Many-qutrit Mermin inequalities with three measurement bases

Jay Lawrence

Department of Physics and Astronomy,
Dartmouth College, Hanover, NH 03755, USA and
The James Franck Institute, University of Chicago, Chicago, IL 60637

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Abstract

Mermin inequalities are derived for systems of three-state particles (qutrits) employing three local measurement bases. These establish perfect correlations which violate local realistic bounds more strongly than those previously reported with two bases. The quantum eigenvalue of the Mermin operator grows as the dimension of the Hilbert space, $3^N$, rather than $2^N$, as obtained with two measurement bases. Unexpectedly, the proof is simplified with three bases because of the increased rotational symmetry of the construction.

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I. INTRODUCTION

This work is motivated by recent experimental breakthroughs on the entanglement of three nonbinary particles [1, 2]. In particular, a three-qutrit GHZ state has recently been produced and documented for the first time [2]. The qutrits are realized as photons with orbital angular momentum, and years of progress leading to the present breakthroughs are reviewed in Ref. [3]. Here we build upon recent theoretical work [4] on qutrits by extending the number of local measurement bases from two to three, the maximum such number for qutrits. This extension enhances the violations of local realism, which now increase in proportion to the dimension of the Hilbert space, $3^N$, of the $N$-qutrit system. Such violations are expressed by Greenberger-Horne-Zeilinger (GHZ) paradoxes [5], or more quantitatively, by Mermin inequalities [6].

Mermin inequalities place an upper bound on a particular weighted sum of observables (called the Mermin function) which must be obeyed by any local hidden variable theory. The quantum value, which exceeds this upper bound, is an eigenvalue of the Mermin function. The definiteness of the quantum value distinguishes Mermin [6] from Bell inequalities [7, 8], giving the former additional power and applicability, beyond their fundamental connection to Einstein-Podolsky-Rosen’s elements of reality [9]. There have been many extensions of Mermin’s original work over the years, as summarized in Ref. [4]. More relevant examples include the extension to qubit graph states for all $N$ up to six [10], and extensions to all even particle dimensions $d$ for GHZ states of all $N \geq 4$ [11]. So it was surprising in this context that no GHZ paradoxes were found for any odd dimensions $d$ until 2013 [12], and no Mermin inequality until 2017 [4]. The reason it took so long, as now understood [13], is that stabilizer sets cannot give rise to GHZ paradoxes in any odd dimensions.

Existing odd-$d$ paradoxes [12, 14] and Mermin inequalities [4] are based on so-called concurrent observables, which are incompatible but share common eigenstates [15]. Despite this difference, the observables employed in both the qubit and qutrit Mermin functions share the crucial common feature of rotational covariance with GHZ states [14].

In the next section we describe the system of GHZ states and rotationally covariant observables that comprise the Mermin operator, and we compare its quantum and classical values. In Section III we prove that the quoted classical values are indeed the maxima over all hidden variable choices. In the concluding section we remark on the physical significance
FIG. 1: (a) GHZ states in Eq. 1 [1] and (b) tensor product observables for $N=3$ and (c) $N=4$. Parentheses denote the number of permutations within tensor products. Black arrows define the subset whose joint eigenstate is $|\Psi_0\rangle$. Red and green arrows relate similarly to $|\Psi_1\rangle$ and $|\Psi_2\rangle$. of the Mermin operator and comment on higher dimensional systems.

II. MERMIN OPERATOR - QUANTUM AND CLASSICAL VALUES

To construct the Mermin operator, first consider the eight related $N$-qutrit GHZ states,

$$|\Psi_k\rangle = \frac{1}{\sqrt{3}}\left(|00...0\rangle + \alpha^k|11...1\rangle + \alpha^{2k}|22...2\rangle\right), \quad (k = 0, 1, ..., 8),$$

where $\alpha = \exp(2\pi i/9)$, so that each $|\Psi_k\rangle$ is generated from $|\Psi_{k-1}\rangle$ by a rotation through $2\pi/9$, as shown in Fig. 1a. It is a defining symmetry of GHZ states [14] that such rotations may be distributed arbitrarily among qutrits (about their respective $\hat{z}$ axes), in increments that add up to the net rotation angle.

Now consider the observable of which $|\Psi_0\rangle$ is an eigenstate with eigenvalue unity,

$$X \equiv X^{\otimes N} = X_1...X_N.$$

Its factors are the standard qutrit Pauli matrices, $X_i = \sum_{n=0}^{2}|n+1\rangle_i\langle n|_i$, acting on the $i$th qutrit. Rotations of these factors through the basic angles $(\pm2\pi/9)$ generate other qutrit matrices which we call, respectively (dropping the index $i$)

$$Y \equiv Z^{1/3}XZ^{-1/3} = \sum_{n=0}^{2}|n+1\rangle\alpha^{(1-3\delta_{n,2})}\langle n|,$$

$$V \equiv Z^{-1/3}XZ^{1/3} = \sum_{n=0}^{2}|n+1\rangle\alpha^{(3\delta_{n,2}-1)}\langle n|,$$

$$1$$
where \( Z = \sum_{n=0}^{2} |n\rangle\omega^n \langle n| \) is the usual diagonal Pauli matrix, which rotates qutrits through \( 2\pi/3 \). With these three one-qutrit matrices, we construct the \( 3^N \) tensor product observables in which each factor can be either \( X \) or \( Y \) or \( V \). Every such tensor product is generated from \( X \) by some combination of one-qutrit rotations (through 0 or \( \pm 2\pi/9 \), respectively), and Figs. 1(b and c) show the net rotations for \( N = 3 \) and 4. Because of rotational covariance, observables at the point \( k \) share the state \( |\Psi_k\rangle \) as an eigenstate with eigenvalue unity.

Now, operators have a periodicity property [14] - a rotational Bloch theorem: If any factor (\( X, Y, \) or \( V \)) is rotated through \( 2\pi/3 \), it is simply multiplied by \( \omega \); that is, \( ZXZ = \omega X \), and similarly for \( Y \) and \( V \). This means that \( \omega XXX \) appears at the point 3 (rotation angle \( 2\pi/3 \)), and it follows that every operator at point 3 has \( |\Psi_0\rangle \) as an eigenstate with eigenvalue \( \omega \). Similarly, every operator at point 6 has \( |\Psi_0\rangle \) as an eigenstate with eigenvalue \( \omega^2 \). Therefore, we may define the Mermin operator corresponding to the state \( |\Psi_0\rangle \) as the weighted sum of operators identified by black arrows on the plot,

\[
M_0 = (\text{sum of operators at } k = 0) + \omega^2(\text{sum of operators at } k = 3) + \omega(\text{sum of operators at } k = 6). \tag{5}
\]

Every term in this expression contributes +1 to the eigenvalue, so that

\[
M_0 |\Psi_0\rangle = 3^{N-1} |\Psi_0\rangle. \tag{6}
\]

One could define different Mermin operators, \( M_1 \) and \( M_2 \), corresponding to \( |\Psi_1\rangle \) and \( |\Psi_2\rangle \) and identified by red and green arrows, respectively. These have identical eigenvalues because each accounts for one-third of all operators appearing on the plot. We focus on \( M_0 \) because its higher symmetry simplifies the analysis.

Let us briefly compare the above quantum result with the classical, or hidden variable result. The assumption embodying local realism, or noncontextuality, is that every local factor, \( X_i, Y_i, V_i \), takes a definite value [eg, \( v(X_i) = 1, \omega, \) or \( \omega^2 \)], and it must take the same value wherever it appears. A given choice produces a classical value, \( v(M_0) \), of the Mermin function. The goal is to find the maximum value, \( M_{HVM} \), over all such choices. We shall simply state here (and prove in the following section) that the maximum is realized when
TABLE I: Quantum values, classical upper bounds and ratio, $A$, for the $N$-qutrit Mermin operator.

| $N$ | $M_Q$ | $M_{HV}$ | $A$     |
|-----|-------|----------|---------|
| 3   | 9     | 6        | $3/2 = 1.5$ |
| 4   | 27    | 15       | $9/5 = 1.8$ |
| 5   | 81    | 36       | $9/4 = 2.25$ |
| 6   | 243   | 90       | $27/10 = 2.70$ |
| 7   | 729   | 225      | $81/25 = 3.24$ |

all local factors take the same value (eg., unity). Then, Eq. 5 reduces to

$$M_{HV} = (\text{number of operators at } k = 0)$$
$$+ (\omega^2 + \omega)(\text{number of operators at } k = 3 \text{ or } k = 6)$$
$$= (\text{number of operators at } k = 0) - (\text{number of operators at } k = 3),$$

(7)

where $\omega^2 + \omega = -1$ was used. The resulting numbers are compared in Table I with the quantum results, $M_Q = 3^{N-1}$.

The alternative Mermin operators, $M_1$ and $M_2$, have the same hidden variable maxima, although these are not realized for the same simple assignments. It is notable, nonetheless, that the HV values found here are integers, which is unlike previous cases for qubits [6, 16], or for qutrits when two measurement bases are used [4]. This reflects the greater symmetry of the present construction.

III. PROOF OF CLASSICAL MAXIMA

We now prove that Eq. 7 indeed provides the maximum hidden variable value of $M_0$, and we derive a closed-form expression. To this end, we note that the Mermin operator is given by the identity,

$$M_0 = \frac{1}{3} \left[ \bigotimes_{i=1}^N (X_i + \alpha^2 Y_i + \alpha^{-2} V_i) + \bigotimes_{i=1}^N (X_i + \omega \alpha^2 Y_i + \omega^2 \alpha^{-2} V_i) \right.$$  
$$+ \bigotimes_{i=1}^N (X_i + \omega^2 \alpha^2 Y_i + \omega \alpha^{-2} V_i) \right].$$

(8)
To verify, note that each term generates a weighted sum of all $3^N$ operators shown on the circle graph. So first, using $\omega^2 + \omega + 1 = 0$, one can show that a given term survives only if the number of $Y$ factors equals the number of $V$ factors mod. 3. This locates surviving terms at the black arrows in Fig. 1. Second, using $\alpha^3 = \omega$, one can show that the multiplying factors of these terms are just those given by Eq. 5.

We shall evaluate the HV value, $v(M_0)$, directly from 8. $v(M_0)$ is a function of the values, $v(X_i)$, etc., assigned to each local factor. But its magnitude, $|v(M_0)|$, depends only on two independent local ratios, which we choose to be

$$R_i = v(Y_i)/v(X_i) \quad \text{and} \quad S_i = v(V_i)/v(X_i). \quad (9)$$

Then, setting an overall phase factor $v(X)$ equal to unity, we have simply

$$v(M_0; R_i, S_i) = \frac{1}{3} \left[ \bigotimes_{i=1}^{N} (1 + \alpha^2 R_i + \alpha^{-2} S_i) + \bigotimes_{i=1}^{N} (1 + \omega \alpha^2 R_i + \omega^2 \alpha^{-2} S_i) \right.$$

$$\left. + \bigotimes_{i=1}^{N} (1 + \omega^2 \alpha^2 R_i + \omega \alpha^{-2} S_i) \right] \equiv \frac{1}{3} \left( \bigotimes_{i=1}^{N} B(R_i, S_i) + \bigotimes_{i=1}^{N} C(R_i, S_i) + \bigotimes_{i=1}^{N} A(R_i, S_i) \right), \quad (10)$$

where the last line simply gives names to the three different sums of individual qutrit factors in Eq. 10, respectively. These names are chosen because, when evaluated at $R_i = S_i = 1$ (the “uniform HV” point),

$$A(1,1) = (1 + \omega^2 \alpha^2 + \omega \alpha^{-2}) = 1 + 2 \cos \frac{2\pi}{9} \equiv A \approx 2.532; \quad (12)$$

$$B(1,1) = (1 + \alpha^2 + \alpha^{-2}) = 1 + 2 \cos \frac{4\pi}{9} \equiv B \approx 1.347; \quad (13)$$

$$C(1,1) = (1 + \omega \alpha^2 + \omega^2 \alpha^{-2}) = 1 + 2 \cos \frac{8\pi}{9} \equiv -C \approx -0.879; \quad (14)$$

their magnitudes are ordered as $A > B > C$, with $C(1,1) \equiv -C$ being negative. The three complex contributions to each of $A$, $B$, and $C$ are shown in Fig. 2.

The hidden variable value of $M_0$ at this “uniform HV” point is given by

$$v(M_0; 1, 1) = \frac{1}{3} \left( A^N + B^N \pm C^N \right), \quad \text{for } N \text{ even/odd}, \quad (15)$$

which duplicates the results of Eq. 7 and Table I. We now show that this expression gives the maximum possible value of $|v(M_0; R_i, S_i)|$ for all $N \geq 3$.

As background, Table II shows how the hidden variable choices ($R$ and $S$) affect the individual qutrit factors appearing in Eq. 11. To derive Table II, observe that the choices...
FIG. 2: Complex contributions to each of $A$, $B$, and $C$ corresponding to Eqs. 12 - 14 respectively. Colors identical to those in Fig. 1.

TABLE II: Transformations of single-qutrit $A$, $B$, and $C$-factors by hidden variable choices. Phase angles, in degrees, are given in parentheses.

| $R$ | $S$ | $A(R,S)$ | $B(R,S)$ | $C(R,S)$ |
|-----|-----|-----------|-----------|-----------|
| 1   | 1   | $A$       | $B$       | $-C$      |
| $\omega$ | 1 | $A(40^\circ)$ | $B(-80^\circ)$ | $C(-20^\circ)$ |
| 1   | $\omega^2$ | $A(-40^\circ)$ | $B(80^\circ)$ | $C(20^\circ)$ |
| $\omega$ | $\omega^2$ | $B$ | $-C$ | $A$ |
| $\omega^2$ | $\omega$ | $-C$ | $A$ | $B$ |
| $\omega^2$ | 1 | $C(20^\circ)$ | $A(-40^\circ)$ | $B(80^\circ)$ |
| 1   | $\omega$ | $C(-20^\circ)$ | $A(40^\circ)$ | $B(-80^\circ)$ |
| $\omega$ | $\omega$ | $B(80^\circ)$ | $C(20^\circ)$ | $A(-40^\circ)$ |
| $\omega^2$ | $\omega^2$ | $B(-80^\circ)$ | $C(-20^\circ)$ | $A(40^\circ)$ |

$R = \omega$ ($\omega^2$) correspond, in Fig. 2, to rotations of each green arrow by $120^\circ$ ($-120^\circ$), while $S = \omega$ ($\omega^2$) correspond to rotations of each red arrow by $120^\circ$ ($-120^\circ$). All entries follow immediately. Note that two choices simply rotate factors in the complex plane. Two others simply permute the three magnitudes. The remaining four choices do a combination of both. Note that every choice preserves the total number of $A$, $B$, and $C$ factors appearing in the sum for $v(\mathcal{M}_0)$.

We can now proceed with the proof. First, Expression 11 is the sum of three terms, each having $N$ factors; in the sum, each factor ($A$, $B$, $C$) appears $N$ times. From this...
alone, it is clear that Eq. 15 gives the maximum of $|v(M_0, R_i, S_i)|$ for any even $N$. For odd $N$, however, we must show that no HV assignment can realign the $C^N$ term without a compensating reduction of $A^N + B^N$. This is easy to show for any assignment producing a net permutation (regardless of phase factors). For, in the trivial case of $N = 3$, the trial value $v(M_0, R_i, S_i)$ cannot exceed $A^2B + B^2C + C^2A$, which is less than $A^3 + B^3 - C^3$. The same clearly holds for all larger $N$.

It remains to consider the pure rotations, which can produce one of two possible outcomes: The $A^N$ and $B^N$ terms can either be aligned (phase angle zero), or ±120° out of phase. The latter case is clearly ruled out. In the former case, the $C^N$ term is always aligned for even $N$, but always 180° out of phase for odd $N$. This completes the proof that $v(M_0; 1, 1) = M_{HVM}$.

Having shown that Eq. 15 gives the HV maximum, it is clear that the large-$N$ asymptote is $M_{HVM} \to \frac{1}{3}(2.532)^N$, so that the quantum to classical ratio increases as

$$\lim_{N \to \infty} \left( \frac{M_Q}{M_{HVM}} \right) \approx \left( \frac{3}{2.532} \right)^N \approx 1.185^N,$$

as compared with $1.064^N$ when two measurement bases are used.

**IV. CONCLUSIONS**

We have derived many-qutrit Mermin inequalities employing three measurement bases. The importance of this extension is shown in Table II, which compares the qutrit cases (with two and three measurement bases), and Mermin’s original qubit proof [6] with two measurement bases (refined by Ardehali [16]). In the two cases where the number of independent measurement bases ($nb$) is maximized at the particle’s dimension $d$, the quantum value $M_Q$ grows as the dimension $D$ of the Hilbert space of the system. It is well known that $D - 1$ is the number of operators which can be diagonalized simultaneously. It is also the number of operators which can share a common eigenstate. In comparison, $M_Q$ is the number of operators which contribute to $M_0$, and this in turn is proportional to the number of independent GHZ contradictions that exist for the given GHZ state [17]. Thus, the number of observables violating local realistic constraints grows as a fixed fraction of the number that can take sharp values simultaneously (be they compatible or merely concurrent). Note that in the qubit case, $M_Q = D/2$, while in the qutrit case, $M_Q = D/3$.

The symmetry of the construction in Fig. 1 extends to systems of higher odd dimensions,
TABLE III: Mermin eigenvalue $\mathcal{M}_Q$ compared with the dimension $\mathcal{D}$ of the Hilbert space. $d$ is particle dimension and $mb$ is number of measurement bases.

| source                  | $d$ | $mb$ | $\mathcal{M}_Q$ | $\mathcal{D}$ |
|-------------------------|-----|------|-----------------|----------------|
| qubits (Ref. [6][16])   | 2   | 2    | $2^{N-1}$       | $2^N$          |
| qutrits (Ref. [4])      | 3   | 2    | $2^{N/3}$       | $3^N$          |
| qutrits (present)       | 3   | 3    | $3^{N-1}$       | $3^N$          |

provided that the maximum number $d$ of measurement bases are employed. On this basis, for $d = 5$, we would expect the quantum value $\mathcal{M}_Q = 5^{(N-1)} = \mathcal{D}/5$, although the hidden variable maximum is less clear-cut. Supposing that the optimal HV assignment is the uniform one, we could guess the asymptotic classical maximum to be $\mathcal{M}_{HV m} \approx 4.6898^N/5$, so that the quantum violation of this bound grows (slowly but exponentially) as $\mathcal{A} \approx 1.066^N$.

There are daunting practical limitations to preparing entangled systems with larger $d$ as well as larger $N$. Regarding fundamental limitations, however, there is no limit on $N$, and only a narrowing of quantum-classical gap (as measured by the ratio $\mathcal{A}$) with dimension $d$. The narrowing found here suggests that hidden variables can better mimic the quantum values as $d$ increases.

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