Poisson $S$-order. Then $\dim M$ is divisible by $\inf \{ p^d(y), y \in \text{Supp}(M) \subset \text{Spec} \, S \}$.

**Proof.** Let us write $M = \bigoplus_{y \in \text{Supp}(M)} M_y$, where $M_y$ is the submodule of elements of $M$ supported on $y$. Observe that $M_y$ is actually a Poisson submodule of $M$. Indeed, let $a \in M_y$, then $(m_y)^p a = 0$, for some $i$. Therefore, for any $a \in S$, $(m_y)^p \{ a, m \} = 0$, since $\{ a, m \} = 0$. Now applying Theorem 2.2 to each $M_y$, we are done. \[ \square \]

Recall the following standard definition.

**Definition 2.4.** A quantization of a Poisson $S$-order $A$ is an $h$-complete flat associative $k[[h]]$-algebra $A'$, such that $A = A'/hA'$ and Poisson bracket of $S$ is induced from the commutator bracket of $S'$.

**Proposition 2.2.** Let $A'$ be a quantization of a Poisson $S$-order $A$. Let $J \subset A'[h^{-1}]$ be a two-sided ideal of finite codimension over $k((h))$, then $\dim_{k((h))} A'[h^{-1}]/J$ is a multiple of $\inf \{ p^d(y), y \in \text{supp} A / ((A' \cap J)/h) \}$.

**Proof.** Let us put $J' = J \cap A'$, then $A'/J'$ is a free $k[[h]]$-module and $A'[h^{-1}]/J = A'/J' \otimes_{k[[h]]} k((h))$. Therefore $\dim_{k((h))} A'[h^{-1}]/J = \dim_{k[[h]]} A'/J'$. Let us put $J'/h = N \subset A$. Then $N$ is a Poisson submodule of $A$, and $A'/J'$ is a quantization of $A/N$. Therefore, $\dim_{k[[h]]} A'/J' = \dim_k A/N$. Thus, applying Corollary 2.1 to $A/N$ we are done. \[ \square \]

## 3. Estimates for filtered algebras

Let $A$ be an associative $k$-algebra equipped with a positive filtration by $k$-subspaces $A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots$, $A_n A_m \subset A_{n+m}, A = \cup A_n$. Recall that the center of the associated graded algebra $\text{gr} A = \oplus A_n/\text{gr} A_n$ becomes equipped with the natural Poisson bracket and $\text{gr} A$ is a Poisson module over it. Indeed, let $d$ be the largest integer such that $[a, b] \subset A_{n+m-d}$ for any $b \in A_m, a \in A_n, \text{gr} a \in Z(\text{gr} A)$. Then one puts $\{ \text{gr} a, \text{gr} b \} = [a, b]/A_{n+m-d-1} \in (\text{gr} A)_{n+m-d-1}$.

In this section we apply results from the previous section to representations of certain class of filtered affine $k$-algebras.

Recall for a filtered algebra $A$ the construction of the Rees algebra $R(A) = \oplus_n A_n h^n \subset A[h]$, where $h$ is an indeterminate. Then $R(A)/hR(A) = \text{gr} A, R(A)/(h - \lambda)R(A) = A, R(A)[h^{-1}] = A[h, h^{-1}]$, for any $\lambda \in k, \lambda \neq 0$. Since $Z(A)$ inherits filtration from $A$, we have $R(Z(A)) = Z(R(A))$. We will fix the embedding $A \subset R(A)[h^{-1}] = A[h, h^{-1}]$. 


More generally, if $Z_0 \subset Z(A)$ is a subalgebra such that $\text{gr} \, A$ is finite over $\text{gr} \, Z_0$, then $R(A)$ is finite over $R(Z_0)$.

We will need the following standard fact.

**Proposition 3.1.** Let $M$ be a nonegatively filtered module over a nonegatively filtered commutative $k$-algebra $B$, such that $\text{gr} \, M$ is a finitely generated Cohen-Macaulay module over $\text{gr} \, B$ and $\text{gr} \, B$ is a finitely generated algebra over $k$. Then both $M, R(M)$ are Cohen-Macaulay modules over $\text{gr} \, B, R(B)$ respectively.

We will recall the proof for the convenience of the reader.

**Proof.** Let us choose algebraically independent homogeneous elements $x_1, \ldots, x_n \in \text{gr} \, B$, such that $\text{gr} \, B$ is finite over $k[x_1, \ldots, x_n]$. Then $\text{gr} \, M$ is a finitely generated $k[x_1, \ldots, x_n]$-module, and since it is a Cohen-Macaulay module it is a projective (by Lemma 2.2, 2.4 [BBG]), and hence by the Quillen-Suslin theorem a free $k[x_1, \ldots, x_n]$-module. Let $y_1 \cdots, y_m \in \text{gr} \, M$ be a homogeneous basis of $\text{gr} \, M$ over $k[x_1, \ldots, x_n]$. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be the any lifts of $x_1, \ldots, x_n, y_1, \ldots, y_m$ in $B, M$ respectively. Then it follows immediately that $a_1, \ldots, a_n$ are algebraically independent and $M, R(M)$ is a free $B, R(B)$-module with basis $b_1, \ldots, b_n$; $b_i h^{\deg b_i}$ respectively.

□

**Proposition 3.2.** Suppose that $A$ is a nonegatively filtered $k$-algebra, such that $\text{gr} \, A$ an affine commutative Cohen-Macaulay domain. Suppose also that $\text{Spec} \, \text{gr} \, A$ consists of finitely many symplectic leaves. If $(\text{gr} \, A)^p \subset \text{gr} \, Z(A)$ then for any character $\chi$ which belongs to the smooth locus of $\text{Spec} \, Z(A)$, a simple $A$-module affording $\chi$ has dimension $p^{\frac{1}{2} \dim A}$.

**Proof.** Applying [I] Lemma 2.4, we get that $\text{gr} \, Z(A) = (\text{gr} \, A)^p$. By [I] Theorem 2.3, the complement of the Azumaya locus of $Z(A)$ has codimension 2, and the largest possible dimension of $A$-module is $p^{\frac{1}{2} \dim A}$. Let $U \subset \text{Spec} \, Z(A)$ be the smooth locus of $\text{Spec} \, Z(A)$. We claim that $A|_U$ is a locally free sheaf over $U$. Indeed, by the assumption $\text{gr} \, A$ is a Cohen-Macaulay module over $(\text{gr} \, A)^p$. But then by [3.1] A is a Cohen-Macaulay module over $Z(A)$, since $\text{gr} \, Z(A) = (\text{gr} \, A)^p$. So $A|_U$ is a Cohen-Macaulay over $U$, and since $U$ is nonsingular, we get that $A|_U$ is locally free over $U$. To summarize, $A|_U$ is locally free and its Azumaya locus has complement of codimension $\geq 2$. Therefore, $A|_U$ is an Azumaya algebra over $U$ by [BG] Lemma 3.6.

□

From now on in this section we will follow very closely Premet and Skryabin [PS]. We will use the following result from [PS].

**Lemma 3.1** [PS Lemma 2.3]. Let $A$ be finite and projective over its central affine subalgebra $Z_0$, and let $L \subset \text{Der}_{Z_0}(A)$ be a restricted Lie subalgebra of derivations. Then for any $i \geq 0$, the set of characters $\chi \in \text{Spec} \, Z_0$, such that $A_\chi$ has an $L$-stable two sided ideal of codimension $i$ is closed.
Of course, $L$-stable two-sided ideal of $A$ is the same as a $D'_L(A)$-submodule of $A$.

From now on in this section, by $L \subset \text{Der}(R(\mathfrak{a}))(R(A))$ we will always denote the restricted Lie algebra generated by all inner derivations and by $h^{-d} \text{ad}(a)$, for all $a \in R(A)$, such that $a/h \in Z(\text{gr} A)$. In this setting, we have the following

**Lemma 3.2.** $D'_L(R(A))/h = D'(\text{gr} A)$ and $D'_L(R(A))[h^{-1}] = A[h, h^{-1}]$.

**Proof.** First equality is immediate. The second equality follows from the fact that $L$ consists of inner derivations in $\text{Der}(A[h, h^{-1}])$.

We will use the following

**Assumption 1.** Let $A$ be a positively filtered $k$-algebra, such that $Z(\text{gr} A)$ is finitely generated over $k$ and $\text{gr} A$ is a finitely generated module over $Z(\text{gr} A)$. Moreover, assume that there in a positive integer $n$, such that $(Z(\text{gr} A))^n \subset \text{gr} Z(A)$.

In what follows, we will use the following standard notations. For a subset $W$ of an affine variety $X$, we will denote by $I(W)$ the reduced ideal of zeros of $W$ in $\mathcal{O}(X)$, and for an ideal $I \subset \mathcal{O}(X)$, $V(I)$ will denote the set of zeros of $I$ in $X$.

We have the following

**Theorem 3.1.** Let algebra $A$ satisfy the assumption \[.\] Let $Z_0 \subset Z(A)$ be a subalgebra such that $R(A)$ is finitely generated projective module over $R(\mathfrak{a})$.

Let $W \subset \text{Spec} Z_0$ be a closed subset consisting of points $\chi \in W$ such that $A_\chi$ has a two sided ideal of codimension $p^j$ where $p$ does not divide $j$, then for any $y' \in V(\text{gr} I(W)) \subset \text{Spec} \text{gr} Z_0$, there is $y \in \text{Spec} Z(\text{gr} A)$ which is in the preimage of $y'$ in $\text{Spec} Z(\text{gr} A) \rightarrow \text{Spec} \text{gr} Z_0$, such that $d(y) \leq i$.

**Proof.** Recall that $R(z_0)[h^{-1}] = Z_0[h, h^{-1}]$. We again denote by $L$ the Lie subalgebra of $\text{Der}(R(A))$ as in above lemma. $\text{Spec} Z_0 \times (A^1 - 0) \hookrightarrow \text{Spec} R(\mathfrak{a})$.

We have a map $\rho : Z(\text{gr} A) \rightarrow HH^1(R(A))$ defined as $\rho(a) = \frac{1}{h^{a'}} \text{ad}(a')$, where $a' \in p^{-1}(a), a \in Z(\text{gr} A)$. and $p : R(A) \rightarrow R(A)/h = \text{gr} A$ is the quotient map. Let $X$ denote the set of all closed points $\chi \in \text{Spec} R(\mathfrak{a})$ such that $R(A)_\chi$ has a two sided $L$-stable ideal of codimension $p^j$. By lemma \[3.1\] $X$ is closed. We claim that $X \cap \{h = 0\}$ is the set of all characters $\chi : \text{gr} Z_0 \rightarrow k$ such that there is a Poisson ideal in $\text{gr} A$ containing $\text{ker}(\chi)$ of codimension $p^j$. Indeed, for two-sided ideal $I \subset \text{Gr}$ to be closed under the map $\{a,\}$ for all $a \in Z(\text{gr} A)$ is the same as ideal $p^{-1}(I)$ being $L$-stable. Similarly, $X \cap \{h \neq 0\}$ consists of characters $\chi : Z_0(A)[h, h^{-1}] \rightarrow k$ such that $A_{\chi}$ has a two-sided ideal of codimension $p^j$. Indeed, given a two sided ideal $I \subset A$ which contains $\text{ker}(\chi) \cap Z_0(A)$ ideal $p^{-1}(I)$ is $L$-stable, since $h$ is invertible on $R(A)/p^{-1}(I)$, where $p : R(A) \rightarrow R(A)/(h - \chi(h)) = A$ is the quotient map.
Therefore, $W \times (A^1 - 0) \subset X$. In particular, $W \times (A^1 - 0) \cap (h = 0) \subset X$. But $W \times (A^1 - 0) \cap (h = 0)$ is precisely $V(\text{gr } I(W))$. Thus for any character $\chi \in V(\text{gr } I(W))$ Algebra $(\text{gr } A)/m_\chi(\text{gr } A)$ has a Poisson ideal of codimension $p^i j$, where $m_\chi$ is the maximal ideal of $(\text{gr } A)^p$ corresponding to $\chi$. Let us choose such an ideal $J \subset (\text{gr } A)/m_\chi^p$. Let $\{y_i, i \in I\}$ be the (finite) preimage of $\chi$ under the map $\text{Spec } \text{gr } A \to \text{Spec } \text{gr } Z_0$. Then $\text{gr } A/J$ is a Poisson $\text{Z}(\text{gr } A)$-module supported on $\{y_i\}$. Applying Corollary 2.1 we are done.

\[ \square \]

Recall that for a character $\chi$, $i(\chi)$ denotes the largest power of $p$ which divides all dimensions of irreducible representations affording $\chi$.

\textbf{Corollary 3.1.} Let $\text{Spec } \text{gr } A$ satisfy the assumption 1. Let $\text{Z}_0$ be a subalgebra of $\text{Z}(\text{gr } A)$ such that $R(\text{gr } A)$ is a finitely generated projective module over $R(\text{Z}_0)$. Let $G \subset \text{Aut}(R(\text{gr } A), R(\text{Z}_0))$ be a subgroup of the group of $k[\hbar]$-algebra automorphisms of $R(\text{gr } A)$ which preserve $R(\text{Z}_0)$. Then, for any $\chi \in \text{Spec } \text{gr } Z_0$, and any $y \in V(\text{gr } I(\text{gr } A)) \subset \text{gr } Z_0$, we have $i(\chi) \geq \inf d(y')$, where $y' \in \text{Spec } \text{Z}(\text{gr } A)$ runs through preimages of $y$ under the map $\text{Spec } \text{Z}(\text{gr } A) \to \text{Spec } \text{gr } Z_0$.

\textbf{Proof.} Let $W \subset \text{Spec } Z_0$ be a set of characters $\chi'$ such that $i(\chi') = i(\chi)$. So $\chi \in W$ and $W$ is closed by Lemma 3.1. Therefore $G\chi \subset W$, and by Corollary 2.1 we are done.

\[ \square \]

\textbf{Corollary 3.2.} Let $\text{Spec } Z(\text{gr } A)$ be a union of algebraic symplectic leaves, and if $\text{Spec } Z(\text{gr } A)$ has at most one (the origin) zero dimensional symplectic leaf, then all but finitely many irreducible representations of $A$ have dimension divisible by $p$.

\textbf{Proof.} Recall that given a finite dimensional $k$-algebra $S$, all simple $S$-modules have dimensions divisible by $p$ if and only if $1 \in [S, S]$. Therefore, we are required to show that the support of $1 \in A/[A, A]$ on $\text{Spec } Z(A)$ is a finite set. Let $Y$ be the support of $1 \in D'(\text{gr } A)/[D'(\text{gr } A), D'(\text{gr } A)]$ on $\text{Spec } Z(D'(\text{gr } A))$. We have that $Y \cap \{h \neq 0\} = Y \cap (A - \{0\})$, where $X$ is the support of $1 \in A/[A, A]$ on $\text{Spec } Z(A)$. On the other hand, $Y \cap \{h = 0\}$ is the support of $1 \in D'(\text{gr } A)/[D'(\text{gr } A), D'(\text{gr } A)]$ in $\text{Spec } Z(D'(\text{gr } A))$. By the following lemma, the letter set is finite, so $X$ has to be finite and we are done.

\textbf{Lemma 3.3.} Under the assumption of the theorem, $1 \in D'(\text{gr } A)/[D'(\text{gr } A), D'(\text{gr } A)]$ is supported on the origin of $\text{Spec } Z(\text{gr } A)^p$.

\textbf{Proof.} Indeed, if $y \in \text{Spec } Z(\text{gr } A)^p$, $y \neq 0$ (where 0 denotes the origin of $\text{Spec } Z(\text{gr } A)$), then all representations of $D'(\text{gr } A)/m_y^p$ are divisible by $p^{d(y)}$ (as in the proof of Theorem 2.2). But since $d(y) > 0$, we conclude that all simple $D'(\text{gr } A)/m_y^p$-modules have dimensions divisible by $p$. Therefore, $1 \in D'(\text{gr } A)/[D'(\text{gr } A), D'(\text{gr } A)] + m_y^p D'(\text{gr } A)$ so $y \not\in \text{Supp}(1)$.

\[ \square \]
In particular, the assumptions of the above are satisfied when \( \text{Spec} \mathbb{Z} (\text{gr} \ A) \) consists of finitely many symplectic leaves. This result can be thought of as a positive characteristic analogue of a result due to Etingof-Schedler [ES], which states that if \( A \) is a nonnegatively filtered algebra over \( \mathbb{C} \) such that \( \text{gr} \ A \) is finite over its center and \( \text{Spec} \mathbb{Z} (\text{gr} \ A) \) is a union of finitely many symplectic leaves, then \( A \) has a finitely many nonisomorphic finite dimensional simple modules.

4. Applications to rational Cherednik algebras

At first, we will recall the decomposition of \( V/W \) into symplectic leaves [[BG1], Proposition 7.4], where \( V \) is a symplectic vector space over \( k \) and \( W \subset \text{Sp}(V) \) is a finite group such that \( p \) does not divide \( |W| \). Given a subgroup \( H \subset W \) denote by \( V^0 \) the set of all \( v \in V \) such that the stabilizer of \( v \) is \( H \). Then we have the decomposition of \( V/W \) into symplectic leaves \( V/W = \bigcup V^0/W \), where \( H \) ranges through all conjugacy classes of subgroups of \( W \). In particular, for \( v \in V/W \) dimension of the symplectic leaf through \( v \) is \( \dim V^0/W \), where \( V^0 \) is (a conjugacy class) the stabilizer of \( v \).

Recall that for a class function \( c : W \to k \), Etingof-Ginzburg defined a symplectic reflection algebra \( H_c(W, V) \). This algebra comes equipped with the natural filtration such that \( \text{gr} H_c(W, V) = k[W] \rtimes \text{Sym} V \), (the PBW property, [EG] Theorem 1.3).

**Proposition 4.1.** Let \( W \subset \text{Sp}(V) \) be a finite group. If \( p \) does not divide \( |W| \), then for a symplectic reflection algebra \( H_c(W, V) \), there are only finitely many irreducible representations whose dimension is not divisible by \( p \).

**Proof.** The center of \( \text{gr} H_{1,c} = k[W] \rtimes \text{Sym} V \) is \( (\text{Sym} V)^\Gamma \), so \( \text{gr} H_c(W, V) \) is finite over its center. Also, by a theorem of Etingof ([BGP], Theorem 9.1.1) \( \text{gr} \mathbb{Z}(H_{1,c}) = ((\text{Sym} V)^\Gamma)^p \). Since as explained above, \( \text{Spec}(\text{Sym} V)^\Gamma = V/W \) is a union of finitely many symplectic leaves, all the assumptions of 3.2 are satisfied and we are done.

Important class of symplectic reflection algebras consists of Rational Cherednik algebras. Let us recall their definition.

Let \( \mathfrak{h} \) be a finite dimensional vector space over \( k \). Given a finite group \( W \subset GL(\mathfrak{h}) \), let \( S \subset \Gamma \) be the set of pseudo-reflections (recall that \( s \in W \) is a pseudo-reflection if \( \text{Im}(\text{Id} - s) \) 1-dimensional). Let \( \alpha_s \in \mathfrak{h}^* \) be the generator of \( \text{Im}(\text{Id} - s)_{|\mathfrak{h}^*} \) and \( \alpha_s' \in \mathfrak{h} \) be the generator of \( \text{Im}(\text{Id} - s) \) such that \( (\alpha_s, \alpha_s') = 2 \). Let \( c : S \to k \) be a \( W \)-invariant function. The rational Cherednik algebra, \( H_c(W, \mathfrak{h}) \), as introduced by Etingof and Ginzburg [EG], is the quotient of the skew group algebra of the tensor algebra, \( k[W] \rtimes T(\mathfrak{h} \oplus \mathfrak{h}^*) \), by the ideal generated by the relations

\[
[x, x'] = 0, [y, y'] = 0, [y, x] = (y, x) - \sum c_s(y, \alpha_s)(x, \alpha_s')s,
\]
\(x, x' \in h, y, y' \in h^*\).

There is a standard filtration on \(H_c\) given by setting \(\deg x = 1, \deg y = 1, \deg(g) = 0\) for all \(x \in V, y \in h^*, g \in \Gamma\). The PBW property of \(H_c(W, h)\) says that \(\text{gr } H_c = k\Gamma \rtimes \text{Sym}(h^* \oplus h^*)\). When \(c\) is identically \(0\) then \(H_0(W, h) = k \rtimes D(h)\), where \(D(h)\) is the ring of (crystalline) differential operators on \(h\).

Algebra \(H_c(W, h)\) has a distinguished central subalgebra \(Z_0 = Z' \otimes Z''\), where \(Z'\) (respectively \(Z''\)) denotes \(((\text{Sym } h^*)^W)^p((\text{Sym } h)^W)^p \) \cite[Proposition 4.2]{BR}.

For a rational Cherednik algebra \(H_c(W, h)\) we have the spherical subalgebra \(B_c = eH_c(W, h)e \ (e = \frac{1}{|W|} \sum g \in W)\). Algebra \(B_c\) inherits a filtration from \(H_c(W, h)\) and \(\text{gr } B_c = (\text{Sym}(h^* \oplus h^*))^W\). Then \(cZ_0 \subset Z(B_c)\).

**Corollary 4.1.** Assume that \(p\) does not divide \(|W|\). Let \(\chi : eZ' \to k\) be a central character. Let us identify \(\chi : e((\text{Sym } h^*)^W)^p \to k\) with a point in \(h/W\). Let \(W_{\chi}\) be the stabilizer in \(W\) of a preimage of \(\chi\) in \(h\). Then all irreducible representations of \(B_c\) affording \(\chi\) have dimensions divisible by \(p|h|_W\).

**Proof.** We need to prove that given a character \(\mu : eZ_0 = eZ' \otimes eZ'' \to k\), such that \(\mu| : eZ' = \chi\), then any \(B_c\)-module affording \(\mu\) has dimension divisible by \(p|h|_W\). Let us write \(\mu = (\chi, \chi'), \chi' \in \text{Spec } eZ''\).

Since, \(\text{Sym}(h^* \oplus h^*)^W = \text{gr } B_c\) is a Cohen-Macaulay algebra, it is a Cohen-Macaulay module over \(\text{gr } eZ_0\), hence by Proposition \[5.1\] \(R(B_{1,c})\) is a Cohen-Macaulay module over \(R(eZ_0)\). However, \(R(eZ_0)\) is a polynomial algebra, therefore \(R(B_{1,c})\) is a projective (actually free by the Quillen-Suslin theorem) \(R(eZ_0)\)-module. \(eZ_0\) is a polynomial algebra, it follows that \(B_c\) is projective (actually free) over \(eZ_0\). We have an action of \(G_m\) on \(H_c(W, h)\) which preserves \(B_c\) corresponding to the grading with \(\deg(x) = 1, \deg(y) = -1, \deg(g) = 0, g \in \Gamma = 0, x \in h, y \in h^*\). This action preserves \(eZ_0\). Therefore we may apply \[5.1\] We need to understand \(V(\text{gr } I(G_{m,\mu})) \subset \text{Spec } e\text{gr } Z_0\) and its preimage in \(\text{Spec } \text{gr } B_c\).

We have \(I(G_{m,\mu}) \cap eZ' = I(G_{m,\chi})\). Then \(\text{gr } I(G_{m,\mu}) \cap eZ' = \text{gr } I(G_{m,\chi})\). Indeed, clearly \(\text{gr } I(G_{m,\chi}) \subset \text{gr } I(G_{m,\mu}) \cap \text{gr } eZ'\). Suppose that \(f \in \text{gr } I(G_{m,\mu}) \cap eZ'\). Therefore, there is \(g \in \text{ gr } eZ''\) such that \(\deg g < \deg f\) and \(f + g \in I(G_{m,\mu})\). Thus, \(f(t\chi) + g(t^{-1}\chi', t) = 0, t \in k^*\). But the letter is a Laurent polynomial with leading term \(t^{\deg f(\chi)}\). Hence \(f(\chi) = 0, \text{ so } f \in \text{ gr } I(G_{m,\chi})\). Therefore \(p(V(\text{gr } I(G_{m,\mu}))) = V(\text{gr } I(G_{m,\chi})) = k\chi\), where \(p : \text{Spec } eZ' / \text{ Spec } eZ'\) is the projection. Thus we conclude that there is \(\chi'' \in \text{Spec } eZ''\), such that \(\chi'' \in \text{ Spec } eZ''\). Then if \(v \in (h^* \times h^*)^W\) is a preimage of \(\chi''\) under the map \(\text{Spec } B_c = (h^* \times h^*)^W \to \text{ Spec } eZ' \otimes eZ''\), then the projection of \(v\) its projection on \(h/W\) is \(\chi\).

We conclude that \(W_{\chi}\) is a subgroup of \(W_{\chi'}\), so \((h^* \times h^*)^W \subset (h^* \times h^*)^W_{\chi'}\).

Therefore, Using the description of symplectic leaves of \((h^* \times h^*)^W\) discussed above, we conclude \(d(v) \geq 2 \dim h^W\). So, applying \[5.1\] \(H_c(W, h)\)-module
which affords character $\chi$ has dimension divisible by $p^\frac{1}{2}d(v)$, therefore it is divisible by $p^\dim h^W x$.

We have a similar result for Cherednik algebras

**Corollary 4.2.** Assume that $p$ does not divide $|W|$. Let $\chi : Z' \to k$ be a character. Then any irreducible representation of $H_c(W, h)$ affording $\chi$ has dimension divisible by $|W/W_\chi| p^\dim h^W x$, where $W_\chi$ is a subgroup fixing an element of $W$-orbit corresponding to $\chi$ viewed as an element of $h/W$.

**Proof.** Since $\gr H_c(W, h)$ is a free $\Sym(h \oplus h^*)$-module, $\gr H_c(W, h)$ is a Cohen-Macauly $\gr(\mathbb{Z}_0)$-module, hence $R(H_c(W, h))$ is a projective $R(\mathbb{Z}_0)$-module. Just like in the proof of 4.1 using 3.1 we obtain that any simple $H_c(W, h_Z)$-module $M$ has dimension divisible by $p^\dim h^W v$. But as $k[h]$-module, $M$ can be written as $M = \bigoplus_{\chi \in \pi_{p^{-1}(\chi)} M_{\chi'}}$, where $p : h \to h/W$ is the projection. Clearly, if $m \in M_{\chi'}$, then $gm \in M_{\chi'}$. So action by elements of $W$ is permuting $M_{\chi'}$. Hence $\dim M_{\chi'} = \dim M_{\chi'}$, $g \in W$. Since $|p^{-1}(\chi)| = |W/W_\chi|$, we conclude that $|W/W_\chi| p^\dim h^W x$ divides $\dim k M$. So, $|W/W_\chi| p^\dim h^W x$ divides $\dim M$.

As pointed out to us by I. Gordon, one can also prove Corollary 4.2 as follows.

As before, let $\chi$ be a character of $Z' = (\Sym h^*)^W$. We will denote by $Z_c$ the center $H_c(W, h)$. Denote by $\overline{H_c(W, h)}_{\chi}$ the completion of $H_c(W, h)$ with respect to $Ker(\chi) \subset (\Sym h^*)^W$. Denote by $Z_{c, \chi}$ the center of $\overline{H_c(W, h)}_{\chi}$. Then $\overline{Z_{c, \chi}}$ is the completion of $Z_c$ with respect to the ideal $Ker(\chi) \subset Z_c$. We have the similar notations for $\chi \in \Spec((\Sym h^*)^W)$. By $0 \in \Spec Z'(\Spec Z^*)$ we will denote the origin. The following result is the characteristic $p$ version of a result by Bezrukavnikov-Etingof [BE], whose prove is identical to the original one.

**Theorem 4.1. ([BE] Theorem 3.2)** For any $\chi \in \Spec((\Sym h^*)^W)$ (respectively $\Spec((\Sym h^*)^W)$), there is an isomorphism of algebras $H_c(W, h)_{\chi} \simeq Mat_{|W|/|W_\chi|}(\overline{H_c(W, h)}_{\chi})$, where $W_\chi$ is the stabilizer of a lift of $\chi$ in $h$ (respectively $h^*$) and $c'$ is the restriction of $c$ on $W_\chi$.

In particular, since $H_c(W_\chi, h) \simeq Mat_{|W/W_\chi|}(D(h^W) \otimes H_c(W_\chi, (h^W)_+^\perp)$ where $(h^W)_+ = ((h^*)^W)^-$ and dimension of any finite dimensional $Mat_{|W/W_\chi|}(D(h^W))$-module is a multiple of $|W/W_\chi| p^\dim h^W x$, we get that 4.1 implies 4.2.

We have the following

**Corollary 4.3.** For any character $\mu \in \Spec Z_c$, there is a subgroup $W' \subset W$ and a character $\mu' \in \Spec Z_{c'}$, $\mu'(\{Z' \otimes Z^*\}_+) = 0$ where $c'$ is the restriction of $c$ on $W'$, such that $H_c(W, h)_\mu \simeq Mat_{|W'|/|W'|}(H_{c'}(W', h))_{\mu'}$. 
Proof. Let $\chi$ be the restriction of $\mu$ on $Z'$. By (4.1) we have $H_c(W, h, \chi) \simeq \text{Mat}_{|W|/|W_\chi|}(H_c(W_\chi, h))$. Let $\mu_1$ be the character of $Z_{c,0}$ corresponding to $\mu$ under the isomorphism $Z_{c,0} \simeq Z_{c,0}$. We will also denote by $\mu_1$ the restriction of $\mu_1$ on $Z_{c,0}$, in particular $\mu_1((\text{Sym} h W)_p) = 0$. Thus, $H_c(W, h, \mu_1) \simeq \text{Mat}_{|W|/|W_\chi|}(H_c(W_\chi, h))$. Therefore, by replacing $\mu$ with $\mu_1$, we may assume that $\chi = 0$. Again using (4.1) we have $H_c(W, h, \chi) \simeq \text{Mat}_{|W|/|W_\chi|}(H_c(W_\chi, h))$. Let $\mu'$ be the corresponding character of $Z'$; then $\mu'((Z' \otimes Z')_p) = 0$ and we are done.

The above corollary implies that for studying irreducible representations of Cherednik algebras it is enough to consider restricted representations, i.e., modules annihilated by $(\text{Sym} h W \otimes \text{Sym} h^* W)_p$.

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