Complete spherical convex bodies

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Abstract. Similarly to the classic notion in Euclidean space, we call a set on the sphere $S^d$ complete, provided adding any extra point increases its diameter. Complete sets are convex bodies on $S^d$. Our main theorem says that on $S^d$ complete bodies of diameter $\delta$ coincide with bodies of constant width $\delta$.

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1. On spherical geometry

Let $S^d$ be the unit sphere in the $(d + 1)$-dimensional Euclidean space $E^{d+1}$, where $d \geq 2$. By a great circle of $S^d$ we mean the intersection of $S^d$ with any two-dimensional subspace of $E^{d+1}$. The common part of the sphere $S^d$ with any hyper-subspace of $E^{d+1}$ is called a $(d-1)$-dimensional great sphere of $S^d$. By a pair of antipodes of $S^d$ we mean any pair of points of intersection of $S^d$ with a straight line through the origin of $E^{d+1}$.

Clearly, if two different points $a, b \in S^d$ are not antipodes, there is exactly one great circle containing them. By the arc $ab$ connecting $a$ with $b$ we mean the shorter part of the great circle containing $a$ and $b$. By the spherical distance $|ab|$, or shortly distance, of these points we mean the length of the arc connecting them. The diameter $\text{diam}(A)$ of a set $A \subset S^d$ is the number $\sup_{a, b \in A} |ab|$. By a spherical ball $B_\rho(r)$ of radius $\rho \in (0, \frac{\pi}{2}]$, or shorter a ball, we mean the set of points of $S^d$ having distance at most $\rho$ from a fixed point, called the center of this ball. Spherical balls of radius $\frac{\pi}{2}$ are called hemispheres. Two hemispheres whose centers are antipodes are called opposite hemispheres.

We say that a subset of $S^d$ is convex if it does not contain any pair of antipodes and if together with every two points $a, b$ it contains the arc $ab$. By a convex body, or shortly body, on $S^d$ we mean any closed convex set with non-empty interior.
Recall a few notions from [6]. If for a hemisphere \( H \) containing a convex body \( C \subset S^d \) we have \( \text{bd}(H) \cap C \neq \emptyset \), then we say that \( H \) supports \( C \). If hemispheres \( G \) and \( H \) of \( S^d \) are different and not opposite, then \( L = G \cap H \) is called a lune of \( S^d \). The \((d - 1)\)-dimensional hemispheres bounding the lune \( L \) and contained in \( G \) and \( H \), respectively, are denoted by \( G/H \) and \( H/G \). We define the thickness of a lune \( L = G \cap H \) as the spherical distance of the centers of \( G/H \) and \( H/G \). For a hemisphere \( H \) supporting a convex body \( C \subset S^d \) we define the width \( \text{width}_H(C) \) of \( C \) determined by \( H \) as the minimum thickness of a lune of the form \( H \cap H' \), where \( H' \) is a hemisphere, containing \( C \). If for all hemispheres \( H \) supporting \( C \) we have \( \text{width}_H(C) = w \), we say that \( C \) is of constant width \( w \).

2. Spherical complete bodies

Similarly to the traditional notion of a complete set in the Euclidean space \( E^d \) (for instance, see [1–3] and [10]) we say that a set \( K \subset S^d \) of diameter \( \delta \in (0, \pi) \) is complete provided \( \text{diam}(K \cup \{x\}) > \delta \) for every \( x \not\in K \).

**Theorem 1.** An arbitrary set of diameter \( \delta \in (0, \pi) \) on the sphere \( S^d \) is a subset of a complete set of diameter \( \delta \) on \( S^d \).

We omit the proof since it is similar to the proof by Lebesgue [9] in \( E^d \) (it is recalled in Part 64 of [1]). Let us add that earlier Pál [12] proved this for \( E^2 \) by a different method.

The following fact permits to use the term a complete convex body for a complete set.

**Lemma 1.** Let \( K \subset S^d \) be a complete set of diameter \( \delta \). Then \( K \) coincides with the intersection of all balls of radius \( \delta \) centered at points of \( K \). Moreover, \( K \) is a convex body.

**Proof.** Denote by \( I \) the intersection of all balls of radius \( \delta \) with centers in \( K \).

Since \( \text{diam}(K) = \delta \), then \( K \) is contained in every ball of radius \( \delta \) whose center is a point of \( K \). Consequently, \( K \subset I \).

Let us show that \( I \subset K \); so let us show that \( x \not\in K \) implies \( x \not\in I \). Really, from \( x \not\in K \) we get \( |xy| > \delta \) for a point \( y \in K \), which means that \( x \) is not in the ball of radius \( \delta \) with center \( y \), and thus \( x \not\in I \).

The first thesis implies that \( K \) is a convex body. \( \square \)

**Lemma 2.** If \( K \subset S^d \) is a complete body of diameter \( \delta \), then for every \( p \in \text{bd}(K) \) there exists \( p' \in K \) such that \( |pp'| = \delta \).

**Proof.** Suppose the contrary, i.e., that \( |pq| < \delta \) for a point \( p \in \text{bd}(K) \) and for every point \( q \in K \). Since \( K \) is compact, there is an \( \varepsilon > 0 \) such that
\[ |pq| \leq \delta - \varepsilon \text{ for every } q \in K. \] Hence there is a point \( s \notin K \) in a positive distance from \( p \) which is smaller than \( \varepsilon \) such that \( |sq| \leq \delta \) for every \( q \in K \). Thus \( \text{diam}(K \cup \{s\}) = \delta \), which contradicts the assumption that \( K \) is complete. Consequently, the thesis of our lemma holds true.

For different points \( a, b \in S^d \) at a distance \( \delta < \pi \) from a point \( c \in S^d \) define the piece of the circle \( P_c(a, b) \) as the set of points \( v \in S^d \) such that \( cv \) has length \( \delta \) and intersects \( ab \).

We show the next lemma for \( S^d \) despite we apply it later only for \( S^2 \).

**Lemma 3.** Let \( K \subset S^d \) be a complete convex body of diameter \( \delta \). Take \( P_c(a, b) \) with \( |ac| \) and \( |bc| \) equal to \( \delta \) such that \( a, b \in K \) and \( c \in S^d \). Then \( P_c(a, b) \subset K \).

**Proof.** First let us show the thesis for a ball \( B \) of radius \( \delta \) in place of \( K \). There is unique \( S^2 \subset S^d \) with \( a, b, c \in S^2 \). Consider the disk \( D = B \cap S^2 \). Take the great circle containing \( P_c(a, b) \) and points \( a^*, b^* \) of its intersection with the circle bounding \( D \). There is a unique \( c^* \in S^2 \) such that \( P_c(a, b) \subset P_{c^*}(a^*, b^*) \). Clearly, \( P_{c^*}(a^*, b^*) \subset D \subset B \). Hence \( P_c(a, b) \subset B \). By the preceding paragraph and Lemma 1 we obtain the thesis of the present lemma.

### 3. Complete and constant width bodies on \( S^d \) coincide

Here is our main result presenting the spherical version of the classic theorem in \( E^d \) proved by Meissner [11] for \( d = 2, 3 \) and by Jessen [5] for arbitrary \( d \).

**Theorem 2.** A body of diameter \( \delta \) on \( S^d \) is complete if and only if it is of constant width \( \delta \).

**Proof.** (\( \Rightarrow \)) Let us prove that if a body \( K \subset S^d \) of diameter \( \delta \) is complete, then \( K \) is of constant width \( \delta \).

Suppose the opposite, i.e., that \( \text{width}_I(K) \neq \delta \) for a hemisphere \( I \) supporting \( K \). By Theorem 3 and Proposition 1 of [6] \( \text{width}_I(K) \leq \delta \). So \( \Delta(K) < \delta \). By lines 1-2 of p. 562 of [6] the thickness of \( K \) is equal to the minimum thickness of a lune containing \( K \). Take such a lune \( L = G \cap H \), where \( G, H \) are different and non-opposite hemispheres. Denote by \( g, h \) the centers of \( G/H \) and \( H/G \), respectively. Of course, \( |gh| < \delta \). By Claim 2 of [6] we have \( g, h \in K \). By Lemma 2 there exists a point \( g' \in K \) in the distance \( \delta \) from \( g \). Since the triangle \( ghg' \) is non-degenerate, there is a unique two-dimensional sphere \( S^2 \subset S^d \) containing \( g, h, g' \). Clearly, \( ghg' \) is a subset of \( M = K \cap S^2 \). Hence \( M \) is a convex body on \( S^2 \). Denote by \( F \) this hemisphere of \( S^2 \) such that \( hg' \subset \text{bd}(F) \) and \( g \in F \). There is a unique \( c \in F \) such that \( |ch| = \delta = |cg'| \). By Lemma 3 for \( d = 2 \) we have \( P_c(h, g') \subset M \).
We intend to show that $c$ is not on the great circle $E$ of $S^2$ through $g$ and $h$. In order to see this, for a while suppose the opposite, i.e. that $c \in E$. Then from $|g'g| = \delta, |g'c| = \delta$ and $|hc| = \delta$ we conclude that $\angle gg'c = \angle hcg'$. So the spherical triangle $g'gc$ is isosceles, which together with $|gg'| = \delta$ gives $|cg| = \delta$. Since $|gh| = \Delta(L) = \Delta(K) > 0$ and $g$ is a point of $ch$ different from $c$, we get a contradiction. Hence, really, $c \notin E$.

By the preceding paragraph $P_c(h, g')$ intersects $bd(M)$ at a point $h'$ different from $h$ and $g'$. So the non-empty set $P_c(h, g') \setminus \{h, h'\}$ is out of $M$. This contradicts the result of the paragraph before the last. Consequently, $K$ is a body of constant width $\delta$.

$(\Leftarrow)$ Let us prove that if $K$ is a spherical body of constant width $\delta$, then $K$ is a complete body of diameter $\delta$. In order to prove this, it is sufficient to take any point $r \notin K$ and to show that $\text{diam}(K \cup \{r\}) > \delta$.

Take the largest ball $B_\rho(r)$ disjoint with the interior of $K$. Since $K$ is convex, $B_\rho(r)$ has exactly one point $p$ in common with $K$. By Theorem 3 of [8] there exists a lune $L \supset K$ of thickness $\delta$ with $p$ as the center of one of the two $(d-1)$-dimensional hemispheres bounding this lune. Denote by $q$ the center of the other $(d-1)$-dimensional hemisphere. By Claim 2 of [6] also $q \in K$. Since $p$ and $q$ are the centers of the two $(d-1)$-dimensional hemispheres bounding $L$, we have $|pq| = \delta$. From the fact that $rp$ and $pq$ are orthogonal to $bd(H)$ at $p$, we see that $p \in rq$. Moreover, $p$ is not an endpoint of $rq$ and $|pq| = \delta$, Hence $|rq| > \delta$. Thus $\text{diam}(K \cup \{r\}) > \delta$. Since $r \notin K$ is arbitrary, $K$ is complete. □

We say that a convex body $D \subset S^d$ is of constant diameter $\delta$ provided $\text{diam}(D) = \delta$ and for every $p \in \text{bd}(D)$ there is a point $p' \in \text{bd}(D)$ with $|pp'| = \delta$ (see [8]). The following fact is analogous to the result in $E^d$ given by Reidemeister [13].

**Theorem 3.** Bodies of constant diameter on $S^d$ coincide with complete bodies.

**Proof.** Take a complete body $D \subset S^d$ of diameter $\delta$. Let $q \in \text{bd}(D)$ and $G$ be a hemisphere supporting $D$ at $q$. By Theorem 2 the body $D$ is of constant width $\delta$. So $\text{width}_G(D) = \delta$ and there exists a hemisphere $H$ such that the lune $G \cap H \supset D$ has thickness $\delta$. By Claim 2 of [6] the centers $h$ of $H/G$ and $g$ of $G/H$ belong to $D$. So $|gh| = \delta$. Thus $D$ is of constant diameter $\delta$.

Consider a body $D \subset S^d$ of constant diameter $\delta$. Let $r \notin D$. Take the largest $B_\rho(r)$ whose interior is disjoint with $D$. Denote by $p$ the common point of $B_\rho(r)$ and $D$. A unique hemisphere $J$ supports $B_\rho(r)$ at $p$. Observe that $D \subset J$ (if not, there is a point $v \in D$ out of $J$; clearly $vp$ passes through $\text{int}B_\rho(r)$, a contradiction). Since $D$ is of constant diameter $\delta$, there is $p' \in D$ with $|pp'| = \delta$. Observe that $\angle rpp' \geq \frac{\pi}{2}$. If it is $\frac{\pi}{2}$, then $|rp'| > \delta$. If it is larger than $\frac{\pi}{2}$, the triangle $rpp'$ is obtuse and then by the law of cosines $|rp'| > |pp'|$ and hence $|rp'| > \delta$. By $|rp'| > \delta$ in both cases we see that $D$ is complete. □
Theorem 2 permits to change “complete” to “constant width” in Theorem 3. This form is proved earlier as follows. In [8] it is shown that any body of constant width \( \delta \) on \( S^d \) is of constant diameter \( \delta \) and the inverse is argued for \( \delta \geq \frac{\pi}{2} \), and for \( \delta < \frac{\pi}{2} \) if \( d = 2 \). By [4] the inverse holds for any \( \delta \). Our short proof of Theorem 3 is quite different from the considerations in [8], [4] and [7].

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