Abstract. For a finite simplicial graph Γ, let $G(\Gamma)$ denote the right-angled Artin group on the complement graph of $\Gamma$. In this article, we introduce the notions of “induced path lifting property” and “semi-induced path lifting property” for immersions between graphs, and obtain graph theoretical criteria for the embeddability between right-angled Artin groups. We recover the result of S.-h. Kim and T. Koberda that an arbitrary $G(\Gamma)$ admits a quasi-isometric group embedding into $G(T)$ for some finite tree $T$. The upper bound on the number of vertices of $T$ is improved from $2^{2^{(m-1)/2}}$ to $m2^{m-1}$, where $m$ is the number of vertices of $\Gamma$. We also show that the upper bound on the number of vertices of $T$ is at least $2^{m/4}$. Lastly, we show that $G(C_m)$ embeds in $G(P_n)$ for $n \geq 2m - 2$, where $C_m$ and $P_n$ denote the cycle and path graphs on $m$ and $n$ vertices, respectively.

Keywords: Right-angled Artin group, quasi-isometry, immersion between graphs, path lifting.

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1. Introduction

For a finite simplicial graph $\Gamma$, let $G(\Gamma)$ denote the right-angled Artin group on the complement graph of $\Gamma$.

This paper is motivated by the recent work [KK13b] of S.-h. Kim and T. Koberda that if $\Gamma$ is a graph with $m$ vertices then $G(\Gamma)$ is embeddable into $G(T)$ for some tree $T$ with at most $2^{2^{(m-1)/2}}$ vertices, hence $G(\Gamma)$ is embeddable into the $n$-strand braid group $B_n$ with $n \leq 2^{2m^2}$. This result has several important corollaries.

At the first glance of their paper, we thought that the double exponential upper bound for the braid index is far from being sharp (one may expect a polynomial upper bound) and that their construction is very interesting and instructive, but it is not simple enough for practical uses.

We have tried to make a new construction which gives a polynomial upper bound on the number of vertices of the tree $T$. Though we were short of this goal, we succeeded in improving their results as follows. First, the double exponential upper bound $2^{2^{(m-1)/2}}$ is improved to an exponential upper bound $m2^m$. Second, we show that an exponential upper bound is unavoidable as long as the embedding $G(\Gamma) \to G(T)$ is induced by an immersion $T \to \Gamma$ as in the construction of Kim and Koberda.

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These come from graph theoretical criteria on immersions between graphs for the embeddability between associated right-angled Artin groups, using path lifting properties of immersions. We think that these properties are simple enough for practical uses. In particular, we show that $G(C_n)$ is embeddable into $G(P_{2n-2})$, where $C_m$ and $P_m$ denote the cycle and path graphs on $m$ vertices, respectively. This generalizes the result of M. Casals-Ruiz, A. Duncan and I. Kazachkov [CDK13] that $G(C_5)$ is embeddable in $G(P_8)$.

1.1. Right-angled Artin groups. Throughout the paper, all graphs are assumed to be undirected and simplicial (that is, without loops or multiple edges). Let $\Gamma$ and $\Gamma_1$ be graphs. We denote by $V(\Gamma)$ and $E(\Gamma)$ the vertex and edge sets of $\Gamma$, respectively. The notation "$\Gamma_1 \leq \Gamma$" means that $\Gamma_1$ is an induced subgraph of $\Gamma$, that is, $V(\Gamma_1) \subseteq V(\Gamma)$ and $E(\Gamma_1) = \{ \{v_1, v_2\} \in E(\Gamma) | v_1, v_2 \in V(\Gamma_1) \}$. For a vertex $v \in V(\Gamma)$, the link of $v$ in $\Gamma$ is the set $Lk_\Gamma(v) = \{ u \in V(\Gamma) | \{v, u\} \in E(\Gamma) \}$. For $A \subseteq V(\Gamma)$, we denote by $\Gamma \setminus A$ the induced subgraph of $\Gamma$ on $V(\Gamma) \setminus A$.

For a finite graph $\Gamma$, the right-angled Artin group (RAAG) $A(\Gamma)$ on $\Gamma$ is defined by the presentation $A(\Gamma) = \langle v \in V(\Gamma) | [v_i, v_j] = 1 \text{ if } \{v_i, v_j\} \in E(\Gamma) \rangle$. In the present paper, we use the opposite convention

$$G(\Gamma) = \langle v \in V(\Gamma) | [v_i, v_j] = 1 \text{ if } \{v_i, v_j\} \notin E(\Gamma) \rangle.$$ 

In other words, $G(\Gamma) = A(\Gamma^c)$, where $\Gamma^c$ denotes the complement graph of $\Gamma$.

For metric spaces $(X, d_X)$ and $(Y, d_Y)$, a map $f : X \to Y$ is a quasi-isometric embedding if there is a constant $C \geq 1$ such that for any $x_1, x_2 \in X$

$$d_X(x_1, x_2)/C - C \leq d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2) + C.$$ 

For finitely generated groups $G$ and $H$, a quasi-isometric group embedding from $G$ to $H$ is an injective homomorphism $f : G \to H$ such that it is a quasi-isometric embedding when $G$ and $H$ are endowed with word-metrics.

1.2. Embeddability of RAAGs into braid groups. An interesting question in the theory of RAAGs is, given a graph $\Gamma$, for which surface $S$ the group $G(\Gamma)$ admits an embedding into the mapping class group $\text{Mod}(S)$. (See [CW07, CLM12, KK13b, KK14] for instance.) It is well-known that if there is an embedding of the graph $\Gamma$ into $S_{g,p}$ (the orientable surface with genus $g$ and $p$ punctures), then $G(\Gamma)$ admits a quasi-isometric embedding into $\text{Mod}(S_{g,q})$ for some $q \geq p$. For example, if $\Gamma$ is a planar graph, then $G(\Gamma)$ is embeddable into the mapping class group of a punctured sphere. Notice that, in this construction, the genus of the surface is at least the graph-genus of $\Gamma$ (i.e. the smallest genus of a surface $S$ which admits an embedding $\Gamma \hookrightarrow S$). Therefore one may ask the following question.

Question. For a finite (non-planar) graph $\Gamma$, does $G(\Gamma)$ admit a quasi-isometric embedding into the mapping class group of a punctured sphere?
It was answered affirmatively by S.-h. Kim and T. Koberda [KK13b].

**Theorem 1.1 (KK13b).** For each finite graph $\Gamma$, there exists a finite tree $T$ such that $G(\Gamma)$ admits a quasi-isometric group embedding into $G(T)$.

This theorem has several interesting corollaries: any RAAG admits quasi-isomorphic group embeddings into a pure braid group and into the area-preserving diffeomorphism groups of the 2-disc and the 2-sphere; every finite-volume hyperbolic 3-manifold group is virtually a quasi-isometrically embedded subgroup of a pure braid group.

1.3. The Kim-Koberda construction. Let us briefly review the construction of Kim and Koberda.

A map of graphs $\phi : \Lambda \rightarrow \Gamma$ consists of a pair of functions, vertices to vertices and edges to edges, preserving the structure, i.e. $\phi$ sends adjacent vertices to adjacent vertices.

Let $\phi : \Lambda \rightarrow \Gamma$ be a map of graphs between finite graphs. Then $\phi$ induces a group homomorphism $\phi^* : G(\Gamma) \rightarrow G(\Lambda)$ defined by

$$\phi^*(v) = \prod_{v' \in \phi^{-1}(v)} v'$$

for $v \in V(\Gamma)$, where the product is defined to be the identity if $\phi^{-1}(v)$ is the empty set. Since $\phi$ is a map of graphs and since $\Gamma$ has no loops, no two vertices of $\phi^{-1}(v)$ are adjacent, hence the product is well-defined.

We say that $\phi : \Lambda \rightarrow \Gamma$ is $F$-surviving for $F \subseteq V(\Lambda)$ if for any $v' \in F$ and for any reduced word $w$ in $G(\Gamma)$, the word $\phi^*(w)$ has no cancellation of $v'$. (See §2 for details.) It is not difficult to observe that if $\phi$ is $F$-surviving for some $F \subseteq V(\Lambda)$ with $\phi(F) = V(\Gamma)$ then $\phi^* : G(\Gamma) \rightarrow G(\Lambda)$ is a quasi-isometric group embedding.

In [KK13b], Kim and Koberda considered the universal cover $p : \hat{\Gamma} \rightarrow \Gamma$, a finite induced subgraph $T \leq \hat{\Gamma}$, and the restriction $\phi = p|_T : T \rightarrow \Gamma$. Here, we may assume that $\Gamma$ and $T$ are connected, hence $T$ is a tree. They showed that if $T$ is sufficiently large, then $\phi$ is $F$-surviving for some $F \subseteq V(T)$ with $\phi(F) = V(\Gamma)$, hence $\phi^* : G(\Gamma) \rightarrow G(T)$ is a quasi-isometric group embedding. In order to achieve $\phi$ being $F$-surviving, they repeatedly enlarged the tree $T$ by taking union with its images under a certain collection of “deck transformations”. They showed that one can take $T$ such that $|V(T)| \leq 2^{2^m}$, where $m = |V(\Gamma)|$.

1.4. Path lifting properties of immersions and our results. A map of graphs $\phi : \Lambda \rightarrow \Gamma$ is called an immersion (or a locally injective map) if the restriction $\phi|_{\text{Lk}_{\Lambda}(v')} \text{ is injective for each } v' \in V(\Lambda)$.

Notice that a map of graphs $\phi : \Lambda \rightarrow \Gamma$ is a covering if $\phi$ is surjective and $\phi|_{\text{Lk}_{\Lambda}(v')} : \text{Lk}_{\Lambda}(v') \rightarrow \text{Lk}_{\Gamma}(\phi(v'))$ is bijective for each $v' \in V(\Lambda)$. It is easy to see that, for finite simplicial graphs $\Lambda$ and $\Gamma$, a map of graphs $\phi : \Lambda \rightarrow \Gamma$ is an immersion if and only if there is a covering $p : \hat{\Gamma} \rightarrow \Gamma$ that extends $\phi$, i.e. $\Lambda$ is an induced subgraph of $\hat{\Gamma}$ such that
\( \phi = p|_{\Lambda} \). (For example, J. Stallings [Sta83, Theorem 6.1] proved this for the case when \( \Gamma \) is a bouquet. Our case can be proved similarly.)

As observed by J. Stallings [Sta83], immersions \( \phi : \Lambda \to \Gamma \) have some of the properties of coverings such as the unique path lifting property and \( \pi_1 \)-injectivity. However, a path in \( \Gamma \) may not be lifted to a path in \( \Lambda \).

In §3, we introduce the notions of semi-induced paths and induced paths in \( \Gamma \). We say that a map of graphs \( \phi : \Lambda \to \Gamma \) has the semi-induced path lifting property (SIPL) for \( F \subseteq V(\Lambda) \) if any semi-induced path in \( \Gamma \) starting from \( \phi(v') \) for some \( v' \in F \) is lifted to a path in \( \Lambda \) starting from \( v' \). The induced path lifting property (IPL) is defined similarly. (See §3 for details.)

**Theorem 3.9.** Let \( \phi : \Lambda \to \Gamma \) be an immersion between finite graphs, and let \( F \) be a finite nonempty subset of \( V(\Lambda) \). If \( \phi \) has SIPL for \( F \), then \( \phi \) is \( F \)-surviving.

The property SIPL plays the role of deck transformation in the Kim-Koberda construction. Using this, we prove the following.

**Theorem 3.12.** For each graph \( \Gamma \) with \( m \) vertices, there exists a tree \( T \) with \( |V(T)| \leq m^{2^{m-1}} \) such that \( G(\Gamma) \) admits a quasi-isometric group embedding into \( G(T) \).

Therefore the upper bound on \( |V(T)| \) is improved from a double exponential function in \( m = |V(\Gamma)| \) in [KK13b] to an exponential function.

The following theorem shows that a partial converse of Theorem 3.9 holds.

**Theorem 3.14.** Let \( \phi : \Lambda \to \Gamma \) be an immersion between finite graphs, and let \( F \) be a finite nonempty subset of \( V(\Lambda) \). If \( \phi \) is \( F \)-surviving, then \( \phi \) has IPL for \( F \).

Using this, we show that the upper bound on \( |V(T)| \) in Theorem 3.12 must be at least an exponential function in \( m \).

**Theorem 3.15.** For each integer \( m \geq 2 \), there exists a graph \( \Gamma_m \) with \( m \) vertices such that if \( \phi : T \to \Gamma_m \) is an immersion of a finite tree \( T \) into \( \Gamma_m \) and \( \phi^* : G(\Gamma_m) \to G(T) \) is injective, then \( |V(T)| \geq 2^m/4 \).

The graphs \( \Gamma_m \) in the above theorem is thanks to Young Soo Kwon, Sang-il Oum and Paul Seymour. The lower bound from our original example was \( |V(T)| \geq 2\sqrt{m} \).

Let \( C_m \) and \( P_n \) denote the cycle and path graphs on \( m \) and \( n \) vertices, respectively. Regard \( P_n \) as a subgraph of the universal cover \( \tilde{C}_m \) of \( C_m \) which is the bi-infinite path graph, and let \( \phi_{n,m} : P_n \to C_m \) be the restriction of the covering map \( \tilde{C}_m \to C_m \). In [CDK13], M. Casals-Ruiz, A. Duncan and I. Kazachkov showed that \( \phi_{8,5}^* : G(C_5) \to G(P_8) \) is injective, which gives a counterexample to the Weakly Chordal Conjecture in [KK13a]. (In [CDK13], \( P_n \) denotes the path graph of length \( n \), hence it is \( P_{n+1} \) in our notation.) Using the induced path lifting property, we establish the following.

**Theorem 3.16.** For each \( m \geq 3 \), \( \phi_{n,m}^* : G(C_m) \to G(P_n) \) is injective if and only if \( n \geq 2m - 2 \).

We close this section with a couple of remarks.
Remark 1.2. Let $\phi : \Lambda \to \Gamma$ be an immersion between finite graphs and $F \subseteq V(\Lambda)$. By Theorems 3.9 and 3.14 we know that

$\phi$ has SIPL for $F \Rightarrow \phi$ is $F$-surviving $\Rightarrow \phi$ has IPL for $F$.

It would be interesting to know whether the converses hold.

Remark 1.3. Suppose that we are given a quasi-isometric group embedding $\phi^* : G(\Gamma) \to G(T)$ for a tree $T$. Then, as observed in [KK13b], $G(T)$ admits a quasi-isometric group embedding into the pure braid group $P_n$ with $n \leq 4|V(T)| + 2$, hence so does the original group $G(\Gamma)$. For the embeddability into pure braid groups, it suffices to require $T$ to be a planar graph, not necessarily a tree. Hence we can use a planar cover $\tilde{\Gamma}$ rather than a universal cover, and it would give a smaller upper bound on $|V(T)|$. For example, if $\Gamma$ is embeddable into a M"obius band (equivalently, in a real projective plane), then its double cover $\tilde{\Gamma}$ is a planar graph. It would be interesting to see whether the upper bound obtained by using planar cover is substantially smaller than the one given in this paper.

2. Preliminaries

For a map of graphs $\phi : \Lambda \to \Gamma$ and induced subgraphs $\Lambda_1 \subseteq \Lambda$ and $\Gamma_1 \subseteq \Gamma$ with $\phi(\Lambda_1) \subseteq \Gamma_1$, we denote by $\phi|_{\Lambda_1} : \Lambda_1 \to \Gamma_1$ the restriction $\phi|_{\Lambda_1}$.

As noted in [11], a map of graphs $\phi : \Lambda \to \Gamma$ induces a well-defined group homomorphism $\phi^* : G(\Gamma) \to G(\Lambda)$ defined by $\phi^*(v) = \prod_{v' \in \phi^{-1}(v)} v'$ for $v \in V(\Gamma)$. Abusing notation, for a word $w$ in $G(\Gamma)$, $\phi^*(w)$ denotes the word defined by the product. For this, we may fix a total order on $V(\Lambda)$ and write each product $\prod_{v' \in \phi^{-1}(v)} v'$ in the increasing order.

The map of graphs $\phi$ considered in [KK13b] is a restriction of the universal cover $p : \tilde{\Gamma} \to \Gamma$ to a subtree $T \subseteq \tilde{\Gamma}$. However, the group homomorphism $\phi^*$ can be defined for any map of graphs $\phi : \Lambda \to \Gamma$, and most of their arguments work for immersions or regular covers. Hence we describe their construction in a little more general setting.

Lemmas 2.4 and 2.6 are the key lemmas in this section. Corollary 2.7 is a simple version of Lemma 2.6. The tree $T$ in [KK13b] is constructed by repeatedly applying Lemma 2.4 and Corollary 2.7. So they are somehow implicit in the proof of Lemma 11 in [KK13b]. Lemma 2.6 is an improved version of Corollary 2.7 which was inspired by examples in [CDK13].

Let $w$ be a word in $G(\Gamma)$ representing $g \in G(\Gamma)$. The word $w$ is reduced if $w$ is a “shortest” word among all words representing $g$. In this case, the length of $w$ is the word length of $g$, denoted by $\|g\|$. The support of $w$, denoted by $\text{supp}(w)$, is the set of all vertices $v \in V(\Gamma)$ such that $v$ or $v^{-1}$ appears in $w$. It is well-known that if $w'$ is another reduced word representing $g$, then $\text{supp}(w) = \text{supp}(w')$.

Let $w$ be a (non-reduced) word in $G(\Gamma)$. A subword $v^\pm w_1 v'^\pm$ of $w$ is a cancellation of $v$ in $w$ if $\text{supp}(w_1) \cap \text{Lk}_T(v) = \emptyset$. If, furthermore, no letter in $w_1$ is equal to $v$ or $v^{-1}$, it is an innermost cancellation of $v$ in $w$. It is known that $w$ is reduced if and only if $w$ has no (innermost) cancellation.
Definition 2.1. We say that $\phi : \Lambda \to \Gamma$ is $v'$-surviving for $v' \in V(\Lambda)$ if for any reduced word $w$ in $G(\Gamma)$, the word $\phi^*(w)$ has no innermost cancellation of $v'$. We say that $\phi$ is $F$-surviving for $F \subseteq V(\Lambda)$ if it is $v'$-surviving for each $v' \in F$.

For example, let $\phi : \Lambda \to \Gamma$ be as in Figure 1. Then $\phi^*(v_1v_4v_1^{-1}) = v'_1v''_1v'_1v''_1^{-1}v'_1^{-1}$ has an innermost cancellation of $v'_1$, hence $\phi$ is not $v'_1$-surviving. In fact, $\phi^* : G(\Gamma) \to G(\Lambda)$ is not injective: $w = v_1v_2v_1^{-1}v_4v_1v_2^{-1}v_1^{-1}v_4^{-1}$ is a reduced word in $G(\Gamma)$ but $\phi^*(w)$ is the identity in $G(\Lambda)$.

Lemma 2.2 ([KK13b] Lemma 9). Let $\phi_1 : \Lambda_1 \to \Gamma$ be a map of graphs. If $\Lambda \subseteq \Lambda_1$ and $\phi = \phi_1|_{\Lambda}$, then $\text{supp}(\phi^*(w)) \subseteq \text{supp}(\phi_1^*(w))$ for any $w \in G(\Gamma)$. In particular, $\ker \phi_1^* \subseteq \ker \phi^*$.

Lemma 2.3 ([KK13b] Lemma 10). If a map of graphs $\phi : \Lambda \to \Gamma$ is $F$-surviving for some $F \subseteq V(\Lambda)$ with $\phi(F) = V(\Gamma)$, then $\phi^* : G(\Gamma) \to G(\Lambda)$ is a quasi-isometric group embedding.

The idea of proof of Lemma 2.2 is as follows: the inclusion map $i : \Lambda \to \Lambda_1$ induces a homomorphism $i^* : G(\Lambda_1) \to G(\Lambda)$ such that $\phi^* = i^* \circ \phi_1^*$ and $i^*$ sends the vertices in $\Lambda_1 \setminus \Lambda$ to the identity.

The idea of proof of Lemma 2.3 is as follows: if $\phi : \Lambda \to \Gamma$ is $F$-surviving for some $F \subseteq V(\Lambda)$ with $\phi(F) = V(\Gamma)$, then $\|\phi^*(w)\| \geq \|w\|$ for any reduced word $w$ in $G(\Gamma)$.

The following lemma shows that, under a certain condition, if $\phi : \Lambda \to \Gamma$ is an immersion such that $\phi^*$ is injective, then $\phi$ can be extended to $\phi_1 : \Lambda_1 \to \Gamma$ such that $\phi_1$ is $F$-surviving.

Lemma 2.4. Let $\phi : \Lambda \to \Gamma$ be an immersion that is a restriction of a regular cover $p : \hat{\Gamma} \to \Gamma$ to a finite induced subgraph $\Lambda$ of $\hat{\Gamma}$. Suppose that $\phi$ is surjective on the sets of vertices. Let $F$ be a finite nonempty subset of $V(\Gamma)$, and let $\Sigma$ be the set of all deck transformations $\sigma : \hat{\Gamma} \to \hat{\Gamma}$ such that $\sigma(\Lambda) \cap F \neq \emptyset$. Let $\Lambda_1$ be the induced subgraph of $\hat{\Gamma}$ on

$$\bigcup_{\sigma \in \Sigma} \sigma(V(\Lambda)).$$

If $\phi^* : G(\Gamma) \to G(\Lambda)$ is injective, then $\phi_1 = p|_{\Lambda_1} : \Lambda_1 \to \Gamma$ is $F$-surviving.
Lemma 2.6. Let \( \phi : \Lambda \to \Gamma \) be an immersion. Let \( v' \) be a vertex of \( \Lambda \) such that \( \phi(\text{Lk}_\Lambda(v')) = \text{Lk}_\Gamma(v) \), where \( v = \phi(v') \). Let \( \text{Lk}_\Gamma(v) = \{x_1, \ldots, x_1\} \) and \( \text{Lk}_\Lambda(v') = \{x_2, \ldots, x_2\} \).

**Proof.** Notice that \( F \subseteq V(\Lambda_1) \) because \( \phi \) is surjective and \( p \) is a regular cover. Assume that \( \phi_1 \) is not \( v' \)-surviving for some \( v' \in F \). Let \( v = \phi_1(v') \). See Figure 2. Let \( w \) be a nontrivial reduced word in \( G(\Gamma) \) such that \( \phi_1^*(w) \) has an innermost cancellation of \( v' \).

Then \( w \) has a subword of the form \( v^1w_1v^\perp \)

such that \( w_1 \) is a word in \( G(\Gamma \backslash v) \) with \( \text{supp}(\phi_1^*(w_1)) \cap \text{Lk}_{\Lambda_1}(v') = \emptyset \).

Since \( \phi^* \) is injective, \( \phi^*(v^1w_1v^\perp) \neq \phi^*(w_1) \). Hence there exists \( v'' \in \phi^1(v) \subseteq V(\Lambda) \) such that \( v'' \in \text{supp}(\phi^*(v^1w_1v^\perp)) \), thus there exists

\[
  x'' \in \text{supp}(\phi^*(w_1)) \cap \text{Lk}_{\Lambda}(v'').
\]

Because \( p : \tilde{\Gamma} \to \Gamma \) is a regular cover, there is a deck transformation \( \sigma \) such that \( \sigma(v'') = v' \).

(Here, \( \sigma \in \Sigma \) because \( v' = \sigma(v'') \in \sigma(\Lambda) \cap F \).) Let \( x' = \sigma(x'') \). By Lemma 2.2,

\[
x' = \sigma(x'') \in \sigma(\text{supp}(\phi^*(w_1))) \cap \text{Lk}_{\phi^1(\Lambda)}(\sigma(v'')) = \text{supp}((\phi \circ \sigma^{-1})^*(w_1)) \cap \text{Lk}_{\phi^1(\Lambda)}(v') \subseteq \text{supp}(\phi_1^*(w_1)) \cap \text{Lk}_{\Lambda_1}(v'),
\]

which contradicts \( \text{supp}(\phi_1^*(w_1)) \cap \text{Lk}_{\Lambda_1}(v') = \emptyset \).

**Remark 2.5.** In Lemma 2.2, if \( \Gamma \) is connected, then

\[
  |\Sigma| \leq |V(\Lambda)| \cdot \max \left\{ |p^{-1}(v) \cap F| : v \in V(\Gamma) \right\}
\]

because a deck transformation \( \sigma \in \Sigma \) is uniquely determined by a vertex \( v' \in V(\Lambda) \) and its image \( \sigma(v') \in p^{-1}(v) \cap F \), where \( v = p(v') \). In particular, if \( |p^{-1}(v) \cap F| \leq 1 \) for all \( v \in V(\Gamma) \), then \( |\Sigma| \leq |V(\Lambda)| \), hence

\[
  |V(\Lambda_1)| \leq |\Sigma| : |V(\Lambda)| \leq |V(\Lambda)|^2.
\]

**Lemma 2.6.** Let \( \phi : \Lambda \to \Gamma \) be an immersion. Let \( v' \) be a vertex of \( \Lambda \) such that \( \phi(\text{Lk}_\Lambda(v')) = \text{Lk}_\Gamma(v) \), where \( v = \phi(v') \). Let \( \text{Lk}_\Gamma(v) = \{x_1, \ldots, x_1\} \) and \( \text{Lk}_\Lambda(v') = \{x_2, \ldots, x_2\} \).
Therefore $G$ is nontrivial in $G \not\in \phi$.

Since $x_i$ is a nontrivial reduced word in $G$, we have the following simple version.

**Proof.** See Figure 3. Assume that $\phi$ is not $v'$-surviving. Then there exists a nontrivial reduced word $w$ in $G(\Gamma)$ with an innermost cancellation of $v'$ in $\phi^*(w)$, hence there is a subword of $w$ of the form

$$v^{i+1}w_1v^{i+1}$$

such that $w_1$ is a word in $G(\Gamma \setminus v)$ with $\supp(\phi^*(w_1)) \cap \text{Lk}_\Lambda(v') = \emptyset$.

Since $w$ is reduced, $\supp(w_1) \cap \text{Lk}_\Gamma(v)$ is nonempty. Let

$$i = \min\{k : x_k \in \supp(w_1) \cap \text{Lk}_\Gamma(v)\}.$$ 

Then $w_1$ is a reduced word in $G(\Gamma_i)$, hence $\phi^*(w_1) = \phi_i^*(w_1)$ and $\supp(\phi^*(w_1)) = \supp(\phi_i^*(w_1))$.

Since $x_i \in \supp(w_1)$ and $\phi_i$ is $x_i'$-surviving, $x_i' \in \supp(\phi_i^*(w_1)) = \supp(\phi^*(w_1))$. Therefore $x_i' \in \supp(\phi^*(w_1)) \cap \text{Lk}_\Lambda(v')$, which contradicts $\supp(\phi^*(w_1)) \cap \text{Lk}_\Lambda(v') = \emptyset$. Therefore $\phi$ is $v'$-surviving.

Now, suppose that $\phi_i^*$ is injective. Let $w$ be a nontrivial reduced word in $G(\Gamma)$. If $v \not\in \supp(w)$, then $w$ belongs to $G(\Gamma) = G(\Gamma \setminus v)$, hence $\phi^*(w) = \phi_i^*(w)$ is nontrivial in $G(\Lambda_i)$. Since $\Lambda_i$ is an induced subgraph of $\Lambda$, $G(\Lambda_i)$ embeds in $G(\Lambda)$, hence $\phi^*(w)$ is nontrivial in $G(\Lambda)$. If $v \in \supp(w)$, then $\phi^*(w)$ is nontrivial because $\phi$ is $v'$-surviving. Therefore $\phi^*$ is injective.

In the above lemma, observe that, for each $1 \leq i \leq l$ and a reduced word $w$ in $G(\Gamma_i)$, if $\phi_i^*(w)$ has a cancellation of $x_i'$ then so does $\phi_i^*(w)$. Thus if $\phi_i$ is $x_i'$-surviving, then so is $\phi_i$. Hence we have the following simple version.

**Figure 3.** The map of graphs $\phi_i$.
Corollary 2.7. Let \( \phi : \Lambda \rightarrow \Gamma \) be an immersion. Let \( v' \) be a vertex of \( \Lambda \) such that \( \phi(\text{Lk}_\Lambda(v')) = \text{Lk}_\Gamma(v) \), where \( v = \phi(v') \). Let
\[
\Gamma_1 = \Gamma \setminus v, \quad \Lambda_1 = \Lambda \setminus \phi^{-1}(v), \quad \phi_1 = \phi(\Lambda_1, \Gamma_1) : \Lambda_1 \rightarrow \Gamma_1.
\]
If \( \phi_1 \) is \( \text{Lk}_\Lambda(v') \)-surviving, then \( \phi \) is \( v' \)-surviving. Furthermore, if \( \phi^*_1 : G(\Gamma_1) \rightarrow G(\Lambda_1) \) is injective, then so is \( \phi^* : G(\Gamma) \rightarrow G(\Lambda) \).

3. Path lifting properties and embedding between RAAGs

In this section we introduce the notions of “induced path lifting property” and “semi-induced path lifting property” for immersions, and apply them to embedability between RAAGs. This will improve some results of Kim and Koberda in [KK13a] and Casals-Ruiz, Duncan and Kazachkov in [CDK13].

Throughout this section, \( \phi : \Lambda \rightarrow \Gamma \) is assumed to be an immersion between finite graphs.

3.1. Path lifting properties.

Definition 3.1. A path in a graph \( \Gamma \) is a tuple \( \alpha = (v_0, v_1, \ldots, v_k) \) of vertices of \( \Gamma \) such that \( \{v_i, v_{i+1}\} \in E(\Gamma) \) for \( 0 \leq i \leq k - 1 \) and \( v_i \neq v_j \) if \( i \neq j \). (In particular, \( \alpha \) is not a loop because \( v_0 \neq v_k \).)

Definition 3.2. A path \( \alpha = (v_0, v_1, \ldots, v_k) \) in \( \Gamma \) is an induced path if \( \{v_i, v_j\} \notin E(\Gamma) \) whenever \( j \geq i + 2 \). (In other words, \( \alpha \) is an induced path if and only if the induced subgraph of \( \Gamma \) on the vertices of \( \alpha \) is a path graph.)

Definition 3.3. Suppose that a total order \( \prec \) is given on \( V(\Gamma) \). A path \( \alpha = (v_0, v_1, \ldots, v_k) \) in \( \Gamma \) is a semi-induced path with respect to \( \prec \) if \( \{v_i, v_j\} \notin E(\Gamma) \) whenever \( j \geq i + 2 \) and \( v_j \prec v_{i+1} \).

If there is no confusion, we will not refer to the total order on \( V(\Gamma) \), assuming that some order is given. In particular, we omit the term “with respect to \( \prec \)” for simplicity.

A path with one or two vertices is both induced and semi-induced. Notice that a path \( \alpha = (v_0, v_1, \ldots, v_k) \) is an induced path if, for each \( j \geq 2 \),
\[
v_j \notin \bigcup_{i=0}^{j-2} \text{Lk}_\Gamma(v_i)
\]
and it is a semi-induced path if, for each \( j \geq 2 \),
\[
v_j \notin \bigcup_{i=0}^{j-2} \{v \in \text{Lk}_\Gamma(v_i) : v \prec v_{i+1}\}.
\]

Let \( \phi : \Lambda \rightarrow \Gamma \) be an immersion. A path \( \tilde{\alpha} = (v'_0, v'_1, \ldots, v'_k) \) in \( \Lambda \) is called a lift of the path \( \alpha = (v_0, v_1, \ldots, v_k) \) in \( \Gamma \) if \( \phi(v'_i) = v_i \) for \( 0 \leq i \leq k \). Notice that the lift \( \tilde{\alpha} \), if exists, is uniquely determined by \( \alpha \) and \( v'_0 \) due to the unique path lifting property for immersions between graphs [Sta83].
Example 3.7. If $\Gamma$ is the complete graph $v$ maximal paths, semi-induced paths and induced paths starting from $v$ in $\Gamma$ such that $\phi$ is a group homomorphism. 

Let $\phi \colon \Lambda \to \Gamma$ be an immersion, and let $F \subseteq V(\Lambda)$. We say that $\phi$ has the induced path lifting property (IPL) for $F$ (resp. semi-induced path lifting property (SIPL) for $F$) if for any $v' \in F$ and for any induced path (resp. semi-induced path) $\alpha$ in $\Gamma$ starting from $\phi(v')$, there is a lift of $\alpha$ starting from $v'$.

It is obvious from the definitions that if $\phi$ has SIPL for $F$, then $\phi$ has IPL for $F$.

Example 3.5. Let $C_5$ be the cycle graph on five vertices as in Figure 4(a). Then $\alpha_1 = (v_0, v_1, v_2, v_3, v_4)$ and $\alpha_2 = (v_0, v_4, v_3, v_2, v_1)$ are maximal paths starting from $v_0$, but they are not induced paths because $\{v_0, v_4\}$ and $\{v_0, v_1\}$ are edges of $C_5$. The paths $(v_0, v_1, v_2, v_3)$ and $(v_0, v_4, v_3, v_2)$ are the only maximal induced paths starting from $v_0$.

Fix a total order on $V(C_5)$, say, $v_0 \prec v_1 \prec v_2 \prec v_3 \prec v_4$. Then $\alpha_2 = (v_0, v_4, v_3, v_2, v_1)$ is not semi-induced because $v_1 \in \{v \in Lk_{C_5}(v_0) : v \prec v_4\} = \{v_1\}$. The paths $\alpha_1 = (v_0, v_1, v_2, v_3, v_4)$ and $\alpha_3 = (v_0, v_4, v_3, v_2)$ are the only maximal semi-induced paths starting from $v_0$. Similarly, $\alpha_4 = (v_1, v_0, v_4, v_3, v_2)$ and $\alpha_5 = (v_1, v_2, v_3, v_4)$ are the only maximal semi-induced paths starting from $v_1$.

Let $P_8$ be the path graph on eight vertices as in Figure 4(b), and let $\phi \colon P_8 \to C_5$ map $v'_i$ and $v''_i$ to $v_i$ for $0 \leq i \leq 4$. Then $\phi$ is an immersion.

The maximal induced paths in $C_5$ starting from $v_0$ are $(v_0, v_1, v_2, v_3)$ and $(v_0, v_4, v_3, v_2)$, and they can be lifted to paths in $P_8$ starting from $v'_0$. Hence $\phi$ has IPL for $v'_0$. In the same way, $\phi$ has IPL and SIPL for $\{v'_0, v'_1\}$. In fact, the map $\phi \colon P_8 \to C_5$ is an example given in [CDK13] such that $\phi^* : G(C_5) \to G(P_8)$ is injective. Theorem 3.16 gives a necessary and sufficient condition on $(m, n)$ for $\phi : P_n \to C_m$, defined as above, to induce an injective group homomorphism.

Example 3.6. Let $\Gamma$ be the complete bipartite graph $K_{2,3}$. See Figure 5 which shows maximal paths, semi-induced paths and induced paths starting from $v_0$ in $\Gamma$.

Example 3.7. If $\Gamma$ is the complete graph $K_n$ on $n$ vertices endowed with the total order $v_0 \prec v_1 \prec \cdots \prec v_{n-1}$, then $\alpha = (v_{i_0}, v_{i_1}, \ldots, v_{i_k})$ is a maximal semi-induced path starting from $v_0$ if and only if $0 = i_0 < i_1 < i_2 < \cdots < i_{k-1} < i_k = n - 1$, hence there are $2^{n-2}$ maximal semi-induced paths starting from $v_0$.

Lemma 3.8. Let $\phi : \Lambda \to \Gamma$ be an immersion, and let $v'$ be a vertex of $\Lambda$ such that $\phi(Lk_{\Lambda}(v')) = Lk_{\Gamma}(v)$, where $v = \phi(v')$. Let $Lk_{\Gamma}(v) = \{x_1, \ldots, x_l\}$ and $Lk_{\Lambda}(v') = \{y_1, \ldots, y_m\}$.

![Figure 4. The map $\phi : P_8 \to C_5$ sends $v'_i$ and $v''_i$ to $v_i$ (0 ≤ i ≤ 4).](image-url)
has SIPL for \( \gamma \).

Then the restriction \( (v, x_1, \ldots, x_l) \) of \( \alpha \) is a lift of the path \( (v, x_1, \ldots, x_l) \) of \( \alpha \) starting from \( x_1 \). Therefore \( \phi_i \) has SIPL for \( x_i \).

Conversely, assume that each \( \phi_i \) has SIPL for \( x_i \). Let \( \alpha \) be a semi-induced path in \( \Gamma \) starting from \( v \), hence \( \alpha \) is of the form \( \alpha = (v, x_i, v_1, \ldots, v_k) \) for some \( 1 \leq i \leq l \) and \( v_1, \ldots, v_k \in V(\Gamma) \).

\[
\{x'_1, \ldots, x'_l\} \quad \text{with} \quad \phi(x'_i) = x_i \quad \text{for} \quad 1 \leq i \leq l.
\]

Suppose that a total order is given on \( V(\Gamma) \) such that \( x_1 < x_2 < \cdots < x_l \). For \( 1 \leq i \leq l \), let

\[
\Gamma_i = \Gamma \setminus \{v, x_1, \ldots, x_i-1\},
\]

\[
\Lambda_i = \Lambda \setminus \phi^{-1}(\{v, x_1, \ldots, x_i-1\}),
\]

\[
\phi_i = \phi(\Lambda_i, \Gamma_i) : \Lambda_i \to \Gamma_i.
\]

Suppose that each \( V(\Gamma_i) \) inherits the total order from \( V(\Gamma) \). Then \( \phi : \Lambda = \Gamma \to \Gamma \) has SIPL for \( v' \) if and only if each \( \phi_i \) has SIPL for \( x'_i \).

**Proof.** Assume that \( \phi \) has SIPL for \( v' \). Fix \( i \in \{1, \ldots, l\} \). Let \( \alpha_1 = (x_i, v_1, \ldots, v_k) \) be a semi-induced path in \( \Gamma_i \) starting from \( x_i \). Then \( \alpha = (v, x_i, v_1, \ldots, v_k) \) is a path in \( \Gamma \) since \( x_i \in Lk_\Gamma(v) \).

Since \( \alpha_1 \) is a path in \( \Gamma_i \), we have \( v_j \notin \{x_1, \ldots, x_{i-1}\} = \{u \in Lk_\Gamma(v) : u < x_i\} \) for \( 1 \leq j \leq k \). Notice that \( \alpha_1 \) is semi-induced also in \( \Gamma \) because \( \Gamma_i \) is an induced subgraph of \( \Gamma \). Hence, for \( 2 \leq j \leq k \), we have \( v_j \notin \{u \in Lk_\Gamma(x_i) : u < x_1\} \) and \( v_j \notin \{u \in Lk_\Gamma(v_1) : u < v_{p+1}\} \) for \( 1 \leq p \leq j-2 \). These imply that \( \alpha \) is a semi-induced path in \( \Gamma \).

Since \( \phi \) has SIPL for \( v' \), there is a lift \( \bar{\alpha} = (v', x''_1, x''_2, \ldots, x''_k) \) starting from \( v' \). Because \( (v', x'_i) \) is a lift of the path \( (v, x_i) \), the unique path lifting property implies that \( x''_i = x_i \).

Then the restriction \( (x'_1, x'_2, \ldots, x'_k) \) of \( \bar{\alpha} \) is a lift of \( \alpha_1 \) to \( \Lambda_i \) starting from \( x'_i \). Therefore \( \phi_i \) has SIPL for \( x'_i \).
Let \( \alpha_1 = (x_i, v_i, \ldots, v_k) \). Since \( \alpha \) is semi-induced in \( \Gamma \), we have \( v_j \notin \{u \in \text{Lk}_\Gamma(v) : u \prec x_i\} \) for \( 1 \leq j \leq k \). Hence \( \alpha_1 \) is a path in \( \Gamma_i \). Since \( \alpha_1 \) is semi-induced in \( \Gamma \), it is semi-induced also in \( \Gamma_i \) because \( V(\Gamma_i) \) inherits the total order from \( V(\Gamma) \).

Since \( \phi_i \) has SIPL for \( x_i \), there is a lift \( \tilde{\alpha}_1 = (x_i', v_1', \ldots, v_k') \) of \( \alpha_1 \) to \( \Lambda_i \) starting from \( x_i' \). Then \( \tilde{\alpha} = (v', x_i', v_1', \ldots, v_k') \) is a lift of \( \alpha \) to \( \Lambda \) starting from \( v' \). Therefore \( \phi \) has SIPL for \( v' \).

3.2. Semi-induced path lifting property and embedding between RAAGs. Compare the similarity between Lemma 3.8 and Lemma 2.6. From this, we have the following theorem almost for free.

**Theorem 3.9.** Let \( \phi : \Lambda \to \Gamma \) be an immersion between finite graphs, and let \( F \) be a finite nonempty subset of \( V(\Lambda) \). If \( \phi \) has SIPL for \( F \), then \( \phi \) is \( F \)-surviving.

**Proof.** Choose any \( v' \in F \) and let \( v = p(v') \). It suffices to show that \( \phi \) is \( v' \)-surviving. Notice that \( \phi(\text{Lk}_\Lambda(v')) = \text{Lk}_\Gamma(v) \) because if \( x \in \text{Lk}_\Gamma(v) \) then \( (v, x) \) is a semi-induced path in \( \Gamma \) hence there is a unique lift \( (v', x') \) of \( (v, x) \) to \( \Lambda \).

Let \( \text{Lk}_\Gamma(v) = \{x_1, \ldots, x_l\} \) and \( \text{Lk}_\Lambda(v') = \{x_1', \ldots, x_l'\} \) with \( \phi(x_i') = x_i \) for \( 1 \leq i \leq l \). Rearranging \( x_i \)'s if necessary, we may assume that the total order on \( V(\Gamma) \) is such that \( x_1 \prec x_2 \prec \cdots \prec x_l \). For \( 1 \leq i \leq l \), let

\[
\begin{align*}
\Gamma_i &= \Gamma \setminus \{v, x_1, \ldots, x_{i-1}\}, \\
\Lambda_i &= \Lambda \setminus \{\phi^{-1}(\{v, x_1, \ldots, x_{i-1}\}\}, \\
\phi_i &= \phi(\Lambda_i, \Gamma_i) : \Lambda_i \to \Gamma_i.
\end{align*}
\]

By Lemma 3.8 each \( \phi_i \) has SIPL for \( x_i' \). Using induction on the number of vertices of \( \Gamma \), we may assume that each \( \phi_i \) is \( x_i' \)-surviving. (If \( |V(\Gamma)| = 1 \), then \( \phi \) is obviously \( v' \)-surviving.) Then \( \phi \) is \( v' \)-surviving by Lemma 2.6.

**Proposition 3.10.** Let \( \Gamma \) be a graph with \( m \) vertices, \( p : \tilde{\Gamma} \to \Gamma \) a covering, and \( F \) a finite nonempty subset of \( V(\tilde{\Gamma}) \). Then there exists an induced subgraph \( \Lambda \) of \( \tilde{\Gamma} \) with \( |V(\Lambda)| \leq |F| \cdot 2^{m-1} \) such that \( \phi = p|_\Lambda : \Lambda \to \Gamma \) has SIPL for \( F \). Furthermore, if the induced subgraph of \( \tilde{\Gamma} \) on \( F \) is connected, then \( \Lambda \) can be chosen to be connected.

**Proof.** For each \( v' \in V(\tilde{\Gamma}) \), let \( \Lambda_{v'} \) be the union of the lifts of all maximal semi-induced paths in \( \Gamma \) starting from \( p(v') \) to \( \tilde{\Gamma} \) starting from \( v' \). Then \( p|_{\Lambda_{v'}} : \Lambda_{v'} \to \Gamma \) has SIPL for \( v' \).

**Claim.** \( |V(\Lambda_{v'})| \leq 2^{m-1} \).

**Proof of Claim.** We use induction on \( m = |V(\Gamma)| \). If \( m = 1 \), then both \( \Gamma \) and \( \Lambda_{v'} \) are the graph with one vertex and no edge, hence the claim holds.

Let \( \text{Lk}_{\tilde{\Gamma}}(v') = \{x_1', \ldots, x_l'\} \) and \( \text{Lk}_\Gamma(v) = \{x_1, \ldots, x_l\} \), where \( v = p(v') \) and \( x_i = p(x_i') \) for \( 1 \leq i \leq l \). Rearranging \( x_i \)'s if necessary, we may assume that the total order on \( V(\Gamma) \) is such that \( x_1 \prec x_2 \prec \cdots \prec x_l \). For \( 1 \leq i \leq l \), let

\[
\Gamma_i = \Gamma \setminus \{v, x_1, \ldots, x_{i-1}\},
\]

and let
and let $\Lambda_i$ be the union of the lifts of all maximal semi-induced paths in $\Gamma_i$ starting from $x_i$ to $\tilde{\Gamma}$ starting from $x'_i$. Then $\Lambda_i$ is defined in the same way as $\Lambda_{i'}$, where $\Gamma_i$ and $x'_i$ play the roles of $\Gamma$ and $v'$, respectively. By induction hypothesis, $|V(\Lambda_i)| \leq 2^{m-i-1}$ because $|V(\Gamma_i)| = |V(\Gamma)| - i = m - i$. Notice that $l \leq m - 1$ and $V(\Lambda_{i'}) = \{v'\} \cup (\cup_{i=1}^l V(\Lambda_i))$. Therefore

$$|V(\Lambda_{i'})| \leq 1 + \sum_{i=1}^l |V(\Lambda_i)| \leq 1 + \sum_{i=1}^{m-1} 2^{m-i-1} = 1 + \sum_{k=0}^{m-2} 2^k = 2^{m-1}. \quad \square$$

Let $\Lambda$ be the induced subgraph of $\tilde{\Gamma}$ on the vertices of $\cup_{i' \in F} \Lambda_{i'}$, and let $\phi = p|_{\Lambda} : \Lambda \to \Gamma$. By the construction, $\phi$ has SIPL for $F$ and $|V(\Lambda)| \leq \sum_{i' \in F} |V(\Lambda_{i'})| \leq |F| \cdot 2^{m-1}$. It is easy to see that if the induced subgraph of $\tilde{\Gamma}$ on $F$ is connected, then $\Lambda$ is connected. \square

In the above theorem, if $p : \tilde{\Gamma} \to \Gamma$ is the universal cover and $\Gamma$ is connected, we can take $F$ as the vertex set of a lift of a maximal tree in $\Gamma$. Then $|F| = m$ and $\Lambda$ is a tree with $|V(\Lambda)| \leq |F| \cdot 2^{m-1} = m2^{m-1}$. Combining this observation with Lemma 2.3 and Theorem 3.9 we have the following.

**Corollary 3.11.** If $\Gamma$ is a connected graph with $m$ vertices and $p : \tilde{\Gamma} \to \Gamma$ is the universal cover, then there exists an induced subtree $T$ of $\tilde{\Gamma}$ with $|V(T)| \leq m2^{m-1}$ such that $\phi^* : G(\Gamma) \to G(T)$ is a quasi-isometric group embedding, where $\phi = p|_T$.

Theorem 3.12 is immediate from the above corollary.

**Theorem 3.12.** For each graph $\Gamma$ with $m$ vertices, there exists a tree $T$ with $|V(T)| \leq m2^{m-1}$ such that $G(\Gamma)$ admits a quasi-isometric group embedding into $G(T)$.

**Proof.** We follow the argument of Kim and Koberda in [KK13b].

If $\Gamma$ is connected, then it is Corollary 3.11.

Suppose that $\Gamma$ is disconnected, hence $\Gamma = \Gamma_1 \coprod \Gamma_2$ with $|V(\Gamma_i)| = m_i \geq 1$ for $i = 1, 2$. Using induction on $|V(\Gamma)|$, we may assume that for each $i = 1, 2$, there exist a tree $T_i$ with $|V(T_i)| \leq m_i2^{m_i-1}$ and a quasi-isometric group embedding of $G(\Gamma_i)$ into $G(T_i)$. Let $T$ be the tree obtained by joining a vertex in $T_1$ and another vertex in $T_2$ by a length 2 path. Since $m_1, m_2 \geq 1$,

$$|V(T)| = |V(T_1)| + |V(T_2)| + 1 \leq m_12^{m_1-1} + m_22^{m_2-1} + 1 \leq (m_1 + m_2)2^{m_1+m_2-1} = m2^{m-1}.$$

Since $G(\Gamma) = G(\Gamma_1) \times G(\Gamma_2)$ and there is a natural quasi-isometric embedding of $G(T_1) \times G(T_2)$ into $G(T)$, there is a quasi-isometric group embedding of $G(\Gamma)$ into $G(T)$. \square

**Remark 3.13.** Let us say that an immersion $\phi : \Lambda \to \Gamma$ has the path lifting property (PL) for $F \subseteq V(\Lambda)$ if, for any $v' \in F$ and for any path $\alpha$ in $\Gamma$ starting from $v = \phi(v')$, there is a lift of $\alpha$ to $\Lambda$ starting from $v'$.

If $\phi$ has PL for $F$, then $\phi$ has SIPL for $F$, by definitions, regardless of the choice of a particular total order on $V(\Gamma)$, hence $\phi$ is $F$-surviving by Theorem 3.9.
The notion of PL is simpler than SIPL, and PL is enough for some cases. For example, we can obtain Corollary 3.11 with $|V(T)| \leq m \cdot m!$ if SIPL in Proposition 3.10 is replaced with PL; the upper bound on $|V(T)|$ increases from $m \cdot 2^{m-1}$ to $m \cdot m!$.

However, the property SIPL is indeed necessary for some cases. For example, the map $\phi : P_8 \to C_5$ in Example 3.5 has SIPL for $v'_0$ but it does not have PL for $v'_0$.

3.3. **Induced path lifting property and embedding between RAAGs.** Recall from Theorem 3.12 that for each graph $\Gamma$ with $m$ vertices, there exists a tree $T$ with $|V(T)| \leq m2^{m-1}$ such that $G(\Gamma)$ admits a quasi-isometric group embedding into $G(T)$. In this subsection, we prove Theorem 3.15 which shows that the above upper bound on $|V(T)|$ is almost optimal: there is a collection of graphs $\Gamma_m$ with $m$ vertices such that if $\phi : T \to \Gamma_m$ is an immersion of a tree $T$ into $\Gamma_m$ and $\phi^* : G(\Gamma_m) \to G(T)$ is injective, then $|V(T)| \geq 2^m/4$.

**Theorem 3.14.** Let $\phi : \Lambda \to \Gamma$ be an immersion between finite graphs, and let $F$ be a finite nonempty subset of $V(\Lambda)$. If $\phi$ is $F$-surviving, then $\phi$ has IPL for $F$.

**Proof.** Let $p : \Gamma \to \Gamma$ be a covering which extends $\phi$, i.e. $\Lambda \leq \Gamma$ and $\phi = p|\Lambda$.

Assume that $\phi$ is $F$-surviving, but does not have IPL for $F$. Then there exist a vertex $v' \in F$ and an induced path $\alpha = (v_0, v_1, \ldots, v_k)$ in $\Gamma$ with $v_0 = \phi(v')$ and $k \geq 1$ such that if $\tilde{\alpha} = (v'_0, v'_1, \ldots, v'_k)$ is the lift of $\alpha$ to $\tilde{\Gamma}$ starting from $v' = v'_0$, then $v'_i \notin \Lambda$ and $v'_i \in \Lambda$ for $0 \leq i \leq k - 1$. Consider the following reduced word

$$w = v_0 \cdots v_{k-1}v_kv_{k-1}^{-1} \cdots v_0^{-1}.$$  

**Claim.** For $i = 0, \ldots, k - 1$, we have

(1) $\supp(\phi^*(v_{i+1} \cdots v_{k-1}v_kv_{k-1}^{-1} \cdots v_{i+1}^{-1})) \cap \text{Lk}_{\Lambda}(v'_i) = \emptyset$.

In particular, $v'_i \notin \supp(\phi^*(v_{i+1} \cdots v_{k-1}v_kv_{k-1}^{-1} \cdots v_{i+1}^{-1}))$.

**Figure 6.** Pictures for the proof of Theorem 3.14
Proof of Claim. Notice that the equality (1) implies that $v'_i$ commutes with each element in $\text{supp}(\phi^*(v_{i+1} \cdots v_{k-1}v_kv_{k-1}^{-1} \cdots v_{i+1}^{-1}))$, hence $v'_i \notin \text{supp}(\phi^*(v_i v_{i+1} \cdots v_{k-1}v_kv_{k-1}^{-1} \cdots v_{i+1}^{-1}))$. We will prove (1) by using reverse induction on $i = 0, \ldots, k - 1$.

For the case $i = k - 1$, assume that there exists $x' \in \text{supp}(\phi^*(v_k)) \cap \text{Lk}_\Lambda(v'_{k-1})$. See Figure 7(a). Since $x' \in \text{Lk}_\Lambda(v'_{k-1})$, $\{v'_{k-1}, x'\}$ is an edge in $\Lambda$. Since $x' \in \text{supp}(\phi^*(v_k)) \subseteq p^{-1}(v_k)$, the edge $\{v'_{k-1}, x'\}$ maps to the edge $\{v_{k-1}, v_k\}$ by $p$. Since $p$ is a covering, $v'_k = x'$ by the unique path lifting property. This is a contradiction because $x' \in \Lambda$ but $v'_k \notin \Lambda$. Therefore $\text{supp}(\phi^*(v_k)) \cap \text{Lk}_\Lambda(v'_{k-1}) = \emptyset$.

Now, assume that the claim is true for $i = j + 1$ for some $0 \leq j \leq k - 2$, hence

$$v'_{j+1} \notin \text{supp}(\phi^*(v_{j+1} \cdots v_{k-1}v_kv_{k-1}^{-1} \cdots v_{j+1}^{-1})).$$

Assume that there exists $x' \in \text{supp}(\phi^*(v_{j+1} \cdots v_{k-1}v_kv_{k-1}^{-1} \cdots v_{j+1}^{-1})) \cap \text{Lk}_\Lambda(v'_j)$. See Figure 7(b). Then

$$x' \neq v'_{j+1}$$

by (2). Since $p(x') \in \text{supp}(v_{j+1} \cdots v_{k-1}v_kv_{k-1}^{-1} \cdots v_{j+1}^{-1}) = \{v_{j+1}, \ldots, v_k\}$, we have $p(x') = v_j$ for some $j + 1 \leq l \leq k$. Since $x' \in \text{Lk}_\Lambda(v'_j)$, $\{v'_j, x'\}$ is an edge in $\Lambda$, hence $\{v_j, v_i\} = \{p(v'_j), p(x')\}$ is an edge in $\Gamma$. If $l \geq j + 2$, this contradicts the assumption that $\alpha$ is an induced path. Therefore $l = j + 1$. Then $x' = v'_{j+1}$ by the unique path lifting property as before. This is a contradiction to $x' \neq v'_{j+1}$. This completes the proof of Claim. □

By the above claim, $v' = v'_0 \notin \text{supp}(\phi^*(w))$, hence $\phi$ is not $v'$-surviving. This contradicts the assumption that $\phi$ is $F$-surviving because $v' \in F$. □

Theorem 3.15. For each integer $m \geq 2$, there exists a graph $\Gamma_m$ with $m$ vertices such that if $\phi : T \rightarrow \Gamma_m$ is an immersion of a finite tree $T$ into $\Gamma_m$ and $\phi^* : G(\Gamma_m) \rightarrow G(T)$ is injective, then $|V(T)| \geq 2m/4$.

Proof. The theorem is obvious for $m = 2$. For $m \geq 3$, let $\Gamma_m$ be the graph defined as

- $V(\Gamma_{2k+1}) = \{u_i, v_i : 1 \leq i \leq k\} \cup \{v_0\}$,
- $V(\Gamma_{2k+2}) = \{u_i, v_i : 1 \leq i \leq k\} \cup \{v_0, v_{k+1}\}$,
- $E(\Gamma_{2k+1}) = \{\{v_i, v_{i+1}\}, \{u_i, u_{i+1}\}, \{v_i, u_{i+1}\}, \{u_i, v_{i+1}\} : 1 \leq i \leq k - 1\}$
  $\cup \{\{v_0, u_1\}, \{v_0, v_1\}\}$,
- $E(\Gamma_{2k+2}) = E(\Gamma_{2k+1}) \cup \{\{v_k, v_{k+1}\}, \{u_k, v_{k+1}\}\}$.
In other words, there is an edge between every pair of vertices with adjacent indices. See Figure 7 for the cases $m = 9, 10$.

Let $\phi : T \rightarrow \Gamma_m$ be an immersion of a tree $T$ into $\Gamma_m$ such that $\phi^* : G(\Gamma_m) \rightarrow G(T)$ is injective. Notice that $\phi$ is surjective on the sets of vertices.

Because $T$ is a tree and $\phi$ is an immersion, we can consider $\phi$ as a restriction of the universal cover $p : \tilde{\Gamma}_m \rightarrow \Gamma_m$ to an induced subtree $T$, i.e. $\phi = p|_T$.

Take any subset $F$ of $V(\tilde{\Gamma}_m)$ such that $|F| = m$ and $p(F) = V(\Gamma_m)$. (For example, $F$ can be chosen to be the set of vertices of a lift of a maximal tree in $\Gamma_m$ to $\tilde{\Gamma}_m$.) Let $\Sigma$ be the set of all deck transformations $\sigma : \tilde{\Gamma}_m \rightarrow \tilde{\Gamma}_m$ such that $\sigma(T) \cap F \neq \emptyset$, and let $T_1$ be the induced subgraph of $\tilde{\Gamma}_m$ on

$$\bigcup_{\sigma \in \Sigma} \sigma(V(T)).$$

Then $F \subseteq V(T_1)$, and $\phi_1 = p|_{T_1} : T_1 \rightarrow \Gamma_m$ is $F$-surviving by Lemma 2.4 hence it has IPL for $F$ by Theorem 3.14. By Remark 2.5

$$|V(T_1)| \leq |\Sigma| \cdot |V(T)| \leq |V(T)|^2.$$ 

Now, we will show that $|V(T_1)| \geq 2^{m/2}$.

First, assume that $m = 2k + 1$. The induced paths $\alpha$ starting from $v_0$ in $\Gamma_{2k+1}$ are of the form $\alpha = (v_0, x_1, x_2, \ldots, x_l)$, where $0 \leq l \leq k$ and $x_i \in \{u_i, v_i\}$ for $1 \leq i \leq l$. There are $2^l$ such paths $\alpha$ of length $l$, and all their lifts to $\tilde{\Gamma}_{2k+1}$ starting from $v'_0$ are contained in $T_1$. If $\alpha_1$ and $\alpha_2$ are distinct induced paths starting from $v_0$ in $\Gamma_{2k+1}$, then their lifts to $\tilde{\Gamma}_{2k+1}$ starting from $v'_0$ have distinct endpoints because $\tilde{\Gamma}_{2k+1}$ is the universal cover. Hence

$$|V(T_1)| \geq 1 + 2 + \cdots + 2^k = 2^{k+1} - 1 = 2^{(m+1)/2} - 1.$$

When $m = 2k + 2$, the same argument as above gives

$$|V(T_1)| \geq 1 + 2 + \cdots + 2^k + 2^k = 2^{k+1} - 1 + 2^k = \frac{3}{2} \cdot 2^{k+1} - 1 = \frac{3}{2} \cdot 2^{m/2} - 1.$$

Therefore, for $m \geq 3$, we have $|V(T_1)| \geq 2^{m/2}$, and hence $|V(T)| \geq |V(T_1)|^{1/2} \geq 2^{m/4}$. □

3.4. Embedding of RAAGs on cycle graphs into RAAGs on path graphs. Let $C_m$ and $P_n$ denote the cycle and path graphs on $m$ and $n$ vertices, respectively. We denote their vertices by

$$V(C_m) = \{v_0, v_1, \ldots, v_{m-1}\} \quad \text{and} \quad V(P_n) = \{v'_0, v'_1, \ldots, v'_{n-1}\}.$$

The edge sets of $C_m$ and $P_n$ are

$$E(C_m) = \{\{v_i, v_{i+1}\} : 0 \leq i \leq m - 2\} \cup \{\{v_{m-1}, v_0\}\},$$

$$E(P_n) = \{\{v'_j, v'_{j+1}\} : 0 \leq j \leq n - 2\}.$$

Let $\phi_{n,m} : P_n \rightarrow C_m$ be the immersion defined by

$$\phi_{n,m}(v'_j) = v_{j \mod m}.$$
Regard $P_n$ as an induced subgraph of the universal cover $\hat{C}_m$ of $C_m$ which is the bi-infinite path graph. Then the map $\phi_{n,m}$ is the restriction of the universal cover $p : \hat{C}_m \to C_m$ to $P_n$.

In [CDK13], M. Casals-Ruiz, A. Duncan and I. Kazachkov showed that $\phi_{5,5}^* : G(C_5) \to G(P_5)$ is injective. Using the induced path lifting property, we generalize their result as follows.

**Theorem 3.16.** For each $m \geq 3$, $\phi_{n,m} : G(C_m) \to G(P_n)$ is injective if and only if $n \geq 2m - 2$.

**Proof.** It suffices to show that $\phi_{2m-2,m}^*$ is injective and that $\phi_{2m-3,m}^*$ is not injective, because $\ker \phi_{k,m}^* \subseteq \ker \phi_{k,m}^*$ for $1 \leq k \leq l$ by Lemma 2.2.

First, we will show that $\phi_{2m-2,m}^*$ is injective. Let $\phi = \phi_{2m-2,m} : P_{2m-2} \to C_m$. Renaming the vertices, we may assume that

$$V(P_{2m-2}) = \{v'_0, v'_1, \ldots, v'_{m-1}, v''_1, v''_2, \ldots, v''_{m-1}\}$$

as in Figure 8 so that $\phi$ maps $v'_i$ and $v''_i$ to $v_i$ for each $i$. Notice that $Lk_{C_m}(v_0) = \{v_1, v_{m-1}\}$ and $Lk_{P_{2m-2}}(v'_0) = \{v'_1, v''_{m-1}\}$. Let

$$\Gamma_1 = C_m \setminus v_0, \quad \Lambda_1 = P_{2m-2} \setminus \phi^{-1}(v_0) = P_{2m-2} \setminus v'_0, \quad \Gamma_2 = C_m \setminus \{v_0, v_1\}, \quad \Lambda_2 = P_{2m-2} \setminus \phi^{-1}(\{v_0, v_1\}) = P_{2m-2} \setminus \{v'_1, v'_2\}.$$

Then $\phi(\Lambda_i) = \Gamma_i$ for $i = 1, 2$. Let $\phi_i = \phi(\Lambda_i, \Gamma_i)$ for $i = 1, 2$.

The graph $\Lambda_1$ has two components. One of them is the path graph on $\{v'_1, \ldots, v'_{m-1}\}$, and it is isomorphic to $\Gamma_1$ under $\phi_1$. Hence $\phi_1$ is $v'_1$-surviving and $\phi_1^*$ is injective. Similarly, $\phi_2$ is $v''_{m-1}$-surviving (and $\phi_2^*$ is injective). Therefore $\phi^* = \phi_{2m-2,m}^*$ is injective by Lemma 2.6.

Now, we will show that $\phi_{2m-3,m}^*$ is not injective. Let $\phi = \phi_{2m-3,m} : P_{2m-3} \to C_m$. Assume that $\phi^*$ is injective. Renaming the vertices, we may assume that

$$V(P_{2m-3}) = \{v'_0, v'_1, \ldots, v'_{m-1}, v''_3, \ldots, v''_{m-1}\}$$

as in Figure 8 so that $\phi$ maps $v'_i$ and $v''_i$ to $v_i$ for each $i$. Let $\Sigma$ be the set of all deck transformations $\sigma : \hat{C}_m \to \hat{C}_m$ such that $\sigma(P_{2m-3}) \cap \{v_0\} \neq \emptyset$. It is easy to see that
\[ \Sigma = \{\text{Id}\}. \] Then \( \phi : P_{2m-3} \to C_m \) is \( v'_0 \)-surviving by Lemma 2.4, hence \( \phi \) has IPL for \( v'_0 \) by Theorem 3.14. Thus there is a path in \( P_{2m-3} \) that is a lift of the induced path \((v_0, v_{m-1}, v_{m-2}, \ldots, v_2)\), which is impossible. Therefore \( \phi^* = \phi^*_{2m-3,m} \) is not injective. \( \square \)

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