Investigating texture six zero lepton mass matrices

Neelu Mahajan¹, Monika Randhawa²,*, Manmohan Gupta³, and P.S. Gill⁴

¹DAV College, Chandigarh, India
²University Institute of Engineering and Technology, Panjab University, Chandigarh, India
³Department of Physics, Centre of Advanced Study, Panjab University, Chandigarh, India
⁴Sri Guru Gobind Singh College, Chandigarh, India
*E-mail: monika@pu.ac.in

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Texture six zero Fritzsch-like as well as non-Fritzsch-like Hermitian lepton mass matrices (144 combinations in all) have been investigated for both Majorana and Dirac neutrinos for their compatibility with current neutrino oscillation data, keeping in mind the hierarchy of neutrino masses. All the combinations considered here for Majorana neutrino masses are ruled out by the existing data in the case of inverted hierarchy and degenerate scenario. For Majorana neutrinos with normal hierarchy, only 16 combinations can accommodate the experimental data. Assuming neutrinos to be Dirac particles, normal hierarchy, inverted hierarchy as well as degenerate neutrinos are ruled out for all combinations of texture 6 zero Hermitian mass matrices.

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1. Introduction

The large value of $\theta_{13}$ determined by the reactor neutrino experiments [1–3] has not only thrown open the door for the search for CP violation in the lepton sector, but has also provided an impetus for the formulation of theories for understanding the origin of neutrino masses and their mixing. Apart from the recent $\theta_{13}$ measurement, there has also been considerable progress in the measurement of neutrino mass square differences and mixing angles $\theta_{12}$ and $\theta_{23}$ in the last few years. However, not much information is available about the leptonic CP violation phase $\delta_l$ from the neutrino oscillation data. Also, the absolute neutrino mass scale is still unknown. Further, the presently available neutrino oscillation data does not throw any light on the neutrino mass hierarchy, which may be a normal or inverted hierarchy and may even be degenerate. Furthermore, the situation becomes complicated when one realizes that neutrino masses are much smaller than charged fermion masses, and it is not yet clear whether neutrinos are Dirac or Majorana particles.

In the absence of a convincing fermion flavor theory, several approaches have been considered to understand the fermion mass generation problem, e.g., texture zeros [4–7], the seesaw mechanism [8–12], radiative mechanisms [13,14], flavor symmetries [15–19], extra dimensions [20–24], etc. In this context, texture-specific mass matrices have received a good deal of attention in the literature; in particular, Fritzsch-like texture-specific mass matrices seem to be very helpful in understanding the pattern of quark mixings and CP violation [25–35]. Taking clues from the success of these matrices in the context of quarks, several attempts have been made to consider texture-specific lepton mass matrices [36–85] for explaining the pattern of neutrino masses and mixings. In the absence of a sufficient amount of data regarding neutrino masses and mixings, it would require a very careful scrutiny...
of all possible textures to find viable structures which are compatible with data and theoretical ideas, so that these are kept in mind while formulating mass matrices at the grand unified theory (GUT) scale.

In the quark sector, both Fritzsch-like as well as non-Fritzsch-like texture six zero mass matrices have been completely ruled out [35]. In the leptonic sector, most of the analyses have been carried out in the flavor basis [57–71]. In the non-flavor basis, where both the charged lepton mass matrix $M_l$ and the neutrino mass matrix $M_\nu$ are of three-zero type, the number of possibilities for texture six zero mass matrices becomes very large. These possibilities have been explored in the literature [72] both for Majorana as well as for Dirac neutrinos; however, adequate attention has not been given to the cases for inverted hierarchy and degenerate scenarios. Also, it is desirable to note that Dirac neutrinos have not yet been ruled out by the experiments [86], and it therefore becomes interesting in the case of texture six zero mass matrices to carry out a detailed comparison for Dirac-like as well as Majorana-like neutrinos in the normal, inverted, and degenerate cases. This exercise becomes all the more interesting in view of the refinements of data and advocacy of quark lepton symmetry [87].

The purpose of this paper is to update the analysis of Zhou and Xing [72] for all possibilities of texture six zero lepton mass matrices as well as to extend this analysis to the case of inverted and degenerate neutrino masses. To preserve the parallelism between quarks and leptons, only those neutrino mass matrices have been considered which are consistent with the requirement for non-zero and distinct neutrino masses. Following our analysis in the quark sector [35], the mass matrices for leptons and neutrinos are taken to be Hermitian. For the sake of completeness, we have also investigated the cases corresponding to charged leptons being in the flavor basis. It would also be a desirable exercise to calculate phenomenological quantities, such as effective neutrino mass $\langle m_{ee} \rangle$ related to neutrinoless double beta decay, Jarlskog’s rephasing invariant parameter in the leptonic sector $J_l$ and the corresponding Dirac-like CP-violating phase $\delta_l$ for the viable cases.

The detailed plan of the paper is as follows. In Sect. 2, we present the essentials of the formalism connecting the mass matrix to the neutrino mixing matrix. Inputs used in the present analysis have been given in Sect. 3. In Sect. 4, various combinations of texture six zero mass matrices have been given. In Sects. 5 and 6, for Majorana and Dirac neutrinos respectively, the detailed calculations pertaining to normal, degenerate, and inverted hierarchies have been discussed. Finally, Sect. 7, summarizes our conclusions.

## 2. Construction of the PMNS matrix from mass matrices

To begin with, we present the Fritzsch-like Hermitian texture six zero lepton mass matrices, e.g.,

$$
M_l = \begin{pmatrix}
0 & A_l & 0 \\
A_l^* & 0 & B_l \\
0 & B_l^* & C_l
\end{pmatrix}, \quad M_{\nu D} = \begin{pmatrix}
0 & A_{\nu D} & 0 \\
A_{\nu D}^* & 0 & B_{\nu D} \\
0 & B_{\nu D}^* & C_{\nu D}
\end{pmatrix}, \quad (1)
$$

$M_l$ and $M_{\nu D}$ respectively corresponding to Dirac-like charged lepton and neutrino mass matrices. It may be noted that each of the above matrices is texture three zero type with $A_{l(\nu D)} = |A_{l(\nu D)}|e^{i\delta_{l(\nu D)}}$ and $B_{l(\nu D)} = |B_{l(\nu D)}|e^{i\delta_{l(\nu D)}}$. For Majorana neutrinos, the neutrino mass matrix $M_{\nu}$ is given by the seesaw mechanism [8–12], for example,

$$
M_{\nu} = -M_{\nu D}^T(M_R)^{-1}M_{\nu D}, \quad (2)
$$

where $M_{\nu D}$ and $M_R$ are, respectively, the Dirac neutrino mass matrix and the right-handed Majorana mass matrix. It may be mentioned that for both Majorana as well as Dirac neutrinos the texture is
imposed only on $M_{\nu D}$, with no such restriction on $M_{\nu}$ for the Majorana case. In the absence of any guidelines for the right-handed Majorana mass matrix $M_R$, to keep the number of parameters under control it would be desirable to keep its structure as simple as possible. Therefore, following Fukugita et al. [88,89], we take $M_R = m_R I$, where $I$ is the unity matrix and $m_R$ denotes a very large mass scale. Here, it is pertinent to mention that any mechanism leading to texture zeros in $M_{\nu D}$ does not necessarily require a diagonal form of $M_R$, and therefore one may as well consider a more general form of $M_R$ with or without texture zeros. However, since the seesaw framework contains more free parameters than can be obtained from the low-energy data, considering a texture zero model for $M_R$ leads to a reduction in the number of parameters and thus enhances the predictive power of the model. In this regard, Fritzsch et al. [90] have recently considered a model wherein a three zero texture structure is imposed on the charged lepton mass matrix $M_l$, the Dirac neutrino mass matrix $M_{\nu D}$, and also on the heavy right-handed Majorana mass matrix $M_R$. Therefore, to widen the scope of the paper as well as for the sake of completeness, in Appendix B we have briefly discussed the cases where we consider a parallel texture three zero structure for $M_R$ and $M_{\nu D}$.

To fix the notations and conventions, as well as to facilitate understanding of the inverted hierarchy case and its relationship to the normal hierarchy case, we detail the formalism connecting the mass matrices to the neutrino mixing matrix. To facilitate the diagonalization of $M_k$, where $k = l, \nu D$, the complex mass matrix $M_k$ can be expressed as

$$M_k = Q_k M_k^r P_k$$ (3)

or

$$M_k^r = Q_k^T M_k P_k^T,$$ (4)

where $M_k^r$ is a real symmetric matrix with real eigenvalues and $Q_k$ and $P_k$ are diagonal phase matrices. For the Hermitian mass matrix, $Q_k = P_k^T$. In general, the real matrix $M_k^r$ is diagonalized by the orthogonal transformation $O_k$, e.g.,

$$M_k^r \text{diag} = O_k^T M_k^r O_k,$$ (5)

which on using Eq. (4) can be rewritten as

$$M_k^r \text{diag} = O_k^T Q_k^T M_k P_k^T O_k.$$ (6)

To facilitate the construction of diagonalization transformations for different hierarchies, we introduce a diagonal phase matrix $\xi_k$ defined as $\text{diag}(1, e^{i\pi}, 1)$ for the case of the normal hierarchy and as $\text{diag}(1, e^{i\pi}, e^{2i\pi})$ for the case of the inverted hierarchy. Equation (6) can now be written as

$$\xi_k M_k^r \text{diag} = O_k^T Q_k^T M_k P_k^T O_k,$$ (7)

which can also be expressed as

$$M_k^r \text{diag} = \xi_k^T O_k^T Q_k^T M_k P_k^T O_k.$$ (8)

Making use of the fact that $O_k^* = O_k$, it can be further expressed as

$$M_k^r \text{diag} = (Q_k O_k \xi_k)^T M_k (P_k^* O_k),$$ (9)

from which one gets

$$M_k = Q_k O_k \xi_k M_k^r \text{diag} O_k^T P_k.$$ (10)

The case of leptons is fairly straightforward, whereas in the case of neutrinos, the diagonalizing transformation is hierarchy specific, as well as requiring some fine-tuning of the phases of the
right-handed mass matrix $M_R$. To clarify this point further, by analogy with Eq. (10), we can express $M_{vD}$ as

$$M_{vD} = Q_{vD} O_{vD} \xi_{vD} M_{vD}^{\text{diag}} O_{vD}^T P_{vD}. \quad (11)$$

Substituting the above value of $M_{vD}$ in Eq. (2) one obtains

$$M_v = -(Q_{vD} O_{vD} \xi_{vD} M_{vD}^{\text{diag}} O_{vD}^T P_{vD}) (M_R)^{-1} (Q_{vD} O_{vD} \xi_{vD} M_{vD}^{\text{diag}} O_{vD}^T P_{vD}). \quad (12)$$

which, on using $P_{vD}^T = P_{vD}$, $Q_{vD}^T = Q_{vD}$, can further be written as

$$M_v = -P_{vD} O_{vD} M_{vD}^{\text{diag}} \xi_{vD} O_{vD}^T Q_{vD}(M_R)^{-1} Q_{vD} O_{vD} \xi_{vD} M_{vD}^{\text{diag}} O_{vD}^T P_{vD}. \quad (13)$$

wherein, assuming fine-tuning, the phase matrices $Q_{vD}^T$ and $Q_{vD}$ along with $-M_R$ can be taken as $m_R \text{ diag}(1, 1, 1)$ and, using the unitarity of $\xi_{vD}$ and orthogonality of $O_{vD}$, the above equation can be expressed as

$$M_v = P_{vD} O_{vD} \left(\frac{M_{vD}^{\text{diag}}}{m_R}\right)^2 O_{vD}^T P_{vD}. \quad (14)$$

The lepton mixing matrix or the Pontecorvo–Maki–Nakagawa– Sakata (PMNS) matrix $[91–93]$ $U$ can be obtained from the matrices used for diagonalizing the mass matrices $M_l$ and $M_v$, and is expressed as

$$U = (Q_l P_l \xi_l)^\dagger (P_{vD} O_{vD}). \quad (15)$$

Eliminating the phase matrix $\xi_l$ by redefinition of the charged lepton phases, the above equation becomes

$$U = O_l^\dagger Q_l P_{vD} O_{vD}, \quad (16)$$

where $Q_l P_{vD}$, without loss of generality, can be taken as $(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$; $\phi_1$, $\phi_2$, and $\phi_3$ are related to the phases of the mass matrices and can be treated as free parameters.

3. Inputs used in the present analysis

Before going into the details of the analysis, we would like to mention some of the essentials pertaining to various inputs. The inputs for the masses and mixing angles used in the present analysis at $3\sigma$ C.L. are as follows $[94]$:

$$\Delta m_{12}^2 = (6.99 - 8.18) \times 10^{-5} \text{ eV}^2, \quad \Delta m_{23}^2 = \begin{cases} (2.19 - 2.62) \times 10^{-3} \text{ eV}^2 & \text{NH} \\ (2.17 - 2.61) \times 10^{-3} \text{ eV}^2 & \text{IH} \end{cases} \quad (17)$$

$$\sin^2 \theta_{12} = 0.259 - 0.359, \quad \sin^2 \theta_{23} = \begin{cases} 0.331 - 0.637 & \text{NH} \\ 0.335 - 0.663 & \text{IH} \end{cases}, \quad \sin^2 \theta_{13} = \begin{cases} 0.0169 - 0.0313 & \text{NH} \\ 0.0171 - 0.0315 & \text{IH} \end{cases} \quad (18)$$

where NH and IH correspond to the normal hierarchy and inverted hierarchy respectively.

For the purpose of calculations, we have taken the lightest neutrino mass and the phases $\phi_1$, $\phi_2$, and $\phi_3$ as free parameters. The other two masses are constrained by $\Delta m_{12}^2 = m_{v2}^2 - m_{v1}^2$ and $\Delta m_{23}^2 = m_{v3}^2 - m_{v2}^2$ in the normal hierarchy case defined as $m_{v1} < m_{v2} < m_{v3}$ and also valid for the degenerate case defined as $m_{v1} \lesssim m_{v2} \sim m_{v3}$, and by $\Delta m_{23}^2 = m_{v2}^2 - m_{v3}^2$ in the inverted hierarchy case defined as $m_{v3} \ll m_{v1} < m_{v2}$. It may be noted that the lightest neutrino mass corresponds
Table 1. Table showing various patterns of texture three zero mass matrices classified into two classes, Class I and II.

|   | Class I                          | Class II                          |
|---|----------------------------------|-----------------------------------|
| a | \[
\begin{pmatrix}
0 & Ae^{ia} & 0 \\
Ae^{-ia} & 0 & Be^{ib} \\
0 & Be^{-ib} & C
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & Ae^{ia} & 0 \\
Ae^{-ia} & B & 0 \\
0 & 0 & C
\end{pmatrix}
\] |
| b | \[
\begin{pmatrix}
0 & 0 & Ae^{ia} \\
0 & C & Be^{ib} \\
Ae^{-ia} & Be^{-ib} & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 0 & Ae^{ia} \\
0 & C & 0 \\
Ae^{-ia} & 0 & B
\end{pmatrix}
\] |
| c | \[
\begin{pmatrix}
0 & Ae^{ia} & Be^{ib} \\
Ae^{-ia} & 0 & C \\
Be^{-ib} & 0 & C
\end{pmatrix}
\] | \[
\begin{pmatrix}
B & Ae^{ia} & 0 \\
0 & Ae^{ia} & 0 \\
0 & 0 & C
\end{pmatrix}
\] |
| d | \[
\begin{pmatrix}
C & Be^{ib} & 0 \\
Be^{-ib} & 0 & Ae^{ia} \\
0 & Ae^{-ia} & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
C & 0 & 0 \\
0 & B & Ae^{ia} \\
0 & 0 & Ae^{-ia}
\end{pmatrix}
\] |
| e | \[
\begin{pmatrix}
0 & Be^{ib} & Ae^{ia} \\
Be^{-ib} & C & 0 \\
Ae^{-ia} & 0 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
B & 0 & Ae^{ia} \\
0 & C & 0 \\
0 & 0 & Ae^{-ia}
\end{pmatrix}
\] |
| f | \[
\begin{pmatrix}
C & 0 & Be^{ib} \\
0 & 0 & Ae^{ia} \\
Be^{-ib} & Ae^{-ia} & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
C & 0 & 0 \\
0 & 0 & Ae^{ia} \\
0 & Ae^{-ia} & B
\end{pmatrix}
\] |

to $m_{\nu_1}$ for the normal hierarchy case and to $m_{\nu_3}$ for the inverted hierarchy case. The explored range of the lightest neutrino mass is taken to be 0.0001 eV $-$ 1.0 eV as our results remain unaffected even if the range is extended further. In the absence of any constraints on the phases $\phi_1$, $\phi_2$, and $\phi_3$, these have been given full variation from 0 to $2\pi$.

4. Texture six zero lepton mass matrices

To begin with, we enumerate the number of possibilities for the texture six zero lepton mass matrices. It is easy to see from Eq. (1) that there are 20 possible patterns of texture three zero Hermitian mass matrices, which differ from each other with regard to the position of zeros in the structure of the mass matrix.

Texture six zero mass matrices are obtained when both $M_l$ and $M_{\nu D}$ are texture three zero type, implying that there will be 400 combinations of texture six zero lepton mass matrices. As mentioned earlier, in this paper we have considered only those mass matrices which lead to non-zero and distinct mass eigenvalues, therefore imposing the trace and determinant condition on the mass matrix, i.e. \( \text{Det } M_{l,\nu D} \neq 0 \) and \( \text{Trace } M_{l,\nu D} \neq 0 \), we are left with 12 patterns classified into 2 distinct classes depending upon the diagonalization equations these satisfy, as given in Table 1. Details of the diagonalization equations for these 12 mass matrices can be found in our earlier work [35]. Matrices $M_l$ and $M_{\nu D}$ can each correspond to any of the 12 patterns, therefore yielding 144 possible combinations of texture six zero lepton mass matrices which in principle can yield the neutrino mixing matrix. These 144 combinations form 4 different categories:

Category 1: $M_l$ from Class I and $M_{\nu D}$ from Class I.
Category 2: $M_l$ from Class II and $M_{\nu D}$ from Class II.
Category 3: $M_l$ from Class I and $M_{\nu D}$ from Class II.
Category 4: $M_l$ from Class II and $M_{vD}$ from Class I.

Each category corresponds to 36 combinations which need exhaustive analysis. For all these combinations, we have considered the cases of normally hierarchical, inversely hierarchical, and degenerate Majorana as well as Dirac neutrinos.

5. Majorana neutrinos

We have analyzed 144 combinations of texture six zero lepton mass matrices corresponding to the normal hierarchy of Majorana neutrinos by confronting their corresponding mixing matrix against the latest neutrino oscillation data given in Sect. 3. It may be mentioned again that for Majorana neutrinos, the texture three zero structure is imposed on $M_l$ above, where both $M_l$ and $M_{vD}$ are non-Fritzsch type and, as mentioned earlier, $M_R$ is diagonal. For example,

$$M_l = \begin{pmatrix} C_l & 0 & 0 \\ 0 & D_l & A_l \\ 0 & A_l^T & 0 \end{pmatrix}, \quad M_{vD} = \begin{pmatrix} 0 & 0 & A_{vD} \\ 0 & C_{vD} & B_{vD} \\ A_{vD}^T & B_{vD}^T & 0 \end{pmatrix}, \quad M_R = m_R \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (19)$$

The diagonalizing transformations for these matrices can easily be obtained in terms of neutrino masses $m_{v_1}, m_{v_2},$ and $m_{v_3}$ for both normal and inverted hierarchies, and are given in Appendix A. The phase matrix $Q_l P_{vD}$ for this particular combination is given as $(e^{i\phi_1}, e^{i\phi_2}, 1)$. The scanned ranges of the lightest neutrino mass $m_{v_1}$ and phases $\phi_1$ and $\phi_2$ have been given in Section 3, and $\Delta m_{12}^2, \Delta m_{23}^2,$ and the mixing angles have been constrained, as given by Eqs. (17) and (18).

Using Eq. (16), the PMNS matrix $U$ obtained for this particular combination is

$$U = \begin{pmatrix} 0.78 - 0.84 & 0.52 - 0.61 & 0.12 - 0.16 \\ 0.39 - 0.49 & 0.40 - 0.53 & 0.72 - 0.79 \\ 0.29 - 0.43 & 0.67 - 0.72 & 0.59 - 0.68 \end{pmatrix}, \quad (20)$$

in good agreement with the ranges of the mixing matrix elements given by Garcia et al. [95] at 3$\sigma$ C.L.

To graphically show the viability of this particular combination of texture six zero mass matrices, in Fig. 1 we have plotted the lightest neutrino mass against the mixing angles $\theta_{12}, \theta_{13},$ and $\theta_{23}$ by giving full variation to the input parameters. The dashed lines depict the limits obtained by assuming the normal hierarchy and the solid horizontal lines show the experimental 3$\sigma$ limits of the plotted mixing angles. A general look at the Fig. 1 shows that the mixing angles are well within their experimental ranges for a common neutrino mass range for the normal hierarchy. Moreover, one can see that the neutrino masses are following a strict normal hierarchy.

In a similar manner, one can check the viability of the above set of mass matrices for inversely hierarchical Majorana neutrinos. In the same Fig. 1, using dot-dashed lines, we have plotted the limits of the mixing angles obtained by assuming an inverted hierarchy against the lightest neutrino mass. It is immediately clear from Figs. 1a and 1c that the obtained ranges of $\theta_{12}$ and $\theta_{23}$ are far from their experimental limits, thus ruling out the inverted hierarchy for this particular combination.

Coming to the degenerate scenarios of Majorana neutrinos characterized by either $m_{v_1} \lesssim m_{v_2} \sim m_{v_3} \sim 0.1$ eV or $m_{v_3} \sim m_{v_1} \lesssim m_{v_2} \sim 0.1$ eV, corresponding to the normal hierarchy and inverted
Fig. 1. Plots showing variation of mixing angles $\theta_{12}$, $\theta_{13}$, and $\theta_{23}$ with the lightest neutrino mass for the $\nu_2\bar{\nu}_3$ combination given in Eq. (19) for Majorana neutrinos. The dashed lines and the dot-dashed lines depict the limits obtained assuming normal and inverted hierarchy respectively; the solid horizontal lines show the experimental $3\sigma$ limits.

Table 2. Calculated ranges of neutrino masses, mixing angles, $\langle m_{ee} \rangle$, $I_1$, and $\delta_1$ for viable combinations of $M_1$ and $M_D$ for normal hierarchy Majorana neutrinos.

| Category | $M_1$ | $M_D$ | Neutrino masses | Mixing angles | $\langle m_{ee} \rangle$ | $I_1$ | $\delta_1$ |
|----------|-------|-------|----------------|---------------|----------------------|------|---------|
| I$^a$I$^a$ | $0$ | $A_1$ | $0$ | $0$ | $A_1$ | $0$ | $m_{v_1} = 0.0007 - 0.0032$ | $\theta_{12} = 31^\circ - 35^\circ$ | $0.0029 - 0.0081$ | $0.0 - 0.014$ | $0 - 20^\circ$ |
| I$^a$I$^a$ | $A_1^T$ | $0$ | $B_1$ | $A_1^T$ | $0$ | $B_1$ | $m_{v_2} = 0.0084 - 0.0093$ | $\theta_{13} = 7.7^\circ - 10.4^\circ$ | $0.0034 - 0.0065$ | $0.0 - 0.015$ | $0 - 19^\circ$ |
| II$^b$I$^b$ | $0$ | $A_1$ | $0$ | $0$ | $A_1$ | $0$ | $m_{v_1} = 0.0007 - 0.0033$ | $\theta_{12} = 31^\circ - 38^\circ$ | $0.0029 - 0.0081$ | $0.0 - 0.014$ | $0 - 20^\circ$ |
| II$^b$I$^b$ | $A_1^T$ | $0$ | $B_1$ | $A_1^T$ | $0$ | $B_1$ | $m_{v_2} = 0.0083 - 0.0098$ | $\theta_{13} = 8^\circ - 11^\circ$ | $0.0034 - 0.0065$ | $0.0 - 0.015$ | $0 - 19^\circ$ |
| III$^c$I$^c$ | $0$ | $B_1$ | $C_1$ | $0$ | $B_1^* C_1^*$ | $0$ | $m_{v_3} = 0.0486 - 0.0516$ | $\theta_{23} = 35^\circ - 43^\circ$ | $0.0029 - 0.0081$ | $0.0 - 0.014$ | $0 - 20^\circ$ |

Hierarchy respectively, one can easily infer from Figs. 1(a) and 1(c) that the degenerate scenario is also ruled out. This can be understood by noting that around 0.1 eV the limits obtained by assuming the normal hierarchy and inverted hierarchy have no overlap with the experimental limits of $\theta_{12}$ and $\theta_{23}$.

The viability of the rest of the 143 combinations can be checked similarly for the normal hierarchy, the inverted hierarchy, and for degenerate Majorana neutinos. For the normal hierarchy, we find that in Category 1 there are 12 viable combinations. Numerical results corresponding to one such parallel combination, labelled as $I_a I_a$, where both $M_1$ and $M_D$ are of type $I_a$, are given in the first row of
Table 2. The spectrum of neutrino masses shows that neutrinos are following a strict normal hierarchy. Further, $\theta_{12}$ and $\theta_{13}$ are spanning their full experimental range; however, the obtained range of $\theta_{23}$ is just below its maximal value. One finds that $\theta_{23}$ is sensitive to variations in the mass squared differences; however, it is not possible to obtain a higher value for $\theta_{23}$ even when the ranges of $\Delta m^2_{12}$ and $\Delta m^2_{23}$ are extended further.

Apart from the mixing angles, we have also calculated Jarlskog’s rephasing invariant in the leptonic sector $J_l$ and the Dirac-like CP-violating phase $\delta_l$ and effective neutrino mass $\langle m_{ee} \rangle$ related to neutrinoless double beta decay $\beta\beta_{0v}$. The parameter $J_l$ has been calculated by using the expression

$$J_l = Im[U_{23}U^*_{33}U_{22}U^*_{32}],$$

where $U_{23}, U_{33}, U_{22}$, and $U_{32}$ are the elements of mixing matrix $U$ given in Eq. (16).

The Dirac-like CP-violating phase $\delta_l$ can be determined from

$$J_l = s_{12}s_{23}s_{13}c_{12}c_{23}c_{13}^2 \sin \delta_l,$$

where $s_{12}, s_{13},$ and $s_{23}$ are sines of the mixing angles $\theta_{12}, \theta_{13},$ and $\theta_{23}$.

The effective Majorana mass to be measured in a $\beta\beta_{0v}$ decay experiment is given as

$$\langle m_{ee} \rangle = |m_1 U^*_{11} + m_2 U^2_{12} + m_3 U^2_{13}|.$$

The obtained ranges of $J_l, \delta_l,$ and $\langle m_{ee} \rangle$ are in agreement with the ranges obtained in other such analyses [73–85,95]. It may be mentioned that the results corresponding to other parallel combinations, e.g. $I_b I_b$, $I_c I_c$, $I_d I_d$, $I_e I_e$, and $I_f I_f$ are exactly the same as $I_a I_a$ and hence are not given in the table. Similarly, Category 1 also has six non-parallel viable combinations such as $I_b I_b$, similar in predictions to combinations $I_b I_a$, $I_c I_f$, $I_f I_c$, $I_d I_c$, $I_d I_e$, given in the second row of Table 2. It is clear from the table that the results for the non-parallel combinations are similar to the parallel combinations except that the obtained range of $\theta_{23}$ is above its maximal value. Further, Categories 2 and 3 do not have any viable combination, because of only two generations mixing in the neutrino mass matrix. Lastly, Category 4 does have four viable combinations such as $I_f I_a$, similar in predictions to $I_b I_b$, and $I_b I_b$, similar in predictions to $I_f I_b$, given respectively in the third and fourth rows of Table 2. The above two sets can be distinguished again on the basis of $\theta_{23}$, as combination $\Pi_f I_a$ gives $\theta_{23}$ below the maximal value while $\Pi_f I_b$ gives $\theta_{23}$ above its maximal value. All four combinations of Category 4 lead to a constrained range of $\theta_{13}$, i.e. $7^\circ–9^\circ$, very much in compliance with the latest data. It may be noted that although most of the phenomenological implications of the abovementioned 16 texture six zero lepton mass matrices are similar, these matrices can still be experimentally distinguished with more precise measurements of $\theta_{23}$ and $\theta_{13}$. The abovementioned texture combinations are found to be compatible with the current data even when the inputs are at their $2\sigma$ C.L.

Similarly, the viability of texture six zero Hermitian lepton mass matrices can be checked for inversely hierarchical Majorana neutrinos. Our analysis shows that inverted hierarchy as well as degenerate neutrinos are completely ruled out for Hermitian texture six zero lepton mass matrices.

6. Dirac neutrinos

Coming to the case of Dirac neutrinos, we have again analyzed 144 combinations for normal hierarchy, inverted hierarchy, and degenerate neutrinos. For comparison with the Majorana neutrino case, we pick up the same non-Fritzsch-like combination given in Eq. (19), to check its compatibility with the latest neutrino mixing data. The diagonalizing transformations for these matrices can easily be obtained in terms of neutrino masses $m_{\nu_1}, m_{\nu_2},$ and $m_{\nu_3}$ for both normal and inverted hierarchies,
and are given in Appendix A. The phase matrix $P_{\nu D}$ and the scanned ranges of lightest neutrino mass and phases $\phi_1$, $\phi_2$, and $\phi_3$ have already been mentioned above.

To check the compatibility of this particular combination for the normal hierarchy of Dirac neutrinos, in Fig. 2 we have plotted the allowed parameter space for $\theta_{12}$ and $\theta_{23}$ in the $m_{\nu_1}$–$\theta_{13}$ plane, represented respectively by dots and crosses. A general look at the figure shows that there is no common parameter space available to $\theta_{12}$ and $\theta_{23}$. Moreover, the obtained $\theta_{13}$ range is well below the experimental range, so that this particular combination is not viable for Dirac neutrinos. This result remains unaffected even if the input parameter ranges are extended further. Thus, normally hierarchical Dirac neutrinos are ruled out for the texture combination given in Eq. (19). Similarly, one can also show that degenerate and inversely hierarchical Dirac neutrinos for this particular combination are also ruled out.

The combinations which are viable for Majorana neutrinos are not viable for Dirac neutrinos primarily because of mixing angle $\theta_{23}$. For example, for parallel combinations given in the first row, as well as the combinations given in the third row of Table 2, $\theta_{23}$ for Dirac neutrinos comes out to be below the experimental limits. Similarly, for the non-parallel combinations given in the second row and the combinations given in the fourth row of Table 2, $\theta_{23}$ lies above its experimental limits.

A similar analysis carried out for the rest of the 128 combinations shows that there are no viable texture six zero lepton mass matrices for normally hierarchical, inversely hierarchical, as well as for degenerate Dirac neutrinos, thus ruling out Dirac neutrinos completely for texture six zero mass matrices.

For the sake of completeness, we have also analyzed the cases corresponding to charged leptons being in the flavor basis for Dirac as well as Majorana neutrinos, and one finds that none of matrices give results within the experimental ranges.

7. Summary and conclusion

To summarize, we have analyzed 144 combinations corresponding to Hermitian texture six zero lepton mass matrices to check for their viability against current neutrino oscillation data. For each
combination of mass matrices, various cases have been considered in the analysis, for example, normal hierarchy, inverted hierarchy, and degenerate neutrinos for both Majorana as well as Dirac neutrinos. For Majorana neutrinos with normal hierarchy, out of 144, only 16 combinations are compatible with current neutrino oscillation data at 3$\sigma$ C.L. The abovementioned texture combinations are found to be compatible with current data even at 2$\sigma$ level. The 16 viable combinations can be grouped in four sets such that the matrices placed in each set are similar with regards to their predictions for lepton masses and flavor mixing parameters. It is important to note that these four sets can be experimentally distinguished from each other with more precise measurements of $\theta_{13}$ and $\theta_{23}$. The ranges of neutrino masses, the PMNS matrix, Jarlskog’s rephasing parameter $J_l$, Dirac-like CP-violating phase $\delta_l$, and effective neutrino mass ($m_{ee}$), calculated for each of the viable combinations, are in agreement with the ranges obtained in other such analyses. We find that inverted hierarchy and degenerate neutrinos are ruled out for texture six zero Majorana neutrinos. Interestingly, for Dirac neutrinos none of the 144 combinations is viable for normal hierarchy, inverted hierarchy, as well as degenerate neutrinos, thus ruling out Dirac neutrinos for texture six zero lepton mass matrices.

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Appendix A. The diagonalization transformations

The diagonalization transformations for the real texture three zero mass matrices given in Eq. (19) are as follows:

(1) The diagonalizing matrix $O_l$ for the real texture three zero mass matrix $M'_l$,

$$M'_l = \begin{pmatrix} C_l & 0 & 0 \\ 0 & D_l & A_l \\ 0 & A_l & 0 \end{pmatrix},$$

is given as

$$O_l = \begin{pmatrix} 1 & 0 & 0 \\ \frac{m_\mu}{m_\mu + m_\tau} & \frac{m_\tau}{m_\mu + m_\tau} & \frac{1}{2} \\ 0 & \frac{m_\tau}{m_\mu + m_\tau} & \frac{1}{2} \end{pmatrix},$$

where $m_\mu$ and $m_\tau$ are the masses of the charged leptons $\mu$ and $\tau$.

(2) The diagonalizing matrix $O_{vD}$ for the real texture three zero mass matrix $M'_{vD}$,

$$M'_{vD} = \begin{pmatrix} 0 & 0 & A_{vD} \\ 0 & C_{vD} & B_{vD} \\ A_{vD} & B_{vD} & 0 \end{pmatrix},$$

for the normal hierarchy of neutrinos, is given as

$$O_{vD} = \begin{pmatrix} m_1 m_2 (m_3 - m_2) & m_1 m_3 (m_1 + m_3) & m_1 m_2 m_3 (m_1 + m_2) & m_1 m_2 m_3 (m_1 + m_2) & m_1 m_2 m_3 (m_1 + m_2) \\ m_1 m_2 (m_3 - m_2) & m_1 m_3 (m_1 + m_3) & m_1 m_2 m_3 (m_1 + m_2) & m_1 m_2 m_3 (m_1 + m_2) & m_1 m_2 m_3 (m_1 + m_2) \\ m_1 m_2 (m_3 - m_2) & m_1 m_3 (m_1 + m_3) & m_1 m_2 m_3 (m_1 + m_2) & m_1 m_2 m_3 (m_1 + m_2) & m_1 m_2 m_3 (m_1 + m_2) \\ m_1 m_2 (m_3 - m_2) & m_1 m_3 (m_1 + m_3) & m_1 m_2 m_3 (m_1 + m_2) & m_1 m_2 m_3 (m_1 + m_2) & m_1 m_2 m_3 (m_1 + m_2) \\ m_1 m_2 (m_3 - m_2) & m_1 m_3 (m_1 + m_3) & m_1 m_2 m_3 (m_1 + m_2) & m_1 m_2 m_3 (m_1 + m_2) & m_1 m_2 m_3 (m_1 + m_2) \end{pmatrix}^{1/2}.$$
Similarly, for the inverted hierarchy of neutrinos, $O_D$ is given as

$$
O_D = \begin{pmatrix}
\left(\frac{m_3 m_1 (m_1 + m_2)}{m_3 (m_1 + m_2)}\right)^{1/2}
- \left(\frac{|m_1 m_3 (m_1 - m_2)|}{m_3 (m_1 + m_2)}\right)^{1/2}
- \left(\frac{m_3 m_1 (m_1 - m_2)}{m_3 (m_1 + m_2)}\right)^{1/2}
- \left(\frac{m_1 m_3 (m_1 - m_2)}{m_3 (m_1 + m_2)}\right)^{1/2},
\end{pmatrix}
$$

where $m_1 = m_{\nu_1}$, $m_2 = m_{\nu_2}$ and $m_3 = m_{\nu_3}$ for Dirac neutrinos and $m_1 = \sqrt{m_{\nu_1} m_R}$, $m_2 = \sqrt{m_{\nu_2} m_R}$ and $m_3 = \sqrt{m_{\nu_3} m_R}$ for Majorana neutrinos.

### Appendix B. $M_R$ with three texture zeros

In this Appendix we present some details of the texture six zero mass matrices, wherein, along with $M_I$ and $M_{V_D}$, we also impose a three zero structure on $M_R$. $M_{I}$ and $M_{V_D}$ can be any of the 12 matrices given in Table 1, while we choose $M_R$ to be similar in texture structure to $M_{V_D}$, thereby yielding 144 combinations of $M_I$ and $M_V$ which must be compared with the neutrino oscillation data given in Eqs. (17) and (18). These 144 combinations can be divided into four categories:

- **Category 1**: $M_I$ from Class I, $M_{V_D}$ and $M_R$ from Class I.
- **Category 2**: $M_I$ from Class II, $M_{V_D}$ and $M_R$ from Class II.
- **Category 3**: $M_I$ from Class I, $M_{V_D}$ and $M_R$ from Class II.
- **Category 4**: $M_I$ from Class II, $M_{V_D}$ and $M_R$ from Class I.

As an example, we choose a texture combination $II_b I\nu_e$ from Category 4, where $M_I$ is of type $II_b$ and $M_{V_D}$ and $M_R$ are of type $I\nu_e$, as given below:

$$
M_I = \begin{pmatrix}
0 & 0 & A_l \\
0 & C_l & 0 \\
A_{l*} & 0 & B_l
\end{pmatrix},
M_{V_D} = \begin{pmatrix}
0 & B_{V_D} & A_{V_D} \\
B_{V_D}^* & C_{V_D} & 0 \\
A_{V_D}^* & 0 & 0
\end{pmatrix},
M_R = \begin{pmatrix}
0 & B_R & A_R \\
B_R & C_R & 0 \\
A_R & 0 & 0
\end{pmatrix}.
$$

(B1)

For simplicity, we have neglected the Majorana phases in the right-handed neutrino mass matrix. Using seesaw Eq. (2), the effective neutrino mass matrix $M_v$ is given as

$$
M_v = \begin{pmatrix}
D_v & B_v & A_v \\
B_v & C_v & 0 \\
A_v & 0 & 0
\end{pmatrix},
$$

(B2)

where

$$
A_v = |A_v| = -\frac{|A_{V_D}|^2}{A_R}
$$

(B3)

$$
B_v = -\frac{A_{BR}B_{V_D}^*C_{V_D} - A_{V_D}^*B_RC_{V_D} + A_{V_D}^*B_{V_D}C_{R}}{A_RC_R}
$$

(B4)

$$
C_v = |C_v| = -\frac{C_{V_D}^2}{C_R}
$$

(B5)

$$
D_v = -\frac{(A_{BR}B_{V_D}^* - A_{V_D}^*B_RC_R)^2}{A_R^2C_R}
$$

(B6)

It is evident that $M_v$ turns out to be of texture two zero type. However, if the condition $A_{BR}B_{V_D}^* = A_{V_D}^*B_R$ is satisfied, it becomes texture three zero, i.e. the texture structure is preserved by the seesaw mechanism in that case. We find that such a simplified assumption does not yield viable results for the matrices given in Eq. (B1).
The matrices $M_l$ and $M_\nu$ given above can easily be diagonalized by bi-unitary transformations, and the corresponding PMNS matrix can be constructed as explained in Sect. 2. The real diagonalizing matrix $O_l$ for $M^\dagger_l$ is given as

$$
O_l = \begin{pmatrix}
\left( \frac{m_e}{m_e + m_\tau} \right)^{\frac{1}{2}} & 0 & \left( \frac{m_\tau}{m_e + m_\tau} \right)^{\frac{1}{2}} \\
0 & 1 & 0 \\
-\left( \frac{m_\tau}{m_e + m_\tau} \right)^{\frac{1}{2}} & 0 & \left( \frac{m_e}{m_e + m_\tau} \right)^{\frac{1}{2}}
\end{pmatrix},
$$

(B7)

where $m_e$ and $m_\tau$ are the masses of the charged leptons $e$ and $\tau$. Similarly, the real diagonalizing matrix $O_\nu$ for $M^\dagger_\nu$ is given as,

$$
O_\nu = \begin{pmatrix}
\left( \frac{m_{11}(C_\nu-m_{11})}{(m_{11}+m_{12})(m_{12}-m_{13})} \right)^{\frac{1}{2}} & \left( \frac{m_{12}(C_\nu-m_{12})}{(m_{11}+m_{12})(m_{12}+m_{13})} \right)^{\frac{1}{2}} & \left( \frac{m_{13}(C_\nu-m_{13})}{(m_{11}-m_{13})(m_{12}+m_{13})} \right)^{\frac{1}{2}} \\
\left( \frac{m_{11}(C_\nu+m_{11})}{(m_{11}+m_{12})(m_{12}+m_{13})} \right)^{\frac{1}{2}} & \left( \frac{m_{12}(C_\nu+m_{12})}{(m_{11}+m_{12})(m_{12}-m_{13})} \right)^{\frac{1}{2}} & \left( \frac{m_{13}(C_\nu+m_{13})}{(m_{11}-m_{13})(m_{12}-m_{13})} \right)^{\frac{1}{2}} \\
\left( \frac{m_{11}(C_\nu-m_{11})}{(m_{11}-m_{13})(m_{12}-m_{13})} \right)^{\frac{1}{2}} & \left( \frac{m_{12}(C_\nu+m_{12})}{(m_{11}-m_{13})(m_{12}+m_{13})} \right)^{\frac{1}{2}} & \left( \frac{m_{13}(C_\nu-m_{13})}{(m_{11}+m_{13})(m_{12}+m_{13})} \right)^{\frac{1}{2}}
\end{pmatrix}
$$

(B8)

The mixing matrix $U$ may be constructed by using the equation,

$$
U = O_l^\dagger P_{1l}O_\nu,
$$

(B9)

where $P_{1l}$ is the phase matrix, which in general can be taken as $(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$, $\phi_1$, $\phi_2$, and $\phi_3$ being related to the phases of the complex mass matrices $M_l$ and $M_\nu$.

Thus, in the mixing matrix, apart from the phases $\phi_1$, $\phi_2$, and $\phi_3$, an additional free parameter $C_\nu$ appears. As before, the phases have been given full variation from 0 to $2\pi$, while we allow $C_\nu$ to vary between $m_{11}$ and $m_{13}$, such that the diagonalizing transformation given in Eq. (B8) remains real. Our results, with regards to mixing angles, CP-violating phase $\delta_l$, and effective neutrino mass $\langle m_{ee}\rangle$, are given as $\theta_{12} \approx 31^\circ-37^\circ$, $\theta_{13} \approx 7^\circ-9^\circ$, $\theta_{23} \approx 44^\circ-45^\circ$, $\delta_l \approx 0-8^\circ$, $\langle m_{ee}\rangle \approx 0.0054 - 0.0074$, indicating that the matrices given in Eq. (B1) accommodate the unsuppressed $\theta_{13}$ very well. The angle $\theta_{23}$ is maximal; however, the CP-violating phase $\delta_l$ is rather small for the combination $\Pi_{I1}I_e$.

Similarly, for Category 4, there are 11 more combinations of $M_l$, $M_\nu$, and $M_R$ which accommodate the data given in Eqs. (17) and (18), for example $\Pi_{I1}I_{d1}I_d$, $\Pi_{I1}I_{d2}I_{d2}$, $\Pi_{I1}I_{d3}I_{d3}$, $\Pi_{I2}I_{d1}I_{d1}$, $\Pi_{I2}I_{d2}I_{d2}$, $\Pi_{I2}I_{d3}I_{d3}$, $\Pi_{I3}I_{d1}I_{d1}$, $\Pi_{I3}I_{d2}I_{d2}$, $\Pi_{I3}I_{d3}I_{d3}$, and $\Pi_{I4}I_{d1}I_{d1}$. Most of these combinations have similar predictions with regards to $\theta_{12}$ and $\theta_{13}$, although the allowed ranges of $\theta_{23}$ and $\delta_l$ vary.

One of the purposes of the above analysis is to compare with the case of diagonal $M_R$ discussed in detail in Sect. 5. We find that with texture three zero $M_R$, apart from the combinations given in Table 2, several new textures, as given above, also become viable. Unlike the diagonal $M_R$ case, we do not get any viable combination from Category 1, while, similarly to the diagonal $M_R$ case, Categories 2 and 3 do not accommodate the experimental data. To summarise this Appendix, we can say that the set of three zero mass matrices for $M_l$, $M_\nu$, and $M_R$ that can explain the oscillation data is not unique. More refined measurements of mixing angles, particularly $\theta_{23}$, and also a measurement of CP violation in the leptonic sector, can help isolate the unique texture structure for charged leptons and neutrinos.

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