Higher direct images of snc ideal sheaves

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October 17, 2023

Abstract

We prove invariance results for the cohomology groups of ideal sheaves of simple normal crossing divisors under (a restricted class of) birational morphisms of pairs in arbitrary characteristic, assuming a conjecture regarding the existence of normal Cohen-Macaulayfications. As an application, we extend some foundational results in the theory of rational pairs that were previously known only in characteristic 0.

1 Introduction

A foundational problem in birational geometry, posed by Grothendieck in his 1958 ICM address [Gro60, Problem B], asked whether for every proper birational morphism of non-singular projective varieties \( f : X \to Y \),

\[
R^i f_* \mathcal{O}_X = 0 \text{ for } i > 0.
\]

In characteristic 0 this was answered affirmatively by Hironaka as a corollary of resolution of singularities [Hir64, §7 Cor. 2]. In characteristic \( p > 0 \), where resolutions of singularities are not known to exist, answering Grothendieck’s question proved much harder, remaining open until 2011 when Chatzistamatiou and Rülling proved the following theorem.

**Theorem 1.1** ([CR11, Thm. 3.2.8], see also [CR15, Thm. 1.1], [Kov22, Thm. 1.19]). Let \( k \) be a perfect field and let \( S \) be a scheme. Suppose \( X \) and \( Y \) are two separated finite type \( S \)-schemes which are (i) smooth over \( k \) and

(ii) **properly birational** over \( S \) in the sense that there is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{r} & Z \\
\downarrow{f} & & \downarrow{s} \\
S & \xrightarrow{g} & Y
\end{array}
\]

with \( r \) and \( s \) proper birational morphisms.

Then, there are isomorphisms of sheaves

\[
R^i f_* \mathcal{O}_X \sim R^i g_* \mathcal{O}_Y \text{ and } R^i f_* \omega_X \sim R^i g_* \omega_Y \text{ for all } i.
\]

One of the primary applications of **Theorem 1.1** was to extend foundational results on rational singularities from characteristic 0 to arbitrary characteristic (for definitions of rational resolutions and rational singularities see **Definition 5.1**).
**Corollary 1.4** ([CR11, Cor. 3.2.10], see also [Kov22, Thm. 1.4]). If \( S \) has a rational resolution, then every resolution of \( S \) is rational.

In this article, we prove analogues of **Theorem 1.1** and **Corollary 1.4** for pairs.

**Definition 1.5** (slightly more general version of [Kol13, Def. 1.5]). In what follows a pair \((X, \Delta_X)\) will mean a reduced, equidimensional excellent scheme \( X \) admitting a dualizing complex together with a \(\mathbb{Q}\)-Weil divisor \( \Delta_X = \sum_i a_i D_i \) on \( X \) such that no irreducible component \( D_i \) of \( \Delta_X \) is contained in \( \text{Sing}(X) \).

**Definition 1.6.** A simple normal crossing pair is an equidimensional, regular excellent scheme \( X \) together with a reduced effective divisor \( \Delta_X = \sum_i D_i \) such that for every subset \( J \subseteq \{1, \ldots, N\} \) the scheme-theoretic intersection

\[
D_J := \cap_{i \in J} D_i \subseteq X
\]

is regular of codimension \( |J| \).

**Remark 1.7.** If \( X \) is regular as in **Definition 1.6** then it admits a dualizing complex. By an amazing result of Kawasaki [Kaw02, Cor. 1.4], a noetherian ring admits a dualizing complex if and only if it is a homomorphic image of a finite-dimensional Gorenstein ring.

As observed in [Kol13, §2.5], to generalize **Corollary 1.4** to pairs we must restrict attention to a special class of thrifty resolutions.

**Definition 1.8.** A stratum of a simple normal crossing pair \((X, \Delta_X = \sum_i D_i)\) is a connected (equivalently, irreducible) component of an intersection \( D_J = \cap_{i \in J} D_i \).

**Definition 1.9** (compare with [Kol13, Def. 2.79-2.80], [KK16, §1, discussion before Def. 10]). Let \((S, \Delta_S = \sum_i D_i)\) be a pair, and assume \( \Delta_S \) is reduced and effective. A separated, finite type birational morphism \( f : X \to S \) is thrifty with respect to \( \Delta_S \) if and only if

1. \( f \) is an isomorphism over the generic point of every stratum of \( \text{snc}(S, \Delta_S) \) and
2. letting \( D_i' = f^{-1}D_i \) for \( i = 1, \ldots, N \) be the strict transforms of the divisors \( D_i \), and setting \( \Delta_X := \sum_i D_i' \), the map \( f \) is an isomorphism at the generic point of every stratum of \( \text{snc}(X, \Delta_X) \).

Philosophically, a resolution \((X, \Delta_X) \to (S, \Delta_S)\) is thrifty if it induces a one-to-one birational correspondence between the strata of \((X, \Delta_X)\) and \((S, \Delta_S)\), thus preserving the combinatorial geometry of the pair \((S, \Delta_S)\).

In order to prove our main result, **Theorem 1.12**, we need to assume a conjecture regarding the existence of normal Cohen-Macaulayfications (**Conjecture 1.11** below). To contextualize this conjecture we first recall that requiring normality, the existence of Cohen-Macaulayfications is known.

**Theorem 1.10** ([Čes21, Thm. 1.6], see also [Kaw00, Thm. 1.1]). For every CM-quasi-excellent noetherian scheme \( X \) there exists a projective birational morphism \( \pi : \tilde{X} \to X \) such that \( \tilde{X} \) is Cohen-Macaulay and \( \pi \) is an isomorphism over the Cohen-Macaulay locus \( CM(X) \subset X \).

The notion of CM-quasi-excellence is a weakening of excellence introduced by Česnavičius, so in particular the theorem applies to excellent noetherian schemes. To prove our results, we require the existence of a \( \tilde{X} \) as appearing in **Theorem 1.10** that is also normal, and it is at least technically very useful for us to require that the associated projective birational morphism \( \pi \) is an isomorphism over the regular locus \( \text{Reg}(X) \).\(^1\) Note that this is weaker than requiring \( \pi \) to be an isomorphism over \( CM(X) \).

**Conjecture 1.11** (see also [Čes21, Conj. 1.1], [Kov22, Conj. 1.14]). For every CM-quasi-excellent noetherian scheme \( X \) there exists a projective birational morphism \( \pi : \tilde{X} \to X \) such that \( \tilde{X} \) is Cohen-Macaulay and normal and \( \pi \) is an isomorphism over the regular locus \( \text{Reg}(X) \subset X \).

Due to the existence of resolution of singularities [Hir64; Tem08], **Conjecture 1.11** is known in characteristic zero. If we drop the requirement that \( \pi \) be an isomorphism over the regular locus \( \text{Reg}(X) \subset X \), **Conjecture 1.11** is known for varieties of dimension at most 4 over algebraically closed fields.

\(^1\)It is however possible that overhauling our proofs would allow for relaxing this condition.
fields by [Bro83, Cor. 1.8]. To the best of our knowledge, Theorem 1.12 is new even in characteristic 0.  

**Theorem 1.12 (Theorem 4.34).** Let $S$ be an excellent noetherian scheme and let $(X, \Delta_X)$ and $(Y, \Delta_Y)$ be simple normal crossing pairs separated and of finite type over $S$. Suppose $(X, \Delta_X)$ and $(Y, \Delta_Y)$ are properly birational over $S$ in the sense that there is a commutative diagram

$$
\begin{array}{ccc}
(X, \Delta_X) & \xrightarrow{r} & (Z, \Delta_Z) \\
| & & | \\
\bigcirc & \xrightarrow{s} & (Y, \Delta_Y) \\
| & & | \\
S & \xleftarrow{f} & \bigcirc & \xrightarrow{g} & S
\end{array}
$$

(1.13)

where $r$, $s$ are proper and birational morphisms, and assume $\Delta_Z = r^{-1}_s \Delta_X = s^{-1}_r \Delta_Y$. If $r$ and $s$ are thrifty and Conjecture 1.11 holds, then there are quasi-isomorphisms

$$Rf_* \Theta_X(-\Delta_X) \simeq Rg_* \Theta_Y(-\Delta_Y) \text{ and } Rf_* \omega_X(\Delta_X) \simeq Rg_* \omega_Y(\Delta_Y).$$

(1.14)

In order to state an analogue of Corollary 1.4, we say what we mean by rational singularities of pairs.

**Definition 1.15** (compare with [Kol13, Def. 2.78]). Let $(S, \Delta_S)$ be a pair as in Definition 1.5 and assume $\Delta_S$ is reduced and effective. A proper birational morphism $f : X \to S$ is a rational resolution if and only if

(i) $X$ is regular and the strict transform $\Delta_X := f^{-1}_\ast \Delta_S$ has simple normal crossings,

(ii) the natural morphism $\Theta_S(-\Delta_S) \to Rf_* \Theta_X(-\Delta_X)$ is a quasi-isomorphism, and letting $\omega_X = h^{-\dim X} \omega_X^*$ where we use $\omega_X^* = f^! \omega_S^*$ as a normalized dualizing complex on $X$,

(iii) $R^if_* \omega_X(\Delta_X) = 0$ for $i > 0$.

The pair $(S, \Delta_S)$ is resolution-rational if and only if it has a thrifty rational resolution.

We wish to emphasize that this is not the only definition of rational pairs available in the literature: Schwede and Takagi adopted a different definition in [ST08]. Here we focus on the variant of rational pairs defined in [Kol13, §2.5], simply because it is the one to which our methods most directly apply, however we view identifying and studying a notion of rational pairs that simultaneously generalizes those of [ST08] and [Kol13] as an interesting question for future work. With Definition 1.15 in hand, the precise statement of our result is:

**Theorem 1.16** ([Kol13, Cor. 2.86] in characteristic 0, Theorem 5.5 in arbitrary characteristic). Let $(S, \Delta_S)$ be a pair with $\Delta_S$ reduced and effective. If $(S, \Delta_S)$ has a thrifty rational resolution $f : (X, \Delta_X) \to (S, \Delta_S)$, and if Conjecture 1.11 holds, then every thrifty resolution $g : (Y, \Delta_Y) \to (S, \Delta_S)$ is rational.

To prove Theorem 1.12 and Theorem 1.16 we take a small step outside of the category of schemes, to that of semi-simplicial schemes (for their definitions and basic properties we refer to section 2). We hope to show that this is a natural approach — for example, every pair $(X, \Delta_X)$ has a naturally associated semi-simplicial scheme categorifying its set of strata. We emphasize that all of the semi-simplicial schemes $X$, we consider have strong finite-dimensionality properties ($X_i = \emptyset$ for $i >> 0$). Some of our ideas could be of interest even in characteristic 0; for example, we obtain a criterion for a pair $(S, \Delta_S)$ to have rational singularities in terms of a resolution of the associated semi-simplicial scheme $S_*$.  

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2 Although it can be proved there without too much difficulty using the machinery developed in [Kol13, §2.5], in particular without using the techniques developed in this paper.
**Lemma 1.17 (Lemma 5.7).** Let \((S, \Delta_S)\) be a pair, with \(\Delta_S\) reduced and effective. Let \(\epsilon^S : S, \to S\) be the associated augmented semi-simplicial scheme and suppose

\[
\begin{array}{ccc}
X & \xrightarrow{f} & S, \\
\downarrow_{\epsilon^X} & & \downarrow_{\epsilon^S} \\
X & \xrightarrow{f} & S
\end{array}
\] (1.18)

is a resolution. Then, \((S, \Delta_S)\) is a rational pair if and only if the sheaf \(\theta_S(-\Delta_S)\) is Cohen-Macaulay and the natural map \(\theta_S(-\Delta_S) \to Rf_*\mathcal{K}\) is a quasi-isomorphism, where \(\mathcal{K}\) is naturally defined complex on \(X\).

A benefit of this lemma is that resolutions of the semi-simplicial scheme \(S\), are arguably easier to construct than thrifty resolutions of pairs: in [Kol13, Thm. 10.45], the existence of thrifty resolutions was proved using the refined log-resolution theorems of [BM97; Sza94]. In contrast, resolving semi-simplicial schemes requires only standard resolution of singularities together with elementary algorithms for inductively constructing semi-simplicial schemes (quite similar to the constructions of hyper-resolutions in the theory of Du Bois singularities [Del71; Del74; Du 81]). Of course, the question of which route is easier is subjective, but in light of Lemma 1.17 we think an overarching “slogan” of this work is:

**Slogan 1.19.** Rationality of a pair \((S, \Delta)\) can be tested using any resolution \(X, \to S\), of a semi-simplicial scheme \(S\), associated to \(S\) (coming from closures of strata of the snc locus snc\((S, \Delta)\)), and thrifty resolutions simply provide one way to obtain such resolutions \(X, \to S\).

### 1.1 Overview

We next provide some motivation for the appearance of semi-simplicial schemes in the proof of Theorem 1.12. To begin we may translate the condition that a birational morphism \(f : (X, \Delta_X) \to (S, \Delta_S)\) is thrifty into the statement that the dual complexes \(\mathcal{D}(\Delta_X)\) and \(\mathcal{D}(\Delta_S)\) are isomorphic. The dual complex \(\mathcal{D}(\Delta_X)\) is usually described as the \(\Delta\)-complex (in the sense of [Hat02, §2.1]) with 0-cells the irreducible components \(D^X_i\) of \(\Delta_X = \sum D^X_i\), 1-cells the components of intersections \(D^X_i \cap D^X_j\) for \(i < j\) with gluing maps corresponding to the inclusions \(D^X_i \cap D^X_j \subseteq D^X_i\) and \(D^X_i \cap D^X_j \subseteq D^X_j\), and so on — in terms of Definition 1.8, the cells of \(\mathcal{D}(\Delta_X)\) correspond to strata of \((X, \Delta_X)\), with gluing maps corresponding to inclusions of strata. The topological properties of \(\mathcal{D}(\Delta_X)\) have been extensively studied, for example in this non-exhaustive list of references: [ABW13; Dan75; FXX17; Ste06]. Upon inspection we see that a \(\Delta\)-complex is precisely a semi-simplicial set, and that \(\mathcal{D}(\Delta_X)\) is the semi-simplicial set obtained by taking \(\pi_0\) (connected components) of a semi-simplicial scheme \(X,\) with

\[
X_i = \prod_{\mid J \mid = i+1} (\cap j \in J D^X_j) \text{ for } i \geq 0
\]

The thriftyness hypotheses of Theorem 1.12 ensure that \((X, \Delta_X)\) and \((Y, \Delta_Y)\) have the same dual complex, which provides enough rigidity to attempt to prove Theorem 1.12 by induction on \(\dim X\) and the number of components of \(\Delta_X\), using Theorem 1.1 as a base case. For example, we have exact sequences

\[
0 \to \theta_X(-\Delta_X) \to \theta_X(-\Delta_X + D^X_1) \to \theta_{D^X_1}(-\Delta_X + D^X_1|_{D^X_1}) \to 0
\]

and similarly on \(Y\). We can even assume by induction the existence of already-defined quasi-isomorphisms

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3 What exactly “resolution” means and how \(\mathcal{K}\) is defined will be made precise in Lemma 5.7.

4 We could relax the condition that both pairs are snc, but it will make this motivational discussion simpler.
in a diagram

\[
\begin{array}{cccccc}
Rf_s \mathcal{O}_X(-\Delta_X) & \rightarrow & Rf_s \mathcal{O}_X(-\Delta_X + D_1^X) & \rightarrow & Rf_s \mathcal{O}_{D_1^X}(-\Delta_X + D_1^X|_{D_1^X}) & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
Rg_s \mathcal{O}_Y(-\Delta_Y) & \rightarrow & Rg_s \mathcal{O}_Y(-\Delta_Y + D_1^Y) & \rightarrow & Rg_s \mathcal{O}_{D_1^Y}(-\Delta_Y + D_1^Y|_{D_1^Y}) & \rightarrow \\
\end{array}
\]  

(1.20)

If the square \((*)\) commutes, then using only the fact that \(D^b_{\text{coh}}(S)\) is a triangulated category we get a quasi-isomorphism \(\alpha\) on the dashed arrow. However, in this approach \(\beta, \gamma\) are themselves defined by induction, and so to know \((*)\) commutes we must take one inductive step further, considering maps of distinguished triangles

\[
\begin{array}{cccccc}
Rf_s \mathcal{O}_X(-\Delta_X + D_1^X) & \rightarrow & Rf_s \mathcal{O}_X(-\Delta_X + D_1^X + D_2^X) & \rightarrow & Rf_s \mathcal{O}_{D_1^X}(-\Delta_X + D_1^X + D_2^X|_{D_1^X}) & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
Rg_s \mathcal{O}_Y(-\Delta_Y + D_1^Y) & \rightarrow & Rg_s \mathcal{O}_Y(-\Delta_Y + D_1^Y + D_2^Y) & \rightarrow & Rg_s \mathcal{O}_{D_1^Y}(-\Delta_Y + D_1^Y + D_2^Y|_{D_1^Y}) & \rightarrow \\
\end{array}
\]  

(1.21)

and

\[
\begin{array}{cccccc}
Rf_s \mathcal{O}_X(-\Delta_X + D_1^X + D_2^X) & \rightarrow & Rf_s \mathcal{O}_X(-\Delta_X + D_1^X + D_2^X + D_3^X) & \rightarrow & Rf_s \mathcal{O}_{D_1^X + D_2^X}(-\Delta_X + D_1^X + D_2^X + D_3^X|_{D_1^X + D_2^X}) & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
Rg_s \mathcal{O}_Y(-\Delta_Y + D_1^Y + D_2^Y) & \rightarrow & Rg_s \mathcal{O}_Y(-\Delta_Y + D_1^Y + D_2^Y + D_3^Y) & \rightarrow & Rg_s \mathcal{O}_{D_1^Y + D_2^Y}(-\Delta_Y + D_1^Y + D_2^Y + D_3^Y|_{D_1^Y + D_2^Y}) & \rightarrow \\
\end{array}
\]  

(1.22)

together with a map from (1.21) to (1.22) including the square \((*)\), and so on. It is certainly possible that the correct induction hypothesis (building in not only quasi-isomorphisms like \(\beta, \gamma\) in (1.20) but also commutativity hypotheses) and some careful analysis of diagrams in \(D^b_{\text{coh}}(S)\) could make this strategy work, but the author had no such luck. A separate technical issue the above approach encounters is that at some point in the base case, we must analyze how the isomorphisms of Theorem 1.1 behave with respect to restrictions, i.e. diagrams of schemes like

\[
\begin{array}{cccccc}
D_1^X & \leftarrow & D_1^Z & \rightarrow & D_1^Y \\
\downarrow & & \downarrow & & \downarrow \\
X & \leftarrow & Z & \rightarrow & Y
\end{array}
\]

delving into the methods of [CR11; CR15; Kov22], this analysis runs into subtle aspects of Grothendieck duality, especially since for this approach to work we do require morphisms in \(D^b_{\text{coh}}(S)\), not simply of cohomology sheaves as in Theorem 1.1.

Despite the aforementioned technical issues, what is clear is that this attempted induction takes place on the semi-simplicial schemes \(X, Y\), underlying the dual complexes \(\mathcal{D}(\Delta_X)\) and \(\mathcal{D}(\Delta_Y)\). Under necessary thriftiness hypotheses, in the situation of Theorem 1.12 we find that there is also an auxiliary semi-simplicial scheme \(Z\), together with morphisms \(X \xleftarrow{f} Z \xrightarrow{g} Y\), which are birational in each simplicial degree. Using the refined forms of Chow’s lemma and resolution of indeterminacies from Conrad’s article on Deligne’s notes on Nagata compactifications [Con07], together with the Conjecture 1.11, we can prove the existence of such a \(Z\), where each scheme \(Z_i\) is Cohen-Macaulay and normal and the morphisms \(X_i \xleftarrow{\sim} Z_i \xrightarrow{\sim} Y_i\) are projective — this occupies Sections 3 and 4. We then make essential use of recent work of Kovács [Kov22, Thm. 1.4] to conclude that there are natural maps \(\theta_{X_i} \rightarrow Rf_{s_i} \mathcal{O}_{Z_i}\) and \(\theta_{Y_i} \rightarrow Rg_{s_i} \mathcal{O}_{Z_i}\) are quasi-isomorphisms for all \(i\). A more detailed overview of this construction is included at the beginning of Section 4.

The remainder of our proof is pure homological algebra: in Section 2 we show that when \((X, \Delta_X)\) is a simple normal crossing pair (more generally, when the components \(D_1^X\) of \(\Delta_X\) form a regular sequence, see Definition 2.6) the ideal sheaf \(\mathcal{O}_X(-\Delta_X)\) admits a Čech-type resolution of the form

\[
\mathcal{O}_X(-\Delta_X) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X|_0 \rightarrow \mathcal{O}_X|_1 \rightarrow \cdots,
\]

in other words we can recover \(\mathcal{O}_X(-\Delta_X)\) from an augmentation morphism \(X \rightarrow X\). Moreover, we can recover the cohomology of \(\mathcal{O}_X(-\Delta_X)\) from a descent-type spectral sequence Corollary 2.14 — the
last major technical ingredient is a comparison of the resulting spectral sequences associated to $X$, $Y$ and $Z$.

Section 5 deals with applications to rational pairs, in particular Theorem 1.16 and Lemma 1.17. Appendix A includes some new examples illustrating the subtleties of thrifty and rational resolutions of pairs, including an affirmative answer to a question of Erickson and Prelli on whether there exists a non-thrifty rational resolution of a pair $(S, \Delta)$ — our $(S, \Delta)$ is even a rational pair, and the resolution is related to the famous Atiyah flop.

1.2 Acknowledgements

I would like to thank Jarod Alper, Chi-yu Cheng, Kristin DeVleming, Gabriel Dorfman-Hopkins, Max Lieblich, Takumi Murayama, Karl Schwede and Tuomas Tajakka for helpful conversations, multiple anonymous reviewers for useful feedback and my advisor Sándor Kovács for proposing the problem of extending results Theorem 1.1 to pairs and for many valuable suggestions.

2 Regular sequences of divisors and descent spectral sequences

2.1 Semi-simplicial schemes and their derived categories

To any simple normal crossing pair we can naturally associate a semi-simplicial scheme. A primary reference for the theory of semi-simplicial schemes is [SGA4II, Vbis]; since many elementary facts about simplicial schemes carry over to semi-semi-simplicial schemes, [Con03], [Ols16, §2.4] and [Stacks, Tag 0162] are also relevant. What follows is a condensed summary of the machinery we need.

Let $\Lambda$ be the category with objects the sets $[i] := \{0, 1, 2, ..., i\}$ for $i \in \mathbb{N}$ and with morphisms the strictly increasing functions $[j] \to [i]$; in particular $\text{Hom}_\Lambda([j], [i]) = \emptyset$ if $j > i$. A semi-simplicial object in a category $\mathcal{C}$ is a functor $\Lambda^{\text{op}} \to \mathcal{C}$; semi-simplicial $\mathcal{C}$-objects naturally form a category, the functor category $\mathcal{C}^{\Lambda^{\text{op}}}$. Any morphism $\varphi : [j] \to [i]$ can be written non-uniquely as a composition of the basic morphisms

$$\delta^i_k : [i - 1] \to [i] \text{ defined by } \delta^i_k(x) = \begin{cases} x & \text{if } x < k \\ x + 1 & \text{otherwise} \end{cases}$$

(so $\delta^i_k$ skips $k$) [Stacks, Tag 0164], and hence a semi-simplicial object $X : \Lambda^{\text{op}} \to \mathcal{C}$ is equivalent to a sequence of objects $X_i := X([i])$ together with morphisms

$$d^i_k := X(\delta^i_k) : X_i \to X_{i-1} \text{ subject to the relations } d^{i-1}_k \circ d^i_k = d^{i-1}_k \circ d^i_k,$$

and all semi-simplicial objects below will be obtained from such an explicit description. In what follows semi-simplicial objects will be denoted with a $\bullet$, e.g. “the semi-simplicial scheme $X$.” (to distinguish them from plain schemes).

When $\mathcal{C}$ is a category of schemes, a sheaf on a semi-simplicial scheme $X$, is the data of a sheaf $\mathcal{F}_i$ on each scheme $X_i$ together with morphisms of sheaves $\delta^i_k : \mathcal{F}_{i-1} \to d^i_k \ast \mathcal{F}_i$ on $X_{i-1}$ satisfying compatibilities coming from (2.1). These sheaves form a topos $\tilde{X}$, such that morphisms of semi-simplicial schemes $f_i : X_i \to Y_i$ induce functorial maps of topoi $\tilde{X} \to \tilde{Y}$, (see [SGA4II, Vbis, Prop. 1.2.15]) — the benefit of the topos-theoretic point of view is that it immediately implies the category of abelian sheaves $\text{Ab}(X_\ast)$ on $X_\ast$ is an abelian category with enough injectives ([Stacks, Tag 01DL]), enables us to define pushforward functors $Rf_\ast : D^+(\text{Ab}(X_\ast)) \to D^+(\text{Ab}(Y_\ast))$ for morphisms of semi-simplicial schemes $f_\ast : X_\ast \to Y_\ast$, and so on.

An augmented semi-simplicial scheme is a morphism of semi-simplicial schemes $e_\ast : X_\ast \to S_\ast$, where $S_\ast$ is a constant semi-simplicial scheme (that is, $S_i = S$ for all $i$ for some fixed scheme $S$, and all
This is equivalent to the data of a semi-simplicial object of $\mathbf{Sch}_S$. For such a constant semi-simplicial scheme $S$, $\mathbf{Ab}(S)$ is equivalent to the category $\mathbf{Ab}(S)^\Lambda$ of co-semi-simplicial sheaves of abelian groups on $S$, that is, sequences of sheaves of abelian groups $\mathcal{G}_i$ on $S$ together with morphisms $\delta^i_k : \mathcal{G}_i \to \mathcal{G}_j$ satisfying compatibilities forced by (2.1). As in the construction of the Čech complex setting $d^i = \sum_k (-1)^k : \mathcal{G}_i \to \mathcal{G}_j$ gives a complex of abelian sheaves on $S$ and hence in particular an abelian sheaf $a(\mathcal{G}_i) := \ker d^0$. Writing $\varepsilon_* := a \circ \varepsilon_*$, the composite derived functor

$$
\xymatrix{
D^+(\mathbf{Ab}(X)) \ar[r]^{R\varepsilon_*} & D^+(\mathbf{Ab}(S)) \ar[r]^{Rg_*} & D^+(\mathbf{Ab}(S))
}
$$

admits the following concrete description: given a sheaf $\mathcal{F}_*$ on $X$, one takes an injective resolution

$$
\mathcal{F}_* \to \mathcal{F}_0^0 \to \mathcal{F}_1^1 \to \mathcal{F}_2^2 \to \ldots \text{ in } \mathbf{Ab}(X),
$$

Here the $\mathcal{F}_j^j$ are in particular sheaves on $X$, with each $\mathcal{F}_j^j$ an injective abelian sheaf on $X_i$ — for further discussion of injective objects in $\mathbf{Ab}(X)$ see [SGA4II, Vbis, Prop. 1.3.10] and [Con03, Lem. 6.4, comments on p. 42]. Then

$$
\varepsilon_* \mathcal{F}_0^0 \to \varepsilon_* \mathcal{F}_1^1 \to \varepsilon_* \mathcal{F}_2^2 \to \ldots \text{ in } \mathbf{Ab}(S),
$$

is a complex of co-semi-simplicial abelian sheaves which via the Čech construction becomes a complex of complexes. Applying the sign trick gives a double complex whose Tot computes $R\varepsilon_\ast \mathcal{F}_\ast$. One of the spectral sequences of this double complex is displayed below. In our calculations it is crucial that this spectral sequence is (at least in a minimal sense) functorial.

**Lemma 2.2** (Descent spectral sequence, [SGA4II, Vbis §2.3], [Con03, Thms. 6.11-6.12]). If $\mathcal{F}_\ast$ is an abelian sheaf on an augmented semi-simplicial scheme $\varepsilon : X_\ast \to S$ then there is a spectral sequence

$$
E_1^{pq} = R^q \varepsilon_\ast \mathcal{F}_p \to R^{p+q} \varepsilon_\ast \mathcal{F}_\ast,
$$

Moreover if $\mathcal{G}_\ast$, is an abelian sheaf on another augmented semi-simplicial scheme $\varepsilon' : Y_\ast \to T$ and

$$
\begin{array}{ccc}
Y \xrightarrow{g_*} X & \xrightarrow{\varepsilon} & S \\
\downarrow{\varepsilon'} & & \downarrow{g} \\
T & \xrightarrow{g} & S
\end{array}
$$

is a map of augmented semi-simplicial schemes together with a map of abelian sheaves $\varphi : \mathcal{F}_\ast \to g_\ast \mathcal{G}_\ast$, on $X_\ast$, then $\varphi$ induces a morphism of spectral sequences

$$
E_1^{pq}(\mathcal{F}_\ast) = R^q \varepsilon_\ast \mathcal{F}_p \to R^{p+q} \varepsilon_\ast \mathcal{G}_\ast = E_1^{pq}(\mathcal{G}_\ast)
$$

converging to the morphism $R\varepsilon_\ast (\varphi) : R\varepsilon_\ast \mathcal{F}_\ast \to R\varepsilon_\ast g_\ast \mathcal{G}_\ast = Rg_* R\varepsilon'_\ast \mathcal{G}_\ast$.

**Proof of the “Moreover...”**. We work with the abelian categories of sheaves of abelian groups on $Y_\ast, X_\ast$. Let $\mathcal{F}_\ast$ be an injective resolution of $\mathcal{G}_\ast$, in $\mathbf{Ab}(Y)_\ast$. Then $f_\ast \mathcal{F}_\ast$ is a complex of injectives (this uses the fact that $f_\ast$ has an exact left adjoint $f^{-1}$), $\mathcal{F}_\ast \to \mathcal{F}_\ast$ is a quasi-isomorphism and we are given a map

$$
\varphi : \mathcal{F}_\ast \to f_\ast \mathcal{G}_\ast \to f_\ast \mathcal{F}_\ast;
$$

By [Stacks, Tag 013P] (see also [Wei94, Thm. 2.2.6]) there is a map of complexes of abelian sheaves on $X_\ast$, extending $\varphi$:

$$
\tilde{\varphi} : \mathcal{F}_\ast \to f_\ast \mathcal{F}_\ast;
$$

Applying $\varepsilon_\ast$ then gives a morphism of complexes of co-semi-simplicial abelian sheaves on $S$ consisting of morphisms

$$
\varepsilon_{p*} \mathcal{F}_p^q \to \varepsilon_{p*} g_{p*} \mathcal{F}_p^q
$$
compatible with both the simplicial sheaf maps (in the $p$ direction) and the injective resolution maps (in the $q$ direction), to which we may apply the Čech construction and sign trick to obtain a map of double complexes. This reduces us to the claim that a map of double complexes (or more generally a filtered map of filtered complexes) induces a map of spectral sequences, which we take as well known.

**Remark 2.3.** The above proof is at least suggested in the last sentence of [Con03, Thm. 6.11]. An alternative method would be to use Deligne’s trick of viewing $\varphi$ as an abelian sheaf on the $\Lambda \times I$ scheme associated to $f$, — for related discussion see [SGA4II, Vbis, §3.1].

**Corollary 2.4.** In the situation of Lemma 2.2 suppose in addition that the morphisms $\varphi_p : \mathcal{F}_p \to Rf_{p*}\mathcal{G}_p$ are quasi-isomorphisms for all $p$. Then, the induced morphism

$$R\varepsilon_*(\varphi) : R\varepsilon_*\mathcal{F}_* \to R\varepsilon_*Rg_*\mathcal{G}_* = Rg_*R\varepsilon'_*\mathcal{G}$$

is a quasi-isomorphism.

**Corollary 2.5.** In the situation of Lemma 2.2, suppose in addition that the scheme $S$ is noetherian, each of the morphisms $X_i \to S$ is proper, and each of the sheaves $\mathcal{F}_i$ on $X_i$ is coherent. Then, sheaf $R^i\varepsilon_*\mathcal{F}_*$ on $S$ is coherent for all $i$ and $R^i\varepsilon_*\mathcal{F}_*$, $= 0$ for $|i| \gg 0$. In other words, the complex $R\varepsilon_*\mathcal{F}_*$ belongs to the bounded derived category of coherent sheaves $D^b_\text{c}(S)$.

### 2.2 Regular sequences of divisors

**Definition 2.6.** Let $X$ be a locally noetherian scheme. A sequence of effective Cartier divisors $D_1, D_2, \ldots, D_N \subseteq X$ is called regular if and only if for each point $x \in X$, letting $f_1, \ldots, f_N \in \mathcal{O}_{X,x}$ be local generators for the ideal sheaves $\mathcal{F}_{D_i}$ at $x$ and letting $I(x) = \{i \mid x \in D_i\}$, the elements $(f_j \in \mathfrak{m}_x \mid j \in I(x))$ form a regular sequence.

This definition is designed to ensure that a permutation of a regular sequence of divisors is again a regular sequence (see [Mat80, §15, Thm. 27], [Stacks, Tag 00LJ]). Let $X$ be a locally noetherian scheme together with a regular sequence of effective Cartier divisors $D_1, D_2, \ldots, D_N \subseteq X$. We define an augmented semi-simplicial scheme $X_s$, as follows: $X_{s-1} = X, X_0 = \coprod \mathring{D}_i$ and for $k > 0$,

$$X_k = \coprod_{I \subseteq \{1, \ldots, N\} \mid |I| = k+1} D_I, \text{ where } D_I = \bigcap_{j \in I} D_j$$

The face maps are defined by the inclusions $d^i_k : D_I \hookrightarrow D_{I \setminus \{i\}}$ for $I = \{i_0, \ldots, i_k\}$ and $0 \leq j \leq i$, as in a Čech complex, and for each $k$ we have an augmentation map $\varepsilon_p : X_k \to X$ obtained from the inclusions $D_I \subseteq X$. In this situation the descent spectral sequence of Lemma 2.2 degenerates: since the $\varepsilon_p : X_p \to X$ are closed immersions and hence affine, $R^q\varepsilon_{p*}\mathcal{O}_{X_p} = 0$ for $q > 0$. It follows that $R^i\varepsilon_*\mathcal{O}_{X_s}$ is the cohomology of the Čech type complex

$$\varepsilon_0_\mathcal{O}_{X_0} \xrightarrow{d^1} \varepsilon_1_\mathcal{O}_{X_1} \xrightarrow{d^2} \cdots \xrightarrow{d^N} \varepsilon_N_\mathcal{O}_{X_N} = \bigoplus_{i} \mathcal{O}_{D_i} \xrightarrow{d^1} \bigoplus_{i<j} \mathcal{O}_{D_i \cap D_j} \xrightarrow{d^2} \cdots \xrightarrow{d^N} \mathcal{O}_{\bigcap D_i} \quad \tag{2.7}$$

**Lemma 2.8.** The complex (2.7) is exact in degrees $i > 0$, with $\ker d^1 \cong \mathcal{O}_{\bigcup \mathring{D}_i}$. Equivalently, the extended complex

$$0 \to \mathcal{O}_{X}(\sum D_i) \to \mathcal{O}_{X} \xrightarrow{\gamma} \bigoplus_{i} \mathcal{O}_{D_i} \xrightarrow{d^1} \bigoplus_{i<j} \mathcal{O}_{D_i \cap D_j} \xrightarrow{d^2} \cdots \xrightarrow{d^N} \mathcal{O}_{\bigcap D_i} \to 0$$

where $\gamma : \mathcal{O}_{X} \to \bigoplus \mathcal{O}_{D_i}$ is restriction in each factor is exact, and hence there is a canonical quasi-isomorphism $\mathcal{O}_{X}(\sum D_i) \cong \text{cone}(\mathcal{O}_{X} \to R\varepsilon_*\mathcal{O}_{X_s})[-1]$. 


Proof. We proceed by induction on the number $N$ of divisors. The base case $N = 0$ is vacuous ($X$ is empty). If that seems too weird, the case $N = 1$ simply says that the sequence $0 \to \mathcal{O}_X(-D_1) \to \mathcal{O}_X \to \mathcal{O}_{D_1} \to 0$ is exact, which is indeed the case as $D_1$ is an effective Cartier divisor.

Suppose now that $N > 1$. Then by the definition of a regular sequence, $D_1 \cap D_2, D_1 \cap D_3, \ldots, D_1 \cap D_N \subsetneq D_1$ is a regular sequence of divisors, and by permutation invariance of regular sequences (for noetherian local rings [Mat80, §15, Thm. 27], [Stacks, Tag 00LJ] — this dictated Definition 2.6) $D_2, \ldots, D_N \subsetneq X$ is a regular sequence. We form a short exact sequence of complexes (with cohomological degrees as indicated)

\[
\begin{array}{cccccccc}
C' : & 0 & \to & \mathcal{O}_{D_1} & \to & \bigoplus_{1 < j} \mathcal{O}_{D_1 \cap D_j} & \to & \bigoplus_{1 < j < k} \mathcal{O}_{D_1 \cap D_j \cap D_k} & \to & \cdots \\
\downarrow & & \downarrow & \alpha & & \downarrow & \alpha & & \\
C : & \mathcal{O}_X & \to & \bigoplus \mathcal{O}_{D_i} & \to & \bigoplus_{1 < j} \mathcal{O}_{D_i \cap D_j} & \to & \bigoplus_{1 < j < k} \mathcal{O}_{D_i \cap D_j \cap D_k} & \to & \cdots \\
\downarrow & \beta & & \downarrow & \beta & & \downarrow & \beta & & \\
C'' : & \mathcal{O}_X & \to & \bigoplus_{1 < i} \mathcal{O}_{D_i} & \to & \bigoplus_{1 < j} \mathcal{O}_{D_i \cap D_j} & \to & \bigoplus_{1 < j < k} \mathcal{O}_{D_i \cap D_j \cap D_k} & \to & \cdots \\
\end{array}
\]

(2.9)

(in fact by comparing ranges of indices we can see the columns are split short exact sequences). By inductive hypotheses,

\[
h^i(C') = \begin{cases}
\mathcal{O}_{D_1}(- \sum_{1 < j} D_j) & \text{if } i = 0 \\
0 & \text{otherwise}
\end{cases}
\text{ and } h^i(C'') = \begin{cases}
\mathcal{O}_X(- \sum_{1 < j} D_j) & \text{if } i = -1 \\
0 & \text{otherwise}
\end{cases}
\]

showing that $h^i(C) = 0$ for $i > 0$, and that in low degrees there is an exact sequence

\[
0 \to h^{-1}(C) \to \mathcal{O}_X(- \sum_{1 < j} D_j) = h^{-1}(C'') \to h^0(C') = \mathcal{O}_{D_1}(- \sum_{1 < j} D_1 \cap D_j) \to 0 \quad (2.10)
\]

To complete the proof, we must verify that the connecting map $\delta$ is indeed restriction of sections, so that (2.10) coincides with the usual exact sequence

\[
0 \to \mathcal{O}_X(- \sum_{j} D_j) \to \mathcal{O}_X(- \sum_{1 < j} D_j) \to \mathcal{O}_{D_1}(- \sum_{1 < j} D_1 \cap D_j) \to 0
\]

and indeed, by the snake lemma construction of the connecting map $\delta$ we lift a local section $\sigma \in \text{ker } \gamma'' \subset \mathcal{O}_X$ along $\beta$, apply $\gamma : \mathcal{O}_X \to \bigoplus \mathcal{O}_{D_i}$ to obtain a local section $(\sigma|_{D_i}) \in \text{ker } \beta \subset \bigoplus \mathcal{O}_{D_i}$, and then lift along $\alpha : \mathcal{O}_{D_1} \to \bigoplus \mathcal{O}_{D_i}$ — the net result is $\sigma|_{D_1}$ as claimed. \qed

Remark 2.11. Here we sketch a different proof of Lemma 2.8, which could potentially shed more light on what happens if $D_1, \ldots, D_N \subsetneq X$ deviates from being a regular sequence. For each $i$ let $\sigma_i : \mathcal{O}_X \to \mathcal{O}_X(D_i)$ be the canonical global section and let $\sigma_i' : \mathcal{O}_X(-D_i) \to \mathcal{O}_X$ be its dual. For each subset $J \subseteq \{1, \ldots, N\}$ let $\mathcal{E}_J := \bigoplus_{i \notin J} \mathcal{O}_X(D_i)$. For each such $J$ we have a section $\sigma_J = (\sigma_i|_{i \in J}) : \mathcal{O}_X \to \mathcal{E}_J$. There’s a map of chain complexes

\[
\begin{array}{cccccccc}
0 = \mathcal{E}_0 & \to & \mathcal{E}_{[1]} & \to & \mathcal{E}_{[2]} & \to & \mathcal{E}_{[3]} & \to & \cdots \\
\uparrow & & \oplus \sigma_1 & & \oplus \sigma_2 & & \oplus \sigma_3 & & \\
\mathcal{O}_X & \to & \mathcal{O}_{[1]} & \to & \mathcal{O}_{[2]} & \to & \mathcal{O}_{[3]} & \to & \cdots \\
\end{array}
\]

where the horizontal differentials are alternating sums of summand inclusions (in effect, they come from the singular co-chain complex of the $N - 1$-simplex $\Delta^{N-1}$) and the vertical maps are induced
by the \( \sigma_j \). Applying the Koszul construction to the individual maps \( \sigma_j : \mathcal{O}_X \to \mathfrak{g}_j \) (along with the usual sign trick) then results in a double complex \( C^{**} \) with \( C^{pq} = \bigoplus_{|j|=p} \wedge^{-q} \mathfrak{g}_j^{\vee} \).

I conjecture\(^6\) that the horizontal complexes

\[
C^{q} : 0 \to \cdots \to 0 \to \bigoplus_{|j|=-q} \wedge^{-q} \mathfrak{g}_j^{\vee} \to \bigoplus_{|j|=-q+1} \wedge^{-q+1} \mathfrak{g}_j^{\vee} \to \cdots \to \bigoplus_{|j|=N} \wedge^{-q} \mathfrak{g}_j^{\vee} = \wedge^{-q} \left( \bigoplus_{i=1}^N \mathcal{O}_X(-D_i) \right)
\]

are exact for \( q > -N \), and hence \( \text{Tot}(C^{**}) \) is quasi-isomorphic to \( \wedge^N \left( \bigoplus_{i=1}^N \mathcal{O}_X(-D_i) \right) = \mathcal{O}_X(-\sum_i D_i) \).

On the other hand, the vertical complexes

\[
C^{p} : 0 \to \cdots \to 0 \to \bigoplus_{|j|=p} \mathfrak{g}_j^{\vee} \to \bigoplus_{|j|=p-1} \mathfrak{g}_j^{\vee} \to \cdots \to \bigoplus_{|j|=0} \mathfrak{g}_j^{\vee} = \bigoplus_{|j|=0} \mathcal{O}_X
\]

are direct sums of Koszul complexes by design, and so their cohomology is

\[
h^q(C^{p}) = \bigoplus_{|j|=p} \mathcal{O}_X \mathcal{O}_D_j,
\]

which reduces to

\[
h^q(C^{p}) = \begin{cases} \bigoplus_{|j|=p} \mathcal{O}_D_j & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}
\]

precisely when the sequence \( D_1, \ldots, D_N \) is regular \([\text{Mat80}, \S 18 \text{ Thm. 43}]\), \([\text{Ful98}, \text{Lem. A.5.3}]\). As a technical aside, this approach might show that \textbf{Lemma 2.8} holds under slightly weaker hypotheses of \textit{Koszul regularity} (see e.g. \([\text{Stacks, Tag 062D}]\)).

### 2.3 Replacing the ideal sheaf with a filtered complex

Let \( X \) be a locally noetherian scheme and let \( D_1, \ldots, D_N \subseteq X \) be a regular sequence of effective Cartier divisors, with sum \( \Delta_X := \sum_{i=1}^N D_i \). By \textbf{Lemma 2.8} the ideal sheaf \( \mathcal{O}_X(-\Delta_X) \) is quasi-isomorphic to \( \text{cone}(\mathcal{O}_X \to R\mathcal{O}_X)[-1] \), which for convenience moving forward we give a name:\(^7\)

\textbf{Definition 2.12.} \( \Omega^0_{X, \Delta_X} : = \text{cone}(\mathcal{O}_X \to R\mathcal{O}_X)[-1] \).

By \textbf{Lemma 2.8} and its proof this complex has the explicit representation

\[
\mathcal{O}_X \longrightarrow \bigoplus_j \mathcal{O}_{D_j} \longrightarrow \bigoplus_{i \leq j} \mathcal{O}_{D_i \cap D_j} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\sum D_i}
\]

\[
0 \quad 1 \quad 2 \quad \ldots \quad N
\]

We can give \( \Omega^0_{X, \Delta_X} \) a descending filtration by truncations

\[
\Omega^0_{X, \Delta_X} = \sigma_{\geq 0} \Omega^0_{X, \Delta_X} \supset \sigma_{\geq 1} \Omega^0_{X, \Delta_X} \supset \sigma_{\geq 2} \Omega^0_{X, \Delta_X} \supset \cdots
\]

where

\[
(\sigma_{\geq j} \Omega^0_{X, \Delta_X}) = \begin{cases} \Omega^0_{X, \Delta_X} & \text{if } j < i \\ 0 & \text{otherwise} \end{cases}
\]

Using this filtration we obtain a spectral sequence for higher direct images.

\(^6\)It seems a proof by induction on \( N \) analogous to the argument in \textbf{Lemma 2.8} works, although it is combinatorially more involved.

\(^7\)This notation is chosen to align with the fact that over \( \mathbb{C} \) and when \( (X, \Delta_X) \) is a simple normal crossing pair, the complex \( (\Omega^0_{X, \Delta_X}) \) represents the 0th graded part of the Du Bois complex of the pair \( (X, \Delta_X) \).
Corollary 2.14. Let $S$ be a locally noetherian scheme and let $f : X \to S$ be a finite type morphism. Let $D_1, \ldots, D_N \subseteq X$ be a regular sequence of effective Cartier divisors, with sum $\Delta_X$. Then there is a filtered complex $(Rf_*\mathcal{O}_{X,\Delta_X}, F)$ whose cohomology computes the higher direct images $R^{i+j}f_*\mathcal{O}_X(-\Delta_X)$. For each $i$ there is a distinguished triangle

$$F^{i+1}Rf_*\mathcal{O}_{X,\Delta_X}^0 \rightarrow F^iRf_*\mathcal{O}_{X,\Delta_X}^0 \rightarrow \prod_{J \subseteq \{1, \ldots, N\} | |J|=i} Rf_*\mathcal{O}_{D_J} \rightarrow \cdots$$

In particular, there is a spectral sequence

$$E_1^{ij} = \prod_{J \subseteq \{1, \ldots, N\} | |J|=i} R^j f_*\mathcal{O}_{D_J} \Rightarrow R^{i+j}f_*\mathcal{O}_X(-\Delta_X)$$

The filtration $F$ is defined as $F = Rf_*\sigma$, and the resulting spectral sequence is just the usual hypercohomology spectral sequence.

Remark 2.15. Viewing $\varepsilon : X_{\sigma} \to X$ as a sort of resolution of the pair $(X, \Delta_X)$, we can consider the spectral sequence of Corollary 2.14 as a sort of descent spectral sequence (see [SGA4II, Vbis], [Con03]).

3 Simple normal crossing divisors and thriftyness

3.1 Definitions and basic properties

Definition 3.1 ([EGAIV, §7.8]). A scheme $X$ is excellent if and only if

- $X$ is locally noetherian,
- for every point $x \in X$ the fibers of the natural map $\text{Spec} \mathcal{O}_{X, x}^\wedge \to \text{Spec} \mathcal{O}_{X, x}$ are regular,
- for every integral $X$-scheme $Z$ that is finite over an affine open of $X$, there is a non-empty regular open subscheme $U \subseteq Z$, and
- every scheme $X$ locally of finite type over $X$ is catenary (that is, if $x \in X$ and $x \rightsquigarrow y$ is a specialization, then any 2 saturated chains of specializations $x = x_0 \rightsquigarrow x_1 \rightsquigarrow \cdots \rightsquigarrow x_n = y$ have the same length).

If $X$ is excellent, then the locus

$$\text{Reg}(X) = \{x \in X | \mathcal{O}_{X, x} \text{ is regular}\}$$

is open [EGAIV, Prop. 7.8.6]; we will make repeated use of this fact.

We first relate the notion of a simple normal crossing pair to the regular sequences of effective Cartier divisors considered in the previous section.

Lemma 3.2. If $(X, \Delta_X = \sum_i D_i)$ is a simple normal crossing pair then $(D_i)$ is a regular sequence of effective Cartier divisors.

Proof. Let $x \in X$ be a point and as above let $I(x) = \{i | x \in D_i\}$. Let $f_j \in \mathfrak{m}_x \subseteq \mathcal{O}_{X, x}^\wedge$ be local generators for the $D_j$, for $j \in I(x)$. By hypothesis for any subset $J \subseteq I(x)$ the quotient $A/(f_j | j \in J)$ is regular, and so by induction we reduce to the commutative algebra statement that if $A$ is a regular local ring, $f \in A$ and $A/f$ is a regular local ring with dimension $\dim A - 1$ then $f$ is a non-0-divisor (see for example [Stacks, Tag 0AGA]).

Lemma 3.3. Let $X$ be an integral excellent scheme with an effective Weil divisor $\Delta_X = \sum_i D_i$, and for each $i$ let $\mathcal{I}_i \subseteq \mathcal{O}_X$ be the ideal sheaf of $D_i$. Then the locus

$$\text{snc}(X, \Delta_X) := \{x \in X | \sum_{i \in I(x)} \mathcal{I}_{i,x}^\wedge \subseteq \mathcal{O}_{X, x}^\wedge \text{ is a simple normal crossing pair}\} \subseteq X$$

is open, and this is the largest open set $U \subseteq X$ such that $(U, \Delta_X|_U)$ is a simple normal crossing pair.
We could alternatively just declare \( \text{snc}(X, \Delta_X) \) to be the largest open set \( U \subseteq X \) such that \( (U, \Delta_X|_U) \) is a simple normal crossing pair; the content of the lemma is that in some sense the snc locus is “already open.”

**Proof.** Suppose \( J \subseteq \{1, \ldots, N\} \), and write \( \mathcal{F}_J = (f_j \in \mathcal{O}_{X,x} \mid j \in J) \subseteq \mathcal{O}_{X,x} \). Consider the co-cartesian diagram of noetherian local rings

\[
\begin{array}{ccc}
\mathcal{O}_{X,x}^\wedge & \longrightarrow & \mathcal{O}_{X,x}^\wedge / \mathcal{F}_J \mathcal{O}_{X,x}^\wedge \\
\uparrow & & \uparrow \\
\mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}_{X,x} / \mathcal{F}_J
\end{array}
\]

The vertical homomorphisms are faithfully flat and by hypothesis \( \mathcal{O}_{X,x}^\wedge / \mathcal{F}_J \mathcal{O}_{X,x}^\wedge \) is regular — since regularity satisfies faithfully flat descent, \( \mathcal{O}_{X,x} / \mathcal{F}_J \) is also regular. Thus \( D_j \) is regular at the point \( x \in D_j \), and as \( x \) is excellent by hypothesis the regular locus of \( D_j \) is open. Letting \( x \in U_j \subseteq X \) be a neighborhood such that \( D_j \cap U_j \subseteq D_j \) is regular and then letting \( U = \cap_j D_j \) gives a neighborhood of \( x \) such that \( (U, (D_j \cap U)) \) is a simple normal crossing pair. \( \square \)

Note that for a simple normal crossing pair \((X, \Delta_X)\), since the intersections \( D_j = \cap_{j \in J} D_j \) are regular, their connected components and irreducible components coincide. For convenience we recall the definitions of strata and thriftiness mentioned in the Introduction.

**Definition 3.4.** A **stratum** of a simple normal crossing pair \((X, \Delta_X = \sum_i D_i)\) is a connected (equivalently, irreducible) component of an intersection \( D_j = \cap_{j \in J} D_j \).

**Definition 3.5** (compare with [Kol13, Def. 2.79-2.80], [KX16, §1, discussion before Def. 10]). Let \((S, \Delta_S = \sum_i D_i)\) be a pair in the sense of **Definition 1.5**, and assume \( \Delta_S \) is reduced and effective. A separated, finite type birational morphism \( f : X \to S \) is **thrift with respect to** \( \Delta_S \) if and only if

1. \( f \) is an isomorphism over the generic point of every stratum of \( \text{snc}(S, \Delta_S) \)

2. letting \( D_i = f^{-1}_* D_i \) for \( i = 1, \ldots, N \) be the strict transforms of the divisors \( D_i \), and setting \( \Delta_X := \sum_i \overline{D_i} \), the map \( f \) is an isomorphism at the generic point of every stratum of \( \text{snc}(X, \Delta_X) \).

The restriction that \( D_i \cap \text{Reg}(S) \neq \emptyset \) for all \( i \) ensures that if \( \eta \in D_i \) is a generic point of a component, then \( \eta \in \text{Reg}(S) \). Since on a regular scheme every Weil divisor is Cartier, and as \( S \) is excellent and \( D_i \) is reduced, by hypothesis, there is a neighborhood \( \eta \in U \subseteq S \) such that \( U, D_i \cap U \) is a simple normal crossing pair. In other words, \( \eta \in \text{snc}(S, \Delta_S) \) is the generic point of a stratum, so (i) implies \( f^{-1}(\eta) \) is a single (non-closed) point. For our purposes the strict transform \( D_i \) can be defined as

\[
D_i := \bigcup_{\eta \in D_i, \text{generic}} f^{-1}(\eta) \subseteq X.
\]

Since \( f \) is an isomorphism over \( \eta \), we also see \( f^{-1}(\eta) \subseteq \text{snc}(X, \Delta_X) \).

**Lemma 3.6.** Let \( S \) be an integral excellent noetherian scheme with a sequence of reduced effective Weil divisors \( D_1, \ldots, D_N \subseteq S \) such that no component of \( \cup_i D_i \) is contained in \( \text{Sing}(X) \), and let \( f : X \to S \) be a separated, finite type birational morphism. Then, \( f \) is thrift if and only if there is a diagram of separated finite type \( S \)-schemes

\[
S \leftarrow U \leftarrow X
\]

with both morphisms (necessarily dense) open immersions, such that \( U \) contains all generic points of strata of \( \text{snc}(S, \Delta_S) \) and \( \text{snc}(X, \Delta_X) \).

**Proof.** Since the existence of a common dense open \( S \leftarrow U \leftarrow X \) as in the statement of the lemma certainly guarantees (i) and (ii), we focus on the “only if,” and in fact we show that one can take \( U = \) the maximal domain of definition of \( f^{-1} : S \to X \). By (i) of **Definition 3.5** this \( U \) contains all generic points of strata of \( \text{snc}(S, \Delta_S) \).
Suppose \( \xi \in \text{snc}(X, \Delta_X) \) is a generic point of a stratum. By hypothesis there is a neighborhood \( \xi \in V \subseteq X \) such that \( f|_V : V \to S \) is an isomorphism onto its image. Then \( W := f(V) \) is a Zariski neighborhood of \( f(\xi) \) and the inverse of \( f|_V \) gives a section of the birational map \( X_W = X \times_S W \to W \).

\[
\begin{array}{c}
V \\
\downarrow f|_V \\
W
\end{array} \xrightarrow{f_W} \begin{array}{c}
X_W \\
\downarrow \\
S
\end{array}
\]

But then the inclusion \( V \hookrightarrow X_W \) is a proper dense open immersion, hence an isomorphism. \( \Box \)

Remark 3.7. It seems that the above proof shows in addition that \( f(\xi) \in S \) is the generic point of a stratum of \( \text{snc}(S, \Delta_S) \).

We will make repeated use of a few blowup lemmas from the construction of Nagata compactifications in Section 4 — here, they are used to show that thrifty morphisms can be dominated by certain admissible blowups.

**Lemma 3.8** ([Con07, Lem. 2.4, Rmk. 2.5, Cor. 2.10]). Let \( S \) be a quasi-compact, quasi-separated scheme.

(i) If \( X \) is a quasi-separated quasi-compact \( S \)-scheme and \( Y \) is a proper \( S \)-scheme, and if \( f : U \to Y \) is an \( S \)-morphism defined on a dense open \( U \subseteq X \), then there exists a \( U \)-admissible blowup \( \tilde{X} \to X \) and an \( S \)-morphism \( \tilde{f} : \tilde{X} \to Y \) extending \( f \).

(ii) Let \( j_i : U \to X_i \) be a finite collection of dense open immersions between finite type separated \( S \)-schemes. Then there exist \( U \)-admissible blowups \( X'_i \to X \) and a separated finite type \( S \)-scheme \( X \), together with open immersions \( X'_i \hookrightarrow X \) over \( S \), such that the \( X'_i \) cover \( X \) and the open immersions \( U \hookrightarrow X'_i \hookrightarrow X \) are all the same.

**Corollary 3.9.** There exist \( U \)-admissible blowups

\[
\begin{array}{ccc}
\tilde{X} & \xleftarrow{\text{open imm}} & S \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & S
\end{array}
\]

In particular if \( f \) is proper then \( X \) and \( S \) have a common \( U \)-admissible blowup.

**Proof.** By Lemma 3.8 there are a separated, finite type \( S \)-scheme \( Y \), \( U \)-admissible blowups \( \tilde{S} \to S \) and \( \tilde{X} \to X \) and dense open immersions \( \tilde{S} \hookrightarrow Y \hookrightarrow \tilde{S} \) over \( S \) such that the diagram

\[
\begin{array}{ccc}
U & \xleftarrow{\tilde{f}} & \tilde{X} \\
\downarrow & & \downarrow \\
S & \xrightarrow{f} & Y
\end{array}
\]

commutes. Since \( \tilde{S} \) is proper over \( S \), the bottom arrow is necessarily an isomorphism, in other words \( Y = \tilde{S} \). If \( f \) is proper then \( \tilde{X} \) is proper over \( S \), so \( Y = \tilde{X} \) as well. \( \Box \)

**Remark 3.10.** If \( (S, \Delta_S) \) is a simple normal crossing pair and \( U \subseteq S \) is an open containing all strata, a \( U \)-admissible blowup \( f : X \to S \) need not be thrifty, see Example A.12.

### 3.2 The regular-to-regular case

Using Corollary 2.14 we can already obtain a restricted form of Theorem 1.12, the case of a thrifty proper birational morphism of simple normal crossing pairs. In the proof we will make use of Grothendieck duality, as formulated in [Con00; R&D].
Theorem 3.11 (Grothendieck duality, [R&D, Cor. VII.3.4], [Con00, Thm. 3.4.4]). Let \( f : X \to Y \) be a proper morphism of finite-dimensional noetherian schemes and assume \( Y \) admits a dualizing complex (for example \( X \) and \( Y \) could be schemes of finite type over \( k \)). Then for any pair of objects \( \mathcal{F}^* \in D^b_c(X) \) and \( \mathcal{G}^* \in D^b_c(Y) \) there is a natural isomorphism
\[
Rf_*R\text{Hom}_Y(\mathcal{F}^*, f^*\mathcal{G}^*) \cong R\text{Hom}_X(Rf_*\mathcal{F}^*, \mathcal{G}^*) \text{ in } D^b_c(Y)
\]
If \( \omega_Y^* \) is a dualizing complex on \( Y \) then \( \omega_X^* := f^!\omega_Y^* \) is a dualizing complex on \( X \) [R&D, §V.10, Cor. VI.3.5], and so in the case \( \mathcal{G} = \omega_Y^* \) we obtain a natural isomorphism
\[
Rf_*R\text{Hom}_Y(\mathcal{F}^*, \omega_Y^*) \cong R\text{Hom}_X(Rf_*\mathcal{F}^*, \omega_Y^*) \text{ in } D^b_c(Y)
\]
When \( X \) is a Cohen-Macaulay \( n \)-dimensional noetherian scheme (so in particular when \( X \) is a regular \( n \)-dimensional noetherian scheme), a dualizing complex \( \omega_X^* \) satisfies \( h^i\omega_X^* = 0 \) for \( i \neq -n \), and the unique non-0 cohomology sheaf is called the dualizing sheaf and denoted \( \omega_X := h^{-n}\omega_X^* \).

Theorem 3.12. Let \( (Y, \Delta_Y) \) be a simple normal crossing pair and let \( f : X \to Y \) be a thrifty proper birational morphism. Assume \( (X, \Delta_X) \) is also a simple normal crossing pair. Then, the natural map
\[
\Theta_Y(-\Delta_Y) \to Rf_*\Theta_X(-\Delta_X) \text{ is a quasi-isomorphism,}
\]
and it follows that \( Rf_*\omega_X(\Delta_X) \cong \omega_Y(\Delta_Y) \).

Proof. Let \( X \) (resp. \( Y \)) be the semi-simplicial scheme associated to \( (X, \Delta_X) \) (resp. \( (Y, \Delta_Y) \)). For any \( J \subseteq \{1, \ldots, N\} \) \( f \) restricts to a morphism \( \cap_{j \in J} \tilde{D}_j \to \cap_{j \in J} D_j \), and in this way we obtain a morphism of semi-simplicial schemes
\[
\begin{array}{ccccccc}
\cdots & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & X \\
\downarrow f_2 & \downarrow f_1 & \downarrow f_2 & \downarrow f & \\
\cdots & Y_2 & \longrightarrow & Y_1 & \longrightarrow & Y_0 & \longrightarrow & Y
\end{array}
\tag{3.13}
\]
The hypothesis that both pairs have simple normal crossings and \( f \) is thrifty implies that for each \( i \), \( f_i : X_i \to Y_i \) is a proper birational morphism of (possibly disconnected) regular schemes over \( k \). By [CR11, Cor. 3.2.10] (or [CR15, Thm. 1.1], [Kov22, Thm. 1.4])
\[
\Theta_{Y_i} \cong Rf_*\Theta_{X_i} \text{ is a quasi-isomorphism for all } i
\tag{3.14}
\]
The diagram (3.13) induces a morphism of filtered complexes \( f^* : \Omega^0_{Y, \Delta_Y} \to Rf_*\Omega^0_{X, \Delta_X} \), and by Lemma 2.8 and Corollary 2.14 it will suffice to show that the resulting map of descent spectral sequences
\[
E_1^{ij}(Y) = \left\{ \prod_{\sigma \in D(\Delta_Y)^{j-1}} \Theta_D(\sigma) \right\}
\]
\[
\begin{cases}
\prod_{\sigma \in D(\Delta_X)^{j-1}} \Theta_D(\sigma) & \text{otherwise} \\
1 & \text{otherwise}
\end{cases}
\]
is an isomorphism, and this last step is a consequence of (3.14).

Finally, once we know \( \Theta_Y(-\Delta_Y) \cong Rf_*\Theta_X(-\Delta_X) \), applying the functor \( R\text{Hom}(\cdot, -) \) we see that
\[
R\text{Hom}(\Theta_Y(-\Delta_Y), \omega_Y^*) \cong R\text{Hom}(Rf_*\Theta_X(-\Delta_X), \omega_Y^*) \cong Rf_*R\text{Hom}(\Theta_X(-\Delta_X), \omega_Y^*)
\tag{3.15}
\]
where the second quasi-isomorphism comes from Theorem 3.11. Since \( (X, \Delta_X) \) and \( (Y, \Delta_Y) \) are simple normal crossing pairs by hypothesis, in particular we know that the divisors \( \Delta_X \) and \( \Delta_Y \) are Cartier, so that
\[
R\text{Hom}(\Theta_Y(-\Delta_Y), \omega_Y^*) \cong \omega_Y^*(\Delta_Y) \text{ and } Rf_*R\text{Hom}(\Theta_X(-\Delta_X), \omega_Y^*) \cong Rf_*\omega_X^*(\Delta_X).
\tag{3.16}
\]

\[\square\]
4 Constructing semi-simplicial projective Cohen-Macaulayfications

4.1 Preliminaries

In the situation of Theorem 1.12, if $Z$ is smooth and $\Delta_Z$ is snc, then Theorem 3.12 applied to both $r$ and $s$ shows

$$Rf_*\mathcal{O}_X(-\Delta_X) \cong Rf_*Rr_*\mathcal{O}_Z(-\Delta_Z) = Rg_*Rs_*\mathcal{O}_Z(-\Delta_Z) \cong Rg_*\mathcal{O}_Y(-\Delta_Y).$$

Of course, $Z$ need not be smooth and in the absence of resolution of singularities away from characteristic $0$, we cannot replace it by a resolution. In characteristic $p > 0$ we could replace $Z$ with an alteration, but only at the cost of allowing $r, s$ to be generically finite but not necessarily birational, and as such using alterations seems incompatible with the strategy of Theorem 3.12. Moreover, to the best of our knowledge at the level of generality Theorem 1.12 is stated, even alterations are unavailable.\(^8\)

Instead, we will replace $Z$ with a mildly singular (specifically Cohen-Macaulay and normal) semi-simplicial scheme $Z$, together with morphisms $X, \xrightarrow{r} Z, \xrightarrow{s} Y$, over $S$ which are term-by-term proper birational equivalences over $S$. It is in this construction that we need Conjecture 1.11, restated here for convenience:

**Conjecture 4.1** (see also [Čes21, Conj. 1.1], [Kov22, Conj. 1.14]). *For every CM-quasi-excellent noetherian scheme $X$ there exists a projective birational morphism $\pi : \tilde{X} \rightarrow X$ such that $\tilde{X}$ is Cohen-Macaulay and normal and $\pi$ is an isomorphism over the regular locus $\text{Reg}(X) \subset X$.*

The usefulness of normal Cohen-Macaulayfications for the problem at hand stems from an extension of the results of Chatzistamatiou-Rülling due to Kovács.

**Theorem 4.2** ([Kov22, Thm. 1.4]). *Let $f : X \rightarrow Y$ be a locally projective birational morphism of excellent Cohen-Macaulay normal schemes. If $Y$ has pseudo-rational singularities then $\omega_Y = Rf_*\mathcal{O}_X$ and $Rf_*\omega_X = \omega_Y$.*

By a result of Lipman-Teissier, if $Y$ is regular (so in particular if it is smooth over $k$) then $Y$ is pseudo-rational [LT81, §4], hence Theorem 4.2 applies when $Y$ is regular.

4.2 Gluing on simplices

In this section we describe an inductive method for constructing a sequence of truncated semi-simplicial schemes converging to $Z$. Here for any $i \in \mathbb{N}$ an $i$-truncated semi-simplicial object in a category $\mathcal{C}$ is a functor $\Lambda^\text{op}_i \rightarrow \mathcal{C}$, where $\Lambda^\text{op}_i$ is the full subcategory of $\Lambda^\text{op}$ generated by the objects $[j]$ with $j \leq i$. Given an $i - 1$-truncated semi-simplicial object $X$, of $\mathcal{C}$, let

$$[i]_k^2 := \{j, k \in [i] \mid j < k\}$$

and define two morphisms

$$\delta_+, \delta_- : X_{i-1}^{[i]} \rightarrow X_{i-2}^{[i]_2}$$

by $\delta_+(x_0, \ldots, x_i) = (d_{i-1}^{[i]}(x_k) \mid j < k)$ and $\delta_-(x_0, \ldots, x_i) = (d_{k-1}^{[i]}(x_j) \mid j < k)$. Assuming $\mathcal{C}$ has finite limits we may form the equalizer

$$E(X_i) := \text{Eq}(\delta_+, \delta_-) \longrightarrow X_{i-1}^{[i]} \xrightarrow{\delta_+} X_{i-2}^{[i]_2}$$

(4.3)

\(^8\)At least at the time of this writing ...

\(^9\)Ditto.
one can check that this construction is functorial in $X$; indeed if $Y$ is another $i - 1$-truncated semi-simplicial object then given a morphism $X \to Y$, we can form a commutative diagram

$$
E(X) := \text{Eq}(\delta_+, \delta_-) \to X^{[i]}_{i-1} \overset{\delta_+}{\underset{\delta_-}{\to}} X^{[i]}_{i-2}
$$

(4.4)

and obtain a unique morphism on the dashed arrow by functoriality of equalizers. Finally, let $I$ denote the category $0 \to 1$ (thought of as the “unit interval”). An object of $\mathcal{C}^I$ is a morphism $f : X \to Y$ in $\mathcal{C}$ and there are 2 functors $s : \mathcal{C}^I \to \mathcal{C}$ defined by $s(f) = X, t(f) = Y$ (source and target).

**Lemma 4.5** (compare with [SGA4II, Vbis, Prop. 5.1.3], [Stacks, Tag 0AMA]). Let $\mathcal{C}$ be a category containing finite limits. The functor

$$
\Phi_i : \mathcal{C}^{\Delta^I_{\leq i}} \to \mathcal{C}^{\Delta^I_{\leq i-1}} \times_{\mathcal{C}} \mathcal{C}^I,
$$

where the right hand side is the 2-fiber product with respect to the functors $E : \mathcal{C}^{\Delta^I_{\leq i-1}} \to \mathcal{C}$ and $t : \mathcal{C}^I \to \mathcal{C}$ that sends an $i$-truncated semi-simplicial object $X$, to the pair $(sk_{i-1}X, X \to E(sk_{i-1}X))$, is an equivalence of categories.

**Proof.** We first check that $\Phi_i$ is fully faithful. For faithfulness, note that for any 2 $i$-truncated semi-simplicial objects $X, Y$, there is an injection

$$
\text{Hom}_{\mathcal{C}^{\Delta^I_{\leq i}}}(X, Y) \hookrightarrow \prod_{j=0}^i \text{Hom}_{\mathcal{C}}(X_j, Y_j)
$$

(4.6)

since a morphism $\alpha : X \to Y$, is equivalent to a sequence of morphisms $\alpha_i : X_i \to Y_i$ commuting with differentials. By the definition of the 2-fiber product, the morphism $\Phi_i(\alpha) : \Phi_i(X) \to \Phi_i(Y)$ induced by $\alpha$ consists of the morphism $sk_{i-1}\alpha : sk_{i-1}X \to sk_{i-1}Y$, and the commutative diagram

$$
X_i \longrightarrow E(sk_{i-1}X) \quad \Phi_i(\alpha) \quad E(sk_{i-1}Y)
$$

This shows that (4.6) factors as

$$
\text{Hom}_{\mathcal{C}^{\Delta^I_{\leq i}}}(X, Y) \overset{\Phi_i}{\longrightarrow} \text{Hom}_{\mathcal{C}^{\Delta^I_{\leq i-1}} \times_{\mathcal{C}} \mathcal{C}^I}(\Phi_i(X), \Phi_i(Y)) \to \prod_{j=0}^i \text{Hom}_{\mathcal{C}}(X_j, Y_j)
$$

(4.7)

hence the first map is injective, or in other words $\Phi_i$ is faithful. On the other hand given an arbitrary morphism $\Phi_i(X) \to \Phi_i(Y)$ consisting of a map $\beta : sk_{i-1}X \to sk_{i-1}Y$, a map $\gamma : X_i \to Y_i$ and a commutative diagram

$$
X_i \longrightarrow E(sk_{i-1}X) \quad E(\beta) \quad E(sk_{i-1}Y)
$$

(4.8)
we may verify commutativity of

![Diagram](image)

as follows: commutativity of (1) is exactly (4.8), and commutativity of (2) can be deduced from that of the left square of (4.4). Hence \( \beta \) and \( \gamma \) define a map \( X_\ast \rightarrow Y_\ast \), and so \( \Phi_i \) is full.

Next we show \( \Phi_i \) is essentially surjective — this argument is inspired by and closely follows the proof of [Stacks, Tag 0186]. For this we consider an object of the 2-fiber product \( E^{\psi \phi}_{X_\ast} \times_{\mathcal{C}^{op}} \mathcal{C}^{op} \) consisting of an \( i-1 \)-truncated semi-simplicial object \( X_\ast \), and object \( Y \) and a morphism \( f : Y \rightarrow E(X_\ast) \), and we must prove that there exists an \( i \)-truncated semi-simplicial object \( Z \), and an isomorphism \( \Phi_i(Z) \simeq (X_\ast, f) \). We first let \( Z_j = X_j \) for \( j < i \) and let \( Z(\varphi) = X(\varphi) \) for any \( \varphi : [j'] \rightarrow [j] \) with \( j' < j < i \). Then we set \( Z_i = Y \), and we must define morphisms \( Z(\varphi) : Z_i = Y \rightarrow X_j = Z_j \) for increasing maps \( [j] \rightarrow [i] \) which are functorial in \( \varphi \), in the sense that for any increasing \( \psi : [j'] \rightarrow [j] \) the diagram

![Diagram](image)

commutes (note that the data of \( X(\psi) \) is already included in \( X_\ast \)). We may assume \( j < i \) (otherwise \( \varphi = \text{id} \) and we must set \( Z(\varphi) = \text{id} \)), and so \( \varphi \) must factor as

\[
[j] \xrightarrow{\psi} [i-1] \xrightarrow{\delta_i^k} [i]
\]

for some \( k \) and some \( \psi \). We define \( Z(\varphi) \) to be the composition

\[
Y \xrightarrow{f} E(X_\ast) \xrightarrow{X(\varphi)} X_{i-1} \xrightarrow{\text{pr}_k} X_{i-1} \xrightarrow{\delta_i^k} X_j
\]

(so in particular we define \( Z(\delta_i^k) = \text{pr}_k \circ f =: f_k \)). To verify this definition is independent of \( \psi \), suppose that there is another factorization

\[
[j] \xrightarrow{\psi'} [i-1] \xrightarrow{\delta_i^k} [i]
\]

Note that if \( j = i-1 \) then \( \psi = \psi' = \text{id} \) and \( k = i \) for trivial reasons, so we may assume \( j < i-1 \) and in that case \( \varphi \) misses both \( k \) and \( i \), so we may factor through \( [i-2] \) as follows:

![Diagram](image)

By the defining property of the equalizer \( E(X_\ast) \), we know \( X(\delta_{i-1}^j \circ f_k) = X(\delta_{i-1}^j \circ f_i) \), and

\[
X(\varphi) \circ X(\delta_{j-1}^i) = X(\psi) \quad \text{and} \quad X(\varphi) \circ X(\delta_{k-1}^i) = X(\psi')
\]
because $X$, is an $i - 1$-truncated semi-simplicial object. It follows that $X(\psi) \circ f_k = X(\psi') \circ f_i$ as desired.

We now prove to prove the commutativity statement in (4.9). Again we may assume $j < i$, since otherwise $\varphi = \text{id}$ and $\psi = \varphi \circ \psi$ so commutativity is implied by the above proof that the $Z(\varphi)$ are well defined. When $j < k$ the map $\varphi$, and hence also $\varphi \circ \psi$ must factor through some $\delta^i_k : [i - 1] \to [i]$ and we obtain the following situation:

\[
\begin{array}{ccc}
[j] & \overset{\psi}{\longrightarrow} & [j] \\
\downarrow \varphi \psi & & \downarrow \varphi \circ \psi \\
[i - 1] & \overset{\rho}{\longrightarrow} & [i]
\end{array}
\]

Now by definition $Z(\varphi) = X(\rho) \circ f_k$ and $Z(\varphi \circ \psi) = X(\rho \circ \psi) \circ f_k$, and since $X$, is an $i - 1$-truncated semi-simplicial object $X(\rho \circ \psi) = X(\varphi) \circ X(\rho)$, so that

$$X(\psi) \circ Z(\varphi) = X(\psi) \circ X(\rho) \circ f_k = X(\rho \circ \psi) \circ f_k = Z(\varphi \circ \psi)$$

as claimed. 

\[\square\]

### 4.3 Common admissible blowups

Using Lemma 4.5 to build the semi-simplicial scheme $Z$, inductively, at each step we encounter the situation of the lemma below.

**Lemma 4.11.** Suppose

\[
\begin{array}{ccc}
X & \overset{i}{\longleftarrow} & U \\
\downarrow \varphi & & \downarrow \rho \\
F & \overset{f}{\longleftarrow} & E \\
\downarrow \psi & & \downarrow \phi \\
& & Y
\end{array}
\]

is a commutative diagram of schemes of finite type over a quasi-compact quasi-separated base scheme $S$, and assume that $f, g, \varphi$ and $\psi$ are proper and $i$ and $j$ are dense open immersions. Then, there is a commutative diagram

\[
\begin{array}{ccc}
X & \overset{r}{\longleftarrow} & Z \\
\downarrow \varphi & & \downarrow \rho \\
F & \overset{f}{\longleftarrow} & E \\
\downarrow \psi & & \downarrow \phi \\
& & Y
\end{array}
\]

where $r$ and $s$ are $U$-admissible blowups (hence in particular projective).

If in addition $S$ is a CM-quasi-excellent noetherian scheme and $U$ is regular, then assuming Conjecture 1.11 we may ensure that $Z$ is Cohen-Macaulay and normal.

**Proof.** First, $X$ and $E$ are proper over the scheme $F$, which is quasi-compact and quasi-separated since it is of finite type over $S$. By the first part of Lemma 3.8 applied to the map of $F$-schemes $\rho^0 : U \to E$ defined on the dense open $U \subseteq X$, there is a $U$-admissible blowup $V_X \to X$ and an $F$-morphism $V_X \to E$ extending $\rho^0$. A similar argument produces a $U$-admissible blowup $V_Y \to Y$. 

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and a $G$-morphism $V_Y \to E$ extending $\rho^0$. The current situation is summarized below:

\begin{center}
\begin{tikzcd}
U \arrow[r, hookrightarrow] \arrow[d, hookrightarrow] & V_Y \arrow[r, hookrightarrow] \arrow[d, hookrightarrow] & E \arrow[dl, hookrightarrow] \\
V_X \arrow[d, hookrightarrow] & G \arrow[u, hookrightarrow] \arrow[d, hookrightarrow] & F \arrow[dl, hookrightarrow] \\
X \arrow[r, hookrightarrow] & \varphi \\
Y & f & \psi & \theta
\end{tikzcd}
\end{center}

Since $V_X, V_Y$ are $U$-admissible blowups of $X, Y$ respectively, they still contain $U$ as a dense open \((\text{[Con07, comments before Lem. 1.1]})\). Note that since $V_X \to X$ is a blowup, $\varphi$ is proper and $f$ is proper the morphism $V_X \to E$ is also proper; similarly $V_Y$ is proper over $E$. Now applying the second part of Lemma 3.8 to $V_X$ and $V_Y$ over $E$ we obtain a separated finite type morphism $\varphi : Z \to E, U$ admissible blowups $\tilde{V}_X \to V_X$ and $\tilde{V}_Y \to V_Y$ and open immersions $\tilde{V}_X \leftarrow Z \leftrightarrow \tilde{V}_Y$ over $E$ such that the diagram

\begin{center}
\begin{tikzcd}
U \arrow[r, hookrightarrow] & \tilde{V}_Y \arrow[d, hookrightarrow] \\
\tilde{V}_X \arrow[d, hookrightarrow] & Z \arrow[u, hookrightarrow] \arrow[d, hookrightarrow]
\end{tikzcd}
\end{center}

commutes and $E = \tilde{V}_X \cup \tilde{V}_Y$. Since $U$ is dense in both $\tilde{V}_X$ and $\tilde{V}_Y$, we see that $\tilde{V}_X$ and $\tilde{V}_Y$ are both dense in $Z$. Then as $\tilde{V}_X \to Z$ is a dense open immersion of separated finite type $E$-schemes where $\tilde{V}_X$ is proper over $E$, it must be that $\tilde{V}_X = Z$; similarly, $\tilde{V}_Y = Z$ (see also the comments following [Con07, Cor. 2.10]). Finally, we define $r$ and $s$ to be the compositions

\begin{center}
Z \rightarrow \tilde{V}_X \rightarrow V_X \rightarrow X \quad \text{and} \quad Z \rightarrow \tilde{V}_Y \rightarrow V_Y \rightarrow Y
\end{center}

Finally if $S$ is CM-quasi-excellent, then since $Z$ is of finite type over $S$ it is also CM-quasi-excellent by [Čes21, Rmk.1.5]. By hypothesis $U \subseteq \text{CM}(Z)$, and by Conjecture 1.11 there is a $\text{Reg}(Z)$-admissible (hence also $U$-admissible) blowup $\tilde{Z} \to Z$ such that $\tilde{Z}$ is Cohen-Macaulay and normal. In this case we replace $Z$ with $\tilde{Z}$. \hfill $\square$

**Lemma 4.12.** Let $S$ be a quasi-compact quasi-separated base scheme and let

\begin{equation}
X, \leftarrow \quad U, \quad \leftarrow \quad Y, \\
\downarrow \quad \downarrow \quad \downarrow \\
X_{-1}, \leftarrow \quad U_{-1}, \quad \leftarrow \quad Y_{-1}
\end{equation}

be morphisms of augmented semi-simplicial schemes of finite type over $S$. Assume that all differentials and augmentations of $X_i$ and $Y_i$ are proper,\(^\text{10}\) and that the morphisms $X_i \leftarrow U_i \rightarrow Y_i$ are dense open immersions for all $i$ (including $i = -1$). If there exists a finite-type $S$-scheme $Z_{-1}$ and $U_{-1}$-admissible blowups $X_{-1} \leftarrow Z_{-1} \rightarrow Y_{-1}$, then there exists an augmented semi-simplicial $S$-scheme $Z, \rightarrow Z_{-1}$ together with morphisms

\begin{equation}
X, \leftarrow \quad Z, \quad \leftarrow \quad Y, \\
\downarrow \quad \downarrow \quad \downarrow \\
X_{-1}, \leftarrow \quad Z_{-1}, \quad \leftarrow \quad Y_{-1}
\end{equation}

\(^\text{10}\)This is equivalent to requiring that $X,_{-1}$ is a semi-semi-simplicial object in the category of proper $X_{-1}$-schemes (and similarly for $Y,_{-1}$).
such that for all $i$ the morphisms $X_i \xleftarrow{r_i} Z_i \xrightarrow{s_i} Y_i$ are $U_i$-admissible blowups (hence in particular projective and birational).

Moreover if $S$ is a CM-quasi-excellent noetherian scheme, each $U_i$ is regular and Conjecture 1.11 holds, then we may ensure that the $Z_i$ are also Cohen-Macaulay and normal.

**Proof.** We construct a sequence of $i$-truncated semi-simplicial $S$-schemes $Z_{i_1}$, converging to $Z_s$, with the additional requirement that the morphisms $sk_{i-1}(U_s) \to sk_{i-1}(X_s)$ and $sk_{i-1}(U_s) \to sk_{i-1}(Y_s)$ factor through $Z_{i_1}$.$^{11}$ The $i = -1$ case is included in the hypotheses. At the inductive step we may assume that there is an $i - 1$-truncated semi-simplicial $S$-scheme $Z_{i-1}$, together with a commutative diagram

$$
\begin{array}{ccc}
sk_{i-1}(U_s), & sk_{i-1}(X_s), \downarrow & sk_{i-1}(Y_s), \\
\downarrow & Z_{i-1} \downarrow & s_{i-1}, \\
X_{i-1} \leftarrow r_{i-1}, & Z_{i-1} \rightarrow Z_{i-1}, & Y_{i-1}.
\end{array}
$$

Next, we verify that (4.16) satisfies the hypotheses of Lemma 4.11, making repeated reference to the constructions in (4.3) and (4.4). Note that the bottom horizontal arrows are proper, since they are obtained as limits of the blowup maps $r_{i-1,j} : Z_{i-1} \to X_j$ and $s_{i-1,j} : Z_{i-1} \to Y_j$ for $j = i-1, i-2$. The vertical maps on the outside edges are proper since the differentials $X(\delta^l_k) : X_i \to X_{i-1}$ and $Y(\delta^l_k) : Y_i \to Y_{i-1}$ are proper by hypothesis. Hence applying Lemma 4.11 we obtain a commutative diagram

$$
\begin{array}{ccc}
X_i \leftarrow i & U_i \xrightarrow{j_i} Y_i, \\
\downarrow (U(\delta^l_k)) & \downarrow E(sk_{i-1}(U_s)) & \downarrow (Y(\delta^l_k)) \\
E(sk_{i-1}(X_s)) \leftarrow e_{i, i-1} & E(Z_{i-1}) \xrightarrow{E(s_{i-1}))} E(sk_{i-1}(Y_s)),
\end{array}
$$

in which the maps $r_{i-1,j} : Z_{i} \to X_j$ and $s_{i-1,j} : Z_{i} \to Y_s$ are $U_i$-admissible blowups. In the case where $S$ is CM-quasi-excellent, $U_i$ is regular and Conjecture 1.11 holds, we apply Lemma 4.11 to ensure that $Z_i$ is Cohen-Macaulay.

$^{11}$I think that this isn’t actually an additional restriction, but including it makes the inductive step easier.
Now Lemma 4.5 implies that there is an i-truncated semi-simplicial S-scheme $Z_i$, such that $sk_{i-1}(Z_{i,j}) = Z_{i-1,j}$, and $Z_{i,j} = Z_i$, together with a commutative diagram

$$
\begin{array}{ccccccccc}
& & & & & & & \text{sk}_i(U_*), & & & \\
& & & & & & & \text{sk}_i(U_*), & & & \\
& & & & & & & \text{sk}_i(U_*), & & & \\
\text{sk}_i(U_*), & & & & & & & \text{sk}_i(U_*), & & & \\
k_i & & & & & & & s_{i,j} & & & \\
\downarrow & & & & & & & \downarrow & & & \\
\text{sk}_i(X_i) & & & & & & & \text{sk}_i(Y_i) & & & \\
& & & & & & & \text{sk}_i(X_i) & & & \\
& & & & & & & \text{sk}_i(Y_i) & & & \\
X_{i-1} & & & & & & & Z_{i-1,j} & & & Y_j
\end{array}
$$

(4.18)

such that for all $j \leq i$ the morphisms $X_{i-1,j} \rightarrow Z_{i-1,j} \rightarrow Y_j$ are $U_j$-admissible blowups. □

**Corollary 4.19.** With the hypotheses of Lemma 4.12, if in addition the base scheme S is CM-quasi-excellent and noetherian, and all the schemes $X_i$ and $Y_i$ have pseudorational singularities, all the schemes $U_i$ are regular and Conjecture 1.11 holds, then:

(i) There is a complex $\mathcal{K}$ in the derived category $D^b_c(Z_{-1})$ together with quasi-isomorphisms

$$
\text{cone}(\mathcal{O}_{X_{-1}} \rightarrow R^\mathcal{X}_{\mathcal{O}_{X_1}}[-1]) \cong Rr_{-1,\mathcal{K}} \text{ and } Rs_{-1,\mathcal{K}} \cong \text{cone}(\mathcal{O}_{Y_{-1}} \rightarrow R^\mathcal{Y}_{\mathcal{O}_{Y_1}})[-1].
$$

(4.20)

(ii) There is a complex $\mathcal{L}$ in the derived category $D^b_c(Z_{-1})$ together with quasi-isomorphisms

$$
R\mathcal{H}\text{om}(\text{cone}(\mathcal{O}_{X_{-1}} \rightarrow R^\mathcal{X}_{\mathcal{O}_{X_1}})[-1], \omega_{X_{-1}}) \cong Rr_{-1,\mathcal{L}} \text{ and } Rs_{-1,\mathcal{L}} \cong R\mathcal{H}\text{om}(\text{cone}(\mathcal{O}_{Y_{-1}} \rightarrow R^\mathcal{Y}_{\mathcal{O}_{Y_1}})[-1], \omega_{Y_{-1}}).
$$

(4.21)

In the special case where $X$ and $Y$, are the semi-simplicial schemes associated to simple normal crossing pairs $(X, \Delta_X)$ and $(Y, \Delta_Y)$ respectively (so that $X_{-1} = X$ and $Y_{-1} = Y$), then there are quasi-isomorphisms

$$
\mathcal{O}_X(\Delta_X) \cong Rr_{-1,\mathcal{K}}, \quad Rs_{-1,\mathcal{K}} \cong \mathcal{O}_Y(\Delta_Y),
$$

$$
\omega_X(\Delta_X) \cong Rr_{-1,\mathcal{L}}, \quad Rs_{-1,\mathcal{L}} \cong \omega_Y(\Delta_Y)
$$

(4.22)

**Proof.** By the “moreover” part of Lemma 4.12 we may ensure $Z_i$ is Cohen-Macaulay for all $i$.

By functoriality, eq. (4.14) induces commutative diagrams of complexes

$$
\begin{array}{c}
\text{cone}(\mathcal{O}_{X_{-1}} \rightarrow R^\mathcal{X}_{\mathcal{O}_{X_1}}[-1]) \\
\downarrow^{\delta^X} \\
\mathcal{O}_{X_{-1}} \\
\downarrow^{\delta^X} \\
Rr_{-1,\mathcal{K}} \\
\downarrow^{\delta^X} \\
Rr_{-1,\mathcal{K}} \\
\downarrow^{\delta^X} \\
\cdots
\end{array}
$$

(4.23)

$$
\begin{array}{c}
\text{cone}(\mathcal{O}_{Y_{-1}} \rightarrow R^\mathcal{Y}_{\mathcal{O}_{Y_1}}[-1]) \\
\downarrow^{\delta^Y} \\
\mathcal{O}_{Y_{-1}} \\
\downarrow^{\delta^Y} \\
Rr_{-1,\mathcal{K}} \\
\downarrow^{\delta^Y} \\
Rr_{-1,\mathcal{K}} \\
\downarrow^{\delta^Y} \\
\cdots
\end{array}
$$

(4.24)

on $X_{-1}$ and an entirely analogous diagram

$$
\begin{array}{c}
\text{cone}(\mathcal{O}_{Z_{-1}} \rightarrow R^\mathcal{Z}_{\mathcal{O}_{Z_1}}[-1]) \\
\downarrow^{\delta^Z} \\
\mathcal{O}_{Z_{-1}} \\
\downarrow^{\delta^Z} \\
Rr_{-1,\mathcal{K}} \\
\downarrow^{\delta^Z} \\
Rr_{-1,\mathcal{K}} \\
\downarrow^{\delta^Z} \\
\cdots
\end{array}
$$

(4.25)

on $Y_{-1}$. This motivates the choice:

$$
\mathcal{K} := \text{cone}(\mathcal{O}_{Z_{-1}} \rightarrow R^\mathcal{Z}_{\mathcal{O}_{Z_1}})[-1],
$$

and one can show that $\mathcal{K}$ has bounded, coherent cohomology using Corollary 2.5. To complete the proof of (i) it will suffice to show that $\alpha^X$ and $\alpha^Y$ are quasi-isomorphisms — we will deal with the case of $\alpha^X$, as that of $\alpha^Y$ is almost identical.
As $X_{-1}$ has rational singularities by hypothesis and by Lemma 4.12 $Z_{-1}$ is Cohen-Macaulay and the map $r_{-1} : Z_{-1} \to X_{-1}$ is projective and birational, Theorem 4.2 implies that $\beta^X$ is a quasi-isomorphism. By commutativity of eq. (4.14), $\gamma^X$ can be identified with the morphism

$$R\ell_1^X \mathcal{O}_Y \to R\ell_1^Z \mathcal{O}_Z.$$  \hspace{1cm} (4.26)

and by Lemma 2.2 the maps on cohomology induced by eq. (4.26) are the abutment of a map of descent spectral sequences; the map of $E_1$ pages reads

$$E_1^{ij}(X_i) = R\ell_1^X \mathcal{O}_{X_i} \to R\ell_1^Z \mathcal{O}_{Z_i} = E_1^{ij}(Z_i).$$ \hspace{1cm} (4.27)

By Theorem 4.2 again, for each $i$ the natural map $\mathcal{O}_{X_i} \to R\ell_1^Z \mathcal{O}_{Z_i}$ is a quasi-isomorphism and from this it follows that eq. (4.27) is an isomorphism of $E_1$ pages; this implies eq. (4.26) is a quasi-isomorphism. At this point we have shown $\beta^X$ and $\gamma^X$ are quasi-isomorphisms, and by the 5-lemma we conclude that so is $\alpha^X$.

For (ii), we essentially apply the functor $R\mathcal{H}om(-, \omega^X_{X_{-1}})$ (resp. $R\mathcal{H}om(-, \omega^X_{Z_{-1}})$) to the first (resp. second) quasi-isomorphism of eq. (4.20), and then use Grothendieck duality (Theorem 3.11). In more detail:

Applying $R\mathcal{H}om(-, \omega^X_{X_{-1}})$ to the first quasi-isomorphism of eq. (4.20) results in a quasi-isomorphism

$$R\mathcal{H}om(\text{cone}(\mathcal{O}_{X_{-1}} \to R\ell_1^X \mathcal{O}_{X_i})[-1], \omega^X_{X_{-1}}) \simeq R\mathcal{H}om(R\ell_1^X \mathcal{K}, \omega^X_{X_{-1}})$$ \hspace{1cm} (4.28)

and by Theorem 3.11 there is a natural quasi-isomorphism

$$R\mathcal{H}om(R\ell_1^X \mathcal{K}, \omega^X_{X_{-1}}) \simeq R\mathcal{H}om(\mathcal{K}, \omega^X_{Z_{-1}}).$$ \hspace{1cm} (4.29)

This motivates our choice of $\mathcal{K} := R\mathcal{H}om(\mathcal{K}, \omega^X_{Z_{-1}})$, and establishes the first quasi-isomorphism of eq. (4.20); the proof of the second is similar. Again one can show that $\mathcal{K}$ has bounded, coherent cohomology using Corollary 2.5.

Finally, when $X_i$ and $Y_i$ are the semi-simplicial schemes associated to simple normal crossing pairs, we know by Lemma 2.8 that there are quasi-isomorphisms

$$\text{cone}(\mathcal{O}_X \to R\ell_1^X \mathcal{O}_{X_i})[-1] \simeq \mathcal{O}_{X_{-1}}(-\Delta_X)$$
$$\text{cone}(\mathcal{O}_Y \to R\ell_1^Y \mathcal{O}_{Y_i})[-1] \simeq \mathcal{O}_{Y_{-1}}(-\Delta_Y).$$ \hspace{1cm} (4.30)

Combining these with (i) and (ii) gives eq. (4.22). \hfill \Box

4.4 Invariance results for cohomology of snc ideal sheaves

**Lemma 4.31.** Let $S$ be an excellent noetherian scheme and let $(X, \Delta_X)$ and $(Y, \Delta_Y)$ be simple normal crossing pairs separated and of finite type over $S$, and let $X \xleftarrow{\iota} Z \xrightarrow{j} Y$ be a thrifty proper birational equivalence over $S$. Let $X_i$ and $Y_i$ be the semi-simplicial schemes associated to $(X, \Delta_X)$ and $(Y, \Delta_Y)$ respectively. Then, there exist morphisms of augmented semi-simplicial schemes over $S$

\[
\begin{array}{ccc}
X_i & \xleftarrow{i_i} & U_i & \xrightarrow{j_i} & Y_i \\
\downarrow & & \downarrow & & \\
X_{-1} & \xleftarrow{i_{-1}} & U_{-1} & \xrightarrow{j_{-1}} & Y_{-1} = Y
\end{array}
\] \hspace{1cm} (4.32)

such that

(i) the morphisms $X_i \xleftarrow{i_i} U_i \xrightarrow{j_i} Y_i$ are dense open immersions for all $i$ and

(ii) there exists a $S$-scheme $Z_{-1}$ and $U_{-1}$-admissible blowups $X_{-1} \xleftarrow{f_{-1}} Z_{-1} \xrightarrow{s_{-1}} Y_{-1}$.

That is, the hypotheses of Lemma 4.12 (including the non-conjectural hypotheses of its “moreover”) are satisfied.
Lemma 3.6 applies, and thus there exist a Cohen-Macaulay scheme \( \text{Section 2} \) and Definition 1.5 eq. (5.6) to the collection of open immersions \( \text{and} \) there is a dense open set \( \text{Lemma 4.12} \) Corollary 4.19 \( \text{Lemma 4.31} \) \( \bar{\mathbb{B}} \) We now let \( \text{with thrifty birational (but not necessarily projective) morphisms} \), \( \text{With the same hypotheses as} \), \( \text{Similarly, applying} \), \( \text{using the functorial quasi-isomorphism} \) \( \text{Lemma 4.38 to the collection of open immersions} \) \( \text{and} \) \( \text{and using the functorial quasi-isomorphism} \), \( \text{Assume} \) \( \text{we ensure} \), \( \text{hence replacing} \) \( \text{we can ensure} \), \( \text{Repeating this construction with} \) \( \text{we may ensure} \), \( \text{is also a} \) \( \text{Lemma 4.34.} \text{With the same hypotheses as} \text{Lemma 4.31}, \text{there are quasi-isomorphisms} \)

\[
R_f s_\mathcal{X}(\Delta_X) \simeq Rg s_\mathcal{Y}(\Delta_Y) \text{ and } Rf s_\omega_X(\Delta_X) \simeq Rg s_\omega_Y(\Delta_Y).
\]

Proof. By Lemma 4.31, the hypotheses of Lemma 4.12, hence also its corollary Corollary 4.19, are satisfied. In particular, since by hypothesis we begin with simple normal crossing pairs \( (X, \Delta_X) \) and \( (Y, \Delta_Y) \), the “special case” of Corollary 4.19 applies, and thus there exist a Cohen-Macaulay scheme \( Z_{-1} \), projective birational morphisms \( X \leftarrow Z_{-1} \rightarrow Y \), and complexes \( \mathcal{K}, \mathcal{L} \) in the derived category \( D^B_c(Z) \) together with quasi-isomorphisms

\[
\begin{align*}
\theta_\mathcal{X}(\Delta_X) &\simeq Rr_{-1}s_\mathcal{K}, \text{ } Rr_{-1}s_\mathcal{K} \simeq \theta_\mathcal{Y}(\Delta_Y), \\
\omega_\mathcal{X}(\Delta_X) &\simeq Rr_{-1}s_\mathcal{L} \text{ and } Rr_{-1}s_\mathcal{L} \simeq \omega_\mathcal{Y}(\Delta_Y).
\end{align*}
\] (4.35)

Applying the pushforward \( R_f s \) (resp. \( Rg s \)) to the first (resp. second) of the above quasi-isomorphisms and using the functorial quasi-isomorphism \( Rf s, Rr_{-1}s_\mathcal{K} \simeq Rg s, Rr_{-1}s_\mathcal{K} \) we see that

\[
Rf s_\theta_\mathcal{X}(\Delta_X) \simeq Rf s Rr_{-1}s_\mathcal{K} \simeq Rg s Rr_{-1}s_\mathcal{K} \simeq Rg s \theta_\mathcal{Y}(\Delta_Y).
\] (4.36)

Similarly, applying \( Rf s \) (resp. \( Rg s \)) to the third (resp. fourth) quasi-isomorphisms of eq. (5.6) and using the functorial quasi-isomorphism \( Rf s Rr_{-1}s_\mathcal{L} \simeq Rg s Rr_{-1}s_\mathcal{L} \) we see that

\[
Rf s_\omega_\mathcal{X}(\Delta_X) \simeq Rf s Rr_{-1}s_\mathcal{L} \simeq Rg s Rr_{-1}s_\mathcal{L} \simeq Rg s \omega_\mathcal{Y}(\Delta_Y).
\] (4.37)

\[\square\]

5 Applications to rational pairs

Definition 5.1 (compare with [Kol13, Def. 2.78]). Let \( (S, \Delta_S) \) be a pair as in Definition 1.5 and assume \( \Delta_S \) is reduced and effective. A proper birational morphism \( f : X \to S \) is a rational resolution if and only if

(i) \( X \) is regular and the strict transform \( \Delta_X := f^{-1}_S \Delta_S \) has simple normal crossings,

(ii) the natural morphism \( \theta_S(-\Delta_S) \to Rf_* \theta_X(-\Delta_X) \) is a quasi-isomorphism, and
letting $\omega_X = h^{-\dim X}\omega_X^*$, where we use $\omega_X^* = f^*\omega_Y^*$ as a normalized dualizing complex on $X$,

(iii) $R^if_*\omega_X(\Delta_X) = 0$ for $i > 0$.

In the situation of Definition 5.1, the map $\Theta_S(-\Delta_S) \to Rf_*\Theta_X(-\Delta_X)$ appearing in condition (ii) is Grothendieck dual to a morphism

$$Rf_*\omega_X^*(\Delta_X) \xrightarrow{(1)} Rf_*R\mathcal{H}om_X(\Theta_X(-\Delta_X), \omega_X^*)$$

$$\xrightarrow{(2)} R\mathcal{H}om_S(Rf_*\Theta_X(-\Delta_X), \omega_X^*) \xrightarrow{(3)} R\mathcal{H}om_S(\Theta_S(-\Delta_S), \omega_S^*)$$

where the equality (1) comes from the fact that $\Delta_X$ is a Cartier divisor ($(X, \Delta_X)$ is snc by hypothesis), the isomorphism (2) comes from Grothendieck duality and the map (3) is obtained from the morphism of (ii) by applying the derived functor $R\mathcal{H}om_S(-, \omega_S^*)$. As $X$ is regular and the dualizing complex $\omega_X^*$ is normalized $h^i\omega_X^* = 0$ for $i \neq -\dim X$; in other words, $\omega_X^* \simeq \omega_X[\dim X]$. Twisting this equation with the Cartier divisor $\Delta_X$ gives $\omega_X(\Delta_X) \simeq \omega_X(\Delta_X)[\dim X]$. If $\Theta_S(-\Delta_S) \to Rf_*\Theta_X(-\Delta_X)$ is a quasi-isomorphism, so is

$$Rf_*\omega_X^*(\Delta_X)[\dim X] \simeq Rf_*\omega_X^*(\Delta_X) \to R\mathcal{H}om_S(\Theta_S(-\Delta_S), \omega_S^*)$$

and taking cohomology sheaves we see that $R^{i-\dim X}f_*\omega_X(\Delta_X) \simeq h^iR\mathcal{H}om_S(\Theta_S(-\Delta_S), \omega_S^*)$ for all $i$.

Thus given conditions (i) and (ii) of Definition 5.1, condition (iii) is equivalent to Cohen-Macaulayness of the sheaf $\Theta_S(-\Delta_S)$. We record these observations as a lemma.

**Lemma 5.3** (compare with [Kol13, Cor. 2.73, Props. 2.82-2.23], [Kov22, Def. 1.3]). With notation and setup as in Definition 5.1, the morphism $f : X \to S$ is a rational resolution if and only if

(i) $X$ is regular and the strict transform $\Delta_X := f^{-1}\Delta_S$ has simple normal crossings,

(ii) the natural morphism $\Theta_S(-\Delta_S) \to Rf_*\Theta_X(-\Delta_X)$ is a quasi-isomorphism, and

(iii) the sheaf $\Theta_S(-\Delta_S)$ is Cohen-Macaulay.

As illustrated in the examples of Appendix A, even simple normal crossing pairs $(S, \Delta_S)$ may have non-rational resolutions in the absence of additional thriftness restrictions, hence the following definition of rational singularities for pairs.

**Definition 5.4.** Let $(S, \Delta_S)$ be a pair such that $\Delta_S$ is a reduced effective Weil divisor. Then, $(S, \Delta_S)$ is resolution-rational if and only if it has a thrifty rational resolution.

### 5.1 Rational resolutions of pairs are all-for-one

In the case where $S$ is a normal variety over a field of characteristic 0, it is known that if $(S, \Delta_S)$ has a thrify rational resolution then every thrify resolution is rational [Kol13, Cor. 2.86]. The proof of this fact shows more generally that if $f : X \to S$ and $g : Y \to S$ are thrify resolutions, then there are isomorphisms $R^if_*\Theta_X(-\Delta_X) \simeq R^ig_*\Theta_Y(-\Delta_Y)$ for all $i$. This remains true in arbitrary characteristic.

**Theorem 5.5** ([Kol13, Cor. 2.86] in characteristic 0). Let $(S, \Delta_S)$ be a pair such that $\Delta_S$ is a reduced effective Weil divisor, let $f : X \to S$ and $g : Y \to S$ be thrify resolutions and assume Conjecture 1.11 holds. Then there are quasi-isomorphisms $Rf_*\Theta_X(-\Delta_X) \simeq Rg_*\Theta_Y(-\Delta_Y)$ and $Rf_*\omega_X(\Delta_X) \simeq Rg_*\omega_Y(\Delta_Y)$. In particular, $f$ is a rational resolution if and only if $g$ is.

Note that this includes Theorem 3.12 as a special case: indeed, if $(S, \Delta_S)$ is a simple normal crossing pair then given any thrify resolution $f : X \to S$ we may choose $g$ to be the identity.

**Proof.** By Lemma 3.6, there are dense open immersions $S \hookrightarrow U_X \hookrightarrow X$ and $S \hookrightarrow U_Y \hookrightarrow Y$ such that $U_X$ (resp. $U_Y$) contains all strata of snc$(S, \Delta_S)$ and $(X, \Delta_X)$ (resp. snc$(S, \Delta_S)$ and $(Y, \Delta_Y)$). Then $U := U_X \cap U_Y$ also contains all strata of snc$(S, \Delta_S)$ — moreover since $f$ and $g$ are thrify, the strata of $(X, \Delta_X)$ and $(Y, \Delta_Y)$ are in one-to-one birational correspondence with those of $(S, \Delta_S)$, so it remains true that $U$ contains all strata of $(X, \Delta_X)$ and $(Y, \Delta_Y)$. Replacing $U$ with $U \cap \text{snc}(X, \Delta_X)$, we may
assume \((U, \Delta_U := \Delta_S \cap U)\) is an snc pair. We now have morphisms \(i : U \hookrightarrow X, j : U \hookrightarrow Y\) which are thrifty and birational, but not necessarily proper.

Now let \(X, \rightarrow X_{-1} =: X, Y, Y_{-1} =: Y\) and \(U, \rightarrow U_{-1} =: U\) be the augmented semi-simplicial schemes associated to these simple normal crossing pairs. The inclusions \(i\) and \(j\) induce a diagram as in (4.13); we proceed to verify that the hypotheses of Lemma 4.12 are satisfied. All schemes in sight are defined over the noetherian and hence quasi-compact quasi-separated \(S\). The differentials and augmentations are all closed immersions and hence proper, and thriftness of \(i\) and \(j\) implies that the morphisms \(X_i \overset{i}{\hookrightarrow} U_i \overset{j}{\hookrightarrow} Y_i\) are dense open immersions for all \(i\). Applying Lemma 3.8 to the collection of \(S\)-schemes \(X, Y\) with the common dense open \(U\) gives a common \(U\)-admissible blowup \(X \rightarrow Z \rightarrow Y\). Finally (for the moreover part of the lemma) \(S\) is excellent by hypothesis, the \(U_i\) are regular and we are assuming Conjecture 1.11 holds.

Applying Lemma 4.12 and Corollary 4.19 we obtain a Cohen-Macaulay scheme \(Z_{-1}\), projective birational morphisms \(X \overset{r_{-1}}{\rightarrow} Z_{-1} \overset{s_{-1}}{\rightarrow} Y\), and complexes \(\mathcal{K}, \mathcal{L}\) in the derived category \(D^b(Z)\) together with quasi-isomorphisms

\[
\begin{align*}
\Theta_X(-\Delta_X) &\cong Rr_{-1}\mathcal{K}, \ Rs_{-1}\mathcal{K} \cong \Theta_Y(-\Delta_Y), \\
\omega_X(\Delta_X) &\cong Rs_{-1}\mathcal{L} \text{ and } Rs_{-1}\mathcal{L} \cong \omega_Y(\Delta_Y).
\end{align*}
\]

As was the case for Theorem 4.34, pushing these quasi-isomorphisms forward with \(Rf_*\) and \(Rg_*\) and using the functorial quasi-isomorphisms \(Rf_*Rr_{-1}\mathcal{K} \cong Rg_*Rs_{-1}\mathcal{K}\) and \(Rf_*Rr_{-1}\mathcal{L} \cong Rg_*Rs_{-1}\mathcal{L}\) completes the proof. \(\square\)

### 5.2 Semi-simplicial versus thrifty resolutions

We can also deduce the following lemma. While it’s statement is quite verbose, it may be of interest as it opens the possibility of defining rational singularities of pairs without thrifty resolutions.

**Lemma 5.7.** Let \((S, \Delta_S)\) be a pair, with \(\Delta_S\) reduced and effective, and suppose \((S, \Delta_S)\) admits a thrifty resolution. Let \(\varepsilon^S : S \rightarrow S\) be the associated semi-simplicial scheme and suppose there exists an augmented semi-simplicial scheme \(\varepsilon^Y : Y \rightarrow Y\) with proper differentials and augmentation, together with a map \(g : Y \rightarrow S\), such that each \(Y_i\) has rational singularities in the sense of [Kov22, Def. 1.3] and all of the morphisms \(Y_i \rightarrow S_i\) are admissible blowups for \(U_{Y,i} \subseteq S_i\), where \(U_Y \subseteq S\) is a dense open containing the generic points of all strata of snc\((S, \Delta_S)\) and \(U_{Y,i}\) is the semi-simplicial scheme associated to \((U_Y, U_Y \cap \Delta_S)\). Let \(\mathcal{H} = \text{cone}(\Theta_Y \rightarrow R\varepsilon^Y_*\Theta_Y)[-1]\) (this is a complex in the derived category of \(Y\)). Then, \((S, \Delta_S)\) is a rational pair if and only the sheaf \(\Theta_S(-\Delta_S)\) is Cohen-Macaulay and the natural map \(\Theta_S(-\Delta_S) \rightarrow Rg_*\mathcal{K}\) is quasi-isomorphism.

**Proof.** Let \(f : X \rightarrow S\) be a thrifty resolution. We observe that the first two paragraphs of the proof of Theorem 5.5 remain valid when \(Y_s\) is defined as in the lemma (as opposed to being obtained as the associated semi-simplicial scheme of an snc pair \((Y, \Delta_Y)\)). Since the \(Y_i\) are assumed to have rational singularities in the sense of [Kov22, Def. 1.3] the spectral sequence argument appearing in Corollary 4.19 remains valid and we conclude

\[
Rf_*\Theta_X(-\Delta_X) \cong Rg_*\mathcal{K}.
\]

Now, if \((S, \Delta_S)\) is a rational pair, by Definition 5.1 and Lemma 5.3 we conclude that \(\Theta_S(-\Delta_S) \cong Rf_*\Theta_X(-\Delta_X)\) and \(\Theta_S(-\Delta_S)\) is Cohen-Macaulay; then from eq. (5.8) we conclude \(\Theta_S(-\Delta_S) \cong Rg_*\mathcal{K}\). On the other hand, if \(\Theta_S(-\Delta_S)\) is Cohen-Macaulay and \(\Theta_S(-\Delta_S) \cong Rg_*\mathcal{K}\) eq. (5.8) shows \(\Theta_S(-\Delta_S) \cong Rf_*\Theta_X(-\Delta_X)\) and Definition 5.1 and Lemma 5.3 imply \(f\) is a rational resolution. \(\square\)

In characteristic \(0\) where we know resolutions exist, we can construct a \(Y_s\), as in the statement of the lemma with each \(Y_i\) smooth, using the methods of Section 4, but substituting Hironaka’s strong resolution of singularities for Conjecture 1.11. Unlike an augmented semi-simplicial resolution obtained from a thrifty resolution \(f : X \rightarrow S\), the differentials of \(Y_s\) need not be closed embeddings. To illustrate this distinction in the simplest possible situation, suppose \(S\) is a smooth variety over \(\mathbb{C}\) and
$D \subset S$ is a (not necessarily smooth) irreducible divisor. In this case a thrifty resolution is simply an embedded resolution of $D$. On the other hand, an augmented semi-simplicial resolution $g : Y \to S$, can be obtained as $\epsilon : \hat{D} \to S$ where $\pi : \hat{D} \to D$ is any resolution of $D$ and $\epsilon$ is the composition $\hat{D} \to D \hookrightarrow S$. Unpacking the definition of $\mathcal{K}$ we see that it is the shifted cone of the morphism $\mathcal{O}_S \to R\pi_*\mathcal{O}_D$, and hence coincides with $\mathcal{O}_S(-D)$ precisely when $R\pi_*\mathcal{O}_D = \mathcal{O}_D$. This recovers the fact that when $S$ is smooth and $D$ is irreducible, $(S, D)$ is a rational pair if and only if $D$ has rational singularities [Kol13, Rmk. 2.85].

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A (Non-)examples of thrift

In this section we work over a field $k$. Our first example is not new, and likely served as the original motivation for considering thrifty morphisms.

Example A.1. Let $S = \mathbb{A}_x^2$ and $\Delta = V(xy)$. Then $f : X = Bl_0 S \to S$ is neither thrifty nor rational. Indeed, letting $D_1 = V(x), D_2 = V(y)$ we see that $\Delta$ is the union of the 2 lines $D_1, D_2$ meeting at the origin. Let $D_i = f^{-1}D_i$ be the strict transforms, $E = f^{-1}(0)$ the exceptional divisor, and $\Delta = D_1 + D_2 - E$. The map $f : X \to S$ fails to be thrifty since it is not an isomorphism over the stratum $0 = D_1 \cap D_2$ of $(S, \Delta)$. We will calculate cohomology to show $f$ isn’t rational either.

Since $S = \mathbb{A}_x^2$ is affine, we can identify the sheaves $R^if_*\mathcal{O}_X(\Delta)$ with the sheaves associated to the $k[x, y]$-modules $H^i(X, \mathcal{O}_X(\Delta))$. Observe that $X$ can be identified with the geometric line bundle $\text{Spec}_k \text{Sym} \mathcal{O}_{\mathbb{P}^1}(1)$ associated to $\mathcal{O}_{\mathbb{P}^1}(-1)$. Under this identification, the projection $\pi : \text{Spec}_k \text{Sym} \mathcal{O}_{\mathbb{P}^1}(1) \to \mathbb{P}^1$ corresponds to the composition $\mathbb{B}_0 S \subseteq \mathbb{A}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$, and the blowup map $f : \text{Spec}_k \text{Sym} \mathcal{O}_{\mathbb{P}^1}(1) \to \mathbb{A}^2$ corresponds to the natural map

$$\text{Spec}_k \text{Sym} \mathcal{O}_{\mathbb{P}^1}(1) \to \text{Spec}_k H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = \text{Spec}_k k[x, y] = \mathbb{A}^2$$

Hence $\Delta = \pi^*(0 + \infty)$. Now since $\pi$ is affine its Leray spectral sequence degenerates to give

$$H^i(X, \mathcal{O}_X(\Delta)) = H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\Delta))$$

and via projection formula

$$\pi_* \mathcal{O}_X(\Delta) = \pi_* \mathcal{O}_X(-\pi^*(0 + \infty)) = (\pi_* \mathcal{O}_X)(-0 - \infty)$$

By the correspondence between affine schemes and sheaves of algebras,

$$\pi_* \mathcal{O}_X = \text{Sym} \mathcal{O}_{\mathbb{P}^1}(1) = \bigoplus_{d \geq 0} \mathcal{O}_{\mathbb{P}^1}(d)$$

Hence $H^i(X, \mathcal{O}_X(\Delta)) = \bigoplus_{d \geq 0} H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 2))$. In particular, when $i = 1$ and $d = 0$, we see $H^1(X, \mathcal{O}_X(\Delta)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1 - 2)) \simeq k$ by [Har77, Thm. III.5.1].

An elaboration of Example A.1 shows in general that if $(S, \Delta)$ is a simple normal crossing pair and $Z \subseteq S$ is a stratum, then $f : X = Bl_Z S \to S$ fails to be thrifty. Localizing at the generic point $\eta \in Z$ we can reduce to the case where $Z$ is replaced by a closed point $\eta \in S$ and $\Delta = V(x_1 x_2 \cdots x_n)$ where $x_1, \ldots, x_n \in m_\eta$ is a regular system of parameters. Then the long exact sequence obtained by pushing forward $\mathcal{O}_E(\Delta - E) \to \mathcal{O}_E(-\Delta) \to \mathcal{O}_E(\Delta |_E)$ ends in

$$R^{n-1} f_* \mathcal{O}_X(\Delta - E) \to R^{n-1} f_* \mathcal{O}_X(\Delta) \to R^{n-1} f_* \mathcal{O}_E(\Delta |_E) \to R^n f_* \mathcal{O}_X(E - \Delta - E) = 0$$

where the vanishing on the right holds since the maximal fiber dimension of $f$ is $n - 1$ [Har77, Cor. III.11.2]. Thus $R^{n-1} f_* \mathcal{O}_X(\Delta - E) \to R^{n-1} f_* \mathcal{O}_E(\Delta |_E) = H^{n-1}(E, \mathcal{O}_E(\Delta |_E))$ is surjective, and identifying $E$ with the projectivized Zariski tangent space $\mathbb{P}(T_S)_{\eta}$ with homogeneous coordinates $x_1, \ldots, x_n$ and $\Delta |_E$ with $V(\prod x_i)$ shows $H^{n-1}(E, \mathcal{O}_E(\Delta |_E)) \approx H^{n-1}(\mathcal{O}_{\mathbb{P}^{n-1}}, \mathcal{O}_{\mathbb{P}^{n-1}}(-n)) \simeq k$. For related discussion see [Kol13, p. 86].

The next example answers (in the affirmative!) a question of Erickson [Eri14b, p.2] and Prelli [Pre17, p.3] about whether there exists a resolution which is rational but not thrifty. In fact, we give such an example where the underlying pair $(S, \Delta)$ is rational.

Example A.2. Let $S = V(xy - zw) \subseteq \mathbb{A}_{xyzw}$, $D_0 = V(x, w)$ and $D_{\infty} = V(y, z)$; finally let $C_\infty = V(w, y)$. We can identify $S = C(\mathbb{P}^1 \times \mathbb{P}^1)$ as the affine cone over the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3_{xyzw}$ given by

$$\begin{bmatrix} x \\ w \\ z \\ y \end{bmatrix} = \begin{bmatrix} s_0 \\ s_1 \\ t_0 \\ t_1 \end{bmatrix} \begin{bmatrix} s_0t_0 & s_0t_1 \\ s_1t_0 & s_1t_1 \end{bmatrix} \quad (A.3)$$

Hence $D_0 = C(\{0\} \times \mathbb{P}^1), D_{\infty} = C(\{\infty\} \times \mathbb{P}^1)$ and $C_\infty = C(\mathbb{P}^1 \times \{0\})$.

Let $\Delta = D_0 + D_{\infty} + C_\infty$. Note that $\Delta$ is not Cartier, as it is not linearly equivalent to any multiple of $C(\{0\} \times \mathbb{P}^1) + C(\mathbb{P}^1 \times \{0\})$ (here $\{0\} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{0\}$ is a hyperplane section of the Segre embedding)
— see e.g. [Har77, Ex. II.6.3], [Kol13, Prop. 3.14]. Since $K_S$ is $Q$-Cartier, it follows that the pair $(S, \Delta = D_0 + D_\infty)$ is not $Q$-Gorenstein — in particular it isn’t dlt, so we are not at risk of violating [Kol13, Thm. 2.87] which implies that a resolution of a dlt pair is thirsty if and only if it is rational.

Now let $f : X = Bl_{D_0} S \to S$ be the blowup at $D_0$, let $D_i = f^{-1}D_i$ for $i = 0, \infty$ and $C_\infty = f^{-1}C_\infty$, and let $\tilde{\Delta} = D_0 + D_\infty + C_\infty$. The map $f$ is a small resolution of $S$ (as mentioned in [KM98, Ex. 2.7]). This means we are not at risk of violating [Eri14b, Prop. 1.6] which states that if a log resolution of a pair is rational then it is thirsty. Indeed, the ambient blowup is described as

$$Bl_{D_0} \mathbb{A}^4 \subseteq \{(x, y, z, w), [u, v] | (x, w) \propto (u, v)\} \subseteq \mathbb{A}^4 \times \mathbb{P}_w^1$$

so on the $D(u)$ patch $(x, w) = \lambda(1, u)$ and

$$xy - zw = \lambda y - z \lambda v = \lambda(y - zv)$$

Since $V(\lambda)$ is the exceptional divisor we see the strict transform $X \subseteq Bl_{D_0} \mathbb{A}^4$ of $S$ is $V(y - zv)$ on the $u = 1$ patch — this is smooth as it’s a graph. By symmetry in $x, w$, we conclude $X$ is smooth.

Even better, this allows us to parametrize $X \cap D(u)$ with coordinates $z, \lambda, v$:

$$\mathbb{A}^5_{z, \lambda, v} \cong Bl_{D_0} S \cap D(u) \subseteq D(u) \cong \mathbb{A}^5_{x, y, z, w, v}$$

sending $(z, \lambda, v) \mapsto (\lambda, zv, z, \lambda v) = (x, y, z, w, v)$ (A.4)

So in particular the restriction of $f$ looks like $(z, \lambda, v) \mapsto (\lambda, zv, z, \lambda v)$ and we see that the exceptional locus is the $v$-axis. In this coordinate patch the strict transforms $D_0$ and $D_\infty$ are $V(\lambda)$ and $V(z)$ respectively, which intersect along the $v$-axis $V(\lambda, z)!$ Thus $\hat{\Delta}$ has a stratum in $\text{Ex}(f)$ and $f$ isn’t thirsty. We also see that on this patch $C_\infty = V(v)$. As a philosophical aside, the blowup coordinates $[u, v]$ correspond to $[x, w] = [s_0, t_0, s_1, t_1] = [t_0, t_1]$ as long as $s_0 \neq 0$, so $\text{Ex} f$ can be viewed as a copy of the $\mathbb{P}_v^1$ appearing in $D_0 = C\{(0) \times \mathbb{P}_v^1\}$.

To show that $f$ is in fact a rational resolution we will use an alternative description of $X$. Starting with the ample invertible sheaf $\mathcal{O}_{\mathbb{P}_x^1 \times \mathbb{P}_t^1}(1, 1)$ we have natural morphisms of relative spectra

$$\text{Spec}_{\mathbb{P}_x^1 \times \mathbb{P}_t^1} \text{Sym}_H^0(\mathcal{O}_{\mathbb{P}_x^1 \times \mathbb{P}_t^1}(1, 1)) \to \text{Spec}_{\mathbb{P}_t^1} \text{Sym}_{\mathcal{O}_{\mathbb{P}_t^1}}(1, 1) \to \text{Spec}_k H^0(\text{Sym}_H^0(\mathcal{O}_{\mathbb{P}_x^1 \times \mathbb{P}_t^1}(1, 1))),$$

where $\mathcal{O}_{\mathbb{P}_x^1 \times \mathbb{P}_t^1} \to \mathcal{O}_{\mathbb{P}_t^1}$ is the projection. It is well known that the scheme on the left can be identified with the blowup $Bl_{D_0} S$, and the scheme on the right is $S$.

Claim A.6. There is an isomorphism of $S \times \mathbb{P}_t^1$-schemes

$$X = Bl_{D_0} S \cong \text{Spec}_{\mathbb{P}_t^1} \text{Sym}_{\mathcal{O}_{\mathbb{P}_t^1}}(1, 1)$$

This can be proven via the universal property. On the other hand, at least when $k$ is algebraically closed, a quick, dirty and more illuminating proof is possible: we have a morphism $f' : \text{Spec}_{\mathbb{P}_t^1} \text{Sym}_{\mathcal{O}_{\mathbb{P}_t^1}}(1, 1) \to S \times \mathbb{P}_t^1$: the first factor is the second map of (A.5), the second is the canonical projection

$$\pi : \text{Spec}_{\mathbb{P}_t^1} \text{Sym}_{\mathcal{O}_{\mathbb{P}_t^1}}(1, 1) \to \mathbb{P}_t^1$$

from the relative Spec construction. $X \subseteq S \times \mathbb{P}_t^1$ by construction, and we can check $\varphi$ maps the $k$-points of $\text{Spec}_{\mathbb{P}_t^1} \text{Sym}_{\mathcal{O}_{\mathbb{P}_t^1}}(1, 1)$ bijectively onto those of $X$. Indeed, the fiber of $\text{Spec}_{\mathbb{P}_t^1} \text{Sym}_{\mathcal{O}_{\mathbb{P}_t^1}}(1, 1)$ over $t \in \mathbb{P}_t^1$ can be described as follows: Note by projection formula

$$\text{Sym}_{\mathcal{O}_{\mathbb{P}_t^1}}(1, 1) \cong H^0(\mathbb{P}_t^1, \mathcal{O}_{\mathbb{P}_t^1}(1)) \otimes_k \mathcal{O}_{\mathbb{P}_t^1}(1),$$

so

$$\text{Sym}_{\mathcal{O}_{\mathbb{P}_t^1}}(1, 1) \cong H^0(\mathbb{P}_t^1, \mathcal{O}_{\mathbb{P}_t^1}(1)) \otimes_k \mathcal{O}_{\mathbb{P}_t^1}(1).$$

Explicitly $H^0(\mathbb{P}_t^1, \mathcal{O}_{\mathbb{P}_t^1}(1)) \otimes_k \mathcal{O}_{\mathbb{P}_t^1}(1) = \mathcal{O}_{\mathbb{P}_t^1}[s_0, s_1]_d \otimes \mathcal{O}_{\mathbb{P}_t^1}(d) = \mathbb{k}[s_0, s_1] \times \text{Sym} \mathcal{O}_{\mathbb{P}_t^1}(1)$ where $\times$ denotes the product of graded rings of [Har77, Ex. II.5.11] and hence for a $k$-point $t$,

$$\pi^{-1}(t) \cong \text{Spec} \mathbb{k}[s_0, s_1], \text{ so that } f'|_{\pi^{-1}(t)} : \pi^{-1}(t) \to S.$$
is a map $\mathbb{A}^2_{\text{aff}} \to S \subseteq \mathbb{A}^4_{xyzw}$. Writing down the map of algebras corresponding to $f^*$ shows that it is none other than the linear transformation of (A.3). Finally, referencing (A.4) we see that the fibers of $X \to \mathbb{P}^1$ have the same description.\footnote{In slogan form: $X = \text{Bl}_{D_0} S$ is a pencil of 2-planes on $S$ corresponding to the pencil of rulings $\mathbb{P}^1 \times \{t\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$.}

Using the claim, we proceed as in Example A.1 using degeneration of the Leray spectral sequence for the affine map $\pi : X \to \mathbb{P}^1$ to calculate

$$H^i(X, \mathcal{O}_X(-\Delta)) = H^i(\mathbb{P}^1, \pi_*\mathcal{O}_X(-\Delta))$$

On $\mathbb{P}^1$, noting that $\mathcal{C}_\infty = \pi^*(\infty)$, the projection formula gives

$$\pi_*\mathcal{O}_X(-\Delta) = (\pi_*\mathcal{O}_X(-\tilde{D}_0 - \tilde{D}_\infty))(-\infty)$$

(A.8)

and $\pi_*\mathcal{O}_X(-\tilde{D}_0 - \tilde{D}_\infty) \subseteq \pi_*\mathcal{O}_X$ is the sheaf of ideals $(s_0 \cdot s_1) \subseteq k[s_0, s_1] \times \text{Sym } \mathcal{O}_{\mathbb{P}^1}(1)$. Letting $(s_0 \cdot s_1)_d \subseteq k[s_0, s_1]$ denote the $d$-th graded part, we see

$$\pi_*\mathcal{O}_X(-\Delta) = \bigoplus_{d \geq 0} (s_0 \cdot s_1)_d \otimes_k \mathcal{O}_{\mathbb{P}^1}(d - 1)$$

where the “-1” comes from the twist “(-∞)” in (A.8). This yields:

$$H^i(\mathbb{P}^1, \pi_*\mathcal{O}_X(-\Delta)) = \bigoplus_{d \geq 0} (s_0 \cdot s_1)_d \otimes_k H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 1))$$

$$= \begin{cases} 
\bigoplus_{d \geq 0} (s_0 \cdot s_1)_d \otimes_k (t_1)_d \subseteq k[s_0, s_1] \times k[t_0, t_1] = H^0(S, \mathcal{O}_S) & \text{if } i = 0 \\
0 & \text{if } i = 1 
\end{cases}$$

(A.9)

the key point being that $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 1)) = 0$ for $d \geq 0$. This calculation shows $f_*\mathcal{O}_X(-\Delta) = \mathcal{O}_S(-\Delta)$ (this holds for more general reasons, namely $S$ is normal [Pre17, Lem. 2.1] and $R^1f_*\mathcal{O}_X(-\Delta) = 0$).

Finally, $(S, \Delta)$ is a rational pair, as a consequence of the theorem below — this was the main reason for including the additional divisor $\mathcal{C}_\infty$. If we had left it out, the above calculations would still show that $f : X \to (S, D_0 + D_\infty)$ is a non-trivial rational resolution, however the pair $(S, D_0 + D_\infty)$ isn’t rational (also by the theorem below).

Theorem A.10 ([Pre17, Thm. 3.2]). Let $(Y, B)$ be a pair such that $Y$ is a normal variety over $k$ and $B$ is a reduced effective Weil divisor on $Y$ (for example a simple normal crossing pair) and let $\mathcal{L}$ be an ample invertible sheaf on $Y$. Let $(C, \mathcal{B})$ be the abstract affine cone over $(Y, B)$ with respect to $\mathcal{L} : CY = \text{Spec}_k H^0(Y, \text{Sym } \mathcal{L})$ and $\mathcal{B}$ is the image of $\text{Spec}_k H^0(B, \text{Sym } \mathcal{L}|_B) \to \text{Spec}_k H^0(Y, \text{Sym } \mathcal{L}) = CY$ with its reduced subscheme structure. Then $(CY, \mathcal{B})$ is a rational pair if and only if $(Y, B)$ is a rational pair and

$$H^i(Y, \mathcal{L}^d(-B)) = 0 \text{ for } i > 0, d \geq 0$$

Applying the theorem to $Y = \mathbb{P}^1 \times \mathbb{P}^1$ with the divisor $B = \{0, \infty\} \times \mathbb{P}^1 + \mathbb{P}^1 \times \{\infty\}$ which has associated invertible sheaf $\mathcal{O}_Y(B) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2, 1)$, together with the ample invertible sheaf $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, 1)$ we calculate (using Künneth)

$$H^i(Y, \mathcal{L}^d(-B)) = H^i(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d - 2, d - 1)) = \bigoplus_{j+k=i} H^j(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 2)) \otimes_k H^k(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 1))$$

(A.11)

Noting that $H^k(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 1)) = 0$ for $k > 0$ and $d \geq 0$, we see that $H^2(Y, \mathcal{L}^d(-B)) = 0$ for $d \geq 0$, and

$$H^1(Y, \mathcal{L}^d(-B)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 2)) \otimes_k H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 1))$$

Now $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 2)) = 0$ for $d \neq 0$, but $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, so the tensor product is always 0.
The last example of this section shows that even when \((S, \Delta)\) is a simple normal crossing pair and \(f : X \to S\) is a \(U\)-admissible blowup for some \(U \subseteq S\) containing all strata, and \(\Delta = f^{-1}_*\Delta\) is snc, \(f\) may still fail to be thrifty. Unfortunately our presentation only makes sense in characteristic 0, but I would be shocked and appalled if this example doesn’t work in any characteristic \(p > 2\).

**Example A.12.** Let \(S = \mathbb{A}^3_{xy}\), let \(\Delta = V((z - x)(z + x))\) and let \(Z = V(x, y); \text{ let } U = S \setminus Z\). Then there is a \(U\)-admissible blowup \(f : X \to S\) such that \(f^{-1}_*\Delta\) is a simple normal crossing divisor but \(f\) is not thrifty.

We first blow up \(Z\) to obtain \(g : \mathbb{Bl}_Z S \to S\), and claim that the strict transform of \(\Delta\) is no longer snc. Letting \(D_\pm = V(z \pm x)\) we can work in blowup coordinates described like

\[
\mathbb{Bl}_Z S = \{(x, y, z, [u, v]) \in \mathbb{A}^3 \mid (x, y) \propto (u, v)\}
\]

so that on the \(D(u)\) patch \((x, y) = \lambda(1, v)\) and

\[
z \pm x = z \pm \lambda, \text{ so in } (z, \lambda, v) \text{ coordinates } D_\pm \cap D(u) = V(z \pm \lambda)
\]

in other words \(\Delta\) is snc on the \(D(u)\) patch (as is expected since on \(D(x) \subseteq \mathbb{A}^3\), \(\Delta\) is smooth). But on the \(D(v)\) patch where \((x, y) = \lambda(u, 1),\)

\[
z \pm x = z \pm \lambda u, \text{ so in } (z, \lambda, u) \text{ coordinates } D_\pm \cap D(v) = V(z \pm \lambda u) \quad (A.13)
\]

and here we see the strict transforms intersect along \(V(\lambda u)\) and hence fail to be snc.

A global description of the situation: \(\mathbb{Bl}_Z S\) is isomorphic to \(\mathbb{A}^1_0 \times \mathbb{Bl}_0 \mathbb{A}^2_{xy}\), and \(D_\pm\) are 2 copies of \(\mathbb{Bl}_0 \mathbb{A}^2_{xy}\) embedded via the maps

\[
(\pm x, \text{id}) : \mathbb{Bl}_0 \mathbb{A}^2_{xy} \to \mathbb{A}^1_0 \times \mathbb{Bl}_0 \mathbb{A}^2_{xy}
\]

where the map \(\pm x : \mathbb{Bl}_0 \mathbb{A}^2_{xy} \to \mathbb{A}^1_0\) really means the composition \(\mathbb{Bl}_0 \mathbb{A}^2_{xy} \to \mathbb{A}^2_{xy} \overset{\pm x}{\to} \mathbb{A}^1_0\). From this perspective \(D_+ \cap D_-\) is the preimage of \(V(x)\) under the blowup map \(\mathbb{Bl}_0 \mathbb{A}^2_{xy} \to \mathbb{A}^2_{xy}\), the union \(\mathbb{P}^1_{xy} \cup \mathbb{A}^1_y\) glued along the points \([0, 1] \in \mathbb{P}^1_{xy}\) and \(0 \in \mathbb{A}^1_y\). Let \(p\) denote the point in \(\mathbb{P}^1_{xy} \cap \mathbb{A}^1_y\). Equivalently \(\text{Sing}(D_+ \cap D_-)\) consists of a single closed point which we call \(p\).

This discussion shows that the snc locus of \((\mathbb{Bl}_Z S, \Delta)\) is

\[
\text{snc}(\mathbb{Bl}_Z S, \Delta) = \mathbb{Bl}_Z S \setminus \{p\}
\]

By work of Szabó and Bierstone-Milman [BM97; Sza94] (this is where we use the characteristic 0 hypothesis) there exists another blowup \(h : X \to \mathbb{Bl}_Z S\) such that \(h^{-1}_*\Delta + \text{Ex } h\) is a simple normal crossing divisor and \(h\) is an isomorphism over \(\text{snc}(\mathbb{Bl}_Z S, \Delta)\), that is, \(h\) must be a snc(\(\mathbb{Bl}_Z S, \Delta\))-admissible blowup. Now by [Har77, Thm. II.7.17] we know that \(f := gh : X \to S\) is a blowup at some closed subscheme \(W \subseteq S\) and since \(g(p) \in Z\) (equivalently) \(g^{-1}(U) \subseteq \text{snc}(\mathbb{Bl}_Z S, \Delta)\), it must be that \(W \subseteq Z\) as closed sets (see also [RG71, Lem. 5.1.4]), hence \(f : X \to S\) is a \(U\)-admissible blowup.

On the other hand, by a proposition of Erickson [Eri14b, Prop. 1.4], since \(h^{-1}_*\Delta + \text{Ex } h\) is snc the map \(h\) is thrifty and so the strata of \(f^{-1}_*\Delta = h^{-1}_*\Delta\) are in 1-1 birational correspondence with those of \(\Delta\), in particular \(f^{-1}_*\Delta\) has a stratum in \(\text{Ex } f\).

While the application of [BM97; Sza94] is heavy-handed for this toy example, we point out that \(h\) is not simply the blowup at \(p\) as one might initially guess: starting from (A.13), blowing up the origin \(0 \in \mathbb{A}^3_0\) and introducing blowup coordinates

\[
\mathbb{Bl}_0 \mathbb{A}^3_{zlu} = \{((z, \lambda, u), [r, s, t]) \in \mathbb{A}^3_{zlu} \times \mathbb{P}^2_{rst} \mid (z, \lambda, u) \propto (r, s, t)\}
\]

we note that since \(V((z - \lambda u) \cdot (z + \lambda u))\) is smooth on \(D(z)\) we can check that the strict transform remains smooth on the \(D(r)\) patch. We will investigate the \(D(s)\) patch — by symmetry of \(\lambda, u\) in the equation \((z - \lambda u) \cdot (z + \lambda u)\) the situation is similar on the \(D(t)\) patch. On \(D(s)\) we have \((z, \lambda, u) = \mu(r, 1, t)\) and so

\[
z \pm \lambda u = \mu r \pm \mu^2 t = \mu(r \pm \mu t)
\]
Here $V(\mu)$ is a copy of the exceptional divisor of $\text{Bl}_0 \mathbb{A}^3_{z,\lambda, u} \to \mathbb{A}^3_{z,\lambda, u}$ but we are still left with strict transforms $(r \pm \mu t)$ of exactly the same form as $z \pm \lambda u$; in other words, blowing up $0 \in \mathbb{A}^3_{z,\lambda, u}$ does not help! This is quite similar to the classical fact that blowing up the origin of the pinch point $V(z^2 - \lambda u^2) \subseteq \mathbb{A}^3_{z,\lambda, u}$ gives another pinch point singularity. In fact, since $(z - \lambda u) \cdot (z + \lambda u) = z^2 - \lambda^2 u^2$ our example is a double cover of the pinch point (that is, it is the preimage of the pinch point with respect to $(z, \lambda, u) \mapsto (z, \lambda^2, u)$).