Article

Unification of the Fixed Point in Integral Type Metric Spaces

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Abstract: In metric fixed point theory, the conditions like “symmetry” and “triangle inequality” play a predominant role. In this paper, we introduce a new kind of metric space by using symmetry, triangle inequality, and other conditions like self-distances are zero. In this paper, we introduce the weaker forms of integral type metric spaces, thereby we establish the existence of unique fixed point theorems. As usual, illustrations and counter examples are provided wherever necessary.

Keywords: weaker forms of generating spaces; weaker forms of integral type metric spaces; cyclic map

MSC: 49Q15; 54H25

1. Introduction and Preliminaries

Sumati et al. [1] introduced the weaker forms of various generating spaces and proved pertinent fixed point theorems. The study of fixed points satisfying cyclic mappings has been at the center of strong research activity in the last decade. In 2003, Kirk et al. [2] introduced the concept of cyclic maps and extended the famous Banach contraction principle.

Definition 1. Let $\mathcal{K}$ and $\mathcal{L}$ be non-empty subsets of a set $X$. A map $\mathcal{U} : \mathcal{K} \cup \mathcal{L} \to \mathcal{K} \cup \mathcal{L}$ is said to be a cyclic map if $\mathcal{U}(\mathcal{K}) \subseteq \mathcal{L}$ and $\mathcal{U}(\mathcal{L}) \subseteq \mathcal{K}$.

Karpagam and Agrawal [3] introduced the predominant concept of cyclic orbital Meir-Keeler contraction maps. Very recently, Karapinar et al. [4] generalized the concept of cyclic orbital Meir-Keeler contraction by introducing an extended cyclic orbital contraction and an extended cyclic orbital-\(F\)-contraction in the setting of extended \(b\)-metric space. Since then, many researchers have continued the investigation in this direction and obtained many more results concerning cyclic maps. For more details, one may refer to [5–13].

On the other hand, there is a strong literature on “generalizations of metric space and related fixed points”, and this has become a very active research field in pure and applied mathematics (see for example [14–29]).

In 1997, Chang et al. [21] introduced the theory of a generating space of a quasi-metric family and established some interesting fixed point theorems. Later, G M Lee et al. [22] established a family of weak quasi-metrics in a generating space of quasi-metric family. He proved some predominant named theorems such as the Takahashi-type minimization theorem, a generalized Ekeland variational principle, and a
general Caristi-type fixed point theorem for set-valued maps. In particular, Sumati et al. [1] introduced weaker forms of various generating spaces and proved pertinent fixed point theorems.

In this paper, we introduce some weaker forms of integral type metric spaces, that is the integral type metric space, the integral type dislocated metric space, and the integral type dislocated quasi-metric space.

**Definition 2.** A function $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is called a sub-additive integrable function iff for all $p, q \in \mathbb{R}^+$,

$$\int_0^{p+q} \Gamma(t)dt \leq \int_0^p \Gamma(t)dt + \int_0^q \Gamma(t)dt.$$ 

**Definition 3.** Let $X$ be a non-empty set and $\{d_\alpha : \alpha \in (0,1]\}$ be a family of mappings $d_\alpha : X \times X \to \mathbb{R}^+$. Consider the following conditions for any $x, y, z \in X$:

(a). $\int_0^{d_\alpha(x,x)} \Gamma(t)dt = 0$;
(b). $\int_0^{d_\alpha(x,y)} \Gamma(t)dt = \int_0^{d_\alpha(y,x)} \Gamma(t)dt$;
(c). $\int_0^{d_\alpha(x,z)} \Gamma(t)dt = \int_0^{d_\alpha(x,y)} \Gamma(t)dt + \int_0^{d_\alpha(y,z)} \Gamma(t)dt$;
(d). $\int_0^{d_\alpha(x,y)} \Gamma(t)dt$ is non-increasing and left continuous in $\alpha$;

where $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping, which is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for any $s > 0$, $\int_0^s \Gamma(t)dt > 0$.

$d_\alpha$ is called:

(i). integral type metric space if $d_\alpha$ satisfies (a) through (c).
(ii). integral type dislocated metric space if $d_\alpha$ satisfies (b) through (c).
(iii). integral type dislocated quasi-metric space if $d_\alpha$ satisfies (c) through (e).

**Definition 4.** A sequence $\{x_n\}$ in integral type metric space is said to be convergent to a point $x \in X$ if:

$$\lim_{n\to\infty} \int_0^{d_\alpha(x_n,x)} \Gamma(t)dt = 0.$$ 

**Definition 5.** A sequence $\{x_n\}$ in integral type metric space $(X, d_\alpha)$ is said to be Cauchy sequence if for $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $m, n \geq n_0$ implies:

$$\int_0^{d_\alpha(x_m,x_n)} \Gamma(t)dt < \epsilon.$$ 

or:
\[
\lim_{n \to \infty} \int_0 \Gamma(t) \, dt = \lim_{n \to \infty} \int_0 \Gamma(t) \, dt = 0.
\]

**Definition 6.** An integral type metric space \((X, d_a)\) is said to be complete if every Cauchy sequence in \(X\) converges to a point in \(X\).

**Lemma 1.** The limit of a convergent sequence in integral type metric space \((X, d_a)\) is unique.

If we put \(d\) instead of \(d_a\) in Definition 3, we get the following definition.

**Definition 7.** Let \(X\) be a non-empty set. Take \(d : X \times X \to [0, \infty)\), and consider the following conditions for any \(x, y, z \in X\):

\[
\begin{align*}
(I_1). & \quad d(x, x) = 0; \\
(I_2). & \quad \int_0 \Gamma(t) \, dt = \int_0 \Gamma(t) \, dt; \\
(I_3). & \quad \int_0 \Gamma(t) \, dt = \int_0 \Gamma(t) \, dt = 0 \text{ implies } x = y; \\
(I_4). & \quad \int_0 \Gamma(t) \, dt \leq \int_0 \Gamma(t) \, dt + \int_0 \Gamma(t) \, dt;
\end{align*}
\]

where \(\Gamma : \mathbb{R}^+ \to \mathbb{R}^+\) is a Lebesgue integrable mapping, which is summable on each compact subset of \(\mathbb{R}^+\), non-negative, and such that for any \(s > 0\), \(\int_0^s \Gamma(t) \, dt > 0\).

d is called:

(a). Integral metric space if \(d\) satisfies \((I_1)\) through \((I_4)\).

(b). Integral dislocated metric space if \(d\) satisfies \((I_2)\) through \((I_4)\).

(c). Integral dislocated quasi-metric space if \(d\) satisfies \((I_3)\) through \((I_4)\).

Encouraged by the success in the pursuit of mathematical properties of some weak forms of generating spaces, we continue the study of weaker forms of various generating spaces \([1]\) by introducing the weaker forms of integral type metric spaces. Furthermore, we establish the existence of unique fixed point theorems in such spaces. Illustrations and counter examples are provided wherever necessary.

2. Main Theorems

In this section, we establish some fixed point results in the weaker forms of integral type metric spaces.

**Theorem 1.** Let \((X, d_a)\) be a complete integral type metric space for \(\nu \in (0, 1)\). Let \(U : X \to X\) be a mapping such that for all \(x, y \in X\) satisfying:

\[
\frac{d_a(x, y)}{\int_0 \Gamma(t) \, dt} \leq \nu \int_0 \Gamma(t) \, dt,
\]

\[
\frac{d_a(x, y)}{\int_0 \Gamma(t) \, dt} \leq \nu \int_0 \Gamma(t) \, dt,
\]

\[
\lim_{n \to \infty} \int_0 \Gamma(t) \, dt = \lim_{n \to \infty} \int_0 \Gamma(t) \, dt = 0.
\]
where $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping, which is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for any $s > 0$, $\int_0^s \Gamma(t)dt > 0$. Then, $\mathcal{U}$ has a unique fixed point in $X$.

**Proof.** The proof is not very difficult, and hence, we omit it. ☐

The following result is a generalization of Theorem 1.

**Theorem 2.** Let $\mathcal{K}$ and $\mathcal{L}$ be non-empty closed subsets of a complete integral type metric space $(X, d_X)$. Let $\mathcal{U} : \mathcal{K} \cup \mathcal{L} \to \mathcal{K} \cup \mathcal{L}$ be a continuous cyclic map that satisfies the condition:

\[
\begin{align*}
\int_0^1 \Gamma(t)dt &\leq \varrho_1 \\
\int_0^1 \Gamma(t)dt + \varrho_2 &\leq \int_0^1 \Gamma(t)dt \\
\int_0^1 \Gamma(t)dt + \varrho_3 &\leq \int_0^1 \Gamma(t)dt \\
\int_0^1 \Gamma(t)dt + \varrho_4 &\leq \int_0^1 \Gamma(t)dt \\
\int_0^1 \Gamma(t)dt + \varrho_5 &\leq \int_0^1 \Gamma(t)dt
\end{align*}
\]

where $\varrho_i \geq 0$ for $i = 1$ to 5 and $\sum_{i=1}^5 \varrho_i < 1$ for all $x \in \mathcal{K}, y \in \mathcal{L}$, and it $\Gamma$ is Lebesgue integrable, then $\mathcal{U}$ has a unique fixed point in $\mathcal{K} \cap \mathcal{L}$.

**Proof.** Let $x_0 \in \mathcal{K}$ be arbitrary. Define a sequence $\{x_n\}$ in $X$ by $x_n = \mathcal{U}^n x_0 = \mathcal{U} x_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for $\mathcal{U}$, and the result is proven. Hence, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Consider,

\[
\begin{align*}
\frac{d_X(U x_0, U^{2n+1} x_0)}{\Gamma(t)dt} &\leq \frac{2}{\Gamma(t)dt} \\
\frac{d_X(U x_0, U^{2n+1} x_0)}{\Gamma(t)dt} &\leq \frac{2}{\Gamma(t)dt} \\
\frac{d_X(U x_0, U^{2n+1} x_0)}{\Gamma(t)dt} &\leq \frac{2}{\Gamma(t)dt}
\end{align*}
\]

This implies,

\[
\begin{align*}
\int_0^1 \Gamma(t)dt &\leq \varrho_3 \\
\int_0^1 \Gamma(t)dt &\leq \varrho_4 \\
\int_0^1 \Gamma(t)dt &\leq \varrho_5
\end{align*}
\]
\[ d_h(1^{2n}, x_0, 1^{2n+1}, x_0) \]

Clearly, \[ \int_0^\Gamma(t) dt \leq h_1 \int_0^\Gamma(t) dt; \text{ where } h_1 = \frac{\epsilon_1 + \epsilon_2}{1 - \epsilon_3} < 1. \]

Similarly,

\[ d_h(1^{2n+1}, x_0, 1^{2n+2}, x_0) \]

\[ = d_h(1^{2n}, x_0, 1^{2n+1}, x_0) \]

\[ + \epsilon_3 \int_0^\Gamma(t) dt + \epsilon_4 \int_0^\Gamma(t) dt + \epsilon_5 \int_0^\Gamma(t) dt \]

\[ \leq \epsilon_3 + \epsilon_4 + \epsilon_5 \]

\[ d_h(1^{2n+1}, x_0, 1^{2n+2}, x_0) \]

\[ = d_h(1^{2n}, x_0, 1^{2n+1}, x_0) \]

Further, this implies, \[ \int_0^\Gamma(t) dt \leq h_2 \int_0^\Gamma(t) dt; \]

where \( h_2 = \frac{\epsilon_3 + \epsilon_5}{1 - \epsilon_4} < 1. \)

For all \( n \in \mathbb{N}, \) we get,

\[ d_h(x_0, x_{n+1}) \]

\[ \leq h^n \]

\[ \int_0^\Gamma(t) dt \]

\[ \text{for all } n \geq 1, \]

where \( h = \max\{h_1, h_2\}. \)

Since \( h < 1 \) and taking the limit as \( n \to \infty, \) we have \( h^n \to 0. \) Hence,

\[ \int_0^\Gamma(t) dt \to 0. \]

by the hypothesis, \( \{x_n\} \) is a Cauchy sequence. Since \( (X, d_h) \) is complete, we see that \( \{1^{2n} x_0\} \) converges to some \( z \in X \) for all \( n \). We note that \( \{1^{2n} x_0\} \) is a sequence in \( \mathcal{X} \) and \( \{1^{2n-1} x_0\} \) is a sequence in \( \mathcal{L} \) such that both sequences tend to the same limit \( z. \) Since \( \mathcal{X} \) and \( \mathcal{L} \) are closed, we have \( z \in \mathcal{X} \cap \mathcal{L}. \)

Now, we shall show that \( \mathcal{U} z = z, \) i.e., \( z \) is a fixed point of \( \mathcal{U}. \)

By the continuity of \( \mathcal{U}, \) we deduce that \( z = \lim_{n \to \infty} 1^{2n} x_0 = \mathcal{U} \lim_{n \to \infty} 1^{2n-1} x_0 = \mathcal{U} z. \)

In order to prove uniqueness, let \( z^* \) and \( z^* \) be two fixed points of \( \mathcal{U}, \) i.e., \( z^* \neq z^*. \)
Example 1. Let $X = [-1, 1], \mathcal{K} = [-1, 0]$ and $\mathcal{L} = [0, 1]$ and $d : X \times X \to \mathbb{R}^+$ such that $d(x, y) = |x - y|$. Then, $(X, d)$ is an integral type metric space, and $\mathcal{U} : \mathcal{K} \cup \mathcal{L} \to \mathcal{K} \cup \mathcal{L}$ is defined by $\mathcal{U}x = \frac{x}{2}$, $\Gamma(t) = t^2$.

Consider,

$$
\int_0^\Gamma(t) dt \leq q_1 + q_2 \int_0^\Gamma(t) dt + q_3 \int_0^\Gamma(t) dt
$$

which implies $(1 - q_1 - \frac{q_2}{2} - q_5)d_\mathcal{U}(z^*, z^*) \leq 0$. Since $q_1 + \frac{q_2}{2} + q_5 < 1$, we get $\int_0^\Gamma(t) dt = 0$.

This implies $d_\mathcal{U}(z^*, z^*) = 0$.

Hence, $z^* = z^*$. This completes the proof of the theorem. \hfill \Box

If we put $d$ instead of $d_\mathcal{U}$ in Theorem 2, we obtain the following corollary in the setting of complete integral metric spaces.

**Corollary 1.** Let $\mathcal{K}$ and $\mathcal{L}$ be non-empty closed subsets of a complete integral metric space $(X, d)$. Let $\mathcal{U} : \mathcal{K} \cup \mathcal{L} \to \mathcal{K} \cup \mathcal{L}$ be a continuous cyclic map that satisfies the condition:

$$
\int_0^\Gamma(t) dt \leq q_1 + q_2 \int_0^\Gamma(t) dt + q_3 \int_0^\Gamma(t) dt + q_4 \int_0^\Gamma(t) dt + q_5 \int_0^\Gamma(t) dt
$$

where $q_1 \geq 0$ for $i = 1$ to $5$ and $\sum_{n=1}^5 q_i < 1$ for all $x \in \mathcal{K}$, $y \in \mathcal{L}$, and $\Gamma$ is Lebesgue integrable. Then, $\mathcal{U}$ has a unique fixed point in $\mathcal{K} \cap \mathcal{L}$.

**Example 1.** Let $X = [-1, 1], \mathcal{K} = [-1, 0]$ and $\mathcal{L} = [0, 1]$ and $d : X \times X \to \mathbb{R}^+$ such that $d(x, y) = |x - y|$. Then, $(X, d)$ is an integral type metric space, and $\mathcal{U} : \mathcal{K} \cup \mathcal{L} \to \mathcal{K} \cup \mathcal{L}$ is defined by $\mathcal{U}x = \frac{x}{2}$, $\Gamma(t) = t^2$.

Consider,

$$
\int_0^\Gamma(t) dt = \int_0^t \frac{2}{3} dt
$$

$$
= \left[ \frac{2t}{3} \right]_0^1
$$

$$
= 8|x - y|^3
$$

$$
= 0.02|x - y|^3
$$
Now, take,

\[
\frac{d(y,1x) + d(x,1y) + d(x,1y) - d(y,1x)}{d(x,1y) + d(y,1x)} \Gamma(t) dt + \phi_2 \frac{d(x,1y) + d(y,1x)}{d(x,1y) + d(y,1x)} \Gamma(t) dt + \phi_3 \frac{d(x,1y) + d(y,1x)}{d(x,1y) + d(y,1x)} \Gamma(t) dt + \phi_4 \frac{d(x,1y) + d(y,1x)}{d(x,1y) + d(y,1x)} \Gamma(t) dt + \phi_5 \frac{d(x,1y) + d(y,1x)}{d(x,1y) + d(y,1x)} \Gamma(t) dt
\]

\[
= \phi_1 \int_0^r r^2 dt + \phi_2 \int_0^r r^2 dt + \phi_3 \int_0^r r^2 dt + \phi_4 \int_0^r r^2 dt + \phi_5 \int_0^r r^2 dt
\]

Take \( \phi_1 = 0.3, \phi_2 = \phi_3 = 0.2 \) and \( \phi_4 = \phi_5 = 0.1 \), then:

\[
= 0.3 \left( \frac{1}{3} \right) \int_0^r \left( |y + \frac{2x}{3}| |x + \frac{2y}{3}|^2 + |x + \frac{2y}{3}| + |y + \frac{2x}{3}|^2 \right) + 0.2 \left( \frac{1}{3} \right) \int_0^r \left( |x + \frac{2y}{3}| + |x + \frac{2y}{3}|^2 \right) + 0.1 \left( \frac{1}{3} \right) \int_0^r \left( |x + \frac{2y}{3}|^2 + |y + \frac{2x}{3}| \right)
\]

\[
= 0.3 \left( \frac{1}{3} \right) \int_0^r \left( |y + \frac{2x}{3}| + |x + \frac{2y}{3}|^2 + |x + \frac{2y}{3}| + |y + \frac{2x}{3}|^2 \right) + 0.2 \left( \frac{1}{3} \right) \int_0^r \left( |x + \frac{2y}{3}| + |x + \frac{2y}{3}|^2 \right) + 0.1 \left( \frac{1}{3} \right) \int_0^r \left( |x + \frac{2y}{3}|^2 + |y + \frac{2x}{3}| \right)
\]

\[
= 0.03 (|x - y|^2 + \theta), \text{ where } \theta > 0. \text{ Hence, } 0.02 |x - y|^2 < 0.03 (|x - y|^2 + \theta). \text{ Thus, all the conditions of the above corollary are satisfied, and zero is the unique fixed point of } U.
\]

**Theorem 3.** Let \( \mathcal{K} \) and \( \mathcal{L} \) be non-empty closed subsets of a complete integral type metric space \( (X,d_a) \). Let \( U : \mathcal{K} \cup \mathcal{L} \to \mathcal{K} \cup \mathcal{L} \) be a continuous cyclic map that satisfies the condition:

\[
\left\{ \begin{array}{l}
d_{a}(x,1y) \leq \phi_1 \int_0^r \Gamma(t) dt + \phi_2 \int_0^r \Gamma(t) dt + \phi_3 \int_0^r \Gamma(t) dt + \phi_4 \int_0^r \Gamma(t) dt + \phi_5 \int_0^r \Gamma(t) dt \\
d_{a}(y,1x) \leq \phi_6 \int_0^r \Gamma(t) dt + \phi_7 \int_0^r \Gamma(t) dt \end{array} \right.
\]

where \( \phi_1 + 2\phi_2 + 2\phi_3 + \phi_4 + \phi_5 + \phi_6 + \phi_7 < 1 \) for all \( x \in \mathcal{K}, y \in \mathcal{L}, \) and as \( \Gamma \) is Lebesgue integrable and sub-additive integrable, then \( U \) has a unique fixed point in \( \mathcal{K} \cap \mathcal{L} \).

**Proof.** Let \( x_0 \in \mathcal{K} \) be arbitrary. Define a sequence \( \{x_n\} \) in \( X \) by \( x_n = U^n x_0 = UX_{n-1} \) for all \( n \in \mathbb{N} \). If \( x_{n+1} = x_n \) for some \( n \in \mathbb{N} \), then \( x = x_n \) is a fixed point for \( T \), and the result is proven. From now on, we suppose that \( x_{n+1} \neq x_n \) for all \( n \in \mathbb{N} \). Consider,
\[
\int_0^{\Gamma(t)}\Gamma(t)dt = \int_0^{\Gamma(t)}\Gamma(t)dt
\]

\[
= \int_0^{\Gamma(t)}\Gamma(t)dt
\]

\[
= \int_0^{\Gamma(t)}\Gamma(t)dt
\]

\[
\leq \epsilon_1 \int_0^{\Gamma(t)}\Gamma(t)dt + \epsilon_2 \int_0^{\Gamma(t)}\Gamma(t)dt + \epsilon_3 \int_0^{\Gamma(t)}\Gamma(t)dt
\]

\[
+ \epsilon_4 \int_0^{\Gamma(t)}\Gamma(t)dt
\]

\[
\leq \epsilon_1 \int_0^{\Gamma(t)}\Gamma(t)dt + \epsilon_3 \int_0^{\Gamma(t)}\Gamma(t)dt + \epsilon_4 \int_0^{\Gamma(t)}\Gamma(t)dt
\]

\[
+ \epsilon_5 \int_0^{\Gamma(t)}\Gamma(t)dt
\]

\[
\leq \epsilon_1 \int_0^{\Gamma(t)}\Gamma(t)dt + \epsilon_3 \int_0^{\Gamma(t)}\Gamma(t)dt + \epsilon_4 \int_0^{\Gamma(t)}\Gamma(t)dt
\]

\[
+ \epsilon_5 \int_0^{\Gamma(t)}\Gamma(t)dt
\]

\[
\leq \epsilon_1 \int_0^{\Gamma(t)}\Gamma(t)dt + \epsilon_3 \int_0^{\Gamma(t)}\Gamma(t)dt + \epsilon_4 \int_0^{\Gamma(t)}\Gamma(t)dt
\]

\[
+ \epsilon_5 \int_0^{\Gamma(t)}\Gamma(t)dt
\]

\[
\leq \epsilon_1 \int_0^{\Gamma(t)}\Gamma(t)dt + \epsilon_3 \int_0^{\Gamma(t)}\Gamma(t)dt + \epsilon_4 \int_0^{\Gamma(t)}\Gamma(t)dt
\]

\[
+ \epsilon_5 \int_0^{\Gamma(t)}\Gamma(t)dt
\]

which implies,

\[
\int_0^{\Gamma(t)}\Gamma(t)dt \leq \frac{\epsilon_1 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6}{1 - \epsilon_3 - \epsilon_7} \int_0^{\Gamma(t)}\Gamma(t)dt
\]

\[
\leq h_1 \int_0^{\Gamma(t)}\Gamma(t)dt
\]

where \(h_1 = \frac{\epsilon_1 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6}{1 - \epsilon_3 - \epsilon_7}\).
Similarly,
\[
\begin{align*}
d_s(1^{2n+1}x_0,1^{2n+2}x_0) & \leq d_s(x_{2n+1},x_{2n+2}) + d_s(x_{2n+1},x_{2n+1}) \\
& \leq \epsilon_1 \int_0^\Gamma(t) dt + \epsilon_2 \int_0^\Gamma(t) dt + \epsilon_3 \int_0^\Gamma(t) dt \\
& + \epsilon_4 \int_0^\Gamma(t) dt + \epsilon_5 \int_0^\Gamma(t) dt + \epsilon_6 \int_0^\Gamma(t) dt \\
& + \epsilon_7 \int_0^\Gamma(t) dt
\end{align*}
\]
which implies,
\[
\begin{align*}
d_s(1^{2n+1}x_0,1^{2n+2}x_0) & \leq \frac{\epsilon_1 + \epsilon_2 + \epsilon_4 + \epsilon_7}{1 - \epsilon_2 - \epsilon_5 - \epsilon_6} \int_0^\Gamma(t) dt \\
& \leq h_2 \int_0^\Gamma(t) dt
\end{align*}
\]
where \( h_2 = \frac{\epsilon_1 + \epsilon_2 + \epsilon_4 + \epsilon_7}{1 - \epsilon_2 - \epsilon_5 - \epsilon_6} \).
On similar proceeding lines, we get:

\[ d_k(x_{n}, x_{n+1}) \quad \int_0^\infty \Gamma(t)dt \leq h^n \quad \int_0^\infty \Gamma(t)dt; \quad \text{for all } n \geq 1, \]

where \( h = \max\{h_1, h_2\} \).

Since \( h < 1 \) and taking the limit \( n \to \infty \), we have \( h^n \to 0 \). Hence, \( \int_0^\infty \Gamma(t)dt \to 0 \), which implies that \( d_k(x_n, x_{n+1}) \to 0 \) as \( n \to \infty \). Hence, by the hypothesis, \( \{x_n\} \) is a Cauchy sequence in complete integral type metric space \( (X, d_k) \). Since \( (X, d_k) \) is complete, we see that \( \{x_n\} \) converges to some \( z \in X \) for all \( \alpha \in (0, 1] \). We note that \( \{x_n\} \) is a sequence in \( \mathcal{K} \) and \( \{x_n\} \) is a sequence in \( \mathcal{L} \) such that both sequences tend to the same limit \( z \). Since \( \mathcal{K} \) and \( \mathcal{L} \) are closed, we have \( z \in \mathcal{K} \cap \mathcal{L} \).

Now, we shall show that \( \mathcal{U}z = z \), i.e., \( z \) is a fixed point of \( \mathcal{U} \). By the continuity of \( \mathcal{U} \), we deduce that \( z = \lim_{n \to \infty} x_n = \lim_{n \to \infty} x_{2n-1} = z_0 \). In order to prove uniqueness, let \( z^* \) and \( z^* \) be two fixed points of \( \mathcal{U} \) such that \( z^* \neq z^* \). Consider,

\[
\frac{d_k(z^*, z^*)}{\Gamma(t)dt} \leq \frac{d_k(z^*, z^*)}{\Gamma(t)dt} + \frac{d_k(z^*, z^*)}{\Gamma(t)dt} \leq (\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) \int_0^\infty \Gamma(t)dt
\]

which implies,

\[
\frac{d_k(z^*, z^*)}{\Gamma(t)dt} \leq (\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) \int_0^\infty \Gamma(t)dt.
\]

Thus, \( \int_0^\infty \Gamma(t)dt = 0 \).

Hence, \( d_k(z^*, z^*) = 0 \), which implies \( z^* = z^* \). This completes the proof of the theorem. \( \Box \)

If we put \( d \) instead of \( d_k \) in the above theorem, we obtain the following corollary in complete integral metric space.

**Corollary 2.** Let \( \mathcal{K} \) and \( \mathcal{L} \) be non-empty closed subsets of a complete integral metric space \( (X, d) \). Let \( \mathcal{U} : \mathcal{K} \cup \mathcal{L} \to \mathcal{K} \cup \mathcal{L} \) be a continuous cyclic map that satisfies the condition:
where \( q_1 + 2q_2 + 2q_3 + q_4 + q_5 + q_6 + q_7 < 1 \) for all \( x \in \mathcal{K}, y \in \mathcal{L} \), and as \( \Gamma \) is Lebesgue integrable and sub-additive integrable, then \( \mathcal{U} \) has a unique fixed point in \( \mathcal{K} \cap \mathcal{B} \).

**Proof.** Let \( x_0 \in \mathcal{K} \) be an arbitrary. Define a sequence \( \{ x_n \} \) in \( X \) by \( x_n = \mathcal{U}^n x_0 = \mathcal{U} x_{n-1} \) for all \( n \in \mathbb{N} \). If \( x_{n+1} = x_n \) for some \( n \in \mathbb{N} \), then \( x = x_n \) is a fixed point for \( T \), and the result is proven. From now on, we suppose that \( x_{n+1} \neq x_n \) for all \( n \in \mathbb{N} \).

Consider,

\[
\begin{align*}
&d(x_{2n}x_{2n+1}) \\
&= \int_0^1 \Gamma(t)dt \\
&\leq q_1 \int_0^1 \Gamma(t)dt + q_2 \int_0^1 \Gamma(t)dt + q_3 \int_0^1 \Gamma(t)dt \\
&\quad + q_4 \int_0^1 \Gamma(t)dt + q_5 \int_0^1 \Gamma(t)dt + q_6 \int_0^1 \Gamma(t)dt \\
&\quad + q_7 \int_0^1 \Gamma(t)dt \\[8]
\end{align*}
\]
This implies,  
\[
\begin{align*}
&\frac{d_s(1^{2n-1}x_0,1^{2n+1}x_0)}{\Gamma(t)} \leq \frac{q_1+q_3+q_5+q_6}{1-\epsilon_3-\epsilon_7} \frac{d_s(1^{2n-1}x_0,1^{2n}x_0)}{\Gamma(t)} \\
&\leq h_1 \frac{d_s(1^{2n-1}x_0,1^{2n+1}x_0)}{\Gamma(t)} dt,
\end{align*}
\]
where \(h_1 = \frac{q_1+q_3+q_5+q_6}{1-\epsilon_3-\epsilon_7}.

Similarly,  
\[
\begin{align*}
&\frac{d_s(1^{2n-1}x_0,1^{2n+1}x_0)}{\Gamma(t)} dt \\
&\frac{d_s(x_{2n-1},x_{2n+2})}{\Gamma(t)} dt \\
&= \frac{d_s(x_{2n-1},x_{2n+1})}{\Gamma(t)} dt \\
&\leq q_1 \frac{d_s(x_{2n-1},x_{2n+1})}{\Gamma(t)} dt + q_2 \frac{d_s(x_{2n},x_{2n+2})}{\Gamma(t)} dt + q_3 \frac{d_s(x_{2n-1},x_{2n+3})}{\Gamma(t)} dt + q_5 \frac{d_s(x_{2n},x_{2n+1})}{\Gamma(t)} dt + q_6 \frac{d_s(x_{2n-1},x_{2n+2})}{\Gamma(t)} dt + q_7 \frac{d_s(x_{2n+1},x_{2n+2})}{\Gamma(t)} dt \\
&\leq q_1 \frac{d_s(x_{2n-1},x_{2n+1})}{\Gamma(t)} dt + q_2 \frac{d_s(x_{2n},x_{2n+2})}{\Gamma(t)} dt + q_3 \frac{d_s(x_{2n-1},x_{2n+3})}{\Gamma(t)} dt + q_4 \frac{d_s(x_{2n-1},x_{2n+2})}{\Gamma(t)} dt + q_5 \frac{d_s(x_{2n},x_{2n+1})}{\Gamma(t)} dt + q_6 \frac{d_s(x_{2n-1},x_{2n+2})}{\Gamma(t)} dt + q_7 \frac{d_s(x_{2n+1},x_{2n+2})}{\Gamma(t)} dt.
\end{align*}
\]
which implies,

\[
d_a(\mathbb{U}^{2n+1}x_0,\mathbb{U}^{2n+2}x_0) \leq \frac{\varrho_1 + \varrho_2 + \varrho_4 + \varrho_7}{1 - \varrho_2 - \varrho_5 - \varrho_6} d_a(\mathbb{U}^{2n}x_0,\mathbb{U}^{2n+1}x_0)
\]

\[
\leq h_2 \int_0^\infty \Gamma(t)dt
\]

where \( h_2 = \frac{\varrho_1 + \varrho_2 + \varrho_4 + \varrho_7}{1 - \varrho_2 - \varrho_5 - \varrho_6} \).

By proceeding same way, we get:

\[
d_a(x_n, x_{n+1}) \leq h^n \int_0^\infty \Gamma(t)dt; \text{ for all } n \geq 1,
\]

where \( h = \max\{h_1, h_2\} \).

Since \( h < 1 \) and taking the limit \( n \to \infty \), we have \( h^n \to 0 \). Hence, \( \int_0^\infty \Gamma(t)dt \to 0 \),

which implies that \( d_a(x_n, x_{n+1}) \to 0 \) as \( n \to 0 \). Hence, by the hypothesis, \( \{x_n\} \) is a Cauchy sequence in complete integral type metric space \((X, d_a)\). Since \((X, d_a)\) is complete, we see that \( \{\mathbb{U}^nx_0\} \) converges to some \( z \in X \) for all \( n \in (0, 1] \). We note that \( \{\mathbb{U}^n x_0\} \) is a sequence in \( \mathcal{K} \) and \( \{\mathbb{U}^{2n-1}x_0\} \) is a sequence in \( \mathcal{L} \) such that both sequences tend to the same limit \( z \). Since \( \mathcal{K} \) and \( \mathcal{L} \) are closed, we have \( z \in \mathcal{K} \cap \mathcal{L} \).

Now, we shall show that \( \mathbb{U}z = z \), i.e., \( z \) is a fixed point of \( \mathbb{U} \). By the continuity of \( \mathbb{U} \), we deduce that \( z = \lim_{n \to \infty} \mathbb{U}^n x_0 = \mathbb{U} \lim_{n \to \infty} \mathbb{U}^{2n-1}x_0 = \mathbb{U}z \). In order to prove uniqueness, let \( z^* \) and \( z^* \) be two fixed points of \( \mathbb{U} \) such that \( z^* \neq z^* \). Consider,

\[
d_a(z, z^*) \leq \varrho_1 \int_0^\infty \Gamma(t)dt + \varrho_2 \int_0^\infty \Gamma(t)dt + \varrho_3 \int_0^\infty \Gamma(t)dt
\]

\[
+ \varrho_4 \int_0^\infty \Gamma(t)dt + \varrho_5 \int_0^\infty \Gamma(t)dt + \varrho_6 \int_0^\infty \Gamma(t)dt
\]

\[
\leq (\varrho_1 + \varrho_2 + \varrho_3 + \varrho_6) \int_0^\infty \Gamma(t)dt
\]
which implies,

\[ (1 - \varrho_1 - \varrho_2 - \varrho_3 - \varrho_6) \int_0^\Gamma(t) dt \leq 0. \]

Thus, \( d_\alpha(z_*, z_\star) \int_0^\Gamma(t) dt = 0. \) Hence, \( d_\alpha(z_*, z_\star) = 0, \) which implies \( z_* = z_\star. \) This completes the proof of the theorem. \( \square \)

If we take \( d \) instead of \( d_\alpha \) in Theorem 3, we obtain the following result in the setting of complete integral metric space.

**Corollary 3.** Let \( K \) and \( L \) be non-empty closed subsets of a complete integral metric space \( (X, d) \). Let \( \mathcal{U} : K \cup L \to K \cup L \) be a continuous cyclic map that satisfies the condition:

\[
\begin{align*}
&\int_0^\Gamma(t) dt \leq \varrho_1 \int_0^\Gamma(t) dt + \varrho_2 \int_0^\Gamma(t) dt + \varrho_3 \int_0^\Gamma(t) dt + \varrho_4 \int_0^\Gamma(t) dt + \varrho_5 \int_0^\Gamma(t) dt + \varrho_6 \int_0^\Gamma(t) dt + \varrho_7 \int_0^\Gamma(t) dt + \varrho_8 \int_0^\Gamma(t) dt,
\end{align*}
\]

where \( \varrho_1 + 2\varrho_2 + 3\varrho_3 + \varrho_4 + \varrho_5 + \varrho_6 + \varrho_7 < 1 \) for all \( x \in K, y \in L \), and as \( \Gamma \) is Lebesgue integrable and sub-additive integrable, then \( \mathcal{U} \) has a unique fixed point in \( K \cap B \).

**Example 2.** Let \( X = [-1, 1], K = [-1, 0] \) and \( L = [0, 1] \). Let \( \mathcal{U} \) be a mapping defined by \( d(x, y) = |x - y| \). Then, \( (X, d) \) is an integral type metric space, and \( \mathcal{U} : K \cup L \to K \cup L \) is defined by \( Ux = -\frac{x}{2} \) and \( \Gamma(t) = \frac{t}{2} \).

Consider,

\[
\int_0^\Gamma(t) dt = \int_0^\frac{|x - y|}{2} dt = \frac{t^2}{4} \bigg|_0^\frac{|x - y|}{2} = \frac{|x - y|^2}{36}. \]
Now, take,

\[
\begin{aligned}
&d(x,y) + d(x,y) + d(y, l x) + d(y, l y) + d(y, B d(x, l x) + B d(x, l y)) \\
&\quad + e_6 + \frac{[x - y] + [x + \frac{y}{2}]}{1 + [x + \frac{y}{2}]} \\
&= e_1 \int_0^t \frac{1}{2} dt + e_2 \int_0^t \frac{1}{2} dt + e_3 \int_0^t \frac{1}{2} dt + e_4 \int_0^t \frac{1}{2} dt + e_5 \int_0^t \frac{1}{2} dt + e_7 \int_0^t \frac{1}{2} dt.
\end{aligned}
\]

Take \(e_1 = 0.5, e_2 = e_3 = 0.01\) and \(e_4 = e_5 = e_6 = e_7 = 0.1\), then:

\[
\begin{aligned}
&d(x,y) + d(x,y) + d(y, l x) + d(y, l y) + d(y, B d(x, l x) + B d(x, l y)) \\
&\quad + e_6 + \frac{[x - y] + [x + \frac{y}{2}]}{1 + [x + \frac{y}{2}]} \\
&= 0.5 \left( \frac{t^2}{4} \right)_{0}^{x-y} + 0.01 \left( \frac{t^2}{4} \right)_{0}^{x+\frac{y}{2}} + 0.01 \left( \frac{t^2}{4} \right)_{0}^{y+\frac{y}{2}} + 0.01 \left( \frac{t^2}{4} \right)_{0}^{y+\frac{y}{2}} + 0.01 \left( \frac{t^2}{4} \right)_{0}^{y+\frac{y}{2}} \\
&\quad + 0.01 \left( \frac{t^2}{4} \right)_{0}^{y+\frac{y}{2}} + 0.01 \left( \frac{t^2}{4} \right)_{0}^{y+\frac{y}{2}} \\
&= 0.5 \left( \frac{|x - y|^2}{4} \right) + 0.01 \left( \frac{|x + \frac{y}{2}|^2}{4} \right) + 0.01 \left( \frac{|y + \frac{y}{2}|^4}{4} \right) + 0.01 \left( \frac{|y + \frac{y}{2}|^4}{4} \right) + 0.01 \left( \frac{|y + \frac{y}{2}|^4}{4} \right) \\
&\quad + 0.01 \left( \frac{|y + \frac{y}{2}|^4}{4} \right) + 0.01 \left( \frac{|y + \frac{y}{2}|^4}{4} \right) \\
&= 0.5 \left( \frac{|x - y|^2}{4} \right) + \text{some value (which is > 0)}.
\end{aligned}
\]

Hence, \(\frac{|x - y|^2}{36} < \frac{|x - y|^2}{8} + \text{some value (which is > 0)}\).
Thus,

\[
\begin{align*}
    d(\theta_1, \theta_2) & \leq \theta_1 \int_0^1 \Gamma(t)dt \leq \theta_2 \int_0^1 \Gamma(t)dt + \theta_3 \int_0^1 \Gamma(t)dt + \theta_4 \int_0^1 \Gamma(t)dt + \theta_5 \int_0^1 \Gamma(t)dt + \theta_6 \int_0^1 \Gamma(t)dt + \theta_7 \int_0^1 \Gamma(t)dt + \theta_8 \int_0^1 \Gamma(t)dt, \\
    & = \sum_{i=1}^{8} \theta_i \\
    & \leq 2 \theta_1 + 2 \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7. 
\end{align*}
\]

where \( \sum_{i=1}^{8} \theta_i < 1 \) and \( \Gamma \) is Lebesgue integrable and sub-additive integrable. Thus, all the conditions of the above corollary are satisfied, and zero is the unique fixed point of \( \mathcal{U} \).

**Theorem 4.** Let \((X, d_X)\) be a complete integral type metric space. Suppose \( \mathcal{K} \) and \( \mathcal{L} \) are non-empty closed subsets of \( X \). Let \( \mathcal{U} : \mathcal{K} \cup \mathcal{L} \to \mathcal{K} \cup \mathcal{L} \) be a continuous cyclic map such that for all \( x \in \mathcal{K}, y \in \mathcal{L} \), the following condition is satisfied:

\[
\begin{align*}
    d_X(\mathcal{U}x, \mathcal{U}y) & \leq \theta_1 \int_0^1 \Gamma(t)dt \leq \theta_2 \int_0^1 \Gamma(t)dt + \theta_3 \int_0^1 \Gamma(t)dt + \theta_4 \int_0^1 \Gamma(t)dt + \theta_5 \int_0^1 \Gamma(t)dt + \theta_6 \int_0^1 \Gamma(t)dt + \theta_7 \int_0^1 \Gamma(t)dt + \theta_8 \int_0^1 \Gamma(t)dt, \\
    & = \sum_{i=1}^{8} \theta_i \\
    & \leq 2 \theta_1 + 2 \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7, \\
    & \leq \sum_{i=1}^{8} \theta_i (1 + \theta_i) < 1, \\
    & \leq \sum_{i=1}^{8} \theta_i (1 + \theta_i)^2 < 1. 
\end{align*}
\]

where \( \sum_{i=1}^{8} \theta_i < 1 \) and \( \Gamma \) is Lebesgue integrable and sub-additive integrable. Then, \( \mathcal{U} \) has a unique fixed point in \( \mathcal{K} \cap \mathcal{L} \).

**Proof.** Let \( x_0 \in \mathcal{K}(fixed) \), and define a sequence \( \{x_n\} \) by using:

\[
\mathcal{U}x_n = x_{n+1}, \quad n = 0, 1, 2, ..., 
\]

First of all, we prove that \( \{x_n\} \) is a Cauchy sequence in \( X \). For this, consider \( d_X(x_{n+1}, x_{n+2}) = d_X(\mathcal{U}x_n, \mathcal{U}x_{n+1}) \). By using the hypothesis of the theorem with \( x = x_{2n+1} \) and \( y = x_{2n-1} \), we get:
\[
\begin{align*}
&d_s((1^{2n-1}x_0,1^{2n+1}x_0)) \\
&= \int_0^1 \Gamma(t)dt \\
&+ \int_0^1 \Gamma(t)dt \\
&\leq \vartheta_1 \int_0^1 \Gamma(t)dt + \vartheta_2 \int_0^1 \Gamma(t)dt + \vartheta_3 \int_0^1 \Gamma(t)dt \\
&+ \vartheta_4 \int_0^1 \Gamma(t)dt + \vartheta_5 \int_0^1 \Gamma(t)dt + \vartheta_6 \int_0^1 \Gamma(t)dt \\
&\leq \vartheta_1 \int_0^1 \Gamma(t)dt + \vartheta_3 \int_0^1 \Gamma(t)dt + \vartheta_5 \int_0^1 \Gamma(t)dt \\
&+ \vartheta_4 \int_0^1 \Gamma(t)dt + \vartheta_6 \int_0^1 \Gamma(t)dt \\
&\leq \vartheta_3 \int_0^1 \Gamma(t)dt + \vartheta_5 \int_0^1 \Gamma(t)dt \\
&+ \vartheta_6 \int_0^1 \Gamma(t)dt
\end{align*}
\]

By using the hypothesis of the theorem, we get:
\[
\begin{align*}
&\int_0^1 \Gamma(t)dt \leq \frac{\vartheta_1 + \vartheta_3 + \vartheta_5}{1 - \vartheta_4 - \vartheta_5} \int_0^1 \Gamma(t)dt \\
&\leq h_1 \int_0^1 \Gamma(t)dt
\end{align*}
\]

where \( h_1 = \frac{\vartheta_1 + \vartheta_3 + \vartheta_5}{1 - \vartheta_4 - \vartheta_5} < 1 \).
Similarly,

\[
d_a(x_{2n+1},x_{2n+2}) \leq d_a(x_{2n},x_{2n+1}) + d_a(x_{2n+1},x_{2n+2}) + \phi_4 \int_0^{\Gamma(t)} dt + \phi_5 \int_0^{\Gamma(t)} dx + \phi_6 \int_0^{\Gamma(t)} dx + \phi_7 \int_0^{\Gamma(t)} dx
\]

Thus,

\[
d_a(x_{2n+1},\ldots,x_{2n+2}) \leq \phi_1 + \phi_2 + \phi_4 + \phi_5 + \phi_7 + \phi_8 \int_0^{\Gamma(t)} dt \leq h_2 \int_0^{\Gamma(t)} dt
\]

where \( h_2 = \frac{\phi_1 + \phi_2 + \phi_4 + \phi_5 + \phi_7 + \phi_8}{1 - \phi_2 - \phi_5 - \phi_6 - \phi_7 - \phi_8} < 1 \).

Hence, \( \int_0^h \Gamma(t) dt \leq h^n \int_0^h \Gamma(t) dt; \) for all \( n \geq 1 \),

where \( h = \max\{h_1, h_2\} \).

Now, for each integer \( k \geq 1 \), we have:
Now, we will prove that $L$ is a Cauchy sequence in $X$. Since $(X, d_A)$ is complete, the sequence $\{x_n\}$ converges to some $v \in X$. We note that $U^{2n}x_0$ is a sequence in $\mathcal{K}$ and $U^{2n-1}x_0$ is in $\mathcal{L}$ such that both sequences tend to some limit $v$. Since $\mathcal{K}$ and $\mathcal{L}$ are closed, we have $v \in X \cap \mathcal{L}$. Now, we will prove that $v$ is a fixed point of $\mathcal{U}$, i.e., $\mathcal{U}v = v$. By the continuity of $\mathcal{U}$, we deduce that $v = \lim_{n \to \infty} U^{2n}x_0 = \lim_{n \to \infty} U^{2n-1}x_0 = \mathcal{U}v$. To prove uniqueness, let $v$ and $v^*$ be two fixed points of $\mathcal{U}$ such that $v \neq v^*$. Consider,

\[
d_A(x_n, x_{n+k}) \leq \frac{d_A(x_{n+1}, x_{n+2}) + d_A(x_{n+k+1})}{\Gamma(t)dt} \leq h^n \frac{d_A(x_{n+1}, x_{n+2}) + d_A(x_{n+k+1})}{\Gamma(t)dt} \leq h^n (1 + h + h^2 + ...) \leq \frac{h^n}{1-h} \frac{d_A(x_{n+1}, x_{n+2})}{\Gamma(t)dt}
\]

Since $h < 1$ and taking the limit as $n \to \infty$, we have $h^n \to 0$. Hence, $\lim_{n \to \infty} \frac{d_A(x_{n+1}, x_{n+2})}{\Gamma(t)dt}$ for each $k \geq 1$. Thus, by the hypothesis, $\{x_n\}$ is a Cauchy sequence in $X$. Now, consider $\mathcal{U}$, we note that $\mathcal{U}^{2n}x_0$ is a sequence in $\mathcal{K}$ and $\mathcal{U}^{2n-1}x_0$ is in $\mathcal{L}$ such that both sequences tend to some limit $v$. Since $\mathcal{K}$ and $\mathcal{L}$ are closed, we have $v \in X \cap \mathcal{L}$. Now, we will prove that $v$ is a fixed point of $\mathcal{U}$, i.e., $\mathcal{U}v = v$. By the continuity of $\mathcal{U}$, we deduce that $v = \lim_{n \to \infty} \mathcal{U}^{2n}x_0 = \lim_{n \to \infty} \mathcal{U}^{2n-1}x_0 = \mathcal{U}v$. To prove uniqueness, let $v$ and $v^*$ be two fixed points of $\mathcal{U}$ such that $v \neq v^*$. Consider,

\[
d_A(v, v^*) \leq \theta_1(v, v^*) \frac{d_A(v, v^*)}{\Gamma(t)dt} + \theta_2(v, v^*) \frac{d_A(v, v^*)}{\Gamma(t)dt} + \theta_3(v, v^*) \frac{d_A(v^*, v^*)}{\Gamma(t)dt} + \theta_4(v, v^*) \frac{d_A(v^*, v^*)}{\Gamma(t)dt} + \theta_5(v, v^*) \frac{d_A(v^*, v^*)}{\Gamma(t)dt} + \theta_6(v, v^*) \frac{d_A(v^*, v^*)}{\Gamma(t)dt} + \theta_7(v, v^*) \frac{d_A(v^*, v^*)}{\Gamma(t)dt} + \theta_8(v, v^*) \frac{d_A(v^*, v^*)}{\Gamma(t)dt} \leq [\theta_1(v, v^*) + \theta_2(v, v^*) + \theta_3(v, v^*) + \theta_5(v, v^*) + \theta_7(v, v^*) + \theta_8(v, v^*)] \frac{d_A(v, v^*)}{\Gamma(t)dt}
\]
⇒ \((1 - h_1) \int_0^\Gamma(t)dt \leq 0,\)

where \(h_1 = \frac{1}{d_4(v,v^*)} \left( \frac{1}{d_4(v,v^*)} + \vartheta_2(v,v^*) + \vartheta_3(v,v^*) + \vartheta_4(v,v^*) + \vartheta_5(v,v^*) + \vartheta_6(v,v^*) \right) < 1.\) Thus, \(\int_0^\Gamma(t)dt = 0.\) Thereby, we get \(v = v^*.\) This completes the proof of the theorem. □

3. Conclusions

Since Kirk’s characterization of the Banach contraction mapping theorem, many characterizations of contraction-type mapping theorems have appeared in the literature. In this article, we introduced the concepts of weaker forms of integral type metric spaces. Furthermore, we established the existence of unique fixed point theorems in such spaces. We have deduced a number of corollaries and presented some examples to illustrate our results in weaker forms of integral type metric spaces.

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